THE BUNDLE OF KMS STATE SPACES FOR FLOWS ON A UNITAL $C^*$-ALGEBRA

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1. Introduction

The collection of KMS state spaces for a flow on a unital $C^*$-algebra can be thought of as a bundle of simplices over the real line. This may seem like a far-fetched analogy to other more established notions of bundles because the fibers of the bundle may be empty or they may all be mutually non-isomorphic, but we will show here that it is in fact a well-behaved and useful concept. Specifically, we use it to show that for any given unital separable infinite-dimensional simple AF algebra $A$ and for any configuration of KMS state spaces which occurs for a flow on a unital separable $C^*$-algebra and has the property that the simplex of 0-KMS states is affinely homeomorphic to the tracial state space of $A$, there is a flow on $A$ with the same configuration. In particular, it follows that for any given closed subset $F$ of real numbers containing 0 there are flows on $A$ whose KMS spectrum is $F$. This removes the lower boundedness condition which occurs in a recent work by the second author, [Th3].

Since we deal with unital AF algebras, there are always 0-KMS states present. For flows on infinite $C^*$-algebras this is not the case, and in a joint work with Y. Sato we have shown that for any unital, nuclear, purely infinite, simple, separable $C^*$-algebra $A$ in the UCT class and with torsion-free $K_1$ group, and for any configuration of KMS state spaces which occurs for a flow on a unital separable $C^*$-algebra without trace states, there is also a flow on $A$ with the same configuration; see [EST]. In both cases we depend on results from the classification of simple $C^*$-algebras.

While the work in [Th3] was based on ideas from [BEH] and [BEK1], in the present paper the underlying ideas are closer to those presented by Bratteli, Elliott and Kishimoto in [BEK2]. In particular, the idea of considering the configuration of KMS simplices as a bundle originates from [BEK2].

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2. Proper simplex bundles

Let $S$ be a second countable locally compact Hausdorff space and $\pi : S \to \mathbb{R}$ a continuous map. If the inverse image $\pi^{-1}(t)$, equipped with the relative topology inherited from $S$, is homeomorphic to a compact metrizable Choquet simplex for all $t \in \mathbb{R}$ we say that $(S, \pi)$ is a simplex bundle. We emphasize that $\pi$ need not be surjective, and we consider therefore also the empty set as a simplex. When $(S, \pi)$ is a simplex bundle we denote by $A(S, \pi)$ the set of continuous functions $f : S \to \mathbb{R}$ with the property that the restriction $f|_{\pi^{-1}(t)}$ of $f$ to $\pi^{-1}(t)$ is affine for all $t \in \mathbb{R}$.

Definition 2.1. (Compare [BEK2].) A simplex bundle $(S, \pi)$ is a proper simplex bundle when

1. $\pi$ is proper; that is $\pi^{-1}(K)$ is compact in $S$ when $K \subseteq \mathbb{R}$ is compact, and
2. $A(S, \pi)$ separate points on $S$; that is for all $x \neq y$ in $S$ there is an $f \in A(S, \pi)$ such that $f(x) \neq f(y)$.

Two proper simplex bundles $(S, \pi)$ and $(S', \pi')$ are isomorphic when there is a homeomorphism $\phi : S \to S'$ such that $\pi' \circ \phi = \pi$ and $\phi : \pi^{-1}(\beta) \to \pi'^{-1}(\beta)$ is affine for all $\beta \in \mathbb{R}$.

2.1. Proper simplex bundles from flows. In this paper all $C^*$-algebras are assumed to be separable and all traces and weights on a $C^*$-algebra are required to be non-zero, densely defined and lower semi-continuous. Let $A$ be a $C^*$-algebra and $\theta$ a flow on $A$. Let $S^\theta_\beta$ be the (possibly empty) set of $\beta$-KMS states for $\theta$. Let $E(A)$ be the state space of $A$, a compact convex set in the weak* topology. Set $S^\theta = \{ (\omega, \beta) \in E(A) \times \mathbb{R} : \omega \in S^\theta_\beta \}$, and equip $S^\theta$ with the relative topology inherited from the product topology of $E(A) \times \mathbb{R}$. Since $S^\theta$ is a closed subset of $E(A) \times \mathbb{R}$ by Proposition 5.3.23 of [BR], it follows that $S^\theta$ is a second countable locally compact Hausdorff space. Denote by $\pi^\theta : S^\theta \to \mathbb{R}$ the projection.
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to the second coordinate. Since the inverse image \( \pi^{-1}(\beta) \) is homeomorphic to \( S^\theta_\beta \), which is a Choquet simplex by Theorem 5.3.30 of [BR], the pair \((S^\theta, \pi^\theta)\) is a simplex bundle. An obvious application of Proposition 5.3.23 of [BR], using the compactness of \( E(A) \), shows that \( \pi^\theta \) is proper. Note that every self-adjoint element \( a \in A \) gives rise to an element \( \hat{a} \in A(S^\theta, \pi^\theta) \) such that \( \hat{a}(\omega, \beta) = \omega(a) \). A continuous function \( f : \mathbb{R} \to \mathbb{R} \) also gives rise to an element of \( A(S^\theta, \pi^\theta) : \)

\[ S^\theta \ni (\omega, \beta) \mapsto f(\beta). \]

It follows that \( A(S^\theta, \pi^\theta) \) separates the points of \( S^\theta \), showing that \((S^\theta, \pi^\theta)\) is a proper simplex bundle, which we shall call the KMS bundle. In general, \( \pi^{-1}(0) \) is the set of \( \theta \)-invariant trace states and hence non-empty if and only if \( A \) has trace states. When \( A \) is AF it is the simplex of all trace states of \( A \).

Remark 2.2. Let \((S, \pi)\) be a proper simplex bundle. Denote by \( A_\mathbb{R}(S, \pi) \) the subset of \( A(S, \pi) \) consisting of the elements that have a limit at infinity. This is a separable real Banach space (in the supremum norm) containing the constant function 1, and its state space \( E(A_\mathbb{R}(S, \pi)) \) is a metrizable compact convex set in the weak* topology. For \( x \in S \), let \( \text{ev}_x \in E(A_\mathbb{R}(S, \pi)) \) denote evaluation at \( x \). For each \( \beta \in \mathbb{R} \), the set \( K_{\beta} := \\{ \text{ev}_x : x \in \pi^{-1}(\beta) \} \) is a closed convex subset of \( E(A_\mathbb{R}(S, \pi)) \). With \( K = E(A_\mathbb{R}(S, \pi)) \), the system \( K_{\beta}, \beta \in \mathbb{R} \), has the properties required in Theorem 2.1 of [BEK1]. Thus, there is a unital, simple, separable, nuclear \( C^* \)-algebra \( A \) equipped with a \( 2\pi \)-periodic flow \( \theta \) such that \( \pi^{-1}(\beta) \) is affinely homeomorphic to the simplex of \( \beta \)-KMS states for all \( \beta \in \mathbb{R} \). In the construction of \( A \) in [BEK1] the algebra appears to depend on \((S, \pi)\) and on the many choices made in the process of its construction, but it was shown in [EST], based on the Kirchberg-Phillips classification result, that when \( \pi^{-1}(0) \) is empty one can take \( A \) to be any given separable, simple, nuclear, purely infinite \( C^* \)-algebra in the UCT class and with torsion free \( K_1 \) group. It follows from the main result we describe next that when \( \pi^{-1}(0) \) is not empty one can take \( A \) to be any infinite dimensional unital simple AF algebra whose tracial state space is affinely homeomorphic to \( \pi^{-1}(0) \).

3. THE MAIN RESULT AND APPLICATIONS

Theorem 3.1. Let \((S, \pi)\) be a proper simplex bundle and let \( A \) be a unital infinite-dimensional simple \( AF \) algebra with a tracial state space
affinely homeomorphic to $\pi^{-1}(0)$. There is a $2\pi$-periodic flow on $A$ whose KMS bundle is isomorphic to $(S, \pi)$.

By definition the KMS spectrum of a flow $\theta$ on a unital $C^*$-algebra is the set of real numbers $\beta$ for which $\theta$ has a $\beta$-KMS state. When the KMS bundle of $\theta$ is isomorphic to a proper simplex bundle $(S, \pi)$, the KMS spectrum of $\theta$ is the range $\pi(S)$ of $\pi$.

