Localisation, delocalisation, and topological transitions in disordered 2D quantum walks

Jonathan M. Edge$^1$ and Janos K. Asboth$^2$

$^1$Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23 106 91 Stockholm, Sweden
$^2$Institute for Solid State Physics and Optics, Wigner Research Centre, Hungarian Academy of Sciences, H-1525 Budapest P.O. Box 49, Hungary

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We investigate time-independent disorder on several two-dimensional discrete-time quantum walks. We find numerically that, contrary to claims in the literature, random onsite phase disorder, spin-dependent or otherwise, cannot localise the Hadamard quantum walk; rather, it induces diffusive spreading of the walker. In contrast, split-step quantum walks are generically localised by phase disorder. We explain this difference by showing that the Hadamard walk is a special case of the split-step quantum walk, with parameters tuned to a critical point at a topological phase transition. We show that the topological phase transition can also be reached by introducing strong disorder in the rotation angles. We determine the critical exponent for the divergence of the localisation length at the topological phase transition, and find $\nu = 2.6$, in both cases. This places the two-dimensional split-step quantum walk in the universality class of the quantum Hall effect.

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Discrete-time quantum walks$^1$, which we will simply refer to as quantum walks, are the quantum analogues of classical random walks. They are model systems which sit at the interface between quantum information theory and condensed matter physics. On the one hand, they form archetypical systems for studying quantum algorithms, e.g. Grover’s search algorithm$^2$. On the other hand, condensed matter physics has recently also shown interest in quantum walks$^3$, in particular ever since it was shown that the topological phases$^6,7,10–14$ can also be realised in quantum walk$^6,9$.

Condensed matter physics has a very wide scope, but one important subject of it is disorder and the associated localisation of single-particle wave functions (for a review, see Ref. 9). Thus, to understand quantum walks from the condensed matter point of view, we need to address the effect of disorder on the propagation of a quantum walker.

One of the interesting aspects of quantum walks is that in the absence of disorder the quantum walker propagates ballistically$^9$, thus much faster than its classical counterpart, which shows diffusive propagation. The ballistic spreading of the quantum walk is related to the quantum speed up of certain quantum algorithms, notably Grover’s search algorithm$^2$, as the quantum walker is able to explore the search space more rapidly than its classical counterpart.

If disorder is introduced into the quantum walk system, it is expected to break the ballistic propagation of the quantum walk, analogously to the way in which disorder introduced into a solid-state system will prevent the ballistic propagation of electrons due to disorder scattering. This could be of relevance to the quantum information applications of quantum walks, as the quantum speed up of quantum algorithms is intimately related to ballistic propagation.

Although the effects disorder on one-dimensional quantum walks have been studied, not much is known about the two-dimensional case. For one-dimensional quantum walks it has been shown that spatial disorder can lead to exponential localisation of all energy eigenstates$^10,11$. It was also found, however, that chiral symmetry can prevent localisation in one dimension$^3$. To the best of our knowledge, the effects of spatial disorder in two-dimensional quantum walks and the impact on the quantum walk propagation was only studied in Ref. 15 for the Hadamard walk. In that paper it was reported that in the disordered system the wave function remains majoritatively close to the starting position, unlike in the clean case, where the amplitude of the wavefunction at the initial site decreases to zero in the long time limit. This concentration of the wave function close to its initial position (which is according to a looser terminology used as the definition of localisation, as, e.g, in Ref. 16) was attributed to Anderson localisation.

In this article we study the effects of spatial disorder on the propagation and localisation of the Hadamard walk and on the broader family of two-dimensional split-step walks to which it belongs. Sec. II collects the definitions of these walks, recalls their connection and their topological phases. In Sec. III we show that phase disorder localises generic split-step walks, but not the Hadamard walk: this latter shows slow diffusion (contrary to the findings of Ref. 15). In Sec. IV we attribute this difference to the fact that the Hadamard walk is critical: it is a split-step walk that is tuned to a topological phase transition point. We demonstrate this phase transition and calculate the corresponding critical exponent, $\nu = 2.6$, which places the split-step walk in the quantum Hall universality class. Finally, in Sec. V we study disorder in the angle parameters of the split-step walks. Based on the previous section, one can expect that if the angle disorder is large enough, the split-step walk can become diffusive even with maximal phase disorder. We show...
that this disorder-induced delocalisation actually takes place, and find for it the same critical exponent $\nu$ as in Sec. III. We also find that angle disorder alone leads to diffusion rather than localisation, which is probably connected to the presence of a particle-hole symmetry in this disordered quantum walk.

