Application of Lie group analysis to functional differential equations

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Abstract

In the present paper the classical point symmetry analysis is extended from partial differential to functional differential equations with functional derivatives. In order to perform the group analysis and deal with the functional derivatives we extend the quantities such as infinitesimal transformations, prolongations and invariant solutions. For the sake of example the procedure is applied to the continuum limit of the heat equation. The method can further lead to significant applications in statistical physics and fluid dynamics.

1 Introduction

In the paper we consider such functional differential equations which can be regarded as the extensions of partial differential equations. The key idea is that the discrete set of independent variables \((y_1, y_2, \ldots, y_n)\) in a partial equation is replaced by a continuous set of infinitely many variables denoted by \([y(x)]\). In order to illustrate the above extension for partial differential equations to functional differential equations we introduce the example

\[
\frac{\partial f}{\partial t} = \sum_{i=1}^{n} y_i \frac{\partial f}{\partial y_i}
\]

where \(f = f(t; y_1, y_2, \ldots, y_n)\). Taking the continuum limit we obtain

\[
\frac{\partial f}{\partial t} = \int y(x) \frac{\delta f}{\delta y(x)} \, dx
\]

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where $f = f(t; [y(x)])$ and the partial derivatives in $\frac{\partial}{\partial y(x)}$ have been replaced by a functional derivative $\frac{\delta}{\delta y(x)}$ which can also be denoted by $\frac{\partial}{\partial y(x)}dx$, (cf. Gelfand & Fomin (1963) for the definition of the functional derivative). The latter notation is more convenient in some situations (Hopf, 1952). The similar procedure of deriving the functional differential equations as the limit case of partial differential equations is used also in the work of Hopf (1952) and Breuer & Petruccione (1994).

Functional equations are used to describe problems of statistical mechanics where the probability distributions of phases and their time evolution are studied. In a discrete medium the phase space at a given time instant $t$ is defined by velocities and positions of all particles $(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_n)$ contained in a considered domain. In a continuum limit (eg. in hydromechanics) the phase becomes a continuous function of spatial variables $u = u(x)$. The time evolution of its probability distribution is determined by a functional equation.

Many examples of the functional equations with functional derivatives can be found in physics. The areas of interests of the authors are the turbulent flows and turbulent reacting flows. The idea of description of turbulence in terms of the characteristic functional has been introduced in a seminal work by E. Hopf (1952). The approach is presented in the book of Monin & Yaglom (1971), we mention also the work of Lewis & Kraichnan (1962). The similar functional formulation can also be derived for the Burgers equation. Functional formulation of Burgers equation is studied in the works of Breuer & Petruccione (1992) and Breuer et al (1996). The authors perform stochastic simulation of the Burgers model. The method is based on a discrete master equation which is equivalent to the Hopf functional equation in the limit of continuous space. The approach has been continued in works of Breuer & Petruccione (1994), Biechele et al. (1999), Friedrich (2002). Another functional differential equation is used for the statistical description of a turbulent premixed flame (see Oberlack et al. 2001).

The approach of Hopf originates from the field theory where the functional equations and functional derivatives are also used (cf. Itzykson & Drouffe 1989, Schweber 1962). The examples of functional equations in field theory are the Schwinger equations (cf. Pester et al. 2002) and Wheeler de Witt equations (cf. DeWitt 1967, Barvinsky & Kiefer 1998).

The symmetry analysis based on the Lie group theory has become a powerful tool of analysing, simplifying and finding solutions of partial differential equations (cf. Ibragimov 1994, 1995, 1996, Cantwell, 2002). The method also gives a deep insight into the underlying physical problems described by the differential equation. Examples of its applications include problems of fluid dynamics, where a broad range of invariant solutions for turbulence statistics were found (Oberlack, 1999, 2001). However, much less attention have been given so far to the symmetry analysis of functional equations. Some of the previous works concern the integro-differential (Zawistowski 2001, Chetverikov & Kudryavtsev 1995, Roberts 1985) and delay differential equations (Zawistowski 2002, Tanthanuch & Meleshko, 2004). However, to the best of the authors knowledge, the type of functional equations considered in the present paper, has not been studied so far in terms of the symmetry analysis. In the present work the classical symmetry analysis is extended to functional differential equations which contain functional derivatives. The structure of the paper is the following: first, we introduce the necessary notation to study functional differential equations. In the main section we extend classical Lie group methods to functional differential equations by extending quantities such as infinitesimal
transformations, prolongations or invariant solutions. Finally, as an example, the new method is applied to a continuum limit of the heat equation. This leads to the transformation groups as well as to invariant solutions of the considered functional equation.

2 Notations

In the present work we study the extension of the partial differential equation for a scalar function \( \Phi(u_1, \ldots, u_n, t_1, \ldots, t_m) \) of \( m+n \) independent variables. The general form of the differential equation describing \( \Phi \) writes:

\[
F(u_1, \ldots, u_n, t_1, \ldots, t_m, \Phi, \Phi_1, \Phi_2, \ldots, \Phi_q) = 0
\]

(3)

where \( \Phi_k \) denote the \( k \)-th derivatives of the function \( \Phi \) with respect to any possible combination of independent variables and \( q \) is the highest order of derivative present in Eq. (3). In the considered continuum limit \( u \) becomes a function of the continuous variable \( x \). For the sake of generality we consider that \( u = u(x) \) where \( u = (u_1, u_2, \ldots, u_p) \) and \( x = (x_1, x_2, \ldots, x_r) \) are vectors in a \( p \)- and \( r \)-dimensional space. In this case \( \Phi \) becomes a functional

\[
\Phi = \Phi([u_1(x)], \ldots, [u_p(x)], t_1, \ldots, t_m) = \Phi([u(x)], t_1, \ldots, t_m)
\]

(4)

and the partial differential equation (3) becomes a functional differential equation:

\[
F([u(x)], t_1, \ldots, t_m, \Phi, \Phi_1, \Phi_2, \ldots, \Phi_q) = 0;
\]

(5)

here again, \( \Phi_k \) denotes all possible derivatives of order \( k \), which can include partial derivatives with respect to \( t_i \) and functional derivatives with respect to \( u_\alpha(x) \). The following, equivalent notation will be used for the first functional derivatives:

\[
\Phi_{,u_\alpha(x)} = \frac{\delta \Phi}{\delta u_\alpha(x)} = \frac{\partial \Phi}{\partial u_\alpha(x)dx}.
\]

(6)

The different types of second order derivatives in Eq. (5) will be denoted by

\[
\Phi_{,t_it_j} = \frac{\partial^2 \Phi}{\partial t_i \partial t_j}, \quad \Phi_{,u_\alpha(x)t_j} = \frac{\partial^2 \Phi}{\partial u_\alpha(x)dx \partial t_j}, \quad \Phi_{,u_\alpha(x)u_\beta(x')} = \frac{\delta^2 \Phi}{\delta u_\alpha(x)\delta u_\beta(x')}.
\]

(7)

while the order of any derivative is commutative. Higher order derivatives can be expressed in an analogous way.

As \( x \) is a vector in a \( r \)-dimensional space, we will use the following, convenient notation for the integrals with respect to \( x \):

\[
\int_G dx = \int \cdots \int_G dx_1dx_2\cdots dx_r.
\]

(7)
where $G$ is a domain in $r$-space to be specified for each particular problem.

For further purposes we recall here some differentiation and integration rules for functionals. We also solve a simple functional equation by the method of characteristics. For the sake of clarity we will first present necessary formulae for the partial differential equation (3) and introduce their counterparts in the continuum limit (5). The two approaches will also be called “classical” and “continuum formulation”, respectively.

