Generation and evaluation of multipartite entanglement with multi-rail encoding in linear optics networks

Jun-Yi Wu

Department of Physics, Tamkang University, 151 Yingzhuang Rd., Tamsui Dist., New Taipei City 25137, Taiwan, ROC

Abstract

A linear optics network (LON) is a multimode interferometer system, where indistinguishable photon inputs can create nonclassical interference that cannot be simulated with classical computers. Such nonclassical interference implies the existence of entanglement among its subsystems, if we divide its modes into different parties. Entanglement in such systems is naturally encoded in multi-rail (multi-mode) quantum registers. For bipartite entanglement, a generation and detection scheme with multi-rail encoding has been theoretically proposed [NJP 19(10):103032, 2017] and experimentally realized [Optica, 7(11):1517, 2020]. In this paper, we will take a step further to establish a theory for the generation and detection of genuine multipartite entanglement (GME) of LONs in multi-rail encoding sources. We show that with indistinguishable single-photon (DV) or squeezed-state (CV) inputs one can always generate GME in LONs. The effect of photon losses is also numerically analyzed for the generation scheme based on CV inputs.

I. INTRODUCTION

For quantum information processing in multipartite systems, multipartite entanglement has its advantages over bipartite entanglement, if the local systems have limited sizes [1]. In particular, genuine multipartite (GM) entanglement [2] plays an important role in quantum computation [3], quantum algorithm [4], quantum secret sharing [5-7], etc. The generation and verification of GM entanglement are therefore essential steps for various quantum information processing. Many theories have been proposed for experimental generation and verification of GM entanglement in multipartite qudit systems [8-15].

However, as a quantum computing processing that has already outperformed its classical counterpart [16-17], the role of entanglement in boson sampling computing [18] is still unclear. The physical systems that perform boson sampling are linear optics networks (LONs). In LONs, the indistinguishability of photons is the prerequisite of quantum supremacy of boson sampling [19]. As boson sampling interferes indistinguishable photons to generate classically non-simulatable photon statistics, one would expect a large amount of entanglement in the system. However, the indistinguishability of photons in LONs tangles the concept of entanglement [20-32], since entanglement should be defined with distinguishable local quantum information registers.

A conventional multipartite qudit system is multiple distinguishable particles. Each of the particles is a well-defined distinguishable quantum information register $|e_{m}\rangle$. In LONs, it is equivalent to a single-photon $M$-mode system.

$$|e_{m}\rangle = |0_{1}, ..., 0_{m-1}, 1_{m}, 0_{m+1}, ..., 0_{M-1}\rangle.$$ (1)

Each mode is a well-defined path in the outputs or inputs of a LON. Such quantum information register is called an $M$-rail qudit. Although one can extract multipartite qudit entanglement from a multipartite multiphoton LON to multipartite $M$-rail qudits [22], we still need a direct method to evaluate the multipartite entanglement in LONs without extraction that changes the testing state.

Here, we employ the intrinsic $M$-rail encoding as the quantum information registers in multiphoton LONs and adopt multi-rail Fock states as the discrete variables $|n\rangle$,

$$|n\rangle := |n_{0}, ..., n_{M-1}\rangle,$$ (2)

where $n_{m}$ is the photon number in the $m$-th mode. The entanglement defined in such $M$-rail encoding was first quantified in [33]. A detection method has been established in [34-35] for bipartite entanglement with the multiphoton $M$-rail encoding. Besides, bipartite entanglement between two multiphoton 3-rail systems has been generated and verified in experiments [36] according to the scheme proposed in [34].

In this paper, we will address a further question about the generation and evaluation of genuine multipartite entanglement (GME) in multiphoton LONS. In particular, our target GME exhibits Heisenberg-Weyl symmetry. Such symmetric states can generate special photon statistics under generalized Hadamard transformation [37-39]. In Section II, we will derive a GME criterion employing a GME verifier [39] that stabilizes the target Heisenberg-Weyl symmetric states. This criterion can be evaluated directly in experiments implemented with local generalized Hadamard transformation. In Section III, we will propose a generation scheme for GME. Our generation scheme allows single-photon sources and continuous-variable sources, such as displaced squeezed vacuum. We will evaluate and compare the GME generated by single-photon and squeezed sources, respectively. We will then conclude the paper in Section IV.
II. CRITERIA FOR GENUINE MULTIPARTITE ENTANGLEMENT IN LONS

In a $P$-partite linear optics network system, where each local system is an $M$-mode LON labeled by $i \in \{1, ..., P\}$, the photon number in each local system is conserved under perfect linear optics operations. We therefore consider a pure state $|\psi_{N_1,N_2,...,N_P}\rangle$ with fixed local photon number $N_i$ in the $i$-th local LON,

$$|\psi_{N_1,N_2,...,N_P}\rangle = \sum_{\mathbf{n}_i|\mathbf{n}_i|=N_i} c_{\mathbf{n}_1,...,\mathbf{n}_P} |\mathbf{n}_1,...,\mathbf{n}_P\rangle. \tag{3}$$

Here, the vector $\mathbf{n}_i = (n_i^{(i)},...,n_{M-1}^{(i)})$ indicates the photon occupation number $n_i^{(i)}$ in the $m$-th mode of the $i$-th local system. In such $P$-partite multiphoton $M$-rail encoding systems, a state $|\phi_{\text{bisep.}}\rangle$ is biseparable if there exists a bipartition, such that $|\phi_{\text{bisep.}}\rangle$ is a product state. A state $\hat{\rho}$ is biproducible, if it can be decomposed as a mixture of biseparable states $|\phi_{\text{bisep.}}\rangle$. On the contrary, it is called GM entangled, if it cannot be decomposed as a mixture of biseparable states $|\phi_{\text{bisep.}}\rangle$.

$$\hat{\rho}_{\text{GME}} \neq \sum_{p_\Phi=|\phi_{\text{bisep.}}\rangle} p_\Phi |\phi_{\text{bisep.}}\rangle \langle \phi_{\text{bisep.}}|. \tag{4}$$

To reveal the physical significance of GM entanglement in multi-rail encoded LONs, one needs to evaluate the complementary properties of multiphoton states in each local LON in complementary Heisenberg-Weyl measurements. Analog to the evaluation of bipartite multi-rail entanglement in LONs, one will need to derive the theoretical boundary on a GM entanglement witness or quantity for biproducible states through a convex roof extension over the irreducible subspaces of local Heisenberg-Weyl operators.

A. GM entangled states with Heisenberg-Weyl symmetry

In multiparticle qudit systems, many GM-entangled states are symmetric under simultaneously Heisenberg-Weyl (HW) transformations, e.g. particular graph states, singlet states, and Dicke states. The GME of these states can be detected in local complementary measurements in mutually unbiased bases associated with Heisenberg-Weyl operators. The multiphoton states that exhibit Heisenberg-Weyl symmetry are therefore of particular interest in our analysis.

In an $M$-mode LON system, a Heisenberg-Weyl operator $\hat{X}^i\hat{Z}^j$ is a phase shifting $\hat{Z}^j$ followed by a mode shifting $\hat{X}^i$, where the mode shifting $\hat{X}$ shifts each photon to its next neighboring mode cyclicly,

$$\hat{X}\hat{a}_m^\dagger\hat{X}^\dagger = \hat{b}_m^\dagger \quad \text{with} \quad m \oplus 1 := (m+1) \quad \text{(mod } M), \tag{5}$$

and the phase shifting $\hat{Z}$ adds a phase $\omega^m$ to each photon in the $m$-th mode,

$$\hat{Z}\hat{a}_m^\dagger\hat{Z}^\dagger = w^m\hat{b}_m^\dagger \quad \text{with} \quad \omega = e^{\pi i/2M}. \tag{6}$$

Here $\hat{a}_m^\dagger$ and $\hat{b}_m^\dagger$ are the creation operators of the $m$-th input and output mode, respectively. The HW operators of particular interest are

$$\hat{\Lambda}_j := \hat{X}\hat{Z}^j \quad \text{with} \quad j = 0, ..., M-1, \tag{7}$$

since they can be exploited to construct mutually unbiased bases for complementary measurements.

