Triangles Formed via Poisson Nearest Neighbors

STEVEN FINCH

December 21, 2017

ABSTRACT. We start with certain joint densities (for sides and for angles) corresponding to pinned Poissonian triangles in the plane, then discuss analogous results for staked and anchored triangles.

A planar triangle is pinned Poissonian if one vertex $A$ is fixed at the origin and the other vertices $B$, $C$ are the two nearest neighboring particles to $(0,0)$ of a unit intensity Poisson process. Hence $\|B\|^2$ and $\|C\|^2 - \|B\|^2$ are independent Exponential($\pi$) variables, where $\|\cdot\|$ denotes Euclidean norm. Let $a$, $b$, $c$ denote the sides opposite the random vertices; clearly $c < b$ and $a < 2b$ due to the triangle inequality. A Jacobian determinant calculation given in Appendix 1 yields

\[
\begin{cases}
8\pi \frac{xyz}{\sqrt{(x+y+z)(-x+y+z)(x-y+z)(x+y-z)}} \exp\left(-\pi y^2\right) \\
0 & \text{if } y-z < x < y+z, \ y > z, \ z > 0
\end{cases}
\]

as the trivariate density $f(x, y, z)$ for $a$, $b$, $c$. As a consequence, the univariate density for $c$ is

\[2\pi x \exp\left(-\pi x^2\right), \quad x > 0,\]

the univariate density for $b$ is

\[2\pi^2 x^3 \exp\left(-\pi x^2\right), \quad x > 0\]

and the univariate density for $a$ is

\[\pi x \text{erfc}\left(\sqrt{\pi} x/2\right), \quad x > 0.\]

Densities for $c$ and $b$ were known earlier [1, 2, 3], but that for $a$ seems to be new. Table 1 contains moments for these variables. For example, the cross-correlation coefficient

\[
\rho(a, b) = \frac{\text{Cov}(a, b)}{\sqrt{\text{Var}(a) \text{Var}(b)}} = \frac{8}{3} \sqrt{\frac{32 - 9\pi}{(-64 + 27\pi)\pi}} \approx 0.636
\]

Copyright © 2017 by Steven R. Finch. All rights reserved.
indicates strong positive dependency. An exact expression for $E(ac) = 0.49181215...$ remains open.

Table 1 Pinned Moments of Angles, Sides and some Products and Ratios

| Variable | Mean       | Mean Square |
|----------|------------|-------------|
| $\alpha$ | $\pi/2$    | $\pi^2/3$   |
| $\beta$  | $\pi/4 + 1/\pi$ | $1 + \pi^2/12$ |
| $\gamma$ | $\pi/4 - 1/\pi$ | $-1/2 + \pi^2/12$ |
| $\alpha\beta$ | $1/4 + \pi^2/12$ | - |
| $\beta\gamma$ | $-1/4 + \pi^2/12$ | - |
| $\gamma\alpha$ | $-1/4 + \pi^2/12$ | - |
| $a$      | $8/(3\pi)$ | $3/\pi$     |
| $b$      | $3/4$      | $2/\pi$     |
| $c$      | $1/2$      | $1/\pi$     |
| $ab$     | $64/(9\pi^2)$ | - |
| $bc$     | $4/(3\pi)$ | - |
| $ca$     | $0.49181215...$ | - |
| $a/b$    | $32/(9\pi)$ | $3/2$     |
| $b/a$    | $4/\pi$    | $\infty$    |
| $b/c$    | $2$        | $\infty$    |
| $c/b$    | $2/3$      | $1/2$       |
| $c/a$    | $(1 + 2G)/\pi$ | $\infty$ |
| $a/c$    | $(5 + 2G)/\pi$ | $\infty$ |
| area     | $4/(3\pi^2)$ | $3/(8\pi^2)$ |

