POSITIVITY OF CANONICAL BASES UNDER COMULTIPLICATION

ZHAOBING FAN AND YIQIANG LI

Abstract. We show the positivity of the canonical basis for a modified quantum affine $\mathfrak{sl}_n$ under the comultiplication. Moreover, we establish the positivity of the $i$-canonical basis in [LW15] with respect to the coideal subalgebra structure.

Introduction

The geometric study of the modified quantum $\mathfrak{sl}_n$ via perverse sheaves on partial flag varieties of type $A$ is initiated in the work [BLM90] of Beilinson, Lusztig and MacPherson. It is then generalized to quantum affine $\mathfrak{sl}_n$ by Ginzburg-Vasserot [GV93] and Lusztig [L99], [L00], independently, by considering the geometry of affine partial flag varieties of type $A$. This line of research is culminated in the work of Schiffmann-Vasserot [SV00] and McGerty [M10] showing that the canonical basis of modified quantum affine $\mathfrak{sl}_n$ defined geometrically via transfer maps can be identified with the one defined algebraically by Lusztig [L93] (see also Kashiwara [K94]). As a consequence, one obtains the positivity of the structure constants of the canonical basis of quantum affine $\mathfrak{sl}_n$ with respect to multiplication, which is conjectured in [L93, 25.4.2].

In a recent remarkable work [BW13] of Bao-Wang, a quantum-Schur-like duality relating a type-$B/C$ Hecke algebra and a coideal subalgebra of quantum $\mathfrak{sl}_n$ is obtained, and moreover an $i$-canonical basis for the representations of coideal subalgebras involved is constructed. The desire to geometrize Bao-Wang’s work and to describe the convolution algebras of certain perverse sheaves on partial flag varieties of classical type lead to the work [BKLW14], where the approach in [BLM] is revived and adapted to give a geometric construction of the (modified) coideal subalgebra of quantum $\mathfrak{gl}_n$ and a stably canonical basis by using certain perverse sheaves of partial flag varieties of type $B/C$. Since a modified coideal subalgebra of quantum $\mathfrak{gl}_n$ can be regarded as a direct sum of infinitely many copies of its $\mathfrak{sl}_n$ version, one obtains infinitely many stably canonical bases of the modified coideal subalgebra of quantum $\mathfrak{sl}_n$. As a consequence, the $i$-canonical basis of the tensor space in the duality in [BW13] admits a geometric incarnation as certain intersection cohomology complexes.

Despite of many favorable properties of the stably canonical bases of modified quantum $\mathfrak{gl}_n$ and its coideal subalgebras, they do not admit positivity with respect to multiplication; counterexamples can be found in [LW15]. Instead, a new basis, called the $i$-canonical basis, of the modified coideal subalgebra of quantum $\mathfrak{sl}_n$ is constructed in loc. cit. following the spirit of [L00] and [M10] (see also [SV00]). This basis can be regarded as an asymptotical version of the stably canonical basis since they coincide asymptotically ([LW15]). It is further shown that the $i$-canonical basis does admit three positivities with respect to the multiplication,

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a bilinear form of geometric origin in *loc. cit.* and its action on the $i$-canonical basis of a tensor space.

In this article, we establish three more positivities of $i$-canonical bases, in addition to the previous ones in [LW15], mainly with respect to the coideal subalgebra structure. To be more precise, let $B$ be the canonical basis of modified quantum $\mathfrak{sl}_n$, say $\hat{U}$, and $B^i$ its coideal analogue in the modified coideal subalgebra $\hat{U}^i$. We note that notations in the introduction are slightly different from the main body of the paper. One can define in a natural way an algebra homomorphism $\Delta^i : \hat{U}^i \to (\hat{U}^i \otimes \hat{U})^\wedge$, where the target is a certain variant of the tensor algebra $\hat{U}^i \otimes \hat{U}$, which is an idempotented version of the coideal structure coming from the comultiplication of quantum $\mathfrak{sl}_n$. (See (59), (77) for precise definitions.) In particular, if $a \in B^i$, one has

$$\Delta^i(a) = \sum_{b \in B^i, c \in B} n_{a}^{b,c} \otimes c, \quad n_{a}^{b,c} \in \mathbb{Z}[v,v^{-1}].$$

The positivity with respect to the idempotented coideal structure further says that

**Positivity A** (Theorem 4.3.1, 5.2.1). *The structure constant $n_{a}^{b,c}$ is in $\mathbb{Z}_{\geq 0}[v,v^{-1}]$.***

A degenerate version of $\Delta^i$ induces an imbedding $i : \hat{U}^i \to (\hat{U})^\wedge$, which reflects the subalgebra structure of the ordinary coideal subalgebra in quantum $\mathfrak{sl}_n$. (See (68), (78).) The positivity with respect to the idempotented subalgebra structure says that

**Positivity B** (Joint with Weiqiang Wang; Theorem 4.4.1, 5.2.2). If $i(a) = \sum_{b \in B} g_{b,a} b$, $\forall a \in B^i$, then $g_{b,a} \in \mathbb{Z}_{\geq 0}[v,v^{-1}]$.***

As a second degeneration of $\Delta^i$, we make a direct connection between the geometric type $A$ duality of [GL92] and type $B/C$ duality of [BKLW14], which reveals yet another positivity:

**Positivity C** (Theorem 3.5.3). *The $i$-canonical basis in a tensor space is a positive sum of the canonical basis in the same tensor space.*

As is shown, these positivities are boiled down to a geometric interpretation of the coideal structure coming from the comultiplication of quantum $\mathfrak{sl}_n$. To this end, we also establish a geometric realization of the comultiplication of quantum affine $\mathfrak{sl}_n$, and we obtain the following positivity on quantum affine $\mathfrak{sl}_n$:

**Positivity D** (Theorem 2.4.2). *The canonical basis of modified quantum affine $\mathfrak{sl}_n$ admits positivity with respect to the idempotented comultiplication.*

The proof of the positivity result on quantum affine $\mathfrak{sl}_n$ consists of two parts since the geometrically defined comultiplication on the affine Schur algebra level is a composition of a hyperbolic localization [B03] and a twist of a certain $v$-power. The positivity on the former is well-known by [B03], (see also [L00], [SV00]), while we show that in the latter it sends a canonical basis to a canonical basis up to a $v$-power. Note that the second step is trivial in the ordinary quantum $\mathfrak{sl}_n$ case, but non-trivial in the affine case as far as we can see: because at some point, we have to invoke the multiplication formula of a semisimple generator of Du-Fu [DF13], for which we provide a new geometric proof. These arguments are contained in the first two sections, with the first section devoted to quantum $\mathfrak{sl}_n$ and the second one to its affine version.
The argument of the proof on quantum affine \( \mathfrak{sl}_n \) also applies with modifications to the various positivities of the i-canonical basis, which occupies the last three sections. The third section treats the results on the i-Schur-algebra level, and the fourth section lifting the results on the i-Schur-algebra level to the projective limit level for \( n \) being odd. The last section collects similar results for \( n \) even. The transfer maps used in [LW15] are constructed geometrically in these sections and the reader can find the proof of [LW15, Lemma 4.3] in Proposition [3.6.1].

Note that we work over the partial flag varieties of type \( B \) for the i-canonical basis and following the treatment of type \( A \) in [L00, M10]. One can obtain the same results via partial flag varieties of type \( C \) by using the principle in [BKLW14].

In [FLLLW], we shall construct and investigate geometrically the i-canonical basis of modified coideal subalgebras of quantum affine \( \mathfrak{sl}_n \) among others.

We refer to [EST13] and [FL14] for the interactions of type \( D \) partial flag varieties, coideal subalgebras and type \( D \) duality. In a forthcoming paper, we will present a type \( D \) picture similar to the positivity results on i-canonical basis in this paper.

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Contents

| Section | Title                                      | Page |
|---------|--------------------------------------------|------|
| 1       | Introduction                               | 1    |
| 1.1     | Convolution                               | 3    |
| 1.2     | Positivity for quantum \( \mathfrak{sl}_n \) | 3    |
| 2       | Positivity for quantum affine \( \mathfrak{sl}_n \) | 13   |
| 3       | Coproduct for the i-Schur algebras         | 21   |
| 4       | Positivity for the modified coideal subalgebra \( \hat{U}_i \) | 35   |
| 5       | \( \iota \)-version                        | 41   |
| References |                                               | 45   |

1. Positivity for quantum \( \mathfrak{sl}_n \)

In this section, we shall present a proof of the positivity of the canonical basis of quantum \( \mathfrak{sl}_n \) with respect to comultiplication.

1.1. Convolution. Let \( G \) be a group, and \( \mathcal{X} \) a \( G \)-set. The \( G \)-action on \( \mathcal{X} \) thus induces a diagonal \( G \)-action on the product \( \mathcal{X} \times \mathcal{X} \). Let \( \mathcal{A} \) be a unital commutative ring. We consider the set \( \mathcal{A}_G(\mathcal{X} \times \mathcal{X}) \) of all \( \mathcal{A} \)-valued \( G \)-invariant functions on \( \mathcal{X} \times \mathcal{X} \) supported on finitely many \( G \)-orbits. Assume that any \( G \)-orbit \( \mathcal{O} \) in \( \mathcal{X} \times \mathcal{X} \) has the property that the set \( \mathcal{X}_\mathcal{O} = \{ y \in \mathcal{X} | (x, y) \in \mathcal{O} \} \) is finite for one and hence any fixed \( x \) in \( \mathcal{X} \). Then \( \mathcal{A}_G(\mathcal{X} \times \mathcal{X}) \) is a free \( \mathcal{A} \)-module with a basis indexed by the \( G \)-orbits in \( \mathcal{X} \times \mathcal{X} \), and further an associative
**Proposition 1.3.1.** The map $\tilde{\Delta}$ in (4) is a well-defined algebra homomorphism over $\mathcal{A}$. 

Let $|W|$ denote the dimension of the vector space $W$ over $\mathbb{F}_q$. We use the notation $W_1 \overset{a}{\subset} W_2$ to denote $W_1 \subset W_2$ and $\dim W_2/W_1 = a$. Similarly, we define the notation $W_1 \overset{a}{\supset} W_2$. We
Lemma 1.3.2. For any \( i \in [1, n-1], a \in [1, n], \)

\[
E_d(V, V') = \begin{cases} 
  v^{-|V'_i/V_i'|}, & \text{if } V_i \subseteq V'_i, V_j = V_{j'}, \forall j \neq i; \\
  0, & \text{otherwise}.
\end{cases}
\]

\[
F_i(V, V') = \begin{cases} 
  v^{-|V_i'/V_i|}, & \text{if } V_i \supseteq V'_i, V_j = V_{j'}, \forall j \neq i; \\
  0, & \text{otherwise}.
\end{cases}
\]

\[
H^\pm_a(V, V') = v^\pm|a/a_0-1|\delta_{V,V'}, \quad \forall V, V' \in X_d.
\]

\[
K_i^\pm = H^\pm_{i+1}H^\pm_i.
\]

The following lemma is due to Lusztig [L00, Lemma 1.6].

**Lemma 1.3.2.** For any \( i \in [1, n-1], \) we have

\[
\tilde{\Delta}(E_i) = E'_i \otimes H''_{i+1} + H'_{i+1} \otimes E''_i, \quad \tilde{\Delta}(F_i) = F'_i \otimes H''_{i-1} + H'_i \otimes F''_i, \quad \tilde{\Delta}(K_i) = K'_i \otimes K''_i.
\]

Note that the functions \( E_i \) and \( F_i \) correspond to the functions in [L99, 2.4] in the notations \( F_i \) and \( E_i \), respectively. Let \( \Lambda_{d,n} = \{ a = (a_1, \ldots, a_n) \in \mathbb{Z}^n_\geq 0 | a_1 + \cdots + a_n = d \} \).

Proposition 1.3.3. The linear map \( \Delta_v \) in (7) is an algebra homomorphism. Moreover,

\[
\Delta_v(E_i) = E'_i \otimes K''_i + 1 \otimes E''_i,
\]

\[
\Delta_v(F_i) = F'_i \otimes 1 + K'_i \otimes F''_i,
\]

\[
\Delta_v(K_i) = K'_i \otimes K''_i, \quad \forall 1 \leq i \leq n-1.
\]

The following is a refinement of Lemma 1.3.2.
Proof. It is straightforward to see that $\Delta_\nu$ is an algebra homomorphism. We proceed to the proof of the equalities in the proposition. Suppose that a quadruple $(b', a', b'', a'')$ satisfies the conditions that $b'_k = a'_k - \delta_{k,i} + 1$ and $b''_k = a''_k$ for some $i$ and for all $1 \leq k \leq n$. Then

$$\sum_{1 \leq i \leq j \leq n} b'_i b'_j - a'_i a'_j = \sum_{1 \leq k \leq n} (b'_k - a'_k) a''_j = -\sum_{i \leq j \leq n} a''_j - \sum_{i+1 \leq j \leq n} a''_j = -a''_j.$$ 

So if $(V', V'', V'''') \in X_{d'} \times X_{d''} \times X_{d'''} \times X_{d''}$, then

$$\Delta_{\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''}(V', V'', V'', V''') = \mathbf{v}^{-a''} \mathbf{E}_i^i \otimes \mathbf{H}_i^{i+1}(V', V'', V'', V''') = \mathbf{E}_i^i \otimes \mathbf{K}_i^{i+1}(V', V'', V'', V''').$$

On the other hand, if $(b', a', b'', a'')$ is a quadruple subject to $b'_k = a'_k$ and $b''_k = a''_k$, then $\sum_{1 \leq k \leq n} b'_i b''_i - a'_i a''_j = a''_i$. Thus if $(V', V'', V''') \in X_{d'} \times X_{d''} \times X_{d'''}$, then

$$\Delta_{\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''}(V', V'', V''') = \mathbf{v}^{a''} \mathbf{H}_i^{i+1} \otimes \mathbf{E}_i^i (V', V'', V'', V''') = 1 \otimes \mathbf{E}_i^i (V', V'', V'', V''').$$

Altogether, we have $\Delta_\nu(E_i) = \mathbf{E}_i^i \otimes \mathbf{K}_i^{i+1} + 1 \otimes \mathbf{E}_i^i$, which is the first equality in the lemma.

If the quadruple $(b', a', b'', a'')$ satisfies that $b'_k = a'_k + \delta_{k,i} - 1$ and $b''_k = a''_k$ for some $i$ and for all $1 \leq k \leq n$, then $\sum_{1 \leq i \leq j \leq n} b'_i b''_i - a'_i a''_j$ is equal to $a''_j$. So after the twist, it makes the first term $\mathbf{H}_i^{i+1} \otimes \mathbf{E}_i^i$ in Lemma 1.3.2 into $\mathbf{F}_i^i \otimes 1$. Meanwhile, if $(b', a', b'', a'')$ is a quadruple subject to $b'_k = a'_k$ and $b''_k = a''_k + \delta_{k,i} - 1$ and $b''_k = a''_k$ for some $i$ and for all $1 \leq k \leq n$, then $\sum_{1 \leq i \leq j \leq n} b'_i b''_i - a'_i a''_j = -a''_i$. Hence after the twist, the second term $\mathbf{H}_i^{i+1} \otimes \mathbf{F}_i^i$ in $\tilde{\Delta}(F_i)$ becomes $\mathbf{K}_i^{i+1} \otimes \mathbf{F}_i^i$. This verifies the second equality in the lemma.

Since the twist is zero if $b' = a'$ and $b'' = a''$, the third equality holds. □

Remark 1.3.4. Note that if we write $(E_i, F_i, K_i)$ as $(F_i, E_i, K_i^{-1})$, we have the conventional comultiplication.

For the rest of this section, we give a second interpretation of $\tilde{\Delta}$ to be used in the proof of Proposition 1.5.3. Fix $V \in X_d(b)$ and set $P_b = \text{Stab}_{G_d}(V)$. Then $P_b$ acts via $G_d$ on $X_d(b)$.

Consider the imbedding

$$i_{b,a} : X_d(a) \to X_d(b) \times X_d(a), \quad V \mapsto (V, \tilde{V}).$$

It induces a bijection between $P_b$-orbits in the domain and $G_d$-orbits in the range of $i_{b,a}$. Hence the pullback (restriction)

$$i_{b,a}^* : \mathcal{A}_{G_d}(X_d(b) \times X_d(a)) \to \mathcal{A}_{P_b}(X_d(a))$$

of the imbedding $i_{b,a}$ is an isomorphism of $A$-modules.

Recall now that we fix a triple $(V, V', V''')$ in the definition of $\Delta$ in (4). We assume that $V \in X_d(b)$, $V' \in X_{d'}(b')$ and $V''' \in X_{d'''}(b'')$ so that $b' + b'' = b$. We also define $P_{b'}$ and $P_{b''}$ similar to $P_b$. Thus we have similar isomorphisms

$$i_{b',a'}^* : \mathcal{A}_{G_{d'}}(X_{d'}(b') \times X_{d'}(a')) \to \mathcal{A}_{P_{b'}}(X_{d'}(a')),$$

$$i_{b'',a''}^* : \mathcal{A}_{G_{d'''}}(X_{d'''}(b'') \times X_{d'''}(a'')) \to \mathcal{A}_{P_{b''}}(X_{d'''}(a'')).$$

Consider the subset of $X_d(a)$:

$$X^+_{a,a',a''} = \{ \tilde{V} \in X_d(a) | \pi'(\tilde{V}) \in X_{d'}(a'), \pi''(\tilde{V}) \in X_{d'''}(a'') \}. $$
Then we have the following diagram
\[
X_d(a) \xleftarrow{\iota} X_{a,a',a''} \xrightarrow{\pi} X_{d'}(a') \times X_{d''}(a''),
\]
where \( \iota \) is the natural inclusion and \( \pi(V) = (\pi'(V), \pi''(V)) \). Thus the composition of the pullback \( \iota^* \) of \( \iota \) followed by the pushforward \( \pi! \) of \( \pi \) defines a linear map
\[
\pi_! \iota^* : A_{P_b}(X_d(a)) \to A_{P_{b'} \times P_{b''}}(X_{d'}(a') \times X_{d''}(a'')),
\]
where \( \pi_! \) is defined by \( \pi_!(f)(V', V'') = \sum_{x \in X_{a,a',a''}^+, \pi(x) = (V', V'')} f(x) \), for all \( V', V'' \). Clearly, we have an isomorphism of \( \mathcal{A} \)-modules
\[
A_{P_{b'} \times P_{b''}}(X_{d'}(a') \times X_{d''}(a'')) \cong A_{P_{b'}}(X_{d'}(a')) \otimes A_{P_{b''}}(X_{d''}(a'')).
\]
The following lemma makes connection between \( \Delta \) and \( \pi_! \iota^* \).

**Lemma 1.3.5.** We have the following commutative diagram.
\[
\begin{array}{ccc}
A_{G_d}(X_d(b) \times X_d(a)) & \xrightarrow{\iota_{b,a}} & A_{P_b}(X_d(a)) \\
\Delta_{b',a',b'',a''} & & \Delta_{b',a',a''} \xrightarrow{\pi_! \iota^*} A_{P_{b'}(X_{d'}(a'))} \otimes A_{P_{b''}(X_{d''}(a''))} \\
A_{G_{d'}}(X_{d'}(b') \times X_{d'}(a')) & \xrightarrow{i_{b',a'}} & A_{P_{b'}(X_{d'}(a'))} \\
A_{G_{d''}}(X_{d''}(b'') \times X_{d''}(a'')) & \xrightarrow{i_{b'',a''}} & A_{P_{b''}(X_{d''}(a''))} \\
\end{array}
\]

**Proof.** For any \( f \in A_{G_d}(X_d(b) \times X_d(a)) \) and \((\tilde{V}', \tilde{V}'') \in X_{d'}(a') \times X_{d''}(a'')\), we have
\[
\pi_! \iota^* \iota_{b,a}^*(f)(\tilde{V}', \tilde{V}'') = \sum_{\tilde{V} \in Z_{\tilde{V}', \tilde{V}''}} \iota_{b,a}^*(f)(\tilde{V}) = \sum_{\tilde{V} \in Z_{\tilde{V}', \tilde{V}''}} f(V, \tilde{V}) = \Delta_{b',a',b'',a''}(f)(V', \tilde{V}', \tilde{V}'', \tilde{V}'') = (\iota_{b',a'}^* \otimes \iota_{b'',a''}^*) \circ \Delta_{b',a',b'',a''}(f)(\tilde{V}', \tilde{V}'').
\]
The lemma is thus proved. \( \square \)

**Remark 1.3.6.** Note that \( \pi \) is a vector bundle of rank \( \sum_{1 \leq i < j \leq n} a'_i a''_j \), which is closely related to the twist in (7).

### 1.4. Transfer map
To a pair \((V, V')\) in \( X_d \), we can associate an \( n \) by \( n \) matrix \( M = (m_{ij}) \) with coefficients in \( \mathbb{Z}_{\geq 0} \) by
\[
m_{ij} = \left| \frac{V_i \cap V'_j}{V_{i-1} \cap V'_j + V_i \cap V'_{j-1}} \right|, \quad \forall i, j \in [1, n].
\]
Let \( \Xi_d \) be the set of all matrices obtained this way. The set \( \Xi_d \) can be characterized by \( M \in \Xi_d \) if and only if \( m_{ij} \in \mathbb{Z}_{\geq 0} \) and \( \sum_{1 \leq i < j \leq n} m_{ij} = d \). It is shown in [BLM90] that the set \( \Xi_d \) parameterizes the \( G_d \)-orbits in \( X_d \times X_d \). Let \( \eta_M \) be the characteristic function of the \( G_d \)-orbit in \( X_d \times X_d \) indexed by \( M \), for any \( M \in \Xi_d \). Let
\[
\chi : S_n \to A
\]
be the algebra homomorphism defined by $\chi(\eta_M) = \det(M)$, for all $M \in \Xi_n$. (Here $d$ is taken to be $n$.) Let $\xi : S_{d-n} \to S_{d-n}$ be the $A$-algebra isomorphism defined by

$$\xi(f)(V, V') = v^{-\sum_{i=1}^n (|V_i| - |V'_i|)} f(V, V'), \quad \forall f \in S_{d-n}, \; V, V' \in X_d.$$  

The transfer map

$$\phi_{d,d-n,v} : S_d \to S_{d-n}, \; \forall d \geq n,$$

is defined to be the composition $S_d \xrightarrow{\Delta_v} S_{d-n} \otimes S_n \xrightarrow{\xi \otimes X} S_{d-n} \otimes A = S_{d-n}$. The following lemma is quoted from [L00 Lemma 1.10].

**Lemma 1.4.1.** For any $i \in [1, n-1]$, we have $\phi_{d,d-n,v}(E_i) = E'_i$, $\phi_{d,d-n,v}(F_i) = F'_i$ and $\phi_{d,d-n,v}(K_i^{\pm 1}) = K_i^{\prime \pm 1}$.

### 1.5. Generic version.

Recall from [BLM90] that one can construct an associative algebra $S_d$ over $A = \mathbb{Z}[v, v^{-1}]$ such that

$$S_d = A \otimes_A S_d,$$

where $A$ is regarded as an $A$-module with $v$ acting as $v$. More precisely, $S_d$ is a free $A$-module spanned by the symbols $\zeta_M$ for $M \in \Xi_d$ in Section 1.4. The multiplication on $S_d$ is defined so that if $\zeta_{M_1} \zeta_{M_2} = \sum_{M \in \Xi_d} c^{M}_{M_1,M_2}(v) \zeta_M$, $c^{M}_{M_1,M_2}(v) \in A$, then $\eta_{M_1} \eta_{M_2} = \sum_{M \in \Xi_d} c^{M}_{M_1,M_2}(v)|_{v=v} \eta_M$, in $S_d$. In particular, $S_d$ has analogous elements of $E_n$, $F_i$ and $K_i^{\pm 1}$, which we will use the same notations to denote them. For a matrix $M = (m_{ij}) \in \Xi_d$, we set

$$\text{ro}(M) = \left( \sum_{j=1}^n m_{ij} \right)_{1 \leq i \leq n} \quad \text{and} \quad \text{co}(M) = \left( \sum_{i=1}^n m_{ij} \right)_{1 \leq j \leq n}.$$  

Then we have a decomposition

$$S_d = \bigoplus_{b,a \in A_{d,n}} S_d(b, a), \quad S_d(b, a) = \text{span}_A \{ \zeta_M | \text{ro}(M) = b, \text{co}(M) = a \}.$$  

Note that $S_d(b, a)$ is nothing but the generic version of $S_d(b, a)$.

