On Sequential Vector Parameter Estimation

Yasin Yilmaz†, George V. Moustakides‡, and Xiaodong Wang†

Abstract

Sequential estimation of a vector of signal amplitudes in a linear signal model is considered. It is known that finding the optimal sequential estimator is an intractable problem with a general stopping time that is adapted to the complete observation history [1]. By properly restricting the search space to stopping times that are adapted to a specific subset of the complete observation history we derive the optimal sequential estimator under two different formulations of the problem. In the first formulation, the (unconditional) covariance of the estimator is used to assess its accuracy, as a common practice. In the second formulation, the conditional covariance is used for the same purpose, which is preferred over the former when there is an auxiliary statistic whose distribution does not depend on the parameters to be estimated [2]. Our analytical results show that the optimal stopping rule in the conditional formulation is a simple one-dimensional threshold rule for any number of parameters to be estimated. On the other hand, the optimal stopping rule in the unconditional formulation for $n$ parameters is characterized by a hypersurface in a $\frac{n^2+n}{2}$-dimensional space, which is found by numerically computing an $n$-dimensional integral. Hence, finding the optimal sequential estimator under the traditional unconditional formulation is not tractable even for a small number of parameters. However, under the conditional formulation the optimal sequential estimator has the same simple form for any number of parameters, allowing for decentralized implementations. We further consider the decentralized version of the conditional problem in a wireless sensor network where each sensor can only send a brief summary of its observations to a fusion center (FC) due to strict bandwidth and energy constraints. Using a non-uniform sampling technique, namely the level-triggered sampling, which provides an effective way of information transmission in decentralized systems, we propose a computationally efficient decentralized sequential estimator.

I. INTRODUCTION

In this paper, we are interested in sequentially estimating a vector of linearly modulated signal amplitudes $X \in \mathbb{R}^n$ at a stopping time $S$, which is random. In sequential estimation, we keep refining our estimator as new samples are observed and stop when a target accuracy level is reached. This is in contrast...
to the fixed-sample-size estimation where one waits until a specific number of samples is obtained and performs estimation at this deterministic time. More precisely, a sequential estimator consists of a pair \((S, \hat{X}_S)\) of the stopping time and the corresponding estimate at the stopping time, whereas a fixed-sample-size estimator is composed of only an estimate \(\hat{X}_t\) corresponding to some deterministic time \(t\). It is known that sequential methods on average need much less samples than their fixed-sample-size counterparts to achieve the same accuracy level \[3, \text{Page 109}\]. On the other hand, computation and analysis of sequential estimators are more demanding tasks than those of fixed-sample-size estimators, as expected. For instance, the optimal fixed-sample-size estimator achieving a certain performance bound, such as the Ziv-Zakai lower bound in \[4\] or the Cramer-Rao lower bound (CRLB) in \[5\], is available in many cases, whereas as shown in \[11\] in most cases finding an optimal sequential estimator is not a tractable problem if \(S\) is adapted to the complete observation history. Alternatively, in \[6\] and more recently in \[7\], \[8\], it was proposed to restrict \(S\) to stopping times that are adapted to a specific subset of the complete observation history, which leads to simple solutions with alternative optimality. Note that this idea of using a restricted stopping time was first advocated in \[6\] although without providing any optimality result. In the case of continuous-time observations, a sequential estimator with a restricted stopping time was shown to achieve the sequential version of the CRLB in \[7\]. In the discrete-time case, a similar sequential estimator was shown to achieve the conditional sequential CRLB in \[8\]. In this paper, with discrete-time observations and using a similar restricted stopping time we present the optimal sequential estimator for the more challenging unconditional problem. Moreover, we treat the vector parameter estimation problem, whereas only the scalar parameter estimation problem was considered in \[6\]–\[8\].

On the other hand, decentralized vector parameter estimation has been studied in a variety of papers, e.g., \[9\]–\[18\]. Among those \[9\], \[10\], \[12\]–\[14\] consider a wireless sensor network with a fusion center (FC), which computes a global estimator using local information received from sensors, whereas \[15\]–\[18\] consider \textit{ad hoc} wireless sensor networks, where sensors in the absence of an FC compute their local estimators and communicate them through the network. Decentralized estimation under both network topologies is reviewed in \[11\]. In \[9\], \[11\], \[14\], \[15\], \[17\], \[18\] a general nonlinear signal model is assumed. The majority of pertinent works in the literature, e.g., \[9\]–\[16\], studies fixed-sample-size estimation. There are a few works, such as \[17\], \[18\], that consider sequential decentralized vector parameter estimation. Nevertheless, \[17\], \[18\] assume that sensors transmit real numbers, which obviously requires quantization in practice. In many applications quantization must be performed using a small number of bits due to the strict energy and bandwidth constraints on sensors. Recently, in a series of papers \[7\], \[8\], \[20\]–\[22\] it was shown that level-triggered sampling, which is a non-uniform sampling
method, meets those strict constraints by infrequently transmitting a single bit from sensors to the FC.
Moreover, decentralized schemes based on level-triggered sampling can significantly outperform their
counterparts based on traditional uniform-in-time sampling [21]. Following the decentralized schemes
based on level-triggered sampling presented for scalar parameter estimation in [7] and [8] we propose in
this paper a computationally efficient decentralized scheme for vector parameter estimation.

We represent scalars with lower-case letters, vectors with upper-case letters and matrices with upper-
case bold letters. The remainder of the paper is organized as follows. The centralized sequential vector
estimation problem is formulated in Section II. In Section III, we derive the optimal centralized sequen-
tial estimators under two different criteria. We then propose a computationally efficient decentralized
sequential estimator in Section IV. Finally, the paper is concluded in Section V.

II. PROBLEM FORMULATION AND BACKGROUND

Consider the following linear signal model,
\[ y_t = H_t^T X + w_t, \quad t \in \mathbb{N}, \]
(1)
where \( y_t \in \mathbb{R} \) is the observed sample, \( X \in \mathbb{R}^n \) is the deterministic but unknown vector of parameters to
be estimated, \( H_t \in \mathbb{R}^n \) is the random vector of scaling coefficients and \( w_t \in \mathbb{R} \) is the additive noise. We
observe, at each time \( t \), the sample \( y_t \) and the coefficient vector \( H_t \). Hence, at each time \( t \), \( \{ (y_\tau, H_\tau) \}_{\tau=1}^t \)
are available. We assume \( \{w_\tau\} \) are i.i.d. with \( \mathbb{E}[w_\tau] = 0 \) and \( \text{Var}(w_\tau) = \sigma^2 \). We are interested in estimating
\( X \).

The least squares (LS) estimator minimizes the sum of squared errors, i.e.,
\[ \hat{X}_t = \arg \min_X \sum_{\tau=1}^t (y_\tau - H_\tau^T X)^2, \]
and is given by
\[ \hat{X}_t = \left( \sum_{\tau=1}^t H_\tau H_\tau^T \right)^{-1} \sum_{\tau=1}^t H_\tau y_\tau = (H_t^T H_t)^{-1} H_t^T Y_t, \]
(3)
where \( H_t = [H_1, \ldots, H_t]^T \) and \( Y_t = [y_1, \ldots, y_t]^T \).

Under the Gaussian noise, \( w_\tau \sim \mathcal{N}(0, \sigma^2) \), the LS estimator coincides with the minimum variance
unbiased estimator (MVUE). That is, the LS estimator achieves the Cramer-Rao lower bound (CRLB),
i.e., \( \text{Cov}(\hat{X}_t|H_t) = \text{CRLB} \). To compute the CRLB we first write, given \( X \) and \( H_t \), the log-likelihood
of the vector \( Y_t \) as
\[ L_t = \log f(Y_t|X, H_t) = -\sum_{\tau=1}^t \frac{(y_\tau - H_\tau^T X)^2}{2\sigma^2} - \frac{t}{2} \log(2\pi\sigma^2). \]
(4)
Then, we have
\[ \text{CRLB} = \left( \mathbb{E} \left[ -\frac{\partial^2}{\partial X^2} L_t | H_t \right] \right)^{-1} = \sigma^2 U_t^{-1}, \] (5)
where \( \mathbb{E} \left[ -\frac{\partial^2}{\partial X^2} L_t | H_t \right] \) is the Fisher information matrix and \( U_t \triangleq H_t^T \Sigma H_t \). Since \( \mathbb{E}[Y_t|H_t] = H_t X \) and \( \text{Cov}(Y_t|H_t) = \sigma^2 I \), from (3) we have \( \mathbb{E}[\hat{X}_t|H_t] = X \) and \( \text{Cov}(\hat{X}_t|H_t) = \sigma^2 U_t^{-1} \), thus from (5) \( \text{Cov}(\hat{X}_t|H_t) = \text{CRLB} \). Note that the maximum likelihood (ML) estimator, that maximizes (4), coincides with the LS estimator in (3).

In general, under a non-Gaussian noise distribution the LS estimator is the best linear unbiased estimator (BLUE). In other words, any linear unbiased estimator of the form \( A_t Y_t \) with \( A_t \in \mathbb{R}^{n \times t} \), where \( \mathbb{E}[A_t Y_t|H_t] = X \), has a covariance no smaller than that of the LS estimator in (3), i.e., \( \text{Cov}(A_t Y_t|H_t) \geq \sigma^2 U_t^{-1} \) in the positive semidefinite sense. To see this result we write \( A_t = (H_t^T H_t)^{-1} H_t^T + B_t \) for some \( B_t \in \mathbb{R}^{n \times t} \), and then \( \text{Cov}(A_t Y_t|H_t) = \sigma^2 U_t^{-1} + \sigma^2 B_t B_t^T \), where \( B_t B_t^T \) is a positive semidefinite matrix.

The recursive least squares (RLS) algorithm enables us to compute \( \hat{X}_t \) in a recursive way as follows
\[
\hat{X}_t = \hat{X}_{t-1} + K_t(y_t - H_t^T \hat{X}_{t-1})
\]
where \( K_t = \frac{P_{t-1} H_t}{1 + H_t^T P_{t-1} H_t} \) and \( P_t = P_{t-1} - K_t H_t^T P_{t-1} \), (6)
where \( K_t \in \mathbb{R}^n \) is a gain vector and \( P_t = U_t^{-1} \). While applying RLS we first initialize \( \hat{X}_0 = 0 \) and \( P_0 = \delta^{-1} I \), where 0 represents a zero vector and \( \delta \) is a small number, and then at each time \( t \) compute \( K_t, \hat{X}_t \) and \( P_t \) as in (6).

