On Stancu type Szász-Mirakyan-Durrmeyer Operators Preserving $e^{2ax}, a > 0$

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Abstract

The present paper deals with the Szász-Mirakyan-Durrmeyer-Stancu operators preserving $e^{2ax}$ for $a > 0$. The uniform convergence of the constructed operators is mentioned in this paper. The rate of convergence is examined by employing two different modulus of continuities. After that, a Voronovskaya-type theorem is investigated for quantitative asymptotic estimation. Finally, a comparison is made theoretically to show that the new constructed operators perform well.

1. INTRODUCTION

In 1985, Mazhar and Totik [1] defined Durrmeyer-type generalization of the Szász-Mirakyan operators. In 2017, Acar et al. [2] introduced a modification of the Szász-Mirakyan operators preserving constants and $e^{2ax}$, $a > 0$. Then Deniz et al. [3] investigated the Szász-Mirakyan-Durrmeyer operators reproducing $e^{2ax}$ for $a > 0$. For $0 \leq \alpha \leq \beta$ and $m > 0$ Stancu type Szász-Mirakyan-Durrmeyer operators are given by Gupta et al. [4]

$$S_{m,r}^{(\alpha,\beta)}(f;x) = m \sum_{k=0}^{\infty} e^{-m\gamma(k)} \left[ \frac{\gamma(k)}{k!} \right] f\left( \frac{mt}{m+\beta} \right) dt, \quad (1)$$

We consider the generalized form of the Szász-Mirakyan-Durrmeyer-Stancu operators

$$S_{m,r}^{\alpha,\beta,\gamma}(f;x) = m \sum_{k=0}^{\infty} e^{-m\gamma(k)} \left[ \frac{\gamma(k)}{k!} \right] f\left( \frac{mt}{m+\beta} \right) dt, \quad (2)$$

where $0 \leq \alpha \leq \beta$, $x \geq 0$ and $m > 0$. For notational convenience, we briefly denote the operators $S_{m,r}^{\alpha,\beta,\gamma}$ as $S_{m,r}^{\gamma}$. In this paper, we study the Szász-Mirakyan-Durrmeyer-Stancu operators preserving $e^{2ax}$ for $a > 0$.

In this situation, the function $\theta(x)$ which satisfies $S_{m,r}^{\theta}(e^{2at};x) = e^{2ax}$ is obtained as follows:

$$e^{2ax} = m \sum_{k=0}^{\infty} e^{-m\gamma(k)} \left[ \frac{\gamma(k)}{k!} \right] f\left( \frac{mt}{m+\beta} \right) dt$$

$$= \left( \frac{m+\beta}{m+\beta-2a} \right)^{r+1} e^{\frac{2a(t+\theta)}{e^{mt}}(\frac{m(m+\beta)}{(m+\beta-2a)}-t)}, \quad m + \beta > 2a.$$
By simple computations, we have
\[
\theta(x) = \frac{m+\beta-2a}{2am} \left( \frac{2a((m+\beta)x-a)}{m+\beta} + (r+1)\ln \left( \frac{m+\beta-2a}{m+\beta} \right) \right), \quad m + \beta > 2a. \tag{3}
\]

The aim of the current paper is to investigate the approximation properties of the Stancu type Szász-Mirakyan-Durrmeyer operators preserving \( e^{2ax} \), \( a > 0 \) defined by (2), with \( \theta(x) \) given in (3). By taking \( \theta(x) = x \) and \( \alpha = \beta = r = 0 \), we obtain the Szász-Mirakyan-Durrmeyer operators \([1]\). Some recent papers are Szász-Mirakyan type operators which fix exponential \([5]\), Szász-Mirakyan operators which preserve exponential functions \([6]\), Baskakov-Szász-Stancu operators which preserve exponential functions \([7]\), Baskakov-Szász-Mirakyan-type operators preserving exponential type functions \([8]\) and Szász-Mirakyan-Kantorovich operators which preserve \( e^{-x} \) \([9]\).

2. SOME AUXILIARY RESULTS

Here, for \( 0 \leq \alpha \leq \beta \) and \( m + \beta > 2a \), we present three lemmas which are necessarily used in the proof of the theorems.

**Lemma 1.** Let \( f(t) = e^{-At} \). Then for the Szász-Mirakyan-Durrmeyer-Stancu operators we have
\[
S_{m,r}^\theta(e^{-At};x) = \left( 1 - \frac{A}{m+\beta+A} \right)^{r+1} e^{-A \left( \frac{m\theta(x)}{m+\beta+A} - \frac{a}{m+\beta} \right)}. \tag{4}
\]
Here, \( \theta(x) \) is given by (3).

**Lemma 2.** Let \( e_k(t) = t^k, k = 0,1,2,3,4 \). Then we have the next equalities:
\[
\begin{align*}
S_{m,r}^\theta(e_0;x) &= 1, \\
S_{m,r}^\theta(e_1;x) &= \frac{m}{m+\beta} \theta(x) + \frac{r+\alpha+1}{m+\beta}, \\
S_{m,r}^\theta(e_2;x) &= \frac{m^2}{(m+\beta)^2} \theta^2(x) + \frac{m(2r+2\alpha+4)}{(m+\beta)^2} \theta(x) + \frac{r^2+3(2\alpha+2)r+\alpha^2+2+2\alpha+2}{(m+\beta)^2} \theta^2(x), \\
S_{m,r}^\theta(e_3;x) &= \frac{m^3}{(m+\beta)^3} \theta^3(x) + \frac{m(3r+3\alpha+6)}{(m+\beta)^3} \theta^2(x) + \frac{(3r^2+12\alpha+18\alpha+12\alpha+3\alpha^2)\beta}{(m+\beta)^3} \theta^2(x), \\
S_{m,r}^\theta(e_4;x) &= \frac{m^4}{(m+\beta)^4} \theta^4(x) + \frac{m(4r+4\alpha+6)}{(m+\beta)^4} \theta^3(x) + \frac{(6r^2+42+12\alpha)r+37+36\alpha+6\alpha^2)}{(m+\beta)^4} \theta^2(x) + \frac{r^3(10+4\alpha)r^3(35+24\alpha+6\alpha^2)r^2+(50+44\alpha+18\alpha+4\alpha^2)r+24+24\alpha+12\alpha+4\alpha^2+4\alpha^2+4\alpha^2+4\alpha^2)}{(m+\beta)^4} \theta(x).
\end{align*}
\]

