Eigenvalue ratios for vibrating String equations with concave densities

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Abstract

In this paper, we prove the optimal lower bound $\frac{\lambda_n}{\lambda_m} \geq \left(\frac{n}{m}\right)^2$ of vibrating string

$$-y'' = \lambda\rho(x)y,$$

with Dirichlet boundary conditions for concave densities. Our approach is based on the method of Huang [Proc. AMS., 1999]. The main argument is to restrict the two consecutive eigenfunction $y_{n-1}$ and $y_n$ between two successive zeros of $y_{n-1}$. We also prove the same result for the Dirichlet Sturm-Liouville problems.

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1 Introduction

We consider the Sturm-Liouville equation acting on $[0, 1]$

$$-(p(x)y')' + q(x)y = \lambda\rho(x)y,$$  \hspace{1cm} (1.1)

with Dirichlet boundary conditions

$$y(0) = y(1) = 0,$$  \hspace{1cm} (1.2)

where $p > 0$, $\rho > 0$ and $q$ (may change sign) are continuous coefficients on $[0, 1]$. Here we limit ourselves to the case $\rho > 0$. The case $\rho < 0$ has been considered for related problems providing different results, we refer to pioneering works [6,7] and some refer therein.

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As is well-known (see [16]), there exist two countable sequences of eigenvalues 
\[ \lambda_1 < \lambda_2 < \cdots < \lambda_n \ldots \infty. \]
The issues of optimal estimates for the eigenvalue ratios \( \frac{\lambda_n}{\lambda_m} \) have attracted a lot of attention (cf. [1, 3, 4, 5, 8, 9, 10, 11, 12, 14]) and references therein. Ashbaugh and Benguria proved in [3] that if \( q \geq 0 \) and \( 0 < k \leq p\rho(x) \leq K \), then the eigenvalues of (1.1) satisfy
\[ \frac{\lambda_n}{\lambda_1} \leq K \frac{n^2}{k}. \]
They also established the following ratio estimate (of two arbitrary eigenvalues)
\[ \frac{\lambda_n}{\lambda_m} \leq K \frac{n^2}{km^2}, \quad n > m \geq 1, \]
with \( q \equiv 0 \) and \( 0 < k \leq p\rho(x) \leq K \). Later, Huang and Law [10] extended the results in [3] to more general boundary conditions. Recently, J. Hedhly [5], showed that
\[ \frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}, \quad n > m \geq 1, \]
for single-barrier potential \( q \) and single-well \( p\rho \). He also established that the eigenvalues for the string equation
\[ -y'' = \lambda\rho(x)y, \quad (1.3) \]
with Dirichlet boundary conditions (1.2) satisfy
\[ \frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}, \quad n > m \geq 1, \]
with single-well density \( \rho \).
Recall that \( f \) is a single-barrier (resp. single-well) function on \([0, 1]\) if there is a point \( x_0 \in [0, 1] \) such that \( f \) is increasing (resp. decreasing) on \([0, x_0]\) and decreasing (resp. increasing) on \([x_0, 1]\) (see [2]).
In this paper, we prove the optimal lower bound \( \frac{\lambda_n}{\lambda_m} \geq \frac{(n^2)}{m^2} \) of (1.3)-(1.2) for concave density \( \rho \). Our approach is based on the method of Huang [Proc. AMS., 1999]. The main argument is to restrict the two consecutive eigenfunction \( y_{n-1} \) and \( y_n \) between two successive zeros of \( y_{n-1} \), say \( x_i \) and \( x_{i+1} \). We prove arguing as in \([x_i, x_{i+1}]\) that
\[ \int_{x_i}^{x_{i+1}} x(y_{n-1}(x, \tau))^2 - y_n(x, \tau)^2 \geq 0. \]
We also prove an result for the Dirichlet Sturm-Liouville problems (1.1)-(1.2). More precisely, we show that \( \frac{\lambda_n}{\lambda_m} \geq \frac{(n^2)}{m^2} \) with \( q \equiv 0 \) and \( p\rho \) concave.
2 Eigenvalue ratio for the vibrating String equations

