CONVERGENCE OF RIEMANNIAN 4-MANIFOLDS WITH $L^2$-CURVATURE BOUNDS

NORMAN ZERGAENGE

Abstract. In this work we prove convergence results of sequences of Riemannian 4-manifolds with almost vanishing $L^2$-norm of a curvature tensor and a non-collapsing bound on the volume of small balls.

In Theorem 1.1, we consider a sequence of closed Riemannian 4-manifolds, whose $L^2$-norm of the Riemannian curvature tensor tends to zero. Under the assumption of a uniform non-collapsing bound and a uniform diameter bound, we prove that there exists a subsequence that converges with respect to the Gromov-Hausdorff topology to a flat manifold.

In Theorem 1.2, we consider a sequence of closed Riemannian 4-manifolds, whose $L^2$-norm of the Riemannian curvature tensor is uniformly bounded from above, and whose $L^2$-norm of the traceless Ricci-tensor tends to zero. Here, under the assumption of a uniform non-collapsing bound, which is very close to the euclidean situation, and a uniform diameter bound, we show that there exists a subsequence which converges in the Gromov-Hausdorff sense to an Einstein manifold.

In order to prove Theorem 1.1 and Theorem 1.2, we use a smoothing technique, which is called $L^2$-curvature flow or $L^2$-flow, introduced by Jeffrey Streets in the series of works [Str08], [Str12b], [Str12a], [Str13a], [Str13b] and [Str16]. In particular, we use his “tubular averaging technique”, which he has introduced in [Str16, Section 3], in order to prove distance estimates of the $L^2$-curvature flow which only depend on significant geometric bounds. This is the content of Theorem 1.3.

Contents
1. Introduction and statement of results 1
2. Distance control under the $L^2$-flow in 4 dimensions 5
  2.1. Tubular neighborhoods 6
  2.2. Forward estimates 11
  2.3. Backward estimates 18
3. Proof of Theorem 1.1 22
4. Proof of Theorem 1.2 24
Appendix A. Auxiliary Results 32
Appendix B. Notation 35
References 36

I. INTRODUCTION AND STATEMENT OF RESULTS

In order to approach minimization problems in Riemannian geometry, it is often useful to know if a minimizing sequence of smooth Riemannian manifolds contains a subsequence that converges with respect to an appropriate topology to a sufficiently smooth space. Here, in general, the minimization problem refers to a
certain geometric functional, for instance the area functional, the total scalar curvature functional, the Willmore functional or the $L^p$-norm of a specific curvature tensor on a Riemannian manifold, to name just a few. Latter functionals are the main interest in this work. That means that we consider sequences of Riemannian manifolds that have a uniform $L^p$-bound on the full curvature tensor, the Ricci tensor and the traceless Ricci tensor respectively.

Naturally, the situation is more transparent, if we have more precise information about the $L^p$-boundedness of curvature tensors of the underlying Riemannian manifolds, that is, that we have a uniform $L^p$-bound, where $p \in [1, \infty]$ is large. In particular, a uniform $L^\infty$-bound should give the most detailed information about geometric quantities.

One of the basic results in this context is stated in [And89, Theorem 2.2, p. 464-466]. Here, for instance, one assumes a uniform $L^\infty$-bound on the full Riemannian curvature tensor, a uniform lower bound on the injectivity radius and a uniform two sided bound on the volume, to show the existence of a subsequence that converges with respect to the $C^{0,\alpha}$-topology to a Riemannian manifold of regularity $C^{1,\alpha}$. The proof uses the fact, that it is possible to find uniform coverings of the underlying manifolds with harmonic charts, which follows from [JK82].

In [Yan92c], Deane Yang has considered sequences of Riemannian manifolds satisfying a suitable uniform $L^p$-bound on their full Riemannian curvature tensors, where $p > \frac{n}{2}$, and a uniform bound on the Sobolev constant. In order to show compactness and diffeomorphism finiteness results, he examines Hamilton’s Ricci flow (cf. [Ham82], [CLN06] and [Top06]) and he shows curvature decay estimates and existence time estimates that only depend on the significant geometric bounds.

In [Yan92a] and [Yan92b], Deane Yang has approached a slightly more general problem. Here, he has considered sequences of Riemannian $n$-manifolds, $n \geq 3$, having a uniform $L^{\frac{n}{2}}$-bound on their full Riemannian curvature tensors and a suitable uniform $L^p$-bound on their Ricci tensors instead of a uniform $L^p$-bound on their full Riemannian curvature tensors, where $p > \frac{n}{2}$. Due to the scale invariance of the bound on the Riemannian curvature tensors - we name such bound a “critical curvature bound”- the situation becomes much more difficult, than in the “supercritical” case, that is, when $p$ is bigger than $\frac{n}{2}$. In particular, in general, it is doubtful whether the global Ricci flow is applicable in this situation.

In [Yan92a], the author has introduced the idea of a “local Ricci flow” which is, by definition, equal to the Ricci flow weighted with a truncation function that is compactly contained in a local region of a manifold. The author shows that on regions, where the local $L^{\frac{n}{2}}$-norm of the full Riemannian curvature tensor is sufficiently small, the local Ricci flow satisfies curvature decay estimates and existence time estimates that only depend on significant local geometric bounds. So, on these “good”, regions one may apply [And89, Theorem 2.2, pp. 464-466] to a slightly mollified metric, to obtain local compactness with respect to the $C^{0,\alpha}$-topology. Since the number of local regions having too large $L^{\frac{n}{2}}$-norm of the full Riemannian curvature tensor is uniformly bounded, the author is able to show that each sequence of closed Riemannian manifolds, satisfying a uniform diameter bound, a uniform non-collapsing bound on the volume of small balls, a uniform bound on the $L^{\frac{n}{2}}$-norm of the full Riemannian curvature tensor and a sufficiently small uniform bound on the $L^p$-norm of the Ricci curvature tensor, where $p > \frac{n}{2}$, contains a subsequence that converges in the Gromov-Hausdorff sense to a metric space, which is, outside of a finite set of points, an open $C^1$-manifold with a Riemannian metric of regularity $C^0$. 
In [Yan92b], the author has used the local Ricci flow to find a suitable harmonic chart around each point in whose neighborhood the local $L^{\frac{2n}{n-2}}$-norm of the full Riemannian curvature tensor and the local $L^p$-norm of the full Riemannian curvature tensor, where $p > \frac{n}{2}$, is not too large. Using these estimates, the author is able to improve the statements about the convergence behavior in the convergence results in [Yan92a] on regions having a sufficiently small curvature concentration.

It seems so, that the reliability of the Ricci flow in [Yan92c], and the local Ricci flow in [Yan92a] and [Yan92b] is based on the appearance of the supercritical curvature bounds. For instance, in order to develop the parabolic Moser iteration in [Yan92c] and [Yan92a] one uses a well-controlled behavior of the Sobolev constant. As shown in [Yan92a, 7, pp. 85-89] this behavior occurs, if one assumes suitable supercritical bounds on the Ricci curvature. The examples in [Aub07, Section 9, pp. 690-694] show that the critical case is completely different.

Another important issue is the absence of important comparison geometry results under critical curvature bounds. In order to understand the rough structure of Riemannian manifolds, satisfying a fixed lower bound on the Ricci tensor, one uses the well-known “Bishop-Gromov volume comparison theorem” (cf. [Pet06, 9.1.2., pp. 268-270]) which allows a one-directed volume comparison of balls in Riemannian manifolds satisfying a fixed lower Ricci curvature bound with the volume of balls in a such called “space form”, (cf. [Lee97, p. 206]), which is a complete, connected Riemannian manifold with constant sectional curvature. Later, in [PW97], Peter Petersen and Guofang Wei have shown that it is possible to generalize this result to the situation, in that an $L^p$-integral of some negative part of the Ricci tensor is sufficiently small. Here the authors assume that $p$ is bigger that $\frac{n}{2}$.

It seems that the treatment of Riemannian manifolds with pure critical curvature bounds needs to be based on methods that are different from the approaches we have just mentioned. Instead of considering the Ricci flow, which is closely related to the gradient flow of the Einstein-Hilbert functional (cf. [CLN06, Chapter 2, Section 4, pp. 104-105]), one could try to deform a Riemannian manifold of dimension 4 into the direction of the negative gradient of the $L^2$-integral of the full curvature tensor, in order to analyze slightly deformed approximations of the initial metric, having a smaller curvature energy concentration. This evolution equation was examined by Jeffrey Streets in [Str08], [Str12b], [Str12a], [Str13a], [Str13b], [Str16]. In this series of works, J. Streets has proved a plenty of properties of this geometric flow and he also shows a couple of applications.

Using J. Streets technique, we show compactness results for Riemannian 4-manifolds, that only assume a uniform diameter bound, a uniform non-collapsing bound on the volume of sufficiently small balls and critical curvature bounds.

In the first theorem, we consider a sequence of Riemannian 4-manifolds having almost vanishing Riemannian curvature tensor in some rough sense and we show that a subsequence converges with respect to the Gromov-Hausdorff topology to a flat Riemannian manifold:

**Theorem 1.1.** Given $D, d_0 > 0$, $\delta \in (0, 1)$ and let $(M_i, g_i)_{i \in \mathbb{N}}$ be a sequence of closed Riemannian 4-manifolds, satisfying the following assumptions:

\[
\begin{align*}
    d_0 &\leq \text{diam}_{g_i}(M_i) \leq D & \forall i \in \mathbb{N} \\
    \text{Vol}_{g_i}(B_{g_i}(x, r)) &\geq \delta \omega_4 r^4 & \forall i \in \mathbb{N}, x \in M_i, \forall r \in [0, 1] \\
    \|Rm_{g_i}\|_{L^2(M_i, g_i)} &\leq \frac{1}{\tau} & \forall i \in \mathbb{N}
\end{align*}
\]

then, there exists a subsequence $(M_{i_j}, d_{g_{i_j}})_{j \in \mathbb{N}}$ that converges in the Gromov-Hausdorff sense to a smooth flat manifold $(M, g)$. 

Throughout, a closed Riemannian is defined to be a smooth, compact and connected oriented Riemannian manifold without boundary.

In the second theorem, we consider a sequence of Riemannian 4-manifolds with uniformly bounded curvature energy and almost vanishing traceless Ricci tensor in some rough sense. Under these assumptions, we show that a subsequence converges with respect to the Gromov-Hausdorff topology to an Einstein manifold, provided that the volume of small balls behaves almost euclidean:

**Theorem 1.2.** Given \( D, d_0, \Lambda > 0 \), there exists a universal constant \( \delta \in (0, 1) \) close to 1 so that if \((M_i, g_i)_{i \in \mathbb{N}}\) is a sequence of closed Riemannian 4-manifolds satisfying the following assumptions:

\[
\begin{align*}
\text{diam}_{g_i}(M_i) &\leq D \quad \forall i \in \mathbb{N} \\
\|\text{Rm}_{g_i}\|_{L^2(M, g_i)} &\leq \Lambda \quad \forall i \in \mathbb{N} \\
\|\hat{\text{Rc}}_{g_i}\|_{L^2(M, g_i)} &\leq \frac{1}{\delta} \quad \forall i \in \mathbb{N} \\
\text{Vol}_{g_i}(B_{g_i}(x, r)) &\geq \delta \omega_4 r^4 \quad \forall i \in \mathbb{N}, x \in M_i, r \in [0, 1]
\end{align*}
\]

then there exists a subsequence \((M_{i_j}, d_{g_{i_j}})_{j \in \mathbb{N}}\) that converges in the Gromov-Hausdorff sense to a smooth Einstein manifold \((M, g)\).

As mentioned above, it is our aim to show these results, using the negative gradient flow of the following functional:

\[
(1.2) \quad \mathcal{F}(g) := \int_M |\text{Rm}_g|^2_g dV_g
\]

That is, on a fixed sequence element \((M^4, g_0)\), we want to evolve the initial metric in the following manner:

\[
(1.3) \quad \begin{cases}
\frac{\partial}{\partial t} g = -\text{grad} \mathcal{F} = -2\delta d\text{Rc}_g + 2\hat{\text{R}}_g - \frac{1}{2}|\text{Rm}_g|^2_g g \\
g(0) = g_0
\end{cases}
\]

where \(\hat{\text{R}}_{ij} := R^k_{ij} - R^k_{ipqr}\) in local coordinates and the gradient formula, which appears in (1.3) can be found in [Bes87, Chapter 4, 4.70 Proposition, p. 134]. Here, \(d\) denotes the exterior derivative acting on the Ricci tensor and \(\delta\) denotes the adjoint of \(d\). The gradient of a differentiable Riemannian functional is defined in [Bes87, Chapter 4, 4.10 Definition, p. 119].

In [Str08, Theorem 3.1, p. 252] J. Streets has proved short time existence of the flow given by (1.3) on closed Riemannian manifolds. The author has also proved the uniqueness of the flow (cf. [Str08, Theorem 3.1, p. 252]). In this regard, the expression “the”, \(L^2\)-flow makes sense. In [Str16, Theorem 1.8, p. 260] J. Streets has proved, that under certain assumptions, the flow given by (1.3) has a solution on a controlled time interval and the solution satisfies certain curvature decay and injectivity radius growth estimates.

In Section 2, we use J. Streets ideas, in order to show that, under certain assumptions, the distance between two points does not change too much along the flow. This allows us to bring the convergence behavior of a slightly mollified manifold back to the initial sequence. That means we will prove the following theorem:
Theorem 1.3. Let $(M^4, g_0)$ be a closed Riemannian 4-manifold. Suppose that $(M, g(t))_{t \in [0,1]}$ is a solution to (1.3) satisfying the following assumptions:

\begin{align}
(1.4) \quad & \int_M |\text{Rm}_{g(t)}|^2 dV_{g(t)} \leq \Lambda \\
(1.5) \quad & \|\text{Rm}_{g(t)}\|_{L^\infty(M,g(t))} \leq K t^{\frac{1}{2}} \quad \forall t \in (0,1] \\
(1.6) \quad & \text{inj}_g(M) \geq t^{\frac{1}{2}} \quad \forall t \in [0,1] \\
(1.7) \quad & \text{diam}_{g(t)}(M) \leq 2(1+D) \quad \forall t \in [0,1] \\
\end{align}

Then we have the following estimate:

\begin{equation}
(1.8) \quad |d(x,y,t_2) - d(x,y,t_1)| \leq C(K,\iota,D) \|A\|_2 \left( t^{\frac{1}{2}} - t^{\frac{1}{8}} \right)^\frac{1}{2} + C(K,\iota,D) \left( t^{\frac{1}{2}} - t^{\frac{1}{8}} \right)^\frac{1}{2}
\end{equation}

for all $t_1, t_2 \in [0,1]$ where $t_1 < t_2$.

These estimates allow one to prove Theorem 1.1 and Theorem 1.2 which are the main goals of Section 3 and Section 4. Here, in Section 3, we may refer to the estimates in [Str16, 1.3, Theorem 1.8, p. 260]. In Section 4, we write down an existence result which allows to apply Theorem 1.3 to the elements of the sequence occurring in Theorem 1.2.

This work is a part of the author’s doctoral thesis ([Zer17]), written under the supervision of Miles Simon at the Otto-von-Guericke-Universität Magdeburg.

2. Distance control under the $L^2$-flow in 4 dimensions

In this section we prove Theorem 1.3. In order to prove this Theorem we use the “tubular averaging technique” from [Str16, Section 3, pp. 269-282]. The method is derived from [Str16, Section 3]. In Subsection 2.3, we apply the “tubular averaging technique” to the time-reversed flow. For the sake of understanding, we give detailed explanations of the steps in the proof, even if the argumentation is based on the content of [Str16, Section 3]. In order to get a very rough feeling for J. Streets “tubular averaging technique” we recommend to read the first paragraph of [Str16, p. 270] and the suggested references.

The proof of Theorem 1.3 is divided in two principal parts:

In the first part of this section we show that, along the flow, the distance between two points in manifold $M$ does not increase too much, i.e.: we derive the estimates of the shape $d(x,y,t) < d(x,y,0) + \epsilon$ for small $t(\epsilon) > 0$. We say that this kind of an estimate is a “forward estimate”.

The second part in this section is concerned with the opposite direction, i.e: we show that, along the flow, the distance between two points does not decay too much, which means that we have $d(x,y,t) > d(x,y,0) - \epsilon$ for $t(\epsilon) > 0$ sufficiently small.

We point out that the estimate of the length change of a vector $v \in TM$ along a geometric flow usually requires an integration of the metric change $|g'(t)|_{g(t)}$ from 0 to a later time point $T$ (cf. (A.2)). With a view to (1.3) and (1.5) we note that, on the first view, this would require and integration of the function $t^{-1}$ from 0 to $T$ which is not possible.

In order to overcome this difficulty, we follow the ideas in [Str16, Section 3], i.e. we introduce some kind of connecting curves which have almost the properties of geodesics. Then we construct an appropriate tube around each of these connecting curves so that the integral $\int_\gamma |\text{grad} F| d\sigma$, which occurs in the estimate of $\left| \frac{d}{dt} L(\gamma,t) \right|$ (cf. (A.1)), can be estimated from above against a well-controlled average integral along the tube plus an error integral which behaves also well with respect to $t$. We point out that we do not widen J. Streets ideas in [Str16, Section 3] by fundamental...
facts, we merely write down detailed information which allow to understand the distance changing behavior of J. Streets $L^2$-flow in a more detailed way.

2.1. **Tubular neighborhoods.** We quote the following definition from [Str16, Definition 3.3., pp. 271-272]

**Definition 2.1.** Let $(M^n, g)$ be a smooth Riemannian manifold without boundary, and let $\gamma : [a, b] \to M$ be an smooth curve. Given $r > 0$, and $s \in [a, b]$ then we define

$$D(\gamma(s), r) := \exp_{\gamma(s)} \left( B(0, r) \cap \langle \dot{\gamma}(s) \rangle^\perp \right)$$

and

$$D(\gamma, r) := \bigcup_{s \in [a, b]} D(\gamma(s), r)$$

We say “$D(\gamma, r)$ is foliated by $(D(\gamma(s), r))_{s \in [a, b]}$” if

$$D(\gamma(s_1), r) \cap D(\gamma(s_2), r) = \emptyset$$

for all $a \leq s_1 < s_2 \leq b$.

