A SURVEY ON THE TURAEV GENUS OF KNOTS

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Abstract. The Turaev genus of a knot is a topological measure of how far a given knot is from being alternating. Recent work by several authors has focused attention on this interesting invariant. We discuss how the Turaev genus is related to other knot invariants, including the Jones polynomial, knot homology theories, and ribbon-graph polynomial invariants.

1. Introduction

Knots and links have been studied using graphs associated to their diagrams since the first knot tables were compiled in the late 1800’s. Separately, ribbon graphs, which are cellulary embedded graphs on a two-dimensional surface, have a long history, not only in graph theory but in the study of Riemann surfaces, Galois theory, quantum field theory and many other subjects (see [21] for example). Only recently, though, have ribbon graphs been associated to knot diagrams in a way that yields powerful new invariants of knots and links. In [12], Dasbach, Futer, Kalfagianni, Lin and Stoltzfus discovered that the Jones polynomial is a specialization of the Bollobás–Riordan–Tutte polynomial of a particular ribbon graph on a surface originally constructed by Turaev, called the Turaev surface of a knot. The minimal genus of such a surface, called the Turaev genus of a knot, is an interesting new invariant that measures how far a given knot (or link) is from being alternating.

The aim of this paper is to give some historical background about the ideas leading to the Turaev surface and Turaev genus, explain the connections to the more well-known graphs associated to knot diagrams, review some modern applications in knot homology theories, and lastly focus on open problems and new research directions related to the Turaev genus. A natural question is how the Turaev genus is related to other diagrammatic, geometric and topological invariants of knots and links.

1.1. Definition. Let \( D \) be a diagram of a link \( L \). For any crossing \( \times \) in \( D \), we obtain the \( A \)-smoothing as \( \times \) and the \( B \)-smoothing as \( \times \). A state \( s \) of \( D \) is a choice of smoothing at every crossing, resulting in a disjoint union of circles in the plane. Let \( |s| \) denote the number of circles in \( s \). Let \( s_A \) denote the all–\( A \) state, for which every crossing of \( D \) has an \( A \)-smoothing. Similarly, \( s_B \) is the all–\( B \) state of \( D \).

Now, at every crossing of \( D \), we put a saddle surface which bounds the \( A \)-smoothing on the top and the \( B \)-smoothing on the bottom as shown in Figure 1. In this way, we get a cobordism between \( s_A \) and \( s_B \), with the link projection at the level of the saddles. See Figure 1.
For any diagram $D$, the Turaev surface $F(D)$ is obtained by attaching $|s_A| + |s_B|$ discs to all boundary circles. The Turaev genus of $D$ is defined by

$$g_T(D) = g(F(D)) = (c(D) + 2 - |s_A| - |s_B|)/2.$$  

The Turaev genus of any non-split link $L$ is defined by

$$g_T(L) = \min\{g_T(D) \mid D \text{ is a diagram of } L\}.$$  

The properties below follow easily from the definitions. See [12] for proofs and figures.

(a) $F(D)$ is an unknotted closed orientable surface in $S^3$; i.e., $S^3 - F(D)$ is a disjoint union of two handlebodies.
(b) $D$ is alternating on $F(D)$.
(c) $L$ is alternating if and only if $g_T(L) = 0$, and if $D$ is an alternating diagram then $F(D) = S^2$.
(d) The projection of $D$ is a 4-valent graph which gives a cell decomposition of $F(D)$, for which the 2-cells can be checkboard colored on $F(D)$, with discs corresponding to $s_A$ and $s_B$ respectively colored white and black.
(e) This cell decomposition is a Morse decomposition of $F(D)$, for which $D$ and the crossing saddles are at height zero, and the $s_A$ and $s_B$ 2-cells are the maxima and minima, respectively.

For example, any non-alternating pretzel knot can be made alternating on the torus as follows:

In fact, the Turaev genus of any non-alternating pretzel knot (or more generally, any non-alternating Montesinos knot) equals one, and its Turaev surface is the torus. We will return to this fact in Section 5. See [16] for a nice animation of Turaev surfaces.

The paper is organized as follows: we discuss the motivation behind the Turaev surface in Section 2. In Section 3, we discuss the Turaev surface in the context of ribbon graphs. In Section 4, we discuss applications to knot homology theories. In Section 5, we discuss
known bounds for the Turaev genus. In Section 6, we discuss research directions and open problems related to the Turaev genus.

2. Tait’s Conjecture

Modern knot theory began in late 1800’s when Tait, Little and others tried to make a periodic table of elements by tabulating knot diagrams by crossing number. The only invariants at this time were of the form, “minimize something among all diagrams,” such as crossing number, unknotting number, bridge number, etc. Such invariants are easy to define but hard to compute: Diagrams that are minimal with respect to one property may not be minimal with respect to other properties.

There is a correspondence between connected link diagrams $D$ and connected plane graphs $G$ with signed edges. It follows from the Jordan Curve Theorem that any link diagram can be checkerboard colored. The Tait graph $G$ of $D$ is obtained by checkerboard coloring complementary regions of $D$, assigning a vertex to every shaded region, an edge to every crossing and a $\pm$ sign to every edge as in Figure 2.

![Image of Tait graphs](https://via.placeholder.com/150)

**Figure 2.** Edge sign convention and Tait graphs (lower figure from [26])

Conversely, we can recover the diagram from any signed planar graph by taking its medial graph, and making crossings according to the sign on each edge:

![Image of diagram recovery](https://via.placeholder.com/150)

Tait graphs for opposite checkerboard colorings are planar duals.

A link diagram is *alternating* if the crossings alternate between overcrossing and undercrossing as one walks along every component of the link. A link is *alternating* if it has a reduced alternating diagram. Tait emphasized the importance of an alternating diagram, for which all the edges in its Tait graph have the same sign, so it corresponds to an unsigned plane graph, determined by the diagram up to planar duality.

**Conjecture 2.1** (Tait). *An alternating link always has an alternating diagram that has minimal crossing number among all diagrams for that link.*
A proof had to wait about 100 years until the Jones polynomial (1984), which led to several new ideas that were used to prove Tait’s conjecture [20, 28, 32]. Below we follow a later proof by Turaev [33] using Turaev surfaces defined above. In Section 4, we discuss the spanning tree expansion for the Jones polynomial introduced in [32], which is of independent interest.

