Determining Lamé coefficients by the elastic Dirichlet-to-Neumann map on a Riemannian manifold

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Abstract

For the Lamé operator $\mathcal{L}_{\lambda, \mu}$ with variable coefficients $\lambda$ and $\mu$ on a smooth compact Riemannian manifold $(M, g)$ with smooth boundary $\partial M$, we give an explicit expression for the full symbol of the elastic Dirichlet-to-Neumann map $\Lambda_{\lambda, \mu}$. We show that $\Lambda_{\lambda, \mu}$ uniquely determines the partial derivatives of all orders of the Lamé coefficients $\lambda$ and $\mu$ on $\partial M$. Moreover, for a nonempty smooth open subset $\Gamma \subset \partial M$, suppose that the manifold and the Lamé coefficients are real analytic up to $\Gamma$, we prove that $\Lambda_{\lambda, \mu}$ uniquely determines the Lamé coefficients on the whole manifold $\bar{M}$.

Keywords: Lamé system, inverse problems, elastic Calderón problem, elastic Dirichlet-to-Neumann map, pseudodifferential operators

1. Introduction

Let $(M, g)$ be a smooth compact Riemannian manifold of dimension $n$ with smooth boundary $\partial M$. In this paper, we consider $M$ as an inhomogeneous, isotropic, elastic medium. Assume that the Lamé coefficients $\lambda, \mu \in C^\infty (\bar{M})$ of this elastic medium satisfy

$$\mu > 0, \quad \lambda + \mu \geq 0. \quad (1.1)$$

In the local coordinates $\{x_i\}_{i=1}^n$, we denote by $\{\frac{\partial}{\partial x_i}\}_{j=1}^n$ and $\{dx_j\}_{j=1}^n$, respectively, the natural basis for the tangent space $T_xM$ and the cotangent space $T^*_xM$ at the point $x \in M$. In what follows, we will use the Einstein summation convention. The Greek indices run from 1 to $n-1$, whereas the Roman indices run from 1 to $n$, unless otherwise specified. Then, the Riemannian metric $g$ is given by $g = g_{ij} dx_i \otimes dx_j$. 

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Let $\nabla_i = \nabla_{\frac{\partial}{\partial x_i}}$ be the covariant derivative with respect to $\frac{\partial}{\partial x_i}$ and $\nabla = g^{\mu\nu} \nabla_{\mu}$, where $[g^{\mu\nu}] = \{g_{\mu\nu}\}^{-1}$. Then, for smooth displacement vector field $u = u^j \frac{\partial}{\partial x_j}$ and smooth tensor field $T = T^i_k \, dx_k \otimes \frac{\partial}{\partial x_i}$ of type $(1,1)$, we denote by $\text{div}$ the divergence operator, i.e.
\[
\text{div} \, u := \nabla_i u^i = \frac{\partial u^i}{\partial x_i} + \Gamma^i_{jk} u^j, \quad (\text{div} \, T)^i := \nabla^k T^i_k.
\] (1.2)

The gradient operator is given by
\[
\text{grad}f := (\nabla^j f) \frac{\partial}{\partial x_j} = g^{\mu\nu} \frac{\partial f}{\partial x_k} \frac{\partial}{\partial x_j}, \quad f \in C^\infty(M).
\] (1.3)

### 1.1. Lamé operator

We first define the Cauchy stress tensor $\tau$ of type $(1,1)$ as follows:
\[
\tau := \lambda (\text{div} \, u) I_n + \mu (Su),
\] (1.4)
where $\lambda$ and $\mu$ are Lamé parameters. Furthermore, we let $\tau^i_j := \lambda (\text{div} \, u)^i_j + \mu (Su)^i_j$, where $I_n$ is the $n \times n$ identity matrix, $\delta^i_j$ is the Kronecker delta, and the strain tensor $S$ (also called the deformation tensor) of type $(1,1)$ is defined by $S^i_j := \nabla_i u_j + \nabla_j u_i$. The Lamé operator $\mathcal{L}_{\lambda,\mu} u$ with variable coefficients $\lambda$ and $\mu$ on the Riemannian manifold is given by
\[
\mathcal{L}_{\lambda,\mu} u := \text{div} \, \tau = \text{grad} (\lambda \text{div} \, u) + \text{div} (\mu (Su))
\] (1.6)
for the displacement $u$. Note that, by (1.2), (1.3) and (1.5), we have
\[
\text{div} \, (\mu (Su)) = \nabla^k (\mu (Su)^i_k) = (\nabla^k \mu) (Su)^i_k + \mu \nabla^k (Su)^i_k,
\]
\[
\nabla^k \nabla_i u_k = \nabla^j \nabla_i u_j + g^{\mu\nu} R^i_{\mu\nu} = \nabla^j (\text{div} \, u) + \text{Ric}(u)^i_j.
\]

Here $\text{Ric}(u)^i_j = g^{\mu\nu} R^i_{\mu\nu}$, where $R^i_{\mu\nu}$ are the components of Ricci tensor of the manifold, i.e.
\[
R^i_{\mu\nu} = \frac{\partial \Gamma^i_{\mu\nu}}{\partial x_j} - \frac{\partial \Gamma^i_{\mu j}}{\partial x_\nu} + \Gamma^i_{\mu m} \Gamma^m_{\nu j} - \Gamma^i_{\nu m} \Gamma^m_{\mu j},
\] (1.7)
the Christoffel symbols
\[
\Gamma^i_{\mu j} = \frac{1}{2} g^{i m} \left( \frac{\partial g_{\mu m}}{\partial x_j} + \frac{\partial g_{\nu m}}{\partial x_\mu} - \frac{\partial g_{\mu \nu}}{\partial x_m} \right).
\]
Thus, (1.6) can be written as
\[
\mathcal{L}_{\lambda,\mu} u := \mu \Delta u + (\lambda + \mu) \text{grad} \, \text{div} \, u + \mu \text{Ric}(u)
\] + (\text{grad} \lambda) \text{div} \, u + (Su)(\text{grad} \mu),
\] (1.8)
where the Bochner Laplacian $(\Delta u)^i_j := \nabla^k \nabla_i u_j$ for $1 \leq j \leq n$.

By a direct computation, we have (see also [41])
\[
(\Delta u)^i_j = \Delta u^i_j + g^{i k} \left( 2 \Gamma^i_{j k} \frac{\partial u^m}{\partial x_l} + \Gamma^i_{j m} \frac{\partial u^k}{\partial x_l} + (\Gamma^i_{j k} \Gamma^m_{k h} - \Gamma^i_{j h} \Gamma^m_{k h}) u^k \right), \quad 1 \leq j \leq n.
\]
where the Laplace–Beltrami operator is given by
\[
\Delta f = g^k \left( \frac{\partial^2 f}{\partial x_j \partial x_k} - \Gamma^k_{jk} \frac{\partial f}{\partial x_l} \right), \quad f \in C^\infty(M).
\] (1.9)

It follows from (1.7) that
\[
(\Delta_x u)^j = \Delta_x u^j - \text{Ric}(u)^j + g^{kl} \left( 2 \Gamma^j_{mk} \frac{\partial u^m}{\partial x_i} + \frac{\partial \Gamma^j_{km}}{\partial x_l} u^n \right), \quad 1 \leq j \leq n.
\]

Therefore, the Lamé operator (1.8) with variable coefficients \( \lambda \) and \( \mu \) on the Riemannian manifold \( M \) can be rewritten as
\[
(L_{\lambda,\mu} u)^j = \mu \Delta_x u^j + (\lambda + \mu) \nabla^j \nabla k u^k + (\nabla^j \lambda) \nabla k u^k + (\nabla^k \mu)(\nabla_j u^k + \nabla^j u_k)
\]
\[
+ \mu g^{kl} \left( 2 \Gamma^j_{km} \frac{\partial u^m}{\partial x_i} + \frac{\partial \Gamma^j_{km}}{\partial x_l} u^n \right), \quad 1 \leq j \leq n.
\] (1.10)

1.2. The elastic Calderón problem on the Riemannian manifold

We first consider the following Dirichlet boundary value problem for the Lamé system on the Riemannian manifold:
\[
\begin{aligned}
L_{\lambda,\mu} u &= 0 \quad \text{in } M, \\
u &= f \quad \text{on } \partial M.
\end{aligned}
\] (1.11)

For any displacement \( f \in [H^{1/2}(\partial M)]^n \) on the boundary, by the theory of elliptic operators, there is a unique solution \( u \in [H^1(M)]^n \) solving the above problem (1.11). Therefore, the elastic Dirichlet-to-Neumann map (also called the displacement-to-traction map) \( \Lambda_{\lambda,\mu} : [H^{1/2}(\partial M)]^n \to [H^{-1/2}(\partial M)]^n \) associated with the operator \( L_{\lambda,\mu} \) is defined by
\[
\Lambda_{\lambda,\mu}(f) := \tau(\nu) = \lambda (\text{div } u) \nu + \mu (\text{Su}) \nu \quad \text{on } \partial M,
\] (1.12)
where \( \nu \) is the outward unit normal vector to the boundary \( \partial M \). Physically, \( \tau(\nu) \) is called the Neumann boundary condition (or free boundary condition, or traction boundary condition) of (1.11). Roughly speaking, the elastic Dirichlet-to-Neumann map \( \Lambda_{\lambda,\mu} \) sends the displacement at the boundary to the corresponding normal component of the stress (i.e. the traction) at the boundary (see [52]). It is clear that the elastic Dirichlet-to-Neumann map \( \Lambda_{\lambda,\mu} \) is an elliptic, self-adjoint pseudodifferential operator of order one defined on the boundary \( \partial M \).

In particular, for a bounded Euclidean domain \( \Omega \subset \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), the Lamé operator with variable coefficients \( \lambda, \mu \in C^\infty(\Omega) \) is given by (see [52])
\[
(Lu)_j := \sum_{k,l,m=1}^n \frac{\partial}{\partial x_k} \left( C_{jklm} \frac{\partial u_l}{\partial x_m} \right), \quad 1 \leq j \leq n,
\] (1.13)
where the isotropic elastic tensor
\[
C_{jklm} = \lambda \delta_{jk} \delta_{lm} + \mu (\delta_{jl} \delta_{km} + \delta_{jm} \delta_{lk}), \quad 1 \leq j, k, l, m \leq n.
\]
Equivalently, (1.13) can be written as (see [53])
\[
Lu = \mu \Delta u + (\lambda + \mu) \nabla (\nabla \cdot u) + (\nabla \cdot u) \nabla \lambda + (\nabla u + (\nabla u)^T) \nabla \mu,
\] (1.14)
where $\Lambda(\nabla u)^T$ denotes the transpose of $\nabla u$. The corresponding elastic Dirichlet-to-Neumann map $\Lambda$ is defined by (see [52, 53])

$$\Lambda(f)_j := \sum_{k, l, m=1}^n \nu_k c_{jklm} \frac{\partial u_l}{\partial x_m} \text{ on } \partial \Omega, \quad 1 \leq j \leq n, \quad (1.15)$$

or

$$\Lambda(f) := \lambda (\nabla \cdot u) \nu + \mu (\nabla u + (\nabla u)^T) \nu \text{ on } \partial \Omega, \quad (1.16)$$

where $\nu_k$ is the $k$th component of the outward unit normal vector $\nu$, and $u$ is the solution to the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial \Omega. \end{cases}$$

In other words, when the manifold $\mathcal{M}$ is a bounded Euclidean domain in $\mathbb{R}^n$, the Lamé operator (1.8) and the elastic Dirichlet-to-Neumann map (1.12) reduce to the corresponding Euclidean cases (1.14) and (1.16), respectively. More particularly, when $\Omega \subset \mathbb{R}^n$ is a bounded Euclidean domain with smooth boundary $\partial \Omega$ and the Lamé coefficients $\lambda, \mu$ are constants, the Lamé operator has the form $\mu \Delta u + (\lambda + \mu) \nabla (\nabla \cdot u)$, and the elastic Dirichlet-to-Neumann map becomes (1.16) (see [32, 33]). We also refer the reader to [1, 8, 11, 18–25, 36, 41, 50–54] for this topic in elasticity.

