UNIMODAL SEQUENCES SHOW LAMBERT W IS BERNSTEIN

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Abstract. We consider a sequence of polynomials appearing in expressions for the derivatives of the Lambert W function. The coefficients of each polynomial are shown to form a positive sequence that is log-concave and unimodal. This property implies that the positive real branch of the Lambert W function is a Bernstein function.

1. Introduction

The Lambert W function was defined and studied in [5]. It is a multivalued function having branches $W_k(z)$, each of which obeys $W_k \exp(W_k) = z$. The principal branch $W_0$ maps the set of positive reals to itself, and is the only branch considered here. Therefore we omit the subscript 0 for brevity. The $n$th derivative of $W$ is given implicitly by

$$
\frac{d^nW(x)}{dx^n} = \frac{\exp(-nW(x))p_n(W(x))}{(1 + W(x))^{2n-1}} \quad \text{for } n \geq 1,
$$

where the polynomials $p_n(w)$ satisfy $p_1(w) = 1$, and the recurrence relation

$$
p_{n+1}(w) = -(nw + 3n - 1)p_n(w) + (1 + w)p'_n(w) \quad \text{for } n \geq 1.
$$

In [6], the first 5 polynomials were printed explicitly:

$$
p_1(w) = 1, \quad p_2(w) = -2 - w, \quad p_3(w) = 9 + 8w + 2w^2, \quad p_4(w) = -64 - 79w - 36w^2 - 6w^3, \quad p_5(w) = 625 + 974w + 622w^2 + 192w^3 + 24w^4.
$$

These initial cases suggest the conjecture that each polynomial $(-1)^{n-1}p_n(w)$ has all positive coefficients, and if this is true, then $dW(x)/dx$ is a completely monotonic function [11]. We here prove the conjecture and prove in addition that the coefficients are unimodal and log-concave.

2. Formulae for the coefficients

In view of the conjecture, we write

$$
p_n(w) = (-1)^{n-1} \sum_{k=0}^{n-1} \beta_{n,k}w^k.
$$

We now give several theorems regarding the coefficients.

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Theorem 2.1. The coefficients $\beta_{n,k}$ defined in (2.1) obey the recurrence relations

\begin{align}
\beta_{n,0} &= n^{n-1}, \quad \beta_{n,1} = 3n^n - (n + 1)^n - n^{n-1} , \\
\beta_{n,n-1} &= (n - 1)! , \quad \beta_{n,n-2} = (2n - 2)(n - 1)! , \\
\beta_{n+1,k} &= (3n - k - 1)\beta_{n,k} + n\beta_{n,k-1} - (k + 1)\beta_{n,k+1} , \quad 2 \leq k \leq n - 3 .
\end{align}

Proof. By substituting (2.1) into (1.2) and equating coefficients. \hfill \Box

Theorem 2.2. An explicit expression for the coefficients $\beta_{n,k}$ is

\begin{equation}
\beta_{n,k} = \sum_{m=0}^{k} \frac{1}{m!} \binom{2n - 1}{k - m} \sum_{q=0}^{m} \binom{m}{q} (-1)^q (q + n)^{m+n-1} .
\end{equation}

Proof. We rewrite (1.1) in the form

\begin{equation}
p_n(W(x)) = (1 + W(x))^{2n-1} e^{nW(x)} \frac{d^n W(x)}{dx^n} .
\end{equation}

From the Taylor series of $W(x)$ around $x = 0$, given in [5], we obtain

\begin{equation}
\frac{d^n W(x)}{dx^n} = \sum_{m=0}^{\infty} \frac{(-m)^{m-1}}{(m-n)!} x^{m-n} .
\end{equation}

Substituting this into the expression of $p_n$, using $x = We^W$ and changing the index of summation, we obtain the equation

\begin{equation}
p_n(w) = (1 + w)^{2n-1} \sum_{s=0}^{\infty} (-1)^{n+s-1} (n+s)^{n+s-1} \frac{w^s}{s!} e^{(n+s)w} .
\end{equation}