III. OPTIMAL SEQUENTIAL ESTIMATORS

In this section we aim to find the optimal pair \( (T, \hat{X}_T) \) of stopping time and estimator corresponding to the optimal sequential estimator. The stopping time for a sequential estimator is defined as the time it first achieves a target accuracy level. We assess the accuracy of an estimator by using either its covariance matrix \( \text{Cov}(\hat{X}_t) \) or conditional covariance matrix \( \text{Cov}(\hat{X}_t|H_t) \). Although traditionally the former is used in general, in the presence of an auxiliary statistic whose distribution does not depend on the parameters to be estimated, such as \( H_t \), the latter was shown to be a better choice in (2) through asymptotic analysis. Specifically, the optimal sequential estimators can be formulated as the following constrained optimization problems,

\[
\min_{T, \hat{X}_T} \mathbb{E}[T|H_T] \quad \text{such that} \quad f \left( \text{Cov}(\hat{X}_T|H_T) \right) \leq C, \quad (7)
\]
and
\[
\min_{T, \hat{X}_T} \mathbb{E}[T] \quad \text{such that} \quad f \left( \text{Cov}(\hat{X}_T) \right) \leq C, \quad (8)
\]
under the conditional and unconditional setups, respectively, where \( f(\cdot) \) is a function from \( \mathbb{R}^{n \times n} \) to \( \mathbb{R} \) and \( C \in \mathbb{R} \) is the target accuracy level.

Note that the constraint in (7) is stricter than the one in (8) since it requires that \( \hat{X}_T \) satisfies the target accuracy level for each realization of \( H_T \), whereas in (8) it is sufficient that \( \hat{X}_T \) satisfies the target accuracy level on average. In other words, in (8) even if for some realizations of \( H_T \) we have \( f(Cov(\hat{X}_T|H_T)) > C \), we can still have \( f(Cov(\hat{X}_T)) \leq C \). In fact, we can always have \( f(Cov(\hat{X}_T)) = C \) by using a probabilistic stopping rule such that we sometimes stop above \( C \), i.e., \( f(Cov(\hat{X}_T|H_T)) > C \), and the rest of the time at or below \( C \), i.e., \( f(Cov(\hat{X}_T|H_T)) \leq C \). On the other hand, in (7) we always have \( f(Cov(\hat{X}_T|H_T)) \leq C \), and moreover since we observe discrete-time samples, in general we have \( f(Cov(\hat{X}_T|H_T)) < C \) for each realization of \( H_T \). Hence, the optimal objective value \( E[T] \) in (8) will in general be smaller than the optimal objective value \( E[T|H_T] \) in (7).

Note that on the other hand, if we observed continuous-time processes with continuous paths, then we could always have \( f(Cov(\hat{X}_T|H_T)) = C \) for each realization of \( H_T \), and thus the optimal objective values of (7) and (8) would be the same.

The accuracy function \( f \) should be a monotonic function of the covariance matrices \( Cov(\hat{X}_T|H_T) \) and \( Cov(\hat{X}_T) \) in order to make consistent accuracy assessments. Two popular and easy-to-compute choices are the trace \( Tr(\cdot) \), which corresponds to the mean squared error (MSE), and the Frobenius norm \( \| \cdot \|_F \). In the sequel we will next treat the two problems in (7) and (8) separately.

### A. The Optimal Conditional Sequential Estimator

Denote \( \{\mathcal{F}_t\} \) as the filtration that corresponds to the samples \( \{y_1, \ldots, y_t\} \) where \( \mathcal{F}_t = \sigma\{y_1, \ldots, y_t\} \) is the \( \sigma \)-algebra generated by the samples observed up to time \( t \) and \( \mathcal{F}_0 \) is the trivial \( \sigma \)-algebra. Similarly we define the filtration \( \{\mathcal{H}_t\} \) where \( \mathcal{H}_t = \sigma\{H_1, \ldots, H_t\} \), i.e., the accumulated history related to the coefficient vectors, and \( \mathcal{H}_0 \) is again the trivial \( \sigma \)-algebra. It is known that, in general, with discrete-time observations and an unrestricted stopping time, that is \( \{\mathcal{F}_t \cup \mathcal{H}_t\} \)-adapted, the sequential CRLB is not attainable under any noise distribution (Gaussian or non-Gaussian) except for the Bernoulli noise [1]. On the other hand, in the case of continuous-time observations with continuous paths the sequential CRLB is attainable by using an \( \{\mathcal{H}_t\} \)-adapted stopping time [7]. Moreover, with an \( \{\mathcal{H}_t\} \)-adapted stopping time, that depends only on \( H_T \), the LS estimator attains the conditional sequential CRLB as we will show in the following Lemma. Hence, in this paper we restrict our attention to \( \{\mathcal{H}_t\} \)-adapted stopping times as in [6]–[8].
**Lemma 1.** With a monotonic accuracy function $f$ and an $\{H_t\}$-adapted stopping time $T$ we can write

$$f \left( \text{Cov} (\hat{X}_T | H_T) \right) \geq f \left( \sigma^2 U_T^{-1} \right)$$

for all unbiased estimators under Gaussian noise, and for all linear unbiased estimators under non-Gaussian noise, and the inequality in (9) holds with equality for the LS estimator.

**Proof:** In the previous section, the LS estimator was shown to be the MVUE under Gaussian noise and the BLUE under non-Gaussian noise. It was also shown that $\text{Cov}(\hat{X}_t|H_t) = \sigma^2 U_t^{-1}$. Hence, we write

$$f \left( \text{Cov} (\hat{X}_T | H_T) \right) = f \left( \mathbb{E} \left[ \sum_{t=1}^{\infty} (\hat{X}_t - X)(\hat{X}_t - X)^T \mathbb{1}_{\{t=T\}} | H_t \right] \right)$$

$$= f \left( \sum_{t=1}^{\infty} \mathbb{E} \left[ (\hat{X}_t - X)(\hat{X}_t - X)^T | H_t \right] \mathbb{1}_{\{t=T\}} \right)$$

$$\geq f \left( \sum_{t=1}^{\infty} \sigma^2 U_t^{-1} \mathbb{1}_{\{t=T\}} \right)$$

$$= f \left( \sigma^2 U_T^{-1} \right),$$

for all unbiased estimators under Gaussian noise and for all linear unbiased estimators under non-Gaussian noise. We used the facts that the event $\{T = t\}$ is $\mathcal{H}_t$-measurable and $\mathbb{E}[(\hat{X}_t - X)(\hat{X}_t - X)^T | H_t] = \text{Cov}(\hat{X}_t|H_t) \geq \sigma^2 U_t^{-1}$ to write (10) and (11), respectively.

Since $T$ is $\{\mathcal{H}_t\}$-adapted, i.e., determined by $H_T$, we have $\mathbb{E}[T | H_T] = T$, and thus from (7) we want to find the first time that a member of our class of estimators (i.e., unbiased estimators under Gaussian noise and linear unbiased estimators under non-Gaussian noise) satisfies the constraint $f \left( \text{Cov}(\hat{X}_T | H_T) \right) \leq C$, as well as the estimator that attains this earliest stopping time. From Lemma 1 it is seen that the LS estimator achieves the earliest stopping time among its competitors. Hence, for the conditional problem the optimal pair of stopping time and estimator is $(T, \hat{X}_T)$ where $T$ is given by

$$T = \min \{ t \in \mathbb{N} : f \left( \sigma^2 U_t^{-1} \right) \leq C \},$$

(13)

and from (3), $\hat{X}_T = U_T^{-1} V_T$ where $V_T \triangleq H_T^T Y_T$, which can be computed recursively as in (6). The recursive computation of $U_t^{-1} = P_t$ in the test statistic in (13) is also given in (6). Note that for an accuracy function $f$ such that $f(\sigma^2 U_t^{-1}) = \sigma^2 f(U_t^{-1})$, e.g., $\text{Tr}(\cdot)$ and $\| \cdot \|_F$, we can use the following stopping time,

$$T = \min \{ t \in \mathbb{N} : f \left( U_t^{-1} \right) \leq C' \},$$

(14)
where \( C' = \frac{C}{\sigma^2} \) is the relative target accuracy with respect to the noise power. Hence, given \( C' \) we do not need to know the noise variance \( \sigma^2 \) to run the test given by (14).

Note that \( U_t = H_t^T H_t \) is a non-decreasing positive semidefinite matrix. Thus, from the monotonicity of \( f \), the test statistic \( f (\sigma^2 U_t^{-1}) \) is a non-increasing scalar function of time. Specifically, for accuracy functions \( \text{Tr}(\cdot) \) and \( \|\cdot\|_F \) we can show that if the minimum eigenvalue of \( U_t \) tends to infinity as \( t \to \infty \), then the stopping time is finite, i.e., \( T < \infty \).

For the special case of scalar parameter estimation, we do not need a function \( f \) to assess the accuracy of the estimator since instead of a covariance matrix we now have a variance \( \sigma^2 u_t \), where \( u_t = \sum_{\tau=1}^t h_\tau^2 \) and \( h_\tau \) is the scaling coefficient in (1). Hence, from (14) the stopping time in the scalar case is given by

\[
T = \min \left\{ t \in \mathbb{N} : u_t \geq \frac{1}{C'} \right\}
\]

(15)

where \( \frac{u_t}{\sigma^2} \) is the Fisher information at time \( t \). This result is in accordance with [8, Eq. (3)].

B. The Optimal Unconditional Sequential Estimator

In this case we assume \( \{H_t\} \) is i.i.d.. From the constrained optimization problem in (8), using a Lagrange multiplier \( \lambda \) we obtain the following unconstrained optimization problem,

\[
\min_{T, \hat{X}_T} E[T] + \lambda f \left( \text{Cov}(\hat{X}_T) \right).
\]

(16)

We are again interested in \( \{H_t\} \)-adapted stopping times to use the optimality property of the LS estimator in the sequential sense. For simplicity assume a linear accuracy function \( f \) so that \( f(E[\cdot]) = E[f(\cdot)] \), e.g., the trace function \( \text{Tr}(\cdot) \). Then, our constraint function becomes the sum of the individual variances, i.e.,

\[
\text{Tr} \left( \text{Cov}(\hat{X}_T) \right) = \sum_{i=1}^n \text{Var}(\hat{x}_i).
\]

Since \( \text{Tr} \left( \text{Cov}(\hat{X}_T) \right) = \text{Tr} \left( E \left[ \text{Cov}(\hat{X}_T | H_T) \right] \right) = E \left[ \text{Tr} \left( \text{Cov}(\hat{X}_T | H_T) \right) \right] \), we rewrite (16) as

\[
\min_{T, \hat{X}_T} E \left[ T + \lambda \text{Tr} \left( \text{Cov}(\hat{X}_T | H_T) \right) \right],
\]

(17)

where the expectation is with respect to \( H_T \).

From Lemma [1] we have \( \text{Tr} \left( \text{Cov}(\hat{X}_T | H_T) \right) \geq \text{Tr} \left( \sigma^2 U_T^{-1} \right) \) where \( \sigma^2 U_T^{-1} \) is the covariance matrix of the LS estimator at time \( t \). Note that \( U_t/\sigma^2 \) is the Fisher information matrix at time \( t \) [cf. (5)]. Using the LS estimator we minimize the objective value in (17). Hence, \( \hat{X}_T = U_T^{-1} V_T \) [cf. (6) for recursive computation] is also the optimal estimator for the unconditional problem.

