ALTERNATING QUIVER HECKE ALGEBRAS

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ABSTRACT. For simply-laced quivers, we consider the fixed-point subalgebra of the quiver Hecke algebra under the homogeneous sign map. This leads to a new family of algebras we call alternating quiver Hecke algebras. We give a basis theorem and a presentation by generators and relations which is strikingly similar to the KLR presentation for quiver Hecke algebras.

INTRODUCTION

The study of quiver Hecke algebras is a recent development in representation theory, dually motivated from studying representations of cyclotomic Hecke algebras and Khovanov-Lauda-Rouquier (KLR) algebras. Whereas cyclotomic Hecke algebras, being related to symmetric groups and classical Iwahori-Hecke algebras, have been studied for some time, KLR algebras are relative newcomers, and the two ostensibly different families were linked by groundbreaking work of Brundan and Kleshchev [4]. Because of this deep and meaningful link, it is becoming customary to refer to KLR algebras as quiver Hecke algebras. In recent years much work has been done to study these algebras in their own right. Mathas [17] gives a particularly erudite survey of such developments.

Quiver Hecke algebras are associative graded algebras whose presentation by generators and relations, and whose grading, depends on the data of a quiver. Particularly interesting is the case when the quiver is simply laced, as quotients of these algebras are isomorphic to cyclotomic Hecke algebras, endowing these algebras with a \(\mathbb{Z}\)-grading. In this paper, we define a new family of algebras called alternating quiver Hecke algebras. For quiver Hecke algebras whose quivers are simply laced, it is possible to define a homogeneous involution on the quiver Hecke algebra; studying the fixed-point subalgebra under this involution gives rise to our family of algebras. We discuss these algebras using a version of Clifford theory for associative algebras, and construct a homogeneous basis and a presentation by homogeneous generators and relations which are reminiscent of Khovanov and Lauda [14] and Rouquier’s [22] theorems for quiver Hecke algebras. Cyclotomic quotients of these algebras are studied in [2] and [3].

This paper is organised as follows. We start by giving the definition of quiver Hecke algebras for arbitrary quivers. Then we discuss the Clifford theory for associative algebras that will give the technical mechanism for most of our proofs; this relies on the construction of the opposite quiver. In Chapter 3 we discuss alternating quiver Hecke algebras, giving a basis theorem for this new family of algebras.

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Finally, in Chapter 4 we prove the main result of this paper, giving a KLR-style presentation for alternating quiver Hecke algebras.

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1. Quiver Hecke algebras and opposite quivers

In this chapter we define quiver Hecke algebras, and give some basic properties. These algebras were introduced by Khovanov and Lauda [14] and Rouquier [22].

Let $\Gamma$ be a quiver with vertex set $I$. Following Kac [13], to the simply-laced quiver $\Gamma$ we attach the usual Lie theoretic data of the positive roots $\{\alpha_i | i \in I\}$, the fundamental weights $\{\Lambda_i | i \in I\}$, the non-degenerate pairing $(\Lambda_i, \alpha_j) = \delta_{ij}$, for $i, j \in I$, and the Cartan matrix $C = (c_{ij})_{i,j \in I}$ where

$$c_{ij} = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } i \leftarrow j \text{ or } i \rightarrow j, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Let $P^+ = \bigoplus_{i \in I} N\alpha_i$ and $Q^+_\Gamma = \bigoplus_{i \in I} N\alpha_i$. The height of $\beta = \sum a_i \alpha_i \in Q^+_\Gamma$ is the non-negative integer $ht \beta = \sum a_i \in \mathbb{N}$. Fix $n \geq 0$ and let $Q^+_n = \{ \beta \in Q^+_\Gamma | ht \beta = n \}$. For $\beta \in Q^+_n$ let

$$I^\beta = \{ \mathbf{i} = (i_1, \ldots, i_n) \in I^n \mid \beta = \alpha_{i_1} + \cdots + \alpha_{i_n} \} . \tag{1.2}$$

In this paper this data plays only a superficial role in describing the combinatorics of the grading on the algebras that we consider.

Let $Z$ be an arbitrary unital associative integral domain.

1.3. **Definition** (Khovanov and Lauda [14] and Rouquier [22]). Let $\Gamma$ be a simply-laced quiver with vertex set $I$ and suppose that $\beta \in Q^+_\Gamma$. The quiver Hecke algebra $\mathcal{R}_\beta(\Gamma) = \mathcal{R}_\beta^Z(\Gamma, Z)$ is the unital associative $Z$-algebra with generators

$$\{\psi_1, \ldots, \psi_{n-1} \} \cup \{y_1, \ldots, y_n \} \cup \{e_\Gamma(\mathbf{i}) \mid \mathbf{i} \in I^\beta\}$$

and relations

$$e_\Gamma(\mathbf{i}) e_\Gamma(\mathbf{j}) = \delta_{\mathbf{i}, \mathbf{j}} e_\Gamma(\mathbf{i}) e_\Gamma(\mathbf{j}), \quad \sum_{\mathbf{i} \in I^\beta} e_\Gamma(\mathbf{i}) = 1,$$

$$y_\Gamma e_\Gamma(\mathbf{i}) = e_\Gamma(\mathbf{i}) y_\Gamma, \quad \psi_\Gamma e_\Gamma(\mathbf{i}) = e(s_\Gamma) \psi_\Gamma, \quad y_\Gamma y_s = y_s y_\Gamma,$$

$$\psi_{r+s} e_\Gamma(\mathbf{i}) = (y_{r+s} \psi_{r+s} + \delta_{i_r, i_{r+s}}) e_\Gamma(\mathbf{i}), \quad y_{r+1} \psi_{r+1} e_\Gamma(\mathbf{i}) = (\psi_{r+1} y_{r+1} + \delta_{i_r, i_{r+1}}) e_\Gamma(\mathbf{i}),$$

$$\psi_s y_s = y_s \psi_s, \quad \psi_s \psi_s = \psi_s \psi_s, \quad \text{if } |r-s| > 1,$$

$$e_\Gamma(\mathbf{i}) e_\Gamma(\mathbf{j}) = \delta_{\mathbf{i}, \mathbf{j}} e_\Gamma(\mathbf{i}) e_\Gamma(\mathbf{j}),$$

$$y_\Gamma e_\Gamma(\mathbf{i}) = e_\Gamma(\mathbf{i}) y_\Gamma, \quad \psi_\Gamma e_\Gamma(\mathbf{i}) = e(s_\Gamma) \psi_\Gamma, \quad y_\Gamma y_s = y_s y_\Gamma,$$

$$\psi_{r+s} e_\Gamma(\mathbf{i}) = (y_{r+s} \psi_{r+s} + \delta_{i_r, i_{r+s}}) e_\Gamma(\mathbf{i}), \quad y_{r+1} \psi_{r+1} e_\Gamma(\mathbf{i}) = (\psi_{r+1} y_{r+1} + \delta_{i_r, i_{r+1}}) e_\Gamma(\mathbf{i}),$$

$$\psi_s y_s = y_s \psi_s, \quad \psi_s \psi_s = \psi_s \psi_s, \quad \text{if } |r-s| > 1,$$
for $i, j \in I^\beta$ and all admissible $r$ and $s$. If $n \geq 0$ then the quiver Hecke algebra is the algebra $R_n = R_n(\Gamma, Z) = \bigoplus_{\beta \in Q_+^\Gamma} R_\beta(\Gamma, Z)$.

