ON A TWISTED CONICAL KÄHLER-RICCI FLOW

YASHAN ZHANG

Abstract. In this paper, we discuss diameter bound and Gromov-Hausdorff convergence of a twisted conical Kähler-Ricci flow on the total spaces of some holomorphic submersions. We also observe that, starting from a model conical Kähler metric with possibly unbounded scalar curvature, the conical Kähler-Ricci flow will instantly have bounded scalar curvature for \( t > 0 \), and the bound is of the form \( \frac{C}{t} \). Several key results will be obtained by direct arguments on the conical equation without passing to a smooth approximation. In the last section, we present several remarks on a twisted Kähler-Ricci flow and its convergence.

1. Introduction

The conical Kähler-Ricci flow is the Kähler-Ricci flow with certain cone singularities, whose existence, regularity and convergence have been widely studied in the recent years, see e.g. [4, 5, 17, 18, 21, 33, 40, 43, 45] and references therein.

In this paper, we shall discuss some properties of the conical Kähler-Ricci flow, focusing on the long time collapsing limits and a phenomenon on scalar curvature.

Let's begin with the general setup as follows. Let \( X \) be an \( n \)-dimensional compact Kähler manifold with a Kähler metric \( \omega_0 \) and \( D = \sum_{i=1}^{l} D_i \) a simple normal crossing divisor on \( X \) with every \( D_i \) an irreducible complex hypersurface and \( \beta_i \in (0, 1) \). Fix a defining section \( S_i \) for \( D_i \) on \( X \) and a Hermitian metric \( h_i \) on the line bundle \( L_i \) associated to \( D_i \). We fix a sufficiently small positive constant \( \delta \) such that

\[
\omega_0^*: = \omega_0 + \delta \sum_{i=1}^{l} \sqrt{-1} \partial \bar{\partial}|S_i|_{h_i}^{2\beta_i}
\]

is a conical Kähler metric on \( X \) with cone angle \( 2\pi \beta_i \) along \( D_i \), \( i = 1, \ldots, l \). We call \( \omega_0^* \) a model conical Kähler metric (with respect to \( D \)). Then consider the conical Kähler-Ricci flow starting from \( \omega_0^* \):

\[
\begin{cases}
\partial_t \omega(t) = -\text{Ric}(\omega(t)) + 2\pi \sum_{i=1}^{l} (1 - \beta_i)[D_i] \\
\omega(0) = \omega_0^*
\end{cases}
\]

(1.1)

By [4, 5, 17, 24] (which generalizes [3, 38, 33]), the conical Kähler-Ricci flow (1.1) has a solution up to

\[
T_{\text{max}} := \{ t > 0 | [\omega_0] + t2\pi(c_1(K_X) + \sum_{i=1}^{l}(1 - \beta_i)[D_i]) > 0 \}.
\]

See subsection 2.2 for a more precise definition of “a solution to the conical Kähler-Ricci flow”.

1
1.1. A general phenomenon on scalar curvature along the conical Kähler-Ricci flow. Firstly, we would like to note that, as a smooth Kähler metric on the open (non-compact) manifold \( X \setminus D \), the scalar curvature of \( \omega_0^* \) on \( X \setminus D \) may not be a bounded function (see e.g. Remark 4.3). However, we observe that, starting from a model conical Kähler metric \( \omega_0^* \) with possibly unbounded scalar curvature, the conical Kähler-Ricci flow will instantly have bounded scalar curvature for \( t \in (0, t_0] \) (\( t_0 \) is a sufficiently small number), and the bound is of the form \( C_t \). Precisely, we have

**Theorem 1.1.** Let \( \omega(t) \in [0, T_{\text{max}}) \) be the solution to (1.1) and \( R(t) := R(\omega(t)) \) the scalar curvature of \( \omega(t) \) on \( X \setminus D \). For any fixed \( t_0 \in (0, T_{\text{max}}) \), there exists a constant \( C \geq 1 \) such that for any \( t \in (0, t_0] \),

\[
\sup_{X \setminus D} |R(t)| \leq \frac{C}{t}.
\]  

If we regard the conical Kähler-Ricci flow (1.1) as a smooth Kähler-Ricci flow on the open (non-compact) Kähler manifold \( X \setminus D \), then it seems our observation in Theorem 1.1 fits also in the study of the Kähler-Ricci flow starting from a Kähler metric with unbounded curvature, where the phenomenon in Theorem 1.1 has usually happened even for Riemannian curvature, i.e. once one can solve the Kähler-Ricci flow from a Kähler metric with unbounded curvature, then Riemannian curvature of solution in some cases has a bound of the form \( \frac{C}{t} \) instantly, see e.g. [25, 12] and references therein for more precise results and discussions. From this viewpoint, it seems interesting to ask: can we prove the analogue result of Theorem 1.1 for Riemannian curvature? Even more, if this is true, then it provides an effective way to produce a (not necessarily model) conical Kähler metric with bounded curvature.

**Remark** We should mention that, in the special case that \( X \) is Fano, a weaker version of Theorem 1.1 (i.e. replacing the bound \( \frac{C}{t} \) in (1.2) by \( \frac{C}{t^2} \)) seems contained in arguments in [17, Proposition 4.1, Lemma 4.6] and [18, Proposition 4.1, Theorem 4.4].

1.2. Diameter bound and convergence. Next, we discuss the diameter bound and convergence of the conical Kähler-Ricci flow when \( X \) additionally admits a holomorphic submersion over another compact Kähler manifold.

Let \( f : X \to Y \) be a holomorphic submersion between two compact Kähler manifolds with \( n = \text{dim}(X) > \text{dim}(Y) = k \geq 1 \) and connected fibers. Fix an irreducible complex hypersurface \( D' \) on \( Y \), a defining section \( S' \) for \( D' \) on \( Y \) and a Hermitian metric \( h' \) on the line bundle \( L' \) associated to \( D' \).

Set \( D := f^*D' \), a smooth irreducible complex hypersurface on \( X \). Then \( S := f^*S' \) is a defining section for \( D \) and \( h := f^*h' \) is a Hermitian metric on \( L \), the line bundle associated to \( D \).

For an arbitrary Kähler metric \( \omega_0 \) on \( X \) and \( \beta \in (0, 1) \), we fix a sufficiently small positive constant \( \delta \) such that

\[
\omega_0^* := \omega_0 + \delta \sqrt{-1} \partial \bar{\partial} |S|_{h}^{2\beta}
\]  

is a model conical Kähler metric on \( X \) with cone angle \( 2\pi \beta \) along \( D \) and

\[
\chi^* = \chi + \delta \sqrt{-1} \partial \bar{\partial} |S'|_{h'}^{2\beta}
\]  

(1.4)
is a model conical Kähler metric on \( Y \) with cone angle \( 2\pi \beta \) along \( D' \). Then, again we consider the following conical Kähler-Ricci flow on \( X \) starting from \( \omega_0^* \):

\[
\begin{aligned}
\partial_t \omega(t) &= -\text{Ric}(\omega(t)) + 2\pi(1 - \beta)[D] \\
\omega(0) &= \omega_0^*,
\end{aligned}
\]  

(1.5)

which has a solution up to

\[ T_{\text{max}} := \{ t > 0 | \omega_0 + t2\pi(c_1(K_X) + (1 - \beta)[D]) > 0 \}. \]

We now assume there exists a Kähler metric \( \chi \) on \( Y \) such that

\[
f^*\chi \in \frac{1}{T_{\text{max}}} [\omega_0] + 2\pi(c_1(K_X) + (1 - \beta)[D]) .
\]

(1.6)

Easily, (1.6) implies

\[
f^*(\chi - (1 - \beta)R_{hr}) \in \frac{1}{T_{\text{max}}} [\omega_0] + 2\pi c_1(K_X).
\]

(1.7)

In this paper, given above setting, we study the convergence of the following twisted conical Kähler-Ricci flow:

\[
\begin{aligned}
\partial_t \omega(t) &= -\text{Ric}(\omega(t)) - \omega(t) + \frac{1}{T_{\text{max}}} \omega_0 + 2\pi(1 - \beta)[D] \\
\omega(0) &= \omega_0^*,
\end{aligned}
\]  

(1.8)

In general, \( 0 < T_{\text{max}} \leq \infty \). If \( T_{\text{max}} = \infty \), then (1.8) is just the usual (normalized) conical Kähler-Ricci flow and the fiber is Calabi-Yau; if \( T_{\text{max}} < \infty \), then (1.8) is really a twisted conical Kähler-Ricci flow and the fiber is Fano. We will study these two different cases unifiedly. In both cases, (1.8) can be solved for \( t \in [0, \infty) \).

One of the most natural problems is to understand the long time behavior of the flow. Our main result can be stated as follows. 

**Theorem 1.2.** Assume (1.6) holds. Let \( \omega(t)_{t \in [0, \infty)} \) be the solution to (1.8) and \((X, d_t)\) the metric completion of \((X \setminus D, \omega(t))\). We have the following conclusions:

(1) There exists a positive constant \( C < \infty \) such that

\[
\sup_{t \in [0, \infty)} \text{diam}(X, d_t) \leq C.
\]

(1.9)

(2) There exists a conical Kähler metric \( \overline{\chi} \) on \( Y \) with cone angle \( 2\pi \beta \) along \( D' \) such that, as \( t \to \infty \),

(2.1) \( \omega(t) \to f^*\overline{\chi} \) as currents on \( X \);

(2.2) there exists an \( \varepsilon_0 \in (0, 1) \) with the following property: for any \( K \subset\subset X \setminus D \) there exists a constant \( C_K \geq 1 \) such that for any \( t \in [1, \infty) \),

\[
|\omega(t) - f^*\overline{\chi}|_{C^0(K, \omega_0)} \leq C_K e^{-\varepsilon_0 t};
\]

(2.3) if additionally \( \text{dim}(Y) = 1 \), then \((X, d_t) \to (Y, \overline{d})\) in Gromov-Hausdorff topology, here \((Y, \overline{d})\) is the metric completion of \((Y \setminus D', \overline{\chi})\).

In our Theorem 1.2 we have focused on the volume collapsing case, i.e. \( \text{dim}(X) > \text{dim}(Y) \). For the volume noncollapsing case, we have convergence results on the conical Kähler-Ricci flow of Chen-Wang [5] when twisted canonical line bundle \( K_X + (1 - \beta)L \) is trivial or ample and Shen [24] when \( K_X + (1 - \beta)L \) is nef and big (also see Liu-X. Zhang [17] on the conical Kähler-Ricci flow on Fano manifolds). Items (2.1) and (2.2) in Theorem 1.2 can be seen as generalizations of Chen-Wang [5] and Shen [24] to the
volume collapsing case. Of course, Theorem 1.2 are also generalizations of results on the Kähler-Ricci flow \[29, 30, 9, 35\] to the conical setting.

In the case \( T_{\text{max}} = \infty \) in (1.8), the conical Kähler metric \( \chi \) on \( Y \) given in Theorem 1.2 (2) will satisfy a generalized conical Kähler-Einstein equation (see Remark 2.1).

In the case \( \beta = 1 \) (i.e. we remove the conical singularity) and \( T_{\text{max}} < \infty \), the equation (1.8) is essentially introduced (in a slightly different form) in \[50\] and that corresponding convergence result is even new for this smooth setting, solving some special case of \[50\] Conjecture 3.1.

We remark that items (2.1) and (2.2) of Theorem 1.2 also hold when \( D' \) (and hence \( D \)) is a simple normal crossing divisor as in subsection 1.1. However, our arguments for item (1) in Theorem 1.2 involve the upper bound for bisectional curvature of the model conical metric \( \omega_0^* \), which seems only known when \( D' \) is of only one irreducible component; we can extend Theorem 1.2 (1) to simple normal crossing divisor case, provided an upper bound for bisectional curvature for the simple normal crossing case is available.

Let’s also take a look at Theorem 1.2 (2.3). If \( \dim(Y) = 1 \), we know the hypersurface \( D' \) on \( Y \) is just a single point, say \( o \in Y \), and \( D = f^*D' \) is a smooth fiber \( X_o \) on \( X \). Then Theorem 1.2 (2.3) means that the twisted conical Kähler-Ricci flow (1.8) collapses not only the generic fiber \( X_y := f^{-1}(y) \) for \( y \in Y \setminus \{o\} \), but also the cone divisor \( X_o \) to a point on the base \( Y \), in Gromov-Hausdorff topology. We hope that the additional condition \( \dim(Y) = 1 \) can be removed in the future work.

We point out that the first result that the conical Kähler-Ricci flow will collapse the cone divisor to a point in Gromov-Hausdorff topology was recently obtained by Edwards in \[7\] on Hirzebruch surfaces (in which Edwards also proved that, in some finite-time noncollapsing case, the conical Kähler-Ricci flow can contract cone divisor to a point in Gromov-Hausdorff topology), under certain symmetry condition on the initial model conical metric. Our Theorem 1.2 (2.3) provides another result on such phenomenon. Note that our result doesn’t need any symmetry condition.

1.3. Organization of this paper. In the remaining part of this paper, we will always be in the setup introduced in subsection 1.2. As we will pointed out later (see subsection 4.1), a proof for Theorem 1.1 is essentially contained in the discussions for setup introduced in subsection 1.2. A more precise organization can be found as follows:

- In Section 2, by using direct arguments on the conical equations, we prove item (2.1) in Theorem 1.2. In fact, we will show in Proposition 2.4 that the convergence takes place exponentially fast at the level of Kähler potentials.
- In Section 3, by using direct arguments on the conical equations, we prove in Proposition 3.4 that the solution is uniformly equivalent to a family of collapsing conical Kähler metric, which will immediately imply item (1) in Theorem 1.2. In the proof of Proposition 3.4, a key step is carried out in Lemma 3.1, which seems can not be proved by passing to a smooth approximation, see Remark 3.2 for more comments.
- In Section 4, we show in Lemma 4.1 that the twisted scalar curvature along the twisted conical Kähler-Ricci flow (1.8) with the assumption (1.6) has a bound of the from \( C_{\text{num}(Y)} \). Compare with the uniform scalar bound obtained in \[6\], our arguments don’t rely on some nontrivial properties of a smooth approximation for the initial model Kähler metric (see Remark 4.2 for more comments). We point out in subsection 4.1 that arguments for Lemma 4.1 can be applied to prove Theorem...
The arguments in this Section need to use a smooth approximation for the twisted conical Kähler-Ricci flow.