I. DEFINITIONS OF THE QUANTUM WALKS

A particle undergoing a quantum walk on a square lattice is represented by a time-dependent two-component wavefunction,

$$|\psi(t)\rangle = \sum_{m,n,s=\pm 1} \psi(t)_{m,n,s} |m,n,s\rangle. \quad (1)$$

Here $m,n \in \mathbb{Z}$ give the horizontal and vertical positions on the lattice, $s \in \{+1,-1\}$ is the value of the internal state that we call spin, and $t \in \mathbb{N}$ denotes the time, which is only allowed to take on discrete values. We take as initial condition a localized state, $|0,0,+1\rangle$, and obtain the time evolution by iterated applications of the time evolution operator $U$ on the state,

$$|\psi(t)\rangle = U^t |0,0,+1\rangle. \quad (2)$$

We will consider different types of quantum walks, with the time evolution operator $U$ consisting of a product of several shift operators and coin operators, to be defined below.

Shift operators displace the walker by one lattice site in a direction that depends on its internal state, but their action is independent of the position of the walker. We consider the quantum walk on a square lattice with the sites labelled by $(m,n)$ and so define the following shift operators,

$$\hat{S}_x = \sum_{m,n,s=\pm 1} |m+s,n,s\rangle \langle m,n,s|; \quad \hat{S}_y = \sum_{m,n,s=\pm 1} |m,n+s\rangle \langle m,n,s|. \quad (3)$$

We use absorbing boundary conditions$^{17}$ in both the $x$ and $y$ directions.

Coin operators act locally on the walker, but can have position-dependent parameters. They can be written in compact forms using the Pauli operators, $\sigma_z |m,n,s\rangle = s |m,n,s\rangle$: $\sigma_x |m,n,s\rangle = |m,n,-s\rangle$; $\sigma_y |m,n,s\rangle = i s |m,n,-s\rangle$; and $\sigma_0 |m,n,s\rangle = |m,n,s\rangle$, for all values of $m,n$ and $s$. We consider the Hadamard coin operator

$$\hat{H} = 2^{-1/2} (\sigma_x + \sigma_z). \quad (4)$$

and the spin rotation operator,

$$\hat{R}[\theta_j] = \sum_{m,n} e^{-i\theta_j^{mn} \sigma_y} |m,n\rangle \langle m,n|, \quad (5)$$

with $\theta_j^{mn}$ denoting the position-dependent rotation angles. The index $j$ differentiates between rotations in one sequence of operations defining the timestep; below, the time evolution operator will contain two spin rotations, and so $j$ will take values 1 and 2. Since $\hat{R}[\theta_j + \pi] = -\hat{R}[\theta_j]$, only angles between $-\pi/2$ and $\pi/2$ give distinct rotation operators (the minus sign is only a phase factor).

The first type of quantum walk we consider is the Hadamard walk, defined through its time evolution operator,

$$U_H = \hat{S}_y \hat{H} \hat{S}_x \hat{H}. \quad (6)$$

It thus consists of a Hadamard coin operation followed by a spin-dependent displacement in the $x$ direction, another Hadamard coin operation rotation, a displacement in the $y$ direction, and a spin-dependent phase operation.

We also consider the split-step quantum walk$^8$, where the time evolution operator is defined as

$$U_s = \hat{S}_y \hat{R}[\theta_2] \hat{S}_x \hat{R}[\theta_1]. \quad (7)$$

As described in Ref. 17 for rotation angles $\theta_j^{mn} = \theta_j$.

FIG. 1. Phase diagram for the topological quantum numbers for the split-step quantum walk defined by eq. (6) without disorder$^8$. As described in Ref. 17 due to the time-periodic nature of the quantum walk, two topological invariants can be defined, only one of which changes in the parameter range under consideration in this paper. The red transparent box shows the range of $\theta_1$ and $\theta_2$ which is accessible at the point $\theta_1 = \theta_2 = \pi/4$ for the parameters in fig. 3 and 5. The blue four-sided star shows the parameter set $\theta_1 = 0.35\pi$, $\theta_2 = 0.15\pi$ which is frequently used throughout this paper. The purple eight-sided star shows the parameters $\theta_1 = \pi/4$, $\theta_2 = -\pi/4$ at which the split-step quantum walk reduces to the Hadamard walk.
independent of position, the system has two topological invariants: the Chern number and the quasienergy winding, which are determined by $\theta_1$ and $\theta_2$. The phase diagram for the topological invariants is reproduced in Fig. 1. The Chern number for this quantum walk is always zero, but, as we will see, the quasienergy winding plays an important role in determining the localisation properties.

