A Generalized Linear Response Theory of Complex Networks with an Application to Renewable Fluctuations in Microgrids

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In this work we study the general linear response theory for the distribution of energy fluctuations through complex networks. We develop the response equations for oscillators coupled on arbitrary, directed and weighted networks, when subjected to stationary fluctuations with arbitrary power spectra.

Guided by the case study of network models for the distributed control and stabilization of turbulent renewable energy fluctuations in power grids, we then develop approximations that capture the most impactful interactions between intrinsic network modes and typical fluctuations found in renewable energies. These cover an intermediate resonant regime where the fluctuations are neither slow enough to cause a homogeneous response of the whole system, nor fast enough to be localized on the network.

Applying these analytic approximations to the question which nodes in a microgrid are particularly vulnerable to fluctuations, we are able to give analytic explanations and expressions for the previously numerically observed network patterns in vulnerability. We see that these effects can only be explained by taking the losses on the lines, and the resulting asymmetry in the effective weighted graph Laplacian, into account. These structural asymmetries give rise to a dynamical asymmetry between nodes that cause a strong response when perturbed (troublemaker nodes), and nodes that always respond strongly whenever the network is somewhere perturbed (excitable nodes).

For the important special case of tree-like networks we derive a simple relation for troublemaker nodes stating that fluctuations are enhanced when going upstream.

The general theory also opens the door to future investigations into the stabilization of networks under correlated distributed fluctuations.

I. INTRODUCTION

A central requirement in the operation of power grids is the stability of the grid frequency at 50 or 60Hz. In the language of theoretical physics this can be phrased as the question whether a networked system of inertial oscillators stays close to synchrony in the presence of fluctuating energy infeed at the nodes [1, 2]. As energy fluctuations are typically localized, we might also ask how fluctuations spread throughout the network.

With the energy transition these questions, and the complex networks perspective on them, gain increasing importance. Higher shares of renewable energy sources (RES) lead to an increase in fluctuating intermittent power infeed characterized by turbulent power spectra and at the same time reduce the grids inertia, meaning that the grid frequency becomes more sensitive to such fluctuations. The rate of the change of frequency at locations corresponding to conventional generators is one of the limiting factors in the integration of renewable energies in island grids already today. Currently about 90% of RES are installed in lower voltage grids. The presence of non-negligible losses in these systems means that experiences and theory from high voltage systems is not directly applicable. From the perspective of complex networks theory, the losses lead to asymmetric effective network Laplacians. The problem naturally lives on a directed network, necessitating new analytical tools.

Motivated by these questions, we develop a general linear response theory for networks of oscillators with asymmetric weighted coupling with respect to stationary fluctuations of arbitrary power spectra. From previous empirical work we know that this level of generality is required. When losses are neglected, or fluctuations with different power spectra are applied, the response of the...
system is not just quantitatively but even qualitatively wrong, see [3, 4] for comparison to the lossless case, [4] for the non-linear impact of losses and [6, 7], where it was shown that white Gaussian noise for example dramatically underestimates the networks response.

Whereas in symmetric networks it makes sense to speak of a node’s susceptibility as a single measure to quantify its fluctuation response, the presence of distinct left and right eigenvectors in the effective networks Laplacian leads to new dynamical phenomena. The bulk response that dominates the networks response to slow fluctuations, which is homogeneous in undirected networks, exhibits a non-trivial network structure when losses are present. In general there appear potentially distinct sets of nodes from which fluctuations spread strongly into the network (troublemakers) and to which fluctuations spread easily (excitable nodes) [3, 4].

The previous theoretical work on fluctuation spreading focused almost exclusively on the case of lossless power grids and symmetric coupling.

Analytical results were given for singular perturbations [8], harmonic perturbations [9], white Gaussian noise [10] and exponentially correlated (coloured) noise [11, 12].

The impact of turbulent wind power fluctuations on the intermittency of the frequency signal has been studied in [13]. In [14] the variance of the increment distributions was calculated for the case of lossless grids. The expressions are derived using a linear response approach similar to our own and where deduced lossless grids. The expressions are derived using increment statistics [13]. In [14] the variance of the intermittent fluctuations was studied for the case of low voltage grids and symmetric coupling.

In [3, 4] three frequency regimes of the network response were identified: a bulk, a resonant and a local regime. In the bulk regime for low frequencies the network is excited as a whole, whereas in contrast, high frequency perturbations stay localized at the fluctuating node and decay exponentially with the network distance (see also [15]). Fluctuations of renewables cover the whole frequency spectrum with a Kolmogorov-like turbulent power spectrum [16, 17], thus potentially touching all regimes.

Using our general expressions for the linear response we derive analytic approximations for the variance of the frequency response, that reveal the key drivers of the networks response in the bulk and the resonant regime. We show that these expressions correctly reproduce the network response patterns empirically observed in previous works [4]. For the practically important case of radial grids with tree topology, we give a simple expression for the new patterns that dominate the low frequency response. We find that fluctuations are enhanced when going upstream.

