DECONVOLUTION OF POISSONIAN IMAGES USING VARIABLE SPLITTING AND AUGMENTED LAGRANGIAN OPTIMIZATION

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ABSTRACT

Although much research has been devoted to the problem of restoring Poissonian images, namely in the fields of medical and astronomical imaging, applying the state of the art regularizers (such as those based on wavelets or total variation) to this class of images is still an open research front. This paper proposes a new image deconvolution approach for images with Poisson statistical models, with the following building blocks: (a) a standard regularization/MAP criterion, combining the Poisson log-likelihood with a regularizer (log-prior) is adopted; (b) the resulting optimization problem (which is difficult, since it involves a non-quadratic and non-separable term plus a non-smooth term) is transformed into an equivalent constrained problem, via a variable splitting procedure; (c) this constrained problem is addressed using an augmented Lagrangian framework. The effectiveness of the resulting algorithm is illustrated in comparison with current state-of-the-art methods.

1. INTRODUCTION

1.1. Poissonian Images

Image restoration is one of the earliest and most classical inverse problems in imaging, dating back to the 1960’s. Much of the work in this field has been devoted to developing regularizers (priors or image models, in a Bayesian perspective) to deal with the ill-conditioning or ill-posedness of the observation operator, and to devising efficient algorithms to solve the resulting optimization problems.

A large fraction of the work on image restoration assumes that the observation operator is linear (often the convolution with some blur point spread function) and the presence of additive Gaussian noise. For this scenario, recent work has lead to a set of state-of-the-art restoration methods, which involve non-smooth convex regularizers (e.g., total-variation, $\ell_1$ norm of frame coefficients) and efficient special-purpose algorithms (see [2], [7], [12], [19], and references therein).

The algorithms developed for the linear/Gaussian observation model cannot be directly applied to other statistical (e.g., Poisson or Gamma) observation models. The Poisson case is well studied and highly relevant in fields such as astronomical [18], biomedical [8], [11], and photographic imaging [13]. A very recent overview of deconvolution methods for Poissonian images can be found in [9], where a state-of-the-art algorithm is also introduced.

Although our approach can be applied to other regularizers, we focus here on total-variation (TV), well-known for its discontinuity preserving ability [3], [16]. The combination of TV regularization with the log-likelihood resulting from the Poissonian observations of a convolved image, leads to an objective function with a non-quadratic non-separable term (the log-likelihood) plus a non-smooth term (TV). This objective function poses the following difficulties to the current state-of-the-art algorithms: (a) the Poisson log-likelihood term doesn’t have a Lipschitz-continuous gradient, which is a necessary condition for the applicability of algorithms of the forward-backward splitting (FBS) class [7], [9]; (b) the presence of a convolution in the observation model precludes the direct application of the Douglas-Rachford splitting methods described in [6]. Moreover, if an FBS algorithm is applied (ignoring that the convergence conditions are not met), it is known to be slow, specially when the observation operator is ill-conditioned, a fact which has stimulated recent research aimed at obtaining faster methods [1], [2], [21].

In this paper, we propose a new approach to tackle the optimization problem referred to in the previous paragraph. Firstly, the original optimization problem is transformed into an equivalent constrained one, via a variable splitting procedure. Secondly, this constrained problem is addressed using an algorithm developed within the augmented Lagrangian framework, for which convergence is guaranteed. The effectiveness of the resulting algorithm is illustrated in comparison with current state-of-the-art alternatives [9], [13], [8].

2. AUGMENTED LAGRANGIAN

In this section, we briefly review the augmented Lagrangian framework, a key building block of our approach. Consider a
convex optimization problem with linear equality constraints

$$\min_{v \in \mathbb{R}^d} E(v)$$

s.t. $$A v = b,$$ (1)

where $$b \in \mathbb{R}^P$$ and $$A \in \mathbb{R}^{P \times d}$$. The so-called augmented Lagrangian function for this problem is defined as

$$L_A(v, \eta, \mu) = E(v) + \eta^T (A v - b) + \frac{\mu}{2} \| A v - b \|_2^2,$$ (2)

where $$\eta \in \mathbb{R}^P$$ is a vector of Lagrange multipliers and $$\mu \geq 0$$ is called the AL penalty parameter [15]. The AL algorithm iterates between minimizing $$L_A(v, \eta, \mu)$$ with respect to $$v$$, while keeping $$\eta$$ fixed, and updating $$\eta$$.

