ON THE WELL-POSEDNESS OF A NONLINEAR DIFFUSIVE SIR EPIDEMIC MODEL

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Abstract. This work considers an extension of the SIR equations from epidemiology that includes a spatial variable. This model, referred to as the Kermack-McKendrick equations (KM), is a pair of diffusive partial differential equations, and methods developed for the Navier-Stokes equations and models of fluid dynamics are adapted to prove that KM is well-posed in the homogenous Sobolev spaces with exponent \( 0 \leq s < 2 \).

1. Introduction

The outbreak of the COVID-19 pandemic galvanized the efforts to improve the predictive power of the mathematics modeling the spread of disease. The most well-known of these models is given by the SIR equations. The present paper considers an extension of these equations that includes a spatial variable. These equations, which we call the Kermack-McKendrick equations (KM), change the SIR model from a set of ordinary differential equations (ODEs) into a coupled pair of diffusive partial differential equations (PDEs). We investigate the well-posedness of KM using methods that were originally developed for fluid dynamics, in particular for the Navier-Stokes equations (NS).

The SIR model was pioneered by W. Kermack and A. McKendrick in [20] and is an example of a compartmental model. In their original formulation, the population is partitioned into the disjoint groups, or compartments, consisting of the susceptible (\( S \)), infectious (\( I \)), and recovered (\( R \)) individuals. These quantities are strictly functions of time \( t \), and the ODEs they give rise to, called the SIR-equations, are given by

\[
\begin{align*}
S'(t) &= -\beta SI, \\
I'(t) &= \beta SI - \mu I, \\
R'(t) &= \mu I.
\end{align*}
\]

(1.1)

Here \( \beta \) and \( \mu \) are constants representing the transmission and recovery rates, respectively. For a detailed description of the basic assumptions of the model and the technical underpinnings that lead to its equations we refer to [5], [18], [21], [27]. For a review of the history of the SIR equations, the interested reader may consult [2].

Since its inception, most of the attention has concentrated on using the SIR model to understand disease transmission, and over the years important applications to public health have been found [29]. A prime example of this is vaccination, where the transition rate between...
compartments is accelerated, since vaccinated individuals can be immediately placed in the $R$, (recovered) compartment. Kermack and McKendrick applied their model to the 1906 bubonic outbreak in Bombay [2], but the model has also been employed in a wide variety of circumstances such as the evolution of the dengue outbreaks in Cuba (1997) and Venezuela (2000) [16], the classical swine flu in the Netherlands (1997-1998) [26], and many others.

Considerable work has been devoted to improving the SIR model itself. Notably, the original model has been expanded by the addition of more compartments. For instance, along with the traditional three compartments, some models also include incubation and latency periods ($E$) [21]. More recently, some models have incorporated compartments to account for immunization and vaccination in populations [11], [30].

The descriptive and predictive power of the model has been applied to the COVID-19 pandemic. The great interest generated by the topic and its timely nature are evidenced by the explosion of the literature on the subject. For a few applications of the SIR model in this context, we refer the readers to [1], [7], [14].

One fundamental issue in using the SIR equations to model a pandemic, however, is that it completely ignores spatial information. As compared to a localized disease outbreak, the location and concentration of affected individuals in a global setting would most certainly contribute to the time evolution of the model. With this issue in mind, a generalized compartmental SIR model is constructed by allowing individuals to move via random walks. For an investigation of random walks in this context, we refer the reader to [25], and for the foundational work on Brownian motion, which lies at the core of these diffusion processes, we refer the reader to [8]. According to these models, an individual moves randomly in a direction with the amplitude of the Brownian motion equalling $D_S$ and $D_I$ for the Susceptible and Infected individuals, respectively. After taking an expected value, these constants become the coefficients in the diffusion linear symbol, and we obtain the equations

$$S_t = D_S \Delta S - \beta SI,$$
$$I_t = D_I \Delta I + \beta SI - \mu I.$$ (1.2)

It is worth noting that if there is no displacement of the individuals, then $D_S = D_I = 0$, and we obtain the original SIR equations. A detailed description of how to obtain (1.2) appears in [6], and is reviewed in [9], [10]. Additionally, the traveling wave solutions of (1.2) have been investigated in [3], [17], [23], [28].

In this work, we will assume the spatial dimension in (1.2) to be one. Additionally, we make the change of variables $u(x,t) = S(x\sqrt{D_S/\beta},t/\beta)$ and $v(x,t) = I(x\sqrt{D_I/\beta},t/\beta)$ to obtain a slightly simpler non-dimensional version of (1.2). In this new version, the coefficient for the linear $v$ term will be $-\mu \beta$ which we will relabel as $-\mu$. We refer to this new system of equations as the Kermack-McKendrick equations (KM), and they are given by

$$u_t + uv - u_{xx} = 0,$$
$$v_t - uv - v_{xx} - \mu v = 0,$$ (1.3)
$$u(x,0) = \varphi(x), \quad v(x,0) = \psi(x), \quad t \in [0,T), x \in \mathbb{R}.$$  

We will approach the initial value problem (ivp) posed by (1.3) using theory developed in the study of diffusion equations in fluid dynamics. To that end, we adapt the methods developed and applied in [4], [16], [19]. Specifically, we investigate the well-posedness of KM in the sense
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of Hadamard. In order to rigorously state what we mean by well-posedness, we must also state precisely what spaces we are taking the initial data and solutions to be in. We will take the initial data \( \varphi, \psi \) to be in the homogeneous Sobolev spaces \( \dot{H}^s \), and the solution \( u \) to be in the intersection of \( \dot{H}^s \) and the time-weighted \( L^4 \) spaces, which we will call \( X^s \). The precise definitions of these spaces are provided for the reader in (2.1) and (2.2), respectively.

The idea of well-posedness was introduced in [15], and we say that the KM equations are well-posed with initial data in \((\varphi, \psi) \in \dot{H}^s \times \dot{H}^s \) and solution \((u, v) = W(\varphi, \psi) \in X^s \times X^s \), if the following three conditions hold:

I) \textbf{Existence.} For any initial data \((\varphi, \psi) \in \dot{H}^s \times \dot{H}^s \), there exists a solution \((u, v) \in X^s \times X^s \) to KM.

II) \textbf{Uniqueness.} The solution \((u, v) = W(\varphi, \psi) \) is unique in the space \( X^s \times X^s \).

III) \textbf{Continuity/Stability.} The solution map \( W : \dot{H}^s \times \dot{H}^s \to X^s \times X^s \) is continuous.

With this definition in mind, we now state the primary result of this work.

