Homoclinic Tubes in Discrete Nonlinear Schrödinger Equation Under Hamiltonian Perturbations

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Abstract

In this paper, we study the discrete cubic nonlinear Schrödinger lattice under Hamiltonian perturbations. First we develop a complete isospectral theory relevant to the hyperbolic structures of the lattice without perturbations. In particular, Bäcklund-Darboux transformations are utilized to generate heteroclinic orbits and Melnikov vectors. Then we give coordinate-expressions for persistent invariant manifolds and Fenichel fibers for the perturbed lattice. Finally based upon the above machinery, existence of codimension 2 transversal homoclinic tubes is established through a Melnikov type calculation and an implicit function argument. We also discuss symbolic dynamics of invariant tubes each of which consists of a doubly infinite sequence of curve segments when the lattice is four dimensional. Structures inside the asymptotic manifolds of the transversal homoclinic tubes are studied, special orbits, in particular homoclinic orbits and heteroclinic orbits when the lattice is four dimensional, are studied.

Keywords: Homoclinic tubes, Bäcklund-Darboux transformations, cubic nonlinear Schrödinger lattice, Melnikov vectors, Fenichel fibers.
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1 Introduction

The concept of a homoclinic tube was introduced by Silnikov [16] in a study on the structure of the neighborhood of a homoclinic tube asymptotic to an invariant torus $\sigma$ under a diffeomorphism $F$ in a finite dimensional phase space. The asymptotic torus is of saddle type. The homoclinic tube consists of a doubly infinite sequence of tori \( \{\sigma_j, \ j = 0, \pm 1, \pm 2, \cdots\} \) in the transversal intersection of the stable and unstable manifolds of $\sigma$, such that $\sigma_{j+1} = F \circ \sigma_j$ for any $j$. It is a generalization of the concept of a transversal homoclinic orbit when the points are replaced by tori. Silnikov obtained a similar theorem on the symbolic dynamics structures in the neighborhood of the homoclinic tube as Smale's theorem for a transversal homoclinic orbit [15] [14].

We are interested in homoclinic tubes for several reasons: 1. Especially in high dimensions, dynamics inside each invariant tubes in the neighborhoods of homoclinic tubes are often chaotic too. We call such chaotic dynamics “chaos in the small”, and the symbolic dynamics of the invariant tubes “chaos in the large”. Such cascade structures are more important than the structures in a neighborhood of a homoclinic orbit, when high or infinite dimensional dynamical systems are studied. 2. Symbolic dynamics structures in the neighborhoods of homoclinic tubes are more observable than in the neighborhoods of homoclinic orbits in numerical and physical experiments due to the large dimensionality and the robustness of the homoclinic tubes. 3. When studying high or infinite dimensional Hamiltonian system (for example, the discrete NLS or NLS equations under Hamiltonian perturbations), each invariant tube contains both KAM tori and stochastic layers (chaos in the small). Thus, not only dynamics inside each stochastic layer is chaotic, all these stochastic layers also move chaotically under Poincaré maps.

In [7], we proved the existence of transversal homoclinic tubes which are asymptotic to locally invariant center manifolds for the cubic nonlinear Schrödinger equation under Hamiltonian perturbations. Due to the locally invariant nature of the center manifolds, the symbolic dynamics in the neighborhood of the homoclinic tube is very difficult to establish. The locally invariant nature is very difficult to control in infinite dimensions. Therefore, we need to seek low dimensionality and hopefully can establish the above mentioned symbolic dynamics for the low dimensional systems which is the first step toward establishing such symbolic dynamics for PDEs. Then naturally we want to study the finite difference discretization of the cubic
nonlinear Schrödinger equation under Hamiltonian perturbations (perturbed discrete NLS). The dimension of the perturbed discrete NLS is \( n \). As shown in this paper, when \( n = 4 \), the center manifold is actually invariant and Silnikov’s theorem implies the symbolic dynamics of line segments.

Denote by \( W^{(c)} \) a normally hyperbolic locally invariant center manifold, by \( W^{(cu)} \) and \( W^{(cs)} \) the locally invariant center-unstable and center-stable manifolds such that \( W^{(c)} \subset W^{(cu)} \cap W^{(cs)} \), and by \( F^t \) the evolution operator of the system. We call \( \mathcal{H} \) a transversal homoclinic tube if \( \mathcal{H} \subset W^{(cu)} \cap W^{(cs)} \), the intersection between \( W^{(cu)} \) and \( W^{(cs)} \) is transversal at \( \mathcal{H} \), and \( \mathcal{H} \) has the same dimension with \( W^{(c)} \). Let \( \Sigma \) be an appropriate Poincaré section, and \( P \) is the Poincaré map induced by the flow \( F^t \); then \( \mathcal{H} \cap \Sigma \) is called a transversal homoclinic tube under the Poincaré map \( P \).

In [10], the discrete cubic nonlinear Schrödinger equation under dissipative perturbations is studied. Existence of a symmetric pair of homoclinic orbits is established through a Melnikov calculation and a geometric argument. In [11], symbolic dynamics in the neighborhood of the symmetric pair of homoclinic orbits is established. In contrast to these previous works, the current work is a study on Hamiltonian perturbations and homoclinic tubes.

The paper is organized as follows: Section 2 is on the formulation of the problem, section 3 is on the isospectral theory for the discrete NLS equation, section 4 is on the coordinatization for invariant submanifolds, section 5 is on the existence of transversal homoclinic tubes, section 6 is on the symbolic dynamics of segments for the case when the perturbed discrete NLS equation is 4-dimensional, section 7 is on structures inside the asymptotic manifolds of the transversal homoclinic tubes, and section 8 is the conclusion.
2 Formulation of the Problem

Consider the discretized cubic nonlinear Schrödinger equation under Hamiltonian perturbations (perturbed discrete NLS),

\[ i\dot{q}_n = \rho_n \partial H / \partial \bar{q}_n , \]  

(2.1)

where

\[ H = H_0 + \varepsilon H_1 , \]

\[ H_0 = \frac{1}{h^2} \sum_{n=0}^{N-1} \left\{ \bar{q}_n (q_{n+1} + q_{n-1}) - \frac{2}{h^2} (1 + \omega^2 h^2) \ln \rho_n \right\} . \]

\( \varepsilon \) is the perturbation parameter, \( \varepsilon H_1 \) is the Hamiltonian perturbation, \( i = \sqrt{-1} \) is the imaginary unit, \( \omega \) is a positive parameter, and \( \rho_n = 1 + h^2 |q_n|^2 \).

\( q_n \) satisfies the periodic and even boundary conditions,

\[ q_{n+N} = q_n , \quad q_{-n} = q_n , \]  

(2.2)

where \( N \) is a positive integer \( N \geq 3 \). (2.1) is a \( 2(M+1) \)-dimensional system, where \( M = N/2 \) (\( N \) even) and \( M = (N-1)/2 \) (\( N \) odd).

**Remark 2.1** When \( \varepsilon = 0 \), \( \sum_{n=0}^{N-1} \left\{ \bar{q}_n (q_{n+1} + q_{n-1}) \right\} \) itself is also a constant of motion. This invariant, together with \( H_0 \), implies that \( \sum_{n=0}^{N-1} \ln \rho_n \) is a constant of motion too. Therefore,

\[ D^2 \equiv \prod_{n=0}^{N-1} \rho_n \]  

(2.3)

is a constant of motion.

The phase space is defined as

\[ \mathcal{S} \equiv \left\{ \bar{q} = \left( q \ \bar{q} \right) \mid r = -\bar{q} , \ q = (q_0, q_1, \ldots, q_{N-1})^T , \ q_{n+N} = q_n , \ q_{N-n} = q_n \right\} . \]  

(2.4)

In \( \mathcal{S} \) (viewed as a vector space over the real numbers), we define the inner product, for any two points \( \bar{q}^{(1)} \) and \( \bar{q}^{(2)} \), as follows:

\[ \langle \bar{q}^{(1)}, \bar{q}^{(2)} \rangle = \sum_{n=0}^{N-1} (q_n^{(1)} q_{n}^{(2)} + \bar{q}_n^{(1)} \bar{q}_n^{(2)}) . \]
And the norm of \( \vec{q} \) is defined as \( \| \vec{q} \|^2 = \langle \vec{q}, \vec{q} \rangle \). We also use the notation \( \vec{q}_n = (q_n, r_n)^T \), where \( r_n = -\bar{q}_n \).

We will study transversal homoclinic tubes in this phase space \( S \). When \( \varepsilon = 0 \), the unperturbed Hamiltonian system is the integrable discrete cubic nonlinear Schrödinger equation,

\[
iq_n = \frac{1}{\hbar^2} [q_{n+1} - 2q_n + q_{n-1}] + |q_n|^2(q_{n+1} + q_{n-1}) - 2\omega^2 q_n.
\]

The Hamiltonian perturbation term \( H_1 \) can be very general. In this paper, for concreteness, we will study the simple example,

\[
H_1 = \frac{1}{\hbar^2} \sum_{n=0}^{N-1} \left\{ \alpha_1(q_n + \bar{q}_n) + \alpha_2(q_n^2 + \bar{q}_n^2) \right\} \ln \rho_n,
\]

where \( \alpha_1 \) and \( \alpha_2 \) are real parameters. The corresponding equation is,

\[
iq_n = \frac{1}{\hbar^2} [q_{n+1} - 2q_n + q_{n-1}] + |q_n|^2(q_{n+1} + q_{n-1})
- 2\omega^2 q_n + \varepsilon \left\{ \alpha_1(q_n + \bar{q}_n) + \alpha_2(q_n^2 + \bar{q}_n^2) \right\} q_n
+ \left[ \alpha_1 + 2\alpha_2 \bar{q}_n \right] \frac{\rho_n}{\hbar^2} \ln \rho_n.
\] (2.5)

We will show that there exist transversal homoclinic tubes for this system in the phase space \( S \) through Melnikov analysis and an implicit function argument.

**Remark 2.2** In the literature [2], for integrable systems under Hamiltonian perturbations, an interesting fact is that the Melnikov function sometimes is identically zero. Consider the NLS equation under Hamiltonian perturbation [2]

\[
iq_t = -q_{xx} - 2|q|^2 q - \varepsilon q_{xxxx} ,
\] (2.6)

with the Hamiltonian

\[
H = H_0 + \varepsilon H_1 = \int_0^1 [|q_x|^2 - |q|^4 - \varepsilon |q_{xx}|^2] dx .
\]

In [2], the Melnikov function \( M \) is built with \( H_0 \),

\[
M = \int_{-\infty}^{\infty} \{H_0, H_1\}(Q) dt
\]
where \( Q \) is a heteroclinic orbit generated through Bäcklund-Darboux transformations, which is asymptotic to a periodic orbit independent of \( x \) in \( |t| \to \infty \) limit. The reason why \( M \) is identically zero is as follows

\[
M = -\frac{1}{\varepsilon} \int_{-\infty}^{\infty} \frac{dH}{dt} Q \, dt = -\frac{1}{\varepsilon} \left[ H(\lim_{t \to \infty} Q) - H(\lim_{t \to -\infty} Q) \right] = 0.
\]

From the study [9], one doubts that \( H_0 \) is the proper invariant to build the Melnikov function, since the center-unstable manifold associated with the unstable mode is really a level set of the invariant \( F_1 \) (3.6). Nevertheless, direct calculation shows that

\[
\frac{\delta F_1}{\delta q}(Q) = \text{linear combination of } \left\{ \frac{\delta H_0}{\delta q}(Q), \frac{\delta H_0}{\delta \bar{q}}(Q), \frac{\delta I}{\delta q}(Q), \frac{\delta I}{\delta \bar{q}}(Q) \right\}
\]

with coefficients only dependent upon \( t \),

where \( I = \int_0^1 |q|^2 dx \) is still invariant under the perturbed flow (2.6). This relation and the above Melnikov function do not imply that the Melnikov function built with \( F_1 \) is identically zero, but offer a hint that it is probably identically zero too. From some trial calculations for this paper, if a perturbation term still keeps

\[
I = \sum_{n=0}^{N-1} [\bar{q}_n(q_{n+1} + q_{n-1})]
\]

invariant, the corresponding term of the Melnikov function built from \( \tilde{F}_1 \) (3.6) is identically zero. Such perturbation terms are for example,

\[
H_1 = \frac{1}{h^2} \sum_{n=0}^{N-1} \tilde{\alpha}_2 |q_n|^2 \ln \rho_n.
\]

When the Melnikov function is identically zero, second order Melnikov function is needed to measure the splitting between center-unstable and center-stable manifolds [8].
3 Isospectral Theory for Discrete NLS Equation

In this section, we will study the isospectral theory for the discrete nonlinear Schrödinger equation

$$i\dot{q}_n = \frac{1}{\hbar^2} [q_{n+1} - 2q_n + q_{n-1}] + |q_n|^2(q_{n+1} + q_{n-1}) - 2\omega^2 q_n,$$  \hspace{1cm} (3.1)

where \( i = \sqrt{-1} \), \( q_n \)'s are complex variables, \( n \in \mathbb{Z} \); under periodic and even boundary conditions,

\[ q_{n+N} = q_n, \quad q_{-n} = q_n. \]

Since it is brand new, we will present the isospectral theory for the discrete NLS equation in details.