We expand the right side around $w = 0$ and equate coefficients of $w$. \hfill \Box

Remark 2.3. The polynomials $p_n(w)$ can be expressed in terms of the diagonal Poisson transform $D_n[f_s; z]$ defined in [10], namely, by (2.6)

\begin{equation}
p_n(w) = (-1)^{n-1} (1 + w)^{2(n-1)} D_n[(n+s)^{n-1}; -w] .
\end{equation}

Theorem 2.4. The coefficients can equivalently be expressed either in terms of shifted $r$-Stirling numbers of the second kind $\{n+r\}_{m+r}$ defined in [3],

\begin{equation}
\beta_{n,k} = \sum_{m=0}^{k} (-1)^m \binom{2n - 1}{k - m} \binom{2n - 1 + m}{n + m} ,
\end{equation}

or in terms of Bernoulli polynomials of higher order $B_n^{(z)}(\lambda)$ defined in [9],

\begin{equation}
\beta_{n,k} = \sum_{m=0}^{k} (-1)^m \binom{2n - 1}{k - m} \binom{m + n - 1}{n - 1} B_n^{(-m)}(n) ,
\end{equation}

or in terms of the forward difference operator $\Delta$ [7] p. 188,

\begin{equation}
\beta_{n,k} = \sum_{m=0}^{k} \binom{2n - 1}{k - m} \frac{(-1)^m}{m!} \Delta^m n^{m+n-1} .
\end{equation}
Proof. We convert (2.5) using identities found in [3] and [8] respectively.

\[
\binom{n+r}{m+r} = \frac{1}{m!} \sum_{q=0}^{m} (-1)^{m-q} \binom{m}{q} (q+r)^n
\]

and

\[
B_{n}^{(-m)}(r) = \frac{n!}{(m+n)!} \sum_{q=0}^{m} (-1)^{m-q} \binom{m}{q} (q+r)^{m+n}.
\]

\[\square\]

3. Properties of the coefficients

We now give theorems regarding the properties of the \(\beta_{n,k}\). We recall the following definitions [12]. A sequence \(c_0, c_1, \ldots, c_n\) of real numbers is said to be unimodal if for some \(0 \leq j \leq n\) we have \(c_0 \leq c_1 \leq \cdots \leq c_j \geq c_{j+1} \geq \cdots \geq c_n\), and it is said to be logarithmically concave (or log-concave for short) if \(c_{k-1}c_{k+1} \leq c_k^2\) for all \(1 \leq k \leq n-1\). We prove that for each fixed \(n\), the \(\beta_{n,k}\) are unimodal and log-concave with respect to \(k\). Since a log-concave sequence of positive terms is unimodal [15], it is convenient to start with the log-concavity property.

Theorem 3.1. For fixed \(n \geq 3\) the sequence \(\{k!\beta_{n,k}\}_{k=n}^{n-1}\) is log-concave.

Proof. Using (2.5) we can write

\[
k!\beta_{n,k} = (2n-1)! \sum_{m=0}^{k} \binom{k}{m} x_m y_{k-m},
\]

where

\[
x_m = \sum_{j=0}^{m} \binom{m}{j} a_j, \quad a_j = (-1)^j (n+j)^{m+n-1},
\]

and \(y_m = 1/(2n-1-m)!\). Since the binomial convolution preserves the log-concavity property [13 14], it is sufficient to show that the sequences \(\{x_m\}\) and \(\{y_m\}\) are log-concave. We have

\[
a_{j-1}a_{j+1} = (-1)^{j-1}(n+j-1)^{m+n-1}a_j + (-1)^{j+1}(n+j)^{m+n-1}a_j
\]

\[
= (-1)^{2j}(n+j)^2 - 1 < (-1)^{2j}(n+j)^2(m+n-1) = a_j^2.
\]

Thus the sequence \(\{a_j\}\) is log-concave and so is \(\{x_m\}\) due to (3.1) and the aforementioned property of the binomial convolution. The sequence \(\{y_m\}\) is log-concave because

\[
y_{m-1}y_{m+1} = \frac{1}{(2n-1-m)!} \frac{1}{(2n-1-m-1)!} - \frac{1}{2n-1-m} \frac{1}{(2n-1-m)!} \frac{1}{(2n-1-m)!} < y_m^2.
\]

\[\square\]

Now we prove that the coefficients \(\beta_{n,k}\) are positive. The following two lemmas are useful.

