CRYSTAL BASES OF MODIFIED QUANTIZED ENVELOPING ALGEBRAS AND A DOUBLE RSK CORRESPONDENCE

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Abstract. The crystal base of the modified quantized enveloping algebras of type $A_{+\infty}$ or $A_{\infty}$ is realized as a set of integral bimatrices. It is obtained by describing the decomposition of the tensor product of a highest weight crystal and a lowest weight crystal into extremal weight crystals, and taking its limit using a tableaux model of extremal weight crystals. This realization induces in a purely combinatorial way a bicrystal structure of the crystal base of the modified quantized enveloping algebras and hence its Peter-Weyl type decomposition generalizing the classical RSK correspondence.

1. Introduction

Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra associated with a symmetrizable Kac-Moody algebra $\mathfrak{g}$. In [15], Lusztig introduced the modified quantized enveloping algebra $\tilde{U}_q(\mathfrak{g}) = \bigoplus_\Lambda U_q(\mathfrak{g}) a_\Lambda$, where $\Lambda$ runs over all integral weight for $\mathfrak{g}$, and proved the existence of its global crystal base or canonical basis. In [8], Kashiwara studied the crystal structure of $\tilde{U}_q(\mathfrak{g})$ in detail, and showed that

$$B(U_q(\mathfrak{g}) a_\Lambda) \simeq B(\infty) \otimes T_\Lambda \otimes B(-\infty),$$

where $B(U_q(\mathfrak{g}) a_\Lambda)$ denotes the crystal base of $U_q(\mathfrak{g}) a_\Lambda$, $B(\pm \infty)$ is the crystal base of the negative (resp. positive) part of $U_q(\mathfrak{g})$ and $T_\Lambda$ is a crystal with a single element $t_\Lambda$ with $\varepsilon_i(t_\Lambda) = \varphi_i(t_\Lambda) = -\infty$. It is also shown that the anti-involution $*$ on $\tilde{U}_q(\mathfrak{g})$ provides the crystal $B(\tilde{U}_q(\mathfrak{g})) = \bigsqcup_\Lambda B(\infty) \otimes T_\Lambda \otimes B(-\infty)$ with another crystal structure called $*$-crystal structure and therefore a regular $(\mathfrak{g}, \mathfrak{g})$-bicrystal structure [8]. With respect to this bicrystal structure, a Peter-Weyl type decomposition for $B(\tilde{U}_q(\mathfrak{g}))$ was obtained when it is of finite type or affine type at non-zero levels by Kashiwara [8] and of affine type at level zero by Beck and Nakajima [1] (see also [17, 18] for partial results). It is also shown in [8] that the crystal graph of the quantized coordinate ring for $\mathfrak{g}$ [7] is a subcrystal of $B(\tilde{U}_q(\mathfrak{g}))$, and equal to $B(\tilde{U}_q(\mathfrak{g}))$ if and only if $\mathfrak{g}$ is of finite type.

The main purpose of this work is to study the crystal structure of the modified quantized enveloping algebras in type $A$ of infinite rank in terms of Young tableaux and understand its connection with the classical RSK correspondence. Note that
the essential ingredient for understanding the structure of $\tilde{U}_q(\mathfrak{g})$ is the notion of extremal weight $U_q(\mathfrak{g})$-module introduced by Kashiwara \cite{Kashiwara93}. An extremal weight module associated with an integral weight $\lambda$ is an integrable $U_q(\mathfrak{g})$-module, which is a generalization of a highest weight and a lowest weight module, and it also has a (global) crystal base. Our approach is based on the combinatorial models for extremal weight crystals of type $A_{1+\infty}$ and $A_{\infty}$ developed in \cite{Kontsevich98, KashiwaraKazhdan98}.

From now on, we denote $\mathfrak{g}$ by $\mathfrak{gl}_{>0}$ and $\mathfrak{gl}_{\infty}$ when it is a general linear Lie algebra of type $A_{1+\infty}$ and $A_{\infty}$, respectively. The main result in this paper gives an explicit combinatorial realization of $B(\infty) \otimes T_\Lambda \otimes B(-\infty)$ for all integral $\mathfrak{gl}_{>0}$-weights and all level zero integral $\mathfrak{gl}_{\infty}$-weights $\Lambda$ as a set of certain bimatrices, which implies directly Peter-Weyl type decompositions of $B(\tilde{U}_q(\mathfrak{gl}_{>0}))$ and the level zero part of $B(\tilde{U}_q(\mathfrak{gl}_{\infty}))$ without using the $*$-crystal structure.

Let us state our results more precisely. Let $M$ be the set of $N \times N$ matrices with finitely many non-negative integral entries. Recall that $M$ has a $\mathfrak{gl}_{>0}$-crystal structure where each row of a matrix in $M$ is identified with a single row Young tableau or a crystal element associated with the symmetric power of the natural representation (cf.\cite{Kash93}). Let $M^\vee = \{ M^\vee \mid M \in M \}$ be the dual crystal of $M$. For each integral weight $\Lambda$, let

$$\tilde{M}_\Lambda = \{ M^\vee \otimes N \mid \text{wt}(N^t) - \text{wt}(M^t) = \Lambda \} \subset M^\vee \otimes M.$$  

Here $\text{wt}$ denotes the weight with respect to $\mathfrak{gl}_{>0}$-crystal structure and $A^t$ denotes the transpose of $A \in M$. Then our main result (Theorem 5.5) is

$$\tilde{M}_\Lambda \simeq B(\infty) \otimes T_\Lambda \otimes B(-\infty).$$

Note that a connected component of $B(\infty) \otimes T_\Lambda \otimes B(-\infty)$ is in general an extremal weight $\mathfrak{gl}_{>0}$-crystal, and an extremal weight $\mathfrak{gl}_{>0}$-crystal is isomorphic to the tensor product of a lowest weight crystal and a highest weight crystal \cite{Kontsevich98}. The crucial step in the proof is the description of the tensor product $B(\Lambda') \otimes B(-\Lambda'')$ for dominant integral weights $\Lambda'$, $\Lambda''$ with $\Lambda = \Lambda' - \Lambda''$ in terms of skew Young bitableaux (Proposition 5.1), and its embedding into $B(\Lambda' + \xi) \otimes B(-\xi - \Lambda'')$ for arbitrary integral dominant weight $\xi$ (Proposition 5.4). In fact, $B(\Lambda' + \xi) \otimes B(-\xi - \Lambda'')$ is realized as a set of skew Young bitableaux whose shapes are almost horizontal strips as $\xi$ goes to infinity. This establishes the above isomorphism and as a consequence

$$B(\tilde{U}_q(\mathfrak{gl}_{>0})) \simeq M^\vee \otimes M,$$

since $\bigsqcup_{\Lambda} \tilde{M}_\Lambda = M^\vee \otimes M$.

Now, for partitions $\mu, \nu$, let $B_{\mu,\nu}$ be the extremal weight crystal with the Weyl group orbit of its extremal weight corresponding to the pair $(\mu,\nu)$. Note that $B_{\mu,\emptyset}$ (resp. $B_{\emptyset,\nu}$) is a highest (resp. lowest) weight crystal and it is shown in \cite{Kontsevich98} that
$\mathcal{B}_{\mu,\nu} \simeq \mathcal{B}_{0,\nu} \otimes \mathcal{B}_{\mu,0}$. Then a $(\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})$-bicrystal structure of $M$ and $M'$ arising from the RSK correspondence naturally induces a $(\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})$-bicrystal structure of $B(\widetilde{U}_q(\mathfrak{gl}_{>0}))$ and the following Peter-Weyl type decomposition (Corollary 5.7)

$$B(\widetilde{U}_q(\mathfrak{gl}_{>0})) \simeq \bigsqcup_{\mu,\nu} \mathcal{B}_{\mu,\nu} \times \mathcal{B}_{\mu,\nu}.$$ 

Hence the decomposition of $B(\widetilde{U}_q(\mathfrak{gl}_{>0}))$ into extremal weight crystals can be understood as the tensor product of two RSK correspondences, which are dual to each other as a $(\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})$-bicrystal.

Next, we prove analogues for the level zero part of $B(\widetilde{U}_q(\mathfrak{gl}_{\infty}))$. This is done by taking the limit of the results in $\mathfrak{gl}_{>0}$. In this case, $M$ is replaced by $\mathbb{Z} \times \mathbb{Z}$-matrices and $\mathcal{B}_{\mu,\nu}$ is replaced by the level zero extremal weight $\mathfrak{gl}_{\infty}$-crystal with the same parameter $(\mu, \nu)$. Finally, we conjecture that the second crystal structures arising from the RSK correspondence is compatible with the dual of $*$-crystal structure.

There are several nice combinatorial descriptions of $B(\infty)$ for $\mathfrak{gl}_{>0}$ and $\mathfrak{gl}_{\infty}$, by which we can understand the structure of the modified quantized enveloping algebras (see e.g. [6, 14, 19, 20]). But our combinatorial description of $B(U_q(\mathfrak{gl}_{>0})a_\Lambda)$ and $B(U_q(\mathfrak{gl}_{\infty})a_\Lambda)$ explains more directly its connected components, projections onto tensor products of a highest weight crystal and a lowest weight crystal, and a bicrystal structure on $B(U_q(\mathfrak{gl}_{>0}))$ and $B(U_q(\mathfrak{gl}_{\infty}))$.

The paper is organized as follows. In Section 2, we give necessary background on crystals. In Section 3, we recall some combinatorics of Littlewood-Richardson tableaux from a viewpoint of crystals, which is necessary for our later arguments. In Section 4, we review a combinatorial model of extremal weight $\mathfrak{gl}_{>0}$-crystals and their non-commutative Littlewood-Richardson rules, and then in Section 5 we prove the main theorem. In Section 6, we recall the combinatorial model for extremal weight $\mathfrak{gl}_{\infty}$-crystals and describe the Littlewood-Richardson rule for the tensor product of a highest weight crystal and a lowest weight crystal into extremal ones. In Section 7, we prove analogues for the level zero part of $B(\widetilde{U}_q(\mathfrak{gl}_{\infty}))$. We remark that the Littlewood-Richardson rule in Section 6 is not necessary for Section 7, but is of independent interest, which completes the discussion on tensor product of extremal weight $\mathfrak{gl}_{\infty}$-crystals in [13].

2. Crystals

2.1. Let $\mathfrak{gl}_{\infty}$ be the Lie algebra of complex matrices $(a_{ij})_{i,j \in \mathbb{Z}}$ with finitely many non-zero entries, which is spanned by $E_{ij}$ ($i, j \in \mathbb{Z}$), the elementary matrix with 1 at the $i$-th row and the $j$-th column and zero elsewhere.

Let $\mathfrak{h} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}E_{ii}$ be the Cartan subalgebra of $\mathfrak{gl}_{\infty}$ and $(\cdot, \cdot)$ denote the natural pairing on $\mathfrak{h}^* \times \mathfrak{h}$. We denote by $\{ h_i = E_{ii} - E_{i+1,i+1} \mid i \in \mathbb{Z} \}$ the set of simple
coroots, and denote by \( \{ \alpha_i = \epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z} \} \) the set of simple roots, where \( \epsilon_i \in \mathfrak{h}^* \) is given by \( \langle \epsilon_i, E_{jj} \rangle = \delta_{ij} \).

Let \( P = Z\Lambda_0 \oplus \bigoplus_{i \in \mathbb{Z}} Z\epsilon_i \) be the weight lattice of \( \mathfrak{g}l_\infty \), where \( \Lambda_0 \) is given by \( \langle \Lambda_0, E_{-j-1,-j+1} \rangle = -\langle \Lambda_0, E_{jj} \rangle = \frac{1}{2} \) \((j \geq 1)\), and \( \Lambda_i = \Lambda_0 + \sum_{k=1}^i \epsilon_k \), \( \Lambda_{-j} = \Lambda_0 - \sum_{k=1}^0 \epsilon_k \) for \( i \geq 1 \). We call \( \Lambda_i \) the \( i \)-th fundamental weight.

For \( k \in \mathbb{Z} \), let \( P_k = k\Lambda_0 + \bigoplus_{i \in \mathbb{Z}} Z\epsilon_i \) be the set of integral weights of level \( k \). Let \( P^+ = \{ \Lambda \in P \mid \langle \Lambda, h_i \rangle \geq 0, i \in \mathbb{Z} \} = \sum_{i \in \mathbb{Z}} Z\Lambda_i \) be the set of dominant integral weights. We put \( P^+_k = P^+ \cap P_k \) for \( k \in \mathbb{Z} \). For \( \Lambda = \sum_{i \in \mathbb{Z}} c_i \Lambda_i \in P \), the level of \( \Lambda \) is \( \sum_{i \in \mathbb{Z}} c_i \). If we put \( \Lambda_\pm = \sum_{i \in \mathbb{Z}} c_i \Lambda_i \), then \( \Lambda = \Lambda_+ - \Lambda_- \) with \( \Lambda_\pm \in P^+ \).

For \( i \in \mathbb{Z} \), let \( r_i \) be the simple reflection given by \( r_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i \) for \( \lambda \in \mathfrak{h}^* \). Let \( W \) be the Weyl group of \( \mathfrak{g}l_\infty \), that is, the subgroup of \( GL(\mathfrak{h}^*) \) generated by \( r_i \) for \( i \in \mathbb{Z} \).

For \( p, q \in \mathbb{Z} \), let \( [p, q] = \{ p, p+1, \ldots, q \} \) \((p < q)\), and \( [p, \infty) = \{ p, p+1, \ldots \} \). For simplicity, we denote \([1,n]\) by \( [n] \) \((n > 1)\). For an interval \( S \) in \( \mathbb{Z} \), let \( \mathfrak{g}l_S \) be the subalgebra of \( \mathfrak{g}l_\infty \) spanned by \( E_{ij} \) for \( i, j \in S \). We denote by \( S^0 \) the index set of simple roots for \( \mathfrak{g}l_S \). For example, \([p,q]^0 = \{ p, \ldots, q-1 \} \).

We also put \( \mathfrak{g}l_\geq = \mathfrak{g}l_{[r+1, \infty)} \) and \( \mathfrak{g}l_\leq = \mathfrak{g}l_{(-\infty, r-1]} \) for \( r \in \mathbb{Z} \).

