CONTINUUM KAC–MOODY ALGEBRAS

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ABSTRACT. We introduce a new class of infinite–dimensional Lie algebras, which we refer to as continuum Kac–Moody algebras. Their construction is closely related to that of usual Kac–Moody algebras, but they feature a continuum root system with no simple roots. Their Cartan datum encodes the topology of a one–dimensional real space and can be thought of as a generalization of a quiver, where vertices are replaced by connected intervals. For these Lie algebras, we prove an analogue of the Gabber–Kac–Serre theorem, providing a complete set of defining relations featuring only quadratic Serre relations. Moreover, we provide an alternative realization as continuum colimits of symmetric Borcherds–Kac–Moody algebras with at most isotropic simple roots. The approach we follow deeply relies on the more general notion of a semigroup Lie algebra and its structural properties.

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1. INTRODUCTION

In the present paper, we introduce a new class of infinite–dimensional Lie algebras, which we refer to as continuum Kac–Moody algebras, associated to a topological generalization of the notion of a quiver, where vertices are replaced by intervals in a real one–dimensional topological space. These Lie algebras do not fall into the realm of Kac–Moody algebras (nor of their several generalizations due to Borcherds [Bor88], Saveliev–Vershik [SV90], and Bozec [Boz16]). Rather, they encode the algebraic structure of certain colimits of symmetric Borcherds–Kac–Moody algebras, corresponding, roughly, to families of quivers with a number of vertices tending to infinity.
The simplest non-trivial examples of continuum Kac–Moody algebras are the Lie algebras of the line and the circle, constructed in [SS19a] together with their quantizations, the quantum groups of the line and the circle. The latter has various geometric realizations via the theory of Hall algebras. Originally, it arises from the Hall algebra of coherent sheaves on the infinite root stack of a pointed curve. In addition, T. Kuwagaki provided in loc. cit. a mirror symmetry type construction of the same quantum group from the (derived infinitesimally wrapped) Fukaya category of the cotangent bundle of the circle. In analogy with these constructions, we expect that continuum Kac–Moody algebras admit various geometric realizations, as classical limits of Hall algebras associated to coherent sheaves on weighted projective lines, coherent persistence modules 1, and Fukaya categories of cotangent bundles.

In the remaining part of the introduction, we provide some motivations and the description of the continuum Kac–Moody algebras. These algebras are introduced as examples of a more general class of Lie algebras, the semigroup Lie algebras, which are naturally associated to (partial) semigroups. Finally, we prove that the Goldman Lie algebra of the torus can be thought of as a remarkable example of such Lie algebras.

**The continuum Kac–Moody algebras of the line and the circle.** In [SS19a], the second and third–named authors introduced the so-called circle quantum group $U_c(\mathfrak{sl}(Q/Z))$ and its classical limit $\mathfrak{sl}(Q/Z)$. Their defining relations are somewhat cumbersome. Roughly, they are generated by infinitely–many $\mathfrak{sl}(2)$–triples indexed by pairs of points (or equivalently intervals 2) in the rational circle $Q/Z$ and commutation relations depending upon their mutual position in $Q/Z$ (cf. Definition 2.7). The origin of such relations resides in the colimit realization, induced by the underlying Hall algebra structure. More precisely, $\mathfrak{sl}(Q/Z)$ can be thought of as a direct limit of standard affine Lie algebras $\mathfrak{sl}(n), n \geq 2$, with system of morphisms given by iterated applications on simple root vectors of the elementary map $\mathfrak{sl}(2) \to \mathfrak{sl}(3)$ identified by the highest root in $\mathfrak{sl}(3)$. This construction endows $\mathfrak{sl}(Q/Z)$ with several rather exotic and interesting features. For example, its root system has no simple roots, since every simple root vector in $\mathfrak{sl}(n)$ can be identified with a commutator in $\mathfrak{sl}(n+1)$, and it is presented by quadratic, apparently non–homogeneous, Serre relations. Several straightforward generalization are at hand. For instance, one may replace $Q/Z$ with $Q$, thus obtaining a Lie algebra isomorphic to a colimit of $\mathfrak{sl}(n), n \geq 2$. Then, one may replace $Q$ and $Q/Z$ with the real line $\mathbb{R}$ and the circle $S^1 := \mathbb{R}/\mathbb{Z}$, respectively, so to obtain the continuum analogues $\mathfrak{sl}(\mathbb{R})$ and $\mathfrak{sl}(S^1)$. In Section 2, we provide a much shorter and concise presentation of these Lie algebras (cf. Corollary 2.10), which allows to think of them as the simplest examples of continuum Kac–Moody algebras.

**Semigroup Lie algebras.** The original defining relations of the Lie algebras $\mathfrak{sl}(\mathbb{R})$ and $\mathfrak{sl}(S^1)$ show some similarities, but also some striking differences with the usual relations appearing in the theory of Kac–Moody algebras. In particular, although the role of vertices in the Dynkin diagram is seemingly played by open–closed intervals, the commutation rules between the elements $e_j$ and $f_j$ above do not quite match the usual Kac–Moody relations. From a purely combinatorial point of view, these new commutation rules depend upon two simple operations on the set of intervals: the concatenation of two adjacent intervals and the truncation of an interval into a smaller one. These operations amount to a partial semigroup structure on the set of intervals.

Motivated by this observation, we develop in Section 3 a general theory of Lie algebras $\mathfrak{g}(S)$ associated to a triple $S = (S,k,\xi_{\pm})$ where $S$ is a partial semigroup and $k,\xi_{\pm} : S \times S \to k$ are functions satisfying some natural conditions. The construction of $\mathfrak{g}(S)$ is close in spirit to that of Kac–Moody algebras. More precisely, we define $\mathfrak{g}(S)$ as the Lie algebra generated by elements

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1See [SS19b] for the notion of coherent persistence modules when $X$ is the line or the circle, and the references therein about the general theory of persistence modules.

2We call interval the image in $Q/Z$ of an open–closed interval $(a,b) \subset Q$ with $b − a \leq 1$. 

$x_a^\pm$ and $\zeta_a$, with $a \in S$, subject to the conditions

$\zeta_{a \oplus \beta} = \zeta_a + \zeta_\beta$

whenever $a \oplus \beta$ is defined, and

$[\zeta_a, \zeta_\beta] = 0$,

$[\zeta_a, x_\beta^\pm] = \pm \kappa(\alpha, \beta)x_\beta^\pm$,

$[x_a^+, x_\beta^-] = \delta_{a, \beta} \zeta_a + \zeta_\alpha(\alpha, \beta)x_{a \ominus \alpha}^+ - \zeta_\alpha(\beta, \alpha)x_{\beta \ominus \alpha}^-$.

Then, we set $\mathfrak{g}(S) := \tilde{\mathfrak{g}}(S)/\tau$, where $\tau$ is the sum of all two–sided graded ideals in $\tilde{\mathfrak{g}}(S)$ having trivial intersection with the Cartan subalgebra generated by the $\zeta_a$’s.

It is natural to ask whether $\mathfrak{g}(S)$ is graded over $S$. This led to the study in Section 4 and 5 of a semigroup analogue of the usual Serre relations of the form

$[x_a^+, x_\beta^-] = \mu_\pm(\alpha, \beta) \cdot x_{a \ominus \beta}^+$,

for some suitable $\mu_\pm : S \times S \rightarrow k$. We identify a list of key properties, encoded by the notion of a good Cartan semigroup, which guarantee the existence of Serre relations for a large class of pairs $(\alpha, \beta)$ (Theorem 5.7).

**Continuum quivers and continuum Kac–Moody algebras.** We apply the structural results on semigroup Lie algebras to study a new class of infinite–dimensional Lie algebras, which we refer to as continuum Kac–Moody algebras. They are naturally associated to a topological datum, which we refer to as continuum quiver in that it can be thought of as a topological generalization of a quiver. Roughly, a continuum quiver is the datum of a one–dimensional real CW complex $X$, which is locally modeled by smooth trees glued with copies of $S^1$, with a bilinear form on the space of locally constant functions (cf. Definition 6.4). More precisely, continuum quivers shall be thought of as good Cartan semigroups with a topological origin and similar properties to that of positive roots for a Kac–Moody algebras.

The notion of interval lifts easily from $R$ to $X$, and the set $\text{Int}(X)$ of intervals in $X$ is therefore naturally endowed with two partially defined operations, i.e., the sum of intervals $\oplus$, given by concatenation, and their difference $\ominus$, given by truncation. The set $\text{Int}(X)$ is naturally endowed with a $\oplus$–bilinear form.

On the space of locally constant, left–continuous functions on $R$ with limited support, we consider a bilinear form given by:

$\langle f, g \rangle := \sum_x f_-(x)(g_-(x) - g_+(x))$,

where $h_\pm(x) := \lim_{t \rightarrow 0, t > 0} h(x \pm t)$. Note that the form $\langle \cdot, \cdot \rangle$ is essentially defined by its values on the characteristic functions of connected intervals. Therefore, the form $(f, g)$ and its symmetrization $(f, g) := \langle f, g \rangle + \langle g, f \rangle$ extend uniquely from $R$ to $X$ by $\oplus$–bilinearity. Moreover, we regard them as forms on $\text{Int}(X)$ by setting $(a, \beta) := (1_a, 1_\beta)$ and $\langle a, \beta \rangle := (1_a, 1_\beta)$. Finally, the continuum quiver is the good Cartan semigroup $Q_X = (\text{Int}(X), \kappa_X, \zeta_X)$ with $\kappa_X, \zeta_X : \text{Int}(X) \times \text{Int}(X) \rightarrow k$ given by $\kappa_X(a, \beta) := (a, \beta)$ and $\zeta_X(a, \beta) := (-1)^{|a|, |\beta|} (a, \beta)$. The datum of $Q_X$ should be interpreted as a topological generalization of the Borcherds–Cartan datum associated to a locally finite quiver with loops.

Given a continuum quiver $Q_X$, the continuum Kac–Moody algebra $\mathfrak{g}_X := \mathfrak{g}(Q_X)$ is by definition the semigroup Lie algebra associated to $Q_X$. Note that its Cartan subalgebra essentially coincides with the algebra of locally constant functions on $X$.

**Complete presentation and colimit realization.** Our main theorem is a continuum analogue of the Gabber–Kac theorem [GK81]. More precisely, we show that the maximal ideal in $\mathfrak{g}_X$ is generated by the semigroup Serre relations described by $Q_X$. This leads to the following explicit description.
Theorem (Theorem 6.19). The continuum Kac–Moody algebra \( g_X \) is generated by the elements \( x_a^\pm \) and \( \zeta_a \) with \( a \in \text{Int}(X) \), subject to the following defining relations:

1. for any \( \alpha, \beta \in \text{Int}(X) \) such that \( \alpha \oplus \beta \) is defined,
   \[ \zeta_{a \oplus \beta} = \zeta_a + \zeta_\beta ; \]

2. Diagonal action: for any \( \alpha, \beta \in \text{Int}(X) \) such that \( \alpha \oplus \beta \) is defined,
   \[ [\zeta_a, \zeta_\beta] = 0 , \quad [\zeta_a, x_\beta^\pm] = \pm (\alpha, \beta) \cdot x_\beta^\pm ; \]

3. Double relations: for any \( \alpha, \beta \in \text{Int}(X) \),
   \[ [x_a^+, x_\beta^-] = \delta_{a,\beta} \zeta_a + (-1)^{(a,\beta)} (\alpha, \beta) \cdot \left( x_a^{- \alpha \oplus \beta} - x_\beta^{+ \alpha \oplus \beta} \right) ; \]

4. Serre relations: if \( (a, \beta) \in S_X \), then
   \[ [x_a^+, x_\beta^-] = (-1)^{(\beta, \alpha)} x_a^{+ \alpha \oplus \beta} , \quad [x_a^-, x_\beta^+] = (-1)^{(\alpha, \beta)} x_a^{- \alpha \oplus \beta} . \]

Roughly, \( S_X \) consists of unordered pairs \( (\alpha, \beta) \in \text{Int}(X) \times \text{Int}(X) \) such that either

- \( \alpha \oplus \beta \) does not exist and \( \alpha \cap \beta = \emptyset \) or
- \( \alpha \) is contractible and, for subintervals \( \alpha' \subseteq \alpha \) and \( \beta' \subseteq \beta \) with \( (\beta', \beta') \neq 0 \) whenever \( \beta' \neq \beta, \alpha' \oplus \beta' \) is either undefined or non–homeomorphic to \( S^1 \).

An immediate consequence is the realization of \( g_X \) as a continuous colimit of symmetric Borcherds–Kac–Moody algebras with at most isotropic simple roots (Theorem 6.23). This is based on the following observation. The semigroup Serre relations do imply the usual Serre relations appearing in the case of quivers with at most one loop in every vertex, suggesting that \( g_X \) can be locally described in terms of standard Borcherds–Kac–Moody algebras.

Let \( J \) be a finite set of intervals \( a \in \text{Int}(X) \), which roughly corresponds to a local description of \( X \) as a CW complex. More precisely, this means that every interval is either contractible or homeomorphic to \( S^1 \), and given two intervals \( a, \beta \in J, a \neq \beta \), one of the following mutually exclusive cases occurs:

(a) \( a \oplus \beta \) exists;
(b) \( a \oplus \beta \) does not exist and \( a \cap \beta = \emptyset ; \)
(c) \( a \simeq S^1 \) and \( \beta \subset a \).

For any such \( J \), we get a symmetric matrix \( A_J := (a, \beta) \in J \) and therefore a quiver \( Q_J \). Note that the diagonal entries of \( A_J \) are either 2 or 0, while the off–diagonal entries are 0, −1, or −2. In the table below, we give few examples.

| Configuration of intervals | Borcherds–Cartan diagram |
|----------------------------|-------------------------|
| \( a_1 \)                  | \( a_1 \)               |
| \( a_2 \)                  | \( a_2 \)               |
| \( a_3 \)                  | \( a_3 \)               |
Note, in particular, that any contractible elementary interval corresponds to a vertex of $Q_J$ without loops, while any interval homeomorphic to $S^1$, corresponds to a vertex having exactly one loop. This correspondence between intervals and quivers describes the relation between $g_X$ and standard Borcherds–Kac–Moody algebras. Indeed, we prove that the Borcherds–Kac–Moody algebra $g_Q$ is isomorphic to the subalgebra $g(J) \subset g_X$ generated by the elements $x_\alpha$ and $\zeta_\alpha$ with $\alpha \in J$.