The integrability of the NLS equation (3.1) is proven with the use of the discretized Lax pair [1]:

\[ \varphi_{n+1} = L_n^{(z)} \varphi_n, \]
\[ \dot{\varphi}_n = B_n^{(z)} \varphi_n, \]

where

\[ L_n^{(z)} \equiv \begin{pmatrix} z & i\hbar q_n \\ i\hbar \bar{q}_n & 1/z \end{pmatrix}, \]
\[ B_n^{(z)} \equiv \frac{i}{\hbar^2} \begin{pmatrix} 1 - z^2 + 2i\lambda h - h^2 \bar{q}_n q_{n-1} + \omega^2 \hbar^2 & -izh \bar{q}_n + (1/z)i\hbar q_{n-1} \\ -iz\bar{q}_n + (1/z)i\hbar \bar{q}_n - 1 + 2i\lambda h + h^2 \bar{q}_n q_{n-1} - \omega^2 \hbar^2 & 1/z^2 - 1 \end{pmatrix}, \]

and where \( z \equiv \exp(i\lambda h) \). Compatibility of the over determined system (3.2,3.3) gives the "Lax representation"

\[ \dot{L}_n = B_{n+1}L_n - L_nB_n \]

of the discrete NLS equation (3.1). Focusing attention upon the discrete spatial flow (3.2), we let \( Y^{(1)}, Y^{(2)} \) be the fundamental solutions of the ODE (3.2), i.e. solutions with the initial conditions:

\[ Y_0^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Y_0^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

The Floquet discriminant:

\[ \Delta : C \times S \mapsto C \]  \hspace{1cm} (3.4)
is defined by
\[ \Delta(z; \vec{q}) \equiv \text{tr}\{M(N; z; \vec{q})\}, \tag{3.5} \]
where \( S \) is the phase space defined in (2.4), and \( M(n; z; \vec{q}) = \text{columns}\{Y_n^{(1)}, Y_n^{(2)}\} \)
is the fundamental matrix of (3.2).

**Remark 3.1** \( \Delta(z; \vec{q}) \) is a constant of motion for the integrable system (3.1) for any \( z \in C \) [13]. Since \( \Delta(z; \vec{q}) \) is a meromorphic function in \( z \) of degree \((+N, -N)\), the Floquet discriminant \( \Delta(z; \vec{q}) \) acts as a generating function for \((M + 1)\) functionally independent constants of motion, and is the key to the complete integrability of the system (3.1), where \( M = N/2 \) (\( N \) even), \( M = (N - 1)/2 \) (\( N \) odd).

The Floquet theory here is not standard as can be seen from the Wronskian relation:
\[ W_N(\psi^+, \psi^-) = D^2 W_0(\psi^+, \psi^-), \]
where \( D \) is defined in (2.3),
\[ W_n(\psi^+, \psi^-) \equiv \psi_n^{(+1)}\psi_n^{(-2)} - \psi_n^{(+2)}\psi_n^{(-1)}, \]
\( \psi^+ \) and \( \psi^- \) are any two solutions to the linear system (3.2). In fact, \( W_{n+1}(\psi^+, \psi^-) = \rho_n W_n(\psi^+, \psi^-) \). Due to this nonstandardness, modifications of the usual definitions of spectral quantities [6] are required. The Floquet spectrum is defined as the closure of the complex \( \lambda \) for which there exists a bounded eigenfunction to the ODE (3.2). In terms of the Floquet discriminant \( \Delta \), this is given by
\[ \sigma(L) = \left\{ z \in C \mid -2D \leq \Delta(z; \vec{q}) \leq 2D \right\}. \]

Periodic and antiperiodic points \( z^s \) are defined by
\[ \Delta(z^s; \vec{q}) = \pm 2D. \]

A critical point \( z^c \) is defined by the condition
\[ \frac{d\Delta}{dz}_{(z^c; \vec{q})} = 0. \]

A multiple point \( z^m \) is a critical point which is also a periodic or antiperiodic point. The *algebraic multiplicity* of \( z^m \) is defined as the order of the zero of \( \Delta(z) \pm 2D \). Usually it is 2, but it can exceed 2; when it does equal
2, we call the multiple point a *double point*, and denote it by $z^d$. The *geometric multiplicity* of $z^m$ is defined as the dimension of the periodic (or antiperiodic) eigenspace of (3.2) at $z^m$, and is either 1 or 2.

The normalized Floquet discriminant $\tilde{\Delta}$ is defined as

$$\tilde{\Delta} = \Delta/D.$$ 

**Remark 3.2 (Continuum Limit)** In the continuum limit (i.e. $h \to 0$), the Hamiltonian has a limit in the manner: $H/h \to H_c$, where $H_c$ is the Hamiltonian for NLS PDE, $H_c = i \int_0^1 \{q_x \bar{q}_x + 2\omega^2|q|^2 - |q|^4\}dx$. The Lax pair (3.2;3.3) also tends to the corresponding Lax pair for NLS PDE with spectral parameter $\lambda (z = e^{i\lambda h})$ [9]. If $Q \equiv \max_n \{|q_n|\}$ is finite, then $\rho_n \to 1$ as $h \to 0$. Therefore, $D^2 \equiv \{\prod_{n=0}^{N-1} \rho_n\} \to 1$ as $h \to 0$. The nonstandard Floquet theory for the spatial part of the Lax pair (3.2) becomes the standard Floquet theory in the continuum limit.

### 3.1 Examples of Floquet Spectra

Consider the uniform solution: $q_n = q_c, \forall n$

$$q_c(t) = a \exp \left\{ -i \left[ 2(a^2 - \omega^2)t - \gamma \right] \right\}.$$ 

The corresponding Bloch functions of the Lax pair are given by:

$$\psi_n^+ = (\sqrt{\rho} e^{i\beta})^n e^{\Omega_+ t} \left( \begin{array}{c}
\left( \frac{1}{z} - \sqrt{\rho} e^{i\beta} \right) \exp \{ -i [(a^2 - \omega^2)t - \gamma/2] \} \\
-iha \exp \{ i [(a^2 - \omega^2)t - \gamma/2] \}
\end{array} \right),$$

$$\psi_n^- = (\sqrt{\rho} e^{-i\beta})^n e^{\Omega_- t} \left( \begin{array}{c}
\left( z - \sqrt{\rho} e^{-i\beta} \right) \exp \{ i [(a^2 - \omega^2)t - \gamma/2] \} \\
-iha \exp \{ i [(a^2 - \omega^2)t - \gamma/2] \}
\end{array} \right),$$

where

$$z = \sqrt{\rho} \cos \beta + \sqrt{\rho} \cos^2 \beta - 1, \quad \rho = 1 + h^2 a^2,$$

$$\Omega_+ = \frac{i}{h^2} \left\{ \left( \frac{1}{z} - z \right) \sqrt{\rho} e^{i\beta} + i2\lambda h \right\},$$

$$\Omega_- = \frac{i}{h^2} \left\{ \left( \frac{1}{z} - z \right) \sqrt{\rho} e^{-i\beta} + i2\lambda h \right\}.$$ 

The Floquet discriminant is given by:

$$\Delta = 2D \cos(N\beta),$$

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where $D = \rho^{N/2}$. Thus the Floquet spectra are given by:

$$-1 \leq \cos(N\beta) \leq 1.$$ 

See figure 3.1 for an illustration of the Floquet spectra. Periodic and antiperiodic points are given by:

$$z_m^{(s)} = \sqrt{\rho \cos \frac{m}{N} \pi} + \sqrt{\rho \cos^2 \frac{m}{N} \pi - 1},$$

where $z_m^{(s)}$ is a periodic point when $m$ is even, and $z_m^{(s)}$ is an antiperiodic point when $m$ is odd. See figure 3.1 for an illustration of the periodic and antiperiodic points. The following facts are obvious,

- If $N$ is odd, all the periodic and antiperiodic points are on the real axis for sufficiently large $|q_c|$.

- If $N$ is even, except the two points $z = \pm i$, all other periodic and antiperiodic points are on the real axis for sufficiently large $|q_c|$.

Derivatives of the Floquet discriminant $\Delta$ with respect to $z$ are given by:

$$d\Delta/dz = 2ND \sin(N\beta)[z\sqrt{\rho \sin \beta}]^{-1} \sqrt{\rho \cos^2 \beta - 1},$$
\[ d^2 \Delta / dz^2 = -2ND[\rho z^2 \sin^3 \beta]^{-1} \left[ N \cos(N\beta) \sin \beta (\rho \cos^2 \beta - 1) + (1 - \rho) \cos \beta \sin(N\beta) + \sqrt{\rho} \sin(N\beta) \sin^2 \beta \sqrt{\rho \cos^2 \beta - 1} \right]. \]

The critical points are given by:

\[ \beta = \frac{m}{N} \pi \quad (\beta \neq 0, \pi), \quad \text{or} \quad \cos^2 \beta = \frac{1}{\rho}. \]

See figure 3.1 for an illustration of the critical points and double points. There can be multiple points with algebraic multiplicity greater than 2. For example, when two symmetric double points on the circle collide at the intersection points \( z = \pm 1 \), we have multiple points of algebraic multiplicity 4. In such case, \( \rho \cos^2 \beta - 1 = 0 \) and \( \sin(N\beta) = 0 \); then \( d^2 \Delta / dz^2 = 0 \), at \( z = \pm 1 \).

### 3.2 An Important Sequence of Invariants

**Definition 1** The sequence of invariants \( \tilde{F}_j \) is defined as:

\[ \tilde{F}_j(\vec{q}) = \tilde{\Delta}(z_j^{(c)}(\vec{q}); \vec{q}) . \] (3.6)

These invariants \( \tilde{F}_j \)'s are perfect candidate for building Melnikov functions. The Melnikov vectors are given by the gradients of these invariants.

