GEODESIC-EINSTEIN METRICS AND NONLINEAR STABILITIES

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Abstract. In this paper, we introduce notions of nonlinear stabilities for a relative ample line bundle over a holomorphic fibration and define the notion of a geodesic-Einstein metric on this line bundle, which generalize the classical stabilities and Hermitian-Einstein metrics of holomorphic vector bundles. We introduce a Donaldson type functional to show that this functional attains its absolute minimum at geodesic-Einstein metrics, and we also discuss the relationships between the existence of geodesic-Einstein metrics and the nonlinear stabilities of the line bundle. As an application, we will prove that a holomorphic vector bundle admits a Finsler-Einstein metric if and only if it admits a Hermitian-Einstein metric, which answers a problem posed by S. Kobayashi.

Contents

Introduction 1
1. Geodesic-Einstein metrics and a Donaldson type functional 3
1.1. Geodesic-Einstein metrics 3
1.2. A Donaldson type functional 5
2. Geodesic-Einstein metrics and notions of stabilities 11
3. A special fibration: the projective bundles 13
3.1. $L^2$ metrics on direct image bundles. 13
3.2. Projective bundles 17
References 20

Introduction

In this paper, we study the triple $(\mathcal{X}, M, L)$, where $\pi : \mathcal{X} \to M$ is a holomorphic fibration with $\dim M = m$ and $\dim \mathcal{X} = m + n$, and $L \to \mathcal{X}$ is a relative ample line bundle over $\mathcal{X}$, i.e. there exists a metric $\phi$ (more precisely, $e^{-\phi}$ is a metric) on $L$ such that $\sqrt{-1} \partial \bar{\partial} \phi > 0$ fiberwisely. Such a metric $\phi$ is called an admissible metric on $L$. We always assume in this paper that $\mathcal{X}$ is compact and $M$ is a compact Kähler manifold with a fixed Kähler form $\omega$.

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For any admissible metric $\phi$ on $L$, the geodesic curvature $c(\phi)$ of $\phi$, which is a horizontal $(1,1)$ form on $\mathcal{X}$, is defined by (cf. [11], Definition 2.1):

\[ c(\phi) = \left( \phi_{\alpha\bar{\beta}} - \phi_{\alpha l} \phi^{kl} \phi_{k\bar{\beta}} \right) \sqrt{-1} dz^\alpha \wedge d\bar{z}^{\beta}, \]

(0.1)

Here the notations $\phi_{\alpha\bar{\beta}}$, $\phi_{\alpha l}$ and $\phi^{kl}$ are defined in the Section 1.1.

The geodesic curvature $c(\phi)$ plays an important role in many aspects (see, e.g. [1], [11], [22], [26]). For the case of canonically polarized family, i.e., each fiber with $c_1 < 0$, the unique Kähler Einstein metric on the fibers define an metric $\phi$ on the relative canonical bundle $K_{\mathcal{X}/M}$.

By proving the positivity of geodesic curvature $c(\phi)$ along any curve, Schumacher [22] proved that $K_{\mathcal{X}/M}$ is a positive line bundle if the family is nowhere infiniesimally trivial.

When $\dim M = 1$, the equation $c(\phi) = 0$ is equivalent to the famous homogenous Monge-Ampère equation \( (\sqrt{-1} \partial \bar{\partial} \phi)^{n+1} = 0 \) (cf. [9], [23]), which plays a crucial role in a lot of related important problems. Note that in this case, the equation $c(\phi) = 0$ can be also written as $tr_\omega c(\phi) = 0$ for any metric $\omega$ on $M$. Inspired by this, for a general holomorphic fibration $\pi : \mathcal{X} \to M$ over a compact Kähler manifold $\left( M, \omega = \sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^{\beta} \right)$, we introduce the notion of a geodesic-Einstein metric on $L$ with respect to $\omega$.

We say that an admissible metric $\phi$ on $L$ is geodesic-Einstein with respect to $\omega$ if

\[ tr_\omega c(\phi) = \lambda, \]

(0.2)

where $\lambda$ is a constant.

In this paper, we mainly study the relationship between the existence of geodesic-Einstein metrics on $L$ and certain notions of stability of $L$.

In Section 1, we first introduce the following Donaldson type functional $\mathcal{L}$ on the space $F^+(L)$ of admissible metrics on $L$: for any fixed $\psi \in F^+(L)$ and any $\phi \in F^+(L)$

\[ \mathcal{L}(\phi, \psi) = \int_M \left( \frac{\lambda}{m} \mathcal{E}(\phi, \psi) \wedge \omega - \frac{1}{n+1} \mathcal{E}_1(\phi, \psi) \right) \frac{\omega^{m-1}}{(m-1)!}, \]

(0.3)

where $\mathcal{E}$ and $\mathcal{E}_1$ are given by (1.12) and (1.13) in this paper. This functional can be viewed as a generalization of the famous Donaldson functional in the family case. By computing the first variation of the functional (0.3), we can show that the critical points of this functional $\mathcal{L}$ coincide with the geodesic-Einstein metrics on $L$ (see Proposition 1.4). Moreover, by using X. Chen’s geodesic approximation lemma (cf. [5], Lemma 7; also [13], Lemma 2.3), we get the following theorem:

**Theorem 0.1.** The functional $\mathcal{L}(\cdot, \psi)$ attains its absolute minimum at the geodesic-Einstein metrics on $L$.

The famous Donaldson-Uhlenbeck-Yau theorem reveals the deep relationship between the stability of a holomorphic vector bundle and the existence of Hermitian-Einstein metrics (cf. [21], [6], [7], [8], [25]). In Section 2, we introduce the notions of the nonlinear semistability (stability) and the nonlinear polystability associated to a triple $(\mathcal{X}, M, L)$ and discuss the relationships between the existence of geodesic-Einstein metrics on $L$ and these stabilities. We have
Theorem 0.2. For a triple \((X, M, L)\), if \(L\) admits a geodesic-Einstein metric, then the triple \((X, M, L)\) is nonlinear semistable and nonlinear polystable.

To get a full understanding of the relationships among the notions of the geodesic-Einstein metric, nonlinear semistability and the nonlinear polystability would be an interesting problem. For example, we could ask whether there exists a geodesic-Einstein metric for a nonlinear polystable triple \((X, M, L)\).

As an application, in Section 3, we study the special triple \((P(E), M, \mathcal{O}_P(E)(1))\) associated to a holomorphic vector bundle \(E \to M\). By the Kobayashi correspondence (cf. [16], [12], [13]), a Finsler metric \(G\) on \(E\) induces a natural admissible metric on \(\mathcal{O}_P(E)(1)\). In this case, we can prove that the induced metric on \(\mathcal{O}_P(E)(1)\) is geodesic-Einstein if and only if \(G\) is Finsler-Einstein. So for a Finsler-Einstein vector bundle \(E \to M\), we know from Theorem 0.2 that the associated triple \((P(E), M, \mathcal{O}_P(E)(1))\) is nonlinear semistable and nonlinear polystable. Also recall that a Finsler-Einstein vector bundle \(E \to M\) is semistable (cf. [13]). Here a natural question is whether a Finsler-Einstein vector bundle admits a Hermitian-Einstein metric.

Theorem 0.3. \(E\) admits a Finsler-Einstein metric if and only if \(E\) admits a Hermitian-Einstein metric. Therefore, the existence of Finsler-Einstein metrics is equivalent to polystable of the holomorphic vector bundle.

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1. Geodesic-Einstein metrics and a Donaldson type functional

In this section, we first introduce the notion of a geodesic-Einstein metric on \(L\), and then introduce a Donaldson type functional on \(F^+(L)\) and prove that this functional attains its absolute minimum at the geodesic-Einstein metrics on \(L\).

1.1. Geodesic-Einstein metrics. Let \(\pi : \mathcal{X} \to M\) be a holomorphic fibration with compact fibres. Let \(L\) be a relative ample line bundle over \(\mathcal{X}\). As usual, we denote by \((z; v) = (z^1, \cdots, z^m; v^1, \cdots, v^n)\) a local admissible holomorphic coordinate system of \(\mathcal{X}\) with \(\pi(z; v) = z\).

For any smooth function \(\phi\) on \(\mathcal{X}\), we denote

\[
\phi_{\alpha} := \frac{\partial \phi}{\partial z^\alpha}, \quad \phi_{\bar{}\beta} := \frac{\partial \phi}{\partial \bar{z}^{\beta}}, \quad \phi_i := \frac{\partial \phi}{\partial v^i}, \quad \phi_{\bar{}\bar{i}} := \frac{\partial \phi}{\partial \bar{v}^{i}},
\]

where \(1 \leq i, j \leq n, 1 \leq \alpha, \beta \leq m\).
Set
\[ F^+(L) := \{ \phi | \phi \text{ is an admissible metric on } L \}. \]

For any \( \phi \in F^+(L) \), set
\[
\frac{\delta}{\delta z^\alpha} := \frac{\partial}{\partial z^\alpha} - \phi_{\alpha j} \phi^{j k} \frac{\partial}{\partial v^k}.
\]

By a routine computation, one can show that \( \{ \frac{\delta}{\delta z^\alpha} \}_{1 \leq \alpha \leq m} \) spans a well-defined horizontal subbundle of \( TX \). In fact, for any two local admissible coordinate neighborhoods \( \{ U_A, (z_A, v_A) \} \) and \( \{ U_B, (z_B, v_B) \} \), if \( U_A \cap U_B \neq \emptyset \), one has the following holomorphic functions:
\[
z_B = z_B(z_A), \quad v_B = v_B(z_A, v_A).
\]