A $P$-partite state is symmetric under simultaneous HW operation, if it is an eigenstate of the operator $\hat{\Lambda}_{j_1} \otimes \cdots \otimes \hat{\Lambda}_{j_P}$, where $(j_1,...,j_P)$ are the indices of local phase shifting. The simultaneous HW symmetry of a state can be described by two indices $(k,\kappa)$ as follows,

$$\hat{\Lambda}_{j_1} \otimes \cdots \otimes \hat{\Lambda}_{j_P} |\psi_{k,\kappa}\rangle = \omega^{k+\kappa l} |\psi_{k,\kappa}\rangle. \tag{8}$$

The state $|\psi_{k,\kappa}\rangle$ is therefore symmetric under the simultaneous HW operator,

$$\hat{\Lambda}_{j_1} \otimes \cdots \otimes \hat{\Lambda}_{j_P} |\psi_{k,\kappa}\rangle = \omega^{k+\kappa l} |\psi_{k,\kappa}\rangle. \tag{9}$$

We call this type of symmetry the $(k,\kappa)$-symmetry. A $(k,\kappa)$-symmetric state must be a superposition of the computational-basis states that satisfy $\oplus_{j_1,j_2,\kappa} \mu(n_i) = \kappa$, 

$$|\psi_{k,\kappa}\rangle = \sum_{n_i: \oplus_{j_1,j_2,\kappa} \mu(n_i) = \kappa} c_{n_1,...,n_P} |n_1,...,n_P\rangle. \tag{10}$$

The quantity $\oplus_{j_1,j_2,\kappa} \mu(n_i)$ is the eigenphase of $\hat{\Lambda}_{j_1} \otimes \cdots \otimes \hat{\Lambda}_{j_P}$, which we call the $(j_1,...,j_P)$-weighted Z-clock label of $|n_1,...,n_P\rangle$.

$$\hat{Z}^{j_1} \otimes \cdots \otimes \hat{Z}^{j_P} |n_1,...,n_P\rangle = \omega^{\oplus_{j_1,j_2,\kappa} \mu(n_i)} |n_1,...,n_P\rangle, \quad \mu(n) := \sum_m n_m m. \tag{11}$$

Since the representation of $\hat{\Lambda}_{j_1} \otimes \cdots \otimes \hat{\Lambda}_{j_P}$ is reducible, the representation of a $(k,\kappa)$-symmetric state can be also reduced to a superposition of $(k,\kappa)$-symmetric states constructed within $\hat{\Lambda}_{j_1} \otimes \cdots \otimes \hat{\Lambda}_{j_P}$-irreducible subclasses $X$. A $(\hat{\Lambda}_{j_1} \otimes \cdots \otimes \hat{\Lambda}_{j_P})$-irreducible subclass $X$ is equivalent to an $\hat{X}^{j_1} \otimes \cdots \otimes \hat{X}^{j_P}$-irreducible subclass, which is generated by the simultaneous mode shifting $\hat{X}^{j_1} \otimes \cdots \otimes \hat{X}^{j_P}$ acting on a $P$-partite $M$-rail Fock state $|n_1,...,n_P\rangle$,

$$X_{n_1,...,n_P} := \left\{ (\hat{X}^{j_1} \otimes \cdots \otimes \hat{X}^{j_P})^m |n_1,...,n_P\rangle \right\}_{m=0,...,M-1}. \tag{12}$$

Within these $(\hat{\Lambda}_{j_1} \otimes \cdots \otimes \hat{\Lambda}_{j_P})$-irreducible subclasses, one can then construct the eigenstates of $\hat{X}^{j_1} \otimes \cdots \otimes \hat{X}^{j_P}$ as a uniform superposition of the states in $X$. 


The simultaneous HW operator \( \hat{A}_{j_1} \otimes \cdots \otimes \hat{A}_{j_P} \) will then induce a phase shift and transform an \( \hat{X} \otimes \hat{P} \) eigenstate as follows,

\[
\hat{A}_{j_1} \otimes \cdots \otimes \hat{A}_{j_P} |E_k(\mathbb{X}_{n_1,\ldots,n_P})\rangle := \omega^{k+\sum j_i \mu(n_i)} |E_k(\mathbb{X}_{n_1,\ldots,n_P})\rangle .
\]

One can then choose the HW indices \( j = (j_1,\ldots,j_P) \) in the following way to turn the state \( |E_k(\mathbb{X})\rangle \) into an eigenstate of the simultaneous HW operator,

\[
j = (j_1,\ldots,j_P) : \bigoplus_i j_i n_i = 0.
\]

The eigenstate \( |E_k(\mathbb{X})\rangle \) in Eq. (13) is therefore \((k,\kappa)\)-symmetric with \( \kappa = \bigoplus_i j_i \mu(n_i) \).

In this case, every element of an \( \mathbb{X} \) class has the same \( j \)-weighted \( Z \)-clock label, which we call the \( j \)-weighted \( Z \)-clock label of a class \( \mathbb{X} \),

\[
\mu_j(\mathbb{X}) := \bigoplus_i j_i \mu(n_i) \quad \text{with} \quad |n_1,\ldots,n_P\rangle \in \mathbb{X}.
\]

As a result, a state \( |\psi_{k,\kappa}\rangle \) that exhibits the \((k,\kappa)\)-symmetry described in Eq. (8) can be expressed as a superposition of the \( \hat{A}_{j_1} \otimes \cdots \otimes \hat{A}_{j_P} \) eigenstates \( |E_k(\mathbb{X})\rangle \) over all \( \hat{X} \otimes \hat{P} \)-irreducible subclasses, of which the \((j_1,\ldots,j_p)\)-weighted \( Z \)-clock label is \( \kappa \)

\[
|\psi_{k,\kappa}\rangle = \sum_{\mu_j(\mathbb{X})=\kappa} c_{\mathbb{X}} |E_k(\mathbb{X})\rangle .
\]

Mathematically, this state is highly GM-entangled. A general \((N_1,\ldots,N_P)\)-photon pure state can be written as a superposition of HW symmetric states

\[
|\Phi_{N_1,\ldots,N_P}\rangle = \sum_{k,\kappa} c_{k,\kappa} |\psi_{k,\kappa}\rangle .
\]

The more unbalanced the superposition in Eq. (18) is, the higher is the GM-entangled. The extremum case is \( c_{k,\kappa} = 1 \), which leads to a unique HW symmetry. One can therefore pick the HW-symmetric component with the highest probability amplitude \( |c_{k,\kappa}|^2 \) as the target GM-entangled state, and construct a corresponding GME detection measurement to verify the GME of \( |\Phi_{N_1,\ldots,N_P}\rangle \).

\section*{B. Detection of GM entanglement}

The entanglement of an HW-symmetric state is the target GME that we want to verify with the theory developed in this section. Each component \( |\Phi_{k}(\mathbb{X})\rangle \) of an HW-symmetric state in Eq. (17) is a highly GM-entangled state. As an intrinsic property of GME, the HW symmetry is a strong indication of GME, if one can reveal its characteristic complementary correlations in the eigenbases of corresponding local HW operators that are mutually unbiased to each other. In such local complementary measurements, one can determine the upper bound on the complementary correlations for biproducible states through the convex roof extension of the upper bounds for different \( \hat{X} \otimes \hat{P} \)-irreducible subspaces.

