Finslerian Isospin–Nonlinear Equations for Pion and Spinor Interactions

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Abstract

The Finslerian curvature is introduced in the three-dimensional isospin space, suggesting that the isospin-pion field transforms according to non-linear Finslerian invariance transformations. The fundamental non-linear realization is associated with the pion field triplet. The Finslerian Lagrangian, as well as the implied pion-field selfinteraction, involves the first but no higher derivatives of the pion field. While the invariance transformations are now nonlinear, the conservation laws are still meaningful and sufficient to get the conservations for the energy-momentum and charges. The Finsler-invariant pion-nucleon Lagrangian function is also proposed. The Finslerian isospin-space symmetry is a pure interaction symmetry not shared by the asymptotic fields. The full Klein-Gordon-Dirac equation is accordingly extended. Various details of the implied Finsleroid-structure are formulated in an explicit way.
Basic Part

Introduction

The Finslerian isospin approach is a representation of three-dimensional invariance group in a curved instead of euclidean isospin-space. Below, we formulate the equations for possible Finsleroid-mediated symmetry in the curved isotopic space, based on the ison triplet of pions. So, the triplets are imagined as vectors in the respective Finsleroid-space. The Finsler-geometry ideas enable us to introduce the curvature in the three-dimensional isospin-space such that the curvature is constant and positive and is dependent on a single characteristic parameter. The resultant Finsleroid-space remains invariant under particular nonlinear pion-dependent rotations, henceforth mentioned as the $F_g$–rotations.

We shall start with formulating respective Finslerian field equations from first principles. The Finsleroid-type geometry of the curved three-dimensional isotopic space replaces now the ordinary euclidean geometry of linear pion-representation isotopic space. We construct the nonlinear Lagrangian such that the Lagrangian is invariant under $F_g$–rotations. We get the Finsleroid-mediated extension of the Klein-Gordon equation, which involves explicitly a self-interaction of pions through dependence of the Finslerian metric tensor components on pion fields.

After that, we introduce appropriately the pion-dependence in the nucleon spinor function $\psi$, thereby extending the Klein-Gordon-Dirac equations. The spinor part is constructed simply by forming isospin-invariant Lagrangian out of the $\psi$, their covariant derivative $D\psi$, and an appropriate pion covariant derivative $D\pi$. The function $\psi$ depends now on both, the space-time point and the pion field, and transforms under combined isospin-space transformations. Remarkably, the right-hand side involves explicitly the characteristic Finslerian curvature tensor (the so-called “Cartan curvature tensor”). In Appendix, the initial ingredients of the Finsler geometry are presented.

Formulation of Equations

Let $\pi^a = \pi^a(x)$ be an isotopic triplet of pions; the components $\{\pi^1(x), \pi^2(x), \pi^3(x)\}$ are assumed to be real. Introducing also the notation $\pi_i^a = \partial_i \pi^a = \partial \pi^a / \partial x^i$ for partial derivatives with respect to the position point $x^i$, we construct the Lagrangian

$$L_{\pi} = \frac{1}{2} r^{ij} g_{pq}(g; \pi^r) \pi^i_p \pi^j_q - \frac{1}{2} m^2 \pi^o_p \pi_p; \quad (1.1)$$

$m$ denotes the rest mass for the pions; $r^{ij} = r_{ij} = \text{diag}(1,-1,-1,-1)$ is a euclidean metric tensor. We may use the Finslerian metric tensor $g_{pq}$ (see Appendix below) and the Finslerian metric function $K$ (see Appendix below), so that

$$\pi^o_p = K^2(g; \pi^r) \quad (1.2)$$

with

$$\pi_p = g_{pq}(g; \pi^r) \pi^q, \quad (1.3)$$

to consider the action integral

$$S\{\pi^r\} = \int L_{\pi} dx. \quad (1.4)$$
The associated Euler-Lagrange derivatives

\[ \mathcal{E}_{\pi q} \overset{\text{def}}{=} \partial_j \frac{\partial L_\pi}{\partial \pi_j} - \frac{\partial L_\pi}{\partial \pi^q} \]  

