A PROOF OF THE REFINED CLASS NUMBER FORMULA OF
GROSS

MINORU HIROSE

Abstract. In 1988, Gross proposed a conjectural congruence between Stickelberger elements and algebraic regulators, which is often referred to as the refined class number formula. In this paper, we prove this congruence.

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Date: December 29, 2021.
1. Introduction

1.1. The conjecture of Gross. Let $F$ be a totally real field and $K$ a finite abelian extension of $F$. Let $S$ and $T$ be disjoint finite sets of places of $F$ such that $S$ contains all the infinite places and all places ramifying in $K/F$. We assume that

$$\ker(\mu_K) \to \prod_{p \in T_K} (\mathcal{O}_K/p)^\times = \{1\}$$

where $\mu_K$ is the group of roots of unity in $K$ and $T_K$ is the set of places of $K$ lying above places in $T$. Put $G := \text{Gal}(K/F)$. Let $G_v \subset G$ be the decomposition group at $v \in v$. Let $\Theta_{S,T,K} \in \mathbb{Z}[G]$ be the Stickelberger element. Define the $(S,T)$-ideal class group by

$$\text{Cl}_{S,T} := \text{coker}(F^\times_{(T)} \xrightarrow{\text{ord}_v} \bigoplus_{w \notin S \cup T} \mathbb{Z})$$

where $F^\times_{(T)}$ is the group of elements of $F^\times$ which are congruent to 1 modulo all places in $T$. Put $h_{S,T} := \#\text{Cl}_{S,T}$. We write $\mathcal{O}_{S,T}^\times$ for the group of $S$-units of $F$ which are congruent to 1 modulo all places in $T$. Put $r := \#S - 1$ and $S = \{v_0, \ldots, v_r\}$. From the condition, $\mathcal{O}_{S,T}^\times$ is a free abelian group of rank $r$. Take a $\mathbb{Z}$-basis $\{u_1, \ldots, u_r\}$ of $\mathcal{O}_{S,T}^\times$ such that $(-1)^r \det(-\log |u_i|_{v_j})_{1 \leq i,j \leq r} > 0$. Put $I := \ker(\mathbb{Z}[G] \to \mathbb{Z})$. Define $R_{G,S,T} \in I'/I^{r+1}$ by

$$R_{G,S,T} := -h_{S,T} \det(\text{rec}_v(u_j) - 1)_{1 \leq i,j \leq r}$$

where rec$_v$ is the composite map $F^\times \to F^\times_v \to G_v \subset G$. In this paper, we prove the following congruence.

Theorem 1 (conjectured by Gross in [6, Conjecture 4.1]).

$$\Theta_{S,T,K} \equiv R_{G,S,T} \pmod{I^{r+1}}$$

This congruence is an analogue of $(S,T)$-version of Dedekind’s class number formula

$$\lim_{s \to 0} s^{-r} \zeta_{F,S,T}(s) = -h_{S,T} \det(-\log |u_i|_{v_j})_{1 \leq i,j \leq r}.$$ 

Let $H$ be the maximal abelian unramified extension of $F$ such that all the places in $S$ split completely at $H$. Assume that $H \subset K$. Put $I_{G_v} := \ker(\mathbb{Z}[G] \to \mathbb{Z}[G/G_v])$, $I_{G_H} := \ker(\mathbb{Z}[G] \to \mathbb{Z}[\text{Gal}(H/F)])$, and $n_{S,T} := -h_{S,T} \#\text{Gal}(H/F) \in \mathbb{Z}$. We lift $R_{G,S,T}$ to $(\prod_{v \in S \setminus \{v_0\}} I_{G_v})/(I_{G_H} \prod_{v \in S \setminus \{v_0\}} I_{G_v})$ by

$$R_{G,S,T} := n_{S,T} \sum_{c \in \text{Gal}(H/F)} [c] \det(\text{rec}_v(u_j) - 1)_{1 \leq i,j \leq r}.$$ 

In fact, we prove the following stronger claim.

Theorem 2.

$$\Theta_{S,T,K} \equiv R_{G,S,T} \pmod{I_{G_H} \prod_{v \in S \setminus \{v_0\}} I_{G_v}}.$$ 

Throughout this paper, we assume that $F \neq \mathbb{Q}$ since it is already known that Theorem 2 holds for $F = \mathbb{Q}$ [1,2].
1.2. The enhancement of the conjecture of Gross. For a place $p$ of $F$, we denote by $\text{ch}(p)$ the residue characteristic of $p$. Let us consider the case of $T = \{q\}$ where $q$ is the prime ideal of $F$ such that $\text{ch}(q) \geq [F : \mathbb{Q}] + 2$. For a finite place $v$ not in $S$, we put $J_v := O_v^\times$ where $O_v$ is the maximal compact subring of $F_v$. For a finite place $v$ in $S$, we fix an enough small open subgroup $J_v$ of $O_v^\times$ such that $J_v \subset \ker(F_v^\times \xrightarrow{\text{rec}} G_v)$. For an infinite place $v$ of $F$, we denote by $J_v$ the identity component of $F_v^\times$ $\simeq \mathbb{R}^\times$. We put $N_v := F_v^\times/J_v$, $N_F := \mathbb{A}_F^\times/\prod_v J_v$, and $N^S := \prod_{v \in S} N_v$. For $v \in S$, we put $I_v := \ker(\mathbb{Z}[N_F] \to \mathbb{Z}[N_F/N_v])$. We write rec for any homomorphism induced from the reciprocity map. We put $I_H := \ker(\mathbb{Z}[N_F] \xrightarrow{\text{rec}} \mathbb{Z}[	ext{Gal}(H/F)])$. Take a $\mathbb{Z}$-basis $\{u_1, \ldots, u_r\}$ of $\mathcal{O}_S^\times$ such that $-\det(-\log |u_i|_v)_{1 \leq i, j \leq r} > 0$. Define $\hat{R}_q := I_{v_1} \cdots I_{v_r}/I_H I_{v_1} \cdots I_{v_r}$ by

$$\hat{R}_q := n_{S,\{q\}} \sum_{c \in N^S/F^\times} [c] \det(f_{v_i}(u_j) - 1)_{1 \leq i, j \leq r}$$

where $f_v$ is the composite map $F^\times \to F_v^\times \to N_v \to N_F$. Then we have $\text{rec}(\hat{R}_q) = R_{G, S, \{q\}}$. We put $I_{F^\times} := \ker(\mathbb{Z}[N_F] \to \mathbb{Z}[N_F/F^\times])$. In this paper, we construct an element $\hat{\Theta}_{S, \{q\}}$ of $(\prod_{v \in S \setminus \{v_0\}} I_v)/(I_{F^\times} \prod_{v \in S \setminus \{v_0\}} I_v)$ which is mapped to $\hat{\Theta}_{S, \{q\}}$ by rec. The following theorem is an enhancement of the $T = \{q\}$ case of Theorem 2.

**Theorem 3.** We have

$$\hat{\Theta}_{S, \{q\}} \equiv \hat{R}_q \pmod{I_H \prod_{v \in S \setminus \{v_0\}} I_v}.$$

The most part of the proofs of Theorem 2 and Theorem 3 overlaps. Theorem 3 is also important as a special case of more generalized conjecture.

1.3. Shintani data. In this paper, we introduce the notion of Shintani data, which plays an important role in the proofs of 2 and 3. Fix $F, K, S, (J_v)_{v}$ as in the previous two subsections, and fix a prime ideal $q$ of $F$ such that $q \notin S$ and $\text{ch}(q) \neq 2$. Let $V$ be a subset of $S$. In this paper, we define the category of Shintani data on $V$. A Shintani datum on $V$ is a quadruple $(B, \mathcal{L}, \vartheta, m)$ satisfying certain conditions. We call $m \in \mathbb{Z}_{\geq 0}$ an integrality of this Shintani datum. For a subset $V'$ of $V$, we can naturally define a functor $|_{V'}$ from the category of Shintani data on $V$ to that on $V'$. For each proper subset $V$ of $S$ and a Shintani datum $Sh$ on $V$, we construct a special element $Q^N(Sh) \in (\prod_{v \in V} I_v)/(I_{F^\times} \prod_{v \in V} I_v)$. Then $Q^N(Sh)$ satisfy the following properties.

1. If the integrality of $Sh$ is $m$ then $\text{rec}(Q^N(Sh)) = m\Theta_{S, \{q\}}$.
2. Let $Sh_1$ and $Sh_2$ be Shintani data, whose integrality are $m_1$ and $m_2$, respectively. If there exists a morphism from $Sh_1$ to $Sh_2$, then

$$\frac{m}{m_1} Q^N(Sh_1) = \frac{m}{m_2} Q^N(Sh_2) \quad (m := \text{lcm}(m_1, m_2)).$$

To illustrate the importance of Shintani data, let us sketch the proof of Theorem 3 for example. Assume that $\text{ch}(q) \geq [F : \mathbb{Q}] + 2$. We construct a Shintani datum $Sh^{v_0}$ on $S \setminus \{v_0\}$ whose integrality is 1, and define $\hat{\Theta}_{S, \{q\}}$ by $Q^N(Sh^{v_0})$. We also construct a Shintani datum $Sh^0$ on $S$ whose integrality is $\text{ch}(q)^{[F : q]}$. In this paper, we introduce a technique to compute

$$Q^N(Sh|_{S \setminus \{v_0\}}) \mod I_{F^\times} I_{v_1} \cdots I_{v_r} + I_{v_0} \cdots I_{v_r}.$$
for a general Shintani datum $\text{Sh}$ on $S$. By using this technique, we prove the following formula
\[
Q^N(\text{Sh}^v|_{S\setminus\{v_0\}}) \equiv \text{ch}(q)^{|F:Q|} \hat{R}_q \pmod{I_F \times I_{v_1} \cdots I_{v_r} + I_{v_0} \cdots I_{v_r}}.
\]
To relate $Q^N(\text{Sh}^v)$ and $Q^N(\text{Sh}^\circ|_{S\setminus\{v_0\}})$, we construct a Shintani datum $\text{Sh}^\circ$ on $S\setminus\{v_0\}$ whose integrality is $\text{ch}(q)^{|F:Q|}$, and morphisms $\text{Sh}^v \rightarrow \text{Sh}^\circ|_{S\setminus\{v_0\}}$. Hence we obtain
\[
\text{ch}(q)^{|F:Q|} \hat{\Theta}_{S,q,S\setminus\{v_0\}} \equiv \text{ch}(q)^{|F:Q|} \hat{R}_q \pmod{I_F \times I_{v_1} \cdots I_{v_r} + I_{v_0} \cdots I_{v_r}},
\]
which implies Theorem 3.

1.4. The outline of the proof. Fix $F$, $K$, $S = \{v_0,\ldots,v_r\}$ and $(J_v)_v$. Let $\Theta_{S,K}$ be the $S$-modified Stickelberger element. We put $\delta_T := \prod_{p \in T} (1 - N(p)\sigma_p^{-1})$. Then we have $\Theta_{S,T,K} = \delta_T \Theta_{S,K}$. If $S \cap T = \emptyset$ and there exists $q \in T$ such that $\text{ch}(q) \neq 2$, then the regulator
\[
R_{G,S,T} \in \left( \prod_{v \in S\setminus\{v_0\}} I_{G_v} \right)/(I_{G_H} \prod_{v \in S\setminus\{v_0\}} I_{G_v})
\]
is well-defined.
We prove Theorem 2 and Theorem 3 by the following steps.

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ix, Theorem 2)

(v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (viii)

(x) $\Rightarrow$ (xi) $\Rightarrow$ (xii, Theorem 3)

where

(i) For a prime ideal $q$ of $F$ such that $q \notin S$ and $\text{ch}(q) \geq |F : Q| + 2$, we construct a Shintani datum $\text{Sh}^v$ on $S\setminus\{v_0\}$ whose integrality is 1.

(ii) From (i), for all prime ideals $q$ of $F$ such that $q \notin S$ and $\text{ch}(q) \geq |F : Q| + 2$, we have
\[
\Theta_{S,\{q\},K} \in \prod_{v \in S\setminus\{v_0\}} I_{G_v}.
\]

(iii) From (ii), we have $\Theta_{S,T,K} \in \prod_{v \in S\setminus\{v_0\}} I_{G_v}$ since $\{1 - N(q)\sigma_q^{-1} | \text{ch}(q) \geq |F : Q| + 2, q \notin S\}$ generates the annihilator ideal $\text{Ann}_Z[G(\mu_K) \supset \delta_T]$.

(iv) From (iii), we have $2\Theta_{S,T,K} \in 2 \prod_{v \in S\setminus\{v_0\}} I_{G_v} \in I_{G_H} \prod_{v \in S\setminus\{v_0\}} I_{G_v}$.

(v) For a prime ideal $q$ of $F$ such that $q \notin S$ and $\text{ch}(q) \neq 2$, we construct a Shintani datum $\text{Sh}^\circ$ on $S$ whose integrality is $\text{ch}(q)^{|F:Q|}$.

(vi) For a prime ideal $q$ of $F$ such that $q \notin S$ and $\text{ch}(q) \neq 2$, we prove that
\[
Q^N(\text{Sh}^\circ|_{S\setminus\{v_0\}}) \equiv \text{ch}(q)^{|F:Q|} \hat{R}_q \pmod{I_H I_{v_1} \cdots I_{v_r}}.
\]

(vii) From (vi), for a prime ideal $q$ of $F$ such that $q \notin S$ and $\text{ch}(q) \neq 2$, we have $\text{ch}(q)^{|F:Q|} \Theta_{S,\{q\},K} \in Z[G]$ and
\[
\text{ch}(q)^{|F:Q|} \Theta_{S,\{q\},K} \equiv \text{ch}(q)^{|F:Q|} R_{G,S,\{q\}} \pmod{I_{G_H} \prod_{v \in S\setminus\{v_0\}} I_{G_v}}.
\]
(viii) Let \( q \) be any element of \( T \) such that \( \text{ch}(q) \neq 2 \). From (vii), we have
\[
\text{ch}(q)^{[F:Q]} \Theta_{S,T,K} \equiv \text{ch}(q)^{[F:Q]} R_{G,S,T} \pmod{I_{G_H} \prod_{v \in S \setminus \{v_0\}} I_{G_v}}
\]
(ix) From (iv) and (viii), we have
\[
\Theta_{S,T,K} \equiv R_{G,S,T} \pmod{I_{G_H} \prod_{v \in S \setminus \{v_0\}} I_{G_v}},
\]
which is a statement of Theorem 2.
(x) For a prime ideal \( q \) of \( F \) such that \( \text{ch}(q) \geq [F : Q] + 2 \), we construct a Shintani datum \( \text{Sh}^{v_0,\phi} \) on \( V \), and morphisms \( \text{Sh}^{v_0} \to \text{Sh}^{v_0,\phi} \to \text{Sh}^\phi |_{S \setminus \{v_0\}} \).
The integrality of \( \text{Sh}^{v_0,\phi} \) is \( \text{ch}(q)^{[F:Q]} \).
(xi) We put \( \hat{\Theta}_{S,q,S \setminus \{v_0\}} = \text{Q}^N(\text{Sh}^{v_0}) \). From (x), we have
\[
\text{ch}(q)^{[F:Q]} \hat{\Theta}_{S,q,S \setminus \{v_0\}} = \text{Q}^N(\text{Sh}^\phi |_{S \setminus \{v_0\}}).
\]
(xii) From (vi) and (xi), we have
\[
\hat{\Theta}_{S,q,S \setminus \{v_0\}} \equiv \hat{R}_q \pmod{I_{H}I_{v_1} \cdots I_{v_n}},
\]
which is a statement of Theorem 3.

The hardest part of this proof is a construction of \( \text{Sh}^{v_0} \), \( \text{Sh}^\phi \), and \( \text{Sh}^{v_0,\phi} \). Section 4, 5 and 6 are devoted to the construction of these Shintani data.

1.5. Setting, notations and remarks. Throughout this paper, we keep all the notations and settings in the previous subsections except that we do not fix \( T \). In addition, the following notations are used throughout this paper.

We denote by \( F_v \) the completion of \( F \) at \( v \). For \( x \in F \) or \( \mathcal{A}_F \), we denote by \( x_v \) the image of \( x \) at \( F_v \). For a prime ideal \( \mathfrak{p} \), write \( \mathcal{O}_\mathfrak{p} \) for the localization of \( \mathcal{O}_F \) at \( \mathfrak{p} \), \( \kappa_{\mathfrak{p}} \) for the residue field \( \mathcal{O}_F / \mathfrak{p} \), \( \mathfrak{p}_F \) for the maximal compact subring of \( F_v \), and \( \pi_\mathfrak{p} \) for a uniformizer of \( \mathcal{O}_\mathfrak{p} \). For a place \( \mathfrak{p} \notin S \) of \( F \), we denote by \( \sigma_{\mathfrak{p}} \in G \) the Frobenius element. For a set \( M \) of places of \( F \), we put \( \mathcal{A}^M_F := \prod_{v \notin M} F_v \), \( N_M := \prod_{v \in M} \mathcal{O}_v \), \( \mathfrak{n}^M := \prod_{v \in M} \mathfrak{n}_v \). For a set \( M \) of infinite places of \( F \), we write \( \mathcal{O}_M \) for the ring of \( M \)-integers of \( F \). We write \( S_\infty \) for the set of infinite places of \( F \). We put \( S_f := S \setminus S_\infty \).