The simplest combinatorial approach to the Jones polynomial is via the Kauffman bracket \( \langle D \rangle \in \mathbb{Z}[A, A^{-1}] \) defined recursively by

1. \( \langle \otimes \rangle = A \langle \circ \rangle + A^{-1} \langle \otimes \rangle \)
2. \( \langle \bigcirc \ D \rangle = \delta \langle D \rangle, \quad \delta = -A^{-2} - A^2 \)
3. \( \langle \bigcirc \rangle = 1 \)

For any link \( L \) with a diagram \( D \) of writhe \( w(D) \), the Jones polynomial is determined by the Kauffman bracket as \( V_L(A^{-4}) = (-A)^{-3w(D)} \langle D \rangle \). We will use \( \langle L \rangle \) or \( V_L(t) \) depending on the context.

Besides this axiomatic definition, Kauffman [20] expressed \( \langle L \rangle \) as a sum over all possible states of \( L \): If \( L \) has \( n \) crossings, all possible \( A \) and \( B \) smoothings yield \( 2^n \) states \( s \). Let \( a(s) \) and \( b(s) \) be the number of \( A \) and \( B \) smoothings, respectively, to get \( s \).

\[
\langle L \rangle = \sum_{\text{states } s} A^{a(s) - b(s)} (-A^2 - A^{-2})^{s-1}
\]

A diagram \( D \) is adequate if (i) \( |s_A| > |s| \) for any state \( s \) with exactly one \( B \)-smoothing, and (ii) \( |s_B| > |s| \) for any state \( s \) with exactly one \( A \)-smoothing. In particular, any reduced alternating diagram is adequate. A link is adequate if it has an adequate diagram.

The proof of Tait’s Conjecture now follows from three claims (see [10]):

(i) Although defined for diagrams, the Jones polynomial \( V_L(t) \) is a link invariant.
(ii) \( s_A \) and \( s_B \) contribute the extreme terms \( \pm t^\alpha \) and \( \pm t^\beta \) of \( V_L(t) \), which determine the span \( V_L(t) = \alpha - \beta \), which is a link invariant. In particular,

\[
\max \deg_A \langle D \rangle - \min \deg_A \langle D \rangle \leq 2(c(D) + |s_A(D)| + |s_B(D)|) - 2
\]

with equality if \( D \) is adequate, hence if \( D \) is alternating.

(iii) By Turaev’s dual-state lemma, \(|s_A(D)| + |s_B(D)| = 2 + c(D) - 2g_T(D)\). Thus,

\[
\max \deg_A \langle D \rangle - \min \deg_A \langle D \rangle \leq 4c(D) - 4g_T(D)
\]

(1)

\[
\text{span } V_L(t) \leq c(L) - g_T(L)
\]

with equality if \( L \) is adequate, hence if \( L \) is alternating. If \( D \) is a prime non-alternating diagram, then \( g_T(D) > 0 \) so we get a strict inequality. Thus, \( \text{span } V_L(t) = c(L) \) if and only if \( L \) is alternating, from which Tait’s Conjecture follows.

Therefore, for any adequate link \( L \) with an adequate diagram \( D \), we have (see [1]):

(2)

\[
g_T(L) = g_T(D) = \frac{1}{2} (c(D) - |s_A(D)| - |s_B(D)|) + 1 = c(L) - \text{span } V_L(t)
\]
2.1. Spanning trees and Jones polynomial. The Tait graph plays a role in an earlier proof of the Tait conjecture via Thistlethwaite’s spanning tree expansion of the Jones polynomial. Thistlethwaite [32] gave an expansion of $V_L(t)$ in terms of the spanning trees of the Tait graph of any diagram of $L$. Every spanning tree contributes a monomial to $V_L(t)$.

For non-alternating diagrams, these monomials may cancel with each other, but for alternating diagrams, such cancellations do not occur. Thus, for alternating links, the number of spanning trees is exactly the $L^1$-norm of coefficients of $V_L(t)$, and the span of $V_L(t)$ is maximal, equal to the crossing number. This gives a different proof of claim (3) above for alternating links. Thistlethwaite also showed that the Jones polynomial of an alternating link can be obtained as a specialization of the Tutte polynomial of its Tait graph.

The Tutte polynomial is a fundamental and ubiquitous invariant of graphs, which can be defined by a state sum over all subgraphs, by contraction-deletion operations, and by a spanning tree expansion, any of which could have led to the Jones polynomial three decades earlier! Tutte’s original definition in [35] used the spanning tree expansion, which relies on the concept of activity of edges with respect to a spanning tree.

In Section 4, we discuss the spanning tree expansion and its applications in more detail. For example, for the figure-eight knot, Figure 6 shows the skein resolution tree in terms of spanning trees.

3. Ribbon graphs and polynomial invariants

In this section we look at two graph-theoretic generalizations of the ideas above.

First, Turaev’s construction gives rise to a graph embedded on a surface, i.e. a ribbon graph, in a way that generalizes the Tait graph of a diagram $D$. When $D$ is non-alternating, the Tait graph must have signs to encode all the crossing information of $D$. Instead, we can construct an un-signed ribbon graph whose topology completely encodes $D$. We will formally define ribbon graphs below, which can be more general graphs on surfaces.

Second, Thistlethwaite’s specialization of the Tutte polynomial to the Jones polynomial of alternating links also generalizes in the ribbon graph setting: a specialization of the Bollobás–Riordan–Tutte polynomial gives the Jones polynomial for any link. Moreover, the spanning trees of a plane graph also have natural counterparts in the ribbon graph setting. We discuss all of these ideas below.

3.1. Ribbon graphs. An oriented ribbon graph is a cellularly embedded graph in an oriented surface (precisely, a multi-graph for which loops and multiple edges are allowed) that is embedded in such a way that its complement is a union of open discs on the surface. A ribbon graph is also described as a band decomposition by thickening the cellularly embedded graph. See Table 1 and Figure 4. The embedding, combined with the orientation on the surface, determines a cyclic order on the edges at every vertex, and also a cell structure for the surface. Terms for the same or closely related objects include: combinatorial maps, fat graphs, cyclic graphs, graphs with rotation systems, ribbon and arrow marked graphs and dessins d’enfant (see [3, 14, 21] and references therein).