(1.5)
can explicitly be calculated from Eqs. (1.1)–(1.5) to yield \( \mathcal{E}_{\pi q} = g_{qp} \mathcal{E}_\pi^p \) with

\[ \mathcal{E}_\pi^p = \Box \pi^p + C^p_q (g; \pi^r) \pi_i^q \pi_j^r r^{ij} + m^2 \pi^p, \]  

(1.6)
where

\[ \Box = r^{ij} \partial_i \partial_j. \]  

(1.7)

Since \( \delta S\{\pi^r\} = 0 \Rightarrow \mathcal{E}_\pi^p = 0 \), we arrive at the following Finslerian pion field equations:

\[ \Box \pi^p + C^p_q (g; \pi^r) \pi_i^q \pi_j^r r^{ij} + m^2 \pi^p = 0, \]  

(1.8)
where \( C^p_q \) are coefficients given in Appendix below. The equations are nonlinear and, therefore, introduce a Finslerian self-interaction of pions.

The associated Hamiltonian is

\[ H\{\pi^r\} = \frac{1}{2} \int \left[ g_{pq}(g; \pi^r) \left( \partial_0 \pi^p \partial_0 \pi^q + \partial_c \pi^p \partial_c \pi^q \right) + m^2 \pi^p \pi^p \right] d^3x \]  

(1.9)
where \( c = 1, 2, 3 \).

Considering the respective energy-momentum tensor:

\[ T_i^j \overset{\text{def}}{=} \pi_i^q \frac{\partial L_\pi}{\partial \pi_j^q} - \delta_i^j L_\pi, \]  

(1.10)
from (1.1) it follows that

\[ T_i^j = g_{pq}(g; \pi^r) \pi_i^q \pi_j^r \]  

(1.11)
The conservation law

\[ \partial_j T_i^j = 0 \]  

(1.12)
can readily be verified by direct calculations. The energy–component reads

\[ T^{00} = \frac{1}{2} g_{pq}(g; \pi^r) \left( \partial^0 \pi^p \partial^0 \pi^q + \partial_c \pi^p \partial_c \pi^q \right) + \frac{1}{2} m^2 \pi^p \pi^p. \]  

(1.13)

Also, for the currents

\[ j_a^q \overset{\text{def}}{=} \pi^q \frac{\partial L_\pi}{\partial \pi_a^q} t_{aq}(g; \pi^r), \]  

(1.14)
we get the conservation law

\[ \partial_n j_a^n = 0. \]  

(1.15)
The isospin-pion charges

\[ Q_a = \int j_a^q d\Sigma_n = \text{const} \]  

(1.16)
are still conserved, independently of whether the Finslerian extension is applied, or not applied.

Let a field \( \psi \) be an isospinor dublet of four-component space-time spinors, so that \( \psi = \{\psi^\alpha\} \). The Greek indices \( \alpha, \beta \ldots \) label the isospinor components and take on the values
1, 2; \(P, Q = 0, 1, 2, 3\). The notation \(T_P\) will be used for the ordinary Pauli matrices, so that
\[
T_P T_Q + T_Q T_P = 2a_{PQ},
\]
(1.17)
where \(a_{PQ} = \text{diag}(1, 1, 1) = \delta_{PQ}\). Considering the argument dependence of the spinor to be of the composite type
\[
\psi = \psi(x, \pi^r(x))
\]
(1.18)
we shall apply the total derivative \(d_j\):
\[
d_j \psi^\beta(x, \pi^r(x)) = \frac{\partial \psi^\beta(x, \pi^r(x))}{\partial x^j} + \partial_j \pi^s \frac{\partial \psi^\beta(x, \pi^r(x))}{\partial \pi^s(x)}.
\]
(1.19)
We introduce, in agreement with the known methods [1], the \(F\)-invariant isospinor derivative
\[
D_j \psi^\beta(x, \pi^r(x)) = d_j \psi^\beta(x, \pi^r(x)) + L^\beta_{\alpha j}(g; \pi^r(x)) \psi^\alpha(x, \pi^r(x))
\]
(1.20)
by the help of the \(F\)-invariant isospinor connection coefficients
\[
L_j = L_p(g; \pi^r) \partial_j \pi^p
\]
(1.21)
with
\[
L_p(g; \pi^r) = -\frac{1}{8} R^{PQ}_{\cdot p}(g; \pi^r)(T_P T_Q - T_Q T_P),
\]
(1.22)
where the associated Finslerian Ricci rotation coefficients
\[
R^{PQ}_{\cdot p}(g; \pi^r) \equiv (\partial_p e^Q_{\cdots q} - C^r_{p \cdot q} e^Q_{\cdots r}) e^P q
\]
(1.23)
have been used; \(\{e^Q_{\cdots q}\}\) is an appropriate triad.

