1. Introduction

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One of the major achievements of XIX century mathematics (mainly due to Klein and Poincare, but completed by Koebe in 1907) was the celebrated
Uniformisation Theorem, claiming that every conformal class of the surface metrics admits complete constant curvature representative. In particular, on a compact surface one can introduce conformally equivalent metric of positive constant curvature if it is a topological sphere, a flat metric if it is a torus and a negative constant curvature metric if it has genus $g \geq 2$.

In dimension three the situation is more complicated. According to the famous Thurston’s Geometrisation Conjecture [35, 49], now proved by Perelman, any compact orientable 3-manifold can be cut in a special way into pieces, admitting one of the 8 special geometric structures, namely the Euclidean $E^3$, spherical $S^3$ and hyperbolic $H^3$, the product type $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$ and three geometries related to the 3D Lie groups: $Nil$, $Sol$ and $SL(2, \mathbb{R})$, where the last group is the universal covering of $SL(2, \mathbb{R})$. Corresponding metrics are locally homogeneous, which means that any two points $x$ and $y$ have isometric neighbourhoods $U$ and $V$, see [44, 49].

Let $(\mathcal{M}^3, g)$ be a compact Riemannian 3-fold admitting one of these geometries, and consider the corresponding geodesic flow. What can we say about its integrability?

In dimension two the answer is well-known: the geodesic flows on round sphere and flat tori are integrable (in any sense), while on surfaces of genus $g > 1$ we have chaotic behaviour with positive entropy.

In this paper we will study the integrability problem for the geodesic flows on 3-folds with $SL(2, \mathbb{R})$-geometry. In this paper we will use instead of the simply-connected group $SL(2, \mathbb{R})$ (which has no finite-dimensional faithful matrix representation) its quotients: the standard matrix Lie group $SL(2, \mathbb{R})$ and $PSL(2, \mathbb{R}) = \overline{SL(2, \mathbb{R})} / \pm I$.

The situation in the other two group cases $Nil$ and $Sol$ were studied earlier by Butler [12] and Bolsinov and Taimanov [9]. In particular, in [9] it was shown that in $Sol$-case on critical level these flows can be described by the chaotic hyperbolic maps of 2D torus, while outside of it we have usual Liouville integrability. As a corollary, this gave the first class of examples of the Liouville integrable (in smooth category) systems with positive topological entropy.

In this paper we show that in the $SL(2, \mathbb{R})$-case the chaos spreads out from the critical level to occupy an open region in the phase space.

More precisely, we consider the class of the manifolds $\mathcal{M}_\Gamma^2 = \Gamma \backslash PSL(2, \mathbb{R})$, where $\Gamma \subset PSL(2, \mathbb{R})$ is a Fuchsian group [26] acting naturally on the hyperbolic plane $\mathbb{H}^2$. We will assume that the quotient $\mathcal{M}_\Gamma^2 = \Gamma \backslash \mathbb{H}^2$ is either compact, or at least has a finite area. A particular example is the modular group $\Gamma = PSL(2, \mathbb{Z})$, which we discuss in more detail.

Topologically $\mathcal{M}_\Gamma^2 = SM_\Gamma^2$ is the unit tangent bundle of the surface $\mathcal{M}_\Gamma^2$ and carries out a class of natural metrics, coming from the left-invariant metrics on $SL(2, \mathbb{R})$, which are also right $SO(2)$ invariant. They are particular case of the two-parameter family of the naturally reductive metrics [22] on $SL(n, \mathbb{R})$, which are left $SL(n, \mathbb{R})$-invariant and right $SO(n)$-invariant.
and determined by the following inner product on the Lie algebra $sl(n, \mathbb{R})$:

\[
\langle X, Y \rangle = \alpha \langle \text{sym} X, \text{sym} Y \rangle + \beta \langle \text{skew} X, \text{skew} Y \rangle, \quad \alpha > 0 > \beta.
\] (1)

Here $(X, Y) := \text{Tr} XY$ is the standard invariant form on $sl(n, \mathbb{R})$, and $X = \text{skew} X + \text{sym} X$ is the Cartan decomposition of $X \in sl(n, \mathbb{R})$:

\[
\text{skew} X := (X - X^\top)/2 \in so(n), \quad \text{sym} X := (X + X^\top)/2.
\]

They are also known as the generalised Sasaki metrics [37], Kaluza-Klein metrics [34] and appeared in the theory of elastoplasticity [31] (see more details in the next section). In particular, we will see that the Sasaki metric [43] corresponds to the most convenient case $\alpha = -\beta = 2$.

To write down the equations of geodesic flow introduce the angular momentum $\Omega := g^{-1} \dot{g} \in \mathfrak{g}$ and the momentum $M = A(\Omega) \in \mathfrak{g}^*$ determined by the relation

\[
(\Omega, A(\Omega)) = \langle \Omega, \Omega \rangle.
\]

Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ using the Cartan-Killing form, we can write

\[
M = \frac{1}{2} (\alpha + \beta) \Omega + \frac{1}{2} (\alpha - \beta) \Omega^\top
\]

and the corresponding Euler equations [4] as

\[
\dot{M} = [M, \Omega] = \frac{\beta - \alpha}{2\alpha\beta} [M, M^\top].
\]

These equations can be easily integrated and the corresponding geodesics on the group can be found explicitly (see e.g. [22, 31] and Section 3 below).

When $n = 2$ introduce the notations by writing the momentum as

\[
M = \alpha \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g}^* \approx \mathfrak{g}.
\] (2)

We have two obvious integrals of the Euler equations: the Hamiltonian

\[
H = \frac{1}{2} \langle \Omega, M \rangle = \frac{\alpha}{4\beta} (\beta [4a^2 + (b + c)^2] - \alpha (b - c)^2)
\]

and the Casimir function $\Delta = -\frac{1}{a^2} \det M = a^2 + bc$. Their natural extension to the cotangent bundle $T^*G$ by the left shifts (which we will denote by the same letters) give two Poisson commuting integrals of the geodesic flow. As the third required for the Liouville integrability integral we can take any non-constant right-invariant function $F$ on $T^*G$.

The situation is different for the quotients $M_3^2 = \Gamma \backslash PSL(2, \mathbb{R})$, since in general $F$ is not invariant under $\Gamma$ acting from the left. It turns out that the third Poisson commuting integral, required for Liouville’s integrability, exists only in an open half of the phase space.

More precisely, we prove the following

**Theorem 1.1.** Geodesic flow on $T^* M_3^2$ is integrable in analytic Liouville’s sense in the open region of the phase space with $\Delta < 0$.

In the region with $\Delta > 0$ there are no smooth integrals independent from $H$ and $\Delta$ and the system has positive topological entropy.
This chaos-integrability split has a natural geometric explanation related to the action of the Fuchsian group $\Gamma$ on the co-adjoint orbits $\Delta = \delta$ of $G$ (see Fig. 2 below). Namely, when $\delta < 0$ we have two-sheeted hyperboloid, giving a model of the hyperbolic plane, where the action of $\Gamma$ is discrete, while when $\delta > 0$ we have one-sheeted hyperboloid with dense orbits of $\Gamma$ (see more detail in Section 5).

This kind of split is well known in the classical theory of magnetic geodesic flow on $M_2^2$, see Hedlund [23], Arnold [3], Paternain et al [11, 39], Taimanov [48]. We show that this is not accidental since the magnetic geodesic flow on $M_2^2$ can be considered as a projection of the geodesic flow on $M_3^3$.

