ON THE MICROSCOPIC SPACETIME CONVEXITY PRINCIPLE OF FULLY NONLINEAR PARABOLIC EQUATIONS I: SPACETIME CONVEX SOLUTIONS

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Abstract. Spacetime convexity is a basic geometric property of the solutions of parabolic equations. In this paper, we study microscopic convexity properties of spacetime convex solutions of fully nonlinear parabolic partial differential equations and give a new simple proof of the constant rank theorem in [11].

1. Introduction

Spacetime convexity is a basic geometric property of the solutions of parabolic equations. In [6, 7, 8], Borell used the Brownian motion to study certain spacetime convexities of the solutions of diffusion equations and the level sets of the solution to heat equations with Schrödinger potential. In this paper we consider the spacetime convexity of the solution of fully nonlinear parabolic equations.

So far as we know, there are two important methods to approach the convexity of the solutions of partial differential equations, which are the microscopic and the macroscopic methods.

For the macroscopic convexity argument, Korevaar made breakthroughs in [22, 23], in which he introduced a concavity maximum principle for a class of quasilinear elliptic equations. Later it was improved by Kennington [20] and Kawhol [19]. The theory was further developed to its great generalization by Alvarez-Lasry-Lions [1]. There are some related results on the spacetime convexity of the solutions of parabolic equations in [21, 29] using the similar elliptic macroscopic convexity technique in Kennington [20] and Kawhol [19].

The key of the study of microscopic convexity is a method called constant rank theorem, which is a very useful tool to produce convex solutions in geometric analysis. By the corresponding homotopic deformation, the existence of convex solution comes from the constant rank theorem. Constant rank theorem was discovered in 2 dimension by Caffarelli-Friedman [9] (see also the work of Singer-Wong-Yau-Yau [30] for a similar approach). Later the result in [9] was generalized to high dimensions by Korevaar-Lewis [25]. Recently the constant rank theorem was generalized to fully nonlinear elliptic and parabolic equations in [10, 2, 3]. For the parabolic equations, the constant rank theorems in [10, 2] are only about the spatial hessian of the solution. For the

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geometric applications of the constant rank theorem, the Christoffel-Minkowski problem and the related prescribing Weingarten curvature problems were studied in [14] [13] [15]. The preservation of convexity for the general geometric flows of hypersurfaces was given in [2]. Soon after the constant rank theorem for the level sets was established in [4], where the result is a microscopic version of [5] (also it was studied in [24]). And the existence of the k-convex hypersurfaces with prescribed mean curvature was given in [17] recently. Also, there are two dimensional results [27, 28].

In this paper, we consider the following fully nonlinear parabolic equation

\[
(1.1) \quad \frac{\partial u}{\partial t} = F(D^2u, Du, u, x, t), \quad (x, t) \in \Omega \times (0, T],
\]

where \( F = F(A, p, u, x, t) \in C^{2,1}(S^n_+ \times \mathbb{R}^n \times \mathbb{R} \times \Omega \times [0, T]), D^2u = \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{u \times n}, Du = (u_{x_1}, \ldots, u_{x_n}), \)

and \( F \) is elliptic in the following sense

\[
(1.2) \quad (\frac{\partial F}{\partial u_{ij}}) > 0, \quad \text{for all} \ (x, t) \in \Omega \times (0, T],
\]

where \( S^n_+ \) denotes the set of all the semipositive definite \( n \times n \) matrices. We say equation (1.1) is parabolic if \( F \) is elliptic in the sense of (1.2).

In [2], Bian-Guan consider the (spatial) Hessian of the (spatial) convex solutions of (1.1) and get the following result.

**Theorem 1.1.** Suppose \( F(A, p, u, x, t) \in C^{2,1}(S^n_+ \times \mathbb{R}^n \times \mathbb{R} \times \Omega \times [0, T]). \) If \( F \) satisfies condition

\[
(1.3) \quad F(A^{-1}, p, u, x, t) \text{ is locally convex in } (A, u, x) \text{ for each pair } (p, t).
\]

If \( u \in C^{2,1}(\Omega \times [0, T]) \) is a convex solution of the equation

\[
\frac{\partial u}{\partial t} = F(D^2u, Du, u, x, t).
\]

For each \( t \in (0, T], \) let \( l(t) \) be the minimal rank of \( (D^2u(x,t)) \) in \( \Omega, \) then the rank of \( (D^2u(x,t)) \)

is constant \( l(t) \) and \( l(s) \leq l(t) \) for all \( s < t < T. \) For each \( 0 < t \leq T, \) \( x_0 \in \Omega \) there exist a neighborhood \( U \) of \( x_0 \) and \( (n - l(t)) \) fixed directions \( V_1, \ldots, V_{n-l(t)} \) such that \( D^2u(x,t)V_j = 0 \) for all \( 1 \leq j \leq n-l(t) \) and \( x \in U. \) Furthermore, for any \( t_0 \in [0, T], \) there is a \( \delta > 0, \) such that the null space of \( (D^2u(x,t)) \) is parallel in \( (x,t) \) for all \( x \in \Omega, \) \( t \in (t_0, t_0 + \delta). \)

Naturally, one can consider the spacetime Hessian of the spacetime convex solutions of (1.1) and establish the corresponding constant rank theorem. Hu-Ma [18] established the spacetime constant rank theorem for the heat equation in \( \mathbb{R}^n, \) and Chen-Hu [11] extended to fully nonlinear parabolic equations with the "inverse convex" structural condition. But the proof of the spacetime constant rank theorem is very complicate due to the choice of spacetime coordinates. As before, we can always choose a suitable (spatial) coordinates such that the matrix (e.g. the Hessian matrix or the second fundamental form of the level sets) is diagonal at each fixed point, which can simplify the calculations. For the spacetime Hessian matrix and the fundamental forms of the spacetime level sets, we cannot diagonalize them by choose the spatial coordinates, and the parabolic equation may change the form if we rotate the spacetime coordinates. In [18] [11] [12], the calculations are
Based on the good choice of spatial coordinates and the fixed equation form, but it is very hard. In fact, the difficulty is essential.

First, we give the definition of the spacetime convexity of a function \( u(x, t) \).

**Definition 1.2.** Suppose \( u \in C^2(\Omega \times (0, T]) \), where \( \Omega \) is a domain in \( \mathbb{R}^n \), then \( u \) is spacetime convex if \( u \) is convex for every \((x, t) \in \Omega \times (0, T]\), i.e

\[
D^2_{x,t}u = \begin{pmatrix} D^2u & (Du_t)^T \\ Du_t & uu_t \end{pmatrix} \geq 0
\]

In this paper and a successive paper, we will rotate the spacetime coordinates such the spacetime matrix is diagonal at each fixed point as before, but there are other difficulties in the calculations. We introduce some new techniques and inequalities to overcome the difficulties. Our main result is the following constant rank theorem, which is also the main theorem of [11].

**Theorem 1.3.** Suppose \( \Omega \) is a domain in \( \mathbb{R}^n \), \( F = F(A, p, u, x, t) \in C^2(S^n_+ \times \mathbb{R}^n \times \mathbb{R} \times \Omega \times (0, T]) \) and \( u \in C^2(\Omega \times (0, T]) \) is a spacetime convex solution of (1.1). If \( F \) satisfies (1.2) and the following condition

(1.4) \( F(A^{-1}, p, u, x, t) \) is locally convex in \((A, u, x, t)\) for each fixed \( p \in \mathbb{R}^n \).

Then \( D^2_{x,t}u \) has a constant rank in \( \Omega \) for each fixed \( t \in (0, T] \). Moreover, let \( l(t) \) be the minimal rank of \( D^2_{x,t}u \) in \( \Omega \), then \( l(s) \leq l(t) \) for all \( 0 < s \leq t \leq T \).

**Remark 1.4.** The techniques can be used to obtain the Constant Rank Theorem of spacetime second fundamental form of spacetime convex level sets.

The rest of the paper is organized as follows. In Section 2, we do some preliminaries. In Section 3, we prove a special case: heat equation, introduce the new ideas and new difficulties. In Section 4, we give a new simple proof of the Constant Rank Theorem [13].

## 2. PRELIMINARIES

In this section, we do some preliminaries.

### 2.1. elementary symmetric functions

First, we recall the definition and some basic properties of elementary symmetric functions, which could be found in [14, 26].

**Definition 2.1.** For any \( k = 1, 2, \ldots, n \), we set

\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}, \quad \text{for any } \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n.
\]

We also set \( \sigma_0 = 1 \) and \( \sigma_k = 0 \) for \( k > n \).