**Lemma 3.** For \( k = 0,1,2,4 \), we briefly denote \( \phi_k^\theta(t) = (t-x)^k \). Then for the central moments we get the equalities as follows:
\[
\begin{align*}
S_{m,r}^\theta(\phi_0^0;x) &= 1, \\
S_{m,r}^\theta(\phi_1^1;x) &= \frac{m}{m+\beta} \theta(x) - \frac{r+\alpha+1}{m+\beta} x, \\
S_{m,r}^\theta(\phi_2^2;x) &= \frac{m^2}{(m+\beta)^2} \theta^2(x) + \frac{m(2r+2\alpha+4)}{(m+\beta)^2} \theta(x) + \frac{r^2+3(2\alpha+2)r+\alpha^2+2+2\alpha+2}{(m+\beta)^2} \theta^2(x), \\
S_{m,r}^\theta(\phi_4^4;x) &= \frac{m^4}{(m+\beta)^4} \theta^4(x) + \frac{m(4r+4\alpha+6)}{(m+\beta)^4} \theta^3(x) + \frac{(6r^2+42+12\alpha)r+37+36\alpha+6\alpha^2)}{(m+\beta)^4} \theta^2(x) + \frac{r^3(10+4\alpha)r^3(35+24\alpha+6\alpha^2)r^2+(50+44\alpha+18\alpha+4\alpha^2)r+24+24\alpha+12\alpha+4\alpha^2+4\alpha^2+4\alpha^2+4\alpha^2+4\alpha^2)}{(m+\beta)^4} \theta(x).
\end{align*}
\]
\[-4x \left( \frac{m^3}{(m+\beta)^3} \theta^3(x) + \frac{(3\alpha+3\beta)m^2}{(m+\beta)^3} \theta^2(x) + \frac{(3\alpha^2+15\alpha+18+6\alpha\beta+12\alpha+3\beta^2)m}{(m+\beta)^3} \theta(x) \right) + \frac{r^3+6r^2+11r+6+3\beta(r^2+3r+2)+3\alpha^2(r+1)\beta}{(m+\beta)^3} \theta^3(x) + \frac{m(2r+2\alpha+4)}{(m+\beta)^2} \theta^2(x) + \frac{r^2+(3+2\alpha)r+\alpha^2+2\alpha+2}{(m+\beta)^2} \theta(x) - 4x^3 \left( \frac{m}{m+\beta} \theta(x) + \frac{r+\alpha+1}{m+\beta} x^2 \right) + x^4.\]

**Proof.** By using the linearity of the $S_{m,r}^0$ operators and Lemma 2, we obtain

\[
S_{m,r}^0(\phi_1^0; x) = S_{m,r}^0(e_0; x), \\
S_{m,r}^0(\phi_1^1; x) = S_{m,r}^0(e_1; x) - xS_{m,r}^0(e_0; x), \\
S_{m,r}^0(\phi_2^0; x) = S_{m,r}^0(e_2; x) - 2xS_{m,r}^0(e_1; x) + x^2S_{m,r}^0(e_0; x), \\
S_{m,r}^0(\phi_4^0; x) = S_{m,r}^0(e_4; x) - 4xS_{m,r}^0(e_2; x) + 6x^2S_{m,r}^0(e_1; x) - 4x^3S_{m,r}^0(e_0; x) + x^4S_{m,r}^0(e_0; x).
\]

**Remark 4.** Taking into consideration the definition of $\theta(x)$, we get the following limit results for each $x \in [0, \infty)$, $m + \beta > 2\alpha$ and $0 \leq \alpha \leq \beta$

\[
\lim_{m \to \infty} mS_{m,r}^0(\phi_1^1; x) = -2ax
\]

and

\[
\lim_{m \to \infty} mS_{m,r}^0(\phi_2^0; x) = 2x.
\]

**3. RESULTS**

Let the subspace of all continuous and real-valued functions on the interval $[0, \infty)$ is denoted by $C' [0, \infty)$ with the condition that $\lim_{x \to \infty} f(x)$ exists and also is finite, equipped with the uniform norm. In 1970, Boyanov and Veselinov [10] demonstrated the uniform convergence of a sequence of linear positive operators. For the new constructed operators (2) with $\theta(x)$ as shown in (3), we present the next theorem according to [10].

**Theorem 5.** If the Stancu-type Szász-Mirakyan-Durrmeyer operators (2) satisfy

\[
\lim_{m \to \infty} S_{m,r}^0(e^{-kt}; x) = e^{-kt}, k = 0, 1, 2.
\]

uniformly in $[0, \infty)$, then for each $f \in C' [0, \infty)$

\[
\lim_{m \to \infty} S_{m,r}^0(f; x) = f(x)
\]

uniformly in $[0, \infty)$.

