Abstract

We quantize a flat cosmological model in the context of $f(T)$ theory of modified gravity. First, we show that the correct study of $f(T)$ gravity should be analyzed using the formalism of Dirac’s Hamiltonian constraint systems. Then, we proceed to quantize this model using the Dirac’s quantization approach for Hamiltonian constraint systems. In this regard, first we obtain the Wheeler-DeWitt equation as the operator equation of the Hamiltonian constraint and solve it for a typical model of $f(T) = T - \Lambda$ cosmology. Then, we interpret the wavefunction of universe to describe an accelerating de Sitter universe. Finally, we study Bohm–de Broglie interpretation of the quantum model.

1 Introduction

The current problems in standard cosmology, such as dark energy, accelerated expansion of the universe, inflation paradigm, and some other related problems have led the people to introduce and develop the modified theories of gravity. There are many ways to develop modified theories of gravity. The simplest way is the modification of Einstein-Hilbert action or corresponding Lagrangian by arbitrary functions of the scalars that live on the spacetime manifold. One such modification is the well-known $f(R)$ modified theory of gravity which includes an arbitrary function of the Ricci scalar $R$\cite{1,2,3,4,5}. Another one is the “Teleparallel Equivalent of General Relativity” (TEGR), so-called $f(T)$ modified theory of gravity, which includes an arbitrary function of the torsion scalar $T$\cite{6,7,8,9,11}. The main dynamical variable of TEGR theory or $f(T)$ gravity is the tetrad or vierbein field, a field of orthonormal basis in the tangent space. The Lagrangian is quadratic in the torsion of the Weitzenböck connection, which is a curvatureless connection that defines a spacetime with absolute parallelism\cite{12}. Because the action of $f(T)$ gravity includes only first derivatives of the vierbein, the dynamical equations are always second order. Thus, at field equations level, $f(T)$ gravity is different from $f(R)$ gravity which contains dynamical equations of fourth order. $f(T)$ theories of gravity have been considered in various cosmological scenarios in which they can both describe an inflationary expansion without resorting to an inflaton field and produce an accelerated expansion at late times\cite{13,14}.

Apart from the cosmological interests in the study of $f(T)$ theories of gravity, some interests have been directed toward the study of $f(T)$ gravity in the context of covariant Hamiltonian formalisms to search the degrees of freedom in this modified gravity theory \cite{15} and investigate its constraint.

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structure, using the well known Dirac formalism of Hamiltonian constraint system \[16\], \[17\]. This line of investigation is particularly relevant to the present paper, as explained below.

The problem of initial conditions is one of the most challenging questions in cosmological models. Unlike ordinary classical systems in which the dynamical equations are solved by implementing some initial conditions, in the case of cosmological models there are no initial conditions external to the universe to be implemented for solving the Einstein equations. This is mainly because there is no time parameter external to the universe. This problem may be solved by resorting to the “quantum cosmology” where the classical Einstein equations are replaced by a quantum Schrödinger-like equation, so-called Wheeler-DeWitt equation, subject to some appropriate boundary conditions \[18\]. Quantum cosmology has been studied in the context of various modified gravity theories such as \(f(R)\) gravity \[19\], massive gravity \[20\], rainbow gravity \[21\], conformally coupled scalar field gravity \[22\], Hořava gravity \[23\] and so on (see Refs.\[24\]).

To the authors knowledge, the quantum cosmology of \(f(T)\) gravity has not been received any attention, so we are motivated in this paper to study the quantum cosmology of \(f(T)\) gravity. It is well known that, in general, the study of quantum cosmology is tightly related to the Dirac formalism of Hamiltonian constraint system. This is because of “Time reparametrization invariance” property of gravitational and cosmological models which make them to be Hamiltonian constraint systems. Therefore, if we intend to study the quantum cosmology of \(f(T)\) gravity, we necessarily need to implement the Dirac formalism of Hamiltonian constraint system on this modified gravity. However, since we are merely concerned about the cosmological variables of \(f(T)\) gravity over a fixed cosmological Friedmann-Robertson-Walker (FRW) background, the implementation of Dirac formalism on this \(f(T)\) cosmological model is straightforward and we may start Dirac formalism from beginning, without engaging in the complications of \[17\].

The outline of this paper is as follows. In section 2, we study the theoretical framework of \(f(T)\) gravity and review the “common” formulation of the model in FRW cosmological background. In section 3, we comment on a subtle inconsistency of this “common” formulation, and implement the Dirac formalism of Hamiltonian constraint systems as a correct formulation of this modified gravity. In section 4, we quantize the model, according to Dirac, by applying the operator equation of Hamiltonian constraint so-called Wheeler-DeWitt equation, and solve it for a specific \(f(T)\) gravity to obtain the corresponding wavefunction of the Universe. Moreover, in this section by using de-Broglie Bohm interpretation of quantum mechanics, we write the Hamiltonian equations in presence of quantum potential. The paper ends with a brief conclusion in section 5.

2 \(f(T)\) gravity

To study of the teleparallel gravity, we use the orthonormal tetrad components \(e_A(x^\mu)\), where an index \(A\) runs over 0, 1, 2, 3 to the tangent space at each point \(x^\mu\) of the manifold. Thus, the relation of the metric \(g_{\mu\nu}\) with tetrad components is given by

\[
g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu, \tag{1}\]

where \(\mu\) and \(\nu\) are coordinate indices on the manifold and also run over 0, 1, 2, 3, and \(e^\mu_A\) forms the tangent vector on the tangent space over which are related to the metric \(\eta_{AB}\).

In the ordinary general relativity we use the torsionless Levi-Civita connection but in the teleparallelism we use the curvatureless Weitzenböck connection \[12\], whose non-null torsion \(T^\rho_{\mu\nu}\) and contorsion \(K^{\mu\nu}_\rho\) are given by

\[
T^\rho_{\mu\nu} \equiv e^\rho_A \left( \partial_{\mu} e^A_\nu - \partial_{\nu} e^A_\mu \right), \tag{2}\]

\[
K^{\mu\nu}_\rho \equiv -\frac{1}{2} \left( T^\rho_{\mu\nu} - T^\nu_{\rho\mu} - T^\mu_{\rho\nu} \right), \tag{3}\]
respectively. Here we can define the torsion scalar $T$ as follows

$$T \equiv S_{\rho}^{\mu \nu} T_{\rho \mu \nu}$$

(4)

where

$$S_{\rho}^{\mu \nu} \equiv \frac{1}{2} \left(K_{\rho}^{\mu \nu} + \delta_{\rho}^{\mu} T_{\alpha \nu}^{\alpha} - \delta_{\rho}^{\nu} T_{\alpha \mu}^{\alpha} \right).$$

(5)

Instead of the Ricci scalar $R$ for the Lagrangian density in general relativity, to define the teleparallel Lagrangian density we use the torsion scalar $T$.