Partial uniqueness results of the elastic Calderón problem for the determination of Lamé coefficients from boundary measurements were obtained [1, 11, 22–24, 51–54]. For a bounded Euclidean domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial \Omega$, Nakamura and Uhlmann [54] used the methods of pseudodifferential operators and examined the symbol of the elastic Dirichlet-to-Neumann operator, they proved that one can determine the full Taylor series of the Lamé coefficients on the boundary in all dimensions $n \geq 2$ and for a generic anisotropic elastic tensor in two dimensions. For a bounded two dimensional Euclidean domain, Akamatsu et al [1] gave an inversion formula for the normal derivatives on the boundary of the Lamé coefficients from the elastic Dirichlet-to-Neumann map. Nakamura and Uhlmann [51] proved that the elastic Dirichlet-to-Neumann map uniquely determines the Lamé coefficients $\lambda, \mu \in W^{31, \infty}(\Omega)$, provided that $\lambda, \mu$ are sufficiently close to positive constants. Imanuvilov and Yamamoto [23] proved the uniqueness by the elastic Dirichlet-to-Neumann map limited to an arbitrary sub-boundary, provided that $\lambda \in C^1(\Omega)$ and $\mu$ is a constant. Imanuvilov and Yamamoto [24] also proved the global uniqueness of the Lamé coefficients $\lambda, \mu \in C^{30}(\Omega)$. In three dimensional Euclidean domains, Nakamura and Uhlmann [52, 53] as well as Eskin andRalston [11] proved the global uniqueness of the Lamé coefficients provided that $\nabla \mu$ is small in a suitable norm. Imanuvilov et al [22] proved that the Lamé coefficients are uniquely recovered from partial Cauchy data, provided that $\lambda_1 = \lambda_2$ on an arbitrary open subset $\Gamma_0 \subset \partial \Omega$ and $\mu_1, \mu_2$ are some constants. However, in dimensions $n \geq 3$, the global uniqueness of the Lamé coefficients $\lambda, \mu \in C^\infty(\Omega)$ without the smallness assumption ($\|\nabla \mu\| < \varepsilon_0$ for some small positive $\varepsilon_0$) remains an open problem (see [25, p 210]).

We briefly recall the classical Calderón problem [5, 66], i.e. whether one can uniquely determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. To be more precise, let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain with smooth boundary and let a bounded positive function $\gamma$ be the electrical conductivity. In the absence of sinks or sources of current, the Dirichlet problem for the conductivity equation is
\[
\begin{align*}
\nabla \cdot (\gamma \nabla u) &= 0 & \text{in } \Omega, \\
\n u &= f & \text{on } \partial \Omega.
\end{align*}
\]

The classical Dirichlet-to-Neumann map (also called the voltage-to-current map) is defined by

\[\Lambda_\gamma(f) := \gamma \frac{\partial u}{\partial \nu} \bigg|_{\partial \Omega}.\]

The classical Calderón problem is to determine the conductivity \(\gamma\) by the classical Dirichlet-to-Neumann map \(\Lambda_\gamma\). This problem has been studied for decades. In dimensions \(n \geq 2\), Kohn and Vogelius [28] proved the famous uniqueness result on the boundary for \(C^\infty\)-conductivities, that is, if \(\Lambda_{\gamma_1} = \Lambda_{\gamma_2}\), then,

\[\frac{\partial^{|J|} \gamma_1}{\partial x^J} \bigg|_{\partial \Omega} = \frac{\partial^{|J|} \gamma_2}{\partial x^J} \bigg|_{\partial \Omega}\]

for all multi-indices \(J \in \mathbb{N}^n\). This settled the uniqueness problem on the boundary in the real analytic category. They extended the uniqueness result to piecewise real analytic conductivities in [29]. In dimensions \(n \geq 3\), in the celebrated paper [62], Sylvester and Uhlmann proved the uniqueness of the \(C^\infty\)-conductivities in a bounded domain \(\Omega \subset \mathbb{R}^n\) with smooth boundary by constructing the complex geometrical optics solutions. Haberman and Tataru [13] proved uniqueness for \(C^1\)-conductivities and Lipschitz conductivities sufficiently close to the identity. Caro and Rogers [6] reduced regularities of conductivities and boundary to Lipschitz. In two dimensional case, many important results were also obtained. Nachmann [48] gave the corresponding uniqueness result for \(W^{2,p}\)-conductivities \((p > 1)\). This uniqueness result be extended significantly by Astala and Päivärinta [2] for \(L^\infty\)-conductivities. The classical Calderón problem have attracted lots of attention for decades (see [3, 10, 15, 21, 28, 29, 34, 47, 48, 55, 56, 60–62] and references therein). We also refer the reader to the survey articles [68, 69] for the classical Calderón problem and related topics.

Let us come back to the elastic Calderón problem on the Riemannian manifold. The study of the elastic Dirichlet-to-Neumann map is of importance in many areas, such as material science, geophysical exploration (exploring the interior structure of the earth by boundary measurements on the surface of the earth in the cases of earthquakes or artificial earthquakes), materials characterization and acoustic emission of many important materials and nondestructive testing (see, for example, [51–54]). Generally, inverse problems consider the problems of determining the interior physical or geometrical properties of a medium from boundary measurements. The uniqueness problem is one of the important aspects of inverse problems, namely the problem of whether the measurements on the boundary (all possible measurements or a proper subset of them) can uniquely determine the interior material properties (for example, the material parameters) of the medium (see [8]). These problems are of clear both theoretical and practical interests, have received a lot of attention from the mathematics and physics communities. These inverse problems are closely related to the method known as electrical impedance tomography (EIT). EIT is widely used in exploration of petroleum and minerals beneath the surface of the earth, and in medical imaging in detecting breast cancer, pulmonary edema, leaks from buried pipes, etc. For these problems, we refer to, for example, [1, 5, 11, 22–24, 51–54, 68, 69].

Various inverse problems occurring in mathematics, physics, and engineering have been studied for decades. Here we do not list all the references about these topics. We refer the reader to [7, 27, 42, 46, 57, 59] for Maxwell’s equations, to [14, 35, 38, 43] for incompressible fluid, to [9, 30, 31, 49, 58] for Schrödinger operator and magnetic Schrödinger operator. For
the studies about other types of Dirichlet-to-Neumann map, we also refer the reader to [37, 39, 40, 44, 45] and references therein.

In this paper, we study the elastic Calderón problem on a Riemannian manifold that is determining variable Lamé coefficients $\lambda$ and $\mu$ by the elastic Dirichlet-to-Neumann map $\Lambda_{\lambda, \mu}$. By giving an explicit expression for $\Lambda_{\lambda, \mu}$ with variable coefficients, we show that $\Lambda_{\lambda, \mu}$ uniquely determines the partial derivatives of all orders of the Lamé coefficients $\lambda$ and $\mu$ on the boundary $\partial M$. Moreover, we prove that $\Lambda_{\lambda, \mu}$ uniquely determines the Lamé coefficients $\lambda$ and $\mu$ on the whole manifold $M$ in the real analytic setting.

Before we state the main results of this paper, we recall some basic concepts about boundary normal coordinates, pseudodifferential operators and symbols.

### 1.3. Boundary normal coordinates

We briefly introduce the construction of geodesic coordinates with respect to the boundary $\partial M$ (see [34], [63, p 532]). For each boundary point $x' \in \partial M$, let $\gamma_{x'} : [0, \varepsilon) \to M$ denote the unit-speed geodesic starting at $x'$ and normal to $\partial M$. If $x' := \{x_1, \ldots, x_{n-1}\}$ are any local coordinates for $\partial M$ near $x_0 \in \partial M$, we can extend them smoothly to functions on a neighborhood of $x_0$ in $M$ by letting them be constant along each normal geodesic $\gamma_{x'}$. If we then define $x_n$ to be the parameter along each $\gamma_{x'}$, it follows easily that $\{x_1, \ldots, x_n\}$ form coordinates for $M$ in some neighborhood of $x_0$, which we call the boundary normal coordinates determined by $\{x_1, \ldots, x_{n-1}\}$. In these coordinates $x_n > 0$ in $M$, and the boundary $\partial M$ is locally characterized by $x_n = 0$. A standard computation shows that the metric has the form $g = g_{\alpha \beta} \, dx_\alpha \, dx_\beta + dx_n^2$.

### 1.4. Pseudodifferential operators and symbols

We recall some concepts of pseudodifferential operators and their symbols (cf [63, chapter 7]). Assuming $U \subset \mathbb{R}^n$ and $m \in \mathbb{R}$, we define $S^m_{1,0} = S^m_{1,0}(U, \mathbb{R}^n)$ to consist of $C^\infty$-functions $p(x, \xi)$ satisfying for every compact set $V \subset U$,

$$
|D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C_{V, \alpha, \beta} |\xi|^{-m-|\alpha|}, \quad x \in V, \xi \in \mathbb{R}^n
$$

for all $\alpha, \beta \in \mathbb{N}^n$, where $D^\alpha = D^{\alpha_1} \cdots D^{\alpha_n}$, $D_\xi = -i \frac{\partial}{\partial \xi}$, and $|\xi| = (1 + |\xi|^2)^{1/2}$. The elements of $S^m_{1,0}$ are called symbols of order $m$. It is clear that $S^m_{1,0}$ is a Fréchet space with semi-norms given by

$$
\|p\|_{V, \alpha, \beta} := \sup_{x \in V} \left| (D_\xi^\beta D_x^\alpha p(x, \xi))(1 + |\xi|)^{-m+|\alpha|} \right|.
$$