**Proof.** As is already known that $\lim_{m \to \infty} S_{m,r}^0(1; x) = 1$. Taking into consideration the equality (4) with $\theta(x)$ given in (3), we write

\[
S_{m,r}^0(e^{-t}; x) = e^{-x} + \frac{(1+2a)xe^{-x}}{m} + O(m^{-2})
\]

and

\[
S_{m,r}^0(e^{-2t}; x) = e^{-2x} + \frac{4(1+a)xe^{-2x}}{m} + O(m^{-2}).
\]
Thus, we prove that
\[
\lim_{m \to \infty} S_m^\theta(e^{-kt}; x) = e^{-kx}, k = 0,1,2.
\]
uniformly in the interval \([0, \infty)\). This proof guarantees that \(\lim_{m \to \infty} S_m^\theta(f; x) = f(x)\) uniformly in the interval \([0, \infty)\) for any \(f \in C'[0, \infty)\).

After Boyanov and Veselinov [10], in 2010 Holhoş, [11] examined the uniform convergence of a sequence of linear positive operators. For a beneficial estimation of the positive and linear operators, the following theorem is presented.

**Theorem 6.** [11] For a sequence of positive and linear operators \(A_m: C^* [0, \infty) \to C^*[0, \infty)\), we get
\[
\|A_m(f; x) - f(x)\|_{[0, \infty)} \leq \|f\|_{[0, \infty)} \delta_m + (2 + \delta_m) \omega^*(f, \sqrt{\delta_m + 2\sigma_m + \rho_m})
\]
for each function \(f \in C^*[0, \infty)\), where
\[
\|A_m(e^0, x) - 1\|_{[0, \infty)} = \delta_m,
\]
\[
\|A_m(e^{-t}, x) - e^{-x}\|_{[0, \infty)} = \sigma_m,
\]
\[
\|A_m(e^{-2t}, x) - e^{-2x}\|_{[0, \infty)} = \rho_m
\]
and the modulus of continuity is denoted by \(\omega^*(f, \eta) = \sup_{|x|,|t| \leq \eta} |f(t) - f(x)|\). In these equalities, \(\delta_m, \sigma_m\) and \(\rho_m\) tend to zero as \(m \to \infty\).

Accordingly, we provide a quantitative estimation of the Szasz-Mirakyan-Durrmeyer-Stancu operators reproducing \(e^{2ax}\) for \(a > 0\) as can be seen:

**Theorem 7.** For \(f \in C^*[0, \infty)\), we get the following inequality
\[
\|S_m^\theta(f) - f\|_{[0, \infty)} \leq 2\omega^*(f, \sqrt{2\sigma_m + \rho_m}),
\]
(11)
where
\[
\|S_m^\theta(e^{-t}, x) - e^{-x}\|_{[0, \infty)} = \sigma_m,
\]
\[
\|S_m^\theta(e^{-2t}, x) - e^{-2x}\|_{[0, \infty)} = \rho_m.
\]
In these equalities, \(\sigma_m\) and \(\rho_m\) tend to zero as \(m \to \infty\). So, \(S_m^\theta f\) converges \(f\) uniformly.

**Proof.** The Szasz-Mirakyan-Durrmeyer-Stancu operators \(S_m^\theta\) preserve constants. So, \(\delta_m = 0\). One can write as
\[
\frac{k-n}{lnk-lnm} < \frac{k+n}{2}
\]
(12)
for \(0 < n < k\). By choosing \(k = e^{-km}\) and \(n = e^{-x}\), we get
\[
e^{-kmx} - e^{-x} < \frac{1-km}{2} (xe^{-km} + xe^{-x}).
\]
Then let us notice that
\[
\max_{x > 0} xe^{-sx} = \frac{1}{es}
\]
for each \( s > 0 \). Therefore, we have
\[
e^{-k_m x} - e^{-x} < \frac{1-k_m}{2} \left( \frac{1}{ek_m} + \frac{1}{e} \right) < \frac{1-k_m}{2ek_m}
\]
In addition, by simple computations, we acquire
\[
S^R_{m,r}(e^{-t}, x) = \left(1 - \frac{1}{m+\beta+1}\right)^{r+1} e^{\frac{m\theta(x) + \alpha}{m+\beta+1}}
\]
\[
= e^{\frac{-\alpha}{m+\beta+1}(1-\frac{m+\beta-2a}{m+\beta+1})} \left(1 - \frac{1}{m+\beta+1}\right)^{r+1} \left(1 + \frac{2a}{m+\beta-2a}\right)^{\frac{(r+1)(m+\beta-2a)}{2a(m+\beta+1)}} e^{\frac{-2a}{m+\beta+2}x}.
\]
Thus, we arrive at
\[
\sigma_m = \|S^R_{m,r}(e^{-t}, x) - e^{-x}\|_{0,\infty} = \|K_m e^{-k_m x} - e^{-x}\|_{0,\infty}
\]
\[
= \|K_m (e^{-k_m x} - e^{-x}) + e^{-x}(K_m - 1)\|_{0,\infty}
\]
\[
< K_m \left( \frac{1-k_m}{2ek_m} \right) + K_m - 1 \to 0
\]
as \( m \to \infty \). Here \( K_m = \frac{m+\beta-2a}{m+\beta+1} \) and
\[
K_m = e^{\frac{-\alpha}{m+\beta+1}(1-\frac{m+\beta-2a}{m+\beta+1})} \left(1 - \frac{1}{m+\beta+1}\right)^{r+1} \left(1 + \frac{2a}{m+\beta-2a}\right)^{\frac{(r+1)(m+\beta-2a)}{2a(m+\beta+1)}} e^{\frac{-2a}{m+\beta+2}x}.
\]
In the same manner, if we choose \( k = e^{-n_m x} \), \( n = e^{-2x} \) in (12) and use (13), we obtain
\[
e^{-n_m x} - e^{-2x} < \frac{2-n_m}{2} xe^{-x} + xe^{-2x} < \frac{2-n_m}{2} \left( \frac{1}{en_m} + \frac{1}{2e} \right) < \frac{4-n_m^2}{4en_m}.
\]
On the other hand,
\[
S^R_{m,'}(e^{-2t}, x) = \left(1 - \frac{2}{m+\beta+2}\right)^{r+1} e^{-\frac{2m\theta(x) - \alpha}{m+\beta+2}}
\]
\[
= e^{\frac{-2\alpha}{m+\beta+2}(1-\frac{m+\beta-2a}{m+\beta+2})} \left(1 - \frac{2}{m+\beta+2}\right)^{r+1} \left(1 + \frac{2a}{m+\beta-2a}\right)^{\frac{(r+1)(m+\beta-2a)}{a(m+\beta+2)}} e^{\frac{2(m+\beta-2a)x}{m+\beta+2}x}
\]
Thus, we find
\[
\rho_m = \|S^R_{m,'}(e^{-2t}, x) - e^{-2x}\|_{0,\infty} = \|M_m e^{-n_m x} - e^{-x}\|_{0,\infty}
\]
\[
= \|M_m (e^{-n_m x} - e^{-x}) + e^{-x}(M_m - 1)\|_{0,\infty}
\]
\[
< M_m \left( \frac{4-n_m^2}{4en_m} \right) + M_m - 1 \to 0,
\]
as \( m \to \infty \). Here \( n_m = \frac{2(m+\beta-2a)}{m+\beta+2} \) and
\[
M_m = \frac{-2a}{m+\beta}(1 + \frac{m+\beta-2a}{m+\beta+2}) (1 + \frac{2a}{m+\beta+2})^{\frac{r+1}{a}} \left( 1 + \frac{2a}{m+\beta-2a} \right). 
\]

