THE BRAIDED GROUP OF A SQUARE-FREE SOLUTION OF
THE YANG-BAXTER EQUATION AND ITS GROUP ALGEBRA

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Abstract. Set-theoretic solutions of the Yang–Baxter equation form a meeting-
ground of mathematical physics, algebra and combinatorics. Such a solution

\((X, r)\)

consists of a set \(X\) and a bijective map \(r : X \times X \to X \times X\) which satis-

fies the braid relations. In this work we study the braided group \(G = G(X, r)\)
of an involutive square-free solution \((X, r)\) of finite order \(n\) and cyclic index

\(p = p(X, r)\) and the group algebra \(k[G]\) over a field \(k\). We show that \(G\) contains

a \(G\)-invariant normal subgroup \(F_p\) of finite index \(p^n\), \(F_p\) is isomorphic to the
free abelian group of rank \(n\). We describe explicitly the quotient braided group

\(\tilde{G} = G/F_p\) of order \(p^n\) and show that \(X\) is embedded in \(\tilde{G}\). We prove that the

group algebra \(k[G]\) is a free left (resp. right) module of finite rank \(p^n\) over its

commutative subalgebra \(k[F_p]\) and give an explicit free basis. The center of

\(k[G]\) contains the subalgebra of symmetric polynomials in \(k[x_1^p, \ldots, x_n^p]\). Clas-
sical results on group rings imply that \(k[G]\) is a left (and right) Noetherian
domain of finite global dimension.

1. Introduction

Let \(V\) be a vector space over a field \(k\). It is well-known that the “Yang–Baxter
equation” on a linear map \(R : V \otimes V \to V \otimes V\), the equation

\[ R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23} \]

(where \(R_{i,j}\) denotes \(R\) acting in the \(i, j\) place in \(V \otimes V \otimes V\)), gives rise to a linear
representation of the braid group on tensor powers of \(V\). When \(R^2 = \text{id}\) one
says that the solution is involutive, and in this case one has a representation of
the symmetric group on tensor powers. A particularly nice class of solutions is
provided by set-theoretic solutions, where \(X\) is a set and \(r : X \times X \to X \times X\) obeys
similar relations on \(X \times X \times X\). [7]. Of course, each such solution extends linearly
to \(V = kX\) with matrices in this natural basis having only entries from 0,1 and
many other nice properties. During the last two decade the study of set-theoretic
solutions and related structures notably intensified, a relevant selection of works for
the interested reader is [19, 8, 4, 6, 23, 14, 26, 28, 18, 15, 11, 10, 16, 12, 11, 25],
and references therein. We shall use the terminology, notation and some results from
[14, 18, 15, 17, 16].

Definition 1.1. Let \(X\) be a nonempty set (not necessarily finite) and let \(r : \)
\(X \times X \to X \times X\) be a bijective map. We use notation \((X, r)\) and refer to it
as a quadratic set. The image of \((x, y)\) under \(r\) is presented as \(r(x, y) = (x^y, y^x)\).

Date: February 6, 2019.
The author was partially supported by the Max Planck Institute for Mathematics in the Sciences (MiS) in Leipzig.
This formula defines a “left action” \( \mathcal{L} : X \times X \rightarrow X \), and a “right action” \( \mathcal{R} : X \times X \rightarrow X \), on \( X \) as:

\[
\mathcal{L}_x(y) = x^y, \quad \mathcal{R}_y(x) = x^y, \quad \forall x, y \in X.
\]

(i) \( r \) is nondegenerate, if the maps \( \mathcal{L}_x \) and \( \mathcal{R}_x \) are bijective, \( \forall x \in X \). (ii) \( r \) is involutive if \( r^2 = \text{id}_{X \times X} \). (iii) \( (X, r) \) is called square-free if \( r(x, x) = (x, x) \), \( \forall x \in X \). (iv) \( r \) is a set-theoretic solution of the Yang–Baxter equation (YBE) if the braid relation

\[
r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23}
\]

holds in \( X \times X \times X \), where \( r_{12} = r \times \text{id}_X \), and \( r_{23} = \text{id}_X \times r \). In this case, see [8] (X, r) is called a braided set. (v) A braided set \( (X, r) \) with \( r \) involutive is called a symmetric set.

**Convention 1.2.** In this paper "a solution" means a nondegenerate symmetric set \( (X, r) \), where \( X \) is a set of arbitrary cardinality. We shall also refer to it as "a symmetric set", keeping the convention that we consider only nondegenerate symmetric sets.

**Definition 1.3.** To each quadratic set \( (X, r) \) we associate canonical algebraic objects generated by \( X \) and with quadratic defining relations \( \mathcal{R} = \mathcal{R}(X, r) \), given by

\[
xy = zt \in \mathcal{R} \text{ if } r(x, y) = (z, t) \text{ and } (x, y) \neq (z, t).
\]

(i) The monoid \( S = S(X, r) = (X; \mathcal{R}) \), with a set of generators \( X \) and a set of defining relations \( \mathcal{R} \), is called the monoid associated with \( (X, r) \). (ii) The group \( G = G(X, r) \) associated with \( (X, r) \) is defined as \( G = G(X, r) = g_{\mathcal{R}}(X; \mathcal{R}) \). Each element \( a \in G \) can be presented as a monomial \( a = \zeta_1\zeta_2\cdots\zeta_m, \quad \zeta_i \in X \cup X^{-1} \). We shall consider a reduced form of \( a \), that is a presentation with minimal length \( m \). By convention, length of \( a \), denoted by \( |a| \) means the length of a reduced form of \( a \). (iii) For arbitrary fixed field \( k \), the \( k \)-algebra associated with \( (X, r) \) is defined as \( A(k, X, r) = kS(X; \mathcal{R}) \). \( A(k, X, r) \) is a quadratic algebra isomorphic to the monoidal algebra \( kS(X, r) \). (iv) Furthermore, to each nondegenerate braided set \( (X, r) \) we also associate a permutation group, denoted \( G = G(X, r) \), see Definition 1.3. If \( (X, r) \) is a solution then \( G = G(X, r) \) has a canonical structure of an involutive braided group, see Facts 2.2. In the paper we shall refer to \( G \) as "the (involutive) braided group associated to \( (X, r) \)”, see Definition 2.3.