The split-step quantum walk can be seen as a generalisation of the Hadamard walk. Since $H = \sigma_x \exp[-i(\pi/4)\sigma_y] = \exp[i(\pi/4)\sigma_x]$, we have

$$U_H = S_y R(-\pi/4) S_x^{-1} R(\pi/4).$$

Thus the Hadamard walk is the a mirror reflected $x \leftrightarrow -x$ version of the split-step walk, with $\theta_1 = -\theta_2 = \pi/4$.

II. THE EFFECT OF PHASE DISORDER

One way to introduce time-independent disorder into quantum walks, is to multiply the wavefunction at the end of each timestep by a random phase factor, which depends on position and spin value, but not on time. For this, we define the phase operators

$$\hat{P}_a[\phi] = \sum_{m,n} e^{i\phi_{mn}} |m,n\rangle \langle m,n|,$$

with $a = 0$ for spin-independent, and $a = z$ for $s$-dependent phase operator. We take the phases $\phi_{mn}$ to have zero mean value, and distributed randomly in the interval $[-\delta\phi/2, \delta\phi/2]$. Intuitively, $\hat{P}_0$ mimics an on-site energy in a tight binding lattice model, while $\hat{P}_z$ can be understood as a disordered magnetic field. As such, these types of disorder favour localisation in non-interacting two-dimensional lattice systems.\textsuperscript{13}

A. Hadamard walk with phase disorder: Disorder-induced diffusion

To add phase disorder to the Hadamard quantum walk, Eq. (1), we define the timestep operator as

$$U_{H,a} = \hat{P}_a \hat{S}_y \hat{H} \hat{S}_x \hat{H}.$$  

For different values of $\delta\phi$ between 0 and 2$\pi$, and different disorder realizations, we initialise the quantum walker at the centre of a 220 $\times$ 220 lattice\textsuperscript{20}, and follow the time evolution for 1000 time steps.

To detect localisation, we will use two of its signatures. First, in the presence of localisation, the wave function in the long time limit should decrease exponentially as a function of the distance from the initial site,

$$\sum_{s=\pm 1} |\Psi(t \to \infty)_{m,n,s}| \propto e^{-2\sqrt{m^2+n^2}/\xi}.$$  

The localisation length $\xi \in \mathbb{R}$ of a localized wavefunction should be well defined (at least in the vicinity of the initial site). Second, in the localised case, the spreading $s(t)$ of the wave function, defined as

$$s^2(t) = \sum_{m,n} (m^2+n^2) |\Psi(t)_{m,n,s}|^2,$$

should saturate, i.e., $\lim_{t \to \infty} s(t) = \text{const}$.

In the Hadamard walk with phase disorder, we find diffusive dynamics instead of localisation. In Fig. 2, we have plotted a cross-section of the probability amplitude squared of the wave function after 1000 timesteps, averaged over 500 disorder realizations. We see that although the wave function is strongly peaked towards the centre, it does not decay exponentially: for both types of quantum walks, it shows a Gaussian profile characteristic of diffusive behaviour\textsuperscript{21}. The inset shows the spreading $s(t)$, which displays no sign of saturation: it is well approximated by $s(t) \propto t^{1/2}$, which again is an indication of diffusion.

Our results contradict those of Ref.\textsuperscript{15} where localisation was found for the disordered Hadamard walk, and also go against the intuitive picture that onsite disorder induces localisation. Although it cannot, in principle, be ruled out that localisation will eventually set in, the 1000 time steps we considered give an already significantly larger timescale than the 20 time steps investigated in...
However, the wave function spreads more slowly, and for increasing the phase disorder, the walk shifts from a delocalised set to a localised (lim $t \to \infty s$) behaviour. Inset: exponent $\alpha$ of $s(t) \propto t^\alpha$ fitted to the curves between $t_{\text{min}} = 10$ and $t_{\text{max}}$ (blue solid line: $t_{\text{max}} = 100$, green dashed line: $t_{\text{max}} = 1000$). For $\delta \theta = 0$, the system behaves ballistically, with $\alpha = 1$. For larger disorder the fitted value of $\alpha$ decreases with time, indicating localisation.