One can check easily that the relations in Definition 1.3 are homogeneous with respect to the following $\mathbb{Z}$-valued degree function:

$$\deg e_\Gamma(i) = 0, \quad \text{for all } i \in I^n,$$

$$\deg y_r = 2, \quad \text{for } 1 \leq r \leq n,$$

$$\deg \psi_i e_\Gamma(i) = -c_{i, i_r+1}, \quad \text{for } 1 \leq r < n \text{ and } i \in I^n.$$  

Therefore, $R_n(\Gamma)$ is a $\mathbb{Z}$-graded algebra.

Using (1.2), if we define

$$e_\Gamma(\alpha) = \sum_{i \in I^\beta} e_\Gamma(i)$$

for $\beta \in Q_+^\Gamma$ then the algebras $R_\beta = e_\Gamma(\beta) R_n e_\Gamma(\beta)$ are actually the blocks of $R_n$ [14, Corollary 2.11].

For each permutation $\omega \in S_n$, fix a reduced expression $\omega = s_{i_1}s_{i_2} \cdots s_{i_n}$ and define

$$\psi_\omega = \psi_{i_1}\psi_{i_2} \cdots \psi_{i_n}.$$  

Importantly, because the braid relations for the elements $\{\psi_1, \psi_2, \ldots, \psi_{n-1}\}$ are more complicated than the symmetric group braid relations, the elements $\psi_\omega$ do depend on the choice of reduced expression. For $a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$, write $y^a \in R_n$ for the monomial $y_1^{a_1}y_2^{a_2} \cdots y_n^{a_n}$. The following theorem appears as [14, Theorem 2.5] and [22, Theorem 3.7].

1.8. **Theorem** (Basis theorem for quiver Hecke algebras). Let $\beta \in Q^+$. Then $R_\beta(\Gamma)$ is a free $\mathbb{Z}$-algebra with homogeneous basis

$$\{\psi_\omega y^a e_\Gamma(i) \mid \omega \in S_n, a \in \mathbb{N}^n, i \in I^\beta\}.$$  

In this paper we will be interested in the following construction on quivers.

1.9. **Definition.** For a quiver $\Gamma$ with vertex set $I$, let $\Gamma'$ be the opposite quiver whose vertex set is $I$ and which has edges $j \rightarrow i$ whenever there is an edge $i \rightarrow j$ in $\Gamma$, for $i, j \in I$.

1.10. **Remark.** Note that a quiver may be isomorphic to its own opposite under a bijection $\tau : I \rightarrow I$. In this case there will be edges $\tau(i) \rightarrow \tau(j)$ in $\Gamma'$ for every edge $i \rightarrow j$ in $\Gamma$. This is why we label idempotents $e_\Gamma(i)$ with the quiver $\Gamma$.

1.11. **Example** Fix an integer $e \in \{0, 3, 4, 5, \ldots\}$ and take $\Gamma$ to be the quiver $\Gamma_e$ with vertex set $I = \mathbb{Z}/e\mathbb{Z}$ and edges $i \rightarrow i + 1$, for $i \in I$. That is, $\Gamma_e$ is the infinite
quiver of type $A_\infty$ if $e = 0$ and the finite quiver of Dynkin type $A_{e-1}$ if $e \geq 3$. Notice that $\Gamma_e'$ is isomorphic to $\Gamma_e$ under the bijection
\[ \tau_e : I \rightarrow I \]
\[ i \mapsto -i \mod e. \]
The results in this paper in the particular case when $\Gamma = \Gamma_e$ are discussed at length in [1, Chapter 5]. This is how the result for general simply-laced $\Gamma$ may be motivated. Throughout this paper we give examples of how our general results work for this special case.

2. Clifford theory for associative algebras

In order to deduce results about our fixed-point subalgebras from the corresponding results for the full algebra, we will use the language of Clifford theory. Clifford theory was initially developed to study the representations of normal subgroups of finite groups [5]. Here we adapt it to cover associative algebras with a $C_2$-graded Clifford system; details in the finite group case can be found in [7], and a slightly different and more general treatment for associative algebras is given in [21]. Below we write the two elements of the cyclic group $C_2$ as $\{+,-\}$, where signs multiply according to the usual rules. Again, let $\mathcal{Z}$ be an arbitrary associative unital integral domain.

2.1. Definition. Let $A$ be a $\mathcal{Z}$-algebra. A $C_2$-graded Clifford system for $A$ is a family $\{A_s \mid s \in C_2\}$ of two $\mathcal{Z}$-submodules of $A$ such that

(i) $A_s A_t = A_{st}$, for $s, t \in C_2$;
(ii) there exists a distinguished central element $\varepsilon \in A$ such that $\varepsilon^2 = 1$ and $A_+ = \varepsilon A_-;
(iii) A = A_+ \oplus A_-; \text{ and}$
(iv) $1 \in A_+$.

$A_+$ is called the even part of the algebra; $A_-$ is called the odd part.

Clifford theory allows us to give a neat description of the representation theory of the subalgebra $A_+$ given knowledge of the representation theory of $A$. Specifically, for an $A_+$-module $M$, we may twist the $A_+$-action by $\varepsilon$ to define the module $M^\varepsilon$, which is $M$ as an $O$-module and where $a \in M$ acts as
\[ a \cdot m = \varepsilon a m. \]
Using Definition 2.1(ii) to realise $C_2$ as $\{1, \varepsilon\}$, the inertia group of $M$ is
\[ I(M) = \{x \in C_2 \mid M^x \cong M\} \leq C_2. \]
The size of the inertia group (either 1 or 2) determines the behaviour of an irreducible representation $N$ restricted from $A$ to $A_+$, which we denote by $\text{Res}^A_{A_+} N$, in the following sense. We refer the reader to [6, pp344–345] for a proof of the following result, noting that inverting 2 is necessary to prove the direct sum decomposition in (ii).