So
\[
\frac{\partial^2 \phi}{\partial v_A^i \partial v_A^j} = \frac{\partial^2 \phi}{\partial v_B^i \partial v_B^j} + \frac{\partial v_B^i}{\partial v_A^j} \frac{\partial v_B^j}{\partial v_A^i} \frac{\partial^2 \phi}{\partial v_A^k \partial v_A^l}
\]
and
\[
\frac{\partial^2 \phi}{\partial z_A^\alpha \partial v_A^j} = \frac{\partial^2 \phi}{\partial z_B^\gamma \partial v_B^j} + \frac{\partial v_B^j}{\partial z_A^\alpha} \frac{\partial^2 \phi}{\partial z_B^\gamma \partial v_A^j} + \frac{\partial v_B^j}{\partial z_A^\alpha} \frac{\partial^2 \phi}{\partial z_B^\gamma \partial v_B^j} \frac{\partial z_B^\gamma}{\partial z_A^\alpha} \frac{\partial^2 \phi}{\partial v_B^k \partial v_B^l}
\]
From (1.3) and (1.4), one has
\[
\frac{\delta}{\delta z^\alpha_A} = \frac{\partial}{\partial z^\alpha_A} - \frac{\partial^2 \phi}{\partial z_A^\alpha \partial v_A^j} \left( \frac{\partial^2 \phi}{\partial v_A^i \partial v_A^j} \right)^{-1} \frac{\partial}{\partial v_A^i}
\]
\[
= \frac{\partial z_B^\gamma}{\partial z_A^\alpha} \frac{\partial}{\partial z_B^\gamma} + \frac{\partial v_B^i}{\partial z_A^\alpha} \frac{\partial}{\partial v_B^i} - \frac{\partial^2 \phi}{\partial z_A^\alpha \partial v_A^j} \left( \frac{\partial^2 \phi}{\partial v_A^i \partial v_A^j} \right)^{-1} \frac{\partial}{\partial v_A^i} \frac{\partial}{\partial v_B^i}
\]
\[
= \frac{\partial z_B^\gamma}{\partial z_A^\alpha} \frac{\partial}{\partial z_B^\gamma} + \frac{\partial v_B^i}{\partial z_A^\alpha} \frac{\partial}{\partial v_B^i} - \frac{\partial^2 \phi}{\partial z_A^\alpha \partial v_A^j} \left( \frac{\partial^2 \phi}{\partial v_A^i \partial v_A^j} \right)^{-1} \frac{\partial}{\partial v_A^i} \frac{\partial}{\partial v_B^i} \frac{\partial^2 \phi}{\partial z_B^\gamma \partial v_B^i}
\]
\[
= \frac{\partial z_B^\gamma}{\partial z_A^\alpha} \frac{\partial}{\partial z_B^\gamma} - \frac{\partial^2 \phi}{\partial v_B^i \partial v_B^j} \left( \frac{\partial^2 \phi}{\partial v_A^i \partial v_A^j} \right)^{-1} \frac{\partial}{\partial v_A^i} \frac{\partial}{\partial v_B^i} \frac{\partial^2 \phi}{\partial z_B^\gamma \partial v_B^i}
\]
Therefore, \( \{ \frac{\delta}{\delta z^\alpha} \}_{1 \leq \alpha \leq m} \) spans a subbundle of \( TX \).

Let \( \{ dz^\alpha; \delta v^k \} \) denote the dual frame of \( \{ \frac{\delta}{\delta z^\alpha}; \frac{\partial}{\partial v^l} \} \). One has
\[
\delta v^k = dv^k + \phi^{kl} \phi_{l\alpha} dz^\alpha.
\]

Moreover, the differential operators
\[
\partial^V = \frac{\partial}{\partial v^i} \otimes \delta v^i, \quad \partial^H = \frac{\delta}{\delta z^\alpha} \otimes dz^\alpha.
\]
are well-defined.

For any \( \phi \in F^+(L) \), the geodesic curvature \( c(\phi) \) of \( \phi \) (cf. [11], Definition 2.1) is defined by
\[
c(\phi) = \left( \phi_{\alpha \beta} - \phi_{\alpha j} \phi^{j \gamma} \phi_{\gamma \beta} \right) \sqrt{-1} dz^\alpha \wedge dz^\beta.
\]
Proposition 1.3. Let $\omega$ be a geodesic-Einstein metric on $\mathcal{X}$. From the following lemma, one sees that the geodesic curvature $c(\phi)$ of $\phi$ is also well-defined.

Lemma 1.1. The following decomposition holds,
\begin{equation}
\sqrt{-1}\partial\bar{\partial}\phi = c(\phi) + \sqrt{-1}\phi_{ij}\delta v^i \wedge \delta \bar{v}^j.
\end{equation}

Proof. By a direct computation, one has
\[
\begin{align*}
\phi + \sqrt{-1}\phi_{ij}(dv^i + \phi_{j\alpha}dz^\alpha) \wedge (d\bar{v}^j + \phi_{ij\beta}d\bar{z}^\beta) \\
= \sqrt{-1}(\phi_{ij}dz^\alpha \wedge d\bar{z}^\beta + \phi_{i\alpha}dz^\alpha \wedge d\bar{\partial}\phi + \phi_{ij}dv^i \wedge d\bar{v}^j + \phi_{ij\beta}d\bar{z}^\beta)
\end{align*}
\]

$\square$

Definition 1.2. Let $\omega = \sqrt{-1}g_{\alpha\beta}dz^\alpha \wedge d\bar{z}^\beta$ be a (fixed) Kähler metric on $M$. A metric $\phi \in F^+(L)$ is called a geodesic-Einstein metric on $L$ with respect to $\omega$ if it satisfies that
\begin{equation}
tr_\omega c(\phi) := g^{\alpha\beta}(\phi_{\alpha\beta} - \phi_{\alpha}\phi^{\alpha} \phi_{\beta}) = \lambda,
\end{equation}
where $\lambda$ is a constant.

Proposition 1.3. Let $\phi$ be a geodesic-Einstein metric on $L$. Then
\begin{equation}
\lambda = \frac{2\pi m}{n + 1}\frac{\langle \omega \rangle^{m-1}c_1(L)^{n+1}[\mathcal{X}]}{\langle \omega \rangle^{m}c_1(L)^{n}[\mathcal{X}]}.
\end{equation}
So $\lambda$ is a topological quantity depending only on the classes $c_1(L)$ and $[\omega]$.

Proof. Since $\phi \in F^+(L)$ is a geodesic-Einstein metric, one has
\begin{equation}
tr_\omega c(\phi)\omega^m \wedge (\sqrt{-1}\partial\bar{\partial}\phi)_n = mc(\phi) \wedge \omega^{m-1} \wedge (\sqrt{-1}\partial\bar{\partial}\phi)_n
\end{equation}
\begin{equation}
= m\omega^{m-1} \wedge (\sqrt{-1}\partial\bar{\partial}\phi)_{n+1},
\end{equation}
where $(\cdot)_r := (\cdot)^r/r!$. Taking integral over $\mathcal{X}$ to the both sides of (111), one gets
\[
\lambda = \frac{m \int_\mathcal{X} \omega^{m-1} \wedge (\sqrt{-1}\partial\bar{\partial}\phi)_{n+1}}{\int_\mathcal{X} \omega^m \wedge (\sqrt{-1}\partial\bar{\partial}\phi)_n} = \frac{2\pi m}{n + 1}\frac{\langle \omega \rangle^{m-1}c_1(L)^{n+1}[\mathcal{X}]}{\langle \omega \rangle^{m}c_1(L)^{n}[\mathcal{X}]},
\]
which depends only on the classes $[\omega]$ and $c_1(L)$.

1.2. A Donaldson type functional. For any fixed metric $\psi \in F^+(L)$ on $L$ and any $\phi \in F^+(L)$, we define the following two functionals $\mathcal{E}, \mathcal{E}_1$:
\begin{equation}
\mathcal{E}(\phi, \psi) = \frac{1}{n+1} \int_{\mathcal{X}/M} (\phi - \psi) \sum_{k=0}^{n} (\sqrt{-1}\partial\bar{\partial}\phi)^k \wedge (\sqrt{-1}\partial\bar{\partial}\psi)^{n-k}
\end{equation}
and
\begin{equation}
\mathcal{E}_1(\phi, \psi) = \frac{1}{n+2} \int_{\mathcal{X}/M} (\phi - \psi) \sum_{k=0}^{n+1} (\sqrt{-1}\partial\bar{\partial}\phi)^k \wedge (\sqrt{-1}\partial\bar{\partial}\psi)^{n+1-k}.
\end{equation}
Note that $\mathcal{E}(\phi, \psi)$ is a smooth function, while $\mathcal{E}_1(\phi, \psi)$ is a smooth real $(1,1)$-form on $M$. 

Geodesic-Einstein metrics and Stability
Now we introduce the following Donaldson type functional $\mathcal{L}$ on $F^+(L)$ by defining

\begin{equation}
\mathcal{L}(\phi, \psi) = \int_M \left( \frac{\lambda}{m} \mathcal{E}(\phi, \psi) \wedge \omega - \frac{1}{n+1} \mathcal{E}_1(\phi, \psi) \right) \frac{\omega^{m-1}}{(m-1)!},
\end{equation}

where $\lambda$ is the constant given by (1.10).

