Surface Casimir densities and induced cosmological constant in higher dimensional braneworlds

Aram A. Saharian
Department of Physics, Yerevan State University,
1 Avel Manoogian Str., 375025 Yerevan, Armenia
and Departamento de Física-CCEN, Universidade Federal da Paraíba,
58.059-970, Caixa Postal 5.008, João Pessoa, PB, Brazil
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We investigate the vacuum expectation value of the surface energy-momentum tensor for a massive scalar field with general curvature coupling parameter obeying the Robin boundary conditions on two codimension one parallel branes in a $(D + 1)$-dimensional background spacetime $\text{AdS}_{D+1} \times \Sigma$ with a warped internal space $\Sigma$. These vacuum densities correspond to a gravitational source of the cosmological constant type for both subspaces of the branes. Using the generalized zeta function technique in combination with contour integral representations, the surface energies on the branes are presented in the form of the sum of single brane and second brane induced parts. For the geometry of a single brane both regions, on the left and on the right of the brane, are considered. At the physical point the corresponding zeta functions contain pole and finite contributions. For an infinitely thin brane taking these regions together, in odd spatial dimensions the pole parts cancel and the total zeta function is finite. The renormalization procedure for the surface energies and the structure of the corresponding counterterms are discussed. The parts in the surface densities generated by the presence of the second brane are finite for all nonzero values of the interbrane separation and are investigated in various asymptotic regions of the parameters. In particular, it is shown that for large distances between the branes the induced surface densities give rise to an exponentially suppressed cosmological constant on the brane. The total energy of the vacuum including the bulk and boundary contributions is evaluated by the zeta function technique and the energy balance between separate parts is discussed.

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I. INTRODUCTION

The braneworld scenario provides an interesting alternative to the standard Kaluza-Klein compactification of the extra dimensions. The simplest phenomenological models describing such a scenario are the five-dimensional Randall-Sundrum type braneworld models (for a review see [1, 2]). From the point of view of embedding these models into a more fundamental theory, such as string/M-theory, one may expect that a more complete version of the scenario must admit the presence of additional extra dimensions compactified on an internal manifold. From a phenomenological point of view, the consideration of more general spacetimes offer a richer geometrical structure and may provide interesting extensions of the Randall-Sundrum mechanism for the geometric origin of the hierarchy. More extra dimensions also relax the fine-tunings of the fundamental parameters. These models can provide a framework in the context of which the stabilization of the radion field naturally takes place. In addition, a richer topological structure of the field configuration in transverse space provides the possibility of more realistic spectrum of chiral fermions localized on the brane. Several variants of the Randall-Sundrum scenario involving cosmic strings and other global defects of various codimensions have been investigated in higher dimensions (see, for instance, [3] and references therein).

Motivated by the problems of the radion stabilization and the generation of cosmological constant, the role of quantum effects in braneworlds has attracted great deal of attention [4–49]. A class of higher dimensional models with the topology $\text{AdS}_{D+1} \times \Sigma$, where $\Sigma$ is a one-parameter compact manifold, and with two branes of codimension one located at the orbifold fixed points, is considered in Refs. [26, 27]. In both cases of the warped and unwarped internal manifold, the quantum effective potential induced by bulk scalar fields is evaluated and it has been shown that this potential can stabilize the hierarchy between the Planck and electroweak scales without fine tuning. In addition to the effective potential, the investigation of local physical characteristics in these models is of considerable interest. Local quantities contain more information on the vacuum fluctuations than the global ones and play an important role in modelling a self-consistent dynamics involving the gravitational field. In the previous papers [43–45] we have studied the Wightman function, the vacuum expectation values of the field square and the energy-momentum tensor for a scalar field with an arbitrary curvature coupling parameter obeying Robin boundary conditions on two codimension one parallel branes embedded in the background spacetime $\text{AdS}_{D+1} \times \Sigma$ with a warped internal space $\Sigma$. For an arbitrary internal space $\Sigma$, the application of the generalized Abel-Plana formula [50] allowed us to extract form

*Electronic address: saharyan@server.physdep.r.am
the vacuum expectation values the part due to the bulk without branes and to present the brane induced parts in terms of exponentially convergent integrals for the points away from the branes.

The braneworld corresponds to a manifold with boundaries and the physical quantities, in general, receive both volume and surface contributions. In particular, the contributions located on the visible brane are of special interest as they are direct observables in the theory. In Ref. [51] the vacuum expectation value of the surface energy-momentum tensor is discussed. The last section of the zeta function technique. The relation between the surface energy-momentum tensor. This function is constructed for the zeta function related to the vacuum expectation value of the surface energy-momentum tensor. This function is presented in the form of the sum of single brane and internal spaces.

The paper is organized as follows. In the next section we describe the corresponding bulk geometry and outline the structure of the eigenmodes and the Kaluza-Klein spectrum. An integral representation is constructed for the zeta function related to the vacuum expectation value of the surface energy-momentum tensor. This function is presented in the form of the sum of single brane and second brane induced parts. The analytic continuation of the zeta function for the model with a single brane is done in Section IV. The surface energy-momentum is investigated on both sides of the branes and the renormalization procedure is discussed. In Section V we study the surface energy density induced by the presence of the second brane. Various limiting cases are considered and the cosmological constant induced on the brane is estimated. The Section VI is devoted to the evaluation of the total vacuum energy in the two-brane setup on the base of the zeta function technique. The relation between the bulk and surface energies is discussed. The last section contains a summary of the work.

II. SURFACE ENERGY-MOMENTUM TENSOR AND THE RELATED ZETA FUNCTION

Consider a massive scalar field $\varphi(x)$ with curvature coupling parameter $\zeta$. The corresponding field equation has the form (we adopt the conventions of Ref. [51] for the metric signature and the curvature tensor)

$$\left(\nabla^M \nabla_M + m^2 + \zeta R\right) \varphi(x) = 0,$$

where $\nabla_M$ is the covariant derivative operator and $R$ is the scalar curvature for a $(D+1)$-dimensional background spacetime. For the most important special cases of minimally and conformally coupled scalars one has $\zeta = 0$ and $\zeta = \zeta_D \equiv (D-1)/4D$, respectively. We will assume that the background spacetime has the topology $\text{AdS}_{D_1+1} \times \Sigma$ and is described by the line element

$$ds^2 = g_{MN} dx^M dx^N = e^{-2k_D y} \left(\eta_{\mu\nu} dx^\mu dx^\nu - \gamma_{ik} dX^i dX^k\right) - dy^2,$$

where $\eta_{\mu\nu}$ is the metric tensor for $D_1$-dimensional Minkowski spacetime, $k_D$ is the inverse AdS radius, and the coordinates $X^i$, $i = 1, \ldots, D_2$, cover the internal manifold $\Sigma$, $D = D_1 + D_2$. The Ricci scalar corresponding to line element is given by formula $R = -D(D + 1)k_D^{-2} - e^{2k_D y} R_\gamma$, with $R_\gamma$ being the scalar curvature for the metric tensor $\gamma_{\gamma k}$. In the discussion below, in addition to the radial coordinate $y$ we will also use the coordinate $z = e^{k_D y}/kd$, in terms of which the line element is written in the form conformally related to the metric in the direct product spacetime $R^{(D_1,1)} \times \Sigma$ by the conformal factor $(k_D z)^{-2}$.

We are interested in one-loop vacuum effects induced by quantum fluctuations of the bulk field $\varphi(x)$ on two parallel branes of codimension one, located at $y = a$ and $y = b$, $a < b$. We assume that on the branes the field obeys Robin boundary conditions

$$\left(\tilde{A}_y + \tilde{B}_y \partial_y\right) \varphi(x) = 0, \quad y = a, b,$$

with constant coefficients $\tilde{A}_y, \tilde{B}_y$. This type of conditions is an extension of Dirichlet and Neumann boundary conditions and appears in a variety of situations, including the considerations of vacuum effects for a confined charged scalar field in external fields, spinor and gauge field theories, quantum gravity and supergravity. Robin boundary conditions naturally arise bulk fields in braneworld models.

The branes divide the space into three regions corresponding to $-\infty < y < a$, $a < y < b$, and $b < y < \infty$. In general, the coefficients in the boundary conditions can be different for separate regions. In a higher dimensional generalization of the Randall-Sundrum braneworld model the coordinate $y$ is compactified on an orbifold $S^1/Z_2$, of length $l$, $-l \leq y \leq l$, and the orbifold fixed points $y = 0$ and $y = l$ are the locations of two branes. The corresponding line-element has the form with the warp factor $e^{-2k_D |y|}$. In these models the region between the branes is employed only. For an untwisted bulk scalar with brane mass terms $c_a$ and $c_b$, the corresponding ratio of the coefficients in the boundary condition is determined by the expression (see, e.g., Refs. [16, 38, 52] for the case of the bulk $\text{AdS}_{D+1}$ and Refs. [26, 43] for the geometry under consideration)

$$\frac{A_y}{B_y} = -\frac{n(c) c_j + 4D \zeta_D k_D}{2}, \quad n(a) = 1, \quad n(b) = -1.$$
In the supersymmetric version of the model \[52\] one has \(c_b = -c_a\) and the boundary conditions are the same for both branes. For a twisted scalar, Dirichlet boundary conditions are obtained on both branes.

For the geometry of two parallel branes in AdS\(D_1 + 1 \times \Sigma\) with boundary conditions \[54\], the Wightman function and the vacuum expectation values (VEVs) of the field square and the bulk energy-momentum tensor are investigated in Refs. \[43, 44\]. On manifolds with boundaries the energy-momentum tensor on the brane at line element \[54\]. For an untwisted bulk scalar in the higher dimensional generalization of the Randall-Sundrum braneworld based on the bulk AdS\(D_1 + 1 \times \Sigma\) the coefficient in Eq. \[52\] is given by the formula

\[
T_{MN}^{(s)} = \delta(x; \partial M)\tau_{MN},
\]

where the "one-sided" delta-function \(\delta(x; \partial M)\) locates this tensor on \(\partial M\) and

\[
\tau_{MN} = \zeta \varphi^2 K_{MN} - (2\zeta - 1/2)h_{MN} \varphi n^L \nabla_L \varphi.
\]

In Eq. \[50\], \(h_{MN} = g_{MN} + n_M n_N\) is the induced metric on the boundary and \(K_{MN} = h^L_M h^P_N \nabla_L n_P\) is the corresponding extrinsic curvature tensor.

By expanding the field operator over a complete set of eigenfunctions \(\{\varphi_\alpha(x), \varphi^*_\alpha(x)\}\) obeying the boundary conditions and using the standard commutation rules, for the VEV of the operator \(\tau_{MN}\) one finds

\[
\langle 0 | \tau_{MN} | 0 \rangle = \sum_{\alpha} \tau_{MN} \{\varphi_\alpha(x), \varphi^*_\alpha(x)\},
\]

where the bilinear form \(\tau_{MN} \{\varphi, \psi\}\) on the right is determined by the classical energy-momentum tensor \[50\] and the collective index \(\alpha\) can contain both discrete and continuous parts. Below in this section we will consider the region between the branes. The corresponding quantities for the regions \(y < a\) and \(y > b\) are obtained as limiting cases and are investigated in the next section. As we have mentioned before, in the orbifolded version of the model, the only bulk is the one between the branes. In this region the inward-pointing normal to the brane at \(y = j\) and \(a, b\), and the corresponding extrinsic curvature tensor are given by the relations

\[
n^{(j)} M = n^{(j)} \delta^M_D, \quad K^{(j)}_{MN} = -n^{(j)} k_D h_{MN},
\]

where \(n^{(j)}\) is defined in formula \[51\]. By using relations \[51\] and the boundary conditions, the VEV of the surface energy-momentum tensor on the brane at \(y = j\) is presented in the form

\[
\langle 0 | \tau_{MN}^{(j)} | 0 \rangle = -h_{MN} n^{(j)} k_D C_j \langle 0 | \varphi^2 | 0 \rangle_{y = j}.
\]

with the notation

\[
C_j = \zeta - (2\zeta - 1/2) \tilde{A}_j/(k_D \tilde{B}_j).
\]

From the point of view of physics on the brane, Eq. \[53\] corresponds to the gravitational source of the cosmological constant type with the surface energy density

\[
\varepsilon_j^{(s)} = \langle 0 | \tau_{0}^{(j)} | 0 \rangle (\text{surface energy per unit physical volume on the brane at } y = j \text{ or brane tension}),
\]

\[
P_j^{(s)} = -(0|\tau_{1}^{(j)}|0), \quad \text{and the equation of state}
\]

\[
\varepsilon_j^{(s)} = -P_j^{(s)}.
\]

It is remarkable that this relation takes place for both subspaces on the brane. It can be seen that this result is valid also for the general metric \(g_{\mu\sigma}\) instead of \(\eta_{\mu\sigma}\) in line element \[54\]. For an untwisted bulk scalar in the geometry of two parallel branes in AdS\(D + 1 \times \Sigma\) the coefficient in Eq. \[52\] is given by the formula

\[
C_j = 4D\zeta(\zeta - \zeta_D) + \frac{\zeta - 1/4}{k_D} n^{(j)} c_j.
\]

In particular, the corresponding surface energy vanishes for minimally and conformally coupled scalar fields with zero brane mass terms.

