Interpolating wave packets in QFT and neutrino oscillation problem

S. E. Korenblit,*a,b,1 and D. V. Taychenachev∗a

aDepartment of Physics, Irkutsk State University, 20 Gagarin blvd, RU-664003, Irkutsk, Russia.
bDzhelepov Laboratory of Nuclear Problems, Joint Institute for Nuclear Research, RU-141980, Dubna, Russia.

E-mail: korenb@ic.isu.ru, dt@sia.ru

Abstract: Consistent constructive generalization of the introduced recently in [1–8] relativistic covariant wave packet is deduced on the grounds of general principles of quantum field theory with careful extension to the higher spins. This state is uniquely defined as a so called interpolating state, which has the both correct limits to the states localized in momentum space and/or in coordinate space. The wave packet is unambiguously determined by analytical properties of Wightman function in complex coordinate space, defining a representation of nonhomogeneous complex Lorentz group. It is shown how this analytical continuation in coordinate space specifies a universal way to wave-packet construction for massive particles with arbitrary spin. It appears as only possible careful and natural relativistic generalization of non relativistic Gaussian wave packet but contains covariant particle (antiparticle) states only with positive (negative) energy sign and propagates without their mixing and without changing of its relativistically invariant width. Its specific simultaneous zero-mass and zero-width limit naturally reduces this packet from 3+1 to 1+1 space-time dimensions, leading to wave packet on the light cone with a one-parametric freedom. Within intermediate wave-packet approach to the neutrino oscillation phenomena the respectively generalized expression for two-flavor oscillations of leptonic charge of electronic neutrino is given. For diagrammatic treatment of oscillation with the use of these wave packets a covariant meaning of “pole integration” [34, 35] is elucidated with the help of Huygens’ principle in terms of the reduced to mass shell composite wave functions closely related to overlap functions for the neutrino creation/detection processes. Their introduction resolves the problems with causality and with covariant equal time prescription. These functions are explicitly calculated for two-packet case. Theirs various exact, approximating, asymptotic and limiting expressions are obtained and investigated. The correspondence of their asymptotic behaviour with the narrow-width approximation of one-packet state and with asymptotic of oscillation amplitude is established.

Keywords: neutrino oscillation, covariant wave packet with spin, composite wave function, wave packet on the light cone.

ArXiv ePrint: 1712.06641v2

*Corresponding author.
1 Introduction

The concept of a wave packet permeates the entire construction of modern quantum mechanics (QM) and quantum field theory (QFT) [1–29]. It plays a central role in achieving their mathematical rigor and provide their interpretation in terms of particles. Wave packets are necessary to obtain the weak asymptotic conditions for the in- and out- asymptotic fields, as well as for the respective asymptotic states, and for the use of LSZ formalism [11–29]. The dominant paradigm governing the use of this concept came out from optics [30], in fact dealing with the wave packets for massless particles with infinite Compton wave length $\lambda = h/(mc)$ only. Thus it declares the arbitrariness of amplitude of profile function
of wave packet in the suitable functional space $L^2(\mathbb{R}^3), S(\mathbb{R}^3)$, etc. For the massive case this is also enough for mathematical rigor [19–24] but it is not enough for consistent physical interpretation [1–8].

Indeed, the local nature of the basic quantum-field interactions, jointly with the hypothesis about their adiabatic switching on and switching off, allows in a final version of the asymptotic formalism of $S$-matrix for these interactions to replace wave packets of asymptotic states by the plane waves that are elements of equipped Hilbert space [19] for given QFT. Having been successfully applied to description of the vast majority of phenomena in particle physics [13–29], this formalism is not suitable for a consistent description of the phenomenon of neutrino oscillations in vacuum [1–8, 31–47] and surely for their oscillations in matter. Being basically a manifestation of non trivial vacuum structure of QFT of electro-weak interactions and the existence of non equivalent representations for its (anti)commutation relations [44–47], this fact reflects also the inability of adiabatic "switching off" the above mechanism of oscillations. Therefore, an attempt to describe this mechanism on the language of ordinary $S$-matrix diagram technique requires to return the wave packet into the theory, as a such basic element of that description, as the plane wave previously was.

For example, trying to understand the neutrino oscillation [31–47], we face the problem of construction a consistent relativistic analog of Gaussian wave packet [1–8], which in fact corresponds to the drastically changed meaning of Heisenberg uncertainty conditions [14–17, 30] in the relativistic QFT. Furthermore, it happens for some invariant parameter of width $\sigma$ this relativistic wave packet, unlike the non relativistic one, directly interpolates between the covariant state with definite 4-momentum at $\sigma \to 0$ and the covariant state with definite 4-coordinate in relativistically covariant sense at $\sigma \to \infty$. The inevitable appearance of two independent invariant dimensional parameters: mass $m$ and width $\sigma$ gives rise to non trivial dimensionless function of their dimensionless combination $\tau \sim (mc/\sigma)^{\epsilon}$, $\epsilon > 0$.

The aim of this paper is to demonstrate also, that up to this non trivial function in normalization factor, the axioms of relativistic QFT unambiguously determine such interpolating relativistic covariant wave-packet state with positive mass for arbitrary spin, and that the above mentioned limiting properties fix unambiguously the only important here asymptotic behaviour of this dimensionless function in normalization factor. Whereas its remaining ambiguity defines the form of averages corresponding to observables. Constancy of invariant width $\sigma_a$ is the evident merit of Lorentz covariant propagation. The physical meaning of parameter $\tau_a$ for $a$-th particle is illuminating by special limit of the wave packet, when $\sigma_a \to 0$, $m_a \to 0$ with fixed $\tau_a$. This limit leads to in fact $1+1$ - dimension covariant wave packet, propagating on the light cone.

The next section 2 reminds the properties of wave packets in quantum mechanic. In section 3 the relativistic Lorentz covariant interpolating wave packet for massive (pseudo) scalar field is constructed. Its connection with analytically continued Wightman function for this field is shown. Its interpolating and limiting properties are demonstrated. Its uniqueness is advocated, and its non spreading off shell approximation [1] in subsection 3.1 is discussed. In section 4 this construction is generalized onto the massive fields with spins $1/2$ and spin 1. The general formulae for averages corresponding to different observables
are justified. Unlike [1–7], our definition is based on the main principles of construction of states with definite mass and definite spin as irreducible representation of extended Lorentz group [27–29].

Sections 5 and 6 contain various applications of suggested wave packets to both the treatments of neutrino oscillations, using intermediate wave packet and macroscopic diagrams. In section 5 an inconsistency of conventional “plane-wave” mixing relations with general principles of QFT for these wave packets is elucidated. The respectively generalized expression for two-flavor oscillations of leptonic charge of electronic neutrino is given.

In order to analyse the connection between the both approaches, in section 6, the notion of composite wave function of the neutrino creation/detection \(\{C/D\}\) processes is shown to be convenient and is closely related to respective overlap function [1, 34, 35]. It is shown how their introduction resolves the problems with causality and covariant equal time prescription. The scalar product of composite wave functions is uniquely defined in accordance with Huygens’ principle. The respective overlap function is explicitly calculated for two-packet case. Its narrow packet approximation is shown to be similar but not exactly the same that was obtained in [1–7]. The well known procedure of “pole integration” [34, 35, 47] is related to respective pole approximation for the off shell composite wave function, converting the latter to the on shell one. Their asymptotic behavior is found in expected form, which is correlated with corresponding behavior of oscillation amplitude, as narrow-width approximation for one-packet state [1–7, 41]. The exact integral representation of two-packet-on-shell composite wave function as the linear superposition of one-packet functions is given. Its various limiting expressions are obtained and their properties are studied.

The results are discussed in section 7. The conclusions are given in section 8. In appendix A the plane-wave limit and non relativistic limit and zero-mass-width limit of suggested wave packet are traced in some details. Some intermediate useful formulas and definitions are collected in appendix B. In appendix C an explicit form of two-packet overlap function is calculated, whose narrow-packet approximation is checked in appendix D.

2 Wave packets in quantum mechanics

As it is well known [9–12], the time in non relativistic quantum mechanics plays the role of parameter. So, the both momentum- and coordinate- ket states make sense at any instant of time \(t\) as formal eigenstates of three-dimensional momentum \(P\)- operator and coordinate \(X\)- operator: \(P|k\rangle = k|k\rangle\), \(\langle k | 1\rangle = \delta_3(k - 1)\) and \(X|x\rangle = x|x\rangle\), \(\langle x | y\rangle = \delta_3(x - y)\),

with \([X_n, P_l] = i\hbar\delta_{nl}\). Then for any state \(|f\rangle\) : \(X = x\), \(P = P_x = -i\hbar \nabla_x\) on the wave function \(f(x) = \langle x | f\rangle\) of its \(x\)-representation; and \(X = X_p = i\hbar \nabla_p\), \(P = p\) on the wave function \(\tilde{f}(p) = \langle p | f\rangle\) of its \(p\)-representation, also at any instant of time \(t\), regardless of the Hamiltonian and the used non-relativistic quantum picture: Schrödinger, Heisenberg,
etc. [9, 10, 17]. Since for $\hbar = 1$:

$$\int d^3k |k\rangle\langle k| = I, \quad \int d^3x |x\rangle\langle x| = I,$$  

(2.1)

$$e^{-i(k \cdot X)} P e^{i(k \cdot X)} = P + k, \quad e^{i(k \cdot X)}|p\rangle = |p + k\rangle,$$  

(2.2)

$$e^{i(y \cdot P)} X e^{-i(y \cdot P)} = X + y, \quad e^{-i(y \cdot P)}|x\rangle = |x + y\rangle,$$  

(2.3)

or: $< p|e^{-i(k \cdot X)}|f > = \exp \{ -i (k \cdot X_p) \} \bar{f}(p) = \bar{f}(p + k)$,  

(2.4)

or: $< x|e^{i(y \cdot P)}|f > = \exp \{ i (y \cdot P_x) \} f(x) = f(x + y)$,  

(2.5)

ey they are connected by direct and inverse 3- dimension Fourier transformation also at any instant of time $t$ as:

$$f(x) = \int d^3k < x|k > \bar{f}(k), \quad \bar{f}(k) = \int d^3k < k|x > f(x),$$  

(2.6)

where: $< x|k >= e^{i(k \cdot x)} / (2\pi)^{3/2}$, and: $< k|x >^* = e^{-i(k \cdot x)} / (2\pi)^{3/2}$,  

(2.7)

is $x$ - representation of the state $|k >$ with definite momentum $k$, and vice versa [9–12]. For Gaussian wave packet, which is localized in the domain $\sigma_x$ around the point $x_a$ in coordinate space and respectively the plane-wave modes $|k >$ with $k$ near momentum $p_a$ in the domain $\sigma_p = 1/\sigma_x$, for $\Delta k = k - p_a$, $\Delta x = x - x_a$, $f(x) \rightarrow \Psi_{p_a, x_a, \sigma}(x)$, $\bar{f}(k) \rightarrow \tilde{\Psi}_{p_a, x_a, \sigma}(k)$, with [9–12]:

$$\tilde{\Psi}_{p_a, x_a, \sigma}(k) = < k|\{ p_a, x_a, \sigma \} > = \left( \frac{1}{\pi \sigma_p^2} \right)^{3/4} \exp \left[ - \frac{(\Delta k)^2}{2\sigma_p^2} - i (\Delta k \cdot x_a) \right],$$  

(2.8)

$$\Psi_{p_a, x_a, \sigma}(x) = < x|\{ p_a, x_a, \sigma \} > = e^{i(p_a \cdot x_a)} \left( \frac{1}{\pi \sigma_x^2} \right)^{3/4} \exp \left[ - \frac{(\Delta x)^2}{2\sigma_x^2} + i (\Delta x \cdot p_a) \right],$$  

(2.9)

these wave functions (2.8), (2.9) define the same wave packet state $|\{ p_a, x_a, \sigma \} >$, normalized to unity $< \{ p_a, x_a, \sigma \}|\{ p_a, x_a, \sigma \} > = 1$ in fact for arbitrary initial instant $t_a$ (A.28) at instant $t = t_a$. The both its limits onto above eigenstates make sense also for arbitrary instant $t_a$:

$$\left( 2\sigma_p \sqrt{\pi} \right)^{-3/2} |\{ p_a, x_a, \sigma \} > \rightarrow |p_a >, \quad \sigma_p \rightarrow 0, \quad (\sigma_x \rightarrow \infty),$$  

(2.10)

$$\left( 2\sigma_x \sqrt{\pi} \right)^{-3/2} |\{ p_a, x_a, \sigma \} > \rightarrow \exp \{ i(p_a \cdot x_a) \}|x_a >, \quad \sigma_x \rightarrow 0, \quad (\sigma_p \rightarrow \infty),$$  

(2.11)

also regardless the Hamiltonian and the quantum picture under consideration. The further fate of the initial state (2.8), (2.9 at $t > t_a$ surely depends on the Hamiltonian. For the harmonic oscillator this wave packet propagates without spreading in coordinate space, while for the non relativistic free one $p^2/(2m)$ it spreads with time with the effective width $\sigma_x^2(t) = \sigma_x^2 + t^2 \sigma_p^2 / m^2$ [4, 9–12].

As it is well known, the Gaussian wave packet (2.8), (2.9) minimizes the Heisenberg uncertainty condition [9, 10]. Nevertheless this profile of wave packet is not the unique possible one as in the non-relativistic quantum mechanics [11, 12] as well in optics [30]:

$$\frac{1}{3} \langle \langle (\Delta P)^2 \rangle \rangle_f \frac{1}{3} \langle \langle (\Delta X)^2 \rangle \rangle_f \geq \frac{\hbar^2}{9} \langle \langle (\Delta k)^2 \rangle \rangle \tilde{\Psi} \langle \langle (\Delta x)^2 \rangle \rangle = \frac{\hbar^2}{4} \frac{\sigma_p^2 \sigma_x^2}{\sigma^2} = \frac{\hbar^2}{4}. \quad (2.12)$$
In the next sections is shown how the axioms of relativistic QFT up to non trivial normalization factor unambiguously determine a relativistically covariant wave packet state for arbitrary spin. It has the limiting properties similar (2.10), (2.11), and in non-relativistic limit it recasts exactly into the Gaussian wave packet (2.8), (2.9) regardless of the remained ambiguities.

3 Relativistic wave packet. Scalar field.

A free real massive quantum scalar field $\varphi(x)$, for $x' = (x_0, \mathbf{x})$, $x_0 = ct$, $k_\mu = (k_0, -\mathbf{k})$ satisfying Klein-Gordon (KG) equation of motion has the form [14–26]:

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^32k^0} \left( a_k f_k(x) + a_k^\dagger f^*_k(x) \right), \quad (\partial^2 + m^2)\varphi(x) = 0, \quad a_k|0\rangle = 0, \quad a_k^\dagger|0\rangle = |k\rangle,$$

where: $k^0 = \frac{E_k}{c} = +\sqrt{k^2 + (mc)^2} \mapsto +\sqrt{k^2 + m^2} = E_k > 0$, and:

$$|k\rangle = \frac{1}{\lambda} \mapsto mc \mapsto m, \quad \text{for} \quad h \mapsto 1, \quad \text{and} \quad c \mapsto 1. \quad (3.4)$$

The creation operator $a_k^\dagger$ creates a state with definite momentum $|k\rangle$ (3.2) by acting on vacuum state $|0\rangle$ and obeys the commutation relation with annihilation operator $a_k$:

$$\left[ a_q, a_k^\dagger \right] = (q|k) = (2\pi)^32k_0\delta_3(k - q), \quad \text{whence:} \quad \langle 0|\varphi(x)|k\rangle = f_k(x) = e^{-ikx} \mapsto \langle x|k\rangle, \quad \langle k|\varphi(x)|0\rangle = f^*_k(x) = f_{-k}(x) = e^{ikx} \mapsto \langle k|x\rangle, \quad (3.7)$$

for $(kx) = k_0x^0$ is a plane-wave solution to KG equation (3.2) and represents by definition [14–26] the coordinate wave function of state with the definite mass $m$, momentum $\mathbf{k}$ and energy $ck^0 = E_k$. The state $|k\rangle$ (3.2) differs from the states $|\mathbf{x}\rangle$, $|\mathbf{k}\rangle$ of (2.1), (2.2), (2.7) not only by its Lorentz invariant normalization condition (3.5) but also by its meaning [16, 35], because:

$$|k\rangle = (2\pi)^{3/2}\sqrt{2k_0} |\mathbf{k}\rangle, \quad k_0 = E_k/c > 0, \quad \text{and:} \quad \int \frac{d^3k|k\rangle\langle k|}{(2\pi)^32k_0} = \mathbf{I}_1, \quad \text{now is the one-particle completeness only.} \quad (3.9)$$

The well known inability in relativistic QFT to localize only one particle at a space domain and time interval, respectively less than $\lambda = h/(mc)$ and $\lambda/c = h/(mc^2)$, makes meaningless also the non covariant states $|\mathbf{x}\rangle$, $|\mathbf{k}\rangle$ of (2.1), (2.7) for any instant of time $t = x^0/c$ as well as corresponding definition of non covariant packet state (2.8) – (2.11), and gives no chances to define a covariant self-adjoint operator of four-dimensional position for this particle [14–29].

The relativistic generalization of self-adjoint three-dimensional position operator [15–19] $\mathbf{X}_p = i\hbar/\sqrt{E_p}\nabla_p(\sqrt{E_p})^{-1}$ only for $t = 0$ has $|\mathbf{p}\rangle|\mathbf{y}\rangle = \sqrt{2E_p} e^{-i|\mathbf{p}|\mathbf{y}}$ as eigenfunction with eigenvalue $\mathbf{y}$: $\mathbf{X}_p|\mathbf{p}\rangle|\mathbf{y}\rangle = |\mathbf{p}\rangle|\mathbf{y}\rangle$. In spite of validity of the same commutation
relation with momentum operator \([\hat{X}_n, P_l] = i\hbar \delta_{nl}\), and the relations (2.2)–(2.5) with \(X_p \to \hat{X}_p\) for \(<p| \to \langle p|\), and the Eq. (3.9) instead of (2.1), this operator loses the above properties of completeness (2.1) and localization (2.6), as well as the orthogonality properties of functions (2.7) for its eigenfunctions \(\langle p|y \rangle \) already at \(t = 0\) [15, 18].

According to (3.6), in QFT the operator creating from the vacuum state \(|0\rangle\) the covariant one-particle state with definite 4-coordinate \(|x\rangle = |x^0, x\rangle\), which is accepted as a state of free particle localized in the 3-point \(x\) only at the instant \(x^0 = 0\) [15–18], is the operator (3.1) of quantized free field itself, and by means of Eq. (3.9) one has (hereafter \(c = 1\), where it is possible):

\[
\varphi(x)|0\rangle = \int \frac{d^3k|k\rangle e^{i(k \cdot x)}}{(2\pi)^{3/2}k^0} = \int \frac{d^3k|k\rangle \langle k|x\rangle}{(2\pi)^{3/2}k^0} = |x\rangle, \quad k^0 = E_k > 0, \quad \text{whence:} \quad (3.10)
\]

\[
|x\rangle \xrightarrow[x \to 0]{} \int \frac{d^3k|k\rangle e^{-i(k \cdot x)}}{(2\pi)^{3/2}k^0} = \int \frac{d^3k|k\rangle > e^{-i(k \cdot x)}}{(2\pi)^{3/2}k^0} = |0, x\rangle \neq |x\rangle = \int \frac{d^3k|k\rangle > e^{-i(k \cdot x)}}{(2\pi)^{3/2}k^0}, \quad (3.11)
\]

or: \(\langle p|x\rangle = f_p^\prime(x) = e^{i(px)} \xrightarrow[x \to 0]{} \langle p|0, x\rangle = e^{-i(px)} \neq \langle p|x\rangle = \sqrt{2E_p} e^{-i(px)}, \quad (3.12)
\]

Only the covariant states in the l.h.s. of inequalities (3.11), (3.12), unlike the non covariant ones in the r.h.s. [15–18], make a safe ground for construction of covariant wave-packet states. Similarly (2.10), (2.11) for the packet (2.8), (2.9), the desired relativistic wave packet \(|\{p_a, x_a, \sigma\}\rangle \) has to interpolate between the covariant state \(|x_a\rangle \) (3.10) with definite 4-coordinate for some \(\sigma \to \infty\) and the covariant state \(|p_a\rangle \) (3.2), (3.8) with definite 4-moment for the \(\sigma \to 0\). So, its similar to (3.6) Lorentz covariant (invariant) wave function in this coordinate-space representation due to conditions of translation symmetry, for \(\vec{z} = x_a - x\), looks like:

\[
F_{p_a, x_a}(x) \equiv \langle x|\{p_a, x_a, \sigma\}\rangle = \langle 0|\varphi(x)|\{p_a, x_a, \sigma\}\rangle = e^{-i(p_a x)} \Phi_\sigma(p_a, x - x_a), \quad \text{or:} \quad (3.13)
\]

\[
F_{p_a, x_a}(x) = \langle 0|\varphi(x)|\{p_a, x_a, \sigma\}\rangle = e^{-i(p_a x)} \Phi_\sigma(p_a, x - x_a), \quad \text{with:} \quad (3.14)
\]

\[
\psi_\sigma(p_a, \vec{x}) = \int \frac{d^3k}{(2\pi)^3 E_k} \phi^\sigma(k, p_a) e^{i(k \cdot \vec{x})} = \int \frac{d^3k}{(2\pi)^3} \phi^\sigma(k, p_a) e^{i(k \cdot \vec{x})} \delta(k^2 - m_a^2), \quad (3.15)
\]

\[
\psi_\sigma(p_a, \vec{x}) = e^{i(p_a \cdot \vec{x})} \Phi_\sigma(p_a, -\vec{x}), \quad \{|p_a, x_a, \sigma\}\rangle = \int \frac{d^3k|k\rangle}{(2\pi)^3 E_k} \phi^\sigma(k, p_a) e^{i(k \cdot p_a - x_a)}, \quad (3.16)
\]

where \(k^0 = E_k, \quad p_a^0 = E_{p_a}, \) and the Lorentz invariant function \(\psi_\sigma(p_a, -\vec{x})\) (3.14), unlike the function \(\Phi_\sigma(p_a, -\vec{x})\), satisfies KG equation (3.2) relative to any of variables \(x, x_a, \vec{x}\).

The scalar function \(\phi^\sigma(k, p_a)\) is to be determined as Lorentz-invariant due to invariance of the measure in (3.15), (3.16). It depends on the invariants: \(\sigma = \sigma_a, \quad m = m_a, \quad \zeta_a^2, \quad (k \zeta_a)\), where the time-like for \(m_a > 0\) 4-vector \(\zeta_a(p_a, \sigma_a)\) carries all the basic properties and “particle-like” vector quantum numbers of the wave packet. Thus in general it has to be a linear combination of the 4-momentum \(p_a\) and the “4-spin” \(\hat{\omega}_a = \hat{s}_a m_a \sqrt{S(S + 1)}\) of wave packet\(^1\), that for \(p_a^2 = m_a^2, \quad (p_a \hat{\omega}_a) = 0, \quad \hat{s}_a^2 = -1\) fully characterize the relativistic one-particle state in the rest frame by its mass \(p_a^0 = (m_a, 0)\) and by the axis of its spin

\(^1\)Here the product of truth (polar) vectors \((\hat{\omega}_a \hat{s}_a)\) only parametrizes the eigenvalue of square of Pauli-Lubanski pseudo vector operator \(W^a\) of type (4.25), unlike the operator product (4.4a) below (see [26–28]).
quantization $\hat{\mathcal{H}}_{\mathbf{a}} = (0, \hat{s}_\mathbf{a})$ [28]:

$$\zeta_\mathbf{a}(p_\mathbf{a}, \sigma_\mathbf{a}) = p_\mathbf{a}g_1(m_\mathbf{a}, \sigma_\mathbf{a}) + \hat{\omega}_\mathbf{a}g_2(m_\mathbf{a}, \sigma_\mathbf{a}), \quad \zeta_\mathbf{a}^2 = m_\mathbf{a}^2 [g_1^2 - S(S + 1)g_2^2], \quad (3.17)$$

where $\zeta_\mathbf{a}^2 > 0$, $c_\mathbf{a}^0 > 0$, if without loss of generality is imposed that $\forall \sigma$: $g_1(m, \sigma) \gg |g_2(m, \sigma)|$. This is meaningful because the spin degrees of freedom should not change drastically the properties of the scalar amplitude $\phi^\sigma(\mathbf{k}, \mathbf{p}_\mathbf{a})$ of wave packet with any spin, and because for the scalar field (3.1): $S = 0$, $\hat{\omega}_\mathbf{a} = 0$, what is equivalent to choosing here $g_2 \equiv 0$. The further requirements onto universal invariant scalar functions $g_1, g_2$ will be given below.

The case of full delocalization in coordinate space is expressed by the limit $\sigma \to 0$. So the wave packet (3.16) reduces precisely to the state with definite momentum $|\mathbf{p}_\mathbf{a}\rangle$:

$$\begin{align*}
|\{p_\mathbf{a}, x_\mathbf{a}, \sigma_\mathbf{a}\}\rangle &\longrightarrow |\mathbf{p}_\mathbf{a}\rangle, \quad \Phi_\sigma(\mathbf{p}_\mathbf{a}, -x) \longrightarrow 1, \quad (3.18) \\
F_{\mathbf{p}_\mathbf{a}x_\mathbf{a}}(x) &\longrightarrow e^{-i(p_\mathbf{a}x)}, \quad \psi(\mathbf{p}_\mathbf{a}, x_\mathbf{a} - x) \longrightarrow e^{i(p_\mathbf{a}(x_\mathbf{a} - x))}, \quad (3.19) \\
\phi^\sigma(\mathbf{k}, \mathbf{p}_\mathbf{a}) &\equiv e^{i(p_\mathbf{a}(\mathbf{k} - x_\mathbf{a}))}\langle\{\{p_\mathbf{a}, x_\mathbf{a}, \sigma_\mathbf{a}\}\rangle \longrightarrow (2\pi)^3 2E_k \delta_3(\mathbf{k} - \mathbf{p}_\mathbf{a}), \quad \text{for:} \quad (3.20) \\
\{g_1(m, \sigma) \gg |g_2(m, \sigma)|, \quad \text{and} \quad \zeta_\mathbf{a}^2, c_\mathbf{a}^0 \longrightarrow +\infty, \quad \text{with} \quad g_2/g_1 \longrightarrow 0. \quad (3.21)
\end{align*}$$

The function $\Phi_\sigma(\mathbf{p}_\mathbf{a}, x - x_\mathbf{a})$ in (3.13) varies with $x$ very smoothly in compare with the plane wave $e^{-i(p_\mathbf{a}x)}$ and in the limit (3.18) does not contribute to the flux density [26] of the state (3.14):

$$\begin{align*}
\mathcal{J}_{\mathbf{p}_\mathbf{a}x_\mathbf{a}}(x) &\equiv F_{\mathbf{p}_\mathbf{a}x_\mathbf{a}}(x) (i \partial^\mu_x)F_{\mathbf{p}_\mathbf{a}x_\mathbf{a}}(x) \longrightarrow 2p_\mathbf{a}^\mu, \quad \text{with:} \quad \partial^\mu_x \equiv \partial^\mu_x - \partial^\mu_0. \quad (3.22)
\end{align*}$$

For state (3.16) the space integration over the center $x_\mathbf{a}$ of wave packet for arbitrary time $x_\mathbf{a}^0$ and $\sigma$ gives again the momentum state (3.18), where for $\phi^\sigma(\mathbf{p}_\mathbf{a}, \mathbf{p}_\mathbf{a}) = \phi_\mathbf{m} = \text{const} \neq 0$:

$$|\mathbf{p}_\mathbf{a}\rangle = \frac{2F_{\mathbf{p}_\mathbf{a}}}{\phi_\mathbf{m}} \int d^3x_\mathbf{a}\{\{p_\mathbf{a}, x_\mathbf{a}, \sigma_\mathbf{a}\}\}. \quad (3.23)$$

The conditions (3.18)–(3.23) are implied for any wave packets in any scattering theory [1–29].

The opposite case $\sigma \to \infty$ corresponds to field excitation fully localized in the point $x_\mathbf{a}$ in above relativistically covariant sense (3.10), (3.11). The packet state should transforms to the state (3.10) up to some normalization factor $N_\infty$ and up to inessential now again phase factor $e^{-i\Theta_\mathbf{a}}$ with invariant phase $\Theta_\mathbf{a} = (p_\mathbf{a}, x_\mathbf{a})$ instead of $\Theta_\mathbf{a} \rightarrow -(p_\mathbf{a}, x_\mathbf{a})$ in Eq. (2.11). So, up to the same factors its wave function (3.14), (3.15) recasts into the matrix element (3.25), because in accordance with relativity of time it assumes the sense of the only meaningful now transition amplitude [15–17] from point $x_\mathbf{a}$ to the point $x$ during the time $T = x^0 - x_\mathbf{a}^0 > 0$:

$$\begin{align*}
|\{p_\mathbf{a}, x_\mathbf{a}, \sigma_\mathbf{a}\}\rangle &\longrightarrow N_\infty e^{-i\Theta_\mathbf{a}}|x_\mathbf{a}\rangle = N_\infty e^{-i\Theta_\mathbf{a}}\varphi(x_\mathbf{a})|0\rangle, \quad (3.24) \\
\psi(\mathbf{p}_\mathbf{a}, x_\mathbf{a} - x) &\longrightarrow \psi(\mathbf{p}_\mathbf{a}, x_\mathbf{a} - x) = N_\infty (0)\varphi(x_\mathbf{a})\varphi(x)|0\rangle, \quad (3.25) \\
\phi^\sigma(\mathbf{k}, \mathbf{p}_\mathbf{a}) &\equiv e^{i(p_\mathbf{a}(\mathbf{k} - x_\mathbf{a}))}\langle\{\{p_\mathbf{a}, x_\mathbf{a}, \sigma_\mathbf{a}\}\rangle \longrightarrow \phi^\infty(\mathbf{k}, \mathbf{p}_\mathbf{a}) = N_\infty, \quad \text{for:} \quad (3.26) \\
\{g_1(m, \sigma) \gg |g_2(m, \sigma)|, \quad \text{and} \quad \zeta_\mathbf{a}^2, c_\mathbf{a}^0 \longrightarrow +0, \quad \text{with} \quad g_2/g_1 \longrightarrow 0. \quad (3.27)
\end{align*}$$
The conditions (3.24)–(3.27) are essentially new. Since the Wightman function in the r.h.s of Eq. (3.25) is the boundary value (B.15) for \( \zeta_a \) (3.27) of analytic function (3.30) of complex 4-vector variable \( z_a = x_a + i \zeta_a \), holomorphic in future tube \( \mathbb{V}^+ \), \( \zeta_a^2, \zeta_a^0 > 0 \) [18–23], and because the existence of asymptotic fields in Haag-Ruelle scattering theory and the reduction formulas for \( \mathcal{S} \) -matrix in QFT [18–27] requires an infinite smoothness of wave packet relative to \( x \): \( F_{p_k x_k}(x^0, x) \in \mathcal{S}(\mathbb{R}^4_x) \) for fixed \( x^0 \), let us consider the function \( \phi^\sigma(k, p_a) \in \mathcal{S}(\mathbb{R}^4_k) \), i.e. in the space of functions infinitely differentiable \( \forall k \in \mathbb{R}^4_k \), decrease faster than any power of \( 1/|k| \) together with all its derivatives [19, 20]. This ensures all such properties of \( F_{p_k x_k}(x) \) with the above vector \( \zeta_a = \zeta_a(p_a, \sigma) \) (3.17), due to \( \zeta_a^2 > 0, k^2 = m^2 > 0 \), with the amplitude:

\[
\phi^\sigma(k, p_a) = N_{\sigma}(m, \zeta_a^2) e^{-(k \zeta_a)}, \quad \text{for: } (k \zeta_a) > 0, \quad N_{\sigma} > 0, \quad \text{whence:} \quad (3.28)
\]

\[
\psi_{\sigma}(p_a, x_a - x) = N_{\sigma}(0) \varphi(x) \varphi(x_a + i \zeta_a(p_a, \sigma)) |0\rangle = N_{\sigma} \frac{1}{i} D_m^-(x - x_a - i \zeta_a(p_a, \sigma)), \quad (3.29)
\]

where:

\[
\frac{1}{i} D_m^-(y) = \int \frac{d^4 k}{(2\pi)^4} e^{-i(k y)}, \quad \text{for: } k^0 = E_k > 0, \quad y = x - z_a, \quad (3.30)
\]

is Wightman function (WF) [18–24] of a free scalar field (3.1). Thus the wave-packet state (3.16) now may be elegantly rewritten in such a form, that makes obvious the limit (3.24)–(3.27):

\[
\langle \{ p_a, x_a, \sigma \} \rangle = N_{\sigma} e^{-i(p_a x_a)} \varphi(x_a + i \zeta_a(p_a, \sigma)) |0\rangle \equiv N_{\sigma} e^{-i(p_a x_a)} \varphi(z_a)|0\rangle, \quad (3.31)
\]

\[
\langle \{ p_a, x_a, \sigma \} \rangle = N_{\sigma} e^{i(p_a x_a)} |0\rangle \varphi(x_a - i \zeta_a(p_a, \sigma)) \equiv N_{\sigma} e^{i(p_a x_a)} |0\rangle \varphi(z_a^*), \quad (3.32)
\]

The well known generalization of this vector-valued function of complex 4-vector variable \( z_a \) onto the case of product of a few fields [21, 22] defines a representation of nonhomogeneous complex Lorentz group. It allows to analyze the main properties of higher WFs and to prove Bargmann-Hall-Wightman theorem, PCT-theorem and dispersion relations [18–23].