Moreover, we prove that such local description gives rise to a global description of $g_X$ as a colimit of Borcherds–Kac–Moody algebras. Indeed, let $\text{Sh}(X)$ be the set of all quivers which arise from a collection of intervals in $X$. Then, we prove that there is a direct system of embeddings $\varphi_{Q, Q'}: g_Q \to g_{Q'}$, indexed by pairs of quivers in $\text{Sh}(X)$, which leads to the following:

**Theorem** (cf. Theorem 6.23). There is a canonical isomorphism $g_X \simeq \text{colim}_{Q \in \text{Sh}(X)} g_Q$.

**Remark.** In [AS20], the first and second–named authors prove that $g_X$ is further endowed with a non–degenerate invariant inner product and a standard structure of quasi–triangular (topological) Lie bialgebra graded over the semigroup $Q_X$. The methods used in the proof extend to the case of a Hopf algebra and allow to construct algebraically an explicit quantization $U_\nu(g_X)$ of $g_X$, which is then called the continuum quantum group of $X$.

**Goldman Lie algebra of the torus and semigroup Lie algebras.** The main motivation behind the theory of semigroup Lie algebras was to provide a solid ground and an abstract approach to the theory of continuum Kac–Moody algebras. The resulting theory is on the other hand extremely rich and features remarkable examples, such as the Goldman Lie algebra of the torus.

Recall that in [MS17] the Goldman Lie algebra of the torus $g_{\mathbb{T}^2}$ is proved to be isomorphic to the Lie algebra generated by the elements $P_v$ with $v := (a, b) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, modulo the relation

$$[P_v, P_w] = \det(v w) P_{v+w},$$

where $[v, w]$ is the order two matrix with columns $v$ and $w$.

This presentation suggests an obvious interpretation of $g_{\mathbb{T}^2}$ as a semigroup Lie algebra. The integral half plane $H := \{(a, b) \in \mathbb{Z}^2 \mid a \geq 1 \text{ or } a = 0, b \geq 1\}$ is naturally endowed with a structure of (total) semigroup with respect to the usual sum of vectors and truncated difference $\ominus$. 

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given by

\[(a, b) \oplus (a', b') := \begin{cases} (a - a', b - b') & \text{if } (a - a', b - b') \in H, \\ 0 & \text{otherwise}. \end{cases} \]

Set \(\kappa(v, w) = 0\) and \(\xi_+(v, w) := \det[wv] = \xi_-(v, w)\) for any \(x, y \in H\). We set \(H = (H, 0, \xi)\) and consider the semigroup Lie algebra \(g(H)\). Note that \(Z^2 \setminus \{(0, 0)\} = H \cup (-H)\) and, since \(\kappa = 0\), the Cartan subalgebra \(\mathcal{Z} \subset g(H)\) is central and we set \(g_H := g(H) / \mathcal{Z}\). Moreover, by Proposition 4.10, the Serre relations hold in \(g(H)\) and one has \([x^+_v, x^-_w] = \det[vw]x^+_v \cdot w, \) for any \(v, w \in H\). This yields to the following:

**Proposition.** There is a surjective homomorphism of Lie algebras \(\psi : g_{\mathbb{R}^2} \rightarrow g_H\) given by the assignment

\[P_{\psi} \mapsto \begin{cases} x^+_v & \text{if } a \geq 1 \text{ or } a = 0, b \geq 1, \\ x^-_{-v} & \text{otherwise,} \end{cases} \]

where \(v := (a, b) \in Z^2 \setminus \{(0, 0)\} \).

We expect \(\psi\) to be an isomorphism.

**Highest weight theory and Fock spaces.** We conclude by outlining a further direction of research, currently under investigation. The usual description of the highest weight theory of Borcherds–Kac–Moody algebras does not extend immediately to continuum Kac–Moody algebras, mainly due to the lack of simple roots and thus of fundamental weights. Nevertheless, we expect the existence of a continuum analog of the theory of highest weight representations, Weyl groups, character formulas, both in the classical and quantum setups.

A first example is given in [SS19b], where the second–named and third–named authors define the Fock space for \(U_v(sl(\mathbb{R}))\), considering a continuum analogue of the usual combinatorial construction of the Fock space of \(U_v(sl(\infty))\) in terms of Maya diagrams. In addition, the quantum group \(U_v(sl(\mathbb{F}))\) act on such a Fock space, in a way similar to the folding procedure of Hayashi–Misra–Miwa. It would be interesting to extend this construction to the case of an arbitrary continuum quiver \(X\), producing a wide class of representations for the continuum quantum group \(U_v(g_X)\), and therefore for the continuum Kac–Moody algebra \(g_X\).

**Outline.** In Section 2 we give a brief review of Kac–Moody algebras, their generalizations introduced by Saveliev and Vershik, and the Lie algebra \(sl(\mathbb{R})\) as a remarkable counterexample. A unifying approach is developed in Section 3 through the notion of semigroup Lie algebra, whose weight lattice is controlled by a partial semigroup. In Section 4 we study a semigroup analogue of the classical Serre relations. We determine necessary and sufficient conditions for such relations to hold and we prove them in several special cases. In Section 5, we identify a list of key properties, encoded by the notion of a good Cartan semigroup, which guarantee the existence of Serre relations (Theorem 5.7). Finally, in Section 6, we apply the abstract framework developed in the previous sections to describe a new class of infinite–dimensional Lie algebras associated with a one–dimensional real space. We introduce the notion of continuum quivers, whose defining data are interpreted in terms of good Cartan semigroup. We define the continuum Kac–Moody algebra of a continuum quiver as a semigroup Lie algebra and prove a continuum analogue of the Gabber–Kac theorem (Theorem 6.19). Moreover, we prove that every continuum Kac–Moody algebra is isomorphic to an uncountable colimit of symmetric Borcherds–Kac–Moody algebras with at most isotropic simple roots (Theorem 6.23). Finally, in Appendix A we include a brief review of the basic definitions and properties of the partial semigroups we are mainly interested in, while in Appendix B, C, D, E we report in details some of the most technical proofs.

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2. KAC–MOODY ALGEBRAS AND THE LIE ALGEBRA OF THE REAL LINE

In this section, we recall the definitions of Kac–Moody algebras and their generalization introduced by Saveliev and Vershik. We then introduce a distinguished Lie algebra \( \mathfrak{s}(\mathbb{R}) \), which first appeared in [SS19a] and whose presentation is controlled by the topology of the real line. We prove that \( \mathfrak{s}(\mathbb{R}) \) admits a simpler presentation à la Kac–Moody, depending upon basic operations on open–closed connected intervals. Henceforth, we fix a base field \( k \) of characteristic zero.

2.1. Kac–Moody algebras. We recall the definition from [Kac90, Chapter 1]. Fix a finite set \( I \) and a matrix \( A = (a_{ij})_{i,j \in I} \) with entries in \( k \). Recall that a realization \( \mathcal{R} := (\mathfrak{h}, \Pi, \Pi') \) of \( A \) is the datum of a finite dimensional \( k \)-vector space \( \mathfrak{h} \), and linearly independent vectors \( \Pi := \{ a_i \}_{i \in I} \subset \mathfrak{h}^* \), \( \Pi' := \{ h_i \}_{i \in I} \subset \mathfrak{h} \) such that \( a_i(h_j) = a_{ij} \). Note that these conditions imply \( \dim \mathfrak{h} \geq 2|I| - \text{rk}(A) \).

Moreover, up to a (non–unique) isomorphism, there is a unique realization of minimal dimension \( 2|I| - \text{rk}(A) \).

Let \( \tilde{\mathfrak{g}}(\mathcal{R}) \) be the Lie algebra generated by \( h, \{ e_i, f_i \}_{i \in I} \) with relations \([h, h] = 0 \) and
\[ [h, e_i] = a_i(h) e_i, \quad [h, f_i] = -a_i(h) f_i, \quad [e_i, f_j] = \delta_{ij} h_i. \]

for any \( h \in \mathfrak{h} \) and \( i, j \in I \). Set
\[ Q_+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} a_i \subset \mathfrak{h}^*, \]
\[ Q := Q_+ \oplus (-Q_+), \]
and denote by \( \tilde{n}_+ \) (resp. \( \tilde{n}_- \)) the subalgebra generated by \( \{ e_i \}_{i \in I} \) (resp. \( \{ f_i \}_{i \in I} \)). Then, as vector spaces, \( \tilde{\mathfrak{g}}(\mathcal{R}) = \tilde{n}_+ \oplus \mathfrak{h} \oplus \tilde{n}_- \) with root space decomposition
\[ \tilde{n}_\pm = \bigoplus_{a \in Q_+ \atop a \neq 0} \tilde{\mathfrak{g}}_{\pm a}, \]
where \( \tilde{\mathfrak{g}}_{\pm a} = \{ X \in \tilde{\mathfrak{g}}(\mathcal{R}) \mid \forall h \in \mathfrak{h}, [h, X] = \pm a(h) X \} \). Note also that \( \tilde{\mathfrak{g}}_0 = \mathfrak{h} \) and \( \dim \tilde{\mathfrak{g}}_{\pm a} < \infty \).

The Kac–Moody algebra associated to the realization \( \mathcal{R} \) is the Lie algebra \( \mathfrak{g}(\mathcal{R}) := \tilde{\mathfrak{g}}(\mathcal{R}) / \tau \), where \( \tau \) is the sum of all two–sided graded ideals in \( \tilde{\mathfrak{g}}(\mathcal{R}) \) having trivial intersection with \( \mathfrak{h} \). In particular, as ideals, \( \tau = \tau_+ \oplus \tau_- \), where \( \tau_{\pm} := \tau \cap \tilde{n}_{\pm} \).

Remark 2.1. If \( A \) is a generalised Cartan matrix (i.e., \( a_{ii} = 2 \), \( a_{ij} \in \mathbb{Z}_{\leq 0}, i \neq j \), and \( a_{ij} = 0 \) implies \( a_{ji} = 0 \)), then \( \tau \) contains the ideal \( s \) generated by the Serre relations
\[ \text{ad}(e_i)^{1 - a_{ij}}(e_j) = 0 = \text{ad}(f_i)^{1 - a_{ij}}(f_j) \quad i \neq j \]
and, if \( A \) is symmetrisable, is generated by \( s \) [GK81]. A similar property holds, more generally, for any symmetrizable A such that \( a_{ij} \in \mathbb{Z}_{\leq 0}, i \neq j \), and \( 2a_{ij}/a_{ii} \in \mathbb{Z} \) whenever \( a_{ii} > 0 \). In this case, \( \mathfrak{g}(A) \) is called a Borcherds–Kac–Moody algebra and the corresponding maximal ideal is generated by the Serre relations
\[ \text{ad}(e_i)^{1 - a_{ij}}(e_j) = 0 = \text{ad}(f_i)^{1 - a_{ij}}(f_j) \quad \text{if} \quad a_{ii} > 0 \] (2.1)
and
\[ [e_i, e_j] = 0 = [f_i, f_j] \quad \text{if} \quad a_{ij} \leq 0 \quad \text{and} \quad a_{ij} = 0 \] (2.2)
for \( i \neq j \) (cf. [Bor88, Corollary 2.6]).

3The terminology differs slightly from the one given in [Kac90] where \( \mathfrak{g}(\mathcal{R}) \) is called a Kac–Moody algebra if \( A \) is a generalised Cartan matrix (cf. Remark 2.1) and \( \mathcal{R} \) is the minimal realization. Note also that in [Kac90, Theorem 1.2] \( \tau \) is set to be the sum of all two–sided ideals, not necessarily graded. However, since the functionals \( a_i \) are linearly independent in \( \mathfrak{h}^* \) by construction, \( \tau \) is automatically graded and satisfies \( \tau = \tau_+ \oplus \tau_- \) (cf. [Kac90, Proposition 1.5]).
Since \( r = r_+ \oplus r_- \), the Lie algebra \( g(\mathcal{R}) \) has an induced triangular decomposition \( g(\mathcal{R}) = n_- \oplus h \oplus n_+ \) (as vector spaces), where
\[
\begin{align*}
n_+ \oplus g_{\pm} := \bigoplus_{a \in Q_+ \setminus \{0\}} g_{\pm a}, \\
n_+ := \{ X \in g(\mathcal{R}) \mid \forall h \in h, [h, X] = a(h) X \}.
\end{align*}
\]
Note that \( \dim g_a < \infty \). The set of positive roots is \( R_+ := \{ a \in Q_+ \setminus \{0\} \mid g_a \neq 0 \} \).

**Remark 2.2.** The derived subalgebra \( g(\mathcal{R})' := \left[ g(\mathcal{R}), g(\mathcal{R}) \right] \) is generated by \( \{e, f, h_i\} \leq \mathbb{N} \) and admits a presentation similar to that of \( g(\mathcal{R}) \). Namely, let \( \tilde{g} \) be the Lie algebra generated by \( \{h_i, e_i, f_i\} \leq \mathbb{N} \) with relations
\[
[h_i, h_j] = 0, \quad [h_i, e_i] = a_i(h_j) e_i, \quad [h_i, f_i] = -a_i(h_j) f_i, \quad [e_i, f_i] = \delta_{ij} h_i.
\]
(2.3)

Then, \( \tilde{g} \) has a \( Q \)-gradation defined by \( \deg(e_i) = a_i \), \( \deg(f_i) = -a_i \), \( \deg(h_i) = 0 \), and \( \tilde{g}_0 = h' \), where the latter is the \( |I| \)-dimensional span of \( \{h_i\} \leq I \). The quotient of \( \tilde{g} \) by the sum of all two-sided graded ideals with non-trivial intersection with \( h' \) is easily seen to be canonically isomorphic to \( g(\mathcal{R})' \).

Recall that a direct sum of vector spaces \( L = L_{-1} \oplus L_0 \oplus L_{+1} \) is a **local Lie algebra** if there are bilinear maps \( L_i \times L_j \to L_{i+j} \) for \( i, j, i + j \leq 1 \), such that antisymmetry and Jacobi identity hold whenever they make sense. We write \( \pm 1 \) instead of \( \pm 1 \). It follows that \( g(\mathcal{R})' := \left[ g(\mathcal{R}), g(\mathcal{R}) \right] \) is freely generated by the local Lie algebra \( L := L_{-1} \oplus L_0 \oplus L_+ \) where
\[
\begin{align*}
L_- &:= \bigoplus_i k \cdot f_i, \\
L_0 &:= \bigoplus_i k \cdot h_i, \\
L_+ &:= \bigoplus_i k \cdot e_i,
\end{align*}
\]
with bracket defined by equation (2.3).