**Lemma 3.1** Let \( z_j^{(c)}(\vec{q}) \) be a simple critical point; then

\[ \frac{\delta \tilde{F}_j}{\delta \vec{q}_n}(\vec{q}) = \frac{\delta \tilde{\Delta}}{\delta \vec{q}_n}(z_j^{(c)}(\vec{q}); \vec{q}) . \] (3.7)

\[ \frac{\delta \tilde{\Delta}}{\delta \vec{q}_n}(z; \vec{q}) = \frac{ih(\zeta - \zeta^{-1})}{2W_n+1} \begin{pmatrix} \psi_{n+1}^{(+,2)} \psi_n^{(-,2)} + \psi_n^{(+,2)} \psi_{n+1}^{(-,2)} \\ \psi_{n+1}^{(+,1)} \psi_n^{(-,1)} + \psi_n^{(+,1)} \psi_{n+1}^{(-,1)} \end{pmatrix} , \] (3.8)

where \( \psi_n^{\pm} = (\psi_n^{(\pm,1)}, \psi_n^{(\pm,2)})^T \) are two Bloch functions of the Lax pair (3.2,3.3), such that

\[ \tilde{\psi}_n^{\pm} = D_n/N \zeta^{\pm n/N} \tilde{\psi}_n^{\pm} , \]

where \( \tilde{\psi}_n^{\pm} \) are periodic in \( n \) with period \( N \), \( W_n = \det (\psi_n^{+}, \psi_n^{-}) \).
Proof: By the definition of critical points,
\[ \tilde{\Delta}'(z^{(c)}_j(q); \tilde{q}) = 0 . \]
Differentiating this equation, we have
\[ \frac{\delta z^{(c)}_j}{\delta \tilde{q}_n} = -\frac{1}{\Delta'' \delta \tilde{q}_n} \delta \tilde{\Delta}' . \]
Since \( z^{(c)}_j(q) \) is a simple critical point, \( z^{(c)}_j \) is a differentiable function. Thus
\[ \frac{\delta \tilde{F}_j}{\delta \tilde{q}_n} = \frac{\delta \tilde{\Delta}}{\delta \tilde{q}_n} \bigg|_{z=z^{(c)}_j} + \frac{\partial \tilde{\Delta}}{\partial z} \bigg|_{z=z^{(c)}_j} \frac{\delta z^{(c)}_j}{\delta \tilde{q}_n} \]
\[ = \frac{\delta \tilde{\Delta}}{\delta \tilde{q}_n} \bigg|_{z=z^{(c)}_j} . \]
This proves equation (3.7). Next we derive the formula (3.8). Let \( M_n \) be the fundamental matrix to the system (3.2), i.e. the matrix solution to (3.2) with initial condition \( M_0 \) being a 2 x 2 identity matrix. Variation of \( \tilde{q} \) leads to the variational equation for the variation of \( M_n \) at fixed \( z \),
\[ \delta M_{n+1} = \begin{pmatrix} z & ih \delta q_n \\ ih \tilde{q}_n & 1/z \end{pmatrix} \delta M_n + \begin{pmatrix} 0 & ih \delta q_n \\ ih \tilde{q}_n & 0 \end{pmatrix} M_n , \]
\[ \delta M_0 = 0 . \]
Let \( \delta M_n = M_n A_n \), where \( A_n \) is a 2 x 2 matrix to be determined, we have
\[ A_{n+1} - A_n = M_{n+1}^{-1} \delta U_n M_n , \]
(3.9)
\[ A_0 = 0 , \]
where
\[ \delta U_n = \begin{pmatrix} 0 & ih \delta q_n \\ ih \tilde{q}_n & 0 \end{pmatrix} . \]
Solving the system (3.9), we have
\[ \delta M_n = M_n \left[ \sum_{j=1}^{n} M_j^{-1} \delta U_{j-1} M_{j-1} \right] , \]
\[ \delta M_0 = 0 . \]
Then,
\[ \delta \Delta(z, \vec{q}) = \text{trace} \left\{ M_N \left[ \sum_{j=1}^{N} M_j^{-1} \delta U_{j-1} M_{j-1} \right] \right\}. \]

Thus,
\[ \frac{\delta \Delta}{\delta q_n} = i\hbar \text{trace} \left\{ M_{n+1}^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} M_n M_N \right\}, \quad (3.10) \]
\[ \frac{\delta \Delta}{\delta r_n} = -i\hbar \text{trace} \left\{ M_{n+1}^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_n M_N \right\}, \quad (3.11) \]

where \( r_n = -\bar{q}_n \). Let \( \psi^+ \) and \( \psi^- \) be two Bloch functions for the discrete Lax pairs (3.2,3.3),
\[ \psi^\pm_n = D^{n/N} \zeta^{\pm n/N} \tilde{\psi}^\pm_n, \quad (3.12) \]
where \( \tilde{\psi}^\pm_n \) are periodic in \( n \) with period \( N \). Let \( B_n \) be the 2 x 2 matrix with \( \psi^+_n \) and \( \psi^-_n \) as the column vectors,
\[ B_n = \begin{pmatrix} \psi^+_n \\ \psi^-_n \end{pmatrix}. \quad (3.13) \]

Then
\[ M_n = B_n B_0^{-1}, \quad M_N = B_0 \begin{pmatrix} D \zeta & 0 \\ 0 & D^{-1} \zeta \end{pmatrix} B_0^{-1}. \quad (3.14) \]

Substitute the representations (3.13) and (3.14) into (3.10,3.11), we have
\[ \frac{\delta \Delta}{\delta \bar{q}_n} = \tilde{\Delta} \frac{\delta \bar{q}_n}{\delta \bar{q}_n} + i\hbar \frac{Dh(\zeta - \zeta^{-1})}{2W_{n+1}} \begin{pmatrix} \psi^{(+,2)}_n \psi^{(-,2)}_n + \psi^{(+,2)}_n \psi^{(-,2)}_{n+1} \\ \psi^{(+,1)}_n \psi^{(-,1)}_n + \psi^{(+,1)}_n \psi^{(-,1)}_{n+1} \end{pmatrix}, \]
where
\[ W_n = \det B_n, \quad \psi^\pm_n = (\psi^{(+,1)}_n, \psi^{(+,2)}_n)^T. \]

Thus,
\[ \frac{\delta \bar{\Delta}}{\delta \bar{q}_n} = \frac{i\hbar (\zeta - \zeta^{-1})}{2W_{n+1}} \begin{pmatrix} \psi^{(+,2)}_n \psi^{(-,2)}_n + \psi^{(+,2)}_n \psi^{(-,2)}_{n+1} \\ \psi^{(+,1)}_n \psi^{(-,1)}_n + \psi^{(+,1)}_n \psi^{(-,1)}_{n+1} \end{pmatrix}, \]
which is the formula (3.8). The lemma is proved. \( \square \)
3.3 Bäcklund-Darboux Transformations

The hyperbolic structure and homoclinic orbits for (3.1) are constructed through the Bäcklund-Darboux transformations, which were built in [6]. First, we present the Bäcklund-Darboux transformations. Then, we show how to construct homoclinic orbits.

Fix a solution \( q_n(t) \) of the system (3.1), for which the linear operator \( L_n \) has a double point \( z^d \) of geometric multiplicity 2, which is not on the unit circle. We denote two linearly independent solutions (Bloch functions) of the discrete Lax pair (3.2;3.3) at \( z = z^d \) by \((\phi_n^+, \phi_n^-)\). Thus, a general solution of the discrete Lax pair (3.2;3.3) at \( (q_n(t), z^d) \) is given by

\[
\phi_n(t; z^d, c^+, c^-) = c^+ \phi_n^+ + c^- \phi_n^-,
\]

where \( c^+ \) and \( c^- \) are complex parameters. We use \( \phi_n \) to define a transformation matrix \( \Gamma_n \) by

\[
\Gamma_n = \begin{pmatrix}
z + (1/z) a_n & b_n \\
c_n & -1/z + zd_n
\end{pmatrix},
\]

where,

\[
a_n = \frac{z^d}{(z^d)^2 \Delta_n} \left[ |\phi_{n2}|^2 + |z^d|^2 |\phi_{n1}|^2 \right],
\]

\[
d_n = -\frac{1}{z^d \Delta_n} \left[ |\phi_{n2}|^2 + |z^d|^2 |\phi_{n1}|^2 \right],
\]

\[
b_n = \frac{|z^d|^4 - 1}{(z^d)^2 \Delta_n} \phi_{n1} \bar{\phi}_{n2},
\]

\[
c_n = \frac{|z^d|^4 - 1}{z^d |z^d|^2 \Delta_n} \bar{\phi}_{n1} \phi_{n2},
\]

\[
\Delta_n = -\frac{1}{z^d} \left[ |\phi_{n1}|^2 + |z^d|^2 |\phi_{n2}|^2 \right].
\]

From these formulae, we see that

\[
\bar{a}_n = -d_n, \quad \bar{b}_n = c_n.
\]

Then we define \( Q_n \) and \( \Psi_n \) by

\[
Q_n = \frac{i}{n} b_{n+1} - a_{n+1} q_n
\] (3.15)
\[ \Psi_n(t; z) \equiv \Gamma_n(z; z^d, \phi_n)\psi_n(t; z) \quad (3.16) \]

where \( \psi_n \) solves the discrete Lax pair (3.2;3.3) at \((q_n(t), z)\). Formulas (3.15) and (3.16) are the Bäcklund-Darboux transformations for the potential and eigenfunctions, respectively. We have the following theorem [6].

**Theorem 3.1 (Bäcklund-Darboux Transformations)** Let \( q_n(t) \) denote a solution of the system (3.1), for which the linear operator \( L_n \) has a double point \( z^d \) of geometric multiplicity 2, which is not on the unit circle and which is associated with an instability. We denote two linearly independent solutions (Bloch functions) of the discrete Lax pair (3.2;3.3) at \((q_n, z^d)\) by \((\phi_n^+, \phi_n^-)\). We define \( Q_n(t) \) and \( \Psi_n(t; z) \) by (3.15) and (3.16). Then

1. \( Q_n(t) \) is also a solution of the system (3.1). (The eveness of \( Q_n \) can be obtained by choosing the complex Bäcklund parameter \( c^+ / c^- \) to lie on a certain curve, as shown in the example below.)

2. \( \Psi_n(t; z) \) solves the discrete Lax pair (3.2;3.3) at \((Q_n(t), z)\).

3. \( \Delta(z; Q_n) = \Delta(z; q_n) \), for all \( z \in \mathbb{C} \).

4. \( Q_n(t) \) is homoclinic to \( q_n(t) \) in the sense that \( Q_n(t) \to e^{i\theta^\pm} q_n(t) \), exponentially as \( \exp(-\sigma|t|) \) as \( t \to \pm\infty \). Here \( \theta^\pm \) are the phase shifts, \( \sigma \) is a nonvanishing growth rate associated to the double point \( z^d \), and explicit formulas can be developed for this growth rate and for the phase shifts \( \theta^\pm \).

This theorem is quite general, constructing homoclinic solutions from a wide class of starting solutions \( q_n(t) \). Its proof is by direct verification [6].