It turns out that the covariant packet (3.28) – (3.32) precisely conforms to the plane-wave limit (3.18)–(3.21). This defines the \( N_{\sigma}(m, \zeta_a^2) > 0 \) up to some dimensionless function with fixed asymptotic behaviour over the one invariant dimensionless variable \( \tau = \tau_a = m \sqrt{\zeta_a^2}(p_a, \sigma) \):

\[
\mathcal{I}(\tau) = \int \frac{d^4 k}{(2\pi)^4} \phi^\sigma(k, p_a) = N_{\sigma} \frac{1}{i} D_m^-(i \zeta_a(p_a, \sigma)) = \quad (3.33)
\]

\[
\psi_{\sigma}(p_a, 0) = \psi_{\sigma}(0, 0) = \frac{N_{\sigma} m^2}{(2\pi)^2} K_1(\tau) \quad \tau \equiv \frac{\mathcal{R}(\tau)}{(2\pi)^2} |\mathcal{R}(\tau)|^2, \quad (3.34)
\]

where: \( \mathcal{R}(\tau) \equiv N_{\sigma} m^2 \) and \( K_1(\tau) \) is Macdonald function (B.19) [49]. The conditions (3.20), (3.21) and (3.26), (3.27) define independently the same dimension of \( N_{\sigma} \) and the corresponding to (B.20)) asymptotic behaviour of unknown smooth dimensionless functions \( \mathcal{I}(\tau) \) and \( \mathcal{R}(\tau) \) as:

\[
\lim_{\tau \to \infty} \mathcal{I}(\tau) = 1, \quad \text{or } \mathcal{R}(\tau) \to 2(2\pi)^3 \tau^{3/2} e^{\tau}, \quad \text{whence } N_{\sigma} \to +\infty, \quad (3.35)
\]

\[
\lim_{\tau \to 0} \tau^2 \mathcal{I}(\tau) = \frac{\mathcal{R}(0)}{(2\pi)^2}, \quad \mathcal{R}(0) > 0, \quad \text{or } N_{\sigma} \to N_{\infty} = \frac{\mathcal{R}(0)}{m^2} > 0. \quad (3.36)
\]
Indeed, from the simple dimensional analysis with \( c \neq 1 \), in order to satisfy both the conditions (3.21) and (3.27), up to rescaling of \( \sigma \) by dimensionless constant, it is enough suppose, that:

\[
(mc)^2 g_{1,2}(m, \sigma) \xrightarrow{\sigma \to 0} C_{1,2} \left( \frac{mc}{\sigma} \right)^{\epsilon - \gamma}, \quad C_1 = 1, \quad \epsilon > \gamma > 0, \quad (3.37)
\]

\[
(mc)^2 g_{1,2}(m, \sigma) \xrightarrow{\sigma \to \infty} C_{1,2} \left( \frac{mc}{\sigma} \right)^{A,B}, \quad C_1 > 0, \quad 0 < A < B, \quad \text{whence:} \quad (3.38)
\]

\[
\tau \mapsto (mc)^2 g_1(m, \sigma) \to +\infty, \quad \text{with} \quad \sigma \to 0, \quad \text{or with} \quad mc \to +\infty, \quad (3.39)
\]

\[
\tau \mapsto (mc)^2 g_1(m, \sigma) \to 0, \quad \text{with} \quad \sigma \to +\infty, \quad \text{or with} \quad mc \to 0. \quad (3.40)
\]

According to (3.21) and (3.37) the limit (3.20) is defined only by properties of the function \( g_1(m, \sigma) \) with \( \tau \mapsto (mc)^2 g_1(m, \sigma) \equiv (p_a \zeta_a) \), and from (3.28), (3.35), for \( c = 1 \), by means of:

\[
2(kp) = 2m^2 - (k^0 - p^0)^2 + (k - p)^2 > 0, \quad k^0 = E_k, \quad p^0 = E_p, \quad (3.41)
\]

for: \( |k| = k, \quad k = km, \quad |p| = p, \quad p = pm_p, \quad n^2_k = n^2_p = 1, \quad (3.42) \)

combining the well known definitions and expressions of delta-functions [20, 52]:

\[
\left\{ \left( \frac{g_1}{2\pi} \right)^{3/2} e^{-(g_1/2)(k - p)^2} \right\} \xrightarrow{g_1 \to \infty} \delta_3(k - p), \quad \left\{ \left( \frac{g_1}{2\pi} \right)^{1/2} e^{-(g_1/2)(k - p)^2} \right\} \xrightarrow{g_1 \to \infty} \delta(k - p), \quad (3.43)
\]

\[
\delta_3(k - p) = \frac{\delta_3(n_k, n_p)}{kp} \delta(k - p), \quad \text{where:} \quad \int d\Omega(n_k)f(n_k)\delta_3(n_k, n_p) = f(n_p), \quad (3.44)
\]

with: \( k^0 - p^0 = E_k - E_p \approx \frac{k}{E_k}(k - p) \), and: \( 1 - \frac{k^2}{E_k^2} = \frac{m^2}{E_k^2} \), for \( p_a = (p^0, p) \), \( (3.45) \)

one finds:

\[
\phi^\sigma(k, p) = e^{i\left(p_a - kx_a\right)} \langle k|\{p_a, x_a, \sigma\}\rangle = N_a e^{-(k\zeta_a)} \xrightarrow{\tau \to \infty} N_a e^{-g_1(kp_a)} \quad (3.46)
\]

\[
\xrightarrow{\tau \to \infty} 2m(2\pi)^3 e^{(g_1/2)(k^0 - p^0)^2} \left\{ \left( \frac{g_1}{2\pi} \right)^{3/2} e^{-(g_1/2)(k - p)^2} \right\}, \quad (3.47)
\]

that by the use of (A.11), (A.13) may be rewritten as:

\[
\xrightarrow{g_1 \to \infty} 2m(2\pi)^3 \frac{\delta_3(n_k, n_p)}{kp} \lim_{g_1 \to \infty} e^{(g_1/2)(E_k - E_p)^2} \left\{ \left( \frac{g_1}{2\pi} \right)^{1/2} e^{-(g_1/2)(k - p)^2} \right\} =
\]

\[
= 2 m(2\pi)^3 \frac{\delta_3(n_k, n_p)}{kp} \lim_{g_1 \to \infty} \left\{ \left( \frac{g_1}{2\pi} \right)^{1/2} \exp \left[ -\frac{g_1 m^2}{2 E_k^2} (k - p)^2 \right] \right\} =
\]

\[
= 2 m(2\pi)^3 \frac{E_k}{m} \frac{\delta_3(n_k, n_p)}{k^2} \delta(k - p) = (2\pi)^3 2E_k \delta_3(k - p). \quad \text{(comp. appendix A)} \quad (3.49)
\]

Starting from quite different reasoning the wave packet like (3.15), (3.28) was introduced earlier in [1–7] and studied in some detail with \( \zeta_a(p_a, \sigma) = p_a g_1(\sigma) = p_a (2\sigma^2)^{-1} \). However the arbitrarily imposed therein condition \( I(\tau) \equiv 1 \) leads to disappearance of \( N_\infty \xrightarrow{\tau \to 0} N(0) \), rendering meaningless the limit (3.24)–(3.27), (3.36) and hiding the relations (3.29)–(3.36). Unlike [1], the derivation (3.46)–(3.49) of the limit (3.20) has nothing to do with the rest frame \( p = 0 \) of wave packet but explicitly requires \( m > 0 \) (see also (A.6), (A.7)). Thus an independent limit \( m \to 0 \) makes sense only for the packet already with zero width \( \sigma = 0 \), i.e. for a plane wave. So, for a massless particle with infinite Compton wave length \( \lambda \) there
are inevitable difficulties [15] with interpretation of (3.10), (3.24), (3.31) as its covariant localizable states. Actually, the lacking of localizability for the massless states manifests in Introduction freedom of the profile of their wave packets [30], which arises in (A.15)–(A.21) due to arbitrariness of \( \tau > 0 \) but nevertheless disappears for the massive case (A.6), (A.7).

On the other hand, according to (3.35), (3.39), (3.46), the non relativistic limit \( c \to \infty \) of the amplitude of profile function (3.28), due to (3.3), has exactly the form of expression (3.47) with \( e^{g_1(E_k-p_\sigma)^2/2} = e^{g_1(E_k-p_\sigma)^2/2c^2} \to 1 \). For \( \epsilon = 2 \) [1, 2], whence \( g_1 = \sigma^{-2} \), this limit differs from the non relativistic profile of wave packet (2.8) just on the multiplier \((2\pi)^32mc(2\sigma\sqrt{\pi})^{-3/2}\) for \( \sigma = \sigma_p \), whereas from (3.11), (3.12), it follows, that \( \sqrt{2mc} \rightarrow \infty \) as \( \epsilon \rightarrow \infty \). According to (A.28) this recasts the wave-packet state (3.16) exactly into Gaussian non relativistic one (2.8), (2.9) at arbitrary initial instant of time \( t_a \).

The difference of localized states in QFT and in QM manifests itself in two different ways of obtaining the wave packet for “infinitely heavy” particle, with \( m_a c \gg |p_a|, |k| \), used, for example, in the model [34, 39] of Kobzarev et al.. Its space-time and four-momentum wave functions would be proportional to \( \exp\{−im_a cx^0\}\delta_\mathcal{H}(x − x_a) \) and \( \delta(k^0 − m_a c) \) respectively. But what about their normalization? By taking firstly the QFT-limit \( \sigma \to \infty \), we will face localized state (3.14), (3.24) – (3.30) for \( p_a^0 = \sqrt{(m_a c)^2 + p_\sigma^2} \), \( k^0 \to k_a^0 = \sqrt{(m_a c)^2 + k^2} \), which then, for \( m_a \to \infty \), and \( p_a^0, k_a^0 \to m_a c \), \( p_a \to 0 \), recurs as:

\[
F^{[\sigma = \infty]}(x) = N_\infty e^{−i(p_a x_a)}/(2\pi)^32k^0 \quad p_a = 0 \to m_a c \quad e^{−im_a cx^0} \delta_\mathcal{H}(x − x_a),
\]

\[
\phi^\sigma(k, p_a) \theta(k^0) \delta(k^2 − (m_a c)^2) = \phi^\sigma(k, p_a) \delta(k^0 − k_a^0) \quad \sigma \to \infty.
\]

On the other hand, starting from the non-relativistic limit \( c \to \infty \) (A.27) of \( \phi^\sigma(k, p_a) \), for the next limit \( \sigma \to \infty \) under the conditions \( p_a = 0, m_a \to \infty \), we arrive to the similar final value \( \exp\{−im_a cx^0\}\delta_\mathcal{H}(x − x_a) \) but with another multiplier regardless the order of making the limit and Fourier transformation (2.6). Thus, the normalization constants appearing here, for the first way, with \( \tau = 0 \) for \( \sigma = \infty \), are:

\[
N_\infty = \frac{N(0)}{(m_a c)^2}, \quad \text{and} \quad \frac{N_\infty}{2m_a c} = \frac{N(0)}{2(m_a c)^3}, \quad \text{but, that is:} \quad \left(\frac{2\pi}{\sigma^2}\right)^{3/2},
\]

for the second contradictory way: with \( \sigma \to \infty \), when \( \tau = \infty \). The both multipliers of the first way in (3.52) are exactly, what were expected from definitions of localized states (3.24) in QFT and (3.11), (3.12) in QM, where \( < x | x_a > = \delta_\mathcal{H}(x − x_a) \). So, the answer given by (3.50), (3.51) for the first way in (3.52) arises as more consequent and more constructive, and it is physically more justified, since it directly connects the norms of localized states in QM and QFT. This illustrates the difference between the meaning of localized states in QM and in QFT according to above discussion. Note, that for both cases the initial instant \( t_a \) disappears for “infinitely heavy” packet only due to limit \( m_a \to \infty \) in (A.28).

The used below narrow-packet approximations (3.63), (6.27), (6.31), (6.36) imply an absence of asymptotic correction of order \( 1/\tau \) at \( \tau \to \infty \) from the function \( \mathcal{I}(\tau) \to 1 \).
This condition conforms with both limits \( \tau \to \infty \) (3.35) and \( \tau \to 0 \) (3.36) e.g. for the choice \( \mathcal{I}(\tau) = 1 + \tau^{-2} \mathcal{N}(0)/(2\pi)^2 \). The further requirement of absence of any asymptotic corrections of higher order \( \tau^{-n} \) also leaves it ambiguous, e.g., for \( \omega_s > 0 \), any \( \ell_s, c_s \neq 0 \), as:

\[
\mathcal{I}(\tau) = \prod_{s=1}^{\infty} \left( \coth \frac{\omega_s}{c_s} \right)^{\ell_s}, \quad \text{if:} \quad \prod_{s=1}^{\infty} (\text{sign}(c_s))^{\ell_s} = 1, \quad \prod_{s=1}^{\infty} (c_s)^{\ell_s} = \frac{\mathcal{N}(0)}{(2\pi)^2}, \quad \sum_{s=1}^{\infty} \omega_s \ell_s = 2.
\]

(3.53)

The first and the second condition come from the first and second limit respectively but the second condition includes the first. Moreover they coincide for \( \mathcal{I}(0) = (2\pi)^2 \) if all \( c_s = \pm 1 \). Nevertheless the infinite numbers of values \( \ell_s \) and \( \omega_s \) remain restricted only by the third condition, which comes from the second limit (3.36). The most “economic” choice is:

\[
\mathcal{I}(\tau) = \left( \coth \frac{\omega_1}{c_1} \right)^{\ell_1}, \quad \text{with:} \quad (c_1)^{\ell_1} = \frac{\mathcal{N}(0)}{(2\pi)^2}, \quad \omega_1 \ell_1 = 2.
\]

(3.54)

The inner product of the states (3.31), (3.32) providing the self-adjointness of operator \( \hat{X}_p \) in the space of solutions to KG equation (3.2) with the same mass \( m_a = m_b \) [15–24] turns out to be naturally consistent with the inner product given by Eq. (8) in Ref. [1], because in accordance with (B.23):

\[
\langle \{ p_b, x_b, \sigma_b \} | \{ p_a, x_a, \sigma_a \} \rangle = N_{\sigma a} N_{\sigma b} e^{i(p_b x_b) - i(p_a x_a)} (0) \mathcal{F}(z_b^* \varphi(z_a) | 0) =
\]

(3.55a)

\[
e^{i(p_b x_b) - i(p_a x_a)} \int \frac{d^3k}{(2\pi)^3} E_k \phi^*_{\sigma b}(k, p_b) \phi^\sigma_{\alpha a}(k, p_a) e^{i(k(x_a - x_b))} = (F_{p_b x_b}, F_{p_a x_a}) =
\]

(3.55b)

\[
= \int d^3x F_{p_b x_b}(x) (i \partial^0) F_{p_a x_a}(x) = \frac{N_{\sigma a} N_{\sigma b}}{i} e^{i(p_b x_b) - i(p_a x_a)} D^- (z_b^* - z_a) =
\]

(3.55c)

\[
= \frac{N_{\sigma a} N_{\sigma b}}{i} e^{i(p_b x_b) - i(p_a x_a)} \int d^3x D^- (z_b^* - x) \partial^0_x D^-(x - z_a).
\]

(3.55d)

It is positively defined for fixed frequency type and does not depend on \( x_0 \) [21]. For \( \sigma_a, \sigma_b \to 0 \) independently it leads to the covariant orthogonality condition (3.5) for solutions (6.6):

\[
(F_{p_b x_b}, F_{p_a x_a}) \to (f_p, f_q) = (p|q) = (2\pi)^3 2E_p b(p - q),
\]

(3.56)

The complex 4-vector \( z_a = z_a(p_a, \sigma_a) = x_a + i \zeta_a(p_a, \sigma_a) \), with time-like imaginary part \( \zeta_a(p_a, \sigma_a) \) (3.17) provides the correct analytical continuation (B.15) to \( V^+ \) for all WFs [19–23] here in Eqs. (3.29), (3.30), (3.33), (3.55c), (3.55d) and below in Eqs. (4.6), (4.16).

Absolute convergence of integral (3.15) due to \( e^{-(k \zeta_a)} \) yields \( | \psi_\sigma(p_a, x) | \leq \mathcal{I}(\tau) \) uniformly relative to \( x_a \). So, non uniqueness of the scalar wave packet (3.28) \( (g_2 = 0) \) would mean an additional polynomial dependence on \( (k - p_a)^2 = 2m_a^2 - 2(k p_a) \), i.e. on the scalar product \( (k p_a) \) for \( N_\sigma \to \tilde{N}_\sigma \). This leads to formally previous expression of wave packet (3.13), (3.14), when the “multiplier” \( \tilde{N}_\sigma \) is pulled out of the integrand as a polynomial on differential operator \( \partial^2_{x_a} \), which acts on the function of previous wave packet: (for any function of \( \zeta^2, \partial^\mu_{\zeta} \to 2\zeta^\mu \partial^\mu_2 \))

\[
\phi^\sigma(k, p_a) \to \tilde{N}_\sigma(m_a, \zeta_a^2, -(k - p_a)^2) e^{-(k \zeta_a)};
\]

(3.57)

\[
F_{p_a x_a}(x) \to \tilde{N}_\sigma(m_a, \zeta_a^2, \partial^2_{x_a}) e^{-i(p_a x_a)} \frac{1}{\ell} D^-_{m_a}(x - z_a),
\]

(3.58)
and the same for the states (3.31), (3.32) and for the scalar product (3.55). Then, from dimensional reasoning in (3.33), (3.34) with ε = m2ζ\(a\)^2 \rightarrow m^2 g_1^2(m, \sigma) = (p_0 \zeta_a)^2 it follows, that for (3.33), (3.34): \(-\partial^2_{x_a} \rightarrow 2[m^2 + (p_0 \partial_{x_a})] \rightarrow 2m^2(1 + \partial_{x_a})\) and \(m^2 \tilde{\Theta}_a \rightarrow \tilde{\Theta}(\tau, \partial^2_{x_a}/m^2) \rightarrow \tilde{\Theta}(\tau, \partial_{x_a})\). The additional dependence on \(\partial^2_{x_a}/m^2\) in (3.58) doesn’t affect the above plane-wave limit \(\sigma \rightarrow 0\) (3.20) and non relativistic limit \(c \rightarrow \infty\) but, for \(\Theta_a = (p_0 x_a)\), leads to inadmissible contributions like \((p_0 \partial_{x_a})\varphi(x_a)|0\rangle\), breaking down localization condition (3.24) at \(\sigma \rightarrow \infty\).

Another possible dimensionless substitution in (3.57): \(m^2 \tilde{\Theta}_a \rightarrow \tilde{\Theta}(\tau, -(k - p_0)^2/\sigma^2)\), contrariwise, is innocent relative to the condition (3.24), but, in general breaks down the limit (3.20), and in non relativistic limit leads to inevitable deformation of the Gaussian profile (2.8) by the polynomial on \((\Delta k)^2/\sigma^2\), and destroys its minimization properties (2.12). Thus, both those dimensionless combinations are excluded, and the expression (3.28) is the only acceptable from both the physical and mathematical points of view [11, 12].

Nevertheless the polynomial \(k\) - dependence arises below naturally for wave packet of the spin particle and for respective scalar product, what finally will fully exclude \(a posteriori\) its arbitrary uncontrollable appearance in Eqs. (3.57), (3.58).

### 3.1 Contracted Relativistic Gaussian approximation

A fully relativistic Gaussian approximation for the (pseudo) scalar wave packet (3.29) was suggested in [1–3] for \(g_2 = 0\), \(\zeta_a(p_0, \sigma_a) = p_0(2\sigma_a^2)^{-1}\), \(\tau^2 = m^2 \zeta_a^2 \rightarrow \infty\). Definitions (3.17), (3.33)–(3.36) with the use of (B.19), (B.20), for: \(m = m_a\), \(\tau = \tau_a\), \(x = x_a - x\),

\[
\psi_\sigma(p_0, x) = N_\sigma \langle 0 | \varphi(x) \varphi(x_a + i \zeta_a) | 0 \rangle = \mathcal{I}(\tau) \frac{h(\tilde{Z}_a(x))}{h(\tau^2)}, \quad \nu_a = \frac{\zeta_a}{\sqrt{\zeta_a^2}}, \quad \nu_a^2 = 1, \tag{3.59}
\]

\[
\tilde{Z}_a(x) = -m^2(x + i \zeta_a)^2 = m^2 \left[ \frac{\zeta_a^2}{\nu_a^2} - 2i (x \zeta_a) - \frac{x^2}{\nu_a^2} \right] = m^2 \left\{ \left[ \sqrt{\zeta_a^2} - i \nu_a \right]^2 + \nu_a^2 \right\}, \tag{3.60}
\]

\[
x_0(x) = (x v_a), \quad x_0^2 = (x v_a)^2 - x^2 \equiv -x_1^2, \quad \text{when:} \quad \zeta_a^2 \gg (x \zeta_a), x^2, \quad \mathcal{I}(\tau) \rightarrow 1, \tag{3.61}
\]

\[
\sqrt{\tilde{Z}_a(x)} \approx m \sqrt{\zeta_a^2} - m \frac{(x \zeta_a)}{\sqrt{\zeta_a^2}} + \frac{m^2}{2} \frac{(x \zeta_a)^2}{\zeta_a^2} \approx \tau - i m x_0^0 + \frac{m^2}{2} x_1^2, \tag{3.62}
\]

\[
\text{give:} \quad \psi_{CRG}^a(p_0, x) = \exp \left\{ i m (x v_a) \right\} \exp \left\{ \frac{3 m (x v_a)}{2 \tau} - \frac{m^2}{2 \tau} [(x v_a)^2 - x^2] \right\}, \tag{3.63}
\]

where:

\[
m^2 \left[(x v_a)^2 - x^2\right] = T_{\alpha}^\beta x_{\beta} x_{\lambda} \equiv (x T_a x) = \frac{m^2}{2 \tau} \left[ (x p_a)^2 - x^2 \right] = \frac{m^2}{2 \tau} \left[ (x p_a)^2 - x^2 \right] + 2 \frac{g_2}{g_1^2} (x p_a)(x \hat{w}_a) + \frac{g_2^2}{g_1^4} \left[ (x \hat{w}_a)^2 - x^2 \hat{w}_a^2 \right]. \tag{3.64}
\]

The values (3.61) define 4-vector \(\mathcal{L}_a^\beta = (x_0, x)\) in the rest frame of time-like vector \(\nu_a (3.59)\). The first imaginary term of last exponential (3.63) was arbitrarily omitted in Eq. (21) in Ref. [1]. In spite of the “hidden spin” asymmetry of the quadratic form (3.64), similar [3], such generalized to \(g_2 \neq 0\) CRG- approximation keeps its non negative definiteness like in [1]. The drawback of CRG- approximation (3.63) is, that due to the Gaussian exponential it obviously can’t be a solution to free KG equation (3.2), i.e. can’t represent an external particle on the mass shell. Thus, strictly speaking, it is not a wave packet in the usual sense,
with definite covariant dispersion equation [12]. However, if it is considered as a suitable approximation to the exact wave packet, then the function (3.63) describes propagation with specific frequency mixing but, without of any spreading [1–3] in the “Gaussian sense” [4]. Indeed, the figure 1 from the ref. [3] shows a very small spreading of exact wave packet near its peak\(^2\), what is analytically approximated in (3.63) as absence of spreading.

\[ |\psi| = \frac{1}{(2\pi)^{3/2}} \delta \left( \frac{1}{2m^2} k^2 - m^2 \right) \exp \left\{ - \frac{1}{2m^2} k^2 \right\}. \]

These properties are clarified by Fourier-image of function (3.63), which was not discussed in refs. [1–3]. For the simplified case of ref. [1], with \( g_2 = 0 \), \( v_a \mapsto u_a = p_a/m \), \( u_a^2 = 1 \), it reads:

\[
\int d^4 \kappa e^{-i(k \cdot x)} \psi^{CRG}_{NN}(p, x) = \int d^4 \kappa e^{i((p - k) \cdot x)} e^{-i \cdot \tau T \cdot \kappa} = (2\pi)^{3/2} \exp \left\{ \frac{\tau}{2m^2} (k^2 - m^2) \right\}, \]

where for calculation of the integral the projective properties of tensor \( T_a^{\beta \lambda} \) are used in the variables \( \kappa^\alpha = (u_a \kappa) \) and \( \kappa^\perp = \Pi_{\alpha a} \kappa \), with: \( k^\perp = \Pi_{\alpha a} k \), \( (\kappa \cdot k) = (\perp \cdot \kappa) = - (x^\alpha \cdot k_\alpha) \).

\(^2\)We thank V.A. Naumov for kind permission to use this picture.
\( \begin{align*}
\mathbf{z}_\perp^2 &= (x \Pi_{ua} x) = -\mathbf{x}_s^2 < 0, \quad k_\perp^2 = k^2 - (u_a k)^2 = -k_s^2 < 0. \quad \text{Here:} \\
T^\beta\lambda_a &= -\frac{m^2}{2\tau} \Pi^\beta\lambda_{ua}, \quad \Pi^\beta\lambda_{ua} = g^\beta\lambda - u_a^\beta \gamma^\lambda = \Pi^\beta\gamma\Pi^\gamma\lambda_{ua}, \quad u_a \Pi_{ua} = \Pi_{ua} u_a = 0, \quad (3.67)
\end{align*} \)

and as: \( \begin{align*}
k_s^0 &= (u_a k), \quad \mathbf{k}_s = \mathbf{k} + u_a \left[ \frac{(u_a \cdot \mathbf{k})}{u_a^0 + 1} - k_s^0 \right] = \mathbf{k} - u_a \frac{k_s^0 + (u_a k) - 1}{u_a^0 + 1}, \quad (3.68)
\end{align*} \)

as well \( \mathbf{z}_s^0, \mathbf{x}_s, \) are defined by the same Lorentz transformations \((3.68)\) to the rest frame of wave packet \( \mathbf{p}_{a*} = 0, \mathbf{p}_{a*}^\lambda = (m, 0). \) Expressions \((3.66b), (3.66c)\) should be compared with the multiplied by \( 2\pi \) l.h.s. of relation \((3.51). \) The dispersion equation now depends on the reference frame as \( k_0^0 = [m + (u_a \cdot \mathbf{k})]/u_a^0. \) Only in the plane-wave limit \( \tau \to \infty \) the both compared expressions coincide with \( (2\pi)^4 \delta_4 (k - p_a) \) in arbitrary reference frame.

The advantage of approximation \((3.63)\) will be seen below in subsection 6.1.1, where it will be naturally reproduced by asymptotic behavior at \( \mathbf{z}_\beta^2 \to \infty \) for defined therein off shell and on shell composite wave functions \( \mathcal{F}_{(C/D)}(\varphi) \) \((6.16), \) \((6.28)\) of neutrino creation/detection vertexes, as well for the amplitude of oscillation \([1]\).