- In Section 5, we use direct arguments and the strategy in [5] to prove item (2.2) in Theorem 1.2.
- In Section 6, we prove Gromov-Hausdorff convergence stated in item (2.3) in Theorem 1.2, in which we will make use of Proposition 3.4 crucially.
- In Section 7, we present several remarks on a twisted Kähler-Ricci flow and its convergence.

2. The weak convergence

In this section, we shall give a proof for item (2.1) in Theorem 1.2.

2.1. Limiting metric \( \overline{\chi} \). We first need to construct the limiting metric \( \overline{\chi} \) on \( Y \). The construction is essentially the same as in [29, 30] and its generalization in [48]. Let’s recall some details. We first fix a smooth real function \( \rho \) on \( X \) such that

\[
\overline{\omega}_0 := \omega_0 + \sqrt{-1}\partial\bar{\partial}\rho \tag{2.1}
\]

on \( X \) is a closed real \((1,1)\)-form and \( \overline{\omega}_0|_y \) is the unique Kähler metric on \( X_y \) in class \([\omega_0]|_y \) with \( \text{Ric}(\overline{\omega}_0|_y) = \frac{1}{T_{\text{max}}}(2\pi\rho)|_y \) for every \( y \in Y \). Secondly we fix a smooth positive volume form \( \Omega \) on \( X \) with \( \sqrt{-1}\partial\bar{\partial}\log \Omega = f^*\chi - \frac{1}{T_{\text{max}}}\omega_0 - (1 - \beta)2\pi f^*R_{h'} \). Then define a positive smooth function \( G := \frac{\Omega}{c_{k,n}^\beta f^*\chi^k} \) on \( X \), which descends to a smooth function on \( Y \). Set \( \overline{\chi} := \chi + \sqrt{-1}\partial\bar{\partial}\psi \) be the unique solution to the following equation:

\[
(\chi + \sqrt{-1}\partial\bar{\partial}\psi)^k = \frac{e^\psi G}{|S^\beta_{h'}|^{2(1-\beta)}} \cdot \chi^k. \tag{2.2}
\]

We know \( \psi \in L^\infty(Y) \cap C^\infty(Y \setminus D') \) \cite{12, 13} and \( \overline{\chi} \) is a conical Kähler metric on \( Y \) with cone angle \( 2\pi\beta \) along \( D' \), see \cite{2, 11}. We will see \( \overline{\chi} \) is the limit of the twisted conical Kähler-Ricci flow \((1.8)\).

**Remark 2.1.** If \( T_{\text{max}} = \infty \), then \( \overline{\chi} \) constructed above satisfies a generalized conical Kähler-Einstein equation (see \cite{29}):

\[
\text{Ric}(\overline{\chi}) = -\overline{\chi} + \omega_{WP} + 2\pi(1 - \beta)|D|,
\]

where \( \omega_{WP} \) is the Weil-Petersson form on \( Y \).

2.2. Reduction to a parabolic complex Monge-Ampère equation. Set \( \omega_t := e^{-t}\omega_0 + (1 - e^{-t})f^*\chi \) and let \( \omega_t = \omega_t + \delta \sqrt{-1}\partial\bar{\partial}|S|^2_{h} + \sqrt{-1}\partial\bar{\partial}\varphi(t) \). It is well-known that \((1.8)\) can be reduced to the following

\[
\left\{ \begin{array}{l}
\partial_t\varphi(t) = \log \frac{e^{(n-k)t}(\omega_t + \delta \sqrt{-1}\partial\bar{\partial}|S|^2_{h} + \sqrt{-1}\partial\bar{\partial}\varphi(t))^{n}}{|S|^2_{h}} - \varphi(t) - \delta |S|^2_{h} \\
\varphi(0) = 0,
\end{array} \right. \tag{2.3}
\]

With the assumption \((1.8)\), by \cite{4, 5, 17, 21, 10} we know \((2.3)\) admits a unique solution \( \varphi(t) \) for \( t \in [0, \infty) \), where \( \varphi(t) \in L^\infty(X) \cap C^\infty((X \setminus D) \times [0, \infty)) \) (in fact \( \varphi(t) \) is Hölder continuous on \( X \) with respect to \( \omega_0 \) for every \( t \geq 0 \)) and \( \varphi(t) \) satisfies \((2.3)\) in current
sense on $X \times [0, \infty)$ and in smooth sense on $(X \setminus D) \times [0, \infty)$; moreover, for any $T < \infty$ one can find a constant $C \geq 1$ such that

$$C^{-1}\omega_0^* \leq \omega(t) \leq C\omega_0^*$$

(2.4)

holds on $(X \setminus D) \times [0, T]$.

We need some estimates along (1.8) and (2.3). We point out that most estimates in this paper will be obtained by working with the conical equations (1.8) and (2.3) directly. Such direct arguments have been also used in [19, Lemmas 2.1, 2.2]. Therefore, we will also provide direct arguments for some existing estimates (see in particular [6]) which have been carried out by passing to a smooth approximation.

2.3. Uniform bound for $\varphi$ and $\partial_t \varphi$. We begin with the $C^0$-estimate. In the remaining part of this paper, we assume without loss of any generality that

$$|S'|^2_{h'} \leq 1.$$  

(2.5)

Moreover, we fix a small positive constant $\lambda_0 \in (0, 1]$ such that for any $\lambda \in [-\lambda_0, \lambda_0]$ there holds on $Y \setminus D'$ that

$$\frac{1}{2} \chi^* \leq \chi^* + \lambda R_{h'} \leq 2\chi^*,$$

(2.6)

and on $X$ that

$$\frac{1}{2} \omega_0 \leq \omega_0 + \lambda f^* R_{h'} \leq 2\omega_0.$$

(2.7)

We also fix a sufficiently large constant $T_0 \geq 1$ such that for any $t \geq T_0$ there holds on $Y \setminus D'$ that

$$\frac{1}{2} \chi^* - e^{-t} \chi \geq \frac{1}{3} \chi^*.$$  

(2.9)

Lemma 2.2. [6] There exists a constant $C \geq 1$ such that

$$\sup_{(X \setminus D) \times [0, \infty)} |\varphi| \leq C.$$  

(2.10)

Proof. This result has been obtained in [6] by using a smooth approximation. We now provide a direct argument without passing to a smooth approximation. Our arguments follow closely the arguments for smooth Kähler-Ricci flow in [29, 9].

We now use an auxiliary function as in [19, Lemmas 2.1, 2.2]: for any given $\lambda \in (0, \lambda_0]$ set $\varphi_{\lambda} := \varphi + \lambda \log |S|^2_{h'}$. We have

$$\partial_t \varphi_{\lambda} = \log \frac{e^{(n-k)t}(\omega_t + \delta \sqrt{-1} \partial \bar{\partial} |S|^2_{h'} + \lambda f^* R_{h'} + \sqrt{-1} \partial \bar{\partial} \varphi_{\lambda})^n}{|S|^{-2(1-\beta)}h} - \varphi - \delta |S|^2_{h}$$

(2.11)

on $X \setminus D$.

For any $T < \infty$, we know $\varphi$ is uniformly bounded on $[0, T]$ (see [24, 10]). Therefore, the maximal value of $\varphi_{\lambda}$ on $X \times [0, T]$ must achieved at some $(x_T, t_T)$ with $x_T \in X \setminus D$. If $t_T = 0$, then $\varphi_{\lambda}(x_T, t_T)$ is bounded from above by a constant independent of $T$; otherwise $t_T > 0$, we can apply maximum principle to (2.11) at $(x_T, t_T)$ to see that

$$\varphi(x_T, t_T) \leq \log \frac{e^{(n-k)t}(\omega_t + \delta \sqrt{-1} \partial \bar{\partial} |S|^2_{h'} + \lambda f^* R_{h'})^n}{|S|^{-2(1-\beta)}h}(x_T, t_T).$$

(2.12)
Using (2.6) gives
\[
\omega_t + \delta \sqrt{-1} \partial \bar{\partial} |S|_{h}^{2\beta} + \lambda f^* R_{h'} = e^{-t} \omega_0 + f^*(\chi^* + \lambda R_{h'}) - e^{-t} f^* \chi \\
\leq e^{-t} \omega_0 + 2 f^* \chi.
\] (2.13)
and hence
\[
(\omega_t + \delta \sqrt{-1} \partial \bar{\partial} |S|_{h}^{2\beta} + \lambda f^* R_{h'})^n \\
\leq (e^{-t} \omega_0 + 2 f^* \chi^*)^n \\
\leq C'|S|_{h}^{-2(1-\beta)} (e^{-(n-k)t} + e^{-(n-k+1)t} + \ldots + e^{-nt})
\] (2.14)
for some uniform constant $C' \geq 1$. Plugging (2.14) into (2.12) implies
\[
\varphi(x_T, t_T) \leq C''
\] for some uniform constant $C''$ independent on $T$ and $\lambda$. In conclusion, we can choose a constant $C \geq 1$ independent on $T \in (0, \infty)$ and $\lambda \in (0, \lambda_0]$ such that
\[
\sup_{(X \setminus D) \times [0, T]} \varphi_\lambda \leq C.
\] (2.15)
Let $T \to \infty$ and $\lambda \to 0$ in (2.14), we get the desired upper bound for $\varphi$:
\[
\sup_{(X \setminus D) \times [0, \infty)} \varphi \leq C.
\] (2.16)
Now let’s look at the lower bound. For any $\lambda \in (0, \lambda]$, set $\varphi_\lambda := \varphi - \lambda \log |S|_{h}^2$. Similarly, we have
\[
\partial_t \varphi_\lambda = \log \frac{e^{(n-k)t}(\omega_t + \delta \sqrt{-1} \partial \bar{\partial} |S|_{h}^{2\beta} - \lambda f^* R_{h'})^n}{|S|_{h}^{-2(1-\beta)} \Omega} - \varphi - \delta |S|_{h}^{2\beta}
\] (2.17)
on $X \setminus D$.
For any $T < \infty$, we know the minimal value of $\varphi_\lambda$ on $X \times [0, T]$ must be achieved at some $(x'_T, t'_T)$ with $x'_T \in X \setminus D$. If $t'_T \in [0, T_0]$, then $\varphi_\lambda(x'_T, t'_T)$ is bounded from below by a constant independent on $T$; otherwise $t'_T > T_0$, we can apply maximum principle to (2.17) at $(x'_T, t'_T)$ to see that
\[
\varphi(x'_T, t'_T) \geq \log \frac{e^{(n-k)t}(\omega_t + \delta \sqrt{-1} \partial \bar{\partial} |S|_{h}^{2\beta} - \lambda f^* R_{h'})^n}{|S|_{h}^{-2(1-\beta)} \Omega}(x'_T, t'_T) - 1.
\] (2.18)
Using (2.6) and (2.9) gives
\[
\omega_t + \delta \sqrt{-1} \partial \bar{\partial} |S|_{h}^{2\beta} - \lambda f^* R_{h'} = e^{-t} \omega_0 + f^*(\chi^* - \lambda R_{h'}) - e^{-t} f^* \chi \\
\geq e^{-t} \omega_0 + \frac{1}{2} f^* \chi^* - e^{-t} f^* \chi \\
\geq e^{-t} \omega_0 + \frac{1}{3} f^* \chi^.
\] (2.19)
and hence
\[
(\omega_t + \delta \sqrt{-1} \partial \bar{\partial} |S|_{h}^{2\beta} - \lambda f^* R_{h'})^n \geq C^{-1} |S|_{h}^{-2(1-\beta)} e^{-(n-k)t} \Omega
\] (2.20)
for some uniform constant $C \geq 1$. Therefore, we find
for some constant $C \geq 1$ independent on $T \in [0, \infty)$ and $\lambda \in (0, \lambda_0]$. Then as before, by letting $T \to \infty$ and $\lambda \to 0$ we conclude

$$\inf_{(X \setminus D) \times [0, \infty) } \varphi \geq -C$$

(2.22)

for some constant $C \geq 1$.

Lemma 2.3 is proved.

**Lemma 2.3.** [6] There exists a constant $C \geq 1$ such that

$$\sup_{(X \setminus D) \times [0, \infty) } |\partial_t \varphi| \leq C.$$  

(2.23)

**Proof.** This result has been obtained in [6] by using a smooth approximation. Again we present a direct proof without passing to a smooth approximation. By computation in [29] there holds on $(X \setminus D) \times [0, \infty)$ that

$$(\partial_t - \Delta_{\omega(t)})(e^t - 1)\partial_t \varphi - \varphi - \delta |S|^{2\beta}_{\ast} = (n - k)e^t + k - tr_{\omega(t)}\omega_0,$$

(2.24)

and so

$$(\partial_t - \Delta_{\omega(t)})(e^t - 1)\partial_t \varphi - \varphi - \delta |S|^{2\beta}_{\ast} - (n - k)e^t - kt = -tr_{\omega(t)}\omega_0.$$  

(2.25)

As in the last lemma, we now use an auxiliary function similar to [19] Lemmas 2.1, 2.2: for any $\lambda \in (0, \lambda_0]$, set $H_\lambda := (e^t - 1)\partial_t \varphi - \varphi - \delta |S|^{2\beta}_{\ast} - (n - k)e^t - kt + \lambda \log |S|^{2\beta}_{\ast}$. Then by (2.25) we have, on $X \setminus D$,

$$(\partial_t - \Delta_{\omega(t)})H_\lambda = -tr_{\omega(t)}(\omega_0 - \lambda f^* R_{h'})$$

$$\leq -\frac{1}{2}tr_{\omega(t)}\omega_0 < 0.$$  

(2.26)

Obviously, for any $T \in (0, \infty)$, since $\partial_t \varphi$ is bounded on $(X \setminus D) \times [0, T]$, the maximal value of $H_\lambda$ on $X \times [0, T]$ must be achieved at some point $(\tilde{x}_T, 0)$ with $\tilde{x}_T \in X \setminus D$ and so $\sup_{(X \setminus D) \times [0, T]} H_\lambda$ is bounded from above by a positive constant independent on $T \in (0, \infty)$ and $\lambda \in (0, \lambda_0]$. Therefore, we have a positive constant $C \geq 1$ such that, on $(X \setminus D) \times [0, \infty)$,

$$(e^t - 1)\partial_t \varphi - \varphi - \delta |S|^{2\beta}_{\ast} - (n - k)e^t - kt \leq C,$$

(2.27)

which, combining with the facts that $\partial_t \varphi$ is uniformly bounded on $(X \setminus D) \times [0, 1]$ and $\varphi$ is uniformly bounded by Lemma 2.2 implies that $\partial_t \varphi$ is uniformly bounded from above on $(X \setminus D) \times [0, \infty)$.