More precisely, we show that the natural projection of the geodesics to the quotient $SL(2, \mathbb{R})/SO(2)$ with constant negative Gaussian curvature $K = -2\alpha^{-1}$ are the curves with constant geodesic curvature $\kappa$ such that

$$C = \frac{\kappa^2}{K^2} = \frac{(b - c)^2}{4a^2 + (b + c)^2} = \frac{\beta H - \alpha \beta \Delta}{\beta H - \alpha^2 \Delta},$$

which is an integral of the system.

It is well-known that in the upper half-plane model these curves are circles, or arcs of circles when $C$ is larger, or less than 1 respectively (see Fig. 1). These curves can also be interpreted as magnetic geodesics on hyperbolic plane (see e.g. [4, 23]). Note that the condition $C > 1$ is equivalent to $\Delta < 0$.

As a concrete example we consider the case of the modular 3-fold $M_3^3$ with $\Gamma = PSL(2, \mathbb{Z})$. It was probably Quillen, who was the first to make an important observation that the quotient

$$PSL(2, \mathbb{Z})/PSL(2, \mathbb{R}) = S^3 \setminus K$$

is topologically equivalent to the complement of the trefoil knot $K$ in 3-sphere (see Milnor [32]).

Ghys used this and Birman-Williams results [7] to establish a remarkable relation between periodic geodesics on $M_3^3$, special class of knots (called modular) and Lorenz attractor [19]. We will see that these geodesics are special case of the periodic geodesics on $M_3^3$ at the ”most chaotic” level of the integral $C = 0$.

We extend this link to the integrable region of the geodesic flow on $M_3^3$.

Theorem 1.2. The periodic geodesics on modular 3-fold $M_3^3$ with sufficiently large values of $C$ represent the trefoil cable knots in $S^3 \setminus K$. Any cable knot of trefoil can be described in such a way.

Recall that the cable knots are special satellite knots, which can be described in the following way (see e.g. [11]). Take a solid torus with torus knot $K_{p,q}$ on the boundary and tie it up as a non-trivial knot $K$. When $K$ is the trefoil knot we get the class of knots called the trefoil cables.

When $C$ is large the projection of the corresponding two-dimensional Liouville torus in $T^*M_3^3$ with frequencies $\omega_1 = \frac{\beta - \alpha}{2\sqrt{\beta}} |b - c|$, $\omega_2 = \sqrt{-\Delta}$ on $M_3^3$ gives an embedding onto a torus close to the trefoil fibre of the projection.
\[ \mathcal{M}_1^3 \to \mathcal{M}_1^2. \] When
\[ \frac{\omega_1}{\omega_2} = \frac{\beta - \alpha |b - c|}{2\beta \sqrt{-\Delta}} = \frac{p}{q} \]
is rational we have the periodic torus filled by the \((p, q)\) cable knots of trefoil.

The appearance of the satellite knots in the integrable region might be expected in view of Thurston’s classification of all knots as hyperbolic, torus or satellite (see e.g. [1]), but in our case these are specifically cable knots of trefoil.

Note that in the case of the principal congruence subgroup \(\Gamma_2\), briefly discussed at the end of Section 6, in the integrable limit we have simply the torus knots \(K_{p,q}\), which is probably the most natural class of “integrable” knots. Their complements in \(S^3\) provide the class of the open 3-folds with \(SL(2, \mathbb{R})\)-geometry and finite volume, which can be explicitly computed, see formula (29) below.

We finish the paper with the discussion of the topological aspects of the problem in relation with known results about topological obstruction to integrability. In particular, in the case of cocompact \(\Gamma\) with \(\mathcal{M}_1^3\) of genus \(g \geq 2\) we have \(\dim H_1(\mathcal{M}_1^3, \mathbb{R}) = 2g > 3 = \dim \mathcal{M}_1^3\), so by the general Taimanov’s result [47] there are no analytically integrable geodesic flows on \(T^*\mathcal{M}_1^3\) with any metric. Our results show that this does not exclude the analytic integrability in a suitable open region of \(T^*\mathcal{M}_1^3\).

2. Geometry and topology of \(SL(2, \mathbb{R})\)

The group \(SL(2, \mathbb{R})\) consisting of \(2 \times 2\) real matrices with determinant 1 has two relatives: its quotient by the centre \(PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I\}\) and the universal covering \(\widetilde{SL}(2, \mathbb{R})\).

It acts by the isometries \(z \to \frac{az + b}{cz + d}\) of the hyperbolic plane \(\mathbb{H}^2\) realised as the upper half-plane \(z = x + iy, y > 0\) with negative constant curvature metric \(ds^2 = \frac{dzd\bar{z}}{y^2}\). Since \(\pm I\) act trivially, the actual hyperbolic isometry group is \(PSL(2, \mathbb{R})\). Stabiliser of a point \(z = i\) is the subgroup \(PSO(2) \subset PSL(2, \mathbb{R})\), so \(PSL(2, \mathbb{R})\) can be naturally identified with the unit tangent bundle of hyperbolic plane \(\mathbb{H}^2 = PSL(2, \mathbb{R})/PSO(2)\).

Explicitly we have the identification
\[ g = \pm \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in PSL(2, \mathbb{R}) \longrightarrow (z = \frac{ai + b}{ci + d}, \xi = \frac{i}{(ci + d)^2}) \in S\mathbb{H}^2, \quad (5) \]
where \(\mathbb{H}^2\) is realised as the upper half-plane \(z = x + iy, y > 0\) with the hyperbolic metric
\[ ds^2 = \frac{dzd\bar{z}}{y^2}. \]
Indeed, it is easy to check that \(\xi \in T_z\mathbb{H}^2\) has the unit norm in this metric.

Denoting the argument of \(\xi\) as \(\varphi\) we can introduce convenient coordinates \(x, y, \varphi\) on \(PSL(2, \mathbb{R})\) with \((x, y) \in \mathbb{H}^2, \varphi \in S^1 = \mathbb{R}/2\pi\mathbb{Z}\).
Thus both $\text{SL}(2, \mathbb{R})$ and $\text{PSL}(2, \mathbb{R})$ topologically are open solid torus $\mathbb{H}^2 \times S^1$ with the fundamental group $\mathbb{Z}$. Their universal cover $\widetilde{\text{SL}(2, \mathbb{R})}$ is simply-connected 3-dimensional Lie group, which is known to have no faithful matrix representations.

In Thurston’s approach it is actually simply connected version $\widetilde{\text{SL}(2, \mathbb{R})}$ considered a model of geometry, but for our purposes we can consider $\text{SL}(2, \mathbb{R})$ and its quotients by discrete subgroups.

As for the metrics we can choose any left-invariant metric on the group, but we will choose the special class, which also right-invariant under the subgroup $\text{SO}(2)$. We cannot choose bi-invariant metric since it is known to be not positive definite.

There is the two parameter family metrics [1], which up to a multiple are determined by the quadratic form

$$|\Omega|^2 = 4(u^2 + vw) + k(v - w)^2, \quad k = 1 - \frac{\beta}{\alpha} > 1 \quad (6)$$

on the Lie algebra

$$\Omega = \begin{pmatrix} u & v \\ w & -u \end{pmatrix} \in \text{sl}(2, \mathbb{R}).$$

We have chosen the normalisation $\alpha = 2$ to make the Gaussian curvature $K = -2/\alpha$ of the quotient to be precisely $-1$ (see below).

