Strong Convergence Rate for Fully Discrete Scheme of Semilinear Stochastic Evolution Equation with Multiplicative Noise

Jialin Hong, Chuying Huang, and Zhihui Liu

Abstract. This paper studies the unconditional strong convergence rate for a fully discrete scheme of semilinear stochastic evolution equations, under a generalized Lipschitz-type condition on both drift and diffusion operators. Applied to the one-dimensional stochastic advection-diffusion-reaction equation with multiplicative white noise, the main theorem shows that the spatial and temporal strong convergence orders are $1/2$ and $1/4$, respectively. This is the first optimal strong approximation result for semilinear SPDEs with gradient term driven by non-trace class noises. Numerical tests are performed to verify theoretical analysis.

1. Introduction

We investigate the strong convergence rate for a fully discrete scheme of the semilinear stochastic evolution equation

\[ dX(t) = (AX(t) + F(X(t)))dt + G(X(t))dW(t), \quad t \in (0, T]; \]

(SEE)

\[ X(0) = X_0 \]

on a Hilbert space $H$. Here $T$ is a fixed positive number, $A$ is a generator of an analytic $C_0$-semigroup $S$ on $H$ and $W = \{W(t) : t \in [0, T]\}$ is a (possibly cylindrical) $Q$-Wiener process.

There have been numerous works on numerical approximations of semilinear stochastic partial differential equation (SPDE); see, e.g., [BJK16, CHL17a, CHL17b, JR12, Kru14] and references therein. Motivated to construct a general semigroup framework for this topic under an assumption (see Assumption 2.1) weaker than the usual Lipschitz condition on drift and diffusion operators $F$ and $G$, we consider a fully discrete scheme (EI) of spectral Galerkin approximation in space and explicit exponential integrator in time. By taking advantages of the semigroup feature, we can handle the case when the covariance operator $Q$ is assumed to be nonnegative definite and bounded and does not need to be of trace class, as soon as

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as an generalized Lipschitz-type condition on the coefficients holds. Consequently, multiplicative white noise and even rough noises are available.

Typical examples are the second order parabolic SPDEs (see Eq. (SHE)), including the stochastic advection-diffusion-reaction equation where there is a gradient term describing the advection phenomenon (see e.g., [BCDS06]). Our main result shows that the scheme (EI) possesses the optimal strong convergence rate, in the sense that the spatial and temporal orders coincide with the Sobolev and Hölder exponents of solutions, respectively, for both the original equation and spatially discretized equations. In particular, for space-time white noise, our spatial and temporal strong orders are 1/2 and 1/4, respectively (see Theorem 4.1). To our best knowledge, this is the first strong approximation result for semilinear SPDEs allowing gradient term driven by non-trace class noises.

Note that [LT13] analyzed the exponential integrator for finite element discretization of Eq. (SEE) with usual Lipschitz continuous operators $F$ and $G$ satisfying (4.11) driven by trace class noise. [ZTRK15] studied the stochastic collocation methods for linear advection-diffusion-reaction equation with finite dimensional Brownian motion. In the case without the gradient term, Eq. (SHE) corresponds to the stochastic heat equation. The main technique of existing literatures about numerical approximations for this equation driven by multiplicative white noise is that they are performed in the framework of Green’s function; see, e.g., [ACQS, Gyö98, Gyö99, DG01, Wal05].

As a matter of fact, it would be quite difficult to obtain a sharp strong convergence rate for numerical approximations of SPDEs with gradient term via estimating the error between continuous and discrete Green’s functions. The method proposed in this paper can derive optimal strong convergence rate for exponential integrator of Eq. (SEE) with gradient term. Furthermore, exponential integrator avoids the CFL-type condition which appears in the usual explicit schemes for SPDEs, such as Euler–Maruyama scheme. For exponential integrator applied to other types of SPDEs, such as stochastic wave equation and stochastic Schrödinger equation, we refer to [AC18, ACLW16, CD17, CLS13, Wan15] and references cited therein.

The rest of this paper is organized as follows. We derive a quantitative continuous dependence result for a sequence of perturbation equations of Eq. (SEE) in the next section, which contains a strong convergence rate of spatially spectral Galerkin approximations for Eq. (SEE). In Section 3, we establish a strong convergence rate of a fully discrete scheme under a temporally uniform Hölder continuous condition for the Galerkin approximate solutions. Concrete examples are given in Section 4, where we verify the main assumptions to derive the strong convergence rate for the scheme under study. Finally, we give several numerical tests to verify our theoretical results in the last section.

2. Perturbation Equations

In this section, we introduce some frequently used notations and analyze a sequence of perturbation equations (SEE$_N$) of the continuous equation (SEE). In particular, we give a strong convergence rate of spatial spectral Galerkin approximations for Eq. (SEE).

2.1. Preliminaries. Let $(H, \| \cdot \|)$ be a separable Hilbert space and denote by $(\mathcal{L}(H), \| \cdot \|_{\mathcal{L}(H)})$ the space of bounded linear operators on $H$. Assume that the
linear operator $A : D(A) \subseteq H \to H$ is the infinitesimal generator of an analytic $C_0$-semigroup $S$ such that the resolvent set of $A$ contains all $\lambda \in \mathbb{C}$ with $\Re[\lambda] \geq 0$. Then the fractional powers $(-A)\theta$ for $\theta \in \mathbb{R}$ of the operator $-A$ are well-defined. We denote by $H^\theta$ the domain of $(-A)^{\theta/2}$ equipped with the $\| \cdot \|_{\theta}$-norm:

$$
\|x\|_{\theta} := \|(A)^{\frac{\theta}{2}}x\|, \quad x \in H^\theta.
$$

It is well known that (see, e.g., [Paz83, Chapter 2.6])

$$
\|(-A)^\nu\|_{L(H)} \leq C, \\
\|(-A)^\nu S(t)\|_{L(H)} \leq CT^{-\mu}, \\
\|(-A)^{-\beta}(S(t) - \mathrm{Id}_H)\|_{L(H)} \leq Ct^{\beta},
$$

for any $0 < t \leq T$, $\nu \leq 0 \leq \mu$ and $0 \leq \beta \leq 1$. Throughout $C$ is a generic constant independent of various approximate parameters.

Let $Q$ be a self-adjoint, nonnegative definite and bounded linear operator and $W := \{W(t) : t \in [0, T]\}$ be a $Q$-Wiener process on another separable Hilbert space $U$ with respect to a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$. We use $L_2^U$ to denote the space of Hilbert-Schmidt operators from $U_0 := Q_{\frac{1}{2}}(U)$ to $H^\theta$. Recall that a predictable stochastic process $X : [0, T] \times \Omega \to H$ is called a mild solution of Eq. (SEE) if $X \in L^\infty(0, T; H)$ almost surely such that

$$
X(t) = S(t)X_0 + S * F(X)(t) + S \circ G(X)(t), \quad t \in [0, T],
$$

where $S * F(X)$ and $S \circ G(X)$ denote the deterministic and stochastic convolutions, respectively:

$$
S * F(X)(\cdot) := \int_0^T S(\cdot - r)F(X(r))dr, \\
S \circ G(X)(\cdot) := \int_0^T S(\cdot - r)G(X(r))dW(r).
$$

To ensure the well-posedness of Eq. (SEE), we propose the following assumption on the data of this equation in terms of [HL, Theorem 2.1]. In the sequel, we use $\theta$ to denote a positive number which characterizes the spatial regularity for the solution of Eq. (SEE).