**Remark 2.3.** It is sometimes convenient to consider Kac–Moody algebras associated to a non-minimal realization (cf. [FZ85, MO19, ATL19b]). Let \( \mathcal{R} = (\mathcal{R}, \mathcal{R}, \mathcal{R}) \) be the realization given by \( \mathcal{R} \cong \mathbb{K}^{2|I|} \) with basis \( \{h_i\}_{i \in I} \cup \{\lambda_i^\vee\}_{i \in I} \), \( \mathcal{R} = \{h_i\}_{i \in I} \) and \( \mathcal{R} = \{a_i\}_{i \in I} \subset \mathcal{R} \), where \( a_i \) is defined by
\[
a_i(h_j) = a_{ji} \quad \text{and} \quad a_i(\lambda_j^\vee) = \delta_{ij}.
\]
We refer to \( \mathcal{R} \) as the canonical realization of \( \mathcal{A} \), and denote by \( \Lambda^\vee \subset \mathcal{R} \) the \( |I| \)-dimensional subspace spanned by \( \{\lambda_i^\vee\}_{i \in I} \). Let \( \mathcal{R}_{\min} \) be a minimal realization of \( \mathcal{A} \). It is easy to check that the Kac–Moody algebra \( g(\mathcal{R}) \) is a central extension of \( g(\mathcal{R}_{\min}) \), i.e., \( g(\mathcal{R}) \cong g(\mathcal{R}_{\min}) \oplus c \), with \( \dim c = r(A) \).

### 2.2. Saveliev–Vershik algebras

In [SV91], Saveliev and Vershik introduced the notion of **continuous Lie algebras**, providing a generalization of Kac–Moody algebras and covering a wide spectrum of examples including Lie algebras arising, for example, from ergodic transformations, crossed products, or infinitesimal area-preserving diffeomorphisms [SV90, SV92, Ver92, Ver92].

Their generalization is entirely different from the one we shall introduce in Section 6. We briefly recall their construction and, in order to avoid confusion, we shall refer to it as Saveliev–Vershik algebras.

Let \( (\mathcal{A}, \cdot, \cdot) \) be an arbitrary associative \( k \)-algebra (possibly infinite-dimensional, non-unital, and non-commutative) endowed with three bilinear mappings \( \kappa_{\pm}, \kappa_0 \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A} \) and \( L \) a vector space of the form \( L \otimes L_0 \oplus L_+ \), where \( L_0 \cong \mathcal{A}, \varepsilon = \pm, 0 \). We denote \( X_\varepsilon(\phi) \) the element in \( L_\varepsilon, \varepsilon = \pm, 0 \), corresponding to \( \phi \in \mathcal{A} \), and the bilinear map given by
\[
\begin{align*}
[X_0(\phi), X_0(\psi)] &= X_0([\phi, \psi]), \\
[X_0(\phi), X_\pm(\psi)] &= X_\pm(\kappa(\phi, \psi)), \\
[X_+(\phi), X_-(\psi)] &= X_0(\kappa_0(\phi, \psi)),
\end{align*}
\]
where \( \phi, \psi \in \mathcal{A} \), \([\phi, \psi] := \phi \cdot \psi - \psi \cdot \phi \). The following is straightforward.
Lemma 2.4 ([SV91]). The vector space $L$ is a local Lie algebra with bracket $[\cdot, \cdot]$ if and only if the maps $\kappa_e$ satisfy the following relations:

$$\kappa_\pm([\phi, \psi], \chi) = \kappa_\pm(\phi, \kappa_\pm(\psi, \chi)) - \kappa_\pm(\psi, \kappa_\pm(\phi, \chi)), \quad (2.4)$$

$$\kappa_0([\phi, \psi], \chi) = \kappa_0(\kappa_+ (\phi, \psi), \chi) + \kappa_0(\kappa_- (\phi, \psi)), \quad (2.5)$$

where $\phi, \psi, \chi \in A$.

Let $(A, \kappa_e)$ be an associative algebra endowed with three bilinear maps $\kappa_e : A \otimes A \to A$, $e = \pm, 0$, satisfying (2.4), (2.5). We denote by $\tilde{g}(A, \kappa_e)$ the Lie algebra freely generated by $L$. Note that $\tilde{g}(A, \kappa_e)$ is $\mathbb{Z}$-graded, i.e.,

$$\tilde{g}(A, \kappa_e) = \bigoplus_{n \in \mathbb{Z}} \tilde{g}_n,$$

with homogeneous components $\tilde{g}_0 := L_0$, and

$$\tilde{g}_n := \begin{cases} [\tilde{g}_{n-1}, L_+] & \text{if } n > 0, \\ [\tilde{g}_{n+1}, L_-] & \text{if } n < 0. \end{cases}$$

Definition 2.5 ([SV91]). The Saveliev–Vershik algebra associated to the datum $(A, \kappa_e)$ is the Lie algebra

$$g(A, \kappa_e) := \tilde{g}(A, \kappa_e) / \mathfrak{r},$$

where $\mathfrak{r}$ is the sum of all two–sided homogeneous ideals in $\tilde{g}(A, \kappa_e)$ having trivial intersection with $L_0$. In particular, $g(A, \kappa_e)$ is $\mathbb{Z}$–graded with $g_0 = L_0$. \hfill $\Box$

The examples given in [SV90, SV92, Ver92, Ver02] belong to a special case of this formulation, where $A$ is a commutative algebra, $\kappa, s : A \to A$ are distinguished linear mappings, and

$$\kappa_\pm(\phi, \psi) := \pm \psi \cdot \kappa(\phi) \quad \text{and} \quad \kappa_0(\phi, \psi) := s(\phi \cdot \psi) \quad (2.6)$$

Note that in this case the conditions (2.4) and (2.5) are automatically satisfied.

Remark 2.6. Definition 2.5 provides a straightforward generalization of derived Kac–Moody algebras as described in Remark 2.2 in terms of the usual 3|1| Chevalley generators. Namely, let $I$ and $A$ be as in Section 2.1. Then, one sees immediately that $g(\mathbb{R})' = g(A, \kappa_e)$, where $A$ is the commutative algebra $\mathbb{R}^{|I|}$ endowed with coordinate multiplication, $(e_i, h_i, f_i) = (X_+(v_i), X_0(v_i), X_-(v_i))$, $i \in I$, where $v_i \in \mathbb{R}^{|I|}$ are the standard unit vectors, and the linear maps $\kappa_e$ are defined as in (2.6) with $\kappa(v) := Av$ and $s(v) := v$ for any $v \in A$.

It is notable that, while the Kac–Moody algebra associated to the minimal realization does not fit in this formalism, the canonical Kac–Moody algebra $g(\mathbb{R})$ does, up to a minor change. Namely, it is enough to consider a local Lie algebra $L$ with $L_0 = A \oplus A$, whose additional elements are denoted $X_\phi^\psi (\phi, \psi) = [X_\phi^\psi (\phi, \psi)] = 0 = [X_\phi^\psi (\phi), X_\psi^\phi (\psi)]$ and $[X_\phi^\psi (\phi), X_{\pm}^\psi (\psi)] = \pm X_\phi^\psi (\phi \cdot \psi).$ \hfill $\Delta$

2.3. The Lie algebra of the line. In [SS19a], the last-two-named authors introduced quantum groups $U_v(sl(\mathbb{K}))$ with $\mathbb{K} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Since the cases of $\mathbb{K} = \mathbb{Z}, \mathbb{Q}$ are easily deduced from that of $\mathbb{K} = \mathbb{R}$, for simplicity, we focus only on the Lie algebra $sl(\mathbb{R})$.

2.3.1. Intervals. Roughly speaking, $sl(\mathbb{R})$ is a Lie algebra generated by elements labeled by intervals of $\mathbb{R}$, subject to some quadratic relations, whose coefficients depend upon a bilinear form on the space of characteristic functions of such intervals. The values of such bilinear form play the same role as the coefficient of a Cartan matrix.

In order to give the precise definition of $sl(\mathbb{R})$, we need to introduce some notation. First, we say that a subset $J \subseteq \mathbb{R}$ is an interval if it is an open–closed interval of the form $J = [a, b] := \{x \in \mathbb{R} \mid a < x \leq b\}$. 
Let $\text{Int}(\mathbb{R})$ be the set of all intervals in $\mathbb{R}$ and define two partial maps $\oplus, \ominus: \text{Int}(\mathbb{R}) \times \text{Int}(\mathbb{R}) \to \text{Int}(\mathbb{R})$ as follows:

$$J \oplus J' := \begin{cases} J \cup J' & \text{if } J \cap J' = \emptyset \text{ and } J \cup J' \text{ is connected}, \\ \text{n.d.} & \text{otherwise} \end{cases} \quad (2.7)$$

$$J \ominus J' := \begin{cases} J \setminus J' & \text{if } J \cap J' = J' \text{ and } J \setminus J' \text{ is connected}, \\ \text{n.d.} & \text{otherwise} \end{cases} \quad (2.8)$$

Denote by $\text{Int}(\mathbb{R})^{(2)}$ the set of pairs $(J, J')$ such that $J \oplus J'$ is defined. Similarly, let $\text{Int}(\mathbb{R})^{(2)}_\ominus$ be the set of pairs $(J, J')$ that $J \ominus J'$ is defined. Set

$$\delta_{J \oplus J'} := \begin{cases} 1 & \text{if } (J, J') \in \text{Int}(\mathbb{R})^{(2)}, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \delta_{J \ominus J'} := \begin{cases} 1 & \text{if } (J, J') \in \text{Int}(\mathbb{R})^{(2)}_\ominus, \\ 0 & \text{otherwise} \end{cases}$$

We adopt the following notation, distinguishing all relative positions of two intervals. For any two intervals $J = [a, b]$ and $J' = [a', b']$, we write

- $J \to J'$ if $b = a'$ (adjacent)
- $J \perp J'$ if $b < a'$ or $b' < a$ (disjoint)
- $J \vdash J'$ if $a = a'$ and $b < b'$ (closed subinterval)
- $J \dashv J'$ if $a' < a$ and $b = b'$ (open subinterval)
- $J < J'$ if $a' < a < b < b'$ (strict subinterval)
- $J \cap J'$ if $a < a' < b < b'$ (overlapping)

In particular we have that

$$\begin{align*}
(J, J') \in \text{Int}(\mathbb{R})^{(2)} & \quad \text{if and only if} \quad J \to J' \text{ or } J' \to J, \\
(J, J') \in \text{Int}(\mathbb{R})^{(2)}_\ominus & \quad \text{if and only if} \quad J \vdash J' \text{ or } J' \vdash J.
\end{align*}$$

2.3.2. *Euler form.* We denote by $\text{fun}(\mathbb{R})$ the algebra of piecewise constant, left–continuous functions $f: \mathbb{R} \to \mathbb{R}$, with bounded support and finitely many points of discontinuity. More explicitly, $f \in \text{fun}(\mathbb{R})$ if and only if $f$ is a linear combination of finitely many characteristic functions $\mathbbm{1}_J$ with $J \in \text{Int}(\mathbb{R})$.

For any $f, g \in \text{fun}(\mathbb{R})$, we set

$$\langle f, g \rangle := \sum_x f_-(x)(g_-(x) - g_+(x)) \quad \text{and} \quad (f, g) := \langle f, g \rangle + \langle g, f \rangle$$

where $h_\pm(x) = \lim_{t \to 0^+} h(x \pm t)$. We have

$$\langle \mathbbm{1}_J, \mathbbm{1}_{J'} \rangle := \begin{cases} 1 & \text{if } J = J', J' \vdash J, J \dashv J', J' \cap J, \\ 0 & \text{if } J \perp J', J' \vdash J, J \dashv J', J \cap J', J' < J, \\ -1 & \text{if } J \vdash J', J' \cap J'. \end{cases} \quad (2.9)$$

The symbol $\vdash$ (resp. $\perp$) should be read as $J$ is a proper subinterval in $J'$ starting from the left (resp. right) endpoint.
Thus,

\[
(\mathbf{1}_I, \mathbf{1}_J) = \begin{cases} 
2 & \text{if } J = J', \\
1 & \text{if } (J, J') \in \text{Int}(\mathbb{R})^2 \cap \mathbb{R}^2 \cap \mathbb{R}^2, \\
-1 & \text{if } (J, J') \in \text{Int}(\mathbb{R})^2, \\
0 & \text{otherwise}.
\end{cases}
\]

2.3.3. The first definition. We are ready to give the definition of \(\mathfrak{sl}(\mathbb{R})\).

**Definition 2.7.** Let \(\mathfrak{sl}(\mathbb{R})\) be the Lie algebra generated by elements \(e_J, f_J, h_J\), with \(J \in \text{Int}(\mathbb{R})\), modulo the following set of relations:

- **Kac–Moody type relations:** for any two intervals \(J_1, J_2\),
  \[
  [h_{J_1}, h_{J_2}] = 0, \\
  [h_{J_1}, e_{J_2}] = (\mathbf{1}_{J_1}, \mathbf{1}_{J_2}) e_{J_2}, \\
  [h_{J_1}, f_{J_2}] = -(\mathbf{1}_{J_1}, \mathbf{1}_{J_2}) f_{J_2},
  \]
  \[
  [e_{J_1}, f_{J_2}] = \begin{cases} 
  h_{J_1} & \text{if } J_1 = J_2, \\
  0 & \text{if } J_1 \perp J_2, J_1 \rightarrow J_2, \text{ or } J_2 \rightarrow J_1.
  \end{cases}
  \]

- **Join relations:** for any two intervals \(J_1, J_2\) with \((J_1, J_2) \in \text{Int}_K(\mathbb{R})^2\),
  \[
  h_{J_1 \oplus J_2} = h_{J_1} + h_{J_2},
  \]
  \[
  e_{J_1 \oplus J_2} = (-1)^{(3_{J_2}, \mathbf{1}_{J_2})} [e_{J_1}, e_{J_2}],
  \]
  \[
  f_{J_1 \oplus J_2} = (-1)^{(3_{J_2}, \mathbf{1}_{J_2})} [f_{J_1}, f_{J_2}],
  \]

- **Nest relations:** for any nested \(J_1, J_2 \in \text{Int}_K(\mathbb{R})\) (that is, such that \(J_1 = J_2, J_1 \perp J_2, J_1 \leq J_2, J_2 < J_1, J_1 \rightarrow J_2, J_2 \rightarrow J_1, \text{ or } J_2 \rightarrow J_1\),
  \[
  [e_{J_1}, e_{J_2}] = 0 \quad \text{and} \quad [f_{J_1}, f_{J_2}] = 0.
  \]

**Remark 2.8.** It is easy to check that the bracket is anti-symmetric and satisfies the Jacobi identity. Note that the join relations are consistent with anti-symmetry, since whenever \(J \oplus J'\) is defined, \((-1)^{(3_{J}, \mathbf{1}_{J})} = (-1)^{(3_{J'}, \mathbf{1}_{J'}),}\) Moreover, the combination of join and nest relations yields the \((\text{type } A)\) Serre relations \((J \neq J')\)

\[
[e_J, [e_J, e_J]] = 0 = [f_J, [f_J, f_J]] \quad \text{if} \quad (\mathbf{1}_J, \mathbf{1}_J) = -1,
\]

\[
[e_J, e_J] = 0 = [f_J, f_J] \quad \text{if} \quad (\mathbf{1}_J, \mathbf{1}_J) = 0.
\]