Lemma 3.2. If a positive sequence \(\{k!c_k\}_{k=0}^{\infty}\) is log-concave, then

(i) \(\{(k+1)c_{k+1}/c_k\}\) is non-increasing;
(ii) \( \{ c_k \} \) is log-concave;
(iii) the terms \( c_k \) satisfy

\[
c_k c_m \geq \binom{k + m}{k} c_0 c_{k+m} \quad (0 \leq m \leq k+1).
\]

**Proof.** The statements (i) and (ii) are obvious. To prove (iii) we apply a method used in \[1\]. Specifically, by (i) we have for \( 0 \leq p \leq k \)

\[
\frac{c_{p+1}}{c_p} \geq \frac{k + p + 1}{p + 1} \frac{c_{k+p+1}}{c_{k}}.
\]

Apply the last inequality for \( p = 0, 1, 2, \ldots, m \) with \( m \leq k+1 \), and form the products of all left-hand and right-hand sides. As a result, after the cancellation we obtain

\[
c_m \geq \frac{k + 1 + 2}{1} \frac{k + m}{2} \frac{c_k}{c_0} \frac{\cdots}{m} \frac{c_{k+m}}{c_k},
\]

which is equivalent to (3.2).

**Lemma 3.3.** If the coefficients \( \beta_{n,k} \) are positive, then for fixed \( n \geq 3 \) they satisfy

\[
\frac{(k+1)\beta_{n,k+1}}{\beta_{n,k}} < n-1.
\]

**Proof.** By Theorem 3.1 and under the assumption of lemma, for fixed \( n \geq 3 \) the sequence \( \{ k!\beta_{n,k} \}_{k=0}^{n-1} \) meet the conditions of Lemma 3.2. Applying the inequality (3.2) with \( m = 1 \) to this sequence gives \( (k+1)\beta_{n,k+1}/\beta_{n,k} \leq \beta_{n,1}/\beta_{n,0} \). Then the lemma follows as due to (2.2)

\[
\beta_{n,1} = 3n - (n + 1)^n - n^{n-1} = 3n - n \left( 1 + \frac{1}{n} \right)^n - 1 < 3n - 2n - 1 = n - 1.
\]

**Theorem 3.4.** The coefficients \( \beta_{n,k} \) are positive.

**Proof.** We prove the statement by induction on \( n \). It is true for \( n \leq 5 \) (see §1). Assume that for some fixed \( n \) all the members of the sequence \( \{ \beta_{n,k} \}_{k=0}^{n-1} \) are positive. Since \( \beta_{n+1,0} = (n+1)^n > 0 \) and \( \beta_{n+1,n} = n! > 0 \) by (2.2) and (2.3), we only need to consider \( k = 1, 2, \ldots, n-1 \).

Substituting inequalities \( \beta_{n,k+1} < (n-1)\beta_{n,k}/(k+1) \) and \( \beta_{n,k-1} > k\beta_{n,k}/(n-1) \), which follow from (3.3), in the recurrence (2.4) immediately gives the result

\[
\beta_{n,k} > \frac{3n-k-1}{n-1} \beta_{n,k} + n \frac{k}{n-1} \beta_{n,k-1} = \left( 2n + \frac{k}{n-1} \right) \beta_{n,k} > 0.
\]

Thus the proof by induction is complete.