**Assumption 2.1.** There exist four nonnegative, Borel measurable functions $K_F, K_{F_\theta}, K_G$ and $K_{G_\theta}$ in $(0, T)$ with

$$
K^*(T) := \int_0^T [K_F(t) + K_G^2(t)]dt < \infty, \quad K^*_\theta(T) := \int_0^T [K_{F_\theta}(t) + K_{G_\theta}^2(t)]dt < \infty,
$$

such that for any $x, y \in H$ and $z \in H^\theta$, it holds that

$$
\|S(t)(F(x) - F(y))\| \leq K_F(t)\|x - y\|, \quad \|S(t)F(z)\|_{\theta} \leq K_{F_\theta}(t)(1 + \|z\|_{\theta}), \\
\|S(t)(G(x) - G(y))\|_{L^2} \leq K_G(t)\|x - y\|, \quad \|S(t)G(z)\|_{L^2} \leq K_{G_\theta}(t)(1 + \|z\|_{\theta}).
$$

**Lemma 2.1.** ([HL, Theorem 2.1]) Let $p \geq 2$ and Assumption 2.1 hold. Assume that $X_0 : \Omega \to H^\theta$ is strongly $\mathcal{F}_0$-measurable and $X_0 \in L^p(\Omega; H^\theta)$. Then Eq. (SEE) admits a unique mild solution $\{X(t) : t \in [0, T]\}$ and there exists a constant $C = C(T, p, K^*_\theta(T))$ such that

$$
\sup_{t \in [0, T]} \mathbb{E}\left[\|X(t)\|_{\theta}^p\right] \leq C \left(1 + \mathbb{E}\left[\|X_0\|_{\theta}^p\right]\right).
$$
2.2. Perturbation Equations. To spatially discretize Eq. (SEE), we consider a family of perturbation equations
\begin{equation}
\text{SEE}_N \quad dX^N(t) = (A^N X(t) + F^N(X^N(t)))dt + G^N(X^N(t))dW(t), \quad t \in (0, T];
\end{equation}
\begin{equation}
X^N(0) = X_0^N,
\end{equation}
where $A^N$ is the infinitesimal generator of an analytic $C_0$-semigroup $S^N(\cdot)$ on $H$, $N \in \mathbb{N}_+$. It stands particularly for the spectral Galerkin approximation for Eq. (SEE) in Section 4.

**Assumption 2.2.** There exist four nonnegative, Borel measurable functions $K_{FN}, K_{GN}, K_{G^*_N}$ and $K_{G^*_{G^*_N}}$ in $(0, T]$ with
\begin{equation}
\tilde{K}^*(T) := \sup_{N \in \mathbb{N}_+} \int_0^T K_{FN}(t) dt + \sup_{N \in \mathbb{N}_+} \int_0^T K_{GN}^2(t) dt < \infty,
\end{equation}
\begin{equation}
\tilde{K}_{\theta}^*(T) := \sup_{N \in \mathbb{N}_+} \int_0^T K_{FN}(t) dt + \sup_{N \in \mathbb{N}_+} \int_0^T K_{G^*_N}^2(t) dt < \infty,
\end{equation}
such that for any $x, y \in H$ and $z \in H^\theta$, it holds that
\begin{align}
\|S^N(t)(F^N(x) - F^N(y))\| &\leq K_{FN}(t)\|x - y\|, \\
\|S^N(t)F^N(z)\|_{\theta} &\leq K_{FN}(t)(1 + \|z\|_{\theta}), \\
\|S^N(t)(G^N(x) - G^N(y))\|_{L_2} &\leq K_{GN}(t)\|x - y\|, \\
\|S^N(t)G^N(z)\|_{L_2} &\leq K_{G^*_N}(t)(1 + \|z\|_{\theta}).
\end{align}

Similarly to the original equation (SEE), the corresponding well-posedness of the perturbation equation (SEE$_N$) is ensured by Assumption 2.2.

**Lemma 2.2.** Let $p \geq 2$ and Assumption 2.2 hold. Assume that $X^N_0 : \Omega \rightarrow H^\theta$ is strongly $\mathcal{F}_0$-measurable and uniformly bounded in $L^p(\Omega; H^\theta)$. Then Eq. (SEE$_N$) admits a sequence of unique mild solutions $\{X^N(t) : t \in [0, T]\}_{N \in \mathbb{N}_+}$ and there exists a constant $C = C(T, p, \tilde{K}_\theta^*(T))$ such that
\begin{equation}
\sup_{N \in \mathbb{N}_+} \sup_{t \in [0, T]} \mathbb{E}\left[\|X^N(t)\|^p_{\theta}\right] \leq C\left(1 + \sup_{N \in \mathbb{N}_+} \mathbb{E}\left[\|X^N_0\|^p_{\theta}\right]\right).
\end{equation}

The next assumption characterizes the convergence rate of the perturbation equations (SEE$_N$) as spatial approximation for Eq. (SEE), which will be explained by concrete examples in Section 4.

**Assumption 2.3.** There exist two nonnegative, Borel measurable functions $R_{FN}, R_{GN}$ in $(0, T]$ and two positive numbers $r_F, r_G$ depending on $\theta$ with
\begin{equation}
\int_0^T R_{FN}(t) dt \leq CN^{-r_F}, \quad \int_0^T R_{GN}^2(t) dt \leq CN^{-2r_G},
\end{equation}
such that for any $z \in H^\theta$, it holds that
\begin{align}
\|S^N(t)F^N(z) - S(t)F(z)\| &\leq R_{FN}(t)(1 + \|z\|_{\theta}), \\
\|S^N(t)G^N(z) - S(t)G(z)\|_{L_2} &\leq R_{GN}(t)(1 + \|z\|_{\theta}).
\end{align}

Our main result of this section is the strong convergence rate between the perturbation equation (SEE$_N$) and the original equation (SEE).
Theorem 2.1. Let $p \geq 2$, $r_0 > 0$ and Assumptions 2.1–2.3 hold. Assume that $X_0, X_0^N : \Omega \to H^p$ are strongly $\mathcal{F}_0$-measurable such that $X_0, X_0^N \in L^p(\Omega; H^p)$ and

\begin{equation}
\sup_{t \in [0,T]} \|S^N(t)X_0^N - S(t)X_0\|_{L^p(\Omega; H)} \leq CN^{-r_0}\|X_0\|_{L^p(\Omega; H^p)}.
\end{equation}

Then there exists a constant $C$ such that

\begin{equation}
\sup_{t \in [0,T]} \|X^N(t) - X(t)\|_{L^p(\Omega; H)} \leq CN^{-(r_0 \wedge r_F \wedge r_G)}.
\end{equation}

Proof. Let $t \in [0, T]$. We decompose the error between $X^N(t)$ and $X(t)$ as

\[ X^N(t) - X(t) = S^N(t)X_0^N - S(t)X_0 + \int_0^t S^N(t-r)F^N(X^N(r))dr - \int_0^t S(t-r)F(X(r))dr + \int_0^t S^N(t-r)G^N(X^N(r))dW(r) - \int_0^t S(t-r)G(X(r))dW(r) =: I_1(t) + I_2(t) + I_3(t). \]

For the first term, it follows from (2.5) that

\[ \|I_1(t)\|_{L^p(\Omega; H)} \leq CN^{-r_0}\|X_0\|_{L^p(\Omega; H^p)}. \]

By Minkowski inequality and Assumptions 2.2-2.3, we get

\[ \|I_2(t)\|_{L^p(\Omega; H)} \leq \int_0^t \|S^N(t-r)F^N(X^N(r)) - S^N(t-r)F^N(X(r))\|_{L^p(\Omega; H)}dr + \int_0^t \|S^N(t-r)F^N(X(r)) - S(t-r)F(X(r))\|_{L^p(\Omega; H)}dr \leq \int_0^t K_{FN}(t-r)\|X^N(r) - X(r)\|_{L^p(\Omega; H)}dr + \int_0^t R_{F\theta}(t-r)(1 + \|X(r)\|_{L^p(\Omega; H^p)})dr. \]