From this, another calculation in Appendix 1 gives

$$
\begin{align*}
\begin{cases}
\frac{2 \sin(x) \sin(x + y)}{\pi} & \text{if } 0 < x < \pi \text{ and } \frac{\pi - x}{2} < y < \pi - x, \\
0 & \text{otherwise}
\end{cases}
\end{align*}
$$

as the bivariate density for angles $\alpha$, $\beta$ opposite sides $a$, $b$. Let $\gamma = \pi - \alpha - \beta$ and let $\varphi$ denote the probability that a pinned Poissonian triangle is obtuse. While $\alpha$ is Uniform$[0, \pi]$, the univariate density for $\beta$ is

$$
\begin{align*}
\begin{cases}
\frac{1}{2\pi} + \frac{1 - 3 \cos(x)^2}{2\pi \sin(x)^2} + \frac{x \cos(x)}{\pi \sin(x)} & \text{if } 0 < x < \pi/2, \\
\frac{1}{\pi \sin(x)^2} + \frac{(\pi - x) \cos(x)}{\pi \sin(x)} & \text{if } \pi/2 < x < \pi
\end{cases}
\end{align*}
$$
and the univariate density for $\gamma$ is
\[
\begin{cases}
(4/\pi) \cos(x)^2 & \text{if } 0 < x < \pi/2, \\
0 & \text{if } \pi/2 < x < \pi.
\end{cases}
\]

These imply that
\[
\varphi = P(\alpha > \pi/2) + P(\beta > \pi/2) + P(\gamma > \pi/2)
= 1/2 + 1/4 + 0 = 3/4
\]
because a triangle can have at most one obtuse angle. See [4, 5] for alternative approaches for computing $\varphi$.

The ratio of a pair of sides is of interest [6]. We find that the univariate density for $a/b$ is
\[
\frac{2x}{\pi} \arccos \left( \frac{x}{2} \right), \quad 0 < x < 2,
\]
the univariate density for $b/a$ is
\[
\frac{1}{x^3} - \frac{2}{\pi x^3} \arcsin \left( \frac{1}{2x} \right), \quad x > \frac{1}{2},
\]
the univariate density for $b/c$ is
\[
\frac{2}{x^3}, \quad x > 1,
\]
the univariate density for $c/b$ is
\[
2x, \quad 0 < x < 1,
\]
the univariate density for $c/a$ is
\[
\begin{cases}
\frac{2x}{(1 - x^2)^3} & \text{if } 0 < x < 1/2, \\
\frac{-2(-1 + x^2) \sqrt{-1 + 4x^2} - \pi x^2 (1 + x^2) + 6x^2 (1 + x^2) \arcsin \left( 1/(2x) \right)}{\pi x (-1 + x^2)^3} & \text{if } x > 1/2
\end{cases}
\]
and the univariate density for $a/c$ is
\[
\begin{cases}
\frac{-2x (1 - x^2) \sqrt{4 - x^2} - \pi (1 + x^2) + 6 (1 + x^2) \arcsin(x/2)}{\pi (1 - x^2)^3} & \text{if } 0 < x < 2, \\
\frac{2x - 1 + x^2}{(-1 + x^2)^3} & \text{if } x > 2.
\end{cases}
\]
It is remarkable that Catalan’s constant \[ G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \]
appears in expressions for both \( \text{E}(c/a) \) and \( \text{E}(a/c) \), as well as in other geometric probability settings \[8, 9\].

A planar triangle is \textbf{staked Poissonian} if one vertex \( A \) is fixed at \((0, 0)\), another vertex \( B \) is fixed at \((1, 0)\) and the third vertex \( C \) is the nearest neighboring particle to \((0, 0)\) of a unit intensity Poisson process. The term \textit{stake} (as in “staking a tent”) was only recently introduced in this context \[10\]. Let \( C = (u, v) \). Clearly

\[
\tan(\alpha) = \frac{v}{1-u}, \quad \tan(\beta) = \frac{v}{u}.
\]

The Jacobian determinant of the transformation \((u, v) \mapsto (\alpha, \beta)\) is

\[
|J| = \begin{vmatrix}
\frac{v}{1-u^2+v^2} & \frac{1-u}{1-u^2+v^2} \\
\frac{-v}{u^2+v^2} & \frac{u}{u^2+v^2}
\end{vmatrix} = \frac{v}{(u^2+v^2)[(1-u)^2+v^2]}.
\]

Solving for \( u, v \) in terms of \( \alpha, \beta \), we obtain

\[
u = \frac{\tan(\alpha)}{\tan(\alpha) + \tan(\beta)}, \quad v = \frac{\tan(\alpha) \tan(\beta)}{\tan(\alpha) + \tan(\beta)}.
\]