By using the monomial basis in [BLM90 Theorem 3.10], one can show that $S_d$ enjoys the same results for $S_d$ from the previous sections. Let us state them for later usages. The following is the generic version of Proposition 1.3.3.

**Proposition 1.5.1.** Suppose that $d' + d'' = d$. There is a unique algebra homomorphism

$$\Delta : S_d \to S_{d'} \otimes S_{d''}$$

such that $A \otimes_A \Delta = \Delta_v$ and satisfies the condition [8].

The following is the generic version of Lemma 1.4.1, which is due to Lusztig [L00].

**Proposition 1.5.2.** There is a unique algebra homomorphism

$$\phi_{d,d-n} : S_d \to S_{d-n}$$

such that $A \otimes_A \phi_{d,d-n} = \phi_{d,d-n,v}$ and

$$\phi_{d,d-n}(E_i) = E'_i, \; \phi_{d,d-n}(F_i) = F'_i, \; \phi_{d,d-n}(K_i^{\pm 1}) = K_i^{\prime \pm 1}, \quad \forall 1 \leq i \leq n - 1,$$
Recall the canonical basis $B_d \equiv \{ \{ B \} \}_d | B \in \Xi_d \}$ of $S_d$ from [BLM90]. We can write
\[ \Delta_{b',a',b'',a''}(\{ B \}) = \sum_{B' \in \Xi_d, B'' \in \Xi_d' } c_{B, B'}^{B', B''} \{ B' \} \otimes \{ B'' \} , \quad c_{B, B'}^{B', B''} \in \mathbb{A}. \]
We have the following proposition.

**Proposition 1.5.3.** $c_{B, B'}^{B', B''} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

**Proof.** It suffices to show the generic version of $\tilde{\Delta}$ has a similar positivity. To establish the latter positivity, we switch from the finite field $\mathbb{F}_q$ to its algebraic closure $\overline{\mathbb{F}_q}$. Let $G_d$ be the general linear group over $\overline{\mathbb{F}_q}$ whose $\mathbb{F}_q$-points form $G_d$. Similarly, we define an algebraic variety $X_d(a)$ over $\overline{\mathbb{F}_q}$ for $X_d(A)$. We set $G_m = GL(1, \overline{\mathbb{F}_q})$. For $d' + d'' = d$, we fix an isomorphism $\overline{\mathbb{F}_q}^d \cong \overline{\mathbb{F}_q}^{d'} \oplus \overline{\mathbb{F}_q}^{d''}$. Via the isomorphism, we fix an imbedding $G_m \rightarrow G_d$ defined by $t \mapsto (1_{\overline{\mathbb{F}_q}^{d''}}, t1_{\overline{\mathbb{F}_q}^{d'}})$. Thus $G_m$ acts on $X_d(a)$ via the imbedding. It is straightforward to see that the fixed-point set of $G_m$ in $X_d(a)$ is $\cup_{\lambda + \nu = \omega} X_d'(a') \times X_d''(a'')$. Moreover, the attracting set of $X_d'(a') \times X_d''(a'')$, i.e., those points $x$ such that $\lim_{t \rightarrow \infty} t \cdot x \in X_d'(a') \times X_d''(a'')$, is exactly the algebraic variety whose $\mathbb{F}_q$-point is $X_{a', a'}^+$ in (10). Thus, the linear map $\pi_1 t^*$ in (11) is the function version of the hyperbolic localization functor attached to the data $(X_d(a), G_m)$ in [B03]. On the other hand, the function $i_{b, a}^* \{ A \}_d$ is nothing but the function version of the intersection cohomology complex attached to the $R_b$-orbit in $X_d(a)$ indexed by $A$. Now the result in [B03] says that a hyperbolic localization functor sends a simple perverse sheaf to a semisimple complex. Therefore, we have the positivity for the generic version of $\tilde{\Delta}$ and therefore the proposition.

1.6. **Positivity for $\hat{U}$**. By definition, the quantum $\mathfrak{sl}_n$, denoted by $U \equiv U(\mathfrak{sl}_n)$, is an associative algebra over $\mathbb{Q}(v)$ generated by the generators:
\[ E_i, F_i, K_i, K_i^{-1}, \quad \forall 1 \leq i \leq n - 1, \]
and subject to the following relations. For $1 \leq i, j \leq n - 1$,
\begin{align*}
K_i K_j^{-1} &= K_j K_i^{-1} = 1, \\
K_i K_j &= K_j K_i, \\
K_i E_j &= v^{2\delta_{i, j} - \delta_{i, j+1} - \delta_{i, j-1}} E_j K_i, \\
K_i F_j &= v^{-2\delta_{i, j} + \delta_{i, j+1} + \delta_{i, j-1}} F_j K_i,
\end{align*}
\begin{align*}
E_i F_j - F_j E_i &= \delta_{i, j} \frac{K_i - K_i^{-1}}{v - v^{-1}}, \\
E_i E_j + F_j E_i &= (v + v^{-1}) E_i E_j E_i, \quad \text{if } |i - j| = 1, \\
F_i F_j + F_j F_i &= (v + v^{-1}) F_i F_j F_i, \quad \text{if } |i - j| = 1, \\
E_i E_j &= E_j E_i, \quad \text{if } |i - j| \neq 1, \\
F_i F_j &= F_j F_i, \quad \text{if } |i - j| \neq 1.
\end{align*}
Moreover, $\mathcal{U}$ admits a Hopf-algebra structure, whose comultiplication is defined by
\[
\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \\
\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \\
\Delta(K_i) = K_i \otimes K_i, \quad \forall 1 \leq i \leq n - 1.
\]

(18)

**Remark 1.6.1.** If one rewrites $E_i, F_i,$ and $K_i$ as $E_i', F_i', K_i'$, respectively, then the resulting presentation is a subalgebra of the quantum $\mathfrak{gl}_n$ used in [BLW14, 4.3].

It is well-known from [BLM90] that the assignments
\[
E_i \mapsto E_i', F_i \mapsto F_i', K_{\pm i} \mapsto K_{\pm i}, \quad \forall 1 \leq i \leq n - 1,
\]
define a surjective algebra homomorphism
\[
\phi_d : \mathcal{U} \to q(v) S_d,
\]
where $q(v) S_d$ is the algebra obtained from $S_d$ in Section 1.5 by extending the ground ring $\mathbb{A}$ to $Q(v)$. By using Proposition 1.5.1 (18) and tracing along the generators, we obtain the following commutative diagram.
\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\Delta} & \mathcal{U} \otimes \mathcal{U} \\
\phi_d \downarrow & & \phi_d \otimes \phi_d' \downarrow \\
q(v) S_d & \xrightarrow{\Delta} & q(v) S_d' \otimes q(v) S_d''
\end{array}
\]

where $d' + d'' = d$ and $\Delta$ for $q(v) S_d$ is defined as in (15).

We define an equivalence relation $\sim$ on $\mathbb{Z}^n$ by $\mu \sim \nu$ if and only if $\mu - \nu = p(1, \cdots, 1)$ for some $p \in \mathbb{Z}$. Let
\[
\mathcal{X} = \mathbb{Z}^n / \sim,
\]
be the set of all equivalence classes. Let $\Xi$ denote the equivalence class of $\mu \in \mathbb{Z}^n$. Let
\[
\mathcal{Y} = \{\nu \in \mathbb{Z}^n | \sum_{1 \leq i \leq n} \nu_i = 0\}.
\]

Then the standard dot product on $\mathbb{Z}^n$ induces a pairing $\cdot : \mathcal{Y} \times \mathcal{X} \to \mathbb{Z}$. Set $I = \{1, \cdots, n - 1\}$. We define two injective maps $I \to \mathcal{Y}$, $I \to \mathcal{X}$, by $i \mapsto -s_i + s_{i+1}, i \mapsto -s_i + s_{i+1}, \quad \forall 1 \leq i \leq n - 1$, respectively, where $s_i$ is the $i$-th standard basis element in $\mathbb{Z}^n$. We thus obtain a root datum of type $A_{n-1}$ in [L93, 2.2]. It is both $\mathcal{X}$-regular and $\mathcal{Y}$-regular.

Recall from [L93, 23.1.1] that $\mathcal{U}$ admits a decomposition $\mathcal{U} = \bigoplus_{\nu \in \mathbb{Z}[I]} \mathcal{U}(\nu)$ defined by the following rules:
\[
\mathcal{U}(\nu') \cup \mathcal{U}(\nu'') \subseteq \mathcal{U}(\nu' + \nu''), \quad K_{\pm i} \in \mathcal{U}(0), \quad E_i \in \mathcal{U}(i), \quad F_i \in \mathcal{U}(-i).
\]

For a triple $\nu', \nu'', \nu$ in $\mathbb{Z}[I]$ such that $\nu' + \nu'' = \nu$, we can have a linear map
\[
\Delta_{\nu', \nu''} : \mathcal{U}(\nu) \to \mathcal{U}(\nu') \otimes \mathcal{U}(\nu''),
\]
on obtained from $\Delta$ by restricting to $\mathcal{U}(\nu)$ and projecting to $\mathcal{U}(\nu') \otimes \mathcal{U}(\nu'')$. Moreover, the restriction of the algebra homomorphism $\phi_d$ in (19) to $\mathcal{U}(\nu)$ induces a linear map, still denoted by $\phi_d$,
\[
\phi_d : \mathcal{U}(\nu) \to \bigoplus_{\mathcal{Y} - \mathcal{X} = \nu} q(v) S_d(b, a).
\]
where $\mathbb{Z}[I]$ is treated as a subset in $\mathcal{X}$ via the imbedding $I \to \mathcal{X}$. 

Lemma 1.6.2. The commutative diagram [20] can be refined to the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{U}(\nu) & \xrightarrow{\Delta_{\nu',\nu''}} & \mathbb{U}(\nu') \otimes \mathbb{U}(\nu'') \\
\phi_d \downarrow & & \downarrow \phi_d \otimes \phi_d'' \\
\oplus_{\mu_\pi' = \nu} q_{(\nu)} S_d(b, a) & \oplus \Delta_{b', a', b'', a''} & \oplus_{\mu_\pi' = \nu'} q_{(\nu')} S_{d'}(b', a') \otimes q_{(\nu'')} S_{d''}(b'', a'')
\end{array}
\]

where \( \Delta_{b', a', b'', a''} \) is defined similar to [7].

Now set

\[
\hat{\mathbb{U}} = \bigoplus_{\mu_\pi' = \nu} \mathbb{U}_{\mu_\pi'},
\]

\[
\hat{\mathbb{U}}_{\mu_\pi'} = \mathbb{U} / \left( \sum_{1 \leq i \leq n-1} (K_i - \nu^{-\mu_i + \mu_{i+1}}) \mathbb{U} + \sum_{1 \leq i \leq n-1} \mathbb{U} (K_i - \nu^{-\lambda_i + \lambda_{i+1}}) \right).
\]

This is the modified/idempotented form of \( \mathbb{U} \) defined in [L93 23.1.1], see also [BLM90]. Recall from [L93 23.1.5], the comultiplication \( \Delta \) induces a linear map

\[
\Delta_{\mu', \nu', \mu'', \nu''} : \mu' \mathbb{U}_{\mu_\pi'} \rightarrow \mu' \mathbb{U}_{\mu_\pi} \otimes \mu'' \mathbb{U}_{\nu_\pi},
\]

and makes the following diagram commutative.

\[
\begin{array}{ccc}
\mathbb{U}(\nu) & \xrightarrow{\Delta_{\nu',\nu''}} & \mathbb{U}(\nu') \otimes \mathbb{U}(\nu'') \\
\pi_{\mu_\pi, \pi} \downarrow & & \downarrow \pi_{\mu_\pi, \mu_\pi} \otimes \pi_{\nu_\pi, \nu_\pi} \\
\pi_{\mu_\pi'} \mathbb{U}_{\mu_\pi'} & \xrightarrow{\Delta_{\mu', \nu', \mu'', \nu''}} & \pi_{\mu_\pi'} \mathbb{U}_{\mu_\pi} \otimes \pi_{\nu_\pi} \mathbb{U}_{\nu_\pi}
\end{array}
\]

where \( \mu - \nu = \nu', \mu_\pi' = \nu', \mu'' - \nu'' = \nu'' \), and \( \pi_{\mu_\pi, \mu_\pi} \) is the projection from \( \mathbb{U} \) to \( \mu' \mathbb{U}_{\mu_\pi'}\).

We write \( 1_{\mu_\pi} = \pi_{\mu_\pi, \mu_\pi}(1) \). It is well-known that \( \mathbb{U} \) and \( q_{(\nu)} S_d \) are \( \mathbb{U} \)-bimodules. So the notations \( \mathbb{E}_{1_{\mu_\pi}} \) and \( \mathbb{F}_{1_{\nu_\pi}} \) in \( \hat{\mathbb{U}} \) are meaningful, and so are \( \mathbb{E}_{\zeta_M}, \mathbb{F}_{\zeta_M} \) in \( q_{(\nu)} S_d \) where the notation \( \zeta_M \) is from Section [L5]. Recall from [L00] (see also [LW15]) that there is a surjective algebra homomorphism \( \tilde{\phi}_d : \hat{\mathbb{U}} \rightarrow q_{(\nu)} S_d \) defined by

\[
\tilde{\phi}_d(1_{\mu_\pi}) = \begin{cases} 
\zeta_{M_a}, & \text{if } \mu_\pi' = \nu, \text{ for some } a \in \Lambda_{d,n}, \\
0, & \text{o.w.}
\end{cases}
\]

\[
\tilde{\phi}_d(\mathbb{E}_{1_{\mu_\pi}}) = \begin{cases} 
\mathbb{E} \zeta_{M_a}, & \text{if } \mu_\pi = \nu, \text{ for some } a \in \Lambda_{d,n}, \\
0, & \text{o.w.}
\end{cases}
\]

\[
\tilde{\phi}_d(\mathbb{F}_{1_{\nu_\pi}}) = \begin{cases} 
\mathbb{F} \zeta_{M_a}, & \text{if } \mu_\pi = \nu, \text{ for some } a \in \Lambda_{d,n}, \\
0, & \text{o.w.}
\end{cases}
\]

where \( M_a \) is the diagonal matrix whose diagonal is \( a \). Moreover, \( \tilde{\phi}_d \) induces a linear map, still denoted by \( \tilde{\phi}_d \),

\[
\tilde{\phi}_d : \mathbb{U}_{\mu_\pi} \rightarrow q_{(\nu)} S_d(b, a).
\]

By definition, we have the following lemma.
Lemma 1.6.3. If $\overline{\mu} = \overline{b}$, $\overline{\lambda} = \overline{a}$ and $\overline{\mu} - \overline{\lambda} = \nu$, then the following diagram is commutative.

\[
\begin{array}{ccc}
\bigtriangledown(\nu) & \longrightarrow & \pi \bigtriangledown \chi \\
\phi_d & \downarrow & \phi_d \\
\oplus_{\overline{\nu} = \nu} q(\nu) S_d(b, a) & \longrightarrow & q(\nu) S_d(b, a)
\end{array}
\] (24)

where the bottom row is the natural projection.

Note that $\overline{b} - \overline{a} \in \mathbb{Z}[I] \subseteq \mathcal{X}$. By piecing together (21), (23) and (24), we have the following cube.

\[
\begin{array}{ccc}
\bigtriangledown(\nu) & \longrightarrow & \bigtriangledown(\nu') \otimes \bigtriangledown(\nu'') \\
\phi_d & \downarrow & \phi_d \\
S_d(b, a) & \longrightarrow & S_d(b', a') \otimes S_d(b'', a'')
\end{array}
\] (25)

where each of the $S$ in the bottom square has a subscript $Q(\nu)$ on the left. In (25), the front square is (21), the top square is (23), the two side squares are (24) and the commutativity of the bottom square is obvious. Since $\pi_{\nu, \chi}$ is surjective and each square is commutative except the one in the back, we immediately have the following proposition by diagram-chasing.

Proposition 1.6.4. The square in the back of the cube (25) is commutative.

\[
\begin{array}{ccc}
\bigtriangledown \chi & \longrightarrow & \bigtriangledown(\nu') \otimes \bigtriangledown(\nu'') \\
\phi_d & \downarrow & \phi_d \\
\oplus S_d(b, a) & \longrightarrow & \oplus S_d(b', a') \otimes S_d(b'', a'')
\end{array}
\] (26)

By using Proposition 1.6.4, we can prove the following positivity of the canonical basis of $\bigtriangledown = \bigtriangledown(st_\mu)$ with respect to the comultiplication. Let $\mathbb{B}$ be the canonical basis of $\bigtriangledown$ defined in [L93, 25.2.4].

Theorem 1.6.5 (Grojnowski). Let $b \in \mathbb{B}$. If $\Delta_{\nu'} \cdot \Delta_{\nu''} \cdot \Delta_{\mu'} \cdot \Delta_{\mu''}(b) = \sum_{b', b''} \hat{m}_{b'}^{b''} b' \otimes b''$, then $\hat{m}_{b'}^{b''} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

Proof. Let $\mathcal{I} = \{(b', b'')| \hat{m}_{b'}^{b''} \neq 0\}$. Clearly, $\# \mathcal{I} < \infty$. By [M10 Proposition 7.8], we can find $d$, $d'$ and $d''$ large enough such that

\[
\tilde{\phi}_d(b) = \{B\}_d, \quad \tilde{\phi}_d(b') = \{B'\}_d', \quad \tilde{\phi}_d(b'') = \{B''\}_d'', \quad \forall (b', b'') \in \mathcal{I},
\]

where $\{B\}_d$, $\{B'\}_d'$ and $\{B''\}_d''$ are certain canonical basis elements in $S_d$, $S_{d'}$ and $S_{d''}$, respectively. Then by (26), we have

\[
(\tilde{\phi}_d \otimes \tilde{\phi}_d') \Delta_{\nu'} \cdot \Delta_{\nu''}(b) = \sum_{(b', b'') \in \mathcal{I}} \hat{m}_{b'}^{b''} \{B'\}_d' \otimes \{B''\}_d'' = \Delta_{\nu'} \cdot \Delta_{\nu''} \cdot \Delta_{\mu'} \cdot \Delta_{\mu''}(\{B\}_d).
\] (27)

By comparing (16) with (27), $\hat{m}_{b'}^{b''} = c_{B'}^{B''}$ and hence the theorem by Proposition 1.5.3. □
Remark 1.6.6. Theorem [1.6.5] is conjectured in [L03, Conjecture 25.4.2] for all symmetric Cartan data.

2. Positivity for quantum affine $\mathfrak{sl}_n$

In this section, we shall lift the positivity result on quantum $\mathfrak{sl}_n$ in the previous section to its affine analogue. We provide a new proof of the multiplication formula of Du-Fu [DF13].

2.1. Results from [L00]. Similar to the previous section, we fix a pair $(d, n)$ of non-negative integers. We set

$$\hat{\Lambda}_{d,n} = \{ \lambda = (\lambda_i)_{i \in \mathbb{Z}} \in \mathbb{Z}^\mathbb{Z}_{\geq 0} | \lambda_i = \lambda_{i+n}, \forall i \in \mathbb{Z}; \sum_{1 \leq i \leq n} \lambda_i = d \}.$$

Let $\hat{\Xi}_d$ be the set of all $\mathbb{Z} \times \mathbb{Z}$ matrices $A = (a_{ij})_{i,j \in \mathbb{Z}}$ such that $a_{ij} \in \mathbb{Z}_{\geq 0}$, $a_{ij} = a_{i+n,j+n}$, and $\sum_{1 \leq i \leq n;j \in \mathbb{Z}} a_{ij} = d$. To each matrix $A \in \hat{\Xi}_d$, we can associate $r(A)$ and $c(A)$ in $\hat{\Lambda}_{d,n}$ by $r(A)_i = \sum_{j \in \mathbb{Z}} a_{ij}$ and $c(A)_j = \sum_{i \in \mathbb{Z}} a_{ij}$ for all $i, j \in \mathbb{Z}$.

We need to switch the ground field from $\mathbb{F}_q$ to the local field $\mathbb{F}_q((\varepsilon))$. Let $\mathbb{F}_q[[\varepsilon]]$ be the subring of $\mathbb{F}_q((\varepsilon))$ of all formal power series over $\mathbb{F}_q$. Suppose that $V$ is a $d$-dimensional vector space over $\mathbb{F}_q((\varepsilon))$. A free $\mathbb{F}_q[[\varepsilon]]$-module $L$ in $V$ is called a lattice if $\mathbb{F}_q((\varepsilon)) \otimes_{\mathbb{F}_q[[\varepsilon]]} L = V$. A lattice chain $L = (L_i)_{i \in \mathbb{Z}}$ of period $n$ is a sequence of lattices $L_i$ in $V$ such that $L_i \subseteq L_{i+1}$ and $L_i = qL_{i+n}$ for all $i \in \mathbb{Z}$. Let $\hat{X}_d$ be the collection of all lattice chains in $V$. Let $\hat{G}_d = GL(V)$ act from the left on $\hat{X}_d$ in the canonical way. Then we can form the algebra

$$\hat{S}_d = A\hat{G}_d(\hat{X}_d \times \hat{X}_d),$$

which is the so-called affine $v$-Schur algebra. It is well-known that the $\hat{G}_d$-orbits in $\hat{X}_d \times \hat{X}_d$ are parameterized by $\hat{\Xi}_d$ via the assignment $(L, L') \mapsto A$, where $a_{ij} = \left| \frac{L_i \cap L'_j}{L_i \cap L_j + L_i \cap L_{j-1}} \right|$ for all $i, j \in \mathbb{Z}$. So we have

$$\hat{S}_d = \text{span}_A(e_A | A \in \hat{\Xi}_d),$$

where $e_A$ is the characteristic function of the $\hat{G}_d$-orbit indexed by $A$. Furthermore, we have $\hat{S}_d = \bigoplus_{b,a \in \hat{\Lambda}_d,} \hat{S}_d(b, a)$ where $\hat{S}_d(b, a)$ is spanned by $e_A$ such that $r(A) = b$ and $c(A) = a$.

If one lifts the functions to the sheaf level, one gets the generic version $\hat{S}_d$ of $\hat{S}_d$ such that $A \otimes A \hat{S}_d = \hat{S}_d$. By abuse of notation, we write $e_A$ for the unique function $x$ in $\hat{S}_d$ such that $A \otimes A x = e_A$ (for all $q$).

The standard basis of $\hat{S}_d$ consists of elements $[A] = v^{-dA}e_A$ where $d_A = \sum_{1 \leq i \leq n} a_{ij} a_{kl}$. Recall the Bruhat order $\preceq$ on $\hat{\Xi}_d$ from [L00]: $A \preceq B$ if and only if

$$\sum_{i \leq r, j \leq s} a_{rs} \leq \sum_{i \leq r, j \leq s} b_{rs}, \quad \forall i < j \in \mathbb{Z}; \quad \sum_{i \leq r, j \geq s} a_{rs} \leq \sum_{i \leq r, j \geq s} b_{rs}, \quad \forall i > j \in \mathbb{Z}.$$

Following [L00], one can associate a bar involution $\bar{\cdot} : \hat{S}_d \to \hat{S}_d$ such that $\overline{[A]} = [A] + \sum_{A' \preceq A, A' \neq A} C_{A,A'} [A']$ where $C_{A,A'} \in \mathbb{A}$.

The canonical basis $\{ \{A\}_d \}$ for all $A \in \hat{\Xi}_d$ of $\hat{S}_d$ is defined by the properties that $\overline{\{A\}_d} = \{A\}_d$ and $\{A\}_d = [A] + \sum_{A' \preceq A, A' \neq A} P_{A,A'} [A']$ where $P_{A,A'} \in v^{-1} \mathbb{Z}[v^{-1}]$. 