The complementary correlations will be evaluated through a GME verifier that stabilizes a target HW-symmetric state. It can be constructed as a projection onto the supporting outputs of corresponding HW measurement settings. The projection onto the subspace that exhibits a symmetry with an eigenphase of \( m \) under the HW transformation \( \hat{A}_{j_1} \otimes \cdots \otimes \hat{A}_{j_P} \) can be constructed as

\[
\hat{S}_L(l,m) := \frac{1}{M} \sum_{m'=0}^{M-1} \omega^{-mm'} \hat{A}_{j_1}^l \otimes \cdots \otimes \hat{A}_{j_P}^{m'} .
\]

This projection is a sum of the projectors \( |E_k(\mathbb{X})\rangle \langle E_k(\mathbb{X})| \) in the corresponding \( \hat{X} \otimes \hat{P} \)-irreducible classes \( \mathbb{X} \), of which the \( j \)-weighted \( Z \)-clock label satisfies \( k + \mu_j(\mathbb{X}) = m \),

\[
\hat{S}_L(l,m) = \sum_{\mathbb{X}: k+\mu_j(\mathbb{X})=m} |E_k(\mathbb{X})\rangle \langle E_k(\mathbb{X})| .
\]

As a result of Eq. (17) and (20), the projector \( \hat{S}_L(l,m) \) with \( m = k+\kappa l \) stabilizes a \((k,\kappa)\)-symmetric state \( |\psi_{k,\kappa}\rangle \),

\[
\hat{S}_L(l,m) |\psi_{k,\kappa}\rangle = \gamma_m^{* k+\kappa l} |\psi_{k,\kappa}\rangle .
\]

According to Theorem 3.1 in [35], the projector \( \hat{S}_L(k,m) \) can be evaluated in the the \( \hat{A}_{j_1} \otimes \cdots \otimes \hat{A}_{j_P} \)-eigenbasis measurement, which is implemented with the local generalized Hadamard transformation \( \hat{H}_{j_1} \otimes \cdots \otimes \hat{H}_{j_P} \) (see Fig. 1 (a)).
where $\hat{H}_j$ is a generalized Hadamard transformation

$$\hat{H}_j |\alpha_m\rangle\hat{H}_j^\dagger = \frac{1}{\sqrt{M}} \omega^{j(\tau(M-m)m)} \sum_{m'} \omega^{m'm'} b_{m'}^\dagger |\alpha_m\rangle. \quad (23)$$

For $j = 0$, $\hat{H}_0$ is the discrete Fourier transformation. The projector $\hat{S}_{\Lambda_{(l,m)}}$ therefore projects input states onto the $\hat{\Lambda}_{j,l} \otimes \cdots \otimes \hat{\Lambda}_{j,P,l}$ eigensubspace that has the correlation of $\bigoplus \mu(n_i) = m$.

On the other hand, in the computational basis, one can also construct a stabilizer of a $(k, \kappa)$-symmetric state

$$\hat{S}_{Z|m} = \sum_{\oplus, i,j \in \rho(n_i) = m} |n_1, \ldots, n_P \rangle \langle n_1, \ldots, n_P|, \quad (24)$$

such that

$$\hat{S}_{Z|m} |\psi_{k,\kappa}\rangle = \delta^m_{\kappa} |\psi_{k,\kappa}\rangle. \quad (25)$$

Mixing the two types of state stabilizers in Eq. 22 and 24, one can construct a GME verifier to verify the GME that exhibits a HW symmetry associated with the indices $(k, \kappa)$,

$$\hat{V}_{k,\kappa} := \frac{1}{1 + |\mathbb{L}|} \left( \hat{S}_{Z|\kappa} + \sum_{l \in \mathbb{L}} \hat{S}_{\Lambda_{(l,k+l)}} \right) \quad (26)$$

where the configuration set $\mathbb{L}$ of the $\bigotimes_i \hat{\Lambda}_{j,i}$ measurements can be chosen as a subset of $\{0, 1, \ldots, M - 1\}$. It is obvious that the GME verifier $\hat{V}_{k,\kappa}$ stabilizes a $(k, \kappa)$-symmetric state $|\psi_{k,\kappa}\rangle$,

$$\hat{V}_{k,\kappa} |\psi_{k',\kappa'}\rangle = \frac{1}{1 + |\mathbb{L}|} \left( \delta^\kappa_{\kappa'} + \sum_{l \in \mathbb{L}} \delta^{k \kappa + k' l} \right) |\psi_{k',\kappa'}\rangle. \quad (27)$$

The configuration $\mathbb{L}$ must be chosen in a way such that the $\bigotimes \hat{\Lambda}_{j,l}$ measurements are complementary to each other, i.e., they have mutually unbiased measurement bases. According to Theorem 2.2 in [35], such a complementary configuration set $\mathbb{L}$ must fulfill

$$\text{gcd} (j_i (l - l') N_i |X_{n_i}| / M, |X_{n_i}|) = 1 \quad (28)$$

for all local systems $i \in \{1, \ldots, P\}, l, l' \in \mathbb{L}$, and

$$\langle n_1, \ldots, n_P |\hat{\rho} |n_1, \ldots, n_P\rangle \neq 0. \quad (29)$$

The choice of $\mathbb{L}$ is adaptive to the measurement statistics in the computational basis. The construction of measurement settings according to Eq. 28 guarantees that the local eigenvectors of $\bigotimes \hat{\Lambda}_{j,l}$ measurements with $l \in \mathbb{L}$ are mutually unbiased within all the local $\Sigma_i$-irreducible subspaces $\bigotimes \text{span}(X_i)$ that supports the testing state $\hat{\rho}$. As a result of the complementarity of the local measurement, one can then detect the GME according to the following theorem.

**Theorem II.1 (GME detection)** For a target HW-symmetric state $|\psi_{k,\kappa}\rangle$ defined in Eq. 8 with fixed local photon numbers $\{N_1, \ldots, N_P\}$, one can construct a set of measurements implemented by local generalized Hadamard transform $\hat{H}_{j,l} \otimes \cdots \otimes \hat{H}_{j,P,l}$, where the indices $\{j_1, \ldots, j_P\}$ satisfy Eq. 15, and $l \in \mathbb{L}$ satisfies Eq. 28. In this set of measurements, one can construct a GME verifier $\hat{V}_{k,\kappa}$ through Eq. 26. The upper bound on the expectation value $\langle \hat{V}_{k,\kappa} \rangle$ for bi-producible states are

$$\langle \hat{V}_{k,\kappa} \rangle \leq \frac{1 + \langle \hat{D} \rangle |\mathbb{L}|}{1 + |\mathbb{L}|}, \text{ for all bi-producible } \hat{\rho}, \quad (30)$$
where $\hat{D}$ is an operator evaluated in the computational basis
\[
\hat{D} = \sum_{n_i | |n_i| = N_i} \frac{1}{\min|\mathcal{X}_{n_i}|} |n_1, \ldots, n_P\rangle\langle n_1, \ldots, n_P|,
\] (31)
and $\min|\mathcal{X}_{n_i}|$ is the minimum cardinality of local $\hat{X}$-irreducible classes $\mathcal{X}_{n_i}$ for the Fock vectors $|n_1, \ldots, n_P\rangle$. 

Proof: See Appendix V A. 

If the expectation value $\langle \hat{V}_{k,\kappa} \rangle$ exceeds this bound, then one can conclude the GME of the testing state. It is obvious that a $(k,\kappa)$-symmetric state $\psi_{k,\kappa}$ has the maximum expectation value,
\[
\langle \psi_{k,\kappa}|\hat{V}_{k,\kappa}|\psi_{k,\kappa}\rangle = 1,
\] (32)
which exceed the upper bound for bi-producible states determined in Eq. (30).

Note that the choice of $\mathbb{L}$ depends on local photon numbers $N_i$, mode number $M$, and HW indices $j$ according to Eq. (28). If one choose $\mathbb{L} = \{0\}$, the condition in Eq. (28) is always fulfilled. The single-index set $\mathbb{L} = \{0\}$ is always a good choice for GME detection. In this construction, one can detect GME with just two measurement settings, one is in the computational basis, the other is in the $\hat{X}^\otimes P$ eigenbasis implemented by the inverse discrete Fourier transformation $\hat{F}^{\otimes P}$. 

