FORMALITY OF CERTAIN CW COMPLEXES

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ABSTRACT. Let $X$ be a simply connected path connected topological space which is formal in the sense of rational homotopy theory. Let $Y = X \cup_{\alpha} \mathbb{D}^n$ where $\alpha : S^{n-1} \to X$ is a non-torsion element. Then we obtain a condition on $\alpha$ for the formality of $Y$. We give several illustrative examples concerning the formality of a finite CW complex having only even dimensional cells.

This is the corrected version of the earlier version which contained a serious error in Theorem 1.4. This theorem, which now Theorem 1.1 of this version, has now been corrected. The proofs of Theorems 1.1, 1.2, and 1.3 of the first version are not valid as they used the erroneous result. In fact, we provide here a counterexample to the assertion of Theorem 1.1. (See Example 3.1 below.) We do not know if the statement of Theorem 1.2, which asserted the formality of Schubert varieties in a generalized flag variety $G/B$, is valid. Theorem 1.3 is correct as stated as it had been proved previously by Panov and Ray using entirely different techniques.

1. Introduction

Any path connected topological space $X$ has a functorial differential graded commutative algebra (dgca) $A_{PL}(X)$ over $\mathbb{Q}$, a minimal model $(\mathcal{M}_X, d)$ (which is a dgca) and a dgc algebra morphism $\rho_X : \mathcal{M}_X \to A_{PL}(X)$ such that $\rho_X$ induces isomorphism in cohomology. The minimal model is unique up to isomorphism. The space $X$ is called formal if there exists a dgca morphism $(\mathcal{M}_X, d) \to (H^*(X; \mathbb{Q}), 0)$ which induces isomorphism in cohomology. If $X$ is a simply-connected space, its rational homotopy type is determined by $\mathcal{M}_X$. In case $X$ is formal, $\mathcal{M}_X$ is determined by $H^*(X; \mathbb{Q})$ and so the rational homotopy type of $X$ is a ‘formal consequence of its cohomology algebra.’

Let $X$ be a simply-connected formal CW complex and let $\alpha : S^{n-1} \to X$ represent an element in the kernel of the Hurewicz homomorphism $\eta : \pi_{n-1}(X) \to H_{n-1}(X; \mathbb{Q})$. Let $Y = X \cup_{\alpha} e^n$. We have the inclusion map $j : Y \hookrightarrow (Y, X)$ and the characteristic map $(D^n, S^{n-1}) \to (Y, X)$. Then $j^* : H^n(Y, X; \mathbb{Z}) \to H^n(Y; \mathbb{Z})$ maps the positive generator $\tilde{u} \in H^n(Y, X; \mathbb{Z}) \cong H^n(D, S^{n-1}) \cong \mathbb{Z}$ to a non-zero element $u$ in $H^n(Y; \mathbb{Z})$. If $\alpha$ represents a torsion element in $\pi_{n-1}(X)$, then $u$ is a non-zero indecomposable element in $H^n(Y; \mathbb{Q})$. In this case, denoting the rationalization of $X$ by $X_0$, we see that $X_0 \cup_{\alpha} e^n$ is homotopically equivalent to $X_0 \vee S^n$. It follows that $Y$ is rationally equivalent to $X \vee S^n$ which is a formal space.

Our main result is the following. Recall that a minimal model of a simply-connected space is isomorphic as a graded algebra to $\Lambda V$ where $V$ is a graded $\mathbb{Q}$-vector space.

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Theorem 1.1. Suppose that $X$ is a simply connected CW complex and is formal. Let $\mathcal{M}_X = \Lambda(V)$ and suppose that $V = \oplus_{k \geq 0} V_k$ is a standard lower gradation of $V$. Let $Y = X \cup_\alpha e^n$. Suppose that $\eta([\alpha]) = 0$ so that $j^*(\bar{u}) =: u \neq 0$. (i) If $[\alpha] \in \pi_{n-1}(X)$ is a torsion element then $u$ is indecomposable and $Y$ is formal. (ii) Let $[\alpha] \neq 0$ in $\pi_{n-1}(X) \otimes \mathbb{Q}$. Suppose that $\langle v, [\alpha] \rangle = 0$ for all $v \in V_k, k \neq 1$, and that $u$ is decomposable in $H^*(Y; \mathbb{Q})$. Then $Y$ is formal. (iii) If $[\alpha] \in \pi_{n-1}(X)$ is not a torsion element and $u$ is not decomposable, then $Y$ is not formal.

Throughout this paper $H^*(X)$ denotes the singular cohomology of $X$ with $\mathbb{Q}$-coefficients. All differential graded commutative algebras will be over $\mathbb{Q}$.

2. Minimal models and formality

In this section we recall the notion of a Sullivan algebra, a model for a cell attachment, and the stepwise construction of the minimal Sullivan model for a differential graded commutative cochain algebra with zero differential. We also prove Theorem 1.1. The reader is referred to [2] for a comprehensive treatment of rational homotopy theory.

2.1. Sullivan algebra. A differential graded commutative algebra (abbreviated dgca) $(M, d)$ is called a Sullivan algebra if the following hold: (i) Freeseness: There exists a graded $\mathbb{Q}$-vector space $V = \oplus_{q \geq 1} V^q$ such that $M$ is freely generated by $V$, that is, $M = \Lambda V := S^*(V^{\text{even}}) \otimes E^*(V^{\text{odd}})$ where $V^{\text{even}} = \oplus_{q \geq 1} V^{2q}$, $V^{\text{odd}} = \oplus_{q \geq 1} V^{2q-1}$. Here $S^*(V)$ denotes the symmetric algebra on $V$ and $E^*(V)$ denotes the exterior algebra on $V$. (ii) Nilpotence: There is a well-ordering on a basis $\{v_\alpha\}$ of $V$ consisting of homogeneous elements such that for each $\alpha$, $d(v_\alpha)$ is a polynomial in the $v_\beta, \beta < \alpha$.