The following definition is based on [Str16, Definition 2.2., p. 267].

**Definition 2.2.** Let $(M^n, g)$ be a closed Riemannian manifold, $k \in \mathbb{N}$ and $x \in M$, then we define

$$f_k(x, g) := \sum_{j=0}^{k} |\nabla^j Rm|_g^{\frac{1}{2}}(x)$$

and

$$f_k(M, g) := \sup_{x \in M} f_k(x, g)$$

At this point we refer to the scaling behavior of $f_k(x, g)$ which is outlined in Lemma A.2.

The following result is a slight modification of [Str16, Lemma 3.4., pp. 272-274]. To be more precise: in this result we allow the considered curve to have a parametrization close to unit-speed, and not alone unit-speed.

**Lemma 2.3.** Given $n, D, K, \iota > 0$ there exists a constant $\beta(n, D, K, \iota) > 0$ and a constant $\mu(n) > 0$ so that if $(M^n, g)$ is a complete Riemannian manifold satisfying

$$\text{diam}_g(M) \leq D$$

$$f_4(M^n, g) \leq K$$

$$\text{inj}_g(M) \geq \iota$$

and $\gamma : [0, L] \to M$ is an injective smooth curve satisfying

$$(2.1) \quad L(\gamma) \leq d(\gamma(0), \gamma(L)) + \beta$$

$$(2.2) \quad |\nabla_\gamma \dot{\gamma}| \leq \beta$$

$$(2.3) \quad \frac{1}{1 + \beta} \leq |\dot{\gamma}| \leq 1 + \beta$$

then $D(\gamma, R)$ is foliated by $(D(\gamma(s), R))_{s \in [0, L]}$ for $R := \mu \min \left\{ \iota, K^{\frac{1}{4}} \right\}$. Furthermore, if

$$\pi : D(\gamma, R) \to \gamma([0, L])$$
is the projection map sending a point \( q \in D(p, R) \), where \( p \in \gamma([0, L]) \), to \( p \), which is well-defined by the foliation property, then
\[
|d\pi| \leq 2 \quad \text{on } D(\gamma, R)
\]
Here \( d\pi \) denotes the differential and \( |d\pi| \) denotes the operator norm of the differential of the projection map.

Proof. Above all, we want to point out, that, due to the injectivity of the curve, we can construct a tubular neighborhood around \( \gamma([0, L]) \). This is a consequence of [O'N83, 26. Proposition, p. 200]. But the size of this neighborhood is not controlled at first. Via radial projection we can ensure that the velocity field of the curve is extendible in the sense of [Lee97, p. 56]. We follow the ideas of the proof of [Str16, Lemma 3.4, pp. 272-274] with some modifications.

Firstly, we describe how \( \mu(n) > 0 \) needs to be chosen in order to ensure that the curve has a suitable foliation which can be used to define the projection map.

Secondly, we show that the desired smallness condition of the derivative of the projection map is valid, i.e.: we show (2.4). Here we allow \( \mu(n) > 0 \) to become smaller.

Let
\[
(2.5) \quad \mu(n) := \min \left\{ \tilde{\mu}(n), \frac{1}{20}, \frac{1}{64C_1(n)C_2(n)} \right\}
\]
where \( \tilde{\mu}(n) > 0 \) and \( C_1(n) > 0 \) are taken from [Str16, Lemma 2.9, p. 268] and \( C_2(n) > 0 \) will be made explicit below. Let
\[
R := \mu \min \left\{ \iota, K^{-\frac{1}{2}} \right\}
\]
Suppose there exists a point \( p \in D(\gamma(s_0), R) \cap D(\gamma(s_1), R) \) where \( s_0, s_1 \in [0, L] \), \( s_0 < s_1 \) and \( s_1 - s_0 \leq 10R \) at first. By definition, there exists a normal chart of radius \( 20R \) around \( p \) (cf. [Lee97, Theorem 6.8., pp. 102-103]). In this chart we have the following estimate
\[
(2.6) \quad \sup_{B_{20R}(p)} \mu K^{-\frac{1}{2}} |\Gamma| \leq \frac{1}{64C_2(n)}
\]
Choosing \( \beta \in (0, 1) \) small enough compared to \( R \) we ensure that \( \gamma([s_0, s_1]) \) lies in this chart. From [Lee97, Theorem 6.8., pp. 102-103] we obtain
\[
\left. \left( \frac{\partial}{\partial r} \gamma \right) \right|_{\gamma(s_0)} = 0
\]
where
\[
(2.7) \quad \left. \frac{\partial}{\partial r} \right|_{\gamma(s)} := \frac{\gamma^1(s)}{r(\gamma(s))} \partial_1|_{\gamma(s)}
\]
and \( \partial_1, ..., \partial_n \) denote the coordinate vector fields and \( \gamma^1, ..., \gamma^n \) denote the coordinates of \( \gamma \) in this normal chart and
\[
r(\gamma(s)) := \sqrt{\sum_{i=1}^{n} (\gamma^i(s))^2}
\]
(cf. [Lee97, Lemma 5.10, (5.10), p. 77]). We show that it is possible to take \( \beta(n, K, \iota) > 0 \) small enough to ensure that
\[
\left. \left( \frac{\partial}{\partial r} \gamma \right) \right|_{\gamma(s)} \neq 0 \quad \forall s \in (s_0, s_1)
\]
This would be a contradiction to the fact that [Lee97, Theorem 6.8., pp. 102-103] also implies

\[(2.8)\]
\[\left\langle \frac{\partial}{\partial \gamma}, \dot{\gamma} \right\rangle \bigg|_{\gamma(s_1)} = 0\]

From [Lee97, Lemma 5.2 (c), p. 67] we infer on \([s_0, s_1]\]

\[\frac{\partial}{\partial s} \left( \frac{\partial}{\partial \gamma} \dot{\gamma} \right) \bigg|_{\gamma(s)} = \left\langle D_s \frac{\partial}{\partial \gamma}, \dot{\gamma} \right\rangle \bigg|_{\gamma(s)} + \left\langle \frac{\partial}{\partial \gamma}, D_s \dot{\gamma} \right\rangle \bigg|_{\gamma(s)} \geq \left\langle D_s \frac{\partial}{\partial \gamma}, \dot{\gamma} \right\rangle \bigg|_{\gamma(s)} - \left\langle \frac{\partial}{\partial \gamma}, D_s \dot{\gamma} \right\rangle \bigg|_{\gamma(s)} \]

\[(2.9)\]

\[ \geq \left\langle D_s \frac{\partial}{\partial \gamma}, \dot{\gamma} \right\rangle \bigg|_{\gamma(s)} - \frac{\partial}{\partial r} \bigg|_{\gamma(s)} \left[ \nabla_{\gamma(s)} \dot{\gamma}(s) \right]_g \]

Using (2.7) together with [Lee97, Lemma 4.9 (b), p. 57] and [Lee97, p. 56 (4.9)] we calculate

\[D_s \frac{\partial}{\partial r} = \frac{\ddot{\gamma} \cdot \dot{r} - \dot{\gamma} \left( \frac{\partial}{\partial \gamma} \dot{r} \right)}{r^2} \partial_t + \frac{\gamma}{r} D_s \partial_t = \frac{\ddot{\gamma}}{r} \partial_t - \frac{\dot{\gamma} \left( \frac{\partial}{\partial \gamma} \dot{r} \right)}{r^2} \partial_t + \frac{\gamma}{r} D_s \partial_t \]

This implies

\[\left\langle D_s \frac{\partial}{\partial r}, \dot{\gamma} \right\rangle = \frac{1}{r} \left[ \gamma \right]^2 - \frac{1}{r^2} \left[ \gamma \frac{\partial}{\partial r} \right] \left[ \gamma \left( \frac{\partial}{\partial \gamma} \dot{r} \right) \right] + \frac{2}{r} \left( D_s \partial_t, \dot{\gamma} \right)\]

\[\overset{(2.7)}{\approx} \frac{1}{r} \left[ \gamma \right]^2 - \frac{1}{r^2} \left[ \gamma \frac{\partial}{\partial r} \right] \left[ \gamma \left( \frac{\partial}{\partial \gamma} \dot{r} \right) \right] - C_2(n) |\gamma|^2 \bigg| \leq 1 - 4C_2(n) |\gamma|^2 \bigg| \]

Here, in order to obtain the first estimate, we refer to Definition B.1 and the fact that

\[\frac{1}{r(\gamma(t))} \sum_{i=1}^{n} |\gamma^i(t)| \leq \hat{C}(n)\]

Hence, (2.9) implies

\[\frac{\partial}{\partial s} \left( \frac{\partial}{\partial \gamma} \dot{\gamma} \right) \geq \frac{1}{4r} \left[ 1 - 8 \left[ \frac{\partial}{\partial \gamma}, \dot{\gamma} \right] - 16C_2(n) r |\Gamma| - 4\beta r \right] \geq \frac{1}{4r} \left[ 1 - 8 \left[ \frac{\partial}{\partial \gamma}, \dot{\gamma} \right] - 16C_2(n) \mu K^{-\frac{1}{2}} |\Gamma| - 4\mu K^{-\frac{1}{2}} \beta \right] \]

\[(2.10)\]

\[\overset{(2.6)}{\approx} \frac{1}{4r} \left[ 1 - 8 \left[ \frac{\partial}{\partial \gamma}, \dot{\gamma} \right] - \frac{1}{4} - 4\mu K^{-\frac{1}{2}} \beta \right] \]

\[\geq \frac{1}{4r} \left[ 1 - 8 \left[ \frac{\partial}{\partial \gamma}, \dot{\gamma} \right] - \frac{1}{4} - \frac{1}{4} \right] \]

\[= \frac{1}{8r} \left[ 1 - 16 \left[ \frac{\partial}{\partial \gamma}, \dot{\gamma} \right] \right]\]
We show that this differential inequality implies the desired contradiction. Let \( w: [s_0, s_1] \to \mathbb{R} \), \( w(s) := \left( \frac{\partial}{\partial s}, \gamma \right) \left( \gamma(s) \right) \). Then (2.10) is equivalent to
\[
w' \geq \frac{1}{8r}(1 - 16|w|)
\]
on \([s_0, s_1]\). Since \( w(s_0) = 0 \), there exists \( \delta > 0 \) such that \( w' > 0 \) on \([s_0, s_0 + \delta] \). This implies \( w > 0 \) on \((s_0, s_0 + \delta)\). We show that we have \( w > 0 \) on \([s_0, s_1]\), which contradicts (2.8). Assumed
\[
\hat{s} := \sup \left\{ s \in [s_0, s_1] \mid w(s) > 0 \right\} < s_1
\]
which implies
\[
(2.11) \quad w(\hat{s}) = 0
\]
Then (2.10) is equivalent to
\[
w' \geq \frac{1}{8r}(1 - 16w)
\]
on \([s_0, \hat{s}]\). The function \( z: [s_0, s_1] \to \mathbb{R} \), \( z(s) := \frac{1}{16}(1 - e^{-\frac{2s - s_0}{r}}) \), satisfies \( z' = \frac{1}{8r}(1 - 16z) \) on \([s_0, s_1]\) and \( z(s_0) = 0 \). Thus we have
\[
(2.12) \quad \begin{cases}
(w - z)' & \geq -\frac{2}{r}(w - z) \\
(w - z)(s_0) & = 0
\end{cases}
on [s_0, \hat{s}]
\]
and we define a new function \( \zeta: [s_0, s_1] \to \mathbb{R} \) as follows \( \zeta(s) := e^{\frac{2s}{r}}(w(s) - z(s)) \). Then (2.12) implies
\[
\zeta'(s) = \frac{2}{r}e^{\frac{2s}{r}}(w(s) - z(s)) + 2e^{\frac{2s}{r}}(w'(s) - z'(s)) \\
\geq \frac{2}{r}e^{\frac{2s}{r}}(w(s) - z(s)) - \frac{2}{r}e^{\frac{2s}{r}}(w(s) - z(s)) = 0
\]
Hence
\[
e^{\frac{2s}{r}}(w(\hat{s}) - z(\hat{s})) = \zeta(\hat{s}) = \int_{s_0}^{\hat{s}} \zeta'(\tau) \, d\tau \geq 0
\]
from this we obtain
\[
w(\hat{s}) \geq z(\hat{s}) = \frac{1}{16}(1 - e^{-\frac{2(\hat{s} - s_0)}{r}}) > 0
\]
which contradicts (2.11). Consequently, we have \( w \geq 0 \) on \([s_0, s_1]\). The same argumentation as above, adapted to the interval \([s_0, s_1]\), implies \( w(s_1) > 0 \) in contradiction to (2.8). This proves that two discs \( D(\gamma(s_0), R) \) and \( D(\gamma(s_1), R) \) cannot intersect, when \(|s_1 - s_0| \leq 10R\).

Now, we show that two discs \( D(\gamma(s_0), R) \) and \( D(\gamma(s_1), R) \) cannot intersect if we assume \( s_0, s_1 \in [0, L], s_0 < s_1 \), to be far away from each other, which means that \( s_1 - s_0 > 10R \) holds.

We suppose that there exists a point \( p \in D(\gamma(s_0), R) \cap D(\gamma(s_1), R) \). As in [Str16, p. 273] we construct a curve \( \alpha \) in the following manner: \( \alpha \) follows \( \gamma \) from \( (0) \) to \( \gamma(s_0), \) next \( \alpha \) connects \( \gamma(s_0) \) and \( p \) by a minimizing geodesic, then \( \alpha \) connects \( p \) and \( \gamma(s_1) \) also by a minimizing geodesic, and finally \( \alpha \) follows \( \gamma \) again from \( \gamma(s_1) \) to \( \gamma(L) \). We infer the following estimate:
\[
d_p(\gamma(0), \gamma(L)) \leq L(\alpha) \leq \int_0^{s_0} |\gamma| \, ds + R + R + \int_{s_1}^L |\gamma| \, ds \tag{2.3}
\]
\[
\leq (1 + \beta)s_0 + 2R + (1 + \beta)(L - s_1)
\]
\[
= (1 + \beta)L + 2R - (1 + \beta)(s_1 - s_0)
\]
\[(2.3) \quad \leq (1 + \beta)^2 \int_0^L |\gamma| \, ds + 2R - (1 + \beta)(s_1 - s_0)\
\leq (1 + \beta)^2 L(\gamma) - 8R
\leq (\beta + 2)(d_\theta(\gamma(0), \gamma(L)) + 4\beta - 8R
\leq (d_\theta(\gamma(0), \gamma(L)) + 3\beta D + 4\beta - 8R
\]

and consequently:
\[0 \leq (3D + 4)\beta - 8R\]
which yields a contradiction when \(\beta(n, D, K, i) > 0\) is chosen small enough. Hence, two discs \(D(\gamma(s_0), R)\) and \(D(\gamma(s_1), R)\) cannot intersect, provided they are not identical. Thus, \(D(\gamma, R)\) is foliated by \((D(\gamma(s), R))_{s \in [0, L]}\).

It remains to show the estimate \((2.4)\). We mentioned at the beginning of the proof, that now, we allow \(\mu\) to become smaller.

As in the proof of [Str16, Lemma 3.4.] we suppose the assertion would be not true, i.e. there exists a sequence of constants \((\mu_i)_{i \in \mathbb{N}}\), where \(\lim_{i \to \infty} \mu_i = 0\), and a sequence of closed Riemannian manifolds \((M_i, g_i)_{i \in \mathbb{N}}\) satisfying
\[f_3(M_i, g_i) \leq K_i \quad \text{and} \quad \text{inj}_g(M_i) \geq \iota_i\]
for all \(i \in \mathbb{N}\), and curves \(\gamma_i : [0, L_i] \to M_i\) satisfying
\[L(\gamma_i) \leq d(\gamma_i(0), \gamma_i(L_i)) + \beta_i, \quad |\nabla\gamma_i| \leq \beta_i \quad \text{and} \quad \frac{1}{1 + \beta_i} \leq |\gamma_i| \leq 1 + \beta_i\]
for all \(i \in \mathbb{N}\), where \((\beta_i)_{i \in \mathbb{N}} \subseteq (0, 1]\), so that for each \(i \in \mathbb{N}\) the tube \(D(\gamma_i, R_i)\) is foliated by \((D(\gamma_i(s), R_i))_{s \in [0, L_i]}\), where \(R_i := \mu_i \min \left\{\iota_i, K_i^{-\frac{3}{2}}\right\}\), but for each \(i \in \mathbb{N}\) there exists a point \(p_i = \gamma_i(s_i)\) and \(y_i \in D(p_i, R_i)\) such that \(|d\pi_i(y_i)| > 2\).

