Fréchet completions of moderate growth old and (somewhat) new results

Nolan R. Wallach

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Abstract

My main goal is to explain the structure of the original proof(s) of Casselman and mine of the Casselman-Wallach theorem. In addition, there are several aspects of my proof that were marred by misprints and convoluted explanations I feel that even though the result is more than twenty years old a clearer exposition is useful. In addition, I include a result related to the work of Bernstein, Krötz that uses theorems of van den Noort to show how one can add (some) dependence on parameters using the methods of the original proof. In particular, this yields a new proof of the meromorphic continuation of $C^\infty$ Eisenstein series. This paper is meant to be a supplement to chapter 11 in my book Real Reductive Groups II.

1 Introduction.

In the past three years there has been new activity related to what has come to be called the Casselman-Wallach Theorem (CW Theorem) notably in the thesis of Vincent van den Noort [vdN] and the paper of Bernstein and Krötz [BK]. These papers are concerned with the problem of inserting dependence on parameters in our theory of Fréchet completions of moderate growth. In [vdN] there is a development of the theory of admissible $(\mathfrak{g}, K)$-modules depending on parameters with a goal of finding criteria for when these families are imbeddable in families of smooth Fréchet completions and in the latter paper the goal give in addition a more elementary proof of the CW Theorem. The complications in “the” original proof in some sense stem from
the fact that there is no joint proof by Casselman and me of the theorem. Casselman’s proof (appearing in [C]) and the proof of [BK] apply only to linear groups my version, which can only be found in my second volume on real reductive groups ([RRGII]), proves the theorem in the context of the class of reductive groups that appear in my book with Borel ([BW]). It had been my hope to include in this paper a proof of the theorem for general semi-simple groups with a finite number of connected components (i.e. allow infinite center for the identity component) but time constraints made it impossible for me to complete the foundational work necessary. The version in this paper implies the theorem for the case of connected semi-simple Lie groups with finite center (e.g. the metaplectic group or the two–fold cover of $SL(n, \mathbb{R})$).

Except for the greater generality of my version the proof in [C] and that in [RRGII] are essentially the same (and which is I think basically the same as that of [BK], except for the handling of spherical representations). In hindsight I must admit that Casselman’s article is more carefully written than my development. Beyond its obvious defects in exposition my argument can be deemed complicated or at least “non-elementary” because it relies on results that are necessary in other contexts in the study of real reductive groups. That is, the asymptotic expansions of generalized matrix coefficients of admissible representations, the theory of intertwining operators developed by Vogen and me ([VW]) and the Langlands quotient theorem. In each case the results are easily stated but their use in a proof can certainly allow one to say that a proof that doesn’t use these results is more elementary (however [BK] uses the Langlands quotient theorem). It is definitely not true that more elementary implies easier to understand (compare Hadamard’s proof of the prime number theorem to Selberg’s elementary proof).

This paper is for the most part an attempt to explain my proof of the theorem with some simplifications and (we hope) clarifications. Emphasizing the places where it differs from that in [C] which can be seen most clearly in the construction of the minimal completion which is given in here detail with several simplifications. Also in the appendix to the construction, there are a few results on $C^\infty$ vectors that I couldn’t find in the literature. In the last section of this paper will show that the beautiful theory of van der Noort combined with an extended version of the argument in subsection 4.3 allows us to add dependence on parameters in particular we show that the $C^\infty$ Eisenstein series has a meromorphic continuation. Another reason for an extended explanation of a 20 year old result is that the unique reference for the general result is in a book that is out of print.
2 A litany of notation

We must first set up the standard notation. We take $G_C$ to be a reductive algebraic group defined over $\mathbb{R}$ and $G_\mathbb{R}$ to be the real points. Then a real reductive group in the sense of my book (and [RRGI]) with Borel is a finite covering of an open subgroup of $G_\mathbb{R}$. This can be described in elementary terms (as in [RRGI]) as follows. Let $G_\mathbb{R}$ be a subgroup of $GL(n, \mathbb{R})$ that is the locus of zeros of polynomial functions on $M_n(\mathbb{R})$ and is invariant under matrix transpose. Then $G$ is a finite covering of an open subgroup of $G_\mathbb{R}$ (for a proof that these notions are the same see e.g. [W]).

It is convenient to assume that $G$ is given by the more elementary definition. Then $K_\mathbb{R} = G_\mathbb{R} \cap O(n)$ is a maximal compact subgroup and if $p: G \rightarrow G_o$ is the covering homomorphism ($G_o \subset G_\mathbb{R}$ open) let $K_o = K_\mathbb{R} \cap G_o$ then $K = p^{-1}(K_\mathbb{R} \cap G_o)$ is a maximal compact subgroup of $G$. We also fix an Iwasawa decomposition $G_\mathbb{R} = K_\mathbb{R} A_\mathbb{R} N_\mathbb{R}$ ($A_\mathbb{R}$ a maximal abelian subgroup consisting of positive definite self adjoint elements and $N_\mathbb{R}$ a maximal unipotent subgroup) so $G_o = K_o A_\mathbb{R} N_\mathbb{R}$ is an Iwasawa decomposition of $G_o$ and if $A$ and $N$ are the identity components of $p^{-1}(A_\mathbb{R})$ and $p^{-1}(N_\mathbb{R})$ respectively then $KAN$ is an Iwasawa decomposition of $G$. Let $M$ denote the centralizer of $A$ in $K$ then $MAN$ is a minimal parabolic subgroup of $G$. As usual, we will call a parabolic subgroup of $G$ containing $MAN$ a standard parabolic subgroup.

We define for $g \in G$, $\|g\|$ to be the Hilbert-Schmidt norm of

$$r(g) = \begin{bmatrix} p(g) & 0 \\ 0 & p(g^{-1})^T \end{bmatrix} \in GL(2n, \mathbb{R}).$$

Then $\|\cdot\|$ is a norm on $G$ which is $K$ invariant on the right and left. That is, if $S \subset \mathbb{R}$ is compact then $\{ g \in G | \|g\| \in S \}$ is compact and $\|xy\| \leq \|x\| \|y\|, x, y \in G$. Furthermore, if $(\pi, H)$ is a strongly continuous Banach representation of $G$ then there exist constants $C$ and $r$ such that $\|g\|_H \leq C \|g\|^r$.

We denote by $\mathfrak{g}$ the complexified Lie algebra of $G$ (thought of as left invariant vector fields), $U(\mathfrak{g})$ the universal enveloping algebra (left invariant differential operators) and $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$ (thought of as bi–invariant differential operators). We define on $Lie(G) = Lie(G_\mathbb{R})$ the form $B(X, Y) = trXY.$ Then $B$ is non-degenerate and negative definite on $Lie(K)$. We set $C$ (resp. $C_K$) equal to the Casimir operator on $G$ (resp. $K$) corresponding to $B$. 

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As usual, a \((\mathfrak{g}, K)\)–module is a module \(V\) for \(\mathfrak{g}\) that is also a \(K\) module with the following compatibility properties

1. Set \(W_v = \text{Span}_\mathbb{C}(Kv)\) then \(\dim W_v < \infty\) the action of \(K\) on \(W_v\) is smooth and the action of \(\text{Lie}(K) \subset \mathfrak{g}\) on \(W_v\) is the same as the action coming from that of \(\mathfrak{g}\).

2. if \(k \in K, X \in \mathfrak{g}\) and \(v \in V\) then \(kXv = (\text{Ad}(k)X)kv\).

We say that a \((\mathfrak{g}, K)\)–module is finitely generated if it is finitely generated as a \(U(\mathfrak{g})\)–module.

We denote by \(\mathcal{H}(\mathfrak{g}, K)\) the category of \((\mathfrak{g}, K)\)–modules that are admissible (if \(W\) is a finite dimensional \(K\)-module then \(\dim \text{Hom}_K(W, V) < \infty\) with morphisms the \(\mathfrak{g}\) and \(K\) module homomorphisms) and finitely generated.

If \(V\) is a \((\mathfrak{g}, K)\)–module and if \(\gamma\) is a class of irreducible representations of \(K\) (that is, an element of \(\hat{K}\)) then \(V_\gamma\) denotes a representative and \(V(\gamma) = \sum TV_\gamma\) the sum over all \(T \in \text{Hom}_K(V_\gamma, V)\).

A smooth Fréchet module, \((\pi, V)\) for \(G\) is a homomorphism of \(G\) to the continuous invertible operators on a Fréchet space \(V\) such that the map

\[ G \times V \to V \]

given by

\[ g, v \mapsto \pi(g)v \]

is of class \(C^\infty\) in \(G\) and jointly continuous. If we have a smooth Fréchet module \((\pi, V)\) then we can differentiate to get a representation of \(\mathfrak{g}\)-module structure on \(V\). We define \(V_K\) to be the space of all \(v \in V\) such that \(\dim \text{Span}_\mathbb{C}(\pi(K)v) < \infty\). Then \(V_K\) is a \((\mathfrak{g}, K)\) module. If \(V_K\) is admissible or finitely generated then we say that \(V\) is admissible or finitely generated.