For a rational prime \( q \), we write \( S_q \) for the set of places of \( F \) lying above \( q \). For a direct product of groups \( A = \prod_{A \in A} A \) and a subset \( A' \subset A \), we sometimes regard \( \prod_{A \in A'} A \) as a subset of \( A \), and sometimes a quotient of \( A \). For a set \( X \), we write \( \text{Sub}(X) \) for the category of subsets of \( X \) whose morphisms are inclusions. For a ring \( R \), we write \( \text{Mod}(R) \) for the category of \( R \)-modules. For a functor \( F : \text{Sub}(X) \to C \) and subsets \( U_1 \subset U_2 \subset X \), we put \( F(U_1) = F(U_2) := F(U_1) \to F(U_2) \) where \( \theta_{U_1}^{U_2} \) is the unique element of \( \text{Hom}_{\text{Sub}(X)}(U_1, U_2) \). If we omit the subscript of \( \otimes \), it means that the tensor product is over \( \mathbb{Z} \). For a group \( A \) and a \( \mathbb{Z}[A] \)-modules \( M_1 \) and \( M_2 \), we regard \( M_1 \otimes M_2 \) as a \( \mathbb{Z}[A] \)-module by the standard way. For a group \( A \) and an \( A \)-module \( M \), we put \( I_A := \ker(Z[A] \to \mathbb{Z}) \) and \( I_AM := \ker(M \to M \otimes \mathbb{Z}[A] \otimes \mathbb{Z}) \). For a set \( X \), we denote by \( 1_X \) the characteristic function of \( X \). We denote \( \text{gcd}(a, b) \) the least common multiple of \( a \) and \( b \). For a complex \( A \to B \to C \), we put \( H(A \to B \to C) := \ker(B \to C) / \text{im}(A \to B) \). We denote by \( S(X) \) the set of Schwartz-Bruhat functions from \( X \) to \( \mathbb{Z} \). We denote by \( S(X, A) \) the set of Schwartz-Bruhat functions from \( X \) to \( \mathbb{Z} \) invariant under the action of \( A \). We denote by \( S_n \) the symmetric group of \( \{1, \ldots, n\} \).
For a group $E$ and an $\mathbb{Z}[E]$-module $M$, we denote by $H_i(E, M)$ the $i$-th group homology. Especially, we have

$$H_0(E, M) = M/I_E M.$$ 

In this paper, we use Shapiro's Lemma

$$H_i(E_2, M \otimes_{\mathbb{Z}[E_1]} \mathbb{Z}[E_2]) = H_i(E_1, M) \quad (E_1 \subset E_2, \ M \text{ is a } \mathbb{Z}[E_1]\text{-module})$$

many times without mention.

2. Stickelberger functions and elements

Let $T$ be a finite set of places of $F$ which is disjoint from $S$. We define the $S$-modified and $(S, T)$-modified Stickelberger functions to be the meromorphic $\mathbb{C}[G]$-valued functions

$$\Theta_{S,K}(s) := \prod_{p \not\in S} (1 - \sigma_p^{-1} N(p)^{-s})^{-1}$$

$$\Theta_{S, T, K}(s) := \prod_{p \not\in S} (1 - \sigma_p^{-1} N(p)^{-s})^{-1} \prod_{p \in T} (1 - \sigma_p^{-1} N(p)^{-1-s}).$$

We put

$$\Theta_{S,K} := \Theta_{S,K}(0) \in \mathbb{Q}[G]$$

$$\Theta_{S, T, K} := \Theta_{S, T, K}(0) \in \mathbb{Q}[G]$$

$$\delta_T := \prod_{p \in T} (1 - N(p)\sigma_p^{-1}) \in \mathbb{Z}[G].$$

Then we have

$$\Theta_{S, T, K} = \delta_T \Theta_{S,K}.$$ 

3. Shintani datum

In this section, we introduce the notion of Shintani data and investigate their properties. Throughout this section, we fix a quadruple $(\mathcal{R}, \Upsilon, \lambda, \theta)$, where $\mathcal{R}$ is a functor from $\text{Sub}(S)$ to $\text{Mod}(\mathbb{Z}[F^\times])$ such that the natural chain

$$\mathcal{R}(0) \rightarrow \prod_{\#W=1}^{W \subset V} \mathcal{R}(W) \rightarrow \prod_{\#W=2}^{W \subset V} \mathcal{R}(W) \rightarrow \cdots \rightarrow \mathcal{R}(V) \rightarrow 0$$

is exact for all $V \subset S$, $\Upsilon$ is a $\mathbb{Z}$-module, $\lambda$ is a homomorphism from $H_0(F^\times, \mathcal{R}(\emptyset))$ to $\Upsilon$, and $\theta$ is an element of $\Upsilon$. For the proof of the main theorems we only use the case where $(\mathcal{R}, \Upsilon, \lambda, \theta)$ is as given at the start of Section 3 but, in this section we consider in a general setting for future research. For $V \subset S$, we write $\mathcal{R}(V)$ for the restriction of $\mathcal{R}$ to $\text{Sub}(V)$. We denote by $\text{Sub}(V) \setminus \{S\}$ the full subcategory of $\text{Sub}(V)$ consisting of $\{W \subset V \mid W \neq S\}$.

**Definition 4.** Let $V$ be a subset of $S$, $B$ a functor from $\text{Sub}(V) \setminus \{S\}$ to $\text{Mod}(\mathbb{Z}[F^\times])$, $L$ a natural transform from $\mathcal{B}$ to $\mathcal{R}|_{\text{Sub}(V) \setminus \{S\}}$, $\vartheta$ an element of $\mathcal{B}(\emptyset)/I_{F^\times} B(\emptyset)$, and $m$ a positive integer. We say that the quadruple $(B, L, \vartheta, m)$ is a Shintani datum for $(\mathcal{R}, \Upsilon, \lambda, \theta)$ on $V$ (or simply a Shintani datum on $V$) if the following conditions are satisfied.

- $H_i(F^\times, B(W)) = 0$ for all $i > 0$ and objects $W$ of $\text{Sub}(V) \setminus \{S\}$.
- $\lambda(\vartheta) = m\theta$. 
\[ r_{\{v\}}^\partial (\vartheta) = 0 \] for all \( v \in V \) where \( r_{\{v\}}^\partial \) denotes a natural map from \( B(\{\emptyset\}) / I_F^V B(\emptyset) \) to \( B(\{v\}) / I_F^V B(\{v\}) \) induced by \( r_{\{v\}}^\partial \).

**Definition 5.** Let \( V \) be a subset of \( S \). We define the category of Shintani data on \( V \) as follows. The objects are Shintani data on \( V \). For Shintani data \( Sh_1 = (B_1, \mathcal{L}_1, \vartheta_1, m_1) \) and \( Sh_2 = (B_2, \mathcal{L}_2, \vartheta_2, m_2) \) on \( V \), we define the set of morphism from \( Sh_1 \) to \( Sh_2 \) to be the set of natural transformation \( \mathcal{F} : B_1 \to B_2 \) such that \[ \frac{\text{lcm}(m_1, m_2)}{m_2} \mathcal{L}_1 = \mathcal{L}_2 \circ \mathcal{F} \] and \( \mathcal{F}(\vartheta_1) = \vartheta_2 \).

### 3.1. Definition of \( Q(Sh) \)

Fix the free resolution of \( \mathbb{Z} \)
\[
\cdots \to \mathcal{I}_2 \xrightarrow{\partial_v} \mathcal{I}_1 \xrightarrow{\partial_j} \mathcal{I}_0 \xrightarrow{\partial_v} \mathbb{Z} \to 0
\]
in the category of \( \mathbb{Z}[F^x] \)-modules defined by \( \mathcal{I}_k := \mathbb{Z}[[F^x]]^{k+1} \) and \( \partial_k([x_1, \ldots, x_k]) := \sum_{j=1}^{k} (-1)^{j-1} x_1, \ldots, \hat{x}_j, \ldots, x_k \).

Put \( \mathcal{I}_{-1} := \mathbb{Z} \) and \( \mathcal{I}_j := 0 \) for \( j \leq -2 \). Let \( (B, \mathcal{L}, \vartheta, m) \) be a Shintani data on \( V \).

Recall that we put \( r = \#S - 1 \). For \( \mathcal{F} \in \{ B, \mathcal{R}^V \} \) and \( k \geq 0 \), we put
\[
\mathcal{F}(k) = \begin{cases} \bigoplus_{W \subset V, \#W = k} \mathcal{F}(W) & k \leq r \\ 0 & k \geq r + 1. \end{cases}
\]

Note that \( \mathcal{F}(r+1) \) is not \( \mathcal{F}(S) \) even if \( V = S \). For \( k \geq 1 \), define a homomorphism \( \partial_h : \mathcal{F}(k-1) \to \mathcal{F}(k) \) by
\[
\partial_h((a_W)_{W \subset V}) = (b_W)_{W \subset V}
\]
where
\[
b_W = \sum_{j=1}^{k} (-1)^{j-1} w_{W \setminus \{v_i\}} (a_{W \setminus \{v_j\}}) \quad (W = \{v_i, v_i, \ldots, v_k\}, i_1 < \cdots < i_k).
\]

We put \( \mathcal{R}^V(-1) := \ker(\mathcal{R}^V(0) \xrightarrow{\partial_h} \mathcal{R}^V(1)) \) and \( \mathcal{R}^V(j) = 0 \) for \( j \leq -2 \). We put \( B(i) = 0 \) for all \( i \leq -1 \). Then we get a chain complex \( ((B(i))_{i \in \mathbb{Z}}, \partial_h) \) and \( ((\mathcal{R}^V(i))_{i \in \mathbb{Z}}, \partial_h) \).

We define two double complex \( (\mathcal{B}_{*,*}, \partial_h, \partial_e) \) and \( (\mathcal{R}^V_{*,*}, \partial_h, \partial_e) \) by \( \mathcal{B}_{i,j} := \mathcal{B}_i \otimes_{\mathbb{Z}[F^x]} \mathcal{I}(j) \) and
\[
\mathcal{R}^V_{i,j} := \begin{cases} \mathcal{R}^V(i) \otimes_{\mathbb{Z}[F^x]} \mathcal{I}(j) & j \neq -1 \\ 0 & j = -1. \end{cases}
\]

Then \( \mathcal{L} \) induces a homomorphism from \( \mathcal{B}_{*,*} \) to \( \mathcal{R}^V_{*,*} \). We regard \( \vartheta \) as an element of \( \ker(H_0(F^x, B(0)) \to H_0(F^x, B(1))) \). For \( \mathcal{F} = B \) or \( \mathcal{F} = \mathcal{R}^V \), we denote by \( (\mathcal{F}[\cdot], \partial_e) \) the total complex of \( \mathcal{F}_{*,*} \), i.e., we put
\[
\mathcal{F}[k] := \bigoplus_{-i+j=k} \mathcal{F}_{i,j},
\]
\[
d_k := \partial_h + (-1)^k \partial_e : \mathcal{F}[k] \to \mathcal{F}[k-1].
\]

The complex \( (\mathcal{B}[\cdot], \partial_e) \) is exact since the vertical chain \( \mathcal{B}_{i,*} \) is exact for \( i \in \mathbb{Z} \).

If \( V \neq S \) then \( \mathcal{R}^V[\cdot], \partial_e \) is exact since the horizontal chain \( \mathcal{R}^V_{*,*} \) is exact.
for \( j \in \mathbb{Z} \). Let us consider the following commutative diagram

\[
\begin{array}{c}
\ker(B_{0,-1} \to B_{1,-1}) & \xrightarrow{i} & B_{0,-1} & \xrightarrow{\partial_h} & B_{1,-1} \\
B[1] & \xrightarrow{d_1} & B[0] & \xrightarrow{d_0} & B[-1] & \xrightarrow{d_{-1}} & B[-2] \\
\mathcal{R}^{(V)}[2] & \xrightarrow{d_2} & \mathcal{R}^{(V)}[1] & \xrightarrow{d_1} & \mathcal{R}^{(V)}[0] & \xrightarrow{d_0} & \mathcal{R}^{(V)}[-1] \\
\mathcal{R}^{(V)}_{-1,1} & \xrightarrow{\partial_e} & \mathcal{R}^{(V)}_{-1,0} & \xrightarrow{q} & \ker(\mathcal{R}^{(V)}_{-1,1} \to \mathcal{R}^{(V)}_{-1,0})
\end{array}
\]

where \( i, i_1, i_2 \) are natural inclusions and \( q, q_1, q_2 \) are natural projections. Since \( \ker(B_{0,-1} \to B_{1,-1}) = \ker(H_0(F^\times, B(0)) \to H_0(F^\times, B(1))) \), we can regard \( \vartheta \) as an element of \( \ker(B_{0,-1} \to B_{1,-1}) \). Note that \( \ker(\mathcal{R}^{(V)}_{-1,1} \to \mathcal{R}^{(V)}_{-1,0}) = H_0(F^\times, \mathcal{R}^{(V)}(-1)) \).

**Definition 6.** Assume that \( V \neq S \). We define \( Q((B, \mathcal{L}, \vartheta, m)) \in H_0(F^\times, \mathcal{R}^{(V)}(-1)) \) as follows. Since \( d_{-1} \circ i_1 \circ i = i_2 \circ \partial_h \circ i = 0 \), there exists \( a \in B[0] \) such that \( d_0(a) = i_1 \circ i(\vartheta) \). Since \( d_0 \circ \mathcal{L}(a) = \mathcal{L} \circ i_1 \circ i(\vartheta) = 0 \), there exists \( b \in \mathcal{R}^{(V)}[1] \) such that \( d_1(b) = \mathcal{L}(a) \). Then \( q \circ q_1(b) \) does not depend on the choice of \( a \) and \( b \). We put \( Q((B, \mathcal{L}, \vartheta, m)) = q \circ q_1(b) \).

Let \( \text{Sh} = (B, \mathcal{L}, \vartheta, m) \) be a Shintani datum on \( V \subset S \), and \( V' \) a proper subset of \( V \). Then \( (B|_{\text{Sub}(V')}, \mathcal{L}|_{\text{Sub}(V')}, \vartheta, m) \) is a Shintani datum on \( V' \). We denote this Shintani datum by \( \text{Sh}|_{V'} \). The next lemma follows from the definition.

**Lemma 7.** The map \( \text{Sh} \mapsto Q(\text{Sh}) \) satisfy the following properties.

(i) \( \lambda(Q(B, \mathcal{L}, \vartheta, m)) = m\vartheta \)

(ii) \( \lambda(Q(\text{Sh}) \mid_{V'}) = Q(\lambda(\text{Sh}) \mid_{V'}) \mod \mathcal{R}^{(V')}(-1)) \).

(iii) \( \lambda(Q(\text{Sh}_1) \mid_{V_1}) = Q(\text{Sh}_1) \mod \mathcal{R}^{(V_1)}(-1)) \).

From (ii), for \( V_2 \subset V_1 \subset S \) and a Shintani \( \text{Sh} \) on \( V_1 \), we have

\[
Q(\text{Sh} \mid_{V_2}) \equiv 0 \mod I_{F^\times} \mathcal{R}^{(V_2)}(-1) + \mathcal{R}^{(V_1)}(-1).
\]

Unfortunately (3.1) does not hold for \( (V_1, V_2) = (S, S \setminus \{v_0\}) \). Instead, we give a way to compute

\[
Q(\text{Sh} \mid_{S \setminus \{v_0\}}) \mod I_{F^\times} \mathcal{R}^{(S \setminus \{v_0\})}(-1) + \mathcal{R}^{(S)}(-1)
\]

in the next section.
3.2. Computation of \( Q(\text{Sh}_{1\setminus \{v_0\}}) \). Let \( \text{Sh} = (\mathcal{B}, \mathcal{L}, \vartheta, m) \) be a Shintani data on \( S \). Note that \( H_r(\mathcal{R}^{(S)}[1] \to \mathcal{R}^{(S)}[0] \to \mathcal{R}^{(S)}[-1]) \) is canonically isomorphic to \( H_r(\mathcal{F}^\times, \mathcal{R}(S)) \). Put \( V = S \setminus \{v_0\} \).

**Definition 8.** We define the map \( \eta_1 : \ker(B_{0,-1} \to B_{1,-1}) \to H_r(\mathcal{F}^\times, \mathcal{R}(S)) \) as follows. Let us consider the following commutative diagram.

\[
\begin{array}{ccc}
\ker(B_{0,-1} \to B_{1,-1}) & \xrightarrow{i} & B_{0,-1} \\
\downarrow & & \downarrow \partial_h \\
\mathcal{B}[1] & \xrightarrow{d_1} & \mathcal{B}[0] \\
\downarrow \mathcal{L} & & \downarrow \mathcal{L} \\
\mathcal{R}^{(S)}[1] & \xrightarrow{d_1^\prime} & \mathcal{R}^{(S)}[0] \\
\downarrow & & \downarrow \\
\mathcal{R}^{(V)}[2] & \xrightarrow{d_2} & \mathcal{R}^{(V)}[1] \\
\downarrow q_2 & & \downarrow q_1 \\
\mathcal{R}_{-1,1} & \xrightarrow{\partial} & \mathcal{R}_{-1,0} \\
\end{array}
\]

Let \( x \in \ker(B_{0,-1} \to B_{1,-1}) \). Then there exists \( a \in \mathcal{B}[0] \) such that \( d_0(a) = i_1 \circ i(x) \). Then \( \mathcal{L}(a) \in \ker(d_0^\prime) \), and \( \langle \mathcal{L}(a) \mod \im(d_1^\prime) \rangle \) does not depend on the choice of \( a \). We put \( \eta_1(x) = \mathcal{L}(a) \in \ker(d_0^\prime) / \im(d_1^\prime) \simeq H_{r,-1}(\mathcal{F}^\times, \mathcal{R}(S)) \).

**Definition 9.** We define the map \( \eta_2 : H_r(\mathcal{F}^\times, \mathcal{R}(S)) \to \mathcal{R}^{(V)}(-1)/\mathcal{R}^{(S)}(-1) + I_{\mathcal{F}^\times, \mathcal{R}^{(V)}}(-1) \) as follows. Let us consider the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{R}^{(S)}[1] & \xrightarrow{d_1^\prime} & \mathcal{R}^{(S)}[0] \\
\downarrow f_1 & & \downarrow f_0 \\
\mathcal{R}^{(V)}[2] & \xrightarrow{d_2} & \mathcal{R}^{(V)}[1] \\
\downarrow q_2 & & \downarrow q_1 \\
\mathcal{R}_{-1,1} & \xrightarrow{\partial} & \mathcal{R}_{-1,0} \\
\end{array}
\]

Let \( y \) be an element of \( H_r(\mathcal{F}^\times, \mathcal{R}(S)) \simeq \ker(d_0^\prime) / \im(d_1^\prime) \). Let \( g \in \mathcal{R}^{(S)}[0] \) be a lift of \( y \). Since \( d_0 \circ f_0(g) = f_{-1} \circ d_0^\prime(g) = 0 \), there exists \( b \in \mathcal{R}^{(V)}[1] \) such that \( d_1^\prime(b) = g \). Then \( q \circ q_1 \circ d_2 = q \circ \partial_\mathcal{L} \circ q_2 = 0 \). We put \( \eta_2(y) = g = q_1(b) \).