A function $\Phi(u_1, \ldots, u_n, t_1, \ldots, t_m)$ of a finite set of variables is differentiable for a particular value of its arguments if its variational (or functional) form $\delta \Phi$ is linear in $\delta u_i$, i.e. the following relation holds (cf. Hopf, 1952)

$$\delta \Phi = \sum_{i=1}^{n} \frac{\partial \Phi}{\partial u_i} \delta u_i + \sum_{i=1}^{m} \frac{\partial \Phi}{\partial t_i} \delta t_i. \quad (8)$$

The analogous formula in the considered continuum limit writes

$$\delta \Phi = \sum_{\alpha=1}^{p} \int_{G} \frac{\delta \Phi}{\delta u_\alpha(x)} \delta u_\alpha(x) dx + \sum_{i=1}^{m} \frac{\partial \Phi}{\partial t_i} \delta t_i. \quad (9)$$

A functional derivative of $\Phi$ exists if its differential form $\delta \Phi$ can be written as the integral (9). Let us consider the following function of a finite number of variables $\Phi = \sum_{i=1}^{n} a_i u_i$. Its derivative with respect to $u_k$ where $1 \leq k \leq n$ is $\delta \Phi/\delta u_k = \sum_{i=1}^{n} a_i \delta_{ik} = a_k$ where $\delta_{ik}$ is a Kronecker delta. We consider the following continuum limit of the function $\Phi$

$$\Phi = \int_G a(x') u(x') dx'. \quad (10)$$

The differential form (9) of this functional writes:

$$\delta \Phi = \int_G a(x') \delta u(x') dx'. \quad (11)$$

Hence, by comparison to (10) the functional derivative $\delta \Phi/\delta u(x) = a(x)$ where $x \in G$. Another functional $\Phi = u(x_1)$ where $x_1$ is a given point in the domain $G$, can also be written in the integral form

$$\Phi = \int_G \delta(x - x_1) u(x) dx \quad \text{and} \quad \delta \Phi = \int_G \delta(x - x_1) u(x) dx, \quad (12)$$

also in this case the functional derivative can be found by comparison with (9), $\delta \Phi/\delta u(x) = \delta(x - x_1)$. Calculation of second order derivatives is presented on another example below. We consider the following function, together with its continuum counterpart

$$\Phi = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} u_i u_j \quad \rightarrow \quad \Phi = \int_G \int_G c(x, x') u(x') u(x) dx \, dx'. \quad (13)$$
Corresponding derivatives of the first and second order write

\[ \frac{\partial \Phi}{\partial u_k} = \sum_{j=1}^{n} c_{kj} u_j + \sum_{i=1}^{n} c_{ki} u_i \quad \rightarrow \quad \frac{\delta \Phi}{\delta u(x'')} = \int_{G} c(x'', x') u(x') \, dx' \]

\[ + \int_{G} c(x, x'') u(x) \, dx. \tag{14} \]

\[ \frac{\partial^2 \Phi}{\partial u_k \partial u_m} = c_{km} + c_{mk} \quad \rightarrow \quad \frac{\delta^2 \Phi}{\delta u(x'') \delta u(x)} = c(x'', x) + c(x, x''). \tag{15} \]

Derivatives of higher orders of the considered function and the corresponding functional are zero.

We now solve the following hyperbolic equation in a classical and continuum formulation by the method of characteristics

\[ \Phi \frac{\partial F}{\partial \Phi} + \sum_{i=1}^{n} \frac{\partial F}{\partial u_i} = 0 \quad \rightarrow \quad \Phi \frac{\partial F}{\partial \Phi} + \int_{a}^{b} \frac{\delta F}{\delta u(x')} \, dx' = 0 \]  

where \( F = F(\Phi, u_1, \ldots, u_n) \) in the classical formulation and \( F = F(\Phi, [u(x)]) \) in the continuum limit; \( u_i \) and \( u(x) \) constitute sets of independent variables and \( \Phi \) is a dependent variable \( \Phi = \Phi(u_1, \ldots, u_n) \) or \( \Phi = \Phi([u(x)]) \). The characteristic equations of (16)

\[ \frac{d\Phi}{\Phi} = du_1 = \cdots = du_n \quad \rightarrow \quad \frac{\delta \Phi}{\Phi} = \delta u(x) \quad \text{for each} \quad x \in (a, b) \tag{17} \]

determine \( n \) integration constants \( C_i \) in the classical formulation and an infinite set of integration constants \( C(x) \) in the continuum formulation. The constants can be employed as new (dependent and independent) variables of \( F \). Corresponding solutions of Eqs (16) have the forms \( F = F(C_1, \ldots, C_n) \) and \( F = F(C_1, [C(x)]) \). A few examples of possible solutions of the characteristic system (17) are presented below. We can e.g. consider equations

\[ \frac{d\Phi}{\Phi} = du_1, \quad du_1 = du_2, \quad du_2 = du_3, \quad \ldots, \quad du_{n-1} = du_n \]

or

\[ \frac{d\Phi}{\Phi} = \frac{1}{n} \sum_{i=1}^{n} du_i, \quad du_1 = du_2, \quad du_1 = du_3, \quad \ldots, \quad du_1 = du_n \]

to obtain the following integration constants

\[ C_1 = \frac{\Phi}{\exp u_1} \quad \text{or} \quad C_1 = \Phi \exp \left(-\frac{1}{n} \sum_{i=1}^{n} u_i \right), \]

\[ C_2 = u_1 - u_2, \quad C_3 = u_2 - u_3, \ldots, \quad C_n = u_{n-1} - u_n, \tag{18} \]

or

\[ C_2 = u_1 - u_2, \quad C_3 = u_1 - u_3, \ldots, \quad C_n = u_1 - u_n. \]
Integration constants \((18)\) have their counterparts in continuum formulation,

\[
C_1 = \Phi \exp \left[ -u(x_1) \right] \quad \text{or} \quad C_1 = \Phi \exp \left( -\frac{1}{b - a} \int_a^b u(x) \, dx \right),
\]

\[
C(x) = \frac{du(x)}{dx} \, dx \quad \text{or} \quad C(x) = u(x_1) - u(x)
\]

where \(x_1\) is a fixed point in the domain \(x \in (a, b)\). Hence, a functional that constitutes a solution of Eq. \((16)\) in its continuum limit may be written as

\[
F = F \left( \Phi \exp (-u(x_1)), [u(x_1) - u(x)] \right).
\]

### 3 Finite and infinitesimal transformations.

In this section we recall the classical symmetry method which can be used to analyse the partial differential equation \((3)\) and present its continuum extension for the functional differential equation \((5)\). By “symmetry transformation” we understand such transformation of variables which does not change the functional form of the considered equation. This means that, for example Eq. \((5)\) in the old variables \(\Phi, t_1 \ldots t_n, u(x)\) and the same equation written in new, transformed variables \(\bar{\Phi}, \bar{t}_1 \ldots \bar{t}_m, \bar{u}(x)\)

\[
F([\bar{u}(x)], \bar{t}_1, \ldots, \bar{t}_m, \bar{\Phi}, \bar{\Phi}_1, \bar{\Phi}_2, \ldots, \bar{\Phi}_q) = 0;
\]

are equivalent. Note that \(x\) is not transformed in the present approach since it constitutes a continuous “counting” parameter, such as in a summation for the classical counterpart. Here, we consider only such transformations of variables which constitute Lie groups, i.e. they depend on a continuous parameter \(\varepsilon\) and satisfy group properties, such as closure, associativity and containing the unitary and inverse elements.

Table 1 presents the comparison of finite one-parameter Lie point transformation for the classical and continuum formulation. As can be seen, the transformed variables \(\bar{\Phi}, \bar{u}(x), \bar{t}_1 \ldots \bar{t}_m\) become functionals in the continuum limit and depend on the infinite set of independent variables \([u(x)]\). It should also be noted that instead of the finite set \(\bar{u}_1 \ldots \bar{u}_n\), in the continuum formulation we define \(\bar{u}(x) = \Phi_x\), which is an explicit function of the variable \(x\), since \(\bar{u}\) defines a new variable at each point \(x\). This has important consequences in the further considerations.