Given any smooth family $\phi_t$ of admissible metrics on $L$, one has

\begin{align*}
\frac{d}{dt} \mathcal{E}(\phi_t, \psi) &= \frac{1}{n+1} \int_{\mathcal{X}/M} \dot{\phi}_t \sum_{k=0}^n (\sqrt{-1} \partial \bar{\partial} \phi_t)^k \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^{n-k} \\
&\quad + \frac{1}{n+1} \int_{\mathcal{X}/M} (\phi_t - \psi) \sum_{k=0}^n k(\sqrt{-1} \partial \bar{\partial} \phi_t)^{k-1} \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^{n-k} \wedge \sqrt{-1} \partial \bar{\partial} \phi_t \\
&= \frac{1}{n+1} \int_{\mathcal{X}/M} \dot{\phi}_t \sum_{k=0}^n (\sqrt{-1} \partial \bar{\partial} \phi_t)^k \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^{n-k} \\
&\quad + \frac{1}{n+1} \int_{\mathcal{X}/M} \dot{\phi}_t \sum_{k=0}^n k(\sqrt{-1} \partial \bar{\partial} \phi_t)^k \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^{n-k} \\
&\quad - \frac{1}{n+1} \int_{\mathcal{X}/M} \dot{\phi}_t \sum_{k=0}^n k(\sqrt{-1} \partial \bar{\partial} \phi_t)^{k-1} \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^{n-k+1} \\
&= \int_{\mathcal{X}/M} \dot{\phi}_t (\sqrt{-1} \partial \bar{\partial} \phi_t)^n.
\end{align*}

Similarly, one has

\begin{equation}
\frac{d}{dt} \mathcal{E}_1(\phi_t, \psi) \equiv \int_{\mathcal{X}/M} \dot{\phi}_t (\sqrt{-1} \partial \bar{\partial} \phi_t)^{n+1}, \mod \text{Im} \partial + \text{Im} \bar{\partial}.
\end{equation}

Therefore, one gets the following first variation of the Donaldson functional $\mathcal{L}(\cdot, \psi)$,

\begin{align*}
-\frac{d}{dt} \mathcal{L}(\phi_t, \psi) &= \int_{\mathcal{X}} \left( \frac{1}{n+1} \dot{\phi}_t (\sqrt{-1} \partial \bar{\partial} \phi_t)^{n+1} - \frac{\lambda}{m} \dot{\phi}_t (\sqrt{-1} \partial \bar{\partial} \phi_t)^n \wedge \omega \right) \frac{\omega^{m-1}}{(m-1)!} \\
&= \int_{\mathcal{X}} \dot{\phi}_t (c(\phi_t) - \frac{\lambda}{m} \omega) \wedge (\sqrt{-1} \partial \bar{\partial} \phi_t)^n \wedge \frac{\omega^{m-1}}{(m-1)!} \\
&= \int_{\mathcal{X}} \dot{\phi}_t (tr \omega c(\phi_t) - \lambda)(\sqrt{-1} \partial \bar{\partial} \phi_t)^n \wedge \frac{\omega^m}{m!}.
\end{align*}

**Proposition 1.4.** $\phi \in F^+(L)$ is a geodesic-Einstein metric if and only if $\phi$ is a critical point of $\mathcal{L}(\cdot, \psi)$ on $F^+(L)$.

**Proof.** One direction is easy. If $\phi \in F^+(L)$ is a geodesic-Einstein metric, then for any smooth curve $\phi_t \in F^+(L)$ with $\phi_0 = \phi$, one has by (1.15),

\[\frac{d}{dt} \bigg|_{t=0} \mathcal{L}(\phi_t, \psi) = 0.\]

So $\phi$ is a critical point of $\mathcal{L}(\cdot, \psi)$ on $F^+(L)$.

Conversely, if $\phi$ is a critical point of $\mathcal{L}(\cdot, \psi)$, then by taking the following variation

\[\dot{\phi}_t = tr \omega c(\phi_t) - \lambda,
\]
one gets

\[ 0 = \int_{\mathcal{X}} (\text{tr}_\omega c(\phi_t) - \lambda)^2 (\sqrt{-1} \partial \bar{\partial} \phi)^n \wedge \frac{\omega^m}{m!}, \]

and so

\[ \text{tr}_\omega c(\phi) - \lambda = 0, \]

i.e. \( \phi \in F^+(L) \) is a geodesic-Einstein metric.

Consider the following geodesic equation

(1.16) \[ c_t(\phi_t) := \ddot{\phi}_t - |\phi_t|^2_{\phi_t} = 0. \]

We view \( t \) as a complex parameter and then \( \phi_t \) in (1.16) does not depend on the imaginary part of \( t \). Moreover, we have

(1.17) \[ (\sqrt{-1} \partial \bar{\partial}_{X/M, t} \phi)^{n+1} = (n + 1)(\ddot{\phi}_t - |\phi_t|^2_{\phi_t})\sqrt{-1} \partial \bar{\partial}_t \wedge df \wedge (\partial \bar{\partial}_{X/M} \phi)^n = 0. \]

Following X. Chen’s method (cf. [5]), one shows easily that there exists a \( C^{1,1} \) solution for the geodesic curve equation (1.17), and moreover, this \( C^{1,1} \) solution can be approximated by a family of smooth solutions of the following equation

(1.18) \[ (\sqrt{-1} \partial \bar{\partial}_{X/M, t} \phi)^{n+1} = \epsilon(n + 1)(\sqrt{-1} \partial \bar{\partial}_X/M \psi)^n \sqrt{-1} \partial \bar{\partial}_t \wedge df \]

for any \( \epsilon > 0 \) and a fixed metric \( \psi \in F^+(L) \). Furthermore, the following analogue of Lemma 7 in [5] holds:

**Lemma 1.5** (X. Chen, [5], Lemma 7). *The equation (1.18) has a smooth solution \( \phi_{t, \epsilon} \in F^+(L) \) for any small \( \epsilon > 0 \) and any two given initial metrics \( \phi_0, \phi_1 \in F^+(L) \). Moreover, \( \phi_{t, \epsilon} \) converges uniformly to a \( C^{1,1} \) solution \( \phi_t \) of the equation (1.17) as \( \epsilon \to 0 \), and \( \phi_{t, \epsilon} \) has a uniformly bound, i.e. there is a constant \( C \) independent of \( t \) and \( \epsilon \) such that \( |\phi_{t, \epsilon} - \psi| < C \).*

For a family of smooth metrics \( \phi_t \in F^+(L) \), one has

(1.19) \[ \mathcal{L}(\phi_t, \psi) = \int_{\mathcal{X}} \left( \frac{\lambda}{n + 1} (\phi_t - \psi) \sum_{k=0}^n (\sqrt{-1} \partial \bar{\partial}_t \phi_t)^k \wedge (\sqrt{-1} \partial \bar{\partial}_\psi)^{n-k} \wedge \frac{\omega^m}{m!} ight. \]
\[ + \left. \frac{1}{(n + 1)(n + 2)} (\phi_t - \psi) \sum_{k=0}^{n+1} (\sqrt{-1} \partial \bar{\partial}_t \phi_t)^k \wedge (\sqrt{-1} \partial \bar{\partial}_\psi)^{n+1-k} \wedge \frac{\omega^{m-1}}{(m - 1)!} \right), \]

and so

(1.20) \[ \sqrt{-1} \partial \bar{\partial}_t \mathcal{L}(\phi_t, \psi) = \int_{\mathcal{X}} \left( \frac{\lambda}{n + 1} (\sqrt{-1} \partial \bar{\partial}_t \phi_t)^{n+1} \wedge \frac{\omega^m}{m!} - \frac{1}{(n + 1)(n + 2)} (\sqrt{-1} \partial \bar{\partial}_t \phi_t)^{n+2} \wedge \frac{\omega^{m-1}}{(m - 1)!} \right) \]
\[ = \int_{\mathcal{X}} \left( |\partial^H \phi_t|^2_{\phi_t} - |\partial^V \phi_t|^2_{\phi_t} (\text{tr}_\omega c(\phi_t) - \lambda) \right) (\sqrt{-1} \partial \bar{\partial}_t \phi_t)^n \frac{\omega^m}{m!} \sqrt{-1} \partial \bar{\partial}_t \wedge df. \]
Since $\phi_t$ is independent of the imaginary part of $t$, one gets
\begin{equation}
\frac{d^2}{dt^2} \mathcal{L}(\phi_t, \psi) = \int_{\mathcal{X}} \left( \partial^H \phi_t |_\omega^2 - (\bar{\phi}_t - \partial^V \phi_t |_{\phi_t}^2)(tr_\omega c(\phi_t) - \lambda) \right) \left( \sqrt{-1} \partial \bar{\partial} \phi_t \right)^n \frac{\omega^m}{m!} \geq - \int_{\mathcal{X}} \left( (\bar{\phi}_t - \partial^V \phi_t |_{\phi_t}^2)(tr_\omega c(\phi_t) - \lambda) \right) \left( \sqrt{-1} \partial \bar{\partial} \phi_t \right)^n \frac{\omega^m}{m!}.
\end{equation}

**Theorem 1.6.** For any fixed metric $\psi \in F^+(L)$, the functional $\mathcal{L}(\cdot, \psi)$ attains the absolute minimum at geodesic-Einstein metrics.

