Top dimensional group of the basic intersection cohomology for singular riemannian foliations.*

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Abstract

It is known that, for a regular riemannian foliation on a compact manifold, the properties of its basic cohomology (non-vanishing of the top-dimensional group and Poincaré duality) and the tautness of the foliation are closely related. If we consider singular riemannian foliations, there is little or no relation between these properties.

We present an example of a singular isometric flow for which the top dimensional basic cohomology group is non-trivial, but its basic cohomology does not satisfy the Poincaré Duality property. We recover this property in the basic intersection cohomology.

It is not by chance that the top dimensional basic intersection cohomology groups of the example are isomorphic to either 0 or \( \mathbb{R} \). We prove in this Note that this holds for any singular riemannian foliation of a compact connected manifold. As a Corollary, we get that the tautness of the regular stratum of the singular riemannian foliation can be detected by the basic intersection cohomology.

For regular riemannian foliations on compact connected manifolds there is a very close relation between tautness (the existence of a bundle-like metric for which all leaves are minimal submanifolds) and the properties of the basic cohomology. In fact, it was shown that for a regular riemannian foliation \( \mathcal{F} \) of codimension \( n \) on a compact connected manifold \( M \) the following conditions are equivalent (cf. [3, 4, 9, 10, 25]):

1. \( \mathcal{F} \) is taut.
2. \( H^n(M/\mathcal{F}) \neq 0 \).
3. \( H^n(M/\mathcal{F}) = \mathbb{R} \).
4. \( H^*(M/\mathcal{F}) \) verifies the Poincaré Duality (PD) property,

*To appear in the Bulletin of the Polish Academy of Sciences.
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under some assumptions of orientability. Although the basic cohomology of a singular riemannian foliation (SRF for short) on a compact manifold is finite dimensional (cf. [26]), the relation between the above conditions is not straightforwardly exportable to the singular framework. As pointed out by the authors in [11], the singular nature of a SRF on a compact manifold prevents any global metric on it from making all the leaves minimal (see also [17]).

The example presented in this paper shows that if, for flows, we replace the condition “taut” by “isometric” - these two conditions are equivalent for regular riemannian flows (see [15]) - we cannot recover the PD property. To recover this property we have to adapt the basic cohomology to the stratification defined by the SRF. With that purpose, we have defined the “basic intersection cohomology” (BIC for short) for SRFs (see [19, 20]). The calculations for the example show that its BIC satisfies the Poincaré Duality. It has been proven in [16] that the BIC of any singular riemannian flow (isometric or not) satisfies this property.

The main part of this Note is dedicated to the proof of the equivalence of conditions (i), (ii) and (iii) for SRFs on compact connected manifolds for the BIC. We also prove that the top dimensional BIC groups are isomorphic to 0 or \( \mathbb{R} \).

The authors would like to thank the referee of the paper for many useful comments which helped improve the paper.

In the sequel \( M \) is a connected, second countable, Haussdorf, without boundary and smooth (of class \( C^\infty \)) manifold of dimension \( m \). All the maps are considered smooth unless something else is indicated. If \( \mathcal{F} \) is a foliation on \( M \) and \( V \) is a saturated submanifold of \( M \) we shall write \((V, \mathcal{F})\) the induced foliated manifold and \( \mathcal{F}_V \) the induced foliation.

1 Presentation of the singular riemannian foliations

We are going to work in the framework of the singular riemannian foliations introduced by Molino.

1.1 The SRF. A singular riemannian foliation (SRF for short) on a manifold \( M \) is a partition \( \mathcal{F} \) by connected immersed submanifolds, called leaves, verifying the following properties:

I- The module of smooth vector fields tangent to the leaves is transitive on each leaf.

II- There exists a riemannian metric \( \mu \) on \( M \), called adapted metric, such that each geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets.

The first condition implies that \((M, \mathcal{F})\) is a singular foliation in the sense of [24] and [23]. Notice that the restriction of \( \mathcal{F} \) to a saturated open subset produces a SRF.

In the next two subsections we recall some basic properties of SRFs which can be found in [13, 1] or easy corollaries of the properties proved there.

1.2 Stratifications. Classifying the points of \( M \) following the dimension of the leaves one gets a stratification \( S_\mathcal{F} \) of \( M \) whose elements are called strata. The strata are smooth embedded submanifolds. The restriction of \( \mathcal{F} \) to a stratum \( S \) is a regular foliation \( \mathcal{F}_S \). The strata are ordered by: \( S_1 \leq S_2 \Leftrightarrow S_1 \subset S_2 \).