If \(V\) is a smooth Fréchet module then we say that \(V\) is of moderate growth if whenever \(v\) is a continuous seminorm on \(G\) then there exists \(\mu\) a continuous semi-norm on \(V\) and \(r\) such that

\[ \nu(\pi(g)v) \leq \|g\|^r \mu(v), g \in G, v \in V. \]

We denote by \(\mathcal{HF}_{\text{mod}}(G)\) the category of all smooth admissible finitely generated Fréchet modules for \(G\) of moderate growth with morphisms continuous intertwiners with images that are direct summands (in the category of Fréchet spaces).

If \((\sigma, F)\) is a continuous finite dimensional representation of \(P\) then we put an inner product \((\; , \; )\) on \(F\) that is \(M\)–invariant and define \(\text{Ind}_M^K(F)\)
to be the corresponding unitarily induced representation. That is functions \( f : K \to F \) such that \( k \mapsto \|f(k)\| \) is in \( L^2(K) \) and we define for \( f, h \in \text{Ind}_M^K(F) \)

\[(f, h) = \int_K (f(k), h(k)) dk.\]

The action of \( K \) is \( \pi(k)f(x) = f(xk) \). We define the Hilbert representation \( \text{Ind}_G^P(F) \) to have underlying Hilbert space \( \text{Ind}_M^K(F) \) and the action of \( G \) is given as follows if \( f \in \text{Ind}_M^K(F) \) then set \( f_\sigma(pk) = \sigma(p)f(k) \). Then \( f_\sigma(pk) \) is defined and (i.e. depends only on \( pk \)). Then \( (\pi(g)f)(x) = f_\sigma(xg) \) for \( g \in G \) and \( x \in K \). The \( C^\infty \) vectors of \( \text{Ind}_G^P(F) \) with respect to \( G \) are the \( C^\infty \) functions in \( \text{Ind}_M^K(F) \) with the topology defined by the seminorms, \( p_k(f) = \| (I + C_K)^{l}f \| \).

3 The theorem and some applications

The theorem referenced as the Casselman–Wallach theorem is

**Theorem 1** The functor \( V \to V_K \) from \( \mathcal{H}_G^\text{mod}(G) \) to \( \mathcal{H}(\mathfrak{g}, K) \) is an equivalence of categories.

In this section we will give some implications. We will sketch the proof in [RRGII] in the next section. First, since there is an inverse functor we see that if \( V \) is an admissible finitely generated \( (\mathfrak{g}, K) \) module then there exists a smooth Fréchet module of moderate growth, \( W \), such that \( W_K \) is isomorphic with \( V \). This is a direct implication of Casselman’s theorem since Banach representations have moderate growth and Casselman has shown that there is a Hilbert space completion of \( V \), \( H \), such that the smooth \( K \)-finite vectors of \( H \) are precisely \( V \). However, the theorem says something much stronger: \( W \) is unique up to isomorphism in the category \( \mathcal{H}_G^\text{mod}(G) \).

The above says that if \( V \in \mathcal{H}(\mathfrak{g}, K) \) and \( (\pi, H) \) is a Banach representation of \( G \) such that \( (H^\infty)_K \) is isomorphic with \( V \) then although there are many different representations in the Banach category with this property there is only one object up to isomorphism in \( \mathcal{H}_G^\text{mod}(G) \) whose \( K \)-finite vectors yield a module isomorphic with \( V \).

The most important implication is just the expansion of what an equivalence of categories means for the two categories in the theorem. Let \( V, W \) be admissible finitely generated \( (\mathfrak{g}, K) \) modules and let \( T : V \to W \) be a morphism. Let \( \overline{V} \) and \( \overline{W} \) be any elements of \( \mathcal{H}_G^\text{mod}(G) \) such that \( \overline{V}_K \) and \( \overline{W}_K \) are
respectively isomorphic with $V$ and $W$ then the induced map $L : V_K \to W_K$ extends to a continuous $G$ intertwining operator from $\overline{V}$ to $\overline{W}$ with closed image that is a direct summand of $\overline{W}$. We will now give several applications. We denote by $V_K$ element of $\mathcal{H}_F \mod (G)$ with $V_K = V$ we will call $V$ the completion of $V$ in $\mathcal{H}_F \mod (G)$.

3.1 Continuous functionals and automorphic forms

In this subsection we give an application to automorphic forms. We first describe an implication of the proof in [RRGII] which characterizes $V'_{V}$ for $V \in H(g, K)$.

Define $A_{mod}(G)$ to be the space of all $C^\infty$ functions $f : G \to \mathbb{C}$ satisfying the following two conditions
1. There exists $d > 0$ and for each $x \in U(g)$, $C_x > 0$ such that $|xf(g)| \leq C_x \|g\|^d$.
2. $\dim Z(g)f < \infty$.
3. $f$ is right $K$–finite.

Theorem 2 Let $V \in \mathcal{H}(g, K)$ and $\lambda \in V^*$ and let $\overline{V}$ be its completion in $\mathcal{H}_F \mod (G)$ then $\lambda$ extends to a continuous functional on $\overline{V}$ if and only if for each $v \in V$ there exists $f \in A_{mod}(G)$ such that $xf(k) = \lambda(kxv)$ for all $k \in K$ and $x \in U(g)$.

We will come back to this theorem in the next section.

We recall $f \in C^\infty(G)$ an automorphic form on $G$ with respect to a discrete subgroup of finite covolume, $\Gamma$, if $f \in A_{mod}(G)$ and $f(\gamma g) = f(g)$ for $\gamma \in \Gamma$. We denote the space of automorphic forms on $\Gamma \backslash G$ by $A_{mod}(\Gamma \backslash G)$.

We now discuss an implication of this theorem to automorphic forms. Let $f \in A_{mod}(\Gamma \backslash G)$, $W$ the span of the right translates of $f$ under $K$ and let $V = U(g)W$. Then $V \in \mathcal{H}(g, K)$. Let $\delta(h) = h(e)$ ($e$ the identity element of $G$). Then $\delta(kxf) = xf(k)$ and $W$ is cyclic so the Theorem implies that $\delta$ extends to $\overline{V}$. For example, this implies that the analytically continued Eisenstein series initially shown to exist for $K$–finite elements of an induced representations from cuspidal parabolic subgroups extend to the $C^\infty$ vectors.

3.2 $C^\infty$ Helgason conjecture and related results

Let $G$ be connected with compact center and let $P = MAN$ be a standard minimal parabolic subgroup with given Langlands decomposition. Here $M \subset$
$K$ is compact, $A$ is the identity component of a split torus over $\mathbb{R}$ and $N$ is the unipotent radical. We say that an irreducible $K$–type, $(\tau, V_\tau)$, is small if $\tau|_M$ is irreducible. Clearly a one dimensional representation is small (e.g. the trivial representation). There are more interesting representations that are small. For example the spin representation for $Spin(2n+1)$ as $K$ for the two fold cover of $SL(2n+1, \mathbb{R})$ or either half spin representation for $Spin(2n)$ as $K$ for the two fold cover of $SL(2n, \mathbb{R})$.

We first describe a general Poisson representation of elements of $A_{mod}(G)$. If $(\sigma, H)$ is a finite dimensional representation of $P$ we set $I_{P,\sigma}^\infty$ equal to the representation $C^\infty$–induced of $\sigma$ on $P$. That is $I_{P,\sigma}^\infty$ is the space of all $C^\infty$ functions from $G$ to $H$ such that $f(pg) = \sigma(p)f(g)$ for $p \in P$ and $g \in G$ and $(\pi_{P,\sigma}(x)f)(g) = f(gx)$. On $I_{P,\sigma}^\infty$ we put the $C^\infty$ topology.

**Theorem 3** Let $f \in A_{mod}(G)$ then there exists a finite dimensional representation, $\sigma$, of $P$, a continuous linear functional $\lambda \in (I_{P,\sigma}^\infty)^*$ and $h \in (I_{P,\sigma}^\infty)_K$ such that $f(g) = \lambda(\pi_{P,\sigma}(g)h) = f(g)$ for all $g \in G$.

