Order Selection Problems in Hiring Pipelines

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Abstract. Motivated by hiring pipelines, we study two order selection problems in which applicants for a finite set of positions must be interviewed or made offers sequentially. There is a finite time budget for interviewing or making offers, and a stochastic realization after each decision, leading to computationally-challenging problems. In the first problem we study sequential interviewing, and show that a computationally-tractable, non-adaptive policy that must make offers immediately after interviewing is approximately optimal, assuming offerees always accept their offers. In the second problem, we assume that applicants have already been interviewed but only accept offers with some probability; we develop a computationally-tractable policy that makes offers for the different positions in parallel, which is approximately optimal even relative to a policy that does not need to make parallel offers.

Our two results both generalize and improve the guarantees in the work of Purohit et al. [18] on hiring algorithms, from $1/2$ and $1/4$ to approximation factors that are at least $1 - 1/e \approx 63.2\%$.

Keywords: hiring, order selection, stochastic probing, adaptivity gap

1 Introduction

Good recruiting policies are central to the success of any enterprise, yet carrying out efficient and timely recruiting processes can prove to be challenging. Several difficulties arise when hiring personnel, such as deciding when to carry out the process, dealing with imperfect information, or the operation of the process at the time it is carried out, just to name a few. This last aspect raises several questions that can be studied through an algorithmic lens.

In a typical recruiting process, applicants submit their resumes to a firm. Information about applicants is imperfect since the firm only has access to their resumes and possibly some complementary material. Once a pool of applicants is formed, the firm starts to carry out interviews with promising applicants, in order to get a better picture of the applicant’s potential value. Once all interviews are conducted, the firm must decide which candidates to make offers to. Once all offers are responded to, the recruiting process ends. Several operational decisions are to be made during this process. The first one is: out of the pool of applicants, who are to be interviewed? This is a relevant question since the pool of candidates can be very large in comparison to the number of positions to be filled by the firm. Moreover, the recruiting process cannot take an arbitrary amount of time, so only a limited amount of interviews are feasible to carry out. Another relevant decision to be made before—or during—the process is: in which order are the interviews going to be carried out? This is important when considering that only a limited amount of interviews can be conducted. If all interviews carried out have been unsuccessful (i.e. promising applicants turned out to be disappointing) then it would be wise to carry on interviewing ‘safe’ applicants: those with a high chance of being satisfactory. On the other hand, if interviews have been successful so far, then it is worth spending a couple of interviews on ‘long shots’: applicants that are risky but have a small probability of being exceptional. Finally, after all interviews have been carried out, a third natural operational question that arises, similar to the previous one, is: in which order are the offers going to be sent to the desirable candidates? Similar to the order of the interviews, if enough offers have been accepted, the firm could opt to send an offer to an exceptional candidate with a low probability of acceptance. On the other hand, if many offers have been turned down and the deadline of the recruitment process approaches, it could be more desirable to go for a safe bet and send an offer to a satisfactory candidate who is likely to accept the offer. Even when dealing with simple and minimalist models, finding the optimal answer to these questions can be computationally intractable. We are also interested in understanding the power of policies that can dynamically decide the order of interviews, defer making offers, avoid parallel interviewing processes, etc., compared to those that cannot.
To answer these questions, we study two models of hiring processes that build off existing work.

**First model: sequential interviewing, a.k.a. ProbeTop-k.** In the first model, we assume that the firm has to hire up to $k$ people from a pool of $n$ applicants. Each applicant has a random value unknown to the firm carrying out the hiring process. The firm has access to distributional knowledge of these values coming from the applicants’ resumes and complementary material submitted. The firm can interview applicants to find out the realization of their values; there is a limit of $T$ on the number of interviews conducted. The realization of the value of the candidate becomes known to the firm immediately after carrying out the interview. We note that this realization should be interpreted as the applicant’s *expected* value to the firm given the interview (and conditional on them accepting the offer); we treat this value as deterministic, which does not lose generality for a risk-neutral firm. We further assume that realizations from interviews are independent across applicants, which is also justifiable in practice. After all interviews are carried out, the firm can choose the best $k$ interviewed candidates to be hired, who are assumed to accept their offers with probability 1. The goal of the firm is to maximize the expected sum of values of the hired personnel.

This problem is exactly the ProbeTop-$k$ problem, as described in Fu et al. [9], which is computationally challenging because the state space is exponential in the number of applicants (one has to track the subset that has been interviewed and their values).

**Classes of policies for the first model.** In this problem we can distinguish between different classes of policies. First, we distinguish between *adaptive* and *non-adaptive* policies. Adaptive policies can decide the order of the interviews on the go, choosing who to interview next depending on the outcomes of previous interviews. Non-adaptive policies, in contrast, have to choose a fixed order before the process starts and stick with it. Using non-adaptive policies is attractive both from a practical and computational perspective. We also distinguish between *committed* and *non-committed* policies. Committed policies have to, after each interview, make an irrevocable decision of whether to hire a candidate or not (given her value). Non-committed policies, in contrast, can carry out all interviews and choose the best $k$ candidates to be hired in hindsight. Using committed policies could be attractive from a practical point of view, as waiting for all interviews to be conducted allows for candidates to accept offers from competing firms in the meantime. We are interested in bounds on how costly it is to restrict the firm to use policies that are non-adaptive and committed.

**Relation to sequential offering model of [13].** Our sequential interviewing model is closely related to the sequential offering model of Purolhit et al. [13]. In this problem, applicants are assumed to have already been interviewed but are not guaranteed to accept an offer. The firm knows, for each candidate, how likely it is that she will accept an offer, and the value she adds to the firm should they accept. The firm has time to send at most $T$ offers and wants to maximize the expected total value of up to $k$ candidates who accept their offers. We show that this setting can be captured by the ProbeTop-$k$ problem with weighted Bernoulli distributions, where the values take a positive realization with some probability (representing an acceptance) or 0 with the remaining probability (representing a rejection). The subtlety is that an accepted offer cannot be turned down by the firm in the sequential offering model, in contrast to the ProbeTop-$k$ problem, where we could choose to not hire a candidate even if their value turns out to be positive. We will show that our algorithm satisfies properties that make it admissible for the sequential offering problem too.

**Relation to free-order prophets.** The free-order prophet problem with $k$-units is the special case of the ProbeTop-$k$ problem when only committed policies are allowed and $T = n$. Note that when $T = n$, the constraint of $T$ interviews is not binding, and hence only committed policies are interesting to study (non-committed policies can interview all $n$ candidates and the order does not matter). Typically in prophet inequalities, one compares to a benchmark who can see all candidates’ values and does not need to commit (i.e. they can hire the $k$ highest-valued candidates in hindsight). We note that such a benchmark is too powerful to compare against in our more general problem if $n$ is much larger than $T$, since the benchmark sees all $n$ realizations while the algorithm can only interview $T$ applicants. Therefore, we instead compare to an optimal (adaptive, non-committed) algorithm that is still bound by $T$ interviews that must be decided without any prophetetic information, which makes our comparison different from prophet inequalities. Our algorithm for ProbeTop-$k$, being non-committed, will however imply an algorithm for free-order prophets that has the same guarantee relative to the prophet benchmark in the special case where $T = n$.

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1 The case where $k = 1$ is called the ProbeMax problem.
2 The latter is quantified in the literature by the widely studied notion of adaptivity gap: the worst-case ratio between the performance of general policies vs. algorithms that are restricted to be non-adaptive. Our results will bound the “adaptivity-commitment gap”, in which the algorithm is restricted to be both non-adaptive and committed.
**Second model: parallel offering.** The second model we study is the parallel offering problem, an extension of the sequential offering problem studied by Purohit et al. [18]. In this model, we again assume that the firm has to hire people in order to fill \( k \) job positions. However, we now allow for heterogeneous positions, where a candidate may have different potential values for different job positions. We assume that all interviews have already been conducted, leaving us with a pool of \( n \) desirable candidates. After conducting all interviews the firm learned, for each candidate, how valuable they are for each of the available positions and how likely it is for each candidate to accept an offer for each of the available positions. The firm must now decide how to send offers in \( T \) parallel offering rounds. At each round, the firm can send an offer for each of the positions that have not yet been filled by a candidate. When a candidate receives an offer, she can either accept or reject it, with the assumption that they will not receive another offer if they reject (and hence candidates do not try to anticipate offers they might receive later). The goal of the firm is to maximize the expected sum of values of hired candidates. We develop a non-adaptive algorithm that can be computed efficiently and performs competitively in comparison to adaptive and even sequential algorithms (we formalize these concepts in the sequel).

**Relation to parallel offering model of [18].** This model generalizes a parallel offering model also introduced in Purohit et al. [18], which is similar but has \( k \) identical instead of heterogeneous positions, justifying the assumption that each candidate can only be offered once. Our parallel offering model is not only more general, but we also derive stronger results.

