THE $Q_p$ CARLESON MEASURE PROBLEM

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In memory of Matts Essén

Abstract. Let $\mu$ be a nonnegative Borel measure on the open unit disk $D \subset \mathbb{C}$. This note shows how to decide that the Möbius invariant space $Q_p$, covering $BMOA$ and $B$, is boundedly (resp., compactly) embedded in the quadratic tent-type space $T_\infty^p(\mu)$. Interestingly, the embedding result can be used to determine the boundedness (resp., the compactness) of the Volterra-type and multiplication operators on $Q_p$.

1. Introduction

Continuing from [30], we answer the following question, i.e., the Carleson measure problem for the holomorphic $Q$-spaces (which are geometric in the sense that they are conformally invariant):

Question 1.1. Let $\mu$ be a nonnegative Borel measure on $D$. What geometric finite (resp., vanishing) property must $\mu$ have in order that $Q_p$ is boundedly (resp., compactly) embedded in $T_\infty^p(\mu)$?

Here, $D = \{z \in \mathbb{C} : |z| < 1\}$ and $p \in (0, \infty)$ are given, and $Q_p$ stands for the space of all holomorphic functions $f$ on $D$ satisfying

$$
\|f\|_{Q_p} = |f(0)| + \sup_{w \in \mathbb{D}} \sqrt{\int_D |f'(z)|^2 (1 - |\sigma_w(z)|^2)^p dm(z) < \infty},
$$

where $\sigma_w(z) = (w - z)/(1 - \bar{w} z)$ and $dm$ are the Möbius map sending $w \in \mathbb{C}$ to 0 and the Lebesgue area measure on $\mathbb{C}$ respectively; see [31] and [32] for an overview of the $Q_p$-theory (from 1995 to 2006) – in particular, $Q_{p_1} \subset Q_{p_2}$ when $0 < p_1 < p_2 \leq 1$ (see Aulaskari-Xiao-Zhao [6]); $Q_1 = BMOA$, John-Nirenberg’s BMO space of holomorphic functions on $D$ (see Baernstein [7]); and $Q_p = B$, $p \in (1, \infty)$, Aulaskari-Lappan’s result in [4] (including Xiao’s $Q_2 = B$ in [29]) regarding the Bloch space $B$ of all holomorphic functions $f$ on $D$ with $\sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2) < \infty$. Meanwhile, $T_\infty^p(\mu)$ denotes the quadratic tent-type space of all $\mu$-measurable functions $f$ on $D$ obeying

$$
\|f\|_{T_\infty^p(\mu)} = \sup_{S(I) \subseteq \mathbb{D}} \left( \int_{S(I)} \left| I \right|^{-p} \left| f \right|^2 d\mu < \infty; \right)
$$

where

$$
|I| = (2\pi)^{-1} \int_I |d\xi| \quad \text{and} \quad S(I) = \{r\xi \in \mathbb{D} : r \in [1 - |I|, 1], \xi \in I\}
$$

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are the normalized length of the subarc \( I \) of the unit circle \( T = \{ z \in \mathbb{C} : |z| = 1 \} \) and the Carleson square in \( \mathbb{D} \) respectively. In particular, \( d\mu(z) = (1 - |z|^2)^p dm(z) \) and \( p \in (0, 1) \) lead to the sphere tent space on \( \mathbb{D} \) extending the disc version of the classic one \( (p = 1) \) on the upper-half Euclidean space discussed in \cite{11} and \cite{10}.

Because the norm of \( f \in \mathcal{Q}_p \) is comparably dominated by the geometric quantity (see also Aulaskari-Stegenga-Xiao \cite{5}):

\[
|f(0)| + \sup_{S(I) \subseteq \mathbb{D}} \sqrt{\frac{\mu(S(I))}{|I|^p \left( \log \frac{2}{|I|} \right)^{-2}}} < \infty \quad \text{(resp., } \lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^p \left( \log \frac{2}{|I|} \right)^{-2}} = 0\text{)}
\]

our answer to Question 1.1 is naturally as follows.