Substituting these expressions into a nonstandard bivariate normal density

\[ \pi \exp \left\{ -\pi \left\{ u^2 + v^2 \right\} \right\} \]

and dividing by \( |J| \) yields

\[ \exp \left[ -\pi \frac{\sin(\alpha)^2}{\sin(\alpha + \beta)^2} \right] \frac{\sin(\alpha) \sin(\beta)}{\sin(\alpha + \beta)^3}. \]

Multiplying by 2 gives the correct normalization. While \( \alpha \) is Uniform\([0, \pi]\), the univariate density for \( \beta \) is decidedly not so. The acuteness probability can be found exactly:

\[
1 - \phi = 2 \int_{0}^{\pi} \int_{0}^{\pi} \exp \left[ -\pi \frac{\sin(\alpha)^2}{\sin(\alpha + \beta)^2} \right] \frac{\sin(\alpha) \sin(\beta)}{\sin(\alpha + \beta)^3} d\beta d\alpha
\]

\[
= \frac{1}{2} \left( e^{-\pi/2} I_0 \left( \frac{\pi}{2} \right) - \text{erfc} \left( \sqrt{\pi} \right) \right) = 0.1725524698...
\]
where $I_0$ is the $0^{th}$ modified Bessel function of the first kind. Densities for sides $a$, $b$ are possible but omitted. Such triangles seem to be mentioned in [6, 11] without further elaboration.

Table 2 Staked Moments of Angles and of a Product

| Variable | Mean   | Mean Square   |
|----------|--------|--------------|
| $\alpha$ | $\pi/2$| $\pi^2/3$    |
| $\beta$  | $0.34306160...$ | $0.20825399...$ |
| $\alpha\beta$ | $0.43825535...$ | - |

A planar triangle is **anchored Poissonian** if one vertex $A$ is fixed at $(-1/2, 0)$, another vertex $B$ is fixed at $(1/2, 0)$ and the third vertex $C$ is the nearest neighboring particle to $(0, 0)$ of a unit intensity Poisson process. The term *anchoring* (as in “anchoring a ship”) again was only recently introduced [10]. Let $C = (u,v)$. Clearly

$$\tan(\alpha) = \frac{v}{\frac{1}{2} - u} = \frac{2v}{1 - 2u}, \quad \tan(\beta) = \frac{v}{\frac{1}{2} + u} = \frac{2v}{1 + 2u}$$

and the corresponding Jacobian determinant is

$$|J| = \begin{vmatrix} 4v & 2(1 - 2u) \\ (1 - 2u)^2 + 4v^2 & (1 - 2u)^2 + 4v^2 \end{vmatrix} = \frac{16v}{[(1 - 2u)^2 + 4v^2][(1 + 2u)^2 + 4v^2]}.$$

Solving for $u$, $v$ in terms of $\alpha$, $\beta$, we obtain

$$u = \frac{1}{2} \frac{\tan(\alpha) - \tan(\beta)}{\tan(\alpha) + \tan(\beta)}, \quad v = \frac{\tan(\alpha) \tan(\beta)}{\tan(\alpha) + \tan(\beta)}.$$

Substituting these expressions into the nonstandard bivariate normal density and dividing by $|J|$ yields

$$\exp \left[ -\frac{\pi}{4} \frac{\sin(\alpha - \beta)^2 + 4 \sin(\alpha)^2 \sin(\beta)^2}{\sin(\alpha + \beta)^2} \right] \frac{\sin(\alpha) \sin(\beta)}{\sin(\alpha + \beta)^3}.$$

Multiplying by $2$ gives the correct normalization. The univariate densities for $\alpha$ and $\beta$ are identical but decidedly not uniform. The acuteness probability can be found exactly:

$$1 - \phi = 2 \int_0^{\pi/2} \int_0^{\pi/2} \exp \left[ -\frac{\pi}{4} \frac{\sin(\alpha - \beta)^2 + 4 \sin(\alpha)^2 \sin(\beta)^2}{\sin(\alpha + \beta)^2} \right] \frac{\sin(\alpha) \sin(\beta)}{\sin(\alpha + \beta)^3} d\beta \, d\alpha$$

$$= e^{-\pi/4} - \text{erfc} \left( \sqrt{\pi}/2 \right) = 0.2458467223...$$
This value is slightly larger for anchored triangles than for staked triangles. Densities for sides $a$, $b$ are possible but again omitted.