From this we construct a blow-up sequence of pointed Riemannian manifolds
\[(M_i, h_i := R_i^{-2}g_i, p_i)_{i \in \mathbb{N}}\]
which satisfies for each \(i \in \mathbb{N}\) and \(x \in M_i\)
\[f_3(x, h_i) = f_3(x, R_i^{-2}g_i) \geq R_i^{-2} f_3(x, g_i) \leq R_i^{2} K_i \leq \mu_i^{2} \frac{1}{i \to \infty} 0\]
and
\[\text{inj}_{h_i}(M_i) = \text{inj}_{R_i^{-2}g_i}(M_i) = R_i^{-1} \text{inj}_{g_i}(M_i) \geq R_i^{-1} \iota_i \geq \mu_i^{-1} \frac{1}{i \to \infty} \infty\]
Hence, using [And89, Theorem 2.2, pp. 464-466], we may extract a subsequence that converges with respect to the pointed \(C^{2,0,0}\)-sense to \((\mathbb{R}^n, g_{can}, 0)\). Next, for each \(i \in \mathbb{N}\) we reparametrize the curve \(\gamma_i\) as follows: Let
\[\tilde{\gamma}_i : [0, \frac{L_i}{R_i}] \to M_i \quad \tilde{\gamma}_i(s) := \gamma_i(R_i s)\]
Then for each \(i \in \mathbb{N}\) we have for all \(s \in [0, \frac{L_i}{R_i}]\)
\[|\tilde{\gamma}_i(s)| h_i\]
from the statement of Proposition / 26. Proposition]. Hence, we conclude that each point $y_i$ be considered as a point in $(\mathbb{R}^n, g_i)$, that each tubular neighborhood is a diffeomorphic image of a neighborhood $\mathcal{N}(y_i)$ of the zero section in the normal bundle on the curve $\gamma_i$. We want to point that it is also possible to deduce Lemma 2.4, Lemma 3.4, p. 272 by use of unit-speed parametrization. On doing so, it can be shown that, under certain assumptions, the curves $\gamma_i$ converge in the $C^2,\alpha$-sense to the euclidean space and the curves $\gamma_i$ converge in the $C^1,\alpha$-sense, the maps $\pi_i$ converge in the $C^1$-sense to a map on the limit space, which will be denoted by $\pi$. Here we have used, that each tubular neighborhood is a diffeomorphic image of a neighborhood of the zero section in the normal bundle on the curve $\gamma_i$ ([OYN83, pp. 199-200, 25. Proposition / 26. Proposition]). Hence, we conclude $|d\pi_i|(y_i) \geq 2$, but the map $\pi$ is explicitly given as $(x^1, ..., x^n) \mapsto (x^1, 0, ..., 0)$ and this map satisfies $|d\pi| \leq 1$, which yields a contradiction.

We want to point that it is also possible to deduce Lemma 2.3 from the statement of [Str16, Lemma 3.4, p. 272] by use of unit-speed parametrization. On doing so, it is possible to avoid the dependence of the constant $\beta > 0$ on the diameter $D > 0$.

2.2. Forward estimates. In this paragraph we show that, under certain assumptions, distances do not increase too much along the $L^2$-flow.

Here, we prove the following estimate:

**Lemma 2.4.** Let $(M^4, g_0)$ be a closed Riemannian 4-manifold and let $(M^4, g(t))_{t \in [0,1]}$ be a solution to the flow given in (1.3) satisfying (1.4), (1.5), (1.6) and (1.7), i.e.:

$$
\int_M |Rm_{g_0}|^2_{g_0} \, dV_{g_0} \leq \Lambda
$$

$$
\|Rm_{g(t)}\|_{L^\infty(M, g(t))} \leq K t^{-\frac{1}{2}}
$$

$$
\text{inj}_{g(t)}(M) \geq \text{dist}_{g(t)}(M)
$$

$$
\text{diam}_{g(t)}(M) \leq 2(1 + D)
$$

for all $t \in (0,1]$. Then we have the following estimate:

$$
(2.14) \ d(x, y, t_2) - d(x, y, t_1) \leq C(K, \iota, D) 2^{\frac{1}{2}} \left( t_2^{\frac{1}{2}} - t_1^{\frac{1}{2}} \right) + C(K, \iota, D) \left( t_2^{\frac{1}{2}} - t_1^{\frac{1}{2}} \right)
$$
for all \( t_1, t_2 \in [0, 1] \) where \( t_1 < t_2 \).

As mentioned at the beginning of this section, we aim to use some kind of connecting curves between two points which are close to geodesics. These curves can be surrounded by a tube such that the projection map has bounded differential (c.f. Lemma 2.3).

The following definition is a modification of [Str16, Definition 3.1., p. 270]. Our definition is slightly stronger in some sense because we also assume a stability estimate of the length of the velocity vectors along the subintervals. We point out that we call the following objects \( \beta \)-quasi-forward-geodesics and not merely \( \beta \)-quasi-geodesics, as in [Str16, Definition 3.1., p. 270]. In Subsection 2.3 we introduce a time-reversed counterpart to these family of curves.

**Definition 2.5.** Let \((M^n, g(t))_{t \in [t_1, t_2]}\) be a family of complete Riemannian manifolds. Given \( \beta > 0 \) and \( x, y \in M \) then we say that a family of curves \((\gamma_t)_{t \in [t_1, t_2]} : [0, 1] \to M\) is a \( \beta \)-quasi-forward-geodesic connecting \( x \) and \( y \) if there is a constant \( S > 0 \) so that:

1. For all \( t \in [t_1, t_2] \) one has \( \gamma_t(0) = x \) and \( \gamma_t(1) = y \)
2. For all \( j \in \mathbb{N}_0 \) such that \( t_1 + jS \leq t_2 \), \( \gamma_{t_1 + jS} \) is a length minimizing geodesic
3. For all \( j \in \mathbb{N}_0 \) such that \( t_1 + jS \leq t_2 \), and all \( t \in [t_1 + jS, t_1 + (j + 1)S] \cap [t_1, t_2] \) one has \( \gamma_t = \gamma_{t_1 + jS} \)
4. For all \( t \in [t_1, t_2] \) one has
   \[
   (2.15) \quad d(x, y, t) \leq L(\gamma_t, t) \leq d(x, y, t) + \beta
   
   (2.16) \quad \frac{1}{1 + \beta} d(x, y, t_1 + jS) \leq |\dot{\gamma}_{t_1 + jS}|_{g(t)} \leq (1 + \beta) d(x, y, t_1 + jS)
   
   (2.17) \quad |g(t) \nabla_{\dot{\gamma}_{t_1 + jS}}|_{g(t)} \leq \beta d^2(x, y, t_1 + jS)
   
   It is our aim to prove the following existence result:

**Lemma 2.6.** Let \((M^n, g(t))_{t \in [t_1, t_2]}\) a smooth family of closed Riemannian manifolds. Given \( \beta > 0 \) and \( x, y \in M \) then there exists a \( \beta \)-quasi-forward-geodesic connecting \( x \) and \( y \).

**Remark 2.7.** The interval length \( S > 0 \) which will be concretized along the following proof has a strong dependency on the given points \( x, y \in M \), \( \beta > 0 \) and the flow itself. As it turns out in the proof of Lemma 2.4, this will not cause problems because estimates on the subintervals will be put together to an estimate on the entire interval \([t_1, t_2]\) via a telescope sum.

**Proof of Lemma 2.6.** In order to obtain the desired existence result, we modify the proof of [Str16, Lemma 3.2., p. 271]. Let

\[
(2.18) \quad A := \max_{t \in [t_1, t_2]} \|g'(t)\|_{L^\infty(M, g(t))} + \max_{t \in [t_1, t_2]} \|\nabla g'(t)\|_{L^\infty(M, g(t))}
\]

At time \( t_1 + jS \) we choose a length minimizing geodesic \( \gamma_{t_1 + jS} : [0, 1] \to M \) with respect to the metric \( g(t_1 + jS) \) connecting \( x \) and \( y \). This curve satisfies

\[
(2.19) \quad |\nabla_{\dot{\gamma}_{t_1 + jS}} \gamma_{t_1 + jS}|_{g(t_1 + jS)} \equiv 0
\]

and

\[
(2.20) \quad |\dot{\gamma}_{t_1 + jS}|_{g(t_1 + jS)} \equiv d(x, y, t_1 + jS)
\]
Firstly, we show that an appropriate choice of $S(\beta, x, y, g) > 0$ implies (2.16). Let $v \in TM$ be an arbitrary vector and $t \in [t_1 + jS, t_1 + (j + 1)S] \cap [t_1, t_2]$. Then, by (A.2), we have

$$\log \left( \frac{|v|_{g(t)}^2}{|v|_{g(t_1+jS)}^2} \right) \leq \int_{t_1+jS}^{t} \|g'(\tau)\|_{(L^\infty(M), g(\tau))} \, d\tau \quad (2.21)$$

Hence, we obtain the estimate

$$\frac{1}{(1 + \beta)^2} |\gamma_{t_1+jS}|_{g(t_1+jS)}^2 \leq |\gamma_{t_1}|_{g(t_1+jS)}^2 \leq (1 + \beta)^2 |\gamma_{g(t_1+jS)}|_{g(t_1+jS)}^2$$

Using (2.20) we infer (2.16) from this. Next we show (2.15). Using (A.1) we obtain

$$\frac{\partial}{\partial t} L(\gamma_t, t) = \frac{\partial}{\partial t} L(\gamma_{t_1+jS}, t) \leq A \cdot L(\gamma_{t_1+jS}, t) = A \cdot L(\gamma_t, t) \quad (2.18)$$

on $(t_1 + jS, t_1 + (j + 1)S) \cap [t_1, t_2]$. This implies \( \frac{\partial}{\partial t} \log L(\gamma_t, t) \leq A \), and we infer

$$d(x, y, t) \leq L(\gamma_t, t) = \frac{L(\gamma_{t_1+jS}, t_{1+jS})}{L(\gamma_{t_1+jS}, t_{1+jS})} L(\gamma_{t_1+jS}, t_1 + jS)$$

$$= \exp \left( \log \left( \frac{L(\gamma_{t_1+jS}, t_{1+jS})}{L(\gamma_{t_1+jS}, t_{1+jS})} \right) \right) L(\gamma_{t_1+jS}, t_1 + jS)$$

$$= \exp \left( \log (L(\gamma_{t_1+jS}, t_1 + jS)) - \log (L(\gamma_{t_1+jS}, t_1 + jS)) \right) L(\gamma_{t_1+jS}, t_1 + jS)$$

$$\leq e^{A(t - (t_1 + jS))} L(\gamma_{t_1+jS}, t_1 + jS) = e^{A(t - (t_1 + jS))} d(x, y, t_1 + jS)$$

In particular, we have

$$d(x, y, t) \leq e^{A(t - t_1)} L(\gamma_{t_1}, t_1) = e^{A(t_1 - t_1)} d(x, y, t) \quad \forall t \in [t_1, t_2] \quad (2.24)$$

From (2.23) we obtain for all $t \in (t_1 + jS, t_1 + (j + 1)S) \cap [t_1, t_2]$

$$d(x, y, t) \leq d(x, y, t_1 + jS) + (e^{AS} - 1)d(x, y, t_1 + jS)$$

$$\leq d(x, y, t_1 + jS) + \frac{\beta}{2} \quad (2.24)$$

In order to prove (2.15) it suffices to show that we can choose $S(\beta, x, y, g) > 0$ small enough to ensure

$$d(x, y, t_1 + jS) \leq d(x, y, t) + \frac{\beta}{2} \quad \forall t \in (t_1 + jS, t_1 + (j + 1)S) \cap [t_1, t_2] \quad (2.25)$$

From (2.21) we conclude for all $v \in TM$

$$e^{-AS} |v|_{g(t_1+jS)}^2 \leq |v|_{g(t)}^2 \leq e^{AS} |v|_{g(t_1+jS)}^2 \quad \forall t \in (t_1 + jS, t_1 + (j + 1)S) \cap [t_1, t_2] \quad (2.26)$$

At time $t$, we choose a length minimizing geodesic $\xi : [0, d(x, y, t)] \rightarrow M$ connecting $x$ and $y$, then:

$$d(x, y, t_1 + jS) \leq L(\xi, t_1 + jS) = \int_{0}^{d(x, y, t)} |\dot{\xi}(s)|_{g(t_1+jS)} \, ds$$

$$\leq e^{AS} \int_{0}^{d(x, y, t)} |\dot{\xi}(s)|_{g(t)} \, ds = e^{AS} L(\xi, t) = e^{AS} d(x, y, t)$$

$$= d(x, y, t) + (e^{AS} - 1)d(x, y, t) \quad (2.24)$$

$$\leq d(x, y, t) + (e^{AS} - 1)e^{A(t_1 - t_1)} d(x, y, t_1) \leq d(x, y, t) + \frac{\beta}{2}$$
It remains to show that, under the assumption that \( S(\beta, x, y, g) > 0 \) is sufficiently small, estimate (2.17) is also valid. From (A.3), (2.18) and (2.16) we conclude for each \( t \in (t_1 + jS, t_1 + (j + 1)S) \cap \{ t_1, t_2 \} \)

\[
\frac{\partial}{\partial t} |\nabla_\gamma \hat{\gamma}_t|^2(t) \leq A|\nabla_\gamma \hat{\gamma}_t|^2(t) + 4AC(n) d^2(x, y, t_1 + jS) |\nabla_\gamma \hat{\gamma}_t|^2(t) 
\]

(2.27)

Now let \( x \in M \) be arbitrary. We assume that

\[
\hat{\gamma} := \sup \left\{ t \in (t_1 + jS, t_1 + (j + 1)S) \cap \{ t_1, t_2 \} | \right. \\
|\nabla_\gamma \hat{\gamma}_t|^2(t) (x, \tau) \leq \min \{ \beta d^2, 1 \} \quad \forall \tau \in [t_1 + jS, t_2] \\
\left. \leq \min \{ t_1 + (j + 1)S, t_2 \} \right\}
\]

where

\[
\beta := \min \frac{d(x, y, t)}{t \in [t_1, t_2] > 0}
\]

Then, (2.27) implies

\[
\frac{\partial}{\partial t} |\nabla_\gamma \hat{\gamma}_t|^2(t) \leq A(1 + 4Ce^{2A(t_2 - t_1)} d^2(x, y, t_1)) \quad \text{on} \quad \{ x \} \times \{ t_1 + jS, \hat{\gamma} \}
\]

Using this, from (2.19), we conclude:

\[
\min \{ \beta d^2, 1 \} = |\nabla_\gamma \hat{\gamma}_t|^2(t) (x, \hat{\gamma}) \\
\leq A(\hat{\gamma} - (t_1 + jS))(1 + 4Ce^{2A(t_2 - t_1)} d^2(x, y, t_1)) \\
\leq AS(1 + 4Ce^{2A(t_2 - t_1)} d^2(x, y, t_1)) \leq \frac{\min \{ \beta d^2, 1 \}}{2}
\]

which yields a contradiction, if \( S(\beta, x, y, g) > 0 \) is small enough. \( \square \)

Now we prove Lemma 2.4. The argumentation is based on [Str16, pp. 277-280].

**Proof of Lemma 2.4.** Let \( x, y \in M \) be fixed and \( t_1, t_2 \in [0, 1] \), \( t_1 < t_2 \). Initially, we construct an appropriate \( \beta \)-quasi-forward geodesic in respect of Lemma 2.3. We choose

\[
(2.28) \quad \beta := \min \beta_t > 0
\]

where

\[
\beta_t := \beta(n, \text{diam}_g(t)(M), f_\delta(M, g(t)), \text{inj}_g(t)(M))
\]

is chosen according to Lemma 2.3 at time \( t \). Next, using Lemma 2.6, we assume the existence of a \( \beta \)-quasi-forward-geodesic

\[
\{ \xi(t) \}_{t \in [t_1, t_2]} : [0, 1] \rightarrow M
\]

connecting \( x \) and \( y \). It is our aim to construct an appropriate tubular neighborhood around each \( \xi_t \) applying Lemma 2.3, the radii \( r_t \) shall be time dependent, where \( r_0 = 0 \), when \( t_1 = 0 \). After doing this, we notice that we are able to estimate the integral \( \int_{[0, t]} |\text{grad} F| \text{d}t \) from above against an average integral of \( |\text{grad} F|^2 \) along the tube plus an error term. Each of these terms is controllable.

By construction of the \( \beta \)-quasi-forward-geodesic, we have a finite set of geodesics denoted by \( \{ \xi_{t_1 + jS} \}_{j \in \{0, \ldots, \lfloor \frac{t_2 - t_1}{S} \rfloor \}} \), where each of these geodesics is parametrized proportional to arc length, i.e.:

\[
|\xi_{t_1 + jS} \rangle |_{g(t_1 + jS)} = d(x, y, t_1 + jS) \quad \text{for all} \quad j \in \{0, \ldots, \lfloor \frac{t_2 - t_1}{S} \rfloor \}
\]
we reparametrize each of these curves with respect to arc length, i.e. for each $j \in \{0, ..., \left\lfloor \frac{2t_{-1}}{S} \right\rfloor \}$ let
\[
\varphi_{t_{1}+jS} : [0, d(x, y, t_{1}+jS)] \rightarrow [0, 1]
\]
\[
\varphi(s) := \frac{s}{d(x, y, t_{1}+jS)}
\]
and let
\[
\gamma_{t_{1}+jS} : [0, d(x, y, t_{1}+jS)] \rightarrow M
\]
\[
\gamma_{t_{1}+jS} := \xi_{t_{1}+jS} \circ \varphi_{t_{1}+jS}
\]
Of course, these curves are satisfying (2.1) (2.2) and (2.3). But we need to get sure that, for each $t \in (t_{1}+jS, t_{1}+(j+1)S) \cap [t_{1}, t_{2}]$, the curve
\[
\gamma_{t} := \xi_{t} \circ \varphi_{t_{1}+jS} : [0, d(x, y, t_{1}+jS)] \rightarrow M
\]
is also satisfying these assumptions. Here $\beta \in (0, 1)$ is defined by (2.28). By construction, using (2.16) for each $t \in (t_{1}+jS, t_{1}+(j+1)S) \cap [t_{1}, t_{2}]$, we have
\[
\frac{1}{1+\beta_{1}} \leq \frac{1}{1+\beta} \leq |\gamma_{t}|_{g(t)} = \frac{1}{d(x, y, t_{1}+jS)}|\dot{\gamma}_{t}|_{g(t)} \leq 1 + \beta \leq 1 + \beta_{t}
\]
and, using (2.17)
\[
|\nabla_{\gamma_{t}} \dot{\gamma}_{t}|_{g(t)} = \frac{1}{d(x, y, t_{1}+jS)^{2}}|\nabla_{\xi_{t}} \dot{\gamma}_{t}|_{g(t)} \leq \beta \leq \beta_{t}
\]
Thus, by Lemma 2.3, for each time $t \in [jS, (j+1)S) \cap [t_{1}, t_{2}]$ the tubular neighborhood $D(\gamma_{t}, \rho_{t})$ is foliated by $(D(\gamma_{t}, \rho_{t}))_{s \in [0, d(x, y, t_{1}+jS)]}$ where
\[
\rho_{t} := \mu \min \left\{ \min_{g(t)}(M), f_{3}(M, g(t))^{-\frac{1}{2}} \right\}
\]
where $\mu > 0$ is fixed and the differential of the projection map satisfies (2.4). For later considerations, we assume that $\mu > 0$ is also chosen compatible to [Str16, Lemma 2.7, p. 268]. Although we have no control on $\beta_{t}$, we can bound $\rho_{t}$ from below if we can bound $f_{3}(M, g(t))^{-\frac{1}{2}}$ from below in the view of (2.29).