The coordinates $u, v, w$ are related to $a, b, c$ of the corresponding momentum [2] by

$$u = a, \quad v = \frac{(2 - k)b - kc}{2(1 - k)}, \quad w = \frac{(2 - k)c - kb}{2(1 - k)}, \quad (7)$$

$$a = u, \quad b = \frac{(2 - k)v + kw}{2}, \quad c = \frac{(2 - k)w + kv}{2} \quad (8)$$

In particular, when $k = 2$ we have particularly simple relation

$$\Omega = \frac{1}{2} M^\top = \begin{pmatrix} a & c \\ b & -a \end{pmatrix}.$$ 

Direct calculation shows that in the coordinates $x, y, \varphi$ this metric has the form

$$ds^2 = \frac{dx^2 + dy^2}{y^2} + (k - 1)(d\varphi + \frac{dx}{y})^2. \quad (9)$$

This means that the projection $\text{PSL}(2, \mathbb{R}) \to \text{PSL}(2, \mathbb{R})/\text{PSO}(2)$ is Riemannian submersion of $\text{PSL}(2, \mathbb{R})$ with metric (6) onto the upper half-plane with the standard hyperbolic metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$ of Gaussian curvature $K = -1$.

One can check that when $k = 2$ the metric (6) is precisely the Sasaki metric [43] on $\mathbb{S}\mathbb{H}^2$, so our class of metrics coincides with the class of the generalised Sasaki metrics considered by Nagy [37].

As it was shown by Nagy, the projection of the corresponding geodesics to $\mathbb{H}^2$ are the curves of constant geodesic curvature, which are either circles or
part of the circles lying in the upper half-plane. We are going to make this more explicit in the next two sections using the Euler-Poincare description of the geodesic flow.

3. GEODESICS ON $SL(n, \mathbb{R})$ WITH NATURALLY REDUCTIVE METRICS

Let $G$ be a semi-simple Lie group $G$ with left-invariant metric defined by the inner product $\langle \ , \ \rangle$ on the Lie algebra $\mathfrak{g}$. We identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ using the Cartan-Killing form $(\ , \ )$.

The Euler-Poincare equations of the corresponding geodesic flow have the following form (see Arnold [4])

$$\dot{M} = [M, \Omega], \quad (10)$$

where $\Omega := \mathfrak{g}^{-1} \dot{\mathfrak{g}} \in \mathfrak{g}$ and the momentum $M \in \mathfrak{g}^* = \mathfrak{g}$ is determined by the relation $(\Omega, M) = \langle \Omega, \Omega \rangle$.

In the case of naturally reductive metrics [1], which are left $G$-invariant and right $K$-invariant on $G = SL(n, \mathbb{R})$ with $K = SO(n)$, we have

$$2M = \alpha(\Omega + \Omega^\top) + \beta(\Omega - \Omega^\top) = (\alpha + \beta)\Omega + (\alpha - \beta)\Omega^\top. \quad (11)$$

We have also that $2M^\top = (\alpha + \beta)\Omega^\top + (\alpha - \beta)\Omega$ and thus

$$\Omega = \frac{(\alpha + \beta)M + (\beta - \alpha)M^\top}{2\alpha\beta}. \quad (12)$$

Substituting this into (10) we have

$$\dot{M} = \frac{\beta - \alpha}{2\alpha\beta} [M, M^\top]. \quad (13)$$

Note that

$$\dot{M}^\top = \frac{\beta - \alpha}{2\alpha\beta} [M, M^\top]^\top = \frac{\beta - \alpha}{2\alpha\beta} [M, M^\top] = \dot{M},$$

which gives the conservation law $\dot{M} - \dot{M}^\top \equiv 0$, related to $SO(n)$-invariance of the metric.

Thus in terms of $\Omega$ the equations are

$$\dot{\Omega} = \frac{\alpha - \beta}{2\alpha} [\Omega^\top, \Omega] = \frac{k}{2} [\Omega^\top, \Omega], \quad (14)$$

where as before $k = 1 - \frac{\beta}{\alpha}$.

These equation can be easily integrated in the following way (see e.g. [22, 31]).

Note first that

$$M = \alpha \text{ sym } \Omega + \beta \text{ skew } \Omega = \alpha[\Omega - k \text{ skew } \Omega], \quad \text{ skew } \Omega = (\Omega - \Omega^\top)/2$$

and introduce the matrices

$$X = \frac{1}{\alpha} M = \Omega - k \text{ skew } \Omega \in \mathfrak{sl}(n), \quad Y = k \text{ skew } \Omega \in \mathfrak{so}(n) \quad (15)$$
with $X + Y = \Omega$. In terms of variables (2) we have

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad Y = \frac{\alpha - \beta}{2\beta} \begin{pmatrix} 0 & b - c \\ c - b & 0 \end{pmatrix} = \frac{k}{2(k-1)} \begin{pmatrix} 0 & c - b \\ b - c & 0 \end{pmatrix}. \tag{16}$$

Theorem 3.1. The geodesic of naturally reductive metric (1) on $SL(n, \mathbb{R})$ with $g(0) = g_0$, $\Omega(0) = \Omega_0$ can be given explicitly by

$$g(t) = g(0)e^{tX_0}e^{tY_0} \tag{17}$$

with $X_0, Y_0$ computed from $\Omega_0$ using formula (15).

Proof. We can check by direct calculation that the Euler-Poincare equations (14) are satisfied. We have

$$\dot{g} = g_0 e^{tX_0}X_0 e^{tY_0} + g_0 e^{tX_0}Y_0 e^{tY_0} = g_0 e^{tX_0} \Omega_0 e^{tY_0},$$

so $\Omega = g^{-1} \dot{g} = e^{-tY_0} \Omega_0 e^{-tY_0}$. Now

$$\dot{\Omega} = e^{-tY_0} \Omega_0 Y_0 e^{tY_0} - e^{-tY_0} \Omega_0 e^{tY_0} = e^{-tY_0} [\Omega_0, Y_0] e^{tY_0} = [e^{-tY_0} \Omega_0 e^{tY_0}, e^{-tY_0} Y_0 e^{tY_0}].$$

Since $e^{tY_0} \in SO(n)$ we have $e^{-tY_0} Y_0 e^{tY_0} = \frac{2}{k} e^{-tY_0} (\Omega_0 - \Omega_0^\top) e^{tY_0} = \frac{2}{k} (\Omega - \Omega^\top).$ Thus we have

$$\dot{\Omega} = [\Omega, \frac{k}{2} (\Omega - \Omega^\top)] = \frac{k}{2} [\Omega^\top, \Omega],$$

which coincides with (14). □

Consider the symmetric space $X_n = G/K = SL(n, \mathbb{R})/SO(n)$ (of type AI in Cartan’s classification [23]) and the natural projection $\pi : G \to M$.

Corollary 3.2. The projection of the geodesics on $SL(n, \mathbb{R})$ with the naturally reductive metrics to the symmetric space $X_n$ have constant geodesic curvature.

