Binary Outcome Copula Regression Model with Sampling Gradient Fitting

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Abstract: Use copula to model dependency of variable extends multivariate gaussian assumption. In this paper we first empirically studied copula regression model with continuous response. Both simulation study and real data study are given. Secondly we give a novel copula regression model with binary outcome, and we propose a score gradient estimation algorithms to fit the model. Both simulation study and real data study are given for our model and fitting algorithm.

Key words and phrases: Copula regression; binary outcome; semi-parametric estimation; gradient estimation; sampling;

1. Introduction

Copula has been a powerful mathematical tool for modelling dependence structure of variables in last decades. Assume \( \mathbf{X} = (X_1, \ldots, X_d)^T \) be a random vector of dimension \( d \geq 1 \) and \( Y \) be a random variable of our most
concern, which means we take $Y$ as our response. Further assume $X_i$ each has cumulative distribution $F_i$ and density $f_i$, while $Y$ has cumulative distribution $F_0$ and density $f_0$. A copula function is defined as $C_\theta(u_1, ..., u_d, v) = \mathbb{P}(F_1(X_1) \leq u_1, ..., F_d(X_d) \leq u_d, F_0(Y) \leq u_0)$. It is quite clear that copula function is joint cumulative function with uniform marginal distribution. Assume the copula function of $(X_1, ..., X_d, Y)$ is $C_\theta(u_1, ..., u_d, v).$ Sklar’s theorem claim joint cumulative function of $(X_1, ..., X_d, Y)$ can be expressed via a composition of a copula function and marginal cumulative function, which means

$$
\mathbb{P}(X_1 \leq x_1, ..., X_d \leq x_d, Y \leq y) = C_\theta(F_1(x_1), ..., F_d(x_d), F_0(y))
$$

. Copula formulation give a clear separation of marginal distribution and dependence structure. Use of Copula has been found in wide variaty of applied science such like quantitative risk control Wu Yijun et al. (2011) and statistical modelling. Intense research on copula based statistical method has been proposed. Chapman et al. (1951) introduced copula at the first time. Sklar, A (1959) give a theoretical background for copula. Most classical results on copula can be found in Nelsen (2000) . Galiani. (2003) propose a copula application for financial derivative products management. Parsa et al. (2011) and Noh et al. (2013) propose a regression model based on copula, which they give name as copula regression. Even in recent years,
copula based regression method have been continuously proposed. Rainer et al. (2012) and Radice et al. (2016) have studied copula regression for binary outcomes.

Joint distributions with special copulas have shown properties different from joint gaussian assumption. So copula has brought researcher a good view to go beyond joint gaussian distribution. Interesting properties like tail dependency of copula has motivated statistical community to persistently do research on copula. Since copula has been used successfully in quantitative risk control. Most work focus on simple modelling dependency and extreme behavior of multivariate variables using copula, few have tried to use copula to do regression inference or classification prediction. An interesting idea is to use copula based method to do regression inference. Parsa et al. (2011) proposes a copula regression method. Noh et al. (2013) propose a copula-based regression method and analyze the asymptotic property. In this paper, will first give a implementation of copula based regression model, and thus we will give and analyze a copula based model estimation with binary outcomes based on latent variable model corresponding real data experiment.
2. Preliminaries

2.1 Copulas and Backgrounds

Assume we have variables \((X_1, ..., X_d, Y)\). Assume variables \(\{X_i | 1 \leq i \leq d\}\) has dependence with response variable \(Y\) while each covariate \(X_i\) and \(X_j\) also has dependence to each other. A elegant formulation of the model is to give a copula dependence among all variables. Assume \((X_1, ..., X_d, Y)\) has Copula \(C_\theta(u_1, ..., u_d, v)\) which represents variables dependence. Assume each variable has marginal distribution \(F_1, ..., F_d, F_0\). Joint cumulative distribution function is naturally

\[
P(X_1 \leq x_1, ..., X_d \leq x_d, Y \leq Y) = C_\theta(F_1(x_1), ..., F_d(x_d), F_0(y))
\]

Before we go further, we give some basic lemma to demonstrate the model.