Thus, the modified teleparallel $f(T)$ gravity is given by

$$I = \int d^4x|e|f(T),$$

(6)

where $|e| = \det(e_{\mu}^{A}) = \sqrt{-g}$ and we have put the units as $c = 16\pi G = 1$. Note that in the action (6), we have omitted any matter contribution in the action. Varying of the action (6) with respect to the tetrad fields $e_{\mu}^{A}$, on can obtain the field equation as

$$\frac{1}{e} \partial_{\mu} \left(e S_{A}^{\mu \nu} f_{T} - e_{A}^{\lambda} T_{\mu \lambda}^{\rho} S_{\rho}^{\mu \nu} f_{T} + S_{A}^{\mu \nu} \partial_{\mu} (T) f_{TT} + \frac{1}{4} e_{A}^{\nu} f = 0, $$

(7)

where $f_{T} = \partial f(T)/\partial T$, $f_{TT} = \partial^2 f(T)/\partial T^2$.

To study of the cosmological scenario, we must take the four-dimensional flat Friedmann-Robertson-Walker (FRW) space-time metric as,

$$ds^2 = -N^2dt^2 + a(t)^2(dx^2 + dy^2 + dz^2),$$

(8)

where $N$ is the lapse function. In this space-time, $g_{\mu \nu} = \text{diag}(-N^2, a^2, a^2, a^2)$ and the tetrad components $e_{\mu}^{A} = (N, a, a, a)$ yield the exact value of torsion scalar

$$T = -6\frac{H^2}{N^2},$$

(9)

where $H = \dot{a}/a$ is the Hubble parameter and the dot denotes for the time derivative.

By choosing $N = 1$ in the flat FRW background, from Eq. (7) the modified Friedmann equations are given by

$$12f_{T}H^2 + f = 0,$$

(10)

$$\dot{H} = \frac{1}{4T f_{TT} + 2f_{T}} \left(-T f_{T} + \frac{f}{2} \right).$$

(11)

The first equation is the “energy constraint” and the second equation is the “field equation” for $H$. Note that the energy constraint should be imposed on the solutions just after (not before) the field equation is solved for a typical $f(T)$. It is known that in $f(T)$ gravity the dynamical equations are always second order. Thus, $f(T)$ gravity is different form the metric $f(R)$ gravity where the gravitational field equation is fourth-order in derivatives. It seems that the theoretical aspects of $f(T)$ gravity are more interesting than $f(R)$ gravity.

2.1 Lagrangian formalism

To consider the $f(T)$ gravity in the FRW background, we can define a canonical point-like Lagrangian $\mathcal{L} = \mathcal{L}(a, \dot{a}, T, \dot{T})$, where $Q = \{a, T\}$ is the configuration space and $\mathcal{T}Q = \{a, \dot{a}, T, \dot{T}\}$ is the related tangent bundle on which $\mathcal{L}$ is defined. However, since we have the equation (9) which relates the variable $T$ to the variable $a$, one can use the method of the Lagrange multipliers to set $T$ as a
constraint of the dynamics. By choosing the suitable Lagrange multiplier and integrating by parts, the Lagrangian $\mathcal{L}$ becomes canonical. In this model, we have

$$I = 2\pi^2 \int dt \, Na^3 \left\{ f(T) - \lambda \left[ T + 6 \left( \frac{\dot{a}^2}{a^2} \right) \right] \right\},$$

where $N$ is the lapse function which together with the Lagrange multiplier $\lambda$, the torsion scalar $T$ and the scale factor $a$ construct the configuration space as $\{a, T, \lambda, N\}$. The common approach for obtaining the dynamical equations of $f(T)$ gravity is as follows. Since the Lapse function is an arbitrary function it is usually fixed to be $N = 1$. Variation with respect to $T$ gives $\lambda = f_T$ which can be put in the action to yield

$$I = 2\pi^2 \int dt \, a^3 \left\{ f - f_T \left[ T + 6 \left( \frac{\dot{a}^2}{a^2} \right) \right] \right\},$$

which is now reduced to the configuration space $\{a, T\}$. Integrating by parts, gives the point-like FRW Lagrangian

$$\mathcal{L} = a^3 (f - f_T T) - 6 f_T a \dot{a}^2,$$

which is a canonical function of two coupled fields, $T$ and $a$, both depending on time $t$. The momenta conjugate to variables $a$ and $T$ are

$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{a}} = -12 f_T a \dot{a},$$

$$p_T = \frac{\partial \mathcal{L}}{\partial \dot{T}} = 0.$$

The equations of motion for $a$ and $T$ are obtained respectively as

$$a^3 f_T T \left( T + 6 \frac{\dot{a}^2}{a^2} \right) = 0,$$

$$-6 f_T H^2 - 12 f_T \frac{\ddot{a}}{a} = 3(f - f_T T) + 12 f_T T \dot{T} H.$$

From the first equation, it turns out that $T$ has no independent dynamics because it is fixed by the dynamics of $a$ through the constraint $T = -6H^2$.

### 2.2 Hamiltonian formalism

The Hamiltonian can be obtained through Legendre transformation as

$$\mathcal{H} = -\frac{p_a^2}{24 a f_T} - a^3(f - f_T T).$$

The Hamilton equations are given by

$$\dot{a} = \{a, \mathcal{H}\} = -\frac{p_a}{12 a f_T},$$

$$\dot{T} = \{T, \mathcal{H}\} = 0,$$

$$\dot{p}_a = \{p_a, \mathcal{H}\} = \frac{p_a^2}{24 a^2 f_T} + 3 a^2 (f - T f_T),$$

$$\dot{p}_T = \{p_T, \mathcal{H}\} = f_T T \left( \frac{p_a^2}{24 a f_T^2} + a^3 T \right).$$
3 \( f(T) \) gravity as a Hamiltonian constraint system

In this section, we show that there is a subtle inconsistency between Lagrangian and Hamiltonian formalisms. The inconsistency has its origin in the fact that the Lagrangian formalism corresponding to the action (12) lacks the independent dynamics for the variable \( T \) because of the constraint \( T = -6H^2 \), whereas the Hamiltonian formalism corresponding to the action (12) determines the independent dynamics \( \dot{T} \) through Eq. (21). This inconsistency between Lagrangian and Hamiltonian formalisms can be resolved by using the Dirac’s formalism of Hamiltonian constraint systems [16].