By the help of the coefficients (1.22) it is convenient to introduce the isospin-derivative
\[
S_p \psi \equiv \frac{\partial \psi}{\partial \pi^p} + L_p \psi.
\]
(1.24)
We have
\[
L_p(g; \pi^r) \pi^p \equiv 0, \quad R^{PQ}_{\cdot p}(g; \pi^r) \pi^p \equiv 0.
\]
(1.25)
The generalized spinor Lagrangian
\[
L_\psi = -\bar{\psi}(\gamma^D + M) \psi
\]
(1.26)
(M denotes the rest mass for the fermions) generates the extension
\[
(-\gamma^D + M) \psi = 0
\]
(1.27)
of the Dirac’s equation. In the component form, the equation (1.26) reads
\[
-\gamma^j D_j \psi^\alpha + M \psi^\alpha = 0.
\]
(1.28)
Finally, we construct the objects
\[
E_q \equiv e^P_q T_P
\]
(1.29)
and
\[
E^p = g^{pq} E_q \equiv e^{Pq} T_P.
\]
(1.30)
For the total Lagrangian

\[ L_{\pi\psi} = L_\pi + L_\psi \]  

(1.31)

we get again the equation (1.28), together with the equation

\[ \Box \pi^p + C_q^{\rho s}(g; \pi^r) \pi^q \pi^s \pi^{ij} + m^2 \pi^p = - \frac{1}{4} \frac{\hbar}{h} S_q^{\rho ts}(g; \pi^r) \bar{\psi} i \gamma^n \pi^q (E^t E^s - E^s E^t) \psi \]  

(1.32)

which extends (1.6) by including a due Finslerian interaction of spinors with pions; \( S_q^{\rho ts} \) is the Finsleroid-geometry curvature tensor (see Appendix below).

Discussion

Thus we have initiated a systematic development of the Finslerian nonlinear-invariant approach to pion-isospin invariance, and thereby to pion-pion and nucleon-pion interactions. It is shown how the Finsler Geometry of the isotopic space should introduce the interactions. The approach involves a convenient nonlinear realization of the isotopic-space rotations. This realization is associated with the pion field and is regarded as a manifestation of a group of invariance in a curved 3-dimensional isospin space of the Finsleroid type, leaving invariant the key Finslerian metric function. Oppositely, we have offered the way of application of the Finsleroid Metric Function to processes involving self-interaction of pions.

It can be hoped that this can provide us in future with new insights into the pion-nucleon interactions, as well as into the modern particle interaction models [2-5].

APPENDIX. Finsleroid-Space \( \mathcal{E}^{PD}_{g} \) of Positive-Definite Type

Suppose we are given an \( N \)-dimensional vector space \( V_N \). Denote by \( R \) the vectors constituting the space, so that \( R \in V_N \). Any given vector \( R \) assigns a particular direction in \( V_N \). Let us fix a member \( R_{(N)} \in V_N \), introduce the straightline \( e_N \) oriented along the vector \( R_{(N)} \), and use this \( e_N \) to serve as a \( R^N \)-coordinate axis in \( V_N \). In this way we get the topological product

\[ V_N = V_{N-1} \times e_N \]  

(2.1)

together with the separation

\[ R = \{ R, R^N \}, \quad R^N \in e_N \quad \text{and} \quad R \in V_{N-1}. \]  

(2.2)

For convenience, we shall frequently use the notation

\[ R^N = Z \]  

(2.3)

and

\[ R = \{ R, Z \}. \]  

(2.4)

Also, we introduce a Euclidean metric

\[ q = q(R) \]  

(2.5)

over the \( (N-1) \)-dimensional vector space \( V_{N-1} \).