For the special case when $M$ is a prime number, the cardinality of all local $\hat{X}$-irreducible classes $\mathcal{X}_{n_i}$ is $M$. Meanwhile, Eq. (28) is always satisfied, if the local photon number $N_i$ is not a multiple of $M$. One can therefore choose an arbitrary subset of $\{0, \ldots, M-1\}$ as the settings $\mathbb{L}$ for complementary measurements. In this case, Theorem II.1 can be simplified as the following corollary.

Corollary II.2 For a multipartite LONs, of which each local mode number $N_i$ is prime, one can construct a set of measurements implemented by local generalized Hadamard transform $\hat{H}_{j_1,1} \otimes \cdots \otimes \hat{H}_{j_P,1}$, where the indices $\{j_1, \ldots, j_P\}$ satisfy Eq. (15), and $l \in \mathbb{L}$ with $\mathbb{L} \subseteq \{0, \ldots, M-1\}$. A GME verifier $\hat{V}_{k,\kappa}$ given in Eq. (26) has then an upper bound
\[
\text{tr} \left( \hat{V}_{k,\kappa} \hat{\rho} \right) \leq \frac{M + |\mathbb{L}|}{M(|\mathbb{L}| + 1)}, \quad \text{for all bi-producible} \ \hat{\rho}. \quad (33)
\]
Proof: For a prime number $M$, any $l \in \{0, \ldots, M-1\}$ satisfy Eq. (28), one can therefore choose any $l$ to construct the set of complementary configuration $\mathbb{L}$. In this case, the expectation value of $\hat{D}$ in Theorem II.1 is always equal to $1/M$, which leads to the upper bound given in (33). 

If one chooses the complete complementary set $\mathbb{L} = \{0, \ldots, M-1\}$, the bound is simply $2/(M + 1)$. In such measurement settings, the expectation value of $\hat{V}_{k,\kappa}$ for a general $(N_1, \ldots, N_P)$-photon pure state $|\Phi_{N_1, \ldots, N_P}\rangle$ in Eq. (18) is given by
\[
\langle \Phi_{N_1, \ldots, N_P}|\hat{V}_{k,\kappa}|\Phi_{N_1, \ldots, N_P}\rangle = \frac{1 + M |c_{k,\kappa}|^2}{M + 1}. \quad (34)
\]
As a result, for a prime-number $M$, one can always detect the GME for a state $|\Phi_{N_1, \ldots, N_P}\rangle$, which has the $(k,\kappa)$ probability amplitude larger than $1/M$,
\[
|c_{k,\kappa}|^2 > \frac{1}{M} \Rightarrow \text{GME}. \quad (35)
\]
Let us consider a state $|\Phi^{(k)}_{N_1, \ldots, N_P}\rangle$ with a particular $\omega^k$ eigenphase symmetry under $\hat{X}^\otimes P$ transformation
\[
|\Phi^{(k)}_{N_1, \ldots, N_P}\rangle = \sum_{\kappa=0}^{M-1} c_{k,\kappa} |\psi_{k,\kappa}\rangle. \quad (36)
\]
It always has a maximum probability amplitude $|c_{k,\kappa}|^2 > 1/M$, unless $|c_{k,\kappa}|^2 = 1/M$ for all $\kappa$. It means that we have very high probability to detect GME of $|\Phi^{(k)}_{N_1, \ldots, N_P}\rangle$ with our measurement settings. For GME generation, it is therefore desirable to create a state which is symmetric under the simultaneous mode shifting $\hat{X}^\otimes P$.

III. GENERATION OF MULTIPARTITE M-RAIL ENTANGLEMENT

In this section, we will consider the generation of GME in multipartite $M$-rail systems through state preparation for $\hat{X}^\otimes P$-symmetry. A generation scheme for multipartite $M$-rail states that exhibit $\hat{X}^\otimes P$-symmetry can be extended from the scheme for bipartite $M$-rail entanglement in [34, 35].

As shown in Fig. 1 (b), $M$ copies of input state $|\varphi\rangle^\otimes M$ are sent into $M$ pieces of multimode splitters with $P$ output modes. A $P$-mode splitter divides the $m$th-input into a uniform superposition of $M$ modes distributed in each local system indexed by the label $m$,
\[
S \hat{a}_m \hat{S}^\dagger = \frac{1}{\sqrt{P}} (\hat{a}_{1,m} + \cdots + \hat{a}_{P,m}^\dagger), \quad (37)
\]
where $\hat{a}_{i,m}$ is the creation operator of the $m$th mode in the $i$-th local system. In general, a single-mode input $|\varphi\rangle$ can be expressed in the $2nd$ quantization formalism as
\[
|\varphi\rangle = \sum_\nu c_\varphi(\nu) |\nu\rangle, \quad (38)
\]
where $|\nu\rangle$ are Fock states. If we have $M$ copies of this state, the input state is a superposition of different Fock-state vectors $|\nu\rangle := |\nu_0, \ldots, \nu_{M-1}\rangle$,
\[
|\varphi\rangle^\otimes M = \sum_\nu c_\varphi(\nu) |\nu\rangle, \quad (39)
\]
where \( c_\varphi (\nu) \) is the product of the probability amplitude \( c_\varphi (\nu_m) \) at each mode

\[
c_\varphi (\nu) := \prod_{m=0}^{M-1} c_\varphi (\nu_m). \tag{40}
\]

The \((N_1, ..., N_P)\)-postselected state \( |\Phi_{N_1, ..., N_P} (\varphi)\rangle \) is a superposition of Fock-vector states \(|n_1, ..., n_P\rangle\) that have local photon numbers \(|n_i| = N_i\),

\[
|\Phi_{N_1, ..., N_P} (\varphi)\rangle = \frac{1}{\sqrt{p_{N_1, ..., N_P}}} \sum_{n_1, ..., n_P} \frac{c_\varphi (n_{\text{tot}})}{\sqrt{P_{N_{\text{tot}}}}} \frac{n_{\text{tot}}!}{n_1! \cdots n_P!} |n_1, ..., n_P\rangle. \tag{41}
\]

Here, \( N_{\text{tot}} := \sum_i N_i \) is the total photon number, \( n_{\text{tot}} := \sum_i n_i \) is the Fock vector component of the input before the multimode splitters, and \( p_{N_1, ..., N_P} \) is the probability of the postselection on the local photon numbers \((N_1, ..., N_P)\),

\[
p_{N_1, ..., N_P} = \sum_{n_1, ..., n_P} \frac{|c_\varphi (n_{\text{tot}})|^2}{P_{N_{\text{tot}}}} \frac{n_{\text{tot}}!}{n_1! \cdots n_P!}. \tag{42}
\]

It is obvious that the state \( |\Phi_{N_1, ..., N_P} (\varphi)\rangle \) is \( \hat{X}^{\otimes P} \)-symmetric,

\[
\hat{X}^{\otimes P} |\Phi_{N_1, ..., N_P} (\varphi)\rangle = |\Phi_{N_1, ..., N_P} (\varphi)\rangle. \tag{43}
\]

It is therefore a superposition of \((0, \kappa)\)-symmetric states

\[
|\Phi_{N_1, ..., N_P} (\varphi)\rangle = \sum_{\kappa} c_\kappa |\psi_{0, \kappa}\rangle. \tag{44}
\]

According to Eq. \((35)\), in a prime-number \( M \)-mode system, for such a \( \hat{X}^{\otimes P} \)-symmetric state, the only case that our method cannot detect GME of \( |\Phi_{N_1, ..., N_P} (\varphi)\rangle \) is when \(|c_\kappa|^2 = 1/M \) for all \( \kappa \in \{0, ..., M-1\} \). Such a uniform superposition with \(|c_{0, \kappa}| = 1/\sqrt{M}\) can be generated with coherent states \( |\varphi\rangle = |\alpha\rangle \) as inputs. In this case, the state \( |\Phi_{N_1, ..., N_P} (\alpha)\rangle \) is fully separable. The expectation value \( \langle \hat{V}_{0, \kappa} |\psi_{0, \kappa}\rangle \) achieves its maximum for bi-producible states.

\[
\langle \Phi_{N_1, ..., N_P} (\alpha) |\hat{V}_{0, \kappa} |\Phi_{N_1, ..., N_P} (\alpha)\rangle = \frac{2}{M+1} \tag{45}
\]

for all \( \kappa \). The more unbalanced is the superposition in Eq. \((14)\), the higher the GME of \( |\Phi_{N_1, ..., N_P} (\varphi)\rangle \) is. With single-photon and squeezed-state inputs, one can always generate \( \hat{X}^{\otimes P} \)-symmetric state with unbalanced superposition in \( \kappa \).