**Theorem 1.** Let \( 0 \leq s < 2 \), \( 0 < T < \frac{1}{6\mu} \) and \((\varphi, \psi) \in \dot{H}^s \times \dot{H}^s \), satisfying the smallness condition

\[
\| (\varphi, \psi) \|_{\dot{H}^s \times \dot{H}^s} \leq \frac{1}{18C_c C_b} \tag{1.4}
\]

where the constants \( C_c \) and \( C_b \) are given in Propositions 1 and 2, respectively. Then the KM ivp (1.3) has a unique solution \((u, v) \in X^s \times X^s \). Moreover, the solution map, \( W : \dot{H}^s \times \dot{H}^s \to X^s \times X^s \), which takes \((\varphi, \psi) \to (u, v)\), is Lipschitz continuous.

The proof of Theorem 1 revolves around the techniques developed in [19] to prove the well-posedness of the Navier-Stokes (NS) equations

\[
\begin{align*}
\frac{D}{Dt} u + (\nabla u) u + \mu \Delta u + \nabla p &= 0, \\
\text{div } u &= 0,
\end{align*}
\tag{1.5}
\]

where \( p \) is the pressure of the fluid, and \( \mu \) its viscosity. Here, the Brownian motion amplitudes in (1.2), \( D_S \) and \( D_I \), act in a similar manner as the viscosity coefficient \( \mu \) in NS. The strategy that was implemented for NS was built on the foundations developed in [12] and consisted in showing that the associated integral operator had a fixed point in a suitable space. These ideas have also been used in other hydrodynamic equations as such the viscous Burgers (vB) equation

\[
\frac{D}{Dt} u + uu_x + \mu u_{xx} = 0, \tag{1.6}
\]

which was examined in [4], and the \( k \)-Burgers equation

\[
\frac{D}{Dt} u + uu_x + \mu u_{xx} = 0, \quad k = 1, 2, 3, \ldots \tag{1.7}
\]

which was investigated in [16].

**Outline of the paper.** The present paper is organized as follows. In section 2, we provide a number of preliminaries, including the definitions of our function spaces as well as linear estimates for the diffusion operator. In section 3, we first reformulate KM as a fixed point problem and then prove that this associated integral operator is a contraction mapping. In section 4, we provide a proof of the bilinear estimate that was needed in order to establish the contraction in section 3.
2. Preliminaries and Linear Estimates

In this section, we set up our notation and collect the basic estimates that will be used in the course of proving the main result.

**Notation.** We say $A \lesssim B$ if there exists a constant $C > 0$ such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$ we write $A \simeq B$.

**Function Spaces.** The spaces that we will use are the combination of homogeneous Sobolev spaces and time-weighted $L^p$ spaces, and we briefly provide a definition of these spaces and their norms.

We begin with the homogenous Sobolev space, $\dot{H}^s$, which is a subspace of the Tempered Distributions where the following norm is finite. We take the the Riesz potential $D_x = (\partial_x^2)^{1/2}$, or equivalently, the Fourier multiplier given by $\hat{D}_x f(\xi) = |\xi|^{2s} \hat{u}(\xi)$, and then define the $\dot{H}^s$-norm as

$$
\|u\|_{\dot{H}^s} \doteq \|D_x^s u\|_{L^2} = \left( \int_{\mathbb{R}} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}.
$$

Next, we define our time-weighted $L^p$ spaces. For any fixed $T, \alpha \geq 0$, we define the subspace $C_0^\alpha((0,T);L^p)$ by

$$
C_0^\alpha((0,T);L^p) = \left\{ u \in C((0,T);L^p) : \sup_{t \in (0,T)} t^{\alpha} \|u\|_{L^p} < \infty \text{ and } \lim_{t \to 0^+} t^{\alpha} \|u\|_{L^p} = 0 \right\}.
$$

For given $s, \alpha, p$ we can now define $X^{s,\alpha,p} = \dot{H}^s \cap C_0^\alpha((0,T);L^p)$; however, in our particular case, there is a relationship between $s$ and $\alpha$ that arises in our proof of the Bilinear Estimate needed for Theorem 1. We thus will restrict our attention to

$$
\alpha = \frac{1}{2} - \frac{s}{4}.
$$

Additionally, the only $p$ that we will utilize is $p = 4$ as it also arises in the Bilinear Estimate after applying the generalized Hölder’s inequality in $L^2$. In view of these choices, we define $X^s = X^{s,\frac{1}{2} - \frac{s}{4},4}$, and take the norm to be

$$
\|u\|_{X^s} \doteq \sup_{t \in (0,T)} \|u\|_{\dot{H}^s} + \sup_{t \in (0,T)} t^{\alpha} \|u\|_{L^4}, \quad \text{where} \quad \alpha = \frac{1}{2} - \frac{s}{4}.
$$

Finally, since we are working with two simultaneous equations, we define our norms on the product spaces in the usual fashion. For pairs, we have

$$
\| (\varphi, \psi) \|_{\dot{H}^s \times \dot{H}^s} \doteq \|\varphi\|_{\dot{H}^s} + \|\psi\|_{\dot{H}^s},
$$

$$
\| (u, v) \|_{X^s \times X^s} \doteq \|u\|_{X^s} + \|v\|_{X^s}.
$$

**The Diffusion Operator and Linear Estimates.** We begin by considering the diffusion equation

$$
\begin{align*}
w_t + w_{xx} &= 0, \\
w(0, x) &= w_0(x).
\end{align*}
$$

We take the solution operator $S$ of (2.3) as the Fourier multiplier given by

$$
\widehat{S(t) \varphi}(\xi) = e^{-t\xi^2} \widehat{\varphi}(\xi).
$$
Before we proceed with our estimates regarding the operator $S$, we state the well-known Hardy-Littlewood-Sobolev inequality for the convenience of the reader. For a proof of this estimate, we refer the reader to [22], §4.3.

**Lemma 1** (Hardy-Littlewood-Sobolev). Suppose that $0 < \alpha < 1$, $1 < \alpha < p < 1/\alpha$ and $q$ satisfies

$$\frac{1}{q} = \frac{1}{p} - \alpha.$$  

If $f \in L^p$, then $D_x^{-\alpha} f \in L^q$ (where $D_x = (-\partial^2_x)^{1/2}$ is the Riesz potential) and there exists a constant $C = C(\alpha, p)$ such that

$$\|D_x^{-\alpha} f\|_{L^q} \leq C\|f\|_{L^p}. \quad (2.4)$$

We now consider three estimates for $S$ that will be used throughout this work. The first is an $L^p$-$L^q$ estimate that can be found in [13].