**Theorem 1.1.** Let \( \mu \) be a nonnegative Borel measure on \( \mathbb{D} \). Then the identity operator \( \mathcal{Q}_p 
\rightarrow \mathcal{T}_p^\infty(\mu) \) is bounded (resp., compact) if and only if

\[
|f(0)| + \sup_{S(I) \subseteq \mathbb{D}} \sqrt{\frac{\mu(S(I))}{|I|^p \left( \log \frac{2}{|I|} \right)^{-2}}} < \infty \quad \text{(resp., } \lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^p \left( \log \frac{2}{|I|} \right)^{-2}} = 0\text{)}
\]

Based on the solution to Question 1.1, we also answer the following problem:

**Question 1.2.** Let \( g \) be holomorphic on \( \mathbb{D} \). What finite (resp., vanishing) property must \( V_g \) have in order that \( V_g \) or \( U_g \) or \( M_g \) is bounded (resp., compact) on \( \mathcal{Q}_p \)?

Here, \( V_g \) and \( U_g \) denote the Volterra-type operators with the holomorphic symbol \( g \) on \( \mathbb{D} \) respectively:

\[
V_g f(z) = \int_0^z g'(w)f(w)dw \quad \text{and} \quad U_g f(z) = \int_0^z g(w)f'(w)dw, \; z \in \mathbb{D}.
\]

At the same time, \( M_g \) is the pointwise multiplication determined by

\[
M_g f(z) = f(z)g(z) = f(0)g(0) + V_g f(z) + U_g f(z), \; z \in \mathbb{D}.
\]

Of course, in the above definition \( f \) is assumed to be holomorphic on \( \mathbb{D} \). Clearly, \( V_g f = U_f g \) and this operator generalizes the classic Cesaro operator \( \mathcal{C}(f)(z) = z^{-1} \int_0^z f(w)(1-w)^{-1}dw \) see also Siskakis \cite{21} for a survey on the study of such operators.

Below is the answer to Question 1.2.

**Theorem 1.2.** Let \( g \) be holomorphic on \( \mathbb{D} \), \( d\mu_{p,g}(z) = (1 - |z|^2)^p |g'(z)|^2 dm(z) \) and \( \|g\|_{\mathcal{H}^\infty} = \sup_{z \in \mathbb{D}} |g(z)| \). Then

(i) \( V_g \) is bounded (resp., compact) on \( \mathcal{Q}_p \) if and only if \( \|\mu_{p,g}\|_{\mathcal{C}} < \infty \) (resp., \( \lim_{|I| \to 0} |I|^{-p} \log^2(2/|I|)\mu_{p,g}(S(I)) = 0 \)).

(ii) \( U_g \) is bounded (resp., compact) on \( \mathcal{Q}_p \) if and only if \( \|g\|_{\mathcal{H}^\infty} < \infty \) (resp., \( g = 0 \)).

(iii) \( M_g \) is bounded (resp., compact) on \( \mathcal{Q}_p \) if and only if \( \|\mu_{p,g}\|_{\mathcal{C}} < \infty \) and \( \|g\|_{\mathcal{H}^\infty} < \infty \) (resp., \( g = 0 \)).

The proofs of Theorems 1.1-1.2 will be given in the subsequent two sections where the symbol \( U \approx V \) will mean that there are two constants \( c_1 \) and \( c_2 \) independent of said or implied variables or functions such that \( c_1 V \leq U \leq c_2 V \), and \( U \leq c_2 V \) will be simply written as \( U \preceq V \).

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2. Proof of Theorem 1.1

Suppose the statement before the if and only if of Theorem 1.1 is true. Given a subarc \( I \) of \( \mathbb{T} \). If \( f_w(z) = \log(1 - \bar{w}z) \) where \( w = (1 - |I|)\zeta \) and \( \zeta \) is the center of \( I \), then

\[
|f_w(z)| \approx \log(2|I|^{-1}), \quad z \in S(I)
\]

and

\[
|I|^{-p} \int_{S(I)} |f_w|^2 d\mu \lesssim \|f_w\|_{\mathcal{D}_p}^2 \lesssim 1.
\]

Accordingly, \( \|\mu\|_{\mathcal{LCM}_p} \lesssim 1 \).

Conversely, let the statement after the if and only if of Theorem 1.1 be true. To approach the desired embedding inequality, we recall that a nonnegative Borel measure \( \nu \) on \( \mathbb{D} \) is said to be a Carleson measure for the weighted Dirichlet space \( \mathcal{D}_p \) of all holomorphic functions \( f \) obeying

\[
\|f\|_{\mathcal{D}_p} = |f(0)| + \sqrt{\int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{\frac{1}{p-1}} dm(z)} < \infty
\]

provided \( \int_{\mathbb{D}} |f|^2 dv \lesssim \|f\|_{\mathcal{D}_p}^2 \) — see also Stegenga [24]. Note that \( p = 1 \) and \( p > 1 \) lead to the Carleson measure for the Hardy space \( \mathcal{H}^2 = \mathcal{D}_1 \) and the weighted Bergman space of all holomorphic functions in the Lebesgue space \( L^2((1 - |z|^2)^{p-1} dm(z)) \) with respect to \( (1 - |z|^2)^{p-1} dm(z) \). The following important result (written as a lemma for our purpose) is due to Carleson [9] (for \( p = 1 \)), Hastings [13] (for \( p = 2 \)), Stegenga [24] (for \( p \in [1, \infty) \)), and Arcozzi-Rochberg-Sawyer [3] (for \( p \in (0, 1) \)).