Using (A.7) and (1.5) we obtain for each $m \in \{1, 2, 3\}$:
\[
\|\nabla^{m} \text{Rm}_{g(t)}\|_{L^{\infty}(M, g(t))} \leq C(m, K) \left( t^{-\frac{3}{2}} \right)^{\frac{2+m}{2}} = C(m, K) t^{-\frac{2+m}{4}}
\]
and consequently
\[
f_{3}(M, g(t)) \leq C(K) t^{-\frac{1}{4}}
\]
Thus, we have for each $t \in [t_{1}, t_{2}]
\[
\rho_{t} \geq \mu \left\{ t^{-\frac{1}{4}}, C^{-\frac{1}{2}}(K) t^{-\frac{1}{2}} \right\} \geq \mu \min \{ t, C^{-\frac{1}{2}}(K) \} \cdot t^{\frac{-1}{2}} =: R(t, K) \cdot t^{\frac{-1}{2}} =: r_{t}(t, K)
\]
Now, we may start to estimate the change of $L(\gamma_{t}, t)$, where $t \in [t_{1}+jS, t_{1}+(j+1)S) \cap [t_{1}, t_{2}]$ and $j \in \left\{ 0, ..., \left\lfloor \frac{2t_{-1}}{S} \right\rfloor \right\}$. From the explicit formula in (1.3) and (2.30) we conclude $|\nabla \text{grad} F_{g(t)}|_{g(t)} \leq C_{3}(K) t^{-\frac{1}{2}}$. Now let $p$ be an arbitrary point on the curve $\gamma_{t_{1}+jS}$ and $q \in D(p, r_{t})$ then we obtain
\[
|\text{grad} F_{g(t)}|_{g(t)}(p) \leq |\text{grad} F_{g(t)}|_{g(t)}(q) + C_{3}(K) r_{t}(t, K) t^{-\frac{1}{2}}
\]
In the following, we write \( r_i \) instead of \( r_i(t, K) \) and \( \mathcal{F} \) instead of \( \mathcal{F}_{g(t)} \).

We infer:

\[
|\text{grad } \mathcal{F}_{g(t)}(p)| = \frac{\int_{D(p, r_i)} |\text{grad } \mathcal{F}_{g(t)}(p)| \, dA}{\text{Area}(D(p, r_i))} \leq \frac{\int_{D(p, r_i)} \left[ |\text{grad } \mathcal{F}_{g(t)}(q) + C_3(K) r_i t^{-\frac{1}{2}} \right] \, dA}{\text{Area}(D(p, r_i))} \]

(2.33)

\[
= \frac{\int_{D(p, r_i)} |\text{grad } \mathcal{F}_{g(t)}(p)| \, dA}{\text{Area}(D(p, r_i))} + C_3(K) R(t, K) t^{-\frac{2}{3}} \leq \left( \frac{\int_{D(p, r_i)} |\text{grad } \mathcal{F}_{g(t)}(p)| \, dA}{\text{Area}^{\frac{1}{2}}(D(p, r_i))} \right)^{\frac{1}{2}} + C_3(K) R(t, K) t^{-\frac{2}{3}} \]

From [Str16, Lemma 2.7, p. 268] we obtain for each \( t \in [t_1, t_2] \) that

\[
\text{Area}(D(\gamma_i(s), r_i)) \geq c r_i^3 = c R_i^3(t, K) t^\frac{2}{3} \]

Inserting this estimate into (2.33), we infer for each \( p \in \gamma_{t_1+jS} \)

\[
|\text{grad } \mathcal{F}_{g(t)}(p)| \leq c^{-\frac{1}{3}} R^{-\frac{1}{3}}(t, K) t^{-\frac{2}{3}} \left( \int_{D(p, r_i)} |\text{grad } \mathcal{F}_{g(t)}(p)| \, dA \right)^{\frac{1}{3}} + C_3(K) R(t, K) t^{-\frac{2}{3}} \]

(2.35)

Hence, on \( (t_1 + jS, t_1 + (j + 1)S) \cap [t_1, t_2] \) we have

\[
\frac{d}{dt} L(\gamma_t, t) = \frac{d}{dt} L(\gamma_{t_1+jS}, t) \overset{(A.1)}{\leq} \int_{\gamma_{t_1+jS}} |\text{grad } \mathcal{F}_{g(t)}(p)| \, ds \overset{(2.35)}{\leq} \left( \int_{D(p, r_i)} |\text{grad } \mathcal{F}_{g(t)}(p)| \, dA \right)^{\frac{1}{3}} + C_3(K) R(t, K) t^{-\frac{2}{3}} L(\gamma_{t_1+jS}, t) \]

\[
\leq c^{-\frac{1}{3}} R^{-\frac{1}{3}}(t, K) t^{-\frac{2}{3}} \sup_{D(\gamma_{t_1+jS}, r_i)} |ds|^{\frac{1}{3}} \left( \int_{M} |\text{grad } \mathcal{F}_{g(t)}(p)| \, dV_{g(t)} \right)^{\frac{1}{3}} + C_3(K) R(t, K) t^{-\frac{2}{3}} L(\gamma_{t_1+jS}, t) \]

(2.4)

\[
\leq c^2 R^{-\frac{3}{2}}(t, K) t^{-\frac{5}{3}} \left( \int_{M} |\text{grad } \mathcal{F}_{g(t)}(p)| \, dV_{g(t)} \right)^{\frac{1}{3}} L^2(\gamma_{t_1+jS}, t) \]

\[
+ C_3(K) R(t, K) t^{-\frac{2}{3}} L(\gamma_{t_1+jS}, t) \]

(2.4)

\[
= c^2 R^{-\frac{3}{2}} R(t, K) t^{-\frac{5}{3}} \left( \int_{M} |\text{grad } \mathcal{F}_{g(t)}(p)| \, dV_{g(t)} \right)^{\frac{1}{3}} L^2(\gamma_t, t) \]

\[
+ C_3(K) R(t, K) t^{-\frac{2}{3}} L(\gamma_t, t) \]

Using

\[
L(\gamma_t, t) \overset{(2.1)}{\leq} d(x, y, t_1 + jS) + 1 \overset{(1.7)}{\leq} 2(1 + D) + 1 = 3 + 2D \]

(2.36)
we conclude
\[
\frac{d}{dt}L(\gamma_t, t) \leq C(D)R^{-\frac{2}{t}}(i, K) t^{-\frac{3}{2t}} \left( \int_M |\text{grad } F|^2_{g(t)} dV_{g(t)} \right)^{\frac{t}{2}} \\
+ C(K, D)R(i, K) t^{-\frac{2}{3t}}
\]
on \left[ t_1 + jS, t_1 + (j + 1)S \right] \cap [t_1, t_2] where \( j \in \{0, ..., \lfloor \frac{t_2 - t_1}{S} \rfloor \} \). Integrating this estimate along \([t_1 + jS, t]\) yields:
\[
d(x, y, t) - d(x, y, t_1 + jS) = d(x, y, t) - L(\gamma_{t_1+jS}, t_1 + jS) \\
\leq L(\gamma_t, t) - L(\gamma_{t_1+jS}, t_1 + jS) \\
\leq C(D)R^{-\frac{2}{t}}(i, K) \int_{t_1 + jS}^{t} s^{-\frac{3}{2t}} \left( \int_M |\text{grad } F|^2_{g(s)} dV_{g(s)} \right)^{\frac{t}{2}} ds \\
+ C(K, D)R(i, K) \int_{t_1 + jS}^{t} s^{-\frac{2}{3t}} ds
\]
for each \( t \in (t_1 + jS, t_1 + (j + 1)S) \cap [t_1, t_2] \). In particular, we obtain for each \( j \in \{0, ..., \lfloor \frac{t_2 - t_1}{S} \rfloor \} \)
\[
d(x, y, t_1 + (j + 1)S) - d(x, y, t_1 + jS) \\
\leq C(D)R^{-\frac{2}{t}}(i, K) \int_{t_1 + jS}^{t_1 + (j + 1)S} s^{-\frac{3}{2t}} \left( \int_M |\text{grad } F|^2_{g(s)} dV_{g(s)} \right)^{\frac{t}{2}} ds \\
+ C(K, D)R(i, K) \int_{t_1 + jS}^{t_1 + (j + 1)S} s^{-\frac{2}{3t}} ds
\]
and
\[
d(x, y, t_2) - d(x, y, t_1 + \lfloor \frac{t_2 - t_1}{S} \rfloor S) \\
\leq C(D)R^{-\frac{2}{t}}(i, K) \int_{t_1 + \lfloor \frac{t_2 - t_1}{S} \rfloor S}^{t_2} s^{-\frac{3}{2t}} \left( \int_M |\text{grad } F|^2_{g(s)} dV_{g(s)} \right)^{\frac{t}{2}} ds \\
+ C(K, D)R(i, K) \int_{t_1 + \lfloor \frac{t_2 - t_1}{S} \rfloor S}^{t_2} s^{-\frac{2}{3t}} ds
\]
and consequently
\[
d(x, y, t_2) - d(x, y, t_1) \\
= \sum_{j=0}^{\lfloor \frac{t_2 - t_1}{S} \rfloor - 1} \left[ d(x, y, t_1 + (j + 1)S) - d(x, y, t_1 + jS) \right] \\
+ \sum_{j=0}^{\lfloor \frac{t_2 - t_1}{S} \rfloor - 1} \left[ d(x, y, t_1 + \lfloor \frac{t_2 - t_1}{S} \rfloor S) - d(x, y, t_2) - d(x, y, t_1 + \lfloor \frac{t_2 - t_1}{S} \rfloor S) \right] \\
\leq C(D)R^{-\frac{2}{t}}(i, K) \int_{t_1}^{t_2} s^{-\frac{3}{2t}} \left( \int_M |\text{grad } F|^2_{g(s)} dV_{g(s)} \right)^{\frac{t}{2}} ds \\
+ C(K, D)R(i, K) \int_{t_1}^{t_2} s^{-\frac{2}{3t}} ds
\]
\[
\leq C(D)R^{-\frac{2}{t}}(i, K) \left( \int_{t_1}^{t_2} s^{-\frac{3}{2t}} ds \right)^{\frac{t}{2}} \left( \int_{t_1}^{t_2} \int_M |\text{grad } F|^2_{g(s)} dV_{g(s)} ds \right)^{\frac{1}{2}} \\
+ C(K, D)R(i, K) \int_{t_1}^{t_2} s^{-\frac{2}{3t}} ds
\]
Using (1.4) and (A.6) we conclude
\[ d(x, y, t_2) - d(x, y, t_1) \leq C(K, \epsilon, D)\Lambda^{\frac{1}{2}} \left( t_2^{\frac{1}{2}} - t_1^{\frac{1}{2}} \right) + C(K, \epsilon, D) \left( t_2^{\frac{1}{4}} - t_1^{\frac{1}{4}} \right) \]

\[ \square \]

2.3. Backward estimates. In this subsection we reverse the ideas from Subsection 2.2 in order to prove that, along the $L^2$-flow, the distance between two points does not become too small when $t > 0$ is small.

**Lemma 2.8.** Let $(M^n, g_0)$ be a closed Riemannian $4$-manifold and let $(M^n, g(t))_{t \in [0, 1]}$ be a solution to the flow given in (1.3) satisfying (1.4), (1.5), (1.6) and (1.7), i.e.:

\[ \int_M |Rm_{g(t)}|^2 dV_{g_0} \leq \Lambda \]

\[ \|Rm_{g(t)}\|_{L^{\infty}(M^n, g(t))} \leq Kt^{\frac{1}{2}} \]

\[ \text{inj}_{g(t)}(M) \geq \frac{1}{t^{\frac{1}{2}}} \]

\[ \text{diam}_{g(t)}(M) \leq 2(1 + D) \]

for all $t \in (0, 1]$. Then we have the following estimate:

\[ (2.37) \]

\[ d(x, y, t_2) - d(x, y, t_1) \geq -C(K, \epsilon, D)\Lambda^{\frac{1}{2}} \left( t_2^{\frac{1}{2}} - t_1^{\frac{1}{2}} \right) - C(K, \epsilon, D) \left( t_2^{\frac{1}{4}} - t_1^{\frac{1}{4}} \right) \]

for all $t_1, t_2 \in [0, 1]$ where $t_1 < t_2$.

The notion of a $\beta$-quasi-backward-geodesic, which is introduced below, is an analogue to the notion of a $\beta$-quasi-forward-geodesic, introduced in Subsection 2.2. The slight difference is that now, the minimizing geodesics are chosen at the subinterval ends.

**Definition 2.9.** Let $(M^n, g(t))_{t \in [t_1, t_2]}$ be a family of complete Riemannian manifolds. Given $\beta > 0$ and $x, y \in M$ then we say that a family of curves $(\gamma_t)_{t \in [t_1, t_2]} : [0, 1] \rightarrow M$ is a $\beta$-quasi-forward-geodesic connecting $x$ and $y$ if $(\gamma_t)_{t \in [t_1, t_2]}$ is a $\beta$-quasi-forward-geodesic connecting $x$ and $y$ with respect to the time-reversed flow $(M^n, g(t_2 + t - t))_{t \in [t_1, t_2]}$, i.e.: there is a constant $S > 0$ so that:

\(1\) For all $t \in [t_1, t_2]$ one has $\gamma_t(0) = x$ and $\gamma_t(1) = y$

\(2\) For all $j \in \mathbb{N}_0$ such that $t_2 - jS \geq t_1$, $\gamma_{t_2 - jS}$ is a minimizing geodesic

\(3\) For all $j \in \mathbb{N}_0$ such that $t_2 - jS \geq t_1$, and all $t \in (t_2 - (j + 1)S, t_2 - jS] \cap [t_1, t_2]$ one has $\gamma_t = \gamma_{t_2 - jS}$

\(4\) For all $t \in [t_1, t_2]$ one has

\[ (2.38) \]

\[ d(x, y, t) \leq L(\gamma_t, t) \leq d(x, y, t) + \beta \]

\(5\) For all $j \in \mathbb{N}_0$ such that $t_2 - jS \geq t_1$, and all $t \in (t_2 - (j + 1)S, t_2 - jS] \cap [t_1, t_2]$ one has

\[ (2.39) \]

\[ \frac{1}{1 + \beta} d(x, y, t_1 - jS) \leq |\gamma_t|_{g(t)} \leq (1 + \beta)d(x, y, t_2 - jS) \]

\[ (2.40) \]

\[ |\nabla_{\gamma_t} |_{g(t)} \leq \beta d^2(x, y, t_2 - jS) \]

Applying Lemma 2.6 to $(M^n, g(t_2 + t - t))_{t \in [t_1, t_2]}$, we infer

**Lemma 2.10.** Let $(M^n, g(t))_{t \in [t_1, t_2]}$ a smooth family of closed Riemannian manifolds. Given $\beta > 0$ and $x, y \in \mathbb{N}$ then there exists a $\beta$-quasi-backward-geodesic connecting $x$ and $y$.

Using this concept, we prove Lemma 2.8:
Proof of Lemma 2.8. The proof is analogous to Lemma 2.4. We choose \( x, y \in M \) and \( t_1, t_2 \in [0, 1] \) where \( t_1 < t_2 \). It is our aim to construct an appropriate backward-geodesic. As in the proof of Lemma 2.4, let
\[
\beta := \min_{t \in [t_1, t_2]} \beta_t > 0
\]
where
\[
\beta_t := \beta(n, \text{diam}_g(t)(M), f_3(M, g(t)), \text{inf}_j \gamma_j(t)(M))
\]
is defined in Lemma 2.3, let \((\xi_t)_{t \in [t_1, t_2]}\) be a \( \beta \)-backward-geodesic, connecting \( x \) and \( y \), whose existence is ensured by Lemma 2.10. As in the proof of Lemma 2.4 we use Lemma 2.3 to construct an appropriate tubular neighborhood around each \( \xi_t \), where \( t \in [t_1, t_2] \), having a time depend radius \( r_t \).