Ref. [15] Why is there no localisation in the disordered Hadamard walk? This is one of the main questions which we will answer below.

### B. Split-step walk with phase disorder: Disorder-induced localisation

To obtain a full picture of phase disorder and its effects on localisation, we now apply disorder to the generic split-step walk, which can be seen as a generalization of the Hadamard walk, cf. Eq. (9). We fix the rotation angles at $\theta_1 = 0.35\pi$ and $\theta_2 = 0.15\pi$. As seen on the phase space of the walk, Fig. 1, this set of parameters is far from the continuous lines along which the quasienergy gap closes. The time evolution operator is then given by

$$U_{s,a} = \hat{P}_a |\phi\rangle S_y R(\theta_2) S_y R(\theta_1),$$

with $a = 0$ for spin-independent, and $a = z$ for spin-dependent disorder. We remark that both types of phase disorder break the particle-hole symmetry of the system, which arose since $U_s$ was real.

Our numerical results indicate that unlike the Hadamard walk, the 2-D split-step quantum walk is localised by phase disorder. As shown in Fig. 3, in the absence of phase disorder, $\delta \phi = 0$, the wave function spreads ballistically, as expected. As $\delta \phi$ is increased, however, the wave function spreads more slowly, and for large values of $\delta \phi$, it seems to saturate indicating localisation. The inset of Fig. 3 shows the localisation transition through the exponent $\alpha$ obtained by fitting $s(t) \propto t^\alpha$ to the numerical results over short ($10 < t < 100$, green dashed) and long ($10 < t < 1000$, blue solid) times. When $\delta \phi = 0$ we observe ballistic propagation, indicated by a time-independent value of $\alpha = 1$. For increasing values of disorder $\alpha$ decreases and, more importantly, decreases as a function of time. This indicates that a power law fit for $s(t)$ does not provide a good fit and that the system is localising. Additional evidence for localisation is furnished by the shape of the wavefunction in the long-time limit, as shown in Fig. 2.

### III. Topological Transition Behind Delocalization

The difference in the effects of phase disorder on the Hadamard walk (diffusion) and the generic split-step quantum walk (localisation), is due to the fact that the Hadamard walk is a special case of the split-step walk, tuned to a topological phase transition point. In this section we expand on this explanation, and investigate it numerically, obtaining the critical exponents corresponding to this phase transition via single parameter scaling.

To make sure that the effect we observe is generic, we also include a small amount of disorder in the angle parameters of the split-step quantum walk. These angles $\theta_{mn}$ will be chosen randomly and independently for each site, from a uniform distribution in the interval $[\theta_1 - \delta \theta, \theta_1 + \delta \theta]$. Thus the first and second rotation have the same disorder $\delta \theta$, which we fix in this section to be $\delta \theta = 0.2\pi$.

In this section we will often refer to the quasienergy winding $W$ of a disordered quantum walk. In a homogeneous quantum walk, with global parameters, the quasienergy winding is a single-valued function of the parameters, $W(\theta_1, \theta_2, \phi)$. Moreover, it is straightforward to show that $W$ does not depend on the global phase $\phi$: a spin-dependent (spin-independent) phase just displaces all eigenstates of the walk in quasienergy (quasimomentum), and thus cannot close the gaps between the quasienergy bands, a prerequisite for changing the topological invariants. Although there is no known way to calculate $W$ in a disordered system, we can expect that it is given as some weighted average of the “local quasienergy windings”, i.e., of the quantities $W(\theta_{mn}^1, \theta_{mn}^2, \phi_{mn})$. If the disorder is such that these local quasienergy windings are all $W$, independent of $m, n$, then we can expect that the topological invariant of the disordered system as a whole is given by this value.

#### A. Topological transition by tuning the mean rotation angles

We locate the topological phase transition, by tuning the parameters of the quantum walk: we gradually in-
crease \( \theta_2 \) from 0 to \( \pi/2 \) while keeping \( \theta_1 + \theta_2 = \pi/2 \) constant, all the while keeping maximal phase disorder, \( \delta \phi = 2\pi \), and a moderate angle disorder, \( \delta \theta = 0.2\pi \). This path is marked by the dashed line in Fig. 1. We characterise the localisation properties for each set of parameter values via the time-dependent diffusion coefficient,

\[
D(t) = \frac{s^2(t)}{t}.
\] (11)

In the long-time limit the diffusion coefficient \( D(t) \) is a constant in regimes governed by diffusion (metallic or possibly critical regimes) and decreases in time in the localised regime (‘non-metallic’ regime). We choose this quantity because it will be a suitable starting point for the scaling analysis of the transition point.