**Lemma 2.1.** Let \( \nu \) be a nonnegative Borel measure on \( \mathbb{D} \).

(i) If \( p \in [1, \infty) \) then \( \nu \) is a Carleson measure for \( \mathcal{D}_p \) if and only if

\[
\|\nu\|_{\mathcal{LCM}_p} = \sup_{S(I) \subseteq \mathbb{D}} \sqrt{\int |I|^{-p\nu(S(I))}} < \infty.
\]

(ii) If \( p \in (0, 1) \), then \( \mu \) is a Carleson measure for \( \mathcal{D}_p \) if and only if

\[
\|\nu\|_{\mathcal{LCM}_p} = \sup_{w \in \mathbb{D}} \sqrt{\int S(w)^{-1} \left( \nu(S(z) \cap S(w))^2 (1 - |z|^2)^{-p-2} dm(z) \right)} < \infty,
\]

where

\[
S(w) = \{ z \in \mathbb{D} : 1 - |z| \leq 1 - |w|, \quad |\arg(wz)| \leq \pi(1 - |w|) \}
\]

and

\[
\bar{S}(w) = \{ z \in \mathbb{D} : 1 - |z| \leq 2(1 - |w|), \quad |\arg(wz)| \leq \pi(1 - |w|) \}
\]

are the Carleson and heightened Carleson boxes with vertex at \( w \in \mathbb{D} \) respectively.

Since \( \|\mu\|_{\mathcal{LCM}_p} < \infty \) and \( \lim_{|I| \to 0} \log(2|I|^{-1}) = \infty \), \( \mu \) is a Carleson measure for \( \mathcal{D}_p \). This fact in the case \( p \in [1, \infty) \) is clear from Lemma 2.1 (i) because of \( \|\mu\|_{\mathcal{LCM}_p} \lesssim \|\mu\|_{\mathcal{LCM}_p} \). If \( p \in (0, 1) \) then this fact is due to Pau-Pelaez [19] and follows from Lemma 2.1 but a short proof is included below for completeness.
Fixing a point $w \in \mathbb{D}$, we use Fubini’s theorem to get
\[
\int_{S(w)} (\mu(S(z) \cap S(w)))^2 (1 - |z|^2)^{-p - 2} dm(z)
\]
\[
\leq \int_{S(w) \setminus S(w)} \frac{\mu(S(z)) \mu(S(w))}{(1 - |z|^2)^{p+2}} dm(z) + \int_{S(w)} \frac{\mu(S(z) \cap S(w)))^2}{(1 - |z|^2)^{p+2}} dm(z)
\]
\[
\lesssim \|\mu\|_{LCMP}^2 \mu(S(w)) \left( \int_{S(w) \setminus S(w)} \frac{dm(z)}{(1 - |z|^2)^2} + \int_{S(w)} \frac{dm(z)}{(1 - |z|^2)^2 \log^2 \frac{2}{1 - |z|^2}} \right)
\]
\[
\lesssim \|\mu\|_{LCMP}^2 \mu(S(w)) \left( 1 + \int_0^{|w|} \int_{\arg w + (1 - |z|)} \frac{dm(z)}{(1 - |z|^2)^2 \log^2 \frac{2}{1 - |z|^2}} \right)
\]
\[
\lesssim \|\mu\|_{LCMP}^2 \mu(S(w)) \left( 1 + \int_0^1 \frac{dt}{(1 - t) \log^2 \frac{2}{1 - |z|^2}} \right)
\]
whence $\|\mu\|_{LCMP} \lesssim \|\mu\|_{LCMP} < \infty$ according to Lemma 2.1 (ii).