**Proof.** Assume $\phi_0$ is a geodesic-Einstein metric on $L$. For any $\phi_1 \in F^+(L)$ and any $\epsilon > 0$, by Lemma 1.5, one can connect $\phi_0$ and $\phi_1$ by a path of solutions $\phi_{t, \epsilon}$ of the equation (1.18). From (1.21), one has
\begin{equation}
\frac{d^2}{dt^2} \mathcal{L}(\phi_{t, \epsilon}, \psi) \geq - \epsilon \int_{\mathcal{X}} (tr_\omega c(\phi_{t, \epsilon}) - \lambda) \left( \sqrt{-1} \partial \bar{\partial} \phi_{t, \epsilon} \right)^n \frac{\omega^m}{m!}.
\end{equation}

Let $\{\rho_A\}$ be a partition of unity subordinate to some open covering $\{U_A\}$ of $\mathcal{X}$, where $1 \leq A \leq N$. Let $\eta$ be a local smooth function satisfying $\eta dV = (\sqrt{-1} \partial \bar{\partial} \psi)^n \frac{\omega^m}{m!}$, where $dV = (\sqrt{-1} dz^1 \wedge \bar{dz}^1) \wedge \cdots \wedge (\sqrt{-1} dz^m \wedge \bar{dz}^m) \wedge (\sqrt{-1} du^1 \wedge \bar{du}^1) \wedge \cdots \wedge (\sqrt{-1} du^n \wedge \bar{du}^n)$. Then
\begin{align*}
\int_{\mathcal{X}} tr_\omega c(\phi_{t, \epsilon}) \left( \sqrt{-1} \partial \bar{\partial} \phi_{t, \epsilon} \right)^n \frac{\omega^m}{m!} &= \sum_{A=1}^{N} \int_{U_A} \rho_A tr_\omega c(\phi_{t, \epsilon}) \eta dV \\
&= \sum_{A=1}^{N} \int_{U_A} g^{\alpha\beta} \left( (\phi_{t, \epsilon})_{\alpha\beta} - (\phi_{t, \epsilon})_{\alpha}^i (\phi_{t, \epsilon})_{i\beta} \right) \rho_A \eta dV \\
&\leq \sum_{A=1}^{N} \int_{U_A} g^{\alpha\beta} (\phi_{t, \epsilon})_{\alpha\beta} \rho_A \eta dV = \sum_{A=1}^{N} \int_{U_A} \phi_{t, \epsilon} \partial_\alpha \partial_\beta (g^{\alpha\beta} \rho_A \eta) dV.
\end{align*}

Since $\phi_{t, \epsilon}$ has a uniform bound, there exists a uniform constant $C_0 > 0$ such that
\begin{equation}
\int_{\mathcal{X}} tr_\omega c(\phi_{t, \epsilon}) \left( \sqrt{-1} \partial \bar{\partial} \phi_{t, \epsilon} \right)^n \frac{\omega^m}{m!} \leq C_0.
\end{equation}

Thus by (1.22), one gets
\begin{equation}
\frac{d^2}{dt^2} \mathcal{L}(\phi_{t, \epsilon}, \psi) \geq - \epsilon \left( C_0 - \lambda (2\pi c_1(L)^n |[\omega]_m| [\mathcal{X}] / (m!)) \right) \geq - \epsilon C_1
\end{equation}
for some $C_1 > 0$. Then by the Taylor expansion of $\mathcal{L}(\phi_t, \psi)$, one obtains that
\begin{equation}
\mathcal{L}(\phi_1, \psi) = \mathcal{L}(\phi_0, \psi) + \frac{d}{dt} \big|_{t=0} \mathcal{L}(\phi_{t, \epsilon}, \psi) + \frac{1}{2!} \left( \frac{d^2}{dt^2} \big|_{t=\xi} \mathcal{L}(\phi_{t, \epsilon}, \psi) \right)
\end{equation}
for some $\xi \in [0, 1]$. Since $\phi_0$ is a geodesic-Einstein metric on $L$, so $\frac{d}{dt} \big|_{t=0} \mathcal{L}(\phi_t, \psi) = 0$ for any smooth curve $\phi_t$. Taking $\epsilon \to 0$ to the both sides of (1.23), one has
\begin{equation}
\mathcal{L}(\phi_1, \psi) \geq \mathcal{L}(\phi_0, \psi).
\end{equation}
\qed
Let $M$ be a projective manifold with an ample line bundle $H$. There exists a Hermitian metric $\| \cdot \|_H$ on $H$ such that first Chern form $c_1(H, \| \cdot \|_H)$ of $H$ is positive. Denote $\omega_H = c_1(H, \| \cdot \|_H)$, which is a Kähler metric on $M$. For any closed hypersurface $V$ in $M$ defined by some holomorphic section $s$ of $H$, we have

**Proposition 1.7.** Let $\pi : \mathcal{X} \to M$ be a holomorphic fibration with $\dim M \geq 2$, $L \to \mathcal{X}$ a relative ample line bundle. Then with respect to $\omega_H$, the Donaldson type functional $\mathcal{L}_M(\phi, \psi)$ and $\mathcal{L}_V(\phi, \psi)$ verifies that

(1.25) \[ \mathcal{L}_M(\phi, \psi) \geq \frac{1}{m-1} \mathcal{L}_V(\phi, \psi) - C \max_{p \in M} \int_{\mathcal{X}_p} (\text{tr}_{\omega_H} c(\phi) - \lambda)^2 (\sqrt{-1} \partial \bar{\partial} \phi)^n - C_1 \]

for some constants $C, C_1$.

**Proof.** Note that by Lelong-Poincaré theorem (cf. [14]), one has

$$\frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log \|s\|^2_H = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log \|s\|^2 + c_1(H, \| \cdot \|_H) = -\text{Div}(s) + c_1(H, \| \cdot \|_H).$$

We may and will assume that $\|s\|_H \leq 1$, this can be done by scaling the metric $\| \cdot \|_H$ suitably. Thus

$$\frac{\sqrt{-1}}{2\pi} \int_M \log \|s\|^2_H \wedge \bar{\partial} \partial \eta = \int_V \eta - \int_M c_1(H, \| \cdot \|_H) \wedge \eta$$

for any $(n-1, n-1)$ form $\eta$. Thus, for any $k \geq 1$, one has

(1.26) \[ ([\omega_H]^k c_1(L)^{m+n-k})[\mathcal{X}] = ([\omega_H]^k \pi_* (c_1(L)^{m+n-k}))[M] \]

\[ = ([\omega_H]^{k-1} \pi_* (c_1(L)^{m+n-k}))[V] \]

\[ = ([\omega_H]^{k-1} c_1(L)^{m+n-k})[\mathcal{X}|_V], \]

where $\mathcal{X}|_V := \pi^{-1}(V)$. From (1.26) and (1.10), one knows that $\lambda = \frac{\lambda_{\mathcal{X}|_V}}{m-1}$, where $\lambda_{\mathcal{X}|_V}$ is defined by (1.10) associated to the holomorphic fibration $\mathcal{X}|_V \to V$.

Therefore, with respect to the Kähler metric $\omega_H$, one has

(1.27) \[ (m-1)! \mathcal{L}_M(\phi, \psi) = \int_M \left( \frac{\lambda}{m} \mathcal{E}(\phi, \psi) \wedge \omega_H - \frac{1}{n+1} \mathcal{E}_1(\phi, \psi) \right) \wedge \omega_H^{m-1} \]

\[ = \int_V \left( \frac{\lambda}{m} \mathcal{E}(\phi, \psi) \wedge \omega_H - \frac{1}{n+1} \mathcal{E}_1(\phi, \psi) \right) \wedge \omega_H^{m-2} \]

\[ - \frac{\sqrt{-1}}{2\pi} \int_M \log \|s\|^2_H \bar{\partial} \partial \left( \frac{\lambda}{m} \mathcal{E}(\phi, \psi) \wedge \omega_H - \frac{1}{n+1} \mathcal{E}_1(\phi, \psi) \right) \wedge \omega_H^{m-2} \]

\[ = (m-2)! \mathcal{L}_V(\phi, \psi) - \frac{\sqrt{-1}}{2\pi} \int_M \log \|s\|^2_H \bar{\partial} \partial \left( \frac{\lambda}{m} \mathcal{E}(\phi, \psi) \wedge \omega_H - \frac{1}{n+1} \mathcal{E}_1(\phi, \psi) \right) \wedge \omega_H^{m-2}. \]