Now, to find the optimal stopping time we need to solve the following optimization problem,

\[
\min_T E \left[ T + \lambda \text{Tr} \left( \sigma^2 U_T^{-1} \right) \right],
\]

(18)
which can be solved by using the *optimal stopping theory*. Writing (18) in the following alternative form

\[
\min_T E \left[ \sum_{t=0}^{T-1} 1 + \lambda \text{Tr} \left( \sigma^2 U^{-1}_T \right) \right],
\]

we see that the term \( \sum_{t=0}^{T-1} 1 \) accounts for the cost of not stopping until time \( T \) and the term \( \lambda \text{Tr} \left( \sigma^2 U^{-1}_T \right) \) represents the cost of stopping at time \( T \). Note that \( U_t = U_{t-1} + H_t H_t^T \) and given \( U_{t-1} \) the current state \( U_t \) is (conditionally) independent of all previous states, hence \( \{U_t\} \) is a Markov process. That is, the optimal stopping time for a Markov process is sought in (19). From (19) the optimal solution is based on solving the following problem:

\[
V(U) = \min \left\{ \lambda \text{Tr} \left( \sigma^2 U^{-1} \right), 1 + E[V(U + H_1 H_1^T)|U] \right\},
\]

where the expectation is with respect to \( H_1 \) and \( V \) is the optimal cost function. The optimal cost function is found by iterating a sequence of functions \( \{V_m\} \) where \( V(U) = \lim_{m \to \infty} V_m(U) \) and

\[
V_m(U) = \min \left\{ \lambda \text{Tr} \left( \sigma^2 U^{-1} \right), 1 + E[V_{m-1}(U + H_1 H_1^T)|U] \right\}.
\]

In the above optimal stopping theory, the original complex optimization problem in (18) is divided into simpler subproblems given by (20). At each time \( t \) we are faced with a subproblem consisting of a stopping cost \( F(U_t) = \lambda \text{Tr} \left( \sigma^2 U^{-1}_t \right) \) and an expected sampling cost \( G(U_t) = 1 + E[V(U_{t+1})|U_t] \) to proceed to time \( t + 1 \). The optimal cost function \( V(U_t) \), selecting the action with minimum cost (i.e., either continue or stop), determines the optimal policy to follow at each time \( t \). That is, we stop the first time the stopping cost is smaller than the average cost of sampling, i.e.,

\[
T = \min \{ t \in \mathbb{N} : V(U_t) = F(U_t) \}.
\]

We obviously need to analyze the structure of \( V(U_t) \), i.e., the cost functions \( F(U_t) \) and \( G(U_t) \), to find the optimal stopping time \( T \).

Note that \( V \), being a function of the symmetric matrix \( U = [u_{ij}] \in \mathbb{R}^{n \times n} \), is a function of \( \frac{n^2 + n}{2} \) variables \( \{u_{ij} : i \leq j\} \). Analyzing a multi-dimensional optimal cost function proves intractable, hence we will first analyze the special case of scalar parameter estimation and then provide some numerical results for the two-dimensional vector case, demonstrating how intractable the higher dimensional problems are.

1) **Scalar case**: For the scalar case, from (20) we have the following one-dimensional optimal cost function,

\[
V(u) = \min \left\{ \frac{\lambda \sigma^2}{u}, 1 + E[V(u + h_1^2)] \right\},
\]

where \( h_1 \) is the step size. This formulation can be solved numerically for various values of \( \sigma^2 \), \( \lambda \), and \( h_1 \). The solution provides insights into the optimal policy and the trade-offs between the stopping cost and the expected sampling cost.
where the expectation is with respect to the scalar coefficient \( h_1 \). Specifically, at time \( t \) the optimal cost function is written as \( V(u_t) = \min \left\{ \frac{\lambda \sigma^2}{u_t}, 1 + E[V(u_{t+1})] \right\} \), where \( u_{t+1} = u_t + h_t^2 \). Writing \( V \) as a function of \( z_t \equiv 1/u_t \) we have \( V(z_t) = \min \left\{ \frac{\lambda \sigma^2 z_t}{G(z)}, 1 + E[V(z_{t+1})] \right\} \), where \( z_{t+1} = \frac{z_t}{1 + z_t h_t^2} \). We need to analyze the cost functions \( F(z) = \lambda \sigma^2 z \) and \( G(z) = 1 + E \left[ V \left( \frac{z}{1 + z h_t^2} \right) \right] \). The former is a line, whereas the latter is in general a nonlinear function of \( z \). We have the following lemma regarding the structure of \( V(z) \) and \( G(z) \). Its proof is given in the Appendix.

**Lemma 2.** The optimal cost \( V \) and the expected sampling cost \( G \), given in (22), are non-decreasing, concave and bounded functions of \( z \).

Following Lemma 2 the theorem below presents the stopping time for the scalar case of the unconditional problem.

**Theorem 1.** The optimal stopping time for the scalar case of the unconditional problem in (8) with \( \operatorname{Tr}(\cdot) \) as the accuracy function is given by

\[
\mathcal{T} = \min \left\{ t \in \mathbb{N} : u_t \geq \frac{1}{C''} \right\},
\]

where \( C'' \) is selected so that \( E \left[ \frac{\sigma^2}{u_T} \right] = C \), i.e., the variance of the estimator exactly hits the target accuracy level \( C \).

**Proof:** The cost functions \( F(z) \) and \( G(z) \) are continuous functions as \( F \) is linear and \( G \) is concave. From (22) we have \( V(0) = \min \{0, 1 + V(0)\} = 0 \), hence \( G(0) = 1 + V(0) = 1 \). Then, using Lemma 2 we illustrate \( F(z) \) and \( G(z) \) in Fig. 1. The optimal cost function \( V(z) \), being the minimum of \( F \) and \( G \) [cf. (22)], is also shown in Fig. 1. Note that as \( t \) increases \( z \) tends from infinity to zero. Hence, we continue until the stopping cost \( F(z_t) \) is lower than the expected sampling cost \( G(z_t) \), i.e., until \( z_t \leq C'' \). The threshold \( C''(\lambda) = \{ z : F(\lambda, z) = G(z) \} \) is determined by the Lagrange multiplier \( \lambda \), which is selected to satisfy the constraint \( \operatorname{Var}(\hat{x}_{\mathcal{T}}) = E \left[ \frac{\sigma^2}{u_T} \right] = C \) [cf. (16)].

Note that the optimal stopping time in (23) is of the same form as that in the scalar case of the conditional problem, where we have \( \mathcal{T} = \min \{ t \in \mathbb{N} : u_t \geq \frac{1}{C} \} \) from (15). In both conditional and
Fig. 1. The structures of the optimal cost function $V(z)$ and the cost functions $F(z)$ and $G(z)$.

**Algorithm 1** The procedure to compute the threshold $C''$ for given $C$

1: Select $C''$
2: Compute $C' = \mathbb{E} \left[ \frac{\sigma^2}{u_T} \right]$ through simulations, where $u_t = \sum_{\tau=1}^{t} h^2_{\tau}$ and $T = \min \{ t \in \mathbb{N} : u_t \geq \frac{1}{C''} \}$
3: if $C' = C$ then
4: return $C''$
5: else
6: if $C' > C$ then
7: Decrease $C''$
8: else
9: Increase $C''$
10: end if
11: Go to line 2
12: end if

unconditional problems the LS estimator

$$\hat{x}_T = \frac{u_T}{u_T}$$ (24)

is the optimal estimator. The fundamental difference between the optimal stopping times in (15) and (23) is that the threshold in the conditional problem is written as $C' = \frac{C}{\sigma^2}$, hence known beforehand; whereas the threshold $C''$ in the unconditional problem needs to be determined through offline simulations following the procedure in Algorithm 1 assuming that some training data $\{h_t\}$ is available or the statistics of $h_t$ is known so that we can generate $\{h_t\}$. We also observe that $C' \leq C''$, hence the optimal objective value $\mathbb{E}[T]$ of the unconditional problem is in general smaller than that of the conditional problem as noted earlier in this section. This is because the upper bound $\sigma^2 C''$ on the conditional variance $\frac{\sigma^2}{u_T}$ [cf. (23)] is also an upper bound for the variance $\mathbb{E}\left[ \frac{\sigma^2}{u_T} \right] = C$, and the threshold $C'$ is given by $C' = \frac{C}{\sigma^2}$. 
2) Two-dimensional case: We will next show that the multi-dimensional cases are intractable by providing some numerical results for the two-dimensional case. In the two-dimensional case, from \((20)\) the optimal cost function is written as

\[
V(u_{11}, u_{12}, u_{22}) = \min \left\{ \lambda \sigma^2 \frac{u_{11}^2 + u_{22}^2}{u_{11} u_{22} - u_{12}^2}, 1 + \mathbb{E} \left[ V(u_{11} + h_{1,1}^2, u_{12} + h_{1,1} h_{1,2}, u_{22} + h_{1,2}^2) \right] \right\},
\]

where \(U = \begin{bmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{bmatrix}\), \(H_1 = \begin{bmatrix} h_{1,1} \\ h_{1,2} \end{bmatrix}\), and the expectation is with respect to \(h_{1,1}\) and \(h_{1,2}\). Changing variables we can write \(V\) as a function of \(z_{11} \equiv 1/u_{11}, \ z_{22} \equiv 1/u_{22}\) and \(\rho \equiv u_{12}/\sqrt{u_{11} u_{22}}\).

\[
V(z_{11}, z_{22}, \rho) = \min \left\{ \lambda \sigma^2 \frac{z_{11} + z_{22}}{1 - \rho^2}, 1 + \mathbb{E} \left[ V \left( \frac{z_{11}}{1 + z_{11} h_{1,1}^2}, \frac{z_{22}}{1 + z_{22} h_{1,2}^2}, \frac{\rho + h_{1,1} h_{1,2} \sqrt{z_{11} z_{22}}}{\sqrt{(1 + z_{11} h_{1,1}^2)(1 + z_{22} h_{1,2}^2)}} \right) \right] \right\}
\]

which can be iteratively computed as follows

\[
V_m(z_{11}, z_{22}, \rho) = \min \left\{ \lambda \sigma^2 \frac{z_{11} + z_{22}}{1 - \rho^2}, 1 + \mathbb{E} \left[ V_{m-1} \left( \frac{z_{11}}{1 + z_{11} h_{1,1}^2}, \frac{z_{22}}{1 + z_{22} h_{1,2}^2}, \frac{\rho + h_{1,1} h_{1,2} \sqrt{z_{11} z_{22}}}{\sqrt{(1 + z_{11} h_{1,1}^2)(1 + z_{22} h_{1,2}^2)}} \right) \right] \right\}
\]

where \(\lim_{m \to \infty} V_m = V\).