Table 3 Anchored Moments of Angles and of a Product

| Variable | Mean         | Mean Square |
|----------|--------------|-------------|
| $\alpha$ | 0.71706372... | 0.92490176... |
| $\beta$  | 0.71706372... | 0.92490176... |
| $\alpha\beta$ | 0.39837926... | -            |

Appendix 2 discusses an evidently unrelated class of random triangles $T$. If $\varphi$, $\psi$ are independent Uniform$[0, \pi]$ variables, then

$$(\alpha, \beta) = \begin{cases} 
(\varphi, \psi) & \text{if } \varphi + \psi < \pi, \\
(\pi - \psi, \pi - \varphi) & \text{if } \varphi + \psi > \pi
\end{cases}$$

are angles of $T$ at vertices $A = (0, 0)$, $B = (1, 0)$. Note that if $\varphi + \psi > \pi$, then

$$(\pi - \psi) + (\pi - \varphi) = \pi + (\pi - \varphi - \psi) < \pi;$$

it follows that $\alpha + \beta < \pi$ always. Upon constructing a line $L_A$ emanating from $A$ with slope $\tan(\alpha)$ and a line $L_B$ emanating from $B$ with slope $-\tan(\beta)$, the remaining vertex

$$C = \left( \frac{\tan(\beta)}{\tan(\alpha) + \tan(\beta)}, \frac{\tan(\alpha) \tan(\beta)}{\tan(\alpha) + \tan(\beta)} \right)$$

is the point $L_A \cap L_B$ of intersection. Such “uniform triangles” have appeared before in the literature [12, 13, 14], although perhaps not with the same specificity as [6].

1. Acknowledgement

I am grateful to Daryl Daley for suggesting the study of pinned Poissonian triangles in $\mathbb{R}^2$. Analogous work for Poisson processes in $\mathbb{R}^d$ for $d > 2$ awaits an interested reader! Thanks are also due to Adrian Baddeley and Rolf Turner for writing an R package spatstat [15], which enables testing of numerical predictions in this essay via simulation [16].

2. Appendix 1

Revisiting the beginning, let $R_1 = \|B\|^2$ and $R_2 = \|C\|^2 - \|B\|^2$. Define $\theta_1$ to be the angle between vector $B$ and the horizontal axis; define $\theta_2$ likewise for vector $C$. The joint density for $(R_1, R_2, \theta_1, \theta_2)$ is

$$\left( \frac{1}{2\pi} \right)^2 \left( \pi e^{-\pi R_1} \right) \left( \pi e^{-\pi R_2} \right) = \frac{1}{4} e^{-\pi (R_1 + R_2)}$$
where \( R_i > 0, \ 0 < \theta_i < 2\pi \) for \( i = 1, 2 \). We rewrite this density in terms of sides \( b, c \). From \( R_1 = c^2, \ R_2 = b^2 - c^2 \) emerges a Jacobian matrix

\[
\begin{pmatrix}
0 & 2c \\
2b & -2c
\end{pmatrix}
\]

with absolute determinant \( 4bc \). Thus the joint density for \( (b, c, \theta_1, \theta_2) \) is

\[ bc e^{-\pi b^2} \]

where \( 0 < c < b \). As in [17], we integrate out \( \theta_1 \) by letting \( \omega = \theta_1 - \theta_2 \) and \( \alpha = |\omega| \), then adding contributions at \( \alpha \) and \( 2\pi - \alpha \). Omitting details, the joint density for \( (\alpha, b, c) \) comes out as

\[ 4\pi b c e^{-\pi b^2} \]

where \( 0 < \alpha < \pi \). We now bring \( a \) into the density, removing \( \alpha \). Differentiating the Law of Cosines

\[ a^2 = b^2 - 2bc \cos(\alpha) + c^2 \]

with respect to \( \alpha \), it is clear that

\[ 2a \, da = 2bc \sin(\alpha) \, d\alpha \]

\[ = \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \, d\alpha \]

by a formula for area, and hence the density becomes

\[ 4\pi b c e^{-\pi b^2} \, d\alpha \, db \, dc \]

\[ = 4\pi b c e^{-\pi b^2} \frac{2a}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}} \, da \, db \, dc \]

as was to be shown.