Let $p(x, \xi) \in S^m_{1,0}$ and $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i \xi \cdot y} u(y) \, dy$ be the Fourier transform of $u$. A pseudodifferential operator in an open set $U$ is essentially defined by a Fourier integral operator

$$
P(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} p(x, \xi) e^{i \xi \cdot \hat{u}(\xi)} \, d\xi
$$

for $u \in C^\infty_0(U)$. In such a case, we say the associated operator $P(x, D)$ belongs to $OPS^m$. We denote $OPS^{-\infty} = \bigcap_m OPS^m$. If there are smooth $p_{m-j}(x, \xi)$, homogeneous in $\xi$ of degree $m - j$ for $|\xi| \geq 1$, that is, $p_{m-j}(x, r\xi) = r^{m-j} p_{m-j}(x, \xi)$ for $r, |\xi| \geq 1$, and if

$$
p(x, \xi) \sim \sum_{j \geq 0} p_{m-j}(x, \xi)
$$

(1.17)
in the sense that
\[ p(x, \xi) - \sum_{j=0}^{N} p_{m-j}(x, \xi) \in S_{1,0}^{m-N-1} \]
for all \( N \), then we say \( p(x, \xi) \in S_{1,0}^{m} \), or just \( p(x, \xi) \in S^{m} \). We denote \( S^{\infty} = \bigcap_{m} S^{m} \). We call \( p_{m}(x, \xi) \) the principal symbol of \( P(x, D) \). We say \( P(x, D) \in OPS^{m} \) is elliptic of order \( m \) if on each compact \( V \subset U \) there are constants \( C_{V} \) and \( r < \infty \) such that
\[ |p(x, \xi)^{-1}| \leq C_{V}(\xi)^{-m}, \quad |\xi| \geq r. \]

We can now define a pseudodifferential operator on the manifold \( M \). In particular,
\[ P : C^{\infty}_{0}(M) \rightarrow C^{\infty}(M) \]
belongs to \( OPS^{m}_{1,0}(M) \) if the kernel of \( P \) is smooth off the diagonal in \( M \times M \) and if for any coordinate neighborhood \( U \subset M \) with \( \Phi : U \rightarrow O \) a diffeomorphism onto an open subset \( O \subset \mathbb{R}^{n} \), the map \( P : C^{\infty}_{0}(O) \rightarrow C^{\infty}(O) \) given by
\[ P_{U} := P(u \circ \Phi) \circ \Phi^{-1} \]
belongs to \( OPS^{m}_{1,0}(O) \). We refer the reader to [12, 17, 65] for more details.

1.5. The main results of this paper

For the sake of simplicity, we denote by \( i = \sqrt{-1} \), \( \xi' = (\xi_{1}, \ldots, \xi_{n-1}) \), \( \xi'' = \gamma^{\alpha \beta} \xi_{\alpha} \), \( |\xi'| = \sqrt{\xi''_{\alpha} \xi''_{\alpha}} \), \( I_{n} \) the \( n \times n \) identity matrix, and
\[
\begin{bmatrix}
[\alpha_{1}] \\
\vdots \\
[\alpha_{n}]
\end{bmatrix} =
\begin{bmatrix}
a_{1} & \cdots & a_{n-1} & a_{n} \\
\vdots & \ddots & \vdots & \vdots \\
a_{n-1} & \cdots & a_{n-1} & a_{n}
\end{bmatrix},
\]
where \( 1 \leq \alpha, \beta \leq n - 1 \).

The main results of this paper are the following three theorems.

**Theorem 1.1.** Let \( (M, g) \) be a smooth compact Riemannian manifold of dimension \( n \) with smooth boundary \( \partial M \). Assume that the Lamé coefficients \( \lambda, \mu \in C^{\infty}(M) \) satisfy \( \mu > 0 \) and \( \lambda + \mu \geq 0 \). Let \( \sigma(\Lambda_{\lambda, \mu}) \sim \sum_{j \leq 1} P_{j}(x, \xi') \) be the full symbol of the elastic Dirichlet-to-Neumann map \( \Lambda_{\lambda, \mu} \). Then, in boundary normal coordinates,
\[
p_{1}(x, \xi') = \mu|\xi'| I_{n-1} + \frac{\mu(\lambda + \mu)}{(\lambda + 3\mu)|\xi'|} \left[ \xi''_{\alpha} \xi_{\beta} \right] - \frac{2i\mu^{2}}{\lambda + 3\mu} |\xi'|,
\]
\[
p_{0}(x, \xi') = \mu I_{n-1} - \frac{2i\mu^{2}}{\lambda + 3\mu} |\xi'|,
\]
\[
p_{-m}(x, \xi') = \mu I_{n-1} - \frac{2i\mu^{2}}{\lambda + 3\mu} |\xi'|,
\]
where \( q_{-m}(x, \xi'), m \geq 0, \) are the remain symbols of a pseudodifferential operator (see (2.13) in section 2).
For the case of constant Lamé coefficients, Liu [41] gave the full symbol of the elastic Dirichlet-to-Neumann map on a Riemannian manifold. For a bounded Euclidean domain $\Omega \subset \mathbb{R}^n$, Zhang [71] studied the elastic wave system and gave the principal symbol of the elastic Dirichlet-to-Neumann map for $n = 3$. For $n \geq 2$, Vodev [70] showed that the elastic Dirichlet-to-Neumann map can be approximated by a pseudodifferential operator on the boundary with a matrix-valued symbol and computed its principal symbol modulo conjugation by unitary matrices.

By studying the full symbol of the elastic Dirichlet-to-Neumann map $\Lambda_{\lambda,\mu}$, we prove the following result:

**Theorem 1.2.** Let $(M, g)$ be a smooth compact Riemannian manifold of dimension $n$ with smooth boundary $\partial M$. Assume that the Lamé coefficients $\lambda, \mu \in C^\infty(\bar{M})$ satisfy $\mu > 0$ and $\lambda + \mu \geq 0$. Then, the elastic Dirichlet-to-Neumann map $\Lambda_{\lambda,\mu}$ uniquely determines $\frac{\partial^{\lambda+\mu}}{\partial x^{\lambda+\mu}}$ on the boundary $\partial M$ for all multi-indices $\lambda$.

Theorem 1.2 generalizes the corresponding result in [54] for Euclidean bounded domains with smooth boundary. By theorem 1.2, the uniqueness result can be extended to the whole manifold for the real analytic setting. Thus, we have the following global uniqueness theorem:

**Theorem 1.3.** Let $(M, g)$ be a compact Riemannian manifold of dimension $n$ with smooth boundary $\partial M$, and let $\Gamma \subset \partial M$ be a nonempty smooth open subset. Suppose that the manifold $M$ is real analytic up to $\Gamma$, assume that the Lamé coefficients $\lambda$ and $\mu$ are also real analytic up to $\Gamma$, and satisfy $\mu > 0$ and $\lambda + \mu \geq 0$. Then, the elastic Dirichlet-to-Neumann map $\Lambda_{\lambda,\mu}$ uniquely determines the Lamé coefficients $\lambda$ and $\mu$ on $\bar{M}$.

Theorem 1.3 shows that the global uniqueness of real analytic Lamé coefficients on a real analytic Riemannian manifold. To the best of our knowledge, this is the first global uniqueness result for Lamé coefficients on a Riemannian manifold. It is clear that theorem 1.3 also holds for any real analytic Euclidean domain with smooth boundary.

### 1.6. The main ideas of this paper

First, Liu [41] established a method such that one can calculate the full symbol of the elastic Dirichlet-to-Neumann map with constant coefficients. By this method, we can also deal with the case for variable coefficients $\lambda, \mu \in C^\infty(M)$. In boundary normal coordinates, we flatten the boundary and induce a Riemannian metric in a neighborhood of the boundary, there is a factorization for the Lamé operator as follows:

$$A^{-1}L_{\lambda,\mu} = I_n \frac{\partial^2}{\partial x_n^2} + B \frac{\partial}{\partial x_n} + C = \left( I_n \frac{\partial}{\partial x_n} + B - Q \right) \left( I_n \frac{\partial}{\partial x_n} + Q \right)$$

modulo a smoothing operator, where $A$ is a matrix given by (2.3) in section 2, $B, C$ are two differential operators (see section 2), and $Q = Q(x, \partial \cdot \cdot \cdot)$ is a pseudodifferential operator of order one in $x' = (x_1, \ldots, x_{n-1})$ depending smoothly on $x_n$ (see Proposition 2.1 below). As a result, we obtain the equation

$$Q^2 - BQ - \left[ I_n \frac{\partial}{\partial x_n}, Q \right] + C = 0$$
modulo a smoothing operator, where \( [I_n \frac{\partial}{\partial x_n}, Q] \) is the commutator. Let \( b = b(x, \xi') \), \( c = c(x, \xi') \), and \( q = q(x, \xi') \sim \sum_{p \leq q} q(x, \xi') \) be the full symbols of the operators \( B, C \), and \( Q \), respectively. Then, the corresponding full symbol equation of the above equation is

\[
\sum_j \frac{(-i)^{|j|}}{j!} \partial_{\xi_j} q \partial_{\xi_j} q - \sum_j \frac{(-i)^{|j|}}{j!} \partial_{\xi_j} b \partial_{\xi_j} q - \frac{\partial q}{\partial x_n} + c = 0
\]  

(1.21)

modulo a smoothing operator, where the sum is over all multi-indices \( J \).

Note that the computations of the full symbols of matrix-valued pseudodifferential operators (i.e. solving equation (1.21)) are quite difficult tasks. There are two major difficulties:

(i) How to solve the unknown matrix \( q_1 \) from the following matrix equation?

\[
q_1^2 - b_1 q_1 + c_2 = 0,
\]

(1.22)

where \( q_1, b_1, \) and \( c_2 \) are the principal symbols of \( Q, B, \) and \( C \), respectively.

(ii) How to solve the unknown matrix \( q_{-m-1}, m \geq -1 \), from the following Sylvester equation?

\[
(q_1 - b_1) q_{-m-1} + q_{-m-1} q_1 = E_{-m}, \quad m \geq -1,
\]

(1.23)

where \( q_{-m-1}, m \geq -1 \), are the remain symbols, and \( E_{-m}, m \geq -1 \), are given by (2.9)–(2.11) below.

For (i), generally, the quadratic matrix equation of the form

\[
X^2 + Y_1 X + Z_1 = 0
\]

(1.24)

can not be solved exactly, where \( X \) is an unknown matrix, \( Y_1 \) and \( Z_1 \) are given matrices. In other words, there is not a general formula for the solution represented by the coefficients of (1.24).

Fortunately, in our setting, this is similar to [41]. \( b_1 \) and \( c_2 \) can be represented as special block matrices, these block matrices can generate a matrix ring. This implies that the \( q_1 \) can also be represented as a block matrix, which is a linear combination of \( I_n \) and a special matrix \( F_1 \) with the property \( F_1^2 = 0 \) (see (2.26) below). By solving a system of the coefficients, we can obtain the explicit expression for \( q_1 \) (see (2.12) below).