### 3.4 Heteroclinic Orbits and Melnikov Vectors

Through Bäcklund-Darboux transformations, heteroclinic orbits \( Q_n \) are generated, with the formula expression (3.15)

\[
Q_n = \left( i h^{-1} b_{n+1} - a_{n+1} q_n \right) \\
= \left[ z^d (|\phi^{(1)}_{n+1}|^2 + |z^d| |\phi^{(2)}_{n+1}|^2) \right]^{-1} \times \\
\left[ \left( i h^{-1} (1 - |z^d|^4) \phi^{(1)}_{n+1} \overline{\phi^{(2)}_{n+1}} + z^d q_n (|\phi^{(2)}_{n+1}|^2 + |z^d|^2 |\phi^{(1)}_{n+1}|^2) \right) \right].
\]
The Melnikov vector field located on this heteroclinic orbit is given by

\[
\frac{\delta \tilde{\Delta}(z^{(d)}; \vec{Q})}{\delta \tilde{Q}_n} = K \frac{W_n}{E_n A_n+1} \begin{pmatrix}
[z^{(d)}]^{-2} \phi_n^{(1)} \\
[z^{(d)}]^{-2} \phi_n^{(2)}
\end{pmatrix},
\]

(3.17)

where

\[
\phi_n = (\phi_n^{(1)}, \phi_n^{(2)})^T = c_+ \psi_n^+ + c_- \psi_n^- ,
\]

\[
W_n = \begin{vmatrix}
\psi_n^+ \\
\psi_n^-
\end{vmatrix},
\]

\[
E_n = |\phi_n^{(1)}|^2 + |z^{(d)}|^2 |\phi_n^{(2)}|^2 ,
\]

\[
A_n = |\phi_n^{(2)}|^2 + |z^{(d)}|^2 |\phi_n^{(1)}|^2 ,
\]

\[
K = - \frac{ihc_+ c_-}{2} |z^{(d)}|^4 (|z^{(d)}|^4 - 1) [\tilde{\Delta}(z^{(d)}; \vec{q})]^{-1} \tilde{\Delta}''(z^{(d)}; \vec{q}) ,
\]

where

\[
\tilde{\Delta}''(z^{(d)}; \vec{q}) = \frac{\partial^2 \tilde{\Delta}(z^{(d)}; \vec{q})}{\partial z^2} .
\]

### 3.4.1 An Important Example

In this subsection, we start with the uniform solution of (3.1)

\[
q_n = q_c , \forall n; \quad q_c = a \exp \left\{ -i [2(a^2 - \omega^2)t - \gamma] \right\} .
\]

(3.18)

We choose the amplitude \( a \) in the range

\[
N \tan \frac{\pi}{N} < a < N \tan \frac{2\pi}{N} , \quad \text{when } N > 3 ,
\]

(3.19)

\[
3 \tan \frac{\pi}{3} < a < \infty , \quad \text{when } N = 3 ;
\]

so that there is only one set of quadruplets of double points which are not on the unit circle, and denote one of them by \( z = z^{(d)} = z^{(c)} \) which corresponds to \( \beta = \pi/N \) (see subsection 3.1 and figure 3.1). The heteroclinic orbit \( Q_n \) is given by

\[
Q_n = q_c (\hat{E}_{n+1})^{-1} \left[ \hat{A}_{n+1} - 2 \cos \beta \sqrt{\rho \cos^2 \beta - 1} \hat{B}_{n+1} \right] ,
\]

(3.20)
and the Melnikov vectors evaluated on this heteroclinic orbit are given by

\[
\frac{\delta \tilde{F}_1}{\delta Q_n} = \hat{K} \left[ \hat{E}_n \hat{A}_{n+1} \right]^{-1} \text{sech} \left[ 2\mu t + 2p \right] \left( \begin{array}{c}
\hat{X}^{(1)}_n \\
-\hat{X}^{(2)}_n
\end{array} \right),
\]

(3.21)

where

\[
\hat{E}_n = ha \cos \beta + \sqrt{\rho \cos^2 \beta - 1} \text{sech} \left[ 2\mu t + 2p \right] \cos \left[ (2n + 1)\beta + \vartheta \right],
\]

\[
\hat{A}_{n+1} = ha \cos \beta + \sqrt{\rho \cos^2 \beta - 1} \text{sech} \left[ 2\mu t + 2p \right] \cos \left[ (2n + 3)\beta + \vartheta \right],
\]

\[
\hat{B}_{n+1} = \cos \varphi + i \sin \varphi \tanh \left[ 2\mu t + 2p \right] + \text{sech} \left[ 2\mu t + 2p \right] \cos \left[ 2(n + 1)\beta + \vartheta \right],
\]

\[
\hat{X}^{(1)}_n = \left[ \cos \beta \text{sech} \left[ 2\mu t + 2p \right] + \cos \left[ (2n + 1)\beta + \vartheta + \varphi \right] - i \tanh \left[ 2\mu t + 2p \right] \sin \left[ (2n + 1)\beta + \vartheta + \varphi \right] \right] e^{i\theta(t)},
\]

\[
\hat{X}^{(2)}_n = \left[ \cos \beta \text{sech} \left[ 2\mu t + 2p \right] + \cos \left[ (2n + 1)\beta + \vartheta - \varphi \right] - i \tanh \left[ 2\mu t + 2p \right] \sin \left[ (2n + 1)\beta + \vartheta - \varphi \right] \right] e^{-i\theta(t)},
\]

\[
\hat{K} = -2Nh^2 a(1 - z^4) \left[ 8\rho^{3/2} z^2 \right]^{-1} \sqrt{\rho \cos^2 \beta - 1},
\]

\[
\beta = \pi/N, \quad \rho = 1 + h^2 a^2, \quad \mu = 2h^{-2} \sqrt{\rho \sin \beta} \sqrt{\rho \cos^2 \beta - 1},
\]

\[
h = 1/N, \quad c_+ / c_- = i e^{2p} e^{i\vartheta}, \quad \vartheta \in [0, 2\pi], \quad p \in (-\infty, \infty),
\]

\[
z = \sqrt{\rho} \cos \beta + \sqrt{\rho \cos^2 \beta - 1}, \quad \theta(t) = (a^2 - \omega^2)t - \gamma/2,
\]

\[
\sqrt{\rho \cos^2 \beta - 1} + i \sqrt{\rho} \sin \beta = hae^{i\varphi},
\]

where \( \varphi = \sin^{-1} \left[ \sqrt{\rho} (ha)^{-1} \sin \beta \right], \varphi \in (0, \pi/2) \).

Next we study the “evenness” condition: \( Q_{-n} = Q_n \). It turns out that the choices \( \vartheta = -\beta, -\beta + \pi \) in the formula of \( Q_n \) lead to the evenness of \( Q_n \) in \( n \). In terms of figure eight structure of \( Q_n \), \( \vartheta = -\beta \) corresponds to one ear of the figure eight, and \( \vartheta = -\beta + \pi \) corresponds to the other ear.

The even formula for \( Q_n \) is given by,

\[
Q_n = q_c \left[ \Gamma / \Lambda_n - 1 \right],
\]

(3.22)
where
\[
\Gamma = 1 - \cos 2\varphi - i \sin 2\varphi \tanh[2\mu t + 2p],
\]
\[
\Lambda_n = 1 \pm \cos \varphi \cos \beta \sech[2\mu t + 2p] \cos[2n\beta],
\]
where \(('+')\) corresponds to \(\vartheta = -\beta\). The Melnikov vectors evaluated on these heteroclinic orbits are not necessarily even and are in fact not even in \(n\). For the purpose of calculating the Melnikov functions, only the even parts of the Melnikov vectors are needed, which are given by
\[
\left. \frac{\delta \tilde{F}_1}{\delta \tilde{Q}_n} \right|_{\text{even}} = \hat{K}^{(e)} \sech[2\mu t + 2p] \Pi_n^{-1} \left( \begin{array}{c} \hat{X}_n^{(1,e)} \\ -\hat{X}_n^{(2,e)} \end{array} \right),
\]
(3.23)
where
\[
\hat{K}^{(e)} = -2N(1 - z^4)[8\alpha \rho^{3/2} z^2]^{-1} \rho \cos^2 \beta - 1,
\]
\[
\Pi_n = \left[ \cos \beta \pm \cos \varphi \sech[2\mu t + 2p] \cos[2(n - 1)\beta] \right] \times \left[ \cos \beta \pm \cos \varphi \sech[2\mu t + 2p] \cos[2(n + 1)\beta] \right],
\]
\[
\hat{X}_n^{(1,e)} = \left[ \cos \beta \sech[2\mu t + 2p] \pm (\cos \varphi \\
- i \sin \varphi \tanh[2\mu t + 2p] \cos[2n\beta]) e^{i2\theta(t)} \right],
\]
\[
\hat{X}_n^{(2,e)} = \left[ \cos \beta \sech[2\mu t + 2p] \pm (\cos \varphi \\
+ i \sin \varphi \tanh[2\mu t + 2p] \cos[2n\beta]) e^{-i2\theta(t)} \right].
\]

The heteroclinic orbit (3.22) represents the figure eight structure as illustrated in Fig.3.2. If we denote by \(S\) the circle, we have the topological identification:
\[
\text{figure 8) \otimes S} = \bigcup_{p \in (-\infty, \infty), \gamma \in [0,2\pi]} Q_n(p, \gamma, a, \omega, \pm, N).
\]
4 Coordinate-Expressions for Invariant Submanifolds

In this section, we will give expressions for invariant submanifolds in coordinates, which enable us to do a Melnikov analysis and an implicit function argument.

4.1 Coordinate-Expressions for Linear Invariant Submanifolds

Consider the discrete cubic integrable nonlinear Schrödinger equation (3.1), denote by $q_c$ the uniform Stokes solution,

$$q_c = ae^{i\theta(t)}, \quad \theta(t) = -[2(a^2 - \omega^2)t - \gamma].$$

Let

$$q_n = [a + \tilde{q}_n]e^{i\theta(t)},$$

and linearize equation (3.1) at $q_c$, we have

$$i\dot{q}_n = \frac{1}{\hbar^2}[\tilde{q}_{n+1} - 2\tilde{q}_n + \tilde{q}_{n-1}]$$
\[ \tilde{q}_n + a^2[\tilde{q}_{n+1} + \tilde{q}_{n-1}] + 2a^2 \tilde{q}_n. \]

Assume that \( \tilde{q}_n \) takes the form,
\[ \tilde{q}_n = \left[ A_j e^{\Omega_j t} + B_j e^{-\Omega_j t} \right] \cos k_j n, \]
where \( k_j = 2j\pi/N, \) \( (j = 0, 1, \ldots, M). \) Then,
\[ \Omega_j(\pm) = \pm 2 \sin k_j \sqrt{a^2 + N^2 \sqrt{a^2 - N^2 \tan^2[k_j/2]}}. \]

In this paper, we only study the case that \( a \) lies in the range given in (3.19), so that only \( \Omega_1(\pm) \) are real and nonzero. In fact, \( \Omega_0(\pm) \) are zero, and \( \Omega_j(\pm) \) are imaginary for \( (j > 1). \) When \( j = 1, \)
\[ \tilde{q}_n = a_1^{(\pm)} e^{\Omega_1^{(\pm)} t} e^{\pm i \vartheta} \cos k_1 n, \]
where \( a_1^{(\pm)} \) are real constants;
\[ \vartheta = -\frac{1}{2} \arctan \left\{ \left( (a^2 + N^2) \sin^2(2\pi/N) \right) \left( a^2 - N^2 \tan^2(\pi/N) \right) \right\}^{1/2} [N^2 - (N^2 + a^2) \cos(2\pi/N)]^{-1} \right\}. \]

Denote by \( B \) the block in the phase space \( S, \)
\[ B \equiv \left\{ \tilde{q} \in S \mid q_n = e^{i\gamma} \left[ a + (b_1 e^{i \vartheta} + b_2 e^{-i \vartheta}) \cos k_1 n \right. \right. \]
\[ \left. + \sum_{j=2}^M c_j \cos k_j n \right], \text{ where } a \in (N \tan[\pi/N], N \tan[2\pi/N]), \gamma \in [0, 2\pi); \]
\[ b_1, b_2 \text{ are real}; \ c_j \text{ is complex}; \text{ and } \vartheta \text{ is given above} \right\} . \]

In terms of the coordinates \( \{a, \gamma, b_1, b_2, c_j(2 \leq j \leq M)\}, \) the linear invariant center manifold \( L^{(c)} \) is given by,
\[ L^{(c)} \equiv \left\{ \tilde{q} \in B \mid b_1 = b_2 = 0 \right\}, \]
the linear invariant center-unstable manifold \( L^{(cu)} \) is given by,
\[ L^{(cu)} \equiv \left\{ \tilde{q} \in B \mid b_2 = 0 \right\}, \]
the linear invariant center-stable manifold \( L^{(cs)} \) is given by,
\[ L^{(cs)} \equiv \left\{ \tilde{q} \in B \mid b_1 = 0 \right\}. \]

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4.2 Coordinate-Expressions for Persistent Locally Invariant Submanifolds