Denote by \( u_n(x) \) be the \( n-th \) eigenfunction of (1.3) corresponding to \( \lambda_n \), normalized so that
\[
\int_0^1 \rho(x) u_n^2(x) dx = 1.
\]

It is well known that the \( u_n(x) \) has exactly \((n-1)\) zeros in the open interval \((0, 1)\). The zeros of the \( n-th \) and \((n+1)st\) eigenfunctions interlace, i.e. between any two successive zeros of the \( n-th \) eigenfunction lies a zero of the \((n+1)st\) eigenfunction. We denote by \((y_i)\), the zeros of \( u_n \) and \((z_i)\), the zeros of \( u_{n-1} \), then in view of the comparison theorem (see [16, Chap.1]), we have \( y_i < z_i \). We may assume that \( u_n(x) > 0 \) and \( u_{n-1}(x) > 0 \) on \((0, y_1)\), then we have \( \frac{u_n(x)}{u_{n-1}(x)} \) is strictly decreasing on \((0, 1)\). Indeed,
\[
\left( \frac{u_n(x)}{u_{n-1}(x)} \right)' = \frac{u''(x)u_{n-1}(x) - u'_n(x)u_n(x)}{u''_{n-1}(x)} = \frac{w(x)}{u''_{n-1}(x)}.
\]
We find
\[
w'(x) = u''_n(x)u_{n-1}(x) - u''_{n-1}(x)u_n(x) = (\lambda_{n-1} - \lambda_n)\rho(x)u_n(x)u_{n-1}(x),
\]
this implies that \( w(x) < 0 \) on \((0, 1)\). Hence \( \frac{u_n(x)}{u_{n-1}(x)} \) is strictly decreasing on \((0, 1)\).

From this, there are points \( x_i \in (y_i, z_i) \) such that
\[
\begin{cases}
  u^2_n(x) > u^2_{n-1}(x), & x \in (x_{2i}, x_{2i+1}), \\
  u^2_n(x) < u^2_{n-1}(x), & x \in (x_{2i+1}, x_{2i+2}).
\end{cases}
\]

Let \( \rho(\cdot, \tau) \) is a one-parameter family of piecewise continuous densities such that \( \frac{\partial \rho(\cdot, \tau)}{\partial \tau} \) exists, and let \( u_n(x, \tau) \) be the \( n-th \) eigenfunction of (1.3) corresponding to \( \lambda_n(\tau) \) of the corresponding String equation (1.3) with \( \rho = \rho(\cdot, \tau) \). From Keller in [13], we get
\[
\frac{d}{d\tau} \lambda_n(\tau) = -\lambda_n(\tau) \int_0^1 \frac{\partial \rho}{\partial \tau}(x, \tau) u^2_n(x, \tau) dx.
\]
By straightforward computation that, yields
\[
\frac{d}{d\tau} \left[ \frac{\lambda_n(\tau)}{\lambda_m(\tau)} \right] = \frac{\lambda_n(\tau)}{\lambda_m(\tau)} \int_0^1 \frac{\partial \rho}{\partial \tau}(x, \tau) (u^2_m(x, \tau) - u^2_n(x, \tau)) dx.
\]

We are now in position to state our main result.

**Theorem 1** Let \( \rho \) a concave density on \([0, 1]\). Then the eigenvalues of the Dirichlet problem (1.3)-(1.2) satisfy
\[
\frac{\lambda_n}{\lambda_m} \geq \left( \frac{n}{m} \right)^2,
\]
with equality if and only if \( \rho \) is constant.
In order to prove Theorem 1 we need some preliminary results, in particular the following result by Huang [8].