Our starting point is the action (12) with the Lagrangian

\[
\mathcal{L} = Na^3 \left\{ f(T) - \lambda \left[ T + 6 \left( \frac{\dot{a}^2}{N^2a^2} \right) \right] \right\},
\]

where the configuration space is reconsidered as \( \{a, T, \lambda, N\} \), with \( T, \lambda \) and \( N \) being unfixed as well as \( a \). The conjugate momenta are obtained as

\[
p_a = \frac{\partial \mathcal{L}}{\partial \dot{a}} = -\frac{12\lambda a\dot{a}}{N},
\]

\[
p_T = \frac{\partial \mathcal{L}}{\partial \dot{T}} = 0,
\]

\[
p_\lambda = \frac{\partial \mathcal{L}}{\partial \dot{\lambda}} = 0,
\]

\[
p_N = \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0.
\]

The Hamiltonian is constructed, using Legendre transformation, as

\[
H_0 = p_a \dot{a} + p_T \dot{T} + p_\lambda \dot{\lambda} + p_N \dot{N} - \mathcal{L} = -N \left( \frac{p_a^2}{24a \lambda} + a^3(f(T) - \lambda T) \right),
\]

which leads to the Hamilton equations given by

\[
\dot{p}_a = \{p_a, H_0\} = \frac{p_a^2}{24a \lambda} + 3a^2(f - T f_T),
\]

\[
\dot{p}_T = \{p_T, H_0\} = Na^3(f_T - \lambda),
\]

\[
\dot{p}_\lambda = \{p_\lambda, H_0\} = -N \left( \frac{p_a^2}{24a \lambda} + a^3T \right),
\]

\[
\dot{p}_N = \{p_N, H_0\} = \frac{p_a^2}{24a \lambda} + a^3(f(T) - \lambda T),
\]

\[
\dot{a} = \{a, H_0\} = -N \frac{p_a}{12a \lambda},
\]

\[
\dot{T} = \{T, H_0\} = 0,
\]

\[
\dot{\lambda} = \{\lambda, H_0\} = 0,
\]

\[
\dot{N} = \{N, H_0\} = 0.
\]

The last three results which determine the dynamics of \( \{T, \lambda, N\} \) are still inconsistent with the physical content of the action (12) which apparently lacks any dynamics of \( \{T, \lambda, N\} \). Hopefully, the inconsistency is removed provided that we consider the \( f(T) \) gravity as a Hamiltonian constraint system and generalize the original Hamiltonian \( H_0 \), by adding some system constraints, according to Dirac’s formalism.
In this regard, we consider the equations (16), (27) and (28) as the Primary constraints
\[ \phi_T = p_T \approx 0, \] (38)
\[ \phi_\lambda = p_\lambda \approx 0, \] (39)
\[ \phi_N = p_N \approx 0. \] (40)
The total Hamiltonian is constructed by adding these primary constraints to the original Hamiltonian
\[ H_{Tot} = H_0 + u^T \phi_T + u^\lambda \phi_\lambda + u^N \phi_N, \] (41)
where \( u^T, u^\lambda, \) and \( u^N \) are arbitrary coefficients. Using \( H_{Tot} \), the Hamilton equations yield
\[ \dot{p}_a = \{ p_a, H_{Tot} \} = \frac{p_a^2}{24a^2 f_T} + 3a^2(f - T f_T), \] \[ \quad (42) \]
\[ \dot{p}_T = \{ p_T, H_{Tot} \} = Na^3(f_T - \lambda), \] \[ \quad (43) \]
\[ \dot{p}_\lambda = \{ p_\lambda, H_{Tot} \} = -N \left( \frac{p_a^2}{24a^2 \lambda} + a^3 T \right), \] \[ \quad (44) \]
\[ \dot{p}_N = \{ p_N, H_{Tot} \} = \frac{p_a^2}{24a^3 \lambda} + a^3(f(T) - \lambda T), \] \[ \quad (45) \]
\[ \dot{a} = \{ a, H_{Tot} \} = -N \frac{p_a}{12a^3}, \] \[ \quad (46) \]
\[ \dot{T} = \{ T, H_{Tot} \} = u^T, \] \[ \quad (47) \]
\[ \dot{\lambda} = \{ \lambda, H_{Tot} \} = u^\lambda, \] \[ \quad (48) \]
\[ \dot{N} = \{ N, H_{Tot} \} = u^N. \] \[ \quad (49) \]
Now, the appearance of arbitrary coefficients \( u^T, u^\lambda, \) and \( u^N \) in the last three equations, compared to (35), (36), and (37), account for the arbitrary dynamics of \( \dot{T}, \dot{\lambda} \) and \( \dot{N} \), in complete agreement with the physical content of the action (12), and hence the above mentioned inconsistency is removed. Note however that the other dynamical equations have not been changed by adding the constraints to the original Hamiltonian. The powerful formalism of Dirac’s Hamiltonian constraint systems is now ready for the full study of \( f(T) \) gravity. The consistency conditions for the primary constraints as
\[ \dot{\phi}_T = \{ \phi_T, H_{Tot} \} \approx 0, \] \[ \quad (50) \]
\[ \dot{\phi}_\lambda = \{ \phi_\lambda, H_{Tot} \} \approx 0, \] \[ \quad (51) \]
\[ \dot{\phi}_N = \{ \phi_N, H_{Tot} \} \approx 0, \] \[ \quad (52) \]
lead to the secondary constraints as
\[ \chi_T = Na^3(f_T - \lambda) \approx 0, \] \[ \quad (53) \]
\[ \chi_\lambda = -N \left( \frac{p_a^2}{24a^2 \lambda} + a^3 T \right) \approx 0, \] \[ \quad (54) \]
\[ \chi_N = \left( \frac{p_a^2}{24a^3 \lambda} + a^3(f(T) - \lambda T) \right) \approx 0. \] \[ \quad (55) \]
Both the primary and secondary constraints can be considered as six constraints \( \phi_j \approx 0, (j = 1, \ldots, 6) \). The consistency conditions for the secondary constraints also lead to the following equations
\[ \chi_T = \{ \chi_T, H_T \} \approx 0 \implies u^m \{ \chi_T, \phi_m \} \approx - \{ \chi_T, H_0 \}, \] \[ \quad (56) \]
\[
\chi_\lambda = \{\chi_\lambda, H_T\} \approx 0 \implies u_m^a \{\chi_\lambda, \phi_m\} \approx -\{\chi_\lambda, H_0\}, \quad (57)
\]
\[
\chi_N = \{\chi_N, H_T\} \approx 0 \implies u_m^a \{\chi_N, \phi_m\} \approx -\{\chi_N, H_0\}, \quad (58)
\]
or
\[
a^2(Nu^T f_T - Nu^N (f_T - \lambda)) \approx \frac{N^2 p_a^2}{4\lambda} (f_T - \lambda), \quad (59)
\]
\[
Nu^T a^3 - Nu^N \left(\frac{p_a^2}{12\lambda^3 a} + u^N \left(\frac{p_a^2}{24\lambda^2 a} + a^3 T\right)\right) \approx \frac{N^2 p_a^2}{4\lambda} (2T - f / \lambda), \quad (60)
\]
\[
u^T a^3 (f_T - \lambda) - u^N \left(\frac{p_a^2}{24\lambda^2 a} + a^3 T\right) \approx 0, \quad (61)
\]
which can be considered as a set of inhomogeneous equations to determine the arbitrary coefficients \(u^T = U^T, u^\lambda = U^\lambda\) and \(u^N = U^N\). These solutions are not unique and one can consider the homogeneous equations
\[
V^m \{\chi_T, \phi_m\} \approx 0, \quad (62)
\]
\[
V^m \{\chi_\lambda, \phi_m\} \approx 0, \quad (63)
\]
\[
V^m \{\chi_N, \phi_m\} \approx 0, \quad (64)
\]
to find the new independent solutions \(V^m, (a = 1, ..., A)\). These solutions can be added through the arbitrary functions \(\nu^a\) to the previous ones to obtain the general solutions \(u^m = U^m + \nu^a V^m\). The determinant of coefficient matrix \(\Delta\) corresponding to the homogeneous equations is vanishing and the \(\Delta\) matrix becomes singular. It also turns out that the \textit{Rank} and \textit{Nullity} of \(\Delta\) matrix is equal to 2 and 1, respectively. Considering all these together, one finds that there is only one nontrivial solution vector \((a = 1)\) for the homogeneous equations as
\[
V^m \equiv (V^T = 0, V^\lambda = 0, V^N = \text{arbitrary}), \quad (65)
\]
which yields
\[
u^m \equiv (U^T, U^\lambda, U^N + \nu V^N). \quad (66)
\]
Up to now, we have 3 \textit{primary} and 3 \textit{secondary} constraints together with 2 determined and 1 undetermined Lagrange coefficients, respectively as \((U^T, U^\lambda)\) and \((U^N + \nu V^N)\). It is time to determine which of them are \textit{first class} and which of them are \textit{second class} constraints. From the structures of all constraints \(\phi_j \approx 0\), it turns out that:

- The \textit{primary} constraint \(\phi_N \approx 0\) has weakly vanishing Poisson brackets with all constraints \(\phi_j \approx 0\) \((j = 1, ..., 6)\). Therefore it is a \textit{first class} constraint.
- All other constraints are \textit{second class} constraints.
- The \textit{first class} constraint \(\phi_N \approx 0\) is the generator of gauge dynamics \(\delta N = e^\eta \{N, \phi_N\} = e^\eta\), where \(e^\eta\) is an arbitrary time dependent coefficient.
- Up to this stage the gauge invariant quantities due to one \textit{first class} constraint \(\phi_N \approx 0\) are \((a, p_a, \lambda, T)\) because of \(\delta a = e^\eta \{a, \phi_N\} = 0, \delta p_a = e^\eta \{p_a, \phi_N\} = 0, \delta \lambda = e^\eta \{\lambda, \phi_N\} = 0\) and \(\delta T = e^\eta \{T, \phi_N\} = 0\), respectively.
- The system of \textit{second class} constraints \(\phi_T \approx 0, \phi_\lambda \approx 0, \chi_T \approx 0\) and \(\chi_\lambda \approx 0\) can be converted to a system of \textit{first class} constraints \(\phi_T \approx 0, \phi_\lambda \approx 0\) supplemented by the gauge conditions \(\chi_T \approx 0\) and \(\chi_\lambda \approx 0\). This conversion is possible if the gauge conditions be invariant under the gauge transformation through the \textit{first class} constraints \(\phi_T \approx 0, \phi_\lambda \approx 0\). This is provided by the followings
\[
\delta \epsilon \chi_T = \epsilon^T \{\chi_T, \phi_T\} + \epsilon^\lambda \{\chi_T, \phi_\lambda\} = 0,
\]
\[
\delta \epsilon \chi_\lambda = \epsilon^T \{\chi_\lambda, \phi_T\} + \epsilon^\lambda \{\chi_\lambda, \phi_\lambda\} = 0,
\]
where use has been made of the second class properties of the constraints $\phi_T \approx 0$, $\phi_\lambda \approx 0$, $\chi_T \approx 0$ and $\chi_\lambda \approx 0$, namely their non-vanishing Poisson brackets, which provided us with the trivial solutions $e^T = e^\lambda = 0$. In principle, the conversion of second class constraints $\phi_T \approx 0$ and $\phi_\lambda \approx 0$ into first class constraints should correspond to the appearance of gauge dynamics for $T$ and $\lambda$, respectively as

$$
\delta T = U^T\{T, \phi_T\} + U^\lambda\{T, \phi_\lambda\} + vV^N\{T, \phi_N\} = U^T,
$$

$$
\delta \lambda = U^\lambda\{\lambda, \phi_\lambda\} + U^T\{\lambda, \phi_T\} + vV^N\{\lambda, \phi_N\} = U^\lambda.
$$

Note that although the coefficients $U^T$ and $U^\lambda$ are considered as determined, nevertheless, since they are fixed by the gauges $\lambda = f_T$ and $T = -6H^2$, respectively, through the inhomogeneous equations (59), (60), and (61), these coefficients can be considered as some sort of “gauge conditions” for fixing those “gauges”. These gauge fixing removes the gauge variables $\lambda$ and $T$ from the system in favor of gauge invariant variables “$a$” and “$p_a$”. Therefore, fixing these gauges in the Hamiltonian $H_0$ is equivalent to imposing all the second class constraints.