\(^1\)For the notions related with riemannian foliations we refer the reader to [13, 25] and for the notions related with singular riemannian foliations we refer the reader to [13, 12, 14, 1].
There are several types of strata. The minimal (resp. maximal) strata are the closed strata (resp. open strata). The open strata are called regular strata and the others are called singular strata. We denote by $S^r_F$ the family of singular strata. In the case of SRFs, the singular strata are of codimension greater than 1, so there is just one regular stratum, if the manifold is connected (cf. [13]). The dimension of the foliation $F$ is the dimension of the biggest leaves of $F$, that is, $\dim F = \dim F_R$. The union of singular strata is the singular part $\Sigma = M \setminus R$.

The stratum $S$ is a boundary stratum if there exists a stratum $S'$ with $S \preceq S'$ and $\text{codim}_M F = \text{codim}_{S'} F - 1$. The reason for this term can be explained by the following example. Take $M = S^4$ and $F$ given by the orbits of the $T^2$-action: $(u, v) \cdot (z_1, z_2, t) = (u \cdot z_1, v \cdot z_2, t)$, where $S^4 = \{(z_1, z_2, t) \in \mathbb{C}^2 \times \mathbb{R} \mid |z_1|^2 + |z_2|^2 + t^2 = 1\}$. Here, the north pole $(0, 0, 1)$, the south pole $(0, 0, -1)$ and the cylinders $\{z_1 \neq 0\}$, $\{z_2 \neq 0\}$ are the boundary strata and we have $M/F = D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. The boundary $\partial(M/F)$ is given by $S^1$, the union of the quotient of the boundary strata. In fact, the link of the maximal boundary strata is a sphere with the one leaf foliation (see for example [19] for the notion of a link).

The depth of $S_F$, written $\text{depth} S_F$, is defined to be the largest $i$ for which there exists a chain of strata $S_0 \prec S_1 \prec \cdots \prec S_i$. So, depth $S_F = 0$ if and only if the foliation $F$ is regular.

1.3 Tubular neighborhood. Any stratum $S \in S_F$ is a proper submanifold of the riemannian manifold $(M, F, \mu)$. So, it possesses a tubular neighborhood $(T_S, \tau_S, S)$. Recall that associated with this neighborhood there are the following smooth maps:

+ The radius map $\rho_S : T_S \to [0, 1]$ defined fiberwise: $z \mapsto |z|$. Each $t \neq 0$ is a regular value of the $\rho_S$. The pre-image $\rho_S^{-1}(0)$ is $S$.

+ The contraction $H_S : T_S \times [0, 1] \to T_S$ defined fiberwise: $(z, r) \mapsto r \cdot z$. The restriction $(H_S)_t : T_S \to T_S$ is an embedding for each $t \neq 0$ and $(H_S)_0 \equiv \tau_S$.

These two maps verify $\rho_S(r \cdot u) = r \rho_S(u)$. This tubular neighborhood can be chosen verifying the two following important properties:

(a) Each $(\rho_S^{-1}(t), F)$ is a SRF, and

(b) Each $(H_S)_t : (T_S, F) \to (T_S, F)$ is a foliated map.

We shall say that $(T_S, \tau_S, S)$ is a foliated tubular neighborhood of $S$. The existence of foliated tubular neighborhoods follows from the homothetic transformation Lemma of [13].

The hypersurface $D_S = \rho_S^{-1}(1/2)$ is the core of the tubular neighborhood. We have the inequality depth $S_{F_D S} < \text{depth} S_{F_{TS}}$.

2 Presentation of the basic intersection cohomology\(^2\)

Goresky and MacPherson introduced the intersection cohomology for the study of the singular manifolds. This cohomology generalizes the usual deRham cohomology for manifolds and possesses similar properties. Following the same principle, the basic intersection cohomology has been introduced for the study of SRFs generalizing the basic cohomology.

We fix for the sequel a manifold $M$ endowed with a SRF $F$. We write $m = \dim M$ and $n = \text{codim}_M F$.