**Lemma 1.** Assume \((X_1, ..., X_d, Y)\) has copula smooth \(C_\theta(u_1, ..., u_d, v)\), and marginal cumulative distribution(density) \(F_i(f_i)\), then \((X_1, ..., X_d, Y)\) has joint density:

\[
f(x_1, ..., x_d, y) = c_\theta(F_1(x_1), ..., F_d(x_d), F_0(y)) \prod_{i=1}^{d} f_i(x_i)
\]

where \(c_\theta(u_1, ..., u_d, v) = \frac{\partial^{d+1} C_\theta(u_1, ..., u_d, v)}{\partial u_1 \partial u_2 ... \partial v}\)

The lemma gives a relation from cumulative distribution function and density function;
Lemma 2. Assume \((X_1, \ldots, X_d, Y)\) has copula smooth \(C_\theta(u_1, \ldots, u_d, v)\), and marginal cumulative distribution(density) \(F_i(f_i)\), then conditional mean of variable \(Y\) given covariates \(\bar{X} = \bar{x}\) is:

\[
m(x) = \mathbb{E}(Y|X) = \frac{\int y C_\theta(F_1(x_1), \ldots, F_0(y)) \Pi f_i(x_i) f_0(y) dy}{C_X(F(\bar{X}))}
\]

where \(C_X(u) = \frac{\partial^d C_\theta(u_1, \ldots, u_d; v=1)}{\partial u_1 \partial u_2 \cdots \partial u_d}\)

For many copula families, \(m(x)\) may have closed or informative expression, we list some copulas for example.

Example 1. Assume \(\rho(corr(Y, X_1), \ldots, corr(Y, X_d))^T\) and \(\Sigma_X\) denote the correlation matrix of \(X\). If the copula of \((Y, X^T)^T\) is Gaussian Copula, then we have:

\[
m(x) = \mathbb{E}[F_0^{-1}(\Phi(u^T \Sigma_X^{-1} \rho + \sqrt{1 - \rho^T \Sigma_X^{-1} \rho} Z))]
\]

where \(u = (\Phi^{-1}(F_1(x_1), \ldots, \Phi^{-1}(F_d(x_d))))^T\) and \(Z \sim \mathcal{N}(0, 1)\)

2.2 General Copula Regression method

Linear models or Generalized linear model are popular statistical model for continuous or categorical response prediction. As for short, we take linear regression model for short, generalized linear model can be viewed as a extension of linear regression model. Linear regression model often model the conditional expectation of \(Y\) given \(X\) as a linear function, however this
may lead to lack of fit because of simplicity of learn function. Another view on linear regression can be derived via generative modelling which leads us to consider copula regression and classification later. Assume variables \((X_1, ..., X_d, Y)\) has a joint gaussian distribution with mean \(\mu = (\mu_x^T, \mu_y)^T\) and covariance matrix \(\Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy} & \Sigma_{yy} \end{pmatrix}\), then

\[
\mathbb{E}(Y|X) = \mu_y - \Sigma_{yx} \Sigma_{xx}^{-1} (\mu_x - X)
\]

So under assumption \((X_1, ..., X_d, Y)\) has joint gaussian distribution, the prediction function is naturally a linear function. Evidences have been proposed that in many real world problem, joint distribution of covariates and response is far from gaussian. Rainer et al. (2012). Embrechts et al. (2002) show how the Pearson correlation coefficient can be misleading when the underlying distributions are not normal. They advise using copulas to model data that are not normal because such models capture a greater variety of relationships (essentially being nonparametric). So there is need for statistic models to capture more complex dependence structure among variables. Noh et al. (2013) have proposed a generic method to use copula dependence and thus fit regression model. In his work he proposed to fit model in a semi-parametric way. Other methods for fitting model have been proposed either. Chib et al. (2007) and Marra et al. (2013) introduced
2.2 General Copula Regression method

Bayesian and likelihood estimation methods based on penalized splines,e, Rainer et al. (2012) discussed a modification of the recursive bivariate probit that maintains the Gaussian assumption for the marginal distributions of the two equations while introducing non-Gaussian dependence between them using the Frank and Clayton copulas. However, these methods only consider bivariate case while multivariate case is largely different from it. In this part, we do an empirical study of copula regression model in Noh et al. (2013) and discuss its pros and cons, while in latter section we will derive our copula regression model with binary outcome motivated by Noh et al. (2013).