Note that

$$\sqrt{-1} \partial \bar{\partial} \mathcal{E}(\phi, \psi) = \frac{1}{n+1} \left( \pi_* ((\sqrt{-1} \partial \bar{\partial} \phi)^{n+1}) - \pi_* ((\sqrt{-1} \partial \bar{\partial} \psi)^{n+1}) \right)$$
and
\[ \sqrt{-1} \partial \bar{\partial} E_1(\phi, \psi) = \frac{1}{n+2} \left( \pi^*_n((\sqrt{-1} \partial \bar{\partial} \phi)^{n+2}) - \pi^*_n((\sqrt{-1} \partial \bar{\partial} \psi)^{n+2}) \right). \]
Therefore, the second term on the right hand side of (1.27) amounts to (1.28)
\[
\int_M \log \|s\|_H^2 \sqrt{-1} \partial \bar{\partial} \left( \frac{\lambda}{m} E(\phi, \psi) \land \omega_H - \frac{1}{n+1} E_1(\phi, \psi) \right) \land \omega_H^{m-2} = \frac{1}{n+1} \int_M \log \|s\|_H^2 \pi^*_n \left( \frac{\lambda}{m} (\sqrt{-1} \partial \bar{\partial} \phi)^{n+1} \land \omega_H^{m-1} - \frac{1}{n+2} (\sqrt{-1} \partial \bar{\partial} \phi)^{n+2} \land \omega_H^{m-2} \right) - \frac{1}{n+1} \int_M \log \|s\|_H^2 \pi^*_n \left( \frac{\lambda}{m} (\sqrt{-1} \partial \bar{\partial} \psi)^{n+1} \land \omega_H^{m-1} - \frac{1}{n+2} (\sqrt{-1} \partial \bar{\partial} \psi)^{n+2} \land \omega_H^{m-2} \right).
\]
Since
\[ \pi^*_n \left( \frac{\lambda}{m} (\sqrt{-1} \partial \bar{\partial} \phi)^{n+1} \land \omega_H^{m-1} \right) = (n+1) \frac{\lambda}{m^2} tr_{\omega_H} c(\phi) (\sqrt{-1} \partial \bar{\partial} \phi)^n \land \omega_H^m, \]
and
\[ \pi^*_n \left( \frac{1}{n+2} (\sqrt{-1} \partial \bar{\partial} \phi)^{n+2} \land \omega_H^{m-2} \right) = \frac{n+1}{2m(m-1)} ((tr_{\omega_H} c(\phi))^2 - |c(\phi)|_{\omega_H}^2) (\sqrt{-1} \partial \bar{\partial} \phi)^n, \]
one gets (1.29)
\[
\frac{1}{n+1} \pi^*_n \left( \frac{\lambda}{m} (\sqrt{-1} \partial \bar{\partial} \phi)^{n+1} \land \omega_H^{m-1} - \frac{1}{n+2} (\sqrt{-1} \partial \bar{\partial} \phi)^{n+2} \land \omega_H^{m-2} \right) / \omega_H^m = \pi^*_n \left( \frac{\lambda}{m^2} tr_{\omega_H} c(\phi) - \frac{1}{2m(m-1)} ((tr_{\omega_H} c(\phi))^2 - |c(\phi)|_{\omega_H}^2) (\sqrt{-1} \partial \bar{\partial} \phi)^n \right) = \pi^*_n \left( \frac{1}{2m^2} (tr_{\omega_H} c(\phi) - \lambda)^2 + \frac{1}{2m^2} (m|c(\phi)|_{\omega_H}^2 - (tr_{\omega_H} c(\phi))^2) + \frac{\lambda^2}{2m^2} \right) (\sqrt{-1} \partial \bar{\partial} \phi)^n \geq - \frac{1}{2m^2} \pi^*_n \left( (tr_{\omega_H} c(\phi) - \lambda)^2 (\sqrt{-1} \partial \bar{\partial} \phi)^n \right),
\]
where the last inequality holds by the mean inequality. From (1.28) and (1.29), one has
\[
\int_M \log \|s\|_H^2 \sqrt{-1} \partial \bar{\partial} \left( \frac{\lambda}{m} E(\phi, \psi) \land \omega_H - \frac{1}{n+1} E_1(\phi, \psi) \right) \land \omega_H^{m-2} \leq \left( \int_M \left( - \frac{1}{2m^2} \log \|s\|_H^2 \omega_H^m \right) \max_{p \in M} \int_{X_p} (tr_{\omega_H} c(\phi) - \lambda)^2 (\sqrt{-1} \partial \bar{\partial} \phi)^n \right) + \frac{1}{n+1} \int_M - \log \|s\|_H^2 \pi^*_n \left( \frac{\lambda}{m} (\sqrt{-1} \partial \bar{\partial} \phi)^{n+1} \land \omega_H^{m-1} - \frac{1}{n+2} (\sqrt{-1} \partial \bar{\partial} \psi)^{n+2} \land \omega_H^{m-2} \right).
\]
By (1.27), it follows that
\[ L_M(\phi, \psi) \geq \frac{1}{m-1} L_V(\phi, \psi) - C \max_{p \in M} \int_{X_p} (tr_{\omega_H} c(\phi) - \lambda)^2 (\sqrt{-1} \partial \bar{\partial} \phi)^n - C_1, \]
where
\[ C = \frac{1}{4\pi m(m!)} \int_M (\log \|s\|_H^2) \omega_H^m. \]
and
\[ C_1 = \frac{1}{2\pi(n+1)((m-1)!)} \int_M -\log \|s\|^2_H \pi^* \left( \frac{\lambda}{m} (\sqrt{-1} \partial \bar{\partial} \psi)^{n+1} \wedge \omega_H^{m-1} - \frac{1}{n+2} (\sqrt{-1} \partial \bar{\partial} \psi)^{n+2} \wedge \omega_H^{m-2} \right). \]

\[ \square \]

2. GEODESIC-EINSTEIN METRICS AND NOTIONS OF STABILITIES

In this section, we introduce certain stabilities of a triple \((\mathcal{X}, M, L)\) and discuss the relationships between geodesic-Einstein metrics and these stabilities.

A fibration \(Y \to M - S\), with \(S\) a closed subvariety in \(M\) of codim \(S \geq 2\), is called a subfibration of the holomorphic fibration \(\mathcal{X} \to M\) if for any \(p \in M - S\), the fiber \(Y_p\) is a complex closed submanifold of the fiber \(\mathcal{X}_p\). Let \(\mathcal{F}\) be the set of sub-fibrations of the holomorphic fibration \(\mathcal{X} \to M\). Set for any \(Y \in \mathcal{F}\),

\[ \lambda_{Y,L} = \frac{2\pi m}{\dim Y/M + 1} \left( [\omega]^{m-1} c_1(L)^{\dim Y/M+1} [Y] \right) \left( [\omega]^{m} c_1(L)^{\dim Y/M} [Y] \right). \]

(2.1)

Note that \(\lambda_{Y,L}\) is well defined and independent of the metrics on \(L\) by the Stoke’s theorem and codim \(S \geq 2\).

Now we introduce some notions of the stability of a triple \((\mathcal{X}, M, L)\).

Definition 2.1. A triple \((\mathcal{X}, M, L)\) is called nonlinear semistable (resp. nonlinear stable) if \(\lambda_{Y,L} \geq \lambda_{X,L}\) (resp. \(\lambda_{Y,L} > \lambda_{X,L}\)) for any sub-fibration \(Y \in \mathcal{F}\) with \(\dim Y < \dim \mathcal{X}\).

We have the following theorem.

Theorem 2.2. If \(L\) admits a geodesic-Einstein metric, then the triple \((\mathcal{X}, M, L)\) is nonlinear semistable.

Proof. For any sub-fibration \(Y \in \mathcal{F}\) with \(\dim Y/M = n'\). We make the following convention for indices:

\[ 1 \leq A, B, C, D \leq n, \quad 1 \leq i, j, k, l \leq n', \]

and use the local admissible coordinate systems \((v^1, \ldots, v^{n'}, \ldots, v^n)\) on the fibres of \(\mathcal{X}\) such that \((v^1, \ldots, v^{n'}, 0, \ldots, 0) \in \mathcal{Y}\). It is clear that such local admissible coordinate systems are always exist.

For any \(\phi \in F^+(L)\), let \(\phi|_Y\) be the restriction of \(\phi\) on the line bundle \(L|_Y\) over the sub-fibration \(Y \to M - S\). Let \(c(\phi|_Y)\) denote the geodesic curvature of \(\phi|_Y\) which is defined similarly to (1.7). By Lemma 2.4, which will be proved later, one has

\[ c(\phi)|_Y - c(\phi|_Y) = \left( (\phi_{\alpha\beta} - \phi_{\alpha\beta}^A \phi_{A\beta}^B) \right) \sqrt{-1} dz^\alpha \wedge d\bar{z}^\beta \]

(2.2)

\[ = (\phi_{\alpha\beta} \phi_{\topi} - \phi_{\alpha\beta} \phi_{A\beta}^B ) \sqrt{-1} dz^\alpha \wedge d\bar{z}^\beta \leq 0, \]

which will be proved later, one has
where the last inequality holds by setting $B = (\phi_{ij}), A_2 = (\phi_{\alpha j}), D = (\phi_{AB}), E = (\phi_{\alpha B})$ in the following Lemma 2.4. Thus, when $\phi$ is geodesic-Einstein, one gets

$$
\lambda_{Y, L} = \frac{2\pi m}{n' + 1} \left( \frac{[\omega^{m-1}c_1(L)^{n'+1}][Y]}{[\omega^m c_1(L)^n][Y]} \right)
\geq \frac{m}{n' + 1} \left( \frac{[\omega^{m-1}(\sqrt{1 - \partial\bar{\partial}(\phi|_Y)})^{n'+1}][Y]}{[\omega^m (\sqrt{1 - \partial\bar{\partial}(\phi|_Y)})^n][Y]} \right)
\geq \frac{(tr\omega c(\phi))_m^m}{[\omega^m (\sqrt{1 - \partial\bar{\partial}(\phi|_Y)})^n][Y]}
= tr\omega c(\phi) = \lambda_{X, L}.
$$

(2.3)

The proof is complete. □

Remark 2.3. Note that from the inequalities in (2.3), we find that if $\phi$ is a geodesic-Einstein metric on $L$, then for any $Y \in \mathcal{I}$ with $\lambda_{Y, L} = \lambda_{X, L}$, the restriction metric $\phi|_Y$ satisfies $c(\phi|_Y) = c(\phi)|_Y$. Moreover, $\phi|_Y$ is also geodesic-Einstein on $L|_Y$, that is, $tr\omega c(\phi|_Y) = \lambda_{Y, L}$.

Lemma 2.4. Consider the following block matrix with $A = \tilde{A}^\top$,

$$
A := \begin{bmatrix}
A_1 & A_2 & A_3 \\
A_2 & B_2 & B_3 \\
A_3 & \tilde{B}_3^\top & C_3
\end{bmatrix} = \begin{bmatrix}
A_1 & E \\
E^\top & D
\end{bmatrix},
$$

where $D$ is a positive definite matrix, then $ED^{-1}E^\top - A_2B_2^{-1}\tilde{A}_2^\top$ is a semipositive definite matrix.

