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ON THE CLOSURE OF THE EXTENDED BICYCLIC SEMIGROUP

Fihel I.R., Gutik O.V. On the closure of the extended bicyclic semigroup, Carpathian Mathematical Publications, 3, 2 (2011), 131–157.

In the paper we study the semigroup $\mathcal{C}_Z$ which is a generalization of the bicyclic semigroup. We describe main algebraic properties of the semigroup $\mathcal{C}_Z$ and prove that every non-trivial congruence $\mathcal{C}$ on the semigroup $\mathcal{C}_Z$ is a group congruence, and moreover the quotient semigroup $\mathcal{C}_Z/\mathcal{C}$ is isomorphic to a cyclic group. Also we show that the semigroup $\mathcal{C}_Z$ as a Hausdorff semitopological semigroup admits only the discrete topology. Next we study the closure $\text{cl}_T(\mathcal{C}_Z)$ of the semigroup $\mathcal{C}_Z$ in a topological semigroup $T$. We show that the non-empty remainder of $\mathcal{C}_Z$ in a topological inverse semigroup $T$ consists of a group of units $H(1_T)$ of $T$ and a two-sided ideal $I$ of $T$ in the case when $H(1_T) \neq \emptyset$ and $I \neq \emptyset$. In the case when $T$ is a locally compact topological inverse semigroup and $I \neq \emptyset$ we prove that an ideal $I$ is topologically isomorphic to the discrete additive group of integers and describe the topology on the subsemigroup $\mathcal{C}_Z \cup I$. Also we show that if the group of units $H(1_T)$ of the semigroup $T$ is non-empty, then $H(1_T)$ is either singleton or $H(1_T)$ is topologically isomorphic to the discrete additive group of integers.

1 INTRODUCTION AND PRELIMINARIES

In this paper all topological spaces are assumed to be Hausdorff. We shall follow the terminology of [6, 7, 9, 10]. If $Y$ is a subspace of a topological space $X$ and $A \subseteq Y$, then by $\text{cl}_Y(A)$ we shall denote the topological closure of $A$ in $Y$. We denote by $\mathbb{N}$ the set of positive integers.

An algebraic semigroup $S$ is called inverse if for any element $x \in S$ there exists the unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element $x^{-1}$ is called the inverse of $x \in S$. If $S$ is an inverse semigroup, then the function $\text{inv}: S \rightarrow S$ which assigns to every element $x$ of $S$ its inverse element $x^{-1}$ is called an inversion.

A congruence $\mathcal{C}$ on a semigroup $S$ is called non-trivial if $\mathcal{C}$ is distinct from universal and identity congruence on $S$, and group if the quotient semigroup $S/\mathcal{C}$ is a group.

If $S$ is a semigroup, then we shall denote the subset of idempotents in $S$ by $E(S)$. If $S$ is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ a band (or the band of $S$). If the band $E(S)$ is a non-empty subset of $S$, then the

2000 Mathematics Subject Classification: 22A15, 20M18, 20M20, 54H15.
Key words and phrases: topological semigroup, semitopological semigroup, topological inverse semigroup, bicyclic semigroup, closure, locally compact space, ideal, group of units.

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semigroup operation on $S$ determines the following partial order $\leq$ on $E(S)$: $e \leq f$ if and only if $ef = fe = e$. This order is called the natural partial order on $E(S)$. A semilattice is a commutative semigroup of idempotents. A semilattice $E$ is called linearly ordered or a chain if its natural order is a linear order.

Let $E$ be a semilattice and $e \in E$. We denote $\downarrow e = \{ f \in E \mid f \leq e \}$ and $\uparrow e = \{ f \in E \mid e \leq f \}$.

If $S$ is a semigroup, then we shall denote by $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{H}$ the Green relations on $S$ (see [7]):

- $a \mathcal{R} b$ if and only if $aS^1 = bS^1$;
- $a \mathcal{L} b$ if and only if $S^1a = S^1b$;
- $a \mathcal{J} b$ if and only if $S^1aS^1 = S^1bS^1$;
- $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$;
- $\mathcal{H} = \mathcal{L} \setminus \mathcal{R}$.

A semigroup $S$ is called simple if $S$ does not contain proper two-sided ideals and bisimple if $S$ has only one $\mathcal{D}$-class.

A semitopological (resp. topological) semigroup is a Hausdorff topological space together with a separately (resp. jointly) continuous semigroup operation [6, 18]. An inverse topological semigroup with the continuous inversion is called a topological inverse semigroup. A topology $\tau$ on a (inverse) semigroup $S$ which turns $S$ to be a topological (inverse) semigroup is called a (inverse) semigroup topology on $S$.

An element $s$ of a topological semigroup $S$ is called topologically periodic if for every open neighbourhood $U(s)$ of $s$ in $S$ there exists a positive integer $n \geq 2$ such that $s^n \in U(s)$. Obviously, if there exists a subgroup $H(e)$ with a neutral element $e$ in $S$, then $s \in H(e)$ is topologically periodic if and only if for every open neighbourhood $U(e)$ of $e$ in $S$ there exists a positive integer $n$ such that $s^n \in U(e)$.

The bicyclic semigroup $C(p, q)$ is the semigroup with the identity $1$ generated by elements $p$ and $q$ subject only to the condition $pq = 1$. The distinct elements of $C(p, q)$ are exhibited in the following useful array:

$$
\begin{array}{ccccccc}
1 & p & p^2 & p^3 & \cdots \\
q & qp & qp^2 & qp^3 & \cdots \\
q^2 & q^2p & q^2p^2 & q^2p^3 & \cdots \\
q^3 & q^3p & q^3p^2 & q^3p^3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

The bicyclic semigroup is bisimple and every one of its congruences is either trivial or a group congruence. Moreover, every non-annihilating homomorphism $h$ of the bicyclic semigroup is either an isomorphism or the image of $C(p, q)$ under $h$ is a cyclic group (see [7, Corollary 1.32]). The bicyclic semigroup plays an important role in algebraic theory of semigroups and in the theory of topological semigroups. For example the well-known Andersen’s result [1] states that a $(0–)$simple semigroup is completely $(0–)$simple if and only if it does
not contain the bicyclic semigroup. The bicyclic semigroup admits only the discrete semigroup topology and a topological semigroup $S$ can contain the bicyclic semigroup $C(p, q)$ as a dense subsemigroup only as an open subset [8]. Also Bertman and West in [5] proved that the bicyclic semigroup as a Hausdorff semitopological semigroup admits only the discrete topology. The problem of an embedding of the bicycle semigroup into compact-like topological semigroups solved in the papers [2, 3, 4, 11, 13] and the closure of the bicycle semigroup in topological semigroups studied in [8].

Let $\mathbb{Z}$ be the additive group of integers. On the Cartesian product $C_\mathbb{Z} = \mathbb{Z} \times \mathbb{Z}$ we define the semigroup operation as follows:

\[(a, b) \cdot (c, d) = \begin{cases} (a - b + c, d), & \text{if } b < c; \\ (a, d), & \text{if } b = c; \\ (a, d + b - c), & \text{if } b > c, \end{cases}\]

for $a, b, c, d \in \mathbb{Z}$. The set $C_\mathbb{Z}$ with such defined operation is called the extended bicycle semigroup [19].

In this paper we study the semigroup $C_\mathbb{Z}$. We describe main algebraic properties of the semigroup $C_\mathbb{Z}$ and prove that every non-trivial congruence $\mathcal{E}$ on the semigroup $C_\mathbb{Z}$ is a group congruence, and moreover the quotient semigroup $C_\mathbb{Z}/\mathcal{E}$ is isomorphic to a cyclic group. Also we show that the semigroup $C_\mathbb{Z}$ as a Hausdorff semitopological semigroup admits only the discrete topology. Next we study the closure $\text{cl}_T(C_\mathbb{Z})$ of the semigroup $C_\mathbb{Z}$ in a topological semigroup $T$. We show that the non-empty remainder of $C_\mathbb{Z}$ in a topological inverse semigroup $T$ consists of a group of units $H(1_T)$ of $T$ and a two-sided ideal $I$ of $T$ in the case when $H(1_T) \neq \emptyset$ and $I \neq \emptyset$. In the case when $T$ is a locally compact topological inverse semigroup and $I \neq \emptyset$ we prove that an ideal $I$ is topologically isomorphic to the discrete additive group of integers and describe the topology on the subsemigroup $C_\mathbb{Z} \cup I$. Also we show that if the group of units $H(1_T)$ of the semigroup $T$ is non-empty, then $H(1_T)$ is either singleton or $H(1_T)$ is topologically isomorphic to the discrete additive group of integers.

2 Algebraic properties of the semigroup $C_\mathbb{Z}$

Proposition 2.1. The following statements hold:

(i) $E(C_\mathbb{Z}) = \{(a, a) \mid a \in \mathbb{Z}\}$, and $(a, a) \leq (b, b)$ in $E(C_\mathbb{Z})$ if and only if $a \geq b$ in $\mathbb{Z}$, and hence $E(C_\mathbb{Z})$ is isomorphic to the linearly ordered semilattice $(\mathbb{Z}, \max)$;

(ii) $C_\mathbb{Z}$ is an inverse semigroup, and the elements $(a, b)$ and $(b, a)$ are inverse in $C_\mathbb{Z}$;

(iii) for any idempotents $e, f \in C_\mathbb{Z}$ there exists $x \in C_\mathbb{Z}$ such that $x \cdot x^{-1} = e$ and $x^{-1} \cdot x = f$;

(iv) elements $(a, b)$ and $(c, d)$ of the semigroup $C_\mathbb{Z}$ are:

(a) $\mathcal{R}$-equivalent if and only if $a = c$;

(b) $\mathcal{L}$-equivalent if and only if $b = d$;
(c) $\mathcal{H}$-equivalent if and only if $a = c$ and $b = d$;
(d) $\mathcal{D}$-equivalent for all $a, b, c, d \in \mathbb{Z}$;
(e) $\mathcal{J}$-equivalent for all $a, b, c, d \in \mathbb{Z}$;

(v) $\mathcal{C}_{\mathbb{Z}}$ is a bisimple semigroup and hence it is simple;

(vi) if $(a, b) \cdot (c, d) = (x, y)$ in $\mathcal{C}_{\mathbb{Z}}$ then $x - y = a - b + c - d$.

(vii) every maximal subgroup of $\mathcal{C}_{\mathbb{Z}}$ is trivial.

(viii) for every integer $n$ the subsemigroup $\mathcal{C}_{\mathbb{Z}}[n] = \{(a, b) \mid a \geq n \& b \geq n\}$ of $\mathcal{C}_{\mathbb{Z}}$ is isomorphic to the bicyclic semigroup $\mathcal{C}(p, q)$, and moreover an isomorphism $h : \mathcal{C}_{\mathbb{Z}}[n] \to \mathcal{C}(p, q)$ is defined by the formula $((a, b)) h = q^{a-n}p^{b-n}$;

(ix) $\mathcal{L}_{\mathcal{C}_{\mathbb{Z}}} = \{ \mathcal{L}^a \mid a \in \mathbb{Z} \}$, where $\mathcal{L}^a = \{(x, y) \in \mathcal{C}_{\mathbb{Z}} \mid y \geq a\}$, is the family of all left ideals of the semigroup $\mathcal{C}_{\mathbb{Z}}$;

(x) $\mathcal{R}_{\mathcal{C}_{\mathbb{Z}}} = \{ \mathcal{R}^a \mid a \in \mathbb{Z} \}$, where $\mathcal{R}^a = \{(x, y) \in \mathcal{C}_{\mathbb{Z}} \mid x \geq a\}$, is the family of all right ideals of the semigroup $\mathcal{C}_{\mathbb{Z}}$.

Proof. The proofs of statements (i), (ii), (iii), (iv), (vi), (vii) and (viii) are trivial. Statement (v) follows from statement (iii) and Lemma 1.1 of [16].

Simple verifications (see: formula (1)) show that

$$(a, b)\mathcal{C}_{\mathbb{Z}} = \{(x, y) \in \mathcal{C}_{\mathbb{Z}} \mid x \geq a\} \quad \text{and} \quad \mathcal{C}_{\mathbb{Z}}(a, b) = \{(x, y) \in \mathcal{C}_{\mathbb{Z}} \mid y \geq b\}$$

for every $(a, b) \in \mathcal{C}_{\mathbb{Z}}$. This completes the proof of statements (ix) and (x).

Proposition 2.2. Every non-trivial congruence $\mathcal{C}$ on the semigroup $\mathcal{C}_{\mathbb{Z}}$ is a group congruence, and moreover the quotient semigroup $\mathcal{C}_{\mathbb{Z}}/\mathcal{C}$ is isomorphic to a cyclic group.