Let $E_{ij}$ be the $\mathbb{Z} \times \mathbb{Z}$ matrix whose $(k, l)$-th entry is 1 if $(k, l) = (i, j) \mod n$, and zero otherwise. For any $i \in \mathbb{Z}$, we define the following elements in $\widehat{S}_d$:

$$E_i = \sum_{A \in E^{i+1}, i \text{ diagonal}} [A], \quad F_i = \sum_{A \in E^{i+1}, i \text{ diagonal}} [A],$$

$$H_i^{\pm 1} = \sum_{A \text{ diagonal}} v^{\pm c(A)} [A], \quad K_i^{\pm 1} = H_i^{\pm 1} H_i^\mp 1, \quad \forall i \in \mathbb{Z}.$$ 

By periodicity, we have $E_i = E_{i+n}, F_i = F_{i+n}, H_i^{\pm 1} = H_{i+n}^{\pm 1}$, and $K_i^{\pm 1} = K_{i+n}^{\pm 1}$, for all $i \in \mathbb{Z}$. The following lemma is from [L00].

**Lemma 2.1.1.** There is an algebra homomorphism $\tilde{\Delta}: \widehat{S}_d \to \widehat{S}_{d'} \otimes \widehat{S}_{d''}$ for $d' + d'' = d$ with

$$\tilde{\Delta}(E_i) = E_i' \otimes H_{i+1}'' + H_i'' \otimes E_i',$$

$$\tilde{\Delta}(F_i) = F_i' \otimes H_{i+1}'' - H_i'' \otimes F_i',$$

$$\tilde{\Delta}(K_i) = K_i' \otimes K_i''', \quad \forall i \in \mathbb{Z}.$$ 

### 2.2. The coproduct $\Delta$

Recall the algebra homomorphism from Lemma 2.1.1. If $b = b' + b''$ and $a = a' + a''$, let $\Delta_{b', a', b'', a'': \widehat{S}_d} : \widehat{S}_d(b, a) \to \widehat{S}_{d'}(b', a') \otimes \widehat{S}_{d''}(b'', a'')$ be the composition of the restriction of $\widehat{S}_d$ to $\widehat{S}_d(b, a)$ and the projection to the component $\widehat{S}_{d'}(b', a') \otimes \widehat{S}_{d''}(b'', a'')$. We set

$$\Delta^\dagger_{b', a', b'', a''} = \sum_{1 \leq i \leq s} b_i b_i' - a_i a_i' \Delta_{b', a', b'', a''}, \quad \Delta^\dagger = \Delta^\dagger_{b', a', b'', a''},$$

where the sum runs over all quadruples $(b', a', b'', a'')$ where $b', a' \in \widehat{S}_{d', n}$ and $b'', a'' \in \widehat{S}_{d''}$. 

**Proposition 2.2.1.** The linear map $\Delta^\dagger$ in (29) is an algebra homomorphism. Moreover,

$$\Delta^\dagger(E_i) = v^{\delta_i, -d'} E_i' \otimes K_i'', + 1 \otimes v^{-\delta_i, -d'} E_i'',$$

$$\Delta^\dagger(F_i) = v^{-\delta_i, -d'} F_i' \otimes 1 + K_i' \otimes v^{\delta_i, -d'} F_i'',$$

$$\Delta^\dagger(K_i) = K_i' \otimes K_i''', \quad \forall i \in [1, n].$$

**Proof.** We now prove the case when $i = n$. Suppose that $b'' = a''$, and $(b', a')$ is chosen such that $b_i = a_i' - \delta_{i,n} + \delta_{i,1}$ for all $1 \leq i \leq n$. Then the twist $\sum_{1 \leq i \leq n} b_i' b_i'' - a_i' a_i''$ is equal to $d'' - a_i''$. This implies that the term $E_i' \otimes H_{i+1}''$ in $\Delta(E_i)$ becomes $v^{\delta_i, -d'} E_i' \otimes K_i''$. For the term $H_i'^{\dagger-1} \otimes E_i''$ in $\Delta(E_i)$, the twist contributes $a_i' - d''$ for a quadruple $(b', a', b'', a'')$ such that $b_i = a_i'$ and $b_i'' = a_i' - \delta_{i,n} + \delta_{i,1}$ for all $1 \leq i \leq n$. The formula for $\Delta^\dagger(E_i)$ is proved.

The proof for the formula $\Delta^\dagger(F_i)$ is entirely similar. The formula for $\Delta^\dagger(K_i)$ is obvious.\hfill $\square$

We set

$$\varepsilon(A) = \sum_{r \leq i < s} a_{r,s} - \sum_{r > i \geq s} a_{r,s}, \quad \forall i \in \mathbb{Z}, A \in \widehat{S}_d.$$ 

We define a linear map

$$\xi_{d,c} : \widehat{S}_d \to \widehat{S}_d, \quad \forall i, c \in \mathbb{Z},$$

such that

$$\xi_{d,c} : \widehat{S}_d \to \widehat{S}_d, \quad \forall i, c \in \mathbb{Z},$$
by \( \xi_{d,i,c}(\{A\}) = v^{e_i(A)}\{A\} \). By [L00], \( \xi_{d,i,c} \) is an algebra isomorphism with inverse \( \xi_{d,i,-c} \). Set (32) 
\[
\Delta : \widehat{S}_d \rightarrow \widehat{S}_{d'} \otimes \widehat{S}_{d''}
\]

\[\xi_d \rightarrow \widehat{S}_d \otimes \widehat{S}_{d'} \rightarrow \widehat{S}_{d'} \otimes \widehat{S}_{d''} \rightarrow \widehat{S}_{d''}.\]

**Proposition 2.2.2.** The linear map \( \Delta \) in (32) is an algebra homomorphism. Moreover, 
\[
\Delta(E_i) = E'_i \otimes K''_i + 1 \otimes E''_i, \\
\Delta(F_i) = F'_i \otimes 1 + K''_i \otimes F''_i, \\
\Delta(K_i) = K'_i \otimes K''_i, \quad \forall i \in \mathbb{Z}.
\]

**Proof.** We have \( \xi_{d,n,c}(E_i) = v^{-c_d,n}E_i, \) \( \xi_{d,n,c}(F_i) = v^{c_d,i}F_i \) and \( \xi_{d,n,c}(K_i) = K_i \). Proposition follows from these computations and the formulas in Proposition [2.2.1].

2.3. The compatibility of \( \xi_{d,i,c} \) and the canonical basis. We have 

**Theorem 2.3.1.** \( \xi_{d,i,c}(\{A\}) = v^{e_i(A)}\{A\} \) where \( \xi_{d,i,c} \) is in (31).

Theorem 2.3.1 follows from the following critical observation.

**Theorem 2.3.2.** Write \( \{A\}_d = \sum_{A' \leq A} P_{A,A'}[A] \). If \( P_{A,A'} \neq 0 \), then \( \varepsilon_i(A) = \varepsilon_i(A') \) for all \( i \).

We make two remarks before we prove Theorem 2.3.2.

**Remark 2.3.3.** The algebra isomorphism \( \prod_{1 \leq i \leq n} \xi_{d,i,-1} \) is the linear map \( \xi \) in [L00, 1.7]. In view of Theorem 2.3.1 we have \( \xi(\{\{A\}\}) = v^{-\sum_{1 \leq i \leq n} \varepsilon_i(A)}\{\{A\}\}_d \).

**Remark 2.3.4.** Even if \( A' \prec A, \varepsilon_i(A') \) may not be the same as \( \varepsilon_i(A) \). For example, take \( A' = 2 \sum_{1 \leq i \leq n} E_i + E_1 + E_{n+1} \) and \( A = \sum_{1 \leq i \leq n} E_i + E_{n+1} \). Then we have \( A' \prec A, \varepsilon_i(A') = 0 \) and \( \varepsilon_i(A) = 1 \) for all \( 1 \leq i \leq n \).

The remaining part of this section is devoted to the proof of Theorem 2.3.2. The main ingredient is a connection between the numerical data \( \varepsilon_i(A) \) in (30) and the multiplication formulas in [DF13], which we shall recall and provide a new proof. Before we state the formula, we need to recall a lemma from [SO6, Section 2.2] as follows.

**Lemma 2.3.5.** Let \( V \) be a finite dimensional vector space over \( \mathbb{F}_q \). Fix a flag \( (V_i)_{1 \leq i \leq n} \) in \( V \) such that \( |V_i/V_{i-1}| = l_i \) for all \( 1 \leq i \leq n \). The number of subspaces \( W \subset V \) such that \( |W \cap V_i| = \sum_{1 \leq i \leq n} a_j \) for all \( 1 \leq i \leq n \) is given by \( q^{-\sum_{n \geq i > j \geq 1} a_i(l_j-a_j)} \prod_{i=1}^{n} \left[ \frac{l_i}{a_i} \right] \) where \( \left[ \frac{l_i}{a_i} \right] = \prod_{1 \leq j \leq a_i} \frac{q^{d_{i+1}-1}-1}{q^{d_i}-1} \).

The following multiplication formula in \( \widehat{S}_d \) is first obtained in [DF13]. To a matrix \( T = (t_{ij})_{i,j \in \mathbb{Z}} \), we set \( \hat{T} = (\hat{t}_{ij})_{i,j \in \mathbb{Z}} \) where \( \hat{t}_{ij} = t_{i-1,j} \).

**Proposition 2.3.6.** (1) Suppose that \( A = (a_{ij}), B = (b_{ij}) \in \widehat{S}_d \) satisfy that \( c(B) = r(A) \) and \( B - \sum_{i=1}^{n} \alpha_i E_i^{i+1} \) is diagonal for some \( \alpha_i \in \mathbb{Z}_{\geq 0} \). We have 
\[
e_B * e_A = \sum_{T} q^{\sum_{1 \leq i \leq n, j > i} (a_{ij} - t_{i-1,j})} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ a_{ij} + t_{ij} - t_{i-1,j} \right] e_{A+T-\hat{T}},
\]
where the sum runs over all \( T = (t_{ij}) \) such that \( t_{i+n,j+n} = t_{ij} \) and \( r(T)_i = \alpha_i \) for all \( 1 \leq i \leq n \).

(2) If \( C \in \hat{\mathbb{C}}_d \) satisfies that \( c(C) = r(A) \) and \( C - \sum_i^{n} \beta_i E^{i+1,i} \) is diagonal, then

\[
e_{C} * e_A = \sum_{T} q^{\sum_{1 \leq i \leq n, j < i} (a_{ij} - t_{ij}) t_{i-1,j}} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ a_{ij} - t_{ij} + t_{i-1,j} \right] e_{A-T+T},
\]

where the sum runs over all \( T \) such that \( t_{ij} = t_{i+n,j+n} \) and \( r(T)_i = \beta_i \) for all \( 1 \leq i \leq n \).

Proof. (1) It suffices to show the similar statement in \( \hat{\mathbb{C}}_d \). Let \( A' = (a'_{ij})_{i,j \in \mathbb{Z}} \) be a matrix in \( \hat{\mathbb{C}}_d \) such that \( r(B) = r(A') \) and \( c(A) = c(A') \). Let \( \mathcal{O}_A \) be the \( \hat{\mathbb{C}}_d \)-orbit in \( \hat{\mathbb{C}}_d \times \hat{\mathbb{C}}_d \) indexed by \( A' \). Fix \( (L, L') \in \mathcal{O}_{A'} \), and we denote

\[ Z = \{ L'' \in \hat{\mathbb{C}}_d \mid L_{i-1} \subset L''_i \subset L_i, \forall 1 \leq i \leq n \}. \]

Note that \((L, L'') \in \mathcal{O}_B \) if and only if \( L'' \in Z \). Clearly, \( Z \) has a partition \( Z = \bigcup_T Z_T \) where

\[ Z_T = \{ L'' \in Z \mid L''_i \cap (L_{i-1} + (L_i \cap L'_j)) / (L''_i \cap (L_{i-1} + (L_i \cap L'_j))) \} = a'_{ij} - t_{ij}, \forall i, j \in \mathbb{Z} \}, \]

and the union runs over all \( T \) such that \( t_{i+n,j+n} = t_{ij} \) and \( r(T)_i = \alpha_i \) for all \( 1 \leq i \leq n \). For each \( L'' \in Z_T \), we have the following identities:

\[
\begin{align*}
    a_{ij} &= |L''_i \cap L'_j / L''_i \cap L'_{j-1}| - |L''_{i-1} \cap L'_j / L''_{i-1} \cap L'_{j-1}|, \\
    a'_{ij} &= |L_i \cap L'_j / L_i \cap L'_{j-1}| - |L_{i-1} \cap L'_j / L_{i-1} \cap L'_{j-1}|, \\
    a'_{ij} - t_{ij} &= |L''_i \cap L'_j / L''_i \cap L'_{j-1}| - |L_{i-1} \cap L'_j / L_{i-1} \cap L'_{j-1}|,
\end{align*}
\]

(33)

where the last identity follows from the definition of \( Z_T \). By (33), we have

\[ t_{ij} = |L_i \cap L'_j / L_i \cap L'_{j-1}| - |L''_i \cap L'_j / L''_i \cap L'_{j-1}|. \]

Thus \( a'_{ij} - t_{ij} = a_{ij} - t_{i-1,j} \), i.e., \( A' = A + T - \check{T} \). Summing up the above analysis, we have

\[
e_{B} * e_A(L, L') = \sum_{L'' \in \hat{\mathbb{C}}_d} e_{B}(L, L'') e_{A}(L'', L') = \sum_{L'' \in Z} e_{A}(L'', L')
\]

(34)

\[ = \sum_{T} \sum_{L'' \in Z_T} e_{A}(L'', L') = \sum_{T} \#Z_T e_{A+T-\check{T}}(L, L'). \]

So it is reduced to compute the cardinality of \( Z_T \). For each \( i \in [1, n] \), we set \( Z(i) = \{ k \in Z \mid k \in [1, i] \mod n \} \) and we define \( Z_T^{[1,i]} \) to be the set of all lattice chains \( L'' = (L''_k)_{k \in Z(i)} \) such that \( L''_k \) satisfies \( |L_{k-1} + L''_k \cap L'_j / L_{k-1} \cap L'_{j-1}| = a'_{ij} - t_{ij} \) for all \( j \in \mathbb{Z} \). Consider

\[ Z_T = Z_T^{[1,i]} \xrightarrow{\pi_n} Z_T^{[1,i-1]} \xrightarrow{\pi_{n-1}} \ldots \xrightarrow{\pi_2} Z_T^{[1,1]} \xrightarrow{\pi_1} \bullet, \]

where \( \pi_i((L''_k)_{k \in Z(i)}) = (L''_k)_{k \in Z(i-1)} \) and the equality is due to \( L''_k \cap (L_{i-1} + (L_i \cap L'_j)) = L_{i-1} \cap L''_i \cap L'_j \) \( \). We observe that the fiber of \( \pi_i \) gets identified with the set of subspaces \( W \) in \( L_i / L_{i-1} \) such that \( |W \cap (L_{i-1} + L_i \cap L'_j) / L_{i-1} \cap (L_{i-1} + L_i \cap L'_{j-1}) / L_{i-1}| = a'_{ij} - t_{ij} \). Observe that \( |(L_{i-1} + L_i \cap L'_j) / L_{i-1}| = |(L_{i-1} + L_i \cap L'_{j-1}) / L_{i-1}| = a'_{ij}, \) and by applying Lemma 2.3.5 we have \( \#\pi^{-1}_i(L^i) = q^{\sum_i (a'_{ij} - t_{ij}) t_{ij}} \prod_{j \in \mathbb{Z}} [a'_{ij} / t_{ij}] \), where \( L^i \) is any element in
positivity under comultiplication

$$Z^{[1,i-1]}_T.$$ So \(\pi_i\) is surjective with constant fiber. Hence

$$(35) \quad \#Z_T = \prod_{1 \leq i \leq n} \#\pi_i^{-1}(L_i) = q^{\sum_{1 \leq i \leq n, 1 \leq j} (a_{ij} - t_{ij}) t_{ij}} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ a'_{ij} \right].$$

The statement (1) follows from (34) and (35).

Let us prove (2). Let \(A'\) be a matrix such that \(r(A') = r(C)\) and \(c(A') = c(A).\) Fix \((L, L') \in \mathcal{O}_A.\) We consider the set \(Y = \{L' \cap L_i, 1 \leq i \leq n \}\). Then \(Y\) admits a partition \(Y = \bigcup Y_T,\) where

$$Y_T = \{L'' \in Y | L_{i-1} + L'' \cap L'_j / L_{i-1} + L'' \cap L'_{j-1} = |t_{i-1,j}, \forall i, j \in \mathbb{Z}\}.$$ 

By applying Lemma 2.3.5 and arguing similar to (1), we have

$$\#Y_T = \prod_{1 \leq i \leq n} q^{\sum_{1 \leq j} \sum_{t_{i-1,j}} (a_{ij} - t_{ij})} \prod_{j \in \mathbb{Z}} \left[ a'_{ij} \right].$$

Moreover, for \(L'' \in Y_T\) such that \((L'', L') \in \mathcal{O}_A\) if and only if \(A' = A - T + \hat{T}\) Therefore, we have (2). The proposition is thus proved.

Remark 2.3.7. If \(n = 1\), Proposition 2.3.6 shows that \(e_B * e_A = e_A * e_B.\) (Here we use \((e_A * e_B)' = e_B'e_A'.\)) This implies that \(\mathbb{S}_d\) is commutative for \(n = 1\), which corresponds to the geometric Satake of type A.

Lemma 2.3.8. Suppose that \([B] * [A] = \sum Q_{B,A}^C [C].\) If \(B = \sum_{1 \leq j \leq n} \beta_j E^{j,j} + \alpha_j E^{j,j+1}\) or \(\sum_{1 \leq j \leq n} \beta_j E^{j,j} + \alpha_j E^{j+1,j},\) and \(Q_{B,A}^C \neq 0\), then \(\varepsilon_i(A) + \varepsilon_i(B) = \varepsilon_i(C)\) for all \(i \in \mathbb{Z}.

Proof. Assume that \(B = \sum_{1 \leq j \leq n} \beta_j E^{j,j} + \alpha_j E^{j,j+1}\). Then we have \(\varepsilon_i(B) = \alpha_i.\) If \(Q_{B,A}^C \neq 0\), then by Proposition 2.3.6 (1), the matrix C is of the form \(A - T + \hat{T}.\) Thus we have

$$\varepsilon_i(C) = \varepsilon_i(A) + \varepsilon_i(T - \hat{T}) = \varepsilon_i(A) + \sum_{r \leq i,s} t_{r,s} - \hat{t}_{r,s} + \sum_{r > i,s} t_{r,s} - \hat{t}_{r,s} = \varepsilon_i(A) + \sum_{r \leq i,s} \hat{t}_{r,s} - \varepsilon_i(A) + \alpha_i = \varepsilon_i(A) + \varepsilon_i(B).$$

Therefore, the lemma holds for \(B = \sum_{1 \leq j \leq n} \beta_j E^{j,j} + \alpha_j E^{j,j+1}\).

For the case when \(B = \sum_{1 \leq j \leq n} \beta_j E^{j,j} + \alpha_j E^{j+1,j},\) then \(\varepsilon_i(B) = -\alpha_i\) and C is of the form \(A - T + \hat{T}\) if \(Q_{B,A}^C \neq 0\) by Proposition 2.3.6 (2). So we have \(\varepsilon_i(C) = \varepsilon_i(A) - \varepsilon_i(T - \hat{T}) = \varepsilon_i(A) - \alpha_i = \varepsilon_i(A) + \varepsilon_i(B).\) Therefore the lemma holds in this case. We are done.

Next we introduce a second numerical data. We define

$$\deg_i \left( \sum_{1 \leq j \leq n} \beta_j E^{j,j} + \alpha_j E^{j,j+1} \right) = -\alpha_i, \quad \deg_i \left( \sum_{1 \leq j \leq n} \beta_j E^{j,j} + \alpha_j E^{j,j+1} \right) = \alpha_i.$$

Suppose that \(M = [A_1] * \cdots * [A_m]\) is a monomial in \([A_j]\) where \(A_j\) is either \(\sum_{1 \leq k \leq n} \beta_{jk} E^{k,k} + \alpha_{jk} E^{k,k+1,1,k},\) or \(\sum_{1 \leq k \leq n} \beta_{jk} E^{k,k} + \alpha_{jk} E^{k,k+1}.\) We define \(\deg_i(M) = \sum_{1 \leq j \leq m} \deg_i(A_j).\) To the same monomial \(M,\) we also define its length \(\ell(M)\) to be \(\ell(M) = \sum_{1 \leq j \leq n} \deg_i(A_j).\) (We define \([A]\) to be a monomial of length zero if \(A\) is diagonal.) Then we have

\(\sum_{1 \leq i \leq n} \beta_{ij} E^{i,j} + \alpha_{ij} E^{i,j+1}\)
Lemma 2.3.9. Let \( M \) be a monomial and write \( M = \sum R_A[A] \). If \( R_A \neq 0 \), then \( \deg_i(M) = \varepsilon_i(A) \) for all \( i \in \mathbb{Z} \).

Proof. We argue by induction on the length \( \ell(M) \) of \( M \). When \( \ell(M) = 1 \), the lemma follows from the definitions. Assume now that \( \ell(M) > 1 \) and the lemma holds for any monomial \( M' \) such that \( \ell(M') < \ell(M) \). We write \( M = [A_1] \ast [M'] \), where \( A \) is either \( \sum_{1 \leq j \leq n} \beta_j E^{i,j} + \alpha_j E^{i,j+1} \) or \( \sum_{1 \leq j \leq n} \beta_j E^{i,j} + \alpha_j E^{i,j+1} \), and \( M' \) is a monomial of the remaining terms in \( M \). Thus \( \ell(M') < \ell(M) \). Suppose that \( M' = \sum R'_{A'}[A'] \), then we have

\[
M = [A_1] \ast M' = \sum_{A'} R'_{A'}[A_1] \ast [A'] = \sum_{A', B} R'_{A'} Q_{A_1, A'}^B[B].
\]

If \( A_1 = \sum_{1 \leq j \leq n} \beta_j E^{i,j} + \alpha_j E^{i,j+1} \), then by Lemma 2.3.8 and induction hypothesis, we have

\[
\deg_i(M) = -\alpha_i + \deg_i(M') = -\alpha_i + \varepsilon_i(A') = \varepsilon_i(B), \quad \text{if } Q_{A_1, A'}^B \neq 0, R_{A'} \neq 0.
\]

Similarly, if \( A_1 = \sum_{1 \leq j \leq n} \beta_j E^{i,j} + \alpha_j E^{i,j+1} \), then

\[
\deg_i(M) = \alpha_i + \deg_i(M') = \alpha_i + \varepsilon_i(A') = \varepsilon_i(B), \quad \text{if } Q_{A_1, A'}^B \neq 0, R_{A'} \neq 0.
\]

Lemma follows.

By a result in [DF13] (see also [LL]), there exists a monomial \( M_A \) such that

\[
M_A = [A] + \sum_{A' \prec A} S_{A,A'}[A'], \quad \text{for some } S_{A,A'} \in \mathbb{Z}[v,v^{-1}].
\]

(36) Since \( [A] \) forms a basis for \( \widehat{S}_d \), the monomial \( M_A \) forms a basis for \( \widehat{S}_d \). In particular,

(37) \[
[A] = M_A + \sum_{A' \prec A} R_{A,A'} M_{A'}, \quad \text{for some } R_{A,A'} \in \mathbb{Z}[v,v^{-1}].
\]

Moreover, we have

Lemma 2.3.10. Suppose that \( R_{A,A'} \neq 0 \) in (36), then \( \varepsilon_i(A) = \varepsilon_i(A') \).

Proof. We prove by induction with respect to \( \preceq \) in descending order. If \( A' = A \), it is trivial. Suppose that for all \( A'' \) such that \( A' \prec A'' \preceq A \), the statement holds. If \( [A'] \) appears in \( M_{A'} \) for some \( A'' \) such that \( A' \prec A'' \), then \( \varepsilon_i(A) = \varepsilon_i(A'') = \deg_i(M_{A''}) = \varepsilon_i(A') \) by induction hypothesis. If \( [A'] \) does not appear in \( M_{A''} \) for all \( A'' \) such that \( A' \prec A'' \), then the coefficient of \( [A'] \) in the right-hand side of (37) is 0, contradicting to the assumption. We are done.

Furthermore, \n
Lemma 2.3.11. Suppose that \( [A] = [A] + \sum_{A' \prec A} C_{A,A'}[A'] \) for some \( C_{A,A'} \in \mathbb{Z}[v,v^{-1}] \). If \( C_{A,A'} \neq 0 \), then \( \varepsilon_i(A) = \varepsilon_i(A') \) for all \( i \in \mathbb{Z} \).