In order to evaluate the expectation value \[53\] we need the corresponding eigenfunctions. These functions can be taken in the decomposed form

\[
\varphi_\alpha(x^M) = \frac{f_n(y) e^{-\eta_{\mu\nu} k^\mu x^\nu}}{\sqrt{2\omega_{\beta,n}(2\pi)^{D-1}}} \psi_\beta(X),
\]

where

\[
k^\mu = (\omega_{\beta,n}, k), \quad \omega_{\beta,n} = \sqrt{k^2 + m_{\beta,n}^2}, \quad k = |k|,
\]

and \(m_{\beta,n}\) are separation constants. The modes \(\psi_\beta(X)\) are eigenfunctions for the internal subspace:

\[
[\Delta(\gamma) + \zeta R(\gamma)] \psi_\beta(X) = -\lambda_{3,\beta}^2 \psi_\beta(X),
\]

with the eigenvalues \(\lambda_{3,\beta}^2\) and the orthonormalization condition

\[
\int d^{D-2} X \sqrt{\gamma} \psi_\beta(X) \psi^*_{\beta'}(X) = \delta_{\beta\beta'}.
\]

In Eq. \[56\], \(\Delta(\gamma)\) is the Laplace-Beltrami operator for the metric \(\gamma_{ij}\). In the consideration below we will assume that \(\lambda_{3,\beta}^2 \geq 0\). The dependence on \(\beta\) and \(n\) in the separation constants is factorized:

\[
m_{3,\beta,n}^2 = m_n^2 + \lambda_{3,\beta}^2,
\]

where the mass spectrum \(m_n\) is determined by the radial equation together with the boundary conditions. From the field equation \[56\], one obtains the equation for the function \(f_n(y)\) with the solution

\[
f_n(y) = C_n e^{Dk_D y_0^2/2} y_0^{(n)}(m_n z_a, m_n z_b),
\]
where the normalization coefficient $C_n$ is determined below, and

$$g^{(j)}_\nu(u, v) = J_\nu(v)Y^{(j)}_\nu(u) - Y_\nu(v)J^{(j)}_\nu(u), \quad j = a, b. \quad (19)$$

Here $J_\nu(x)$, $Y_\nu(x)$ are the Bessel and Neumann functions of the order

$$\nu = \sqrt{D^2/4 - D(D + 1)\zeta + m^2/k_D^2}, \quad (20)$$

and the z-coordinates of the branes are denoted by $z_j = e^{k_j z}/k_D$, $j = a, b$. In formula (18) and in what follows for a given function $F(x)$ we use the notation

$$\tilde{F}^{(j)}(x) = A_jF(x) + B_jxF'(x), \quad j = a, b, \quad (21)$$

with the coefficients

$$A_j = \tilde{A}_j + \tilde{B}_j k_D D/2, \quad B_j = \tilde{B}_j k_D. \quad (22)$$

In the discussion below we will assume values of the curvature coupling parameter for which $\nu$ is real. For imaginary $\nu$ the ground state becomes unstable [51, 52]. Note that for conformally and minimally coupled massless scalar fields one has $\nu = 1/2$ and $\nu = D/2$, respectively.

The function (18) satisfies the boundary condition on the brane at $y = a$. Imposing the boundary condition on the brane $y = b$ we find that the eigenvalues $m_n$ are solutions to the equation

$$g^{(ab)}_\nu(m_n z_a, m_n z_b) = 0. \quad (23)$$

where the function on the left is defined by

$$g^{(ab)}_\nu(u, v) = \bar{J}^{(a)}_\nu(u)Y^{(b)}_\nu(v) - \bar{Y}^{(a)}_\nu(u)\bar{J}^{(b)}_\nu(v). \quad (24)$$

Equation (23) determines the Kaluza-Klein (KK) spectrum along the transverse dimension. We denote by $u = \gamma_{\nu,n}$, $n = 1, 2, \ldots$, the positive zeros of the function $g^{(ab)}_\nu(u, uz_b/z_a)$, arranged in the ascending order, $\gamma_{\nu,n} < \gamma_{\nu,n+1}$. The eigenvalues for $m_n$ are related to these zeros as

$$m_n = k_D \gamma_{\nu,n} e^{-k_D a} = \gamma_{\nu,n}/z_a. \quad (25)$$

From the orthonormality condition of the radial functions,

$$\int_0^b dy e^{-(2-D)k_D y} f_n(y)f_n'(y) = \delta_{nn'}, \quad (26)$$

for the coefficient $C_n$ in Eq. (18) one finds

$$C_n^2 = \frac{\pi m_n \bar{J}^{(b)}_\nu(m_n z_b)\bar{Y}^{(a)}_\nu(m_n z_a)}{k_D \frac{\partial}{\partial \nu} g^{(ab)}_\nu(u z_a, u z_b)|_{u = m_n}}. \quad (27)$$

As it follows from formula (11), the VEV of the surface energy-momentum tensor can be obtained from the VEV of the field square evaluated on the branes. The VEV of the field square on the bulk is derived from the corresponding Wightman function in the coincidence limit.

By using the generalized Abel-Plana formula [53] for the summation over the zeros $\gamma_{\nu,n}$, in Ref. [43] this VEV is presented in the form

$$\langle 0|\phi^2|0 \rangle = \langle \phi^2 \rangle^{(0)} + \langle \phi^2 \rangle^{(j)} - 2k_D^{D-1}z^D \beta D_1 \times \sum_\beta \int_{\lambda_\beta}^{\infty} du u^2 - \chi_\beta^2 D_1/2 - 1 \times \Omega J_{\nu}(uz_a, uz_b)G^{(j)}_{\nu}(uz_j, uz), \quad (28)$$

where $j = a$ and $j = b$ provide two equivalent representations and

$$\beta D_1 = \frac{1}{(4\pi)^{D/2} \Gamma(D/2)}. \quad (29)$$

Here and in what follows we use the notations

$$\Omega_{\nu\nu}(u, v) = \frac{\tilde{K}^{(b)}_\nu(u)|\bar{K}^{(a)}_\nu(u)G^{(ab)}_\nu(u, v)|^2}{\tilde{K}^{(a)}_\nu(u)|\bar{K}^{(b)}_\nu(u)|^2}, \quad (30)$$

$$\Omega_{\nu\nu}(u, v) = \frac{\bar{I}^{(b)}_\nu(u)|\bar{I}^{(a)}_\nu(u)G^{(ab)}_\nu(u, v)|^2}{\bar{I}^{(a)}_\nu(u)|\bar{I}^{(b)}_\nu(u)|^2}. \quad (31)$$

with the modified Bessel functions $I_\nu(u), K_\nu(u)$, and

$$G^{(j)}_\nu(u, v) = I_\nu(v)K^{(j)}_\nu(u) - K^{(j)}_\nu(u)I^{(j)}_\nu(u), \quad (32)$$

$$G^{(ab)}_\nu(u, v) = K^{(a)}_\nu(u)\bar{I}^{(b)}_\nu(v) - I^{(a)}_\nu(u)\bar{K}^{(b)}_\nu(v). \quad (33)$$

The first term on the right of Eq. (28), $\langle \phi^2 \rangle^{(0)}$, corresponds to the VEV in AdS$_{D+1} \times \Sigma$ spacetime without branes, $\langle \phi^2 \rangle^{(j)}$ is induced by the presence of the second brane. After the standard renormalization of the boundary free part, the VEV given by Eq. (28) is finite in the bulk and diverges on the branes. As a result, the VEVs of the field square on the branes cannot be obtained directly from this formula and an additional renormalization procedure is necessary. One possibility is to remove from (28) the terms which diverge for the points on the branes. An alternative way is to restrict from the beginning to the fluctuations on the brane and to use the dimensional regularization or the generalized zeta function technique (for a discussion of the relation between different methods see, for instance, Ref. [41]). Here we will follow the last method.

Substituting the eigenfunctions (18) into the corresponding mode sum and integrating over the angular part of the vector $k$, for the expectation value of the energy density on the brane at $y = j$ we obtain

$$\langle \varepsilon^{(s)}_j \rangle = -2n^{(j)}l_j k_D^{D+2} B_j \beta D_1 \int_0^\infty dk k^{D_1-2} \times \sum_\beta \sum_{n=1}^\infty \frac{\lambda_\beta^{\omega_\beta,n} |\partial_{\nu} g^{(ab)}_\nu(uz_a, uz_b)|_{u = m_n}}{\omega_\beta,n}, \quad (34)$$
where \( j, l = a, b \), and \( l \neq j \). \( \omega_{\beta, n} \) is defined by formulas (34), (37), and we have used the relation \( g^{(j)}_\nu(u, u) = 2B_j/\pi \). To regularize the divergent expression on the right of this formula we define the function

\[
\Phi_j(s) = 2z_j^DB_j^\beta D_{\beta j}^{-1} \mu^{1+s} \sum_{\beta} |\psi_\beta(X)|^2 \int_0^\infty dk k^{D_{\beta j} - 2} \times \sum_{n=1}^\infty \omega_{\beta, n}^s \frac{m_n g^{(1)}_\nu(m_n z_l, m_n z_j)}{\partial \nu g^{(ab)}_\nu(u z_a, u z_b)|u = m_n},
\]

(35)

where an arbitrary mass scale \( \mu \) is introduced to keep the dimension of the expression. After the evaluation of the integral over \( k \), this expression can be presented in the form

\[
\Phi_j(s) = \frac{z_j^DB_j}{(4\pi)^{(D-1)/2}} \sum_{\beta} |\psi_\beta(X)|^2 \zeta_{j\beta}(s),
\]

(36)

where the generalized partial zeta function

\[
\zeta_{j\beta}(s) = \frac{\Gamma(-\alpha_s)}{\Gamma(-s/2)\mu^{\alpha_s}} \sum_{n=1}^\infty (m_n^2 + \lambda_\beta^2)^{\alpha_s} \times \frac{m_n g^{(1)}_\nu(m_n z_l, m_n z_j)}{\partial \nu g^{(ab)}_\nu(u z_a, u z_b)|u = m_n},
\]

(37)

is introduced with the notation

\[
\alpha_s = (D_1 + s - 1)/2.
\]

The computation of the VEV of the surface energy-momentum tensor requires the analytic continuation of the function \( \Phi_j(s) \) to the value \( s = -1 \) (here and below \( |s - 1| \) is understood in the sense of the analytic continuation),

\[
\zeta_j^{(s)} = -n^{(j)}k_B^2 C_j \Phi_j(s)|_{s = -1}.
\]

(39)

In order to obtain this analytic continuation we will follow the procedure multiply used for the evaluation of the Casimir energy (see, for instance, [44]). The starting point is the representation of the function \( \Phi_j(s) \) in terms of the contour integral

\[
\zeta_{j\beta}(s) = \frac{\Gamma(-\alpha_s)}{2\pi i \Gamma(-s/2)} \int_C du (u^2 + \lambda_\beta^2)^{\alpha_s} \times \frac{g^{(1)}_\nu(u z_l, u z_j)}{g^{(ab)}_\nu(u z_a, u z_b)},
\]

(40)

where \( C \) is a closed counterclockwise contour in the complex \( u \) plane enclosing all zeros \( m_n \). The location of these zeros enables one to deform the contour \( C \) into a segment of the imaginary axis \((-iR, iR)\) and a semicircle of radius \( R, R \rightarrow \infty \), in the right half-plane. We will also assume that the branch points \( \pm i\lambda_j \) are avoided by small semicircles \( C^\pm_\beta \) in the right half-plane with small radius \( \rho \). Here we assume that there is no zero mode corresponding to \( m_n = 0 \). The changes introduced by the presence of the zero mode will be discussed below in this section. For negative \( \text{Re} \ s \) with sufficiently large absolute value the integral over the large semicircle in Eq. (40) tends to zero in the limit \( R \rightarrow \infty \), and the expression on the right can be transformed to

\[
\zeta_{j\beta}(s) = \frac{\Gamma(-\alpha_s)}{\Gamma(-s/2)\mu^{\alpha_s}} \int C_\rho du (u^2 + \lambda_\beta^2)^{\alpha_s} \times \frac{g^{(1)}_\nu(u z_l, u z_j)}{g^{(ab)}_\nu(u z_a, u z_b)} \times \left( \frac{\mu^{-s-1}B_j}{\Gamma(-s/2)\Gamma(\alpha_s + 1)} \times \int_{\lambda_\beta}^\infty \frac{du}{u^2 + \lambda_\beta^2} \times \frac{\Omega_{j\nu}(u z_a, u z_b)}{\Omega_{j\nu}(u z_a, u z_b)} \right).
\]

(41)

For \(-D_1 - 1 < \text{Re} \ s < -D_1 \) the second integral on the right is finite in the limit \( \rho \rightarrow 0 \), whereas the first integral vanishes as \( \mu^{\alpha_s + 1} \). As a result, by using the formula \( \Gamma(-\alpha_s) \sin \pi \alpha_s = -\pi/\Gamma(\alpha_s + 1) \), in this strip of complex plane \( s \) we have the following integral representation

\[
\zeta_{j\beta}(s) = \frac{n^{(j)}\mu^{-s-1}}{\Gamma(-s/2)\Gamma(\alpha_s + 1)} \times \int_{\lambda_\beta}^\infty \frac{du}{u^2 - \lambda_\beta^2} \times \frac{\Omega_{j\nu}(u z_a, u z_b)}{\Omega_{j\nu}(u z_a, u z_b)},
\]

(42)

where

\[
\zeta_{j\beta}^{(s)} = -n^{(j)}k_B^2 C_j \Phi_j(s)|_{s = -1}.
\]

(43)

In the last formula we use the notation \( F = K \) for \( j = a \) and \( F = I \) for \( j = b \). The contribution of the second term on the right of Eq. (42) is finite at \( s = -1 \) and vanishes in the limits \( z_a \rightarrow 0 \) or \( z_b \rightarrow \infty \). The first term corresponds to the contribution of a single brane at \( z = z_j \) when the second brane is absent. The surface energy density corresponding to this term is located on the surface \( y = a + 0 \) for the brane at \( y = a \) and on the surface \( y = b - 0 \) for the brane at \( y = b \). To distinguish on which side of the brane is located the corresponding term we use the superscript \( J=L \) for the left side and \( J=R \) for the right side. As we consider the region between the branes, in formula (43) \( J=L \) for \( j = a \) and \( J=R \) for \( j = b \). The further analytic continuation is needed for the function \( \zeta_{j\beta}^{(s)}(s) \) only and this is done in the next section.