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\(^2\)For the notions related with the basic cohomology we refer the reader to [15 26], for the notions related with the basic intersection cohomology we refer the reader to [19 20] and for the notions related with the intersection cohomology we refer the reader to [9 2].
2.1 The BIC. A perversity is a map $\overline{\tau}: S^c_F \to \mathbb{Z} = \mathbb{Z} \cup \{-\infty, \infty\}$. There are several particular perversities,

- the constant perversity $\overline{\tau}$, defined by $\overline{\tau}(S) = c$, where $c \in \mathbb{Z}$,

- the (basic) top perversity $\overline{t}$, defined by $\overline{t}(S) = n - \text{codim}_S F_S - 2$, (cf. 1.2) and

- the boundary perversity $\overline{\partial}$, defined by $\overline{\partial} = \min(\overline{0}, \overline{t})$.

The basic intersection cohomology (BIC for short) $H^*_\overline{\tau}(M/F)$ is the cohomology of the complex $\Omega^*_\overline{\tau}(M/F)$ of $\overline{\tau}$-intersection basic forms. A $\overline{\tau}$-intersection basic form is a basic form defined on $R$ possessing a vertical degree $\|\omega\|_S$, relatively to a foliated tubular neighborhood $(T_S, \tau_S, S)$, lower than $\overline{\tau}(S)$ and this for each singular stratum $S$ (see [19, 20] for the exact definition). Recall that $\|\omega\|_S \leq i$ when $\omega(v_0, \ldots, v_i, -) = 0$ for each family $\{v_0, \ldots, v_i\}$ of vectors tangent to the fibers of $\tau_S$.

When depth $S_F = 0$ then we have $H^*_\overline{\tau}(M/F) = H^*(M/F)$ for any perversity.

2.2 Mayer-Vietoris. A covering $\{U, V\}$ of $M$ by saturated open subsets possesses a subordinated partition of the unity made up of basic functions (see Lemma below).

For a such covering we have the Mayer-Vietoris short sequence

$$0 \to \Omega^*_\overline{\tau}(M/F) \to \Omega^*_\overline{\tau}(U/F) \oplus \Omega^*_\overline{\tau}(V/F) \to \Omega^*_\overline{\tau}((U \cap V)/F) \to 0,$$

where the maps are defined by restriction. The third map is onto since the elements of the partition of the unity are $\overline{0}$-basic functions and $\Omega^*_\overline{\tau}(\bullet/F) \cdot \Omega^*_\overline{\tau}(\bullet/F) \subset \Omega^*_\overline{\tau}(\bullet/F)$. Thus, the sequence is exact. This result is not longer true for more general coverings.

For the existence of the above Mayer-Vietoris sequence we need the following folk result, well-known for compact Lie group actions and regular riemannian foliations.

Lemma 2.2.1 A covering $\{U, V\}$ of $M$ by saturated open subsets possesses a subordinated partition of the unity made up of basic functions.

2.3 Compact supports. In this Note we need to work with the BIC with compact supports. The support of a differential form $\omega \in \Omega^*_\overline{\tau}(M/F)$, written $\text{supp} \ \omega$, is the closure in $M$ of $\{x \in M \mid \omega(x) \neq 0\}$. We denote by $\Omega^*_{\overline{\tau},c}(M/F)$ the complex of $\overline{\tau}$-intersection basic forms with compact support. Its cohomology is $H^*_{\overline{\tau},c}(M/F)$. When $M$ is compact, we have $H^*_{\overline{\tau},c}(M/F) = H^*_{\overline{\tau}}(M/F)$ and when depth $S_F = 0$ then we have $H^*_\overline{\tau,c}(M/F) = H^*_\overline{\tau}(M/F)$, for any perversity.

Associated to a saturated open covering $\{U, V\}$ of $M$ we have the Mayer-Vietoris short sequence (see Lemma above)

$$0 \to \Omega^*_{\overline{\tau},c}((U \cap V)/F) \to \Omega^*_{\overline{\tau},c}(U/F) \oplus \Omega^*_{\overline{\tau},c}(V/F) \to \Omega^*_{\overline{\tau},c}(M/F) \to 0,$$

where the map are defined by inclusion. The third map is onto since the elements of the partition of the unity are $\overline{0}$-basic functions. Thus, the sequence is exact.
2.4 Example. Us consider the isometric action \( \Phi: \mathbb{R} \times S^{2d+2} \rightarrow S^{2d+2} \) given by the formula

\[
\Phi(t, (z_0, \ldots, z_d, x)) = (e^{0\pi i t} \cdot z_0, \ldots, e^{a_d \pi i t} \cdot z_d, x),
\]