Table 1 contains a comparison of the previously mentioned models.

| Action performed          | ProbeTop-k | Free-order Prophets | Sequential Offering | Parallel Offering |
|---------------------------|------------|---------------------|---------------------|-------------------|
| Action per time step      | Interview  | Interview           | Send offer          | Send offer        |
| Result of action          | Observe value of candidate | Observe value of candidate | Observe accept/reject decision | Observe accept/reject decision(s) |
| Moment of hiring          | After last interview | After each interview (irrevocable) | Upon acceptance of offer | Upon acceptance of offer |
| Type of positions         | Identical  | Identical           | Identical           | Heterogeneous     |

Table 1. Comparison between models. Each model comes with with \( n \) applicants/candidates, \( k \) positions to fill, and \( T \) time steps to perform actions. The Free-order prophets is additionally constrained to have \( T = n \) time steps to perform actions.

### 1.1 Our Results and Proof Techniques

For \( \alpha \leq 1 \), we say that an algorithm is \( \alpha \)-approximate if its expected total value collected is at least \( \alpha \) times that of an optimal algorithm. \( \alpha \) is referred to as the approximation factor.

**Sequential interviewing a.k.a. ProbeTop-k problem.** For the ProbeTop-k problem, we develop a polynomial-time, non-adaptive, and committed algorithm that achieves a \( (1 - e^{-k^2/k!}) \) approximation factor relative to an optimal (adaptive, non-committed) algorithm (Theorem 1, Section 3). We note that the approximation factor of \( (1 - e^{-k^2/k!}) \) is always at least \( 1 - 1/e \approx 0.632 \) (when \( k = 1 \), and increases to \( 100\% \) as \( k \to \infty \)). We also note that it is tight relative to the LP (Section 3.5). Our results are significant for the following reasons. First, the previously best-known approximation factor was \( 1/2 \), obtained by both [18] and [9], so our lower bound is not only stronger for all values of \( k \), but demonstrates asymptotic optimality in \( k \). Second, the 1/2-approximate algorithm in [9] is not polynomial-time and not committed, while the polynomial-time 1/2-approximate algorithm in [18] needs to be adaptive, only compares against policies that are committed, and also focuses on the special case of sequential offering (we elaborate on this in Section 3.4). Finally, our abstract treatment leads to an algorithm that works in the free-order prophets setting (because it is committed), hence our result extends to a free-order prophet inequality that allows an additional constraint of \( T \) on the number of interviews.

Our algorithm and analysis work as follows. First, our algorithm solves a natural linear programming relaxation introduced by Gupta and Nagarajan [11] which we denote by LP$_{ptk}$ (Section 5.1). We exploit a
property about the basic feasible solutions of this linear program to develop a simple dependent rounding scheme that allows us to randomize between at most two subsets of applicants to interview (Lemma 2, Section 3.1, Section 3.2). Once the subset is selected, we again use the linear programming solution to determine a fixed order in which to conduct the interviews, together with (possibly randomized) thresholds for hiring freshly interviewed candidates on the spot. We use the commitment property of the algorithm to show that the expected reward can be lower bounded by the expectation of a weighted rank function of a uniform matroid, where the expectation is taken over a distribution that includes each element independently (Lemma 3, Section 3.3). We further show that the objective function of LP_{par} can also be expressed as the expectation of the same weighted rank function, but where the inclusion of elements can be correlated. We use the correlation gap results by Yan [21] to conclude our approximation result for our problem.

**Parallel offering problem.** For the parallel offering model, we develop an algorithm that is non-adaptive and achieves a \((1 - 1/e)\) approximation factor relative to an optimal (adaptive) algorithm (Theorem 2, Section 4). This result also both improves and generalizes existing results, in this case, the 1/4-approximate algorithm of Purohit et al. [18] that works in the special case where all positions are identical. It is also tight relative to the LP (Section 4).

Our algorithm again solves a linear programming relaxation which we refer to as LP_{par} (Section 4.1). The algorithm then rounds the solution using the dependent rounding scheme by Gandhi et al. [10]. The resulting rounded solution will help us form one separate candidate list for each position. These lists will satisfy: (a) each candidate is included in at most one list, and (b) each list will include at most \(T\) candidates. Our algorithm then runs each of these lists in parallel, where candidates on each list are ordered by descending value for that position. For each position, we show that the value captured by the corresponding list is not less than the expected maximum value of a random subset of candidates, where each candidate is included in the set independently (Lemma 7, Section 4.2). We further decompose the objective value of the linear program by positions and show that each of these components can be expressed as the maximum value of a random subset of candidates, where the inclusion of candidates in the random subset is negatively associated. We again show that the performance in this negatively associated setting is no worse than the performance in an independent setting, although we emphasize that here this fact is only true when \(k = 1\) (see Example 1 in Section 4.2 for a counterexample when \(k = 2\)). We then apply the correlation gap result by Yan [21] to bound, position by position, that the algorithm obtains at least \(1 - 1/e\) times the corresponding term in the objective function.

We further show that our LP_{par} is equivalent to its sequential offering counterpart LP_{seq}, when we allow for \(kT\) time steps instead of \(T\). This implies that our algorithm obtains at least \((1 - 1/e)\) times what an optimal algorithm would obtain in the sequential offering problem with \(kT\) time steps, establishing a lower bound on the value of a batched solution (Corollary 2, Section 4.3). This quantifies how costly it can be to send offers in batches instead of sequentially, which loses efficiency due to the lack of ability to respond individually, but has the logistical benefit of requiring only \(T\) parallel time steps instead of \(kT\) sequential time steps.

### 1.2 Further Related Work

**Stochastic probing and matching.** Our work is closely related to the general stochastic probing problem studied in Gupta and Nagarajan [11] and Gupta et al. [12]. These papers study the problem of sequentially probing elements in order to maximize the sum of the weights of a selected subset. In their setting they consider more general sets of ‘outer’ constraints to be satisfied by the probed elements, and ‘inner’ constraints to be satisfied by the selected elements. In this language, the ProbeTop-k considers the outer constraint to be the \(T\)-uniform matroid and the inner constraint to be the \(k\)-uniform matroid. Their work is different from ours in that it only allows the probe to have two outcomes (active or inactive) and that active probes must be irrevocably included in the final subset. Bradac et al. [6] introduce the multiple-type general stochastic probing problem. In this work, they only work with outer constraints and include the inner constraints by allowing submodular functions instead of modular functions.

In Bradac et al. [6] they show that the adaptivity gap is exactly \(1/2\) for the stochastic probing problem with monotone submodular functions under prefix-closed probing constraints. Their proof is not constructive.

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This is defined as the worst-case ratio between the performance of adaptive and non-adaptive policies.
in the sense that the algorithm for their lower bound requires the optimal decision tree as an input. The best-known non-adaptive algorithm that can be computed in polynomial time is due by Gupta et al. \cite{13}, which achieves a 1/3 approximation for submodular and XOS functions under prefix-closed probing constraints.

We note that these stochastic probing problems were heavily inspired by the stochastic matching problem with patience constraints, originally studied in Chen and Immorlica \cite{7} and Bansal et al. \cite{4}. Our parallel offering problem has the flavor of a stochastic matching problem, although it is heavily constrained and simpler.

**PTAS-type results.** Fu et al. \cite{9} develop several PTASs for a class of dynamic programs that includes the ProbeTop-k problem. They propose an adaptive PTAS for ProbeTop-k and another one for the committed version of the problem. Segev and Singla \cite{20} develop EPTASs for several related problems, also including the non-adaptive version of the ProbeTop-k problem. Their EPTAS works differently when \(k\) is small or large. They actually use the same linear programming relaxation for the large \(k\) case, although their rounding scheme and analysis are substantially different. We note that none of these PTAS-type results use non-adaptive, committed policies to approximate the best adaptive, non-committed policies, as in our paper.

**Prophet inequalities.** Our work has the flavor of prophet inequalities in that we are trying to decide online whether to accept values drawn from known distributions, and comparing against a supernatural benchmark. Classical works in prophet inequalities \cite{16, 17, 19, 15, 8} assume that the order is beyond our control, while we are studying the free-order variant where the order can be decided \cite{14, 15, 2}. However, due to the constraint of \(T\) time steps, as discussed earlier, it is important that the benchmark we are comparing against is also bound by \(T\) time steps, so in essence we are not comparing against (and it is impossible to compare against) the true prophet. For sequential interviewing, the approximation factor we derive of \((1 - e^{-k}k^k/k!))\) has also been established \cite{21} for free-order and extended in \cite{3} to random-order prophets, but neither of these works allows for a constraint of \(T\) time steps.

## 2 Problem Formulations and Preliminaries

In this section, we formally state the problems studied throughout the paper. We first state the ProbeTop-k problem in Section 2.1. We then state the Parallel Offering problem in Section 2.2. We close with Section 2.3, where we mention some results in the literature that will be useful for both constructing our algorithms and carrying out our analyses.