In the discussion above we have assumed that there is no zero mode corresponding to \( m_n = 0 \). For special values of the parameters of the model, satisfying the condition

\[
(A_a + B_a \nu)(A_b - B_b \nu) \left( \frac{z_a}{z_b} \right)^{2\nu} = (A_a - B_a \nu)(A_b + B_b \nu),
\]

(44)
the value \( m_n = 0 \) is a solution to the eigenvalue equation and the zero mode is present. We will denote this mode by \( m_0 = 0 \). The corresponding contribution to the VEV of the surface energy-momentum tensor and to the zeta function defined above is taken into account if we assume that the summation over \( n \) in the corresponding formulas includes the summand with \( n = 0 \) as well. The expression for this term is obtained from the formula for general \( n \) taking the limit \( m_n \to 0 \). In particular, the contribution of the zero mode to the partial zeta function \( \zeta_{j\beta}(0) \) is given by

\[
\zeta_{j\beta}(0) = \frac{\Gamma(-\alpha_s)}{\Gamma(-s/2)} \frac{\lambda_j^2 \rho_{j}^{(a)}(0,0)}{(2\pi)^{2s+1} g^2_{\rho}(u_{z_a},u_{z_b})|_{u=0}}. \tag{45}
\]

As before, the contribution of the nonzero modes, can be presented as the contour integral given by Eq. \( \text{(10)} \), but now the contour \( C \) contains an additional semicircle \( C_p(0) \) of small radius which avoids the origin in the right half-plane. Consequently, this contribution is given by the right hand side of Eq. \( \text{(11)} \) plus the integral over the contour \( C_p(0) \). Evaluating the latter integral we can see that it cancels the contribution coming from the zero mode presented by Eq. \( \text{(14)} \). Hence, we conclude that the integral representation for the partial zeta function given by Eqs. \( \text{(12)}, \text{(13)} \) is valid for the case of the presence of the zero mode as well.

### III. ENERGY DENSITY ON A SINGLE BRANE

In this section we will consider the geometry of a single brane placed at \( y = j \). The orbifolded version of this model corresponds to the higher dimensional generalization of the Randall-Sundrum 1-brane model with the brane location at the orbifold fixed point \( y = 0 \). The corresponding partial zeta function is given by Eq. \( \text{(18)} \). Now in this formula \( J = L, R \) for the left and right sides of the brane, respectively, and \( F = K \) for \( J = R \) and \( F = I \) for \( J = L \). In addition, the replacement \( n^{(J)} \to n^{(J)} \) should be done with \( n^{(R)} = 1 \) and \( n^{(L)} = -1 \). The integral representation \( \text{(13)} \) for a single brane partial zeta function is valid in the strip \(-D_1 - 1 < \text{Re} \, s < -D_1 \) and under the assumption that the function \( F_{\nu}^{(J)}(u) \) has no real zeros. For the analytic continuity to \( s = -1 \) we employ the asymptotic expansions of the modified Bessel functions for large values of the argument. For \( B_J \neq 0 \) from these expansions one has

\[
\frac{F_{\nu}(u)}{F_{\nu}^{(J)}(u)} \sim \frac{1}{B_J} \sum_{l=1}^{\infty} v_l^{(F,J)} u^l, \tag{46}
\]

where the coefficients \( v_l^{(F,J)} \) are combinations of the corresponding coefficients in the expansions for the functions \( F_{\nu}(u) \) and \( F_{\nu}^{(J)}(u) \). Note that one has the relation

\[
v_l^{(K,J)} = (-1)^l v_l^{(I,J)}, \tag{47}
\]

assuming that the coefficients in the boundary conditions are the same for both sides of the brane. For the nonzero modes along the internal space \( \Sigma \) we subtract and add to the integrand in \( \text{(18)} \) the \( N \) leading terms of the corresponding asymptotic expansion and exactly integrate the asymptotic part:

\[
\zeta_{j\beta}^{(1)}(s) = \frac{-n(J) \lambda_j^2 \rho_{j}^{(a)}(0,0)}{\Gamma(-s/2)} \left[ \int_{\lambda_j z_j}^{\infty} du \frac{u^2 - \lambda_j^2 z_j^2}{\Gamma(\alpha_s + 1)} \right. \times \left( \frac{F_{\nu}(u)}{F_{\nu}^{(J)}(u)} - \sum_{l=1}^{N} \frac{v_l^{(F,J)}}{B_J u^l} \right) + \sum_{l=1}^{N} \frac{v_l^{(F,J)}}{2B_J} \times \left( \frac{\lambda_j^2}{\Gamma(l/2)} \right) \left( \frac{l}{2} - \alpha_s - 1 \right) \right]. \tag{48}
\]

For the zero mode we first separate the integral over the interval \((0,1)\) and apply the described procedure to the corresponding asymptotic expansion and exactly integrate the asymptotic part:

\[
\Phi_{j}^{(1)}(s) = \frac{z_j^P B_J}{(4\pi)^{D_1-1/2}} \sum_{\beta} |\psi_{\beta}(X)|^2 \zeta_{j\beta}^{(1)}(s), \tag{49}
\]

is written in the form

\[
\Phi_{j}^{(1)}(s) = \frac{(4\pi)^{1-D_1/2} n(J) \lambda_j^2}{\Gamma(-s/2)^{D_1}} \left( \sum_{\beta} |\psi_{\beta}(X)|^2 \left[ \delta_{0\lambda_\beta} B_J \int_{0}^{1} du \frac{D_1 u^{D_1+s-1}}{\Gamma(l/2)} \frac{F_{\nu}(u)}{F_{\nu}^{(J)}(u)} \right] \right.
\]

\[
+ \int_{u_{\beta}}^{\infty} du \frac{u^2 - \lambda_j^2 z_j^2}{\alpha_s} \left( B_J \frac{F_{\nu}(u)}{F_{\nu}^{(J)}(u)} - \sum_{l=1}^{N} \frac{v_l^{(F,J)}}{B_J u^l} \right) - \sum_{l=1}^{N} \frac{|\psi_{0}(X)|^2 v_{l}^{(F,J)}}{D_1 + s - l + 1} \left. \right) + \frac{1}{2} \Gamma(\alpha_s + 1) \sum_{l=1}^{N} \frac{v_l^{(F,J)}}{B_J} \frac{z_j^2}{\Gamma(l/2)} \left( \frac{l}{2} - \alpha_s - 1 \right) \zeta_{j\beta}^{(1)} \left( \frac{l}{2} - \alpha_s - 1, X \right), \tag{50}
\]

where \( u_{\beta} = \lambda_j \beta_j + \delta_{0\lambda_\beta} \) and \( \alpha_s \) is defined by formula \( \text{(38)} \). In Eq. \( \text{(50)} \) we have introduced the local spectral zeta function associated with the massless laplacian defined
on the internal subspace Σ:

\[
\zeta_{\Sigma}(z, X) = \sum_\beta |\psi_\beta(X)|^2 \lambda_\beta^{-2s},
\]

where the prime on the summation sign means that the zero mode should be omitted. Both integrals in Eq. (51) are finite at \( s = -1 \) for \( N \geq D_1 - 1 \). For large values \( \lambda_\beta \) the second integral behaves as \( \lambda_\beta^{D_1+s-N} \) and the series over \( \beta \) in Eq. (50) is convergent at \( s = -1 \) for \( N > D - 1 \). For these values \( N \) the poles at \( s = -1 \) are contained only in the last two terms on the right.

The zero mode part has a simple pole at \( s = -1 \) presented by the summand \( \ell = D - 1 \) of the second sum in figure braces. The pole part corresponding to the nonzero modes is extracted from the pole structure of the local zeta function (51). The latter is given by the formula

\[
\Gamma(z)\zeta_{\Sigma}(z, X)|_{z=p} = \frac{C_{D_2/2-p}(X)}{\zeta - p} + \Omega_p(X) + \cdots,
\]

where \( p \) is a half integer, the coefficients \( C_{D_2/2-p}(X) \) are related to the Seeley-DeWitt or heat kernel coefficients for the corresponding non-minimal laplacian, and the dots denote the terms vanishing at \( z = p \). In the way similar to that used in Ref. [27], it can be seen that the coefficients \( C_p(X) \) are related to the corresponding coefficients \( C_p(X, m) \) for the massive zeta function

\[
\zeta_{\Sigma}(s, X; m) = \sum_\beta \frac{|\psi_\beta(X)|^2}{(\lambda_\beta^2 + m^2)^s},
\]

\[
C_p(X) = C_p(X, 0) - |\psi_0(X)|^2 \delta_p, D_2/2.
\]

The VEV of the energy density on a single brane is derived from

\[
\bar{\xi}^{(j)} = -n^{(j)} k_B^2 C_j \Phi^{(j)}(s)|_{s=1}.
\]

By using relation (52), the energy density is written as a sum of pole and finite parts:

\[
\bar{\xi}^{(j)} = \bar{\xi}^{(j)}_{j, p} + \bar{\xi}^{(j)}_{j, l}.
\]

Laurent-expanding the expression on the right of Eq. (51) near \( s = -1 \), one finds

\[
\bar{\xi}^{(j)}_{j, l} = \frac{-2k_B^2 C_j}{(4\pi)^{D_1/2}(s + 1)} \sum_{l=1}^{D} v^{(F)}(j) z_l^{D-1} \times [C_{(D-l)/2}(X) + |\psi_0(X)|^2 \delta_{D_l}] 
\]

for the finite part, with \( \psi(x) \) being the diagamma function. In this formula the prime on the summation sign means that the term with \( l = D_1 \) should be omitted and it is understood that \( C_p(X) = 0 \) for \( p < 0 \). In the pole part the second term in the square braces comes from the zero mode along \( \Sigma \) and this term is cancelled by the delta term on the right of Eq. (51). For a one parameter internal manifold \( \Sigma \) with the length scale \( L \) one has \( \lambda_\beta \sim 1/L \) and \( \psi_\beta(X) \sim 1/L^{D_2} \). In this case instead of the zeta function (51) we could introduce the dimensionless function \( \tilde{\zeta}_{\Sigma}(s, X) = L^{D_2-2s} \zeta_{\Sigma}(s, X) \). The corresponding dimensionless coefficients \( \tilde{C}_{D_2/2-p}(X) \) and \( \tilde{\Omega}_p(X) \) in the formula analog to Eq. (52), are related to the coefficients in Eq. (52) by the formulas

\[
C_{D_2/2-p}(X) = L^{2p-D_2} \tilde{C}_{D_2/2-p}(X),
\]

\[
\Omega_p(X) = L^{2p-D_2} \left[ \tilde{\Omega}_p(X) + 2 \tilde{C}_{D_2/2-p}(X) \ln L \right],
\]
and do not depend on $L$. Now we see that the term with $\ln L$ is combined with the term $\ln \mu$ of Eq. (53) in the form $\ln(\mu L)$.