Now suppose that $(\tau, H)$ is a small $K$–type. Let $\sigma = \tau|_M$. Let $\nu \in Lie(A)^*_C$ and set $\sigma_{\nu}(man) = e^{(\nu + \rho)(\log(a))}\sigma(m)$ for $m \in M$, $a \in A$ and $n \in N$ (here $\exp(\log(a)) = a$ and $\log(a) \in Lie(A)$, $\rho(h) = tr(ad(h)|_{Lie(N)})$ for $h \in Lie(A)$). Let $I_{P,\sigma,\nu}^\infty = I_{P,\sigma}^\nu$. Set $I_{P,\sigma,\nu} = (I_{P,\sigma,\nu}^\infty)_K$. Then Frobenious reciprocity implies that $\dim Hom_K(H_\tau, I_{P,\sigma,\nu}^\infty) = 1$. This implies that if $x \in U(\mathfrak{g})^K$ then $\pi_{P,\sigma,\nu}(x)f = \gamma_{\tau,\nu}(x)f$ for $f$ in the $\tau$ isotypic component of $I_{P,\sigma,\nu}^\infty$. One checks easily that $\gamma_{\tau,\nu}(x)$ is polynomial in $\nu$ and a homomorphism of $U(\mathfrak{g})^K$ into $\mathbb{C}$. Furthermore since $\sigma = \tau|_M$ we see that if $k \in N_K(M)$ (normalizer of $M$ in $K$) the representation $\sigma^k(m) = \sigma(k^{-1}mk) = \tau(k^{-1})\sigma(m)\tau(k)$ this implies that $\gamma_{\tau,\nu} = \gamma_{\tau,s\nu}$ for $s \in W(A) = N_K(A)/M$. The analogue of the $C^\infty$–Helgason conjecture is

**Theorem 4** Let $f \in A_{mod}(G)(\tau)$ be such that $xf = \gamma_{\tau,\nu}(x)f$ for $x \in U(\mathfrak{g})^K$ then if $I_{P,\sigma,\nu} = U(\mathfrak{g})I_{P,\sigma,\nu}(\tau)$ there exists $\lambda \in (I_{P,\sigma,\nu})'$ and $u \in I_{P,\sigma,\nu}(\tau)$ such that $f(g) = \lambda(\pi_{P,\sigma,\nu}(g)u)$ for $g \in G$.

We note that if $\tau$ is the trivial representation then Kostant [K] has shown that the condition of the theorem is satisfied for $\Re\nu$ in the closed positive Weyl chamber. This proves the distribution version of the Helgason Conjecture due to [OS]. We note that using the fact that the analytic vectors of $I_{P,\sigma,\nu}^\infty$ are the real analytic elements of $I_{P,\sigma,\nu}^\infty$ we see that the distributional
version of the conjecture implies the hyperfunction version that is the same assertion but with \( f \in C^\infty(G) \) satisfying the \( K \)-condition and \( xf = \gamma_{\tau,\nu}(x)f \) for \( x \in U(g)^K \) and \( \lambda \) in the continuous dual of the analytically induced representation.

In his work on split groups over \( \mathbb{R} \), Seung Lee [Le] has shown that if \( \tau \) is small, \( \dim V_\tau > 1 \) and \( G \) is simply laced then the analogue of Kostant’s result is true and so the above theorem completely describes \( \mathcal{A}_{\text{mod}}(G)(\tau) \). In the non-simply laced case there are some exceptions see [Le] for a complete discussion.

4 The main points of our proof

We divide the proof into the construction of a left exact and a right exact functor and then prove that the two functors are the same. The theorems involved in the left exact functor will only be sketched. It is the argument in the existence of the right exact functor that will be given in detail since it plays a role in the sequel to this paper and since in the original version in [RRGII] it was rather convoluted and had some misprints that made it hard to understand.

4.1 Step one: A left exact functor

The first step is to construct what will be an inverse functor to the \( K \)-finite functor. Let \( P = MAN \) be a minimal parabolic subgroup of \( G \) (as in section 2). Let \( n = \text{Lie}(N) \) and let \( V \in \mathcal{H}(g, K) \). We first sketch the proof of

**Theorem 5** Let \( V \in \mathcal{H}(g, K) \). Then there exists an object \( \overline{V} \in \mathcal{HF}_{\text{mod}}(G) \) such that

1. \( \overline{V}_K \) is isomorphic with \( V \).
2. If \( W \in \mathcal{H}(g, K) \) and \( T : W \to V \) is a \( (g, K) \)-homomorphism and if \( X \in \mathcal{HF}_{\text{mod}}(G) \) is such that \( X_K \) is isomorphic with \( W \) then there exists a \( \mathcal{HF}_{\text{mod}}(G) \)-morphism \( S : X \to \overline{V} \) in such that \( S(X_K) = T(W) \).

Using standard methods the proof is reduced to the case when \( G \) is connected, simple with finite center. This is the part of the proof that we will sketch. We consider the simply connected (complex) Lie group with Lie algebra \( g_C, G_C \) and the connected subgroup, \( G_\mathbb{R} \), corresponding to \( g \). Then we may assume that \( G \) is a finite covering group of \( G_\mathbb{R} \) with the kernel of the
covering homomorphism $Z$. By our assumption $Z$ is a finite abelian group.

We first prove

**Theorem 6** If $\chi \in \hat{Z}$ then there exists $\tau$ an irreducible representation of $K$ such that $\tau|_Z = \chi I$ and $\tau|_M$ is irreducible (i.e $\tau$ is small).

A complete classification of representations of $K$ whose restriction to $M$ is irreducible was given in Seung Lee’s thesis [Le]. If $(\sigma, V)$ is an irreducible finite dimensional representation of $P$ then we denote by $\text{Ind}_P^K(\sigma)\infty$ the smooth induced representation from $P$ to $G$ and by $\text{Ind}_P^K(\sigma)_K$ the $K$–finite induced representation of $P$ to $G$. If $\mu$ is a finite dimensional unitary representation of $M$ and if $\nu \in a^*_C$ then $I_{P,\mu,\nu}$ is the (normalized) smooth principal series representation of $G$ corresponding to these parameters and

Using this result we next prove

**Lemma 7** Let $(\sigma, V)$ be an irreducible finite dimensional representation of $P$ such that $\sigma|_Z = \chi I$. Let $\tau$ be a small representation of $K$ with $\tau(z) = \chi(z)I$ and set $\mu = \tau|_M$. Then there exist finite dimensional $G$–representations $F_1, \ldots, F_r$ and $\nu_1, \ldots, \nu_r \in a^*_C$ such that $\text{Ind}_P^K(\sigma)_K$ is equivalent to a quotient of $\oplus I_{P,\mu,\nu_i} \otimes F_i$.

Using Casselman’s imbedding theorem and the Artin-Rees Lemma we see that if $V \in \mathcal{H}(g, K)$ then $\dim V/n^k V < \infty$ and $\cap_k n^k V = \{0\}$. This implies that the natural morphism

$$T_k : V \to \text{Ind}_P^K(V/n^k V)\infty$$

and for some $k_o$, $T_k$, is injective for all $k \geq k_o$. We will take $k_o$ to be the minimal choice of such a $k$. Using this and the theory of intertwining operators we prove that if $V \in \mathcal{H}(g, K)$ and if $k_o$ is as above then if $W$ is any element of $\mathcal{F}_{mod}(G)$ such that $W_K$ is isomorphic with $V$ then the map $T_k$ extends to a homomorphism of $W$ into $\text{Ind}_P^K(V/n^k V)\infty$. Using this we take $\nabla V$ to be the closure of $T_{k_o}(V)$ in $\text{Ind}_P^K(V/n^k V)\infty$. We then use the theory of asymptotic expansions of elements of $\mathcal{A}_{mod}(G)$ restricted to $MA$ to prove what we call the automatic continuity theorem which implies

**Theorem 8** If $V \in \mathcal{H}\mathcal{F}_{mod}(G)$ and if $F$ is a finite dimensional representation of $P$ and if $T : V_K \to \text{Ind}_P^K(F)\infty$ is a morphism in $\mathcal{H}(g, K)$ then $T$ extends to a morphism $V \to \text{Ind}_P^K(F)\infty$. 

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This result has as a consequence

**Corollary 9** If \( V, W \in \mathcal{H}(g, K) \) and if \( T : W \to V \) is an injective morphism then the closure of \( W \) in \( V \) is isomorphic with \( W \) in \( \mathcal{HF}_{\text{mod}}(G) \).

This easily implies 2. in Theorem 5

This completes the sketch of the existence of the maximal completion in \( \mathcal{F}_{\text{mod}}(G) \). We note that all one needs from the theory of intertwining operators is Schiffman’s work [Sc] which stems from the theory for rank one groups.

We note that for any \( V \in \mathcal{H}(g, K) \), there exists a Hilbert completion of \( V, H \) such that the \( K-C^\infty \) vectors of \( H \) are the same as the \( G-C^\infty \) vectors and \( H^\infty = V \). This implies that \( V \to V \) defines a left exact faithful functor from \( \mathcal{H}(g, K) \) to \( \mathcal{HF}_{\text{mod}}(G) \).

### 4.2 Step two: A right exact functor

The next step is the dual assertion.

**Theorem 10** Let \( V \in \mathcal{H}(g, K) \) then there exists an object \( \overline{V} \in \mathcal{HF}_{\text{mod}}(G) \) such that \( \overline{V}_K \) is isomorphic with \( V \) and if \( W \in \mathcal{HF}_{\text{mod}}(G) \) and \( T : W_K \to \overline{V}_K \) is a surjective morphism then \( T \) extends to continuous surjection of \( W \) onto \( \overline{V} \).

We note that there is up to isomorphism only one object in \( \mathcal{HF}_{\text{mod}}(G) \) with the property above enjoyed by \( \overline{V} \). Also \( V \to \overline{V} \) is a right exact functor from \( \mathcal{H}(g, K) \) to the category \( \mathcal{HF}_{\text{mod}}(G) \). The CW Theorem will therefore be proved if we can show that \( V \) and \( \overline{V} \) are isomorphic in \( \mathcal{HF}_{\text{mod}}(G) \). If this condition is satisfied for \( V \) then we say that \( V \) is good. We now devote a subsection to the proof of the above theorem.