The nilpotence condition can be restated as follows: There is an increasing filtration $V = \cup_{k \geq 0} V(k)$, such that $d(V(k)) \subseteq \Lambda(V(k))$ and there exists a subspace $V_k \subseteq V(k)$ such that $d(V_k) \subseteq \Lambda(V(k - 1))$ and $\Lambda(V(k)) = \Lambda(V_k) \otimes \Lambda(V(k - 1))$. This filtration is referred to as the lower filtration of $M$.

A Sullivan model $(M, d)$ for a dgca $(A, d)$ is a Sullivan algebra $(M, d)$ together with a dgca morphism $f : M \to A$ which is a quasi-isomorphism. Thus, $f$ is a dgca morphism which induces an isomorphism $f^* : H^*(M, d) \to H^*(A, d)$. A Sullivan model $(M, d)$ is called a minimal model if $d(M) \subseteq M^+.M^+$, the ideal of decomposable elements. A dgc algebra $A$ with $H^0(A) = \mathbb{Q}$ has a unique minimal model up to isomorphism. Such a dgc algebra is called formal if there exists a dgca morphism $\Phi : (\mathcal{M}_A, d) \to (H^*(A), 0)$ which is a quasi-isomorphism where $\mathcal{M}_A$ denotes the minimal model of $A$.

Suppose that $X$ is a path connected topological space. Sullivan [6] constructed a natural dgc algebra $(A_{PL}(X), d)$ over $\mathbb{Q}$ called the polynomial differential forms on $X$ over $\mathbb{Q}$,
which is contravariant in $X$. Its cohomology $H^*(A_{PL}(X),d)$ is naturally isomorphic to $H^*(X;\mathbb{Q})$. See [2, §10] for details of construction of the functor $A_{PL}(\cdot)$.

A Sullivan model (resp. minimal model) for $X$ is by definition a Sullivan model (resp. minimal model) for $A_{PL}(X)$. Any path connected space $X$ has a minimal Sullivan model $\mathcal{M}_X$. See [2, Proposition 12.1]. Minimal models of $X$ are unique up to isomorphism provided $H^1(X) = 0$. If $X$ is simply connected, then $\text{Hom}(\pi_k(X),\mathbb{Q}) \cong V^k$ where $\mathcal{M}_X = \Lambda(V), V = \oplus_{k \geq 2} V^k$ denotes the minimal model of $X$. If $X$ and $Y$ are simply connected and have the same rational homotopy type, then their minimal models are isomorphic (as dgc algebras). In fact we have a bijection between the collection of all the rational homotopy types of simply connected spaces and the collection of all isomorphism classes of minimal Sullivan algebras over $\mathbb{Q}$. Assume that $X$ and $Y$ are simply connected and that their rational cohomology algebras are of finite type. Then we have an isomorphism of sets: $[X_0,Y_0] \to [\mathcal{M}_Y,\mathcal{M}_X]$ where $X_0$ denotes the rationalization of $X$, $[A,B]$ denotes the homotopy classes of dgca morphisms $A \to B$ between Sullivan algebras and $\mathcal{M}_X := \mathcal{M}_{A_{PL}(X)}$ is the minimal model of $X$. Observe that $\mathcal{M}_X$ and $\mathcal{M}_{X_0}$ are naturally isomorphic since $X \subset X_0$ is a rational homotopy equivalence. The isomorphism is obtained by sending $[f] \in [X_0,Y_0]$ to the homotopy class of any lift $\theta : \mathcal{M}_Y \to \mathcal{M}_X$ of $A_{PL}(f) : A_{PL}(Y_0) \to A_{PL}(X_0)$ so that the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
\mathcal{M}_Y & \xrightarrow{\theta} & \mathcal{M}_X \\
\rho_Y & \downarrow & \downarrow \rho_X \\
A_{PL}(Y_0) & \xrightarrow{A_{PL}(f)} & A_{PL}(X_0).
\end{array}
\]

\textit{Notations.} If $V$ is a graded vector space, then $V^{\leq k}$ (resp. $V^{<k}$) denotes the subspace consisting of elements of degree at most $k$ (resp. $k$). If $A$ is a differential graded algebra, $A^{\leq k}$ (resp. $A^{<k}$) denotes the differential graded subalgebra of $A$ generated by elements of degree at most (resp. less than) $k$.

If $A$ is a dgca, we denote by $\delta(A)$ (or simply $\delta$ if $A$ is clear from the context) the ideal of decomposable elements in $A$. By abuse of notation we write $A^n/\delta$ to mean $A^n/\delta \cap A^n \subset A/\delta$.

Two dgca algebras $(A,d)$ and $(B,d)$ are quasi-isomorphic if there is a finite sequence of dgca morphisms $f := \{f_i\}$ where $A_0 \xrightarrow{f_0} A_1 \xleftarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xleftarrow{f_{2n-1}} A_{2n}$ with $(A_0,d) = (A,d), (A_{2n},d) = (B,d)$ such that induced morphisms in cohomology are all isomorphisms. In this case we write $(A,d) \xleftarrow{f} (B,d)$ or $(A,d) \simeq (B,d)$. We denote by $f^* : H^*(A,d) \to H^*(B,d)$ the composition of isomorphisms $(f^*_n)^{-1} \circ f_0^*$.  

\section{2.2. Minimal model of $(A,0)$.} We refer the reader to [2] for construction of the minimal model $(\mathcal{M}_A,d)$ for a dgca $(A,d)$. For our purposes, we need only consider minimal model for a dgca with zero differential satisfying $A^0 = H^0(A) = \mathbb{Q}, A^1 = H^1(A) = 0$. We shall particularly use the description given in [3, §3].