In this situation we have a finite set of geodesics \((\xi_{t_2 - jS})_{j \in \{0, ..., \lfloor \frac{t_2 - t_1}{S} \rfloor \}}\) satisfying
\[
|\xi_{t_2 - jS}|_{g(t_2 - jS)} = d(x, y, t_2 - jS) \text{ for all } j \in \{0, ..., \lfloor \frac{t_2 - t_1}{S} \rfloor \}
\]
Analogous to the proof of Lemma 2.4, we reparametrize these curves with respect to arc length, i.e.: for each \( j \in \{0, ..., \lfloor \frac{t_2 - t_1}{S} \rfloor \} \) we define
\[
\varphi_{t_2 - jS} : [0, d(x, y, t_2 - jS)] \rightarrow [0, 1]
\]
\[
\varphi(s) := \frac{s}{d(x, y, t_2 - jS)}
\]
\[
\gamma_{t_2 - jS} : [0, d(x, y, t_2 - jS)] \rightarrow M
\]
\[
\gamma_{t_2 - jS} := \xi_{t_2 - jS} \circ \varphi_{t_2 - jS}
\]
and for each \( t \in (t_2 - (j + 1)S, t_2 - jS] \cap [t_1, t_2] \) we define
\[
\gamma_t := \xi_t \circ \varphi_{t_2 - jS} : [0, d(x, y, t_2 - jS)] \rightarrow M
\]
so that, for each \( t \in [t_1, t_2] \) the curve \( \gamma_t \) satisfies (2.1) (2.2) and (2.3) with respect to \( \beta_t \). Hence, following Lemma 2.3, at each time \( t \in (t_2 - (j + 1)S, t_2 - jS] \cap [t_1, t_2] \) the tubular neighborhood \( D(\gamma_t, \rho_t) \) around \( \gamma_t \) is foliated by \( \{D(\gamma_t(s), \rho_t)\}_{s \in [0, d(x, y, t_2 - jS)]} \)
where \( \rho_t := \mu \min \left\{ \text{inf}_j \gamma_j(t)(M), f_3(M, g(t))^{-\frac{1}{2}} \right\} \), again \( \mu > 0 \) shall also satisfy the requirements of Lemma 2.3. Using the same arguments as in the proof of Lemma 2.4 we also obtain (2.31) and (2.32), i.e.: \[
\rho_t \geq R(t, K) \frac{\mathbb{R}^4}{t^\frac{\delta}{2}} =: r_t(i, K) \text{ for each } t \in [t_1, t_2]
\]
and
\[
|\text{grad} \, F|_{g(t)}(p) \leq |\text{grad} \, F|_{g(t)}(q) + C_3(K) r_t(i, K) t^{-\frac{\delta}{2}}
\]
for each \( p \in \gamma_t = \gamma_{t_2 - jS} \) and \( q \in D(p, r_t) \) where \( t \in (t_2 - (j + 1)S, t_2 - jS] \cap [t_1, t_2] \) and \( j \in \{0, ..., \lfloor \frac{t_2 - t_1}{S} \rfloor \} \). From this we also obtain (2.33), i.e.:
\[
|\text{grad} \, F|_{g(t)}(p) \leq \left( \int_{D(p, r_t)} |\text{grad} \, F|_{g(t)}^2(q) \, dA(q) \right)^{\frac{1}{2}}
\]
\[
\frac{1}{\text{Area}^2(D(p, r_t))} + C_3(K) R(t, K) t^{-\frac{\delta}{2}}
\]
Using [Str16, Lemma 2.7, p. 268] we obtain (2.34), i.e.:
\[
\text{Area}(D(\gamma_t(s), r_t)) \geq cr_t^3 = cR^3 t^{\frac{\delta}{2}}
\]
for all \( t \in (t_2 - (j + 1)S, t_2 - jS] \cap [t_1, t_2] \). Hence, for each \( j \in \{0, ..., \lfloor \frac{t_2 - t_1}{S} \rfloor \} \) we infer on \((t_2 - (j + 1)S, t_2 - jS) \cap [t_1, t_2]\) the following estimate
\[
\frac{d}{dt} L(\gamma_t, t) = \frac{d}{dt} L(t_2 - jS, t) \overset{(A.1)}{\geq} - \int_{\gamma_{t_2 - jS}} |\text{grad} \, F|_{g(t)} \, d\sigma
\]
\[ \geq -C(\delta) R^{-\frac{2}{3}}(t, K)^{-\frac{1}{2}} \int_{\gamma_{t_2 - j}^{-}} \left( \int_{D(p, r_1)} |\text{grad } F_{g(t)}^2 | \, dA \right)^{\frac{1}{2}} \, d\sigma \\
- C(\delta) R(t, K)^{-\frac{1}{2}} L(\gamma_{t_2 - jS}, t) \]

\[ \geq -C(\delta) R^{-\frac{2}{3}}(t, K)^{-\frac{1}{2}} \left( \int_{\gamma_{t_2 - j}^{-}} \left( \int_{D(p, r_1)} |\text{grad } F_{g(t)}^2 | \, dA \right)^{\frac{1}{2}} \, L(\gamma_{t_2 - jS}, t) \right) \\
- C(\delta) R(t, K)^{-\frac{1}{2}} L(\gamma_{t_2 - jS}, t) \]

\[ \geq -C(D) R^{-\frac{2}{3}}(t, K)^{-\frac{1}{2}} \left( \int_{\gamma_{t_2 - j}^{-}} \left( \int_{M} |\text{grad } F_{g(t)}^2 | \, dV_{g(t)} \right)^{\frac{1}{2}} \, L(\gamma_{t_2 - jS}, t) \right) - C(K, D) R(t, K)^{-\frac{1}{2}} L(\gamma_{t_2 - jS}, t) \]

Here we have used the fact that \(\gamma_t\) is nearly length minimizing and that the diameter is bounded (cf. (2.36)). By integration along \([t, t_2 - j S]\) we conclude for each \(t \in (t_2 - (j + 1)S, t_2 - jS) \cap [t_1, t_2]\)

\[ d(x, y, t_2 - jS) - d(x, y, t) = L(\gamma_{t_2 - jS}, t_2 - jS) - d(x, y, t) \]

\[ \geq -C(D) R^{-\frac{2}{3}}(t, K) \int_{t}^{t_2 - jS} \gamma_{s}^{-\frac{1}{2}} \left( \int_{M} |\text{grad } F_{g(s)}^2 | \, dV_{g(s)} \right)^{\frac{1}{2}} \, ds \\
- C(K, D) R(t, K) \int_{t}^{t_2 - jS} \gamma_{s}^{-\frac{1}{2}} \, ds \]

In particular, we have for each \(j \in \{0, \ldots, \left\lfloor \frac{t_2 - t_1}{S} \right\rfloor \} \)

\[ d(x, y, t_2 - jS) - d(x, y, t_2 - (j + 1)S) \]

\[ \geq -C(D) R^{-\frac{2}{3}}(t, K) \int_{t_2 - (j + 1)S}^{t_2 - jS} \gamma_{s}^{-\frac{1}{2}} \left( \int_{M} |\text{grad } F_{g(s)}^2 | \, dV_{g(s)} \right)^{\frac{1}{2}} \, ds \\
- C(K, D) R(t, K) \int_{t_2 - (j + 1)S}^{t_2 - jS} \gamma_{s}^{-\frac{1}{2}} \, ds \]

and also

\[ d(x, y, t_2 - \left\lfloor \frac{t_2 - t_1}{S} \right\rfloor S) - d(x, y, t_1) \]

\[ \geq -C(D) R^{-\frac{2}{3}}(t, K) \int_{t_1}^{t_2 - \left\lfloor \frac{t_2 - t_1}{S} \right\rfloor S} \gamma_{s}^{-\frac{1}{2}} \left( \int_{M} |\text{grad } F_{g(s)}^2 | \, dV_{g(s)} \right)^{\frac{1}{2}} \, ds \\
- C(K, D) R(t, K) \int_{t_1}^{t_2 - \left\lfloor \frac{t_2 - t_1}{S} \right\rfloor S} \gamma_{s}^{-\frac{1}{2}} \, ds \]

and finally

\[ d(x, y, t_2) - d(x, y, t_1) \]
\[
\begin{align*}
&\frac{t_2 - t_1}{S} - \frac{1}{S} \sum_{j=0}^{t_2 - t_1} \left[ d(x, y, t_2 - jS) - d(x, y, t_2 - (j+1)S) \right] \\
&+ d(x, y, t_2 - \frac{t_2 - t_1}{S} S) - d(x, y, t_1) \\
&\geq -C(D)R^{-\frac{7}{2}}(t, K) \int_{t_1}^{t_2} s^{-\frac{7}{8}} \left( \int_M |\text{grad } F|_{g(s)}^2 \, dV_{g(s)} \right)^{\frac{1}{2}} \, ds \\
&- C(K, D)R(t, K) \int_{t_1}^{t_2} s^{-\frac{7}{4}} \, ds \\
&\geq -C(K, D)R(t, K) \left( \int_{t_1}^{t_2} s^{-\frac{7}{8}} \, ds \right)^{\frac{1}{2}} \left( \int_{t_1}^{t_2} \int_M |\text{grad } F|_{g(s)}^2 \, dV_{g(s)} \, ds \right)^{\frac{1}{2}} \\
&- C(K, D)R(t, K) \int_{t_1}^{t_2} s^{-\frac{7}{4}} \, ds
\end{align*}
\]

we infer
\[
\begin{align*}
d(x, y, t_2) - d(x, y, t_1) &\geq -C(K, \iota, D)A^\frac{1}{2} \left( \frac{1}{2} - \frac{1}{8} \right)^{\frac{1}{2}} - C(K, \iota, D) \left( \frac{1}{2} - \frac{1}{8} \right) \\
&\geq -C(K, \iota, D)A^\frac{1}{2} \left( \frac{1}{2} - \frac{1}{8} \right)^{\frac{1}{2}} - C(K, \iota, D) \left( \frac{1}{2} - \frac{1}{8} \right)
\end{align*}
\]

\[\square\]

Finally, (2.14) and (2.37) together imply (1.8), which finishes the proof of Theorem 1.3. Using Theorem 1.3, the following result

**Corollary 2.11.** Let \((M^4, g(t)))_{t \in [0, 1]}\), where \(M^4\) is a closed Riemannian 4-manifold, be a solution to (1.3) satisfying the assumptions, (1.4), (1.5), (1.6) and (1.7), then for each \(k \in \mathbb{N}\) there exists \(j(k, A, K, \iota, D) \in \mathbb{N}\) such that

\[
d_{GH}((M, d_g), (M, d_{g(t)})) < \frac{1}{k}
\]

for all \(t \in [0, 1/j]\)

is a consequence of the following Lemma

**Lemma 2.12.** Let \(M^n\) be a closed manifold. Given two metrics \(g_1\) and \(g_2\) on \(M\) satisfying

\[
\sup_{x, y \in M} |d_{g_1}(x, y) - d_{g_2}(x, y)| < \epsilon
\]

then we have

\[
d_{GH}((M, d_{g_1}), (M, d_{g_2})) < \frac{\epsilon}{2}
\]

**Proof.** The set \(\mathfrak{R} := \{(x, x) \in M \times M | x \in M\}\) is a correspondence between \(M\) and \(M\) itself (cf. [BBI01, Definition 7.3.17., p. 256]) and the distorsion of \(\mathfrak{R}\) (cf. [BBI01, Definition 7.3.21., p. 257]) is:

\[
\text{dis} \mathfrak{R} = \sup_{x, y \in M} |d_{g_1}(x, y) - d_{g_2}(x, y)| < \epsilon
\]

From [BBI01, Theorem 7.3.25., p. 257] we obtain

\[
d_{GH}((M, d_{g_1}), (M, d_{g_2})) \leq \frac{1}{2} \text{dis} \mathfrak{R} < \frac{1}{2} \epsilon
\]

\[\square\]
3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 using Corollary 2.11. The conditions (1.4), (1.5), (1.6) and (1.7) are ensured by the following result

**Theorem 3.1.** (cf. [Str16, Theorem 1.8, p. 260]) Given \( \delta \in (0,1) \), there are constants \( \epsilon(\delta), \epsilon(\delta), A(\delta) > 0 \) so that if \( (M^4, g_0) \) is a closed Riemannian manifold satisfying the following conditions

\[
\mathcal{F}_{g_0} \leq \epsilon
\]

(3.1) \( \text{Vol}_{g_0}(B_{g_0}(x, r)) \geq \delta \omega_4 r^4 \quad \forall x \in M, r \in [0,1] \)

then the flow given in (1.3) with initial metric \( g_0 \) has a solution on \([0,1]\) and we have the following estimates:

\[
\|Rm_{g(t)}\|_{L^\infty(M,g(t))} \leq A \mathcal{F}_{g(0)}^{\frac{1}{4}} t^{-\frac{1}{4}} \quad \text{inj}_{g(t)}(M) \geq A \mathcal{F}_{g(0)}^{\frac{1}{4}} t^{-\frac{1}{4}} \quad \text{diam}_{g(t)}(M) \leq 2(1 + \text{diam}_{g(0)}(M))
\]

for all \( t \in (0,1] \).

From these estimates we may conclude the following precompactness result, at first

**Corollary 3.2.** Given \( D, \delta > 0 \). Then there exists \( \epsilon(\delta) > 0 \) so that the space \( \mathcal{M}^4(D, \delta, \epsilon(\delta)) \) which consists of the set of all closed Riemannian 4-manifolds \( (M, g) \) satisfying

\[
\text{diam}_g(M) \leq D \quad \text{Vol}_g(B_g(x, r)) \geq \delta \omega_4 r^4 \quad \forall x \in M, r \in [0,1] \quad \|Rm_g\|_{L^2(M,g)} \leq \epsilon^2
\]

are mutually disjoint and the balls \( B_g(x_1, r), ..., B_g(x_N, r) \) cover \( M \). Using the non-collapsing assumption (cf. (3.1)), we infer

\[
N \omega_4 (\frac{r}{2})^4 \leq \sum_{k=1}^{N} \text{Vol}_g(B_g(x_k, \frac{r}{2})) \quad \text{Vol}_g(B_g(x_1, r), ..., B_g(x_N, r)) \leq \text{Vol}_g(M) \leq V_0(D)
\]
This implies that the number of elements in such an \( r \)-net is bounded from above by a natural number \( N(r, \delta, D) \). The assertion follows from [BBI01, Theorem 7.4.15, p. 264].

**Proof of Theorem 1.1.** As in the proof of Corollary 3.2, we know that for each \( i \in \mathbb{N} \) the \( L^2 \)-flow with initial metric \( g_i \) exists on \([0, 1]\) and that this flow satisfies the following estimates

\[
\|Rm_{g_i(t)}\|_{L^\infty(M, g_i(t))} \leq A F_{g_i(t)}^{\frac{1}{2}} t^{-\frac{1}{2}} \tag{A.5}
\]

in \( \text{inj}_{g_i(t)}(M) \geq t^{\frac{1}{2}} \)

\( \text{diam}_{g_i(t)}(M) \leq 2(1 + D) \)

for all \( t \in (0, 1] \). Using Corollary 2.11, we may choose a monotone decreasing sequence \( (t_j) \in \mathbb{N} \subseteq (0, 1] \) that converges to zero and that satisfies

\[
d_{GH}((M_i, g_i), (M_i, g_i(t_j))) < \frac{1}{3j} \quad \forall i, j \in \mathbb{N}
\]

Estimate (A.7) implies, that for each \( m \in \mathbb{N} \)

\[
\left\|\nabla^m Rm_{g_i(t_j)}\right\|_{L^\infty(M, g_i(t_j))} \leq C(m)t_j^{\frac{2m}{3}} \quad \forall i, j \in \mathbb{N}
\]

As in the proof of Corollary 3.2 we also have

\[
v_0(D, \delta) \leq \text{Vol}_{g_i(t_j)}(M_i) = \text{Vol}_{g_i(1)}(M_i) \leq v_0(D)
\]

where we have used the non-collapsing assumption in order to prove the lower bound. Hence, at each time \( t_j \), we are able to apply [And89, Theorem 2.2, pp. 464-466] to the sequence of manifolds \((M_i, g_i(t_j))) \in \mathbb{N} \), i.e.: for all \( j \in \mathbb{N} \) there exists a subsequence \((M_{i(j,k)}, g_{i(j,k)}(t_j))) \subseteq \mathbb{N} \) converging in the \( C^{m,\alpha}-\)sense, where \( m \in \mathbb{N} \) is arbitrary, to a smooth manifold \((N_j, h_j)\) as \( k \) tends to infinity. We may assume that the selection process is organized so that each sequence \((M_{i(j,k)}, g_{i(j,k)}(t_j))) \subseteq \mathbb{N} \) is a subsequence of \((M_{i(j-1,k)}, g_{i(j-1,k)}(t_j))) \subseteq \mathbb{N} \). The smooth convergence together with (3.3) implies \( Rm_{h_j} \equiv 0 \) for each \( j \in \mathbb{N} \).

In order to apply [And89, Theorem 2.2, pp. 464-466] to the sequence \((N_j, h_j)) \subseteq \mathbb{N} \), we need an argument for a uniform lower bound on the injectivity radius because the injectivity radius estimate in (3.3) is not convenient. To overcome this issue, we recall that the volume of balls does not decay to quickly along the flow (cf. Lemma A.5) and the convergence is smooth. So, the volume of suitable balls is well-controlled from below. Since \((N_j, h_j)\) is flat, we are able to apply [CGT+82, Theorem 4.7, pp. 47-48], which yields a uniform lower bound on the injectivity radius for each \((N_j, h_j)\). Hence, there exists a subsequence of \((N_j, h_j)) \subseteq \mathbb{N} \) that converges in the \( C^\infty\)-sense, to a flat manifold \((M, g)\). Finally we need to get sure that \((M_i, g_i)) \subseteq \mathbb{N} \) contains a subsequence that also converges to \((M, g)\), at least in the Gromov-Hausdorff sense. For each \( m \in \mathbb{N} \), we choose \( j(m) \geq m \) so that

\[
d_{GH}((N_{j(m)}), (h_{j(m)})) \leq \frac{1}{3m}
\]

and \( k(m) \in \mathbb{N} \) so that

\[
d_{GH}((N_{j(m)}), (h_{j(m)}), (M_{i(j(m), k(m))}, g_{i(j(m), k(m))}(t_{j(m)}))) \leq \frac{1}{3m}
\]

This implies

\[
d_{GH}((M, g), (M_{i(j(m), k(m))}, g_{i(j(m), k(m))})) \leq d_{GH}((M, g), (N_{j(m)}), (h_{j(m)})) + d_{GH}((N_{j(m)}), (h_{j(m)}), (M_{i(j(m), k(m))}, g_{i(j(m), k(m))}(t_{j(m)})))
\]
\[ + d_{GH}(\langle M_{ij(m), k(m)} \rangle, \langle g_{ij(m), k(m)} \rangle) \]
\[ \leq \frac{1}{3m} + \frac{1}{3m} + \frac{1}{3j(m)} \leq \frac{1}{3m} + \frac{1}{3m} + \frac{1}{3m} = \frac{1}{m} \]
and this implies, that the sequence \( \langle M_{ij(m), k(m)} \rangle, \langle g_{ij(m), k(m)} \rangle \rangle \) converges with respect to the Gromov-Hausdorff topology to \((M, g)\) as \(m\) tends to infinity. \(\square\)

4. Proof of Theorem 1.2

In order to apply Theorem 1.3 to the situation in Theorem 1.2 we give a proof of the following existence result

**Theorem 4.1.** Let \(D, \Lambda > 0\). Then there are universal constants \(\delta \in (0, 1), K > 0\) and constants \(\epsilon(\Lambda), T(\Lambda) > 0\) satisfying the following property: Let \((M, g)\) be a closed Riemannian 4-manifold satisfying

\[
\text{diam}_g(M) \leq D \\
\|Rm_g\|_{L^2(M, g)} \leq \Lambda \\
\text{Vol}_g(B_g(x, r)) \geq \delta \omega_4 r^4 \quad \forall x \in M, r \in [0, 1] \\
\|\tilde{Rc}_g\|_{L^2(M, g)} \leq \epsilon
\]

then the \(L^2\)-flow exists on \([0, T]\), and we have the following estimates:

\[
\|Rm_{g(t)}\|_{L^\infty(M, g(t))} \leq K t^{-\frac{1}{2}} \\
\text{inj}_{g(t)}(M) \geq t^\frac{1}{2}
\]
and

\[
\text{diam}_{g(t)}(M) \leq 2(1 + D)
\]

for all \(t \in (0, T]\).