As a consequence, \( \sigma_m \) and \( \rho_m \) tend to zero as \( m \to \infty \).

Section 4 investigates the rate of convergence with the help of the modulus of continuity.

4. THE MODULUS OF CONTINUITY

With the norm \( ||f||_{C_B} = \sup_{x \in [0, \infty)} |f(x)| \), \( C_B[0, \infty) \) denotes the class of all uniform continuous and bounded functions \( f \) on \( [0, \infty) \). For \( f \in C_B[0, \infty) \),

\[
\omega(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x, x+h \in [0, \infty)} |f(x+h) - f(x)|
\]

presents the modulus of continuity.

\[
\omega_2(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x, x+h, x+2h \in [0, \infty)} |f(x+2h) - 2f(x+h) + f(x)|
\]

defines the second order modulus of continuity of the function \( f \in C_B[0, \infty) \) for \( \delta > 0 \). Peetre’s K-functional are given by

\[
K_2(f, \delta) = \inf_{g \in C_B[0, \infty)} \{ ||f - g||_{C_B[0, \infty)} + \delta ||g||_{C_B[0, \infty)} \}
\]

Here, \( C_B[0, \infty) \) describes the space of the functions, where \( f, f' \) and \( f'' \) belong to \( C_B[0, \infty) \). The relationship between Peetre’s K-functional and second order modulus of continuity is defined by [12],

\[
K_2(f, \delta) \leq M \omega_2(f, \sqrt{\delta})
\]

for \( M > 0 \).

**Lemma 8.** For \( f \in C_B[0, \infty) \), we obtain \( |S^\theta_{m,r}(f; x)| \leq ||f|| \).

**Theorem 9.** For \( f \in C_B[0, \infty) \) and for all \( x \in [0, \infty) \), there exists a constant \( M > 0 \), such that

\[
|S^\theta_{m,r}(f; x) - f(x)| \leq M \omega_2(f, \sqrt{\mu_m}) + \omega \left( f, \frac{m\theta(x)+r+\alpha+1}{m+\beta} - x \right).
\]

where

\[
\mu_m = \frac{2r^2}{m+\beta} \theta^2(x) + 2m \left( \frac{2r+2\alpha+3}{(m+\beta)^2} - \frac{2x}{m+\beta} \right) \theta(x) + 2x^2 \left( \frac{4x(r+\alpha+1)}{m+\beta} + \frac{2r^2+(4\alpha+5)r+2\alpha^2+4\alpha+3}{(m+\beta)^2} \right).
\]

Here, \( \theta(x) \) is as shown in (3).

**Proof.** We define \( S^\theta_{m,r}: C_B[0, \infty) \to C_B[0, \infty) \) auxiliary operators as follows

\[
S^\theta_{m,r}(g; x) = S^\theta_{m,r}(g; x) + g(x) - g \left( \frac{m\theta(x)+r+\alpha+1}{m+\beta} \right).
\]

where Eqn. (3) gives \( \theta(x) \). It is important to notice that the operators given by (16) are linear and positive. From the Taylor expansion, we have for \( g \in C_B[0, \infty) \)
g(t) = g(x) + (t - x)g'(x) + \int_{x}^{t} (t-u)g''(u)du, \quad x, t \in [0, \infty). \quad (17)

When $\hat{S}_{m,r}^{\theta}$ operators are applied to the equation (17) and then Lemma 3 is used, we get

$$\|\hat{S}_{m,r}^{\theta}(g; x) - g(x)\| \leq \|S_{m,r}^{\theta}(f)\| \leq \|f\| \leq \|f\| \leq 2\|f\| \leq 3\|f\|. \quad (18)$$

Further,

$$\|\hat{S}_{m,r}^{\theta}(f; x) - g(x)\| \leq \|S_{m,r}^{\theta}(f; x)\| \leq \|f\| \leq \|f\| \leq 2\|f\| \leq 3\|f\|. \quad (19)$$

and

$$\left(\frac{m^2(x) + r + \alpha + 1}{m + \beta} - u \right) g''(u)du \leq \|g''\| \left(\frac{m^2(x) + r + \alpha + 1}{m + \beta} - x \right)^2. \quad (20)$$