2.2. Proposition (Clifford’s theorem for $C_2$-graded associative algebras). Let 2 be invertible in $\mathcal{Z}$ and let $A$ be an associative $\mathcal{Z}$-algebra with a $C_2$-graded Clifford system. Let $N$ be an irreducible $A$-module. Then
(i) If $\mathcal{I}(\text{Res}_{A_+}^A N) = C_2$ then $\text{Res}_{A_+}^A N \cong \left(\text{Res}_{A_+}^A N\right)^\tau$ is an irreducible $A_+$-module.

(ii) If $\mathcal{I}(\text{Res}_{A_+}^A N) = 1$ then $\text{Res}_{A_+}^A N = M_+ \oplus M_-$ is the direct sum of two irreducible $A_+$-modules, related under the conjugation map (i.e. $M_+ = M_-^\tau$).

Moreover, all irreducible $A_+$-modules arise in one of these two ways.

3. Alternating quiver Hecke algebras

Given a result concerning symmetric groups, it is natural to ask what this result implies for its index-2 subgroup, the alternating group. It is this construction and line of enquiry which motivates our definition of alternating quiver Hecke algebras, which are related to quiver Hecke algebras in much the same way. Indeed, as the alternating group algebra is nothing more than the subgroup of the symmetric group algebra of points fixed by the sign map [12, §2.1], our alternating quiver Hecke algebras will be subalgebras of fixed points under an analogous map, this time a homogeneous involution. Recall the definition of the quiver Hecke algebra $R_n(\Gamma)$ from §1. We now extend the original definition of Kleshchev, Mathas and Ram [15] of the graded sign map.

3.1. Definition. The graded sign map $\text{sgn} : R_n(\Gamma) \to R_n(\Gamma')$ is the map defined on generators as

$$e_\Gamma(i) \mapsto e_{\Gamma'}(i), \quad y_r \mapsto -y_r, \quad \psi_r \mapsto -\psi_r.$$

3.2. Proposition. [15] The graded sign map is a well-defined homogeneous algebra homomorphism $R_n(\Gamma) \to R_n(\Gamma')$.

Proof. That the map is homogeneous is clear from its definition; checking it is a well-defined algebra homomorphism amounts to checking it respects the list of relations in Definition 1.3. Once the notation is correctly interpreted, remembering that in the assignment $e_\Gamma(i) \xrightarrow{\text{sgn}} e_{n'}(i)$, the $i$ on the right is a sequences of vertices in the opposite quiver, whose edges are reversed, this is a straightforward exercise which we leave to the reader. \qed

3.3. Definition. Let $\Gamma$ be a simply-laced quiver and $\Gamma'$ its opposite. We write $\tau : I \to I$ for the map for which $i \to j$ in $\Gamma$ if and only if $\tau(j) \to \tau(i)$ in $\Gamma'$. $\tau$ is called the reversal map.

We can extend the reversal map $\tau$ to $Q_\Gamma^+$ as follows. Let $\alpha = \alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_n} \in Q_\Gamma^+$; then define $\tau(\alpha)$ to be the root $\alpha_{\tau(i_1)} + \alpha_{\tau(i_2)} + \ldots + \alpha_{\tau(i_n)}$. The following lemma gives the consequences of our notation.

3.4. Lemma. There is an isomorphism of algebras $R_\alpha(\Gamma) \cong R_{\tau(\alpha)}(\Gamma')$. Under this isomorphism $e_\Gamma(i)$ is identified with $e_{\Gamma'}(\tau(i))$.

Notice that, on the level of blocks, the $\text{sgn}$ map takes $R_\alpha(\Gamma)$ to $R_\alpha(\Gamma')$ and so is an involution on the direct sum algebra $R_\alpha(\Gamma) \oplus R_\alpha(\Gamma')$. Where it is convenient, we often use Lemma 3.4 to abuse notation without comment in the sequel. We are now ready to define alternating quiver Hecke algebras, which we will study for the remainder of this paper.

3.5. Definition. We write $[\alpha]$ for the equivalence class of $\alpha \in Q_\Gamma^+$ under the equivalence relation $\sim$ generated by $\alpha \sim \beta$ if $\beta = \tau(\alpha)$. We write $(Q_\Gamma^+)_\tau$ for the set of all equivalence classes.
3.6. Definition. For a simply-laced quiver $\Gamma$ and $\alpha \in Q^+_\Gamma$, the alternating quiver Hecke algebra is the fixed-point subalgebra

$$R_\alpha(\Gamma/\Gamma')^{sgn} = \left[ R_\alpha(\Gamma) \oplus R_\alpha(\Gamma') \right]^{sgn}$$

of the quiver Hecke algebra under the graded sign map. The alternating quiver Hecke algebra $R^{sgn}_n$ is the algebra

$$R^{sgn}_n = \bigoplus_{[\alpha] \in (Q^+_\Gamma)^\tau} R_\alpha(\Gamma/\Gamma')^{sgn}$$

(3.7)

Note that this decomposition is independent of our choice of equivalence class representative $\alpha \in [\alpha]$ by Lemma 3.4; by the same lemma we can identify $R^{sgn}_n$ with a subalgebra of $R_n$.

3.8. Remark. It is important to note the summands appearing on the right-hand side of (3.7) in general are not the blocks of the alternating quiver Hecke algebras, as they are not guaranteed to be indecomposable (although most of them are). It would be an interesting problem to classify the blocks of alternating quiver Hecke algebras.

We also want to work with equivalence classes of $I^n$ under the involution on residue sequences $i$ induced by the reversal map $\tau$. More precisely, for $i,j \in I^n$ let $\sim$ be the equivalence relation on $I^n$ generated by $i \sim j$ if $i = \tau(j)$, where $\tau(i) = (\tau(i_1), \tau(i_2), \ldots, \tau(i_n))$ for $i = (i_1, \ldots, i_n)$. From this we obtain a partition of the set $I^n$ into equivalence classes of size 1 or 2, depending on whether or not $i = \tau(i)$. We denote the equivalence class containing a sequence $i$ by $[i]$. We denote the set of equivalence classes by $I^n_{\tau}$ and in each equivalence class we choose a representative $i_\tau \in [i]$. Similarly we write $I^n_{\tau}$ for the set of $\tau$-equivalence classes in $I^n$, noting that $I^n_{\tau} = I^n_{\tau(\alpha)}$.