### A. Generation of GME with single-photon input sources

For single-photon inputs \(|\varphi\rangle = |1\rangle\), the \((N_1, ..., N_P)\)-postselected state of GME generation in Fig. \([1]\) (b) is

\[
|\Phi_{N_1, ..., N_P}\rangle = \frac{1}{\sqrt{N_1 \cdots N_P}} \sum_{n_1, ..., n_P} |n_1, ..., n_P\rangle. \tag{46}
\]

To detect the GME of this state, one can choose the HW-indices \( j = (1, ..., 1) \), which fulfill the Eq. \((15)\), to construct the GME detection measurements. The state \( |\Phi_{N_1, ..., N_P}\rangle \) exhibits the \((k, \kappa)\)-symmetry with \( k = 0 \) and \( \kappa = M(M-1)/2 \),

\[
\hat{X}^{\otimes P} |\Phi_{N_1, ..., N_P}\rangle = |\Phi_{N_1, ..., N_P}\rangle, \tag{47}
\]

\[
\hat{Z}^{\otimes P} |\Phi_{N_1, ..., N_P}\rangle = \omega^{M(M-1)/2} |\Phi_{N_1, ..., N_P}\rangle. \tag{48}
\]

For a general mode number \( M \), which has \( d_M \) as its minimum prime-number divider, one can construct the complementary measurements for GME detection according to Theorem \([1]\). Since \( l \in \{0, ..., d_M-1\} \) fulfills Eq. \((28)\), one can choose any subset can include additional measurements in the \( \Lambda_l \)-eigenbasis with \( L \subseteq \{0, ..., d_M-1\} \) as the set of complementary measurement configurations. The GME verifier in Eq. \((26)\), associated with these measurement settings is then given by

\[
\hat{V}_{0, \kappa} = \frac{1}{1+|L|} \left( \hat{S}_{0, \kappa} + \sum_{l \in |L|} \hat{S}_{\Lambda_l(0, \kappa)} \right), \tag{49}
\]

with \( \kappa = M(M-1)/2 \). The expectation value of \( \hat{V}_{0, \kappa} \) for the state \( |\Phi_{N_1, ..., N_P}\rangle \) is unity

\[
\langle \Phi_{N_1, ..., N_P} |\hat{V}_{0, \kappa} |\Phi_{N_1, ..., N_P}\rangle = 1 > \beta_{\text{biprod}}, \tag{50}
\]

where the upper bound \( \beta_{\text{biprod}} \) for bi-producible states can be determined by Eq. \((30)\). If the mode number \( M \) is prime, the upper bound is simply determined by Eq. \((33)\).

In a more general scheme, one can also employ Fock states \(|\varphi\rangle\) with photon number \( \nu \) larger than one as the inputs in the GME generation shown in Fig. \([1]\) (b). If \( \nu \) is not a divider of \( M \), the corresponding measurement construction of GME detection is identical to the one constructed for single-photon inputs. On the other hand, if \( \nu \) is a divider of \( M \), Eq. \((28)\) does not hold for any \( l \neq 0 \). In this case, the GME detection measurement can only be constructed with the measurement in the computational basis and an additional measurement in the \( \hat{X}^{\otimes P} \) eigenbasis.

From Eq. \((50)\), one can see that indistinguishable Fock-state inputs result in perfect HW symmetric states, which have the optimum GME signature. Besides, another advantage of GME generation with Fock-state inputs is its robustness of GME generation against photon.
losses, since the postselection of the measurement outputs on local photon numbers \(\{N_1, \ldots, N_P\}\), of which the total photon number \(\sum_i N_i\) is equal to the input total photon number, already excludes the photon losses in the statistics. However, the scalability of such a generation scheme is a challenging issue in practice due to the difficulty of preparation of indistinguishable single-photon sources.

### B. Generation of GME with displaced squeezed vacuum

A solution of the scalability of GME generation scheme in Fig. 1(b) is to replace the single-photon inputs by continuous-variable sources, such as squeezed states. Here, we consider the \(x\)-displaced \(r\)-squeezed vacuum state \(|\varphi\rangle = |\sigma(r, x)\rangle\) with

\[
|\sigma(r, x)\rangle = (1 - \gamma^2)^{1/4} e^{-2z(x)^2} \sum_n \frac{\sqrt{\gamma^n}}{n!} h_n(2\zeta x) |n\rangle,
\]

where \(h_n\) are the probabilists’ Hermite polynomial, and \(\gamma\) and \(\zeta\) are determined by the squeezing factor \(r\) [45]

\[
\gamma = \tanh(r) \quad \text{and} \quad \zeta = \sqrt{\frac{1}{1 - e^{-4r}}}.
\]

According to Eq. 11, the \(\{N_1, \ldots, N_P\}\)-postselected quantum state in this generation scheme is

\[
|\Phi_{N_1, \ldots, N_P}(r, x)\rangle = \frac{1}{\sqrt{R_{N_1, \ldots, N_P}}} \sum_{n_1, \ldots, n_P} \frac{h_{n_{\text{max}}}(2\zeta x)}{\sqrt{n_{\text{max}}!} \cdots n_P!} |n_1, \ldots, n_P\rangle
\]

where \(h_{n}(x) = \prod_m h_{n_m}(x)\) is a product of the Hermite polynomials \(h_{n_m}(x)\), and \(R_{N_1, \ldots, N_P}\) is a normalization factor.

In Eq. 53, one can see that the squeezing factor \(r\) only affects the factor \(\zeta\), which only changes the scaling of the displacement \(x\). Changing the squeezing factor \(r\) therefore does not change the behavior of the expectation value of GME verifier \(\tilde{V}_{0,\kappa}\) against the displacement \(x\). For a large enough \(r\), the scaling factor is approximately given by

\[
\zeta(r) \approx 1 + \frac{1}{2} e^{-4r}.
\]

For \(r \gtrapprox 0.576(5dB)\), the scaling factor \(\zeta - 1 \lesssim 0.05\), which means increasing the squeezing factor \(r\) does not introduce a significant change to the post-selected state anymore. However, one should note that it still affect the efficiency of the post-selection on \(\{N_1, \ldots, N_P\}\) local photon numbers.

If the squeezing factor \(r = 0\), the input states are simply coherent states, which will create a fully separable state. Since the photon statistics of a displaced squeezed vacuum is approximately equal to the photon statistics of coherent states for a large displacement, one expects that the GME signature detected by the GME verifier \(\tilde{V}_{0,\kappa}\) asymptotically diminishes as the displacement increases.

Here, we take a tripartite (5, 5, 5)-mode and (2, 1, 1)-photon system as an example. The testing state \(|\Phi_{2,1,1}(r, x)\rangle\) is a (2, 1, 1)-photon GM entangled state. To detect the GME, we first choose the HW indices as \((j_A, j_B, j_C) = (1, 4, 4)\). The complementary measurement settings are implemented with the generalized Hadamard transformation \(\hat{H}_l \otimes \hat{H}_{4l} \otimes \hat{H}_{4l}\) with \(l \in \mathbb{L} = \{0, \ldots, M - 1\}\). Together with the measurement in the computational basis, one will implement six measurements. The measurement statistics of \(|\Phi_{2,1,1}(r, x)\rangle\) is numerically simulated in Fig. 3 for \(x = 0\).