We denote by \( \sigma_k(\lambda |i) \) the symmetric function with \( \lambda_i = 0 \) and \( \sigma_k(\lambda |ij) \) the symmetric function with \( \lambda_i = \lambda_j = 0 \).

The definition can be extended to symmetric matrices by letting \( \sigma_k(W) = \sigma_k(\lambda(W)) \), where \( \lambda(W) = (\lambda_1(W), \lambda_2(W), \cdots, \lambda_n(W)) \) are the eigenvalues of the symmetric matrix \( W \). We also
denote by \( \sigma_k(W | i) \) the symmetric function with \( W \) deleting the \( i \)-row and \( i \)-column and \( \sigma_k(W | ij) \) the symmetric function with \( W \) deleting the \( i, j \)-rows and \( i, j \)-columns. Then we have the following identities.

**Proposition 2.2.** Suppose \( W = (W_{ij}) \) is diagonal, and \( m \) is a positive integer, then

\[
\frac{\partial \sigma_m(W)}{\partial W_{ij}} = \begin{cases} 
\sigma_{m-1}(W | i), & \text{if } i = j, \\
0, & \text{if } i \neq j.
\end{cases}
\]

and

\[
\frac{\partial^2 \sigma_m(W)}{\partial W_{ij} \partial W_{kl}} = \begin{cases} 
\sigma_{m-2}(W | ik), & \text{if } i = j, k = l, i \neq k, \\
-\sigma_{m-2}(W | ik), & \text{if } i = l, j = k, i \neq j, \\
0, & \text{otherwise}.
\end{cases}
\]

We need the following standard formulas of elementary symmetric functions.

**Proposition 2.3.** Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \) and \( k = 0, 1, \ldots, n \), then

\[
\sigma_k(\lambda) = \sigma_k(\lambda | i) + \lambda_i \sigma_{k-1}(\lambda | i), \quad \forall 1 \leq i \leq n,
\]

\[
\sum_i \lambda_i \sigma_{k-1}(\lambda | i) = k \sigma_k(\lambda),
\]

\[
\sum_i \sigma_k(\lambda | i) = (n-k) \sigma_k(\lambda).
\]

### 2.2. rank of spacetime Hessian

To study the rank of the spacetime Hessian \( D^2_{x,t}u \), we need the following simple lemma.

**Lemma 2.4.** Suppose \( D^2_{x,t}u \geq 0 \), and \( l = \text{rank}\{D^2_{x,t}u(x_0, t_0)\} \), and \( D^2u(x_0, t_0) \) is diagonal with \( u_{11} \geq u_{22} \geq \cdots \geq u_{nn} \), then at \((x_0, t_0)\), there is a positive constant \( C_0 \) such that

**CASE 1:**

\[
u_{11} \geq \cdots \geq u_{l-1} \geq C_0, \quad u_{ll} = \cdots = u_{nn} = 0,
\]

\[
u_{ll} - \sum_{i=1}^{l-1} \frac{u_{ll}^2}{u_{ii}} \geq C_0, \quad u_{il} = 0, \quad l \leq i \leq n.
\]

In particular, \( \sigma_l(D^2u(x_0, t_0)) = 0 \).

**CASE 2:**

\[
u_{11} \geq \cdots \geq u_{ll} \geq C_0, \quad u_{l+1} = \cdots = u_{nn} = 0,
\]

\[
u_{ll} = \sum_{i=1}^{l} \frac{u_{ll}^2}{u_{ii}}, \quad u_{il} = 0, \quad l + 1 \leq i \leq n.
\]

In particular, \( \sigma_l(D^2u(x_0, t_0)) > 0 \).

**Proof:** Set \( M = D^2u(x_0, t_0) = \text{diag}(u_{11}, u_{22}, \cdots, u_{nn}) \geq 0 \) and we can assume \( \text{Rank}\{M\} = k \), then we can obtain \( k = l - 1 \) or \( k = l \). Otherwise, if \( k < l - 1 \), we know

\[
u_{l-1l-1} = \cdots = u_{nn} = 0 \text{ at } (x_0, t_0),
\]
and from $D^{2}_{x,t}u(x_0,t_0) \geq 0$, we get
\[ u_{l-1t} = \cdots = u_{nt} = 0 \text{ at } (x_0,t_0). \]
So $\text{Rank}\{D^{2}_{x,t}u\} \leq l - 1$, contradiction. If $k > l$, we have
\[ l = \text{Rank}\{D^{2}_{x,t}u\} \geq \text{Rank}\{M\} = k \geq l + 1 \text{ at } (x_0,t_0). \]
This is impossible.
For $k = l - 1$, we know at $(x_0,t_0)$
\[ u_{11} \geq \cdots \geq u_{l-1l-1} > 0, \quad u_{ll} = \cdots = u_{nn} = 0, \]
and due to $D^{2}_{x,t}u(x_0,t_0) \geq 0$, we get
\[ u_{lt} = \cdots = u_{nt} = 0. \]
Since $\text{Rank}\{D^{2}_{x,t}u\} = l$, then $\sigma_l(D^{2}_{x,t}u) > 0$. Direct computation yields
\[ \sigma_l(D^{2}_{x,t}u) = u_{tt}\sigma_{l-1}(M) - \sum_{i=1}^{l-1} u_{ti}u_{it}\sigma_{l-2}(M|i) = \sigma_{l-1}(M)[u_{tt} - \sum_{i=1}^{l-1} \frac{u_{ti}^2}{u_{ii}}] > 0, \]
so we have
\[ u_{tt} - \sum_{i=1}^{l-1} \frac{u_{ti}^2}{u_{ii}} > 0, \]
This is CASE 1.
For $k = l$, we know at $(x_0,t_0)$
\[ u_{11} \geq \cdots \geq u_{ll} > 0, \quad u_{l+1l+1} = \cdots = u_{nn} = 0, \]
and due to $D^{2}_{x,t}u(x_0,t_0) \geq 0$, we get
\[ u_{l+1t} = \cdots = u_{nt} = 0. \]
Since $\text{Rank}\{D^{2}_{x,t}u\} = l$, then $\sigma_{l+1}(D^{2}_{x,t}u) = 0$. Direct computation yields
\[ \sigma_{l+1}(D^{2}_{x,t}u) = u_{tt}\sigma_{l}(M) - \sum_{i=1}^{l} u_{ti}u_{it}\sigma_{l-1}(M|i) = \sigma_{l}(M)[u_{tt} - \sum_{i=1}^{l} \frac{u_{ti}^2}{u_{ii}}] = 0, \]
so we have
\[ u_{tt} - \sum_{i=1}^{l} \frac{u_{ti}^2}{u_{ii}} = 0, \]
This is CASE 2. \qed
2.3. structure condition \([1.4]\). Now we discuss the structure condition \([1.4]\). For any given \(\vec{X} = ((X_{\alpha \beta}), Y, (Z_i), D) \in S^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}\), we define a quadratic form

\[
Q^*(\vec{X}, \vec{X}) = \sum_{a,b,c,d=1}^{n} F_{a,b,c,d} X_{a,b,c,d} + 2 \sum_{a,b,c,d=1}^{n} F_{a,b}^{c,d} X_{a,b} X_{c,d} + 2 \sum_{a,b=1}^{n} F_{a,b}^{u} X_{a,b} Y \\
+ 2 \sum_{a,b=1}^{n} \sum_{i=1}^{n} F_{a,b}^{x_i} X_{a,b} Z_i + 2 \sum_{a,b=1}^{n} F_{a,b}^{t} X_{a,b} D + F_{a,b}^{u} Y^2 + 2 \sum_{i=1}^{n} F_{x_i}^{t} Y Z_i 
\]

(2.6)

where the derivative functions of \(F\) are evaluated at \((A, p, u, x, t)\) and \((A^{ab}) = A^{-1}\).

Through direct calculations, we can get

**Lemma 2.5.** \(F\) satisfies the condition \([1.4]\) if and only if for each \(p \in \mathbb{R}^n\)

\[
Q^*(\vec{X}, \vec{X}) \geq 0, \quad \forall \quad \vec{X} = ((X_{\alpha \beta}), Y, (Z_i), D) \in S^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R},
\]

where the derivative functions of \(F\) are evaluated at \((A, p, u, x, t)\), and \(Q^*\) is defined in (2.6).

The proof of Lemma 2.5 is similar to the discussion in [2], and we omit it.