For the lower bound, by a computation similar to [29] we have

$$(\partial_t - \Delta_{\omega(t)})(\partial_t \varphi + 2\varphi + \delta |S|^{2\beta}_{\ast}) = \partial_t \varphi + tr_{\omega(t)}(\omega_t + \delta \sqrt{-1}\partial \bar{\partial}|S|^{2\beta}_{\ast} + f^* \chi) - n - k,$$

(2.28)

and so, for any $\lambda \in (0, \lambda_0]$,

$$(\partial_t - \Delta_{\omega(t)})(\partial_t \varphi + 2\varphi + \delta |S|^{2\beta}_{\ast} - \lambda \log |S|^{2\beta}_{\ast})$$

$$= \partial_t \varphi + tr_{\omega(t)}(\omega_t + \delta \sqrt{-1}\partial \bar{\partial}|S|^{2\beta}_{\ast} + f^* \chi - \lambda f^* R_{h'}) - n - k,$$

(2.29)

on $(X \setminus D) \times [0, \infty)$. By (2.26) we have
\[ \omega_t + \delta \sqrt{-1} \partial \overline{\partial} |S_h^{2\beta}| + f^* \chi - \lambda f^* R_{h^t} = e^{-t} \omega_0 + f^* \chi^* + (1 - e^{-t}) f^* \chi - \lambda f^* R_{h^t} \]
\[ \geq e^{-t} \omega_0 + f^* (\chi^* - \lambda R_{h^t}) \]
\[ \geq e^{-t} \omega_0 + \frac{1}{2} f^* \chi^* , \]  
which implies
\[ tr_{\omega(t)} (\omega_t + \delta \sqrt{-1} \partial \overline{\partial} |S_h^{2\beta}| + f^* \chi - \lambda f^* R_{h^t}) \geq tr_{\omega(t)} (e^{-t} \omega_0 + \frac{1}{2} f^* \chi^*) \]
\[ \geq n \left( \frac{(e^{-t} \omega_0 + \frac{1}{2} f^* \chi^*)^n}{\omega(t)^n} \right)^{\frac{1}{n}} \]
\[ \geq n \left( \frac{C^{-1} e^{-\frac{\delta \varphi}{n}} |S_h^{2\beta}|^{-2(1-\beta)} \Omega}{e^{-\frac{n-k}{2}} |S_h^{2\beta}|^{-2(1-\beta)} \delta \varphi + \delta |S_h^{2\beta}| \Omega} \right)^{\frac{1}{n}} \]
\[ \geq C^{-1} e^{-\frac{\delta \varphi}{n}} , \]  
for some uniform positive constant \( C \) independent on \( \lambda \in (0, \lambda_0] \), where we have used \( \varphi \) is uniformly bounded by Lemma \( \ref{2.2} \). Plugging \( \ref{2.31} \) into \( \ref{2.29} \) gives
\[(\partial_t - \Delta_{\omega(t)}) (\partial_t \varphi + 2 \varphi + \delta |S_h^{2\beta}| - \lambda \log |S_h^2|) \geq \partial_t \varphi + C^{-1} e^{-\frac{\delta \varphi}{n}} - n - k \]  
\[ \text{on } (X \setminus D) \times [0, \infty) \]. Now by maximum principle we can find a constant \( C \geq 1 \) such that for any \( \lambda \in (0, \lambda_0] \)
\[ \inf_{(X \setminus D) \times [0, \infty)} \partial_t \varphi + 2 \varphi + \delta |S_h^{2\beta}| - \lambda \log |S_h^2| \geq -C. \]
Letting \( \lambda \to 0 \) gives the desired uniform lower bound for \( \partial_t \varphi \) on \( (X \setminus D) \times [0, \infty) \).
Lemma \( \ref{2.3} \) is proved. \( \square \)

2.4. Convergence of Kähler potentials. We come to the main result of this section.

**Proposition 2.4.** There exists a constant \( C \geq 1 \) such that for any \( t \in [0, \infty) \),
\[ \sup_{X \setminus D} |\varphi (t) + \delta |S_h^{2\beta}| - f^* \psi| \leq C e^{-\frac{\delta t}{2}} . \]  
**Proof.** Set \( V := \varphi + \delta |S_h^{2\beta}| - f^* \psi - e^{-t} \rho \), where \( \psi \) is the unique solution to \( \ref{2.2} \) and \( \rho \) is the function in \( \ref{2.1} \). By \( \ref{2.2} \) and \( \ref{2.3} \) we have (see \( \ref{2.9} \))
\[ \partial_t V = \log \frac{e^{(n-k)t} (e^{-t} \omega_0 + f^* \chi - e^{-t} f^* \chi + \sqrt{-1} \partial \overline{\partial} V)^n}{C_k n \omega_0^{-k} \wedge f^* \chi^k} - V \]  
\[ \text{on } (X \setminus D) \times [0, \infty) \]. For any \( \lambda \in (0, \lambda_0] \) we set \( V_\lambda := V + e^{-t} \lambda \log |S_h^2| \). By \( \ref{2.34} \) one easily has
\[ \partial_t V_\lambda = \log \frac{e^{(n-k)t} (e^{-t} \omega_0 + f^* \chi - e^{-t} f^* (\chi + \lambda R_{h^t}) + \sqrt{-1} \partial \overline{\partial} V_\lambda)^n}{C_k n \omega_0^{-k} \wedge f^* \chi^k} - V_\lambda \]  
\[ \text{on } (X \setminus D) \times [0, \infty) \]. For any \( t \in [0, \infty) \), the maximal value \( V_{\lambda_{\text{max}}}(t) \) of \( V_\lambda(t) \) must be achieved at some point in \( X \setminus D \). Hence, applying the maximum principle in \( \ref{2.35} \) and then using \( \ref{2.7} \) give
\[ \partial_t V_{\lambda,\max} + V_{\lambda,\max} \leq \log \frac{e^{(n-k)t}(e^{-t}\overline{w}_0 + f^*\overline{x} - e^{-t}f^*(\chi + \lambda R_k'))^n}{C_k^n \overline{w}_0^{n-k} \wedge f^*\overline{x}^k} \]
\[ \leq \log \frac{e^{(n-k)t}(e^{-t}\overline{w}_0 + f^*\overline{x})^n}{C_k^n \overline{w}_0^{n-k} \wedge f^*\overline{x}^k} \]
\[ \leq \log(1 + Ce^{-t}) \leq Ce^{-t}, \] (2.36)

i.e.,
\[ \partial_t (e^t V_{\lambda,\max} - Ct) \leq 0, \] (2.37)
and so
\[ V_{\lambda,\max} \leq C'e^{-t}(t + 1) \leq Ce^{-\frac{3}{4}t} \]

for some uniform constant \( C \geq 1 \) independent on \( \lambda \in (0, \lambda_0] \). Let \( \lambda \to 0 \), we find a constant \( C \geq 1 \) such that
\[ \inf_{(X\setminus D) \times (0, \infty)} V \leq Ce^{-\frac{3}{4}t}. \] (2.38)

For the lower bound we define \( V^\lambda := V - e^{-t\lambda}\log |S_\lambda'|^2 \) for any \( \lambda \in (0, \lambda_0] \). We know the minimal value \( V^\lambda_{\min}(t) \) of \( V^\lambda(t) \) must be achieved at some point in \( X \setminus D \). Similarly, we have
\[ \partial_t V_{\min}^\lambda + V_{\min}^\lambda \geq \log \frac{e^{(n-k)t}(e^{-t}\overline{w}_0 + f^*\overline{x} - e^{-t}f^*(\chi - \lambda R_k'))^n}{C_k^n \overline{w}_0^{n-k} \wedge f^*\overline{x}^k} \] (2.39)

We now fix a large \( T_1 \) such that \( e^{-t}\overline{w}_0 + f^*\overline{x} - 2e^{-t}f^*\chi > 0 \) for any \( t \geq T_1 \) (in particular, \( T_1 \) does not depend on \( \lambda \in (0, \lambda_0] \)). Then, for any \( t \geq T_1 \),
\[ \log \frac{e^{(n-k)t}(e^{-t}\overline{w}_0 + f^*\overline{x} - e^{-t}f^*(\chi - \lambda R_k'))^n}{C_k^n \overline{w}_0^{n-k} \wedge f^*\overline{x}^k} \geq \log \frac{e^{(n-k)t}(e^{-t}\overline{w}_0 + f^*\overline{x} - 2e^{-t}f^*\chi)^n}{C_k^n \overline{w}_0^{n-k} \wedge f^*\overline{x}^k} \]
\[ \geq \log(1 - Ce^{-t}) \geq -Ce^{-t}, \] (2.40)

where \( C \geq 1 \) is a uniform positive constant independent on \( t \in [0, \infty) \) and \( \lambda \in (0, \lambda_0] \), and we may increase \( T_1 \) to make sure that \( 1 - Ce^{-t} > 0 \) for any \( t \geq T_1 \). Therefore, we arrive at
\[ \partial_t (e^t V_{\min}^\lambda + Ct) \geq 0, \] (2.41)
on \( t \in [T_1, \infty) \), from which we conclude
\[ \inf_{(X\setminus D) \times [T_1, \infty)} V^\lambda \geq -Ce^{-\frac{3}{4}t} \] (2.42)

for some constant \( C \geq 1 \) independent on \( \lambda \in (0, \lambda_0] \). By letting \( \lambda \to 0 \), we find
\[ \inf_{(X\setminus D) \times [T_1, \infty)} V \geq -Ce^{-\frac{3}{4}t}. \]
Moreover, since $V$ is uniformly bounded on $(X \setminus D) \times [0, T_1]$, after possibly increasing $C$ in (2.42) we have

$$\inf_{(X \setminus D) \times [0, \infty)} V \geq -Ce^{-\frac{1}{4}t}.$$  \hspace{1cm} (2.43)

By combining (4.31), (2.43) and the fact that $\rho$ is a bounded function on $X$ we conclude (2.33) and Proposition 2.4 is proved.  \hfill \Box

2.5. **Proof of Theorem 1.2 (2.1).** Consequently, we have

*Proof of Theorem 1.2 (2.1).* By Lemma 2.2 and Proposition 2.4 we conclude that, as $t \to \infty$, $\varphi + \delta|S_h|^{2/3} \to f^*\psi$ in $L^1(X, \Omega)$-topology and so $\omega(t) \to f^*\chi$ as currents on $X$.  \hfill \Box

3. **uniform equivalence to a family of collapsing conical Kähler metrics**

Recall (2.4) that for any $T < \infty$, there exists a constant $A = A(T) \geq 1$ such that

$$A^{-1}\omega^*_0 \leq \omega(t) \leq A\omega^*_0$$ \hspace{1cm} (3.1)

holds on $(X \setminus D) \times [0, T]$. In our setting, we know the Kähler class $[\omega(t)] = e^{-t}[\omega_0] + (1 - e^{-t})[f^*\chi]$ along the twisted conical Kähler-Ricci flow (1.8) will converge to $[f^*\chi]$, which lies on the boundary of Kähler cone of $X$ and is not strictly positive. Therefore, $A = A(T)$ in (3.1) should go to $\infty$ as $T \to \infty$, i.e. $\omega(t)$ should not be uniformly equivalent to a fixed conical Kähler metric on $(X \setminus D) \times [0, \infty)$.

In this section, we will show that $\omega(t)$ is uniformly equivalent to a family of conical Kähler metrics on $X$ whose Kähler classes will degenerate to $[f^*\chi]$ (see Proposition 3.4). Let’s begin by the following key lemma, which will play an important role in our later discussions. Recall that, defined by (1.4), $\chi^*$ is a model conical metric on $Y$ with cone angle $2\pi\beta$ along $D'$.

**Lemma 3.1 (key lemma).** There exists a constant $C \geq 1$ such that there holds on $(X \setminus D) \times [0, \infty)$ that

$$tr_{\omega(t)}f^*\chi^* \leq C.$$ \hspace{1cm} (3.2)

*Proof.* The proof makes use of Lemmas 2.2, 2.3 a parabolic Schwarz lemma argument 29, 41 and, especially, an upper bound for bisectional curvature of $\chi^*$ in [13, Proposition A.1]. Firstly, we note that by (3.1) $tr_{\omega(t)}f^*\chi^*$ is uniformly bounded on $(X \setminus D) \times [0, 1]$. Secondly, for any given point $x \in X \setminus D$, we choose normal coordinates for $\omega(t)$ around $x$ and $\chi^*$ around $f(x)$, in which we write $tr_{\omega(t)}f^*\chi^* = g^i\tilde{f}_i\tilde{f}^j\chi^*_{\alpha\beta}$. A standard computation gives

$$(\partial_t - \Delta_{\omega(t)})tr_{\omega(t)}f^*\chi^*$$

$$= tr_{\omega(t)}f^*\chi^* - \left(\frac{1}{T_{\text{max}}}\omega_0, f^*\chi^*\right)_{\omega(t)} + \tilde{g}^i\tilde{g}^j\tilde{Rm}(\chi^*)_{\alpha\beta\gamma\delta}\tilde{f}_i\tilde{f}_j\chi^*_{\alpha\beta} - \tilde{g}^i\tilde{g}^j(\partial_\gamma f^\alpha_i)(\partial_\delta f^\beta_j)\chi^*_{\alpha\beta}$$

$$\leq tr_{\omega(t)}f^*\chi^* + B(tr_{\omega(t)}f^*\chi^*)^2 - \tilde{g}^i\tilde{g}^j(\partial_\gamma f^\alpha_i)(\partial_\delta f^\beta_j)\chi^*_{\alpha\beta},$$ \hspace{1cm} (3.3)

and so

$$(\partial_t - \Delta_{\omega(t)})\log tr_{\omega(t)}f^*\chi^* \leq Btr_{\omega(t)}f^*\chi^* + 1$$ \hspace{1cm} (3.4)
on \((X\setminus D) \times [0, \infty)\), where \(B\) is an upper bound for bisectional curvature of \(\chi^*\) on \(Y \setminus D'\) given by [13, Proposition A.1] (also see [11, Section 7] and [13, Lemma 2.3] for the case \(\beta \in (0, 1/2)\)) and we have used

\[
|\nabla \text{tr}_{\omega(t)} f^* \chi^*|^2 - (\text{tr}_{\omega(t)} f^* \chi^*) g^{\alpha \bar{\beta}} \partial_\alpha f^* \partial_\beta f^* \chi^*_{\alpha \bar{\beta}} \leq 0, \tag{3.5}
\]