Proof. Since $D$ is a positive definite matrix, $B_2$ is also a positive definite matrix. By a direct computation, one has

$$
\begin{bmatrix}
I & 0 \\
-B_2 & B_3^{-1}B_3
\end{bmatrix} = \begin{bmatrix}
B_2 & 0 \\
0 & C_3 - B_3^{-1}B_3
\end{bmatrix}.
$$

So $C_3 - \tilde{B}_3^\top B_2^{-1}B_3$ is a positive definite matrix and

$$
D^{-1} = \begin{bmatrix}
I & 0 \\
0 & B_3^{-1}B_3
\end{bmatrix} \begin{bmatrix}
C_3 - B_3^{-1}B_3
\end{bmatrix}^{-1} \begin{bmatrix}
I & 0 \\
-B_2 & B_2^{-1}B_2
\end{bmatrix}.
$$

Therefore,

$$
ED^{-1}E^\top = [A_2, A_3] \begin{bmatrix}
I & 0 \\
0 & B_3^{-1}B_3
\end{bmatrix} \begin{bmatrix}
0 \\
C_3 - B_3^{-1}B_3
\end{bmatrix}^{-1} \begin{bmatrix}
I & 0 \\
-B_3 & B_3^{-1}B_3
\end{bmatrix} [A_2, A_3]^\top
= A_2B_2^{-1}A_2^\top + (A_3 - A_2B_2^{-1}B_3)(C_3 - \tilde{B}_3^\top B_2^{-1}B_3)^{-1}(A_3 - A_2B_2^{-1}B_3)^T.
$$

So

(2.4) $ED^{-1}E^\top - A_2B_2^{-1}\tilde{A}_2^\top = (A_3 - A_2B_2^{-1}B_3)(C_3 - \tilde{B}_3^\top B_2^{-1}B_3)^{-1}(A_3 - A_2B_2^{-1}B_3)^T$

is a semipositive definite matrix. □
Definition 2.5. The triple \((X, M, L)\) is called nonlinear polystable if there exists a filtration
\[(2.5)\]
\[X_0 := X \supset X_1 \supset \cdots \supset X_N,\]
where \((X_i, S_i, L)\) is a maximal nonlinear semistable sub-fibration of \((X_{i-1}, S_{i-1}, L)\) with \(\lambda_{X_i, L} = \lambda_{X_{i-1}, L}\) for \(1 \leq i \leq N\), and \((X_N, M - S_N, L)\) is also nonlinear stable, and moreover, there exists a family of metrics \(\phi_{i-1} \in F^+(L|_{X_{i-1}})\) such that \(c(\phi_{i-1})|_{X_i} = c(\phi_{i-1}|_{X_i})\).

Clearly, a nonlinear stable triple \((X, M, L)\) must be nonlinear polystable. Moreover, we have

Theorem 2.6. If \(L\) admits a geodesic-Einstein metric, then the triple \((X, M, L)\) is nonlinear polystable.

Proof. Since \(L\) admits a geodesic-Einstein metric, so by Theorem 2.2, it is nonlinear semistable. If \(\lambda_{Y, L} > \lambda_{X, L}\) for all \(Y \in \mathcal{F}\), then the triple \((X, M, L)\) is nonlinear stable and so it is nonlinear polystable; otherwise, there exists a maximal sub-fibration \(X_1 \subset X\) such that \(\lambda_{X_1, L} = \lambda_{X, L}\). Thus, by Remark 2.3 one has
\[c(\phi)|_{X_1} = c(\phi|_{X_1}).\]

Now by induction, the proof is complete. \(\square\)

3. A special fibration: the projective bundles

The curvature of direct image sheaf has been computed in (\([2, 3, 4, 20]\)). In this section, we will first review some works of Berndtsson on \(L^2\) metrics and then discuss the relationships of the geodesic-Einstein metrics on a relative ample bundle \(L\) over a holomorphic fibration and the \(L^2\) metrics on the related direct image bundles. Finally, we study the special holomorphic fibration \(\pi : P(E) := (E - \{0\})/\mathbb{C}^* \to M\) associated to a holomorphic vector bundle \(E \to (M, \omega)\).

3.1. \(L^2\) metrics on direct image bundles. For any admissible metric \(\phi\) on a relative ample line bundle \(L\) over a holomorphic fibration \(\pi : \mathcal{X} \to M\), we consider the direct image sheaf \(E := \pi_*(K_{\mathcal{X}/M} + L)\). Then \(E\) is a holomorphic vector bundle. In fact, for any point \(p \in M\), taking a local coordinate neighborhood \((U; \{z^\alpha\})\) of \(p\), then \(\phi + \beta \sum_{\alpha=1}^{m} |z^\alpha|^2\) is a metric on the line bundle \(L \to \mathcal{X}|_U\), whose curvature is
\[\sqrt{-1}\partial\bar{\partial}\phi + \beta \sqrt{-1} \sum_{\alpha=1}^{m} dz^\alpha \wedge d\bar{z}^\beta.\]

By taking \(\beta\) large enough, the curvature of \(\phi + \beta \sum_{\alpha=1}^{m} |z^\alpha|^2\) is positive. The same argument as in \([2, \S 4, \text{page 542}]\), there exists a local holomorphic frame for \(E\). So \(E\) is a holomorphic vector bundle.

Following Berndtsson (cf. \([2, 3, 4]\)), we define the following \(L^2\) metric on the direct image bundle \(E := \pi_*(K_{\mathcal{X}/M} + L)\): for any \(u \in E_p \equiv H^0(\mathcal{X}_p, (L + K_{\mathcal{X}/M})_p), \ p \in M\), then we define
\[(3.1) \quad ||u||^2 = \int_{\mathcal{X}_p} |u|^2 e^{-\phi}.\]
Note that $u$ can be written locally as $u = f dv \wedge e$, where $e$ is a local holomorphic frame for $L|_{\mathcal{X}}$, and so locally
\[ |u|^2 e^{-\phi} = (\sqrt{-1})^{n^2} |f|^2 |e|^2 dv \wedge d\bar{v} = (\sqrt{-1})^{n^2} |f|^2 e^{-\phi} dv \wedge d\bar{v}, \]
where $dv = dv^1 \wedge \cdots \wedge dv^n$ is the fiber volume.

The following theorem actually was proved by Berndtsson in [4] Theorem 1.2], here we will give a proof for reader’s convenience.

**Theorem 3.1** ([4]). For any $p \in M$ and Let $u \in E_p$, one has
\[ (\sqrt{-1} \Theta^{\bar{E}} u, u) = \int_{\pi^{-1}(p)} c(\phi) |u|^2 e^{-\phi} + \langle (1 + \Box')^{-1} i\bar{\partial}v \pi^* u, i\bar{\partial}v \pi^* u \rangle \sqrt{-1} dz^\alpha \wedge d\bar{z}^\beta, \]
where $\Theta^{\bar{E}}$ denotes the curvature of the Chern connection on $E$ with the $L^{2}$ metric defined above, here $\Box' = \nabla^* \nabla + \nabla^* \nabla$ is the Laplacian on $L|_{\pi^{-1}(p)}$-valued forms on $\pi^{-1}(p)$ defined by the $(1,0)$-part of the Chern connection on $L|_{\pi^{-1}(p)}$.

**Proof.** For any local holomorphic section $u$ of $E$, following Berndtsson,
\[ \bar{\partial} u = dz^\alpha \wedge \eta_\alpha, \quad D' u = (\Pi_{\text{holo}}) u dz^\alpha, \]
where $\bar{\partial} u = dz^\alpha \wedge \mu_\alpha$, $\bar{\partial} \phi = e^\phi \bar{\partial} e^{-\phi}$, $\Pi_{\text{holo}}$ is the projection on the space of holomorphic sections.

By a direct computation, we have
\[ \sqrt{-1} \bar{\partial} \bar{\partial} |u|^2 = \sqrt{-1} \bar{\partial} \bar{\partial} \pi_*(\langle \sqrt{-1} \rangle^{n^2} u \wedge \bar{u} e^{-\phi} ) = -\pi_*(\bar{\partial} \bar{\partial} \phi \langle \sqrt{-1} \rangle^{n^2} u \wedge \bar{u} e^{-\phi} ) + \pi_*(\langle \sqrt{-1} \rangle^{n^2} \mu_\alpha \wedge \bar{\mu}_\beta e^{-\phi} ) \sqrt{-1} dz^\alpha \wedge d\bar{z}^\beta + \pi_*(\langle \sqrt{-1} \rangle^{n^2} \eta_\alpha \wedge \bar{\eta}_\beta e^{-\phi} ) \sqrt{-1} dz^\alpha \wedge d\bar{z}^\beta. \]

For any element $u$ of $E$, we can take a canonical representation $u$ of $u$, $u = u' \delta v \otimes e$, where $u'$ is a local smooth function and holomorphic when restricting on each fiber, $\delta v = \delta v^1 \wedge \cdots \wedge \delta v^n$ and $e$ is a local frame of $L$. Then
\[ u = e^{iN}(u^0), \]
where $N = \phi^i \phi_0^j dz^\alpha \otimes \frac{\partial}{\partial z^i}$, $e^{iN} = \sum_{k=0}^{\infty} \frac{i^k}{k!} u^0 = u' dv \otimes e$. So
\[ \bar{\partial} u = \bar{\partial} e^{iN}(u^0) = e^{iN}(\bar{\partial} u^0 + i\partial_N u^0). \]
Comparing with (3.3), one has
\[ \bar{\partial} u = e^{iN}(i\partial_N u^0) = -dz^\alpha \wedge e^{iN}(i\partial_{\pi^*} u^0), \]
so $\eta_\alpha = -e^{iN}(i\partial_{\pi^*} u^0)$.

When restricting on each fiber, $\eta_\alpha = i\bar{\partial}v \frac{\delta}{\delta z^i} u^0$ and so
\[ \eta \wedge \bar{\partial} \bar{\partial} \phi = i\bar{\partial}v \frac{\delta}{\delta z^i} u^0 \wedge \bar{\partial} \bar{\partial} \phi = -u^0 \wedge \bar{\partial} \bar{\partial} \phi = 0, \]
i.e. $\eta$ is primitive on each fiber. It follows that
\[ \pi_*(\langle \sqrt{-1} \rangle^{n^2} \eta_\alpha \wedge \bar{\pi}_\beta e^{-\phi} ) \sqrt{-1} dz^\alpha \wedge d\bar{z}^\beta = -\langle \eta_\alpha, \eta_\beta \rangle, \]
where $(\langle \cdot , \cdot \rangle)$ denotes the Hermitian inner induced by the Hermitian line bundle $(L, \phi)$ and $(\pi^{-1}(p), \partial \bar{\partial} \phi|_{\pi^{-1}(p)})$.

