Nonlinear Dynamics and Chaos in Two Coupled Nanomechanical Resonators

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Two elastically coupled nanomechanical resonators driven independently near their resonance frequencies show intricate nonlinear dynamics. The dynamics provide a scheme for realizing a nanomechanical system with tunable frequency and nonlinear properties. For large vibration amplitudes the system develops spontaneous oscillations of amplitude modulation that also show period doubling transitions and chaos. The complex nonlinear dynamics are quantitatively predicted by a simple theoretical model.

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Resonant nanoelectromechanical systems (NEMS) are attracting interest in a broad variety of research areas and for many possible applications due to their remarkable combination of properties: small mass, high operating frequency, large quality factor, and easily accessible nonlinearity. Further development of NEMS applications such as superior mass, force and charge sensors, or reaching the quantum limit of detection in mechanical systems, requires addressing several important challenges. For example, the nonlinearity of the devices has to be either minimized or utilized for improving performance. In addition, the large-scale integration of nanodevices demands a detailed understanding of the behavior of coupled devices in NEMS arrays.

In this paper we demonstrate complex nonlinear behavior of a pair of coupled nanomechanical devices, and show that this can be quantitatively understood from the basic physics of the devices. We show that the linear and weakly nonlinear response of one oscillation can be modified by driving the second oscillation, and, for some ranges of parameters of the devices, that the linear response range of the first oscillation can be significantly extended. When both oscillations are driven into their strongly nonlinear range more complicated frequency-sweep response curves are found, corresponding to the well known bistability of driven anharmonic “Duffing” resonators, but now with switching between a variety of different stable states of the coupled pair. Spontaneous amplitude modulation oscillations may develop, with frequencies characteristic of the dissipation rates rather than of the intrinsic frequencies or their sums and differences. These amplitude modulations show period doubling bifurcations and chaos. The complex dynamics are reproduced quantitatively by a simple theoretical model, giving us confidence that the nonlinear behavior of coupled nanomechanical devices can be understood and controlled.

We study a system of two strongly coupled nonlinear nanoelectromechanical resonators using a structure of doubly-clamped beams with a shared mechanical ledge shown in Fig. 1a. The devices consist of a stack of three layers of gallium arsenide (GaAs): a 100nm highly n-doped layer, a 50nm insulating layer, and another 50nm layer that is highly p-doped. The piezoelectric property of GaAs results in a highly efficient integrated actuation mechanism described in [12]. A preliminary 120nm deep etch step is done to isolate the actuation electrodes of the two beams, so that the two beams can be addressed separately while retaining strong elastic coupling. Optical interferometry is used for the motion transduction [13]. The laser beam is adjusted so that both beams are in the illuminated spot.

We model the behavior of the two strongly interacting nonlinear resonators by a system of coupled equations of motion for the beam displacements $x_1, x_2$ in their fundamental modes

\[
\begin{align*}
\ddot{x}_1 + \gamma_1 \dot{x}_1 + \omega_1^2 x_1 + \alpha_1 x_1^3 + D(x_1 - x_2) &= g_{D1}(t), \\
\ddot{x}_2 + \gamma_2 \dot{x}_2 + \omega_2^2 x_2 + \alpha_2 x_2^3 + D(x_2 - x_1) &= g_{D2}(t).
\end{align*}
\]

As well as the usual terms describing the resonant frequencies, damping, and Duffing nonlinearity, we include a linear coupling term in the displacements of strength $D$. The terms on the right hand side are the external drives applied to the two beams, which are controlled independently. The linear terms in the equations, ignoring for now the drive and dissipation, give two modes with frequencies $\omega_1$ and $\omega_{II}$ and corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_{II}$ [14]. The frequency difference of the modes results from both the intrinsic frequency difference $\omega_1 - \omega_2$ and the coupling $D$.

To investigate the behavior of the system, the two beams are connected to two different sources with independent frequencies and amplitudes. The monitored output variables are the amplitudes and phases of a linear combination of the mechanical displacements of the two beams at or near the two drive frequencies [21]. By applying various combinations of small amplitude signals to the two beams at frequencies near the mode resonances, the linear coupling parameters can be determined. For the particular device shown in Fig. 1a the mode frequencies are determined to be $\omega_1 / 2\pi = 16.79$ MHz and $\omega_{II} / 2\pi = 17.25$ MHz and the eigenvectors...
and a corresponding equation for the plex mode amplitudes. Briefly, we introduce the slowly varying corrections \[9, 10, 11\], where hysteretic switches between different states in frequency sweeps were also predicted. Briefly, we introduce the slowly varying complex mode amplitudes \(A_I, A_{II}\) and forces \(F_I, F_{II}\) using \(x_{II} = \text{Re}(A_{II}e^{i\omega_{II}t})\) and \(\theta_{DI} = \text{Re}(F_{II}(t)e^{i\omega_{II}t})\) (where \([I]\) stands for either \(I\) or \(II\)), substitute into the equations of motion, and retain only the near resonant terms. This reduces the equations of motion to