The total Hamiltonian is now obtained by using $u^m \equiv (U^T, U^\lambda, U_N + vV^N)$, and imposing the gauges $\lambda = f_T$, $T = g(a, p_a)$ on $H_0$ to obtain

$$
H_{\text{tot}} = -N\mathcal{H} + U^T\phi_T + U^\lambda\phi_\lambda + vV^N\phi_N,
$$

where

$$
\mathcal{H} = \left(\frac{p_a^2}{24a f_T} + a^3(f(T) - f_T)\right)|_{T=g(a, p_a)}.
$$

In this form, the constraints $\phi_T \approx 0$ and $\phi_\lambda \approx 0$ which were considered as second class constraints are reconsidered as equivalent first class constraints, and $U^T$ and $U^\lambda$ which were considered as determined coefficients corresponding to the second class constraints, are reconsidered as determined coefficients corresponding to the gauge fixings $\lambda = f_T$ and $T = -6H^2$, respectively. This gauge property of $U^T$ and $U^\lambda$ lets us to interpret them equally as undetermined coefficients of the first class constraints $\phi_T \approx 0$ and $\phi_\lambda \approx 0$.

Now, time independence of the first class constraint $\phi_N$ leads to

$$
\dot{\phi}_N = \{\phi_N, \bar{H}_{\text{tot}}\} = \mathcal{H} \approx 0,
$$

where $\mathcal{H} \approx 0$ is considered as a first class constraint, so called “Hamiltonian constraint”, which involves just the physical variables $(a, p_a)$. The equations of motion for the physical variables are now obtained as

$$
\dot{a} = \{a, \bar{H}_{\text{tot}}\} = -N\frac{p_a}{12a f_T},
$$

$$
\dot{p}_a = \{p_a, \bar{H}_{\text{tot}}\} = -N\left(\frac{p_a^2}{24a^2 f_T} - 3a^2(f - T f_T)\right),
$$

$$
\dot{T} = \{T, \bar{H}_{\text{tot}}\} = U^T,
$$

$$
\dot{\lambda} = \{\lambda, \bar{H}_{\text{tot}}\} = U^\lambda,
$$

$$
\dot{N} = \{N, \bar{H}_{\text{tot}}\} = vV^N.
$$

The last three equations show that $T$, $\lambda$ and $N$ have gauge dynamics, due to the gauge property of $U^T$ and $U^\lambda$ and arbitrariness of $vV^N$, in complete agreement with the essence of the first class constraints $\phi_T \approx 0$, $\phi_\lambda \approx 0$ and $\phi_N \approx 0$. Especially, gauge dynamics of the lapse function accounts for the “Time Reparametrization Invariance” of the gravitational model.
4 Quantization of the Hamiltonian constraint systems

The method of quantization is essentially involves restricting the Hilbert space in the quantum theory to ensure that constraints are obeyed by the state vectors. This is called “Dirac quantization of constraint systems”. State vectors which satisfy this property are called physical states and the sector of the original Hilbert space spanned by these physical states is called the physical state space. In our case, such property is satisfied by the operator version of the first class constraints as

\[ \hat{\phi}_T |\Psi\rangle = 0, \]  
\[ \hat{\phi}_\lambda |\Psi\rangle = 0, \]  
\[ \hat{\phi}_N |\Psi\rangle = 0, \]  
\[ \hat{H} |\Psi\rangle = 0, \]  

where \( |\Psi\rangle \) is the physical state. The first three equations, using \( p_T \to -i \frac{\partial}{\partial T}, p_\lambda \to -i \frac{\partial}{\partial \lambda}, p_N \to -i \frac{\partial}{\partial N} \), guarantee that \( |\Psi\rangle \) is independent of \( \{T, \lambda, N\} \) and so it is just a function of the scale factor \( a \).

In the context of quantum cosmology, the operator equation (78) and the physical state \( |\Psi\rangle \) are considered as the Wheeler-DeWitt equation and the wavefunction \( \Psi(a) = \langle a |\Psi\rangle \), respectively, in the “Minisuperspace”. The Wheeler-DeWitt equation takes the following form

\[ \frac{\partial^2 \Psi(a)}{\partial a^2} + \frac{q}{a} \frac{\partial \Psi(a)}{\partial a} - 12a^4 f_T \Psi(a) = 0, \]  

where use has been made of \( p_a \to -i \frac{\partial}{\partial a} \), and \( q \) is the operator ordering parameter. To continue, we need to suggest the appropriate forms of the function \( f(T) \). There are some candidates for \( f(T) \) gravity, from different points of view, especially as alternatives to other modified gravity theories like \( f(T) \) gravity. Here, for simplicity of solving the Wheeler-DeWitt equation, we confine ourselves to a simple model in agreement with the observations considering the current accelerating phase of the universe. In this regard, we take the following case with a cosmological term \( \Lambda \), as:

- \( f(T) = T - 2\Lambda \)

This \( f(T) \) gravity is equivalent to the general relativity with a cosmological constant, namely \( R - 2\Lambda \). It is well known that this cosmology describes a de Sitter universe where the universe experiences an accelerating phase. This behavior can also be derived by putting \( f(T) = T - 2\Lambda \) in the equation (11), which results in

\[ 2\dot{H} - 3H^2 + \Lambda = 0, \]  

and has a solution \( H = \sqrt{\frac{\Lambda}{3}} \) expressing de Sitter expansion. One of the interesting topics in quantum cosmology is the prediction of classical limit. In this regard, we shall study the quantum cosmology of \( f(T) = T - 2\Lambda \) and try to interpret the corresponding wavefunction of the universe that can describe an accelerating classical universe.