For the last term, Burkholder–Davis–Gundy inequality and Assumptions 2.2-2.3 yield that

\[ \|I_3(t)\|_{L^p(\Omega; H)}^2 \leq 2 \int_0^t \|S^N(t-r)G^N(X^N(r)) - S^N(t-r)G^N(X(r))\|^2_{L^p(\Omega; L^2)}dr + 2 \int_0^t \|S^N(t-r)G^N(X(r)) - S(t-r)G(X(r))\|^2_{L^p(\Omega; L^2)}dr \leq 2 \int_0^t K_{GN}^2(t-r)\|X^N(r) - X(r)\|^2_{L^p(\Omega; H)}dr + 2 \int_0^t R_{G\theta}^2(t-r)(1 + \|X(r)\|_{L^p(\Omega; H^p)})^2dr. \]
Combining the previous estimations and Minkowski and Hölder inequalities, we obtain
\[
\|X^N(t) - X(t)\|^2_{L^p(\Omega; H)} \\
\leq CN^{-2\rho_0}\|X_0\|^2_{L^p(\Omega; H^s)} \\
+ 4\left(1 + \sup_{t\in[0,T]} \|X(t)\|_{L^p(\Omega; H^s)}\right)^2 \left(\int_0^T R_{F^N}(t)dt\right)^2 \\
+ 8\left(1 + \sup_{t\in[0,T]} \|X(t)\|_{L^p(\Omega; H^s)}\right)^2 \left(\int_0^T R_{G^N}(t)dt\right)^2 \\
+ 4\tilde{K}^*(T) \int_0^t K_{F^N}(t-r)\|X^N(r) - X(r)\|^2_{L^p(\Omega; H)} dr \\
+ 8\int_0^t K_{G^N}^2(t-r)\|X^N(r) - X(r)\|^2_{L^p(\Omega; H)} dr.
\]
Define \(M_N(t) := 4\tilde{K}^*(T)K_{F^N}(t) + 8K_{G^N}^2(t), \ t \in [0, T]\). It is clear that \(M_N\) is uniformly integrable in \((0, T)\) for any \(N\) according to Assumption 2.2. By Lemma 2.1 and Assumption 2.3, we get
\[
\|X^N(t) - X(t)\|^2_{L^p(\Omega; H)} \\
\leq CN^{-2(\rho_0/\rho_{F^N}\wedge\rho_{G^N})} + C \int_0^t M_N(t-r)\|X^N(r) - X(r)\|^2_{L^p(\Omega; H)} dr,
\]
from which we conclude (2.6) by the Gronwall inequality with singular kernel in [HL, Lemma 3.1].

3. Fully Discrete Scheme

To construct a fully discrete approximation of Eq. (SEE), we propose the explicit exponential integrator for spatially discretized equation (SEE)\_N.

Let \(K \in \mathbb{N}_+\) and denote \(\mathbb{Z}_K = \{0, 1, \ldots, K\}\). Divide the temporal interval \((0, T]\) into \(K\) equidistant subinterval \(\{(t_i, t_{i+1}) : i \in \mathbb{Z}_K\}\), i.e., \(t_i = i\tau\) for \(i \in \mathbb{Z}_K\) with time step \(\tau = T/K\). The fully discrete scheme is
\[
(\text{EI}) \quad X^N_{k+1} = S^N(\tau)X^N_k + \tau S^N(\tau)F^N(X^N_k) + S^N(\tau)G^N(X^N_k)\Delta W_{k+1},
\]
with the same initial datum as Eq. (SEE)\_N, where \(\Delta W_{k+1} = W(t_{k+1}) - W(t_k), \ k \in \mathbb{Z}_{K-1}\). By iteration, we get
\[
X^N_k = S^N(t_k)X^N_0 + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} S^N(t_k - t_i)F^N(X^N_t) dt \\
+ \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} S^N(t_k - t_i)G^N(X^N_t) dW(r), \quad k \in \mathbb{Z}_K \setminus \{0\}.
\]

To derive a strong convergence rate of the scheme (EI), we propose the following assumption on \(S^N, F^N\) and \(G^N\).
ASSUMPTION 3.1. There exist two nonnegative, Borel measurable functions $R_{F_0\tau}$ and $R_{G_0\tau}$ in $(0,T)$ and two positive numbers $\eta_F, \eta_G$ depending on $\theta$ with
\[
\sup_{N \in \mathbb{N}_+} \int_0^T R_{F_0\tau}(t)dt \leq C\tau^{\eta_F} \quad \text{and} \quad \sup_{N \in \mathbb{N}_+} \int_0^T R_{G_0\tau}^2(t)dt \leq C\tau^{2\eta_G},
\]
such that for any $z \in H^\theta$ and $t \in (t_i, t_{i+1}]$, $i \in \mathbb{Z}_{K-1}$, it holds that
\[
\|(S^N(t) - S^N(t_{i+1})F^N(z))\| \leq R_{F_0\tau}(t)(1 + \|z\|_\theta),
\]
\[
\|(S^N(t) - S^N(t_{i+1})G^N(z))\|_L^2 \leq R_{G_0\tau}(t)(1 + \|z\|_\theta).
\]

Besides the above assumption on the data of the perturbation equations (SEE$_N$), we also need a uniform Hölder regularity of their solutions.

ASSUMPTION 3.2. The uniform Hölder exponent of \{$X^N(t)$ : $t \in [0,T]$\}$_{N \in \mathbb{N}_+}$ in $L^p(\Omega; H)$ is $\gamma$ for some $\gamma \in (0,1/2]$, i.e.,
\[
\sup_{N \in \mathbb{N}_+} \sup_{0 \leq s < t \leq T} \frac{\|X^N(t) - X^N(s)\|_{L^p(\Omega; H)}}{\theta(t-s)} < \infty.
\]

REMARK 3.1. The Hölder exponent in Assumption 3.2 is essentially not larger than $1/2$ due to the temporal regularity of $Q$-Wiener processes.

We begin with a discrete version of the Gronwall inequality with singular kernel given in [HL, Lemma 3.1]. To make it complete, we give the proof of this discrete Gronwall inequality.

LEMMA 3.1. Let $m > 0$, $N \in \mathbb{N}_+$ and $R^N : (0,T) \to \mathbb{R}_+$ be a sequence of nonnegative, Borel measurable functions such that
\[
(3.1) \quad \alpha(T) := \sup_{N \in \mathbb{N}_+} \sup_{K \in \mathbb{N}_+} \sum_{i=1}^K R^N(t_i)\tau < \infty.
\]
For any $N \in \mathbb{N}_+$, assume that \{$f^N(i)$\}$_{i \in \mathbb{Z}_K}$ is a nonnegative sequence such that
\[
f^N(k) \leq m + \sum_{i=0}^{k-1} R^N(t_k - t_i)f^N(i)\tau, \quad k \in \mathbb{Z}_K.
\]
Then there exists a constant $\mu$ independent of $K$ and $N$ such that
\[
\sup_{N \in \mathbb{N}_+} f^N(k) \leq 2me^{\mu\tau}, \quad k \in \mathbb{Z}_K.
\]

PROOF. For any $\mu \geq 0$, it holds that for $k \in \mathbb{Z}_K$,
\[
e^{-\mu\tau}f^N(k) \leq m + \sum_{i=0}^{k-1} e^{-\mu(t_k - t_i)}R^N(t_k - t_i)e^{-\mu\tau}f^N(i)\tau.
\]
Denoting $f^N_{\mu}(i) := e^{-\mu\tau}f^N(i)$ and $R^N_{\mu}(t) := e^{-\mu\tau}R^N(t)$, we have
\[
f^N_{\mu}(k) \leq m + \sum_{i=0}^{k-1} R^N_{\mu}(t_k - t_i)f^N_{\mu}(i)\tau.
\]
From (3.1), we have
\[
\alpha_{\mu}(T) := \sup_{N \in \mathbb{N}_+} \sup_{K \in \mathbb{N}_+} \sum_{i=1}^K R^N_{\mu}(t_i)\tau
\]
decreases with respect to $\mu$. Since
\[
\lim_{\mu \to 0} \alpha_\mu(T) \leq \alpha(T), \quad \lim_{\mu \to \infty} \alpha_\mu(T) = 0,
\]
there exists a $\mu_0$ independent of $K$ and $N$ such that $\alpha_\mu(T) \leq \frac{1}{2}$. As a consequence,
\[
f_{\mu_0}(k) \leq m + \alpha_\mu(T) \sup_{i \in \mathbb{Z}_k} f_{\mu_0}(i) \leq m + \frac{1}{2} \sup_{i \in \mathbb{Z}_k} f_{\mu_0}(i),
\]
from which we obtain $\sup_{i \in \mathbb{Z}_k} f_{\mu_0}(i) \leq 2m$. This completes the proof. 