Assume \((X_1, \ldots, X_d, Y)\) has copula smooth \(C_\theta(u_1, \ldots, u_d, v)\), and marginal cumulative distribution(density) \(F_i(f_i)\). Noh2013 proposed to esitmiate marginal distribution with non-parametric method while fit maximum likelihood for parametric copula family. More clearly, they use kernel smoothed estimation

\[
\bar{F}_j(x_i) = \frac{1}{n} \sum \hat{K}_j\left(\frac{x_i - X_{i,j}}{h}\right)
\]

to be estimation of marginal distribution. As for copula estimation, there are a bunch of method which can estimate copula’s parameter. Nonparametric methods for estimating \(c\) include kernel smoothing estimators (see for example Gijbels et al. (1990), Charpentier et al. (2006) and Chen et al.
2.3 Simulation Study for Copula Regression

(2010)) and Bernstein estimator (see Chen et al. (2013)). In spite of the
great flexibility of nonparametric methods, they are typically affected by the
curse of dimensionality and they come with the difficult problem of select-
ing a good smoothing parameter. On the other hand imposing a parametric
structure on both the copula and marginal distributions can lead to severely
biased and inconsistent (fully parametric) estimator in case of misspecifi-
cation. So a non-parametric marginal together with a parametric copula
estimation is supposed to be considered.

2.3 Simulation Study for Copula Regression

The objective of this section is to compare the semi-parametric copula re-
gression estimator proposed by Noh2013 with OLS both when the true
copula family is known and when the copula family and its parameters are
adaptively selected using the data. To this end, we consider the follow-
data generating procedures (DGPs):

- **DGP I.a** \((F_0(Y), F_1(X_1)) \sim \text{Clayton copula with parameter } \delta = 1; Y \sim \mathcal{N}(\mu_Y = 1, \sigma_Y^2 = 1), X_1 \sim \mathcal{N}(\mu_{X_1} = 0, \sigma_{X_1}^2 = 1)\). The
  resulting regression function is \(m(x_1) = \mu_Y + \mathbb{E}[\sigma_Y \Phi^{-1}(T^{-1/\delta})]\), where
  \(T \sim f_T(t) = (1/\delta + 1)(1 + \xi)^{(1/\delta+1)}/(t + \xi)^{(1/\delta+2)}\) for \(t > 1\) and \(\xi = F_{X_1}(x_1)^{-\delta} - 1\).
2.3 Simulation Study for Copula Regression

- **DGP I.b** \((F_0(Y), F_1(X_1)) \sim \text{FGM copula with parameter } \theta = 0.8;\)
  
  \(Y \sim \mathcal{N}(\mu_Y = 0, \sigma_Y^2 = 1), X_1 \text{ is generated from the Gumbel distribution } F_{X_1}(x_1) = 1 - \exp(-\exp(x_1)).\) The resulting regression function
  
  \[m(x_1) = \mu_Y - \frac{\theta}{\sqrt{\pi}} \sigma_Y + 2\frac{\theta}{\sqrt{\pi}} \sigma_Y F_1(x_1).
  \]

- **DGP I.c** \((F_0(Y), F_1(X_1), \ldots, F_d(X_d)) \sim \text{Gaussian copula with correlation matrix } \Sigma = \begin{bmatrix} 1 & \rho^T & \rho \\ \rho & \Sigma_X & \Sigma_X \\ \rho & \Sigma_X & \Sigma_X \end{bmatrix} \text{ where } \rho \text{ is a d-dimensional vector}; Y \sim \mathcal{U}(0, 1); X_j \sim \mathcal{N}(\mu_{X_j} = 0, \sigma_{X_j}^2 = 1), j = 1, \ldots, d = 3.\) We choose correlation matrix as
  
  \[\Sigma = \begin{bmatrix} 1 & 0.23 & 0.23 & 0.90 & 0.67 \\ 0.23 & 1 & 0.90 & 0.51 & 0.49 \\ 0.23 & 0.90 & 0.51 & 1 & 0.49 \\ 0.23 & 0.90 & 0.51 & 0.49 & 1 \\ 0.23 & 0.90 & 0.51 & 0.49 & 1 \end{bmatrix} \text{.}\] The resulting regression function
  
  \[m(x) = \Phi(\sum_{j=1}^{d} a_j \frac{\sigma_Y}{\sqrt{2-\rho^T \rho}} \Phi^{-1}(F_j(x_j))),\] where
  
  \[a = (a_1, \ldots, a_d)^T \equiv \Sigma_X^{-1} \rho.\]