### 2.1 ProbeTop-k Problem

Consider the ProbeTop-k (ptk) problem. In this problem a firm faces the challenge of filling \(k\) positions out of a pool of \(n\) candidates. Each candidate \(i\) has a random, non-negative value \(V_i \sim F_i\), and \(F_i\) is known to the firm. Before the firm hires a candidate, an interview must be conducted. When the firm interviews candidate \(i\), the realization of \(V_i\) becomes known to the firm. The firm can conduct at most \(T\) interviews in total, and then can hire any \(k\) interviewed candidates to be hired. The goal of the firm is to maximize the sum of the values of the hired candidates. An instance of the problem is characterized by a tuple \(I = (k, T, n, F)\), where \(1 \leq k \leq T \leq n\), and \(F = \{F_i\}_{i \in [n]}\) is a collection of probability distribution functions. In this work we focus on distributions supported on a finite set \(\{r_1, \ldots, r_J\}\) of polynomial size and we use \(q_{ij}\) to denote \(P(V_i = r_j)\). We use \(I_{ptk}\) to denote the set of all possible instances for the ProbeTop-k problem. We further use \(I_{ptk}^k\) to denote \(\{I \in I_{ptk} : \text{amount of positions is } k\}\). Let \(P_{ptk}\) be the set of all policies for ptk. For a policy \(\pi \in P_{ptk}\) and an instance \(I \in I_{ptk}\) let \(R_{\pi}(I)\) be the expected reward of using policy \(\pi\) on the instance in question. For any instance \(I \in I_{ptk}\) define \(OPT_{ptk}(I) := \sup_{\pi \in P_{ptk}} R_{\pi}(I)\) as the expected reward of using the best possible policy on instance \(I\). We say that a policy \(\pi \in P_{ptk}\) is an \(\alpha\)-approximation if

\[
\inf_{I \in I_{ptk}} \frac{R_{\pi}(I)}{OPT_{ptk}(I)} \geq \alpha.
\]

\footnote{The committed version is where only committed policies are allowed. Adaptive policies are still allowed.}

\footnote{In this paper they call it the Top-\(r\) ProbeMax problem.}

\footnote{We use \([n]\) to denote \(\{1, \ldots, n\}\).}
We distinguish between adaptive and non-adaptive policies. A non-adaptive policy has to decide an order in which to conduct the interviews before the process starts. An adaptive policy, in contrast, conducts the interviews sequentially and can use the outcomes of previous interviews to decide which candidate to interview next. We also distinguish between committed and non-committed policies. After each interview, committed policies must irrevocably decide whether to hire the candidate or not. Non-committed policies, in contrast, can interview \( T \) candidates and then choose the \( k \) highest realizations among them. We are interested in how well can the firm perform when restricted to using non-adaptive policies, committed policies, or both. To quantify this, we are interested in how good an approximation factor can be achieved by restricting the firm to using policies that are both non-adaptive and committed.

### 2.2 Parallel Offering Problem

The second setting we study in this paper is the Parallel Offering problem (par). Again consider the case of a firm that has to hire candidates in order to fill \( k \) positions. Instead of conducting interviews, the firm has to send offers to candidates. The positions of the firm are now not identical, and for each candidate \( i \in [n] \) and position \( j \in [k] \) the firm knows \( p_{ij} \); the probability that the candidate accepts an offer for the position, and \( v_{ij} \): the reward collected by the firm if the candidate accepts an offer for the position. The firm can carry out at most \( T \) offering rounds. At each offering round the firm can send an offer for each position that is yet to be occupied by a candidate (hence the name parallel offering). Each candidate can receive at most one offer in total. The goal of the firm is to maximize the expectation of the sum of the values of the accepted offers. An instance for this problem is defined by a tuple \( I = (k, T, n, p, v) \), where \( 1 \leq k \leq T \leq n \), \( p \in [0,1]^{n \times k} \) and \( v \in \mathbb{R}^{n \times k} \). Let \( I_{\text{par}} \) denote the set of all instances and \( \Pi_{\text{par}} \) denote the set of all policies for the parallel offering model. Again let \( R_{\pi}(I) \) denote the expected reward of using policy \( \pi \in \Pi_{\text{par}} \) on instance \( I \in I_{\text{par}} \). Let \( \text{OPT}_{\text{par}}(I) := \sup_{\pi \in \Pi_{\text{par}}} R_{\pi}(I) \) be the best possible expected reward that can be obtained from an instance \( I \in I_{\text{par}} \). We say that a policy \( \pi \in \Pi_{\text{par}} \) is an \( \alpha \)-approximation if

\[
\inf_{I \in I_{\text{par}}} \frac{R_{\pi}(I)}{\text{OPT}_{\text{par}}(I)} \geq \alpha.
\]

In this model, we assume that before the offering rounds start, each candidate has already decided which offers she would accept. We allow for these decisions to be correlated across different positions for a single candidate, but we assume independence across candidates. In this context, it is not clear how to define non-adaptive policies. A natural class of policies that could be considered non-adaptive are the ones that construct one list with a fixed order for each position and runs the lists in parallel. The algorithm we develop for this problem falls in this class of non-adaptive policies, establishing a lower bound on how well these algorithms can perform.

### 2.3 Useful Known Results

We conclude this section by enlisting some concepts and known results that will be used in our algorithms and analyses. We start by defining weighted rank functions for \( k \)-uniform matroids, which have a close connection to both the expected rewards of our algorithms and our linear programming benchmarks. We use correlation gaps for this family of functions to relate these two quantities. We also review the dependent rounding scheme developed by Gandhi et al. \cite{gandhi2007}, which we use both in the implementation of our algorithms and as a tool for analysis.

**Weighted rank functions and Correlations Gaps.** Let \( \{w_i\}_{i \in [n]} \) be a set of weights and \( D \subset [n] \.

Define the weighted rank function for the \( k \)-uniform matroid \( w^* : 2^{|D|} \to [0,\infty) \) as

\[
w^*(D) := \max_{R \subset D, |R| \leq k} \sum_{i \in R} w_i.
\]

Loosely speaking, the reward obtained by our algorithms can be expressed as weighted rank functions of \( k \)-uniform matroids. Here, \( D \) is to be interpreted as the set of candidates eligible for hire, the weights are to

\footnote{Our notation \( D \subset [n] \) allows \( D = [n] \).}
be interpreted as the expected reward collected from each candidate conditional on being eligible, and the weighted rank function simply selects the $k$ highest rewards from the set $D$.

Let $D \sim \mathcal{D}$, with $\mathcal{D}$ a distribution over $2^n$, and $q_i = \mathbb{P}(i \in D)$. Let $\tilde{D} \sim \mathcal{I}(\mathcal{D})$, where $\mathcal{I}(\mathcal{D})$ is a distribution over $2^n$ in which each $i \in D$ independently with probability $q_i$. The correlation gap of a set function $f : 2^n \to [0, \infty)$ is defined by

$$\sup_{\mathcal{D}} \frac{\mathbb{E}_{D \sim \mathcal{D}}(f(D))}{\mathbb{E}_{D \sim \mathcal{I}(\mathcal{D})}(f(D))}.$$  

This concept was first formalized by Agarwal et al. [1]. In words, it quantifies how much we can win by correlating the outcome of $\{i \in D\}$ for $i \in [n]$ while maintaining the marginal probabilities. We use a result from Yan [21] concerning the correlation gap of weighted rank functions for $k$-uniform matroids.

**Proposition 1 (Lemma 4.4 from [21]).** For any $n, k \geq 1$, the correlation gap of the weighted rank function of a $k$-uniform matroid of size $n$ is at most $(1 - \frac{\log k}{k})^{-1}$.

**GKPS dependent rounding.** Gandhi et al. [10] developed a dependent rounding scheme that we use for establishing our results. This dependent rounding scheme receives as an input a bipartite graph $(A, B, E)$ with weights $x_{ij} \in [0, 1]$ for all edges $(i, j) \in E$. The output is, for each edge, a rounded binary number $X_{ij} \in \{0, 1\}$. For any vertex $i \in A$, define the fractional degree $\ell_i = \sum_{j: (i, j) \in E} x_{ij}$. Analogously, for $j \in B$ define $\ell_j = \sum_{i: (i, j) \in E} x_{ij}$. The outputs satisfies the following three properties:

(P1) Marginal distribution: $E(X_{ij}) = x_{ij}$ for every $(i, j) \in E$,

(P2) Degree preservation: with probability 1 it holds that $\sum_{j: (i, j) \in E} X_{ij} \in \{\lfloor \ell_i \rfloor, \lceil \ell_i \rceil\}$ for all $i \in A$ and $\sum_{i: (i, j) \in E} X_{ij} \in \{\lfloor \ell_j \rfloor, \lceil \ell_j \rceil\}$ for all $j \in B$,

(P3) Negative correlation: For any vertex $i \in A \cup B$, any subset $S$ of edges incident in $i$ and any $b \in \{0, 1\}$, it holds that

$$\mathbb{P}\left(\bigcap_{e \in S} \{X_e = b\}\right) \leq \prod_{e \in S} \mathbb{P}(X_e = b).$$

For the special case when the bipartite graph is a star graph (i.e. $A = [n], |B| = 1$) the input are simply weights $x_i \in [0, 1]$ for all $i \in [n]$, and the negative correlation property can be stated as: for any subset $M \subseteq [n]$ and any $b \in \{0, 1\}$ it holds that

$$\mathbb{P}\left(\bigcap_{i \in M} \{X_i = b\}\right) \leq \prod_{i \in M} \mathbb{P}(X_i = b).$$

### 3 ProbeTop-k Problem

In this section, we provide our algorithm and analysis for the ProbeTop-k problem, where candidates are sequentially interviewed. We develop a non-adaptive, committed policy that achieves a $(1 - e^{-k/k!})$ approximation of any policy (adaptive, non-committed). The algorithm first solves an LP relaxation that upper bounds the performance of any (adaptive, non-committed) algorithm, and uses the LP solution as an input to decide the candidates that will be interviewed, in which order, and whether to hire an interviewed candidate, given her value.