\(\triangle\)

2.3.4. A new presentation. The datum \((\text{Int}(\mathbb{R}), \oplus, \odot)\) has the role of a continuum root system of \(\mathfrak{sl}(\mathbb{R})\). It is therefore natural to ask whether \(\mathfrak{sl}(\mathbb{R})\) is an example of a Saveliev–Vershik algebra. Namely, set \(\mathcal{A} := \text{fun}(\mathbb{R})\), and define the maps \(\kappa_0, \kappa_\pm : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}\) by setting

\[
\kappa_0(\mathbf{1}_J, \mathbf{1}_J) := \delta_{J,J'} \mathbf{1}_{J'} \quad \text{and} \quad \kappa_\pm(\mathbf{1}_J, \mathbf{1}_J) := \pm (\mathbf{1}_J, \mathbf{1}_J) \mathbf{1}_{J'}.
\]

The maps \(\kappa_0\) and \(\kappa_\pm\) satisfy the relations (2.4) and (2.5). Thus there exists a continuum Lie algebra \(\mathfrak{g}(\text{fun}(\mathbb{R}), \kappa_0, \kappa_\pm)\). Nonetheless, we shall show in this section that \(\mathfrak{g}(\text{fun}(\mathbb{R}), \kappa_0, \kappa_\pm)\) and \(\mathfrak{sl}(\mathbb{R})\) are not the same. More generally, the latter cannot be a Saveliev–Vershik algebra. The first result we need is the following.
Proof. It is clear that (2.16) reduces to (2.10), while (2.17) reduces to (2.12), (2.13), and (2.14), since $J_1 \oplus J_2$ is not defined whenever $J_1$ and $J_2$ are nested, and thus the RHS of (2.17) equals zero. Conversely, we shall prove that (2.16) and (2.17) hold in $\mathfrak{sl}(\mathbb{R})$. The proof is based on a case–by–case inspection described in Appendix B. \qed

We are now able to give a more efficient presentation of $\mathfrak{sl}(\mathbb{R})$. In order to stress the analogy with the definition provided in Section 6, we adopt a slightly different notation. Set $\zeta_f := h_f$, $x_f^+ := e_f$, $x_f^- := f_f$ for any interval $f$.

**Corollary 2.10.** $\mathfrak{sl}(\mathbb{R})$ is presented on the generators $x_f^\pm$, $\zeta_f$, $f \in \text{Int}(\mathbb{R})$, with relations

\[
\begin{align*}
\zeta_{f \oplus f'} &= \delta_{f \oplus f'}(\zeta_f + \zeta_{f'} ), \\
[\zeta_f, \zeta_{f'}] &= 0 , \\
[\zeta_f, x_{f'}^\pm] &= \pm (1_{f'} 1_f) x_{f' \oplus f}^\pm , \\
[x_f^+, x_{f'}^-] &= 0 , \\
[x_f^+, x_{f'}^+ - x_{f' \ominus f}^-] &= \delta_{f, f'}(1_{f'} 1_f) (x_{f \oplus f}^+ - x_{f' \ominus f}^-) , \\
[x_f^-, x_{f'}^- + x_{f' \ominus f}^+] &= \pm (-1)^{(1_{f'} 1_f)} x_{f' \ominus f}^\pm ,
\end{align*}
\]

where we assume that $x_{f \odot f}^\pm = 0$ whenever $f_1 \odot f_2$ is not defined, for $\odot = \oplus, \ominus$ and $e = \pm, 0$.

The above presentation of $\mathfrak{sl}(\mathbb{R})$ makes clear that the main reason for which $\mathfrak{sl}(\mathbb{R})$ cannot coincide with $\mathfrak{g}(\text{fun}(\mathbb{R}), \kappa_0, \kappa_{\pm})$ is the relation (2.18). Indeed, the latter Lie algebra is, by definition, generated by a local Lie algebra $L$ (cf. Section 2.2), while this is not possible $\mathfrak{sl}(\mathbb{R})$. As we shall see in the following sections, we need to consider a wider class of Lie algebras, containing both Saveliev–Vershik Lie algebras (hence, also Kac–Moody algebras) and $\mathfrak{sl}(\mathbb{R})$ by relaxing the condition of locality in the spirit of relation (2.18).

**Remark 2.11.** As we pointed out at the beginning of this section, the description of the Lie algebras $\mathfrak{sl}(\mathbb{Z})$ and $\mathfrak{sl}(\mathbb{Q})$ are easily deduced from that of $\mathfrak{sl}(\mathbb{R})$. More precisely, one can consider the subset $\text{Int}_K(\mathbb{R}) \subset \text{Int}(\mathbb{R})$ consisting of intervals with boundaries in $K = \mathbb{Z}, \mathbb{Q}$. The Lie algebra $\mathfrak{sl}(K)$, $K = \mathbb{Z}, \mathbb{Q}$, is then realized as the subalgebra in $\mathfrak{sl}(\mathbb{R})$ generated by $x_f^\pm$ and $\zeta_f$ with $f \in \text{Int}_K(\mathbb{R})$. In particular, there is a canonical chain of embeddings $\mathfrak{sl}(\mathbb{Z}) \subset \mathfrak{sl}(\mathbb{Q}) \subset \mathfrak{sl}(\mathbb{R})$. \triangle

**Remark 2.12.** We conclude this section by observing that $\mathfrak{sl}(\mathbb{R})$ should be rather thought of as a continuum analogue of the familiar Lie algebra $\mathfrak{sl}(\infty)$. Let

\[
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
1 & 2 & 3 & 4 & n-2 & n-1
\end{array}
\]

be the Dynkin diagram of type $A_n$. We can consider two different limits for $n \to +\infty$: the infinite Dynkin diagram

\[
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

which gives rise to the infinite–dimensional Lie algebra $\mathfrak{sl}(+\infty)$, and the infinite Dynkin diagram

\[
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]
which corresponds to the infinite dimensional Lie algebra sl(∞). One can show that sl(+∞) coincides with sl(∞) (cf. [DP99, Section 2]). The Lie algebra sl(∞) admits a Kac–Moody algebra type description with generators $e_i, f_i, h_i$ with $i \in \mathbb{Z}$ and the infinite Cartan matrix $A = (a_{ij})_{i, j \in \mathbb{Z}}$ with $a_{ii} = 2$, $a_{ij} = -1$, if $|i - j| = 1$, and $a_{ij} = 0$ otherwise.

In view of the Serre relations (2.15), there is a canonical embedding $sl(\infty) \to sl(\mathbb{Z})$ given by
\[
e_i \mapsto e_{(i, i + 1)}, \quad f_i \mapsto f_{(i, i + 1)}, \quad h_i \mapsto h_{(i, i + 1)}.
\]
Moreover, since $\{e_{(i, i + 1)}, f_{(i, i + 1)}, h_{(i, i + 1)} \mid \forall i \in \mathbb{Z}\}$ is a minimal set of generators, this gives a canonical isomorphism $sl(\infty) \simeq sl(\mathbb{Z})$. \(\triangle\)

3. Semigroup Lie algebras

In this section, we introduce a semi–local version of contragredient Lie algebras [FZ85], as a straightforward generalization of the new presentation of $sl(\mathbb{R})$ given in Corollary 2.10. We describe a special class of examples, whose combinatorics depends upon the choice of a partial semigroups.

3.1. Semi–local continuum Lie algebras. It is clear from the previous section that the Lie algebra $sl(\mathbb{R})$ does not fit in the usual Kac–Moody framework (nor in the one introduced by Saveliev–Vershik). In particular, it is not generated by a local Lie algebra and its Cartan elements satisfy the bracket $[\mathfrak{sl}, \mathfrak{sl}]$ coincides with $\mathfrak{sl}(\mathbb{Z})$ canonical isomorphism $\delta$. We shall think of the tuple $(\mathcal{A}, \kappa_\epsilon, \xi_\epsilon)$ as a Cartan datum and develop its corresponding Kac–Moody theory.
Definition 3.3. The tuple \((A, \kappa_e, \xi_e)\) is a continuum Cartan datum if \(\kappa_e\) satisfies (2.4), (2.5), and \(\kappa_e, \xi_e\) satisfy (3.1), (3.5). Then, the (semi–local) continuum Lie algebra associated to \((A, \kappa_e, \xi_e)\) is the Lie algebra
\[ g(A, \kappa_e, \xi_e) := \tilde{g}(A, \kappa_e, \xi_e) / \tau , \]
where \(\tilde{g}(A, \kappa_e, \xi_e)\) is the Lie algebra freely generated by the semi–local Lie algebra \(L\) and \(\tau \subset \tilde{g}(A, \kappa_e, \xi_e)\) is the sum of all two–sided ideals having trivial intersection with \(L_0\).

Let \(n_\pm \subset g(A, \kappa_e, \xi_e)\) be the Lie subalgebra generated by \(\{x_\pm^k \mid k \in A\}\). We shall make use of the following standard result.

Lemma 3.4. Let \(S\) be an index set, let \(\{X_a\}\) be a collection of elements of \(n_\pm\) indexed by \(a \in S\), and let \(X_S = \text{span}\{X_a \mid a \in S\}\). If \([X_S, x_\pm^k] \subseteq X_S\) for any \(k \in A\), then \(X_S = 0\).

Proof. Assume \(X_S \subseteq n_+\) and set \(\tau_X := \sum_{a \in S} \text{ad}(L_+) / X_S\). \(\tau_X\) is a subspace of \(n_+\), which is clearly invariant under \(\text{ad}(L_0)\) and \(\text{ad}(L_+)\). Moreover, since \(\text{ad}(L_-) X_S \subseteq X_S\), one also has \(\text{ad}(L_-) \tau_X \subseteq \tau_X\). In particular, whenever \(X_S \neq \{0\}\), the subspace \(\tau_X\) is a non–zero ideal trivially intersecting \(L_0\). Therefore, necessarily, \(\tau_X = 0\) and \(X_S = 0\). The case \(X_S \subseteq n_-\) is similar.

3.2. Cartan semigroups. We shall give a combinatorial description of certain semi–local continuum Lie algebras, whose defining relations (3.2), (3.3), (3.4) are controlled by a class of partial semigroups, which we refer to as Cartan semigroups.

Henceforth, with a slight abuse of terminology, by a semigroup we mean a positive, commutative, partial semigroup with a maximal cancellation law (cf. Appendix A).

Let \(S\) be a semigroup with commutative product \(\oplus\) and cancellation law \(\ominus\). Let \(\kappa : S \times S \rightarrow k\) be a function such that
\[ \kappa(a \oplus \beta, \gamma) = \delta_{a \oplus \beta} (\kappa(a, \gamma) + \kappa(\beta, \gamma)) , \tag{3.6} \]
where \(\delta_{a \oplus \beta}\) is the characteristic function\(^5\) of \(S_0^{(2)}\) and, by convention, \(\kappa(a \oplus \beta, \gamma) = 0 = \kappa(a, \beta \oplus \gamma)\) whenever \(a \oplus \beta\) and \(\beta \oplus \gamma\) are not defined. Set \(L^S = L^S_+ \oplus L^S_0 \oplus L^S_-,\)
and \(N^S\) is the subspace spanned by the elements of the form
\[ \xi_{a \oplus \beta} - \delta_{a \oplus \beta} (\xi_a + \xi_\beta) , \]
where we assume that \(x_{a \oplus \beta}^\pm = 0 = \xi_{a \oplus \beta}\) if \((a, \beta) \not\in S_0^{(2)}\). By a slight abuse of notation, we will denote the class of \(\xi_a\) in \(L^S_0\) by the same symbol. Given two functions \(\xi_{\pm} : S \times S \rightarrow k\), we define a bilinear map \([,\,] : L^S \times L^S \rightarrow L^S\) (whose dependence by \(\xi_{\pm}\) is omitted) by
\[ [\xi_a, \xi_\beta] = 0 , \]
\[ [\xi_a, x_{\beta}^\pm] = \pm \kappa(a, \beta) x_{\beta}^\pm , \]
\[ [x_{\beta}^+, x_{\beta}^-] = \delta_{a \ominus \beta} \xi_a + \xi_\beta (a, \beta) x_{a \ominus \beta}^+ - \xi_\beta (\beta, a) x_{a \ominus \beta}^- \tag{3.7} \]
where we assume that \(x_{a \ominus \beta}^\pm = 0 = \xi_{a \ominus \beta}\) if \((a, \beta) \not\in S_0^{(2)}\). Note that the condition (3.6) is equivalent to require \([N^S, L^S] = 0\). Therefore, the map \([,\,]\) is well–defined. The following is straightforward (cf. Lemmas 2.4 and 3.2).

\(^5\)This means that \(\delta_{a \oplus \beta}\) takes value one if \((a, \beta) \in S_0^{(2)}\), and it takes value zero otherwise.
Lemma 3.5. The vector space $L^S$ is a semi–local Lie algebra with bracket $[\cdot, \cdot]$ if and only if
\[ \xi_{\pm}(\beta, \gamma)\kappa(\alpha, \beta \ominus \gamma) = \delta_{\beta \ominus \gamma} \xi_{\pm}(\beta, \gamma) \left( \kappa(\alpha, \beta) - \kappa(\alpha, \gamma) \right) \] (3.9)
where we assume that $\kappa(\alpha, \beta \ominus \gamma) = 0$ if $(\beta, \gamma) \not\in S^{(2)}_\ominus$.

Remark 3.6. Note that if $\kappa: S \times S \to k$ is symmetric or satisfies
\[ \kappa(\alpha, \beta \ominus \gamma) = \delta_{\beta \ominus \gamma} \left( \kappa(\alpha, \beta) + \kappa(\alpha, \gamma) \right) \] (3.10)
then equation (3.9) is automatically satisfied for any choice of $\xi_{\pm}$.

We shall regard the tuple $(S, \kappa, \xi_{\pm})$ as a generalization of the usual Cartan datum.

Definition 3.7. A Cartan semigroup is a tuple $S = (S, \kappa, \xi_{\pm})$, where $S$ is a semigroup, $\kappa: S \times S \to k$ is a function satisfying (3.6) and (3.10), and $\xi_{\pm}: S \times S \to k$ are two arbitrary functions. We denote by $\tilde{g}(S)$ the Lie algebra freely generated by $L^S$.