**Lemma 2.** Let $S(t)$ be the solution operator for the heat equation (2.3) with initial data $\varphi$ and $1 \leq q \leq p \leq \infty$. Then for $t > 0$ we have the estimate

$$\|S(t)\varphi\|_{L^p} \leq (4\pi t)^{-\frac{1}{2}}\|\varphi\|_{L^q}. \quad (2.5)$$

**Lemma 3.** For $s \geq 0$, there exists a positive constant $c_s$ such that

$$\|D_x^s S(t)\varphi\|_{L^2} \leq c_s t^{-s/2}\|\varphi\|_{L^2}.$$

**Proof.** By Parseval’s identity and Hölder’s inequality, we have

$$\|D_x^s S(t)\varphi\|_{L^2} = \frac{1}{2\pi} \|\xi^s e^{-\xi^2 t} \hat{\varphi}(\xi)\|_{L^2} \leq \frac{1}{2\pi} \|\xi^s e^{-\xi^2 t}\|_{L^\infty} \|\hat{\varphi}\|_{L^2}. \quad (2.6)$$

We see that the $L^\infty$-norm of $\xi^s e^{-\xi^2 t}$ is now easily bounded by

$$\|\xi^s e^{-\xi^2 t}\|_{L^\infty} \leq \left(\frac{s}{2\pi t}\right)^{\frac{1}{2}} t^{-\frac{s}{2}}. \quad (2.7)$$

We can then continue bounding (2.6) by using (2.7) to get

$$\|D_x^s S(t)\varphi\|_{L^2} \leq \frac{1}{2\pi} \left(\frac{s}{2\pi t}\right)^{\frac{1}{2}} t^{-\frac{s}{2}} \|\hat{\varphi}\|_{L^2} = \left(\frac{s}{2\pi t}\right)^{\frac{1}{2}} t^{-\frac{s}{2}} \|\varphi\|_{L^2}.$$  

We see then that taking $c_s = \left(\frac{s}{2\pi t}\right)^{\frac{1}{2}} t^{-\frac{s}{2}}$ we obtain our desired estimate.  

**Proposition 1** (Linear Estimate). For $0 \leq s < 2$, the mapping $\phi \mapsto S(t)\phi$, where $S$ is the solution operator to the diffusion equation, continuously maps $H^s \to X^s$, and we have the inequality

$$\|S(t)\varphi\|_{X^s} \leq C_t\|\varphi\|_{H^s}. \quad (2.8)$$

where the constant $C_t$ depends on $s$ and $T$.

**Proof.** From the definition of the $X^s$-norm, we have

$$\|S(t)\varphi\|_{X^s} = \sup_{t \in (0, T)} \|S(t)\varphi\|_{H^s} + \sup_{t \in (0, T)} t^{\alpha}\|S(t)\varphi\|_{L^4}.$$
The first term is handled with the usual methods using Plancherel’s Theorem and the definition of the diffusion solution operator $S$. Indeed, we have

$$\|S(t)\varphi\|_{H^s}^2 = \int_{\mathbb{R}} |\xi|^{2s}e^{-t|\xi|^2} |\hat{\varphi}(\xi)|^2 d\xi \leq \int_{\mathbb{R}} |\xi|^{2s} |\hat{\varphi}(\xi)|^2 d\xi = \|\varphi\|_{H^s}^2.$$  

Thus, the first term has the upper bound

$$\sup_{t \in (0,T)} \|S(t)\varphi\|_{H^s} \leq \sup_{t \in (0,T)} \|\varphi\|_{H^s}.$$  

For the second term, we begin by restricting our attention to the quantity under the $L^4$-norm. Applying the Riesz Derivative and its inverse with exponent $1/4$ gives us

$$\|S(t)\varphi\|_{L^4} = \|D_x^{-1/4}D_x^{1/4}S(t)\varphi\|_{L^4}.$$  

Now we apply the Hardy-Littlewood-Sobolev Lemma, or Lemma 1, with $p = 2, q = 4$ and $\alpha = 1/4$ to get

$$\|D_x^{-1/4}D_x^{1/4}S(t)\varphi\|_{L^4} \lesssim \|D_x^{1/4}S(t)\varphi\|_{L^2}.$$  

Next, we again apply the Riesz Derivative along with its inverse, both with exponent $s/2$, after which we can apply Lemma 3 with the corresponding $\alpha = 1/4 - s/2$. Note that this Lemma requires $\alpha \geq 0$, which corresponds to our hypothesis $0 \leq s \leq 1/2$. We thus get

$$\|D_x^{1/4}S(t)\varphi\|_{L^2} = \|D_x^{1/4-s/2}D_x^{s/2}S(t)\varphi\|_{L^2} \lesssim t^{1/8-s/4}\|D_x^{s/2}\varphi\|_{L^2} = t^{1/8-s/4}\|\varphi\|_{H^s}.$$  

Using this upper bound for the $L^4$-norm of $S(t)\phi$, we get the time-weighted norm to be

$$\sup_{t \in (0,T)} t^{\alpha} \|S(t)\varphi\|_{L^4} \lesssim \sup_{t \in (0,T)} t^{\alpha} t^{-1/8+s/4}\|\varphi\|_{H^s}.$$  

From the definition of the $X^s$ space, we have $\alpha = 1/2 - s/4$ which implies that the exponent for $t$ is $\alpha - \frac{1}{8} + \frac{s}{4} = \frac{3}{8}$. Thus, we have the upper bound

$$\sup_{t \in (0,T)} t^{\alpha-1/8+s/4}\|\varphi\|_{H^s} \leq T^{3/8}\|\varphi\|_{H^s}.$$  

Putting our results together for each term gives the desired inequality (2.8), where the associated constant with the upper bound depends on $s$ and $T$.  

\[\square\]

3. PROOF OF THEOREM 1

We begin by first reformulating the KM equations as a fixed point problem and then proving that the associated integral operator is a contraction. The Banach Contraction Mapping Theorem then implies the existence and uniqueness of solutions to the KM ivp. After this task is accomplished, we prove that the solution map $W$ is Lipschitz continuous.

Reformulating (1.3) as a fixed point problem. We begin by taking the Fourier transform in the spatial variable of both sides of the $u$ and $v$ equations in (1.3). Additionally, using the properties $\partial_x u = i\xi \hat{u}$ and $\partial_t \hat{u} = \partial_t \hat{u}$, we get

$$\partial_t \hat{u} + \hat{u} + \xi^2 \hat{u} = 0,$$

$$\partial_t \hat{v} - \hat{u} + \xi^2 \hat{v} + \mu \hat{v} = 0.$$
Next, we isolate the terms corresponding to the linear heat equation on the left-hand side and multiply by the integrating factor $e^{t\xi^2}$. The integrating factor allows us to then write the left-hand side of each equation as an exact derivative as for instance $\partial_t(e^{t\xi^2}\hat{u}) = e^{t\xi^2}(\hat{u}_t + \xi^2\hat{u})$. We thus obtain

\[ \partial_t(e^{t\xi^2}\hat{u}) = -e^{t\xi^2}\hat{uv}, \]
\[ \partial_t(e^{t\xi^2}\hat{v}) = e^{t\xi^2}(\hat{uv} - \mu\hat{v}). \]