Let \( \Delta = (a+b+c)(-a+b+c)(a-b+c)(a+b-c) \). The natural transformation \( (\alpha, \beta, c) \mapsto (a, b, c) \) appearing in [17] has Jacobian determinant \( ab \). Using the identities

\[ \frac{a}{c} = \frac{\sin(\alpha)}{\sin(\alpha + \beta)}, \quad \frac{b}{c} = \frac{\sin(\beta)}{\sin(\alpha + \beta)}, \quad \frac{\sqrt{\Delta}}{2c^2} = \frac{\sin(\alpha) \sin(\beta)}{\sin(\alpha + \beta)} \]

we have

\[ \sin(\alpha + \beta) = \frac{c \sqrt{\Delta}}{ab 2c^2} = \frac{\sqrt{\Delta}}{2ab} \]
thus the pinned angle density can be rewritten as

\[ 8\pi \frac{a^2 b^2 c}{\sqrt{\Delta}} \exp \left[ -\pi b^2 \right] \]
\[ = 8\pi c^5 \frac{\sin(\alpha)^2 \sin(\beta)^2}{\sin(\alpha + \beta)^4 \sqrt{\Delta}} \exp \left[ -\pi c^2 \frac{\sin(\beta)^2}{\sin(\alpha + \beta)^2} \right] \]
\[ = 4\pi c^3 \frac{\sin(\alpha) \sin(\beta)}{\sin(\alpha + \beta)^3} \exp \left[ -\pi c^2 \frac{\sin(\beta)^2}{\sin(\alpha + \beta)^2} \right]. \]

Integrating out \( c \) is facilitated by observing that

\[ \int_0^\infty c^3 \exp \left( -\pi c^2 r \right) dc = \frac{1}{2\pi^2 r^2} \]

for \( r > 0 \), therefore the density for \((\alpha, \beta)\) is

\[ \frac{2 \sin(\alpha) \sin(\beta)}{\pi} \frac{\left( \sin(\alpha + \beta)^2 \right)^2}{\sin(\beta)^2} = \frac{2 \sin(\alpha) \sin(\alpha + \beta)}{\pi} \frac{\sin(\beta)^2}{\sin(\beta)^2}. \]

The restriction \( \beta > (\pi - \alpha)/2 \) is implied by \( b > c \), equivalently, \( \sin(\beta) > \sin(\alpha + \beta) \).

We note that the joint density for \((a, b)\) is

\[ 4\pi a b \exp \left( -\pi b^2 \right) \arccos \left( \frac{a}{2b} \right) \]

for \( 0 < a < 2b \) and the joint density for \((b, c)\) is

\[ 4\pi^2 b c \exp \left( -\pi b^2 \right) \]

for \( 0 < c < b \). No closed-form expression of the joint density \( f \) for \((a, c)\) is known. If \( 0 < a < 2c \), then \( a - c < c \) and

\[ f(a, c) = 8\pi a c \int_c^{a+c} \frac{b \exp \left( -\pi b^2 \right)}{\sqrt{[(a + c)^2 - b^2] [b^2 - (a - c)^2]}} db; \]

if \( 0 < 2c < a \), then \( c < a - c \) and

\[ f(a, c) = 8\pi a c \int_{a-c}^{a+c} \frac{b \exp \left( -\pi b^2 \right)}{\sqrt{[(a + c)^2 - b^2] [b^2 - (a - c)^2]}} db. \]
The density $g$ for the ratio $z = a/c$ is

$$g(z) = \int_0^\infty c \, f(z \, c, c) \, dc$$

$$= \begin{cases} \frac{8\pi}{\sqrt{\pi \pi}} \int_0^\infty c^3 \int_0^{(z+1)c} b \exp(-\pi b^2) \frac{db \, dc}{\sqrt{[(z+1)^2 c^2 - b^2] \, [b^2 - (z-1)^2 c^2]}} & \text{if } 0 < z < 2, \\ \frac{8\pi}{\sqrt{\pi \pi}} \int_0^\infty c^3 \int_0^{(z-1)c} b \exp(-\pi b^2) \frac{db \, dc}{\sqrt{[(z+1)^2 c^2 - b^2] \, [b^2 - (z-1)^2 c^2]}} & \text{if } z > 2 \end{cases}$$