To exhibit possible ways to work with proper simplex bundles and to illustrate how Theorem 3.1 can be applied let us use it to prove the following statement.

**Corollary 3.2.** Let $A$ be a unital infinite-dimensional simple AF algebra and let $F$ be a closed subset of real numbers containing 0.

- There is a $2\pi$-periodic flow on $A$ whose KMS spectrum is $F$ and such that there is a unique $\beta$-KMS state for all $\beta \in F \setminus \{0\}$.
- There is a $2\pi$-periodic flow $\theta$ on $A$ whose KMS spectrum is $F$ and such that $S''_{\beta}$ is not affinely homeomorphic to $S''_{\beta'}$ when $\beta, \beta' \in F \setminus \{0\}$ and $\beta \neq \beta'$.

**Proof.** Given a proper simplex bundle $(S, \pi)$, set $S_F = \pi^{-1}(F)$ and denote by $\pi_F$ the restriction of $\pi$ to $\pi^{-1}(F)$. The pair $(S_F, \pi_F)$ is again a proper simplex bundle.

For the first item, let $T(A)$ be the tracial state space of $A$ and fix an element $\omega_0 \in T(A)$. Let $S$ be the subset

$$(T(A) \times \{0\}) \cup \{ (\omega_0, t) : t \in \mathbb{R} \setminus \{0\} \}$$

of the topological product $T(A) \times \mathbb{R}$. Let $\pi : S \to \mathbb{R}$ be the canonical projection. Then $(S, \pi)$ is a proper simplex bundle such that $\pi^{-1}(\beta) = \{ (\omega_0, \beta) \}$ when $\beta \neq 0$ and such that $\pi^{-1}(0)$ is a copy of $T(A)$. The existence of the desired flow follows from Theorem 3.1 by applying it to the bundle $(S_F, \pi_F)$.

For the second item, note that the KMS bundle of the flow described in Theorem 1.1 of [Th2] is a proper simplex bundle $(S', \pi')$ such that $\pi'^{-1}(0)$ contains only one point, $\pi'(S) = \mathbb{R}$ and such that $\pi'^{-1}(\beta)$ is not affinely homeomorphic to $\pi'^{-1}(\beta')$ when $\beta \neq \beta'$. With $(S, \pi)$ the bundle defined above set

$$S'' = \{ (x, y) \in S' \times S : \pi'(x) = \pi(y) \} .$$

Define $\pi'' : S'' \to \mathbb{R}$ by $\pi''(x, y) = \pi(y)$. Then $(S'', \pi'')$ is a proper simplex bundle and an application of Theorem 3.1 this time to $(S'_F, \pi'_F)$, gives the desired flow. \( \square \)

4. **Proof of the main result**

4.1. **Tools.** The following lemma is an immediate consequence of Lemma 3.1 of [Th3] and Theorem 2.4 of [Th1].
Lemma 4.1. Let $D$ be a $C^*$-algebra. Denote by $\rho \in \text{Aut}(D)$ an automorphism of $D$ and let $q \in D$ a projection in $D$ which is full in $D \rtimes \rho \mathbb{Z}$. Let $\hat{\rho}$ denote the restriction to $q(D \rtimes \rho \mathbb{Z})q$ of the dual action on $D \rtimes \rho \mathbb{Z}$ considered as a $2\pi$-periodic flow. Let $P : D \rtimes \rho \mathbb{Z} \to D$ denote the canonical conditional expectation. For each $\beta \in \mathbb{R}$, the map $\tau \mapsto \tau \circ P|_{q(D \rtimes \rho \mathbb{Z})q}$ is an affine homeomorphism from the set of traces $\tau$ on $D$ that satisfy
$$\tau \circ \rho = e^{-\beta} \tau \text{ and } \tau(q) = 1,$$  (4.1)
on onto the simplex of $\beta$-KMS states for $\hat{\rho}$.

In this lemma the topology on the set of traces of $D$ with the properties (4.1) is given by pointwise convergence on elements from the corner $qDq$ of $D$.

When $D$ is an AF algebra it is well-known that the set of its traces can be identified, via the map $\tau \mapsto \tau^*$, with the set $\text{Hom}^+(K_0(D), \mathbb{R})$ of non-zero positive homomorphisms $\phi : K_0(D) \to \mathbb{R}$; a fact stated as Lemma 3.5 in [Th3]. In the setting of Lemma 4.1 this implies that when $D$ is an AF algebra the KMS spectrum and the structure of the KMS states for $\hat{\rho}$ can be determined directly from the pair $(K_0(D), \rho^*)$.

To see how, we note that by Remark 3.3 in [LN] every $\beta$-KMS state $\tau$ for $\hat{\rho}$ on $q(D \rtimes \rho \mathbb{Z})q$ extends uniquely to a $\beta$-KMS weight $\hat{\tau}$ for the dual action. Since $D$ is the fixed point algebra for the dual action the restriction of $\hat{\tau}$ to $D$ is a trace on $D$, yielding a map
$$\tau \mapsto (\hat{\tau}|_D)_+$$  (4.2)
from the set of $\beta$-KMS states $\tau$ for $\hat{\rho}$ to $\text{Hom}^+(K_0(D), \mathbb{R})$. Therefore Lemma 4.1 has the following consequence.

Corollary 4.2. In the setting of Lemma 4.1 assume that $D$ is an AF algebra. For each $\beta \in \mathbb{R}$ the map (4.2) is an affine homeomorphism from the set of $\beta$-KMS states for $\hat{\rho}$ on $q(D \rtimes \rho \mathbb{Z})q$ onto the set of positive homomorphisms $\phi \in \text{Hom}^+(K_0(D), \mathbb{R})$ that satisfy
$$\phi \circ \rho_* = e^{-\beta} \phi \text{ and } \phi([q]) = 1.$$  (4.3)

Here the topology on the elements from $\text{Hom}^+(K_0(D), \mathbb{R})$ with the properties (4.3) is given by pointwise convergence on $\{x \in K_0(D) : 0 \leq x \leq [q]\}$.

Corollary 4.2 will be complemented by the following lemma which helps to control the Elliott invariant of $q(D \rtimes \rho \mathbb{Z})q$, and to ensure that it is classified by it. It follows from Lemma 3.4 of [Th3], which is based on arguments from [Sa], [MS1] and [MS2].

\footnote{Recall that a projection $q$ in a $C^*$-algebra $A$ is full when $\overline{AqA} = A$. This is automatic when $q \neq 0$ and $A$ is simple.}
Lemma 4.3. Let $D$ be a stable AF algebra such that $K_0(D)$ has large denominators. For any order automorphism $\alpha \in \text{Aut}(K_0(D))$ of $K_0(D)$ there is an automorphism $\gamma \in \text{Aut}(D)$ of $D$ such that

(a) $\gamma_* = \alpha$ on $K_0(D)$,
(b) the restriction map $\mu \mapsto \mu|_D$ is a bijection from traces $\mu$ on $D \cong \mathbb{Z}$ onto the $\gamma$-invariant traces on $D$, and
(c) $D \cong \mathbb{Z}$ is $\mathbb{Z}$-stable; that is, $(D \cong \mathbb{Z}) \otimes \mathbb{Z} \simeq D \otimes \mathbb{Z}$ where $\mathbb{Z}$ denotes the Jiang-Su algebra, [JS].

Given a proper simplex bundle $(S, \pi)$ and a closed subset $F \subseteq \mathbb{R}$ we denote by $(S_F, \pi_F)$ the proper simplex bundle with $S_F = \pi^{-1}(F)$ and $\pi_F$ is the restriction of $\pi$ to $S_F$. The following lemma relates $\mathcal{A}(S_F, \pi_F)$ to $\mathcal{A}(S, \pi)$ and it will be a crucial tool in the following.

Lemma 4.4. Let $(S, \pi)$ be a proper simplex bundle and $F \subseteq \mathbb{R}$ a closed subset.