**Proposition 10.** We have

\[
Q(\text{Sh}_{1\setminus \{v_0\}}) \equiv \eta_2(\eta_1(\vartheta)) \quad (\text{mod } I_{\mathcal{F}^\times, \mathcal{R}^{(V)}}(-1) + \mathcal{R}^{(S)}(-1)).
\]

**Proof.** Let us see the diagrams in Definition 8 and 9. There exists \( a \in \mathcal{B}[0] \) such that \( d_0(a) = i_1 \circ i(\vartheta) \). Then there exists \( b \in \mathcal{R}^{(V)}[1] \) such that \( d_1^\prime(b) = f_0 \circ \mathcal{L}(a) \). From the definition, we have \( \eta_2(\eta_1(\vartheta)) = q \circ q_1(b) \). Since \( d_0(a) = i_1 \circ i(\vartheta) \) and \( f_0 \circ \mathcal{L}(a) = d_1(b) \), we have

\[
q \circ q_1(b) \equiv Q(\text{Sh}_{1\setminus \{v_0\}}) \quad (\text{mod } I_{\mathcal{F}^\times, \mathcal{R}^{(V)}}(-1) + \mathcal{R}^{(S)}(-1)).
\]

Thus the proposition is proved. \( \square \)

**Lemma 11.** Let \( A \) be a subgroup of \( \mathcal{F}^\times \). Assume that \( H_r(A, \mathcal{B}(W)) = 0 \) for all positive integer \( i \) and proper subset \( W \) of \( S \), and that there exists a lift \( x \in \mathcal{B}(\vartheta) \) of \( \vartheta \) such that \( i_1^\vartheta(x) \in IA(B(\vartheta)) \) for \( v \in S \). Then \( \eta_1(\vartheta) \) is contained in \( \im(H_r(A, \mathcal{R}(S)) \to H_r(\mathcal{F}^\times, \mathcal{R}(S))) \).
In this section, we define a subset $Y$ and $K$ generated by the formal symbol $V$. Denote vectors in $W$ by $u$. We say that a subset $C$ is an integer $n$-dimensional vector space $V$.

Consider the following commutative diagram.

Let $\bar{B}$ be an $n$-dimensional vector space $A$ corresponding to a vector space $V$. For $\bar{B} \in B(\emptyset)$, we denote by $\mathcal{A} \in A$ and construct a certain exact sequence (Proposition 18).

A certain $\mathbb{Z}$-module corresponding to a vector space. Fix a positive integer $n$ and an $n$-dimensional vector space $V$ over $\mathbb{Q}$. For $x_1, \ldots, x_k \in V$, we denote by $C(x_1, \ldots, x_k)$ the open cone generated by $x_1, \ldots, x_k$ in $V$ or $V \otimes \mathbb{R}$.

**Definition 12.** We say that a subset $U$ of $V \setminus \{0\}$ is fat if $U$ cannot be covered by any finite union of proper subspaces of $V$.

Fix a fat subset $U$ of $V \setminus \{0\}$. For $k \geq 1$, we denote by $X_k(U)$ the $\mathbb{Z}$-module generated by the formal symbol $[x_1, \ldots, x_k]$ where $x_1, \ldots, x_k$ are linearly independent vectors in $U$. We put $X(U) := \bigoplus_{k=1}^{n} X_k(U)$ and $X_{\text{low}}(U) := \bigoplus_{k=1}^{n-1} X_k(U) \subset X(U)$. We define a homomorphism $\mathcal{L}_{\infty} : X(U) \to \text{Map}(V, \mathbb{Z})$ by $\mathcal{L}_{\infty}([x_1, \ldots, x_k]) := 1_{C(x_1, \ldots, x_k)}$.

We put $K'(U) := \text{im}(X(U) \xrightarrow{\mathcal{L}_{\infty}} \text{Map}(V, \mathbb{Z}))$ and $K(U) := K'(U) / (K'(U) \cap 1_{V \setminus \{0\}} \mathbb{Z})$.

In this section, we define a subset $Y(U)$ of $X(U)$, prove that $Y(U) \subset \ker(X(U) \xrightarrow{\mathcal{L}_{\infty}} K(U))$ (Proposition 18), and construct a certain exact sequence (Proposition 18).

For $m \geq 0$, we denote by $C_m(U)$ the $\mathbb{Z}$-module generated by the formal symbol $(x_1, \ldots, x_m)$.
where \( x_1, \ldots, x_m \in U \) are in general position. We define \( \partial_m : C_m(U) \to C_{m-1}(U) \) by
\[
\partial_m((x_1, \ldots, x_m)) = \sum_{j=1}^{m} (-1)^{j-1}(x_1, \ldots, \hat{x}_j, \ldots, x_m).
\]

Since \( U \) is fat, the following sequence is exact.
\[
\cdots \to C_3(U) \to C_2(U) \to C_1(U) \to C_0(U) \to 0.
\]

Let us fix a map \( r : V^n \to \{0, 1, -1\} \) such that
- \( r(x_1, \ldots, x_n) \neq 0 \) if and only if \( x_1, \ldots, x_n \) are linearly independent,
- \( r(f x_1, \ldots, f x_n) = \text{sgn}(\text{det}(f)) r(x_1, \ldots, x_n) \) for all automorphisms \( f \) of \( V \).

We call such a map an orientation of \( V \). We define the homomorphism \( \psi : C_{n+1}(U) \to X(U) \) by
\[
\psi((x_1, \ldots, x_{n+1})) = \sum_{u \in \{\pm 1\}} \sum_{k=1}^{n} \sum_{i_{k+1}, \ldots, i_k} u[x_{i_1}, \ldots, x_{i_k}]
\]
where \( 1 \leq i_1 < \cdots < i_k \leq n+1 \) runs all tuples such that
\[
(-1)^{j-1} r(x_1, \ldots, \hat{x}_j, \ldots, x_{n+1}) = u
\]
for all \( j \in \{1, \ldots, n+1\} \setminus \{i_1, \ldots, i_k\} \). We put \( Y(U) = \text{Image}(\psi) \subset X(U) \) and \( Z(U) = X(U)/Y(U) \). Note that \( Y(U) \) does not depend on the choice of \( r \). We say that \( Q \in V \otimes \mathbb{R} \) is an irrational vector if \( Q \) is not contained in any proper subspace of \( V \otimes \mathbb{R} \) spanned by vectors in \( V \). For an irrational vector \( Q \), define the homomorphism \( \varphi^Q : C_n(U) \to X(U) \) by
\[
\varphi^Q((x_1, \ldots, x_n)) = r(x_1, \ldots, x_n) \sum_{k=1}^{n} \sum_{i_{k+1}, \ldots, i_k} [x_{i_1}, \ldots, x_{i_k}]
\]
where \( 1 \leq i_1 < \cdots < i_k \leq n \) runs all tuple such that
\[
\frac{r_{x_j \to Q}(x_1, \ldots, x_n)}{r(x_1, \ldots, x_n)} > 0
\]
for all \( j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\} \), where \( r_{x_j \to Q}(x_1, \ldots, x_n) \) denotes \( r(x_1, \ldots, x_{j-1}, Q, x_{j+1}, \ldots, x_n) \).

**Lemma 13.** Let \( x_1, \ldots, x_{n+1}, y \in V \otimes \mathbb{R} \) be vectors in general position. Then we have
\[
\sum_{j=1}^{n+1} (-1)^{j-1} r(x_1, \ldots, \hat{x}_j, \ldots, x_{n+1}) 1_{C(x_1, \ldots, x_{n+1})}(y)
\]
\[
= \begin{cases} 
1 & \text{for all } j = 1, \ldots, n+1 \ 
-1 & \text{for all } j = 1, \ldots, n+1 \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** It was proved in [7, Proposition 2].

**Lemma 14.** Let \( Q \in V \otimes \mathbb{R} \) be an irrational vector. We have \( \psi(a) = \varphi^Q(\partial a) \) for all \( a \in C_{n+1}(U) \).
Proof. Put \( a = (x_1, \ldots, x_{n+1}) \). We put \( I = \{(i_1, \ldots, i_k) \mid 1 \leq i_1 < \cdots < i_k \leq n+1 \} \). From the definition of \( \varphi^Q \) and \( \partial a \), we have

\[
\varphi^Q(\partial a) = \sum_{(i_1, \ldots, i_k) \in I} w(i_1, \ldots, i_k)[i_1, \ldots, i_k]
\]

where \( w(i_1, \ldots, i_k) \) is an integer defined by

\[
w(i_1, \ldots, i_k) := \sum_m (-1)^{m-1}r(x_1, \ldots, \widehat{x_m}, \ldots, x_{n+1})
\]

where \( m \) runs all integers between 1 and \( n+1 \) such that \( m \notin \{i_1, \ldots, i_k\} \) and

\[
(4.1) \quad \frac{r_{x_j - Q}(x_1, \ldots, \widehat{x_m}, \ldots, x_{n+1})}{r(x_1, \ldots, \widehat{x_m}, \ldots, x_{n+1})} > 0
\]

for all \( j \in \{1, \ldots, n+1\} \setminus \{i_1, \ldots, i_k, m\} \). Fix \( (i_1, \ldots, i_k) \in I \). Put \( V' := V/(x_i, Q + \cdots + x_i, Q) \). For \( x \in V' \), we denote by \( \bar{x} \) the image of \( x \) in \( V' \). Then the condition (4.1) is equivalent to

\[
(4.2) \quad \bar{Q} \in C((\bar{x}_j)_{j \neq i_1\ldots,i_k,m}).
\]

Put

\[
\{j_1 \leq \cdots \leq j_{n+1-k}\} := \{1, \ldots, n+1\} \setminus \{i_1, \ldots, i_k\},
\]

\( m = j_c \) and \( y_c := x_{j_c} \). Then (4.2) is equivalent to

\[
\bar{Q} \in C(y_1, \ldots, \widehat{y_c}, \ldots, y_{n+1-k}).
\]

Thus we have

\[
w(i_1, \ldots, i_k) = \sum_{c=1}^{n+1-k} (-1)^{j_c-1}r(x_1, \ldots, \widehat{x_{j_c}}, \ldots, x_{n+1})1_{C(y_1, \ldots, \widehat{y_c}, \ldots, y_{n+1-k})}(\bar{Q}).
\]

Take an orientation \( r' \) of \( V' \) such that

\[
(-1)^{c-1}r'(y_1, \ldots, \widehat{y_c}, \ldots, y_{n+1-k}) = (-1)^{j_c-1}r(x_1, x_2, \ldots, \widehat{x_{j_c}}, \ldots, x_n, x_{n+1}) \quad (1 \leq c \leq n+1-k).
\]

Then we have

\[
w(i_1, \ldots, i_k) = \sum_{c=1}^{n+1-k} (-1)^{c-1}r'(y_1, \ldots, \widehat{y_c}, \ldots, y_{n+1-k})1_{C(y_1, \ldots, \widehat{y_c}, \ldots, y_{n+1-k})}(\bar{Q}).
\]

Thus from Lemma \( \text{[13]} \) we have

\[
w(i_1, \ldots, i_k) = \begin{cases} 1 & (\text{for all } c) \\ -1 & (\text{for all } c) \\ 0 & (\text{otherwise}) \end{cases} \quad (\text{for all } j \notin \{i_1, \ldots, i_k\}) \quad (\text{and}) \quad (\text{for all } j \notin \{i_1, \ldots, i_k\})
\]

Thus the claim is proved. \( \square \)

**Proposition 15.** We have \( Y(U) \subset \ker(X(U) \xrightarrow{\zeta_{\infty}} \mathcal{K}(U)) \).
Proof. From Lemma 13 it is enough to prove that
\[ \mathcal{L}_\infty(\varphi^n(\partial a)) \in 1_{V \setminus \{0\}} \]
for all \( a \in C_{n+1}(U) \). For \((x_1, \ldots, x_n) \in C_n(U)\), we have
\[ \mathcal{L}_\infty(\varphi^n((x_1, \ldots, x_n)) = r(x_1, \ldots, x_n) : f_{Q,(x_1, \ldots, x_n)} \]
where \( f_{Q,(x_1, \ldots, x_n)} : V \to \mathbb{Z} \) is a map defined by
\[ f_{Q,(x_1, \ldots, x_n)}(z) := \begin{cases} \lim_{\epsilon \to +0} 1_{C(x_1, \ldots, x_n)}(z + \epsilon Q) & z \neq 0 \\ 0 & z = 0. \end{cases} \]
Thus we have
\[ \mathcal{L}_\infty(\varphi^n(\partial(x_1, \ldots, x_{n+1}))(z) = \sum_{j=1}^{n+1} (-1)^j r(x_1, \ldots, x_j, \ldots, x_{n+1}) 1_{C(x_1, \ldots, x_j, \ldots, x_{n+1})}(z + \epsilon Q) \]
from Lemma 13. Thus the claim is proved. \( \square \)

Definition 16. For \( a \in \ker(C_n(U) \to C_{n-1}(U)) \), we define \( \varphi(a) \in X(U) \) by \( \psi(b) \)
where \( b \) is any element of \( C_{n+1}(U) \) such that \( \partial b = a \). This definition does not depend on the choice of \( b \).

Lemma 17. The natural map \( X_{\text{low}}(U) \to Z(U) \) is injective.

Proof. The claim is equivalent to \( X_{\text{low}}(U) \cap Y(U) = \{0\} \). Let \( a \in C_{n+1}(U) \)
that \( \psi(a) \in X_{\text{low}}(U) \). Since \( \psi(a) \in X_{\text{low}}(U) \), we have \( \partial_{n+1}(a) = 0 \). Let \( Q \) be any irrational vector. Then we have
\[ \psi(a) = \varphi^n(\partial_{n+1}(a)) = \varphi^n(0) = 0. \]
Thus the claim is proved. \( \square \)

Define a homomorphism \( \phi : X(U) \to C_n(U) \) by
\[ \phi([x_1, \ldots, x_m]) = \begin{cases} r(x_1, \ldots, x_n) : (x_1, \ldots, x_n) & m = n \\ 0 & m < n. \end{cases} \]
Put \( \bar{C}_n(U) = C_n(U)/\ker(\partial_n) \). Then \( \varphi \) induces an isomorphism from \( Z(U)/X_{\text{low}}(U) \)
to \( \bar{C}_n(U) \). Thus we get the following proposition.

Proposition 18. The following sequence is exact:
\[ 0 \to X_{\text{low}}(U) \to Z(U) \xrightarrow{\phi} \bar{C}_n(U) \to 0. \]

4.2. The group action on \( V_{\text{ad}} \). Let \( E \) be a group which acts linearly on \( V \)
freely on \( V \setminus \{0\} \). Let \( U \subset V \) be a fat subset closed under the action of \( E \). For \( \epsilon \in E \), put \( \text{sgn}(\epsilon) := \text{sgn}(\det(\epsilon : V \to V)) \). The \( \mathbb{Z} \)-module \( C_m(U) \) has a two kind of structure of \( \mathbb{Z}[E] \)-module. The one is defined by
\[ [\epsilon](x_1, \ldots, x_m) = (\epsilon x_1, \ldots, \epsilon x_m) \quad (\epsilon \in E, x_1, \ldots, x_m \in U), \]
and the other is defined by
\[ [\epsilon](x_1, \ldots, x_m) = \text{sgn}(\epsilon)(\epsilon x_1, \ldots, \epsilon x_m) \quad (\epsilon \in E, x_1, \ldots, x_m \in U). \]
We write $C^+_m(U)$ for the first $\mathbb{Z}[E]$-module and $C^-_m(U)$ for the second one. Let $\mathbb{Z}^-$ be the $\mathbb{Z}[E]$-module whose underlying $\mathbb{Z}$-module is $\mathbb{Z}$, and define the action of $E$ to $\mathbb{Z}^-$ by
\[ cn = \text{sgn}(\epsilon) n \quad (\epsilon \in E, \ n \in \mathbb{Z}^-). \]

Note that we have $C^+_0(V_{\text{ad}}) \cong \mathbb{Z}$ and $C^-_0(V_{\text{ad}}) \cong \mathbb{Z}^-$. The purpose of this subsection is to define the homomorphism
\[ \Omega : H_{n-1}(E, \mathbb{Z}^-) \to H_0(E, \mathbb{Z}(U)) = \mathbb{Z}(U)/I_E \mathbb{Z}(U). \]

**Definition 19.** For $x = (x_1, \ldots, x_k) \in U^k$, we denote by $A_m(x)$ (resp. $A'_m(x)$) the subgroup of $C_m(U)$ spanned by
\[ (y_1, \ldots, y_m) \in C_m(U) \]
where $y_1, \ldots, y_n$ are elements of $U$ such that $(x_1, \ldots, x_k)$ is (resp. is not) a subsequence of $(y_1, \ldots, y_m)$.

From the definition, we have
\[ C_m(U) = A_m(x) \oplus A'_m(x). \]

**Definition 20.** For $x = (x_1, \ldots, x_k) \in U^k$ and $z \in X(U)$, we denote by $\text{coeff}(x, z) \in \mathbb{Z}$ the coefficient of $[x_1, \ldots, x_k]$ in $z$.

**Definition 21.** For $x \in U^k$ and $a \in C_n(U)$ such that $\partial a \in A'_{n-1}(x)$, define $\text{coeff}^#(x, a) \in \mathbb{Z}$ by
\[ \text{coeff}^#(x, a) := \text{coeff}(x, \varphi(a - a')) \]
where $a'$ is an element of $A'_n(x)$ such that $\partial(a - a') = 0$. Such an element $a'$ always exists and $\text{coeff}^#(x, a)$ does not depend on the choice of $a'$.

In other words, $\text{coeff}^#(x, -)$ is a unique homomorphism from $\partial^{-1}(A'_{n-1}(x))$ to $\mathbb{Z}$ such that
\[ \text{coeff}^#(x, a) = \text{coeff}(x, \varphi(a)) \quad (a \in \ker \partial) \]
\[ \text{coeff}^#(x, a) = 0 \quad (a \in A'_n(x)). \]

**Definition 22.** Let $x \in U^k$ and $a \in \partial^{-1}_{n-1}(I_E C_{n-1}(U))$. We define $\text{coeff}^#_E(x, a) \in \mathbb{Z}$ by
\[ \text{coeff}^#_E(x, a, E) := \lim_{\substack{Z \to \mathbb{Z}^+ \ \#Z < \infty \ \text{finite}}} \text{coeff}^#(x, \sum_{\epsilon \in Z} \epsilon a). \]

Here the right hand side means $\text{coeff}^#(x, \sum_{\epsilon \in Z_0} \epsilon a)$ where $Z_0$ is an enough large finite subset of $E$ such that $\sum_{\epsilon \in Z} \epsilon a \in \ker \partial$ and
\[ \text{coeff}^#(x, \sum_{\epsilon \in Z_0} \epsilon a) = \text{coeff}^#(x, \sum_{\epsilon \in Z} \epsilon a) \]
for all finite subset $Z$ of $E$ which contains $Z_0$.