- **DGP II.a** the same as DGP I.c

- **DGP II.b** \((F_0(Y), F_1(X_1), \ldots, F_d(X_d)) \sim \text{R-Vine copula with the same structure and parameters as the illustrating example in the help page of function RVineMatrix of R package VineCopula; } Y \sim \mathcal{U}(0, 1); X_j \sim \mathcal{N}(\mu_{X_j} = 0, \sigma_{X_j}^2 = 1), j = 1, \ldots, d = 4.\)

- **DGP II.c** \((F_0(Y), F_1(X_1), \ldots, F_d(X_d)) \sim \text{Clayton copula with parameter } \delta = 1; Y \text{ is generated from the Beta distribution with parameters } \alpha = 0.5, \beta = 0.5; X_j \sim \mathcal{N}(\mu_{X_j} = 0, \sigma_{X_j}^2 = 1), j = 1, \ldots, d = 2.\)

- **DGP II.d** \((F_0(Y), F_1(X_1), \ldots, F_d(X_d)) \sim T \text{ copula with correlation}\)
2.3 Simulation Study for Copula Regression

Matrix $\Sigma$ as DGP I.c and degree of freedom $df = 5$; $Y$ is generated from the Beta distribution with parameters $\alpha = 0.5, \beta = 0.5$; $X_j \sim \mathcal{N}(\mu_{X_j} = 0, \sigma^2_{X_j} = 1), j = 1, \ldots, d$, $d = 3$.

As mentioned above, we conduct two set of simulation studies. In the first part, data are generated from DGP I.a to DGP I.c, and when we estimate copula regression, we know the true copula structure and estimate copula parameters using pseudo-MLE. In the second part, data are generated from DGP II.a to DGP II.d, and when we estimate copula regression, we adaptively select copula structure and parameters from data. In all seven experiments, we do simulations $N = 200$ times, each time with a data sample of $n = 100$ observations. Then we calculate IMSE, IBIAS and IVAR in a fixed evaluation set with $I = 150$ observations in each experiment as follows:

$$IMSE = \frac{1}{N} \sum_{l=1}^{N} ISE(\hat{m}^{(l)}) \equiv \frac{1}{N} \sum_{l=1}^{N} \left[ \frac{1}{I} \sum_{i=1}^{I} (\hat{m}^{(l)}(x_i) - m(x_i))^2 \right]$$

$$= \frac{1}{I} \sum_{i=1}^{I} (m(x_i) - \hat{m}(x_i))^2 + \frac{1}{I} \sum_{i=1}^{I} \left[ \frac{1}{N} \sum_{l=1}^{N} (\hat{m}^{(l)}(x_i) - \hat{m}(x_i))^2 \right] \equiv IBIAS + IVAR$$

where $\{(y_i, x_i), i = 1, \ldots, I\}$ is a fixed evaluation set, which corresponds to a random sample of size $I = 150$ generated from the DGP, $\hat{m}^{(l)}(.)$ is the estimated regression function from the $l$th data sample and $\hat{m}(x_i) = N^{-1} \sum_{l=1}^{N} \hat{m}^{(l)}(x_i)$. We should point out that in the second part of simu-
lation study, multi-dimensional $X$ makes it difficult to calculate the true regression function $m(x)$. So we replace $m(x_i)$ with $y_i$ in the calculation of IMSE and IBIAS. Then the variances of error terms $y_i - m(x_i)$ are included in IMSE and IBIAS, making them larger than their counterparts in the first part of simulation study.

2.3.1 Simulation Study I: Known Copula Structure

In this part, data are generated from DGP I.a to DGP I.c and when we estimate copula regression, we know the true copula structure and then estimate parameters using pseudo-MLE. Table 1 shows the IMSE together with the IBIAS and the IVAR of copula regression and OLS (with intercept). In all three settings, copula regression has much lower bias but higher variance than OLS. Together, copula regression attains lower IMSE. This simulation study reveals the potential of copula regression. But to apply it practically, we need to adaptively select copula structure and parameters from data, which is dealt with in the following section.