Our results for the diffusion coefficient \( D \) for various times, as the rotation angles \( \theta_1 \) are tuned across topological phase transition, are shown in Fig. 1. At most values of the angles, the calculated values of \( D(t) \) decrease with time \( t \), and we can infer that the quantum walk is localised. At the point \( \theta_1 = \theta_2 = \pi/4 \), however, the curves of \( D(t) \) corresponding to various times overlap, and so the system is diffusive. This is a delocalisation transition.

We attribute the delocalisation at \( \theta_1 = \theta_2 = \pi/4 \) to the occurrence of a topological phase transition. As explained above, we expect that at the endpoints of the path, the quantum walk has topological invariants \((-1,0)\), and \((+1,0)\), respectively. Thus, somewhere along the path a topological phase transition has to occur. At the phase transition point, the average topological invariant of the disordered system is not well defined. As shown by the red transparent box in Fig. 1 at the special point the probabilities for the two phases are equal. Thus for symmetry reasons the transition has to occur at \( \theta_1 = \theta_2 = \pi/4 \).

Another angle from which to understand the delocalisation at \( \theta_1 = \theta_2 = \pi/4 \) is the following. At the interface between two domains of the quantum walk with different topological phases there are edge states\(^{25}\). If both possible topological quantum numbers occur locally with equal probability, a percolating network of edge states appears. This network can be thought of as a realisation of the Chalker-Coddington network model for the integer quantum Hall effect\(^{23}\), tuned to the plateau transition point. At this point a non-zero conductance appears, which in this case is signalled by a diffusively spreading wavefunction.

\[ \xi = A|\eta|^{-\nu}; \] (12)

Here \( \eta \) is the distance from the critical point, \( A \) is a constant of proportionality and \( \nu \) is the critical exponent\(^9\). When this transition is obtained as explained above, along the line \( \theta_1 + \theta_2 = \pi/2 \), the role of \( \eta \) is played by

\[ \eta = \theta_2 - \theta_1. \] (13)

Instead of measuring the localisation length \( \xi \) directly, we find \( \nu \) by assuming single-parameter scaling of the diffusion coefficient \( D(t) \) of Eq. (11). Taking finite-time corrections\(^{20}\) into account, we have

\[ \ln D(t) = F(t^{1/2\nu} u) + t^{-\nu} G(t^{1/2\nu} u); \] (14)

\[ u = \eta + O(\eta^2). \] (15)

Here the scaling functions \( F(z) \), \( G(z) \), and \( u(z) \), as well as the exponents \( y \) and \( \nu \) are to be fitted to the numerical data.

We computed the high accuracy data for the fitting procedure by simulating the quantum walk on an 800 \times 800 lattice for varying number of timesteps over many

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FIG. 4. Diffusion coefficient as a function of \( \theta_2 - \theta_1 \) with \( \theta_1 + \theta_2 = \pi/2 \) for \( \delta \theta = 0.2\pi \), \( \delta \phi = 2\pi \), obtained by averaging over 100 disorder realisations on a 220x220 lattice.
disorder realizations. A large number of disorder realizations was used for the runs at shorter times (4001, 4001, 2001, 1001, to obtain $D(t)$ at $t = 32, 80, 203, 512$, respectively), whereas due to self averaging, fewer disorder realizations already provided enough accuracy for the runs at longer times (200 realizations for $t = 8192, 3250, 1290$). The resulting values of $D(t)$ were then fitted with the scaling Ansatz, Eq. (14), using a Taylor series expansion of the functions $F, G,$ and $v$ to various orders. Instead of converging to a single solution, we obtained a good fit to the data for different forms of the scaling functions, and also different values of the exponents – an example is shown in Fig. 5 lower panel. To represent our estimate of the critical exponent $\nu$, we define an estimator function $E(\nu)$, whose integral between any two values $\nu_{\text{min}}$ and $\nu_{\text{max}}$ reflects our degree of confidence that $\nu_{\text{min}} < \nu < \nu_{\text{max}}$. The construction of this function, along with the details of the fitting procedure, are explained in Appendix A.