Using these equivalence classes and distinguished elements, we now define an element $\varepsilon \in R_n$ which will be very important in studying alternating quiver Hecke algebras and their cyclotomic quotients. Define $\varepsilon$ to be the element

$$\varepsilon = \sum_{[i] \in I^n_{\tau}} \left( e_\Gamma(i_\tau) - e_\Gamma(\tau(i_\tau)) \right) \in R_n.$$  

(3.9)

Notice that by Lemma 3.4 we can also identify $\varepsilon$ with an element of the direct sum algebra $R_n(\Gamma) \oplus R_n(\Gamma')$. It also has the important properties

$$\text{sgn}(\varepsilon) = -\varepsilon \quad \text{and} \quad \varepsilon^2 = 1.$$  

Recall the idempotents $e_\Gamma(\alpha)$ from (1.7), which define the blocks of quiver Hecke algebras. We now define new idempotents

$$e_{[\alpha]} = e_\Gamma(\alpha) + e_{\Gamma'}(\alpha),$$

(3.10)

noting that since $[e_\Gamma(\alpha)]^{sgn} = e_{\Gamma'}(\alpha)$, $e_{[\alpha]}$ is $\text{sgn}$-invariant; in particular by Lemma 3.4, $e_{[\alpha]}$ is independent of the choice of equivalence class representative of $[\alpha]$. Finally, for an equivalence class $[i] \in I^n_{\tau}$ we write

$$e[i] = e_\Gamma(i) + e_{\Gamma'}(i)$$

(3.11)

which again does not depend on the choice of equivalence class representative.
We now give a basis theorem for alternating quiver Hecke algebras analogous to Theorem 1.8 for quiver Hecke algebras. Recall that \((Q^+_\Gamma)_\tau\) is the set of all equivalence classes \([\alpha]\). For \(a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n\), let \(|a| = \sum_{i=1}^n a_i\).

3.12. Theorem (Basis theorem for alternating quiver Hecke algebras). Let \(\Gamma\) be a simply-laced quiver and let \(2\) be invertible in \(\mathbb{Z}\). For \([\alpha] \in (Q^+_\Gamma)_\tau\), the alternating quiver Hecke algebra \(\mathcal{R}_\alpha(\Gamma/\Gamma')_{\text{sgn}}\) has homogeneous basis

\[
\left\{ \psi_\omega y^a e[i] \mid \omega \in \mathfrak{S}_\alpha, \quad a \in \mathbb{N}^n, \quad |i| \in I^\alpha, \quad b \in \{0, 1\}, \quad \ell(\omega) + |a| + b \equiv 0 \mod 2 \right\}
\]

Proof. Let \([\alpha] \in (Q^+_\Gamma)_\tau\). To observe that the specified set spans \(\mathcal{R}_\alpha(\Gamma) \oplus \mathcal{R}_\alpha(\Gamma') \cong \mathcal{R}_\alpha(\Gamma) \oplus \mathcal{R}_{\tau(\alpha)}(\Gamma)\) as a finite sum

\[
x = \sum_{\omega, a, i} \lambda_{\omega, a, i} \psi_\omega y^a e_\Gamma(i),
\]

for \(\omega \in \mathfrak{S}_\alpha, a \in \mathbb{N}^n, i \in I^\alpha \cup I^{\tau(\alpha)}\) and \(\lambda_{\omega, a, i} \in \mathbb{Z}\) using Theorem 1.8 (note we are implicitly using Lemma 3.4 here to write this as an element of \(\mathcal{R}_\alpha(\Gamma)\)). In order that \(x\) be an element of \(\mathcal{R}_\alpha^\text{sgn}\) we require \(x_{\text{sgn}} = x\), i.e. that

\[
\sum_{\omega, a, i} (-1)^{\ell(\omega)+|a|} \lambda_{\omega, a, i} \psi_\omega y^a e_\Gamma(i) = \sum_{\omega, a, i} \lambda_{\omega, a, i} \psi_\omega y^a e_\Gamma(i)
\]

which implies that the given elements span by applying Theorem 1.8 and equating coefficients. For linear independence, take a linear combination

\[
\sum_{|i| \in I^\alpha} \left( \sum_{\omega \in \mathfrak{S}_\alpha, a \in \mathbb{N}^n} \lambda_{\omega, a, i} \psi_\omega y^a e[i] + \sum_{\omega \in \mathfrak{S}_\alpha, a \in \mathbb{N}^n} \lambda_{\omega, a, i} \psi_\omega y^a e[i] \right) = 0
\]

for coefficients \(\lambda_{\omega, a, i} \in \mathbb{Z}\) and project the above sum separately onto the idempotents \(e_\Gamma(i)\) and \(e_{\Gamma'}(i)\) for each \(i \in I^\alpha\); we thus obtain sums of basis vectors of the form in Theorem 1.8 which are linearly independent. \(\square\)

We can obtain a \(C_2\)-graded Clifford decomposition for direct sums of blocks of quiver Hecke algebras using the element \(\varepsilon\) defined in (3.9).

3.13. Proposition. Let \(\Gamma\) be a simply-laced quiver and let \(2\) be invertible in \(\mathbb{Z}\). Then we have the \(C_2\)-graded Clifford decomposition

\[
\bigoplus_{\beta \in [\alpha]} \mathcal{R}_\beta \cong \mathcal{R}_\alpha(\Gamma/\Gamma')_{\text{sgn}} \oplus \varepsilon \mathcal{R}_\alpha(\Gamma/\Gamma')_{\text{sgn}}.
\]

Proof. To demonstrate the Clifford decomposition, we check the requirements in Definition 2.1. Condition (i) follows easily; since \(\varepsilon_{\text{sgn}} = -\varepsilon\), if \(x \in \mathcal{R}_\alpha^\text{sgn}\) and \(y = \varepsilon z \in \varepsilon \mathcal{R}_\alpha^\text{sgn}\) then \((xy)_{\text{sgn}} = -xy\) giving the required multiplicative property. Conditions (ii) and (iv) follow by definition, so it remains to demonstrate the direct sum decomposition. Since any \(x \in \mathcal{R}_\alpha \oplus \mathcal{R}_\alpha'\) may be written as

\[
x = \sum_{i \in I^\alpha} x e_\Gamma(i) = \sum_{i \in I^\alpha} x e_\Gamma(i) + e_\Gamma(\tau(i))|_{i = 2}\text{is invertible in } \mathbb{Z},\text{ we can write}
\]

\[
x = \frac{1}{2} \sum_{i \in I^\alpha} (x + x_{\text{sgn}}) [e_\Gamma(i) + e_\Gamma(\tau(i))] + \frac{1}{2} \varepsilon \sum_{i \in I^\alpha} (x - x_{\text{sgn}}) [e_\Gamma(i) + e_\Gamma(\tau(i))]
\]

which gives the required decomposition. \(\square\)
3.14. Remark. The reader may have noticed the dependence of our element \( \varepsilon \) on choices of equivalence class representatives for each class \([i] \in I^*\). Indeed, there are many choices of Clifford element which give rise to different Clifford decompositions, much in the way that different choices of coset representatives give rise to different Clifford decompositions of representations of finite groups with respect to normal subgroups [6].