2.3.2 Simulation Study II: Unknown Copula Structure

In this part, data are generated from DGP II.a to DGP II.d and we adaptively select copula structure and parameters from data. This step may be
2.3 Simulation Study for Copula Regression

Table 1: Copula regression and OLS: Known Copula Structure

| Y margin | copula | IMSE copula | IMSE OLS | IBIAS copula | IBIAS OLS | IVAR copula | IVAR OLS |
|----------|--------|-------------|----------|--------------|-----------|-------------|----------|
| normal   | Clayton| 0.0197      | 0.0611   | 0.0031       | 0.0463    | 0.0166      | 0.0147   |
|          | FGM    | 0.0159      | 0.0241   | 0.0006       | 0.0068    | 0.0153      | 0.0173   |
| Uniform  | Gaussian| 0.0010     | 0.0037   | 0.0001       | 0.0033    | 0.0009      | 0.0004   |

difficult especially when the number of covariates is large. The reason is that the set of high-dimensional copulas available in the literature is limited to very special and restrictive copula families such as elliptical copulas and Archimedean copulas. For this reason, we make use of the recent available work about the simplified pair-copula decomposition. The main idea is to decompose a multivariate copula to a cascade of bivariate copulas so that we can take advantage of the relative simplicity of bivariate copula selection and estimation. In our simulation, we choose one decomposition (R-Vine structure) for the data and use R package VineCopula to select copula structure and estimate parameters.

Again, we compare copula regression with OLS (with intercept). In all four settings, copula regression has lower bias but higher variance than OLS. Together, copula regression attains lower IMSE. Note that now the variances
of error terms are also included in IMSE and IBIAS, so they are higher in the Gaussian setting compared to I.c. This simulation study illustrates that copula regression may have higher prediction power than OLS practically and in the next section, we will show some evidence in the real data.

Table 2: Copula regression and OLS: Unknown Copula Structure

| Y margin | copula | IMSE copula | IMSE OLS | IBIAS copula | IBIAS OLS | IVAR copula | IVAR OLS |
|----------|--------|-------------|----------|--------------|-----------|-------------|----------|
| Uniform  | Gaussian | 0.0077 | 0.0078 | 0.0054 | 0.0074 | 0.0023 | 0.0004 |
|          | R-Vine  | 0.0116 | 0.0136 | 0.0083 | 0.0129 | 0.0033 | 0.0007 |
| Beta(0.5,0.5) | Clayton | 0.0887 | 0.0910 | 0.0854 | 0.0881 | 0.0033 | 0.0029 |
|          | T       | 0.0126 | 0.0204 | 0.0082 | 0.0191 | 0.0043 | 0.0013 |

2.4 Real Data Study for Copula Regression

In this section, we analyze Boston Housing Data. The data consist of 506 observations with 14 variables. The dependent variable is MEDV, the median value of owner-occupied homes in $1000’s. The independent variables include per capita crime rate, nitric oxides concentration, weighted distances to five Boston employment centres, average number of rooms per dwelling, index of accessibility to radial highways, property-tax rate, pupil-
2.4 Real Data Study for Copula Regression

teacher ratio etc. To estimate regression function, we consider 4 methods:

- (OLS) Least-squre estimator
- (GAM) Generalized additive model estimator
- (CART) Classification and Regression Tree
- (CR) Copula regression method

We use a smoothing spline to fit GAM using function `gam` of R package `mgcv`. For CART, we use R package `rpart` and choose complexity parameter equals 0.001. As an evaluation measure of each estimator, we randomly split the data into training set (\(n = 337\)) and testing set (\(n = 169\)) for 100 times and calculate the mean and standard error of MSE in the test set for each estimator. Table 3 shows that, copula regression can attain much lower prediction error than OLS, slightly lower than CART and almost the same as GAM. As for the stability of prediction power, copula regression has slightly higher MSE standard error than OLS and CART, but much lower than that of GAM. All together, copula regression can balance prediction precision and stability and do fairly good job in the real data.
Table 3: Real Data Comparison of OLS, GAM, CART and CR

|        | MSE  | OLS  | GAM  | CART | CR  |
|--------|------|------|------|------|-----|
| mean   | 0.465| 0.295| 0.318| 0.296|     |
| sd     | 0.0670| 0.2302| 0.0711| 0.0781|     |