As seen in Fig. 5, our estimator of the critical exponent $\nu$ is a bi-modal function, with a peak around $\nu_m = 2.616$ (full width at half the maximum of 0.125), and a second peak at $\nu_2 = 2.384$. The value corresponding to the larger peak, $\nu_m = 2.616$, is very close to the quantum Hall critical exponent of $2.593 \pm 0.003$. The smaller peak is close to previous estimates of the exponent of the quantum Hall transition, which are now attributed to bi-stability of the fitting procedure, possibly related to finite-size effect. To summarize, the transition which we observe is compatible with the integer quantum Hall transition universality class.

IV. DISORDER IN THE ROTATION ANGLES OF THE SPLIT-STEP WALK

We already introduced disorder to the rotation angles of the split-step quantum walk, although with a small value of $\delta \theta = 0.2\pi$, in the previous sections. We now examine what happens to the quantum walk as this disorder grows. We first consider a split-step quantum walk that is localised by maximal phase disorder. As we turn on the angle disorder $\delta \theta$, we will find that at special values of $\delta \theta$, the walk delocalises. We then consider a split-step walk with no phase disorder, only angle disorder. We find that, contrary to what one might expect, angle disorder does not induce localisation.

A. Competition of phase and angle disorder: Disorder-induced delocalisation

We now consider what happens if we first localise a quantum walk by phase disorder, as in section IV B, and then increase the disorder in the rotation angles $\delta \theta = \delta \theta_1 = \delta \theta_2$ to $\pi$. At this maximal value, as well as at $\delta \theta = \pi/2$, all values of the rotation angles are equally likely, and therefore we expect that the topological invariant of the quantum walk cannot be well defined. By the arguments of the previous section we thus expect localised behaviour at these values of the rotation angle disorder.

Our numerics clearly show the disorder-induced delocalisation, at both $\delta \theta = \pi/2$ and $\delta \theta = \pi$. We plot the time-dependent diffusion coefficient $D(t)$ in Fig. 5 as a function of $\delta \theta$, at fixed mean values of the rotation angles, $\theta_1 = 0.35\pi$, $\theta_2 = 0.15\pi$, and maximal phase disorder, $\delta \phi = 2\pi$. The diffusion coefficient decreases with time, indicating localised dynamics, except near the points of maximal disorder, $\delta \theta = \pi/2$ and $\pi$: there the system is diffusive.

We believe that the disorder-induced delocalisation we observe here accompanies a topological phase transition, much like in the case of Fig. 4. Indeed, for $\delta \theta < \pi/2$ the majority of sites have parameters corresponding to topological invariants of $(+1, 0)$, whereas for $\pi/2 < \delta \theta < \pi$,
the majority topological invariant is $(-1, 0)$. At the point $\delta \theta = \pi/2$ all rotation angles have the same probability and thus the system cannot be assigned a global topological invariant. We performed a scaling analysis on this transition, with now the control parameter being $\eta = \delta \theta - \pi/2$. We obtained consistent results of $\nu = 2.58 \pm 0.05$, in agreement with the mode of the distribution of $\nu_{\text{rms}} = 2.6$ shown in Fig. 6. This confirms that the exponent $\nu$ is universal: its value does not depend on the method we use to drive the system across the transition.

**B. Diffusive behaviour in the presence of only rotation angle disorder**

Finally we investigate the spreading of the split step quantum walk in the presence of only rotation angle disorder. We fix the mean rotation angles to $\theta_1 = 0.35\pi$ and $\theta_2 = 0.15\pi$. This choice of the mean rotation angles places the system in an insulating phase with topological invariants $(+1,0)$, as shown by the blue four-sided star in Fig. 6.

In Fig. 7 we show the result of increasing the rotation angle disorder from $\delta \theta = 0$ to $\delta \theta = 2\pi$. We observe the expected ballistic behaviour at $\delta \theta = 0$, and already for rather small values of $\delta \theta$ we see the crossover to the diffusive regime with $s \propto t^{1/2}$. Unlike in the case with phase disorder, though, we don’t observe any signs of localisation here.