Algorithm AL
1. Set $$k = 0$$, choose $$\mu > 0$$, $$v_0$$, and $$\eta_0$$.
2. repeat
3. $$v_{k+1} \leftarrow \arg \min_v L_A(v, \eta_k, \mu)$$
4. $$\eta_{k+1} \leftarrow \eta_k + \mu (A v_{k+1} - b)$$
5. $$k \leftarrow k + 1$$
6. until stopping criterion is satisfied.

It is possible (in some cases recommended) to update the value of $$\mu$$ at each iteration [15]. Notice, however, that it is not necessary to take $$\mu$$ to infinity to guarantee convergence to the solution of the constrained problem (1). In this paper, we will consider only the case of fixed $$\mu$$.

After a straightforward manipulation, the terms added to $$E(v)$$ in $$L_A(v, \eta_k, \mu)$$ (see (2)) can be written as a single quadratic term, leading to the following alternative form for the AL algorithm:

Algorithm AL (version 2)
1. Set $$k = 0$$, choose $$\mu > 0$$, $$v_0$$, and $$d_0$$.
2. repeat
3. $$v_{k+1} \leftarrow \arg \min_v E(v) + \frac{\mu}{2} \| A v - d_k \|_2^2$$
4. $$d_{k+1} \leftarrow d_k - (A v_{k+1} - b)$$
5. $$k \leftarrow k + 1$$
6. until stopping criterion is satisfied.

This form of the AL algorithm makes clear its equivalence with the recently introduced Bregman iterative method [22].

3. PROBLEM FORMULATION

Let $$y = (y_1, ..., y_n) \in \mathbb{N}_0^n$$ denote an $$n$$-elements observed image or signal of counts, assumed to be a sample of a random image $$Y = (Y_1, ..., Y_n) \in \mathbb{N}_0^n$$ composed of $$n$$ independent Poisson variables

$$P[Y = y | \lambda] = \prod_{i=1}^n \frac{\lambda^{y_i} e^{-\lambda_i}}{y_i!},$$ (3)

where $$\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}_+^n$$ is the underlying mean signal, assumed to be a blurred version of an unknown $$x$$, i.e.,

$$\lambda = K x,$$ (4)

where $$K$$ is the matrix representation of the blur operator, which is herein assumed to be a convolution. When dealing with images, we adopt the usual vector notation obtained by stacking the pixels into an $$n$$-vector using, e.g., lexicographic order. Combining (3) and (4), we can write

$$\log P[Y = y | x] = \sum_{i=1}^n y_i \log ((K x)_i) - (K x)_i - \log(y_i!)$$

where $$(K x)_i$$ denotes the $$i$$-th component of $$K x$$.

Under the regularization or the Bayesian maximum a posteriori (MAP) criterion, the original image $$x$$ is inferred by solving a minimization problem with the form

$$\min_x L(x)$$

s.t. $$x \geq 0.$$ (6)

The objective $$L(x)$$ is the penalized negative log-likelihood, (7)

$$L(x) = - \log P[Y = y | x] + \tau \phi(x),$$

$$= \sum_{i=1}^n (K x)_i - y_i \log ((K x)_i) + \tau \phi(x),$$ (8)

where $$\phi : \mathbb{R}^n \to \mathbb{R}$$ is the penalty/regularizer (negative of the log-prior, from the Bayesian perspective), and $$\tau \in \mathbb{R}_+$$ is the regularization parameter. Notice that the non-negativity constraint on $$x$$ guarantees that $$\lambda = K x$$ is also non-negative, if all the entries in $$K$$ are non-negative (as is the case in most convolution kernels modeling a variety of blur mechanisms).