2.4. an auxiliary lemma. Similarly to the Lemma 2.5 in Bian-Guan [2], we have

**Lemma 2.6.** Suppose \(W(x) = (W_{ij}(x)) \geq 0\) for every \(x \in \Omega \subset \mathbb{R}^n\), and \(W_{ij}(x) \in C^{1,1}(\Omega)\), then for every \(\Omega \subset \subset \Omega\), there exists a positive constant \(C\) depending only on \(\text{dist}\{\Omega, \partial \Omega\}\) and \(\|W\|_{C^{1,1}(\Omega)}\) such that

\[
|\nabla W_{ij}| \leq C(W_{ii} W_{jj})^{\frac{1}{2}},
\]

(2.8)

for every \(x \in \Omega\) and \(1 \leq i, j \leq n\).

**Proof:** The same arguments as in the proof of Lemma 2.5 in [2] carry through with a small modification since \(W\) is a general matrix instead of a Hessian of a convex function.

It’s known that for any nonnegative \(C^{1,1}\) function \(h\), \(|\nabla h(x)| \leq Ch^*(x)\) for all \(x \in \Omega\), where \(C\) depends only on \(\|h\|_{C^{1,1}(\Omega)}\) and \(\text{dist}\{\Omega, \partial \Omega\}\) (see \([31]\)).

Since \(W(x) \geq 0\), so we choose \(h(x) = W_{ii}(x) \geq 0\). Then we can get from the above argument

\[
|\nabla W_{ii}| \leq C_1(W_{ii}) \leq C_1(W_{ii} W_{ii})^{\frac{1}{2}},
\]

(2.8) holds for \(i = j\).

Similarly, for \(i \neq j\), we choose \(h = \sqrt{W_{ii} W_{jj}} \geq 0\), then we get

\[
|\nabla \sqrt{W_{ii} W_{jj}}| \leq C_2(\sqrt{W_{ii} W_{jj}})^{\frac{1}{2}} = C_2(W_{ii} W_{jj})^{\frac{1}{4}},
\]

(2.9)

And for \(h = \sqrt{W_{ii} W_{jj}} - W_{ij}\), we have

\[
|\nabla (\sqrt{W_{ii} W_{jj}} - W_{ij})| \leq C_3(\sqrt{W_{ii} W_{jj}} - W_{ij})^{\frac{1}{2}} \leq C_3(W_{ii} W_{jj})^{\frac{1}{4}}.
\]

(2.10)
So from (2.9) and (2.10), we get
\[ |\nabla W_{ij}| = |\nabla \sqrt{W_{ii}W_{jj}} - \nabla (\sqrt{W_{ii}W_{jj}} - W_{ij})| \leq |\nabla \sqrt{W_{ii}W_{jj}}| + |\nabla (\sqrt{W_{ii}W_{jj}} - W_{ij})| \leq (C_2 + C_3)(W_{ii}W_{jj})^{\frac{1}{4}}. \]
So (2.8) holds for \( i \neq j \). \( \square \)

3. The Constant Rank Theorem for the Heat Equation

In this section, we consider a special case: heat equation. This is a new proof of the main result in [18], and the idea is from [16].

Our main result is the following theorem.

**Theorem 3.1.** Suppose \( \Omega \) is a domain in \( \mathbb{R}^n \), and \( u \in C^2(\Omega \times (0, T]) \) is a spacetime convex solution of the heat equation
\[ u_t = \Delta u \quad \text{in} \quad \Omega \times (0, T]. \]
Then \( D^2_{x,t}u \) has a constant rank in \( \Omega \) for each fixed \( t \in (0, T] \). Moreover, let \( l(t) \) be the minimal rank of \( D^2_{x,t}u \) in \( \Omega \), then \( l(s) \leq l(t) \) for all \( 0 < s < t < T \).

**Proof.** Following the assumptions of Theorem 3.1, we know \( D^2_{x,t}u \geq 0 \). By the regularity theory, we can get \( u \in C^3(\Omega \times (0, T]) \). Suppose \( D^2_{x,t}u \) attains its minimal rank \( l \) at some point \((x_0, t_0) \in \Omega \times (0, T)\). We pick a small open neighborhood \( \mathcal{O} \times (t_0 - \delta, t_0] \) of \( (x_0, t_0) \). And for any fixed point \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0] \), we rotate coordinates \((x, t)\) with \( y = (y_1, \cdots, y_n, y_{n+1}) = (x, t)P \), such that the matrix \( D^2_{x,t}u \) is diagonal, where \( P = (P_{\alpha\beta})_{n+1 \times n+1} \) is an orthogonal matrix. For convenience, we will use \( i, j, k, l = 1, \cdots, n \) to represent the \( x \) coordinates, \( t \) still the time coordinate, and \( \alpha, \beta, \gamma, \eta = 1, \cdots, n + 1 \) the \( y \) coordinates. And we have
\[ \frac{\partial y_\alpha}{\partial x_i} = P_{\alpha i}, \quad \frac{\partial y_\alpha}{\partial t} = P_{\alpha(n+1)}, \]
In the following, we always denote
\[ u_i = \frac{\partial u}{\partial x_i}, u_t = \frac{\partial u}{\partial t}, u_\alpha = \frac{\partial u}{\partial y_\alpha}, u_{n+1} = \frac{\partial u}{\partial y_{n+1}}, \]
\[ u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, u_{it} = \frac{\partial^2 u}{\partial x_i \partial t}, u_{\alpha t} = \frac{\partial^2 u}{\partial x_\alpha \partial t}, u_{\alpha\beta} = \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}, \text{ etc.} \]

At \((x, t)\), the matrix \( D^2_{x,t}u \) is diagonal in the \( y \) coordinates, so without loss of generality we assume \( u_{\alpha\alpha} \geq u_{\beta\beta} \) for arbitrary \( 1 \leq \alpha < \beta \leq n + 1 \). Then there is a positive constant \( C > 0 \)
depending only on \( \|u\|_{C^{3,1}} \), such that \( \frac{\partial^2 \phi}{\partial y_1^2} \geq \cdots \geq \frac{\partial^2 \phi}{\partial y_n^2} \geq C > 0 \) for all \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0)\).

For convenience we denote \( G = \{1, \ldots, l\} \) and \( B = \{l + 1, \ldots, n, n + 1\} \) which means good terms and bad ones in indices respectively. Without confusion we will also simply denote \( G = \{\frac{\partial^2 \phi}{\partial y_1^2}, \ldots, \frac{\partial^2 \phi}{\partial y_n^2}\} \) and \( B = \{\frac{\partial^2 \phi}{\partial y_l^2}, \ldots, \frac{\partial^2 \phi}{\partial y_n^2}, \frac{\partial^2 \phi}{\partial y_{n+1}^2}\} \).

Set
\[
\phi = \sigma_{l+1}(D_x^2 u),
\]

In the following, we will prove a differential inequality
\[
\Delta_x \phi - \phi_t \leq C(\phi + |\nabla_x \phi|) \quad \text{in } \mathcal{O} \times (t_0 - \delta, t_0].
\]

Then by the strong maximum principle and the method of continuity, we can prove the theorem.

In the \( y \) coordinates, we have
\[
\phi = \sigma_{l+1}(D_x^2 u) \geq \sigma_1(G)\sigma_1(B) \geq 0,
\]
so we get
\[
u_{aa} = O(\phi), \quad a \in B.
\]

Taking the first derivatives of \( \phi \) in \( x, t \), we have
\[
\phi_t = \frac{\partial \phi}{\partial x_t} = \sum_{a=1}^{n+1} \sigma_l(D_y^2 u)u_{aa} = \sigma_l(G) \sum_{a \in B} u_{aa} + O(\phi),
\]
\[
\phi_t = \frac{\partial \phi}{\partial t} = \sum_{a=1}^{n+1} \sigma_l(D_y^2 u)u_{a} = \sigma_l(G) \sum_{a \in B} u_{aa} + O(\phi),
\]
so from (3.7), we get
\[
\sum_{a \in B} u_{aa} = O(\phi + |\nabla_x \phi|).
\]