## 4 Fermionic wave packet. Higher spins.

The relativistic fermionic wave packet is constructed in completely similar way to described above scalar packet. A free massive quantum Fermi-Dirac-field \( \psi(x) \) has the form \([14–26]\):

\( \psi_{\alpha}(x) = \sum_{r=\pm 1/2} \int \frac{d^3k}{(2\pi)^32k^0} \left( u_\alpha^+(k, r) f^x_k(x) b_{k,r}^{(+)} + u_\alpha^-(k, r) f_{-k}(x) b_{k,r}^{(-)} \right), \quad (4.1) \)

where: \( k_0^0 = E_k > 0, \quad b_{k,s}^{(\xi,0)}(0) = 0, \quad b_{k,s}^{(\xi,0)}(0) = |(\xi); \mathbf{k}, s\rangle, \quad (4.2) \)

and respectively for the Dirac-conjugated field \( \overline{\psi}(x) = \psi^\dagger(x) \gamma^0. \) The bispinors \( u^\xi(k, r) \) are solutions to the free Dirac equations \( [i(x) - \xi m] u^\xi(k, r) = 0, \) with “positive and negative” energy according to the index \( \xi = \pm \) and, for \( k^\mu = (E_k, \mathbf{k}) \) with \( f_{\xi k}(x) = e^{-i\xi(xk)}, \gamma^0 = u^\dagger \gamma^0, \) define analogically to \((3.6)\) the matrix elements \([14–26]\), as corresponding wave functions for initial and final states with mass \( m, \) momentum \( \mathbf{k} \) and spin \( s = \pm 1/2 \) along its quantization 4- axis \( \hat{s}^\mu: \)

\( \begin{align*}
\langle 0 | \psi(x) | (+); \mathbf{k}, s \rangle &= u^{(+)}(k, s) f_k(x) = U^{(+)}_{k,s}(x), \quad (4.3a)

\langle \mathbf{k}, s; (+) | \overline{\psi}(x) | 0 \rangle &= \overline{u}^{(+)}(k, s) f_k^*(x) = \overline{U}^{(+)}_{k,s}(x), \quad (4.3b)

\langle \mathbf{k}, s; (-) | \psi(x) | 0 \rangle &= u^{(-)}(k, s) f_{-k}(x) = U^{(-)}_{k,s}(x), \quad (4.3c)

\langle 0 | \overline{\psi}(x) | (-); \mathbf{k}, s \rangle &= \overline{u}^{(-)}(k, s) f^*_k(x) = \overline{U}^{(-)}_{k,s}(x), \quad (4.3d)
\end{align*} \)
where for any $\gamma^\mu$- representation and any such axis $[26-28]$ $\bar{s}^2 = -1$, $(\bar{s}k) = 0$ (cf. (4.25)):

\[
(W, \bar{s}) \mapsto -\frac{1}{2} \gamma^5 (\bar{s}) \xi (\gamma^k), \quad u^\xi (k, s) \bar{u}^\xi (k, s) = [i (\gamma^k) + i \xi m] \frac{1}{2} \left[ 1 + 2 s \gamma^5 (\bar{s}) \right], \quad (4.4a)
\]

\[
\bar{u}^\xi (k, s) u^\xi (k, r) = 2 m \xi \delta_{\xi r} \delta_{s r}, \quad u^\xi (k^0, \xi k; s) u^\xi (k^0, \eta k; r) = 2 E_k \delta_{\xi r} \delta_{\eta r}, \quad \xi, \eta = \pm, \quad (4.4b)
\]

\[
\left\{ b^\xi_\eta, b^{\xi \eta}_r \right\} = \{ q_r, (\xi)(\eta); k, s \} = (2 \pi)^3 2 E_k \delta_\eta (k - q) \delta_{s r} \delta_{\xi r} = \left( U^\xi_{q, s} U^\eta_{q, r} \right) \quad (4.4c)
\]

\[
\left( \begin{array}{cc} U^\xi_{q, s} & U^\xi_{q, r} \end{array} \right) = \int d^3 x U^{\xi}_{q, s} (x) U^{\eta}_{q, r} (x) = \int d^3 x \left( U^{\xi}_{q, s} (x) \right) \gamma^0 \left( U^{\eta}_{q, r} (x) \right). \quad (4.4d)
\]

Following the scalar case (3.31), (3.32), the wave-packet state is created by operators:

\[
\begin{align*}
\left| + \right> : \{ p_a, x_a, s_a \} & = \hat{N}_{\sigma a} \psi (x_a + i \zeta a) \left| 0 \right> \left( U^\xi_{p_a, s_a} (x_a) = \hat{N}_{\sigma a} \bar{\psi} (z_a) \left| 0 \right> \right), \quad (4.5a) \\
\left| - \right> : \{ p_a, x_a, s_a \} & = \hat{N}_{\sigma a} \bar{U}^{-\xi}_{p_a, s_a} (x_a) \psi (x_a + i \zeta a) \left| 0 \right> = \hat{N}_{\sigma a} \bar{U}^{-\xi}_{p_a, s_a} (x_a) \psi (z_a) \left| 0 \right>, \quad (4.5b)
\end{align*}
\]

with: $\hat{N}_{\sigma a} = \frac{\xi N_{\sigma a}}{2m_a}, \quad \hat{N}_{\sigma a}^\ast = \frac{\bar{N}_{\sigma a}}{2m_a}, \quad \hat{N}_{\sigma a} \bar{U}^{\xi}_{p_a, s_a} (x_a) \rightarrow U^{\xi}_{p_a, s_a} (x), \quad \hat{N}_{\sigma a} \bar{U}^{-\xi}_{p_a, s_a} (x_a) \rightarrow U^{-\xi}_{p_a, s_a} (x), \quad (4.5c)$

where the wave functions of these states and the respective Dirac-conjugated ones, according to (4.3) and similarly (3.14), (3.29), for $x = x_a - x, \xi A = \sigma a, z_a = x_a + i \zeta a, \xi a^r = x_a - i \zeta a, \xi = \pm$, with the same $N_{\sigma a}$ as for the scalar case (3.33)–(3.36), are defined as bispinors:

\[
\begin{align*}
\Xi^{\xi \ast}_{p_a, x_a, s_a, \sigma a} (x) & = \langle 0 | \psi (x) | + \rangle : \{ p_a, x_a, s_a \} = \hat{N}_{\sigma a} \bar{\psi} (x_a + i \zeta a) \left| 0 \right> \left( U^\xi_{p_a, s_a} (x_a) = \hat{N}_{\sigma a} \bar{\psi} (z_a) \left| 0 \right> \right), \quad (4.6a) \\
\Xi^{-\xi \ast}_{p_a, x_a, s_a, \sigma a} (x) & = \langle 0 | \bar{\psi} (x) | - \rangle : \{ p_a, x_a, s_a \} = (i) \hat{N}_{\sigma a} \bar{U}^{-\xi}_{p_a, s_a} (x_a) \bar{\psi} (x_a + i \zeta a) \left| 0 \right> - \hat{N}_{\sigma a} \bar{U}^{-\xi}_{p_a, s_a} (x_a) \bar{\psi} (z_a) \left| 0 \right>, \quad (4.6b)
\end{align*}
\]

\[
\begin{align*}
\Xi^{\xi \ast}_{p_a, x_a, s_a, \sigma a, \alpha} (x) & = \langle 0 | \psi \alpha (x) | + \rangle : \{ p_a, x_a, s_a \} = \hat{N}_{\sigma a} \left( \psi \alpha (x_a - i \zeta a) \psi \alpha (x) \right) \left| 0 \right> \left( U^\xi_{p_a, s_a, \alpha} (x_a) = \hat{N}_{\sigma a} \left( \psi \alpha (z_a - i \zeta a) \psi \alpha (x) \right) \left| 0 \right> \right), \quad (4.6c) \\
\Xi^{-\xi \ast}_{p_a, x_a, s_a, \sigma a, \alpha} (x) & = \langle 0 | \bar{\psi} \beta (x) | - \rangle : \{ p_a, x_a, s_a \} = (i) \hat{N}_{\sigma a} \bar{U}^{-\xi}_{p_a, s_a, \alpha} (x_a) \bar{\psi} \beta (x_a + i \zeta a) \left| 0 \right> \left( U^{-\xi}_{p_a, s_a, \alpha} (x_a) = \hat{N}_{\sigma a} \bar{U}^{-\xi}_{p_a, s_a, \alpha} (x_a) \bar{\psi} \beta (z_a + i \zeta a) \left| 0 \right> \right), \quad (4.6d)
\end{align*}
\]

The fermionicWFs $S^\xi (x)$ are connected mutually and with causal propagator $S^\xi (x)$, and with the scalar WF $D^\xi (x)$ (3.30), (B.15), for $(D^{-\xi} (x))^\ast = -D^\xi (x)$, and $D^\xi (x)$ (B.17), by the relations $[15-24]$, containing charge conjugation matrix $C^T = -C, C \gamma^\mu C^{-1} = -\gamma^\mu$:

\[
\begin{align*}
\frac{1}{i} \left( S^{-\xi} (x - y) = \int \frac{d^4 k}{(2 \pi)^3} \theta (k^0) \left( \delta (k^2 - m^2) e^{-i \xi (k(x - y))} \right) \right) \left[ (i \gamma^k + i \xi m) \right], \quad (4.7a) \\
S^\xi (x) = -C S^{-\xi} (-x) C^{-1}, \quad S^\xi (x) = C S^{\xi T} (-x) C^{-1}, \quad (4.7b) \\
(\text{see (B.9)--(B.12)}), \quad \text{but with } 0 = \left( \psi (x) \psi (y) \right) \equiv 0, \quad (4.7c)
\end{align*}
\]

and instead of (B.23), for $\xi, \eta = \pm, \phi^\mu = \left( \phi^0, \rho \right)$ satisfy $\forall \phi^0$ or space-like $\Sigma (\phi)$ $[15-24]$:

\[
\begin{align*}
\delta_{\xi \eta} \left( \frac{1}{i} S^\xi (x - y) = \int d^3 \rho \left( \frac{1}{i} S^\xi (x - \rho) \gamma^0 \frac{1}{i} S^\eta (\rho - y) \right) = \int d \Sigma (\phi) \left( \frac{1}{i} S^\xi (x - \phi) \gamma^\mu \frac{1}{i} S^\eta (\phi - y) \right) \right), \quad (4.8)
\end{align*}
\]
also for complex values of $x,y$. So, the similar (4.4c), (4.4d) inner product for the packets (4.6) reads:

\[
\langle \{ \Phi_a, x_a, s_a \}; (\eta) | \langle \xi \rangle ; \{ \Phi_c, x_c, s_c \} \rangle = \delta_{\xi,\eta} \int d^3x \left\{ \frac{\Xi^{(+)}_{\Phi_a, x_a, s_a} (x) \gamma^0 \Xi^{(+)}_{\Phi_c, x_c, s_c} (x)}{\Xi^{(-)}_{\Phi_a, x_a, s_a} (x)} \right\} = (4.9a)
\]

\[
\Xi^{\eta}_{\{a/c\}} \Xi^{\xi}_{\{a/c\}} = \delta_{\xi,\eta} \hat{N}_{\Phi_a} \hat{N}_{\Phi_c} \left\{ \frac{U^{(+)}_{\Phi_a, s_a, \alpha} (x_a) (0) \psi_\alpha (z_a^+) \overline{\psi}_\beta (z_c) | 0 \rangle U^{(+)}_{\Phi_c, s_c, \beta} (x_c) \langle 0 |}{U^{(-)}_{\Phi_a, s_a, \alpha} (x_a) (0) \overline{\psi}_\beta (z_a^+) \psi_\alpha (z_c) | 0 \rangle U^{(-)}_{\Phi_c, s_c, \beta} (x_c) \langle 0 |} \right\} = (4.9b)
\]

\[
= \delta_{\xi,\eta} \hat{N}_{\Phi_a} \hat{N}_{\Phi_c} \left\{ \frac{U^{(+)}_{\Phi_a, s_a, \alpha} (x_a)}{U^{(-)}_{\Phi_c, s_c, \alpha} (x_c)} \right\} \frac{1}{i} S^{-\xi} \left( \xi (z_a^+ - z_c) \right) \left\{ \frac{U^{(+)}_{\Phi_c, s_c, \beta} (x_c)}{U^{(-)}_{\Phi_c, s_c, \beta} (x_a)} \right\} = (4.9c)
\]

\[
= \int d^3x \Xi^{\eta}_{\{a/c\}} (x) \gamma^0 \int d^3y \Xi^{\xi}_{\{a/c\}} (y) \gamma^0 \Xi^{\xi}_{\{a/c\}} (y), \quad \forall x_a^0, x_c^0, \quad \xi (x^0 - y^0) > 0. \quad (4.9d)
\]

The expressions (3.55c), (4.9c) are independent of the choice of $x^0, y^0$ in (3.55d), (4.9) respectively. However, according to (3.25) and Huygens’ principle, for $x_a^0 > x_c^0$ it may be considered as the projections (4.9d) of “final” packet “a/c” onto the result of causal evolution of the “initial” packet “c/a” according to causal propagator $S^c(x-y)$, where both these packets, due to (B.11), (4.8), have select natural causal sequence of events, which for both ($\pm$) cases is reduced to the same chosen one:

\[
\theta(x_a^0 - x_c^0) \theta(x_c^0 - y^0) \theta(y^0 - x_c^0) \mapsto \theta(x_a^0 - x_c^0), \quad \text{or} \quad \theta(x_a^0 - x_c^0) \theta(y^0 - x_c^0) \theta(x_a^0 - y^0) \mapsto \theta(x_a^0 - x_c^0),
\]

if the time-ordering $\theta$-functions for specific process are formally assigned to the packets itself [15].

From (4.9c) for $a = c$, $\xi = \eta$, by making use of (3.55b), (4.3), (4.4a), (3.28)–(3.30), (3.33), (3.34), with $m^2 \zeta^2 = \tau^2$, $(p_a s_a) = 0$, for normalization of the Fermi packet with spin $2s_a = \pm 1$, one has:

\[
A_2^a = \langle \{ \Phi_a, x_a, s_a \}; (\xi) | \langle \xi \rangle ; \{ \Phi_a, x_a, s_a \} \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{|\phi^\sigma (k, p_a) |^2}{2E_k} \left( \frac{\gamma k}{2m^2} \right)^2 \frac{1}{i} \left( \frac{\gamma k}{2m^2} \right)^2 \Xi^{\xi}_{\{a\}} (x_a) = (4.10)
\]

\[
\Xi^{\xi}_{\{a\}} (x_a) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{-2(kc_a (p_a, s_a))} \left( \frac{\gamma k}{2m^2} \right)^2 \Xi^{\xi}_{\{a\}} (k_a) \Xi^{\xi}_{\{a\}} (k_a)}{2m^2} = (4.11)
\]

\[
\Xi^{\xi}_{\{a\}} (x_a) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{-2(kc_a (p_a, s_a))} \left( \frac{\gamma k}{2m^2} \right)^2 \Xi^{\xi}_{\{a\}} (k_a) \Xi^{\xi}_{\{a\}} (k_a)}{2m^2} = (4.12)
\]

For the scalar packet (3.55b) all the square brackets here should be omitted. The absence of $\hat{s}_a$ - contribution indicates an independence of this result also of the spin averaging and suggests again to neglect $g_2$ in (3.17), whence $(p_a \zeta_a) \mapsto \tau$, and from definitions (3.34), (49) it follows:

\[
A_2^a \big\vert_{s = \pm 1} \overset{N_2^a (\tau)}{m^2 4\pi} h(4\tau^2) = \overset{N_2^a (\tau)}{m^2 4\pi} K_1 (2\tau) + K_2 (2\tau) = \overset{N_2^a (\tau)}{m^2 4\pi} \left( \frac{K_1 (2\tau)}{4\tau} + \frac{K_2 (2\tau)}{4\tau} \right). \quad (4.13)
\]
This last normalization of Fermi packet, with definite spin, conforms with the first normalization of scalar packet from (3.55c) for $b = a$ and coincides with it for $\sigma \to 0$, $\tau \to \infty$ [49]: $A^2_\sigma \to A^2_0$. Although such consistency will be saved also in the case of general dependence (3.57), (3.58), we’ll emphasize, the form (3.28), (3.34)–(3.36) of the function $\phi^\sigma(k, p_a)$ is already uniquely defined for any spin by uniqueness condition of analytical continuation [21–23] of state vectors (3.31), (3.32), (4.5) and respective WFs [19–23] (cf. (B.15)) as a wave-packet functions (3.29), (4.6), by the limiting conditions (3.20), (3.21) and (3.26), (3.27), by described above transformation of (3.47) into (2.8) in nonrelativistic limit, and by the conditions of consistency (4.5e) with Fermi packet. Whereas the appearance of any polynomial $(kp_a)$ - dependence in (4.11) is fully regulated by the spin of wave packet, and actually takes place evidently [20, 26] for any spin. For the free real massive vector field $B^\mu(x) (S = 1)$, with the WF $D^\mu_{\nu}(x)$, defined by (B.14), and the state wave function with definite momentum and polarization $A^\mu_{k, \lambda}(x) = \epsilon^\mu_{(\lambda)}(k)f_k(x)$, the wave-packet state and its vector wave function are:

$$
|\{p_a, x_a, \lambda_a\}\rangle = N_{\sigma a}B_{\nu}(x_a + i\zeta_a)|0\rangle A^\nu_{\sigma p_a, \lambda_a}(x_a) = N_{\sigma a}B_{\nu}(z_a)|0\rangle A^\nu_{\sigma p_a, \lambda_a}(x_a),
$$

$$
F^\mu_{p_a, x_a, \lambda_a}(x) = (0|B^\nu(x)|\{p_a, x_a, \lambda_a\}\rangle = N_{\sigma a}(0|B^\nu(x)B_{\nu}(z_a + i\zeta_a)|0\rangle A^\nu_{\sigma p_a, \lambda_a}(x_a) =
$$

$$
= iN_{\sigma a}g^{\mu\nu}D_{\beta\nu}(x - z_a)A^\nu_{\sigma p_a, \lambda_a}(x_a), \quad \text{with: } F^\mu_{p_a, x_a, \lambda_a}(x) \to A^\mu_{\sigma p_a, \lambda_a}(x).
$$

This way of construction of wave packets by means of analytical continuation in coordinate space is differ from the one used in [1, 31] by its universality for any spin. On the other hand it ensures their wave functions (3.14), (4.6), (4.15) by correct tensor dimension and space is differ from the one used in [1, 31] by its universality for any spin. This way of construction of wave packets by means of analytical continuation in coordinate space is differ from the one used in [1, 31] by its universality for any spin. For the free real massive vector field $B^\mu(x) (S = 1)$, with the WF $D^\mu_{\nu}(x)$, defined by (B.14), and the state wave function with definite momentum and polarization $A^\mu_{k, \lambda}(x) = \epsilon^\mu_{(\lambda)}(k)f_k(x)$, the wave-packet state and its vector wave function are:

$$
|\{p_a, x_a, \lambda_a\}\rangle = N_{\sigma a}B_{\nu}(x_a + i\zeta_a)|0\rangle A^\nu_{\sigma p_a, \lambda_a}(x_a) = N_{\sigma a}B_{\nu}(z_a)|0\rangle A^\nu_{\sigma p_a, \lambda_a}(x_a),
$$

$$
F^\mu_{p_a, x_a, \lambda_a}(x) = (0|B^\nu(x)|\{p_a, x_a, \lambda_a\}\rangle = N_{\sigma a}(0|B^\nu(x)B_{\nu}(z_a + i\zeta_a)|0\rangle A^\nu_{\sigma p_a, \lambda_a}(x_a) =
$$

$$
= iN_{\sigma a}g^{\mu\nu}D_{\beta\nu}(x - z_a)A^\nu_{\sigma p_a, \lambda_a}(x_a), \quad \text{with: } F^\mu_{p_a, x_a, \lambda_a}(x) \to A^\mu_{\sigma p_a, \lambda_a}(x).
$$

This way of construction of wave packets by means of analytical continuation in coordinate space is differ from the one used in [1, 31] by its universality for any spin. On the other hand it ensures their wave functions (3.14), (4.6), (4.15) by correct tensor dimension and space is differ from the one used in [1, 31] by its universality for any spin. The remaining ambiguity of functions $g_{1, 2}(m, \sigma)$, with asymptotics (3.37), (3.38), concerns only to some inessential details of dependence of $\tau = \tau(m/\sigma)$, being absorbed to redefinition of width $\sigma$. While the remaining unavoidable ambiguity in a choice of dimensionless function $\delta(\tau)$ or $\mathcal{I}(\tau)$ in (3.35)–(3.36) imposes the general form of observables, being average of an arbitrary operator $\mathcal{O}$ over packet state, because it is canceled only in the ratio:

$$
\langle\mathcal{O}\rangle \equiv \frac{\langle\{p_a, x_a, s_a\}; \xi\rangle\mathcal{O}\{|\xi\};\{p_a, x_a, s_a\}\rangle}{A^2_\sigma}.
$$

For example, for the mass of scalar wave packet as average of operator $\mathcal{P}^2 = -\partial^2_x$, with $P_\mu = i\partial_\mu = i(\partial_0, \nabla_a)$, for $k^2 = p_0^2 = m_a^2$, from (4.17) following to (3.55), one has:

$$
\langle\mathcal{P}^2\rangle \equiv \frac{\langle\{p_a, x_a, \sigma\};|\mathcal{P}^2|\{p_a, x_a, \sigma\}\rangle}{\langle\{p_a, x_a, \sigma\};\{p_a, x_a, \sigma\}\rangle} = \frac{F_{p_a x_a}, (-\partial^2_x)F_{p_a x_a}}{F_{p_a x_a}, F_{p_a x_a}} \equiv \frac{1}{A^2_\sigma} \int \frac{d^3k}{(2\pi)^3} 2E_k |\phi^\sigma(k, p_a)|^2 = m_a^2,
$$

and exactly the same for Fermi wave packets (4.6) with the help of (4.10)–(4.12) and changing $A^2_\sigma \to A^2_0$ (4.13). Therefore any predictions for the averages of such kind with scalar wave packets will be the same as in [1, 2] where the choice $\mathcal{I}(\tau) \equiv 1$ was made.
The difference may arise for averaging of matrix operators with the spin wave packets. For the Fermi packet by using (4.17) or (4.4) and (B.1)–(B.5), (B.9):

\[
\langle \langle \gamma \mathcal{P} \rangle \rangle \equiv \frac{1}{A_2^2} \int \frac{d^3k}{(2\pi)^3 2E_k} |\phi^\sigma(k, \mathbf{p}_a)|^2 \left[ \mathcal{O}_{\mathbf{p}_a, s_a}(x_a) \gamma(k) + \xi m \gamma(k) U_{\mathbf{p}_a, s_a}(x_a) \right] = m_a. \tag{4.22}
\]

According to (3.20), (3.21), (3.22) and (3.24), (3.26), the meaning of quantum numbers of wave packet manifests only at respective limit. For any \( \mathcal{O}(P^\mu) \) by means of (3.33)–(3.35), (3.47)–(3.49), (4.13), for scalar packet follows:

\[
\langle \langle \mathcal{O}(P^\mu) \rangle \rangle \equiv \frac{(F_{\mathbf{p}_a x_a}, \mathcal{O}(i\partial^\mu) F_{\mathbf{p}_a x_a})}{(F_{\mathbf{p}_a x_a}, F_{\mathbf{p}_a x_a})} = \frac{N^2_\sigma}{A_2^2} \int \frac{d^3k}{(2\pi)^3 2E_k} e^{-2(k_\sigma(p_a, \sigma))} \rightarrow \mathcal{O}(p^\mu), \tag{4.23}
\]

where:

\[\frac{N^2_\sigma}{A_2^2} e^{-2(k_\sigma(p_a, \sigma))} \rightarrow 2m(2\pi)^3 e^{g_1(E_k - E_p)^2} \left\{ \left( \frac{g_1}{\pi} \right)^{3/2} e^{-g_1(k-p)^2} \right\} \rightarrow \frac{(2\pi)^3 2E_k \delta(k - p)}{g_1 \rightarrow \infty}, \tag{4.24}\]

because this differ from (3.47) only by replacement \( g_1 \rightarrow 2g_1 \), and the same takes place for Fermi wave packets with formal changing \( A_2^2 \rightarrow A_3^2 \) (4.13). For Pauli-Lubanski operator \( \mathcal{W}^\mu \) and any space-like vector \( \mathbf{S} \), similarly (4.10)–(4.12) by the use of (B.6)–(B.8) for upper sign of \( 2s_a = 1 \), with \( \mathcal{O} = -(\mathcal{W}\mathbf{S})/m_a \) one has:

\[
4(\mathcal{W}\mathbf{S}) \equiv 4\mathcal{W}^\mu S_\mu = -\gamma^5 \left[ \gamma^\mu, \gamma^\nu \right] S_\mu P_\nu = -2\gamma^5 \left\{ (\gamma \mathcal{S})(\gamma \mathcal{P}) - (\mathcal{S} \mathcal{P}) \right\}, \tag{4.25}
\]

\[
\left\langle \left\langle \frac{-(\mathcal{W}\mathbf{S})}{m_a} \right\rangle \right\rangle = \frac{1}{2} \left\langle \left\langle \frac{\gamma^5}{m_a} [(\gamma \mathcal{S})(\gamma \mathcal{P}) - (\mathcal{S} \mathcal{P})] \right\rangle \right\rangle = \frac{1}{2} \left( \hat{s}_a \mathbf{S} \right) + \frac{1}{2} N^2_\sigma \int \frac{d^3k e^{-2(k_\sigma)} (2\pi)^3 2k^0}{2m_a^2} \left[ \frac{(k_\sigma)(k \mathcal{S}) + (k \sigma)(p_a \mathcal{S})}{2m_a^2} \right]. \tag{4.26}
\]

For the pure polarized state with \( \hat{s}_a^2 = -1 \), \( p_a \mathbf{S} = 0 \), if \( \mathcal{S} \rightarrow \hat{s}_a \), then (comp. (4.4a)):

\[
\left\langle \left\langle \frac{-(\mathcal{W}\mathbf{S})}{m_a} \right\rangle \right\rangle \rightarrow \frac{1}{2} + \frac{1}{2} N^2_\sigma \int \frac{d^3k e^{-2(k_\sigma)} (k \sigma)^2}{(2\pi)^3 2k^0} \frac{1}{2m_a^2} \rightarrow \frac{1}{2}. \tag{4.28}
\]

In the plane-wave limit (4.24) the last summands in (4.27), (4.28) obviously disappear as it should. For the states with mixed polarization, \( -1 < \hat{s}_a^2 < 0 \), that are correlated with \( \mathcal{S} \) by the rule:

\[
\hat{s}_a^2 \mathcal{S}^\sigma = \frac{g^{\mu\nu}}{4} (s_a \mathcal{S}), \quad \text{one finds:} \quad \left\langle \left\langle \frac{-(\mathcal{W}\mathbf{S})}{m_a} \right\rangle \right\rangle = -\frac{1}{2} (\hat{s}_a \mathcal{S})^{\frac{3}{4}}. \tag{4.29}
\]

5 Neutrino oscillations. Intermediate wave packets.

Spin degrees of freedom are potentially important [31] in the intermediate wave packet approach to the neutrino oscillations problem [32]. Respective generalization of the method of Blasone [44–46] onto above Fermi wave packets by the use of respectively generalized
calculations leads to the following expression for two-flavor oscillations of leptonic charge for the electronic neutrino $\nu_e$ (with redefinition in (4.1) $b_{k,r}^{(+)} = a_{k,r}$, $b_{k,r}^{(-)} = b_{k,r}$):

$$Q_e(t) = G_{\vartheta}^{-1}(t)Q_1 G_{\vartheta}(t) = \int \frac{d^3k}{(2\pi)^3 2E_{k,1}} \sum_{r = \pm 1/2} \left[ a_{k,r}^{\dagger}(t) a_{k,r}^{\dagger}(t) - b_{-k,r}^{\dagger}(t) b_{-k,r}^{\dagger}(t) \right],$$  \hspace{1cm} (5.1)

$$\langle Q_e(t) \rangle_e = \frac{\langle \nu_e \{ P_1, x_1, s_1 \} | Q_e(t) | \nu_e \{ P_1, x_1, s_1 \} \rangle}{A_{\vartheta e}} = \frac{N_{\vartheta e}^2}{A_{\vartheta e}^2} \int \frac{d^3k e^{-2(k_1 E_{k,1})}}{(2\pi)^3 2E_{k,1}} \left[ \frac{1}{2} + \frac{(kp_1)}{2m_1^2} \right] Q_{k,e}(t), \text{ with: }$$

$$Q_{k,e}(t) = 1 - \sin^2 2\vartheta \left[ |U_k|^2 \sin^2 \left( \frac{E_{k,1} - E_{k,2} t}{2} \right) + |V_k|^2 \sin^2 \left( \frac{E_{k,1} + E_{k,2} t}{2} \right) \right],$$ \hspace{1cm} (5.2)

where: $|U_k|^2 + |V_k|^2 = 1$, with: $|U_k| = \frac{|k|^2 + (E_{k,1} + m_1)(E_{k,2} + m_2)}{2\sqrt{E_{k,1}E_{k,2}(E_{k,1} + m_1)(E_{k,2} + m_2)}}$, and:

$$|V_k| = \frac{(E_{k,2} + m_2) - (E_{k,1} + m_1)}{2\sqrt{E_{k,1}E_{k,2}(E_{k,1} + m_1)(E_{k,2} + m_2)}} |k|,$$ \hspace{1cm} (5.3)

$$\lambda_{\vartheta e} = \langle \nu_e \{ P_1, x_1, s_1 \} | \nu_{\varrho} \{ P_1, x_1, s_1 \} \rangle = \langle \nu_1 \{ P_1, x_1, s_1 \} | \nu_{1} \{ P_1, x_1, s_1 \} \rangle = A_{\vartheta e}^2, \text{ for: }$$

$$\epsilon_{k,r}^{\dagger}(x) = G_{\vartheta}^{-1}(x) \epsilon_{k,r}^{\dagger}, G_{\vartheta}(x), \text{ and: } |\nu_e \{ P_1, x_1, s_1 \} \rangle = G_{\vartheta}^{-1}(x) |\nu_1 \{ P_1, x_1, s_1 \} \rangle,$$ \hspace{1cm} (5.4)

is the wave-packet state with definite electronic flavor $e$, defined according to Ref. [45] by the same improper transformation [44–46] (5.1), (5.6) of the above packet states (4.5) for initial massive field (4.1), (4.2) $\psi(x_1) = \nu_1(x_1)$ with mass $m_1$ at the same instant of time $x^0 = x^0_1$, and normalized by the same constant (4.12). The conserved charge $Q_1$ [26, 46] is defined by expression (5.1) for $\vartheta = 0$ via operators $a_{k,r}^{\dagger}, b_{k,r}^{\dagger}$ of the field (4.1). Notice the time $t$ of operator $Q(t)$ (5.1) has nothing to do with the local packet time $x^0_a = x^0_1$ in (5.2), (5.6), is automatically canceled in Eq. (5.3). Finally for each spin $s_1$, from (5.3), (5.4) one has:

$$\langle Q_e(t) \rangle_e = 1 + \frac{N_{\vartheta e}^2}{A_{\vartheta e}^2} \int \frac{d^3k e^{-2(k_1 E_{k,1})}}{(2\pi)^3 2E_{k,1}} \left[ \frac{1}{2} + \frac{(kp_1)}{2m_1^2} \right] [Q_{k,e}(t) - 1],$$ \hspace{1cm} (5.5)

what due to (4.24) surely reproduces the plane-wave limit $\langle Q_e(t) \rangle_e \underset{\vartheta \to 0}{\longrightarrow} Q_{p_e} e(t)$ and the other cases of [44–46], and after time averaging of (5.4) yields to the well known result: $\langle Q_e(t) \rangle_e = 1 - (1/2) \sin^2 2\vartheta$, where the mixing angle $\vartheta$ below defines the mixing matrix $U$.

It should be stressed, since the first relation in (5.6) takes place only in a weak sense [44–46], according to the second relation (5.6) it generates wave packets of flavor states for $\{\ell, j\} = \{e, 1\} \text{ or } \{\mu, 2\}$, with $\zeta_{aj}(p_{aj}, \sigma_j) = p_{aj} g_1(m_j, \sigma_j) + \bar{w}_{aj} g_2(m_j, \sigma_j)$, $p^0_{aj} = E_{p_{a,j}} = +\sqrt{p^2_{a,j} + m^2_j}$ only over respective flavor vacuum $|0(x^0)\rangle_{e,\mu}$ by making use of the improper transformation $G_{\vartheta}(x^0)$ of operators and states at the same instant $x^0_a$ with $a^\# = a, a^\dagger, b^\# = $
b, \bar{b}:

\begin{equation}
G_{\theta}(x^0) = \exp \left[ \theta \int d^3x \left( \nu_1^\dagger(x)\nu_2(x) - \nu_1^\dagger(x)\nu_1(x) \right) \right], \quad U = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}, \quad (5.8)
\end{equation}

\begin{equation}
\left( a_{k,r}^\#(x^0), b_{k,r}^\#(x^0) \right) = G_{\theta}^{-1}(x^0) \left( a_{k,r}^\#(0), b_{k,r}^\#(0) \right) G_{\theta}(x^0), \quad |0(x^0)\rangle_{e,\mu} = G_{\theta}^{-1}(x^0)|0\rangle_{1,2}, \quad (5.9)
\end{equation}

with the help of:

\begin{equation}
\begin{aligned}
\langle \bar{\nu}_k^+(x_a; \zeta_{\alpha j}) | 0(x_b^0) \rangle_{e,\mu} &= G_{\theta}^{-1}(x_a^0) \bar{\nu}_j(x_a + i\zeta_{\alpha j}) |0\rangle_{1,2}, \quad \text{where:} \\
\langle \bar{\nu}_k^-(x_a; \zeta_{\alpha j}) | 0(x_b^0) \rangle_{e,\mu} &= G_{\theta}^{-1}(x_a^0) \nu_j(x_a + i\zeta_{\alpha j}) |0\rangle_{1,2},
\end{aligned} \quad (5.10)
\end{equation}

\begin{equation}
\begin{aligned}
\bar{\nu}_k^+(x_a; \zeta_{\alpha j}) &\langle \nu_k^-(x_a; \zeta_{\alpha j}) \rangle = \sum_{r = \pm 1/2} \frac{d^3k}{(2\pi)^3} \int \frac{d^3k e^{i(kx_a) - (k\zeta_{\alpha j})}}{2E_{k,j}} \left\{ \begin{array}{c}
\bar{\nu}_k^+(k, r) \nu_k^+(x_a) \\
\nu_k^-(k, r) b_{k,r}^\#(x_a)
\end{array} \right\}, \quad k^0 = E_{k,j}, \quad (5.11)
\end{aligned}
\end{equation}

Here the Eq. (5.11) only for $\zeta_{\alpha j} = 0$ recovers a “negative frequency parts” of local field operators with definite flavor $\nu_\ell(x_a)$ as a mix of massive fields at the same point $x_a$ [44]:

\begin{equation}
\begin{aligned}
\bar{\nu}_\ell(x_a) &= \bar{\nu}_\ell^+(x_a; 0) + \nu_\ell^-(x_a; 0) = G_{\theta}^{-1}(x_a^0) \bar{\nu}_j(x_a) G_{\theta}(x_a^0) = \sum_n U_{\ell n} \nu_n(x_a), \quad (5.12) \\
\nu_\ell(x_a) &= G_{\theta}^{-1}(x_a^0) \nu_j(x_a) G_{\theta}(x_a^0) = \sum_n U_{\ell n} \nu_n(x_a), \quad j, n = 1, 2, \quad (5.13)
\end{aligned}
\end{equation}

where $\equiv$ means equality in a weak sense. However the conventional mixing relations for the states follow from there only by neglecting the difference (5.9) between the flavor and massive vacua [44, 45]: $|0(0)\rangle_{e,\mu} \rightarrow |0\rangle_{1,2}$ thus ignoring the weak sense of (5.12), (5.13), and strictly speaking, only for the states with definite momentum $|k\rangle$, implying the plane-wave limit (3.21), (3.49) of states (4.5), (5.6), (5.10), (5.11), when $\sigma_j \rightarrow 0$, $\zeta_{\alpha j} \rightarrow \infty$. Corresponding mixing relations for the states of wave packets with definite mass $m_j$ require Lorentz invariant profile functions $\phi^\theta(k, p_a)$ to be independent of $j$ [31, 46], what, as explained in Introduction, is impossible for the interpolating wave-packets states (3.28), (4.6) and gives the another reason, why the conventional mixing relations for wave-packet states [31, 32]:

\begin{equation}
|\nu_\ell\{p_a, x_a, s_a\}\rangle \equiv \sum_n U_{\ell n} \nu_n\{p_a, x_a, s_a\}, \quad (5.14)
\end{equation}

are inconsistent with general principles of QFT [44] already for two-flavor case. However, at the same time, for $\{\ell, j\} = \{e, 1\}$, $\{\mu, 2\}$:

\begin{equation}
|\nu_\ell\{p_a, x_a, s_a\}\rangle = \hat{N}_{\sigma j}^\xi \nu_\ell^+(x_a; \zeta_{\alpha j}) |0(x_a)\rangle_{e,\mu} = G_{\theta}^{-1}(x_a^0) \hat{N}_{\sigma j}^\xi \bar{\nu}_j(x_a + i\zeta_{\alpha j}) |0\rangle_{1,2}.
\end{equation}

Eventually only Eqs. (5.6), (5.10), (5.11), (5.14), with the expressions (4.5) for $\psi(z_a)$ $\rightarrow$ $\nu_j(z_a)$, are meaningful for the states with flavor wave packet in the approach with intermediate wave packets.

