ON ONE PROPERTY OF TIKHONOV REGULARIZATION ALGORITHM

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Abstract. For linear inverse problem with Gaussian random noise we show that Tikhonov regularization algorithm is minimax in the class of linear estimators and is asymptotically minimax in the sense of sharp asymptotic in the class of all estimators. The results are valid if some a priori information on Fourier coefficients of solution is provided. For trigonometric basis this a priori information implies that the solution belongs to a ball in Besov space $B^{s}_{2,\infty}$.

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1. Introduction

Tikhonov regularization algorithm (TRA) is very popular [16, 13] thanks to many remarkable properties. We mention only two of them. TRA is minimax for deterministic noise [12] and is Bayes estimator [17, 3] in the problems with Gaussian random noise and Gaussian a priori measure. In paper we explore the minimax properties of TRA in linear inverse problems with Gaussian random noise. We show that TRA is minimax in the class of linear estimator and asymptotically minimax in the class of all estimators. In these setups a priori information is provided that Fourier coefficients of solution satisfies the same restrictions as Fourier coefficients of functions in a ball in Besov space $B^{s}_{2,\infty}$ for the case of trigonometric basis.

Such a form of a priori information is rather natural.

This is rather reasonable information on a solution smoothness.
For the most nonparametric estimators these sets are the largest sets with a given rates of convergence [15].

For linear statistical estimators these sets are the largest sets with a given rate of convergence [13].

The asymptotic minimaxity of TRA is proved in the sense of sharp asymptotic. The asymptotically minimax nonparametric estimators in the sense of sharp asymptotic has been obtained earlier, only if a priori information is provided that a solution belongs to ellipsoid in $L_2$, in particular, a ball in Sobolev space $W^{1,2}[14,15,13,17]$. There are numerous research on sharp adaptive minimax estimation [3, 7, 2, 18, 11]. The results on adaptive estimation in Pinsker model [4, 18] are easily carried over on paper setup.

In what follows we shall denote letters $c, C$ positive constants and $a_r \asymp b_r$ implies $c < a_r/b_r < C$.

2. Main Results

Let $H$ be separable Hilbert space and let $A : H \to H$ be known self-adjoint linear bounded operator.

We wish to estimate a solution of linear equation

$$f = Ax, \quad x \in H,$$

on observation $Y = f + \epsilon \xi$ where $\xi$ is Gaussian random error and $\epsilon > 0$ defines the level of noise.

Let $\{a_i\}_{i=1}^\infty$ and $\{\phi_i\}_{i=1}^\infty$ be eigenvalues and eigenvectors of operator $A$ respectively. Then we can rewrite the vector $Y = \{y_i\}_{i=1}^\infty$ in the following form

$$y_i = a_i x_i + \epsilon \sigma_i \xi_i$$

where $x_i = \langle x, \phi_i \rangle, y_i = \langle Y, \phi_i \rangle$ and $\sigma_i = \sigma_i^{-1} = \langle \xi, \phi_i \rangle, 1 \leq i < \infty$. $\text{Var}[\xi_i] = 1$. We suppose that $\sigma_i, 1 \leq i < \infty$ are i.i.d.r.v.'s and $E[\xi_i] = 0$. The representation (2.1) holds in particular if $\xi$ is Gaussian white noise.

Here $a_r \asymp b_r$ denote inner product of vectors $a, b \in H$. For any $a \in H$ denote $||a|| = \langle a, a \rangle^{1/2}$.

Suppose a priori information is provided that

$$x \in B_{2r}^r = \left\{ \theta = \{\theta_i\}_{i=1}^\infty : \sup_k k^{2r} \sum_i \theta_i^2 \leq P_0 \right\} \quad (2.2)$$

with $r > 0$.