A ribbon graph $G$ can be considered both as a geometric and as a combinatorial object. The combinatorial definition is given as follows: let $(\sigma_0, \sigma_1, \sigma_2)$ be permutations of $\{1, \ldots, 2n\}$, such that $\sigma_1$ is a fixed-point free involution and $\sigma_0 \sigma_1 \sigma_2 = 1$. We define the
orbits of $\sigma_0$ to be the vertex set $V(G)$, the orbits of $\sigma_1$ to be the edge set $E(G)$, and the orbits of $\sigma_2$ to be the face set $F(G)$. Let $v(G)$, $e(G)$ and $f(G)$ be the numbers of vertices, edges and faces of $G$. The preceding data determine an embedding of $G$ on a closed orientable surface, denoted $S(G)$, as a cell complex. The set \{1, \ldots, 2n\} can be identified with the directed edges (or half-edges) of $G$. Thus, $G$ is connected if and only if the group generated by $\sigma_0$, $\sigma_1$, $\sigma_2$ acts transitively on \{1, \ldots, 2n\}. The genus of $S(G)$ is called the genus of $G$, $g(G)$. If $G$ has $k(G)$ components, $2g(G) = 2k(G) - v(G) + e(G) - f(G) = k(G) + n(G) - f(G)$, where $n(G) = e(G) - v(G) + k(G)$ denotes the nullity of $G$. Henceforth, we assume that $G$ is a connected, orientable ribbon graph. See Table 1 for an example of distinct ribbon graphs with the same underlying graph.

\begin{table}[h]
\centering
\begin{tabular}{cccc}
$\sigma_0$ & $\sigma_1$ & $\sigma_2$
\hline
(1234)(56) & (14)(25)(36) & (246)(35) \\
(1234)(56) & (13)(26)(45) & (152364)
\end{tabular}
\caption{Ribbon graphs described as graphs on surfaces, as combinatorial maps and as permutations.}
\end{table}

A ribbon graph $\mathbb{H}$ is a ribbon subgraph of $G$ if $\mathbb{H}$ can be obtained by deleting vertices and edges of $G$. A ribbon subgraph $\mathbb{H} \subset G$ is called a ribbon spanning subgraph if $V(\mathbb{H}) = V(G)$. Note that the surface on which $\mathbb{H}$ is cellularly embedded need not be the same surface on which $G$ is cellularly embedded (i.e. $S(\mathbb{H})$ need not be the same as $S(G)$), and $g(\mathbb{H}) \leq g(G)$.

Bollobás and Riordan [3] extended the Tutte polynomial to a polynomial invariant of oriented ribbon graphs $C(G) \in \mathbb{Z}[X,Y,Z]$ in a way that takes into account the topology of the ribbon graph $G$. The Bollobás–Riordan–Tutte polynomial has a spanning ribbon subgraph expansion given by the following sum:

$$C(G) = \sum_{\mathbb{H} \subseteq G} (X - 1)^{k(\mathbb{H}) - k(G)} Y^{n(\mathbb{H})} Z^{g(\mathbb{H})}.$$ 

In [4], they generalized it to a four-variable polynomial invariant $R(G)$ of non-orientable ribbon graphs.

3.2. Ribbon graphs from link diagrams. Turaev’s construction gives rise to a ribbon graph in a way that generalizes the Tait graph of a diagram $D$. The projection of $D$ can be checkerboard colored on the Turaev surface $F(D)$ with $|s_A|$ white regions (at height > 0), and $|s_B|$ black regions (at height < 0). Let $G_A$ (and similarly $G_B$) be the graph on $F(D)$ obtained by assigning a vertex to every white region (respectively, black region), and an edge to every crossing as we do for the Tait graph. Then the complementary regions of $G_A$
(similarly of $G_B$) on $F(D)$ are the $s_B$ circles (respectively, $s_A$ circles) which bound discs on $F(D)$. Note that

$$v(G_A) = |s_A| = f(G_B), \quad e(G_A) = e(G_B) = e(D), \quad f(G_A) = |s_B| = v(G_B)$$

Thus $G_A$, $G_B$ are dual ribbon graphs embedded in $F(D)$ with $g(G_A) = g(G_B) = g_T(D)$. If $D$ is alternating, $G_A$ and $G_B$ are Tait graphs which are planar duals on $F(D) = S^2$. If $D$ is $A$–adequate ($B$–adequate), as defined in Section 2, then $G_A$ ($G_B$) has no loops.

We can also obtain $G_A$ directly from the link diagram as follows:

1. For a given diagram $D$, use the $A$–smoothing of every crossing to obtain the state $s_A$, add a ribbon edge (band) joining the two arcs at every smoothed crossing.
2. Checkerboard color complementary regions of the circles of $s_A$, and orient the circles as the oriented boundary of the black regions.
3. Collapse each state circle of $s_A$ to a vertex of $G_A$, preserving the cyclic order of the ribbon edges.
4. Order the half edges at each vertex using the ordering on the crossings. The ordering gives us the permutations describing $G_A$.

This is illustrated in Figure 3. $G_B$ can be obtained similarly by starting from the all-B state $s_B$.

![Figure 3. Ribbon graph $G_A$ for a four-crossing diagram of the trefoil knot](image)

Thistlethwaite [32] showed that if $L$ is alternating, then $V_L(t) \doteq T_G(-t, -1/t)$, where $G$ is the Tait graph of $L$ and $T_G(x, y)$ is its Tutte polynomial. In [12], it was shown that the Jones polynomial $V_L(t)$ is a specialization of the Bollobás–Riordan–Tutte polynomial of the all–$A$ ribbon graph $G_A$. Chmutov [9] extended these ideas to virtual links and non-orientable ribbon graphs, and to links given as a diagram on a surface. In [14], a unified description is given for all these knot and ribbon graph polynomial invariants using the four-variable polynomial $R(G)$:

**Theorem 3.1** ([12, 9]). Let $D$ be either a classical link diagram, a link diagram on a surface, or a virtual link diagram and $G$ be the all–$A$ ribbon graph of $D$. Then

$$\langle D \rangle = \delta^{k(G) - 1} A^{n(G) - r(G)} R(G; -A^4, A^{-2} \delta, \delta^{-1}, 1)$$

where $\langle D \rangle$ is the Kauffman bracket of $D$ and $\delta = -A^2 - A^{-2}$.