Taking the second derivatives of \( \phi \) in \( y \) coordinates, we have
\[
\phi_{aa} = \frac{\partial^2 \phi}{\partial y_a \partial y_a} = \sum_{l=1}^{n+1} \frac{\partial \sigma_{l+1}(D_x^2 u)}{\partial u_{\gamma \gamma}} u_{\gamma \gamma a} + \sum_{l=1}^{n+1} \frac{\partial^2 \sigma_{l+1}(D_x^2 u)}{\partial u_{\gamma \gamma} \partial u_{\eta \eta}} u_{\gamma \gamma a} u_{\eta \eta a} + \sum_{l=1}^{n+1} \frac{\partial^2 \sigma_{l+1}(D_x^2 u)}{\partial u_{\gamma \gamma} \partial u_{\eta \eta}} u_{\gamma \gamma a} u_{\eta \eta a} - \sum_{l=1}^{n+1} (D_x^2 u) u_{\gamma \gamma a} u_{\eta \eta a},
\]
where
\[
\sum_{l=1}^{n+1} \sigma_l(D_y^2 u) u_{\gamma \gamma a} = \sum_{a \in B} \sigma_l(D_y^2 u) u_{\gamma \gamma a} + \sum_{a \in G} \sigma_l(D_y^2 u) u_{\gamma \gamma a} = 0,
\]
so we get
\[
\phi_{aa} = \sigma_l(G) \sum_{a \in B} u_{aa} + O(\phi).
\]
\[
\sum_{\gamma \neq \eta} \sigma_{l-1}(D^2_y u|\gamma\eta)u_{\gamma\gamma\alpha}u_{\eta\eta\beta} = \sum_{\gamma \neq \eta} \sigma_{l-1}(D^2_y u|\gamma\eta)u_{\gamma\gamma\alpha}u_{\eta\eta\beta} + \sum_{\gamma \neq \eta} \sigma_{l-1}(D^2_y u|\gamma\eta)u_{\gamma\gamma\alpha}u_{\eta\eta\beta} \\
+ \sum_{\gamma \neq \eta} \sigma_{l-1}(D^2_y u|\gamma\eta)u_{\gamma\gamma\alpha}u_{\eta\eta\beta} + \sum_{\gamma \neq \eta} \sigma_{l-1}(D^2_y u|\gamma\eta)u_{\gamma\gamma\alpha}u_{\eta\eta\beta} \\
= O(\phi) + \sum_{\gamma \neq \eta} \sigma_{l-1}(G|\gamma\eta)u_{\gamma\eta\beta} \sum_{\gamma \neq \eta} u_{\gamma\gamma\alpha} + \sum_{\gamma \neq \eta} \sigma_{l-1}(G|\gamma\eta)u_{\gamma\gamma\alpha} \sum_{\eta \neq \gamma} u_{\eta\eta\beta} \\
(3.12) \\
= \sigma_l(G)[\sum_{\gamma \neq \eta} u_{\gamma\eta\beta} \sum_{\eta \neq \gamma} u_{\gamma\gamma\alpha} + \sum_{\gamma \neq \eta} u_{\gamma\gamma\alpha} \sum_{\eta \neq \gamma} u_{\eta\eta\beta}] + O(\phi),
\]

and
\[
\sum_{\gamma \neq \eta} \sigma_{l-1}(D^2_y u|\gamma\eta)u_{\gamma\gamma\alpha}u_{\eta\eta\beta} = \sum_{\gamma \neq \eta} \sigma_{l-1}(D^2_y u|\gamma\eta)u_{\gamma\gamma\alpha}u_{\eta\eta\beta} \\
+ \sum_{\gamma \neq \eta} \sigma_{l-1}(D^2_y u|\gamma\eta)u_{\gamma\gamma\alpha}u_{\eta\eta\beta} \\
= O(\phi) + \sum_{\gamma \neq \eta} \sigma_{l-1}(G|\gamma\eta)u_{\gamma\eta\alpha}u_{\eta\eta\beta} + \sum_{\gamma \neq \eta} \sigma_{l-1}(G|\gamma\eta)u_{\gamma\eta\alpha}u_{\eta\eta\beta} \\
(3.13) \\
= 2\sigma_l(G) \sum_{\gamma \neq \eta} \frac{u_{\gamma\eta\alpha}u_{\eta\eta\beta}}{u_{\eta\eta}} + O(\phi).
\]

So from (3.10) - (3.13), we get
\[
\phi_{\alpha\beta} = \sigma_l(G) \sum_{\gamma \in B} u_{\gamma\alpha\beta} - 2\sigma_l(G) \sum_{\gamma \in B} \frac{u_{\gamma\eta\alpha}u_{\eta\eta\beta}}{u_{\eta\eta}} \\
+ \sigma_l(G)[\sum_{\eta \in G} u_{\eta\eta\beta} \sum_{\eta \neq \gamma} u_{\gamma\gamma\alpha} + \sum_{\gamma \neq \eta} u_{\gamma\gamma\alpha} \sum_{\eta \neq \gamma} u_{\eta\eta\beta}] + O(\phi). \\
(3.14)
\]

Then we get
\[
\Delta_x \phi = \sum_{i=1}^n \phi_{ii} = \sum_{i=1}^n \sum_{\alpha\beta=1}^{n+1} P_{i\alpha} P_{i\beta} \phi_{\alpha\beta} \\
= \sigma_l(G) \sum_{i=1}^n \sum_{\gamma \in B} \left[ \sum_{\alpha=1}^{n+1} P_{i\alpha} P_{i\beta} u_{\gamma\alpha\beta} \right] - 2\sigma_l(G) \sum_{i=1}^n \sum_{\gamma \in B} \left\{ \left[ \sum_{\alpha=1}^{n+1} P_{i\alpha} u_{\gamma\alpha} \right] \left[ \sum_{\beta=1}^{n+1} P_{i\beta} u_{\eta\eta\beta} \right] \right\} \\
+ \sigma_l(G) \sum_{i=1}^n \sum_{\eta \in G} \left[ \sum_{\beta=1}^{n+1} P_{i\beta} u_{\eta\eta\beta} \right] \sum_{\gamma \in G} P_{i\alpha} u_{\gamma\alpha} + \sum_{\gamma \in G} P_{i\alpha} u_{\gamma\alpha} \sum_{\eta \in B} \sum_{\beta=1}^{n+1} P_{i\beta} u_{\eta\eta\beta} + O(\phi) \\
= \sigma_l(G) \sum_{i=1}^n \sum_{\gamma \in B} u_{\gamma\gamma\alpha} - 2\sigma_l(G) \sum_{i=1}^n \sum_{\gamma \in B} \frac{u_{\gamma\eta\alpha} u_{\eta\eta\gamma}}{u_{\eta\eta}} + 2\sigma_l(G) \sum_{i=1}^n \sum_{\eta \in G} \frac{u_{\eta\eta\gamma}}{u_{\eta\eta}} \sum_{\gamma \in B} u_{\gamma\gamma\alpha} + O(\phi). 
\]
By (3.9), it holds

\[
\Delta x \phi = \sigma_l(G) \sum_{\gamma \in B} \left[ \Delta x u_{\gamma \gamma} - 2 \sum_{\eta \in G} \sum_{i=1}^n \frac{u_{\gamma \eta i}^2}{u_{\eta \eta}} \right] + O(\phi + |\nabla x \phi|)
\]

(3.15)

By (3.8), (3.15) and the equation (3.1), we can obtain

\[
\Delta x \phi - \phi_t = \sigma_l(G) \sum_{\gamma \in B} \left[ (\Delta x u_{\gamma \gamma} - u_{\gamma \gamma t}) - 2 \sum_{\eta \in G} \sum_{i=1}^n \frac{u_{\gamma \eta i}^2}{u_{\eta \eta}} \right] + O(\phi + |\nabla x \phi|)
\]

\[
= -2\sigma_l(G) \sum_{\gamma \in B} \sum_{\eta \in G} \sum_{i=1}^n \frac{u_{\gamma \eta i}^2}{u_{\eta \eta}} + O(\phi + |\nabla x \phi|)
\]

(3.16)

\[
\leq C(\phi + |\nabla x \phi|).
\]

Then (3.5) holds, and we prove Theorem 3.1. \(\square\)

Remark 3.2. In the proof, we rotate the spacetime coordinates \((x, t)\) such that \(D^2_{x,t} u\) is diagonal, and the heat equation in \((x, t)\) is a new linear equation in \(y\) coordinates. While, the idea in [18] is only to rotate the spatial coordinates \(x\) such that \(D^2_x u\) is diagonal, and keep the heat equation invariant. This proof is easier than [18], and it can be applied to linear parabolic equations.