with \((a_0, \ldots, a_d) \neq (0, \ldots, 0)\). Here, \(S^{2d+2} = \{(z_0, \ldots, z_d, x) \in \mathbb{C}^d \times \mathbb{R} \mid |z_0|^2 + \cdots + |z_d|^2 + x^2 = 1\}\). There are two singular strata: the north pole \(S_1 = (0, \ldots, 0, 1)\) and the south pole \(S_2 = (0, \ldots, -1)\). The regular stratum is \(R = S^{2d+1} \times [-1, 1]\). Let \(r: R \rightarrow \mathbb{R}\) a smooth map, depending just on \(|1, 1|\), with \(r \equiv 0\) on \([0, 1/4] \cup [3/4, 1]\) and \(\int_0^1 r = 1\). The basic cohomology \(H^*(S^{2d+2}/\mathcal{F})\) of the foliation defined by this flow is the following:

| \(i\) | \(i = 0\) | \(i = 1\) | \(i = 2\) | \(i = 3\) | \(i = 4\) | \(i = 5\) | \(i = \cdots\) | \(i = 2d\) | \(i = 2d + 1\) |
|---|---|---|---|---|---|---|---|---|---|
| \(\text{r} \in \Omega^2(S^{2d+2}/\mathcal{F})\) | is an Euler form (cf. \([7]\)). These calculations come directly from the equalities:

(3) \(\|r\|_{S_k} = 0, \|dr\|_{S_k} = -\infty, \|e^j\|_{S_k} = \|re^j\|_{S_k} = 2^j, \|dr \wedge e^j\|_{S_k} = -\infty\) and \(dr \wedge e^j = d(re^j)\),

for \(k = 1, 2\) and \(j \in \{1, \ldots, d\}\).

We notice that the top dimensional basic cohomology group is isomorphic to \(\mathbb{R}\), but this cohomology does not have the Poincaré Duality property in spite of the fact that the flow is isometric! The classical basic cohomology does not take into account the stratification \(S_F\). However, even for the SRF, that basic cohomology is finite dimensional (cf. \([26]\)).

If we consider the BIC of our example the picture changes. The following table presents the BIC \(H_p^k(S^{2d+2}/\mathcal{F})\) for the constant perversities:

| \(i\) | \(0\) | \(1\) | \(2\) | \(3\) | \(4\) | \(5\) | \(6\) | \(7\) | \(\cdots\) | \(k-2\) | \(k-1\) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \(p < 0\) | \(0\) | \([dr]\) | \(0\) | \([e \wedge dr]\) | \(0\) | \([e^2 \wedge dr]\) | \(0\) | \([e^3 \wedge dr]\) | \(\cdots\) | \(0\) | \([e^d \wedge dr]\) |
| \(p = 0, 1\) | \(1\) | \(0\) | \(0\) | \([e \wedge dr]\) | \(0\) | \([e^2 \wedge dr]\) | \(0\) | \([e^3 \wedge dr]\) | \(\cdots\) | \(0\) | \([e^d \wedge dr]\) |
| \(p = 2, 3\) | \(0\) | \(0\) | \([e]\) | \(0\) | \(0\) | \([e^2 \wedge dr]\) | \(0\) | \([e^3 \wedge dr]\) | \(\cdots\) | \(0\) | \([e^d \wedge dr]\) |
| \(p = 4, 5\) | \(1\) | \(0\) | \([e]\) | \(0\) | \([e^2]\) | \(0\) | \(0\) | \([e^3 \wedge dr]\) | \(\cdots\) | \(0\) | \([e^d \wedge dr]\) |
| \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) |
| \(p = k-4, k-3\) | \(1\) | \(0\) | \([e]\) | \(0\) | \([e^2]\) | \(0\) | \([e^3]\) | \(0\) | \(\cdots\) | \(0\) | \([e^d \wedge dr]\) |
| \(p \geq k-2\) | \(1\) | \(0\) | \([e]\) | \(0\) | \([e^2]\) | \(0\) | \([e^3]\) | \(0\) | \(\cdots\) | \([e^d]\) | \(0\) |
These calculations come directly from the equalities \([16]\).

We notice that the top dimensional basic cohomology group is isomorphic either to 0 or \(\mathbb{R}\). These cohomology groups are finite dimensional. We recover the Poincaré Duality in the perverse sense:

\[
H^k_p(\mathbb{S}^k/F) \cong H^{n-k}_p(\mathbb{S}^k/F)
\]

for two complementary perversities: \(p + q = l = k - 3\).