3.3. **Ribbon graphs for dual states.** States $s$ and $\bar{s}$ are called dual states if the smoothing at every crossing in $\bar{s}$ is opposite to that in $s$, e.g. $s_A$ and $s_B$ are dual states. The Turaev surface construction applies to any pair of dual states to obtain the surface $F(D_s)$, with
the projection of $D$ embedded as a 4-valent cellular graph. As before, we get dual ribbon graphs $G_s$ and $G_\tau$ as Tait graphs of the embedding of $D$ in $F(D_s)$.

We can also obtain $G_s$ (and $G_\tau$) directly from the link diagram as described above. We add a + sign for the edges obtained by a $B$-smoothing and a − sign for the edges obtained by an $A$-smoothing. This gives us a signed ribbon graph which keeps track of the smoothings. Note that the sign convention is chosen so that the signs on edges agree with those for the Tait graph in the case when the state is the “Tait” state i.e $F(D_s) = S^2$. See Figure 4 [26].

![Figure 4](image)

**Figure 4.** Ribbon graph corresponding to any state of a link diagram (figure from [26]).

With this construction, any link diagram $D$ with $n$ crossings gives rise to a set of $2^n$ (signed) ribbon graphs associated to the states of $D$. It turns out that all of these ribbon graphs are partial duals of each other, in the sense of Chmutov [9]. It follows that a ribbon graph represents a link diagram if and only if it is a partial dual of a plane graph. Recently in [27], Moffatt used this fact to completely characterize such ribbon graphs in terms of three excluded minors.

3.4. **Quasi-trees.** A quasi-tree $Q$ of a ribbon graph $G$ is a ribbon spanning subgraph with $f(Q) = 1$. So a quasi-tree is a spanning ribbon subgraph whose regular neighborhood on $S(G)$ has one boundary component. This generalizes the analogous defining property of a spanning tree of a plane graph. See Figure 5.

![Figure 5](image)

**Figure 5.** Regular neighborhoods of spanning trees and quasi-trees

The Tutte polynomial counts the number of spanning trees of a connected graph $G$ by the specialization $T_G(1,1)$. (For any alternating knot, this is exactly the determinant of
the knot.) For any ribbon graph, we proved with Stoltzfus that the Bollobás–Riordan–Tutte polynomial also counts the number of quasi-trees of every genus by a specialization as follows.

**Proposition 3.2** ([8]). Let \( q(G; t, Y) = C(G; 1, Y, tY^{-2}) \). Then \( q(G; t, Y) \) is a polynomial in \( t \) and \( Y \) such that

\[
q(G; t) := q(G; t, 0) = \sum_j a_j t^j
\]

where \( a_j \) is the number of quasi-trees of genus \( j \).

Consequently, \( q(G; 1) \) equals the number of quasi-trees of \( G \).

**Question 1.** Let \( G \) be the all–A ribbon graph for a diagram \( D \) of a knot \( K \). If \( g_T(D) = g_T(K) \), is \( q(G; t) \) an invariant of \( K \)?

4. Turaev genus and knot homology

In his theorem mentioned above, Thistlethwaite [32] gave an expansion of the Jones polynomial \( V_L(t) \) in terms of spanning trees of any Tait graph \( G \) of \( L \). In [5], for any connected link diagram \( D \), we defined the spanning tree complex \( C(D) = \{ C^v(D), \partial \} \), whose generators correspond to spanning trees \( T \) of \( G \), and whose homology is the reduced Khovanov homology. As described precisely below, \( C(D) \) is at most \( g_T(L) + 1 \)-thick, where \( g_T(L) \) is the Turaev genus.

4.1. Spanning tree expansion. We first describe the spanning tree expansion for the Jones polynomial, and then the spanning tree expansion for Khovanov homology, which is also similar to the one for knot Floer homology.

Fix an order on the edges of \( G \). For every spanning tree \( T \) of \( G \), each edge \( e \in G \) has an activity with respect to \( T \), as follows. If \( e \in T \), \( \text{cut}(T, e) \) is the set of edges that connect \( T \setminus e \). If \( f \notin T \), \( \text{cyc}(T, f) \) is the set of edges in the unique cycle of \( T \cup f \). Note \( f \in \text{cut}(T, e) \) if and only if \( e \in \text{cyc}(T, f) \). An edge \( e \in T \) (resp. \( e \notin T \)) is *live* if it is the lowest edge in its cut (resp. cycle), and otherwise it is *dead*.

For any spanning tree \( T \) of \( G \), the *activity word* \( W(T) \) gives the activity of each edge of \( G \) with respect to \( T \). The letters of \( W(T) \) are as follows: \( L, D, \ell, d \) denote a positive edge that is live in \( T \), dead in \( T \), live in \( G - T \), dead in \( G - T \), respectively; \( L, D, \ell, d \) denote activities for a negative edge. Note that \( T \) is given by the capital letters of \( W(T) \).

Thistlethwaite assigned a monomial \( \mu(T) \) to each \( T \) as follows:

\[
L^p D^q \ell^r d^s L^x D^y \ell^z d^w \Rightarrow \mu(T) = (-1)^{p+r+x+z} A^{-3p+q+3r-s+3z-y-3z+w}
\]

**Theorem 4.1** ([32]). Let \( G \) be the Tait graph of any connected link diagram \( D \) with any order on its edges. Let \( \langle D \rangle \) denote the Kauffman bracket polynomial of \( D \). Summing over all spanning trees \( T \) of \( G \), \( \langle D \rangle = \sum_T \mu(T) \).