Under the perturbed flow (2.5), the linear invariant submanifolds \( \mathcal{L}_c \), \( \mathcal{L}_{cu} \) and \( \mathcal{L}_{cs} \) perturb into locally invariant submanifolds \( W_c \), \( W_{cu} \) and \( W_{cs} \) (local invariance means that orbits can only enter or leave the submanifolds through their boundaries). For references on the proof of such results, see for example [4] [10] [12]. In terms of the coordinates \( \{a, \gamma, b_1, b_2, c_j \ (2 \leq j \leq M)\} \), for any small \( \delta_1 > 0 \), there exist \( \delta_l > 0 \) (\( l = 0, 2, 3 \)), such that \( W_c \) has the expression,

\[
\begin{align*}
\begin{cases}
    b_1 = f_1^{(c)}(a, \gamma, \vec{c}; \varepsilon, \alpha, \omega), \\
    b_2 = f_2^{(c)}(a, \gamma, \vec{c}; \varepsilon, \alpha, \omega);
\end{cases}
\end{align*}
\]

\( W_{cu} \) has the expression,

\[
b_2 = f^{(u)}(a, \gamma, b_1, \vec{c}; \varepsilon, \alpha, \omega);
\]

and \( W_{cs} \) has the expression,

\[
b_1 = f^{(s)}(a, \gamma, b_2, \vec{c}; \varepsilon, \alpha, \omega);
\]

where \( f_l^{(c)} \ (l = 1, 2) \), \( f^{(u)} \) and \( f^{(s)} \) are \( C^n \) smooth functions for some large \( n \), \( \alpha = (\alpha_1, \alpha_2) \), \( |\varepsilon| < \delta_0 \), \( \gamma \in [0, 2\pi) \), \( |b_k| < \delta_2 \) (\( k = 1, 2 \)), \( \omega \in E_{\delta_1} \), \( a \in E_{\delta_1} \), \( E_{\delta_1} = (N \tan(\pi/N) + \delta_1, N \tan(2\pi/N) - \delta_1) \) when \( N > 4 \) and \( = (N \tan(\pi/N) + \delta_1, \Lambda) \) when \( N = 3, 4 \), where \( \Lambda \) is a fixed large constant, \( \vec{c} = (c_2, \cdots, c_M) \), \( ||c_j|| < \delta_3 \) (\( 2 \leq j \leq M \)). Denote by \( \mathcal{A} \) the annulus,

\[
\mathcal{A} \equiv \left\{ \vec{q} \in B \mid b_1 = b_2 = \vec{c} = 0, \ a \in E_{\delta_1}, \ \gamma \in [0, 2\pi) \right\}.
\]

Then, \( \mathcal{A} \subset \mathcal{L}_c \), and \( \mathcal{A} \subset W_c \); thus,

\[
f_l^{(c)}(a, \gamma, 0; \varepsilon, \alpha, \omega) = 0, \quad (l = 1, 2).
\]

Notice also that \( W^{(c)} = W^{(cu)} \cap W^{(cs)} \); then,

\[
\begin{align*}
    f^{(u)}(a, \gamma, f_1^{(c)}(a, \gamma, \vec{c}; \varepsilon, \alpha, \omega), \vec{c}; \varepsilon, \alpha, \omega) &= f_2^{(c)}(a, \gamma, \vec{c}; \varepsilon, \alpha, \omega), \\
    f^{(s)}(a, \gamma, f_2^{(c)}(a, \gamma, \vec{c}; \varepsilon, \alpha, \omega), \vec{c}; \varepsilon, \alpha, \omega) &= f_1^{(c)}(a, \gamma, \vec{c}; \varepsilon, \alpha, \omega).
\end{align*}
\]
4.3 Coordinate-Expressions for Fenichel Fibers

The center-unstable and center-stable manifolds $W^{(cu)}$ and $W^{(cs)}$ admit fibration through Fenichel fibers. For references on the proof of such results, see for example [10] [12]. Take unstable Fenichel fibers, for example, they are a family of curves in $W^{(cu)}$, each of them is labeled by a base point in $W^{(c)}$. $W^{(cu)}$ is a union of the family of curves over $W^{(c)}$. In this sense, Fenichel fibers are coordinates for $W^{(cu)}$. Indeed, they are very good coordinates. Fenichel fibers depend $C^{n-1}$ smoothly on their base points. In backward time, orbits starting from points on the same fiber (in particular, from the base point) approach each other exponentially.

The unstable Fenichel fibers $\{\mathcal{F}_q^{(u)}\}$ have the coordinate-expressions:

\[
\begin{align*}
  a &= a^{(u)}(b_1; a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega), \\
  \gamma &= \gamma^{(u)}(b_1; a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega), \\
  \vec{c} &= \vec{c}^{(u)}(b_1; a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega), \\
  b_2 &= f^{(u)}(a^{(u)}, \gamma^{(u)}, b_1, \vec{c}^{(u)}; \varepsilon, \alpha, \omega);
\end{align*}
\]

which are $C^n$ smooth in $b_1$, and $C^{n-1}$ smooth in $(a_0, \gamma_0, \vec{c}_0)$ and $(\varepsilon, \alpha, \omega)$, where $b_1 \in (-\delta_2, \delta_2)$. The base points $q$ are given by,

\[
\begin{align*}
  a^{(u)}(f_1^{(c)}; a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega) &= a_0, \\
  \gamma^{(u)}(f_1^{(c)}; a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega) &= \gamma_0, \\
  \vec{c}^{(u)}(f_1^{(c)}; a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega) &= \vec{c}_0, \\
  f^{(u)}(a^{(u)}, \gamma^{(u)}, f_1^{(c)}, \vec{c}^{(u)}; \varepsilon, \alpha, \omega) &= f^{(u)}(a_0, \gamma_0, f_1^{(c)}, \vec{c}_0; \varepsilon, \alpha, \omega) \\
  &= f_2^{(c)}(a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega),
\end{align*}
\]

where $f_1^{(c)} = f_1^{(c)}(a_0, \gamma_0; \vec{c}_0; \varepsilon, \alpha, \omega)$.

The stable Fenichel fibers $\{\mathcal{F}_q^{(s)}\}$ have the coordinate-expressions:

\[
\begin{align*}
  a &= a^{(s)}(b_2; a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega), \\
  \gamma &= \gamma^{(s)}(b_2; a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega), \\
  \vec{c} &= \vec{c}^{(s)}(b_2; a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega), \\
  b_1 &= f^{(s)}(a^{(s)}, \gamma^{(s)}, b_2, \vec{c}^{(s)}; \varepsilon, \alpha, \omega);
\end{align*}
\]
which are \( C^n \) smooth in \( b_2 \), and \( C^{n-1} \) smooth in \((a_0, \gamma_0, \vec{c}_0)\) and \((\varepsilon, \alpha, \omega)\), where \( b_2 \in (-\delta_2, \delta_2) \). The base points \( q \) are given by,

\[
\begin{align*}
    \alpha^{(s)}(f^{(c)}_2; a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega) &= a_0, \\
    \gamma^{(s)}(f^{(c)}_2; a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega) &= \gamma_0, \\
    \vec{c}^{(s)}(f^{(c)}_2; a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega) &= \vec{c}_0, \\
    f^{(s)}(\alpha^{(s)}, \gamma^{(s)}, f^{(c)}_2, \vec{c}^{(s)}; \varepsilon, \alpha, \omega) &= f^{(s)}(a_0, \gamma_0, f^{(c)}_2; \varepsilon, \alpha, \omega) \\
    &= f^{(c)}_1(a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega),
\end{align*}
\]

where \( f^{(c)}_2 = f^{(c)}_2(a_0, \gamma_0, \vec{c}_0; \varepsilon, \alpha, \omega) \).
5 Existence of Transversal Homoclinic Tubes

In this subsection, we will establish the existence of transversal homoclinic tubes, based upon the coordinatization in the previous subsection, a Melnikov function calculation, and an implicit function argument.

Define the region $\mathcal{U}$ as follows,

$$\mathcal{U} \equiv \{ \mathbf{q} \in B \mid b_2 \in \left( \frac{1}{2} \delta_2, \delta_2 \right) \}.$$ 

There exists $T > 0$ such that $F^T \circ W^{cu}$ intersects the region $\mathcal{U}$ (where $F^T$ is the evolution operator of the system (2.5)), the intersection $(F^T \circ W^{cu}) \cap \mathcal{U}$ has the expression,

$$b_1 = f^{(u)}_T(a, \gamma, b_2, \mathbf{c}; \varepsilon, \alpha, \omega).$$

Since for any fixed finite $T$, $F^T$ is a $C^\infty$ diffeomorphism, $f^{(u)}_T$ is also $C^\infty$ smooth. Notice also that when $\varepsilon = 0$, $W^{cu} = W^{cs}$, we have

$$f^{(u)}_T(a, \gamma, b_2, \mathbf{c}; 0, \alpha, \omega) = f^{(s)}(a, \gamma, b_2, \mathbf{c}; 0, \alpha, \omega).$$

Define the function,

$$\tilde{\Delta}(a_0, \gamma_0, \mathbf{c}_0, b_2; \varepsilon, \alpha, \omega) = f^{(u)}_T(a^{(s)}, \gamma^{(s)}(s), b_2, \mathbf{c}^{(s)}; \varepsilon, \alpha, \omega) - f^{(s)}(a^{(s)}, \gamma^{(s)}, b_2, \mathbf{c}^{(s)}; \varepsilon, \alpha, \omega),$$

where

$$a^{(s)} = a^{(s)}(b_2; a_0, \gamma_0, \mathbf{c}_0; \varepsilon, \alpha, \omega),$$

$$\gamma^{(s)} = \gamma^{(s)}(b_2; a_0, \gamma_0, \mathbf{c}_0; \varepsilon, \alpha, \omega),$$

$$\mathbf{c}^{(s)} = \mathbf{c}^{(s)}(b_2; a_0, \gamma_0, \mathbf{c}_0; \varepsilon, \alpha, \omega);$$

are defined in the coordinate-expression (4.2) for the stable Fenichel fibers, and $\tilde{\Delta}$ is a $C^{n-1}$ smooth function. Setting $b_2 = \frac{3}{4} \delta_2$, for sufficiently small $\varepsilon$ and $\|\mathbf{c}_0\| \left( \|\mathbf{c}_0\| = \sqrt{\sum_{j=2}^{M} |c_j|^2} \right)$, we have

$$\tilde{\Delta}(a_0, \gamma_0, \mathbf{c}_0, \frac{3}{4} \delta_2; \varepsilon, \alpha, \omega) = \varepsilon M_1(a_0, \gamma_0; \alpha, \omega) + \varepsilon R_1(a_0, \gamma_0, \mathbf{c}_0; \varepsilon, \alpha, \omega) + \varepsilon^2 R_2(a_0, \gamma_0, \mathbf{c}_0; \varepsilon, \alpha, \omega),$$

with
where $M_{\tilde{F}_1}(a_0, \gamma_0; \alpha, \omega)$ is the Melnikov function,

$$R_1 \sim O(\|\tilde{c}_0\|), \quad \text{as } \|\tilde{c}_0\| \to 0.$$  

For a derivation on this result, see for example [10]. Define the function $\Delta$ as follows,

$$\Delta = \Delta(a_0, \gamma_0, \tilde{c}_0; \varepsilon, \alpha, \omega) = \frac{1}{\varepsilon} \tilde{\Delta}(a_0, \gamma_0, 3\frac{3}{4}\delta_2; \varepsilon, \alpha, \omega)$$

$$= M_{\tilde{F}_1}(a_0, \gamma_0; \alpha, \omega) + R_1(a_0, \gamma_0, \tilde{c}_0; \varepsilon, \alpha, \omega) + \varepsilon R_2(a_0, \gamma_0, \tilde{c}_0; \varepsilon, \alpha, \omega) \quad (5.1)$$