Rewrite (19) and (20) in (18), then we have

$$\left(\frac{m^2(x) + r + \alpha + 1}{m + \beta} - u \right) g''(u)du \leq \|g''\| \left(\frac{m^2(x) + r + \alpha + 1}{m + \beta} - x \right)^2. \quad (21)$$

where

$$\mu_m = \frac{2m^2}{(m+\beta)^2} \theta(x) + 2m \left(\frac{2r+2\alpha+3}{(m+\beta)^2} - \frac{2x}{m+\beta}\right) \theta(x) + 2x^2 - \frac{4x(r+\alpha+1)}{m+\beta} + \frac{2r^2+(4\alpha+5)r+2\alpha^2+4\alpha+3}{(m+\beta)^2}. \quad (22)$$

By using the auxiliary operators (16) and Lemma 8, we get

$$\|S_{m,r}^{\theta}(f; x)\| \leq \|S_{m,r}^{\theta}(x; x)\| + 2\|f\| \leq 3\|f\|. \quad (23)$$

With the help of (16), (21) and (23), for each $g \in C_0^\theta[0, \infty)$ we obtain

$$\left|\hat{S}_{m,r}^{\theta}(f; x) - g(x)\right| \leq \left|\hat{S}_{m,r}^{\theta}(f; x) - f(x)\right| + \left|\hat{S}_{m,r}^{\theta}(x; x) - f(x)\right| + \left|\hat{S}_{m,r}^{\theta}(g; x) - g(x)\right|$$

$$\leq \left|\hat{S}_{m,r}^{\theta}(f; x) - f(x)\right| + \left|\hat{S}_{m,r}^{\theta}(x; x) - f(x)\right| + \left|\hat{S}_{m,r}^{\theta}(g; x) - g(x)\right|$$

$$\leq 4\|f - g\| + \|g''\|\mu_m + \left|\int_{x}^{t} (t-u)g''(u)du\right|$$

$$\leq K_2(f, \mu_m) + \omega \left(\int_{x}^{t} (t-u)g''(u)du\right)$$

$$\leq \omega \left(\int_{x}^{t} (t-u)g''(u)du\right). \quad (24)$$

**Remark 10.** We see that $\mu_m = \frac{2x}{m} + O(m^{-2}) \to 0$, when $m \to \infty$. This result guarantees the convergence of the Theorem 9.
Section 5 investigates the rate of convergence with the help of exponential modulus of continuity.

5. THE EXPONENTIAL MODULUS OF CONTINUITY

The exponential growth of order \( B > 0 \) is given by

\[
||f||_{B^*} = \sup_{x \in [0, \infty)} |f(x)e^{-Bx}| < \infty
\]

for \( f \in C[0, \infty) \). Also,

\[
\omega_1(f, \delta, B) = \sup_{x \in [0, \infty), h > 0} |f(x) - f(x + h)|e^{-Bx}
\]

(26)

gives the first order modulus of continuity of functions \( f \) with the exponential growth. Let \( K \) be a subspace of continuous functions space on \([0, \infty)\), which includes functions \( f \) with exponential growth with \( ||f||_{B^*} < \infty \).

Assume that the function \( f \) belong to Lipschitz class. So, for every \( \delta < 1 \) and \( 0 < c \leq 1 \)

\[
\omega_1(f, \delta, B) \leq M\delta^c.
\]

(27)

**Theorem 11.** Let \( S^0_{m,r}: K \rightarrow C[0, \infty) \) be the sequence of positive and linear operators reproducing \( e^{2ax} \) for \( a > 0 \). It is assumed that \( S^0_{m,r} \) give

\[
S^0_{m,r}((t - x)^2 e^{Bt} x) \leq C_a(B, x)S^0_{m,r}(\phi^2_2 x),
\]

(28)

for \( 0 < B < x < \frac{m}{2B^2} \). Additionally, if \( f \in C^2[0, \infty) \cap K, 0 < c \leq 1 \) and \( f'' \in \text{Lip}(c, B) \), then for \( 0 < B < x < \frac{m}{2B^2} \), we obtain

\[
\left| S^0_{m,r}(f; x) - f(x) - f'(x)(t - x) + \frac{1}{2} f''(x)(\phi^2_2 x) - \frac{1}{2} f''(x)(\phi^2_2 x) - \frac{\sqrt{C_a(2Bx)} + C_a(Bx)}{2} e^{2Bx}\omega_1(f', B, \phi^2_2 x, B) \right|
\]

\[
\leq S^0_{m,r}(\phi^2_2 x) \left( \sqrt{\frac{C_a(2Bx)}{2}} + \frac{C_a(Bx)}{2} e^{2Bx}\omega_1(f', B, \phi^2_2 x, B) \right),
\]

where \( C_a(B, x) = M_{e^{2Bx}+1} \).