Proof. By (37) and Lemma 2.3.10 we have \( [A] = M_A + \sum_{A' \prec A, \varepsilon_i(A') = \varepsilon_i(A)} R_{A,A'} M_{A'} \). By Lemma 2.3.9, we have \( M_{A'} = \sum_{A'' \preceq A', \varepsilon_i(A'') = \varepsilon_i(A')} S'_{A', A''}[A''] \). The lemma follows by putting the previous two identities together.
Finally, we are ready to prove Theorem 2.3.2. We set \( \phi = \{ A' | P_{A,A'} \neq 0, \varepsilon_i(A') \neq \varepsilon_i(A) \} \). We only need to show that \( \phi = \emptyset \). Pick an element \( B \) in \( \phi \) that is maximal with respect to the partial order \( \preceq \). Clearly, we have \( B \neq A \). We rewrite \( \{ A \} \) as follows.

\[
\{ A \} = P_{A,B}[B] + \left( \sum_{B < A'} + \sum_{A' < B} + \sum_{A' \not\in B, B \not\in A'} \right) P_{A,A'}[A']
\]

Apply the bar operation to the above equality, we have

\[
\{ A \} = \{ A \} = \overline{P_{A,B}[B]} + \left( \sum_{B < A'} + \sum_{A' < B} + \sum_{A' \not\in B, B \not\in A'} \right) \overline{P_{A,A'}[A']}
\]

By Lemma 2.3.11, we know that the coefficient of \( [B] \) in \( \overline{A} \) for \( B < A' \) is zero. Notice \( [B] \) will not appear in the rest of the terms, except \( \overline{[B]} \). Hence, by comparing the coefficients of \( [B] \) in the previous two equalities, we must have \( P_{A,B} = \overline{P_{A,B}} \). But \( P_{A,B} \in v^{-1} \mathbb{Z}[v^{-1}] \) forces \( P_{A,B} = 0 \), a contradiction to the definition of \( \phi \). Hence \( \phi = \emptyset \). Theorem 2.3.2 follows.

2.4. Positivity of \( \wt{\Delta} \). We set \( \wt{X}_d(a) = \{ L \in \wt{X}_d|L_i/L_i-1| = a_i, \ \forall i \} \) and \( \wt{P}_a = \text{Stab}_{G_d}(L) \) for a fixed chain \( L \in \wt{X}_d \). We still have the same commutative diagram as in Lemma 1.3.5:

\[
\begin{array}{c}
\mathcal{A}_{G_d}(\wt{X}_d(b) \times \wt{X}_d(a)) \xrightarrow{i^{b,a}_{b,a}} \mathcal{A}_{\wt{R}_b}(\wt{X}_d(a)) \\
\end{array}
\]

So the positivity of \( \wt{\Delta}_{b',a',b'',a''} \) is reduced to the positivity of \( \pi_{i*} \).

Fix \( b', b'' \) such that \( b' + b'' = b \). Let \( d' = |b'| \) and \( d'' = |b''| \). Let \( V = T \oplus W \) and \( L_b = L_{b'} \oplus L_{b''} \). Thus we have \( \pi'(L_b) = L_{b'}, \pi''(L_b) = L_{b''} \). Let \( L_i \) be the \( i \)-th lattice in \( L_a \). We consider the following subset in \( \wt{X}_d(a) \).

\[
Y^{L_{0,p}}_a := \{ \wt{L} \in \wt{X}_d(a) | \varepsilon^p L_0 \subseteq \wt{L}_0 \subseteq \varepsilon^{-p} L_0 \}, \ \forall p \in \mathbb{Z}_{\geq 0}
\]

It is well-known that \( Y^{L_{0,p}}_a \) for various \( p \) is a \( G_{L_{0,p}} \)-invariant algebraic variety over \( \overline{F}_q \) if we replace the ground field \( \mathbb{F}_q((\varepsilon)) \) by \( \overline{\mathbb{F}}_q((\varepsilon)) \), which we shall assume now and for the rest of this section. Moreover, there exists a \( p_0 \) such that

\[
X^{L_{0,p}}_A := \{ L' | (L_b, L') \in \mathcal{O}_A \} \subseteq Y^{L_{0,p}}_a, \ \ p \geq p_0.
\]

Indeed, we have \( a_{0,p} = 0 \) and \( a_{p,0} \) for \( p >> 0 \) due to the fact that \( \sum_{ij \in \mathbb{Z}} a_{ij} < \infty \). The first condition implies that \( L_0 \subseteq \wt{L}_p \), if \( \wt{L} \in X^L_A \), while \( \wt{L}_0 \subseteq L_p \), if \( \wt{L} \in X^L_A \), follows from the second. Fix an \( l \) such that \( p < ln \). Then we have

\[
\varepsilon^l L_0 \subseteq \wt{L}_0 \subseteq \varepsilon^{-l} L_0.
\]

Set \( p_0 = l \), then we have \( X^{L_{0,p}}_A \subseteq Y^{L_{0,p}}_a \) for \( p \geq p_0 \).
Now we fix a 1-parameter subgroup of $P_{L_0}$:

$$\lambda : \text{GL}(1, \mathbb{F}_q) \to P_{L_0}, t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & t.1_{W} \end{pmatrix}.$$  

The fixed point set $(Y^{L_0}_{a})^{\text{GL}(1, \mathbb{F}_q)} = \sqcup_{a',a''} Y^{L_0}_{a'} Y^{L_0}_{a''}$, and the attracting set associated to $Y^{L_0}_{a'} Y^{L_0}_{a''}$ is

$$Y^{L_0}_{a',a''} = \{ \tilde{L} \in Y^{L_0}_{a'}, \pi'(\tilde{L}) \in Y^{L_0}_{a''}, \pi''(\tilde{L}) \in Y^{L_0}_{a''} \}.$$  

Hence we have the following cartesian diagram.

$$
\begin{array}{ccc}
Y^{L_0}_{a} & \xleftarrow{c_1} & Y^{L_0}_{a',a''} \\
\downarrow & & \downarrow \\
\hat{\mathcal{X}}_d(a) & \xleftarrow{\iota} & \hat{\mathcal{X}}^{+}_{a',a''}
\end{array}
\quad
\begin{array}{ccc}
Y^{L_0}_{a'} Y^{L_0}_{a''} & \xrightarrow{\pi_1} & Y^{L_0}_{a'} Y^{L_0}_{a''} \\
\downarrow & & \downarrow \\
\hat{\mathcal{X}}_d(a') \times \hat{\mathcal{X}}_d(a'') & \xrightarrow{\pi} & \hat{\mathcal{X}}_d(a') \times \hat{\mathcal{X}}_d(a'')
\end{array}
$$

where the vertical maps are natural inclusion and the top horizontal maps are induced from the corresponding bottom ones.

Hence the positivity of $\pi_{l_1}^*$ is boiled down to that of $\pi_{l_1 l_1}^*$, which follows from Braden's work [B03], since all objects involved are in the category of algebraic varieties over $\mathbb{F}_q$.

**Proposition 2.4.1.** If $\hat{\Delta}_{b',b'',b',b''}^A,\{\{A\}\} = \sum \hat{m} B_{A'} \{ B \} \odot \{ C \} \odot \{ D \}$, then $\hat{m} B_{A'} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

**Theorem 2.4.2.** If $\Delta_{b',b'',b',b''}^A,\{\{A\}\} = \sum \hat{m} B_{A'} \{ B \} \odot \{ C \} \odot \{ D \}$, then $\hat{m} B_{A'} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

Following [L00], the transfer map $\hat{\Delta}_{d,d-n} : \hat{S}_d \to \hat{S}_{d-n}$ is the composition of

$$
\begin{array}{ccc}
\hat{S}_d & \xrightarrow{\hat{A}} & \hat{S}_{d-n} \odot \hat{S}_n \\
\xi^{-1} \otimes \chi & \xrightarrow{} & \hat{S}_{d-n} \odot \hat{A} \cong \hat{S}_{d-n},
\end{array}
$$

where $\xi$ is in Remark 2.3.3 and $\chi$ is the signed representation of $\hat{S}_n$ defined in [L00] 1.8.

Note that by [L00] 1.12 and an argument similar to [L01] 3.3, $\chi$ sends a canonical basis element to 1 or 0. By Remark 2.3.3 and Proposition 2.4.1 we have

**Corollary 2.4.3.** $\hat{\Delta}_{d,d-n} \{\{A\}\} = \sum c_{A,A'} \{ A' \} \odot \{ D \} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

2.5. **Positivity in quantum affine $\mathfrak{sl}_n$.** Let $\hat{S}_n$ be the set of all $a = (a_i)_{i \in \mathbb{Z}}$ such that $a_i \in \mathbb{Z}$ and $a_i = a_{i+n}$ for all $i \in \mathbb{Z}$. Let $\hat{Y} = \{a \in \hat{S}_n| \sum_{1 \leq i \leq n} a_i = 0 \}$. We define an equivalence relation $\sim$ on $\hat{S}_n$ by declaring $a \sim b$ if there is a $z$ in $\mathbb{Z}$ such that $a_i - b_i = z$ for all $i$. Let $\hat{\mathfrak{g}}$ be the set $\hat{S}_n/ \sim$ of all equivalence classes in $\hat{S}_n$ with respect to $\sim$. Let $\mathfrak{a}$ denote the class of $a$. Both $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{y}}$ admit a natural abelian group structure with the component-wise addition. Moreover, we have a bilinear form

$$\langle -, - \rangle : \hat{\mathfrak{y}} \times \hat{\mathfrak{g}} \to \mathbb{Z}, \quad \langle b, a \rangle = \sum_{1 \leq i \leq n} b_i a_i.$$  

Now set $\hat{\mathfrak{g}} = \mathbb{Z}/n\mathbb{Z}$. For each $i \in \hat{\mathfrak{g}}$, we associate an element, still denoted by $i$, in $\hat{\mathfrak{y}}$ whose value is 1 for each integer in the equivalence class $i$ and zero otherwise. This defines a map $\hat{\mathfrak{g}} \to \hat{\mathfrak{y}}$. The same map induces a map $\hat{\mathfrak{g}} \to \hat{\mathfrak{k}}$ which sends $i \in \hat{\mathfrak{g}}$ to the equivalence class
We fix a non-degenerate symmetric bilinear form $Q$. The data $(\widehat{Y}, \widehat{X}, \langle - , - \rangle, \widehat{I} \subset \widehat{Y}, \widehat{I} \subset \widehat{X})$ is thus a Cartan datum of affine type $A_{n-1}$, which is neither $\widehat{X}$-regular nor $\widehat{Y}$-regular.

By definition, the quantum affine $\mathfrak{sl}_n$ attached to the above root datum, denoted by $\mathbb{U} (\widehat{\mathfrak{sl}_n})$, is an associative algebra over $\mathbb{Q}(v)$ generated by the generators: $E_i, F_i, K_\mu$ for all $i \in \widehat{I}, \mu \in \widehat{Y}$, and subject to the relations $K_1 K_2 \cdots K_n = 1$ and (17), for all $i, j \in \widehat{I}$. Note that the first defining relation of $\mathbb{U} (\widehat{\mathfrak{sl}_n})$ is due to the degeneracy of the Cartan datum.

Moreover, $\mathbb{U} (\widehat{\mathfrak{sl}_n})$ admits a Hopf-algebra structure, whose comultiplication is defined by the following rules.

\begin{equation}
\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i, \quad \forall i \in \widehat{I}.
\end{equation}

Let $\mathbb{U} (\widehat{\mathfrak{sl}_n})$ be Lusztig’s idempotented algebra associated to $\mathbb{U} (\widehat{\mathfrak{sl}_n})$. It is defined similar to that of quantum $\mathfrak{sl}_n$ in Section [4.6]. Similar to the finite case, $\Delta$ then induces a linear map

\begin{equation}
\Delta_{\mathfrak{m}, \mathfrak{n}, \mathfrak{m}', \mathfrak{n}'} : \mathbb{U}_X(\mathfrak{sl}_n) \to \mathbb{U}_X(\widehat{\mathfrak{sl}_n}) \otimes \mathbb{U}_X(\widehat{\mathfrak{sl}_n}),
\end{equation}

where $\mathbb{U}_X(\widehat{\mathfrak{sl}_n})$ is defined similar to $\mathbb{U}_X(\mathfrak{sl}_n)$ in finite case and $\mathfrak{m} = \mathfrak{m}' + \mathfrak{n}', \mathfrak{n} = \mathfrak{m} + \mathfrak{n}'$ in $\widehat{X}$.

By the same definition as $\widehat{\phi}_d$ in (19), we still have an algebra homomorphism

$$
\widehat{\phi}_d : \mathbb{U}(\widehat{\mathfrak{sl}_n}) \to \mathcal{S}_d.
$$

But this time $\widehat{\phi}_d$ is not surjective anymore. Then the rest of the result in finite case can be transported to affine case. In particular, we have

**Theorem 2.5.1.** Let $b$ be a canonical basis element of $\mathbb{U} (\widehat{\mathfrak{sl}_n})$. If $\Delta_{\mathfrak{m}, \mathfrak{n}, \mathfrak{m}', \mathfrak{n}'}(b) = \sum b', b'' \otimes m_{b'} b''$, then $m_{b'} b'' \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

By [L93 25.2.2], Theorem 2.5.1 remains valid over other root datum of affine type $A_{n-1}$.

### 3. Coproduct for the $j$Schur Algebras

In this section, we define the comultiplication on the $j$Schur algebra level and show that it gives rise to the transfer map used in [LW15]. We shall also show that the comultiplication degenerates to an imbedding of a $j$Schur algebra to an ordinary Schur algebra and establish a direct connection of the type A geometric duality of Grojnowski-Lusztig [GL92] and the type $B/C$ geometric duality in [BKLW14].

#### 3.1. The $j$Schur algebra $S_d$.

In this section, we assume that $n$ and $D$ are odd, i.e.,

$$
n = 2r + 1 \quad \text{and} \quad D = 2d + 1.
$$

We fix a non-degenerate symmetric bilinear form $Q^j : \mathbb{F}_q^D \times \mathbb{F}_q^D \to \mathbb{F}_q$. Let $W^\perp$ stand for the orthogonal complement of the vector subspace $W$ in $\mathbb{F}_q^D$ with respect to the form $Q^j$. By convention, $W$ is called isotropic if $W \subseteq W^\perp$. Recall the set $X_d$ from Section [1.2] Consider the subset $X^*_d$ of $X_d$ defined by

$$
X^*_d = \{ V \in X_d | V_i = V_j^\perp, \text{ if } i + j = n \}.
$$

Let $G^j_d$ be the orthogonal group attached to $Q^j$, i.e.,

$$
G^j_d = \{ g \in G_d | Q^j(gu, gu') = Q^j(u, u'), \forall u, u' \in \mathbb{F}_q^D \}.
$$
The group $G_q^2$ acts from the left on $X_q^2$. It induces a diagonal action on $X_d^2 \times X_d^2$. By the general construction in Section 1.1 we have a unital associative algebra
\[(40)\quad S_q^d = A_{G_q^2}(X_d^2 \times X_d^2).\]
This is the algebra first appeared in [BKLW14]. See also [G97] and [DS00].

3.2. Coproduct on $S_q^d$. We set $D = \mathbb{F}_q^D$. We need the following auxiliary lemma.

**Lemma 3.2.1.** Suppose that $D'$ is an isotropic subspace of $D$ and $L = (L_i|0 \leq i \leq n) \in X_d^2$. Then we can find a pair $(T, W)$ of subspaces in $D$ such that
(a) $D = D' \oplus T \oplus W$, $(D')^\perp = D' \oplus T$,
(b) $W$ is isotropic and $T \perp W$,
(c) There exists bases $\{z_1, \cdots, z_s\}$ and $\{w_1, \cdots, w_s\}$ of $D'$ and $W$, respectively, such that $Q(z_i, w_j) = \delta_{ij}$ for any $i, j \in [1, s]$,
(d) $L_i = (L_i \cap D') \oplus (L_i \cap T) \oplus (L_i \cap W)$, for any $1 \leq i \leq n - 1$.

**Proof.** Assume that $n = 3$. We can use an induction process to find a subspace $T' \subset (D')^\perp$ such that $(D')^\perp = D' \oplus T'$ and
\[L_i \cap (D')^\perp = (L_i \cap D') \oplus (L_i \cap T'), \quad \forall 1 \leq i \leq n - 1.\]

Moreover, the restriction of the bilinear form $Q$ to $T'$ is automatically non-degenerate. Next, we can find a subspace $W_1 \subseteq L_1$ such that $L_1 = (L_1 \cap (D')^\perp) \oplus W_1$. Similarly, we can find subspaces $U_2$ and $T_2$ such that
\[L_2 \cap D' = (L_1 \cap D') \oplus U_2, \quad L_2 \cap T' = (L_1 \cap T') \oplus T_2.\]

Via the natural projection $L_2 \to L_2/L_1$, we can regard $U_2 \oplus T_2$ as subspaces in $L_2/L_1$. Now $L_2/L_1$ inherits a non-degenerate bilinear form from that of $D$. Moreover, $U_2 \oplus T_2$ is the orthogonal complement of $U_2$ with respect to the form on $L_2/L_1$. By a well-known fact, say [J85] Theorem 6.11, we can find an isotropic subspace $W_2'$ such that $L_2/L_1 = U_2 \oplus T_2 \oplus W_2'$, $T_2 \perp W_2'$ and $\dim U_2 = \dim W_2'$. Furthermore, the restriction of the form to $U_2 + W_2'$ is non-degenerate. Now take a subspace $W_2$ in $L_2$ such that it gets sent to $W'_2$ via the projection map. Then by comparing the dimensions, we have
\[L_2 = (L_2 \cap (D')^\perp) \oplus (W_1 \oplus W_2).\]

It is clear that $W_1 \oplus W_2$ is an isotropic subspace in $L_2$ and $(W_1 \oplus W_2) \perp (L_2 \cap T')$.

Note that $T'$ is not necessarily perpendicular to $W_1 \oplus W_2$. We consider the subspace $D' \oplus T' \oplus W_1 \oplus W_2$. We can find a subspace $U_1$ in $V'$ such that $U_1 \cap (L_2 \cap D') = \{0\}$ and the restriction of the bilinear form to $U_1 \oplus W_1$ is non-degenerate. The latter implies that we can find bases $\{u_1, \cdots, u_s\}$ and $\{w_1, \cdots, w_s\}$ in $U_1$ and $W_1$, respectively, such that $(u_i, w_j) = \delta_{ij}$. Recall that we have bases $\{u_{i+1}, \cdots, u_{s+1}\}$ and $\{w_{i+1}, \cdots, w_{s+1}\}$ for $U_2$ and $W_2$ such that $(u_{i+1}, w_{j+1}) = \delta_{ij}$. Fix a basis $\{t'_i\}$ for $T'$ such that $\{t'_i\} \cap (L_2 \cap T')$ is a basis of $L_2 \cap T'$. Let $T$ be the subspace spanned by the elements $t_i = t'_i - \sum_{1 \leq j \leq s+1} (t'_i, w_j)w_j$. We thus have $T \perp (W_1 \oplus W_2)$ and $T$ satisfies all properties $T'$ has with respect to the flag $L$.

By [J85] Theorem 6.11, we can extend $W_1 \oplus W_2$ to a subspace $W$ satisfying the required properties, by extending the subspace $(D')^{\perp} \oplus W_1 \oplus W_2$ to the whole space $D$. So the pair $(T, W)$ satisfies the desired properties. The lemma follows for $n = 3$. For general $n$, it can be shown by a similar argument inductively.
Suppose that $\mathcal{D}''$ is an isotropic subspace of $\mathcal{D}$ of dimension $d''$. We set $\mathcal{D}' = (\mathcal{D}'')^\perp / \mathcal{D}''$, and denote by $D'$ its dimension $D - 2d'' = 2d' + 1$. Thus $\mathcal{D}'$ admits a non-degenerate bilinear form inherited from that of $\mathcal{D}$. Given any subspace $C \subseteq D$, it induces a subspace $\pi^i(C) \in \mathcal{D}'$ defined by

$$\pi^i(C) = \frac{C \cap (\mathcal{D}'')^\perp + \mathcal{D}''}{\mathcal{D}''}.$$  

Recall the operation $\pi''$ from Section 1.3. For any $L \in X_d'$, we have that $\pi'^i(L) \in X_d'$ and $\pi''(L) \in X_d'$. For any pair $(L', L'') \in X_d' \times X_d''$, we set

$$(41) \quad Z_{L', L''}^i = \{ L \in X_d' | \pi^i(L) = L', \pi''(L) = L'' \}.$$  

We also set $\bar{Z}$ to be the set of all pairs $(T, W)$ subject to the conditions (1), (2) and (3) in Lemma 3.2.1. To a pair $(T, W) \in \bar{Z}$, we have an isomorphism $\pi : T \to \mathcal{D}'$. Define a map $\bar{Z} \to Z_{L', L''}^{i, r}$ by sending $(T, W) \to L^{T, W}$, where

$$L^{T, W}_i = L_i'' \oplus \pi^{-1}(L_i') \oplus (L_{n-i}'')^\# \oplus (L_{n-i}'')^\#, \quad (L_{n-i}'')^\# = \{ w \in W | (w, L_{n-i}'') = 0 \}, \quad \forall 1 \leq i \leq n.$$  

By Lemma 3.2.1, we see that the map $\bar{Z} \to Z_{L', L''}^{i, r}$ is surjective. Let

$$\mathcal{U} = \{ g \in G_d | g(v) = v, \forall v \in \mathcal{D}'', g(v_1) - v_1 \in \mathcal{D}'', \forall v_1 \in (\mathcal{D}'')^\perp \}.$$  

Clearly $\mathcal{U}$ acts on $\bar{Z}$ and $Z_{L', L''}^{i, r}$. Moreover, it can be checked that $\mathcal{U}$ acts transitively on $\bar{Z}$ and is compatible with the surjective map $\bar{Z} \to Z_{L', L''}^{i, r}$. Therefore, we have the following lemma, analogous to [L00, Lemma 1.4].

**Lemma 3.2.2.** The group $\mathcal{U}$ acts transitively on the set $Z_{L', L''}^{i, r}$.

Recall the algebra $S_d$ from Section 1.2 (5). We are now ready to define the comultiplication $\Delta^j$. This is a map

$$(42) \quad \widetilde{\Delta}^j : S_d \to S_d \otimes S_d, \quad \forall d' + d'' = d,$$

defined by

$$\widetilde{\Delta}^j(f)(L', \bar{L}', L'', \bar{L}'') = \sum_{L \in Z_{L', \bar{L}''}^{i, r}} f(L, \bar{L}), \quad \forall L', \bar{L}' \in X_d', L'', \bar{L}'' \in X_d'',$$

where $L$ is a fixed element in $Z_{L', \bar{L}''}^{i, r}$ (See (41) for notations). By Lemma 3.2.2, we see that the definition of $\widetilde{\Delta}^j$ is independent of the choice of $L$. Moreover, by an argument exactly the same way as that of Proposition 1.5 in [L00], we have

**Proposition 3.2.3.** The map $\widetilde{\Delta}^j : S_d \rightarrow S_d \otimes S_d$ is an algebra homomorphism.

For any $i \in [1, r], a \in [1, r + 1]$, we define the following functions in $S_d^j$

$$e_i(L, L') = \begin{cases} v^{-|L_{i+1}/L_{i}|}, & \text{if } L_i \subseteq L_{i+1}, L_j = L_j', \forall j \in [1, r] \setminus \{i\}; \\ 0, & \text{otherwise}. \end{cases}$$

$$f_i(L, L') = \begin{cases} v^{-|L_{i}/L_{i-1}|}, & \text{if } L_i \supset L_{i-1}, L_j = L_{j}', \forall j \in [1, r] \setminus \{i\}; \\ 0, & \text{otherwise}. \end{cases}$$

$$h_a^{\pm 1}(L, L') = v^{\pm |L_a/L_{a-1}|} \delta_{L, L'}, \quad k_i^{\pm 1} = h_{i+1}^{\pm 1} h_i^{\pm 1},$$
for any \( L, L' \in X^j_i \). We write \( e'_i, f'_i \) and \( h'_{+i} \) for the elements in \( S^e \) analogous to \( e_i, f_i \) and \( h_{+i} \) in \( S^d \), respectively. Similarly, we use the notations \( E''_i, F''_i \), and \( K''_i \) for \( 1 \leq i \leq n - 1 \), and \( H''_{+i} \) for \( 1 \leq i \leq n \), for elements in \( S_{d'} \) defined in Section 1.3. We now study the effect of the application of \( \tilde{\Delta}^j \) to the generators.