Now we can give and prove our main result in this section, which shows the strong convergence rate between the fully discrete scheme (EI) and Eq. (SEE). In the general multiplicative noise case, the uniform Hölder continuous assumption (see Assumption 3.2) is essential in our error analysis. We also note that in the additive noise case, i.e., there exists a constant operator $G$ such that $G(z) \equiv G$ for all $z \in H$, the strong error analysis becomes independent of Assumption 3.2.

**Theorem 3.1.** Let $p \geq 2$ and Assumptions 2.1-2.3, 3.1-3.2 hold with $K_{pN} + K_{G,N}^2$ satisfying condition (3.1). Assume that $X_0$ and $X_0^N$ are strongly $\mathcal{F}_0$-measurable such that $X_0, X_0^N \in \mathbb{L}^p(\Omega; H^0)$ and (2.5) holds. Then there exists a constant $C$ such that
\[
\max_{k \in \mathbb{Z}_K} \|X(t_k) - X_k^N\|_{\mathbb{L}^p(\Omega; H)} \leq C(N^{-\left(r_0 \wedge r_F \wedge r_G\right)} + \tau^{\gamma \wedge \eta_F \wedge \eta_G}).
\]

**Proof.** In terms of Theorem 2.1 and the triangle inequality, it suffices to show that
\[
\max_{k \in \mathbb{Z}_K} \|X(t_k) - X_k^N - J_1 + J_2\|_{\mathbb{L}^p(\Omega; H)} \leq C_T^{\gamma \wedge \eta_F \wedge \eta_G}.
\]
Let $k \in \mathbb{Z}_K$. Notice that the error can be expressed by
\[
X(t_k) - X_k^N = J_1 + J_2,
\]
where
\[
J_1 = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \left[ S^N(t_k - r)F^N(X(t_k)) - S^N(t_k - t_i)F^N(X_{t_i}^N) \right]dr,
\]
\[
J_2 = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \left[ S^N(t_k - r)G^N(X(t_k)) - S^N(t_k - t_i)G^N(X_{t_i}^N) \right]dW(r).
\]
Then it follows that
\[
\|X(t_k) - X_k^N\|_{\mathbb{L}^p(\Omega; H)} \leq \|J_1\|_{\mathbb{L}^p(\Omega; H)} + \|J_2\|_{\mathbb{L}^p(\Omega; H)}.
\]
We begin with the estimation of $\|J_1\|_{\mathbb{L}^p(\Omega; H)}$. By the Minkowski inequality, we get
\[
\|J_1\|_{\mathbb{L}^p(\Omega; H)} \leq \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \|S^N(t_k - r) - S^N(t_k - t_i)\|_{\mathbb{L}^p(\Omega; H)} dr
\]
\[
+ \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \|S^N(t_k - t_i)(F^N(X(t_k)) - F^N(X(t_i)))\|_{\mathbb{L}^p(\Omega; H)} dr
\]
\[
+ \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \|S^N(t_k - t_i)(F^N(X(t_i)) - F^N(X_{t_i}^N))\|_{\mathbb{L}^p(\Omega; H)} dr.
\]
\[ J_11 \leq \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \| (S^N(r) - S^N(t_{i+1}))F^N(X^N(t_k - r)) \|_{L^p(\Omega;H)} dr \]
\[ \leq \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} R_{F^N,\tau}(r) (1 + \| X^N(t_k - r) \|_{L^p(\Omega;H)}) dr \]
\[ \leq \left( \int_0^{t_k} R_{F^N,\tau}(r) dr \right) \left( 1 + \sup_{t \in [0,T]} \| X^N(t) \|_{L^p(\Omega;H)} \right) \leq C\tau^{\eta_p}. \]

The Minkowski inequality in combination with Assumptions 2.2 and 3.2 ensures that

\[ J_{12} \leq \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} K_{F^N(t_k - t_i)} \| X^N(r) - X^N(t_i) \|_{L^p(\Omega;H)} dr \]
\[ \leq C\tau \left( \sum_{i=1}^k K_{F^N(t_i)} \tau \right). \]

For \( J_{13} \), according to the Minkowski and Hölder inequalities and Assumption 2.2, we have

\[ J_{13} \leq \sum_{i=0}^{k-1} K_{F^N(t_k - t_i)} \| X^N(t_i) - X^N_i \|_{L^p(\Omega;H)} \tau \]
\[ \leq C \left( \sum_{i=1}^k K_{F^N(t_i)} \tau \right) \left( \sum_{i=0}^{k-1} K_{F^N(t_k - t_i)} \| X^N(t_i) - X^N_i \|_{L^p(\Omega;H)}^2 \right)^{\frac{1}{2}} \]
\[ \leq C \left( \sum_{i=0}^{k-1} K_{F^N(t_k - t_i)} \| X^N(t_i) - X^N_i \|_{L^p(\Omega;H)}^2 \right)^{\frac{1}{2}}. \]

Combining the above estimations of \( J_{11}, J_{12} \) and \( J_{13} \), we get

\[ (3.4) \quad \| J_3 \|_{L^p(\Omega;H)} \leq C\tau^{2(\gamma \wedge \eta_p)} + C \sum_{i=0}^{k-1} K_{F^N(t_k - t_i)} \| X^N(t_i) - X^N_i \|_{L^p(\Omega;H)}^2. \]

Next we estimate \( \| J_2 \|_{L^p(\Omega;H)} \). It follows from the Minkowski and Burkholder–Davis–Gundy inequalities and similar arguments as in the estimation of \( J_{11}, J_{12} \) and \( J_{13} \) that

\[ \| J_2 \|_{L^p(\Omega;H)}^2 \leq 3 \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \| (S^N(t_k - r) - S^N(t_k - t_i))G^N(X^N(r)) \|_{L^p(\Omega;L^p_2)} dr \]
\[ + 3 \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \| S^N(t_k - t_i)(G^N(X^N(r)) - G^N(X^N(t_i))) \|_{L^p(\Omega;L^p_2)}^2 dr \]
\[ + 3 \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \| S^N(t_k - t_i)(G^N(X^N(t_i)) - G^N(X^N(t_i))) \|_{L^p(\Omega;L^p_2)}^2 dr. \]
\[ \leq C \left( \int_0^{L_k} R_{t^N_h r}^2(r)dr \right) \left( 1 + \sup_{t \in [0,T]} \|X_N(t)\|^2_{L^p(\Omega; H^s)} \right) + C\tau^{2\gamma} \left( \sum_{i=1}^{k} K_{G_N}^2(t_i) \right) + C\sum_{i=0}^{k-1} R_N(t_k - t_i) \|X_N(t_i) - X_i^N\|^2_{L^p(\Omega; H^s)} \tau. \]

As a consequence,

\[ \|J_2\|^2_{L^p(\Omega; H^s)} \leq C\tau^{2(\gamma \wedge \eta_G)} + C\sum_{i=0}^{k-1} R_N(t_k - t_i) \|X_N(t_i) - X_i^N\|^2_{L^p(\Omega; H^s)} \tau, \]

where \( R_N = K_{F_N} + K_{G_N}^2, N \in \mathbb{N}_+ \). Finally, we conclude (3.2) by Lemma 3.1. \( \square \)

**Remark 3.2.** Compared with the spatial Sobolev regularity and temporal Hölder regularity for the solution of Eq. (SHE), the strong convergence rate of the spectral Galerkin-exponential integrator is sharp. Moreover, the temporal step size \( \tau \) and the dimension \( N \) of spectral approximate space do not suffer from any CFL-type condition.

### 4. Example: Parabolic SPDE with Gradient Term

We illustrate our results by the following second-order semilinear parabolic SPDE with gradient term and multiplicative noise

\[
\begin{align*}
    dX(t, \xi) &= (\Delta X(t, \xi) + \nabla f(X(t, \xi)) + \tilde{f}(X(t, \xi)))dt + g(X(t, \xi))dW(t, \xi), \\
    \text{(SHE)} & \quad X(t, \xi) = 0, \quad (t, \xi) \in [0, T] \times \partial \Omega, \\
    X(0, \xi) &= X_0(\xi), \quad \xi \in \partial \subset \mathbb{R}^d,
\end{align*}
\]

where \( \partial \subset \mathbb{R}^d \) is a bounded open set with regular boundary.