If \( (X, r) \) is a braided set, and \( G = G(X, r) \), then the assignment \( x \mapsto \mathcal{L}_x \) for \( x \in X \) extends canonically to a group homomorphism \( \mathcal{L} : G \rightarrow \text{Sym}(X) \), which defines the canonical left action of \( G \) on the set \( X \). Analogously, there is a canonical right action of \( G \) on \( X \), [8].

**Definition 1.4.** (i) The image \( \mathcal{L}(G(X, r)) \) is a subgroup of \( \text{Sym}(X) \), that is, a permutation group. We denote it by \( \mathcal{G} = \mathcal{G}(X, r) \), and call it the permutation group (of left actions) of \( (X, r) \), [13] [17]. \( \mathcal{G} \) is generated by the set \( \{ \mathcal{L}_x \mid x \in X \} \).

(ii) If \( (X, r) \) is a finite solution then the least common multiple of all orders of permutations \( \mathcal{L}_x \in \mathcal{G}, x \in X \), is called the cyclic degree of \( (X, r) \) and denoted by \( p = p(X, r) \), see [13], Def 3.17. One has \( (\mathcal{L}_x)^p = \text{id}_X \) for all \( x \in X \), and \( p \) is the minimal integer with this property.
Example 1.5. For arbitrary set $X$, $|X| \geq 2$, denote by $\tau_X = \tau$ the flip map $\tau(x, y) = (y, x)$ for all $x, y \in X$. Then $(X, \tau)$ is a square-free solution called the trivial solution on $X$. It is clear that an involutive quadratic set $(X, r)$ is the trivial solution if and only if $y^2 = y$, for all $x, y \in X$, or equivalently $L_x = \text{id}_X$ for all $x \in X$. In this case $S(X, r)$ is the free abelian monoid generated by $X$, $G(X, r)$ is the free abelian group, $A(k, X, r)$ is the algebra of commutative polynomials in $X$, and $G(X, r) = \{\text{id}_X\}$ is the trivial group.

The results of [19] show the close relation among various mathematical notions and theories: the set-theoretic solutions of the Yang-Baxter equation, the semigroups of I-type, the skew abelian polynomial rings, introduced by the author in [13], and the theory of Bieberbach groups. We shall use the following results.

Theorem 1.6. (I) [19]. Every $X$-generated binomial skew-polynomial ring $A = A(k, X, r) = k[X | \mathcal{R}]$ defines a square-free solution $(X, r)$. In this case the map $r$ is defined canonically via the set of quadratic defining relations $\mathcal{R}$ of $A$. Moreover, $A$ is an Artin-Schelter regular Noetherian domain of global dimension $n$.

(II) [14], [20]. Conversely, suppose $(X, r)$ is a finite square-free solution then the set $X$ can be ordered $X = \{x_1 < x_2 < \cdots < x_n\}$, so that the $k$-algebra $A = A(k, X, r) = k[X | \mathcal{R}]$ over arbitrary field $k$, is a skew polynomial ring with binomial relations in the sense of [13], or shortly a binomial skew-polynomial rings (see Def. 5.1).

The main results of the paper are Theorems A, B, and C which are proven under the following assumptions, notation and convention.

Convention-Notation 1.7. $(X, r)$ is a square-free solution of finite order $|X| = n$, and with a cyclic degree $p = p(X, r)$, see Def[1.3] $k$ is a field. We fix an enumeration $X = \{x_1 < x_2 < \cdots < x_n\}$ on $X$, so that the associated Yang-Baxter $k$-algebra $A = A(k, X, r) = k[X | \mathcal{R}]$ is a skew polynomial ring with binomial relations, $N$ is the normal $k$-basis of $A$, see Definition 3.1 and Theorem 1.6 $G = G(X, r)$, $G = G(X, r)$, and $S = S(X, r)$ are the associated groups and monoid, see Def[1.3] $(G, r_G)$, and $(G, r_S)$ are the corresponding involutive braided groups (see Facts 2.2).

We introduce the sets

\begin{align*}
Y &:= \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid 0 \leq \alpha_i \leq p - 1\} \subset N; \\
X_p &:= \{x_1^{p}, x_2^{p}, \cdots, x_n^{p}\} \subset N; \\
N_p &:= [X_p] \subseteq S \text{ is the submonoid of } S \text{ generated by } X_p. \\
F_p &:= \{g(x_1^{p}, \cdots, x_n^{p}) \mid G \text{ is the subgroup of } G \text{ generated by } X_p. \\
A_p &:= kN_p \text{ is the semigroup algebra of } N_p, \text{ equivalently,} \\
A_p &\text{ is the subalgebra of } A \text{ generated by } X_p.
\end{align*}

Each element $W \in N_p$ will be considered in its normal form, so we identify (as sets) $N_p$ and $N_p := N_p \cap N$.

Theorem A. Let $(X, r)$ be a square-free solution of finite order $|X| = n$ and cyclic degree $p = p(X, r)$, let $G = G(X, r)$.

(1) Every element $u \in G$ has unique presentation as $u = yW$, where $y \in Y$, $W \in F_p$. 

(2) In particular, for every pair \( y, z \in Y \) there exist uniquely determined \( t \in Y \), and \( W \in \mathcal{F}_p \), such that the equality
\[
y.z = t.W \text{ holds in } G.
\]
(1.2)

(3) \( G \) splits as a union of \( p^n \) disjoint subsets
\[
G = \bigcup_{y \in Y} y\mathcal{F}_p, \quad \text{where } (y\mathcal{F}_p) \cap (t\mathcal{F}_p) = \emptyset, \quad \forall \ y \neq t, \ y, t \in Y.
\]

(1.3)

**Theorem B.** Let \( (X, r) \) be a square-free solution of finite order \( |X| = n \) and cyclic degree \( p = p(X, r) \), let \( (G, \tau_G) \) be the associated braided group, where \( G = G(X, r) \).

1. \( \mathcal{F}_p \) is an ideal of \((G, \tau_G)\) and there is a chain of ideals \( \mathcal{F}_p \triangleleft \Gamma \triangleleft G \), in \( G \).
2. The quotient group \( \tilde{G} = G/\mathcal{F}_p \) is an involutive braided group of finite order \( p^n \). More precisely, the canonical projection
\[
\pi : G \rightarrow \tilde{G} = G/\mathcal{F}_p, \quad g = yW \mapsto \tilde{y} = \tilde{g}
\]
induces the structure of an involutive braided group on the finite set
\[
\tilde{G} = \tilde{Y} := \{\tilde{y} \mid y \in Y\},
\]
such that the induced operation \( \cdot \triangleq \) on \( \tilde{Y} \) is well-defined via \( (\tilde{L}, \tilde{Z}) \) as:
\[
\tilde{y} \cdot \tilde{z} := \tilde{t}, \quad \text{where } y.z = t.W, \text{ holds in } G, \ y, z, t \in Y, \ W \in \mathcal{F}_p.
\]
(1.4)

The restriction \( \pi|_Y : Y \rightarrow \tilde{Y} = \tilde{G} \) is a bijective map of sets of order \( p^n \).