Note that \(\rho\) is the correlation coefficient, hence we have \(\rho \in [-1, 1]\). Following the procedure in Algorithm \(2\) we numerically compute \(V\) from \((27)\) and find the boundary surface

\[
\mathcal{S}(\lambda) = \{(z_{11}, z_{22}, \rho) : F(\lambda, z_{11}, z_{22}, \rho) = G(z_{11}, z_{22}, \rho)\},
\]

that defines the stopping rule. In Algorithm \(2\) firstly the three-dimensional grid \((n_1 dz, n_2 dz, n_3 dr)\), \(n_1, n_2 = 0, \ldots, \frac{R_z}{dz}, n_3 = -\frac{1}{dr}, \ldots, \frac{1}{dr}\) is constructed. Then, in lines 5-7 the stopping cost \(F\) [cf. \((26)\)] and in line 8 the first iteration of the optimal cost function \(V_1\) with \(V_0 = 0\) are computed over the grid. In lines 10-34, the optimal cost function \(V\) is computed for each point in the grid by iterating \(V_m\) [cf. \((27)\)] until no significant change occurs between \(V_m\) and \(V_{m+1}\). In each iteration, the double integral with respect to \(h_{1,1}\) and \(h_{1,2}\) corresponding to the expectation in \((27)\) is computed in lines 15-27. While computing the integral, since the updated (future) \((z_{11}, z_{22}, \rho)\) values, i.e., the arguments of \(V_{m-1}\) in \((27)\), in general may not correspond to a grid point, we average the \(V_{m-1}\) values of eight neighboring grid points with appropriate weights in lines 21-24 to obtain the desired \(V_{m-1}\) value.
Algorithm 2  The procedure to compute the boundary surface $\mathcal{S}$ for given $\lambda$

1: Set $dz, R_z, dr, dh, Rh$; $Nz = \frac{Rz}{dz} + 1$; $N_r = \frac{2}{dr} + 1$; $Nh = \frac{2Rh}{dh} + 1$
2: $z_1 = [0 : dz : Rz]$; $z_2 = z_1$; $\rho = [-1 : dr : 1]$ {all row vectors}
3: $Z_1 = 1_{Nz} z_1$; $Z_2 = Z_1^T$ {1_{Nz}: column vector of ones in $\mathbb{R}^{Nz}$}
4: $h = [-Rh : dh : Rh]$
5: for $i = 1 : Nr$ do
6: $F(:, :, i) = \lambda \frac{Z_1 + Z_2}{\rho(i)}$ {stopping cost over the 3D grid}
7: end for
8: $V = \min(F, 1)$ {start with $V_0 = 0$}
9: dif = $\infty$; Fr = $\|V\|_F$
10: while dif > $\delta$ Fr {\delta: a small threshold} do
11: for $i = 1 : Nz^2$ do
12: $z_{11} = Z_1(i)$; $z_{22} = Z_2(i)$ {linear indexing in matrices}
13: for $j = 1 : Nr$ do
14: $G = 0$ {initialize continuing cost}
15: for $k = 1 : Nh$ do
16: $h_1 = h(k)$ {scalar}; $h_2 = h$ {vector}
17: $z_{11}^* = z_{11}/(1 + z_{11} h_2^T)$ {scalar}; $Z_{11}^* = z_{11}^* 1_{Nh}$ {vector}
18: $Z_{22}^* = Z_{22}/(1 + z_{22} h_2^{-2})$ {vector; dot denotes elementwise operation}
19: $\rho' = \rho(j) + h_1 h_2 \sqrt{z_{11} + z_{22}} / \sqrt{(1 + z_{11} h_2^T)(1 + z_{22} h_2^{-2})}$ {vector}
20: $I_1 = Z_{11}^* dz + 1$; $I_2 = Z_{22}^* dz + 1$; $I_3 = (\rho' + 1)/dr + 1$ {fractional indices}
21: $J_n^{8 \times Nh} = $ linear indices of 8 neighbor points using $[I_n], [I_n], n = 1, 2, 3$
22: $D_n = [I_n] - I_n$; $\overline{D}_n = 1 - D_n$, $n = 1, 2, 3$ {distances to neighbor indices}
23: $W^{8 \times Nh} = $ weights for neighbors as 8 multiplicative combinations of $D_n, \overline{D}_n, n = 1, 2, 3$
24: $V^{Nh \times 1} = \text{diag}(W^T V(J))$ {average the neighbor $V$ values}
25: $E^{1 \times Nh} = \frac{1}{Nh} \exp(-\frac{1}{2} (h_1^2 + h_2^2)) dh^2$ {weights in the integral}
26: $G = G + E \cdot V$ {update continuing cost}
27: end for
28: $\ell = i + (j - 1)Nz^2$ {linear index of the point the 3D grid}
29: $V'(\ell) = \min(F(\ell), 1 + G)$ {new optimal cost function}
30: end for
31: end for
32: dif = $\|V' - V\|_F$; Fr = $\|V\|_F$
33: $V = V'$ {update the optimal cost function}
34: end while
35: Find the points where transition occurs between regions $V = F$ and $V \neq F$, i.e., $\mathcal{S}$.

The results for $\lambda \in \{0.01, 1, 100\}$, $\sigma^2 = 1$ and $h_{1,1}, h_{1,2} \sim \mathcal{N}(0,1)$ are shown in Fig. 2 and Fig. 3.

For $\lambda = 1$, the dome-shaped surface in Fig. 2 separates the stopping region from the continuing region. Outside the “dome” $V = G$, hence we continue. As time progresses $z_{t,11}$ and $z_{t,22}$ decrease, so we move towards the “dome”. And whenever we are inside the “dome”, we stop, i.e., $V = F$. We obtain similar
Fig. 2. The surface that defines the stopping rule for $\lambda = 1$, $\sigma^2 = 1$ and $h_{t,1}, h_{t,2} \sim \mathcal{N}(0, 1)$ in the two-dimensional case.

Fig. 3. The stopping regions for $\rho_t = 0$, $\sigma^2 = 1$ and $h_{t,1}, h_{t,2} \sim \mathcal{N}(0, 1), \forall t$ in the unconditional problem with (a) $\lambda = 0.01$, (b) $\lambda = 1$, (c) $\lambda = 100$. That of the conditional problem is also shown in (c).

Dome-shaped surfaces for different $\lambda$ values. However, the cross-sections of the “domes” at specific $\rho_t$ values differ significantly. In particular, we investigate the case of $\rho_t = 0$, where the scaling coefficients $h_{t,1}$ and $h_{t,2}$ are uncorrelated. For small values of $\lambda$, e.g., $\lambda = 0.01$, the boundary that separates the stopping and the continuing regions is highly nonlinear as shown in Fig. 3(a). In Fig. 3(b) and 3(c), it is seen that the boundary tends to become more and more linear as $\lambda$ increases.

Now let us explain the meaning of the $\lambda$ value. Firstly, note from (26) that $F$ and $G$ are functions of $z_{11}$, $z_{22}$ for fixed $\rho$, and the boundary is the solution to $F(\lambda, z_{11}, z_{22}) = G(z_{11}, z_{22})$. When $\lambda$ is small, the region where $F < G$, i.e., the stopping region, is large, hence we stop early as shown in Fig. 3(a).\[1\]

\[1\]Note that the axis scales in Fig. 3(a) are on the order of hundreds and $z_{t,11}, z_{t,22}$ decrease as $t$ increases.
Algorithm 3 The procedure to compute the boundary surface $\mathcal{S}$

1: Select $\lambda$

2: Compute $\mathcal{S}(\lambda)$ as in Algorithm 2

3: Compute $C = E\left[\sigma^2 \frac{z_{t,11}^2 + z_{t,22}^2}{1 - \rho_t^2}\right]$ through simulations, where $z_{t,11} = 1/u_{t,11}$, $z_{t,22} = 1/u_{t,22}$, $\rho_t = u_{t,12}/\sqrt{u_{t,11}u_{t,22}}$ and $T = \min\{t \in \mathbb{N} : (z_{t,11}, z_{t,22}, \rho_t) \text{ is between } \mathcal{S} \text{ and the origin}\}$

4: if $\mathcal{C} = C$ then

5: return $\mathcal{S}$

6: else

7: if $\mathcal{C} > C$ then

8: Increase $\lambda$

9: else

10: Decrease $\lambda$

11: end if

12: Go to line 2

13: end if

Conversely, for large $\lambda$ the stopping region is small, hence the stopping time is large [cf. Fig. 3(c)]. In fact, the Lagrange multiplier $\lambda$ is selected through simulations following the procedure in Algorithm 3 so that the constraint $\text{Tr}(E[\sigma^2 U_T^{-1}]) = E\left[\sigma^2 \frac{z_{t,11}^2 + z_{t,22}^2}{1 - \rho_t^2}\right] = C$ is satisfied. Note that line 2 of Algorithm 3 uses Algorithm 2 to compute the boundary surface $\mathcal{S}$.

In general, in the unconditional problem we need to numerically compute the stopping rule offline, i.e., the hypersurface that separates the stopping and the continuing regions, for a given target accuracy level $C$. This becomes a quite intractable task as the dimension $n$ of the vector to be estimated increases since the computation of $G$ in (26) involves computing an $n$-dimensional integral, which is then used to find the separating hypersurface in a $\frac{n^2 + n}{2}$-dimensional space. On the other hand, in the conditional problem we have a simple stopping rule given in (14), which uses the target accuracy level $C/\sigma^2$ as its threshold, hence known beforehand for any $n$. Specifically, in the two-dimensional case of the conditional problem the optimal stopping time is given by $T = \min\{t \in \mathbb{N} : \frac{z_{t,11} + z_{t,22}}{1 - \rho_t^2} \leq \frac{C}{\sigma^2}\}$, which is a function of $z_{t,11} + z_{t,22}$ for fixed $\rho_t$. In Fig. 3(c), where $\rho_t = 0$ and $\sigma^2 = 1$, the stopping region of the conditional problem, which is characterized by a line, is shown to be smaller than that of the unconditional problem due to the same reasoning in the scalar case.

IV. DECENTRALIZED SEQUENTIAL ESTIMATOR

In this section, we propose an efficient scheme based on level-triggered sampling to implement the conditional sequential estimator in a decentralized way. Consider a network of $K$ distributed sensors and a fusion center (FC) which is responsible for computing the stopping time and the estimate. In practice, due
to the stringent bandwidth and energy constraints, sensors should infrequently convey low-rate information to the FC, which is the main concern in the design of a decentralized sequential estimator.