In Section 3.1 we introduce the linear program in question. In Section 3.2 we introduce a simple dependent rounding scheme that will be used in our approximation algorithm. In Section 3.3 we introduce our approximation algorithm and show that it attains the mentioned approximation guarantee. In Section 3.4 we treat the special case when values are distributed as weighted Bernoulli random variables. In Section 3.5 we show that the guarantee that our algorithm attains is tight.
3.1 Linear program

Consider linear program $LP_{ptk}$, which for any instance of ProbeTop-$k$, upper bounds the performance of all possible policies. In the linear program, variable $y_i$ is to be interpreted as the probability that candidate $i$ is interviewed. Variable $x_{ij}$ is to be interpreted as the probability that candidate $i$ is hired and $V_i = r_j$.

$$LP_{ptk}(I) = \max \sum_{i=1}^{n} \sum_{j=1}^{J} r_{ij} x_{ij}$$

s.t. $x_{ij} \leq y_i q_{ij}$ \quad $\forall i \in [n], j \in [J]$ (1)

$$\sum_{i=1}^{n} y_i \leq T$$ (2)

$$\sum_{i=1}^{n} \sum_{j=1}^{J} x_{ij} \leq k$$ (3)

$$x_{ij} \geq 0 \quad \forall i \in [n], j \in [J]$$

$$0 \leq y_i \leq 1 \quad \forall i \in [n].$$

We first show that this linear program upper bounds the optimal (adaptive, non-committed) algorithm.

**Lemma 1.** For any instance $I \in \mathcal{I}_{ptk}$, $LP_{ptk}(I) \geq \text{OPT}_{ptk}(I)$.

**Proof.** Consider any algorithm and let $Y_i$ be the indicator of candidate $i$ gets and interview and $X_{ij}$ the indicator of $i$ being hired and $V_i = r_j$. Also let $Q_{ij}$ be the indicator of $V_i = r_j$. We will show that $x_{ij} = \mathbb{E}(X_{ij})$ and $y_i = \mathbb{E}(Y_i)$ is feasible, and that the objective function is equal to the expected reward of the algorithm.

In order to hire a candidate $i$ with $V_i = r_j$, we need to interview the candidate and that the candidate has value $r_j$. This translates to $X_{ij} \leq Y_i Q_{ij}$, and by taking expectation and using that $Y_i$ and $Q_{ij}$ are independent we get constraint (1). The algorithm can interview at most $k$ candidates, so $\sum_i Y_i \leq T$. Again taking expectation we get constraint (2). The algorithm can hire at most $k$ candidates, so $\sum_{i,j} X_{ij} \leq k$, by taking expectation we get constraint (3). The remaining constraints are clearly satisfied.

Finally, we have that the reward of the algorithm equals $\sum_{i,j} X_{ij} r_j$, so the expected reward of the algorithm is equal to the objective function of the linear program. \qed

The following lemma establishes a convenient fact about the basic feasible solutions of the linear program. This structure will let us develop a simple rounding scheme for selecting a subset of candidates to be interviewed.

**Lemma 2.** Let $I \in \mathcal{I}_{ptk}$ and let $y = (y_i)_{i \in [n]}, x = (x_{ij})_{i \in [n], j \in [J]}$ be a basic feasible solution of $LP_{ptk}(I)$. Then $y$ has at most 2 non-integer components. If it has 2 non-integer components, then they add up to 1.

**Proof.** The linear program has $Jn + n$ variables and $2Jn + 2n + 2$ constraints. This proof relies two observations, the first one being that constraints $y_i \geq 0$ and $y_i \leq 1$ cannot be tight simultaneously. The second observation is that constraints $x_{ij} \geq 0$ and $x_{ij} \leq y_i q_{ij}$ cannot be tight simultaneously unless $y_i = 0$. If that is the case, then we have that constraints

$$y_i \geq 0, \quad x_{ij} \geq 0, \quad x_{ij} \leq q_{ij} y_i$$

are tight, but linearly dependent. With these two observations we conclude that

(i) for each $i \in [n]$, we can only count one of $y_i \geq 0$ and $y_i \leq 1$ as a linearly independent tight constraint,

(ii) for each $(i, j) \in [n] \times [J]$, we can only count one of $x_{ij} \geq 0$ and $x_{ij} \leq q_{ij} y_i$ as a linearly independent tight constraint.

As we need $Jn + n$ linearly independent tight constraints for $x, y$ to be a basic feasible solution and we only have two other constraints, we can drop at most 2 tight constraints out of the ones listed in points (i) and (ii). This shows that at most 2 components of $y$ are different to 0 or 1.
We still need to show that if there are two fractional components, they will add up to 1. This holds because if there are two fractional components, then constraint (2) is necessarily tight. This implies that $T - 1$ components of $y$ are equal to 1, as setting $T - 2$ or less would contradict the tightness of (2) and setting $T$ would contradict the two fractional components. Combining this with the tightness of (2) gives

$$T = \sum_{i:y_i=1} y_i + \sum_{i:0<y_i<1} y_i = T - 1 + \sum_{i:0<y_i<1} y_i,$$

from which we conclude that

$$\sum_{i:0<y_i<1} y_i = 1.$$

**3.2 Dependent rounding**

We develop a simple dependent rounding scheme (which we call DR for short) that will be used as a subroutine of our approximation algorithm. DR receives an optimal solution $y = (y_i)_{i \in [n]}$ of LP$_{ptk}$ as an input, and returns a (possibly random) vector $Y \in \{0,1\}^n$. DR works differently depending on the amount of fractional components of the optimal solution $y$. In any case, for all $i$ such that $y_i \in \{0,1\}$, it sets $Y_i = y_i$ with probability 1. If $y$ is integral, then $Y$ is deterministic. If there is exactly one fractional component $i'$, then it sets $Y_{i'} = 1$ with probability $y_{i'}$ (and it sets $Y_{i'} = 0$ otherwise). If there are exactly two fractional components $i_1$ and $i_2$, then it sets $Y_{i_1} = 1, Y_{i_2} = 0$ with probability $y_{i_1}$ and sets $Y_{i_1} = 0, Y_{i_2} = 1$ with probability $1 - y_{i_1} = y_{i_2}$. It is easy to see that the output of DR satisfies the following two properties:

(P1) $E(Y_i) = y_i$ for every $i \in [n]$.

(P2) $\sum_{i \in [n]} Y_i \leq T$ with probability 1.

As a remark, the output of DR will also satisfy the negative correlation property of the dependent rounding developed by Gandhi et al. [10], although we do not make use of it explicitly.

**3.3 Approximation algorithm**

We propose the following algorithm for the ProbeTop-$k$ problem, which we call ALG$_{ptk}$. Given an instance, we first solve LP$_{ptk}(I)$. Let $x, y$ be an optimal solution. Define

$$p_i = \sum_j \frac{x_{ij}}{y_i}, \quad v_i = \sum_j \frac{r_j x_{ij}}{\sum_j x_{ij}}.$$

If any of the denominators is 0, define these values as 0. For reasons that will soon become clear, $p_i$ is to be interpreted as the probability that we accept candidate $i$ given that she gets interviewed, and $v_i$ is to be interpreted as the expected value of candidate $i$ given that she gets hired. Assume that we relabel the candidates after solving the LP so that $v_1 \geq v_2 \geq \cdots \geq v_n$.

The algorithm first chooses the candidates to interview and the order in which the interviews are going to take place. For the first task we use the dependent rounding DR just introduced and the order in which the offers are sent is decreasing in $v_i$. After deciding the candidates to interview and the order, the interviews are conducted. After interviewing candidate $i$ and observing her value $r_j$, hire the candidate with probability $x_{ij}/(y_i q_{ij})$. As we are using DR to decide which candidates get interviews we again have that, if the optimal solution $y$ has zero or one fractional components, candidates are independently selected for receiving interviews. If there are two fractional components, then the two corresponding candidates have a perfect negative correlation (meaning that one is selected for interviewing if and only if the other one is not). We will show that the algorithm performs better with this negative correlation than it would if each candidate was selected to be interviewed independently with probability $y_i$.