Let \( H = L^2(\Omega) \) and define \( A = \Delta \) with domain \( \text{Dom}(A) = H^1_0(\Omega) \cap H^2(\Omega) \), where \( H^1_0(\Omega) := \{ f \in H^1(\Omega) : f|_{\partial \Omega} = 0 \} \). Then the self-adjoint and positive definite operator \((-A)\) possesses an eigensystem \( \{ (\lambda_j, e_j) \} \) with \( \{ \lambda_j \} \) being an increasing sequence and \( \{ e_j \} \) forming an orthonormal basis of \( H \). It is known from Weyl’s law (see, e.g., (Gri02)) that

\[ \lambda_j \sim j^{2} \quad \text{and} \quad \|e_j\|_{L^\infty(\Omega)} \leq C\lambda_j^{d-1} \leq Cj^{d-1}. \]

Define the operators \( F_f : H \to H^{-1}, F_{\tilde{f}} : H \to H \) and \( G : H \to L(U, H) \) by Nemytskii operators associated with \( f, \tilde{f} \) and \( g \), respectively:

\[
\begin{align*}
    F_f(z)(\xi) &= \nabla f(z(\xi)), \quad F_{\tilde{f}}(z)(\xi) := \tilde{f}(z(\xi)), \quad G(z)u(\xi) := g(z(\xi))u(\xi),
\end{align*}
\]

where \( z \in H, \xi \in \partial \) and \( u \in U \). Denote \( F(z) := F_f(z) + F_{\tilde{f}}(z) \). Then the stochastic advection-diffusion-reaction (SHE) is equivalent to Eq. (SEE).
Assume that \( f, \tilde{f}, g : \mathbb{R} \to \mathbb{R} \) are Lipschitz continuous functions with Lipschitz constants \( L_f, L_{\tilde{f}}, L_g > 0 \), i.e., for any \( \xi_1, \xi_2 \in \mathbb{R} \),
\[
| f(\xi_1) - f(\xi_2) | \leq L_f | \xi_1 - \xi_2 |,
\]
\[
| \tilde{f}(\xi_1) - \tilde{f}(\xi_2) | \leq L_{\tilde{f}} | \xi_1 - \xi_2 |,
\]
\[
| g(\xi_1) - g(\xi_2) | \leq L_g | \xi_1 - \xi_2 |.
\]
Then the generalized Lipschitz condition of \( F \) in Assumption 2.1 is valid, which had been verified in [HL, Section 2.3] with
\[
K_F(t) = C \left( \| (-A)^{1/2} S(t) \|_{L(H)} + \| S(t) \|_{L(H)} \right) \leq C(1 + t^{-1/2}), \quad t \in (0, T].
\]
Indeed, by the Lipschitz continuity of \( f, \tilde{f} \) and dual argument we have
\[
\| S(t) F(z) \|_\theta \leq \| (-A)^{1/2} S(t) (-A)^{-\frac{1}{2}} F_f(z) \| + \| (-A)^{\frac{1}{2}} S(t) F_{\tilde{f}}(z) \|
\]
\[
\leq \| (-A)^{1/2} S(t) \|_{L(H)} \| F_f(z) \|_{L^1} + \| (-A)^{\frac{1}{2}} S(t) \|_{L(H)} \| F_{\tilde{f}}(z) \|
\]
\[
\leq C(t^{-1/2} + t^{-\frac{1}{2}})(1 + \| z \|),
\]
for any \( z \in H \). Defining the integrable function by \( K_{F_\theta}(t) := C(t^{-1/2} + t^{-\frac{1}{2}}) \) in \((0, T] \) for any \( \theta \in (0, 1) \), we get
\[
\| S(t) F(z) \|_\theta \leq K_{F_\theta}(t)(1 + \| z \|).
\]
Thus combined with (2.1), the growth condition of \( F \) in Assumption 2.1 is valid.

We perform the spectral Galerkin method to construct the perturbation equations. For \( N \in \mathbb{N}_+ \), let \( V_N = \text{span}\{ e_1, \cdots, e_N \} \) and \( P_N \) be the projection from \( H \) to \( V_N \). Set \( X_N^0 = P_N X_0, A^N = A P_N, F^N = P_N F \) and \( G^N = P_N G, N \in \mathbb{N}_+ \). Since \( P_N \) is a contraction operator for each \( N \in \mathbb{N}_+ \), one can take \( K_{F^N} = K_F, K_{G^N} = K_G \) and \( K_{G^N} = K_G, N \in \mathbb{N}_+ \). Thus Assumption 2.2 holds with \( K_{F^N} + K_{G^N} \) satisfying condition (3.1).

Our aim is to verify the assumptions in Section 3. We begin with the following result which ensures Assumption 3.2.

**Lemma 4.1.** Let \( p \geq 2, \theta \in (0, 2) \) and \( X_0 \in \mathcal{L}^p(\Omega; H^\theta) \). Assume in addition to Assumption 2.1 that there exist two positive constants \( \gamma_1 \leq 1 \) and \( \gamma_2 \leq 1/2 \) such that
\[
\int_0^t K_F(r)dr \leq C t^{\gamma_1}, \quad \int_0^t K_G^2(r)dr \leq C t^{\gamma_2}, \quad t \in (0, T].
\]
Then \( \{ X_N(t) : t \in [0, T] \} \) is uniformly Hölder continuous with exponent \( \gamma = \frac{\alpha}{2} \wedge \gamma_1 \wedge \gamma_2 \) in Assumption 3.2.

**Proof.** Let \( p \geq 2 \) and \( 0 \leq s < t \leq T \). By the Minkowski and Burkholder–Davis–Gundy inequalities, we get
\[
\| X_N(t) - X_N(s) \|_{\mathcal{L}^p(\Omega; H)}
\]
\[
\leq \| (S(t) - S(s)) X_0^N \|_{\mathcal{L}^p(\Omega; H)}
\]
\[
+ \int_s^t \| (S(t-r) F^N (X^N(r))) \|_{\mathcal{L}^p(\Omega; H)} dr
\]
\[
+ \int_0^s \| (S(t-s) - 1_{H^\theta}) S(s-r) F^N (X^N(r)) \|_{\mathcal{L}^p(\Omega; H)} dr.
\]
\[ + C \left( \int_s^t \| S(t-r)G^N(X^N(r)) \|_{L^p(\Omega; \mathcal{L}_2)}^2 \, dr \right)^{\frac{1}{2}} \]

\[ + C \left( \int_0^s \| (S(t-s) - \text{Id}_H)S(s-r)G^N(X^N(r)) \|_{L^p(\Omega; \mathcal{L}_2)}^2 \, dr \right)^{\frac{1}{2}} \]

\[ =: I_1 + I_2 + I_3 + I_4 + I_5. \]

The smoothing effect \((2.1)\) of the semigroup \(S\) yields that

\[ I_1 \leq C \|(-A)^{-\left(\frac{\gamma}{2} \wedge 1\right)}(S(t-s) - \text{Id}_H)\|_{\mathcal{L}(H)} \| S(s) \|_{\mathcal{L}(H)} \| X_0^N \|_{L^p(\Omega; H^p)} \]

\[ \leq C(t-s)^{\frac{\gamma}{2} \wedge 1} \| X_0 \|_{L^p(\Omega; H^p)} \leq C(t-s)^{\frac{\gamma}{2} \wedge 1}. \]