With respect to an admissible coordinate basis \( \{ e_a \} \) in \( V_{N-1} \), we obtain the coordinate representations

\[ R = \{ R^a \} = \{ R^1, \ldots, R^{N-1} \} \]  

(2.6)
and
\[ R = \{ R^p \} = \{ R^a, R^N \} \equiv \{ R^a, Z \}, \]  \hfill (2.7)

together with
\[ q(R) = \sqrt{r_{ab} R^a R^b}, \]  \hfill (2.8)

where \( r_{ab} \) are the components of a symmetric positive-definite tensor defined over \( V_{N-1} \). The indices \( (a, b, \ldots) \) and \( (p, q, \ldots) \) will be specified over the ranges \( (1, \ldots, N - 1) \) and \( (1, \ldots, N) \), respectively; vector indices are up, co-vector indices are down; repeated up-down indices are automatically summed; the notation \( \delta^a_b \) will stand for the Kronecker symbol. The variables
\[ w^a = R^a / Z, \quad w_a = r_{ab} w^b, \quad w = q / Z, \]  \hfill (2.9)

where
\[ w \in (-\infty, \infty), \]  \hfill (2.10)

are convenient whenever \( Z \neq 0 \). Sometimes we shall mention the associated metric tensor
\[ r_{pq} = \{ r_{NN} = 1, r_{N a} = 0, r_{ab} \} \]  \hfill (2.11)

meaningful over the whole vector space \( V_N \).

Given a parameter \( g \) subject to the inequality
\[ -2 < g < 2, \]  \hfill (2.12)

we introduce the convenient notation
\[ h = \sqrt{1 - \frac{1}{4} g^2}, \]  \hfill (2.13)

\[ G = g / h, \]  \hfill (2.14)

\[ g_+ = \frac{1}{2} g + h, \quad g_- = \frac{1}{2} g - h, \]  \hfill (2.15)

\[ g^+ = -\frac{1}{2} g + h, \quad g^- = -\frac{1}{2} g - h, \]  \hfill (2.16)

so that
\[ g_+ + g_- = g, \quad g_+ - g_- = 2h, \]  \hfill (2.17)
\[ g^+ + g^- = -g, \quad g^+ - g^- = 2h, \]  \hfill (2.18)
\[ (g_+)^2 + (g_-)^2 = 2, \]  \hfill (2.19)
\[ (g^+)^2 + (g^-)^2 = 2, \]  \hfill (2.20)

and
\[ g_+ \xrightarrow{g \rightarrow -g} -g_-, \quad g^+ \xrightarrow{g \rightarrow -g} -g^-. \]  \hfill (2.21)
The characteristic quadratic form

\[ B(g; R) = Z^2 + gqZ + q^2 \equiv \frac{1}{2} \left[ (Z + g_+q)^2 + (Z + g_-q)^2 \right] > 0 \]  \hspace{1cm} (2.22)

is of the negative discriminant, namely

\[ D_{\{B\}} = -4h^2 < 0, \] \hspace{1cm} (2.23)

because of Eqs. (2.12) and (2.13). Whenever \( Z \neq 0 \), it is also convenient to use the quadratic form

\[ Q(g; w) \overset{\text{def}}{=} B/(Z)^2, \] \hspace{1cm} (2.24)

obtaining

\[ Q(g; w) = 1 + gw + w^2 > 0, \] \hspace{1cm} (2.25)

together with the function

\[ E(g; w) \overset{\text{def}}{=} 1 + \frac{1}{2}gw. \] \hspace{1cm} (2.26)

The identity

\[ E^2 + h^2w^2 = Q \] \hspace{1cm} (2.27)

can readily be verified. In the limit \( g \to 0 \), the definition (2.22) degenerates to the quadratic form of the input metric tensor (2.11):

\[ B\Big|_{g=0} = r_{pq}R^pR^q. \] \hspace{1cm} (2.28)

Also

\[ Q\Big|_{g=0} = 1 + w^2. \] \hspace{1cm} (2.29)