3.15. Example. We finish this chapter with an example of our results when \( \Gamma = \Gamma_e \). First suppose \( n = 1 \) and that \( 2 < e < \infty \). Set \( I = \mathbb{Z}/e\mathbb{Z} \) and notice that in this case our quiver Hecke algebra \( \mathcal{R}_1(\Gamma_e, \mathbb{Z}) \) has a particularly simple presentation:

\[
\mathcal{R}_1(\Gamma_e) = \left\langle y, e(i) \mid i \in I, \sum_{i \in I} e(i) = 1, e(i)e(j) = \delta_{ij} e(i) \right\rangle \\
\cong \bigoplus_{i \in I} \mathbb{Z}[y]e(i),
\]

where \( \deg y = 2 \) and \( \deg e(i) = 0 \). This has the obvious homogeneous basis

\( \{y^k e(i) \mid k \geq 0, i \in I\} \).

Suppose now that \( e = 3 \) and that \( \text{char}(\mathbb{Z}) \neq 2 \); then it is not too hard to see that the following is a homogeneous basis for the fixed-point subalgebra \( \mathcal{R}_1(\Gamma_e)_{\text{sgn}} \), where we have grouped basis vectors by degree:

\[
\begin{align*}
\text{deg 0} & : 2e(0), e(1) + e(2) \\
\text{deg 2} & : y \left(e(1) - e(2)\right) \\
\text{deg 4} & : 2y^2 e(0), y^2 \left(e(1) + e(2)\right) \\
\text{deg 6} & : y^3 \left(e(1) - e(2)\right) \\
\end{align*}
\]

where \( y = y_1 \). For general \( e > 2 \), we have the homogeneous basis

\[
\left\{y^k \left(e(i) + e(-i)\right) \mid k \text{ even}\right\} \cup \left\{y^k \left(e(i) - e(-i)\right) \mid k \text{ odd, } i \neq 0\right\}
\]

for \( \mathcal{R}_n(\Gamma_e)_{\text{sgn}} \). Note that

\[
\dim \left( [\mathcal{R}_n(\Gamma_e)_{\text{sgn}}]_{2k} \oplus [\mathcal{R}_n(\Gamma_e)_{\text{sgn}}]_{2(k+1)} \right)
\]

\[
= \frac{1}{2} \dim \left( [\mathcal{R}_n(\Gamma_e)]_{2k} \oplus [\mathcal{R}_n(\Gamma_e)]_{2(k+1)} \right)
\]

for all \( k > 0 \). \( \diamond \)

4. Generators and relations for alternating quiver Hecke algebras

Using the \( C_2 \)-graded Clifford decomposition from Theorem 3.12, we can give a presentation for the alternating quiver Hecke algebra \( \mathcal{R}_n(\Gamma/\Gamma')_{\text{sgn}} \) by homogeneous generators and relations, akin to the Khovanov-Lauda presentation for the quiver Hecke algebras (Definition 1.3). The following elements will play the role of the generators from Definition 1.3 for alternating quiver Hecke algebras.
4.1. **Definition** (Generators for alternating quiver Hecke algebras). Let $\Gamma$ be a simply-laced quiver and $R_n(\Gamma/\Gamma')^\text{sgn}$ be an alternating quiver Hecke algebra. For $1 \leq r < n$, define $\Psi_r = \psi_r \epsilon \in R_n(\Gamma/\Gamma')^\text{sgn}$, and for $1 \leq r \leq n$, define $\gamma_r = y_r \epsilon \in R_n(\Gamma/\Gamma')^\text{sgn}$. Finally, for $[i] \in I^r_n$ recall the definition of $e[i] \in R_n(\Gamma/\Gamma')^\text{sgn}$ from (3.11).

As a corollary to Theorem 3.12 we see that the elements defined above generate the alternating quiver Hecke algebras.

4.2. **Corollary.** Let $\Gamma$ be a simply-laced quiver, let $n > 1$ and $e > 2$ and suppose 2 is invertible in $\mathbb{Z}$. For $[\alpha] \in (Q^+_\Gamma)^r$, the alternating quiver Hecke algebra $R_n(\Gamma/\Gamma')^\text{sgn}$ is generated by the collection of elements

$$\{\Psi_1, \Psi_2, \ldots, \Psi_n\} \cup \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \cup \{e[i] \mid [i] \in I^r_n\}.$$  

**Proof.** Suppose $[\alpha] \in (Q^+_\Gamma)^r$. We proceed to write each basis vector from Theorem 3.12 in terms of the proposed generators; since the $y_r$ commute with the $e_r(i)$ for all admissible $r$ and $i$ by the relations in Definition 1.3, and since $e^2 = 1$,

$$y_1^{a_1} \cdots y_n^{a_n} = \begin{cases} \gamma_1^{a_1} \cdots \gamma_n^{a_n}, & \text{if } |\alpha| \equiv 0 \mod 2 \\
\epsilon \gamma_1^{a_1} \cdots \gamma_n^{a_n}, & \text{if } |\alpha| \equiv 1 \mod 2 \end{cases}$$

Moreover, since $\psi_\omega e_r(i) = e_r(\omega \cdot i) \psi_\omega$ by the relations in Definition 1.3,

$$\psi_\omega = \begin{cases} \Psi_\omega, & \text{if } \ell(\omega) \equiv 0 \mod 2 \\
\Psi_\omega \epsilon, & \text{if } \ell(\omega) \equiv 1 \mod 2 \end{cases}$$  

There are four cases to consider; we just give two as illustration of the method of proof. If $|\alpha| \equiv 0 \mod 2$ and $\ell(\omega) \equiv 0 \mod 2$, then the basis vector $\psi_\omega y_1^{a_1} \cdots y_n^{a_n} e[i]$ is equal to $\Psi_{i_1} \cdots \Psi_{i_\ell} \gamma_1^{a_1} \cdots \gamma_n^{a_n} e[i]$ where $\omega = s_{i_1} \cdots s_{i_\ell}$. If $|\alpha| \equiv 1 \mod 2$ and $\ell(\omega) \equiv 0 \mod 2$ then the basis vector

$$\psi_\omega y_1^{a_1} \cdots y_n^{a_n} [e_r(i) - e_r(\tau(i))] = \psi_\omega y_1^{a_1} \cdots y_n^{a_n} \epsilon e[i]$$

$$= \Psi_{i_1} \cdots \Psi_{i_\ell} \epsilon \gamma_1^{a_1} \cdots \gamma_n^{a_n} e[i]$$

$$= \Psi_{i_1} \cdots \Psi_{i_\ell} \gamma_1^{a_1} \cdots \gamma_n^{a_n} e[i]$$

since $\epsilon$ commutes with the $\gamma_r$ and squares to 1.  