### 4.3 The proof of Theorem 10

If \( V \in \mathcal{H}(g, K) \) we denote by \( \hat{V} \) its conjugate dual module. That is \( \hat{V} \) consists of all of the real linear functionals, \( \lambda \), on \( V \) satisfying

1. \( \lambda(zv) = \bar{z}\lambda(v) \) for \( z \in \mathbb{C} \).
2. Set \( k\lambda = \lambda \circ k^{-1} \) then \( \dim \text{Span}\mathbb{C}K\lambda < \infty \).
If we act an \( \hat{V} \) by \( X\lambda = -\lambda \circ X \) for \( X \in \hat{V} \) then \( \hat{V} \) is in \( \mathcal{H}(g, K) \).

\( \hat{V} \) is clearly a vector space over \( \mathbb{C} \). If \((\pi, H)\) is a Hilbert representation of \( G \) then we define the conjugate dual representation to \((\pi, H)\) to be \((\hat{\pi}, H)\) with \( \hat{\pi}(g) = \pi(g^{-1})^* \). Obviously, if \( \pi \) is unitary \( \hat{\pi} = \pi \). If \((\pi, H)\) is a Hilbert representation of \( G \) then we denote by \((\pi, H^\infty)\) the \( G-C^\infty \) vectors and by \((\pi, H^\infty_K)\) the \( K-C^\infty \) vectors.

We first relate Theorem 2 to Theorem 10. We note that if \( V \in \mathcal{H}_\text{mod}(G) \) and if \( \lambda \in V' \) and \( v \in V_k \) then \( f(g) = \lambda(gv) \) defines an element of \( A_\text{mod}(G) \). If \( V \in \mathcal{H}(g, K) \) then we denote by \( V_\text{mod}^* \) the space of all \( \lambda \in V^* \) satisfying the hypotheses of Theorem 2. Thus \( V_\text{mod}^* \) contains \( Z'_V \) for any object of \( \mathcal{H}_\text{mod}(G) \) such that \( Z_K = V \). We start with

**Lemma 11** Let \( V \in \mathcal{H}(g, K) \) then there exists a Hilbert representation of \( G \), \((\pi, H)\), such that \( H^\infty_K \) is isomorphic with \( V \), \((\pi, H^\infty) = (\pi, H^\infty_K)\), \((\hat{\pi}, H^\infty) = (\hat{\pi}, H^\infty_K)\) and \((\hat{\pi}, H^\infty)\) is isomorphic with \((V)\).

**Proof.** We note that according to the discussion at the end of subsection 4.1 (and in the notation thereof) \((\hat{V})\) is isomorphic with the space of \( C^\infty \) vectors of the closure of \( T_k(\hat{V}) \) in the induced Hilbert representation \( \text{Ind}_{\tilde{g}}^G(\hat{V}/n^k\hat{V}) \) for appropriate \( k \). The \( C^\infty \) vectors in this representation are the \( K-C^\infty \) vectors. Since taking \( C^\infty \) vectors is an exact functor the \( C^\infty \)–vectors in the closure of \( T_k(\hat{V}) \) in \( \text{Ind}_{\tilde{g}}^G(\hat{V}/n^k\hat{V}) \) are the \( K-C^\infty \) vectors. We take \( H \) to be the closure of \( T_k(\hat{V}) \). We now observe that the conjugate dual representation of \( G \) is the representation induced representation \( \text{Ind}_{\tilde{g}}^G((\hat{V}/n^k\hat{V}) \otimes \delta_P) \) with an underlying Hilbert space \( \text{Ind}_{\tilde{g}}^G(\hat{V}/n^k\hat{V}) \) and the \( G-C^\infty \) vectors are the same as the \( K-C^\infty \). Let \( \langle \ldots | \ldots \rangle \) denote the conjugate linear \( G \)-invariant pairing between \( \text{Ind}_{\tilde{g}}^G(\hat{V}/n^k\hat{V}) \) and \( \text{Ind}_{\tilde{g}}^G((\hat{V}/n^k\hat{V}) \otimes \delta_P) \) and set \( Z = H^\perp \) then as a Hilbert space the conjugate dual of \( H \) is \( \text{Ind}_{\tilde{g}}^G((\hat{V}/n^k\hat{V}) \otimes \delta_P)/Z \) which is isomorphic with \( H \) and if \((\pi, H)\) is the corresponding Hilbert representation of \( G \) then the underlying object of \( \mathcal{H}(g, K) \) is isomorphic with \( V \) and the action of \( G \) on the closure of \( T_k(\hat{V}) \) defines \((\hat{\pi}, H)\).

The crux of the matter is the following theorem which will be proved after we give two implications.
Theorem 12 Let $V \in \mathcal{H}(\mathfrak{g}, K)$ then if $(\pi, H)$ is a Hilbert representation such that $V$ is isomorphic with $H^K_\infty$. Assume that $(\tilde{\pi}, H^K_\infty)$ is as an object in $\mathcal{H}\mathcal{F}_{\text{mod}}(G)$ isomorphic with $(\widehat{V})$ then denoting the $(\pi, H^K_\infty)$ by $Z$ we have $Z^{\prime}|_V = V^{\ast}_{\text{mod}}$.

Corollary 13 If $X, Y \in \mathcal{H}\mathcal{F}_{\text{mod}}(G)$ are such that $X_K$ is isomorphic with $Y_K$ and $X^{\prime}_{|X_K} = (X_K)^{\ast}_{\text{mod}}, Y^{\prime}_{|X_K} = (Y_K)^{\ast}_{\text{mod}}$ then $X$ is isomorphic with $Y$ in $\mathcal{H}\mathcal{F}_{\text{mod}}(G)$.

Proof. For this we use a theorem of Banach which implies that if $Z$ and $W$ are Fréchet spaces, if $u : Z \rightarrow W$ is a continuous linear map and we set $u^T(\lambda) = \lambda \circ u$ for $\lambda \in W^{\prime}$ so $u^T : W^{\prime} \rightarrow Z^{\prime}$. Then $u$ is surjective if $u^T$ is injective and the image of $u^T$ is weakly closed in $Z^{\prime}$ (see Treves [T,3,7.2]).

We will identify $X_K$ and $Y_K$ and call them both $V$. Theorem 5 implies that the identity map $V \rightarrow V$ induces continuous morphisms $A : X \rightarrow \overline{V}$ and $B : Y \rightarrow \overline{V}$. Let $Z = A(X) \cap A(Y)$ with topology given by the seminorms for both $X$ and $Y$. Then $Z$ is complete, $G$–invariant and hence an element of $\mathcal{H}\mathcal{F}_{\text{mod}}(G)$ and $Z_K = V$. There are two morphisms $\alpha : Z \rightarrow X$ and $\beta : Z \rightarrow Y$ in $\mathcal{H}\mathcal{F}_{\text{mod}}(G)$. Now $Z^{\prime}|_V \supset X^{\prime}_{|V} = Y^{\prime}_{|V} = V^{\ast}_{\text{mod}} \supset Z^{\prime}|_V$ thus $\alpha^T$ and $\beta^T$ are bijective. Hence, $\alpha$ and $\beta$ are surjective by Banach’s theorem. This implies $X = Y$ (under our identification of $K$–finite vectors).

We note that the hypotheses of the theorem above are satisfied for some $(\pi, H)$ by the lemma above. Thus if $V \in \mathcal{H}(\mathfrak{g}, K)$ then there is up to isomorphism exactly one $Z \in \mathcal{H}\mathcal{F}_{\text{mod}}(G)$ with $Z_K$ isomorphic with $V$ and $Z^{\prime}_{|Z_K} = (Z_K)^{\ast}_{\text{mod}}$. We will denote one choice by $\overline{V}$. Then $V \rightarrow \overline{V}$ defines a functor from $\mathcal{H}(\mathfrak{g}, K)$ to $\mathcal{H}\mathcal{F}_{\text{mod}}(G)$. We will show that it has the property described in Theorem 10 after we prove the theorem above.

The idea of the proof of Theorem 12 is to show that if $\lambda \in (H_K)^{\ast}_{\text{mod}}$ then there exists a Hilbert representation of $G$, $(\pi_1, H_1)$ such that $(H_1)_K = \hat{V}$ and so $(\hat{\pi}_1, H_1)_K$ is isomorphic with $V$, under the isomorphism $T$, and there exists an element $u \in H_1$ with inner product $\langle ..., ... \rangle$ so that $\langle Tv, u \rangle = \lambda(v)$ for $v \in V$. In other words there is a Hilbert completion of $V$ so that $\lambda$ is an element of the conjugate dual Hilbert representation. We will give many details since we will use the technique in later sections. We write $\langle ..., ... \rangle$ for the inner product on $H$ which we assume is $K$–invariant. If $\mu \in V^{\ast}_{\text{mod}}$, $v \in V$ we will use the notation $f_{\mu,v}$ for the element of $A_{\text{mod}}(G)$ such that $xf(k) = \mu(kxv)$ for $x \in U(\mathfrak{g})$ and $k \in K$. 