Then $\Delta$ is $C^{n-2}$ smooth in $(a_0, \gamma_0, \tilde{c}_0; \varepsilon, \alpha, \omega)$. The Melnikov function $M_{\tilde{F}_1} = M_{\tilde{F}_1}(a, \gamma; \alpha, \omega)$ is given as:

$$M_{\tilde{F}_1} = i \int_{-\infty}^{\infty} \{\tilde{F}_1, H_1\} \bar{Q} dt = 2 \int_{-\infty}^{\infty} \sum_{n=0}^{N-1} \text{Im} \left\{ \frac{\partial \tilde{F}_1}{\partial r_n} \rho_n \frac{\partial H_1}{\partial q_n} \right\} \bar{Q} dt,$$

where

$$\{\tilde{F}_1, H_1\} = \sum_{n=0}^{N-1} \left[ \frac{\partial \tilde{F}_1}{\partial q_n} \rho_n \frac{\partial H_1}{\partial r_n} - \frac{\partial \tilde{F}_1}{\partial r_n} \rho_n \frac{\partial H_1}{\partial q_n} \right],$$

and $\bar{Q}$ is given in (3.22) and $\text{grad} \tilde{F}_1(\bar{Q})$ is given in (3.23). Thus,

$$\frac{1}{2} M_{\tilde{F}_1} = \alpha_1 \int_{-\infty}^{\infty} \sum_{n=0}^{N-1} \text{Im} \left\{ \frac{\partial \tilde{F}_1}{\partial r_n} \left[ \frac{\rho_n}{h^2} \ln \rho_n + (q_n + \bar{q}_n) q_n \rho_n \right] \right\} \bar{Q} dt$$

$$+ \alpha_2 \int_{-\infty}^{\infty} \sum_{n=0}^{N-1} \text{Im} \left\{ \frac{\partial \tilde{F}_1}{\partial r_n} \left[ 2q_n \frac{\rho_n}{h^2} \ln \rho_n + (q_n^2 + \bar{q}_n^2) q_n \right] \right\} \bar{Q} dt \quad (5.2)$$

$$\equiv f_1(a, \gamma; \omega, N) \alpha_1 + f_2(a, \gamma; \omega, N) \alpha_2 .$$

Setting $M_{\tilde{F}_1} = 0$ in (5.2), we have an equation of the form,

$$\alpha_1 - 4\omega \kappa \alpha_2 = 0 , \quad (5.3)$$

where

$$\kappa = \kappa(a, \gamma; \omega, N) = -\frac{1}{4\omega} f_2(a, \gamma; \omega, N) .$$
Figure 5.1: The graph of $\kappa$ for a fixed value of $\omega$.

Figure 5.2: The graph of $\kappa$ when $a = \omega$. 

Figure 5.3: The graph of $\frac{\partial M}{\partial \gamma_0}$ for a fixed value of $\omega$.

The graphs of $\kappa$ are shown in figures 5.1 and 5.2. Figure 5.2 is for the case when $q_c$ is a circle of fixed points. For the convenience of later argument, we denote $M_{F_1}$ simply by $M$ and denote the surface defined by (5.3) by $S_\gamma$:

$$S_\gamma : \gamma_0 = \Gamma^{(0)}(a_0; \alpha, \omega). \quad (5.4)$$

**Theorem 5.1** There exist a positive constant $\varepsilon_0 > 0$ and a region $E$ for $(\alpha, \omega)$, such that for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and any $(\alpha, \omega) \in E$, there exists a codimension 2 transversal homoclinic tube asymptotic to the codimension 2 center manifold $W^{(c)}$.

Proof: There exists a region $\mathcal{V}$ on the surface $S_\gamma$ in (5.4), such that

$$\frac{\partial}{\partial \gamma_0} M(a_0, \gamma_0; \alpha, \omega) \neq 0$$

and is bounded. See figures (5.3) and (5.4) for the corresponding graphs of $\frac{\partial M}{\partial \gamma_0}$. Next we want to solve the equation (5.1) by the implicit function theorem [[3],p265]. For any $(a_0^*, \gamma_0^*; \alpha^*, \omega^*) \in \mathcal{V}$,

$$\frac{\partial}{\partial \gamma_0} \Delta(a_0^*, \gamma_0^*, 0; 0, \alpha^*, \omega^*) = \frac{\partial}{\partial \gamma_0} M(a_0^*, \gamma_0^*; \alpha^*, \omega^*) \neq 0 \quad (5.5)$$

and is bounded. Then by the implicit function theorem [[3],p265], there is a neighborhood $\mathcal{W}^{(s)}$ of $(a_0^*, \bar{c}_0 = 0; \varepsilon = 0, \alpha^*, \omega^*)$ and a unique $C^{n-2}$ function,

$$\gamma_0 = \Gamma^{(s)}(a_0, \bar{c}_0; \varepsilon, \alpha, \omega)$$

defined in $\mathcal{W}^{(s)}$, such that

$$\Gamma^{(s)}(a_0^*, \bar{c}_0 = 0; \varepsilon = 0, \alpha^*, \omega^*) = \gamma_0^*.$$
Figure 5.4: The graph of $\frac{\partial M}{\partial \gamma_0}$ when $a = \omega$.

and

$$\Delta(a_0, \Gamma^{(s)}(a_0, \vec{c}_0; \varepsilon, \alpha, \omega), \vec{c}_0; \varepsilon, \alpha, \omega) = 0.$$ 

Since $\Delta$ is a $C^{n-2}$ smooth function, by relation (5.5), we have

$$\frac{\partial}{\partial \gamma_0} \Delta(a_0, \Gamma^{(s)}(a_0, \vec{c}_0; \varepsilon, \alpha, \omega), \vec{c}_0; \varepsilon, \alpha, \omega) \neq 0,$$

and is bounded for $(a_0, \vec{c}_0; \varepsilon, \alpha, \omega) \in W^{(s)}$. Thus, the center-unstable manifold $W^{(cu)}$ and the center-stable manifold $W^{(cs)}$ have a transversal intersection at the neighborhood $W^{(s)}$. Let

$$W = \bigcup_{(a_0, \gamma_0^*; \alpha^*, \omega^*) \in V} W^{(s)};$$

then there is a unique $C^{n-2}$ function,

$$\gamma_0 = \Gamma(a_0, \vec{c}_0; \varepsilon, \alpha, \omega) \quad (5.6)$$

defined in $W$, such that

$$\Gamma(a_0, \vec{c}_0 = 0; \varepsilon = 0, \alpha, \omega) = \Gamma^{(0)}(a_0; \alpha, \omega),$$

and

$$\Delta(a_0, \Gamma(a_0, \vec{c}_0; \varepsilon, \alpha, \omega), \vec{c}_0; \varepsilon, \alpha, \omega) = 0.$$

Notice that (5.6) defines a codimension 1 submanifold $W^{(c)}_b$ of the center manifold $W^{(c)}$, which has the expression,

$$b_1 = f_1^{(c)}(a_0, \Gamma(a_0, \vec{c}_0; \varepsilon, \alpha, \omega), \vec{c}_0; \varepsilon, \alpha, \omega),$$

$$b_2 = f_2^{(c)}(a_0, \Gamma(a_0, \vec{c}_0; \varepsilon, \alpha, \omega), \vec{c}_0; \varepsilon, \alpha, \omega),$$

$$\gamma_0 = \Gamma(a_0, \vec{c}_0; \varepsilon, \alpha, \omega);$$

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where \((a_0, \bar{c}_0; \varepsilon, \alpha, \omega) \in W\). Define \(\mathcal{H}\) as follows,

\[
\mathcal{H} = \bigcup_{t \in (-\infty, \infty)} F^t \circ \bigcup_{q \in W_b^{(c)}} \mathcal{F}_q^{(s)},
\]

where \(F^t\) is the evolution operator for the system (2.5). Then \(\mathcal{H}\) is the codimension 2 transversal homoclinic tube asymptotic to \(W^{(c)}\). \(\square\)

For an illustration of the homoclinic tube, see figure 5.5. Studies on the symbolic dynamics in the neighborhood of this homoclinic tube are topics of future works.
6 Symbolic Dynamics of Segments for the Case (N = 3, Nonresonant Region)

Denote by Π the plane,

\[ \Pi \equiv \left\{ \vec{q} \in \mathcal{S} \mid q_n = q \ \forall n \right\}. \tag{6.1} \]

Π is invariant under the flow governed by (2.5). The annulus \( \mathcal{A} \) defined in (4.1) is a subset of Π. Away from the resonant circle:

\[ S_{\gamma} \equiv \left\{ \vec{q} \in \mathcal{B} \mid b_1 = b_2 = \vec{c} = 0, \ a = \omega, \ \gamma \in [0, 2\pi) \right\}, \tag{6.2} \]

the dynamics is given by periodic motions, and denote by \( \hat{\mathcal{A}} \) such an annulus with boundaries which are periodic orbits. When \( N = 3 \), the Melnikov analysis shows that there exists a transversal homoclinic tube asymptotic to Π. By the invariance of the Hamiltonian \( H \), each orbit inside the homoclinic tube, approaching a periodic orbit in either the forward or the backward time, has to approach the same periodic orbit in both forward and backward time. Thus, there exists a homoclinic tube asymptotic to the annulus \( \hat{\mathcal{A}} \), as illustrated in figure 6.1.

**Definition 2** Define a Poincaré section \( \Sigma \) as follows,

\[ \Sigma = \left\{ \vec{q} \in \mathcal{B} \mid \gamma = 0 \right\}, \]

and let \( P \) be the Poincaré map,

\[ P : \mathcal{D} \subset \Sigma \mapsto \Sigma, \]

induced by the flow, where \( \mathcal{D} \) is the domain of definition for \( P \).

The transversal homoclinic tube \( H_P \) for the Poincaré map \( P \) is illustrated in figure 6.2. This transversal homoclinic tube \( H_P \) consists of segments, and is asymptotic to a segment \( s \) in the annulus \( \hat{\mathcal{A}} \). The segment \( s \) consists of fixed points of \( P \). Invariant tubes in the neighborhood of \( H_P \) also consist of segments. We have the following symbolic dynamics theorem for the homoclinic tube \( H_P \).
Figure 6.1: A homoclinic tube asymptotic to a nonresonant annulus when $N = 3$.

Figure 6.2: A homoclinic tube asymptotic to a segment under the Poincaré map $P$ when $N = 3$. 
Theorem 6.1 (Silnikov [16])  When $N = 3$, the set of invariant tubes of the Poincaré map $P$ lying wholly within a sufficiently small neighborhood of the transversal homoclinic tube $H_P$ is in one-to-one correspondence with the set of all doubly infinite sequences,

$$J = (\cdots, j_{-1}, j_0, j_1, \cdots),$$

where $j_l \geq j_*$ for all $l$ and some large integer $j_*$. 

Structures inside the transversal homoclinic tubes can be very complicated. Since we are studying near integrable Hamiltonian systems, the asymptotic manifold (i.e. the center manifold) $W^{(c)}$ often contains both KAM tori and stochastic layers. Studies on such structures are topics of future works. In this section, we study interesting structures on an invariant plane inside the center manifold $W^{(c)}$, and special orbits inside the transversal homoclinic tubes.