A. The model

The oscillator model that we consider is the Kuramoto model with inertia, also called second-order synchronous machine model for our case study. It accurately captures the short time behavior of existing power grids driven by synchronous machines [19], but also can be used to approximate the dynamics of grid-forming inverters that are expected to drive the future power grid [20]. Generally speaking it is the lowest order approximation of inertial oscillators coupled through the lowest harmonic on the limit cycle, and thus plays a paradigmatic role in many different physical systems. The system of equations is given by:

\[ \dot{\phi}_i = \omega_i, \]

\[ M_i \ddot{\omega}_i = P_i + \delta P_i(t) - D_i \dot{\omega}_i - \sum_{j=1}^{N} P_{ij}, \]

\[ P_{ij} = K_{ij} \left( \sin (\alpha_{ij}) + \sin (\phi_i - \phi_j - \alpha_{ij}) \right). \]

Here \( P_i \) is the power injected/consumed at node \( i \), \( \delta P_i(t) \) its fluctuation and \( P_{ij} \) the power flowing from node \( i \) to node \( j \). This power flow is given in terms of the voltage phase difference and \( Y_{ij} = \exp(i\omega_{ij}) \) of the links, with \( \alpha \) typically positive. The admittance is the inverse of the complex impedance consisting of the reactance and the resistance \( Y_{ij} = \frac{1}{R_{ij} + jX_{ij}} \) and therefore we have \( \tan(\alpha_{ij}) = \frac{R_{ij}}{X_{ij}} \). The maximal power flow on a link is given by the capacity \( K_{ij} = U_i|Y_{ij}|U_j \). The dynamical parameters \( D_i \) and \( M_i \) characterize the power’s response to frequency changes and the inertia. The exact parametrization of the model for the case of low voltage power grids is discussed in section [11].

For concreteness we will focus on the power grid model [1]. However, the linear response theory presented here is valid for arbitrary coupled inertial oscillators with fluctuating power infeed though. Any system

\[ \dot{\phi}_i = P_i + \delta P_i(t) - D_i \dot{\phi}_i - \sum_{j=1}^{N} G_{ij}(\phi_i, \phi_j), \]

that admits a fixed point with all \( \frac{\partial G_{ij}(\phi^*, \phi^*)}{\partial \phi_i} \) and \( \frac{\partial G_{ij}(\phi^*, \phi^*)}{\partial \phi_j} \geq 0 \) will have a linearisation amenable to our analysis.

The intermittent fluctuations \( \delta P(t) \) we will study in the concrete example are given by synthetic models for wind and solar power fluctuations derived and tuned to real data. They are turbulent with a power spectrum scaling as \( \frac{5}{3} \) and smoothed at around the second scale, in accordance with observed data [3, 18]. Thus it is appropriate to model them as time varying power infeed rather than as a stochastic differential equation.

II. ANALYTIC FOUNDATION

A. Linear Response Theory

In the absence of power fluctuations \( \delta P_i(t) = 0 \), we assume that there exists a global synchronous state in
the dynamical system [1] with \( \omega_i(t) = \omega_{\text{global}} \) \( \forall i \) and constant phase angles \( \phi_i(t) = \phi_i^* \). We analyse the grid dynamics in the co-rotating frame, i.e. \( \omega_{\text{global}} = 0 \). The phase angles \( \phi_i^* \) are then determined by the lossy nonlinear power flow equations

\[
P_i = \sum_{j=1}^{N} K_{ij} \left[ \sin(\alpha_{ij}) + \sin(\phi_i^* - \phi_j^* - \alpha_{ij}) \right].
\]

This synchronous state corresponds to the operating state of the grid. In the following we will analyse the influence of power fluctuations \( \delta P_i(t) \) close to this operating point. Assuming these fluctuations to be small, the dynamics is sufficiently described by the linearised system. Linearising equation (1) at the operating point \( \phi_i(t) = \phi_i^* + \delta \phi_i(t) \) yields

\[
\begin{bmatrix}
\delta \dot{\phi}_i \\
\dot{\omega}_i
\end{bmatrix} = \begin{bmatrix}
0_{N \times N} & 1_{N \times N} \\
L & D
\end{bmatrix}\begin{bmatrix}
\delta \phi_i \\
\omega_i
\end{bmatrix} + \begin{bmatrix}
0_N \\
\delta P_i(t)
\end{bmatrix},
\]

(3)

with the parameter matrices \( M = \text{diag}(M_i) \), \( D = -\text{diag}(D_i) \) and the Laplacian matrix

\[
L_{ij} = \begin{cases}
-\sum_{l=1}^{L} K_{ii} \cos(\phi_i^* - \phi_j^* - \alpha_{ii}) & \text{for } i = j, \\
K_{ij} \cos(\phi_i^* - \phi_j^* - \alpha_{ij}) & \text{for } i \neq j.
\end{cases}
\]

Defining the Jacobian matrix as

\[
J = \begin{bmatrix}
\frac{\partial \phi_i}{\partial \phi_j} & \frac{\partial \phi_i}{\partial \phi_l} \\
\frac{\partial \phi_j}{\partial \phi_l} & \frac{\partial \phi_l}{\partial \phi_l}
\end{bmatrix} = \begin{bmatrix}
0_{N \times N} & 1_{N \times N} \\
M^{-1}L & M^{-1}D
\end{bmatrix},
\]

(4)

the solution of the linearised system [3] can be written as

\[
\begin{bmatrix}
\delta \phi(t) \\
\omega(t)
\end{bmatrix} = \int_{-\infty}^{t} e^{J(t-t')} \begin{bmatrix}
0_N \\
M^{-1} \delta P(t')
\end{bmatrix} dt'.
\]

The Jacobian is an asymmetric matrix having distinct left and right eigenvectors \( v_l^{(n)} \), \( v_r^{(n)} \) for every eigenvalue \( \sigma_n \). The left and right eigenvectors are orthonormal to each other but not orthonormal within themselves

\[
v_r^{(n)} \cdot v_l^{(m)} = \delta_{nm}.
\]

With this, the Jacobian matrix can be decomposed as

\[
J = v_r \cdot \Sigma \cdot v_l^T
\]

where \( \Sigma = \text{diag}(\sigma_n) \). Using the fact that the matrix exponential \( e^J \) has the same eigenvectors as the Jacobian itself,

\[
e^J v_r^{(n)} = e^{\sigma_n} v_r^{(n)}
\]

\[
e^J v_l^{(n)} = v_l^{(n)} e^{\sigma_n},
\]

the frequency fluctuations at the nodes can then be expressed as

\[
\omega(t) = \sum_n w_r^{(n)} \int_{-\infty}^{t} e^{\sigma_n (t-t')} \omega_i^{(n)} \cdot M^{-1} \delta P(t') dt',
\]