This is more general. We have proved that the basic intersection cohomology is finite dimensional and verifies this perverse Poincaré Duality for the linear foliations (cf. \([20, 21]\)) and for the riemannian flows (cf. \([16]\)).

### 3 Top class

The top group \(H^n(M/F)\) of the basic cohomology of a regular riemannian foliation \(F\) defined on a connected compact manifold \(M\) is \(\mathbb{R}\) or 0. Here \(n = \text{codim}_M F\). We prove the same result for the top group \(H^n(M/F)\) when \(F\) is a SRF defined on a connected compact manifold \(M\). To do so, we need three Lemmas.

**Lemma 3.1** Consider \((T_S, \tau_S, S)\) a foliated tubular neighborhood of a singular stratum \(S \in \mathcal{S}_F\). Fix \(f : ]0, 1[ \rightarrow [0, 1] a smooth function with with \(f \equiv 0\) on \([0, 1/4]\) and \(f \equiv 0\) on \([3/4, 1]\). The map \([\omega] \mapsto [df \wedge \omega]\) defines the isomorphism

\[
H^{i-1}_{\tau, c}(D_S/F) \cong H^i_{\tau, c}(D_S \times [0, 1/[F \times I])
\]

**Proof.** Consider the following differential complexes:

- \(\mathbb{A}^* = \{ \omega \in \Omega_c^*(D_S \times [0, 3/4]/F \times I) / \left[ \text{supp } \omega \subset K \times [c, 3/4] \right. \text{ for a compact } K \subset D_S \text{ and } 0 < c < 3/4 \} \},

- \(\mathbb{B}^* = \{ \omega \in \Omega_c^*(D_S \times [1/4, 1]/F \times I) / \left[ \text{supp } \omega \subset K \times [1/4, c] \right. \text{ for a compact } K \subset D_S \text{ and } 1/4 < c < 1 \} \},

- \(\mathbb{C}^* = \{ \omega \in \Omega_c^*(D_S \times [1/4, 3/4]/F \times I) / \left[ \text{supp } \omega \subset K \times [1/4, 3/4] \right. \text{ for a compact } K \subset D_S \} \}.

Proceeding as in \([2]\) we get the short exact sequence

\[
0 \rightarrow \Omega^*_{\tau, c}(D_S \times [0, 1/[F \times I]) \rightarrow \mathbb{A}^* \oplus \mathbb{B}^* \rightarrow \mathbb{C}^* \rightarrow 0.
\]

The associated long exact sequence is

\[
\cdots \rightarrow H^{i-1}(\mathbb{C}^*) \xrightarrow{\delta} H^i_{\tau, c}(D_S \times [0, 1/[F \times I]) \rightarrow H^i(\mathbb{A}^*) \oplus H^i(\mathbb{B}^*) \rightarrow H^i(\mathbb{C}^*) \rightarrow \cdots,
\]

where the connecting morphism is \(\delta([\omega]) = [df \wedge \omega]\).

Before executing the calculation let us introduce some notation. Let \(\beta\) be a differential form on \(\Omega^i(D_S \times [a, b])\) which does not include the \(dt\) factor. By \(\int_c^s \beta(s) \wedge ds\) and \(\int_c^s \beta(s) \wedge ds\) we denote the forms on \(\Omega^i(D_S \times [a, b])\) obtained from \(\beta\) by integration with respect to \(s\), that is, \(\left(\int_c^s \beta(s) \wedge ds\right)(x, t)(\vec{v}_1, \ldots, \vec{v}_i) = \int_t^c (\beta(x, s)(\vec{v}_1, \ldots, \vec{v}_i)) \, ds\) and on the other hand
We proceed in several steps. Top in the BIC inclusion $\Omega$ Lemma 3.2 Consider we have seen in [20] that we can identify $\Omega$ and $(\vec{v},)$ $H$ $\delta$ Here $\parallel - \parallel$ is an isomorphism. Each differential form $\omega \in A^*, B^*$ and $C^*$ can be written $\omega = \alpha + dt \wedge \beta$ where $\alpha$ and $\beta$ do not contain $dt$. Consider a cycle $\omega = \alpha + dt \wedge \beta \in A^i$ with $\text{supp } \omega \subset K \times [c, 3/4]$ for a compact $K \subset D_S$ and $0 < c < 3/4$. We have $\omega = -d \left( \int_{c/2}^{c/2} \beta(s) \wedge ds \right)$. Since $\text{supp } \int_{c/2}^{c/2} \beta(s) \wedge ds \subset K \times [c, 3/4]$ then we get $H^*(A^i) = 0$. In the same way, we get $H^*(B^*) = 0$. We conclude that

$$\delta: H^*-1(C) \rightarrow \mathbb{H}^*_{p,c}(D_S \times ]0,1[ / \mathcal{F} \times \mathcal{I})$$

is an isomorphism.