2.2. Let us briefly recall the notion of crystals (see [9] for a general review and references therein).

Let \( S \) be an interval in \( \mathbb{Z} \). A \( \mathfrak{g}l_S \)-crystal is a set \( B \) together with the maps \( \text{wt} : B \to P, \epsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{ -\infty \} \) and \( \bar{e}_i, \bar{f}_i : B \to B \cup \{ 0 \} \) \((i \in S^0)\) such that for \( b \in B \)

\[
\begin{align*}
1. & \quad \varphi_i(b) = \langle \text{wt}(b), h_i \rangle + \epsilon_i(b), \\
2. & \quad \epsilon_i(\bar{e}_i b) = \epsilon_i(b) - 1, \varphi_i(\bar{e}_i b) = \varphi_i(b) + 1, \text{wt}(\bar{e}_i b) = \text{wt}(b) + \alpha_i \text{ if } \bar{e}_i b \neq 0, \\
3. & \quad \epsilon_i(\bar{f}_i b) = \epsilon_i(b) + 1, \varphi_i(\bar{f}_i b) = \varphi_i(b) - 1, \text{wt}(\bar{f}_i b) = \text{wt}(b) - \alpha_i \text{ if } \bar{f}_i b \neq 0, \\
4. & \quad \bar{f}_i b = b' \text{ if and only if } b = \bar{e}_i b' \text{ for } b, b' \in B, \\
5. & \quad \bar{e}_i b = \bar{f}_i b = 0 \text{ if } \varphi_i(b) = -\infty,
\end{align*}
\]

where \( 0 \) is a formal symbol and \( -\infty \) is the smallest element in \( \mathbb{Z} \cup \{ -\infty \} \) such that \( -\infty + n = -\infty \) for all \( n \in \mathbb{Z} \). Note that \( B \) is equipped with a colored oriented graph structure, where \( b \xrightarrow{\epsilon_i} b' \) if and only if \( b' = \bar{f}_i b \) for \( b, b' \in B \) and \( i \in S^0 \). We call \( B \) connected if it is connected as a graph. We call \( B \) regular if \( \epsilon_i(b) = \max \{ k \mid \bar{e}_i^k b \neq 0 \} \) and \( \varphi_i(b) = \max \{ k \mid \bar{f}_i^k b \neq 0 \} \) for \( b \in B \) and \( i \in S^0 \). The dual crystal \( B^\vee \) of \( B \) is defined to be the set \( \{ b^\vee \mid b \in B \} \) with \( \text{wt}(b^\vee) = -\text{wt}(b), \epsilon_i(b^\vee) = \varphi_i(b), \varphi_i(b^\vee) = \epsilon_i(b), \bar{e}_i(b^\vee) = (\bar{f}_i b)^\vee \) and \( \bar{f}_i(b^\vee) = (\bar{e}_i b)^\vee \) for \( b \in B \) and \( i \in S^0 \). We assume that \( 0^\vee = 0 \).

Let \( B_1 \) and \( B_2 \) be crystals. A morphism \( \psi : B_1 \to B_2 \) is a map from \( B_1 \cup \{ 0 \} \) to \( B_2 \cup \{ 0 \} \) such that for \( b \in B_1 \) and \( i \in S^0 \)
1. \(\psi(0) = 0\),
2. \(\text{wt}(\psi(b)) = \text{wt}(b), \varepsilon_i(\psi(b)) = \varepsilon_i(b), \text{and } \varphi_i(\psi(b)) = \varphi_i(b) \text{ if } \psi(b) \neq 0\),
3. \(\tilde{\varepsilon}_i(b) = \varepsilon_i(\psi(b)) \text{ if } \psi(b) \neq 0 \text{ and } \tilde{\varepsilon}_i(b) \neq 0\),
4. \(\tilde{f}_i(b) = f_i(\psi(b)) \text{ if } \psi(b) \neq 0 \text{ and } \tilde{f}_i(b) \neq 0\).

We call \(\psi\) an embedding and \(B_1\) a subcrystal of \(B_2\) when \(\psi\) is injective, and call \(\psi\) strict if \(\psi : B_1 \cup \{0\} \to B_2 \cup \{0\}\) commutes with \(\tilde{e}_i\) and \(\tilde{f}_i\) for \(i \in S^o\), where we assume that \(\tilde{e}_i0 = \tilde{f}_i0 = 0\). If \(\psi\) is a strict embedding, then \(B_2\) is isomorphic to \(B_1 \cup (B_2 \setminus B_1)\). Note that an embedding between regular crystals is always strict.

For \(b_1 \in B_i\) \((i = 1, 2)\), we say that \(b_1\) is \((\mathfrak{gl}_S^-)\)-equivalent to \(b_2\), and write \(b_1 \equiv b_2\) if there exists an isomorphism of crystals \(C(b_1) \to C(b_2)\) sending \(b_1\) to \(b_2\), where \(C(b_i)\) denote the connected component of \(B_i\) including \(b_i\).

A tensor product of \(B_1\) and \(B_2\) is defined to be the set \(B_1 \otimes B_2 = \{ b_1 \otimes b_2 \mid b_i \in B_i \ (i = 1, 2) \}\) with

\[
\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2),
\]

\[
\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2)) - \langle\text{wt}(b_1), h_i\rangle,
\]

\[
\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_1) + \langle\text{wt}(b_2), h_i\rangle, \varphi_i(b_2)),
\]

\[
\tilde{\varepsilon}_i(b_1 \otimes b_2) = \begin{cases} 
\tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\
b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2),
\end{cases}
\]

\[
\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} 
\tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\
b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2),
\end{cases}
\]

for \(i \in S^o\) and \(b_1 \otimes b_2 \in B_1 \otimes B_2\), where we assume that \(0 \otimes b_2 = b_1 \otimes 0 = 0\).

Then \(B_1 \otimes B_2\) is a crystal. If \(B_1\) and \(B_2\) are regular, then so is \(B_1 \otimes B_2\). Note that \((B_1 \otimes B_2)^\vee \cong B_2^\vee \otimes B_1^\vee\).

For a crystal \(B\) and \(m \in \mathbb{Z}_{\geq 0}\), we denote by \(B^{\otimes m}\) the disjoint union \(B_1 \sqcup \cdots \sqcup B_m\) with \(B_i \simeq B\), where \(B^{\otimes 0}\) means the empty set.

2.3. Let \(U_q(\mathfrak{gl}_S)\) be the quantized enveloping algebra of \(\mathfrak{gl}_S\). For \(\Lambda \in P\), let \(B(\Lambda)\) be the crystal base of the extremal weight \(U_q(\mathfrak{gl}_S)\)-module with extremal weight vector \(u_\Lambda\) of weight \(\Lambda\), which is a regular \(\mathfrak{gl}_S\)-crystal. We refer the reader to [8, 10] for more details. When \(\pm \Lambda\) is a dominant weight for \(\mathfrak{gl}_S\), \(B(\Lambda)\) is a crystal base of the integrable highest (resp. lowest) weight \(U_q(\mathfrak{gl}_S)\)-module with highest (resp. lowest) weight \(\Lambda\). Also we have \(B(\Lambda) \simeq B(w\Lambda)\) for \(w \in W\). Hence, when \(S\) is finite, \(B(\Lambda)\) is always isomorphic to the crystal base of a highest weight module and in particular it is connected. When \(S\) is infinite, it is shown in [12] Proposition 3.1] and [13, Proposition 4.1] that \(B(\Lambda)\) is also connected.

Let \(B(\pm \infty)\) be the crystal base of the negative (resp. positive) part of \(U_q(\mathfrak{gl}_S)\) with the highest (resp. lowest) weight vector \(u_{\pm \infty}\) and let \(T_\Lambda = \{ t_\Lambda \} \ (\Lambda \in P)\)
be the crystal with \( \tilde{e}_i t_\Lambda = \tilde{f}_i t_\Lambda = 0 \), \( \text{wt}(t_\Lambda) = \Lambda \) and \( \varepsilon_i(t_\Lambda) = \varphi_i(t_\Lambda) = -\infty \) for \( i \in S^0 \). There is a strict embedding of \( B(\Lambda) \) into \( B(\infty) \otimes T_\Lambda \otimes B(-\infty) \) sending \( u_\Lambda \) to \( u_\infty \otimes t_\Lambda \otimes u_{-\infty} \). Hence \( B(\Lambda) \) is isomorphic to \( C(u_\infty \otimes t_\Lambda \otimes u_{-\infty}) \) since \( B(\Lambda) \) is connected. Note that \( B(\infty) \otimes T_\Lambda \otimes B(-\infty) \) is regular.

There is an embedding \( B(\Lambda_+) \to B(\infty) \otimes T_{\Lambda_+} \) (resp. \( B(-\Lambda_-) \to T_{\Lambda_-} \otimes B(-\infty) \)) sending \( u_{\Lambda_+} \) to \( u_\infty \otimes t_{\Lambda_+} \) (resp. \( u_{\Lambda_-} \) to \( t_{\Lambda_-} \otimes u_{-\infty} \)). This gives a strict embedding

\[
(2.1) \quad t_{\Lambda_+,-\Lambda_-} : B(\Lambda_+) \otimes B(-\Lambda_-) \to B(\infty) \otimes T_\Lambda \otimes B(-\infty)
\]

sending \( u_{\Lambda_+} \otimes u_{-\Lambda_-} \) to \( u_\infty \otimes t_\Lambda \otimes u_{-\infty} \) since \( t_\Lambda \equiv t_{\Lambda_+} \otimes t_{\Lambda_-} \). For a \( \mathfrak{gl}_s \)-dominant weight \( \xi \in P \), let

\[
(2.2) \quad t_{\Lambda_+,-\Lambda_-}^\xi : B(\Lambda_+) \otimes B(-\Lambda_-) \to B(\Lambda_+ + \xi) \otimes B(-\xi - \Lambda_-)
\]

be an embedding given by the composition of the following two morphisms

\[
B(\Lambda_+) \otimes B(-\Lambda_-) \to B(\Lambda_+) \otimes B(\xi) \otimes B(-\xi) \otimes B(-\Lambda_-)
\to B(\Lambda_+ + \xi) \otimes B(-\xi - \Lambda_-),
\]

where

\[
\tilde{f}_i_1 \cdots \tilde{f}_i_r u_{\Lambda_+} \otimes \tilde{e}_j_1 \cdots \tilde{e}_j_s u_{-\Lambda_-} \mapsto \left( \tilde{f}_i_1 \cdots \tilde{f}_i_r u_{\Lambda_+} \right) \otimes u_\xi \otimes u_{-\xi} \otimes \left( \tilde{e}_j_1 \cdots \tilde{e}_j_s u_{-\Lambda_-} \right)
\]

\[
\mapsto \tilde{f}_i_1 \cdots \tilde{f}_i_r u_{\Lambda_+ + \xi} \otimes \tilde{e}_j_1 \cdots \tilde{e}_j_s u_{-\xi - \Lambda_-}.
\]

Note that

\[
\tilde{f}_i_1 \cdots \tilde{f}_i_r u_{\Lambda_+ + \xi} \equiv \left( \tilde{f}_i_1 \cdots \tilde{f}_i_r u_{\Lambda_+} \right) \otimes u_\xi, \quad \text{if } \tilde{f}_i_1 \cdots \tilde{f}_i_r u_{\Lambda_+} \neq 0,
\]

\[
\tilde{e}_j_1 \cdots \tilde{e}_j_s u_{-\xi - \Lambda_-} \equiv u_{-\xi} \otimes \left( \tilde{e}_j_1 \cdots \tilde{e}_j_s u_{-\Lambda_-} \right), \quad \text{if } \tilde{e}_j_1 \cdots \tilde{e}_j_s u_{-\Lambda_-} \neq 0.
\]

Since

\[
B(\infty) \otimes T_\Lambda \otimes B(-\infty) = \bigcup_{\Lambda', \Lambda'' : \mathfrak{gl}_s\text{-dominant}} \text{Im}(t_{\Lambda', \Lambda''}),
\]

\[
(2.3) \quad t_{\Lambda', \Lambda''} = t_{\Lambda' + \xi, \Lambda'' + \xi} \circ t_{\Lambda', \Lambda''}^\xi,
\]

\{ \( B(\Lambda') \otimes B(-\Lambda'') \mid \Lambda', \Lambda'' : \mathfrak{gl}_s\text{-dominant with } \Lambda = \Lambda' - \Lambda'' \} \text{ forms a direct system,}

whose limit is \( B(\infty) \otimes T_\Lambda \otimes B(-\infty) \). Note that \( B(\Lambda) \) is isomorphic to \( C(u_{\Lambda_+} + \xi \otimes u_{-\xi - \Lambda_-}) \) in \( B(\Lambda_+ + \xi) \otimes B(-\xi - \Lambda_-) \) for any \( \mathfrak{gl}_s\)-dominant weight \( \xi \).
3. Young and Littlewood-Richardson tableaux

3.1. Let \( \mathcal{P} \) denote the set of partitions. We identify a partition \( \lambda = (\lambda_i)_{i \geq 1} \) with a Young diagram or a subset \( \{ (i, j) \mid 1 \leq j \leq \lambda_i \} \) of \( \mathbb{N} \times \mathbb{N} \) following [10]. Let \( \ell(\lambda) = |\{ i \mid \lambda_i \neq 0 \}| \) and \( |\lambda| = \sum_{i \geq 1} \lambda_i \). We denote by \( \lambda' = (\lambda'_i)_{i \geq 1} \) the conjugate partition of \( \lambda \) whose Young diagram is \( \{ (i, j) \mid (j, i) \in \lambda \} \). For \( \mu, \nu \in \mathcal{P} \), \( \mu \cup \nu \) is the partition obtained by rearranging \( \{ \mu_i, \nu_i \mid i \geq 1 \} \), and \( \mu + \nu = (\mu_i + \nu_i)_{i \geq 1} \).

Let \( \mathcal{A} \) be a linearly ordered set and \( \lambda/\mu \) a skew Young diagram. A tableau \( T \) obtained by filling \( \lambda/\mu \) with entries in \( \mathcal{A} \) is called a semistandard tableau or Young tableau of shape \( \lambda/\mu \) if the entries in each row (resp. column) are weakly (resp. strictly) increasing from left to right (resp. from top to bottom). We denote by \( \text{SST}_\mathcal{A}(\lambda/\mu) \) the set of all semistandard tableaux of shape \( \lambda/\mu \) with entries in \( \mathcal{A} \).