(5)

where \( w_r^{(n)} \) and \( w_l^{(n)} \) are the lower halves of the Jacobian’s eigenvectors \( v_l^{(n)} \) and \( v_r^{(n)} \). We define a network response matrix as

\[
\chi(t) = \sum_n \chi^{(n)}(t) = \sum_n \theta(t) e^{\sigma_n} w_r^{(n)} w_l^{(n)} \cdot M^{-1},
\]

(6)

where \( \theta(t) \) is the Heaviside function. The factor \( w_r^{(n)} w_l^{(n)} \) is the outer product of the right and left eigenvectors and thus \( \chi(t) \) is a matrix valued function which can also be written as a superposition of the single mode response functions \( \chi^{(n)}(t) \). Using the network response matrix \( \chi(t) \), equation (5) can be rewritten as

\[
\omega(t) = \int_{-\infty}^{\infty} \chi(t-t') \cdot \delta P(t') dt'.
\]

(7)

**B. Power Spectral Densities**

By applying the convolution theorem to (7), it follows that in Fourier space the frequencies at the single nodes are simply given by the product of the Fourier transformed network response matrix and the power fluctuations

\[
\tilde{\omega}(\nu) = \tilde{\chi}(\nu) \cdot \delta \tilde{P}(\nu).
\]

The power spectral density \( S_{\omega_i}(\nu) = |\tilde{\omega}(\nu)|^2 \) of the frequency at a single node \( i \) is then given by the line-wise square of the right hand side

\[
S_{\omega_i}(\nu) = \sum_{j,l} \tilde{\chi}_{ij}(\nu) \tilde{\chi}_{lj}(\nu) S_{P_{ij}}(\nu).
\]

(8)

Here, \( S_{P_{ij}} = \delta \tilde{P}_i(\nu) \delta \tilde{P}_j(\nu) \) are the cross-spectral densities of the fluctuations \( \delta P(t) \) at different nodes.

This expression gives the response of a networked oscillator system to arbitrary, even correlated, fluctuations at various nodes. If the cross spectral densities, and thus the cross-correlations vanish, the response is given by the sum of the responses to perturbation at individual nodes. In the following we will focus on this case, however, this formula can also serves as a natural starting point for analysing the response to correlated fluctuations. In particular one can ask for network structures that are adapted to known anti-correlations in the input, using the network itself as a filter that can cancel out fluctuations.

**C. Variance of Frequency Fluctuations**

In the following, we want to study how the frequency of a node \( i \) is affected by a single node fluctuation at another node \( j \) as depicted in Figure 1. With this assumption, equation (8) reduces to

\[
S_{\omega_i}(\nu) = |\tilde{\chi}_{ij}(\nu)|^2 S_{P_j}(\nu).
\]

(9)
Assuming stationary fluctuations with average $\langle \omega_i(t) \rangle_t = 0$, we can quantify the strength of the frequency fluctuations in terms of the variance

$$\text{Var}[\omega_i(t)] = \langle (\omega_i(t))^2 \rangle_t = C_\omega(\tau = 0),$$

where $C_\omega(\tau) = \langle \omega_i(t) \omega_i(t + \tau) \rangle_t$ is the auto-correlation function of the frequency signal at node $i$. From the Wiener-Khinchin-theorem it follows that the Fourier transform of the correlation function is equal to the power spectral density $S_{\omega_i}(\nu)$ and thus the frequency variance is given by

$$\text{Var}[\omega_i(t)] = \int_{-\infty}^{\infty} S_{\omega_i}(\nu) d\nu.$$ 

Inserting equation (9) and decomposing the network response into the single mode responses finally yields

$$\text{Var}[\omega_i(t)] = \sum_{n,m} \int_{-\infty}^{\infty} \overline{\chi_{ij}^{(n)}(\nu)} S_{\omega_i}(\nu) \nu d\nu.$$  

Here, the susceptibilities $\chi_{ij}^{(n)}(\nu)$ are the Fourier transforms of the response functions $\chi_{ij}^{(n)}(t)$ defined in (6) and take the form

$$\chi_{ij}^{(n)}(\nu) = \frac{w_{ri}^{(n)} w_{ij}^{(n)} M_j^{-1}}{\text{Re}(\sigma_n) - i(\nu - \text{Im}(\sigma_n))}.$$ 

D. Approximations

The exact formula (10) cleanly separates the network response matrix and the power spectrum of the fluctuations. It is an excellent starting point for approximating the networks response in order to find the dominant contributions to the systems response to turbulent fluctuations. Our main aim with these approximations will be to eliminate the integral to further distill what properties of the spectrum affect the network response.

While the first of the approximations we will explore is explicitly controlled, the subsequent ones are heuristic. The validity of these approximations will be explored in the concrete example of section III and the supplemental information. For coherence of the presentation we will already introduce them here though.