For the subsequent purpose of symmetry analysis all variables of equation \((3)\), i.e. the sets \(t_1, \ldots, t_m, u_1, \ldots, u_n\), as well as \(\Phi\) and all its possible derivatives of any order will be treated as
Table 1: Comparison of one-parameter Lie point transformation for the classical and continuum formulation.

| classical formulation | continuum formulation |
|-----------------------|-----------------------|
| $i. \quad \Phi = \psi(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m, \varepsilon)$ | $\Phi = \psi(\Phi, [u(x)], t_1, \ldots, t_m, \varepsilon)$ |
| $ii. \quad \bar{u}_1 = \phi_1(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m, \varepsilon)$ | $\bar{u}(x) = \phi_x(\Phi, [u(x')], x, t_1, \ldots, t_m, \varepsilon)$ |
| $\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$ | $\quad x \in G$ |
| $iii. \quad \bar{t}_1 = \phi_{t_1}(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m, \varepsilon)$ | $\quad \bar{t}_1 = \phi_{t_1}(\Phi, [u(x)], t_1, \ldots, t_m, \varepsilon)$ |
| $\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$ | $\quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$ |
| $\bar{t}_m = \phi_{t_m}(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m, \varepsilon)$ | $\bar{t}_m = \phi_{t_m}(\Phi, [u(x)], t_1, \ldots, t_m, \varepsilon)$ |

independent variables. Now, the following, new differential operators are introduced:

$$
\frac{\mathcal{D}}{\mathcal{D}t_k} = \frac{\partial}{\partial t_k} + \Phi_{,tk} \frac{\partial}{\partial \Phi} + \sum_{j=1}^{m} \Phi_{,tk} \frac{\partial}{\partial \Phi_{,tj}} + \sum_{j=1}^{n} \Phi_{,tk} \frac{\partial}{\partial \Phi_{,uj}} + \cdots,
$$

(22)  \quad k = 1, \ldots, m

$$
\frac{\mathcal{D}}{\mathcal{D}u_i} = \frac{\partial}{\partial u_i} + \Phi_{,ui} \frac{\partial}{\partial \Phi} + \sum_{j=1}^{n} \Phi_{,ui} \frac{\partial}{\partial \Phi_{,uj}} + \sum_{j=1}^{m} \Phi_{,ui} \frac{\partial}{\partial \Phi_{,tj}} + \cdots,
$$

(23)  \quad i = 1, \ldots, n.

The partial derivatives e.g. of the form $\partial / \partial t_i$ will act only on terms which depends explicitly on $t_i$. Within this formulation the derivatives of $\Phi$ can be expressed as:

$$
\Phi_{,t_k} = \frac{\mathcal{D}\Phi}{\mathcal{D}t_k}, \quad \Phi_{,ui} = \frac{\mathcal{D}\Phi}{\mathcal{D}u_i}, \quad \Phi_{,u_{itj}} = \frac{\mathcal{D}}{\mathcal{D}u_i} \frac{\mathcal{D}}{\mathcal{D}t_j},
$$

(24)

$$
\Phi_{,u_{itk}} = \frac{\mathcal{D}}{\mathcal{D}u_i} \frac{\mathcal{D}}{\mathcal{D}t_k}, \quad \Phi_{,tjt_k} = \frac{\mathcal{D}}{\mathcal{D}t_j} \frac{\mathcal{D}}{\mathcal{D}t_k}.
$$

(25)

Analogous definitions will apply in the continuum limit (5). The derivatives (22) have the following
counterparts

\[
\frac{\mathcal{D}}{\mathcal{D}t_k} = \frac{\partial}{\partial t_k} + \Phi_{t_j} \frac{\partial}{\partial \Phi} + \sum_{j=1}^{m} \Phi_{t_k t_j} \frac{\partial}{\partial \Phi_{t_j}} \\
+ \sum_{\alpha=1}^{p} \int_G d\mathbf{x} \Phi_{t_k u_\alpha(x)} \frac{\delta}{\delta \Phi_{u_\alpha(x)}} + \cdots,
\]

\[
\frac{\mathcal{D}}{\mathcal{D}u_\alpha(x) d\mathbf{x}} = \frac{\delta}{\delta u_\alpha(x)} + \Phi_{u_\alpha(x)} \frac{\partial}{\partial \Phi} + \sum_{j=1}^{m} \Phi_{u_\alpha(x) t_j} \frac{\partial}{\partial \Phi_{t_j}} \\
+ \sum_{\beta=1}^{p} \int_G d\mathbf{x}' \Phi_{u_\alpha(x) u_\beta(x')} \frac{\delta}{\delta \Phi_{u_\beta(x')}} + \cdots,
\]

(for \(\alpha = 1, \ldots, p, \mathbf{x} \in G\)) and the derivatives of \(\Phi\) in terms of the new differential operators write:

\[
\Phi_{t_j} = \frac{\mathcal{D} \Phi}{\mathcal{D} t_j}, \quad \Phi_{u_\alpha(x)} = \frac{\mathcal{D} \Phi}{\mathcal{D} u_\alpha(x) d\mathbf{x}},
\]

\[
\Phi_{u_\alpha(x) u_\beta(x')} = \frac{\mathcal{D}}{\mathcal{D} u_\alpha(x) d\mathbf{x} \mathcal{D} u_\beta(x') d\mathbf{x}'},
\]

\[
\Phi_{u_\alpha(x) t_j} = \frac{\mathcal{D}}{\mathcal{D} u_\alpha(x) d\mathbf{x} \mathcal{D} t_j}, \quad \Phi_{t_j t_k} = \frac{\mathcal{D}}{\mathcal{D} t_j} \frac{\mathcal{D} \Phi}{\mathcal{D} t_k}.
\]

Note that all derivatives in (27) are commutative. Also it is important to distinguish between \(x\) and \(x'\) which denote different integration indices such as \(i\) and \(j\) in two consecutive summations. The quantities given by formulae \((i), (ii), (iii)\) and derivatives of \(\Phi\) can be written in a Taylor series expansion about \(\varepsilon = 0\). Their infinitesimal forms, after neglecting terms of order \(O(\varepsilon^2)\) are given in Table 2. A few notation particularities should be noted. Indices of \(\zeta\) are separated by a semicolon to distinguish it from derivatives. Functional \(\xi_\mathbf{x}\) which denote infinitesimals corresponding to \(\mathbf{u}\) is an explicit function of \(\mathbf{x}\). The same dependence holds true for infinitesimals corresponding to the functional derivatives of \(\Phi\), such as \(\zeta_{u_\alpha(x)}\). In these cases the index of the set \([\mathbf{u}(\mathbf{x})]\) has been given a different name such as \([\mathbf{u}(\mathbf{x}')]\) to avoid confusion with the parameter \(\mathbf{x}\). The remaining infinitesimals do not depend on \(\mathbf{x}\).

The key property of Lie group method is that the finite transformations, given by the formulae \((i)–(iii)\) can be computed from their infinitesimal forms \((iv)–(vi)\) (cf. Bluman and Kumei, 1989). According to the Lie’s first theorem, the finite form of the transformation can be obtained by integrating the first order system of equations. For the continuum formulation this system takes the form:

\[
\frac{d\Phi}{d\varepsilon} = \eta, \quad \frac{d\xi_i}{d\varepsilon} = \xi_i, \quad i = 1, \ldots, m, \quad \frac{d u_\alpha(x)}{d\varepsilon} = \xi_{\alpha x}, \quad \alpha = 1, \ldots, p, \quad \mathbf{x} \in G;
\]

where the latter equations should be integrated with the initial condition

\[
\varepsilon = 0: \quad \Phi = \Phi, \quad \xi_i = t_i, \quad u_\alpha(x) = u_\alpha(x).
\]
Table 2: Comparison of infinitesimal transformations for the classical and continuum formulation.