Our goal now is to obtain a set of relations for the generators from Corollary 4.2. Let us start by defining a new abstract algebra with a slightly different presentation and the additional structure of a $\mathbb{Z} \times C_2$-grading; as in §5.1, let us realise $C_2$ as the sign group $\{+, -\}$ with usual multiplication of signs.

4.3. **Definition.** Let $n \geq 0$. The **signed quiver Hecke algebra** of type $\Gamma$ and corresponding to $[\alpha] \in (Q^+_\Gamma)^r$ is the unital associative $\mathbb{Z}$-algebra $R_{[\alpha]}(\Gamma)^r = R_{[\alpha]}(\Gamma, \mathbb{Z})$ with generators

$$\{\psi_1, \psi_2', \ldots, \psi_{n-1}'\} \cup \{y_1', y_2', \ldots, y_n'\} \cup \{\varepsilon_{a(i)} \mid i \in \bigcup_{\beta \in [\alpha]} I^\beta, a \in C_2\}$$
subject to the relations

\[ \varepsilon_a(i) \varepsilon_b(j) = \delta_{ij} \varepsilon_{ab}(i), \quad \sum_{i \in I^{[n]}} \varepsilon^+(i) = 1 \]

\[ \varepsilon_a(i) = a \varepsilon_a(\tau(i)) \]

\[ y'_r \varepsilon_a(i) = \varepsilon_a(i) y'_r \]

\[ \psi'_a \varepsilon_a(i) = \varepsilon_a(s_r \cdot i) \psi'_r \]

\[ y'_r y'_s \varepsilon_{-}(i) = y'_s y'_r \varepsilon_{-}(i) \]

\[ \psi'_r y'_{r+1} \varepsilon_a(i) = \begin{cases} 
(\psi'_r y'_r + 1) \varepsilon_a(i), & \text{if } i_r = i_{r+1} \\
\psi'_r y'_s \varepsilon_a(i), & \text{if } i_r \neq i_{r+1}
\end{cases} \]

\[ y'_{r+1} \psi'_r \varepsilon_a(i) = \begin{cases} 
0, & \text{if } i_r = i_{r+1} \\
(\psi'_r y'_r - 1) \varepsilon_a(i), & \text{if } i_r \rightarrow i_{r+1} \\
(\psi'_r y'_r - 1) \varepsilon_a(i), & \text{if } i_r \leftarrow i_{r+1}
\end{cases} \]

\[ (\psi'_r)^2 \varepsilon_{-}(i) = \begin{cases} 
0, & \text{if } i_r = i_{r+1} \\
(y'_r - y'_{r+1}) \varepsilon_{+}(i), & \text{if } i_r \rightarrow i_{r+1} \\
(\psi'_r y'_r - 1) \varepsilon_{+}(i), & \text{if } i_r \leftarrow i_{r+1} \\
\varepsilon_{-}(i), & \text{otherwise}
\end{cases} \]

\[ \psi'_r y'_{r+1} \psi'_r \varepsilon_{+}(i) = \begin{cases} 
\psi'_{r+1} y'_r \psi'_r y'_{r+1} \varepsilon_{+}(i) + \varepsilon_{-}(i), & \text{if } i_{r+2} = i_r \rightarrow i_{r+1} \\
\psi'_{r+1} y'_r \psi'_r y'_{r+1} \varepsilon_{+}(i) - \varepsilon_{-}(i), & \text{if } i_{r+2} = i_r \leftarrow i_{r+1} \\
\psi'_{r+1} y'_r \psi'_r y'_{r+1} \varepsilon_{+}(i), & \text{otherwise}
\end{cases} \]

for \( i, j \in I^{[n]}, a, b \in C_2 \) and all admissible \( r \) and \( s \).

4.4. Remark.

(i) As usual, the algebra \( \mathcal{R}'_{[n]} \) is a block of the full signed quiver Hecke algebra \( \mathcal{R}'_n \), which decomposes as

\[ \mathcal{R}_n(\Gamma)' = \mathcal{R}'_n(\Gamma, Z) = \bigoplus_{[\alpha] \in (Q^+_r)} \mathcal{R}'_{[\alpha]} \]

and \( \mathcal{R}'_n = \varepsilon^+_n \mathcal{R}'_n \) where \( \varepsilon^+_n = \sum_{i \in I^n} \varepsilon^+_i \).

(ii) The generators in Definition 4.3 are somewhat superfluous: for example we can use the relations to see that \( \varepsilon_a(i) = 0 \) whenever \( i = \tau(i) \). These extra generators do however mean we can more easily compare the algebra \( \mathcal{R}'_{[n]} \) with the quiver Hecke algebra, which we are about to do.

The proof of the following lemma requires nothing more than an inspection of the relations in Definition 4.3 which we leave to the reader (see §2.1 for general remarks on algebras graded by a finite group \( G \)).

4.5. Lemma. The relations in Definition 4.3 are homogeneous with respect to the degree function \( \deg_2 : \mathcal{R}'_n \rightarrow (\mathbb{Z} \times C_2) \) given by

\[ \deg_2 \varepsilon_a(i) = (0, a), \quad \deg_2 y'_r = (2, -), \quad \deg_2 \psi'_r \varepsilon_{+}(i) = (-c_{i, i_{r+1}}, -) \]

\[ \deg_2 \psi'_r \varepsilon_{-}(i) = (-c_{i, i_{r+1}}, +) \]

for all \( i \in I^n, a \in C_2 \) and \( 1 \leq r \leq n \). In particular, \( \mathcal{R}'_n \) is a \((\mathbb{Z} \times C_2)\)-graded \( Z \)-algebra.
4.6. Remark. By forgetting the $C_2$-grading (but not the $\mathbb{Z}$-grading) on $\mathcal{R}'_\alpha$, we obtain a $\mathbb{Z}$-graded algebra.

4.7. Proposition. Let $n > 0$ and suppose $e > 2$. If 2 is invertible in $\mathbb{Z}$, then for $[\alpha] \in (Q^+_r)_r$,

$$\mathcal{R}'_{[\alpha]}(\Gamma, \mathbb{Z}) \cong \bigoplus_{\beta \in [\alpha]} \mathcal{R}_\beta(\Gamma, \mathbb{Z})$$

as $\mathbb{Z}$-graded $\mathbb{Z}$-algebras.