\(^8\) If $y_i = 0$, then $\sum_j x_{ij} = 0$ too. If $y_i = 0$ then the algorithm will never interview applicant $i$, so the definition of these values is not relevant for the analysis.
To analyze the algorithm let \( Y_i \) be the indicator that candidate \( i \) is included in the list of candidates to receive an interview. As in the proof of Lemma 1 let \( Q_{ij} \) be the indicator of \( V_j = r_j \). Let \( P_i \) be the indicator that candidate \( i \) would be hired if she was interviewed while still having unfilled positions. We colloquially refer to this event as candidate \( i \) ‘making the cut’. We have

\[
\mathbb{E}(P_i) = \sum_{j=1}^{J} \mathbb{E}(P_i | Q_{ij}) q_{ij} = \sum_{j=1}^{J} \frac{x_{ij}}{q_{ij}y_i} q_{ij} = p_i.
\]

Define \( Z_i = Y_i \cdot P_i \) as the indicator that \( i \) is included in the list of candidates to receive an interview and that the algorithm would respond with a hire decision upon revealing \( i \)’s value. Note that \( Y_i \) is necessarily independent from \( P_i \). Let \( N_{i-1} = \sum_{\ell=1}^{i-1} Z_{\ell} \) be the amount of candidates that are interviewed before candidate \( i \) that would be hired if there are still positions available. We can therefore write the reward of our algorithm as

\[
\sum_i V_i \mathbb{1}\{N_{i-1} < k\} Z_i.
\]

When taking expectations we can derive the following simplifying auxiliary expression:

\[
\mathbb{E}(V_i \mathbb{1}\{N_{i-1} < k\} Z_i) = \mathbb{E}(V_i | \mathbb{1}\{N_{i-1} < k\} Z_i) = v_i \mathbb{E}(\mathbb{1}\{N_{i-1} < k\} Z_i).
\]

This expression holds because \( \mathbb{1}\{N_{i-1} < k\} Z_i = 1 \) is equivalent to candidate \( i \) being hired and \( v_i \) is the expected value of candidate \( i \) given that she received an interview and the algorithm would decide that she is hired:

\[
\mathbb{E}(V_i | \mathbb{1}\{N_{i-1} < k\} Z_i) = \mathbb{E}(V_i | Z_i = 1)
\]

\[
= \sum_{j=1}^{J} r_j \mathbb{P}(Q_{ij} = 1 | Z_i = 1)
\]

\[
= \sum_{j=1}^{J} r_j \mathbb{P}(Z_i = 1 | Q_{ij} = 1) \frac{\mathbb{P}(Q_{ij} = 1)}{\mathbb{P}(Z_i = 1)}
\]

\[
= \sum_{j=1}^{J} r_j \mathbb{P}(Z_i = 1 | Q_{ij} = 1) \frac{q_{ij}}{p_i y_i}
\]

\[
= \sum_{j=1}^{J} r_j y_i \mathbb{P}(P_i = 1 | Q_{ij} = 1) \frac{q_{ij}}{y_i p_i}
\]

\[
= \sum_{j=1}^{J} r_j y_i \frac{x_{ij}}{y_i q_{ij} y_i p_i}
\]

\[
= \frac{\sum_{j=1}^{J} r_i x_{ij}}{\sum_{j=1}^{J} x_{ij}} = v_i.
\]

In the first line we simply use that the value of a candidate is independent of the remaining positions left when she would be interviewed. In the third line we use Bayes’ rule. In the fourth line we replaced the known probabilities in the fraction: \( \mathbb{P}(Q_{ij} = 1) = q_{ij} \) and \( \mathbb{P}(Z_i = 1) = \mathbb{P}(P_i = 1) \mathbb{P}(Y_i = 1) = p_i y_i \), since they are independent. In the fifth line we use use that \( Y_i \) and \( P_i \) are independent. In the sixth line we replace \( \mathbb{P}(Y_i = 1 | Q_{ij} = 1) = \mathbb{P}(Y_i = 1) = y_i \), since \( Y_i \) is to be decided before knowing the realization of the values. In the seventh line we replace \( \mathbb{P}(P_i = 1 | Q_{ij} = 1) = x_{ij} / (y_i q_{ij}) \). In the last line we simply replace the definition of \( p_i \) and rearrange to obtain the desired equality.
With this equivalence we get the following expression for the reward of ALG_{ptk}:

\[ \sum_{i=1}^{n} v_i \mathbb{E}(1 \{N_{i-1} < k\} Z_i). \]

This expression implies that our algorithm has the same expected reward as an algorithm that would first sample \( Z_i = Y_i \mathbb{P}_i \) for all candidates and then collect the \( k \) highest values \( v_i \) among those candidates with \( Z_i = 1 \). This interpretation is only possible because ALG_{ptk} has the committed property. The same expressions can be derived for an algorithm that instead of sampling \( Y_i \) using our dependent rounding scheme, samples \( \tilde{Y}_i \) with the same marginal probabilities, but independently. These expressions are useful for showing that our algorithm outperforms a hypothetical algorithm that decides to interview each candidate \( i \) using the independent indicators \( \tilde{Y}_i \) instead of the correlated indicators \( Y_i \), potentially interviewing \( T + 1 \) candidates. The correlation induced by the constraint of \( T \) interviews on the actual algorithm only works in our favor.

**Lemma 3.** Let \( y = (y_i)_{i \in [n]} \), \( x = (x_{ij})_{i \in [n], j \in [T]} \) be a basic feasible solution of \( \text{LP}_{\text{ptk}}(I) \) with two fractional components \( i_1 \) and \( i_2 \). Let \( \tilde{Y}_i \) be independent Bernoulli random variables with mean \( y_i \). Define \( \tilde{Z}_i = P_i \tilde{Y}_i \) and \( \bar{N}_i = \min\{k, \sum_{i=1}^{n} \tilde{Z}_i\} \). Then

\[ \sum_{j=i}^{n} v_j \mathbb{E}(Z_j | {N_{i-1} < k}) \geq \sum_{i=1}^{n} v_i \mathbb{E}(\tilde{Z}_i | {\bar{N}_{i-1} < k}). \]

**Proof.** The right hand side of (4) can be interpreted as the reward of a policy that samples from \( \tilde{Y} \) instead of \( Y \) for deciding the candidates that will receive an interview. Notice that ALG_{ptk} selects a set of \( T \) candidates to make offers to, but the dependent analogue might select an order of \( T + 1 \) candidates (if \( \tilde{Y}_{i_1} = \tilde{Y}_{i_2} \)). Define \( V := \sum_{i=1}^{n} v_i \tilde{Z}_i | \{N_{i-1} < k\} \) and \( V := \sum_{i=1}^{n} v_i \tilde{Z}_i | \{\bar{N}_{i-1} < k\} \). Let \( \mathcal{F} \) be the sigma algebra generated by \( \{P_i\}_{i \in [n]} \) (i.e. all the randomness except \( Y_{i_1}, Y_{i_2}, \tilde{Y}_{i_1}, \tilde{Y}_{i_2} \), as the remaining \( Y_i \) and \( \tilde{Y}_i \) are deterministic).

For proving the lemma we show that \( \mathbb{E}(V - \tilde{V} | \mathcal{F}) \geq 0 \), which implies the result by using the law of iterated expectations. We do this by an exhaustive sample path analysis establishing the inequality for all possible events in \( \mathcal{F} \). The first thing to notice is that if \( P_{i_1} = P_{i_2} = 0 \), then \( V \) and \( \tilde{V} \) are identical. Indeed, both \( i_1 \) and \( i_2 \) will not be hired, and \( Z_i = \tilde{Z}_i \) for the remaining candidates. If it is the case that \( P_{i_1} = 1 \) and \( P_{i_2} = 0 \), then \( V \) and \( \tilde{V} \) have the same expectation. Indeed, the values of \( Y_{i_2} \) and \( \tilde{Y}_{i_2} \) won’t have any effect on the candidates hired by the algorithm. The only difference in the selection could be made by \( Y_{i_1} \) and \( \tilde{Y}_{i_1} \), which have the same marginal distribution, so the expectation is equal. The same argument works to establish that \( V \) and \( \tilde{V} \) have the same expectation when \( P_{i_1} = 0 \) and \( P_{i_2} = 1 \).

We still need to analyze the case where \( P_{i_1} = P_{i_2} = 1 \). To analyze this case, define \( \phi := \{i \in [n] \setminus \{i_1, i_2\} : i < i_2, Z_i = 1\} \), so \( |\phi| \) is the amount of candidates other than \( i_1 \) scheduled to get an interview before \( i_2 \) and would make the cut. Clearly \( \phi \in \mathcal{F} \). Observe that in the event of \( |\phi| \geq k \) we have \( \mathbb{E}(V - \tilde{V} | \mathcal{F}) \geq 0 \). Indeed, as \( |\phi| \geq k \), then no matter what happens with candidate \( i_1 \), candidate \( i_2 \) will never get an interview. The only difference can be made by different values of \( Y_{i_1} \) and \( \tilde{Y}_{i_1} \), whose marginal distributions are identical, so the expectation must also be.