In this work, we adopt the TV regularizer [3], [16], i.e.,

$$\phi(x) = TV(x) = \sum_{s=1}^n \sqrt{\Delta h^2 x^2 + \Delta v^2 x^2},$$ (9)

where $$(\Delta h x)$$ and $$(\Delta v x)$$ denote the horizontal and vertical first order differences at pixel $$s \in \{1, ..., n\}$$, respectively.

Each term $$(K x)_i - y_i \log ((K x)_i)$$ of (8), corresponding to the negative log-likelihood, is convex, thus so is their sum. If the space of constant images $$\{x = \alpha (1, 1, ..., 1), \alpha \in \mathbb{R}\}$$, for which TV is zero, does not belong to the null space of $$K$$, and the counts $$(y_1, ..., y_n)$$ are all non-zero, then the objective function $$L$$ is coercive and strictly convex thus possessing a unique minimizer [27].

4. PROPOSED APPROACH

4.1. Variable Splitting

The core of our approach consists in rewriting the optimization problem defined by (5–9) as the following equivalent
constrained problem:

$$\min_{x, z, u} \sum_{i=1}^{n} (z_i - y_i \log z_i) + \tau \phi(u)$$ \hspace{1cm} (10)

s. t. \hspace{1cm} \mathbf{K} x = z \hspace{1cm} (11)
\hspace{1cm} x = u. \hspace{1cm} (12)

Notice that we have dropped the non-negativity constraint \(u\); this constraint could be applied to either \(x\), \(z\), or \(u\) (as long as all elements of \(\mathbf{K}\) are non-negative). However, as shown below, if applied to \(z\), this constraint will be automatically satisfied during the execution of the algorithm, thus can be dropped. Notice that this problem can be written compactly in the form (1), using the translation table

$$\mathbf{v} = \begin{bmatrix} x \\ z \\ u \end{bmatrix}, \hspace{1cm} b = 0, \hspace{1cm} \mathbf{A} = \begin{bmatrix} \mathbf{K} & -\mathbf{I} & 0 \\ \mathbf{I} & 0 & -\mathbf{I} \end{bmatrix}, \hspace{1cm} (13)$$

and with

$$E(\mathbf{v}) = E(x, z, u) = \sum_{i=1}^{n} (z_i - y_i \log z_i) + \tau \phi(u).$$ \hspace{1cm} (14)

### 4.2. Applying the AL Algorithm

The application of Step 3 of the AL (version 2) algorithm to the problem just described requires the solution of a joint minimization with respect to \(x\), \(z\), and \(u\), which is still a non-trivial problem. Observing that each partial minimization (e.g., with respect to \(x\), while keeping \(z\) and \(u\) fixed) is computationally treatable suggests that this joint minimization can be addressed using a non-linear block Gauss-Seidel (NLBGS) iterative scheme. Of course, this raises the question of whether such a scheme converges, and of how much computational effort (i.e., iterations) should be spent in solving this minimization in each step of the AL algorithm. Experimental evidence (e.g., (14)) suggests that good results are obtained by running just one NLBGS step in each step of the AL algorithm. In fact, it has been shown that the AL algorithm with a single NLBGS step per iteration does converge (10), (17). Remarkably, the only condition required is that the objective function be proper and convex.

Finally, applying AL (version 2), with a single NLBGS step per iteration, to the constrained problem presented in the previous subsection leads to our proposed algorithm, termed PIDAL (Poisson image deconvolution by AL). The algorithm is presented in Fig. 1

The minimization with respect to \(z\) (line 5) is given by

$$x_{k+1} = (\mathbf{K}^T \mathbf{K} + \mathbf{I})^{-1} (\mathbf{K}^T x' + x'').$$ \hspace{1cm} (15)

Step 7 is separable and has closed form: for each \(z_i\), it amounts to computing the non-negative root of the second order polynomial \(\mu z_i^2 + (1 - \mu z_i^2)z_i - y_i\), given by

$$z_{i, k+1} = \left( \mu z_i' - 1 + \left( (\mu z_i' - 1)^2 + 4 \mu y_i \right)^{1/2} ight)/(2\mu).$$ \hspace{1cm} (16)

Notice that this is always a non-negative value, thus justifying the statement made above that the constraint \(z \geq 0\) is automatically satisfied by the algorithm.