6 Neutrino oscillations. Diagrammatic treatment.

The method of “macroscopic Feynman diagrams” [1–8, 33–41] describes oscillation as a scattering of initial particles in neutrino source on the initial particles in neutrino detector and, unlike the previous approach, uses the wave packets only for the external particles relatively to the intermediate massive t-channel neutrino. Thus, the weak sense of the Eqs. (5.12), (5.13) as well as difference between the vacua (5.9) are ignored again because these
flavor fields have not appearing at all. This method also emphasizes the suggested form
of wave packet (3.14), (3.29), (4.6) and (4.16). According to [34, 35] the wave packet of
intermediate neutrino of previous approach arises in the expression of amplitude of such a
tree macro-diagram:

\[ A_{DC}^{ij} = \int d^4x \, \Psi_{DF}^*(x) \Psi_{DI}(x) \tilde{M}_{jD} \int d^4y \, \frac{1}{i} \, S_{j}^{-\xi}(x-y) \, \tilde{M}_{jC} \, \Psi_{CF}^*(y) \Psi_{CI}(y) = \sum_{\xi=\pm} A_{DC}^{ij(\xi)}, \] (6.1)

by making use of the “pole integration”, which for \(|x^0 - y^0| \gg 1/m_j\) is equivalent
to replacement of the causal propagator \(S_{j}^{-\xi}(x-y)\) onto its relevant on-shell frequency part
\(S_{j}^{-\xi}(x-y)\) of its Lorentz invariant time-ordered decomposition (B.11), (B.22) into the WFs
(4.7), (B.9) (or (3.30) for scalar case) with \(\xi = +\) for neutrino, \(\xi = -\) for antineutrino:

\[ A_{DC}^{ij} = \int d^4x \, \Psi_{DF}^*(x) \Psi_{DI}(x) \tilde{M}_{jD} \int d^4y \, \theta(\xi(x^0 - y^0)) \, \frac{1}{i} \, S_{j}^{-\xi}(x-y) \, \tilde{M}_{jC} \, \Psi_{CF}^*(y) \Psi_{CI}(y). \] (6.2)

This differs from Eq. (4.9d) with the same replacement, due to Eq. (4.8) exactly applicable
therein, only by changing of three-dimensional integrals onto the four-dimensional ones and
by replacement of the “source”, being factor with only one wave packet of free intermediate
neutrino \(\gamma^0 \, \Xi_{(c/a)j}^\xi(y)\), onto the source as a product of respective vertex \(\tilde{M}_{jC}\) or \(\tilde{M}_{jD}\) with
such wave packets \(\Psi_{CF}(y)\) or \(\Psi_{DI}(x)\) for incoming and with \(\Psi_{CF}(y)\) or \(\Psi_{DF}(x)\) for outgoing
particles [1, 2, 33–35] at respective neutrino creation 4-point \(y\) or neutrino detection 4-point
\(x\), but vice versa \(x \rightleftharpoons y\) for the antineutrino with simultaneous interchange of indexes
\(D \rightleftharpoons C\) [16, 17]. This conforms also with discussion of Huygens’ principle after Eq. (4.9d),
because in agreement with [13] and Eqs. (6.49), (6.50) of [25], from (B.11) and the solution
to Cauchy problem:

\[ \Xi_{(c/a)j}^\xi(x) \, \delta_{\xi\eta} = \frac{1}{i} \int d^3y \, S_{j}^{-\eta}(x-y) \, \gamma^0 \, \Xi_{(c/a)j}^\xi(y), \text{ for any } x^0, y^0, \] (6.3a)

it follows:

\[ \theta(\xi(x^0 - y^0)) \Xi_{(c/a)j}^\xi(x) = \frac{1}{i} \int d^3y \, S_{j}^{-\xi}(x-y) \, \gamma^0 \, \Xi_{(c/a)j}^\xi(y) = \] (6.3b)

\[ = \theta(\xi(x^0 - y^0)) \frac{1}{i} \int d^3y \, S_{j}^{-\xi}(x-y) \, \gamma^0 \, \Xi_{(c/a)j}^\xi(y), \text{ or with: } d^3y \gamma^0 \rightarrow d\Sigma(y) \gamma^\mu. \] (6.3c)

So, the group property (4.8) of WF enable to recast amplitude (6.2) into the form of Lorentz
-invariant scalar product similar to (4.9), for any \(\xi \, q^0 \in (\xi \, x^0, \xi \, y^0)\) or infinite space-like
hypersurface \(\Sigma(q), q^\mu = (q^0, \rho)\) with element \(d\Sigma_{\mu}(q)\), \(d^3y = dy^\mu d\Sigma_{\mu}(y)\), and with indexes
\(\{C/D\}\), that are not correlated at all with indexes \(\xi = \pm\) for the off shell composite wave
functions:

\[ \hat{\Gamma}_{(C/D)j}^{\xi}(\rho) = \int d^2y \, \theta(\xi(q^0 - y^0)) \frac{1}{i} \, S_{j}^{-\xi}(\rho - y) \tilde{M}_{j(C/D)} \Psi_{(C/D)F}^*(\rho) \Psi_{(C/D)I}(\rho), \] (6.4)

\[ \hat{\Gamma}_{(D/C)j}^{\xi}(\rho) = \int d^2x \, \Psi_{(D/C)I}(x) \Psi_{(D/C)F}(x) \tilde{M}_{j(D/C)} \theta(\xi(x^0 - \rho^0)) \frac{1}{i} \, S_{j}^{-\xi}(x - \rho), \] (6.5)

\[ A_{DC}^{ij(\xi)} \, \delta_{\xi\eta} = \xi \, \int d^3\rho \, \hat{\Gamma}_{(D/C)j}^{\xi}(\rho) \, \gamma^\mu \, \hat{\Gamma}_{(C/D)j}^{\xi}(\rho) = \xi \, \int d\Sigma_{\mu}(\rho) \, \hat{\Gamma}_{(D/C)j}^{\xi}(\rho) \, \gamma^\mu \, \hat{\Gamma}_{(C/D)j}^{\xi}(\rho), \] (6.6)

where: \(\theta(\xi(q^0 - y^0)) \rightarrow \theta(\xi(n\Sigma(q - y)))\), for: \(d^3\rho \rightarrow d\Sigma(\rho) = n_\Sigma d\Sigma, \quad n_\Sigma^2 = 1, \)
with: \(\theta(t) \theta(-t) = 0, \quad \theta^2(t) = \theta(t), \text{ so: } \theta(\xi(q^0 - \rho^0)) \theta(\xi(q^0 - y^0)) \rightleftharpoons \theta(\xi(x^0 - y^0)), \) (6.7)
at least for \( \xi g^0 = \xi (x^0 + y^0) / 2 \in (\xi x^0, \xi y^0) \), and thus for all admissible \( g^0 \) due to Lorentz invariance of Eqs. (6.6), (6.8). Essentially different approximate qualitative way to transform the product of external wave packets into the scalar analog (B.35) of neutrino composite wave functions (6.4), (6.5) was shown previously in [35]. The Lorentz covariance of definitions (6.4)–(6.8) in fact with arbitrary \( g^0 \) and time-like \( n^\mu \), immediately resolves the both problems with causality and with covariant equal time prescription, that were pointed out in paragraph 5.2 of Ref. [34]. Eventually it means that the transformation of amplitude (6.1) into the scalar product similar (4.9a) implies the reduction:

\[ A^I_{DC} \mapsto A^I_{DC}^{(\xi)}, \quad \hat{\Psi}^{(\xi)}_{(C/D)j}(\xi) \mapsto \Xi^{(\xi)}_{(C/D)j}(\xi), \quad \hat{\Psi}^{(\xi)}_{(D/C)j}(\xi) \mapsto \Xi^{(\xi)}_{(D/C)j}(\xi), \quad (6.9) \]

into the on shell wave functions with \( \xi = + \) for neutrino and \( \xi = - \) for antineutrino, hat now become correlated with indexes \( \{C\} \), what effectively looks like the reduction of the sources:

\[ \theta(\xi (q^0 - y^0)) \hat{M}^{(C/D)}_{(y)} \Psi^{(C/D)}_{(y)}(y) \Psi_{(C/D)}^{(y)}(y) \mapsto \delta(q^0 - y^0) \Xi^{(C/D)}_{(y)}, \quad (6.10) \]

\[ \theta(\xi (q^0 - y^0)) \Psi^{(C/D)}_{(x)}(x) \Psi_{(C/D)}^{(x)}(x) \mapsto \delta(q^0 - y^0) \Xi^{(C/D)}_{(x)}, \quad (6.11) \]

In accordance with above discussions of Huygens’ principle and Eq. (4.9d), this means the wave functions (6.4), (6.5) become again solutions to free Dirac Eqs. (4.17) only after omitting the time-ordering \( \theta \)-functions: \( \hat{\Psi} \mapsto \Psi \). This is well justified for amplitudes (6.1), (6.2) only when the sources in the l.h.s. of Eqs. (6.10), (6.11) are well localized in time near the fixed points \( Y^0_{(C/D)}, X^0_{(D/C)} \) with \( \xi Y^0_{(C/D)} \ll \xi q^0 \ll \xi X^0_{(D/C)} \) for each macro-diagram (6.1). The approximate reduction like (6.9) reproducing (6.10), (6.11) also without the temporal factors was shown previously in [32, 34] for non relativistic Gaussian profile (2.8) of \( \phi^0(k, p_n) \) with \( x_n = 0 \) (follows from (3.28), as is shown above, only at \( c \to \infty \) for all external wave packets \( \Psi^{(C/D)}_{(x)} \)). In spite of they remain to be exact solutions to free relativistic equations of motion (4.17), they propagate with inevitable frequency mixing and usual Gaussian spreading [4, 25, 26, 31].

### 6.1 Composite wave function in QFT

For the decay vertexes \( \pi^+ \to \mu^+ \nu_\mu \) and \( \pi^- \to \mu^- \bar{\nu}_\mu \) (and similarly for the case \( \pi^+ \to e^+ \nu_e \) or \( \pi^- \to e^- \bar{\nu}_e \)) the wave functions (6.4), (6.5) of creation process at point \( \{C\} \) become as:

\[ \hat{\Psi}^{(+)}_{(C/Dj)}(\xi) = \frac{1}{i} \int d^4 y \theta(q^0 - y^0) S_{j}^{-}(q - y) \hat{M}_{jC} \Xi^{(-)}_{p_n, Y_{\mu, s_n}}(y) F^{(+)}_{p_n, Y_{\mu}}(y), \text{ for } \nu_\mu(e)(\nu_j), \quad (6.12) \]

\[ \hat{\Psi}^{(-)}_{(C/Dj)}(\xi) = \frac{1}{i} \int d^4 x F_{p_n, Y_{\mu}}^{-}(x) \Xi^{(+)}_{p_n, Y_{\mu, s_n}}(x) \hat{M}_{jC} \theta(q^0 - x^0) S_{j}^{+}(x - q), \text{ for } \bar{\nu}_\mu(e)(\bar{\nu}_j), \quad (6.13) \]

where, according to definitions of scalar wave packets (3.14), (3.28)–(3.30) with their scalar products (3.55), (B.23), and due to the absence of Dirac sea [18–26] for (pseudo) scalar charged pions \( \pi^\pm \):

\[ e^{(p_n Y_n)} F^{\pi^\pm}_{p_n, Y_n}(y) = (\mp i) N_{\sigma \pi} D_{m^\pm}(\pm(y - Y_\pi - i\xi_\pi)) = -i N_{\sigma \pi} D_{m^\pm}(y - Y_\pi - i\xi_\pi). \quad (6.14) \]
that is explicitly calculated by Fourier transformation, with

\[ \theta(t) \]

Then, by dropping temporal factors \( \theta(t) \) and using \( y \)-independent matrix \( \widetilde{M}_{JC} \), these candidates into the free (anti-) neutrino wave packets recast into the following solutions of free Dirac equations (4.17), \( \tilde{Y} \rightarrow Y \):

\[
\begin{align*}
Y_{\{C\}j}^{(+)}(q) \
\tilde{Y}_{\{C\}j}^{(-)}(q)
\end{align*}
\]

\[
e^{i(p_\mu y_\nu - i(p_\mu y_\nu)} \left\{ \left[ i(\gamma \partial_\mu) + m_j \right] \widetilde{M}_{JC} \left[ m_\mu - i(\gamma \partial_\nu) \right] u(-) \left( p_\mu, s_\mu \right) \right\} \frac{G_{\{C\}j}(q)}{2m_\mu},
\]

\[ G_{\{C\}j}(q) = i N_{\sigma_\mu} N_{\sigma_\pi} \int d^4 \! y \, D_{m_j}(-q - y) D_{m_\mu}^{-1}(Y_\mu - y - i\zeta_\mu) D_{m_\pi}^{-1}(y - Y_\pi - i\zeta_\pi), \]

that is explicitly calculated by Fourier transformation, with \( Z_\pi = Y_\pi + i\zeta_\pi, Z_\mu = Y_\mu + i\zeta_\mu; \)

\[
G_{\{C\}j}(q) = \frac{2\pi i}{(2\pi)^4} e^{-i(q\eta)} \theta(q^0) \delta(q^2 - m_j^2) \tilde{V}_{\{C\}}(q), \quad \text{where, } \forall.q:
\]

\[
\tilde{V}_{\{C\}}(q) = i N_{\sigma_\mu} N_{\sigma_\pi} \int d^4 \! y \, e^{i(q\eta)} D_{m_\mu}^{-1}(Z_\mu^0 - y) D_{m_\mu}^{-1}(y - Z_\pi), \quad \text{or more generally:}
\]

\[
\tilde{V}_{\{C/D\}}(q) = (-i) \int d^4 \! y \, e^{+i(q\eta)} \psi_{\sigma_\rho_\pi}(p_\delta F, Y_{\delta F[C/D]} - y) \psi_{\sigma_\rho_\pi}(p_\delta F, Y_{\delta F[C/D]} - y),
\]

are the “overlap functions” [1, 34] of creation/annihilation processes. For the independent plane-wave limit (3.19) of packet functions (3.29) with \( \sigma_\pi, \sigma_\mu \rightarrow 0 \) they keep energy-momentum conservation for 4-vector \( k \equiv p_\pi - p_\mu \), giving \( \forall.q; \)

\[
\tilde{V}_{\{C\}}(q) \longrightarrow (-i)(2\pi)^4 e^{i(p_\mu y_\nu - i(p_\mu y_\nu)} \delta_4(q - k),
\]

with the function: \( G_{\{C\}j}(q) \rightarrow 2\pi e^{i(p_\mu y_\nu - i(p_\mu y_\nu)} e^{-i(k\eta)} \theta(k^0) \delta(k^2 - m_j^2), \)

and the wave functions:

\[
\begin{align*}
Y_{\{C\}j}^{(+)}(q) \
\tilde{Y}_{\{C\}j}^{(-)}(q)
\end{align*}
\]

\[
\rightarrow 2\pi e^{-i(k\eta)} \theta(k^0) \delta(k^2 - m_j^2) \sum_{s = \pm 1/2} \left\{ u_{j_s}^{(+)}(k, s) \left( \begin{array}{c} \tilde{u}_{j_s}^{(+)}(k, s) \tilde{M}_{JC} \tilde{u}_{j_s}^{(-)}(p_\mu, s_\mu) \end{array} \right) \right\}.
\]

The matrix \( \widetilde{M}_{JC} \) can eliminate the sum over \( s \) selecting here only one value of the (anti-) neutrino spin [1, 34]. Thus the plane-wave limit for external packets automatically leads to plane wave for internal packet on the respective mass shell. The explicit value (C.21)–(C.23), for \( \Delta = \Delta(q) \), reads:

\[
\tilde{V}_{\{C\}}(q) = \frac{N_{\sigma_\mu} N_{\sigma_\pi}}{2\pi \Delta} \exp \{ i(qZ_\eta) \} \left\{ \frac{\theta(\Delta W)}{W} \exp (\chi_\Delta W) - \theta(q^0) \exp (-\varepsilon(q^0)\chi_\Delta W) \right\},
\]

\[
\chi_\Delta = \frac{\Delta^{1/2}(q)}{2q^2}, \quad \chi_0 = \frac{m_\pi^2 - m_\mu^2}{2q^2}, \quad \eta_\pi = \frac{1}{2} \pm \chi_0, \quad Z_\eta = \eta_2 Z_\pi + \eta_1 Z_{\mu}^* = Y_\eta + i\zeta_\eta,
\]

\[
\begin{cases}
Y_\eta = \eta_2 Y_\pi + \eta_1 Y_\mu \\
\zeta_\eta = \eta_2 \zeta_\pi - \eta_1 \zeta_\mu
\end{cases}, \quad Z_\eta = Z_\pi - Z_{\mu}^* = Y_\eta + i\zeta_\eta, \quad \begin{cases}
Y_{\pm} = Y_\pi \pm Y_\mu \\
\zeta_{\pm} = \zeta_\pi \pm \zeta_\mu
\end{cases}, \quad W = w^{1/2},
\]

\[
\Delta(q) = [(m_\pi + m_\mu)^2 - q^2] \left[ (m_\pi - m_\mu)^2 - q^2 \right] > 0, \quad \text{for } q^2 \rightarrow m_j^2 \ll m_\mu^2 < m_\pi^2.
\]
\[ \chi_0 \simeq 7/2, \chi_\Delta \to 0, \text{ for } \sqrt{q^2} \to m_\pi - m_\mu, \text{ but } \chi_0 \simeq 10^{17}, \chi_0 - \chi_\Delta \simeq 1, 79 \text{ for } \sqrt{q^2} = 0, 2 \text{ eV}. \]

Calculations of (6.23) with \( W(w) = \left[ q^2 Z^2 - (q Z) - q^2 \right]^{1/2} \), for \( q^2 \leq (m_\pi - m_\mu)^2 \), are given in appendix C. In appendix D it is shown how the plane-wave limit (6.20) of Eq. (6.23) is realized without any addition constraints onto the universal functions \( I(\tau_a) \), \( g_1(m_\alpha, \sigma_\alpha) \), \( a = \pi, \mu \).

Note, that unlike the common belief \([34, 35]\) the overlap functions (6.19), (6.23) explicitly depend on the centers of initial and final wave packets, and thus from the effective points \( X_{(D/C)} \), \( Y_{(C/D)} \) of (anti-) neutrino creation/detection, that in turn, define eventually the time \( T = \xi \left( X_{(D/C)}^0 - Y_{(C/D)}^0 \right) \) and the length \( L = \xi \left( X_{(D/C)} - Y_{(C/D)} \right) \) of oscillations \([1-7]\). Due to opposite signs of \( \gamma \) for initial and final wave packets the properties of Eq. (6.18) are essentially different from the ones of usual two-particle phase volume \([19, 48]\) (cf. appendix C). Similarly \([1]\) its extreme properties define the points \( X_{(D/C)} \), \( Y_{(C/D)} \) (cf. appendix D). This becomes transparent from the approximate expression (D.14) in fact representing the effective narrow wave-packet of intermediate (anti-) neutrino, when external packets are close to plane waves. From this approximation it is clear, that up to the small corrections of order \( O(q - k) \) and \( O(g_{1\alpha}^{-1}) \) one can replace the Eqs. (6.15)–(6.19) on the following simple ones:

\[
\begin{align*}
Y_{(C)j}^{(+)}(q) & \rightarrow \pm \left\{ m_j + i(\gamma \partial_q) \right\} \frac{M_j C u^(-)}{m_j - i(\gamma \partial_q)} \frac{F_{(C)j}(q)}{\left| F_{(C)j}(q) \right|^2}, \\
Y_{(C)j}^{(-)}(q) & \rightarrow \left\{ m_j + i(\gamma \partial_q) \right\} \frac{M_j C u^(-)}{m_j - i(\gamma \partial_q)} \frac{F_{(C)j}(q)}{\left| F_{(C)j}(q) \right|^2}, \\
\text{with: } & \quad F_{(C)j}(q) = e^{i(p_a Y_{(D)}) - i(p_a Y_{(C)})} G_{(C)j}(q) \equiv e^{-i\Phi(k)} G_{(C)j}(q), \\
\Phi(k) & \equiv \Phi_{(C)j}(k), \\
F_{(C)j}(q) & = \int \frac{d^4 q}{(2\pi)^4} e^{-i(q \theta)} 2\pi i \theta(q^0) \delta(q^2 - m_j^2) \hat{V}_{(C)}(q) \equiv \int \frac{d^4 q}{(2\pi)^4} e^{-i(q \theta)} H_{(C)j}(q),
\end{align*}
\]

where, \( \forall q : \hat{V}_{(C)}(q) = e^{i(p_a Y_{(D)}) - i(p_a Y_{(C)})} \hat{V}_{(C)}(q) \equiv e^{-i\Phi(k)} \hat{V}_{(C)}(q) \)

is the “reduced overlap function” approximately defined for \( q^2 \geq 0 \) by Eq. (D.14) as:

\[
\hat{V}_{(C)}(q) \approx (2\pi)^4 \frac{\Theta(\Delta)}{i} \hat{\delta}_{(C)}(q - k) e^{i(q - k)Y_{(C)}} e^{\Sigma_k(Y_{-})} \hat{\delta}_{(C)}(k) = \left| \frac{F_{(C)j}(q)}{F_{(C)j}(k)} \right|^2 e^{-\kappa T \kappa},
\]

with: \( \kappa = q - k, \quad k \equiv k_{(C)}, \quad T \equiv T_{(C)}, \quad \Sigma_k(Y_{-}) \equiv \Sigma_{k}^{(C)}(Y_{-}) = \frac{1}{2g^2} \left[ \frac{\left( P_{\Pi K} Y_{(\Pi K)} - (P_{\Pi K} Y_{(\Pi K)})^2 \right)}{P_{\Pi K}^2} \right] > 0, \quad (6.33)
\]

because \( (k Y_{-}) = (P_{\Pi K} Y_{-}) = 0, \) and for \( P = p_\pi + p_\mu, \) either \( k \) or \( P_{\Pi K} = \Pi K P \) is time-like. The function \( \hat{\delta}_{(C)}(k) \) (6.31) for symmetrical positively defined tensor \( \tilde{\mathcal{T}}_{(C)}^{(\beta \alpha)} \) (D.16) similarly \([1]\) keeps the approximate energy-momentum conservation according to Eq. (D.15). The point \( Y_{(C)} \) (D.9) depends on the widths \( \sigma_\pi, \sigma_\mu \) only via dimensionless combination \(|\varsigma| < 1:\)

\[
2Y_{(C)}^\lambda = Y_{(C)}^{\lambda} + \partial_q^\lambda(Q(q) Y_{-})|_{q=k} = \sum_{a=\pi, \mu} \mathcal{O}_{(a)}^\lambda (a)^{\beta} Y_a^{\beta}, \quad \text{with: } \varsigma = \frac{g_\pi - g_\mu}{g_\pi + g_\mu} = \frac{g_-}{g_+}, \quad \mathcal{O}_{(a)}^\lambda (1 \mp \varsigma) \pm \frac{k^\lambda P_{\Pi K}^\beta}{k^2} \pm \frac{(k - \varsigma P)^\lambda P_{\Pi K}^\beta k^2 + (P - \varsigma k)(\Pi \kappa)^{k} \Pi K^2}{\Delta(k)}, \quad (6.34)
\]

\[
\mathcal{O}_{(a)}^\lambda = g_\lambda (1 \mp \varsigma) \pm \frac{k^\lambda P_{\Pi K}^\beta}{k^2} \pm \frac{(k - \varsigma P)^\lambda P_{\Pi K}^\beta k^2 + (P - \varsigma k)(\Pi \kappa)^{k} \Pi K^2}{\Delta(k)}. \quad (6.35)
\]
The geometric suppression factor \( \exp \{ \Sigma_k(Y_-) \} \) here due to (6.33) depends only on the relative distance \( Y_- \equiv Y_-^{(C)} = Y_\pi - Y_\rho \) between initial and final wave packets, but not on their distance from effective (“impact” [1]) point \( Y_{(C)} \), and only its transverse \( k \) and \( p_{\Pi k} \) space-like two-dimension part \( Y_- \) (6.32) is significant for suppression. In spite of exactly the same physical meaning of obtained approximation (6.31) as in [1], with exactly the same structure of obtained approximation (6.31) as in [1], their explicit expressions (6.32) – (6.35) may be essentially different from respective \( \delta_{(C)}(K) \), \( \Sigma_C \), \( Y_C \), \( k_C \), \( \mathbf{R}_C^{-1}/4 \) given therein.

Indeed, substitution of Eqs. (3.63), (3.64) to Eq. (6.19) with arbitrary number of packets in initial \( \{ I \} \) and final \( \{ F \} \) states for \( a_C \in \{ I \oplus F \}, \ Y_\alpha^\beta \mapsto \frac{C_\beta}{C_\alpha}, \) by means of Gaussian integration over Minkowski space [53], with \( R^{\lambda} = X^{\lambda}_{D} - Y^{\lambda}_{C} = (T_{,L}) \) (see also [1–3, 35, 41]), gives:

\[
\tilde{\nu}_{(C)}(q) \approx \tilde{\nu}^{C_{RG}}_{(C)}(q) = (-i)(2\pi)^4 \tilde{\delta}_{C}(K) \exp \{ \Sigma_C \} \exp \{ i(KY_C) \}, \quad K = q - \tilde{k}_C, \quad (6.36)
\]

with: \( \tilde{\delta}_{C}(K) = \left| \frac{R_C^{-1/2}}{(4\pi)^2} \right| \left[ -\frac{1}{4} \left( K \frac{\Sigma_C^{-1}}{K} \right) \right] \mapsto \delta_{L}(K), \) and: \( K \mapsto C \mapsto -K, \quad (6.37) \)

\[
\tilde{k}_C = \sum_{a_C=1}^{I_C} \tilde{p}^u_{a_C} - \sum_{a_C=1}^{F_C} \tilde{p}^v_{a_C}, \quad \tilde{k}_D = \sum_{a_{D,1}}^{F_{D,1}} \tilde{p}^{u}_{a_{D,1}} - \sum_{a_{D,1}}^{I_{D,1}} \tilde{p}^{v}_{a_{D,1}}, \quad \mathbf{R}_C = \sum_{a_C} T_{C a_C}, \quad (6.38)
\]

\[
T_{C a_C} \mapsto T^\lambda_{\alpha} = \frac{n_{a}^2}{2\pi_a} \left[ v^\alpha_{a} v^\beta_{a} - g^\lambda_{\alpha} \right], \quad \tilde{p}_{a} = m_a v_a \left( 1 + \frac{3}{2\pi_a} \right), \quad v_{a} \mapsto v_{a} \mapsto u_{a} = \frac{p_{a}}{m_a}, \quad (6.39)
\]

\[
\Sigma_{C} = -\sum_{a_{C}} \left( (x_{a_C}^{C} - Y_{C}) T_{C a_C} (x_{a_C}^{C} - Y_{C}) \right), \quad \text{or:} \quad C \mapsto D, \quad Y_C = X_C \mapsto X_D = Y_D, \quad (6.40)
\]

\[
Y_C = \mathbf{R}_C^{-1} \sum_{a_C} T_{C a_C} x_{a_C}^{C}, \quad \text{i.e.} \quad \sum_{a_C} T_{C a_C} (x_{a_C}^{C} - Y_{C}) = 0, \quad \sum_{a_C} \partial^{x_{a_C}^{C}}_{\beta} \Sigma_{C} = 0, \quad (6.41)
\]

\[
\sum_{a_{C,D}} \left( \mp \partial^{x_{a_C}^{C}}_{\beta} \right) \left( \mp Y_{C/D}^{\beta} \right) = \frac{1}{2} \left( \sum_{a_{D}} \partial^{x_{a_D}^{C}}_{\beta} - \sum_{a_{C}} \partial^{x_{a_C}^{C}}_{\beta} \right) (X_{D}^{\beta} - Y_{C}^{\beta}) = \partial^{R} \delta_{C}^{\lambda} = \delta_{C}^{\lambda}. \quad (6.42)
\]

Unlike the quadratic form with tensor \( T_{(C)} \) (D.15) – (D.22), the forms with tensors \( \mathbf{R}_{C,D}^{-1} \) (6.38) for two or more external packets are positively defined only almost everywhere [1]. Moreover, founding only on the different leading terms for powers of \( x \) and \( x^2 \) in (3.63), one can’t neglect with the same accuracy of order \( O(\tau_a^{-1}) = O(\tilde{g}_{Ia}^{-1}) \) the difference between \( \tilde{p}_{a} \) and \( p_{a} = m_{a} u_{a} \) in (6.39) even for \( g_2 = 0 \) [1] and so one can’t neglect the difference between \( \tilde{k}_{C,D} \) (6.38) and \( k_{C,D} \). Nevertheless, for two-packet case, the expressions (6.33) and (6.40) of geometric suppression factor \( \Sigma_C \) coincide at specific reference frame, given in Eq. (81) Ref. [41].