3. Binary outcome model

3.1 Latent Variable formulation

In this section, we give our formulation of Binary outcome regression model along with our proposed sampling based fitting method. Recap last section, we give a copula regression model and corresponding semi-parametric fitting method. Evidence have shown the model can perform well if both covariate and response are continuous. In real world applications, variables with binary outcome play an important role. In medical field, doctor uses patients’ observed variable to predict whether a patient has certain disease. In individual credit risk management field, bank uses customers’ observed variable to judge if a customer may default in near future or not. There are cases revealed the importance of prediction for binary response. Classical model such as logistic regression or linear discriminant analysis has been proposed for prediction of binary variables. However the simplicity of linear
fomula reduces the dependency structure for variables and will lead to lack of fit. In this paper, we consider use a latent variable model with copula to model variable dependency. Assume we have \( (X_1, ..., X_n, Y) \) are observed data where \( Y \) takes value from \{0, 1\} and \( X_i \) are continuous variable taking values in \( \mathcal{R} \). Assume the relation ship between \( X \) and \( Y \) are connected from one latent variable \( Z \in [0, 1] \), for which:

\[
Y|X, Z \equiv Y|Z \sim Ber(Z)
\]

\[
(X, Z) \sim c_\theta(F_X(x), F_Z(z))f_X(x)f_Z(z)
\]

where \( c_\theta(u, v) \) is the Copula Density of \( (X, Z) \). The nature of the model can be interpreted as, the response \( Y \) is determined by latent probability \( Z \) via a Bernoulli experiment \( Y|X \sim Ber(Z) \), while the covariates \( X \) and latent probability \( Z \) share joint distribution density \( c_\theta(F_X(x), F_Z(z))f_X(x)f_Z(z) \).

Our attempt to use latent variable to reveal the connection between covariates and binary outcome is not the first one. Rainer et al. (2012) and Radice et al. (2016) have proposed a latent probit model with copula dependency, but they assumed a Gaussian latent variable \( Z \) which does not have much explainable meaning. To our best knowledge, we are the first one to use a Bernoulli response with latent probability variable with copula to represent dependency. It is beneficial to use Bernoulli response to model the
binary outcomes. One benefit is if we assume the marginal distribution of latent probability has a beta distribution form, the nature property of one peak for beta distribution can interpret the prior response strength, while covaraiates $X$ are then to adjust the response strength. The second benefit is once we fit the parameter of the model, the natural conditional mean can be $E(Z|X)$ is the prediction probability for an individual observation. The function $m(x) = E(Z|X = x)$ can not only predict response but also the probability with which the response will take 1 or 0. The probability is of great importance in many statistical application such as individual credit scoring or customer click rate prediction. In the following part we will give our proposed methods for fitting the model.

### 3.2 Fitting Algorithm

Assume variables $(X, Y), X \in \mathcal{R}, Y \in \{0, 1\}$ are observed variable, $Z \in [0, 1]$ is the latent variable. Assume $Y|Z, X \sim Ber(Z)$ and $(X, Z) \sim c_{\theta}(F_X(x), F_\phi(z)) f_X(x) f_\phi(z)$. The joint density of $(X, Y, Z)$ is

$$P(x, y, z) = p(x, z)p(y|z) = c_{\theta}(F_X(x), F_\phi(z)) f_X(x) f_\phi(z) \times z^y(1 - z)^{1 - y}$$

The likelihood for parameter $(\theta, \phi)$ is

$$L(\theta, \phi) = p(x, y) = \int c_{\theta}(F_X(x), F_\phi(z)) f_X(x) f_\phi(z) \times z^y(1 - z)^{1 - y} dz$$
The derivative for likelihood wrt parameters are under regularity condition:

\[
\frac{\partial L(\theta, \phi)}{\partial \theta} = \int \frac{\partial c_{\theta}(F_X(x), F_\phi(z))}{\partial \theta} f_X(x)f_\phi(z) \times z^y(1-z)^{1-y} dz
\]

\[
\frac{\partial L(\theta, \phi)}{\partial \phi} = \int \left[ \frac{\partial c_{\theta}(u, v) \partial F_\phi(z)}{\partial v} \bigg|_{v=F_\phi(z)} + c_\theta(F_X(x), F_\phi(z)) \frac{\partial f_\phi(z)}{\partial \phi} \bigg] \times f_X(x)z^y(1-z)^{1-y} f_\phi(z) dz
\]

For \( F_X(.) \), we can use non-parametric method like kernel smoothing method to estimate. It is clear in most case the integral will not have explicit formula, but one fortunate thing is that because of good structure of model, the integral can be interpreted as a expectation for \( Z \) if \( Z \) have density \( f_\phi(z) \). The fact means under current parameters \((\theta, \phi)\), one can sample \((z_1, ..., z_K) \sim f_\phi(z)\) and use sample mean to estimate true parameters.