Proof. Define a map $\vartheta : \mathcal{R}'_{[\alpha]}(\Gamma, \mathbb{Z}) \to \bigoplus_{\beta \in [\alpha]} \mathcal{R}_\beta(\Gamma, \mathbb{Z})$ on generators by

$$\vartheta(y_r') = y_r$$
$$\vartheta(\psi_s') = \psi_s$$
$$\vartheta(\varepsilon_a(i)) = \begin{cases} e(i) + e(\tau(i)), & \text{if } a = + \\ e(i) - e(\tau(i)), & \text{if } a = - \end{cases}$$

for $1 \leq r \leq n$ and $1 \leq s < n$ and $i \in I^\alpha$. We must check $\vartheta$ is an algebra homomorphism of degree zero; this amounts to the largely tedious and straightforward calculation of checking it preserves the relations. We check two relations so the reader can obtain a taste for how they are done. For example, let $i, j \in I^\alpha$. Then we have

$$\vartheta(\varepsilon_a(i)e_b(j)) = \vartheta(\delta_{ij}e_{ab}(i))$$
$$= \begin{cases} \delta_{ij}(e(i) + e(\tau(i))), & \text{if } a = b \\ \delta_{ij}(e(i) - e(\tau(i))), & \text{if } a \neq b \end{cases}$$
$$= \begin{cases} \delta_{ij}(e(i) + e(\tau(i))), & \text{if } a = b \\ \delta_{ij}(e(i) - e(\tau(i))), & \text{if } a \neq b \end{cases}$$
$$= \vartheta(\varepsilon_a(i))\vartheta(e_b(j)).$$

Now suppose that $i \in I^\alpha$ is such that $i_{r+2} = i_r \to i_{r+1}$. Then

$$\vartheta(\psi_r\psi_{r+1}\psi_r', \varepsilon_+) = \vartheta(\psi_{r+1}\psi_r\psi_r', \varepsilon_+)$$
$$= \psi_{r+1}\psi_r\psi_{r+1}(e(i) + e(\tau(i))) + e(i) - e(\tau(i))$$
$$= (\psi_{r+1}\psi_r\psi_r')e(i) + (\psi_{r+1}\psi_r\psi_{r+1} - 1)e(\tau(i))$$
$$= \vartheta(\psi_r')\vartheta(\psi_{r+1})\vartheta(\psi_r)\vartheta(e_+)$$

as required. We leave the remaining checks to the reader; this amounts to proving $\vartheta$ is surjective. Similarly, define a map $\zeta : \bigoplus_{\beta \in [\alpha]} \mathcal{R}_\beta \to \mathcal{R}'_{[\alpha]}$ on generators by

$$\zeta(y_r') = y_r$$
$$\zeta(\psi_s') = \psi_s$$
$$\zeta(e(i)) = \frac{1}{2}(\varepsilon_+(i) + \varepsilon_-(i))$$

for $1 \leq r \leq n$, $1 \leq s < n$ and $i \in \bigcup_{\beta \in [\alpha]} I^\beta$. We must check that $\zeta$ extends to an algebra homomorphism, again by checking it preserves all relations: note that since $\varepsilon_a(i)e_b(i) = \varepsilon_{a+b}(i)$ for all $i \in I^n$ and $a, b \in \mathbb{Z}_2$, we can multiply all the relations
These, together with the original list of relations, allow one to check that $\varsigma$ respects all of the relations. For example, if we compute

$$\varsigma(y, \psi e(i)) = \frac{1}{2} \left( y'_t \psi'_s e_+(i) + y'_t \psi'_s e_-(i) \right)$$

we can see that both terms on the right-hand side are indeed relations in $\mathcal{R}'_{[\alpha]}$ (with the first term being a relation from Definition 4.3 multiplied on the right by $\varepsilon_-(i)$). Since $\varsigma$ is also homogeneous of degree zero and since we clearly have $\vartheta \circ \varsigma = \text{id}$ and $\varsigma \circ \vartheta = \text{id}$, this establishes the required isomorphism of $\mathbb{Z}$-graded algebras.

Using the $C_2$-grading, we can write the decomposition of $\mathcal{R}'_{n}$ into odd and even parts afforded by its $C_2$ grading as $\mathcal{R}'_{n} = (\mathcal{R}'_{n})_+ \oplus (\mathcal{R}'_{n})_-$. This, combined with Theorem 3.12, gives the following corollary regarding the alternating quiver Hecke algebra.

4.8. Corollary. Let $n \geq 0$, let $\Gamma$ be a simply-laced quiver, and let 2 be invertible in $\mathcal{Z}$. Then $(\mathcal{R}_{n}(\Gamma'))_{+} \cong \mathcal{R}_{n}(\Gamma / \Gamma')^{sgn}$. 

4.9. Theorem (Generators and relations for alternating quiver Hecke algebras). Let $n > 1$, let $(\Gamma, \tau)$ be a reversible quiver and suppose 2 is invertible in $\mathcal{Z}$. Let $[\alpha] \in Q_{+}^{\uparrow}$. Then the alternating quiver Hecke algebra $\mathcal{R}_{n}(\Gamma / \Gamma')^{sgn}$ is generated by the elements

$$\{\Psi_{1}, \Psi_{2}, \ldots, \Psi_{n-1}\} \cup \{\mathcal{Y}_{1}, \mathcal{Y}_{2}, \ldots, \mathcal{Y}_{n}\} \cup \{e[i] \mid [i] \in I_{[\alpha]}^{\uparrow}\}$$

subject to the relations

$$e[i]e[j] = \delta_{i+j}e[i], \quad \sum_{[i] \in I_{[\alpha]}^{\uparrow}} e[i] = 1$$

$$\mathcal{Y}_{t}e[i] = e[i]\mathcal{Y}_{t}, \quad \Psi_{t}e[i] = e[s_{t} \cdot i]\Psi_{t}, \quad \mathcal{Y}_{t}\mathcal{Y}_{s} = \mathcal{Y}_{s}\mathcal{Y}_{t}$$

$$\Psi_{t}\Psi_{s}e[i] = \mathcal{Y}_{s}\Psi_{t}e[i], \quad \text{if } s \neq r, r + 1$$

$$\Psi_{t}\mathcal{Y}_{s}e[i] = e[i], \quad \text{if } |r - s| > 1$$

$$\Psi_{t}\mathcal{Y}_{r+1}e[i] = \begin{cases} (\mathcal{Y}_{r}\Psi_{t} + 1)e[i], & \text{if } i_{r} = i_{r+1} \\ \mathcal{Y}_{r}\Psi_{t}e[i], & \text{if } i_{r} \neq i_{r+1} \end{cases}$$