Remark 3.3. For fully nonlinear parabolic equations, the above test function \(\phi = \sigma_l(D^2_{x,t} u)\) is not good enough. If we choose Bian-Guan’s test function \(\phi = \sigma_{l+1}(D^2_{x,t} u) + \sigma_{l+1}(D^2_{x,t} u) + \sigma_{l+1}(D^2_{x,t} u)\) as in [2] and rotate the spacetime coordinates \((x, t)\) such that \(D^2_{x,t} u\) is diagonal, we should use \(|\nabla x \phi|\), not \(|\nabla y \phi| = |D_{x,t} u|\) to control the bad terms \(\sum_{\alpha \beta \in B} |u_{\alpha \beta \gamma}|\). This is very easy.

4. Proof of Theorem 1.3

In this section, we will prove the constant rank theorem of spacetime Hessian for the fully nonlinear parabolic equations, Theorem 1.3. In fact, we will use a new idea to simplify the calculations. The key is to use the constant rank properties of the spatial Hessian \(D^2 u\).

Suppose \(W(x, t) = D^2_{x,t} u\) attains the minimal rank \(l\) at some point \((x_0, t_0) \in \Omega \times (0, T]\). We may assume \(l \leq n\), otherwise there is nothing to prove. From lemma 2.4 we can transform the \(x\) coordinates such that \(D^2 u(x_0, t_0)\) is diagonal with \(u_{11} \geq u_{22} \geq \cdots \geq u_{nn}\), then at \((x_0, t_0)\), there is a positive constant \(C_0\) such that

**CASE 1:**

\[
\begin{align*}
    u_{11} &\geq \cdots \geq u_{l-1} & \geq C_0, \\
    u_{ll} & = \cdots = u_{nn} = 0, \\
    u_{ll} - \sum_{i=1}^{l-1} \frac{u_{ii}^2}{u_{ii}} & \geq C_0, \\
    u_{ii} & = 0, & l \leq i \leq n.
\end{align*}
\]

In particular, \(\sigma_l(D^2 u(x_0, t_0)) = 0\).
CASE 2:

\[ u_{11} \geq \cdots \geq u_{ll} \geq C_0, \quad u_{l+1,l+1} = \cdots = u_{nn} = 0, \]
\[ u_{tt} = \sum_{i=1}^{l} \frac{u_{tt}}{u_{ii}}, \quad u_{it} = 0, \quad l + 1 \leq i \leq n. \]

In particular, \( \sigma_l(D^2u(x_0, t_0)) > 0 \).

In the following, we denote

\[
F^{ij} = \frac{\partial F}{\partial u_{ij}}, \quad F^{u_i} = \frac{\partial F}{\partial u_i}, \quad F^u = \frac{\partial F}{\partial t}, \quad F^t = \frac{\partial F}{\partial t},
\]
\[
F^{ij,kl} = \frac{\partial^2 F}{\partial u_{ij} \partial u_{kl}}, \quad F^{ij,u_k} = \frac{\partial^2 F}{\partial u_{ij} \partial u_k}, \quad F^{ij,u} = \frac{\partial^2 F}{\partial u_{ij} \partial u},
\]
\[
F^{u_{i1},u_{j1}} = \frac{\partial^2 F}{\partial u_{i1} \partial u_{j1}}, \quad F^{u_{i1},u} = \frac{\partial^2 F}{\partial u_{i1} \partial u}, \quad F^{u, u} = \frac{\partial^2 F}{\partial u \partial u},
\]

where \( 1 \leq i, j, k, l \leq n. \)

In order to prove Theorem 1.3, we will firstly consider the constant rank theorem of \( D^2u \), which is all from \([2]\), and state some important results. Then we prove Theorem 1.3 under the above CASE 1 and CASE 2, respectively.

4.1. The constant rank properties of \( D^2u \). Following the assumptions of Theorem 1.3, we know \( D^2u \geq 0 \). If \( F \) satisfies (1.3), then \( F \) satisfies (1.3) and Theorem 1.1 holds. Suppose \( D^2u \) attains its minimal rank \( l \) at some point \((x_0, t_0) \in \Omega \times (0, T). \) We pick a small open neighborhood \( \Omega \times (t_0 - \delta, t_0] \) of \((x_0, t_0)\), and for any fixed point \((x, t) \in \Omega \times (t_0 - \delta, t_0)\), we rotate the \( x \) coordinates such that the matrix \( D^2u(x, t) \) is diagonal and without loss of generality we assume \( u_{11} \geq u_{22} \geq \cdots \geq u_{nn} \). Then there is a positive constant \( C > 0 \) depending only on \( \|u\|_{C^{3,1}} \), such that \( u_{11} \geq \cdots \geq u_{ll} \geq C > 0 \) for all \((x, t) \in \Omega \times (t_0 - \delta, t_0)\). For convenience we denote \( G = \{1, \ldots, l\} \) and \( B = \{l + 1, \ldots, n\} \) which means good terms and bad ones in indices respectively. Without confusion we will also simply denote \( G = \{u_{11}, \ldots, u_{ll}\} \) and \( B = \{u_{l+1,l+1}, \ldots, u_{nn}\} \).

Set

\[
q(W) = \begin{cases} \frac{\sigma_{l+2}(W)}{\sigma_{l+1}(W)} & \text{if } \sigma_{l+1}(W) > 0, \\ 0 & \text{if } \sigma_{l+1}(W) = 0. \end{cases}
\]

And denote

\[
\phi = \sigma_{l+1}(D^2u) + q(D^2u).
\]

In \([2]\), Bian-Guan got the following differential inequality,

\[
\sum_{i,j=1}^{n} F^{ij} \phi_{ij}(x, t) - \phi_t(x, t) \leq C(\phi(x, t) + |\nabla \phi(x, t)|) - C_2 \sum_{i,j \in B} |\nabla u_{ij}|,
\]

where \( C_1, C_2 \) are two positive constants and \((x, t) \in \Omega \times (t_0 - \delta, t_0]\). Together with

\[
\phi(x, t) \geq 0, \quad (x, t) \in \Omega \times (t_0 - \delta, t_0], \quad \phi(x, t_0) = 0,
\]

\[ (4.3) \]

\[ (4.4) \]
we can apply the strong maximum principle of parabolic equations, and we obtain
\begin{equation}
\phi(x, t) \equiv 0, \quad (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0],
\end{equation}
and
\begin{equation}
\sum_{i,j \in B} |\nabla u_{ij}| \equiv 0, \quad (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0].
\end{equation}
By the argument in [2], the null space of $D^2u$ is parallel for all $x \in \Omega, t \in (t_0 - \delta, t_0]$. So we can fix $e_{l+1}, \cdots, e_n$ in $(x, t) \in \Omega \times (t_0 - \delta, t_0]$ such that $u_{ii}(x, t) \equiv 0$, for any $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ and $i \in B$.

So we can get the following constant rank properties.

**Proposition 4.1.** Under above assumptions, we can get
\begin{equation}
\sum_{i,j \in B} |\nabla u_{ij}|(x, t) \equiv 0, \quad \text{for} \quad (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0].
\end{equation}

4.2 **CASE 1.** In this subsection, we will prove Theorem 1.3 under CASE 1. Suppose $W(x, t) = D^2_{x,t}u$ attains the minimal rank $l$ at some point $(x_0, t_0) \in \mathcal{O} \times (0, T]$. We may assume $l \leq n$, otherwise there is nothing to prove. Then from lemma 2.4 there is a neighborhood $\mathcal{O} \times (t_0 - \delta, t_0]$ of $(x_0, t_0)$, such that $u_{11} \geq \cdots \geq u_{l-1l-1} \geq C > 0$ and $u_{ll} - \sum_{i=1}^{l-1} u_{ii}^2 u_{ii}u_{ll} \geq C$ for all $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$.

And for any fixed point $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$, we rotate the $x$ coordinate such that the matrix $D^2u$ is diagonal, and without loss of generality we assume $u_{11} \geq u_{22} \geq \cdots \geq u_{nn}$. We can denote $G = \{1, \cdots, l-1\}$ and $B = \{l, \cdots, n\}$.