\[
\hat{L}(\theta, \phi) = \frac{1}{K} \sum_{k=1}^{K} \frac{\partial c_{\theta}(F_X(x), F_\phi(z_k))}{\partial \theta} f_X(x) \times z_k^y(1-z_k)^{1-y}
\]

\[
\hat{L}(\theta, \phi) = \frac{1}{K} \sum_{k=1}^{K} \frac{\partial c_{\theta}(u, v) \partial F_\phi(z_k)}{\partial v} \bigg|_{v=F_\phi(z_k)} + c_\theta(F_X(x), F_\phi(z_k)) \frac{\partial f_\phi(z_k)}{f_\phi(z_k)} \times f_X(x)z_k^y(1-z_k)^{1-y}
\]

In practice, the likelihood is strictly bounded in \([0, 1]\), thus the derivative may vanish because of computation accuracy and prevent the algorithms to converge. It is a usual practice to estimate gradients of log
3.2 Fitting Algorithm

likelihood, and we briefly give the formula here:

\[
\frac{\partial l(\theta, \phi)}{\partial \theta} = \int \frac{\partial c_\theta(F_X(x), F_\phi(z))}{\partial \theta} f_\phi(z) \times z^y(1-z)^{1-y} dz
\]

\[
\frac{\partial l(\theta, \phi)}{\partial \phi} = \int \left[ \frac{\partial c_\theta(u,v)}{\partial v} \frac{\partial F_\phi(z)}{\partial \phi} \bigg|_{u=F_X(x), v=F_\phi(z)} + c_\theta(F_X(x), F_\phi(z)) \frac{\partial f_\phi(z)}{\partial \phi} \right] \times z^y(1-z)^{1-y} f_\phi(z) dz
\]

where \( l(\theta, \phi) = \log \left[ \int c_\theta(F_X(x), F_\phi(z)) f_\phi(z) \times f_X(x) z^y(1-z)^{1-y} dz \right] \)

Our algorithm is show as:

**Input:** Observed Data \((X_i, Y_i), 1 \leq i \leq N\)

**Output:** Fitted Parameter \((\hat{\theta}, \hat{\phi})\), stepsize \(\epsilon\)

1. initialize \((\theta^0, \phi^0)\)

2. for \( t \) in 1 : Max_Iter do

3. Sample \( N \times K \) samples \((z_{nk})\)

4. \( \hat{\theta} = \frac{\sum_{nk} \frac{\partial c_\theta(F_X(x_n), F_\phi(z_{nk}))}{\partial \theta} \times z_{nk}^y(1-z_{nk})^{1-y_n}}{\sum_{nk} c_\theta(F_X(x_n), F_\phi(z_{nk})) \times z_{nk}^y(1-z_{nk})^{1-y_n}} \)

5. \( \hat{\phi} = \frac{\sum_{nk} \left[ \frac{\partial c_\theta(u,v)}{\partial v} \frac{\partial F_\phi(z_{nk})}{\partial \phi} \right]_{u=F_X(x_n), v=F_\phi(z_{nk})} \times z_{nk}^y(1-z_{nk})^{1-y_n}}{\sum_{nk} c_\theta(F_X(x_n), F_\phi(z_{nk})) \times z_{nk}^y(1-z_{nk})^{1-y_n}} \)

6. \( \theta^{t+1} = \theta^t + \epsilon \hat{\theta} \)

7. \( \phi^{t+1} = \phi^t + \epsilon \hat{\phi} \)

8. end

**Algorithm 1:** Sampling Gradient Fitting Algorithm
3.3 Some examples of the method

In this section, we give some examples with different $c_\theta(u, v)$ and $f_\phi(z)$ to get explicit update formula with our algorithm.

**Proposition I** When $c_\theta$ is Gaussian copula, we have

$$
\frac{\partial l(\Sigma, \phi)}{\partial \Sigma} = \frac{\int z^y(1-z)^{1-y}c_\Sigma(F_X(x), F_\phi(z))[\frac{1}{2} \Sigma^{-1}tt^T\Sigma^{-1} - \frac{1}{2} \Sigma^{-1}]f_\phi(z)dz}{\int z^y(1-z)^{1-y}c_\Sigma(F_X(x), F_\phi(z))f_\phi(z)dz}
$$

where $t = (\Phi^{-1}(F_\phi(z)), \Phi^{-1}(F_{X_1}(x_1)), ...\Phi^{-1}(F_{X_d}(x_d)))^T$, $\Sigma$ is the correlation matrix of Gaussian copula.