We now integrate both equations from 0 to $t$ using the Fundamental Theorem of Calculus and the initial data specified in (1.3) to get

\[ e^{t\xi^2}\hat{u} - \hat{\varphi} = -\int_0^te^{t'\xi^2}\hat{uv}dt', \]
\[ e^{t\xi^2}\hat{v} - \hat{\psi} = \int_0^t e^{t'\xi^2}((\hat{uv} - \mu\hat{v})dt'). \]

Next we move the initial data to the right-hand side of each equation and multiply both sides of both equations by $e^{-t\xi^2}$. Finally, applying the inverse Fourier Transform gives us

\[ u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi}e^{-t\xi^2}\hat{\varphi}d\xi - \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi}e^{-t\xi^2}(\int_0^t e^{t'\xi^2}\hat{uv}dt')d\xi, \]
\[ v(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi}e^{-t\xi^2}\hat{\psi}d\xi + \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi}e^{-t\xi^2}(\int_0^t e^{t'\xi^2}(\hat{uv} - \mu\hat{v})dt')d\xi. \]

We see further that using the solution operator to the heat equation $S$, we can also rewrite these equations as

\[ u(x,t) = S(t)\varphi - \int_0^t S(t - t')(uv)dt', \]
\[ v(x,t) = S(t)\psi + \int_0^t S(t - t')(uv - \mu v)dt'. \]

With the desired form of our equations obtained, we define the operators

\[ T_1(u,v) \equiv S(t)\varphi - \int_0^t S(t - t')(uv)dt', \quad \text{ (3.1)} \]
\[ T_2(u,v) \equiv S(t)\psi + \int_0^t S(t - t')(uv - \mu v)dt'. \quad \text{ (3.2)} \]

We note here that the initial data $\varphi, \psi$ are assumed to be fixed in this construction. Should the dependence on the initial data be needed, such as in the proof of Lipschitz continuity, we will denote the operator with an appropriate subscript, for instance $T_1 = T_{1,\varphi}$. Finally, we take $T$ to be the two component operator

\[ T(u, v) = (T_1(u, v), T_2(u, v)). \quad \text{ (3.3)} \]

We now can rewrite our equation as

\[ (u, v) = T(u, v). \quad \text{ (3.4)} \]

Our next objective will be to demonstrate that $T$ has a fixed point using the Banach Fixed-Point Theorem. One of the key ingredients we will need to establish a contraction is the following Bilinear Estimate, which is proved in the next section.
**Proposition 2 (Bilinear Estimate).** Let $0 \leq s < 2$. Then there exists a constant $C_b > 0$ such that for $f, g \in X^s$ we have

$$
\left\| \int_0^t S(t-t')(fg)dt' \right\|_{X^s} \leq C_b \|f\|_{X^s} \|g\|_{X^s}.
$$

We now show the contraction, which completes the proof of well-posedness under a smallness assumption.

**Proposition 3 (Contraction).** Let $0 \leq s < 2$, $T < \frac{1}{6\mu}$, $\rho = \frac{1}{3C_b}$, $(\varphi, \psi) \in \dot{H}^s \times \dot{H}^s$ satisfying the smallness condition

$$
\| (\varphi, \psi) \|_{\dot{H}^s \times \dot{H}^s} \leq \frac{1}{18C'\ell C_b},
$$

where the constants $C_\ell$ and $C_b$ are given in Propositions 1 and 2, respectively. Then the operator $T$ defined in (3.3) maps $T : B(0, \rho) \to B(0, \rho)$, where $B(0, \rho) = \{(u, v) \in X^s \times X^s : \|(u, v)\|_{X^s \times X^s} \leq \rho\}$ and is a contraction mapping.

The key to proving this proposition lies in the Bilinear Estimate, Proposition 2, whose proof can be found in the next section. We further note that the hypothesis on $s$ comes from the Bilinear Estimate.

**Proof.** We begin by first demonstrating that $T : B(0, \rho) \to B(0, \rho)$. Noting that

$$
\|T(u, v)\|_{X^s \times X^s} = \|T_1(u, v)\|_{X^s} + \|T_2(u, v)\|_{X^s},
$$

we proceed by examining the terms on the right-hand side separately.

**Estimating** $\|T_1(u, v)\|_{X^s}$. We have

$$
\|T_1(u, v)\|_{X^s} = \|S(t)\varphi - \int_0^t S(t-t')(uv)dt'\|_{X^s},
$$

$$
\leq \|S(t)\varphi\|_{X^s} + \left\| \int_0^t S(t-t')(uv)dt' \right\|_{X^s}.
$$

For the first term, we use Proposition 1, which gives us

$$
\|S(t)\varphi\|_{X^s} \leq C_\ell \|\varphi\|_{\dot{H}^s}.
$$

The second term is bounded using Proposition 2, giving us

$$
\left\| \int_0^t S(t-t')(uv)dt' \right\|_{X^s} \leq C_b \|u\|_{X^s} \|v\|_{X^s} = C_b \rho^2.
$$

Therefore, we get

$$
\|T_1(u, v)\|_{X^s} \leq C_\ell \|\varphi\|_{\dot{H}^s} + C_b \rho^2.
$$

**Estimating** $T_2(u, v)$. We have

$$
\|T_2(u, v)\|_{X^s} = \|S(t)\psi + \int_0^t S(t-t')(uv - \mu v)dt'\|_{X^s},
$$

$$
\leq \|S(t)\psi\|_{X^s} + \left\| \int_0^t S(t-t')(uv)dt' \right\|_{X^s} + \mu \left\| \int_0^t S(t-t')vd't' \right\|_{X^s}.
$$

Again, the linear term is estimated using Proposition 1, giving us

$$
\|S(t)\psi\|_{X^s} \leq C_\ell \|\psi\|_{\dot{H}^s}.
$$

(3.9)
Taking the supremum over $t$

For the time-weighted integral. We then apply the

Again, we will apply the Mean-Value Theorem, giving us for some $t^* \in (0, T)$, we have

Putting these results together, (3.9) and (3.11), we get

The second term was already estimated in (3.7). Thus, we will restrict our attention to the third term, leaving out the coefficient of $\mu$ for the moment. We begin with the definition of the $X^s$-norm, noting that we will reinsert the constant $\mu$ at the end of the computation. We have

Looking at the expression under the supremum, we see that for the Sobolev term we have

We can now apply the Mean-Value Theorem, which tells us that for some $t^* \in (0, T)$, we have

Taking the supremum over $t \in (0, T)$ thus gives us

For the time-weighted $L^4$-term, we first obtain an upper bound by passing the norm into the integral. We then apply the $L^p-L^q$ Linear Estimate, Lemma 2, with $p = q = 4$.