In Fig. 2 (a), we fix the squeezing factors as \(r = 0.058(0.5dB)\) and numerically evaluate the expectation values of the GME verifiers \(\tilde{V}_{0,\kappa}\) for different displacements \(x\) (the blue solid line for \(\kappa = 0\), and blue dashed line for \(\kappa = 1, \ldots, 4\)). The expectation value \(\langle \tilde{V}_{0,0}\rangle\) has its maximum at \(x = 0\) and decreases as \(x\) increases until a turning point, meanwhile, the expectation values of \(\langle \tilde{V}_{0,\kappa=1,\ldots,4}\rangle\) have their minimum at \(x = 0\) and increases as \(x\) increases until the same tuning point. There are two turning points before \(\langle \tilde{V}_{0,\kappa}\rangle\) asymptotically approaches the bi-producible bound 1/3.

Except for the two crossing points \(x \approx 0.127\) and \(x \approx 0.257\), where \(\langle \tilde{V}_{0,\kappa=0}\rangle = \langle \tilde{V}_{0,\kappa>0}\rangle = 1/3\), there exists at least a \(\langle \tilde{V}_{0,\kappa}\rangle\) greater than the bi-producible bound 1/3. It means that GME of \(|\Phi_{2,1,1}(r, x)\rangle\) can be always detected for \(x \neq 0.127\) or 0.257 with the measurement settings constructed in the \(\Lambda_l \otimes \Lambda_{4l} \otimes \Lambda_{4l}\) eigenbasis. For these two crossing points, one can verify its GME with the measurement settings constructed in another HW indices, e.g. \((j_A, j_B, j_C) = (1, 1, 2)\). The expectation value \(\langle \tilde{V}_{0,\kappa}\rangle\) evaluated in the \(\Lambda_l \otimes \Lambda_l \otimes \Lambda_{2l}\)-eigenbasis measurements are plotted with orange solid and dashed lines for \(\kappa = 0\) and \(\kappa > 0\), respectively. One can see that the GME of \(|\Phi_{2,1,1}(r, x)\rangle\) at \(x = 0.127\) and \(x = 0.257\) is verified by the GME verifier \(\tilde{V}_{0,\kappa>0}\) in the \(\Lambda_l \otimes \Lambda_l \otimes \Lambda_{2l}\)-eigenbasis measurement settings.

In Fig. 2 (b), the expectation value \(\langle \tilde{V}_{0,\kappa}\rangle\) evaluated in the \(\Lambda_l \otimes \Lambda_{4l} \otimes \Lambda_{4l}\)-eigenbasis measurements are plotted for different squeezing factors. One can see that the squeezing factor changes the scale of the asymptotic behavior of \(\langle \tilde{V}_{0,\kappa}\rangle\) against the displacement. As the squeezing factor increases, the value of \(\langle \tilde{V}_{0,\kappa}\rangle\) converges to a function of the displacement \(x\), which is \(r\)-independent.

Compare to single-photon inputs, the signature of the GME generated with CV inputs is not as significant as the GME generated with single photons. However, its scalability is much better than the scheme with single-photon sources. Besides these two aspects, photons losses play an important role in the GME generation with CV.
squares, as the postselection on local photon numbers can not exclude the photon-loss events from the statistics.

C. Effect of photon losses

In the GME generation and evaluation scheme of Fig. [4] photons in each mode may be lost in the waveguide medium, at the beam splitters, or during the photon number resolving detection. For the GME generation with single-photon sources in Section III A, the effect of photon losses is excluded in the post-selected measurement statistics, since we post-select on an output total photon number which is equal to the input total photon number. For the GME generation with CV sources, photon losses will change the measurement statistics and diminish the GME signature. In this case, one therefore has to take photon losses into account.

In linear optics networks, the mechanism of photon losses can be modeled as each path mode interferes with an external mode in the environment that is not accessible. Such lossy interference can be modeled as a beam splitter with a reflection coefficient \( \sqrt{\epsilon} \). A photon has a probability of \( \epsilon \) being reflected and lost in the environment (see Appendix V B for detailed analysis). For simplicity, we assume a uniform photon loss rate \( \epsilon \) in each mode. Employing the above lossy model, we obtain a lossy input state

\[
\hat{\rho}_{N_1, \ldots, N_P}(r, x; \epsilon) = \sum_{\nu_1, \ldots, \nu_P} p_{\nu_1|N_1}(\epsilon) |\Phi_{\nu_1|N_1}(r, x)\rangle \langle \Phi_{\nu_1|N_1}(r, x)|.
\]

where \( p_{\nu_1|N_1}(r, x; \epsilon) \) is the probability of losing \( \nu_1 \) photons in the \( i \)-th local system, and the pure state component is

\[
|\Phi_{\nu_1|N_1}(r, x)\rangle = \frac{1}{\sqrt{R_{\nu_1|N_1}}} \sum_{n_1|n_1|N_1} h_{n_1|n_1|N_1}^{(2)} |n_1, \ldots, n_P\rangle.
\]

with \( R_{\nu_1|N_1} \) being the normalization factor. The expectation value of the lossy input state is therefore

\[
\text{tr} \left( \hat{\rho}_{N_1, \ldots, N_P}(r, x; \epsilon) \hat{V}_{0, \kappa} \right) = \sum_{\nu_1, \ldots, \nu_P} p_{\nu_1|N_1}(r, x; \epsilon) \left< \Phi_{\nu_1|N_1}(r, x) | \hat{V}_{0, \kappa} | \Phi_{\nu_1|N_1}(r, x) \right>,
\]

where \( \langle \hat{V}_{0, \kappa} \rangle \) of each component is

\[
\left< \Phi_{\nu_1|N_1}(r, x) | \Phi_{\nu_1|N_1}(r, x) \right> = \frac{1 + M |c_{0, \kappa}|^2}{M + 1}.
\]

The expectation value \( \langle \hat{V}_{0, \kappa} \rangle \) of two state \( |\Phi_{\nu_1|N_1}\rangle \) and \( |\Phi_{\nu_1|N_1}\rangle \) are identical if \( |\nu_{\text{tot}}\rangle \) and \( |\nu_{\text{tot}}'\rangle \) belong to the same \( \tilde{X} \)-irreducible class,

\[
\exists m \in \{0, \ldots, M - 1\}, \quad \text{s.t.} \quad |\nu_{\text{tot}}\rangle = \tilde{X}^m |\nu_{\text{tot}}'\rangle.
\]

In general, it holds that

\[
\left\{ \begin{array}{ll}
\sum_{\kappa} |c_{0, \kappa}|^2 < 1, & \text{for all } |\nu_{\text{tot}}\rangle \neq \tilde{X} |\nu_{\text{tot}}'\rangle \\
\sum_{\kappa} |c_{0, \kappa}|^2 = 1, & \text{for all } |\nu_{\text{tot}}\rangle = \tilde{X} |\nu_{\text{tot}}'\rangle,
\end{array} \right.
\]

which means that the sum of the expectation value for different \( \kappa \) under photon losses is in general smaller than \( 2M/(M + 1) \) for non-zero photon losses,

\[
\sum_{\kappa} \text{tr} \left( \hat{\rho}_{N_1, \ldots, N_P}(r, x; \epsilon) \hat{V}_{0, \kappa} \right) < \frac{2M}{M + 1} \quad \text{for } \epsilon > 0.
\]
As the photon loss rate increases, more non-$\hat{X}^{\otimes P}$-symmetric components $\{|\Phi_{\nu_i}|_N\}$ with $X|\nu_{\text{tot}}\rangle \neq |\nu_{\text{tot}}\rangle$ are mixed in the state $\bar{\rho}_{N_1,...,N_P}$. The significance of GME detection is therefore decreased by photon losses.