and the density $h$ for the ratio $w = c/a$ is

$$h(w) = \int_0^\infty a \, f(a, w \, a) \, da$$

$$= \begin{cases} \frac{8\pi}{\sqrt{\pi \pi}} \int_0^\infty a^3 \int_0^{(w+1)a} b \exp(-\pi b^2) \frac{db \, da}{\sqrt{[(w+1)^2 a^2 - b^2] \, [b^2 - (w-1)^2 a^2]}} & \text{if } w > 1/2, \\ \frac{8\pi}{\sqrt{\pi \pi}} \int_0^\infty a^3 \int_0^{(1-w)a} b \exp(-\pi b^2) \frac{db \, da}{\sqrt{[(w+1)^2 a^2 - b^2] \, [b^2 - (w-1)^2 a^2]}} & \text{if } 0 < w < 1/2 \end{cases}$$

Both $g(z)$ and $h(w)$ are readily evaluated. The circumstances are less advantageous
in the following:

\[ E(a, c) = \int_0^\infty \int_0^{2c} a c f(a, c) \, da \, dc + \int_0^\infty \int_0^\infty a c f(a, c) \, dc \, da \]

for which the first integral becomes

\[
\int_0^\infty \int_0^{2c} 8\pi a^2 c^2 \int_c^{a+c} \frac{b \exp(-\pi b^2)}{\sqrt{[(a+c)^2 - b^2][b^2 - (a-c)^2]}} \, db \, da \, dc
\]

\[
= 8\pi \int_0^{\infty} \int_0^{3c} \exp(-\pi b^2) \int_{c-b}^{c} \frac{a^2}{\sqrt{[(a+c)^2 - b^2][b^2 - (a-c)^2]}} \, da \, db \, dc
\]

\[
= 8\pi \int_0^{\infty} b \exp(-\pi b^2) \int_0^{c} \int_{\frac{c-b}{2}}^{\frac{c+b}{2}} \frac{a^2}{\sqrt{[(a+c)^2 - b^2][b^2 - (a-c)^2]}} \, da \, dc \, db
\]

and the second integral becomes

\[
\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 8\pi a^2 c^2 \int_{\frac{a-c}{2}}^{\frac{a+c}{2}} \frac{b \exp(-\pi b^2)}{\sqrt{[(a+c)^2 - b^2][b^2 - (a-c)^2]}} \, db \, dc \, da
\]

\[
= 8\pi \int_0^{\infty} a^2 \int_0^{\frac{\pi}{2}} \exp(-\pi b^2) \int_{\frac{a-c}{2}}^{\frac{a+c}{2}} \frac{c^2}{\sqrt{[(a+c)^2 - b^2][b^2 - (a-c)^2]}} \, dc \, db \, da
\]

\[
= 8\pi \int_0^{\infty} b \exp(-\pi b^2) \int_0^{\frac{2b}{2}} \int_{\frac{a}{2}}^{\frac{2b}{2}} \frac{c^2}{\sqrt{[(a+c)^2 - b^2][b^2 - (a-c)^2]}} \, dc \, da \, db.
\]

Both triple integrals can be reduced to double integrals, each involving an incomplete elliptic integral of the third kind, but further simplification does not seem to be feasible.

3. **Appendix 2**

Starting from the joint density for angles in \( T \):

\[
\left\{ \begin{array}{ll}
\frac{2}{\pi^2} & \text{if } 0 < \alpha < \pi, 0 < \beta < \pi \text{ and } \alpha + \beta < \pi, \\
0 & \text{otherwise}
\end{array} \right.
\]
we find the joint density for sides

$$k(a, b) = \begin{cases} 
\frac{2}{\pi^2 a b} & \text{if } |1 - a| < b < 1 + a \text{ and } a > 0, \\
0 & \text{otherwise}
\end{cases}$$

which is true because $c = 1$ and since the natural transformation $(\alpha, \beta) \mapsto (a, b)$ has Jacobian determinant $a b$. Integrating out $b$, the marginal density for $a$ is

$$k_a(a) = \frac{2}{\pi^2} \frac{\ln(1 + a) - \ln|1 - a|}{a}, \quad a > 0.$$ 

Note the singularity at $a = 1$. The density for $z = a/b$ is

$$\int_0^\infty b k(z b, b) db = \int_{\frac{1}{1 + z}}^{\frac{1}{1 - z}} \frac{2}{\pi^2 z} db = \frac{2}{\pi^2} \frac{\ln(1 + z) - \ln|1 - z|}{z}$$