**Proof.** By considering Taylor expansion of the function \( f \in C^2[0, \infty) \) at \( x \in (0, \infty) \), we obtain

\[
f(t) = f(x) + f'(x)(t - x) + \frac{1}{2} f''(x)(t - x)^2 + H_2(f; t, x).
\]

Here the remainder term is \( H_2(f; t, x) = \frac{(t - x)^2}{2} (f''(x) - f'')(x) \), and \( \eta \) is between \( t \) and \( x \). Applying the operators \( S^0_{m,r} \) to the equality (29), we get

\[
\left| S^0_{m,r}(f; x) - f(x) - f'(x)S^0_{m,r}(\phi^2_2 x) - \frac{1}{2} f''(x)S^0_{m,r}(\phi^2_2 x) \right| = |S^0_{m,r}(H_2(f; t, x); x)|
\]

(30)

\[
\leq S^0_{m,r}(H_2(f; t, x); x).
\]

Additionally,

\[
H_2(f; t, x) = \frac{(t - x)^2}{2} (f''(x) - f'')(x) \leq \frac{(t - x)^2}{2} (e^{Bx}\omega_1(f'', h, B), |t - x| \leq h)
\]
It was proved by Tachev et al. [13] that

\[ \omega_1(f, kh, B) \leq k e^{B(k-1)h} \omega_1(f, h, B) \]  

(31)

for each \( h > 0 \) and \( k \in \mathbb{N} \). With the help of the inequality (31), we obtain

\[
\frac{(t-x)^2e^{Bx}}{2} \omega_1(f', kh, B) \leq \frac{(t-x)^2e^{Bx}}{2}k e^{B(k-1)h} \omega_1(f'', h, B) \\
\leq \frac{(t-x)^2}{2} \left(\frac{|t-x|^2}{h} + 1\right) e^{Bh} e^{Bt-x} \omega_1(f', h, B) \\
\leq \frac{(t-x)^2}{2} \left(\frac{|t-x|^2}{h} + 1\right) \left( e^{Bt} + e^{2Bx} \right) \omega_1(f'', h, B).
\]

Thusly,

\[
|H_2(f; t, x)| \leq \frac{(t-x)^2}{2} \left(\frac{|t-x|^2}{h} + 1\right) \left( e^{Bt} + e^{2Bx} \right) \omega_1(f'', h, B). 
\]

(32)

Applying the operators \( S^0_{m,r} \) to the inequality (32), we write

\[
S^0_{m,r}(|H_2(f; t, x)|; x) \leq \frac{1}{2h} S^0_{m,r}(\left(\frac{|t-x|^2}{h} + |t-x|^2\right) \left( e^{Bt} + e^{2Bx}\right); x) \omega_1(f', h, B) \\
= \frac{1}{2h} S^0_{m,r}(\left(|t-x|^2 e^{Bt}; x\right) + \frac{1}{2h} S^0_{m,r}(\left(|t-x|^2 e^{2Bx}; x\right) \\
+ \frac{e^{2Bx}}{2h} S^0_{m,r}(\left(|t-x|^2; x\right) + \frac{e^{2Bx}}{2h} S^0_{m,r}(\left(|t-x|^2; x\right) \omega_1(f', h, B).
\]

With some calculations we get

\[
S^0_{m,r}(\left(|t-x|^2 e^{Bt}; x\right) = e^{Bx} \left(1 + \frac{B^2x}{2h} \left(\frac{3}{Bx} - \frac{6a}{B} + 3\right) \right) \\
\]

\[
+ \frac{1}{2h} \left(\frac{B^2x}{m} \right)^2 \left(\frac{1-3r}{B^3x^2} + \frac{6B+9B+6ar-3B-6B+6a+12a}{B^3x^2} \right) \\
+ \frac{1}{3h} \left(\frac{B^2x}{m} \right)^3 \left(\frac{1-3/2r^2+3ar+3ar+6a}{B^5x^5} + \frac{6B+9B+6ar-3B-6B+6a+12a}{B^5x^5} \right) \\
+ \frac{18B^2-12B^2B-3B^2-3B^2-4B^2+36B^2-1B^2-36B^2+6B^2+66arB}{B^5x^2} \\
+ \frac{108B^2-72B^2B-36B^2-180B^2+120B^2+216B^2B-8(15+12a)}{B^5x^2}
\]

\[m \geq 72\]

\[\text{EARLY VIEW}\]
\[ S_{m,r}^0((t-x)^2 e^{Bt}; x) = e^{Bx} M_0 + \frac{B^2 x}{m} M_1 + \frac{1}{2!} \left( \frac{B^2 x}{m} \right)^2 M_2 + \frac{1}{3!} \left( \frac{B^2 x}{m} \right)^3 M_3 + O(m^{-4}) \]

\[ S_{m,r}^0((t-x)^2 e^{Bt}; x) = e^{Bx} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{B^2 x}{m} \right)^k M_k \]

Let us choose \( M_0 = 1 \) and \( M = \max\{M_0, M_1, M_2, \ldots\} \). Therefore, we have

\[ S_{m,r}^0((t-x)^2 e^{Bt}; x) \leq e^{Bx} M \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{B^2 x}{m} \right)^k S_{m,r}^0(\phi_x^2; x) \]

\[ = Me^{Bx} \frac{x}{m} S_{m,r}^0(\phi_x^2; x). \]

Since \( 0 < B < x < \frac{m}{B^2} \),

\[ S_{m,r}^0((t-x)^2 e^{Bt}; x) \leq C_a(B, x) S_{m,r}^0(\phi_x^2; x), \]

(33)

where \( C_a(B, x) = Me^{Bx+1} \). By employing Cauchy-Schwarz inequality, we obtain the next inequalities

\[ S_{m,r}^0((t-x)^3 e^{Bt}; x) \leq C_a(2B, x) \frac{\theta^2}{\theta^2} S_{m,r}^0(\phi_x^3; x) \]

(34)

\[ S_{m,r}^0((t-x)^3; x) \leq \sqrt{S_{m,r}^0((t-x)^2; x) S_{m,r}^0((t-x); x) \sqrt{S_{m,r}^0(\phi_x^3; x)}} \]

(35)