Given any subarc $I$ of $\mathbb{T}$, let $w = (1 - |I|) \zeta$ and $\zeta$ be the center of $I$. Then
\[
|f(w)| \lesssim \|f\|_{Q_p} \log(2)|I|^{-1}, \quad f \in Q_p
\]
and
\[
(1 - |w|^2)/|1 - \bar{w}z|^2 \approx |I|^{-1}, \quad z \in S(I).
\]
Consequently, the above-verified fact that $\mu$ is a Carleson measure for $D_p$ yields
\[
|I|^{-p} \int_{S(I)} |f|^2 d\mu
\]
\[
\lesssim |I|^{-p} \left( \int_{S(I)} |f(z) - f(w)|^2 d\mu(z) + |f(w)|^2 \mu(S(I)) \right)
\]
\[
\lesssim (1 - |w|^2)^p \int_D \left| f(z) - f(w) \right|^2 \frac{d\mu(z)}{(1 - |w|^2)^p} + \|f\|^2_{Q_p} \|\mu\|_{LCMP}^2
\]
\[
\lesssim \|\mu\|^2_{LCMP} \left( \frac{|f(0) - f(w)|^2 + \int_D \frac{d\mu}{(1 - |w|^2)^p} \left( |f(z) - f(w)| \right|^2 \frac{dm(z)}{(1 - |z|^2)^{p}} + \|f\|^2_{Q_p}}{(1 - |w|^2)^{p}} \right)
\]
\[
\lesssim \|\mu\|^2_{LCMP} \left( \|f\|^2_{Q_p} \left( 1 + \frac{\log \frac{2}{1 - |w|^2}}{2} \right)^2 + \int_D \frac{d\mu}{(1 - |w|^2)^p} \left( |f(z) - f(w)| \right|^2 \frac{dm(z)}{(1 - |z|^2)^{p}} \right)
\]
\[
\lesssim \|\mu\|^2_{LCMP} \left( \|f\|^2_{Q_p} \int_D \left| f(z) - f(w) \right|^2 (1 - |\sigma_w(z)|^2)^p dm(z) \right)
\]
\[
\lesssim \|\mu\|^2_{LCMP} \|f\|^2_{Q_p}.
\]
In the last inequality we have used the following estimate:
\[
\Lambda(f, w, p) = \int_D \left| f(z) - f(w) \right|^2 (1 - |\sigma_w(z)|^2)^p dm(z) \lesssim \|f\|_{Q_p}.
\]
To check this estimate we extend largely Pau-Perlaez’s argument in [19] from $p \in (0, 1)$ to $p \in (0, \infty)$. Choosing $0 < \eta < p/2$ and $w = \sigma_\eta(v)$, we get by Zhu’s [36]
Theorem 1.12 (1)] that for any \( z \in \mathbb{D} \),

\[
\int_D \frac{(1 - |u|^2)^p - n}{|1 - zu|^{2n} |1 - wu|^2} dm(u) = \frac{1 - |z|^2}{|1 - w|^2} \int_D \frac{(1 - |u|^2)^p - n}{|1 - \bar{v}\sigma_z(w)|^2} dm(v) \leq \frac{(1 - |z|^2)^n}{|1 - w|^2} \int_D \frac{(1 - |u|^2)^n}{|1 - \bar{v}\sigma_z(w)|^2} dm(v) \leq \frac{(1 - |z|^2)^n}{|1 - w|^2}.
\]

The previous estimate, together with Rochberg-Wu-Zhu’s formula (see for example \[20\] (2.1) and \[35\], p. 75: Ex. 11), Cauchy-Bunyakovsky-Schwarz’s inequality, Fubini’s theorem and Zhu’s \[36\] Theorem 1.12 (3), implies a series of estimates below:

\[
\Lambda(f, w, p) = \int_D |f \circ \sigma_w(z) - f \circ \sigma_w(0)|^2 |1 - \bar{w}z|^{-2} (1 - |z|^2)^p dm(z) \approx \int_D |\int_D (f \circ \sigma_w)(u)(1 - |u|^2)^{1+p} \frac{1 - (1 - \bar{u}z)^{2+p}}{|1 - \bar{u}z|^{2+p}} dm(u)|^2 (1 - |z|^2)^p dm(z) \leq \int_D |(f \circ \sigma_w)(u)|^2 (1 - |u|^2)^{2+p+n} \frac{1 - (1 - |u|^2)^{2+p}}{|1 - |u|^2|^{2+p}} dm(u) \leq \int_D |f \circ \sigma_w(z)|^2 (1 - |z|^2)^{2+p+\eta} dm(z) \leq \int_D |f \circ \sigma_w(z)|^2 (1 - |z|^2)^{2+p+\eta} \leq \|f\|_{Q_p}^2.
\]

as desired.