Again, we will apply the Mean-Value Theorem, giving us for some $t^* \in (0, T)$,

To bound this above and remove the dependence on $t^*$, we note that $t^* = (t^*)^\alpha(t^*)^{1-\alpha}$ and $0 \leq \alpha \leq 1$. Then we use the fact that $t^\alpha \leq T^\alpha$ and $(t^*)^{1-\alpha} \leq T^{1-\alpha}$. Thus we get

Thus we can conclude

Putting these results together, (3.9) and (3.11), we get

(3.12)
Estimating $\|T(u,v)\|_{X^s \times X^s}$. Putting our results together, (3.7) and (3.12), we get
\[
\|T(u,v)\|_{X^s \times X^s} = \|T_1(u,v)\|_{X^s} + \|T_2(u,v)\|_{X^s},
\]
\[
\leq C_\ell \|(\varphi, \psi)\|_{\dot{H}^s \times H^s} + 2C_b \rho^2 + \mu T \rho,
\]
where $C_\ell$ is the constant associated with the upper bound from Lemma 1 and $C_b$ is the constant associated with the Bilinear Estimate, Proposition 2. Now using the hypothesis, $\rho = \frac{1}{3C_b}$ and $T < \frac{1}{6\mu}$, we see that
\[
C_\ell \|(\varphi, \psi)\|_{\dot{H}^s \times H^s} + 2C_b \rho^2 + \mu T \rho < C_\ell \|(\varphi, \psi)\|_{\dot{H}^s \times H^s} + 2C_b \left( \frac{1}{3C_b} \right) \rho + \mu \frac{1}{6\mu} \rho,
\]
\[
= C_\ell \|(\varphi, \psi)\|_{\dot{H}^s \times H^s} + \frac{5}{6} \rho.
\]
Thus, in order for $T$ to map $B(0, \rho)$ into $B(0, \rho)$, we must have
\[
C_\ell \|(\varphi, \psi)\|_{\dot{H}^s \times H^s} \leq \frac{1}{6} \rho,
\]
which is equivalent to our hypothesis
\[
\|(\varphi, \psi)\|_{\dot{H}^s \times H^s} \leq \frac{1}{18C_\ell C_b}.
\]

Contraction. Recalling that the initial data $(\varphi, \psi)$ are fixed, we take two pairs of functions $(u_1, v_1), (u_2, v_2) \in X^s \times X^s$ and compute the norm of $T(u_1, v_1) - T(u_2, v_2)$. We break this computation down by first examining the component operators $T_1$ and $T_2$.

Estimating the $T_1$ difference. We add and subtract a mixed term $u_1v_2$ and regroup terms to obtain
\[
T_1(u_1, v_1) - T_1(u_2, v_2) = - \int_0^t S(t-t') (u_1 v_1 - u_2 v_2) \, dt',
\]
\[
= - \int_0^t S(t-t') (u_1 (v_1 - v_2)) \, dt' - \int_0^t S(t-t') (v_2 (u_1 - u_2)) \, dt'.
\]
We now examine this difference in the $X^s$ norm. After applying the triangle inequality, we use Bilinear Estimate, Proposition 2, to further bound the nonlinearities. We get
\[
\|T_1(u_1, v_1) - T_1(u_2, v_2)\|_{X^s} \leq \|\int_0^t S(t-t') (u_1(v_1 - v_2)) \, dt'\|_{X^s} + \|\int_0^t S(t-t') (v_2(u_1 - u_2)) \, dt'\|_{X^s},
\]
\[
\leq C_b \|u_1\|_{X^s} \|v_1 - v_2\|_{X^s} + C_b \|v_2\|_{X^s} \|u_1 - u_2\|_{X^s}.
\]
Now using the hypothesis that both $\|u_1\|_{X^s}$ and $\|v_2\|_{X^s}$ are bounded by the constant $\rho$, and recalling that the norm on the product space is the sum of the norms, we get
\[
\|T_1(u_1, v_1) - T_1(u_2, v_2)\|_{X^s} \leq \rho C_b \|u_1 - u_2, v_1 - v_2\|_{X^s \times X^s}.
\]

Estimating the $T_2$ difference. We first break up the integral to separate the linear term, giving us
\[
T_2(u_1, v_1) - T_2(u_2, v_2)
\]
\[
= \int_0^t S(t-t') (u_1 v_1 - u_2 v_2) \, dt - \int_0^t S(t-t') \mu (v_1 - v_2) \, dt'.
\]
The first term under the $X^s$ norm is precisely the same as that of $T_1$. Thus, adding and subtracting a mixed term and then applying the Bilinear estimate leads to the same result. Thus we get
\[
\|T_2(u_1, v_1) - T_2(u_2, v_2)\|_{X^s} \\
\leq \rho C_b \|(u_1 - u_2, v_1 - v_2)\|_{X^s \times X^s} + \mu \left\| \int_0^t S(t - t')(v_1 - v_2)dt' \right\|_{X^s}.
\] (3.14)
To estimate the linear term, we begin with the definition of the $X^s$-norm, noting that we will reinsert the constant $\mu$ at the end of the computation. We have
\[
\left\| \int_0^t S(t - t')(v_1 - v_2)dt' \right\|_{X^s} = \sup_{t \in (0, T)} \left\| \int_0^t S(t - t')(v_1 - v_2)dt' \right\|_{\dot{H}^s} + \sup_{t \in (0, T)} t^\alpha \left\| \int_0^t S(t - t')(v_1 - v_2)dt' \right\|_{L^4}.
\]
The expression under the supremum can be bounded in precisely the same fashion as (3.10), giving us
\[
\left\| \int_0^t S(t - t')(v_1 - v_2)dt' \right\|_{\dot{H}^s} \leq \int_0^t \|v_1 - v_2\|_{\dot{H}^s} dt'.
\]
We can now apply the Mean-Value Theorem, which tells us that for some $t^* \in (0, T)$, we have
\[
\int_0^t \|v_1 - v_2\|_{\dot{H}^s} dt' = t^* \|v_1(t^*) - v_2(t^*)\|_{\dot{H}^s}^2.
\]
Taking the supremum over $t \in (0, T)$ thus gives us
\[
\sup_{t \in (0, T)} \left\| \int_0^t S(t - t')(v_1 - v_2)dt' \right\|_{\dot{H}^s} \leq \sup_{t \in (0, T)} t^* \|v_1(t^*) - v_2(t^*)\|_{\dot{H}^s},
\]
\[
\leq T \sup_{t \in (0, T)} \|v_1 - v_2\|_{\dot{H}^s}.
\] (3.15)
For the time-weighted $L^4$-term, we apply the $L^p - L^q$ Linear Estimate, Lemma 2, with $p = q = 4$.
\[
\sup_{t \in (0, T)} t^\alpha \left\| \int_0^t S(t - t')(v_1 - v_2)dt' \right\|_{L^4} \leq \sup_{t \in (0, T)} t^\alpha \int_0^t \|S(t - t')(v_1 - v_2)\|_{L^4} dt',
\]
\[
\leq \sup_{t \in (0, T)} t^\alpha \int_0^t \|v_1 - v_2\|_{L^4} dt'.
\]
Again, we will apply the Mean-Value Theorem, giving us for some $t^* \in (0, T)$,
\[
\sup_{t \in (0, T)} t^\alpha \int_0^t \|v_1 - v_2\|_{L^4} dt' = \sup_{t \in (0, T)} t^\alpha t^* \|v_1(t^*) - v_2(t^*)\|_{L^4} dt'.
\]
To bound this above and remove the dependence on $t^*$, we note that $t^* = (t^*)^\alpha (t^*)^{1 - \alpha}$ and $0 \leq \alpha \leq 1$. Then we use the fact that $t^\alpha \leq T^\alpha$ and $(t^*)^{1 - \alpha} \leq T^{1 - \alpha}$. Thus we get
\[
\sup_{t \in (0, T)} t^\alpha t^* \|v_1(t^*) - v_2(t^*)\|_{L^4} \leq T \sup_{t \in (0, T)} (t^*)^\alpha \|v_1(t^*) - v_2(t^*)\|_{L^4},
\]
\[
\leq T \sup_{t \in (0, T)} t^\alpha \|v_1 - v_2\|_{L^4}.
\] (3.16)
Putting everything together, (3.15) and (3.16), we get the estimate for the term of $T_2$ as