Although we do not have a complete explanation for this absence of localisation, we believe it is related to the particle-hole symmetry of the system, that is not broken by rotation angle disorder. The time evolution operator has only real elements in position basis, and thus, the effective Hamiltonian possesses particle-hole symmetry represented by complex conjugation. In time-independent lattice systems, the presence of this symme-

**V. SUMMARY AND CONCLUSION**

To summarise, we have found that the Hadamard walk is not localised by phase disorder, while generic split-step quantum walks are. We gave an intuitive physical explanation for this difference, namely, that the Hadamard walk is a split-step walk tuned to a topological phase transition. We corroborated this picture by numerically demonstrating that this transition can be reached through angle disorder as well, at precisely the value that this explanation predicts. We determined the critical exponent for the divergence of the localisation length for both of these routes to criticality, and found $\nu = 2.6$, which places the split-step quantum walk with phase disorder in the universality class of the quantum Hall effect. We have also found that angle disorder alone does not localise the split-step quantum walk, which may be due to the fact that this disorder does not break the particle-hole symmetry of the system.

A useful next step to strengthen our interpretation of the localisation effects of disorder would be the calculation of the topological invariant of the disordered split-step walk, the quasienergy winding. Here, any of the existing approaches to the Chern number in disordered systems can be of use. One could extend the definition of the quasienergy winding using noncommutative geometry, or measure the winding number of the scattering matrix.

Our interpretation of the localization phenomena relied on qualitative similarity with disordered quantum
Here the function $F$ is related to $\tilde{F}$ as

$$F(z) = \tilde{F}(z^{2\nu}/A^2),$$

(A4)

where $A$ is the constant from Eq. (12). The function $G$ takes into account finite-time corrections, with $y$ denoting the first subleading exponent. We expand the formulae for $\ln D$ and $u$ of Eq. (A2) in Taylor series,

$$\ln D = \sum_{j=0}^{J} f_j(t^{1/2\nu}u)^j + \sum_{k=1}^{K} t^{-y} g_k(t^{1/2\nu}u)^{k-1}$$

$$u = \eta + \sum_{l=3}^{L} u_l l^l.$$

Since the function $\ln D$ must be even, $j$ may only take even values. In contrast, $k$ and $l$ may only take odd values, though $k = 0$ is also allowed, $k = 0$ corresponding to the absence of finite-time corrections. We choose the order of the approximation by fixing $J, K, L \in \mathbb{N}$, and then fit the Taylor coefficients $f_j, g_k, u_l$, and the exponents $y$ and $\nu$ to the $D(t)$ data. This allows us to obtain an estimate for the critical exponent $\nu$.

We fitted the numerically obtained data for $D(t)$ with 90 different functions, defined by different values of $J \in \{2 \ldots 10\}$, $K \in \{0, 1, 3 \ldots 9\}$, and $L \in \{1, 3, 5\}$. In a first approach, we systematically increased the order of the approximation, i.e., the values of $J, K$, and $L$, until we obtained a reasonable goodness of fit. Unfortunately, this did not yield a uniform convergence, neither when the standard $\chi^2$-test was used (value of $\chi^2$ per degree of freedom ($\chi^2/ndf$) of order 1), nor when the more sophisticated goodness of fit measure ($Q$, was used. We thus resorted to an alternative approach, as explained below.

We represent our results for the critical exponent $\nu$, by use of an estimator function $E(\nu)$, obtained by the following procedure. Out of the 90 different fitting functions $F_i$, we reject those which gave a value of $\chi^2$ outside of an acceptance range, $0.5 < \chi^2/ndf < 2$. For the remaining 60 functions $F_i$, with critical exponents $\nu_i$, we used the bootstrap method to evaluate the goodness of fit $Q_i$, and the 68% confidence interval for the critical exponent: $\nu \in [\nu_i - \sigma_i, \nu_i + \sigma_i]$. The estimator $E(\nu)$ is then defined as

$$E(\nu) = \frac{1}{N_{\text{max}}^2} \sum_{i=1}^{N_{\text{max}}} \sqrt{2\pi} \sigma_i \exp \left( -\frac{(\nu - \nu_i)^2}{2\sigma_i^2} \right),$$

(A5)

with $N_{\text{max}} = 60$, and $\sigma_i = (\sigma_i^+ + \sigma_i^-)/2$. This is the probability density of the critical exponent, if we deem all acceptable outcomes of our fitting procedure equally likely.

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We choose $\ln D$ instead of $D$ to perform scaling on, since this makes the fitting simpler, as only a lower order expansion of the fitting function is required. This follows Ref. [24].

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