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Let \( v_1, \ldots, v_n \) be an orthonormal basis of a \( K \) and \( Z(g) \)-invariant subspace of \( W \) in \( V \) such that \( V = U(g)W \). Fix \( \lambda \in (H_K)^* \). Let \( d \) be such that

1. \( \|\hat{\pi}(g)\| \leq C \|g\|^d \) for some \( C > 0 \) and
2. \( |f_{\lambda, v_i}(g)| \leq C_{v_i} \|g\|^d \) for \( i = 1, \ldots, n \).

We also choose \( d_o \) such that

\[
\int_G \|g\|^{-d_o} \, dg < \infty.
\]

If \( v, w \in H \) then we define a new inner product

\[
\langle v, w \rangle = \sum_{i=1}^n \int_G (v_i, \hat{\pi}(g)w)(\hat{\pi}(g)v, v_i) \|g\|^{-2d-d_o} \, dg.
\]

We note that since

\[
|(v_i, \hat{\pi}(g)v)(\hat{\pi}(g)w, v_i)| \leq \|v_i\|^2 \|v\| \|w\| \|\hat{\pi}(g)\|^2 \leq C^2 \|v_i\|^2 \|v\| \|w\| \|g\|^{2d}
\]

the above integral converges for all \( v, w \in H \) and defines a new inner product on \( H \) that is \( K \)-invariant (since \( \|gk\| = \|g\| \) for \( g \in G \) and \( k \in K \)). Furthermore, if we set \( \|v\|_1^2 = \langle v, v \rangle \) for \( v \in H \) then

\[
\|v\|_1 \leq C_1 \|v\| \quad (*)
\]

with \( C_1 = C \sqrt{n \int_G \|g\|^{-d_o} \, dg} \). We set \( H_1 \) equal to the Hilbert space completion of \( H \) relative to \( \|\ldots\|_1 \). The above inequality implies that \( H \) imbeds continuously into \( H_1 \) via the canonical injection \( (v \mapsto v) \) and that \( \hat{V} \) is dense in \( H_1 \). Hence \( (H_1)_K = \hat{V} \).

We next observe that if \( v \in H \)

\[
\|\hat{\pi}(x)v\|_1^2 = \sum_{i=1}^n \int_G |(\hat{\pi}(gx)v, v_i)|^2 \|g\|^{-2d-d_o} \, dg =
\]

\[
\sum_{i=1}^n \int_G |(\hat{\pi}(g)v, v_i)|^2 \|gx^{-1}\|^{-2d-d_o} \, dg \leq \|x\|^{2d+d_o} \|v\|_1^2
\]

Here we use

\[
\|g\| = \|gx^{-1}\| \leq \|gx^{-1}\| \|x\|
\]

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so
\[ \|gx^{-1}\|^{-2d-d_o} \leq \|g\|^{-2d-d_o} \|x\|^{2d+d_o}. \]

This implies that \( \hat{\pi}(g) \) on \( H \) extends to a strongly continuous representation \((\pi_1, H_1)\) of \( G \). Let \( X \in \text{Lie}(G) \) then if \( v \) is a \( C^\infty \) vector in \( H \) relative to \( \hat{\pi} \) then using (*) above we have

\[ \left\| \frac{\pi_1(\exp(tX))v - v}{t} - d\hat{\pi}(X)v \right\|_1 \leq C_1 \left\| \frac{\hat{\pi}(\exp(tX))v - v}{t} - d\hat{\pi}(X)v \right\|_1. \]

Thus, \( d\pi_1(X)v = d\hat{\pi}(X)v \). So \((\pi_1, H_1)\) is a Hilbert completion of \( \hat{V} \). Iterating this argument we see that if \( Z \subset H \) is the space of \( C^\infty \) vectors relative to \( \hat{\pi} \) with the usual Fréchet topology then the imbedding of \( Z \) into \( H_1 \) maps it continuously into \( H_1^{\infty} \). On the other hand, we have \( Z \) is isomorphic to \( H_2^{\infty} \) by \( H_2 \) so Theorem 1 implies that the identity map on \( \hat{V} \) induces a continuous isomorphism of \( H_1^{\infty} \) into \( Z \). This implies that \( Z = H_1^{\infty} \).

If \( \gamma \in \hat{K} \) let \( E_{\gamma} \) denote the orthogonal projection to the \( K \)-isotypic component \( H_1(\gamma) = \hat{V}(\gamma) \) we note that \( E_{\gamma}|_H \) is also the orthogonal projection onto \( H(\gamma) \). If \( \mu \in V^* \) then denote \( \mu \circ E_{\gamma} \) by \( \mu_{\gamma} \), then there is a unique element \( \tau(\mu_{\gamma}) \in V(\gamma) \) such that

\[ \mu_{\gamma}(v) = (v, \tau(\mu_{\gamma})), v \in V. \]

Note that \( \tau \) is conjugate linear and we will identify \( \mu_{\gamma} \) with \( \tau(\mu_{\gamma}) \) as an element of \( \hat{V}(\gamma) \). Let for \( \mu \in V^*_{\text{mod}} \) and \( v \in V \), \( f_{\mu,v} \in \mathcal{A}_{\text{mod}}(G) \) be as in the beginning of this proof. We also note that if \( \mu \in V(\gamma)^* \) then

\[ f_{\mu,v}(g) = (\pi(g)v, \tau(\mu)) = (v, \hat{\pi}(g)^{-1}\tau(\mu)). \]

We now return to the element \( \lambda \) as in the beginning of the proof. We will drop the \( \tau \) and consider \( \lambda_{\gamma} \) an element of \( \hat{V}(\gamma) \). We assert that \( \sum \lambda_{\gamma} \) converges in \( H_1 \). We prove this assertion starting with the observation that (2) above implies

\[ \int_G |f_{\lambda,v}(g)|^2 \|g\|^{-2d-d_o} dg < \infty. \]

We note that if \( \chi_{\gamma} \) is the character of \( \gamma \) then

\[ f_{\lambda_{\gamma},v}(g) = d(\gamma) \int_K \chi(k^{-1}) f_{\lambda}(kg)dk. \]
Thus using fact that $\|kg\| = \|g\|$ for all $k \in K$ and $g \in G$ and the Schur orthogonality relations we have

$$\int_G |f_{\lambda, v_i}(g)|^2 \|g\|^{-2d-d_o} \, dg = \sum_{\gamma \in \hat{K}} \int_G |f_{\lambda, v_i}(g)|^2 \|g\|^{-2d-d_o} \, dg.$$

This implies that

$$\infty > \sum_{i=1}^n \sum_{\gamma \in \hat{K}} \int_G |f_{\lambda, v_i}(g)|^2 \|g\|^{-2d-d_o} \, dg =$$

$$\sum_{i=1}^n \sum_{\gamma \in \hat{K}} \int_G |(v_i, \hat{\pi}(g)^{-1} \lambda_{\gamma})|^2 \|g\|^{-2d-d_o} \, dg =$$

$$\sum_{i=1}^n \sum_{\gamma \in \hat{K}} \int_G |(v_i, \hat{\pi}(g) \lambda_{\gamma})|^2 \|g^{-1}\|^{-2d-d_o} \, dg =$$

$$\sum_{i=1}^n \sum_{\gamma \in \hat{K}} \int_G |(v_i, \hat{\pi}(g) \lambda_{\gamma})|^2 \|g\|^{-2d-d_o} \, dg = \sum \|\lambda_{\gamma}\|^2_1.$$

**Lemma 14** The topology of $H^\infty_1$ is given by the seminorms

$$p_l(v) = \|(1 + C_K)^l v\|_1.$$

We will give a proof in the appendix below, there is also a more elementary proof in [RRGII]. In the appendix we prove that this is also true for any Hilbert representation for which the Casimir operator of $G$ acts by a scalar on the $C^\infty$-vectors. We now continue with the proof.

We note that $p_l(v) \leq p_{l+1}(v)$ for all $v \in H^\infty_1$. Thus since $(\hat{\pi}, H^\infty) = H^\infty_1$ we see that $\|...\|$ defines a continuous norm on $H^\infty_1$. This implies that there exists $l$ and $B > 0$ such that

$$\|v\| \leq B \|(1 + C_K)^l v\|_1.$$

Let $C_K$ act on $V_{\gamma}$ by $\mu(\gamma)I$. Then the above inequality implies that

$$(1 + \mu(\gamma))^{-l} \|\lambda_{\gamma}\| \leq B \|\lambda_{\gamma}\|_1.$$
This implies that
\[ \sum (1 + \mu(\gamma))^{-l}\lambda_{\gamma} \]
converges in \( H \), to an element \( u \). Thus if \( v \) then
\[ ((I + C_K)^lv, u) = \lambda(v). \]
Hence
\[ |\lambda(v)| \leq B_1 \|(1 + C_k)^lv\| . \]
Thus \( \lambda \) extends to a continuous functional on \((\pi, H^\infty)\). This completes the proof.

