ANALYTIC EXTENSIONS OF CONSTANT MEAN CURVATURE ONE GEOMETRIC CATENOIDS IN DE SITTER 3-SPACE

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Abstract. We show that a certain simply-stated notion of “analytic completeness” of the image of a real analytic map implies the map admits no analytic extension. We also give a useful criterion for that notion of analytic completeness by defining arc-properness of continuous maps, which can be considered as a very weak version of properness. As an application, we judge the analytic completeness of a certain class of constant mean curvature surfaces (the so-called “G-catenoids”) or their analytic extensions in the de Sitter 3-space.

Introduction

In the authors’ previous works [5]–[13], [23]–[25] on zero mean curvature surfaces in the Lorentz-Minkowski 3-space $\mathbb{R}^3_1$ and constant mean curvature one surfaces (i.e. CMC-1 surfaces) in de Sitter 3-space $S^3_1$, a number of concrete examples were constructed. Such surfaces in $\mathbb{R}^3_1$ or $S^3_1$ have singularities, in general. Moreover, some of them have non-trivial analytic extensions (cf. Definition 2.2). To find such extensions, one sometimes needs further techniques beyond the usual reflection principle.

Thinking of $\mathbb{R}^3_1$ with signature $(-+)$, the analytic map $f_K: \mathbb{R} \times T^1 \to (\mathbb{R}^3_1; t, x, y)$ defined by

\begin{equation}
(0.1) \quad f_K(u, v) = (\sinh u \cos v, u, \cosh u \cos v) \quad (u \in \mathbb{R}, \ v \in T^1),
\end{equation}

gives a typical such example (the subscript “K” stands for “Kobayashi” [19]), where

\begin{equation}
(0.2) \quad T^1 := \mathbb{R}/2\pi \mathbb{Z}
\end{equation}

is a 1-dimensional torus. This map $f_K$ gives a parametrization of a maximal surface derived from the Weierstrass-type representation formula and has two disjoint singular sets along the lines $v = 0, \pi$. The map is a proper (i.e. the inverse image...
of a compact set is compact) analytic map, but is a two-to-one mapping away from its singular set (consisting of fold singular points). Osamu Kobayashi [19] found that the image of $f_K$ is a proper subset of the entire (zero-mean curvature) graph of the function $t = y \tanh x$. The complement of the image of $f_K$ within this graph contains two disjoint time-like zero mean curvature surfaces, that is, the graph of the function $t = y \tanh x$ can be considered as an analytic extension of the image of $f_K$ (cf. Theorem 2.25, the precise definition of analytic extension is given in Definition 2.22).

Up to now several space-like maximal surfaces admitting analytic extensions have been found by several geometers, including the authors, see [2, 3, 10, 11, 7, 15], which have motivated the authors to find criteria that ensure the resulting surfaces satisfy “analytic completeness”, namely, they have no further non-trivial analytic extensions (cf. Proposition 2.23).

For example, space-like maximal surfaces in $\mathbb{R}^3$ which are invariant under a one-parameter family of isometries with a common fixed point (i.e. $G$-catenoids, see below) are congruent to one of the following three surfaces up to a homothety:

- the elliptic $G$-catenoid

(0.3) \[ f_E(u, v) = (u, \sinh u, u \cos v, \sinh u \sin v) \quad (u, v) \in \mathbb{R} \times T^1, \]

- the parabolic $G$-catenoid

(0.4) \[ f_P(u, v) := \left( v - \frac{v^3}{3} + u^2v, v + \frac{v^3}{3} - u^2v, 2uv \right) \quad (u, v) \in \mathbb{R}^2, \]

- the hyperbolic $G$-catenoid

(0.5) \[ f_H(u, v) = (\cosh u \sin v, v, \sinh u \sin v) \quad (u, v) \in \mathbb{R}^2. \]

The image of $f_E$ coincides with the subset

(0.6) \[ \mathcal{E} = \{(t, x, y) \in \mathbb{R}^3_1; x^2 + y^2 = \sinh^2 t\}, \]

and is rotationally symmetric with respect to the time-like axis, and has a cone-like singular point. Moreover, $f_E$ has no analytic extensions (in fact, the map is proper and admits only one cone-like singular point, and we can apply Proposition 1.5 and conclude that $f_E$ is analytically complete).

As for $f_P$ in (0.4), its image is not closed. If we set

(0.7) \[ \mathcal{P} := \{(t, x, y) \in \mathbb{R}^3; 12(x^2 - t^2) = (x + t)^4 - 12y^2\}, \]
then it has a cone-like singular point and satisfies

\[(0.8) \quad \mathcal{P} = f_P(\mathbb{R}^2) \cup L, \quad L := \{(t, -t, 0) : t \in \mathbb{R}\}.\]

The inclusion map associated with \(\mathcal{P}\) can be considered as an analytic extension of the map \(f_P\) (cf. Theorem 2.23). Similarly, \(f_H\) also has an analytic extension. In fact, by a suitable symmetry \(T\) in \(\mathbb{R}^3\), we can write

\[(0.9) \quad \mathcal{H} := \{(t, x, y) \in \mathbb{R}^3 : \sin^2 x + y^2 - t^2 = 0\} = f_H(\mathbb{R}^2) \cup T \circ f_H(\mathbb{R}^2),\]

which is a singly-periodic surface with a countably infinite number of cone-like singular points, giving an analytic extension of \(f_H\) (see Examples 2.19 and Theorem 2.25).

The figures of \(\mathcal{P}\) and \(\mathcal{H}\) are found in [11, Fig. 4 (left) and Fig. 1 (left)], and they can be expressed as the zero sets of the single analytic functions

\[
\tau_P(t, x, y) := 12(x^2 - t^2) - (x + t)^4 + 12y^2, \\
\tau_H(t, x, y) := \sin^2 x + y^2 - t^2
\]

respectively. By this fact, we can show that they are analytically complete (cf. Proposition 1.5). However, in contrast to this characterization of \(\mathcal{P}\) and \(\mathcal{H}\) as the zero sets of analytic functions, analytically complete subsets might not be expressed as the zero set of a certain analytic function, in general: For example, the image of the map

\[(0.10) \quad f_0(t) := (t, e^{1/t}) \quad (t > 0)\]

is an analytically complete subset of \(\mathbb{R}^2\) (cf. Example 3.9) although it cannot be characterized as the zero set of any real analytic function. In fact, \(f_0((0, \infty))\) is not a closed subset of \(\mathbb{R}^2\).

In this paper, we define analytic completeness for subsets of real analytic manifolds, which is essentially the same concept as the arc-symmetric property introduced by Kurdyka [20] (see also Nash [21] for arc-structures). Moreover, inspired by [20], we introduce “arc-properness” for continuous maps to an analytic manifold (cf. Definition 3.6). One may regard arc-properness as a very weak version of properness of maps (cf. Proposition 3.7). In fact, proper maps are arc-proper, but the converse is not true, in general. This concept plays an important role in describing our criterion for analytic completeness (cf. Theorem 3.10). In the remainder of this paper, we shall demonstrate that this criterion is actually applicable to concrete examples. In fact, in the last two sections of this paper, it is applied to show the analytic completeness of analytic extensions of constant mean curvature one catenoids in de Sitter 3-space, as follows: Let \(\mathbb{R}^4_1\) be the Lorentz-Minkowski 4-space with the metric \((\cdot, \cdot)\) of signature \((-+++)\). Then

\[S^3_1 = \left\{X = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 : ((X, X) = -(x_0)^2 + \sum_{j=1}^3 (x_j)^2 = 1)\right\},\]

with the metric induced from \(\mathbb{R}^4_1\), is a simply-connected Lorentzian 3-manifold with constant sectional curvature 1, called the de Sitter 3-space.

Space-like constant mean curvature one surfaces in \(S^3_1\) have a Weierstrass-type representation formula like space-like maximal surfaces in Lorentz-Minkowski space \(\mathbb{R}^4_1\) (see [11, 5] and [12]). We consider constant mean curvature one surfaces in \(S^3_1\) and zero mean curvature surfaces in \(\mathbb{R}^3_1\) with either of the following properties:

(G) The surface is invariant under a one-parameter family of isometries with a common fixed point.

(W) The surface is weakly complete with Gauss map of degree 1 and Hopf differential having a pole of order 2 at each of two ends.
We call them geometric catenoids (G-catenoids in short) if they satisfy (G), and catenoids with the same holomorphic data as the Euclidean catenoid (W-catenoids in short) if they satisfy (W). Here the subscript “W” stands for “Weierstrass-type representation formulas” for space-like constant mean curvature one surfaces in $S^3_1$ and zero mean curvature surfaces in $R^3_1$.

As mentioned above, G-catenoids in $R^3_1$ are space-like maximal surfaces that are congruent to one of $f_E$, $f_P$, $f_H$ up to a homothety, and the last two maps have analytic extensions as mentioned above. W-catenoids in $R^3_1$ are congruent to either $f_E$ or $f_K$ up to homotheties. (We have already seen that $f_K$ can be extended as an entire graph.)

On the other hand, in [25] and [6], all G-catenoids and W-catenoids in $S^3_1$ are classified. In light of this, it is interesting that there are so many different types of G-catenoids and W-catenoids in $S^3_1$. G-catenoids in $S^3_1$ admit only cone-like singular points like as in the case of catenoids in $R^3_1$. On the other hand, W-catenoids in $S^3_1$ admit various kinds of singular points.

In this paper, we focus on G-catenoids, each of which is induced by a family of isometries fixing a geodesic of $S^3_1$. (Analytic completeness of W-catenoids will be discussed in the forthcoming paper [9] by the last author’s classification [25].) Geometric catenoids are divided into the following eight classes:

- G-catenoids of type TE, TP or TH (which are special cases of W-catenoids with elliptic, parabolic or hyperbolic monodromies, respectively),
- G-catenoids of type SE, SP or SH (exceptional W-catenoids of type II belong to a special class of G-catenoids of type SE),
- G-catenoids of type LE or LH,

where $T, S$ and $L$ denote whether the geodesic fixed by the 1-parameter group of isometries is time-like, space-like or light-like, and $E, P$ and $H$ stand for elliptic, parabolic and hyperbolic, respectively. We show that, except for G-catenoids of type TE, TP and TH which are already analytically complete, the other five types of G-catenoids have analytic extensions which are analytically complete.

To formulate the notion of analytic extension of analytic maps, we define the concept “double-cone manifold”. In fact, all analytic extensions of G-catenoids admit only cone-like singular points as their singular points, and can be parametrized by analytic maps defined on double-cone manifolds.

In Section 1, we define analytic completeness. In Section 2, we define analytic maps on double-cone manifolds and show their fundamental properties. All analytic extensions of G-catenoids can be interpreted as DC-immersions (i.e. double-cone immersions, that is immersions of DC-manifolds). In Section 3, we give a criterion for analytic completeness of real analytic DC-immersions. In Section 4, we prove that G-catenoids belonging to the above five types SE, SP, SH, LE or LH can be extended as real analytic DC-immersions from certain DC-manifolds of dimension 2.

The analytic completeness of maximal catenoids or their particular analytic extensions in $R^3_1$ is also discussed in Section 2 and Appendix A.

1. Preliminaries

Throughout this paper, we fix a real analytic manifold $N^n$ of dimension $n(\geq 1)$. We also fix a positive integer $m(\leq n)$. We begin by defining analytic completeness as follows:

**Definition 1.1.** A non-empty subset $S$ of $N^n$ is said to be **analytically complete** if it satisfies the following property:

Any real analytic map $\Gamma : [0, 1] \rightarrow N^n$ satisfying $\Gamma([0, \varepsilon)) \subset S$ for some $\varepsilon \in (0, 1)$ satisfies $\Gamma([0, 1]) \subset S$.
Analytic completeness is the same concept as the “arc-symmetric property” given in [20]. For example:

- Any affine subspace \(A^m \subseteq \mathbb{R}^n\) is an analytically complete subset of \(\mathbb{R}^n\) (cf. Example 1.3 and Proposition 1.5).
- the interval \(I := [0, 1]\) is not an analytically complete subset of \(\mathbb{R}\). In fact, let \(\Gamma : [0, 1] \to \mathbb{R}\) be a real analytic map defined by \(G(t) = 2t\) which maps \([0, 1/2)\) to \(I\) but \(\Gamma(I) \not\subseteq I\).
- Let \(S = \{(t, e^{-1/t}) \in \mathbb{R}^2; 0 < t < \infty\}\). Then \(S\) is an analytically complete subset of \(\mathbb{R}^2\), although \(S\) has a \(C^\infty\)-extension (cf. Example 8.9).

**Definition 1.2.** A subset \(S\) of \(\mathbb{N}^n\) is called **globally analytic** if there exists a positive integer \(l\) and a real analytic map \(\tau : N^n \to \mathbb{R}^l\) such that the zero set

\[
Z(\tau) := \{P \in N^n; \tau(P) = (0, \ldots, 0)\}
\]

of \(\tau\) coincides with \(S\).

**Example 1.3.** If \(S\) is an \(m\)-dimensional linear subspace in \(\mathbb{R}^n\), then \(S\) is a globally analytic subset, as there is a linear map \(F : \mathbb{R}^n \to \mathbb{R}^l\) \((l \geq 1)\) such that \(S\) is the kernel of \(F\).

**Example 1.4.** For \(m \geq 2\), the set

\[
C^n_m := \{(x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1}; \sum_{j=1}^{m} x_j^2 = x_{m+1}^2, \ x_{m+1} \geq 0\}
\]

is called the **\(m\)-dimensional standard cone**, which is a subset of

\[
(1.1) \quad C^n_m := \{(x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1}; \sum_{j=1}^{m} x_j^2 = x_{m+1}^2\}.
\]

This \(C^n_m\) is called the **\(m\)-dimensional standard double-cone** and is a globally analytic subset of \(\mathbb{R}^{m+1}\). Moreover, \(C^n_m\) is the image of the analytic map

\[
\pi_{DC}^m : S^{m-1} \times \mathbb{R} \ni (x, t) \mapsto (tx, t) \in \mathbb{R}^{m+1},
\]

where \(S^{m-1}\) is the unit sphere centered at the origin in \(\mathbb{R}^m\). We call the origin \(0 \in C^n_m\) the **standard double-cone point**.

As pointed out in the introduction, the subsets \(E, \mathcal{P}\) and \(\mathcal{H}\) of \(\mathbb{R}^1\) given by \([0, 0]\), \([0, 1]\) and \([1, 1]\) are globally analytic. The following assertion holds:

**Proposition 1.5.** Let \(S\) be a subset of \(\mathbb{N}^n\). If \(S\) is a globally analytic subset of \(\mathbb{N}^n\), then it is analytically complete.

**Proof.** Suppose that \(S\) is globally analytic. Then there exists a positive integer \(l\) and a real analytic map \(\tau : \mathbb{N}^n \to \mathbb{R}^l\) such that \(S = Z(\tau)\). Let \(\Gamma : [0, 1] \to \mathbb{N}^n\) be a real analytic map such that \(\Gamma([0, \varepsilon]) \subset S \subset Z(\tau))\) for some sufficiently small \(\varepsilon > 0\). Then the real analyticity of \(\Gamma\) and \(\tau\) implies \(\Gamma([0, 1]) \subset S\). So \(S\) is analytically complete. \(\square\)

2. **Double-cone manifolds (DC-manifolds)**

In this section, we consider real analytic maps as well as smooth maps. So, by \(C^r\)-map, we mean a real analytic (resp. smooth) map if \(r = \omega\) (resp. \(r = \infty\)). In the previous section, we defined the standard double-cone \(C^n_m\). Using this, we define double-cone manifolds (i.e. DC-manifolds): Related to these are the concepts of “orbifold” and “conifold”, but the first one requires a group action and the second one is considered on complex manifolds or on real manifolds modeled on
the standard cone $C_m^1$. As far as the authors know, there is no similar definition of manifolds modeled on the standard double-cone $C_m^2$. We first define topological double-cone manifolds (i.e. DC-manifolds) as follows:

**Definition 2.1.** Fix an integer $m(\geq 2)$. A Hausdorff space $X$ with the second axiom of countability is called a topological DC-manifold of dimension $m$ if each point of $X$ has a neighborhood which is homeomorphic to an open subset of $C_m^1$. Moreover, $p \in X$ is called a DC-point if $p$ corresponds to the origin of $C_m^1$.

