Nonlinear visco-elastic-plastic model of impact

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Abstract. A nonlinear model of the impact of a visco-elastic-plastic body against a fixed obstacle is investigated. This model is reduced to a nonlinear second-order differential equation. The solution of the body motion equation using the special Lambert function is obtained. Within the framework of the model, an elastic and absolutely inelastic impact are possible. In the case of an elastic impact, the coefficient of restitution decreases when the collision velocity is increasing.

1. Introduction
The paper considers the most simple case of a body collision against a fixed surface (obstacle) when the body motion is translational along the same axis as before and so on after the collision. The shape of the body and the obstacle can be different, but the impact forces of their interaction are reduced to the resultant force directed along this axis. The line of action of the resultant force is directed passes through the body center of mass. It is assumed that the impact forces of interaction are significantly greater than other forces and the action of the latter can be neglected\textsuperscript{[1-9]}. The coefficient of restitution during the impact is the ratio of the modules of the body velocities after and before the collision\textsuperscript{[1-9]}

\[
k = \frac{V^+}{V^-} = \frac{V^+}{V^-}.
\]

Kinetic energy lost during the impact

\[
\Delta T = \frac{m(V^-)^2}{2} - \frac{m(V^+)^2}{2} = \frac{m(V^-)^2}{2} - (1 - k^2)
\]

within a constant value of the coefficient of restitution is proportional to the square of the collision velocity $V^-$. The most strong model of the impact is associated with the investigation of the motion dynamics of a visco-elastic-plastic deformable bodies\textsuperscript{[1-3]} that is requires a large amount of numerical calculations. Newton's impact model\textsuperscript{[1-3]} is based on the assumption that the impact time is infinitely short and the movement of the body during the impact process can be neglected. Newton made the assumption that the coefficient of restitution is determined by the material from which the bodies are made, and does not depend on the collision velocity. He broke the impact process into two phases: the approach (or compression) phase and the restitution phase. Poisson introduced a different definition of the coefficient of restitution, as the ratio of the pulses of impact interaction force in the phases of...
restitution and compression. In the problem under consideration, these two definitions are equivalent. With an oblique impact of the body against a fixed obstacle, these definitions are not equivalent and Poisson’s definition should be used [1–4]. Newton’s model of impact does not allow one to determine many important parameters of the impact: its duration, the maximum value of the force of interaction of bodies, their deformation, etc.

Kelvin-Voigt linear viscoelastic model of impact [1-3] has become widespread. In this model it is assumed that the contact force of colliding bodies interaction reduces to a linear elastic force and a linear dissipative force. The body motion equation during the impact is a linear differential equation with constant coefficients that has an analytical solution. The coefficient of restitution in this case is constant. This model contradicts to the natural physical concepts. The force of bodies interaction at the beginning and the end of collision is equal to dissipative force and is non-zero. If in the collision process the contact patch changes, then it seems unnatural to assume that elastic and dissipative interaction forces are linear.

Experimental data [1, 5] refute the assumption of a constant coefficient of restitution and show that with an increase in the velocity of a body collision, the restitution coefficient monotonically decreases.

In the wave theory of impact [1-3], bodies are elastic, and there is no residual deformation of bodies. The loss of energy due to an impact is caused by the elastic acoustic waves of a deformation. The speed of propagation of these waves is equal to the speed of sound and depends on the properties of the material. In engineering practice, wave theory is used to calculate the impact of rods against an obstacle.

In the investigation of this model of impact, Hunt and Crossley [7] carried out a numerical integration of the nonlinear equation of motion of the body during the impact. This model is a development of the Hertz model in the case when the body and the obstacle obey the laws of viscoelastic deformation. The model was built under the assumption that wave processes can be neglected, deformations upon impact are small, the magnitude of the residual deformation can be neglected. This model is valid for compact bodies made of a sufficiently rigid material, with relatively small (up to several meters per second) impact speeds. The disadvantages of this impact model are not the possibility of an absolutely inelastic impact and the
tendency of the recovery coefficient to unity when the impact velocity tends to zero, regardless of the material from which the bodies are made.