We say that linear estimator $\hat{x}_\epsilon = \{\hat{x}_{ij}\}_{i=1}^\infty$ is minimax linear estimator if

$$\sup_{x \in B_{2r}^r} E_x ||\hat{x}_\epsilon - x||^2 = \inf_{\lambda \in B_{2r}^r} \sup_{x \in B_{2r}^r} E_x ||\hat{x}_\epsilon \lambda - x||^2. \quad (2.3)$$

where $\lambda = \{\lambda_j\}_{j=1}^\infty, \hat{x}_{\epsilon \lambda} = \{\hat{x}_{ij} \lambda_j\}_{i=1}^\infty, \hat{x}_{ij} \lambda_j = \lambda_j y_j, \lambda_j \in R^1, 1 \leq j < \infty$.

We say that the estimator $\hat{x}_\epsilon$ is asymptotically minimax if

$$\sup_{x \in B_{2r}^r} E_x ||\hat{x}_\epsilon - x||^2 = \inf_{\hat{x}_\epsilon \in \Psi} \sup_{x \in B_{2r}^r} E_x ||\hat{x}_\epsilon - x||^2 (1 + o(1)) \quad (2.4)$$

as $\epsilon \to 0$. Here $\Psi$ is the set of all estimators.

The minimaxity of TRA in the class of linear estimators will be proved if the following assumption holds.

A. For all $j > 1$

$$\frac{\sigma_j^2 \sigma_{j-1}^2 ((j-1)^{-2r} - j^{-2r})}{\sigma_{j-1}^2 \sigma_j^2 (j^{-2r} - (j+1)^{-2r})} > 1. \quad (2.5)$$
Theorem 2.1. Assume A. Then TRA is minimax on the set of all linear estimators with
\[ \lambda_j = a_j^{-1} \frac{a_j^2 P_0(j^{-2r} - (j + 1)^{-2r})}{a_j^2 P_0(j^{-2r} - (j + 1)^{-2r}) + \epsilon^2 \sigma_j^2}. \] (2.6)
The asymptotically minimax risk equals
\[ R_t = \epsilon^2 (1 + o(1)) \sum_{j=1}^{\infty} \frac{\sigma_j^2 P_0(j^{-2r} - (j - 1)^{-2r})}{a_j^2 P_0(j^{-2r} - (j - 1)^{-2r}) + \epsilon^2 \sigma_j^2}. \] (2.7)
The asymptotic minimaxity of TRA will be proved if the following assumptions hold.
B1. For \( j > j_0 \), there holds \( |a_j/a_{j+1}| \leq 1 \).
B2. There holds \( 0 < c < \sigma_j^2 < C < \infty \).
B3. For all \( j > j_0 \)
\[ \frac{\sigma_{j+1}^2 a_j^2}{\sigma_j^2 a_{j+1}^2} j^{2r+1} > 1. \] (2.8)

Theorem 2.2. Assume B1-B3. Then TRA is asymptotically minimax on the set of all estimators with
\[ \lambda_j = a_j^{-1} \frac{a_j^2}{a_j^2 + (2r P_0)^{-1} \epsilon^2 \sigma_j^2 j^{2r+1}}. \] (2.9)
The asymptotically minimax risk equals
\[ R_t = \epsilon^2 (1 + o(1)) \sum_{j=1}^{\infty} \frac{\sigma_j^2}{a_j^2 + (2r P_0)^{-1} \epsilon^2 \sigma_j^2 j^{2r+1}}. \] (2.10)

Example 2.1. Let \(|a_j| = C j^{-\gamma} (1 + o(1)) \) and \( \sigma_j = 1 \). Then
\[ R_t = \epsilon^{2r+2} \sigma_j^2 P_0^{2r+2} (2r P_0) \pi^{2r+1} C^{-2r+2} (1 + o(1)). \] (2.11)

Example 2.2. Let \(|a_j| = C j^{-\alpha} \exp\{-B j^\gamma\} \) and \( \sigma_j = 1 \). Then
\[ R_t = |\log \epsilon|^{-2r+2} P_0 B^{2r+2} (1 + o(1)). \] (2.12)