Proof. Recall that the geodesic curvature $\kappa$ of the curve $\gamma(s)$ parametrised by the arc length $s$ in a Riemannian manifold $M$ is defined as the norm of the covariant derivative of the velocity vector field in the Levi-Civita connection

$$\kappa = \|D\dot{\gamma}/ds\|$$

(see e.g. [16]). From the formula (17) we see that, since $e^{tY_0} \in SO(n)$ acts trivially on $M_0$, the projections of the geodesics are the orbits of the one-parametric groups $e^{tX_0} \in SL(n, \mathbb{R})$. Since $SL(n, \mathbb{R})$ acts on $M$ by isometries, the geodesic curvature is constant along the orbits. □

4. $SL(2, \mathbb{R})$ Case and Hyperbolic Magnetic Geodesics

For $n = 2$, the space $SL(2, \mathbb{R})/SO(2)$ is the hyperbolic plane $\mathbb{H}^2$, so the projection of the geodesics are constant geodesic curvature curves in $\mathbb{H}^2$. In the Poincare model on the upper half-plane these curves are known to be either the usual circles, or their arcs lying in the upper half-plane, depending
on whether the geodesic curvature $\kappa$ is larger or smaller than the Gaussian curvature $K$ (see e.g. Hedlund [23] and Arnold [4]).

To make all this more explicit, let us use the formulae (17) for the geodesics on $SL(2,\mathbb{R})$ with metric (6), assuming for convenience that $k = 2$ (which corresponds to the Sasaki metric). In that case

$$X = \Omega - 2 \text{ skew } \Omega = \Omega^\top = \frac{1}{2} M, \quad Y = \Omega - \Omega^\top,$$

or, explicitly, in terms of variables (2)

$$\Omega = \begin{pmatrix} a & c \\ b & -a \end{pmatrix}, \quad X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & c-b \\ b-c & 0 \end{pmatrix}. \quad (18)$$

Note that $X^2 = \Delta I$, $Y^2 = -(b-c)^2 I$, where $\Delta = a^2 + bc$, so when $\Delta \neq 0$ we have

$$e^{Xt} = \cosh \sqrt{\Delta} t I + \frac{\sinh \sqrt{\Delta} t}{\sqrt{\Delta}} X, \quad e^{Yt} = \begin{pmatrix} \cos (b-c)t & -\sin (b-c)t \\ \sin (b-c)t & \cos (b-c)t \end{pmatrix}.$$  

When $\Delta = 0$, then $X^2 = 0$ and $e^{Xt} = I + Xt$.

Substituting this into the equation (17), we have the explicit formula for the geodesic on the group $G = SL(2,\mathbb{R})$ passing through $g(0) = g_0$ in direction $\dot{g}(0) = \dot{g_0} \Omega \in T_{g_0}G$:

$$g(t) = g_0 (\cosh \sqrt{\Delta} t I + \frac{\sinh \sqrt{\Delta} t}{\sqrt{\Delta}} X) \begin{pmatrix} \cos (b-c)t & -\sin (b-c)t \\ \sin (b-c)t & \cos (b-c)t \end{pmatrix}. \quad (19)$$

Assume for simplicity that $g_0 = I$, then acting by the right-hand side on $i \in \mathbb{H}^2$ gives the projection of the geodesics to $\mathbb{H}^2$ explicitly as

$$z(t) = \frac{i \left(e^{2 \sqrt{\Delta} t \sqrt{\Delta}} - ibe^{2 \sqrt{\Delta} t} + e^{2 \sqrt{\Delta} t}a + \sqrt{\Delta} - a + ib \right)}{e^{2 \sqrt{\Delta} t \sqrt{\Delta}} + ice^{2 \sqrt{\Delta} t} - e^{2 \sqrt{\Delta} t}a + \sqrt{\Delta} + a - ci}. \quad (20)$$

Assuming that $c \neq 0$ one can check that

$$z(t) - \left(\frac{a}{c} + \frac{c-b}{2c} i\right) = - \left(\frac{a}{c} - \frac{b+c}{2c} i\right) \frac{Z(t)}{Z(t)} \quad Z(t) = e^{2 \sqrt{\Delta} t \sqrt{\Delta}} - e^{2 \sqrt{\Delta} t}a - ice^{2 \sqrt{\Delta} t} + \sqrt{\Delta} + a + ic.$$

Thus

$$\left| z(t) - \frac{2a + (c-b)i}{2c} \right|^2 = \left| \frac{2a - (b+c)i}{2c} \right|^2 = \frac{4a^2 + (b+c)^2}{4c^2},$$

which gives a Euclidean circle (or the arc of it belonging to the upper half-plane). In the case of $c = 0$, we have the (part of the) straight line

$$z(t) = \frac{b(e^{2 \sqrt{\Delta} t} - 1) + 2ai e^{2 \sqrt{\Delta} t}}{2a}.$$

Summarising we have the following result.
Theorem 4.1. The projection to $\mathbb{H}^2$ of the geodesic on $SL(2, \mathbb{R})$ passing through the identity in the direction $\Omega$ given by (7) with $c \neq 0$ and $\Delta = a^2 + bc < 0$ is the Euclidean circle centred at

$$z_0 = \frac{(2a + (c - b)i)}{2c}$$

with radius

$$R = \sqrt{4a^2 + (b + c)^2}/2c.$$

If $\Delta > 0$ the projections are the arcs of these circles lying in the upper half-plane. When $c = 0$ we have the part of the straight line $y = \frac{2a}{b}x + 1$ belonging to the upper half-plane.

The three different cases depending on the sign of $\Delta$ are shown on Fig.1.

Since the Euclidian circles are known to be the curves of constant geodesic curvature, it is natural to consider this relation in more detail.

Computation of the Christoffel symbols $[18]$

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(g_{i,j}^l + g_{j,i}^l - g_{ij,l}^l)$$

for the hyperbolic metric $ds^2 = \frac{1}{y}(dx^2 + dy^2)$ gives (with $x^1 = x, x^2 = y$)

$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\frac{1}{y^2}, \ \Gamma^2_{11} = \frac{1}{y^2},$$

with all other symbols to be zero. The equation for geodesics

$$\frac{D\dot{x}}{ds} = \ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$$

in our case with $z = x + iy$ becomes

$$\frac{D\dot{z}}{ds} = \ddot{z} + \frac{i\dot{z}^2}{y} = 0.$$  \hspace{1cm} (21)

The equation for the curves $z(s)$ of constant geodesic curvature $\kappa$ in the arc length parameter $s$ is

$$\frac{d^2z}{ds^2} + \frac{i}{y}\left(\frac{dz}{ds}\right)^2 = i\kappa\frac{dz}{ds},$$
where by definition the velocity $\|\frac{dz}{dt}\|$ (computed in the hyperbolic metric) is equal to 1.

In the parameter $t$ when this velocity $V = \|\frac{dz}{dt}\|$ is constant (so $s = Vt$), we have the equation

$$\frac{d^2z}{dt^2} + \frac{i}{y} \left( \frac{dz}{dt} \right)^2 = i\kappa \frac{dz}{dt},$$

(22)

Note that the right-hand side can also be interpreted as the Lorenz force in the magnetic field defined by the 2-form $Bd\sigma = Bdx \wedge dy/y^2$ with the density $B = \kappa V$, so the same equations describe the magnetic geodesics on the hyperbolic plane (see e.g. [4], [33]).