In order to prove the main theorem, we just need to prove
\begin{equation}
\sigma_{l+1}(D^2_{x,t}u) \equiv 0, \quad \text{for every} \quad (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0].
\end{equation}
In fact, when $D^2u$ is diagonal at $(x, t)$, we have
\begin{equation}
\sigma_{l+1}(D^2_{x,t}u) = \sigma_{l+1}(D^2u) + u_{ll} \sigma_l(D^2u) - \sum_{i=1}^{l-1} u_{ii}^2 \sigma_{l-1}(D^2u|i)
\leq \sigma_{l+1}(D^2u) + u_{ll} \sigma_l(D^2u).
\end{equation}
Under CASE 1, the spatial Hessian $D^2u$ attains the rank $l - 1$. From [2], the constant rank theorem holds for the spatial Hessian $D^2u$ of the solution $u$ for the equation $u_t = F(D^2u, Du, u, x, t)$, so we can get,
\begin{equation}
\sigma_{l+1}(D^2u) = \sigma_l(D^2u) \equiv 0, \quad \text{for every} \quad (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0].
\end{equation}
Then
\begin{equation}
0 \leq \sigma_{l+1}(D^2_{x,t}u) \leq \sigma_{l+1}(D^2u) + u_{ll} \sigma_l(D^2u) = 0.
\end{equation}
Hence (4.9) holds.

By the continuity method, Theorem 1.3 holds under CASE 1.
4.3. CASE 2. In this subsection, we will prove Theorem 1.3 under CASE 2. Suppose \( W(x, t) = D^2_x,t u \) attains the minimal rank \( l \) at some point \( (x_0, t_0) \in \Omega \times (0, T) \). We may assume \( l \leq n \), otherwise there is nothing to prove. Under CASE 2, \( l \) is also the minimal rank of \( D^2 u \) in \( \Omega \times (t_0 - \delta, t_0] \), we obtain from the discussions in Subsection 4.1, we can fix \( e_{i+1}, \cdots, e_n \) in \( (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0] \) such that \( u_{\alpha i}(x, t) \equiv 0 \), for any \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0] \) and \( i = l + 1, \cdots, n \). For each fixed \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0] \), we choose the coordinates \( \bar{e}_1, \cdots, \bar{e}_i, \bar{e}_{i+1}, \cdots, e_n, \bar{e}_{n+1} \) so that \( D^2_{x, \bar{y}} u \) is diagonal in the coordinates \( \{e_1, \cdots, \bar{e}_i, e_{i+1}, \cdots, e_n, \bar{e}_{n+1} \} \). In fact, the new coordinate is

\[
y = (y_1, \cdots, y_n, y_{n+1}) = (x, t) P,
\]

where \( P \) is an orthonormal matrix with

\[
P = (P_{\alpha \beta})_{n+1 \times n+1} = \begin{pmatrix}
P_{11} & \cdots & P_{1l} & 0 & \cdots & 0 & P_{1n+1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
P_{l1} & \cdots & P_{ll} & 0 & \cdots & 0 & P_{ln+1} \\
0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\
P_{n+11} & \cdots & P_{n+l} & 0 & \cdots & 0 & P_{n+1n+1}
\end{pmatrix}.
\]

Without loss of generality, we can assume \( \frac{\partial^2 u}{\partial y_1 \partial y_1} \geq \cdots \geq \frac{\partial^2 u}{\partial y_l \partial y_l} \geq C > 0 \) for all \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0] \), where the positive constant \( C > 0 \) depending only on \( \|u\|_{C^{3,1}} \). For convenience we denote \( G = \{1, \cdots, l\} \) and \( B = \{l + 1, \cdots, n\} \) which means good terms and bad ones in indices respectively. Without confusion we will also simply denote \( G = \{\frac{\partial^2 u}{\partial y_1 \partial y_1}, \cdots, \frac{\partial^2 u}{\partial y_l \partial y_l}\} \) and \( B = \{\frac{\partial^2 u}{\partial y_{l+1} \partial y_{l+1}}, \cdots, \frac{\partial^2 u}{\partial y_n \partial y_n}\} \).

For simplicity, we will use \( i, j, k, l = 1, \cdots, n \) to represent the \( x \) coordinates, \( t \) still the time coordinate, and \( \alpha, \beta, \gamma, \eta = 1, \cdots, n + 1 \) the \( y \) coordinates. And we have

\[
\frac{\partial y_\alpha}{\partial x_i} = P_{\alpha i},
\]

\[
\frac{\partial y_\alpha}{\partial t} = P_{\alpha n+1}.
\]

In the following, we always denote

\[
\begin{align*}
u_i &= \frac{\partial u}{\partial x_i}, u_t &= \frac{\partial u}{\partial t}, u_\alpha &= \frac{\partial u}{\partial y_\alpha}, u_{n+1} &= \frac{\partial u}{\partial y_{n+1}}, \\
u_{ij} &= \frac{\partial^2 u}{\partial x_i \partial x_j}, u_{it} &= \frac{\partial^2 u}{\partial x_i \partial t}, u_{tt} &= \frac{\partial^2 u}{\partial t^2}, u_{\alpha \alpha} &= \frac{\partial^2 u}{\partial y_\alpha \partial y_\alpha},
\end{align*}
\]

\[
u_{\alpha \beta} = \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}, \text{etc.}
\]

From the discussion in Subsection 4.1,

\[
u_{\alpha \alpha} = \frac{\partial^2 u}{\partial y_\alpha \partial y_\alpha} = \frac{\partial^2 u}{\partial x_\alpha \partial x_\alpha} = 0, \quad \forall \alpha \in B.
\]
Set
(4.18) \[ \phi = \sigma_{t+1}(D^2_{x,t}u), \]

In the following, we will prove a differential inequality
(4.19) \[ \sum_{ij=1}^{n} F^{ij} \phi_{ij} - \phi_t \leq C(\phi + |\nabla_x \phi|) \text{ in } \mathcal{O} \times (t_0 - \delta, t_0]. \]

Then by the strong maximum principle and the method of continuity, we can prove Theorem 1.3 under CASE 2.

In the \( y \) coordinates, we have from (4.17)
\[ \phi = \sigma_l(D^2_{y}u) = \sigma_l(G)u_{y_{n+1}y_{n+1}} \geq 0, \]
so we have
(4.20) \[ u_{y_{n+1}y_{n+1}} = O(\phi). \]

Taking the first derivatives of \( \phi \) in \( x \), we have
\[ \phi_x = \frac{\partial \phi}{\partial x_i} = \sum_{\alpha=1}^{n+1} \sigma_l(D^2_{y}u|\alpha)u_{\alpha x_i} \]
\[ = \sum_{\alpha \in G} \sigma_l(D^2_{y}u|\alpha)u_{\alpha x_i} + \sum_{\alpha \in B} \sigma_l(D^2_{y}u|\alpha)u_{\alpha x_i} + \sum_{\alpha = n+1} \sigma_l(D^2_{y}u|\alpha)u_{\alpha x_i} \]
\[ = \sigma_l(G)u_{y_{n+1}y_{n+1}x_i} + O(\phi), \]
so
(4.21) \[ u_{y_{n+1}y_{n+1}x_i} = O(\phi + |\nabla_x \phi|), \]

Similarly, taking the first derivatives of \( \phi \) in \( t \), we have
(4.22) \[ \phi_t = \frac{\partial \phi}{\partial t} = \sum_{\alpha=1}^{n+1} \sigma_l(D^2_{y}u|\alpha)u_{\alpha x_i} = \sigma_l(G)u_{y_{n+1}y_{n+1}t} + O(\phi), \]