In [8], the first integral and the solution of the equation of body motion in quadratures were obtained for the Hunt-Crossley impact model. The dependence of the coefficient of restitution on the collision velocity is analytically constructed.

In [9], a nonlinear elastoplastic model of a collinear impact of a body against a fixed obstacle was considered, in which it is assumed that the friction between particles of the bodies deformed during the impact is dry friction. In this model of impact, absolutely inelastic impact is possible. The coefficient of restitution in the case of elastic impact does not depend on the velocity of the collision. The latter result does not agree with the experimental data.

In this paper, a visco-elastic-plastic impact model is investigated, which is a generalization of the Hertz and Hunt-Crossley impact models.

2. Nonlinear visco-elastic-plastic impact model

Consider a model of impact similar to the models of Hertz and Hunt-Crossley [3, 6, 7–9], but suppose that between particles of a bodies deformed upon impact there is both viscous and dry friction. Following the works [7–9], let us assume that, as a result of an increase in the contact patch, the forces of viscous and dry friction increase, in proportion to the elastic force. The contact force of the interaction of the body and the obstacle is determined by the ratio

\[ F = F(x, \dot{x}) = -f(x) - bf(x)\dot{x} - df(x) \text{sgn} \dot{x}, \]

where \( x \) – the movement of the body in the process of impact (deformation); \( f(x) \) – the elastic force of interaction of bodies at impact; \( b \) – viscous friction constant, \( d \) – dry friction constant.

During the impact deformation \( x \geq 0 \). At the beginning and at the end of the impact \( x = 0 \). The elastic force of interaction of colliding bodies is zero at the beginning and end of the impact \( f(0) = 0 \), and it is an increasing function of the deformation \( x \).

In the Hertz and Hunt-Crossley models of impact, it is assumed that the elastic force of interaction of colliding bodies is equal \( f(x) = cx^n \), where \( c \) – the coefficient of elasticity, \( n \) – the constant that is determined by the shape of the body surface and the obstacles in the vicinity of the point of contact.

We denote by \( V = \dot{x} \) the body velocity. The equation of body motion in the compression phase (when \( V = \dot{x} > 0 \)) has the form

\[ m\ddot{x} = F(x, \dot{x}) = -f(x)(1+bV+d) \quad \text{,} \]

where \( m \) – the body mass.

At the end of the compression phase, the body velocity \( V = \dot{x} = 0 \). If the dry friction constant \( d \geq 1 \), then at the end of the compression phase the body stops. Contact force of interaction is zero. Impact is absolutely inelastic. Coefficient of restitution is zero.

If \( d < 1 \), then the impact is elastic and in the restitution phase (when \( V = \dot{x} < 0 \)) the equation of motion is

\[ m\ddot{x} = F(x, \dot{x}) = -f(x)(1+bV-d) \quad \text{.} \]

It is easy to see that in the case of elastic impact, \( V = -(1-d)b^{-1} = \text{const} < 0 \) is the solution of the differential equation of motion (4). This solution on the phase plane corresponds to a phase trajectory, which is a straight line and divides the phase plane into two half-planes. By virtue of the principle of determinism, phase trajectories do not intersect and lie completely in one of these half-planes. At the initial moment of time \( t = 0 \) the velocity of the collision \( V^- > 0 > -(1-d)b^{-1} \). Then at any moment of time \( t \) the inequality \( V > -(1-d)b^{-1} \) holds true.