Assuming that the overlaps of the susceptibilities of the different modes are small, the variance of the frequency fluctuations is dominated by the absolute squares of the individual modes

$$\text{Var}[\omega_i(t)] \approx \sum_{n} \int_{-\infty}^{\infty} |\chi_{ij}^{(n)}(\nu)|^2 S_{\omega_i}(\nu) d\nu.$$  

This assumption is valid as long as the gap between the different modes is significantly larger than the linewidth of the modes, or the overlap of the eigenvectors is small. A bound on the approximation in terms of the gap suppression is given in the supplemental material. The squared susceptibilities of the single modes then have a Lorentzian shape

$$|\chi_{ij}^{(n)}(\nu)|^2 \approx \frac{w_{ri}^{(n)} w_{ij}^{(n)} M_j^{-2}}{\text{Re}(\sigma_n)^2 + (\nu - \text{Im}(\sigma_n))^2}.$$ 

The peak position is determined by the imaginary part and the width determined by the real part of the Jacobian eigenvalues. The height of the peaks is determined by the

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**FIG. 1. Single-node fluctuations in an example microgrid** Left: Power fluctuation time series $\delta P(t)$ jointly generated by solar and wind models that capture their intermittent behavior [6, 18]. Center: Randomly generated radial microgrid. Right: Frequency time series for all network nodes for single-node fluctuations at nodes a, b and c.
For each mode we define a spectral excitation factor as

\[ SEF^{(n)} = \int_{-\infty}^{\infty} \frac{S_P(\nu)}{\text{Re}(\sigma_n)^2 + (\nu - \text{Im}(\sigma_n))^2} d\nu. \quad (12) \]

With this the approximation for the variance of the frequency fluctuations can be written as

\[ \text{Var}[\omega_i(t)] \approx M_j^{-2} \sum_n |w_{rij}^{(n)} w_{rij}^{(n)}|^2 SEF^{(n)}. \quad (13) \]

For white noise the power spectrum \( S_P(\nu) \) is a constant, and the spectral excitation factor can easily be calculated by integrating the Lorentzian functions. However, for correlated (e.g. coloured) noise the integral (12) is generally hard to solve. In the following sections we will derive approximations of the spectral excitation factor for different kinds of network modes excited by turbulent-like noise.

### E. Harmonic Modes

When the ratio of damping to inertia is assumed to be constant in the network \( \gamma := \frac{D_i}{M_i} \), the eigenvalues of the Jacobian (4) can be calculated explicitly. Assume that \( u_l^{(n)}, u_r^{(n)} \) and \( w_l^{(n)}, w_r^{(n)} \) are the upper and lower halves of the Jacobian eigenvectors \( v_l^{(n)}, v_r^{(n)} \). The eigenvalue for the Jacobian matrix is then given by

\[
\begin{bmatrix}
0_{N \times N} & 1_{N \times N} \\
M^{-1}L & M^{-1}D
\end{bmatrix}
\begin{bmatrix}
u_l^{(n)} \\
u_r^{(n)}
\end{bmatrix}
= \sigma_n
\begin{bmatrix}
u_l^{(n)} \\
u_r^{(n)}
\end{bmatrix},
\]

\[
\begin{bmatrix}
u_l^{(n)} \\
u_l^{(n)}
\end{bmatrix}
\begin{bmatrix}
0_{N \times N} & 1_{N \times N} \\
M^{-1}L & M^{-1}D
\end{bmatrix}
= \sigma_n
\begin{bmatrix}
u_l^{(n)} \\
u_l^{(n)}
\end{bmatrix}.
\]

From this we can derive the following equations

\[ w_l^{(n)} \cdot M^{-1}L = \sigma_n (\sigma_n + \gamma) w_l^{(n)} \]

\[ M^{-1}L \cdot w_r^{(n)} = \sigma_n (\sigma_n + \gamma) w_r^{(n)}. \]

These imply that in case of a homogeneous ratio of damping and inertia the lower halves of the Jacobian eigenvectors are proportional to the eigenvectors \( u^{(n)} \) of \( M^{-1}L \) and the Jacobian eigenvalues are given by

\[ \sigma_n = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} + \lambda_n}, \]

where \( \lambda_n \) are the eigenvalues of \( M^{-1}L \). For the harmonic modes \( |\lambda_n| > \frac{\gamma^2}{4} \) we get

\[ \text{Re}(\sigma_n) = -\frac{\gamma}{2}, \quad \text{Im}(\sigma_n) = \pm \sqrt{\frac{\gamma^2}{4} + \lambda_n}. \]

For modes that have a small linewidth compared to the variation of the power spectrum \( S_P(\nu) \) in that specific frequency range, the integral in (12) has only significant contributions in the neighbourhood of \( \nu = \text{Im}(\sigma_n) \). We therefore make the approximation \( S_P(\nu) \approx S_P(\text{Im}(\sigma_n)) \) such that (12) reduces to an integral over the Lorentzian function and yields the spectral excitation factor

\[ SEF^{(n)} \approx \frac{2 \pi}{\text{Re}(\sigma_n)^2} \int_0^{\text{Re}(\sigma_n)} S_P(\nu) d\nu. \quad (14) \]

The inertia weighted Laplacian is an asymmetric matrix and hence, its left and right eigenvectors are distinct. This asymmetry arises from the heterogeneity of the inertia parameter \( M_i \) as well as from the loss parameter \( \alpha_{ij} \). From (13) it follows that this leads to an asymmetric dynamics. The nodes at which the network is particularly susceptible to external fluctuations are not necessarily those which are strongly oscillating when the network modes are excited and vice versa.

### F. Overdamped Modes

For the overdamped modes \( |\lambda_n| < \frac{\gamma^2}{4} \) we get

\[ \text{Re}(\sigma_n) = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} + \lambda_n}, \quad \text{Im}(\sigma_n) = 0. \]

The spectral excitation factor for these modes is given by the integral

\[ SEF^{(n)} = \int_{-\infty}^{\infty} \frac{S_P(\nu)}{\nu^2 + \text{Re}(\sigma_n)^2} d\nu. \quad (15) \]

The power spectrum of turbulent noise is typically strongest for very low frequencies. The kernel we are integrating against has fallen of to half its peak at \( \nu = \text{Re}(\sigma_n) \). Assuming that the power spectrum is not varying much in this regime, we can approximate the power spectrum by its average in this regime,

\[ S_P(\nu) \approx \frac{2}{\text{Re}(\sigma_n)^2} \int_0^{\text{Re}(\sigma_n)} S_P(\nu') d\nu' \]

and assume that larger frequencies do not significantly contribute to the integral (12). With this the spectral excitation factor can be approximated by

\[ SEF^{(n)} \approx \frac{2 \pi}{\text{Re}(\sigma_n)^2} \int_0^{\text{Re}(\sigma_n)} S_P(\nu) d\nu. \]