| classical formulation | continuum formulation |
|-----------------------|-----------------------|
| \( \Phi = \Phi + \eta(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m) \epsilon \) | \( \Phi = \Phi + \eta(\Phi, [u(x)], t_1, \ldots, t_m) \epsilon \) |
| \( \bar{u}_1 = u_1 + \xi_1(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m) \epsilon \) | \( \bar{u}(x) = u(x) \) |
| \( \vdots \) | \( + \xi_x(\Phi, [u(x')], x, t_1, \ldots, t_m) \epsilon \) |
| \( \bar{u}_n = u_n + \xi_n(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m) \epsilon \) | \( \vdots \) |
| \( \bar{t}_1 = t_1 + \xi_1(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m) \epsilon \) | \( \bar{t}_1 = t_1 + \xi_t(\Phi, [u(x)], t_1, \ldots, t_m) \epsilon \) |
| \( \vdots \) | \( \vdots \) |
| \( \bar{t}_m = t_m + \xi_t(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m) \epsilon \) | \( \bar{t}_m = t_m + \xi_t(\Phi, [u(x)], t_1, \ldots, t_m) \epsilon \) |
| \( \bar{\Phi}_{x_1} = \Phi_{x_1} + \zeta_{x_1}(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m) \epsilon \) | \( \bar{\Phi}_{t_1} = \Phi_{t_1} + \zeta_{t_1}(\Phi, [u(x')], t_1, \ldots, t_m) \epsilon \) |
| \( \vdots \) | \( \vdots \) |
| \( \bar{\Phi}_{x_m} = \Phi_{x_m} + \zeta_{x_m}(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m) \epsilon \) | \( \bar{\Phi}_{t_m} = \Phi_{t_m} + \zeta_{t_m}(\Phi, [u(x')], t_1, \ldots, t_m) \epsilon \) |
| \( \vdots \) | \( \vdots \) |
| \( \bar{\Phi}_{,x_1} = \Phi_{,x_1} + \zeta_{,x_1}(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m) \epsilon \) | \( \bar{\Phi}_{,t_1} = \Phi_{,t_1} + \zeta_{,t_1}(\Phi, [u(x')], t_1, \ldots, t_m) \epsilon \) |
| \( \vdots \) | \( \vdots \) |
| \( \bar{\Phi}_{,x_m} = \Phi_{,x_m} + \zeta_{,x_m}(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m) \epsilon \) | \( \bar{\Phi}_{,t_m} = \Phi_{,t_m} + \zeta_{,t_m}(\Phi, [u(x')], t_1, \ldots, t_m) \epsilon \) |
| \( \vdots \) | \( \vdots \) |
| \( \bar{\Phi}_{,x_1} = \Phi_{,x_1} + \zeta_{,x_1}(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m) \epsilon \) | \( \bar{\Phi}_{,x_m} = \Phi_{,x_m} + \zeta_{,x_m}(\Phi, [u(x')], t_1, \ldots, t_m) \epsilon \) |
| \( \vdots \) | \( \vdots \) |
| \( \bar{\Phi}_{,x_1} = \Phi_{,x_1} + \zeta_{,x_1}(\Phi, u_1, \ldots, u_n, t_1, \ldots, t_m) \epsilon \) | \( \bar{\Phi}_{,x_m} = \Phi_{,x_m} + \zeta_{,x_m}(\Phi, [u(x')], x, t_1, \ldots, t_m) \epsilon \) |
| \( \vdots \) | \( \vdots \) |
Now, our aim is to find the infinitesimals forms $\eta, \xi_t, \zeta$. To do this we should first express the infinitesimals $\zeta$ in terms of $\eta, \xi_t, \zeta$ and independent variables $t_1, \ldots, t_m, [u(x)], \Phi$. In the classical formulation the following relation is used for this purpose

\[
\frac{D\psi}{Du_i} = \sum_{k=1}^{n} \frac{D\phi_k}{Du_i} \frac{D\Phi}{D\bar{u}_k} + \sum_{k=1}^{m} \frac{D\phi_{tk}}{Du_i} \frac{D\Phi}{D\bar{u}_k} = \sum_{k=1}^{n} \Phi_{\alpha k} \frac{D\phi_k}{Du_i} + \sum_{k=1}^{m} \Phi_{tk} \frac{D\phi_{tk}}{Du_i};
\]

(30)

its continuum counterpart writes

\[
\frac{D\psi}{Du_\alpha(x)dx} = \sum_{\beta=1}^{p} \int_G \Phi_{\alpha \beta(x')} \frac{D\phi_{\beta x'}}{Du_\alpha(x)dx} dx' + \sum_{k=1}^{m} \Phi_{tk} \frac{D\phi_{tk}}{Du_\alpha(x)dx}.
\]

(31)

When the infinitesimal forms $\text{(iv)}$–$\text{(viii)}$, are introduced into equation (31) we obtain:

\[
\frac{D(\Phi + \eta \varepsilon)}{Du_\alpha(x)dx} = \sum_{\beta=1}^{p} \int_G \Phi_{\alpha \beta(x')} \frac{D(u_\beta(x') + \zeta_{\alpha \beta(x')} \varepsilon)}{Du_\alpha(x)dx} dx' + \sum_{k=1}^{m} \Phi_{tk} + \zeta_{tk} \varepsilon \frac{D(t_k + \xi_{tk})}{Du_\alpha(x)dx}.
\]

(32)

Equation (32) can be further split into two equations, containing terms $O(1)$ and $O(\varepsilon)$, respectively. The first of the two gives the identity

\[
\frac{D\Phi}{Du_\alpha(x)dx} = \sum_{\beta=1}^{p} \int_G \Phi_{\alpha \beta(x')} \frac{Du_\beta(x')}{Du_\alpha(x)dx} dx' = \sum_{\beta=1}^{p} \delta_{\alpha \beta} \int_G \Phi_{\alpha \beta(x')} \delta(x - x') dx' = \Phi_{\alpha u_\alpha(x)}
\]

(33)

where $Du_\beta(x')/Du_\alpha(x)dx = \delta_{\alpha \beta} \delta(x - x')$ has been used where $\delta_{\alpha \beta}$ and $\delta(x - x') = \delta(x_1 - x'_1) \cdot \cdots \delta(x_r - x'_r)$ denote the Kronecker and Dirac delta respectively. From $O(\varepsilon)$ we obtain a formula for the infinitesimals $\zeta_{\alpha u_\alpha(x)}$

\[
\frac{D\eta}{Du_\alpha(x)dx} = \sum_{\beta=1}^{p} \int_G \left[ \Phi_{\alpha \beta(x')} \frac{D\zeta_{\beta x'}}{Du_\alpha(x)dx} + \zeta_{\alpha \beta(x')} \frac{Du_\beta(x')}{Du_\alpha(x)dx} \right] dx' + \sum_{k=1}^{m} \Phi_{tk} \frac{D\zeta_{tk}}{Du_\alpha(x)dx},
\]

(34)

hence,

\[
\zeta_{\alpha u_\alpha(x)} = \frac{D\eta}{Du_\alpha(x)dx} - \sum_{\beta=1}^{p} \int_G \Phi_{\alpha \beta(x')} \frac{D\zeta_{\beta x'}}{Du_\alpha(x)dx} dx' - \sum_{k=1}^{m} \Phi_{tk} \frac{D\zeta_{tk}}{Du_\alpha(x)dx}.
\]

(35)
By analogy, formula for the infinitesimals $\zeta_{ti}$ can be found
\[
\zeta_{ti} = \frac{D\eta}{Dt_i} - \sum_{j=1}^{p} \int_{G} \Phi_{,u_{\alpha}(x')} \frac{D\xi_{\beta}}{Dt_i} dx' - \sum_{k=1}^{m} \Phi_{,t_k} \frac{D\xi_{t_k}}{Dt_i}.
\]

The infinitesimals of higher orders will follow from the following, recursive formulae:
\[
\frac{D\psi_{,u_{\alpha}(x^{(1)}),...,u_{\beta}(x^{(s-1)})}}{Du_{\gamma}(x^{(s)})} = \sum_{\delta=1}^{p} \int_{G} \Phi_{,u_{\alpha}(x^{(1)}),...,u_{\beta}(x^{(s-1)}),u_{\delta}(x)} \frac{D\phi_{,t_k}}{Du_{\gamma}(x^{(s)})} dx + \sum_{k=1}^{m} \Phi_{,u_{\alpha}(x^{(1)}),...,u_{\beta}(x^{(s-1)}),t_k} \frac{D\phi_{,t_k}}{Du_{\gamma}(x^{(s)})},
\]

or
\[
\frac{D\psi_{,u_{\alpha}(x^{(1)}),...,u_{\beta}(x^{(s-1)})}}{Dt_j} = \sum_{\delta=1}^{p} \int_{G} \Phi_{,u_{\alpha}(x^{(1)}),...,u_{\beta}(x^{(s-1)}),u_{\delta}(x)} \frac{D\phi_{,t_k}}{Dt_j} dx + \sum_{k=1}^{m} \Phi_{,u_{\alpha}(x^{(1)}),...,u_{\beta}(x^{(s-1)}),t_k} \frac{D\phi_{,t_k}}{Dt_j}.
\]