In terms of this notation, we propose the Finslerian Metric Function

\[ K(g; R) = \sqrt{B(g; R)J(g; R)}, \] \hspace{1cm} (2.30)

where

\[ J(g; R) = e_{\frac{1}{2}}^{\frac{1}{2}G\Phi(g; R)}, \] \hspace{1cm} (2.31)

\[ \Phi(g; R) = \frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \left( \frac{q}{hZ} + \frac{G}{2} \right), \quad \text{if} \quad Z \geq 0, \] \hspace{1cm} (2.32)

\[ \Phi(g; R) = -\frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \left( \frac{q}{hZ} + \frac{G}{2} \right), \quad \text{if} \quad Z \leq 0, \] \hspace{1cm} (2.33)

or in other convenient forms,

\[ \Phi(g; R) = \frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \left( \frac{L(g; R)}{hZ} \right), \quad \text{if} \quad Z \geq 0, \] \hspace{1cm} (2.34)

\[ \Phi(g; R) = -\frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \left( \frac{L(g; R)}{hZ} \right), \quad \text{if} \quad Z \leq 0, \] \hspace{1cm} (2.35)

where

\[ L(g; R) = q + \frac{g}{2}Z. \] \hspace{1cm} (2.36)
and
\[ \Phi(g; R) = \frac{\pi}{2} - \arctan \frac{hq}{A(g; R)}, \quad \text{if} \quad Z \geq 0, \] (2.37)
\[ \Phi(g; R) = -\frac{\pi}{2} - \arctan \frac{hq}{A(g; R)}, \quad \text{if} \quad Z \leq 0, \] (2.38)
where
\[ A(g; R) = Z + \frac{1}{2}gq. \] (2.39)

This Finslerian Metric Function has been normalized to show the handy properties
\[ -\frac{\pi}{2} \leq \Phi \leq \frac{\pi}{2}, \] (2.40)
\[ \Phi = \frac{\pi}{2}, \quad \text{if} \quad q = 0 \quad \text{and} \quad Z > 0; \quad \Phi = -\frac{\pi}{2}, \quad \text{if} \quad q = 0 \quad \text{and} \quad Z < 0. \] (2.41)

We also have
\[ \cot \Phi = \frac{hq}{A}, \quad \Phi|_{Z=0} = \arctan \frac{G}{2}. \] (2.42)

It is often convenient to use the indicator of sign \( \epsilon_Z \) for the argument \( Z \):
\[ \epsilon_Z = 1, \quad \text{if} \quad Z > 0; \quad \epsilon_Z = -1, \quad \text{if} \quad Z < 0; \] (2.43)

Under these conditions, we call the considered space the \( \mathcal{E}_g^{PD} \)-space:
\[ \mathcal{E}_g^{PD} = \{ V_N = V_{N-1} \times e_N; \ R \in V_N; \ K(g; R); \ g \}. \] (2.44)

The right–hand part of the definition (2.30) can be considered to be a function \( \tilde{K} \) of the arguments \( \{g; q, Z\} \), such that
\[ \tilde{K}(g; q, Z) = K(g; R). \] (2.45)

We observe that
\[ \tilde{K}(g; q, -Z) \neq \tilde{K}(g; q, Z), \quad \text{unless} \quad g = 0. \] (2.46)

Instead, the function \( \tilde{K} \) shows the property of \( gZ \)-parity
\[ \tilde{K}(-g; q, -Z) = \tilde{K}(g; q, Z). \] (2.47)

The \( (N-1) \)-space reflection invariance holds true
\[ K(g; R)^{R^a} \leftrightarrow -R^a K(g; R). \] (2.48)

It is frequently convenient to rewrite the representation (2.30) in the form
\[ K(g; R) = |Z|V(g; w), \] (2.49)
whenever $Z \neq 0$, with the generating metric function

$$V(g; w) = \sqrt{Q(g; w)} j(g; w). \quad (2.50)$$

We have

$$j(g; w) = J(g; 1, w).$$

Using (2.25) and (2.31)–(2.35), we obtain

$$V' = wV/Q, \quad V'' = V/Q^2, \quad (2.51)$$

$$\frac{(V^2/Q)'}{Q} = -gV^2/Q^2, \quad \frac{(V^2/Q^2)'}{Q} = -2(g + w)V^2/Q^3, \quad (2.52)$$

$$j' = -\frac{1}{2}gj/Q, \quad (2.53)$$

and also

$$\frac{1}{2}(V^2)' = wV^2/Q, \quad \frac{1}{2}(V^2)'' = (Q - gw)V^2/Q^2, \quad (2.54)$$

$$\frac{1}{4}(V^2)''' = -gV^2/Q^3, \quad (2.55)$$

together with

$$\Phi' = -h/Q, \quad (2.56)$$

where the prime (’) denotes the differentiation with respect to $w$.