### G. The Bulk Mode

We now assume the network to be connected and hence, if the system has no further symmetry, the inertia weighted Laplacian has exactly one zero eigenvalue \( \lambda_0 = 0 \). The corresponding eigenvalues of the Jacobian
are $\sigma_0^+ = 0$ and $\sigma_0^- = -\gamma$. Here, $\sigma_0^+$ corresponds to the symmetry of homogeneous phase shifts that have no impact on the dynamics, whereas $\sigma_0^-$ corresponds to homogeneous frequency shifts leading to an exponentially decaying response of the nodes’ frequencies with rate $-\gamma$. This mode therefore corresponds to a bulk behaviour of the nodes with

$$\text{Re}(\sigma_n) = -\gamma, \quad \text{Im}(\sigma_n) = 0.$$  

The spectral excitation factor for the bulk mode is given by the integral

$$SEF^{(0)} = \int_{-\infty}^{\infty} S_{P_{ij}}(\nu) \, d\nu.$$  

With the same approximation as in the case of overdamped modes we get

$$SEF^{(0)} \approx \frac{2\pi}{\gamma^2} \int_0^{\gamma} S_{P_{ij}}(\nu) \, d\nu.$$  

The right eigenvector corresponding to the bulk mode is homogeneous, i.e. $v_i^{0,0} = v_j^{0,0} \forall i,j$. The left eigenvector, however, is generally heterogeneous and fulfils the condition

$$0 = \sum_i v_i^{0,0} [M^{-1}L]_{ij}$$  

$$= \sum_i K_{ij} \left[ \frac{v_i^{0,0}}{M_j} \cos(\phi_{ji} - \alpha_{ij}) - \frac{v_i^{0,0}}{M_i} \cos(\phi_{ij}^* - \alpha_{ij}) \right].$$  

In tree-like networks there exist no closed loops, therefore every single summand has to be zero itself. From this condition, we can derive the following equation for the ratio of the eigenvector entries of two neighbouring nodes

$$\frac{v_i^{0,0}}{v_j^{0,0}} \approx \frac{M_i}{M_j} \left( 1 + \frac{R_{ij}}{X_{ij}} \tan(\phi_{ij}) \right). \quad (16)$$  

In the DC approximation with small phase angle differences $\phi_{ij}$, (16) simplifies to

$$\frac{v_i^{0,0}}{v_j^{0,0}} = \frac{M_i}{M_j} \left( 1 - 2\phi_{ij}^* R_{ij} X_{ij} \right). \quad (17)$$  

From this, it follows that even in the case of identical node dynamics ($M_i = M_j \forall i,j$) in a tree-network, the left eigenvector corresponding to the bulk mode has non-identical entries if there are losses on the line ($R_{ij} > 0$). This means that although all nodes are equally excited by this mode, they are not equally susceptible to external fluctuations driving this mode.

With the full set of approximations for the spectral excitation factors $SEF$, the approximation (13) is now given purely in terms of a sum over modes. For the model discussed below, Figure 2 shows the various spectra. We see that despite the rather radical simplifications the peak approximation is capable of accurately capturing the qualitative behaviour of the system in the bulk and the resonant regime.

**FIG. 2. Power spectral density of fluctuation series, network kernel and network response.** Upper Left: Turbulent power spectrum of a power fluctuation time series generated with the stochastic model described in section IIIA. Lower left: Network response spectrum for a certain pair of input and output nodes and the dynamical parameters $D = 0.01 \text{s}$ and $M = 0.1 \text{s}^2$. The vertical lines correspond to the eigenfrequencies of the network. Right: Power spectral density of the network response with different approximations. The diagonal approximation is the assumption that modes couple only with themselves and there is no off-diagonal interaction between different modes, as discussed in section IID. The peak approximation is given by equation (13) with all approximations of the spectral excitation factor discussed in sections II D, II E, II F, and II G.
III. CASE STUDY: AC MICROGRIDS

The linear response theory presented in the previous section is rather general and valid for power grids disturbed with arbitrary noise. In the following we will study the quality of our results and approximations by comparing them to numerical simulations capturing the full nonlinear dynamics given by equation (1). Concretely we will study the paradigmatic model case of an inverter based AC microgrid with high share of renewable energies. We will test a range of different dynamical regimes to explore the range of validity of our approximations.

A. Model Setup

The following model case represents an islanded mid-voltage microgrid. Islanded microgrids play a crucial role for the decentralized provision of energy, but also as part of a safety and stability strategy to localize faults by partitioning the grid into autonomous units. A mid-voltage microgrid has a non-hierarchical grid topology, every node represents a mix of distributed RES, storage and consumers, connected directly or via low voltage feeders.

This model case is kept at a conceptual level to study the effect of local fluctuations on dynamic grid stability and isolate the influence of the network structure.

a. Microgrid modelling

Germany has 4,500 MV distribution networks that connect 500,000 LV distribution networks [21]. Thus the microgrid is chosen as a network of 100 nodes [22] to represent an average German grid at medium-voltage (MV) level. The MV level is a good testing case for modelling power grids with a high renewable energy share, since most PV power plants are connected to low-voltage- (LV) or MV levels. An islanded microgrid must be internally power-balanced and not connected to a higher grid level.