**Corollary 3.5.** The sequence \( \{ \beta_{n,k} \}_{k=0}^{n-1} \) is unimodal for \( n \geq 3 \).

**Proof.** By Theorem 3.4 the sequence \( \{ \beta_{n,k} \}_{k=0}^{n-1} \) is positive, therefore by Theorem 3.1 and Lemma 3.2(i) it is log-concave and, hence, unimodal.
4. Relation to Carlitz’s numbers

There is a relation between the coefficients $\beta_{n,k}$ and numbers $B(\kappa, j, \lambda)$ introduced by Carlitz in [3]. Comparing the formulae (2.8) and (2.9) with the corresponding [4] eq.(6.3) and [4] eq.(2.9), taking into account that he uses the notation $R(n, m, r) = \binom{n+r}{m+r}$, we find

$$\beta_{n,k} = (-1)^k B(n-1, n-1-k, n).$$  

(4.1)

It follows that for $n \geq 3$, the sequence $\{B(n-1, k, n)\}_{k=0}^{n-1}$ is log-concave together with $\{\beta_{n,k}\}_{k=0}^{n-1}$.

Using the property [4] eq.(2.7) that $\sum_{j=0}^{n} B(\kappa, j, \lambda) = (2\kappa-1)!!$, we can compute $p_n(w)$ at the singular point where $W = -1$ (cf. (1.1)). Thus, substituting $w = -1$ in (2.4) gives $p_n(-1) = (2^{-n+1}(2n-3))!$. Thus $w = -1$ is not a zero of $p_n(w)$.

We also note that the numbers $B(\kappa, j, \lambda)$ are polynomials of $\lambda$ and satisfy a three-term recurrence [4] eq. (2.4)]

$$B(\kappa, j, \lambda) = (\kappa + j - \lambda)B(\kappa - 1, j, \lambda) + (\kappa - j + \lambda)B(\kappa - 1, j - 1, \lambda)$$  

(4.2)

with $B(\kappa, 0, \lambda) = (1 - \lambda)^k$, $B(0, j, \lambda) = \delta_{j,0}$. This gives one more way to compute the coefficients $\beta_{n,k}$, specifically, for given $n$ and $k$ we find a polynomial $B(n-1, n-1-k, \lambda)$ using (4.2) and then set $\lambda = n$ to use (4.1).

5. Concluding remarks

It has been established that the coefficients of the polynomials $(-1)^{n-1}p_n(w)$ are positive, unimodal and log-concave. These properties imply an important property of $W$. In particular, it follows from formula (1.1) and Theorem 3.3 that $(-1)^{n-1}(dW/dx)^{(n-1)} > 0$ for $n \geq 1$. Since $W(x)$ is positive for all positive $x$, this means that the derivative $W'$ is completely monotonic and $W$ itself is a Bernstein function [2].

Some additional identities can be obtained from the results above. For example, computing $\beta_{n,n-1}$ by (2.8) and comparing with (2.9) gives

$$\sum_{m=0}^{n-1} (-1)^{m} \binom{2n-1}{n-m} \binom{2n-1+m}{n+m} = (n-1)!.$$  

A relation between $\binom{2n-1+m}{n+m}$ and $B^{(-m)}_n(n)$ can be obtained from (2.8) and (2.9), but this is a special case of [4] eq. (7.5)]. It is finally interesting to note that (2.8) and (2.9) can be inverted. Indeed, in these formulae for fixed $n$, the sequence $(-1)^k \beta_{n,k}$ is a convolution of two sequences and the corresponding relation between their generating functions is $G(w) = (1-w)^{2n-1}F(w)$. Since $F(w) = G(w)(1-w)^{-(2n-1)} = G(w) \sum_{k \geq 0} \binom{2n-2+k}{2n-2}w^k$, the inverse of, for example, (2.8) is

$$\sum_{k=0}^{n-1} (-1)^k \beta_{n,k} \binom{2n-2+m-k}{2n-2}.$$  

(5.1)

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