The renormalization of the surface energy density can be done modifying the procedure used previously for the renormalization of the Casimir energy in the Randall-Sundrum model [2, 12, 17] and in its higher-dimensional generalizations with compact internal spaces [20, 27]. The form of the counterterms needed for the renormalization is determined by the pole part of the surface energy density on a single brane given by Eq. (57). For an internal manifold with no boundaries, this part has the structure $\sum_{l=0}^{[(D-1)/2]} a^{(s)}_l (z_j/L)^{2l}$. By taking into account that the intrinsic scalar curvature $R_j$ for the brane at $y = j$ contains the factor $(z_j/L)^2$, we see that the pole part can be absorbed by adding to the brane action counterterms of the form

$$\int d^D x \sqrt{|h|} \sum_{l=0}^{[(D-1)/2]} b^{(s)}_l R^l_j,$$  

where the square brackets in the upper limit of summation mean the integer part of the enclosed expression. By taking into account that there is the freedom to perform finite renormalizations, we see that the renormalized surface energy density on a single brane is given by the formula

$$\varepsilon_{j,\text{ren}}^{(3)} = \varepsilon_{j,3}^{(3)} + \sum_{l=0}^{[(D-1)/2]} c^{(s)}_l (z_j/L)^{2l}. \quad (61)$$

The coefficients $c^{(s)}_l$ in the finite renormalization terms are not computable within the framework of the model under consideration and their values should be fixed by additional renormalization conditions.

The total surface energy density for a single brane at $y = j$ is obtained by summing the contributions from the left and right sides:

$$\varepsilon_{j}^{(LR)} = \varepsilon_{j}^{(L)} + \varepsilon_{j}^{(R)}. \quad (62)$$

In formulas (61), (62) we should take $F = I$ for $J = L$ and $F = K$ for $J = R$. Now we see that, assuming the same boundary conditions on both sides of the brane, the coefficients $C_p(X, 0)$ enter into the sum of pole terms in the form

$$2 \sum_{l=1}^{[D/2]} \frac{v_{1l}^{(I)}}{\Gamma(l)} (z_j)^{D-2l} C_{D/2-1}(X, 0). \quad (63)$$

If the internal manifold contains no boundaries and $D$ is an odd number, one has $C_{D/2-1}(X, 0) = 0$ and, hence, the pole parts coming from the left and right sides cancel out. In this case for the total surface energy density one obtains the formula

$$\varepsilon_{j}^{(LR)} = 2k_D \sum_{l=0}^{[(D-1)/2]} a^{(s)}_l (z_j/L)^{2l} \left[ \delta_{0l_0\lambda} \int_{u_{\lambda}}^{\infty} du u^{D-1} \left( \frac{I_{\nu}(u)}{K_{\nu}(u)} + \frac{K_{\nu}(u)}{I_{\nu}(u)} \right) \right] + \int_{u_{\lambda}}^{\infty} du (u^2 - \lambda^2 z_j^2) u_0^{2l-1} \left[ \frac{I_{\nu}(u)}{K_{\nu}(u)} + \frac{K_{\nu}(u)}{I_{\nu}(u)} - \frac{2}{\nu_2} \sum_{l=1}^{[N/2]} \frac{v_{2l}^{(I)}}{d^{2l-1}} \right]$$

$$-2|\psi_0(X)|^2 \sum_{l=1}^{[N/2]} \frac{v_{2l}^{(I)}}{d^{2l-1}} + \Gamma \left( \frac{d_1}{2} \sum_{l=1}^{[N/2]} \frac{v_{2l}^{(I)}}{\Gamma(l)} \frac{\Omega_{D/2-1}(X)}{z_j^{D-2l}} \right), \quad (64)$$

where the coefficients $v_{1l}^{(I,J)}$ are defined by relation (60).

Note that this quantity does not depend on the renormalization scale $\mu$. In the orbifolded version of the model with a single brane at $y = 0$, which corresponds to the higher dimensional generalization of the Randall-Sundrum 1-brane model, the bulk is symmetric under the reflection $y \rightarrow -y$. In this model the surface densities on the left and right sides of the brane are the same and coincide with $\varepsilon_{j}^{(R)}$. In particular, here the abovementioned cancellation of the pole parts from left and right sides does not take place.

For a one parameter internal space of size $L$ the surface energy density on the brane at $y = j$ is a function on the ratio $L/z_j$ only. Note that in the case of the AdS bulk the corresponding quantity does not depend on the brane position. To discuss the physics from the point of view of an observer residing on the brane, it is convenient to introduce rescaled coordinates

$$x^M = e^{-k_0 j} x^M, \quad M = 0, 1, \ldots, D - 1. \quad (65)$$

With this coordinates the warp factor in the metric is equal to 1 on the brane and they are physical coordinates for an observer on the brane. For this observer the
physical size of the subspace $\Sigma$ is $L_j = L e^{-k D j}$ and the corresponding KK masses are rescaled by the warp factor: $\lambda^{(j)} = \lambda_j e^{k D j}$. Now we see that the surface energy density is a function on the ratio $L_j/(1/k_D)$ of the physical size for the internal space (for an observer residing on the brane) to the AdS curvature radius.

As an application of the general results presented above, we can consider a simple example with $\Sigma = S^1$. In this case the bulk corresponds to the AdS$_{D+1}$ spacetime with one compactified dimension $X$. The corresponding normalized eigenfunctions and eigenvalues are as follows

$$\psi^j_\beta(X) = \frac{1}{\sqrt{L}} e^{2\pi i \beta X/L}, \quad \beta = 0, \pm 1, \pm 2, \ldots, \quad (66)$$

where $L$ is the length of the compactified dimension. The surface energy density induced on the brane is obtained from general formulas by the replacements

$$\sum_\beta |\psi^j_\beta(X)|^2 \to \frac{2}{L} \sum_{\beta=0}^\infty \lambda^j_{\beta} \to \frac{2\pi}{L} |\beta|, \quad D_2 = 1, \quad (67)$$

where the prime means that the summand $\beta = 0$ should be taken with the weight $1/2$. For the local zeta function from Eq. (61) one has

$$\zeta_\Sigma(s, X) = \frac{2}{L} \sum_{\beta=1}^\infty \left(\frac{2\pi}{L}\right)^{-2s} \frac{2L^{2s-1}}{(2\pi)^{2s}} \zeta_R(2s) \quad (68)$$

where $\zeta_R(z)$ is the Riemann zeta function. Now the only poles of the function $\Gamma(z) \zeta_\Sigma(z, X)$ are the points $z = 0, 1/2$. By using the standard formulas for the gamma function and the Riemann zeta function (see, for instance, [71]), it can be seen that one has

$$C_{1/2}(X) = -1/L, \quad C_0(X) = 1/2\sqrt{\pi}, \quad (69)$$

for the residues appearing in (62) and

$$\Omega_0(X) = \frac{\gamma - 2 \ln L}{L}, \quad \Omega_\pm(X) = \frac{\gamma + 2 \ln(L/4\pi)}{2\sqrt{\pi}}, \quad \Omega_\nu(X) = \frac{2L^{2p-1}}{(2\pi)^{2p}} \Gamma(p)\zeta_R(2p), \quad p \neq 0, 1/2, \quad (70)$$

for the finite parts, with $\gamma$ being the Euler constant.

IV. TWO-BRANE GEOMETRY AND INDUCED COSMOLOGICAL CONSTANT

As it has been shown in Section II the partial zeta function related to the surface energy density on the brane at $y = j$ is presented in the form [72], where the second term on the right is finite at the physical point $s = -1$. By taking into account formulas [60], [61], for two-brane geometry the VEV of the surface energy density on the brane at $y = j$ is presented as the sum

$$\varepsilon^{(s)}_j = \varepsilon^{(1)}_j + \Delta \varepsilon^{(s)}_j. \quad (71)$$

The first term on the right is the energy density induced on a single brane when the second brane is absent. The second term is induced by the presence of the second brane and is given by the formula

$$\Delta \varepsilon^{(s)}_j = 2C_j R^{(j)}(k_D z_j) B_j^\nu \beta_D \sum_\beta |\psi_\beta(X)|^2 \int_{\lambda_\beta}^{\infty} du u \times (u^2 - \lambda_{\beta}^2) D_{1/2-1} \Omega_\nu(u z_a, u z_b). \quad (72)$$

By taking into account relation [9] between the VEV of the field square and the surface energy-momentum tensor, this formula can also be obtained from the last term of Eq. [83] evaluated at the brane $z = z_j$. As we consider the region $a \leq y \leq b$, the energy density [74] is located on the surface $y = a + 0$ for the left brane and on the surface $y = b - 0$ for the right brane. Consequently, in formula (71) we take $J = R$ for $j = a$ and $J = L$ for $j = b$. The energy densities on the surfaces $y = a - 0$ and $y = b + 0$ are the same as for the corresponding single brane geometry. The expression on the right of Eq. (72) is finite for all nonzero distances between the branes and is not touched by the renormalization procedure. For a given value of the AdS energy scale $k_D$ and one parameter manifold $\Sigma$ with the length scale $L$, it is a function on the ratios $z_b/z_a$ and $L/z_a$. The first ratio is related to the proper distance between the branes,

$$z_b/z_a = e^{k_D \theta (b-a)}, \quad (73)$$

and the second one is the ratio of the size of the internal space, measured by an observer residing on the brane at $y = a$, to the AdS curvature radius $k_D^{-1}$. The expression [74] for the surface energy density can be presented in another equivalent form:

$$\Delta \varepsilon^{(s)}_j = \sum_\beta |\psi^j_\beta(X)|^2 \int_{\lambda_\beta}^{\infty} du u (u^2 - \lambda_{\beta}^2) D_{1/2-1} \times \frac{2k_D^2 z_j^{D+1} B_j^\nu \beta_D C_j}{B_j^\nu(u^2 z_j^2 + u^2)} - A_j^\nu, \quad (74)$$

which will be used below in the discussion of the relations between the bulk and surface energy densities.

For the comparison with the case of the bulk spacetime AdS$_{D+1}$ when the internal space is absent, it is useful in addition to the VEV [72] to consider the corresponding quantity integrated over the subspace $\Sigma$:

$$\Delta \varepsilon^{(s)}_{D,j} = \int_{\Sigma} d^D X \sqrt{\gamma} \Delta \varepsilon^{(s)}_j e^{-D_2 k D j} = e^{-D_2 k D j} \sum_\beta \Delta \varepsilon^{(s)}_{j,\beta}, \quad (75)$$
where \( \Delta \varepsilon_{j\beta}^{(s)} \) is defined by the relation
\[
\Delta \varepsilon_{j\beta}^{(s)} = \sum_{\beta} |\psi_\beta(X)|^2 \Delta \varepsilon_{j\beta}^{(s)}. \tag{76}
\]
Comparing this integrated VEV with the corresponding formula from Ref. \[35\], we see that the contribution of the zero KK mode (\( \lambda_0 = 0 \) in Eq. \[65\]) differs from the VEV of the energy density in the bulk AdS\(D_{1}+1\) by the order of the modified Bessel functions; for the latter case \( \nu \to \nu_1 \) with \( \nu_1 \) defined by Eq. \[26\] with the replacement \( D \to D_1 \). Note that for \( \zeta \leq \zeta_{D+D_{1}+1} \) one has \( \nu \geq \nu_1 \). In particular, this is the case for minimally and conformally coupled scalar fields.

Now we turn to the investigation of the part in the surface energy density in asymptotic regions of the parameters. For large values of AdS radius compared with the interbrane distance, \( k_D (b-a) \ll 1 \), the main contribution to the integral on the right of Eq. \[72\] comes from large values of \( u_{2a} \sim [k_D (b-a)]^{-1} \). Assuming that \( B_\alpha/(b-a) \) and \( m(b-a) \) are fixed, we see that the order of the Bessel modified functions is large. Replacing these functions by their uniform asymptotic expansions for large values of the order \( \beta \), one obtains
\[
\Delta \varepsilon_{j\beta}^{(s)} \approx 2n^{(j)} (1 - 4\zeta) \tilde{A}_j B_\beta D_1 \sum_{\beta} |\psi_\beta(X)|^2 \int_{m_\beta}^\infty du \, u^2 \\
\times \left( \frac{u^2 - m_\beta^2}{[\tilde{c}_a(u) \tilde{c}_b(u) e^{2u(b-a)} - 1]} \right)^{-1}, \tag{77}
\]
where \( m_\beta = \sqrt{m^2 + \lambda_\beta^2} \) and we have introduced the notation
\[
\tilde{c}_j(u) = \frac{\tilde{A}_j - n^{(j)} \tilde{B}_j u}{\tilde{A}_j + n^{(j)} \tilde{B}_j u}, \quad j = a, b. \tag{78}
\]
The expression on the right of Eq. \[77\] is the corresponding surface energy on the brane in the bulk geometry \( R^{(D_1-1,1)} \times \Sigma \).