The activity word \( W(T) \) contains much more information than just \( \mu(T) \). A *twisted unknot* \( U \) is a diagram of the unknot obtained from the round unknot using only Reidemeister I moves. \( W(T) \) determines a twisted unknot \( U(T) \) by changing the crossings of \( D \) according to Table 2 for dead edges, and leaving the crossings unchanged for live edges (Lemma 1 [5]). In Table 2, the sign of the crossing in \( U(T) \) is indicated for unsmoothed crossings, and Kauffman state markers are indicated for smoothed crossings.
We can also consider each \( U(T) \) as a partial smoothing of \( D \) determined by \( W(T) \). In fact, there exists a skein resolution tree for \( D \) whose leaves are exactly all the partial resolutions \( U(T) \), for each spanning tree \( T \) of \( G \). Let \( \sigma(U) = \#A\text{-smoothings} - \#B\text{-smoothings} \), and let \( w(U) \) be the writhe. If \( U \) corresponds to \( T \), then \( \mu(T) = A^{\sigma(U)}(-A)^{3w(U)} \) is exactly the monomial above Theorem 4.1. As Louis Kauffman pointed out, this is how humans would compute \( \langle D \rangle \): Instead of smoothing all the way to the final Kauffman states, a human would stop upon reaching any twisted unknot \( U \), and use the formula \( \mu(T) \). We illustrate all of this for the figure-eight knot diagram in Figure 6 and Table 3.

### Table 2. Activity word for a spanning tree determines a twisted unknot

| \( L \) | \( D \) | \( \ell \) | \( d \) | \( \bar{L} \) | \( \bar{D} \) | \( \bar{\ell} \) | \( \bar{d} \) |
|---|---|---|---|---|---|---|---|
| \( - \) | \( A \) | \( + \) | \( B \) | \( + \) | \( B \) | \( - \) | \( A \) |

For any connected link diagram \( D \), we choose the checkerboard coloring such that its Tait graph \( G \) has more positive edges than negative edges, and in case of equality that the unbounded region is unshaded. In [5], we defined the spanning tree complex \( \mathcal{C}(D) = \{ C^u_v(D), \partial \} \), whose generators correspond to spanning trees \( T \) of \( G \). The \( u \) and \( v \)-grading are determined by \( W(T) \) as follows:

\[
u(T) = \#L - \#\ell - \#\bar{L} + \#\bar{\ell} \quad \text{and} \quad v(T) = \#L + \#D = e_+(T)
\]

**Theorem 4.2** ([5]). For any connected link diagram \( D \), there exists a spanning tree complex \( \mathcal{C}(D) = \{ C^u_v(D), \partial \} \) with \( \partial \) of bi-degree \((-1, -1)\) that is a deformation retract of the reduced Khovanov complex.

### 4.2. Turaev genus and Khovanov homology

The key idea for relating Khovanov homology to Turaev surfaces, Turaev genus and ribbon graphs is our observation that there is a one-to-one correspondence between spanning trees of the Tait graph and quasi-trees of the all-A ribbon graph [7].
Figure 6. Twisted unknots corresponding to spanning trees in Table 3. Crossings are smoothed in reverse order of the crossings of the diagram. At every node of the skein resolution tree, $A$–smoothings are on the left, and $B$–smoothings are on the right.

Let $D$ be a connected link diagram, $G$ be its Tait graph for which the number of positive edges is greater than or equal to the number of negative edges and let $G$ be the all-A ribbon graph. In [7], we proved with Stoltzfus

**Theorem 4.3 ([7]).** Quasi-trees of $G$ are in one-one correspondence with spanning trees of $G$:

$$Q_j \leftrightarrow T_v \quad \text{where} \quad v + j = \frac{v(G) + e_+(G) - v(G)}{2}$$

$q_j$ denotes a quasi-tree of genus $j$, $T_v$ denotes a spanning tree with $v$ positive edges, and $e_+(G)$ equals the number of positive edges in $G$.  

To construct the spanning tree chain complex in [5], every spanning tree $T$ of the Tait graph $G$ was given a bigrading $(u(T), v(T))$. By Theorem 4.3, the $v$-grading, which is the number of positive edges in $T$, is determined by the genus of the corresponding quasi-tree $Q$. The $u$-grading, which was defined using activities in the sense of Tutte, also has a quasi-tree analogue in terms of the ordered chord diagram for $Q$.

If $D$ has $n$ ordered crossings, let $G$ be given by permutations $(\sigma_0, \sigma_1, \sigma_2)$ of the set $\{1, \ldots, 2n\}$, such that the $i$-th crossing corresponds to half-edges $\{2i-1, 2i\}$, which are marked on the components of the all–A state of $D$. We give the components of the all–A state of $D$ the admissible orientation for which outer ones are oriented counterclockwise (see [12]). In this way, every component has a well-defined positive direction.

The orbits of $\sigma_0$ form the vertex set. In particular, $\sigma_0$ is given by noting the half-edge marks when going in the positive direction around the components of the all–A state of $D$. The other permutations are given by $\sigma_1 = \prod_{i=1}^{n} (2i-1, 2i)$ and $\sigma_2 = \sigma_1 \circ \sigma_0^{-1}$.

Let an ordered chord diagram denote a circle marked with $\{1, \ldots, 2n\}$ in some order, and chords joining all pairs $\{2i-1, 2i\}$. By Proposition 1 of [7], every quasi-tree $Q$ corresponds to the ordered chord diagram $C_Q$ with consecutive markings in the positive direction given by the permutation:

$$\sigma(i) = \begin{cases} 
\sigma_0(i) & i \notin Q \\
\sigma_2^{-1}(i) & i \in Q
\end{cases}$$

For example see Figure 7.

Using $\min(i, \sigma_1(i))$, there is an induced total order on the chords of $C_Q$. A chord is live if it does not intersect lower-ordered chords, and otherwise it is dead. For any quasi-tree $Q$, an edge $e$ is live or dead when the corresponding chord of $C_Q$ is live or dead. In Figure 7, we show $C_Q$ such that the only edge live with respect to $Q$ is (12).

To compute the genus $g(Q)$ from $C_Q$, let $C$ be the sub-chord diagram of chords that correspond to edges in $Q$. Then $g(Q)$ is half the rank of the adjacency matrix of the intersection graph of $C$ [3].