1. The map $\mathcal{A}(S, \pi) \to \mathcal{A}(S_F, \pi_F)$ given by restriction is surjective.
2. Let $f_1, f_2, g_1, g_2 \in \mathcal{A}(S, \pi)$ such that $f_1(x) < g_1(x)$ for all $x \in S$ and all $i, j \in \{1, 2\}$. Assume that there is an element $h^F \in \mathcal{A}(S_F, \pi_F)$ such that

$$f_i(x) < h^F(x) < g_j(x) \quad \forall x \in S_F, \forall i, j \in \{1, 2\}.$$ 

There is an element $h \in \mathcal{A}(S, \pi)$ such that $h(y) = h^F(y)$ for all $y \in S_F$ and

$$f_i(x) < h(x) < g_j(x) \quad \forall x \in S, \forall i, j \in \{1, 2\}.$$

Proof. (1) Let $h \in \mathcal{A}(S_F, \pi_F)$. For each $n \in \mathbb{N}$ the pair $(S_{[-n,n]}, \pi_{[-n,n]})$ is a compact simplex bundle in the sense of [BEK2] and it follows from Lemma 2.2 in [BEK2] that there are elements $f_n \in \mathcal{A}(S_{[-n,n]}, \pi_{[-n,n]})$ such that the restriction $f_n|_{S_{F\cap [-n,n]}}$ of $f_n$ to $S_{F\cap [-n,n]}$ agrees with $h|_{S_{F\cap [-n,n]}}$. For $n \in \mathbb{N}$ choose a continuous function $\chi_n : \mathbb{R} \to [0, 1]$ such that $\chi_n(t) = 1$ for $t \leq n - \frac{1}{2}$ and $\chi_n(t) = 0$ for $t \geq n$. Define $f'_n : S_{[-n,n]} \to \mathbb{R}$ recursively by

$$f'_1(x) = (1 - \chi_1(|\pi(x)|))f_2(x) + \chi_1(|\pi(x)|)f_1(x),$$

and then $f'_n$ for $n \geq 2$ such that $f'_n(x) = f'_{n-1}(x)$ when $x \in \pi^{-1([-n+1,n-1])}$ and $f'_n(x) = (1 - \chi_n(|\pi(x)|))f_{n+1}(x) + \chi_n(|\pi(x)|)f_n(x)$ when $x \in \pi^{-1}([-n+1,n-1]) \cup [-n+1]$. Then $f'_n|_{S_{F\cap [-n,n]}} = h|_{S_{F\cap [-n,n]}}$ and since $f'_{n+1}$ extends $f'_n$ for all $n$, there is an element $f \in \mathcal{A}(S, \pi)$ such that $f|_{S_{[-n,n]}} = f'_n$. This element $f$ extends $h$.

The assertion (2) follows in a similar way on using Lemma 2.3 in [BEK2].

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2 An ordered group $(G, G^+)$ has large denominators when the following condition holds: For any $a \in G^+$ and any $n \in \mathbb{N}$ there are an element $b \in G$ and an $m \in \mathbb{N}$ such that $nb \leq a \leq mb$; see [Ni].
4.2. Dimension groups from proper simplex bundles. Given a Choquet simplex $\Delta$, as usual denote by $\text{Aff}\Delta$ the set of real-valued continuous and affine functions on $\Delta$. Fix a proper simplex bundle $(S, \pi)$ with $\pi^{-1}(0)$ non-empty. Let $H$ be a torsion free abelian group and $\theta : H \to \text{Aff}\pi^{-1}(0)$ a homomorphism. We assume

1. there is an element $u \in H$ such that $\theta(u) = 1$, and
2. $\theta(H)$ is dense in $\text{Aff}\pi^{-1}(0)$.

Set $H^* = \{h \in H : \theta(h)(x) > 0 \ \forall x \in \pi^{-1}(0)\} \cup \{0\}$. Then $(H, H^*)$ is a simple dimension group; see [EHS].

It follows from (1) of Lemma 4.4 that the map $r : A(S, \pi) \to \text{Aff}\pi^{-1}(0)$ given by restriction is surjective and we can therefore choose a linear map $L : \text{Aff}\pi^{-1}(0) \to A(\pi, S)$ such that $r \circ L = \text{id}$. We arrange, as we can, that $L(1) = 1$. Define $\hat{L} : \bigoplus_{\mathbb{Z}} H \to A(S, \pi)$ by

$$\hat{L}((h_n)_{n \in \mathbb{Z}})(x) = \sum_{n \in \mathbb{Z}} L(\theta(h_n))(x)e^{n\pi(x)}.$$

Let $\mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]$ denote the $\mathbb{Q}$-linear span of the functions defined on $S \setminus \pi^{-1}(0)$ by

$$x \mapsto e^{n\pi(x)}(1 - e^{-\pi(x)})^l$$

for some $n, l \in \mathbb{Z}$. For each $k \in \mathbb{N}$, choose continuous functions $\psi_k^0, \psi_k^+ : \mathbb{R} \to [0, 1]$ such that

$$\psi_k^0(t) = 1, \quad -\frac{1}{2k} \leq t \leq \frac{1}{2k},$$

$$\psi_k^+(t) = 1, \quad t \leq -\frac{1}{k},$$

$$\psi_k^+(t) = 1, \quad t \geq \frac{1}{k},$$

and

$$\psi_k^+(t) + \psi_k^0(t) + \psi_k^-(t) = 1 \text{ for all } t \in \mathbb{R}.$$

Consider the countable subgroup of $A(S, \pi)$

$$G_k := \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]\psi_k^0 \circ \pi + \hat{L}(\bigoplus_{\mathbb{Z}} H)\psi_k^0 \circ \pi + \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]\psi_k^+ \circ \pi.$$

In what follows we denote the support of a real-valued function $f$ by $\text{supp} f$. Let $A_{00}(S, \pi)$ denote the set of elements $f$ from $A(S, \pi)$ for which $\text{supp} f$ is compact and contained in $S \setminus \pi^{-1}(0)$. Since the topology of $S$ is second countable we can choose a countable subgroup $G_{00}$ of $A_{00}(S, \pi)$ with the following density property:

**Property 4.5.** For all $N \in \mathbb{N}$, all $\epsilon > 0$ and all $f \in A_{00}(S, \pi)$ with $\text{supp} f \in \pi^{-1}(\mathbb{Z} - N, N\setminus\{0\})$, there is $g \in G_{00}$ such that

$$\sup_{x \in S}|f(x) - g(x)| < \epsilon$$

and $\text{supp} g \subseteq \pi^{-1}(\mathbb{Z} - N, N\setminus\{0\})$.

When $f_1 \in \hat{L}(\bigoplus_{\mathbb{Z}} H)$ and $f_2^+ \in \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]$ the difference between

$$f_2^+ \psi_k^0 \circ \pi + f_1 \psi_k^0 \circ \pi + f_2^+ \psi_k^+ \circ \pi$$
and
\[ f_2 \psi_{k+1}^* \circ \pi + f_1 \psi_{k+1}^0 \circ \pi + f_2^* \psi_{k+1}^* \circ \pi \]
is an element of \( A_0(S, \pi) \), and so by enlarging \( G_{00} \) we can ensure that
\[ G_k + G_{00} \subseteq G_{k+1} + G_{00}. \] (4.5)

Let \( \alpha_0 \in \operatorname{Aut} A(S, \pi) \) be defined by
\[ \alpha_0(f)(x) = e^{-\pi(x)} f(x). \]
Since \( \alpha_0(A_{00}) = A_{00} \) and since \( (1 - e^{-\pi})^{-1} A_{00}(S, \pi) \subseteq A_0(S, \pi) \), we can enlarge \( G_{00} \) further to achieve that
\[ \alpha_0(G_{00}) = G_{00} \] (4.6)
and that
\[ (\operatorname{id} - \alpha_0)(G_{00}) = G_{00}. \] (4.7)

We define
\[ G = \bigcup_{k=1}^{\infty} (G_k + G_{00}). \]

Let \( \sigma \in \operatorname{Aut} (\bigoplus_{n \in \mathbb{Z}} H) \) denote the shift:
\[ \sigma((h_n)_{n \in \mathbb{Z}}) = (h_{n+1})_{n \in \mathbb{Z}}. \]
Then \( \alpha_0 \circ \hat{L} = \hat{L} \circ \sigma \), which implies that \( \alpha_0(G_k) = G_k \) and hence
\[ \alpha_0(G) = G. \]

Set
\[ A(S, \pi)^+ = \{ f \in A(S, \pi) : f(x) > 0 \ \forall x \in S \} \cup \{0\} \]
and
\[ G^+ = G \cap A(S, \pi)^+. \]

**Lemma 4.6.** The pair \((G, G^+)\) has the following properties.

1. \( G^+ \cap (-G^+) = \{0\} \).
2. \( G = G^+ - G^+ \).
3. \((G, G^+)\) is unperforated, i.e., \( n \in \mathbb{N} \setminus \{0\}, \ g \in G, \ ng \in G^+ \Rightarrow g \in G^+ \).
4. \((G, G^+)\) has the strong Riesz interpolation property, i.e., if \( f_1, f_2, g_1, g_2 \in G \) and \( f_i < g_j \) in \( G \) for all \( i, j \in \{1, 2\} \), then there is an element \( h \in G \) such that
   \[ f_i < h < g_j \]
   for all \( i, j \in \{1, 2\} \).