Also,

$$(A(g; R))^2 + h^2q^2 = B(g; R) \quad (2.57a)$$

and

$$(L(g; R))^2 + h^2Z^2 = B(g; R). \quad (2.57b)$$

Sometimes it is convenient to use the function

$$E(g; w) \overset{\text{def}}{=} 1 + \frac{1}{2}gw. \quad (2.58)$$

The simple results for these derivatives reduce the task of computing the components of the associated Finslerian Metric Tensor to an easy exercise, indeed:

$$R_p = \frac{1}{2} \frac{\partial K^2(g; R)}{\partial R_p}; \quad (2.59)$$

$$R_a = r_{ab}R_b^{K^2/B}, \quad R_N = (Z + gq)\frac{K^2}{B};$$

$$g_{pq}(g; R) \overset{\text{def}}{=} \frac{1}{2} \frac{\partial^2 K^2(g; R)}{\partial R_p \partial R_q} = \frac{\partial R_p(g; R)}{\partial R_q};$$
\[ g_{NN}(g; R) = \left[(Z + gq)^2 + q^2\right] \frac{K^2}{B^2}, \quad g_{Na}(g; R) = gqr_{ab}R^b \frac{K^2}{B^2}, \quad (2.60) \]

\[ g_{ab}(g; R) = \frac{K^2}{B} r_{ab} - g \frac{r_{ad}R^d r_{be}R^e Z K^2}{q} \frac{1}{B^2}. \quad (2.61) \]

The reciprocal tensor components are

\[ g^{NN}(g; R) = (Z^2 + q^2) \frac{1}{K^2}, \quad g^{Na}(g; R) = -gqR^a \frac{1}{K^2}, \quad (2.62) \]

\[ g^{ab}(g; R) = \frac{B}{K^2} r^{ab} + g(Z + gq) \frac{R^a R^b}{q} \frac{1}{K^2}. \quad (2.63) \]

The determinant of the Finslerian Metric Tensor given by Eqs. (2.59)–(2.60) can readily be found in the form

\[ \det(g_{pq}(g; R)) = [J(g; R)]^{2N} \det(r_{ab}) \quad (2.64) \]

which shows, on noting (2.31)–(2.33), that

\[ \det(g_{pq}) > 0 \quad \text{over all the definition range} \quad V_N \setminus 0. \quad (2.65) \]

The associated angular metric tensor

\[ h_{pq} \overset{\text{def}}{=} g_{pq} - R_p R_q \frac{1}{K^2} \]

proves to be given by the components

\[ h_{NN}(g; R) = q^2 \frac{K^2}{B^2}, \quad h_{Na}(g; R) = -Z r_{ab} R^b \frac{K^2}{B^2}, \]

\[ h_{ab}(g; R) = \frac{K^2}{B} r_{ab} - (gZ + q) \frac{r_{ad}R^d r_{be}R^e K^2}{q} \frac{1}{B^2}, \]

which entails

\[ \det(h_{ab}) = \det(g_{pq}) \frac{1}{V^2}. \]

The use of the components of the Cartant tensor (given explicitly in the end of the present section) leads, after rather tedious straightforward calculations, to the following simple and remarkable result.
PROPOSITION 1. The Cartan tensor associated with the Finslerian Metric Function (2.30) is of the following special algebraical form:

\[ C_{pqr} = \frac{1}{N} \left( h_{pq}C_r + h_{pr}C_q + h_{qr}C_p - \frac{1}{C_sC_s} C_tC_tC_r \right) \]  

with

\[ C_tC_r = \frac{N^2}{4K^2} g^2. \]

By the help of (2.65), elucidating the structure of the curvature tensor

\[ S_{pqrs} \overset{\text{def}}{=} (C_{tqr}C_{p\text{t}} - C_{tqs}C_{p\text{t}}) \]

results in the simple representation

\[ S_{pqrs} = -\frac{C_tC_r}{N^2} (h_{pr}h_{qs} - h_{ps}h_{qr}). \]

Inserting here (2.66), we are led to

PROPOSITION 2. The curvature tensor of the space \( E_g^{PD} \) is of the special type

\[ S_{pqrs} = S^* (h_{pr}h_{qs} - h_{ps}h_{qr})/K^2 \]

with

\[ S^* = -\frac{1}{4} g^2. \]

DEFINITION. The Finslerian Metric Function (2.30) introduces an \((N - 1)\)-dimensional indicatrix hypersurface according to the equation

\[ K(g; R) = 1. \]

We call this particular hypersurface the Finsleroid, to be denoted as \( F_g^{PD} \).