$$\mathcal{Y}_{r+1}\Psi_{t}e[i] = \begin{cases} (\Psi_{t}\mathcal{Y}_{r} + 1)e[i], & \text{if } i_{r} = i_{r+1} \\ \Psi_{t}\mathcal{Y}_{r}e[i], & \text{if } i_{r} \neq i_{r+1} \end{cases}$$
\[ \Psi_{r,2} \varepsilon[i] = \begin{cases} 0, & \text{if } i_r = i_{r+1} \\ (\Psi_r - Y_{r+1})\varepsilon[i], & \text{if } i_r \rightarrow i_{r+1} \\ (Y_{r+1} - \Psi_r)\varepsilon[i], & \text{if } i_r \leftarrow i_{r+1} \\ \varepsilon[i], & \text{otherwise} \end{cases} \]

\[ \Psi_r \Psi_{r+1} \varepsilon[i] = \begin{cases} (\Psi_{r+1} - Y_{r+1})\varepsilon[i], & \text{if } i_r = i_{r+2} \rightarrow i_{r+1} \\ (\Psi_{r+1} \Psi_{r+1} + 1)\varepsilon[i], & \text{if } i_r = i_{r+2} \leftarrow i_{r+1} \\ \Psi_{r+1} \Psi_{r+1} \varepsilon[i], \end{cases} \]

for all \([i] \in I_r^{[a]}\) and all admissible \(r\) and \(s\). Moreover, \(R_{\alpha}(1/1')^{\text{ep}}\) is \(\mathbb{Z}\)-graded with degree function

\[ \deg \Psi_r \varepsilon[i] = -a_{r,r+1}, \quad \deg Y_s = 2, \quad \deg \varepsilon[i] = 0. \]

for all \(1 \leq r < n, 1 \leq s \leq n\) and \([i] \in I_r^{[a]}\).

**Proof.** First we note that all the requirements on the residue sequences \(i\) in fact depend only on the equivalence class of the sequence because of the symmetry of the Cartan matrix in (1.1). By Corollary 4.8, it is enough to prove that the abstract algebra \(A_{[a]}\) defined in the statement of the theorem is isomorphic to \((R_{\alpha}').^+\).

Define a map \(\varphi : A_{[a]} \rightarrow (R_{\alpha}').^+\) by

\[ \varepsilon[i] \mapsto \varepsilon_+(i), \quad Y_r \mapsto y_r' \varepsilon_-(i), \quad \Psi_s \mapsto \psi_s' \varepsilon_-(i), \]

for all \([i] \in I_r^{[a]}\), \(1 \leq r < n\) and \(1 \leq s < n\). It is a straightforward check that all the relations of \(A_{[a]}\) are satisfied in \((R_{\alpha}').^+\), so \(\varphi\) determines a well-defined homogeneous algebra homomorphism of degree zero.

By definition, \((R_{\alpha}').^+\) is generated by even words in the generators of \(R_{\alpha}'.\) However the only even generators of \(R_{\alpha}'\) are the idempotents \(\varepsilon_+(i)\) for \(i \in I^a \cup I^{a'}\), so \((R_{\alpha}').^+\) is generated by these idempotents together with all words of even length in the odd generators of \(R_{\alpha}'\). It is now easy to see that \((R_{\alpha}').^+\) is generated by the images of the generators of \(A_{[a]}\) under \(\varphi\) and so \(\varphi\) is surjective.

The algebra \(R_{\alpha}'\) is defined by generators and relations, so \((R_{\alpha}').^+\) is the subalgebra of \(R_{\alpha}'\) generated by the even words in the generators of \(R_{\alpha}'\) modulo the even part of the relation ideal defining \(R_{\alpha}'\), which is the set of all linear combinations of arbitrary products of even relations multiplied by even products of odd relations. However the only even relations in \(R_{\alpha}'\) are given by idempotents and commutation relations. Therefore, the even part of the relation ideal for \(R_{\alpha}'\) is generated by the even relations in \(R_{\alpha}'\) together with all products of the odd relations in \(R_{\alpha}'\). It follows that all the remaining relations are generated by even products of odd relations, together with odd relations multiplied by \(\varepsilon\): in this way we obtain the complete set of relations for \((R_{\alpha}').^+\). One checks these are precisely the relations written above for \(A_{[a]}\); for example multiplying the relation

\[ y_{r+1} \psi_r' \varepsilon_-(i) = \begin{cases} (\psi_r' y_r' + 1) \varepsilon_-(i), & \text{if } i_r = i_{r+1} \\ \psi_r' y_r' \varepsilon_-(i), & \text{if } i_r \neq i_{r+1} \end{cases} \]
by \( \varepsilon_-(i) \) and using the idempotent relations in \( R'_{\alpha_i} \) gives the relation

\[
y'_{r+1} \varepsilon_-(i) \psi_r \varepsilon_-(i) \varepsilon(r) = \begin{cases} \psi_r \varepsilon_-(r) \varepsilon(r) + 1, & \text{if } i_r = i_{r+1} \\ \psi_r \varepsilon_-(r) \varepsilon(r), & \text{if } i_r \neq i_{r+1} \end{cases}
\]

which is precisely the image of the relation

\[
Y_{r+1} \Psi_r e[i] = \begin{cases} \Psi_r Y_r + 1 e[i], & \text{if } i_r = i_{r+1} \\ \Psi_r Y_r e[i], & \text{if } i_r \neq i_{r+1} \end{cases}
\]

under \( \varrho \). Continuing in this way we see \( \varrho \) is an isomorphism.

Finally, we note that the given degree function is well-defined; since the Cartan matrix \((c_{ij})_{i,j \in I}\) is symmetric, the entries only depend on equivalence classes of residue sequences under \( \tau \).

\[ \square \]

4.10. Remark.

(i) Since we have given an isomorphism between the alternating quiver Hecke algebra and the even part of the abstract algebra \( R'_{\alpha_i} \), for which we gave an abstract presentation by generators and relations, we see that the generators from Definition 4.1 do not depend on the choice of equivalence class representatives \( i \tau \) (see p6).

(ii) In [1, §5.4] and [3], cyclotomic quotients of the algebras from this paper are studied. In particular, [3, Theorem B] gives a presentation for a cyclotomic quotient of the alternating quiver Hecke algebras by generators and relations very similar to those in this paper, in the special case when \( \Gamma = \Gamma_e \) and the cyclotomic relation is simple. In the cyclotomic case when the cyclotomic parameter \( \Lambda = \Lambda_0 \), it is shown in [1, Chapter 6] that alternating cyclotomic quiver Hecke algebras are isomorphic to alternating Hecke algebras in the sense of Mitsuhashi [19], giving an analogue of Brundan and Kleshchev’s result [4].

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