For large KK masses along \( \Sigma \), \( z_\beta \lambda_\beta \gg 1 \), \( \lambda_\beta \gg 1 \), we can replace the modified Bessel functions by the corresponding asymptotic expansions for large values of the argument. For the contribution of a given KK mode to the leading order this gives
\[
\Delta \varepsilon_{j\beta}^{(s)} \approx 4n^{(j)} k_D^2 z_\beta^{D+1} B_j^2 C_j \beta D_1 \int_{\lambda_\beta}^\infty du \, u^2 \\
\times \left( \frac{(A_j^2 - B_j^2 u^2 z_\beta^2)^{-1} (u^2 - \lambda_\beta^2)^{\lambda_\beta^2 - 1}}{c_a(u z_\beta) c_b(u z_\beta) e^{2u(b-a)} - 1} \right), \tag{79}
\]
where
\[
c_j(u) = \frac{A_j - n^{(j)} B_j u}{A_j + n^{(j)} B_j u}, \quad j = a, b. \tag{80}
\]
If in addition one has the condition \( \lambda_\beta (z_\beta - z_a) \gg 1 \), the dominant contribution into the \( u \)-integral comes from the lower limit and we have the formula
\[
\Delta \varepsilon_{j\beta}^{(s)} \approx \frac{2n^{(j)} B_j C_j}{A_j^2} \left( \frac{k_D^2}{D_1+1} \right) \left( \frac{4\pi}{D_1/2} \right)^2 \\
\times \frac{\lambda_\beta^{2+1}}{\lambda_\beta^{2+2}} e^{-2\lambda_\beta (z_\beta - z_a)}. \tag{81}
\]
In particular, for sufficiently small length scale of the internal space this formula is valid for all nonzero KK masses and the main contribution to the surface densities comes from the zero KK mode. In the opposite limit of large internal space, to the leading order we obtain the corresponding result for parallel branes in \( AdS_{D+1} \) bulk \[39\].

For large values of the mass with \( m \gg k_D, m(z_\beta - z_a) \gg 1 \), and \( m \gg \lambda_\beta \), one has \( \nu \sim m/k_D \gg 1 \). Introducing a new integration variable \( u = \nu y \) and using the uniform asymptotic expansions for the modified Bessel functions, it can be seen that the corresponding asymptotic formula for \( \Delta \varepsilon_{j\beta}^{(s)} \) is obtained from \[51\] by the replacement \( \lambda_\beta \to m_\beta \). If both masses \( m \) and \( \lambda_\beta \) are of the same order, the asymptotic behavior for \( \Delta \varepsilon_{j\beta}^{(s)} \) is described by Eq. \[51\] replacing \( \lambda_\beta \to m_\beta \). As we could expect in this case the induced surface densities are exponentially suppressed.

For small interbrane distances, \( k_D (b-a) \ll 1 \), which is equivalent to \( z_\beta z_\alpha - 1 \ll 1 \), the main contribution into the integral on the right of Eq. \[72\] comes from large values \( u \) and to the leading order we obtain formula \[73\]. If in addition one has \( \lambda_\beta (z_\beta - z_a) \ll 1 \) or equivalently \( \lambda_\beta^{(a)} (b-a) \ll 1 \), we can put in this formula \( \lambda_\beta = 0 \). Assuming \( (b-a) \ll \tilde{B}_j/\tilde{A}_j \), for \( \tilde{B}_j \neq 0 \) to the leading order one finds
\[
\Delta \varepsilon_{j\beta}^{(s)} \approx -4k_D \sigma_j n^{(j)} C_j \Gamma \left( \frac{D_1-1}{2} \right) \zeta_R (D_1-1) \left( \frac{4\pi}{D_1+1/2} \right)^2 \left( \frac{b}{D_1-1} \right)^{D_2 k_D j}, \tag{82}
\]
where \( \sigma_j = 1 \) for \([B_a/A_a]|B_b/A_b| \gg k_D (b-a)\), and \( \sigma_j = 2^{D-2} - 1 \) for \([B_a/A_a] \gg k_D (b-a)\) and \( B_b/A_b = 0 \), with \( l = b \) for \( j = a \) and \( l = a \) for \( j = b \). We see that for small interbrane distances the sign of the induced surface energy density is determined by the coefficient \( C_j \) and this sign is different for two cases of \( \sigma_j \).

Now we consider the limit \( \lambda_\beta z_\beta \gg 1 \) assuming that \( \lambda_\beta z_\beta \lesssim 1 \). Using the asymptotic formulas for the Bessel modified functions containing in the argument \( z_\beta \) for the contribution of a given nonzero KK mode we find the following results
\[
\Delta \varepsilon_{\alpha\beta}^{(s)} \approx \frac{k_D^2}{D_1+1} \left( \frac{D_2}{2} \right) \left( \frac{4\pi}{D_1/2} \right)^2 \left( \frac{b}{D_1-1} \right)^{D_2 k_D j} \\
\times \frac{\lambda_\beta^{2+1}}{\lambda_\beta^{2+2}} e^{-2\lambda_\beta (z_\beta - z_a)}. \tag{83}
\]
\[
\Delta \varepsilon_{\beta\alpha}^{(s)} \approx -\frac{k_D^2}{D_1+1} \left( \frac{D_2}{2} \right) \left( \frac{4\pi}{D_1/2} \right)^2 \left( \frac{b}{D_1-1} \right)^{D_2 k_D j} \\
\times \frac{\lambda_\beta^{2+1}}{\lambda_\beta^{2+2}} e^{-2\lambda_\beta (z_\beta - z_a)}. \tag{84}
\]
This limit corresponds to the interbrane distances much larger compared with the AdS curvature radius and with the inverse KK masses measured by an observer on the left brane: \((b-a) \gg 1/k_D, 1/\lambda_\beta\). For a single parameter manifold \(\Sigma\) with length scale \(L\) and \((b-a) \gg L_a\) these conditions are satisfied for all nonzero KK modes.

In the limit \(z_a \lambda_\beta \ll 1\) for fixed \(z_b \lambda_\beta\), by using the asymptotic formulas for the modified Bessel functions for small values of the argument and assuming \(|A_a| \neq |B_a|\), one finds

\[
\Delta s_{a\beta}^{(s)} \approx \frac{k_D^2}{2^{2\nu-3} \Gamma(\nu)} \int_{\lambda_{b\Sigma}}^{\infty} du \frac{u^{2\nu+1}}{f^{(b)}_\nu(u z_b)} \left( u^2 - \lambda_\beta^2 \right) D_1/2 - 1 \left( u^2 - \lambda_\beta^2 \right) D_2/2 - 1.
\]

For \(|A_a| = |B_a|\) we should take into account the next terms in the corresponding expansions of the modified Bessel functions. The integral in Eq. (87) is negative for small values of the ratio \(A_b/B_b\) and is positive for large values of this ratio. As it follows from Eq. (87), for large interbrane separations the sign of the quantity \(\Delta s_{b}^{(s)}\) is determined by the combination \((D_b^2 - \sqrt{A_b^2})\) of the coefficients in the boundary conditions. In the limit under consideration the KK masses measured by an observer on the brane at \(y = a\) are much less than the AdS energy scale, \(\lambda_\beta \ll k_D\), and the interbrane distance is much larger than the AdS curvature radius. In particular, substituting \(\lambda_\beta = 0\), from these formulas we obtain the asymptotic behavior for the contribution of the zero mode to the surface energy density induced by the second brane in the limit \(z_a/z_b \ll 1\). Now combining the corresponding result with formulas (38), (39), we see that under the conditions \((b-a) \gg 1/k_D, L_a, L_b k_D \gtrsim 1\), the contribution of the nonzero KK modes along \(\Sigma\) is suppressed with respect to the contribution of the zero mode by the factor \((z_b/z_a)^{2\nu+1}D_1/2 + \sqrt{A_b} \approx \lambda_\beta z_b\).

From the analysis given above it follows that in the limit when the right brane tends to the AdS horizon, \(z_b \to \infty\), the energy density \(\Delta s_{a\beta}^{(s)}\) vanishes as \(e^{-2\lambda_\beta z_b}/z_b^{D_1/2}\) for the nonzero KK mode along \(\Sigma\) and as \(z_b^{D_1/2}\) for the zero mode. The energy density on the right brane, \(\Delta s_{b\beta}^{(s)}\), vanishes as \(z_b^{D_2+D_1/2+1}e^{-2\lambda_\beta z_b}\) for the nonzero KK mode and behaves like \(z_b^{D_2+2\nu}\) for the zero mode. In the limit when the left brane tends to the AdS boundary, \(z_a \to 0\), the contribution of a given KK mode vanishes as \(z_b^{D_2+2\nu}\) for \(\Delta s_{a\beta}^{(s)}\) and as \(z_b^{2\nu}\) for \(\Delta s_{b\beta}^{(s)}\). For small values of the AdS curvature radius corresponding to strong gravitational fields, assuming \(\lambda_\beta z_a \gg 1\) and \(\lambda_\beta(z_b - z_a) \gg 1\), we can estimate the contribution of the nonzero KK modes to the induced energy densities by formula (31). In particular, for the case of a single parameter internal space with the length scale \(L\), under the assumed conditions the length scale of the internal space measured by an observer on the brane at \(y = a\) is much smaller compared to the AdS curvature radius, \(L_a \ll k_D^{-1}\). If \(L_a \gtrsim k_D^{-1}\) one has \(\lambda_\beta z_a \gtrsim 1\) and to estimate the contribution of the induced surface densities we can use formulas (38) and (39), and the suppression is stronger compared with the previous case. For the zero KK mode, under the condition \(k_D(b-a) \gg 1\) we have \(z_a/z_b \ll 1\) and to the leading order the corresponding energy densities are described by relations (38) and (39). From these formulae it follows that the induced energy densities integrated over the internal space behave as \(k_D^{-1} \exp[(D_1^2 + 2\nu) k_D(b-a)]\) for the brane at \(y = j\) and are exponentially suppressed. Note that in the model without the internal space we have similar behavior with \(\nu\) replaced by \(\nu_1\) and for a scalar field with \(\zeta < \zeta_D + D_1 + 1\) the suppression is relatively weaker.

In Fig. 1 and Fig. 2 we have plotted the energy densities induced on the left and right branes, respectively, by the presence of the second brane as functions on the size of the internal space \(\Sigma = S^1\) and interbrane distance for a \(D = 5\) minimally coupled massless scalar field with the Robin coefficients \(A_a/B_a = -1, A_b/B_b = 5\). The corresponding expressions are obtained from the general formulas of this section by the substitutions (67).

![FIG. 1: Surface energy density, \(\Delta s_{a\beta}^{(s)}/k_D^2\), induced on the brane at \(y = a\) as a function of \(z_a/z_b\) and \(L_a/z_b\) for a \(D = 5\) minimally coupled massless scalar field in the model with \(\Sigma = S^1\) and with the Robin coefficients \(A_a/B_a = -1, A_b/B_b = 5\).](image-url)
are related by the formula

\[ G_{D_{1}j} = \frac{(D - 2)k_{D}G_{D+1}}{V_{\Sigma j}} e^{(D-2)k_{D}(b-a)} - 1 \]  

where

\[ V_{\Sigma j} = e^{-D_{2}k_{D}j} \int d^{D_{2}}X \sqrt{\gamma}, \]

is the volume of the internal space measured by the same observer. In the orbifolded version of the model an additional factor 2 appears in the denominator of the expression on the right. Formula (88) explicitly shows two possibilities for the hierarchy generation by the redshift and large volume effects. Note that the ratio of the Newton constants on the branes, \( G_{D_{1}b}/G_{D_{1}a} = e^{(2-D_{1})k_{D}(b-a)}, \) is the same as in the model without an internal space. For large interbrane distances one has \( G_{D_{1}a} \sim k_{D}G_{D+1}/V_{\Sigma a}, \) \( G_{D_{1}b} \sim k_{D}G_{D+1}e^{(2-D_{1})k_{D}(b-a)}/V_{\Sigma b}, \) and the gravitational interactions on the brane \( y = b \) are exponentially suppressed. This feature is used in the Randall-Sundrum model to address the hierarchy problem. As we will see below this mechanism also allows to obtain a naturally small cosmological constant generated by the vacuum quantum fluctuations (for the discussion of the cosmological constant problem within the framework of braneworld models see references given in [35]).