$$
\mu \left\| \int_0^t S(t - t')(v_1 - v_2)dt' \right\|_{X^s} \leq \mu T \left( \sup_{t \in (0,T)} \|v_1 - v_2\|_{\dot{H}^s} + \sup_{t \in (0,T)} t^\alpha \|v_1 - v_2\|_{L^1} \right),
$$

$$
\leq 2\mu T \|v_1 - v_2\|_{X^s}.
$$

(3.17)

Finally, we obtain the full estimate for $T_2$ as (3.14), where the second term is bounded by (3.17). Hence

$$
\|T_2(u_1, v_1) - T_2(u_2, v_2)\|_{X^s} \leq (\rho C_b + 2\mu T)(\|u_1 - u_2, v_1 - v_2\|_{X^s \times X^s}).
$$

(3.18)

**Combined Estimate for the $T$ difference.** Putting together our estimates for $T_1$ and $T_2$, namely, (3.13) and (3.18), we have

$$
\|T(u_1, v_1) - T(u_2, v_2)\|_{X^s \times X^s} = \|T_1(u_1, v_1) - T_1(u_2, v_2)\|_{X^s} + \|T_2(u_1, v_1) - T_2(u_2, v_2)\|_{X^s},
$$

$$
\leq (2\rho C_b + 2\mu T)(\|u_1 - u_2, v_1 - v_2\|_{X^s \times X^s}).
$$

(3.19)

Our hypotheses that $\rho = \frac{1}{3C_b}$ and $T < \frac{1}{6\mu}$ now allow us to conclude that

$$
2\rho C_b + 2\mu T < 1,
$$

and thus $T$ is indeed a contraction mapping. \hfill \square

With the proof of Proposition 3 complete, we now are ready to prove Theorem 1.

**Proof of Theorem 1.** We see that the fixed point from Proposition 3 gives us the existence and uniqueness of the solutions to KM ivp (1.3), thus we restrict our attention to proving that the solution map $W$ is Lipschitz continuous.

Let $(\varphi, \psi_1), (\varphi_2, \psi_2) \in \dot{H}^s \times \dot{H}^s$, and $(u_1, v_1), (u_2, v_2) \in X^s \times X^s$ be the corresponding solutions to the KM initial value problem with these initial data respectively. Thus, for $W$ the solution operator to KM, our objective is to estimate the difference

$$
\|W(\varphi, \psi_1) - W(\varphi_2, \psi_2)\|_{X^s \times X^s} = \|(u_1, v_1) - (u_2, v_2)\|_{X^s \times X^s}.
$$

(3.20)

We see that we can reframe this question using the operator $T$ established above in (3.3). To use $T$, however, we must have fixed initial data. Thus we will assume that $T$ in this instance uses the initial data $(\varphi, \psi) = (0, 0)$. Thus, we can rewrite the difference inside of the norm in (3.20) as

$$
W(\varphi, \psi_1) - W(\varphi_2, \psi_2) = (S(t)(\varphi_1 - \varphi_2), S(t)(\psi_1 - \psi_2)) + (T(u_1, v_1) - T(u_2, v_2)).
$$

We therefore can now rewrite (3.20) as

$$
\|W(\varphi, \psi_1) - W(\varphi_2, \psi_2)\|_{X^s \times X^s}
\leq \|(S(t)(\varphi_1 - \varphi_2), S(t)(\psi_1 - \psi_2))\|_{X^s \times X^s} + \|(T(u_1, v_1) - T(u_2, v_2))\|_{X^s \times X^s}.
$$

For the first term on the right-hand side, we use our linear estimate from Lemma 1, giving us

$$
\|(S(t)(\varphi_1 - \varphi_2), S(t)(\psi_1 - \psi_2))\|_{X^s \times X^s} \leq C_t\|(\varphi_1 - \varphi_2, \psi_1 - \psi_2)\|_{\dot{H}^s \times \dot{H}^s}.
$$
For the second term, we will use the work from proving that \( T \) is a contraction. Setting our contraction constant as \( C = 2\rho C_b + 2\mu T < 1 \), we use (3.19) to give us
\[
\|(T(u_1, v_1) - T(u_2, v_2))\|_{X^s \times X^s} \leq C\|(u_1 - u_2, v_1 - v_2)\|_{X^s \times X^s},
\]
\[
= C\|W(\varphi_1, \psi_1) - W(\varphi_2, \psi_2)\|_{X^s \times X^s}.
\]
Thus, we get
\[
\|W(\varphi_1, \psi_1) - W(\varphi_2, \psi_2)\|_{X^s \times X^s} \leq C\|\varphi_1 - \varphi_2, \psi_1 - \psi_2\|_{\dot{H}^s \times \dot{H}^s} + C\|W(\varphi_1, \psi_1) - W(\varphi_2, \psi_2)\|_{X^s \times X^s}.
\]
Simplifying this inequality, gives us
\[
\|W(\varphi_1, \psi_1) - W(\varphi_2, \psi_2)\|_{X^s \times X^s} \leq \frac{C\ell}{1 - C}\|\varphi_1 - \varphi_2, \psi_1 - \psi_2\|_{\dot{H}^s \times \dot{H}^s},
\]
which implies that \( W \) is Lipschitz continuous.