We let $\Sigma$ be the set of all DC-points of a topological DC-manifold $X$ of dimension $m$. Then, by definition, $X \setminus \Sigma$ is a topological manifold of dimension $m$ and $\Sigma$ is a discrete subset of $X$.

**Definition 2.2.** Let $X$ be a topological DC-manifold of dimension $m$, and let $\varphi$ be a homeomorphism of an open subset $\hat{U}$ of $C_m^2$ onto a connected open set $O(\subset X)$. Since $\pi_m^{DC}$ (cf. (1.2)) is a continuous map, $U := (\pi_m^{DC})^{-1}(\hat{U})$ is an open subset of $S^{m-1} \times \mathbb{R}$. We consider a continuous map $\Phi : U \rightarrow O$ given by $\Phi := \varphi \circ \pi_m^{DC}$, and call the pair $(U, \Phi)$ an inverse DC-coordinate system (or a parametrization system). We then call the pair $(O, \varphi^{-1})$ a coordinate system associated with $(U, \Phi)$.

**Definition 2.3.** A differentiable DC-structure $F$ of class $C^r$ on an $m$-dimensional topological DC-manifold $X$ is a collection of inverse coordinate systems

$$\{(U_\lambda, \Phi_\lambda) : \lambda \in \Lambda\}$$

satisfying the following properties:

(i) By setting $O_\lambda := \Phi_\lambda(U_\lambda)$, the family $\{O_\lambda\}_{\lambda \in \Lambda}$ is an open covering of $X$.

(ii) Whenever $O_\lambda \cap O_\mu \neq \emptyset$, there exists a $C^r$-diffeomorphism

$$\Phi_{\mu \lambda} : \Phi^{-1}_\mu(O_\mu) \cap U_\lambda \rightarrow \Phi^{-1}_\lambda(O_\lambda) \cap U_\mu$$

satisfying $\Phi_{\mu} \circ \Phi_{\mu \lambda} = \Phi_{\lambda}$ (see Figure 2).

(iii) The collection $F$ is maximal with respect to (ii).

Moreover, an $m$-dimensional topological DC-manifold $X$ with a differentiable DC-structure of class $C^r$ is called a (differentiable) $m$-dimensional DC-manifold. In this setting, each $(U_\lambda, \Phi_\lambda) (\lambda \in \Lambda)$ is an inverse DC-coordinate system of $X$, and $(O_\lambda, \varphi^{-1}_\lambda)$ is a local coordinate system of $X$ associated with $(U_\lambda, \Phi_\lambda)$.

**Figure 2.** The commutative diagram related to the maps $\Phi_\lambda$ and $\Phi_\mu$.

Usual manifolds can be considered as DC-manifolds without DC-points: We let $M^m$ be an $m$-dimensional manifold, and $(O, \psi)$ a local coordinate neighborhood of
a point \( q \in M^n \). Since \( S^{m-1} \times (0, \infty) \) is diffeomorphic to \( \mathbb{R}^m \setminus \{0\} \), we may assume that \( \psi(O) \) is a subset of \( S^{m-1} \times (0, \infty) \). Then
\[
(2.1) \quad (\psi(O), \Phi) \quad (\Phi := \psi^{-1})
\]
gives a local inverse DC-coordinate system of \( M^n \) as a DC-manifold.

The standard double-cone \( C_2^m \) is a typical example of a \( C^r \)-differentiable DC-manifold with a single local inverse DC-coordinate system \( \Phi := \pi^m_{DC} : S^{m-1} \times \mathbb{R} \to C_2^m \).

**Definition 2.4.** Let \( X^m \) and \( Y^n \) be two \( C^r \)-differentiable DC-manifolds whose DC-structures are \( \{(U_\alpha, \Phi_\alpha)\}_{\alpha \in A} \) and \( \{(V_\beta, \Psi_\beta)\}_{\beta \in B} \), respectively. A map \( f : X^m \to Y^n \) is called a \( C^r \)-differentiable DC-map if for each pair of indices \( (\alpha, \beta) \in A \times B \), there exists a \( C^r \)-map
\[
F_{\alpha, \beta} : U_\alpha \cap \Phi_\alpha^{-1} \left( f^{-1}(\Psi_\beta(V_\beta)) \right) \to V_\beta
\]
satisfying \( \Psi_\beta \circ F_{\alpha, \beta} = f \circ \Phi_\alpha \). Moreover, if \( f \) is bijective and \( f^{-1} \) is also a \( C^r \)-differentiable DC-map, then \( f \) is called a \( (C^r \)-differentiable) DC-diffeomorphism.

**Remark 2.5.** Let \( (O, \psi) \) be a local coordinate system of a DC-manifold \( X^m \), then \( \psi : O \to \psi(O) (\subset C_2^m) \) gives a typical example of a DC-diffeomorphism.

We next define “DC-immersions” as follows:

**Definition 2.6.** Let \( X^m \) be a \( C^r \)-differentiable DC-manifold, and let \( N^n \) be a \( C^r \)-manifold. A map \( f : X^m \to N^n (n > m) \) is called a DC-immersion if, for each \( p \in X \), there exist
- a local DC-coordinate system \((O, \psi)\) containing \( p \) and
- a \( C^r \)-diffeomorphism \( F \) defined on a neighborhood \( \Omega_0 \) of the origin \( 0 \) of \( \mathbb{R}^n \) onto a neighborhood \( \Omega \) of \( f(p) \) such that
\[
F(a, 0_{n-m-1}) = f \circ \psi^{-1}(a) \quad (a \in \psi(O) \subset C_2^m),
\]
where \( 0_{n-m-1} \) is the zero vector of \( \mathbb{R}^{n-m-1} \).

By definition, we have the following:

**Proposition 2.7.** A DC-immersion is a DC-map.

The following assertion is also obvious:

**Proposition 2.8.** Let \( M^m \) be a \( C^r \)-manifold, and let \( f : M^m \to N^n \) be a \( C^r \)-immersion into a \( C^r \)-manifold \( N^n \) of dimension \( n(> m) \). Then \( f \) is a DC-immersion.

As a consequence, we have the following:

**Corollary 2.9.** Let \( X^m \) be a \( C^r \)-differentiable DC-manifold and \( f : X^m \to N^n \) (\( m < n \)) a DC-immersion. If \( g : N^n \to L^l (l \geq n) \) is a \( C^r \)-immersion into a \( C^r \)-manifold of dimension \( l \), then \( g \circ f \) is also a DC-immersion.

**Definition 2.10.** A subset \( X \) of a \( C^r \)-manifold \( N^n \) is said to be admissible if there exists a discrete subset \( \Gamma \) of \( X \) such that \( X \setminus \Gamma \) is an \( m \)-dimensional \( C^r \)-submanifold of \( N^n \), where \( m < n \). We call such \( \Gamma \) a pre-residual subset of \( X \). We consider the intersection of all pre-residual subsets of \( X \), which gives the smallest pre-residual subset of \( X \). We denote it by \( \Sigma \), and call it the residual set of \( X \).

By definition, \( X \setminus \Sigma \) is an \( m \)-dimensional \( C^r \)-submanifold of \( N^n \).

**Definition 2.11.** Let \( X \) be an admissible subset of a \( C^r \)-manifold \( N^n (n > m) \), and let \( \Omega(\subset \mathbb{R}^n) \) be an open subset so that
\[
\hat{U} := (C_2^n \times \{0_{n-m-1}\}) \cap \Omega
\]
is non-empty. A $C^r$-diffeomorphism $F$ from $\Omega$ into $\mathbb{R}^n$ is called a $C^r$-differentiable extended parametrization of $X$ if $\hat{U}$ satisfies the following properties;

1. $F(\hat{U})$ is contained in $X$,
2. if $\hat{U}$ contains the origin $0$, then $F(0)$ belongs to the residual set $\Sigma$ of $X$, and
3. $F$ gives a $C^r$-map from the $C^r$-submanifold $\hat{U} \setminus \{0\}$ of $\mathbb{R}^{m+1} \times \{0, \ldots, 0\}$ into the $C^r$-submanifold $X \setminus \Sigma$.

Regarding $C^2_2$ as an admissible subset of $\mathbb{R}^{m+1}$, we prepare the following:

**Lemma 2.12.** Let $\Omega_i$ ($i = 1, 2$) be open subsets containing the origin of $C^2_2$. Suppose that $F : \Omega_1 \to \mathbb{R}^{m+1}$ is a $C^r$-differentiable extended parametrization of $C^2_2$, such that $F(\Omega_1) = \Omega_2$. Then the restriction $f := F|_{\Omega_1 \cap C^2_2}$ is a $C^r$-differentiable DC-diffeomorphism between $\Omega_1 \cap C^2_2$ and $\Omega_2 \cap C^2_2$.

**Proof.** Since we can switch the roles of $F$ and $F^{-1}$, it is sufficient to show that $f$ is a DC-map at each point $p \in C^2_2 \cap \Omega_1$. If $p$ is not the origin $0$ in $\mathbb{R}^m$, the assertion is obvious. So we may assume that $p = 0$. By (2) of Definition 2.11, we have $f(p) = 0$.

We set $U_i := (\pi_{DC}^m)^{-1}(\Omega_i)$ ($i = 1, 2$). Since $0$ belongs to $\Omega_1$ and $\Omega_2$, the subset $S^{m-1} \times \{0\}$ is contained in $U_1$ and $U_2$. We set $G := F \circ \pi_{DC}^m$, which is a $C^r$-map from $U_1$ to $\mathbb{R}^{m+1}$. Using the central projection $\pi_C : C^m_2 \setminus \{0\} \ni (x, t) \mapsto x/t \in S^{m-1}$, we set

$$g : U_1 \setminus (S^{m-1} \times \{0\}) \ni (x, t) \mapsto \pi_C \circ G(x, t) \in S^{m-1}.$$ 

If we set

$$((\lambda(x, t), \tau(x, t)) := G(x, t) \in C^m_2(\subset \mathbb{R}^{m+1}),$$

then

$$\alpha(x, t) := \frac{\lambda(x, t)}{t}, \quad \beta(x, t) := \frac{\tau(x, t)}{t}$$

are both $C^r$-differentiable maps at $t = 0$ (cf. ([22], Appendix A)). Since $t \mapsto G(x, t)$ is a regular curve, neither

$$\alpha(x, 0) = \frac{\partial \lambda(x, 0)}{\partial t} \quad \text{nor} \quad \beta(x, 0) = \frac{\partial \tau(x, 0)}{\partial t}$$

vanishes at $t = 0$. Moreover, since $\lambda(x, t)/\tau(x, t)$ is a unit vector for $t \neq 0$, the order of the map $t \mapsto \lambda(x, t)$ at $t = 0$ coincides with that of $\tau(x, t)$. In particular, $\partial \alpha(x, 0)/\partial t$ and $\partial \beta(x, 0)/\partial t$ are both non-zero. Thus, the map

$$\Phi : U_1 \ni (x, t) \mapsto \frac{\lambda(x, t)}{\tau(x, t)}, \tau(x, t) \in U_2$$

is a $C^r$-map, and satisfies $\pi_{DC}^m \circ \Phi = G$, which implies that $f$ is $C^r$-differentiable on $\pi_{DC}^m(U_1)$, proving the assertion. \hfill $\Box$

**Definition 2.13.** Let $\mathbb{R}^n$ be an $n$-dimensional $C^r$-manifold and $X$ an admissible subset of $\mathbb{R}^n$. Then $X$ is said to be a DC-submanifold of $\mathbb{R}^n$ if $X$ has an $m$-dimensional DC-structure ($m < n$) such that, for any extended parametrization $F : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}^n$ of $X$ (cf. Definition 2.11), the pair

$$\left((\pi_{DC}^m)^{-1}(\Omega), F \circ \pi_{DC}^m\right)$$

gives a local inverse DC-coordinate system of $X$, where

$$\pi_{DC}^m : S^{m-1} \times \mathbb{R} \ni a \mapsto (\pi_{DC}^m(a), 0) \in C^m_2 \times \mathbb{R}^{n-m-1} \quad (a \in S^{m-1} \times \mathbb{R}).$$

By definition, the following assertion is obvious:

**Proposition 2.14.** The inclusion map of a DC-submanifold is a DC-immersion.

We next prepare the following:
Lemma 2.15. Let \( N^n \) be an \( n \)-dimensional \( C^r \)-manifold and \( X \) an admissible subset of \( N^n \). Suppose that, for each \( p \in X \), there exist

1. an open subset \( \Omega_p \) of \( \mathbb{R}^n \), and
2. a \( C^r \)-differentiable extended parametrization \( F_p : \Omega_p \to N^n \) of \( X \) such that

\( p \in F_p(\Omega_p) \).

Then, \( X \) has the structure of a \( C^r \)-differentiable DC-submanifold of \( N^n \) such that the residual set is the set of DC-points and the inclusion map \( X \hookrightarrow N^n \) is a DC-immersion.

Proof. We take \( p,q \in X \) such that \( O_{p,q} := F_p(\Omega_p) \cap F_q(\Omega_q) \) is non-empty. Then \( G := F_q^{-1} \circ F_p \) is an extended parametrization of \( \hat{C}_2^n \) (cf. Definition 2.11) which maps \( F_p^{-1}(O_{p,q}) \) onto \( F_q^{-1}(O_{p,q}) \). By Lemma 2.12, the restriction of \( G \) is a \( C^r \)-differentiable DC-diffeomorphism between \( \Omega_p \cap G^{-1}(\Omega_q) \cap \hat{C}_2^n \) and \( \Omega_q \cap G(\Omega_p) \cap \hat{C}_2^n \). Thus, by setting \( U_p := (\hat{\pi}_m)^{-1} (\Omega_p) \cap \hat{C}_2^n \), the family \( \{ (U_p, F_p \circ \hat{\pi}_m|_{U_p}) \}_{p \in X} \) gives the structure of a \( C^r \)-differentiable \( m \)-dimensional DC-manifold on \( X \), which can be considered as a DC-submanifold of \( N^n \). It is obvious that the inclusion map \( X \hookrightarrow N^n \) is a \( C^r \)-differentiable DC-immersion. \( \square \)

We then prove the following:

Proposition 2.16. Let \( X \) be an admissible subset (with the residual set \( \Sigma \)) of a \( C^r \)-manifold \( N^n \) of dimension \( n \). Suppose that for each \( p \in \Sigma \), there exist

1. an open subset \( \Omega_p \) of \( \mathbb{R}^n \), and
2. a \( C^r \)-differentiable extended parametrization \( F : \Omega_p \to N^n \) of \( X \) such that

\( p \in F(\Omega_p) \).

Then \( X \) has the structure of a \( C^r \)-differentiable DC-submanifold of \( N^n \) such that the residual set is the set of DC-points and the inclusion map \( X \hookrightarrow N^n \) is a DC-immersion.