Proof. First we shall show that if two distinct idempotents $(a, a)$ and $(b, b)$ of $\mathcal{C}_{\mathbb{Z}}$ are $\mathcal{C}$-equivalent then the quotient semigroup $\mathcal{C}_{\mathbb{Z}}/\mathcal{C}$ is a group. Without loss of generality we can assume that $(a, a) \leq (b, b)$, i.e., $a \geq b$ in $\mathbb{Z}$. Then we have that

$$(a, b) \cdot (b, b) \cdot (b, a) = (a, a);$$

$$(a, b) \cdot (a, a) \cdot (b, a) = (a + (a - b), a + (a - b));$$

$$(a, b) \cdot (a + (a - b), a + (a - b)) \cdot (b, a) = (a + 2(a - b), a + 2(a - b));$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$(a, b) \cdot (a + j(a - b), a + j(a - b)) \cdot (b, a) = (a + (j + 1)(a - b), a + (j + 1)(a - b));$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

This implies that for every non-negative integers $i$ and $j$ we have that

$$(a + i(a - b), a + i(a - b)) \mathcal{C} (a + j(a - b), a + j(a - b)).$$
If \( b \geq k \) in \( \mathbb{Z} \) for some integer \( k \), then by Proposition 2.1(viii) we get that any two distinct idempotents of the subsemigroup \( C_n[k] \) of \( C_\mathbb{Z} \) are \( C \) -equivalent and hence Proposition 2.1(viii) and Corollary 1.32 from [7] imply that for every integer \( n \) all idempotents of the subsemigroup \( C_n[n] \) are \( C \) -equivalent. This implies that all idempotents of the subsemigroup \( C_n[n] \) are \( C \) -equivalent. Since the semigroup \( C_\mathbb{Z} \) is the semigroup that \( \langle (1) \rangle \) contains only one idempotent and hence by Lemma II.1.10 from [17] the semigroup \( C_\mathbb{Z}/C \) is a group.

Suppose that two distinct elements \((a, b)\) and \((c, d)\) of the semigroup \( C_\mathbb{Z} \) are \( C \) -equivalent. Since \( C_\mathbb{Z} \) is an inverse semigroup, Lemma III.1.1 from [17] implies that \((a, a)C(c, c)\) and \((b, b)C(d, d)\). Since \((a, b) \neq (c, d)\) we have that either \((a, a) \neq (c, c)\) or \((b, b) \neq (d, d)\), and hence by the first part of the proof we get that all idempotents of the semigroup \( C_\mathbb{Z} \) are \( C \) -equivalent.

Next we shall show that if \( C_{mg} \) be a least group congruence on the semigroup \( C_\mathbb{Z} \), then the quotient semigroup \( C_\mathbb{Z}/C_{mg} \) is isomorphic to the additive group of integers \( \mathbb{Z} \).

By Proposition 2.1(i) and Lemma III.5.2 from [17] we have that elements \((a, b)\) and \((c, d)\) are \( C_{mg} \) -equivalent in \( C_\mathbb{Z} \) if and only if there exists an integer \( n \) such that \((a, b) \cdot (n, n) = (c, d) \cdot (n, n)\). Then Proposition 2.1(i) implies that \((a, b) \cdot (g, g) = (c, d) \cdot (g, g)\) for any integer \( g \) such that \( g \geq n \) in \( \mathbb{Z} \). If \( g \geq b \) and \( g \geq d \) in \( \mathbb{Z} \), then the semigroup operation in \( C_\mathbb{Z} \) implies that \((a, b) \cdot (g, g) = (g - b + a, g)\) and \((c, d) \cdot (g, g) = (g - d + c, g)\), and since \( \mathbb{Z} \) is the additive group of integers we get that \( a - b = c - d \). Converse, suppose that \((a, b)\) and \((c, d)\) are elements of the semigroup \( C_\mathbb{Z} \) such that \( a - b = c - d \). Then for any element \( g \in \mathbb{Z} \) such that \( g \geq b \) and \( g \geq d \) in \( \mathbb{Z} \) we have that \((a, b) \cdot (g, g) = (g - b + a, g)\) and \((c, d) \cdot (g, g) = (g - d + c, g)\), and since \( a - b = c - d \) we get that \((a, b)C_{mg}(c, d)\). Therefore, \((a, b)C_{mg}(c, d)\) in \( C_\mathbb{Z} \) if and only if \( a - b = c - d \).

We determine a map \( f: C_\mathbb{Z} \to \mathbb{Z} \) by the formula \( ((a, b)) f = a - b \), for \( a, b \in \mathbb{Z} \). Proposition 2.1(vi) implies that such defined map \( f: C_\mathbb{Z} \to \mathbb{Z} \) is a homomorphism. Then we have that \((a, b)C_{mg}(c, d)\) if and only if \(((a, b)) f = ((c, d)) f\), for \((a, b)\), \((c, d)\) \( \in C_\mathbb{Z} \), and hence the homomorphism \( f \) generates the least group congruence \( C_{mg} \) on the semigroup \( C_\mathbb{Z} \).

If \( C \) is any congruence on the semigroup \( C_\mathbb{Z} \) then the mapping \( C \mapsto C \cap C_{mg} \) maps the congruence \( C \) onto a group congruence \( C \cap C_{mg} \), where \( C_{mg} \) is the least group congruence on the semigroup \( C_\mathbb{Z} \) (cf. [17, Section III]). Therefore every homomorphic image of the semigroup \( C_\mathbb{Z} \) is a homomorphic image of the quotient semigroup \( C_\mathbb{Z}/C \), i.e., it is a homomorphic image of the additive group of integers \( \mathbb{Z} \). This completes the proof of the theorem.

3 The semigroup \( C_\mathbb{Z} \): topologizations and closures of \( C_\mathbb{Z} \) in topological semigroups

**Theorem 1.** Every Hausdorff topology \( \tau \) on the semigroup \( C_\mathbb{Z} \) such that \((C_\mathbb{Z}, \tau)\) is a semitopological semigroup is discrete, and hence \( C_\mathbb{Z} \) is a discrete subspace of any semitopological semigroup which contains \( C_\mathbb{Z} \) as a subsemigroup.

**Proof.** We fix an arbitrary idempotent \((a, a)\) of the semigroup \( C_\mathbb{Z} \) and suppose that \((a, a)\) is a non-isolated point of the topological space \((C_\mathbb{Z}, \tau)\). Since the maps \( \lambda_{(a, a)}: C_\mathbb{Z} \to C_\mathbb{Z} \) and
\( \rho(a,a) \colon \mathcal{C}_Z \to \mathcal{C}_Z \) defined by the formulae \((x,y)) \lambda(a,a) = (a,a) \cdot (x,y)\) and \((x,y)) \rho(a,a) = (x,y) \cdot (a,a)\) are continuous retractions we conclude that \((a,a)\mathcal{C}_Z\) and \(\mathcal{C}_Z(a,a)\) are closed subsets in the topological space \((\mathcal{C}_Z, \tau)\). We put
\[
\text{DL}_{(a,a)} [(a,a)] = \{(x,y) \in \mathcal{C}_Z \mid (x,y) \cdot (a,a) = (a,a)\}.
\]

Simple verifications show that
\[
\text{DL}_{(a,a)} [(a,a)] = \{(x,x) \in \mathcal{C}_Z \mid x \leq a \text{ in } \mathbb{Z}\},
\]
and since right translations are continuous maps in \((\mathcal{C}_Z, \tau)\) we get that \(\text{DL}_{(a,a)} [(a,a)]\) is a closed subset of the topological space \((\mathcal{C}_Z, \tau)\). Then there exists an open neighbourhood \(W_{(a,a)}\) of the point \((a,a)\) in the topological space \((\mathcal{C}_Z, \tau)\) such that
\[
W_{(a,a)} \subseteq \mathcal{C}_Z \setminus \{(a + 1, a + 1) \mathcal{C}_Z \cup \mathcal{C}_Z(a + 1, a + 1) \cup \text{DL}_{(a-1,a-1)}(a - 1, a - 1)\}.
\]

Since \((\mathcal{C}_Z, \tau)\) is a semitopological semigroup we conclude that there exists an open neighbourhood \(V_{(a,a)}\) of the idempotent \((a,a)\) in the topological space \((\mathcal{C}_Z, \tau)\) such that the following conditions hold:
\[
V_{(a,a)} \subseteq W_{(a,a)}, \quad (a,a) \cdot V_{(a,a)} \subseteq W_{(a,a)} \quad \text{and} \quad V_{(a,a)} \cdot (a,a) \subseteq W_{(a,a)}.
\]
Hence at least one of the following conditions holds:

(a) the neighbourhood \(V_{(a,a)}\) contains infinitely many points \((x,y) \in \mathcal{C}_Z\) such that \(x < y \leq a\);

or

(b) the neighbourhood \(V_{(a,a)}\) contains infinitely many points \((x,y) \in \mathcal{C}_Z\) such that \(y < x \leq a\).

In case (a) we have that
\[
(a,a) \cdot (x,y) = (a,a + (y-x)) \notin W_{(a,a)},
\]
because \(y - x \geq 1\), and in case (b) we have that
\[
(x,y) \cdot (a,a) = (a + (x-y), a) \notin W_{(a,a)},
\]
because \(x - y \geq 1\), a contradiction. The obtained contradiction implies that the set \(V_{(a,a)}\) is singleton, and hence the idempotent \((a,a)\) is an isolated point of the topological space \((\mathcal{C}_Z, \tau)\).

Let \((a,b)\) be an arbitrary element of the semigroup \(\mathcal{C}_Z\) and suppose that \((a,b)\) is a non-isolated point of the topological space \((\mathcal{C}_Z, \tau)\). Since all right translations are continuous maps in \((\mathcal{C}_Z, \tau)\) and every idempotent \((a,a)\) of \(\mathcal{C}_Z\) is an isolated point of the topological space \((\mathcal{C}_Z, \tau)\) we conclude that
\[
\text{DL}_{(b,a)} [(a,a)] = \{(x,y) \in \mathcal{C}_Z \mid (x,y) \cdot (b,a) = (a,a)\}
\]
is a closed-and-open subset of the topological space \((\mathcal{C}_Z, \tau)\). Simple verifications show that
\[
\text{DL}_{(b,a)} [(a,a)] = \{(x,y) \in \mathcal{C}_Z \mid x - y = a - b \text{ and } x \leq a\}.
\]
Then we have that
\[
\{(a, b)\} = DL_{(b, a)} [(a, a)] \setminus DL_{(b-1, a-1)} [(a - 1, a - 1)],
\]
and hence \((a, b)\) is an isolated point of the topological space \((\mathcal{C}_Z, \tau)\). This completes the proof of the theorem.

Theorem 1 implies the following:

**Corollary 3.1.** Every Hausdorff semigroup topology \(\tau\) on \(\mathcal{C}_Z\) is discrete, and hence \(\mathcal{C}_Z\) is a discrete subspace of any topological semigroup which contains \(\mathcal{C}_Z\) as a subsemigroup.

Since every discrete topological space is locally compact, Theorem 1 and Theorem 3.3.9 from [9] imply the following:

**Corollary 3.2.** Let \(T\) be a semitopological semigroup which contains \(\mathcal{C}_Z\) as a subsemigroup. Then \(\mathcal{C}_Z\) is an open subsemigroup of \(T\).

**Lemma 3.1.** Let \(T\) be a Hausdorff semitopological semigroup which contains \(\mathcal{C}_Z\) as a dense subsemigroup. Let \(f \in T \setminus \mathcal{C}_Z\) be an idempotent of the semigroup \(T\) which satisfies the property: there exists an idempotent \((n, n) \in \mathcal{C}_Z, n \in \mathbb{Z}\), such that \((n, n) \leq f\). Then the following statements hold:

1. there exists an open neighbourhood \(U(f)\) of \(f\) in \(T\) such that \(U(f) \cap \mathcal{C}_Z \subseteq E(\mathcal{C}_Z)\);
2. \(f\) is the unit of \(T\).

**Proof.** (i) Let \(W(f)\) be an arbitrary open neighbourhood of the idempotent \(f\) in \(T\). We fix an arbitrary element \((n, n) \in \mathcal{C}_Z, n \in \mathbb{Z}\). By Corollary 3.2 the element \((n, n)\) is an isolated point in \(T\), and since \(T\) is a semitopological semigroup we have that there exists an open neighbourhood \(U(f)\) of \(f\) in \(T\) such that
\[
U(f) \subseteq W(f), \quad U(f) \cdot \{(n, n)\} = \{(n, n)\} \quad \text{and} \quad \{(n, n)\} \cdot U(f) = \{(n, n)\}.
\]
If the set \(U(f)\) contains a non-idempotent element \((x, y) \in \mathcal{C}_Z\), then Proposition 2.1(iii) implies that \((x, y) \cdot (n, n), (n, n) \cdot (x, y) \notin E(\mathcal{C}_Z)\), a contradiction. The obtained contradiction implies the statement of the assertion.

(ii) First we show that \(f \cdot (k, l) = (k, l) \cdot f = (k, l)\) for every \((k, l) \in \mathcal{C}_Z\).

Suppose the contrary: there exists an element \((k, l) \in \mathcal{C}_Z\) such that \(x = f \cdot (k, l) \neq (k, l)\) for some \(x \in T\). Let \(U(x)\) be an open neighbourhood of \(x\) in \(T\) such that \((k, l) \notin U(x)\). Since \(T\) is a semitopological semigroup we get that there exists an open neighbourhood \(V(f)\) of \(f\) in \(T\) such that \(V(f) \cdot \{(k, l)\} \subseteq U(x)\). Again, since for an arbitrary integer \(a\) the maps \(\lambda_{(a, a)}: \mathcal{C}_Z \to \mathcal{C}_Z\) and \(\rho_{(a, a)}: \mathcal{C}_Z \to \mathcal{C}_Z\) defined by the formulae \(((x, y)) \lambda_{(a, a)} = (a, a) \cdot (x, y)\) and \(((x, y)) \rho_{(a, a)} = (x, y) \cdot (a, a)\) are continuous retractions we conclude that statement (i) implies that there exists an open neighbourhood \(W(f)\) of \(f\) in \(T\) such that \(W(f) \subseteq V(f)\). \(W(f) \cap \mathcal{C}_Z \subseteq E(\mathcal{C}_Z)\) and the following condition holds:
\[
(p, p) \in W(f) \cap \mathcal{C}_Z \quad \text{if and only if} \quad p \geq k.
\]
Then \((p, p) \cdot (k, l) = (k, l) \notin U(x)\) for every \((p, p) \in W(f) \cap \mathcal{C}_Z\), a contradiction. The obtained contradiction implies that \(f \cdot (k, l) = (k, l)\) for every \((k, l) \in \mathcal{C}_Z\). Similar arguments show that \((k, l) \cdot f = (k, l)\) for every \((k, l) \in \mathcal{C}_Z\).