We now complete the proof of Theorem 10. Let \( S : V \to W \) be a surjective morphism in \( \mathcal{H}(g, K) \) then we have \( S : \hat{V} \to \hat{W} \) is an injective morphism. Let \( F = \hat{V}/\mathfrak{n}^{k_0}\hat{V} \) as a \( P \) module then \( \hat{V} \) is the closure of \( T_{k_0}(\hat{V}) \) in \((\pi, H^\infty)\) where \( H \) is the closure of \( T_{k_0}(\hat{V}) \) in the Hilbert representation \( \text{Ind}_P^G(F) \). For simplicity we replace \( \hat{V} \) with \( T_{k_0}(\hat{V}) \) and \( \hat{W} \) with it’s image in \( \hat{V} \). Let \( H_1 \) be the closure of \( \hat{W} \) in \( H \) and if \( \pi_1 \) is the corresponding action of \( G \) on \( H_1 \) then \((\pi_1, H_1^\infty)\) is equivalent with \( \hat{W} \). Now \((\hat{\pi}, H)\) and \((\hat{\pi}_1, H_1)\) are Hilbert realizations of \( V \) and \( W \) respectively. The corresponding \( C^\infty \) vectors are therefore respectively \( \overline{V} \) and \( \overline{W} \) by Theorem 12. If \( v \in H \) then \( \lambda_v(w) = (v, w) \) defines an element of the conjugate dual. The restriction of \( \lambda_v, v \in H \) to \( H_1 \) yields a Hilbert representation surjection of \((\hat{\pi}, H)\) to \((\hat{\pi}_1, H_1)\) taking \( C^\infty \) vectors completes the proof.

4.3.1 Appendix: A proof of Lemma 14

Let \( C \) and \( C_K \) be the Casimir operators of \( G \) and \( K \) respectively (as in Section 2) we set \( \Delta = C - 2C_K \). Noting that \( \langle X, Y \rangle = \text{tr}(XY^T) \) defines an inner product on \( \text{Lie}(G) \), \( \Delta = \sum X_i^2 \) for \( X_1, ..., X_m \) an orthonormal basis of \( \text{Lie}(G) \). We note that \( \Delta \) is an elliptic operator on \( G \). Let \( (\pi, H) \) be a Hilbert representation of \( G \) such that \( V = H_K^\infty \) is in \( \mathcal{H}(g, K) \). Let \( Z \) be the completion of \( V \) relative to the seminorms \( q_l(v) = \|\Delta^lv\|, l = 0, 1, 2, ... \) Then since \( q_0 = \|...\| \), \( Z \) can be looked upon as a subspace of \( H \). Also \( H_K^\infty \) is the completion of \( V \) using the seminorms \( s_x(v) = \|xv\| \) with \( x \in U(g) \). Thus \( Z \supset H^\infty \).

**Lemma 15** \( Z = H^\infty \). Furthermore, the topology on \( H^\infty \) is given by the semi-norms \( q_l \).
Proof. We note that the second assertion is a direct consequence of the open mapping theorem. We will now prove the first assertion. Let \( v \in Z \subset H \). We must prove that \( v \in H^\infty \). Let \( v_j \in V \) be a sequence converging to \( v \) in the topology of \( Z \). Let \( w \in H \) then for all \( j \)
\[
\Delta^k(\pi(g)v_j, w) = (\pi(g)\Delta^k v_j, w).
\]
Furthermore, since \( q_l(\Delta^k v) = q_{l+k}(v) \) and \( q_l(v) \leq q_{l+1}(v) \) we see that for fixed \( k \) the sequence \( \Delta^k v_j \) converges to \( u_k \) in \( Z \).

We assert that the function \( g \mapsto (\pi(g)v, w) \) is \( C^\infty \). Since \( w \in H \) is arbitrary this would imply that the map \( g \mapsto \pi(g)v \) is weakly \( C^\infty \). But a weakly \( C^\infty \) map of a finite dimensional manifold into a Hilbert space is strongly \( C^\infty \) (c.f. [S]). This is exactly the statement that \( v \) is a \( C^\infty \) vector. We now prove the assertion. We first show that if we look upon the continuous function \( h(g) = (\pi(g)v, w) \) as a distribution on \( G \) (using the Haar measure on \( G \)) then in the distribution sense
\[
\Delta^k h(g) = (\pi(g)u_k, w).
\]
Indeed, let \( f \in C^\infty_c(G) \) then
\[
\int_G h(g)\Delta^k f(g)dg = \lim_{j \to \infty} \int_G (\pi(g)v_j, w)\Delta^k f(g)dg = \\
\lim_{j \to \infty} \int_G \Delta^k(\pi(g)v_j, w)f(g)dg = \lim_{j \to \infty} \int_G (\pi(g)\Delta^k v_j, w)f(g)dg = \\
\int_G (\pi(g)u_k, w)f(g)dg
\]
as asserted. Since \( \Delta \) is elliptic, local Sobolev theory (c.f. [F, Chapter 6]) implies that \( h \in C^\infty(G) \). \( \blacksquare \)

We will now prove Lemma [14]. We note that
\[
\Delta^k = \sum_{j=0}^{k} (-2)^j \binom{k}{j} C^{k-j} K^j.
\]
If \( v \in \hat{V} \) then
\[
\| Cv \|_1^2 = \sum_{i=1}^{n} \int_G |(v_i, \widetilde{\pi}(g)d\widetilde{\pi}(C)v)|^2 \| g \|^{-2d-d_o} =
\]
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Now \( d\pi(C)v_i = \sum a_{ij}v_j \) hence \((d\pi(C)v_i, \hat{\pi}(g)v) = \sum a_{ji}(v_j, \hat{\pi}(g)v)\). Hence setting \( A = \max_{ij} \{|a_{ij}|\} \)

\[
|(d\pi(C)v_i, \hat{\pi}(g)v)| = |\sum a_{ji}(v_j, \hat{\pi}(g)v)| \leq A \sum_j |(v_j, \hat{\pi}(g)v)|
\]

so

\[
\sum_i (d\pi(C)v_i, \hat{\pi}(g)v))^2 \leq nA^2 \left( \sum_j |(v_j, \hat{\pi}(g)v)| \right)^2 \leq n(A^2 + 1) \sum_i |(v_i, \hat{\pi}(g)v)|^2.
\]

Thus

\[
\|Cv\|_1 \leq \sqrt{n(A^2 + 1)} \|v\|_1.
\]

Set \( B = \sqrt{n(A^2 + 1)} \). Then

\[
\|\Delta^k v\|_1 = \left\| \sum_{j=0}^k \binom{k}{j} C^{k-j} C^j v \right\|_1 \leq \sum_{j=0}^k (2)^j \binom{k}{j} B^{k-j} \|C^j v\|_1 \leq \sum_{j=0}^k (2)^j \binom{k}{j} B^{k-j} \|(1 + C^j) v\|_1
\]

We note that in the general case we have

**Proposition 16** If \((\pi, H)\) is a Hilbert representation of \(G\) such that \(C\) acts as the scalar \(c\) on \(H^\infty\) then topology of \(H^\infty\) is given by the semi-norms \( p_l(v) = \| (I + C^j)^l v \|, l = 0, 1, 2, \ldots \)

**Proof.** Notice that in the proof of Lemma 15 the assumption of admissibility is never used. Under the assumption that \(C\) acts by \(c\) on \(H^\infty\) implies that if \(v \in H^\infty\)

\[
\|\Delta^k v\| = \left\| \sum_{j=0}^k \binom{k}{j} C^{k-j} C^j v \right\| \leq \sum_{j=0}^k (2)^j \binom{k}{j} |c|^{k-j} \|C^j v\| \leq \sum_{j=0}^k (2)^j \binom{k}{j} |c|^{k-j} \|(1 + C^j) v\|.
\]

\[\Box\]
4.4 Step 3: All objects in $\mathcal{H}(\mathfrak{g}, K)$ are good.

First some functorial properties of goodness (see [RRGII,11.7.2], [C,7.16]) Casselman uses the term regular for our good and uses a single bar for our double bar completion and vice-versa).

Lemma 17 Let $V \in \mathcal{H}(\mathfrak{g}, K)$

1. If $V$ is good then $\hat{V}$ is good
2. If $V$ is good and if $W$ is a summand of $V$ then $W$ is good.
3. If $V \in \mathcal{H}(\mathfrak{g}, K)$ and if every irreducible subquotient of $V$ is good then $V$ is good.

There are no new ideas in the proof of this lemma. Also using similar arguments to those in the previous section we have (see [RRGII,11.7.3])

Lemma 18 Let $Q = L_Q N_Q$ be a standard parabolic subgroup of $G$ (i.e. $Q \supset P$) with $L_Q = Q \cap \theta(Q)$ and $N_Q$ the unipotent radical of $Q$. Then if $W$ is a good object in $\mathcal{H}(\text{Lie}(L_Q), K \cap L_Q)$ and $N_Q$ acts locally finitely then the corresponding induced $(\mathfrak{g}, K)$-module is good.

This combined with the proof that square integrable representations are good ([RRGII,11.7.4]) implies that tempered representations are good.

Finally we have here we defer to Casselman’s much cleaner version of the end game([C,Section 9]), which involves the Langland’s quotient theorem and a deformation argument due to him.