Proof. We let \( q \in X \setminus \Sigma \). Since \( X \setminus \Sigma \) is a \( C^r \)-submanifold of \( N^n \), we can take a sufficiently small local coordinate neighborhood \( (W, \psi) \subset (X \setminus \Sigma) \) of \( q \) satisfying \( W \subset X \setminus \Sigma \). Since \( \psi(W) \) is an open subset of \( \mathbb{R}^n \), this can be considered as an open subset of \( \hat{C}_2^n \). Moreover, without loss of generality, we may assume that there exist

- an open subset \( \Omega_0(\subset \mathbb{R}^n) \) satisfying \( \Omega_0 \cap \hat{C}_2^n = \psi(W) \),
- an open subset \( \Omega \) of \( N^{n+1} \) such that \( \Omega \cap X = W \), and
- an extended parametrization \( F : \Omega_0 \to \Omega \) of \( X \)

such that \( q \in F(\Omega_0) \). So the assertion follows from Lemma 2.15. \( \square \)

Here, we show that the image \( \mathcal{E} \) and the extended images \( \mathcal{P} \) and \( \mathcal{H} \) of geometric catenoids in \( \mathbb{R}^3 \) given in the introduction can be considered as DC-submanifolds in \( \mathbb{R}^3 \) as follows:

Example 2.17. The subset \( \mathcal{E} \) of \( \mathbb{R}^3 \) given in (2.10) coincides with the image of the elliptic \( G \)-catenoid \( f_E \), which can be characterized as the zero set of the function

\( \tau_E := x^2 + y^2 - \sinh^2 t. \)

The exterior derivative of \( \tau_E \) vanishes only at the origin \( 0 \) of \( \mathbb{R}^3 \). So \( \mathcal{E} \setminus \{0\} \) has the structure of a (usual) real analytic 2-manifold and \( \mathcal{E} \) is an admissible subset of \( \mathbb{R}^3 \). Moreover, the map defined by

\[ F_E : \mathbb{R}^3 \ni (x_1, x_2, x_3) \to (\sinh^{-1} x_3, x_1, x_2) \in \mathbb{R}^3 \]

is an extended parametrization of \( \mathcal{E} \) (cf. Definition 2.11), where \( \sinh^{-1} \) is the inverse function of sinh. By Proposition 2.16, \( \mathcal{E} \) is a real analytic DC-submanifold of \( \mathbb{R}^3 \).
Example 2.18. We consider the subset $\mathcal{P}$ given by (1.7) of $\mathbb{R}^3_1$, which is the zero-level set of the real analytic function $\tau_{\mathcal{P}} = 12(x^2 - t^2) - (x + t)^4 + 12y^2$. The exterior derivative of $\tau_{\mathcal{P}}$ vanishes only at the origin $0$ of $\mathbb{R}^3_1$. So $\mathcal{P} \setminus \{(0,0,0)\}$ has the structure of a (usual) real analytic 2-manifold and $\mathcal{P}$ is an admissible subset of $\mathbb{R}^3$. If we set

$$x_1 + x_3 := x + t, \quad x_1 - x_3 := 12(x - t) - (x + t)^3, \quad x_2 := 2\sqrt{3}y,$$

then $\tau_{\mathcal{P}} = 0$ is equivalent to the relation $(x_1)^2 + (x_2)^2 = (x_3)^2$. Then the map $F_{p}: \mathbb{R}^3 \to \mathbb{R}^3_1$ defined by

$$F_{p}(x_1, x_2, x_3) := \frac{1}{24}(11x_1 + 13x_3 - (x_1 + x_3)^3, 13x_1 + 11x_3 + (x_1 + x_3)^3, 4\sqrt{3}x_2)$$

is an extended parametrization of $\mathcal{P}$ (cf. Definition 2.11), and by Proposition 2.16 $\mathcal{P}$ is a real analytic DC-submanifold of $\mathbb{R}^3_1$.

Example 2.19. We consider the subset $\mathcal{H}$ given by (0.9), which is the zero-level set of the real analytic function $\tau_{\mathcal{H}} = \sin^2 x + y^2 - t^2$. It can be easily checked that $\mathcal{H}$ is an admissible subset of $\mathbb{R}^3_1$ whose residual set is $\Sigma = \{(0, \kappa \pi, 0) ; k \in \mathbb{Z}\}$. We set

$$\Omega_0 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 ; |x_1| < 1\}$$

and, for every $k \in \mathbb{Z}$, consider the map

$$F_k : \Omega_0 \ni (x_1, x_2, x_3) \mapsto (x_3, k\pi + \arcsin x_1, x_2) \in \mathbb{R}^3_1,$$

which is an extended parametrization of $\mathcal{H}$ (cf. Definition 2.11) such that $(0, k\pi, 0) \in F_k(\Omega_0)$. By Proposition 2.16 $\mathcal{H}$ is a real analytic DC-submanifold of $\mathbb{R}^3$.

Definition 2.20. Let $X^m$ be a connected $m$-dimensional DC-manifold, and $N^n$ be an $n$-dimensional real analytic manifold. A DC-map $f : X^m \to N^n$ is said to be $m$-dimensional if there exists an open dense subset $O$ of $X^m$ consisting of non-DC-points of $X^m$ such that $f$ is an immersion on $O$.

By definition, DC-immersion from a connected $m$-dimensional DC-manifold is a special case of $m$-dimensional DC-map. From here we restrict our attention to the real analytic case. The following assertion holds:

Proposition 2.21. Let $X^m$ be a connected DC-manifold, and $f : X^m \to N^n$ ($m \leq n$) be a real analytic DC-map (Definition 2.24). If there exists a non-DC-point $p \in X^m$ at which $(df)_p$ is injective, then $f$ is an $m$-dimensional DC-map.

Proof. We denote by $\Sigma$ the set of DC-points of $X^m$. Then $X^m \setminus \Sigma$ is a union of connected real analytic manifolds $\{C_\lambda\}_{\lambda \in \Lambda}$. We can take $\mu_0 \in \Lambda$ so that $p \in C_{\mu_0}$. Since $(df)_p$ is injective, $f|_{C_{\mu_0}}$ gives an immersion on a dense subset of $C_{\mu_0}$. It is sufficient to show that each $f|_{C_\lambda}$ ($\lambda \in \Lambda$) gives an immersion on a dense subset of $C_\lambda$. We suppose that it fails. Then there exists $\lambda_0$ such that the rank of $(df)_p$ is less than $m$ for each $p \in C_{\lambda_0}$. Since $X^m$ is connected, there exists a continuous map $\gamma : [0,1] \to X^m$ such that $\gamma(0) = p$ and $\gamma(1) \in C_{\lambda_0}$. Since $\Sigma$ is discrete, $\gamma$ passes through only finitely many connected components of $X \setminus \Sigma$. So we may assume that $C_{\mu_0}$ is adjacent to $C_{\lambda_0}$, that is, there exists a point $q \in \Sigma$ which connects to $C_{\lambda_0}$ with $C_{\mu_0}$, without loss of generality. We let $\Phi$ be a local inverse DC-coordinate system such that $q \in \Phi(U)$ and $U$ is connected. Then, there exists $x_0 \in U$ such that $\Phi(x_0) = q$. We can take open subsets $V, W \subset U$ such that $\Phi(V) \subset C_{\lambda_0}$ and $\Phi(W) \subset C_{\mu_0}$. By our choice of $\lambda_0$, the rank of the differential of $f \circ \Phi$ is less than $m$ on $V$. Then, by the real analyticity of $f$, the rank of the differential of $f \circ \Phi$ must be less than $m$ on $U$ because of the connectedness of $U$. Since $\Phi(W) \subset C_{\mu_0}$, this contradicts the fact that $f|_{C_{\mu_0}}$ gives an immersion on a dense subset of $C_{\mu_0}$. □
Definition 2.22. Let $X^m$ be connected DC-manifolds of dimension $m$, and $f_i : X_i^m \to N^n$ ($i = 1, 2$) be two real analytic DC-analytic maps. Then $f_2$ is called an analytic extension of $f_1$ if there exists a DC-analytic map $\varphi : X_1^m \to X_2^m$ such that $f_1 = f_2 \circ \varphi$ and $f_2(X_1^m)$ is a proper subset of $f_2(X_2^m)$.

Proposition 2.23. Let $f : X^m \to N^n$ be a real analytic $m$-dimensional DC-map. If $f(X^m)$ is analytically complete, then $f$ does not admit any analytic extension.

To prove this, we prepare the following:

Lemma 2.24. Let $S$ be an analytically complete subset of $N^n$, and let $g : M^m \to N^n$ be an $m$-dimensional real analytic map defined on a connected $m$-dimensional real analytic manifold $M^m$ (cf. Definition 2.22). If there exists a non-empty open subset $W$ of $M^m$ satisfying $g(W) \subset S$, then $g(M^m)$ is also contained in $S$.

Proof. Since $g$ is $m$-dimensional, we can find a regular point $p \in U$ of $g$. We fix an arbitrary point $q \in M^m$. Since $M^m$ is connected, there exists a real analytic map $\sigma : [0, 1] \to M^m$ such that $\sigma(0) = p$ and $\sigma(1) = q$ (cf. 10 the footnote on page 402]). Since $W(\subset M^m)$ is an open subset, there exists $\varepsilon > 0$ such that $\sigma([0, \varepsilon])$ is contained in $W$. Then $g \circ \sigma([0, \varepsilon]) \subset S$, and the analytic completeness of $S$ implies that $g \circ \sigma(1)$ also lies in $S$. \qed

Proof of Proposition 2.23. By way of contradiction, we suppose that $g : Y^m \to N^n$ is a DC-map giving an analytic extension of $f$. By definition of analytic extension, there exists a DC-analytic map $\varphi : X_1^m \to X_2^m$ such that $f = g \circ \varphi$ on $X_1^m$. Since $f$ is $m$-dimensional, applying Proposition 2.21 one can easily observe that $\varphi$ and $g$ are both $m$-dimensional DC-maps. Since $f$ is $m$-dimensional, there exists a non-DC-point $p_0$ and a neighborhood $W$ of $p_0$ so that $f|_W$ is an immersion. Since $\Sigma$ is discrete, we may assume that $\varphi(W)$ does not contain any DC-points. In particular, $\varphi|_W$ is an immersion between the same dimensional manifolds, and so, we may assume that $\varphi$ is a $C^\infty$ diffeomorphism between $W$ and $\varphi(W)$.

Since $g$ is an analytic extension of $f$, we can find a point $q \in Y^m$ such that $g(q) \notin f(X^m)$. Since $Y^m$ is connected, there exists a continuous map $\gamma : [0, 1] \to Y^m$ such that $\gamma(0) = p_0$ and $\gamma(1) = q$. Since the set of DC-points $\Sigma$ of $Y^m$ are discrete, the map $\gamma$ passes through only finitely many DC-points $y_1, \ldots, y_k$. We let $(U_j, \Phi_j)$ ($j = 1, \ldots, k$) be a local inverse DC-coordinate system so that $y_j \in \Phi_j(U_j)$ and $U_j$ is connected. By changing the order of $y_1, \ldots, y_k$ if necessary, there exist $k$ points $t_1, \ldots, t_k$ such that $\gamma(t_j) = y_j$ ($j = 1, \ldots, k$) and

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1.$$ 

Then there exist connected components $C_1, \ldots, C_{k+1}$ of $Y^m \setminus \Sigma$ such that $\gamma([t_{j-1}, t_j]) \subset C_j$ ($j = 1, \ldots, k+1$).

By Proposition 2.21, each $g|_{C_j}$ ($j = 1, \ldots, k+1$) is an $m$-dimensional analytic map. By Proposition 2.21, the fact $g(\varphi(W)) \subset f(X^m)$ implies that $g(C_1) \subset f(X^m)$. Since $g(q) \notin f(X^m)$, we have $g(C_{k+1}) \notin f(M^m)$. So we can find an index $j \in \{1, \ldots, k\}$ such that $g(C_j) \subset f(X^m)$ and $g(C_{j+1}) \notin f(X^m)$. Taking a sufficiently small $\varepsilon > 0$ so that

$$\gamma([t_j - \varepsilon, t_j + \varepsilon]) \subset \Phi(U_j),$$

we can choose $x_0, z_0 \in U_j$ so that $\Phi_j(x_0) = \gamma(t_j - \varepsilon)$ and $\Phi_j(z_0) = \gamma(t_j + \varepsilon)$. Moreover, we can also choose open subsets $V, W$ of $U_j$ satisfying

$$x_0 \in V, \quad z_0 \in W, \quad \pi^m_{DC}(V) \subset C_j, \quad \pi^m_{DC}(W) \subset C_{j+1}.$$ 

Since $h := g \circ \pi^m_{DC}$ is real analytic on the connected set $U_j$, the fact $h(V) \subset f(X^m)$ and Lemma 2.24 imply that $h(U_j) \subset f(X^m)$. In particular, $g(W) \subset f(X^m)$.
holds. Since $\pi^m_{DC}(W)$ is an open subset of $C$, by Lemma 2.24 we can conclude $g(C) \subset f(X^m)$, a contradiction. \hfill \Box

For the analytic extension of G-catenoids and W-catenoids in $R^3_1$, the following can be shown.

**Theorem 2.25.** The image of the elliptic G-catenoid $f_E$ in $R^3_1$ (given in the introduction) is analytically complete. On the other hand, the image of the other G-catenoids $f_P$ and $f_H$ and the W-catenoid $f_K$ in $R^3_1$ admit analytic extensions, which are analytically complete. Moreover, after taking the analytic completions, all of the images of these catenoids are DC-submanifolds of $R^3_1$.

**Proof.** By the classification of G-catenoids and W-catenoids in $R^3_1$ (cf. Appendix A), we know that G-catenoids are $f_E, f_P, f_H$ and W-catenoids are $f_E, f_K$. The image of $f_E$ is the set $\mathcal{E}$ in Example 2.17 which is globally analytic. So $\mathcal{E}$ is analytically complete.

On the other hand, as mentioned in the introduction, the image of $f_K$ is a proper subset of the entire (zero-mean curvature) graph of the function $t = y \tanh x$. Since, the entire graph is globally analytic (cf. Proposition 1.5), it gives an analytic extension of $f_K$, which is analytically complete.

As shown in Example 2.18 the set $\mathcal{P}$ given in (0.7) has the structure of a DC-submanifold whose inclusion map can be considered as an analytic extension of the map $f_P$ given in (0.3). Since $\mathcal{P}$ is globally analytic, it is analytically complete (cf. Proposition 1.5). Similarly, the set $\mathcal{H}$ given in (0.9) has the structure of DC-submanifold (cf. Example 2.24), whose inclusion map can be considered as an analytic extension of the map $f_H$ given in (0.5). Since $\mathcal{H}$ is globally analytic, it is analytically complete (cf. Proposition 1.5). \hfill \Box

### 3. Analytic Completeness for the Images of Real Analytic DC-maps

In this section, we give a criterion which we can apply to show the analytic completeness of G-catenoids in Section 4. Throughout this section, we fix a real analytic $m$-dimensional DC-manifold $X^m$. We first prepare several lemmas:

**Lemma 3.1.** Let $X^m$ be a real analytic DC-manifold and $f : X^m \to N^n$ a DC-immersion. Then for each $p \in X^m$, there exists a triple $(W, \Omega, \tau)$ consisting of

- a relatively compact neighborhood $W(\subset X^m)$ of $p$,
- a neighborhood $\Omega(\subset N^n)$ of $f(p)$, and
- a real analytic function $\tau : \Omega \to R^{n-m}$ such that $Z(\tau) = f(W)$ and $f|_{\overline{W}}$ is a homeomorphism between $W$ and $f(W)$, where $Z(\tau)$ is the zero set of the function $\tau$.

**Proof.** We fix $p \in X$ arbitrarily. Then there exist

- an open subset $\Omega$ of $R^n$,
- a real analytic diffeomorphism $F : \Omega \to N^n$, and
- a local inverse DC-coordinate system $(U, \Phi)$ satisfying $p \in \Phi(U)$

such that $\Omega \cap C^m = \hat{x}^m_{DC}(U \times \{0_{n-m-1}\})$ and $\Phi = F \circ \hat{x}^m_{DC}|_{U \times \{0_{n-m-1}\}}$ holds on $U$ by identifying $U$ with $U \times \{0_{n-m-1}\}$ (cf. Definition 2.3). Since

$$C^m_2 = \left\{ (x_1, \ldots, x_n) \in R^n : \sum_{i=1}^m x_i^2 = x_{m+1}^2, x_{m+2} = \cdots = x_n \right\},$$

the map

$$\tau := (h_0 \circ F^{-1}, x_{m+2} \circ F^{-1}, \ldots, x_n \circ F^{-1})$$

$$h_0 := -x_{m+1}^2 + \sum_{i=1}^m x_i^2$$

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gives the desired real analytic function. In fact, we can take a relatively compact neighborhood \( W \) of \( p \) so that \( f(W) \subset \Omega \). Moreover, we may assume that \( F \) is defined on \( \Omega \) such that \( f(W) = \Omega \cap Z(\tau) \). Since a bijective continuous map from a compact set to a Hausdorff space is a homeomorphism, \( f|_W \) is a homeomorphism between \( W \) and \( f(W) \).

Lemma 3.2. Let \( f : X^m \to N^n \) be a DC-immersion. Suppose that \( I \) is an open interval of \( R \) and that \( \Gamma : I \to N^n \) is a real analytic map satisfying \( \Gamma(I) \subset f(X^m) \). Then there exists a continuous map \( \gamma : J \to X^m \) defined on an open subinterval \( J \) of \( I \) such that \( \Gamma = f \circ \gamma \) on \( J \).