Similarly (6.15) the Feynman amplitude (6.1) admits an exact operator factorization of its spin structure by means of representations \( (4.6), (B.9), (B.12) \), of translation invariance
of $\psi_{p}(p_{a}, z)$ (3.29) in overlap function (6.19) and by integration by parts over $x$ and/or $y$: 

$$A_{DC}^{I} = \mathcal{U}_{\{\mathcal{P}_{f}, i\}}(\{x_{f}^{D}\}) \left( \left[ m_{\ell} + \eta' i(\gamma \partial_{x} D_{\eta}) \right] \tilde{M}_{jD} \left[ m_{j} + i(\gamma \partial_{j} D_{\eta}) \right] \tilde{M}_{jC} \left[ m_{\mu} + \eta(\gamma \partial_{j} D_{\mu}) \right] \times \right.$$  

$$\left. \times \frac{1}{2m_{\ell}} \frac{1}{2m_{\mu}} \int \frac{d^{3}q}{(2\pi)^{3}} \tilde{V}_{D}(q) \frac{i}{q^{2} - m_{j}^{2} + i0} \tilde{V}_{C}(q) \mathbf{u}_{\{\mathcal{P}_{f}, i\}}(\{x_{f}^{C}\}) \right).$$

(6.43)

Here: $\tilde{\mathcal{O}}$ is the any one of differential operators defined in Eq. (6.42); all the differentiations act on $\tilde{V}_{D/C}(q)$ only; $\mathbf{u}_{\{\mathcal{P}_{f}, i\}}(\{p_{f}^{C}\})$ or $\mathbf{u}_{\{\mathcal{P}_{f}, i\}}(\{x_{f}^{C}\})$ is a product of respective bispinorial plane waves (4.3) of external particles at the creation (detection) point $C/D$. Similarly [1, 2] the approximation of narrow packets (6.27) gives:

$$A_{DC}^{I} \approx \mathcal{U}_{\{\mathcal{P}_{f}, i\}}(\{p_{f}^{D/C}\}) \tilde{M}_{jD} i \int \frac{d^{4}q}{(2\pi)^{4}} \tilde{V}_{D}(q) \frac{(\gamma q) + m_{j} - \tilde{V}_{C}(q) \tilde{M}_{jC} \mathbf{u}_{\{\mathcal{P}_{f}, i\}}(\{p_{f}^{D/C}\}), \right.$$  

(6.44)

The final result of this factorization for approximations (6.25), (6.30) – (6.33) or (6.36) – (6.41) depends on the actual “pole integration” of propagator, selecting neutrino (antineutrino) pole with definite $\xi = \pm$ and, similarly (6.22), [1, 34, 35, 47], for $k_{D/C} \rightarrow k_{C} \approx k_{D} \rightarrow k_{D/C}$ reads:

$$A_{DC}^{I} \rightarrow \mathbf{u}_{\{\mathcal{P}_{f}, i\}}(\{p_{f}^{D/C}\}) \tilde{M}_{jD/C} \sum_{s=\pm 1/2} u_{i}^{\xi}(k_{C}, s) \pi_{f}^{\xi}(k_{C}, s) \tilde{M}_{jC/D} \mathbf{u}_{\{\mathcal{P}_{f}, i\}}(\{p_{f}^{D/C}\}) \tilde{A}_{DC}^{I(\xi)},$$

(6.45)

where $\ell, \mu, \eta, \eta'$ discriminate corresponding reaction, and the amplitudes $\tilde{A}_{DC}^{I(\xi)}$ (B.33), like (6.2) for (6.1), are respective fractions of corresponding to (6.43) scalar Feynman amplitude [1, 2, 33–35, 47]:

$$\tilde{A}_{DC}^{I} = \int \frac{d^{4}q}{(2\pi)^{4}} \tilde{V}_{D}(q) \frac{i}{q^{2} - m_{j}^{2} + i0} \tilde{V}_{C}(q).$$

(6.46)

Detailed calculation of oscillation probabilities for amplitudes (6.44), (6.46) were described in [1–8, 34–41] and the references therein. Of course the possibility of reduction $A_{DC}^{I} \rightarrow A_{DC}^{I(\xi)}$ like (6.45) depends on subsidiary kinematical conditions [38] in the both vertexes of macro-diagram (6.1). Here we will to focus on the further properties of composite wave functions.

Assuming below, only for unification, only two external wave packets (6.19) also for detection point $\{D\}$ [34, 35], one can connect similarly [19] the off-shell composite wave functions (6.12), (6.13) with the on-shell ones (6.15) by a sort of “dispersion relation”. For the scalar version of Eq. (6.4), written now for the wave function $\tilde{F}_{\{C\}j}^{(\pm)}(q)$ instead of $F_{\{C\}j}(q)$ (6.28), (6.29), with $\tilde{G}_{\{C\}j}^{(\pm)}(q)$ instead of $G_{\{C\}j}(q)$ (6.16), (6.17), (6.23), one has, replacing $\pi \rightarrow I$, $\mu \rightarrow F$, for $\pm$ correlated here with $[C/D]$, as in (6.19), also in accordance
\[G_{\mathcal{C}/D\mathcal{J}}^{(+)}(\theta) = \int d^4 y \, \Phi(k) \, G_{\mathcal{C}/D\mathcal{J}}^{(+)}(\theta) = \int \frac{d^4 q}{(2\pi)^4} e^{i(q\cdot L)} \, \hat{G}_{\mathcal{C}/D\mathcal{J}}^{(+)}(\theta),\]  
\[\hat{F}_{\mathcal{C}/D\mathcal{J}}^{(+)}(\theta) = e^{-i\Phi(k)} \hat{G}_{\mathcal{C}/D\mathcal{J}}^{(+)}(\theta) = \int \frac{d^4 p}{(2\pi)^4} \, \hat{F}_{\mathcal{C}/D\mathcal{J}}^{(+)}(\theta) = i e^{-i\Phi(k)} N_{\sigma F} N_{\sigma I} 2i\pi \int \frac{d^4 p}{(2\pi)^4} \theta(p^0) \delta(p^2 - m_j^2) \cdot \left(\frac{1}{2\pi} \int \frac{d\omega}{\omega - i0} \int d^4 q e^{i((q-p)\cdot \omega)\pm \omega \cdot q} \int d^4 y e^{i(y\cdot q)\pm \omega y} \, D_{m_j}(Z_F - y) \, D_{m_j}(y - Z_I)\right) = \hat{V}_{\mathcal{C}/D\mathcal{J}}(q) \int \frac{d^4 p}{p^0 - q^0 - i0} \theta(p^0) \delta(p^0)^2 - q^2 \right) = \frac{\hat{V}_{\mathcal{C}/D\mathcal{J}}(q)}{2E_{\mathcal{C}j}(E_{\mathcal{C}j} - q^0 - i0)} = \frac{\hat{V}_{\mathcal{C}/D\mathcal{J}}(q) \, PP}{2} \frac{1}{(E_{\mathcal{C}j} - q^0) \, PP}. \]
This explanation of equivalence of the “pole integration” for the Feynman amplitude (6.46) to the similar pole approximation for the off-shell composite wave functions (6.47) – (6.49), that transform them into respective free on-shell ones (6.16), (6.17), (6.28), surely takes place also for the spinor case (6.4), (6.5): \( \hat{\Psi}_{\{C\}}(q) \rightarrow \hat{\Psi}_{\{D\}}(q) \), as (6.12), (6.13) \( \rightarrow \) (6.15). Moreover the self-consistency of both these pole approximations conforms with Lorentz invariance, leading automatically to independence of \( q^0 \) for amplitude (6.51) – (6.52). This also simplifies an asymptotic analysis of the off-shell composite wave functions (6.47) – (6.49) with fixed \( q^\lambda \). For \( q^\lambda = (q^0, \mathbf{q}) \), \( q = q^{\mathbf{q}} \), \( \mathbf{K}_j(q^0) = (q_0^2 - m_j^2)^{1/2} \), \( E_{qj} \equiv E_j(q) = (q^2 + m_j^2)^{1/2} \), with the main branches of square roots and with \( \mp i(q^0) = \mp iq_0q_0 \pm i(q, \rho) \), \( \rho = |\rho|, \rho = \mathbf{n}_\rho \), that is:

\[
\hat{F}^{(+)}_{\{C\}|D}(q) = \int \frac{d^4q}{(2\pi)^4} \frac{e^{\mp i(q^0)^q} \hat{\Psi}_{\{C\}/\{D\}}(q)}{2E_{qj}(E_{qj} - q^0 - i0)} = \int \frac{d^4q}{(2\pi)^4} \frac{e^{\mp i(q^0)^q} \hat{\Psi}_{\{C\}/\{D\}}(q) 1}{2 \left[ 1 + \frac{q_0}{E_{qj}} \right]}.
\]  

(6.55)

Separating the rapidly oscillating exponential functions of \( q \) in (6.31) or in (6.36), for \( k_{\{i\}} = k_{\{C\}D} \), \( Y_{\{i\}} = Y_{\{C\}/\{D\}j} \) (as \( q - Y_{\{i\}} \)) = \( R_{\{i\}}(R^0, R) = R^\lambda \) and \( R = Rn \equiv R_{\{i\}}n_{\{i\}} \), we rewrite (6.55) as:

\[
\hat{\Psi}_{\{C\}/\{D\}}(q) = e^{\mp i(q^0)^{Y_{\{i\}}}} \hat{\Psi}_{\{C\}/\{D\}}(q),
\]

(6.56)

\[
\hat{\Psi}_{\{C\}/\{D\}}(q) \approx (2\pi)^4 \frac{\theta(\Delta)}{i} \hat{\delta}_{\{C\}/\{D\}}(q - k_{\{i\}}) e^{i\Sigma_k(Y_{\{i\}} - \mp i(k_{\{i\}})}.
\]

(6.57)

\[
\hat{F}^{(+)}_{\{C\}|D}(q) = \int \frac{d^4q}{(2\pi)^4} \frac{e^{\mp i(q^0)^{R}} \hat{\Psi}_{\{C\}/\{D\}}(q)}{2E_{qj}(E_{qj} - q^0 - i0)} = \int \frac{d^4q}{(2\pi)^4} \frac{e^{\mp i(q^0)^{R}} \hat{\Psi}_{\{C\}/\{D\}}(q) 1}{2 \left[ 1 + \frac{q_0}{E_{qj}} \right]},
\]

(6.58)

\[
\rightarrow \int_{R \to \infty} \frac{d^4q_0 e^{\mp i\theta R_{\{i\}}}}{2(2\pi)^2} \frac{\theta(q_0)}{\theta(K_j^2)} \frac{e^{i\Sigma_k R_{\{i\}}}}{R} \hat{\Psi}_{\{C\}/\{D\}}(q_0; \pm nK_j) \pm \frac{R_{\{i\}}}{T} \frac{T}{T \to \infty}
\]

(6.59)

\[
\int_{R \to \infty} \frac{d^4q_0 e^{\mp i\theta R_{\{i\}}}}{2(2\pi)^2} \frac{\theta(q_0)}{\theta(K_j^2)} \frac{e^{i\Sigma_k R_{\{i\}}}}{R} \hat{\Psi}_{\{C\}/\{D\}}(q_0; \pm nK_j), \text{ with } \theta(q_0) = \frac{1}{2} \left[ 1 + \frac{q_0}{|q_0|} \right].
\]

(6.60)

The asymptotic expression (6.59) follows at \( R \to \infty \) (\( |Y_{\{i\}}| \to \infty \)) from the second equality (6.58) by using the Grimus-Stockinger theorem [40]. The first \( \theta \) function comes here from the square brackets, while the second \( \theta \) function comes from the conditions of theorem. The asymptotic expression (6.60) appears from the first equality (6.58) due to Jacob-Sachs theorem at \( \pm R^0 = T \to \infty \) [34]. Then its final form (6.61) is obtained at \( R \to \infty \) by using the well known asymptotics of plane wave (B.29), integration over solid angle in \( d^3q = q^2 dq d\Omega(\omega) \), and change of variable \( E_j(K_j(q^0)) = q^0 \). All these theorems are adduced in appendix B, Eqs. (B.28) – (B.31). Contribution of converging spherical wave (B.29) is omitted in (6.61) due to faster oscillations of the total exponential for this term and since it furnishes the spatial vector \( q \) of \( q^\lambda = (q^0, \mathbf{q}) \) with wrong direction \( \mp \mathbf{n}_{\{i\}} \). The momentum \( q \) is globally defined for amplitude of macrodiagram (6.43), (6.46), (6.52) and for composite wave functions (6.19), (6.36), (6.37), (6.49) of both \{C\} and \{D\} vertexes. For the (+) case the momentum \( q \) leaks from point \( Y_{\{C\}} \) to \( Y_{\{D\}} \), when \(-\infty \leftarrow Y_{\{C\}}^0 \ll q^0 \ll Y_{\{D\}}^0 \to +\infty \).
Coincidence of expressions (6.59) and (6.61) means, that \( R_{\{l\}} \to +\infty \) for \( \pm R_{\{C/D\}} = \pm (\rho - Y_{\{C/D\}}) = \pm R_{\{C/D\}} n_{\{C/D\}} \approx R_{\{l\}} n_{\{C\}} \), already implies \( \pm R_{\{C/D\}} = T \to +\infty \). Thus, the \( j \)-dependence of asymptotical integrand of the off-shell composite wave functions may be changed by using above simple change of variable:

\[
\int_{0}^{\infty} q d\mathbf{q} e^{-iT_{E_{\text{qj}}}} e^{iq_{\mathbf{R}}} \Psi_{\{C/D\}}(E_{\text{qj}}; \pm n_{\mathbf{q}}) = \int_{m_{j}}^{\infty} d\mathbf{q} e^{-iq_{\mathbf{R}}} e^{iK_{r}R} \Psi_{\{C/D\}}(q_{0}; \pm n_{K_{r}}) = (6.62)
\]

\[
= m_{j} \int_{0}^{\infty} d\beta \sinh \beta e^{-iT_{m_{j}} \cosh \beta} \frac{e^{iR_{m_{j}} \sinh \beta}}{2(2\pi)^{2}} \Psi_{\{C/D\}}(m_{j} \cosh \beta; \pm n_{\{C/D\}} m_{j} \sinh \beta), \quad (6.63)
\]

where \( n_{\{C\}} \mapsto n_{p} \), only if coordinate system origin is placed to the point \( Y_{\{C\}} \). These expressions show that the often discussed difference between both “equal energy” and “equal momentum” scenarios [34–40] may be considered on the same footing and in fact is very conditionally. The taking into account in (6.61) the above omitted contribution of converging spherical wave (B.29) replaces the first expression (6.62) to the following one (see [9] §130):

\[
\hat{\mathcal{F}}_{\{C/D\}j}(\mathcal{q}) \xrightarrow{R \to \infty} \int_{-\infty}^{\infty} q d\mathbf{q} e^{-iT_{E_{\text{qj}}}} e^{iq_{\mathbf{R}}} \Psi_{\{C/D\}}(E_{\text{qj}}; \pm q n_{\{C/D\}}), \quad (6.64)
\]

where one can again put \( q = m_{j} \sinh \beta \). Eqs. (6.62) implies only contribution of positive saddle point for the further approximation of function \( \Psi_{\{C/D\}}(q_{0}; \pm n_{\mathbf{q}}) \), with \( q_{0}, q_{j} > 0 \) near \( q \approx k_{1} \), \( q_{0} \approx k_{0} \) with additional assumption about its ultrarelativistic position \( q_{j}, q_{0} \gg m_{j} \). Unlike (6.62), the dominant contribution of saddle point for Eq. (6.64) is not restricted by these assumptions and may be estimated here without them. Nevertheless we will adopt these assumptions below in order to compare different ways of estimations and to see the correspondence with previous works. According to the obtained above by different ways the same structure (6.31), (6.36) of reduced overlap function with the narrow wave packets of external particles, we adopt the approximation (6.56), (6.57) for general case. Below in this subsection we consider corresponding saddle-point estimations for the wave function and amplitude following the spirit and interpretation of the seminal works [1–7, 41].

To this end we unify and simplify our notations. We omit subscript \( \{\} \) where it is possible and define 4-vectors and 4-tensors: \( \ell^{\lambda} = (\ell_{i}^{\lambda}) = (1,1) \), or \( \ell^{\lambda} = (1,1) \), \( \ell^{2} = \ell^{2} = 0 \), \( l^{2} = l^{2} = 1 \), for \( l \equiv l_{1} = \pm n_{1} \), or for \( X_{D} - Y_{C} = L = Ll \); with \( H_{\{l\}} = (kT)^{\lambda}_{\{l\}} \) and \( U_{\{l\}} = (\ell T)^{\lambda}_{\{l\}} \); or with \( \overline{T}^{\lambda}_{\{l\}} = H_{\{l\}} + H_{\{l\}} \); \( \overline{T} \equiv T_{DC} = T_{D} + T_{C} \); \( \overline{\lambda} = (\ell T)^{\lambda} \); \( Q_{\lambda} = Q_{\lambda} \), or \( \overline{Q} = \overline{T}^{\vee} \); \( Q^{2} = \overline{Q}^{2} = 0 \), \( Q_{\{l\}} = (H\ell)/(\ell\ell) \), \( \overline{Q} = (\overline{H}\ell)/(\overline{\ell}\overline{\ell}) \). Besides this we define: \( q_{j}^{\lambda} = (q_{0}^{\lambda}, q_{j}l)_{\{l\}} = Q^{\lambda} + w_{\lambda}, \) or \( \overline{q}_{j}^{\lambda} = (\overline{q}_{0}^{\lambda}, \overline{q}_{j}l) = \overline{Q}^{\lambda} + \overline{w}_{\lambda}, q_{j}^{2} = \overline{q}_{j}^{2} = m_{j}^{2} \). Up to next to the leading order \( O(m_{j}) \), these definitions mean that \( (\ell w_{j}) = m_{j}^{2}/(2Q) \), or \( (\overline{\ell} w_{j}) = m_{j}^{2}/(2\overline{Q}) \).
The following expansions for \(q^0 = q_0^0 + (q^0 - q_j^0)\), \(q = q_j + (q - q_j)\) are also useful:

\[
\begin{align*}
K_j(q^0) &\approx q_j + \frac{1}{\nu_j} (q^0 - q_j^0) - \frac{m_j^2}{2q_j^0} (q^0 - q_j^0)^2, \quad q_j = K_j(q^0), \quad \nu_j = \frac{q_j}{q_j^0}, \quad \nu_j = \frac{q_j}{q_j^0}, \quad (6.65) \\
E_j(q) &\approx q_j^0 + \nu_j (q - q_j) + \frac{m_j^2}{2(q_j^0)^3} (q - q_j)^2, \quad q_j^0 = E_j(q_j), \quad \text{and so on.} \quad (6.66)
\end{align*}
\]

### 6.1.1 Asymptotic of wave function

For \(q^\lambda = (q^0, q_1)\) the quadratic form of \(\delta_{\{1\}} (q - k_1)\) in Eqs. (6.31), (6.57), (6.62) reads:

\[
(q-k)^2 (q-k)_{\{1\}} = (q\mathcal{T} q) - 2(qH) + (k\mathcal{T} k) \implies f_{\{1\}}(q^0, q), \quad (6.67)
\]

\[
f_{\{1\}}(q_0^0, q) = q_0^3 \mathcal{T} 0 + q^2 (\mathcal{T} 1) - 2q_0^0 q_0 (\mathcal{T} \cdot 1) - 2[q^0 H^0 - q_0 (H \cdot 1)] + f_{\{1\}}(0, 0), \quad (6.68)
\]

\[
\mathcal{T} = \left( \begin{array}{cc}
\mathcal{T}^{00} & \mathcal{T}^{01} \\
\mathcal{T}^{10} & \mathcal{T}^{11}
\end{array} \right) \equiv \mathcal{T}^{\beta \lambda} = \left( \begin{array}{cc}
\mathcal{T}^{00} & \mathcal{T}^{0r} \\
\mathcal{T}^{r0} & \mathcal{T}^{rr}
\end{array} \right), \quad s, r = 1, 2, 3; \quad f_{\{1\}}(0, 0) \equiv (k\mathcal{T} k), \quad (6.69)
\]

whence: \(F_j(q) = f_{\{1\}}(E_j(q), q)\), while \(\tilde{F}_j(q^0) = f_{\{1\}}(q^0, K_j(q^0))\),

respectively for the first and second expression (6.62). The extremum condition of zero first derivatives of these functions: \(F_j'(q) = 0\) and \(\tilde{F}_j'(q^0) = 0\), gives the same equation for saddle point \(q^\lambda_j\), similar [41], but leads to different expressions for the second derivatives and to different complex “effective widths” \(D_j\) of Gaussians integrals, appearing by the use of expansions\(^3\) (6.65), (6.66) for powers of exponentials of Eqs. (6.62), (6.64):

\[
\begin{align*}
q^0 &\equiv m_j \sqrt{1 - \nu^2} = \frac{\nu H^0 - (H \cdot 1)}{v(U\ell) - (1 - \nu)^2 (T \cdot 1)}, \quad \nu \equiv \frac{q^0}{q^0}, \quad \nu \rightarrow \nu_j, \quad q \rightarrow q_j, \quad (6.71) \\
1 \frac{D_j^2}{2} &\equiv F_j''(q_j) + i T \frac{m_j^2}{q_j^0 \nu_j^3}, \quad \frac{q^d q}{2E_j(q)} \frac{e^{i R q_j - i T E_j(q)}}{R} e^{-F_j(q)} \approx \frac{v_j}{2R} e^{i R q_j - i T q_j^0} e^{-F_j(q_j^0)} \sqrt{2 \pi D_j^2 e^{-(D_j^2)/2(R - T v_j)^2}}, \quad (6.73)
\end{align*}
\]

The second representation (6.62) exactly leads to the same expression with the replacement \(D_j^2 \rightarrow \tilde{D}_j^2\). Since \(q = q_0^0\) but \(dq^d = v dq\), these complex effective widths (6.72) will coincide only after substitution of \(R \rightarrow T v_j\). Taking into account the Lorentz invariance of:

\[
\begin{align*}
& i [q_j R - q_j^0 T]_{\{1\}} = -i (q_j R)_{\{1\}} , \quad (q_j^0 R - q_j T)_{\{1\}} \equiv [(q_j R)^2 - m_j^2 R^2]_{\{1\}}, \quad (6.74) \\
& F_j(q_j) = \tilde{F}_j(q_j^0) = f_{\{1\}}(q_j^0, q_j) = ((q_j - k) \mathcal{T} (q_j - k))_{\{1\}}, \quad \text{with:} \quad (6.75) \\
& \Omega_{\{1\}}(R) = i(q_j R)_{\{1\}} + \frac{D_j^2}{2(q_j^0)^2} [(q_j R)^2 - m_j^2 R^2]_{\{1\}} = i(q_j R) + \frac{m_j^2}{2q_j} [(u_j R)^2 - R^2], \quad (6.76)
\end{align*}
\]

for: \(u_j = \frac{q_j}{m_j}\), \(\tau_j = \frac{(q_j^0)^2}{D_j^2} = (q_j^0)^2 F_j'(q_j) + i T \frac{m_j^2}{q_j^0}, \quad \text{or with} \quad T \rightarrow R \frac{v_j}{v_j}, \quad (6.77)\)

\(^3\)That are not the expansions on power of \(m_j\).
one obtains an “almost” Lorentz covariant asymptotics:

$$\tilde{F}^{(s)}_{[C,D]}(q) \overset{R \to \infty}{\sim} \frac{(2\pi)^{5/2}}{i} e^{+i(k_{11}Y_{11})} e^{+i(k_{11}Y_{11})} \delta_{\{q_{1}-k_{1}\}} \frac{v_{j}}{2R} D_{j} e^{-\Omega_{j}(R)}.$$  \hspace{1cm} (6.78)

First of all this result should be compared with evolution of neutrinos emitted by classical sources, given by Eq. (11.20) of Ref. 43. We observe that the spherical wave $e^{-i(q_{1}R)/R}$ given therein, now is modulated by the same Gaussian factor that appears in CRG approximation of narrow width for one-packet state (3.63) with $g = -R_{11}$. Equation (6.78) confirms qualitative reasons leading to similar form of intermediate neutrino wave function introduced in [1–3]. But here this function is defined by parameters of external wave packets interacting only at one vertex: \{C\} or \{D\} and has necessary representation variable $q$. It is worth to note that the expressions (6.65) – (6.78) fulfil without any assumption about value of $m_{j}$, but leads to ambiguity (6.72), (6.77) of imaginary part of effective width, which besides this, does not has consistent physical interpretation [1] of Gaussian oscillations thus appearing.

Assumption of small $m_{j}$ allows to avoid this ambiguity and leads to unique fully invariant expression for effective width. Indeed, from Eqs. (6.67) – (6.72), with $q_{j}^{\lambda} = Q^{\lambda} + \omega_{j}^{\lambda}$, $(U \ell \equiv (i \ell T \ell) > 0, U^{2} < 0, H^{2} < 0$, up to the next to the leading order $O(m_{j}^{2})$ it follows for:

$$F_{j}(q) \approx f_{j}(0,0) - Q(H \ell) + (U \ell)(q - Q)^{2} + m_{j}^{2} \left[ U^{0} - H^{0} \right], \hspace{0.5cm} r_{j} \equiv \frac{m_{j}^{2}}{2Q^{2}} \ll 1,$$  \hspace{0.5cm} (6.79)

that: $v_{j} \approx 1 - r_{j}, \hspace{0.5cm} w_{j}^{\lambda} \approx - \frac{r_{j}}{(U \ell)} \left[ (H \cdot l), H^{0}l \right], \hspace{0.5cm} (H w_{j}) = 0, \hspace{0.5cm} \tau_{j} \simeq \tau_{1}$),

$$\frac{1}{D_{j}} \approx 2(U \ell) \left\{ 1 - \frac{r_{j}}{(H \ell)} \left[ 2H^{0} - iT_{1} \right] \right\} \simeq 2(U \ell) > 0, \hspace{0.5cm} \tau_{1} = 2Q(H \ell) \equiv 2 \left( \frac{H \ell}{U \ell} \right)^{2},$$  \hspace{0.5cm} (6.80)

where only the main contribution is saved. This also allows to avoid some inconsistence of the used saddle-point calculation [1], because Eq. (6.72) implies an “exact” saddle point $q_{j}$, which in fact is defined by zero derivative of full power of exponential of integrand in (6.73): $F_{j}(q) + i(Tv(q) - R) = 0$. Whence $q_{j}$, acquires an inadmissible imaginary part, which is generated by value $i(Tv(q) - R)$. To extract the $j$- dependence, the following relation, ensuing with the same accuracy from (6.75), (6.79), (6.80), is observed:

$$F_{j_{1}}(q_{j}) = (\langle q_{j} - k \rangle T(q_{j} - k)_{1}) = ((Q - k)T(Q - k)_{1}) + \Theta_{j_{1}}, \hspace{0.5cm} \text{where:}$$  \hspace{0.5cm} (6.82)

$$\Theta_{j_{1}} \approx m_{j}^{2} \left[ U^{0} - H^{0} \right] = \frac{m_{j}^{2}}{(H \ell)} \left[ H^{0}(U \ell - U^{0}(H \cdot l) \right], \hspace{0.5cm} \text{whence:}$$  \hspace{0.5cm} (6.83)

$$\tilde{F}^{(s)}_{[C,D]}(q) \overset{R \to \infty}{\sim} \frac{(2\pi)^{5/2}}{i} e^{+i(k_{11}Y_{11})} e^{+i(k_{11}Y_{11})} \delta_{\{Q - k\}} \frac{v_{j}}{2R} D_{j} e^{-\Omega_{j}(R) - \Theta_{j_{1}}}.$$

Similarly [1] the value of $\Theta_{j_{1}}$ is negligible due to smeared delta-function of approximate energy-momentum conservation, keeping $k^{\lambda} \simeq Kl^{\lambda}, \hspace{0.5cm} (H \ell) \simeq K(U \ell)$, when $\Theta_{j_{1}} = 0$.

### 6.1.2 Asymptotic of oscillation amplitude

Now let us show how the definitions of scalar products (6.51) – (6.54) of composite wave functions reproduce the previously obtained [1] asymptotic of oscillation amplitude. Unfortunately the time derivative $\partial_{\varrho}^{0}$ and $d^{3}p$- integration in Eq. (6.51) is not uniform with
respect to asymptotical expansions over $T$ and $R$, and substitution of the results (6.78), (6.84) into (6.51) is not correct. Starting from substitution of (6.49), (6.57) into (6.52), we perform at first the integration over $q^0$ at the limit $Y^0_B \to +\infty$, i.e. $T_D \to +\infty$ with the help of Jacob-Sachs (JS) theorem (B.31). Thus we arrive to very similar to (6.58) $d^4q$-integral, which may be estimated by the same two ways (6.59), (6.61) as above. The use for example again of JS theorem for $q^0$, integration at $Y^0_C \to -\infty$, i.e. $T_C \to +\infty$, with $T = T_D + T_C$, $R^\lambda = (T, L)$, $L = iL$, $\tilde{T}^\lambda = (1, l)$, gives:

$$
\tilde{A}_{DC}^{(\pm)}(\frac{Y^0_C \to -\infty}{Y^0_D \to +\infty}) (-1) \int \frac{d^4q}{(2\pi)^3} e^{-iE_qT} 2E_{qj} e^{i(q \cdot L)} \tilde{\Psi}_{[D]}(E_{qj}; q) \tilde{\Psi}_{[C]}(E_{qj}; q), \quad (6.85)
$$

what again up to a sign is equal to substitution of $\tilde{\Psi}_{[D]}(q)$ (6.57) into the on-shell Eq. (6.53). Similarly (6.62), (6.64), the use again of the plane-wave asymptotic (B.29) leads to integral:

$$
\tilde{A}_{DC}^{(\pm)}(\frac{L \to \infty}{T \to -\infty}) \int_{-\infty}^{\infty} \frac{q dq}{(2\pi)^2} e^{-iE_j(q)} e^{iqL} \tilde{\Psi}_{[D]}(E_j(q); qL) \tilde{\Psi}_{[C]}(E_j(q); qL), \quad (6.86)
$$

whose saddle-point estimation with substitution of (6.57) repeats all the above steps with simple replacing of all defining above values onto the same but with the bar: $H \to \overline{H}$, $U \to \overline{U}$, $Q \to \overline{Q}$, $q \to \overline{q}$, $\tau \to \overline{\tau}$, $q^j \to \overline{q}^j$, $w_j \to \overline{w}_j$, $1 \to l$. Thus, with $q^\lambda = (q^0, q^\nu)$ one has for quadratic form $((q - kC)T_C(q - kC)) + ((q - kD)T_D(q - kD)) \to \overline{\mathbf{F}}(\overline{q}^0, \overline{q})$:

$$
\mathbf{F}_j(q) = \overline{\mathbf{F}}(E_j(q), q) = F_{jC}(q) + f_{jD}(q), \quad \mathbf{F}_j(q) = 0, \quad \text{so: } q \to \overline{q}_j, \quad \text{with: } (6.87)
$$

$$
q^0 \equiv \sqrt{1 - \overline{v}^2} = \frac{\overline{v}T^0 - (\overline{H} \cdot l)}{\sqrt{1 - \overline{v}^2}}, \quad \overline{v} \equiv \frac{q^0}{q^0}, \quad \overline{v}_j = \overline{q}_j \overline{m}_j, \quad (6.88)
$$

$$
\frac{1}{D_j} \equiv \overline{F}_j'(\overline{q}_j) + iT \frac{m_j^2}{(\overline{q}^j)^2}, \quad \overline{\tau}_j = \frac{(\overline{e}_j^0)^2}{D_j}, \quad \overline{\Pi}_j(R) = i(\overline{q}_j R) + \frac{m_j^2}{2\overline{\tau}_j} [(\overline{q}_j R)^2 - R^2] : (6.89)
$$

$$
\tilde{A}_{DC}^{(\pm)}(\frac{L \to \infty}{T \to -\infty}) i(2\pi)^6 e^{i\Theta_{DC}} e^{\Sigma_{DC} \delta_{(D)}(\overline{q}_j - kD) \delta_{(C)}(q_j - kC)} \sqrt{2\pi} \frac{\overline{V}_j}{2\overline{L}} \overline{D}_j e^{-\overline{\Pi}_j(R)}: (6.90)
$$

where: $\Theta_{DC} = (kD X_D) - (kC Y_C)$, $\Sigma_{DC} = \Sigma_{kD}(X_D^U) + \Sigma_{kC}(Y_C^U)$. (6.91)

This, up to a sign, is exactly the result (39) of Ref. [1] and of course it may be also simplified as above for small $m_j$ to the form given in Ref. [41] for $\overline{q}_j^\lambda = (\overline{q}_j^0, \overline{q}_j^\nu) = \overline{Q}^\lambda + \overline{w}_j^\lambda$:

$$
\tilde{A}_{DC}^{(\pm)}(\frac{L \to \infty}{T \to -\infty}) i(2\pi)^6 e^{i\Theta_{DC}} e^{\Sigma_{DC} \delta_{(D)}(\overline{Q} - kD) \delta_{(C)}(\overline{Q} - kC)} \sqrt{2\pi} \frac{\overline{V}_j}{2\overline{L}} \overline{D}_j e^{-\overline{\Pi}_j(R)} - e^{-\overline{\Theta}_j}, (6.92)
$$

with: $\overline{\tau}_j \equiv \frac{m_j^2}{2\overline{Q}^2} \ll 1$, $\overline{\tau}_j \approx 1 - \overline{\tau}_j$, $\overline{w}_j^\lambda \approx - \frac{\overline{\tau}_j}{(\overline{U} \overline{l})} \left( (\overline{H} \cdot l), \overline{H} \overline{0} l \right)$, $(\overline{H} \overline{w}_j) = 0$, (6.93)

$$
\frac{1}{D_j^2} \approx 2(\overline{U} \overline{l}) \left( 1 - \frac{\overline{\tau}_j}{(\overline{H} \overline{t})} \left[ 2\overline{H}^0 - i\overline{r} \right] \right) \simeq 2(\overline{U} \overline{l}) \equiv 2(\overline{i} \overline{r}) > 0, \quad \overline{\tau}_j \approx \overline{\tau} = 2\overline{Q}(\overline{H} \overline{l}), (6.94)
$$

or: $\overline{\tau} = 2(\overline{H} \overline{l})^2 > 0$, $\overline{\Theta}_j \approx m_j^2 \left[ \overline{U}^0 - \overline{H}^0 \right] \overline{Q} = \frac{m_j^2}{(\overline{H} \overline{l})} \left[ \overline{H}^0 (\overline{U} \cdot l) - \overline{U}^0 (\overline{H} \cdot l) \right]$. (6.95)
The value $\Theta_j$ disappears now for $k_C^\lambda = k_D^\lambda \simeq K\ell^\lambda$, [41]. Moreover, for this case: $H^\lambda \simeq KU^\lambda$ and so on, and $Q \simeq \overline{Q} \simeq K$. Nevertheless: $1 \neq l$ and $w_j \neq \overline{w}_j$. Here the assumption of small $m_j$ [1] repairs again the self-consistency of our calculations and again leads to the invariant width (6.81), (6.94) of the approximate non-spreading CRG- wave function similar (3.63). The imaginary contribution to the width should be neglected as artifact of our estimation method because it depends on the used integration variable. As above, it also implies inadmissible complex value of saddle point $\overline{\tau}_j^\lambda$, depending on time $T$ and length $L$. The additional smallness of $\overline{\tau}_j(H\ell)^{-1}$ in (6.94) also indicates such neglect.