Theorem 19 If $V \in \mathcal{H}(\mathfrak{g}, K)$ then $V$ is good.

5 Adding parameters

5.1 Some results of van der Noort

Let $\Omega$ be an open subset of $\mathbb{C}^n$ then following [vdN] we will first define a holomorphic family of objects in $\mathcal{H}(\mathfrak{g}, K)$ in terms of two conditions. We start with pair $(\pi, V)$ with $V$ a $(\text{Lie}(K) \otimes \mathbb{C}, K)$-module and

1. $\pi : \Omega \times \mathfrak{g} \to \text{End}(V)$ such that each $z \in \Omega$ the operators $\pi(z, X)$ for $X \in \mathfrak{g}$ define a representation of $\mathfrak{g}$ compatible with the $K$-action.

Thus we have a map $\pi : \Omega \times U(\mathfrak{g}) \to \text{End}(V)$. 

Lemma 20 If $W \subset U(\mathfrak{g})$ and $v \in V$ is a finite dimensional subspace then $\dim \text{Span}_\mathbb{C} \pi(\Omega, W)v < \infty$.

Proof. We may assume that $W$ is invariant under $Ad(K)$. Let $Z = \text{span}_\mathbb{C} K v$ then $\dim Z < \infty$. Let $W \otimes Z = \bigoplus_{\gamma \in S} (W \otimes Z)(\gamma)$ be its isotypic decomposition with $S$ the finite subset of $\hat{K}$ consisting of those $\gamma$ such that $(W \otimes Z)(\gamma) \neq \{0\}$. Then

$$\text{Span}_\mathbb{C} \pi(\Omega, W)v \subset \bigoplus_{\gamma \in S} V(\gamma)$$

which is finite dimensional. ■

In light of this we can define the holomorphy condition to be

2. For each $W \subset U(\mathfrak{g}), L \subset V$ finite dimensional the map $\Omega \times W \times L \to \text{span}_\mathbb{C} \pi(\Omega, W)L$ given by $(z, w, u) \mapsto \pi(z, w)u$ is holomorphic.

A triple $(\Omega, \pi, V)$ will be called a holomorphic family of objects in $\mathcal{H}(\mathfrak{g}, K)$ if $\Omega$ is open in $\mathbb{C}^n$ for some $n$, $V$ is an admissible $(\text{Lie}(K) \otimes \mathbb{C}, K)$-module and $\pi$ satisfies 1.and 2. above.

Example 21 We first note that we can define the same concept for a standard parabolic subgroup $Q$ of $G$ and $Q \cap K$. Then if we have a holomorphic family of $\mathcal{H}(\text{Lie}(Q) \otimes \mathbb{C}, K \cap Q)$-modules, $(\sigma, W)$ on $\Omega$ then if we form the $K$-finite induced representation $V = \text{Ind}_{M \cap Q}^K(W)$ then let $\pi(z, X)$ be the action on $\text{Ind}_Q^G(\sigma(z, \cdot))$. We ill call this example a parabolically induced holomorphic family. This includes the parabolically induced representations $I_{Q, \sigma, v}$ with $\sigma$ admissible for $M_Q$ and $\Omega = \text{Lie}(A_Q)^{\ast}$.

We will now state one of the main results in [vdN] which is his Theorem 3.2.11.

Theorem 22 Let $(\pi, V, \Omega)$ be a holomorphic family of objects in $\mathcal{H}(\mathfrak{g}, K)$ then for each $z_0 \in \Omega$ there exists $U \subset \Omega$ an open neighborhood of $z_0$ and $W \subset V$ a finite dimensional subspace of $V$ such that $\pi(z, U(\mathfrak{g}))W = V$ for all $z \in U$.

van der Noort’s theorem is stronger but this is all we will need.

Fix a Cartan subalgebra, $\mathfrak{h}$, of $\mathfrak{g}$. Denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. We use the Harish-Chandra parametrization of homomorphisms of $Z(\mathfrak{g})$ to $\mathbb{C}$. That is $\chi = \chi_\Lambda$ with $\Lambda \in \mathfrak{h}^\ast$ (determined up to the action of the Weyl group). As part of his proof of this result he proves (Proposition 3.2.5 in [vdN])
Theorem 23 If $X$ is a compact subset of $\Omega$ then the set of Harish-Chandra parameters of the generalized infinitesimal characters of the $\pi(z, \cdot)$ is contained in a compact subset of $\mathfrak{h}^*$.

5.2 Families of Hilbert representations

A holomorphic family of admissible Hilbert representations of $G$ is a triple $(\Omega, \pi, H)$ with $\Omega$ an open subset of $\mathbb{C}^n$ invariant under complex conjugation, $(\tau, H)$ a unitary representation of $K$ such that the $K$-finite vectors form an admissible $(\text{Lie}(K) \otimes \mathbb{C}, K)$-module and $\pi$ is a strongly continuous map of $\Omega \times G$ to the bounded operators on $H$ such that $g \mapsto \pi(z, g)$ defines an admissible finitely generated representation of $G$ for each $z \in \Omega$ and if $(..., ...,)$ is the inner product on $H$ then the map $z \mapsto (\pi(z, g)v, w)$ is holomorphic for all $g \in G, v, w \in H$. We define the conjugate dual family to be $\hat{\pi}(z, g) = \pi(z, g^{-1})^*$ we note that this family is antiholomorphic (i.e. $\hat{\pi}(\bar{z}, g)$ defines a holomorphic family).

For simplicity we will assume that on the $K$–finite vectors of $H$, $d\pi(z, C)$ acts by a scalar. Thus if $V$ is the space of $K$–finite vectors then the space of $C^\infty$ vectors for $\pi(z, \cdot)$ is equal to the completion of $V$ relative to the seminorms $p_l(v) = \| (I + C_K)^l v \|$, $l = 1, 2, ...$ (see Proposition 16). We leave it to the reader to give the analogue of Example 21 in this context we will call this a parabolically induced family..

Let $d\pi(z, x)$ denote the action of $x \in U(\mathfrak{g})$ on $H^\infty$ and on $V$.

Proposition 24 1. $(\Omega, d\pi, V)$ is a holomorphic family of objects in $\mathcal{H}(\mathfrak{g}, K)$.

2. If $\lambda \in (H^\infty)'$, $v \in H^\infty$ then the map $z, g \mapsto \lambda(\pi(z, g)v)$ is $C^\infty$ and holomorphic in $\Omega$.

Proof. The second assertion follows from the fact that there exists $l \in \mathbb{Z}_{\geq 0}$ and $u \in H$ such that $\lambda(v) = ((I + C_K)^l v, u)$ for $v \in H^\infty$. ■

We will say that a holomorphic family of Hilbert representations, $(\Omega, \pi, H)$ is locally of uniform moderate growth if for each $z_0 \in \Omega$ there exist $d_{z_0}, C_{z_0}$ and $U \subset \Omega$ an open neighborhood of $z_0$ such that $\| \pi(z, g) \| \leq C_{z_0} \| g \|^{d_{z_0}}$ for $z \in U, g \in G$. One can check that a parabolically induced family satisfies this condition.
5.3 Theorem 2 with parameters

We now know that if $V \in H\mathcal{F}_{\text{mod}}(G)$ then $V$ is isomorphic with $(V_K)$ which
is isomorphic with $(V_K)$. The theorem there for now says

**Theorem 25** Let $V \in H\mathcal{F}_{\text{mod}}(G)$ then $\overline{V}_{V_K} = (V_K)^{\ast}_{\text{mod}}$.

Let $(\Omega, \sigma, V)$ be a Holomorphic family of objects in $\mathcal{H}(g, K)$. We define
$V_z \in H\mathcal{F}_{\text{mod}}(G)$ to be $(\sigma(z, \cdot), V)$ for $z \in \Omega$. If $\lambda_z \in (V_z)^{\ast}_{\text{mod}}$ for each $z \in \Omega$ then $z \mapsto \lambda_z$
will be called holomorphic if the correspondence $z \mapsto \lambda_z(v)$ is holomorphic
for all $v \in V$. We will say that this family is locally of uniform growth on $U$
if for each $v \in V$ and $z_0 \in \Omega$ there is $U \subset \Omega$ an open neighborhood of $z_0$ and
$C_{U,v}$ and $d_{U,v}$ such that

$$|f_{\lambda_z,v}(g)| \leq C_{U,v} \|g\|^{d_{U,v}}, \quad z \in U.$$  

Our main result is

**Theorem 26** Assume that $(\Omega, \sigma, V)$ is a Holomorphic family of objects in $\mathcal{H}(g, K)$ such that there is a holomorphic family of admissible Hilbert representations $(\Omega, \pi, H)$ of local uniform moderate growth such that $\sigma = d\pi$ and $V = H_{|K}$. If $z \mapsto \lambda_z \in (V_z)^{\ast}_{\text{mod}}$ is a holomorphic on $\Omega$ of local uniform moderate growth on $\Omega$ and if we (also) denote the extension of $\lambda_z$ to $H^\infty$
then $z \mapsto \lambda_z$ is weakly holomorphic from $\Omega$ to $(H^\infty)'$.