Next we show that \(f \cdot x = x \cdot f = x\) for every \(x \in T \setminus \mathcal{C}_Z\). Suppose the contrary: there exists an element \(x \in T \setminus \mathcal{C}_Z\) such that \(y = f \cdot x \neq x\) for some \(y \in T\). Let \(U(x)\) and \(U(y)\) be open neighbourhoods of \(x\) and \(y\) in \(T\), respectively, such that \(U(x) \cap U(y) = \emptyset\). Since \(T\) is a semitopological semigroup we get that there exists an open neighbourhood \(V(x)\) of \(x\) in \(T\) such that \(V(x) \subseteq U(x)\) and \(f \cdot V(x) \subseteq U(y)\). Again, since \(x \in T \setminus \mathcal{C}_Z\) we have that the set \(V(x) \cap \mathcal{C}_Z\) is infinite, and the previous part of the proof of the statement implies that \(f \cdot (V(x) \cap \mathcal{C}_Z) \subseteq (V(x) \cap \mathcal{C}_Z)\). But we have that \(V(x) \cap U(y) = \emptyset\), a contradiction. The obtained contradiction implies the equality \(f \cdot x = x\). Similar arguments show that \(x \cdot f = x\) for every \(x \in T \setminus \mathcal{C}_Z\).

\(\square\)

**Remark 3.1.** We observe that the assertion (i) of Lemma 3.1 holds for right-topological and left-topological monoids.

**Lemma 3.2.** Let \(T\) be a Hausdorff topological monoid with the unit \(1_T\) which contains \(\mathcal{C}_Z\) as a dense subsemigroup. Then the following assertions hold:

(i) there exists an open neighbourhood \(U(1_T)\) of the unit \(1_T\) in \(T\) such that \(U(1_T) \cap \mathcal{C}_Z \subseteq E(\mathcal{C}_Z)\);

and if the group of units \(H(1_T)\) of \(T\) is non-singleton, then:

(ii) for every \(x \in H(1_T)\) there exists an open neighbourhood \(U(x)\) in \(T\) such that \(a - b = c - d\) for all \((a, b), (c, d) \in U(x) \cap \mathcal{C}_Z\);

(iii) for distinct \(x, y \in H(1_T)\) there exist open neighbourhoods \(U(x)\) and \(U(y)\) of \(x\) and \(y\) in \(T\), respectively, such that \(a - b \neq c - d\) for every \((a, b) \in U(x) \cap \mathcal{C}_Z\) and for every \((c, d) \in U(y) \cap \mathcal{C}_Z\);

(iv) the group \(H(1_T)\) is torsion free;

(v) the group of units \(H(1_T)\) of \(T\) is a discrete subgroup in \(T\);

(vi) the group of units \(H(1_T)\) of \(T\) is isomorphic to the infinite cyclic group;

(vii) every non-identity element of the group of units \(H(1_T)\) in the semigroup \(T\) is not topologically periodic.

**Proof.** Statement (i) follows from Lemma 3.1(i).

(ii) In the case \(H(1_T) = \{1_T\}\) statement (i) implies our assertion. Hence we suppose that \(H(1_T) \neq \{1_T\}\) and let \(x \in H(1_T) \setminus \{1_T\}\). By statement (i) there exists an open neighbourhood \(U(1_T)\) of the unit \(1_T\) in \(T\) such that \(U(1_T) \cap \mathcal{C}_Z \subseteq E(\mathcal{C}_Z)\). Then the continuity of the semigroup operation in \(T\) implies that there exist open neighbourhoods \(U(x)\) and \(U(x^{-1})\) in the topological space \(T\) of \(x\) and the inverse element \(x^{-1}\) of \(x\) in \(H(1_T)\), respectively, such that

\[U(x) \cdot U(x^{-1}) \subseteq U(1_T) \quad \text{and} \quad U(x^{-1}) \cdot U(x) \subseteq U(1_T).\]
Since $U(1_T) \cap \mathcal{C}_2 \subseteq E(\mathcal{C}_2)$ we have that Proposition 2.1(vi) implies that $a - b + u - v = c - d + u - v$ for all $(a, b), (c, d) \in U(x) \cap \mathcal{C}_2$ and some $(u, v) \in U(x^{-1}) \cap \mathcal{C}_2$, and hence $a - b = c - d$.

(iii) Suppose the contrary: there exist distinct $x, y \in H(1_T)$ and for all open neighbourhoods $U(x)$ and $U(y)$ of $x$ and $y$ in $T$, respectively, there are $(a, b) \in U(x) \cap \mathcal{C}_2$ and $(c, d) \in U(y) \cap \mathcal{C}_2$ such that $a - b = c - d$. The Hausdorffness of $T$ implies that without loss of generality we can assume that $U(x) \cap U(y) = \emptyset$. Then statement (i) and the continuity of the semigroup operation in $T$ imply that there exist open neighbourhoods $V(1_T), V(x)$ and $V(y)$ of $1_T, x$ and $y$ in $T$, respectively, such that

$$V(1_T) \cap \mathcal{C}_2 \subseteq E(\mathcal{C}_2), V(x) \subseteq U(x), V(y) \subseteq U(y), V(1_T) \cdot V(x) \subseteq U(x)$$

and

$$V(1_T) \cdot V(y) \subseteq U(y).$$

Since by Theorem 1.7 from [6, Vol. 1] the sets $(a, a)T$ and $T(a, a)$ are closed in $T$ for every idempotent $(a, a) \in \mathcal{C}_2$ and both neighbourhoods $V(x)$ and $V(y)$ contain infinitely many elements of the semigroup $\mathcal{C}_2$, we conclude that for every $(p, p) \in V(1_T) \cap \mathcal{C}_2$ there exist $(k, l) \in V(x) \cap \mathcal{C}_2$ and $(m, n) \in V(y) \cap \mathcal{C}_2$ such that

$$p > k > m, \quad p > l > n \quad \text{and} \quad k - l = m - n.$$ 

Then we get that

$$(p, p) \cdot (k, l) = (p, p + (l - k)) \quad \text{and} \quad (p, p) \cdot (m, n) = (p, p + (n - m)),$$

a contradiction. The obtained contradiction implies our assertion.

(iv) Suppose the contrary: there exist $x \in H(1_T) \setminus \{1_T\}$ and a positive integer $n$ such that $x^n = 1_T$. Then by statement (i) there exists an open neighbourhood $U(1_T)$ of the unit $1_T$ in $T$ such that $U(1_T) \cap \mathcal{C}_2 \subseteq E(\mathcal{C}_2)$. The continuity of the semigroup operation in $T$ and statement (ii) imply that there exists an open neighbourhood $V(x)$ of $x$ in $T$ such that $a - b = c - d$ for all $(a, b), (c, d) \in V(x) \cap \mathcal{C}_2$ and $V(x) \cdot \ldots \cdot V(x) \subseteq U(1_T)$. We fix an arbitrary element $(a, b) \in V(x) \cap \mathcal{C}_2$. If $(a, b)^n = (x, y)$, then Proposition 2.1(vi) implies that $x - y = n \cdot (a - b)$ and since $x \neq 1_T$ we get that $(x, y) \notin U(1_T)$, a contradiction. The obtained contradiction implies statement (iv).

(v) Statement (iv) implies that the group of units $H(1_T)$ is infinite.

We fix an arbitrary $x \in H(1_T)$ and suppose that $x$ is not an isolated point of $H(1_T)$. Then by statement (ii) there exists an open neighbourhood $U(x)$ in $T$ such that $a - b = c - d$ for all $(a, b), (c, d) \in U(x) \cap \mathcal{C}_2$. Since the point $x$ is not isolated in $H(1_T)$ we conclude that there exists $y \in H(1_T)$ such that $y \in U(x)$. Hence the set $U(x)$ is an open neighbourhood of $y$ in $T$. Statement (iii) implies that there exist open neighbourhoods $W(x) \subseteq U(x)$ and $W(y) \subseteq U(x)$ for $x$ and $y$ in $T$, respectively, such that $a - b \neq c - d$ for every $(a, b) \in W(x) \cap \mathcal{C}_2$ and for every $(c, d) \in W(y) \cap \mathcal{C}_2$. This contradicts the choice of the neighbourhood $U(x)$. The obtained contradiction implies that every $x \in H(1_T)$ is an isolated point of $H(1_T)$.

(vi) Since the group of units $H(1_T)$ is non-trivial, i.e., the group $H(1_T)$ is non-singleton, we fix an arbitrary $x \in H(1_T) \setminus \{1_T\}$. Then by statement (iv) we have that $x^n \neq 1_T$ for any
positive integer \( n \). Statement (\( ii \)) implies that there exists an open neighbourhood \( U(x) \) in \( T \) such that \( a - b = c - d \) for all \( (a, b), (c, d) \in U(x) \cap G \). We define the map \( \varphi : H(1_T) \to \mathbb{Z} \) by the following way: \( (x) \varphi = k \) if and only if \( a - b = k \) for every \( (a, b) \in U(x) \cap G \). Then statement (\( iv \)) and Proposition 2.1(\( vi \)) imply that the map \( \varphi : H(1_T) \to \mathbb{Z} \) is an injective homomorphism. Obviously that \( (H(1_T)) \varphi \) is a subgroup in the additive group of integers. We fix the least positive integer \( p \in (H(1_T)) \varphi \). Then the element \( p \) generates the subgroup \( (H(1_T)) \varphi \) in the additive group of integers \( \mathbb{Z} \), and hence the group \( (H(1_T)) \varphi \) is cyclic.

(\( vii \)) We fix an arbitrary element \( x \in H(1_T) \setminus \{1_T\} \). Suppose the contrary: \( x \) is a topologically periodic element of \( S \). Then there exist open neighbourhoods \( U(1_T) \) and \( U(x) \) of \( 1_T \) and \( x \) in \( T \), respectively, such that \( U(1_T) \cap U(x) = \emptyset \). Statements (\( i \)) and (\( iii \)) imply that without loss of generality we can assume that \( U(1_T) \cap G \subseteq E(G) \), and \( a - b = c - d \neq 0 \) for all \( (a, b), (c, d) \in U(x) \cap G \). Then the topologically periodicity of \( x \) implies that there exists a positive integer \( n \) such that \( x^n \in U(1_T) \). Since the semigroup operation in \( T \) is continuous we conclude that there exists an open neighbourhood \( V(x) \) of \( x \) in \( T \) such that \( \bigcup_{n \text{-times}} V(x) \subseteq U(1_T) \). We fix an arbitrary element \( (a, b) \in V(x) \cap G \). Then we have that

\[
(a, b)^n \in U(1_T) \cap G \quad \text{and hence} \quad n(a - b) = 0, \quad \text{a contradiction.} \]

The obtained contradiction implies assertion (\( vii \)).

**Proposition 3.1.** Let \( G \) be non-trivial subgroup of the additive group of integers \( \mathbb{Z} \) and \( n \in \mathbb{Z} \). Then the subsemigroup \( H \) which is generated by the set \( \{n\} \cup G \) is a cyclic subgroup of \( \mathbb{Z} \).

**Proof.** Without loss of generality we can assume that \( n \in \mathbb{Z} \setminus G \) and \( n > 0 \).

Since every subgroup of a cyclic group is cyclic (see [14, P. 47]), we have that \( G \) is a cyclic subgroup in \( \mathbb{Z} \). We fix a generating element \( k \) of \( G \) such that \( k > 0 \). Then we have that

\[
(n + \cdots + n) - (k + \cdots + k) + n = 0,
\]

and hence we have that \(-n \in H\). Since \( \mathbb{Z} \) is a commutative group we conclude that \( H \) is a subgroup in \( \mathbb{Z} \), which is generated by elements \( n \) and \( k \), and hence \( H \) is a cyclic subgroup in \( \mathbb{Z} \).