For (ii), the matrix equation of the form

\[
Y_2 X + X Z_2 = W
\]

(1.25)

is called the Sylvester equation (see [4, chapter 9]), where \( X \) is an unknown matrix, \( Y_2, Z_2 \), and \( W \) are given matrices. Let

\[
A = \begin{bmatrix}
  a_1^1 & a_1^2 & \cdots & a_1^n \\
  a_2^1 & a_2^2 & \cdots & a_2^n \\
  \vdots & \vdots & \ddots & \vdots \\
  a_n^1 & a_n^2 & \cdots & a_n^n
\end{bmatrix}
\]

The vectorization vec \( A \) of the matrix \( A \) is a column vector, which is defined by (see [4, chapter 9])

\[
\text{vec} \ A := (a_1^1, a_2^1, \ldots, a_1^n, a_2^n, \ldots, a_n^1, a_1^1, \ldots, a_n^n)^T.
\]

(1.26)
The Kronecker product $A \otimes B$ of two matrices $A$ and $B$ is defined by (see [4, chapter 9])

$$A \otimes B := \begin{bmatrix}
a_1^1 B & a_1^2 B & \cdots & a_1^n B \\
a_2^1 B & a_2^2 B & \cdots & a_2^n B \\
\vdots & \vdots & \ddots & \vdots \\
a_n^1 B & a_n^2 B & \cdots & a_n^n B
\end{bmatrix}.$$  \hspace{1cm} (1.27)

There are some properties of Kronecker product and vectorization as follows (see [4, chapter 9]):

$$(A + B) \otimes C = A \otimes C + B \otimes C,$$  \hspace{1cm} (1.28)

$$(C \otimes (A + B)) = C \otimes A + C \otimes B,$$  \hspace{1cm} (1.29)

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1},$$  \hspace{1cm} (1.30)

$$(A \otimes B)(C \otimes D) = AC \otimes BD,$$  \hspace{1cm} (1.31)

$$\text{vec}(AB) = (I_n \otimes A) \text{vec} B,$$  \hspace{1cm} (1.32)

$$\text{vec}(BC) = (C^T \otimes I_n) \text{vec} B,$$  \hspace{1cm} (1.33)

$$\text{vec}(ABC) = (C^T \otimes A) \text{vec} B.$$  \hspace{1cm} (1.34)

It follows from (1.25), (1.32), and (1.33) that

$$\text{vec} W = \text{vec}(Y_2 X + X Z_2)$$
$$= (I_n \otimes Y_2 + (Z^T \otimes I_n)) \text{vec} X$$
$$: = G \text{vec} X.$$  \hspace{1cm} (1.35)

Therefore, (1.25) has a unique solution if and only if $G$ is invertible and $\text{vec} X = G^{-1} \text{vec} W$.

Thus, we can obtain $X$ from $\text{vec} X$. Finally, using this method, we solve (1.23) and obtain the symbols $q_{-m-1}$ for $m \geq -1$. Therefore, we get the full symbol $q(x, \xi') \sim \sum_{j \leq 1} q_j(x, \xi')$ of the pseudodifferential operator $Q$.

Next, in boundary normal coordinates, the operator $-\frac{\partial}{\partial x_n} \big|_{\partial M}$ can be represented as the pseudodifferential operator $Q$ modulo a smoothing operator. We give a local representation for the elastic Dirichlet-to-Neumann map $\Lambda_{\lambda, \mu}$ in boundary normal coordinates, that is, (see (2.37) below)

$$\Lambda_{\lambda, \mu} = (AQ - K) \big|_{\partial M}$$

modulo a smoothing operator, where $K$ is a matrix given by (2.38) below.

Finally, we obtain the full symbol of the elastic Dirichlet-to-Neumann map $\Lambda_{\lambda, \mu}$, which contain the information about the Lamé coefficients $\lambda$, $\mu$, and their derivatives on the boundary $\partial M$. Thus, we can prove that $\Lambda_{\lambda, \mu}$ uniquely determines the Lamé coefficients $\lambda$ and $\mu$ on the boundary $\partial M$. Furthermore, we show that the Lamé coefficients $\lambda$ and $\mu$ can be uniquely determined on the whole manifold $M$ by $\Lambda_{\lambda, \mu}$ in the real analytic setting.

This paper is organized as follows. In section 2, we derive a factorization of the Lamé operator $\mathcal{L}_{\lambda, \mu}$ with variable coefficients, and compute the full symbol of the pseudodifferential operator $Q$, we then give the explicit expression for the elastic Dirichlet-to-Neumann map $\Lambda_{\lambda, \mu}$ in boundary normal coordinates. In section 3, we prove theorem 1.1 and theorem 1.2.
for boundary determination. Finally, section 4 is devoted to proving theorem 1.3 for global uniqueness.

2. Symbols of the pseudodifferential operators

In boundary normal coordinates, we write the Laplace–Beltrami operator as

$$\Delta_x = \frac{\partial^2}{\partial x_n^2} + \Gamma^\alpha_{\beta n} \frac{\partial}{\partial x_n} + g^{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} + \left( g^{\alpha\beta} \Gamma^\gamma_{\gamma \alpha} + \frac{\partial g^{\alpha\beta}}{\partial x_\alpha} \right) \frac{\partial}{\partial x_\beta}. \tag{2.1}$$

According to (1.10), we deduce that

$$A^{-1} L_{\lambda, \mu} = L_n \frac{\partial^2}{\partial x_n^2} + B \frac{\partial}{\partial x_n} + C, \tag{2.2}$$

where

$$A = \begin{bmatrix} \mu I_{n-1} & 0 \\ 0 & \lambda + 2\mu \end{bmatrix}, \tag{2.3}$$

$$B = B_1 + B_0, \quad C = C_2 + C_1 + C_0,$$

and

$$B_1 = (\lambda + \mu) \begin{bmatrix} 0 & \frac{1}{\mu} \left[ g^{\alpha\beta} \frac{\partial}{\partial x_\beta} \right] \\ \frac{1}{\lambda + 2\mu} \left[ \frac{\partial}{\partial x_\beta} \right] & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} \Gamma_{\alpha n} I_{n-1} + 2\Gamma_{\beta n}^\alpha & 0 \\ \frac{\lambda + \mu}{\lambda + 2\mu} \left[ \Gamma^\alpha_{\beta n} \right] & \Gamma^\alpha_{\alpha n} \end{bmatrix} + \begin{bmatrix} \frac{1}{\mu} \left[ \frac{\partial}{\partial x_\alpha} \right] & \frac{1}{\mu} \left[ \nabla^\alpha \lambda \right] \\ \frac{1}{\lambda + 2\mu} \left[ \frac{\partial}{\partial x_\beta} \right] & \frac{1}{\lambda + 2\mu} \left[ \frac{\partial\left(\lambda + 2\mu\right)}{\partial x_\alpha} \right] \end{bmatrix},$$

$$C_2 = \begin{bmatrix} \left( g^{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \right) I_{n-1} + \frac{\lambda + \mu}{\mu} \left[ g^{\alpha\gamma} \frac{\partial^2}{\partial x_\gamma \partial x_\beta} \right] & 0 \\ 0 & \frac{\mu}{\lambda + 2\mu} g^{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \end{bmatrix},$$

$$C_1 = \begin{bmatrix} \left( g^{\alpha\beta} \Gamma_{\alpha \gamma} + \frac{\partial g^{\alpha\beta}}{\partial x_\alpha} \right) \frac{\partial}{\partial x_\beta} I_{n-1} & 0 \\ 0 & \frac{\lambda + \mu}{\mu} \left[ g^{\alpha\gamma} \Gamma_{\rho \gamma} + \frac{\partial g^{\alpha\beta}}{\partial x_\alpha} \right] \frac{\partial}{\partial x_\beta} \end{bmatrix} + \frac{\lambda + \mu}{\mu} \begin{bmatrix} \left[ g^{\alpha\gamma} \Gamma_{\rho \gamma} + \frac{\partial g^{\alpha\beta}}{\partial x_\alpha} \right] \frac{\partial}{\partial x_\beta} \\ 0 \end{bmatrix} + 2 \begin{bmatrix} \left[ \frac{g^{\rho\beta} \Gamma^{\alpha \beta}}{\partial x_\gamma} \right] \frac{\partial}{\partial x_\beta} \\ \frac{\lambda + 2\mu}{\mu} \left[ \frac{g^{\rho\beta} \Gamma^{\alpha \beta}}{\partial x_\gamma} \right] \frac{\partial}{\partial x_\beta} \end{bmatrix} + \begin{bmatrix} \frac{1}{\mu} \left[ \nabla^\alpha \mu \right] \frac{\partial}{\partial x_\alpha} \\ 0 \end{bmatrix} \frac{1}{\lambda + 2\mu} \frac{\partial}{\partial x_\beta}.$$
Let
\[ b(x, \xi') = b_1(x, \xi') + b_0(x, \xi') \]
and
\[ c(x, \xi') = c_2(x, \xi') + c_1(x, \xi') + c_0(x, \xi') \]
be the full symbols of \( B \) and \( C \), respectively, where \( b_j(x, \xi') \) and \( c_j(x, \xi') \) are homogeneous of degree \( j \) in \( \xi' \). We simply write
\[ \xi^\alpha = g^{\alpha\beta} \xi_\beta, \quad |\xi'|^2 = \xi^\alpha \xi_\alpha = g^{\alpha\beta} \xi_\alpha \xi_\beta. \]

Thus, we obtain
\[
b_1(x, \xi') = i(\lambda + \mu) \left[ \begin{array}{cc} 0 & \frac{1}{\lambda + 2\mu} [\xi^\alpha] \\ \frac{1}{\lambda + 2\mu} [\xi_\beta] & 0 \end{array} \right],
\]
(2.4)
\[
b_0(x, \xi') = B_0, \tag{2.5}
\]
\[
c_2(x, \xi') = - \left[ \begin{array}{cc} \xi'^\alpha \xi_\alpha \partial \mu + \lambda + \mu & 0 \\ 0 & \frac{\lambda + \mu}{2\mu} |\xi'|^2 \end{array} \right],
\]
(2.6)
\[
c_1(x, \xi') = i \left[ \begin{array}{cc} \xi^\alpha \Gamma_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial \xi_\alpha} & 0 \\ 0 & \frac{\lambda + 2\mu}{2\mu} \left( \xi^\alpha \Gamma_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial \xi_\alpha} \right) \end{array} \right] + \frac{i(\lambda + \mu)}{\mu} \left[ \begin{array}{cc} \xi^\alpha \Gamma_{\alpha\gamma} & \Gamma_{\alpha\beta} \xi^\gamma \\ 0 & \frac{\lambda + 2\mu}{2\mu} \left( \xi^\alpha \Gamma_{\alpha\gamma} \right) \end{array} \right] + 2i \left[ \begin{array}{cc} \xi^\alpha \Gamma_{\alpha\beta} & 0 \\ \frac{\lambda + 2\mu}{2\mu} \left( \xi^\alpha \Gamma_{\alpha\gamma} \right) & 0 \end{array} \right].
\]
(2.7)
\[
c_0(x, \xi') = C_0. \tag{2.8}
\]
For the convenience of stating the following proposition, we define

\[
E_1 := i \sum_\alpha \frac{\partial(q_1 - b_1)}{\partial \xi_\alpha} \frac{\partial q_1}{\partial x_\alpha} + b_0 q_1 + \frac{\partial q_1}{\partial x_n} - c_1, \\
E_0 := i \sum_\alpha \left( \frac{\partial(q_1 - b_1)}{\partial \xi_\alpha} \frac{\partial q_0}{\partial x_\alpha} + \frac{\partial q_0}{\partial \xi_\alpha} \frac{\partial q_1}{\partial x_\alpha} \right) + \frac{1}{2} \sum_{\alpha, \beta} \frac{\partial^2 q_1}{\partial \xi_\alpha} \frac{\partial^2 q_1}{\partial \xi_\beta} \left( \frac{\partial q_0}{\partial x_\alpha} \frac{\partial q_0}{\partial x_\beta} - \frac{\partial q_0}{\partial \xi_\alpha} \frac{\partial q_0}{\partial \xi_\beta} \right) - q_0^2 + b_0 q_0 + \frac{\partial q_0}{\partial x_n} c_0,
\]

(2.9)

\[
E_{-m} := b_0 q_{-m} + \frac{\partial q_{-m}}{\partial x_n} - i \sum_\alpha \frac{\partial b_1}{\partial \xi_\alpha} \frac{\partial q_{-m}}{\partial x_\alpha} - \sum_{-m \leq j, k \leq 1} \frac{(-i)^{|j|}}{j!} \frac{\partial^j}{\partial \xi^j} q_j \frac{\partial^k}{\partial x^k} q_k
\]

(2.10)

(2.11)

for \( m \geq 1 \), where \( q_j = q_j(x, \xi') \), \( b_j = b_j(x, \xi') \), and \( c_j = c_j(x, \xi') \).