Proof. We let \( \{(U_p, \Omega_p, \tau_p)\}_{p \in X^m} \) be a family of triples determined by the previous lemma. Since \( X^m \) satisfies the second axiom of countability, there exists a sequence \( \{p_j\}_{j=1}^\infty \) of points in \( X^m \) such that \( \{U_p\}_{j=1}^\infty \) is an open covering of \( X^m \). We set

\[
W_j := \Gamma^{-1}(\Omega_p) \subset I
\]

and consider the map \( \lambda_j := \tau_j \circ \gamma : W_j \to R^l \) \((j = 1, 2, \ldots)\). We denote by \( Z_j \) the zero set of \( \lambda_j \) on \( W_j \). Since \( \Gamma(I) \subset f(X^m) \) and \( f(X^m) \subset \bigcup_{j=1}^\infty Z(\tau_j), \) we have \( I = \bigcup_{j=1}^\infty Z_j \). By Baire’s category theorem, there exists a number \( j_0 \) such that \( Z_{j_0} \) has an interior point. So there exists an open subinterval \( J \) of \( I \) such that \( J \subset Z_{j_0} \), that is, \( (J) \subset Z(\tau_{j_0}) \). Since \( f(\Omega_p) \) is a homeomorphism, we can set \( \varphi := (f|_{U_{j_0}})^{-1} \circ \Gamma \), and hence \( f \circ \varphi = \Gamma \) holds on \( J \).

Lemma 3.3. Let \( f : X^m \to N^n \) be a DC-immersion. Suppose that \( I \) is an open interval containing \( \theta \in R \), and \( \gamma_i : I \to X^m \) \((i = 1, 2)\) are two real analytic maps so that \( f \circ \gamma_i(t) = f \circ \gamma_j(t) \) for all \( t \in \Gamma \). If \( \gamma_1(0) = \gamma_2(0) \), then \( \gamma_1 \) coincides with \( \gamma_2 \) as maps on the interval \( I \).

Proof. We consider the subset \( A := \{t \in I : \gamma_1(t) = \gamma_2(t)\} \). Since \( \gamma_1(0) = \gamma_2(0) \), we have \( 0 \in A \), that is, \( A \) is a non-empty set. Obviously, \( A \) is a closed subset. We fix \( t_0 \in A \) arbitrarily. By Lemma 3.1, there exist a neighborhood \( O(\subset X^m) \) of \( \gamma_j(t_0) \), a neighborhood \( \Omega(\subset N^n) \) of \( p_0 := f \circ \gamma_1(t_0) = f \circ \gamma_2(t_0) \) \((j = 1, 2)\), and a real analytic function \( \tau : \Omega \to R^l \) \((l \geq 1)\) such that \( Z(\tau) = f(O) \) and \( f|_O \) gives a homeomorphism between \( O \) and \( f(O) \). By the continuity of \( \gamma_j \), \((j = 1, 2)\), there exists \( \delta > 0 \) such that \( \gamma_j(t) \in O \) for \( |t - t_0| < \delta \) for \( j = 1, 2 \). By Lemma 3.1, the inverse map \( g \) of \( f|_O \) is defined and \( \gamma_1(t) = g \circ f \circ \gamma_2(t) \) holds when \( t \in (t_0 - \delta, t_0) \). Then \( \gamma_1(t) = g \circ f \circ \gamma_2(t) \) holds for \( t \in (t_0 - \delta, t_0 + \delta) \), because \( \gamma_1 \) and \( \gamma_2 \) are real analytic. So \( (t_0 - \delta, t_0 + \delta) \) is a subset of \( A \), and \( A \) is an open subset of \( I \). Since \( I \) is connected, we have \( A = I \), proving the assertion.

We prepare one more lemma:

Lemma 3.4. Let \( f : X^m \to N^n \) be a DC-immersion, and let \( \Gamma : [a, b] \to N^n \) \((a < b)\) be a real analytic map. If there exist \( c \in (a, b) \), a continuous map \( \gamma : [a, c) \to X^m \) and a sequence \( \{t_n\} \) on \([a, c)\) converging to \( c \) such that \( \Gamma = f \circ \gamma \) on \([a, c)\) and \((p :=) \lim_{n \to \infty} \gamma(t_n) \) exists, then there exists \( \theta < \varepsilon \leq b - c \) and a continuous map \( \tilde{\gamma} : [a, c + \varepsilon) \to X^m \) such that \( f \circ \tilde{\gamma} = \Gamma \) and \( \tilde{\gamma}(t) = \gamma(t) \) for \( t \in [a, c) \).

Proof. By the continuity of \( f \), we have

\[
f(p) = f(\lim_{k \to \infty} \gamma(t_k)) = \lim_{k \to \infty} f(\gamma(t_k)) = \lim_{k \to \infty} \Gamma(t_k) = \Gamma(c).
\]
We then take a triple \((U, \Omega, \tau)\) at \(p\) as in Lemma \ref{lem:5.1} so that \(\overline{U}\) is compact and \(f(\overline{U})\) coincides with the zero set of the \(\mathbf{R}^{n-m}\)-valued function \(\tau\). By the continuity of \(\Gamma\), there exists a sufficiently small \(\varepsilon(>0)\) such that \(\Gamma(c-\varepsilon, c+\varepsilon) \subset \Omega\). Since
\[
\Gamma([a, c]) = f \circ \gamma([a, c]) \subset f(X) = Z(\tau),
\]
\(\tau \circ \Gamma(t)\) is equal to zero for \(t \in [a, c]\), by the real analyticity of \(\tau \circ \Gamma\), we have \(\tau \circ \Gamma(t) = 0\) for \(t \in (c - \varepsilon, c + \varepsilon)\), which implies \(\Gamma((c - \varepsilon, c + \varepsilon)) \subset f(\overline{U})\). Since \(f|_{\overline{U}}\) is a homeomorphism (cf. Lemma \ref{lem:6.1}), \((f|_{\overline{U}})^{-1} \circ \Gamma(t)\) \((t \in (c - \varepsilon, c + \varepsilon))\) is a continuous map which coincides with \(\gamma\) on \((c - \varepsilon, c)\), and so \(\gamma\) can be extended to \([a, c + \varepsilon]\).

To define “arc-properness”, we give the following definition:

**Definition 3.5.** Let \(X\) be a topological space, and let \(f : X \to \mathbf{R}^n\) be a continuous map. A continuous map \(\sigma : [0, 1) \to X\) is said to be \((C^r, f)\)-extendable if \(f \circ \sigma(t)\) can be extended to a \(C^r\)-map defined on \([0, 1]\), where \(r\) is a non-negative integer or \(\infty\) or \(r = \omega\).

We now define the concept “arc-properness” as follows:

**Definition 3.6.** Let \(X\) be a locally compact Hausdorff space satisfying the second axiom of countability (as a consequence, \(X\) is metrizable, affecting convergence of sequences of points on \(X\)). We let \(f : X \to \mathbf{R}^n\) be a continuous map. We fix \(r\) as a non-negative integer or \(\infty\) or \(r = \omega\). Then \(f\) is called \(C^r\)-arc-proper if for each \((C^r, f)\)-extendable continuous map \(\sigma : [0, 1) \to X\), there exists a sequence \(\{t_k\}_{k=1}^{\infty}\) on \([0, 1)\) converging to 1 such that \(\lim_{k \to \infty} \sigma(t_k)\) exists.

**Proposition 3.7.** Properness implies \(C^r\)-arc-properness for \(r \in \{0, 1, \ldots\} \cup \{\infty, \omega\}\).

**Proof.** Let \(f : X \to \mathbf{R}^n\) be a proper continuous map. We fix a \((C^r, f)\)-extendable continuous map \(\sigma : [0, 1) \to X\). Then \(f \circ \sigma(t)\) can be extended to a \(C^r\)-map \(\Gamma : [0, 1] \to \mathbf{R}^n\). Since \(K := \Gamma([0, 1])\) is compact, the assumption that \(f\) is proper implies \(f^{-1}(K)\) is compact. We let \(\{t_k\}_{k=1}^{\infty}\) be a sequence on \([0, 1)\) converging to 1. Since \(f^{-1}(K)\) is compact, the sequence \(\{\sigma(t_k)\}_{k=1}^{\infty}\) has a convergent subsequence.

**Example 3.8.** We consider the image \(f(\mathbf{R})\) of the real analytic immersion
\[
(3.1) \quad f : \mathbf{R} \ni t \mapsto (t, \sqrt{2}t) \in \mathbf{T}^2 := \mathbf{R}^2/\mathbf{Z}^2.
\]
Although \(f\) is not a proper map, as will be seen below, \(f\) is \(C^0\)-arc-proper. Let \(\Gamma : [0, 1] \to \mathbf{T}^2\) be a continuous map satisfying \(\Gamma([0, 1]) \subset f(\mathbf{R})\). We denote by \(\pi : \mathbf{R}^2 \to \mathbf{T}^2\) the canonical covering projection. Then by the homotopy lifting property on covering spaces, there exists a continuous map \(\tilde{\Gamma} : I \to \mathbf{R}^2\) such that \(\pi \circ \tilde{\Gamma} = \Gamma\) on \(I\) and
\[
\tilde{\Gamma}(0) \in A, \quad A := \{(x, y) \in \mathbf{R}^2 : y = \sqrt{2}x\}.
\]
Since \(A\) is a closed subset of \(\mathbf{R}^2\), we have \(\tilde{\Gamma}(1) \in A\) and so
\[
\Gamma(1) = \pi \circ \tilde{\Gamma}(1) \in \pi(A) = f(\mathbf{R}).
\]
Thus, \(f\) is \(C^0\)-arc-proper.

**Example 3.9.** Consider the analytic map \(f(t) := (t, e^{-1/t})\) \((t > 0)\) in the \(xy\)-plane. As will be shown below, \(f\) is a \(C^\omega\)-arc-proper immersion. We set \(S := f([0, \infty))\). Let \(\sigma : [0, 1) \to (0, \infty)\) be a \((C^\omega, f)\)-extendable continuous map. Then there exists a real analytic map \(\Gamma : [0, 1] \to \mathbf{R}^2\) such that \(\Gamma(s) = f \circ \sigma(s)\) holds for \(s \in [0, 1]\).
We can write $\Gamma(s) = (x(s), y(s))$ $(s \in [0, 1])$. Suppose that $\Gamma(1) \notin S$. Since $S \setminus S = \{(0,0)\}$, we have $\Gamma(1) = (0,0)$. Moreover, since $\Gamma([0,1]) \subset S$, we have

$$y(s) = e^{-1/x(s)}, \quad x(s) > 0 \quad (s \in [0,1]).$$

Then one can inductively check that

$$\lim_{s \to 1^-} \frac{d^k y}{ds^k}(s) = 0 \quad (k = 0,1,2,\ldots)$$

holds. This contradicts the fact that $y(s)$ is a real analytic function at $s = 1$. Thus, we have $\Gamma(1) \in S$. Then $x(1) > 0$ and

$$(x(s), y(s)) = \Gamma(s) = f \circ \sigma(s) = (\sigma(s), e^{-1/\sigma(s)})$$

hold. In particular, $\sigma(s) = x(s)$ converges to $x(1)$ as $s$ tends to 1. Thus, $f$ is a $C^\omega$-arc proper map. By the reflection with respect to the $y$-axis, the map $f$ can be extended as a $C^\omega$-map defined by

$$F(t) := \begin{cases} f(t) & (t > 0), \\ (0,0) & (t = 0), \\ (t, e^{t^2}) & (t < 0). \end{cases}$$

Then the map $F(t)$ itself is a smooth map on $\mathbb{R}^2$, and $\sigma(t) := t$ $(t > 0)$ satisfies $f \circ \sigma(t) = F(t)$ for $t > 0$. Since $\lim_{t \to 0} \sigma(t) = 0$ holds and the origin 0 does not belong to $(0,\infty)$, $f$ is not $C^\omega$-arc proper. So $f$ gives an example which is not $C^\omega$-arc proper but is $C^\omega$-arc proper.

**Theorem 3.10.** Let $f : X^m \to N^n$ be a DC-immersion which is $C^\omega$-arc-proper as a continuous map. Then $f(X^m)$ is analytically complete. In particular, if $f$ is a proper map, then $f(X^m)$ is analytically complete.

We will apply this to discuss the analytic extension of G-catenoids in Section 4.

**Proof.** Let $\Gamma : [0,1] \to N^n$ be an analytic map such that $\Gamma([0,\varepsilon)) \subset f(X^m)$ for some $\varepsilon \in (0,1]$. It is sufficient to show that $\Gamma(1) \in f(X^m)$. Given $0 \leq a < b \leq 1$, a continuous map $\sigma : [a,b] \to X^m$ is called a lift of $\Gamma$ on the interval $[a,b]$ if $f \circ \sigma(t) = \Gamma(t)$ for all $t \in [a,b]$. By Lemma 3.3 if two lifts $\sigma_1$ and $\sigma_2$ of $\Gamma$ on $[a,b]$ satisfy $\sigma_1(a) = \sigma_2(a)$, then $\sigma_1(t) = \sigma_2(t)$ for all $t \in [a,b]$. By Lemma 3.2 there exists $s_0, c_0$ with $0 < s_0 < c_0 < \varepsilon$ and a lift $\sigma_0 : [s_0, c_0] \to X^m$ of $\Gamma$ on $[s_0, c_0]$. Let

$$C := \{c_1 \in (s_0,1] ; \text{there exists a lift } \sigma \text{ of } \Gamma \text{ on } [s_0, c_1) \text{ with } \sigma(s_0) = \sigma(0)\}.$$

Then $C$ is nonempty, since $c_0 \in C$. Let $c$ be the supremum of $C$. By definition, for each positive integer $k$, there exists a lift $\sigma_k : I_k \to X^m$ of $\Gamma$ defined on an interval $I_k$ containing $(0,s_0-1/k)$ such that $\sigma_k(s_0) = \sigma_0(s_0)$. By Lemma 3.3 $\sigma_k = \sigma_{k+1}$ holds on $(0,s_0-1/k)$. So there exists a unique continuous map $\gamma : (0,s_1) \to X$ satisfying $\gamma = \sigma_k$ on $[s_0, s_1 - 1/k)$. If $s_1 < 1$, then by Lemma 3.3 $\gamma$ can be continuously extended on $[s_1, c + \varepsilon_1]$ for some $\varepsilon_1 > 0$ such that $f \circ \gamma = \Gamma$, which contradicts the maximality of $c$. So $s_1 = 1$. Taking a sequence $\{t_k\}_{k=1}^\infty$ on $[s_0,1)$ converging to 1, we have

$$\Gamma(1) = \lim_{k \to \infty} \Gamma(t_k) = f(\lim_{k \to \infty} \gamma(t_k)) \in f(X^m),$$

proving the assertion. \hfill \Box

As a consequence, we get the following:

**Corollary 3.11.** Let $X^m$ be a connected $C^\omega$-manifold and $f : X^m \to N^n$ a $C^\omega$-immersion. If $f$ is a $C^\omega$-arc-proper map, then $f(X^m)$ is analytically complete.

Moreover, the following assertion holds:
Corollary 3.12. Let $X^m$ be a connected real analytic DC-manifold of dimension $m$ and $f : X^m \to N^n$ a real analytic DC-immersion. Suppose that $N^n$ is properly $C^\infty$-embedded in a $C^\infty$-manifold $L^j (j \geq n)$, and denote by $i : N^n \hookrightarrow L^j$ the inclusion map. Then $f(X^m)$ is analytically complete in $N^n$ if and only if it is analytically complete in $L^j$.

Proof. The “if” part is obvious. We suppose that $f(X^m)$ is analytically complete in $N^n$. We consider a real analytic map $\Gamma : [0,1] \to L^j$ such that $\Gamma([0,\varepsilon)) \subset f(X^m)$ for sufficiently small $\varepsilon > 0$. Then $\Gamma([0,\varepsilon)) \subset f(X^m) \subset N^n$ holds. Since $N^n$ is analytically complete in $L^j$, we have $\Gamma([0,1]) \subset N^n$. Since $f(X^m)$ is an analytically complete subset of $N^n$, we also have $\Gamma([0,1]) \subset f(X^m)$. Thus $f(X^m)$ is analytically complete in $L^j$. □

4. Analytic extensions of catenoids in $S^3_k$

In this section, we define “real analytic space-like CMC-1 DC-immersions”, and we show that each G-catenoid or its appropriate analytic extension in the de Sitter 3-space $S^3_1$ belongs to this class and that all of the images of these maps are analytically complete.

