FREE BANACH SPACES AND THE APPROXIMATION PROPERTIES

GILLES GODEFROY AND NARUTAKA OZAWA

Abstract. We characterize the metric spaces whose free spaces have the bounded approximation property through a Lipschitz analogue of the local reflexivity principle. We show that there exist compact metric spaces whose free spaces fail the approximation property.

1. Introduction.

Let $M$ be a pointed metric space, that is, a metric space equipped with a distinguished point denoted $0$. We denote by $\text{Lip}_0(M)$ the Banach space of all real-valued Lipschitz functions defined on $M$ which vanish at $0$, equipped with the natural Lipschitz norm

$$\|f\|_L = \sup \left\{ \frac{|f(x) - f(y)|}{\|x - y\|} ; \; (x, y) \in M^2, \; x \neq y \right\}.$$ 

For all $x \in M$, the Dirac measure $\delta(x)$ defines a continuous linear form on $\text{Lip}_0(M)$. Equicontinuity shows that the closed unit ball of $\text{Lip}_0(M)$ is compact for pointwise convergence on $M$, and thus the closed linear span of $\{\delta(x) \ ; \; x \in M\}$ in $\text{Lip}_0(M)^*$ is an isometric predual of $\text{Lip}_0(M)$. This predual is called the Arens-Eells space of $M$ in [We], and (when $M$ is a Banach space) the Lipschitz-free space over $M$ in [GK], denoted by $\mathfrak{F}(M)$. We will use this notation, and simply call $\mathfrak{F}(M)$ the free space over $M$. When $M$ is separable, the Banach space $\mathfrak{F}(M)$ is separable as well, since the set $\{\delta(x) \ ; \; x \in M\}$ equipped with the distance induced by $\text{Lip}_0(M)^*$ is isometric to $M$.

The free spaces over separable metric spaces $M$ constitute a fairly natural family of separable Banach spaces, which are moreover very useful in non-linear geometry of Banach spaces (see [Ka2]). However, they are far from being well-understood at this point and some basic questions remain unanswered. We recall that a Banach space $X$ has the approximation property (AP) if the identity $\text{id}_X$ of $X$ is in the closure of the finite rank operators on $X$ for the topology of uniform convergence on compact sets. The $\lambda$-bounded approximation property ($\lambda$-BAP) means that there are approximating finite rank operators with norm less than $\lambda$, and the (1-BAP) is called the metric approximation property (MAP). This note is devoted to the following problem: for which metric spaces $M$ does the space $\mathfrak{F}(M)$ have (AP), or (BAP), or (MAP)? For motivating this query, recall that real-valued Lipschitz functions defined on subsets of metric spaces extend with the same

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Lipschitz constant through the usual inf-convolution formula. However, approximation properties for free spaces are related with the existence of linear extension operators for Lipschitz functions defined on subsets (see [Ba], and Proposition 6 below).

It is already known that some free spaces fail AP: indeed one of the main results of [GK] asserts that if $X$ is an arbitrary Banach space and $\lambda \geq 1$, then $X$ has $\lambda$-BAP if and only if $\mathcal{F}(X)$ has $\lambda$-BAP. Since moreover any separable Banach space $X$ is isometric to a 1-complemented subspace of $\mathcal{F}(X)$ ([GK], Theorem 3.1), it follows that $\mathcal{F}(X)$ fails AP when $X$ does.

This note provides further examples of metric spaces whose free spaces fail AP. We show in particular that some spaces $\mathcal{F}(K)$, with $K$ compact metric spaces, fail AP although MAP holds for “small” Cantor sets.

Section 2 gives a characterization of the $\lambda$-BAP for $\mathcal{F}(M)$ through weak*-approximation of Lipschitz functions from $M$ into bidual spaces, somewhat similar to the local reflexivity principle. In section 3 a method used in [GK] and localized in [DL] is shown to provide the existence of compact convex sets $K$ with $\mathcal{F}(K)$ failing AP. Several open questions conclude the note.

### 2. Lipschitz Local Reflexivity

For metric spaces $M$ and $X$, we denote by $\text{Lip}^\lambda(M, X)$ the set of $\lambda$-Lipschitz maps from $M$ into $X$. We assume that $M$ is separable and $X$ is complete. Fix a dense sequence $(x_n)_n$ in $M$ and define a metric $d$ on $\text{Lip}^\lambda(M, X)$ by

$$d(f, g) = \sum_{n=1}^{\infty} \min\{d(f(x_n), g(x_n)), 2^{-n}\}.$$  

Then, $d$ is a complete metric on $\text{Lip}^\lambda(M, X)$ whose topology coincides with the pointwise convergence topology.