Putting this \( f(T) \) into the equation (79) we obtain

\[ \frac{\partial^2 \Psi(a)}{\partial a^2} + \frac{q}{a} \frac{\partial \Psi(a)}{\partial a} - 24\Lambda a^4 \Psi(a) = 0. \]  

The analytic solutions of Eq. (81) can be expressed in terms of the Bessel functions \( J \) and \( Y \) as follows

\[ \Psi(a) = \left(\frac{2}{3}\Lambda\right)^{1-q} a^{1-q} \left[ c_1 J_{\frac{1-q}{2}} \left(2 \sqrt{\frac{2\Lambda}{3}} a^3\right) + c_2 Y_{\frac{1-q}{2}} \left(2 \sqrt{\frac{2\Lambda}{3}} a^3\right)\right]. \]  

According to [26], its nonsingular boundary is the line \( a = 0 \), while at the singular boundary this variable is infinite. Now, we impose the boundary condition on the above solutions such that at \( a = 0 \)
the wave function vanishes to avoid the singularity [26]. This yields $c_2 = 0$, and by choosing $c_1 = 1$ we arrive at the unique solution

$$
\Psi(a) = \left(\frac{2}{3} \Lambda \right)^{\frac{1-q}{12}} a^{\frac{1-q}{2}} J_{\frac{1-q}{6}} \left(2 \sqrt{\frac{2\Lambda}{3}} a^3\right). \tag{83}
$$

It is worth mentioning that Eq.(81) is a Schrödinger-like equation which describes the motion of a fictitious particle with zero energy under the superpotential $U(a) = -24\Lambda a^4$. In general, and for a typical superpotential $U(a)$, the minisuperspace may be divided into two regions, $U(a) > 0$ and $U(a) < 0$, which can be termed as the classically forbidden and classically allowed regions, respectively. The classically forbidden region corresponds to the exponential behavior of the wavefunction, while in the classically allowed region the wavefunction has oscillatory behavior. The division of minisuperspace into classically forbidden and classically allowed regions makes it possible that the Universe can tunnel from “nothing” to the “existence”, similar to the tunneling effect through a potential barrier in the sense of usual quantum mechanics [26].

In our model, however, there is no possibility of quantum tunneling because the superpotential is always negative and there is no a potential barrier through which the Universe can tunnel from “nothing” to “existence”. Therefore, the wave function always exhibits oscillatory behavior to mimic a classical evolution of the Universe. The appearance of cosmological constant $\Lambda$ both in the amplitude and the argument of Bessel functions $J$ in the solution (83) is of particular importance which deserves further discussion in the following.

In the figures 1 and 2, we have plotted the square of wavefunction (83) for the typical values $(q = -1, \Lambda = 1)$ and $(q = -1, \Lambda = 8)$, respectively.

![Figure 1: The square of the wave function for the quantum universe with $q = -1$ and $\Lambda = 1$.](image1)

![Figure 2: The square of the wave function for the quantum universe with $q = -1$ and $\Lambda = 8$.](image2)

The following general properties are seen in the figures:

- The wavefunction has a well-defined behavior near $a = 0$ and describes a universe, without singularity problem, emerging out of nothing without any tunneling.
For large cosmological constants, the locations of all amplitudes are shifted towards \( a = 0 \) and the frequency of oscillation is increased. This property causes the amplitude to more decrease at large scale factors.

### 4.1 Classical limit

One of the most challenging topics in quantum cosmology is the mechanisms through which the classical cosmology can be predicted by quantum cosmology. Most of the suggestions in resolving this problem use the properties of wavefunction. In this regard, we try to find the suitable interpretation of the wavefunction, using its properties, that can describe an accelerating classical universe. Considering the above mentioned second property of the wavefunction, we easily find that this wavefunction describes appropriately a classical universe which tends to be realized (from nothing) at smaller scale factors for larger values of cosmological constants. In other words, the probability of “realization from nothing” becomes larger for larger values of cosmological constants, in agreement with the results obtained for the probability of “tunneling from nothing” [27]. This property coincides with the inflationary scenario in that the universe having a large cosmological constant emerges from nothing with large probability, at small scales, and this is just the right initial condition for inflation, namely once the universe with large cosmological constant is realized from nothing at small scale, it immediately begins a de Sitter inflationary expansion. The accelerating behavior of de Sitter expansion is manifested within the “decreasing amplitude” and the “increasing frequency” of the wavefunction, in terms of the scale factor, in both figures. These behaviors mimic the accelerating motion of a zero-energy particle under a negative gravitational potential.

### 4.2 Bohm–de Broglie interpretation of the quantum model

In the context of the Bohm–de Broglie interpretation of quantum mechanics and also its application in quantum cosmology, we may use the polar form of the wave function \( \Psi(a) = \Omega(a) e^{iS(a)} \) in the corresponding wave equation to obtain the modified Hamilton-Jacobi equation as

\[
\mathcal{H} \left( q_i, p_i = \frac{\partial S}{\partial q_i} \right) + Q = 0, \quad (84)
\]

where \( p_i \) and \( Q \) are the momentum conjugate to the dynamical variables \( q_i \) and the quantum potential, respectively. In this context, for the wave equation (79) we can write

\[
-\frac{1}{24af_T} \left( \frac{\partial S}{\partial a} \right)^2 + a^3(f - Tf_T) + Q = 0, \quad (85)
\]

where the quantum potential is defined as

\[
Q = \frac{1}{24af_T\Omega} \frac{\partial^2 \Omega}{\partial a^2} + \frac{q}{24a^2f_T\Omega} \frac{\partial \Omega}{\partial a}. \quad (86)
\]

Thus, the quantum Hamiltonian is given by

\[
\mathcal{H}_Q = \mathcal{H} + Q, \quad (87)
\]

where \( \mathcal{H} \) is the gauge fixed Hamiltonian over the reduced phase space \( (a, p_a) \). The quantum equations of motion over the reduced phase space are obtained as

\[
\dot{a} = \{ a, \mathcal{H}_Q \} = -\frac{p_a}{12af_T}, \quad (88)
\]

\[
\dot{p}_a = \{ p_a, \mathcal{H}_Q \} = -\frac{p_a^2}{24a^2f_T} - 3a^2(f - Tf_T) - \frac{\partial Q}{\partial a}. \quad (89)
\]
From the above equations we obtain

\[ p_a = -12a \dot{a} f_T, \]  
\[ \dot{H} = \frac{1}{4T f_{TT}} \left( -T f_T + 2f_T + \frac{f}{2} - \frac{1}{6a^2} \frac{\partial Q}{\partial a} \right). \] (90) (91)

Putting \( f(T) = T - 2\Lambda \) in the equation (91) results in

\[ 2\dot{H} - 3H^2 + \left( \Lambda + \frac{1}{6a^2} \frac{\partial Q}{\partial a} \right) = 0, \] (92)

which shows that the quantum potential can alter the contribution of cosmological constant in the cosmic dynamics of de Sitter expansion. Also, the quantum Hamiltonian constraint \( \mathcal{H}_Q = 0 \) leads to

\[ 12f_T H^2 + f + \frac{Q}{a^3} = 0. \] (93)

Both equations (92) and (93) indicate that the contribution of quantum potential to the cosmic dynamics, in the context of Bohm–de Broglie interpretation, is vanishing at large scale factors (when the universe is considered as a classical system) and is very important at early universe (when the universe is considered as a quantum system).