As we have already mentioned, surface energy density corresponds to the gravitational source of the cosmological constant type induced on the brane at \( y = j \) by the presence of the second brane. For an observer living on the brane at \( y = j \) the corresponding effective \( D_{1} \)-dimensional cosmological constant is determined by the relation

\[ \Lambda_{D_{1}j} = 8\pi G_{D_{1}j} \Delta \varepsilon_{D_{1}j} = \frac{8\pi \Delta \varepsilon_{D_{1}j}}{M_{D_{1}j}^{D_{1}-2}}, \]

where \( M_{D_{1}j} \) is the \( D_{1} \)-dimensional effective Planck mass scale for the same observer and \( \Delta \varepsilon_{D_{1}j} \) is defined by Eq. (75). Denoting by \( M_{D+1} \) the fundamental \((D + 1)\)-dimensional Planck mass, \( G_{D+1} = M_{D+1}^{-2} \), from Eq. (88) one has the following relation

\[ \left( \frac{M_{D_{1}j}}{M_{D+1}} \right)^{D_{1}-2} = \frac{(z_{b}/z_{a})^{D_{2}-2} - 1}{(D - 2)(z_{b}/z_{a})^{D_{2}} - 2} V_{\Sigma j} k_{D} M_{D+1}^{D_{1}+1}, \]

for the ratio of the effective and fundamental Planck scales. From the asymptotic analysis for the induced vacuum densities given above it follows that for large interbrane distances, \( z_{b}/z_{a} \gg 1 \), the contribution of the modes with \( \lambda_{b}z_{b} \gg 1, \lambda_{a}z_{a} \lesssim 1 \) is suppressed by the factor \( (z_{b}/z_{a})^{2D_{2}+D_{1}}/z_{b}^{2} \exp(-2\lambda_{b}z_{b}) \) with respect to the contribution of the zero mode. For the contribution of the modes with \( z_{a}\lambda_{b} \lesssim 1, z_{b}\lambda_{a} \ll 1 \) from Eqs. (89), (90) one has \( \Delta \varepsilon_{D_{1}j} \sim k_{D}z_{a}^{D_{2}+D_{1}}/z_{b}^{2} \). By using these relations, one obtains the following estimate for the ratio of the induced cosmological constant (90) to the corresponding Planck scale quantity in the brane universe:

\[ h_{j} = \frac{\Lambda_{D_{1}j}}{8\pi G_{D_{1}j} M_{D_{1}j}} \sim \left( \frac{z_{a}}{z_{b}} \right)^{2D_{2}+D_{1}+1} \left( \frac{k_{D}}{M_{D_{1}j}} \right)^{D_{1}}. \]

By taking into account relation (91), this can also be written in the form

\[ h_{j} \sim \left( \frac{k_{D}^{D_{1}+1}}{V_{\Sigma j} M_{D+1}^{D_{1}+1}} \right)^{D_{1}/(D_{2}-2)} \times \exp \left[ k_{D}(a-b) \left( 2D_{2}+D_{1}+1 \right) \sum_{\beta} f_{\beta}^{(b)} \right]. \]

For the model without an internal space this ratio is of the same order of magnitude for both branes. In the higher dimensional version of the Randall-Sundrum braneworld the brane at \( z = z_{a} \) corresponds to the visible brane. For large interbrane distances, by taking into account Eq. (89), for the ratio of the induced cosmological constant (90) to the Planck scale quantity in the corresponding brane universe one obtains

\[ h_{b} \approx -\frac{1}{\epsilon_{\nu}(\nu)} \left( \frac{k_{D}}{M_{D_{1}b}} \right)^{D_{1}} \left( \frac{z_{a}}{z_{b}} \right)^{2D_{2}+D_{1}+1} \sum_{\beta} f_{\beta}^{(b)}, \]

where the function \( f_{\beta}^{(b)} \) is defined by Eq. (74). In Fig. 6 we have plotted the coefficient in this formula as a function of \( B_{b} \) and \( L/2z_{b} \) for a \( D = 5 \) minimally coupled massless scalar field in the model with the internal space \( \Sigma = S^{1} \) and \( A_{b} = 1 \). We recall that \( L/2z_{b} = k_{D}L_{b}, \) where \( L_{b} \) is the physical size of the internal space for an observer living on the brane \( y = b \).

Using relation (91) with \( j = b \), we can express the corresponding interbrane distance in terms of the ratio of the Planck scales

\[ \frac{z_{a}}{z_{b}} \sim \left[ M_{D+1}^{D_{1}+1} V_{\Sigma b} \left( \frac{M_{D+1}}{M_{D_{1}b}} \right)^{D_{1}+1} \right]^{1/(D_{2}-2)}. \]
Substituting this into Eq. (94), for the ratio of the cosmological constant on the brane at \( j = b \) to the corresponding Planck scale quantity one finds

\[
h_b \approx -\frac{1}{c_a(\nu)} \left( \frac{k_D}{M_{D+1}} \right)^{D_1-\tilde{\nu}} (V_{2k} M_{D_2})^{\tilde{\nu}} \times \left( \frac{M_{D+1}}{M_{D_1}} \right)^{D_1+(D_1-2)} \sum_{\beta} f_{\nu\beta},
\]

with \( \tilde{\nu} = 2\nu/(D - 2) \). The higher dimensional Planck mass \( M_{D+1} \) and AdS inverse radius \( k_D \) are two fundamental energy scales in the theory which in the Randall-Sundrum model are usually assumed to be of the same order, \( k_D \sim M_{D+1} \) (see, e.g., [1]). In this case one obtains the induced cosmological constant which is exponentially suppressed compared with the corresponding Planck scale quantity on the brane. In the model with \( D_1 = 4, k_D \sim M_{D+1} \sim 1 \) TeV, \( M_{D_1,b} = M_{Pl} \sim 10^{16} \) TeV, assuming that the compactification scale on the visible brane is close to the fundamental Planck scale, \( V_{2k} M_{D_1} \sim 1 \), for the ratio of the induced cosmological constant to the Planck scale quantity on the visible brane we find the estimate \( h_b \approx 10^{-32(2+\nu)} \). From (95) one has \( k_D(b - a) \approx 74/(D_2 + 2) \) and the corresponding interbrane distances generating the required hierarchy between the electroweak and Planck scales are smaller than those for the model without an internal space. In the model proposed in Ref. [26], a separation between the fundamental Planck scale and curvature scale is assumed: \( k_D \sim M_{D+1} z_b/z_a \sim 1 \) TeV. Under the assumption \( V_{2k} M_{D_2} \sim 1 \), in this model we have \( h_b \approx 10^{-64[1+\nu/(D+1)]} \) and \( k_D(b - a) \approx 74/(D_2 + 3) \).

V. TOTAL VACUUM ENERGY AND THE ENERGY BALANCE

On background of manifolds with boundaries the total vacuum energy is splitted into bulk and boundary parts. In the region between two branes the bulk energy per unit coordinate volume in the \( D_1 \)-dimensional subspace is obtained by the integration of the \( \delta \)-component of the energy-momentum tensor over this region:

\[
E^{(v)} = \int d^{D_2}X d\gamma \sqrt{|g|} \langle 0 | T^{(v)0}_0 | 0 \rangle.
\]

The surface energy per unit coordinate volume in the \( D_1 \)-dimensional subspace, \( E^{(s)} \), is related to the surface densities evaluated in previous sections by the formula

\[
E^{(s)} = \sum_{j=a,b} \frac{\zeta_j^{(s)}}{(k_D z_j)^{D_1}}.
\]

Now by making use of the formula for the volume energy-momentum tensor from Ref. [44] and the formula (94) for the surface densities, it can be seen that the following formal relation takes place for the unrenormalized VEVs:

\[
E = E^{(v)} + E^{(s)},
\]

where

\[
E = \frac{1}{2} \int d^{D_1-1}k (2\pi)^{D_1-1} \sum_{\beta} \sum_{n=1}^{\infty} (k^2 + m_{\beta n}^2 + \lambda_{\beta}^2)^{-s/2},
\]

is the total vacuum energy per unit coordinate volume of the \( D_1 \)-dimensional subspace, evaluated as the sum of zero-point energies of elementary oscillators.

The total vacuum energy within the framework of the Randall-Sundrum braneworld is evaluated in Refs. [7, 9, 16] by the dimensional regularization method and in Ref. [12] by the zeta function technique. Refs. [6, 41, 12] consider the case of a minimally coupled scalar field in \( D = 4 \), and the case of arbitrary \( \zeta \) and \( D \) with zero mass terms \( c_a \) and \( c_b \) is calculated in Ref. [18]. For the orbifolded version of the model under consideration with \( D_1 = 4 \) and zero mass terms on the branes, the vacuum energy is investigated in [26] by using the dimensional regularization. Here we briefly outline the zeta function approach in the general case.

We consider the zeta function related to the vacuum energy (100):

\[
\zeta(s) = \mu^{s+1} \int \frac{d^{D_1-1}k}{(2\pi)^{D_1-1}} \sum_{\beta} \sum_{n=1}^{\infty} (k^2 + m_{\beta n}^2 + \lambda_{\beta}^2)^{-s/2},
\]

where, as before, the parameter \( \mu \) with dimension of mass is introduced by dimensional reasons. The vacuum energy in the region between the branes is obtained by the
analytic continuation of this zeta function to the value $s = -1$:

$$E = \frac{1}{2} \zeta(s)|_{s=-1}. \quad (102)$$

After the evaluation of the integral over $k$ one obtains the formula

$$\zeta(s) = \frac{\mu^{s+1}}{(4\pi)^{(D_1-1)/2}} \frac{\Gamma(s_1/2)}{\Gamma(s/2)} \sum_{\beta} \zeta_\beta(s_1), \quad (103)$$

with the partial zeta function

$$\zeta_\beta(s_1) = \sum_{n=1}^{\infty} \left( m_n^2 + \lambda_\beta^2 \right)^{-s_1/2}, \quad s_1 = s + 1 - D_1. \quad (104)$$

To evaluate the vacuum energy we need to perform the analytic continuation of the function $\zeta(s_1)$ to the neighborhood of $s = -1$. This corresponds to the analytic continuation for $\zeta_\beta(s_1)$ to the point $s_1 = -D_1$. For this we present the partial zeta function (104) as the contour integral

$$\zeta_\beta(s) = \frac{1}{2\pi i} \int_C du \left( u^2 + \lambda_\beta^2 \right)^{-s/2} \frac{d}{du} \ln g_{\nu}^{(ab)}(uz_a, uz_b), \quad (105)$$

where the integration contour is the same as in formula (10). Now by making use of the standard properties of the Bessel functions, we see that the parts of the integrals over $(0, \pm i\lambda_\beta)$ cancel and we find the following integral representation

$$\zeta_\beta(s) = \frac{1}{\pi} \sin \frac{\pi}{2} s \int_{\lambda_\beta}^{\infty} du \left( u^2 - \lambda_\beta^2 \right)^{-s/2} \times \frac{d}{du} \ln G_{\nu}^{(ab)}(uz_a, uz_b). \quad (106)$$

For the further discussion it is convenient to write the partial zeta function in the decomposed form

$$\zeta_\beta(s) = \sum_{j=a,b} \zeta^{(j)}_\beta(s) + \Delta \zeta_\beta(s), \quad (107)$$

with the separate terms

$$\zeta^{(j)}_\beta(s) = \frac{1}{\pi} \sin \frac{\pi}{2} s \int_{\lambda_\beta}^{\infty} du \left( u^2 - \lambda_\beta^2 \right)^{-s/2} \times \frac{d}{du} \ln \left[ u^{(j)}_{\nu} F^{(j)}(uz_j) \right], \quad (108)$$

$$\Delta \zeta_\beta(s) = \frac{1}{\pi} \sin \frac{\pi}{2} s \int_{\lambda_\beta}^{\infty} du \left( u^2 - \lambda_\beta^2 \right)^{-s/2} \times \frac{d}{du} \ln \left[ 1 - \frac{\bar{F}_\nu^{(a)}(uz_a) F_\nu^{(b)}(uz_b)}{\bar{K}_\nu^{(a)}(uz_a) K_\nu^{(b)}(uz_b)} \right]. \quad (109)$$

where $F = K$ for $j = a$ and $F = I$ for $j = b$. The part $\Delta \zeta_\beta(s)$ is finite at $s = -D_1$ and the analytic continuation is necessary for the parts $\zeta^{(j)}_\beta(s)$, $j = a, b$. The latter are the partial zeta functions for the geometry of a single brane in the regions $y \geq a$ and $y \leq b$, respectively. On the base of Eq. (107) similar decomposition can be given for the total zeta function

$$\zeta(s) = \sum_{j=a,b} \zeta^{(j)}(s) + \Delta \zeta(s), \quad (110)$$

where separate parts are related to the corresponding partial zeta functions by the relations similar to Eq. (103) with $s_1$ defined by Eq. (104). In deriving the integral representation (106) we have assumed that there is no zero mode corresponding to $m_n = 0$. The contribution of the possible zero mode to the partial zeta function $\zeta_\beta(s)$ is $\lambda_\beta^s$. As regards to the nonzero modes, their contribution, as before, is given by the contour integral on the right of Eq. (105), but now the point $u = 0$ has to be avoided by the small semicircle in the right half-plane. The integrals over the parts of the imaginary axis are transformed to the expression given by the right hand side of Eq. (109), whereas the integral over the small semicircle is equal to $-\lambda_\beta^{-s}$ and cancels the contribution from the zero mode. Hence, the integral representation (106) for the partial zeta function is valid in the case of presence of the zero mode as well.