Lemma 1 [8] If $g$ is three times differentiable and $u$ satisfies

$$-y'' = \lambda \rho(x) y, \quad 0 \leq x \leq 1, \quad y(0) = y(1) = 0,$$

where $\rho$ is differentiable, then

$$g(1)y'(1)^2 - g(0)y'(0)^2 = \int_0^1 \left[ 2\lambda g'(x)\rho(x) + \lambda g(x)\rho'(x) + \frac{1}{2}g'''(x) \right] y^2(x) dx.$$

Lemma 2 Consider the one-parameter family of linear densities $\rho(x, \tau) = \tau x + b$, where $t > 0$ and $b$ is a positive constant. Let $\lambda_n(\tau)$ be the $n$th eigenvalue of (1.3)–(1.2) with $\rho = \rho(x, \tau)$. Then the ratio $\frac{\lambda_n(\tau)}{\lambda_{n-1}(\tau)}$ is a strictly increasing function of $t$.

Proof From (2.1)

$$\frac{d}{d\tau} \left[ \frac{\lambda_n(\tau)}{\lambda_{n-1}(\tau)} \right] = \frac{\lambda_n(\tau)}{\lambda_{n-1}(\tau)} \int_0^1 x(y_{n-1}^2(x, \tau) - y_n^2(x, \tau)) dx.$$

So, we have to show that

$$\ll x(\tau) \gg = \int_0^1 x(y_{n-1}^2(x, \tau) - y_n^2(x, \tau)) dx \geq 0, \quad (2.3)$$

for all $\tau > 0$.

Firstly notice that

$$\ll x(\tau) \gg = \int_0^1 x(y_{n-1}^2(x, \tau) - y_n^2(x, \tau)) dx$$

$$= \sum_{i=1}^{n-1} \int_{z_i}^{z_{i+1}} x(y_{n-1}^2(x, \tau) - y_n^2(x, \tau)) dx.$$

To show (2.3), it suffices to show that

$$\ll x(\tau) \gg_i = \int_{z_i}^{z_{i+1}} x(y_{n-1}^2(x, \tau) - y_n^2(x, \tau)) d\tau \geq 0$$

Taking $g(x) = x$ in Lemma 1 we get

$$y_n'(z_{i+1}, \tau)^2 = \lambda_n \int_{z_i}^{z_{i+1}} (3\tau x + 2b) + y_n^2(x, \tau) dx,$$

and, with $g(x) = x^2$,

$$y_n'(z_{i+1}, \tau)^2 = \lambda_n \int_{z_i}^{z_{i+1}} (5\tau x^2 + 4bx) + y_n^2(x, \tau) dx.$$
Therefore,
\[ 5t \int_{z_i}^{z_{i+1}} x^2 y_n^2(x, \tau) \, dx = (3\tau - 4b) \int_{z_i}^{z_{i+1}} xy_n^2(x, \tau) \, dx + 2ab \int_{z_i}^{z_{i+1}} y_n^2(x, \tau) \, dx. \] (2.4)
By the normalization condition of \( y \), we obtain
\[ \int_0^1 y_n^2(x, \tau) \, dx = \frac{1}{b} - \frac{\tau}{b} \int_0^1 xy_n^2(x, \tau) \, dx, \]
yields
\[ \int_0^1 y_n^2(x, \tau) \, dx = \frac{1}{nb} - \frac{\tau}{b} \int_{z_i}^{z_{i+1}} xy_n^2(x, \tau) \, dx. \]
Then
\[ \int_{z_i}^{z_{i+1}} y_n^2(x, \tau) \, dx = \frac{1}{nb} - \frac{\tau}{b} \int_{z_i}^{z_{i+1}} xy_n^2(x, \tau) \, dx. \]
Thus, from (2.4), we get
\[ \int_{z_i}^{z_{i+1}} x^2 y_n^2(x, \tau) \, dx = \frac{2}{5\tau} + \frac{\tau - 4b}{b} \int_{z_i}^{z_{i+1}} xy_n^2(x, \tau) \, dx. \]
From this, it follows that
\[ \int_{z_i}^{z_{i+1}} [y_{n-1}^2(x, \tau) - y_n^2(x, \tau)^2] \, dx = \frac{-\tau}{b} \ll x(\tau) \gg_i \] (2.5)
and
\[ \int_{z_i}^{z_{i+1}} x^2 [y_{n-1}^2(x, \tau) - y_n^2(x, \tau)^2] \, dx = \frac{\tau - 4b}{5\tau} \ll x(\tau) \gg_i. \] (2.6)
First of all notice that \( \ll x(\tau) \gg_i \neq 0 \) for all \( \tau > 0 \). For, if for some \( t, \ll x(\tau) \gg_i = 0 \) then from (2.5) and (2.6), we obtain
\[ \int_{z_i}^{z_{i+1}} (Ax^2 + Bx + C) [y_{n-1}^2(x, \tau) - y_n^2(x, \tau)^2] \, dx = 0, \]
where \( A, B \) and \( C \) are arbitrary constants, which is impossible because there are a points \( z_i(\tau) < x_{2i+1}(\tau) < x_{2i+2}(\tau) < z_{i+1}(\tau) \) such that
\[ \begin{align*}
  & y_n^2(x, \tau) > y_{n-1}^2(x, \tau), \quad x \in (z_i(\tau), x_{2i+1}(\tau)) \cup (x_{2i+2}(\tau), z_{i+1}(\tau)), \\
  & y_n^2(x, \tau) < y_{n-1}^2(x, \tau), \quad x \in (x_{2i+1}(\tau), x_{2i+2}(\tau)).
\end{align*} \] (2.7)
therefore, \[ x(\tau) \rightrightarrows_i \neq 0 \] for all \( \tau > 0 \). Then, according to the continuity of \( x(\tau) \rightrightarrows_i \), we either have \( x(\tau) \rightrightarrows_i < 0 \) or \( x(\tau) \rightrightarrows_i > 0 \). We assume the contrary that \( x(\tau) \rightrightarrows_i < 0 \). Then, from this together with \((2.5)\) and \((2.6)\), we obtain