\textbf{Lemma 2.1.} Let $(\mathcal{M}_A,d) = (\Lambda V,d)$ be a minimal model of a dgca $(A,0)$ with zero differential. Let $\rho_A$ (or more briefly $\rho$) denote a quasi-isomorphism $\mathcal{M}_A \to A$ inducing
identity in cohomology. Then there exists a lower gradation $V = \oplus_{k \geq 0} V_k$ such that (i) $\rho(V_k) = 0$ for all $k \geq 1$, and, (ii) $d(V_k) \subset \Lambda(V_0). \Lambda^+(V_1 \oplus \cdots \oplus V_{k-1})$ for $k \geq 2$.

Proof. The existence of a lower gradation $V_k, k \geq 1$, such that $\rho(V_k) = 0$ is well-known. Indeed (i) holds by the construction of the minimal model of $A$ given in [3, §3]. We start with such a lower gradation $V_k, k \geq 0$ and modify this to obtain a new lower gradation $V'_k$ so as to meet both our requirements. We set $V'_k = V_k \cap V^n$.

Let $\{y_\gamma\}_{\gamma \in J_{k,n}}$ be a basis for $V^k_2$. Write $dy_\gamma = u_0 + u_1$ where $u_0 \in \Lambda(V_0)^{k+1}$ and $u_1 \in \Lambda(V_0). \Lambda^+(V_1)$. Then $\rho(u_1) = 0$ using $\rho(V_1) = 0$ and the fact that $\rho$ is an algebra homomorphism. Therefore, $0 = d\rho(y_\gamma) = \rho(dy_\gamma) = \rho(u_0)$ implies that $u_0 = \sum f_i dv_i = d(\sum f_i v_i)$ where $f_i \in \Lambda(V_0), v_i \in V_1$ since $u_0 \in \Lambda(V_0)$ and $\rho$ induces isomorphism in cohomology. Now let $y'_\gamma = y_\gamma - \sum f_i v_i$. Then $dy'_\gamma = u_1 \in \Lambda(V_0). \Lambda^+(V_1)$ and $\rho(y'_\gamma) = \rho(y_\gamma) - \sum \rho(f_i) \rho(v_i) = 0 \neq \rho(V_2) = 0 = \rho(V_1)$. We define $V^k_2 \subset \Lambda(V_0 + V_1)^{k} \oplus V^k_2$ to be the space spanned by $y'_\gamma, \gamma \in J_{k,2}$. Set $V'_2 = \oplus_{k \geq 3} V^k_2$. Note that $V'_2 \cap (V_0 + V_1) = 0$, $V(2) = V_0 + V_1 + V'_2$, $\rho(V'_2) = 0$ and $d(V'_2) \subset \Lambda(V_0). \Lambda^+(V_1)$.

We now proceed by induction. Assume that $V'_j, 2 \leq j < n$, have been constructed satisfying (i) and (ii) such that $V_0 + V_1 + V'_2 + \cdots + V'_{n-1} = V(n-1)$. It is convenient to set $V'_1 := V_1$. Let $\{y_\gamma\}_{\gamma \in J_{k,n}}$ be a basis for $V^k_n$. Write $dy_\gamma = z_0 + z_1$ where $z_0 \in \Lambda(V_0)^{k}$ and $z_1 \in \Lambda(V_0). \Lambda^+(V_1 + V'_2 + \cdots + V'_{n-1})$. Then $\rho(z_1) = 0$ using $\rho(V'_j) = 0, j \geq 1$. Therefore, $0 = d\rho(y_\gamma) = \rho(dy_\gamma) = \rho(z_0)$ implies that $z_0 = d(\sum f_j x_j)$ where $f_j \in \Lambda(V_0), x_j \in \Lambda^+(V_1)$ since $z_0 \in \Lambda(V_0)$. Set $y'_\gamma = y_\gamma - \sum f_j x_j$. Then $dy'_\gamma = z_1 \in \Lambda(V_0). \Lambda^+(V_1 + \cdots + V'_{n-1})$ and $\rho(y'_\gamma) = 0$ as $\rho(y_\gamma) = 0$ and $\rho(V_j) = 0, 1 \leq j < n$. Thus $V'_n := \oplus_{k \geq 3} (\oplus_{\gamma \in J_{k,n}} \Lambda y'_\gamma)$ satisfies (i) and (ii). Furthermore $V(n) = V(n-1) + V'_n, V_n \cap V(n-1) = 0$. This completes the induction step and we see that $V_0, V_1, V'_j, j \geq 2$, yield a lower gradation for $V$ that meets our requirements. \hfill \Box

Definition 2.2. Let $\mathcal{M}_A$ be a minimal model of $(A, 0)$. We say that a lower gradation $V = \oplus_{k \geq 0} V_k$ of $\mathcal{M}_A = \Lambda(V)$ is standard if it satisfies conditions (i) and (ii) of Lemma 2.1.

2.3. A model for cell attachment. Let $X$ be a simply connected topological space. Let $Y = X \cup_{\alpha} e^n$ where $\alpha : S^{n-1} \to X$ represents an element $[\alpha] \in \pi_{n-1}(X)$. We assume that $n \geq 2$ so that $Y$ is also simply connected. We recall the following proposition which will play a crucial role in our proofs. Let $m_X : (\mathcal{M}_X, d) \to (A_{PL}(X), d)$ be a minimal Sullivan model for $X$. Suppose that $\mathcal{M}_X = \Lambda(V)$ so that $V = \oplus_{k \geq 2} V^k$. (Note that $V^1 = 0$ since $X$ is simply connected.) Recall that $V^k \cong \text{Hom}(\pi_k(X), \mathbb{Q})$; thus we have the pairing $\langle -, - \rangle : V^k \times \pi^Q_k(X) \to \mathbb{Q}$ defined by evaluation. If $n = 2$, then $[\alpha] = 0$ as $X$ is simply connected. It follows that $Y \simeq X \vee S^2$ which is formal.