Phase trajectories of the system upon impact, i.e. at \( x \geq 0 \) and \( V^- > 0 \), are given in fig. 1. The dashed line corresponds to the phase trajectory \( V = -(1-d)b^{-1} = \text{const} \), unrealizable upon impact.
We denote by $\Pi(x)$ – the potential energy of elastic deformation $\Pi(x) = \int_{0}^{x} f(x) \, dx$. In particular

$$\Pi(x) = \frac{cx^{n+1}}{n+1} \quad \text{when} \quad f(x) = cx^n.$$

We exclude time $t$ from the differential equation of motion in the compression phase (3) using a transformation

$$\dot{x} = \frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt} = V \frac{dV}{dx}.$$

Separating the variables in the resulting equation and integrating it from the initial position to the final one, we obtain the first integral of the equations of motion

$$\left(1 + d\right) \ln \left(1 + \frac{v}{1 + d}\right) - v = \left(1 + d\right) \ln \left(1 + \frac{v^*}{1 + d}\right) - v^* + \frac{b^2 \Pi(x)}{m},$$

where

$$v = bV,$$

(5)

(6)

dimensionless velocity.

Equation (5) can be solved for relatively dimensionless velocity $v$. To do this, convert it to form

$$\ln \phi_{1} - \phi_{1} = \eta_{1},$$

where

$$\phi_{1} = 1 + \frac{v}{1 + d}, \quad \eta_{1}(x) = \ln \left(1 + \frac{v}{1 + d}\right) - 1 - \frac{v^*}{1 + d} + \frac{b^2 \Pi(x)}{m}.$$  

(7)

(8)

The transcendental equation (7) is solved using the special W Lambert function, which gained fame after its introduction into the computer algebra system MAPLE [10]. Previously, this function was used by a number of authors, starting with Euler, in solving various mathematical problems. This function is differentiable, integrable. Effective procedures for calculating its values on a computer are built. With the help of the Lambert function, an analytical solution of a number of problems in mathematics and mechanics was obtained [10].

Figure 1. Phase trajectories.
Figure 2. W Lambert function. The solid line shows the main real branch \( W_0(x) \), the dotted line shows the second real branch \( W_{-1}(x) \).

The Lambert function \( W(x) \) is given implicitly as a solution to the equation \( We^W = x \). Over the field of complex numbers, this equation has a countable set of solutions, which correspond to the branches of the Lambert function \( W_k(x) \), where \( k = 0; \pm 1; \pm 2; \ldots \). Above the field of real numbers, the Lambert function has two branches, which are shown in Fig. 2. The solid line shows the main real branch \( W_0(x) \), the dotted line– the second real branch \( W_{-1}(x) \). The Lambert function allows us to obtain a solution to some transcendental equations, including equations (7), in which it is required to determine \( \varphi(x) \) that

\[
\ln \varphi - \varphi = x \Leftrightarrow \ln (\varphi e^{-\varphi}) = x \Leftrightarrow -\varphi e^{-\varphi} = -e^x \Leftrightarrow \varphi = -W(-e^x)
\]

Taking into account that in the compression phase \( v > 0 \), we obtain

\[
v = v(x) = -(1 + d)\left[1 + W_{-1}\left(-\exp(\eta_1(x))\right)\right],
\]

where \( W_{-1} \) is the second real branch of the Lambert function.

We denote by \( x_{\max} \) – the maximum displacement of the body during impact or the value of \( x \) at the end of the compression phase. Value of \( x_{\max} \) is defined as a solution to the equation \( V(x_{\max}) = 0 \iff W_{-1}\left(-\exp(\eta_1(x_{\max}))\right) = -1 \iff \eta_1(x_{\max}) = -1 \). As a result, by (8) \( x_{\max} \) is a solution to the equation

\[
(1 + d)\ln\left(1 + \frac{v^-}{1 + d}\right) - v^- + b^2 \frac{\Pi(x_{\max})}{m} = 0.
\]

When \( f(x) = cx^n \) potential energy \( \Pi(x) = \frac{cx^{n+1}}{n + 1} \), and the solution of equation (10) is

\[
x_{\max} = \left[\frac{(n + 1)m}{cb^2} \left(v^- - (1 + d)\ln\left(1 + \frac{v^-}{1 + d}\right)\right)\right]^\frac{1}{n+1}.
\]
Relations (6), (8), (9) allow to obtain a solution of the motion equation in the compression phase (3) in quadratures, as a solution to an equation with separable variables \( \dot{x} = V(x) \)

\[
\int_{0}^{x} \frac{dx}{1 + W_{-1}[\exp(\eta(x))] - \exp(\eta(x))} = -\frac{1 + d}{b} t.
\]  