On the other hand, by Lemma 2.3 and (2.9) we have a uniform constant \(\bar{C} \geq 1\) such that

\[
(\partial_t - \Delta_{\omega(t)}) \varphi = \partial_t \varphi - \text{tr}_{\omega(t)}(\omega(t) - e^{-t}\omega_0) + \text{tr}_{\omega(t)} f^*(\chi^* - e^{-t}\chi) \\
\geq \text{tr}_{\omega(t)} f^*(\chi^* - e^{-t}\chi) - \bar{C} \\
\geq \frac{5}{6} \text{tr}_{\omega(t)} f^* \chi^* - \bar{C}, \tag{3.6}
\]
on \((X \setminus D) \times [T_0, \infty)\), here \(T_0\) is given in (2.9). Therefore, we arrive at

\[
(\partial_t - \Delta_{\omega(t)}) (\log \text{tr}_{\omega(t)} f^* \chi^* - \frac{6}{5}(B + 1)\varphi) \leq -\text{tr}_{\omega(t)} f^* \chi^* + 1 + \frac{6}{5} \bar{C}(B + 1) \tag{3.7}
\]
and so, for any \(\lambda \in (0, \lambda_0)\),

\[
(\partial_t - \Delta_{\omega(t)}) (\log \text{tr}_{\omega(t)} f^* \chi^* - \frac{6}{5}(B + 1)\varphi + \lambda \log |S|^2_\alpha) \\
\leq -\text{tr}_{\omega(t)} f^*(\chi^* + \lambda R_{\omega(t)}) + 1 + \frac{6}{5} \bar{C}(B + 1) \\
\leq -\frac{1}{2} \text{tr}_{\omega(t)} f^* \chi^* + \frac{6}{5} \bar{C}(B + 1). \tag{3.8}
\]
on \((X \setminus D) \times [T_0, \infty)\). Now by applying the maximum principle in (3.8), Lemma 2.2 and (3.1), we can find a constant \(C \geq 1\) such that for any \(\lambda \in (0, \lambda_0)\) we have

\[
\log \text{tr}_{\omega(t)} f^* \chi^* + \lambda \log |S|^2_\alpha \leq C, \tag{3.9}
\]
which implies

\[
\text{tr}_{\omega(t)} f^* \chi^* \leq C \tag{3.10}
\]
on \((X \setminus D) \times [T_0, \infty)\). By (3.1), after possibly increasing \(C\) in (3.10), we have

\[
\sup_{(X \setminus D) \times [0, \infty)} \text{tr}_{\omega(t)} f^* \chi^* \leq C. \tag{3.11}
\]
Lemma 3.1 is proved. \(\square\)

We would like to make a remark on Lemma 3.1 and its proof.

**Remark 3.2.** The key Lemma 3.1 seems can not be proved by the usual smooth approximation arguments in e.g. [6, 17]. A key point is that, while \(\chi^*\) has an upper bound for bisectional curvature, it seems unclear how to approximate \(\chi^*\) (in a suitable sense) by a family of smooth Kähler metrics with bisectional curvature uniformly bounded from above. In our argument, we don’t need to pass to a smooth approximation of conical equation and the upper bound for bisectional curvature of \(\chi^*\) itself is enough to carry out our Lemma 3.1.

Next we make use of Lemma 3.1 and an argument in [29] to show

**Lemma 3.3.** There exists a constant \(C \geq 1\) such that there holds on \((X \setminus D) \times [0, \infty)\) that

\[
\text{tr}_{\omega(t)}(e^{-t}\omega_0) \leq C. \tag{3.12}
\]
Proof. As before, by (3.1) we know \( tr_{\omega(t)}(e^{-t}\omega_0) \) is uniformly bounded on \( (X \setminus D) \times [0, 1] \).

Firstly, we have two positive constants \( B_0 \) (an upper bound for bisectional curvature of \( \omega_0 \)) and \( C \) such that
\[
(\partial_t - \Delta_{\omega(t)}) \log tr_{\omega(t)}(e^{-t}\omega_0) \leq B_0 tr_{\omega(t)}\omega_0 + C
\]
on \( (X \setminus D) \times [0, \infty) \).

Next, following \([29]\), we define a function \( \varphi \) on \( (Y \setminus D') \times [0, \infty) \): for any \( (y, t) \in (Y \setminus D') \times [0, \infty) \),
\[
\varphi(y, t) := \frac{\int_{X_y} \varphi(t)(\omega_0|_{X_y})^{n-k}}{\int_{X_y} (\omega_0|_{X_y})^{n-k}}.
\]
Then we have
\[
(\omega_0|_{X_y} + \sqrt{-1}\partial\bar{\partial}(e^{t}(\varphi|_{X_y} - \varphi(y, t))))^{n-k} = (e^{t}\omega(t)|_{X_y})^{n-k},
\]
and so
\[
\frac{(\omega_0|_{X_y} + \sqrt{-1}\partial\bar{\partial}(e^{t}(\varphi|_{X_y} - \varphi)))^{n-k}}{(\omega_0|_{X_y})^{n-k}} = \frac{e^{(n-k)t}\omega(t)^n \wedge f^*(\chi^*)^k}{\omega_0^{n-k} \wedge f^*(\chi^*)^k} = \frac{\omega(t)^n \wedge f^*(\chi^*)^k}{\omega(t)^n} \cdot \frac{e^{(n-k)t}\omega(t)^n}{\omega_0^{n-k} \wedge f^*(\chi^*)^k}
\]
\[
\leq C^k (tr_{\omega(t)}f^*\chi^*)^k
\]
where we have used Lemmas \([22, 23, 24]\) and \([31]\) and \( C \) in the last line is a positive constant uniform for \( (x, t) \in (X \setminus D) \times [0, \infty) \) (we point out that Lemma \([31]\) plays a crucial role in this step). Then by Yau \([42]\) we get a constant \( C \geq 1 \) such that for all \( (y, t) \in (Y \setminus D') \times [0, \infty) \),
\[
\sup_{X_y} |e^{t}(\varphi - \varphi)| \leq C.
\]
Note that \( |S|_h^{2\beta} = f^*|S|_h^{2\beta} \) is constant along every fiber \( X_y \) and hence
\[
(\varphi + \delta|S|_h^{2\beta}) - (\varphi + \bar{\partial}\bar{\partial}(e^{t}(\varphi - \varphi))) = \varphi - \varphi.
\]
Now by using arguments in \([29]\) (also see \([9]\)), Lemma \([23]\) and Lemma \([31]\) (in fact a weaker version that \( tr_{\omega(t)}f^*\chi \leq C \) is enough) we have a constant \( C \geq 1 \) such that
\[
(\partial_t - \Delta_{\omega(t)})\left(e^{t}(\varphi - \varphi)\right) = (\partial_t - \Delta_{\omega(t)}) \left(e^{t}((\varphi + \delta|S|_h^{2\beta}) - (\varphi + \bar{\partial}\bar{\partial}(e^{t}(\varphi - \varphi))))\right)
\]
\[
\geq tr_{\omega(t)}\omega_0 - Ce^t
\]
holds on \( (X \setminus D) \times [0, \infty) \). Combining (3.13) and (3.18) gives
\[
(\partial_t - \Delta_{\omega(t)})(\log tr_{\omega(t)}(e^{-t}\omega_0) - (B_0 + 1)e^{t}(\varphi - \varphi)) \leq -tr_{\omega(t)}\omega_0 + Ce^t
\]
Lemma 3.3 is proved. \[\square\]

By applying the maximum principle in (3.20) and using (3.17) we find a constant \(C \geq 1\) such that for any \(\lambda \in (0, \lambda_0]\),

\[
\sup_{(X \setminus D) \times [0, \infty)} (\log tr_{\omega(t)}(e^{-t}\omega_0) - (B_0 + 1)e^t(\varphi - \overline{\varphi}) + \lambda \log |S|^2_h) \leq C,
\]

from which, by letting \(\lambda \to 0\) and using (3.17) again, we conclude a constant \(C \geq 1\) such that

\[
\sup_{(X \setminus D) \times [0, \infty)} tr_{\omega(t)}(e^{-t}\omega_0) \leq C.
\]  

Lemma 3.3 is proved. \[\square\]

The main result of this section is the following, which says \(\omega(t)\) is uniformly equivalent to a family of collapsing conical Kähler metrics.

**Proposition 3.4.** There exists a constant \(C \geq 1\) such that

\[
C^{-1}(e^{-t}\omega_0 + f^*\chi^*) \leq \omega(t) \leq C(e^{-t}\omega_0 + f^*\chi^*)
\]  

holds on \((X \setminus D) \times [0, \infty)\).

**Proof.** The left hand side follows from Lemmas 3.1 and 3.3 directly. For the right hand side, we look at

\[
tr(e^{-t}\omega_0 + f^*\chi^*) \omega(t) \leq \frac{1}{(n-1)!}(tr_{\omega(t)}(e^{-t}\omega_0 + f^*\chi^*))^{n-1} \cdot \frac{\omega(t)^n}{(e^{-t}\omega_0 + f^*\chi^*)^n}
\]

\[
\leq C \frac{e^{-(n-k)t}e^{\partial_t \varphi + \overline{\varphi} + \delta}\|S_h\|_h^{2\beta} |S|^2_h^{-2(1-\beta)}}{e^{-(n-k)t}e^{-\Delta t} \omega_0^{n-k} \wedge f^*(\chi^*)^k}
\]

\[
\leq C,
\]

on \((X \setminus D) \times [0, \infty)\), where we have used left hand side of (3.22) and upper bound of \(\partial_t \varphi, \varphi\) given by Lemmas 2.2 and 2.3. Then we conclude that

\[
\omega(t) \leq C(e^{-t}\omega_0 + f^*\chi^*)
\]

on \((X \setminus D) \times [0, \infty)\).

Proposition 3.4 is proved. \[\square\]

**Remark 3.5.** Before moving to the next step, we would like to provide a more direct argument for a special case of Lemma 3.3, i.e., the case that \(T_{\text{max}} = \infty, X = E \times Y\) with \(E\) a smooth \((n-k)\)-dimensional torus and \(f : X = E \times Y \to Y\) is the projection. Denote \(\hat{f} : X \to E\) be the projection to the factor \(E\) and \(\omega_E\) a fixed flat Kähler metric on \(E\). Then we shall show that there exists a constant \(C \geq 1\) such that for \((x, t) \in (X \setminus D) \times [0, \infty)\),

\[
tr_{\omega(t)}(e^{-t}\hat{f}^*\omega_E) \leq C.
\]  

In fact, using \(\omega_E\) is flat, we easily have

\[
(\partial_t - \Delta_{\omega(t)})tr_{\omega(t)}(e^{-t}\hat{f}^*\omega_E) \leq 0
\]

and so

\[
(\partial_t - \Delta_{\omega(t)})(tr_{\omega(t)}(e^{-t}\hat{f}^*\omega_E) + \lambda \log |S|^2_h) \leq \lambda tr_{\omega(t)}R_h \leq \lambda\tilde{C} tr_{\omega(t)}f^*\chi \leq \lambda\tilde{C}_1,
\]
namely,
\[(\partial_t - \Delta_{\omega(t)}) (tr_{\omega(t)} (e^{-t} \tilde{f}^* \omega_E) + \lambda \log |S|_h^2 - \lambda \tilde{C}_1 t) \leq 0 \] (3.25)
on \((X \setminus D) \times [0, \infty)\). For any \(T < \infty\), we can apply the maximal principle in (3.25) to see\[
sup_{(X \setminus D) \times [0, T]} (tr_{\omega(t)} (e^{-t} \tilde{f}^* \omega_E) + \lambda \log |S|_h^2 - \lambda \tilde{C}_1 t) \leq C,
\]
where \(C\) does not depend on \(T \in (0, \infty)\) and \(\lambda \in (0, \lambda_0]\). Letting \(T \to \infty\) and \(\lambda \to 0\) gives the desired result (3.24). Having (3.24), we can use arguments in Proposition 3.4 to see that \(\omega(t)\) is uniformly equivalent to \(e^{-t} \tilde{f}^* \omega_E + f^* \chi\) on \((X \setminus D) \times [0, \infty)\).

3.1. Proof of Theorem 1.2 (1). We now end this section by proving diameter upper bound of \((X, \omega(t))\), i.e. item (1) of Theorem 1.2.

Proof of Theorem 1.2 (1). By the right hand side of (3.22) in Proposition 3.4 we can find a constant \(C \geq 1\) such that \(\omega(t) \leq C \omega_0\) holds on \((X \setminus D) \times [0, \infty)\), where \(\omega_0\) is defined in (1.3), the reference conical Kähler metric on \(X\) with cone angle \(2\pi \beta\) along \(D\). Let \((X, d_0)\) be the metric completion of \((X, \omega_0)\), which is of finite diameter. Therefore, \(\text{diam}(X, d_t) \leq C \cdot \text{diam}(X, d_0)\) is uniformly bounded from above.

Remark 3.6. Easily, the left hand side of (3.22) in Proposition 3.4 also implies a uniform positive lower bound for \(\text{diam}(X, d_t)\).

4. A bound for the twisted scalar curvature

For later discussions, we need to bound the twisted scalar curvature along the twisted conical Kähler-Ricci flow (1.8).

We define the twisted Ricci curvature by
\[\tilde{Ric}(\omega(t)) := Ric(\omega(t)) - \frac{1}{T_{\text{max}}} \omega_0\] (4.1)
and the twisted scalar curvature by
\[\tilde{R}(\omega(t)) := tr_{\omega(t)} \tilde{Ric}(\omega(t)) = R(\omega(t)) - \frac{1}{T_{\text{max}}} tr_{\omega(t)} \omega_0.\] (4.2)

Note that if \(T_{\text{max}} = \infty\), the twisted term vanishes and the twisted curvatures defined above are the usual curvatures.

We will need the following result.