4 Generator $X$ and its prolongations.

Once all the necessary infinitesimal forms are obtained they can be substituted into the equations (3) or (5) written in the transformed variables. In order to simplify notation we will assume that Eqs (3) and (5) only contain derivatives up to the second order. The generalization of the following relations to the case of higher order derivations is straightforward. After expansion in Taylor series about $\varepsilon = 0$ in both, classical and continuum formulation the expanded equation has the form:
\[
F + \varepsilon X^{(2)}F + \frac{\varepsilon^2}{2} [X^{(2)}]^2 F + O(\varepsilon^3) = 0
\]
where $X^{(2)}$ in the classical formulation is given by the formula
\[
X^{(2)} = \eta \frac{\partial}{\partial \Phi} + \sum_{j=1}^{m} \xi_{t_j} \frac{\partial}{\partial t_j} + \sum_{j=1}^{n} \xi_{u_j} \frac{\partial}{\partial u_j} + \sum_{j=1}^{m} \zeta_{t_j} \frac{\partial}{\partial \Phi_{,t_j}} + \sum_{j=1}^{n} \zeta_{u_j} \frac{\partial}{\partial \Phi_{,u_j}}
\]
\[
+ \sum_{j=1}^{m} \sum_{k=1}^{m} \zeta_{t_j t_k} \frac{\partial}{\partial \Phi_{,t_j t_k}} + \sum_{j=1}^{n} \sum_{k=1}^{n} \zeta_{u_j u_k} \frac{\partial}{\partial \Phi_{,u_j u_k}}
\]

and is called the prolongation of the generator $X$
\[
X = \eta \frac{\partial}{\partial \Phi} + \sum_{j=1}^{n} \xi_{t_j} \frac{\partial}{\partial t_j} + \sum_{j=1}^{n} \xi_{u_j} \frac{\partial}{\partial u_j}
\]
of the second order. The corresponding formulae for the continuum limit write

$$X = \eta \frac{\partial}{\partial \Phi} + \sum_{j=1}^{m} \xi_{tj} \frac{\partial}{\partial t_j} + \sum_{\alpha=1}^{p} \int_{G} dx \xi_{\alpha x} \frac{\delta}{\delta u_{\alpha}(x)}$$

(42)

and

$$X^{(2)} = \eta \frac{\partial}{\partial \Phi} + \sum_{j=1}^{m} \xi_{tj} \frac{\partial}{\partial t_j} + \sum_{\alpha=1}^{p} \int_{G} dx \xi_{\alpha x} \frac{\delta}{\delta u_{\alpha}(x)} + \sum_{j=1}^{m} \zeta_{tj} \frac{\partial}{\partial \Phi} + \sum_{\alpha=1}^{p} \int_{G} d\Phi, t_j \frac{\partial}{\partial \Phi}$$

$$+ \sum_{\alpha=1}^{p} \int_{G} dx \xi_{u_{\alpha}(x)} \frac{\delta}{\delta \Phi, u_{\alpha}(x)} + \sum_{j=1}^{m} \sum_{k=1}^{m} \zeta_{tj t_k} \frac{\partial}{\partial \Phi}$$

$$+ \sum_{\alpha=1}^{p} \sum_{k=1}^{m} \int_{G} dx \xi_{u_{\alpha}(x)} \frac{\delta}{\delta \Phi, u_{\alpha}(x)}$$

$$+ \sum_{\alpha=1}^{p} \sum_{\beta=1}^{p} \int_{G} \int_{G} d\Phi, d\Phi' \xi_{u_{\alpha}(x) u_{\beta}(x')} \frac{\delta}{\delta \Phi, u_{\alpha}(x) u_{\beta}(x')}.$$  (43)

The first term in Eq. (39) equals zero, as follows from the equations (3) or (5). All the remaining terms $[X^{(2)}]^n$ in (39), representing a successive application of $X^{(2)}$ will be zero if the following relation holds

$$X^{(2)} F = 0.$$  (44)

In order to find the infinitesimal transformations we use the condition

$$[X^{(2)} F] |_{F=0} = 0$$  (45)

where, in the continuum formulation, the prolongation $X^{(2)}$ is expressed by the formula (43) and the forms of infinitesimals $\zeta$ are found from relations (35)–(38). The resulting condition constitutes an overdetermined system of linear differential equations. In the continuum limit we obtain a set of functional differential equations. This system can be further solved for the infinitesimals $\eta$, $\xi_{tj}$ and $\xi_{\alpha x}$.

5 Invariant solutions

If the functional differential equation (3) admits a symmetry given by the generator (42), then a solution $\Phi = \Theta(t_1, \ldots, t_m, [u(x)])$ of this equation is called an invariant solution if it satisfies the relation

$$X [\Phi - \Theta(t_1, \ldots, t_m, [u(x)])] = 0.$$  (46)
After employing (42) and expanding the derivatives, from (46) the following, hyperbolic functional equation is obtained

\[ \sum_{i=1}^{m} \xi_{t_i} \frac{\partial \Theta}{\partial t_i} + \sum_{\alpha=1}^{p} \int_{G} \xi_{\alpha x} \frac{\delta \Theta}{\delta u_{\alpha}(x)} \, dx = \eta \]  

(47)

This equation can be solved by the method of characteristics. The corresponding system of equations writes

\[ \frac{dt_1}{\xi_{t_1}} = \cdots = \frac{dt_m}{\xi_{t_m}} = \frac{d\Phi}{\eta} = \frac{\delta u_{\alpha}(x)}{\xi_{\alpha x}} \quad \text{for each} \quad x \in G \quad \text{and} \quad \alpha = 1, \ldots, p. \]  

(48)

Above, \( \Theta \) has been replaced by \( \Phi \). Note that the last term in fact corresponds to an infinite set of equations for each \( \alpha \) and each point in \( G \). The infinite set of constants, which is a solution of the above system, can be employed as new variables in Eq. (45). As in the considered case one of them, say \( C_1 \), will be a dependent variable and the rest will constitute a set of independent variables, the following relation holds

\[ C_1 = H(C_2, \ldots, C_m, [C_1(x)], \ldots, [C_p(x)]). \]  

(49)

After the process of solving characteristic system for a partial differential equation with a finite set of variables, the number of independent variables is reduced by one. In the case of functional differential equations, in formula (49) one point of the considered domain will be excluded from a set \([C_i(x)]\), \( x \in G/\{x_{j_1}\} \).

6 Example

For the sake of clarity we will consider, as an example, the continuum limit of a heat equation in infinite many dimension. In a classical formulation we take \( m = 1 \). Hence, the equation for a function of \( n + 1 \) variables \( \Phi = \Phi(u_1, \ldots, u_n, t) \) writes

\[ \frac{\partial \Phi}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial u_i^2} \]  

(50)

while in the continuum limit we consider

\[ \frac{\partial \Phi}{\partial t} = \int_{a}^{b} \frac{\partial}{\partial u(x)} \frac{\partial \Phi}{\partial u(x)} \, dx \quad \text{or} \quad \Phi, t = \int_{a}^{b} \Phi, u(x)u(x) \, dx \]  

(51)

where \( \Phi = \Phi([u(x)], t) \). The generator \( X \) of Eq. (51) (cf. Eq. (41)) has the form

\[ X = \eta \frac{\partial}{\partial \Phi} + \xi_{t} \frac{\partial}{\partial t} + \int_{a}^{b} dx \xi_{x} \frac{\delta}{\delta u(x)}. \]  

(52)
The second prolongation \( X^{(2)} \) of \( X \) necessary for (51) (cf. Eq. (40)) is defined as

\[
X^{(2)} = \eta \frac{\partial}{\partial \Phi} + \xi_t \frac{\partial}{\partial t} + \int_a^b dx \xi_x \frac{\delta}{\delta u(x)} + \zeta_t \frac{\partial}{\partial \Phi_t},
\]

where any unneeded \( \zeta \) has been omitted. Applying (53) to (51) the first three terms of (51) have no effect, while the fourth one acts on \( \Phi_t \) to lead to \( \zeta_t \). The last term in (53), acting on (51) can be written as

\[
\int_a^b dx \xi_{u(x)u(x)} \frac{\delta}{\delta \Phi_{u(x)u(x)}} \int_a^b \Phi_{u(x')}u(x') dx' = (54)
\]

\[
\int_a^b dx \xi_{u(x)u(x)} \int_a^b \delta(x-x') dx' = \int_a^b \zeta_{u(x)u(x)} dx.
\]

As a result we obtain

\[
\zeta_t - \int_a^b \zeta_{u(x)u(x)} dx = 0.
\]