Suppose that \( \mathcal{A} \) is an interval in \( \mathbb{Z} \) with a usual linear ordering. Then \( \mathcal{A} \) is a \( \mathfrak{gl}_A \)-crystal associated with the natural representation of \( U_q(\mathfrak{gl}_A) \), where \( \text{wt}(i) = \epsilon_i \) and \( i \mapsto i + 1 \) for \( i \in \mathcal{A}^0 \). The image of \( \text{SST}_\mathcal{A}(\lambda/\mu) \) in \( \mathcal{A}^{\otimes r} \) under the map \( T \mapsto w(T)_{\text{col}} = w_1 \ldots w_r \) together with \( \{ 0 \} \) is invariant under \( \tilde{e}_i, \tilde{f}_i \), where \( w(T)_{\text{col}} \) is the word obtained by reading the entries column by column from right to left, and in each column from top to bottom. Hence \( \text{SST}_\mathcal{A}(\lambda/\mu) \) is a subcrystal of \( \mathcal{A}^{\otimes r} \) [11]. We may identify \( T^\vee \in \text{SST}_\mathcal{A}(\lambda/\mu)^\vee \) with the tableau obtained from \( T \) by 180°-rotation and replacing each entry \( t \) with \( t^\vee \). So we have \( \text{SST}_\mathcal{A}(\lambda/\mu)^\vee \simeq \text{SST}_\mathcal{A}(\lambda/\mu)^\vee \), where \( a^\vee < b^\vee \) if and only if \( b < a \) for \( a, b \in \mathcal{A} \) and \( (\lambda/\mu)^\vee \) is the skew Young diagram obtained from \( \lambda/\mu \) by 180°-rotation. We use the convention \( (t^\vee)^\vee = t \) and hence \( (T^\vee)^\vee = T \).

3.2. For \( \lambda, \mu, \nu \in \mathcal{P} \) with \( |\lambda| = |\mu| + |\nu| \), let \( \text{LR}_{\mu \nu}^\lambda \) be the set of tableaux \( U \) in \( \text{SST}_\mathbb{N}(\lambda/\mu) \) such that

1. the number of occurrences of each \( i \geq 1 \) in \( U \) is \( \nu_i \),
2. for \( 1 \leq k \leq |\nu| \), the number of occurrences of each \( i \geq 1 \) in \( w_1 \ldots w_k \) is no less than that of \( i + 1 \) in \( w_1 \ldots w_k \), where \( w(U)_{\text{col}} = w_1 \ldots w_{|\nu|} \).

We call \( \text{LR}_{\mu \nu}^\lambda \) the set of Littlewood-Richardson tableaux of shape \( \lambda/\mu \) with content \( \nu \) and put \( c_{\mu \nu}^\lambda = |\text{LR}_{\mu \nu}^\lambda| \) [10]. We introduce a variation of \( \text{LR}_{\mu \nu}^\lambda \), which is necessary for our later arguments. Let \( \text{LR}_{\mu \nu}^\lambda \) be the set of tableaux \( U \) in \( \text{SST}_{-\mathbb{N}}(\lambda/\mu) \) such that

1. the number of occurrences of each \( -i \leq -1 \) in \( U \) is \( \nu_i \),
2. for \( 1 \leq k \leq |\nu| \), the number of occurrences of each \( -i \leq -1 \) in \( w_k \ldots w_{|\nu|} \) is no less than that of \( -(i + 1) \) in \( w_k \ldots w_{|\nu|} \), where \( w(U)_{\text{col}} = w_1 \ldots w_{|\nu|} \).
There are other characterizations of $\text{LR}_{\mu\nu}^\lambda$ and $\overline{\text{LR}}_{\mu\nu}^\lambda$ using crystals. For $U \in \text{SST}_N(\lambda/\mu)$, we have $U \in \text{LR}_{\mu\nu}^\lambda$ if and only if $U$ is $\mathfrak{gl}_{>0}$-equivalent (or Knuth equivalent) to the highest weight element $H_\nu$ in $\text{SST}_N(\nu)$, that is, $H_\nu(i,j) = i$ for $(i,j) \in \nu$. Similarly, we have for $U \in \text{SST}_N^{-}(\lambda/\mu), U \in \overline{\text{LR}}_{\mu\nu}^\lambda$ if and only if $U$ is $\mathfrak{gl}_{<0}$-equivalent (or Knuth equivalent) to the lowest weight element $L_\nu$ in $\text{SST}_N^{-}(\nu)$, that is, $L_\nu(i,j) = -\nu_j + i - 1$ for $(i,j) \in \nu$.

There is a one-to-one correspondence with the set of $V \in \text{SST}_N^+(\nu)$ such that $H_\mu \otimes V \equiv H_\lambda$ and $\text{LR}_{\mu\nu}^\lambda$. Indeed, $V$ corresponds to $\iota(V) = U \in \text{LR}_{\mu\nu}^\lambda$, where the number of $k$'s in the $i$-th row of $V$ is equal to the number of $i$'s in the $k$-th row of $U$ for $i,k \geq 1$.

**Example 3.1.**

\[\text{SST}_N((3,3,2)) \ni \begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 & 3 \\
3 & 4 \\
\end{array} \rightarrow \begin{array}{c}
\bullet & \bullet & \bullet & 1 & 1 \\
\bullet & 1 & 2 & 2 \\
2 & 3 \\
\end{array} \in \text{LR}_{(3,1)}^{(5,4,2,1)}((3,3,2))\]

3.3. Next, let us briefly recall the switching algorithm [2]. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two linearly ordered sets. Let $\lambda/\mu$ be a skew Young diagram. Let $U$ be a tableau of shape $\lambda/\mu$ with entries in $\mathcal{A} \cup \mathcal{B}$, satisfying the following conditions;

(S1) $U(i,j) \leq U(i',j')$ whenever $U(i,j), U(i',j') \in \mathcal{X}$ for $(i,j), (i',j') \in \lambda/\mu$ with $i \leq i'$ and $j \leq j'$,

(S2) in each column of $U$, entries in $\mathcal{X}$ increase strictly from top to bottom, where $\mathcal{X} = \mathcal{A} \cup \mathcal{B}$. Suppose that $a \in \mathcal{B}$ and $b \in \mathcal{A}$ are two adjacent entries in $U$ such that $a$ is placed above or to the left of $b$. Interchanging $a$ and $b$ is called a switching if the resulting tableau still satisfies the conditions (S1) and (S2).

Let $\lambda/\mu$ and $\mu/\eta$ be two skew Young diagrams. For $S \in \text{SST}_A(\mu/\eta)$ and $T \in \text{SST}_B(\lambda/\mu)$, we denote by $S \ast T$ the tableau of shape $\lambda/\eta$ with entries $\mathcal{A} \cup \mathcal{B}$ obtained by gluing $S$ and $T$. Let $U$ be a tableau obtained from $S \ast T$ by applying switching procedures as far as possible. Then it is shown in [2] Theorems 2.2 and 3.1] that

1. $U = T' \ast S'$, where $T' \in \text{SST}_A(\nu/\eta)$ and $S' \in \text{SST}_B(\lambda/\nu)$ for some $\nu$,
2. $U$ is uniquely determined by $S$ and $T$,
3. $w(S)_{\text{col}}$ (resp. $w(T)_{\text{col}}$) is Knuth equivalent to $w(S')_{\text{col}}$ (resp. $w(T')_{\text{col}}$),

Suppose that $\eta = \emptyset$ and $S = H_\mu \in \text{SST}_N(\mu)$. We put

\[j(T) = T', \quad j(T)_R = S'.\]

Then we have the following.
Proposition 3.2. Suppose that $A$ is an interval in $\mathbb{Z}$. The map sending $T$ to $(j(T), j(T)_R)$ is an isomorphism of $\mathfrak{gl}_A$-crystals

$$SST_A(\lambda/\mu) \rightarrow \bigcup_{\nu \in \mathcal{P}} SST_A(\nu) \times LR^\lambda_{\nu, \mu},$$

where $\overline{\epsilon}_i(T', S') = (\overline{\epsilon}_i T', S')$ for $i \in A^c$ and $x = e, f$ on the righthand side. In particular, the map $Q \mapsto j(Q)_R$ restricts to a bijection from $LR^\lambda_{\mu, \nu}$ to $LR^\lambda_{\nu, \mu}$, and from $LR^\lambda_{\mu, \nu}$ to $LR^\lambda_{\nu, \mu}$ when $A = \pm N$, respectively.

Proof. The map is clearly a bijection by [2, Theorem 3.1]. Moreover, $j(T)$ is $\mathfrak{gl}_A$-equivalent to $T$ and $j(T)_R$ is invariant under $\overline{\epsilon}_i$ and $\overline{f}_i$ for $i \in A^c$ (cf.[5, Theorem 5.9]). Hence the bijection is an isomorphism of $\mathfrak{gl}_A$-crystal, where on the right-hand side the operators $\overline{\epsilon}_i, \overline{f}_i$ act on the first component. $\square$

Remark 3.3. The inverse of the isomorphism in Proposition 3.2 is given directly by applying the switching process in a reverse way.

4. Extremal weight crystals of type $A_{+\infty}$

Note that for $r \in \mathbb{Z}$ the $\mathfrak{gl}_{r+1}$-crystals $[r+1, \infty)$ and $[r+1, \infty)^\vee$ are given by

$$r + 1 \longrightarrow r + 2 \longrightarrow r + 3 \longrightarrow \cdots,$$

$$\cdots \longrightarrow (r + 3)^\vee \longrightarrow (r + 2)^\vee \longrightarrow (r + 1)^\vee .$$

For $\mu \in \mathcal{P}$, let

$$B^r_{\mu} = SST_{[r+1, \infty]}(\mu).$$

Then $B^r_{\mu}$ is the highest weight $\mathfrak{gl}_{r}$-crystal with highest weight element $H^r_{\mu}$ of weight $\sum_{i \geq 1} \lambda_i \epsilon_{r+i}$, where $H^r_{\mu}(i, j) = r + i$ for $(i, j) \in \mu$. We identify $(B^r_{\mu})^\vee$ with $SST_{[r+1, \infty)^\vee}(\mu^\vee)$.

For $\nu \in \mathcal{P}$ and $s \geq \ell(\nu)$, let $E^r_{\nu}(s) \in (B^r_{\nu})^\vee$ be an element given by

$$(E^r_{\nu}(s))^\vee (i, j) = s - \nu'_j + i$$

for $(i, j) \in \nu$. For $s \geq \ell(\mu) + \ell(\nu)$, let

$$B^r_{\mu, \nu} = C (H^r_{\mu} \otimes E^r_{\nu}(s)) \subset B^r_{\mu} \otimes (B^r_{\nu})^\vee,$$

de the connected component including $H^r_{\mu} \otimes E^r_{\nu}(s)$ as a $\mathfrak{gl}_{r}$-crystal. Then we have the following by [12, Proposition 3.4] and [12, Theorem 3.5].

Theorem 4.1. For $\mu, \nu \in \mathcal{P}$,

1. $B^r_{\mu, \nu}$ is the set of $S \otimes T \in B^r_{\mu} \otimes (B^r_{\nu})^\vee$ such that for each $k \geq 1$,

$$\left| \{ i \mid S(i, 1) \leq r + k \} \right| + \left| \{ i \mid T^\vee(i, 1) \leq r + k \} \right| \leq k,$$
(2) $B_{\mu,\nu}^{>r}$ is isomorphic to an extremal weight $\mathfrak{gl}_{>r}$-crystal with extremal weight

$$\ell(\mu) + \sum_{i=1}^{r} \mu_{i} r + i \ - \ \sum_{j=1}^{r} \nu_{j} r + j + 1.$$

Note that $B_{\mu,\nu}^{>r}$ does not depend on the choice of $s$ and $\{ B_{\mu,\nu}^{>r} \mid \mu, \nu \in \mathcal{P} \}$ is a complete list of pairwise non-isomorphic extremal weight $\mathfrak{gl}_{>r}$-crystals [12, Theorem 3.5 and Lemma 5.1] and the tensor product of extremal weight $\mathfrak{gl}_{>r}$-crystals is isomorphic to a finite disjoint union of extremal weight crystals [12, Theorem 4.10].

To describe the tensor product of extremal weight $\mathfrak{gl}_{>r}$-crystals, let us review an insertion algorithm for extremal weight crystal elements [12], which is an infinite analogue of [22]. Recall that for $a \in A$ and $T \in \text{SST}_{\lambda}(\lambda)$, $a \rightarrow T$ (resp. $T \leftarrow a$) denotes the tableau obtained by the Schensted column (resp. row) insertion, where $A$ is a linearly ordered set and $\lambda \in \mathcal{P}$ (see for example [4, Appendix A.2]).

We denote $S \otimes T \in B_{\mu,\nu}^{>r}$ by $(S, T)$. For $a \in [r+1, \infty)$, we define $a \rightarrow (S, T)$ in the following way:

Suppose first that $S$ is empty and $T$ is a single column tableau. Let $(T', a')$ be the pair obtained by the following process;

1. If $T$ contains $a^{\vee}, (a + 1)^{\vee}, \ldots, (b - 1)^{\vee}$ but not $b^{\vee}$, then $T'$ is the tableau obtained from $T$ by replacing $a^{\vee}, (a + 1)^{\vee}, \ldots, (b - 1)^{\vee}$ with $(a + 1)^{\vee}, (a + 2)^{\vee}, \ldots, b^{\vee}$, and put $a' = b$.
2. If $T$ does not contain $a^{\vee}$, then leave $T$ unchanged and put $a' = a$. 

Now, we suppose that $S$ and $T$ are arbitrary.

1. Apply the above process to the leftmost column of $T$ with $a$.
2. Repeat (1) with $a'$ and the next column to the right.
3. Continue this process to the right-most column of $T$ to get a tableau $T'$ and $a''$.
4. Define $a \rightarrow (S, T)$ to be $((a'' \rightarrow S), T')$.

Then $(a \rightarrow (S, T)) \in B_{\sigma, \nu}^{s, r}$ for some $\sigma \in \mathcal{P}$ with $|\sigma| - |\mu| = 1$. For a finite word $w = w_{1} \ldots w_{n}$ with letters in $[r+1, \infty)$, we let $(w \rightarrow (S, T)) = (w_{n} \rightarrow (w_{n-1} \rightarrow (w_{n-2} \rightarrow \cdots (w_{1} \rightarrow (S, T)) \cdots)))$.