The remaining cases are when \( P_{i_1} = P_{i_2} = 1 \) and \( |\phi| < k \). We first treat the case \( |\phi| = k - 1 \). Here, \( V \) will select everything in \( \phi \) plus either \( i_1 \) or \( i_2 \). On the other hand, \( \tilde{V} \) might select only \( i_1 \), only \( i_2 \), both, or neither. In the last case, the last selected index in \( \tilde{V} \) will be \( i^* := \inf\{i > i_2 : Z_i = 1\} \). Notice that \( i^* \) might not exist, in that case we say \( i^* = n + 1 \) and \( v_{n+1} = 0 \). This way we write

\[ \mathbb{E}(V | P_{i_1} = P_{i_2} = 1, \phi, |\phi| = k - 1) = \sum_{i \in \phi} v_i + y_i v_{i_1} + y_i v_{i_2}, \]

\[ \mathbb{E}(\tilde{V} | P_{i_1} = P_{i_2} = 1, \phi, |\phi| = k - 1) = \sum_{i \in \phi} v_i + y_i v_{i_1} + (1 - y_i)(y_{i_2} v_{i_2} + (1 - y_{i_2}) v_{i^*}), \]

so the difference is

\[ \mathbb{E}(V - \tilde{V} | P_{i_1} = P_{i_2} = 1, \phi, |\phi| = k - 1) = y_{i_2} v_{i_2} - y_{i_2} (y_{i_2} v_{i_2} + y_{i_1} v_{i^*}) \]

\[ \geq y_{i_2} (v_{i_2} - v_{i_2}) = 0, \]
as \( x_{ij} + y_{ij} = 1 \) and \( x_{ij} \geq y_{ij} \).

Finally, we treat the case \(|\phi| \leq k - 2\). Define \( \phi' := \{i \neq i_1, i_2 : \sum_{j=1}^i Z_j \leq k - 2\} \) to be the indices that will be certainly selected both in \( V \) and in \( \tilde{V} \). Define \( j_1 := \inf \{i \in [n] \setminus \{i_1, i_2\} : Z_i = 1, \sum_{j=i}^i Z_j = k - 1\} \). If the set defining \( j_1 \) is empty we again say \( j_1 = T + 1 \). Notice that \( j_1 \geq i_2 \) because \(|\phi| \leq k - 2\). Index \( j_1 \) will always be selected by \( V \), as it will select the \( k - 2 \) elements in \( \phi' \), either \( i_1 \) or \( i_2 \), and \( j_1 \), which will be the next best index remaining. On the other hand, \( \tilde{V} \) will also select all indices in \( \phi' \), but the remaining two positions can be filled in more ways. It can be the case that \( \tilde{V} \) picks either \( i_1 \) or \( i_2 \) plus \( j_1 \), but it also might happen that it picks both \( i_1 \) and \( i_2 \) and not pick \( j_1 \), or it can also pick neither of \( i_1 \) and \( i_2 \) and pick instead \( j_1 \) plus another index which we define as \( j_2 := \inf \{i > j_1 : Z_i = 1\} \). With this in mind we can express

\[
\mathbb{E}(V|P_{i_1} = P_{i_2} = 1, \phi', |\phi| \leq k - 2) = \sum_{i \in \phi'} v_i + y_{i_1}v_{i_1} + y_{i_2}v_{i_2} + v_{j_1},
\]

\[
\mathbb{E}(\tilde{V}|P_{i_1} = P_{i_2} = 1, \phi', |\phi| \leq k - 2) = \sum_{i \in \phi'} v_i + y_{i_1}v_{i_2} + v_{i_2} + v_{j_1} + (y_{i_1})^2(v_{i_1} + v_{j_1}) + (y_{i_2})^2(v_{i_2} + v_{j_1}) + y_{i_1}y_{i_2}(v_{j_1} + v_{j_2})
\]

\[
= \sum_{i \in \phi'} v_i + y_{i_1}v_{i_1} + y_{i_2}v_{i_2} + v_{j_1} + y_{i_1}v_{j_1} + y_{i_2}v_{j_2}.
\]

The difference is

\[
\mathbb{E}(V - \tilde{V}|P_{i_1} = P_{i_2} = 1, \phi', |\phi| \leq k - 2) = v_{j_1} - y_{i_1}v_{j_1} - y_{i_2}v_{j_2} \geq v_{j_1} - v_{j_1} = 0,
\]

\[\square\]

To conclude our analysis we relate the performance of the hypothetical algorithm (with independent indicators) and the objective function of the linear program, through the weighted rank function of \( k \)-uniform matroids. For a set \( S \subseteq [n] \) we define the weighted rank function for the \( k \)-uniform matroid with weights \( v \) as

\[
v^*(S) = \max_{R \subseteq [n], |R| \leq k} \sum_{i \in R} v_i.
\]

Both the reward of the independent version of our algorithm and the objective of LP\(_{ptk}\) can be expressed as expectations of weighted rank functions of \( k \)-uniform matroids. We use this together with the correlation gap results by Yan [21] to show the main result of the section.

**Theorem 1.** For any instance \( I \in T^k_{ptk} \),

\[
R_{ALG_{ptk}}(I) \geq \left(1 - \frac{e^{-k^k/k!}}{k!}\right) LP_{ptk}(I).
\]

**Proof.** To show the result we first show how to express the expected reward of the independent version of our algorithm, ALG\(_{ptk}\) and the objective function of LP\(_{ptk}\) as expectations of weighted rank functions.

For the first one the interpretation of the algorithm collecting value \( v_i \) when candidates make the cut. Let \( M = \{i : Z_i = 1\} \). As the algorithm will interview candidates in decreasing order of \( v_i \), we can express the expected reward as \( R_{ALG_{ptk}} = v^*(M) \). Notice that \( P(i \in M) = E(Z_i) = y_ip_i \).

For expressing the objective function of the linear program as a weighted rank function we rewrite

\[
\sum_i \sum_j r_{ij}x_{ij} = \sum_i y_i \sum_j r_{ij}x_{ij} \sum_{j=1}^n x_{ij} y_i = \sum_{i=1}^n y_i v_i p_i,
\]

This way, we can use the dependent rounding procedure presented in Gandhi et al. [10] to generate indicators \( W_i \) with probabilities \( w_i = y_ip_i \). Constraint (3) can be re-written as

\[
\sum_{i,j} x_{ij} = \sum_i y_i \sum_j \frac{x_{ij}}{y_i} = y_ip_i \leq k,
\]
so the dependent rounding procedure ensures that $D = \{i \in [n] : W_i = 1\}$ has cardinality at most $k$. With this we can use the linearity of the expectation to write
\[
\sum_i v_i y_i p_i = \sum_i v_i P(W_i = 1) = \max_{R \subseteq D, |R| \leq k} \sum_{i \in R} v_i = v^*(D).
\]
The second equality holds because $D$ has cardinality of at most $k$.

With these expressions, for any instance $I \in I_{ptk}$ we can write
\[
\frac{R_{ALG'}_{ptk}(I)}{LP_{ptk}(I)} = \frac{v^*(M)}{v^*(D)} \geq 1 - \frac{e^{-k^k/k}}{k!}.
\]
The inequality follows from Proposition 1 by Yan [21], since $M$ and $D$ have the same marginal distributions.

The last step is to invoke Lemma 3 for the inequality $R_{ALG_{ptk}}(I) \geq R_{ALG'}_{ptk}(I)$. \(\square\)

It is worth noting that this algorithm can be de-randomized. Indeed, the algorithm will randomize between at most two fixed orders, so both of them can be evaluated and the best among them can be selected. Thus, we have a deterministic, non-adaptive, committed algorithm whose reward will be higher than a factor $(1 - e^{-k^k/k})!$ of the expected reward of the optimal algorithm. Since the algorithm is non-adaptive and committed, it establishes a lower bound on how good an approximation factor can be achieved by restricting to these classes of policies.

### 3.4 Weighted Bernoulli Values and Sequential Offering Problem

ProbeTop-$k$ is closely related to the Sequential Offering problem ($\text{seq}$) studied by Purohit et al. [13]. In this problem, the hiring firm has already interviewed $n$ candidates and must decide the order in which to send offers to them. Each candidate $i$ has a probability $q_i$ of accepting an offer, and adds a value $r_i$ to the firm if she accepts an offer. The firm can hire at most $k$ candidates and send at most $T$ offers in total. This model can be captured by ProbeTop-$k$ by considering the special case of $\text{ptk}$ where the value of candidate $i$ takes value $r_i$ with probability $q_i$ (representing acceptance of an offer), and 0 with probability $(1 - q_i)$ (representing rejection of an offer). The subtlety making $\text{ptk}$ different from $\text{seq}$ is that policies for $\text{ptk}$ can reject a candidate $i$ even if they had a realization to value $r_i$, which is not allowed in $\text{seq}$. We show, however, that $ALG_{ptk}$ can be made to hire any candidate when her realized value is $r_i$ without sacrificing performance, therefore making it an admissible algorithm for the sequential offering model with the same guarantee.

We can rewrite $LP_{ptk}$ for this special case as
\[
\begin{align*}
\max & \sum_{i=1}^n r_i x_i \\
\text{s.t.} & x_i \leq y_i q_i & \forall i \in [n] \\
& \sum_{i=1}^n y_i \leq T \\
& \sum_{i=1}^n x_i \leq k \\
& x_i \geq 0 & \forall i \in [n], \\
& 0 \leq y_i \leq 1 & \forall i \in [n],
\end{align*}
\]
where $x_i$ is to be interpreted as the probability of hiring the candidate and $V_i = r_i$. (We can omit the variable corresponding to $V_i = 0$ since it does not weigh in the objective and it can only make the constraints tighter.) The following lemma will let us restrict $ALG_{ptk}$ to be admissible for the Sequential Offering problem.