We point out that J. Streets has proved this result as a part of the proof of [Str16, Theorem 1.21] (cf. [Str16, pp. 285-287]). For sake of completeness, we also want to give a proof here, under the viewpoint of the dependence of \(\epsilon\) and \(T\) on given parameters and that \((4.2)\) is also satisfied.

**Proof.** We follow the lines of [Str16, pp. 285-286], giving further details. At first, we allow \(\delta \in (0, 1)\) and \(K > 0\) to be arbitrary but fixed. Along the proof, we concretize these constants. We argue by contradiction.

Suppose, there is a sequence of closed Riemannian 4-manifolds \((M_i, g_i)_{i \in \mathbb{N}}\) so that for all \(i \in \mathbb{N}\) we have the following estimates:

\[
\int_{M_i} |Rm_{g_i}|^2 g_i \, dV_{g_i} \leq \Lambda \\
\text{Vol}_{g_i}(B_{g_i}(x, r)) \geq \delta \omega_4 r^4 \quad \forall r \in [0, 1]
\]

and

\[
\int_{M_i} |\tilde{Rc}_{g_i}|^2 g_i \, dV_{g_i} \leq \frac{1}{i}
\]

but the estimates \((4.1)\) hold on a maximal interval \([0, T_i]\) where \(\lim_{i \to \infty} T_i = 0\). We consider the following sequence of rescaled metrics:

\[
\overline{g}_i(t) := T_i^{-\frac{1}{2}} g_i(T_i t)
\]
Then, for each $i \in \mathbb{N}$ the solution of the $L^2$-flow exists on $[0, 1]$ and satisfies:

\begin{equation}
\|Rm_{\gamma_i(t)}\|_{L^2(M, \gamma_i(t))} = T_i^2 \|Rm_{\gamma_i(T_i t)}\|_{L^2(M, \gamma_i(T_i t))} \leq T_i^2 K(T_i t)^{-\frac{1}{2}} K t^{-\frac{1}{2}} \nonumber
\end{equation}

\begin{equation}
in j_{\gamma_i(t)}(M_i) = T_i^{-\frac{1}{2}} i n j_{\gamma_i(T_i t)}(M_i) \geq T_i^{-\frac{1}{2}} (T_i t)^{\frac{1}{2}} = t^{\frac{1}{2}}
\end{equation}
onumber

on $[0, 1]$, which means that the estimates (4.1) are formally preserved under this kind of rescaling.

By assumption, for each $i \in \mathbb{N}$, one of the inequalities in (4.3) is an equality at time $t = 1$. In respect of the generalized Gauss-Bonnet Theorem (cf. [Sim15, Appendix A]), i.e.:

\begin{equation}
\int_M |Rm|^2 dV_g = c_0 \pi^2 \chi(M) + 4 \int_M |Rc|^2 dV_g - \int_M R^2 dV_g \nonumber
\end{equation}

\begin{equation}
= c_0 \pi^2 \chi(M) + 4 \int_M |Rc|^2 dV_g \nonumber
\end{equation}

we have used

\begin{equation}
|Rc|^2 = \left| R - \frac{1}{4} Rg \right|^2 = |Rc|^2 - \frac{1}{2} (Rc, Rg) + \frac{1}{16} R^2 |g|^2 \nonumber
\end{equation}

\begin{equation}
= |Rc|^2 - \frac{1}{2} R tr(Rc) + \frac{1}{4} R^2 = |Rc|^2 - \frac{1}{2} R^2 + \frac{1}{4} R^2 \nonumber
\end{equation}

\begin{equation}
= |Rc|^2 - \frac{1}{4} R^2 \nonumber
\end{equation}

we introduce the following functional

\begin{equation}
\mathcal{G}_g := \int_M |\bar{Rc}|_g^2 dV_g \nonumber
\end{equation}

From (4.4) and [Bes87, 4.10 Definition, p. 119] we infer

\begin{equation}
\text{grad} \, \mathcal{F} \equiv 4 \text{ grad} \, \mathcal{G} \nonumber
\end{equation}

As in the proof of Lemma A.3 we obtain for each $i \in \mathbb{N}$ and $t \in [0, T_i]$

\begin{equation}
\mathcal{G}_{\gamma_i(0)} - \mathcal{G}_{\gamma_i(t)} = \int_0^t \int_{M_i} |\text{grad} \, \mathcal{G}_{\gamma_i(s)}|_{\gamma_i(s)}^2 dV_{\gamma_i(s)} ds \geq 0 \nonumber
\end{equation}

which implies $\mathcal{G}_{\gamma_i(t)} \leq \frac{1}{t}$ for each $i \in \mathbb{N}$ and $t \in [0, T_i]$. Due to the scale invariance of the functional $\mathcal{G}$, we have in particular

\begin{equation}
\mathcal{G}_{\gamma_i(1)} \leq \frac{1}{i} \quad \text{for all} \quad i \in \mathbb{N} \nonumber
\end{equation}

As already stated, (4.3) implies

\begin{equation}
\|Rm_{\gamma_i(1)}\|_{L^2(M, \gamma_i(1))} = K \quad \text{or} \quad \text{inj}_{\gamma_i(1)}(M_i) = 1 \nonumber
\end{equation}

for each $i \in \mathbb{N}$.

At first, we assume that there is a subsequence $(M_i, \gamma_i(t))_{i \in \mathbb{N}}$ (we do not change the index) satisfying

\begin{equation}
\left\{ \begin{array}{l}
\|Rm_{\gamma_i(1)}\|_{L^2(M, \gamma_i(1))} = K \\
\text{inj}_{\gamma_i(1)}(M_i) \geq 1
\end{array} \right. \nonumber
\end{equation}

for each $i \in \mathbb{N}$. Using the compactness, for each $j \in \mathbb{N}$ we may choose a point $p_i \in M_i$ satisfying $|Rm_{\gamma_i(1)}(p_i)|_{\gamma_i(1)} = K$. From [Str13b, Corollary 1.5, p. 42] we conclude that there exists a subsequence of manifolds, also index by $i$, and a
complete pointed 4-manifold \((M_\infty, p_\infty)\) together with a 1-parametrized family of Riemannian metrics \((g_\infty(t))_{t \in [1/2, 1]}\) on \(M_\infty\) such that for each \(t \in [1/2, 1] \)
\[
(M_\infty, \overline{g}_i(t), p_i) \overset{\|\cdot\|^{L_\infty}}{\to} (M_\infty, g_\infty(t), p_\infty)
\]
in the sense of \(C^\infty\)-local submersions (cf. \cite[Definition 2.4, p. 45]{Str13b}), and
\[
\|\text{Rm}_{g_\infty(1)}\|_{L_\infty(M_\infty, g_\infty(1))} = |\text{Rm}_{g_\infty(1)}(p_\infty)|_{g_\infty(1)} = K
\]
as well as, using \cite[Theorem]{Sak83}
\[
inj_{g_\infty(1)}(M_\infty) \geq 1
\]
Since \(\lim_{i \to \infty} G_{i(1)} = 0\) we conclude that \((M_\infty, g_\infty(1), p_\infty)\) needs to be an Einstein manifold satisfying
\[
\int_{M_\infty} |\text{Rm}_{g_\infty(1)}|^2_{g_\infty(1)} \, dV_{g_\infty(1)} \leq A
\]
In particular, \cite[Proposition 7.8, p. 125]{Lee97} implies that the scalar curvature is constant. On the other hand, from the non-collapsing condition and \eqref{A.9} we obtain that \(\text{Vol}_{g_\infty(1)}(M_i)\) tends to infinity as \(i \in \mathbb{N}\) tends to infinity. Then, estimate \eqref{4.5} implies that the scalar curvature needs to vanish on \((M_\infty, \overline{g}_\infty(1))\), hence \((M_\infty, \overline{g}_\infty(1))\) is a Ricci-flat manifold. From Lemma \ref{A.7} we obtain
\[
\|\text{Rm}_{\overline{g}_\infty(1)}\|_{L_\infty(M_\infty, \overline{g}_\infty(1))} \leq C
\]
where \(C\) is a universal constant, since the space dimension is fixed and the injectivity radius is bounded from below by 1. Choosing \(K = C + 1\) we obtain a contradiction to \(|\text{Rm}_{\overline{g}_\infty(1)}(p_\infty)|_{\overline{g}_\infty(1)} = k\). This finishes the part of the proof that \(\|\text{Rm}_{g_i(T_i)}\|_{L_\infty(M_i, g_i(T_i))} = KT_i^{-\frac{1}{2}}\) can only be valid for a finite number of \(i \in \mathbb{N}\).

Now we assume that, after taking a subsequence, we are in the following situation
\[
\left\{ \begin{array}{l}
\|\text{Rm}_{\overline{g}_\infty(1)}\|_{L_\infty(M_i, \overline{g}_\infty(1))} \leq K \\
inj_{\overline{g}_\infty(1)}(M_i) = 1
\end{array} \right.
\]
Then, the non-collapsing assumption of the initial sequence implies the following non-collapsing condition concerning the rescaled metrics
\[
\text{Vol}_{\overline{g}_i(0)}(B_{\overline{g}_i(0)}(x, r)) \geq \omega_4 r^4 \quad \forall x \in M_i, r \in [0, T_i^{-\frac{1}{2}}]
\]
Hence, for each \(\sigma \geq 1\) there exists \(i_0(\sigma) \in \mathbb{N}\) so that
\[
\text{Vol}_{\overline{g}_i(0)}(B_{\overline{g}_i(0)}(x, r)) \geq \omega_4 r^4 \quad \forall x \in M_i, r \in (0, \sigma]
\]
for all \(i \geq i_0(\sigma)\). Now let \(\lambda \in (0, 1)\) be fixed. This constant will be made explicit below. Using \eqref{A.10} we obtain for \(i \geq i_0(\sigma, \lambda, \delta)\)
\[
\left[\text{Vol}_{\overline{g}_i(1)}(B_{\overline{g}_i(0)}(x, \lambda \sigma))\right]^{\frac{1}{2}} \geq \left[\text{Vol}_{\overline{g}_i(0)}(B_{\overline{g}_i(0)}(x, \lambda \sigma))\right]^{\frac{1}{2}} - C \left(\frac{1}{i}\right)^{\frac{1}{2}}
\]
\[
\geq \left[\delta \omega_4 (\lambda \sigma)^4\right]^{\frac{1}{2}} - C \left(\frac{1}{i}\right)^{\frac{1}{2}} = \left[\left(1 - (1 - \delta)\right)\omega_4 \lambda^4 \sigma^4\right]^{\frac{1}{2}} - C \left(\frac{1}{i}\right)^{\frac{1}{2}}
\]
where the last estimate does not use that \(i_0\) depends on \(\sigma\), because, in order to choose \(i_0 \in \mathbb{N}\) large enough one may fix \(\sigma = 1\) at first. Afterwards, one may multiply the inequality by \(\sigma^2\). Since \(\sigma \geq 1\), the desired estimate follows.
It is our intention to prove that
\[
B_{\overline{g}_i(0)}(x, \lambda \sigma) \subseteq B_{\overline{g}_i(1)}(x, \sigma) \quad \forall i \geq i_0(\sigma, \lambda, \delta), \forall x \in M_i
\]
Before proving this, we demonstrate that this fact implies a contradiction.
For each $i \in \mathbb{N}$ we choose a point $p_i \in M_i$ satisfying
\[ \inf_{\gamma_{t(1)}}(M_i, p_i) = \inf_{\gamma_{t(1)}}(M_i) = 1 \]
As above, using [Str13b, Corollary 1.5, p. 42], we may assume that there exists a subsequence of manifolds, again indexed by $i$, and a complete pointed 4-manifold $(M_\infty, p_\infty)$ as well as a 1-parametrized family of Riemannian metrics $(g_\infty(t))_{t \in [1/2, 1]}$ on $M_\infty$ so that for each $t \in [1/2, 1]$
\[ (M_i, \overline{\gamma}_t(t), p_i) \xrightarrow{t \to \infty} (M_\infty, g_\infty(t), p_\infty) \]
in the sense of $C^\infty$-local submersions. Using [Sak83, Theorem] we infer
\[ \inf_{\gamma_{\infty(1)}}(M_\infty, p_\infty) = 1 \]
Let $\zeta > 0$ be equal to the non-collapsing parameter in [And90, Gap Lemma 3.1, p. 440] which is denoted by "c" in that work and only depends on the space dimension $n = 4$. We assume $\delta \in (0, 1)$ and $\lambda \in (0, 1)$ to be close enough to 1 so that
\[ (1 - 2(1 - \delta))\lambda^4 \geq 1 - \zeta \]
Assumed (4.8) is valid, then for each for $i \geq i_0(\sigma, \lambda, \delta)$ we obtain the following estimate
\[ Vol_{\gamma_{\infty(1)}}(B_{\gamma_{\infty(1)}}(p_i, \sigma)) \geq Vol_{\gamma_{\infty(1)}}(B_{\gamma_{\infty(1)}}(p_i, \lambda \sigma)) \geq (1 - \zeta)\omega_4 \sigma^4 \]
and finally, as $i \in \mathbb{N}$ tends to infinity
\[ Vol_{\gamma_{\infty(1)}}(B_{\gamma_{\infty(1)}}(p_\infty, \sigma)) \geq (1 - \zeta)\omega_4 \sigma^4 \quad \forall \sigma \geq 1 \]
Then [And90, Gap Lemma 3.1, p. 440] implies that $(M_\infty, g_{\infty(1)})$ is isometric to $(\mathbb{R}^4, g_{can})$ which contradicts (4.9).