It is worth to note, that exactly the same mechanism [9–12] of Gaussian integration, (6.72) – (6.78) or (6.89), (6.90), leads to usual inevitable spreading in time of free non relativistic Gaussian wave packet (2.9) discussed in Section 2. The usually discussed spreading of relativistic wave packet also is based on the using of Gaussian approximation and on the calculation of quantities directly inspired by Gaussian distribution [4, 25, 26, 31]. A very instructive calculation of dispersion for wave packets of general form is given in ref. [4].

On the other hand, the shown below form-invariance (7.7), (7.8) of covariant on shell one-packet wave function, similar (6.3), assures its propagation without change of its relativistically invariant width $\sigma$ or $g_{1,2}(m,\sigma)$. This in turn manifests in CRG- approximation of subsection 3.1, as the non spreading at all packet wave function (see also figure 1). We think, this difference is also the manifestation of discussed in section 3 the different meaning of time in QM and QFT, that indicates its different meaning at different scales.

Evidently, the performing in (6.58), (6.57) at first the plane-wave limit $\sigma_{\pi,\mu} \to 0$, (D.15), (6.20) of reduced overlap function (6.31), (6.57) will fully eliminate its dependence on “impact” points $Y_{(C/D)}$, and the above asymptotic limits become impossible to take at all. Moreover, now unlike $q^\lambda$ in (6.63), (6.64), 4-vector $k^\lambda = k_{(C/D)}^\lambda$ will be in principle off the mass shell:

$$\hat{V}_{(C/D)}(q)\big|_{p.w.} = \frac{(2\pi)^4}{i} \delta_4(q - k), \quad \hat{F}_{(C/D)}^{(\pm)}(\varrho)\big|_{p.w.} = \frac{(-i) e^{\mp i(k\varrho)}}{2E_{kj}(E_{kj} - k^0 - i0)}. \quad (6.96)$$

This also illuminates the above main difficulty of the problem under consideration: the asymptotical expansion of $\hat{F}_{(C/D)}^{(\pm)}(\varrho)$ over parameters $T, R$ is not uniform with respect to the limit of parameters $\sigma_{a, m_j}$ and with respect to the time derivative $\partial^0_\varrho$ and $d^3\rho$-integration in Eq. (6.51). That means we need more uniform representation of composite wave-packet state. Such representation for on-shell composite wave function is given below.

### 6.2 “One-packet” representation of two-packet state

To trace the another limiting properties of the on shell composite wave function (6.15) – (6.17) it is convenient to use for the last multipliers of Eq. (6.23) the integral representation # 6.677.6 [50] (or # 1.13(47) [51]), which factorizes its different $q^j$-dependences for $q^2 > 0$,
$q^0 > 0$, $\Delta(q) > 0$, $\chi_\Delta > 0$ via Bessel function $J_0(r)$, with the main branch of square roots:

$$
\sin\left(\frac{\alpha \sqrt{\beta^2 + y^2}}{\sqrt{\beta^2 + y^2}}\right) = \frac{1}{2} \int dx \, e^{i \beta x} J_0\left(\beta \sqrt{\alpha^2 - x^2}\right), \quad \text{for arbitrary } \beta, y \text{ and } \alpha > 0, \quad (6.97)
$$

$$
\sinh\left(\frac{\chi_\Delta \left[q^2 Z_+^2 - (q Z_-)^2\right]^{1/2}}{\left[q^2 Z_+^2 - (q Z_-)^2\right]^{1/2}}\right) = \frac{1}{2} \int dx \, e^{i \chi(q Z_-)} J_0\left(\left[-q^2 Z_+^2\right]^{1/2} \left(\chi_\Delta^2 - \chi^2\right)^{1/2}\right). \quad (6.98)
$$

Therefore the function (6.17) with $q^2 = m_j^2$ becomes Lorentz invariant linear superposition of interpolating wave packets (3.29), (6.14) with the same fixed mass $m_j$ but with various centers $Y_\chi$, various on shell momentums $p_{\chi j}^2 = m_j^2$, $p_{\chi j} = m_j v_\chi$, $v_\chi = \zeta_\chi / \sqrt{\zeta_\chi^2 - 1}$, and various invariant widths $\sigma_{\chi j}$, defined by e.g. $g_{\chi j} \rightarrow 1 / \sigma_{\chi j}^2$, with $\tau_{\chi j}^2 = m_j^2 \sigma_{\chi j}^2 = m_j^2 g_{\chi j}^2$:

$$
G_{(\chi j)}(q) = \frac{N_{\chi \mu} N_{\sigma \tau}}{4\pi} \left(\frac{\chi_\Delta}{\chi_{\Delta \chi}}\right) \int \frac{d\chi \sqrt{m_j} D_{m_j}(q - \chi^0)}{\sqrt{\alpha^2 - \chi^2}} \left.\frac{N_{\chi \mu}}{i} D_{m_j}(q - \chi^0)\right|_{\chi^0} \quad (6.99)
$$

$$
\frac{N_{\chi \mu}}{i} D_{m_j}(q - \chi^0) = \psi_{\sigma_{\chi j}}(p_{\chi j}, Y_\chi - \theta) = e^{i p_{\chi j} Y_\chi} F_{p_{\chi j}}(Y_\chi - \theta) \quad (6.100)
$$

for: $N_{\chi \mu} = \frac{(2\pi)^2}{m_j^2} \frac{\mathcal{Z}(\tau_{\chi j})}{h(\tau_{\chi j})} \tau_{\chi j} \rightarrow \infty \frac{2}{m_j^2} \left(2\pi\right)^{3/2} \tau_{\chi j}^{3/2} \exp \left\{i \tau_{\chi j}\right\}, \quad (6.101)

$$
Z_\chi = Z_\eta + \chi Z_- = \eta_+ Z_\pi + \eta_- Z_\mu^* = Y_\chi + i \zeta_\chi, \quad \eta_\pm = \eta_\pm(\chi) = \frac{1}{2} \pm (\chi_0 + \chi), \quad (6.102)
$$

$$
Y_\chi = \eta_+ Y_\pi + \eta_- Y_\mu = \frac{Y_\pi}{2} + (\chi_0 + \chi) Y_-^0, \quad \zeta_\chi = \eta_+ \zeta_\pi - \eta_- \zeta_\mu = \frac{\zeta_\pi}{2} + (\chi_0 + \chi) \zeta_\chi, \quad (6.103)
$$

with: $\zeta_\chi = \frac{g_+}{2} \left\{p_\chi \left[\frac{\zeta_\pi}{2} + (\chi_0 + \chi)\right] + \left[1 + \xi(\chi_0 + \chi)\right]\right\}$, for $g_{2a} = 0; \quad (6.104)$

where $\tau_{\chi j} \rightarrow \infty$ means $g_+ \rightarrow \infty$. For the opposite limit, with independent localizations of pion and muon, $\sigma_{\pi}, \sigma_{\mu} \rightarrow \infty$, we have all these values $g_{1,2a}, g_{1,2a}, \zeta_\pi, \zeta_\mu, \zeta_\chi \rightarrow 0$, i.e. $\tau_{\chi j} \rightarrow 0$. This transforms (6.99) into superposition of localized packets (3.25) with different centers only:

$$
G_{(\chi j)}(q) \rightarrow \frac{(N(0))^2}{m_j^2 m_\mu^2} \frac{\chi_\Delta}{8\pi} \int dx \, J_0\left(\frac{-m_j^2 Y_\pi^2}{2} \left(\chi_\Delta^2 - \chi^2\right)^{1/2}\right) \frac{1}{i} D_{m_j}(q - Y_\pi) \quad (6.105)
$$

Coincidence of centers $Y_\pi \rightarrow 0$ of $\pi$ and $\mu$ packets in point $Y_\pi$ leads to localization (3.25) in this point $Y_\chi \rightarrow Y_\pi$ for the full on-shell composite wave packet as a single-packet function:

$$
G_{(\chi j)}^{(Y_\pi \rightarrow Y_\pi)}(q) = \frac{(N(0))^2}{m_j^2 m_\mu^2} \frac{\chi_\Delta}{4\pi} \frac{1}{i} D_{m_j}(q - Y_\pi) = \frac{(N(0))^2}{8\pi} \frac{\Delta(0)(m_j)}{m_j^2 m_\mu^2} \psi_\infty(p_j, Y_\pi - q). \quad (6.106)
$$

Its further zero-mass limit $m_j \rightarrow 0$ repeats (A.10) and also implies the change of normalization. So, the above obtained limiting properties of on-shell composite wave function for intermediate neutrino wave packets (6.15), or (6.27), (6.28) carefully replicate these properties folowing to external interpolating wave packets, as given by Eqs. (3.19), (3.25), (A.10).
However some differences appear for another cases. To consider at first the limit $m_j \to 0$, we note that the 4-vector (6.104) is reduced to $\zeta \chi \mapsto (g+/4)k(1-\varsigma^2)$ only for $\chi_0+\chi \mapsto -\varsigma/2$.

For $m_j \geq 0$ this requires a value of $\chi$, which is unachievable for the integrand in (6.99). Indeed, with substitution $\chi = \gamma - \chi_\Delta$, $\gamma > 0$ one has $\chi_0 + \chi = \gamma + \epsilon$, where:

$$\bar{\epsilon} \equiv \chi_0 - \chi_\Delta \mapsto \frac{1}{2} \left( \frac{m_\pi^2 + m_\mu^2}{m_\pi^2 - m_\mu^2} \right) + O(m_j^2) \approx 1,79, \text{ for } m_j \to 0, \chi_0, \chi_\Delta \to +\infty. \quad (6.107)$$

With replacing $\chi_0 + \chi = \gamma + \bar{\epsilon}$ for $Z_\chi = Z_\gamma$ and with respective redefinition of (6.102) – (6.104), (A.10), the finite limit of expression (6.99) reads:

$$G\{C\}_{j}(\varrho)\bigg|_{m_j=0} = \frac{N_{\sigma\mu}N_{\sigma\pi}}{4\pi} \frac{1}{2i} \int_0^\infty d\gamma J_0 \left( \left[ -Z^2 (m_\pi^2 - m_\mu^2) \right]^{1/2} \sqrt{\gamma} \right) D_0 (\varrho - Z_\gamma). \quad (6.108)$$

Unlike the single-packet case (A.10), such superposition of these massless solutions to KG equation arises without any change of normalization. The integral exists for $(Y - \zeta) = 0$, when $-Z^2 \mapsto \zeta^2 - Y^2 > 0$. For the rest frame of time-like vector $\zeta_0^\perp \mapsto (\zeta_0^\perp, 0)$, that means arbitrary $Y_\perp = (0, Y_\perp)$. The using of formula # 6.532.4 [50] (or # 7.14.(58) [49]) reduces this to the form admiting analytical continuation to arbitrary $Y_\perp$. If $a > 0$, $Re \beta > 0$, as:

$$\int_0^\infty \frac{tdtJ_0(\varrho t)}{t^2 + \beta^2} = K_0(a\beta), \quad \text{and thus, for } \mathcal{R} = \varrho - \frac{Z_+}{2}, \quad Z_\mp = Y_\mp + i\zeta_\mp, \quad (6.109)$$

$$\mathcal{L} = \left[ (Z_\perp \mathcal{R})^2 - Z^2 \mathcal{R}^2 \right]^{1/2}, \quad \Lambda_\perp = \left\{ (m_\pi^2 - m_\mu^2) \left[ (Z_\perp \mathcal{R}) \mp \mathcal{L} - Z^2 \bar{\epsilon} \right] \right\}^{1/2}, \quad (6.110)$$

one obtains: $G\{C\}_{j}(\varrho)\bigg|_{m_j=0} = \frac{N_{\sigma\mu}N_{\sigma\pi}}{4(2\pi)^3} \left[ K_0(\Lambda_-) - K_0(\Lambda_+) \right] \mathcal{L}, \quad (6.111)$

what has the form similar to finite-difference-analog of relation (B.27) for $N=2, S=1$, with the step, proportional to the value of $\mathcal{L} = [-Z^2 \mathcal{R}^2 \mathcal{R}^2]^{1/2}$ in (6.110) for small $\mathcal{L}$.

### 7 Discussion

The principal differences of suggested here wave packet from the one suggested in [1–7] and marked below as NN, manifest in spin degrees of freedom and thus in the Lorentz transformation and localization properties (3.24) – (3.27), and normalization. According to our definitions (4.6), (4.7), and (3.17), (3.28), (3.33), (3.34) but unlike the formulas (6),
(7) of [1], for $k_0 = E_k = \sqrt{k^2 + m^2}$, \( \vec{x} = x_a - x \), \( \zeta_a = \zeta_a(p_a, \sigma_a) \), \( \xi = +: \)

\[
\Xi^{(+)}_{p_a, x_a, s_a}(x) \equiv \Xi^{(+)}_{\{a\}}(x) = \frac{1}{i} \hat{N}_{\sigma a} S^-(x - z_a) \mathcal{U}^{(+)}_{p_a, s_a}(x_a) = \hat{N}_{\sigma a} \langle 0 | \psi(x) \bar{\psi}(x_a + i\zeta_a) | 0 \rangle u^{(+)}(p_a, s_a) e^{-i(p_a x_a)} = \langle 0 | \psi(x)| (+); \{p_a, x_a, s_a\} \rangle = \int \frac{d^4k}{(2\pi)^3} 2E_k \phi^\sigma(k, p_a) \frac{(\gamma k) + m_a}{2m_a} u^{(+)}(p_a, s_a) e^{-i(p_a x_a)},
\]

but that is: \( \neq \Xi^{(+)}_{\{a\}}^{NN}(x) = \langle 0 | \psi(x)| (+); \{p_a, x_a, s_a\}^{NN} \rangle = \int \frac{d^4k}{(2\pi)^3} 2E_k \phi^\sigma_{NN}(k, p_a) u^{(+)}(k, s_a) e^{-i(p_a x_a)} \), for \( \phi^\sigma_{NN}(k, p_a) \) with: \( \mathcal{T}^{NN}(\tau) \equiv 1 \), \( N_{\sigma}^{NN}(m_a, \zeta_a^2) = \frac{(2\pi)^2}{m_a^2 h(\tau^2)} \), \( g_1 = \frac{1}{2\sigma^2} \), \( g_2 \equiv 0 \). (7.6)

According to [14–29], if the packet realizes a state with fixed momentum \( p_a \) and spin \( s_a \) as irreducible representation of Lorentz group, its spin-quantization 4- axis \( \hat{s}_a \) in (4.4a) is fixed by \( (p_a \hat{s}_a) = 0 \) with the fixed momentum \( p_a \) of wave packet. That is exactly the case of (7.1) – (7.3), containing only the external bispinorial plane wave \( \mathcal{U}^{(+)}_{p_a, s_a}(x_a) = u^{(+)}(p_a, s_a) e^{-i(p_a x_a)} \). For the packet (7.5) the fixed spin \( s_a \) of state “sits” on a running momentum \( k^\lambda \) of its internal plane waves via bispinor \( u^{(+)}(k, s_a) \), \( (k \hat{s}_a) = 0 \), i.e. \( \hat{s}_a = \hat{s}_a(k) \), what can’t be fulfilled simultaneously with \( (p_a \hat{s}_a) = 0 \).

Nevertheless, according to (6.3), the evident from the relations (4.6) with (B.11), (4.8), (4.9d) form-invariant propagation, without changing of relativistically invariant width \( \sigma \), for the packet (7.1) – (7.3), due to the definitions (B.11), (4.7), (B.9), (4.3) and the second orthogonality relation (4.4b) [16, 17], takes place also for the packet (7.5), also as covariant solution to free Eqs. (4.17). Similar assertion, where \( D^\xi(x) \) is even function (B.17), takes place for the respective scalar wave packet (3.14), (3.29), \( \forall x_a \), with \( \xi(x^0 - y^0) > 0 \), for both the initial and final states (emp. [13], pp. 101, 102). So:

\[
\Xi^\xi_{\{a\}}(x) = \frac{\xi}{i} \int d^3y S^*(x - y) \gamma^0 \Xi^\xi_{\{a\}}(y), \quad F_{p_a x_a}(x) = \frac{1}{i} \int d^3y D^\xi(x - y)(i \partial_y^0) F_{p_a x_a}(y),
\]

\[
\Xi^\xi_{\{a\}}(x) = \frac{\xi}{i} \int d^3y \Xi^\xi_{\{a\}}(y) \gamma^0 S^*(y - x), \quad F^{*\xi}_{p_a x_a}(x) = \frac{1}{i} \int d^3y F^{*\xi}_{p_a x_a}(y)(i \partial_y^0) D^\xi(y - x),
\]

and the same with the changes \( d^3y \gamma^0 \rightarrow d\Sigma_\nu(y) \gamma^\nu, d^3y \partial^0_y \rightarrow d\Sigma_\nu(y) \partial^\nu_y \), and \( \xi(x^0 - y^0) \rightarrow \xi(n_\Sigma(x - y)) \) (6.7), and/or the same for \( \Xi^\xi_{\{a\}} \rightarrow \Xi^{\xi_{NN}}_{\{a\}} \). Free propagation (7.7), (7.8) can not change the invariant width \( \sigma \) and the value of \( \tau \) of covariant free wave packets, because such changing would be in contradiction with the local nature of Lorentz covariance of this propagation [13]. Nevertheless, the shown on figure 1 usual Gaussian spreading will inevitably take place at any fixed reference frame [4].

Due to dimension reduction of free Green functions [52] and due to conservation of orthogonality relation (4.4b) also for \( m = 0 \), the similar picture with \( d^3x \rightarrow dx_\parallel \) takes place also for evolution (7.7), (7.8) of the massless wave packets, defined for fixed \( \tau > 0 \) by (A.15),
on the light cone. From (B.24), (B.25): (A.17), (A.26), may be considered as inverse dimensionless width of these massless packets $m$

The normalization conditions (4.4c), (4.9) of all bispinors leads to their finite limits with $m_a \to 0$ [17, 27]. The parameter $\tau$, originated by (3.37), (3.39), in the rough approximations (A.17), (A.26), may be considered as inverse dimensionless width of these massless packets on the light cone. From (B.24), (B.25):

$$F_{p_a,x_a}(x) \overset{m_a \to 0}{\underset{\sigma \to 0}{\longrightarrow}} F_{p_a,x_a}^{(\tau)}(x) = e^{-i(p_a \cdot x_a)} \psi(\hat{p}_a, \xi) = \frac{\hbar}{(2\pi)^2} e^{-i(p_a \cdot x_a)} h(\tau^2 - 2i\tau(p_a \cdot \xi)), \quad (7.9a)$$

$$\Xi^{\xi}_{\{a\}}(x) \overset{m_a \to 0}{\underset{\sigma \to 0}{\longrightarrow}} \Xi^{\xi(\tau)}_{\{a\}}(x) = \lim_{m_a \to 0} i\frac{\gamma_0 \partial_x + m_a}{2m_a} F_{\xi p_a,x_a}^{(\tau)}(x) u_{m_a}^{\xi}(p_a, s_a) = \frac{\hbar}{(2\pi)^2} e^{-i\xi(p_a \cdot x)} u_{m_a}^{\xi}(p_a, s_a) = e^{-i\xi(p_a \cdot x)} u_{m_a}^{\xi}(p_a, s_a) = U_{\xi, p_a, s_a, 0}(x). \quad (7.9d)$$

The normalization conditions (4.4c), (4.9) of all bispinors leads to their finite limits with $m_a \to 0$ [17, 27]. The parameter $\tau$, originated by (3.37), (3.39), in the rough approximations (A.17), (A.26), may be considered as inverse dimensionless width of these massless packets on the light cone. From (B.24), (B.25):

$$F_{p_a,x_a}(\pi) = \frac{1}{i} \int_{-\infty}^{\infty} dy \|D(\pi - y)\|^{+} F_{p_a,x_a}^{(\tau)}(\pi), \quad \Xi^{\xi(\tau)}_{\{a\}}(\pi) = \frac{\hbar}{i \int_{-\infty}^{\infty} dy \|S(\pi - y)\|^{+}\Xi^{\xi(\tau)}_{\{a\}}(y), \quad (7.10)$$

where for $p' = p(1, n_p)$, all functions of $x$ with $m_a = 0$ in Eqs. (7.9), (7.10) as well as in Eqs. (A.20) – (A.23) in fact depend only on 2-dimensional $\pi' = (x^0, x^i)$ with $x^i = (n_p, x)$.

It is not difficult to construct the wave packet of type (7.1) – (7.3) for Majorana neutrino for both massive and massless cases [42, 43], with the same properties of normalization constant.

8 Conclusions

Due to the relativity of time only the transition amplitude (3.25) from point $x_a$ to point $x$ during the time interval $T = x^0 - x^0_a > 0$ is meaningful in relativistic QFT, unlike the non relativistic quantum-mechanical probability amplitude of finding the particle been localized in any point $x$ at any instant of time $t$. This relativity is reflected by respective formulation of Huygens’ principle [25].

Here it is shown how the consistent use of QFT axioms [18–24] fixes a general form of interpolating relativistic covariant wave-packet states for the fields of free massive particles also with higher spins. They are simply expressed via corresponded field operators, what elucidates profound physical meaning of analytical continuation of the Wightman functions for these fields into the complex Minkowski space and the meaning of remained arbitrariness of normalization of state. Interpolating wave packet contains covariant particle (antiparticle) states [15] only with positive (negative) energy (frequency) sign without their mixing. This packet propagates without change of its relativistically invariant width. It has non-relativistic Gaussian wave packet (2.8) as a precise non-relativistic limit (3.47), (A.28), independent of the remained normalization arbitrariness (3.33), (3.34), (3.35). Thus, the conventional belief about inevitable mixing of states with opposite energy sign, starting to propagate inside the pure non-relativistic Gaussian wave packet [25, 26, 31], has the only
limited sense. A similar assertion concerns relativistic CRG- approximation [1–7] for the wave packet, which by definition describes evolution off the mass shell (3.66).

Implementation of the interpolating wave packets to neutrino oscillation problem reveals their natural appearance in amplitude of macroscopic Feynman diagram by making use of any kind of “pole integration” [34, 35, 47], which naturally arises in any space-time-asymtotic regime. It arises as equivalence of the on-shell reduction for the off-shell composite wave function (6.9), by omitting the time-ordering $\theta$-functions: $\hat{\Upsilon} \mapsto \Upsilon$. The notion of composite wave function allows to transform the amplitude of Feynman diagram into the form of scalar product, similar to that is used in intermediate wave-packet picture of neutrino oscillation [31, 32, 44–46]. Moreover, unlike the non relativistic Gaussian packets and the CRG- approximation [1–3], formula (4.9d) demonstrates, that the usual causal evolution (6.3) for any such free relativistic wave packet [13] does not mix the states of particles with different energy sign (with antiparticles). However, as elucidated here, exactly for those wave packets (3.28), (3.29), (4.6) the causal ordering, naturally arising for the actual macroscopic scattering processes (6.1) similarly (4.9d) and according to Huygens’ principle, leads to the natural transformation (6.1)–(6.9) of integrated causal neutrino propagator $S_c^j(x−y)$ into the composite off-shell wave function (6.12), (6.13), and then into the composite on-shell one (6.15)–(6.17), as the linear superposition of free interpolating wave-packet states with the same neutrino mass $m_j$ but with different widths. The last state already may be fully localized for $V^+ \ni \zeta_a \to 0$ as (6.106). Non accidentally both these functions are just the differently defined boundary values in complex Minkowski space (B.15), (B.16) and (B.17), (B.18) or (B.9) and (B.12) respectively for scalar or spinor case in fact of the same invariant analytical function $h(Z)$ [20], defined by (3.34), (B.19).

Thus, unlike the non relativistic quantum mechanics and optics, the relativistic QFT admits the massive wave-packets interpolating in above covariant sense only in the form of Eqs. (3.28), (3.29) and (4.5), (4.6), (4.15)–(4.16), respectively. This wave packet has also a finite limit of $m \to 0$, $\sigma \to 0$ simultaneously, with fixed $\tau \sim \sqrt{m c / \sigma} \epsilon$, $\epsilon > 0$ (7.9), (7.9c). It separates naturally the light-cone degrees of freedom and elucidates the origin of possible arbitrariness of wave packet for the massless case.

The given here exact calculations of composite wave functions and of their universal asymptotic behavior may be successfully repeated for three-packet vertexes of a “little donkey” diagram [36–38] of the process like $\ell + A \oplus B \to A' \oplus (B' + \ell')$. To this end one can use the wave packets only for incoming particles $I_{C\oplus D} \in \ell, A, B$ and/or for one of outgoing $B'$, using the plane waves at least for one of outgoing unregistered particles $F_{C\oplus D} \in A', B', \ell'$ in the both $\{C\}$ and $\{D\}$ vertexes, or otherwise, by using the wave function (3.51) of “infinitely heavy nuclei” for some of particles $A, A', B, B'$ [39, 40]. That may be the subject of subsequent works.

The recent results of refs. [5–8] demonstrate the possibility of direct experimental manifestation of the wave packets of external particles also for the small-distance effects, such as the reactor antineutrino anomaly [55].
Acknowledgments

Authors thank V. Naumov and D. Naumov for usefull discussions and A. N. Vall, I. F. Ginzburg, E. Kh. Akhmedov, N. V. Ilyin, E. G. Aman for important comments.

A Various limiting cases of covariant wave packet.