**Lemma 4.** There is an optimal solution for $LP_{ptk}$ in the special weighted Bernoulli instance with $x_i = y_i q_i$ for all $i \in [n]$. 

Proof. Let \( x, y \) be an optimal solution with \( x_i < y_i q_i \) for some \( i \). Then construct \( \tilde{x}, \tilde{y} \) with \( \tilde{x}_i = x_i \) and \( \tilde{y}_i = x_i / q_i \) for all \( i \in [n] \). It is clear to see that \( \tilde{x}, \tilde{y} \) is feasible and that the value of the objective function remains unchanged. \( \square \)

Recall that after interviewing candidate \( i \), ALG ptk will hire her with probability \( x_i / (y_i q_i) \). Therefore, if we restrict to solutions with \( x_i = y_i q_i \), then ALG ptk will hire candidate \( i \) with probability 1 if \( V_i = r_i \), making it admissible for the sequential offering problem.

We can compute
\[
p_i = \frac{x_i y_i}{x_i q_i} = q_i, \quad v_i = \frac{r_i x_i}{x_i} = r_i.
\]
We can further simplify the linear program by removing variables \( x_i \) and express it in terms of \( p_i \) and \( v_i \), leading to \( \text{LP}_{\text{seq}} \):
\[
\text{LP}_{\text{seq}}(I) = \max \sum_{i=1}^{n} v_i y_i p_i \\
\text{s.t.} \sum_{i=1}^{n} y_i \leq T \\
\sum_{i=1}^{n} y_i p_i \leq k \\
0 \leq y_i \leq 1 \quad \forall i \in [n].
\]

Being consistent with the previously introduced notation, let \( I_{\text{seq}} \) be the set of all instances for the sequential offering problem. Let \( I_{\text{seq}}^k \) be the set of all instances of seq that have exactly \( k \) positions to fill. Let ALG seq be the modified version of ALG ptk that, when faced to instances of Weighted Bernoulli random variables, modifies the solution of the linear program such that the algorithm is admissible for seq. We have the following corollary of Theorem 1.

**Corollary 1.** For any instance \( I \in I_{\text{seq}}^k \),
\[
R_{\text{ALG}_{\text{seq}}}(I) \geq \left( 1 - \frac{e^{-k} k^k}{k!} \right) \text{LP}_{\text{seq}}(I).
\]

### 3.5 Tightness of Guarantee

We close the section by showing that the guarantee in Theorem 1 is tight. Indeed, construct an instance with \( n \) identical candidates with weighted Bernoulli values. In particular, \( r_i = 1 \) and \( q_i = k/n \) for all \( i \). There is no time constraint for this algorithm, i.e. \( T = n \). Denote the described instance by \( I_{UB}^{ptk} \).

**Proposition 2.** For any \( \pi \in I_{UB}^{ptk} \),
\[
R_{\pi}(I_{UB}^{ptk}) \leq \left( 1 - \frac{e^{-k} k^k}{k!} \right) \text{LP}_{ptk}(I_{UB}^{ptk}).
\]

**Proof.** Since we are working with an instance of weighted Bernoulli values we can work with \( \text{LP}_{\text{seq}} \) without loss of generality. An optimal solution for \( \text{LP}_{\text{seq}}(I_{UB}^{ptk}) \) will have \( y_i = 1 \) for all \( i \), and attain an objective value of \( k \). On the other hand, any reasonable algorithm will interview all candidates until \( k \) of them have value 1 and are hired. The expected reward of the algorithm will be \( \mathbb{E}(\min\{k, B_{n,k}\}) \), where \( B_{n,k} \sim Bin(n, k/n) \). If we make \( n \) go to infinity, the expected reward will converge to
\[
\mathbb{E}(\min\{\text{Poisson}(k), k\}) = k \left( 1 - \frac{e^{-k} k^k}{k!} \right).
\]
\( \square \)

It is worth noting that this tightness is only with respect to \( \text{LP}_{ptk} \). Indeed, ALG ptk is actually optimal in this instance.
4 Parallel Offering

We now turn our focus to the parallel offering model defined in Section 2.2. We provide an algorithm that achieves a \((1 - 1/e)\) approximation of the optimal policy. The algorithm works by solving a linear programming relaxation and rounding its solution to decide who to offer which position, and in which order.

In Section 4.1, we introduce the linear program in question. In 4.2, we present and analyze our approximation algorithm. In Section 4.3, we study the special case with identical positions introduced by Purohit et al. [18] and establish a connection between the parallel and sequential offering problems. We conclude in Section 4.4 by establishing that the bound achieved by our algorithm is tight.

4.1 Linear Program

For an instance \(I \in \mathcal{I}_{\text{par}}\), we introduce \(LP_{\text{par}}(I)\), with LP variables \(y_{ij}\) for \(i \in [n]\) and \(j \in [k]\). Variable \(y_{ij}\) is to be interpreted as the probability that candidate \(i\) receives an offer for position \(j\). As with \(LP_{\text{ptk}}\), this linear program only enforces that the problem’s constraints are satisfied in expectation.

\[
LP_{\text{par}}(I) = \max \sum_{i=1}^{n} \sum_{j=1}^{k} v_{ij} y_{ij} p_{ij}
\]

s.t.

\[
\sum_{i=1}^{n} y_{ij} \leq T \quad \forall j \in [k] \tag{5}
\]

\[
\sum_{i=1}^{n} y_{ij} p_{ij} \leq 1 \quad \forall j \in [k] \tag{6}
\]

\[
\sum_{j=1}^{k} y_{ij} \leq 1 \quad \forall i \in [n] \tag{7}
\]

\[
0 \leq y_{ij} \leq 1 \quad \forall (i, j) \in [n] \times [k].
\]

We start by formally proving that the optimal value of the linear program upper bounds the expected reward of any algorithm.

**Lemma 5.** For any instance \(I \in \mathcal{I}_{\text{par}}\), \(LP_{\text{par}}(I) \geq OPT_{\text{par}}(I)\).

**Proof.** For an arbitrary policy let \(Y_{ij}\) be the indicator that candidate \(i\) receives an offer for position \(j\). Let \(P_{ij}\) be the indicator that candidate \(i\) would accept an offer for position \(j\) should she receive one.

Any policy satisfies that for each position \(j\), at most \(T\) offers can be sent. In terms of the indicators:

\[
\sum_{i=1}^{n} Y_{ij} \leq T,
\]

so constraint (5) follows by taking expectation. Similarly, each candidate can receive at most one offer:

\[
\sum_{j=1}^{k} Y_{ij} \leq 1,
\]

so constraint (7) follows by expectation. Next, for each position \(j\), at most one candidate can be hired. In terms of our indicators:

\[
\sum_{i=1}^{n} Y_{ij} P_{ij} \leq 1,
\]

so constraint (6) is obtained by taking expectation since we assume that the acceptance/rejection decisions are made before the offering rounds start. The last constraint is clearly satisfied.
Finally, the reward collected by the policy is
\[ \sum_{j=1}^{k} \sum_{i=1}^{n} v_{ij} Y_{ij} P_{ij}. \]
Again by taking expectation and using the independence of \( Y_{ij} \) and \( P_{ij} \) we obtain that the expected reward of the policy is equal to the objective function of the linear program. \( \square \)

### 4.2 Approximation algorithm

We now present our algorithm for the parallel offering model, which we refer to as \( \text{ALG}_{\text{par}} \). The algorithm first solves \( \text{LP}_{\text{par}} \) to produce an optimal solution \( y \). With the solution at hand, the algorithm will round it to obtain a random binary matrix \( Y = (Y_{ij})_{(i,j) \in [n] \times [k]} \). To round our solution here, we use the dependent rounding scheme developed by Gandhi et al. [10] that is mentioned in Section 2.3. After the solution is rounded, the algorithm uses the rounded solution to make a sequential offering list for each position. Specifically, candidate \( i \) is included in the list for position \( j \) if \( Y_{ij} = 1 \). The properties of the depending rounding scheme by Gandhi et al. [10] combined with the constraints of \( \text{LP}_{\text{par}} \) will ensure that: (a) each list has at most \( T \) candidates, and (b) each candidate is included in at most one list. After the lists are formed, each list is run in parallel, and the order in which the offers are sent for position \( j \) is decreasing in \( v_{ij} \).