3.3. Semigroup Lie algebras. We denote by $\varphi_a$ the element of the standard basis of $Z^S$ for $a \in S$. Then, we set $Q^S = Z^S / \sim$, where $\sim$ is the relation $\varphi_{a \oplus \beta} = \delta_{a \oplus \beta}(\varphi_a + \varphi_\beta)$, and
\[ Q^S_\pm := \text{span}_{Z_{\delta}} \{ \varphi_a \mid a \in S \} \subset Q^S. \]
Set $Q^S_- := -Q^S_+$, so that $Q^S = Q^S_+ \oplus Q^S_-$. For $\lambda, \mu \in Q^S$, we say that $\mu \preceq \lambda$ if and only if $\lambda - \mu \in Q^S_+$. The following is standard.

Proposition 3.8.

1. As vector spaces, $\tilde{g}(S) = \tilde{n}_+ \oplus L^S_0 \oplus \tilde{n}_-$, where $\tilde{n}_\pm$ is the subalgebra generated by the elements $x^\pm_a$, $a \in S$. Moreover, $\tilde{n}_\pm$ is freely generated.
2. There is a natural $Q^S_-$–gradation on $\tilde{g}(S)$ given by $\deg(x^\pm_a) = \pm \varphi_a$ and $\deg(\tilde{\kappa}_a) = 0$. In particular,
\[ \tilde{g}(S) = \left( \bigoplus_{\mu \in Q^S_\pm \setminus \{ 0 \}} \tilde{g}_\mu \right) \oplus \left( \bigoplus_{\mu \in Q^S_\pm \setminus \{ 0 \}} \tilde{g}_\mu \right) \]
and $\tilde{g}_{\pm \mu} \subseteq \tilde{n}_\pm$.

Definition 3.9. The semigroup Lie algebra with Cartan datum $S$ is the Lie algebra
\[ g(S) := \tilde{g}(S, \kappa, \xi_{\pm}) / \tau, \]
where $\tau$ is the sum of all two–sided $Q^S_-$–graded ideals in $\tilde{g}(S)$ having trivial intersection with $L^S_0$.

In particular, it follows immediately from Proposition 3.8 that $g(S)$ inherits the triangular decomposition $g(S) = n_+ \oplus L^S_0 \oplus n_-$, where $n_\pm \subset g(S)$ denotes the subalgebra generated by $x^\pm_a$, $a \in S$, and the $Q^S_-$–gradation
\[ g(S) = \left( \bigoplus_{\mu \in Q^S_\pm \setminus \{ 0 \}} g_\mu \right) \oplus \left( \bigoplus_{\mu \in Q^S_\pm \setminus \{ 0 \}} g_\mu \right) \]
where $g_{\pm \mu} \subseteq n_\pm$.

Definition 3.10. We call root an element $\mu \in Q^S \setminus \{ 0 \}$ such that $g_\mu \neq 0$. We say that a root $\mu$ is positive (resp. negative) if $\mu > 0$ (resp. $\mu < 0$). The set of roots (resp. positive, negative roots) is denoted by $R^S$ (resp. $R^S_+, R^S_-$).

Remark 3.11. Let $S$ be a Cartan semigroup and let $f: S \to (k, +, -)$ be a homomorphism of partial semigroups. Then, we may define a gradation with respect to $f$ by setting
\[ \deg x^\pm_a := \pm f(a) \quad \text{and} \quad \deg \tilde{\kappa}_a := 0. \]
Remark 3.12. As expected, a semigroup Lie algebra is a special cases of a semi–local continuum Lie algebra as in Definition 3.3. Namely, let $S$ be a Cartan semigroup, $A_S = k[S]$ the algebra of regular functions over the set $S$, and $I_\alpha \in A_S$ the characteristic function of $\alpha \in S$. For any $\alpha, \beta \in S$, define

$$\kappa_{S,0}(1_\alpha, 1_\beta) = \delta_{\alpha, \beta}1_\alpha \quad \text{and} \quad \kappa_{S,\pm}(1_\alpha, 1_\beta) = \pm \kappa(\alpha, \beta)1_\beta$$

and

$$\xi_{S,0}(1_\alpha, 1_\beta) = \delta_{\alpha, \beta}1_{\alpha \ominus \beta} \quad \text{and} \quad \xi_{S,\pm}(1_\alpha, 1_\beta) = \pm \xi(\alpha, \beta)1_{\alpha \ominus \beta}$$

where we assume that $1_{\alpha \ominus \beta} = 0$ whenever $(\alpha, \beta) \notin S^{(2)}$, $\ominus = \oplus, \ominus$. Then, the assignment $x^\pm_\alpha \rightarrow X_\pm(1_\alpha)$ and $\zeta_\alpha \rightarrow X_0(1_\alpha)$ gives a Lie algebra isomorphism

$$g(S) \simeq \tilde{g}(A_S, \kappa_{S,\cdot}, \xi_{S,\cdot})/\bar{\rho}$$

where $\bar{\rho}$ is the sum of all two–sided graded ideals in $\tilde{g}(A_S, \kappa_{S,\cdot}, \xi_{S,\cdot})$ having trivial intersection with its Cartan subalgebra. In particular, $g(S)$ is a graded semi–local continuum Lie algebra. \(\triangle\)

3.4. Derived Kac–Moody algebras and semigroups. Derived Kac–Moody algebras are easily realized as degenerate examples of semigroup Lie algebras. We use the notation from Section 2. Let $S$ be the trivial semigroup with underlying set $\Pi = \{a_i | i \in I\}$ and $S^{(2)} = \emptyset = S^{(2)}$. Then, $S = (S, \kappa_A, 0)$, with $\kappa_A(a_i, a_j) = a_{ij}, \xi(i, j) = 0$, is a Cartan semigroup and the assignment

$$(x^+_\alpha, \xi_\alpha, x^-_\alpha) \mapsto (e_\mu, h_i, f_i)$$

defines a Lie algebra isomorphism $g(S) \simeq g(A)'$.

Symmetric Borcherds–Kac–Moody algebras can also be described in terms of a more interesting semigroup structure. More precisely, we show in Proposition 6.18 and Theorem 6.19 that Borcherds–Kac–Moody algebras corresponding to quivers with at most one loop on each vertex and at most two arrows between any two vertices, can be realised as Lie subalgebras of semigroup Lie algebras $g(S)$ for some non–trivial semigroups of topological origin.

4. SEMIGROUP SERRE RELATIONS

In this section, we study the necessary and sufficient conditions for the occurrence of distinguished quadratic relations in a semigroup Lie algebra $g(S)$. Such relations, which we refer to as semigroup Serre relations, are clearly inspired by the case of $\mathfrak{sl}(\mathbb{R})$ as described in Corollary 2.10.

4.1. Serre relations. In analogy with the case of Kac–Moody algebras and $\mathfrak{sl}(\mathbb{R})$, it is desirable to have in $g(S)$ certain quadratic Serre relations of the form

$$[x^\pm_\alpha, x^\pm_\beta] = \mu_{\pm}(\alpha, \beta) \cdot x^\pm_{\alpha \ominus \beta},$$

for some $\mu_{\pm} : S \times S \rightarrow k$. The next result describes the necessary and sufficient conditions for such relations to hold. To this end, we shall define recursively the set $S_{\leq a} \subseteq S$ of partitions of $\alpha \in S$ as follows. We set

$$S^{(0)}_{\leq a} := \{a\},$$

$$S^{(n)}_{\leq a} := \{\beta \ominus \gamma \mid \gamma \in S, \beta \in S^{(n-1)}\} \quad \text{for } n \geq 1,$$

and $S_{\leq a} := \bigcup_{n \geq 0} S^{(n)}_{\leq a}$. More precisely, $\alpha' \in S_{\leq a}$ if and only if there exist a sequence

$$\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1} = \alpha'$$
such that \((\alpha_i, \alpha_{i+1}) \in S^{(2)}_{\leq 1}\) for any \(1 \leq i \leq n\), so that
\[
a = (\beta_1 \oplus (\beta_2 \oplus \cdots \oplus (\beta_n \oplus a') \cdot \cdots )
\]
where \(\beta_i = \alpha_i \oplus \alpha_{i+1}\). We shall call such a sequence an **partition** of \(a\) at \(a'\) and we write \(a' \preceq a\) if \(a' \in S_{\leq a}\). Finally, we denote by \(S^{\pm}_{\leq a}\) the subset of all elements \(a'\) in \(S_{\leq a}\) for which there exists a partition \(a = a_1, a_2, \ldots, a_n, a_{n+1} = a'\) such that, if \(n > 1\) we have \(\xi^\pm(\alpha_i, \alpha_{i+1}) \neq 0\) for any \(1 \leq i \leq n\).

**Proposition 4.1.** Let \(\mu : S \times S \to k\) be two functions and \(a, b \in S, a \neq b\). Then,
\[
[x^\pm_a, x^\pm_b] = \mu_\pm(a, b) x^\pm_{a \oplus b}
\]
holds in \(g(S)\) if and only if the following relations hold.

1. For any \(a \in S^{\pm}_{\leq a}, b \in S^{\pm}_{\leq b}\),
\[
\xi_\pm(a \oplus b, a) = \mu_\pm(a, b) = -\xi_\pm(a \oplus b, b),
\]
\[
\xi_\pm(a, b) = \xi_\pm(a \oplus b, a) - (\delta_{a \oplus b} + \delta_{b \oplus a})\xi_\pm(b, a) - \xi_\pm(b, b) = \kappa(b, a).
\]
2. For any \(a \in S^{\pm}_{\leq a}, b \in S^{\pm}_{\leq b}, c \in S\),
\[
\delta_{c \ominus (a \oplus b)}\xi_\pm(a \oplus b, a)\xi_\pm(c, a \oplus b) = \\
\delta_{(c \ominus a) \ominus b}\xi_\pm(c, a)\xi_\pm(c \ominus a, b) - \delta_{(b \ominus c) \ominus a}\xi_\pm(c, b)\xi_\pm(c \ominus b, a).
\]
3. For any \(a \in S^{\pm}_{\leq a}, b \in S^{\pm}_{\leq b}, c \in S, c \neq a, b\),
\[
\xi_\pm(a, c)\xi_\pm((a \ominus c) \ominus b, b) - \xi_\pm(b, c)\xi_\pm(a \ominus (b \ominus c), a) = \\
\delta_{b \ominus (c \ominus a)}\xi_\pm(c, a)\xi_\pm(c \ominus a, b) - \delta_{c \ominus (b \ominus a)}\xi_\pm(b, c)\xi_\pm(b \ominus c, a) = \\
-\delta_{a \ominus (b \ominus c)}\xi_\pm(a \ominus b, a)\xi_\pm(a \ominus b, c).
\]

**Proof.** The result follows as a straightforward application of Lemma 3.4. Namely, let \(X_{\pm, a, b}\) be the element defined by the equation (4.1) for the pair \((a, b)\). We shall prove that \(X_{\pm, a, b}\) generates an ideal in \(\tilde{n}_\pm\), having trivial intersection with \(L_0^S\). Then, \(X_{\pm, a, b} = 0\) in \(g(S)\). By direct inspection, one sees easily that this holds if and only if the elements \(X_{\pm, a, b}\), for any \((a, b) \in S^{\pm}_{\leq a} \times S^{\pm}_{\leq b}\), are also added to the generating set of the ideal and the functions \(\kappa\) and \(\xi\) satisfy the relations listed above. In Appendix C, we carry out this computation in full details.

In Sections 4.2 and 4.3 below, we study two special cases for which the conditions (4.2)-(4.5) are particularly simple to describe.

### 4.2. Orthogonality
Recall that, in a Kac–Moody algebra, generators corresponding to **orthogonal vertices** in the Dynkin diagram (i.e., \(a_{ij} = 0\)) commute. In the case of **trivial** Cartan semigroup (i.e., with \(\bar{\xi}_\pm = 0\)), Proposition 4.1 reduces precisely to an analogue of the Serre relations for \(a_{ij} = 0\) in a Kac–Moody algebra.

**Corollary 4.2.** If the functions \(\xi\) are trivial, then
\[
[x^\pm_a, x^\pm_b] = 0 \quad \text{if and only if} \quad \kappa(a, b) = 0 = \kappa(b, a),
\]
where \(a, b \in S\).

More in general, Corollary 4.2 suggests to introduce a suitable notion of orthogonal subsemigroups, so that the corresponding generators automatically commute.
Definition 4.3. Let $S', S'' \subseteq S$ be two saturated\(^6\) sub–semigroups. We say that $S'$ and $S''$ are orthogonally, and we write $S' \perp S''$, if they satisfy the following property: for any $a \in S'$, $b \in S''$ and $c \in S$,

1. $a \oplus b$ is not defined;
2. $\kappa(a, b) = 0 = \kappa(b, a)$;
3. $\xi_{\pm}(c, b) = \xi_{\pm}(c \ominus a, b)$ and $\xi_{\pm}(b, c) = \xi_{\pm}(b \ominus c, a)$ whenever all terms are defined.

If $S' \perp S''$, $a \in S'$, and $b \in S''$, we also write $a \perp b$.

Remark 4.4.

(i) Note that if $a \perp b$, then the elements $a \ominus b$, $b \ominus a$ are never defined. Indeed, since $S'$ is saturated and $a \in S'$, then $a \ominus b$ should belong to $S'$. Therefore the sum $(a \ominus b) \oplus b = a$ would be defined, contradicting condition (1) above.

(ii) Note also that, by equation (3.6), $\kappa$ already satisfies the analogue of condition (3) above. Namely, if $a \perp b$, then it follows from condition (2) above that

$$
\kappa(c, a) = \kappa((c \ominus b) \ominus b, a) = \kappa(c \ominus b, a) + \kappa(b, a) = \kappa(c \ominus b, a),
$$

$$
\kappa(a, c) = \kappa(a, (c \ominus b) \ominus b) = \kappa(a, c \ominus b) + \kappa(b, a) = \kappa(a, c \ominus b),
$$

whenever $c \ominus b$ is defined.

This notion of orthogonality produces the desired outcome.

Corollary 4.5. Let $a, \beta \in S$ be such that $a \perp \beta$. Then $[x_a^+, x_{\beta}^+] = 0$.

Proof. Let $S', S''$ be the smallest orthogonal saturated sub–semigroups containing $a, \beta$, respectively. Note that $S_{\perp a}^+ \subseteq S'$ and $S_{\perp \beta}^+ \subseteq S''$. Thus, (4.2) and (4.3) are automatically satisfied in view of Remark 4.4–(i) and Definition 4.3–(1) and (2). The condition (4.4) follows immediately from Definition 4.3–(3). Finally, (4.5) follows from Remark 4.4–(i).

4.3. Degenerate elements and Serre relations. Another particularly simple case is given by elements which, roughly, do not interact with any other element.