3. Proof of Theorem 2.1
We begin with the proof of lower bound. Denote \( \theta_j^2 = P_0(j^{-2r} - (j + 1)^{-2r}), \theta = \{\theta_j\}_{j=1}^{\infty} \).
We have
\[ \inf_{\lambda} \sup_{x \in B_{2r}^{\infty}} E_x ||\hat{x}_\lambda - x||^2 \geq \inf_{\lambda} E_0 ||\hat{\theta}_\lambda - \theta||^2 = \epsilon^2 \sum_{j=1}^{\infty} \frac{\theta_j^2 \sigma_j^2}{\theta_j^2 a_j^2 + \epsilon^2 \sigma_j^2}. \] (3.1)
and infimum is attained for
\[ \lambda_j = \frac{a_j^{-1}}{\theta_j^2 a_j^2 + \epsilon^2 \sigma_j^2}. \]
Proof of upper bound is based on the following reasoning. Let \( x = \{x_j\}_{j=1}^{\infty} \in B_{2r}^{\infty} \).
For all \( k \) denote
\[ u_k = k^{2r} \sum_{j=k}^{\infty} x_j^2. \]
Then \( x_k^2 = k^{-2r} u_k - (k + 1)^{-2r} u_{k+1} \).
For the sequence of $\lambda_j$ defined in Theorem 2.1, we have

$$
E_x \sum_{j=1}^{\infty} (\lambda_j y_j - x_j)^2 = c^2 \sum_{j=1}^{\infty} \lambda_j^2 \sigma_j^2 a_j^{-2} + \sum_{j=1}^{\infty} (1 - a_j \lambda_j)^2 x_j^2
$$

$$
= c^2 \sum_{j=1}^{\infty} \lambda_j^2 \sigma_j^2 a_j^{-2} + \sum_{j=1}^{\infty} \left( \frac{1}{\theta_j^2 \sigma_j^2 a_j^{-2} + 1} \right)^2 (j^{-2r} u_j - (j + 1)^{-2r} u_{j+1})
$$

$$
= c^2 \sum_{j=1}^{\infty} \lambda_j^2 \sigma_j^2 a_j^{-2} + \left( \frac{1}{\theta_j^2 \sigma_j^2 a_j^{-2} + 1} \right)^2 u_1
$$

$$
- \sum_{j=2}^{\infty} u_j j^{-2r} \left( (\theta_j^{-1} \sigma_j^{-2} a_j^{-1} + 1)^{-2} - (\theta_j^2 \sigma_j^{-2} a_j^{-2} + 1)^{-2} \right).
$$

By A, the last addendums in the right hand-side of (3.2) are negative. Therefore the supremum of right hand-side of (3.2) is attained for $u_j = P_0$, $1 \leq j < \infty$. This completes the proof of Theorem 2.1.

4. Proof of Theorem 2.2

The upper bound follows from Theorem 2.1. Below the proof of lower bound will be provided. This proof has a lot of common feachers with the proof of lower bound in Pinsker Theorem [7, 14, 18].

Fix values $\delta, 0 < \delta < 1$, and $\delta_1, 0 < \delta_1 < P_0$. Define a family of natural numbers $k_\epsilon, \epsilon > 0$, such that $a_j^2 \epsilon^{-2} \sigma_j^2 P_0 (2r)^{-1} = \lambda_j^{-1} k_\epsilon^{-2r-1} = 1 + o(1)$ as $\epsilon \to 0$. Define sequence $\eta = \{\eta_j\}_{j=1}^{\infty}$ of Gaussian i.i.d.r.v.'s $\eta_j = \eta_{j\delta_1}, E[\eta_j] = 0, \text{Var}[\eta_j] = (P_0 - \delta_1)(2r)^{-1} j^{2r-1}$, if $\delta k_\epsilon \leq j \leq \delta^{-1} k_\epsilon$, and $\eta_j = 0$ for $j < \delta k_\epsilon$ and $j > \delta^{-1} k_\epsilon$.