Being power balanced, we assume that there are 50 net producers and 50 net consumers with $P_i = \pm 0.2 MW$ power infeed before losses. The power infeeds are chosen homogeneously to focus on topology and network effects in the model. As there is no connection to upper grid levels, losses are compensated locally at each node, and the net power infeed is given by $P_i = (P_i + P_{loss}/N)$. Mathematically this is equivalent to switching to the co-rotating frame.

b. The grid parametrization

follows from the voltage level. The line impedance for typical MV grids with 20kV base voltage equals $Z = R + jX \approx (0.4 + 0.3i)\Omega/km$ [23]. The coupling strength between a node pair $(i, j)$ then equals

$$K_{ij} = U_i|Y_{ij}|U_j$$

where $|U_i| = 20 \text{kV}$. For simplicity all power, voltage and impedance values are transformed into per unit with a base voltage of 20kV and a base power of 1MW, which are typical values for MV grids [23, 24]. The absolute impedance of each line scales with the geographic distance $l$ between linked nodes and consequently differs per link. The average line length, according to [23], is 23.7km. The inclusion of resistive lines leads to line losses and thus introduces a phase shift of $\alpha_{ij} = \text{arctan}(\frac{R_{ij}}{X_{ij}}) \approx 0.93$.

Further, the model case is assumed to be dominated by inverters to analyze a scenario with high RES penetration. Wind and solar power plants are connected to the grid via inverters. In an islanded scenario some of these inverters will need to be grid forming to ensure frequency stability. As mentioned above, network nodes are aggregated with a mix of grid-forming inverters, grid-feeding inverters and demand [20]. Since grid-feeding inverters do not contribute any inertia, the effective nodes have inertia much lower than nodes fully consisting of grid-forming inverters would have.

Grid forming inverters are modeled following [20] with a droop controlled frequency based on a low pass filtered power measurement. The virtual inertia and damping for the network model is then given by the low-pass filter exponent $\tau_p$ and the droop control parameter $k_p$ from grid-forming inverters: $M = \tau_p/k_p$, $D = 1/k_p$, $\forall i$ with $i = 1, ..., N$. Standard parameters for the droop and time constants of grid-forming inverters are in the range $k_p = [0.1, \ldots, 10]\text{s}^{-1}$ and $\tau_p = [0.1, \ldots, 10]\text{s}$ [25, 26]. In the low inertia case, with only a few low powered grid forming inverters at each node, we assume a weakly reacting, strongly smoothed system. This lead us to consider $D = [0.01, \ldots, 0.1]\text{s}$ and $M = [0.1, \ldots, 1.0]\text{s}^2$.

c. Intermittent Power infeed

In the following simulations intermittent time series for solar and wind power fluctuations were generated by a clear sky index model, based on a combination of a Langevin and a jump process, developed in [18], and a Non-Markovian Langevin type model developed in [6], respectively. An example time series, $\Delta P(t)$, of the combined wind and solar power fluctuations, $\Delta P_W(t)$ and $\Delta P_S(t)$ respectively, is shown in Fig.1 (left)

$$\Delta P(t) = 0.5\Delta P_W(t) + 0.5\Delta P_S(t). \quad (19)$$

Both the PDF of the fluctuations and their increment time series are fat tailed (the tails are not exponentially bounded [27] and thus intermittent. The power generation from wind and solar power plants has a power spectrum that is power-lawed with the Kolmogorov exponent of turbulence [16, 18]. Thus, these time series show long-term temporal correlations [28, 31].

B. Predictors and Simulation Results

For an unobstructed operation of AC power grids the grid frequency is required to stay within certain bounds. Thus the time the frequency response to a stochastic infeed of power is outside of the predefined bounds should
Variance

having a much higher frequency variance for a significant
dynamics. Notably, there are a couple of output nodes
there are certain characteristics missing in the linearised
linear system with the linearised system shows that
capture the variance structure of the linear system.
proximation for the variance of frequency fluctuations.
mode plot.
corresponds to the continuous horizontal lines in the bulk
geneous depending on the losses of the network. This
whereas the left eigenvector entries can be very inhomo-
right eigenvector of this mode has homogeneous entries
the bulk mode. As shown in the previous chapter, the
in the dynamics due to line losses becomes apparent. It
can be seen that this asymmetry is very distinct for the
output nodes, respectively. Here, the strong asymmetry
for single node fluctuations. The vertical and horizontal
series do not differ considerably in shape.
probability distribution functions of the frequency time
deviation and the 90th percentiles, suggests that the
ance are also expected to have a large exceedance. This,
together with the strong linear correlation of standard
such plots contain simulations of the full nonlinear system (8)
for every pair of input and output nodes in the network. The inverter
parameters are $M = 1s$ and $D = 0.1s^2$. Exceedance is defined as the fraction of time the frequency signal is above the
threshold 0.1 Hz.

Although this might be the proper measure for practical applications, the response statistic most easily calculated is the variance. Figure 3 depicts the relation between the variance and exceedance for every pair of nodes in the network. It shows that these two measure are indeed highly correlated and thus node pairs with a large variance are also expected to have a large exceedance. This, together with the strong linear correlation of standard deviation and the 90th percentiles, suggests that the probability distribution functions of the frequency time series do not differ considerably in shape.

Figure 4 depicts the variance of the frequency signal for single node fluctuations. The vertical and horizontal lines correspond to large frequency variances of input and output nodes, respectively. Here, the strong asymmetry in the dynamics due to line losses becomes apparent. It can be seen that this asymmetry is very distinct for the the bulk mode. As shown in the previous chapter, the right eigenvector of this mode has homogeneous entries whereas the left eigenvector entries can be very inhomogeneous depending on the losses of the network. This corresponds to the continuous horizontal lines in the bulk mode plot.

In the previous section we derived an analytical approximation for the variance of frequency fluctuations. Figure 6 shows that this peak approximation is able to capture the variance structure of the linear system.