**Proposition 3.2.4.** For any \( i \in [1, r] \), we have

\[
\tilde{\Delta}^j(e_i) = e'_i \otimes H''_{i+1} + h''_{i+1} \otimes E''_i H''_{n-i} + h''_{i+1} \otimes F''_{n-i} H''_{i+1}.
\]

\[
\tilde{\Delta}^j(f_i) = f'_i \otimes H''_{i+1} H''_{n-i-1} + h'_i \otimes F''_{n-i} H''_{i+1} + h''_{i+1} \otimes E''_{n-i} H''_{i+1}.
\]

\[
\tilde{\Delta}^j(k_i) = k'_i \otimes K''_i K''_{n-i}.
\]

**Proof.** By definition, we have

\[
\tilde{\Delta}^j(e_i)(L', L'', L''', L''') = v^{-|L'''+L''|} S,
\]

where \( S = \{ \tilde{L} = Z_{L', L''} \mid L_i < \tilde{L}_i, |L_i/L_i| = 1, L_j = \tilde{L}_j, \forall 1 \leq j \neq i \leq r \} \). The set \( S \) is nonempty only when the quadruple \( (L', \tilde{L}, L'', L''') \) is in one of the following three cases.

(i) \( L'_i \subset \tilde{L}'_i, |L'_i/L'_i| = 1, L'_j = \tilde{L}'_j \) for all \( 1 \leq j \neq i \leq r, L''_j = \tilde{L}''_j \) for all \( j \).

(ii) \( L''_j = \tilde{L}''_j \) for all \( j \), \( L''_i \subset \tilde{L}''_i, |L''_i/L''_i| = 1, L''_j = \tilde{L}''_j \) for all \( j \neq i \).

(iii) \( L''_j = \tilde{L}''_j \) for all \( j, L''_{n-i} \supset \tilde{L}''_{n-i}, |L''_{n-i}/L''_{n-i}| = 1, L''_j = \tilde{L}''_j \) for all \( j \neq n-i \).

We now compute the number \#S in case (i). This amounts to count all possible lines \( < u > \), spanned by the vector \( u \), such that \( L_i < u > \) is in \( S \). Since we want \( L_i < u > \supset L_i+1 \), we should find \( u \) in \( L_i+1 \). Since we need \( \pi(L_i+1 < u >) = L'_i \), we need to find those \( u \) such that \( \pi(u) = u' \), where \( u' \) is a fixed element in \( D' \) such that \( L'_i = L_{i'+1} < u' > \). Fix a pair \((T, W)\) in \( D' \) such that it satisfies all conditions in Lemma 3.2.1 with respect to the flag \( L \). In particular, \( L_{i+1} = L''_{i+1} \oplus (L_{i+1} \cap T) \oplus (L_{i+1} \cap W) \). Since \( T \) gets identified with \( D' \) via the canonical projection, there is a unique \( t \in T \) sending to \( u' \). So we need to look for those \( u \) such that at component \( L_{i+1} \cap T, u = t \), and at component \( L_{i+1} \cap W, u = 0 \). Thus \( u \) is of the form \( t + w \) where \( w \in L''_{i+1} \). Since adding \( w \) by any vector in \( L''_{i+1} \) does not change the resulting space \( L_{i'+1} < u > \), we see that the freedom of choice for \( w \) is \( L''_{i+1} \mod L'_i \), i.e., \( L''_{i+1}/L'_i \). So we see that the value of \( \tilde{\Delta}^j(e_i)(L', \tilde{L}', L'', L''') \) is equal to

\[
v^{-|L''+L''|} v^{-|L''+L''|} v^{-|L''+L''|} = (e'_i \otimes H''_{i+1} H''_{n-i}) (L', \tilde{L}', L'', L'''),
\]

where we use \( |L''+L''| = |L''_{i+1}/\tilde{L}'_i| + |L''_{i+1}/\tilde{L}'_i| + |L''_{n-i}/\tilde{L}'_{n-i}| \).

For case (ii), \( S \) consists of only one element, i.e., the \( \tilde{L} \) such that \( \tilde{L}_i = L_j \) for \( 1 \leq j \neq i \leq r \), and \( \tilde{L}_i = L_i + L''_i \). (Since \( L''_i \subset \tilde{L}''_{n-i}, \tilde{L}_i \) is isotropic.) So the value of \( \tilde{\Delta}^j(e_i)(L', \tilde{L}', L'', L''') \) in case (ii) is equal to

\[
v^{-|L''+L''|} v^{-|L''+L''|} v^{-|L''+L''|} = (h''_{i+1} \otimes E''_i H''_{n-i}) (L', \tilde{L}', L'', L''').
\]

For case (iii), we need to consider two situations, i.e., \( i = r \) or \( i \neq r \). For \( i = r \), the set \( S \) gets identified with the set \( S_r = \{ l \in L_{r+1}/L_r : \tilde{U}_{r+1} \subset l^{\perp}, U_{r+1} \subset l^{\perp} \} \), via \( \tilde{L} \mapsto \tilde{L}_r/L_r \), where \( \tilde{U}_{r+1} = (\tilde{L}_r^{\perp} + L_r)/L_r \) and \( U_{r+1} = (L_r^{\perp} + L_r)/L_r \). Set \( S_r = \{ W \subset L_{r+1}/L_r \mid \text{W isotropic}, \tilde{U}_{r+1} \subset W, \dim W/\tilde{U}_{r+1} = 1, W + U_{r+1} \text{ not isotropic} \} \).
We define a map $\bar{S}_r \to \bar{S}_r$ by $l \mapsto l + \bar{U}_{r+1}$. It is clear that this is a surjective map and its fiber is isomorphic to $\bar{U}_{r+1}$. The set $\bar{S}_r$ can be broken into the difference of the two sets
\[
\bar{S}_r = \{ W \mid W \text{ isotropic } \bar{U}_{r+1} \subset W, \dim W/\bar{U}_{r+1} = 1 \}
\]
\[- \{ W \mid W \text{ isotropic } \bar{U}_{r+1} \subset W, \dim W/\bar{U}_{r+1} = 1, W + \bar{U}_{r+1} \text{ isotropic} \}.
\]

For the first set, its order is equal to $q^{\frac{|L_{r+1}/L_r| - 2|L''_{r+1}/L''_r| - 1}{q-1}}$ because $\bar{U}_{r+1} \cong \bar{L}_r''$. For the second set, it is the union of $\{ W = U_{r+1} \}$ and the subset $\{ \dim W + U_{r+1}/\bar{U}_{r+1} = 2 \}$. The latter has a surjection onto the set of isotropic lines in $U_{r+1}/\bar{U}_{r+1}$ with fiber isomorphic to $\mathbb{F}_q$, via $W \mapsto W + U_{r+1}/\bar{U}_{r+1}$. Thus the order of the second set is equal to $1 + q^{\frac{|L_{r+1}/L_r| - 2|L''_{r+1}/L''_r| - 1}{q-1}}$.

So we have
\[
\#S = \#\bar{U}_{r+1}(q^{\frac{|L_{r+1}/L_r| - 2|L''_{r+1}/L''_r| - 1}{q-1}} - 1 - q^{\frac{|L_{r+1}/L_r| - 2|L''_{r+1}/L''_r| - 1}{q-1}}).
\]

So we see that the value of $\tilde{\Delta}(e_i)(L', \bar{L}', L'', \bar{L}'')$, for $i = r$ in case (iii), is equal to
\[
v^{-|L_{r+1}/L_r|}q^{L''_{r+1}/L''_r + |L'_{r+1}/L'_r|} = v^{L'_{r+1}/L'_r} = (h_{i+1} \otimes F''_{n-i}H_{i+1})(L', \bar{L}', L'', \bar{L}'').
\]

For $i \neq r$, the set $S$ gets identified with the set $S'$ of isotropic lines $l$ in $L_{n-i}/L_i$ such that $l \subseteq L_{i+1}/L_i, \bar{U} \subset l^\perp, U \not\subset l^\perp$,

where $\bar{U} = \bar{L}_{n-i}' + L_i/L_i$ and $U = L''_{n-i}' + L_i/L_i$. Notice $S'$ is the difference of the two sets:
\[
S' = \{ |l| \subseteq L_{i+1}/L_i, \bar{U} \subset l^\perp \} - \{ |l| \subseteq L_{i+1}/L_i, U \subset l^\perp \}.
\]

We can use a similar arguments as in (i) to compute the two sets and we get
\[
\#S = q^{\frac{|L''_{r+1}/L''_r + |L'_{r+1}/L'_r|}{q-1}} + q^{\frac{|L''_{i+1}/L''_i + |L'_{i+1}/L'_i|}{q-1}} = q^{L''_{i+1}/L''_i + |L'_{i+1}/L'_i|}.
\]

So we see that the value of $\tilde{\Delta}(e_i)(L', \bar{L}', L'', \bar{L}'')$, for $i \neq r$ in case (iii), is equal to
\[
v^{-|L_{i+1}/L_i|}q^{L''_{i+1}/L''_i + |L'_{i+1}/L'_i|} = v^{L'_{i+1}/L'_i}v^{-|L''_{n-i}/L''_{n-i-1} + |L'_{n-i}/L'_n|}
\]
\[
= (h_{i+1} \otimes F''_{n-i}H_{i+1})(L', \bar{L}', L'', \bar{L}'').
\]

We have the first identity.

Next, we determine $\tilde{\Delta}(f_i)$. By definition, we have
\[
\tilde{\Delta}(f_i)(L', \bar{L}', L'', \bar{L}'') = v^{-|\bar{L}_i/\bar{L}_{i-1}|}\#R
\]

where $R = \{ \bar{L} \in \mathbb{Z}_{L'/\bar{L}'}, |L_i/\bar{L}_i| = 1, L_j = \bar{L}_j, \forall 1 \leq j \leq r, j \neq r \}$. Now the set $R$ is empty unless the quadruple $(L', \bar{L}', L'', \bar{L}'')$ is in one of the following cases.

(iv) $L_i' \supset \bar{L}_i', |L_i'/\bar{L}_i'| = 1, L_j' = \bar{L}_j', \forall 1 \leq j \leq r, j \neq i, L_j'' = \bar{L}_j''$ for all $j$.

(v) $L_j'' = \bar{L}_j''$ for all $j$, $L_i'' \supset \bar{L}_i''$, $|L_i''/\bar{L}_i''| = 1, L_j'' = \bar{L}_j'', \forall 1 \leq j \neq i \leq n$.

(vi) $L_j'' = \bar{L}_j''$ for all $j$, $L''_{n-i} \subset \bar{L}_{n-i}'$, $|L''_{n-i}/\bar{L}_{n-i}'| = 1, L_j'' = \bar{L}_j''$, $\forall 1 \leq j \neq n - i \leq n$. 
In these cases, $\tilde{L}$ differs from $L$ only at $i$ and $n - i$. Thus we can identify $\tilde{L}$ with $\tilde{L}_i$.

In case (iv), to count the number of elements in $R$, we break it into two steps. We first determine all possible choices of $\tilde{L}_i \cap (D')^\perp$ for $\tilde{L}_i \in R$. Since $\tilde{L}_i'' = L''_i$, we have only one choice, i.e., $\tilde{L}_i = L''_i + T$, where $T$ is any subspace (of dimension $|L'_i|$) in $L_i \cap (D')^\perp$ maps onto $\tilde{L}_i'$ via the canonical projection. We next want to determine the number of choices of $W \subseteq L_i$ such that $W + \tilde{L}_i \in R$. We first observe that if $\tilde{L} \in R$, then the projection, say $\tilde{L}_i''$, of $\tilde{L}_i$ to $D/(D')^\perp$ is the same as that of $L_i$. Since $|L_i \cap (D')^\perp|/|L_i \cap (D')^\perp| = 1$ and $L_{i-1} \subseteq \tilde{L}_i$, we see that all possible choices for $W \subseteq L_i$ such that $W + \tilde{L}_i \in R$ is bijective to the space

$$\tilde{L}''_i/L''_{i-1} \cong (\tilde{L}'_{n-i})^\#/(\tilde{L}'_{n-i+1})^\# \cong \tilde{L}'_{n-i+1}/L''_{n-i},$$

where $L''_{i-1}$ is the projection of $L_{i-1}$ to $D/(D')^\perp$. Thus, in case (iv), the left hand side of (43) is equal to $v^{-|L_i/L_{i-1}|}q^{\tilde{L}'_{n-i+1}/L''_{n-i}} = h'_i \otimes H''_{n+1-i}(L', \tilde{L}'', L''', \tilde{L}'').$

In case (v), to build a subspace $\tilde{L}_i$ in $L_i$ such that it is in $R$, there are $\tilde{L}_i'/\tilde{L}'_{i-1}$ choices to build the component $\tilde{L}_i = \tilde{L}_i \cap (D')^\perp$. This is done by using a similar argument as in the first step of case (iv) since $L''_i \subseteq L''_i$ and $|L''_i/L''_i| = 1$. By a similar argument as the step two in case (iv), we see that the number of choices for a subspace $W$ in $L_i$ such that $W + \tilde{L}_i \in R$ is again $\tilde{L}_n''/\tilde{L}_n'$ for a fixed subspace from the first step. Thus the value of (43) in case (v) is equal to

$$v^{-|L_i/L_{i-1}|}q^{\tilde{L}_i'/\tilde{L}'_{i-1}}q^{\tilde{L}'_{n-i+1}/L''_{n-i}} = h'_i \otimes F''_{n+1-i}(L', \tilde{L}'', L''', \tilde{L}'').$$

In case (vi), there is only one element in $R$. First of all, $\tilde{L}_i \cap (D')^\perp = L_i \cap (D')^\perp$ for $\tilde{L}_i \in R$. Second of all, by fixing a decomposition of $D = D' \oplus T \oplus W$ as in Lemma 3.2.1, we see that if $\tilde{L}_i \in R$, then the projection of $\tilde{L}_i$ to $D/(D')^\perp$ is $(\tilde{L}'_{n-i})^\#$. Thus by an argument similar to the second step of case (iv), we see that there is only one $\tilde{L}_i$ in $R$. This implies that the value of (43) in case (vi) is equal to $v^{-|L_i/L_{i-1}|} = h_{i-1} \otimes E''_{n-i}H''_{n-i}(L', \tilde{L}'', L''', \tilde{L}')$. We see that the second identity follows from the above computations.

Finally the last identity follows from the definitions and $\tilde{\Delta}^j(h_i) = h'_i \otimes H''_{n+1-i}$. \qed

**Corollary 3.2.5.** We have $(1 \otimes \tilde{\Delta})\tilde{\Delta}^j = (\tilde{\Delta}^j \otimes 1)\tilde{\Delta}^j$.

The corollary follows by checking if the relation holds for generators, which is immediate.

### 3.3. Renormalization

Given a pair $(b, a)$ in $\Lambda_{d,n}$ for $n = 2r + 1$ (see (6)), we set

$$u(b, a) = \frac{1}{2} \left( \sum_{i+j \geq n+1} b_i b_j - a_i a_j + \sum_{i \geq r+1} a_i - b_i \right)$$

$$= \sum_{i+j, i+j \geq n+1} b_i b_j - a_i a_j + \frac{1}{2} \left( \sum_{i \geq r+1} b_i^2 - a_i^2 + a_i - b_i \right) \in \mathbb{Z}.$$  

Consider the subset $\Lambda^j_{d,n}$ of $\Lambda_{2d+1,n}$ defined by

$$\Lambda^j_{d,n} = \{ a \in \Lambda_{2d+1,n} | a_i = a_{n+i-1}, \forall 1 \leq i \leq n \}.$$

Similar to $X_d$, the set $X^j_d$ admits a decomposition:

$$X^j_d = \bigcup_{a \in X^j_{d,n}} X^j_d(a), \quad X^j_d(a) = \{ V \in X^j_d | \delta \in V \} = a_i, \forall 1 \leq i \leq n \}.$$


Let $S_d^j(b, a)$ be the subspace of $S_d^j$ spanned by all functions supported on $X_d^j(b) \times X_d^j(a)$. We have

$$S_d^j = \bigoplus_{b, a \in X_d^j} S_d^j(b, a),$$

such that $S_d^j(c, b) \ast S_d^j(b, a) \subseteq S_d^j(c, a)$. So $\Delta^j$ in (42) can be decomposed as

$$\Delta^j = \bigoplus_{b', a', b'', a''} \Delta^j_{b', a', b'', a''},$$

where $\Delta^j_{b', a', b'', a''}$ is the component from $S_d^j(b, a)$ to $S_d^j(b', a') \otimes S(b'', a'')$ such that

$$b_i = b'_i + b''_{n+1-i}, \quad a_i = a'_i + a''_i, \quad \forall 1 \leq i \leq n.$$

We renormalize $\Delta^j_\nu$ in (42) as follows.

$$(45) \quad \Delta^j_\nu = \bigoplus_{b, a, b', a', b'', a''} \Delta^j_{b, a, b', a', b'', a''}.$$ 

Assume that we have a quadruple $(b', a', b'', a'')$ such that $b'_k = a'_k - \delta_{k,i} + \delta_{k,n-i} - \delta_{k,n+1-i}$ and $b''_k = a''_k$ for all $k \in \{1, n\}$. We have $u(b'', a'') = 0$ and $\sum_{i \leq j \leq n} b'_i a''_{n-i} = b''_n = a''_n$. So, after the twist, the first term on the right of $\Delta^j_\nu(e_i)$ in Proposition 3.2.4 becomes $e'_i \otimes H_{n-i}^{d-1} |_{b', a', b''} = e'_i \otimes K_{n-i}^{d-1} |_{b', a', b''}$, where the notation $f_{b', a', b''}$ is the restriction of $f$ to $X_d^j(b') \times X_d^j(a') \times X_d^j(b'') \times X_d^j(a'')$.

Assume that we have a quadruple $(b', a', b'', a'')$ such that $b'_k = a'_k$ and $b''_k = a''_k - \delta_{k,i} + \delta_{k,n+1-i}$ for all $k \in \{1, n\}$. Then we have $\sum_{1 \leq k \leq n} b'_k a''_k = a''_n = b''_n$. Thus, after the twist, the second term on the right of $\Delta^j_\nu(e_i)$ in Proposition 3.2.4 is equal to $h_{n-i}^{d-1} \otimes E_i^{d-1} |_{b', a', b''}$.

The first equality in the proposition follows from the above analysis.

Assume that we have a quadruple $(b', a', b'', a'')$ such that $b'_k = a'_k + \delta_{k,i} - \delta_{k,n-i} + \delta_{k,n+1-i}$ and $b''_k = a''_k$ for all $k \in \{1, n\}$. Then we have $\sum_{1 \leq k \leq n} b'_k a''_k = a''_n$. So, after the twist, the first term on the right of $\Delta^j_\nu(f_i)$ in Proposition 3.2.4 becomes $f'_i \otimes H_{n-i}^{d-1} |_{b', a', b''} = f'_i \otimes K_{n-i}^{d-1} |_{b', a', b''}$. 

The second term on the right of $\Delta^j_\nu(f_i)$ in Proposition 3.2.4 is equal to $h_{n-i}^{d-1} \otimes E_i^{d-1} |_{b', a', b''}$. 

Assume that we have a quadruple $(b', a', b'', a'')$ such that $b'_k = a'_k$ and $b''_k = a''_k - \delta_{k,i} + \delta_{k,n+1-i}$ for all $k \in \{1, n\}$. Then we have $\sum_{1 \leq k \leq n} b'_k a''_k = a''_n$. So, after the twist, the second term on the right of $\Delta^j_\nu(f_i)$ in Proposition 3.2.4 becomes $f'_i \otimes H_{n-i}^{d-1} |_{b', a', b''} = f'_i \otimes K_{n-i}^{d-1} |_{b', a', b''}$.
Lemma 3.4.3. \( S \) is linearly independent in position 4.5. The only difference is an involution the same manner, and skipped.

Proof. By Proposition 3.3.1, we have \( \Delta \) commutative diagram for the quadruple \( d, d', d'', d''' \) such that \( b_k' = a_k' - \delta_{k,n-i} + \delta_{k,n+1-i} \) for all \( k \in [1, n] \). Then we have \( \sum_{1 \leq k < j \leq n} b_k' b_j' = a_k' a_j' = a_k' + \delta_{k,n-i} + \delta_{k,n+1-i} \) and \( u(b', a'') = a'' \). So, after the twist, the third term on the right of \( \Delta \) in Proposition 3.2.4 becomes \( h_i' \otimes F'' H''_n - a_i' - a_i' + \delta_{i,r} \otimes V_{b', a', b'', a''} = k_i' \otimes K''_n - a_i' + \delta_{i,r} \otimes V_{b', a', b'', a''} \).

Assume that we have a quadruple \( \langle b', a', b'', a'' \rangle \) such that \( b_k' = a_k' - \delta_{k,n-i} + \delta_{k,n+1-i} \) for all \( k \in [1, n] \). Then we have \( \sum_{1 \leq k < j \leq n} b_k' b_j' = a_k' a_j' = a_k' + \delta_{k,n-i} + \delta_{k,n+1-i} \) and \( u(b', a'') = a'' \).

The above analysis implies the second equality in the proposition. Since the twist will not affect the original term when \( b' = a' \) and \( b'' = a'' \), we have the third equality. \hfill \Box

Moreover, we have

**Proposition 3.3.2.** \((1 \otimes \Delta) \Delta = (\Delta \otimes 1) \Delta\). More precisely, we have the following commutative diagram for the quadruple \( d, d', d'', d''' \) such that \( d = d' + d'' + d''' \):

\[
\begin{array}{ccc}
S^j_d & \xrightarrow{\Delta} & S^j_{d'} \otimes S_{d'' + d'''} \\
\Delta \downarrow & & \downarrow 1 \otimes \Delta \\
S^j_{d' + d''} \otimes S_{d'''} & \xrightarrow{\Delta \otimes 1} & S^j_{d'} \otimes S_{d''} \otimes S_{d'''}.
\end{array}
\]

3.4. The imbedding \( j_{d,v} : S^j_d \to S_d \). In this section, we set \( d = 0 \) and \( d'' = d \), then the comultiplication \( \Delta \) in (42) becomes \( \Delta : S^j_d \to S^j_0 \otimes S_d \). Observe that \( S^j_0 \) consists of only one basis element, so we have \( S^j_0 \simeq A \). Thus the coproduct \( \Delta \) becomes the following algebra homomorphism, denoted by \( j_{d,v} \),

\[(46) \quad j_{d,v} : S^j_d \to S_d.\]

The following corollary is by Proposition 3.3.1 \( e_i = 0, f_i = 0 \) and \( k_i' = v^{b', r} \) in \( S^j_0 \).

**Corollary 3.4.1.** We have

\[
\begin{align*}
j_{d,v}(e_i) &= E_i + K_i F_{n-i}, \\
j_{d,v}(f_i) &= F_i K_{n-i} + E_{n-i}, \\
j_{d,v}(k_i) &= v^{b', r} K_i K_{n-i}^{-1}, \quad \forall 1 \leq i \leq r.
\end{align*}
\]

**Proof.** By Proposition 3.3.1, we have

\[
j_{d,v}(e_i) = e_i' \otimes K_i'' + 1 \otimes E_i'' + k_i' \otimes F_{n-i}'' K_i'' = 0 + E_i'' + v^{b', r} F_{n-i}'' K_i'' = E_i'' + K_i'' F_{n-i}''
\]

which is the first identity if we skip the superscripts. The rest two are obtained in exactly the same manner, and skipped. \hfill \Box

**Remark 3.4.2.** The homomorphism \( j_{d,v} \) matches with the imbedding \( j \) in [BKLW14 Proposition 4.5]. The only difference is an involution \( \omega \) on \( U \) defined by \( (E_i, F_i, K_i) \mapsto (F_i, E_i, K_i^{-1}) \).