In [7], we proved that if the spanning tree $T$ corresponds to $Q$, as in Theorem 4.3, then the chord diagram $C_Q$ parametrizes the regular neighborhood of $T$ formed by the appropriate smoothings of $D$. Consequently, we proved that the $i$-th edge of $G$ is live with respect to $Q$ if and only if the $i$-th edge of $G$ is live with respect to $T$. This is the essential reason that
the spanning tree complex can be expressed entirely as a bigraded quasi-tree complex, with the Turaev genus as one of the gradings:

**Theorem 4.4** ([7]). For a knot diagram $D$ with all-$A$ ribbon graph $\mathcal{G}$, there exists a quasi-tree complex $\mathcal{C}(\mathcal{G}) = \{C_u^v(\mathcal{G}), \partial\}$ that is a deformation retract of the reduced Khovanov complex, where $C_u^v(\mathcal{G}) = \mathbb{Z}(Q \subset \mathcal{G} | u = u(Q), v = -g(Q))$.

**Corollary 4.5** ([7]). For any knot $K$, the width of its reduced Khovanov homology

$$w_{KH}(K) \leq 1 + g_T(K).$$

The proof follows from the fact that $w_{KH}(K) \leq \max_{T \subset \mathcal{G}} v(T) - \min_{T \subset \mathcal{G}} v(T) + 1$, where $G$ is the Tait graph of any diagram of $K$, and that for the all-$A$ ribbon graph $\mathcal{G}$,

$$g(\mathcal{G}) = \max_{Q \subset \mathcal{G}} g(Q) - \min_{Q \subset \mathcal{G}} g(Q) = \max_{T \subset \mathcal{G}} E_+(T) - \min_{T \subset \mathcal{G}} E_+(T) = \max_{T \subset \mathcal{G}} v(T) - \min_{T \subset \mathcal{G}} v(T).$$

Note that $u(Q) = -w(U(T))$, where the twisted unknot $U(T)$ comes from the spanning tree corresponding to the quasi-tree $Q$. The grading $v(Q)$ is related to Rasmussen’s $\delta$–grading for Khovanov homology as $\delta = 2v + k$, where $k$ is a constant that depends only on $D$. Because the $(u, v)$ gradings are linear combinations of Khovanov’s $(i, j)$ gradings, the width refers to the diagonals of Khovanov’s complex, as in the following figure from [11]:

![Diagram](image)

The rational Khovanov homology of $(3, q)$–torus links was computed in [34], where it was shown that the width of the Khovanov homology of torus knots of type $(3, 3N + 1)$ and $(3, 3N + 2)$ is exactly $N + 2$. Corollary 4.5 gives a family of links with unbounded Turaev genus.

**Corollary 4.6** ([7]). The Turaev genus of $(3, q)$–torus knots is unbounded.

Lowrance [22] and Watson [36] have proved that the width of Khovanov homology remains unchanged after replacing a crossing in a link diagram with an alternating rational tangle, provided the crossing satisfies certain conditions. Using Corollary 4.6, this generates many families of knots with unbounded Turaev genus.
4.3. Turaev genus and knot Floer homology. Finally, we turn briefly to knot Floer homology \( \hat{HFK} \) (see the recent surveys \([19, 24]\)). Lowrance \([23]\) proved the analogous bound to Corollary 4.5.

**Theorem 4.7** \([23]\). Let \( K \subset S^3 \) be a knot. The knot Floer width of \( K \) is bounded by the Turaev genus of \( K \) plus one: \( w_{HF}(K) \leq g_T(K) + 1 \).

This bound follows from the same idea as for the Khovanov homology. In \([31]\), Ozsváth and Szabó showed that for any diagram \( D \) of a knot \( K \), there exists a complex whose generators are in one-to-one correspondence with spanning trees of the Tait graph of \( D \) and whose homology is the knot Floer homology of \( K \). The \( \delta \)-grading on \( \hat{HFK} \) is defined as the difference of the Alexander and homological gradings. The \( \delta \)-grading of a spanning tree when considered in the reduced Khovanov complex is the same as the \( \delta \)-grading of that spanning tree when considered in the knot Floer complex (see \([13]\)). Thus, by Theorem 4.3, the width of the spanning tree complex giving knot Floer homology is also bounded by \( g_T(K) + 1 \), hence so is the width of the homology.

5. Bounds for the Turaev genus

The first bound that was discovered, equation (1) in Section 2, motivated interest in the Turaev genus. Corollary 4.5 and Theorem 4.7 give lower bounds for the Turaev genus in terms of homological width. Here, we survey other known bounds for the Turaev genus.

5.1. Alternating embeddings of link diagrams. Let \( g_F(L) \) be the minimal genus of an unknotted closed orientable surface \( F \) on which \( L \) can be cellularly embedded on \( F \), i.e. the complementary regions \( F - L \) are discs, and such that \( L \) has an alternating diagram on \( F \). We saw in Section 1 that any non-alternating pretzel link \( P \) can be made alternating on the torus, so \( g_F(P) = 1 \). Colin Adams called such links “toroidally alternating.” Using Turaev’s construction it’s easy to see that \( g_T(P) = 1 \) for such pretzel links. (See \([17]\) for properties of cellularly embedded links with alternating diagrams on higher genus surfaces.) In general,

\[
g_F(L) \leq g_T(L)
\]

There are examples due to Adam Lowrance of a family of links \( L_n \) for which \( g_F(L_n) = 1 \) but \( g_T(L_n) \to \infty \) as \( n \to \infty \). The idea is to start with torus links \( T_n \) for which \( w_{KH}(T_n) \to \infty \), and then to insert a small alternating tangle to get \( L_n \) so that \( g_F(L_n) = 1 \) but \( w_{KH}(T_n) = w_{KH}(L_n) \).

Note that these provide examples of cellularly embedded alternating links on an unknotted closed orientable surface, but they do not satisfy the Morse decomposition property of the Turaev surface, which is property (e) listed in Section 1.

5.2. Dealternating number. Let \( dalt(L) \) be the dealternating number, which is defined to be the minimal number of crossing changes needed to make a diagram of \( L \) into an alternating diagram. When \( dalt(L) = 1 \), the link is called almost alternating. Abe and Kishimoto \([2]\) proved

\[
g_T(L) \leq dalt(L).
\]

Moreover, the almost alternating torus knots are exactly the only two knots which are both torus and pretzel knots, \( T(3, 4) = P(3, 3, -2) \) and \( T(3, 5) = P(5, 3, -2) \).
5.3. **Homological invariants.** For a knot $K$, let $\sigma(K)$, $s(K)$, and $\tau(K)$ denote the signature of $K$, the Rasmussen $s$–invariant of $K$ which comes from Khovanov homology, and the $\tau$–invariant of $K$ which comes from the knot Floer homology. If $K$ is any alternating knot, then $2\tau(K) = s(K) = -\sigma(K)$.