Lemma 4.1. [6] There exists a constant \(C \geq 1\) such that for any \(t \in (0, \infty)\),
\[
\sup_{X \setminus D} |\tilde{R}(t)| \leq \frac{C}{\min\{t, 1\}}.
\] (4.3)

In the case \(T_{\text{max}} = \infty\), the twisted term \(\frac{1}{T_{\text{max}}} \omega_0\) vanishes and the twisted scalar is just the scalar curvature, which is uniformly bounded by [6, Theorem 1.1] using a smooth approximation argument (this is a generalization of Song and Tian’s result on the Kähler-Ricci flow [31]). In fact, while our Lemma 4.1 only provides uniform bound for \(t \geq 1\), [6, Theorem 1.1] says the bound for scalar curvature can be uniform for all \(t \in [0, \infty)\) (i.e. \(\sup_{(X \setminus D) \times [0, \infty)} |\tilde{R}| \leq C < \infty\)), where some properties of a smooth approximation for \(\omega_0^*\)
is involved (see discussions next to (4.9), (4.20) and (4.31) for more details). Though bounding scalar curvature along the conical Kähler-Ricci flow has been studied in some different settings, see e.g. [6, 21], we still would like to provide an argument for the above Lemma 4.1. Let’s list in the following Remark 4.2 the main reasons for doing this.

Remark 4.2. We provide an argument for Lemma 4.1 mainly due to twofold reasons: (1) we will use slightly different quantum to apply the maximum principle (see footnotes next to (4.22) and (4.25) for more details); (2) we can avoid using some properties of a smooth approximation $\omega^\epsilon_0$ (given in (4.4)) for $\omega_0$, which seems needed in [6, Theorem 1.1, Sections 4,7,8] and [21, Theorem A, Sections 6,7] (see discussions next to (4.9), (4.26) and (4.41) for more details).

Remark 4.3. It seems in general we can not improve the bound $C_t$ to a uniform bound $C$ near $t = 0$, as the initial model conical metric $\omega_0$ may have unbounded scalar curvature. For example, as pointed out in [22, Lemma 3.14], if $\beta \in (\frac{1}{2}, 1)$, the computation in [13, Proposition A.1] provides not only the existence of an upper bound for bisectional curvature, but also the nonexistence of a lower bound for bisectional curvature, and hence the nonexistence of a lower bound for scalar curvature.

Proof of Lemma 4.1. We need to use a smooth approximation for the conical equations (1.8) and (2.3), introduced in [17, 40] and also used in e.g. [24, 6]. Following [2] we define

$$\eta_\epsilon := \beta \int_{|S^3_h|^2} \left( \frac{(r + \epsilon^2)^\beta - \epsilon^{2\beta}}{r} \right) dr,$$

and

$$\omega^\epsilon_0 := \omega_0 + \sqrt{-1} \partial \bar{\partial} \eta_\epsilon.$$  

Then one can approximate (2.3) by

$$\left\{ \begin{array}{l}
\partial_t \varphi_\epsilon(t) = \epsilon^{(n-k)t} (\omega_\epsilon + \delta \sqrt{-1} \partial \bar{\partial} \eta_\epsilon + \sqrt{-1} \partial \bar{\partial} \omega_\epsilon(t)) - \varphi_\epsilon(t) - \delta \eta_\epsilon \\
\varphi_\epsilon(0) = 0,
\end{array} \right.$$  

and, correspondingly, if we let $\omega_\epsilon(t) := \omega_\epsilon + \delta \sqrt{-1} \partial \bar{\partial} \eta_\epsilon + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon(t)$, then we have

$$\left\{ \begin{array}{l}
\partial_t \omega_\epsilon(t) = -Ric(\omega_\epsilon(t)) - \omega_\epsilon(t) + \frac{1}{T_{max}} \omega_0 + (1 - \beta)(\sqrt{-1} \partial \bar{\partial} \log(|S^3_h|^2 + \epsilon^2) + R_h) \\
\omega(0) = \omega^\epsilon_0,
\end{array} \right.$$  

Set $A_\epsilon := (1 - \beta) (\sqrt{-1} \partial \bar{\partial} \log(|S^3_h|^2 + \epsilon^2) + R_h)$, then it suffices to get a bound for $\tilde{R}_\epsilon := tr_{\omega_\epsilon(t)} (Ric(\omega_\epsilon(t)) - \frac{1}{T_{max}} \omega_0 - A_\epsilon)$. Let’s first look at the easier part, i.e. lower bound.

Claim 1: For any $\epsilon > 0$ and $t > 0$,

$$\inf_{\chi} \tilde{R}_\epsilon \geq -\frac{3n}{\min\{t,1\}}.$$  

Proof of Claim 1. First we have the evolution of $\tilde{R}_\epsilon$ along (4.6) (see e.g. [6, Proposition 4.1]):

$$(\partial_t - \Delta_{\omega_\epsilon(t)}) \tilde{R}_\epsilon = |Ric(\omega_\epsilon(t)) - \frac{1}{T_{max}} \omega_0 - A_\epsilon|_{\omega_\epsilon(t)}^2 + \tilde{R}_\epsilon,$$

which, combining with $|Ric(\omega_\epsilon(t)) - \frac{1}{T_{max}} \omega_0 - A_\epsilon|_{\omega_\epsilon(t)}^2 \geq \frac{1}{n} \tilde{R}_\epsilon^2$, implies

$$(\partial_t - \Delta_{\omega_\epsilon(t)}) (e^\epsilon (\tilde{R}_\epsilon + n)) \geq \frac{1}{n} e^\epsilon (\tilde{R}_\epsilon + n)^2.$$
To avoid involving the uniform bound for $\tilde{R}_t(0)$, we use a trick similar to e.g. [32, Lemma 3.2] and further consider

$$(\partial_t - \Delta_{\omega(t)})(te^t(\tilde{R}_e + n)) \geq \frac{1}{n}te^t(\tilde{R}_e + n)^2 + e^t(\tilde{R}_e + n)$$

(4.10)

We now apply the maximum principle in (4.10). Assume $te^t(\tilde{R}_e + n)$ achieves the minimal value on $X \times [0, 1]$ at $(x', t') \in X \times [0, 1]$. If $t' = 0$, $(te^t(\tilde{R}_e + n))(x', t') = 0$ is uniformly bounded from below; otherwise $t' > 0$, then by (4.10) we know, at $(x', t')$,

$$t(\tilde{R}_e + n)^2 \leq -n(\tilde{R}_e + n),$$

(4.11)

which in particular implies $(\tilde{R}_e + n)(x', t') \leq 0$. We may assume

$$(\tilde{R}_e + n)(x', t') < 0$$

(otherwise we are done), and so by (4.11) we find

$$t(\tilde{R}_e + n) \geq -n,$$

(4.12)

and so $(te^t(\tilde{R}_e + n))(x', t') \geq -en$. In conclusion, we have

$$\inf_{X \times [0, 1]} (te^t(\tilde{R}_e + n)) \geq -en,$$

(4.13)

which implies

$$\inf_{X \times [0, 1]} (t(\tilde{R}_e + n)) \geq -en,$$

(4.14)

and so $\inf_X (\tilde{R}_e + n))(1) \geq -n$. Now, back to (4.13), we easily find that for any $t \geq 1$,

$$\inf_{X} (e^t(\tilde{R}_e + n))(t) \geq \inf_{X}(e^t(\tilde{R}_e + n))(1) \geq -en,$$

and

$$\inf_{X \times [1, \infty]} (\tilde{R}_e + n) \geq -n.$$

(4.15)

Combining (4.14) and (4.15), Claim 1 is proved. \qed

Next we try to get an upper bound for $\tilde{R}_e$. Recall that [6, Propositions 3.1,3.2,6.1] provides a constant $C \geq 1$ such that for any small $\epsilon > 0$, there holds

$$\sup_{X \times [0, \infty]} (|\varphi_\epsilon| + |\partial_t \varphi_\epsilon| + tr_{\omega(t)} f^* \chi) \leq C.$$

(4.16)

It suffices to remark that the twisted term $\frac{1}{\max \omega_0}$ is smooth and semipositive and hence does not cause any trouble.

We also recall some useful inequalities. Since $\chi$ is a Kähler metric on $Y$, which in particular has bounded bisectional curvature, by arguments similar to Lemma 3.1 (or see [6, Section 6]) we have a constant $C \geq 1$ such that for all positive $\epsilon$,

$$(\partial_t - \Delta_{\omega(t)})tr_{\omega(t)} f^* \chi \leq C - C^{-1}|\nabla tr_{\omega(t)} f^* \chi|_{\omega(t)}^2,$$

(4.17)

where we have used $A_\epsilon \geq -C f^* \chi$ and $\frac{1}{\max \omega_0} \geq 0$.

Set $u_\epsilon := \partial_t \varphi_\epsilon + \varphi_\epsilon + \delta \eta_\epsilon$. Then by a direct computation we have, on $X \times [0, \infty)$,

$$(\partial_t - \Delta) u_\epsilon = tr_{\omega(t)} f^* \chi - k \leq C.$$

(4.18)

To simplify the notation we set $| \cdot | := | \cdot |_{\omega(t)}$. 
We first bound $|\nabla u_\epsilon|$. A direct computation gives

$$(\partial_t - \Delta)|\nabla u_\epsilon|^2 = |\nabla u_\epsilon|^2 + 2 \Re \langle \nabla \text{tr}_\omega(t)f^*\chi, \nabla u_\epsilon \rangle_{\omega(t)} - |\nabla \nabla u_\epsilon|^2 - |\nabla \nabla u_\epsilon|^2 - (\frac{1}{T_{\text{max}}} \omega_0 + A_\epsilon)(\nabla u_\epsilon, \nabla u_\epsilon),$$

which, combining with $A_\epsilon \geq -Cf^*\chi$ and $\frac{1}{T_{\text{max}}} \omega_0 \geq 0$, in particular implies that

$$(\partial_t - \Delta)|\nabla u_\epsilon|^2 \leq -|\nabla \nabla u_\epsilon|^2 - |\nabla \nabla u_\epsilon|^2 + C|\nabla u_\epsilon|^2 + |\nabla \text{tr}_\omega(t)f^*\chi|^2. \quad (4.20)$$

Now we fix a sufficiently large constant $B > 4$ such that

$$2|u_\epsilon| \leq B,$$ \hspace{1cm} (4.21)

and so,

$$2 < \frac{B}{2} \leq B - u_\epsilon \leq 2B. \quad (4.22)$$

Fix a constant $s \in (0, 1)$ and set $E_\epsilon := \frac{|\nabla u_\epsilon|^2}{(B - u_\epsilon)^s}$. Compute

$$(\partial_t - \Delta_{\omega(t)})E_\epsilon = \frac{(\partial_t - \Delta_{\omega(t)})|\nabla u_\epsilon|^2}{(B - u_\epsilon)^s} + \frac{2s|\nabla u_\epsilon|^2(\partial_t - \Delta_{\omega(t)})u_\epsilon}{(B - u_\epsilon)^{s+1}} - \frac{2s \Re \langle \nabla |\nabla u_\epsilon|^2, \nabla u_\epsilon \rangle_{\omega(t)}}{(B - u_\epsilon)^{s+1}} - \frac{s(s+1)|\nabla u_\epsilon|^4}{(B - u_\epsilon)^{s+2}}$$

$$\leq -\frac{|\nabla \nabla u_\epsilon|^2 - |\nabla \nabla u_\epsilon|^2 + 2|\nabla u_\epsilon|^2 + |\nabla \text{tr}_\omega(t)f^*\chi|^2}{(B - u_\epsilon)^s} + \frac{sC|\nabla u_\epsilon|^2}{(B - u_\epsilon)^{s+1}} + \frac{2s|\nabla u_\epsilon|^2}{(B - u_\epsilon)^{s+1}} - \frac{s(s+1)|\nabla u_\epsilon|^4}{(B - u_\epsilon)^{s+2}}. \quad (4.23)$$

Note that

$$\frac{2s|\nabla |\nabla u_\epsilon|^2, \nabla u_\epsilon \rangle_{\omega(t)}}{(B - u_\epsilon)^{s+1}} \leq \frac{2s|\nabla u_\epsilon|^2(|\nabla u_\epsilon| + |\nabla u_\epsilon - \epsilon|)}{(B - u_\epsilon)^{s+1}}$$

$$\leq s \left( \frac{2(|\nabla u_\epsilon|^2 + |\nabla u_\epsilon|^2)}{2s(B - u_\epsilon)^s} + \frac{2s|\nabla u_\epsilon|^4}{(B - u_\epsilon)^{s+2}} \right)$$

$$= \frac{|\nabla \nabla u_\epsilon|^2 + |\nabla \nabla u_\epsilon|^2}{(B - u_\epsilon)^s} + \frac{2s|\nabla u_\epsilon|^4}{(B - u_\epsilon)^{s+2}}. \quad (4.24)$$

Putting (4.24) into (4.23) gives

$$\frac{2s|\nabla \text{tr}_\omega(t)f^*\chi|}{(B - u_\epsilon)^{s+1}} \leq \frac{2s|\nabla u_\epsilon|^2(|\nabla u_\epsilon| + |\nabla u_\epsilon - \epsilon|)}{(B - u_\epsilon)^{s+1}}$$

$$\leq \frac{2(|\nabla u_\epsilon|^2 + |\nabla u_\epsilon|^2)}{2s(B - u_\epsilon)^s} + \frac{2s|\nabla u_\epsilon|^4}{(B - u_\epsilon)^{s+2}}$$

$$= \frac{|\nabla \text{tr}_\omega(t)f^*\chi|^2}{(B - u_\epsilon)^s} + \frac{2s|\nabla u_\epsilon|^4}{(B - u_\epsilon)^{s+2}}. \quad (4.25)$$

Now consider $F_\epsilon := E_\epsilon + C_1 t \text{tr}_\omega(t)f^*\chi$. Combining (4.17) and (4.25) gives

$$(\partial_t - \Delta_{\omega(t)})F_\epsilon \leq C + C|\nabla u_\epsilon|^2 - (2B)^{-s+2} - s|\nabla u_\epsilon|^4. \quad (4.26)$$

To get an upper bound for $F_\epsilon$, one may like to apply the maximum principle in (4.26) (see last step in [6, Section 7] or [21, Section 6]). Note that, arguing in such way, since the

1We point out that in previous works on bounding scalar curvature along Kähler-Ricci flow (see [23, 31, 46, 4]), people usually choose $s = 1$ to define this quantity $E$. Here we choose $s < 1$ to slightly simplify some arguments.