For the analysis, let \( L_j \) denote the expected reward collected from list \( j \). Let \( L^*_j = \sum_{i=1}^{n} v_{ij} y_{ij} p_{ij} \), so that the objective function of \( \text{LP}_{\text{par}} \) can be expressed as \( \sum_{j=1}^{k} L^*_j \). We will show that for each position \( j \) it holds that \( L_j \geq (1-1/e) L^*_j \). Define \( Z_{ij} = P_{ij} Y_{ij} \) and \( \tilde{Z}_{ij} = \tilde{P}_{ij} \tilde{Y}_{ij} \), where \( \tilde{Y}_{ij} \) are independent Bernoulli random variables with mean \( y_{ij} \). Let \( D_j = \{ i : Z_{ij} = 1 \} \) and \( \tilde{D}_j = \{ i : \tilde{Z}_{ij} = 1 \} \). Let \( v^*_j(S) = \max_{S \subseteq S} v_{ij} \). It is clear to see that \( L_j = \mathbb{E}(v^*_j(D)) \).

The first step for showing the guarantee of \( \text{ALG}_{\text{par}} \) is to show that the reward collected by a list is not lower than what we would collect if we rounded each component of \( y \) independently. This is a consequence of the negative correlation property (P3) of the dependent rounding scheme by Gandhi et al. [10].

**Lemma 6.** For all \( j \in [k] \), \( \mathbb{E}(v^*_j(D)) \geq \mathbb{E}(v^*_j(\tilde{D})) \).

**Proof.** For simplicity assume that \( v_{1,j} \geq v_{2,j} \geq \cdots \geq v_{n,j} \). We can write
\[
v^*_j(D) = v_{1,j} + \sum_{i=2}^{n} (v_{i-1,j} - v_{i,j}) \mathbb{I}\{ \cap_{\ell=1}^{i-1} \{ \ell \in D \} \}.\]

By taking expectation we get
\[
\mathbb{E}(v^*_j(D)) = v_{1,j} + \sum_{i=2}^{n} (v_{i,j} - v_{i+1,j}) \mathbb{P}(\cap_{\ell=1}^{i-1} \{ \ell \in D \}).
\]

Now, using the negative correlation property (P3) together with (P1) and \( (v_{i-1,j} - v_{i,j}) \leq 0 \) we can conclude
\[
\mathbb{E}(v^*_j(D)) \geq v_{1,j} + \sum_{i=2}^{n} (v_{i,j} - v_{i+1,j}) \prod_{\ell=1}^{i-1} \mathbb{P}(\{ \ell \in D \})
\]
\[
= v_{1,j} + \sum_{i=2}^{n} (v_{i,j} - v_{i+1,j}) \prod_{\ell=1}^{i-1} \mathbb{P}(\{ \ell \in \tilde{D} \})
\]
\[
= \mathbb{E}(v^*_j(\tilde{D})).
\]

\( \square \)

Notice that in the proof of Lemma 6 we only use (P1) and (P3) from the rounding scheme by Gandhi et al. [10]. These properties are not enough if we want to show that the dependent rounding outperforms the independent rounding when selecting the \( k \geq 2 \) elements with the highest weights (which holds, for instance, in Lemma 5). Consider the following counterexample.
Example 1. Consider four elements \(a, b, c\) and \(d\), all with identical weights equal to 1. Consider the following rounding scheme: pick any subset from \(\{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}\) with equal probability. This implies that any element will be included in the set of eligible elements \(D\) with probability 1/2. It is not hard to see that this rounding scheme satisfies (P3)\(\textsuperscript{3}\). The expected reward collected by choosing the \(k = 2\) highest weights with this rounding scheme is 3/2: with probability 1/2 we will have one element in the subset, with probability 1/2 we will have three elements in the subset, of which we can only choose 2. However, if we formed the eligible subset by including each of the elements independently with probability 1/2, the reward collected would be

\[
\mathbb{E}(\min\{B_{4,1/2}, 2\}) = 1 \cdot \frac{4}{16} + 2 \cdot \left(\frac{6}{16} + \frac{4}{16} + \frac{1}{16}\right) = \frac{13}{8} > \frac{3}{2},
\]

where \(B_{4,1/2}\) refers to a Binomial random variable with 4 independent trials of probability 1/2.

Continuing with the main result, the following lemma will help establish the desired bound for each list, in which only the single candidate \((k = 1)\) with the highest weight will be hired. Both the correlation gap and the GKPS rounding scheme mentioned in Section 2.3 are used in the analysis.

Lemma 7. For all \(j \in [k]\), \(\mathbb{E}(v_j^*(\tilde{D})) \geq (1 - 1/e)L_j^*\).

Proof. Let \(W_{ij}\) be the output of the Gandhi et al. \[10\] dependent rounding with input \(y_{ij}p_{ij}\). These indicators satisfy \(\mathbb{E}(W_{ij}) = y_{ij}p_{ij}\) and \(\sum_{i=1}^n W_{ij} \leq 1\) with probability 1, since \(y\) satisfies constraint \(\text{(O)}\). Let \(M_j = \{i : W_{ij} = 1\}\). We can then express

\[
L_j^* = \sum_{i=1}^{n} y_{ij}p_{ij} = \mathbb{E}(\max_{i \in M} v_{ij}) = \mathbb{E}(v_j^*(M)).
\]

The lemma follows from Proposition 1. \(\square\)

By combining lemmas 6 and 7 we obtain the main result of the section.

Theorem 2. For any instance \(I \in \mathcal{I}_{\text{par}}\), \(\text{ALG}_{\text{par}}(I) \geq (1 - 1/e)\text{LP}_{\text{par}}(I)\).

4.3 Identical Positions and Cost of Batching

We now turn our focus to the special case where all positions are identical (i.e. \(v_{ij} = v_i\) and \(p_{ij} = p_i\) for all \(i \in [n]\) and \(j \in [k]\)). For this case we can obtain a connection between the sequential offering model and the parallel offering model through their respective linear programs.

Let \(I \in \mathcal{I}_{\text{par}}\) be an instance for the parallel offering model in which all \(k\) positions are identical, we have \(n\) candidates, and there are \(T\) offering rounds. Construct \(I' \in \mathcal{I}_{\text{seq}}\) to be the same instance as \(I\) except \(kT\) sequential offers can be sent in total. We obtain the following lemma.

Lemma 8. \(\text{LP}_{\text{par}}(I) = \text{LP}_{\text{seq}}(I')\).

Proof. Let \(y\) be a solution for \(\text{LP}_{\text{seq}}\). Let us construct \(y'\), a solution for \(\text{LP}_{\text{par}}\). In particular, we let \(y'_{ij} = y_{ij}/k\) for all \((i, j) \in [n] \times [k]\). It is straightforward to see that the solution is feasible and that the values of the objective functions are equal. For the other direction, let \(z\) be a solution for \(\text{LP}_{\text{par}}\). We can construct \(z'\) by setting \(z'_{ij} = \sum_{j=1}^{k} z_{ij}\). It is again straightforward that this solution is feasible in \(\text{LP}_{\text{seq}}\) and that the values of the objective functions are equal.

This lemma implies the following corollary, which we refer to as the cost of batching.

Corollary 2. For any \(I \in \mathcal{I}_{\text{par}}\), \(\text{ALG}_{\text{par}}(I) \geq (1 - 1/e)\text{OPT}_{\text{seq}}(I')\).

This corollary helps us understand how costly it can be to send offers in batches instead of one by one, like in the sequential offering problem. By reducing to \(T\) parallel offering rounds instead of \(kT\) sequential offering rounds, we know that we are not going to be worse than a factor of \((1 - 1/e)\) from the optimal sequential policy.

\(\textsuperscript{3}\) For subsets of cardinality 1, the right hand side of (P3) is 1/2. The left hand side is also 1/2: it can be obtained by choosing the corresponding singleton, plus either of the three subsets of cardinality 3 that contain the element in question. By the same simple counting argument, the property will also be satisfied with equality for subsets of size 2 and 3. For the full set, however, the right hand side of (P3) will be 1/16, while the left hand side will be 0.
4.4 Tightness of Guarantee

We conclude the section by showing that the guarantee obtained in Theorem \(2\) is tight. Consider an instance with \(k\) identical positions and \(kT\) identical candidates. In particular, \(v_{ij} = 1\) and \(p_{ij} = 1/T\) for all \((i, j) \in [kT] \times [k]\). Denote this instance by \(I_{UB}^{par}\).

**Proposition 3.** For any \(\pi \in \Pi_{UB}^{par}\),

\[
R_\pi(I_{UB}^{par}) \leq \left(1 - \frac{1}{e}\right) LP_{UB}(I_{UB}^{par})
\]

**Proof.** A feasible solution for \(LP_{UB}^{par}\) is to set \(y_{ij} = 1/k\) for all \((i, j) \in [kT] \times [k]\), achieving an objective of \(k\). On the other hand, any reasonable algorithm would form \(k\) parallel lists with \(T\) candidates each, and run each list in parallel. The reward collected by a list is

\[
E(\min\{B_{T,1/T}, 1/T\}) \to T \to \infty E(\min\{\text{Poisson}(1), 1\}) = 1 - \frac{1}{e},
\]

where \(B_{T,1/T} \sim \text{Bin}(T, 1/T)\). Since there are \(k\) identical lists, the ratio between the performance of the algorithm and the linear program is exactly \((1 - 1/e)\). \(\square\)

We again remark that in this instance, \(\text{ALG}_{UB}^{par}\) is actually optimal.

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