In [13], Dasbach and Lowrance proved the same lower bounds for the Turaev genus that Abe had proved for the dealternating number:

$$(3) \quad |\tau(K) + \frac{\sigma(K)}{2}| \leq g_T(K), \quad |s(K) + \frac{\sigma(K)}{2}| \leq g_T(K), \quad |\tau(K) - \frac{s(K)}{2}| \leq g_T(K)$$

These are known to be equalities for alternating knots (when $g_T(K) = 0$). Dasbach and Lowrance gave certain examples of $(3,q)$–torus knots for which they are not sharp.

For any link $L$, let $L^{(r)}$ be its $r$–fold parallel. Huggett, Moffatt and Virdee [18] gave an upper bound on the Turaev genus of $L^{(r)}$,

$$g_T(L^{(r)}) \leq (r+1) \cdot g_T(L) + r^2c - r$$

where $c$ is the crossing number of any diagram $D$ for which $g_T(D) = g_T(L)$.

6. **Open questions and research directions**

6.1. **Adequate knots.** Equations (1) and (2) in Section 2, focused attention on adequate diagrams because it follows that they have minimal crossing number. These are much more general than alternating diagrams. For example, the $r$–fold parallel of an adequate diagram is adequate.

Recall that we say that a diagram $D$ is $A$–adequate if $|s_A| > |s|$ for any state $s$ with exactly one $B$–smoothing, and is $B$–adequate if $|s_B| > |s|$ for any state $s$ with exactly one $A$–smoothing. In terms of ribbon graphs, $D$ is $A$–adequate ($B$–adequate) if $G_A$ ($G_B$) has no loops. $D$ is adequate if it is both $A$–adequate and $B$–adequate. A knot or link is adequate if it has an adequate diagram.

For an adequate knot, Abe [1] proved that the inequality in Corollary 4.5 is an equality. Thus, extending equation (2) for an adequate knot $K$,

$$g_T(K) = w_{KH}(K) - 1 = c(K) - \text{span} V_K(t).$$

**Question 2.** Is Lowrance’s analogous inequality for knot Floer homology an equality for adequate knots?

**Question 3.** For any two knots $K$ and $K'$, is $g_T(K \# K') = g_T(K) + g_T(K')$?

**Question 4.** If $K$ and $K'$ are mutant knots, is $g_T(K) = g_T(K')$?

In [1], Abe answered both questions for adequate knots, proving that the Turaev genus is additive under connect sum, and is invariant under mutation for adequate knots. (Mutation of an adequate diagram preserves adequacy.)

It is not known whether the other inequalities above are equalities for adequate knots:

**Question 5.** If $K$ is an adequate knot, is $g_T(K) = g_F(K)$?

**Question 6.** Are the inequalities (3) equalities for adequate knots?
In [6], we considered an operation on diagrams to extend a twist on two strands by any rational tangle, as defined by Conway. This operation can change the link type, but it preserves the properties of the diagram $D$ being alternating, adequate, or quasi-alternating [6], and it also preserves $g_F(D)$ and $g_T(D)$.  

**Question 7.** Let $L_1$ and $L_2$ be links whose diagrams are related by extending a twist on two strands by some rational tangle. Is $g_F(L_1) = g_F(L_2)$ and is $g_T(L_1) = g_T(L_2)$?  

We now turn to a related open problem in knot homology. Because knot Floer homology detects the Seifert genus, it can detect mutation; the Seifert genus of the Conway knot is 3, and that of the Kinoshita-Teresaka knot is 2. It is not known whether Khovanov homology is invariant under mutation (odd Khovanov homology is known to be invariant). For both homology theories, though, the rank of the homology in each $\delta$–grading (i.e., the $v$–grading discussed above for Khovanov homology) is conjectured to be invariant under mutation (see Conjecture 3 of [19]). Following Abe’s results, and by the proofs of Corollary 4.5 and the similar result by Lowrance for knot Floer homology, we are led naturally to the following conjecture. For an adequate knot, it would imply the mutation invariance of the ranks of both homology theories in each $\delta$–grading.

**Conjecture 6.1.** Let $K$ be an adequate knot.

1. For any adequate diagram of $K$, the ranks of both Khovanov and knot Floer homology in each $\delta$–grading is given by $q(\mathbb{G}; t)$, as in Proposition 3.2.
2. If $K'$ is any mutant of $K$, then for any adequate diagrams of $K$ and $K'$, $q(\mathbb{G}; t) = q(\mathbb{G}'; t)$.

6.2. Quasi-alternating links. Quasi-alternating links were first defined in [30] and it was shown in [25, 30] that, like alternating links, they are homologically thin with respect to both Khovanov and knot Floer homology. As a result, the homological bounds on the Turaev genus discussed above vanish for quasi-alternating links. Examples of quasi-alternating links of Turaev genus one include non-alternating pretzel links and, more generally, non-alternating Montesinos links (see [6]). To answer the following question requires a new kind of lower bound for the Turaev genus.

**Question 8.** For any $g > 1$, do there exist quasi-alternating links with Turaev genus equal to $g$?

6.3. Geometry of knot complements. For any Kauffman state $s$ of a knot $K$, the state surface $F_s$ is constructed like a Seifert surface: state circles bound disjoint disks, which are connected by half-twisted bands such that $\partial F_s = K$. The ribbon graph $\mathbb{G}_s$ embeds as a spine of the surface $F_s$. Let $F_A$ ($F_B$) denote the all–$A$ (all–$B$) state surface. Ozawa [29] proved that if $D$ is $A$–adequate ($B$–adequate), then $F_A$ ($F_B$) is essential in $S^3 - K$. (Ozawa’s result actually holds for more general diagrams, which have a state that is both adequate and homogeneous.) Ozawa’s theorem opens the door to geometric results. Futer, Kalfagianni and Purcell [15] related certain stable coefficients of colored Jones polynomials to fibering data and hyperbolic volume bounds using essential state surfaces.