Using the explicit formula (20) one can check that the projection of the corresponding geodesic has constant velocity $V = \sqrt{4a^2 + (b+c)^2}$ and satisfies the equation (22) with

$$\kappa = \frac{b - c}{\sqrt{4a^2 + (b+c)^2}},$$

where we assigned the sign to the geodesic curvature using equation (22).

**Theorem 4.2.** The projection of the geodesic on $SL(2, \mathbb{R})$ with metric (6) and $M$ given by (2) to $\mathbb{H}^2$ is a curve of constant geodesic curvature

$$\kappa = \frac{b - c}{\sqrt{4a^2 + (b+c)^2}}$$

(23)

parametrised by $t = s/\sqrt{4a^2 + (b+c)^2}$, where $s$ is the arc length.

Equivalently, it can be described as the magnetic geodesic in the constant magnetic field with density $B = b - c$.

For the non-normalised metrics (1) the quotient $SL(2, \mathbb{R})/SO(2)$ has Gaussian curvature $K = -2\alpha^{-1}$ and

$$\kappa = K \frac{c - b}{\sqrt{4a^2 + (b+c)^2}},$$

so the ratio

$$\mathcal{C} := \frac{\kappa^2}{K^2} = \frac{(b - c)^2}{4a^2 + (b+c)^2}$$

is independent on the choice of the metric.

Depending on the values of $\mathcal{C}$ the corresponding curves on the hyperbolic plane are called hypercycles when $\mathcal{C} < 1$, horocycles when $\mathcal{C} = 1$ and hyperbolic circles when $\mathcal{C} > 1$ (see Hedlund [23]). The special case of hypercycles with $\mathcal{C} = 0$ are the usual geodesics with zero geodesic curvature.

Hedlund was probably the first who studied the properties of these curves on the quotient $\mathcal{M}^2_\Gamma = \Gamma \backslash \mathbb{H}^2$ of hyperbolic plane by the Fuchsian groups $\Gamma \subset PSL(2, \mathbb{R})$ of the first kind (for the theory of Fuchsian groups we refer to S. Katok [26]). There was studied later by Arnold [4] and Anosov and Sinai [2], see also more recent papers by Paternain et al. [11, 39], Taimanov [38] and Miranda [33] and references therein.
Recall that the geodesic flow on the surface $M^2$ is called *transitive* if there exists a geodesic which is everywhere dense on the unit tangent bundle of $M^2$.

Let $\Gamma \subset PSL(2, \mathbb{R})$ be *cocompact Fuchsian group*, so that the quotient $M^2_\Gamma = \Gamma \backslash \mathbb{H}^2$ is compact.

We believe that the following result is true for all *cofinite Fuchsian groups* $\Gamma$ with the quotients $M^2_\Gamma$ of finite area, but we cannot find a proper reference.

**Theorem 4.3.** *(Hedlund [23], Arnold [4])** The magnetic geodesic flow (21) on $M^2_\Gamma$ is transitive if and only if $\kappa \leq 1$. The metric entropy $h(\kappa)$ of such flow is

$$h(\kappa) = \sqrt{1 - \kappa^2}.$$

**Corollary 4.4.** The magnetic geodesic flow with $\kappa < 1$ has the positive Lyapunov exponent

$$\lambda = \sqrt{1 - \kappa^2}.$$ (24)

Recall that the *Lyapunov exponent* of the flow $f_t$, $t \in \mathbb{R}$ on a Riemannian manifold $M$ can be defined (see e.g. [40]) as

$$\lambda(f) = \limsup_{t \to +\infty} \frac{1}{t} \ln ||df_t(x)||,$$

which in ergodic case is constant independent on $x \in M$, describing the rate of separation of infinitesimally close orbits. In our case we can apply Pesin’s formula [40], which says that

$$\lambda(f) = h(f) = \sqrt{1 - \kappa^2} > 0.$$

The general hypercyclic case $\kappa < 1$ can be reduced to the geodesic flow case with $\kappa = 0$ since the hypercycles are known to be the curves equidistant from the geodesics (see [3, 23]). The most subtle horocyclic case $\kappa = 1$ was sorted by Hedlund in [23]. In the case $\kappa > 1$ all the orbits are periodic.

Taimanov [48] interpreted these results in terms of the integrability of the corresponding magnetic geodesic flow on $M^2_\Gamma$. Indeed, since the geodesic curvature of the magnetic geodesics is $\kappa = B/V$, where $B$ is the density of the magnetic field and $V$ is the velocity, the integrability condition $\kappa > 1$ is equivalent to the condition $V < B$.

We will use these results now to study the geodesic flow on three-folds $M^3_\Gamma = \Gamma \backslash PSL(2, \mathbb{R})$.

5. **Geodesic flow on Fuchsian quotients of $PSL(2, \mathbb{R})$**

Let $\Gamma \subset G = PSL(2, \mathbb{R})$ be a finitely generated Fuchsian group with the compact quotient $\Gamma \backslash \mathbb{H}^2 = M^2_\Gamma$ and consider $SL(2, \mathbb{R})$-manifold

$$M^3_\Gamma = \Gamma \backslash G.$$

It is known (see e.g. Thurston [49]) that in the compact case

$$M^3_\Gamma = SM^2_\Gamma \subset T(M^2_\Gamma).$$
is the unit tangent bundle of the corresponding surface $M_2^\Gamma$ of genus $g \geq 2$. In the non-compact case (for example, for $\Gamma = \text{PSL}(2, \mathbb{Z})$) the quotient could be an orbifold, so one should be a bit more careful here.

Let $\Omega = g^{-1}\dot{g} \in \mathfrak{g}$, $M = A(\Omega) = \alpha \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g}^*$ be the left angular velocity and left momentum as before. Note that $a, b, c$ can be considered as left invariant functions on $T^*\text{SL}(2, \mathbb{R})$.

Introduce now the right momentum

$$m = gMg^{-1} = \alpha \begin{pmatrix} u & v \\ w & -u \end{pmatrix} \in \mathfrak{g}^*.$$  

It is well-known (see Arnold [4]) that $m$ is preserved: $\dot{m} = 0$, which can be also easily checked directly using Euler-Poincare equations (10).

In terms of these functions, using relation (18), the Hamiltonian of the geodesic flow on $T^*M_2^\Gamma = \Gamma \backslash T^*G$ for the metric (1) can be written as

$$H = \frac{1}{2} (M, \Omega) = \frac{\alpha}{4\beta} (\beta(4a^2 + (b + c)^2) - \alpha(b - c)^2),$$

which is well defined function on $T^*M_2^\Gamma$.

Another obvious integral is the Casimir function of the corresponding Poisson bracket on $\mathfrak{g}^*$

$$\Delta = -\det M = -\det m = a^2 + bc = u^2 + vw.$$  

For Liouville integrability we need one more analytic integral. On the group $G$ we can take, for example, $F = v - w$ corresponding to the left $SO(2)$ invariance of the metric, but it is not $\Gamma$-invariant.

To find the invariants we need to study the action of $\Gamma \subset G$ on the space of the conserved momenta $m$, which is $\mathfrak{g}^*$. The action of the group $G$ preserves $\Delta$, which defines the pseudo-Euclidean structure on $\mathfrak{g}^*$ and thus an isomorphism of $\text{PSL}(2, \mathbb{R})$ with the identity connected component $SO(2, 1)_0$ of the group $SO(2, 1)$.