**Definition 23.** For $a \in \partial^{-1}_{n-1}(I_E C_{n-1}(U))/I_E C_n(U)$, we define $\varphi_E(a) \in H_0(E, X(U))$ by
\[ \sum_{k=1}^n \sum_{(x_1, \ldots, x_k) \in U^k/E} \text{coeff}^#_E((x_1, \ldots, x_k), a) \cdot [x_1, \ldots, x_k]. \]
Lemma 24. For all \( a \in \ker \partial_n \), we have
\[
\varphi_E(a) \equiv \varphi(a) \pmod{I_E X(U)}.
\]

Proof. For \( x = (x_1, \ldots, x_k) \in U_k/E \) and \( z \in H_0(E, X(U)) \), we put
\[
\text{coeff}_E(x, z) = \lim_{Z \to X} \text{coeff}(x, \sum_{\epsilon \in Z} \epsilon z).
\]
Then it is enough to prove that
\[
\text{coeff}_E(x, \varphi_E(a)) = \text{coeff}_E(x, \varphi(a)).
\]
The left hand side is equal to
\[
\text{coeff}_E^\#((x_1, \ldots, x_k), a) = \lim_{Z \to E} \text{coeff}_E^\#((x, \sum_{\epsilon \in Z} \epsilon a)).
\]

Thus the lemma is proved. \( \square \)

Since
\[
\cdots \to C_2^{-}(U) \to C_1^{-}(U) \to C_0^{-}(U) \simeq Z^{-} \to 0
\]
is a free resolution of \( Z^- \) in the category of \( Z[E] \)-module, there exists a natural isomorphism
\[
H_{n-1}(E, Z^-) \simeq \partial_n^{-1}(I_E C_{n-1}^{-}(U))/(\ker \partial_n + I_E C_{n-1}^{-}(U)).
\]
By composing this isomorphism and \( \varphi_E \), we get the homomorphism
\[
\Omega_{E,U} : H_{n-1}(E, Z^-) \to H_0(E, Z(U)).
\]
Note that the diagram
\[
\begin{array}{ccc}
H_{n-1}(E_1, Z^-) & \xrightarrow{\Omega_{E_1,U_1}} & H_0(E_1, Z(U_1)) \\
\downarrow & & \downarrow \\
H_{n-1}(E_2, Z^-) & \xrightarrow{\Omega_{E_2,U_2}} & H_0(E_2, Z(U_2))
\end{array}
\]
commutes for \( E_1 \subset E_2 \) and \( U_1 \subset U_2 \) where two vertical arrows in the diagram are natural maps.

Lemma 25. Let \( Q \in F \otimes \mathbb{R} \) be an irrational vector such that \( \epsilon Q \in \mathbb{R} \) for all \( \epsilon \in E \). For \( a \in \partial_n^{-1}(I_E C_{n-1}(U)) \), we have
\[
\varphi_E(a) \equiv \varphi^Q(a) \pmod{I_E Z(U)}.
\]

Proof. Fix \( x = (x_1, \ldots, x_k) \in U_k \) and \( a \in \partial_n^{-1}(I_E C_{n-1}(U)) \). It is enough to prove that
\[
(4.3) \quad \text{coeff}_E(x, \varphi_E(a)) = \text{coeff}_E(x, \varphi^Q(a)).
\]
Since \( a \in \partial_n^{-1}(I_E C_{n-1}(U)) \), if \( Z \) is large enough, there exists \( b_Z \in A_n'(x) \) such that
\[
\sum_{\epsilon \in Z} \epsilon a - b_Z \in \ker(\partial_n).
The left hand side of (4.3) is equal to
\[ \epsilon \phi \]
Note that we have
\[ \text{side of (4.3) is equal to} \]
\[ i \]
\[ \text{denote by} \]
\[ p \]
be a homomorphism induced by
\[ F \]
Lemma 26.
Let
\[ \Box \]
Hence (4.3) is proved.

The next lemma follows from a simple calculation.

Lemma 26. Let \( p \) be a \( \mathbb{Q}[E] \)-automorphism of \( V \). Let \( U_1 \) and \( U_2 \) be fat subsets of \( F^\times \) such that \( pU_1 \subset U_2 \). Let
\[ p' : H_0(E, Z(U_1)) \to H_0(E, Z(U_2)) \]
be a homomorphism induced by \( p \). For \( x \in H_{n-1}(E, Z) \), we have
\[ p' \circ \Omega_{E,U_1}(x) = \text{sgn}(\det(p)) \times \Omega_{E,U_2}(x). \]

4.3. The module \( Z(U, W) \). For \( W \subset S_\infty \), put \( X_W = \prod_{v \in S_\infty \setminus W} F_v^\times / \mathbb{R}_{>0} \). We denote by \( i_W \) the natural diagonal map from \( F^\times \) to \( X_W \). For groups \( E_1 \subset E_2 \) and an \( \mathbb{Z}[E_2] \)-module \( M \), let \( \alpha_{E_1, E_2} : H_0(E_1, M) \to H_0(E_2, M) \) be a natural homomorphism.

Let \( U \) be a subset of \( F^\times \) such that \( U \cap i_q^{-1}(g) \) is fat for all \( g \in X_\emptyset \). For example, this condition is satisfied if \( U \) is dense in \( F \otimes \mathbb{R} \). For \( W \subset S_\infty \) and \( g \in X_W \), we put \( U_g := U \cap i_W^{-1}(g) \). We define the functors \( Z(U, -) \) from \( \text{Sub}(S_\infty) \) to \( \text{Mod}(\mathbb{Z}) \) by
\[ Z(U, W) = \bigoplus_{g \in X_W} Z(U_g). \]

Proposition 27. Let \( E \) be a subgroup of \( O_F^\times \). Assume that \( U \) is closed under the action of \( E \). For \( W \subset S_\infty \) and \( i > 0 \), we have
\[ H_i(E, Z(U, W)) = 0 \]
if \( W \subset S_\infty \) or \(-1 \notin E \).

Proof. Fix \( i > 0 \). Put \( E' = E \cap i_W^{-1}(1) \). Then \( X_W \) can be written as
\[ X_W = \bigcup_{\epsilon \in E/E'} \epsilon C \]
where $C$ is a subset of $X_W$. Put
\[
Z_C = \bigoplus_{g \in C} Z(U_g).
\]
Then we have
\[
Z(U,W) = Z_C \otimes_{\mathbb{Z}[E']} \mathbb{Z}[E].
\]
Thus we have
\[
H_i(E, Z(U,W)) = H_i(E', Z_C)
\]
\[
= \bigoplus_{g \in C} H_i(E', Z(U_g)).
\]
Fix $g \in C$. From Proposition 18, there exists an exact sequence of $\mathbb{Z}[E']$-module:
\[
0 \to X_{\text{low}}(U_g) \to Z(U_g) \xrightarrow{\phi_r} \bar{C}_n^-(U_g) \to 0.
\]
Since $X_{\text{low}}(U_g)$ is a free $E'$-module, we have
\[
H_i(E', X_{\text{low}}(U_g)) = 0 \quad (i > 0).
\]
Thus it is enough to prove that $H_i(E', \bar{C}_n^-(U_g)) = 0$ for $i > 0$. Put $E'' := \ker(\text{sgn} : E' \to \{\pm 1\})$. Note that $E'$ and $E''$ are free abelian groups of rank $n - 1$ from the condition $(W \neq S_{\infty}) \vee (-1 \notin E)$. In the case $E'' = E'$, we have
\[
H_i(E', \bar{C}_n^-(U_g)) = H_{i+n-1}(E', \mathbb{Z}) = 0.
\]
In the case $E'' \neq E'$, there exists an exact sequence
\[
0 \to \bar{C}_n^-(U_g) \to \bar{C}_n^+(U_g) \otimes_{\mathbb{Z}[E']} \mathbb{Z}[E'] \to \bar{C}_n^+(U_g) \to 0.
\]
For $i > 0$, we have
\[
H_i(E', \bar{C}_n^+(U_g)) = H_{i+n-1}(E', \mathbb{Z}) = 0
\]
and
\[
H_i(E', \bar{C}_n^+(U_g) \otimes_{\mathbb{Z}[E']} \mathbb{Z}[E']) = H_i(E'', \bar{C}_n^+(U_g))
\]
\[
= H_{i+n-1}(E'', \mathbb{Z})
\]
\[
= 0.
\]
Thus the claim is proved. \hfill \Box

Let $\rho_1, \ldots, \rho_n$ be the distinct real embeddings of $F$. Define an orientation $r : F^n \to \{0, 1, -1\}$ by $r(x_1, \ldots, x_n) := \text{sgn}(\det(\rho_i(x_j))_{i,j=1}^n)$. From the definition, $\mathcal{L}_{\infty}$ induces a map
\[
Z(U,W) \to \bigoplus_{g \in X_W} \mathcal{K}(U_g).
\]
If $W \neq S_{\infty}$ then $\bigoplus_{g \in X_W} \mathcal{K}(U_g)$ can be viewed as a subgroup of $\text{Map}(F^\times, \mathbb{Z})$ by the obvious way. Therefore we obtain a natural map from $Z(U,W)$ to $\text{Map}(F^\times, \mathbb{Z})$ for $W \neq S_{\infty}$. By abuse of notation, we denote this map by $\mathcal{L}_{\infty}$. We denote by $\mathcal{O}_{F, +}$ the group of totally positive units of $F$. Let $E$ be a finite index subgroup of $\mathcal{O}_{F, +}$. Then $H_{n-1}(E, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. The canonical generator $\eta_E$ of $H_{n-1}(E, \mathbb{Z})$ is defined by the following Lemma.
Lemma 28. There exists a generator $\eta_E$ of $H_{n-1}(E, \mathbb{Z})$ such that

$$\sum_{\epsilon \in E} \epsilon L_\infty(a) = \text{sgn}(g)1_{i_0^{-1}(g)}$$

for any element $g$ of $X_0$ and lift $a \in Z(U_g)$ of $\Omega_{E,U_g}(\eta_E) \in H_0(E, Z(U_g))$.

Proof. We put $D_{k,g} := \mathbb{Z}[(i_0^{-1}(g))^k]$. Then $C_k(U_g)$ can be naturally embedded in $D_{k,g}$. Take a $\mathbb{Z}$-basis $(\epsilon_1, \ldots, \epsilon_{n-1})$ of $E$ such that $\det(\log(\rho_i(\epsilon_j)))_{i,j=1}^{n-1} > 0$. For $\sigma \in \mathfrak{S}_{n-1}$ and $0 \leq j \leq n - 1$, we put

$$\epsilon_{j,\sigma} := \prod_{k=1}^{j} \epsilon_{\sigma(k)}.$$ 

For $x \in i_0^{-1}(g)$, we put

$$F(x) := \sum_{\sigma \in \mathfrak{S}_{n-1}} (x\epsilon_{0,\sigma}, \ldots, x\epsilon_{n-1,\sigma}) \in D_{n,g}.$$ 

Then we have $F(x) \in \ker(D_{n,g} \to D_{n-1,g})$. Let $\eta_E \in H_{n-1}(E, \mathbb{Z})$ be the image of $F(x)$ under the composite map

$$\ker(D_{n,g} \to D_{n-1,g}) \to \ker(D_{n,g} \to D_{n-1,g})/\text{im}(D_{n+1,g} \to D_{n,g}) \simeq H_{n-1}(E, \mathbb{Z}).$$

Then $\eta_E$ does not depend on the choice of $g$ and $x$. Fix an irrational vector $Q \in F \otimes \mathbb{R}$ such that $\rho_j(Q) = 0$ for $1 \leq j \leq n - 1$. We define $\varphi^Q : D_{n,g} \to K(F_+)$ by

$$\varphi^Q((x_1, \ldots, x_n)) := \begin{cases} \varphi^Q((x_1, \ldots, x_n)) & \text{if } x_1, \ldots, x_n \text{ are in general position} \\ 0 & \text{otherwise.} \end{cases}$$

Fix $x \in i_0^{-1}(g)$. Then it is known that

$$\sum_{\epsilon \in E} \epsilon L(\varphi^Q(F(x))) = \text{sgn}(g)1_{i_0^{-1}(g)},$$

and $\varphi^Q(a) = 0$ for $a \in \ker(D_{n,g} \to D_{n-1,g})$. Let $z \in C_n(U_g)$ be a lift of $\eta_E$. Since

$$F(x) - z \in I_ED_{n,g} + \ker(D_{n,g} \to D_{n-1,g}),$$

we have

$$\sum_{\epsilon \in E} \epsilon L(\varphi^Q(F(x) - z)) = 0.$$ 

Thus we have

$$\sum_{\epsilon \in E} \epsilon L(\varphi^Q(z)) = \text{sgn}(g)1_{i_0^{-1}(g)}.$$ 

Since $\varphi^Q(z) = \varphi^Q(z) = \varphi_E(z)$ from Lemma 25 $\varphi^Q(z) - a \in I_EZ(U_g)$. Thus the claim is proved.

For a subgroup $E'$ of $E$ of finite index, we have

$$\alpha_{E,E'}(\eta_{E'}) = \#(E/E') \cdot \eta_E.$$ 

Definition 29. Let $E$ be a subgroup of finite index of $O_{E,+}^\times$. Let $U$ be a subset of $F^\times$ such that $U \cap i_0^{-1}(g)$ is fat for all $g \in X_0$. Assume that $U$ is closed under the action of $E$. For $g \in X_0$, we define $\vartheta_{\infty,g}(U, E) \in H_0(E, Z(U_g))$ by

$$\vartheta_{\infty,g}(U, E) = \text{sgn}(g)\Omega_{E,U,g}(\eta_E).$$
Definition 30. Let $E$ be a subgroup of finite index of $\mathcal{O}_K^\times$. Let $U$ be a subset of $F^\times$ such that $U \cap i_{\varepsilon}^{-1}(g)$ is fat for all $g \in X_\emptyset$. Assume that $U$ is closed under the action of $E$. Put $E_+ = E \cap \mathcal{O}_K^\times$. Let $f_g : Z(U_g) \to Z(U, \emptyset)$ be the natural inclusion. Put

$$a_g = \alpha_{E_+E}(f_g(\vartheta_{\infty,g}(U, E_+))) \in H_0(E, Z(U, W)).$$

Note that $a_{eg} = a_g$ for all $g \in X_\emptyset$ and $e \in E$. We define $\vartheta_{\infty}(U, E) \in H_0(E, Z(U, W))$ by

$$\vartheta_{\infty}(U, E) = \sum_{g \in X_\emptyset/(E/E_+)} a_g.$$

Proposition 31. Let $D \in \mathcal{B}_{\infty}(0)$ be a lift of $\vartheta_{\infty}(U, E)$. Then we have

$$\sum_{e \in E} \epsilon L_{\infty}(D) = 1_{F^\times}.$$

Proof. It follows from Lemma

Proposition 32. Fix $v \in S_{\infty}$. Let $h : H_0(E, Z(U, \emptyset)) \to H_0(E, Z(U, \{v\}))$ be the map induced from the natural map $Z(U, \emptyset) \to Z(U, \{v\})$. Then we have

$$h(\vartheta_{\infty}(U, E)) = 0.$$

Proof. The case $(F, E) = \{1\}$ is obvious from the definition. Therefore we assume that $F \neq Q$. Put $E_+ = E \cap \mathcal{O}_K^\times$ and $E' = E \cap i_{\varepsilon}^{-1}(1)$. For $g \in X_\emptyset$, consider the commutative diagram

$$H_{n-1}(E_+, Z^-) \xrightarrow{\Omega_{E_+, U_g}} H_0(E_+, Z(U_g)) \xrightarrow{f_g} H_0(E_+, Z(U, \emptyset)) \xrightarrow{\alpha_{E_+E}} H_0(E, Z(U, \emptyset)) \xrightarrow{h}$$

where $q(g) \in X_{\{v\}}$ is the image of $g$, and $f_g, f_q(g)$ are natural maps. From the definition, we have

$$\vartheta_{\infty}(U, E) = \sum_{g \in X_\emptyset/(E/E_+)} a_g$$

where we put

$$a_g := \text{sgn}(g) \times \alpha_{E_+, E} \circ f_g \circ \Omega_{E_+, U_g}(\eta_{E_+}).$$

We have

$$h(a_g) = \text{sgn}(g) \times \alpha_{E_+, E} \circ f_q(g) \circ \Omega_{E_+, U_q(g)}(\eta_{E_+}).$$

(The case $E_+ = E'$). Let $e$ be a unique element of $i_{\varepsilon}^{-1}(1) \setminus \{1\}$. Then we have

$$\vartheta_{\infty}(U, E) = \sum_{g \in X_\emptyset/(E/E_+)/(1, e)} (a_g + a_{ge}).$$

Since $q(ge) = q(g)$ and $\text{sgn}(ge) = -\text{sgn}(g)$, we have

$$h(a_g + a_{ge}) = 0 \quad (g \in X_\emptyset).$$

Thus we have

$$h(\vartheta_{\infty}(U, E)) = 0.$$
Let \( A \) be Shintani zeta functions. This section does not contain any new result. We denote \( H \).

Hence we have \( H_0 \).