Moreover, if the holomorphic section $u$ of $E$ satisfies $D' u = 0$ at the point $p$, that is, $\mu_\alpha$ is orthogonal to holomorphic forms. When restricting on the center fiber $\pi^{-1}(p)$,

\begin{equation}
\begin{aligned}
dz^\alpha \wedge \bar{\partial} \mu_\alpha &= -\bar{\partial} \bar{\partial} \phi u = \partial \bar{\partial} u = -dz^\alpha \wedge \bar{\partial} \eta_\alpha = -dz^\alpha \wedge \nabla' \eta_\alpha,
\end{aligned}
\end{equation}

so $\bar{\partial} \mu_\alpha = -\nabla' \eta_\alpha$. Since $\mu_\alpha$ is the $L^2$ minimal solution, so

\begin{equation}
\mu_\alpha = -\bar{\partial} (\bar{\partial}'')^{-1} \nabla' \eta_\alpha.
\end{equation}

Then

\begin{equation}
\langle \eta_\alpha, \eta_\beta \rangle - \langle \mu_\alpha, \mu_\beta \rangle = \langle \eta_\alpha, \eta_\beta \rangle - \langle -\bar{\partial} (\bar{\partial}'')^{-1} \nabla' \eta_\alpha, \mu_\beta \rangle = \langle \eta_\alpha, \eta_\beta \rangle - \langle (\bar{\partial}' + 1)^{-1} \nabla' \eta_\alpha, \nabla' \eta_\beta \rangle.
\end{equation}

\begin{equation}
= \langle \eta_\alpha, \eta_\beta \rangle - \langle (\bar{\partial}' + 1)^{-1} \nabla' \eta_\alpha, \eta_\beta \rangle = \langle (\bar{\partial}' + 1)^{-1} \eta_\alpha, \eta_\beta \rangle.
\end{equation}

On the other hand, by Lemma 1.1 we have

\begin{equation}
\sqrt{-1} \pi_*(\partial \bar{\partial} \phi (\sqrt{-1})^n u \wedge u e^{-\phi}) = \int_{\pi^{-1}(p)} c(\phi) |u|^2 e^{-\phi}.
\end{equation}

For any element $u$ of $E_p$, one can take a local holomorphic extension of $u$ and such that $D' u = 0$ at $p$. From (3.4), (3.9), (3.12) and (3.13), one has at the point $p$,

\begin{equation}
\langle \sqrt{-1} \Theta^E u, u \rangle = -\sqrt{-1} \bar{\partial} \|u\|^2
\end{equation}

\begin{equation}
= \int_{\pi^{-1}(p)} c(\phi) |u|^2 e^{-\phi} + \langle (1 + \bar{\partial}')^{-1} i \bar{\partial} \bar{\partial} \frac{s}{2} u, i \bar{\partial} \bar{\partial} \frac{s}{2} u \rangle \sqrt{-1} d\zeta^\alpha \wedge \bar{d} \zeta^\beta,
\end{equation}

which completes the proof. \hfill $\square$

Taking trace to both sides of (3.2) and by the semipositivity of the second term in the right hand side of (3.4), one has

\begin{equation}
g^{\alpha \beta} (\Theta^E u, u) \geq \int_{\pi^{-1}(p)} \text{tr}_\omega c(\phi) |u|^2 e^{-\phi}.
\end{equation}

From this we have the following proposition.

**Proposition 3.2.** If $L$ admits a geodesic-Einstein metric $\phi$, then one has

\begin{equation}
\frac{\deg E}{\text{rank} E} \geq \frac{\text{Vol}_\omega(M)}{2 \pi m} \lambda_{\chi, L},
\end{equation}

where $\deg E := (c_1(E) \wedge [\omega]^{m-1})[M], \text{Vol}_\omega(M) = \omega^m [M]$.

**Proof.** Let $\{u_A\}, 1 \leq A \leq \text{rank} E$, be a local holomorphic frame of $E$, and $h$ denote the $L^2$ metric on $E$. Set

\begin{equation}
h_{AB} = \langle u_A, u_B \rangle = \int_{\chi^p} u_A u_B e^{-\phi}.
\end{equation}

Since $\phi$ is geodesic-Einstein, then from (3.15), one has for any $u = \sum_{A=1}^{\text{rank} E} a^A u_A \in E$,

\begin{equation}
g^{\alpha \beta} \Theta_{\alpha \beta A B} a^A a^B \geq \lambda_{\chi, L} h_{AB} a^A a^B,
\end{equation}

Geodesic-Einstein metrics and Stability
and then by taking trace with respect to the $L^2$ metric $h$, one gets
\[ g^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}} := g^{\alpha\bar{\beta}} \Theta_{\alpha\bar{\beta} AB} h^{AB} \geq \lambda_{\chi, L} \cdot \text{rank} E. \]

Integrating both sides of the above inequality over $M$, one has
\[ \lambda_{\chi, L} \leq \frac{\deg E}{\text{rank} E \cdot \text{Vol}_\omega(M)}, \]
from which the proposition follows. \( \square \)

From Proposition 3.2, we have the following corollary.

**Corollary 3.3.** If $\phi$ is a geodesic-Einstein metric on $L$, then
\[ \deg E \cdot \text{rank} E = \text{Vol}_\omega(M) \cdot 2\pi m \cdot \lambda_{\chi, L}, \tag{3.17} \]
if and only if the induced $L^2$ metric (3.1) is a Hermitian-Einstein metric on $E = \pi_* (L + K_{\chi/M})$.

**Proof.** If (3.17) holds, then all equalities in the proof of Proposition 3.2 hold. In particular, one has
\[ g^{\alpha\bar{\beta}} \Theta_{\alpha\bar{\beta} AB} = \lambda h^{AB}, \tag{3.18} \]
that is, the induced $L^2$ metric (3.1) on $E$ is a Hermitian-Einstein metric. The converse part is obvious. \( \square \)

Now we consider the asymptotic behavior of the corresponding quantities $\frac{\deg E^{(k)}}{\text{rank} E^{(k)}}$ when replacing $L$ by $L^k$ as $k \to \infty$. Firstly, by Bergman Kernel expansion, one has
\[ \text{rank}(\pi_* (L^k + K_{\chi/M})) = \dim H^0(\chi_y, L^k + K_{\chi/M}|_y) = b_0 k^n + b_1 k^{n-1} + O(k^{n-2}), \tag{3.19} \]
where
\[ b_0 := \pi_* \left( \frac{c_1(L)^n}{n!} \right), \quad b_1 = \pi_* \left( \frac{c_1(L)^{n-1} c_1(K_{\chi/M})}{2(n-1)!} \right). \tag{3.20} \]

One the other hand, one has for any complex vector bundle $E$,
\[ ch(E) = \text{rank} E + c_1 + \frac{1}{2} (c_1^2 - 2c_2) + \frac{1}{6} (c_1^3 - 3c_1 c_2 + c_3) + \cdots \]
and
\[ Td(E) = 1 + \frac{1}{2} c_1 + \frac{1}{12} (c_1^2 + c_2) + \frac{1}{24} c_1 c_2 + \cdots. \]
So from the Grothendieck-Riemann-Roch Theorem and Kodaira vanishing theorem, one gets
\[ c_1(\pi_*(L^k + K_{\chi/M})) = c_1(\pi_!(L^k + K_{\chi/M})) \]
\[ = \left\{ \pi_*(ch(L^k + K_{\chi/M}) Td(T_{\chi/M})) \right\}^{(1,1)} \]
\[ = \pi_* \left( \frac{c_1(L)^{n+1}}{(n+1)!} \right) k^{n+1} + \pi_* \left( \frac{1}{2} \frac{c_1(L)^n}{n!} c_1(K_{\chi/M}) \right) k^n + O(k^{n-1}). \]

Now by setting $E^{(k)} := \pi_*(L^k + K_{\chi/M})$, one has
\[ \deg E^{(k)} = (c_1(\pi_*(L^k + K_{\chi/M})) [\omega]^{m-1}) [M] = a_0 k^{n+1} + a_1 k^n + O(k^{n-1}), \tag{3.21} \]
By Proposition 2.2 in \[17\], one has that $\pi$ is relative ample over the holomorphic fibration $P$, one knows that $L$ satisfies $\nabla$ for $k$, which can be viewed as an analogue of the Donaldson-Futaki invariant (cf. \[10\]) on the obstruction for the existence of geodesic-Einstein metrics.

**Proposition 3.4.** If $DF(\mathcal{X}, L) < 0$, then there exists no geodesic-Einstein metrics on $L$.

**Proof.** By (3.23), if $DF(\mathcal{X}, L) < 0$, one has

$$
\frac{\deg E^{(k)}}{\text{rank} E^{(k)}} < \frac{1}{n+1} \frac{c_1(L)^{n+1} \omega^{m-1} [\mathcal{X}]}{c_1(L)^n [\mathcal{X}/M]} k - \frac{1}{n+1} \frac{c_1(L)^n [\mathcal{X}/M]}{c_1(L)^n [\mathcal{X}/M]} k = \frac{\text{Vol}_\omega(M)}{2\pi} \lambda_{\mathcal{X}, L^k}
$$

for $k > 0$ large enough. Therefore, by Proposition 3.2, there exists no geodesic-Einstein metric on $L^k$ and so on $L$. In fact, if $\phi$ is a geodesic-Einstein metric on $L$, then induced metric $k\phi$ on $L^k$ satisfies

$$
tr_\omega c(k\phi) = ktr_\omega c(\phi) = k\lambda,
$$

i.e. $k\phi$ is a geodesic-Einstein metric on $L^k$ and this yields a contradiction. \qed

### 3.2. Projective bundles.

For any holomorphic vector bundle $E \to M$ of rank $r$, let $\pi : P(E^*) \to M$ be the associated projective fibre bundle to the dual bundle $E^* \to M$, and $\mathcal{O}_{P(E^*)}(1)$ the hyperplane line bundle over $P(E^*)$. Clearly, the line bundle $\mathcal{O}_{P(E^*)}(1) \to P(E^*)$ is relative ample over the holomorphic fibration $\pi : P(E^*) \to M$. Also by Lemma 5.37 in [24], one knows that

$$
E = \pi_*(\mathcal{O}_{P(E^*)}(1)).
$$

By Proposition 2.2 in [17], one has

$$
K_{\mathcal{X}/M} = K_{P(E^*)/M} = \pi^*(\det E) - r\mathcal{O}_{P(E^*)}(1).
$$