**Lemma 3.4.3.** \( j_{d,v} \) in (46) is injective.

**Proof.** Recall from [BKLW14 Theorem 3.10] that \( S^j_d \) has a monomial basis \( m^j_A \) indexed by \( A \in \Xi^j_d \) (which is denoted \( m_A \) therein). It is enough to show that the set \( \{ j_{d,v}(m^j_A) | A \in \Xi^j_d \} \) is linearly independent in \( S^j_d \). We set

\[
\deg(1, \lambda) = 0, \quad \deg(e_i, 1, \lambda) = i, \quad \deg(f_i, 1, \lambda) = n - i, \quad \forall \lambda \in \Lambda^j_{d,n}, 1 \leq i \leq r.
\]
Similarly, we define
\[ \deg(1_\lambda) = 0, \quad \deg(E_{1\lambda}) = i, \quad \deg(F_{1\lambda}) = -i, \quad \forall \lambda \in \Lambda_{d,n}, 1 \leq i \leq n. \]

We write \( \nu' < \nu \) if \( \nu'_i < \nu_i \) for all \( i \) and \( \nu'_0 < \nu_0 \) for some \( i_0 \). Suppose that \( \deg(m^2_A) = \nu \in \mathbb{Z}_{\geq 0}[I] \). By Corollary 3.4.1, we have
\[ j_{d,\nu}(m^2_A) \in \oplus_{b - \pi = \nu} S_d(b, a) \oplus \oplus_{d - \pi < \nu} S_d(d, c). \]

For \( A = (a_{ij}) \in \Xi_d \), we set
\[ \Xi_d(A) = \{ B = (b_{ij}) \in \Xi_d \mid b_{ij} = 0, \forall i < j, b_{ij} = a_{ij}, \forall i > j, \text{co}(B) \vdash \text{co}(A) \}, \]
where \( B \vdash A \) if \( b_{i} + b_{i+1} - a_{i} = 0 \) for all \( 1 \leq i \leq n \). By Corollary 3.4.1, we see that
\[ j_{d,\nu}(m^2_A) = \sum_{B \in \Xi_d(A)} m_B + \text{lower terms}, \]
where \( m_B \) denotes the monomial attached to \( B \) in [BLM90, Proposition 3.9] and ‘lower term’ is the remaining summand in \( \oplus_{d - \pi < \nu} S_d(d, c) \). Now suppose that we have
\[ \sum_{A \in \Xi_d} c_{A \xi_{d,\nu}}(m^2_A) = 0, \quad c_A \in A. \]

Let \( \mathcal{M} \) be the set of maximal \( \nu \in \mathbb{Z}[I] \) in the set \( \{ \deg(m^2_A) \mid A \in \Xi_d \} \) with respect to the natural partial order in \( \mathbb{Z}[I] \), i.e., \( \nu' \leq \nu \) if and only if \( \nu'_i \leq \nu_i \) for all \( i \). We have
\[ 0 = \sum_{A \in \Xi_d} c_{A \xi_{d,\nu}}(m^2_A) = \sum_{A : \deg(m^2_A) \in \mathcal{M}} c_{A \xi_{d,\nu}}(m^2_A) + \text{lower term}. \]

So we have \( \sum_{A : \deg(m^2_A) \in \mathcal{M}} c_{A \xi_{d,\nu}}(m^2_A) = 0 \). By [BLM] Proposition 3.9] and the fact that \( \Xi_d(A) \cap \Xi_d(A') = \emptyset \) if \( A \neq A' \), the set \( \{ m_B \mid B \in \Xi_d(A) \} \), where \( A \) runs over all matrices in \( \Xi_d \) such that \( \deg(m^2_A) \in \mathcal{M} \), is linearly independent in \( S_d \). Thus, \( c_A = 0 \) for all \( A \in \Xi_d \) such that \( \deg(m^2_A) \in \mathcal{M} \). Inductively, \( c_A = 0 \) for all \( A \in \Xi_d \). Therefore, the set \( \{ j_{d,\nu}(m^2_A) \mid A \in \Xi_d \} \) is linearly independent. \( \square \)

The following is nothing but a special case of Proposition 3.3.2.

**Corollary 3.4.4.** Suppose that \( d' + d'' = d \). We have the following commutative diagram.

\[
\begin{array}{ccc}
S_d^j & \xrightarrow{\Delta^j} & S_d^j \otimes S_{d''}^j \\
\downarrow j_{d,\nu} & & \downarrow j_{d',\nu} \otimes 1 \\
S_d & \xrightarrow{\Delta^\nu} & S_{d'} \otimes S_{d''}.
\end{array}
\]

**Remark 3.4.5.** \( S_d^j \) can be regarded as a ‘coideal’ subalgebra of \( S_d \) in view of Lemma 3.4.3 and Corollary 3.4.4.
3.5. Type A duality vs type B duality. In this section, we use the algebra homomorphism $J_{d,v}$ to establish a direct connection between the geometric type A duality in [GL92] and the geometric type B duality in [BKLW14].

For any nonnegative integers $a, b$, we write $1^a0^b$ for the sequence $(1, \ldots, 1, 0, \ldots, 0)$ containing $a$ copies of 1’s and $b$ copies of 0’s. Similarly, we can define $1^a0^b1^c$, etc.

Recall $X_d, X_d(b)$ for $b \in \Lambda_{d,n}$ from Section 1.2 and 1.3. We set

$$T_{d,n} = \mathcal{A}_{G_d}(X_d \times X_d(1^d)), \quad \text{and} \quad H_{A_d} = \mathcal{A}_{G_d}(X_d(1^d) \times X_d(1^d)).$$

By [GL92], we know that $H_{A_d}$ is a Hecke algebra of type $A_d$ and $T_{d,n}$ is a tensor space $V_n^{\otimes d}$ where $V_n$ is a free $A$-module of rank $n$. Now the standard convolution defines commuting actions of $S_d$ and $H_{A_d}$ on $T_{d,n}$ from the left and the right, respectively, which is captured in the following diagram.

$$S_d \times T_{d,n} \to T_{d,n} \xleftarrow{T_d} T_{d,n} \times H_{A_d}.$$  

Moreover, the two actions centralize each other.

We shall recall a similar picture in [BKLW14] if the $X_d$ is replaced by its $j$-analogue. Recall $X_d^j, X_d^j(b)$ for $b \in \Lambda_{d,n}^j$ from Section 3.1 and 3.3. We set

$$T_{d,n}^j = \mathcal{A}_{G_d^j}(X_d^j \times X_d^j(1^{2d+1})) \quad \text{and} \quad H_{B_d} = \mathcal{A}_{G_d^j}(X_d^j(1^{2d+1}) \times X_d^j(1^{2d+1})).$$

Then we have that $T_{d,n}^j$ is also isomorphic to the tensor space $V_n^{\otimes d}$, and the following diagram of commuting actions.

$$S_d^j \times T_{d,n}^j \to T_{d,n}^j \xleftarrow{T_d^j} T_{d,n}^j \times H_{B_d}^j.$$  

A slight variant of the imbedding $J_{d,v}$ yields the following linear map

$$\zeta_{d,v} : \mathcal{A}_{G_d}(X_d(b) \times X_d(1^{2d+1})) \to \oplus_{b' \vdash b, a'' \vdash 1^{2d+1}} \mathcal{A}_{G_d}(X_d(b'') \times X_d(a'')), \forall b \in \Lambda_{d,n}^j,$$

where $b'' \vdash b$ stands for $b_i = b''_i + b''_{n+1-i} + \delta_{i,r+1}$ for all $i$. For $a'', b'' \vdash 1^{2d+1}$, we set

$$T_{d,n}^{a''} = \mathcal{A}_{G_d}(X_d(1^{2d+1})) \quad \text{and} \quad H_{A_d} = \mathcal{A}_{G_d}(X_d(b'') \times X_d(1^d)).$$

Let $\zeta_{d,b,v}$ denote the composition of $\zeta_{b,v}$ with the projection to the components of $a'' = 1^{d0^d+1}$, i.e.,

$$\zeta_{d,b,v} : \mathcal{A}_{G_d}(X_d(b) \times X_d(1^{2d+1})) \to \oplus_{b'' \vdash b} \mathcal{A}_{G_d}(X_d(b'') \times X_d(1^d)),$$

where we identify $X_d(1^{d0^d+1})$ with $X_d(1^d)$. Summing over all $b \in \Lambda_{d,n}^j$, we get a linear map

$$\zeta_{d,v} \equiv \oplus_{b \in \Lambda_{d,n}} \zeta_{d,b,v} : T_{d,n}^j \to T_{d,n}.$$  

Take $n = 2d + 1, b = 1^{2d+1}$, we obtain a linear map

$$\zeta_{d,v}^1 : H_{B_d} \to \oplus_{b'' \vdash 1^{2d+1}} b'' H_{A_d},$$

which is not necessarily an algebra homomorphism. Note that we identify $T_{d,n}^{1^{d0^d+1}}$ and $1^{d0^d+1} H_{A_d}$ with $T_{d,n}$ and $H_{A_d}$, respectively.
Proposition 3.5.1. We have the following commutative diagram relating the geometric type
A duality with the geometric type B duality.

\[
\begin{array}{ccc}
S_d \times T_{d,n}^J & \longrightarrow & T_{d,n}^J \\
\downarrow_{\zeta_{d,v} \times \zeta_{d,v}} & & \downarrow_{\zeta_{d,v}} \\
S_d \times T_{d,n} & \longrightarrow & T_{d,n} \oplus a^n_{ij=1}^{d+1} T_{d,n}^a \times \oplus b_{ij=1}^{d+1} b^{r'} H_{A_d} \leftrightarrow T_{d,n} \times H_{A_d}.
\end{array}
\]

where \( \psi_2 \) is the natural imbedding and \( \psi_2 \psi_1 \) is the \( \psi \) in (47).

We can describe the linear map \( \zeta_{d,v} \) explicitly. Let \( \Pi_{d,n} \) be the set of \( n \times d \) matrices \( A \) such that \( a_{ij} \in \{0, 1\} \) and \( \sum_{1 \leq i \leq n} a_{ij} = 1 \) for all \( 1 \leq j \leq d \). Then we have

\[
(50) \quad T_{d,n} = \text{span}_A \{ a[A] | A \in \Pi_{d,n} \}
\]

where \( a[A] = v^d A \zeta_A \) and \( d_A = \sum_{i \geq k, j < l} a_{ij} a_{kl} \).

Let \( \Pi'_{d,n} \) be the subset of \( \Pi_{2d+1,n} \) such that \( a_{ij} = a_{n+i-1,2d+2-j} \) for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq 2d+1 \). (In particular, we have \( a_{r+1,d+1} = 1 \).) We have

\[
(51) \quad T_{d,n}^J = \text{span}_A \{ [A] | A \in \Pi'_{d,n} \},
\]

where \( [A] = v^{\ell A} \zeta_A' \) and \( \ell_A = \frac{1}{2} \left( \sum_{i \geq j, k < l} a_{ij} a_{kl} - \sum_{i \geq n+1, d+1 > j} a_{ij} \right) \).

Let \( J_m \) be the \( m \times m \) matrix whose \((i,j)\)-th component is \( \delta_{i,n+1-j} \) for all \( 1 \leq i, j \leq m \). To a matrix \( A \in \Pi_{d,n} \), we define a matrix

\[
A' = (A|\epsilon_{r+1}|J_n AJ_d),
\]

where \( \epsilon_{r+1} \) is the column vector whose entries are zero except at \( r+1 \) which is 1. Then the assignment \( A \mapsto A' \) defines a bijection \( \Pi_{d,n} \overset{\sim}{\to} \Pi'_{d,n} \).

Proposition 3.5.2. \( \zeta_{d,v}([A']) = [a[A]], \) for all \( A \in \Pi_{d,n} \).

Proof. Suppose that \( \text{ro}(A') = b' \). We set \( a'' = 1^{d_0+1}, b' = 0^{r+1}0^r \) and \( a' = 0^d1^{d-1}0^d \). Then by the definition of \( \zeta_{d,v} \), we have

\[
(52) \quad \zeta_{d,v}([A']) = v^{b'} \Delta_{b',a',b',a'}([A'])
\]

where

\[
t_{b''} = \sum_{1 \leq i \leq j \leq n} b''_{ij} - \sum_{1 \leq i \leq j \leq 2d+1} a'_{ij} a''_{ij} + \frac{1}{2} \left( \sum_{i+j \geq n+1} b''_{ij} - \sum_{i+j \geq r+1} b''_{ij} + \sum_{i \geq d+1} a''_{ij} \right).
\]

Note that the following formula \( t_{b''} \) is compatible with the twist in (45), since we need to rescale from \( n \) components to \( 2d+1 \) components for \( a' \) and \( a'' \). Now using the fact that \( a'' = 1^{d_0+1}, b' = 0^{r+1}0^r \) and \( a' = 0^d1^{d-1}0^d \), the twist \( t_{b''} \) can be simplified to

\[
t_{b''} = \frac{1}{2} \left( \sum_{i+j \geq n+1} b''_{ij} - \sum_{i \geq r+1} b''_{ij} \right).
\]
By the definition of $A^J$, we can also simplify the numeric $\ell_{A^J}$ as follows.

$$\ell_{A^J} = \frac{1}{2} \left( \sum_{i \geq k, j < l} a_{ij}^I a_{kl}^J - \sum_{i \geq r+1, d+1 > j} a_{ij}^J \right), \quad A^J = (a_{ij}^J)$$

(53)

$$= \frac{1}{2} \left( \sum_{i \geq k, j < l < d+1} + \sum_{i \geq k, j < d+1 \leq l} + \sum_{i \geq k, d+1 \leq j < l} \right)a_{ij}^J a_{kl}^J - \sum_{i \geq r+1, d+1 > j} a_{ij}.$$

The first sum simplifies to $\sum_{i \geq k, j < l < d+1} a_{ij} a_{kl}$. The third sum simplifies to

$$\sum_{i \geq k, d+1 \leq j < l} a_{ij}^I a_{kl}^J = \sum_{i \geq k, j < d+1 \leq l} a_{n+1-i,2d+2-j} a_{n+1-k,2d+2-l} = \sum_{i \geq k, j < l < d+1} a_{ij} a_{kl} + \sum_{i \geq r+1, d+1 > j} a_{ij}.$$

The second sum is reduced to

$$\sum_{i \geq k, j < d+1 \leq l} a_{ij}^I a_{kl}^J = \sum_{i \geq k, j < d+1 \leq l} a_{ij} a_{kl} + \sum_{i \geq r+1, d+1 > j} a_{ij} = 2t_{\text{ro}(A)}.$$

So we get $t_{b^\nu} - \ell_{A^J} = -d_A + t_{b^\nu} - t_{\text{ro}(A)}$. Thus the identity (52) can be rewritten as

$$\zeta_{A^J} = v^{-d_A + t_{b^\nu} - t_{\text{ro}(A)}} \Delta_{b^\nu, a^I, b^\nu, a^J}^J (\zeta_{A^J})$$

where $\zeta_{A^J}$ denote the characteristic function attached to the $G^J_d$-orbit indexed by $A^J$.

Recall that $a^I = 1^{d+1}$. This implies that for any $\tilde{L}_i \in X_d(a^I)$, we have $Z_{d,\nu, \tilde{L}_i}$ consists of only one point, i.e., the flag $\tilde{L}$ such that $\tilde{L}_i = \tilde{L}_i''$ for all $i \leq r$ and $\tilde{L}_i = (\tilde{L}_i''_{n+1-i})$ for all $i \geq r + 1$. Furthermore, if $(\tilde{L}_i'', L_i''') \in \mathcal{A}$, then $(L, \tilde{L}) \in \mathcal{A}$ for any $L \in Z_{d,\nu, \tilde{L}_i''}$ and $\tilde{L} \in Z_{d,\nu, \tilde{L}_i''}$ because $L_i \cap \tilde{L}_j = L_i'' \cap \tilde{L}_j''$. Hence we have $\Delta_{b^\nu, a^I, b^\nu, a^J}^J (\zeta_{A^J}) = \delta_{b^\nu, \text{ro}(A)} \zeta_{A^J}$. The proposition is proved. \hfill \square

By Proposition 3.5.2 we have

**Theorem 3.5.3.** For all $A \in \Pi_{d,n}$, we have $\zeta_{\nu, \gamma}(\{A^J\}) = a\{A\} + \sum_{B^J, B^J \neq \text{ro}(A)} c_{B,A} a\{B\}$ where $c_{B,A} \in \mathbb{Z}_{\geq 0}[\nu, \nu^{-1}]$.

Recall the parabolic Kazhdan-Lusztig polynomials $P_{B^J, A^J}$ and $P_{B_A}$ of type $B_d$ and $A_d$, respectively.

**Corollary 3.5.4.** $P_{B^J, A^J} = P_{B_A}$ if $\text{ro}(B) = \text{ro}(A)$.

More generally, we have the following commutative diagram of algebras.

$$\begin{array}{ccc}
H_{B_d} = \mathcal{A}_{G_d}(X_d'(1^{2d+1}) \times X_d'(1^{2d+1})) & \xrightarrow{\zeta_{d,\nu}} & \oplus_{b^\nu, a^I, d+1} \mathcal{A}_{G_d}(X_d(b^\nu) \times X_d(a^I)) \\
\downarrow & & \downarrow \\
S_{d,2d+1} & \xrightarrow{\chi_{d,\nu}} & S_{d,2d+1}.
\end{array}$$

3.6. **Transfer maps on $S_d^I$.** The transfer map

$$\phi_{d,d-n,\nu}^I : S_d^I \rightarrow S_{d-n}^I$$

is defined to be the composition: $S_d^I \xrightarrow{\chi^I} S_{d-n}^I \otimes S_n \xrightarrow{1 \otimes \chi} S_{d-n}^I \otimes A \equiv S_{d-n}^I$, where $\chi$ is in [13]. It is clear that $\phi_{d,d-n,\nu}^I$ is an algebra homomorphism. Moreover, we have
Proposition 3.6.1. \( \phi^j_{d,d-n,v}(e_i) = e'_i, \phi^j_{d,d-n,v}(f_i) = f'_i, \) and \( \phi^j_{d,d-n,v}(k_i^{\pm 1}) = k_i^{\pm 1} \), for any \( i \in [1, r] \).

Proof. By definitions, we have \( \chi(E''_i) = 0, \chi(F''_i) = 0, \) and \( \chi(H''_i) = v \). So we have

\[
\phi^j_{d,d-n,v}(e_i) = e'_i \chi(H''_{i+1}H''_{n-i}) + h''_{i+1} \chi(E''_{i}H''_{n-i}) + h''_{i} \chi(F''_{n-i}H''_{i+1}) = e'_i,
\]

\[
\phi^j_{d,d-n,v}(f_i) = f'_i \chi(H''_{i}H''_{n-i}) + h''_{i} \chi(F''_{n-i}H''_{i}) + h''_{i-1} \chi(E''_{n-i}H''_{i}) = f'_i,
\]

\[
\phi^j_{d,d-n,v}(k_i) = k'_i \chi(K''_{i}K''_{n-i}^{-1}) = k'_i.
\]

The lemma is proved. \( \square \)

Together with Theorem 3.10 in [BKLW14], we have

Corollary 3.6.2. The homomorphism \( \phi^j_{d,d-n,v} \) is surjective.

3.7. Generic version. Recall the definition of \( v = \sqrt{q} \) from \([2]\) and \( \mathcal{A} = \mathbb{Z}[v, v^{-1}] \). Recall from [BKLW14] that one can construct an associative algebra \( S'_d \) over \( \mathcal{A} \) such that

\[ S'_d = \mathcal{A} \otimes_{\mathcal{A}} S''_d, \]

where \( \mathcal{A} \) is regarded as an \( \mathcal{A} \)-module with \( v \) acting as \( v \). Let us make the algebra \( S'_d \) more precise. Recall \( \Xi'_d \) from Section 1.4. Consider the set

\[ (54) \quad \Xi'_d = \{ M \in \Xi'_d | m_{ij} = m_{n+1-i, n+1-j}, \forall 1 \leq i, j \leq n \}. \]

Then \( S'_d \) is a free \( \mathcal{A} \)-module with basis \( \zeta'_M \) for any \( M \in \Xi'_d \) whose multiplication is defined by the condition that if \( \zeta'_M, \zeta'_M' \) then \( \eta'_M, \eta'_M' \) is the characteristic function of the \( G'_d \)-orbit in \( X'_d \times X'_d \) indexed by \( M \) via \([12]\). Let \( S'_d(a) = \operatorname{span}_{\mathcal{A}} \{ \zeta_M | \operatorname{co}(M) = a \} \). We have \( S'_d(c, b) S'_d(b, a) \subseteq \delta_{b', b} S'_d(c, a) \).

By using the monomial basis in [BKLW14] Theorem 3.10], one can show that \( S'_d \) enjoys the same results for \( S'_d \) from the previous sections. The following is a generic version of Proposition 3.3.1

Proposition 3.7.1. There is a unique algebra homomorphism

\[ (55) \quad \Delta^j : S'_d \rightarrow S'_{d'} \otimes S''_{d''} \]

such that \( \mathcal{A} \otimes_{\mathcal{A}} \Delta^j = \Delta^j \) and

\[
\Delta^j(e_i) = e_i' \otimes K''_i + 1 \otimes E''_{n-i} + k'_i \otimes F''_{n-i} K''_i,
\]

\[
\Delta^j(f_i) = f_i' \otimes K''_{n-i} + k''_{i-1} \otimes K''_{n-i} F''_i + 1 \otimes E''_{n-i},
\]

\[
\Delta^j(k_i) = k'_i \otimes K''_{n-i}^{-1}, \quad \forall 1 \leq i \leq r.
\]

Recall the canonical basis \( \{ \{ M \}_d | M \in \Xi'_d \} \) of \( S'_d \) from [BKLW14] 3.6. We have the following positivity result of the canonical basis of \( S'_d \) with respect to the coproduct \( \Delta^j \).

Proposition 3.7.2. If \( \Delta^j(\{ M \}_d) = \sum_{M' \in \Xi'_{d'}, M'' \in \Xi''_{d''}} h_{M', M''}^M \{ M' \}_d' \otimes \{ M'' \}_d'' \), then we have

\[ h_{M', M''}^M \in \mathbb{Z}_{\geq 0}[v, v^{-1}]. \]
There is a unique algebra homomorphism $\phi^{i,j}_{d,d-n}$ such that $A \otimes_{A'} \phi^{i,j}_{d,d-n} = \phi^{i,j}_{d,d-n,v}$ and $\phi^{i,j}_{d,d-n}(e_i) = e'_i$, $\phi^{i,j}_{d,d-n}(f_i) = f'_i$, and $\phi^{i,j}_{d,d-n}(k^{-1}_{i}) = k^{\pm 1}_i$, $\forall 1 \leq i \leq r$.