Since the diagram commutes, we have

\[
H_n^{-1}(E', Z^-) \xrightarrow{\alpha_{E', Z}} H_0(E', Z(U_{q(g)})) \xrightarrow{f_{q(g)}} H_0(E, Z(U, \{v\}))
\]

Proposition 33. Fix \( p \in F^\times \). Assume that \( pU \subset U \). Then we have

\[
[p] \vartheta_\infty(U, E) = \vartheta_\infty(U, E).
\]

Proof. Let \( (a_g)_{g \in X_0} \) be as in Definition 30. Take \( g \in X_0 \). Let us consider the diagram

\[
H_n^{-1}(E', Z^-) \xrightarrow{\alpha_{E', Z}} H_0(E', Z(U_{q(g)})) \xrightarrow{f_g} H_0(E, Z(U, \{v\})) \xrightarrow{\alpha_{E, Z}} H_0(E, Z(U, \{v\}))
\]

where \( f_g \) and \( f_{pg} \) are natural maps. From Lemma 26 we have

\[
\Omega_{E', U_{q(g)}} = (\times \sgn(p)[p]) \circ \Omega_{E, U_{pg}}.
\]

Thus this diagram commutes. From the definition, we have

\[
a_g = \sgn(g) \times \alpha_{E, E} \circ f_g \circ \Omega_{E, U_{q(g)}}(\eta_E)
\]

\[
a_{pg} = \sgn(pg) \times \alpha_{E, E} \circ f_{pg} \circ \Omega_{E, U_{pg}}(\eta_E).
\]

Since the diagram commutes, we have

\[
[p] a_g = a_{pg}.
\]

Hence we have

\[
[p] \vartheta_\infty(U, E) = [p] \sum_g a_g = \sum_g a_g = \vartheta_\infty(U, E).
\]

5. Shintani zeta functions

In this section, we view some basic definitions and results concerning to the Shintani zeta functions. This section does not contain any new result. We denote by \( A_{\text{finite}} \) the finite adele ring of \( \mathbb{Q} \).
5.1. The Shintani zeta functions and Solomon-Hu pairings. Let $V$ be an $n$-dimensional vector space over $\mathbb{Q}$ and $\mathcal{U}$ a subset of $V$. We say that $\Phi \in S(V \otimes \mathbb{A}_{\text{finite}})$ is $\mathcal{U}$-smooth if

$$\int_{\mathbb{A}_{\text{finite}}} \Phi(x + tu)dt = 0 \quad \forall (x, u) \in V \times \mathcal{U}. $$

We define the action of $\mathbb{Z}[V]$ to $\text{Map}(V, \mathbb{Z})$ by $([y]f)(x) = f(x - y)$. Let $\varepsilon : \mathbb{Z}[V] \rightarrow \mathbb{Z}$ be the augmentation map. We view $\mathbb{Q}$ as a $\mathbb{Z}[V]$-module by the action

$$\Delta m = \varepsilon(\Delta)m \quad (\Delta \in \mathbb{Z}[V], m \in \mathbb{Q}).$$

We denote by $\delta \in \text{Map}(V, \mathbb{Z})$ the Kronecker delta function, i.e., $\delta(0) = 1$ and $\delta(x) = 0$ for $x \neq 0$. We denote by $A'(V)$ the set of maps from $V$ to $\mathbb{Z}$ which has a finite support. We put

$$A(V) = \{ f : V \rightarrow \mathbb{Z} \mid \exists \Delta \in X(V), \varepsilon(\Delta) \neq 0 \text{ and } \Delta f \in A'(V) \}. $$

**Lemma 34.** For $f \in K'(\mathcal{U})$ and $\mathcal{U}$-smooth $\Phi$, we have $f \cdot \Phi \in A(V)$.

**Proof.** It is enough to only consider the case $f = 1_{C(u_1, \ldots, u_m)}$ where $u_1, \ldots, u_m \in \mathcal{U}$. There exists a positive integer $k$ such that

$$\text{supp}(\Phi) \cap (\mathbb{Q}u_1 + \cdots + \mathbb{Q}u_m) \subset \frac{1}{k}\mathbb{Z}u_1 + \cdots + \frac{1}{k}\mathbb{Z}u_m$$

$$\Phi(x + ku_j) = \Phi(x) \quad (x \in V, j = 1, \ldots, m).$$

If we put $\Delta = \prod_{j=1}^{m} \sum_{c=0}^{k^2-1} [cu_j/k] \in X(V)$ then $\varepsilon(\Delta) = k^{2m} \neq 0$ and $\Delta (f \cdot \Phi) \in A'(V)$.

Let $\varpi : A(V) \rightarrow \mathbb{Q}$ be the unique $\mathbb{Z}[V]$-homomorphism such that $\varpi(\delta) = 1$. For $f \in K(\mathcal{U})$ and $\mathcal{U}$-smooth $\Phi$, we define a (constant term of) Solomon-Hu pairing by $\langle f, \Phi \rangle = \varpi(f \cdot \Phi)$. The importance of this pairing comes from the following Lemma:

**Lemma 35.** Let $m$ be a positive integer and $\rho_1, \ldots, \rho_m$ homomorphisms from $V$ to $\mathbb{R}$ such that $\rho_j(u) > 0$ for all $u \in \mathcal{U}$ and $j = 1, \ldots, m$. Take $f \in K(\mathcal{U})$ and $\mathcal{U}$-smooth $\Phi$. Then the series

$$\zeta((s_1, \ldots, s_m), \Phi, f) := \sum_{f(x) \neq 0} f(x)\Phi(x)\rho_1(x)^{-s_1} \cdots \rho_m(x)^{-s_m}$$

converges absolutely if the real part of $s_1 + \cdots + s_m$ are larger than $n$. Moreover $\zeta((s_1, \ldots, s_m), \Phi, f)$ is analytically continued to the whole $\mathbb{C}^m$ and satisfies

$$\zeta((0, \ldots, 0), \Phi, f) = \langle f, \Phi \rangle. $$

**Proof.** The absolute convergence is obvious. Let $M$ be the field of meromorphic functions on $\mathbb{C}^m$. We view $M$ as a $\mathbb{Z}[V]$-module by the action

$$([y]f)(t_1, \ldots, t_m) = \exp(-\rho_1(y)t_1 - \cdots - \rho_m(y)t_m) \cdot f(t_1, \ldots, t_m) \quad (f \in M).$$

Let $\Omega : A(V) \rightarrow M$ be the unique $\mathbb{Z}[V]$-homomorphism such that $\Omega(\delta) = 1$. Put $h = \Omega(f \cdot \Phi) \in M$. Then we have

$$\zeta((s_1, \ldots, s_m), \Phi, f)$$

$$= \prod_{j=1}^{m} \int_{0}^{\infty} \frac{1}{\Gamma(s_j)} \int_{0}^{\infty} \cdots \int_{0}^{\infty} h(t_1, \ldots, t_m)t_1^{s_1-1} \cdots t_m^{s_m-1} dt_1 \cdots dt_m$$
if the real parts of \( s_1, \ldots, s_m \) are large enough. Thus
\[
\prod_{j=1}^{m} \frac{1}{\Gamma(s_j)(e^{2\pi i s_j} - 1)} \int_{C} \cdots \int_{C} h(t_1, \ldots, t_m)t_1^{s_1-1} \cdots t_m^{s_m-1} dt_1 \cdots dt_m
\]
gives an analytic continuation of \( \zeta((s_1, \ldots, s_m), \Phi, f) \) where \( C \) is the Hankel contour.
From the simple residue calculus, we obtain
\[
\zeta((0, \ldots, 0), \Phi, f) = h(0, \ldots, 0) = \langle f, \Phi \rangle.
\]

5.2. Some integrality results concerning Solomon-Hu pairings. Let \( q \) be a rational prime. Fix two \( \mathbb{Z}_q \)-lattices \( \Gamma' \subset \Gamma \subset V \otimes \mathbb{Q}_p \). In this section, we consider the case \( \mathcal{U} = (\Gamma \setminus \Gamma') \cap V \). For \( x \in \Gamma' \), define \( h_x \in S(V \otimes \mathbb{Q}_p) \) by \( h_x = 1_{\mathbb{R}^{-1}(x)} - 1_{\mathbb{R}^{-1}(0)} \) where \( \pi : \Gamma \to \Gamma' \) is the projection. We write \( S_0 \) for the submodule of \( S(V \otimes \mathbb{Q}_p) \) spanned by \( \{h_x \mid x \in \Gamma' \} \), and \( S_1 \) for the submodule of \( S(V \otimes \mathbb{Q}_p) \) spanned by \( \sum_{x \in \Gamma' \cap \Gamma} h_x \).

**Lemma 36.** Let \( \Phi \in S(V \otimes \prod_{p \neq q} \mathbb{Q}_p) \otimes S_0 \subset S(V \otimes \mathbb{A}_{\text{finite}}) \) and \( f \in K(\mathcal{U}) \). Then \( f \cdot \Phi \in A(V) \) and
\[
\langle f, \Phi \rangle \in \frac{1}{q^n} \mathbb{Z}.
\]

**Proof.** It is enough to prove that \( \omega(1_{C(u_1, \ldots, u_m)} \cdot (\Phi' \otimes \Phi'')) \) \( \in \frac{1}{q^n} \mathbb{Z} \) for all \( u_1, \ldots, u_m \in U \), \( \Phi' \in S(V \otimes \prod_{p \neq q} \mathbb{Q}_p) \) and \( \Phi'' \in S_0 \). Put \( \Phi'' = q(1_{C(u_1, \ldots, u_m)} \cdot (\Phi' \otimes \Phi'')) \). Take a positive integer \( k \in \mathbb{Z} \setminus q\mathbb{Z} \) such that \( \Phi' \) is periodic with respect to \( \sum_{i=1}^{m} \mathbb{Z}k u_i \). Put \( \Delta = \prod_{j=1}^{m} \sum_{q=1}^{m-1} \epsilon(q u_j) \in \mathbb{Z}[V] \). Then \( \Delta h \in A'(V) \). Since \( \epsilon(\Delta) = q^m \), the lemma is proved. \( \square \)

The following lemma was proved in [4 Proposition 4.1].

**Lemma 37.** Let \( \Phi \in S(V \otimes \prod_{p \neq q} \mathbb{Q}_p) \otimes S_1 \subset S(V \otimes \mathbb{A}_{\text{finite}}) \) and \( f \in K(\mathcal{U}) \). If \( q \geq n + 2 \) then
\[
\langle f, \Phi \rangle \in \mathbb{Z}.
\]

**Proof.** It is enough to prove that \( \omega(1_{C(u_1, \ldots, u_m)} \cdot (\Phi' \otimes \Phi'')) \) \( \in \frac{1}{q^n} \mathbb{Z} \) for all \( u_1, \ldots, u_m \in U \), \( \Phi' \in S(V \otimes \prod_{p \neq q} \mathbb{Q}_p) \) and \( \Phi'' = q(1_{C(u_1, \ldots, u_m)} \cdot (\Phi' \otimes \Phi'')) \). Take a positive integer \( k \in \mathbb{Z} \setminus q\mathbb{Z} \) such that \( \Phi' \) is periodic with respect to \( \sum_{j=1}^{m} \mathbb{Z}k u_j \). Put \( w_j = ku_j \) for \( j = 1, \ldots, m \). Let \( \zeta_q \) be a primitive \( q \)-th root of unity. We denote by \( M \) the set of non-trivial group homomorphism from \( \Gamma/\Gamma' \) to \( \mathbb{Z}[\zeta_q]^\times \). For \( \varphi \in M \), we put
\[
h_\varphi := \sum_{\varphi \in M} 1_{C(u_1, \ldots, u_m)} \cdot (\Phi' \otimes \varphi).
\]
Then we have \( h = \sum_{\varphi \in M} h_\varphi \). For \( \varphi \in M \), we have
\[
\omega(h_\varphi) \in (1 - \zeta_q)^{-m} \mathbb{Z}[\zeta_q]
\]
since \( \Delta_\varphi h_\varphi \in A'(V) \otimes \mathbb{Z}[\zeta_q] \) and \( 1/\epsilon(\Delta_\varphi) \in (1 - \zeta_q)^{-m} \mathbb{Z}[\zeta_q] \) where
\[
\Delta_\varphi := \prod_{j=1}^{m} (1 - \varphi(w_j)[w_j]).
\]
Hence, from the condition \( q \geq n + 2 \), we have
\[
\varpi(h) = \sum_{\varphi \in M} \varpi(h_\varphi) \in (1 - \zeta_q)^{-m} \mathbb{Z}[\zeta_q] \cap \mathbb{Q} = \mathbb{Z}.
\]

\[\square\]

6. Construction of the Shintani data \( \text{Sh}^{v_0}, \text{Sh}^\circ, \text{Sh}^{v_0,\circ} \)

For \( W \subset S \), let \( \mathbb{R}[[s_v]_{v \in S \setminus W}] \) be the formal power series ring in \( \#(S \setminus W) \) variables. We regard \( \mathbb{R}[[s_v]_{v \in S \setminus W}] \) as a \( \mathbb{Z}[F^\times] \)-module by the action
\[
[x]p = p \prod_{v \in S \setminus W} |x|_v^{-s_v} \quad (x \in F^\times, p \in \mathbb{R}[[s_v]_{v \in S \setminus W}]).
\]

We denote by \( \mathbb{F}_W \) the submodule of \( \mathbb{R}[[s_v]_{v \in S \setminus W}] \) consisting of formal power series whose constant terms are in \( \mathbb{Z} \). Let \( \mathcal{R} \) be the functor from \( \text{Sub}(S) \to \text{Mod}(\mathbb{Z}[F^\times]) \) defined by \( \mathcal{R}(W) := \mathbb{F}_W \otimes \mathbb{Z}[N_F/\prod_{v \in W} N_v] \). Put \( \Upsilon = \mathbb{R}[[s]] \otimes \mathbb{Z}[G] \). We define \( \lambda : \mathcal{R}(\emptyset) \to \Upsilon \) by
\[
\lambda(P(s_0, \ldots, s_v) \otimes [u]) = P(s, \ldots, s) \prod_{v \notin S} \left| u_v \right|_v^{-s_v} \otimes [\text{rec}(u)].
\]

Put \( \vartheta = \Theta_{S,\{q\},K}(s) \in \Upsilon \).

Let us fix a prime ideal \( q \notin S \). Put \( q = ch(q) \). We put \( \mathbb{F}_q = \mathbb{Z}/q\mathbb{Z} \subset \kappa_q \).

The purpose of this section is to construct three Shintani data \( \text{Sh}^{v_0}, \text{Sh}^\circ, \text{Sh}^{v_0,\circ} \) for \( (\mathcal{R}, \Upsilon, \lambda, \theta) \). First, we define a triple \( (B_{R,M}, L_{R,M}, \vartheta_{R,M}) \) for a subset \( R \subset S_f \) and a subgroup \( M \subset \mathbb{F}_q^\times \). We define \( \text{Sh}^{v_0}, \text{Sh}^\circ, \text{Sh}^{v_0,\circ} \) by using \( (B_{R,M}, L_{R,M}, \vartheta_{R,M}) \) for \( (R, M) = (S_f \cap \{v_0\}, \mathbb{F}_q^\times), (\emptyset, 1), (S_f \cap \{v_0\}, 1) \) respectively.

6.1. Definition of \( \mathcal{A}_{R,G}(W) \). Fix a subset \( R \subset S_f \) and a subgroup \( M \subset \mathbb{F}_q^\times \). We put \( \tilde{M} := \ker(O_q^\times \to \kappa_q^\times \to \kappa_q^\times / M) \).

Definition 38. For \( x \in \mathbb{F}_q \subset \kappa_q \), define \( h_x \in \mathcal{S}(\mathbb{F}_q) \) by \( h_x = 1_{\pi^{-1}(x)} - 1_{\pi^{-1}(0)} \) where \( \pi : O_q \to O_q/qO_q = \kappa_q \) is the natural composite map.

For a prime ideal \( p \), we put \( V_p = 1 + p^mO_p \) where \( m \) is a least positive integer such that \(-1 \notin 1 + p^mO_p \).

Definition 39. For \( W \subset S_f \setminus R \), define \( f_{R,M,W} \in \mathcal{S}(\mathbb{A}_F^{S^\infty}_E) \) by
\[
f_{R,M,W}(x) := \prod_{p \notin S^\infty} f_p(x_p)
\]
where \( f_p \in \mathcal{S}(\mathbb{F}_p) \) is defined by
\[
f_p = \begin{cases} 
\sum_{x \in M} h_x & p = q \\
1_{V_p} & p \in R \\
1_{O_p^\times} & p \in S_f \setminus (W \cup R) \\
1_{O_p} & p \in W \\
1_{O_p} & p \notin S_f \cup \{q\}.
\end{cases}
\]

For \( x \in \mathbb{A}_F^{S^\infty} \) and \( f \in \mathcal{S}(\mathbb{A}_F^{S^\infty}) \), define \( xf \in \mathcal{S}(\mathbb{A}_F^{S^\infty}) \) by \((xf)(xy) = f(y)\). The function \( f_{R,M,W} \) satisfies the following properties:

- \( f_{R,M,W} \) is invariant under the action of \( \prod_{p \notin R \cup \{q\}} O_p^\times \times \prod_{p \in R} V_p \times \tilde{M} \).
• For \( p \in S_f \setminus (W \cup R) \),
\[
\sum_{x \in O_p^\times / V_p} xf_{(p) \cup R,M,W} = f_{R,M,W}
\]
\( 6.1 \)
• For \( M_1 \subset M_2 \),
\[
\sum_{x \in M_2 / M_1} xf_{R,M_1,W} = f_{R,M_2,W}
\]
\( 6.2 \)
• For \( p \in S_f / (R \cup W) \),
\[
(p^{-1}_p - 1) f_{R,M,W \cup \{p\}} = f_{R,M,W}
\]
\( 6.3 \)

**Definition 40.** For \( W \subset S_f \setminus R \), we denote by \( A_{R,M}(W) \) the subgroup of \( S(k_{R \cap \mathbb{F}_q}^\infty) \) spanned by \( \{xf_{R,M,W} \mid x \in k_{R \cap \mathbb{F}_q}^\infty \} \).

**Definition 41.** We define subgroups \( F_{R,M}, \mathcal{E}_{R,M} \) and a submonoid \( \mathcal{E}_{R,M}^0 \) of \( F^\times \) by
\[
F_{R,M} := F^\times \cap (\tilde{M} \times \prod_{p \in R} V_p)
\]
\[
\mathcal{E}_{R,M} := F_{R,M}^\times \cap \left( \bigcap_{p \in S_f \cup S_q} O_p^\times \right)
\]
\[
\mathcal{E}_{R,M}^0 := F_{R,M}^\times \cap \left( \bigcap_{p \in S_q} O_p \right).
\]

We have \( \mathcal{E}_{R,M} \subset \mathcal{E}_{R,M}^0 \subset F_{R,M} \subset F^\times \). Note that \( A_{R,M}(W) \) is isomorphic to a direct sum of copies of \( \mathbb{Z}[\mathcal{E}_{R,M} / (\mathcal{E}_{R,M} \cap O_{F_p}^\times)] \).

6.2. **Definition of the functor** \( B_{R_M} \). Fix a subset \( R \) of \( S_f \) and a subgroup \( M \) of \( \mathbb{F}_q^\times \). In this section, we define a functor \( B_{R,M} \) from \( \text{Sub}(S \setminus R) \) to \( \text{Sub}(\mathbb{Z}[F^\times]) \).

We denote by \( U \) the set of \( x \in F^\times \) which satisfy the following conditions.

- For all \( p \in S_q \setminus \{q\} \), \( \text{ord}_p(x) \geq \min\{m \in \mathbb{Z}_{\geq 1} \mid 1 + \pi_p^m O_p \subset J_p \} \).
- \( x_q \in \pi^{-1}(F_q^\times) \) where \( \pi : O_q \to k_q \) is the natural projection.