In order to obtain an analytic continuation for the function $\zeta^{(j)}_\beta(s)$, in the integrand for the non-zero modes we subtract and add the term which is obtained replacing the function $\bar{F}^{(j)}(uz_j)$ by the first $N$ terms of the corresponding asymptotic expansion for large values $u$. In the zero mode part we split the integral in Eq. (108) into the integrals over $(0, 1)$ and $(1, \infty)$. The contribution of the first integral to the zeta function is finite at $s = -D_1$ and the part with the second integral can be treated in the way similar to that used for the non-zero modes. After the explicit integration of the asymptotic part and introducing the notation

$$\Sigma^{(F,j)}_{\nu}(u) = -n^{(j)} \sqrt{\frac{2q_F}{u}} e^{n^{(j)} u} \frac{\sqrt{2}}{B_j} F^{(j)}(u), \quad (111)$$

with $q_1 = \pi$ and $q_K = 1/\pi$, for the corresponding zeta function the following formula is obtained
\[ \zeta^{(j)}(s) = \frac{\mu^{s+1}(4\pi)^{1-D}u_{s_j}^{s+1}}{\Gamma(s/2)\Gamma(1-s_1/2)} \left\{ \sum_{\beta} \delta_{\lambda\beta} \int_{0}^{1} du u^{-s_1} \frac{d}{du} \ln\left( \Sigma_{u}^{(F,j)}(u) \right) + \int_{u_\beta}^{\infty} du (u^2 - \lambda_{\beta}^2)^{-s_2/2} \right\} \]

where, as before, \( u_\beta = \lambda_{\beta} z_j + \delta_{\lambda\beta} \), and we have used the relation \( \sin(\pi z)\Gamma(1-z) = \pi/\Gamma(z) \) for the gamma function. In formula \( 112 \),

\begin{align*}
\tilde{w}_{-1}^{(F,j)} &= \frac{w^{(j)}_0}{2\sqrt{\pi}}, \quad \tilde{w}_0^{(F,j)} = -\frac{1 + 2n^{(j)}_\nu}{4}, \\
\tilde{w}_l^{(F,j)} &= \frac{w_l^{(F,j)}}{\Gamma(l/2)}, \quad l = 1, 2, \ldots, \tag{113}
\end{align*}

with \( w_l^{(F,j)} \) being the coefficients in the asymptotic expansion of the function \( \ln[\Sigma_{u}^{(F,j)}(u)] \) for large values of the argument:

\[ \ln[\Sigma_{u}^{(F,j)}(u)] \sim \sum_{l=1}^{\infty} \frac{w_l^{(F,j)}}{u^l}, \quad j = a, b, \tag{114} \]

and we have defined the global zeta function:

\[ \zeta_{\Sigma}(s) = \sum_{\beta} \lambda_{\beta}^{-2s}. \tag{115} \]

Note that one has the relation \( w_l^{(K,j)} = (-1)^lw_l^{(L,j)} \), assuming that the coefficients in the boundary conditions are the same, and these coefficients are related to the coefficients in the similar expansions for the modified Bessel functions (see, for instance, \[ 57 \]). For \( N > D_l \) both integrals on the right of formula \( 112 \) are finite at the physical point \( s = -1 \). For large values of \( \lambda_{\beta} \) the second integral behaves as \( (\lambda_{\beta} z_j)^{-N-s_1} \), and, hence, the series over \( \beta \) converges at the point \( s_1 = -D_l \) if \( N > D_l \). As a result, for these values of \( N \) the only poles in the expression \( 112 \) are those for the function \( \Gamma(z)\zeta_{\Sigma}(z) \) and the pole in the last sum corresponding to the term with \( l = D_l \). Noting that \( N = \infty \) we obtain the expansion of the zeta function over \( 1/z_j \).

In order to separate the pole and finite parts of the zeta function \( \zeta^{(j)}(s) \) at the physical point \( s = -1 \), we employ the global analog of formula \( 154 \):

\[ \Gamma(z)\zeta_{\Sigma}(z)|_{z=p} = \frac{C_{D_l/2-p}}{z-p} + \Omega_p + \cdots. \tag{116} \]

As it has been shown in Ref. \[ 27 \], the coefficients \( C_p \) in this formula are related to the integrated Seeley-DeWitt coefficients \( C_p(m) \) for the massive zeta function \( \zeta_{\Sigma}(s; m) = \sum_{\beta}(\lambda_{\beta}^2 + m^2)^{-s} \) by the formula

\[ C_p = C_p(0) - \delta_{p,D_l/2}, \tag{117} \]

which is obtained from \( 65 \) by the integration over the internal space. For the model with the internal space \( \Sigma = S^1 \) the zeta function \( \zeta_{\Sigma}(z) \) is related to the corresponding local zeta function discussed in Section \( \ref{section} \) by the formula \( \zeta_{\Sigma}(z) = L\zeta_{\Sigma}(z, X) \) and the expressions for the coefficients \( C_{D_l/2-p} \) and \( \Omega_p \) are directly obtained from \( 89 \) and \( 70 \).

By making use of formula \( 116 \), near the point \( s = -1 \) the zeta function \( \zeta^{(j)}(s) \) is presented in the form of the sum of pole and finite parts:

\[ \zeta^{(j)}(s)|_{s=-1} = \frac{\zeta^{(j)}}{s+1} + \zeta^{(j)}_0, \tag{118} \]

with the residue

\[ \zeta^{(j)}_{-1} = \frac{2}{(4\pi)^{1-D_l/2}} \sum_{l=-1}^{N} \frac{w_l^{(F,j)}}{z_j^l} [C_{(D_l-1)/2} + \delta_{lD_l}]. \tag{119} \]

In formula \( 119 \) the part with the first term in the square brackets comes from the nonzero KK modes along \( \Sigma \) and the part with the second term comes from the zero mode. Now by taking into account formula \( 117 \), we see that the zero mode part is cancelled by the delta term in Eq. \( 117 \). For the finite part of the zeta function we obtain the formula
where the prime on the summation sign means that the term \( l = D_1 \) should be omitted. Note that taking in this formula \( N = \infty \) we obtain the expansion of the finite part over \( 1/z_j \). For an internal space with the length scale \( L \) we can introduce the dimensionless zeta function \( \tilde{\zeta}_\Sigma(s) = L^{-2s}\zeta_\Sigma(s) \) with the coefficients \( \tilde{C}_{D_2/2-p} \) and \( \tilde{\Omega}_p \) defined by the relation similar to Eq. (116). Now it can be seen that in terms of these coefficients the last term on the right of formula (120) is rewritten as

\[
\frac{L^{-D_1}}{(4\pi)^{D_1/2}} \sum_{l=1}^N \tilde{W}_l^{(F,j)} \left( \frac{L}{z_j} \right)^l \left[ 2\tilde{C}_{(D-1)/2} \right. \\
\times \left( \ln(\mu L) - \frac{1}{2} \psi \left( -\frac{1}{2} \right) \right) + \tilde{\Omega}_{(D-D_1)/2}. \right. \tag{121}
\]

This explicitly shows that the combination \( \zeta_0^{(j)} z_j^{D_1} \) is a function on the ratio \( L/z_j \) only.

Similar to the case of the surface energy, for a single brane at \( y = j \) we denote by \( E_j^{(j)} \) the total vacuum energy in the region \( y \geq j \) for \( J = R \) and in the region \( y \leq j \) for \( J = L \). This energy is related to the zeta function considered above by the formula

\[
E_j^{(j)} = \frac{1}{2} \zeta^{(j)}(s) |_{s=-1}, \tag{122}
\]

where in the formulas for the function \( \zeta^{(j)}(s) \) it should be taken \( F = I \) for \( J = L \) and \( F = K \) for \( J = R \). In accordance with Eq. (118) the regularized vacuum energy contains pole and finite parts. The structure of the residue \( \zeta^{(j)}_1 \) determines the form of the counterterms which should be added to the action for the renormalization of the vacuum energy. From formula (119) it follows that for an internal space without boundaries the pole terms can be absorbed by adding to the brane action the counterterm

\[
\int d^D x \sqrt{|h|} \sum_{l=0}^{|(D+1)/2|} \alpha_l R_{j,l}^l, \tag{123}
\]

where \( R_{j} \) is the intrinsic curvature scalar for the brane at \( y = j \). Note that this counterterm has the same structure as that for the surface energy. The only difference is the additional summand corresponding to the term \( l = [(D + 1)/2] \). Hence, by adding the counterterm (123) we can absorb the pole terms for both total and surface energies. Now by taking into account the possibility for the appearance of the finite renormalization terms, the renormalized vacuum energy for the geometry of a single brane is presented in the form

\[
E_j^{(j, ren)} = E_j^{(j)} + \sum_{l=0}^{|(D+1)/2|} \alpha_l (z_j/L)^{2l}, \tag{124}
\]

where \( E_j^{(j)} = \zeta^{(j)}_0/2 \) and the coefficients \( \alpha_l \) in the finite renormalization terms should be fixed by imposing renormalization conditions.

As in the case of the surface energies, now we see that for the internal manifold with no boundaries in the calculation of the total vacuum energy for a single brane at \( y = j \), including the contributions from the left and right regions, the pole parts of the energies cancel out for odd values \( D \) (assuming that the coefficients in the boundary conditions (8) on the right and left surfaces are the same) and we obtain a finite result. For this energy one receives
with \( M > (D - 1)/2 \). Note that the energy per unit physical volume in the \( D_1 \)-dimensional subspace is determined as \((kD\; z_j)^{D_1} E^{(j)}\) and is a function on the ratio \( L/z_j \) only.

On the base of the analysis given above, for the geometry of two branes the total energy in the region \( a \leq y \leq b \) is presented in the form

\[
E = E_a^{(R)} + E_b^{(L)} + \Delta E,
\]

where the interference term \( \Delta E \) in this formula is finite for all nonzero values of the interbrane separation and is obtained directly from the part \( \Delta K(s) \) substituting \( s = -1 \). After the integration by parts this term is presented in the form

\[
\Delta E = \beta D_1 \sum_{\beta} \int_{m_3}^{\infty} du \; u^2 (u^2 - \lambda_\beta^2)^{D_1/2 - 1} \times \ln \left| 1 - \frac{\tilde{T}_\nu^{(a)}(u z_a) \tilde{R}_\nu^{(b)}(u z_b)}{\tilde{K}_\nu^{(a)}(u z_a) \tilde{T}_\nu^{(b)}(u z_b)} \right|,
\]

and is not affected by finite renormalizations.

Now we briefly discuss the behavior of the interference part in the vacuum energy in the asymptotic regions of the parameters. For large values of the AdS radius compared with the interbrane distance, \( kD(b - a) \ll 1 \), by making use the uniform asymptotic expansions for the modified Bessel functions, to the leading order one finds the result for two parallel branes on the bulk \( R^{(D_1-1.1)} \times \Sigma \):

\[
\Delta E \approx \beta D_1 \sum_{\beta} \int_{m_3}^{\infty} du \; u^2 (u^2 - \lambda_\beta^2)^{D_1/2 - 1} \times \ln \left| 1 - \frac{e^{-2u(b-a)}}{\tilde{c}_\beta(u) \bar{c}_\beta(u)} \right|,
\]

where \( m_\beta = \sqrt{m^2 + \lambda_\beta^2} \) and \( \tilde{c}_\beta(u) \) is defined by formula (78). For large KK masses along the internal space, \( z_a \lambda_\beta \gg 1 \), for the contribution of the mode with a given \( \beta \) one has

\[
\Delta E_\beta \approx \beta D_1 \int_{\lambda_\beta}^{\infty} du \; u^2 (u^2 - \lambda_\beta^2)^{D_1/2 - 1} \times \ln \left| 1 - \frac{e^{-2u(z_b-z_a)}}{\tilde{c}_\beta(u z_a) \bar{c}_\beta(u z_b)} \right|,
\]

with \( c_j(u) \) defined by Eq. (59). This contribution is exponentially suppressed if \( \lambda_\beta(z_b - z_a) \gg 1 \):

\[
\Delta E_\beta \approx -\frac{1}{2(4\pi D_1)^{1/2}} \left( \frac{\lambda_\beta}{z_b - z_a} \right)^{D_1/2} e^{-2\lambda_\beta(z_b - z_a)}/c_\beta(\lambda_\beta z_a)c_\beta(\lambda_\beta z_b),
\]

(130)

For small interbrane distances, satisfying the conditions \( kD(b - a) \ll 1 \) and \( \lambda_\beta(z_b - z_a) \ll 1 \) to the leading order we have the formula

\[
\Delta E_\beta \approx -\sigma \tilde{c}_\beta(D_1 + 1) e^{-D_1 kD_{b-a}}/((4\pi D_1)^{1/2}(b - a)^{D_1}/2),
\]

(131)

where \( \sigma = 1 \) for \( c_\beta(\lambda_\beta z_a)c_\beta(\lambda_\beta z_b) > 0 \) and \( \sigma = 2 - D_1 - 1 \) for \( c_\beta(\lambda_\beta z_a)c_\beta(\lambda_\beta z_b) < 0 \). Noting that the renormalized vacuum energies for single branes are finite in the limit \( a \to b \), we see that for sufficiently small interbrane distances the total vacuum energy is dominated by the interference part. In particular, it follows from here that for the case \( \sigma = 1 \) any fixation of the interbrane distance can be stable only locally.