(3) The map \( X \rightarrow \tilde{X} \subseteq \tilde{G} \) is an embedding of \( X \) into the braided group \( \tilde{G} \), we shall identify the sets \( X \) and \( \tilde{X} \). Thus, \( X \) is a set of generators of the braided group \( \tilde{G} \) of order \( p^n \).

(4) There is a canonical epimorphism of braided groups \( \tilde{G} \rightarrow G \simeq G/\Gamma \), so the order \( |G| \) divides \( p^n \). Moreover, if \( p \) is a prime number, then \( \tilde{G} \) is a \( p \)-group.

**Theorem C.** Let \( (X, r) \) be a square-free solution of finite order \( |X| = n \) and cyclic degree \( p = p(X, r) \), \( G = G(X, r) \). Let \( k[G] \) and \( k[\mathcal{F}_p] \) be the corresponding group algebras over a field \( k \). The following conditions are in force.

1. \( k[\mathcal{F}_p] \) is a commutative Noetherian domain, \( k[G] \) is a free left (resp. right) module of finite rank \( p^n \) over the algebra \( k[\mathcal{F}_p] \) with a free basis the set \( Y \).
2. Let \( \mathcal{S} \) be the subalgebra of \( k[G] \) (and of \( A \)) consisting of all symmetric polynomials \( f(x_1^n, \ldots, x_p^n) \in A_p \) in \( n \) variables \( x_1^n, \ldots, x_p^n \). Then \( \mathcal{S} \) is an \( n \)-generated subalgebra contained in the center of \( k[G] \) (and of \( A \)).
3. \( k[G] \) is a left and a right Noetherian domain. Moreover, \( k[G] \) has finite global dimension, whenever \( k \) is a field of characteristic 0.

The paper is organized as follows. In Section 2 we recall some basic definitions and results which will be used throughout the paper. In Sec 3.1 we give a short introduction to square-free solutions \( (X, r) \) and their algebraic objects. In particular, we present some useful formulae. In Section 3.2, we study the braided group \( G(X, r) \) and prove Theorems A and B. The group ring \( k[G] \) over a field \( k \) is studied in Section 3.3, where we prove theorem C.
2. Preliminaries on braided groups

In this section \((X, r)\) denotes a nondegenerate braided set, where the set \(X\) has arbitrary cardinality. Lu, Yan and Zhu \cite{23} proposed a general way of constructing set-theoretical solutions of the Yang-Baxter equation using braiding operators on groups, or essentially, the matched pairs of groups. In his survey Takeuchi \cite{28} gave an introduction to the ESS-LYZ theory, reviewing the main results in \cite{23, 8}, from a matched pair of groups point of view, among them a good way to think about the properties of the group \(G(X, r)\) universally generated from \((X, r)\). In particular, it is known that the group \(G(X, r)\) is itself a braided set in an induced manner.

**Definition 2.1.** \cite{23} A braided group is a pair \((G, \sigma)\), where \(G\) is a group and

\[
\sigma : G \times G \rightarrow G \times G, \quad \sigma(a, u) = (\sigma^1 a, a^u)
\]

is a bijective map, such that the left and the right actions induced by \(\sigma\) via (2.1) satisfy the following conditions for all \(a, b, u, v \in G\):

- **ML0**: \(a1 = 1, \ 1u = u\),
- **ML1**: \(abu = a(bu)\),
- **ML2**: \(a(uv) = (a^u)^v\),
- **MR0**: \(1^a = 1, \ a^1 = a\),
- **MR1**: \(a^w = (a^v)^r\),
- **MR2**: \((ab)^u = (a^bu)(b^u)\),

and the compatibility condition **M3**:

\[
M3 : \quad uv = (u^v)(av), \quad \text{an equality in } G.
\]

If the map \(\sigma\) is involutive \((\sigma^2 = id_{G\times G})\) then \((G, \sigma)\) is an involutive braided group, also called “a symmetric group”, see \cite{28}, and \cite{16}. The notion of a “symmetric group”, was suggested by Takeuchi, as a group analogue of a symmetric set \cite{28}.

**Facts 2.2.** (1) \cite{23} If \((G, \sigma)\) is a braided group, then \(\sigma\) satisfies the braid relations and is nondegenerate. So \((G, \sigma)\) forms a nondegenerate braided set in the sense of \cite{8}, \((G, \sigma)\) is a symmetric set, whenever \(\sigma\) is involutive.

Assume that \((X, r)\) is a nondegenerate braided set, \(G = G(X, r)\) and \(S = S(X, r)\).

(2) \cite{23} There is unique braiding operator \(r_G : G \times G \rightarrow G \times G\), such that the restriction of \(r_G\) on \(X \times X\) is exactly the map \(r\). Furthermore, \((r_G)^2 = id_{G\times G}\) iff \(r^2 = id_{X \times X}\), so \((G, r_G)\) is a symmetric group iff \((X, r)\) is a symmetric set.

(3) \cite{18} There is unique braiding operator \(r_S : S \times S \rightarrow S \times S\), such that the restriction of \(r_S\) on \(X \times X\) is exactly the map \(r\), so \((S, r_S)\) is a braided monoid, and, in particular, a braided set.

**Definition 2.3.** We shall refer to the group \((G, r_G)\) as ”the (involutive) braided group associated to \((X, r)\)” or ”the symmetric group associated to \((X, r)\)”, see \cite{16}.

**Remark 2.4.** It is proven in \cite{19} that if \((X, r)\) is a finite symmetric set then the monoid \(S = (X, r)\) is of I-type, satisfies cancellation law and the Ore conditions. \(X\) is embedded in \(S\) and \(S\) is embedded in \(G = G(X, r)\) which is its group of quotients of \(S\). By \cite{19}, Theor. 1.4, \(G\) is a Bieberbach group, or in other words, \(G\) is finitely generated, torsion-free and abelian-by-finite. \cite{3}. Moreover, \(G(X, r)\) is a solvable group, \cite{8}, see also \cite{16}, Remark 3.10. Valuable systematic information on the properties of \(S(X, r)\) and \(G(X, r)\) can be found in \cite{20}.