Note that \( U \) is closed under the action of \( \mathcal{E}_{R,M}^0 \). For \( W \subset S_{\infty} \), we put
\[
Q(W) := S(k_{R \cap \mathbb{F}_q}^\infty) \otimes \mathbb{Z}(F^\times, W) \otimes \mathbb{Z}[N^S].
\]

**Definition 42.** For \( W \subset S \setminus R \), we define a \( \mathbb{Z}[\mathcal{E}_{R,M}] \)-submodule \( B_{R,M}(W) \) of \( Q(W \cap S_{\infty}) \) by
\[
B_{R,M}(W) := A_{R,M}(W \cap S_f) \otimes (U, W \cap S_{\infty}) \otimes \mathbb{Z}[N^S].
\]

**Definition 43.** For \( W \subset S \setminus R \), we define a \( \mathbb{Z}[F^\times] \)-module \( B_{R,M}(W) \) by
\[
B_{R,M}(W) := B_{R,M}(W) \otimes_{\mathbb{Z}[\mathcal{E}_{R,M}]} \mathbb{Z}[F^\times].
\]

Note that the natural map
\[
B_{R,M}(W) \otimes_{\mathbb{Z}[\mathcal{E}_{R,M}]} \mathbb{Z}[F^\times] \to Q(W \cap S_{\infty})
\]
is injective since
\[
A_{R,M}(W) \otimes_{\mathbb{Z}[\mathcal{E}_{R,M}]} \mathbb{Z}[F^\times] \to S(k_{R \cap \mathbb{F}_q}^\infty)
\]
is injective. From now we regard \( B_{R,M}(W) \) as a \( \mathbb{Z}[F^\times] \)-submodule of \( Q(W \cap S_{\infty}) \).
Proof. It is enough to consider the case \(c \neq 0\). If \(c\) is infinite then the claim is obvious. We assume that \(c\) is finite. It is enough to prove that \(xf_{R,M,W_1} \otimes b \otimes c \in B_{R,M}(W_2)\) for \(x \in A_{F}^{S_1 \cup S_2 \cup S_3 \cup S_4}, b \in Z(U, W_1 \cap S_\infty),\) and \(c \in Z[N^S]\). Let \(p\) be an any element of \(F_{R,M} \cap (\pi_v O_\infty) \cap O_{S_1 \cup S_2 \cup S_3 \cup S_4} \neq 0\). Then we have \(pb \in b \in Z(U, W_1 \cap S_\infty)\) since \(p \in \mathcal{E}_{R,M}^\infty\). Thus we have
\[
xf_{R,M,W_1} \otimes b \otimes c = x(\pi_v^{-1} - 1)f_{R,M,W_2} \otimes b \otimes c
\]
\[
= [p^{-1}] (x \pi_v^{-1}f_{R,M,W_2} \otimes pb \otimes pc) - xf_{R,M,W_2} \otimes b \otimes 1.
\]
Since \(A_{R,M,W_2}\) is invariant under the action of \(xp^{-1}\), we have \(xp^{-1}f_{R,M,W_2} \in A_{R,M}(W_2)\). Thus the claim is proved.

From Proposition 44 we can regard \(B_{R,M}\) as a functor from \(\text{Sub}(S \setminus R)\) to \(\text{Mod}(Z[F^\times])\).

Lemma 45. Let \(A\) be an abelian group, \(A'\) a subgroup of \(A\), and \(X\) a \(Z[A]\)-module. Then we have
\[
H_i(A, X \otimes Z[A/A']) = H_i(A', X).
\]

Proof. Put \(Y := \text{Fl}(X) \otimes Z[A]/Z[A]\) where \(\text{Fl}\) is the forgetful functor from \(\text{Sub}(Z[A])\) to \(\text{Sub}(Z[A'])\). Define \(f \in \text{Hom}_{Z[\mathbb{A}]}(Y, X \otimes Z[A/A'])\) by
\[
f(x \otimes a) = ax \otimes a \quad (x \in \text{Fl}(X), a \in Z[A]).
\]
Then \(f\) is an isomorphism. Thus we have
\[
H_i(A, X \otimes Z[A/A']) = H_i(A, Y)
\]
\[
= H_i(A', \text{Fl}(X)).
\]
Thus the lemma is proved.

Proposition 46. Let \(E\) be a subgroup of \(F^\times\), and \(W\) a subset of \(S \setminus R\). If \(-1 \notin F_{R,M} \cap E\) or \(S_\infty \neq W \cap S_\infty\), we have
\[
H_i(E, B_{R,M}(W)) = 0
\]
for \(i > 0\).

Proof. Since
\[
B_{R,M}(W) = B_{R,M}^*(W) \otimes Z[\mathcal{E}_{R,M}^*] Z[F^\times]
\]
\[
= (B_{R,M}^*(W) \otimes Z[\mathcal{E}_{R,M}^*] Z[\mathcal{E}_{R,M} \cdot E]) \otimes Z[\mathcal{E}_{R,M} \cdot E] Z[F^\times]
\]
\[
= (B_{R,M}^*(W) \otimes Z[\mathcal{E}_{R,M} \cap E] Z[E]) \otimes Z[\mathcal{E}_{R,M} \cdot E] Z[F^\times]
\]
is isomorphic to a direct sum of copies of \(B_{R,M}^*(W) \otimes Z[\mathcal{E}_{R,M} \cap E] Z[E]\) as a \(Z[E]\)-module, it is enough to prove that
\[
H_i(\mathcal{E}_{R,M} \cap E, B_{R,M}^*(W)) = 0.
\]
Put \(U_S = \mathcal{E}_{R,M} \cap E \cap O_S^\times\) and \(U_F = \mathcal{E}_{R,M} \cap E \cap O_S^\times\). Since \(Z[N^S]\) is isomorphic to a direct sum of copies of \(Z[(\mathcal{E}_{R,M} \cap E)/U_S]\) as a \(Z[\mathcal{E}_{R,M} \cap E]\)-module, it is enough to prove that
\[
H_i(\mathcal{E}_{R,M} \cap E, A_{R,M}(W \cap S_j) \otimes Z(U, W \cap S_\infty) \otimes Z[(\mathcal{E}_{R,M} \cap E)/U_S]) = 0.
\]
From Lemma 45, we have
\[ H_1(E_{R,M} \cap E, A_{R,M}(W \cap S_f)) \otimes \mathbb{Z}(U, W \cap S_\infty) \otimes \mathbb{Z}[(E_{R,M} \cap E)/U_S]) = H_1(U_S, A_{R,M}(W \cap S_f)) \otimes \mathbb{Z}(U, W \cap S_\infty). \]
Since \( A_{R,M}(W \cap S_f) \) is isomorphic to a direct sum of copies of \( \mathbb{Z}[U_S/U_F] \) as a \( \mathbb{Z}[U_S] \)-module, it is enough to prove that
\[ H_1(U_S, \mathbb{Z}[U_S/U_F] \otimes \mathbb{Z}(U, W \cap S_\infty)) = 0. \]
From Lemma 45, we have
\[ H_1(U_S, \mathbb{Z}[U_S/U_F] \otimes \mathbb{Z}(U, W \cap S_\infty)) = H_1(U_F, \mathbb{Z}(U, W \cap S_\infty)). \]
From Proposition 27, we have
\[ H_1(U_F, \mathbb{Z}(U, W \cap S_\infty)) = 0. \]
Thus the claim is proved. \( \square \)

**Lemma 47.** For \( R_1 \subset R_2 \subset S_f \), \( M_2 \subset M_1 \subset \mathbb{F}_q^\times \) and \( W \subset S \setminus R_2 \), we have
\[ B_{R_1,M_1}(W) \subset B_{R_2,M_2}(W). \]

**Proof.** It is enough to prove that \( x f_{R_1,M_1,W \cap S_f} \otimes b \otimes c \subset B_{R_2,M_2}(W) \) for \( x \in \mathbb{F}_q^{S_\infty \cup S_f \cup S_q, \times}, b \in \mathbb{Z}(U, W_1 \cap S_\infty), \) and \( c \in \mathbb{Z}[N^S] \). Put
\[ X := (M_1/M_2) \times \prod_{p \in R_2 \setminus R_1} (O_p^\times / V_p). \]
From (6.1) and (6.2), we have
\[ f_{R_1,M_1,W \cap S_f} = \sum_{y \in X} y f_{R_2,M_2,W \cap S_f}. \]
Thus it is enough to prove that \( y x f_{R_1,M_1,W \cap S_f} \otimes b \otimes c \subset B_{R_2,M_2}(W) \) for \( y \in X \). Since the natural map \( E_{R_1,M_1} \to X \) is surjective, there exists lift \( p \in E_{R_1,M_1} \) of \( y \in X \). Then we have
\[ y x f_{R_1,M_1,W \cap S_f} \otimes b \otimes c := (p^{-1} y f_{R_1,M_1,W \cap S_f} \otimes p^{-1} b \otimes p^{-1} c) \otimes [p]. \]
Since \( p^{-1} y f_{R_1,M_1,W \cap S_f} \otimes p^{-1} b \otimes p^{-1} c \in B_{R_2,M_2}^*(W) \), the lemma is proved. \( \square \)

**Definition 48.** For \( R_1 \subset R_2 \subset S_f \) and \( M_2 \subset M_1 \subset \mathbb{F}_q^\times \), we define a natural transformation \( \iota_{R_1,M_1}^{R_2,M_2} \) from \( B_{R_1,M_1} \big|_{\text{Sub}(S \setminus R_2)} \) to \( B_{R_2,M_2} \) by the natural inclusion map.

### 6.3. Definition and properties of \( \vartheta_{R,M} \in H_0(F^x, B_{R,M}(\emptyset)) \)

Fix a subset \( R \) of \( S_f \) and a subgroup \( M \) of \( \mathbb{F}_q^\times \). We put \( \text{Cl}_{R,M} := N^{S \cup S_q}/E_{R,M} \). Note that \( \text{Cl}_{R,M} \) is a finite group.

**Definition 49.** For \( c \in \text{Cl}_{R,M}, \) we define \( \vartheta_{R,M,c} \in H_0(F^x, B_{R,M}(\emptyset)) \) as follows. Let \( u \in N^{S \cup S_q} \) be a lift of \( c \) and \( z \in Z(U, \emptyset) \) a lift of \( \vartheta_{\infty}(U, F_{R,M} \cap O_F^x) \) in \( H_0(F_{R,M} \cap O_F^x, Z(U, \emptyset)) \). Let \( \vartheta_{R,M,c} \) be the image of \( u f_{R,M,\emptyset} \otimes z \otimes [u] \in B_{R,M}(\emptyset) \). This definition does not depend on the choice of \( u \) and \( z \).

**Definition 50.** Define \( \vartheta_{R,M} \in H_0(F^x, B_{R,M}(\emptyset)) \) by
\[ \vartheta_{R,M} := \sum_{c \in \text{Cl}_{R,M}} \vartheta_{R,M,c}. \]
Proposition 51. There exists a lift $\alpha \in H_0(F_{R,M}^x \cap \mathcal{O}_S^x, B_{R,M}(\emptyset))$ of $\vartheta_{R,M}$ such that $r_{(\emptyset)}^0(\alpha) \in H_0(F_{R,M}^x \cap \mathcal{O}_S^x, B_{R,M}([\emptyset]))$ vanishes for all $v \in S \setminus R$.

Proof. Put $E_S = F_{R,M}^x \cap \mathcal{O}_S^x$, $E := F_{R,M}^x \cap \mathcal{O}_F^x$, $X := \prod_{v \in S \setminus R}(F_v^x / \mathcal{O}_v^x)$ and $Y := \prod_{v \notin S \setminus R \cup \{\emptyset\}}(F_v^x / \mathcal{O}_v^x)$. For $W \subset S \setminus R$ and $u \in N_{S \setminus S_\emptyset}$, we put

$$g_W(u) := uf_{R,M,W} \otimes t \otimes u \in H_0(E, B_{R,M}(W))$$

where $z$ is a lift of $\vartheta_{\emptyset}(U, E)$. For each $v \in S \setminus R$, fix an element $a(v)$ of

$$F_{R,M}^x \cap (\pi_v \mathcal{O}_v^x) \cap \left( \bigcap_{p \in (S \cup S_f) \setminus \{v\}} \mathcal{O}_p^x \right).$$

From (6.3) and Proposition 52 for $v \in S \setminus R$, we have

$$(6.4) \quad r_{(\emptyset)}^0(g_{(\emptyset)}(u)) = g_{(\emptyset)}(v) \otimes [a(v)] - g_{(\emptyset)}((v)) \otimes [1]$$

where $w \in N_{S \setminus S_\emptyset}$ is the image of $a(v)$. From Proposition 52 for $v \in S_\emptyset$, we have

$$r_{(\emptyset)}^0(g_{(\emptyset)}(u)) = 0 \in H_0(E, B_{R,M}([\emptyset])).$$  

For $u \in N_{S \setminus S_\emptyset}$ and $C \in \text{im}(X \to Y / F_{R,M}^x)$, we define $h_W(C, u) \in H_0(E_S, B_{R,M}(W))$ as the image of

$$g_W(tu) \otimes [b] \in H_0(E, B_{R,M}(W))$$

where $t \in N_{S \setminus S_\emptyset} \subset Y$ is a lift of $C$ and $b$ is an element of $F_{R,M}^x$, such that $bt \in \text{im}(X \to Y)$. This definition does not depend on the choice of $t$ and $b$. From (6.4), for $v \in S \setminus R$, we have

$$h_{(\emptyset)}(C, u) = g_{(\emptyset)}(tu) \otimes [b]$$

$$(6.6) \quad g_{(\emptyset)}((v)tu) \otimes [a(v)] - g_{(\emptyset)}((v)tu) \otimes [1]$$

Then $\alpha$ is a lift of $\vartheta_{R,M}$. For all $v \in S \setminus R$, we have $r_{(\emptyset)}^0(\alpha) = 0$ since $r_{(\emptyset)}^0(h(u)) = 0$ from (6.5) and (6.6). Thus the claim is proved.

□

Proposition 52. For all $v \in S \setminus R$, we have

$$\vartheta_{R,M} \in \ker(r_{(\emptyset)}^0 : H_0(F^x, \mathcal{O}_R \cup \{\emptyset\}) \to H_0(F^x, \mathcal{O}_R \cup \{\emptyset\})).$$

Proof. The claim follows from Proposition 51. □

Proposition 53. We have $i_{R_1,M_1}^{R_2,M_2}(\vartheta_{R_1,M_1}) = \vartheta_{R_2,M_2}$. □
Proof. Let \( j : F^\times \to N^{S \cup S_q} \) be a natural projection. Put \( E_i = \mathcal{E}_{R_i,M_i} \), \( E_i' = E_i \cap O^\times_F \), and \( E_i'' = j(E_i) \) for \( i = 1, 2 \). Then the natural sequence
\[
1 \to E_1'/E_2' \to E_1/E_2 \xrightarrow{j} E_1''/E_2''
\]
is exact. For \( u \in N^{S \cup S_q}/E_2'' \) and \( z \in H_0(E_2, Z(\mathcal{U}, \emptyset)) \), we put
\[
a(u, z) := uf_{R_2,M_2} \otimes z \otimes [u] \in H_0(F^\times, \mathcal{B}_{R_2,M_2}(\emptyset)).
\]
Fix a lift \( y \in H_0(E_2, Z(\mathcal{U}, \emptyset)) \) of \( \vartheta_\infty(\mathcal{U}, E_1) \). Since
\[
f_{R_1,M_1} = \sum_{g \in E_1/E_2} g^{-1}j(g)uf_{R_2,M_2},
\]
for \( u \in N^{S \cup S_q} \) we have
\[
uf_{R_1,M_1} \otimes y \otimes [u] = \sum_{g \in E_1/E_2} (j(g)uf_{R_2,M_2} \otimes gy \otimes [gu]) \otimes [g^{-1}]
\]
\[
\equiv \sum_{g \in E_1/E_2} a(j(g)u, gy)
\]
\[
\equiv \sum_{g' \in E'_1/E'_2} \sum_{g \in E_1/E_2} a(g'u, gy)
\]
\[
\equiv \sum_{g' \in E'_1/E'_2} a(g'u, \vartheta_\infty(\mathcal{U}, E_2)) \quad (\text{mod } I_{F^\times} \mathcal{B}_{R_2,M_2}(\emptyset)).
\]
Thus we have
\[
f_{R_1,M_1}(\vartheta_{R_1,M_1}) \equiv \sum_{u \in N^{S \cup S_q}/E_1''} uf_{R_1,M_1} \otimes y \otimes [u]
\]
\[
\equiv \sum_{u \in N^{S \cup S_q}/E_1''} a(u, \vartheta_\infty(\mathcal{U}, E_2))
\]
\[
\equiv \vartheta_{R_2,M_2} \quad (\text{mod } I_{F^\times} \mathcal{B}_{R_2,M_2}(\emptyset)).
\]
Thus the claim is proved. \( \square \)

6.4. Evaluation map. We denote by \( \mathbb{Q} \otimes \mathbb{R} \) the functor from \( \text{Sub}(S) \) to \( \text{Mod}(Z[F^\times]) \) defined by \( (\mathbb{Q} \otimes \mathbb{R})(W) = \mathbb{Q} \otimes \mathbb{R}(W) \). Fix a subset \( R \) of \( S_f \) and a subgroup \( M \) of \( F^\times_q \). Let \( W \) be a subset of \( S \setminus R \). In this section, we define a natural transformation \( L_{R,M}|_{\text{Sub}(S \setminus R)}(S) \) from \( \mathcal{B}_{R,M} \) to \( (\mathbb{Q} \otimes \mathbb{R})|_{\text{Sub}(S \setminus R)}(S) \). Put \( W_\infty := W \cap S_\infty \) and \( W_f := W \cap S_f \). For \( a \in \mathcal{A}_{R,M}(W_f) \) and \( e \in N_{S_f \setminus W_f} \), define \( \Phi_{a,e} \in \mathcal{S}(\mathbb{A}_F^{S_\infty}) \) by
\[
\Phi_{a,e}(x) := \begin{cases} a(x) & \text{if } y \in j^{-1}(e) \\ 0 & \text{if } y \notin j^{-1}(e) \end{cases}
\]
where \( j : \mathbb{A}_F^{S_\infty} \to N_{S_f \setminus W_f} \) is the projection map.