Other than this connection, very little is known about the geometry of the knot complement and the Turaev genus. An exciting direction to explore may be the extent to which the Turaev genus measures how the geometry differs from that of alternating knots.
Question 9. Given a knot diagram, does the Turaev genus provide any additional constraint on the geometry of the knot complement?

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References

1. Tetsuya Abe, *The Turaev genus of an adequate knot*, Topology Appl. 156 (2009), no. 17, 2704–2712.
2. Tetsuya Abe and Kengo Kishimoto, *The dealternating number and the alternation number of a closed 3-braid*, J. Knot Theory Ramifications 19 (2010), no. 9, 1157–1181.
3. B. Bollobás and O. Riordan, *A polynomial invariant of graphs on orientable surfaces*, Proc. London Math. Soc. (3) 83 (2001), no. 3, 513–531.
4. ______, *A polynomial of graphs on surfaces*, Math. Ann. 323 (2002), no. 1, 81–96.
5. Abhijit Champanerkar and Ilya Kofman, *Spanning trees and Khovanov homology*, Proc. Amer. Math. Soc. 137 (2009), no. 6, 2157–2167.
6. ______, *Twisting quasi-alternating links*, Proc. Amer. Math. Soc. 137 (2009), no. 7, 2451–2458.
7. Abhijit Champanerkar, Ilya Kofman, and Neal Stoltzfus, *Graphs on surfaces and Khovanov homology*, Algebr. Geom. Topol. 7 (2007), 1531–1540.
8. ______, *Quasi-tree expansion for the Bollobás-Riordan-Tutte polynomial*, Bull. Lond. Math. Soc. 43 (2011), no. 5, 972–984.
9. Sergei Chmutov, *Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial*, J. Combin. Theory Ser. B 99 (2009), no. 3, 617–638.
10. Peter R. Cromwell, *Knots and links*, Cambridge University Press, Cambridge, 2004.
11. Oliver Dasbach and Adam Lowrance, *A Turaev surface approach to Khovanov homology*, (arXiv:1107.2344 [math.GT]).
12. Oliver T. Dasbach, David Futer, Efstratia Kalfagianni, Xiao-Song Lin, and Neal W. Stoltzfus, *The Jones polynomial and graphs on surfaces*, J. Combin. Theory Ser. B 98 (2008), no. 2, 384–399.
13. Oliver T. Dasbach and Adam M. Lowrance, *Turaev genus, knot signature, and the knot homology concordance invariants*, Proc. Amer. Math. Soc. 139 (2011), no. 7, 2631–2645.
14. Joanna A. Ellis-Monaghan and Iain Moffatt, *Graphs on surfaces: Dualities, polynomials, and knots*, Springer Briefs in Mathematics, Springer, New York, 2013.
15. David Futer, Efstratia Kalfagianni, and Jessica Purcell, *Guts of surfaces and the colored Jones polynomial*, Lecture Notes in Mathematics, vol. 2069, Springer, Heidelberg, 2013.
16. M. Hajij, *Turaev Surface Borromean rings - REMIX*, http://www.youtube.com/watch?v=j43LionQD9w.
17. Chuichiro Hayashi, *Links with alternating diagrams on closed surfaces of positive genus*, Math. Proc. Cambridge Philos. Soc. 117 (1995), no. 1, 113–128.
18. Stephen Huggett, Iain Moffatt, and Natalia Virdee, *On the Seifert graphs of a link diagram and its parallels*, Math. Proc. Cambridge Philos. Soc. 153 (2012), no. 1, 123–145.
19. A. Juhász, *A survey of Heegaard Floer homology*, (arXiv:1310.3418 [math.GT] (2013)).
20. L. Kauffman, *State models and the Jones polynomial*, Topology 26 (1987), no. 3, 395–407.
21. S. Lando and A. Zvonkin, *Graphs on surfaces and their applications*, Encyclopaedia of Mathematical Sciences, vol. 141, Springer-Verlag, 2004.
22. Adam Lowrance, *The Khovanov width of twisted links and closed 3-braids*, Comment. Math. Helv. 86 (2011), no. 3, 675–706.
23. Adam M. Lowrance, *On knot Floer width and Turaev genus*, Algebr. Geom. Topol. 8 (2008), no. 2, 1141–1162.
24. Ciprian Manolescu, *An introduction to knot Floer homology*, (arXiv:1401.7107 [math.GT] (2014)).
25. Ciprian Manolescu and Peter Ozsváth, *On the Khovanov and knot Floer homologies of quasi-alternating links*, Proceedings of Gökova Geometry-Topology Conference 2007, Gökova Geometry/Topology Conference (GGT), Gökova, 2008, pp. 60–81.
26. Iain Moffatt, *Partials duals of plane graphs, separability and the graphs of knots*, Algebr. Geom. Topol. 12 (2012), 1099–1136.
27. excluded minors and the ribbon graphs of knots, (arXiv:1311.2160).
28. Kunio Murasugi, Jones polynomials and classical conjectures in knot theory, Topology 26 (1987), no. 2, 187–194.
29. Makoto Ozawa, Essential state surfaces for knots and links, J. Aust. Math. Soc. 91 (2011), no. 3, 391–404.
30. P. Ozsváth and Z. Szabó, On the Heegaard Floer homology of branched double-covers, Adv. Math. 194 (2005).
31. Peter Ozsváth and Zoltán Szabó, Heegaard Floer homology and alternating knots, Geom. Topol. 7 (2003), 225–254 (electronic).
32. M. Thistlethwaite, A spanning tree expansion of the Jones polynomial, Topology 26 (1987), no. 3, 297–309.
33. V. G. Turaev, A simple proof of the Murasugi and Kauffman theorems on alternating links, Enseign. Math. (2) 33 (1987), no. 3-4, 203–225.
34. Paul Turner, A spectral sequence for Khovanov homology with an application to (3, q)-torus links, Algebr. Geom. Topol. 8 (2008), no. 2, 869–884.
35. W. Tutte, A contribution to the theory of chromatic polynomials, Canadian J. Math. 6 (1954), 80–91.
36. Liam Watson, Surgery obstructions from khovanov homology, Selecta Mathematica 18 (2012), no. 2, 417–472 (English).

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