for $z > 0$, which interestingly is the same as that for $a, b$ and $b/a$ as well! Similar general formulas [18] are applicable to $x = \max\{a, b\}$:

$$\int_0^x k(x, b) db + \int_0^x k(a, x) da = \frac{4}{\pi^2} \frac{\ln(x) - \ln|1 - x|}{x}$$

for $x > 1/2$ and $y = \min\{a, b\}$:

$$k_a(y) + k_b(y) - \int_0^y k(y, b) db + \int_0^y k(a, y) da = \begin{cases} 
\frac{4}{\pi^2} \frac{\ln(1 + y) - \ln(1 - y)}{y} & \text{if } 0 < y < 1/2, \\
\frac{4}{\pi^2} \frac{\ln(1 + y) - \ln(y)}{y} & \text{if } y > 1/2.
\end{cases}$$

Our proofs are simpler than those in [6]. We have not examined, however, the complicated density for the area of $T$.

4. Appendix 3
Here, for completeness’ sake, are R simulation output results (histograms in blue) graphed against density expressions found in this paper (curves in red).

References
[1] D. Stoyan and H. Stoyan, Fractals, Random Shapes and Point Fields: Methods of Geometrical Statistics, Wiley, 1994, pp. 212–215; MR1297125 (95h:60016).

[2] F. Haken, Quantum Signatures of Chaos, 2nd ed., Springer-Verlag, 2004, pp. 345–346, 388–389; MR2242927 (2008h:81055).
[3] J. Sakhr and J. M. Nieminen, Wigner surmises and the two-dimensional homogeneous Poisson point process, *Phys. Rev. E* 73 (2006) 047202.

[4] B. Eisenberg and R. Sullivan, Random triangles in $n$ dimensions, *Amer. Math. Monthly* 103 (1996) 308–318; MR1383668 (96m:60025).

[5] S. R. Finch, Random Gaussian tetrahedra, [http://arxiv.org/abs/1005.1033](http://arxiv.org/abs/1005.1033).

[6] D. J. Daley, S. Ebert and R. J. Swift, Size distributions in random triangles, *J. Appl. Probab.* 51A (2014) 283–295; MR3317364.

[7] S. R. Finch, Catalan’s constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 53–59; MR2003519 (2004i:00001).

[8] S. R. Finch, Correlation between angle and side, [http://arxiv.org/abs/1012.0781](http://arxiv.org/abs/1012.0781).

[9] S. R. Finch, Rank-3 projections of a 4-cube, [http://arxiv.org/abs/1204.3468](http://arxiv.org/abs/1204.3468).

[10] S. R. Finch, Pins, stakes, anchors and Gaussian triangles, [http://arxiv.org/abs/1410.6742](http://arxiv.org/abs/1410.6742).

[11] D. J. Daley, S. Ebert and G. Last, Two lilypond systems of finite line-segments, *Probab. Math. Statist.* 36 (2016) 221–246; [http://arxiv.org/abs/1406.0096](http://arxiv.org/abs/1406.0096); MR3593022.

[12] D. Griffiths, Uniform distributions and random triangles, *Math. Gazette*. 67 (1983) 38–42.

[13] T. Moore, Random triangle problem (long summary), [http://mathforum.org/kb/plaintext.jspa?messageID=86196](http://mathforum.org/kb/plaintext.jspa?messageID=86196).

[14] A. Edelman and G. Strang, Random triangle theory with geometry and applications, *Found. Comput. Math.* 15 (2015) 681–713; [http://arxiv.org/abs/1501.03053](http://arxiv.org/abs/1501.03053); MR3348170.

[15] A. Baddeley, E. Rubak and R. Turner, *Spatial Point Patterns: Methodology and Applications with R*, Chapman and Hall/CRC Press, 2015; [http://spatstat.org/](http://spatstat.org/).

[16] S. R. Finch, Simulations in R involving triangles and tetrahedra, unpublished software code (2017).

[17] S. R. Finch, Random triangles, unpublished note (2010).

[18] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, 1965, pp. 187–206; MR0176501 (31 #773).
Figure 1: Pinned densities for Poissonian triangle sides and angles.
Figure 2: Pinned densities for Poissonian triangle side ratios.
Figure 3: Staked densities for Poissonian triangle sides and angles.
Figure 4: Anchored densities for Poissonian triangle sides and angles.