Lemma 54. For all \( \mathfrak{p} \in S_q \setminus \{q\} \) and \( u \in \mathcal{U} \), \( \Phi_{a,e} \) is invariant under the action of \( uO_\mathfrak{p} \).

Proof. Assume that \( \mathfrak{p} \notin S_f \setminus W \). Then \( a \in \mathcal{S}(\mathbb{A}_F^{S_\infty}) \) is invariant under the action of \( \mathfrak{p}O_\mathfrak{p} \), and \( j^{-1}(e) \) is invariant under the action of \( F_\mathfrak{p} \). Thus \( \Phi_{a,e} \) is invariant under the action of \( \mathfrak{p}O_\mathfrak{p} \). Assume that \( \mathfrak{p} \in S_f \setminus W \). Then \( a \in \mathcal{S}(\mathbb{A}_F^{S_\infty}) \) is invariant under the action of \( \mathfrak{p}O_\mathfrak{p} \). If \( \text{ord}_\mathfrak{p}(x) \neq 0 \) then \( \Phi_{a,e} \) is invariant under the action of \( \mathfrak{p}O_\mathfrak{p} \). If \( \text{ord}_\mathfrak{p}(x) \neq 0 \) then \( j^{-1}(e) \)
is invariant under the action of $uO_p$. Thus $\Phi_{a,e}$ is invariant under the action of $pO_p \cap uO_p = uO_p$. Thus the lemma is proved. \qed

**Lemma 55.** $\Phi_{a,e}$ is $U$-smooth for all $a$ and $e$.

**Proof.** Let $u \in U$ and $x \in \mathcal{K}^S_{\infty}$. It is enough to prove that

$$\int_{\mathbb{Z}_q} \Phi_{a,e}(x + tu) dt = 0.$$ 

From Lemma 54, we have

$$\int_{\mathbb{Z}_q} \Phi_{a,e}(x + tu) dt = \int_{\mathbb{Z}_q} \Phi_{a,e}(x + tu_q) dt.$$ 

We denote by $A$ the subgroup of $\mathcal{S}(F_q)$ generated by $\{h_x \mid x \in \mathbb{F}_q \subset \kappa_q\}$ where $h_x = 1_{x^{-1}(x)} - 1_{x^{-1}(0)}$ and $\pi : O_q \rightarrow O_q / qO_q = \kappa_q$ is a natural projection. From the definition, $\Phi_{a,e}$ is in $\mathcal{S}(\mathbb{A}_{\infty} \cup \{q\}) \cap A$. Thus it is enough to prove that

$$\int_{\mathbb{Z}_q} \Phi(x' + tu_p) dt = 0$$

for all $\Phi \in A$ and $x' \in F_q$. This follows from the fact that $u_p \in \pi^{-1}(F_q^\times)$. \qed

**Definition 56.** We define a natural transformation $L_{R,M}$ from $B_{R,M}|_{\text{Sub}(S \setminus R) \setminus \{S\}}$ to $\mathbb{C}^{|\text{Sub}(S \setminus R) \setminus \{S\}|}$ as follows. Fix $W \subset S \setminus R$ such that $W \neq S$. (The case $W_\infty = S_\infty$ ). For $a \in \mathcal{A}_{R,M}(W_f)$, $b = (b_g)_{g \in N_{S_\infty \setminus W_\infty}} \in \mathcal{Z}(U, W_\infty)$, $g \in N_{S_\infty \setminus W_\infty}$ and $x \in N_{S_\infty \setminus W_\infty}$, we put

$$\zeta_{a,b,g,x}(s) = \sum_{g \in N_{S_\infty \setminus W_\infty}} \sum_{x \in N_{S_f \setminus W_f}} \sum_{y \in F^x} L_{\infty}(b)(y) \Phi_{a,x}(y) \prod_{v \in S \setminus W} |y|_v^{-s_v} \ (s = (s_v)_{v \in S \setminus W}).$$

Then $\zeta_{a,b,g,x}(s)$ is analytically continued to the whole $\mathbb{C}^{|\text{Sub}(S \setminus W)|}$. Let $Z_{a,b} \in \mathbb{R}[\{(s_v)_{v \in S \setminus W}\}$ be a taylor series at $s = 0$. We put

$$L_{R,M}(W)(a \otimes b \otimes c) = c \sum_{g \in N_{S_\infty \setminus W_\infty}} \sum_{x \in N_{S_f \setminus W_f}} Z_{a,b,g,x} \otimes [gx].$$

(The case $W_\infty = S_\infty$). We put

$$L_{R,M}(W)(a \otimes b \otimes c) := c \sum_{x \in N_{S_f \setminus W_f}} \langle \widetilde{L}_{\infty}(b), \Phi_{a,x} \rangle \prod_{v \in S \setminus W_f} |x|_v^{-s_v} \otimes [x]$$

where $a \in \mathcal{A}_{R,M}(W \cap S_f)$, $b \in \mathcal{Z}(U) = \mathcal{Z}(U, S_\infty)$, $c \in \mathbb{Z}[N^S]$ and $\widetilde{L}_{\infty}(b) \in \mathcal{K}'(U)$. This definition does not depend on the choice of $L_{\infty}(b)$ because $\langle 1_{F^x}, \Phi_{a,x} \rangle = 0$ from $W_f \neq S_f$ and $\Phi_{a,x}(0) = 0$.

From the definition, we have

$$L_{R_2,M_2} \circ i_{R_2,M_2} = L_{R_1,M_1}.$$ 

**Proposition 57.** We have $\lambda(L_{R,M}(\varphi_{R,M})) = \Theta_{S_\infty \cup \{q\}, \mathcal{K}(s)}$. 

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**A Proof of the Refined Class Number Formula of Gross**

29
First, we need to prove that

\[ \text{Proof.} \]

\[ (\text{V} 6.5. \text{Definition of Lemma 58.}) \]

Let \( a \in J \), we write \( \sigma_a \) := \( \text{rec}(\lambda^{-1}(a)) \) ∈ \( \text{Gal}(K/F) \). We put \( A := \{(x, a) \mid a \in J, x \in F^\times \} = J \times F^\times \). We define the action of \( F^\times \) to \( A \) by \( p(x, a) = (px, pa) \). We define \( \text{ev} \in \text{Hom}_{\mathbb{Z}(F^\times)}(\mathcal{B}_{R,M}(\emptyset) \rightarrow \text{Map}(A, \mathbb{Z})) \) by

\[
(\text{ev}(a \otimes b \otimes c))(x, a) := \begin{cases} a(x) \times L_\infty(b)(x) & a = \lambda(c) \\ 0 & a \neq \lambda(c), \end{cases}
\]

where \( a \in \mathcal{S}(A^S_F) \), \( b \in \mathcal{Z}(F^\times, \emptyset) \), \( c \in \mathbb{Z}[N^S] \), and \( (x, a) \in A \). Then we have

\[
L(h) = \sum_{(x, a) \in A} \text{ev}(h)(x, a) \left( \prod_{v \in S} |x_v|^{-s_v} \right) \otimes [\lambda^{-1}(a)] \prod_{v \in S} x_v.
\]

and

\[
\lambda(L(h)) = \sum_{(x, a) \in A} \text{ev}(h)(x, a) \left( \prod_{v \in S} |x_v|^{-s_v} \right) N(a)^s \otimes [\sigma_{x^{-1}a}].
\]

Define \( D \in \text{Map}(A, \mathbb{Z}) \) by

\[
D((x, a)) = \begin{cases} 1 & x \in a \setminus aq \\ N(q) - 1 & x \in aq \\ 0 & x \notin a, \end{cases}
\]

where \( a \) and \( aq \) are fractional ideals of \( \mathcal{O}_S \). Let \( t \in B_{R,M}(\emptyset) \) be a lift of \( \vartheta \). Then from Proposition [34] we have

\[
\sum_{p \in F^\times} |p| \text{ev}(t) = D.
\]

Thus we have

\[
\lambda(L(t)) = \sum_{(x, a) \in A} D(x, a) \left( \prod_{v \in S} |x_v|^{-s_v} \right) N(a)^s \otimes [\sigma_{x^{-1}a}]
\]

\[
= \sum_{a \in J} D(1, a) N(a)^s \otimes [\sigma_a]
\]

\[
= \Theta_{S, \{q\}, K}(s).
\]

6.5. Definition of \( \text{Sh}^{\text{v}_0} \), \( \text{Sh}^0 \) and \( \text{Sh}^{\text{v}_0, \text{v}} \).

Lemma 58. Let \( m \) be a positive integer defined by

\[
m := \begin{cases} 1 & M = F_q^\times \text{ and } \text{ch}(q) \geq [F : \mathbb{Q}] + 2 \\ \text{ch}(q)^{|F : \mathbb{Q}|} & \text{otherwise.} \end{cases}
\]

Let \( V \) be a subset of \( S \setminus R \). If \( M = 1 \), \( S_\infty \cap V \neq S_\infty \) or \( R \neq \emptyset \) then \( \text{Sh}_{R,M,V} := (\mathcal{B}_{R,M}|_{\text{Sub}(V) \setminus \{S\}}, m\mathcal{L}_{R,M}|_{\text{Sub}(V) \setminus \{S\}}, \vartheta_{R,M} , m) \) is a Shintani datum on \( V \) for \( (\mathcal{R}, \mathcal{Y}, \lambda, \theta) \).

Proof. First, we need to prove that

\[
m\mathcal{L}_{R,M}(W)(\mathcal{B}_{R,M}(W)) \subset \mathcal{R}(W) \quad (W \subset V, \ W \neq S).
\]
Let \( a \in \mathcal{A}_{R,M}(W \cap S_f) \), \( b = (b_g)g \in \prod_{g \in N_{S_f} \setminus W_f} \mathbb{Z}(U_g) \) and \( c \in \mathbb{Z}[N^S] \). Since the constant term of \( mL_{R,M}(W)(a \otimes b \otimes c) \) is given by
\[
 c \sum_{g \in N_{S_f} \setminus W_f} \sum_{x \in N_{S_f} \setminus W_f} m\langle L_{\infty}(b_g), \Phi_{a,x} \rangle [x],
\]
it is enough to prove that
\[
 (6.9) \quad m\langle L_{\infty}(b_g), \Phi_{a,x} \rangle \in \mathbb{Z}. \tag{6.9}
\]
The claim (6.9) follows from Lemma 37 and Proposition 52 and 57 and Definition 59.

**Definition 59.** For \( q \notin S \) such that \( \text{ch}(q) \geq [F : \mathbb{Q}] + 2 \), we define Shintani data \( \text{Sh}^{v_0} \) and \( \text{Sh}^{v_0,\circ} \) on \( S \setminus \{v_0\} \) by
\[
\begin{align*}
\text{Sh}^{v_0} & = \text{Sh}_{S_f \cap \{v_0\}, F^\times_q S \setminus \{v_0\}} \\
\text{Sh}^{v_0,\circ} & = \text{Sh}_{S_f \cap \{v_0\}, (1), S \setminus \{v_0\}}.
\end{align*}
\]
For \( q \notin S \) such that \( \text{ch}(q) \neq 2 \), we define a Shintani datum \( \text{Sh}^\circ \) on \( S \) by
\[
\text{Sh}^\circ = \text{Sh}_{0,1,S}.
\]

From Proposition 63 and relation (6.7), the natural transformations \( i^{S_f \cap \{v_0\}, F^\times_q} \) and \( i^{0,1}_{S_f \cap \{v_0\}, (1)} \) give morphisms \( \text{Sh}^{v_0} \to \text{Sh}^{v_0,\circ} \) and \( \text{Sh}^\circ \to \text{Sh}^{v_0,\circ} \) respectively.

### 7. Proof of the main theorems

We put
\[
X := \{q \notin S, \text{ch}(q) \geq [F : \mathbb{Q}] + 2\},
\]
\[
Y := \{q \notin S, \text{ch}(q) \neq 2\}.
\]
In Section 6, we constructed a Shintani datum \( \text{Sh}^{v_0} \) for \( q \in X \) and Shintani data \( \text{Sh}^\circ, \text{Sh}^{v_0,\circ} \) for \( q \in Y \). To avoid confusion, we write \( \text{Sh}^{v_0}_q, \text{Sh}^\circ_q \) and \( \text{Sh}^{v_0,\circ}_q \) for these Shintani data. For \( v \in S \), put \( J_v := \ker(R(\emptyset) \to R(\{v\})) \).

**Definition 60.** For a Shintani datum on \( W \subseteq S \), we define
\[
Q^N(\text{Sh}) \in \left( \prod_{v \in W} I_v \right) / \left( I_{F^\times} \prod_{v \in W} I_v \right)
\]
as follows. The homomorphism
\[
R(\emptyset) \to \mathbb{Z}[N_F] ; P(s_{v_0}, \ldots, s_{v_n}) \otimes [c] \mapsto P(0, \ldots, 0)[c]
\]
naturally induces the homomorphism
\[
\left( \prod_{v \in W} J_v \right) / \left( I_{F^\times} \prod_{v \in W} J_v \right) \to \left( \prod_{v \in W} I_v \right) / \left( I_{F^\times} \prod_{v \in W} I_v \right).
\]
Then \( Q^N(\text{Sh}) \) the image of \( Q(\text{Sh}) \) under this homomorphism.

In this section, we put
\[
V := S \setminus \{v_0\}.
\]

**Definition 61.** We define \( \hat{\Theta}_{S, q, K} \) by
\[
\hat{\Theta}_{S, q, K} := Q^N(\text{Sh}^{v_0}_q).
\]
Lemma 62. Let $T$ be a finite set of places of $F$ such that $S \cap T = \emptyset$. The conditions
\[
\ker(\mu_K) \rightarrow \prod_{p \in T_K} (O_K/p)^\times = \{1\}
\]
and
\[
\delta_T \in \text{Ann}_{\mathbb{Z}[G]}(\mu_K)
\]
are equivalent.

Proof. Put $m = \#\mu_K$. The both conditions are equivalent to the condition
\[
\forall p \mid m, \exists p \in T, \text{ch}(p) \neq p.
\]
\[\square\]

Lemma 63. Let $J \subset \mathbb{Z}[G]$ be the ideal spanned by
\[
\{1 - N(q)\sigma_q^{-1} \mid q \in X\}.
\]
Then we have $J = \text{Ann}_{\mathbb{Z}[G]}(\mu_K)$.

Proof. Put $m = \#\mu_K$ and $n = \gcd\{1 - N(q) \mid q \notin X, \sigma_q|K = 1\}$. Let us prove that $m = n$. It is obvious that $m \mid n$. Assume that $n > m$. Then $K(\zeta_n)/K$ is not a trivial extension. Let $\sigma \in \text{Gal}(K(\zeta_n)/K) \subset \text{Gal}(K(\zeta_n)/F)$ be a non trivial element. From Chebotarev’s density theorem, there exists $q \in X$ such that
\[
\sigma_q|\text{Gal}(K(\zeta_n)/F) = \sigma.
\]
Then $\sigma_q|K = \sigma|K = id$. From the norm functoriality of the reciprocity map, we have
\[
\sigma_{N(q)} \neq 1 \in \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}).
\]
Therefore, $n \nmid 1 - N(q)$. This contradicts to the assumption that $n = \gcd\{1 - N(q) \mid q \in X, \sigma_q = 1\}$. Therefore the assumption $n > m$ must be false. Thus we have $n = m$ and $m \in J$. From Chebotarev’s density theorem, for all $\sigma \in G$, there exists $q \in X$ such that $\sigma_q = \sigma$. Thus we have $J = \text{Ann}_{\mathbb{Z}[G]}(\mu_K)$. \[\square\]

Lemma 64. For $q \in X$, we have
\[
(1 - N(q)\sigma_q^{-1})\Theta_{S,K} \in \prod_{v \in V} I_{G_v}.
\]

Proof. It follows from
\[
\text{rec}(\Theta_{S,q,K}) = (1 - N(q)\sigma_q^{-1})\Theta_{S,K}
\]
and
\[
\Theta_{S,q,K} \in \prod_{v \in V} I_v.
\]
\[\square\]

Lemma 65 (Vanishing order part of Gross conjecture). Let $T$ be a finite set of places of $F$ such that $S \cap T = \emptyset$. If $\ker(\mu_K) \rightarrow \prod_{p \in T_K} (O_K/p)^\times = \{1\}$ then we have
\[
\Theta_{S,T,K} \in \prod_{v \in V} I_{G_v}.
\]
Proof. From Lemma 63 and 64 we have

$$J\Theta_{S,K} \subseteq \prod_{v \in V} I_{G_v}.$$ 

Since $\delta_T \in J$ from Lemma 62, we have

$$\Theta_{S,T,K} = \delta_T \Theta_{S,K} \in \prod_{v \in V} I_{G_v}.$$ 

□

Lemma 66. We have

$$2 \prod_{v \in V} I_v \subseteq \left( \sum_{v \in V} I_v \right) \prod_{v \in V} I_v$$

and

$$2 \prod_{v \in V} I_{G_v} \subseteq \left( \sum_{v \in V} I_{G_v} \right) \prod_{v \in V} I_{G_v}.$$ 

Proof. Since $F \neq \mathbb{Q}$, $S_\infty \setminus \{v_0\} \neq \emptyset$. Let $v'$ be any place in $S_\infty \setminus \{v_0\}$. Then we have $2I_v \subseteq I_{v'}$ (resp. $2I_{G_v} \subseteq I_{G_{v'}}$) since

$$2([1] - [x]) = ([1] - [x])^2$$

for all $x \in N_{v'}$ (resp. $x \in G_{v'}$). Thus the lemma is proved. □

Lemma 67. Let $T$ be a finite set of places of $F$ such that $S \cap T = \emptyset$. If $\ker(\mu_K \to \prod_{p \in T} (O_K/p)\times) = \{1\}$ then we have

$$2\Theta_{S,T,K} \in I_{G_H} \prod_{v \in V} I_{G_v}.$$ 

Proof. The claim follows from Lemma 65 and Lemma 66. □

Recall that $\mathcal{R}(\emptyset) = \mathcal{P}_0 \otimes \mathbb{Z}[N_F]$. For $v \in S$, define $g_v : F^\times \to \mathcal{R}(\emptyset)^\times$ by

$$g_v(x) = [x]_v^{-s_v} \otimes [f_v(x)].$$

For $v \in S$, put $J_v := \ker(\mathcal{R}(\emptyset) \to \mathcal{R}(\{v\}))$.