Taking the second derivatives of \( \phi \) in \( y \) coordinates, we have
\[ \phi_{\alpha \beta} = \frac{\partial^2 \phi}{\partial y_{\alpha} \partial y_{\beta}} \]
\[ = \sum_{\gamma=1}^{n+1} \frac{\partial \sigma_{t+1}(D^2_{y}u)}{\partial u_{\gamma \gamma}} u_{\gamma \gamma \alpha \beta} + \sum_{\gamma \neq \eta} \frac{\partial^2 \sigma_{t+1}}{\partial u_{\gamma \gamma} \partial u_{\eta \eta}} u_{\gamma \gamma \alpha} u_{\eta \alpha \beta} + \sum_{\gamma \neq \eta} \frac{\partial^2 \sigma_{t+1}}{\partial u_{\gamma \gamma} \partial u_{\eta \gamma}} u_{\gamma \gamma \alpha} u_{\eta \gamma \beta} \]
\[ = \sum_{\gamma=1}^{n+1} \sigma_l(D^2_{y}u|\gamma)u_{\gamma \gamma \alpha \beta} + \sum_{\gamma \neq \eta} \sigma_{t-1}(D^2_{y}u|\gamma \eta)u_{\gamma \gamma \alpha} u_{\eta \alpha \beta} - \sum_{\gamma \neq \eta} \sigma_{t-1}(D^2_{y}u|\gamma \eta)u_{\gamma \gamma \alpha} u_{\eta \gamma \beta} \]
\[ \text{where} \]
\[ \sum_{\gamma=1}^{n+1} \sigma_l(D^2_{y}u|\gamma)u_{\gamma \gamma \alpha \beta} = \sum_{\gamma \in G} \sigma_l(D^2_{y}u|\gamma)u_{\gamma \gamma \alpha \beta} + \sum_{\gamma = n+1} \sigma_l(D^2_{y}u|\gamma)u_{\gamma \gamma \alpha \beta} \]
\[ = \sigma_l(G)u_{n+1n+1\alpha \beta} + O(\phi), \]
(4.24)
\[
\sum_{\gamma \neq \eta} \sigma_{l-1}(D^2_y u|\gamma \eta) u_{\gamma \gamma \alpha} u_{\eta \eta \beta} = \sum_{\gamma \neq \eta \in G} \sigma_{l-1}(D^2_y u|\gamma \eta) u_{\gamma \gamma \alpha} u_{\eta \eta \beta} + \sum_{\gamma = n+1} \sigma_{l-1}(D^2_y u|\gamma \eta) u_{\gamma \gamma \alpha} u_{\eta \eta \beta} \\
+ \sum_{\gamma \neq \eta \in G} \sigma_{l-1}(D^2_y u|\gamma \eta) u_{\gamma \gamma \alpha} u_{\eta \eta \beta} \\
= O(\phi) + \sum_{\eta \in G} \sigma_{l-1}(G|\eta) u_{n+1 \eta \eta} u_{n+1 \eta} + \sum_{\gamma \in G} \sigma_{l-1}(G|\gamma) u_{n+1 \gamma} u_{n+1 \gamma} \\
= \sigma_l(G)\left[\sum_{\eta \in G} u_{n+1 \eta \eta} u_{n+1 \eta} + \sum_{\gamma \in G} u_{\gamma \gamma} u_{n+1 \gamma} \right] + O(\phi),
\]

(4.25)

and

\[
\sum_{\gamma \neq \eta} \sigma_{l-1}(D^2_y u|\gamma \eta) u_{\eta \eta \alpha} u_{\eta \eta \beta} = \sum_{\gamma \neq \eta \in G} \sigma_{l-1}(D^2_y u|\gamma \eta) u_{\eta \eta \alpha} u_{\eta \eta \beta} + \sum_{\gamma = n+1} \sigma_{l-1}(D^2_y u|\gamma \eta) u_{\eta \eta \alpha} u_{\eta \eta \beta} \\
+ \sum_{\gamma \neq \eta \in G} \sigma_{l-1}(D^2_y u|\gamma \eta) u_{\eta \eta \alpha} u_{\eta \eta \beta} \\
= O(\phi) + \sum_{\eta \in G} \sigma_{l-1}(G|\eta) u_{n+1 \eta \eta} u_{n+1 \eta} + \sum_{\gamma \in G} \sigma_{l-1}(G|\gamma) u_{n+1 \gamma} u_{n+1 \gamma} \\
= 2\sigma_l(G)\sum_{\eta \in G} u_{n+1 \eta \eta} u_{n+1 \eta} + O(\phi).
\]

(4.26)

So we have

\[
\phi_{\alpha \beta} = \sigma_l(G) u_{n+1 \alpha}, \quad \sigma_l(G) \sum_{\eta \in G} u_{n+1 \eta \eta} u_{n+1 \eta} \\
+ \sigma_l(G)\left[\sum_{\eta \in G} u_{\eta \eta \eta} u_{n+1 \eta} + \sum_{\gamma \in G} u_{\gamma \gamma} u_{n+1 \gamma} \right] + O(\phi).
\]

(4.27)
Then

\[
\sum_{ij=1}^{n} F_{ij} \phi_{ij} = \sum_{ij=1}^{n} F_{ij} \sum_{\alpha\beta=1}^{n+1} P_{i\alpha} P_{j\beta} \phi_{\alpha\beta}
\]

\[
= \sigma_l(G) \sum_{ij=1}^{n} F_{ij} \sum_{\alpha\beta=1}^{n+1} P_{i\alpha} P_{j\beta} u_{n+1n+1\alpha\beta} - 2\sigma_l(G) \sum_{ij=1}^{n} F_{ij} \sum_{\eta\in\mathbb{G}} \sum_{\alpha=1}^{n+1} P_{i\alpha} u_{n+1n+1\alpha} \sum_{\gamma\in\mathbb{G}} \sum_{\beta=1}^{n+1} P_{j\beta} u_{n+1n+1\gamma} + O(\phi)
\]

\[
(4.28)
\]

\[
= \sigma_l(G) \sum_{ij=1}^{n} F_{ij} u_{n+1n+1i} - 2\sigma_l(G) \sum_{ij=1}^{n} F_{ij} \sum_{\eta\in\mathbb{G}} \frac{u_{n+1n+1i}}{u_{\eta \eta}} + O(\phi)
\]

By (4.21), we have

\[
(4.29) \quad \sum_{ij=1}^{n} F_{ij} \phi_{ij} = \sigma_l(G) \left[ \sum_{ij=1}^{n} F_{ij} u_{n+1n+1i} - 2 \sum_{ij=1}^{n} F_{ij} \sum_{\eta\in\mathbb{G}} \frac{u_{n+1n+1i}}{u_{\eta \eta}} \right] + O(\phi + |\nabla x \phi|)
\]

From (4.22) and (4.29), we have

\[
(4.30) \quad \sum_{ij=1}^{n} F_{ij} \phi_{ij} - \phi_t = \sigma_l(G) \left[ \left( \sum_{ij=1}^{n} F_{ij} u_{n+1n+1i} - u_{n+1n+1t} \right) - 2 \sum_{ij=1}^{n} F_{ij} \sum_{\eta\in\mathbb{G}} \frac{u_{n+1n+1i} u_{\eta \eta}}{u_{\eta \eta}} \right] + O(\phi + |\nabla x \phi|)
\]

For the first term in the right hand side of (4.30), we use the equation

\[
u_t = F(\nabla^2 u, \nabla u, u, x, t)
\]
Taking the second derivative in $y_{n+1}$, we have

\begin{equation}
\begin{split}
u_{n+1n+1t} &= \sum_{ij=1}^{n} F^{ij} u_{n+1n+1ij} + \sum_{i=1}^{n} F^{ui} u_{n+1n+1i} + F^u u_{n+1n+1} \\
&+ \sum_{ijkl=1}^{n} F^{ijkl} u_{ij\gamma} u_{k\gamma} + 2 \sum_{ij=1}^{n} F^{ij,uk} u_{ij\gamma} + 2 \sum_{ij=1}^{n} F^{ij,n} u_{ijn+1} u_{n+1} \\
&+ 2 \sum_{jk=1}^{n} F^{ij,xk} u_{ijn+1} \frac{\partial x_k}{\partial y_{n+1}} + 2 \sum_{ij=1}^{n} F^{ij,t} u_{ijn+1} \frac{\partial t}{\partial y_{n+1}} + \sum_{ij=1}^{n} F^{ui,u_j} u_{in+1} u_{jn+1} \\
&+ 2 \sum_{i=1}^{n} F^{ui,uk} u_{in+1} u_{n+1} + 2 \sum_{i=1}^{n} F^{ui,xk} u_{in+1} \frac{\partial x_k}{\partial y_{n+1}} + 2 \sum_{i=1}^{n} F^{ui,t} u_{in+1} \frac{\partial t}{\partial y_{n+1}} \\
&+ 2 \sum_{ik=1}^{n} F^{\varepsilon,x_k} u_{n+1} \frac{\partial x_i}{\partial y_{n+1}} \frac{\partial x_k}{\partial y_{n+1}} + 2 \sum_{i=1}^{n} F^{\varepsilon,t} \frac{\partial x_i}{\partial y_{n+1}} \frac{\partial t}{\partial y_{n+1}} + F^{t,t} \left( \frac{\partial t}{\partial y_{n+1}} \right)^2
\end{split}
\end{equation}

From (4.17), (4.20) and (4.21), we have

\begin{equation}
(4.32) \quad u_{n+1n+1t} = O(\phi + |\nabla_x \phi|), \forall i = 1, \cdots, n;
\end{equation}

\begin{equation}
(4.33) \quad u_{\alpha\alpha} = 0, \forall \alpha \in B; u_{n+1n+1} = O(\phi),
\end{equation}

\begin{equation}
(4.34) \quad u_{in+1} = \frac{\partial u_{n+1}}{\partial x_i} = \sum_{\eta=1}^{n+1} \frac{\partial u_{n+1}}{\partial y_{\eta}} \frac{\partial y_{\eta}}{\partial x_i} = \sum_{\eta=1}^{n+1} u_{n+1\eta} P_{\eta} + u_{n+1n+1} P_{n+1} = O(\phi).
\end{equation}