4. BILINEAR ESTIMATES

We begin with a proof of Proposition 2. This proof in turn requires analogous estimates in \( \dot{H}^s \) and the time-weighted \( L^4 \)-space, which were proved in Lemma 5.

**Proof of Proposition 2.** Starting with the definition of the \( X^s \)-norm, and passing the norms inside the integrals to bound above, we get
\[
\left\| \int_0^t S(t - t')(fg)dt' \right\|_{X^s} \leq \sup_{t \in (0, T)} \int_0^t \left\| S(t - t')(fg) \right\|_{\dot{H}^s} dt' + \sup_{t \in (0, T)} t^\alpha \int_0^t \left\| S(t - t')(fg)dt' \right\|_{L^4} dt'.
\]
We now apply Lemma 5 to each term on the right-hand side of this equation. For the first term, we get
\[
\sup_{t \in (0, T)} \int_0^t \left\| S(t - t')(fg) \right\|_{\dot{H}^s} dt' \lesssim \sup_{t \in (0, T)} t^{2\alpha} \|f\|_{L^4} \|g\|_{L^4}. \tag{4.1}
\]
For the second term, we have
\[
\sup_{t \in (0, T)} t^\alpha \int_0^t \left\| S(t - t')(fg) \right\|_{L^4} dt' \lesssim \sup_{t \in (0, T)} t^{2\alpha} \|f\|_{L^4} \|g\|_{L^4}. \tag{4.2}
\]
Combining (4.1) and (4.2), we get
\[
\left\| \int_0^t S(t - t')(fg)dt' \right\|_{X^s} \lesssim \sup_{t \in (0, T)} t^{2\alpha} \|f\|_{L^4} \|g\|_{L^4},
\]
\[
\leq \left( \sup_{t \in (0, T)} t^\alpha \|f\|_{L^4} \right) \left( \sup_{t \in (0, T)} t^\alpha \|g\|_{L^4} \right),
\]
\[
\leq \|f\|_{X^s} \|g\|_{X^s}.
\]
The constant associated with the upper bound comes from Lemma 5, and as its particular value is used to establish the Contraction in Theorem 1, we label it as \( C_b \).

To complete the argument for Proposition 2, we use the following beautiful calculus estimate related to the Beta distribution. A proof is given in [16], A.2.
Lemma 4. Let $0 < a, b < 1$ and $r = a + b - 1$ with $0 \leq r \leq 1$, then we have the following bound

$$B(a, b, t) = \int_0^t (t - y)^{-b} y^{-a} dy \leq c_{a,b} t^{-r}. \quad (4.3)$$

With this estimate in hand, we now proceed to compute the following estimates on the components of the $X^s$-norm, which completes our arguments.

Lemma 5. Let $f$ and $g \in X^s$, with $0 \leq s < 2$ and $0 \leq t < T < \infty$. Then the following inequalities hold

$$\sup_{t \in (0,T)} \int_0^t \left\| S(t - t')(fg) \right\|_{H^s} dt' \lesssim \sup_{t \in (0,T)} t^{2\alpha} \|f\|_{L^4} \|g\|_{L^4}, \quad (4.4)$$

$$\sup_{t \in (0,T)} \int_0^t \left\| S(t - t')(fg) \right\|_{L^4} dt' \lesssim \sup_{t \in (0,T)} t^{2\alpha} \|f\|_{L^4} \|g\|_{L^4}. \quad (4.5)$$

Proof. The argument for each of these estimates follows similar lines, but we examine each one separately.

Estimating (4.4). Using the definition of the Riesz Derivative, we begin with

$$\int_0^t \left\| S(t - t')(fg) \right\|_{H^s} dt' = \int_0^t \left\| D_x^s S(t - t')(fg) \right\|_{L^2} dt'$$

We now apply Lemma 3, which requires $s \geq 0$, to obtain

$$\int_0^t \left\| D_x^s S(t - t')(fg) \right\|_{L^2} dt' \leq \int_0^t (t - t')^{-\frac{s}{2}} \|f(t')g(t')\|_{L^2} dt'.$$

Next, the generalized Hölder inequality allows us to break the $L^2$ norm in the integrand into a product of $L^4$ norms, giving us

$$\int_0^t (t - t')^{-\frac{s}{2}} \|f(t')g(t')\|_{L^2} dt' \leq \int_0^t (t - t')^{-\frac{s}{2}} \|f(t')\|_{L^4} \|g(t')\|_{L^4} dt'.$$

We now multiply and divide by $(t')^{2\alpha}$ and then pull out the factor $(t')^{2\alpha} \|f(t')\|_{L^4} \|g(t')\|_{L^4}$ by taking a supremum over time. This gives us the upper bound

$$\int_0^t (t - t')^{-\frac{s}{2}} \|f(t')\|_{L^4} \|g(t')\|_{L^4} dt' \leq \sup_{t \in (0,T)} t^{2\alpha} \|f(t)\|_{L^4} \|g(t)\|_{L^4} \int_0^t (t - t')^{-\frac{s}{2}} (t')^{-2\alpha} dt'. \quad (4.6)$$

The remaining integral can now be handled by Lemma 4. The choice of multiplying and dividing by $(t')^{2\alpha}$ is now apparent as this allows us to satisfy the hypothesis for $r = \frac{s}{2} + 2\alpha - 1$. Noting that we require $0 \leq r \leq 1$, we see that our definition of $\alpha = \frac{1}{2} - \frac{s}{4}$ gives us

$$r = \frac{s}{2} + 2\alpha - 1 = \frac{s}{2} + 2\left(\frac{1}{2} - \frac{s}{4}\right) - 1 = 0.$$

We see here that this $r$ restriction is always satisfied regardless of the value of $s$. The additional requirements of Lemma 4 are satisfied so long as we take $0 \leq s < 2$ and $0 < \alpha \leq 1/2$. Thus for a constant that only depends on $s$, as we have $\alpha$ a function of $s$, we get

$$\int_0^t (t - t')^{-\frac{s}{2}} (t')^{-2\alpha} dt' < c_s.$$
We therefore further bound (4.6) by

$$\sup_{t \in (0, T)} t^{2\alpha} \|f(t)\|_{L^4} \|g(t)\|_{L^4} \int_0^t (t-t')^{-\frac{7}{8}} (t')^{-2\alpha} dt' \leq c_{\alpha} \sup_{t \in (0, T)} t^{2\alpha} \|f(t)\|_{L^4} \|g(t)\|_{L^4}.$$

This chain of inequalities thus establishes (4.4).