Lemma 68. For $x_1, \ldots, x_r \in F^\times$, we have

$$\det(-1 + \prod_{s=j}^r g_v(x_i)^r_{i,j=1} \equiv \det(-1 + g_v(x_i))^r_{i,j=1} \pmod{(J_{v_1} + \cdots + J_{v_r})J_{v_1} \cdots J_{v_r}}.$$ 

Proof. For $k \in \{0, 1, \ldots, r\}$ and $y_1, \ldots, y_k \in F^\times$, we put

$$a_k(y_1, \ldots, y_k) := \det(-1 + \prod_{s=j}^k g_v(y_i))^{r}_{i,j=1}$$

and

$$b_k(y_1, \ldots, y_k) := \det(-1 + g_v(y_i))^r_{i,j=1}. $$

Let $P(k)$ be the following statement:

For any $y_1, \ldots, y_k \in F^\times$, we have

$$a_k(y_1, \ldots, y_k) \equiv b_k(y_1, \ldots, y_k) \pmod{(J_{v_1} + \cdots + J_{v_k})J_{v_1} \cdots J_{v_k}}.$$
Note that \( P(k) \) implies \( a_k(y_1, \ldots, y_k) \in J_{v_1} \cdots J_{v_k} \). We prove \( P(k) \) for \( k = 0, \ldots, r \) by the induction on \( k \). Assume that \( P(k-1) \) holds. We have

\[
a_k(y_1, \ldots, y_k) = \sum_{i=1}^{k} (-1)^{i-1} a_{k-1}(y_1, \ldots, \hat{y}_i, \ldots, y_k)(-1 + g_{v_k}(y_i)) \prod_{i' \neq i} g_{v_k}(y_{i'}). \]

Since \( a_{k-1}(y_1, \ldots, \hat{y}_i, \ldots, y_k) \in J_{v_1} \cdots J_{v_{k-1}} \) and \( 1 - \prod_{i' \neq i} g_{v_k}(y_{i'}) \in J_{v_k} \), we have

\[
a_k(y_1, \ldots, y_k) \equiv \sum_{i=1}^{k} (-1)^{i-1} a_{k-1}(y_1, \ldots, \hat{y}_i, \ldots, y_k)(1 + g_{v_k}(y_i)) \pmod{J_{v_1} \cdots J_{v_{k-1}} J_{v_k}^2}.
\]

Since \( -1 + g_{v_k}(y_i) \in J_{v_k} \) and

\[
a_{k-1}(y_1, \ldots, \hat{y}_i, \ldots, y_k) \equiv b_{k-1}(y_1, \ldots, \hat{y}_i, \ldots, y_k) \pmod{(J_{v_1} + \cdots + J_{v_k}) J_{v_1} \cdots J_{v_k}},
\]

we have

\[
a_k(y_1, \ldots, y_k) \equiv \sum_{i=1}^{k} (-1)^{i-1} b_{k-1}(y_1, \ldots, \hat{y}_i, \ldots, y_k)(1 + g_{v_k}(y_i)) \pmod{(J_{v_1} + \cdots + J_{v_k}) J_{v_1} \cdots J_{v_k}}.
\]

From (7.1) and (7.2), we have

\[
a_k(y_1, \ldots, y_k) \equiv \sum_{i=1}^{k} (-1)^{i-1} b_{k-1}(y_1, \ldots, \hat{y}_i, \ldots, y_k)(1 + g_{v_k}(y_i)) \equiv b_k(y_1, \ldots, y_k) \pmod{(J_{v_1} + \cdots + J_{v_k}) J_{v_1} \cdots J_{v_k}}.
\]

Hence \( P(k) \) holds. Thus \( P(0), \ldots, P(r) \) are proved by the induction. The lemma is equivalent to \( P(r) \). \( \square \)

**Lemma 69.** Let \( T \) be a finite set of places of \( F \) such that \( S \cap T = \emptyset \) and \( Y \cap T \neq \emptyset \). Let \( \langle u_1, \ldots, u_r \rangle \) be a \( \mathbb{Z} \)-basis of \( \mathcal{O}_{S,T}^\times \) such that \((-1)^#T \det(- \log |u_i|_{v_i})_{1 \leq i,j \leq r} > 0 \). Then we have

\[
\lim_{s \to 0} s^{-r} \Theta_{S,T,H}(s) = n_{S,T} \det(- \log |u_j|_{v_i})_{i,j=1}^{r} \sum_{\sigma \in \text{Gal}(H/F)} [\sigma].
\]

**Proof.** The claim follows from the functional equation of \( \Theta_{S,T,H}(s) \). \( \square \)

**Lemma 70.** For \( q \in Y \), we have

\[
Q^N(\text{Sh}^\circ|_\nu) \equiv \text{ch}(q)^{[F:Q]} \tilde{R}_q \pmod{I_H I_{v_1} \cdots I_{v_r}}.
\]

**Proof.** Fix \( q \in Y \) and put \( \text{Sh}^\circ : = (\mathcal{B}^\circ, \mathcal{L}^\circ, \vartheta^\circ, \text{ch}(q)^{[F:Q]}) \). From Lemma [10] we have

\[
Q(\text{Sh}^\circ|_\nu) = \eta_2(\eta_1(\vartheta^\circ)).
\]

Put \( E = \mathcal{O}_{S,\{q\}}^\times \). Take a \( \mathbb{Z} \)-basis \( \langle u_1, \ldots, u_r \rangle \) of \( E \) such that \(- \det(- \log |u_i|_{v_i})_{1 \leq i,j \leq r} > 0 \). Note that \( \mathcal{R}(S) = \mathbb{Z}[N^S] \). From Lemma [11] Proposition [10] and Proposition [51] we have

\[
\eta_1(\vartheta^\circ) \in \text{im}(H_r(E, \mathbb{Z}[N^S]) \to H_r(F^\times, \mathbb{Z}[N^S])).
\]

Since the action of \( E \) to \( N^S \) is trivial, we have

\[
H_r(E, \mathbb{Z}[N^S]) \simeq H_r(E, \mathbb{Z}) \otimes \mathbb{Z}[N^S].
\]
Therefore there exists $A \in \mathbb{Z}[N^S]$ such that $\eta_1(\overline{q})$ is the image of

$$A \otimes \sum_{\sigma \in \mathfrak{c}_H} \text{sgn}(\sigma) [1, u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(1)} \cdots u_{\sigma(r)}] \in \mathbb{Z}[N^S] \otimes \mathbb{Z}[F^\times] \mathcal{I}_r.$$ 

Let $\omega : H(F^\times, N^S) \cong H(R^{(S)}[-1] \to R^{(S)}[0] \to R^{(S)}[1])$ be a natural isomorphism. Then we have

$$\omega(\eta_1(\overline{q})) = y$$

where

$$y = (y_0, \ldots, y_r) \in \prod_{i=0}^{r} \mathcal{R}_{i,i}^{(S)} = \mathcal{R}^{(S)}[0]$$

and

$$y_k = A \sum_{\sigma \in \mathfrak{c}_H} \text{sgn}(\sigma) \prod_{i=k+1}^{r} \left(-1 + \sum_{s=i}^{r} g_{v_s}(x_{\sigma(i)}) \right) \otimes [1, x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(1)} \cdots x_{\sigma(k)}],$$

$$\in \mathcal{R}((v_0, \ldots, v_{k-1})) \otimes \mathcal{I}_k.$$

Let $f$ be a natural homomorphism from $\mathcal{R}^{(S)}[0] \to \mathcal{R}^{(V)}[0]$. Put

$$b_0 = A \det(-1 + \prod_{s=j}^{r} g_{v_s}(x_{\sigma(i)})i_{i,j=1} \in \mathcal{R}^{(V)}_{-1,0}$$

and

$$b := (b_0, 0, \ldots, 0) \in \prod_{i=0}^{r+1} \mathcal{R}^{(V)}_{i-1,i} = \mathcal{R}^{(V)}[1].$$

Then $f(y_k) = d(b)$. Therefore we have

$$\eta_2(\eta_1(\overline{q})) = b_0$$

from the definition. By using Lemma 68, we have

$$b_0 \equiv A \det(g_{v_i}(u_j) - 1)_{1 \leq i, j \leq r} \mod (I_{F^\times} \mathcal{R}(\emptyset) + J_{v_0} + \cdots + J_{v_r}) \prod_{v \in V} J_v.$$

Thus we have

$$(7.3)$$

$$Q(\text{Sh}^n|_V) \equiv A \det(g_{v_i}(u_j) - 1)_{1 \leq i, j \leq r} \mod (I_{F^\times} \mathcal{R}(\emptyset) + J_{v_0} + \cdots + J_{v_r}) \prod_{v \in V} J_v.$$

Let $\tilde{A} \in \mathbb{Z}[N^S/F^\times] = \mathbb{Z}[\text{Gal}(H/F)]$ be the image of $A$. From (7.3), we have

$$\text{ch}(q)^{[F:Q]} \Theta_{S,(q),H}(s) \equiv s^r \det(- \log |u_j|_{v_i})_{i,j=1}^r \tilde{A} \mod s^{r+1}.$$ 

Therefore, from Lemma 69, we have

$$\tilde{A} = n_{S,T} \sum_{c \in N^S/F^\times} [c]$$

Hence, from (7.3), we have

$$Q^n(\text{Sh}^n|_V) \equiv \text{ch}(q)^{[F:Q]} n_{S,T} \sum_{c \in N^S/F^\times} [c] \det(f_{v_i}(u_j) - 1)_{1 \leq i, j \leq r} \mod I_H I_{v_1} \cdots I_{v_r}$$

$$= \text{ch}(q)^{[F:Q]} \tilde{R}_q.$$ 

$\square$
Lemma 71. For \( q \in Y \), we have
\[
\text{ch}(q)^{[F:Q]}\Theta_{S,\{q\},K} \equiv \text{ch}(q)^{[F:Q]}R_{G,S,\{q\}} \pmod{I_{G,v} \prod_{v \in V} I_{G_v}}.
\]

Proof. By applying \( \text{rec} \) to the both hand sides of Lemma 70, we obtain the claim. \( \square \)

Lemma 72. Let \( q \) be an element of \( T \) such that \( \text{ch}(q) \neq 2 \). Then we have
\[
\text{ch}(q)^{[F:Q]}\Theta_{S,T,K} \equiv \text{ch}(q)^{[F:Q]}R_{G,S,T} \pmod{I_{G,h} \prod_{v \in V} I_{G_v}}.
\]

Proof. From the definition, we have
\[
\Theta_{S,T,K} = \Theta_{S,\{q\},K} \prod_{p \in T \setminus \{q\}} (1 - N(q)\sigma_q^{-1})
\]
and
\[
R_{G,S,T} = R_{G,S,\{q\}} \prod_{p \in T \setminus \{q\}} (1 - N(q)).
\]

Thus the claim follows from Lemma 71. \( \square \)

Let us prove Theorem 2. Assume that \( T \) satisfies the condition
\[
\ker(\mu_K) \to \prod_{p \in T_K} (O_K/p)^* = \{1\}.
\]

Lemma 72 implies that
\[
\text{ch}(q)^{[F:Q]}\Theta_{S,T,K} \equiv \text{ch}(q)^{[F:Q]}R_{G,S,T} \pmod{I_{G,v} \prod_{v \in V} I_{G_v}}.
\]

From Lemma 67 we have \( 2\Theta_{S,T,K} \in I_{G,h} \prod_{v \in V} I_{G_v} \). From Lemma 66 we have \( 2R_{G,S,T} \in I_{H} \prod_{v \in V} I_{G_v} \). Thus we have
\[
\Theta_{S,T,K} \equiv R_{G,S,T} \pmod{I_{G,v} \prod_{v \in V} I_{G_v}},
\]
which completes the proof of Theorem 2.

Let \( q \) be prime ideal of \( F \) such that \( \text{ch}(q) \geq n + 2 \). Since there exists a morphism from \( \text{Sh}^{pv} \) to \( \text{Sh}^{pv,0} \), we have
\[
\text{ch}(q)^n Q(\text{Sh}^{pv}) \equiv Q(\text{Sh}^{pv,0})
\]
Since there exists a morphism from \( \text{Sh}^0 \) to \( \text{Sh}^{pv,0} \), we have
\[
Q(\text{Sh}^{pv,0}) \equiv Q(\text{Sh}^0)
\]
\[
\equiv \text{ch}(q)^n \hat{R}_q.
\]
Thus we have
\[
\text{ch}(q)^n Q(\text{Sh}^{pv}) \equiv \text{ch}(q)^n \hat{R}_q \pmod{I_{H} \prod_{v \in V} I_{v}}.
\]
Since \( 2Q(\text{Sh}^{pv}) \) and \( 2\hat{R}_q \) are in \( I_{H} \prod_{v \in V} I_{v} \) from Lemma 66, we have
\[
\hat{\Theta}_{S,a,V} \equiv \hat{R}_q.
\]
which completes the proof of Theorem 3.

8. Some example

In this section, we present a certain example of $\hat{\Theta}_{S,q,S\setminus\{v_0\}}$ and Theorem 3. The reader who is not interested in such an example can skip this section.

Let us consider the case $F = \mathbb{Q}(\sqrt{5})$, $S = S_\infty$ and $q = (\sqrt{5})$. Let $v_0$ and $v_1$ be the infinite places corresponding to the real embeddings $a + b\sqrt{5} \mapsto a + b\sqrt{5}$ and $a + b\sqrt{5} \mapsto a - b\sqrt{5}$ respectively. Then we have $\text{Sh}_{\nu_0} = (B, \mathcal{L}, \vartheta, 1)$ where we put

\begin{align*}
B := & \quad B_{0,q,S} |_{\text{Sub}(S\setminus\{v_0\})} \\
\mathcal{L} := & \quad \mathcal{L}_{0,q,S}
\end{align*}

$\vartheta := \vartheta_{0,q,S}$.

Note that $B(W) = B_{0,q,S}(W)$ is the certain $\mathbb{Z}[F^\times]$ submodule of $S(\mathbb{A}_{F}^{\times}) \otimes \mathbb{Z}(F^\times, W) \otimes \mathbb{Z}[\mathbb{N}_S]$ defined in Section 3. We define $f \in S(\mathbb{A}_{F}^{\times})$ by $f(x) := \prod_v f_v(x_v)$ where $f_v = 1_{O_v}$ for $v \neq q$ and

\[ f_q(x_q) = \begin{cases} 
1 & x_q \in O_q^\times \\
-4 & x_q \in qO_q \\
0 & x_q \notin qO_q
\end{cases} \]

We put $\epsilon = \frac{1 + \sqrt{5}}{2}$ and

\begin{align*}
D_0 := & \quad [1] + [1, c^2] \in \mathbb{Z}(F^\times, \emptyset) \\
D_1 := & \quad [1, \epsilon] \in \mathbb{Z}(F^\times, \{v_1\})
\end{align*}

Then we have $f \otimes D_0 \otimes [1] \in B(\emptyset)$, and $\vartheta \in H_0(F^\times, B(\emptyset))$ is the image of $f \otimes D_0 \otimes [1]$. Define $a = (a_0, a_1) \in B_{0,0} \oplus B_{1,1} = B[0]$ by

\begin{align*}
a_0 := & \quad f \otimes D_0 \otimes [1] \in B(\emptyset) \\
a_1 := & \quad [1, \epsilon] \otimes (f \otimes D_1 \otimes [1]) \in \mathcal{L}_1 \otimes B(\{v_1\})
\end{align*}

Then we have

\[ d_1(a) = (\vartheta, 0) \in B_{0,-1} \oplus B_{1,0}. \]

For $W \subset S_\infty$, we denote the element $(x_v)_{v \in S \setminus W} \in N_{S_\infty \setminus W}$ by

\[(x_0, x_1) \quad (x_j \in N_{O_j} \cup \{\ast\})\]

where $x_i = x_{v_i}$ for $v_i \in S \setminus W$ and $x_i = \ast$ for $v_i \in W$. We have

\begin{align*}
\mathcal{L}(a_0) = & \quad Z_0(s_0, s_1) \otimes [(+1, +1)] \\
\mathcal{L}(a_1) = & \quad [1, \epsilon] \otimes (Z_1(s_0) \otimes [(+1, \ast)])
\end{align*}

where $Z_0(s_0, s_1)$ and $Z_1(s_0)$ are Maclaurin series of

\[ \sum_{x \in C(1, c^2) \cup C(\epsilon^2)} f(x) |x|^{-s_0}_{v_0} |x|^{-s_1}_{v_1} \]

and

\[ \sum_{x \in C(1, \epsilon)} f(x) |x|^{-s_0}_{v_0} \]
respectively. Define \( b = (b_0, b_1) \in \mathcal{B}_{-1,0} \oplus \mathcal{B}_{0,1} = \mathcal{B}[1] \) by
\[
\begin{align*}
    b_1 &= [1, \epsilon] \otimes (Z_1(s_0) \otimes [(+1, +1)]) \\
    b_0 &= (Z_0(s_0, s_1) - Z_1(s_0)) \otimes [(+1, +1)] + |\epsilon|_{v_0}^{-s_0} |\epsilon|_{v_1}^{-s_1} Z_1(s_0) \otimes [(+1, -1)].
\end{align*}
\]
Since \( d(b) = a \), we have
\[
Q(\text{Sh}^{v_0}) = b_0.
\]
Since the constant term of \( Z_0(s_0, s_1) \) (resp. \( Z_1(s_0) \)) is equal to 0 (resp. \(-1\)), we have
\[
\hat{\Theta}_{S,q,\{v_1\}} = Q^N(\text{Sh}^{v_0}) = [(+1, +1)] - [(+1, -1)] \in I_{v_1}/(I_F \times I_{v_1}).
\]
Note that we have \( H = F \) and \( n_{S,T} = -1 \). Since \( u = -\epsilon^{-2} \) is a generator of \( \mathcal{O}^S_{S,q} \) such that \(-(-\log |u|_{v_1}) > 0\), we have
\[
\hat{R}_q = -([1, u] - [1, 1]) = [(+1, +1)] - [(+1, -1)] \in I_{v_1}/I_H I_{v_1}.
\]
Thus we have
\[
\hat{\Theta}_{S,q,\{v_1\}} \equiv \hat{R}_q \pmod{I_H I_{v_1}}.
\]
Theorem 3 says that such congruences holds in more general settings. Note that Theorem 2 says nothing in this case because there exists no non-trivial extension \( K \) of \( F \) unramified outside \( S_\infty \).

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Department of Mathematics, Kyoto University, Kitashirakawa Oiwake-cho, Sakyo-ku, Kyoto, 606-8502, Japan

E-mail address: hirose@math.kyoto-u.ac.jp