**Proposition 3.2.** Let \( T \) be a Hausdorff topological monoid with the unit \( 1_T \) which contains \( G \) as a dense subsemigroup. Then the following assertions hold:

(i) if the set \( L_G = \{x \in T \setminus G \mid \text{there exists} \ y \in G \text{ such that} \ x \cdot y \in G \} \) is non-empty, then \( L_G \) is a subsemigroup of \( T \), and moreover if \( a \in L_G \), then there exists an open neighbourhood \( U(a) \) of \( a \) in \( T \) such that \( n_1 - m_1 = n_2 - m_2 \) for all \( (n_1, m_1), (n_2, m_2) \in U(a) \cap G \);

(ii) if the set \( R_G = \{x \in T \setminus G \mid \text{there exists} \ y \in G \text{ such that} \ y \cdot x \in G \} \) is non-empty, then \( R_G \) is a subsemigroup of \( T \), and moreover if \( a \in R_G \), then there exists an open neighbourhood \( U(a) \) of \( a \) in \( T \) such that \( n_1 - m_1 = n_2 - m_2 \) for all \( (n_1, m_1), (n_2, m_2) \in U(a) \cap G \);
(iii) if the set \( L_{\mathcal{E}_z} \) (resp., \( R_{\mathcal{E}_z} \)) is non-empty, then for every \( a \in L_{\mathcal{E}_z} \) (resp., \( a \in R_{\mathcal{E}_z} \)) there exist an open neighbourhood \( U(a) \) of \( a \) in \( T \) and an integer \( n_a \) such that \( p \leq n_a \) and \( q \leq n_a \) for all \( (p, q) \in U(a) \cap \mathcal{C}_z \);

(iv) \( L_{\mathcal{E}_z} = R_{\mathcal{E}_z} \);

(v) \( \uparrow \mathcal{C}_z = \mathcal{C}_z \cup L_{\mathcal{E}_z} \) is a subsemigroup of \( T \) and \( \mathcal{C}_z \) is a minimal ideal in \( \uparrow \mathcal{C}_z \);

(vi) if for an element \( a \in T \setminus \mathcal{C}_z \) there is an open neighbourhood \( U(a) \) of \( a \) in \( T \) and the following conditions hold:

(a) \( m_1 - m_2 = n_1 - n_2 \) for all \( (m_1, n_1), (m_2, n_2) \in U(a) \cap \mathcal{C}_z \); and

(b) there exists an integer \( n_a \) such that \( n \leq n_a \) and \( m \leq n_a \) for every \( (m, n) \in U(a) \cap \mathcal{C}_z \),

then \( a \in L_{\mathcal{E}_z} \);

(vii) if \( I = T \setminus \uparrow \mathcal{C}_z \neq \emptyset \), then \( I \) is an ideal of \( T \);

(viii) the set

\[
\uparrow(a, b) = \{ x \in T \mid x \cdot (b, b) = (a, b) \} = \{ x \in T \mid (a, a) \cdot x = (a, b) \} = \{ x \in T \mid (a, a) \cdot (b, b) = (a, b) \}
\]

is closed-and-open in \( T \) for every \( (a, b) \in \mathcal{C}_z \);

(ix) the set \( \uparrow(a, b) \cap L_{\mathcal{E}_z} \) is either singleton or empty;

(x) \( L_{\mathcal{E}_z} \) is isomorphic to a submonoid of the additive group of integers \( \mathbb{Z} \), and moreover if a maximal subgroup of \( L_{\mathcal{E}_z} \) is non-singleton, then \( L_{\mathcal{E}_z} \) is isomorphic to the additive group of integers \( \mathbb{Z} \);

(xi) \( \uparrow \mathcal{C}_z \) is an open subset in \( T \), and hence if \( I = T \setminus \uparrow \mathcal{C}_z \neq \emptyset \), then the ideal \( I \) is a closed subset in \( T \);

(xii) if the semigroup \( T \) contains a non-singleton group of units \( H(1_T) \), then \( H(1_T) = T \setminus (\mathcal{C}_z \cup I) \).

**Proof.** (i) We observe that since \( \mathcal{C}_z \) is an inverse semigroup we conclude that \( x \in L_{\mathcal{E}_z} \) if and only if there exists an idempotent \( e \in \mathcal{C}_z \) such that \( x \cdot e \in \mathcal{C}_z \), for \( x \in T \).

We fix an arbitrary \( x \in L_{\mathcal{E}_z} \). Let \( (n, n) \) be an idempotent in \( \mathcal{C}_z \) such that \( (a, b) = x \cdot (n, n) \in \mathcal{C}_z \). Then by Corollary 3.1 we have that \( (n, n) \) and \( (a, b) \) are isolated points in \( T \), and the continuity of the semigroup operation in \( T \) implies that there exists an open neighbourhood \( U(x) \) of \( x \) in \( T \) such that

\[
U(x) \cdot \{(n, n)\} = \{(a, b)\} \in \mathcal{C}_z.
\]
Then Proposition 2.1(vi) implies that \( p - q = a - b \) for all \((p, q) \in U(x) \cap \mathcal{C}_Z\). Also, since

\[
(p, q)(n, n) = \begin{cases} 
(p - q + n, n), & \text{if } q \leq n; \\
(p, q), & \text{if } q \geq n
\end{cases}
\]

(2)

we have that \( q \leq n = b \).

Suppose that \( x, y \in L_{\mathcal{C}_Z} \), and \((i, i)\) and \((j, j)\) are idempotents in \( \mathcal{C}_Z \) such that \( x \cdot (i, i) = (k, l) \in \mathcal{C}_Z \) and \( y \cdot (j, j) \in \mathcal{C}_Z \), \( i, j, k, l \in \mathbb{Z} \). We fix an arbitrary integer \( d \) such that \( d \geq \max\{k, j\} \). Then we have that

\[
(y \cdot x) \cdot ((i, i) \cdot (l, k) \cdot (d, d)) = y \cdot (x \cdot (i, i) \cdot (l, k) \cdot (d, d))
\]

\[
= y \cdot ((k, l) \cdot (l, k) \cdot (d, d))
\]

\[
= y \cdot (k, k) \cdot (d, d)
\]

\[
= y \cdot ((j, j) \cdot (d, d))
\]

\[
= (y \cdot (j, j)) \cdot (d, d) \in \mathcal{C}_Z.
\]

This implies that \( L_{\mathcal{C}_Z} \) is a subsemigroup of \( T \) and completes the proof of our assertion.

The proof of assertion (ii) is similar to (i).

Statement (i) and formula (2) imply assertion (iii). In the case \( a \in R_{\mathcal{C}_Z} \) the proof is similar.

(iv) Let be \( L_{\mathcal{C}_Z} \neq \emptyset \). We fix an arbitrary element \( a \in L_{\mathcal{C}_Z} \). Then there exists an idempotent \((i_a, i_a) \in \mathcal{C}_Z\) such that \( a \cdot (i_a, i_a) = (i, j) \in \mathcal{C}_Z\). Assertion (iii) implies that there exist an open neighbourhood \( U(a) \) of \( a \) in \( T \) and an integer \( n_a \) such that \( n - m = i - j \), \( n \leq n_a \) and \( m \leq n_a \) for all \((n, m) \in U(a) \cap \mathcal{C}_Z\). Without loss of generality we can assume that \( i_a \geq n_a \).

We shall show that \((i_a, i_a) \cdot a \in \mathcal{C}_Z\). Suppose the contrary: \((i_a, i_a) \cdot a = b \in T \setminus \mathcal{C}_Z\). Assertion (iii) implies that there exist integers

\[
n_0(a) = \max\{n \mid (n, m) \in U(a) \cap \mathcal{C}_Z\} \quad \text{and} \quad m_0(a) = \max\{m \mid (n, m) \in U(a) \cap \mathcal{C}_Z\}.
\]

Since \( i_a \geq n_a \) we have that

\[
(i_a, i_a) \cdot (n_0(a), m_0(a)) = (i_a, i_a - n_0(a) + m_0(a)).
\]

Let \( W(b) \) be an open neighbourhood of \( b \) in \( T \) such that \((i_a, i_a - n_0(a) + m_0(a)) \notin W(b)\). Then the continuity of the semigroup operation in \( T \) implies that there exists an open neighbourhood \( V(a) \) of \( a \) in \( T \) such that

\[
V(a) \subseteq U(a) \quad \text{and} \quad \{(i_a, i_a)\} \cdot V(a) \subseteq W(b).
\]

We fix an arbitrary element \((n, m) \in V(a) \cap \mathcal{C}_Z\). Then we have that

\[
(i_a, i_a) \cdot (n, m) = (i_a, i_a - n + m) = (i_a, i_a - n_0(a) + m_0(a)),
\]

a contradiction. The obtained contradiction implies that \( a \in R_{\mathcal{C}_Z} \), and hence we have that \( L_{\mathcal{C}_Z} \subseteq R_{\mathcal{C}_Z} \).
The proof of the inclusion $R_{\mathbb{C}_Z} \subseteq L_{\mathbb{C}_Z}$ is similar.

Statement (v) follows from statements (i) – (iv) and Proposition 2.1(v).

(vi) Let $U(a)$ be an open neighbourhood of $a$ in $T$ such that conditions (a) and (b) hold, and let $n_a$ be such integer as in condition (b). Then for all $(m_1, n_1), (m_2, n_2) \in U(a) \cap \mathbb{C}_Z$ we have that

$$(m_1, n_1) \cdot (n_a, n_a) = (m_1 - n_1 + n_a, n_a) = (m_2 - n_2 + n_a, n_a) = (m_2, n_2) \cdot (n_a, n_a),$$

and hence the continuity of the semigroup operation in $T$ implies that $a \in L_{\mathbb{C}_Z}$.

(vii) Statements (i) and (iii) imply that $a \cdot (m, n) \in I$ and $(m, n) \cdot a \in I$ for all $a \in I$ and $(m, n) \in \mathbb{C}_Z$.

Fix arbitrary elements $a, b \in I$. We consider the following two cases:

1) $a \cdot b \in \mathbb{C}_Z$ and 2) $a \cdot b \in L_{\mathbb{C}_Z}$.

In case 1) we put $a \cdot b = (m, n) \in \mathbb{C}_Z$. Then the continuity of the semigroup operation in $T$ implies that there exist open neighbourhoods $U(a)$ and $U(b)$ of $a$ and $b$ in $T$, respectively, such that

$$U(a) \cdot U(b) = \{(m, n)\}.$$ 

Since $a$ and $b$ are accumulation points of $\mathbb{C}_Z$ in $T$, we conclude that there exist $(m_a, n_a) \in U(a) \cap \mathbb{C}_Z$ and $(m_b, n_b) \in U(b) \cap \mathbb{C}_Z$. Hence we have that

$$(m_a, n_a) \cdot b \in \{(m_a, n_a)\} \cdot U(b) \subseteq U(a) \cdot U(b) = \{(m, n)\}$$

and

$$a \cdot (m_b, n_b) \in U(a) \cdot \{(m_b, n_b)\} \subseteq U(a) \cdot U(b) = \{(m, n)\}$$

This implies that $a, b \in L_{\mathbb{C}_Z}$, a contradiction.

Suppose case 2) holds and $a \cdot b = x \in L_{\mathbb{C}_Z}$. Then by statements (i) and (iii) we have that there exist an open neighbourhood $U(x)$ of $x$ in $T$ and an integer $n_x$ such that $m_1 - n_1 = m_2 - n_2$, $m_1 \leq n_x$ and $n_1 \leq n_x$ for all $(m_1, n_1), (m_2, n_2) \in U(x) \cap \mathbb{C}_Z$. Also, the continuity of the semigroup operation in $T$ implies that there exist open neighbourhoods $U(a)$ and $U(b)$ of $a$ and $b$ in $T$, respectively, such that

$$U(a) \cdot U(b) \subseteq U(x).$$

Since $U(a) \cap \mathbb{C}_Z \neq \emptyset$ and $U(b) \cap \mathbb{C}_Z \neq \emptyset$, we can find arbitrary elements $(m_a, n_a) \in U(a) \cap \mathbb{C}_Z$ and $(m_b, n_b) \in U(b) \cap \mathbb{C}_Z$. Then by Proposition 2.1(vi) we have that

$$x_a - y_a + m_b - n_b = m_1 - n_1$$

and

$$m_a - n_a + x_b - y_b = m_1 - n_1$$

for all $(x_a, y_a) \in U(a) \cap \mathbb{C}_Z$ and $(x_b, y_b) \in U(b) \cap \mathbb{C}_Z$. This implies that there exist integers $k_a$ and $k_b$ such that

$$x_a - y_a = k_a$$

and

$$x_b - y_b = k_b$$

for all $(x_a, y_a) \in U(a) \cap \mathbb{C}_Z$ and $(x_b, y_b) \in U(b) \cap \mathbb{C}_Z$. Then by statement (vi) we have that $a, b \in L_{\mathbb{C}_Z}$, a contradiction.
The obtained contradictions imply that \( a \cdot b \in I \), and hence we get that the set \( I \) is an ideal of \( T \).

\((viii)\) Proposition 2.1\( (vi)\) and assertion \((vi)\) imply the following equalities:

\[
\{ x \in T \mid x \cdot (b, b) = (a, b) \} = \{ x \in T \mid (a, a) \cdot x = (a, b) \} = \{ x \in T \mid (a, a) \cdot x \cdot (b, b) = (a, b) \}.
\]

Since by Corollary 3.1 every element \((a, b)\) of the semigroup \( \mathcal{C}_Z \) is an isolated point in \( T \), the continuity of the semigroup operation in \( T \) implies that \( \uparrow(a, b) \) is a closed-and-open subset in \( T \).

\((ix)\) Suppose that the set \( \uparrow(a, b) \cap L_{\mathcal{C}_Z} \) is non-empty. Assuming that the set \( \uparrow(a, b) \cap L_{\mathcal{C}_Z} \) is non-singleton implies that there exist distinct \( x, y \in \uparrow(a, b) \cap L_{\mathcal{C}_Z} \). Then the Hausdorffness of \( T \) implies that there exist disjoint open neighbourhoods \( U(x) \) and \( U(y) \) of \( x \) and \( y \) in \( T \), respectively. By the continuity of the semigroup operation in \( T \) we can find open neighbourhoods \( V(1_T), V(x) \) and \( V(y) \) of \( 1_T, x \) and \( y \) in \( T \), respectively, such that the following conditions hold:

\[
V(x) \subseteq U(x), \quad V(y) \subseteq U(y), \quad V(1_T) \cdot V(x) \subseteq U(x) \quad \text{and} \quad V(1_T) \cdot V(y) \subseteq U(y).
\]

By assertions \((i) - (iii)\) we can find the integers \( n, n_1, n_2, m_1 \) and \( m_2 \) such that

\[
(n, n) \in V(1_T), \quad (n_1, n_2) \in V(x), \quad (m_1, m_2) \in V(y), \quad n_1 - n_2 = m_1 - m_2, \quad n \geq n_1 \quad \text{and} \quad n \geq m_1.
\]

Then we have that

\[
(n, n) \cdot (n_1, n_2) = (n, n - n_1 + n_2) = (n, n - m_1 + m_2) = (n, n) \cdot (m_1, m_2),
\]

and hence \( (V(1_T) \cdot V(x)) \cdot (V(1_T) \cdot V(y)) \neq \emptyset \), a contradiction. The obtained contradiction implies that \( x = y \).