4.1. Space-like CMC-1 DC-immersions. We give the following definition.

Definition 4.1. Let $f : X^2 \to S^3_1$ be a real analytic DC-immersion defined on a connected 2-dimensional real analytic DC-manifold $X^2$. Then $f$ is called a real analytic space-like CMC-1 DC-immersion if there exists an open dense subset $O$ of $X^2 \setminus \Sigma$ such that the restriction of $f$ to $O$ is a space-like constant mean curvature one (i.e. CMC-1) immersion, where $\Sigma$ is the set of DC-points in $X^2$.

Similarly, real analytic ZMC DC-immersion in $R^3_1$ can be defined (see Appendix A for details). We prove the following:

Proposition 4.2. Let $f : X^2 \to S^3_1$ be a real analytic DC-immersion defined on a connected 2-dimensional real analytic DC-manifold $X^2$. Suppose that there exists a non-empty open subset $O$ of $X^2$ such that the restriction $f|_O : O \to S^3_1$ gives a real analytic space-like CMC-1 DC-immersion. Then $f$ is a real analytic space-like CMC-1 DC-immersion.

Proof. We let $\Sigma$ be the set of DC-points in $X^2$. We fix a point $p \in X^2 \setminus \Sigma$ arbitrarily. Since $X^2$ is connected, there exists a continuous map $\gamma : [0,1] \to X^2$ such that $\gamma(0) \in O$ and $\gamma(1) = p$. Since $\gamma([0,1])$ is compact, we can take a partition $0 = t_0 < t_1 < \ldots < t_k = 1$ and a family of inverse DC-coordinate systems $\{(U_i, \Phi_i)\}_{i=0}^k$ such that

- $U_i$ ($i = 0, \ldots, k$) are connected and $\gamma(t_i) \in \Phi_i(U_i)$,
- $\Phi_i(U_0) \subset O$, and
- $\Phi_{i-1}(U_{i-1}) \cap \Phi_i(U_i)$ is non-empty for $i = 1, \ldots, k$.

We know that $f|_{\Phi_0(U_0)}$ is a space-like CMC-1 DC-immersion on $\Phi_0(U_0)$. If the assertion fails, then there exists a number $j(\geq 1)$ so that $f|_{\Phi_{j-1}(U_{j-1})}$ is a space-like CMC-1 DC-immersion, but $f|_{\Phi_j(U_j)}$ is not a space-like CMC-1 DC-immersion. Since $U_{j-1}$ and $U_j$ are open submanifolds of $S^3 \times R$, we may assume that they have local coordinate systems. Let $A_i$ and $B_i$ ($i = 0, \ldots, n$) be the two particular analytic functions defined on $U_i$ in [17] (2.3) (in [17], $\alpha$, $\beta$ correspond to $A_i$ and $B_i$ in our paper), which have the property that the mean curvature $H$ of $f_i := f \circ \Phi_i$ satisfies

$$H = \pm A_i/(2B_i)^{3/2} \quad (i = 0, \ldots, n)$$

on an open dense subset of $U_i$, and $B_i$ is positive (resp. zero, negative) at a space-like (resp. light-like, time-like) point. We take a non-empty open subset $W$ of $X^2$
such that \( W \subset \Phi_{j-1}(U_{j-1}) \cap \Phi_j(U_j) \). Since \( \Sigma \) is a discrete subset of \( X^2 \), we may assume that \( W \) does not contain any DC-points. Since \( f \) is a real analytic space-like DC-immersion, by taking \( W \) sufficiently small, we may assume that \( f_{j-1} \) gives a space-like CMC-1 immersion on \( \Phi_j^{-1}(W) \). In particular, \( B_{j-1} \) is positive-valued on \( \Phi_j^{-1}(W) \). So, \( (4.1) \) implies

\[
A_{j-1}(x)^2 = 4B_{j-1}(x)^3 \quad (x \in \Phi_j^{-1}(W)).
\]

Since the property that “space-like with \( |H| = 1 \)” does not depend on the choice of local coordinate system, we have

\[
A_j(x)^2 = 4B_j(x)^3 \quad (x \in \Phi_j^{-1}(W)).
\]

Since \( \Phi_j^{-1}(W) \subset U_j \) and \( U_j \) is connected, the real analyticity of \( f_j \) implies that \( A_j^3 = 4B_j^3 \) holds on \( U_j \). As a consequence, \( B_j \geq 0 \) holds on \( U_j \) and \( f \) gives a space-like CMC-1 immersion on an open dense subset of \( \Phi_j(U_j) \), a contradiction. \( \square \)

**Remark 4.3.** As a consequence of [24, Theorem D], if a real analytic CMC-1 immersion \( f : (U; u, v) \to S^3_1 \) admits a light-like point \( p \in U \), then there exists a regular map \( \sigma : (-\varepsilon, \varepsilon) \to U \) \((\varepsilon > 0)\) such that

- \( \sigma(0) = p \),
- \( \sigma(t) \) \((|t| < \varepsilon)\) are light-like points of \( f \), and
- \( f \circ \sigma \) is a light-like geodesic segment in \( S^3_1 \).

This is the reason why the following analytic extensions of CMC-1 catenoids in \( S^3_1 \) often contain light-like geodesic segments.

### 4.2. The classification of G-catenoids.

A geometric catenoid (or a G-catenoid for short) is a space-like CMC-1 face in de Sitter 3-space \( S^3_1 \) (cf. [5] and [12]) which is invariant under the action of a one-parameter subgroup \( G \) of isometries of \( S^3_1 \) with a common fixed point. Such a subgroup \( G \) fixes one or two geodesics, which are called the axes of the subgroup \( G \). We say that a G-catenoid is of type \( T \) (resp. \( L, S \)) if the corresponding axes are time-like (resp. light-like, space-like) geodesics. The analytic part of a G-catenoid can be considered as a CMC-1 face in the sense of [6], that is, it can be expressed as

\[
f = (x_0, x_1, x_2, x_3), \quad \begin{pmatrix} x_0 + x_3 \\ x_1 + ix_2 \\ x_0 - x_3 \end{pmatrix} = F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F^*,
\]

where \( F \) is a holomorphic immersion into \( \text{SL}(2, \mathbb{C}) \) defined on a certain Riemann surface. Such an \( F \) is called the holomorphic lift of the surface (cf. [6] for details). G-catenoids were completely classified by the last author [25] as in Table 1. We remark that a surface of type LE (resp. LH) with holomorphic lift \( F_L(\alpha, z) \) is congruent to \( F_L(1, z) \) (resp. \( F_L(-1, z) \)).

Since the monodromy matrix of the secondary Gauss map of a G-catenoid takes values in \( \text{SU}(1, 1) \) whose trace is real and less than (resp. greater than, equal to) two, we call such a matrix elliptic (resp. hyperbolic, parabolic). So we call \( f_{TE}^\alpha \) a G-catenoid of type TE, since the monodromy of its secondary Gauss map is elliptic.

From now on, we show that each G-catenoid or its suitable analytic extension is analytically complete in \( S^3_1 \). Since \( S^3_1 \) is a properly embedded real analytic submanifold of \( \mathbb{R}^4 \), all those surfaces are also analytically complete in \( \mathbb{R}^4 \) (cf. Corollary 6.12).

### 4.3. G-catenoids of type T.

As mentioned above, there are three subclasses of G-catenoids of type T. In this subsection, we show that the images of all of them are analytically complete.
| axis       | E-type | P-type | H-type |
|------------|--------|--------|--------|
| time-like  | \(F_T(\frac{1-\mu^2}{2}, \mu z)\) | \(F_T(0, z)\) | \(F_T(\frac{1-\mu^2}{2}, \mu z)\) |
| light-like | \(F_L(\alpha, z)\) | — | \(F_L(\alpha, z)\) |
| space-like | \(F_S(\frac{1-\mu^2}{2}, \mu z)\) | \(F_S(0, z)\) | \(F_S(\frac{1-\mu^2}{2}, \mu z)\) |

Table 1. The classification of G-catenoids in [25]. \(F_T, F_L, F_S\) denote the holomorphic lifts of surfaces, which are defined in [25, Lemma 4.2, Section 4].

4.3.1. G-catenoids of type TE. We set \(T^1 := \mathbb{R}/2\pi\mathbb{Z}\). For each \(\mu \in \mathbb{R}_+ \setminus \{1\}\), we consider a real analytic map

\[
f^\mu_{TE}: \mathbb{R} \times T^1 \ni (s, \theta) \mapsto (x_0(s), x_1(s, \theta), x_2(s, \theta), x_3(s)) \in S^3_1,
\]

which gives a G-catenoid of type TE, where

\[
f^\mu_{TE}(s, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Gamma^\mu_{TE}(s), \quad \Gamma^\mu_{TE}(s) := \begin{pmatrix} x_0(s) \\ \frac{1-\mu^2}{2\mu} \sinh s \mu z \\ 0 \\ x_3(s) \end{pmatrix},
\]

and

\[
x_0(s) = \sinh s \cosh \mu s - \frac{(\mu^2 + 1) \cosh s \sinh \mu s}{2\mu},
\]

\[
x_3(s) = \cosh s \sinh \mu s - \frac{(\mu^2 + 1) \sinh s \sinh \mu s}{2\mu}.
\]

Here \(\Gamma^\mu_{TE}(s) (s \in \mathbb{R})\) is a profile curve of \(f^\mu_{TE}\) lying on \(S^3_1 = S^3_1 \cap \{x_2 = 0\}\). The axes of \(f^\mu_{TE}\) are \(\{(\sinh t, 0, 0, \pm \cosh t) : t \in \mathbb{R}\}\). The singular point set of \(f^\mu_{TE}\) is \(\{(0, \theta) \in \mathbb{R} \times T^1\}\), whose image consists of one point \(\{(0, 0, 0, 1)\}\) (see Figure [3]).

Figure 3. G-catenoids of type TE with \(0 < \mu < 1\) (left) and \(\mu > 1\) (right) in the stereographic hollow ball model (cf. [8, (1)]).

The following assertion can be checked easily:

**Lemma 4.4.** For each \(\mu \in \mathbb{R}_+ \setminus \{1\}\), the plane curve \(s \mapsto (x_0(s), x_3(s))\) is a regular curve on \(\mathbb{R}\).
Proof. In fact, we have
\[
\begin{pmatrix}
x_0'(s) \\
x_3'(s)
\end{pmatrix} = \frac{\mu^2 - 1}{2\mu} \begin{pmatrix}
cosh s & \sinh s \\
\sinh s & \cosh s
\end{pmatrix} \begin{pmatrix}
-\mu \cosh s \\
\sinh s
\end{pmatrix},
\]
which implies the assertion. \(\square\)

Proposition 4.5. The map \(f^\mu_{\text{TE}}\) is a real analytic space-like CMC-1 DC-immersion, and its image is analytically complete.

Note that \(f^\mu_{\text{TE}}\) is a W-catenoid with elliptic monodromy as in [6].

Proof. For each \(\mu \in \mathbb{R}_+ \setminus \{1\}\), we set
\[
X^\mu_{\text{TE}} := \left\{ (\xi, \eta, s) \in \mathbb{R}^3 \mid \xi^2 + \eta^2 = \left( \frac{1 - \mu^2}{2\mu} \right)^2 \sinh^2 s \right\}.
\]
It can be easily checked that \(X^\mu_{\text{TE}} \setminus \{0\}\) is a real analytic submanifold of \(\mathbb{R}^3\), and so it is an admissible subset of \(\mathbb{R}^3\). Since the map
\[
\mathbb{R}^3 \ni (x, y, z) \mapsto \left( x, y, \frac{1}{\mu} \sinh^{-1} \left( \frac{2\mu z}{1 - \mu^2} \right) \right) \in \mathbb{R}^3
\]
gives a surjective extended parametrization of \(X^\mu_{\text{TE}}\), by Proposition 2.10, \(X^\mu_{\text{TE}}\) has the structure of a DC-submanifold of \(\mathbb{R}^3\) so that \(0 \in X^\mu_{\text{TE}}\) is a DC-point. Moreover, the inclusion map \(\iota : X^\mu_{\text{TE}} \hookrightarrow \mathbb{R}^3\) is a DC-immersion. By Lemma 4.3, the map \(\Psi : \mathbb{R}^3 \ni (\xi, \eta, s) \mapsto (x_0(s), \xi, \eta, x_3(s)) \in S^3_1\) is an immersion, and so the composition \(\tilde{f}_{\text{TE}} := \Psi \circ \iota\) is also a DC-immersion (cf. Corollary 2.9). Since
\[
f^\mu_{\text{TE}}(s, \theta) = \tilde{f}^\mu_{\text{TE}}(x_1(s, \theta), x_2(s, \theta), s),
\]
the image of \(\tilde{f}^\mu_{\text{TE}}\) coincides with that of \(f^\mu_{\text{TE}}\). The map \(\tilde{f}^\mu_{\text{TE}}\) is a real analytic space-like CMC-1 DC-immersion. It can be easily checked that \(\tilde{f}^\mu_{\text{TE}}\) is a proper map. So, Theorem 3.10 yields that the image of \(\tilde{f}^\mu_{\text{TE}}\) is analytically complete. \(\square\)

4.3.2. G-catenoids of type TP. A G-catenoid of type TP can be expressed as
\[
f_{\text{TP}} : \mathbb{R} \times T^1 \ni (s, \theta) \mapsto (x_0(s), x_1(s, \theta), x_2(s, \theta), x_3(s)) \in S^3_1,
\]
where
\[
f_{\text{TP}}(s, \theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \Gamma_{\text{TP}}(s), \quad \Gamma_{\text{TP}}(s) := \begin{pmatrix}
x_0(s) \\
(s/2) \\
0 \\
x_3(s)
\end{pmatrix},
\]
and
\[
(4.5) \quad x_0(s) = \sinh s - \frac{s}{2} \cosh s, \quad x_3(s) = \cosh s - \frac{s}{2} \sinh s.
\]
Since the monodromy matrix of the secondary Gauss map of \(f_{\text{TP}}\) is parabolic, we call \(f_{\text{TP}}\) a G-catenoid of type TP. Here \(\Gamma_{\text{TP}}(s) (s \in \mathbb{R})\) is the profile curve of \(f_{\text{TP}}\) lying on \(S^3_1 = S^3_1 \cap \{x_2 = 0\}\). The axes of \(f_{\text{TP}}\) are the same as the axes of \(f^\mu_{\text{TE}}\). The singular point set of \(f_{\text{TP}}\) is \(\{(0, \theta) \in \mathbb{R} \times T^1\}\), whose image consists of one point \(\{(0, 0, 0, 1)\}\). Note that \(f_{\text{TP}}\) is the limit of \(f^\mu_{\text{TE}}\) as \(\mu \to 0\). We set
\[
X_{\text{TP}} := \left\{ (\xi, \eta, s) \in \mathbb{R}^3 \mid \xi^2 + \eta^2 = \frac{s^2}{4} \right\}.
\]
Like as in the case of \(X_{\text{TE}}\), the subset \(X_{\text{TP}}\) has the structure of a DC-submanifold of \(\mathbb{R}^3\) such that \(0, 0, 0\) is a DC-point. It can be easily checked that \(\mathbb{R} \ni s \mapsto (x_0(s), x_3(s)) \in \mathbb{R}^2\) is a regular curve. So
\[
\Psi : \mathbb{R}^3 \ni (\xi, \eta, s) \mapsto (x_0(s), \xi, \eta, x_3(s)) \in S^3_1
\]
is an immersion. If we denote the inclusion map by \( \iota : X_{\AP} \hookrightarrow \mathbb{R}^3 \), then the composition \( f_{\AP} := \Psi \circ \iota \) is a DC-immersion. Since \( f_{\AP}(s, \theta) = (x_1(s, \theta), x_2(s, \theta), s) \), the image of \( f_{\AP} \) coincides with that of \( f_{\TP} \). On the other hand, it is easily checked that \( f_{\AP} \) is a proper map. Applying Theorem 3.10 yields the following:

**Proposition 4.6.** The map \( f_{\TP} \) is a real analytic space-like CMC-1 DC-immersion, and its image is analytically complete.