**Proof.** (1) and (3) are obvious. (2): Let \( f \in G \). Then \( f \in G_k + G_{00} \) for some \( k \in \mathbb{N} \) and we can write
\[ f = h^- \psi_k^* \circ \pi + f_0 \psi_k^0 \circ \pi + h^+ \psi_k^0 \circ \pi + g \]
where \( h^\pm \in \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}], \ f_0 \in \hat{L}(\bigoplus_{n \in \mathbb{Z}} H) \) and \( g \in G_{00} \). By definition of \( \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}] \) there are \( n, m \in \mathbb{N} \) and \( M > 0 \) such that
\[ h^-(x) \leq e^{-n \pi(x)} \ \forall x \in \pi^{-1}([-\infty, -M]) \],
\[ h^+(x) \leq e^{\pi(x)} \quad \forall x \in \pi^{-1}([M, \infty]) . \]

Since \( f_0 \psi_k^0 \circ \pi \) and \( g \) are compactly supported there is \( K \in \mathbb{N} \) such that \( f(x) < g(x) \) for all \( x \in S \), where

\[ g = K \left( e^{-\pi \psi_k^0} \circ \pi + \psi_k^0 \circ \pi + e^{\pi \psi_k^+} \circ \pi \right) \in G^+ . \]

Note that \( f = g - (g - f) \in G^+ - G^+ \).

(4): Since \( \theta(H) \) has the Riesz interpolation property for the strict order by Lemma 3.1 in [EHS], there is \( h_0 \in \theta(H) \) such that \( f_i(x) < h_0(x) < g_j(x) \) for all \( i, j \) and all \( x \in \pi^{-1}(0) \). We claim that there are elements \( h^+ \in \mathbb{Q}[-e^{-\pi}, 1-e^{-\pi}] \) and \( K^+ \in \mathbb{N} \) such that \( f_i(x) < h^-(x) < g_j(x) \) when \( \pi(x) \leq -K^+ \) and \( f_i(x) < h^+(x) < g_j(x) \) when \( \pi(x) \geq K^+ \). To find \( h^+ \) and \( K^+ \), note that we may assume that \( \pi(S) \) contains arbitrarily large positive numbers; otherwise we take \( K^+ \) larger that \( \sup \pi(S) \) and \( h^+ = 0 \). By definition of \( G \) there is \( N \in \mathbb{N} \) so large that there are polynomials \( p_1, p_2, q_1, q_2 \) with rational coefficients such that

\[ \left( e^{\pi(x)}(e^{\pi(x)} - 1) \right)^N f_i(x) = p_i(e^{\pi(x)}) \]

and

\[ \left( e^{\pi(x)}(e^{\pi(x)} - 1) \right)^N g_j(x) = q_j(e^{\pi(x)}) \]

for all \( i, j \) and all large \( x \). Then \( p_i(y) < q_j(y) \) for all large elements \( y \) of \( e^{\pi(S)} \) and it follows therefore from Lemma 4.7 that there is a polynomial \( h' \) with rational coefficients such that \( p_i(x) < h'(x) < q_j(x) \) for all \( i, j \) and all large \( x \). Set

\[ h^+(x) = \left( e^{\pi(x)}(e^{\pi(x)} - 1) \right)^N h'(e^{\pi(x)}) . \]

Then \( h^+ \in \mathbb{Q}[-e^{-\pi}, 1-e^{-\pi}] \) and if \( K^+ \) is large enough \( f_i(x) < h^+(x) < g_j(x) \) for all \( i, j \) and all \( x \in \pi^{-1}([K^+, \infty)) \). The pair \( h^-, K^- \) is constructed in a similar way: Multiply each of the functions from \( \{f_1, f_2, g_1, g_2\} \) with the same function of the form

\[ (e^{-\pi})^N (e^{-\pi} - 1)^N \]

to get them into the form \( x \mapsto p(e^{-\pi(x)}) \) for \( p \) a polynomial with rational coefficients and apply Lemma 4.7.

Having \( h^+ \) and \( K^+ \) we use the statement (2) of Lemma 4.4 to find \( H \in \mathcal{A}(S, \pi) \) such that \( H(x) = h^-(x) \) when \( \pi(x) \leq -K^- \), \( H(x) = h^+(x) \) when \( \pi(x) \geq K^+ \), \( H(x) = h_0(x) \) when \( x \in \pi^{-1}(0) \), and \( f_i(x) < H(x) < g_j(x) \) for all \( x \in S \) and all \( i, j \). Set

\[ H'(x) = H \psi_k^0 \circ \pi + L(h_0) \psi_k^0 \circ \pi + H \psi_k^+ \circ \pi . \]

If \( k \) is large enough we have

\[ f_i(x) < H'(x) < g_j(x) \]

for all \( x \in S \) and all \( i, j \). Set

\[ H''(x) = H'(x) - L(h_0) \psi_k^0 \circ \pi - h^- \psi_k^0 \circ \pi - h^+ \psi_k^+ \circ \pi , \]
and note that sup $H'' \subseteq [K^- , K^+[\{0\}]$. Set $K = \max\{K^- , K^+\}$ and let $\delta > 0$ be given, smaller than $g_j(x) - H'(x)$ and $H'(x) - f_i(x)$ for all $i, j$ and all $x \in \pi^{-1}(\{ -K, K\})$. By Property 4.5 there is an element $g' \in \mathbb{G}_{00}$ such that

$$\text{supp } g' \subseteq \pi^{-1}(\{ -K, K[\{0\}]$$

and $\sup_{x \in S} |g'(x) - H''(x)| < \frac{\delta }{2}$. Then

$$h = g' + \mu (h_0) \psi^0_k \circ \pi + h^{-} \psi^{-}_k \circ \pi + h^{+} \psi^{+}_k \circ \pi \in G$$

and $f_i(x) < h(x) < g_j(x)$ for all $i, j$ and all $x \in S$. \hfill $\Box$

**Lemma 4.7.** Let $p_i, q_j$, $i, j \in \{1, 2\}$, be polynomials with rational coefficients. Assume that there is a sequence $\{x_n\}$ in $\mathbb{R}$ such that $\lim_{n \to \infty} x_n = \infty$ and such that $p_i(x_n) < q_j(x_n)$ for all $i, j, n$. It follows that there is a polynomial $h$ with rational coefficients and a $K > 0$ such that

$$p_i(x) < h(x) < q_j(x)$$

for all $i, j \in \{1, 2\}$ and all $x \geq K$.