2Note that choosing the positive number $s$ to be in $(0, 1)$ helps to make sure the factor $s - s^2$ appeared in (4.25) is positive and avoid an additional trick used in [31, 4].
maximal value may be achieved at \( t = 0 \), one may need a uniform bound for \( |\nabla u_\epsilon(0)| = |\nabla \log \frac{\omega_{\epsilon}}{(g_{\epsilon}^0 + \epsilon \omega_{\epsilon})^{1/2}}| \). To avoid involving such bound, here we use a trick similar to e.g. [32, Lemma 3.2] and further consider

\[
(\partial_t - \Delta_{\omega_0(t)})(tF_\epsilon) \leq F_\epsilon + t(C + C|\nabla u_\epsilon|^2 - (2B)^{-(s+2)}(s-s^2)|\nabla u_\epsilon|^4)
\leq C(t+1) + C(t+1)|\nabla u_\epsilon|^2 - t(2B)^{-(s+2)}(s-s^2)|\nabla u_\epsilon|^4.
\]  (4.27)

Assume the maximal value of \( tF_\epsilon \) on \( X \times [0,1] \) is achieved at \((x', t') \in X \times [0,1] \). If \( t' = 0 \), \((tF_\epsilon)(x', t') = 0 \) is bounded from above; otherwise \( t' > 0 \), by (4.27) we find a constant \( C \geq 1 \) independent on \( \epsilon \) such that, at \((x', t') \),

\[
t|\nabla u_\epsilon|^4 \leq C(t+1)|\nabla u_\epsilon|^2 + C(t+1).
\]

Note that \( t' \in (0,1] \). We have, at \((x', t') \),

\[
(t|\nabla u_\epsilon|^2)^2 \leq Ct(t+1) + Ct(t+1)|\nabla u_\epsilon|^2
\leq C + C(t|\nabla u_\epsilon|^2)
\]  (4.28)

and so

\[
(t|\nabla u_\epsilon|^2)(x', t') \leq C.
\]

Plugging into \( tF_\epsilon \) we find

\[
(tF_\epsilon)(x', t') = \left( \frac{t|\nabla u_\epsilon|^2}{(B - u_\epsilon)^s} + tC_1 \text{tr}_{\omega_0(t)} f^* \chi \right)(x', t') \leq C.
\]

Therefore, we have find a constant \( C \geq 1 \) independent on \( \epsilon \) such that

\[
\sup_{X \times [0,1]} (tF_\epsilon) \leq C,
\]  (4.29)

which implies

\[
\sup_{X \times [0,1]} (t|\nabla u_\epsilon|^2) \leq C.
\]  (4.30)

In particular, we have a constant \( C \geq 1 \) independent on \( \epsilon \) such that

\[
\sup_X F_\epsilon(1) \leq C.
\]  (4.31)

Having (4.31), we can now apply the maximum principle in (4.26) to conclude that

\[
\sup_{X \times [1,\infty)} F_\epsilon \leq C
\]

and so

\[
\sup_{X \times [1,\infty)} |\nabla u_\epsilon|^2 \leq C
\]  (4.32)

for some constant \( C \geq 1 \) independent on \( \epsilon \).

Combining (4.17), (4.20) and (4.32), we have constants \( C_1, C \geq 1 \) independent on \( \epsilon \) such that

\[
(\partial_t - \Delta_{\omega_0(t)})(t(|\nabla u_\epsilon|^2 + C_1 \text{tr}_{\omega_0(t)} f^* \chi)) \leq -t|\nabla \nabla u_\epsilon|^2 - t|\nabla \nabla u_\epsilon|^2 + C.
\]  (4.33)

on \((X \setminus D) \times [0,1]\), and

\[
(\partial_t - \Delta_{\omega_0(t)})(|\nabla u_\epsilon|^2 + C_1 \text{tr}_{\omega_0(t)} f^* \chi) \leq -|\nabla \nabla u_\epsilon|^2 - |\nabla \nabla u_\epsilon|^2 + C.
\]  (4.34)

on \((X \setminus D) \times [1,\infty)\).
Next we try to bound $-\Delta_{\omega_0(t)}$ from above. Recall that

\[(\partial_t - \Delta_{\omega_0(t)})\Delta_{\omega_0(t)}u_e = \Delta_{\omega_0(t)}u_e + \langle Ric(\omega_0(t)) - \frac{1}{T_{\max}}\omega_0 - A_\varepsilon, \sqrt{-1}\partial\bar{\partial}u_e\rangle_{\omega_0} + \Delta_{\omega_0(t)}tr_{\omega_0(t)}f^*\chi.\]

(4.35)

Using

\[\sqrt{-1}\partial\bar{\partial}u_e = -Ric(\omega_0) - f^*\chi + \frac{1}{T_{\max}}\omega_0 + A_\varepsilon,\]

we see that the second term in (4.35)

\[\langle Ric(\omega_0(t)) - \frac{1}{T_{\max}}\omega_0 - A_\varepsilon, \sqrt{-1}\partial\bar{\partial}u_e\rangle_{\omega_0(t)} = \langle -\sqrt{-1}\partial\bar{\partial}u_e - f^*\chi, \sqrt{-1}\partial\bar{\partial}u_e\rangle_{\omega_0(t)}\]

\[= -|\nabla\nabla u_e|^2 - \langle f^*\chi, \sqrt{-1}\partial\bar{\partial}u_e\rangle_{\omega_0(t)}\]

\[\geq -\frac{3}{2}|\nabla\nabla u_e|^2 - \frac{1}{2}|f^*\chi|^2\]

\[\geq -\frac{3}{2}|\nabla\nabla u_e|^2 - \frac{n}{2}(tr_{\omega_0(t)}f^*\chi)^2\]

\[\geq -\frac{3}{2}|\nabla\nabla u_e|^2 - C,\]

(4.36)

and the third term in (4.35)

\[\Delta_{\omega_0(t)}tr_{\omega_0(t)}f^*\chi = -(\partial_t - \Delta_{\omega_0(t)})tr_{\omega_0(t)}f^*\chi + \partial_tr_{\omega_0(t)}f^*\chi\]

\[\geq -C + C_{\lambda_{\omega_0(t)}}^{-1}|\nabla tr_{\omega_0(t)}f^*\chi|^2 + \langle Ric(\omega_0(t)) - \frac{1}{T_{\max}}\omega_0 - A_\varepsilon, f^*\chi\rangle_{\omega_0(t)} + tr_{\omega_0(t)}f^*\chi\]

\[\geq -C + \langle Ric(\omega_0(t)) - \frac{1}{T_{\max}}\omega_0 - A_\varepsilon, f^*\chi\rangle_{\omega_0(t)}\]

\[= -C + \langle -\sqrt{-1}\partial\bar{\partial}u_e - f^*\chi, f^*\chi\rangle_{\omega_0(t)}\]

\[\geq -C - \frac{1}{2}|\nabla\nabla u_e|^2 - \frac{3}{2}|f^*\chi|^2\]

\[\geq -C - \frac{1}{2}|\nabla\nabla u_e|^2.\]

(4.37)

Then we arrive at

\[(\partial_t - \Delta_{\omega_0(t)})\Delta_{\omega_0(t)}u_e \geq \Delta_{\omega_0(t)}u_e - 2|\nabla\nabla u_e|^2 - C.\]

(4.38)

(4.39)

Using (4.33) and (4.34), we see

\[(\partial_t - \Delta_{\omega_0(t)}) \left( t(\Delta_{\omega_0(t)}u_e - 3(|\nabla u_e|^2 + C_1tr_{\omega_0(t)}f^*\chi)) \right) \]

\[\geq (t + 1)\Delta_{\omega_0(t)}u_e + t|\nabla\nabla u_e|^2 - C\]

\[\geq (t + 1)\Delta_{\omega_0(t)}u_e + \frac{1}{n}(\Delta_{\omega_0(t)}u_e)^2 - C.\]

(4.40)

on $X \times [0, 1]$, and

\[(\partial_t - \Delta_{\omega_0(t)}) \left( \Delta_{\omega_0(t)}u_e - 3(|\nabla u_e|^2 + C_1tr_{\omega_0(t)}f^*\chi) \right) \]

\[\geq \Delta_{\omega_0(t)}u_e + |\nabla\nabla u_e|^2 - C\]

\[\geq \Delta_{\omega_0(t)}u_e + \frac{1}{n}(\Delta_{\omega_0(t)}u_e)^2 - C\]

(4.41)

on $X \times [1, \infty)$. To bound $\Delta_{\omega_0(t)}u_e$ from below (see [6, Section 8] or [21, Section 7] for related discussions), we also try to avoid using a uniform bound for $(\Delta_{\omega_0(t)}u_e)(0) =$
\[ \Delta_{\omega_t} \log \frac{(\omega_t^n)^n}{(S^n/n)^n} \] For convenience, we set \( H_\epsilon := t(\Delta_{\omega_t} u_\epsilon - 3(|\nabla u_\epsilon|^2 + C_1 \text{tr}_{\omega_t} f^* \chi)) \), which by (4.46) satisfies

\[
(\partial_t - \Delta_{\omega_t}) \left( t(\Delta_{\omega_t} u_\epsilon - 3(|\nabla u_\epsilon|^2 + C_1 \text{tr}_{\omega_t} f^* \chi)) \right) \geq (t + 1)\Delta_{\omega_t} u_\epsilon + \frac{1}{n} t(\Delta_{\omega_t} u_\epsilon)^2 - C \tag{4.42}
\]

\( X \times [0, 1] \). We assume the minimal value of \( H_\epsilon \) on \( X \times [0, 1] \) is achieved at \((x', t') \in X \times [0, 1] \). If \( t' = 0 \), \( H_\epsilon(x', t') = 0 \) is bounded from below; otherwise \( t' > 0 \), then by maximum principle in (4.42) we find, at \((x', t') \),

\[
t(\Delta_{\omega_t} u_\epsilon)^2 \leq -(t + 1)\Delta_{\omega_t} u_\epsilon + C,
\]

and so

\[
(t\Delta_{\omega_t} u_\epsilon)^2 \leq -(t + 1)(t\Delta_{\omega_t} u_\epsilon) + Ct.
\]

We may assume \((\Delta_{\omega_t} u_\epsilon)(x', t') < 0 \) (otherwise we are done); then by noting that \( t \in (0, 1] \)

we have

\[
(t\Delta_{\omega_t} u_\epsilon)^2 \leq -2(t\Delta_{\omega_t} u_\epsilon) + C,
\]

from which we conclude that

\[
(t\Delta_{\omega_t} u_\epsilon)(x', t') \geq -C.
\]

Plugging into \( H_\epsilon \) gives

\[
H_\epsilon(x', t') = (t(\Delta_{\omega_t} u_\epsilon - 3(|\nabla u_\epsilon|^2 + C_1 \text{tr}_{\omega_t} f^* \chi)))(x', t') \geq -C.
\]

Therefore, we can choose a constant \( C \geq 1 \) independent on \( \epsilon \) such that

\[
\inf_{X \times [0, 1]} H_\epsilon \geq -C,
\]

which, combining with (4.30), implies

\[
\inf_{X \times [0, 1]} (t\Delta_{\omega_t} u_\epsilon) \geq -C. \tag{4.43}
\]

In particular,

\[
\inf_{X}(\Delta_{\omega_t} u_\epsilon - 3(|\nabla u_\epsilon|^2 + C_1 \text{tr}_{\omega_t} f^* \chi))(1) \geq -C. \tag{4.44}
\]

Having (4.44), we can easily apply the maximum principle in (4.41) to find a constant \( C \geq 1 \) independent on \( \epsilon \) such that

\[
\inf_{X \times [1, \infty)} \Delta_{\omega_t} u_\epsilon \geq -C. \tag{4.45}
\]

Combining (4.43) and (4.45) gives, for any \( t > 0 \),

\[
\inf_{X} \Delta_{\omega_t} u_\epsilon \geq -\frac{C}{\min\{t, 1\}}. \tag{4.46}
\]

Finally, recall from (4.30) that

\[
\tilde{R}_\epsilon = -\text{tr}_{\omega_t} f^* \chi - \Delta_{\omega_t} u_\epsilon. \tag{4.47}
\]

Therefore, by (4.46) there exists a constant \( C \geq 1 \) independent on \( \epsilon \) such that for any \( t > 0 \)

\[
\sup_{X} \tilde{R}_\epsilon \leq \frac{C}{\min\{t, 1\}}. \tag{4.48}
\]
Combining (4.7) and (4.48) gives a constant $C \geq 1$ independent on $\epsilon$ such that for any $t > 0$,
\[ \sup_X |\tilde{R}_\epsilon| \leq \frac{C}{\min\{t, 1\}}. \] (4.49)
By letting $\epsilon \to 0$, Lemma 4.1 is proved.

4.1. Proof of Theorem 1.1
In this subsection, we explain how to apply the above arguments for Lemma 4.1 to prove Theorem 1.1.

Proof of Theorem 1.1. It suffices to prove the conclusion in Theorem 1.1 for the following normalized version of (1.5):
\[ \begin{cases} 
\partial_t \omega(t) = -Ric(\omega(t)) - \omega(t) + 2\pi(1 - \beta)[D] \\
\omega(0) = \omega_0^*, 
\end{cases} \] (4.50)

here, without loss of any generality we assume there is only one irreducible component $D$ in the cone divisor. We have the smooth approximation (4.6) and (4.5) as before. For the fixed $t_0 \in (0, T_{\text{max}})$, we fix a Kähler metric $\hat{\omega}_{t_0} \in e^{-t_0}[\omega_0] + (1 - e^{-t_0})2\pi(c_1(K_X) + (1 - \beta)[D])$. Also note that $\varphi_\epsilon$ and $\partial_t \varphi_\epsilon$ are uniformly bounded on $X \times [0, t_0]$ (see [24, 40]). In the above arguments for Lemma 4.1 we replace $Y$ by $X$, $f$ by the identity map $X \to X$ and $\chi$ by $\hat{\omega}_{t_0}$, and then those arguments apply and Theorem 1.1 follows. □

5. Local $C^0$-convergence away from cone divisor

In this section, we will prove $C^0$-convergence away from cone divisor $D$. The strategy used here is taken from Tosatti-Weinkove-Yang [35].