Thus, by using the inequalities (33), (34) and (35) in (30), we write

\[ \left| \frac{S_{m,r}(f; x) - f(x) S_{m,r}(\phi_x^2; x) - \frac{1}{2} f''(x) S_{m,r}(\phi_x^2; x)}{S_{m,r}^0(\phi_x^2; x)} \right| \]

\[ \leq \left( \frac{1}{2h} \right)^{\frac{1}{2}} \left[ C_a(2B, x) \frac{\theta^2}{\theta^2} S_{m,r}^0(\phi_x^3; x) \right] \frac{\theta^2}{\theta^2} S_{m,r}^0(\phi_x^3; x) + \frac{1}{2} C_a(B, x) S_{m,r}^0(\phi_x^2; x) \]

\[ + \frac{e^{2Bx}}{2h} \left( \frac{\theta^2}{\theta^2} S_{m,r}^0(\phi_x^3; x) + \frac{e^{2Bx}}{2} S_{m,r}^0(\phi_x^2; x) \right) \omega_1 \left( f'', (f', h, B) \right). \]

(36)

Lastly, when \( h = \frac{S_{m,r}(\phi_x^2; x)}{S_{m,r}^0(\phi_x^2; x)} \) is chosen and substituted in (36), we get

\[ \left| \frac{S_{m,r}(f; x) - f(x) S_{m,r}(\phi_x^2; x) - \frac{1}{2} f''(x) S_{m,r}(\phi_x^2; x)}{S_{m,r}^0(\phi_x^2; x)} \right| \]

\[ \leq S_{m,r}^0(\phi_x^2; x) \left( \frac{C_a(2B, x)}{2} + \frac{C_a(B, x)}{2} + e^{2Bx} \right) \omega_1 \left( f'', \frac{S_{m,r}(\phi_x^2; x)}{S_{m,r}^0(\phi_x^2; x)}, B \right). \]

It must be noticed that for fixed \( x \in (0, \infty) \), \( \frac{S_{m,r}(\phi_x^2; x)}{S_{m,r}^0(\phi_x^2; x)} = \frac{6x}{m} + O(m^{-2}) \to 0 \) as \( m \to \infty \), guarantees the convergence of Theorem 11.
In section 6, in order to investigate the asymptotic behaviour of the constructed operators (2), the Voronovskaya-type theorem is given.

6. VORONOVSKA-Y-TYPE THEOREM

Theorem 12. For \( f, f', f'' \in C^*([0, \infty)) \) and \( x \in [0, \infty) \), we get

\[
|f(S_{m,r}^\theta(f;x) - f(x)) - 2axf''(x) - x f''(x)| \leq \left| r_m(x) \right| |f''(x)| + \left| t_m(x) \right| |f''(x)| + 2(2t_m(x) + 2x + z_m(x)) \omega^*(f'', m^{-1/2})
\]

where

\[
r_m(x) = mS_{m,r}^\theta(\phi_k^1; x) + 2ax,
\]

\[
t_m(x) = \frac{m}{2} S_{m,r}^\theta(\phi_k^2; x) - x,
\]

\[
z_m(x) = m^2 \sqrt{S_{m,r}^\theta((e^{-x} - e^{-t})/4; x)} \sqrt{S_{m,r}^\theta(\phi_k^2; x)}.
\]

Proof. By considering the Taylor expansion, we get

\[
f(t) = f(x) + (t - x)f'(x) + \frac{(t-x)^2}{2} f''(x) + k(t, x) (t - x)^2.
\]

(37)

Here, the remainder term \( k(t, x) \) can be written as

\[
k(t, x) = \frac{1}{2} (f''(\xi) - f''(x)).
\]

Also, the remainder term is \( k(t, x) \) and \( \xi \) is a number between \( x \) and \( t \). When we apply the \( S_{m,r}^\theta \) operators to (37), we have

\[
S_{m,r}^\theta(f; x) - f(x) = f'(x) S_{m,r}^\theta(\phi_k^1; x) + \frac{1}{2} f''(x) S_{m,r}^\theta(\phi_k^2; x) + S_{m,r}^\theta(k(t, x) \phi_k^2; x).
\]

Then

\[
|S_{m,r}^\theta(f; x) - f(x)| \leq \left| r_m(x) \right| |f'(x)| + \left| t_m(x) \right| |f''(x)| + \frac{1}{2} m S_{m,r}^\theta(\phi_k^2; x) - 2x |f''(x)| + \left| S_{m,r}^\theta(k(t, x) \phi_k^2; x) \right|.
\]

It is briefly symbolized that \( r_m(x) = mS_{m,r}^\theta(\phi_k^1; x) + 2ax \) and \( t_m(x) = \frac{m}{2} S_{m,r}^\theta(\phi_k^2; x) - x \). Thus,

\[
|S_{m,r}^\theta(f; x) - f(x)| \leq \left| r_m(x) \right| |f'(x)| + \left| t_m(x) \right| |f''(x)| + \left| S_{m,r}^\theta(k(t, x) \phi_k^2; x) \right|.
\]

Note that from (5) and (6), we see that \( r_m(x) \) and \( t_m(x) \) go to zero as \( m \to \infty \). Now, we study the term \( |S_{m,r}^\theta(k(t, x) \phi_k^2; x)\).

\[
|f(t) - f(x)| \leq \left( 1 + \frac{(e^{-x} - e^{-t})^2}{\eta^2} \right) \omega^*(f, \eta).
\]

By employing this inequality, we get

\[
|k(t, x)| \leq \left( 1 + \frac{(e^{-x} - e^{-t})^2}{\eta^2} \right) \omega^*(f'', \eta).
\]
For \( \eta > 0 \), if \( |e^{-x} - e^{-t}| \leq \eta \), then \( |k(t, x)| \leq 2\omega^*(f'', \eta) \) and if \( |e^{-x} - e^{-t}| > \eta \), then \( |k(t, x)| \leq \frac{2(e^{-x} - e^{-t})^2}{\eta^2} \omega^*(f'', \eta) \). Thusly, we have \( |k(t, x)| \leq 2\left( \frac{(e^{-x} - e^{-t})^2}{\eta^2} + 1 \right) \omega^*(f'', \eta) \).