**Estimating (4.5).** Before estimating the full expression on the left-hand side of (4.5) we first estimate its integrand. To accomplish this, we begin by using Lemma 2 which is the $L^p$-$L^q$ Heat Kernel Estimate and the generalized Hölder inequality to get

$$\|S(t-t')(fg)\|_{L^4} \lesssim (t-t')^{-\frac{7}{8}} \|f(t')g(t')\|_{L^2} \lesssim (t-t')^{-\frac{1}{8}} \|f(t')\|_{L^4} \|g(t')\|_{L^4}.$$

This upper bound for $\|S(t-t')(fg)\|_{L^4}$ thus gives us

$$\sup_{t \in (0, T)} t^{\alpha} \int_0^t \left\|S(t-t')(fg)\right\|_{L^4} dt' \leq \sup_{t \in (0, T)} t^{\alpha} \int_0^t (t-t')^{-\frac{7}{8}} \|f(t')\|_{L^4} \|g(t')\|_{L^4} dt'.$$

(4.7)

While our construction would suggest examining $0 < \alpha \leq \frac{1}{2}$, we in fact are able to prove this estimate for $0 < \alpha \leq \frac{7}{8}$; though the values of $\alpha > \frac{1}{2}$ are unused. To continue estimating (4.7), our argument splits based into the cases where $0 < \alpha \leq \frac{7}{16}$ and $\frac{7}{16} < \alpha \leq \frac{7}{8}$. The strategy in both cases is similar, with the difference lying in the quantity we multiply and divide by in order to utilize Lemma 4.

**The case $0 < \alpha \leq \frac{7}{16}$.** In this case, we multiply and divide by $(t')^{-\frac{7}{8}}$ inside the integrand and then pull out the factor of $(t')^{-\frac{7}{8}} \|f(t')\|_{L^4} \|g(t')\|_{L^4}$ by taking a supremum over time. We thus get

$$\sup_{t \in (0, T)} t^{\alpha} \int_0^t (t-t')^{-\frac{1}{8}} \|f(t')\|_{L^4} \|g(t')\|_{L^4} dt'$$

$$\leq \sup_{t \in (0, T)} t^{\alpha} \left[ \sup_{t' \in (0, t)} (t')^{-\frac{7}{8}} \|f(t')\|_{L^4} \|g(t')\|_{L^4} \right] \int_0^t (t-t')^{-\frac{1}{8}} (t')^{-\frac{7}{8}} dt'.$$

(4.8)

The integral in (4.8), can now be handled with the Beta distribution estimate, Lemma 4. We see that the corresponding $r$ will be $r = 1/8 + 7/8 - 1 = 0$, and therefore

$$\int_0^t (t-t')^{-\frac{1}{8}} (t')^{-\frac{7}{8}} dt' \lesssim 1.$$

To handle the composition of suprema, we first note that as $\alpha > 0$, we have $t^{\alpha} \leq T^{\alpha}$. Then the interior supremum can be bounded above by taking the supremum to be over the full interval $(0, T)$. Putting these estimates together gives us

$$\sup_{t \in (0, T)} t^{\alpha} \left[ \sup_{t' \in (0, t)} (t')^{-\frac{7}{8}} \|f(t')\|_{L^4} \|g(t')\|_{L^4} \right] \int_0^t (t-t')^{-\frac{1}{8}} (t')^{-\frac{7}{8}} dt'$$

$$\lesssim T^{\alpha} \sup_{t \in (0, T)} t^\frac{7}{8} \|f(t)\|_{L^4} \|g(t')\|_{L^4}.$$

Thus, so long as $0 < 2\alpha \leq 7/8$ we see that $t^{\frac{7}{8} - 2\alpha}$ will be positive, and we thus continue our estimation by multiplying and dividing by $t^{2\alpha}$. After bounding above by pulling out the factor of $t^{\frac{7}{8} - 2\alpha}$, we get
We therefore get
\[ T^\alpha \sup_{t \in (0,T)} t^{\frac{7}{8}} \| f(t) \|_{L^4} \| g(t) \|_{L^4} \leq T^{\frac{7}{8} - \alpha} \sup_{t \in (0,T)} t^{2\alpha} \| f(t) \|_{L^4} \| g(t) \|_{L^4}. \]

Chaining these inequalities gives us (4.5).

**The case** \( \frac{7}{16} < \alpha \leq \frac{7}{8} \). To continue bounding (4.7), we follow a similar approach to the above case but alternatively multiply and divide by \((t')^{2\alpha}\). We thus obtain
\[
\sup_{t \in (0,T)} t^\alpha \int_0^t (t-t')^{-\frac{1}{8}} \| f(t') \|_{L^4} \| g(t') \|_{L^4} \, dt' \\
\leq \sup_{t \in (0,T)} t^\alpha \left[ \sup_{t' \in (0,t)} (t')^{2\alpha} \| f(t') \|_{L^4} \| g(t') \|_{L^4} \right] \int_0^t (t-t')^{-\frac{1}{8}} (t')^{-2\alpha} \, dt'.
\]
(4.9)

Our hypothesis of \( \frac{7}{16} < \alpha \leq \frac{7}{8} \) in this case satisfies the requirement of Lemma 4. We see that for \( r = \frac{1}{8} + 2\alpha - 1 \) to satisfy \( 0 \leq r \leq 1 \) is equivalent to \( \frac{7}{16} \leq \alpha \leq \frac{15}{16} \). Thus the Beta distribution estimate implies that we have
\[
\int_0^t (t-t')^{-\frac{1}{8}} (t')^{-2\alpha} \, dt' \lesssim t^{-\frac{1}{8} - 2\alpha + 1}.
\]
Continuing our estimation on (4.9), and noting that \( \alpha - r = \frac{7}{8} - \alpha \), we get
\[
\sup_{t \in (0,T)} t^\alpha \left[ \sup_{t' \in (0,t)} (t')^{2\alpha} \| f(t') \|_{L^4} \| g(t') \|_{L^4} \right] \int_0^t (t-t')^{-\frac{1}{8}} (t')^{-2\alpha} \, dt' \\
\lesssim \sup_{t \in (0,T)} t^{\frac{7}{8} - \alpha} \left[ \sup_{t' \in (0,t)} (t')^{2\alpha} \| f(t') \|_{L^4} \| g(t') \|_{L^4} \right] \]
(4.10)

To bound the exterior supremum, we see that our hypothesis in this case implies that \( \frac{7}{8} - \alpha \geq 0 \). We therefore get
\[
\sup_{t \in (0,T)} t^{\frac{7}{8} - \alpha} \left[ \sup_{t' \in (0,t)} (t')^{2\alpha} \| f(t') \|_{L^4} \| g(t') \|_{L^4} \right] \leq T^{\frac{7}{8} - \alpha} \sup_{t \in (0,T)} t^{2\alpha} \| f(t) \|_{L^4} \| g(t) \|_{L^4}.
\]
Chaining these inequalities thus gives us (4.5).

\[\square\]

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