\((x)\) Statement \((vii)\) implies that \( T \setminus (I \cup \mathcal{C}_Z) = L_{\mathcal{C}_Z} \). Let \( \mathbb{Z} \) be the additive group of integers. We define a map \( \mathfrak{h} : L_{\mathcal{C}_Z} \to \mathbb{Z} \) as follows:

\[
(x) \mathfrak{h} = n \quad \text{if and only if there exists a neighbourhood } U(x) \text{ of } x \text{ in } T \text{ such that } a - b = n, \quad \text{for all } (a, b) \in U(x) \cap \mathcal{C}_Z,
\]

where \( x \in L_{\mathcal{C}_Z} \). We observe that assertions \((i) - (v)\) imply that the map \( \mathfrak{h} \) is well defined. Also, Proposition 2.1 implies that \( \mathfrak{h} : L_{\mathcal{C}_Z} \to \mathbb{Z} \) is a monomorphism, and hence \( L_{\mathcal{C}_Z} \) is a submonoid of \( \mathbb{Z} \). In the case when a maximal subgroup of \( L_{\mathcal{C}_Z} \) is non-singleton Proposition 3.1 implies that \( (L_{\mathcal{C}_Z})_{\mathfrak{h}} \) is a cyclic subgroup of \( \mathbb{Z} \). This completes the proof of our assertion.

\((xi)\) Assertion \((v)\) implies that

\[
\uparrow \mathcal{C}_Z = \{ x \in T \mid \text{there exists } y \in \mathcal{C}_Z \text{ such that } x \cdot y \in \mathcal{C}_Z \} = \bigcup_{(a, b) \in \mathcal{C}_Z} \uparrow(a, b).
\]

Then assertion \((viii)\) implies that \( \uparrow \mathcal{C}_Z \) is an open subset in \( T \) and hence by assertion \((vii)\) we get that the ideal \( I \) is a closed subset of \( T \).

Assertion \((xii)\) follows from \((x)\). \( \square \)
4 On a closure of the semigroup $C_\mathbb{Z}$ in a locally compact topological inverse semigroup

For every non-negative integer $k$ by $k\mathbb{Z}$ we denote a subgroup of the additive group of integers $\mathbb{Z}$ which is generated by an element $k \in \mathbb{Z}$. We observe if $k = 0$ then the group $k\mathbb{Z}$ is trivial. Also, we denote $G_0 = \mathbb{Z}$ and $G_1(k) = k\mathbb{Z}$ for a positive integer $k$.

The following five examples illustrate distinct structures of a closure of the semigroup $C_\mathbb{Z}$ in a locally compact topological inverse semigroup.

**Example 1.** Let be $S_1 = G_1(0) \sqcup C_\mathbb{Z}$. Then $G_1(0)$ is a trivial group and we put $\{e_1\} = G_1(0)$. We extend the semigroup operation from $C_\mathbb{Z}$ onto $S_1$ as follows:

$$e_1 \cdot (a, b) = (a, b) \cdot e_1 = (a, b) \in C_\mathbb{Z} \quad \text{and} \quad e_1 \cdot e_1 = e_1,$$

i.e., $S_1$ is the semigroup $C_\mathbb{Z}$ with the adjoined unit $e_1$. We fix an arbitrary decreasing sequence $\{m_i\} \in \mathbb{N}$ of negative integers and for every positive integer $n$ we put

$$U_n(e_1) = \{e_1\} \cup \{(m_i, m_i) \in C_\mathbb{Z} \mid i \geq n\}.$$

Then we determine a topology $\tau_1$ on $S_1$ as follows:

1) all elements of the semigroup $C_\mathbb{Z}$ are isolated points in $(S_1, \tau_1)$; and

2) the family $\mathcal{B}_1(e_1) = \{U_n(e_1) \mid n \in \mathbb{N}\}$ is a base of the topology $\tau_1$ at the point $e_1 \in G_1(0) \subseteq S_1$.

Then for every positive integer $n$ we have that

$$U_n(e_1) \cdot U_n(e_1) = U_n(e_1) \quad \text{and} \quad (U_n(e_1))^{-1} = U_n(e_1).$$

Let $(m, n)$ be an arbitrary element of the semigroup $C_\mathbb{Z}$. We fix a positive integer $i_{(m,n)}$ such that $m_{i_{(m,n)}} \leq m$ and $m_{i_{(m,n)}} \leq n$. Then we have that

$$U_{i_{(m,n)}}(e_1) \cdot \{(m, n)\} = \{(m, n)\} \quad \text{and} \quad \{(m, n)\} \cdot U_{i_{(m,n)}}(e_1) = \{(m, n)\}.$$

Hence we get that $(S_1, \tau_1)$ is a topological inverse semigroup. Obviously, $(S_1, \tau_1)$ is a Hausdorff locally compact space.

**Example 2.** Let $k$ and $n$ be any positive integers such that $n \in \{1, \ldots, k\}$ is a divisor of $k$ and we put $k = n \cdot s$, where $s$ is some positive integer. We put $S_2 = G_1(k) \sqcup C_\mathbb{Z}$. Later an element of the group $G_1(k) = k\mathbb{Z}$ will be denote by $ki$, where $i \in \mathbb{Z}$. We extend the semigroup operation from $C_\mathbb{Z}$ onto $S_2$ by the following way:

$$ki \cdot (a, b) = (-ki + a, b) \in C_\mathbb{Z} \quad \text{and} \quad (a, b) \cdot ki = (a, b + ki) \in C_\mathbb{Z},$$

for arbitrary $(a, b) \in C_\mathbb{Z}$ and $ki \in G_1(k)$. To see that the extended binary operation is associative we need only check six possibilities, the other being evident.

Then for arbitrary $ki_1, ki_2 \in G_1(k)$ and $(a, b), (c, d) \in C_\mathbb{Z}$ we have that:
1) \((ki_1 \cdot ki_2) \cdot (a, b) = (ki_1 + ki_2)(a, b) = (-ki_1 - ki_2 + a, b) = ki_1 \cdot (-ki_2 + a, b) = ki_1 \cdot (ki_2 \cdot (a, b))\);

2) \((a, b) \cdot (ki_1 \cdot ki_2) = (a, b) \cdot (ki_1 + ki_2) = (a, b + ki_1 + ki_2) = (a, b + ki_1) \cdot ki_2 = ((a, b) \cdot ki_1) \cdot ki_2;\)

3) \((ki_1 \cdot (a, b)) \cdot ki_2 = (-ki_1 + a, b) \cdot ki_2 = (-ki_1 + a, b + ki_2) = ki_1 \cdot (a, b + ki_2) = ki_1 \cdot ((a, b) \cdot ki_2);\)

4) \((ki_1 \cdot (a, b)) \cdot (c, d) = (-ki_1 + a, b) \cdot (c, d) = \begin{cases} (-ki_1 + a - b + c, d), & \text{if } b \leq c; \\ (-ki_1 + a, b - c + d), & \text{if } b \geq c \end{cases} = ki_1 \cdot ((a, b) \cdot (c, d));\)

5) \(((a, b) \cdot (c, d)) \cdot ki_1 = \begin{cases} (a - b + c, d) \cdot ki_1, & \text{if } b \leq c; \\ (a, b - c + d) \cdot ki_1, & \text{if } b \geq c \end{cases} = (a, b) \cdot (c, d + ki_1) = (a, b) \cdot ((c, d) \cdot ki_1);\)

6) \(((a, b) \cdot ki_1) \cdot (c, d) = (a, b + ki_1) \cdot (c, d) = \begin{cases} (a - b - ki_1 + c, d), & \text{if } b + ki_1 \leq c; \\ (a, b + ki_1 - c + d), & \text{if } b + ki_1 \geq c \end{cases} = (a, b) \cdot (-ki_1 + c, d) = (a, b) \cdot (ki_1 \cdot (c, d)).\)

Also simple verifications show that \(S_2\) is an inverse semigroup.

Let \(ki\) be an arbitrary element of the group \(G_1(k)\). For every positive integer \(j\) we denote 
\[
U^n_j(ki) = \{ki\} \cup \{(-nq, -nq + ki) \mid q \geq j, q \in \mathbb{N}\}.
\]

We determine a topology \(\tau_2\) on \(S_2\) as follows:

1) all elements of the semigroup \(C_{Z}\) are isolated points in \((S_2, \tau_2)\); and

2) the family \(\mathcal{B}_2(ki) = \{U^n_j(ki) \mid j \in \mathbb{N}\}\) is a base of the topology \(\tau_2\) at the point \(ki \in G_1(k) \subseteq S_2\).

Then for every positive integer \(j\) we have that
\[
U^n_j(ki_1) \cdot U^n_{j-i_1}(ki_2) \subseteq U^n_j(ki_1 + ki_2) \quad \text{and} \quad (U^n_j(ki_1))^{-1} = U^n_j(-ki_1),
\]
for \(ki_1, ki_2 \in G_1(k)\).

Let \((a, b)\) be an arbitrary element of the semigroup \(C_{Z}\) and \(ki \in G_1(k)\). Then we have that
\[
U^n_j(ki) \cdot \{(a, b)\} = \{(a - ki, b)\} \quad \text{and} \quad \{(a, b)\} \cdot U^n_j(ki) = \{(a, b + ki)\},
\]
for every positive integer \(j\) such that \(nj \geq \max\{-b; ki - a\}\).

Therefore \((S_2, \tau_2)\) is a topological inverse semigroup, and moreover the topological space \((S_2, \tau_2)\) is Hausdorff and locally compact.
Example 3. We put \( S_3 = \mathcal{C}_Z \cup G_0 \) and extend the semigroup operation from the semigroup \( \mathcal{C}_Z \) onto \( S_3 \) by the following way:

\[
(a, b) \cdot n = n \cdot (a, b) = n + b - a \in G_0,
\]

for all \((a, b) \in \mathcal{C}_Z \text{ and } n \in G_0\). To see that the extended binary operation is associative we need only check two possibilities, the other being evident.

Then for arbitrary \( m, n \in G_0 \) and \((a, b), (c, d) \in \mathcal{C}_Z \) we have that:

1) \( (n \cdot (a, b)) \cdot (c, d) = (n + b - a) \cdot (c, d) = n + b - a + d - c = \begin{cases} n \cdot (a + b + c, d), & \text{ if } b \leq c; \\ n \cdot (a, b + c + d), & \text{ if } b \geq c \end{cases} = n \cdot ((a, b) \cdot (c, d)); \)

2) \( (m \cdot n) \cdot (a, b) = m + n + b - a = m \cdot (n + b - a) = m \cdot (n \cdot (a, b)). \)

This completes the proof of the associativity of such defined binary operation on \( S_3 \). Also, we observe that \( S_3 \) with such defined semigroup operation is an inverse semigroup.

For every positive integer \( n \) and every element \( k \in G_0 \) we put:

\[
U_n(k) = \begin{cases} \{ k \} \cup \{ (a, a + k) \mid a = n, n + 1, n + 2, \ldots \}, & \text{ if } k \geq 0; \\ \{ k \} \cup \{ (a - k, a) \mid a = n, n + 1, n + 2, \ldots \}, & \text{ if } k \leq 0. \end{cases}
\]

We determine a topology \( \tau_3 \) on \( S_3 \) as follows:

1) all elements of the semigroup \( \mathcal{C}_Z \) are isolated points in \((S_3, \tau_3)\); 

2) the family \( \mathcal{B}_3(k) = \{ U_n(k) \mid n \in \mathbb{N} \} \) is a base of the topology \( \tau_3 \) at the point \( k \in G_0 \subseteq S_3 \).

Then for all \( k_1, k_2 \in G_0 \) we have that

\[
U_{2n}(k_1) \cdot U_{2n}(k_2) \subseteq U_n(k_1 + k_2),
\]

for every positive integer \( n \geq \max \{ \mid k_1 \mid, \mid k_2 \mid \} \), and

\[
(U_i(k_1))^{-1} = U_i(-k_1),
\]

for every positive integer \( i \). Also, for arbitrary \((a, b) \in \mathcal{C}_Z \) and \( k \in G_0 \) we have that

\[
(a, b) \cdot U_{2n}(k) \subseteq U_n(k + b - a) \quad \text{ and } \quad U_{2n}(k) \cdot (a, b) \subseteq U_n(k + b - a),
\]

for every positive integer \( n \geq \max \{ \mid a \mid, \mid b \mid, \mid k \mid \} \).

This completes the proof that \((S_3, \tau_3)\) is a topological inverse semigroup. Obviously, \((S_3, \tau_3)\) is a Hausdorff locally compact space.

Example 4. Let \( S_4 = G_1(0) \cup S_3 \), where the group \( G_1(0) \) and the semigroup \( S_3 \) are defined in Example 1 and Example 3, respectively. We extend the semigroup operation from \( S_3 \) onto \( S_4 \) as follows:

\[
e_1 \cdot x = x \cdot e_1 = x \in \mathcal{C}_Z \quad \text{ and } \quad e_1 \cdot e_1 = e_1,
\]
i.e., $S_4$ is the semigroup $S_3$ with the adjoined unit $e_1$.