By Assumption 2.1 and condition \((4.5)\), we get

\[ I_2 \leq \left( \int_0^{t-s} (K_{F_S}(r) + C) \, dr \right) \left( 1 + \sup_{t \in [0,T]} \| X^N(t) \|_{L^p(\Omega; H)} \right) \]

\[ \leq C(t-s)^{\gamma_1} + C(t-s), \]

\[ I_4 \leq C \left( \int_0^{t-s} (K_{G^N}(r) + C) \, dr \right)^{\frac{1}{2}} \left( 1 + \sup_{t \in [0,T]} \| X^N(t) \|_{L^p(\Omega; H)} \right) \]

\[ \leq C(t-s)^{\gamma_2} + C(t-s)^{\frac{\gamma}{2}}. \]

For the third term \(I_3\) and the last term \(I_5\), we derive by the smoothing effect \((2.1)\) of the semigroup \(S\) and Assumption 2.1 that

\[ I_3 \leq \int_0^s \|(-A)^{-\left(\frac{\gamma}{2} \wedge 1\right)}(S(t-s) - \text{Id}_H)\|_{\mathcal{L}(H)} \| S(s-r)F^N(X^N(r)) \|_{L^p(\Omega; H^p)} \, dr \]

\[ \leq C(t-s)^{\frac{\gamma}{2} \wedge 1} \left( \int_0^s K_{F_S} \, dr \right) \left( 1 + \sup_{t \in [0,T]} \| X^N(t) \|_{L^p(\Omega; H^p)} \right) \leq C(t-s)^{\frac{\gamma}{2} \wedge 1}, \]

\[ I_5 \leq C \|(-A)^{-\left(\frac{\gamma}{2} \wedge 1\right)}(S(t-s) - \text{Id}_H)\|_{\mathcal{L}(H)} \left( \int_0^s \| S(s-r)G^N(X^N(r)) \|_{L^p(\Omega; \mathcal{L}_2)}^2 \, dr \right)^{\frac{1}{2}} \]

\[ \leq C(t-s)^{\frac{\gamma}{2} \wedge 1} \left( \int_0^s K_{G^N}^2 \, dr \right)^{\frac{1}{2}} \left( 1 + \sup_{t \in [0,T]} \| X^N(t) \|_{L^p(\Omega; H^p)} \right) \leq C(t-s)^{\frac{\gamma}{2} \wedge 1}. \]

Combining the above five estimations, we get that Assumption 3.2 holds with \(\gamma = \frac{6}{2} \wedge \gamma_1 \wedge \gamma_2\).

In the following, we give related rates for two types of frequently used noises.

### 4.1. Equation Driven by Space-time White Noise

In this case, \(Q = \text{Id}_H\) and Eq. \((\text{SHE})\) is driven by the cylindrical Wiener process. Recall that the generalized Lipschitz condition of \(G\) in Assumption 2.1 is verified in [HL, Section 2.3] in one-dimension with

\[ K_G(t) = \left( C \sum_{j=1}^{\infty} e^{-2\lambda_j t} \right)^{\frac{1}{2}} \leq Ct^{-\frac{1}{2}}, \quad t \in (0, T], \]

which is square integrable in \((0, T]\). Moreover, Eq. \((\text{SHE})\) driven by space-time white noise possesses a mild solution if and only if \(d = 1\). Thus we focus on the one dimensional case in this part.
Next we verify the growth condition of $G$. The uniform $L^\infty$-boundedness of $\{e_j\}_{j \in \mathbb{N}}$, and Lipschitz continuity of $g$ yield that

$$\|S(t)G(z)\|_{L^2}\leq C\sum_{j=1}^{\infty} \lambda_j^\theta e^{-2\lambda_j t} \|G(z)e_j\|^2 \leq C\sum_{j=1}^{\infty} \lambda_j^\theta e^{-2\lambda_j t}(1 + \|z\|)^2 \leq C\sum_{j=1}^{\infty} \lambda_j^\theta e^{-2\lambda_j t}(1 + \|z\|)^2.$$

Then we obtain the growth condition of $G$ in Assumption 2.1 with

$$K_{G_0}(t) = \left( C\sum_{j=1}^{\infty} \lambda_j^\theta e^{-2\lambda_j t} \right)^{\frac{1}{2}}, \quad t \in (0, T],$$

which is square integrable in $(0, T]$ for any $\theta \in (0, 1/2)$. By Lemma 2.1, Eq. (SEE) possesses a unique mild solution $\{X(t) : t \in [0, T]\}$ in $C([0, T]; L^p(\Omega; H^\theta))$ as soon as $X_0 \in L^p(\Omega; H^\theta)$ for some $p \geq 2$ and $\theta \in (0, 1/2)$.

To obtain the optimal convergence rate of the fully discrete scheme (EI), it remains to give the maximal values of $r_0$ in (2.5), $r_F$, $r_G$ in Assumption 2.3, $\eta_F$, $\eta_G$ in Assumption 3.1 and $\gamma$ in Assumption 3.2.

**Lemma 4.2.** Let $p \geq 2$ and $\theta \in (0, 1/2]$. Assume that $X_0 : \Omega \to H^\theta$ is strongly $\mathcal{F}_0$-measurable such that $X_0 \in L^p(\Omega; H^\theta)$, $f, f, g : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions and $Q = \text{Id}_H$. Then (2.5) and Assumptions 2.3, 3.1-3.2 hold for Eq. (SHE) with

$$r_0 = \theta, \quad r_F = 1 - \epsilon, \quad r_G = \frac{1}{2}, \quad \eta_F = \frac{1}{2} - \epsilon, \quad \eta_G = \frac{1}{4}, \quad \gamma = \frac{\theta}{2},$$

where $\epsilon$ is a sufficiently small positive number.

**Proof.** To show (2.5) and Assumption 2.3, we mainly use the standard estimation

$$\|(P_N - \text{Id}_H)z\|_\nu \leq \lambda_{N+1}^{-\frac{\nu-\theta_2}{2}} \|z\|_{\theta_2}, \quad z \in H^{\theta_2}, \quad \nu, \theta_2 \in \mathbb{R}.$$

It follows that

$$\|S_N(t)X_0^N - S(t)X_0\| \leq C\|(P_N - \text{Id}_H)X_0\| \leq C\lambda_{N+1}^{-\frac{\theta}{2}} \|X_0\|_{\theta},$$

i.e., the inequality (2.5) holds with $r_0 = \theta$.

The estimation (4.4) and (4.9) imply that

$$\|S^N(t)F^N(z) - S(t)F(z)\| = \|(P_N - \text{Id}_H)S(t)F(z)\| \leq C\lambda_{N+1}^{-\frac{\theta}{2}} \|S(t)F(z)\| \leq C\lambda_{N+1}^{-\frac{\theta}{2}} K_{F_\theta}(t)(1 + \|z\|).$$

Note that $K_{F_\theta}$ defined by (4.4) is integrable in $(0, T]$ for any $\theta \in (0, 1)$, then one can take $r_F$ to be any positive number less than 1. Analogously, for the diffusion term, we have

$$\|S_N(t)G^N(z) - S(t)G(z)\|^2 = \sum_{j=N+1}^{\infty} e^{-2\lambda_j t} \|G(z)e_j\|^2 \leq C\sum_{j=N+1}^{\infty} e^{-2\lambda_j t}(1 + \|z\|)^2.$$
Since
\[ \int_0^T \sum_{j=0}^{\infty} e^{-2\lambda_j t} dt \leq C \sum_{j=0}^{\infty} j^{-2} \leq CN^{-1}, \]
onel one can take \( r_G = 1/2 \). In particular, Assumption 2.3 holds with \( r_F = 1 - \epsilon \) for any \( \epsilon \in (0, 1] \) and \( r_G = 1/2 \).