For a small photon loss rate $\epsilon$, the expectation value in Eq. (57) can be approximately evaluated with the sum over the photon-loss vector up to a small total photon loss. In Fig. 3 we numerically evaluate the expectation value $\langle \hat{V}_{0,\kappa} \rangle$ for $(5,5,5)$-mode $(2,1,1)$-photon GME entanglement generated with the input $x$-displaced (5dB)-squeezed state under the consideration of a uniform photon loss rate $\epsilon$ at each mode. Fig. 3(a) and (b) show the $\langle \hat{V}_{0,\kappa} \rangle$ of different components $\{|\Phi_{\nu_i}|_N\}$ with $\nu_{\text{tot}} \in \{00000, 10000, 20000, 11000, 10100\}$ for $\kappa = 0$ and $\kappa > 0$, respectively. For a small enough photon loss rate ($\epsilon \leq 0.25$), the expectation value $\langle \hat{V}_{0,\kappa} \rangle$ of the mixed state $\rho_{N_1,...,N_P}(r, x; \epsilon)$ for $\kappa = 0$ and $\kappa > 0$ can be approximately given by the convex combination of the values in Fig. 3(a) and (b), respectively. The sum of the expectation value for different $\kappa$ under photon losses is in general smaller than $5/3$:

$$\sum_{\kappa} \text{tr} \left( \bar{\rho}_{N_1,...,N_P}(r, x; \epsilon) \hat{V}_{0,\kappa} \right) < 5/3 \quad \text{for} \quad \epsilon > 0. \tag{62}$$

The sum $\sum_{\kappa} \langle \hat{V}_{0,\kappa} \rangle$ decreases as the photon loss rate increases. For $\epsilon \geq 0.2$ one can only detect GME for $x < 0.265$.

IV. DISCUSSION AND CONCLUSION

In this paper, we have derived a method for detecting genuine multipartite entanglement (GME) among multimode multimode linear optics networks in Theorem [11.1] which employs GME verifiers for a target GME-entangled state to reveal the signature for GME in experiments. The expectation value of a GME verifier exceeding its bi-producible bounds signifies the existence of GME. In general, the bi-producible bounds depend on the dimension of local mode-shifting irreducible subspace, which can be determined adaptively to the measurement in the computational basis. For prime-number local modes, the upper bounds on the state verifiers within all local mode-shifting-irreducible subspaces are uniform. In this case, the upper bounds can be simplified as is shown in Corollary [11.2].

A scheme for GME generation employing multimode splitters with indistinguishable single-photon or CV sources has been also proposed. Although the GME generation with single-photon sources can provide high entanglement signature, which is also robust to photon losses, the scalability of such GME generation requires a large amount of single-photon sources with good indistinguishability. This obstacle the scalability of GME generation. On the other hand, although the GME generation with CV sources provides less GME signature, which is also diminished by photon losses, the generation scheme with CV sources is much more scalable than the one with single-photon sources.

The GME signature of $(2,1,1)$-photon $(5,5,5)$-mode GME generation with displaced squeezed vacuum inputs is numerically investigated. It tells us that one can obtain the most significant GME with a squeezed vacuum without displacement. For a larger displacement, one will not get GME signature with detectable contrast anymore. Although the photon losses diminish the GME signature, one can still obtain a good contrast of GME for squeezed vacuum input under photon losses up to 25%. Since the GME evaluation method does not require any calculation of permanent, one can efficiently determine the bound for GME detection for a particular generation scheme in practice. With the same GME detection setup, one can evaluate the GME for different postselection on local photon numbers.

Our results provide a possibility to generate GME in boson sampling systems with single-photon sources or CV sources. As GME is a result of photon indistinguishability in linear optics networks, the GME can be reversely exploited to characterize the input sources.

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V. APPENDIX

A. Proof of Theorem II.1

According to [35], the GM verifier $\hat{V}_{k,\kappa}$ is diagonal with respect to $\hat{X}$-irreducible subspaces,

$$\hat{V}_{k,\kappa} = \sum_{X_1,...,X_P} \pi_{X_1,...,X_P} \hat{V}_{k,\kappa} \pi_{X_1,...,X_P} \tag{63}$$

where $\pi_{X_1,...,X_P}$ is a projector onto the local $\hat{X}$-irreducible subspaces $\otimes_i \text{span}(X_i)$,

$$\pi_{X_1,...,X_P} := \sum_{|n_1,...,n_P\rangle} |n_1,...,n_P\rangle \langle n_1,...,n_P| \tag{64}$$

The expectation value $\langle \hat{V}_{k,\kappa} \rangle$ of a multimode state $\hat{\rho}$ is therefore equal to the convex mixture of the expectation value $\langle \hat{V}_{k,\kappa} \rangle$ of its projected component $\hat{R}_{X_1,...,X_P}(\hat{\rho})$:

$$\text{tr} \left( \hat{V}_{k,\kappa} \hat{\rho} \right) = \sum_{X_1,...,X_P} p_{X_1,...,X_P}(\hat{\rho}) \text{tr} \left( \hat{V}_{k,\kappa} \hat{R}_{X_1,...,X_P}(\hat{\rho}) \right), \tag{65}$$

where $\hat{R}_{X_1,...,X_P}(\hat{\rho})$ is the state projected from $\hat{\rho}$ onto the subspace $\otimes_i \text{span}(X_i)$, and $p_{X_1,...,X_P}$ is probability of the
The upper bound $\beta$ on $\hat{V}_{k,\kappa}$ for bi-producible states is therefore the convex extension of the upper bound $\beta_{X_1,\ldots,X_P}$ on $\hat{V}_{k,\kappa}$ bi-producible states within each $\hat{X}$-irreducible subspace $\bigotimes_i \text{span}(X_i)$,

$$\beta(\rho) := \max_{|\psi\rangle\text{is bi-separable}} \langle \psi | \hat{V}_{k,\kappa} | \psi \rangle = \sum_{X_1,\ldots,X_P} p_{X_1,\ldots,X_P}(\rho) \beta_{X_1,\ldots,X_P}. \quad (68)$$

The upper bound on $\langle \hat{V}_{0,\kappa} \rangle$ for a bi-separable pure state in an $\hat{X}$-irreducible subspace $\bigotimes_i \text{span}(X_i)$ is determined as follows. Let $A \cup B$ be a bipartition of the $P$ local parties with $A \subset \{1,\ldots,P\}$ and $B = \{1,\ldots,M\} \setminus A$. For a bi-separable pure state $|\psi_A,\psi_B\rangle \in \bigotimes_i \text{span}(X_i)$, the expectation value $\langle \hat{V}_{k,\kappa} \rangle$ is given as

$$\langle \hat{V}_{k,\kappa} \rangle = \frac{1}{1 + ||L||} \sum_{k',\kappa'} (\delta_{k',\kappa'}^{\kappa} + \sum_{l} \delta_{k'+\kappa'}^{k+\kappa} |c_{k',\kappa'}|)^2, \quad (69)$$

where the probability amplitude $|c_{k',\kappa'}|$ is equal to

$$|c_{k',\kappa'}|^2 = \sum_{\mu_j(X) = \kappa', X' \subseteq \bigotimes_i X_i} |\langle \mathbb{E}_{k'}(X')|\psi_A,\psi_B \rangle|^2. \quad (70)$$

It holds that

$$\delta_{k',\kappa'}^{\kappa} + \sum_{l} \delta_{k'+\kappa'}^{k+\kappa} \left\{ \begin{array}{ll} 1 & \text{for } (k',\kappa') \neq (k,\kappa); \\
1 + ||L|| & \text{for } (k',\kappa') = (k,\kappa). \end{array} \right. \quad (71)$$

which leads to the following inequality,

$$\langle \hat{V}_{k,\kappa} \rangle \leq \frac{1 + ||L|| |c_{k,\kappa}|^2}{1 + ||L||}. \quad (72)$$