**Proof.** Since polynomials only have finitely many zeros there is a $K' > 0$ such that $p_i(x) < q_j(x)$ for all $i, j$ and all $x \geq K'$. Write $q_j(x) = a_{0,j} + a_{1,j}x + a_{2,j}x^2 + \cdots + a_{N,j}x^N$ and $p_i(x) = b_{0,j} + b_{1,j}x + b_{2,j}x^2 + \cdots + b_{N,j}x^N$ for some $N$ larger than the degree of any of the four given polynomials, and set

$$\xi_i = (b_{N,i}, b_{N-1,i}, \cdots, b_{0,i}) \in \mathbb{Q}^{N+1}$$

and

$$\eta_j = (a_{N,j}, a_{N-1,j}, \cdots, a_{0,j}) \in \mathbb{Q}^{N+1}. $$

Since $p_i(x) < q_j(x)$ for all large $x$, we have that $\xi_i <_{\text{lex}} \eta_j$ for all $i, j$ with respect to the lexicographic order $<_{\text{lex}}$. Since $\mathbb{Q}^{N+1}$ is totally ordered in the lexicographic order there is an element $c = (c_N, c_{N-1}, c_{N-2}, \cdots, c_0) \in \mathbb{Q}^{N+1}$ such that $\xi_i <_{\text{lex}} c <_{\text{lex}} \eta_j$ for all $i, j$. By changing $c_0$ by a small amount we can arrange that $c \notin \{\xi_1, \xi_2, \eta_1, \eta_2\}$. Then the polynomial

$$h(x) = c_0 + c_1x + c_2x^2 + \cdots + c_Nx^N$$

will have the desired property. \hfill $\Box$

Consider the subset $\Gamma$ of $(\oplus \mathbb{Z}) \oplus G$ consisting of the elements $(\xi, g) \in (\oplus \mathbb{Z}) \oplus G$ with the property that there is an $\epsilon > 0$ such that

$$\hat{L}(\xi) (x) = g(x) \quad \forall x \in \pi^{-1}(\{ -\epsilon, \epsilon\}) . \quad (4.8)$$

$\Gamma$ is a subgroup of $(\oplus \mathbb{Z}) \oplus G$.

**Lemma 4.8.** The projection $\Gamma \to G$ is surjective.

**Proof.** Let $g \in G$. By definition of $G$ there is an element $\xi \in \oplus \mathbb{Z}$ and $k \in \mathbb{N}$ such that $g(x) = \hat{L}(\xi)$ on $\pi^{-1}(\{ -\frac{1}{2k}, \frac{1}{2k}\})$. \hfill $\Box$
Set
\[ \Gamma^+ = \{ (\xi, g) \in \Gamma : g \in G^+ \{0\} \cup \{0\} \}. \]

By combining Lemma 4.7 and Lemma 4.8 above with Lemma 3.1 and Lemma 3.2 in \[\text{EHS}\] we conclude that \((\Gamma, \Gamma^+)\) is a dimension group.

Given an element \(h \in H\) we denote in the following by \(h^{(0)}\) the element of \(\bigoplus \mathbb{Z} H\) defined by \((h^{(0)})_0 = h\) and \((h^{(0)})_n = 0\) when \(n \neq 0\). Define \(\Sigma : \bigoplus \mathbb{Z} H \to H\) by
\[ \Sigma((h_n)_{n \in \mathbb{Z}}) = \sum_{n \in \mathbb{Z}} h_n. \]

**Lemma 4.9.** \((\Gamma, \Gamma^+)\) has large denominators; that is for all \(x \in \Gamma^+\) and \(m \in \mathbb{N}\) there is an element \(y \in \Gamma^+\) and an \(n \in \mathbb{N}\) such that \(my \leq x \leq ny\).

**Proof.** Let \(x = (\xi, g) \in \Gamma^+ \{0\}\) and \(m \in \mathbb{N}\) be given. Then \(\Sigma(\xi) \in H^+ \{0\}\) and since \(H\) has large denominators by \[\text{[4.7]}\], there is an element \(b \in \bigoplus \mathbb{Z} H\) such that \(mb < \Sigma(\xi) < nb\) for some \(n \in \mathbb{N}\), \(n > m + 2\). Since \(L(\theta(\Sigma(\xi)))\) agrees with \(g\) on \(\pi^{-1}(0)\), there is a compact neighborhood \(U\) of \(0\) such that
\[ mL(\theta(b))(x) < g(x) < nL(\theta(b))(x) \]
for all \(x \in \pi^{-1}(U)\). There is also a \(K \in \mathbb{N}\) such that \(U \subseteq [-K, K]\) and functions \(f^+ \in \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]\) such that \(g(x) = f^-(x)\), \(x \leq -K\), and \(g(x) = f^+(x), x \geq K\). It follows from (2) of Lemma 4.7 that there is an element \(a \in \mathcal{A}(S, \pi)\) such that
\[ \frac{1}{m}g(x) < a(x) < \frac{1}{m}g(x) \quad \forall x \in S, \]
\[ a(x) = L(\theta(b))(x) \quad \text{for all } x \in \pi^{-1}(U), \]
\[ a(x) = \frac{1}{m+1}f^+(x) \quad \text{for all } x \geq K. \]

Choose \(k \in \mathbb{N}\) so large that \([-\frac{1}{K}, \frac{1}{K}] \subseteq U\) and note that
\[ a(x) = L(\theta(b))(x)\psi_0^0 \circ \pi(x) + a(x)\psi_k^0 \circ \pi(x) + a(x)\psi_k^- \circ \pi(x). \]

Then the function
\[ a' = a - L(\theta(b))\psi_k^0 \circ \pi - \frac{1}{m+1}f^+\psi_k^+ \circ \pi - \frac{1}{m+1}f^-\psi_k^- \circ \pi \]
is supported in \([-K, K]\) and \(\frac{1}{m}g(x) - a(x)\) and \(a(x) - \frac{1}{m}g(x)\) for all \(x \in \pi^{-1}([-K, K])\). By Property 4.3 we can find \(c \in G_{00}\) such that \(\supp c \subseteq [-K, K]\) and \(|c(x) - a'(x)| < \delta\) for all \(x \in S\). Then
\[ g' = c + L(\theta(b))\psi_k^0 \circ \pi + \frac{1}{m+1}f^+\psi_k^+ \circ \pi + \frac{1}{m+1}f^-\psi_k^- \circ \pi \quad \epsilon \quad G \]
and \(mg'(x) < g(x) < ng'(x)\) for all \(x \in S\). It follows that \(y = (b^{(0)}, g) \in \Gamma^+\) has the desired property.

Since \(\hat{\mathcal{L}} \circ \sigma = \alpha_0 \circ \hat{\mathcal{L}}\) we can define \(\alpha \in \text{Aut} \Gamma\) by
\[ \alpha = \sigma \oplus \alpha_0. \]
Lemma 4.10. The only order ideals $I$ in $\Gamma$ such that $\alpha(I) = I$ are $I = \{0\}$ and $I = \Gamma$.

Proof. Recall that an order ideal $I$ in $\Gamma$ is a subgroup such that
(a) $I = I \cap \Gamma^+$ and $I \cap \Gamma^*$, and
(b) when $0 \leq y \leq x$ in $\Gamma$ and $x \in I$, then $y \in I$.

Let $I$ be a non-zero order ideal such that $\alpha(I) = I$. Since $I \cap \Gamma^* \neq \{0\}$ there is an element $g \in G^* \setminus \{0\}$ and an element $\xi \in \bigoplus_{n \in \mathbb{Z}} H$ such that $(\xi, g) \in I$. Set $h = \Sigma(\xi)$. By definition of $G^* \setminus \{0\}$ there are natural numbers $n, m, k, K \in \mathbb{N}$ such that the function

$$g' = L(\theta(h))\psi^0_k \circ \pi + e^{n\pi} \psi^-_k \circ \pi + e^{-m\pi} \psi^+_k \circ \pi$$

has the property that

$$0 < g'(x) < K g(x) \quad \forall x \in S.$$

It follows that $(h(0), g') \in I \cap \Gamma^+$. Note that

$$\alpha^l_1(g') = e^{-l\pi} L(\theta(h))\psi^0_k \circ \pi + e^{(n-l)\pi} \psi^-_k \circ \pi + e^{-(m+l)\pi} \psi^+_k \circ \pi$$

for all $l \in \mathbb{Z}$. Consider an arbitrary element $(\xi', f) \in \Gamma^* \setminus \{0\}$. We can then find $l_1, l_2 \in \mathbb{Z}$ and $M \in \mathbb{N}$ such that

$$f(x) < M \left(\alpha^{l_1}_1((h(0), g')) + \alpha^{l_2}_1((h(0), g'))\right) \quad \forall x \in S.$$

Since

$$\alpha^{l_1}_1((h(0), g')) + \alpha^{l_2}_1((h(0), g')) \in I \cap \Gamma^+,$$

it follows that $(\xi', f) \in I$ and we conclude therefore that $I = \Gamma$. □

4.3. Some homomorphisms $\Gamma \rightarrow \mathbb{R}$. Note that the constant function $1$ is in $G$ and that, with $v := (u(0), 1)$, $v \in \Gamma^+$. Let $\beta \in \mathbb{R}$ and $\omega \in \pi^{-1}(\beta)$. As in Remark 2.2 we denote in the sequel by $A_{\mathbb{R}}(S, \pi)$ the real Banach space consisting of the elements of $A(S, \pi)$ that have a limit at infinity.