Let $n \geq 3$. Let $M_\alpha = \Lambda(V_\alpha)$ be the dgca defined as follows: $V_\alpha := V \oplus \mathbb{Q} u_\alpha, \deg(u_\alpha) = n, u_\alpha^2 = 0, v = 0, v \in V$, with differential $d_\alpha$ where $d_\alpha(u_\alpha) = 0$ and $d_\alpha(v) = dv + \langle v, \alpha \rangle u_\alpha, v \in V$. 

Proposition 2.3. The dgca \((M_\alpha, d_\alpha)\) defined above is a model for \(Y = X \cup_\alpha e^n\). Moreover, one has the following diagram of dgca algebras in which the rows are exact and the vertical arrows are quasi-isomorphism:

\[
\begin{array}{cccc}
0 & \to & \mathbb{Q} & \overset{u_\alpha}{\to} & M_\alpha & \overset{\lambda}{\to} & M_X & \to 0 \\
\downarrow & & \uparrow & & \downarrow & & \downarrow & \\
0 & \to & A_{PL}(Y, X) & \overset{\Phi}{\to} & A_{PL}(Y) & \overset{\Phi}{\to} & A_{PL}(X) & \to 0 \\
\downarrow & & \uparrow & & \downarrow & & \downarrow & \\
& & A_{PL}(D^n, S^{n-1}) & & & & & \\
\end{array}
\]

where \(i : X \hookrightarrow Y\) and \(j : Y \hookrightarrow (Y, X)\) are inclusions and \(\lambda\) is induced by projection \(V_\alpha \to V\). The induced diagram

\[
\begin{array}{cccc}
0 & \to & \mathbb{Q} & \overset{\eta}{\to} & H^*(M_\alpha) & \overset{\lambda^*}{\to} & H^*(M_X) & \to 0 \\
\downarrow & & \uparrow & & \downarrow & & \downarrow & \\
0 & \to & H^*(Y, X) & \overset{i^*}{\to} & H^*(Y) & \overset{i^*}{\to} & H^*(X) & \to 0 \\
\downarrow & & \uparrow & & \downarrow & & \downarrow & \\
& & H^*(D^n, S^{n-1}) & & & & & \\
\end{array}
\]

is commutative with exact rows in which the vertical arrows are all isomorphisms.

We refer the reader to [2, Chapter 13] for a proof.

Remark 2.4. The dgca \(M_\alpha\) is not a minimal model for \(Y\) most often. Indeed it is not free except in the case \(V = 0\) and \(n\) odd, since \(u_\alpha^2 = 0\) and the relation \(u_\alpha v = 0\) holds for \(v \in V\).

2.4. Proof of Theorem 1.1. We keep the notations and set-up of §2.3. It is understood that a base point for \(X\) is chosen and fixed and it serves as the point for \(Y = X \cup_\alpha e^n\) as well; the homotopy groups are defined with respect to this choice and will be suppressed in the notation \(\pi_k(X)\), etc. Recall that \(i\) (resp. \(j\)) denotes the inclusion \(X \hookrightarrow Y\) (resp. \(Y \hookrightarrow (Y, X)\)). Also \(V\) and \(W\) are graded vector spaces so that \(M_X = \Lambda(V)\) and \(M_Y = \Lambda(W)\). One has a morphism of dgca \(\phi : M_Y \to M_X\) which is a lift of \(A_{PL}(i) : A_{PL}(Y) \to A_{PL}(X)\). The linear part \(Q(\phi) : W \to V\) of \(\phi\) is defined by the requirement that \(\phi(w) - Q(\phi(w)) \in \Lambda^{2\geq2}V\); it induces \(i^* : \text{Hom}(\pi_k(Y), \mathbb{Q}) \to \text{Hom}(\pi_k(X), \mathbb{Q})\) for all \(k\) under the isomorphisms \(V^k \cong \text{Hom}(\pi_k(X), \mathbb{Q})\) and \(W^k \cong \text{Hom}(\pi_k(Y), \mathbb{Q})\).

Recall that \(X\) is simply connected. By the relative Hurewicz theorem, we obtain that \(\eta : \pi_n(Y, X) \cong H_n(Y, X; \mathbb{Z}) \cong \mathbb{Z}\). The group \(\pi_n(Y, X) = \mathbb{Z}\) is generated by the homotopy class of the characteristic map \(\tilde{\alpha} : (D^n, S^{n-1}) \to (Y, X)\) of the cell \(e_\alpha\). The homomorphism \(\partial : \pi_n(Y, X) \to \pi_{n-1}(X)\) maps \([\tilde{\alpha}]\) to \([\alpha]\). Denoting by \(\pi^0_k\) the rational homotopy group functor \(\pi_k(-) \otimes \mathbb{Q}\), we have the commuting diagram

\[
\begin{array}{cccc}
\pi^0_n(Y) & \overset{i^*}{\to} & \pi^0_n(Y, X) & \overset{\partial}{\to} & \pi^0_{n-1}(X) \\
\eta \downarrow & & \eta \downarrow & & \eta \downarrow \\
H_n(X) & \to & H_n(Y) & \overset{j^*}{\to} & H_{n-1}(X) \\
\end{array}
\]
where $\eta$ denotes the Hurewicz homomorphism, with the middle one being an isomorphism.

Suppose that $\eta([\alpha]) \neq 0$. Then $\partial(\eta([\alpha])) = \eta([\alpha]) \neq 0$ and since $H_n(Y, X) \cong \mathbb{Q}$, we conclude that $i_* = 0$ and $i_*: H_*(X) \to H_*(Y)$ is an isomorphism since $H_{n+1}(Y, X) = 0$. Therefore $i^*: H^n(Y) \to H^n(X)$ is also an isomorphism.