Hence, in order to prove the existence result and the validity of (4.1), it remains to prove (4.8). From here on we do not write the subindex $i \in \mathbb{N}$. The following considerations shall be understood with $i \in \mathbb{N}$ fixed. That means that $p$ is one of the points $p_i$ and $\overline{\gamma}(t)$ is the metric $\overline{\gamma}(t)$ on $M = M_i$ with the same index. Let
\[ y \in B_{\overline{\gamma}(0)}(p, \lambda \sigma) \]
be an arbitrary point. As in the proof of Lemma 2.4 we construct a suitable forward-geodesic: Let
\[ \beta := \min_{t \in [0, 1]} \beta_t > 0 \]
where
\[ \beta_t := \beta(4, \operatorname{diam}_{\gamma(t)}(M), f_3(M, \overline{\gamma}(t)), \inf_{\gamma(t)}(M)) \]
is chosen according to Lemma 2.3. Next, using Lemma 2.6, we construct a $\beta$-forward-geodesic connecting $p$ and $y$ which is denoted by $\{\xi_t\}_{t \in [0, 1]}$. Hence, we have a finite set of geodesics $\{\xi_j S\}_{j \in \{0, ..., [\frac{1}{S}]\}}$ which are parametrized proportional to arc length, i.e.:
\[ \left[ \xi_j S \right]_{p(jS)} \equiv d(p, y, jS) \quad \text{for all } j \in \{0, ..., \left[\frac{1}{S}\right]\} \]
Furthermore, for each $j \in \{0, ..., \left[\frac{1}{S}\right]\}$ let
\[ \varphi : [0, d(p, y, jS)] \longrightarrow [0, 1] \]
\[ \varphi(s) = \frac{s}{d(p, y, jS)} \]
and let
\[ \gamma_t := \xi_j S \circ \varphi_{jS} \quad \text{for each } t \in [jS, (j + 1)S) \cap [0, 1] \]
Applying the same argumentation as in the proof of Lemma 2.4 we ensure that for each \( j \in \{0, \ldots, \lfloor \frac{1}{S} \rfloor \} \) and \( t \in [jS, (j + 1)S] \cap [0, 1] \) the tubular neighborhood \( D(\gamma_t, \rho_t) \) is foliated by \( (D(\gamma_t(s), \rho_t))_{s \in [0, d(p, y, jS)]} \) where
\[
\rho_t := \mu \min \left\{ \min_{\mathcal{M}(t)}(M), f_0(M, \gamma(t))^{-\frac{1}{2}} \right\}
\]
and the differential of the projection map satisfies (2.4). Here \( \mu > 0 \) is chosen fixed but also compatible to [Str16, Lemma 2.7, p. 268]. We want to give a controlled lower bound on \( \rho_t \). The curvature decay estimate from (4.3) together with (A.7) implies for each \( m \in \{1, 2, 3\} \):
\[
\|D_m R_{\gamma(t)}\|_{L^\infty(M, g(t))} \leq C(m) t^{-\frac{2 + m}{4}} \text{ for all } t \in (0, 1]
\]
From this, we infer
\[
\rho_t \geq \mu \left\{ \min \left\{ \frac{1}{2}, C^{-\frac{1}{2}} t^2 \right\} \right\} \geq \mu \min \{1, C^{-\frac{1}{2}} t^2 \} \frac{1}{\rho_t} =: R t \frac{1}{\rho_t} =: \rho_t
\]
we also obtain the estimate
\[
\frac{d}{dt} L(\gamma_t, t) \leq C_2 R^{-\frac{1}{2}} t^{-\frac{1}{2}} \left( \int_M |\nabla F(\gamma(t))|^2 dV_{\gamma(t)} \right)^{\frac{1}{2}} L^2(\gamma_t, t)
\]
\[
+ C_2 R t^{-\frac{1}{2}} L(\gamma_t, t)
\]
on \( (jS, (j + 1)S) \cap [0, 1) \) where \( j \in \{1, \ldots, \lfloor \frac{1}{S} \rfloor \} \). Now we assume that
\[
j_0 := \min \left\{ j \in \{1, \ldots, \lfloor \frac{1}{S} \rfloor \} \mid \exists t \in [jS, (j + 1)S] \cap (0, 1) \text{ s. th. } L(\gamma_t, t) = \sigma \right\}
\]
exists, and let
\[
t_0 := \sup \{ t \in [j_0S, (j_0 + 1)S] \cap (0, 1) \mid L(\gamma_t, t) \leq \sigma \ \forall t \in [j_0S, t] \}
\]
Then, for each \( j \in \{0, \ldots, j_0 - 1\} \) and \( t \in (jS, (j + 1)S) \cap (0, t_0) \) estimate (4.13) implies
\[
\frac{d}{dt} L(\gamma_t, t) \leq \sigma \left[ C_2 R^{-\frac{1}{2}} t^{-\frac{1}{2}} \left( \int_M |\nabla F(\gamma(t))|^2 dV_{\gamma(t)} \right)^{\frac{1}{2}} + C_2 R t^{-\frac{1}{2}} \right]
\]
and consequently
\[
d(p, y, t) - d(p, y, jS) \leq L(\gamma_t, t) - L(\gamma_{jS}, jS)
\]
\[
\leq \sigma C_2 R^{-\frac{1}{2}} \int_{jS}^t s^{-\frac{1}{2}} \left( \int_M |\nabla F(\gamma(s))|^2 dV_{\gamma(s)} \right)^{\frac{1}{2}} ds + \sigma C_2 R \int_{jS}^t s^{-\frac{1}{2}} \frac{ds}{s}
\]
In particular, for each \( j \in \{0, \ldots, j_0 - 1\} \) we infer
\[
d(p, y, (j + 1)S) - d(p, y, jS)
\]
\[
\leq \sigma C_2 R^{-\frac{1}{2}} \int_{jS}^{(j + 1)S} s^{-\frac{1}{2}} \left( \int_M |\nabla F(\gamma(s))|^2 dV_{\gamma(s)} \right)^{\frac{1}{2}} ds + \sigma C_2 R \int_{jS}^{(j + 1)S} s^{-\frac{1}{2}} \frac{ds}{s}
\]
and
\[
L(\gamma_{t_0}, t_0) = d(p, y, j_0S)
\]
\[
\leq L(\gamma_{t_0}, t_0) - L(\gamma_{j_0S}, j_0S)
\]
\[
\leq \sigma C_2 R^{-\frac{1}{2}} \int_{j_0 S}^{t_0} s^{-\frac{1}{2}} \left( \int_M |\nabla F_{\gamma(s)}|^2 dV_{\gamma(s)} \right)^{\frac{1}{2}} ds + \sigma C_2 R \int_{j_0 S}^{t_0} s^{-\frac{3}{2}} ds
\]
and finally
\[
L(\gamma_{t_0}, t_0) - d(p, y, 0)
\]
\[
\leq L(\gamma_{t_0}, t_0) - d(p, y, j_0 S) + \sum_{j=0}^{j_0-1} [d(p, y, (j+1)S) - d(p, y, jS)]
\]
\[
\leq \sigma \left[ C_2 R^{-\frac{1}{2}} \int_{j_0 S}^{t_0} s^{-\frac{1}{2}} \left( \int_M |\nabla F_{\gamma(s)}|^2 dV_{\gamma(s)} \right)^{\frac{1}{2}} ds + C_2 R \int_{j_0 S}^{t_0} s^{-\frac{3}{2}} ds \right]
\]
\[
\leq \sigma \left[ C_2 R^{-\frac{1}{2}} \left( \int_{j_0 S}^{1} s^{-\frac{1}{2}} ds \right)^{\frac{1}{2}} \left( \int_{j_0 S}^{1} \int_M |\nabla F_{\gamma(s)}|^2 dV_{\gamma(s)} ds \right)^{\frac{1}{2}} + C_2 R \right]
\]
\[
\leq \sigma \left[ C_4 R^{-\frac{1}{2}} \left( \int_{j_0 S}^{1} \int_M |\nabla F_{\gamma(s)}|^2 dV_{\gamma(s)} ds \right)^{\frac{1}{2}} + C_4 R \right]
\]
\[
\leq \sigma \left[ C_4 R^{-\frac{1}{2}} G_{\gamma(0)}^{\frac{1}{2}} + C_4 R \right]
\]
\[
= \sigma \left[ C_4 R^{-\frac{1}{2}} G_{\gamma(0)}^{\frac{1}{2}} + C_4 R \right]
\]
Together with (4.11) we obtain
\[
L(\gamma_{t_0}, t_0) < \sigma \left[ \lambda + C_4 R^{-\frac{1}{2}} G_{\gamma(0)}^{\frac{1}{2}} + C_4 R \right]
\]
Throughout, we may assume that \( R > 0 \) is small enough compared to \( C_4 > 0 \) and \( \lambda > 0 \) in order to ensure that
\[
C_4 R \leq \frac{1 - \lambda}{2}
\]
and we may assume that \( i \in \mathbb{N} \) is chosen large enough, so that \( G_{\gamma(0)} = G_{\gamma_i} \leq \frac{1}{4} \) is small enough compared to \( \lambda > 0 \), \( R(\lambda) > 0 \) and \( C_4 > 0 \) so that
\[
C_4 R^{-\frac{1}{2}} G_{\gamma(0)}^{\frac{1}{2}} \leq \frac{1 - \lambda}{2}
\]
Hence, we have \( L(\gamma_{t_0}, t_0) < \sigma \), which contradicts \( L(\gamma_{t_0}, t_0) = \sigma \). This implies that \( L(\gamma_0, t) < \sigma \) is valid for each \( t \in [0, 1] \) and consequently \( d(p, y, 1) < \sigma \). This finishes the proof of (4.8).

We have proved the existence time estimate as well as the curvature decay estimate and the injectivity radius growth estimate.

It remains to show the diameter estimate (4.2). The argumentation is based on [Str16, p. 281] but we are in a different situation. Let \( x, y \in M \) so that \( d(x, y, 1) = \text{diam}_{\gamma(1)}(M) \). As above, there exists \( \beta > 0 \), \( S > 0 \) and a family of curves \( (\gamma_t)_{t \in [0, T]} \) so that
• for each \( j \in \{0, \ldots, \lceil T \rceil \} \)
  \[ \gamma_{jS} : [0, d(x, y, jS)] \to M \]
  is a unit-speed length minimizing geodesic

• for each \( j \in \{0, \ldots, \lceil T \rceil \} \) and \( t \in [jS, (j + 1)S) \cap [0, T] \) the curve
  \[ \gamma_t : [0, d(x, y, jS)] \to M \]
  satisfies
  \[ L(\gamma_t, t) \leq d(x, y, t) + \beta \]

• for each \( j \in \{0, \ldots, \lceil T \rceil \} \) and \( t \in [jS, (j + 1)S) \cap [0, T] \) the tubular neighborhood \( D(\gamma_t, r_t) \) is foliated by \( (D(\gamma_t(s), r_t))_{s \in \{0, d(x, y, jS)\}} \)

\[ r_t := R\frac{\theta}{\pi} := \min \{1, C^{-\frac{2}{3}} \} t^{\frac{23}{7}} \]

Furthermore, the projection map \( \pi \) satisfies (2.4), i.e.
\[ |d\pi| \leq 2 \text{ for all } x \in D(\gamma, r_t) \]

Using these conditions we obtain (4.13), i.e.:
\[
\frac{d}{dt}L(\gamma_t, t) \leq C_2 R^{-\frac{2}{3}} t^{-\frac{23}{7}} \left( \int_M |\text{grad } F_{g(t)}|^2 \, dV_{g(t)} \right)^{\frac{1}{2}} L^+_{\gamma_{jS}, t} + C_2 R^{-\frac{23}{7}} L(\gamma_{jS}, t)
\]
on \( (jS, (j + 1)S) \cap [0, T) \) where \( j \in \{1, \ldots, \lceil T \rceil \} \). In this situation we assume that
\[ j_0 := \min \left\{ j \in \{1, \ldots, \left\lfloor \frac{T}{2} \right\rfloor \} \mid \exists t \in [j_0S, (j_0 + 1)S) \cap (0, T] \text{ s. th. } L(\gamma_t, t) = 2(1 + D) \right\} \]
exists, and we define
\[ t_0 := \sup \{ t \in [j_0S, (j_0 + 1)S) \cap (0, T] \mid L(\gamma_t, \tau) \leq 2(1 + D) \forall \tau \in [j_0S, t] \} \]

Thus, for each \( j \in \{0, \ldots, j_0\} \) we obtain
\[
\frac{d}{dt}L(\gamma_t, t) \leq C_3 R^{-\frac{2}{3}} t^{-\frac{23}{7}} \left( \int_M |\text{grad } F_{g(t)}|^2 \, dV_{g(t)} \right)^{\frac{1}{2}} (1 + D)^{\frac{1}{2}}
\]
on \( (jS, (j + 1)S) \cap (0, t_0) \). From this, we infer
\[
L(\gamma_{t_0}, t_0) - d(x, y, 0)
\]
\[ \leq L(\gamma_{t_0}, t_0) - d(x, y, j_0S) + \sum_{j=0}^{j_0-1} \left[ d(x, y, (j + 1)S) - d(x, y, jS) \right] \]
\[ \leq (1 + D)C_3 \left[ R^{-\frac{2}{3}} \int_0^{t_0} s^{-\frac{23}{7}} \left( \int_M |\text{grad } F_{g(s)}|^2 \, dV_{g(s)} \right)^{\frac{1}{2}} \, ds + R \int_0^{t_0} s^{-\frac{23}{7}} \, ds \right] \]
\[ \leq (1 + D)C_3 \left[ R^{-\frac{2}{3}} \int_0^{1} s^{-\frac{23}{7}} \left( \int_M |\text{grad } F_{g(s)}|^2 \, dV_{g(s)} \right)^{\frac{1}{2}} \, ds + R \int_0^{t_0} s^{-\frac{23}{7}} \, ds \right] \]
\[ \leq (1 + D)C_3 R^{-\frac{2}{3}} \left( \int_0^{1} s^{-\frac{23}{7}} \, ds \right)^{\frac{1}{2}} \left( \int_0^{1} \int_M |\text{grad } F_{g(s)}|^2 \, dV_{g(s)} \, ds \right)^{\frac{1}{2}}
\]
\[ + (1 + D)C_3 R \int_0^{t_0} s^{-\frac{23}{7}} \, ds \]
\[ \leq (1 + D)C_4 R^{-\frac{3}{2}} \left( \int_0^1 s^{-\frac{3}{2}} ds \right)^\frac{1}{2} \left( \int_M \| \nabla G_{g(s)} \|^2 dV_{g(s)} ds \right)^\frac{1}{2} + (1 + D)C_4 R \int_0^R s^{-\frac{3}{2}} ds \]

\[ \leq (1 + D)C_4 \left[ R^{-\frac{3}{2}} \left( \int_0^1 s^{-\frac{3}{2}} ds \right)^\frac{1}{2} G_{g(0)}^\frac{3}{2} + R \int_0^1 s^{-\frac{3}{2}} ds \right] \]

\[ \leq (1 + D)C_5 \left[ R^{-\frac{3}{2}} G_{g(0)}^\frac{3}{2} + R \right] < 1 + D \]

Here, we have assumed that $G_{g(0)}^\frac{3}{2}$ and $R > 0$ are sufficiently small with respect to universal constants. Finally, we obtain

\[ L(\gamma_{t_0}, t_0) < d(x, y, 0) + 1 + D = D + 1 + D < 2(1 + D) \]

contradicting $L(\gamma_{t_0}, t_0) = 2(1 + D)$. This shows, that we have $\text{diam}_{g(t)}(M) \leq 2(1 + D)$ for all $t \in [0, T]$.

This existence result allows us to prove the following diffeomorphism finiteness result:

**Corollary 4.2.** Let $D, \Lambda > 0$. There exists $\epsilon(\Lambda) > 0$ and a universal constant $\delta \in (0, 1)$ so that there are only finitely many diffeomorphism types of closed Riemannian 4-manifolds $(M, g)$ satisfying

\[ \text{diam}_g(M) \leq D \]
\[ \| Rm_g \|_{L^\infty(M, g)} \leq \Lambda \]
\[ \text{Vol}_g(B_g(x, r)) \geq \delta \omega_4 r^4 \quad \forall x \in M, r \in [0, 1] \]
\[ \| Rc_g \|_{L^\infty(M, g)} \leq \epsilon \]

**Proof.** Suppose there exists a sequence of Riemannian 4-manifolds $(M_i, g_i)_{i \in \mathbb{N}}$ satisfying the desired properties but the elements in this sequence are pairwise not diffeomorphic. Using Theorem 4.1 we may smooth out each of these manifolds, then we may apply [And89, Theorem 2.2, pp. 464-466] at a fixed later time point which yields a contradiction. \qed

**Proof of Theorem 1.2.** The proof is nearly analogous to the proof of Theorem 1.1 but the argumentation is slightly different. Throughout, using Corollary 4.2, we assume that $M_i = M$ for all $i \in \mathbb{N}$, applying Theorem 4.1, we may assume, that for each $i \in \mathbb{N}$ the $L^2$-flow on $M$ with initial data $g_i$ exists on $[0, T]$ and satisfies (1.4), (1.5), (1.6) and (1.7). Using Corollary 2.11, we choose a monotone decreasing sequence $(t_j)_{j \in \mathbb{N}} \subseteq (0, 1]$ converging to zero, so that

\[ d_{GH}((M, g_i), (M, g_j(t_j))) < \frac{1}{3j} \quad \forall i, j \in \mathbb{N} \]

(1.5) and (A.7) together imply

\[ \| \nabla^m Rm_{g(t_j)} \|_{L^\infty(M, g(t_j))} \leq C(m)t_j^{-\frac{2m+8}{4}} \quad \forall i, j \in \mathbb{N} \]

for each $m \in \mathbb{N}$, (1.6) implies

\[ \text{inj}_{g(t_j)}(M) \geq t_j^\frac{4}{3} \quad \forall i, j \in \mathbb{N} \]

Applying the same argumentation as in the proof of Theorem 1.1 we infer

\[ v_0(\delta) \leq \text{Vol}_{g(t_j)}(M) \leq V_0(D, \Lambda) \]
for all \(i, j \in \mathbb{N}\). We want to point out that \(\delta > 0\) only depends on the space dimension which is constant. Using the flow convergence result in [Str13b, Corollary 1.5, p. 42] on each time interval \([t_j+1, t_j]\), starting with \(t_0\), we obtain a subsequence \((M_{i(j,k)}, g_{i(j,k)}(t_j))_{k \in \mathbb{N}}\) as well as a family of Riemannian manifolds \((M, g_{\infty,j}(t))_{t \in [t_j+1, t_j]}\) so that for each \(t \in [t_j+1, t_j]\) the sequence of Riemannian manifolds \((M, g_{i(j,k)}(t))_{k \in \mathbb{N}}\) converges smoothly to \((M, g_{\infty,j}(t))\) and the family of manifolds \((M_{\infty,j}, g_{\infty,j}(t))_{t \in [t_j+1, t_j]}\) is also a solution to the \(L^2\)-flow in the sense of [Str13b, Corollary 1.5, p. 42]. Since \(G_{g_{i(t)}} \leq G_{g_{\infty,j}} \leq \frac{1}{\delta}\) for all \(i \in \mathbb{N}\), we conclude that \(G_{g_{\infty,j}}(t) = 0\) for all \(t \in [t_{j+1}, t_j]\). Hence, at infinity, the metric does not change along the interval \([t_{j+1}, t_j]\), which means that the manifold \((M_{\infty,j}, g_{\infty,j}(t_j)) = (M_{\infty,j}, g_{\infty,j}(t_{j+1})) = (M, g)\) is an Einstein manifold. Inductively, we obtain for each \(j \in \mathbb{N}\) a sequence \((M_{i(j,k)}, g_{i(j,k)}(t_j))_{k \in \mathbb{N}}\) that is a subsequence from \((M_{i(j-1,k)}, g_{i(j-1,k)}(t_j))_{k \in \mathbb{N}}, \) so that the sequence \((M_{i(j,k)}, g_{i(j,k)}(t_j))_{k \in \mathbb{N}}\) converges to the Einstein manifold \((M, g)\). Using the same diagonal choice as in the Proof of Theorem 1.1, we infer that there exists a subsequence of \((M_i, g_i)_{i \in \mathbb{N}}\) that also converges in the Gromov-Hausdorff topology to \((M, g)\).

\[\square\]

**Appendix A. Auxiliary Results**

In this paragraph we present some results which we have used in this work. Most of them are quoted from [Str16].

**Lemma A.1.** Let \((M^n, g(t))_{t \in [t_1, t_2]}\) be a smooth family of Riemannian manifolds and let \(\gamma : [0, L] \to M\) be a smooth curve. Then we have the estimates:

\[
\begin{align*}
(\text{A.1}) & \quad \left| \frac{d}{dt} L(\gamma, t) \right| \leq \int_\gamma |g'(t)|_{g(t)} d\sigma_t \\
(\text{A.2}) & \quad \left| \log \left( \frac{|v|^2_{g(t)}}{|v|^2_{g(t_1)}} \right) \right| \leq \int_{t_1}^{t_2} \|g'(t)\|_{L^\infty(M, g(t))} dt \quad \forall v \in TM \\
(\text{A.3}) & \quad \left| \frac{\partial}{\partial t} |\nabla_\gamma \gamma|_{g(t)}^2 \right| \leq |g'(t)|_{g(t)} |\nabla_\gamma \gamma|_{g(t)}^2 + C(n)|\gamma|_{g(t)}^2 |\nabla_\gamma \gamma|_{g(t)} \|\nabla g'\|_{g(t)} \\
\end{align*}
\]

on \(M \times (t_1, t_2)\).