Let $\tau_4$ be a topology on $S_4$ which is generated by the family $\tau_1 \cup \tau_3$ (see Examples 1 and 3). Then for every element $k_0 \in G_0$ and every positive integers $n_1$ and $n_0$ we have that the following inclusions hold:

$$U_{n_1}(e_1) \cdot U_{n_0}(k_0) \subseteq U_{n_0}(k_0) \quad \text{and} \quad U_{n_0}(k_0) \cdot U_{n_1}(e_1) \subseteq U_{n_0}(k_0),$$

where $U_{n_1}(e_1) \in \mathscr{B}_1(e_1)$ and $U_{n_0}(k_0) \in \mathscr{B}_3(k_0)$ (see Examples 1 and 3). These inclusions and Examples 1 and 3 imply that $(S_4, \tau_4)$ is a Hausdorff topological inverse semigroup. Obviously, $(S_4, \tau_4)$ is a locally compact space.

**Example 5.** Let $k$ and $n$ be such positive integers as in Example 2. We put $S_5 = G_1(k) \cup \mathcal{C}_Z \cup G_0$ and extend semigroup operation from $S_2$ and $S_3$ onto $S_5$ as follows. Later we denote elements of groups $G_1(K)$ and $G_0$ by $(ki)^1$ and $(n)^0$, respectively. We put

$$(ki)^1 \cdot (n)^0 = (n)^0 \cdot (ki)^1 = (ki + n)^0 \in G_0,$$

for all $(ki)^1 \in G_1(k)$ and $(n)^0 \in G_0$. To see that the extended binary operation is associative we need only check twelve possibilities, the other either are evident or are proved in Examples 2 and 3.

Then for arbitrary $(ki_1)^1, (ki_2)^1 \in G_1(k)$, $(n_1)^0, (n_2)^0 \in G_0$ and $(a, b) \in \mathcal{C}_Z$ we have that:

1) $((n_1)^0 \cdot (n_2)^0) \cdot (ki_1)^1 = (n_1 + n_2)^0 \cdot (ki_1)^1 = (n_1 + n_2 + ki_1)^0 = (n_1)^0 \cdot (n_2 + ki_1)^0$

$= (n_1)^0 \cdot ((n_2)^0 \cdot (ki_1)^1)$;

2) $((n_1)^0 \cdot (ki_1)^1) \cdot (n_2)^0 = (n_1 + ki_1)^0 \cdot (n_2)^0 = (n_1 + ki_1 + n_2)^0 = (n_1)^0 \cdot (ki_1 + n_2)^0$

$= (n_1)^0 \cdot ((ki_1)^1 \cdot (n_2)^0)$;

3) $((n_1)^0 \cdot (ki_1)^1) \cdot (ki_2)^1 = (n_1 + ki_1)^0 \cdot (ki_2)^1 = (n_1 + ki_1 + ki_2)^0 = (n_1)^0 \cdot (ki_1 + ki_2)^1$

$= (n_1)^0 \cdot ((ki_1)^1 \cdot (ki_2)^1)$;

4) $((n_1)^0 \cdot (ki_1)^1) \cdot (a, b) = (n_1 + ki_1)^0 \cdot (a, b) = (n_1 + ki_1 + b - a)^0 = (n_1)^0 \cdot (ki_1 + a, b)$

$= (n_1)^0 \cdot ((ki_1)^1 \cdot (a, b))$;

5) $((n_1)^0 \cdot (a, b)) \cdot (ki_1)^1 = (n_1 + b - a)^0 \cdot (ki_1)^1 = (n_1 + b - a + ki_1)^0 = (n_1)^0 \cdot (a, b + ki_1)$

$= (n_1)^0 \cdot ((a, b) \cdot (ki_1)^1)$;

6) $((ki_1)^1 \cdot (n_1)^0) \cdot (n_2)^0 = (ki_1 + n_1)^0 \cdot (n_2)^0 = (ki_1 + n_1 + n_2)^0 = (ki_1)^1 \cdot (n_1 + n_2)^0$

$= (ki_1)^1 \cdot ((n_1)^0 \cdot (n_2)^0)$;

7) $((ki_1)^1 \cdot (n_1)^0) \cdot (ki_2)^1 = (ki_1 + n_1)^0 \cdot (ki_2)^1 = (ki_1 + n_1 + ki_2)^0 = (ki_1)^1 \cdot (n_1 + ki_2)^0$

$= (ki_1)^1 \cdot ((n_1)^0 \cdot (ki_2)^1)$;

8) $((ki_1)^1 \cdot (n_1)^0) \cdot (a, b) = (ki_1 + n_1)^0 \cdot (a, b) = (ki_1 + n_1 + b - a)^0 = (ki_1)^1 \cdot (n_1 + b - a)^0$

$= (ki_1)^1 \cdot ((n_1)^0 \cdot (a, b))$;

9) $((ki_1)^1 \cdot (ki_2)^1) \cdot (n_1)^0 = (ki_1 + ki_2)^1 \cdot (n_1)^0 = (ki_1 + ki_2 + n_1)^0 = (ki_1)^1 \cdot (ki_2 + n_1)^0$

$= (ki_1)^1 \cdot ((ki_2)^1 \cdot (n_1)^0)$;
10) \((ki_1)^4 \cdot (a, b) \cdot (n_1)^0 = (-ki_1 + a, b) \cdot (n_1)^0 = (ki_1 + b - a + n_1)^0 \)
\[= (ki_1)^1 \cdot (b - a + n_1)^0 = (ki_1)^1 \cdot ((a, b) \cdot (n_1)^0)\];

11) \((a, b) \cdot (n_1)^0 \cdot (ki_1)^1 = (b - a + n_1)^0 = (ki_1 + b - a + n_1)^0 = (a, b) \cdot (n_1 + ki_1)^0 \)
\[= (a, b) \cdot (n_1)^0 \cdot (ki_1)^1\];

12) \((a, b) \cdot (ki_1)^1 \cdot (n_1)^0 = (b + ki_1 - a + n_1)^0 = (a, b) \cdot (ki_1 + n_1)^0 \)
\[= (a, b) \cdot ((ki_1)^1 \cdot (n_1)^0)\).

This completes the proof of the associativity of such defined binary operation on \(S_5\). Also, we observe that \(S_5\) with such defined semigroup operation is an inverse semigroup.

Let \(\tau_5\) be a topology on \(S_5\) which is generated by the family \(\tau_2 \cup \tau_3\) (see Examples 2 and 3). Also Examples 2 and 3 imply that it is sufficient to show that the semigroup operation in \(S_5\) is continuous in cases \((ki_1)^1 \cdot (n_0)^0\) and \((n_0)^0 \cdot (ki_1)^1\), where \((n_0)^0 \in G_0\) and \((ki_1)^1 \in G_1(k)\). Then for every positive integer \(p \geq \max \{|ki_1|, |n_0|\}\) we have that

\[U_{2p}((ki_1)^1) \cdot U_{2p}((n_0)^0) \subseteq U_p((ki_1 + n_0)^0)\]
and

\[U_{2p}((n_0)^0) \cdot U_{2p}((ki_1)^1) \subseteq U_p((ki_1 + n_0)^0)\].

This completes the proof that \((S_5, \tau_3)\) is a topological inverse semigroup. Obviously, \((S_5, \tau_3)\) is a locally compact space.

**Theorem 2.** Let \(T\) be a Hausdorff topological inverse semigroup. If \(T\) contains \(\mathcal{C}_Z\) as a dense subsemigroup and \(I = T \setminus \uparrow \mathcal{C}_Z \neq \emptyset\), then the following assertions hold:

(i) \(E(T)\) is a countable linearly ordered semilattice;

(ii) \(E(T) \cap (T \setminus \uparrow \mathcal{C}_Z)\) is a singleton set;

(iii) \(T \setminus \uparrow \mathcal{C}_Z\) is a subgroup in \(T\).

**Proof.** (i) By Proposition II.3 from [8] we have that \(\text{cl}_T(E(\mathcal{C}_Z)) = E(T)\) and since the closure of a linearly ordered semilattice in a topological semilattice is a linearly ordered subsemilattice too (see [12, Lemma 1]) we get that \(E(T)\) is a linearly ordered semilattice. Then the semilattice operation in \(E(T)\) implies that the sets \(E(T) \setminus \bigcup_{e \in \mathcal{C}_Z} \downarrow e\) and \(E(T) \setminus \bigcup_{e \in \mathcal{C}_Z} \uparrow e\) are either singleton or empty. This completes the proof of our assertion.

Assertion (ii) follows from assertion (i).

(iii) Since \(T\) is an inverse semigroup and \(\pi\) is a minimal idempotent in \(E(T)\) we conclude that the \(\mathcal{H}\)-class \(H_\pi\) which contains \(\pi\) coincides with the ideal \(I = T \setminus \uparrow \mathcal{C}_Z\). Indeed, if there exist \(x \in I\) and an \(\mathcal{H}\)-class \(H_x \subseteq I\) in \(T\) such that \(x \in H_x \neq H_\pi\), then since \(T\) is an inverse semigroup we have that there exists an idempotent \(e \in T\) such that either \(xx^{-1} = e \in \uparrow \mathcal{C}_Z\) or \(x^{-1}x = e \in \uparrow \mathcal{C}_Z\). If \(xx^{-1} = e \in \uparrow \mathcal{C}_Z\), then we have that \(x = xx^{-1}x = ex \in eT\), and since \(T\) is an inverse semigroup Theorem 1.17 from [7] implies \(e \in xT\), a contradiction. Similar arguments show that \(x^{-1}x \neq e \in \uparrow \mathcal{C}_Z\). Hence assertion (ii) implies that \(xx^{-1} = x^{-1}x = \pi\) and hence \(x \in H_x = H_\pi\). \(\square\)
The following theorem describes the structure of a closure of the semigroup $C_Z$ in a locally compact topological inverse semigroup $T$, i.e., it gives the description of the non-empty ideal $I = T \setminus \uparrow C_Z$ in the remainder of $C_Z$ in $T$.

**Theorem 3.** Let $T$ be a Hausdorff locally compact topological inverse semigroup. If $T$ contains $C_Z$ as a dense subsemigroup and $I = T \setminus \uparrow C_Z \neq \emptyset$, then the following assertions hold:

(i) $\downarrow e_n$ is a compact subsemilattice in $E(T)$ for every idempotent $e_n = (n, n) \in C_Z$, $n \in \mathbb{Z}$;

(ii) $T \setminus \uparrow C_Z$ is isomorphic to the discrete additive group of integers;

(iii) if $\tau$ is a unit of $T \setminus \uparrow C_Z$, then the map $\mathfrak{h}: C_Z \rightarrow T \setminus \uparrow C_Z$ which is defined by the formula $((a, b))\mathfrak{h} = (a, b) \cdot \tau$ is the natural homomorphism generated by the minimal group congruence $C_{mg}$ on the semigroup $C_Z$;

(iv) the subsemigroup $S = C_Z \cup I$ is topologically isomorphic to the topological inverse semigroup $(S_3, \tau_3)$ from Example 3.

**Proof.** (i) We show that $\downarrow e_0$ is a compact subset in $E(T)$ for $e_0 = (0, 0)$. By assertion (ii) of Theorem 2 we get that the set $E(T) \cap (T \setminus \uparrow C_Z)$ is singleton and we put $\{\tau\} = E(T) \cap (T \setminus \uparrow C_Z)$. Then $\tau$ is a smallest idempotent in $E(T)$. By Theorem 1.5 from [6, Vol. 1] we have that $E(T)$ is a closed subset in $T$, and hence by Theorem 3.3.9 from [9] we get that $E(T)$ is a locally compact space. Suppose the contrary: $\downarrow e_0$ is not a compact subset in $E(T)$. Since Corollary 3.1 implies that every element of the semigroup $C_Z$ is an isolated point in $T$ and hence so it is in $E(T)$, we get that there exists an open neighbourhood $U(\tau)$ of $\tau$ in $E(T)$ such that the set $\downarrow e_0 \setminus U(\tau)$ is an infinite discrete subspace of $E(T)$, $U(\tau) \subseteq E(T) \setminus \uparrow e_0$ and $cl_{E(T)}(U(\tau)) = U(\tau)$ is a compact subset of $E(T)$. Then for every positive integer $i$ there exists an integer $j > i$ such that $(j, j) \notin U(\tau)$ and $(j + 1, j + 1) \in U(\tau)$. Then the semigroup operation in $C_Z$ implies that by induction we can construct an infinite subset $M \subseteq \downarrow e_0 \setminus \{\tau\}$ of $E(T)$ such that $M \subseteq U(\tau) \setminus \{\tau\}$ and $M \cdot \{(0, 1)\} \cdot M \cdot \{(1, 0)\} \subseteq \downarrow e_0 \setminus U(\tau)$. Since the set $U(\tau)$ is compact and the set $M \subseteq U(\tau) \setminus \{\tau\}$ contains only isolated points from $E(C_Z)$, we conclude that $\tau \in cl(T)(M)$. Since $\downarrow e_0 \setminus U(\tau)$ is a closed subset of $E(T)$ we have that the continuity of the semigroup operation in $T$ and Proposition 1.4.1 from [9] imply that

$$\tau \in \{(0, 1)\} \cdot cl_T(M) \cdot \{(1, 0)\} \subseteq cl_T(M \cdot \{(0, 1)\} \cdot M \cdot \{(1, 0)\}) \subseteq \downarrow e_0 \setminus U(\tau),$$

which contradicts $\tau \in U(\tau)$. The obtained contradiction implies that the set $\downarrow e_0 \setminus U(\tau)$ is finite, and hence the set $\downarrow e_0$ is compact. Since for every integer $n$ the set $\downarrow e_n \setminus \downarrow e_0$ is either finite or empty and $e_n$ is an isolated point in $E(T)$ we conclude that $\downarrow e_n$ is a compact subsemilattice of $E(T)$.