Comparing the variance structures of the full nonlinear system with the linearised system shows that there are certain characteristics missing in the linearised dynamics. Notably, there are a couple of output nodes having a much higher frequency variance for a significant number of different input nodes. This is due to nonlinear effects that cannot be captured in the linear response analysis.

Following [19] we call input nodes that are very susceptible for external fluctuations troubleshooter nodes and output nodes showing large frequency variances excitable nodes. In order to identify these node classes we define measures based on our theoretical analysis. The troublemaker index $TM$ is defined as the average output variance for the various input nodes

$$TM_j = \frac{1}{N} \sum_i \text{Var}[\omega_i(t)]_j ,$$

the excitable node index $EN$ is given by the average response to single node fluctuations

$$EN_i = \frac{1}{N} \sum_j \text{Var}[\omega_i(t)]_j .$$

With the approximation [13] this becomes:

$$TM^{\text{peak}} \propto M_j^{-2} \sum_n |w_{ij}^{(n)}|^2 \text{SEF}^{(n)} \sum_i |w_{ri}^{(n)}|^2 .$$

and

$$EN^{\text{peak}} \propto \sum_n |w_{ri}^{(n)}|^2 \text{SEF}^{(n)} \sum_j M_j^{-2} |w_{ij}^{(n)}|^2 .$$

Note that if we are only interested in one of these, the internal sums $\sum_i |w_{ri}^{(n)}|^2$ and $\sum_j M_j^{-2} |w_{ij}^{(n)}|^2$ can serve as natural normalization of the left or right eigenvectors. However, due to their mutual orthogonality we cannot have both at the same time.

In Figure 5 it can be seen that for the input nodes there is a very strong correlation between the analytical

FIG. 3. Comparison of different measures for the spread of the node frequency fluctuations. The scatter plots contain simulations of the full nonlinear system (1) for every pair of input and output nodes in the network. The inverter
predictor and the simulation results. This shows that for certain parameter regimes we are able to analytically predict the vulnerability of the nodes to the fluctuation power infeed solely based on the network structure, the control parameters and the power spectrum of the external fluctuations. The robustness of these results for different regimes of the control parameters is shown in the supplements.

For the output nodes we see that vast majority of nodes correlates with the analytical predictor. However, there are certain outliers that have a much higher variance in the simulations of the nonlinear system. These correspond to the vertical lines in Figure 4. This means that the highest excitability of single nodes corresponds to nonlinear effects and the linear response is therefore not suitable to identify the excitable nodes with the highest frequency variance.

When the typical timescale of the fluctuation is larger then the timescale \( \tau = \frac{M}{D} \) defined by the inverter parameters, we find that the network response is entirely dominated by the bulk mode. In this regime all nodes are equally excited but certain nodes have much higher susceptibility for spreading fluctuations through the network. This susceptibility is proportional to the left eigenvector of the bulk mode. Even if we are not directly in this regime the bulk mode can form a substantial contribution to the eventual variance as can be seen in Figure 4.

**FIG. 4.** Colour plot of the variance of frequency fluctuations at single nodes. Upper left: Variance of the frequency signals from simulations of the full nonlinear system \( \text{(1)} \) for each pair of input and output nodes. The inverter parameters are \( M = 0.1 \text{s} \) and \( D = 0.01 \text{s}^2 \). Upper right: Contribution of the bulk mode to the frequency variance calculated by a numeric integral according to \( \text{(11)} \). Lower left: Frequency variance of the linearised System. Lower right: Approximated frequency variance calculated with equation \( \text{(13)} \) and the approximations of the spectral excitation factor as discussed in sections \[ \text{II} \text{E}, \text{II} \text{F}, \text{II} \text{G} \]
FIG. 5. Correlation of predictors with variance data from simulations. The inverter parameters are $M = 0.1\text{s}$ and $D = 0.01\text{s}^2$. The predictors of troublemaker nodes and excitable nodes are defined in (22) and (23).

This is due to the turbulent power spectra of wind and solar power fluctuations which are strongest for very low frequencies (see Figure 2).

From equation (17) it follows that the entries of the left bulk mode eigenvector are monotonously increasing with the power flow. Consequently, when this mode is dominating, the troublemaker index (22) is increasing along the power flow with the troublemaker nodes located at the sinks of the flow. In Figure 5 it can clearly be seen that the network branch where the power is flowing from the center towards the outlying nodes is much more vulnerable than the network branches where the power is flowing towards the center. This explains Figure 5 in [19] showing the relation between the troublemaker index and the closeness centrality of the network.

**IV. DISCUSSION AND OUTLOOK**

We have seen that the analysis of the power spectrum of frequency fluctuations provides a powerful tool to understand how fluctuating infeeds spread in complex power grids. This is especially true for the resonant regime in which no good analytic approximations were previously known.

Our analysis is completely general in terms of fluctuations, and studies inertial oscillators that are widely used in physics as a paradigmatic example for oscillations, as well as in power engineering to understand the frequency and control behaviour of power grids. Crucially we did not require the assumption of lossless power lines. This allowed us to unveil novel patterns in the susceptibility of nodes, and reveal network patterns in the bulk response of coupled oscillators. Despite our focus on the dynamics of inertial oscillators we expect our approach to be generalizable to a wide array of linear dynamical systems on networks.

This work focuses on single node fluctuations. However, the formulation of the response in terms of power spectra also provides an elegant starting point for future investigations of multi-node fluctuations. In particular, we see that the cross-spectral densities play a crucial role for understanding the dynamical interactions of distributed fluctuations.

With some straightforward assumptions on the power spectrum of the power fluctuations we were able to derive reasonable approximations for the network response spectrum and thereby for the variance of frequency fluctuations. It should be noted that the approximations we used here, while successful at describing the observed phenomena in the particular regime of interest, are mostly uncontrolled and very rough. Assuming specific analytic models of the power spectrum of input fluctuations would enable us to derive much more sophisticated approximations from the mode expansion of the network response matrix.