For $a \in [r+1, \infty)$ and $(S, T) \in B_{\mu,\nu}^{>r}$, we define $(S, T) \leftarrow a^{\vee}$ to be the pair $(S', T')$ obtained in the following way:

1. If the pair $(S, (T' \leftarrow a)^{\vee})$ satisfies the condition in Theorem 4.1 (1), then put $S' = S$ and $T' = (T' \leftarrow a)^{\vee}$.
2. Otherwise, choose the smallest $k$ such that $a_{k}$ is bumped out of the $k$-th row in the row insertion of $a$ into $T^{\vee}$ and the insertion of $a_{k}$ into the $(k+1)$-th row violates the condition in Theorem 4.1 (1).
(2-a) Stop the row insertion of \( a \) into \( T' \) when \( a_k \) is bumped out and let \( T' \) be the resulting tableau after taking \( \lor \).

(2-b) Remove \( a_k \) in the left-most column of \( S \), which necessarily exists, and then apply the \textit{jeu de taquin} (see for example [4, Section 1.2]) to obtain a tableau \( S' \).

In this case, \( ((S, T) \leftarrow a^\lor) \in B_{\sigma, \tau}^{\geq r} \), where either (1) \( |\mu| - |\sigma| = 1 \) and \( \tau = \nu \), or (2) \( \sigma = \mu \) and \( |\tau| - |\nu| = 1 \). For a finite word \( w = w_1 \ldots w_n \) with letters in \([r + 1, \infty)\), we let \( ((S, T) \leftarrow w) = ((\cdots ((S, T) \leftarrow w_1) \cdots) \leftarrow w_n) \).

Let \( \mu, \nu, \sigma, \tau \in \mathcal{P} \) be given. For \( (S, T) \in B_{\mu, \nu}^{\geq r} \) and \( (S', T') \in B_{\sigma, \tau}^{\geq r} \), we define

\[
((S', T') \rightarrow (S, T)) = (w(S')_{\text{col}} \rightarrow (S, T)) \leftarrow w(T')_{\text{col}}.
\]

Then \( ((S', T') \rightarrow (S, T)) \in B_{\zeta, \eta}^{\geq r} \) for some \( \zeta, \eta \in \mathcal{P} \). Assume that \( w(S')_{\text{col}} = w_1 \ldots w_s \) and \( w(T')_{\text{col}} = w_{s+1} \ldots w_{s+t} \). For \( 1 \leq i \leq s + t \), let

\[
(S^i, T^i) = \begin{cases} 
    w_1 \cdots w_i \rightarrow (S, T), & \text{if } 1 \leq i \leq s, \\
    (S^s, T^s) \leftarrow w_{s+1} \cdots w_i, & \text{if } s + 1 \leq i \leq s + t,
\end{cases}
\]

and \( (S^0, T^0) = (S, T) \). We define

\[
((S', T') \rightarrow (S, T))_R = (U, V),
\]

where \( (U, V) \) is the pair of tableaux with entries in \( \mathbb{Z} \setminus \{0\} \) determined by the following process;

1. \( U \) is of shape \( \sigma \) and \( V \) is of shape \( \tau \).
2. Let \( 1 \leq i \leq s \). If \( w_i \) is inserted into \((S^{i-1}, T^{i-1})\) to create a dot (or box) in the \( k \)-th row of the shape of \( S^{i-1} \), then we fill the dot in \( \sigma \) corresponding to \( w_i \) with \( k \).
3. Let \( s + 1 \leq i \leq s + t \). If \( w_i \) is inserted into \((S^{i-1}, T^{i-1})\) to create a dot in the \( k \)-th row (from the bottom) of the shape of \( T^{i-1} \), then we fill the dot in \( \tau \) corresponding to \( w_i \) with \( -k \). If \( w_i \) is inserted into \((S^{i-1}, T^{i-1})\) to remove a dot in the \( k \)-th row of the shape of \( S^{i-1} \), then we fill the corresponding dot in \( \tau \) with \( k \).

We call \( ((S', T') \rightarrow (S, T))_R \) the recording tableau of \( ((S', T') \rightarrow (S, T)) \). By [12, Theorem 4.10], we have the following.

**Proposition 4.2.** Under the above hypothesis, we have

1. \( ((S', T') \rightarrow (S, T)) \equiv (S, T) \otimes (S', T') \),
2. \( ((S', T') \rightarrow (S, T))_R \in \mathcal{SST}_N(\sigma) \times \mathcal{SST}_z(\tau) \), where \( \mathcal{Z} \) is the set of non-zero integers with a linear ordering \( 1 < 2 < 3 < \cdots < -3 < -2 < -1 \),
3. the recording tableaux are constant on the connected component of \( B_{\mu, \nu}^{\geq r} \otimes B_{\sigma, \tau}^{\geq r} \) including \( (S, T) \otimes (S', T') \).
Suppose that \( \mu, \nu \in \mathcal{P} \) and \( W \in SST_{\mathbb{Z}}(\nu) \) are given with \( w(W)_{\text{col}} = w_n \ldots w_1 \).

Let \((\alpha^0, \beta^0), (\alpha^1, \beta^1), \ldots, (\alpha^n, \beta^n)\) be the sequence where
\( \alpha^i = (\alpha^i_j)_{j \geq 1} \) and \( \beta^i = (\beta^i_j)_{j \geq 1} \) (1 \( \leq i \leq n \)) are sequences of integers defined inductively as follows;

1. If \( w_i \) is positive, then \( \alpha^i \) is obtained by subtracting 1 in the \( w_i \)-th part of \( \alpha^{i-1} \), and \( \beta^i = \beta^{i-1} \). If \( w_i \) is negative, then \( \alpha^i = \alpha^{i-1} \) and \( \beta^i \) is obtained by adding 1 in the \((-w_i)\)-th part of \( \beta^{i-1} \).

Then for \( \sigma, \tau \in \mathcal{P} \) we define \( \mathcal{C}^{(\mu, \nu)}_{(\sigma, \tau)} \) to be the set of \( W \in SST_{\mathbb{Z}}(\nu) \) such that (1) \( \alpha^i, \beta^i \in \mathcal{P} \) for \( 1 \leq i \leq n \), and (2) \( \alpha^n = \sigma, \beta^n = \tau \).

As a particular case of \([12, \text{Theorem 4.10}]\), we have the following.

**Proposition 4.3.** For \( \mu, \nu \in \mathcal{P} \), we have an isomorphism of \( \mathfrak{gl}_r \)-crystals

\[
B^>_{\mu} \otimes (B^>_\nu)^\vee \rightarrow \bigsqcup_{\sigma, \tau \in \mathcal{P}} B^>_{\sigma, \tau} \times \mathcal{C}^{(\mu, \nu)}_{(\sigma, \tau)},
\]

where \( S \otimes T \) is sent to \(((\emptyset, T) \rightarrow (S, \emptyset)), ((\emptyset, T) \rightarrow (S, \emptyset))_R)\).

Further, we can characterize \( \mathcal{C}^{(\mu, \nu)}_{(\sigma, \tau)} \) as follows.

**Proposition 4.4.** For \( \mu, \nu, \sigma, \tau \in \mathcal{P} \), there exists a bijection

\[
\mathcal{C}^{(\mu, \nu)}_{(\sigma, \tau)} \rightarrow \bigsqcup_{\lambda \in \mathcal{P}} LR^\mu_{\sigma \lambda} \times LR^\nu_{\tau \lambda}.
\]

**Proof.** Suppose that \( W \in \mathcal{C}^{(\mu, \nu)}_{(\sigma, \tau)} \) is given. Let \( W^+ \) (resp. \( W^- \)) be the subtableau in \( W \) consisting of positive (resp. negative) entries.

We have \( W^+ \in SST_{\mathbb{N}}(\lambda) \) and \( W^- \in SST_{\mathbb{Z}}(\nu/\lambda) \) for some \( \lambda \subset \nu \). By definition of \( W \in \mathcal{C}^{(\mu, \nu)}_{(\sigma, \tau)} \), we have \( \iota(W^+) \in LR^\mu_{\sigma \lambda} \) and \( W^- \in LR^\nu_{\lambda \tau} \), hence \( j(W^-)_R \in LR^\nu_{\lambda \tau} \) by Proposition 3.2.

We can check that the correspondence \( W \mapsto (W_1, W_2) = (\iota(W^+), j(W^-)_R) \) is reversible and hence gives a bijection \( \mathcal{C}^{(\mu, \nu)}_{(\sigma, \tau)} \rightarrow \bigsqcup_{\lambda \in \mathcal{P}} LR^\mu_{\sigma \lambda} \times LR^\nu_{\tau \lambda}. \) □

**Example 4.5.** Consider

\[
S = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & \end{pmatrix}, \quad T = \begin{pmatrix} 4^\vee \\ 2^\vee \end{pmatrix}, \quad T = \begin{pmatrix} 2^\vee \\ 3^\vee \end{pmatrix}\in (B^>_{(3,2)})^\vee.
\]

Then we have

\[
\begin{pmatrix} 1 & 1 & 2 , \emptyset \end{pmatrix} \leftarrow 4^\vee = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & \end{pmatrix} \begin{pmatrix} 4^\vee \\ \end{pmatrix} = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \end{pmatrix} -1
\]
\[
\begin{pmatrix}
1 & 1 & 2 \\
2 & 3 & 4^V
\end{pmatrix}
\quad \leftarrow \quad 2^V = \begin{pmatrix}
1 & 1 & 2 \\
3 & 4^V
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 2 \\
3 & 4^V
\end{pmatrix}
\quad \leftarrow \quad 1^V = \begin{pmatrix}
1 & 2 \\
3 & 4^V
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 \\
3 & 4^V
\end{pmatrix}
\quad \leftarrow \quad 3^V = \begin{pmatrix}
1 & 2 \\
3 & 3^V
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 \\
3 & 3^V
\end{pmatrix}
\quad \leftarrow \quad 2^V = \begin{pmatrix}
1 & 2 \\
3 & 2^V
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 \\
2^V
\end{pmatrix}
\quad \leftarrow \quad 2^V = \begin{pmatrix}
1 & 2 \\
2^V & 2^V
\end{pmatrix}
\]

Hence,
\[
((\emptyset, T) \rightarrow (S, \emptyset)) = \begin{pmatrix}
1 & 2 \\
2^V & 2^V
\end{pmatrix}
\in B_{(2),(2,1)}^{>0},
\]

\[
((\emptyset, T) \rightarrow (S, \emptyset))_R = \begin{pmatrix}
1 & 2 & -1 \\
2 & -2 & -1
\end{pmatrix}
\in C_{(2),(2,1)}^{(3,2),(3,2,1)}.
\]

If we put \(W = ((\emptyset, T) \rightarrow (S, \emptyset))_R\), then
\[
W_+ = \begin{pmatrix}
1 & 2 \\
2 & -1
\end{pmatrix}, \quad W_- = \begin{pmatrix}
-2 & -1
\end{pmatrix}.
\]

Since
\[
\iota(W_+) = \begin{pmatrix}
1 & 2 \\
1 & 2
\end{pmatrix}, \quad \jmath(W_-) = \begin{pmatrix}
-2 & -1 \\
-1 & -1
\end{pmatrix}, \quad \jmath(W_-)_R = \begin{pmatrix}
1 & 2 \\
1 & 1
\end{pmatrix}
\]

(see Proposition 3.2 (2)), we have
\[
(W_1, W_2) = \begin{pmatrix}
\bullet & \bullet & 1 \\
1 & 2 & 1
\end{pmatrix}
\in LR_{(2),(2,1)}^{(3,2)} \times LR_{(2),(2,1)}^{(3,2,1)}.
\]
Now, the multiplicity of each connected component can be written in terms of Littlewood-Richardson coefficients as follows. We remark that it was already given in [12, Corollary 7.3], while Proposition 4.4 gives a bijective proof of it.

**Corollary 4.6.** For $\mu, \nu \in \mathcal{P}$, we have

$$B_{\mu}^r \otimes (B_{\nu}^r)^{\vee} \simeq \bigsqcup_{\sigma, \tau \in \mathcal{P}} (B_{\sigma, \tau}^r)^{\otimes c_{(\sigma, \tau)}^{(\mu, \nu)}},$$

where

$$c_{(\sigma, \tau)}^{(\mu, \nu)} = \sum_{\lambda \in \mathcal{P}} c_{\sigma \lambda}^{\mu} c_{\tau \lambda}^{\nu}.$$

**Proposition 4.7.** For $\mu, \nu \in \mathcal{P}$, we have an isomorphism of $gl_{\geq r}$-crystals

$$(B_{\nu}^r)^{\vee} \otimes B_{\mu}^r \longrightarrow B_{\mu, \nu}^r,$$

where $T \otimes S$ is mapped to $((S, \emptyset) \rightarrow (\emptyset, T))$.

**Proof.** For $T \otimes S \in (B_{\nu}^r)^{\vee} \otimes B_{\mu}^r$, it follows from Proposition 4.2 (2) that

1. $((S, \emptyset) \rightarrow (\emptyset, T))_R = (H_{\mu}, \emptyset)$.
2. $((S, \emptyset) \rightarrow (\emptyset, T)) \in B_{\mu, \nu}^r$.

Therefore, by [12, Theorem 4.10] the map

$$(B_{\nu}^r)^{\vee} \otimes B_{\mu}^r \longrightarrow B_{\mu, \nu}^r \times \{ (H_{\mu}, \emptyset) \}$$

sending $T \otimes S$ to $(((S, \emptyset) \rightarrow (\emptyset, T))_R$, $((S, \emptyset) \rightarrow (\emptyset, T))_R$ is an isomorphism of $gl_{\geq r}$-crystals.