By setting $d' = 0$ in (55), we have the following generic version of Corollary 3.4.1:

**Proposition 3.7.5.** There is a unique algebra imbedding $j_d : S_d^j \rightarrow S_d$ such that $A \otimes_{A'} j_d = j_{d,v}$ and

\[
j_d(e_i) = E_i + K_i F_{n-i},
\]

\[
j_d(f_i) = F_i K_{n-i} + E_{n-i},
\]

\[
j_d(k_i) = v^{\phi^{i,j}} K_{i}^{-1} K_{n-i}, \quad \forall 1 \leq i \leq r.
\]
Suppose that \( \{B\}_d \) is a canonical basis element in \( S_d^j \), we have
\[
\Delta(B)_d = \sum g_{B,A}\{A\}_d
\]
where the sum runs over the set of canonical basis elements \( \{A\}_d \) in \( S_d \). By Propositions 3.7.2 and 3.7.5, we have

**Corollary 3.7.6.** \( g_{B,A} \in \mathbb{Z}_{\geq 0}[v, v^{-1}] \).

### 4. Positivity for the modified coideal subalgebra \( \mathbb{U}^j \)

#### 4.1. The coideal subalgebra \( \mathbb{U} \)

By definition, \( \mathbb{U} \equiv \mathbb{U}(\mathfrak{sl}_n) \) is an associative algebra over \( \mathbb{Q}(v) \) generated by \( e_i, f_i, k_{\pm i} \) for \( 1 \leq i \leq r \) and subject to the following defining relations. For any \( 1 \leq i, j \leq r \) and \( a_{ij} = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1} \),
\[
\begin{align*}
\kappa_i \kappa_j &= \kappa_j \kappa_i, \\
\kappa_i \kappa_i^{-1} &= \kappa_i^{-1} \kappa_i = 1, \\
k_i e_i &= v^{a_{ij} + \delta_i, \delta_j} e_j k_i, \\
k_i f_i &= v^{-a_{ij} - \delta_i, \delta_j} f_j k_i, \\
e_i f_i - f_i e_i &= \delta_{i,j} \frac{\kappa_i - \kappa_i^{-1}}{v - v^{-1}}, \quad \text{if } (i, j) \neq (r, r), \\
e_i^2 f_r + f_r e_i^2 &= (v + v^{-1})(e_r f_i e_r - e_r (v k_r + v^{-1} k_r^{-1})), \\
f_i^2 e_r + e_r f_i^2 &= (v + v^{-1})(f_r e_i f_r - (v k_r + v^{-1} k_r^{-1}) f_r), \\
e_i e_j &= e_j e_i, \quad \text{if } |i - j| > 1, \\
f_i f_j &= f_j f_i, \quad \text{if } |i - j| > 1, \\
e_i^2 e_j + e_j e_i^2 &= (v + v^{-1}) e_i e_j e_i, \quad \text{if } |i - j| = 1, \\
f_i^2 f_j + f_j f_i^2 &= (v + v^{-1}) f_i f_j f_i, \quad \text{if } |i - j| = 1.
\end{align*}
\]

Recall the algebra \( \mathbb{U} \) from Section 1.6 from [BKLW14, Proposition 4.5], see also [BW13], we have an injective algebra homomorphism
\[
\Delta: \mathbb{U}^j \rightarrow \mathbb{U},
\]
defined by
\[
\Delta(e_i) = e_i + k_i f_{n-i}, \quad \Delta(f_i) = f_i k_{n-i} + e_{n-i}, \quad \Delta(k_i) = v^{\delta_{i,r}} k_i k_{n-i}^{-1}, \quad \forall 1 \leq i \leq r.
\]
Here \( n = 2r + 1 \). By composing \( \Delta \) with \( \Delta \) in (18), we have an algebra homomorphism
\[
\Delta^j: \mathbb{U}^j \rightarrow \mathbb{U} \otimes \mathbb{U} \text{ defined by }
\]
\[
\begin{align*}
\Delta^j(e_i) &= e_i \otimes k_i + k_i \otimes e_i, \\
\Delta^j(f_i) &= f_i \otimes k_{n-i} + k_i \otimes f_i + k_i \otimes e_i, \\
\Delta^j(k_i) &= k_i \otimes k_i k_{n-i}^{-1}, \quad \forall 1 \leq i \leq r.
\end{align*}
\]
4.2. The algebra $\hat{U}$. On $\mathbb{Z}^n$, we define an equivalence relation “$\approx$” by $\mu \approx \lambda$ if and only if $\mu - \lambda = m(2, \cdots, 2)$ for some $m \in \mathbb{Z}$. Let $\hat{\mu}$ denote the equivalence class of $\mu$ with respect to $\approx$. Consider the following subset in the set $\mathbb{Z}^n/\approx$ of equivalence classes.

$$\mathcal{X} = \{\hat{\mu} \in \mathbb{Z}^n/\approx | \mu_i = \mu_{n+1-i}, \forall 1 \leq i \leq n, \mu_{r+1} \text{ is odd}\}.$$  

We define

$$\hat{U} = \bigoplus_{\hat{\mu} \in \mathcal{X}} \hat{\mu} \hat{U}^j,$$

$$\hat{\mu} \hat{U}^j = \mathcal{U}^j / \left( \sum_{1 \leq i \leq r} (k_i - v^{-\mu_i + \mu_{i+1}}) \mathcal{U}^j + \sum_{1 \leq i \leq r} \mathcal{U}^j (k_i - v^{-\lambda_i + \lambda_{i+1}}) \right).$$

The algebra $\hat{U}$ is the modified form of $U$ (see [BKLW14, 4.6] for the $\mathfrak{g}I_n$ version). Let

$$\pi_{\hat{\mu}, \hat{\lambda}} : \mathcal{U} \to \hat{\mu} \hat{U}^j$$

be the natural projection.

Recall the set $\mathcal{X}$ from (57) and $s_i$ the $i$-standard basis element of $\mathbb{Z}^n$. We define an abelian group structure on $\mathcal{X}$ by $\hat{\mu} + \hat{\lambda} = \hat{\pi}$, with $\pi = \mu + \lambda - s_{r+1}$. We set

$$P = \{1, \cdots, r\}.$$  

The assignment $i \mapsto -s_i + s_{i+1} + s_{n-i} - s_{n+1-i} + s_{r+1}, \forall i \in P$, defines an embedding of abelian groups $\mathbb{Z}[P] \hookrightarrow \mathcal{X}$. We shall identify elements in $\mathbb{Z}[P]$ with their images in $\mathcal{X}$. Then the algebra $\mathcal{U}$ in Section 4.1 admits a $\mathbb{Z}[P]$-graded decomposition $\mathcal{U} = \bigoplus \omega \in \mathbb{Z}[P] \mathcal{U}(\omega)$ defined by

$$e_i \in \mathcal{U}(i), \ f_i \in \mathcal{U}(-i), \ k_i^{\pm 1} \in \mathcal{U}(0), \ \mathcal{U}(\omega) \mathcal{U}(\omega') \subseteq \mathcal{U}(\omega + \omega'), \ \forall \omega \in \mathbb{Z}[P].$$

Let $\hat{\mu} \mathcal{U}^j(\omega) = \pi_{\hat{\mu}, \hat{\lambda}}(\mathcal{U}(\omega))$. By a standard argument, we have



**Lemma 4.2.1.** $\hat{\mu} \mathcal{U}^j(\omega) = 0$ unless $\hat{\mu} - \hat{\lambda} = \omega \in \mathbb{Z}[P] \subseteq \mathcal{X}$.

4.3. Positivity with respect to $\Delta^j$. We introduce the following notations to simplify the presentation. For any $\mu, \mu', \mu'' \in \mathbb{Z}^n$, we write

$$(\mu', \mu'') \vdash \mu,$$

if and only if $\mu_i' + \mu_{n+1-i}'' = \mu_i$, for all $1 \leq i \leq n$. If $\mu' = s_{r+1}$, we simply write $\mu'' \vdash \mu$.

Assume that $(\mu', \mu'') \vdash \mu$, then by definition we have

$$\Delta^j (k_i - v^{\mu_i + \mu_{i+1}}) = (k_i - v^{\mu_i' + \mu_{i+1}'}) \otimes k_i^{\mu_{n+1-i}'} \otimes (k_i - v^{\mu_i'' + \mu_{i+1}'}) k_i^{-\mu_{n+1-i}'},$$

$$+ (k_i - v^{\mu_i'' + \mu_{i+1}'}) \otimes (k_i^{\mu_{n+1-i}''}) k_i^{-1}.$$  

This induces a unique linear map

$$\Delta^j_{\mu', \mu'', \lambda, \lambda'} : \hat{\mu} \hat{U}^j \to \hat{\mu} \hat{U}^j \otimes \hat{\mu}' \hat{U}^j \otimes \hat{\mu}'' \hat{U}^j,$$

such that the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{U}^j & \xrightarrow{\Delta^j} & \mathcal{U}^j \\
\downarrow & & \downarrow \\
\hat{\mu} \hat{U}^j & \xrightarrow{\Delta^j_{\mu', \mu'', \lambda, \lambda'}} & \hat{\mu} \hat{U}^j \otimes \hat{\mu}' \hat{U}^j \otimes \hat{\mu}'' \hat{U}^j
\end{array}$$

(58)
Recall $\mathcal{B}$ is the canonical basis for $\hat{U}$. Let $\mathcal{B}'$ be the canonical basis for $\hat{U}'$ defined in [LW15].

**Theorem 4.3.1.** Let $a \in \mathcal{B}'$. If $h^\lambda_{\mu',\lambda'}(a) = \sum_{b \in \mathcal{B}', c \in \mathcal{B}} n^{b,c}_a b \otimes c$, then $n^{b,c}_a \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

The rest of this section is devoted to the proof of Theorem 4.3.1. For $\omega, \omega' \in \mathbb{Z}[P]$ and $\nu \in \mathbb{Z}[I]$, we write

\[(\omega', \nu) \models \omega,\]

if and only if $\omega'_i + \nu_i - \nu_{n-i} = \omega_i$, for all $1 \leq i \leq r$. If $\omega' = 0$, we simply write $\nu \models \omega$. By the definition of $\mathcal{B}'$, we have

**Lemma 4.3.2.** For any $\omega \in \mathbb{Z}[P]$, $\mathcal{M}(\mathcal{U}(\omega)) \subseteq \bigoplus_{(\omega', \nu) \models \omega} \mathcal{U}(\omega') \otimes \mathcal{U}(\nu)$.

The following lemma is a refinement of (59), and follows from Lemmas 4.3.2 and 4.2.1

**Lemma 4.3.3.** Assume that $\mu, \lambda, \mu', \lambda' \in \mathcal{X}$, $\omega, \omega' \in \mathbb{Z}[P]$, $\mu', \lambda' \in \mathcal{X}$, and $\nu \in \mathbb{Z}[I]$ such that $\mu - \lambda = \omega$, $\mu' - \lambda' = \omega'$, $\mu_{\nu} - \lambda_{\nu} = \nu$, $(\mu', \nu') \vdash \mu$, $(\lambda', \nu') \vdash \lambda$, $(\omega', \nu) \models \omega$. The following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{U}(\omega) & \xrightarrow{h^\lambda_{\mu',\lambda'}} & \mathcal{U}(\omega') \otimes \mathcal{U}(\nu) \\
\pi_{\mu,\lambda} \downarrow & & \downarrow \pi_{\mu',\lambda'} \otimes \pi_{\nu,\lambda'} \\
\mu \mathcal{U}_\lambda^2 \xrightarrow{h^\lambda_{\mu',\lambda'}} \mu' \mathcal{U}_{\lambda'}^2 \otimes \mu_{\nu} \mathcal{U}_{\lambda'}^2
\end{array}
\]

The assignment of sending generators of $\mathcal{U}'$ to the respective generators of $\mathcal{Q}(\nu) S^d_d$ defines an algebra homomorphism, denoted by

\[(61) \phi_d^\lambda : \mathcal{U}' \rightarrow \mathcal{Q}(\nu) S^d_d.
\]

Moreover, this algebra homomorphism is compatible with the gradings. In particular,

\[(62) \phi_d^\lambda(\mathcal{U}(\omega)) \subseteq \bigoplus_{b,a \in \Lambda_{d,n}, \mathfrak{B} - \mathfrak{B} = \omega} \mathcal{Q}(\nu) S^d_d(b, a), \quad \forall \omega \in \mathbb{Z}[P].
\]

On the other hand, we have

\[(63) h^\lambda (\mathcal{Q}(\nu) S^d_d(b, a)) \subseteq \bigoplus_{(b', a') \in \mathcal{Q}(\nu) S^{d'}_{d'}(b', a')} \mathcal{Q}(\nu) S^{d'}_{d'}(b'', a'')
\]

where $d = d' + d''$. By Lemma 4.3.2, (62), and (63), we have the following diagram.

**Lemma 4.3.4.** Assume that $b, a \in \Lambda^1_{d,n}$, $\omega, \omega' \in \mathbb{Z}[P]$, $\nu \in \mathbb{Z}[I]$ such that $(\omega', \nu) \models \omega$. The following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{U}(\omega) & \xrightarrow{h^\lambda_{\mu',\lambda'}} & \mathcal{U}(\omega') \otimes \mathcal{U}(\nu) \\
\phi_d^\lambda \downarrow & & \downarrow \phi_d^{\omega'} \otimes \phi_d^{\nu} \\
\bigoplus_{b,a \in \Lambda^1_{d,n}, \mathfrak{B} - \mathfrak{B} = \omega} \mathcal{Q}(\nu) S^d_d(b, a) & \xrightarrow{h^\lambda_{\mu',\lambda'}} & \bigoplus_{(b', a') \in \mathcal{Q}(\nu) S^{d'}_{d'}(b', a')} \mathcal{Q}(\nu) S^{d'}_{d'}(b'', a'')
\end{array}
\]
Recall from [LW15] that we have an algebra homomorphism
\[
\tilde{\phi}_d : \hat{U}^\gamma \rightarrow q(v) S_d^J
\]
defined by
\[
\tilde{\phi}_d^J(1_\chi) = \begin{cases} 
\zeta_{M_a}, & \text{if } \hat{\lambda} = \tilde{a}, \ a \in \Lambda_{d,n}^J, \\
0, & \text{o.w.}
\end{cases}
\]
\[
\tilde{\phi}_d^J(e_i 1_\chi) = \begin{cases} 
\epsilon_i \zeta_{M_a}, & \text{if } \hat{\lambda} = \tilde{a}, \ a \in \Lambda_{d,n}^J, \\
0, & \text{o.w.}
\end{cases}
\]
By restricting to \(\hat{\mu} U_{\lambda}^J\), it induces a linear map
\[
\tilde{\phi}_d^J : \hat{\mu} U_{\lambda}^J \rightarrow q(v) S_d^J(b, a), \text{ if } \hat{\mu} = \hat{b}, \hat{\lambda} = \hat{a}.
\]
In particular, we have the following lemma.

**Lemma 4.3.5.** Assume that \(b^0, a^0 \in \Lambda_{d,n}^J\) such that \(\hat{b}^0 - \tilde{a}^0 = \omega \in \mathbb{Z}[J]\). The following diagram is commutative.

\[
\begin{array}{ccc}
U^J(\omega) & \xrightarrow{\pi_{b^0, a^0}} & \hat{b}^0 U_{\lambda}^J_{a^0} \\
\downarrow \phi_d^J & & \downarrow \phi_d^J \\
\oplus_{\hat{b} - \tilde{a} = \omega} q(v) S_d^J(b, a) & \longrightarrow & q(v) S_d^J(b^0, a^0)
\end{array}
\]

where the arrow in the bottom is the natural projection.

By putting together Lemmas 4.3.4, 4.3.3 and (66), we have the following cube.

\[
\begin{array}{ccc}
\hat{\mu} U_{\lambda}^J & \xrightarrow{\pi_{\hat{\mu}, \hat{\lambda}}} & \hat{\mu} U_{\lambda}^J \otimes \hat{\mu} U_{\lambda}^J \\
\downarrow \phi_d^J & & \downarrow \phi_d^J \\
S_d^J(b, a) & \longrightarrow & S_d^J(b', a') \otimes S_d^J(b'', a'')
\end{array}
\]

where the sum on the bottom left is over all \(b, a \in \Lambda_{d,n}^J\) such that \(\hat{b} - \tilde{a} = \omega\), while the sum on the bottom right is over \(b', a' \in \Lambda_{d',n}^J\) and \(b'', a'' \in \Lambda_{d'',n}^J\) such that \(\hat{b}', \hat{a}' \vdash b', (a', a'') \vdash a, \hat{b}' - \tilde{a}' = \omega, \text{ and } \hat{b}'' - \tilde{a}'' = \nu\).

From (67) and the surjectivity of \(\pi_{\hat{\mu}, \hat{\lambda}}\), we have the following proposition.

**Proposition 4.3.6.** The square in the back of (67) is commutative.

\[
\begin{array}{ccc}
\hat{\mu} U_{\lambda}^J & \xrightarrow{\pi_{\hat{\mu}, \hat{\lambda}}} & \hat{\mu} U_{\lambda}^J \otimes \hat{\mu} U_{\lambda}^J \\
& & \downarrow \phi_d^J \\
S_d^J(b, a) & \longrightarrow & S_d^J(b', a') \otimes S_d^J(b'', a'')
\end{array}
\]

By using Proposition 4.3.6, we deduce Theorem 4.3.1 via a similar argument for Theorem 1.6.5.
4.4. Positivity with respect to \( \gamma \). Notice that for \( \mu, \mu'' \in \mathbb{Z}^n \) such that \( \mu'' \vdash \mu \), we have

\[
    j((k_i - v^{-\mu_i+\mu_{i+1}})U) \subseteq \sum_{1 \leq i \leq n} (k_i - v^{-\mu''_i+\mu''_{i+1}})U.
\]

Similarly, for any \( \lambda, \lambda'' \in \mathbb{Z}^n \) such that \( \lambda'' \vdash \lambda \) we have

\[
    j(U((k_i - v^{-\lambda_i+\lambda_{i+1}}))) \subseteq \sum_{1 \leq i \leq n} U((k_i - v^{-\lambda''_i+\lambda''_{i+1}})).
\]

The above observations induce a linear map

\[ j_{\hat{\mu}, \hat{\lambda}, \mu''}, \lambda'' : \hat{\mu}U_{\lambda} \rightarrow \mu''U_{\lambda''}, \forall \mu'' \vdash \mu, \lambda'' \vdash \lambda \]

such that the following diagram commutes.

\[
\begin{array}{ccc}
U_{\lambda} & \xrightarrow{\gamma} & U \\
\downarrow{\pi_{\hat{\mu}, \hat{\lambda}}} & & \downarrow{\pi_{\mu''}, \lambda''} \\
\hat{\mu}U_{\lambda} & \xrightarrow{j_{\hat{\mu}, \hat{\lambda}, \mu'', \lambda''}} & \mu''U_{\lambda''}
\end{array}
\]

**Theorem 4.4.1.** Let \( b \in \mathcal{B} \). If \( j_{\hat{\mu}, \hat{\lambda}, \mu''}, \lambda'' (b) = \sum_{a \in \mathcal{B}} g(b, a) a \), then \( g(b, a) \in \mathbb{Z}_{\geq 0}[v, v^{-1}] \).

The proof of Theorem 4.4.1 is a degenerate version of the proof of Theorem 4.3.1. For the sake of completeness, we provide it here. The following lemma is due to the fact that \( \gamma(e_i) \in U(i) + U((-n - i)) \), \( \gamma(f_i) \in U(-i) + U(n - i) \), \( \gamma(k_i^\pm 1) \in U(0) \).

**Lemma 4.4.2.** For any \( \omega \in \mathbb{Z}[P] \), \( j(U(\omega)) \subseteq \bigoplus_{\nu \vdash \omega} U(\nu) \).

From Lemmas 4.4.2 and 4.2.1, we have the following refinement of (69).

**Lemma 4.4.3.** Assume that \( \hat{\mu}, \hat{\lambda} \in \mathcal{X} \), \( \omega \in \mathbb{Z}[P] \), \( \mu'', \lambda'' \in \mathcal{X} \), and \( \nu \in \mathbb{Z}[I] \) such that

\[
\hat{\mu} - \hat{\lambda} = \omega, \mu'' - \lambda'' = \nu, \mu'' \vdash \mu, \lambda'' \vdash \lambda, \nu \vdash \omega.
\]

The following diagram commutes.

\[
\begin{array}{ccc}
U(\omega) & \xrightarrow{j_{\omega, \nu}} & U(\nu) \\
\downarrow{\pi_{\hat{\mu}, \hat{\lambda}}} & & \downarrow{\pi_{\mu''}, \lambda''} \\
\hat{\mu}U_{\lambda} & \xrightarrow{j_{\hat{\mu}, \hat{\lambda}, \mu'', \lambda''}} & \mu''U_{\lambda''}
\end{array}
\]

where \( j_{\omega, \nu} \) is the one induced from \( j \) by restricting to \( U(\omega) \) and projecting down to \( U(\nu) \).

Note that we have

\[
    j_d(q(v)S_d^I(b, a)) \subseteq \bigoplus_{b''+b, a''+a} q(v)S_d(b'', a''),
\]

By Lemma 4.4.2, (62), and (70), we have the following commutative diagram.
**Lemma 4.4.4.** Assume that \( b, a \in \Lambda_{d,n}, \omega \in \mathbb{Z}[I], \nu \in \mathbb{Z}[I] \) such that \( \nu_i - \nu_{n-i} = \omega_i \) for any \( i \in I \). The following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{U}(\nu) & \xrightarrow{\phi_d} & \mathcal{U}(\omega) \\
\downarrow & & \downarrow \\
\oplus_{b,a \in \Lambda_{d,n}} S_d(b, a) & \xrightarrow{Q(\nu)} & \oplus_{b'', a''} S_d(b'', a'')
\end{array}
\]

where the condition (*) is \( b'' - a'' = \nu, b'' \vdash b \) and \( a'' \vdash a \).

By putting together Lemmas 4.4.4, 4.4.3 and (66), we have the following cube.

\[
\begin{array}{ccc}
\mathcal{U}(\omega) & \xrightarrow{\pi_{\hat{\mu}, \hat{\lambda}}} & \mathcal{U}(\nu) \\
\downarrow & & \downarrow \\
\oplus_{b,a} S_d(b, a) & \xrightarrow{Q(\nu)} & \oplus_{b'', a''} S_d(b'', a'')
\end{array}
\]

From (71) and the surjectivity of \( \pi_{\hat{\mu}, \hat{\lambda}} \), we have the following proposition.

**Proposition 4.4.5.** The square in the back of (71) is commutative.

\[
\begin{array}{ccc}
\mathcal{U}(\omega) & \xrightarrow{\pi_{\hat{\mu}, \hat{\lambda}}} & \mathcal{U}(\nu) \\
\downarrow & & \downarrow \\
\oplus_{b,a} S_d(b, a) & \xrightarrow{Q(\nu)} & \oplus_{b'', a''} S_d(b'', a'')
\end{array}
\]

Recall that for any \( b \in \mathbb{B}^d \), we suppose that

\[
\phi_d^J(b) = \sum_{a \in \mathbb{B}} g_{b,a}a,
\]

where \( g_{b,a} \in \mathbb{Z}[v, v^{-1}] \) is zero except for finitely many terms. Let \( S = \{ a | g_{b,a} \neq 0 \} \). Since the set \( S \) is finite, we can find a large enough \( d \) using \[LW15\] and \[M10\] such that

\[
\phi_d^J(b) = \{ B \}_d, \phi_d^J(a) = \{ A \}_d, \forall a \in S
\]

where \( \{ B \}_d \) and \( \{ A \}_d \) are certain canonical basis elements in \( S_d \) and \( S_d \), respectively. Applying \( \phi_d^J \) and Lemma 4.4.5, we have

\[
\phi_d^J(b) = \phi_d^J(\phi_d^J(b)) = \phi_d^J(b)\pi_{\hat{\mu}, \hat{\lambda}}(\nu) = \sum_{a \in \mathbb{B}} g_{b,a}a = \sum_{a \in \mathbb{B}} g_{b,a}\{ A \}_d.
\]

By comparing the above with (56), we have \( g_{b,a} = g_{B,A} \). So we have \( g_{b,a} \in \mathbb{Z}_{\geq 0}[v, v^{-1}] \) by Corollary 3.7.6. Theorem 4.4.1 follows.
4.5. The imbedding $\tilde{j}$. For any pair $(\mu, \lambda)$ in $\mathcal{H}$, we define

$$j_{\tilde{\mu}, \tilde{\lambda}} = \prod_{\tilde{\mu}, \tilde{\lambda}, \tilde{\mu}''} j_{\tilde{\mu}, \tilde{\lambda}, \tilde{\mu}''} : \tilde{\mu} \cup \tilde{\lambda} \rightarrow \prod_{\tilde{\mu}, \tilde{\lambda}} \tilde{\mu} \cup \tilde{\lambda},$$

where the product runs over all $\tilde{\mu}''$, $\tilde{\lambda}$ in $\mathcal{H}$ such that $\mu'' + \mu$ and $\lambda'' + \lambda$. We set

$$\tilde{j} \equiv \bigoplus_{\mu, \lambda} j_{\tilde{\mu}, \tilde{\lambda}} : \tilde{\mu} \cup \tilde{\lambda} \rightarrow \bigoplus_{\tilde{\mu}, \tilde{\lambda}} \tilde{\mu} \cup \tilde{\lambda}.$$

Proposition 4.5.1. $\tilde{j}$ is injective.

Proof. It suffices to show that for any nonzero element $x$ in $\tilde{\mu} \cup \tilde{\lambda}$, there is $\tilde{\mu}''$ and $\tilde{\lambda}$ such that $j_{\tilde{\mu}', \tilde{\lambda}}(x)$ is nonzero. Suppose that $\tilde{\mu} - \tilde{\lambda} = \omega$. Let us pick an element $u \in \mathcal{U}(\omega)$ such that $\tilde{\mu}_u(\omega) = x$. Since $\tilde{j}$ is injective, we have $j(u) \neq 0 \in \bigoplus_{\nu} \mathcal{U}(\nu)$. Thus there is $\nu$ such that the $\nu$-component $j(u)_{\nu}$ of $j(u)$ is nonzero. It is well-known (see [L00]) that we can then find a large enough $d$ such that $\phi_d(j(u)_{\nu}) \neq 0$. In particular, there is a pair $b''$, $a''$ in $\Lambda_{a,n}$ such that the $(b'', a'')$-component of $\phi_d(j(u)_{\nu})$ is nonzero. Take $\tilde{\mu}'' = B''$ and $\tilde{\lambda}'' = a''$. By chasing along the cube (71), we see immediately that $j_{\tilde{\mu}', \tilde{\lambda}', \tilde{\mu}''} \neq 0$. \hfill $\square$

Remark 4.5.2. $\tilde{j}$ can be regarded as an idempotented version of $j$.