**Proof.** Using a unit-speed-parametrization of \(\gamma\) we infer (A.1). Estimate (A.2) is proved in [Ham82, 14.2 Lemma, p. 279]. Estimate (A.3) is stated in [Str16, p. 271]. \(\square\)

A simple calculation shows the following scaling behavior of the quantity \(f_k\) (cf. Definition 2.2).

**Lemma A.2.** Let \((M^n, g)\) be a closed Riemannian manifold, \(k \in \mathbb{N}\), \(x \in M\) and \(c > 0\). Then we have the following equality

\[
(\text{A.4}) \quad f_k(x, cg) = c^{-1} f_k(x, g)
\]

From the definition of the gradient in [Bes87, 4.10 Definition, p. 119] we obtain:

**Lemma A.3.** Let \((M^n, g(t))_{t \in [0, T]}\) be a smooth solution to the flow given in (1.3) then we have:

\[
(\text{A.5}) \quad \int_0^t \int_M |\nabla F_{g(s)}|^2 dV_{g(s)} ds = F(g(0)) - F(g(t))
\]

for all \(t \in [0, T]\).
In particular, we can see that the energy \( \mathcal{F}(g(t)) \) is monotone decreasing under the flow given in (1.3), and

\[
\int_0^t \int_M |\nabla \mathcal{F}_{g(s)}|^2 dV_{g(s)} \, ds \leq \epsilon
\]

for all \( t \in [0, T] \) under the assumption that \( \mathcal{F}(g_0) \leq \epsilon \nabla^n Rm_{g(t)} \subset L^\infty(M, g(t)) \leq A \) for all \( t \in (0, T) \).

\[
\|\nabla^n Rm_{g(t)}\|_{L^\infty(M, g(t))} \leq C \left( (A + 1)t^{-\frac{n}{4}} \right)^{1 + \frac{n}{2}}
\]

**Theorem A.4.** ([Str16, Lemma 2.11, p. 269]) Fix \( m, n \geq 0 \). There exists a constant \( C(n, m) > 0 \) so that if \((M^n, g(t))_{t \in [0, T]}\) is a complete solution to the \( L^2 \)-flow satisfying

\[
(Vol(M^n, g(t)))^{\frac{4}{n}} \leq \epsilon
\]

then for all \( t \in (0, T), \)

\[
\|\nabla^n Rm_{g(t)}\|_{L^\infty(M, g(t))} \leq C \left( (A + 1)t^{-\frac{n}{4}} \right)^{1 + \frac{n}{2}}
\]

**Lemma A.5.** Let \( M^4 \) be a closed Riemannian manifold and \((M, g(t))_{t \in [0, T]}\) be a solution to the \( L^2 \)-flow. We have the following estimates

\[
Vol_{g(t)}(M) = Vol_{g(0)}(M) \quad \text{for all } t \in (0, T]
\]

and

\[
Vol_{g(t)}(U) = Vol_{g(0)}(U) - C t^{\frac{n}{4}} \left( \int_0^t \int_U |\nabla F_{g(s)}|^2 dV_{g(s)} \, ds \right)^{\frac{1}{2}}
\]

for all \( t \in (0, T] \) and \( U \subseteq M \) open.

**Proof.** The equation (A.9) is a special case of the first equation in [Str13b, p. 44]. Furthermore

\[
\left[ Vol_{g(t)}(U) \right]^{\frac{1}{2}} - \left[ Vol_{g(0)}(U) \right]^{\frac{1}{2}} = \int_0^t \frac{d}{ds} \left[ Vol_{g(s)}(U) \right]^{\frac{1}{2}} \, ds = \frac{1}{2} \int_0^t \frac{d}{dt} \left[ Vol_{g(t)}(U) \right]^{\frac{1}{2}} \, dt
\]

\[
= -\frac{1}{4} \int_0^t \int_U \text{tr}_{g(s)} \nabla F_{g(s)} dV_{g(s)} \, ds
\]

\[
\geq -\frac{1}{4} \int_0^t \left( \int_U \text{tr}_{g(s)} \nabla F_{g(s)} \right)^2 dV_{g(s)} \, ds - \frac{1}{2} \left[ Vol_{g(t)}(U) \right]^{\frac{1}{2}} \, ds
\]

\[
\geq -C \int_0^t \left( \int_U |\nabla F_{g(s)}|^2 dV_{g(s)} \right)^{\frac{1}{2}} \, ds
\]

\[
\geq -C t^{\frac{n}{4}} \left( \int_0^t \int_U |\nabla F_{g(s)}|^2 dV_{g(s)} \, ds \right)^{\frac{1}{2}}
\]

**Lemma A.6.** ([Str16, Lemma 2.8, p. 268]) Let \((M, g)\) and \((N, h)\) be smooth Riemannian manifolds and let \( F : M \to N \) be a smooth submersion. Furthermore, let \( \phi : M \to [0, \infty) \) be a smooth function, then one has:

\[
\int_M \phi \, dV_g = \int_{g \in N} \int_{x \in F^{-1}(y)} \frac{\phi(x)}{\text{NJac} F(x)} dF^{-1}(y) \, dV_h
\]

where \( \text{NJac} F(x) \) is the determinant of the derivative restricted to the orthogonal complement of its kernel. This quantity is also called “normal Jacobian”.

\[\square\]
Lemma A.7. Let \( n \in \mathbb{N}, \iota > 0 \) and let \((M^n, g)\) be a complete \( n \)-dimensional Riemannian manifold such that the following is true

\[
\begin{align*}
Rc_g & \equiv 0 \\
\|Rm_g\|_{L^\infty(M^n, g)} & < \infty \\
inj_g(M) & \geq \iota
\end{align*}
\]

then

\[
\|Rm_g\|_{L^\infty(M^n, g)} \leq C(n, \iota).
\]

Proof. We argue by contradiction. Suppose this statement would be wrong, then we could find a sequence of complete \( n \)-dimensional Ricci-flat manifolds \((M_i, g_i)\) \( i \in \mathbb{N} \) so that

\[
inj_{g_i}(M_i) \geq \iota\]

and

\[
\|Rm_{g_i}\|_{L^\infty(M_i, g_i)} = C_i
\]

where

\[
\lim_{i \to \infty} C_i = \infty
\]

We construct a blow-up sequence as follows: for each \( i \in \mathbb{N} \) let

\[
h_i := C_i \cdot g_i
\]

so that

\[
inj_{h_i}(M_i) \geq \sqrt{C_i} \iota
\]

and

\[
\|Rm_{h_i}\|_{L^\infty(M_i, h_i)} = 1
\]

For each \( i \in \mathbb{N} \) we choose a fixed point \( p_i \in M_i \), so that \( |Rm_{h_i}(p_i)|_{h_i} \geq \frac{1}{2} \). Using \( Rc_{h_i} \equiv 0 \), the first equation on [And89, p. 461] or [Ham82, 7., 7.1. Theorem, p. 274] implies

(A.12) \[
\Delta_{h_i} Rm_{h_i} = Rm_{h_i} \ast Rm_{h_i}
\]

and consequently

\[
\|\Delta_{h_i} Rm_{h_i}\|_{L^\infty(M_i, h_i)} \leq K(n)
\]

Furthermore, from [HKW77, Lemma 1], we obtain uniform \( C^0 \)-bounds on the metrics \((h_i)\) \( i \in \mathbb{N} \) in normal coordinates. Hence, an iterative application of the theory of linear elliptic equations of second order to (A.12), following the arguments of [And89, p. 478, second paragraph], we obtain uniform higher order estimates, i.e.: for all \( i, k \in \mathbb{N} \). Hence, [And89, Theorem 2.2, pp. 464-466] implies that there exists a subsequence \((M_i, g_i, p_i)\) \( i \in \mathbb{N} \) that converges in the pointed \( C^{k,\alpha} \)-sense, where \( k \in \mathbb{N} \) is arbitrary, to a smooth manifold \((X, h, p)\) satisfying

\[
|Rm_h(p)|_h \geq \frac{1}{2}
\]

and, using [Sak83, Theorem]

\[
inj_{h}(X, p) = \infty
\]

An iterative application of [CG+71, Theorem 2] implies that \((X, h, p) = (\mathbb{R}^n, g_{euc}, 0)\) which yields a contradiction. \( \square \)
Appendix B. Notation

Here, we give an overview of the notation that we are using in this work. Sometimes it is clear that a quantity depends on a certain metric. In this situation we often omit the dependency in the notation, i.e. $Rm_g = Rm$ for instance.

- For $i \in \{1,\ldots,n\}$ $\partial_i = \frac{\partial}{\partial x^i}$ denotes a coordinate vector in a local coordinate system
- $g_{ij}$ is a Riemannian metric in a local coordinate system and $g^{ij}$ is the inverse of the Riemannian metric
- $dV_g = dV$ is the volume form induced by a Riemannian metric $g$
- $Vol_g(\cdot) = Vol(\cdot)$ is the $n$-dimensional volume of a set in a Riemannian manifold $(M, g)$
- $dA_g = dA$ is the $n-1$-dimensional volume form induced by a Riemannian metric $g$
- $Area_g(\cdot) = Area(\cdot)$ is the $n-1$-dimensional volume of a set in a Riemannian manifold $(M, g)$
- $\omega_3$ is the euclidean volume of a euclidean unit ball
- $Rm_g = Rm$ is the Riemannian curvature tensor. As in [Str08], in local coordinates, the sign convention is consistent with [CLN06, p. 5], i.e. $R_{ijkl} = R_{ijkl}^m g_{ml}$.
- $Rc_g = Rc$ is the Ricci tensor
- $R_g = R$ is the scalar curvature
- $\frac{\partial}{\partial t} g = g'$ is the time derivative of the metric
- $\text{grad} \, F_g$ is the gradient of the functional $F_g$ with respect to $g$ (cf. [Bes87, Chapter 4, 4.10 Definition, p. 119])
- $Rc_g = Rc$ is the traceless Ricci tensor, i.e.: $Rc_g = Rc_g - \frac{1}{n} R g$
- $g^T T = \nabla T$ is the covariant derivative of a tensor $T$ with respect to $g$
- $g^{\mu n} T = \nabla^m T$ is the covariant derivative of order $m$
- $\langle T, S \rangle_g = \langle T, S \rangle$ is the inner product of two tensors
- $|T|_g = |T|$ is the norm of a tensor, i.e. $|T|_g := \sqrt{\langle T, T \rangle_g}$
- $\text{diam}_g(\cdot) = \text{diam}(\cdot)$ is the diameter of a set in a Riemannian manifold
- inj$_g(M, x)$ is the injectivity radius in a point of a Riemannian manifold
- inj$_g(M)$ is the injectivity radius of a Riemannian manifold
- $d_g(x, y) = d(x, y)$ is the distance between the points $x$ and $y$ in a Riemannian manifold
- $B_d(x, r) = B(x, r)$ is the ball of radius $r > 0$ around $x$ in a metric space
- $d_g$ is the metric which is induced by a Riemannian metric $g$
- $B_g(x, r) = B_d(x, r)$ is a metric ball in a Riemannian manifold
- $d(x, y, t)$ is the distance between the points $x$ and $y$ in a Riemannian manifold $(M, g(t))$
- $L(\gamma, t)$ is the length of a curve $\gamma$ in a Riemannian manifold $(M, g(t))$
- The notation $d\sigma$, which occurs in an integral like $\int_{\gamma} |\text{grad} \, F| \, d\sigma$, refers to the integration with respect to arc length
- $D(\gamma(t), r) / D(\gamma, r)$ is a normal disc around a point in a curve $\gamma$ / a (normal) tube around a curve $\gamma$ with radius $r$ (cf. Definition 2.1)
- $f_k(x, g) / f_k(M, g)$ is introduced in Definition 2.2
- $d\sigma$ denotes the push forward and $|d\sigma|$ denotes the operator norm of the push forward of the projection map in the context of Theorem 2.3
- $\Gamma$ denotes the local bilinear form in Definition B.1, $|\Gamma|$ is the norm of this bilinear form which is also introduced in Definition B.1

The following definition is based on [Kau76, (1), p. 261]
Definition B.1. Let \((M^n, g)\) be a smooth Riemannian manifold \(p \in M, U \subseteq M\) a star-shaped neighborhood around \(p\), and \(\varphi : U \to V\) a normal chart centered at \(p\), then for each \(q \in U\) we define a symmetric, bilinear map \(\Gamma\) as follows:

\[
\Gamma : T_q M \times T_q M \to T_q M
\]

\[
(u, v) \mapsto \Gamma^k_{ij} u^i v^j \partial_k
\]

and \(|\Gamma|\) is defined to be the smallest value \(C > 0\) so that

\[
|\Gamma(u, v)|_g \leq C|u|_g|v|_g
\]

for all \(u, v \in T_p M\).

References

[And89] Michael T Anderson. Ricci curvature bounds and Einstein metrics on compact manifolds. Journal of the American Mathematical Society, pages 455–490, 1989.

[And90] Michael T Anderson. Convergence and rigidity of manifolds under Ricci curvature bounds. Inventiones mathematicae, 102(1):429–445, 1990.

[Aub07] Erwann Aubry. Finiteness of \(\pi_1\) and geometric inequalities in almost positive Ricci curvature. Annales Scientifiques de l’École Normale Supérieure, 40(4):675–695, 2007.

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. A course in metric geometry, volume 33. American Mathematical Society Providence, 2001.

[Bes87] Arthur L Besse. Einstein manifolds. Springer Science & Business Media, 1987. First Reprint 2002.

[CG+71] Jeff Cheeger, Detlef Gromoll, et al. The splitting theorem for manifolds of nonnegative Ricci curvature. Journal of Differential Geometry, 6(1):119–128, 1971.

[CGT+82] Jeff Cheeger, Mikhail Gromov, Michael Taylor, et al. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. Journal of Differential Geometry, 17(1):15–53, 1982.

[CLN06] Bennett Chow, Peng Lu, and Lei Ni. Hamilton’s Ricci flow, volume 77. American Mathematical Soc., 2006.

[Ham82] Richard S Hamilton. Three-manifolds with positive Ricci curvature. Journal of Differential Geometry, 17(2):255–306, 1982.

[HKW77] Stefan Hildebrandt, Helmut Kaul, and Kjell-Ove Widman. An existence theorem for harmonic mappings of Riemannian manifolds. Acta Mathematica, 138(1):1–16, 1977.

[JK82] Jürgen Jost and Hermann Karcher. Geometrische Methoden zur Gewinnung von a-priori-Schranken für harmonische Abbildungen. manuscripta mathematica, 40(1):27–77, 1982.

[Kau76] Helmut Kaul. Schranken für die Christoffelsymbole. manuscripta mathematica, 20(3):261–273, 1976.

[Lee97] John M Lee. Riemannian manifolds: An Introduction to Curvature, volume 176. Springer Science & Business Media, 1997.

[O’N83] Barrett O’Neill. Semi-Riemannian Geometry With Applications to Relativity, 103, volume 103. Academic press, 1983.

[Pet06] Peter Petersen. Riemannian geometry, volume 171. Springer, 2006.

[PW97] Peter Petersen and Guofang Wei. Relative volume comparison with integral curvature bounds. Geometric & Functional Analysis GAFA, 7(6):1031–1045, 1997.

[Sak83] Takashi Sakai. On continuity of injectivity radius function. Math. J. Okayama Univ, 25(1):91–97, 1983.

[Sim15] Miles Simon. Some integral curvature estimates for the Ricci flow in four dimensions. arXiv preprint arXiv:1504.02623, 2015. version 1. URL: https://arxiv.org/pdf/1504.02623.pdf.

[Str08] Jeffrey D Streets. The gradient flow of \(\int_M |Rm|^2\). Journal of Geometric Analysis, 18(1):249–271, 2008.

[Str12a] Jeffrey Streets. The gradient flow of the \(L^2\) curvature energy near the round sphere. Advances in Mathematics, 231(1):328–356, 2012.

[Str12b] Jeffrey Streets. The gradient flow of the \(L^2\) curvature functional with small initial energy. Journal of Geometric Analysis, 22(3):691–725, 2012.

[Str13a] Jeffrey Streets. Collapsing in the \(L^2\) Curvature Flow. Communications in Partial Differential Equations, 38(6):985–1014, 2013.

[Str13b] Jeffrey Streets. The long time behavior of fourth order curvature flows. Calculus of Variations and Partial Differential Equations, 46(1-2):39–54, 2013.
[Str16] Jeffrey Streets. A Concentration-Collapse Decomposition for $L^2$ Flow Singularities. *Communications on Pure and Applied Mathematics*, 69(2):257–322, 2016.

[Top06] Peter Topping. *Lectures on the Ricci flow*, volume 325. Cambridge University Press, 2006.

[Yan92a] Deane Yang. Convergence of Riemannian manifolds with integral bounds on curvature. I. *Annales scientifiques de l’École normale supérieure*, 25(1):77–105, 1992.

[Yan92b] Deane Yang. Convergence of Riemannian manifolds with integral bounds on curvature. II. *Annales scientifiques de l’École normale supérieure*, 25(2):179–199, 1992.

[Yan92c] Deane Yang. $L^p$ pinching and compactness theorems for compact Riemannian manifolds. *Forum Mathematicum*, 4(4):323–334, 1992.

[Zer17] Norman Zergäng. Convergence of Riemannian manifolds with critical curvature bounds. PhD thesis, Otto-von-Guericke-Universität Magdeburg, 9 2017. URL: http://edoc2.bibliothek.uni-halle.de/hs/urn/urn:nbn:de:gbv:ma9:1-10313.