Next, we verify the compactness part. According to Lemma 2.10 in \[25\], it suffices to show that any bounded sequence \( \{f_j\} \) in \( Q_p \) with \( f_j \rightarrow 0 \) being uniform on compacta of \( \mathbb{D} \) must obey \( \|f_j\|_{T_p^\infty(\mu)} \rightarrow 0 \).

Assume first the vanishing condition in Theorem 1.14 holds. For \( r \in (0, 1) \), define the cut-off measure \( d\mu_r = 1_{|z| \leq r} d\mu \), where \( 1_E \) denotes the characteristic function of a set \( E \subseteq \mathbb{D} \). If \( r \rightarrow 1 \) then

\[
\sup_{S(I) \subseteq \mathbb{D}} \frac{\mu_r(S(I))}{|I|^p \left( \log \frac{2}{|I|} \right)^{-2}} \leq \sup_{|I| \leq 1-r} \frac{\mu(S(I))}{|I|^p \left( \log \frac{2}{|I|} \right)^{-2}} \rightarrow 0.
\]

Suppose \( \|f_j\|_{Q_p} \leq 1 \) and \( f_j \rightarrow 0 \) uniformly on compacta of \( \mathbb{D} \). Then the limit

\[
\lim_{j \rightarrow -\infty} \|f_j\|_{T_p^\infty(\mu)} = 0
\]

follows from

\[
\int_{S(I)} |f_j|^2 d\mu \leq \int_{S(I)} |f_j|^2 1_{|z| \leq r} d\mu + \|f_j\|_{Q_p}^2 \log^2(2|I|^{-1}) \mu_r(S(I)).
\]

On the other hand, suppose \( Q_p \mapsto T_p^\infty(\mu) \) is compact. Let \( \{I_j\} \) be a sequence of subarcs of \( \mathbb{T} \) such that \( |I_j| \rightarrow 0 \). If \( \zeta_j \) is the center of \( I_j \), \( w_j = (1 - |I_j|) \zeta_j \), and

\[
f_j(z) = \left( \log \frac{1}{1 - w_j^2} \right)^{-1} \left( \log \frac{1}{1 - \bar{z}w_j} \right)^2,
\]

then

\[
f_j(z) = \left( \log \frac{1}{1 - w_j^2} \right)^{-1} \left( \log \frac{1}{1 - \bar{z}w_j} \right)^2.
\]
then $\|f_j\|_{p,g} \leq 1$ and $f_j \to 0$ uniformly on compacta of $\mathbb{D}$. By the compactness of $I$, we achieve that if $j \to \infty$ then

$$0 \leq \|f_j\|_{p,g}^2 \geq |I_j|^{-p} \int_{S(I_j)} |f_j|^2 d\mu \geq \frac{\mu(S(I_j))}{|I_j|^p \left(\log \frac{2}{|I_j|}\right)^2}.$$ 

In other words, the desired vanishing condition is valid.

**Remark 2.2.** Using [28, Theorem 6] or [32, Theorem 2.5.2] we can readily prove

$$\|f\|_{Q_p} \lesssim |f(0)| + \sqrt{\sup_{w \in \mathbb{D}} \int_{\mathbb{D}} \left|\frac{f(z) - f(w)}{1 - |z|^2}\right|^2 (1 - |\sigma_w(z)|^2) \rho dm(z)},$$

which is slightly different from the conjectured-inequality:

$$\|f\|_{Q_p} \lesssim |f(0)| + \sup_{w \in \mathbb{D}} \sqrt{\Lambda(f, w, p)}.$$ 

If this last estimate is true, then a new derivative-free characterization of $Q_p$ is discovered.

### 3. Proof of Theorem 1.2

(i) Note that $(V_g f)'(z) = f(z)g'(z)$. So, the boundedness part of Theorem 1.1 implies that $V_g$ maps boundedly $Q_p$ into itself is equivalent to $\|\mu_{p,g}\|_{LCM_p} < \infty$, as desired. The corresponding compactness can be demonstrated similarly. Nevertheless, in the sequel we provide a different argument which seems to be of independent interest. We begin with the following density result.