**Proposition 2.1.** Let \( Q(x, \partial_x') \) be a pseudodifferential operator of order one in \( x' \) depending smoothly on \( x_n \) such that

\[
A^{-1} \mathcal{L}_{\lambda, \mu} = \left( I_n \frac{\partial}{\partial x_n} + B - Q \right) \left( I_n \frac{\partial}{\partial x_n} + Q \right)
\]

modulo a smoothing operator. Let \( q(x, \xi', \lambda, \mu) \sim \sum_{j \leq 1} q_j(x, \xi') \) be the full symbol of \( Q \), where \( q_j(x, \xi') \in \mathcal{S}' \) are homogeneous of degree \( j \) in \( \xi' \in \mathbb{R}^{n-1} \). Then, in boundary normal coordinates,

\[
q_1(x, \xi') = |\xi'| I_n + \frac{\lambda + \mu}{\lambda + 3 \mu} F_1,
\]

(2.12)

\[
q_{-m-1}(x, \xi') = \frac{1}{2|\xi'|} E_{-m} - \frac{\lambda + \mu}{4(\lambda + 3 \mu)|\xi'|^2} (F_2 E_{-m} + E_{-m} F_1) + \frac{(\lambda + \mu)^2}{4(\lambda + 3 \mu)^2|\xi'|^4} F_2 E_{-m} F_1, \quad m \geq -1,
\]

(2.13)

where \( E_{-m}, m \geq -1 \), are given by (2.9)–(2.11), and

\[
F_1 = \begin{bmatrix}
\frac{1}{|\xi'|} [\xi^\alpha \xi_\beta] & i [\xi^\alpha] \\
i [\xi_\beta] & -|\xi'|
\end{bmatrix},
\]

(2.14)

\[
F_2 = \begin{bmatrix}
\frac{1}{|\xi'|} [\xi^\alpha \xi_\beta] & -i (\lambda + 2 \mu) [\xi^\alpha] \\
\frac{i \mu}{\lambda + 2 \mu} [\xi_\beta] & -|\xi'|
\end{bmatrix}.
\]

(2.15)
Proof. It follows from (2.2) that
\[ A^{-1} L_{\lambda,\mu} = I_n \frac{\partial^2}{\partial x_n^2} + B \frac{\partial}{\partial x_n} + C \]
\[ = \left( I_n \frac{\partial}{\partial x_n} + B - Q \right) \left( I_n \frac{\partial}{\partial x_n} + Q \right) \]
modulo a smoothing operator. Equivalently,
\[ Q^2 - BQ - \left[ I_n \frac{\partial}{\partial x_n}, Q \right] + C = 0 \]  
(2.16)
modulo a smoothing operator, where the commutator \( I_n \frac{\partial}{\partial x_n}, Q \) is defined by, for any \( v \in C^\infty(M) \),
\[ I_n \frac{\partial}{\partial x_n} Q v = I_n \frac{\partial}{\partial x_n} (Qv) - Q \left( I_n \frac{\partial}{\partial x_n} v \right). \]
Recall that if \( G_1 \) and \( G_2 \) are two pseudodifferential operators with full symbols \( g_1 = g_1(x, \xi) \) and \( g_1 = g_2(x, \xi) \), respectively, then the full symbol \( \sigma(G_1G_2) \) of the operator \( G_1G_2 \) is given by (see [63, p11], [17, p71], and also [12, 67])
\[ \sigma(G_1G_2) \sim \sum_J \frac{(-i)^{|J|}}{|J|!} \partial^J \xi g_1 \partial^J \xi g_2, \]
where the sum is over all multi-indices \( J \). Hence, we get the full symbol equation of (2.16) as follows:
\[ \sum_J \frac{(-i)^{|J|}}{|J|!} \partial^J \xi q \partial^J \xi - \sum_J \frac{(-i)^{|J|}}{|J|!} \partial^J \xi b \partial^J \xi - \partial q \partial x_n + c = 0, \]  
(2.17)
where the sum is over all multi-indices \( J \).

We shall determine \( q_j = q_j(x, \xi'), j \leq 1 \), recursively, so that (2.17) holds modulo \( S^{-\infty} \). Grouping the homogeneous terms of degree two in (2.17), we have
\[ q_1^2 - b_1 q_1 + c_2 = 0. \]  
(2.18)
Recall that (see (2.4))
\[ b_1 = i(\lambda + \mu) \begin{bmatrix} 0 & 1 \\ \mu & \frac{1}{\lambda + 2\mu} \end{bmatrix} \]
\[ \xi \]
It follows from (2.6) that \( c_2 \) can be rewritten as
\[ c_2 = -|\xi'|^2 I_n - \begin{bmatrix} \frac{\lambda + \mu}{\mu} \xi \xi_b & 0 \\ 0 & -\frac{\lambda + \mu}{\lambda + 2\mu} |\xi'|^2 \end{bmatrix}. \]  
(2.19)
In our notations, \([\xi^\alpha] = (\xi^1, \ldots, \xi^{n-1})^T\) is a column vector, \([\xi_\beta] = (\xi_1, \ldots, \xi_{n-1})\) is a row vector, and \([\xi^\alpha \xi_\beta]\) is an \((n-1) \times (n-1)\) matrix. Thus,
\[
[\xi^\alpha] \cdot [\xi_\beta] = [\xi^\alpha \xi_\beta],
\]
\[
[\xi_\beta] \cdot [\xi^\alpha] = [\xi^\alpha],
\]
\[
[\xi^\alpha \xi_\beta] \cdot [\xi^\alpha] = [\xi^\alpha]^2[\xi^\alpha],
\]
\[
[\xi_\beta] \cdot [\xi^\alpha \xi_\beta] = [\xi^\alpha]^2[\xi_\beta],
\]
\[
[\xi^\alpha \xi_\beta] \cdot [\xi^\alpha \xi_\beta] = [\xi^\alpha]^2[\xi^\alpha \xi_\beta].
\]

We find that
\[
\frac{1}{|\xi'|} \begin{bmatrix}
[\xi^\alpha \xi_\beta] & 0 \\
0 & |\xi'|
\end{bmatrix}^2 = \begin{bmatrix}
[\xi_\beta] & 0 \\
0 & |\xi'|
\end{bmatrix}^2 = \begin{bmatrix}
[\xi^\alpha \xi_\beta] & 0 \\
0 & |\xi'|^2
\end{bmatrix},
\]
\[
\frac{1}{|\xi'|} \begin{bmatrix}
[\xi^\alpha \xi_\beta] & 0 \\
0 & |\xi'|
\end{bmatrix} \begin{bmatrix}
0 & [\xi^\alpha] \\
0 & [\xi_\beta]
\end{bmatrix} = \begin{bmatrix}
0 & [\xi^\alpha]
\end{bmatrix} \begin{bmatrix}
0 & |\xi'|^2
\end{bmatrix} = |\xi'| \begin{bmatrix}
0 & [\xi^\alpha]
\end{bmatrix}.
\]

Let \(\mathcal{C}\) denote the set of all diagonal matrices of the form
\[
\begin{bmatrix}
a_1I_{n-1} & 0 \\
0 & a_2
\end{bmatrix}, \quad a_1, a_2 \in \mathbb{C}.
\]

Obviously, \(\mathcal{C}\) is a commutative ring according to addition and multiplication of matrices (see also [41]). The following three matrices
\[
|\xi'|I_n, \quad \frac{1}{|\xi'|} \begin{bmatrix}
[\xi^\alpha \xi_\beta] & 0 \\
0 & |\xi'|
\end{bmatrix}, \quad \begin{bmatrix}
0 & [\xi^\alpha]
\end{bmatrix}
\]
generate a matrix algebra \(\mathfrak{S}\) according to the following operations:

(i) for any \(e \in \mathcal{C}\) and any \(h \in \mathfrak{S}\), \(eh \in \mathfrak{S}\).

(ii) for any \(h_1, h_2 \in \mathfrak{S}\), \(h_1 + h_2 \in \mathfrak{S}\) and \(h_1h_2 \in \mathfrak{S}\).

Clearly, \(\mathcal{C}\) is a basis of the matrix algebra \(\mathfrak{S}\).

In view of the special forms of \(b_1, c_2\), and the above properties of block matrices, we see that the solution \(q_1\) should have the following form:

\[
q_1 = s_1|\xi'|I_n + \begin{bmatrix}
s_2 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
\frac{1}{|\xi'|}[\xi^\alpha \xi_\beta] \\
0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
0 \\
[\xi_\beta] \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
is_3 \\
is_4 \\
is_5
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

(2.20)

\[
= s_1|\xi'|I_n + \begin{bmatrix}
s_2 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
\frac{1}{|\xi'|}[\xi^\alpha \xi_\beta] \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
is_3[\xi^\alpha]
\end{bmatrix} \begin{bmatrix}
0 \\
is_4[\xi_\beta] \\
0 \\
0
\end{bmatrix}
\]

(2.20)

\[
= s_1|\xi'|I_n + \begin{bmatrix}
s_2 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
\frac{1}{|\xi'|}[\xi^\alpha \xi_\beta] \\
0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
is_3[\xi^\alpha] \\
is_4[\xi_\beta] \\
is_5[\xi^\alpha] \\
is_5[\xi_\beta]
\end{bmatrix}.
\]