Let $Z$ be a Banach subspace of $Y$ and denote the quotient map by $Q: Y \to Y/Z$. We say $Z$ is an M-ideal with an approximate unit, or an M-iwau in short, if there are nets of operators $\phi_i: Y \to Z$ and $\psi_i: Y \to Y$ such that $\phi_i(z) \to z$ for every $z \in Z$, $Q \circ \psi_i = Q$ for all $i$, $\phi_i + \psi_i \to \text{id}_Y$ pointwise, and $\|\phi_i(x) + \psi_i(y)\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in Y$ and $i$. We note that $\psi_i \to 0$ on $Z$ and $\|Q(y)\| = \lim \|\psi_i(y)\|$.

**Example A.** Let $X$ be a separable Banach space and $X_n$ be an increasing sequence of finite-dimensional subspaces whose union is dense. Then, we define

$$Y = \{(x_n)_n \in (\prod_n X_n)_\infty \; ; \; \text{the sequence } (x_n)_n \text{ is convergent in } X\}.$$  

Then, $Y$ is a Banach space with the MAP with the metric surjection $Q: Y \to X$ given by the limit. The subspace $\ker Q$ is an M-iwau, with $\phi_k((x_n)_n) = (x_1, \ldots, x_k, 0, 0, \ldots)$.

**Example B.** Every closed two-sided ideal $I$ in a $C^*$-algebra is an M-iwau.
Lemma 1 (cf. [Ar], Theorem 6). Let $Z \subset Y$ be an $M$-iwau and $M$ be a separable metric space. Then, for every $\lambda \geq 1$ the set
\[ \{ Q \circ f : f \in \text{Lip}^\lambda(M,Y) \} \subset \text{Lip}^\lambda(M,Y/Z) \]
is closed under the pointwise convergence topology.

Proof. Let $(f_n)_n$ be a sequence in $\text{Lip}^\lambda(M,Y)$ such that $Q \circ f_n$ converge to $F \in \text{Lip}^\lambda(M,Y/Z)$.
To prove that $F$ lifts, we may assume that $d(Q \circ f_n, Q \circ f_{n+1}) < 2^{-n}$. We will recursively construct $g_n$ such that $Q \circ g_n = Q \circ f_n$ and $d(g_n, g_{n+1}) < 2^{-n}$. Then, the sequence $(g_n)_n$ converges and its limit is a lift of $F$. For $g_{n+1}$, we define
\[ g_{n+1,i} = \phi_i \circ g_n + \psi_i \circ f_{n+1}. \]
Then, $g_{n+1,i} \in \text{Lip}^\lambda(M,Y)$, $Q \circ g_{n+1,i} = Q \circ f_{n+1}$ and
\[ \lim_i d(g_n, g_{n+1,i}) = \lim_i d(\psi_i \circ g_n, \psi_i \circ f_{n+1}) = d(Q \circ g_n, Q \circ f_{n+1}) < 2^{-n}. \]
Thus, there is $i$ such that $g_{n+1} := g_{n+1,i}$ works. \qed

Theorem 2. Let $M$ be a separable metric space and $\lambda \geq 1$. Then, the free space $\mathfrak{F}(M)$ has the $\lambda$-BAP if and only if $M$ has the following property: For any Banach space $Y$ and any $f \in \text{Lip}^1(M,Y^{**})$, there is a net in $\text{Lip}^\lambda(M,Y)$ which converges to $f$ in the pointwise-weak* topology.

Proof. Suppose $\mathfrak{F}(M)$ has the $\lambda$-BAP, and $f \in \text{Lip}^1(M,Y^{**})$ is given. Then, $f$ extends to a linear contraction $\hat{f} : \mathfrak{F}(M) \to Y^{**}$. Since $\mathfrak{F}(M)$ has the $\lambda$-BAP, the local reflexivity principle yields a net of operators $T_i : \mathfrak{F}(M) \to Y$ with norm $\leq \lambda$ which weak* converges to $\hat{f}$ pointwise. Restricting it to $M$, we obtain a desired net.