Lemma 4.11. Let $f \in A_{\mathbb{R}}(S, \pi)$ and let $\epsilon > 0$ be given. There is an element $g \in G$ such that $\sup_{x \in S} |f(x) - g(x)| \leq \epsilon$.

Proof. An initial approximation gives us an element $f_1 \in A(S, \pi)$ which is compactly supported and a real number $r \in \mathbb{R}$ such that

$$\sup_{x \in S} |f(x) - f_1(x) - r| \leq \frac{\epsilon}{2}.$$

Let $q \in \mathbb{Q}$ such that $|q - r| < \frac{\epsilon}{6}$ and choose an element $h \in H$ such that $|\theta(h)(y) - f_1(y) - r| < \frac{\epsilon}{6}$ for all $y \in \pi^{-1}(0)$. There is a $k \in \mathbb{N}$ such that $|L(\theta(h))(x) - f_1(x) - r| < \frac{\epsilon}{6}$ for all $x \in \pi^{-1}([-\frac{1}{2}, \frac{1}{2}])$. Since $f_1 \psi^-_k \circ \pi + f_1 \psi^+_k \circ \pi$ is compactly supported in $S \setminus \pi^{-1}(0)$ it follows from Property 4.5 that there is an element $g' \in G_{00}$ such that

$$\sup_{x \in S} |g'(x) - f_1(x)\psi^-_k \circ \pi(x) - f_1(x)\psi^+_k \circ \pi(x)| \leq \frac{\epsilon}{6}.$$

Then

$$g = L(\theta(h))\psi^0_k \circ \pi + q \psi^+_k \circ \pi + q \psi^-_k \circ \pi + g' \in G$$

as desired.
is an element with the desired property.

Let $\beta \in \mathbb{R}$. For each $\omega \in \pi^{-1}(\beta)$, define $\omega_\beta : \Gamma \to \mathbb{R}$ by

$$\omega_\beta(\xi, g) = g(\omega).$$

Then $\omega_\beta(\Gamma^+) \subseteq [0, \infty)$, $\omega_\beta(v) = 1$, and $\omega_\beta \circ \alpha = e^{-\beta} \omega_\beta$.

**Lemma 4.12.** Let $\phi : \Gamma \to \mathbb{R}$ be a positive homomorphism with the properties that $\phi(v) = 1$ and $\phi \circ \alpha = s \phi$ for some $s > 0$. Set $\beta = - \log s$. There is an element $\omega \in \pi^{-1}(\beta)$ such that $\phi = \omega_\beta$.

**Proof.** The projection $p : \Gamma \to G$ is surjective by Lemma 4.8. Assume that $(\xi, g) \in \Gamma$ and $p(\xi, g) = g = 0$. Since $(\Gamma, \Gamma^+)$ has large denominators by Lemma 4.9 there is for each $n \in \mathbb{N}$ an element $(\xi_n, g_n) \in \Gamma^+$ and a natural number $k_n$ such that $n(\xi_n, g_n) \leq v \leq k_n(\xi_n, g_n)$. Then $\pm (\xi, g) \leq (\xi_n, g_n)$ in $\Gamma$ and hence $\pm \phi(\xi, g) \leq \phi(\xi_n, g_n) \leq \frac{1}{n}$. It follows that $\phi(\xi, g) = 0$ and we conclude that there is a homomorphism $\phi' : G \to \mathbb{R}$ such that $\phi' \circ p = \phi$. Let $g \in G$ and assume that $g(x) > 0$ for all $x \in S$, and then also a natural number $k \in \mathbb{N}$ such that $0 < L(\theta(h_n))(x) \psi_k^0 \circ \pi(x) + \frac{1}{2n} \psi_k^+ \circ \pi(x)$ for all $x \in S$. Then

$$g_k^* := L(\theta(h_n)) \psi_k^0 \circ \pi + \frac{1}{2n} \psi_k^+ \circ \pi + \frac{1}{2n} \psi_k^- \circ \pi \in G^+$$

and $0 \leq n(h_n^{(0)}, g_n') \leq v$ in $\Gamma$. Hence $0 \leq \phi'(g_k^*) \leq \frac{1}{n}$. Let $\xi \in \bigoplus_{n \in \mathbb{Z}} H$ be an element such that $(\xi, g) \in \Gamma$. Then $(h_n^{(0)} + \xi, g_n' + g) \in \Gamma^+$ and hence $0 \leq \phi'(g_n^* + g) \leq \phi'(g) + \frac{1}{n}$. Letting $n$ tend to infinity we find that $\phi'(g) \geq 0$, proving that $\phi'$ is positive on $G$. Let $g \in G$ and $n, m \in \mathbb{N}$ satisfy $|g(x)| < \frac{m}{n}$ for all $x \in S$. Then $-n < mg(x) < n$ for all $x \in S$ and since $\phi'(1) = 1$ this leads to the conclusion that $|\phi'(g)| \leq \frac{m}{2n}$. Combined with Lemma 4.11 it follows from the last estimate that $\phi'$ extends by continuity to a linear map $\phi' : \mathcal{A}_G(S, \pi) \to \mathbb{R}$ such that $|\phi'(f)| \leq \sup_{x \in S} |f(x)|$. Using a Hahn-Banach theorem we extend $\phi'$ in a norm-preserving way to the space of all continuous real-valued functions on $S$ with a limit at infinity. Since $\phi'(1) = 1$ the extension is positive. It follows that there is a bounded Borel measure $m$ on $S$ such that

$$\phi'(f) = \int_S f(x) \, dm$$

for all $f \in \mathcal{A}_0(S, \pi)$, where $\mathcal{A}_0(S, \pi)$ denotes the space of elements in $\mathcal{A}_G(S, \pi)$ that vanish at infinity. Let $C_c(\mathbb{R})$ denote the set of continuous real-valued compactly supported functions on $\mathbb{R}$ and note that $C_c(\mathbb{R})$ is mapped into $\mathcal{A}_0(S, \pi)$ by the formula $F \mapsto F \circ \pi$. Since $\phi \circ \alpha = s \phi$ by assumption it follows that the measure $m \circ \pi^{-1}$ on $\mathbb{R}$ satisfies

$$\int_\mathbb{R} e^{-t} F(t) \, dm \circ \pi^{-1}(t) = s \int_\mathbb{R} F(t) \, dm \circ \pi^{-1}(t) \quad \forall F \in C_c(\mathbb{R}).$$
It follows that \( m \circ \pi^{-1} \) is concentrated at the point \( \beta = -\log s \) and hence that \( m \) is concentrated on \( \pi^{-1}(\beta) \). We can therefore define a linear functional \( \phi' : \text{Aff} \pi^{-1}(\beta) \to \mathbb{R} \) by
\[
\phi'(f) = \phi(\hat{f}) = \int_S \hat{f}(x) \, dm(x) ,
\]
where \( \hat{f} \in A_0(S, \pi) \) is any element with \( \hat{f}|_{\pi^{-1}(\beta)} = f \), which exists by (1) in Lemma 4.4. If \( f \geq 0 \) it follows from (2) of Lemma 4.4 that \( \hat{f} \) can be chosen such that \( \hat{f} \geq -\epsilon \) for any \( \epsilon > 0 \) and we see therefore that \( \phi' \) is a positive linear functional. Since every state of \( \text{Aff} \pi^{-1}(\beta) \) is given by evaluation at a point in \( \pi^{-1}(\beta) \) it follows in this way that there is an \( \omega \in \pi^{-1}(\beta) \) and a number \( \lambda \geq 0 \) such that
\[
\phi'(g) = \lambda g(\omega) \quad (4.9)
\]
for all \( g \in A_0(S, \pi) \). In particular, this conclusion holds for all \( g \in G \cap A_0(S, \pi) \). A general element \( f \in G \) can be write as a sum
\[
f = f_- + f_0 + f_+ ,
\]
where \( f_-, f_0 \in G, f_0 \) has compact support and there are natural numbers \( n_+ \in \mathbb{N} \) such that \( e^{n_+} f_+ \in A_0(S, \pi) \) and \( e^{-n_-} f_- \in A_0(S, \pi) \). Then
\[
\phi'(f_0) = \lambda f_0(\omega) ,
\]
\[
\phi'(f_-) = \phi'(\alpha^n(e^{-n_+} f_-)) = s^n \phi'(e^{-n_+} f_-) = s^n \lambda e^{-n_+}(\omega) f_-(\omega) = \lambda f_-(\omega) ,
\]
and similarly, \( \phi'(f_+) = \lambda f_+(\omega) \). It follows that \( \phi(f) = \lambda f(\omega) \). Inserting \( f = 1 \) we find that \( \lambda = 1 \) and the proof is complete.