(ii) By assertion (i) we have that $\tau$ is an accumulation point of the subsemigroup $C_N[0]$ in $T$. Since by Theorem 3.3.9 from [9] a closed subset of a locally compact space is a locally compact subspace too, and by Proposition 2.1(viii) the semigroup $C_N[0]$ is isomorphic to the bicyclic semigroup, Proposition V.3 from [8] implies that the subset $cl_T(C_N[0]) \setminus C_N[0]$
is a non-singleton subgroup of $T$. By Corollary 3.1 we get that $G_Z$ is an open discrete subsemigroup of $T$ and hence we get that $\text{cl}_T(G_N[0]) \setminus G_N[0] \subseteq \text{cl}_T(G_Z) \setminus G_Z$.

By assertion $(iii)$ of Theorem 2 we have that $I = T \setminus \uparrow G_Z$ is a non-singleton subgroup in $T$. Since $T$ is a topological inverse semigroup we get that $I$ is a topological group. Then by Proposition 3.2$(xi)$ we have that $I$ is a closed subset of $T$ and hence by Theorem 3.3.9 from [9] we get that $I$ is a locally compact topological group.

Later we show that $(a, b) \cdot \bar{e} = \bar{e} \cdot (a, b)$ for every $(a, b) \in G_Z$. Suppose the contrary: there exists $(a, b) \in G_Z$ such that $(a, b) \cdot \bar{e} \neq \bar{e} \cdot (a, b)$. Without loss of generality we can assume that $a \leq b$ in $\mathbb{Z}$. Then the Hausdorffness of the space $T$ implies that there exist open neighbourhoods $U((a, b) \cdot \bar{e})$ and $U(\bar{e} \cdot (a, b))$ of the points $(a, b) \cdot \bar{e}$ and $\bar{e} \cdot (a, b)$ in $T$ such that $U((a, b) \cdot \bar{e}) \cap U(\bar{e} \cdot (a, b)) = \emptyset$. Then the continuity of the semigroup operation of $T$ implies that there exists an open neighbourhood $V(\bar{e})$ of $\bar{e}$ in $T$ such that the following conditions hold:

$$\{(a, b) \cdot V(\bar{e}) \subseteq U((a, b) \cdot \bar{e}) \text{ and } V(\bar{e}) \cdot \{(a, b) \subseteq U(\bar{e} \cdot (a, b))\}.$$

By assertion $(i)$ we get that without loss of generality we can assume that $V(\bar{e}) \cap E(T)$ is a compact subset in $T$ and there exists a positive integer $n_0 > \max\{a, b\}$ such that $(n, n) \in V(\bar{e}) \cap E(T)$ for all integers $n \geq n_0$. Then for $n = 2n_0 - a$ and $k = 2n_0 - b$ we get that $(n, n), (k, k) \in V(\bar{e}) \cap E(T)$. But we have

$$(a, b) \cdot (n, n) = (a, b) \cdot (2n_0 - a, 2n_0 - a) = (2n_0 - a - b + a, 2n_0 - a) = (2n_0 - b, 2n_0 - a)$$

and

$$(k, k) \cdot (a, b) = (2n_0 - b, 2n_0 - b) \cdot (a, b) = (2n_0 - b - 2n_0 - b - a + b) = (2n_0 - b, 2n_0 - a),$$

which contradicts $U((a, b) \cdot \bar{e}) \cap U(\bar{e} \cdot (a, b)) = \emptyset$. The obtained contradiction implies that $(a, b) \cdot \bar{e} = \bar{e} \cdot (a, b)$ for every $(a, b) \in G_Z$.

Next we show that $x \cdot \bar{e} = \bar{e} \cdot x$ for every $x \in T \setminus G_Z$. Suppose contrary: there exists $x \in T \setminus G_Z$ such that $x \cdot \bar{e} \neq \bar{e} \cdot x$. Then the Hausdorffness of the space $T$ implies that there exist open neighbourhoods $U(x \cdot \bar{e})$ and $U(\bar{e} \cdot x)$ of the points $x \cdot \bar{e}$ and $\bar{e} \cdot x$ in $T$ such that $U(x \cdot \bar{e}) \cap U(\bar{e} \cdot x) = \emptyset$. The continuity of the semigroup operation of $T$ implies that there exists an open neighbourhood $V(x)$ of $x$ in $T$ such that the following conditions hold:

$$V(x) \cdot \{\bar{e}\} \subseteq U(x \cdot \bar{e}) \text{ and } \{\bar{e}\} \cdot V(x) \subseteq U(\bar{e} \cdot x).$$

Since $G_Z$ is a dense subsemigroup of $T$ we conclude that there exists $(a, b) \in G_Z$ such that $(a, b) \in V(x)$. Then we get that $(a, b) \cdot \bar{e} = \bar{e} \cdot (a, b)$, which contradicts $U(x \cdot \bar{e}) \cap U(\bar{e} \cdot x) = \emptyset$. The obtained contradiction implies that $x \cdot \bar{e} = \bar{e} \cdot x$ for every $x \in T$.

We define a map $\mathfrak{h} : T \rightarrow I$ by the formula $(x) \mathfrak{h} = x \cdot \bar{e}$. Since $x \cdot \bar{e} = \bar{e} \cdot x$ for every $x \in T$ we get that $\mathfrak{h}$ is a homomorphism. Since $G_Z$ is a dense subsemigroup of $T$, Proposition 2.2 and assertion $(iii)$ of Theorem 2 imply that the topological group $I$ contains a dense cyclic subgroup. Since $I$ is a locally compact topological group, Pontryagin-Weil Theorem (see [15, p. 71, Theorem 19]) implies that either $I$ is compact or $I$ is discrete. If $I$ is compact, then by Proposition 3.2$(viii)$ we get that

$$S = T \setminus \bigcup_{(a, b) \notin G_N[0]} \uparrow (a, b).$$
is a closed subset in $T$. Then by Theorem 3.3.9 from [9] $S$ is a locally compact space. Obviously, $S = \mathcal{C}_0[0] \cup I$. Since $I$ is a locally compact ideal in $T$, Proposition 2.1(viii) and Proposition II.4 from [8] imply that the Rees quotient semigroup $S/I$ with the quotient topology is locally compact topological inverse semigroup which is isomorphic to the bicyclic semigroup with an adjoined zero. This contradicts Proposition V.3 from [8]. The obtained contradiction implies that the group $I$ is discrete and hence $I$ is a discrete additive group of integers.

(iii) Let $(a, b), (c, d) \in \mathcal{C}_Z$ such that $(a, b)\mathcal{E}_{mg}(c, d)$. Then there exists an idempotent $(n, n) \in \mathcal{C}_Z$ such that $(a, b) \cdot (n, n) = (c, d) \cdot (n, n)$. Since $(i, i) \cdot \overline{e} = \overline{e}$ for every idempotent $(i, i) \in \mathcal{C}_Z$ we get that $((a, b)) \mathcal{H} = ((c, d)) \mathcal{H}$.

Let $(a, b), (c, d) \in \mathcal{C}_Z$ such that $((a, b)) \mathcal{H} = ((c, d)) \mathcal{H}$. Suppose the contrary: $(a, b) \cdot (n, n) \neq (c, d) \cdot (n, n)$ for any idempotent $(n, n) \in \mathcal{C}_Z$. If $(a, b) \cdot (n_1, n_1) = (c, d) \cdot (n_2, n_2)$ for some idempotents $(n_1, n_1), (n_2, n_2) \in \mathcal{C}_Z$, then we have that

$$(a, b) \cdot (n_1, n_1) \cdot (n_2, n_2) = (a, b) \cdot (n_1, n_1) \cdot (n_2, n_2) = (c, d) \cdot (n_2, n_2) \cdot (n_1, n_1) \cdot (n_2, n_2) = (c, d) \cdot (n_1, n_1) \cdot (n_2, n_2).$$

Therefore we get that $(a, b) \cdot (n_1, n_1) \neq (c, d) \cdot (n_2, n_2)$ for all idempotents $(n_1, n_1), (n_2, n_2) \in \mathcal{C}_Z$. Then Proposition 2.1(vi) implies that $b - a \neq d - c$, and hence by the proof of Proposition 2.2 we get that the congruence on the semigroup $\mathcal{C}_Z$ which is generated by the homomorphism $\mathcal{H}$ distincts from the minimal group congruence $\mathcal{E}_{mg}$ on $\mathcal{C}_Z$. Then the ideal $I$ is not isomorphic to the additive group of integers $\mathbb{Z}$ and hence by Proposition 2.2 we have that the ideal $I$ contains a finite cyclic group. This contradicts assertion (ii). The obtained contradiction implies our assertion.

(iv) Assertions (ii) and (iii) imply that the subsemigroup $S = \mathcal{C}_Z \cup I$ of $T$ is algebraically isomorphic to the inverse semigroup $S_3$ from Example 3. We identify the group $I$ with $G_0$ and put $\overline{e} = 0 \in G_0$.

By $\tau$ we denote the topology of the topological inverse semigroup $T$. Since $G_0$ is a discrete subgroup of $T$, assertion (i) implies that there exists a compact open neighbourhood $U(0)$ of $0$ in $T$ with the following property:

$$U(0) \subseteq E(T) \text{ and there is a positive integer } n_0 \text{ such that } n_0 = \max\{(n, n) \in E(\mathcal{C}_Z) \mid (n, n) \in U(0)\} \text{ and } (i, i) \in U(0) \text{ for all integers } i \geq n_0.$$ 

Hence, we get that $\mathcal{B}_\delta(0) = \{U_n(0) \mid n \in \mathbb{N}\}$ is a base of the topology of the space $T$ at the point $0 \in G_0 \subseteq T$, where $U_n(0) = \{0\} \cup \{(n + i, n + i) \mid i \in \mathbb{N}\}$.

We fix an arbitrary element $k \in G_0$. Without loss of generality we can assume that $k \geq 0$. Then $k^{-1} = -k \in \mathbb{Z} = G_0$. Since $G_0$ is a discrete subgroup of $T$, the continuity of the homomorphism $\mathcal{H}: T \to G_0: x \mapsto x \cdot \overline{e} = x \cdot 0$ implies that $(k)\mathcal{H}^{-1}$ is an open subset in $T$. We observe that, since the homomorphism $\mathcal{H}$ generates the minimal group congruence on $\mathcal{C}_Z$ (see assertion (iii)) we get that $(k)\mathcal{H}^{-1} \cap \mathcal{C}_Z = \{(a, b) \in \mathcal{C}_Z \mid b - a = k\}$. Also, since

$$\uparrow(a, b) = \{(x, y) \in \mathcal{C}_Z \mid (x, y) \cdot (b, b) = (a, b)\},$$

$$\uparrow(a, b) = \{(x, y) \in \mathcal{C}_Z \mid (x, y) \cdot (b, b) = (a, b)\},$$

$$\uparrow(a, b) = \{(x, y) \in \mathcal{C}_Z \mid (x, y) \cdot (b, b) = (a, b)\}.$$
for every \((a, b) \in \mathcal{C}_Z\), Proposition 3.2\((viii)\) implies that \(\uparrow(a, b)\) is a closed-and-open subset in \(T\) for every \((a, b) \in \mathcal{C}_Z\). Hence we get that \(\{k\} \cup \{(i, i + k) \in \mathcal{C}_Z \mid i = 1, 2, 3, \ldots\}\) is an open subset in \(T\).

We fix an arbitrary positive integer \(i\). Since \((i + k, i) \cdot k = 0 \in G_0\), the continuity of the semigroup operation in \(T\) implies that for every \(U_i(0) \in \mathcal{B}_3(0)\) there exists an open neighbourhood

\[V(k) \subseteq \{k\} \cup \{(i, i + k) \in \mathcal{C}_Z \mid i = 1, 2, 3, \ldots\}\]

of \(k\) in \(T\) such that \((i + k, i) \cdot V(k) \subseteq U_i(0)\). Then the semigroup operation of \(\mathcal{C}_Z\) implies that \(V(k) \subseteq U_i(k)\) for \(U_i(k) \in \mathcal{B}_3(k)\).

We observe that for every \(k \in G_0\) and for every positive integer \(i\) we have that

\[0 \cdot (i, k + i) = k \quad \text{and} \quad U_i(0) \cdot \{(i, i + k)\} = U_i(k),\]

where \(U_i(0) \in \mathcal{B}_3(0)\) and \(U_i(k) \in \mathcal{B}_3(k)\). Then the continuity of the semigroup operation in \(T\) implies that for every open neighbourhood \(W(k)\) of \(k\) in \(T\) there exists \(U_i(0) \in \mathcal{B}_3(0)\) such that

\[U_i(0) \cdot \{(i, i + k)\} = U_i(k) \subseteq W(k).\]

This implies that the bases of topologies \(\tau\) and \(\tau_3\) at the point \(k \in T\) coincide.

In the case when \(k < 0\) the proof is similar. This completes the proof of our assertion.

Theorem 3 implies the following:

**Corollary 4.1.** Let \(T\) be a Hausdorff locally compact topological inverse semigroup. If \(T\) contains \(\mathcal{C}_Z\) as a dense subsemigroup such that \(I = T \setminus \uparrow \mathcal{C}_Z \neq \emptyset\) and \(\uparrow \mathcal{C}_Z = \mathcal{C}_Z\), then \(T\) is topologically isomorphic to the topological inverse semigroup \((S_3, \tau_3)\) from Example 3.