Into the above equation we substitute the infinitesimals \( \zeta_t, \zeta_{u(x)u(x)} \), found from Eqs (35)–(37). Their forms, without derivation, are given below

\[
\zeta_t = \frac{D\eta}{Dt} - \int_a^b \Phi_{u(x)} \frac{D\xi_x}{Dt} dx - \Phi_{,t} \frac{D\xi_t}{Dt},
\]

\[
\zeta_{u(x)u(x)} = - \int_a^b \Phi_{,u(x)u(x')} \frac{D\xi_{x'}}{Dx} dx' - \Phi_{,u(x)u(x)} \frac{D\xi_t}{Du} dx - \frac{D}{Du} \left( \Phi_{,t} \frac{D\xi_t}{Du} dx \right).
\]
If the definitions of differential operators (26) are used in the above equations one obtains

\[ \zeta_t = \frac{\partial \eta}{\partial t} + \Phi \frac{\partial \eta}{\partial \Phi} - \Phi_t \frac{\partial \xi_t}{\partial t} - \Phi^2 \frac{\partial \xi_t}{\partial \Phi} - \int_a^b \left( \Phi_{,u(x)} \frac{\partial \xi_x}{\partial t} + \Phi_{,u(x)} \Phi \frac{\partial \xi_x}{\partial \Phi} \right) dx \]  

and

\[ \zeta_{,u(x)u(x)} = \frac{\partial^2 \eta}{\partial u(x)^2} + 2\Phi_{,u(x)} \frac{\partial \eta}{\partial \Phi} + \Phi^2_{,u(x)} \frac{\partial^2 \eta}{\partial \Phi^2} + \int_a^b \Phi_{,u(x)u(x)} \left( \frac{\partial \xi_x}{\partial t} + \Phi \frac{\partial \xi_x}{\partial \Phi} \right) dx' \]

Next, we use Eq. (51) in order to substitute for the derivative \( \Phi_t \)

\[ \Phi_t = \int_a^b \Phi_{,u(x)u(x)} dx. \]

The resulting equation has the form

\[ \frac{\partial \eta}{\partial t} = \int_a^b \frac{\partial^2 \eta}{\partial u(x)^2} dx - \int_a^b \Phi_{,u(x)} \left( \frac{\partial \xi_x}{\partial t} + 2\Phi \frac{\partial \eta}{\partial \Phi} \right) dx - \int_a^b \frac{\partial^2 \xi_x}{\partial u(x)^2} dx' \]

\[ + \int_a^b \Phi_{,u(x)u(x)} \left( \int_a^b \frac{\partial \xi_x}{\partial \Phi} dx' - \frac{\partial \xi_t}{\partial t} \right) dx + 2 \int_a^b \int_a^b \Phi_{,u(x)u(x)} \frac{\partial \xi_x}{\partial \Phi} dx dx' \]

\[ + 2 \int_a^b \int_a^b \Phi_{,u(x)} \Phi_{,u(x)} \frac{\partial^2 \xi_x}{\partial \Phi \partial \Phi} dx dx' - \int_a^b \Phi_{,u(x)} \frac{\partial^2 \eta}{\partial \Phi^2} dx \]

\[ + 2 \int_a^b \int_a^b \Phi_{,u(x)u(x)} \frac{\partial \xi_x}{\partial \Phi} dx dx' + 2 \int_a^b \int_a^b \Phi_{,u(x)u(x)} \frac{\partial^2 \xi_x}{\partial \Phi \partial \Phi} dx dx' \]

\[ + \int_a^b \int_a^b \Phi_{,u(x)} \Phi_{,u(x)} \frac{\partial^2 \xi_x}{\partial \Phi^2} dx dx' + \int_a^b \int_a^b \Phi_{,u(x)u(x)} \Phi_{,u(x)} \frac{\partial^2 \xi_x}{\partial \Phi \partial \Phi} dx dx' \]

\[ + 2 \int_a^b \Phi_{,u(x)} \frac{\delta \xi_t}{\delta u(x)} dx + 2 \int_a^b \Phi_{,u(x)} \frac{\delta \xi_t}{\delta \Phi} dx = 0. \]
As the terms \( \eta, \xi, \xi_x \) do not depend on the derivatives of \( \Phi \), from Eq. (58) the following system of differential equations can be obtained, where on the left hand side the coefficient function is written:

\[
\begin{align*}
\Phi_{,u(x)} : & \quad \frac{\partial \xi_t}{\partial \Phi} = 0, \\
\Phi_{,u(x)t} : & \quad \delta \xi_t = 0, \quad \text{for each } x \in (a, b), \\
\Phi_{,u(x)u(x')} : & \quad \frac{\partial \xi_x}{\Phi u(x')} = 0, \quad \text{for each } x \in (a, b), \\
\Phi^2_{,u(x)} : & \quad \frac{\partial^2 \eta}{\partial \Phi^2} = 0, \\
\Phi_{,u(x)u(x)} : & \quad \int_a^b \frac{\delta^2 \xi_t}{\partial u(x')^2} \, dx' - \frac{\delta \xi_t}{\partial t} + 2 \frac{\delta \xi_x}{\partial u(x)} = 0, \\
\Phi_{,u(x)u(x')} : & \quad \frac{\delta \xi_x}{\partial u(x')} + \frac{\delta \xi_x'}{\partial u(x)} = 0, \\
\Phi_{,u(x)} : & \quad \frac{\partial \xi_x}{\partial t} + 2 \frac{\partial \eta}{\partial \Phi} \frac{\partial u(x)}{\partial x} - \int_a^b \frac{\delta^2 \xi_t}{\partial u(x')^2} \, dx' = 0, \\
1 : & \quad \frac{\partial \eta}{\partial t} - \int_a^b \frac{\delta^2 \eta}{\partial u(x)^2} \, dx = 0.
\end{align*}
\]

Both equations (58) and (63) follow from the second line of Eq. (58). To see that this is true we will differentiate Eq. (58) with respect to \( \delta / \delta \Phi_{,u(x)u(x'')} \). Since \( \Phi_{,u(x)u(x')} \) is an explicit function of two variables it follows that

\[
\frac{\delta \Phi_{,u(x)u(x')} \delta (x-x', x'-x''')}{\delta \Phi_{,u(x'')} \delta (x-x''')} = \delta (x-x', x'-x''') \delta (x', x''').
\]

(67)

On the other hand, \( \Phi_{,u(x)u(x)} \) is a function of variable \( x \) only, say \( \Phi_{,u(x)u(x)} = g(x) \), hence

\[
\frac{\delta \Phi_{,u(x)u(x)} \delta g(x)}{\delta \Phi_{,u(x'')} \delta g(x'')} = \delta (x-x').
\]

(68)

The functional derivative of \( \Phi_{,u(x)u(x)} \) with respect to \( \Phi_{,u(x'')} \) is zero if \( x'' = x''' \). Finally, after differentiation of (68) with respect to \( \Phi_{,u(x'')}u(x'''') \), for \( x'' = x''' \) we obtain

\[
\int_a^b \delta (x-x'') \, dx \left( \int_a^b \frac{\delta^2 \xi_t}{\partial u(x')^2} \, dx' - \frac{\delta \xi_t}{\partial t} \right) + 2 \int_a^b \int_a^b \delta (x-x'') \delta (x-x'''') \frac{\delta \xi_x'}{\partial u(x')} \, dx \, dx' = 0
\]

(69)

whereas in case when \( x'' = x''' \) we have

\[
2 \int_a^b \int_a^b \delta (x-x'') \delta (x-x''') \frac{\delta \xi_x'}{\partial u(x')} + \delta (x-x'') \delta (x-x'''') \frac{\delta \xi_x}{\partial u(x')} \, dx \, dx' = 0.
\]
Note that the presence of two terms inside the integral is a consequence of an equality \( \Phi_{,u(x)}u(x') = \Phi_{,u(x')}u(x) \). From the above equations formulae (63) and (64) are clearly obtained.