Accordingly,
\[
|mS_m^\theta (k(t, x) \phi_X^2; x)| \leq mS_m^\theta ((k(t, x) | \phi_X^2; x) \\
\leq 2m\omega^*(f'', \eta)S_m^\theta (\phi_X^2; x) + \frac{2m}{\eta^2} \omega^*(f'', \eta)S_m^\theta ((e^{-x} - e^{-t})^2 \phi_X^2; x) \\
\leq 2m\omega^*(f'', \eta)S_m^\theta (\phi_X^2; x) + \frac{2m}{\eta^2} \omega^*(f'', \eta)S_m^\theta ((e^{-x} - e^{-t})^4; x) \frac{S_m^\theta (\phi_X^2; x)}{\sqrt{\eta}}.
\]

If we choose \( \eta = 1/\sqrt{m} \) and \( z_m \) = \( \sqrt{m^2S_m^\theta ((e^{-x} - e^{-t})^4; x)} \), we get
\[
|m(S_m^\theta (f; x) - f(x)) + 2axf'(x) - xf''(x)| \leq |r_m(x)||f'(x)| + |t_m(x)||f''(x)| \\
+ (4t_m(x) + 4x + 2z_m(x)) \omega^*(f''(m^{-1/2})).
\]

**Remark 13.** After some calculations the following limit result is obtained:
\[
\lim_{m \to \infty} m^2S_m^\theta ((e^{-t} - e^{-x})^4; x) = 12x^2.
\] (38)

In addition, we get the result as follows:
\[
\lim_{m \to \infty} m^2S_m^\theta ((e^{-t} - e^{-x})^4; x) = 12x^2e^{-4x}.
\] (39)

**Proof.** We have after some calculations
\[
m^2S_m^\theta (\phi_X^2; x) = 12x^2 + \frac{12x(1-x+6ax+2ax^2+3ax^2-2\alpha)}{m^2} \\
+ \frac{3r_m^2 - 96a^3x^3 + 16a^4x^4 - 4r_m(3 - 26ax - 9\beta x + 18a^2x^2 - 3\alpha)}{m^2} \\
+ \frac{72a^2x^2(-1 + 2\beta x + 2\alpha)}{m^2} \\
+ \frac{\text{other terms}}{m^2} + O(m^{-3}).
\]

So,
\[
\lim_{m \to \infty} m^2S_m^\theta (\phi_X^2; x) = 12x^2.
\]

In the same manner, we have
\[
m^2S_m^\theta ((e^{-t} - e^{-x})^4; x) = 12x^2e^{-4x} + \frac{4xe^{-4x}(3r^2-6(5+2a+\beta)x(65+60a+12a^2)x^2)}{m^2} \\
+ \frac{4xe^{-4x}(3r(-3+2(5+2a)x)-6(1+\alpha))}{m} + O(m^{-2}).
\]

Thus,
\[
\lim_{m \to \infty} m^2S_m^\theta ((e^{-t} - e^{-x})^4; x) = 12x^2e^{-4x}.
\]

The next corollary is given as a consequence of Theorem 12 and Remark 13 as follows:
Corollary 14. Assume that $x \in [0, \infty)$ and $f, f'' \in C^* [0, \infty)$. Thus,
\[
\lim_{m \to \infty} m \left( S_{m,r}^0 (f;x) - f(x) \right) = -2axf'(x) + xf''(x)
\] (40)
holds.

Now, we investigate that our new constructed Szász-Mirakyan-Durrmeyer-Stancu operators which reproduce $e^{2ax}$ for $a > 0$ approximate better than Szász-Mirakyan operators preserving $e^{2ax}$ which is taken into consideration by Acar et al. [2].

Theorem 15. Let $f \in C^2 [0, \infty)$ be an increasing and convex function. Assume that for all $m \geq m_0$, $x \in [0, \infty)$ there is a number $m_0 \in \mathbb{N}$ such that
\[
f(x) \leq S_{m,r}^0 (f;x) \leq R_m (f;x).
\] (41)
Then
\[
xf''(x) \geq 2axf'(x) \geq 0.
\] (42)
Contrarily, if inequality (42) holds with strict inequalities at $x \in [0, \infty)$, then there is a number $m_0 \in \mathbb{N}$ such that for $m \geq m_0$
\[
f(x) < S_{m,r}^0 (f;x) < R_m (f;x).
\] (43)
Proof. From the inequality (41) we have for all $m \geq m_0$ and $x \in [0, \infty)$ that
\[
0 \leq m \left( S_{m,r}^0 (f;x) - f(x) \right) \leq m \left( R_m (f;x) - f(x) \right).
\] (44)
By using the Voronovskaya-type theorem for Szász-Mirakyan operators preserving $e^{2ax}$, $a > 0$ which is obtained by Acar et al. [2], we get
\[
\lim_{m \to \infty} m (R_m^* (f;x) - f(x)) = -axf'(x) + \frac{x}{2} f''(x).
\] (45)
After that, by taking the limit of the inequality (44) as $m \to \infty$ and using Equation (40) and Equation (45), we get
\[
0 \leq -2axf'(x) + xf''(x) \leq -axf'(x) + \frac{x}{2} f''(x). \] (46)
Thus, we directly achieve inequality (42). Contrarily, if inequality (42) holds with strict at $x \in [0, \infty)$, then
\[
0 < -2axf'(x) + xf''(x) < -axf'(x) + \frac{x}{2} f''(x). \] (47)
Finally, by using Equation (40) and Equation (45) we obtain the desired result.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.
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