For Assumption 3.1, let \( t_i < t \leq t_{i+1} \) for some \( i \in \mathbb{Z}_{K-1} \). It follows from (2.1) and (4.4) that
\[ \| (S^N(t) - S^N(t_{i+1})) F^N(z) \| = \| P^N (\text{Id}_H - S(t_{i+1} - t)) S(t) F(z) \| \]
\[ \leq C (t_{i+1} - t)^{\frac{d}{2}} \| S(t) F(z) \|_\theta \leq C \tau^{\frac{d}{2}} K_F (t) (1 + \| z \|) \]
for any \( \tilde{\theta} \in (0, 1) \) and thus \( \eta_F = 1/2 - \epsilon \). Similarly,
\[ \| (S^N(t) - S^N(t_{i+1})) G^N(z) \|^2_{L^0_2} = \sum_{j=1}^N (e^{-\lambda_j t} - e^{-\lambda_j t_{i+1}})^2 \| G(z) e_j \|^2 \]
\[ \leq C \sum_{j=1}^{\infty} (e^{-\lambda_j t} - e^{-\lambda_j t_{i+1}})^2 (1 + \| z \|)^2. \]

Since
\[ \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \sum_{j=1}^{\infty} (e^{-\lambda_j t} - e^{-\lambda_j t_{i+1}})^2 dt \leq C r^{1/2}, \]
we have \( \eta_G = 1/4 \). Thus Assumption 3.1 holds with \( \eta_F = 1/2 - \epsilon \) for any \( \epsilon \in (0, 1/2] \) and \( \eta_G = 1/4 \).

Finally, we prove the uniform \( \frac{d}{2} \)-Hölder regularity of \( \{ X^N \} \) in \( L^p(\Omega; H) \) as required in Assumption 3.2:
\[ \sup_{N \in \mathbb{N}^+} \sup_{0 \leq s, t \leq T} \| X^N(t) - X^N(s) \|_{L^p(\Omega; H)} \quad \leq \quad C. \]

Note that in this case \( K_F \) and \( K_G \) are given by (4.3) and (4.6), respectively. Together with representations (4.4) and (4.7), arguments in the proof of Lemma 4.1 make sense for \( \gamma_1 = 1/2 \) and \( \gamma_2 = 1/4 \) in condition (4.5), and then it suffices to refine the term \( I_5 \) there.

By the definition of the \( L^0_2 \)-norm and the Lipschitz continuity (and thus linear growth) of \( g \), we obtain
\[ I_5 \leq C \left( \int_0^t \| (S(t-r) - S(s-r)) G(X^N(r)) \|^2_{L^p(\Omega; L^0_2)} dr \right)^{\frac{1}{2}} \]
\[ \leq C \left( \int_0^t \sum_{j=1}^{\infty} (e^{-\lambda_j (t-r)} - e^{-\lambda_j (s-r)})^2 \| g(X^N(r)) e_j \|^2_{L^p(\Omega; H)} dr \right)^{\frac{1}{2}} \]
\[ \leq C \left( \int_0^t \sum_{j=1}^{\infty} (e^{-\lambda_j (t-r)} - e^{-\lambda_j (s-r)})^2 dr \right)^{\frac{1}{2}} \]
\[ \times \left( 1 + \sup_{t \in [0, T]} \| X^N(t) \|_{L^p(\Omega; H)} \right) \leq C (t - s)^{\frac{d}{2}}. \]

Thus we conclude (4.10) with \( \gamma = \frac{d}{2} \). \( \square \)
Remark 4.1. If we use the approach in Lemma 4.1 to deal with \( I_5 \), we can only derive the uniform \( (\frac{1}{4} - \epsilon) \wedge \frac{\theta}{2} \)-Hölder continuity of \( \{ X^N \} \) in \( L^p(\Omega; H) \) with an infinitesimal factor \( \epsilon \).

Combining Lemma 4.2 and Theorem 3.1, we derive our main result on sharp strong convergence rate for scheme (EI) applied to Eq. (SHE) driven by space-time white noise.

Theorem 4.1. Let \( p \geq 2 \) and \( \beta \in (0, \frac{1}{2}] \). Assume that \( X_0 : \Omega \to H^\beta \) is strongly \( \mathcal{F}_0 \)-measurable such that \( X_0 \in L^p(\Omega; H^\beta) \), \( f, \tilde{f}, g : \mathbb{R} \to \mathbb{R} \) are Lipschitz continuous functions and \( Q = \text{Id}_H \) in Eq. (SHE). Then there exists a constant \( C \) such that

\[
\max_{k \in \mathbb{Z}_N} \| X(t_k) - X^N_k \|_{L^p(\Omega; H)} \leq C(N^{-\beta} + \tau^{\frac{\theta}{2}}).
\]

Remark 4.2. (1) Compared with the optimal spatial regularity for the solution of Eq. (SHE), the strong convergence order of the spectral Galerkin approximation is superconvergent. In the case \( \beta = \frac{1}{2} \), our order is \( \frac{1}{2} \).

Moreover, the convergence rate in temporal direction is consistent with the optimal Hölder regularity under the \( L^p(\Omega; H) \)-norm in Lemma 4.2 and thus sharp.

(2) To our best knowledge, Theorem 4.1 is the first result on strong convergence rates of numerical approximations applied to Eq. (SHE) with gradient term driven by space-time white noise. In [ACQS, Gyö99, Wal05] the authors derived strong convergence rate for numerical approximations of Eq. (SHE) with regular initial datum and without gradient term by using the Green’s function framework. Note that it is rather difficult to derive a sharp strong convergence rate for numerical approximations of SPDEs with gradient term via estimating the error between continuous and discrete Green’s functions.

4.2. Equation Driven by General Q-Wiener Process. As remarked in the previous subsection, Eq. (SHE) driven by space-time white noise possesses a mild solution if and only if \( d = 1 \). In this part, we assume that the diffusion operator \( G \) and the covariance operator \( Q \) of the proposed infinite dimensional Wiener process are as general as possible such that Assumption 2.1 holds for some \( \theta > 0 \).

In this case, one can handle both one-dimensional SPDEs driven by noises rougher than space-time white noise and higher dimensional SPDEs driven by colored noises, and in particular, trace class noises.

Examples such that Assumption 2.1 holds in the case of trace class noise are given in Remark 4.4. Under this assumption, the inequality (2.5) holds with \( r_0 = \theta/d \) by Weyl’s law (4.1) and similar argument in the previous subsection. For the drift and diffusion terms, we have

\[
\| S^N(t)F^N(z) - S(t)F(z) \| \leq \lambda_{N+1}^{-\frac{\theta}{2}} K_{F_\theta}(t)(1 + \| z \|_\theta) := R_{F_\theta}(t)(1 + \| z \|_\theta),
\]

and

\[
\| S^N(t)G^N(z) - S(t)G(z) \|_{L^2} \leq \lambda_{N+1}^{-\frac{\theta}{2}} K_{G_\theta}(t)(1 + \| z \|_\theta) := R_{G_\theta}(t)(1 + \| z \|_\theta),
\]

which shows \( r_F = r_G = \theta/d \) in Assumption 2.3.
To obtain the rate $\eta_F$ and $\eta_G$ in Assumption 3.1, let us fix $z \in H^\theta$ and $t \in (t_i, t_{i+1})$ for some $i \in \mathbb{Z}_{K-1}$. It follows from (2.1) that

$$\|(S^N(t) - S^N(t_{i+1}))F^N(z)\|$$

$$\leq \|(-A)^{-\frac{\nu}{2}}(\text{Id}_H - S(t_{i+1} - t))(-A)^{\frac{\nu}{2}}S(t)F(z)\|$$

$$\leq C\tau^\frac{\nu}{2}K_F(t)(1 + \|z\|) := R_{F\nu,\tau}(t)(1 + \|z\|),$$

and

$$\|(S^N(t) - S^N(t_{i+1}))G^N(z)\|_{\mathcal{L}^2_\nu}$$

$$\leq \|(-A)^{-\frac{\nu}{2}}(\text{Id}_H - S(t_{i+1} - t))(-A)^{\frac{\nu}{2}}S(t)G(z)\|_{\mathcal{L}^2_\nu}$$

$$\leq C\tau^\frac{\nu}{2}K_G(t)(1 + \|z\|) := R_{G\nu,\tau}(t)(1 + \|z\|).$$

Therefore,

$$\sup_N \int_0^T R_{F\nu,\tau}(t)dt \leq C\tau^\frac{\nu}{2}, \quad \sup_N \int_0^T R_{G\nu,\tau}(t)dt \leq C\tau^\frac{\nu}{2},$$

which shows $\eta_F = \eta_G = \frac{\nu}{2} \wedge 1$.