\[
\int_{z_i}^{z_{i+1}} (A x^2 + B x + c) [y_{n-1}^2(x, \tau) - y_n^2(x, \tau)] < 0,
\]

(2.8) for all \( \tau < 4b \). But if we choose \( A < 0 \), \( B > 0 \) and \( C < 0 \) we get

\[
A x^2 + B x + c = -(x - x_{2i+1}(\tau))(x - x_{2i+2}(\tau)).
\]

by \((2.7)\), we find that \((2.8)\) is positive. It is a contradiction with the hypothesis \( x(\tau) \rightrightarrows_i > 0 \). □

Lemma 3 Let \( \hat{\rho} > 0 \) be function continuous on \([0, 1]\) such that \( \hat{\rho}(x) = a_i x + b \) for \( x \in [z_i, z_{i+1}] \). Then the eigenvalues of Problem \((1.3) - (1.2)\) with \( \rho = \hat{\rho} \), satisfy

\[
\frac{\lambda_n(\hat{\rho})}{\lambda_m(\hat{\rho})} \geq \left( \frac{n}{m} \right)^2.
\]

(2.9)

Equality holds iff \( \hat{\rho} \) is constant in \([0, 1]\).

Proof According to Lemma 2

\[
\frac{d}{d\tau} \left[ \frac{\lambda_n(\tau)}{\lambda_{n-1}(\tau)} \right] = \frac{\lambda_n(\tau)}{\lambda_{n-1}(\tau)} \int_0^1 x(y_{n-1}^2(x, \tau) - y_n^2(x, \tau)) dx \geq 0.
\]

Then,

\[
\frac{\lambda_n(\tau)}{\lambda_{n-1}(\tau)} \geq \frac{\lambda_n(0)}{\lambda_{n-1}(0)} = \left( \frac{n}{n-1} \right)^2.
\]

Then

\[
\frac{\lambda_n(\hat{\rho})}{\lambda_m(\hat{\rho})} \geq \left( \frac{n}{m} \right)^2.
\]

We are now ready to prove Theorem 1.