Recalling the known formula \( R = 1 + S^* \) for the indicatrix curvature, from (2.70) we conclude that

\[ R_{\text{Finsleroid}} = h^2 = 1 - \frac{1}{4} g^2, \quad 0 < R_{\text{Finsleroid}} \leq 1. \]

Geometrically, the fact that the quantity (2.70) is independent of vectors \( R \) means that the indicatrix curvature is constant. Therefore, we have arrived at

PROPOSITION 3. The Finsleroid \( F_g^{PD} \) is a constant-curvature space with the positive curvature value (2.72).

Also, on comparing between the result (2.72) and Eqs. (2.22)–(2.23), we obtain

PROPOSITION 4. The Finsleroid curvature relates to the discriminant of the input characteristic quadratic form (2.22) simply as

\[ R_{\text{Finsleroid}} = -\frac{1}{4} D_{\{B\}}. \]
Last, we write down the explicit components of the relevant Cartan tensor

\[ C_{pqr} \overset{\text{def}}{=} \frac{1}{2} \frac{\partial g_{pq}}{\partial R^r}; \]

\[ R^N C_{NNN} = gw^3 V^2 Q^{-3}, \quad R^N C_{aNN} = -gw w_a V^2 Q^{-3}, \]

\[ R^N C_{abN} = \frac{1}{2} gw V^2 Q^{-2} r_{ab} + \frac{1}{2} g(1 - gw - w^2) w_a w_b w^{-1} V^2 Q^{-3}, \]

\[ R^N C_{abc} = -\frac{1}{2} g V^2 Q^{-2} w^{-1} (r_{ab} w_c + r_{ac} w_b + r_{bc} w_a) + gw w_a w_b w_c w^{-3} \left( \frac{1}{2} Q + gw + w^2 \right) V^2 Q^{-3}, \]

and

\[ R^N C_{N}^{N N} = gw^3 / Q^2, \quad R^N C_{a}^{N N} = -gw w_a / Q^2, \]

\[ R^N C_{N}^{a N} = -gw (1 + gw) w^a / Q^2; \]

\[ R^N C_{N}^{b a} = \frac{1}{2} gw r_{ab} / Q + \frac{1}{2} g(1 - gw - w^2) w_a w_b / w Q^2, \]

\[ R^N C_{a}^{N b} = \frac{1}{2} gw \delta_b^a / Q + \frac{1}{2} g(1 + gw - w^2) w^a w_b / w Q^2, \]

\[ R^N C_{a}^{b c} = -\frac{1}{2} g \left( \delta_b^c w_a + \delta_c^a w_a + (1 + gw) r_{ac} w_b \right) / w Q + \frac{1}{2} g(g w Q + Q + 2 w^2) w_a w_b w_c / w^3 Q^2. \]

The components have been calculated by the help of the formulae (2.50)–(2.53).

The use of the contractions

\[ R^N C_{b}^{a c r^{ac}} = -g w^b \frac{1 + gw}{w} \left( \frac{N - 2}{2} + \frac{1}{Q} \right) \]

and

\[ R^N C_{a}^{b c} w^a w^c = -g w^b \frac{w}{Q^2} (1 + gw) w^b \]

is handy in many calculations.

Also

\[ R^N C_{N} = \frac{N}{2} gw Q^{-1}, \quad R^N C_{a} = -\frac{N}{2} g(w_a / w) Q^{-1}, \]

\[ R^N C^{N} = \frac{N}{2} gw / V^2, \quad R^N C^{a} = -\frac{N}{2} gw^a (1 + gw) / w V^2, \]
\[ C^N = \frac{N}{2} gwR^K K^{-2}, \quad C^a = -\frac{N}{2} gw^a(1 + gw)w^{-1}R^K K^{-2}, \]

\[ C'_p C^p = \frac{N^2}{4K^2g^2}. \]

For a better understanding of structure of the Finsleroid space, the reader is referred to [6].

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