In this section, similar to discussions in Sections 2 and 3, we will directly work with the conical equations without passing to a smooth approximation. Let’s begin with the following

Lemma 5.1. There exists a constant $C \geq 1$ such that for any $t \in [1, \infty)$,
\[ \sup_{X \setminus D} |\partial_t \varphi + \varphi(t) + \delta[S]^{2\beta}_h - f^*\psi| \leq C e^{-\frac{1}{4}t}. \] (5.1)

Proof. Recall Proposition 2.4 there exists a constant $\tilde{C}$ such that for any $t \in [0, \infty)$
\[ \sup_{X \setminus D} |\varphi + \delta[S]^{2\beta}_h - f^*\psi| \leq \tilde{C} e^{-\frac{1}{4}t}. \] (5.2)

Also recall
\[ \partial_t(\partial_t \varphi) = \partial_t \varphi - \tilde{R} - k. \] (5.3)

So, by Lemma 4.1 we can fix a constant $C_0 \geq 1$ such that $\sup_{(X \setminus D) \times [1, \infty)} |\partial_t(\partial_t \varphi)| \leq C_0$. We set $C_1 := \sqrt{8\tilde{C}CC_0 + 1}$. It suffices to show
\[ \sup_{X \setminus D} |\partial_t \varphi| \leq C_1 e^{-\frac{1}{4}t}. \] (5.4)

To this end, we claim that

Claim For any $\lambda \in (0, \lambda_0]$ and $t \in [1, \infty)$,
\[ \sup_{X \setminus D} (\partial_t \varphi + \lambda \log |S|^{2\beta}_h) \leq C_1 e^{-\frac{1}{4}t}. \] (5.5)
Proof of Claim. We make use of an argument in [35] to conclude. Suppose there exists a sequence \((x_k, t_k) \in (X \setminus D) \times [1, \infty)\) with \(t_k \to \infty\) as \(k \to \infty\) such that

\[
(\partial_t \varphi + \lambda \log |S|_h^2)(x_k, t_k) \geq C_1 e^{-\frac{1}{4}t_k},
\]

which in particular implies that \(x_k\) in fact will be contained in some fixed \(K \subset \subset X \setminus D\), since \(\partial_t \varphi\) is uniformly bounded. Set \(\theta_k := \frac{1}{2C_0} e^{-\frac{1}{4}t_k}\), where \(C_0\) is a constant such that

\[
\sup_{(X \setminus D) \times [1, \infty)} |\partial_t(\partial_t \varphi)| \leq C_0 \text{ defined as before. Then, } (\partial_t \varphi + \lambda \log |S|_h^2)(x, t) \geq \frac{C_1}{2} e^{-\frac{1}{4}t_k}
\]

for all \(t \in [t_k, t_k + \theta_k]\). On the one hand we have

\[
(\varphi + \delta|S|_h^{2\beta} - f^*\psi)(x_k, t_k + \theta_k) - (\varphi + \delta|S|_h^{2\beta} - f^*\psi)(x_k, t_k)
\]

\[
\leq \sup_{X \setminus D} |\varphi + \delta|S|_h^{2\beta} - f^*\psi|(t_k + \theta_k) + \sup_{X \setminus D} |\varphi + \delta|S|_h^{2\beta} - f^*\psi|(t_k)
\]

\[
\leq \tilde{C}(e^{-\frac{1}{4}(t_k + \theta_k)} + e^{-\frac{1}{4}t_k});
\]

\[
\leq 2\tilde{C} e^{-\frac{3}{4}t_k}. \tag{5.6}
\]

On the other hand,

\[
(\varphi + \delta|S|_h^{2\beta} - f^*\psi)(x_k, t_k + \theta_k) - (\varphi + \delta|S|_h^{2\beta} - f^*\psi)(x_k, t_k)
\]

\[
\geq \int_{t_k}^{t_k + \theta_k} (\partial_t \varphi)(x_k, t)dt
\]

\[
\geq \int_{t_k}^{t_k + \theta_k} (\partial_t \varphi + \lambda \log |S|_h^2)(x_k, t)dt
\]

\[
\geq \theta_k \cdot C_1 e^{-\frac{1}{4}t_k}
\]

\[
= \frac{C_1^2}{4C_0} e^{-\frac{1}{4}t_k}
\]

\[
\geq \frac{C_1^2}{4C_0} e^{-\frac{3}{4}t_k}. \tag{5.7}
\]

By combining with (5.6) and (5.7) we get a contradiction:

\[
8\tilde{C}C_0 + 1 = C_1^2 \leq 8\tilde{C}C_0.
\]

Therefore, Claim is proved. \(\square\)

Let \(\lambda \to 0\) in (5.5), we get, for all \(t \in [1, \infty)\),

\[
\sup_{X \setminus D} \partial_t \varphi \leq C_1 e^{-\frac{1}{4}t}. \tag{5.8}
\]

Similarly, we can get, for all \(t \in [1, \infty)\),

\[
\inf_{X \setminus D} \partial_t \varphi \geq -C_1 e^{-\frac{1}{4}t}. \tag{5.9}
\]

Having (5.10) and (5.9), (5.4) follows and Lemma 5.1 is proved. \(\square\)

Lemma 5.2. There exist two positive constants \(C, \gamma\) such that for any \(t \in [1, \infty)\),

\[
\sup_{X \setminus D} (|S|_h^{2\gamma} (tr_\omega(t)f^*\chi - k)) \leq C e^{-\frac{1}{4}t}. \tag{5.10}
\]
Proof. Recall we have
\[(\partial_t - \Delta_{\omega(t)})(\partial_t \varphi + \varphi + \delta |S_h^{(2)} - f^* \psi|) = \text{tr}_{\omega(t)} f^* \bar{\chi} - k\]
and, for some \(\gamma > 0\),
\[(\partial_t - \Delta_{\omega(t)}) \text{tr}_{\omega(t)} \bar{\chi} \leq \text{tr}_{\omega(t)} \bar{\chi} \leq +|S_h^{(2)}(\text{tr}_{\omega(t)} \bar{\chi})|^2 \leq C|S_h^{(2)}|,\]
where \(|S_h^{(2)}|\) is an uniform upper bound for bisectional curvature of \(\bar{\chi}\) on \(Y \setminus D'\) and we have used \(\text{tr}_{\omega(t)} f^* \bar{\chi}\) is uniformly bounded by Lemma 3.1 (note that \(\bar{\chi}\) and \(\chi^*\) is uniformly equivalent). Then one can apply arguments in [35, Lemma 3.4] to conclude (5.10).

Lemma 6.1 is proved. \(\square\)

Lemma 5.3. For any given \(K' \subset Y \setminus D'\), and \(l \in \mathbb{Z}_{\geq 1}\), there exists a constant \(C = C_{K,l} \geq 1\) such that for any \(y \in K'\) and \(t \in [1, \infty)\),
\[|e^t \omega(t)|_{X_y}|_{C^0(X_y, \omega_0)} \leq C_{K,l}.\] (5.11)

Proof. This lemma can be checked by the same arguments in [36, Theorem 1.1], as we have Proposition 5.4. To see that arguments in [36, page 2933-2934] work in our twisted case, it suffices to observe that, after pulling back the twisted term \(\frac{1}{T_{\text{max}}} \omega_0\) by the ”stretching map” \(F_k\) defined in [36, page 2933], we get a smooth \((1, 1)\)-form \(\frac{1}{T_{\text{max}}} F_k^* \omega_0\), whose \(C^1\)-norm with respect to a local Euclidean metric is uniformly bounded. Then we can get the desired conclusions. \(\square\)

With all the above preparations, we now conclude the main result in this section.

Proposition 5.4. There exists a positive constant \(\varepsilon_0\) such that, for any \(K \subset X \setminus D\) there exists a constant \(C_K \geq 1\) such that, for any \(t \in [1, \infty)\),
\[|\omega(t) - f^* \bar{\chi}|_{C^0(K, \omega_0)} \leq C_K e^{-\varepsilon_0 t}.\] (5.12)

Proof. Having the above results, one can first prove
\[|e^t \omega(t)|_{X_y}|_{C^0(X_y, \omega_0)} \leq C Ke^{-\varepsilon t}\]
for any \(y \in K\), and then
\[|\omega(t) - (e^{-t \omega_0} + f^* \bar{\chi})|_{C^0(K, \omega_0)} \leq C Ke^{-\varepsilon_0 t}\]
by the same arguments in [35, Section 2.5]. We omit details here. Therefore, Proposition 5.4 follows. \(\square\)

6. GROMOV-HAUSDORFF CONVERGENCE

We are going to prove item (2.3) in Theorem 1.2 i.e. Gromov-Hausdorff convergence. Throughout this section, we will always assume \(\dim(Y) = 1\) and so \(D'\) is a single point \(o \in Y\). To obtain Gromov-Hausdorff convergence, the key point is to apply Proposition 3.4 to bound diameter of the cone divisor \(X_o\). We set \(B_5 := \{y \in Y | d_\chi(y, o) < \delta\}\) and \(B_\delta := f^{-1}(B_5)\). Assume without loss of generality that \(\delta\) is always small enough such that \(B_\delta\) is the standard disc in \(C\) and \(o = 0 \in C\). Fix an integer \(L \geq 1\) such that
\[\text{diam}(B_\delta, \delta) \leq \frac{\varepsilon}{4}.\] (6.1)
The key observation is the following

Lemma 6.1. There exists a constant \(C \geq 1\) such that, for any \(\varepsilon > 0\), there exists a positive constant \(T = T_{\varepsilon}\) such that for all \(t \geq T\),
\[\text{diam}(B_{t \varepsilon}, d_t) \leq C \varepsilon.\] (6.2)
In conclusion, by combining (6.3), (6.4) and (6.5), we can choose a positive constant \( \delta \). The proof will make use of Proposition 3.4 crucially. Firstly, since \( X_o \) is compact, we choose a family of local charts \( \{ U_i, (y, z^1, \ldots, z^{n-1}) \}_{i=1}^m \), such that the chart \( U_i \) is centered at some point \( x_i \in X_o \). By Proposition 3.4, \( f(y, z^1, \ldots, z^{n-1}) = y \). For any two points \( x', x'' \in B_\ell \setminus X_o \), we may assume \( x' = (y', z') \in U_1 \) and \( x'' = (y'', z'') \in U_2 \). Then when we restrict to the subset \( Y_{z'} := \{ (y, z') \in U_1 | y \in B_\ell \} \), we know from Proposition 3.4 that \( \omega(t) \mid_{Y_{z'}} \) is a conical metric on \( Y_{z'} \) with cone angle \( 2\pi\beta \) at \((0, z')\). Therefore, we can easily choose a point \( y' \in \partial B_\ell \), and set \( \bar{x} := (y', z') \in Y_{z'} \), such that
\[
d_t(x', \bar{x}) \leq \hat{C} \delta, \tag{6.3}
\]

here \( \hat{C} \) is a positive constant only depends on the constant \( C \) in Proposition 3.4 (3.22) (in particular, \( \hat{C} \) does not depend on \( x', x'' \) and \( t \)). Similarly, one can find a point \( \bar{x}'' := (y'', z'') \) with \( y'' \in \partial B_\ell \) such that
\[
d_t(x'', \bar{x}'') \leq \hat{C} \delta, \tag{6.4}
\]

On the other hand, since \( \chi \) is a conical Kähler metric, the diameters of fibers \( X_y \) away from \( X_o \) uniformly collapse in the rate \( e^{-\frac{1}{t}} \) by Proposition 3.4 and \( \omega(t) \to f^{-1}\chi \) in \( C^0(f^{-1}(Y \setminus B_\ell \setminus \omega_0)) \)-topology by Proposition 5.4, one can use arguments in [47, Section 3] to connect \( \bar{x}', \bar{x}'' \) by a piecewise smooth curve \( \sigma \subset f^{-1}(\partial B_\ell) \) such that
\[
L_{dt}(\sigma) \leq \varepsilon + \hat{C} e^{-\frac{1}{2t}}. \tag{6.5}
\]

In conclusion, by combining (6.3), (6.4) and (6.5), we can choose a positive constant \( T \) (independent on \( x', x'' \)) such that for any \( t \geq T \),
\[
d_t(x', x'') \leq (\hat{C} + 2) \varepsilon. \tag{6.6}
\]

Lemma 6.1 is proved.

Proof of Theorem 1.2 (2.3). Having Lemma 6.1, we can easily apply arguments in [47, Section 3] (also see [46, Section 4]) to conclude Theorem 1.2 (2.3). \( \Box \)

7. Remarks on the twisted Kähler-Ricci flow

The last section provides more remarks on the twisted Kähler-Ricci flow we have studied. Here we remove the conical singularity. Given a projective manifold \( X \) with a rational Kähler metric \( \omega_0 \in 2\pi c_1(H) \) (\( H \) is some ample line bundle on \( X \)), we have the smooth Kähler-Ricci flow starting from \( \omega_0 \) on \( X \):
\[
\begin{aligned}
\partial_t \omega(t) &= -\text{Ric}(\omega(t)) \\
\omega(0) &= \omega_0,
\end{aligned} \tag{7.1}
\]

By [3, 38, 34], the Kähler-Ricci flow (7.1) has a smooth solution up to
\[
T_{\text{max}} := \{ t > 0 | [\omega_0] + t 2\pi c_1(K_X) > 0 \} = \{ t > 0 | H + tK_X > 0 \}.
\]

Then \( T_{\text{max}} = \infty \) if and only if \( X \) is a smooth minimal model.

Assume \( T_{\text{max}} < \infty \). Then \( T \in \mathbb{Q} \) and \( H + T_{\text{max}} K_X \) is semi-ample (see e.g. [20]). By semi-ample fibration theorem there exists a holomorphic map
\[
f : X \to f(X) \subset \mathbb{C}P^N, \tag{7.2}
\]
with the image \( Y := f(X) \) an irreducible normal projective variety of dimension \( 0 \leq k \leq n \), connected fibers and
\[
f^*\mathcal{O}_{\mathbb{C}P^N}(1) = H + T_{\text{max}} K_X. \tag{7.3}
\]
Let’s first recall some basic ideas of Song and Tian [32, Section 6.2] on contracting or collapsing certain positive part of $c_1(X)$ by the Kähler-Ricci flow, which indicates a deep relation between finite-time singularity of the Kähler-Ricci flow and the Minimal Model Program in algebraic geometry.

1. If $k = 0$, then $H = -T_{\text{max}} K_X$, i.e. $X$ is a Fano manifold with $\omega_0 \in T_{\text{max}} c_1(X)$. The Kähler-Ricci flow in this case has been studied by many works, see e.g. discussions next to [32, Conjecture 6.6] and references therein for more comments. Here we will focus on the $k > 0$ case.

2. If $1 \leq k \leq n$, then it is conjectured in [32] that, as $t \to T_{\text{max}}$, $(X, \omega(t)) \to (Y, d_Y)$ in Gromov-Hausdorff topology, here $d_Y$ is some compact metric on $Y$ and should be induced by a smooth Kähler metric on $Y$ outside a subvariety.