Applying Theorem 3.1, we derive our main result on strong convergence rate for the scheme (EI) of Eq. (SHE) driven by general $\mathbf{Q}$-Wiener processes.

**Theorem 4.2.** Let $p \geq 2$, $\theta > 0$, $0 < \gamma \leq \frac{\theta}{2}$, Assumptions 2.1 and 3.2 hold. Assume that $X_0 : \Omega \rightarrow H^\theta$ is strongly $\mathcal{F}_0$-measurable and $X_0 \in L^p(\Omega; H^\theta)$. Then there exists a constant $C$ such that

$$\max_{k \in \mathbb{Z}_K} \|X(t_k) - X_k^N\|_{L^p(\Omega; H)} \leq C(N^{-\frac{\nu}{2}} + \tau^\frac{\nu}{2} \wedge 1).$$

**Remark 4.3.** An example of SPDE driven by a noise rougher than the space-time white noise is given by Eq. (5.1) with the $\mathbf{Q}$-Wiener process of the form (5.3) with $\nu \in (0,1/2)$ in Section 5. It is clear that Assumptions 2.1 and 3.2 hold by using previous approach.

**Remark 4.4.** Several examples in the case of trace class noises are given in [JR12, Section 4], under the classical conditions on $F$ and $G$:

$$\|F(x) - F(y)\| \leq C\|x - y\|, \quad \|G(x) - G(y)\| \leq C\|x - y\|,$$

$$\|F(x)\|_\nu \leq C(1 + \|z\|_\nu), \quad \|G(x)\|_{\mathcal{L}^2_\nu} \leq C(1 + \|z\|_\nu),$$

for some $\nu > 0$. In this case, Assumption 2.1 is satisfied with $\theta = \nu + 1 - \epsilon$ for any $\epsilon \in (0,1]$. Indeed, the uniform boundedness (2.1) of the semigroup leads to

$$\|S(t)F(z)\|_{\nu+1-\epsilon} \leq Ct^{-\frac{\nu+1-\epsilon}{2}} \|F(z)\|_\nu \leq Ct^{-\frac{\nu+1-\epsilon}{2}}(1 + \|z\|_{\nu+1-\epsilon}),$$

$$\|S(t)G(z)\|_{\mathcal{L}^2_{\nu+1-\epsilon}} \leq Ct^{-\frac{\nu+1-\epsilon}{2}} \|G(z)\|_{\mathcal{L}^2_\nu} \leq Ct^{-\frac{\nu+1-\epsilon}{2}}(1 + \|z\|_{\nu+1-\epsilon}),$$

for any $z \in H^{\nu+1-\epsilon}$. By Lemma 4.1 with $\gamma_1 = 1, \gamma_2 = 1/2$, we obtain that scheme (EI) applied to Eq. (SEE) with initial datum $X_0 \in H^\theta$, whose drift and diffusion operators satisfy (4.11), possesses the strong convergence rate $O(N^{-\theta/d} + \tau^{(\nu+1)/2})$. In addition, notice that there is a technique assumption $\nu \in (0,\theta/10)$ in [LT13, Theorem 2.8] for similar results.
5. Numerical Experiments

In this section, we give several numerical tests to verify our main result, Theorem 3.1, on convergence rate of the fully discrete scheme (EI). More precisely, we apply scheme (EI) to the following one-dimensional stochastic advection-diffusion-reaction equation (where we take \( f(v) = -v \), \( f(v) = -v/(1 + |v|) \) and \( g(v) = (1 + v)/8 \) for \( v \in \mathbb{R} \)):

\[
\begin{aligned}
dX(t, \xi) &= \left( \Delta X(t, \xi) - \nabla X(t, \xi) - \frac{X(t, \xi)}{1 + |X(t, \xi)|} \right) dt \\
&+ \frac{1 + X(t, \xi)}{8} dW(t, \xi),
\end{aligned}
\]

(5.1)

with homogeneous Dirichlet boundary condition, in the time-space domain \( (0, T] \times \mathcal{O} = (0, 1] \times (0, 1) \). We refer to [BCDS06] and references therein for relevant applications.

In order to illustrate the effect of regularity of the initial datum \( X_0 \) and covariance operator \( \mathbf{Q} \) on the strong convergence rate of the fully discrete scheme (EI), we take the initial datum and the \( \mathbf{Q} \)-Wiener process are of the forms

\[
X(0, \xi) = X_0(\xi) = \sum_{j=1}^{\infty} \frac{e_j(\xi)}{j^{\beta + 1/2}}, \quad \xi \in (0, 1),
\]

(5.2)

and

\[
W(t, \xi) = \sum_{j=1}^{\infty} \frac{e_j(\xi)}{j^{\nu - 1/2}} \beta_j(t), \quad (t, \xi) \in (0, 1] \times (0, 1),
\]

(5.3)

for some \( \beta > 0 \) and \( \nu > 0 \). Here \( \{\beta_j\}_{j \in \mathbb{N}_+} \) is a sequence of independent standard \( \mathcal{F}_t \)-Brownian motions and \( \{e_j(\cdot) = \sqrt{2} \sin(j \pi \cdot)\}_{j \in \mathbb{N}_+} \) is a sequence of eigenfunctions of the Laplacian operator \( \Delta \) which forms an orthonormal basis of \( L^2(0, 1) \). Then the solution of Eq. (5.1) with initial datum (5.2) and \( \mathbf{Q} \)-Wiener process (5.3) possess a unique mild solution in \( C([0, T]; L^p(\Omega; H^\theta)) \) for any \( p \geq 2 \) and \( \theta < \beta \wedge \nu \). In our numerical tests, we always take \( \beta = \nu \) for simplicity.

When \( \beta = 0.5 \), the proposed \( \mathbf{Q} \)-Wiener process is the \( L^2(0, 1) \)-valued cylindrical Wiener process, which corresponds to the space-time white noise. By Theorem 4.1, the spatial and temporal strong orders are 0.5 and 0.25, respectively. These mean-square convergence rates (i.e., \( p = 2 \)) are shown numerically in Figure 1.

In the next three tests, we take three \( \mathbf{Q} \)-Wiener processes with the first one weaker than the \( L^2(0, 1) \)-valued cylindrical Wiener process and the last two smoother than the \( L^2(0, 1) \)-valued cylindrical Wiener process. They correspond to (5.3) with \( \beta = 0.4, 1 \) and 1.6, respectively. By Theorem 4.2 and Remark 4.3, the proposed scheme (EI) possesses the sharp strong convergence rate \( O(N^{-0.4} + \tau^{0.2}) \), \( O(N^{-1} + \tau^{0.5}) \) and \( O(N^{-1.6} + \tau^{0.5}) \), respectively. The numerical results for these three cases are presented in Figures 2–4.

To simulate the ‘exact’ solution, we perform the fully discrete scheme by \( N = 2^9 - 1 \) for the dimension of spectral Galerkin approximation and \( \tau = 2^{-13} \) for the time step size of the exponential integrator, where the solution is obtained from the fast Fourier transform algorithm with space mesh size \( h = 2^{-9} \). The expectation is approximated from the average of 200 sample paths. The series of (5.3) is truncated as a finite summation up to \( 2^9 - 1 \) terms. Taking \( \tau = 2^i \) with \( i = 7, 8, 9, 10 \) and \( N = 2^9 - 1 \), we have the mean-square convergence rates in temporal direction
shown in the left pictures of Figure 1-4. Choosing four different spatial dimensions $N = 2^i - 1$ with $i = 4, 5, 6, 7$ and fixing $\tau = 2^{-13}$, we get the convergence rates in spatial direction, which are presented in the right pictures of Figure 1-4.

From these numerical experiments, it is clear that the spatial and temporal strong convergence orders of the scheme (EI) applied to Eq. (5.1) are $\beta$ and $(1 \wedge \beta)/2$, respectively. This confirms the theoretical result of Theorem 3.1.

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Figure 3. Temporal (left) and spatial (right) convergence rates with $\beta = 1$.

Figure 4. Temporal (left) and spatial (right) convergence rates with $\beta = 1.6$.

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