Set

$$
L = \mathcal{O}_{P(E^*)}(1) - K_{\mathcal{X}/M} = (r + 1)\mathcal{O}_{P(E^*)}(1) - \pi^*(\det E),
$$

which is also a relative ample line bundle over $P(E^*)$. Then

$$
E = \pi_*(\mathcal{O}_{P(E^*)}(1)) = \pi_*(L + K_{\mathcal{X}/M}).
$$
By a direct computation, one has
\[
\frac{\pi_* (c_1(L)^r)}{\pi_* (c_1(L)^{r-1})} = \frac{\pi_* (((r+1)c_1(O_{P(E^*)}(1)) - c_1(\det E))^r)}{(r+1)^{r-1}\pi_* (c_1(O_{P(E^*)}(1))^{r-1})}
\]
(3.29)
\[
= \frac{(r+1)^r\pi_* (c_1(O_{P(E^*)}(1))^r) - r(r+1)^{r-1}\pi_* (c_1(O_{P(E^*)}(1))^{r-1})c_1(E)}{(r+1)^{r-1}\pi_* (c_1(O_{P(E^*)}(1))^{r-1})}
\]
\[
= c_1(E).
\]
From above equality, one obtains that
\[
\frac{\deg E}{\text{rank} E} = \frac{(c_1(L)^r \wedge [\omega]^{m-1})[P(E^*)]}{r(c_1(L)^{r-1})[P(E^*)]/M}
\]
(3.30)
\[
= \frac{\text{Vol}_\omega(M)}{2\pi m} \lambda_{P(E^*),L}.
\]

For any metric \( \phi \) on \( O_{P(E^*)}(1) \), the first Chern class \( c_1(E) \) can be written as (cf. [12])
\[
c_1(E) = \pi_* c_1(O_{P(E^*)}(1))^r = \left( \frac{1}{2\pi} \right)^r \pi_* (rc(\phi)(\sqrt{-1}\partial\bar{\partial}\phi)^{r-1})
\].

Also by a conformal transformation (cf. [18], Proposition 2.2.23), one can find some metric \( \phi_E \) such that
\[
c_1(\det E, \phi_E) = \left( \frac{1}{2\pi} \right)^r \pi_* (rc(\phi)(\sqrt{-1}\partial\bar{\partial}\phi)^{r-1}).
\]

When \( \phi \) is geodesic-Einstein, we have
\[
tr_{\omega} c_1(\det E, \phi_E) = \frac{1}{2\pi} tr_{\omega} c(\phi) = \text{constant}.
\]
So from (3.27), the induced \( \phi_L = (r+1)\phi - \phi_E \) on \( L \) is also geodesic-Einstein. In fact, since \( \phi_E \) is independent of fibers, so
\[
tr_{\omega} c(\phi_L) = tr_{\omega} c((r+1)\phi - \phi_E) = (r+1)tr_{\omega} c(\phi) - tr_{\omega} c_1(\det E, \phi_E) = \text{constant}.
\]
By (3.28), (3.30), (3.31) and Corollary 3.3 we have

**Proposition 3.5.** If \( O_{P(E^*)}(1) \) admits a geodesic-Einstein metric, then the induced \( L^2 \) metric on \( E \) is a Hermitian-Einstein metric.

For a holomorphic vector bundle \( E \to M \), Kobayashi [14] established a correspondence between the Finsler metrics \( G \) on \( E \) and the Hermitian metrics \( \phi \) on the hyperplane line bundle \( O_{P(E)}(1) \to P(E) \) with \( \partial \bar{\partial} \phi = \partial \bar{\partial} \log G \) and defined the notion of the Finsler-Einstein metric on \( E \) over a Kähler manifold manifold \( M \) (more details, cf. [16], [12], [13]). As applications, we have

**Lemma 3.6.** A strongly pseudo-convex Finsler metric \( G \) on \( E \) is Finsler-Einstein if and only if the corresponding metric \( \phi \) on \( O_{P(E^*)}(1) \) is geodesic-Einstein.

**Proof.** For any Finsler metric \( G \) on \( E \), let \( \phi \) be the corresponding Hermitian metric on the line bundle \( O_{P(E)}(1) \), which is also an admissible metric on \( O_{P(E)}(1) \) (cf. [15], [12], [13]). With respect to a holomorphic trivialization of \( E \to M \), by the standard procedure, one gets a local homogeneous holomorphic coordinate system \( \{ [\zeta] | \zeta = (\zeta^1, \cdots, \zeta^r) \neq 0 \} \) on the fibres of \( P(E) \).
By a direct computation, one gets

\[ \frac{\partial}{\partial \zeta^i} = -\frac{\zeta^a}{(\zeta^i)^2} \frac{\partial}{\partial v^a}, \quad 2 \leq a \leq r. \]

Denote by \((\log G)_{\alpha \beta}^{ab} \) \(2 \leq a, b \leq r \) (resp. \((G^{ij})_{1 \leq i, j \leq r} \) the inverse of the matrix \( (\frac{\partial^2 \log G}{\partial \alpha^i \beta^j})_{2 \leq a, b \leq r} \)

(resp. \( (\frac{\partial^2 G}{\partial \alpha^i \beta^j})_{1 \leq i, j \leq r} \)). Then

\[ (\log G)^{ba} = \frac{G}{|\zeta|^2} \left( -\frac{\zeta^a}{\zeta^1} G^{1b} + G^{ba} + \frac{\bar{\zeta}^b}{|\zeta|^2} G^{11} \right). \]

By a direct computation, one gets

\[ c(\phi) = -\sqrt{-1}(\phi_{a \bar{b}} \phi^{\bar{a} \bar{b}}) dz^a \wedge d\bar{z}^\beta = -\sqrt{-1} K_{\bar{i} \bar{j} \bar{a} \bar{b}} \frac{\bar{v}^i \bar{v}^j}{G} dz^a \wedge d\bar{z}^\beta := -\Psi, \]

where \( \Psi := K_{\bar{i} \bar{j} \bar{a} \bar{b}} \frac{\bar{v}^i \bar{v}^j}{G} \) is the Kobayashi curvature of the Finsler metric \( G \) (cf. [12], (1.21)).

Now the lemma is follows directly from (3.32) and the definitions of the Finsler-Einstein metric (cf. [12], Definition 3.1) and the geodesic-Einstein metric. \( \square \)

**Theorem 3.7.** \( E \) admits a Finsler-Einstein metric if and only if \( E \) admits a Hermitian-Einstein metric.

**Proof.** One direction is obvious, since a Hermitian-Einstein metric must be a Finsler-Einstein metric.

Conversely, if \( E \) admits a Finsler-Einstein metric \( G \), then by Lemma 3.6 the induced metric \( \phi \) is a geodesic-Einstein metric on \( \mathcal{O}_{P(E)}(1) \). So from Proposition 3.5 the induced \( L^2 \) metric is a Hermitian-Einstein metric on \( E^* \). Therefore, the dual Hermitian metric on \( E \) is also a Hermitian-Einstein metric. \( \square \)

Next, we will discuss some relations between the nonlinear stability of the holomorphic fibration \( (P(E), M, \mathcal{O}_{P(E)}(1)) \) and the stability of the holomorphic vector bundle \( E \).

**Proposition 3.8.**  
1. If \( (P(E), M, \mathcal{O}_{P(E)}(1)) \) is nonlinear semistable (resp. nonlinear stable), then \( E \) is semistable (resp. stable).
2. If \( E \) is polystable, then \( (P(E), M, \mathcal{O}_{P(E)}(1)) \) is nonlinear polystable.

**Proof.**  
1. For any coherent subsheaf \( F \) of \( \mathcal{O}(E) \), \( F = F|_{M-S} \) is actually a holomorphic vector bundle over \( M-S \) for some subvariety \( S \) of \( M \) of \( \text{codim}(S) \geq 2 \). So \( P(F) \rightarrow M-S \) is a subfibration of \( P(E) \rightarrow M \). Then

\[
\deg_{\omega} F = \frac{\int_M c_1(F) \wedge \omega^{m-1}}{\text{rank } F}
= -\frac{([\omega]^{m-1} c_1(\mathcal{O}_{P(F)}(1)) \text{rank } F)[P(F)]}{\text{rank } F}
= -\frac{\text{Vol}_{\omega}(M)}{2\pi m} \lambda_{P(F), \mathcal{O}_{P(F)}(1)}.\]

2. If \( E \) is polystable, then \( (P(E), M, \mathcal{O}_{P(E)}(1)) \) is nonlinear polystable.
If \((P(E), M, \mathcal{O}_{P(E)}(1))\) is nonlinear semistable (resp. nonlinear stable), then
\[
\lambda_{P(E), \mathcal{O}_{P(E)}(1)} \geq (\text{resp.} > )\lambda_{P(E), \mathcal{O}_{P(E)}(1)}.
\]
From (3.33) and (3.34), one has
\[
\deg F \leq (\text{resp.} < ) \frac{\deg E}{\text{rank } E}.
\]
So \(E\) is semistable (resp. stable).

(2) If \(E\) is a holomorphic ploystable vector bundle over \(M\), by Donaldson-Uhlenbeck-Yau Theorem (cf. [25]), there exists a Hermitian-Einstein metric on \(E\). From Theorem 3.7, Lemma 3.6 and Theorem 2.6, one knows that \((P(E), M, \mathcal{O}_{P(E)}(1))\) is nonlinear polystable. 

□

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