The above system of equations is now solved in order to find the form of infinitesimals \( \eta, \xi_t \) and \( \xi_x \). From relations (59)–(62) we obtain:

\[
\begin{align*}
\xi_t &= \xi_t(t), \\
\xi_x &= \xi_x([u(x')], x, t), \\
\eta &= f_1([u(x')], t) \Phi + f_2([u(x')], t),
\end{align*}
\]

(70) (71) (72)

Next, we consider formula (63). We recall that \( \xi_x \) is an explicit function of \( x \), hence, the general solution of (63) has the form

\[
\xi_x = \frac{1}{2} \frac{d\xi_t}{dt} \int_a^b u(x') \, dx' + H(x, t, [u(x')]) + f_3(x, t)
\]

(73)

where \( H(x, t, [u(x')]) \) is a functional depending explicitly on \( x \), such that its functional derivative \( \frac{\delta H(x, t, [u(x')])}{\delta u(x)} = 0 \). Without an explicit dependence of \( \xi_x \) on \( x \) the second term on the RHS would disappear as we would obtain \( H(t, [u(x')]) = 0 \). Now, the form (63) is introduced into Eq. (64) to give

\[
\frac{1}{2} \frac{d\xi_t}{dt} + \frac{\delta H(x, t, [u(x')])}{\delta u(x')} = -\frac{1}{2} \frac{d\xi_t}{dt} - \frac{\delta H(x', t, [u(x'')])}{\delta u(x)}, \quad \text{for} \quad x \neq x'.
\]

(74)

Below, we prove that the functional derivative of \( H(x, t, [u(x')]) \) with respect to \( u(x') \) is a function of three variables \( G(x, x', t) \). First, we differentiate Eq. (74) with respect to \( u(x') \) and use the property \( \frac{\delta H(x', t, [u(x'')])}{\delta u(x')} = 0 \) to obtain

\[
\frac{\delta^2 H(x, t, [u(x'')])}{\delta u(x')^2} = 0.
\]

(75)

Let us now differentiate Eq. (74) with respect to \( u(x_1) \), such that \( x_1 \neq x \neq x' \). We obtain

\[
\frac{\delta^2 H(x, t, [u(x'')])}{\delta u(x') \delta u(x_1)} = -\frac{\delta^2 H(x', t, [u(x'')])}{\delta u(x) \delta u(x_1)}.
\]

(76)

Now, if we note that Eq. (74) can be written also for pairs of variables \( x_1, x \) and \( x_1, x' \), the LHS of the above formula equals

\[
\frac{\delta}{\delta u(x')} \left( \frac{\delta H(x, t, [u(x'')])}{\delta u(x_1)} \right) = -\frac{\delta}{\delta u(x')} \left( \frac{\delta H(x_1, t, [u(x'')])}{\delta u(x)} \right)
\]

(77)

and the RHS is

\[
-\frac{\delta}{\delta u(x)} \left( \frac{\delta H(x', t, [u(x'')])}{\delta u(x_1)} \right) = \frac{\delta}{\delta u(x)} \left( \frac{\delta H(x_1, t, [u(x'')])}{\delta u(x')} \right)
\]

(78)
which is an expression with opposite sign to that in (77). It follows that the second order functional
derivative of $H$ is always zero, hence
\[ \frac{\delta H(x, t, [u(x'')])}{\delta u(x')} = G(x, x', t). \] (79)

From Eq. (73) we obtain the relation $G(x, x', t) = -G(x', x, t) - d\xi_t/dt$, additionally $G(x, x, t) = 0$, as the first derivative of $H(x, t, [u(x'')])$ with respect to $u(x)$ is zero. Now, the formula (73) for the
infinitesimal $\xi_x$ writes
\[ \xi_x = \frac{1}{2} \frac{d\xi_t}{dt} \int_a^b u(x') dx' + \int_a^b G(x, x', t)u(x') dx' + f_3(x, t) \] (80)

The above form can be rearranged as follows
\[ \xi_x = \int_a^b C(x, x', t)u(x') dx' + f_3(x, t) \] (81)

where
\[ C(x, x', t) = -C(x', x, t). \quad \text{for} \quad x \neq x' \] (82)
\[ C(x, x, t) = \frac{1}{2} \frac{d\xi_t}{dt}, \] (83)

This result, together with (72) can be further substituted into (65) to give
\[ \frac{\delta f_1([u(x')], t)}{\delta u(x)} = \frac{1}{2} \int_a^b \frac{\partial C(x, x', t)}{\partial t} u(x') u(x) dx' - \frac{1}{2} \frac{\partial f_3}{\partial t}. \] (84)

After integration we obtain
\[ f_1([u(x')], t) = -\frac{1}{4} \int_a^b \int_a^b \frac{\partial C(x, x', t)}{\partial t} u(x') u(x) dx' dx \]
\[ -\frac{1}{2} \int_a^b \frac{\partial f_3}{\partial t} u(x') dx' + f_4(t). \] (85)

If we recall the differentiation rules presented in Section 2 we find that the functional derivative of
the first RHS term in the above equation is a sum of two integrals (cf. Eq. (11)) and is equal to
the corresponding term in Eq. (84) only if for $x \neq x'$, $\partial C(x, x', t)/\partial t = \partial C(x', x, t)/\partial t$. It further
follows from (82) that the only possibility is $\partial C(x, x', t)/\partial t = 0$ for $x \neq x'$. Finally, the form (72) is
substituted into equation (66) to give
\[ \Phi \left( -\frac{1}{8} \frac{d^3 \xi_t}{dx^3} \int_a^b \int_a^b C(x, x') u(x) u(x') dx' dx' \right. \]
\[ -\frac{1}{2} \int_a^b \frac{\partial^2 f_3}{\partial t^2} u(x) dx + \frac{d\xi_t}{dt} + \frac{1}{4} \frac{d^2 \xi_t}{dx^2} (b - a) \left. \right) \]
\[ + \frac{\partial f_2}{\partial t} - \int_a^b \frac{\partial^2 f_3}{\partial u(x)^2} dx = 0 \] (86)

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where \( c(x, x) = 1 \) and \( c(x, x') = 0 \) for \( x \neq x' \). As \( \xi_t, f_3 \) and \( f_4 \) neither depend on \( \Phi \) nor on \( u(x) \), from the above equation it follows that

\[
\frac{d^3 \xi_t}{dt^3} = 0 \quad (87)
\]

\[
\frac{\partial^2 f_3(x, t)}{\partial t^2} = 0 \quad (88)
\]

\[
\frac{df_4}{dt} + \frac{(b - a) d^2 \xi_t}{4 dt^2} = 0 \quad (89)
\]

\[
\frac{\partial f_2}{\partial t} - \int_a^b \frac{\delta^2 f_2}{\delta u(x)^2} dx = 0. \quad (90)
\]

Hence, the final solution of the system \((59)–(66)\) writes

\[
\xi_t = a_1 t^2 + a_2 t + a_3 \quad (91)
\]

\[
\xi_x = \int_a^b C(x, x', t) u(x') dx' + a_4(x) t + a_5(x) \quad (92)
\]

\[
\eta = - \left( \frac{1}{4} a_1 \int_a^b \int_a^b c(x, x') u(x) u(x') dx dx' + \left( \frac{1}{2} \int_a^b a_4(x) u(x) dx + \frac{(b - a)}{2} a_1 t + a_6 \right) \right) \Phi + f_2([u(x)], t) \quad (93)
\]

where \( C(x, x, t) = \frac{1}{2} (2a_1 t + a_2) \), \( C(x, x', t) = C(x, x') = -C(x', x) \), and \( f_2 \) is a solution of equation \((51)\).