Note that \( f_{\AP} \) is a W-catenoid with parabolic monodromy as in [6].

### 4.3.3. G-catenoids of type TH

For each \( \nu \in \mathbb{R}_+ \setminus \{1\} \), we consider the following real analytic map

\[
f_{\TP}^{\nu} : \mathbb{R} \times T^1 \ni (s, \theta) \mapsto (x_0(s), x_1(s, \theta), x_2(s, \theta), x_3(s)) \in S^3_1
\]

which gives a G-catenoid of type TH, where

\[
f_{\TP}^{\nu}(s, \theta) := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\( \Gamma_{\TP}^{\nu}(s), \Gamma_{\TP}^{\nu}(s) := \begin{pmatrix}
x_0(s) \\
\frac{\nu^2+1}{2\nu} \cos(\nu s) \\
0 \\
x_3(s)
\end{pmatrix},
\]

and

\[
x_0(s) := \sin(\nu s) \sinh s - \frac{\nu^2 - 1}{2\nu} \cos(\nu s) \cosh s,
\]

\[
x_3(s) := \sin(\nu s) \cosh s - \frac{\nu^2 - 1}{2\nu} \cos(\nu s) \sinh s.
\]

Since the monodromy matrix of the secondary Gauss map of \( f_{\TP}^{\nu} \) is hyperbolic, we call \( f_{\TP}^{\nu} \) a G-catenoid of type TH. Here \( \Gamma_{\TP}^{\nu}(s) \in \mathbb{R} \) is a profile curve of \( f_{\TP}^{\nu} \) lying on \( S^3_1 = S^3_1 \cap \{x_2 = 0\} \). The axes of \( f_{\TP}^{\nu} \) are the same as the axes of \( f_{\TP}^{\mu} \).

The singular point set of \( f_{\TP}^{\nu} \) is given by

\[
\Sigma := \bigcup_{k \in \mathbb{Z}} \Sigma_k, \quad \Sigma_k := \left\{ \left( \frac{1}{2} + k \right) \frac{\pi}{\nu}, \theta \right\} \in T^1 \right\},
\]

and the image of each connected component \( \Sigma_k \) consists of one point

\[
\left( -1 \right)^{k+1} \sinh \left( \frac{1}{2} + k \right) \pi, 0, 0, \left( -1 \right)^{k+1} \cosh \left( \frac{1}{2} + k \right) \pi \right).
\]

Unlike the case of \( f_{\TP}^{\mu} \) and \( f_{\TP} \), the map \( f_{\TP}^{\nu} : \mathbb{R} \times T^1 \rightarrow S^3_1 \) is not a proper map. In fact, the image of the map accumulates to the set

\[
\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_-, \quad \mathcal{L}_\pm := \{(x_0, x_1, x_2, x_3) \in S^3_1 ; x_0 \mp x_3 = 0\}.
\]

Figures for the image of \( f_{\TP}^{\nu} \) are given in [6] Fig. 3 (page 35)]. Like as in Example 2.19, the set

\[
\chi_{\TP}^{\nu} := \left\{ (\xi, \eta, s) \in \mathbb{R}^3 ; \xi^2 + \eta^2 = \frac{(1 + \nu^2)^2 \cos^2(\nu s)}{4\nu^2} \right\}
\]

has the structure of a DC-submanifold of \( \mathbb{R}^3 \) whose DC-points are \( (0, 0, (\pi/2 + \pi k)/\nu) \) \((k \in \mathbb{Z})\). Using the functions \( \nu \), we define

\[
\Psi : \mathbb{R}^3 \ni (\xi, \eta, s) \mapsto (x_0(s), \xi, \eta, x_3(s)) \in S^3_1,
\]

which is a real analytic diffeomorphism into \( S^3_1 \) by the following lemma:

**Lemma 4.7.** For each \( \nu \in \mathbb{R}_+ \setminus \{1\} \), the plane curve \( s \mapsto (x_0(s), x_3(s)) \) is a regular curve on \( \mathbb{R} \).
By this lemma, \( \tilde{f}^\nu_{\text{TH}} := \Psi|_{\tilde{X}_{\text{TH}}} \) is a DC-immersion. Since

\[
\tilde{f}^\nu_{\text{TH}}(s, \theta) = f^\nu_{\text{TH}}(x_1(s, \theta), x_2(s, \theta), s),
\]

the image of \( \tilde{f}^\nu_{\text{TH}} \) coincides with that of \( f^\nu_{\text{TH}} \). So \( f^\nu_{\text{TH}} \) is a space-like CMC-1 immersion on an open dense subset in \( X^\nu_{\text{TH}} \). So we can prove the following:

**Proposition 4.8.** The map \( f^\nu_{\text{TH}} \) is a real analytic space-like CMC-1 DC-immersion and its image is analytically complete.

**Proof.** Although \( \tilde{f}^\nu_{\text{TH}} \) is not a proper map, \( \tilde{f}^\nu_{\text{TH}} \) is a \( C^0 \)-arc-proper map: In fact, let \( \gamma : [0, 1] \to S_1^2 \) be a continuous map, and suppose that \( \gamma(t) = (\xi(t), \eta(t), s(t)) \) \( (t \in [0, 1]) \) satisfies \( \gamma(t) = \tilde{f}^\nu_{\text{TH}} \circ \gamma(t) \) for each \( t \in [0, 1] \). Since the second and the third components of \( \gamma(t) \) are \( \xi(t), \eta(t) \), the limits \( \lim_{t \to 1^{-}} \xi(t) \) and \( \lim_{t \to 1^{-}} \eta(t) \) exist.

Since

\[
\xi(t)^2 + \eta(t)^2 = \frac{1 + \nu^2}{4\nu^2} \cos^2 s(t) \quad (t \in [0, 1]),
\]

\[ \lim_{t \to 1^{-}} \cos s(t) \text{ exists. By Lemma } [37] \text{ in the second appendix, } \lim_{t \to 1^{-}} s(t) \text{ also exists. Thus, we can show the existence of the limit } \lim_{t \to 1^{-}} \gamma(t), \text{ proving the } C^0 \text{-arc-properness of } \tilde{f}^\nu_{\text{TH}}. \text{ So the assertion follows from Theorem } \text{S.13} \text{ because } \tilde{f}^\nu_{\text{TH}} \text{ is a DC-immersion.} \]

**Remark 4.9.** If \( \nu = 1 \), we have that

\[
(x_0 = \sin s \sinh s, \quad x_1 = \cos \theta \cos s, \quad x_2 = \sin \theta \cos s, \quad x_3 = \sin s \cosh s).
\]

Then \( f^\nu_{\text{TH}}(k\pi, \theta) = (0, (-1)^k \cos \theta, (-1)^k \sin \theta, 0) \) \( (\nu = 1) \) for each \( k \in \mathbb{Z} \). In particular, \( f^\nu_{\text{TH}}(\nu = 1) \) takes the same value infinitely many times.

From now on, we consider G-catenoids of type S and P, all of which have nontrivial analytic extensions:

### 4.4. G-catenoids of type S.

#### 4.4.1. G-catenoids of type SE.

For each \( \mu \in \mathbb{R} \setminus \{1\} \), we consider a map

\[
f^\mu_{\text{SE}} : \mathbb{R}^2 \ni (s, \theta) \mapsto (x_0(s, \theta), x_1(\theta), x_2(\theta), x_3(s, \theta)) \in S_1^3
\]

given by

\[
f^\mu_{\text{SE}}(s, \theta) = \begin{pmatrix} \cosh s & 0 & 0 & \sinh s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh s & 0 & 0 & \cosh s \end{pmatrix} \Gamma^\mu_{\text{SE}}(\theta), \quad \Gamma^\mu_{\text{SE}}(\theta) := \begin{pmatrix} -\frac{\nu^2 - 1}{2\nu} \cos \mu \theta \\ x_1(\theta) \\ x_2(\theta) \\ 0 \end{pmatrix},
\]

where

\[
\begin{align*}
x_1(\theta) &= \frac{(\mu^2 + 1) \cos \theta \cos \mu \theta}{2\mu} - \sin \theta \sin \mu \theta, \\
x_2(\theta) &= \frac{(\mu^2 + 1) \sin \theta \cos \mu \theta}{2\mu} + \cos \theta \sin \mu \theta.
\end{align*}
\]
Then this gives a G-catenoid of type SE. Here, $\Gamma_{SE}^\mu(\theta) (\theta \in \mathbb{R})$ is the profile curve of $f_{SE}^\mu$ lying on $S_1^2 = S_1^3 \cap \{x_3 = 0\}$. The axis of $f_{SE}^\mu$ is $\{(0, \cos t, \sin t, 0); t \in T^1\}$. The singular point set of $f_{SE}^\mu$ is

$$\Sigma := \{(s, \theta); \cos \mu \theta = 0\} = \bigcup_{k \in \mathbb{Z}} \Sigma_k, \quad \Sigma_k := \left\{\left(s, \frac{1}{\mu} \left(\frac{\theta}{2} + k\pi\right)\right); s \in \mathbb{R}\right\},$$

and the image of each $\Sigma_k$ consists of one point

$$(4.8) \quad P_k := f_{SE}^\mu(\Sigma_k) = \left(0, \gamma_{SE}^\mu \left(1 - \frac{1}{\mu} \left(\frac{\pi}{2} + k\pi\right)\right), 0\right),$$

where $\gamma_{SE}^\mu(\theta) := (x_1(\theta), x_2(\theta)) \in \mathbb{R}^2$. The limit point set of $f_{SE}^\mu$ is

$$\mathcal{L} := \bigcup_{k \in \mathbb{Z}} \mathcal{L}_k, \quad \mathcal{L}_k := \left\{(u, \gamma_{SE}^\mu \left(\frac{\pi}{2\mu}(2k + 1)\right), \pm u) \in S_1^3; u \in \mathbb{R}\right\}.$$

We prove that the image of this map $f_{SE}^\mu$ has an analytic extension: For any $\mu \in \mathbb{R}_+ \setminus \{1\}$, define a subset $\mathcal{X}_{SE}^\mu \subset \mathbb{R}^3$ by

$$\mathcal{X}_{SE}^\mu := \left\{(\xi, \eta, \theta) \in \mathbb{R}^3; \xi\eta = \left(\frac{(\mu^2 - 1)\cos \mu \theta}{2\mu}\right)^2\right\}.$$  

By $(4.7)$, we have

$$|\gamma_{SE}^\mu(\theta)|^2 = 1 + \left(\frac{\mu^2 - 1}{2\mu}\right)^2 \cos^2 \mu \theta.$$  

Thus, if we set

$$\xi := x - y, \quad \eta := x + y, \quad z := \left(\frac{\mu^2 - 1}{2\mu}\right) \cos \mu \theta,$$

then $-x^2 + y^2 + z^2 = 0$ holds. Using this expression, it can be easily checked that $\mathcal{X}_{SE}^\mu$ has the structure of a DC-submanifold with a countably infinite number of DC-points, whose inclusion map is a DC-immersion. Using the functions $(4.7)$, we set

$$\tilde{f}_{SE}^\mu: \mathcal{X}_{SE}^\mu \ni (\xi, \eta, \theta) \mapsto \left(\frac{\xi + \eta}{2}, x_1(\theta), \frac{\xi - \eta}{2}\right) \in S_1^3.$$  

It holds that

$$f_{SE}^\mu(s, \theta) = \tilde{f}_{SE}^\mu \left(-\frac{(\mu^2 - 1)e^s \cos \mu \theta}{2\mu}, -\frac{(\mu^2 - 1)e^{-s} \cos \mu \theta}{2\mu}, \theta\right).$$

If we set

$$\tilde{f}_{SE}^\mu(s, \theta) = \tilde{f}_{SE}^\mu \left(\frac{(\mu^2 - 1)e^s \cos \mu \theta}{2\mu}, \frac{(\mu^2 - 1)e^{-s} \cos \mu \theta}{2\mu}, \theta\right),$$

then we have

$$\tilde{f}_{SE}^\mu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} f_{SE}^\mu,$$

which implies the image of $\tilde{f}_{SE}^\mu$ is congruent to that of $f_{SE}^\mu$ in $S_1^3$. It can be easily checked that

$$\tilde{f}_{SE}^\mu(\mathcal{X}_{SE}^\mu) = f_{SE}^\mu(\mathbb{R}^2) \cup \mathcal{L} \cup \tilde{f}_{SE}^\mu(\mathbb{R}^2).$$

We prove the following:

**Lemma 4.10.** The plane curve $\theta \mapsto (x_1(\theta), x_2(\theta))$ is a regular curve on $\mathbb{R}$.  


Proof. In fact, the assertion follows from the following identity
\[
\begin{pmatrix} x_1'(\theta) \\ x_2'(\theta) \end{pmatrix} = \frac{\mu^2 - 1}{2\mu} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \mu \sin \mu \theta \\ \cos \mu \theta \end{pmatrix}.
\]
\[\square\]

We obtain the following:

**Proposition 4.11.** The map \( f_{\mu}^{SE} \) is a real analytic space-like CMC-1 DC-immersion whose image is analytically complete.

Proof. By Lemma 4.10 the curve \( \gamma_{\mu}^{SE} \) is regular, hence the map
\[
\Psi : \mathbb{R}^3 \ni (\xi, \eta, \theta) \mapsto \left( \frac{\xi + \eta}{2}, x_1(\theta), x_2(\theta), \frac{\xi - \eta}{2} \right) \in S_1^3
\]
is an immersion. Since the inclusion map \( \iota : \mathcal{X}_{\mu}^{SE} \hookrightarrow \mathbb{R}^3 \) is a DC-immersion, the composition \( f_{\mu}^{SE} = \Psi \circ \iota \) is also a DC-immersion. Then (4.11) implies that \( f_{\mu}^{SE} \) is an analytic extension of \( f_{\mu}^{SE} \). Moreover, by (4.11), \( f_{\mu}^{SE} \) is a real analytic space-like CMC-1 DC-immersion.

To obtain the analytic completeness of \( f_{\mu}^{SE} \), it is sufficient to show that the map \( f_{\mu}^{SE} \) is \( C^0 \)-arc-proper, where \( f_{\mu}^{SE} \) is real analytic. In fact, let \( \Gamma : [0, 1] \to S_1^3 \) be a continuous map, and suppose that
\[
\sigma(t) := (\xi(t), \eta(t), \theta(t)) \quad (t \in [0, 1])
satisfies \( \Gamma(t) = f \circ \sigma(t) \) for each \( t \in [0, 1] \). Since the second and third components of \( \Gamma(t) \) are \( \xi(t) \pm \eta(t) \), the limits \( \lim_{t \to 1^{-}} \xi(t) \) and \( \lim_{t \to 1^{-}} \eta(t) \) exist. Since
\[
\xi(t)\eta(t) + 1 = |\gamma_{\mu}^{SE}(\theta(t))|^2 = 1 + \left( \frac{\mu^2 - 1}{2\mu} \right)^2 \cos^2 \mu \theta(t) \quad (t \in [0, 1]),
\]
the limit \( \lim_{t \to 1^{-}} \cos \mu \theta(t) \) exists, and so \( \lim_{t \to 1^{-}} \theta(t) \) also exists (applying Lemma B.1 in the second appendix). Thus \( \lim_{t \to 1^{-}} \sigma(t) \) exists, which implies the weak \( C^0 \)-arc-properness of \( f \). So the conclusion follows from Theorem 3.10. \[\square\]

The map \( f_{\mu}^{SE} \) is not a proper map, since its image is not closed. However, if \( \mu \in \mathbb{R}_+ \setminus \{1\} \) is an integer, \( f_{\mu}^{SE} \) induces a map \( f_{\mu}^{SE} : \mathcal{X}_{\mu}^{SE} \to S_1^3 \), where
\[
\mathcal{X}_{\mu}^{SE} := \{(\xi, \eta, \theta) \in \mathbb{R}^2 \times T^1 : -\xi \eta + |\gamma_{\mu}^{SE}(\theta)|^2 = 1\}
\]
is a real analytic 2-dimensional DC-manifold with finitely many DC-points. It can be easily checked that \( f_{\mu}^{SE} \) is a proper map, and its image is congruent to the image of the exceptional W-catenoid \( f_{\mu}^{SE} \) given in [8].