As in (1) each sensor $k$ observes

$$y^k_t = (H^k_t)^T X + w^k_t, \quad t \in \mathbb{N}, \quad k = 1, \ldots, K$$

(28)

as well as the coefficient vector $H^k_t = [h^k_{t,1}, \ldots, h^k_{t,n}]^T$ at time $t$, where $\{w^k_t\}_{k,t}$ are independent, zero-mean, i.e., $E[w^k_t] = 0$, $\forall k, t$, and $\text{Var}(w^k_t) = \sigma^2_k$, $\forall t$. In other words, we allow for different noise variances at different sensors. Then, similar to (3) the weighted least squares (WLS) estimator

$$\hat{X}_t = \arg \min_X \sum_{k=1}^{K} \sum_{\tau=1}^{t} \frac{(y^k_\tau - (H^k_\tau)^T X)^2}{\sigma^2_k}$$

is given by

$$\hat{X}_t = \left( \sum_{k=1}^{K} \sum_{\tau=1}^{t} \frac{H^k_\tau (H^k_\tau)^T}{\sigma^2_k} \right)^{-1} \sum_{k=1}^{K} \sum_{\tau=1}^{t} \frac{H^k_\tau y^k_\tau}{\sigma^2_k} = \bar{U}_t^{-1} \bar{V}_t$$

(29)

where $\bar{U}_t \triangleq \frac{1}{\sigma^2_k} \sum_{\tau=1}^{t} H^k_\tau (H^k_\tau)^T$, $\bar{V}_t \triangleq \frac{1}{\sigma^2_k} \sum_{\tau=1}^{t} H^k_\tau y^k_\tau$, $\bar{U}_t = \sum_{k=1}^{K} \bar{U}_t^k$ and $\bar{V}_t = \sum_{k=1}^{K} \bar{V}_t^k$. As before it can be shown that the WLS estimator $\hat{X}_t$ in (29) is the BLUE under the general noise distributions. Moreover, in the Gaussian noise case, where $w^k_t \sim \mathcal{N}(0, \sigma^2_k)$ $\forall t$ for each $k$, $\hat{X}_t$ is also the MVUE. Note that $\hat{X}_t$ in (29) coincides the ML estimator in the Gaussian case.

Following the steps in Section III-A it is straightforward to show that $(\mathcal{T}, \hat{X}_\mathcal{T})$ is the optimal sequential estimator where

$$\mathcal{T} = \min \left\{ t \in \mathbb{N} : f \left( \bar{U}_t^{-1} \right) \leq C \right\},$$

(30)

is the stopping time and the estimator $\hat{X}_t$ is given in (29). Note that $(\mathcal{T}, \hat{X}_\mathcal{T})$ is achievable only in the centralized case, where all local observations until time $t$, i.e., $\{(y^k_\tau, H^k_\tau)\}_{k,\tau}$ are available to the FC. Local processes $\{\bar{U}_t^k\}_{k,t}$ and $\{\bar{V}_t^k\}_{k,t}$ are used to compute the stopping time and the estimator as in (30) and (29), respectively. On the other hand, in a decentralized system the FC can compute approximations $\bar{U}_t^k$ and $\bar{V}_t^k$ at each time $t$, and then use these approximations to compute the stopping time and the estimator as in (30) and (29), respectively.

2The subscripts $k$ and $t$ in the set notation denote $k = 1, \ldots, K$ and $t \in \mathbb{N}$.

3The subscript $\tau$ in the set notation denotes $\tau = 1, \ldots, t$. 
A. Key Approximations in Decentralized Approach

Before proceeding to explain the proposed decentralized scheme, we note that our optimal estimator, i.e., the LS estimator, is a special case of the Kalman filter, which is easily decentralized through its inverse covariance form, namely the information filter. In the information filter, an $n \times n$ matrix, namely the information matrix, and an $n \times 1$ vector, namely the information vector, are used at each time, similar to $\bar{U}_t$ and $\bar{V}_t$ in the LS estimator, respectively. Decentralized implementations of the information filter in the literature, e.g., [23], require the transmission of local information matrices and information vectors from sensors to the FC at each time $t$, which may not be practical, especially for large $n$, since it requires transmission of $O(n^2)$ terms.

Considering $\text{Tr}(\cdot)$ as the accuracy function $f$ in (30), that is, assuming an MSE constraint, we propose to transmit only the $n$ diagonal entries of $\bar{U}_t^k$ for each $k$, giving us a computationally tractable scheme with linear complexity, i.e., $O(n)$, instead of quadratic complexity, i.e., $O(n^2)$. Using the diagonal entries of $\bar{U}_t$ we define the diagonal matrix

$$D_t \triangleq \text{diag}(d_{t,1}, \ldots, d_{t,n})$$

where

$$d_{t,i} = \sum_{k=1}^{K} \sum_{\tau=1}^{t} \frac{(h_{\tau,i}^k)^2}{\sigma_k^2}, \quad i = 1, \ldots, n.$$  \hspace{1cm} (31)

The simplified decentralized scheme that we propose in this section is motivated from the special case where $E[h_{t,i}^k h_{t,j}^k] = 0$, $\forall k, i, j = 1, \ldots, n$, $i \neq j$. In this case, by the law of large numbers for sufficiently large $t$ the off-diagonal elements of $\bar{U}_t$ vanish, and thus we have $\bar{U}_t \approx D_t$ and $\text{Tr}(\bar{U}_t^{-1}) \approx \text{Tr}(D_t^{-1})$.

For the general case where we might have $E[h_{t,i}^k h_{t,j}^k] \neq 0$ for some $k$ and $i \neq j$, using the diagonal matrix $D_t$ we write

$$\text{Tr} \left( \bar{U}_t^{-1} \right) = \text{Tr} \left( \left( D_t^{1/2} \bar{U}_t D_t^{-1/2} \right)^{-1} \right)$$

$$= \text{Tr} \left( D_t^{-1/2} R_t^{-1} D_t^{-1/2} \right)$$

$$= \text{Tr} \left( D_t^{-1} R_t^{-1} \right).$$ \hspace{1cm} (32) \hspace{1cm} (33)

Note that each entry $r_{t,ij}$ of the newly defined matrix $R_t$ is a normalized version of the corresponding entry $\bar{u}_{t,ij}$ of $\bar{U}_t$. Specifically, $r_{t,ij} = \frac{\bar{u}_{t,ij}}{\sqrt{d_{t,i} d_{t,j}}} = \frac{\bar{u}_{t,ij}}{\sqrt{\bar{u}_{t,ii} \bar{u}_{t,jj}}}, \quad i, j = 1, \ldots, n$, where the last equality...
follows from the definition of \( d_{t,i} \). Hence, \( R_t \) is given by

\[
R_t = \begin{bmatrix}
1 & r_{t,12} & \cdots & r_{t,1n} \\
r_{t,12} & 1 & \cdots & r_{t,2n} \\
\vdots & \vdots & \ddots & \vdots \\
r_{t,1n} & r_{t,2n} & \cdots & 1
\end{bmatrix},
\]

(34)

where \( r_{t,ij} = \frac{\sum_{k=1}^{K} \sum_{\tau=1}^{t} \frac{h_{t,i}^k h_{t,j}^k}{\sigma_k^2}}{\sqrt{\sum_{k=1}^{K} \sum_{\tau=1}^{t} \frac{(h_{t,i}^k)^2}{\sigma_k^2} \sum_{k=1}^{K} \sum_{\tau=1}^{t} \frac{(h_{t,j}^k)^2}{\sigma_k^2}}} \), \( i, j = 1, \ldots, n \).

For sufficiently large \( t \), by the law of large numbers

\[
r_{t,ij} \approx r_{ij} = \frac{\sum_{k=1}^{K} E[h_{t,i}^k h_{t,j}^k]}{\sqrt{\sum_{k=1}^{K} E[(h_{t,i}^k)^2] \sum_{k=1}^{K} E[(h_{t,j}^k)^2]}}
\]

(35)

and \( R_t \approx R \), where \( R \) is written as in (34) by replacing the entries \( \{r_{t,ij}\} \) with \( \{r_{ij}\} \). Hence, for sufficiently large \( t \) we can make the following approximations,

\[
\bar{U}_t \approx D_t^{1/2} R D_t^{1/2}
\]

\[
\text{and } \text{Tr} \left( \bar{U}_t^{-1} \right) \approx \text{Tr} \left( D_t^{-1} R^{-1} \right)
\]

(36)

using (32) and (33), respectively.

Then, assuming that the FC knows a priori the correlation matrix \( R \), i.e., \( \{E[h_{t,i}^k h_{t,j}^k]\}_{i,j,k} \) \(^4\) and \( \{\sigma_k^2\} \) [cf. (35)], it can compute the approximations in (36) if sensors report their local processes \( \{D_t^k\}_{k,t} \) to the FC, where \( D_t = \sum_{k=1}^{K} D_t^k \). Note that each local process \( \{D_t^k\}_t \) is \( n \)-dimensional, and its entries at time \( t \) are given by \( \{d_{t,i}^k = \sum_{\tau=1}^{t} \frac{(h_{t,i}^k)^2}{\sigma_k^2}\}_i \) [cf. (31)]. Hence, we propose that each sensor \( k \) sequentially reports the local processes \( \{D_t^k\}_t \) and \( \{V_t^k\}_t \) to the FC, achieving linear complexity \( O(n) \). On the other side, the FC, using the information received from sensors, computes the approximations \( \{\bar{D}_t\} \) and \( \{\bar{V}_t\} \), which are then used to compute the stopping time

\[
\bar{T} = \min \left\{ t \in \mathbb{N} : \text{Tr} \left( \bar{U}_t^{-1} \right) \leq \bar{C} \right\},
\]

(37)

\(^4\)The subscripts \( i \) and \( j \) in the set notation denote \( i = 1, \ldots, n \) and \( j = i, \ldots, n \). In the special case where \( E[(h_{t,i}^k)^2] = E[(h_{t,j}^m)^2] \), \( k, m = 1, \ldots, K, i = 1, \ldots, n \), the correlation coefficients

\[
\left\{ \xi_{ij}^k = \frac{E[h_{t,i}^k h_{t,j}^k]}{\sqrt{E[(h_{t,i}^k)^2] E[(h_{t,j}^k)^2]}} : i = 1, \ldots, n - 1, \ j = i + 1, \ldots, n \right\}_k
\]

together with \( \{\sigma_k^2\} \) are sufficient statistics since \( r_{ij} = \frac{\sum_{k=1}^{K} \xi_{ij}^k / \sigma_k^2}{\sum_{k=1}^{K} 1 / \sigma_k^2} \) from (35).
and the estimator

$$\tilde{X}_{\tilde{T}} = \tilde{U}_{\tilde{T}}^{-1} \tilde{V}_{\tilde{T}}$$

(38)

similar to (30) and (29), respectively. The approximations \( \text{Tr}(\tilde{U}_{\tilde{T}}^{-1}) \) in (37) and \( \tilde{U}_{\tilde{T}} \) in (38) are computed using \( \tilde{D}_t \) as in (36). The threshold \( \tilde{C} \) is selected through simulations to satisfy the constraint in (7) with equality, i.e., \( \text{Tr}\left(\text{Cov}(\tilde{X}_{\tilde{T}}|H_{\tilde{T}})\right) = C \).

B. Decentralized Sequential Estimator Based on Level-triggered Sampling

Level-triggered sampling provides a very convenient way of information transmission in decentralized systems as recently shown in [7], [8], [20]–[22]. Methods based on level-triggered sampling, sending infrequently small number of bits, e.g., one bit, from sensors to the FC, enables highly accurate approximations and thus high performance tests at the FC. They significantly outperform canonical sampling-and-quantizing methods which sample local processes using the conventional uniform-in-time sampling and send the quantized versions of samples to the FC [21]. On the other hand, level-triggered sampling, which is a non-uniform sampling technique, naturally outputs single-bit information, eliminating the need for quantization. Moreover, the FC can exactly recover the samples of local processes by using these single-bit information from sensors.