The minimization with respect to \(u\) (line 9) is, by definition, the Moreau proximity mapping \(\mathbf{P}_{\tau \phi} \mathbb{R}^n \rightarrow \mathbb{R}^n\) of the regularizer \(\tau \phi\) (7). In this paper, the adopted regularizer is the TV norm (9), thus \(u_{k+1}\) is obtained by applying TV-based denoising to \(u'\). To implement this denoising operation, we use Chambolle’s well-known algorithm (3), although other fast methods are also available (19).

Notice how the variable splitting, followed by the augmented Lagrangian approach, converted a difficult problem (5–9), involving a non-quadratic and non-separable term plus a (non-smooth) TV regularizer, into a sequence of three simpler problems: (a) quadratic problem with a linear solution (line 5); (b) a separable problem with closed-form solution (line 7); (c) a TV-based denoising problem (line 9), for which efficient algorithms exist.

### 5. EXPERIMENTS

We now report experiments where PIDAL is compared with two state-of-the-art methods (9), (13). All the experiments use synthetic data produced according to (3–4), where \(x\) is the Cameraman image and \(\mathbf{K}\) represents a uniform blur. In Experiment 1 (following (13)), the blur is \(9 \times 9\), and the
original image is scaled to a maximum value of 17600; this is a high SNR situation. In Experiment 2 (following [9]) the blur is \(7 \times 7\), and the maximum value of \(x\) belongs to \([5, 30, 100, 255]\); this represents low SNR situations.

Parameter \(\mu\) of the PIDAL algorithm affects its convergence speed, but its adaptive choice is a topic beyond the scope of this paper. In all the experiments, we use \(\mu = \tau/50\), found to be a good rule of thumb. PIDAL is initialized with \(x_0 = y, z_0 = K x_0, u_0 = x_0, d^{(1)}_0 = 0\), and \(d^{(2)}_0 = 0\).

In Experiment 1, the regularization parameter \(\tau\) was set to \(6 \times 10^{-4}\); since our goal is to propose a new algorithm, not a new deconvolution criterion, we didn’t spend time fine tuning \(\tau\) or using methods to adaptively estimate it from the data. Since the method in [13] includes a set of adjustable parameters which need to be hand tuned, the comparison remains fair. The improvement in SNR (ISNR) obtained by PIDAL was 6.96dB (average over 10 runs), better than the 6.61dB reported in [13]. This result is more remarkable if we notice that the TV regularizer is considerably simpler than the locally adaptive approximation techniques used in [13].

For Experiment 2, we downloaded the code available at www.greyc.ensicaen.fr/~fdupe/. Although the regularizer is not the same, we used the same values of \(\tau\) found in that code; if anything, this constitutes a disadvantage for PIDAL. Following [9], the accuracy of an image estimate \(\hat{x}\) is assessed by the mean absolute error

\[
MAE = \frac{\|\hat{x} - x\|_1}{n}.
\]

Table 1 shows the MAE values achieved by PIDAL and the algorithm from [9], for the several values of the maximum original image intensity, showing that PIDAL always yields lower MAE. In our experiments, each run of the algorithm from [9] takes roughly 10 times longer than PIDAL.

### 6. CONCLUDING REMARKS

We have proposed an approach to TV deconvolution of Poissonian images, by exploiting a variable splitting procedure and augmented Lagrangian optimization. In the experiments reported in the paper, the proposed algorithm exhibited state-of-the-art performance. We are currently working on extending our methods to other regularizers, such as those based on frame-based sparse representations.

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