**Example 4.8.** Let

$$(U, V) = \begin{pmatrix} 1 & 2 \\ 2^\vee & 4^\vee \end{pmatrix} \in B_{(2,1), (2,1)}^{>0}$$

be as in Example 4.5. If we put

$$\tilde{V} \otimes \tilde{U} = \begin{pmatrix} 4^\vee \\ 1^\vee \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in B_{(2,1)}^{>0} \otimes B_{(2)}^{>0},$$

then

$$\left( (\tilde{U}, \emptyset) \rightarrow (\emptyset, \tilde{V}) \right) = (U, V).$$

5. **Combinatorial description of $B(\tilde{U}_q(glt_{>0}))$**

In this section, we give a combinatorial realization of $B(\infty) \otimes T_{\Lambda} \otimes B(-\infty)$ for an integral weight $\Lambda$ in case of $glt_{>0}$, and then $B(\tilde{U}_q(glt_{>0}))$. 
5.1. For simplicity, we put for a skew Young diagram $\lambda/\mu$

$$\mathcal{B}_{\lambda/\mu} = \text{SST}_N(\lambda/\mu),$$

and for $\mu, \nu \in \mathcal{P}$

$$\mathcal{B}_{\mu, \nu} = \mathcal{B}_{\mu, \nu}^{>0}.$$ For $S \otimes T \in \mathcal{B}_\mu \otimes \mathcal{B}_\nu$, suppose that

$$(U, V) = ((\emptyset, T) \rightarrow (S, \emptyset)) \in \mathcal{B}_{\sigma, \tau},$$

$$W = ((\emptyset, T) \rightarrow (S, \emptyset))_R \in \mathcal{C}_{(\sigma, \tau)},$$

for some $\sigma, \tau \in \mathcal{P}$. By Proposition 4.7, there exist unique $\widetilde{U} \in \mathcal{B}_\sigma$ and $\widetilde{V} \in \mathcal{B}_\tau^\vee$ such that $\widetilde{V} \otimes \widetilde{U} \equiv (U, V)$. By Proposition 4.4, we have

$$W \leftrightarrow (W_1, W_2) \in \text{LR}_{\sigma, \lambda}^\mu \times \text{LR}_{\tau, \lambda}^\nu$$

for some $\lambda \in \mathcal{P}$. By Proposition 3.2, there exists unique $X \in \mathcal{B}_{\mu/\lambda}$ and $Y \in \mathcal{B}_{\nu/\lambda}$ such that

$$j(X) = \widetilde{U}, \quad j(X)_R = W_1,$$

$$j(Y)^\vee = \widetilde{V}, \quad j(Y)_R = W_2.$$ Now, we define

(5.1) $$\psi_{\mu, \nu}(S \otimes T) = Y^\vee \otimes X \in \mathcal{B}^\vee_{\nu/\lambda} \otimes \mathcal{B}_{\mu/\lambda}.$$ By construction, $\psi_{\mu, \nu}$ is bijective and commutes with $\tilde{x}_i$ for $x = e, f$ and $i \geq 1$. Hence we have the following.

Proposition 5.1. For $\mu, \nu \in \mathcal{P}$, the map

$$\psi_{\mu, \nu} : \mathcal{B}_\mu \otimes \mathcal{B}_\nu^\vee \rightarrow \bigsqcup_{\lambda \subseteq \mu, \nu} \mathcal{B}^\vee_{\nu/\lambda} \otimes \mathcal{B}_{\mu/\lambda}$$

is an isomorphism of $\mathfrak{gl}_{>0}$-crystals.

Example 5.2. Let $S$ and $T$ be the tableaux in Example 4.5. Let

$$X = \begin{array}{cc}
\bullet & \bullet & 1 \\
\bullet & 1 & \\
\end{array}, \quad Y = \begin{array}{cc}
\bullet & 2 \\
1 & 4 \\
\end{array}.$$ Following the above notations, we have

$$H_{(2,1)} * X = \begin{array}{ccc}
1 & 1 & 1 \\
2 & 1 & \\
\end{array}$$

$$\text{switching}$$

$$\begin{array}{ccc}
1 & 1 & 1 \\
1 & 2 & \\
\end{array} = j(X) * j(X)_R = \widetilde{U} * W_1,$$

$$H_{(2,1)} * Y = \begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & \\
4 & & \\
\end{array}$$

$$\text{switching}$$

$$\begin{array}{ccc}
1 & 2 & 1 \\
4 & 2 & \\
1 & & \\
\end{array} = j(Y) * j(Y)_R = \widetilde{V} * W_2.$$
where $\bar{U}, \bar{V}, W_i \ (i = 1, 2)$ are as in Examples 4.5 and 4.8. Hence,

$$
\psi_{\mu, \nu}(S \otimes T) = Y^\vee \otimes X
$$

$$
= \begin{pmatrix}
\bullet & \bullet & 1 \\
\bullet & 2 \\
4
\end{pmatrix} \otimes 
\begin{pmatrix}
\bullet & \bullet & 1 \\
1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
4^\vee
\end{pmatrix} \otimes 
\begin{pmatrix}
2^\vee & \bullet \\
\bullet & 1 \\
1^\vee
\end{pmatrix}.
$$

For a skew Young diagram and $\lambda/\mu$ and $k \geq 1$, we define

(5.2)

$$
\kappa_k : \mathcal{B}_{\lambda/\mu} \rightarrow \mathcal{B}_{(\lambda+(1^k))/(\mu+(1^k))}
$$

by $\kappa_k(S) = S'$ with

$$
S'(i, j) = \begin{cases}
S(i, j), & \text{if } i > k, \\
S(i, j - 1), & \text{if } i \leq k.
\end{cases}
$$

Example 5.3.

$$
\kappa_1 \begin{pmatrix}
\bullet & \bullet & 1 \\
\bullet & 2 \\
1
\end{pmatrix}
= \begin{pmatrix}
\bullet & \bullet & 1 \\
2 \\
1
\end{pmatrix}
$$

$$
\kappa_2 \begin{pmatrix}
\bullet & \bullet & 1 \\
2 \\
1
\end{pmatrix}
= \begin{pmatrix}
\bullet & \bullet & 1 \\
2 \\
1
\end{pmatrix}
$$

For $k \geq 1$ and $\lambda \in \mathcal{P}$, we put

$$
\omega_k = \epsilon_1 + \cdots + \epsilon_k,
$$

$$
\omega_\lambda = \lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \cdots.
$$

Now, we have the following combinatorial interpretation of the embedding (2.2) in terms of \textit{sliding} skew tableaux horizontally. It will play a crucial role in proving our main theorem.

Proposition 5.4. For $\mu, \nu \in \mathcal{P}$ and $k \geq 1$, we have the following commutative diagram of $\mathfrak{gl}_{>0}$-crystal morphisms

$$
\begin{array}{cc}
\mathcal{B}_{\mu} \otimes \mathcal{B}_{\nu} & \rightarrow & \mathcal{B}_{\mu+(1^k)} \otimes \mathcal{B}_{\nu+(1^k)} \\
\psi_{\mu, \nu} & \downarrow & \psi_{\mu+(1^k), \nu+(1^k)} \\
\bigcup_\lambda \mathcal{B}_{\nu/\lambda} \otimes \mathcal{B}_{\mu/\lambda} & \rightarrow & \bigcup_\eta \mathcal{B}_{(\nu+(1^k))/\eta} \otimes \mathcal{B}_{(\mu+(1^k))/\eta}
\end{array}
$$

where $\psi_{\omega_\mu, \omega_\nu}$ is the canonical embedding in (2.2) and $\nu_k = \vee \circ \kappa_k \circ \vee$. 
Proof. Let $S \otimes T \in \mathcal{B}_\mu \otimes \mathcal{B}_\nu^\vee$ be given. We keep the previous notations. Note that

$$S \otimes u_{\omega_k} = S \otimes H^{(1)}_{(k)} = S\{k\} := (1 \cdots k \to S) \in \mathcal{B}_{\mu+(1)^k},$$

$$u_{-\omega_k} \otimes T = H^{(1)}_{(k)} \otimes T \equiv T\{k\} := (1 \cdots k \to T^\vee) \in \mathcal{B}_\nu^{(1)^k}.$$ 

Hence by \cite{22} we have $\iota_{\omega_k}^k(S \otimes T) = S\{k\} \otimes T\{k\}$. Since $S\{k\} \otimes T\{k\} \equiv S \otimes T$, we have

$$(U\{k\}, V\{k\}) := ((\emptyset, T\{k\}) \to (S\{k\}, \emptyset)) \equiv ((\emptyset, T) \to (S, \emptyset)) = (U, V),$$

which implies that $(U\{k\}, V\{k\}) = (U, V)$ by \cite{12} Lemma 5.1. Put

$$W\{k\} = ((\emptyset, T\{k\}) \to (S\{k\}, \emptyset))_R,$$

and suppose that

$$W\{k\} \leftrightarrow (W_1\{k\}, W_2\{k\}) \in \text{LR}_{\sigma+\lambda+(1)^k} \times \text{LR}_{\nu+\lambda+(1)^k},$$

Since $W$ is invariant under $\tilde{e}_i$ and $\tilde{f}_i$ ($i \geq 1$), we may assume that $(U, V) = (H^0_\sigma, E^0_\tau(n))$ for a sufficiently large $n > k$ (see \cite{22}). As a $\mathfrak{g}_{(\lambda)}$-crystal element, $(U, V)$ is a highest weight element, and $\sigma^p_\mu(U, V) = H^0_\zeta$, where $p \geq \tau_1$ and $\zeta = \sigma + (p - \tau_{n-1} + 1)_{i \geq 1}$ (see \cite{12} Section 4.1 for the definition of the map $\sigma_n$). This also implies that $S = H^0_\mu$. By \cite{22} Lemma 7.6, $W\{k\}$ is obtained from

$$\sigma^p_\nu_\mu \left[ (\sigma^p_\mu(\emptyset, T\{k\}) \to (S\{k\}, \emptyset))_R \right]$$

by 180°-rotation and ignoring $\vee$’s in the entries. Since $S\{k\} = H^0_{\mu+(1)^k}$, we have $$(\sigma^p_\mu(\emptyset, T\{k\}) \to (S\{k\}, \emptyset))_R = \sigma^p_\mu(\emptyset, T\{k\}).$$ Now, it is straightforward to check that

$$\begin{align*}
W\{k\} &= *_{k} W.
\end{align*}$$

This implies that

$$W_1\{k\} = W_1 * \Sigma_k, \quad W_2\{k\} = W_2 * \Sigma'_k,$$

where $\Sigma_k$ and $\Sigma'_k$ are vertical strips of shape $(\mu + (1)^k)/\mu$ and $(\nu + (1)^k)/\nu$ filled with 1, \ldots, $k$, respectively. Now, we have

$$\begin{align*}
\tilde{U} * W_1\{k\} &= \tilde{U} * W_1 * \Sigma_k \rightsquigarrow H_\lambda * X * \Sigma_k \quad \text{(switching $\tilde{U}$ and $W_1$)} \\
&\rightsquigarrow H_{\lambda+(1)^k} * \kappa_k(X) \quad \text{(switching $X$ and $\Sigma_k$)}, \\
\tilde{V} * W_2\{k\} &= \tilde{V} * W_2 * \Sigma'_k \rightsquigarrow H_\lambda * Y * \Sigma'_k \quad \text{(switching $\tilde{V}$ and $W_2$)} \\
&\rightsquigarrow H_{\lambda+(1)^k} * \kappa_k(Y). \quad \text{(switching $Y$ and $\Sigma'_k$)}
\end{align*}$$
Therefore, it follows that
\[ \psi_{\mu+(1^k),\nu+(1^k)}(\omega_{\mu,\nu}^\omega(S \otimes T)) = \psi_{\mu+(1^k),\nu+(1^k)}(S\{k\} \otimes T\{k\}) \]
\[ = \kappa_k(Y)^\vee \otimes \kappa_k(X) \]
\[ = \kappa_k^\vee \otimes \kappa_k(\psi_{\mu,\nu}(S \otimes T)) . \]

5.2. Let $M$ be the set of $\mathbb{N} \times \mathbb{N}$ matrices $A = (a_{ij})$ such that $a_{ij} \in \mathbb{Z}_{\geq 0}$ and $\sum_{i,j \geq 1} a_{ij} < \infty$. Let $A = (a_{ij}) \in M$ be given. For $i \geq 1$, the $i$-th row $A_i = (a_{ij})_{j \geq 1}$ is naturally identified with a unique semistandard tableau in $B_{(m_i)}$, where $m_i = \sum_{j \geq 1} a_{ij}$ and $\text{wt}(A_i) = \sum_{j \geq 1} a_{ij} \epsilon_j$. Hence $A$ can be viewed as an element in $B_{(m_1)} \otimes \ldots \otimes B_{(m_r)}$ for some $r \geq 0$. This defines a $\mathfrak{gl}_{>0}$-crystal structure on $M$.

Now, we put
\[ (5.4) \quad \widetilde{M} = M^\vee \times M, \]
which can be viewed as a tensor product of $\mathfrak{gl}_{>0}$-crystals. Let $\mathcal{P} = \bigoplus_{i \geq 1} \mathbb{Z} \epsilon_i$ be the integral weight lattice for $\mathfrak{gl}_{>0}$. For $\omega \in \mathcal{P}$, let
\[ \widetilde{M}_\omega = \{ (M^\vee, N) \in \widetilde{M} | \text{wt}(N^\dagger) - \text{wt}(M^\dagger) = \omega \}. \]

Here $A^\dagger$ denotes the transpose of $A \in M$. Then $\widetilde{M}_\omega$ is a subcrystal of $\widetilde{M}$. Now, we can state the main result in this section.