All of the references above that propose schemes based on level-triggered sampling deal with one-dimensional, i.e., scalar, local processes. However, in our case \( \{\tilde{V}_t^k\}_t \) is \( n \)-dimensional and even worse \( \{\tilde{U}_t^k\}_t \) is \( \frac{n^2+n}{2} \)-dimensional \( \text{[5]} \) for each \( k \). Applying a scheme that was proposed for one-dimensional processes in a straightforward fashion for each dimension, i.e., entry, of \( \tilde{U}_t^k \) becomes cumbersome, especially when \( n \) is large. Hence, in this paper we propose to use the approximations introduced in the previous subsection, achieving linear complexity \( O(n) \).

We will next describe the decentralized scheme based on level-triggered sampling in which each sensor non-uniformly samples the local processes \( \{D_t^k\}_t \) and \( \{\tilde{V}_t^k\}_t \), transmits single bit of information for each sample to the FC, and the FC computes \( \{\tilde{D}_t\} \) and \( \{\tilde{V}_t\} \) using received bits. Our scheme employs a novel algorithm to transmit a one-dimensional process, separately for each dimension of \( \{D_t^k\}_t \) and \( \{\tilde{V}_t^k\}_t \). In other words, our scheme at each sensor \( k \) consists of, in total, \( 2n \) level-triggered samplers running in parallel.

\( ^{5} \text{This is because } \tilde{U}_t^k \text{ is symmetric.} \)
1) **Sampling and Recovery of** $D^k_t$: Each sensor $k$ samples each entry $d_{t,i}^k$ of $D^k_t$ at a sequence of random times $\{s_{k,m,i}^k\}_{m \in \mathbb{N}^+}$ given by

$$s_{k,m,i}^k \triangleq \min \left\{ t \in \mathbb{N} : d_{t,i}^k - d_{s_{m-1},i}^k \geq \Delta_i^k \right\}, \quad s_{0,i}^k = 0,$$

where $d_{t,i}^k = \sum_{\tau=1}^{t} \frac{(h_{k,i}^\tau)^2}{\sigma_k^2}$, $d_{0,i}^k = 0$ and $\Delta_i^k > 0$ is a constant threshold that controls the average sampling interval. Note that the sampling times $\{s_{m,i}^k\}_{m}$ in (39) are dynamically determined by the signal to be sampled, i.e., realizations of $d_{t,i}^k$. Hence, they are random, whereas sampling times in the conventional uniform-in-time sampling are deterministic with a certain period. According to the sampling rule in (39), a sample is taken whenever the signal level $d_{t,i}^k$ increases by at least $\Delta_i^k$ since the last sampling time. Note that $d_{t,i}^k = \sum_{\tau=1}^{t} \frac{(h_{k,i}^\tau)^2}{\sigma_k^2}$ is non-decreasing in $t$.

At each sampling time $s_{m,i}^k$, sensor $k$ transmits a single bit to the FC at time $t_{m,i}^k \triangleq s_{m,i}^k + \delta_{m,i}^k$, indicating that $d_{t,i}^k$ has increased by at least $\Delta_i^k$ since the last sampling time $s_{m-1,i}^k$. The value of the transmitted bit is not important since it only indicates $d_{s_{m-1},i}^k - d_{s_{m-1},i}^k \geq \Delta_i^k$. The delay $\delta_{m,i}^k$ between the transmission time and the sampling time is used to linearly encode the overshoot

$$q_{m,i}^k \triangleq \left( d_{s_{m-1},i}^k - d_{s_{m-1},i}^k \right) - \Delta_i^k < \theta_d, \quad \forall k, m, i$$

and given by

$$\delta_{m,i}^k = \frac{d_{m-1,i}^k}{\phi_d} \in [0, 1).$$

The subscript $m$ in the set notation denotes $m \in \mathbb{N}^+$. 

---

*Fig. 4. Illustration of sampling time, transmission time, transmission delay and overshoot.*
where $\phi_d^{-1}$ is the slope of the linear encoding function (cf. Fig. 4), known to sensors and the FC.

Assume a global clock, that is, the time index $t \in \mathbb{N}$ is the same for all sensors and the FC, meaning that the FC knows the potential sampling times. Assume further ideal channels with no delay or deterministic delays between sensors and the FC. Then, the FC can measure the transmission delay $\delta_{m,i}^k$ if it is bounded by unit time, i.e., $\delta_{m,i}^k \in [0, 1)$. To ensure this, from (41), we need to have $\phi_d > q_{m,i}^k \forall k, m, i$. Assuming a bound for overshoots, i.e., $q_{m,i}^k < \theta_d \forall k, m, i$, we can achieve this by setting $\phi_d > \theta_d$. Consequently, the FC can uniquely decode the overshoot by computing $q_{m,i}^k = \phi_d \delta_{m,i}^k$ (cf. Fig. 4), using which it can also find the increment occurred in $d_{s_{m,i}}^k$ during the interval $[s_{m-1,i}, s_{m,i}]$ as $d_{s_{m-1,i}}^k - d_{s_{m,i}}^k = \Delta_i^k + q_{m,i}^k$ from (40). It is then possible to reach the signal level $d_{s_{m,i}}^k$ by accumulating the increments occurred until the $m$th sampling time, i.e.,

$$d_{s_{m,i}}^k = \sum_{\ell=1}^m \left( \Delta_i^k + q_{\ell,i}^k \right) = m \Delta_i^k + \sum_{\ell=1}^m q_{\ell,i}^k. \quad (42)$$

Using $\{d_{s_{m,i}}^k\}_m$ the FC computes the staircase approximation $\tilde{d}_{t,i}^k$ as

$$\tilde{d}_{t,i}^k = d_{s_{m,i}}^k, \quad t \in [t_{m,i}^k, t_{m+1,i}^k), \quad (43)$$

which is updated when a new bit is received from sensor $k$, otherwise kept constant. Such approximate local signals of different sensors are next combined to obtain the approximate global signal $\tilde{d}_{t,i}$ as

$$\tilde{d}_{t,i} = \sum_{k=1}^K \tilde{d}_{t,i}^k. \quad (44)$$

In practice, when the $m$th bit in the global order regarding dimension $i$ is received from sensor $k_m$ at time $t_{m,i}$, instead of computing (42)–(44) the FC only updates $\tilde{d}_{t,i}$ as

$$\tilde{d}_{t,m,i} = \tilde{d}_{t_{m-1},i} + \Delta_i^m + q_{m,i}, \quad \tilde{d}_{0,i} = \epsilon, \quad (45)$$

and keeps it constant when no bit arrives. We initialize $\tilde{d}_{t,i}$ to a small constant $\epsilon$ to prevent dividing by zero while computing the test statistic [cf. (46)]. Note that in general $\tilde{d}_{t_{m,i},i} \neq d_{s_{m,i}}^k$ unlike (43) since all sensors do not necessarily sample and transmit at the same time. The approximations $\{\tilde{d}_{t,i}\}_t$ form $\tilde{D}_t = \text{diag}(\tilde{d}_{t,1}, \ldots, \tilde{d}_{t,n})$, which is used in (37) and (38) to compute the stopping time and the estimator, respectively. Note that to determine the stopping time as in (37) we need to compute $\text{Tr} \left( \hat{U}_t^{-1} \right)$ using (36) at times $\{t_m\}_m$ when a bit is received from any sensor regarding any dimension. Fortunately, when

\[7\]
In fact, by setting the slope $\phi_d$ arbitrarily large we can make the transmission delay arbitrarily small.

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the \( m \)th bit in the global order is received from sensor \( k_m \) at time \( t_m \) regarding dimension \( i_m \) we can compute \( \text{Tr} \left( \tilde{U}_m^{-1} \right) \) recursively as follows

\[
\text{Tr} \left( \tilde{U}_m^{-1} \right) = \text{Tr} \left( \tilde{U}_{m-1}^{-1} \right) - \frac{\kappa_i (\Delta_{i_m} k_m + q_m)}{d_{m,i_m} d_{m-1,i_m}} \quad \text{Tr} \left( \tilde{U}_0^{-1} \right) = \sum_{i=1}^{n} \frac{\kappa_i}{\epsilon}, \tag{46}
\]

where \( \kappa_i \) is the \( i \)th diagonal element of the inverse correlation matrix \( \mathbf{R}^{-1} \), known to the FC. In (46) bit arrival times are assumed to be distinct for the sake of simplicity. In case multiple bits arrive at the same time, the update rule will be similar to (46) except that it will consider all new arrivals together.

2) Sampling and Recovery of \( \tilde{V}_k^m \): Similar to (39) each sensor \( k \) samples each entry \( \tilde{v}_{t,i}^k \) of \( \tilde{V}_k^m \) at a sequence of random times \( \{\alpha_{m,i}^k\}_m \) written as

\[
\alpha_{m,i}^k \triangleq \min \left\{ t \in \mathbb{N} : |\tilde{v}_{t,i}^k - \tilde{v}_{\alpha_{m-1,i}^k}^k| \geq \gamma_i^k \right\}, \quad \alpha_{0,i}^k = 0, \tag{47}
\]

where \( \tilde{v}_{t,i}^k = \sum_{\tau=1}^{t} \frac{h_k^\tau \gamma_i^k}{\sigma_i^k} \) and \( \gamma_i^k \) is a constant threshold, available to both sensor \( k \) and the FC. It has been shown in [21, Section IV-B] that \( \gamma_i^k \) is determined by

\[
\gamma_i^k \tanh(\gamma_i^k / 2) = T \sum_{K=1}^{K} \frac{E[\tilde{v}_{t,i}^k]}{K}
\]
to ensure the average sampling interval \( T \). Since \( \tilde{v}_{t,i}^k \) is neither increasing nor decreasing, we use two thresholds \( \gamma_i^k \) and \( -\gamma_i^k \) in the sampling rule given in (47). Specifically, a sample is taken whenever \( \tilde{v}_{t,i}^k \) increases or decreases by at least \( \gamma_i^k \) since the last sampling time. Then, sensor \( k \) at time \( \tau_{m,i}^k \triangleq \alpha_{m,i}^k + \beta_{m,i}^k \) transmits a single bit \( b_{m,i}^k \) to the FC, indicating whether \( \tilde{v}_{t,i}^k \) has changed by at least \( \gamma_i^k \) or \( -\gamma_i^k \) since the last sampling time \( \alpha_{m-1,i}^k \). We can simply write \( b_{m,i}^k \) as

\[
b_{m,i}^k = \text{sign} \left( \tilde{v}_{\alpha_{m,i}^k}^k - \tilde{v}_{\alpha_{m-1,i}^k}^k \right), \tag{48}
\]