The another way to check the Eq. (3.49) uses the relations for \(|k − p| ≪ E_p = \sqrt{p^2 + m^2}:

\[ E_k \approx E_p + (v_p \cdot (k - p)) + \frac{1}{2E_p}(k - p)^j (\delta^j - v^j_p v^j_p)(k - p)^j, \]

with: \(v_p = \frac{p}{E_p}\), (A.1)

\((k_p) = E_kE_p - (k \cdot p) \approx m^2 + \frac{1}{2}(k - p)^j (\delta^j - v^j_p v^j_p)(k - p)^j, \quad p \equiv |p|n_p, \)

whence: \(\phi^j(k, p) = e^{i(p_a - k)x_a)}(p\{p_a, x_a, \sigma\}) = N\sigma e^{-(k\xi_a)} \to \frac{N\sigma e^{-g_1(k_p)}}{\tau \to \infty}, \) or (A.3)

\((for \ g_2 = 0, \ \tau = m^2g_1) = \frac{\kappa(\tau)}{m^2} e^{-g_1(k_p)} \to (near \ the \ maximum: \ |k - p| ≪ E_p) \)

\[ \to \frac{(2\pi)^2}{m^2} \frac{I(\tau)}{h(\tau^2)} e^{-((k-p)T(k-p))} \to 2m(2\pi)^3 \frac{E_p}{m} \left[ \frac{|T|}{\pi^j} \right]^{\frac{1}{2}} e^{-((k-p)T(k-p))} = (A.5) \]

\[ = 2m(2\pi)^3 \frac{E_p}{m} \left[ \frac{g_1 m^2}{2\pi E_p^2} \right]^{\frac{1}{2}} \exp \left[ -\frac{g_1 m^2}{2 E_p^2} (k || - p ||)^2 \right] \left\{ \frac{g_1}{2\pi} e^{-(g_1/2)(k_1 \perp - p_1 \perp)^2} \right\} (A.6) \]

\[ \to (2\pi)^3 2E_p \delta (k_1 || - p_1 ||) \delta_1 (k_1 \perp - p_1 \perp) = (2\pi)^3 2E_p \delta_2 (k - p), \quad k = n_p k || + k \perp, \quad (A.7) \]

\((k - p)^2 = (k || - p ||)^2 + (k_1 \perp - p_1 \perp)^2, \) with: \(p \perp \to 0, \ p_ \parallel = |p|, \ k_\parallel = (k \cdot n_p), \)

and for: \((T^j)^j = \frac{g_1}{2} (\delta^j - v^j_p v^j_p), \quad |T| = \text{det}\{T\} = \left( \frac{g_1}{2} \right)^3 (1 - v_\parallel^2) = \left( \frac{g_1}{2} \right)^3 \frac{m^2}{E_p}, \)

may be easy obtained by direct calculation as rotationally invariant determinant of separable positively defined operator (matrix). Surely the nonzero “independent” limit \(m \to 0\) of (3.48) or (A.6) already implies \(\sigma = 0\) i.e. plane wave (3.49), (A.7). Whereas the usual solution to free massless KG equation (4.17) with \(\sigma \neq 0, \ z_a = x_a + i\zeta_\sigma(p_a, \sigma)\) requires changed normalization:

\[ m^2 \psi_\sigma(p_a, x_a - x) \to (−i)\mathcal{R}(0)D_0^-(x - z_a) = (−1)\mathcal{R}(0)(2\pi)^{-2} (x - z_a)^{-2}. \] (A.10)

As it is easy to see the Eq. (A.4) admits also the limit \(\sigma \to 0, \ g_1 \to \infty\) jointly with \(m \to 0\) for fixed \(\tau = m^2g_1, \ p_a^\nu = (p^0, p), \ p^0 = E_p \to |p| = p, \ k^\nu = (k^0, k), \ k^0 = E_k \to |k| = k, \)

so,
that \( p'_\sigma \rightarrow p(1, n_p) \), \( k' \rightarrow k(1, n_k) \), with the help of the relations:

\[
\lim_{g_1 \rightarrow \infty} g_1 e^{-g_1 Y} = \delta(Y), \quad (Y \geq 0), \quad \text{where, for } g_1 \gg g_2:
\]

\[
(k'_\sigma) = g_1 (k p_\sigma) \equiv g_1 [E_k E_p - (k \cdot p)] \approx g_1 kp [1 - (n_k \cdot n_p)] + \frac{\tau}{2} \left( \frac{k}{p} + \frac{p}{k} \right),
\]

\[
\delta(1 - (n_k \cdot n_p)) \rightarrow 2\pi \delta_\Omega(n_k, n_p), \quad \text{under } d\Omega(n_k).
\]

Whence:

\[
\phi^\sigma(k, p) = \frac{\mathcal{N}(\tau)}{\tau} g_1 e^{-g_1 (k p_\sigma) / \sigma_0} \phi_\tau(k, p),
\]

\[
\phi_\tau(k, p) = 2\pi \frac{\mathcal{N}(\tau)}{\tau} \exp \left[ -\frac{\tau}{2} \left( \frac{k}{p} + \frac{p}{k} \right) \right] \delta_\Omega(n_k, n_p),
\]

or:

\[
\phi_\tau(k, p) = (2\pi)^3 \frac{I(\tau)}{\tau e^\tau h(\tau^2)} \exp \left[ -\frac{\tau}{2kp} (k - p)^2 \right] \frac{\delta_\Omega(n_k, n_p)}{kp},
\]

giving:

\[
\phi_\tau(k, p) \rightarrow (2\pi)^3 \left( \frac{2\tau}{\pi} \right)^{1/2} \exp \left[ -\frac{\tau}{2kp} (k - p)^2 \right] \frac{\delta_\Omega(n_k, n_p)}{kp},
\]

for \( \tau \gg 1 \) with the use of Eqs. (3.33) – (3.35), or (B.19), (B.20). According to (3.43), (3.44), for \( \tau \rightarrow \infty \) this returns us to Eq. (A.7). For \( \tau \ll 1 \) Eqs. (A.15), (3.36) gives:

\[
\phi_\tau(k, p) \rightarrow 2\pi \frac{\mathcal{N}(0)}{\tau} \frac{\delta_\Omega(n_k, n_p)}{kp}.
\]

Since \( d\Omega(n_k) \equiv d^2 k_\perp / (kk_\parallel) \), then for \( n_k \simeq n_p \), \( k_\parallel \simeq k \) in Eqs. (A.15) – (A.18) we can replace:

\[
\frac{\delta_\Omega(n_k, n_p)}{kp} \rightarrow \frac{k}{p} \delta_2(k_\perp).
\]

The obtained longitudinal in fact non Gaussian profile with arbitrary \( \tau \) in (A.15), (A.16), (A.17) reflects above mentioned freedom of profile function of wave packet appearing for interpolating wave packet of massless particle unlike the massive one in (A.6), (A.7).

In spite of the singularity of the measure, the limit (A.15) is uniform with respect to integral of Lorentz invariant Fourier-representation (3.15) \( \rightarrow (3.29) \), (3.30). So, the dimensionless function (3.59) of dimensionless argument (3.60) for fixed \( \tau = m^2 g_1, \bar{z} = x_a - x \), \( p'_\sigma \rightarrow p(1, n_p) \) with \( p'_0 = 0 \), \( p^0 = p \), \( (p_0, \bar{z}) \equiv p (\bar{z}^0 - (n_p \cdot \bar{x})) \equiv p (\bar{z}^0 - \bar{x}_\parallel) \equiv p \bar{x}_\parallel \):

\[
\psi_\sigma(p_a, \bar{z}) \rightarrow \frac{\mathcal{N}(\tau)}{h(\tau^2) h(2\tau^2)} \exp \left[ -\frac{\tau}{2kp} (k - p)^2 \right] \frac{\delta_\Omega(n_k, n_p)}{kp},
\]

is still a Fourier-image (3.15) of limit \( \phi_\tau(k, p) \) (A.15), (A.16) of its momentum profile (3.28). Substitution of above to representation (3.15) leads to the integral:

\[
\psi^\tau(p, x) = \frac{\mathcal{N}(\tau)}{(2\pi)^2 2\pi \tau} \int_0^\infty dk \exp \left[ ik (x^0 - (n_p \cdot x)) - \frac{\tau}{2} \left( \frac{k}{p} + \frac{p}{k} \right) \right],
\]

which due to (B.19) with \( k = 2p\tau t \) coincides with (A.20). This reflects a drastic change of Lorentz symmetry [27, 29] for massless states since the problem in fact is reduced from...
3+1 to 1+1 dimensions. Indeed, substituting the massless wave packet (7.9a) into 3D scalar product (3.55c) one immediately shows the necessity of condition \( n_{p_b} = n_{p_a} \) for its existence. Thus the inner product for these light-cone wave packets (7.9a) should be of the following 1D form for \( \mathcal{F}'' = (x_0, x_1) \):

\[
\left( F^{(\tau_b)}_{p_b x_b}, F^{(\tau_a)}_{p_a x_a} \right) = \int_{-\infty}^{\infty} dx_1 \left| F^{+\tau_b}_{p_b x_b}(\mathcal{F}) \left( i \partial_{x_1}^{+\tau_b} \right) F^{(\tau_a)}_{p_a x_a}(\mathcal{F}) \right| = \mathcal{M}_{ba} e^{i(p_{b}x_{b}) - i(p_{a}x_{a})} (-\partial_{\mathcal{F}''}) h(Z_{ba}^b), \quad (A.22)
\]

where: \( Z_{ba}^b = \tau_b^2 + \tau_a^2 + \tau_a \tau_b \left( \frac{p_b}{p_a} + \frac{p_a}{p_b} \right) - 2i(\tau_b p_b + \tau_a p_a)(x_b - x_a) \rightarrow 4\pi^2, \quad (A.23) \)

\[
\mathcal{M}_{ba} = \frac{8 N_b^2(\tau_b) N_a(\tau_a)}{(2\pi)^3}, \quad (A.24)
\]

so the norm: \( \left( F^{(\tau)}_{p_a x_a}, F^{(\tau)}_{p_a x_a} \right) = \frac{N^2(\tau)}{\tau^3} \frac{\partial}{\partial \tau} \frac{h(4\tau^2)}{(2\pi)^3} \rightarrow 2\sqrt{\pi}\tau, \quad (A.25) \)

due to (3.35). This dimensionless norm differs from them in 3+1 D case (4.12), (4.13). The approximation (A.5), (A.6) also admits the joint limit \( \sigma \rightarrow 0 \) with \( m \rightarrow 0 \) for fixed \( \tau \):

\[
(A.6) \lim_{m \rightarrow 0} \left( \frac{2\pi}{\sigma} \right)^3 \frac{\mathcal{I}(\tau)}{h(\tau^2)} \delta_2(k_\perp). \quad (A.26)
\]

For \( \tau \rightarrow \infty \) it also returns to Eq. (A.7). This approximation, unlike \( \phi_r(k, p) \) (A.15) with correct Fourier-image (A.21) = (A.20), is very rough and leads to divergent integral.

For the non relativistic limit: \( c \rightarrow \infty \), with \( x^a = c t_a, m \rightarrow mc, v^a \rightarrow \frac{v^a}{c^2} \rightarrow 0 \), and from (3.39), (3.47) or (A.2) – (A.5), (A.9), one has: \( (T)^{ij} \rightarrow (g_{1/2})^{ij}, c(k^0 - p^0) = E_k - E_p = \varepsilon_k - \varepsilon_p, E_p = mc^2 + \varepsilon_p, \varepsilon_p = p^2/(2m) \). Whence, Eqs. (2.8), (2.10) imply: \( g_1 = \sigma^{-2} \) for \( \sigma = \sigma_p \), with:

\[
\phi^\sigma(k, p) \rightarrow (2\pi)^3 2mc \left\{ \left( \frac{g_1}{2\pi} \right)^{3/2} e^{-(g_{1/2}/k-p)^2} \right\} = (2\pi)^3 2mc \frac{\left| k \right| \left| p, 0, \sigma \right|}{(2\sigma \sqrt{\pi})^{3/2}}, \quad (A.27)
\]

and thus: \( \lim_{c \rightarrow \infty} \frac{\left| k \right| \left\{ p_a, x_a, \sigma \right\}}{2mc} = \frac{(2\pi)^3}{(2\sigma \sqrt{\pi})^{3/2}} e^{i\alpha(\varepsilon_k - \varepsilon_p)} \left| k \right| \left| p_a, x_a, \sigma \right| >. \quad (A.28)\)

**B Some useful intermediate results, formulas and definitions.**

For \( k^0 = E_k > 0, \left( k \hat{s} \right) = 0, \xi = \pm 1, p^2 = m^2, (p_a \hat{s}_a) = 0, \) with arbitrary \( S \), one has:

\[
(\gamma k)^2 = k^2 = m^2, \quad (\gamma \cdot k)^2 = -k^2, \quad \hat{E}(k) = \gamma^0 (\gamma \cdot k) + \gamma^0 m = \hat{E}^\dagger(k), \quad (B.1)
\]

\[
(\hat{E}(\xi k))^2 = E_k^2 = k^2 + m^2, \quad (\gamma \cdot k) + \xi m = [E_k + \xi \hat{E}(\xi k)] \gamma^0 = \gamma^0 [E_k + \xi \hat{E}(-\xi k)], \quad (B.2)
\]

\[
[(\gamma k) + \xi m] \gamma^0 [(\gamma k) + \xi m] = 2E_k [(\gamma k) + \xi m], \quad (B.3)
\]

\[
[(\gamma k) + \xi m] \gamma^0 \xi(\xi k) [(\gamma k) + \xi m] = 2E_k [(\gamma k) + \xi m] m, \quad (B.4)
\]

\[
[(\gamma k) + \xi m] \gamma^0 \xi(\hat{s}) [(\gamma k) + \xi m] = 2E_k [(\gamma k) + \xi m] \gamma^0 (\hat{s}), \quad (B.5)
\]

\[
\text{Sp} \left\{ \frac{[(\gamma k) + m\xi] \gamma^0 \xi(\hat{s}) [(\gamma k) - (k \hat{s})]}{m\xi} \right\} = \pm 8k^0 \left\{ \xi_{\alpha \beta} \hat{g}_{\alpha \beta} - \xi_{\mu \nu} \hat{g}_{\alpha \beta} \right\} \hat{S}_{\alpha}^{\nu} (k^0 k^b + k^0 p^3_b) \quad (B.6)
\]

\[
= \pm 8k^0 \left\{ -\hat{s}_a \hat{S} + (kp_a) + (k \hat{s}_a)(k \hat{s}) + (k \hat{s}_a)(p_a \hat{S}) \right\}, \quad (B.7)
\]
For Dirac fields in (4.6) [19], for $k^0 = E_k > 0$:

$$\frac{1}{i} S^{-\xi}(x - y) = \int \frac{d^3k}{(2\pi)^3} 2k^0 \sum_{r = \pm 1/2} U_{k,r}(x) D_{k,r}^\xi(y) = \left[ i(\gamma \partial_x) + m \right] \frac{1}{i} D_m^{-\xi}(x - y), \quad (B.9)$$

$$\{ \psi(x), \overline{\psi}(y) \} = \frac{1}{i} S(x - y) = \frac{1}{i} \sum_{x_{\xi} = \pm} S^{-\xi}(x - y), \quad S(x - y) \big|_{x^0 = y^0} = i\gamma^0 \delta_3(\mathbf{x} - \mathbf{y}). \quad (B.10)$$

$$\langle 0 | T (\psi(x) \overline{\psi}(y)) | 0 \rangle = \frac{1}{i} S^c(x - y) = \sum_{x_{\xi} = \pm} \theta (\xi(x^0 - y^0)) \frac{\xi}{i} S^{-\xi}(x - y) = \quad (B.11)$$

$$= i \int \frac{d^4q}{(2\pi)^4} e^{-i(q(x-y))} \left( \left( \frac{\gamma q}{m^2} + m \right) \frac{1}{i} D_m^c(x - y) \right). \quad (B.12)$$

For vector fields in (4.14)–(4.16) and $A_{k,\lambda}^\nu(x) = \epsilon_{\nu}^\lambda(k) f_k(x)$ with polarization vector $\epsilon_{\nu}^\lambda(k)$:

$$\left( \epsilon_{\lambda}^\nu(k) \epsilon_{(\sigma)}(k) \right) = -\delta_{\lambda\sigma}, \quad \sum_{\lambda = 1}^{3} \epsilon_{(\lambda)}(k) \epsilon_{(\lambda)}^\nu(k) = -g_{\mu\nu} + \frac{k_{\mu} k_{\nu}}{m^2}, \quad k^0 = E_k > 0, \quad (B.13)$$

$$\frac{1}{i} D_{\mu\nu}^{-}(x - y) = -\int \frac{d^3k}{(2\pi)^3} 2k^0 \sum_{\lambda = 1}^{3} A_{k,\lambda}(x) A_{k,\lambda}^{*\nu}(y) \left( g_{\mu\nu} + \frac{\partial_{\mu} \partial_{\nu}}{m^2} \right) \frac{1}{i} D_m^{-}(x - y), \quad (B.14)$$

where: $D_m^+ = \pm \left( \frac{m^2}{(2\pi)^2} \right) h \left( -m^2 (x + i\zeta)^2 \right), \quad \text{for} \quad \zeta^2, \zeta^0 \to +0, \quad (B.15)$

with: $-Z/m^2 = (x + i\zeta)^2 = x^2 + 2i(x_\zeta) - \zeta^2 \to x^2 + 2ix_\zeta \zeta^0 \to x^2 \neq 0 \zeta(x^0), \quad (B.16)$

$$D_m^c(x) = \theta(x_0^0) D_m^-(x) + \theta(-x_0^0) D_m^-(x) = D_m^-(x) = \frac{1}{i} \int d\zeta \ h \left( \frac{m^2 (i\zeta - x^0)}{m^2 - x^2} \right), \quad (B.17)$$

with: $-Z/m^2 = x^2 - i0$, so: $D_m^c(x) \equiv D_m(x_0^0; \mathbf{x}), \quad D_m^c(x) = D_m(|x^0|; \mathbf{x}), \quad (B.18)$

for the analytical function $h(Z)$, defined for the main branch $\sqrt{Z} > 0$ at $Z > 0$, as [20, 49]:

$$h(Z) = \frac{K_1(\sqrt{Z})}{\sqrt{Z}} = \int_0^\infty dt \exp \left\{ -\frac{1}{4t} - tZ \right\} \equiv \quad (B.19)$$

$$\int_0^\infty dt e^{f(t)} \approx \left[ \frac{2\pi}{-f''(\bar{t})} \right]^{1/2} e^{\frac{f(t)}{2}} = \sqrt{\frac{\pi}{2}} Z^{-3/4} e^{-\sqrt{Z}}, \quad (B.20)$$

for $|Z| \to \infty$, $|\arg Z| < 3\pi$ [49], where for the saddle point: $f'(\bar{t}) = 0, \quad 1/\bar{t} = 2\sqrt{Z}$.

The following relations are also useful for our aims:

$$\frac{1}{i} D_m^{-\xi}(x - y) = \frac{\xi}{i} D_m^{-\xi}(x - y) = \xi \int \frac{d^3k}{(2\pi)^3} \theta(k^0) \delta(k^2 - m^2) e^{-i\xi(k(x - y))}, \quad (B.21)$$

$$\frac{1}{i} D_m^c(x - y) = \sum_{x_{\xi} = \pm} \theta (\xi(x^0 - y^0)) \frac{\xi}{i} D_m^{-\xi}(x - y), \quad \theta(t) = \frac{1}{2i\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{\im \omega t}}{\omega - i0}, \quad (B.22)$$

$$\int d\Sigma_\lambda (x) \frac{1}{i} D_m^{\eta}(x_{\eta}^0 - x) \left( i \partial_{\xi}^x \right) \frac{1}{i} D_m^{-\xi}(x - z_{\alpha}) = \delta_{\eta\xi} \frac{1}{i} D_m^{-\xi}(x_{\eta}^0 - x_{\alpha}), \quad \xi, \eta = \pm. \quad (B.23)$$
Let’s consider integrals over \( y_\perp \) of the massless propagators. For \((\gamma \partial_x) = \gamma^0 \partial^0 + (\gamma \nabla_x)\), with \( p'_0 = p(1, n_p) \) they will depend only on 2-dimensional variable \( \pi^\nu = (x^0, x^\|) \), with \( x_\| = (n_p \cdot x) \), for \( x_{\perp} = x - n_p x_\| \), \( \partial_{x_{\perp}} = (n_p \cdot \nabla_x) \), \( \gamma_\| = (n_p \cdot \gamma) \), \( \gamma \partial_{x_{\|}} = \gamma^0 \partial^0 + \gamma_\| \partial_{x_{\|}} \), \( (\gamma \partial_{\pi}) = \gamma^0 \partial^0 + \gamma_\| \partial_{x_{\|}} \), \((x - y)^2 = (x^0 - y^0)^2 - (x^\| - y^\|)^2 - (x_{\perp} - y_{\perp})^2 \), where \((\gamma \partial_x) \mapsto (\gamma \partial_{\pi})\):

\[
D_{(2)}^c(\pi - \gamma) = \int d^2 y_{\perp} D_{(4)}^c(x - y) = \int \frac{i d^2 y_{\perp}}{(2\pi)^2[0 - (x - y)^2]} = \frac{1}{4\pi i} \ln \left[ \frac{i0 - (\pi - \gamma)^2}{R^2} \right],
\]

(B.24)

\[
S_{(2)}^c(\pi - \gamma) = \int d^2 y_{\perp} S_{(4)}^c(x - y) = i(\gamma \partial_{x_{\perp}}) D_{(2)}^c(\pi - \gamma) = \frac{(\gamma, \pi - \gamma)}{2\pi[(\pi - \gamma)^2 - i0]}.
\]

(B.25)

Here \( R^2_{\perp} \mapsto \mu^{-2} \) is arbitrary scale of infrared regularization for massless 1+1-dimensional case [20]. Note, that according to [19, 20, 49, 50, 52], for \( N = N - 1 + 1 \equiv 2\Lambda + 2\)-dimensional case it is easy to observe the inversion of (B.24) for the causal propagator with \( \varrho = x - y \):

\[
D_{(N)m}^c(\varrho) = \int \frac{d^N q}{(2\pi)^N} \frac{e^{-iq\varrho}}{(m^2 - q^2 - i0)} = i \left( \frac{m}{i0 - \varrho^2} \right)^{\Lambda} \Lambda K_{\pm\Lambda}(m(i0 - \varrho^2)^{1/2}) \frac{(2\pi)^{\Lambda+1}}{(2\pi)^{\Lambda+1}},
\]

(B.26)

so: \( D_{(2)m}^c(\varrho) = \frac{i}{2\pi} K_0(m(i0 - \varrho^2)^{1/2}) \), \( D_{(N+2S)m}^c(\varrho) = \left( \frac{1}{\pi} \frac{\partial}{\partial \varrho^2} \right)^S D_{(N)m}^c(\varrho) \). (B.27)

with respectively changed dimension of \( \varrho \) for any integer \( S \) and Macdonald function \( K_{\Lambda}(z) \) [49].

So called Grimus-Stockinger theorem [40], for \( R = Rn, n^2 = 1 \), and a sufficiently smooth (non oscillating) function \( \Phi(q) \in C^3 \) decreases at least like \( 1/q^2 \) together with its first and second derivatives, gives the leading asymptotic behavior with \( R = |R| \mapsto \infty \) for the integral:

\[
\mathcal{J}(\pm R) = \int \frac{d^3 q}{(2\pi)^3} \frac{e^{\pm i(q \cdot R)\Phi(q)}}{(q^2 - K^2 - i0)} \bigg|_{R \to \infty} = \frac{e^{ikR}}{4\pi R} \Phi(\pm Kn) \left( 1 + O(R^{-1/2}) \right),
\]

(B.28)

where \( K^2 > 0 \) (otherwise for \( K^2 < 0 \) it fails at least as \( O(R^{-2}) \)). Its most simple explanation for analytical \( \Phi(q) \) is given in §130 of [9] or in [10–12] with the help of the relation in the sense of distributions for \( q = \omega q \):

\[
e^{\pm i(q \cdot R)} \bigg|_{R \to \infty} = \frac{2\pi i}{q R} \left( e^{-i q R} \delta_\Omega(\omega, \pm n) - e^{i q R} \delta_\Omega(\omega, \pm n) \right) + O(R^{-2}) \]

(B.29)

\[
= \frac{2\pi i}{q R} \left( e^{\mp i q R} \delta_\Omega(\omega, -n) - e^{\mp i q R} \delta_\Omega(\omega, n) \right) + O(R^{-2}).
\]

(B.30)

In fact Jacob-Sachs theorem [34, 54] states, that for sufficiently smooth (non oscillating) function \( \Psi(q) = \Psi(q_0; q) \), distinct from zero only within certain finite bounds on \( q^2 = q_0^2 - q^2 \) for \( M_1^2 < q^2 < M_2^2 \), \( q_0 > 0 \), where it is taken to be infinitely differentiable, the integral \( I(T) \) has the following asymptotic behavior at \( T \to +\infty \):

\[
I(T) = \int \frac{dq_0}{2\pi} \frac{e^{-iTq_0} \Psi(q)}{2E_{q_j}(E_{q_j} - q^0 - i0)} \bigg|_{T \to +\infty} \mapsto \frac{i e^{-iT\dot{E}_{q_j}}}{2E_{q_j}} \Psi(E_{q_j}; q).
\]

(B.31)
B.1 Scalar composite wave function

In order to outline and underline the differences between scalar and spinor case of Eqs. (6.46) – (6.54), we give here a brief scalar version of formalism of composite wave function, following to the fermionic case (6.1) – (6.9) of Section 5. We start from Feynman amplitude (6.46) in configuration space with omitted at further up to a sign the matrix factors \( \tilde{M}_{J(D/C)} \):

\[
\tilde{A}_{DC}^{(\xi)} = \int d^4 x \Psi^*_D(x) \Psi_{DI}(x) \int d^4 y \frac{1}{i} D^\xi_m(x-y) \Psi^*_C(y) \Psi_{CI}(y) = \sum_{\xi=\pm} \tilde{A}_{DC}^{(\xi)}. \tag{B.32}
\]

\[
\tilde{A}_{DC}^{(\xi)} = \int d^4 x \Psi^*_D(x) \Psi_{DI}(x) \int d^4 y \theta(x^0 - y^0) \frac{\xi}{i} D^{-\xi}_m(x-y) \Psi^*_C(y) \Psi_{CI}(y), \tag{B.33}
\]

\[
\delta_{\text{sgn}} \tilde{A}_{DC}^{(\xi)} = \xi \int d^4 p \tilde{\mathcal{F}}_{(D)}^{(\xi)}(p) (i \partial_0^{\xi}) \tilde{\mathcal{F}}_{(C)}^{(\xi)}(p) = \xi \int d\Sigma_{\lambda}(q) \tilde{\mathcal{F}}_{(D)}^{(\xi)}(p) (i \partial_0^{\xi}) \tilde{\mathcal{F}}_{(C)}^{(\xi)}(p), \tag{B.34}
\]

\[
\tilde{\mathcal{F}}_{(C)}^{(\xi)}(q) = \int d^4 y \theta([\pm]\xi(q^0 - y^0)) \frac{1}{i} D^{-\xi}_m([\pm](q-y)) \Psi^*_C(y) \Psi_{C/D}^I(y), \tag{B.35}
\]

where: the first two equations follow from decomposition (B.22), the third equation reflects the scalar group property (B.23), and the last Eq. (B.33), for \([C/D]\) correlated with \([\pm]\), combines both the two fermionic relations (6.4), (6.5), since the scalar propagator is undirected. Thus, from (6.46), (B.32), due to (B.26), (B.21), one has “overlap function”, which for the products of arbitrary number of initial and final wave packets \( \Psi^*_C(y) \Psi_{C/D}^I(y) \), gives the amplitude (B.33) on mass shell, i.e. without \( \theta(t) \)- function as (cmp. with (6.48) – (6.54)):

\[
\tilde{V}_{(C/D)}(q) = (-i) \int d^4 y e^{[\pm]iqy} \Psi^*_C(y) \Psi_{C/D}^I(y), \tag{B.36}
\]

\[
\tilde{A}_{DC}^{(\xi)} \xrightarrow{\theta(t)=1} (-1) \int \frac{d^4 q}{(2\pi)^4} \tilde{V}_{(D)}(\xi q) \theta(q^0) (q^2 - m_j^2) \tilde{V}_{(C)}(\xi q), \tag{B.37}
\]

\[
\tilde{\mathcal{F}}_{(C)}^{(\xi)}(q) = \int \frac{d^4 q}{(2\pi)^4} e^{-[\pm]iq(q^0)} \tilde{\mathcal{F}}_{(C/D)}^{(\xi)}(q), \tag{B.38}
\]

\[
\tilde{\mathcal{H}}_{(C)}^{(\xi)}(q) = \int d^4 q e^{[\pm]iq(q^0)} \tilde{\mathcal{F}}_{(C/D)}^{(\xi)}(q) = \frac{\xi \tilde{V}_{(C/D)}^{(\xi)}(q)}{2E_{aq}(E_{aq} - q^0 - i\delta)}, \tag{B.39}
\]

\[
\tilde{A}_{DC}^{(\xi)} = \xi \int \frac{d^4 q}{(2\pi)^4} \int \frac{d\theta_0}{2\pi} e^{i\theta_0(q^0 - q)} \tilde{\mathcal{H}}_{(D)}^{(\xi)}(p^0, q) \tilde{\mathcal{H}}_{(C)}^{(\xi)}(q^0, q). \tag{B.40}
\]

As well as the two spinor composite wave functions \( \tilde{\mathcal{F}}_{(C/D)}^{(\pm)}(q) \) (6.12) and \( \tilde{\mathcal{F}}_{(C/D)}^{(\pm)}(q) \) (6.13) via (6.15), (6.28), (6.48) are associated with the same scalar one \( \tilde{\mathcal{F}}_{(C/D)}^{(\pm)}(q) \), the two spinor ones \( \tilde{\mathcal{F}}_{(C/D)}^{(\pm)}(q) \) (6.4) and \( \tilde{\mathcal{F}}_{(C/D)}^{(\pm)}(q) \) (6.5), rather then \( \tilde{\mathcal{F}}_{(C/D)}^{(\pm)}(q) \), are associated with the scalar composite wave function \( \tilde{\mathcal{F}}_{(C/D)}^{(\pm)}(q) \).

C Calculation of two-packet overlap function.

Calculation of overlap function \( \tilde{V}_{(C)}(q) \) (6.18) is simplified by the following hint. Substitution of representation (3.30) in explicitly invariant form (3.29), (3.15) with definitions
\( q \) leads to the following structure of integral in \((6.18)\) with some function \( \Phi(a, b, \ldots) \): 
\[
\mathfrak{I}(q) = \int d l \, \theta(-l^0) \delta(l^2 - m^2_\mu) \int d r \, \theta(r^0) \delta(r^2 - m^2_\mu) \delta_4(q - l - r) \Phi((lQ), (rQ'), \ldots), \tag{C.1}
\]
for: \( r = \eta_2 p + \kappa, \quad l = \eta_1 p - \kappa, \quad d_4 d^4 r = d^4 p d^4 \kappa, \) with: \( \eta_1 + \eta_2 \equiv 1, \quad p = l + r, \tag{C.2} \)
\[
\eta_2 - \eta_1 = \frac{m^2_\mu - m^2_\nu}{q^2}, \quad \sigma = \frac{\Delta(q)}{4q^2}, \quad \delta(\omega) \delta(v) = 2 \delta(\omega - v) \delta(\omega + v), \quad \forall \omega, v, \tag{C.3}
\]
reads: \( \mathfrak{I}(q) = \frac{1}{2} \int d^4 \kappa \, \delta((\kappa q)) \, \delta(\kappa^2 + \sigma) \, \theta(\kappa^0 - \eta_1 q^0) \, \theta(\kappa^0 + \eta_2 q^0) \Phi((\kappa q), (\kappa q'), \ldots). \tag{C.4} \)
\( q^2 > 0 \) gives \( \eta_2 > 0, \sigma > 0, \kappa^2 < 0, \kappa^0 \rightarrow 0, q^0 > 0, \eta_1 < 0. \) In the rest frame of time-like vector \( q^\mu_0 = (q^0, q_\mu = 0) \) there are the following invariant substitutions, with any external vectors \( Q, Q' \), for \( \mathfrak{I}(q) \rightarrow \mathfrak{I}(q) \):
\[
q^0_0 \rightarrow \sqrt{-q^2}, \quad Q^0_0 \rightarrow \frac{(qQ)}{\sqrt{-q^2}}, \quad Q^2_0 \rightarrow \frac{(qQ)^2 - q^2 Q^2}{q^2}, \quad (\kappa Q) \rightarrow - (\kappa \cdot Q_0), \quad n^2 = 1, \tag{C.5}
\]
with: \( \delta((\kappa q)) \, \delta(\kappa^2 + \sigma) \rightarrow \delta((\kappa q)) \, \delta(\sigma - \kappa^2), \) for: \( d^4 \kappa = d\kappa^0 |\kappa|/2 \, d\kappa^2 d\Omega_3(n_\kappa), \tag{C.6} \)
giving: \( \mathfrak{I}(q) = \frac{\Theta[\Delta, q]}{4} \sqrt{-q^2} \int d\Omega_3(n_\kappa) \Phi( - (\kappa \cdot Q_0), - (\kappa \cdot Q'_0), \ldots), \quad \kappa = n_\kappa \sqrt{-\sigma}, \tag{C.7} \)
with: \( \int d\Omega_N(n) \, n^{a_1 \ldots a_2 \ldots a_{2l-1} \ldots a_2} = \frac{\Omega_N}{C_N(l)} \sum_{\varphi = 1}^{(2l-1)!!} \{ \delta \ldots \delta \}^{a_1 a_2 \ldots a_{2l-1} a_2}, \tag{C.8} \)
where: \( \int d\Omega_N(n) = \Omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}, \quad C_N(l) = \prod_{k=0}^{l-1} (N + 2k) = \frac{2^l \Gamma(l + N/2)}{\Gamma(N/2)}, \tag{C.9} \)
for \( N \) - dimensional space, with \( C_N(0) = 1, \quad C_N(1) = N, \quad C_3(l) = (2l + 1)!!, \quad \Omega_3 = 4\pi, \) and sum stands for permutations \( \varphi \) symmetrizing all the indices \( a_1 \ldots a_{2l} \). From \((C.4)\) for \( m_\pi > m_\mu \) with the conditions \((6.24) - (6.26)\): \( \Theta[\Delta, q] = \theta(\Delta) \theta(q^2) \theta(q^0), \quad \theta(\Delta) = \theta(|m_\pi - m_\mu|^2 - q^2). \)

The transverseness of \( \kappa \) with \( q \) becomes very convenient when \( \mathfrak{I}^{\mu}(q) \) contains tensor structure, because for transverse tensor \( q^2 \Pi^\mu_\nu = g^{\mu\nu}q^2 - q^\sigma q^\sigma \nu, \quad \Pi^\mu_\nu Q_\nu = Q^\mu_0, \quad (qQ \Pi_\mu_\nu) \equiv 0: \)
\[
\Pi^\mu_\nu = \Pi^\mu_\nu \lambda^\nu \lambda^\mu, \quad q^2 Q^2 - (qQ)^2 = q^2 Q_\mu \Pi^\mu_\nu Q_\nu = q^2 (QQ \Pi_\mu_\nu) = q^2 Q^2 \Pi_\mu_\nu = - q^2 Q^2, \tag{C.10}
\]
\[
\mathfrak{I}^{\mu}(q) = \frac{1}{2} \int d^4 \kappa \, k^\mu \, \delta((\kappa q)) \, \delta(\kappa^2 + \sigma) \, \theta(\kappa^0 + \eta_2 q^0) \, \theta(\kappa^0 - \eta_1 q^0) \Phi((\kappa q)), \tag{C.11}
\]
and so on, where calculation of \( \mathfrak{I}_1(q) \) is reduced to previous case by multiplying on \( Q_\mu \). With the more complicated tensor structure, choosing for \( q^2, q^0 > 0, Q^2_\Pi < 0 \) one of \( Q^\mu_0 = 0, \) it is useful to transcribe the measure \((C.6)\) for \( \kappa^2 = \kappa^2_3 + \kappa^2_4, \quad \kappa^\mu = (\kappa^0, \kappa^1_\perp, \kappa^3, \varphi), \) as: \( d^4 \kappa = d\kappa^0 \kappa^3_1 \frac{1}{2} d\kappa^1_\perp d\varphi, \quad \int d\kappa^1_\perp \, \delta(\sigma - \kappa^2_3 - \kappa^2_\perp) = \theta(\sigma - \kappa^2_3), \quad |\kappa_\perp| = \sqrt{\sigma - \kappa^2_3}, \)
so that: \( \mathfrak{I}^{\mu\nu}(q) = \frac{\Theta[\Delta, q]}{4\sqrt{-q^2}} \int d\kappa^3_\perp \, 2\pi \int_0^{2\pi} d\varphi \, k^\mu k^\nu \Phi(\kappa_3 Q^3_\perp, \ldots), \) where: \( \int d\varphi \, k^\mu k^\nu \Phi, \tag{C.12} \)
is immediately expressed only via convolutions of tensors $\Pi_{q}^{\mu\alpha}$ and $\Pi_{q}^{\nu\beta}$ with all existing external vectors $Q, Q'$, and $\Pi_{q}^{\mu\nu}$ itself, and similarly for tensor structures of higher rank.