**Theorem 4.** Let \((T, \tau)\) be a Hausdorff locally compact topological inverse monoid with unit \(1_T\). If \(\mathcal{C}_Z\) is a dense subsemigroup of \(T\) such that \(\uparrow \mathcal{C}_Z = T\) and the group of units of \(T\) is singleton, then there exists a decreasing sequence of negative integers \(\{m_i\}_{i \in \mathbb{N}}\) such that \((T, \tau)\) is topologically isomorphic to the semigroup \((S_1, \tau_1)\) from Example 1.

**Proof.** By the assumption of the theorem we have that \(T \setminus \mathcal{C}_Z = \{1_T\}\). Then Lemma 3.2\((i)\) implies that there exists a base \(\mathcal{B}(1_T)\) of the topology \(\tau\) at the unit \(1_T\) such that \(U(1_T) \subseteq E(\mathcal{C}_Z)\) for any \(U(1_T) \in \mathcal{B}(1_T)\). Also statements \((c)\) and \((d)\) of Theorem 1.7 from [6, Vol. 1] imply that we can assume that \((n, n) \in U(1_T)\) if and only if \(n\) is a negative integer. Since by Corollary 3.1 every element of the semigroup \(\mathcal{C}_Z\) is an isolated point of \(T\), without loss of generality we can assume that all elements of the base \(\mathcal{B}(1_T)\) are closed-and-open subsets of \(T\). Also, the local compactness of \(T\) implies that without loss of generality we can assume that the base \(\mathcal{B}(1_T)\) consists of compact subsets, and Corollary 3.3.6 from [9] implies that the base \(\mathcal{B}(1_T)\) is countable.

We suppose that \(\mathcal{B}(1_T) = \{U_n(1_T) \mid n = 1, 2, 3, \ldots\}\). We put

\[W_1(1_T) = U_1(1_T) \quad \text{and} \quad W_i(1_T) = W_{i-1}(1_T) \cap U_i(1_T),\]

for all \(i = 2, 3, 4, \ldots\). We observe that \(\mathcal{B}(1_T) = \{W_n(1_T) \mid n = 1, 2, 3, \ldots\}\) is a base of the topology \(\tau\) at the unit \(1_T\) of \(T\) such that \(W_{n+1}(1_T) \subseteq W_n(1_T)\) for every positive integer.
n. Then the compactness of $U_i(1_T)$, $i = 1, 2, 3, \ldots$, and the discreteness of the space $C_Z$ imply that the family $\mathcal{B}(1_T)$ consists of compact-and-open subsets of $T$. Let $\{m_i\}_{i \in \mathbb{N}}$ be a decreasing sequence of negative integers such that $\bigcup_{i=1}^{\infty} \{(m_i, m_i)\} = W(1_T) \setminus \{1_T\}$. We put $V_n = \{1_T\} \cup \{(m_i, m_i) \in C_Z \mid i \geq n\}$ for every positive integer $n$. Since every element of the family $\mathcal{B}(1_T)$ is a compact subset of $T$, Corollary 3.1 implies that the family

$$\mathcal{B}(1_T) = \{V_n \mid n = 1, 2, 3, \ldots\}$$

is a base of the topology $\tau$ at $1_T$ of $T$ and this completes the proof of our theorem.

Theorems 3 and 4 imply the following:

**Corollary 4.2.** Let $(T, \tau)$ be a Hausdorff locally compact topological inverse semigroup. If $C_Z$ is a dense subsemigroup of $T$ such that the group of units of $T$ is singleton, then there exists a decreasing sequence of negative integers $\{m_i\}_{i \in \mathbb{N}}$ such that $(T, \tau)$ is topologically isomorphic either to the semigroup $(S_1, \tau_1)$ from Example 1 or to the semigroup $(S_4, \tau_4)$ from Example 4.

**Theorem 5.** Let $(T, \tau)$ be a Hausdorff locally compact topological inverse monoid with unit $1_T$. Suppose that $C_Z$ is a dense subsemigroup of $T$ such that the following conditions hold:

(i) $\uparrow C_Z = T$;

(ii) the group of units $H(1_T)$ of $T$ is non-singleton; and

(iii) there exists an integer $j$ such that $K = \{1_T\} \cup \{(i, i) \in C_Z \mid i \geq j\}$ is a compact subset of $T$.

Then there exists a decreasing sequence of negative integers $\{m_i\}_{i \in \mathbb{N}}$ such that $m_{i+1} = m_i - 1$ for every positive integer $i$ and $(T, \tau)$ is topologically isomorphic to the semigroup $(S_2, \tau_2)$ for $n = 1$ from Example 2.

**Proof.** As in the proof of Theorem 4 we construct a decreasing sequence of negative integers $\{m_i\}_{i \in \mathbb{N}}$ such that the family

$$\mathcal{B}(1_T) = \{U_i(1_T) \mid i = 1, 2, 3, \ldots\}$$

determines a base of the topology $\tau$ at the point $1_T$ of $T$, where

$$U_i(1_T) = \{1_T\} \cup \{(m_i, m_i) \in C_Z \mid i \geq j\}.$$

The compactness of the set $K$ implies that we can construct a sequence of negative integers $\{m_i\}_{i \in \mathbb{N}}$ such that $m_{i+1} = m_i - 1$ for every positive integer $i$.

Then for every element $x$ of the group of units $H(1_T)$ left and right translations $\lambda_x : T \to T : s \mapsto x \cdot s$ and $\rho_x : T \to T : s \mapsto s \cdot x$ are homeomorphisms of the topological space $T$ (see [6, Vol. 1, P. 19]), and hence the following families

$$\mathcal{B}_i(x) = \{x \cdot U_i(1_T) \mid U_i(1_T) \in \mathcal{B}(1_T)\}$$
and

\[ B_r(x) = \{ U_i(1_T) \cdot x \mid U_i(1_T) \in B(1_T) \} \]

are bases of the topology \( \tau \) at the point \( 1_T \) of \( T \). Also, we observe that the family

\[ B(x) = \{ U \cap V \mid U \in B_t(x) \text{ and } V \in B_r(x) \} \]

is a base of the topology \( \tau \) at the point \( 1_T \) of \( T \).

Then Lemma 3.2 and Proposition 3.2 imply that the group of units \( H(1_T) \) of \( T \) is topologically isomorphic to the discrete additive group of integers \( \mathbb{Z}_+ \). Let \( g \) be a generator of \( \mathbb{Z}_+ \). Then by Lemma 3.2(iii) there exist an open neighbourhood \( U(g) \) of the point \( g \) in \( T \) and an integer \( k \) such that \( a - b = k \) for all \((a, b) \in U(g) \cap \mathcal{C}_\mathbb{Z}\). Without loss of generality we can assume that \( g \) is a positive integer and \( k < 0 \). Then we have that

\[ g \cdot U_i(1_T) = \{ (m_i + k, m_i) \mid (m_i, m_i) \in U_i(1_T) \} \cup \{ g \} \quad (3) \]

and

\[ U_i(1_T) \cdot g = \{ (m_i, m_i - k) \mid (m_i, m_i) \in U_i(1_T) \} \cup \{ g \} \quad (4) \]

We shall show that equality (4) holds. Let be \((m_i, m_i) \in U_i(1_T)\). Then we get

\[ ((m_i, m_i) \cdot g) \cdot ((m_i, m_i) \cdot g)^{-1} = (m_i, m_i) \cdot g \cdot g^{-1} \cdot (m_i, m_i)^{-1} = (m_i, m_i) \cdot 1_T \cdot (m_i, m_i) = (m_i, m_i). \]

Since \((m_i, m_i) \cdot g \in \mathcal{C}_\mathbb{Z}\) and \( \mathcal{C}_\mathbb{Z}\) is an inverse semigroup we conclude that \((m_i, m_i) \cdot g = (m_i, a)\) for some integer \( a \), and by Lemma 3.2(vi) we have that \((m_i, m_i) \cdot g = (m_i, m_i - k)\). This completes the proof of equality (4). The proof of equality (3) is similar. Then Lemma 3.2(vi), equalities (3) and (4) imply that \( T \) is topologically isomorphic to the semigroup \((S_2, \tau_2)\) for \( n = 1 \) from Example 2. This completes the proof of the theorem. \( \square \)

Theorems 3 and 5 imply the following:

**Corollary 4.3.** Let \((T, \tau)\) be a Hausdorff locally compact topological inverse monoid with unit \( 1_T \). Suppose that \( \mathcal{C}_\mathbb{Z}\) is a dense subsemigroup of \( T \) such that the following conditions hold:

(i) the group of units \( H(1_T) \) of \( T \) is non-singleton; and

(ii) there exists an integer \( j \) such that \( K = \{ 1_T \} \cup \{(i, i) \in \mathcal{C}_\mathbb{Z} \mid i \geq j \} \) is a compact subset of \( T \).

Then there exists a decreasing sequence of negative integers \( \{ m_i \}_{i \in \mathbb{N}} \) such that \( m_{i+1} = m_i - 1 \) for every positive integer \( i \) and \((T, \tau)\) is topologically isomorphic either to the semigroup \((S_2, \tau_2)\) from Example 2 or to the semigroup \((S_5, \tau_5)\) from Example 5.
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Received 19.10.2011
Фігель І.Р., Гутик О.В. Про замикання розширеної біцикличної напівгрупи // Карпатські математичні публікації. — 2011. — Т.3, №2. — С. 131–157.

У статті вивчається напівгрупа \( \mathbb{C}_2 \), яка є узагальненням біцикличної напівгрупи. Описано основні алгебраїчні властивості напівгрупи \( \mathbb{C}_2 \), зокрема доведено, що кожна нетривіальна конгруенція \( \mathcal{C} \) на напівгрупі \( \mathbb{C}_2 \) є груповою, і більше того, фактор-напівгрупа \( \mathbb{C}_2/\mathcal{C} \) ізоморфна цикличній групі. Показано, що на напівгрупі \( \mathbb{C}_2 \) не існує відмінних від дискретної хаусдорфових топологій \( \tau \) таких, що \( (\mathbb{C}_2, \tau) \) — напівтопологічна напівгрупа. Також вивчається замикання напівгрупи \( \mathbb{C}_2 \) у топологічній інверсній напівгрупі \( T \). Показано, що непорожній наріст напівгрупи \( \mathbb{C}_2 \) у напівгрупі \( T \) складається з групи одиниць \( H(1_T) \) напівгрупи \( T \) та двобічного ідеалу \( I \) в \( T \), якщо \( H(1_T) \neq \emptyset \) та \( I \neq \emptyset \). У випадку, коли \( T \) є локально компактною топологічною інверсною напівгрупою, показано, що на напівгрупі \( \mathbb{C}_2 \) не існує відмінних від дискретної топології таких, що \( (\mathbb{C}_2, \tau) \) — хаусдорфова полутопологічна полугрупа. Також вивчається замикання полугрупи \( \mathbb{C}_2 \) в топологічній інверсній полугрупі \( T \). Показано, що непустий наріст полугрупи \( \mathbb{C}_2 \) в полугрупі \( T \) состоит із групи одиниць \( H(1_T) \) полугрупи \( T \) і идеалу \( I \) в \( T \), коли \( H(1_T) \neq \emptyset \) та \( I \neq \emptyset \). У випадку, коли \( T \) є локально компактною топологічною інверсною полугрупою, ідеаль \( I \) в \( T \), доказано, що идеал \( I \) топологічно ізоморфний дискретній аддитивній групі цілих чисел, та описано топологію на піднапівгрупі \( \mathbb{C}_2 \cup I \). Також показано, що група одиниць \( H(1_T) \) в \( T \) є одноточковою множиною, або група \( H(1_T) \) топологічно ізоморфна дискретній аддитивній групі цілих чисел.

Фігель І.Р., Гутик О.В. О замыкании расширенной бициклической полугруппы // Карпатские математические публикации. — 2011. — Т.3, №2. — С. 131–157.

В работе изучается полугруппа \( \mathbb{C}_2 \), которая является обобщением бициклической полугруппы. Описаны основные алгебраические свойства полугруппы \( \mathbb{C}_2 \), в частности доказано, что каждая нетривиальная конгруэнция \( \mathcal{C} \) на \( \mathbb{C}_2 \) является групповой, и более того, фактор-полугруппа \( \mathbb{C}_2/\mathcal{C} \) изоморфна циклической группе. Показано, что на полугруппе \( \mathbb{C}_2 \) не существует отличных от дискретной топологий \( \tau \) таких, что \( (\mathbb{C}_2, \tau) \) — хаусдорфова полутопологическая полугруппа. Также изучается замыкание полугруппы \( \mathbb{C}_2 \) в топологической инверсной полугруппе \( T \). Показано, что непустой наріст полугрупи \( \mathbb{C}_2 \) в полугрупі \( T \) состоит із групи одиниць \( H(1_T) \) полугрупи \( T \) и идеалу \( I \) в \( T \), коли \( H(1_T) \neq \emptyset \) та \( I \neq \emptyset \). У випадку, коли \( T \) є локально компактною топологічною инверсною полугрупою, ідеал \( I \) в \( T \), доказано, що идеал \( I \) топологічно ізоморфний дискретній аддитивній групі цілих чисел, та описано топологію на піднапівгрупі \( \mathbb{C}_2 \cup I \). Також показано, що група одиниць \( H(1_T) \) в \( T \) є одноточковою множиною, або група \( H(1_T) \) топологічно ізоморфна дискретній аддитивній групі цілих чисел.