4.4.2. \( G \)-catenoids of type \( SH \). For each \( \nu \in \mathbb{R}_+ \), we consider a map
\[
f_{\mu}^{SH} : \mathbb{R}^2 \ni (s, \theta) \mapsto (x_0(s, \theta), x_1(\theta), x_2(\theta), x_3(s, \theta)) \in S_1^3,
\]
defined by
\[
f_{\mu}^{SH}(s, \theta) = \begin{pmatrix} \cosh s & 0 & 0 & \sinh s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh s & 0 & 0 & \cosh s \end{pmatrix} \Gamma_{\mu}^{SH}(\theta), \quad \Gamma_{\mu}^{SH}(\theta) := \begin{pmatrix} \frac{\nu^2 - 1}{2\nu} \sinh \nu \theta \\ x_1(\theta) \\ x_2(\theta) \\ 0 \end{pmatrix},
\]
where \( \nu \in \mathbb{R}_+ \) and
\[
\begin{aligned}
x_1(\theta) &= \frac{(\nu^2 - 1) \cos \theta \sin \nu \theta}{2\nu} + \sin \theta \cosh \nu \theta, \\
x_2(\theta) &= \frac{(\nu^2 - 1) \sin \theta \sin \nu \theta}{2\nu} - \cos \theta \cosh \nu \theta.
\end{aligned}
\]
Here $\Gamma_{\nu}^\nu(\theta) (\theta \in \mathbb{R})$ is the profile curve of $f^\nu_{\nu} \mid_{S^1}$ lying on $S^1 \times \{x_3 = 0\}$. Then $f^\nu_{\nu}$ gives a G-cateno of type SH. The axis of $f^\nu_{\nu}$ is the same as the axis of $f^\nu_{\nu}$.

The singular point set of $f^\nu_{\nu}$ is $\{\theta = 0\}$, and its image consists of one point $(0,0,-1,0)$. The limit point set of $f^\nu_{\nu}$ is $L := \{(u,0,-1,\pm u) : u \in \mathbb{R}\} \subset S^1 \times \{x_3 = 0\}$. Let

\[(4.13) \quad \mathcal{X}^\nu_{\nu} := \left\{ (\xi, \eta, \theta) \in \mathbb{R}^3 : \xi = \left( \frac{(\nu^2 + 1) \sinh(\nu \theta)}{2\nu} \right)^2 \right\} \subset \mathbb{R}^3.\]

Like as for $\mathcal{X}^\nu_{\nu}$, the set $\mathcal{X}^\nu_{\nu}$ has the structure of a DC-submanifold of $\mathbb{R}^3$, whose DC-point is $(0,0,0)$. Using the functions in (4.12), we set $f^\nu_{\nu}(x_1(\theta), x_2(\theta), \xi - \eta) \in S^1$.

Then we have

\[f^\nu_{\nu}(s, \theta) = f^\nu_{\nu} \left( \frac{(\nu^2 + 1)e^s}{\nu} \sinh(\nu \theta), \frac{(\nu^2 + 1)e^{-s}}{\nu} \sinh(\nu \theta), \theta \right).\]

By setting

\[f^\nu_{\nu}(s, \theta) := f^\nu_{\nu} \left( \frac{(\nu^2 + 1)e^s}{\nu} \sinh(\nu \theta), -\frac{(\nu^2 + 1)e^{-s}}{\nu} \sinh(\nu \theta), \theta \right),\]

we have

\[f^\nu_{\nu}(s, \theta) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} f^\nu_{\nu}(s, \theta),\]

which implies the image of $f^\nu_{\nu}$ is congruent to that of $f^\nu_{\nu}$. It holds that

\[f^\nu_{\nu}(\mathcal{X}^\nu_{\nu}) = f^\nu_{\nu}(\mathbb{R}^2) \cup L \cup f^\nu_{\nu}(\mathbb{R}^2).\]

We prove the following:

**Lemma 4.12.** The plane curve $\theta \mapsto (x_1(\theta), x_2(\theta))$ is a regular curve on $\mathbb{R}$.

**Proof.** In fact, the assertion follows from the following identity

\[\begin{pmatrix} x_1(\theta) \\ x_2(\theta) \end{pmatrix} = \frac{\nu^2 + 1}{2\nu} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \nu \cosh \nu \theta \\ -\sinh \nu \theta \end{pmatrix}.\]

\[\square\]

By this lemma, we can conclude that

\[\Psi : \mathbb{R}^3 \ni (\xi, \eta, \theta) \mapsto \left( \frac{\xi + \eta}{2}, x_1(\theta), x_2(\theta), \frac{\xi - \eta}{2} \right) \in S^1 \times \{x_3 = 0\}\]

is an immersion. Since the inclusion map $\iota : \mathcal{X}^\nu_{\nu} \rightarrow \mathbb{R}^3$ is a DC-immersion, the composition $f = \Psi \circ \iota$ is also a DC-immersion. The relation (4.14) implies that $f^\nu_{\nu}$ is an analytic extension of $f^\nu_{\nu}$. Moreover, by Proposition 4.12, $f^\nu_{\nu}$ is a real analytic space-like CMC-1 DC-immersion. We set $\gamma^\nu_{\nu}(\theta) = (x_1(\theta), x_2(\theta))$. Since

\[|\gamma^\nu_{\nu}(\theta)|^2 = 1 + \frac{(\nu^2 + 1)}{2\nu} \sinh^2 \nu \theta,\]

the function $|\gamma^\nu_{\nu}(\theta)|$ diverges as $\theta \rightarrow \pm \infty$. Since the curve $\gamma^\nu_{\nu}$ is a proper map, $f^\nu_{\nu}$ is also a proper map. So we can apply Theorem 3.10 and obtain the following:

**Proposition 4.13.** The map $f^\nu_{\nu}$ is a real analytic space-like CMC-1 DC-immersion whose image is analytically complete.
4.4.3. G-catenoids of type SP. A G-catenoid of type SP can be expressed as
\[ f_{\text{SP}}: \mathbb{R}^2 \ni (s, t) \mapsto (x_0(s, t), x_1(t), x_2(t), x_3(s, t)) \in S_1^3, \]
where
\[
f_{\text{SP}}(s, t) = \begin{pmatrix}
\cosh s & 0 & 0 & \sinh s \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sinh s & 0 & 0 & \cosh s
\end{pmatrix} \Gamma_{\text{SP}}(t), \quad \Gamma_{\text{SP}}(t) := \begin{pmatrix}
\frac{1}{2} t \cos t - \sin t \\
\frac{1}{2} t \sin t + \cos t \\
0
\end{pmatrix}.
\]
Here \( \Gamma_{\text{SP}}(t) \) \((t \in \mathbb{R})\) is the profile curve of \( f_{\text{SP}} \) lying on \( S_1^3 = S_1^3 \cap \{ x_3 = 0 \} \). The axis of \( f_{\text{SP}} \) is the same as the axis of \( f_{\text{SE}}^\mu \). The singular point set of \( f_{\text{SP}} \) is \( \{ t = 0 \} \), and its image consists of one point \((0, 0, 1, 0)\). On the other hand, the limit point set of \( f_{\text{SP}} \) in \( S_1^3 \) is \( \mathcal{L} := \{(u, 0, 1, \pm v) : u \in \mathbb{R} \} \). This map has an analytic extension as follows: Like as \( \mathcal{X}_{\text{SE}}^\mu \) and \( \mathcal{X}_{\text{SH}}^\nu \), the set
\[
(4.15) \quad \mathcal{X}_{\text{SP}} := \{ (\xi, \eta, t) : \xi \eta = t^2/4 \} \subset \mathbb{R}^3
\]
is a DC-submanifold in \( \mathbb{R}^3 \) admitting only one cone point at \((0, 0, 0)\). We can define a real analytic DC-map on the DC-manifold \( \mathcal{X}_{\text{SP}} \) by
\[
\tilde{f}_{\text{SP}}: \mathcal{X}_{\text{SP}} \ni (\xi, \eta, t) \mapsto \frac{1}{2} \left( \xi + \eta, t \cos t - 2 \sin t, t \sin t + 2 \cos t, \xi - \eta \right) \in S_1^3.
\]
Then, we have \( f_{\text{SP}}(s, t) = \tilde{f}_{\text{SP}}(-e^t/2, -e^{-t}/2, t) \). If we set
\[
\tilde{f}_{\text{SP}}(s, t) := \tilde{f}_{\text{SP}} \left( \frac{e^t}{2}, \frac{e^{-t}}{2}, t \right),
\]
then
\[
(4.16) \quad \tilde{f}_{\text{SP}} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} f_{\text{SP}}
\]
and
\[
(4.17) \quad \tilde{f}_{\text{SP}}(\mathcal{X}_{\text{SP}}) = f_{\text{SP}}(\mathbb{R}^2) \cup \mathcal{L} \cup \tilde{f}_{\text{SP}}(\mathbb{R}^2).
\]
The relation \( (4.10) \) implies that \( \tilde{f}_{\text{SP}} \) is congruent to \( f_{\text{SP}} \). It can be easily checked that \( R \ni t \mapsto (x_1(t), x_2(t)) \in \mathbb{R}^2 \) is an immersion, and
\[
\Psi: \mathbb{R}^3 \ni (\xi, \eta, t) \mapsto \frac{1}{2} \left( \xi + \eta, t \cos t - 2 \sin t, t \sin t + 2 \cos t, \xi - \eta \right) \in S_1^3
\]
also gives an immersion. Since the inclusion map \( \iota: \mathcal{X}_{\text{SP}} \hookrightarrow \mathbb{R}^3 \) is a DC-immersion, the composition \( \tilde{f}_{\text{SP}} = \Psi \circ \iota \) is also a DC-immersion. On the other hand, since \( R \ni t \mapsto (x_1(t), x_2(t)) \in \mathbb{R}^2 \) is a proper map, so is \( \tilde{f}_{\text{SP}} \). In particular, we obtain the following:

**Proposition 4.14.** The map \( \tilde{f}_{\text{SP}} \) is a real analytic space-like CMC-1 DC-immersion whose image is analytically complete.

4.5. G-catenoids of type L.

4.5.1. G-catenoids of type LH. A G-catenoid of type LH can be expressed as
\[ f_{\text{LH}}: \mathbb{R}^2 \ni (u, v) \mapsto P(u) \Gamma_{\text{LH}}(v) \in S_1^3, \]
where
\[
P(u) = \begin{pmatrix}
1 + \frac{u^2}{2} & u & 0 & -\frac{u^2}{2} \\
0 & 1 & 0 & -u \\
0 & 0 & 1 & 0 \\
\frac{u^2}{2} & u & 0 & 1 - \frac{u^2}{2}
\end{pmatrix}, \quad \Gamma_{\text{LH}}(v) := \begin{pmatrix}
v \cos 2v + \frac{1}{2} v^2 \sin 2v \\
0 \\
cos 2v + v \sin 2v \\
v \cos 2v + \frac{1}{2} (u^2 - 2) \sin 2v
\end{pmatrix}.
\]
that is, \( f_{\text{LH}} = (x_0, x_1, x_2, x_3) \) is given by

\[
\begin{align*}
  x_0 &= v \cos 2v + \frac{u^2 + v^2}{2} \sin 2v, \\
  x_1 &= u \sin 2v, \\
  x_2 &= y_{\text{LH}}(v) := \cos 2v + v \sin 2v, \\
  x_3 &= v \cos 2v + \frac{u^2 + v^2 - 2}{2} \sin 2v.
\end{align*}
\]

(4.18)

This \( f_{\text{LH}} \) is generated by the profile curve \( \Gamma_{\text{LH}}(v) \) (\( v \in \mathbb{R} \)) lying on \( S_3^1 = S_1^3 \cap \{ x_1 = 0 \} \). The singular point set is \( \{ 2v \equiv 0 \text{ (mod } \pi) \} \), and the image of each connected component of the singular set consists of one point. The limit point set of \( f_{\text{LH}}: \mathbb{R}^2 \to S_3^1 \) consists of two light-like lines \( L^\pm := \{ (t, 0, \pm 1, t) \in S_3^1 ; t \in \mathbb{R} \} \), which are the axes of \( f_{\text{LH}} \). The figure of the image of \( f_{\text{LH}} \) is complicated (cf. Figure 4, left).

**Figure 4.** G-catenoids of type LH (left) and LE (right)

Let \( \mathcal{X}_{\text{LH}} := \{ (x, \xi, v) ; -\xi \sin 2v + x^2 + (\cos^2 2v - v \sin^2 2v)^2 - 1 = 0 \} \subset \mathbb{R}^3 \), and set

\[
\tilde{f}_{\text{LH}}: \mathcal{X}_{\text{LH}} \ni (x, \xi, v) \mapsto \left( \frac{\xi + \sin 2v}{2}, x, y_{\text{LH}}(v), \frac{\xi - \sin 2v}{2}) \right) \in S_3^1.
\]

Then it holds that

\[
f_{\text{LH}}(u, v) = \tilde{f}_{\text{LH}} \left( \frac{(u^2 + v^2 + 1) \sin 2v}{2}, u \sin 2v, v \right)
\]

and

\[
(4.20) \quad \tilde{f}_{\text{LH}}(\mathcal{X}_{\text{LH}}) = f_{\text{LH}}(\mathbb{R}^2) \cup L_+ \cup L_-
\]

**Lemma 4.15.** The subset \( \mathcal{X}_{\text{LH}} \) has the structure of a DC-submanifold of \( \mathbb{R}^3 \) such that the residual set is the set of DC-points and the inclusion map \( \mathcal{X}_{\text{LH}} \hookrightarrow \mathbb{R}^3 \) is a DC-immersion.

**Proof.** Here \( (x, \xi, v) = (0, 0, \pi k) \in \mathbb{R}^3 \) (\( k \in \mathbb{Z} \)) are the singular points of \( \mathcal{X}_{\text{LH}} \). We set \( \varphi := \sqrt{1 + v(2 - (1 + v) \sin^2(2v))} \), which can be locally considered as a function of \( \sin 2v \), so we denote it by \( \varphi(w) \) (\( w := \sin 2v \)). Then, we can write

\[
-\xi \sin 2v + (\cos^2 2v - v \sin^2 2v)^2 - 1 = \left( \frac{\xi}{2\varphi(w)} \right)^2 - \left( \frac{\xi}{2\varphi(w)} + w\varphi(w) \right)^2.
\]

So if we set

\[
Y := \frac{\xi}{2\varphi(w)}, \quad Z := \frac{\xi}{2\varphi(w)} + w\varphi(w),
\]

then we have

\[
-\xi \sin 2v + (\cos^2 2v - v \sin^2 2v)^2 - 1 = Y^2 - Z^2.
\]
Here,

\[ \Phi(x, w, x) := (x, Y(x, w), Z(x, w)) \]

gives a local \( C^\infty \)-diffeomorphism around \((0, 0, 0)\). By identifying \( v + k\pi \) with \( w \) by a local \( C^\infty \)-diffeomorphism \( v \mapsto \sin v \), the local inverse map \( \Phi^{-1} \) can be considered as a \( C^\infty \)-diffeomorphism from a neighborhood of the origin of \( \mathbb{R}^3 \) to a neighborhood of \((x, \xi, v) = (0, 0, \pi k)\), which maps the cone \( x^2 + y^2 - z^2 = 0 \) near \((0, 0, 0)\) to an open subset of \( X_{LH} \). Thus, by Proposition 2.10 we obtain the conclusion. \( \square \)

By [1,20], \( \tilde{f}_{LH} \) is an analytic extension of \( f_{LH} \), and is a real analytic space-like CMC-1 DC-immersion, giving an analytic extension of the image of \( f_{LH} \). It can be easily checked that

\[ \mathbb{R}^3 \ni (x, \xi, v) \mapsto \left( \frac{\xi + \sin 2v}{2}, x, y_{LH}(v), \frac{\xi - \sin 2v}{2} \right) \in S^3_1 \]

is an immersion. Since \( \tilde{f}_{LH} \) is the restriction of this map to \( X_{LH} \), it is a DC-immersion. The map \( \tilde{f}_{LH} \) is not proper. In fact, the sequence \( \{ (0, 0, k\pi) \}_{k=1}^\infty \) lies in \( X_{LH} \) and diverges in \( \mathbb{R}^3 \), although its image under \( \tilde{f}_{LH} \) is the point set \( \{ (0, 0, 1, 0) \} \).