5 Conclusions

We have quantized a flat cosmological model in the context of \( f(T) \) theory of modified gravity. First, we have shown that the correct study of \( f(T) \) gravity should be analyzed using the formalism of Dirac’s Hamiltonian constraint systems. Then, we have proceed to quantize this model using the Dirac’s quantization approach for Hamiltonian constraint systems. We have obtained the Wheeler-DeWitt equation, solved it for a typical model of \( f(T) = T - \Lambda \) cosmology, and interpreted the obtained wavefunction to describe an accelerating universe. Finally, we have studied Bohm–de Broglie interpretation of the quantum model.

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References

[1] S. Nojiri and S. D. Odintsov, Phys. Rev. D 68 (2003) 123512;
S. Nojiri and S. D. Odintsov, Phys. Rev. D 74 (2006) 086005;
S. Capozziello, S. Nojiri, S.D. Odintsov, A. Troisi, Phys. Lett. B 639 (2006) 135;
S. Nojiri, S. D. Odintsov, Phys. Rev. D 77 (2008) 026007;
K. Atazadeh and H.R. Sepangi, Int. J. Mod. Phys. D 16 (2007) 687.

[2] S. Capozziello, V. F. Cardone, S. Carloni, A. Troisi, Int. J. Mod. Phys. D 12 (2003) 1969;
W. Hu, I. Sawicki, Phys. Rev. D 76 (2007) 064004;
S. M. Carroll, V. Duvvuri, M. Trodden, M. S. Turner, Phys. Rev. D 70 (2004) 043528;
S. Capozziello, Int. J. Mod. Phys. D 11 (2002) 483;
K. Atazadeh, M. Farhoudi, H. R. Sepangi, Phys. Lett. B 660 (2008) 275;
A. S. Sefiedgar, K. Atazadeh, H. R. Sepangi, Phys. Rev. D 80 (2009) 064010.
[3] O. Bertolami, R. Rosenfeld, Int. J. Mod. Phys. A 23 (2008) 4817;
A. Capolupo, S. Capozziello, G. Vitiello, Int. J. Mod. Phys. A 23 (2008) 4979;
P. K. S. Dunsby, E. Elizalde, R. Goswami, S. Odintsov, D. S. Gomez, Phys. Rev. D 82 (2010) 023519.

[4] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani, S. Zerbini, Phys. Rev. D 77 (2008) 046009;
K. Bamba, Chao-Qiang Geng, Chung-Chi Lee, JCAP 1008 (2010) 021;
S. Nojiri, S. D. Odintsov, D. Saez-Gomez, Phys. Lett. B 681 (2009) 74.

[5] S. Capozziello, V. F. Cardone, A. Troisi, Phys. Rev. D 71 (2005) 043503;
J. C. de Souza, Valerio Faraoni, Class. Quant. Grav. 24 (2007) 3637;
V. Faraoni, Phys. Rev. D 74 (2006) 104017;
G. J. Olmo, Phys. Rev. Lett. 95 (2005) 261102;
G. J. Olmo, Phys. Rev. D 75 (2007) 023511;
K. Bamba, S. Nojiri, S. D. Odintsov, JCAP 0810 (2008) 045;
S. A. Appleby, R. A. Battye, A. A. Starobinsky, JCAP 1006 (2010) 005;
S. A. Appleby, R. A. Battye, Phys. Lett. B 654 (2007) 7;
S. A. Appleby, R. A. Battye, JCAP 0805 (2008) 019;
V. Faraoni, Phys. Rev. D 75 (2007) 067302.

[6] E. V. Linder, Phys. Rev. D 81 (2010) 127301.