(11)

Similarly, in the restitution phase, the equation of motion (4) has the first integral

\[
(1 - d)\ln\left(1 + \frac{v}{1 - d}\right) - v = b^2 \frac{\Pi(x) - \Pi(x_{\text{max}})}{m},
\]

and then, taking into account (10), we obtain

\[
(1 - d)\ln\left(1 + \frac{v}{1 - d}\right) - v = (1 + d)\ln\left(1 + \frac{v^{-}}{1 + d}\right) - v^{-} + b^2 \frac{\Pi(x)}{m} .
\]

(12)

This equation can be solved for relatively dimensionless velocity \( v \). To do this, convert it to form

\[
\ln \varphi_{2} - \varphi_{2} = \eta_{2},
\]

(13)

where

\[
\varphi_{2} = 1 + \frac{v}{1 - d}, \quad \eta_{2} = \eta_{2}(x) = 1 + d \ln\left(1 + \frac{v^{-}}{1 + d}\right) - 1 - \frac{v^{-}}{1 + d} + \frac{b^2}{1 - d} \frac{\Pi(x)}{m}.
\]

(14)

The transcendental equation (14) is solved with the help of the W Lambert function. Taking into account that in the restitution phase \(-1 < -(1 - d) < v < 0\), we obtain

\[
v = v(x) = -(1 - d)\left[1 + W_{0}\left(\exp\left(\eta_{2}(x)\right)\right)\right],
\]

where \( W_{0} \) is the main real branch of the Lambert function.

Equations (6), (8), (11), (15) allows to obtain a solution of the motion equation in the restitution phase (4) in quadratures, as a solution to an equation with separable variables \( \dot{x} = V(x) \)

\[
1 - d \int_{0}^{x_{\text{max}}} \frac{dx}{1 + W_{-1}[\exp(\eta(x))] - \exp(\eta(x))} + \int_{x_{\text{max}}}^{x} \frac{dx}{1 + W_{0}[\exp(\eta(x))] - \exp(\eta(x))} = \frac{1 - d}{b} t ,
\]

(16)

where is the solution of equation (10).

Equations (11), (16) implicitly determine the law of body motion during impact.

3. Coefficient of restitution and lost kinetic energy

From the first integral of the motion equation in the restitution phase (12), due to the fact that at the beginning and end of the impact \( x = 0 \), we obtain that the initial and final dimensionless velocity at impact are related by

\[
(1 - d)\ln\left(1 + \frac{v^{+}}{1 - d}\right) - v^{+} = (1 + d)\ln\left(1 + \frac{v^{-}}{1 + d}\right) - v^{-} ,
\]

(17)

where \( v^{-} \in (0, +\infty) \), \( v^{+} \in (-1, 0) \).

By virtue of (1), (6) \( v^{+} = -kv^{-} \), where \( k \) – the coefficient of restitution. Substituting this relation into (17), we obtain that the coefficient of restitution is a solution of the transcendental equation

\[
v^{-}(1 + k) + (1 - d)\ln\left(1 - \frac{kv^{-}}{1 - d}\right) - (1 + d)\ln\left(1 + \frac{v^{-}}{1 + d}\right) = 0 .
\]

(18)

From (14), (15), (18) follows
Graphs of the dependence of the coefficient of restitution on the dimensionless $v^-$ velocity before impact at various values of the dry friction constant $d = 0.1; 0.2; 0.4; 0.6$ and $0.8$ are given in fig. 3. Note that the coefficient of restitution does not depend on the type of function $f(x)$, i.e., on the type of dependence of the elastic component of the contact force of interaction. At the same time, $f(x)$ significantly affects on the other parameters of the impact: the maximum deformation, the maximum value of the contact force of interaction of the colliding bodies, the duration of the impact. The coefficient of restitution depends only on the dry friction constant $d$ and the dimensionless velocity of collision $v^-=bV^-$, where $b$ – the viscous friction constant, $V^-$ – the velocity of collision.