FIG. 3. The expectation value $\langle \hat{V}_{0,\kappa} \rangle$ with input squeezing factor $r = 0.576(5dB)$. (a) The expectation value $\langle \hat{V}_{0,\kappa} \rangle$ of $|\Phi_{\nu}(2,1,1)(r,x)\rangle$ with $\kappa = 0$ and $\nu_{\text{tot}} \in \{00000, 10000, 20000, 11000, 10100\}$. (b) The expectation value $\langle \hat{V}_{0,\kappa} \rangle$ of $|\Phi_{\nu}(2,1,1)(r,x)\rangle$ with $\kappa > 0$ and $\nu_{\text{tot}} \in \{00000, 10000, 20000, 11000, 10100\}$. (c) The expectation value $\langle \hat{V}_{0,\kappa} \rangle$ of $\hat{\rho}_{2,1,1}(r,x;\epsilon)$ for different loss rates.
FIG. 4. Measurement statistics of the state \(|\Phi_{2,1}(r, x)\rangle\) with a squeezing factor of \(r = 0.058\) (0.5 dB) and a displacement of \(x = 0\). (a) Measurement in the computational basis. (b) The measurement in the $\hat{\Lambda}_0 \otimes \hat{\Lambda}_0 \otimes \hat{\Lambda}_0$ eigenbasis. (c) The measurement in the $\hat{\Lambda}_{jA} \otimes \hat{\Lambda}_{jB} \otimes \hat{\Lambda}_{jC}$ eigenbasis, where the HW indices are chosen as \((j_A, j_B, j_C) = (1, 4, 4)\).
According to Eq. (13), an HW-symmetric state $|E_k(X')\rangle$ can be written as Schmidt decomposition in the computational basis $|n_j\rangle \in X_j$

$$|E_k(X)\rangle = \frac{1}{\sqrt{|X_j|}} \sum_{|n_j\rangle \in X_j} |\varphi_1(n_j), ..., n_j, ..., \varphi_P(n_j)\rangle .$$  \hspace{1cm} (73)

In this composition, the set of states $\{|\varphi_i(n_j)\rangle : n_j \in X_j\}$ are orthonormal. It has been shown in [12] that

$$|\langle E_k(X')|psi_A, psi_B\rangle|^2 \leq \frac{1}{|X_j|}$$  \hspace{1cm} (74)

for all bi-separable $|psi_A, psi_B\rangle$. The upper bound for bi-producible state in $\otimes_i \text{span}(X_i)$ is therefore

$$\beta_{X_1,...,X_F} = \frac{1 + |L|/ \min_i |X_i|}{1 + |L|} .$$ \hspace{1cm} (75)

As a result of the convex roof extension in Eq. (68), we obtain the upper bound

$$\beta(\rho) = \sum_{n_1, ..., n_P} \frac{\langle n_1, ..., n_P | \hat{\rho} | n_1, ..., n_P \rangle}{\min_i |X_i|} ,$$ \hspace{1cm} (76)

which completes the proof.

### B. A lossy model in linear optics networks

As it is shown in Fig. 5 (a), the photon losses in PNR detectors of an $M'$-mode linear optics network can be modeled as $M'$ links connected to $M'$ external modes in the environment. Each link represents a beam splitter with a reflection coefficient $\sqrt{\epsilon_m}$. The circuit in Fig. 5 (a) is equivalent to the circuit in Fig. 5 (b),

$$\prod_m \hat{L}_m(\epsilon_m) \left( \hat{U}_\text{sys} \otimes 1_{\text{env}} \right) = \left( \hat{U}_\text{sys} \otimes \hat{U}_\text{env} \right) \prod_m \hat{L}_m(\epsilon_m) .$$ \hspace{1cm} (77)

Here $\hat{U}_\text{sys}$ and $\hat{U}_\text{env}$ are the unitary in the $M'$-mode system and environment, respectively. The operator $\hat{L}_m$ is the beam-splitter transformation which is responsible for the photon losses in the $m$-th mode,

$$\hat{L}_m \hat{a}_{\text{sys},m}^\dagger \hat{L}_m^\dagger = \sqrt{1 - \epsilon_m} \hat{a}_{\text{sys},m}^\dagger + \sqrt{\epsilon_m} \hat{a}_{\text{env},m}^\dagger ,$$ \hspace{1cm} (78)

where $\hat{a}_{\text{sys},m}$ and $\hat{a}_{\text{env},m}^\dagger$ are the creation operator in the $m$-th mode of the system and environment, respectively.

The state at the output of Fig. 5 (a) is

$$|\Phi_{SE}(\epsilon_1, ..., \epsilon_{M'})\rangle := \prod_m \hat{L}_m(\epsilon_m) \left( \hat{U}_\text{sys} \otimes 1_{\text{env}} \right) |\Phi, \text{vac}\rangle .$$ \hspace{1cm} (79)

According to Eq. (77), the photon losses can be shifted to input side (see Fig. 5 (b))

$$|\Phi_{SE}\rangle = \hat{U}_\text{sys} \otimes \hat{U}_\text{env} |\Phi_{SE}\rangle ,$$ \hspace{1cm} (80)

where

$$\bar{\Phi}_{SE}(\epsilon_1, ..., \epsilon_{M'}) := \prod_m \hat{L}_m(\epsilon_m) |\Phi, \text{vac}\rangle .$$ \hspace{1cm} (81)

As a result, the output state in the LON system in Fig. 5 (a) is equivalent to the $\hat{U}$ transformation of input state with photon losses

$$\text{tr}_\text{env} \left( |\Phi_{SE}\rangle \langle \Phi_{SE}| \right) = \hat{U} \text{tr}_\text{env} \left( |\Phi_{SE}\rangle \langle \Phi_{SE}| \right) \hat{U} .$$ \hspace{1cm} (82)

The right-hand side of this equation is equivalent to Fig. 5 (c), where the order of photon losses and LON transformation in Fig. 5 (a) is exchanged. The effect of photon losses in a LON system can be therefore simply shifted to the input state $|\Phi_{SE}\rangle$. It leads to a mixed state input

$$\tilde{\rho}_\Phi(\epsilon_1, ..., \epsilon_{M'}) := \text{tr}_\text{env} \left( \bar{\Phi}_{SE} \langle \Phi_{SE}| \right)$$

$$= \sum_\nu p_\nu |\Phi_\nu\rangle \langle \Phi_\nu| ,$$ \hspace{1cm} (83)

where $\nu$ is the Fock vector of lost photons, $p_\nu$ is the probability of losing $\nu$ photons,

$$p_\nu = \text{tr} \left( \nu |\Phi_{SE}\rangle \langle \Phi_{SE}| \nu \right) ,$$ \hspace{1cm} (84)

and

$$|\Phi_\nu\rangle = \frac{1}{p_\nu} \langle \nu |\Phi_{SE}\rangle \langle \Phi_{SE}| \nu \rangle .$$ \hspace{1cm} (85)

For small photon loss rate, the lossy input state $\tilde{\rho}_\Phi(\epsilon_1, ..., \epsilon_{M'})$ can be approximated through a cut-off up to a lost photon number $|\nu| \leq V_{\text{cutoff}}$.

$$\tilde{\rho}_\Phi \approx \sum_{\nu:|\nu|\leq V_{\text{cutoff}}} p_\nu |\Phi_\nu\rangle \langle \Phi_\nu| .$$ \hspace{1cm} (86)

One can then numerically estimate the effects of photon losses through this approximation.
(a) A lossy PNR measurement in LONs modeled by beam splitters interfering with the environment. (b) A circuit equivalent to Fig. (a). (c) A circuit equivalent to Fig. (b) after tracing out the environment.

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