**Lemma 3.1.** If $\mathcal{L}Q_p$ and $\mathcal{L}Q_{p,0}$ denote the spaces of all holomorphic functions $g$ on $\mathbb{D}$ such that $\|\mu_{p,g}\|_{LCM_p} < \infty$ and $\lim_{|I| \to 0} |I|^{-p} \log^2 (2/|I|) \|\mu_{p,g}(S(I))\| = 0$ respectively, then $g \in \mathcal{L}Q_{p,0}$ if and only if $g \in \mathcal{L}Q_p$ and $\lim_{r \to 1} \|\mu_{p,g} - g\|_{LCM_p} = 0$, where $g_r(z) = g(rz)$ is the $(0, 1) \ni r$-dilation of $g$.

In fact, as in [30, Remark 1.6] we have that $g \in \mathcal{L}Q_p$, respectively, $g \in \mathcal{L}Q_{p,0}$, is equivalent to

$$\sup_{w \in \mathbb{D}} \left(\log \frac{2}{|w|} \right)^2 \int_{\mathbb{D}} \left(\frac{1 - |w|^2}{1 - w\zeta} \right)^p d\mu_{p,g}(\zeta) < \infty,$$

respectively,

$$g \in \mathcal{L}Q_p$$

and

$$\lim_{|w| \to 1} \left(\log \frac{2}{|w|} \right)^2 \int_{\mathbb{D}} \left(\frac{1 - |w|^2}{1 - w\zeta} \right)^p d\mu_{p,g}(\zeta) = 0.$$ 

Following the arguments for [22, Lemma 3.5] and [27, Theorem 2.1], we can reach the assertion described in Lemma 3.1.

Assume now $V_g$ is compact on $Q_p$. For each natural number $j$ let $I_j$ be the subarc of $T$ with center $\zeta_j$ and limit $|I_j| \to 0$. If $w_j = (1 - |I_j|)\zeta_j$, then $\{w_j\}$ has a cluster point $w_0 \in T$. Passing to a subsequence we may assume that $w_j \to w_0 \in T$ and $f_j \to f_0$ uniformly on compacta of $\mathbb{D}$, where

$$f_j(z) = \log(1 - \overline{w_j}z) \quad \text{and} \quad f_0(z) = \log(1 - \overline{w_0}z).$$
Accordingly, $V$ is compact on $Q_p$. As a matter of fact, let \( \{ f_j \} \) be any sequence with \( \| f_j \|_{Q_p} \leq 1 \) and \( f_j \to 0 \) uniformly on compacta of \( \mathbb{D} \). Then for the polynomial \( p_k \), the number \( r \in (0, 1) \), and the cut-off measure \( dm_{p,r}(z) = (1 - |z|^2)^p 1_{\{ z \in \mathbb{D} : |z| > r \}} \), we use the boundedness part
Choosing function \( w \) Conversely, suppose \( \| f \|_{\mathcal{H}^{1}} \) According to the boundedness part of (ii), the compactness of \( U_g \) on \( \mathcal{Q}_p \) derives the compactness of \( V_g \) on \( \mathcal{Q}_p \).

(ii) If \( \| g \|_{\mathcal{H}^{\infty}} < \infty \), then the boundedness of \( U_g \) follows from

\[
\| U_g f \|_{\mathcal{Q}_p} \lesssim \| g \|_{\mathcal{H}^{\infty}} \| f \|_{\mathcal{Q}_p}, \quad f \in \mathcal{Q}_p.
\]

Conversely, suppose \( U_g \) is bounded on \( \mathcal{Q}_p \). Then its operator norm \( \| U_g \| < \infty \). Given a nonzero point \( w \in \mathbb{D} \), there exists a Carleson square \( S(I) \) such that

\[
\{ z \in \mathbb{D} : |\sigma_w(z)| < 2^{-1} \} \subset S(I) \quad \text{and} \quad 1 - |w|^2 \approx |I|.
\]

Choosing \( f_w(z) = (\bar{w})^{-1} \log(1 - \bar{w}z) \) we employ the boundedness of \( U_g \) to obtain \( \| f_w \|_{\mathcal{Q}_p} \approx 1 \) and

\[
|I|^p |g(w)|^2 \lesssim \int_{\{ z \in \mathbb{D} : |\sigma_w(z)| < 2^{-1} \}} \frac{|g(z)|^2(1 - |z|^2)^p}{|1 - \bar{w}z|^2} \, dm(z)
\]

\[
\lesssim \int_{S(I)} \frac{|g(z)|^2(1 - |z|^2)^p}{|1 - \bar{w}z|^2} \, dm(z)
\]

\[
\lesssim \| U_g(f_w) \|_{\mathcal{Q}_p}^2 |I|^p
\]

\[
\lesssim \| U_g \|_{\mathcal{Q}_p}^2 |I|^p,
\]

and consequently, \( g \in \mathcal{H}^{\infty} \).