\begin{equation}
(19)
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Dependence of the coefficient of restitution and the relative magnitude of the lost kinetic energy on the dimensionless velocity of collision at the values of the dry friction constant $d = 0; 0.2; 0.4; 0.6; 0.8$.}
\end{figure}

Let's investigate this dependence for small values. Using the Maclaurin series expansions of the function $\ln(1+\alpha) = \alpha - \frac{1}{2}\alpha^2 + \frac{1}{3}\alpha^3 - \ldots$, in equation (18) leave only the terms up to the third order of smallness in $v^-$. Then

\begin{equation}
(20)
\end{equation}

\begin{equation}
(21)
\end{equation}

From (20) it follows that for small $v^-$ the coefficient of restitution is equal

\begin{equation}
(21)
\end{equation}
where \( \delta \) is a small positive value. Substitute (21) into (20), then, up to the first-order terms of smallness, obtain

\[
k = \frac{1 - d}{1 + d} - \frac{v^-}{3(1 + d)} \left[ 1 + \sqrt{1 + d} \right].
\]

The kinetic energy lost on impact \( \Delta T \) is determined by the relation (2). Denote by \( T^- \) the kinetic energy of the body before the impact, then the relative value of the lost kinetic energy is equal to

\[
\frac{\Delta T}{T^-} = 1 - k^2.
\]

Its dependence on the dimensionless velocity before impact \( v^- \) at different values of the dry friction constant is shown in fig. 3.

4. Results of mathematical simulation

If the surface of the body and the obstacles in the vicinity of the point of contact are spherical, then the force of elastic deformation in accordance with the Hertz results [1–3, 6] is

\[
f(x) = cx^2^\frac{1}{2},
\]

where

\[
\frac{1}{c} = \frac{3}{4} \left( \frac{1 - \mu_2^2}{E_1} + \frac{1 - \mu_1^2}{E_2} \right) \sqrt{\frac{1}{R_1} + \frac{1}{R_2}},
\]

\( E_{1,2} \) – modules of elasticity, \( \mu_{1,2} \) – Poisson’s ratios, \( R_{1,2} \) – radii of the body surface and obstacles.

As an example, consider the impact of a steel ball with a mass \( m = 0.1 \text{ kg} \) on a massive steel plate. Set for steel \( \rho = 7800 \text{ kg/m}^3 \), \( E = 2 \times 10^{11} \text{ N/m}^2 \), \( \mu = 0.25 \) then the radius of the ball \( R = 1.425 \text{ sm} \), the coefficient of elasticity \( c = 1.7135 \times 10^{10} \text{ N/m}^{3/2} \). In the calculations, we assume that the viscous friction constant \( b = 0.075 \text{ s/m} \), the dry friction constant \( d = 0.077 \). These values were obtained by the least squares method, as a result of processing the digitized data from the experimental dependence [1] of the coefficient of restitution on the impact velocity for two balls of hardened steel.

![Figure 4](image-url)  
**Figure 4.** Dependence on the time of deformation and contact force at \( V^- = 1; 2; 3; 4 \) and \( 5 \text{ m/s} \).
5. Conclusion
On the basis of the Hertz and Hunt-Crossley impact models, a nonlinear visco-elastic-plastic model of the body’s collinear impact against a fixed obstacle was constructed. It is assumed that there are both viscous and dry friction between particles of bodies deformed during the impact. The first integrals of the equations of motion in the phases of deformation and restitution are obtained. The solution to the equation of body motion in the process of impact in quadratures is obtained. In the framework of the constructed model of impact, absolute inelastic and elastic impact are possible. With elastic impact,
the coefficient of restitution decreases with increasing collision velocity. When the collision velocity tends to zero, the coefficient of restitution tends to a limit value that is less than or equal to one and its value decreases with increasing dry friction constant.

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