where \( b_{m,i}^k = 1 \) implies that \( \tilde{v}_{\alpha_{m,i}^k}^k - \tilde{v}_{\alpha_{m-1,i}^k}^k \geq \gamma_i^k \) and \( b_{m,i}^k = -1 \) indicates that \( \tilde{v}_{\alpha_{m,i}^k}^k - \tilde{v}_{\alpha_{m-1,i}^k}^k \leq -\gamma_i^k \). The overshoot \( \eta_{m,i}^k \triangleq |\tilde{v}_{\alpha_{m,i}^k}^k - \tilde{v}_{\alpha_{m-1,i}^k}^k| - \gamma_i^k \) is linearly encoded in the transmission delay as before. Similar to (41) the transmission delay is written as \( \beta_{m,i}^k = \frac{\eta_{m,i}^k}{\phi_v^{-1}} \), where \( \phi_v^{-1} \) is the slope of the encoding function, available to sensors and the FC. Assume again that (i) there exists a global clock among sensors and the FC; (ii) channels between sensors and the FC are ideal with no or deterministic delay, and (iii) overshoots are bounded by a constant, i.e., \( \eta_{m,i}^k < \theta_v \), \( \forall k, m, i \), and we set \( \phi_v > \theta_v \). With these assumptions we ensure that the FC can measure the transmission delay \( \beta_{m,i}^k \), and accordingly decode the overshoot as \( \eta_{m,i}^k = \phi_v \beta_{m,i}^k \). Then, upon receiving the \( m \)th bit \( b_{m,i}^k \) regarding dimension \( i \) from sensor \( k_m \) at time \( \tau_{m,i}^k \) the FC performs the following update,

\[
\tilde{v}_{\tau_{m,i}^k}^k = \tilde{v}_{\tau_{m-1,i}^k}^k + b_{m,i}^k (\gamma_i^k + \eta_{m,i}^k), \tag{49}
\]
The level-triggered sampling procedure at the $k$th sensor for the $i$th dimension is summarized in Algorithm 4. Each sensor $k$ runs $n$ of these procedures in parallel. The sequential estimation procedure at the FC is also summarized in Algorithm 5. We assumed, for the sake of clarity, that each sensor transmits bits to the FC for each dimension through a separate channel, i.e., parallel architecture. On the other hand, in practice the number of parallel channels can be decreased to two by using identical sampling thresholds $\Delta$ and $\gamma$ for all sensors and for all dimensions in (39) and (47), respectively. Moreover, sensors can even employ a single channel to convey information about local processes $\{d_{k,t,i}\}$ and $\{\bar{v}_{k,t,i}\}$ by sending ternary digits to the FC. This is possible since bits transmitted for $\{d_{k,t,i}\}$ are unsigned.

## C. Simulation Results

We next provide simulation results to compare the performances of the proposed scheme with linear complexity, given in Algorithm 4 and Algorithm 5, the unsimplified version of the proposed scheme with quadratic complexity and the optimal centralized scheme. A wireless sensor network with 10 identical sensors and an FC is considered to estimate a five-dimensional deterministic vector of parameters, i.e.,
Algorithm 5 The sequential estimation procedure at the fusion center

1: Initialization: \( T_r \leftarrow \sum_{i=1}^{n} \kappa_i / \epsilon \), \( m \leftarrow 1 \), \( \ell \leftarrow 1 \), \( \tilde{d}_i \leftarrow 0 \) \( \forall i \)
2: while \( T_r < \tilde{C} \) do
3: \hspace{1em} Listen to the channels \( \{ch_{k,i}^d\}_{k,i} \) and \( \{ch_{k,i}^s\}_{k,i} \), and wait to receive a bit
4: \hspace{1em} if \( m \)th bit arrives through \( ch_{k,m,i}^d \) at time \( t_m \) then
5: \hspace{2em} \( q_m = \phi_d(t_m - [t_m]) \)
6: \hspace{2em} \( T_r \leftarrow T_r - \kappa_{im} (\Delta_{km} + q_m) \)
7: \hspace{2em} \( \tilde{d}_{im} = \tilde{d}_{im} + \Delta_{km} + q_m \)
8: \hspace{1em} end if
9: \hspace{1em} if \( \ell \)th bit \( b_{\ell} \) arrives through \( ch_{p,\ell,j}^v \) at time \( \tau_\ell \) then
10: \hspace{2em} \( \eta_\ell = \phi_v(\tau_\ell - [\tau_\ell]) \)
11: \hspace{2em} \( \bar{v}_{\ell} = \bar{v}_{\ell} + b_{\ell} (\gamma_{p,\ell} + \eta_\ell) \)
12: \hspace{2em} \( \ell \leftarrow \ell + 1 \)
13: end if
14: end while
15: \hspace{1em} Stop at time \( \tilde{T} = t_m \)
16: \hspace{1em} \( \tilde{D} = \text{diag}(\tilde{d}_1, \ldots, \tilde{d}_n) \), \( \tilde{U}^{-1} = \tilde{D}^{-1/2} R^{-1} \tilde{D}^{-1/2} \), \( \tilde{V} = [\tilde{v}_1, \ldots, \tilde{v}_n]^T \)
17: end

\( n = 5 \). We assume i.i.d. Gaussian noise with unit variance at all sensors, i.e., \( w^k_t \sim \mathcal{N}(0,1), \forall k, t \). We set the correlation coefficients \( \{r_{ij}\} \) [cf. (35)] of the vector \( H^k_t \) to different values in \( [0, 1) \). In Fig. 5 and Fig. 6 they are set to 0 and 0.5 to test the performance of the proposed scheme in the uncorrelated and correlated cases, respectively. We compare the average stopping time performance of the proposed scheme with linear complexity to those of the other two schemes for different MSE values. In Fig. 5 and Fig. 6 the horizontal axis represents the MSE normalized by the square of the Euclidean norm of the vector to be estimated, i.e., \( \text{nMSE} = \frac{\text{MSE}}{\|X\|_2^2} \).

In the uncorrelated case, where \( r_{ij} = 0 \), \( \forall i, j, i \neq j \), the proposed scheme with linear complexity nearly attains the performance of the unsimplified scheme with quadratic complexity as seen in Fig. 5. This result is rather expected since in this case \( \tilde{U}_t \cong D_t \) for sufficiently large \( t \), where \( \tilde{U}_t \) and \( D_t \) are used to compute the stopping time and the estimator in the unsimplified and simplified schemes, respectively. Strikingly the decentralized schemes (simplified and unsimplified) achieve very close performances to that of the optimal centralized scheme, which is obviously unattainable in a decentralized system, thanks to the efficient information transmission through level-triggered sampling. It is seen in Fig. 6 that the proposed simplified scheme exhibits an average stopping time performance close to those of the unsimplified
scheme and the optimal centralized scheme even when the scaling coefficients \( \{h_{k,t,i}^k\}_i \) are correlated with \( r_{ij} = 0, \forall i, j \), justifying the simplification proposed in Section IV-A to obtain linear complexity.

Finally, in Fig. 7 we fix the normalized MSE value at \( 10^{-2} \) and plot average stopping time against the correlation coefficient \( r \) where \( r_{ij} = r, \forall i, j, i \neq j \). We observe an exponential growth in average stopping time of each scheme as \( r \) increases. The average stopping time of each scheme becomes infinite at \( r = 1 \) since in this case only some multiples of a certain linear combination of the parameters to be estimated, i.e., \( h_{k,t,1} \sum_{i=1}^n c_i x_i \), are observed under the noise \( w_{k,t}^k \) at each sensor \( k \) at each time \( t \), hence it is not possible to recover the individual parameters. Specifically, it can be shown that \( c_i = \sqrt{\frac{E[(h_{k,t,i}^k)^2]}{E[(h_{k,t,1}^k)^2]}} \), which is the same for all sensors as we assume identical sensors. To see the mechanism that causes the exponential growth consider the computation of \( \text{Tr}(\bar{U}_{i,t}^{-1}) \), which is used to determine the stopping time in the optimal centralized scheme. From (50) we write

\[
\text{Tr}(\bar{U}_{i,t}^{-1}) \approx \text{Tr}(D_t^{-1}R^{-1}) = \sum_{i=1}^n \frac{\kappa_i}{d_{t,i}}
\]

for sufficiently large \( t \), where \( d_{t,i} \) and \( \kappa_i \) are the \( i \)th diagonal elements of the matrices \( D_t \) and \( R^{-1} \), respectively. For instance, we have \( \kappa_i = 1, \forall i, \kappa_i = 8.0435, \forall i \) and \( \kappa_i = \infty \) when \( r = 0, r = 0.9 \).
Fig. 6. Average stopping time performances of the optimal centralized scheme and the decentralized schemes based on level-triggered sampling with quadratic and linear complexity vs. normalized MSE values when scaling coefficients are correlated with $r_{ij} = 0.5, \forall i, j$.

and $r = 1$, respectively. Assuming that the scaling coefficients have the same mean and variance when $r = 0$ and $r = 0.9$, we have similar $d_{t,i}$ values [cf. (31)] in (50), hence the stopping time of $r = 0.9$ is approximately 8 times that of $r = 0$ for the same accuracy level. Since $\text{MSE} = \mathbb{E}\left[\|\hat{X}_T - X\|_2^2\right] = \text{Tr}(\bar{U}_T^{-1})$ in the centralized scheme, using $\kappa_i$ for different $r$ values we can approximately know how the average stopping time changes as $r$ increases for a given MSE value. As shown in Fig. 7 with the label “Theory” this theoretical curve is in a good match with the numerical result. The small discrepancy at high $r$ values is due to the high sensitivity of the WLS estimator in (49) to numerical errors when the stopping time is large. The high sensitivity is due to multiplying the matrix $\bar{U}_T^{-1}$ with very small entries by the vector $\bar{V}_T$ with very large entries while computing the estimator $\hat{X}_T$ in (29) for a large $T$. The decentralized schemes suffer from a similar high sensitivity problem [cf. (38)] much more than the centralized scheme since making error is inherent in a decentralized system. Moreover, in the decentralized schemes the MSE is not given by the stopping time statistic $\text{Tr}(\bar{U}_T^{-1})$, hence “Theory” does not match well the curves for the decentralized schemes. Although it cannot be used to estimate the rates of the exponential growths of the decentralized schemes, it is still useful to explain the mechanism behind them as the decentralized schemes are derived from the centralized scheme.

To summarize, with identical sensors any estimator (centralized or decentralized) experiences an
exponential growth in its average stopping time as the correlation between scaling coefficients increases since in the extreme case of full correlation, i.e., $r = 1$, each sensor $k$, at each time $t$, observes a noisy sample of the linear combination $\sum_{i=1}^{n} x_{i} \sqrt{E\left[h_{k,t,i}^{2}\right]}$, and thus the stopping time is infinite. As a result of exponentially growing stopping time, the WLS estimator, which is also the ML estimator and the MVUE, i.e., the optimal estimator, in our case, and the decentralized estimators derived from it become highly sensitive to errors as $r$ increases. In either uncorrelated or mildly correlated cases, which are of practical importance, the proposed decentralized scheme with linear complexity performs very close to the optimal centralized scheme as shown in Fig. 5 and Fig. 6 respectively.