### 4.4. Application of the Pimsner-Voiculescu exact sequence.
Let \( B \) be a stable AF algebra with \( (K_0(B), K_0(B)^+) = (\Gamma, \Gamma^+) \) and let \( \gamma \) be an automorphism of \( B \) such that \( \gamma_* = \alpha \); see [EI].

**Additional properties** 4.13. By Lemma 4.3 we can arrange that \( \gamma \) has the following additional properties:

(A) The restriction map \( \mu \mapsto \mu|_B \) is a bijection from traces \( \mu \) on \( B \times_\gamma \mathbb{Z} \) onto the \( \gamma \)-invariant traces on \( B \), and

(B) \( B \times_\gamma \mathbb{Z} \) is \( \mathbb{Z} \)-stable; that is \( (B \times_\gamma \mathbb{Z}) \otimes \mathbb{Z} = B \times_\gamma \mathbb{Z} \) where \( \mathbb{Z} \) denotes the Jiang-Su algebra, [JS].

Set
\[
C = B \times_\gamma \mathbb{Z} .
\]

It follows from Lemma 4.10 and [EI] that \( B \) is \( \gamma \)-simple and hence from [E2] (see also [KH]) that \( C \) is simple. It follows from the Pimsner-Voiculescu exact sequence, [PV], that we can identify \( K_0(C) \), as a group, with the quotient
\[
\Gamma/(\text{id} - \alpha)(\Gamma) ,
\]
in such a way that the map \( \iota : K_0(B) \to K_0(C) \) induced by the inclusion \( \iota : B \to C \) becomes the quotient map

\[
q : \Gamma \to \Gamma / (\id - \alpha)(\Gamma).
\]

Define \( S_0 : \Gamma \to H \) such that

\[
S_0(\xi, g) = \Sigma(\xi).
\]

**Lemma 4.14.** ker \( S_0 = (\id - \alpha)(\Gamma) \).

**Proof.** Since \( (\id - \alpha)(\Gamma) = (\id - \sigma) \oplus (\id - \alpha_0) \) and \( \Sigma \circ (\id - \sigma) = 0 \), we find that \( (\id - \alpha)(\Gamma) \subseteq \ker S_0 \). Let \( (\xi, g) \in \Gamma \), and assume that \( S_0(\xi, g) = \Sigma(\xi) = 0 \). By Lemma 4.6 of [Th3] there is an element \( \xi' \in \bigoplus H \) such that \( (\id - \sigma)(\xi') = \xi \). By the definition of \( \Gamma \) there is \( \epsilon > 0 \) such that \( \hat{L}(\xi) \) and \( g \) agree on \( \pi^{-1}(]-\epsilon, \epsilon[) \), and so when \( k \geq \epsilon^{-1} \) we have

\[
g = \hat{L}(\xi) \psi_k^0 \circ \pi + h^\sigma \psi_k^1 \circ \pi + h^\sigma \psi_k^1 \circ \pi + g_0
\]

for some \( h^\sigma \in \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}] \) and some \( g_0 \in G_{00} \). By the definition of \( \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}] \) there are elements \( f^\sigma \in \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}] \) such that \( h^\sigma = (\id - \alpha_0)(f^\sigma) \) and by (11.7) there is an element \( g' \in G_{00} \) such that \( g_0 = (\id - \alpha_0)(g') \). Define

\[
g'' : = \hat{L}(\xi') \psi_k^0 \circ \pi + f^\sigma \psi_k^1 \circ \pi + f^\sigma \psi_k^1 \circ \pi + g' \in G.
\]

Since

\[
(\id - \alpha_0)(\hat{L}(\xi') \psi_k^0 \circ \pi) = L((\id - \sigma)(\xi')) \psi_k^0 \circ \pi = \hat{L}(\xi) \psi_k^0 \circ \pi,
\]

it follows that \( g = (\id - \alpha_0)(g'') \) and hence that \( (\xi, g) = (\id - \alpha)(\xi', g'') \).

It follows from Lemma 4.14 that \( S_0 \) induces an isomorphism

\[
S : K_0(C) = \Gamma / (\id - \alpha)(\Gamma) \to H
\]

such that \( S \circ q = S_0 \).

**Lemma 4.15.** \( S(K_0(C)^+) = H^+ \).

**Proof.** Let \( h \in H^+ \setminus \{0\} \). There is then a \( k \) so big that \( L(\theta(h))(x) > 0 \) for all \( x \in \pi^{-1}\left(\left[ \frac{-1}{k}, \frac{1}{k} \right] \right) \). Define

\[
g := L(\theta(h)) \psi_k^0 \circ \pi + \psi_k^1 \circ \pi + \psi_k^1 \circ \pi \in G_k.
\]

Then \( (h^{00}, g) \in \Gamma^+, \ q((h^{00}, g)) \in K_0(C)^+, \) and \( S(q((h^{00}, g))) = h \). Hence, \( S(K_0(C)^+) \supseteq H^+ \). Consider an element \( x \in K_0(C)^+ \setminus \{0\} \) and write \( x = q(\xi, g) \) for some \( (\xi, g) \in \Gamma \). Let \( \omega \in \pi^{-1}(0) \). Since \( \omega_0 \circ \alpha = \omega_0 \), there is a \( \gamma \)-invariant trace \( \tau_\omega \) on \( B \) such that \( \tau_\omega \circ \gamma = \omega_0 \); see Lemma 3.5 in [Th3]. Denote by \( P : C \to B \) the canonical conditional expectation and note that \( \tau_\omega \circ P \) is a trace on \( C \). Since \( x \in K_0(C)^+ \setminus \{0\} \) and \( C \) is simple it follows that

\[
(\tau_\omega \circ P)_\omega(x) > 0.
\]

Since \( (\tau_\omega \circ P)_\omega(x) = \Sigma(\xi)(\omega) \), and \( \omega \in \pi^{-1}(0) \) was arbitrary, it follows that \( S(x) = \Sigma(\xi) \in H^+ \setminus \{0\} \). Hence, \( S(K_0(C)^+) \subseteq H^+ \).
Lemma 4.16. $K_1(C) = 0$.

Proof. To establish this from the Pimsner-Voiculescu exact sequence, we must show that $\text{id} - \alpha$ is injective. Let $(\xi, g) \in \Gamma$ and assume that $\alpha(\xi, g) = (\xi, g)$. Then $\sigma(\xi) = \xi$, implying that $\xi = 0$ and hence that $g|_{\pi^{-1}(0)} = 0$. Since $(1 - e^{-\pi(x)})g(x) = 0$ for all $x \in S$, it follows that $g = 0$. □

Let $e \in B$ be a projection such that $[e] = \nu$ in $K_0(B) = \Gamma$. Since $eCe$ is stably isomorphic to $C$ by [31], it follows that $(K_0(eCe)), K_0(eCe)^+ ) = (K_0(C), K_0(C)^+)$.  