Roughly speaking, (7.3) means there are certain ”positive parts” in $c_1(X)$ (and so $X$ is not a minimal model), and the picture in the above item (2) means one can use the Kähler-Ricci flow to contract/collapse these ”positive parts” in Gromov-Hausdorff topology and arrive at a new space, which is closer to a minimal model in some sense (see [32] Section 6.2 for more precise discussions). There are several progresses on above item (2), see e.g. [8, 10, 26, 28, 37]. However, it seems in general the finite time convergence of the Kähler-Ricci flow are still unclear.

La Nave and Tian [15] proposed a continuity method approach to achieve the above picture, see [15, 16, 47, 48, 46] for more discussions and results on this direction.

Here we would like to discuss the possibility of deforming $X$ to $Y$ (given $f : X \to Y$ as in above (7.2)) by using a twisted Kähler-Ricci flow. Given the above setting, we consider, for an arbitrary Kähler metric $\omega_X$,

\[
\left\{ \begin{array}{l}
\partial_t \omega(t) = -\text{Ric}(\omega(t)) - \omega(t) - \frac{1}{T_{\text{max}}} \omega_0 \\
\omega(0) = \omega_X,
\end{array} \right. \tag{7.4}
\]

If we fix a Kähler metric $\chi_Y$ on $Y$ with $f^* \chi_Y \in \frac{1}{T_{\text{max}}} [\chi_0] + 2\pi c_1(K_X)$, then we easily see that the Kähler class along (7.4) satisfies

\[ [\omega(t)] = e^{-t} [\omega_X] + (1 - e^{-t}) [f^* \chi], \]

which stays positive for any $t \in [0, \infty)$. Therefore, (7.4) has a smooth long time solution $\omega(t)$ on $X \times [0, \infty)$. Then one can prove $\omega(t)$ converges to $f^* \chi_Y$ as currents on $X$, where $\chi_Y$ is a Kähler current solved by a possibly singular complex Monge-Ampère equation on $Y$. Precisely, we set $V$ be the singular set of $Y$ together with critical values of $f$ and fix a $(1, 1)$-form $\omega_X \in [\omega_X]$ on $X \setminus f^{-1}(V)$ such that $\omega_X|_{X_y}$ is a Kähler metric on $X_y$ and $\text{Ric}(\omega_X|_{X_y}) = \frac{1}{T_{\text{max}}} \omega_0|_{X_y}$ for all $y \in Y \setminus V$. Also fix a smooth volume form $\Omega$ on $X$ with $\sqrt{-1} \partial \bar{\partial} \log \Omega = f^* \chi_Y - \frac{1}{T_{\text{max}}} \omega_0$. Then $\chi_Y := \chi_Y + \sqrt{-1} \partial \bar{\partial} \psi$ is solved by

\[ (\chi_Y + \sqrt{-1} \partial \bar{\partial} \psi)^k = e^\psi \frac{\Omega}{C^n(\omega_X)^{n-k} f^* \chi_Y}. \]

Since $\chi_Y$ is rational and $0 < C^{-1} \leq \frac{\Omega}{C^n(\omega_X)^{n-k} f^* \chi_Y} \in L^{1+\epsilon}(Y, \chi_Y^k)$ (see [30, Proposition 3.2]), we can find a unique $\psi \in PSH(Y, \chi_Y) \cap L^\infty(Y) \cap C^\infty(Y \setminus V)$ solving the above equation (see [30, Section 3.2, Theorem 3.2]).

(a) Case $0 < k < n$. If additionally $Y$ is smooth and $f : X \to Y$ is a submersion, then $\chi_Y$ is a Kähler metric on $Y$ and $\omega(t) \to f^* \chi_Y$ in $C^0(X, \omega_0)$-topology exponentially
fast (see Theorem 1.2 (2.2)) and hence \((X, \omega(t)) \to (Y, \bar{\omega}_Y)\) in Gromov-Hausdorff topology. This provides an alternative way (using the twisted Kähler-Ricci flow) to deform a Fano bundle to the base in Gromov-Hausdorff topology (compare with the works in \([8, 10, 26]\)).

(b) Case \(k = n\). In this case \(\bar{\omega}_Y\) is in fact solved by
\[
\left(f^*\chi_Y + \sqrt{-1}\partial\bar{\partial}\psi\right)^n = e^\psi \Omega
\]
on \(X\). Then \(\bar{\omega}_Y\) is a smooth Kähler metric on \(X \setminus f^{-1}(V)\) and \(\omega(t) \to \bar{\omega}_Y\) smoothly on \(X \setminus V\). In fact, it can be checked that \(\bar{\omega}_Y\) coincides with the limit of the continuity method studied in \([16, \text{Theorems 1.1, 1.2}]\), and so the metric completion of \((X \setminus f^{-1}(V), \bar{\omega}_Y)\) is a compact metric space homeomorphic to \(Y\) by \([16, \text{Theorem 1.2}]\); denote this limit space by \((Y, d_Y)\). On the other hand, since the twisted term \(\frac{1}{T_{\max}} \omega_0\) is nonnegative, it is very likely that one can extend a generalized Perelman’s no-local-collapsing theorem of Wang \([39, \text{Theorem 1.1}]\) or Q. Zhang \([44, \text{Theorem 6.3.2}]\) to our twisted setting. Also note that we have a uniform bound for the twisted scalar curvature \(tr_{\omega(t)}(Ric(\omega(t)) - \frac{1}{T_{\max}} \omega_0)\), see Section 4.

Therefore, by Wang’s argument in \([39, \text{Theorem 8.2}]\), we may obtain a uniform diameter upper bound along the twisted Kähler-Ricci flow \((7.4)\) in this volume noncollapsing case (i.e. \(k = n\)). The remaining question in this case is: can we prove \((X, \omega(t)) \to (Y, d_Y)\) in Gromov-Hausdorff topology? We will study this question in the future work.

Acknowledgements

The author thanks Prof. Huai-Dong Cao, Prof. Gang Tian, Prof. Zhenlei Zhang and Dr. Shaochuang Huang for useful discussions, Dr. Jiawei Liu for valuable comments and the referee for careful reading and valuable suggestions and comments. Part of this work was carried out while the author was visiting University of Macau and Capital Normal University, which he would like to thank for the hospitality.

References

[1] Brendle, S., Ricci flat Kähler metrics with edge singularities, Int. Math. Res. Not. IMRN 2013, no. 24, 5727-5766
[2] Campana, F., Guenancia, H. and Paun, M., Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields, Ann. Sci. Ec. Norm. Super. 46, 879-916 (2013)
[3] Cao, H.-D., Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds, Invent. Math. 81, 359-372 (1985)
[4] Chen, X. and Wang, Y., Bessel functions, heat kernel and the conical Kähler-Ricci flow, J. Funct. Anal. 269, no. 2, 551-632 (2015)
[5] Chen, X. and Wang, Y., On the long time behavior of the conical Kähler-Ricci flow, to appear in J. Reine Angew. Math., DOI:10.1515/crelle-2015-0103
[6] Edwards, G., A scalar curvature bound along the conical Kähler-Ricci flow, J. Geom. Anal. 28, 225-252 (2018)
[7] Edwards, G., Metric contraction of the cone divisor by the conical Kähler-Ricci flow, arXiv: 1704.00360 (2017)
[8] Fong, F., Kähler–Ricci flow on projective bundles over Kähler-Einstein manifolds, Trans. Amer. Math. Soc. 366, 563-598 (2014)
[9] Fong, F. and Zhang, Z., The collapsing rate of the Kähler-Ricci flow with regular infinite time singularity, J. Reine Angew. Math. 703, 95-113 (2015)
[10] Fu, X. and Zhang, S., The Kähler-Ricci flow on Fano bundles, Math. Z. 286, no. 3-4, 1605-1626 (2017)
[11] Guenancia, H. and Paun, M., Conic singularities metrics with prescribed Ricci curvature: general cone angles along normal crossing divisors. J. Differential Geom. 103, no. 1, 15-57 (2016)
[12] Huang, S. and Tam, L.-F., Kähler-Ricci flow with unbounded curvature, Amer. J. Math. 140, no. 1, 189-220 (2018)
[13] Jeffres, T.D., Mazzeo, R. and Rubinstein, Y.A., Kähler-Einstein metrics with edge singularities, with an appendix by C. Li and Y. A. Rubinstein, Ann. of Math. 183, 95-176 (2016)
[14] Kolodziej, S., The complex Monge-Ampère equation, Acta Math. 180, no. 1, 69-117 (1998)
[15] La Nave, G. and Tian, G., A continuity method to construct canonical metrics, Math. Ann. 365, 911-921 (2016)
[16] La Nave, G., Tian, G. and Zhang, Z.L., Bounding diameter of singular Kähler metric, Amer. J. Math. 139, no. 6, 1693-1731 (2017)
[17] Liu, J. and Zhang, X., The conical Kähler-Ricci flow on Fano manifolds, Adv. Math. 307, 1324-1371 (2017)
[18] Liu, J. and Zhang, X., The conical Kähler-Ricci flow with weak initial data on Fano manifold, Int. Math. Res. Not. IMRN, no. 17, 5343-5384 (2017)
[19] Liu, J. and Zhang, X., Cusp Kähler-Ricci flow on compact Kähler manifold, arXiv: 1705.05129 (2017)
[20] Matsuki, K., Introduction to the Mori program, Universitext, Springer-Verlag, New York (2002)
[21] Nomura, R., Blow-up behavior of the scalar curvature along the conical Kähler-Ricci flow with finite time singularities, Differential Geom. Appl. 58, 1-16, (2018)
[22] Rubinstein, Y.A., Smooth and singular Kähler-Einstein metrics, Contemp. Math. 630, AMS and Centre Recherches Mathematiques, 45-138 (2014)
[23] Sesum, N. and Tian, G., Bounding scalar curvature and diameter along the Kähler Ricci flow, J. Inst. of Math. Jussieu 7(3), 575-587 (2008)
[24] Shen, L., Maximal time existence of unnormalized conical Kähler-Ricci flow, arXiv:1411.7284 (2014)
[25] Simon, M., Deformation of $C^0$ Riemannian metrics in the direction of their Ricci curvature, Anal. Geom. 10, no. 5, 1033-1074 (2002)
[26] Song, J., Szekelyhidi, G. and Weinkove, B., The Kähler-Ricci flow on projective bundles, Int. Math. Res. Not., no. 2, 243-257 (2013)
[27] Song, J. and Weinkove, B., The Kähler-Ricci flow on Hirzebruch surfaces, J. Reine Angew. Math. 659, 141-168 (2011)
[28] Song, J. and Weinkove, B., Contracting exceptional divisor by the Kähler-Ricci flow, Duke Math. J. 162, no. 2, 367-415 (2013)
[29] Song, J. and Tian, G., The Kähler-Ricci flow on surfaces of positive Kodaira dimension, Invent. Math., 170, 699-653 (2007)
[30] Song, J. and Tian, G., Canonical measures and Kähler-Ricci flow. J. Amer. Math. Soc. 25, no. 2, 303-353 (2012)
[31] Song, J. and Tian, G., Bounding scalar curvature for global solutions of the Kähler-Ricci flow, Amer. J. Math. 138, no. 3, 683-695 (2016)
[32] Song, J. and Tian, G., The Kähler-Ricci flow through singularities, Invent. Math. 207, no. 2, 519-595 (2017)
[33] Takahashi, R., Smooth approximation of the modified conical Kähler-Ricci flow, arXiv:1704.01879 (2017)
[34] Tian, G. and Zhang, Z., On the Kähler-Ricci flow on projective manifolds of general type, Chinese Ann. Math. Ser. B 27, no. 2, 179-192 (2006)
[35] Tosatti, V., Weinkove, B. and Yang, X., The Kähler-Ricci flow, Ricci-flat metrics and collapsing limits, Amer. J. Math. 140 (2018), no. 3, 653-698
[36] Tosatti, V. and Zhang, Y.G., Infinite-time singularities of the Kähler-Ricci flow, Geom. Topol. 19, 2925-2948 (2015)
[37] Tosatti, V. and Zhang, Y.G., Finite time collapsing of the Kähler-Ricci flow on threefolds, to appear in Ann. Sc. Norm Super. Pisa Cl. Sci., arXiv: 1507.08397
[38] Tsuji, H., Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type, Math. Ann. 281, 123-133 (1988)
[39] Wang, B., The local entropy along Ricci flow - part A: the no-local-collapsing theorems, arXiv:1706.08485, to appear in Cambridge Journal of Mathematics
[40] Wang, Y., Smooth approximation of the conical Kähler-Ricci flows, Math. Ann. 365, no. 1-2, 835-856 (2016)
[41] Yau, S.-T., A general Schwarz lemma for Kähler manifolds, Amer. J. Math. 100, no. 1, 197-203 (1978)
[42] Yau, S.-T., On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, Comm. Pure Appl. Math. 31, 339-411 (1978)
[43] Yin, H., Ricci flow on surfaces with conical singularities, J. Geom. Anal., 20 (4), 970-995 (2010)
[44] Zhang, Q.S., Sobolev inequalities, heat kernels under Ricci flow, and the Poincaré conjecture, CRC Press, Boca Raton, FL (2011)
[45] Zhang, Y.S., A note on conical Kähler-Ricci flow on minimal elliptic Kähler surfaces, Acta Math. Sci. Ser. B Engl. Ed. 38, no. 1, 169-176 (2018)
[46] Zhang, Y.S., Collapsing limits of the Kähler-Ricci flow and the continuity method, Math. Ann., https://doi.org/10.1007/s00208-018-1676-x (to appear)
[47] Zhang, Y.S. and Zhang, Z.L., The continuity method on minimal elliptic Kähler surfaces, Int. Math. Res. Not., https://doi.org/10.1093/imrn/rnx209 (to appear)
[48] Zhang, Y.S. and Zhang, Z.L., The continuity method on Fano fibrations, [arXiv:1612.01348] (2016)
[49] Zhang, Z., Scalar curvature bound for Kähler-Ricci flows over minimal manifolds of general type, Int. Math. Res. Not. IMRN, no. 20, 3901-3912 (2009)
[50] Zhang, Z., Globally existing Kähler-Ricci flows, Rev. Roumaine Math. Pures Appl. 60, no. 4, 551-560 (2015)

Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China
E-mail address: yashanzh@pku.edu.cn