V. CONCLUSIONS

We have treated the problem of sequential vector parameter estimation under both centralized and decentralized settings. In the centralized setting two different formulations, which use unconditional and conditional covariances of the estimator respectively to assess the estimation accuracy, are considered and the corresponding sequential estimators are derived. The conditional formulation, having a simple stopping rule for any number of parameters, was shown to be preferable to the unconditional formulation, whose stopping rule can only be found by complicated numerical computations even for a small number of parameters. Moreover, following the optimal conditional sequential estimator we have developed a
computationally efficient decentralized sequential estimator based on level-triggered sampling. Simulation results have shown that the proposed scheme with linear complexity has a similar performance to that of the optimal centralized scheme.

APPENDIX: PROOF OF LEMMA

We will first prove that if $V(z)$ is non-decreasing, concave and bounded, then so is $G(z) = 1 + \mathbb{E} \left[ V \left( \frac{z}{1 + z \lambda z^2} \right) \right]$. That is, assume $V(z)$ satisfies: (a) $\frac{d}{dz} V(z) \geq 0$, (b) $\frac{d^2}{dz^2} V(z) < 0$, (c) $V(z) < c < \infty$, $\forall z$. Then by (c) we have

$$1 + V \left( \frac{z}{1 + z \lambda z^2} \right) < 1 + c, \forall z,$$

hence $G(z) < 1 + c$ is bounded. Moreover,

$$\frac{d}{dz} V \left( \frac{z}{1 + z \lambda z^2} \right) = \frac{d}{dz} \frac{V(z)}{(1 + z \lambda z^2)^2} > 0, \forall z$$

by (a), and thus $G(z)$ is non-decreasing. Furthermore,

$$\frac{d^2}{dz^2} G(z) = \mathbb{E} \left[ \frac{d^2}{dz^2} V \left( \frac{z}{1 + z \lambda z^2} \right) \right] = \mathbb{E} \left[ \frac{d^2}{dz^2} \frac{V(z)}{(1 + z \lambda z^2)^2} + \frac{d}{dz} \frac{V(z)}{(1 + z \lambda z^2)^2} (1 + z \lambda z^2)^3 / 2 \right], \forall z,$$

hence $G(z)$ is concave, concluding the first part of the proof.

Now, it is sufficient to show that $V(z)$ is non-decreasing, concave and bounded. Assume that the limit $\lim_{m \to \infty} V_m(z) = V(z)$ exists. We will prove the existence of the limit later. First, we will show that $V(z)$ is non-decreasing and concave by iterating the functions \{V_m(z)\}. Start with $V_0(z) = 0$. Then,

$$V_1(z) = \min \left\{ \lambda \sigma^2 z, 1 + \mathbb{E} \left[ V_0 \left( \frac{z}{1 + z \lambda z^2} \right) \right] \right\} = \min \{ \lambda \sigma^2 z, 1 \},$$

which is non-decreasing and concave as shown in Fig. Similarly we write

$$V_2(z) = \min \left\{ \lambda \sigma^2 z, 1 + \mathbb{E} \left[ V_1 \left( \frac{z}{1 + z \lambda z^2} \right) \right] \right\},$$

where $1 + \mathbb{E} \left[ V_1 \left( \frac{z}{1 + z \lambda z^2} \right) \right]$ is non-decreasing and concave since $V_1(z)$ is non-decreasing and concave. Hence, $V_2(z)$ is non-decreasing and concave since pointwise minimum of non-decreasing and concave functions is again non-decreasing and concave. We can show in the same way that $V_m(z)$ is non-decreasing and concave for $m > 2$, i.e., $V(z) = V_\infty(z)$ is non-decreasing and concave.

Next, we will show that $V(z)$ is bounded. Assume that

$$V(z) < \min \{ \lambda \sigma^2 z, c \} = \lambda \sigma^2 z 1_{\{ \lambda \sigma^2 z \leq c \}} + c 1_{\{ \lambda \sigma^2 z > c \}}.$$

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The function $V_1(z)$ is non-decreasing and concave.

Then, from the definition of $V(z)$ we have $1 + E \left[ V \left( \frac{z}{1 + z h_1^2} \right) \right] < c$. Since $V(z)$ is non-decreasing, $E \left[ V \left( \frac{z}{1 + z h_1^2} \right) \right] \leq E \left[ V \left( \frac{1}{h_1^2} \right) \right]$. From (56) we can write

$$1 + E \left[ V \left( \frac{z}{1 + z h_1^2} \right) \right] \leq 1 + E \left[ V \left( \frac{1}{h_1^2} \right) \right] < 1 + E \left[ \frac{\lambda \sigma^2}{h_1^2} \mathbb{1}_{\{\frac{\lambda \sigma^2}{h_1^2} \leq c\}} \right] + c \Pr \left( \frac{\lambda \sigma^2}{h_1^2} > c \right),$$

(57)

Recalling $1 + E \left[ V \left( \frac{z}{1 + z h_1^2} \right) \right] < c$ we want to find a $c$ such that

$$1 + E \left[ \frac{\lambda \sigma^2}{h_1^2} \mathbb{1}_{\{\frac{\lambda \sigma^2}{h_1^2} \leq c\}} \right] + c \Pr \left( \frac{\lambda \sigma^2}{h_1^2} > c \right) < c.$$

(58)

For such a $c$ we have

$$1 < c \Pr \left( \frac{\lambda \sigma^2}{h_1^2} \leq c \right) - E \left[ \frac{\lambda \sigma^2}{h_1^2} \mathbb{1}_{\{\frac{\lambda \sigma^2}{h_1^2} \leq c\}} \right]$$

$$= E \left[ \left( c - \frac{\lambda \sigma^2}{h_1^2} \right) \mathbb{1}_{\{\frac{\lambda \sigma^2}{h_1^2} \leq c\}} \right] = E \left[ \left( c - \frac{\lambda \sigma^2}{h_1^2} \right)^+ \right],$$

(59)

where $(\cdot)^+$ is the positive part operator. We need to show that there exists a $c$ satisfying $E \left[ \left( c - \frac{\lambda \sigma^2}{h_1^2} \right)^+ \right] > 1$. Note that we can write

$$E \left[ \left( c - \frac{\lambda \sigma^2}{h_1^2} \right)^+ \right] \geq E \left[ \left( c - \frac{\lambda \sigma^2}{h_1^2} \right)^+ \mathbb{1}_{\{h_1^2 > c\}} \right]$$

$$> E \left[ \left( c - \frac{\lambda \sigma^2}{\epsilon} \right)^+ \mathbb{1}_{\{h_1^2 > c\}} \right]$$

$$= \left( c - \frac{\lambda \sigma^2}{\epsilon} \right)^+ \Pr(h_1^2 > \epsilon),$$

(60)
where \( (c - \frac{\lambda \sigma^2}{\epsilon})^+ \to \infty \) as \( c \to \infty \) since \( \lambda \) and \( \epsilon \) are constants. If \( \mathbb{P}(h_1^2 > \epsilon) > 0 \), which is always true except the trivial case where \( h_1 = 0 \) deterministically, then the desired \( c \) exists.

Now, what remains is to justify our initial assumption \( \mathcal{V}(z) < \min\{\lambda \sigma^2 z, c\} \). We will use induction to show that the assumption holds with the \( c \) found above. From (54), we have \( \mathcal{V}_1(z) = \min\{\lambda \sigma^2 z, 1\} < \min\{\lambda \sigma^2 z, c\} \) since \( c > 1 \). Then, assume that

\[
\mathcal{V}_{m-1}(z) < \min\{\lambda \sigma^2 z, c\} = \lambda \sigma^2 z 1_{\{\lambda \sigma^2 z \leq c\}} + c 1_{\{\lambda \sigma^2 z > c\}}.
\]

(61)

We need to show that \( \mathcal{V}_m(z) < \min\{\lambda \sigma^2 z, c\} \), where \( \mathcal{V}_m(z) = \min\{\lambda \sigma^2 z, 1 + E[\mathcal{V}_{m-1}\left(\frac{z}{1+zh^2}\right)]\} \). Note that \( 1 + E[\mathcal{V}_{m-1}\left(\frac{z}{1+zh^2}\right)] < 1 + E[\mathcal{V}_{m-1}\left(\frac{z}{h_1^2}\right)] \) since \( \mathcal{V}_{m-1}(z) \) is non-decreasing. Similar to (57), from (61) we have

\[
1 + E[\mathcal{V}_{m-1}\left(\frac{1}{h_1^2}\right)] < 1 + E\left[\frac{\lambda \sigma^2}{h_1^2} 1_{\{\lambda \sigma^2 z \leq c\}}\right] + c \mathbb{P}\left(\frac{\lambda \sigma^2}{h_1^2} > c\right) < c,
\]

(62)

where the last inequality follows from (58). Hence,

\[
\mathcal{V}_m(z) < \min\{\lambda \sigma^2 z, c\}, \forall m,
\]

(63)

showing that \( \mathcal{V}(z) < \min\{\lambda \sigma^2 z, c\} \), which is the assumption in (56).

We showed that \( \mathcal{V}(z) \) is non-decreasing, concave and bounded if it exists, i.e., the limit \( \lim_{m \to \infty} \mathcal{V}_m(z) \) exists. Note that we showed in (63) that the sequence \( \{\mathcal{V}_m\} \) is bounded. If we also show that \( \{\mathcal{V}_m\} \) is monotonic, e.g., non-decreasing, then \( \{\mathcal{V}_m\} \) converges to a finite limit \( \mathcal{V}(z) \). We will again use induction to show the monotonicity for \( \{\mathcal{V}_m\} \). From (54) we write \( \mathcal{V}_1(z) = \min\{\lambda \sigma^2 z, 1\} \geq \mathcal{V}_0(z) = 0 \). Assuming \( \mathcal{V}_{m-1}(z) \geq \mathcal{V}_{m-2}(z) \) we need to show that \( \mathcal{V}_m(z) \geq \mathcal{V}_{m-1}(z) \). Using their definitions we write \( \mathcal{V}_m(z) = \min\{\lambda \sigma^2 z, 1 + E[\mathcal{V}_{m-1}\left(\frac{z}{1+zh^2}\right)]\} \) and \( \mathcal{V}_{m-1}(z) = \min\{\lambda \sigma^2 z, 1 + E[\mathcal{V}_{m-2}\left(\frac{z}{1+zh^2}\right)]\} \). We have

\[
1 + E[\mathcal{V}_{m-1}\left(\frac{z}{1+zh^2}\right)] \geq 1 + E[\mathcal{V}_{m-2}\left(\frac{z}{1+zh^2}\right)]
\]

due to the assumption \( \mathcal{V}_{m-1}(z) \geq \mathcal{V}_{m-2}(z) \), hence \( \mathcal{V}_m(z) \geq \mathcal{V}_{m-1}(z) \).

To conclude, we proved that \( \mathcal{V}_m(z) \) is non-decreasing and bounded in \( m \), thus the limit \( \mathcal{V}(z) \) exists, which was also shown to be non-decreasing, concave and bounded. Hence, \( G(z) \) is non-decreasing, concave and bounded.

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