The infinitesimals \((91)–(93)\) may directly be compared to those of the classical heat equation \((50)\) (cf. Ibragimov 1995)

\[
\xi_t = a_1 t^2 + a_2 t + a_3 \quad (94)
\]

\[
\xi_i = \frac{1}{2} (2a_1 t + a_2) u + \sum_{j=1}^n C_{ij} u_j + a_i t + a_5, \quad i = 1, \ldots, n \quad (95)
\]

\[
\eta = - \left( \frac{1}{4} a_1 \sum_{i=1}^n u_i^2 + \frac{1}{2} \sum_{i=1}^n a_i u_i + \frac{1}{2} \sum_{i=1}^n a_i u_i + \frac{1}{2} a_1 t + a_6 \right) \Phi + f_2(u_1, \ldots, u_n, t) \quad (96)
\]

where \( C_{ij} = -C_{ji} \) for \( i \neq j \) and \( C_{ii} = 0 \). As it is seen, the form of infinitesimal \( \xi_t \), Eq. \((94)\), is the same as in the continuum limit. In formula \((95)\) the two first terms on the RHS can be rearranged as \( 1/2(2a_1 t + a_2) \sum_{i=1}^n C_{ij} u_j \) with \( C_{ij}' = -C_{ji}' \) for \( i \neq j \) and \( C_{ii}' = 1 \). Then, in the continuum limit we obtain the first RHS term of \((92)\). The constants \( a_{i4} \) and \( a_{i5} \) become in \((92)\) functions of a continuous parameter \( x \). The double integral in Eq. \((93)\) is a continuum counterpart of the first RHS term in \((96)\). We note that the second order derivatives of these terms, present in Eq. \((96)\) and its classical
counterpart, are the same in both formulations
\[
\begin{align*}
df1
\frac{\partial f_1}{\partial u_k} &= -\frac{1}{2}a_1 u_k - \frac{1}{2}a_{k4}, \\
df1
\frac{\partial^2 f_1}{\partial u_k^2} &= -\frac{1}{2}a_1, \\
\frac{\delta f_1}{\delta u(x)} &= -\frac{1}{4}a_1 \int_a^b c(x, x') \, dx' - \frac{1}{4}a_1 \int_a^b c(x', x) \, dx' - \frac{1}{2}a_4(x), \\
\frac{\delta^2 f_1}{\delta u(x)^2} &= -\frac{1}{2}a_1.
\end{align*}
\]

The second RHS term in Eq. (96) becomes an integral in Eq. (93). In the third RHS term, instead of the number of variables \(n\), in the continuum formulation (93) we obtain the length of the considered integration domain which equals \((b - a)\).

From Eq. (91)-(93) we can distinguish the following symmetry groups, admitted by the considered Eq. (51)
\[
\begin{align*}
X_t &= \frac{\partial}{\partial t}, \quad (97) \\
X_{u(x)} &= \delta^{u(x)}, \quad \text{for each } x \in (a, b), \quad (98) \\
X_{u(x)u(x')} &= u(x') \frac{\delta}{\delta u(x)} - u(x) \frac{\delta}{\delta u(x')}, \\
&\quad \text{for each } x, x' \in (a, b) \text{ and } x \neq x', \quad (99) \\
X_{u(x)\Phi} &= 2t \frac{\delta}{\delta u(x)} - u(x) \Phi \frac{\partial}{\partial \Phi}, \quad \text{for each } x \in (a, b), \quad (100) \\
X_{tu(x)} &= 2t \frac{\partial}{\partial t} + u(x) \frac{\delta}{\delta u(x)}, \quad \text{for each } x \in (a, b), \quad (101) \\
X_{\Phi} &= \Phi \frac{\partial}{\partial \Phi}, \quad (102) \\
X_{u(x)\Phi} &= t^2 \frac{\partial}{\partial t} + tu(x) \frac{\delta}{\delta u(x)} - \frac{1}{4} \int_a^b \int_a^b c(x, x') u(x) u(x') \, dx \, dx' \Phi \frac{\partial}{\partial \Phi} \\
&\quad - \frac{(b - a)}{2} t \Phi \frac{\partial}{\partial \Phi}, \quad \text{for each } x \in (a, b). \quad (103)
\end{align*}
\]

We will investigate the invariant solutions of Eq. (51) under the combined symmetries (98), (100) for each \(x \in (a, b)\) and (102). The invariant solution of Eq. (51) satisfies the following, hyperbolic equation (cf. formula (17))
\[
\int_a^b \left( a_4(x) t + a_5(x) \right) \frac{\delta \Phi}{\delta u(x)} \, dx = -a_6 \Phi - \frac{1}{2} \Phi \int_a^b a_4(x') u(x') \, dx' \quad (104)
\]
the characteristic system writes
\[
\frac{dt}{0} = \frac{d\Phi}{-a_6\Phi - \frac{1}{2}\Phi \int_{a}^{b} a_4(x') u(x') dx'} = \frac{\delta u(x)}{a_4(x) t + a_5(x)}
\]
We will solve the following equations
\[
dt = 0, \\
\frac{\delta u(x)}{a_4(x) t + a_5(x)} = \frac{\delta u(x_1)}{a_4(x_1) t + a_5(x_1)}, \quad \text{for fixed } x_1
\]
\[
\frac{d\Phi}{\Phi} = -\frac{\int_{a}^{b} \delta u(x) \left(2a_6 + \int_{a}^{b} a_4(x') u(x') dx'\right) dx}{2 \int_{a}^{b} (a_4(x) t + a_5(x)) dx}
\]
which give a set of integration constants
\[
t = \tau, \\
C(x) = \frac{u(x)}{a_4(x) t + a_5(x)} - \frac{u(x_1)}{a_4(x_1) t + a_5(x_1)} \quad \text{for each } x \in (a, b), \\
\Phi = C_\Phi \exp \left[-\frac{4a_6 \int_{a}^{b} u(x) dx + \int_{a}^{b} \int_{a}^{b} a_4(x') u(x') u(x') dx' dx'}{4 \int_{a}^{b} (a_4(x) \tau + a_5(x)) dx}\right].
\]
The constants constitute a new set of variables. We treat \(C_\Phi\) as a new dependent variable, hence, from a relation \(C_\Phi = h(\tau, [C(x)])\) we find a formula for \(\Phi\)
\[
\Phi = h(\tau, [C(x)]) \exp \left[-\frac{4a_6 \int_{a}^{b} u(x) dx + \int_{a}^{b} \int_{a}^{b} a_4(x') u(x') u(x') dx' dx'}{4 \int_{a}^{b} (a_4(x) \tau + a_5(x)) dx}\right] = h(\tau, [C(x)]) \exp (H)
\]
where for brevity the term in the exponent is denoted by \(H\). This result can be further substituted into Eq. [51] to find the final form of the invariant solution. First, we compute the necessary derivatives in terms of new variables
\[
\frac{\partial \Phi}{\partial t} = \frac{\partial h}{\partial \tau} \exp (H) + h \exp (H) H \frac{\int_{a}^{b} a_4(x) dx}{\int_{a}^{b} (a_4(x) \tau + a_5(x)) dx},
\]
the expression for the second order functional derivative of \(\Phi\) is lengthy, however, after integration with respect to \(x\) all terms that contain functional derivatives of \(h\) will cancel. Below, we write the final form, after integration
\[
\int_{a}^{b} \frac{\delta^2 \Phi}{\delta u(x)^2} dx = h \exp (H) \left[\frac{2a_6 + \int_{a}^{b} a_4(x') u(x') dx'}{2 \int_{a}^{b} (a_4(x) \tau + a_5(x)) dx}\right]^2
\]
\[
- h \exp (H) \frac{\int_{a}^{b} a_4(x) dx}{2 \int_{a}^{b} (a_4(x) \tau + a_5(x)) dx}
\]
We can now compare (113) with (114) and perform the time integration to obtain the form of functional $h$. Finally, the solution of Eq. (51), invariant under the considered symmetries writes

$$
\Phi(t, [u(x)]) = \frac{g(C(x))}{\sqrt{t \int_a^b a_4(x)dx + \int_a^b a_5(x)dx}} \exp \left[ -\frac{(2a_6 + \int_a^b a_4(x)u(x)dx)^2}{4\int_a^b a_4(x)dx(t \int_a^b a_4(x)dx + \int_a^b a_5(x)dx)} \right].
$$

7 Conclusions and perspectives.

In the present work the classical, point symmetry group analysis is extended from partial differential equations to their counterparts in the continuum limit. In particular, we introduce the procedure of applying symmetry analysis to the case when functional derivatives are present in the equation. As example the method is further applied to the continuum limit of a heat equation and the Lie point symmetries, admitted by this equations are derived. From the infinitesimal transformations one can also find the invariant solutions of the considered equations.

The presented extension of the Lie groups can be a useful tool for analysing the functional equations. Though we have only given the heat equation as an application of the method we believe that the new approach is highly relevant to a variety of important functional differential equations (FDE) in mathematical physics. Generally speaking, very little is known on how to analytically treat and solve FDEs (numerical treatment is difficult anyway because of the high dimensionality). Hence, the methods may give a chance to treat equations which so far have been put aside because of the missing analytical methods. In fact, the benefit is twofold since the symmetries not only allow for analytical solutions but are also useful in itself since symmetries illuminate the axiomatic properties of the physical model equations.

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