The foliated homotopy $L: D_S \times ]1/4, 3/4[ \times [0, 1] \rightarrow D_S \times ]1/4, 3/4[$, defined by $L((u, t), s) = (u, (1-s)t + s/2)$, verifies $L(K \times ]1/4, 3/4[ \times [0, 1]) \subset K \times ]1/4, 3/4[$ for each compact $K \subset D_S$. So, we get that the map $[\alpha] \mapsto [\alpha]$ induces the isomorphism

$$I: \mathbb{H}^*_{p,c}(D_S / \mathcal{F}) \rightarrow H^*(C).$$

The isomorphism $\delta I$ gives the result.

Lemma 3.2 Consider $T_S$ a foliated tubular neighborhood of a singular stratum $S \in S_F$. The inclusion $T_S \setminus S \hookrightarrow T_S$ induces the onto map

$$\iota: \mathbb{H}^n_{p,c}((T_S \setminus S) / \mathcal{F}) \rightarrow \mathbb{H}^n_{p,c}(T_S / \mathcal{F}).$$

Moreover, if $\overline{p}(S) \leq \overline{t}(S)$ then $\iota$ is an isomorphism.

Proof. We proceed in several steps.

(a) Rewriting $\mathbb{H}^n_{p,c}(T_S / \mathcal{F})$.

We have seen in [20] that we can identify $\Omega^*_{p,c}(T_S / \mathcal{F})$ with the complex

$$\left\{ \omega \in \Omega^*_{p,c}(D_S \times ]0,1[ / \mathcal{F} \times \mathcal{I}) \mid \| \omega \|_{D_S \times \{0\}} \tau_S \leq \overline{p}(S) \text{ and } \| d\omega \|_{D_S \times \{0\}} \tau_S \leq \overline{p}(S) \right\}.$$

Here $\| - \|_{\tau_S}$ denotes the vertical degree relatively to the fibration $\tau_S: D_S \equiv D_S \times \{0\} \rightarrow S$. Recall that $\| 0 \|_{\tau_S} = -\infty$. Under this identification, the complex $\mathbb{H}^n_{p,c}((T_S \setminus S) / \mathcal{F})$ becomes $\Omega^*_{p,c}(D_S \times ]0,1[ \times \mathcal{F} \times \mathcal{I})$.

(b) Chasing $\mathbb{H}^n_{p,c}(T_S / \mathcal{F})$.

Consider the complex

$$\mathbb{D}^* = \left\{ \omega \in \Omega^*_{p,c}(D_S \times ]0,3/4[ / \mathcal{F} \times \mathcal{I}) \mid \begin{cases} \| \omega \|_{D_S \times \{0\}} \tau_S \leq \overline{p}(S), \\ \| d\omega \|_{D_S \times \{0\}} \tau_S \leq \overline{p}(S) \text{ and } \\ \text{supp } \omega \subset K \times ]0,3/4[ \\ \text{for a compact } K \subset D_S \end{cases} \right\}.$$
and the complexes $\mathbb{B}^*, \mathbb{C}^*$ as in the proof of Lemma 3.1. The short exact sequence

$$0 \rightarrow \Omega_{p,c}^*(T_S/F) \rightarrow D^* \oplus \mathbb{B}^* \rightarrow \mathbb{C}^* \rightarrow 0$$

produces the long exact sequence

$$\rightarrow H^{n-1}(D^*) \oplus H^{n-1}(\mathbb{B}^*) \rightarrow H^{n-1}(\mathbb{C}^*) \rightarrow H^n(D^*) \oplus H^n(\mathbb{B}^*) \rightarrow H^n(\mathbb{C}^*) \rightarrow 0$$

where the connecting morphism is $\delta('/: [\omega] = [df \wedge \omega]$ for a smooth map $f: [0, 1[ \rightarrow [0, 1]$ with $f \equiv 0$ on $[0, 1/4]$ and $f \equiv 1$ on $[3/4, 1]$.

(c) Relating $H^*(\mathbb{C})$ and $H^*(D)$.