4.5. Completing the proof of Theorem 3.1 via classification theory. Let $(S, \pi)$ and $A$ be as in Theorem 3.1. With $H = K_0(A)$ and the assumed identification of the tracial state space $T(A)$ of $A$ with $\pi^{-1}(0)$ we get the homomorphism $\theta : H \to \text{Aff} \pi^{-1}(0)$ from the canonical map $K_0(A) \to \text{Aff}T(A)$. It follows from Theorem 4.11 in [GH] that $\theta(\Gamma)$ is dense in $\text{Aff} \pi^{-1}(0)$ and that

$$K_0(A)^+ = \{ h \in K_0(A) : \theta(h)(x) > 0 \ \forall x \in \pi^{-1}(0) \} \cup \{ 0 \}.$$ 

We can therefore apply the preceding with $H = K_0(A)$, $H^+ = K_0(A)^+$ and $u = [1]$.

Let $\tau$ be a trace state on $eCe$. Then $\tau_* S^{-1} : H \to \mathbb{R}$ is a positive homomorphism such that $\tau_* S^{-1}(u) = \tau_* (q(v)) = \tau(e) = 1$, and there is therefore a unique trace state $\tau'$ on $A$ such that $\tau'_* = \tau_* S^{-1}$ on $K_0(A) = H$.

Lemma 4.17. The map $\tau \to \tau'$ is an affine homeomorphism from $T(eCe)$ onto $T(A)$.

Proof. The map is clearly affine. To show that it is continuous, assume that $\{ \tau_n \}$ is a convergent sequence in $T(eCe)$ and let $\tau = \lim_{n \to \infty} \tau_n$. Then $\lim_{n \to \infty} \tau_n \circ S^{-1}(h) = \tau \circ S^{-1}(h)$ for all $h \in H$. Since $A$ is AF this implies that $\lim_{n \to \infty} \tau'_n = \tau'$ in $T(A)$. To see that the map is surjective, let $\tau \in T(A)$. Then $\tau_* : H \to \mathbb{R}$ is given by evaluation at a point $\omega \in \pi^{-1}(0)$, and $\tau_1 = \tau_\omega \circ P$ is a trace state on $eCe$ such that $\tau'_1 = \tau$. To see that the map is also injective, consider $\tau_1, \tau_2 \in T(eCe)$. If $\tau'_1 = \tau'_2$, it follows that $\tau_1 = \tau_2$. Since $\tau_1 \circ \iota_* = \tau_2 \circ \iota_*$ and $B$ is AF it follows that $\tau_1|_B = \tau_2|_B$. Thanks to (A) from Additional properties 4.13 this implies that $\tau_1 = \tau_2$. □

Lemma 4.18. $eCe$ is $*$-isomorphic to $A$.

Proof. Since $A$ is AF the $K_1$ group of $A$ is trivial, and by Lemma 4.16 the same is true for $eCe$ since $eCe$ is stably isomorphic to $C$. The affine homeomorphism $\tau \to \tau'$ of Lemma 4.17 is compatible with the isomorphism of ordered groups $S : K_0(eCe) \to K_0(A)$ from Lemma
in the sense that \( \tau' \circ S = \tau \), resulting in an isomorphism from the Elliott invariant of \( eCe \) onto that of \( A \). Both algebras, \( A \) and \( eCe \), are separable, simple, unital, nuclear and in the UCT class. It is well known that all infinite-dimensional unital simple AF algebras are approximately divisible and hence \( Z \)-absorbing by Theorem 2.3 of [TW]; in particular, \( A \) is \( Z \)-absorbing. Since \( C \) is \( Z \)-absorbing thanks to (B) in Additional properties \[113\], it follows from Corollary 3.2 of [TW] that \( eCe \) is \( Z \)-absorbing. Therefore \( eCe \) is isomorphic to \( A \) by Corollary D of [CETWW], which in turn is based on [GLN1], [GLN2], [EGLN] and [TWW]. (In the case where \( A \) is a UHF algebra there is an alternative route through the literature to the same effect. See Remark 4.12 in [Th3].) □

We consider the dual action on \( C = B \rtimes \mathbb{Z} \) as a \( 2\pi \)-periodic flow and we denote by \( \theta \) the restriction of this flow to \( eCe \).

**Lemma 4.19.** The KMS bundle \((S^\theta, \pi^\theta)\) of \( \theta \) is isomorphic to \((S, \pi)\).

**Proof.** Let \((\omega, \beta) \in S^\theta\). By Corollary \[1.12\] \( (\hat{\omega} | _B) \) is a positive homomorphism \( \Gamma \to \mathbb{R} \) such that \( (\hat{\omega} | _B)^*(v) = 1 \) and \( (\hat{\omega} | _B)^* \circ \alpha = e^{\beta}(\hat{\omega} | _B)^* \). By Lemma \[1.12\] there is \( \mu \in \pi^{-1}(\beta) \) such that \( (\hat{\omega} | _B)^*(\xi, g) = g(\mu) \) for all \( (\xi, g) \in \Gamma \). \( \mu \) is unique since \( G \) separates the points of \( S \) by Lemma \[4.11\]. We define \( \Phi : S^\theta \to S \) by \( \Phi(\omega, \beta) = \mu \). By combining Lemma \[1.12\] and Corollary \[4.2\] we conclude that \( \Phi \) restricts to an affine bijection from \( \pi^{-1}(\beta) \) onto \( \pi^{-1}(\beta) \) for every \( \beta \in \mathbb{R} \). It follows in particular that \( \Phi \) is surjective. If \( (\omega_1, \beta_1) \in S^\theta, i = 1, 2 \), are such that \( \Phi((\omega_1, \beta_1)) = (\omega_2, \beta_2) \), it follows that \( \beta_1 = \pi(\Phi((\omega_1, \beta_1))) = \pi(\Phi((\omega_2, \beta_2))) = \beta_2 \) and hence that \( (\omega_1, \beta_1) = (\omega_2, \beta_2) \). Thus, \( \Phi \) is a bijection. Since \( \pi \circ \Phi = \pi^\theta \), and \( \pi \) and \( \pi^\theta \) are both proper maps, it suffices to show that \( \Phi^{-1} \) is continuous. Let therefore \( \{\omega^n\} \) be a sequence in \( S \) such that \( \lim_{n \to \infty} \omega^n = \omega \) in \( S \). Set \( \beta_1 = \pi(\omega^n) \) and note that \( \lim_{n \to \infty} \beta_1 = \beta \), where \( \beta = \pi(\omega) \). It follows that \( \lim_{n \to \infty} \omega^n_\beta_1(x) = \omega_\beta(x) \) for all \( x \in \Gamma \). Let \( \tau^n \) and \( \tau \) be the traces on \( B \) determined by the conditions that \( \tau^n_\beta = \omega^n_\beta \) and \( \tau = \omega_\beta \). Then \( \Phi^{-1}(\omega^n) = (\tau^n \circ P_{eCe, \beta_1}) \) and \( \Phi^{-1}(\omega) = (\tau \circ P_{eCe, \beta}) \). It suffices therefore to show that \( \lim_{n \to \infty} \tau^n \circ P(exe) = \tau \circ P(exe) \) for all \( x \in C \). Since \( \tau^n \circ P(e) = \tau \circ P(e) = 1 \), it suffices to check for \( x \) in a dense subset of \( C \). If \( w \) is the canonical unitary in the multiplier algebra of \( C \) coming from the construction of \( C \) as a crossed product, it suffices to show that \( \lim_{n \to \infty} \tau^n \circ P(ebw^k e) = \tau \circ P(ebw^k e) \) for all \( k \in \mathbb{Z} \) and all \( b \in B \). Since \( P(ebw^k e) = 0 \) when \( k \neq 0 \) it suffices to consider the case \( k = 0 \); that is, it suffices to show that \( \lim_{n \to \infty} \tau^n(ebe) = \tau(ebe) \). By approximating \( ebe \) by a linear combination of projections from \( eBe \) it suffices to show that \( \lim_{n \to \infty} \tau^n(p) = \tau(p) \) when \( p \) is a projection in \( eBe \). This holds because

\[
\lim_{n \to \infty} \tau^n(p) = \lim_{n \to \infty} \omega^n_\beta([p]) = \omega_\beta([p]) = \tau(p).
\]

□
The proof of Theorem 3.1 is complete.

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