Suppose that $\eta([\alpha]) = 0$. This happens, for example, when $\alpha$ is a torsion element or when $H_{n-1}(X) = 0$. (However $\eta[\alpha] = 0$ does not imply that $\alpha$ is of finite order. For example, one can choose $\alpha$ to be an element of infinite order in $\pi_{4m-1}(\mathbb{S}^{2m})$.) There exists an element $\gamma \in H_n(Y)$ such that $j_*(\gamma) = \eta([\alpha])$. Let $\tilde{u}$ denote the generator of $H^n(Y, X) = \text{Hom}(H_n(Y, X), \mathbb{Q}) \cong \mathbb{Q}$ such that $\langle \tilde{u}, \eta([\alpha]) \rangle = 1$. Then $j^*(\tilde{u}) = u$ is a non-zero element of $H^n(Y)$ and we have $\langle u, \gamma \rangle = 1$. Therefore, using the exact sequence $H^{n-1}(Y, X) \xrightarrow{j_*} H^n(Y) \xrightarrow{i_*} H^n(X, Y) \xrightarrow{j^*} H^n(Y) \xrightarrow{i^*} H^n(X) \to H^{n+1}(Y, X) = 0$, we have

$$H^n(Y) \cong \begin{cases} H^n(X) \oplus \mathbb{Q} & \text{if } \eta([\alpha]) = 0, \\ H^n(X) & \text{if } \eta([\alpha]) \neq 0, \end{cases}$$

(3)

and

$$H^{n-1}(X) \cong \begin{cases} H^{n-1}(Y) & \text{if } \eta([\alpha]) = 0, \\ H^{n-1}(Y) \oplus \mathbb{Q} \tilde{u} & \text{if } \eta([\alpha]) \neq 0. \end{cases}$$

(4)

Since $\text{Hom}(\pi_{n-1}(X), \mathbb{Q}) \cong V^{n-1}$, using the exactness of the sequence $\pi^Q_n(Y, X) \xrightarrow{\partial} \pi^Q_{n-1}(X) \xrightarrow{i_*} \pi^Q_{n-1}(Y, X) = 0$ we see that

$$V^{n-1} \cong \begin{cases} W^{n-1} \oplus \mathbb{Q} & \text{if } [\alpha] \neq 0, \\ W^{n-1} & \text{if } [\alpha] = 0, \end{cases}$$

(5)

via the restriction of $Q(\phi)$.

Summarizing the above discussion we obtain the following.

**Lemma 2.5.** (i) Suppose that $\eta[\alpha] = 0$. Then $j^*(\tilde{u}) = u \neq 0$ and $H^n(Y) \cong H^n(X) \oplus \mathbb{Q}u$, $H^k(Y) \cong H^k(X), k \neq n$. If $[\alpha] \neq 0$ in $\pi^Q_{n-1}(X)$, then $V^{n-1} \cong W^{n-1} \oplus \mathbb{Q}$.

(ii) Suppose that $\eta[\alpha] \neq 0$. Then $j^*(\tilde{u}) = 0$ and $H^k(Y) \cong H^k(X), k \neq n-1, H^{n-1}(X) \cong H^{n-1}(Y) \oplus \mathbb{Q}[\tilde{u}]$. Moreover $V^{n-1} \cong W^{n-1} \oplus \mathbb{Q}$.

We now establish Theorem 1.1.

**Proof of Theorem 1.1.** (i) If $[\alpha] \in \pi_{n-1}(X)$ is a torsion element then $Y$ is rational homotopically equivalent to $X_0 \vee S^n$. Hence $Y$ is formal.

(ii) In this case $Q(\phi): W^k \to V^k$ is an isomorphism for $k \leq n-2$ and is a monomorphism when $k = n-1$. Moreover, $Q(\phi)(W^{n-1}) = \ker([\alpha]) \subset V^{n-1}$ has codimension 1. Write $Q(\phi)(W^{n-1}) \oplus \mathbb{Q}v_\alpha = V^{n-1}$ where $v_\alpha \in V^{n-1}$ is an element such that $\langle v_\alpha, [\alpha] \rangle = 1$. (We shall presently make a more specific choice of $v_\alpha$.) Write $u = P(\overline{v}_1, \ldots, \overline{v}_r)$ with $\overline{v}_q \in H^{<n-1}(X) \cong H^{<n-1}(Y)$ where $\overline{v}_q$ are indecomposable elements. Since $\Lambda(V_0) \to H^*(X)$ is onto, we choose cocycles $v_q \in V_0 \subset \mathcal{M}_X$ so that $v_q \mapsto \overline{v}_q$. We set $w = P(v_1, \ldots, v_r) \in \mathcal{M}_X$. Since $X$ is formal we have a quasi-isomorphism $\Phi: (\mathcal{M}_X, d) \to (H^*(X), 0)$. Since $i^*(u) = 0$ we have $\Phi(P(v_1, \ldots, v_r)) = 0$ in $H^n(X)$. That is, $P(v_1, \ldots, v_r) =: w \in \ker(\Phi)$. Note that $w \in \mathcal{M}_X$ is a cocycle. Since $\Phi^*$ is a monomorphism, $w = d_X(v_\alpha)$ for some
\[ v_\alpha \in AV(1) \subset M_{X}^{n-1}. \] We claim that \( \langle v_\alpha, [\alpha] \rangle \neq 0. \) Indeed, since \( \mu : (M_\alpha, d_\alpha) \leftrightarrow A_{PL}(Y) \) is a quasi-isomorphism we have, using the commutative diagram (1) of [1], that \( \mu^*([w]) = P(\overline{v}_1, \ldots, \overline{v}_r) = u \in H^n(Y) \) is non-zero. So \( [w] \neq 0 \) in \( H^n(M_\alpha) \). If \( \langle v_\alpha, [\alpha] \rangle = 0, \) then \( d_\alpha(v_\alpha) = d_X(v_\alpha) = w, \) whence \( \mu^*([w]) = 0 \) in \( H^n(M_\alpha) \), a contradiction. Therefore \( \langle v_\alpha, [\alpha] \rangle \neq 0. \) Now this implies that \( v_\alpha \notin Q(\phi)(W) = \ker([\alpha]) \).