The dynamics on $\Pi$ (6.1) is governed by

$$i\dot{q} = 2[|q|^2 - \omega^2]q + \varepsilon \left\{ [\alpha_1 (q + \bar{q}) + \alpha_2 (q^2 + \bar{q}^2)]q + [\alpha_1 + 2\alpha_2 \bar{q}] \frac{\rho}{h^2} \ln \rho \right\}, \quad (7.1)$$

where $\rho = 1 + h^2 |q|^2$. Let $q = I e^{i\xi}$, where $I$ is the modulus of $q$; then,

$$\frac{dI}{dt} = -\varepsilon \sin \xi \left[ \alpha_1 + 4\alpha_2 I \cos \xi \right] \frac{\rho}{h^2} \ln \rho,$$

$$\frac{d\xi}{dt} = -2[I^2 - \omega^2] - \varepsilon \left\{ 2\alpha_1 I \cos \xi + 2\alpha_2 I^2 \cos 2\xi + \left[ \alpha_1 \cos \xi I + 2\alpha_2 \cos 2\xi \right] \frac{\rho}{h^2} \ln \rho \right\}. \quad (7.1)$$

The dynamics away from the neighborhood of $I = \omega$ is given by periodic motions. The dynamics in the neighborhood of $I = \omega$ is very interesting. Let $I = \omega + \eta y$, $\eta = \sqrt{\varepsilon}$, and define the resonant annulus $\tilde{A}$,

$$\tilde{A} \equiv \left\{ (y, \xi) \mid \xi \in [0, 2\pi], y \text{ is bounded} \right\}.$$

Inside the resonant annulus $\tilde{A}$, the dynamics is governed by,

$$\frac{dy}{d\tau} = f_1,$$

$$\frac{d\xi}{d\tau} = f_2,$$

where

$$f_1 = -\sin \xi \left[ \alpha_1 + 4\alpha_2 (\omega + \eta y) \cos \xi \right] \frac{\rho}{h^2} \ln \rho,$$

$$f_2 = -4\omega y - \eta \left\{ 2y^2 + 2\alpha_1 (\omega + \eta y) \cos \xi + 2\alpha_2 \cos 2\xi (\omega + \eta y)^2 + \left[ \frac{\alpha_1 \cos \xi}{\omega + \eta y} + 2\alpha_2 \cos 2\xi \right] \frac{\rho}{h^2} \ln \rho \right\}. \quad (7.2)$$
where $\tau = \eta t$.

### 7.1 First Order Dynamics in the Resonant Annulus

Setting $\eta = 0$ in (7.2), we have the system,

$$
\begin{align*}
\frac{dy}{d\tau} &= -\Omega \sin \xi \left[ \alpha_1 + 4\alpha_2 \omega \cos \xi \right], \\
\frac{d\xi}{d\tau} &= -4\omega y,
\end{align*}
$$

where $\Omega = h^{-2} \rho_0 \ln \rho_0$ and $\rho_0 = 1 + h^2 \omega^2$. This first order system (7.3) is also a Hamiltonian system,

$$
\begin{align*}
\frac{dy}{d\tau} &= \frac{\partial \tilde{H}}{\partial \xi}, \\
\frac{d\xi}{d\tau} &= -\frac{\partial \tilde{H}}{\partial y};
\end{align*}
$$

where $\tilde{H} = 2\omega y^2 + \Omega (\alpha_1 \cos \xi + \alpha_2 \omega \cos 2\xi)$. The fixed points of the system (7.3) are as follows:

- When $\left| \frac{\alpha_1}{4\alpha_2 \omega} \right| < 1$, there are four fixed points,

  $$
  \begin{align*}
  y_0^{(j)} &= 0, & (j = 1, 2, 3, 4) ; \\
  \xi_0^{(1)} &= 0, & \xi_0^{(2)} = \pi, & \xi_0^{(3)} = -\xi_0^{(4)} = \arccos \left[ -\frac{\alpha_1}{4\alpha_2 \omega} \right].
  \end{align*}
  $$

- When $\left| \frac{\alpha_1}{4\alpha_2 \omega} \right| > 1$, there are two fixed points,

  $$
  \begin{align*}
  y_0^{(j)} &= 0, & (j = 1, 2) ; \\
  \xi_0^{(1)} &= 0, & \xi_0^{(2)} = \pi.
  \end{align*}
  $$

The phase diagrams are shown in figure 7.1 when $\left| \frac{\alpha_1}{4\alpha_2 \omega} \right| < 1$, and in figure 7.2 when $\left| \frac{\alpha_1}{4\alpha_2 \omega} \right| > 1$. 

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Figure 7.1: Phase diagrams in the resonant annulus when $\left| \frac{\alpha_1}{\delta_2 \omega} \right| < 1$, (a). $\alpha_2 > 0$, $\alpha_1 > 0$; (b). $\alpha_2 > 0$, $\alpha_1 < 0$; (c). $\alpha_2 < 0$, $\alpha_1 > 0$; (d). $\alpha_2 < 0$, $\alpha_1 < 0$.

Figure 7.2: Phase diagrams in the resonant annulus when $\left| \frac{\alpha_1}{\delta_2 \omega} \right| > 1$, (a). $\alpha_1 > 0$; (b). $\alpha_1 < 0$. 

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### 7.2 The Full Dynamics in the Resonant Annulus

First we prove the existence of fixed points for the system (7.2) governing the full dynamics in the resonant annulus.

**Lemma 7.1** For any large \( \Lambda > 0 \), any small \( \delta_0 > 0 \), there exists \( \eta_0 > 0 \), such that there exist fixed points for the system (7.2),

\[
y^{(j)} = y^{(j)}(\eta, \alpha_1, \alpha_2, \omega), \quad \xi^{(j)} = \xi^{(j)}(\eta, \alpha_1, \alpha_2, \omega);
\]

which are \( C^1 \) smooth, where \((j = 1, 2, 3, 4; \text{ when } \left| \frac{\alpha_1}{4\omega_2} \right| < 1 - \delta_0) \) and \((j = 1, 2; \text{ when } \left| \frac{\alpha_1}{4\omega_2} \right| > 1 + \delta_0)\), \( |\eta| < \eta_0, \delta_0 < |\alpha_j| < \Lambda, (j = 1, 2), \delta_0 < |\omega| < \Lambda; \) and

\[
y^{(j)}(0, \alpha_1, \alpha_2, \omega) = y_0^{(j)} = y_0^{(j)}(\alpha_1, \alpha_2, \omega),
\]

\[
\xi^{(j)}(0, \alpha_1, \alpha_2, \omega) = \xi_0^{(j)} = \xi_0^{(j)}(\alpha_1, \alpha_2, \omega).
\]

**Proof:** We want to solve the equations

\[
f_1(y, \xi; \eta, \alpha_1, \alpha_2, \omega) = 0, \quad f_2(y, \xi; \eta, \alpha_1, \alpha_2, \omega) = 0;
\]

in the neighborhoods of the fixed points

\[
X_0^{(j)} = (y_0^{(j)}, \xi_0^{(j)}; 0, \alpha_1, \alpha_2, \omega).
\] (7.6)

Notice that

\[
\frac{\partial f_1}{\partial y}(y, \xi; 0, \alpha_1, \alpha_2, \omega) = 0, \\
\frac{\partial f_1}{\partial \xi}(y, \xi; 0, \alpha_1, \alpha_2, \omega) \\
= \frac{\rho_0}{h^2} \ln \rho_0 \left[ 4\alpha_2 \omega \sin^2 \xi - \cos \xi (\alpha_1 + 4\omega \alpha_2 \cos \xi) \right], \\
\frac{\partial f_2}{\partial y}(y, \xi; 0, \alpha_1, \alpha_2, \omega) = -4\omega, \\
\frac{\partial f_2}{\partial \xi}(y, \xi; 0, \alpha_1, \alpha_2, \omega) = 0.
\]
Then for any $\delta_0 > 0$,

$$det \begin{vmatrix} \frac{\partial f_1}{\partial y}(X_0^{(j)}) & \frac{\partial f_1}{\partial \xi}(X_0^{(j)}) \\ \frac{\partial f_2}{\partial y}(X_0^{(j)}) & \frac{\partial f_2}{\partial \xi}(X_0^{(j)}) \end{vmatrix} = \frac{4 \omega \rho_0}{h^2} \ln \rho_0 \left[ 4 \alpha_2 \omega \sin^2 \xi_0^{(j)} - \cos \xi_0^{(j)} \left( \alpha_1 + 4 \alpha_2 \omega \cos \xi_0^{(j)} \right) \right] \neq 0 ,$$

when $\left| \frac{\alpha_1}{4 \omega \alpha_2} \right| < 1 - \delta_0$ or $\left| \frac{\alpha_1}{4 \omega \alpha_2} \right| > 1 + \delta_0$. Then

$$\begin{pmatrix} \frac{\partial f_1}{\partial y}(X_0^{(j)}) & \frac{\partial f_1}{\partial \xi}(X_0^{(j)}) \\ \frac{\partial f_2}{\partial y}(X_0^{(j)}) & \frac{\partial f_2}{\partial \xi}(X_0^{(j)}) \end{pmatrix}$$

define linear homeomorphisms. Notice that $f_1$ and $f_2$ are $C^1$ functions; then by the implicit function theorem [3], pp265, for any fixed value of $X_0^{(j)}$ which satisfies the restrictions given in the Lemma, there is a neighborhood $V_Y^{(j)}$ of $Y = (\eta = 0, \alpha_1, \alpha_2, \omega)$, and unique functions

$$y^{(j)} = y^{(j)}(\eta, \alpha_1, \alpha_2, \omega), \quad \xi^{(j)} = \xi^{(j)}(\eta, \alpha_1, \alpha_2, \omega); \quad (7.7)$$

which are $C^1$ functions defined in $V_Y^{(j)}$, such that

$$f_l(y^{(j)}, \xi^{(j)}; \eta, \alpha_1, \alpha_2, \omega) = 0, \quad (l = 1, 2) .$$

Let $V^{(j)} = \bigcup V_Y^{(j)}$; then the functions (7.7) are uniquely defined in $V^{(j)}$.

There exists $\eta_0 > 0$, such that $|\eta| < \eta_0$ and the rest of restrictions in the Lemma define subregions of $V^{(j)}$. $\square$

In fact, we have

$$\xi^{(1)} = 0, \quad \xi^{(2)} = \pi ,$$

and $y^{(j)}$ ($j = 1, 2$) satisfy the equations,

$$I^2 - \omega^2 + \eta^2 \left[ \alpha_2 I^2 \pm \alpha_1 I + \left( \alpha_2 \pm \frac{\alpha_1}{2I} \right) \frac{\rho}{h^2} \ln \rho \right] = 0 ,$$

where $I = \omega + \eta y$, ‘+’ for $j = 1$ and ‘-’ for $j = 2$;

$$\xi^{(3)} = -\xi^{(4)} = \arccos \left[ -\frac{\alpha_1}{4 \alpha_2 I} \right] ,$$

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and $y^{(3)} = y^{(4)}$ satisfies the equation,

$$(1 - \eta^2 \alpha_2) I^2 - (\omega^2 + \eta^2 \alpha_1^2) - \eta^2 \alpha_2 \frac{\rho}{\hbar^2} \ln \rho = 0,$$

where $I = \omega + \eta y$. Approximate expressions of these fixed points can be obtained:

$$y^{(j)} = \eta y^{(j)}_1 + O(\eta^2), \quad (j = 1, 2); \quad (7.8)$$

where

$$y^{(1)}_1 = -\frac{1}{4\omega} \left[ 2\omega(\alpha_2 \omega + \alpha_1) + (2\alpha_2 + \alpha_1/\omega) \frac{\rho_0}{\hbar^2} \ln \rho_0 \right],$$

$$y^{(2)}_1 = -\frac{1}{4\omega} \left[ 2\omega(\alpha_2 \omega - \alpha_1) + (2\alpha_2 - \alpha_1/\omega) \frac{\rho_0}{\hbar^2} \ln \rho_0 \right].$$

When $\left| \frac{\alpha_1}{4\omega \alpha_2} \right| < 1$,

$$y^{(l)} = \eta y^{(l)}_1 + O(\eta^2), \quad (l = 3, 4); \quad (7.9)$$

where

$$y^{(3)}_1 = y^{(4)}_1 = \frac{1}{16\alpha_2 \omega} \left( 8\alpha_2^2 \omega^2 + \alpha_1^2 \right) + \frac{\alpha_2}{2\omega} \frac{\rho_0}{\hbar^2} \ln \rho_0.$$

$$\xi^{(3)} = -\xi^{(4)} = \arccos \left[ -\frac{\alpha_1}{4\alpha_2 \omega} + \eta^2 \psi + O(\eta^3) \right]$$

$$\xi^{(3)}_0 - \eta^2 \left[ 1 - \frac{\alpha_1^2}{16\alpha_2^2 \omega^2} \right]^{-1/2} \psi + O(\eta^3),$$

where $\psi = \frac{\alpha_1}{4\omega \alpha_2} y^{(3)}_1$. Let $X^{(j)} = (y^{(j)}, \xi^{(j)}; \eta, \alpha_1, \alpha_2, \omega)$ denote these fixed points and notice that $X^{(j)}_0 = (y^{(j)}_0, \xi^{(j)}_0; \eta = 0, \alpha_1, \alpha_2, \omega)$ a notation used before in $(7.6)$.