**Proof.** We will follow the proof of Theorem 12 which can be found in section
4.3 we will use notation from that section. Let $z_0 \in \Omega$ and let $U \subset \Omega$ be an
open neighborhood of $z_0$ such that there exists $d$ and $C > 0$ such that for
$z \in U$
1. $\|\hat{\pi}(z,g)\| \leq C \|g\|^d$ for some $C > 0$ and
2. $v_1$, ..., $v_n$ an orthonormal basis of a finite sum of $K$–isotypic components
$W \subset V$ such that $V = d\pi(z, U(g))W$ for $z \in U$.
3. $|f_{\lambda_z,v_i}(g)| \leq C_{v_i} \|g\|^d$ for $i = 1, ..., n$.

We note that 2. is possible by Theorem 22 of van den Noort quoted above. 3. is a consequence of the local uniformity.

If $z \in U, v, w \in H$ we define $(d_o$ is as in section 4.3)

$$\langle v, w \rangle_z = \sum_{i=1}^{n} \int_{G} (v_i, \hat{\pi}(z,g)v)(v_i, \hat{\pi}(z,g)w) \|g\|^{-2d-o} dg.$$
We observe that if we argue as in Theorem 12 and \( \|v\|_z \) is the corresponding norm we have for all \( z \in U, v \in H \)

\[
\|v\|_z \leq C_1 \|v\| \quad (\ast)
\]

with \( C \sqrt{n \int_G \|g\|^{-d_0} dg} \). The inequality (\ast) implies that \((H^\infty_z)^K\) is isomorphic with \( \hat{\mathcal{V}}_z \). As in the proof of the theorem without parameters if \( \lambda_{z,\gamma} = \lambda_z \circ E_\gamma \) then we have

\[
\sum_{\gamma \in \hat{K}} \|\lambda_{z,\gamma}\|_z^2 \leq C_2 < \infty \quad (\ast\ast)
\]

with \( C_2 = n \max \{ C_i^2 \} \int_G \|g\|^{-d_0} dg \). Thus \( \lambda_z \) defines an element of \( H_z \). We now note that \( \langle v, w \rangle_z = (A_z v, w) \) for \( v, w \in H \) and \( A_z \) is a bounded self adjoint positive operator on \( H \). We also note that \( z \mapsto A_z \) is real analytic in the strong operator topology (this is because weak analyticity is the same as strong analyticity) here use

\[
\langle v, w \rangle_z = \sum_{i=1}^n \int_G (\pi(z, g)v_i, v)(\pi(z, g)v_i, w) \|g\|^{-2d-d_0} dg.
\]

Also, since the topology of \( (\hat{\pi}(z), H^\infty_z) \) is given by the seminorms \( v \mapsto \|(I + C_K)^{1/2}v\|_z = \|(I + C_K)^{1/2}A_z^{1/2}v\| \) or by the semi-norms \( v \mapsto \|(I + C_K)^{1/2}v\| \) this implies that \( B_z = A_z^{1/2} \) defines an isomorphism of \( H^\infty_z \). The upshot is

\[
\sum B_z \lambda_{z,\gamma} = u_z \in H.
\]

The map \( z \mapsto u_z \) is real analytic. We therefore have \( \lambda_z = B_z^{-1} u_z \) so it is real analytic from \( U \) into \( (H^\infty)^\prime \). Since \( z \mapsto \lambda_z(v) \) is holomorphic for \( v \) in the dense set \( V \) we see that \( z \mapsto \lambda_z \) is weakly holomorphic from \( U \) into \( (H^\infty)^\prime \).

\[\blacksquare\]

5.3.1 Holomorphic families of automorphic forms and Eisenstein series.

We assume that \( G \) has compact center consider the example of automorphic forms in subsection 3.1. Let \( \Gamma \) be subgroup of \( G \) of finite covolume and let

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Ω be open in \( \mathbb{C}^n \). If \( f(z, g), z \in \Omega \) is holomorphic in \( z \) and automorphic on \( G \) relative to \( \Gamma \) then the theory of the constant term implies the local uniformity of the previous theorem.

We note that the next result will follow from the work in Langlands [L] so one only needs to assume that \( G \) and \( \Gamma \) satisfy the hypotheses in the first chapter of [L]. This material is difficult and (amazingly) the fact that \( h \) is properties are actually possible is not proved until the end of (the notorious) Chapter 7. We therefore work in the simpler context assuming that \( G \) is the set of real points of an algebraic group defined over \( \mathbb{Q} \) and \( \Gamma \) arithmetic with respect to this \( \mathbb{Q} \)-structure. We choose \( K \) so that \( \mathbb{Q} \) is \( K \)-standard. Before we can state the result for Eisenstein series we need some notation. Let \( Q \) be the real points of a parabolic subgroup of \( G \) defined over \( \mathbb{Q} \). Let \( U \) be the unipotent radical of \( Q \) and let \( Q = LA_Q\) be a standard \( \mathbb{Q} \)-Langlands decomposition of \( Q \) (i.e. \( A_Q \) is the real points of a maximal \( \mathbb{Q} \)-split torus of \( Q \)). We identify \( L \) with \( Q/A_QU \). Set \( \Gamma_L = (\Gamma \cap Q) / (\Gamma \cap (A_QU)) \) and define \( I(\Gamma_L) \) to be the space of all \( f \in C^\infty(\Gamma_L \setminus L \times K) \) such that

1. If \( x \in L, u \in K \cap L, k \in K \) then \( f(xu, u^{-1}k) = f(x, k) \).
2. If we let \( Z(\text{Lie}(L)) \) act on the first factor then \( \dim Z(\text{Lie}(L))f < \infty \).
3. Letting \( U(\text{Lie}(L)) \) act on the first factor and \( U(\text{Lie}(K)) \) act on the second here exists \( C_{f,x,y} \) and \( d_f \) so that
   \[
   |xyf(u, k)| \leq C_{f,x,y}|u|^{d_f}, x \in U(\text{Lie}(L)), y \in U(\text{Lie}(K)).
   \]
4. For each \( k \in K \) the \( u \mapsto f(u, k) \) is a \( \Gamma \)-cusp form that is if \( P \) is the real points of parabolic subgroup of \( L \) defined over \( \mathbb{Q} \) with unipotent radical \( U_P \) then
   \[
   \int_{U_Q \setminus U} f(u, k)du = 0
   \]
   for all \( x \in L \) and \( k \in K \).

Let \( \mu : Z(\text{Lie}(L)) \rightarrow \mathbb{C} \) be a homomorphism and let \( I_\mu(\Gamma_L) \) be the subspace of \( I(\Gamma_L) \) consisting of elements \( h \) such that \( uh = \mu(u)h \). We define an inner product on \( I_\mu(\Gamma_L) \):

\[
(h_1, h_2) = \int_{\Gamma_L \setminus L \times K} h_1(u, k) \overline{h_2(u, k)}dudk.
\]

Defining a unitary representation of \( K \) on the Hilbert space completion \( H_\mu(\Gamma_L) \) of \( I_\mu(\Gamma_L) \) by right translation in the second variable. For \( \nu \in \)
Lie(\(A_Q\))\(^*_C\) and \(f \in I_\mu(\Gamma_L)\) we define \(\tilde{f}_\nu(ua.ak) = a^{\nu+\rho_Q}f(x,k)\) for \(u \in U, a \in A_Q, x \in L\) and \(k \in K\). Then \(\tilde{f}_\nu \in C^\infty(G)\). We define \(\pi(\nu, x)f(u, k) = R(x)\tilde{f}_\nu(uk)\) (here \(R(x)\phi(g) = \phi(gx)\)). This defines a holomorphic family of Hilbert representations on \(H_{\mu,\nu}(\Gamma_L)\) of local uniform (in \(\nu\)) moderate growth.

We note that \((A_QU \cap \Gamma)/(U \cap \Gamma)\) is finite. If \(f \in I_\mu(\Gamma_L)\) we define

\[
f_\nu(g) = \sum_{\gamma \in (A_QU \cap \Gamma)/(U \cap \Gamma)} \tilde{f}_\nu(\gamma g).
\]

Finally, the corresponding Eisenstein series is

\[
E(Q, f, \nu)(g) = \sum_{\gamma \in \Gamma \cap Q \backslash \Gamma} f_\nu(\gamma g).
\]

Langlands has proven a meromorphic continuation of \(K\)-finite Eisenstein series ([L]) (that is \(f\) is right \(K\)-finite in the \(K\) variable). We can think of these series as giving a meromorphic family of elements if \((\pi(\nu), H_\mu(\Gamma_L))_{K}^*\) we have

**Theorem 27** If \(f \in I(\Gamma_L)\) then the Eisenstein series \(E(Q, f, \nu)\) initially defined and holomorphic for \(\text{Re} \, \nu(\breve{\alpha}) > C \rho_Q(\breve{\alpha})\) for some \(C\) depending on \(f\) and \(Q\) and all positive roots appearing in the unipotent radical of \(Q\) has a meromorphic continuation to \((\text{Lie}(A) \otimes \mathbb{C})^*\).

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