Definition 4.6. We say that $a \in S$ is

- degenerate if, for any $\beta \in S$, one of the following holds:
  1. $a \oplus \beta$, $a \ominus \beta$, $\beta \ominus a$ are not defined;
  2. if (1) does not hold, then $\xi_{\pm}(a, \beta), \xi_{\pm}(\beta, a), \xi_{\pm}(a \ominus \beta, \beta), \xi_{\pm}(\beta, a \ominus \beta)$ vanish.

- locally degenerate if there exists a saturated sub–semigroup $S'$ such that $a$ is degenerate in $S'$.

We denote by $D(S)$ (resp. $D^{loc}(S)$) the subset of degenerate (resp. locally degenerate) elements in $S$.

Remark 4.7. Let $a \in D^{loc}(S)$ and $S'$ a saturated sub–semigroup such that $a \in D(S')$. Then, $S'_{\leq a} \subseteq S'$ and $S'_{\leq a} = \{a\}$.

The generators $x_{\pm a}^+$, for $a \in D(S)$, satisfy simple commutation relations in $g(S)$.

Proposition 4.8.

\(^6\)The notion of saturated sub–semigroup is given in Definition A.4.
(1) Let \( a \in D(S) \) and \( \beta \in S, \beta \neq a \). Then
\[
[x_a^+, x\beta^+] = 0
\]
if and only if \( \kappa(a, \gamma), \kappa(\gamma, a) \) vanish for any \( \gamma \in S_{<\beta}^+ \).

(2) For any \( a, \beta \in D(S), a \neq \beta, \) such that \( \kappa(a, a) = 2, \kappa(\alpha, \beta) \in \mathbb{Z}_{<0}, \) and \( \kappa(a, \beta) = 0 \) if and only if \( \kappa(\beta, a) = 0 \). Then,
\[
ad(x_a^+)^{1-\kappa(a, \beta)}(x\beta_+^+) = 0
\]
Proof. (1) We proceed as in Proposition 4.1. Let \( S_{\pm}^{\beta} \) be the subspace spanned by \( [x_a^+, x\beta^+] \) and all elements of the form \( [x_a^+, x\gamma_+^+] \) for some \( \gamma \in S \). Then, one readily sees that, since \( a \in D(S) \), the identities (C.1), (C.2) are trivial and, since \( \kappa(\alpha, \beta) = 0 = \kappa(\beta, a) \), (C.3) reduces to \( \zeta(\beta, \gamma)[x_a^+, x\beta_+^+] \). The result follows as usual from Lemma 3.4.

(2) The proof follows closely [Kac90, Section 3.3]. We first observe that \( (x_a^+, \zeta_+, x\beta^-) \) is an sl(2)–triple, since \( \kappa(a, a) = 2 \), which acts on \( g(S) \) by restriction.

Set \( v_k = (x_a^+)k \cdot x\beta^-_k, k \geq 0 \). Then, \( \zeta_k \cdot v_k = (x(\kappa(\alpha, \beta) - 2k)v_k, \) and
\[
x_a^+ \cdot v_k = k(-\kappa(\alpha, \beta) - k + 1)v_k.
\]
The last relation follows from the fact that \( a \in D(S) \) and therefore \( [x_a^+, x\beta^-] = 0 \).

Set \( \theta_{\alpha\beta} = (x_a^+)^{1-\kappa(a, \beta)} \cdot x\beta^- \). By (4.6), \( [x_a^+, \theta_{\alpha\beta}] = 0 \). Note that, for any \( k \geq 1 \), one has
\[
[x\beta^-_k, (x_a^+)^{k-1} \cdot x\beta^-] = (x_a^+)^k \cdot x\beta^-_k = (x_a^-)^k = \kappa(\beta, a)(x_a^-)^{k-1} \cdot x\beta^-.
\]
For \( k = 1 - \kappa(\alpha, \beta) \), this implies \([x\beta^-_k, \theta_{\alpha\beta}] = 0 \), since if \( \kappa(\alpha, \beta) = 0 \) then \( \kappa(\beta, a) = 0 \) and if \( \kappa(\alpha, \beta) \neq 0 \) then \( k > 1 \).

Finally, since \( a, \beta \in D(S) \), for any \( \gamma \neq a, \beta \), one has \([x\gamma^+, x\beta^-] = 0 = [x\gamma^+, x\beta^-] \) and therefore \([x\gamma^+, \theta_{\gamma\beta}] = 0 \). By Lemma 3.4, \( \theta_{\alpha\beta} = 0 \).

Proposition 4.8–(1) is easily generalized at the level of saturated sub–semigroups and we get the following corollary.

Corollary 4.9. Let \( a \) be degenerate in a saturated sub–semigroup \( S' \), for which \( \kappa(a, \gamma) = 0 = \kappa(\gamma, a) \) for any \( \gamma \in S' \). Then \([x\gamma^+, x\beta^-] = 0 \) for any \( \beta \in S_{<a} \).

4.4. Total semigroups. We conclude this section by showing that the situation becomes particularly simple for an actual semigroup with operations which are always defined. Let \( S = (S, \kappa, \zeta_\pm) \) be a Cartan semigroup satisfying the following additional conditions:

(1) \( S \) is total with respect to \( \oplus \), i.e., \( S^{(2)}_\oplus = S \times S \);

(2) \( S \) is total with respect to \( \triangledown \), i.e., for any \( a, \beta \in S, a \neq \beta, \) either \( (a, \beta) \) or \( (\beta, a) \) is in \( S^{(2)}_\triangledown \);

(3) \( \kappa = 0 \);

(4) \( \zeta_+ = \zeta_- = \zeta \) is antisymmetric and S-linear.

In this case, the quadratic Serre relations hold globally for every pair of elements in \( S \).

Proposition 4.10. For any \( a, \beta \in S \), it holds \( [x\gamma^+, x\beta^-] = \zeta(\beta, a) \cdot x_{\alpha \oplus \beta}^\pm \) in \( g(S) \).

Proof. The result follows from a straightforward application of Proposition 4.1, based on a case–by–case analysis. Note that there is a standard strict partial order \(< \) on \( S \) given by
\[
a < \beta \quad \text{if and only if} \quad \beta \ominus \alpha \in S,
\]
for \(a, \beta \in S\). First, note that equation (4.2) follows from the antisymmetry of \(\xi\), while, because of condition (2), equation (4.3) holds if and only if \(\kappa = 0\). Moreover, equation (4.4) is either trivial or all the terms are defined since:

\[
a \oplus b < c \iff a < c \text{ and } b < c \ominus a \iff b < c \text{ and } a < c \ominus b.
\]

In the latter case, a straightforward check show that equation (4.4) holds. Finally, we have to check that relation (4.5) holds in the following mutually exclusive cases:

- \(c < a \oplus b, c < a, \text{ and } c < b\);
- \(c < a \oplus b, c < a, \text{ and } b < c\);
- \(c < a \oplus b, a < c, \text{ and } b < c\);
- \(c < a \oplus b, c < b, \text{ and } a < c\);
- \(a \oplus b < c\).

Note that in the last case, both sides of (4.5) are zero. The other cases are proved with a simple direct inspection. \(\square\)

\section{Good Cartan Semigroups and Serre Relations}

In this section, we introduce the notion of good Cartan semigroup. The definition is tailored around the conditions highlighted by Proposition 4.1 and aims to provide a list of simple properties a Cartan semigroup \(S\) should satisfy so to induce Serre relations in the Lie algebra \(g(S)\).

\subsection{Good Cartan Semigroups}

Let \(S = (S, \kappa, \xi_{\pm})\) be a Cartan semigroup. Locally degenerate elements are semigroup analogues of imaginary roots. This leads to the following definition.

**Definition 5.1.** We say that \(a \in S\) is imaginary if there exists \(a' \in S_{\leq a}\) which is locally degenerate; while \(a\) is real if it is not imaginary. We denote by \(S^{im}\) (resp. \(S^{re}\)) the set of imaginary (resp. real) elements of \(S\).

**Remark 5.2.** Note that, if \(a \in S^{re}\), then \(S_{\leq a} \subseteq S^{re}\).

The following notion of good Cartan semigroup lists five crucial properties concerning the basic semigroup operations, the real elements and the functions \(\kappa\) and \(\xi_{\pm}\).

**Definition 5.3.** A Cartan semigroup \(S\) is good if the following conditions hold.

1. **Multiplicity free.** For any \(a, \beta \in S\), at most one between the elements \(a \oplus \beta, a \ominus \beta\), and \(\beta \ominus a\) is defined (\(S_{\oplus}^{(2)} \cap S_{\ominus}^{(2)} = \emptyset\)).
2. **Locality.** The following holds:
   (L1) if \(a \not\perp \beta\), then \((\gamma \ominus a) \ominus \beta\) is defined only if \(a \oplus \beta\) is defined;
   (L2) if \((a \ominus \beta) \ominus \gamma\) is defined and \(a \perp \gamma\), then \(\beta \ominus \gamma\) is defined.
3. **Real elements.** If \(a \in S^{re}\) and \(\gamma \ominus a\) is defined for some \(\gamma \notin S^{re}\), then there exists \(\gamma' \in S^{re}\) such that \(\gamma \ominus \gamma'\) is defined and \(a \perp (\gamma \ominus \gamma')\).
4. \(\xi_{+}(a, \beta) = \xi_{-}(\beta, a)\) and satisfies the following properties (\(\xi := \xi_{+}\)):
   - for any \(a, \beta \in S\),
   - \[
   \xi(a \oplus \beta, a) = -\xi(a \oplus \beta, \beta), \quad \xi(a, a \ominus \beta) = -\xi(\beta, a \ominus \beta);
   \]
   - for any \(a, \beta, \gamma \in S\) such that \(a \oplus \beta, a \ominus \gamma\) and \(\beta \ominus \gamma\) are defined;
that in the case of a good

Remark 5.4. Note that, by (L2) and (3), it follows that, if \( a \) is a real element in \( S_{\leq \gamma }^{(1)} \) with \( \gamma \not\in S^e \), then there exists a real element \( \gamma' \in S_{\leq \gamma}^{(1)} \) such that \( a \in S_{\leq \gamma'}^{(1)} \). \( \triangle \)

5.2. Admissible pairs and triples. We shall prove in Theorem 5.7 that in the case of a good Cartan semigroup certain semigroup Serre relations occur for suitable pairs of elements in \( S \). This leads to the following definitions of admissible pairs and triples.

Definition 5.5. Let \( S \) be a good Cartan semigroup.

(1) Let \( (a, \beta, \gamma) \in S \times S \times S \). The triple \((a, \beta, \gamma)\) is admissible if \((a, \beta, \gamma) \in S^e \times S^e \times S^e\) or \((a, \beta, \gamma) \in S^e \times S^m \times S^m\), with \( a \nmid \beta \), then

(a) either none or exactly two elements among \( \gamma \oplus (a \oplus \beta) \), \( (\gamma \ominus a) \ominus \beta \), and \( (\gamma \ominus \beta) \ominus a \) can be simultaneously defined;

(b) either none or exactly two elements among \( (a \ominus \beta) \ominus \gamma \), \( (a \ominus \gamma) \ominus \beta \), \( a \ominus (\beta \ominus \gamma) \), \( a \ominus (\gamma \ominus \beta) \), and \( \beta \ominus (\gamma \ominus a) \) are simultaneously defined.

(2) Let \((a, \beta)\) be an unordered pair of elements in \( S \) with \( a \in S^e \). We say that \((a, \beta)\) is a admissible pair if for any \( a \in S^e \), \( b \in S^m \), we have that

(a) for any \( c \) in \( S \), the triple \((a, b, c)\) is admissible;

(b) either \( (a, b) \not\in S_{\oplus}^{(2)} \) or \( a \ominus b \not\in D_{\text{loc}}(S) \).

We denote by \( \text{Serre}(S)_{\text{adm}} \) the set of admissible pairs.

Remark 5.6. Note that, if \((a, \beta) \in \text{Serre}(S)_{\text{adm}}\), then \( S_{\leq a}^e \times S_{\leq \beta}^e \subseteq \text{Serre}(S)_{\text{adm}} \). Moreover, if \( a, \beta \in S^e \), the element \( a \oplus \beta \) is necessarily real, whenever defined. Finally, if \((a, \beta) \in \text{Serre}(S)_{\text{adm}}\), then \((a, a \ominus \beta) \in \text{Serre}(S)_{\text{adm}}\) whenever \( a \ominus \beta \) is defined. \( \triangle \)

5.3. Serre relations. Let \( \text{Serre}(S) \) be the union of \( \text{Serre}(S)_{\text{adm}} \) and the set of orthogonal pairs.

Theorem 5.7. For any \((a, \beta) \in \text{Serre}(S)\), \( [x_{a}^{\pm}, x_{\beta}^{\pm}] = \xi_{\pm}(a \oplus \beta, a) \cdot x_{a \oplus \beta}^{\pm} \).
Proof. It is clear that, if \( \alpha \perp \beta \), then Corollary 4.5 implies the result. Therefore, we have to prove the Serre relations for any pair of elements in \( \text{Serre}(S)^{adm} \). As before, we proceed by showing that for any such pair the conditions (4.2), (4.3), (4.4), and (4.5) from Proposition 4.1 hold. The first two conditions follow immediately from the properties (5.1), (5.2), and (5.5) of good Cartan semigroups. This simple observation can also be thought of as the main motivation behind these properties. The proof of the remaining two conditions is longer and more technical, but it can be easily carried out on a case–by–case analysis, whose detailed computations motivate the other properties featured in Definition 5.3. The full proof is given in Appendix D. \( \square \)

From Corollary 4.5, Remark 5.6 and Theorem 5.7, we obtain the following result, which can be thought of as a semigroup analogue of the Serre relations for Borchers-Kac-Moody algebras in the case \( a_{ij} = -1,0 \) respectively (cf. Equations (2.1) and (2.2)).

Corollary 5.8. Let \( \alpha, \beta \in S \).

1. If \((\alpha, \beta) \in \text{Serre}(S)^{adm}, then [x^\pm_{\alpha}, [x^\pm_{\beta}, x^\pm_{\beta}]] = 0.
2. If \( \alpha \perp \beta \), then \([x^\pm_{\alpha}, x^\mp_{\beta}] = 0.\)

4.5. Locally nilpotent adjoint actions. We conclude this section by proving that in the case of a good Cartan semigroup the generators \( x^\pm_{\alpha} \) associated to real elements act locally nilpotently on \( \mathfrak{g}(S, \kappa, \xi) \).