By hypothesis \( \langle v, [\alpha] \rangle = 0 \) \( \forall v \in V_{0}^{n-1} + (\oplus_{k \geq 2} V_{k}^{n-1}). \)

The surjective homomorphism \( i^* : H^n(Y) \to H^n(X) \) induces an isomorphism \( H^n(Y)/\mathcal{D} \to H^n(X)/\mathcal{D} \cong V_0^n. \) (Here \( \mathcal{D} \) stands for the space of decomposable elements.) Choose a linear map \( \theta^* : V_{0}^{n} \to H^n(Y) \) such that \( \Phi(v) = i^*(\theta^*(v)), v \in V_0^n \) and extend it to a linear map \( \theta : V^n \to H^n(Y) \) by setting \( \theta(v) = 0 \) for \( v \in \oplus_{k \geq 1} V_{k}^{n}. \)

Define a vector space homomorphism \( \psi : V \oplus Qu_\alpha \to H^*(Y) \) as follows:

\[
\psi(v) = \begin{cases} 
(i^* \circ \Phi(v)) & \text{if } v \in V^k, \ k \neq n, \\
\theta(\Phi(v)) & \text{if } v \in V^n, \\
-u/\langle v_\alpha, [\alpha] \rangle & \text{if } v = u_\alpha.
\end{cases}
\]

This extends to a homomorphism \( M_\alpha \to H^*(Y), \) again denoted by \( \psi, \) of the graded commutative algebra \( M_\alpha \) because the relations \( u^2 = 0, u.z = 0 \) for all \( z \in H^*(Y) \) hold. Note that \( \psi(w) = u. \)

We claim that \( \psi \) is a dgca morphism, that is, \( \psi \circ d_\alpha = 0. \) Clearly \( \psi(d_\alpha(u_\alpha)) = \psi(0) = 0. \) Let \( v \in V^k, k \neq n - 1. \) Then \( d_\alpha(v) = d_Xv \) and so \( \psi(d_\alpha(v)) = (i^*)^{-1}(\Phi(d_Xv)) = 0. \) If \( v \in V_{j}^{n-1}, j \neq 1, \) then \( d_\alpha(v) = d_Xv \) since \( v \in \ker([\alpha]). \) If \( j = 0, \) then \( d_Xv = 0 \) whence \( \psi(d_\alpha v) = 0. \) Assume that \( j > 1. \) Since \( V = \oplus_{k \geq 0} V^k \) is a standard lower gradation, we see that \( d_Xv \) is a sum of monomials in each of which there is a factor belonging to \( V_i, 1 \leq i < j, \) present by Lemma 2.1. Therefore \( \psi(d_Xv) = 0. \)

Finally, let \( v \in V_{1}^{n-1} = V_{1}^{n-1} \cap \ker([\alpha]) \oplus Qv_\alpha. \) Suppose \( v \in \ker([\alpha]). \) Then \( d_\alpha(v) = d_Xv = f(w_1, \ldots, w_s). \) Since \( \mu^* : H^*(M_\alpha) \to H^*(Y) \) is an isomorphism of algebras which agrees with \( H^*(M_X) \to H^*(X) \) in degrees less than \( n - 1, \) we see that \( 0 = [d_\alpha v] \) in \( H^*(M_\alpha) \) implies that \( [d_Xv] = 0 \) in \( H^n(Y). \) On the other hand \( \psi(d_Xv) = \psi(f(w_1, \ldots, w_s)) = f(\psi(w_1), \ldots, \psi(w_s)) \) is the image of the element \( f(w_1, \ldots, w_s) \) under \( \mu^*. \) As \( f(w_1, \ldots, w_s) = d_\alpha v, \) we conclude that \( \psi(d_\alpha v) = 0. \)

It remains to consider the case \( v = v_\alpha. \) Then \( d_\alpha(v_\alpha) = d_Xv_\alpha + \langle v_\alpha, [\alpha] \rangle u_\alpha = w + \langle v_\alpha, [\alpha] \rangle u. \) It follows that \( \psi(d_\alpha v_\alpha) = \psi(w) = u = 0. \) It is clear that \( \psi \) induces isomorphism in cohomology.

Since \( M_\alpha \simeq M_Y, \) there exists a dgca morphism \( h : M_Y \to M_\alpha \) which induces isomorphism in cohomology. Then \( \psi \circ h : M_Y \to H^*(Y) \) induces isomorphism in cohomology.

(iii) Let \( [\alpha] \neq 0 \) in \( \pi_{n-1}^Q(X). \) Assume that \( j^*(\tilde{u}) = u \) is not decomposable, and that \( Y \) is formal. We shall arrive at a contradiction. Recall that by Proposition 2.3, \( \mu : M_\alpha \leftrightarrow A_{PL}(Y) \) is a quasi-isomorphism.

Let \( \nu : M_Y \to M_\alpha \) and \( \Psi : M_Y \to H^*(Y) \) be quasi-isomorphisms so that \( \mu^* \circ \nu^* = \Psi^*. \) Let \( \lambda : M_\alpha \to M_X \) be the dgca morphism considered in Proposition 2.3. Then \( \phi = \lambda \circ \nu : \)
\[ \mathcal{M}_Y \to \mathcal{M}_X \] is a lift of \( A_{\text{PL}}(i) : A_{\text{PL}}(Y) \to A_{\text{PL}}(X) \) and induces \( i^* : H^*(Y) \to H^*(X) \). From (5), we have an isomorphism \( V^{n-1} \cong W^{n-1} \oplus \mathbb{Q} \) given by \( Q(\phi) \). Also \( V^k \cong W^k \) if \( k \leq n - 2 \).