3.4 Simulation Studies

This section aims to evaluate the proposed binary output copula regression method. We compare this new method with logit regression to gain some insight about its performance. To this end, we consider the following data generating procedures (DGPs):

- **DGP III.a** $(F_0(Z), F_1(X_1), ...F_d(X_d)) \sim$ Clayton copula with parameter $\delta = 1; Z \sim U(0, 1), X_j \sim N(\mu_{X_j} = 0, \sigma_{X_j}^2 = 1), d = 3$. The outcome $Y$ is subject to Bernoulli distribution with success rate $Z$.

- **DGP III.b** The structure of $(Z, X_1, ...X_d)$ is the same as DGP I.c. The outcome $Y$ is subject to Bernoulli distribution with success rate $Z$. 

• **DGP III.c** \((X_1, \ldots X_d), d = 4\) is generated from multivariate normal distribution which has correlation matrix as in DGP I.c and standard normal marginal distribution. Then we generate \(Z = \text{sigmoid}(X\beta)\), where \(\beta = (1, -1, -1, 1)^T\). The outcome \(Y\) is subject to Bernoulli distribution with success rate \(Z\).

We first generate a data sample of \(n = 300\) observations from DGP III.a to DGP III.c. After that, we randomly split the data sample into training set (200 obs) and testing set (100 obs). We estimate binary-outcome copula regression and logit regression on the training set and then calculate AUC and KS-value for them on the testing set. Table 4 below shows the average AUC and KS-value for the two method. On average, our binary-outcome copula regression method attains slightly better AUC and KS-value than logit regression.

### 3.5 Real Data Studies

In this section, we analyze Breast-Cancer-Wisconsin Data. The data consist of 699 observations with 10 variables. The outcome variable is CLASS, whether the cancer is benign or malignant. The independent variables include clump thickness, uniformity of cell size and shape, marginal adhesion etc. We consider 4 classification algorithms:
Table 4: BOCR and Logit

|       | AUC  |       | KS value |
|-------|------|-------|----------|
| logit | 0.639| BOCR  | 0.651    |
|       |      | logit | 0.284    |
|       |      | BOCR  | 0.295    |
| Gaussian | 0.747 | BOCR  | 0.742    |
|       |      | logit | 0.375    |
|       |      | BOCR  | 0.408    |
| Logit | 0.604| BOCR  | 0.612    |
|       |      | logit | 0.218    |
|       |      | BOCR  | 0.242    |

- (Logit) Logit regression
- (CART) Classification and Regression Tree
- (SVM) Supporting Vector Machine
- (BOCR) Binary-Outcome Copula Regression

We use R package \textit{rpart} to estimate CART and package \textit{e1071} to estimate SVM. We randomly split the data into training set (n=466) and testing set (n=233) and calculate AUC and KS-value in the testing set for each method. Table 5 shows that, BOCR can attain similar AUC and KS-value as the other three methods. From the results, it seems that it is fairly easy to classify cancer to benign or malignant since all four methods attain KS-value higher than 0.9!
4. Conclusion and Future Work

In this paper, we empirically investigated copula regression model and proposed a binary outcome copula regression model. We give a sampling gradient method for fitting the model together with simulation study and real data study of the model. Evidence has show our model can overwhelm logistic regression model and machine learning models such as CART and SVM. To our best knowledge, our model is the first copula based model to deal with multivariate variables with single binary outcome. Also we are the first attempt to introduce Sampling Gradient Estimation in fitting such models. However, there are still many work to do in future. We briefly discuss two aspects.

One future work may be evaluation of various copula functions under our model framework. In our paper we mainly focus on Gaussian Copula as a template, which means nearly all other copulas can be evaluated in the

Table 5: Real Data Analysis for BOCR

|        | logit | CART | SVM  | BOCR |
|--------|-------|------|------|------|
| AUC    | 0.9956| 0.9780| 0.9964| 0.9964|
| KS value| 0.9482| 0.9350| 0.9539| 0.9548|
same way we do.

Another future work may be Variance Reduction in Sample Gradient Estimation Procedure. One knows their are bunches of methods to reduce variance for Monte Carlo Estimation, some of them include importance sampling, Rao-Blackwellization et-al. The application of variance reduction tricks in our model will be an interesting research direction.

Supplementary Material

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