**Theorem 5.5.** For $\omega \in \mathcal{P}$, we have
\[ \widetilde{M}_\omega \simeq B(\infty) \otimes T_\omega \otimes B(-\infty). \]

**Proof.** Let $\mu, \nu \in \mathcal{P}$ be such that $\omega = \omega_\mu - \omega_\nu$. Suppose that $\psi_{\mu,\nu}(S \otimes T) = Y^\vee \otimes X$ for $S \otimes T \in B_\mu \otimes B_\nu^\vee$, where $\psi_{\mu,\nu}$ is the isomorphism in Proposition 5.1. Let $M = (m_{ij})$ (resp. $N = (n_{ij})$) be the unique matrix in $M$ such that the $i$-th row of $M$ (resp. $N$) is $\mathfrak{gl}_{>0}$-equivalent to the $i$-th row of $Y$ (resp. $X$). Since $\sum_{j \geq 1} m_{ij}$ (resp. $\sum_{j \geq 1} n_{ij}$) is equal to $x_i$ (resp. $y_i$) the number of dots or boxes in the $i$-th row of $X$ (resp. $Y$) for $i \geq 1$ and $\omega = \sum_{i \geq 1} (x_i - y_i) \epsilon_i$ by Proposition 5.1, we have $\text{wt}(N^\dagger) - \text{wt}(M^\dagger) = \omega$. Then we define
\[ t_{\mu,\nu}' : B_\mu \otimes B_\nu^\vee \rightarrow \widetilde{M}_\omega \]
by $t_{\mu,\nu}'(S \otimes T) = (M^\vee, N)$. By Proposition 5.1, it is easy to see that $t_{\mu,\nu}'$ is an embedding and
\[ \widetilde{M}_\omega = \bigcup_{\substack{\mu,\nu \in \mathcal{P} \\mu - \omega_\mu = \omega}} \text{Im} t_{\mu,\nu}'. \]
For \(k \geq 1\), we have \(\ell'_{\mu, \nu} = \ell'_{\mu+(1^k), \nu+(1^k)} \circ \lambda_{\omega_{\mu, \omega_{\nu}}}\) by Proposition 5.4. Using induction, we have
\[
\ell'_{\mu, \nu} = \ell'_{\mu+\xi, \nu+\xi} \circ \lambda_{\omega_{\mu, \omega_{\nu}}} \quad (\xi \in \mathcal{P}).
\]
Therefore, by (2.3), it follows that \(\tilde{M}_\omega \simeq B(\infty) \otimes T_\omega \otimes B(-\infty). \) \(\square\)

Corollary 5.6. As a \(\mathfrak{gl}_{>0}\)-crystal, we have
\[
B(\tilde{U}_q(\mathfrak{gl}_{>0})) \simeq \tilde{M}.
\]

Proof. It follows from \(\tilde{M} = \bigsqcup_{\omega \in \mathcal{P}} \tilde{M}_\omega.\) \(\square\)

For \(A \in M\) and \(i \geq 1\), we also define
\[
\tilde{e}_i^t A = (\tilde{e}_i A^t)^t, \quad \tilde{f}_i^t A = (\tilde{f}_i A^t)^t.
\]

Then \(M\) has another \(\mathfrak{gl}_{>0}\)-crystal structure with respect to \(\tilde{e}_i^t, \tilde{f}_i^t\) and \(\text{wt}^t\), where \(\text{wt}^t(A) = \text{wt}(A^t)\). By [3], \(M\) is a \((\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})\)-bicrystal, that is, \(\tilde{e}_i, \tilde{f}_i\) on \(M \cup \{0\}\) commute with \(\tilde{e}_i^t, \tilde{f}_i^t\) for \(i,j \geq 1\), and so is the tensor product \(\tilde{M} = M^\vee \times M\). Now we have the following Peter-Weyl type decomposition.

Corollary 5.7. As a \((\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})\)-bicrystal, we have
\[
B(\tilde{U}_q(\mathfrak{gl}_{>0})) \simeq \bigsqcup_{\mu, \nu \in \mathcal{P}} B_{\mu, \nu} \times B_{\mu, \nu}.
\]

Proof. Note that the usual RSK correspondence gives an isomorphism of \((\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})\)-bicrystals \(M \simeq \bigsqcup_{\lambda \in \mathcal{P}} B_\lambda \times B_\lambda\). We assume that \(\tilde{e}_i, \tilde{f}_i\) act on the first component, and \(\tilde{e}_j^t, \tilde{f}_j^t\) act on the second component. The decomposition of \(B(\tilde{U}_q(\mathfrak{gl}_{>0}))\) follows from Proposition 1.7. \(\square\)

6. Extremal weight crystals of type \(A_\infty\)

In this section, we describe the tensor product of \(\mathfrak{gl}_{\infty}\)-crystals \(B(\Lambda) \otimes B(-\Lambda')\) for \(\Lambda, \Lambda' \in P^+\) in terms of extremal weight crystals.

6.1. For a skew Young diagram \(\lambda/\mu\), we put
\[
B_{\lambda/\mu} = SST_{\mathbb{Z}}(\lambda/\mu),
\]
and we identify \(B_{\lambda/\mu}^\vee\) with \(SST_{\mathbb{Z}'}((\lambda/\mu)')\). Note that for \(\mu \in \mathcal{P}\), \(B_\mu\) has neither highest nor lowest weight vector. It is shown in [13] that for \(\mu, \nu \in \mathcal{P}\), \(B_\mu \otimes B_\nu^\vee\) is connected, \(B_\mu \otimes B_\nu^\vee \simeq B_\nu^\vee \otimes B_\mu\), and \(B_\mu \otimes B_\nu^\vee \simeq B_\sigma \otimes B_\tau^\vee\) if and only if \((\mu, \nu) = (\sigma, \tau)\).

Put
\[
B_{\mu, \nu} = B_\mu \otimes B_\nu^\vee.
\]

Note that \(B_{\mu, \nu}\) can be viewed as a limit of \(B_{\mu, \nu}^{r} (r \to -\infty)\) since \(B_{\mu, \nu}^{r} \simeq (B_\nu^\vee)^r \otimes B_\mu^{\geq r}\).
Let $\mathbb{Z}_n^+ = \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n | \lambda_1 \geq \cdots \geq \lambda_n \}$ be the set of generalized partitions of length $n$. For $\lambda \in \mathbb{Z}_n^+$, we put

$$\Lambda_\lambda = \Lambda\lambda_1 + \cdots + \Lambda\lambda_n \in P_+^n.$$ 

**Theorem 6.1** (Theorem 4.6 in [13]). For $\Lambda \in P_n \ (n \geq 0)$, there exist unique $\lambda \in \mathbb{Z}_n^+$ and $\mu, \nu \in \mathcal{P}$ such that

$$B(\Lambda) \simeq B_{\mu, \nu} \otimes B(\Lambda) .$$

Here we assume that $\Lambda_\lambda = 0$ when $n = 0$.

Note that $\{B_{\mu, \nu} \otimes B(\Lambda) | \Lambda \in P^+, \ \mu, \nu \in \mathcal{P}\}$ forms a complete list of extremal weight crystals of non-negative level up to isomorphism [13, Proposition 3.12].

6.2. For intervals $I, J$ in $\mathbb{Z}$, let $M_{I,J}$ be the set of $I \times J$ matrices $A = (a_{ij})$ with $a_{ij} \in \{0, 1\}$. We denote by $A_i$ the $i$-th row of $A$ for $i \in I$.

Suppose that $A \in M_{I,J}$ is given. For $k \in J^\circ$ and $i \in I$, we define

$$f_k A_i = \begin{cases} A_i - E_{ik} + E_{ik+1}, & \text{if } (a_{ik}, a_{i, k+1}) = (1, 0), \\ 0, & \text{otherwise}. \end{cases} \quad (6.3)$$

Then we can regard $A_i$ as an element of a regular $\mathfrak{gl}_{(k, k+1)}$ crystal with highest weight $\omega \in \{0, \varepsilon_k, \varepsilon_k + \varepsilon_{k+1}\}$. Consider the sequence $(\varepsilon_k(\Lambda_i))_{i \in I}$. We say that $A$ is $k$-admissible if there exist $L, L' \in I$ such that (1) $\varepsilon_k(\Lambda_i) \neq 1$ for all $i < L$, and (2) $\varepsilon_k(\Lambda_i) \neq -1$ for all $i > L'$. Note that if $I$ is finite, then $A$ is $k$-admissible for all $k \in J$. Suppose that $A$ is $k$-admissible. Then we can define $\tilde{f_k}A$ by regarding $A$ as $\tilde{\otimes}_{i \in I} A_i$, and applying tensor product rule of crystal, where the index $i$ in the tensor product is increasing from left to right.

Let $\rho : M_{I,J} \rightarrow M_{-I,-J}$ be a bijection given by $\rho(A) = (a_{-j,i}) \in M_{-I,-J}$, where $-J = \{-j \ | \ j \in J\}$. For $l \in I^\circ$, we say that $A$ is $l$-admissible if $\rho(A)$ is $l$-admissible in the above sense. If $A$ is $l$-admissible, then we define

$$\tilde{F}_l(A) = \rho^{-1} \left( \tilde{e}_l \rho(A) \right), \quad \tilde{E}_l(A) = \rho^{-1} \left( \tilde{f}_l \rho(A) \right). \quad (6.4)$$

If $A$ is both $k$-admissible and $l$-admissible for some $l \in I^\circ$ and $k \in J^\circ$, then

$$\tilde{x}_k \tilde{X}_l A = \tilde{X}_l \tilde{x}_k A, \quad (6.5)$$

where $x = e, f$ and $X = E, F$ [13, Lemma 3.1].

For convenience, let us say that $A$ is $J$-admissible (resp. $I$-admissible) if $A$ is $k$-admissible (resp. $l$-admissible) for all $k \in J^\circ$ (resp. $l \in I^\circ$). Suppose that $A$ is $J$-admissible and $l$-admissible for some $l \in I^\circ$. Then both $A$ and $\tilde{X}_l A$ generate the same $J$-colored oriented graphs with respect to $\tilde{e}_k$ and $\tilde{f}_k$ for $k \in J^\circ$ whenever
\( \tilde{X}^t A \neq 0 \) \((X = E, F) \) [13] Lemma 3.2. A similar fact holds when \( A \) is \( I \)-admissible and \( k \)-admissible for some \( k \in J^o \).

If \( I \) and \( J \) are finite, then \( M_{t, J} \) is a \((\mathfrak{gl}_t, \mathfrak{gl}_J)\)-bicrystal, where the \( \mathfrak{gl}_t \)-weight (resp. \( \mathfrak{gl}_J \)-weight) of \( A = (a_{ij}) \in M_{t, J} \) is given by \( \sum_{i \in I} \sum_{j \in J} a_{ij} \epsilon_i \) (resp. \( \sum_{j \in J} \sum_{i \in I} a_{ij} \epsilon_j \)).

6.3. For \( n \geq 1 \), let \( \mathcal{E}^n \) be the subset of \( M_{[n],Z} \) consisting of matrices \( A = (a_{ij}) \) such that \( \sum_{i,j} a_{ij} < \infty \). It is clear that \( A \) is \( \mathbb{Z} \)-admissible for \( A \in \mathcal{E}^n \). If we define \( \text{wt}(A) = \sum_{j \in \mathbb{Z}} \left( \sum_{i \in [n]} a_{ij} \right) \epsilon_j \), then \( \mathcal{E}^n \) is a regular \( \mathfrak{gl}_\infty \)-crystal with respect to \( \tilde{e}_k, \tilde{f}_k \) \((k \in \mathbb{Z})\) and \( \text{wt} \). For \( r \in \mathbb{Z} \) and \( \lambda \in \mathcal{P} \) with \( \lambda_i \leq n \), let \( A_\lambda^0(r) = (a_{ij}^0) \in \mathcal{E}^n \) and \( A_\lambda^\circ(r) = (a_{ij}^\circ) \in \mathcal{E}^n \) be such that for \( i \in [n] \) and \( j \in \mathbb{Z} \):

\[
\begin{align*}
a_{ij}^0 &= 1 \iff 1 + r \leq j \leq \lambda_i + 1 + r, \\
a_{ij}^\circ &= 1 \iff r - \lambda_i + 1 \leq j \leq r.
\end{align*}
\]

Then \( C(A_\lambda^0(r)) \cong B_\lambda \) \((= o, o)\).

For \( n \geq 1 \), let \( \mathcal{F}^n \) be the set of matrices \( A = (a_{ij}) \in M_{[n],Z} \) such that for each \( i \in [n] \), \( a_{ij} = 1 \) if \( j \ll 0 \) and \( a_{ij} = 0 \) if \( j \gg 0 \). Note that \( A \) is \( \mathbb{Z} \)-admissible for \( A \in \mathcal{F}^n \). If we define \( \text{wt}(A) = n \Lambda_0 + \sum_{j > 0} \sum_{i \in [n]} a_{ij} \epsilon_j + \sum_{j \leq 0} \sum_{i \in [n]} (a_{ij} - 1) \epsilon_j \), then \( \mathcal{F}^n \) is a regular \( \mathfrak{gl}_\infty \)-crystal with respect to \( \tilde{e}_k, \tilde{f}_k \) \((k \in \mathbb{Z})\) and \( \text{wt} \). For \( \lambda \in \mathbb{Z}_+^n \), let \( A_\lambda^0 = (a_{ij}^0) \in \mathcal{F}^n \) and \( A_\lambda = (a_{ij}^\circ) \in \mathcal{F}^n \) be such that for \( i \in [n] \) and \( j \in \mathbb{Z} \):

\[
\begin{align*}
a_{ij}^0 &= 1 \iff j \leq \lambda_{n-i+1}, \\
a_{ij}^\circ &= 1 \iff j \leq \lambda_i.
\end{align*}
\]

Then \( C(A_\lambda^0) \cong B(\Lambda_\lambda) \) \((= o, o)\).

On the other hand, for \( A = (a_{ij}) \in \mathcal{E}^n \) or \( \mathcal{F}^n \), \( A \) is \([n]\)-admissible. Hence, \( \tilde{E}_l \) and \( \tilde{F}_l \) \((l = 1, \ldots, n - 1)\) commute with \( \tilde{e}_k \) and \( \tilde{f}_k \) \((k \in \mathbb{Z})\).

For \( A = (a_{ij}) \in \mathcal{E}^n \) or \( \mathcal{F}^n \), we will identify its dual crystal element \( A^\vee \) with the matrix \((1 - a_{ij})\) since \( A^\vee \) and \((1 - a_{ij})\) generate the same \( \mathbb{Z} \)-colored graph.

6.4. Let \( m, n \) be non-negative integers with \( m \geq n \). In the rest of this section, we fix \( \mu \in \mathbb{Z}_+^m \) and \( \nu \in \mathbb{Z}_+^n \). We assume that \( B(\Lambda_\mu) = C(A^0_\mu) \subset \mathcal{F}^m \), \( B(-\Lambda_\nu) = C((A^\circ_\nu)^\vee) \subset (\mathcal{F}^n)^\vee \) and hence

\[
B(\Lambda_\mu) \otimes B(-\Lambda_\nu) \subset \mathcal{F}^m \otimes (\mathcal{F}^n)^\vee \subset M_{[m+n],Z}.
\]

By [13] Proposition 4.5, \( \mathcal{F}^m \otimes (\mathcal{F}^n)^\vee \) is a disjoint union of extremal weight \( \mathfrak{gl}_\infty \)-crystals of level \( m - n \), and hence so is \( B(\Lambda_\mu) \otimes B(-\Lambda_\nu) \).