Let \((X, r)\) be a symmetric set, with \(G = G(X, r)\), and let \(Y \subset X\) be a nonempty subset of \(X\). \(Y\) is \(r\)-invariant if \(r(Y \times Y) \subseteq Y \times Y\). In this case \(r\) induces a solution
(Y, r_Y), where r_Y is the restriction r_Y \circ Y \circ Y. (Y, r_Y) is called the induced solution (on Y). The set Y is a (left) G-invariant subset of X, if it is invariant under the left action of G. Right G-invariant subsets are defined analogously. Note that Y is left G-invariant iff it is right G-invariant, so we shall refer to it simply as a G-invariant subset. Each G-invariant subset Y of X is also r-invariant. Given a symmetric group (G, r) one defines G-invariant subsets of G, and r-invariant subsets of G analogously.

Let (G, r) be a symmetric group (i.e. an involutive braided group), and let Γ = Γ_l be the kernel of the left action of G upon itself. The subgroup Γ is called the socle of (G, r). It satisfies the following conditions: (i) Γ is an abelian normal subgroup of G, (ii) Γ is invariant with respect to the left and the right actions of G upon itself, in other words Γ is an ideal of the braided group (G, r), see Definition [16]. In particular, Γ is r-invariant. Note that in a symmetric set (X, r) one has L_x = L_y iff R_x = R_y, therefore Γ coincides with the kernel Γ_r of the right action of G:

(2.2) \[ Γ = Γ_l = \{ a \in G \mid a^u = u, \forall u \in G \} = \{ a \in G \mid u^a = u, \forall u \in G \} = Γ_r. \]

Definition 2.5. [16] Let (G, r) be an involutive braided group. A subgroup H < G is an ideal of G if H is a normal subgroup of G which is G-invariant, that is \( GH := \{ gh \mid g \in G, h \in H \} \subseteq H \) (or, equivalently, \( H^G := \{ h^g \mid g \in G, h \in H \} \subseteq H \)).

Recall that if H is an ideal of the involutive braided group (G, r), then the quotient group \( G/H \) has also a canonical structure of an involutive braided group \((G/H, r_G)\) induced from \((G, r)\), [16].

Remark 2.6. It is known that when (X, r) is a solution with braided group (G, r_G), where \( G = G(X, r) \), there is a group isomorphism \( G/Γ \simeq G = G(X, r) \), moreover, the group \( G(X, r) \) is finite, whenever X is a finite set, [27]. In this case Γ is a normal abelian subgroup of finite index in G, so the group G is abelian-by-finite. Furthermore, see [28] [16], the braided structure on \((G, r_G)\) induces the structure of a braided group on the quotient \( G/Γ \simeq G = G(X, r) \), we use notation \((G, r_G)\). The map \( r_G \) is involutive, since r is involutive, so \((G, r_G)\) is also a symmetric group.

3. The symmetric group \( G(X, r) \) of a finite square-free symmetric set \( (X, r) \) and its group algebra \( k[G] \)

In this section we prove the main results of the paper, Theorems A, B and C. From now on till the end of the paper we shall work assuming Convention-Notation [1.7].

3.1. Square-free solutions and their algebraic objects. Especially interesting with their symmetries are solutions satisfying additional combinatorial properties, such as square-free solutions, or solutions with condition lri, or with cyclic conditions (defined and studied in [13] [14] [15]). The finite square-free solutions \((X, r)\) are closely related with a special class of quadratic PBW algebras called binomial skew polynomial ring, see Theorem [1.6].

Definition 3.1. [13] A binomial skew polynomial ring is an n-generated quadratic algebra \( A = k(x_1, \ldots, x_n)/(R) \) with precisely \( \binom{n}{2} \) defining relations \( R = \{ x_i x_j - c_{ij} x_j x_i \mid 1 \leq i < j \leq n \} \),
such that: (a) For every pair $i, j, 1 \leq i < j \leq n$, the relation $x_jx_i - c_{ij}x_i x_j \in \mathbb{R}$, satisfies $c_{ij} \in k^*$, $j > i'$, $i' < j'$; (b) Every ordered monomial $x_ix_j$, with $1 \leq i < j \leq n$ occurs in the right hand side of some relation in $\mathbb{R}$; (c) $\mathbb{R}$ is the reduced Gröbner basis of the two-sided ideal $(\mathbb{R})$, with respect to the degree-lexicographic order $< \mathit{on} (X)$, induced by $x_1 < x_2 < \cdots < x_n$, or equivalently the ambiguities $x_kx_jx_i$, with $k > j > i$ do not give rise to new relations in $A$. By [2] condition (c) may be rephrased by saying that (c') $A = k \langle x_1, \cdots, x_n \rangle / (\mathbb{R})$ is a PBW algebra with a $k$-basis the set of ordered monomials:

$$\mathcal{N} = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_i \geq 0, 1 \leq i \leq n\}.\tag{3.1}$$

Recall from Gröbner bases theory that $\mathcal{N}$ is the set of normal monomials, and each monomial $u \in S = S(X, r)$ has unique normal form $u_0 \in \mathcal{N}$, [2] [13]. The monoid $S$ (considered as a set) can be identified with the set $\mathcal{N}$.

Algebraic and homological properties of binomial skew polynomial rings and their close relation to YBE were studied in [13] [15] [19], et al. Recall that each binomial skew polynomial algebra $A$ is a free left (resp. right) module over $A_p$ of finite rank $p^n$ and with a free basis the set $Y$, so $A$ is left and right Noetherian, see [13]. Moreover, by [19], $A$ is an Artin-Schelter regular algebra of global dimension $n$, $A$ is Koszul and a domain.

**Remark 3.2.** [14] [18]. Every square-free solution $(X, r)$ satisfies the condition $\mathbf{lri}$: $\mathbf{lri}$: $(xy)^x = y = x^{y^x}$ for all $x, y \in X$.