Proposition 5.9. Let \( \alpha \in S^\kappa, \beta \in S \) be such that \((\alpha, \beta) \notin \text{Serre}(S)^{adm}, \alpha \perp \beta, \) and there exists a sequence of elements \( \beta_i, i = 1, \ldots, k, \) such that

1. \( \beta = (\cdots (\beta_1 \oplus \beta_2) \oplus \cdots \oplus \beta_k); \)
2. \((\cdots (\beta_1 \oplus \beta_2) \oplus \cdots \oplus \beta_k), \beta_{i+1}) \in \text{Serre}(S)^{adm} \) for any \( i = 1, \ldots, k-1; \)
3. either \( \alpha \perp \beta_i \) or \((\alpha, \beta_i) \in \text{Serre}(S)^{adm} \) for any \( i. \)

Then \( \text{ad}(x^\pm_{\alpha})^{k+1}(x^\pm_{\beta}) = 0. \)

Proof. By conditions (1) and (2), we can iteratively apply Theorem 5.7 to the elements \( \beta_1, \ldots, \beta_k, \) and get

\begin{equation}
\begin{aligned}
x^\pm_{\beta} = c \cdot [x^\pm_{\beta \oplus \beta_1}, x^\pm_{\beta_1}] = c \cdot [\cdots [x^\pm_{\beta \oplus \beta_1}, x^\pm_{\beta_2}], x^\pm_{\beta_2}, \ldots], x^\pm_{\beta_k}]
\end{aligned}
\end{equation}

for some constant \( c. \) Therefore,

\begin{equation}
\text{ad}(x^\pm_{\alpha})^{k+1}(x^\pm_{\beta}) = c \cdot \text{ad}(x^\pm_{\alpha})^{k+1}([x^\pm_{\beta \oplus \beta_1}, x^\pm_{\beta_1}]) = c \cdot \sum_{j=1}^{k+1} [\text{ad}(x^\pm_{\alpha})^{k+1-j}(x^\pm_{\beta \oplus \beta_1}), \text{ad}(x^\pm_{\alpha})^j(x^\pm_{\beta_1})]
\end{equation}

From Corollary 5.8, \( \text{ad}(x^\pm_{\alpha})^l(x^\pm_{\beta}) = 0 \) for \( l \geq 2. \) Thus,

\begin{equation}
\text{ad}(x^\pm_{\alpha})^{k+1}(x^\pm_{\beta}) = c \cdot [\text{ad}(x^\pm_{\alpha})^{k+1}(x^\pm_{\beta \oplus \beta_1}), x^\pm_{\beta_1}] + c \cdot [\text{ad}(x^\pm_{\alpha})^k(x^\pm_{\beta \oplus \beta_1}), \text{ad}(x^\pm_{\alpha})^j(x^\pm_{\beta_1})].
\end{equation}

Therefore, we are reduced to prove the statement for \( k = 2. \) In this case, we have

\begin{equation}
\text{ad}(x^\pm_{\alpha})^3(x^\pm_{\beta}) = c \cdot [\text{ad}(x^\pm_{\alpha})^2(x^\pm_{\beta \oplus \beta_1}), x^\pm_{\beta_1}] + c \cdot [\text{ad}(x^\pm_{\alpha})^3(x^\pm_{\beta_1}), \text{ad}(x^\pm_{\alpha})^2(x^\pm_{\beta_1})] = 0
\end{equation}

where the last equality follows directly from Corollary 5.8. \( \square \)

Corollary 5.10. Assume that every \( \beta \in S \) admits a decomposition as in Proposition 5.9. Then, every element \( x^\pm_{\alpha} \) with \( \alpha \in S^\kappa \) acts locally nilpotently on \( \mathfrak{g}(S, \kappa, \xi). \)

Proof. Note that, for any \( \alpha, \beta \in S, \) one has \( \text{ad}(x^\pm_{\alpha})^3(x^\pm_{\beta}) = 0 \) and \( \text{ad}(x^\pm_{\alpha})^2(\xi_{\beta}) = 0. \) Moreover, \( \text{ad}(x^\pm_{\alpha})^2(x^\pm_{\beta}) = 0 \) for \( \beta \neq \alpha, \) since the elements \( \alpha \ominus (\beta \ominus \alpha) \) and \( (\beta \ominus \alpha) \ominus \alpha \) are never defined, therefore \( \text{ad}(x^\pm_{\alpha})^2(\xi_{\beta \ominus \alpha}) = 0 = \text{ad}(x^\pm_{\alpha})^2(x^\pm_{\alpha \ominus \beta}). \) The result then follows from Proposition 5.9. \( \square \)
6. CONTINUUM KAC–MOODY ALGEBRAS

In this section, we introduce the notion of a continuum quiver as a distinguished semigroup associated to a one–dimensional real space. We prove that continuum quivers are examples of good Cartan semigroup. Relying on the results obtained in Section 5, the corresponding semigroup Lie algebras, which we refer to as continuum Kac–Moody algebras, are explicitly presented in terms of generators and relations and proved to be isomorphic to a colimit of symmetric Borcherds–Kac–Moody algebras.

6.1. Vertex spaces. We introduce the class of topological spaces we are mainly interested in. These can be thought of as smoothings of one–dimensional real CW complexes.

Definition 6.1. Let \( X \) be a Hausdorff topological space. We say that \( X \) is a vertex space if for any \( x \in X \), there exists a chart \( (U, A, \phi) \) around \( x \) such that

1. \( U \) is an open neighborhood of \( x \),
2. \( A = \{ A_i \} \) is a family of closed subsets \( A_i \subseteq U \) containing \( x \), such that \( U = \bigcup_i A_i \),
3. \( \phi = \{ \phi_i \} \) is a family of continuous maps \( \phi_i : A_i \to \mathbb{R} \) which are homeomorphisms onto open intervals of \( \mathbb{R} \), such that if the intersection between \( A_i \) and \( A_j \) strictly contains the point \( x \), then \( \phi_i |_{A_i \cap A_j} = \phi_j |_{A_i \cap A_j} \) and \( \phi_i |_{A_i \cap A_j} \) induces a homeomorphism between \( A_i \cap A_j \) and a closed interval of \( \mathbb{R} \).

We say that \( x \) is

- a regular point if the exists a chart such that \( A = \{ U \} \),
- a critical point if there exists a chart such that the boundary \( \partial(A_i \cap A_j) \) of \( A_i \cap A_j \), as a subset of \( U \), contains \( x \) for any \( i, j \).

Remark 6.2. Let \( x \) be a critical point with a chart \( (U, A, \phi) \) such that \( x \in \partial(A_i \cap A_j) \) for any \( i, j \). Then \( x \in \partial A_i \) for any \( i \).

Example 6.3. Simple examples of a vertex space, beyond \( \mathbb{R} \) and \( S^1 \), are given below:

\[
\begin{array}{c}
\vdots \\
\vdots
\end{array}
\]

6.2. Semigroup of intervals. We lift the notion of open–closed interval on \( \mathbb{R} \) to an arbitrary vertex space \( X \).

Definition 6.4. Let \( X \) be a vertex space and let \( \alpha \) be a subset of \( X \). We say that \( \alpha \) is an elementary interval if there exists a chart \( (U, A, \phi) \) for which \( \alpha \subseteq A_i \) for some \( i \) and \( \phi_i(\alpha) \) is a open-closed interval of \( \mathbb{R} \). A sequence of elementary intervals \( (\alpha_1, \ldots, \alpha_n) \), \( n > 0 \), is admissible if

(a) \( (\alpha_1 \cup \cdots \cup \alpha_i) \cap \alpha_{i+1} = \emptyset \) and \( (\alpha_1 \cup \cdots \cup \alpha_i) \cup \alpha_{i+1} \) is connected for any \( i = 1, \ldots, n - 1 \);
(b) for any \( i = 1, \ldots, n - 1 \), there exist \( x \in X \) and a chart \( (U, A, \phi) \) around \( x \) for which \( U \supseteq (\alpha_1 \cup \cdots \cup \alpha_i) \cup \alpha_{i+1} \) and \( ((\alpha_1 \cup \cdots \cup \alpha_i) \cup \alpha_{i+1}) \cap A_k \) is either empty or an elementary interval for any \( k \).
An interval of $X$ is a subset $a$ of the form $a_1 \cup \cdots \cup a_n$, where $(a_1, \ldots, a_n)$ is an admissible sequence of elementary intervals. We denote by $\text{Int}(X)$ the set of all intervals in $X$.

The set of intervals $\text{Int}(X)$ carries a natural semigroup structure induced by the local structure on $\mathbb{R}$ described in Section 2.3. For any $a, \beta \in \text{Int}(X)$, set

$$a \oplus \beta := \begin{cases} a \cup \beta & \text{if } a \cap \beta = \emptyset \text{ and } a \cup \beta \in \text{Int}(X), \\ \text{n.d.} & \text{otherwise}, \end{cases}$$

$$a \ominus \beta := \begin{cases} a \setminus \beta & \text{if } a \cap \beta = \beta \text{ and } a \setminus \beta \in \text{Int}(X), \\ \text{n.d.} & \text{otherwise}. \end{cases}$$

We refer to $\oplus$ and $\ominus$ as the sum and difference of intervals, respectively. The following is straightforward.

**Lemma 6.5.**

1. Every contractible interval is homeomorphic to a finite oriented tree such that any vertex is the target of at most one edge.
2. Every non–contractible interval is homeomorphic to an interval of the form

$$S^1 \oplus \bigoplus_{k=1}^N T_k := (\cdots (S^1 \oplus T_1) \oplus T_2) \cdots \oplus T_N)$$

for some pairwise disjoint contractible intervals $T_k$, with $N \geq 0$.

**Proof.** We assume for simplicity that $X$ is connected. Note that if $X$ has no critical points, $X$ reduces to either $\mathbb{R}$ or $S^1$. In the former case, every interval is elementary, while in the latter the only non–contractible interval is $S^1$ itself. Assume that there exists at least one critical point in $X$. If $a$ does not contain any critical point, it is elementary. If $a$ contains a critical point and it is contractible, then it must be a finite sum of elementary intervals, hence it corresponds to oriented tree such that any vertex is the target of at most one edge. On the other hand, if $a$ is not contractible, by definition, it must be either homeomorphic to $S^1$ or a sum of one copy of $S^1$ with at least one necessarily contractible interval. \hfill $\square$

### 6.3. Euler forms

We denote by $\text{fun}(X)$ the $\mathbb{Z}$-span of the characteristic functions $1_a$ for all interval $a$ of $X$. Note that $1_{a \oplus \beta} = 1_a + 1_{\beta}$ for a given $(a, \beta) \in \text{Int}(X)^{(2)}$. We call support of a function $f \in \text{fun}(X)$ the set $\text{supp}(f) := \{x \in X | f(x) \neq 0\}$. It is a disjoint union of finitely many intervals of $X$.

Define a bilinear form $\langle \cdot, \cdot \rangle$ on $\text{fun}(X)$ in the following way. Let $f, g \in \text{fun}(X)$, and assume that there exists a point $x$ with a chart $(U, A, \phi)$ for which the supports of $f$ and $g$ are contained in $A_i$ for some $i$, then we set

$$\langle f, g \rangle := \sum_{x \in A_i} f^-(x)(g^-(x) - g^+(x)).$$

Since we can always decompose an interval into a sum of elementary subintervals (and we can do similarly with supports of functions of $\text{fun}(X)$), we extend $\langle \cdot, \cdot \rangle$ with respect to $\oplus$ by imposing the condition that $\langle 1_a, 1_{\beta} \rangle = 0$ for two elementary intervals $a, \beta$ for which there does not exist a common $A_i$ containing both.

As a consequence of the definition, the bilinear form $\langle \cdot, \cdot \rangle$ is compatible with the concatenation of intervals, by Lemma 6.5, it is entirely determined by its values on contractible elements.

**Remark 6.6.** Thanks to Definition 6.4-(b), one can easily verify that

- if $\beta$ is a non–contractible sub–interval of $a$, then $\langle 1_a, 1_{\beta} \rangle = \langle 1_{a \ominus \beta}, 1_{\beta} \rangle$ whenever $a \ominus \beta$ is defined;
- if $(a, \beta) \notin \text{Int}(X)^{(2)}$ and $a \cap \beta = \emptyset$, then $\langle 1_a, 1_{\beta} \rangle = 0$. 
Remark 6.7. Let $\alpha$ be a contractible interval given as $\alpha = \alpha_1 \cup \cdots \cup \alpha_n$ for some admissible sequence $(\alpha_1, \ldots, \alpha_n)$ of elementary intervals. One has $\langle 1_{n}, 1_{n} \rangle = 1$, indeed if $n = 1$, this follows from equation (2.9). Assume that the result holds for all intervals given by admissible sequences consisting of $n - 1$ elementary intervals. Let us prove for $n$: we have

$$\langle 1_{n}, 1_{n} \rangle = \langle 1_{n_{1}} \cup \cdots \cup 1_{n_{k}}, 1_{n_{1}} \cup \cdots \cup 1_{n_{k}} \rangle = \langle 1_{n_{1}} \cup \cdots \cup 1_{n_{k}}, 1_{n_{k+1}} \rangle + \langle 1_{n_{k+1}}, 1_{n_{1}} \cup \cdots \cup 1_{n_{k}} \rangle + \langle 1_{n_{k+1}}, 1_{n_{k+1}} \rangle + \langle 1_{n_{k+1}}, 1_{n_{1}} \cup \cdots \cup 1_{n_{k}} \rangle.$$ 

Now, the first summand of the second formula is one by the inductive hypothesis. On the other hand, thanks to Definition 6.4-(b), we can reduce the computations of the other summands to the case of elementary intervals, and rely on equation (2.9). Hence, we get the assertion.

Let $\alpha$ be a non–contractible interval. Assume that $\alpha$ is of the form

$$S^1 \oplus \bigoplus_{k=1}^{N} T_k = (\cdots (S^1 \oplus T_1) \oplus T_2) \cdots \oplus T_N)$$

for some pairwise disjoint contractible intervals $T_k$. Then by the previous remark, we get

$$\langle 1_{n}, 1_{n} \rangle = \langle 1_{S^1}, 1_{S^1} \rangle + \sum_{k=1}^{N} \langle 1_{T_k}, 1_{T_k} \rangle + \sum_{k=1}^{N} \langle 1_{S^1}, 1_{T_k} \rangle + \sum_{k=1}^{N} \langle 1_{T_k}, 1_{S^1} \rangle = 0 ,$$

since $\langle 1_{S^1}, 1_{S^1} \rangle = 0$, the second summand equals $N$ by the previous computation; while, the computation of the last two summands can be obtained by reducing to the case of elementary intervals thanks to Definition 6.4-(b): we get $-N$ and zero, respectively. \(\triangle\)

Set $(f,g) := \langle f,g \rangle + \langle g,f \rangle$ for $f,g \in \text{fun}(X)$. It follows from an easy generalization of the computations carried out in Section 2.3 that, if $\alpha,\beta \in \text{Int}(X)$ are contractible, then

$$\langle 1_{\alpha}, 1_{\beta} \rangle = \begin{cases} 2 & \text{if } \alpha = \beta , \\ 1 & \text{if } (\alpha, \beta) \in \text{Int}(X)^{(2)}_{(2)} \text{ or } (\beta, \alpha) \in \text{Int}(X)^{(2)}_{(2)}, \\ 0 & \text{if } (\alpha, \beta) \notin \text{Int}(X)^{(2)}_{(2)} \text{ and } \alpha \cap \beta = \emptyset, \\ -1 & \text{if } (\alpha, \beta) \in \text{Int}(X)^{(2)}_{(2)} \text{ and } \alpha \oplus \beta \text{ is contractible}, \\ -2 & \text{if } (\alpha, \beta) \in \text{Int}(X)^{(2)}_{(2)} \text{ and } \alpha \oplus \beta \text{ is non–contractible}. \end{cases}$$

All other cases follow therein. Note in particular that, if $\alpha$ is non–contractible, $(1_{\alpha}, 1_{\alpha}) = 0$.