In the limit \( \lambda_\beta z_b \gg 1 \) assuming that \( \lambda_\beta z_a \ll 1 \), for the contribution of a given KK mode one has the estimate

\[
\Delta E_\beta \approx -\pi \left( \frac{\lambda_\beta}{4\pi z_b} \right)^{D_1/2} e^{-2\lambda_\beta z_a}/c_\beta(\lambda_\beta z_b) K_\nu^{(a)}(\lambda_\beta z_a)/K_\nu^{(a)}(\lambda_\beta z_b),
\]

(132)

and the contribution from this part of KK spectrum is exponentially small. For \( \lambda_\beta z_b \ll 1 \), and for fixed \( \lambda_\beta z_b \) we find

\[
\Delta E_\beta \approx -\frac{1}{2^{1-2\nu} kD_{z_b}^{2\nu}} \int_{\lambda_\beta}^{\infty} du \; u^{2\nu+1} \times (u^2 - \lambda_\beta^2)^{D_1/2 - 1} \tilde{K}_\nu^{(b)}(u z_b)/\tilde{I}_\nu^{(b)}(u z_b),
\]

(133)

From the asymptotic analysis given above it follows that when the right brane tends to the AdS horizon, \( z_b \to \infty \), one has the estimate \( \Delta E_\beta \sim e^{-2\lambda_\beta z_a}/z_b^{D_1/2} \) for the nonzero KK modes and \( \Delta E_\beta \sim z_b^{-D_1-2\nu} \) for the zero KK mode. When the left brane tends to the AdS boundary, \( z_a \to 0 \), the interference part of the vacuum energy behaves as \( \Delta E \sim z_a^{2\nu} \). For small values of the AdS curvature radius corresponding to strong gravitational fields the main contribution comes from the zero
mode and the interference part is suppressed by the factor $e^{2\nu (a-b)} / z_b^2$.

In figures we have presented the dependence of the interference part of the vacuum energy on the size of the internal space $\Sigma = S^1$ and interbrane distance for a $D = 5$ minimally coupled massless scalar field with the Robin coefficients $\tilde{A}_a/B_a = -1, \tilde{A}_b/B_b = 5$ (Fig. 4) and $B_a = 0, \tilde{A}_b/B_b = 5$ (Fig. 5). Note that, in accordance with the asymptotic analysis, for small interbrane distances this part is negative for the first case and positive for the second one. In the second example the interference part of the vacuum energy has minimum with respect to both variables $\tilde{A}_a/B_a = -1, \tilde{A}_b/B_b = 5$.

![FIG. 4: Interference part of the vacuum energy, $\Delta E/k^{D+1}$, as a function of $\tilde{A}_a/z_b$ and $L/z_b$ for a $D = 5$ minimally coupled massless scalar field in the model with $\Sigma = S^1$ and with the Robin coefficients $\tilde{A}_a/B_a = -1, \tilde{A}_b/B_b = 5$.](image)

![FIG. 5: The same as in Fig. 4 for the values of the Robin coefficients $B_a = 0, \tilde{A}_b/B_b = 5$.](image)

Now let us check that for the separate parts of the vacuum energy the standard energy balance equation takes places. We denote by $P$ the perpendicular vacuum stress on the brane integrated over the internal space. This stress is determined by the vacuum expectation value of the $D_1$-component of the bulk energy-momentum tensor:

$$ P = -\int d^D z \sqrt{\gamma} \langle \mathcal{E}_{D}^{(s)} \rangle / \gamma_D^{D} |0\rangle. $$

We expect that in the presence of the surface energy the energy balance equation will be in the form

$$ dE = -PdV + \sum_{j=a,b} E_j^{(s)}dS_j^{(s)}, \quad E_j^{(s)} = \int d^D z \sqrt{\gamma} \gamma_j^{(s)}, $$

where $V$ is the $(D+1)$-volume in the bulk and $S_j^{(s)}$ is the $D$-volume on the brane $y = j$ per unit coordinate volume in the $D_1$-dimensional subspace,

$$ V = \int_0^b dy e^{-DkDy} \int d^D z \sqrt{\gamma}, $$

$$ S_j^{(s)} = e^{-DkDj} \int d^D z \sqrt{\gamma}, \quad j = a, b. $$

Combining equations (134)-(136), one obtains

$$ \frac{\partial E}{\partial z_j} = \frac{n^{(s)} P_j^{(s)} - DkDE_j^{(s)}}{(kDz_j)^{D+1}}, $$

with $P_j^{(s)}$ being the perpendicular vacuum stress on the brane at $y = j$. The vacuum stresses normal to the branes can be presented in the form $P_j^{(s)} = \nu_j^{(s)} p_j^{(s)}$, $j = a, b$, where $p_j^{(s)}$ is the stress for a single brane at $y = j$ when the second brane is absent and $p_j^{(s)}$ is induced by the presence of the second brane. The latter determines the interaction forces between the branes and is defined by the formula given in Ref. 44. Here we consider the corresponding quantity integrated over the internal space:

$$ P_j^{(s)}(\text{int}) = \int d^D z \sqrt{\gamma} P_j^{(s)}(\text{int}) = k_D^{D+1} \nu_j^{(s)} D_1 \beta D_1 \frac{A_j^{D+1}}{A_j^{D+1}} \int_0^\infty du \sqrt{u} \gamma_j^{(s)}, $$

with the notation

$$ \nu_j^{(s)}(u) = \left( u^2 - \nu^2 + \frac{m^2}{k_D^2} \right) B_j^2 - D(4\zeta - 1) A_j B_j - A_j^2. $$

As in the case of the induced surface energy density, the formula for the interaction force can also be presented in the form

$$ P_j^{(s)}(\text{int}) = n^{(s)}(k_Dz_j)^{D+1} \beta D_1 \frac{A_j^{D+1}}{A_j^{D+1}} \int_0^\infty du \sqrt{u} \gamma_j^{(s)} \right) \times \frac{\partial}{\partial z_j} \ln \left[ 1 - \frac{V_j^{(s)}(u_{za}) K_j^{(s)}(u_{za})}{K_j^{(s)}(u_{za}) V_j^{(s)}(u_{za})} \right]. $$

Now by taking into account expressions 74, 127, 140, it can be explicitly checked that the interference
parts in the vacuum energies and effective pressures on the branes obey the energy balance equation

\[ \frac{\partial \Delta E}{\partial z_j} = \frac{n^{(j)} P^{(j)}_{(\text{int})} - Dk_D \Delta E^{(s)}_j}{(k_D z_j)^{D+1}}. \]  

(141)

The second term in the nominator on the right of this formula corresponds to the additional pressure acting on the brane due to the nonzero external curvature.

**VI. CONCLUSION**

Continuing our previous work [12, 14], in the present paper we have investigated the expectation value of the surface energy-momentum tensor induced by the vacuum fluctuations of a bulk scalar field with an arbitrary curvature coupling parameter satisfying Robin boundary conditions on two parallel branes in background spacetime $\text{AdS}_{D_1+1} \times \Sigma$ with a warped internal space $\Sigma$. Vacuum stresses on the brane are the same for both subspaces and the energy-momentum tensor on the brane corresponds to the source of the cosmological constant type in the brane universe. It is remarkable that the latter property is valid also for the more general model with the metric $g_{\mu \nu}$ instead of $\eta_{\mu \nu}$ in line element \( g_{(2)} \). As an regularization procedure for the surface energy density we employ the zeta function technique. By using the residue theorem we have constructed an integral representations for the partial zeta function corresponding to a given KK mode along the internal subspace. This function is presented as the sum of single brane and second brane induced parts [see formula (12)]. The latter is finite at the physical point and the further analytical continuation is necessary for the first term only. We have done this in Section \( \text{III} \). As the first step we subtract and add to the integrand the leading terms of the corresponding asymptotic expansion for large values of the argument and explicitly integrate the asymptotic part. Further, for the regularization of the sum over the modes along the internal space we use the local zeta function related to these modes. By making use of the formula for the pole structure of this function, we have presented the energy density on a single brane as the sum of the pole and finite parts. These separate parts are determined by formulas (62) and (55). The pole parts in the surface energy density are absorbed by adding to the brane action the counterterms having the structure given by Eq. (30). The renormalized energy density on the corresponding surface of a single brane is determined by formula (61), where the second term on the right presents the finite renormalization part. The coefficients in this part cannot be determined within the model under consideration and their values should be fixed by additional renormalization conditions which relate them to observables.

Unlike to the single brane part, the surface energy density induced by the presence of the second brane contains no renormalization ambiguities and is investigated in Section \( \text{IV} \). This part is given by formula (72), or equivalently by formula (74). We have investigated the induced energy density in various asymptotic regions for the parameters of the model. In the limit $k_D \to 0$ we have obtained the result for the geometry of two boundaries in the bulk $R^{(D_1-1,1)} \times \Sigma$. For the modes along $\Sigma$ with large KK masses, the induced energy density is exponentially small. In particular, for sufficiently small length scales of the internal space this is the case for all nonzero KK modes and the main contribution comes from the zero mode. For small interbrane distances, to the leading order, the induced energy density is given by formula (52) and the contribution of the induced part dominates the single brane parts. When the right brane tends to the AdS horizon, $z_b \to \infty$, the induced energy density on the left brane vanishes as $e^{-2\lambda_D z_b}/z_b^{D_1/2}$ for the nonzero KK mode and like $z_b^{D_1-2
u}$ for the zero mode. In the same limit the corresponding energy density on the right brane behaves as $z_b^{D_1+D_2/2+1} e^{-2\lambda_D z_b}$ for the nonzero KK mode and like $z_b^{D_2-2\nu}$ for the zero mode. In the limit when the left brane tends to the AdS boundary the contribution of a given KK mode into the induced energy density vanishes as $z_a^{D_1+2
u}$ and as $z_a^{2\nu}$ for the left and right branes, respectively. For small values of the AdS curvature radius corresponding to strong gravitational fields, for nonzero KK modes under the conditions $\lambda_D z_a \gg 1$ and $\lambda_D (z_b - z_a) \gg 1$, the contribution to the induced energy density is suppressed by the factor $e^{-2\lambda_D z_b}/z_b^{D_1/2}$. For the zero KK mode, the corresponding energy density integrated over the internal space behaves as $k_D^{D_1+1} \exp[(D_1 \delta^D_2 + 2\nu) k_D (a-b)]$ for the brane at $y = j$ and is exponentially small. In the model without the internal space we have similar behavior with $\nu$ replaced by $\nu_1$ and for a scalar field with $\zeta < \zeta_{D_2+1}$ the suppression is relatively weaker.

In the model under discussion the hierarchy between the fundamental Planck scale and the effective Planck scale in the brane universe is generated by the combination of redshift and large volume effects. The corresponding effective Newton’s constant on the brane at $y = j$ is related to the higher-dimensional fundamental Newton’s constant by formula (55) (see also Eq. (11)) for the ratio of the corresponding Planck masses) and for large interbrane separations is exponentially small on the brane $y = b$. We show that this mechanism also allows obtaining a naturally small cosmological constant generated by the vacuum quantum fluctuations of a bulk scalar. For large interbrane distances the ratio of the induced cosmological constant to the the corresponding Planck scale quantity in the brane universe is estimated by formula (53) and is exponentially small. For the visible brane in the higher dimensional generalization of the Randall-Sundrum two-brane model, this ratio is given in terms of the effective and fundamental Planck masses by Eq. (60). We have considered two classes of models with the compactification scale on the visible brane close to the fundamental Planck scale. For the first one the higher di-
In Section V we have considered the total vacuum energy in the region between the branes, evaluated as the sum of zero-point energies for elementary oscillators. We have shown that this energy is equal to the sum of surface and volume energies. The latter is evaluated as the integral of the bulk energy density. Similar to the case of the surface energy, for the evaluation of the total vacuum energy we have used the zeta function regularization technique. This energy is presented as the sum of the single brane and interference parts. We have extracted the pole parts of the single brane vacuum energies and have discussed the corresponding renormalization procedure. The divergences can be absorbed by adding the counterterms to the brane action which have the structure similar to that for the surface energy. The interference part in the vacuum energy is given by formula (127) and is regular for all nonzero interbrane distances. We have investigated the asymptotic behavior of this part in various limiting cases. For sufficiently small interbrane distances the total vacuum energy is dominated by the interference part given by formula (134). In particular, from this formula it follows that in the case $\sigma = 1$ any minimum of the vacuum energy is local and the fixation of the interbrane distance by the corresponding effective potential is stable only locally. Further, we have shown that the induced vacuum densities and vacuum effective pressures on the branes satisfy the standard energy balance equation (134) with the inclusion of the surface terms.

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