\[
2i\omega_I \dot{A}_I + i\omega_I^2 A_I + \alpha_I |A_I|^2 A_I + \beta_I |A_I|^2 A_I = F_I(t), \tag{2}
\]

and a corresponding equation for \(A_{II}\). The mode nonlinearity parameters \(\alpha_I\) and \(\beta_I\) are calculated from \(\alpha_1, \alpha_2\) and the eigenvectors \(e_I, e_{II}\) so that all the parameters in \([2]\) are known from linear measurements and the beam geometry and material constants.

We first look at the case where each mode responds at the drive frequency \(\omega_{DI}\) which is set near the resonant frequency \(\omega_I\) so that \(A_I \approx e^{i(\omega_{DI} - \omega_I)t}\), with

\[
|A_I|^2 = \left|\frac{F_I^2}{2\omega_I(\omega_{DI} - \omega_I) - \alpha_I |A_I|^2 - \beta_I |A_{II}|^2} + \omega_I^2 \gamma_I^2\right|^2, \tag{3}
\]

etc. For a single drive (e.g. \(A_{II} = F_{II} = 0\)), so that the cross-mode nonlinear coupling proportional to \(\beta_I\) is not involved, this expression reproduces the regular Duffing response curve \[10\]. Prominent features are the shift of the frequency of the maximum response to larger values for positive \(\alpha\) (nonlinear spring stiffening), and bistability and hysteresis that develop above a critical drive strength. An experimental example of upward and downward frequency sweeps for a drive strength 4.3 times critical is shown in Fig. 1b.

The response of the system driven near resonance and for small dissipation and driving can be calculated from \([1]\) using the standard methods of secular perturbation theory \([2]\). This approach has previously used for the case of parametrically driven nanomechanical devices \([4, 10, 11]\), where hysteretic switches between different states in frequency sweeps were also predicted. Briefly, we introduce the slowly varying complex mode amplitudes \(A_I, A_{II}\) and forces \(F_I, F_{II}\) using \(x_{II} = \text{Re}(A_{II}e^{i\omega_{II}t})\) and \(\theta_{DI} = \text{Re}(F_{II}(t)e^{i\omega_{II}t})\) (where \([I]\) stands for either \(I\) or \(II\)), substitute into the equations of motion, and retain only the near resonant terms. This reduces the equations of motion to

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the motion of the first mode depending on the excitation strength of the second mode through the last term in $\bar{\alpha}_I$.

To test the frequency tuning we excite the second mode at a drive level approximately 4.3 times the critical value so that the spectral response is the strongly nonlinear Duffing curve (see Fig. 1b). As the actuation frequency of the second mode is steadily increased in small steps its vibration amplitude rises over a wide frequency range until it drops to the lower amplitude state beyond the maximum. The evolution of the spectral response of the first mode is monitored at a driving level approximately four times lower than the critical value for this mode using a network analyzer. The dependence of the first mode frequency shift on the vibration amplitude of the second mode is shown in Fig. 1c. The experimental results for frequency shift on the vibration amplitude of the second mode are shown in Fig. 2. The plots show the shape of the resonance curve tilts to the left, as opposed to the usual case for a doubly clamped beam where the peak leans to the right. It also means that the nonlinear coefficient vanishes for some drive strength and frequency of the second mode.

Some experimental results for the first mode driven at twice the critical strength illustrating this effect are shown in Fig. 2. The plots show the shape of the resonance peak for three values of the drive frequency of the second mode. In panel (a), the amplitude $A_{II}$ of the second mode is low and the first mode spectral response has the regular nonlinear Duffing shape leaning to the right. For larger $A_{II}$ as in (b), the first mode resonance peak shape assumes a form close to a Lorentzian with little nonlinearity apparent. For even larger $A_{II}$ as in (c) the sign of the effective Duffing coefficient becomes slightly negative, causing the spectral response peak to lean to the left. The quenching of the nonlinearity in (b) could be used to enhance the limited dynamic range of nanomechanical devices.

If the drive level of the first mode is increased further a variety of new effects can be observed. Under these conditions the dynamics of the system is not fully explained by the steady state solutions as in (3), and so we solve for the expected behavior by numerically integrating the time-dependent coupled equations (2). Now the behavior is more complex, as the response of the first mode becomes large enough to cause transitions in the second mode response through the nonlinear coupling. As a result the spectral response curves acquire peculiar nontrivial shapes, as shown in Fig. 3. Numerical simulations of the equations give good predictions for the complex phenomena.