The cone $\Delta = u^2 + vw = 0$ splits $\mathfrak{g}^*$ into two open regions: $\Delta > 0$ and $\Delta < 0$, foliated by the one- and two-sheeted hyperboloids $\Delta = \delta$ respectively (see Figure 2).

A sheet of two-sheeted hyperboloid $H_\delta$ given by $\Delta = \delta, \delta < 0$ represents a well-known model of the hyperbolic plane $\mathbb{H}^2$, so the action of $\Gamma$ is discrete here. From the theory of automorphic functions it follows that there exists a non-constant real analytic $\Gamma$-invariant function $F$ on $H_\delta$. We can extend it by homogeneity to the function $F(m)$ defined on the domain $\{\Delta < 0\} \subset \mathfrak{g}^*$.

In contrast, on one-sheeted hyperboloid $\Delta = \delta, \delta > 0$ the generic orbits of $\Gamma$ are known to be dense (see e.g. [14], Section VI, Property 2.12) Let us sketch the proof of this, kindly provided to us by John Parker [38].
First, we remind the following correspondence due to Klein [27] between the one sheeted hyperboloid and the double cover of the space of geodesics in the hyperbolic plane.

Recall that Klein’s model of the hyperbolic plane is the interior of the conic on the real projective plane. The points outside the conic correspond by polarity to the straight chords inside the conic, which are the lines in Klein’s model. Representing this conic by $\Delta = 0$ we have the correspondence required. According to Hedlund [23] almost every hyperbolic geodesic is dense in the quotient, so the same is true for the generic orbits of $\Gamma$ on the one sheeted hyperboloids $H_\delta$ with $\delta > 0$.

Summarising we have the following

Theorem 5.1. The geodesic flow on $T^*M_3^\Gamma$ has no smooth right-invariant integrals $F$ independent from $\Delta$ in the part of the phase space $T^*M_3^\Gamma$ with $\Delta \geq 0$.

We can relax now the assumption that the integral $F$ must be right-invariant proving our main result.

Recall that the topological entropy $h_{top}(\phi)$ of the geodesic flow $\phi_t$ on $M = SM$ is defined as

$$h_{top}(\phi) := \lim_{\epsilon \to 0} \limsup_{T \to +\infty} \frac{\log N(\phi, T, \epsilon)}{T},$$

where $N(\phi, T, \epsilon)$ be the cardinality of the minimal finite set $X = X(\epsilon) \subset M$ such that for any $v \in M$ there exists $w \in X$ such that

$$\sup_{0 \leq t \leq T} d(\phi_tv, \phi_tw) < \epsilon$$

(see the details and the history of the notion in Katok’s review [25]).

The topological entropy satisfies the inequality

$$h_{top}(\phi) \geq h_{\mu}(\phi)$$
for metric entropy $h(\phi)$ defined with respect to any ergodic $\phi$-invariant measure $\mu$. It has also the following properties (see e.g. [29]).

Let $\phi_t$ and $\psi_t$ be the flows on the compact manifolds $X$ and $Y$ respectively and $f : X \to Y$ be a map such that $\psi_t \circ f = f \circ \phi_t$. Then
- If $f$ is injective then $h_{\text{top}}(\phi) \leq h_{\text{top}}(\psi)$,
- If $f$ is surjective then $h_{\text{top}}(\phi) \geq h_{\text{top}}(\psi)$.

**Theorem 5.2.** Geodesic flow on $T^*\mathcal{M}_3^\Gamma$ is integrable in analytic Liouville’s sense in the open region of the phase space with $\Delta = a^2 + bc < 0$.

In the region with $\Delta > 0$ there are no smooth integrals independent from $H$ and $\Delta$. At the integral level $C < 1$ the system has positive topological entropy

$$h_{\text{top}} \geq \sqrt{1 - C}.$$  \hfill (26)

**Proof.** As we have seen, the level $\Delta = \delta < 0$ defines in $g^*$ two-sheeted hyperboloid $H_\delta$, one sheet of which can be identified with the hyperbolic plane. Let $F$ be as before real analytic $\Gamma$-invariant function on $H_\delta$ extended to the open region $\{\Delta < 0\} \subset g^*$ by homogeneity, and extend it further to the corresponding domain of $T^*G$ by the right shifts as $F(m)$. Since $F(m)$ is $\Gamma$-invariant this gives us a well-defined function $F$ on the quotient $T^*\mathcal{M}_3^\Gamma = \Gamma \backslash T^*G$.

It is easy to see that $H$, $\Delta$ and $F$ are 3 independent analytic integrals on $T^*\mathcal{M}_3^\Gamma$, which are Poisson commuting. Thus the geodesic flow is Liouville-integrable in this domain, even in the analytic category. Note that we have one more integral by taking an independent from $F$ real analytic $\Gamma$-invariant function (which will not Poisson commute with $F$, of course), so the Liouville tori are two-dimensional, in agreement with the explicit description of geodesics above.

The problem is however that writing down explicitly the automorphic integral $F(m)$ only possible in special cases (see the next section). This is related with the non-effectiveness of the solution of the uniformisation problem in the compact hyperbolic case.

When $\Delta > 0$ we can prove a stronger result that in fact the geodesic flow has no invariant 3-dimensional Liouville tori with fixed $H$ and $\Delta = \delta > 0$. Indeed, let us assume that we have such a torus $T^3 \subset T^*\mathcal{M}_3^\Gamma$ and consider its projection to $\mathcal{M}_3^\Gamma$. Because in that case $C < 1$ the system has positive Lyapunov exponent and the projections diverge exponentially fast, which is impossible since on the Liouville torus they are simply winding lines.

To show that the topological entropy satisfy the inequality (26) consider the geodesic flows on the unit tangent bundles of $\mathcal{M}_3^\Gamma$ and $\mathcal{M}_2^\Gamma$. We have seen that the geodesics on $\mathcal{M}_3^\Gamma$ at the integral level $C = \kappa^2 < 1$ project into magnetic geodesics on $\mathcal{M}_2^\Gamma$ with geodesic curvature $\kappa$. Let $X(\kappa) \subset \mathcal{M}_2^\Gamma$ be the union of all these geodesics, and $f$ be its projection onto $Y = \mathcal{M}_2^\Gamma$. Since the corresponding magnetic geodesic flow on $Y$ has metric entropy $h = \sqrt{1 - \kappa^2} = \sqrt{1 - C}$, by the properties of the topological entropy the geodesic flow on $X(\kappa)$ we have the inequality (26). □
To derive now the Theorem 1.1 for the cofinite Fuchsian groups $\Gamma$ with the non-compact quotients $M^2_\Gamma$ we can use the results by Gurevich and Katok [21], who proved that the (properly understood) topological entropy of the geodesic flow on $SM^2_\Gamma$ in this case also equals 1 (see Theorem 12 in [21]).

6. Geodesics on the modular 3-fold and knots

Consider now the special modular case of the group $\Gamma = PSL(2, \mathbb{Z})$. In that case the quotient $M^2 = PSL(2, \mathbb{Z}) \backslash \mathbb{H}^2$ is the orbifold with two orbifold points $i$ and $e^{i\pi/3}$ corresponding to the elliptic elements in $PSL(2, \mathbb{Z})$ of order 2 and 3 respectively (see Fig. 3).