In other words, $\mathbf{lri}$ holds if and only if $(X, r)$ is nondegenerate, $\mathcal{R}_x = \mathcal{L}_x^{-1}$ and $\mathcal{L}_x = \mathcal{R}_x^{-1}$. In particular, $(X, r)$ is uniquely determined by the left action:

$$r(x, y) = (\mathcal{L}_x(y), \mathcal{L}_y^{-1}(x)).$$

Define the set $X^*$ as

$$(3.2)\quad X^* := X \bigcup X^{-1}, \quad \text{where} \quad X^{-1} = \{x^{-1} \mid x \in X\}.$$ 

We shall need the following useful lemma which is extracted from Proposition 7.6. in [16]. It generalizes the cyclic conditions and $\mathbf{lri}$.

**Lemma 3.3.** Let $(X, r)$ be a symmetric set with $\mathbf{lri}$, and $G = G(X, r)$. Then the following equalities hold in $G$ for all $k \in \mathbb{N}$, $x \in X^*$, $a \in G$.

- $a(x^a) = x = (x^a)^a$, $a^{-1} x = x^a$, $x^{a^{-1}} = a x$.
- $(x^a)^{-1} = (x^{-1})^a$, $(a x)^{-1} = a(x^{-1})$.
- $a^x x = a_x$, $x^a = x^a$, $(a^x)^{x^a} = x^a$, $x^{(x^a)} = x^a$.

$$a(x^k) = (a x)^k, \quad (x^k)^a = (x^a)^k.\tag{3.3}$$

### 3.2. The braided group $G = G(X, r)$.

**Lemma 3.4.** Let $(X, r)$ be a square-free solution, of order $|X| = n$ and cyclic degree $p = p(X, r)$, $S = S(X, r)$ is the associated monoid, and $(G, r_G)$ is the associated braided group.

1. For every $v \in S$ there exist monomials $y \in Y$ and $W, W' \in N_p$, such that $v = y.W = W'.y$ hold in $S$. 
(2) If \(y_1, y_2 \in Y\), \(W_1, W_2 \in N_p\), then the equality \(y_1W_1 = y_2W_2\) holds in \(S\) iff \(y_1 = y_2\) and \(W_1 = W_2\) are equalities of (normal) words in \(N\).

(3) \(N_p\) is the free abelian monoid, with a set of free generators \(X_p\). The group \(F_p\) is isomorphic to the free abelian group in \(n\) generators, \(F_p\) is the group of quotient of the monoid \(N_p\).

(4) \(F_p\) and \(N_p\) are invariant under the left action of \(G\) upon itself.

(5) \(F_p\) is a subgroup of \(\Gamma\), where \(\Gamma\) is the socle of \(G\) (see [22]).

**Proof.** Part (1) and (2) of our lemma are analogous to [13], Lemma 4.19, (1) and (3). Note that the lemma in [13] is proven for \(p = n!\) but the argument there needs only the equality \((L_x)^p = id_X\) for all \(x \in X\), which is true whenever \(p = p(X, r)\) is the cyclic degree of \((X, r)\). (3). It is clear that \(N_p\) is the free abelian monoid with a set of free generators \(X_p\), \(N_p\) is (canonically) embedded in \(F_p\) which is its group of quotients. So \(F_p\) is isomorphic to the free abelian group generated by \(X_p\). Every \(W \in F_p\) can be written uniquely as

\[
W = x_1^{(k_1p)}x_2^{(k_2p)} \cdots x_n^{(k_np)}, \quad \text{where} \ k_i \in \mathbb{Z}, \ 1 \leq i \leq n.
\]

(4). We have to show that \(N_p\) and \(F_p\) are invariant under the left action of \(G\) upon itself. We use equality \([3,3]\), in Lemma \([3,3]\) in which we replace \(k\) with \(p\) to obtain

\[
a(x^p) = (^a x)^p, \quad \forall x \in X^*, \ a \in G.
\]

Lemmas \([3,3]\) implies also that for every pair \(a \in G, x \in X^*\), one has \(^a(x^{-1}) = (^x a)^{-1}\). Due to the nondegeneracy, the left action of \(G\) on \(X\) permutes the elements \(x_1, \ldots, x_{n}\), so there are equalities of sets

\[
(a X) = \{a_1, a_2, \ldots, a_n\} = \{x_1, x_2, \ldots, x_n\} = X, \quad \forall a \in G
\]

\[
a(X_p) = \{a(x_1), a(x_2), \ldots, a(x_n)\} = \{x_1^p, x_2^p, \ldots, x_n^p\} = X_p, \quad \forall a \in G
\]

\[
a(X^{-1}) = \{a((x_1)^{-1}), \ldots, a((x_n)^{-1})\} = \{(x_1)^p, \ldots, (x_n)^p\} = X_p, \quad \forall a \in G.
\]

Hence the sets \(X_p\) and \(X_p^{-1} = \{(x_1)^p, \ldots, (x_n)^p\} = \{x_1^{-p}, \ldots, x_n^{-p}\}\) are invariant under the left action of \(G\). Now we apply ML2 to yield

\[
^a W \in N_p, \quad \forall W \in N_p, \ a \in G, \quad \text{and} \quad ^a W \in F_p, \ \forall W \in F_p, \ a \in G.
\]

It follows that the submonoid \(N_p\) and the subgroup \(F_p\) of \(G\) are \(G\)-invariant.

(5). We have to show that \(F_p \leq \Gamma\), where \(\Gamma\) is the socle of \(G\). We shall use induction on the length \(m = |a|\) of \(a \in G\) (see Definition \([1,2]\) (ii)) to prove

\[
W a = a, \quad \forall W \in F_p, \ a \in G.
\]

Every element of \(G\) is a word in the alphabet \(X^*\). For the base of the induction, \(m = 1\), we have to show

\[
W x = x, \quad \forall W \in F, \ x \in X^*.
\]

Let \(x \in X\). Recall that each square-free solution \((X, r)\) satisfies \(lri\), so the hypothesis of Lemma \([3,3]\) is satisfied. By the choice of \(p\), and condition \(lri\) one has \((L_x)^p = id_X = (R_x)^p\). Using Lemma \([3,3]\) and \(lri\) again we show that the following equalities hold in \(G\), for all \(k \in \mathbb{Z}, x, y \in X^*\):

\[
(y^p)x = x = x^{(y^p)} , \quad (y^{-1}) x = (y^p)^{-1} x = x^{(y^p)} = x, \quad (y^{kp}) x = x.
\]

Suppose \(W \in F_p\) then \(W = x_1^{k_1p}x_2^{k_2p} \cdots x_n^{k_np}\), for some \(k_i \in \mathbb{Z}\), so condition ML1 and (3.8) imply \([3,7]\) which gives the base for the induction. Suppose \(W a = a\),
for all $W \in \mathcal{F}_p$, and all $a \in G, \ |a| \leq m$. Let $u \in G$, has length $|u| = m + 1$, so $u = x.a, x \in X^*, |a| = m$. We apply ML2, and the inductive assumption to yield
\[
Wu = W(x.a) = (Wx)(Wa) = x.(Wx a) = x.a = u.
\]
Therefore $Wu = u, \forall u \in G$, so $W \in \Gamma$, which implies $\mathcal{F}_p \leq \Gamma$.