An obvious difference between the experimental and theoretical plots in Fig. 3 is the noisy regions on the theoretical curves near the up and down transitions. This difference actually results from the different ways the plots are generated in theory and experiment, and a more careful investigations shows consistent and interesting dynamics in both experiment and theory: for the drive frequencies near the transition points the fast (about 17 MHz) oscillating response becomes amplitude modulated with a frequency of about 10 to 20 kHz. The theoretical equations (2) show that the frequency for the amplitude modulation is determined by the line width, which in our experimental setup is about 8 kHz, and is not related to sum and difference frequencies of the two modes.

Examples of simulated and measured amplitude modulation dynamics for three different sets of drive parameters are shown in Fig. 4 for a second device with different parameters to the one used for Figs. 1-3. The first column shows examples of numerically calculated phase portraits of Re $A_{II}$ versus Re $A_I$ obtained by solving the time de-
at frequencies that correspond to double and quadruple
and the spectrum reveals amplitude modulation peaks
visible in both theoretical and experimental trajectories,
iddle row of Fig. 4. The complicated loop structures are
An example of period quadrupling is shown in the mid-
or four revolutions in order to complete the cycle [19].
titude modulation, where the phase trajectory takes two
observed period doubling or quadrupling in the ampli-
corresponding to the anharmonic amplitude modulation.
of the measured mechanical signal shows satellite peaks
mode at small amplitude and
vice versa. The spectrum
mechanical signal shows satellite peaks corresponding to the anharmonic amplitude modulation.
As the parameters of the system are changed, we have
observed period doubling or quadrupling in the amplitude modulation, where the phase trajectory takes two
or four revolutions in order to complete the cycle [19].
An example of period quadrupling is shown in the mid-
le row of Fig. 4. The complicated loop structures are visible in both theoretical and experimental trajectories,
and the spectrum reveals amplitude modulation peaks at frequencies that correspond to double and quadruple
periods. Period doubling transitions are often associated
with chaotic dynamics [19], and indeed for other param-
eter values we observe chaos in our coupled nanoelec-
tromechanical system and in the theoretical model, as
shown in the bottom row of Fig. 4. The evidence for the
chaotic dynamics in the experiment is the broad band
component to the spectrum (evident in the shoulders to
the amplitude modulation peaks), and a phase portrait
trajectory that does not form a closed loop [23]. The
theoretical model shows a similar phase portrait.

Our detailed study of two elastically coupled, indepen-
dently driven, nanomechanical beam resonators reveals
complex nonlinear dynamics with a number of potential
applications. For example, driving one of the modes can
be used to tune the effective nonlinearity of the other
mode. This can be used to significantly increase the dy-
namic range of the resonator by quenching the effective
nonlinearity. In a first approximation, the motion of one
mode couples quadratically to the resonance frequency of
the other mode, a phenomenon that has been proposed
for quantum nondemolition measurements in nanome-
chanical systems. For larger vibration amplitudes spont-
aneous oscillations of amplitude modulation develop, at
a frequency determined by the resonator ring-down time.
These oscillations show period doubling and chaos char-
acteristic of strongly nonlinear systems. The full range
of complex dynamics investigated is quantitatively re-
produced by theory. Our success at predicting and sub-
sequently observing quite delicate features of the non-
linear dynamics is strong evidence that the nonlinearity
and coupling in arrays of nanomechanical devices can be
quantitatively understood and controlled.

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FIG. 4: Complex dynamics of two strongly coupled nanome-
chanical resonators. The three rows correspond to different
input parameters (drive frequencies and amplitudes). The
first column shows the theoretical calculation of the phase
portrait, the second shows its experimental measurement, and
the third column shows the corresponding experimental power
spectrum of the optical measurement near one of the drive
frequencies.

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[21] The combination measured depends on the geometry of the optical spot relative to the beams, which we determine from the linear experiments using a variety of drive combinations.
[22] The structure of the dynamical system depends on the ratio $\alpha_I/\alpha_{II}$ which is close to unity for our nearly identical beams. The absolute magnitude of $\alpha$ (approximately 0.0072 (MHz/nm)$^2$) calibrates the amplitude of the response in nm.
[23] The theoretical points, which are the end values of a long numerical simulation, depend on the phase of the amplitude modulation at the end of each run, whereas this modulation leads to a drop in the experimentally measured amplitude since RF power is transferred outside the measurement bandwidth.
[24] The full phase space is four dimensional, $\text{Re } A_I, \text{Re } A_{II}, \text{Im } A_I, \text{Im } A_{II}$; we plot a two dimensional projection.
[25] Previous experiments that suggested chaos in a nanomechanical system demonstrated a complex behavior of the response at one of the drive frequencies as that parameter was swept, rather than complex dynamics for fixed system parameters as shown in our work.