Consider the dgca morphism \( \iota : \Lambda(W^{\leq n-1}) \to M_\alpha \). Then \( \iota^* \) is an isomorphism in dimension \( \leq n - 2 \) and the cokernel of \( \iota^* : H^n(\Lambda(W^{\leq n-1})) \to H^n(M_\alpha) \cong H^n(Y) \) is isomorphic to \( \ker(d_Y) \cap W^n \). Since \( [dv_\alpha] = u \) and since \( dv_\alpha \in \Lambda(V^{\leq n-2}) = \Lambda(W^{\leq n-2}) \), we see that \( u \) belongs to the image of \( (\iota^*)^\vee \). Since \( u \) is indecomposable we see that \( \dim(\operatorname{coker}(\iota^*)) < \dim(H^n(M_\alpha)/\mathcal{D}) = \dim(H^n(Y)/\mathcal{D}) \). Therefore \( \dim(\ker(d_Y) \cap W^n) < \dim(H^n(Y)/\mathcal{D}) \). On the other hand, since \( Y \) is formal, \( \dim(H^n(Y)/\mathcal{D}) = \dim(ker(d_Y) \cap W^n) \). Therefore we conclude that \( Y \) cannot be formal. \( \square \)

3. Examples

In this section we construct various illustrative examples.

In view of the above Theorem 1.1, we call \( \alpha : S^{n-1} \to X \) or the element \([\alpha] \in \pi_{n-1}(X) \otimes \mathbb{Q}\) special if \([\alpha]\) is in the kernel of the Hurewicz homomorphism and \( \langle v, [\alpha] \rangle = 0 \) for all \( v \in V_k \cap V^{n-1}, k \neq 1 \).

Our first example shows that merely assuming \( u \) to be decomposable is not sufficient to conclude formality of \( Y \) as claimed in Theorem 1.4 of [1].

**Example 3.1.** (1) Let \( X = X_0 \cup X_1 \) where \( X_0 = S^2 \cup S^2 \cup S^2, X_1 = \{ (x, y, z) \in S^2 \times S^2 \times S^2 \mid x \text{ or } y \text{ or } z \text{ equals } * \} \) where * denotes the base point of \( S^2 \). Then \( X_0 \) is formal, being a wedge of formal spaces. The space \( X_1 \) is formal since it is the 4-skeleton of the formal \( S^2 \times S^2 \times S^2 \) where the 2-sphere is given the cell structure with one 0-cell and one 2-cell. Therefore \( X \) is formal. One has \( H^*(X_0; \mathbb{Q}) = \mathbb{Q}[a_1, a_2, a_3]/(a_1^2, a_2^2, a_3^2, a_1a_2, a_2a_3, a_3a_1), |a_i| = 2 \) and \( H^*(X_1; \mathbb{Q}) = \mathbb{Q}[x_1, x_2, x_3]/(x_1^2, x_2^2, x_3^2, x_1x_2x_3), |x_i| = 2 \). The cohomology algebra of \( X \) is readily computed from this. The minimal model \( (\mathcal{M}_X, d) = (\Lambda(V), d) \) of \( X \) can be computed from the description of \( H^*(X; \mathbb{Q}) \) since \( X \) is formal. We obtain that \( V_0 = V^2 = H^2(X; \mathbb{Q}) \) is six dimensional, with basis \( a_1, a_2, a_3, x_1, x_2, x_3 \) all in degree 2 where \( d(a_i) = d(x_i) = 0 \). \( V^3 \subset V_1 \) has basis \( c_{ij}, v_i, w_{ij}, 1 \leq i, j \leq 3 \) where \( dc_{ij} = a_ia_j, dv_i = x_i^2, dw_{ij} = a_ix_j \). Also note that there are elements \( f_{ij} \in V^4, i \neq j \) such that \( df_{ij} = a_ic_{ij} - a_jc_{ii}, g_{ij} \in V^5, i \neq j \) such that \( dg_{ij} = a_jf_{ij} - a_if_{ji} + c_{ii}c_{jj} \) and there is an element \( z \in V_0, |z| = 5 \), with \( dz = x_1x_2x_3 \). Let \( \alpha : S^5 \to X \) be such that \( \langle g_{12}, [\alpha] \rangle = l, \langle z, [\alpha] \rangle = m \) where \( l, m \) are non-zero integers. Then \( y := mdg_{12} - lx \) vanishes on \([\alpha], d_\alpha y = mdg_{12} - lx_1x_2x_3, d_\alpha z = x_1x_2x_3 + mu_\alpha \). So, in \( H^*(M_\alpha) \cong H^*(Y) \) we obtain \( [u_\alpha] = u = (-1/m)x_1x_2x_3 \), a decomposable element. Note that \( g_{12} \in V_3 \cap V^5 \) and \( \langle -, [\alpha] \rangle : V_3 \to \mathbb{Q} \) is non-zero.

We claim that \( Y \) is not formal. Suppose that \( Y \) is formal then it is readily seen that, writing \( \mathcal{M}_Y = \Lambda W, W^j = V^j, j \leq 4, \) and \( V^5 \cong W^5 \oplus \mathbb{Q}z \) where \( W^5 \) is identified with the kernel of \([\alpha]\). In particular \( y \in W^5 \). Let \( \Psi : \mathcal{M}_Y \to H^*(Y) \) be a dgca morphism that induces isomorphism in cohomology. Then \( \Psi|W^j = \Phi|W^j, j \leq 3, \) where \( \Phi : \mathcal{M}_X \to H^*(X) \) is a suitable quasi-isomorphism. Since \( \Psi(a_i) = \Phi(a_i) = a_i \) and since \( a_i.H^4(Y; \mathbb{Q}) = 0 \) we conclude that \( Y \) cannot be formal.
0 we get \( 0 = \Psi(d_Y(y)) = \Psi(md a_{12} - lx_1x_2x_3) = \Psi(m(a_j f_{ij} - a_i f_{ji} - c_i c_{ij}) - lx_1x_2x_3) = -lx_1x_2x_3 \neq 0 \), a contradiction. Hence \( Y \) is not formal.