**Proof of Theorem A.** (1) The monoid $S$ has cancelation law, satisfies Ore conditions, and is embedded in $G = G(X, r)$, which is its group of quotients. So every element $u \in G$ has the shape $u = ab^{-1}$, where $a, b \in S$. By Lemma 3.4 part (1), the elements $a, b$ can be presented as $a = y_1W_1$ and $b = W_2y_2$, where $y_1, y_2 \in Y$, and $W_1, W_2 \in \mathcal{N}_p$. Then $b^{-1} = (y_2)^{-1}W_2^{-1}$, where $W_2^{-1} \in \mathcal{F}_p$. First we shall present $(y_2)^{-1}$ in the form $(y_2)^{-1} = tW'$, with $t \in S, W' \in \mathcal{F}_p$. Since $y_2 \in Y$ one has
\[
y_2 = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad 0 \leq \alpha_i \leq p - 1, \ 1 \leq i \leq n.
\]
Note that for any $x \in X$, and any integer $\alpha$ with $0 \leq \alpha \leq p - 1$, there is an obvious equality
\[
x^{-\alpha} = x^{p - \alpha} x^{-p} = x^\beta x^{-\beta} = x^p x^{-p}, \quad 0 \leq \beta = p - \alpha \leq p - 1.
\]
This, together with Lemma 3.4 part (1) imply
\[
y_2^{-1} = x_1^{-\alpha_1} \cdots x_n^{-\alpha_n} = (x_1^{p - \alpha_1} x_n^{-1})(x_1^{p - \alpha_1} x_n^{-1}) = (x_1^{\beta_1} x_n^{\beta_n}) \cdots (x_1^{\beta_1} x_n^{\beta_n}) = (x_1^{\beta_1} \cdots x_1^{\beta_n})W' = tW', \quad t \in Y, W \in \mathcal{F}_p.
\]
Therefore
\[
b^{-1} = y_2^{-1}W_2^{-1} = (tW)W_2^{-1} = t(W.W_2^{-1}) = tW_3, \quad t \in Y, W_3 = W.W_2^{-1} \in \mathcal{F}_p.
\]
The following equalities hold in $G$:
\[
ab^{-1} = (y_1W_1)(tW_3) = y_1(W_1t)W_3 = y_1(W_1t)((W_1t)^t)W_3, \quad (W_1t)^t, W_3 \in \mathcal{F}.
\]
\[
= (y_1W_1), \quad W_4 = ((W_1t)^t)W_3 \in \mathcal{F}_p, \quad \text{since } W_4t = t.
\]
\[
= (yW_4), \quad yW_0 := y_1t, \quad y \in Y, W_0 \in \mathcal{N}_p, \quad \text{by Lemma 3.4}
\]
\[
yW, \quad y \in Y, W := W_0W_4 \in \mathcal{F}_p.
\]
We have shown that every element $u \in G$ can be presented as $u = yW, \ y \in Y, W \in \mathcal{F}_p$. We claim that this presentation is unique. Suppose $y_1W_1 = y_2W_2$ holds in $G$, where $y_1, y_2 \in Y$, and $W_1, W_2 \in \mathcal{F}_p$. Every element $W \in \mathcal{F}_p$ has a presentation as $W = uv^{-1}$, where $u, v \in \mathcal{N}_p$. Without loss of generality we may assume $u \in \mathcal{N}_p$. We use the equalities $W_1 = u_1v_1^{-1}, W_2 = u_2v_2^{-1}$, where $u_2, v_2 \in \mathcal{N}_p, i = 1, 2$, and obtain
\[
y_1u_1v_1^{-1} = y_1W_1 = y_2W_2 = y_2u_2v_2^{-1},
\]
\[
y_1u_1v_1^{-1} = y_2u_2v_2^{-1}.
\]
The elements of $\mathcal{F}_p$ is commute, hence multiplying both sides of this equality by $v_1v_2$, we obtain
\[
y_1(u_1v_2) = y_2(u_2v_1), \quad \text{where } y_1, y_2 \in Y, \text{ and } u_1v_2, u_2v_1 \in \mathcal{N}_p.
\]
It follows then from Lemma 3.4 that the equalities $y_1 = y_2$, and $u_1v_2 = u_2v_1$ hold in $\mathcal{N}_p$, and in $\mathcal{F}_p$. Next we multiply both sides of $u_1v_2 = u_2v_1$ by $v_2^{-1}v_1^{-1}$, to obtain $W_1 = u_1v_1^{-1} = u_2v_2^{-1} = W_2 \in \mathcal{F}_p$, which proves part (1). Parts (2) and (3) are straightforward, in particular (1, 3) is in force.
Proof of Theorem B. (1) We have to show that $\mathcal{F}_p$ is an ideal of $G$ that is, a $G$-invariant normal subgroup of $G$. By Lemma 3.3, $\mathcal{F}_p$ is a $G$-invariant subgroup of $\Gamma$. It remains to prove that $\mathcal{F}_p$ is a normal subgroup of $G$. Suppose $W \in \mathcal{F}_p, a \in G$.