The modular surface is naturally the moduli space of the elliptic curves with the classical function (known as \textit{Hauptmodul}) $j(\tau) : M^2 \rightarrow \mathbb{C}$, establishing its equivalence to $\mathbb{C}$.

The geodesics on modular surface $M^2$ were studied in detail by Artin in the seminal paper [5], who used the symbolic dynamics and the theory of continued fractions to describe them.

In particular, Artin showed that the periodic geodesics on the modular surface correspond to the lines in Klein’s model determined by the equations with integer coefficients, which are invariant under suitable hyperbolic element $g \in PSL(2, \mathbb{Z})$. Any such element preserves an indefinite binary quadratic form $Q$ with integer coefficients, so we have one-to-one correspondence between periodic geodesics on $M^2$ and the set of integer indefinite binary quadratic forms, considered up to proportionality.

Consider now the modular 3-fold $M^3 = PSL(2, \mathbb{Z}) \backslash PSL(2, \mathbb{R})$. In this case we can produce the integrals in the domain of $SM^3$ with $\Delta < 0$, additional to $H$ and $\Delta$, explicitly by taking the real and imaginary parts of the Hauptmodul $j$ considered as the function on the one sheet of the hyperboloid $\{ \Delta = -1 \} \subset \mathfrak{g}^*$ identified with the upper half-plane $\mathbb{H}^2$. 

![Figure 3. The fundamental domains of $\Gamma$ and $\Gamma_2$](image-url)
The non-integrability of the geodesic flow in the domain with $\Delta > 0$ can be shown using the same arguments as before with the positivity of the topological entropy following from the results of Gurevich and Katok [21].

The important observation, usually attributed to Quillen (see Milnor [32]), is that topologically

$$M^3 \cong S^3 \setminus K$$

(27)

is equivalent to the complement in $S^3$ to trefoil knot $K$, which is the $(2, 3)$-torus knot. Note that the fundamental group of this complement is the braid group $B_3$, so that the modular 3-fold is the quotient

$$M^3 = B_3 \setminus \widetilde{SL}(2, \mathbb{R}).$$

Recall (see e.g. [1]) that a $(p, q)$-torus knot $K_{p,q}$ is a special type of knots specified by a pair of coprime integers $p$ and $q$. It can be realised on the surface of the solid torus in $\mathbb{R}^3$, winding $p$ times around the axis of rotation of the torus and $q$ times around the central circle of the solid torus. When $p = 2, q = 3$ we have the trefoil knot shown on Fig. 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{trefoil_knot}
\caption{Trefoil knot $K$ and its $(2,33)$ cable knot}
\end{figure}

To prove the homeomorphism (27) note that the quotient

$$PSL(2, \mathbb{Z}) \setminus PSL(2, \mathbb{R}) = SL(2, \mathbb{Z}) \setminus SL(2, \mathbb{R})$$

can be interpreted as the moduli space of the lattices $\mathcal{L}$ on the Euclidean plane with fixed area of the quotient, or equivalently the space of the elliptic curves $\mathcal{C}/\mathcal{L}$ up to real scaling. The corresponding $\wp$-function satisfies the Weierstrass equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3,$$

which defines the elliptic curves only if the discriminant $D = g_2^3 - 27g_3^2 \neq 0$. The intersection of the unit sphere $S^3 \subset \mathbb{C}^2$ with coordinates $(g_2, g_3)$ with the set $D = 0$ is $(2, 3)$-torus knot, known as trefoil knot (see Fig. 4).

Alternatively, one can argue that the natural map $M^3 \to M^2$ is the Seifert fibration with two singular fibres over orbifold points of order 2 and

\footnote{We are very grateful to Graeme Segal for the discussions of this remarkable fact and the history of its discovery.}
3, which implies that the missing fibre over infinity is a \((2,3)\)-torus knot. More interesting relations can be found in a nice article \([36]\) by Mostovoy.

Note that there is a deep theorem by Gordon and Luecke \([20]\) saying that any knot in the 3-sphere is determined up to isomorphism by the homeomorphism-type of its complement (it is not true though for the links).

The homeomorphism \([27]\) plays a key role in the remarkable work by Ghys \([19]\), linking the closed geodesics on the modular surface \(M^2\) with the periodic orbits in the celebrated Lorenz system \([30]\)

\[
\begin{align*}
\dot{x} &= \sigma(-x + y) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= -bz + xy
\end{align*}
\]

depending on positive real parameters \(\sigma, r, b, b\), where the most essential for us parameter \(r\) has the physical meaning of relative Rayleigh number.

Historically this was one of the first computer observation of the chaotic attractor. Lorenz had chosen the values \(\sigma = 10, b = 8/3\) and \(r = 28\) to observe the peculiar behaviour known now as "Lorenz butterfly" shown on the left of Fig. 5.

![Figure 5. The Lorenz trajectories for \(r = 28, 10000\) and \(r = \infty\)](image)

It is known (see e.g. Strogatz \([46]\)) that in the limit \(r \to \infty\) the Lorenz dynamics becomes periodic, so the chaos disappears for large \(r\). This fact was first shown by Yudovich \([50]\) and independently by Robbins \([41]\).

Birman and Williams \([7]\) studied periodic trajectories of Lorenz system in the chaotic region from topological point of view. They observed that these trajectories are knotted in a very special way and described the corresponding class of knots.

Remarkably, as it was proved by Ghys \([19]\), the same class of knots appears as the canonical lifts to \(M^3 \cong S^3 \setminus \mathcal{K}\) of the periodic geodesics on the modular surface! Ghys called them modular knots and proved that their linking number with the trefoil knot \(\mathcal{K}\) are given by the number-theoretic Rademacher function, which allowed Sarnak \([42]\) to count geodesics by the linking number.
From our perspective all this is about the zero level of the integral

$$\mathcal{C} = \frac{(b - c)^2}{4a^2 + (b + c)^2} = 0$$

of our geodesic flow on $S\mathcal{M}^3$. The corresponding geodesics have the form

$$g(t) = g_0e^{tX}, \quad X = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$ 

The geodesics considered by Ghys correspond to $a = 1$, $b = 0$. In the general case, the periodic geodesics give equivalent knots, so all periodic geodesics on $\mathcal{M}^3$ with $\mathcal{C} = 0$ form modular knots.

Note that $\mathcal{C} = 0$ is the most chaotic level with maximal entropy $h = 1$.

Let us consider now the most integrable limit when $\mathcal{C} \to \infty$.

In that case the projection of the corresponding geodesics on $\mathcal{M}^3$ are small circles on $\mathcal{M}^2$, turning to the points when $\mathcal{C} = \infty$, which is the case when $a = b + c = 0$. In that case the geodesics on $\mathcal{M}^3$ become fibres of the Seifert fibration $\mathcal{M}^3 \to \mathcal{M}^2$, which topologically are trefoil knots (as well as $K$, which is the fibre over infinity). In the Lorenz system they correspond to the periodic orbits in the limit $r = \infty$.

This means that for large values of $\mathcal{C}$, the projections of the corresponding invariant 2-dimensional tori are embeddings and knotted as trefoil knots. The frequencies of the motion are

$$\omega_1 = \frac{\beta - \alpha}{2\beta} |b - c|, \quad \omega_2 = \sqrt{-\Delta},$$

where $\omega_1$ is the frequency along the trefoil fibre, while $\omega_2$ is the frequency along the hyperbolic circle.