As with the compactness, it is enough to verify that if \( U_g : \mathcal{Q}_p \to \mathcal{Q}_p \) is compact then \( g = 0 \). According to the boundedness part of (ii), the compactness of \( U_g \) on \( \mathcal{Q}_p \) implies \( g \in \mathcal{H}^{\infty} \). Now, assume \( g \) is not identically equal to 0. Then the boundary value function \( g|_{\partial} \) cannot be identically the zero function thanks to the maximum principle. Accordingly, there is a positive constant \( \delta \) and a sequence \( \{ w_j \} \) in \( \mathbb{D} \) such
that \( w_j \to w_0 \in \mathbb{T} \) and \( |g(w_j)| > \delta \). Using the classical Schwarz’s lemma for \( \mathcal{H}^\infty \), we readily get

\[
|g(z_1) - g(z_2)| \leq 2\|g\|_{\mathcal{H}^\infty} |\sigma_{z_1}(z_2)|, \quad z_1, z_2 \in \mathbb{D}.
\]

This inequality implies that there is a sufficiently small number \( r > 0 \) such that \( |g(z)| \geq \delta/2 \) for all \( j \) and \( z \) obeying \( |\sigma_{w_j}(z)| < r \). Note that each pseudo-hyperbolic ball \( \{ z \in \mathbb{D} : |\sigma_{w_j}(z)| < r \} \) is contained in a Carleson box \( S(I_j) \) with \( |I_j| \approx 1 - |w_j|^2 \).

So, if

\[
f_j(z) = \log(1 - \overline{w_j}z) \quad \text{and} \quad f_0(z) = \log(1 - \overline{w_0}z),
\]

then \( f_j - f_0 \to 0 \) uniformly on compacta of \( \mathbb{D} \). By making a change of variable \( z = \sigma_{w_j}(u) \) and noticing \( |\sigma_{w_j}(w_j)| = 1 \), we obtain a series of estimates below:

\[
\begin{align*}
\|U_g(f_j - f_0)\|_{\mathcal{Q}_p}^2 & \geq |I_j|^{-p} \int_{S(I_j)} |f_j(z) - f_0(z)|^2 |g(z)|^2(1 - |z|^2)^p \, dm(z) \\
& \approx |I_j|^{-p} \int_{S(I_j)} \left| \frac{\overline{w_j} - \overline{w_0}}{1 - \overline{w_j}z(1 - \overline{w_0}z)} \right|^2 |g(z)|^2(1 - |z|^2)^p \, dm(z) \\
& \geq \delta^2 |I_j|^{-p} \int_{\{z \in \mathbb{D} : |\sigma_{w_j}(z)| < r \}} \left| \frac{\overline{w_j} - \overline{w_0}}{1 - \overline{w_j}z(1 - \overline{w_0}z)} \right|^2 (1 - |z|^2)^p \, dm(z) \\
& \geq \delta^2 |I_j|^{-p}(1 - |w_j|^2)^p \int_{\{u \in \mathbb{D} : |u| < r \}} \frac{(1 - |u|^2)^p |\sigma_{w_j}(w_j)|^2}{|1 - \overline{w_j}u|^2 |1 - u\overline{w_0}|^2} \, dm(u) \\
& \geq \delta^2 \int_{\{u \in \mathbb{D} : |u| < r \}} (1 - |u|)^p \, dm(u).
\end{align*}
\]

However, the compactness of \( U_g \) on \( \mathcal{Q}_p \) forces \( \|U_g(f_j - f_0)\|_{\mathcal{Q}_p}^2 \to 0 \), and consequently, \( \delta = 0 \), contradicting \( \delta > 0 \). Therefore, \( g \) must be the zero function.

(iii) The “if” part follows immediately from the corresponding ones of (i) and (ii). To see the “only if” part, note that \( f_u(z) = \log(2/(1 - \overline{w}z)) \) belongs to \( \mathcal{Q}_p \) uniformly, i.e., \( \|f_u\|_{\mathcal{Q}_p} \lesssim 1 \) and any function \( f \in \mathcal{Q}_p \) has the growth

\[
|f(z)| \lesssim \|f\|_{\mathcal{Q}_p} \log \frac{2}{1 - |z|^2}, \quad z \in \mathbb{D}.
\]