We use the compatibility condition M3, and the equality $a^W = a$ (since $\mathcal{F}_p < \Gamma$) to yield:

$$(aW)a^{-1} = (aW)(aW)a^{-1} = (aW)(aa^{-1}) = aW \in \mathcal{F}_p.$$ 

So $\mathcal{F}_p$ is a normal subgroup of $G$, and therefore it is an ideal of $G$. Clearly, $\mathcal{F}_p < \Gamma \triangleleft G$ is a chain of ideals in the braided group $G$. (2) We have shown that $\mathcal{F}_p$ is an ideal of $G$, hence the quotient group $\tilde{G} = G/\mathcal{F}_p$ is an involutive braided group of order $|\tilde{G}| = |G : \mathcal{F}_p|$. It follows straightforwardly from Theorem A that the index of $\mathcal{F}_p$ in $G$ is exactly $|G : \mathcal{F}_p| = |Y| = p^n$, hence $|\tilde{G}| = p^n$. Moreover, [16], the canonical projection $\pi : G \to \tilde{G} = G/\mathcal{F}_p$, $g = yW \mapsto \tilde{g} = \bar{y}$, induces the structure of an involutive braided group on the finite set

$$\tilde{G} = \bar{Y} := \{\bar{y} \mid y \in Y\},$$

and the induced operation $\cdot$ on $\bar{Y}$ satisfies (14). It is clear that the restriction $\pi_{|Y} : Y \to \bar{Y} = \tilde{G}$ is a bijective map, in other words $Y$ is embedded in $\tilde{G}$, and so is $X < Y$. We identify $X$ with its image $\tilde{X}$ in $\tilde{G}$. Clearly the set $\tilde{X}$ generates $\tilde{G}$. We have proven parts (2) and (3). (4) Consider the chain of ideals $\mathcal{F}_p < \Gamma < G$.

The 3rd isomorphism theorem for braided groups, see [16], Remark 4.13, and the well-known group isomorphism $\bar{G} \simeq (G/\Gamma)$ imply

$$\bar{G} \simeq (G/\Gamma) \simeq (G/\mathcal{F}_p)/(\Gamma/\mathcal{F}_p),$$

and $|G : \Gamma||\Gamma : \mathcal{F}_p| = |G : \mathcal{F}_p| = p^n$.

It follows that the order $|\tilde{G}| = |G : \Gamma|$ divides $p^n$. In particular, if $p$ is a prime, then the permutation group $\bar{G}$ is a $p$-group.

□

Corollary 3.5. Let $(X, r)$ be a finite square-free solution of order $|X| = n$ with cyclic degree $p = p(X, r)$. Then there exists a finite involutive braided group $(B, \sigma)$ of order $p^n$, such that $X$ is embedded in $B$ and generates the group $B$. Moreover, $X$ is $B$-invariant, and $r = \sigma_{\pi_X \times X}$ is the restriction of $\sigma$ on $X \times X$.

Theorem C (4) shows that the group ring $k[\bar{G}]$ is left and right Noetherian which implies the following.

Corollary 3.6. $G$ is a Noetherian group, that is every subgroup of $G$ is finitely generated.

The following result is well-known, it can be extracted from Remarks 2.4 and 2.6, but here it is an independent and straightforward consequence from Theorem B.

Corollary 3.7. Under the hypothesis of Theorem B. The group $G = G(X, r)$ is a finitely-generated torsion-free, abelian-by-finite solvable group, and therefore $G(X, r)$ is torsion-free poly-$\mathbb{Z}$-by-finite.

3.3. The group algebra $k[\bar{G}]$ over a field $k$. One of the famous and still open problems about abstract group rings is the Kaplansky zero divisor conjecture which is closely related to two more conjectures, the unit conjecture (U) and the idempotent conjecture (I) given below.

Conjectures 3.8. (I. Kaplansky) Let $k$ be a field, let $G$ be a torsion-free group.
Farkas and Snider showed that if \( G \) is a ring already classical results from the theory of group rings. Hall proved that the group algebra \( k[G] \) is a field of characteristic 0, then (1) the group ring \( k[G] \) is a torsion-free finitely generated abelian-by-finite solvable group. Therefore see [9], Main Theorem; and (2) the canonical embedding \( k[G] \) is a torsion-free solvable-by-finite groups, and for an arbitrary field \( k \). It follows from the above discussion that for every field \( k \) torsion-free solvable-by-finite groups, and for an arbitrary field \( k \) zero divisor conjecture for a large class of torsion-free groups which contains all ("Syzygy" theorem). More generally, Kropholler, Linell and Moody, prove the polycyclic. In fact, the group \( G \) is a left (respectively right) Noetherian domain. Moreover, \( k[G] \) is a left module of finite rank over the group algebra \( k[G] \). One has \( A = kS \hookrightarrow k[G] \twoheadrightarrow Q(A) \), where \( Q(A) \) is the canonical right quotient ring of the domain \( A \).

**Lemma 3.11.** Let \( G = G(X, r) \) be the braided group of a finite nondegenerate symmetric set \( (X, r) \), let \( k \) be a field. Then the group algebra \( k[G] \) is a left (respectively right) noetherian domain. Moreover, \( k[G] \) has finite global dimension, whenever the field \( k \) has characteristic 0.

**Proof.** We discussed, see Remarks [22] and [26] that the braided group \( G = G(X, r) \) is a torsion-free finitely generated abelian-by-finite solvable group. Therefore \( G \) is polycyclic. In fact, the group \( G \) is (poly-\( \mathbb{Z} \))-by-finite. Now we simply apply several already classical results from the theory of group rings. Hall proved that the group ring \( k[G] \) of a polycyclic group \( G \) is right (resp. left) Noetherian, [21] Theorem 1. Farkas and Snider, showed that if \( G \) is a torsion free, polycyclic-by-finite group and \( k \) is a field of characteristic 0, then (1) the group ring \( k[G] \) has no zero divisors, see [9], Main Theorem; and (2) \( k[G] \) has finite global dimension, [9], Lemma 1 ("Syzygy" theorem). More generally, Kropholler, Linell and Moody, prove the zero divisor conjecture for a large class of torsion-free groups which contains all torsion-free soluble-by-finite groups, and for an arbitrary field \( k \), see [22], Theorem 1.3. It follows from the above discussion that for every field \( k \) the group ring \( k[G] \) is a left (respectively right) Noetherian domain. \( k[G] \) has finite global dimension, whenever \( k \) is a field of characteristic 0.

**Proof of Theorem C.** \( \{1\} \). \( \mathcal{F}_p \) is the free abelian group in \( n \)-generators \( x_1^p, \ldots, x_n^p \), then it is well-known the group ring \( k[\mathcal{F}_p] \) is a Noetherian domain. It follows from Theorem A that the group algebra \( k[G] \) is a left module of finite rank over the algebra \( k[\mathcal{F}_p] \), generated by \( Y = \{y_i \mid 1 \leq i \leq p^n\} \), and therefore \( k[G] \) is left Noetherian. Analogously one shows that \( k[G] \) is right Noetherian. We have to prove that \( k[G] \) is a free left \( k[\mathcal{F}_p] \)-module with a free basis \( Y \). To simplify notation set \( q = p^n \), and assume that there is a relation:

\[
g_1y_1 + g_2y_2 + \cdots + g_qy_q = 0, \quad \text{where} \ g_i \in k[\mathcal{F}_p], \ 1 \leq i \leq q.
\]

\( (Z) \) The group ring \( k[G] \) is a domain.