We have already seen that $H^*(\mathbb{B}) = 0$. Consider a cycle $\omega = \alpha + dt \wedge \beta \in \mathbb{B}^*$. For degree reasons, we have $\alpha(0) = 0$ and then $\omega = d \left( \int_0^1 \beta(s) \wedge ds \right)$.

Since $\text{supp} \int_0^1 \beta(s) \wedge ds \subset K \times [0, 3/4|$ we get $H^n(D ([0, 3/4]) = 0$. From the above long exact sequence, we obtain the exact sequence

$$H^{n-1}(D^*) \rightarrow H^{n-1}(\mathbb{C}^*) \rightarrow H^n(D^*) \oplus H^n(\mathbb{B}^*) \rightarrow H^n(\mathbb{C}^*) \rightarrow 0.$$

(d) Conclusion.

Since the map $I$ is an isomorphism (cf. [11]) the composition $\delta'^{I}: H^{n-1}(D_S/F) \rightarrow H^n(T_S/F)$ is an onto map. The Lemma 3.1 gives that $\iota$ is an onto map. It remains to prove that the map $\iota$ is an isomorphism when $\mathcal{P}(S) \leq \mathcal{I}(S)$. We prove $H^{n-1}(\mathbb{D}) = 0$. Proceeding as in (c) it suffices to consider a cycle $\omega = \alpha + dt \wedge \beta \in D^{n-1}([0, 3/4])$ and show that $\alpha(0) = 0$.

Let us suppose that $\alpha(0) \neq 0$. The foliation $\mathcal{F}$ induces a foliation $\mathcal{F}_{\tau_S}$ tangent to the fibers of $\tau_S: D_S \rightarrow S$ such that $\dim \mathcal{F} = \dim \mathcal{F}_{\tau_S} + \dim \mathcal{F}_S$ (cf. [13] [19]). By degree reasons, since $\alpha(0) \in \Omega_{p,c}^{n-1}(D_S/F)$, we can write

$$\mathcal{I}(S) \geq \mathcal{P}(S) \geq ||\alpha(0)||_{\tau_S} = \frac{(\dim M - \dim S) - 1}{\dim \mathcal{F}_{\tau_S}} - \frac{\dim \mathcal{F} - \dim \mathcal{F}_S}{\dim \mathcal{F}_{\tau_S}} = \mathcal{I}(S) + 1.$$

This contradiction gives $\alpha(0) = 0$.

Lemma 3.3 Let $N$ be a compact manifold endowed with a RF $\mathcal{N}$. If $O$ is a connected saturated open subset of $N$, then $H^c_n(O/\mathcal{N}) = 0$ or $\mathbb{R}$.

Proof. We proceed in two steps.

The foliation $\mathcal{N}$ is transversally orientable. The result comes essentially from the basic Poincaré duality theorem for non-compact manifolds (cf. [22]). The foliation $\mathcal{N}_O$ is a transversally orientable complete foliation since $O$ is a saturated open subset of the compact manifold $N$. Then we have the isomorphism $H^c_n(O/\mathcal{N}) \cong H^n(\mathcal{N}_O, \mathcal{P})$ where $\mathcal{P}$ is the homological orientation sheaf of $\mathcal{N}_O$. Since the manifold $O$ is connected, the sheaf $\mathcal{P}$ is locally trivial and the stalk is $\mathbb{R}$ then $H^c_n(O/\mathcal{N}) = 0$ or $\mathbb{R}$.

General case Consider the transverse orientation covering $\pi: (N^*, N^*) \rightarrow (N, \mathcal{N})$ (see [8]). The covering is given by a foliated action of $\mathbb{Z}_2$. The foliation $\mathcal{N}^*$ is a transversally orientable RF. We have the equality $H^c_n(O/\mathcal{F}) = (H^c_n(\pi^{-1}(O)/\mathcal{F}^*)) \mathbb{Z}_2$. The subset $\pi^{-1}(O)$ is a saturated open subset of $N^*$. If $\pi^{-1}(O)$ is connected then the result comes from the previous case. If $\pi^{-1}(O)$ is
not connected then $\pi^{-1}(O)$ has two connected components foliated diffeomorphics to $O$ and the $\mathbb{Z}_2$-action interchanges them. The result comes now from the previous case.

The first result of this Note is the following.