So, if \( M_g \) is bounded on \( \mathcal{Q}_p \), then for fixed \( w \in \mathbb{D} \),

\[
|f_u(z)g(z)| \lesssim \|M_g f_u\|_{\mathcal{Q}_p} \log \frac{2}{1 - |z|^2} \lesssim \|M_g\| \log \frac{2}{1 - |z|^2}, \quad z \in \mathbb{D},
\]

and hence \( |g(w)| \lesssim \|M_g\| \) (upon taking \( z = w \) in the last estimate), that is, \( \|g\|_{\mathcal{H}^\infty} < \infty \). Equivalently, \( U_g \) is bounded on \( \mathcal{Q}_p \) by the boundedness part of (ii). Consequently, \( V_g f = M_g f - f(0)g(0) - U_g f \) gives the boundedness of \( V_g \) on \( \mathcal{Q}_p \) and then \( \|M_{f,g}\|_{C\mathbb{C}M_p} < \infty \).

Suppose now \( M_g \) is compact on \( \mathcal{Q}_p \). Then this operator is bounded and hence \( \|g\|_{\mathcal{H}^\infty} < \infty \). For any nonzero sequence \( \{w_j\} \) in \( \mathbb{D} \) let

\[
f_j(z) = \left( \log \frac{1}{1 - |w_j|^2} \right)^{-1} \left( \log \frac{1}{1 - \overline{w_j}z} \right)^2.
\]
Assume $|w_j| → 1$. Then $\|f_j\|_{Q_p} ≤ 1$ and $f_j → 0$ uniformly on any compacta of $D$. So, $\|M_g(f_j)\|_{Q_p} → 0$. Because of

$$|g(z)f_j(z)| = |M_g(f_j)(z)| \lesssim \|M_g(f_j)\|_{Q_p} \log \frac{2}{1 - |z|^2}, \quad z \in D,$$

we get (by letting $z = w_j$)

$$|g(w_j)| \log \frac{1}{1 - |w_j|^2} \lesssim \|M_g(f_j)\|_{Q_p} \log \frac{2}{1 - |w_j|^2},$$

whence $g(w_j) → 0$. Since $g$ is bounded holomorphic function on $D$, it follows that $g = 0$. We are done.

**Remark 3.2.** The boundedness part of $M_g$, plus Xiao’s Theorem 1.3 (i) and Corollary 1.4 in [30], implies that

$$g \in M(Q_p) = \{g \in Q_p : M_g(Q_p) \subseteq Q_p\}$$

if and only if

$$g \in H^\infty \text{ and } \sup_{w \in D} \left( \log \frac{2}{1 - |w|^2} \right)^2 \int_D |g'(z)|^2(1 - |\sigma_w(z)|^2)^pdm(z) < \infty.$$

This equivalence proves Conjecture 1.5 in [30] (where $p \in (0, 1)$) which has been verified in Pau-Pelaez’s recent work [19]. Moreover, this equivalence and [30, Theorem 3.3] indicate that the Carleson’s corona decomposition theorem is valid for the multipliers of $Q_p$, $p \in (0, 1)$; that is, for finitely many of holomorphic functions $g_1, ..., g_n$ the following operator:

$$M_{(g_1, ..., g_n)}(f_1, ..., f_n) = \sum_{j=1}^n f_j g_j$$

maps $M(Q_p) \times \cdots \times M(Q_p)$ onto $M(Q_p)$ is completely determined by the following two conditions:

$$\inf_{z \in D} \sum_{j=1}^n |g_j(z)| > 0 \quad \text{and} \quad (g_1, ..., g_n) \in M(Q_p) \times \cdots \times M(Q_p).$$

This result has been proved recently by Pau in [18] using [32, Theorem 2.5.2]. Note that the case $p = 1$ of the result is due to Tolokonnikov [26], but the case $p > 1$ remains open. Of course, the corona type decompositions for all $Q_p$ are known; see also [30] (for $p \in (0, 1)$), [16] (for $p = 1$), and [17] (for $p \in (1, \infty)$). Furthermore, the results on boundedness and compactness of $V_g$ on $BMOA$ and $B$ can be seen from Siskakis-Zhao [22], Zhao [33] and MacCluer-Zhao [14]; while the boundedness result of $U_g$ acting on $BMOA$ is due to Danikas [12]. Finally, the boundedness descriptions of $M_g$ acting on $BMOA$ and $B$ can be found in Stegenga [23], Arazy [2], Brown-Shields [8] and Zhu [34]. Meanwhile, $M_g : B → B$ is never compact unless $g = 0$; see also Ohno-Zhao [15] and Zhu [37].

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