For $q^2 < 0$, $\Delta(q) > 0$, $\sigma = -|\sigma|$ the Breit frame of space-like vector $q^\mu = (0, 0, q^3)$, with $\kappa^\mu = (\kappa^0, \kappa_+, \kappa_-) \mapsto \kappa^\mu(0, \varphi)$, $\kappa^0 > 0$, $\kappa_+ = |\kappa_+| n_+$, $d\varphi = d\Omega_2(n_+)$, leads to the following invariant substitutions with any external vector $Q$ and integrals (C.4), (C.11) with $\hat{I}(q) \mapsto \hat{I}(q)$:

$$\begin{align*}
\lambda_3 &\mapsto \sqrt{-q^2}, \quad \lambda_3 \mapsto \frac{(qQ)}{\sqrt{-q^2}}, \\
\lambda_0^2 - \lambda_2^2 &\mapsto Q^2 + \lambda_0^2 \mapsto \frac{(qQ)^2}{q^2} = Q^2, \\
\delta((q\kappa)) \delta(\kappa^2 + \sigma) &\mapsto \frac{\delta(\kappa^3)}{\sqrt{-q^2}} \delta(\kappa_0 - |\sigma| - \kappa_-^2), \quad \text{for:} \quad d^4\kappa = d\kappa_0 d\kappa_3 \frac{1}{2} \frac{d\kappa_+^2 d\Omega_2(n_+)}{2}, \\
\int d\kappa^2_+ \delta(\kappa_0^2 - |\sigma| - \kappa_-^2) &\mapsto \theta(\kappa_0^2 - |\sigma|), \quad \text{and:} \quad |\kappa_-| = \sqrt{\kappa_0^2 - |\sigma|}, \\
giving: \hat{I}(q) &\mapsto \frac{\theta(-q^2)}{4 \sqrt{-q^2}} \int_{0}^{2\pi} d\kappa_0 \int 0 d\Omega_2(n_+) \Phi((\kappa Q) \mapsto \kappa^0 Q^0 - (\kappa_+^0, \sigma) \ldots). \quad (C.15)
\end{align*}$$

For $\hat{\Phi} = \exp\{i(rZ_\pi) - i(lZ)\} \mapsto \Phi = \exp\{i(qZ_\mu)\} \exp\{i(\kappa Z_-)\}$ with the help of (C.8), (C.9) and modified Bessel function $I_\lambda(r)$ [49], for any complex vector $R_N \neq R^*_N$, it leads to:

$$\begin{align*}
\int d\Omega_N(n) \exp\{n \cdot R_N\} &= 2\pi^{N/2} \sum_{l=0}^{\infty} \frac{(R_N^* \times)^l}{2^{2l} l! (l + N/2)} = \\
&= 2\pi^{N/2} \left(\frac{2}{R_N^* \times}^{-1/2}\right) \frac{N}{2} I_{N/2 - 1} \left((R_N^* \times)^{1/2}\right) = \begin{cases} 4\pi \frac{\sinh (R_N^* \times)^{1/2}}{(R_N^* \times)^{1/2}}, & (N = 3), \\
2\pi I_0 \left((R_N^* \times)^{1/2}\right), & (N = 2), \end{cases} \quad (C.17)
\end{align*}$$

$$\begin{align*}
R_3 &\mapsto -iZ_+ - \sqrt{\sigma} \quad \mapsto \hat{\Phi}_+ \sqrt{\kappa_0^2 - |\sigma|}, \\
R_2 &\mapsto -iZ_+ - \sqrt{\sigma} \quad \mapsto \hat{\Phi}_- \sqrt{\kappa_0^2 - |\sigma|}. \quad (C.18)
\end{align*}$$

$$\begin{align*}
\hat{I}(q) &= \frac{2\pi}{4} e^{i(qZ_\mu)} \left[\frac{\Theta|\Delta, q|}{2} \left[-q^2 Z_\mu^2 - \frac{1}{2}\right] \sinh \left(\chi \Delta \left[-q^2 Z_\mu^2 - \frac{1}{2}\right]\right) \sqrt{\sigma} = \chi \Delta \sqrt{q^2}, \\
\hat{I}(q) &= \frac{2\pi}{4} e^{i(qZ_\mu)} \left[\frac{\Theta|\Delta, q|}{2} \left[-q^2 Z_\mu^2 - \frac{1}{2}\right] \sinh \left(\chi \Delta \left[-q^2 Z_\mu^2 - \frac{1}{2}\right]\right) \sqrt{\sigma} = \chi \Delta \sqrt{q^2}, \quad (C.19)
\end{align*}$$

By using (C.13) together with the discussed below integral (C.24) and its analytic continuations (C.25), (C.26), one finds, if for the both pieces the same main branch Re $W > 0$ of
square root is used for the value

\[ W(w) = w^{1/2} = [q^2 Z^2 - (qZ_{-})^2]^{1/2} \equiv [q^2 Z^2_{\Pi_{-}}]^{1/2} \rightarrow \{ -q^2 \}^{1/2} \rightarrow \{ -Z^2_{\Pi_{-}} \}^{1/2} : \quad (C.20) \]

\[ \hat{V}_{\{C\}}(q) = \frac{N_{\sigma_{\mu}}N_{\sigma_{\pi}}}{(2\pi)^2} \hat{I}(q), \quad \text{with:} \quad \hat{I}(q) = \hat{I}(q) + \hat{I}(q) = 2\pi e^{i(qZ_n)} / 4 \times \]

\[ \times \left\{ \Theta[\Delta, q] \frac{\sinh \left( \chi_{\Delta} \left[ q^2 Z^2_{\Pi_{-}} \right]^{1/2} \right)}{\left[ q^2 Z^2_{\Pi_{-}} \right]^{1/2}} + \Theta(-q^2) \exp \left( \chi_{\Delta} \left[ (q^2) \left( -Z^2_{\Pi_{-}} \right) \right]^{1/2} \right) \right\} \]

\[ = 2\pi e^{i(qZ_n)} \frac{\Theta(\Delta)}{W} \left\{ \exp \left( \chi_{\Delta} \theta \right) - \theta(q^2) \exp \left( -\varepsilon(0) \chi_{\Delta} \theta \right) \right\}, \quad (C.22) \]

since for \( g_a \rightarrow +\infty \), i.e. \( \zeta_{\pm}^0, \zeta_{\pm}^+ \rightarrow +\infty \), or equivalently \( Y_{-} \rightarrow 0 \), the argument of this square root: \( w = q^2 Z^2_{\Pi_{-}} \rightarrow w_{\zeta} = -q^2 Z^2_{\Pi_{+}} > 0 \) for both cases \( q^2 \geq 0 \), because only either \( q \) or \( \zeta_{\Pi_{+}} \) (D.4), is a time-like vector. Here of course \( \Theta(\Delta) = \Theta[(m_{\pi} - m_{\mu})^2 - q^2] \), \( \varepsilon(0) = \text{sign}(q^0) \).

This essential difference of overlap function (C.1) from usual two-particle phase volume (D.4) is due to opposite sign in \( \Delta \) contributions for both cases (C.23) contributes for both cases \( q^2 \geq 0 \).

In order to obtain the answer (C.22) from (C.19) the formula \# 6.646.2 \[50\], or formula \# 4.17(5) \[51\] is used, at first for \( \alpha > 0 \), \( Re \gamma > |Re \beta| \) in the form:

\[ \int_{-\infty}^{\infty} dx e^{-\gamma x} I_0 \left( \beta \sqrt{x^2 - \alpha^2} \right) = \frac{\exp \left( -\alpha \sqrt{\gamma^2 - \beta^2} \right)}{\sqrt{\gamma^2 - \beta^2}}, \quad (C.24) \]

because for time-like \( \zeta_{\pm}^+ \): \( Re \gamma = Re (\zeta_{\pm}^0 \zeta_{\pm}^+ - \sqrt{\zeta_{\pm}^0}) \geq |Re \left( -Z^2_{\Pi_{-}} \right)^{1/2}| = |Re \beta| \), when \( g_a \rightarrow \infty \). Then that answer is understanding in the sense of analytic continuation with respect to both variables \( \beta \) and \( \gamma \). For \( \beta = \pm ib \) it is given by the formula \#4.15(9) \[51\] with \( \alpha > 0 \), \( Re \gamma > |Im \beta| \):

\[ \int_{-\infty}^{\infty} dx e^{-\gamma x} J_0 \left( \beta \sqrt{x^2 - \alpha^2} \right) = \frac{\exp \left( -\alpha \sqrt{\gamma^2 + b^2} \right)}{\sqrt{\gamma^2 + b^2}}, \quad \text{and then:} \quad (C.25) \]

\[ \int_{-\infty}^{\infty} dx e^{\pm i \gamma x} J_0 \left( \beta \sqrt{x^2 - \alpha^2} \right) = \theta(b - y) \frac{\exp \left( -\alpha \sqrt{b^2 - y^2} \right)}{\sqrt{b^2 - y^2}} + \]

\[ + \theta(y - b) \frac{\exp \left( \mp i \alpha \sqrt{y^2 - b^2} \right)}{\pm i \sqrt{y^2 - b^2}}, \quad (C.26) \]

for \( \gamma = \pm iy \) with \( b, y > 0 \), whence: \( (\gamma^2 + b^2)^{1/2} = \theta(b - y) \sqrt{b^2 - y^2} \pm i \theta(y - b) \sqrt{y^2 - b^2} \).
whence: \( \text{Im} \left[ \mp \left( \mathbf{Z}_z \right)^2 \right]^{1/2} = 0 \), with \( \text{Re} \beta = \left| \text{Re} \right| \left[-(\mathbf{Z}_z)^2 \right]^{1/2} \) and

\[
\text{Re} \left[ \mp \left( \mathbf{Z}_z \right)^2 \right]^{1/2} = \sqrt{\left( \mathbf{Y}_z \right)^2 - \left( \mathbf{Z}_z \right)^2} = \text{Re} \beta.
\]

### D Narrow-packet approximation of exact overlap function.

To see how the plane-wave limit (6.20)–(6.22) of function (6.23) is realized, we put below for \( g_{2a} = 0 \) for \( a = \pi, \mu \) and \( g_{a} = g_{1}(m_{a}, \sigma_{a}) \), and note, that for definitions (C.10), (6.24)–(6.26), for \( q^{2} \geq 0 \), with \( P = p_{\pi} + p_{\mu}, \ k = p_{\pi} - p_{\mu} = k_{(C)} \), \( \chi_{0} = (kP)/(2q^{2}) \), \( g_{\pm} = g_{\pi} \pm g_{\mu}, \)

\[ \zeta_{\pm}, \zeta_{+} > 0, \text{and:} \]

\[
\zeta = \frac{g_{\pi}}{2} \frac{P + \zeta k}{k + \zeta P}, \quad \zeta = \frac{g_{\mu}}{2} + \zeta \mu, \quad Y_{\eta} = \frac{Y_{\pi}}{2} + \zeta \mu Y_{\eta},
\]

\[
q^{2} \Pi_{\eta}^{\lambda} = g^{\lambda \nu} q^{2} - q^{\lambda} q^{\nu}, \quad Z_{\Pi_{\eta}^{-}} = (\Pi_{\eta} Z_{-})^{\lambda} = Y_{\eta}^{\lambda} + i \zeta_{\eta}^{\lambda} + i \zeta_{\eta}^{\lambda}, \quad (D.1)
\]

\[
q^{2} \Pi_{\eta}^{-} = q^{2} Z_{\Pi_{\eta}^{-}}^{2} - (q Z_{-})^{2}, \quad Z_{\Pi_{\eta}^{-}} = -\zeta_{\eta}^{\lambda} + 2i(\zeta_{\eta} Y_{-}) + Y_{\eta}^{2}, \quad (D.2)
\]

\[
(q Y_{\eta}) = (q \Pi_{\eta} +) = 0, \quad \zeta_{\eta}^{2} \Pi_{\eta}^{2} + (\zeta_{\eta} + \Pi_{\eta} \zeta_{\eta}) \leq 0, \quad -q^{2} \zeta_{\eta}^{2} > 0, \quad (D.3)
\]

\[
q^{2} N_{\eta}^{2} = -1, \quad (q N_{\eta}) = 0, \quad \text{for} \quad g_{a}, g_{+} \rightarrow +\infty : \quad (D.5)
\]

\[
W = \left[ q^{2} Z_{\Pi_{\eta}^{-}}^{2} \right]^{1/2} = \left[ q^{2} \zeta_{\eta}^{2} \right]^{1/2} + i (N_{\eta} Y_{-}) q^{2} + \frac{Y_{\eta}^{2} + q^{2} (N_{\eta} Y_{-})^{2}}{2 q^{2} \zeta_{\eta}^{2}} q^{2}, \quad (D.6)
\]

and from (C.21)–(C.23), the narrow-packet approximation for overlap function (6.23) follows as:

\[
\hat{V}_{(C)}(q) \approx (2\pi)^{4} \theta(\Delta) \left[ \frac{m_{\pi}^{2} m_{\mu}^{2} (g_{\pi} g_{\mu})^{3}}{2 \zeta_{\eta}^{2} \Pi_{\eta}^{-} + (\Pi_{\eta} \zeta_{\eta})^{2}} \right]^{1/2} e^{i \Phi(q) L(q)} e^{-\Sigma(q Y_{-})}, \text{with:} \quad \chi_{\Delta} = \frac{\Delta^{1/2}(q)}{2q^{2}}, \quad (D.7)
\]

\[
2 \Phi(q) = (q Y_{+}) + (Q(q) Y_{-}), \quad Q(q) = 2[q \chi_{0} + N_{\Pi\Pi} \chi_{\Delta}], \quad Q^{2}(q) = 2(m_{\pi}^{2} + m_{\mu}^{2}) - q^{2}, \quad (D.8)
\]

\[
\Phi(k) = \Phi_{(C)}(k) = (p_{\pi} Y_{-}) - (p_{\mu} Y_{+}), \quad Q(k) = P, \quad 2 Y_{(C)}^{\lambda} = Y_{+}^{\lambda} + \partial_{\eta}^{\lambda}(Q(q) Y_{-}) |_{q=k}, \quad (D.9)
\]

\[
L(q) = g_{\pi} m_{\pi}^{2} + g_{\mu} m_{\mu}^{2} - (q \zeta_{0}) + \chi_{\Delta}[q^{2} \zeta_{+}^{2}]^{1/2}, \quad L(k) = 0, \quad \partial_{\eta}^{\lambda} L(q) |_{q=k} = 0, \quad (D.10)
\]
where, for this extremum point \( q \mapsto k \):
\[
\Sigma_q(Y_\mp) = \frac{Y_{\mp}^2 + q^2(N_\Pi Y_{\mp})^2}{2-\frac{q^2}{\sqrt{\Delta(k)}}} q^2 \chi_{\Delta} \mapsto \Sigma_k(Y_\mp) = \frac{Y_{\mp}^2 + k^2(N_\Pi Y_{\mp})^2}{2g_+}, \quad \Pi_q \mapsto \Pi_k, \quad (D.11)
\]
and for: \( \Pi_k P = P_{\Pi k}, \quad \zeta_{\Pi q} \mapsto \frac{g_{\pm}}{2} P_{\Pi q}, \quad N_\Pi q \mapsto \frac{P_{\Pi q}}{\sqrt{\Delta(k)}}, \quad \kappa = q - k, \quad u_\pi = \frac{P_\pi}{m_\pi}, \quad (D.12)
\]
one has: \( \Phi(q) \approx \Phi(k) + (\kappa Y_{(C)}), \quad L(q) \approx -(\kappa T \kappa), \quad T^{\beta\lambda} = -\frac{1}{2} \partial_\beta \partial_\lambda L(q)|_{q=k}, \quad (D.13)\)
\[
\hat{V}_{(C)}(q) \approx (2\pi)^4 \frac{\theta(\Delta(q))}{i} e^{i\Phi(k)} e^{i\Sigma_k(Y_\mp)} \left[ \frac{(g_\pi g_\mu)^3 (2\pi)^{-\frac{d}{2}}}{g_+^2 [(u_\pi u_\mu)^2 - 1]} \right]^{1/2} e^{i(\kappa Y_{(C)})} e^{-(\kappa T \kappa)}, \quad (D.14)
\]
with: \( |T| \equiv \text{det}\{T^{\beta\lambda}\} = \frac{(g_\pi g_\mu)^3 2^{-d}}{g_+^2 [(u_\pi u_\mu)^2 - 1]} \), \( \lim_{T \to +\infty} \left| \frac{|T|^1/2}{\pi^4} \right| e^{-(\kappa T \kappa)} = \delta_4(\kappa), \quad (D.15)\)
\[
\text{for: } \frac{4}{g_+} T^{\beta\lambda} = \frac{k^2 k_{\lambda}^\lambda}{k^2} T_0 + \hat{\Pi}^{\beta\lambda}_\perp T_d - \frac{P_{\Pi}^\beta P_{\Pi}^\lambda}{P_{\Pi}^\mu} T_{dd} - \frac{k^2 P_{\Pi}^\beta + k_{\lambda} P_{\Pi}^\lambda}{P_{\Pi}^\mu} T_{0d}, \quad (D.16)
\]
with: \( \hat{\Pi}^{\beta\lambda}_\perp = \frac{P_{\Pi}^\beta P_{\Pi}^\lambda}{P_{\Pi}^\mu} - \Pi^{\beta\lambda}_{\Pi} = -\Pi_{\Pi} \Pi_{\Pi}^{\beta\lambda}, \quad \text{and where for } \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} m_{\pi}^2 \pm m_{\mu}^2 \\ k^2 \end{pmatrix} : \quad (D.17)\)
\[
T_0 = \frac{(k^2)^2}{\Delta(k)} \left[ (1 + b^2)(a + c\beta) - 2b(b + c\alpha) \right], \quad T_d = \frac{(1 - \sqrt{2})}{2} > 0, \quad (D.18)
\]
\[
T_{dd} = (a + cb), \quad T_{0d} = b(a + cb) - (b + c\alpha), \quad T_0, \quad T_{dd} \equiv 0, \quad \text{with } k^2 \geq 0, \quad (D.19)
\]
\[
\Delta(k) = -k^2 P_{\Pi}^\beta = 4m_{\pi}^2 m_{\mu}^2 [(u_\pi u_\mu)^2 - 1] \quad \begin{cases} 4m_{\pi}^2 m_{\mu}^2 (v_\mu - v_\pi)^2, & |P_{\mu} | \ll m_\pi, \\ 4P_{\pi}^2 [1 - (n_\pi \cdot n_\mu)]^2, & m_\pi \to 0. \end{cases} \quad (D.20)
\]
Because for \( k^2 \geq 0 \) it follows \( P_{\Pi}^\beta \leq 0 \), one has at the respective rest frames (*) of vector \( k \) or vector \( P_{\Pi} \):
\[
k_\lambda = (\sqrt{k^2}, 0), \quad P_{\Pi}^\lambda = (0, 0, P_{\Pi}^\beta, P_{\Pi}^\lambda) \quad \text{with } (P_{\Pi}^\beta)^2 = P_{\Pi}^\mu = -P_{\Pi}^\mu = -(P_{\Pi}^\beta P_{\Pi}^\lambda) > 0 \quad \text{for } k^2 > 0, \quad \text{or } P_{\Pi}^\lambda = (\sqrt{P_{\Pi}^\beta}, 0), \quad k_\lambda = (0, 0, k^3) \quad \text{for } (k^3)^2 = k^2 = -k^2 > 0 \quad \text{otherwise, and}
\]
\[
\begin{align*}
\frac{4}{g_+} T^{00} & \equiv T_0, \quad \text{or } \star \equiv -T_{dd}, \quad \frac{4}{g_+} T^{33} \equiv \frac{k^2}{\sqrt{\Delta(k)}} T_{0d}, \quad \text{or } \star \equiv -T_0, \quad (D.21) \\
\frac{4}{g_+} T^{11} + \frac{4}{g_+} T^{22} & \equiv T_d, \quad \frac{4}{g_+} T^{03} \equiv \frac{4}{g_+} T^{30} \equiv \frac{k^2}{\sqrt{\Delta(k)}} T_{0d} \quad \text{otherwise, and}
\end{align*}
\]
where for both cases, with (D.17): \( \hat{\Pi}^{\beta\lambda}_\perp \mapsto \delta^{\beta\lambda}_\perp \), i.e. it is \( \neq 0 \) for \( \beta, \lambda = 1, 2 \) only. So, one finds:
\[
|T| = \varepsilon_{\beta\lambda\nu\sigma} T^{0\beta} T^{1\lambda} T^{2\nu} T^{3\sigma} \begin{vmatrix}
T^{00} & 0 & 0 & T^{03} \\
0 & T^{11} & 0 & 0 \\
0 & 0 & T^{22} & 0 \\
T^{30} & 0 & 0 & T^{33}
\end{vmatrix} = \frac{4}{g_+} T^{03} = \frac{k^2}{\Delta(k)} T_{0d}, \quad (D.23)
\]
what for both cases \( k^2 \geq 0 \) is reduced to the same value (D.15). So, the Lorentz invariant form \( (\kappa T \kappa) \) is strongly positively defined, since \( T^{00}, T^{11}, T^{22}, T^{33}, |T| \) > 0 for both cases.
References

[1] D. V. Naumov, V. A. Naumov. *A diagrammatic treatment of neutrino oscillations*. J. Phys. G: Nucl. Part. Phys. 37 (2010) 105014. (arXiv:hep-ph/1008.0306v2).

[2] D. V. Naumov, V. A. Naumov. *Relativistic wave packets in a field theoretical approach to neutrino oscillations*. Russ. Phys. J. 53 (2010) P.549

[3] V. A. Naumov, D. S. Shkirmanov. *Covariant asymmetric wave packet for a field-theoretical description of neutrino oscillations*. Mod. Phys. Let. A 30, No. 24 (2015) 1550110. (arXiv:1409.4669v2 [hep-ph])

[4] D. V. Naumov. *On the theory of wave packets* Phys. Part. Nucl. Lett. 10 (2013) 642-650. (arXiv:hep-ph/1309.1717).

[5] V. A. Naumov, D. S. Shkirmanov. *Extended Grimus - Stockinger theorem and inverse-square law violation in quantum field theory*. Eur. Phys. J. C 73 (2013) 2627

[6] D.V. Naumov, V.A. Naumov, D.S. Shkirmanov, *Inverse-square law violation and reactor antineutrino anomaly*. Fiz. Elem. Chast. Atom. Yadra 47 N 6 (2016) 1884-1897. [Phys. Part. Nucl. 48 N 1 (2017) 12-20] (arXiv: 1507.04573 [hep-ph])

[7] D.V. Naumov, V.A. Naumov, D.S. Shkirmanov, *Quantum field theoretical description of neutrino oscillations and reactor antineutrino anomaly*. Phys. Part. Nucl. 48 N 6, (2017) 1007-1010.

[8] S. E. Korenblit, D. V. Taychenachev. *Extension of Grimus-Stockinger formula from operator expansion of free Green function* Mod. Phys. Let. A 30, No. 14 (2015) 1550074. (arXiv: 1401.4031v4 [math-ph])

[9] L. D. Landau, E. M. Lifshitz. *Quantum Mechanics*. M. Nauka, 1974.

[10] A. Messiah *Quantum Mechanics*. NY 1961.

[11] J. R. Taylor. *Scattering Theory*. J. Wiley & Sons Inc. NY, 1972.

[12] M. Goldberger, K. Watson. *Collision theory*. NY 1964.

[13] R.P. Feynman. *Quantum Electrodynamics*. NY 1961.

[14] V. B. Beresteckij, E. M. Lifshitz, L. P. Pitaevskij. *Quantum Electrodynamics*. M. Nauka, 1980.

[15] W. E. Thirring. *Principles of Quantum Electrodynamics*. NY, 1958.

[16] M. E. Poskin, D. V. Schroeder. *An Introduction to Quantum Field Theory*, Addison-Wesley Pub. Comp., 1995.

[17] S. N. Vergeles. *Lectures on Quantum Electrodynamics*. M. Fizmatlit, 2006.

[18] S. S. Schweber. *An Introduction to Relativistic Quantum Field Theory*, M. IL, 1963.

[19] N. N. Bogoliubov, A. A. Logunov, I. T. Todorov. *Foundations of axiomatic approach in quantum field theory*. M. Nauka, 1969.

[20] N. N. Bogoliubov, A. A. Logunov, A. I. Oksak, I. T. Todorov. *General Principles of Quantum Field Theory*, Kluwer, 1990.

[21] R. Jost. *The General Theory of Quantized Fields*. M. Mir, 1967;

[22] R. F. Streater, A. S. Wightman. *PCT, Spin and Statistic and all That*. NY, 1966.
[23] F. Strocchi. An Introduction to Non-Perturbative Foundations of Quantum Field Theory. Oxford Univ. Press, 2013.

[24] B. S. DeWitt. Dynamical Theory of Groups and Fields. M. Nauka 1987. (in Russian)

[25] J. D. Bjorken, S. D. Drell, Relativistic Quantum Theoty. Vol. 1,2 Mc Graw-Hill Book Comp. 1978 [M. Nauka, 1978.]

[26] C. Itzykson, J-B. Zuber. Quantum Field Theory. Vol. 1,2. Mcgraw-Hill Int. Book Comp. 1978.

[27] Yu.V. Novozhilov. Introduction to the theory of elementary particles M. Nauka, 1972

[28] S. M. Bilenky. Introduction to the Feynman’s diagrams technique M. Atomizdat, 1971.

[29] S. Weinberg. The Quantum Theory of Fields. Vol 1 Cambridge Univ. Press, 2000

[30] S. Ya. Kilin. Quantum Optics. Fields and their detection. URSS, Moscow, 2003.

[31] A. E. Bernardini, S. De Leo. Flavor and chiral oscillations with Dirac wave packets. Phys. Rev. D 71, (2005) 076008, (arXiv:hep-ph/0504239).

[32] C. Giunti. Neutrino Wave Packets in Quantum Field Theory. JHEP 11, (2002) 017.

[33] J. Rich. Quantum mechanics of neutrino oscillations. Phys.Rev. D 48, (1993) pp. 4318-4325.

[34] M. Beuthe. Oscillations of neutrinos and mesons in quantum field theory. Phys. Rep. 375, pp. 105-218 (2003);

[35] E. Kh. Akhmedov, J. Kopp. Neutrino oscillations: quantum mechanics vs. quantum field theory JHEP 04 (2010) 008

[36] A. D. Dolgov, L. B. Okun, M. V. Rotaev, M. G. Schepkin. Oscillations of neutrinos produced by a beam of electrons. (arXiv: hep-ph/0407189v2).

[37] A. D. Dolgov, O. V. Lychkovskiy, A. A. Manonov, L. B. Okun, M. V. Rotaev, M. G. Schepkin. Oscillations of neutrinos produced and detected in crystals. Nucl. Phys. B 729, (2005) 79 [arXiv: hep-ph/0505251].

[38] C. C. Nishi, First quantized approaches to neutrino oscillations and second quantization. Phys. Rev. D 73, (2006) 053013. (arXiv: hep-ph/0506109v3)

[39] A. S. Bilenky, C. Giunti, W. Grimus. Phenomenology of Neutrino Oscillations Prog. Part. Nucl. Phys. 43, p. 86 (1999). (arXiv: hep-ph/9812360 v4)

[40] V. A. Naumov. Neutrinos in Physics and Astrophysics (Lectures given for V-VI course students of Moscow Institute of Physics and Technology (State University) and postgraduates of the JINR University Center, Dubna, in 2007-2016.) JINR, Full Term 2016 (http://theor.jinr.ru/ vnaumov/Eng/JINR_Lectures/Lectures_files/NPA2017.pdf)

[41] M. Blasone, P. A. Henning, G. Vitiello. The exact formula for neutrino oscillations. Phys. Lett. B 451, 140-145 (1999). (arXiv:hep-th/9803157).
[45] M. Blasone, P. P. Pacheco, H. W. C. Tseung. *Neutrino oscillations from relativistic flavor currents*. Phys. Rev. D 67, (2003) 073011

[46] M. Blasone, G. Vitiello. *Quantum Field Theory of Particle Mixing and Oscillations*. Symmetries in Science XI 2005, pp. 105-128

[47] K. Fujii, N. Toyota. *Expectation values of flavor-neutrino numbers with respect to neutrino-source hadron states*. (arXiv: 1408.1518v2 [hep-ph])

[48] E. Byckling, K. Kajantie. *Particle Kinematics* J. Wiley and Sons, London, 1973.

[49] G. Bateman, A. Erdelyi. *Higher transcendental functions*. NY 1953.

[50] I. S. Gradshteyn, I. M. Ryzhik. *Tables of Integrals, Series, and Products*, 7th edition, Academic Press, San Diego U.S.A. (2007).

[51] G. Bateman, A. Erdelyi. *Tables of integral transforms. Vol. 1*. NY 1954.

[52] I. M. Gel’fand, G. E. Shilov. *Generalized Functions. Vol. 1*. Moscow, “Nauka”, 1959.

[53] S. E. Korenblit, D. V. Taychenachev. *Higher order corrections to the Grimus-Stockinger formula* Phys. Part. Nucl. Lett., 10, No. 7 (2013) pp. 634. (arXiv: 1304.5192v1 [hep-th])

[54] R. Jacob, R.G. Sachs, Mass and lifetime of unstable particles, Phys. Rev. 121 (1961) p. 350.

[55] G. Mention, et al., *The Reactor Antineutrino Anomaly* Phys. Rev. D 83 (2011) 073006. (arXiv: 1101.2755 [hep-ex])