Conversely, suppose $M$ satisfies the property stated in Theorem 2. We apply the construction described in Example A to $\mathfrak{F}(M)$ and obtain $Q : Y \to \mathfrak{F}(M)$. Since $Z = \ker Q$ is an $M$-ideal, one has a canonical identification $Y^{**} = Z^{**} \oplus_\infty \mathfrak{F}(M)^{**}$. In particular, $M \hookrightarrow Y^{**}$ naturally. By assumption, there is a net $f_i \in \text{Lip}^\lambda(M,Y)$ which approximates the above inclusion. Since $Q \circ f_i \in \text{Lip}^\lambda(M,\mathfrak{F}(M))$ converge to $id_M$ in the point-weak topology, by taking convex combinations if necessary, we may assume that they converge in the point-norm topology. Thus by Lemma 1, $id_M : M \hookrightarrow \mathfrak{F}(M)$ lifts to a function $f \in \text{Lip}^\lambda(M,Y)$. The function $f$ extends to $\hat{f} : \mathfrak{F}(M) \to Y$, which is a lift of $id_{\mathfrak{F}(M)}$. Since $Y$ has the MAP, $\mathfrak{F}(M)$ has the $\lambda$-BAP. \qed

Corollary 3. A separable Banach space $X$ has the $\lambda$-BAP if and only if for any Banach space $Y$ and any $f \in \text{Lip}^1(X,Y^{**})$, there is a net in $\text{Lip}^\lambda(X,Y)$ which converges to $f$ in the pointwise-weak* topology.

This follows immediately from Theorem 2 since $X$ has $\lambda$-BAP if and only if $\mathfrak{F}(X)$ has this same property ([GK], Theorem 5.3). Note that we can replace “Lipschitz maps” by “linear operators” in Corollary 3 and reach the same conclusion. In this case, our argument boils down to a method due to Ando ([An], see [HWW], section II.2).
We first prove:

**Theorem 4.** Let \(X\) be a separable Banach space, and let \(C\) be a closed convex set containing 0 such that \(\text{span}[C] = X\). Then \(X\) is isometric to a 1-complemented subspace of \(\mathfrak{F}(C)\).

**Proof.** The proof relies on a modification from [DL] (see Lemma 2.1 in that paper) of the proof of ([GK], Theorem 3.1). We first recall that since every real-valued Lipschitz map on \(C\) extends a Lipschitz map on \(X\) with the same Lipschitz constant, the canonical injection from \(C\) into \(X\) extends to an isometric injection from \(\mathfrak{F}(C)\) into \(\mathfrak{F}(X)\) (see [GK], Lemma 2.3). Thus we simply consider \(\mathfrak{F}(C)\) as a subspace of \(\mathfrak{F}(X)\).

Let \((x_i)_{i \geq 1}\) be a linearly independent sequence of vectors in \(C/2\) such that \(\text{span}\{x_i ; i \geq 1\} = X\) and \(\|x_i\| = 2^{-i}\) for all \(i\). We let \(E = \text{span}\{x_i ; i \geq 1\}\). We denote by \(H = [0, 1]^N\) the Hilbert cube, by \(t = (t_j)_j\) a generic element of \(H\), and by \(\lambda\) the product of the Lebesgue measures on each factor of \(H\). Of course, \(\lambda\) is a probability measure on \(H\). Moreover, for any \(n \in \mathbb{N}\), we denote \(H_n = [0, 1]^N \setminus \{n\}\) and \(\lambda_n\) the similar probability measure on \(H_n\).

We denote \(R : E \to \mathfrak{F}(X)\) the unique linear map which satisfies for all \(n \geq 1\) and all \(f \in \text{Lip}_0(X)\)

\[R(x_n)(f) = \int_{H_n} [f(x_n + \sum_{j \neq n} t_j x_j) - f(\sum_{j \neq n} t_j x_j)] d\lambda_n(t).\]

It is clear that the map \(R\) actually takes its values in the subspace \(\mathfrak{F}(C)\) of \(\mathfrak{F}(X)\). If \(f\) is Gâteaux-differentiable, then Fubini’s theorem shows that

\[R(x)(f) = \int_H \{\nabla f\}(\sum_j t_j x_j), x) \, d\lambda(t)\]

and thus \(|R(x)(f)| \leq \|x\| \|f\|_L\). Since the subset of the unit ball of \(\text{Lip}_0(X)\) consisting of functions which are Gâteaux-differentiable is uniformly dense in this unit ball (see [BL], Corollary 6.43), it follows that \(\|R\| \leq 1\). Since \(E\) is dense in \(X\), the map \(R\) extends to a linear operator of norm 1 from \(X\) to \(\mathfrak{F}(C)\), which we still denote by \(R\).