\( (I) \) \( k[G] \) does not contain any non-trivial idempotents: if \( a^2 = a \), then \( a = 1 \) or \( a = 0 \).

\( (U) \) \( k[G] \) has no non-trivial units: if \( ab = 1 \) in \( k[G] \), then \( a = \alpha g \) for some \( \alpha \in k^\times \) and \( g \in G \).

It is well-known that \((U) \implies (Z) \implies (I)\). There are numerous positive results, in particular, the zero divisor conjecture is known to hold for all virtually solvable group.
But \( \{g_1, \ldots, g_s\} \subset k[F_p] \) is a finite set, and each of its elements is (a finite) linear combination of words \( W \in F_p \), therefore one can find a monomial \( W_0 \in N_p \), such that \( W_0.g_i = a_i \in k[S_p] = A_p \), \( 1 \leq i \leq q \). We multiply (3.11) (on the left) by \( W_0 \), and obtain the following relation in the monoidal algebra \( A = kS \):

\[
(3.11) \quad a_1y_1 + a_2y_2 + \cdots + a_qy_q = 0, \quad \text{where } a_i \in k[S_p], \ 1 \leq i \leq q.
\]

We have proven that the algebra \( A \) is a free left module over \( A_p = kS_p \) with a free-basis \( Y \), see [13], Theorem I, (for more details see the proof of Lemma (4.20)). Therefore \( a_i = 0, \ 1 \leq i \leq q \). But \( a_i = W_0.g_i \), for all \( i \)'s, and since \( k[G] \) has no zero divisors (by Lemma Lemma 3.11), we obtain \( g_i = 0, \ 1 \leq i \leq q \). It follows that \( Y \) is a free basis of \( k[G] \), considered as a left \( k[F_p]- \) module. (2).

Recall that a polynomial in \( n \) variables \( f(X_1, X_2, \ldots, X_n) \in k[X_1, X_2, \ldots, X_n] \) is a symmetric polynomial if for any permutation \( \sigma \) of the subscripts \( 1, 2, \ldots, n \) one has \( f(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)}) = f(X_1, X_2, \ldots, X_n) \). It is well-known that any symmetric polynomial can be presented as a polynomial expression with coefficients from \( k \) in the first \( n \) power sum symmetric polynomials \( s_k(X_1, X_2, \ldots, X_n), 1 \leq k \leq n \), where

\[
s_k(X_1, X_2, \ldots, X_n) = X_1^k + X_2^k + \cdots + X_n^k, \quad k \geq 1.
\]

In particular, the remaining power sum polynomials \( s_k(X_1, X_2, \ldots, X_n) \) for \( k > n \), can also be expressed in the first \( n \) power sum polynomials. So, the set of all symmetric polynomials in \( k[X_1, X_2, \ldots, X_n] \) is an \( n \)-generated subalgebra \( \mathcal{S} = k\langle s_1, \ldots, s_n \rangle \) of the commutative polynomial ring \( k[X_1, X_2, \ldots, X_n] \). We know that the subalgebra \( A_p = k[x_1^p, x_2^p, \ldots, x_n^p] \subset kS \) is isomorphic to the commutative polynomial ring \( k[X_1, X_2, \ldots, X_n] \) in \( n \) variables (we may consider the isomorphism extending the assignment \( x_1^p \rightarrow X_1, 1 \leq i \leq n \)). Consider now the subalgebra \( \mathcal{S} = \mathcal{S}(x_1^p, \ldots, x_n^p) \) consisting of all symmetric polynomials \( f(x_1^p, \ldots, x_n^p) \in A_p \). \( \mathcal{S} \) is generated (as a \( k \)-algebra) by the power sums

\[
s_k = s_k((x_1)^p, \ldots, (x_n)^p) = (x_1^p)^k + \cdots + (x_n^p)^k = (x_1)^{kp} + \cdots + (x_n)^{kp}.
\]

To show that \( \mathcal{S} \) is in the center of \( k[G] \) it suffices to verify the following equalities for all \( y \in X, 1 \leq k \leq n \):

\[
s_k(x_1^p, \ldots, x_n^p).y = y.s_k(x_1^p, \ldots, x_n^p), \quad s_k(x_1^p, \ldots, x_n^p).y^{(1)} = y^{(1)}.s_k(x_1^p, \ldots, x_n^p).
\]

By the nondegeneracy of the right action there is an equality of sets \( \{x_1^p, \ldots, x_n^p\} = X, \forall y \in X \). The following equalities hold in \( k[G] \):

\[
s_k.y = ((x_1)^{kp} + \cdots + (x_n)^{kp}).y = ((x_1)^{kp})y + \cdots + ((x_n)^{kp})y \quad \text{by } M3
\]

\[
= (y)^{kp}.((x_1)^{kp}) + \cdots + (y)^{kp}.((x_n)^{kp})
\]

\[
= y.(y)^{kp} + \cdots + y.(x_n)^{kp} \quad \text{by } x^{kp}y = y, \forall x \in X
\]

\[
= y.(x_1)^{kp} + \cdots + ((x_n)^{kp})y = y.s_k,
\]

which verify \( s_k.y = y.s_k \), for all \( k \geq 1 \) and all \( y \in X \). In particular, this implies that the subalgebra \( \mathcal{S} \) is in the centre of the monoidal algebra \( A = kS \). The proof of the equality \( s_k.y^{(1)} = y^{(1)}.s_k \), \( y \in X, k \geq 1 \), is analogous. In this case one uses the additional relations given by Lemma 3.3 more specifically \( a(y^{(1)}) = (a^y)^{−1} \) and \( a(y^{−1}) = a^y, \forall y \in X, a \in G \). This proves part (2). Part (3) follows from Lemma 3.11. Note that we have shown in part (1) that \( k[G] \) is left and right Noetherian, independently from Lemma 3.11. \(\square\)
Acknowledgments. This paper was written during my visit at the Max Planck Institute for Mathematics in the Sciences (MiS) in Leipzig, in 2018-2019. It is my pleasant duty to thank MiS, and Bernd Sturmfels for the support and the creative, inspiring atmosphere. I thank all colleagues and friends, the people from the wonderful MiS library, and Saskia Gutzschebauch for the cooperation.

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