[7] S.H. Chen, J. B. Dent, S. Dutta and E. N. Saridakis, Phys. Rev. D 83 (2011) 023508;
R.-J. Yang, Europhys. Lett. 93 (2011) 60001;
J. B. Dent, S. Dutta, E. N. Saridakis, JCAP 1101 (2011) 009;
Y. Zhang, H. Li, Y. Gong, Z.-H. Zhu, JCAP 1107 (2011) 015;
Y.-F. Cai, S.-H. Chen, J. B. Dent, S. Dutta, E. N. Saridakis, Class. Quant. Grav. 28 (2011) 2150011;
M. Sharif, S. Rani, Mod. Phys. Lett. A26 (2011) 1657;
S. Capozziello, V. F. Cardone, H. Farajollahi and A. Ravanpak, Phys. Rev. D 84 (2011) 043527;
K. Bamba and C.-Q. Geng, JCAP 1111 (2011) 008;
C.-Q. Geng, C.-C. Lee, E. N. Saridakis, Y.-P. Wu, Phys. Lett. B704 (2011) 384;
H. Wei, Phys. Lett. B 712 (2012) 430;
C.-Q. Geng, C.-C. Lee, E. N. Saridakis, JCAP 1201 (2012) 002;
Y.-P. Wu and C.-Q. Geng, Phys. Rev. D 86 (2012) 104058;
C. G. Boehmer, T. Harko and F. S. N. Lobo, Phys. Rev. D 85 (2012) 044033;
H. Farajollahi, A. Ravanpak and P. Wu, Astrophys. Space Sci. 338 (2012) 33;
M. Jamil, D. Momeni, N. S. Serikbayev and R. Myrzakulov, Astrophys. Space Sci. 339 (2012) 37;
K. Karami and A. Abdolmaleki, JCAP 1204 (2012) 007;
C. Xu, E. N. Saridakis and G. Leon, JCAP 1207 (2012) 005;
H. Dong, Y.-b. Wang and X.-h. Meng, Eur. Phys. J. C 72 (2012) 2002;
N. Tamanini and C. G. Boehmer, Phys. Rev. D 86 (2012) 044009;
K. Bamba, S. Capozziello, S. Nojiri and S. D. Odintsov, Astrophys. Space Sci. 342 (2012) 155;
A. Behboodi, S. Akhshabi and K. Nozari, Phys. Lett. B 718 (2012) 30;
D. Liu and M. J. Reboucas, Phys. Rev. D 86 (2012) 083515;
M. E. Rodrigues, M. J. S. Houndjo, D. Saez-Gomez and F. Rahaman, Phys. Rev. D 86 (2012) 104059;
S. Chattopadhyay and A. Pasqua, Astrophys. Space Sci. 344 (2013) 269;
M. Jamil, D. Momeni and R. Myrzakulov, Gen. Rel. Grav. 45 (2013) 263;
K. Bamba, J. de Haro and S. D. Odintsov, JCAP 1302 (2013) 008;
M. Jamil, D. Momeni and R. Myrzakulov, Eur. Phys. J. C 72 (2012) 2267;
J. -T. Li, C. -C. Lee and C. -Q. Geng, Eur. Phys. J. C 73 (2013) 2315;
H. M. Sadjadi, Phys. Rev. D 87, 064028 (2013);
A. Aviles, A. Bravetti, S. Capozziello and O. Luongo, Phys. Rev. D 87 (2013) 064025;
Y. C. Ong, K. Izumi, J. M. Nester and P. Chen, Phys. Rev. D 88 (2013) 024019;
J. Amoros, J. de Haro and S. D. Odintsov, Phys. Rev. D 87 (2013) 104037;
G. Otalora, Phys. Rev. D 88 (2013) 063505;
C. -Q. Geng, J. -A. Gu and C. -C. Lee, Phys. Rev. D 88 (2013) 024030;
F. Darabi, M. Mousavi and K. Atazadeh, Phys. Rev. D 91 (2015) 084023;
K. Atazadeh and F. Darabi, Eur. Phys. J. C 72 (2012) 2016;
K. Atazadeh and M. Mousavi, Eur. Phys. J. C 73 (2013) 2272;
A. Paliathanasis, S. Basilakos, E. N. Saridakis, S. Capozziello, K. Atazadeh, F. Darabi, M. Tsamparlis, Phys. Rev. D 89 (2014) 104042;
K. Atazadeh and A. Eghbali, Phys. Scr. 90 (2015) 045001.

[8] R. Ferraro, F. Fiorini, Phys. Lett. B 702 (2011) 75.

[9] P. Wu, H. W. Yu, Phys. Lett. B 693 (2010) 415;
G. R. Bengochea, Phys. Lett. B 695 (2011) 405.

[10] S. Nesseris, S. Basilakos, E. N. Saridakis and L. Perivolaropoulos, Phys. Rev. D 88 (2013) 103010.

[11] L. Iorio and E. N. Saridakis, Mon. Not. Roy. Astron. Soc. 427 (2012) 1555.

[12] R. Weitzenböck, Invarianten Theorie, (Nordhoff, Groningen, 1923).

[13] R. Ferraro and F. Fiorini, Phys. Rev. D 75 (2007) 084031;
R. Ferraro, F. Fiorini, Phys. Rev. D78 (2008) 124019.

[14] G. R. Bengochea and R. Ferraro, Phys. Rev. D 79 (2009) 124019.

[15] R. Ferraro, M. J. Guzmán, Phys. Rev. D 97 (2018) 104028.

[16] P. A. M. Dirac, “Lectures on quantum mechanics”, Yeshiva University (Academic press, New York, 1967).

[17] J. L. Anderson and P. G. Bergmann, Phys. Rev. 83 (1951) 1018;
K. Sundermeyer, Lect. Notes Phys. 169 (1982) 1;
K. Sundermeyer, Symmetries in fundamental physics, (Springer, Cham, Switzerland, 2014).

[18] B. S. DeWitt, Phys. Rev. 160 (1967) 1113;
C.W. Misner, Phys. Rev. 186 (1969) 1319;
C. Kiefer, Quantum Gravity (Oxford University Press, New York, 2007).

[19] B. Vakili, Phys. Lett. B 669 (2008)206;
A. Shojai, F. Shojai, Gen. Rel. Grav. 40 (2008) 1967.

[20] B. Vakili, N. Khosravi, Phys. Rev. D 85 (2012) 083529;
F. Darabi, M. Mousavi, Phys. Lett. B 761 (2016) 269.

[21] B. Majumder, Int. J. Mod. Phys. D 22 (2013) 1342021.

[22] P. Pedram, Phys. Lett. B 671 (2009) 1.

[23] B. Vakili, V. Kord, Gen. Rel. Grav. 45 (2013) 1313;
H. Ardehali, P. Pedram and B. Vakili, Acta Phys. Pol. B 48 (2017) 827.
[24] F. Darabi, W. N. Sajko and P. S. Wesson, Class. Quant. Grav. 17 (2000) 4357;
F. Darabi, A. Rastkar, Gen. Rel. Grav. 38 (2006) 1355;
F. Darabi, Int. J. Theor. Phys. 48 (2009) 961;
B. Vakili, Phys. Rev. D 83 (2011) 103505;
S. S. Gousheh, H. R. Sepangi, Phys. Lett. A 272 (2000) 304;
B. Vakili, S. Jalalzadeh, H. R. Sepangi, JCAP 0505 (2005) 006;
B. Vakili, H. R. Sepangi, JCAP 0509 (2005) 008.

[25] G. R. Bengochea and R. Ferraro, Phys. Rev. D 79 (2009) 124019.

[26] A. Vilenkin, Phys. Rev. D 37 (1988) 888;
A. Vilenkin, Phys. Rev. D 33 (1986) 3560.

[27] A. Vilenkin, Annals of the New York Academy of Sciences, (1993) 271;
A. Vilenkin, AIP Conference Proceedings 478 (1999) 23;
M. A. Jafarizadeh, F. Darabi, A. Rezaei-Aghdam, A. R. Rastegar, Phys. Rev. D60 (1999) 063514.