We give below an example of a finite CW complex with only even dimensional cells which is not formal.

**Example 3.2.** (i) Let \( X = S^2 \vee S^2 \vee S^2 \). Then \( X \) is formal. Computing the minimal model \( \mathcal{M}_X = \Lambda(V) \) of \( X \) up to degree five we have the following table.

| deg i | dim V | basis | differential |
|-------|-------|-------|--------------|
| 2     | 3     | \( a_1, a_2, a_3 \) | \( da_i = 0 \) |
| 3     | 6     | \( b_1, b_2, b_3 \) | \( db_i = a_i^2 \) |
|       |       | \( b_{12}, b_{23}, b_{13} \) | \( db_{ij} = a_i a_j \) |
| 4     | 6     | \( c_{ij}, i \neq j \) | \( dc_{ij} = b_i a_j - a_i b_{ij} \) |
| 5     | 3     | \( k_{12}, k_{23}, k_{13} \) | \( dk_{ij} = a_j c_{ij} - a_i c_{ji} + b_i b_{ij} \) |

Let \( \alpha \in \pi_5(X) \) be such that \( \langle k_{12}, \alpha \rangle = 1 \). Let \( Y = X \cup_\alpha e^6 \). Then \( Y \) is not formal by Theorem 1.1 because the class \( [u] \in H^6(Y; \mathbb{Q}) \cong \mathbb{Q} \) is indecomposable.

(ii) Consider the same space \( Y \) regarded as a subcomplex of \( \tilde{Y} = CP^3 \times CP^3 \times CP^3 \cup_\alpha e^6 \) where we regard \( X \) as the 2-skeleton of \( \tilde{X} := CP^3 \times CP^3 \times CP^3 \). Note that \( \pi_5(\tilde{X}) = 0 \) and so \( [\alpha] = 0 \). It follows that \( \tilde{Y} \) is formal. Since \( Y \) is not formal we see that a subcomplex of a formal space with only even-dimensional cell is not necessarily formal.

**Remark 3.3.** (i) By a result of Halperin and Stasheff [3, Theorem 1.5] a nilpotent finite CW complex with only odd dimensional cells in positive dimensions is formal. Such a CW complex is in fact rationally equivalent to a bouquet of odd-dimensional spheres. Halperin and Stasheff point out that this result has also been obtained independently by Baus.

(ii) Papadima [5] has obtained a criterion for the formality of cell attachments. He also considers spaces \( X \) whose cohomology algebra is generated by degree \( k \) elements and remarks that formality of such spaces can be obtained under the hypothesis that \( k \geq c \) (resp. \( c - 1 \)) where \( c \) is the rational cup-length of \( X \) (resp. when \( X \) is a Poincaré duality space).

### 3.1. Formality of certain CW complexes

As an application of Theorem 1.1 we shall prove the following theorem. (This is the corrected version of Theorem 1.1 of [1] which asserted the formality of \( X \) without the dimension restriction.)

**Theorem 3.4.** Let \( X \) be a connected finite CW complex having cells only in dimensions. If \( H^*(X) \) is generated by \( H^{2k}(X) \) and \( \dim X \leq 4k \), then \( X \) is formal.

**Proof.** The 2k-skeleton \( X^{(2k)} \) of \( X \) is bouquet of 2k-dimensional spheres and hence is formal. We may assume that \( X \) is of dimension 4k. The minimal model \( \mathcal{M} = \Lambda(V) \) of \( X^{(2k)} \) has the property that for any standard lower gradation \( V = \oplus_{j \geq 0} V_j \), we have \( V_0 = V^{2k}, V_{\geq 2} \cap V^{4k-1} = 0 \). So any element in \( \pi_{4-1}(X^{2k}) \) is special. (Note that the Hurewicz homomorphism in dimension \( 4k - 1 \) vanishes since \( H^{4k-1}(X) = 0 \).) It follows that attaching any 4k-cell to \( X^{(2k)} \) results in a formal space. Note that any sub complex of \( X \) again has the property that its rational cohomology algebra is generated by degree
2k elements. We may regard \( X = X^{(4k)} \) as obtained from successive cell-attachments \( X_1, \ldots, X_s \) where \( X_{j+1} \) is obtained from \( X_j \) (with \( X_0 := X^{(2k)} \)) by attaching a 4k-cell, \( s \) being the number of 4k-cells in \( X \). We have just shown that \( X_1 \) is formal.

Inductively assume that \( X_j \) is formal. Writing \( X_{j+1} = X_j \cup_\alpha e^{4k} \) for a suitable \( \alpha \in \pi_{4k-1}(X_j) \), we need only show that \( \alpha \) is special. We shall again write \( \mathcal{M} = \Lambda V \) for the minimal model of \( X_j \). Once again \( V_0 = V^{2k} \). We claim that \( V_2 \cap V^{4k-1} = 0 \) (and consequently \( V_j \cap V^{4k-1} = 0 \) for all \( j > 2 \)). Indeed for dimension reasons \( V_1 \cap V^p = 0 \) for all \( p < 4k - 1 \). It follows that there are no relations involving elements of \( V_0 = V^{2k} \) and \( V_1 \cap V^{4k-1} \) in dimensions less than \( 6k - 1 \). Thus any element of \( \pi_{4k-1}(X_j) \) is special. It follows by Theorem 1.1 that \( X_{j+1} \) is special. QED

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