And from (4.17) and Lemma 2.6, we have for $i$ or $j \in B$

\begin{equation}
|u_{ijn+1}| \leq C(u_{ii}u_{jj})^{\frac{1}{2}} = 0,
\end{equation}

so we have

\begin{equation}
(4.35) \quad u_{ijn+1} = 0.
\end{equation}
Then

\[
\frac{u_{n+1;+1} - \sum_{ij=1}^{n} F_{ij} u_{n+1;+1 ij}}{n} = \sum_{ijkl=1}^{n} F_{ijkl} u_{ij;ij} + 2 \sum_{ij=1}^{n} F_{ij;ij} u_{ij;ij} + 2 \sum_{ijkl=1}^{n} F_{ijkl} u_{ij;ij} + \frac{\partial x_k}{\partial y_{n+1}} + 2 \sum_{ijkl=1}^{n} F_{ijkl} u_{ij;ij}
\]

\[
+ 2 \sum_{ij=1}^{n} F_{ij;ij} u_{ij;ij} + 2 \sum_{k=1}^{n} F_{ik;ik} u_{ij;ij} + 2 \sum_{ijkl=1}^{n} F_{ijkl} u_{ij;ij} + 2 \sum_{ijkl=1}^{n} F_{ijkl} u_{ij;ij}
\]

\[
+ \sum_{ijkl=1}^{n} F_{ijkl} u_{ij;ij} + 2 \sum_{ij=1}^{n} F_{ij;ij} u_{ij;ij} + 2 \sum_{ijkl=1}^{n} F_{ijkl} u_{ij;ij} + \frac{\partial x_k}{\partial y_{n+1}} + 2 \sum_{ijkl=1}^{n} F_{ijkl} u_{ij;ij}
\]

(4.36)

For the second part in the right hand side of (4.30), we have the following CLAIM:

(4.37) \[\sum_{n \in G} \sum_{ij=1}^{n} F_{ij} u_{n+1;+1} u_{ij} \geq \sum_{ijkl=1}^{n} F_{ijkl} u_{ij;ij} + O(\phi + |\nabla x|)
\]

If the CLAIM holds, denote

\[Q = \sum_{ijkl=1}^{n} F_{ijkl} u_{ij;ij} + 2 \sum_{ij=1}^{n} F_{ij;ij} u_{ij;ij} + 2 \sum_{ijkl=1}^{n} F_{ijkl} u_{ij;ij} + \frac{\partial x_k}{\partial y_{n+1}} + 2 \sum_{ijkl=1}^{n} F_{ijkl} u_{ij;ij}
\]

(4.38)

By the structural condition (1.4) (that is Lemma 2.5), we have

(4.39) \[Q \geq 0.
\]

Then by (4.30), (4.36), (4.37) and (4.39), we have

(4.40) \[\sum_{ij=1}^{n} F_{ij} \phi_{ij} - \phi_i \leq -\sigma_i(G) Q + O(\phi + |\nabla x|) \leq C(\phi + |\nabla x|).
\]

So (4.19) holds, and Theorem 1.3 holds under CASE 2. \[\square\]
4.4. **Proof of the CLAIM** (4.37). Now we give the proof of the CLAIM as follows.

First, we consider a special case: \( F^{ij} = \delta_{ij} \). That is, we need to prove

\[
\sum_{\eta \in G} \frac{u_{n+1}^{\eta \eta}}{u_{n+1}^{\eta}} \geq \sum_{k \in G} \frac{u_{n+1}^{k \eta}}{u_{n+1}^{k \eta}} u_{n+1}^{k \eta} + O(\phi + |\nabla_x \phi|).
\]

From (4.33) and (4.21), we have

\[
u_{n+1}^{\eta \eta} = 0, \eta \in B \text{ or } i \in B,
\]

\[
u_{n+1}^{i i} = O(\phi + |\nabla_x \phi|).
\]

Since \( D_y^2 u \) is diagonal, by the approximation, we have for \( i \in G \)

\[
\sum_{\eta \in G} \frac{u_{n+1}^{\eta \eta} u_{n+1}^{\eta}}{u_{n+1}^{\eta \eta}} = \lim_{\epsilon \to 0^+} (D_y u_{n+1}) (D_y^2 u + \epsilon I)^{-1} (D_y u_{n+1})^T + O(\phi + |\nabla_x \phi|),
\]

where

\[
(D_y u_{n+1}) (D_y^2 u + \epsilon I)^{-1} (D_y u_{n+1})^T = (D_{x, t} u_{n+1}) P^T (D_y^2 u + \epsilon I)^{-1} P (D_{x, t} u_{n+1})^T
\]

\[
= (D_{x, t} u_{n+1}) (D_y^2 u + \epsilon I)^{-1} (D_{x, t} u_{n+1})^T.
\]

Denote

\[
C := \eta + \epsilon - \sum_{i=1}^{l} \frac{u_{x, i}^2}{u_{x, i}^2 + \epsilon} > 0,
\]

then

\[
(D_{x, t} u + \epsilon I)^{-1} = \text{diag} \left( \frac{1}{u_{x, 1} + \epsilon}, \ldots, \frac{1}{u_{x, i} + \epsilon}, \frac{1}{\epsilon}, \ldots, \frac{1}{\epsilon}, 0 \right)
\]

\[
+ \frac{1}{C} \left( \frac{-u_{x, 1}^i}{u_{x, 1} + \epsilon}, \ldots, \frac{-u_{x, i}^i}{u_{x, i} + \epsilon}, 0, \ldots, 0, 1 \right)^T \left( \frac{-u_{x, 1}^i}{u_{x, 1} + \epsilon}, \ldots, \frac{-u_{x, i}^i}{u_{x, i} + \epsilon}, 0, \ldots, 0, 1 \right)
\]

\[
\geq \text{diag} \left( \frac{1}{u_{x, 1} + \epsilon}, \ldots, \frac{1}{u_{x, i} + \epsilon}, 0, \ldots, 0, 0 \right).
\]

So

\[
(D_y u_{n+1}) (D_y^2 u + \epsilon I)^{-1} (D_y u_{n+1})^T \geq \sum_{k \in G} \frac{u_{n+1}^{k i} u_{n+1}^{i k}}{u_{x, i} + \epsilon}.
\]

Then we have for \( i \in G \)

\[
\sum_{\eta \in G} \frac{u_{n+1 \eta} u_{n+1 \eta}}{u_{n+1 \eta}} \geq \lim_{\epsilon \to 0^+} \sum_{k \in G} \frac{u_{n+1}^{k i} u_{n+1}^{i k}}{u_{x, i} + \epsilon} + O(\phi + |\nabla_x \phi|)
\]

\[
= \sum_{k \in G} \frac{u_{n+1}^{k i} u_{n+1}^{i k}}{u_{x, i} + \epsilon} + O(\phi + |\nabla_x \phi|)
\]

\[
= \sum_{k \in G} u_{n+1}^{k i} u_{n+1}^{i k} + O(\phi + |\nabla_x \phi|).
\]

Hence, (4.41) holds.

For the general case, the CLAIM also holds following the above proof.
5. Discussions

In fact, there are many equations satisfying the conditions (1.4).

Proposition 5.1. (1) All the linear operators satisfy conditions (1.4).

(2) The Hessian operators \( \sigma_k^l \) and \( (\frac{\sigma_k}{\sigma_l})^{1-l/k} \) \((k > l > 0)\) satisfy the condition (1.4) for the convex admissible solutions (that is \( D^2u \geq 0 \), and \( D^2u \in \Gamma_k \) on \( \Omega \times (0,T) \)).

(3) If \( g \) is a non-decreasing and convex function and \( F_1, \cdots, F_m \) satisfy condition (1.4), then \( F = g(F_1, \cdots, F_m) \) also satisfies condition (1.4). In particular, if \( F_1 \) and \( F_2 \) are in the class, so are \( F_1 + F_2 \) and \( F_1^\alpha \) \( (\text{where } F_1 > 0) \) for any \( \alpha \geq 1 \).

Through a direct calculation and using (2.7), we can get the proof. Also we can find the proof of Proposition 5.1 easily from [2, 3, 11].

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