(2.20)
where $s_j \in \mathbb{C}$, $1 \leq j \leq 5$, are coefficients to be determined. Substituting (2.4), (2.20) and (2.19) into (2.18), we get

$$s_1^2 |\xi'|^2 I_n + \begin{bmatrix} (s_2^2 - s_3 s_4) |\xi'^\alpha_\lambda \xi_\beta| & is_3 (s_2 - s_5) |\xi'| |\xi'^\alpha| \\ is_4 (s_2 - s_5) |\xi'| |\xi_\beta| & (s_2^2 - s_3 s_4) |\xi'|^2 \end{bmatrix} + 2s_4 \begin{bmatrix} s_2 |\xi'^\alpha \xi_\beta| & is_3 |\xi'| |\xi'^\alpha| \\ is_4 |\xi'| |\xi_\beta| & -s_5 |\xi'|^2 \end{bmatrix}$$

$$- i(\lambda + \mu) \begin{bmatrix} 0 & 1/\mu |\xi'| |\xi'^\alpha| \\ \frac{1}{\lambda + 2\mu} |\xi'| |\xi_\beta| & 0 \end{bmatrix} + \begin{bmatrix} is_4/\mu |\xi'^\alpha \xi_\beta| & -s_5/\mu |\xi'| |\xi'^\alpha| \\ s_2/\lambda + 2\mu |\xi'| |\xi_\beta| & is_3/\lambda + 2\mu |\xi'|^2 \end{bmatrix}$$

$$- |\xi'|^2 I_n - \begin{bmatrix} \frac{\lambda + \mu}{\mu} |\xi'^\alpha \xi_\beta| & 0 \\ 0 & \frac{-\lambda + \mu}{\lambda + 2\mu} |\xi'|^2 \end{bmatrix} = 0.$$ 

Therefore, we have the following equations of coefficients:

$$\begin{cases} s_1^2 - 1 = 0, \\ s_2^2 - s_3 s_4 + 2s_1 s_2 + \frac{\lambda + \mu}{\mu} (s_4 - 1) = 0, \\ s_3 (s_2 - s_5) + 2s_1 s_3 + \frac{\lambda + \mu}{\mu} (s_5 - s_1) = 0, \\ s_4 (s_2 - s_5) + 2s_1 s_4 - \frac{\lambda + \mu}{\lambda + 2\mu} (s_2 + s_1) = 0, \\ s_5 (s_2 - s_5) - \frac{\lambda + \mu}{\lambda + 2\mu} (s_3 + 1) = 0. \end{cases} \tag{2.21}$$

Because we have chosen the unit outer normal vector $\nu$ on the boundary $\partial M$, we should take

$$\begin{cases} s_1 = 1, \\ s_2 \geq 0, \\ s_1 - s_3 > 0, \end{cases} \tag{2.22}$$

which implies that the real part of the matrix $q_1$ is positive definite. Solving (2.21) with the conditions (1.1) and (2.22), we then get

$$\begin{cases} s_1 = 1, \\ s_2 = s_3 = s_4 = s_5 = \frac{\lambda + \mu}{\lambda + 3\mu}. \end{cases} \tag{2.23}$$

Thus, substituting (2.23) into (2.20), we immediately obtain

$$q_1 (x, \xi') = |\xi'| I_n + \frac{\lambda + \mu}{\lambda + 3\mu} \begin{bmatrix} 1/|\xi'| |\xi'^\alpha \xi_\beta| & i|\xi'| \\ i|\xi_\beta| & -|\xi'| \end{bmatrix}.$$ 

Grouping the homogeneous terms of degree $-m$ ($m \geq -1$) in (2.17), we have

$$(q_1 - b_1) q_{-m-1} + q_{-m-1} q_1 = E_{-m}, \tag{2.24}$$

where $E_{-m}$, $m \geq -1$, are given by (2.9)–(2.11). Equation (2.24) is called the Sylvester equation (see [4, chapter 9]). By (2.4), (2.12) and (2.23), we get

$$q_1 - b_1 = |\xi'| I_n + s_2 F_2, \tag{2.25}$$

where $F_2$ is a matrix.
where $F_2$ is given by (2.15). From (2.14) and (2.15), we find that

$$
F_2^2 = F_2 = 0. \quad (2.26)
$$

Recall that (1.26) and (1.27), we denote by vec and $\otimes$ the vectorization and the Kronecker product of matrices, respectively. It follows from (1.35) that

$$
\text{vec} E_{-m} = \text{vec} \left((q_1 - b_1)q_{-m-1} + q_{-m-1}q_1\right) = H \text{vec} q_{-m-1}, \quad m \geq -1, \quad (2.27)
$$

where

$$
H = (I_n \otimes (q_1 - b_1)) + (q_1^T \otimes I_n). \quad (2.28)
$$

Combining (2.25) and (2.28), and using the properties (1.28) and (1.29), we obtain

$$
H = I_n \otimes (q_1^T | \xi_7| s_2 F_2) + (q_1^T | \xi_7| s_2 F_1) \otimes I_n
$$

$$
= 2(| \xi_7^2|^2 s_6 I_n \otimes F_2 + s_7 F_1^T \otimes I_n). \quad (2.29)
$$

By (1.31) and (2.26), we have

$$
(I_n \otimes F_2)(I_n \otimes F_2) = I_n \otimes F_2^2 = 0, \quad (2.30)
$$

$$
(F_1^T \otimes I_n)(F_1^T \otimes I_n) = (F_1^T)^2 \otimes I_n = 0, \quad (2.31)
$$

$$
(I_n \otimes F_2)(F_1^T \otimes I_n) = F_1^T \otimes F_2. \quad (2.32)
$$

In view of the above properties and $H$ is of order one in $\xi_7$, thus, we set $H^{-1}$ has the form

$$
H^{-1} = \frac{1}{2|\xi_7|^2} I_n \otimes I_n + \frac{1}{|\xi_7|^2} \left(s_6 I_n \otimes F_2 + s_7 F_1^T \otimes I_n\right) + \frac{s_8}{|\xi_7|^3} (F_1^T \otimes F_2), \quad (2.33)
$$

where $s_6, s_7, s_8 \in \mathbb{C}$ are coefficients to be determined. From (2.27), we have

$$
\text{vec} q_{-m-1} = H^{-1} \text{vec} E_{-m}, \quad m \geq -1. \quad (2.34)
$$

Combining (2.33), (2.34), and (1.32)–(1.34), we obtain, for $m \geq -1$,

$$
q_{-m-1} = \frac{1}{2|\xi_7|^2} E_{-m} + \frac{1}{|\xi_7|^2} \left(s_6 F_2 E_{-m} + s_7 E_{-m} F_1\right) + \frac{s_8}{|\xi_7|^3} F_2 E_{-m} F_1. \quad (2.35)
$$

It follows from (1.31), (2.26) and (2.29)–(2.33) that

$$
I_{\omega_3} = HH^{-1}
$$

$$
= I_n \otimes I_n + \frac{2}{|\xi_7|^2} \left(s_6 I_n \otimes F_2 + s_7 F_1^T \otimes I_n\right) + \frac{2s_8}{|\xi_7|^2} (F_1^T \otimes F_2)
$$

$$
+ \frac{s_2}{2|\xi_7|^2} \left(I_n \otimes F_2 + F_1^T \otimes I_n\right) + \frac{s_2}{2|\xi_7|^2} \left(s_6 I_n \otimes F_2^2 + s_7 (F_1^T)^2 \otimes I_n\right)
$$

$$
+ \left(s_6 + s_7\right) F_1^T \otimes F_2 + \frac{s_2 s_8}{|\xi_7|^3} (F_1^T \otimes F_2^2 + (F_1^T)^2 \otimes F_2)
$$

$$
= I_n \otimes I_n + \left(2s_6 + \frac{s_3}{2}\right) \frac{1}{|\xi_7|^2} I_n \otimes F_2 + \left(2s_7 + \frac{s_3}{2}\right) \frac{1}{|\xi_7|^2} F_1^T \otimes I_n
$$

$$
+ \left(2s_8 + s_2 (s_6 + s_7)\right) \frac{1}{|\xi_7|^2} (F_1^T \otimes F_2). \quad (2.36)
$$
In boundary normal coordinates, the elastic Dirichlet-to-Neumann map

\begin{align*}
\begin{cases}
2x_6 + \frac{s_2}{2} = 0, \\
2x_7 + \frac{s_2}{2} = 0, \\
2x_8 + s_2(x_6 + x_7) = 0.
\end{cases}
\end{align*}

Recall that \( s_2 = \frac{\lambda + \mu}{\lambda + \mu} \) by (2.23). Solving the above equations, we get

\begin{align}
&x_6 = \frac{s_2}{2} = -\frac{\lambda + \mu}{4(\lambda + 3\mu)}, \\
&x_7 = \frac{s_2}{2} = \frac{(\lambda + \mu)^2}{4(\lambda + 3\mu)^2}.
\end{align}

Substituting (2.36) into (2.35), we immediately obtain

\begin{align*}
q_{m-1}(x, \xi') = 
\frac{1}{2|\xi'|} E_{-m} - \frac{\lambda + \mu}{4(\lambda + 3\mu)|\xi'|^2} (F_2 E_{-m} + E_{-m} F_1) \\
+ \frac{(\lambda + \mu)^2}{4(\lambda + 3\mu)^2 |\xi'|^3} F_2 E_{-m} F_1, \quad m \geq -1.
\end{align*}

In boundary normal coordinates, the operator \(-\frac{\partial}{\partial n}\big|_{\partial M}\) can be represented as the pseudodifferential operator \(Q\) modulo a smoothing operator (see (2.39) below).

**Proposition 2.2.** In boundary normal coordinates, the elastic Dirichlet-to-Neumann map \(\Lambda_{\lambda, \mu}\) can be represented as

\begin{align}
\Lambda_{\lambda, \mu} = (AQ - K)|_{\partial M}
\end{align}

modulo a smoothing operator, where \(A\) is given by (2.3), and

\begin{align}
K = \begin{bmatrix} 0 & \mu \left[ \delta_{\alpha\beta} \frac{\partial}{\partial x_\beta} + \Gamma_{\alpha\beta}^{\gamma} \right] \\
\lambda \left[ \frac{\partial}{\partial x_\beta} + \Gamma_{\alpha\beta}^{\gamma} \right] & \lambda \Gamma_{\alpha\beta}^{\gamma} \end{bmatrix}.
\end{align}

**Proof.** We use the boundary normal coordinates \((x', x_n)\) with \(x_n \in [0, T]\). Since the principal symbol of the Lamé operator \(\mathcal{L}_{\lambda, \mu}\) is negative definite, the hyperplane \(x_n = 0\) is non-characteristic. Hence, \(\mathcal{L}_{\lambda, \mu}\) is partially hypoelliptic with respect to this boundary (see [16, p 107]). Therefore, the solution to the equation \(\mathcal{L}_{\lambda, \mu} u = 0\) is smooth in normal variable, that is, \(u \in \mathcal{C}^\infty([0, T]; \mathcal{D}'(\mathbb{R}^{n-1}))^n\) locally. From proposition 2.1, we see that (1.11) is locally equivalent to the following system of equations for \(u, w \in \mathcal{C}^\infty([0, T]; \mathcal{D}'(\mathbb{R}^{n-1}))^n\):

\begin{align*}
\left( I_n \frac{\partial}{\partial x_n} + Q \right) u &= w, \quad u|_{x_n=0} = f, \\
\left( I_n \frac{\partial}{\partial x_n} + B - Q \right) w &= y \in \mathcal{C}^\infty([0, T] \times \mathbb{R}^{n-1}))^n.
\end{align*}