**Theorem 3.4** Let $M$ be a connected compact manifold endowed with a SRF $F$. If $n = \text{codim}_M F$ and $\overline{p}$ a perversity on $M$, then

$$\mathbb{H}^n(\overline{F}) = 0 \text{ or } \mathbb{R},$$

**Proof.** For each $i \in \mathbb{Z}$ we write:

- $\Sigma_i \subset M$ the union of strata whose dimension is less or equal than $i$,
- $T_i$ the tubular neighborhood of $\Sigma_i$ in $M \setminus \Sigma_{i-1}$.

We have $M \setminus \Sigma_{-1} = M$ and $M \setminus \Sigma_{m-1} = R$, where $m = \dim M$. The Lemma 3.3 gives $H^c_n(R/F) = 0$ or $\mathbb{R}$. We get the result if we prove that the inclusion $M \setminus \Sigma_i \hookrightarrow M \setminus \Sigma_{i-1}$ induces an onto map $H^c_n((M \setminus \Sigma_i)/F) \to H^c_n((M \setminus \Sigma_{i-1})/F)$, for $i \in \{0, \ldots, m-1\}$.

From the open covering $\left\{ M \setminus \Sigma_i, \bigcup_{\dim S = i} T_S \right\}$ of $M \setminus \Sigma_{i-1}$, we obtain the Mayer-Vietoris sequence

$$\bigoplus_{\dim S = i} H^c_n((T_S \setminus S)/F) \to H^c_n((M \setminus \Sigma_i)/F) \oplus \bigoplus_{\dim S = i} H^c_n(T_S/F) \to H^c_n((M \setminus \Sigma_{i-1})/F) \to 0.$$

Now, the Lemma 3.2 gives the result.

Combining Theorem 3.4 with the tautness characterization of [17], we get the following Corollary.

**Corollary 3.5** Let $M$ be a connected compact manifold endowed with a SRF $F$. Let us suppose that $F$ is transversally orientable. Consider $\overline{p}$ a perversity on $M$ with $\overline{p} \leq \overline{t}$. If $n = \text{codim}_M F$, then the two following statements are equivalent:

(a) The foliation $F_R$ is taut, where $R$ is the regular stratum of $(M, F)$.

(b) The cohomology group $H^c_n(\overline{F})$ is $\mathbb{R}$.

**Proof.** We know from [17] that the condition (a) is equivalent to $H^c_n(R/F) = \mathbb{R}$. So, it suffices to prove that $H^c_n(R/F) \cong H^c_n(\overline{F})$ (cf. Lemma 3.3). We proceed as in the proof of the previous Theorem changing “onto map” by “isomorphism”.

3.6 Remarks.

(a) The perversity $\overline{p} = -\infty$ verifies $\overline{p} \leq \overline{t}$. In this case the group $H^c_n(\overline{F})$ becomes $H^c_n(M,F,\Sigma/F)$. Here, the relative basic cohomology $H^c_n(M,F,\Sigma/F)$ is computed from the relative basic complex $\Omega^*(M,F,\Sigma/F) = \{ \omega \in \Omega^*(M/F) \mid \omega \equiv 0 \text{ on } \Sigma \}$.

(b) The boundary perversity $\overline{\partial}$ verifies $\overline{\partial} \leq \overline{t}$. In this case the group $H^c_n(\overline{F})$ becomes $H^c_n(M,F,\partial(M,F))$. Here, the relative basic cohomology $H^c_n(M,F,\partial(M,F))$ is computed from
the relative basic complex $\Omega^*(\mathcal{M}/\mathcal{F}, \partial(\mathcal{M}/\mathcal{F})) = \{ \omega \in \Omega^*(\mathcal{M}/\mathcal{F}) | \omega \equiv 0 \text{ on the boundary strata} \}$. In particular, when the boundary strata do not appear then $H^n_\mathcal{F}(\mathcal{M}/\mathcal{F}) = H^n(\mathcal{M}/\mathcal{F})$.

(c) When $\mathring{p} \not\in \mathcal{T}$ then the group $H^n_\mathcal{F}(\mathcal{M}/\mathcal{F})$ does not establish the tautness of $(R, \mathcal{F})$. For example, we always have $H^n_{\mathcal{F}^{\mathcal{T}}} = H^n((\mathcal{M} \setminus \Sigma)/\mathcal{F}) = 0$ if $\Sigma \neq \emptyset$.

(d) In the Theorem 3.4 and the Corollary 3.5 we can suppose that the manifold $M$ is not compact and replace $H^n_\mathcal{F}(\mathcal{M}/\mathcal{F})$ by $H^n_{\mathcal{F}^{\mathcal{C}}}(\mathcal{M}/\mathcal{F})$.

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