The approximations were numerically validated for the case of an islanded microgrid with high renewable penetration. With these we were able to explain structures in the node vulnerability that have been observed in numerical simulations [3, 4] and to show that these vulnerability patterns originate from the losses on power lines. The losses induce an asymmetry in the dynamics from which two classes of nodes are emerging: troublemaker nodes at which the network is particularly vulnerable to power fluctuations and excitable nodes that generally feature large frequency spreads.

In the case of low-inertia grids (large damping to inertia ratio) with radial structure (tree networks) the location of troublemaker nodes is related to the power flow throughout the network. In particular, fluctuations at
net power flow sinks results in strong frequency fluctuations at all nodes in the network. The fluctuations are enhanced when going upstream. For microgrids grids with a very unbalanced power production we therefore expect branches that are net consumers to be much more vulnerable to turbulent infeed of renewable power. This effect is direct consequence of the interplay between losses on the power lines and the correlated nature of renewable power fluctuations.

Considering the generality of our theoretical approach, it should be mentioned that the application to power grids is not limited to fluctuation of RES but might also be the basis for studying the impact of demand fluctuations on grids.

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Let us assume that the mode susceptibilities take the form
\[ \chi_{ij}(\nu) = \sum_{n} \int_{-\infty}^{\infty} |\chi_{ij}^{(n)}(\nu)|^2 S_{\nu}(\nu) \, d\nu + \sum_{n \neq m} \int_{-\infty}^{\infty} \chi_{ij}^{(n)}(\nu) \chi_{ij}^{*(m)}(\nu) S_{\nu}(\nu) \, d\nu. \]

In the following we will show under which conditions the off-diagonal mode terms \((n \neq m)\) are negligible. It should be recalled that the mode susceptibilities take the form
\[ \chi_{ij}^{(n)}(\nu) = \frac{w_{ri}^{(n)} w_{ij}^{(n)} M_{ij}^{-1}}{\text{Re}(\sigma_n) - i(\nu - \text{Im}(\sigma_n))}. \]

Let us denote the real and imaginary parts of \(\sigma\) as \(\rho\) and \(\eta\) such that \(\sigma = \rho + i \cdot \eta\), and the numerator \(w_{ri}^{(n)} w_{ij}^{(n)} M_{ij}^{-1}\) as \(W_{ij}^{(n)}\). The maximum of the diagonal contributions to the network transfer function is given by \(\nu = \eta_n\)
\[ \max_{\nu} \left| \chi_{ij}^{(n)}(\nu) \right|^2 = \frac{|W_{ij}^{(n)}|^2}{\rho_n^2}. \]

Let us assume that \(\rho_i = \rho_j = \rho\), then off diagonal contributions are given by
\[ \chi_{ij}^{(n)}(\nu) \chi_{ij}^{*(m)}(\nu) = \frac{W_{ij}^{(n)} W_{ij}^{*(m)}}{|\rho - i(\nu - \eta_m)| |\rho + i(\nu - \eta_m)|} \]

The magnitude of the denominator reaches its extremum at
\[ 0 = (2\nu - \eta_n - \eta_m) (\rho^2 + (\nu - \eta_m)(\nu - \eta_n)) \]

If \(\rho^2 > \frac{|\eta_n - \eta_m|^2}{\eta_n + \eta_m}\), there is only one minimum of the denominator at \(\nu_{\text{min}} = \frac{\eta_n + \eta_m}{2}\). If we instead have \(\rho^2 < \frac{|\eta_n - \eta_m|^2}{\eta_n + \eta_m}\), this minimum turns into a local maximum and the new local minima are given by
\[ \nu_{\text{min}} = \frac{1}{2} \left( \pm \sqrt{\eta_n - \eta_m} + 4\rho^2 + \eta_n + \eta_m \right) \]
Therefore, in this case, the maximum of the magnitude of the off-diagonal term is given by

\[
\hat{\chi}^{(n)}_{ij}(\nu_{\min})\hat{\chi}^{*(m)}_{ij}(\nu_{\min}) = \frac{W^{(n)}_{ij}W^{*(m)}_{ij}}{|\rho||\eta_n - \eta_m|}
\]

We see that if \(|\eta_n - \eta_m| \gg |\rho|\) the off-diagonal mode terms are suppressed relative to the diagonal terms. It follows that the diagonal approximation in section IID is valid if the spectral gap between the modes is much larger than the damping to inertia ratio.

Parameter Regimes

In the sections IIE, IIF and IIG we introduced an approximation for the linearised system perturbed with turbulent noise which we call the peak approximation. In the main part of the paper we have chosen the inverter parameters such that the bulk mode is very dominant in the network response in order to emphasise the impact of the line losses. However, the approximation is also working well for different parameter regimes. Figure S1 shows a color plot of the single nodes frequencies for a parameter regime were the harmonic modes are predominant compared to the bulk mode. Finally, figure S2 shows for four different parameter regimes that there is a clear correlation between the peak approximation and the variance of the linearised system.
FIG. S1. **Colour plot of the variance of frequency fluctuations at single nodes.** This is the same plot as Fig. 4 with different inverter parameters ($M = 1.0s$ and $D = 0.1s^2$). Upper left: Variance of the frequency signals from simulations of the full nonlinear system (1). Upper right: Contribution of the bulk mode to the frequency variance calculated by a numeric integral according to (11). Lower left: Frequency variance of the linearised System. Lower right: Approximated frequency variance calculated with equation (13) and the peak approximation of the spectral excitation factor.
FIG. S2. Correlation between the variance of the linearised system and the peak approximation. The scatter plots show the real and the approximated variance of the frequency signal for each pair of input and output nodes in the network.