If \(\beta\) denotes the canonical quotient map from \(\mathfrak{F}(X)\) onto \(X\) (see [GK], Lemma 2.4), we have \(\beta R = Id_X\) and thus \(R(X)\) is a subspace of \(\mathfrak{F}(C)\) isometric to \(X\) and 1-complemented by the projection \(R\beta\). \(\square\)

The main corollary of this result is the following

**Corollary 5.** There exists a compact metric space \(K\) such that \(\mathfrak{F}(K)\) fails AP.

**Proof.** Let \(X\) be a separable Banach space failing the AP. It is classical and easily seen that there is a compact convex set \(K\) containing 0 such that \(\text{span}[K] = X\). By Theorem 3 the space \(\mathfrak{F}(K)\) contains a complemented subspace failing AP and thus \(\mathfrak{F}(K)\) itself fails AP. \(\square\)

**Example C.** This result emphasizes the need to decide for which metric spaces \(M\) - and in particular for which compact metric spaces - the corresponding free space has the AP.
It is well-known that MAP holds when $K$ is an interval of the real line since then $\mathfrak{F}(K)$ is isometric to $L^1$, and more generally if $M$ is any subset of the real line since then $\mathfrak{F}(M)$ is 1-complemented in $L^1$. If $C$ is a closed convex subset of the Hilbert space $\ell_2$, then $\mathfrak{F}(C)$ has MAP. Indeed $C$ is a 1-Lipschitz retract of $\ell_2$ and thus $\mathfrak{F}(C)$ is 1-complemented in $\mathfrak{F}(\ell_2)$ which has MAP by (GK, Theorem 5.3). A metric space $M$ is isometric to a subset of a metric tree $T$ if and only if $\mathfrak{F}(M)$ embeds isometrically into $L^1$ (GK). It follows from (Mat) that for any such $M$ the space $\mathfrak{F}(M)$ has BAP. Finally, it is shown in [LP] among other things that for any $n \geq 1$ the space $\mathfrak{F}(\mathbb{R}^n)$ has a basis, and that $\mathfrak{F}(M)$ has (BAP) for any doubling metric space $M$.

We now observe that “small” Cantor sets yield to free spaces with MAP.

**Proposition 6.** Let $K$ be a compact metric space such that there exist a sequence $(\epsilon_n)_n$ tending to 0, a real number $\rho < 1/2$ and finite $\epsilon_n$-separated subsets $N_n$ of $K$ which are $\rho \epsilon_n$-dense in $K$, then $\mathfrak{F}(K)$ has MAP.

**Proof.** It follows from ([Bo], Theorem 4) that if $M$ is a separable metric space and $(M_n)_n$ is an increasing sequence of finite subsets of $M$ whose union is dense in $M$, then $\mathfrak{F}(M)$ has BAP if and only if there is a uniformly bounded sequence of linear operators $E_n$ : Lip$(M_n) \to$ Lip$(M)$ such that if $R_n$ denotes the restriction operator to $M_n$, then for every $f \in$ Lip$(M)$ the sequence $f_n = E_n R_n(f)$ converges pointwise to $f$. Our assumptions imply the existence of $\lambda$-Lipschitz retractions $P_n$ from $K$ onto $N_n$, with $\lambda = (1 - 2 \rho)^{-1}$, and then $E_n(f) = f \circ P_n$ shows that $\mathfrak{F}(K)$ has BAP. To conclude the proof, we observe that in the notation of ([We], Definition 3.2.1), the little Lipschitz space lip$_0(K)$ uniformly separates the points in $K$ (use the characteristic functions of the balls of radius $\rho \epsilon_n$ centered at points in $N_n$), and thus by ([We], Theorem 3.3.3) the space $\mathfrak{F}(K)$ is isometric to the dual space of lip$_0(K)$. Now Grothendieck’s theorem shows that $\mathfrak{F}(K)$ has MAP since it is a separable dual with BAP.