Inspired by [41] (see also [34]), if we substitute \(t = T - x_n\) into the second equation above, then we get a backwards generalized heat equation

\begin{align*}
\frac{\partial w}{\partial t} - (B - Q) w &= -y.
\end{align*}

Since \(u\) is smooth in the interior of the manifold \(M\) by interior regularity for elliptic operator \(\mathcal{L}_{\lambda, \mu}\), it follows that \(w\) is also smooth in the interior of \(M\), and so \(w|_{x_n=T}\) is smooth. In view of the real part
of \( q_1 \) (the principal symbol of \( Q \)) is positive definite (see (2.12)), we get that the solution operator for this heat equation is smooth for \( t > 0 \) (see [67, p 134]). Therefore,

\[
\frac{\partial u}{\partial x_n} + Qu = w \in [C^\infty ([0, T] \times \mathbb{R}^{n-1})]^n
\]

locally. If we set \( Rf = w \big|_{\partial M} \), this shows that \( R \) is a smoothing operator and

\[
\frac{\partial u}{\partial x_n} \bigg|_{\partial M} = -Qu \bigg|_{\partial M} + Rf.
\]

In order to get (2.37) and (2.38), we need to calculate the local expression for the elastic Dirichlet-to-Neumann map \( \Lambda_{\lambda, \mu} \) (see (1.12)). It follows from (1.5) that the \( j \)-th component of \( (Su)_\nu \) is

\[
((Su)_\nu)^j = (\nabla^j u_k + \nabla_k u^j)_\nu^k, \quad 1 \leq j \leq n.
\]

In boundary normal coordinates, we take \( \nu = (0, \ldots, 0, -1)^T \) and \( \partial_\nu = -\partial_{x_n} \). In particular, \( u_n = u^a \) since \( g_{jn} = \delta_{jn} \) in boundary normal coordinates. We have

\[
((Su)_\nu)^j = -\left( \nabla^j u_n + \nabla_n u^j \right), \quad 1 \leq j \leq n.
\]

Note that, in boundary normal coordinates,

\[
\Gamma_{nk} = \Gamma_{nk} = 0,
\]

\[
g^{\alpha \beta} \Gamma_{\beta \gamma} + \Gamma_{\alpha \gamma} = 0.
\]

Thus, we get

\[
((Su)_\nu)^j = -\left( \nabla^j u_n + \nabla_n u^j \right) = -g^{\alpha \beta} \frac{\partial u^\alpha}{\partial x_\beta} - \frac{\partial u^\alpha}{\partial x_n}, \quad 1 \leq \alpha \leq n - 1,
\]

(2.40)

\[
((Su)_\nu)^n = -\left( \nabla^j u_n + \nabla_n u^j \right) = -\frac{\partial u^a}{\partial x_n}.
\]

(2.41)

Hence, substituting (1.2), (2.39)–(2.41) into (1.12), we immediately obtain (2.37) holds modulo a smoothing operator.

\[ \square \]

### 3. Determining Lamé coefficients on the boundary

In this section, we will give the full symbol of the elastic Dirichlet-to-Neumann map \( \Lambda_{\lambda, \mu} \), and prove the uniqueness results for the Lamé coefficients \( \lambda, \mu \), and their derivatives on the boundary \( \partial M \).

**Proof of theorem 1.1.** Let

\[
\sigma(\Lambda_{\lambda, \mu}) \sim \sum_{j \in I} p_j(x, \xi')
\]

be the full symbol of the elastic Dirichlet-to-Neumann map \( \Lambda_{\lambda, \mu} \), where \( p_j(x, \xi') \) are homogeneous of degree \( j \) in \( \xi' \). Recall that \( q(x, \xi') \sim \sum_{j \in I} q_j(x, \xi') \) is the full symbol of \( Q \). According to (2.37) and (2.38), we have
\[ p_1(x, \xi') = Aq_1(x, \xi') - k_1, \quad (3.1) \]
\[ p_0(x, \xi') = Aq_0(x, \xi') - k_0, \quad (3.2) \]
\[ p_{-m}(x, \xi') = Aq_{-m}(x, \xi'), \quad m \geq 1, \quad (3.3) \]

where \( A \) is given by (2.3), and
\[
k_1 = \begin{bmatrix} 0 & i\mu [\xi^\alpha] \\ i\lambda [\xi^\beta] & 0 \end{bmatrix}, \quad k_0 = \begin{bmatrix} 0 & 0 \\ \lambda [\Gamma^\alpha_{\alpha\beta}] & \lambda \Gamma^\alpha_{\alpha\alpha} \end{bmatrix}.
\]

Therefore, substituting (2.3), (2.12), and the above matrix \( k_1 \) into (3.1), we obtain
\[
p_1(x, \xi') = \begin{bmatrix} \mu [\xi'] [I_{n-1} + \frac{\mu(\lambda + \mu)}{(\lambda + 3\mu)[\xi']^2} [\xi^\alpha \xi^\beta] - \frac{2i\mu^2}{\lambda + 3\mu} [\xi^\alpha] \\ \frac{2\mu^2}{\lambda + 3\mu} [\xi^\beta] & \frac{2\mu(\lambda + 2\mu)}{\lambda + 3\mu} |\xi'| \end{bmatrix}.
\]

Similarly, substituting (2.3) and the above matrix \( k_0 \) into (3.2) and (3.3), respectively, we immediately obtain
\[
p_0(x, \xi') = \begin{bmatrix} \mu [\xi'] [I_{n-1} + \frac{\mu(\lambda + \mu)}{(\lambda + 3\mu)[\xi']^2} [\xi^\alpha \xi^\beta] \lambda [\Gamma^\alpha_{\alpha\beta}] & \lambda \Gamma^\alpha_{\alpha\alpha} \\ 0 & 0 \end{bmatrix}, \quad (3.2)
\]
\[
p_{-m}(x, \xi') = \begin{bmatrix} \mu [\xi'] [I_{n-1} + \frac{\mu(\lambda + \mu)}{(\lambda + 3\mu)[\xi']^2} [\xi^\alpha \xi^\beta] \lambda [\Gamma^\alpha_{\alpha\beta}] & \lambda \Gamma^\alpha_{\alpha\alpha} \\ 0 & 0 \end{bmatrix} q_{-m}(x, \xi'), \quad m \geq 1.
\]

Now, we prove the uniqueness results for the Lamé coefficients \( \lambda, \mu \), and their derivatives on the boundary \( \partial M \) by the full symbol of the elastic Dirichlet-to-Neumann map \( \Lambda_{\lambda, \mu} \).

**Proof of theorem 1.2.** It follows from (1.18) that the \((n, \beta)\)-entry \((p_1)_\beta\) and the \((n, n)\)-entry \((p_1)_n^n\) of \( p_1 \) are, respectively,
\[
(p_1)_\beta = if_1 \xi_\beta, \\
(p_1)_n^n = f_2 |\xi'|,
\]

where
\[
f_1 = \frac{2\mu^2}{\lambda + 3\mu}, \\
f_2 = \frac{2\mu(\lambda + 2\mu)}{\lambda + 3\mu}.
\]

By (1.1), we see that \( f_1 > 0 \) and \( f_2 > 0 \). It is easy to calculate that
\[
\begin{cases} 
\mu = \frac{1}{2} (f_1 + f_2), \\
\lambda = \frac{1}{2} (f_1 + f_2) \left( \frac{f_2}{f_1} - 2 \right).
\end{cases}
\]

This shows that \( p_1 \) uniquely determines \( \lambda \) and \( \mu \) on the boundary \( \partial M \). Furthermore, the tangential derivatives \( \frac{\partial p_1}{\partial x_\alpha} \) and \( \frac{\partial p_1}{\partial x_n} \) for \( 1 \leq \alpha \leq n-1 \) can also be uniquely determined by \( p_1 \) on the boundary \( \partial M \).

According to the above discussion, the boundary values of \( \lambda \) and \( \mu \) have been uniquely determined, we see from (3.2) that \( q_0 \) is uniquely determined by \( p_0 \). By (2.24), we can determine \( E_1 \) from
the knowledge of \( q_0 \). For \( k \geq 0 \), we denote by \( T_{-k} = T_{-k}(\lambda, \mu) \) the terms that only involve the boundary values of \( \lambda, \mu \), and their normal derivatives of order at most \( k \) (which have been uniquely determined). Note that \( T_{-k} \) may be different in different expressions. From (2.9), we have

\[
E_1 = b_0 q_1 + \frac{\partial q_1}{\partial \alpha} - c_1 + T_0. \tag{3.4}
\]

We calculate the \((\alpha, n)\)-entry \((E_1)^{\alpha}\) and the \((n, n)\)-entry \((E_1)^{\alpha}\) of \(E_1\),

\[
(E_1)^{\alpha} = i f_1 \xi^{\alpha} + T_0, \tag{3.5}
\]

\[
(E_1)^{\alpha} = f_4 |\xi^1| + T_0, \tag{3.6}
\]

where

\[
f_3 = \frac{2}{(\lambda + 3 \mu)^3} \left( \frac{\partial \lambda}{\partial \alpha} - (2 \lambda + 3 \mu) \frac{\partial \mu}{\partial \alpha} \right),
\]

\[
f_4 = \frac{2}{(\lambda + 2 \mu)(\lambda + 3 \mu)^2} \left( \mu^2 \frac{\partial \lambda}{\partial \alpha} + (\lambda^2 + 4 \lambda \mu + 6 \mu^2) \frac{\partial \mu}{\partial \alpha} \right).
\]

Since \( \mu > 0 \) and \( \lambda + \mu \geq 0 \) by (1.1), we have

\[
\det \begin{bmatrix} \mu & -(2 \lambda + 3 \mu) \\ \mu^2 & \lambda^2 + 4 \lambda \mu + 6 \mu^2 \end{bmatrix} = \mu (\lambda + 3 \mu)^2 \neq 0.
\]

This implies that \( p_0 \) uniquely determines \( \frac{\partial \lambda}{\partial \alpha} \) and \( \frac{\partial \mu}{\partial \alpha} \) on the boundary \( \partial M \).