Proof We define \( \rho(x, \tau) = \tau \rho(x) + (1 - \tau)\hat{\rho}(x) \), then from \((2.1)\)

\[
\frac{d}{d\tau} \left[ \frac{\lambda_n(\tau)}{\lambda_{n-1}(\tau)} \right] = \frac{\lambda_n(\tau)}{\lambda_{n-1}(\tau)} \int_0^1 [\rho(x) - \hat{\rho}(x)](y_{n-1}^2(x, \tau) - y_n^2(x, \tau)) dx \geq 0
\]

\[
= \frac{\lambda_n(\tau)}{\lambda_{n-1}(\tau)} \sum_{i=0}^{n} \int_{z_i}^{z_{i+1}} [\rho(x) - \hat{\rho}(x)](y_{n-1}^2(x, \tau) - y_n^2(x, \tau)) dx.
\]

We notice that

\[
\int_{z_i}^{z_{i+1}} [\rho(x) - \hat{\rho}(x)](y_{n-1}^2(x, \tau) - y_n^2(x, \tau)) dx \geq 0.
\]
It then follows that,

\[
\frac{d}{d\tau} \left[ \frac{\lambda_n(\tau)}{\lambda_{n-1}(\tau)} \right] \geq 0.
\]

Thus, by the continuity of eigenvalues, we obtain

\[
\frac{\lambda_n(\rho)}{\lambda_{n-1}(\rho)} = \frac{\lambda_n(1)}{\lambda_{n-1}(1)} \geq \frac{\lambda_n(0)}{\lambda_{n-1}(0)} = \frac{\lambda_n(\hat{\rho})}{\lambda_{n-1}(\hat{\rho})}.
\]

And hence

\[
\frac{\lambda_n(\rho)}{\lambda_m(\rho)} \geq \left( \frac{n}{m} \right)^2.
\]

According to Lemma [2], equality holds, if \( \rho = \hat{\rho} = cte \).

\[\square\]

**Corollary 1** Let \( \rho \) a concave density on \([0, 1]\). Then the eigenvalues of the Dirichlet problem (1.3)-(1.2) satisfy

\[
\lambda_n - \lambda_m \geq \left( \frac{n}{m} \right)^2 (m\pi)^2 \rho_M,
\]

where \((\rho)_M = \max_{x \in [0, 1]} \rho(x)\).

Equality if and only if \( \rho \) is constant.

### 3 Eigenvalue ratios for Sturm-Liouville problems with \( q \equiv 0 \).

In this section, we derive the more general bounds on eigenvalue ratios that can be obtained in the absence of the potential \( q \).

**Theorem 2** Consider the regular Sturm-Liouville problem \(-(p(x)y')' = \lambda \rho(x)y\) with Dirichlet boundary conditions (1.2). If \( pp \) a concave function on \([0, 1]\) then

\[
\frac{\lambda_n}{\lambda_m} \geq \left( \frac{n}{m} \right)^2.
\]

Equality holds iff \( pp \) is constant in \([0, 1]\).

**Proof** By use the Legendre substitution [15] pp. 227-228

\[
t(x) = \frac{1}{\sigma} \int_0^x \frac{1}{p(z)}dz, \quad \sigma = \int_0^1 \frac{1}{p(z)}dz,
\]

Equation (1.1) can be rewritten in the string equation

\[
-\ddot{y} = \lambda \sigma^2 \tilde{p}(t) \tilde{\rho}(t)y,
\]

where \( \tilde{p}(t) = p(x) \) and \( \tilde{\rho}(t) = \rho(x) \). Thus the estimate (3.1) is direct consequence of Theorem [1] \[\square\]
Corollary 2 Consider the regular Sturm-Liouville problem \(- (p(x)y')' = \lambda \rho(x)y\) with Dirichlet boundary conditions (1.2). If \(p\rho\) a concave function on \([0, 1]\) then

\[
\lambda_n - \lambda_m \geq \left(\frac{n}{m}\right)^2 - 1 \frac{(m\pi)^2}{(pp)_M},
\]

(3.3)

where \((pp)_M = \max_{x \in [0,1]} (p(x)\rho(x))\).

Equality holds iff \(pp\) is constant in \([0, 1]\).

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