We prepare the following:

**Lemma 4.16.** The set \( X_{LH} (\subset \mathbb{R}^3) \) has the structure of a real analytic 2-dimensional DC-manifold with a countably infinite number of DC-points. Moreover, \( f_{LH} \) is a DC-immersion, giving the analytic extension of \( f_{LH} \).

**Proof.** We set \( F := -\xi \sin 2v + x^2 + y_{LH}(v)^2 - 1 \). It can be easily checked that \( F = F_x = F_y = 0 \) if and only if

\[ v = v_0, \quad \xi = 2\varepsilon_0 v_0, \quad x = 0 \quad (\varepsilon_0 := \cos(2v_0) \in \{ \pm 1 \}, \quad v_0 \in \frac{\pi}{2} \mathbb{Z}) \]

So we fix such a \( v_0 \). We set \( \tilde{v} := \sin 2v \), and then we can write

\[ v - v_0 = \tilde{v}\varphi(\tilde{v}), \quad \cos 2v = \varepsilon_0 + \tilde{v}\psi(\tilde{v}), \]

where \( \varphi(\tilde{v}) \) and \( \psi(\tilde{v}) \) are functions defined on a neighborhood of \( \tilde{v} = 0 \). So if we set

\[ \tilde{\xi} := \xi - \xi_0 + \tilde{v}(\varphi(\tilde{v})^2 + 2\varepsilon_0 \varphi(\tilde{v}) + 1), \]

then we can write \( F = -\tilde{\xi}\tilde{v} + x^2 \). Using this expression, one can conclude that the zero set of \( F \) has a cone-like singular point at \((\xi_0, 0, v_0)\) of \( f_{LH} \). Since the map

\[ S^3_1 \ni (x_0, x_1, x_2, x_3) \mapsto \left( x_0 + x_3, x_1, \frac{\sin^{-1}(x_0 - x_3)}{2} \right) \in \mathbb{R}^3 \]

gives a local inverse map of the immersion

\[ \mathbb{R}^3 \ni (\xi, x, v) \mapsto \left( \frac{\xi + \sin 2v}{2}, x, y_{LH}(v), \frac{\xi - \sin 2v}{2} \right) \in S^3_1, \]

\( \tilde{f}_{LH} \) is a DC-immersion. By [1,20], it also gives an analytic extension of \( f_{LH} \). \( \square \)

**Proposition 4.17.** The map \( \tilde{f}_{LH} \) is a real analytic space-like CMC-1 DC-immersion whose image is analytically complete.

**Proof.** Applying Proposition [1,2] we can conclude that \( \tilde{f}_{LH} \) is a real analytic space-like CMC-1 DC-immersion. The map \( \tilde{f}_{LH} \) is not a proper map, since \( \tilde{f}_{LH}(\xi/2, 0, 1, \xi/2) = (\xi/2, 0, 1, \xi/2) \) for \( m \in \mathbb{Z} \). By Theorem [3,10] it is sufficient to show that \( \tilde{f}_{LH} \) is \( C^0 \)-arc-proper. We consider a continuous curve \( \Gamma : [0, 1] \to S^3_1 \) such that \( \Gamma([0, 1]) \subset \tilde{f}_{LH}(X_{LH}) \). Let \( \sigma(t) = ((\xi(t), x(t), v(t))) \) (0 \( \leq t < 1 \)) be a continuous curve on \( X_{LH} \) such that

\[ \tilde{f}_{LH} \circ \sigma(t) = \Gamma(t) \quad (0 \leq t < 1). \]
Since $\Gamma(1)$ exists, the three limits
\[
\lim_{t \to 1^-} \xi(t), \quad \lim_{t \to 1^-} x(t), \quad \lim_{t \to 1^-} \sin 2v(t)
\]
exist. Then $\lim_{t \to 1^-} v(t)$ also exists (cf. Lemma B.1 in the second appendix), proving the assertion.}\]

4.5.2. $G$-catenoids of type LE. On the other hand, a G-catenoid of type LE can be expressed as
\[
f_{\text{LE}} : \mathbb{R}^2 \ni (u,v) \mapsto P(u)\Gamma_{\text{LE}}(v) \in S_1^3,
\]
where
\[
P(u) = \begin{pmatrix}
1 + u^2 & u & 0 & -u^2 \\
0 & 1 & 0 & -u \\
-\frac{u^2}{2} & u & 0 & 1 - \frac{u^2}{2}
\end{pmatrix}, \quad \Gamma_{\text{LE}}(v) := \begin{pmatrix}
v \cosh 2v - \frac{1}{2}(u^2 + 2) \sinh 2v \\
0 \\
v \cosh 2v - v \sinh 2v \\
v \cosh 2v - \frac{1}{2}u^2 \sinh 2v
\end{pmatrix},
\]
that is, $f_{\text{LE}} = (x_0, x_1, x_2, x_3)$ is given by
\[
x_0 = v \cosh 2v - \frac{u^2 + u^2 + 2}{2} \sinh 2v, \quad x_1 = -u \sinh 2v, \quad x_2 = y_{\text{LE}}(v) := \cosh 2v - v \sinh 2v, \quad x_3 = v \cosh 2v - \frac{u^2 + v^2}{2} \sinh 2v.
\]
Here $\Gamma_{\text{LE}}(v)$ ($v \in \mathbb{R}$) is a profile curve of $f_{\text{LE}}$ lying on $S_1^3 := S_1^3 \cap \{x_1 = 0\}$. The axes of $f_{\text{LE}}$ are the same as the axes of $f_{\text{LE}}$. The singular point set is $\{v = 0\}$, whose image consists of one point $(0,0,1,0)$. The limit point set of $f_{\text{LE}} : \mathbb{R}^2 \to S_1^3$ consists of a light-like line $L := \{(t,0,1,t) \in S_1^3; \ t \in \mathbb{R}\}$. Set
\[
\mathcal{X}_{\text{LE}} := \left\{ (\xi, x, v) ; \xi \sinh 2v + x^2 + y_{\text{LE}}(v)^2 = 1 \right\} \subset \mathbb{R}^3,
\]
and let
\[
f_{\text{LE}} : \mathcal{X}_{\text{LE}} \ni (\xi, x, v) \mapsto \left( \frac{\xi - \sinh 2v}{2}, x, y_{\text{LE}}(v), \frac{\xi + \sinh 2v}{2} \right) \in S_1^3.
\]
Since the expression of $\mathcal{X}_{\text{LE}}$ is obtained by replacing $\sin 2v$ and $\cosh 2v$ by $\sinh 2v$ and $\cosh 2v$, like in Lemma 4.15, one can show that the set $\mathcal{X}_{\text{LE}} \subset \mathbb{R}^3$ has the structure of a real analytic 2-dimensional DC-manifold with a DC-point, and the residual set is the set of DC-points and the inclusion map $\mathcal{X}_{\text{LE}} \hookrightarrow \mathbb{R}^3$ is a DC-immersion. Moreover, it can be easily checked that $f_{\text{LE}}$ is a DC-immersion defined on the DC-manifold $\mathcal{X}_{\text{LE}}$, and is a proper map. Moreover, we have
\[
f_{\text{LE}}(u,v) = \tilde{f}_{\text{LE}} \left( v \cosh 2v - \frac{u^2 + v^2 + 1}{2} \sinh 2v, -u \sinh 2v, v \right)
\]
and $\tilde{f}_{\text{LE}}(\mathcal{X}_{\text{LE}}) = f_{\text{LE}}(\mathbb{R}^2) \cup L$. In particular, $\tilde{f}_{\text{LE}}$ is an analytic extension of $f_{\text{LE}}$. Since
\[
\mathbb{R}^3 \ni (\xi, x, v) \mapsto \left( \frac{\xi - \sinh 2v}{2}, x, y_{\text{LE}}(v), \frac{\xi + \sinh 2v}{2} \right) \in S_1^3
\]
is an immersion, $\tilde{f}_{\text{LE}}$ is a real analytic DC-immersion. Moreover, by Proposition 1.2, it is a real analytic space-like CMC-1 DC-immersion giving an analytic extension of $f_{\text{LE}}$. The image of $\tilde{f}_{\text{LE}}$ can be expressed as the intersection of $S_1^3$ and the graph of
\[
x_2 = \sqrt{1 + (x_0 - x_3)^2} - \frac{1}{2} (x_0 - x_3) \sinh^{-1}(x_0 - x_3).
\]
By Proposition 1.2, we obtain the following:

**Proposition 4.18.** The map $\tilde{f}_{\text{LE}}$ is a real analytic space-like CMC-1 DC-immersion whose image is analytically complete.
Finally, summarizing the results in this section, we obtained the following:

**Theorem 4.19.** The images of G-catenoids of type $T$ are analytically complete and those of other types admit analytic extensions, which are analytically complete. Moreover, after taking the analytic completions, all of them are the images of real analytic space-like CMC-1 DC-immersions into $S^3_1$.

**Appendix A. Maximal catenoids in $\mathbb{R}^3_1$**

This appendix gives an overview of G-catenoids and W-catenoids which are space-like maximal surfaces in $\mathbb{R}^3_1$.

We give the following definition:

**Definition A.1.** Let $f : X^2 \rightarrow \mathbb{R}^3_1$ be a real analytic DC-immersion defined on a connected 2-dimensional real analytic DC-manifold $X^2$. Then $f$ is called a ZMC DC-immersion if there exists an open dense subset $O$ of $X^2 \setminus \Sigma$ such that the restriction $f|_O$ is a zero mean curvature immersion on $O$, where $\Sigma$ is the set of DC-points in $X^2$.

We first recall the property of real analytic ZMC DC-immersions from [4] as follows: Let $U$ be a domain in the $uv$-plane $\mathbb{R}^2$, and let $f : U \rightarrow \mathbb{R}^3_1$ be a real analytic map into the Lorentz-Minkowski 3-space $\mathbb{R}^3_1$ of signature $(-+++)$. We set $B_f := \det(P)$, $P := \begin{pmatrix} f_u \cdot f_u & f_u \cdot f_v \\ f_v \cdot f_u & f_v \cdot f_v \end{pmatrix}$, where $\cdot$ denotes the canonical Lorentzian inner product of $\mathbb{R}^3_1$. We set $Q := \begin{pmatrix} f_{uu} \cdot \tilde{\nu} & f_{uv} \cdot \tilde{\nu} \\ f_{vu} \cdot \tilde{\nu} & f_{vv} \cdot \tilde{\nu} \end{pmatrix}$, where $\tilde{\nu} := f_u \times_L f_v$ and $\times_L$ is the canonical Lorentzian vector product of $\mathbb{R}^3_1$. In this situation, $f$ is called a zero mean curvature map if $A_f := \text{trace}(\tilde{P}Q)$ vanishes identically, where $\tilde{P}$ is the cofactor matrix of $P$. If $f$ is a ZMC DC-immersion, then it is a zero mean curvature map. So, imitating the proof of Proposition 4.2, we can prove the following (although real analytic space-like CMC-1 DC-immersions never change type from space-like to time-like, real analytic zero mean curvature immersions may change type):

**Proposition A.2.** Let $f : X^2 \rightarrow \mathbb{R}^3_1$ be a DC-immersion defined on a connected 2-dimensional real analytic DC-manifold $X^2$. Suppose that there exists a non-empty local inverse DC-coordinate system $(U, \varphi)$ such that $f \circ \varphi : U \rightarrow \mathbb{R}^3_1$ gives a zero mean curvature immersion on an open dense subset of $U$. Then $f$ is a ZMC DC-immersion.

Osamu Kobayashi [19] showed that classified G-catenoids are

- the elliptic G-catenoid (cf. (0.3))
- the parabolic G-catenoid (cf. (0.4)) and
- the hyperbolic G-catenoid (cf. (0.5)).

The image of the elliptic G-catenoid $f_E$ (cf. (0.3)) can be considered as the image of a ZMC DC-immersion (cf. Example 2.17). The image $E$ of $f_E$ is analytically complete (cf. Theorem 2.25). We next consider the parabolic G-catenoid $f_P$ in (0.4). We let $P$ be the subset of $\mathbb{R}^3_1$ defined by (0.7). Then $P$ gives an analytic extension of $f_P$, which is analytically complete (cf. Theorem 2.25). On the other hand, the analytic extension $H$ of the image of $f_H$ is given by (1.9), which is a singly periodic maximal surface with cone-like singular points at $\Sigma := \{(0, n\pi, 0) ; n \in \mathbb{Z}\}$ with respect to the inclusion map: As shown in Theorem 2.25 $H$ is analytically complete.
It is known that there are only two congruence classes of W-catenoids up to a homothety (cf. Imaizumi-Kato [15]). One is represented by the elliptic G-catenoid \( f_E \), which is analytically complete, as mentioned above. The other is represented by the maximal surface \( f_K \) given in [11], which is the Kobayashi surface of order 2 of type \((0, 0, \pi, \pi)\) in [7]. This surface has an analytic extension as the graph of \( t = x \tanh y \), which is analytically complete (cf. Theorem 2.25).

**Appendix B. A Property of Continuous Functions**

In this appendix, we point out the following property of continuous functions, which is used in Section 4:

**Lemma B.1.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a non-constant real analytic periodic function, and let \( \varphi : [0, 1) \to \mathbb{R} \) be a continuous function. If \( \lim_{t \to 1^-} f \circ \varphi(t) \) exists, then \( \lim_{t \to 1^-} \varphi(t) \) also exists.

**Proof.** Since \( f \) is non-constant, there exists a closed interval \([a, b]\) (\(a < b\)) which coincides with \( f(\mathbb{R})\). If \( \varphi(t) \) is unbounded as \( t \to 1^- \), then \( f \circ \varphi(t) \) takes all values in \([a, b]\) infinitely many times, contradicting the existence of the limit \( \lim_{t \to 1^-} f \circ \varphi(t) \).

So \( \varphi(t) \) is bounded. Suppose that \( \lim_{t \to 1^-} \varphi(t) \) does not exist and so there exist two distinct accumulation points \( \alpha, \beta \in \mathbb{R} \) (\( \alpha < \beta \)) of \( \varphi \). Then there are two distinct sequences \( \{t_k\}_{k=1}^{\infty} \) and \( \{s_k\}_{k=1}^{\infty} \) in \([0, 1)\) converging to 1 such that
\[
\lim_{k \to \infty} \varphi(t_k) = \alpha, \quad \lim_{k \to \infty} \varphi(s_k) = \beta.
\]
By taking a subsequence of \( \{s_k\}_{k=1}^{\infty} \) and a subsequence of \( \{t_k\}_{k=1}^{\infty} \) if necessary, we may assume that \( t_k < s_k < t_{k+1} \) for each positive integer \( k \). We can find a positive number \( \delta(<(\beta - \alpha)/3) \) and a positive integer \( N \) such that
\[
|\varphi(t_k) - \alpha|, \quad |\varphi(s_k) - \beta| < \delta \quad (k \geq N).
\]
Then \([\alpha + \delta, \beta - \delta] \subset \varphi([t_k, s_k]) \) holds for each \( k \geq N \). We denote by \( M \) (resp. \( m \)) the maximum (resp. minimum) value of \( f \) on the interval \([\alpha + \delta, \beta - \delta] \). Since \( f \) is a non-constant real analytic function, we have \( m < M \). Then, by the intermediate value theorem, there exist \( u_k, v_k \in (t_k, s_k) \) such that \( f \circ \varphi(u_k) = m \) and \( f \circ \varphi(v_k) = M \) for each \( k \). Since \( s_k \) and \( t_k \) are converging to 1, so are the two values \( u_k \) and \( v_k \). Then
\[
m = \lim_{k \to \infty} f \circ \varphi(u_k) = \lim_{k \to \infty} f \circ \varphi(v_k) = M,
\]
a contradiction. Hence, \( \lim_{t \to 1^-} \varphi(t) \) exists. \( \square \)

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