In the particular case when the ratio

$$\frac{\omega_1}{\omega_2} = \frac{\beta - \alpha |b - c|}{2\beta \sqrt{-\Delta}} = \frac{p}{q}, \quad p, q \in \mathbb{Z},$$

(28)

is rational, the trajectories are periodic and form a special case of satellite knots of trefoil, called cable knots [1]. They can be realised as the torus knots $K_{p,q}$ on the solid torus knotted as the trefoil knot (see Fig. 4).

Note that by $a, b, c$ as well as parameters of the metric $\alpha, \beta$ we can realise any values of $p, q$, so any trefoil cable knot can be realised in this way.

**Theorem 6.1.** The periodic geodesics on modular 3-fold $M^3_\Gamma$ with sufficiently large values of $\mathcal{C}$ fill 2-dimensional tori with frequencies satisfying (28). They represent the trefoil cable knots in $S^3 \setminus K$ with parameters $p, q$, having linking number $l = 6p$ with $K$.

Any cable knot of trefoil can be realised in such a way.

Thus in the integrable limit the class of modular knots is replaced by the class of cable knots of trefoil. In the Lorenz systems this change is shown on Fig. 5.
It is instructive to see also what happens in the case when
\[ \Gamma = \Gamma_2/\{\pm I\} \subset PSL(2,\mathbb{Z}), \]
where \( \Gamma_2 \subset SL(2,\mathbb{Z}) \) is the principal congruence subgroup, consisting of the matrices congruent to the identity modulo 2. The quotient surface \( M_2^\Gamma_2 \) in this case is the sphere punctured at three points with the fundamental domain being the pair of ideal triangles.

The corresponding 3-fold can be represented topologically as the complement
\[ M_3^\Gamma_2 \cong S^3 \setminus \mathcal{L}, \]
where \( \mathcal{L} \) is the link of 3 Hopf fibres with pairwise linking numbers 1 shown on Fig. 6 (not to be confused with the famous Borromeo link with the pairwise linking numbers being zero). Note that in this case such a representation is not unique, since the topology of the complement to the link does not determine the link, in contrast to the knot case [20].

The periodic geodesics for large \( C \) are the cable knots of the Hopf fibres, which are simply torus knots in this case.

![Figure 6. The Hopf 3-link \( \mathcal{L} \)](image)

Recall that the knot \( K \subset S^3 \) is called hyperbolic, if its complement admits hyperbolic structure (complete constant negative curvature metric). In 1978 Thurston proved the only knots which are not hyperbolic are torus or satellite knots (see e.g. [1]).

All the modular knots are known to be hyperbolic, so one can ask the question about the volume of the corresponding hyperbolic complement (see the recent papers [6][10] about the volumes of the modular knot complements).

In the integrable limit for \( \Gamma_2 \) we have the torus knots \( K_{p,q} \) with the complements admitting \( SL(2,\mathbb{R}) \)-geometry. To see this we simply replace the modular surface with two orbifold points of order 2 and 3 by the hyperbolic surface \( M_{p,q}^2 \) with one cusp and two orbifold points of order \( p \) and \( q \).

The corresponding 3-fold \( SM_{p,q}^2 \) is the complement \( S^3 \setminus K_{p,q} \), which thus admits \( SL(2,\mathbb{R}) \)-geometry. Since the hyperbolic area of \( M_{p,q}^2 \) is
\[ \text{Area} M_{p,q}^2 = 2\left(1 - \frac{1}{p} - \frac{1}{q}\right)\pi, \]

\( 20 \)
the volume of the corresponding complement with $SL(2, \mathbb{R})$-metric is equal to
\[
\text{Vol}(S^3 \setminus K_{p,q}) = 4(k - 1)(1 - \frac{1}{p} - \frac{1}{q})\pi^2
\]
and depends on the parameter $k$ of the metric. In particular, the volume of the modular 3-fold $M^3$ is
\[
\text{Vol} M^3 = \text{Vol}(S^3 \setminus K_{2,3}) = \frac{2(k - 1)\pi^2}{3},
\]
while for the $\Gamma_2$ quotient
\[
\text{Vol} M^3_{\Gamma_2} = \text{Vol}(S^3 \setminus \mathcal{L}) = 4(k - 1)\pi^2 = 6 \text{Vol} M^3.
\]

Note that for the modular 3-fold in the integrable region we have the cable knots of trefoil, which is a particular subclass of the satellite knots. Their complements are known not to admit any geometric structure in Thurston’s sense [17].

7. Topological point of view

Let’s discuss our problem from the topological point of view. The existence of the metric on the topological manifold $M^n$ with the integrable geodesic flow imposes some restriction on the topology of $M^n$.

The first such restrictions were found by Kozlov, who proved that a two-dimensional closed oriented manifold $M^2_g$ of genus $g > 1$ does not admit such metric with an analytic additional integral [28].

The multi-dimensional case was studied by Taimanov [47], who proved that if $M^n$ admits a geodesic flow which is Liouville integrable in real analytic sense, then $\dim H_1(M^n, \mathbb{R})) \leq n$ and the fundamental group $\pi_1(M^n)$ is almost commutative (i.e. contains a commutative subgroup of finite index).

In the case of $M^3 = S\mathcal{M}^2_g$ of unit tangent bundle of genus $g$ surface $\mathcal{M}^2_g$ both Taimanov’s conditions fail. Indeed, the fundamental group $\pi_1(M^3)$ is generated by $a_1, \ldots, a_g, b_1, \ldots, b_g, \gamma$ with the relations
\[
a_i \gamma = \gamma a_i, b_i \gamma = \gamma b_i (i = 1, \ldots, g), \quad \gamma^{2-2g} = (a_1 b_1 a_1^{-1} b_1^{-1}) \ldots (a_g b_g a_g^{-1} b_g^{-1})
\]
(see e.g. Chapter 1, Section 4 in [18]). This implies that $\pi_1(M^3)$ is not almost commutative and
\[
\dim H_1(M^3, \mathbb{R}) = 2g > 3 = \dim M^3,
\]
so by Taimanov’s theorem there are no analytically integrable geodesic flows on $M^3$, which agrees with our result.

This also agrees with Dinaburg’s theorem [15], claiming that if the fundamental group $\pi_1(M^n)$ of a manifold $M^n$ has an exponential growth, then the topological entropy of the geodesic flow of any Riemannian metric on $M^n$ is positive.

We should also mention an important result of Butler [13] which states that if a 3-fold $M^3$ admits an integrable geodesic flow with smooth (not necessarily analytic) integrals and a “tame” singular set, then the fundamental
group $\pi_1(M^3)$ contains a finite–index polycyclic subgroup of step length at most 4. This means that a closed $SL(2, \mathbb{R})$-manifold does not admit integrable geodesic flows of this type even in smooth category. For a review of other results on topological obstructions to integrability we refer to [8].

Note that the first examples of Liouville integrable geodesic flows with smooth integrals having positive topological entropy found by Bolsinov and Taimanov [9] are the Sol-manifolds, which are topologically the torus bundles over circles, twisted by hyperbolic elements $A \in SL(2, \mathbb{Z})$. It is interesting that they were also historically the first non-trivial examples of 3-folds considered by Poincaré in 1892, see [15].

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