We refer to [Ka1] for more on little Lipschitz spaces and the “snowflaking” operation. On the other hand, Corollary 5 provides a negative result. It should be noted that the existence of finite nested metric spaces with no “good” extension operator for Lipschitz functions is known (see Lemma 10.5 in [BB]). This is obtained below from Corollary 5 by abstract nonsense. Conversely, it would be interesting to exhibit spaces failing (AP) from combinatorial considerations on finite metric spaces.

**Proposition 7.** For any $\lambda \geq 1$, there exist a finite metric space $H_\lambda$ and a subset $G_\lambda$ of $H_\lambda$ such that if $E$ : Lip$(G_\lambda) \to$ Lip$(H_\lambda)$ is a linear operator such that $RE = id_{\text{Lip}(G_\lambda)}$ (where $R$ is the operator of restriction to $G_\lambda$) then $\|E\| \geq \lambda$.

**Proof.** Let $K$ be a compact metric space such that $\mathfrak{F}(K)$ fails AP and let $(G_n)_n$ be an increasing sequence of finite subsets of $K$ whose union is dense in $K$. Assume that Proposition 7 fails for some $\lambda_0 \in \mathbb{R}$, and thus that extension operators with norm bounded by $\lambda_0$ exist for all pairs $(G, H)$ of finite metric spaces with $G \subset H$. For any given $n$, we can apply this to $(G_n, G_k)$ with $k \geq n$ and use a diagonal argument to get an operator
$E_n : \text{Lip}(G_n) \to \text{Lip}(K)$ with $R_n E_n = \text{id}_{\text{Lip}(G_n)}$ (where $R_n$ is the operator of restriction to $G_n$) and $\|E_n\| \leq \lambda_0$. The operator $E_n$ is conjugate to a projection from $\mathcal{F}(K)$ onto $\mathcal{F}(G_n)$, and it follows that $\mathcal{F}(K)$ has $\lambda_0$-BAP (with a sequence of projections), contradicting our assumption on $K$. \hfill \square

Our work leads to a number of natural questions. We conclude this note by stating some of them. The first one is due to N. J. Kalton (see [Ka3], Problem 1):

**Question 1.** Let $M$ be an arbitrary uniformly discrete metric space, that is, there exists $\theta > 0$ such that $d(x, y) \geq \theta$ for all $x \neq y$ in $M$. Does $\mathcal{F}(M)$ have the BAP? Note that AP holds by ([Ka1], Proposition 4.4). Proposition 7 shows that a simple step-by-step approach could not suffice. A positive answer to Question 1 would imply that every separable Banach space $X$ is approximable, that is, the identity $\text{id}_X$ is pointwise limit of an equi-uniformly continuous sequence of maps with relatively compact range. Note that by ([Ka3], Theorem 4.6) it is indeed so for $X$ and $X^*$ when $X^*$ is separable. On the other hand, a negative answer to Question 1 would provide an equivalent norm on $\ell_1$ failing MAP and this would solve two classical problems in approximation theory ([Ca], Problems 3.12 and 3.8).

**Question 2.** Is there a countable compact space $K$ such that $\mathcal{F}(K)$ fails (AP)?

**Question 3.** Let $X$ be a separable Banach space. Does there exist a compact convex subset $K$ of $X$ containing 0 such that $\text{span}[K] = X$ and moreover $K$ is a Lipschitz retract of $X$? Note that when it is so, $X$ has BAP if and only if $\mathcal{F}(K)$ has BAP. The answer to this question is positive when $X$ has an unconditional basis: indeed if all the coordinates of $x \in X$ are strictly positive, the order interval $[-x, x] = K$ works since truncation by $x$ shows that $K$ is a Lipschitz retract.

**Question 4.** According to [GK], Definition 5.2., a separable Banach space $X$ has the $\lambda$-Lipschitz BAP if $\text{id}_X$ is the pointwise limit of a sequence $F_n$ of $\lambda$-Lipschitz maps with finite-dimensional range, and this property is shown in [GK] to be equivalent with the usual $\lambda$-BAP. Is it possible to dispense with the assumption that the $F_n$’s have finite-dimensional range and still reach the conclusion? Corollary 5 suggests that this improvement should not be straightforward.

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Institut de Mathématiques de Jussieu, 4 Place Jussieu 75005 Paris
E-mail address: goddefroy@math.jussieu.fr

RIMS, Kyoto University, 606-8502 Kyoto
E-mail address: narutaka@kurims.kyoto-u.ac.jp