**Lemma 7.2** The fixed points $X^{(j)}$ have the same saddle or center nature as $X^{(j)}_0$

Proof:

$$\frac{\partial f_1}{\partial y} = -\sin \xi \left[ 4\alpha_2 \sqrt{\varepsilon} \cos \xi \right] \frac{\rho}{\hbar^2} \ln \rho$$

$$- \sin \xi \left[ \alpha_1 + 4\alpha_2 (\omega + \sqrt{\varepsilon} y) \cos \xi \right] \frac{d}{dy} \left( \frac{\rho}{\hbar^2} \ln \rho \right).$$

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\[
\frac{\partial f_2}{\partial \xi} = \sin \xi \left[ 4\alpha_2 \sqrt{\varepsilon} \cos \xi \right] \frac{\rho}{h^2} \ln \rho \\
+ \sqrt{\varepsilon} \sin \xi \left[ \alpha_1 + 4\alpha_2 (\omega + \sqrt{\varepsilon} y) \cos \xi \right] \\
\left\{ 2(\omega + \sqrt{\varepsilon} y) + \frac{1}{\omega + \sqrt{\varepsilon} y h^2} \ln \rho \right\}.
\]

Then,
\[
\frac{\partial f_1}{\partial y} (X^{(j)}) + \frac{\partial f_2}{\partial \xi} (X^{(j)}) = 0 ,
\]
(7.10)
in fact, when \( j = 1, 2 \), \( \frac{\partial f_1}{\partial y} (X^{(j)}) = \frac{\partial f_2}{\partial \xi} (X^{(j)}) = 0 \). Linearize (7.2) at \( X^{(j)} \), we have
\[
\frac{d}{d\tau} \begin{pmatrix} \tilde{y} \\ \tilde{\xi} \end{pmatrix} = L_j \begin{pmatrix} \tilde{y} \\ \tilde{\xi} \end{pmatrix} ,
\]
(7.11)
where
\[
L_j = \begin{pmatrix}
\frac{\partial f_1}{\partial y} (X^{(j)}) & \frac{\partial f_1}{\partial \xi} (X^{(j)}) \\
\frac{\partial f_2}{\partial y} (X^{(j)}) & \frac{\partial f_2}{\partial \xi} (X^{(j)})
\end{pmatrix},
\]
where
\[
\frac{\partial f_1}{\partial y} (X^{(j)}) = \frac{\partial f_1}{\partial y} (X_0^{(j)}) + O(\eta) , \quad \text{for } l = 1, 2
\]
(7.12)
\[
\frac{\partial f_1}{\partial \xi} (X^{(j)}) = \frac{\partial f_1}{\partial \xi} (X_0^{(j)}) + O(\eta) .
\]
The eigenvalues of \( L_j \) satisfy
\[
\lambda^2 - \text{trace}(L_j)\lambda + \text{det}(L_j) = 0.
\]
By relations (7.10) and (7.12),
\[
\text{trace}(L_j) = 0 ,
\]
\[
\text{det}(L_j) = \text{det} \begin{pmatrix}
\frac{\partial f_1}{\partial y} (X_0^{(j)}) & \frac{\partial f_1}{\partial \xi} (X_0^{(j)}) \\
\frac{\partial f_2}{\partial y} (X_0^{(j)}) & \frac{\partial f_2}{\partial \xi} (X_0^{(j)})
\end{pmatrix} + O(\eta) ;
\]
thus, when \( \eta \) is sufficiently small, the fixed points \( X^{(j)} \) have the same types of stability as \( X_0^{(j)} \). \( \Box \)
The phase diagram is given by the level sets of the rescaled Hamiltonian,

\[
\hat{H} = \frac{\rho_0}{2N\omega\eta^2} \left[ H - \frac{2N}{\hbar^2} (\omega^2 - \rho_0^2 \ln \rho_0) \right],
\]

where \( H \) is the restriction of the Hamiltonian to the resonant annulus \( \tilde{A} \),

\[
H = \frac{2N}{\hbar^2} \left\{ \left[ (\omega + \eta y)^2 - \rho_0 \hbar / \rho_0 \right] + \eta^2 \left[ \alpha_1 (\omega + \eta y) \cos \xi + \alpha_2 (\omega + \eta y)^2 \cos 2\xi \right] \ln \rho \right\}.
\]

\( \hat{H} \) is smooth in \( \eta \) and has the approximate expression,

\[
\hat{H} = 2\omega y^2 + [\alpha_1 \cos \xi + \alpha_2 \omega \cos 2\xi] \rho_0 / \hbar^2 \ln \rho_0 + O(\eta).
\]

Setting \( \eta = 0 \), we have the phase diagrams as shown in figures 7.1 and 7.2. When \( \eta \neq 0 \), invariant manifolds of the saddles perturb smoothly, and are given by level sets of \( \hat{H} \); thus, the figure eight loops do not break. By Lemmas 7.1 and 7.2, in the resonant annulus \( \tilde{A} \), the phase diagrams when \( \eta \neq 0 \) are topologically equivalent to those when \( \eta = 0 \).

### 7.3 Special Orbits Inside the Transversal Homoclinic Tubes

As a consequence of Theorem 5.1, there exist orbits asymptotic to fixed points in the resonant annulus \( \tilde{A} \) in forward or backward time. See figures 7.3 and 7.4 for some examples. Whether or not these orbits are actually homoclinic orbits or heteroclinic orbits is a very difficult open question.
Figure 7.3: Special orbits asymptotic to fixed points in the resonant annulus in forward or backward time when \( \left| \frac{\alpha_1}{\alpha_2 \omega} \right| < 1, \) (a). \( \alpha_2 > 0, \alpha_1 > 0; \) (b). \( \alpha_2 > 0, \alpha_1 < 0; \) (c). \( \alpha_2 < 0, \alpha_1 > 0; \) (d). \( \alpha_2 < 0, \alpha_1 < 0. \)

Figure 7.4: Special orbits asymptotic to fixed points in the resonant annulus in forward or backward time when \( \left| \frac{\alpha_1}{\alpha_2 \omega} \right| > 1, \) (a). \( \alpha_1 > 0; \) (b). \( \alpha_1 < 0. \)
Figure 7.5: Homoclinic and heteroclinic orbits for the case (N = 3, resonant region).

7.4 Homoclinic and Heteroclinic Orbits for the Case (N = 3, Resonant Region)

Inside the resonant annulus ˜A, the dynamics for both the full system (7.2) and the first order system (7.3) is shown in figures 7.1 and 7.2. As a consequence of Theorem 5.1, there exist orbits asymptotic to the saddles in forward time. For fixed (ε, α, ω), these orbits correspond to the intersection points between the surface (5.6) and the figure eight level sets of the saddles. In backward time, these orbits approach the resonant annulus ˜A. Since the Hamiltonian is conserved and the figure eights are given as the level sets of the Hamiltonian, in backward time, these orbits also approach the saddles on the same figure eight level sets. Thereby, we have homoclinic or heteroclinic orbits as shown in figure 7.5 for example. In drawing these figures, the fact that the quantity

\[
I_2 = \sum_{n=0}^{N-1} [\bar{q}_n(q_{n+1} + q_{n-1})]
\]

only changes \(O(\varepsilon)\) in finite time interval, and the fibers are smooth with respect to their base points and \(\varepsilon\).
8 Conclusion

In this paper, we have proved the existence of homoclinic tubes for the discrete cubic nonlinear Schrödinger equation under Hamiltonian perturbations, and discuss the symbolic dynamics of invariant tubes in the neighborhoods of such homoclinic tubes when the system is four dimensional. In future works, we will study such symbolic dynamics structures when the dimension is higher. Since the system that we study here is a near-integrable Hamiltonian system, we are also interested in studying KAM tori and stochastic layer structures inside each invariant tube, in particular, the homoclinic tube, and the connection with the symbolic dynamics of invariant tubes. Due to the normally hyperbolic nature of the center manifold, the relevant work on such KAM tori is that of Graff [5].

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References

[1] M. J. Ablowitz and J. F. Ladik. A Nonlinear Difference Scheme and Inverse Scattering. *Stud. Appl. Math.*, 55:213, 1976.

[2] A. Calini, N. M. Ercolani, D. W. McLaughlin, and C. M. Schober. Melnikov Analysis of Numerically Induced Chaos in the Nonlinear Schrödinger Equation. *Phys. D*, 89, no.3-4:227–260, 1996.

[3] J. Dieudonne. *Foundations of Modern Analysis*. Academic Press, 1960.

[4] N. Fenichel. Geometric Singular Perturbation Theory for Ordinary Differential Equations. *J Diff Eqns*, 31:53–98, 1979.

[5] S. M. Graff. On the Conservation of Hyperbolic Invariant Tori for Hamiltonian Systems. *J. Diff. Eqs.*, 15:1–69, 1974.

[6] Y. Li. Backlund Transformations and Homoclinic Structures for the NLS Equation. *Phys. Letters A*, 163:181–187, 1992.

[7] Y. Li. Homoclinic Tubes in Nonlinear Schrödinger Equation Under Hamiltonian Perturbations. *Progress of Theoretical Physics*, 101, No. 3(4):559–577, 1999.

[8] Y. Li. On 2D Euler Equations: Part II. Lax Pairs and Homoclinic Structures. *Submitted*, 2001.

[9] Y. Li and D. W. McLaughlin. Morse and Melnikov Functions for NLS Pdes. *Comm. Math. Phys.*, 162:175–214, 1994.

[10] Y. Li and D. W. McLaughlin. Homoclinic Orbits and Chaos in Perturbed Discrete NLS System. Part I Homoclinic Orbits. *Journal of Nonlinear Sciences*, 7:211–269, 1997.

[11] Y. Li and S. Wiggins. Homoclinic Orbits and Chaos in Perturbed Discrete NLS System. Part II Symbolic Dynamics. *Journal of Nonlinear Sciences*, 7:315–370, 1997.

[12] Y. Li and S. Wiggins. *Invariant Manifolds and Fibrations for Perturbed Nonlinear Schrödinger Equations*, volume 128. Springer-Verlag, Applied Mathematical Sciences, 1997.
[13] D.W. McLaughlin and E. A. Overman. Whiskered Tori for Integrable Pdes and Chaotic Behavior in Near Integrable Pdes. *Surveys in Appl. Math.* 1, 1993.

[14] K. J. Palmer. Exponential Dichotomies, the Shadowing Lemma and Transversal Homoclinic Points. *Dynamics Reported*, 1:265–306, 1988.

[15] L. P. Silnikov. The Existence of a Countable Set of Periodic Motions in the Neighborhood of a Homoclinic Curve. *Soviet Math. Dokl.*, 8:102–106, 1967.

[16] L. P. Silnikov. Structure of the Neighborhood of a Homoclinic Tube of an Invariant Torus. *Soviet Math. Dokl.*, 9, No.3:624–628, 1968.