5. $\nu$-version

In this section, we show the positivity of the $i$-canonical basis of the modified coideal subalgebra of quantum $\mathfrak{sl}_{\ell}$ for $\ell$ even. Since the arguments are more or less the same as the $n$ odd situation, the presentation will be brief.

5.1. $\nu$-Schur algebras and related results. Recall $n = 2r + 1$ and $D = 2d + 1$. We set

$$\ell = n - 1.$$

Recall $\Xi^d$ from (54). Let $\Xi^d = \{ A \in \Xi^d | a_{i+1,j} = \delta_{j,r+1}, a_{i,r+1} = 0 \}$. Let $j = \sum[A]_{d}$ where the sum runs over all diagonal matrices in $\Xi^d$. Let $S_{d,\ell} = jS_{d,n}$. It is a subalgebra in $S_{d,n}$ and admits a basis $[A]_{d}$ for all $A \in \Xi^d$. In particular, $S_{d,\ell}$ contains the following elements.

$$e_{i,d} = je_{i}, \quad f_{i,d} = jf_{i}, \quad k_{i,d} = jk_{i}, \quad \forall i \in [1, r - 1], \quad h_{a,d} = jh_{a}, \quad \forall a \in [1, r],$$

(72)

$$t_{d} = j(f_{e} + k_{e} - k_{e}^{-1})j.$$
Notice that we have $JH_{r+1}J = 1$ and $\hat{K}_{r,d} = \hat{H}_{r,d}^{-1}\hat{H}_{r+1,d}$.

**Lemma 5.1.1.** Let $d' + d'' = d$. $\bar{\Delta}^j(S_{d,\ell}^i) \subseteq S_{d',\ell}^i \otimes S_{d'',\ell}$. Moreover, for all $i \in [1, r - 1]$.

\[
\begin{align*}
\bar{\Delta}^j(\bar{e}_{i,d}) &= \bar{e}_{i,d'} \otimes \hat{H}_{r+1,d}^{-1}\hat{H}_{r-1,d'}^{-1} + \bar{h}_{i+1,d'}^{-1} \otimes \bar{e}_{i,d'} \hat{H}_{r-1,d'}^{-1} + \bar{h}_{i+1,d'}^{-1} \otimes \bar{f}_{i,d'} \hat{H}_{r+1,d'}^{-1}, \\
\bar{\Delta}^j(\bar{f}_{i,d}) &= \bar{f}_{i,d'} \otimes \hat{H}_{r-1,d}^{-1}\hat{H}_{r+1,d'}^{-1} + \bar{h}_{i+1,d'}^{-1} \otimes \bar{f}_{i,d'} \hat{H}_{r+1,d'}^{-1} + \bar{h}_{i+1,d'}^{-1} \otimes \bar{e}_{i,d'} \hat{H}_{r-1,d'}^{-1}.
\end{align*}
\]

(73)

\[
\bar{\Delta}^j(\bar{k}_{i,d}) = \bar{k}_{i,d'} \otimes \bar{K}_{i,d''}^{-1} \hat{K}_{r+1,d'}^{-1}.
\]

\[
\bar{\Delta}^j(\bar{t}_{i,d}) = \bar{t}_{d} \otimes \bar{K}_{r,d''}^{-1} + v^2 \bar{k}_{r,d'}^{-1} \otimes \hat{H}_{r+1,d''}^{-1} \bar{F}_{r,d''} + v^{-2} \bar{k}_{r,d'}^{-1} \otimes \hat{H}_{r,d''}^{-1} \bar{E}_{r,d''}.
\]

**Proof.** For convenience, we shall drop the subscript $d$ and replace $d', d''$ by superscript 1 and 2 respectively in the proof. The first three equalities are from definitions and $\bar{\Delta}^j(j) = j' \otimes J''$.

We now show the last one. By using $\bar{f}_{j,1} = 0$ and $\bar{f}_{e,1} = 0$, we have

\[
\bar{\Delta}^j(\bar{f}_{j,e,1}) = j' \otimes \bar{K}_{r,d}'' + \bar{k}_{r,d'}^{-1} \otimes \hat{H}_{r+1,d''}^{-1} \bar{F}_{r,d''} + v^{-2} \bar{k}_{r,d'}^{-1} \otimes \hat{H}_{r,d''}^{-1} \bar{E}_{r,d''}.
\]

We observe that $\bar{j}_{r,1} \bar{h}_{r,1} = \bar{k}_{r}$ and $\bar{j}_{r,1} \bar{h}_{r,1} = v^{-2} \bar{k}_{r}^{-1}$. We further observe that

\[
\begin{align*}
JF_r H_{r-1} E_r H_{r+1} J &= \hat{H}_{r+1} \hat{H}_{r} - \hat{H}_{r}^{-1}, \\
JE_r H_{r+1} H_{r}^{-1} F_{r+1} H_{r+1} J &= \hat{H}_{r+1} \hat{H}_{r} - \hat{H}_{r}^{-1}, \\
JE_r H_{r+1} H_{r}^{-1} E_r H_{r+1} J &= \hat{H}_{r+1} \hat{H}_{r} - \hat{H}_{r}^{-1}.
\end{align*}
\]

So we have

\[
\bar{\Delta}^j(\bar{t}_{i,d}) = \bar{t}_{d} \otimes \bar{K}_{r,d''}'' + v^2 \bar{k}_{r,d'}^{-1} \otimes \hat{H}_{r+1,d''}^{-1} \bar{F}_{r,d''} + v^{-2} \bar{k}_{r,d'}^{-1} \otimes \hat{H}_{r,d''}^{-1} \bar{E}_{r,d''} + R,
\]

where the remainder $R$ is equal to

\[
- \frac{\bar{k}_{r} - \bar{k}_{r}^{-1}}{v - v^{-1}} \otimes \bar{K}_{r} + \bar{k}_{r}^{-1} \otimes \hat{H}_{r+1} \frac{\hat{H}_{r} - \hat{H}_{r}^{-1}}{v - v^{-1}} + \bar{k}_{r} \otimes \hat{H}_{r} \frac{\hat{H}_{r+1} - \hat{H}_{r}^{-1}}{v - v^{-1}}
\]

\[
+ \frac{\bar{k}_{r} \otimes \hat{H}_{r} \hat{H}_{r+1}^{-1} - \bar{k}_{r}^{-1} \otimes \hat{H}_{r+1} \hat{H}_{r}}{v - v^{-1}}.
\]

We combine the terms with $\bar{K}_{r}$ together and we get zero. So is the case when we combine the terms with $\bar{K}_{r}^{-1}$. Hence $R$ is zero. Therefore, we have the last equality in the lemma. \[\Box\]

We define the transfer map

\[
\phi_{d,d-\ell}^i : S_{d,\ell}^i \to S_{d-\ell,\ell}^i
\]

to be the composition $S_{d,\ell}^i \xrightarrow{\bar{\Delta}^j} S_{d-\ell,\ell}^i \otimes S_{\ell,\ell}^1 \xrightarrow{1 \times \chi_{\ell}} S_{d-\ell,\ell}^i$, where $\chi_{\ell} : S_{\ell,\ell} \to A$ is the signed representation. By Lemma 5.1.1, we have

**Lemma 5.1.2.** $\phi_{d,d-\ell}(\bar{e}_{i,d}) = \bar{e}_{i,d-\ell}$, $\phi_{d,d-\ell}(\bar{f}_{i,d}) = \bar{f}_{i,d-\ell}$, $\phi_{d,d-\ell}(\bar{k}_{i,d}) = \bar{k}_{i,d-\ell}$ and $\phi_{d,d-\ell}(\bar{t}_{i,d}) = \bar{t}_{d-\ell}$ for all $i \in [1, r - 1]$. 

(74)
Now we handle the case of $\Delta^j$.

**Proposition 5.1.3.** For $i \in [1, r - 1]$,

$$\Delta^j(\mathbf{e}_{i,d}) = \mathbf{e}_{i,d'} \otimes \hat{K}_{i,d''} + 1 \otimes \hat{E}_{i,d'} + \hat{K}_{i,d'} \otimes \hat{F}_{\ell-i,d''} \hat{K}_{i,d''},$$

$$\Delta^j(\mathbf{f}_{i,d}) = \mathbf{f}_{i,d'} \otimes \hat{K}_{\ell-i,d''} + \hat{K}_{i,d'} \otimes \hat{F}_{i,d''} + 1 \otimes \hat{E}_{\ell-i,d''},$$

$$\Delta^j(\mathbf{k}_{i,d}) = \hat{K}_{i,d'} \otimes \hat{K}_{i,d''} \hat{K}_{\ell-i,d''},$$

$$\Delta^j(\mathbf{t}_{d}) = \mathbf{t}_{d'} \otimes \hat{K}_{r,d''} + 1 \otimes v \hat{K}_{r,d''} \mathbf{F}_{r,d''} + 1 \otimes \hat{E}_{r,d''}. \tag{75}$$

**Proof.** Again only the last equality is nontrivial, and we drop the subscript $d$ and $d', d''$ are replaced by $'$ and $''$, respectively. Suppose that we have a quadruple $(b', a', b'', a'')$ such that $b_k' = a_k'$ and $b_k'' = a_k''$ for all $k$, then the twists $\sum_{1 \leq k \leq j \leq n} b_k' b_j'' - a_k' a_j''$ and $u(b'', a'')$ are zero. Hence we have the first term $\mathbf{t} \otimes \hat{K}_r$ after the twist.

Suppose that we have a quadruple $(b', a', b'', a''')$ such that $b_k' = a_k'$ and $b_k'' = a_k'' + \delta_{k,r} - \delta_{k,r+2}$, then we have $\sum_{1 \leq k \leq j \leq n} b_k' b_j'' - a_k' a_j'' = -(a_{r+2}' + 1)$ and $u(b'', a''') = -a_{r+2}'$. So after the twist, we have $u^2 k_{\ell-1}^r \otimes H_{r+1}^n \mathbf{F}_{r'}^{|b',a',b'',a'''}|v^{-a_{r+2}'''' - 1} = 1 \otimes v \hat{K}_r\mathbf{F}_{r'}|b',a',b'',a'''}$. Whence we obtain the second term.

Suppose that we have a quadruple $(b', a', b'', a''')$ such that $b_k' = a_k'$ and $b_k'' = a_k'' + \delta_{k,r} + \delta_{k,r+2}$. Then we have $\sum_{1 \leq k \leq j \leq n} b_k' b_j'' - a_k' a_j'' = a_{r+2}' + 1$ and $u(b'', a''') = a_{r+2}'$. Thus, adding the twists, we have $v^{-2} k_{\ell-1}^r \otimes H_{r+1}^n \mathbf{F}_{r'}|b',a',b'',a'''} = 1 \otimes v \hat{K}_r\mathbf{F}_{r'}|b',a',b'',a'''}$. Whence we obtain the third term.

By the above analysis, we have the last equality. The proposition is proved. \hfill \Box

Now we take care of the degenerate version when $d' = 0$ and $d'' = d$. In this case, $\Delta^j$ degenerates to an algebra homomorphism

$$i_d : S^1_{d,\ell} \to S_{d,\ell},$$

since $S_{0,n-1} \simeq \mathcal{A}$. Observe that $\mathbf{e}_{i,0} = 0$, $\mathbf{f}_{i,0} = 0$ and $\mathbf{k}_{i,0} = v^{b_i}$, for all $i \in [1, r]$, and $\mathbf{t}_0 = 1$ in $S_{0,n-1}$ from which the statements in Proposition 5.1.3 now read as follows.

**Corollary 5.1.4.** For all $i \in [1, r - 1]$,

$$i_d(\mathbf{e}_{i,d}) = \hat{E}_{i,d} + \hat{K}_{i,d} \hat{F}_{\ell-i,d'}, \quad i_d(\mathbf{f}_{i,d}) = \hat{E}_{\ell-i,d} + \hat{K}_{\ell-i,d} \hat{F}_{i,d'}, \quad i_d(\mathbf{k}_{i,d}) = \hat{K}_{i,d} \hat{K}_{\ell-i,d},$$

$$i_d(\mathbf{t}_{d}) = \hat{K}_{r,d} \hat{F}_{r,d} + \hat{K}_{r,d}. \tag{76}$$

**Proposition 5.1.5.** $i_d$ is injective.

**Proof.** This is because $j_d$ is injective by Proposition 3.7.5. \hfill \Box

Let $\Delta^i : S^i_{d,\ell} \to S^i_{d',\ell} \otimes S^i_{d'',\ell}$ be the homomorphism induced from $\Delta^j$. By Proposition 3.7.2

**Proposition 5.1.6.** Let $M \in \Xi_d$. If $\mathcal{A}^j(M) = \sum_{M' \in \Xi_{d'}, M'' \in \Xi_{d''}} h_{M',M''} M' \otimes M'', \quad \text{then we have } h_{M,M''} \in \mathbb{Z}_{\geq 0}[v, v^{-1}].$

Write $i_d(\{B\}) = \sum_{A \in \Xi_{d,\ell}} g_{B,A} \{A\}$. By Propositions 5.1.6, we have

**Corollary 5.1.7.** $g_{B,A} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$. 
Recall $T^d_{n,n}$ and $\Pi^d_{n,n}$ from ([51]). Note that $T^d_{n,n}$ is defined over $A$, but can be lifted to its generic version $T^d_{n,n}$. Let $\Pi^d_{d,\ell}$ be the subset of $\Pi^d_{n,n}$ defined by $a_{r+1,d+1} = 1$. Let $T^d_{d,\ell}$ be the space of $T^d_{n,n}$ spanned by $[A]_d$ where $A \in \Pi^d_{d,\ell}$. In the same fashion, let $T^d_{d,n}$ be the generic version of $T^d_{d,n}$ in ([50]), and let $\Pi^d_{d,\ell}$ be the subset of $\Pi^d_{d,n}$ defined by $a_{r+1,d+1} = 1$. Similarly, we have $T^d_{a''}$.

Let $H_A$ and $H_B$ be the generic version of the Hecke algebra $H_{A_d}$ and $H_{B_d}$ used in Proposition [3.5.1]. The following is the $\gamma$-analogue of Proposition [3.5.1] obtained by restricting the diagram therein to the desired subspaces.

**Proposition 5.1.8.** We have the following commutative diagram.

$$
\begin{array}{ccc}
S^d_{d,\ell} \times T^d_{d,\ell} & \longrightarrow & T^d_{d,\ell} \\
\downarrow_{\delta_{d,\ell} \times \delta_{d,\ell}} & & \downarrow_{\delta_{d,\ell} \times \delta_{d,\ell}} \\
S^d_{d,\ell} \times T^d_{d,\ell} & \longrightarrow & T^d_{d,\ell} \\
\end{array}
$$

Moreover, $\delta_{d,\ell}([A^J]_d) = [A]_d$ for all $A \in \Pi^d_{d,\ell}$.

5.2. **Positivity in the projective limit.** Consider the projective system $(S^d_{d,\ell}, \delta_{d,\ell})_{d \in \mathbb{Z}_{\geq 0}}$ of associative algebras. We define an element $e_i$ in the projective system whose $d$-th component is $e_{i,d}$ for all $d \in \mathbb{Z}_{\geq 0}$. This is well-defined by Lemma 5.1.2. Similarly, we can define $f_i$, $\bar{t}_i$ and $\bar{t}_i \pm 1$.

Let $U^d_{\ell}$ be the subalgebra of the projective system generated by $e_i, f_i, \bar{t}_i \pm 1$ for all $i \in [1, r-1]$ and $\bar{t}_i$. By [LW15], this is a coideal subalgebra of the quantum $s_t$ for $n$ even. A presentation of this algebra by generators and relations can also be found in [LW15].

Set $\Xi_{\ell}^d = \bigcup_{d \in \mathbb{Z}_{\geq 0}} \Xi_d$. We say two matrices are equivalent if they differ by an even multiply of the identity matrix $I_d$. We denote $\Xi_{\ell}^d \approx \Xi_{\ell}$ for the set of equivalence classes.

By [LW15], we know that $\phi_{d,\ell}^d(A) = \{A - 2I_d\}$ if the diagonal entries of $A$ are large enough. To an element $A \in \Xi_{\ell}^d \approx$, we define an element $b_A$ in the projective system whose $d$-th component is $\{A + pI_d\}$ for some $p$ if $d$ is big enough.

Let $\hat{U}^d \equiv \hat{U}^d_{\ell}$ be the space spanned by $b_A$ for $A \in \Xi_{\ell}^d \approx$. By [LW15], $\hat{U}^d$ is an associative algebra, the idempotented version of $U^d_{\ell}$ and $b_A$ forms the canonical basis $B^d$ of $\hat{U}^d$ defined in [LW15]. Let $\mathcal{X}^d_{\ell}$ be the subset of $A \in \Xi_{\ell}^d \approx$ parametrized by the diagonal matrices. This algebra admits a decomposition $\hat{U}^d = \bigoplus_{\mu, \lambda} \hat{U}^d_{\mu, \lambda}$ where $\hat{U}^d_{\mu, \lambda} = b_{\mu} \hat{U} \hat{U} \hat{U}^d_{\lambda}$.

Replace the projective system $(S^d_{d,\ell}, \phi_{d,\ell}^d)$ by $(S^d_{d,\ell}, \phi_{d,\ell}^d)$, we can define the elements $\bar{t}_i$ and $\bar{t}_i \pm 1$ in this projective system and they generate over $Q(t)$ the quantum $s_t$: $\hat{U}^d$.

Set $\Xi_{\ell}^d = \bigcup_{d \in \mathbb{Z}_{\geq 0}} \Xi_d$. We say two matrices are equivalent if they differ by a multiply of the identity matrix $I_d$. We denote $\Xi_{\ell}^d \approx \Xi_{\ell}$ for the set of equivalence classes. To an element $A \in \Xi_{\ell}^d \approx$, we define an element $b_A$ in the projective system whose $d$-th component is $\{A + pI_d\}$ for some $p$ if $d$ is big enough. Then the space $\hat{U}^d$ spanned by $b_A$ for all $A \in \Xi_{\ell}^d \approx$ is an associative algebra, the idempotented version of $U^d_{\ell}$ by [M10] and $b_A$ forms the canonical basis $B^d$. Let $\mathcal{X}^d_{\ell}$ be the subset of $\Xi_{\ell}^d \approx \sim$ consisting of all diagonal matrices. $\hat{U}^d = \bigoplus_{\mu, \lambda} \hat{U}^d_{\mu, \lambda}$, where $\hat{U}^d_{\mu, \lambda} = b_{\mu} \hat{U} \hat{U}$.

The linear map $\Delta^d_{\mu, \lambda}$ on the iSchur algebra level induces a linear map

$$
\Delta^d_{\mu, \lambda, \mu'', \lambda''} : \hat{U}^d_{\mu, \lambda} \to \hat{U}^d_{\mu, \lambda} \otimes \hat{U}^d_{\lambda, \lambda''}, \quad \forall (\mu', \mu'') \vdash \mu, (\lambda', \lambda'') \vdash \lambda
$$
where $\vdash$ is defined similar to [58] with row vectors replaced by diagonal matrices. Write 

$$\Delta_{\mu''}^{\mu} : \mu \mapsto \mu'' \mu''' = \sum_{b \in B^t, c \in B} n^b_a c \otimes c,$$

for all $a \in B^t$, then we have the $\iota$-analogue of Theorem 4.3.1 whose proof is the same as the Theorem using Proposition 5.1.6.

**Theorem 5.2.1.** $n^b_a c \in Z_{\geq 0}[v, v^{-1}]$.

The linear map $\iota_d$ induces a linear map

$$i_{\tilde{\mu}, \tilde{\lambda}, \tilde{\pi}, \tilde{\sigma}} : \mu \mapsto \mu'' \mu''' \mapsto \mu'' \mu''', \quad \forall \mu'' \vdash \mu, \lambda'' \vdash \lambda$$

We have the $\iota$-analogue of Theorem 4.4.1 by using Corollary 5.1.7.

**Theorem 5.2.2.** Let $b \in B^t$. If $i_{\tilde{\mu}, \tilde{\lambda}, \tilde{\pi}, \tilde{\sigma}}(b) = \sum_{a \in B^t} g_{b,a} a$, then $g_{b,a} \in \mathcal{Z}_{\geq 0}[v, v^{-1}]$.

**REFERENCES**

[BDPW] B. Deng, J. Du, B. Parshall and J. Wang, *Finite dimensional algebras and quantum groups*. Mathematical Surveys and Monographs, 150. American Mathematical Society, Providence, RI, 2008.

[BLM] A. Beilinson, G. Lusztig and R. MacPherson, *A geometric setting for quantum deformations of $GL_n$*, Duke Math. J. 61 (1990), 655–677.

[BW13] H. Bao and W. Wang, *A new approach to Kazhdan-Lusztig theory of type $B$ via quantum symmetric pairs*, arXiv:1310.0103.

[BKLW14] H. Bao, J. Kujawa, Y. Li and W. Wang, *Geometric Schur duality of classical type, with an appendix by H. Bao, Y. Li and W. Wang*, arXiv:1404.4000v3.

[BBD82] A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, Asterisque 100 (1982).

[BLM90] A. Beilinson, G. Lusztig and R. MacPherson, *A geometric setting for the quantum deformation of $GL_n$*, Duke Math. J., 61 (1990), 655-677.

[B03] T. Braden, *Hyperbolic localization of intersection cohomology*, Transf. Groups 8 (2003), 209-216.

[DF13] J. Du and Q. Fu, *Quantum affine $\mathfrak{gl}_n$ via Hecke algebras*, arXiv:1311.1868.

[DS00] J. Du and L. Scott, *The $q$-Schur algebra*, Transctions of AMS, 352 (2000), 4355-4369.

[ES13] M. Ehrig and C. Stroppel, *Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality*, arXiv:1310.1972.

[FL14] Z. Fan, C. Lai, Y. Li, L. Luo, and W. Wang, in preparation.

[GL92] I. Grojnowski and G. Lusztig, *On bases of irreducible representations of quantum $GL_n$, in Kazhdan-Lusztig theory and related topics* (Chicago, IL, 1989), 167-174, Contemp. Math., 139, Amer. Math. Soc., Providence, RI, 1992.

[J95] M. Kashiwara, *Crystal bases of modified quantized enveloping algebra*, Duke Math. J. 73 (1994) 383-413.

[KL79] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. 53 (1979), no. 2, 165-184.

[L93] G. Lusztig, *Introduction to quantum groups*, Progress in Math. 110. Birkhäuser 1993.

[L94] G. Lusztig, *Cells in affine Weyl groups and tensor categories*, Adv. Math. 129 (1997), no. 1, 85-98.

[L99] G. Lusztig, *Aperiodicity in quantum affine $\mathfrak{sl}_n$, Asian J. Math. 3 (1999), 147-178.

[L00] G. Lusztig *Transfer maps for quantum affine $\mathfrak{sl}_n$, in Representations and quantizations (Shanghai, 1998)*, 341-356, China High. Educ. Press, Beijing, 2000.

[M10] K. McGerty, *On the geometric realization of the inner product and canonical basis for quantum affine $\mathfrak{sl}_n$, Algebra and Number Theory 6 (2012), 1097–1313.*
[S06] O. Schiffmann, *Lectures on Hall algebras*, arXiv:math/0611617.

[SV00] O. Schiffmann and E. Vasserot, *Geometric construction of the global base of the quantum modified algebra of $\hat{\mathfrak{gl}}_n$,* Transform. Groups 5 (2000), 351–360.

School of science, Harbin Engineering University, Harbin, China 150001

*E-mail address:* fanz@math.ksu.edu (Z. Fan)

Department of Mathematics, University at Buffalo, SUNY, Buffalo, NY 14260

*E-mail address:* yiqiang@buffalo.edu (Y. Li)