By (3.3) and (2.24), we know that \( q_{-1} \) is uniquely determined by \( p_{-1} \), and \( E_0 \) can be determined from the knowledge of \( q_{-1} \). From (2.10), we see that

\[
E_0 = \frac{\partial q_0}{\partial \alpha} + T_{-1},
\]

By (2.24), we have

\[
(q_1 - b_1) \frac{\partial q_0}{\partial \alpha} + \frac{\partial q_0}{\partial \alpha} q_1 = \frac{\partial E_1}{\partial \alpha} + T_{-1}.
\]

This implies that \( \frac{\partial E_1}{\partial \alpha} \) can be determined from the knowledge of \( \frac{\partial q_0}{\partial \alpha} \). Thus, it follows from (3.5) and (3.6) that the \((\alpha, n)\)-entry \((\frac{\partial E_1}{\partial \alpha})^{\alpha}\) and the \((n, n)\)-entry \((\frac{\partial E_1}{\partial \alpha})^{\alpha}\) of \(\frac{\partial E_1}{\partial \alpha}\) are, respectively,

\[
\left( \frac{\partial E_1}{\partial \alpha} \right)^{\alpha} = i \frac{\partial f_3}{\partial \alpha} \xi^{\alpha} + T_{-1},
\]

\[
\left( \frac{\partial E_1}{\partial \alpha} \right)^{\alpha} = \frac{\partial f_4}{\partial \alpha} |\xi^1| + T_{-1},
\]

where

\[
\frac{\partial f_3}{\partial \alpha} = \frac{2}{(\lambda + 3 \mu)^3} \left( \frac{\partial^2 \lambda}{\partial \alpha^2} - (2 \lambda + 3 \mu) \frac{\partial^2 \mu}{\partial \alpha^2} \right) + T_{-1},
\]

\[
\frac{\partial f_4}{\partial \alpha} = \frac{2}{(\lambda + 2 \mu)(\lambda + 3 \mu)^2} \left( \mu^2 \frac{\partial^2 \lambda}{\partial \alpha^2} + (\lambda^2 + 4 \lambda \mu + 6 \mu^2) \frac{\partial^2 \mu}{\partial \alpha^2} \right) + T_{-1}.
\]

Similarly, this implies that \( p_{-1} \) uniquely determines \( \frac{\partial^2 \lambda}{\partial \alpha^2} \) and \( \frac{\partial^2 \mu}{\partial \alpha^2} \) on the boundary \( \partial M \).

Finally, we consider \( p_{-m-1} \) for \( m \geq 1 \). By (3.3) and (2.24), we have \( p_{-m-1} \) uniquely determines \( q_{-m-1} \), and \( E_{-m} \) can be determined from the knowledge of \( q_{-m-1} \). From (2.11), we see that

\[
E_{-m} = \frac{\partial q_{-m}}{\partial \alpha} + T_{-m-1}.
\]
We see from (2.24) that
\[
(q_1 - b_1) \frac{\partial q_{-m}}{\partial x_n} + \frac{\partial q_{-m}}{\partial x_n} q_1 = \frac{\partial E_{-m+1}}{\partial x_n} + T_{-m-1}.
\]
This implies that \(\frac{\partial E_{-m+1}}{\partial x_n}\) can be determined from the knowledge of \(\frac{\partial q_{-m}}{\partial x_n}\).

We end this proof by induction. For the \((\alpha, n)\)-entry \(\left(\frac{\partial f_1}{\partial x_n}\right)^\alpha\) and the \((n, n)\)-entry \(\left(\frac{\partial f_2}{\partial x_n}\right)^n\), \(1 \leq j \leq m\), suppose we have shown that
\[
\left(\frac{\partial^j f_1}{\partial x_n^j}\right)^\alpha = \frac{\partial^j f_1}{\partial x_n^j} \eta_\alpha + T_j,
\]
\[
\left(\frac{\partial^j f_2}{\partial x_n^j}\right)^n = \frac{\partial^j f_2}{\partial x_n^j} \xi_j + T_j,
\]
where
\[
\frac{\partial^j f_1}{\partial x_n^j} = \frac{2}{(\lambda + 3\mu)^j} \left( \mu \frac{\partial^{j+1} \lambda}{\partial x_n^{j+1}} - (2\lambda + 3\mu) \frac{\partial^{j+1} \mu}{\partial x_n^{j+1}} \right) + T_j,
\]
\[
\frac{\partial^j f_2}{\partial x_n^j} = \frac{2}{(\lambda + 2\mu)(\lambda + 3\mu)^j} \left( \mu^2 \frac{\partial^{j+1} \lambda}{\partial x_n^{j+1}} + (\lambda^2 + 4\lambda + 6\mu^2) \frac{\partial^{j+1} \mu}{\partial x_n^{j+1}} \right) + T_j.
\]
By (3.3) and (2.24), we see that \(p_{-j}\) uniquely determines \(q_{-j}\), and \(E_{-j+1}\) can be determined from the knowledge of \(q_{-j}\). By iteration, we have \(\frac{\partial q_{-m}}{\partial x_n}\) can be determined from the knowledge of \(\frac{\partial^{m+1} f_1}{\partial x_n^{m+1}}\).

Then, the \((\alpha, n)\)-entry \(\left(\frac{\partial^{m+1} f_1}{\partial x_n^{m+1}}\right)^\alpha\) and the \((n, n)\)-entry \(\left(\frac{\partial^{m+1} f_2}{\partial x_n^{m+1}}\right)^n\) are, respectively,
\[
\left(\frac{\partial^{m+1} f_1}{\partial x_n^{m+1}}\right)^\alpha = \frac{\partial^{m+1} f_1}{\partial x_n^{m+1}} \eta_\alpha + T_{-m-1},
\]
\[
\left(\frac{\partial^{m+1} f_2}{\partial x_n^{m+1}}\right)^n = \frac{\partial^{m+1} f_2}{\partial x_n^{m+1}} \xi_j + T_{-m-1},
\]
where
\[
\frac{\partial^{m+1} f_1}{\partial x_n^{m+1}} = \frac{2}{(\lambda + 3\mu)^{m+1}} \left( \mu \frac{\partial^{m+2} \lambda}{\partial x_n^{m+2}} - (2\lambda + 3\mu) \frac{\partial^{m+2} \mu}{\partial x_n^{m+2}} \right) + T_{-m-1},
\]
\[
\frac{\partial^{m+1} f_2}{\partial x_n^{m+1}} = \frac{2}{(\lambda + 2\mu)(\lambda + 3\mu)^{m+1}} \left( \mu^2 \frac{\partial^{m+2} \lambda}{\partial x_n^{m+2}} + (\lambda^2 + 4\lambda + 6\mu^2) \frac{\partial^{m+2} \mu}{\partial x_n^{m+2}} \right) + T_{-m-1}.
\]
By the same argument, we see that \(p_{-m-1}\) uniquely determines \(\frac{\partial^{m+1} \lambda}{\partial x_n^{m+1}}\) and \(\frac{\partial^{m+1} \mu}{\partial x_n^{m+1}}\) on the boundary \(\partial M\). Therefore, we conclude that the elastic Dirichlet-to-Neumann map \(\Lambda_{\lambda, \mu}\) uniquely determines \(\frac{\partial^{m+1} \lambda}{\partial x^n}\) and \(\frac{\partial^{m+1} \mu}{\partial x^n}\) on the boundary \(\partial M\) for all multi-indices \(J\).

\[
\square
\]

4. Global uniqueness of the real analytic Lamé coefficients

This section is devoted to proving the global uniqueness of the real analytic Lamé coefficients on a real analytic manifold. More precisely, we shall prove that the elastic Dirichlet-to-Neumann map \(\Lambda_{\lambda, \mu}\) uniquely determines the real analytic Lamé coefficients on the whole manifold \(M\).

We recall the definitions of real analytic functions and real analytic hypersurfaces of a Riemannian manifold. Let \(f(x)\) be a real-valued function defined on an open set \(\Omega \subset \mathbb{R}^n\).
For \( y \in \Omega \) we call \( f(x) \) real analytic at \( y \) if there exist \( a_J \in \mathbb{R} \) and a neighborhood \( N_y \) of \( y \) (all depending on \( y \)) such that \[
f(x) = \sum_J a_J (x-y)^J
\]
for all \( x \in N_y \) and \( J \in \mathbb{N}^n \). We say \( f(x) \) is real analytic in \( \Omega \) if \( f(x) \) is real analytic at each \( y \in \Omega \).

Let \((M, g)\) be a Riemannian manifold of dimension \( n \). A subset \( U \) of \( M \) is said to be an \((n-1)\)-dimensional real analytic hypersurface if \( U \) is nonempty and if for every point \( x \in U \), there is a real analytic diffeomorphism of a unit open ball \( B(0, 1) \subseteq \mathbb{R}^n \) onto an open neighborhood \( N_x \) of \( x \) such that \( B(0, 1) \cap \{ x \in \mathbb{R}^n | x_0 = 0 \} \) maps onto \( N_x \cap U \).

In order to prove theorem 1.3, we need the following result for real analytic functions (see [26, p65]).

**Lemma 4.1 (Unique continuation of real analytic functions).** Let \( \Omega \subset \mathbb{R}^n \) be a connected open set and \( f(x) \) be a real analytic function defined on \( \Omega \). Let \( y \in \Omega \). Then, \( f(x) \) is uniquely determined in \( \Omega \) if we know \( \frac{∂^{|J|} f(x)}{∂ x^J} \) for all \( J \in \mathbb{N}^n \). In particular, \( f(x) \) is uniquely determined in \( \Omega \) by its values in any nonempty open subset of \( \Omega \).

Note that lemma 4.1 still holds for any real analytic function defined on a real analytic manifold.

**Proof of theorem 1.3.** Let \( \Gamma \subset \partial M \) be a nonempty smooth open subset. Suppose that the manifold \( M \) is real analytic up to \( \Gamma \), assume that the Lamé coefficients \( \lambda \) and \( \mu \) are also real analytic up to \( \Gamma \). According to theorem 1.2, we see that the elastic Dirichlet-to-Neumann map \( \Lambda_{\lambda, \mu} \) uniquely determines \( \frac{∂^{|J|} \lambda}{∂ x^J} \) and \( \frac{∂^{|J|} \mu}{∂ x^J} \) on the boundary \( \partial M \) for all multi-indices \( J \in \mathbb{N}^n \). Hence, for any point \( x_0 \in \Gamma \), there is a neighborhood \( N_{x_0} \) of \( x_0 \), by the analyticity of the Lamé coefficients in \( M \cap N_{x_0} \), the Taylor series
\[
\sum_J 1 \frac{∂^{|J|} \lambda}{∂ x^J} (x_0)(x-x_0)^J \quad \text{and} \quad \sum_J 1 \frac{∂^{|J|} \mu}{∂ x^J} (x_0)(x-x_0)^J
\]
converge to \( \lambda \) and \( \mu \) for all \( x \in M \cap N_{x_0} \), respectively. It follows from lemma 4.1 that \( \lambda \) and \( \mu \) can be uniquely determined in \( M \). Therefore, we conclude that \( \lambda \) and \( \mu \) can be uniquely determined on the whole manifold \( M \) by the elastic Dirichlet-to-Neumann map \( \Lambda_{\lambda, \mu} \).

It is clear that theorem 1.3 also holds for any real analytic Euclidean domain with smooth boundary.

**Corollary 4.2.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary \( \partial \Omega \), and let \( \Sigma \subset \partial \Omega \) be a nonempty smooth open subset. Suppose that \( \Omega \) is real analytic up to \( \Sigma \), assume that the Lamé coefficients \( \lambda \) and \( \mu \) are also real analytic up to \( \Sigma \), and satisfy \( \mu > 0 \) and \( \lambda + \mu > 0 \). Then, the elastic Dirichlet-to-Neumann map uniquely determines the Lamé coefficients on \( \Omega \).

**Remark 4.3.** By applying the method of Kohn and Vogelius [29], we can also prove that the elastic Dirichlet-to-Neumann map \( \Lambda_{\lambda, \mu} \) uniquely determines the Lamé coefficients \( \lambda \) and \( \mu \) on \( M \) provided the manifold and the Lamé coefficients are piecewise analytic.

**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).
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