For \( r \in \mathbb{Z} \), we define \( B^{m \circ r}(\mu, \nu) \) to be the set of \( A = (a_{ij}) \in M_{[m+n],Z} \) such that

\[
a_{ij} = \begin{cases} 1, & \text{for } i \in [n] \text{ and } j \leq r, \\ 0, & \text{for } i \in m + [n] \text{ and } j \leq r. \end{cases}
\]
We have
\[
    B^{>\tau+1}(\mu, \nu) \subset B^{>\tau}(\mu, \nu),
\]
\[
    B(\Lambda_\mu) \otimes B(-\Lambda_\nu) = \bigcup_{r \in \mathbb{Z}} B^{>\tau}(\mu, \nu).
\]

Let \( r \) be such that \( r < \min\{\mu_m, \nu_n\} \) so that \( \mu - (r^m) = (\mu_i - r)_{1 \leq i \leq m} \) and \( \nu - (r^n) = (\nu_i - r)_{1 \leq i \leq n} \) are partitions. Then \( B^{>\tau}(\mu, \nu) \neq \emptyset \) and as a \( \mathfrak{gl}_{>\tau} \)-crystal,
\[
    B^{>\tau}(\mu, \nu) \simeq B^{>\tau}_{(\mu-(r^m)^\prime)} \otimes \left( B^{>\tau}_{(\nu-(r^n)^\prime)} \right)^\vee.
\]

Now, let \( A \in B^{>\tau}(\mu, \nu) \) be given and \( C^{>\tau}(A) \) the connected component in \( B^{>\tau}(\mu, \nu) \) including \( A \) as a \( \mathfrak{gl}_{>\tau} \)-crystal. By Proposition 4.6 we have
\[
    C^{>\tau}(A) \simeq B^{>\tau}_{\sigma, \tau}
\]
for some \( \sigma, \tau \in \mathcal{P} \) with \( \sigma_1 \leq m \) and \( \tau_1 \leq n \). Let \( C(A) \) be the connected component in \( B(\Lambda_\mu) \otimes B(-\Lambda_\nu) \) including \( A \) as a \( \mathfrak{gl}_{\infty} \)-crystal. Then
\[
    C(A) \simeq B_{\xi, \eta} \otimes B(\Lambda_{\xi})
\]
for some \( \xi, \eta \in \mathcal{P} \) and \( \xi \in \mathbb{Z}^{m-n} \) by Theorem 6.1.

**Lemma 6.2.** Under the above hypothesis, we have
\[
    \xi = (\sigma_{m-n+1}', \cdots, \sigma_m'),
\]
\[
    \eta = \tau,
\]
\[
    \xi = (\sigma_1', \cdots, \sigma_{m-n}') + (r^{m-n}).
\]

**Proof.** Let \( A \) be as above. For intervals \( I, J \subset \mathbb{Z} \), let \( A_{I, J} \) denote the \( I \times J \)-submatrix of \( A \). Choose \( s \gg r \) so that
\[
    a_{ij} = \begin{cases} 
          0, & \text{if } i \in [m] \text{ and } j > s, \\
          1, & \text{if } i \in [m+n] \text{ and } j > s.
        \end{cases}
\]

Note that \( A \) is \( [m+n] \)-admissible. Considering \( A_{[m+n],[r+1, s]} \) as an element of a \((\mathfrak{gl}_{[r+1, s]}, \mathfrak{gl}_{[m+n]})\)-bicrystal, \( A \) is connected to a unique matrix \( A' = (a'_{ij}) \) satisfying
\[
    \begin{cases} 
          a'_{ij} = a_{ij}, & \text{for } i \in [m+n] \text{ and } j \notin [r+1, s], \\
          a'_{i-1,j} = 0, & \text{if } a'_{ij} = 0 \text{ for } i \neq 1 \text{ and } j \in [r+1, s], \\
          a'_{i,j+1} = 0, & \text{if } a'_{ij} = 0 \text{ for } i \in [m+n] \text{ and } j+1 \in [r+1, s].
        \end{cases}
\]

Equivalently, \( A' \) is a \( \mathfrak{gl}_{[r+1, s]} \)-highest weight element and a \( \mathfrak{gl}_{[m+n]} \)-lowest weight element. Then we have
\[
    C^{>\tau}(A'_{[m],[r],Z}) \simeq B^{>\tau}_\alpha,
\]
\[
    C^{>\tau}(A'_{m+[n],[r],Z}) \simeq \left( B^{>\tau}_\beta \right)^\vee,
\]
\[
    C^{>\tau}(A'_{m+[n],[r],Z}) \simeq \left( B^{>\tau}_\beta \right)^\vee.
\]
where $\alpha = (\alpha_k)_{k \geq 1}$ and $\beta = (\beta_k)_{k \geq 1} \in \mathcal{P}$ are given by $\alpha_k = \sum_{i=1}^{m} a_i^r + k$ for $1 \leq k \leq s - r$ and $\beta_k = \sum_{i=1}^{n} (1 - a_i^{r+s-k+1})$ for $1 \leq k \leq s - r$. Since $\mathcal{A}_{[m+n],[r+1,\infty)}$ is $\mathfrak{gl}_{\geq r}$-equivalent to $H_{\alpha}^\tau \otimes E_{\beta}^{(s)}(s)$ (see (4.12)), we have $C^{>r}(A') \simeq B_{\alpha,\beta}$ by Theorem 4.1 and hence $(\alpha, \beta) = (\sigma, \tau)$ since $C^{>r}(A') \simeq C^{>r}(A) \simeq B_{\alpha,\beta}^{\geq r}$.

Let $A'' = (a''_{ij}) \in M_{[m+n],[z]}$ be such that \[A''_{[n],z} = A''_{\alpha}(r) \in \mathbb{E}^{n},\]
\[A''_{n+[n],z} = (A''_{\eta}((s)))^\vee \in (\mathbb{E}^{n})^\vee,\]
\[A''_{2n+[m-n],z} = A''_{\xi} \in \mathbb{F}^{m-n},\]
where $\xi = (\sigma_{m-n+1}, \ldots, \sigma_{m})$ and $\eta = \tau'$ and $\xi = (\sigma_1, \ldots, \sigma_{m-n}) + (r_{m-n})$ (see (6.6) and (6.7)). Then for $L \ll 0 \ll L'$, we have \[S_w S_w(A''_{[m+n],[L,L']}) = A''_{[m+n],[L,L']},\]
where
\[w = (r_{n-2} \cdots r_1) \cdots (r_{m+n-2} \cdots r_{m-1}) (r_{m+n-1} \cdots r_{m}),\]
\[w' = (r_{2n} \cdots r_{m+n}) \cdots (r_{n+2} \cdots r_{m+2}) (r_{n+1} \cdots r_{m+1}),\]
and $S_w, S_w'$ are the corresponding operators on a regular $\mathfrak{gl}_{[m+n]}$-crystal $M_{[m+n],[L,L']}$ with respect to $\mathcal{E}_{i}$, $\mathcal{F}_{j}$'s. This implies that $A'$ is $\mathfrak{gl}_{[L,L']}$-equivalent to $A''$. Since $L$ and $L'$ are arbitrary, $A'$ is $\mathfrak{gl}_{\infty}$-equivalent to $A''$. Since \[C(A''_{[2n],z}) \simeq B_{\xi,\eta},\]
\[C(A''_{2n+[m-n],z}) \simeq B(A_{\xi}),\]
we have \[C(A) \simeq C(A') \simeq C(A'') \simeq B_{\xi,\eta} \otimes B(A_{\xi}).\]
This completes the proof. \hfill \Box

For $\xi, \eta \in \mathcal{P}$ and $\xi \in \mathbb{Z}^{m-n}_{+}$, let $m_{(\xi,\eta,\xi)}^{(\mu,\nu)}(r)$ be the number of connected components $C$ in $B(\Lambda_{\mu}) \otimes B(-\Lambda_{\nu})$ such that
(1) $C \cap B^{>r}(\mu, \nu) \neq \emptyset$,
(2) $C \simeq B_{\xi,\eta} \otimes B(A_{\xi}).$

**Corollary 6.3.** Under the above hypothesis,
(1) if $\xi_{m-n} < r$, then $m_{(\xi,\eta,\xi)}^{(\mu,\nu)}(r) = 0$,
(2) if $\xi_{m-n} \geq r$, then
\[m_{(\xi,\eta,\xi)}^{(\mu,\nu)}(r) = c_{(\sigma,\eta)}^{((\mu-(r^m)),(\nu-(r^n)))},\]
where $\sigma = [(\xi - (r^{m-n})) \cup \zeta]$. 

Proof. It follows from (6.8) and Lemma 6.2. □

The following lemma shows that \( m_{(\zeta, \eta, \xi)}(r) \) stabilizes as \( r \to -\infty \).

**Lemma 6.4.** For \( \zeta, \eta \in P \) and \( \xi \in \mathbb{Z}_+^{m-n} \), there exists \( r_0 \in \mathbb{Z} \) such that

\[
m_{(\zeta, \eta, \xi)}(r) = m_{(\zeta, \eta, \xi)}(r_0),
\]

for \( r \leq r_0 \).

**Proof.** For \( r \in \mathbb{Z} \) with \( r \leq \min\{m, n\} \), put

\[
\Theta^{(\mu, \nu)}_{(\zeta, \eta, \xi)}(r) = \bigcup_{\lambda \in \mathcal{P}} \text{LR}_{\sigma \lambda}^{(\mu-(m-n))'} \times \text{LR}_{\eta \lambda}^{(\nu-(n))'},
\]

where \( \sigma = [(\xi - (m-n)) \cup \zeta]'. \) Then

\[
\Theta^{(\mu, \nu)}_{(\zeta, \eta, \xi)}(r-1) = \bigcup_{\lambda \in \mathcal{P}} \text{LR}_{\sigma \lambda \cup \{m\}}^{(\mu-(m-n))'(\nu-(r-n))'} \times \text{LR}_{\eta \lambda \cup \{n\}}^{(\nu-(r-n))'},
\]

where \( \mathfrak{S} = [(\xi - (m-n)) + (1^{m-n})] \cup \zeta' \).

By Proposition 4.6 and Corollary 6.3 we have

\[
\left| \Theta^{(\mu, \nu)}_{(\zeta, \eta, \xi)}(r) \right| = c^{((\mu-(m-n))',(\nu-(n))')}(r) = m_{(\zeta, \eta, \xi)}(r).
\]

For a sufficiently small \( r \), we define a map

\[
\theta_r : \Theta^{(\mu, \nu)}_{(\zeta, \eta, \xi)}(r) \longrightarrow \Theta^{(\mu, \nu)}_{(\zeta, \eta, \xi)}(r-1).
\]

as follows;

**STEP 1.** Suppose that \( S_1 \in \text{LR}_{\sigma \lambda}^{(\mu-(m-n))'} \) is given. Put \( \ell = \xi_{m-n} - r \).

Define \( T_1 \) to be the tableau in \( \text{LR}_{\mathfrak{S} \cup \{m\}}^{(\mu-(m-n))'(\nu-(r-n))'} \), which is obtained from \( S_1 \) as follows;

(1) The entries of \( T_1 \) in the \( i \)-th row \( (1 \leq i \leq \ell) \) is equal to those in \( S_1 \).

(2) The entries of \( T_1 \) in the \( (\ell+1) \)-st row is given by

\[
a_1 + 1 \leq a_2 + 1 \leq \cdots \leq a_n + 1,
\]

where \( a_1 \leq a_2 \leq \cdots \leq a_n \) are the entries in the \( \ell \)th row in \( S_1 \).

(3) Let \( S'_1 \) (resp. \( T'_1 \)) be the subtableau of \( S_1 \) (resp. \( T_1 \)) consisting of its \( i \)-th row for \( \ell < i \) (resp. \( \ell + 1 < i \)). Then we define

\[
T'_1(p+1,q) = \begin{cases} 
S'_1(p,q), & \text{if } S'_1(p,q) \leq a_1, \\
S'_1(p,q) + 1, & \text{if } S'_1(p,q) > a_1,
\end{cases}
\]

for \( (p,q) \) in the shape of \( S_1 \).
Step 2. Let \( S_2 \in \text{LR}^{(\nu-(r^n))'}_{\eta\lambda} \) be given. Applying the same argument as in Step 1 (when \( m = n \)), we obtain \( T_2 \in \text{LR}^{(\nu-(r^n))'/\cup\{n\}}_{\eta\lambda'} \). Now we define

\[
\theta_r(S_1, S_2) = (T_1, T_2) \in \mathcal{O}^{(\mu,\nu)}_{(\xi,\eta,\xi)}(r - 1).
\]

By definition of \( \theta_r \), it is not difficult to see that \( \theta_r \) is one-to-one. Also, we observe that for \( \lambda \in \mathcal{P} \)

\[
\text{LR}^{(\mu-(r^m))'}_{\sigma\lambda} \times \text{LR}^{(\nu-(r^m))'}_{\eta\lambda} \neq \emptyset \iff \text{LR}^{(\mu-(r^m))'/\cup\{n\}}_{\lambda\cup\{n\}} \times \text{LR}^{(\nu-(r^m))'/\cup\{n\}}_{\eta\lambda\cup\{n\}} \neq \emptyset.
\]

If \( r \) is sufficiently small, then we have \( (n) \subset \lambda \) for \( \lambda \in \mathcal{P} \) such that \( \text{LR}^{(\mu-(r^m))'/\cup\{m\}}_{\sigma\lambda} \times \text{LR}^{(\nu-(r^m))'/\cup\{n\}}_{\eta\lambda} \neq \emptyset \), which implies that \( \theta_r \) is onto. Therefore, \( \theta_r \) is a bijection and \( m^{(\mu,\nu)}(r) \) stabilizes as \( r \to -\infty \). \( \square \)

**Theorem 6.5.** Suppose that \( m \geq n \). For \( \mu \in \mathbb{Z}^{\mu}_{+} \) and \( \nu \in \mathbb{Z}^{\nu}_{+} \), we have

\[
\text{B}(\Lambda_{\mu}) \otimes \text{B}(-\Lambda_{\nu}) \simeq \bigsqcup_{\zeta,\eta \in \mathcal{P}} \bigsqcup_{\xi \in \mathbb{Z}^{m-n}_{+}} \text{B}_{\zeta,\eta} \otimes \text{B}(\Lambda_{\xi})^{\oplus m^{(\mu,\nu)}_{(\zeta,\eta,\xi)}}
\]

with

\[
m^{(\mu,\nu)}_{(\zeta,\eta,\xi)} = \sum_{\lambda \in \mathcal{P}} c^{\mu+(k^m)}_{\sigma\lambda} c^{\nu+(k^n)}_{\eta\lambda},
\]

where \( k \) is a sufficiently large integer and \( \sigma = (\xi + (k^m-n)) \cup \zeta' \).