Denote $\mu$ the probability measure of random vector $\eta$. Define $\tilde{x}$ Bayes estimator with a prior probability measure $\nu_5$.

Define the conditional probability measure $\nu_5$ of random vector $\eta$ given $\eta \in B_2^{\infty}(P_0)$. Define $\tilde{x}$ Bayes estimator $x$ with a priori measure $\nu_5$. Denote $\theta$ the random variable having probability measure $\nu_5$.

For any estimator $\tilde{x}$ we have

$$
\sup_{x \in B_2^{\infty}} E_x ||\tilde{x} - x||^2 \geq E_{\nu_5} E_{\theta} ||\tilde{x} - \theta||^2
$$

$$
\geq E_{\mu} E_{\eta} ||\tilde{x} - \eta||^2 - E_{\mu} E_{\eta}(||\tilde{x} - \eta||^2, \eta \notin B_2^{\infty}) P_\mu^{-1}(\eta \in B_2^{\infty}).
$$

We have

$$
E_{\mu} E_{\eta} ||\tilde{x} - \eta||^2 = c^2(1+o(1)) \sum_{j=1}^{l_2} \frac{\sigma_j^2}{a_j^2 + (2r(P_0 - \delta_1))^{-1} \epsilon^2 \sigma_j^2 j^{2r+1}} \geq I(P_0 - \delta_1)
$$

with $l_1 = [\delta k_\epsilon]$ and $l_2 = \lfloor \delta^{-1} k_\epsilon \rfloor$. Here $[a]$ denotes whole part of a number $a \in R^1$.

Since

$$
||\tilde{x}||^2 \leq \sup_{x \in B_2^{\infty}} ||x||^2 \leq P_0,
$$

we have

$$
E_{\mu} E_{\eta}(||\tilde{x} - \eta||^2, \eta \notin B_2^{\infty}) \leq 2E_{\mu} E_{\eta}(||\tilde{x}||^2 + ||\eta||^2, \eta \notin B_2^{\infty})
$$

$$
\leq 2P_0 P_\mu(\eta \notin B_2^{\infty}) + \sum_{j=1}^{l_2} (E_{\mu} \eta_j^4)^{1/2} P_\mu^{1/2}(\eta \notin B_2^{\infty}).
$$
Since \( E_{\mu}[\eta_j^4] \leq C j^{-2r-2} \), we have
\[
\sum_{j=1}^{l_2} (E_{\mu} \eta_j^4)^{1/2} \leq C \delta^{-r} k_c^{-2r}.
\]
(4.4)

It remains to estimate
\[
P_{\mu}(\eta \notin B_{2\infty}^r) = P(\max_{l_1 \leq i \leq l_2} \epsilon^2 \sum_{j=1}^{l_2} \eta_j^2 - P_0(1 - \delta_1/2) > P_0 \delta_1/2) \leq \sum_{j=1}^{l_2} J_i
\]
with
\[
J_i = P \left( \epsilon^2 \sum_{j=1}^{l_2} \eta_j^2 - P_0(1 - \delta_1/2) > P_0 \delta_1/2 \right)
\]
(4.5)

To complete the proof it remains to estimate \( R_c - I(P_0 - \delta_1) \).

By straightforward estimation, it is easy to verify that
\[
|I(P_0) - I(P_0 - \delta_1)| < C \delta_1 I(P_0)
\]
(4.10)

We have
\[
e^2 \sum_{j=1}^{l_1} \sigma_j^2 a_j^2 + (2r P_0)^{-1} e^2 \sigma_j^2 j^{2r+1} = e^2 \sum_{j=1}^{l_1} \sigma_j^2 a_j^{-2}
\]
(4.11)

We have
\[
e^2 (1 + o(1)) \sum_{j=1}^{l_2} \sigma_j^2 a_j^2 + (2r P_0)^{-1} e^2 \sigma_j^2 j^{2r+1} = \sum_{j=1}^{l_2} j^{-2r-1}
\]
(4.12)
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