**Proof.** For \( \zeta, \eta \in \mathcal{P} \) and \( \xi \in \mathbb{Z}^{m-n}_{+} \), let \( m^{(\mu,\nu)}_{(\zeta,\eta,\xi)} \) be the number of connected components in \( \text{B}(\Lambda_{\mu}) \otimes \text{B}(-\Lambda_{\nu}) \) isomorphic to \( \text{B}_{\zeta,\eta} \otimes \text{B}(\Lambda_{\xi}) \). Then by Lemma 6.4, we have

\[
m^{(\mu,\nu)}_{(\zeta,\eta,\xi)} = m^{(\mu,\nu)}_{(\zeta,\eta,\xi)}(r)
\]

for some \( r \in \mathbb{Z} \). By Corollary 4.6 and Corollary 6.3, we have

\[
m^{(\mu,\nu)}_{(\zeta,\eta,\xi)} = \sum_{\lambda \in \mathcal{P}} c^{\mu+(k^m)}_{\sigma\lambda} c^{\nu+(k^n)}_{\eta\lambda},
\]

where \( k = -r \) and \( \sigma = (\xi + (k^m-n)) \cup \zeta' \). \( \square \)

The decomposition when \( m \leq n \) can be obtained by taking the dual crystal of the decomposition in Theorem 6.5.

7. **Combinatorial description of the Level zero part of \( \text{B}(\tilde{U}_q(\mathfrak{g}_{1\infty})) \)**

7.1. For \( \mu, \nu \in \mathbb{Z}^{\mu}_{+} \) (\( n \geq 1 \)), let us describe the decomposition of \( \text{B}(\Lambda_{\mu}) \otimes \text{B}(-\Lambda_{\nu}) \) in a bijective way. We assume that \( \text{B}(\Lambda_{\mu}) = C(A^\mu_{\mu}) \subset \mathfrak{f}^m \), \( \text{B}(-\Lambda_{\nu}) = C((A^\nu_{\nu})^\vee) \subset (\mathfrak{f}^m)^\vee \)

Suppose that \( A \in \text{B}(\Lambda_{\mu}) \) and \( A' \in \text{B}(-\Lambda_{\nu}) \) are given. Choose \( r \in \mathbb{Z} \) such that \( A \otimes A' \in \text{B}^{>r}(\mu,\nu) \). Let \( S^{>r} \otimes T^{>r} \in \text{B}^{>r}_{(\mu-(r^m))'} \otimes (\text{B}^{>r}_{(\nu-(r^n))'})^\vee \) correspond to
where \( \psi_{\mu, \nu}^r(A) \) denotes the map in Proposition 7.1 corresponding to \( \mathfrak{gl}_{>r} \)-crystals.

**Proposition 7.1.** For \( \mu, \nu \in \mathbb{Z}_+^n \), the map

\[
\psi_{\mu, \nu}^\infty : B(\Lambda_\mu) \otimes B(-\Lambda_\nu) \rightarrow \bigsqcup_{\alpha, \beta} B_\alpha \otimes B_\beta
\]

is an isomorphism of \( \mathfrak{gl}_{\infty} \)-crystals, where the union is over all skew Young diagrams \( \alpha \) and \( \beta \) such that \( \alpha = (\nu - (r^n))' / \lambda \) and \( \beta = (\mu - (r^n))' / \lambda \) for some \( r \leq \min\{\mu_n, \nu_n\} \) and \( \lambda \in \mathcal{P} \).

**Proof.** It suffices to show that \( \psi_{\mu, \nu}^\infty(A) \) does not depend on the choice of \( r \).

Keeping the above notations, we have

\[
(U^{>r}, V^{>r}) = (0, T^{>r}) \rightarrow (S^{>r}, 0) \in B_{\sigma, r}^{>r},
\]

\[
W^{>r} = (0, T^{>r}) \rightarrow (S^{>r}, 0) \in e_{(\mu, \nu)}^{(\mu, \nu)}.
\]

for some \( \sigma, r \in \mathcal{P} \). By Proposition 4.7, there exist unique \( \tilde{U}^{>r} \in B_{\sigma}^{>r} \) and \( \tilde{V}^{>r} \in (B_{\tau}^{>r})^{\vee} \) such that \( \tilde{V}^{>r} \otimes \tilde{U}^{>r} \equiv (U^{>r}, V^{>r}) \), and by Proposition 4.3

\[
W^{>r} \leftrightarrow (W_1^{>r}, W_2^{>r}) \in LR_{\sigma, \lambda}^{(\mu, \nu)} \times LR_{\tau, \lambda}^{(\nu, \mu)}
\]

for some \( \lambda \in \mathcal{P} \). Then by definition of \( \psi_{\mu, \nu}^r(A) \), we have

\[
\psi_{\mu, \nu}^\infty(A) = Y^{\vee} \otimes X \in B_{(\mu, \nu)}^{(\nu, \mu)} / \lambda \otimes B_{(\nu, \mu)}^{(\mu, \nu)} / \lambda,
\]

where

\[
\mathcal{J}(X) = \tilde{U}^{>r}, \quad \mathcal{J}(X)_R = W_1^{>r},
\]

\[
\mathcal{J}(Y) = \tilde{V}^{>r}, \quad \mathcal{J}(Y)_R = W_2^{>r}.
\]

Now, suppose that

\[
S^{>r-1} \otimes T^{>r-1} \in B_{(\mu, \nu)}^{(\nu, \mu)} \otimes (B_{(\mu, \nu)}^{(\nu, \mu)})^{\vee}
\]

is \( \mathfrak{gl}_{>r-1} \)-equivalent to \( A \otimes A' \). Then

\[
S^{>r-1} = (r - 1 \cdots (r - 1) / S^{>r}, \quad T^{>r-1} = T^{>r} \ast ((r - 1)^{\vee} \cdots (r - 1)^{\vee})
\]

and

\[
((\emptyset, T^{>r-1}) \rightarrow (S^{>r-1}, 0)) = ((\emptyset, T^{>r}) \rightarrow (S^{>r}, 0)) = (U^{>r}, V^{>r}).
\]
This implies that \((U^{>r-1}, V^{>r-1}) = (U^{>r}, V^{>r})\).

Suppose that \(W^{>r} = W_+^{>r} * W_-^{>r}\), where \(W_+^{>r}\) (resp. \(W_-^{>r}\)) is the subtableau of \(W^{>r}\) consisting of positive (resp. negative) entries. By definition of the insertion, it is straightforward to check that

1. \(W_+^{>r-1} = W_+^{>r}\),
2. \(W_-^{>r-1} = (\sigma'_n + 1 \cdots \sigma'_1 + 1) * W_+^{>r}[1],\)

where \(W_+^{>r}[1]\) is the tableau obtained from \(W_+^{>r}\) by increasing each entry by 1. Since \(i(W_-^{>r-1}) = W_+^{>r-1}\), we have

\[
W_+^{>r-1} = \Sigma_n * W_+^{>r}[1],
\]

where \(\Sigma_n\) is the horizontal strip of shape \(\sigma \cup \{n\} / \sigma\) filled with 1, and \(W_+^{>r}[1]\) is the tableau obtained from \(W_+^{>r}\) by increasing each entry by 1. Here, we assume that the shape of \(W_+^{>r}\) is \((\mu - (r^n))' \cup \{n\} / \sigma \cup \{n\}\). Now, we have

\[
\tilde{U}^{>r} * W_+^{>r-1} = \tilde{U}^{>r} * \Sigma_n * W_+^{>r}[1] \\
\sim (1 \cdots 1) * \tilde{U}^{>r} * W_+^{>r}[1] \quad \text{(switching \(\tilde{U}^{>r}\) and \(\Sigma_n\))}
\]

\[
\sim (1 \cdots 1) * H_\lambda[1] * X \quad \text{(switching \(\tilde{U}^{>r}\) and \(W_+^{>r}[1]\))}
\]

\[
= H_{\lambda \cup \{n\}} * X.
\]

This implies that \(X\) does not depend on \(r\). Similarly, we have

\[
W_+^{>r-1} = \Sigma'_n * W_2^{>r}[1],
\]

where \(\Sigma'_n\) is the horizontal strip of shape \(\tau \cup \{n\} / \tau\) filled with 1, and

\[
\tilde{V}^{>r} * W_2^{>r-1} = \tilde{V}^{>r} * \Sigma'_n * W_2^{>r}[1] \\
\sim (1 \cdots 1) * \tilde{V}^{>r} * W_2^{>r}[1] \quad \text{(switching \(\tilde{V}^{>r}\) and \(\Sigma'_n\))}
\]

\[
\sim (1 \cdots 1) * H_\lambda[1] * Y \quad \text{(switching \(\tilde{V}^{>r}\) and \(W_2^{>r}[1]\))}
\]

\[
= H_{\lambda \cup \{n\}} * Y.
\]

This also implies that \(Y\) does not depend on \(r\). Therefore, \(\psi_{\infty}^{\mu,\nu}\) is well-defined.

Since \(\psi_{\infty}^{\mu,\nu}\) is one-to-one and commutes with \(\tilde{e}_i\) and \(\tilde{f}_i\) \((i \in \mathbb{Z})\) by construction, it is an isomorphism of \(\mathfrak{gl}_\infty\)-crystals. \(\square\)
Example 7.2. Let $\mu = (2, 2, 1)$ and $\nu = (3, 2, 1)$. Consider

$$A = \begin{array}{cccccc}
-3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array} \in \mathcal{B}(\Lambda_\mu) \subset \mathfrak{j}^3,$$

$$A' = \begin{array}{cccccc}
-3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array} \in \mathcal{B}(-\Lambda_\nu) \subset (\mathfrak{j}^3)^{\vee},$$

where $\bullet$ and $\cdot$ denote 1 and 0 in matrix, respectively. Then $A \otimes A' \in \mathcal{B}^{>0}(\mu, \nu)$. Suppose that as a $\mathfrak{gl}_{>0}$-crystal element $A$ (resp. $A'$) is equivalent to $S^{>0}$ and $T^{>0}$. Then $S^{>0} = S$ and $T^{>0} = T$, where $S$ and $T$ are tableaux in Example 4.5. Hence, by Example 5.2, we have

$$\psi^{\infty}_{\mu, \nu}(A \otimes A') = \begin{array}{cccc}
4 & 1 \\
2 & 1 \\
1 & 1 \\
\end{array} \otimes \begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}.$$
Let $M$ be the set of $\mathbb{Z} \times \mathbb{Z}$ matrices $A = (a_{ij})$ such that $a_{ij} \in \mathbb{Z}_{\geq 0}$ and $\sum_{i,j \in \mathbb{Z}} a_{ij} < \infty$. Let $A = (a_{ij}) \in M$ be given. As in Section 5.2 we have a $(\mathfrak{gl}_\infty, \mathfrak{gl}_\infty)$-bicrystal structure on $M$ with respect to $\tilde{e}_i, \tilde{f}_i$ and $\tilde{e}_j, \tilde{f}_j$ for $i, j \in \mathbb{Z}$. Now, we put

$$\tilde{M} = M^\vee \times M,$$

$$\tilde{M}_\Lambda = \{ (M^\vee, N) \in \tilde{M} | \text{wt}(N^t) - \text{wt}(M^t) = \Lambda \} \quad (\Lambda \in P_0).$$

Note that $\tilde{M}$ can be viewed as a tensor product of $(\mathfrak{gl}_\infty, \mathfrak{gl}_\infty)$-bicrystals and $\tilde{M}_\Lambda$ is a subcrystal of $\tilde{M}$ with respect to $\tilde{e}_i, \tilde{f}_i$. By Proposition 7.3 we have the following combinatorial realization, which is our second main result. The proof is almost the same as in Theorem 5.3.

**Theorem 7.4.** For $\Lambda \in P_0$, we have

$$\tilde{M}_\Lambda \simeq B(\infty) \otimes T_\Lambda \otimes B(-\infty).$$

Let $B(\widetilde{U}_q(\mathfrak{gl}_\infty))_0 = \bigsqcup_{\Lambda \in P_0} B(\infty) \otimes T_\Lambda \otimes B(-\infty)$ be the level zero part of $B(\widetilde{U}_q(\mathfrak{gl}_\infty))$. Since $\tilde{M} = \bigsqcup_{\Lambda \in P_0} \tilde{M}_\Lambda$ and $M \simeq \bigsqcup_{\mu, \nu \in \mathcal{P}} B_\mu \times B_\nu$ as a $(\mathfrak{gl}_\infty, \mathfrak{gl}_\infty)$-bicrystal, we obtain the following immediately.

**Corollary 7.5.** As a $\mathfrak{gl}_\infty$-crystal, we have

$$B(\widetilde{U}_q(\mathfrak{gl}_\infty))_0 \simeq \tilde{M}.$$  

**Corollary 7.6.** As a $(\mathfrak{gl}_\infty, \mathfrak{gl}_\infty)$-bicrystal, we have

$$B(\widetilde{U}_q(\mathfrak{gl}_\infty))_0 \simeq \bigsqcup_{\mu, \nu} B_{\mu, \nu} \times B_{\mu, \nu}.$$

In [1], Beck and Nakajima proved a Peter-Weyl type decomposition of the level zero part of $B(\widetilde{U}_q(\mathfrak{gl}))$ for a quantum affine algebra $\mathfrak{g}$ of finite rank, where the bicrystal structure is given by star crystal structure, say $\tilde{e}_i^*$ and $\tilde{f}_i^*$, induced from the involution on $\tilde{U}_q(\mathfrak{g})$, usually denoted by $^*$. Based on some computation, we give the following conjecture.

**Conjecture 7.7.** The crystal structure on $B(\widetilde{U}_q(\mathfrak{gl}_{>0}))$ and $B(\widetilde{U}_q(\mathfrak{gl}_\infty))_0$ of type $A_{\infty}$ and $A_{\infty}$ with respect to $\tilde{e}_i^*$ and $\tilde{f}_i^*$ is compatible with the dual of the $^*$-crystal structure with respect to $\tilde{e}_i^*$ and $\tilde{f}_i^*$. That is, $\tilde{e}_i^* = \tilde{f}_i^*$ and $\tilde{f}_i^* = \tilde{e}_i^*$ for all $i$.

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