6.4. Continuum quivers. For any $\alpha,\beta \in \text{Int}(X)$, we set

$$\kappa_{X}(\alpha, \beta) := \langle 1_{\alpha}, 1_{\beta} \rangle \quad \text{and} \quad \xi_{X}(\alpha, \beta) := (\langle 1_{\alpha}, 1_{\beta} \rangle) \langle 1_{\alpha}, 1_{\beta} \rangle.$$ 

One checks immediately that $\kappa_{X}$ satisfies the conditions (3.6) and (3.10). Therefore, the datum $Q_{X} := (\text{Int}(X), \kappa_{X}, \xi_{X})$ is a Cartan semigroup, which we refer to as the continuum quiver of $X$.

Remark 6.8. Recall that, given a quiver $Q$ with adjacency matrix $B_{Q}$, its Cartan matrix is the symmetric matrix $A_{Q} = 2 \cdot \text{id} - B_{Q} - B_{Q}^{t}$. Analogously, we think of $\text{Int}(X)$ as a set of vertices and of $\kappa_{X}$ as a generalized Cartan matrix. \(\triangle\)

The description of locally degenerate, imaginary and real elements in $Q_{X}$ is easily obtained. Specifically, we have the following.

Lemma 6.9. Let $\alpha,\beta$ be two intervals.

1. $\alpha$ is a locally degenerate element if and only if it is homomorphic to $S^{1}$.

2. $\alpha$ is an imaginary (resp. real) element if and only if it is non–contractible (resp. contractible).

3. $\alpha$ and $\beta$ are perpendicular if and only if $(\alpha, \beta) \notin \text{Int}(X)^{(2)}_{(2)}$ and $\alpha \cap \beta = \emptyset$. 

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Remark 6.10. It follows immediately from Remark 6.7 that
\[ \kappa_X(\alpha, \alpha) = \begin{cases} 2 & \text{if } \alpha \text{ is real}, \\ 0 & \text{if } \alpha \text{ is imaginary}. \end{cases} \]

Continuum quivers provide a large class of examples of the theory developed in Sections 5.

Proposition 6.11. The continuum quiver $Q_X$ is a good Cartan semigroup.

Proof. We shall show that $Q_X$ satisfies the conditions (1)–(5) from Definition 5.3. We first observe that $\text{Int}(X)$ satisfies (1), i.e., at most one among $\alpha \oplus \beta$, $\alpha \ominus \beta$, and $\beta \ominus \alpha$ is defined. An easy check shows that the conditions (2) and (3) hold.

It remains to prove that the functions $\zeta_X$ and $\kappa_X$ satisfy (4) and (5). Note that $\kappa_X$ is symmetric and satisfies the condition (5.4) by definition. The proof of conditions (5.1), (5.2), (5.3), and (5.5), which is rather tedious and technical, is carried out in full details in Appendix E. \( \square \)

6.5. Continuum Kac–Moody algebras and Serre relations. We study in greater detail the case of semigroup Lie algebras associated with continuum quivers, providing an explicit description of an essential subset of Serre pairs.

Definition 6.12. Let $Q_X$ be a continuum quiver. The continuum Kac–Moody algebra of $X$ is the semigroup Lie algebra $g_X := g(Q_X)$.

Set $\text{Serre}(X) := \text{Serre}(Q_X)$. We provide a list of pairs in $\text{Serre}(X)$ by studying the admissibility conditions from Definition 5.5 in the context of continuum quivers. Note that, in the two simplest cases of the real line and the circle, one checks easily that

\[ \text{Serre}(\mathbb{R}) = \text{Int}(\mathbb{R}) \times \text{Int}(\mathbb{R}) \]
\[ \text{Serre}(S^1) = \text{Int}(S^1) \times \text{Int}(S^1) \setminus \{(\alpha, \beta) \mid \alpha, \beta \neq S^1, \alpha \oplus \beta = S^1\} \]

The general case is roughly the same, but it is necessary to exclude the appearance of $S^1$–partitions at the level of subintervals.

Proposition 6.13. Let $S_X$ be the set of unordered pairs of intervals $(\alpha, \beta)$ such that either $\alpha \perp \beta$ or $\alpha \not\perp \beta$, a contractible, and the following conditions hold:

1. For any subinterval $\alpha' \subseteq \alpha$ and $\beta' \subseteq \beta$ with $\kappa_X(\beta', \beta) \neq 0$ whenever $\beta' \neq \beta$, the element $\alpha' \oplus \beta'$ is either not defined or not homeomorphic to $S^1$;
2. If $\alpha$ is not elementary, $\alpha \cap \beta$ does not contain any critical point.

Then, $S_X \subseteq \text{Serre}(X)$.

Proof. By definition, if $\alpha \perp \beta$, then $(\alpha, \beta) \in \text{Serre}(X)$. Assume $\alpha \not\perp \beta$. Then, since locally degenerate elements are homeomorphic to copies of $S^1$ in $X$, condition (1) is equivalent to condition (2b) in Definition 5.5. Condition (2) is explained as follows. We consider the following configuration of disjoint intervals

```
\begin{tikzpicture}
    \draw[->, thick] (0,0) -- (1,1) node[pos=.5,above] {w} -- (2,2) node[pos=.5,above] {z} -- (2,0) node[pos=.5,below] {y} -- (1,1) -- (0,0);
    \draw[->, thick] (1,1) -- (2,0) node[pos=.5,above] {x} -- (1,1);
\end{tikzpicture}
```
Set $a := w \cup x \cup y \cup z$, $b := x$, and $c := w \cup x$. It is easy to check that the triple $(a, b, c)$ is not admissible, since it does not satisfy the property (1b) from Definition 5.5. In particular, $(a, b)$ is not an admissible pair. Condition (2) prevents this configuration to appear and implies that the pair $(a, \beta)$ satisfies the property (2a) from Definition 5.5. The result follows. □

Remark 6.14. We shall not need to prove that $\mathcal{S}_X = \text{Serre}(X)$. In fact, in the proof of Theorem 6.19, we show that the relations indexed by $\mathcal{S}_X$ generates the maximal ideal $r_X \subset \mathfrak{g}(Q_X)$. In particular, $\mathcal{S}_X$ contains enough information to recover $\text{Serre}(X)$ in any case.

Note that, if $(\alpha, \beta) \in \mathcal{S}_X$ and $\alpha \oplus \beta$ is defined, one has

$$\xi_X(\alpha \oplus \beta, \alpha) = (-1)^{(1_\alpha \oplus 1_\beta)} (1_\alpha \oplus 1_\beta, 1_\alpha) = (-1)^{1 + (1_\beta \cdot 1_\alpha)} (2 + (1_\beta \cdot 1_\alpha)) = (-1)^{(1_\alpha \cdot 1_\beta)},$$

since $\alpha$ is contractible and $(1_\beta, 1_\alpha) = -1$. Similarly, $\xi_X(\alpha, \alpha \oplus \beta) = (-1)^{(1_\beta \cdot 1_\alpha)}$. Therefore, from Corollary 4.5 and 4.9, and Theorem 5.7, we get the following.

Corollary 6.15. Let $\alpha, \beta$ be two intervals of $X$. The following relations hold in $g_X$.

(1) If $\alpha \perp \beta$ (i.e., $\alpha \cap \beta = \emptyset$ and $\alpha \oplus \beta$ is not defined), then $[x^+_\alpha, x^+_\beta] = 0$.

(2) More in general, if $(\alpha, \beta) \in \mathcal{S}_X$, then

$$[x^+_\alpha, x^+_\beta] = (-1)^{(1_\beta \cdot 1_\alpha)} x^+_\alpha \cdot 1_\beta,$$

$$[x^-_\alpha, x^-_\beta] = (-1)^{(1_\alpha \cdot 1_\beta)} x^-_\alpha \cdot 1_\beta.$$  

Remark 6.16.

(1) If $\beta \simeq S^1$ and $\alpha \subseteq \beta$, then $(\alpha, \beta) \in \mathcal{S}_X$. Hence, by (2) above $[x^+_\alpha, x^+_\beta] = 0$.

(2) For $X = \mathbb{R}$, the relations above coincide with Equations (2.17).

6.6. Borcherds–Kac–Moody subalgebras. Our main goal is to provide an explicit description of the maximal ideal $r_X \subset \mathfrak{g}(Q_X)$, thus providing a complete presentation by generators and relations of $g_X$. To this end, it will be crucial to consider symmetric Borcherds–Kac–Moody subalgebras in $g_X$ associated with certain finite configurations of disjoint intervals.

Definition 6.17. Let $J = \{a_k\}_k$ be a finite set of intervals $a_k \in \text{Int}(X)$. We say that $J$ is irreducible if the following conditions hold:

(1) every interval is elementary;

(2) given two intervals $\alpha, \beta \in J$, $\alpha \neq \beta$, one of the following mutually exclusive cases occurs:

(a) $\alpha \oplus \beta$ is defined;

(b) $\alpha \perp \beta$;

(c) $\alpha \simeq S^1$ and $\beta \subset \alpha$.

Assume henceforth that $J$ is an irreducible set of intervals. Let $A_J$ be the matrix given by the values of $\kappa_X$ on $J$, i.e.,

$$A_J = (\kappa_X(\alpha, \beta))_{\alpha, \beta \in J}.$$  

Note that the diagonal entries of $A_J$ are either 2 or 0, while off–diagonal the only possible entries are 0, −1, −2. Let $Q_J$ be the corresponding quiver with Cartan matrix $A_J$. Note that a contractible elementary interval in $J$ corresponds to a vertex of $Q_J$ without loops at it. For example, we obtain the following quivers.
Instead, an interval of $\mathcal{J}$ homeomorphic to $S^1$ corresponds in $Q$ to a vertex having exactly one loop at it, as in the following examples.

| Configuration of intervals | Borcherds–Cartan diagram |
|-----------------------------|--------------------------|
| $\alpha_1$ $\alpha_2$ $\alpha_3$ $\alpha_4$ | $\alpha_1$ $\alpha_2$ $\alpha_3$ $\alpha_4$ |
| $\alpha_1$ $\alpha_2$ | $\alpha_1$ $\alpha_2$ |
| $\alpha_1$ $\alpha_3$ $\alpha_4$ $\alpha_5$ $\alpha_6$ | $\alpha_1$ $\alpha_2$ $\alpha_3$ $\alpha_4$ $\alpha_5$ $\alpha_6$ |

$(\alpha_3$ is a full circle)
We shall consider two Lie algebras associated to \( \mathcal{J} \). First, let \( g(\mathcal{J}) \) be the Lie subalgebra of \( g_X \) generated by the elements \( x_\alpha^+ \) and \( \zeta_\alpha \) with \( \alpha \in \mathcal{J} \). Then, let \( g_{Q,\mathcal{J}} \) be the symmetric Borcherds–Kac–Moody algebra of \( Q,\mathcal{J} \) (i.e., the derived Lie algebra of \( g(A,\mathcal{J}) \) — cf. Section 2).

**Proposition 6.18.** The assignment

\[
e_\alpha \mapsto x_\alpha^+ \quad \text{and} \quad f_\alpha \mapsto x_\alpha^- \quad \text{and} \quad h_\alpha \mapsto \zeta_\alpha
\]

for any \( \alpha \in \mathcal{J} \), defines a surjective homomorphism of Lie algebras \( \Phi_{\mathcal{J}} : g_{Q,\mathcal{J}} \to g(\mathcal{J}) \).

**Proof.** It is enough to show that \( \Phi_{\mathcal{J}} \) is a Lie algebra map. The surjectivity is clear. Recall that \( g_{Q,\mathcal{J}} \) is generated by \( \{ e_\alpha, h_\alpha, f_\alpha \mid \alpha \in \mathcal{J} \} \) with the following defining relations:

\[
\begin{align*}
[h_\alpha, e_\beta] &= \kappa_X(\alpha, \beta) e_\beta, \\
[h_\alpha, f_\beta] &= -\kappa_X(\alpha, \beta) e_\beta, \\
[e_\alpha, f_\beta] &= \delta_{\alpha\beta} h_\alpha, \\
\text{ad}(e_\alpha)^{-(\alpha, \beta)}(e_\beta) &= 0 = \text{ad}(f_\alpha)^{-(\alpha, \beta)}(f_\beta) \quad \text{if} \quad \alpha \nmid S^1, \\
[e_\alpha, e_\beta] &= 0 = [f_\alpha, f_\beta] \quad \text{if} \quad \alpha \simeq S^1 \quad \text{and} \quad (\alpha, \beta) = 0.
\end{align*}
\]

Note that, for any \( \alpha, \beta \in \mathcal{J} \), their difference \( \alpha \ominus \beta \) is defined only in the case (c), thus necessarily \( \kappa_X(\alpha, \beta) = 0 \). Therefore, the relation (6.1) is easily seen to be satisfied in \( g(\mathcal{J}) \). We shall prove that the (standard) Serre relations (6.2) and (6.3) hold in \( g(\mathcal{J}) \).

Let \( \alpha, \beta \in \mathcal{J} \), \( \alpha \nmid \beta \). First, note that, by construction, \( (\alpha, \beta) = 0 \) if and only if we are either in case (b), *i.e.*, \( \alpha, \beta \) are perpendicular, or in case (c), *i.e.*, \( \alpha \simeq S^1 \) and \( \beta \subset S^1 \). Thus, by Corollary 6.15–(1) and (2) (and Remark 6.16–(1)), \( [x_\alpha^+, x_\beta^-] = 0 \). Therefore, relations (6.2) (for \( (\alpha, \beta) = 0 \)) and equation (6.3) hold.

Assume now that \( (\alpha, \beta) \neq 0 \) (*i.e.*, \( (\alpha, \beta) = -1, -2 \)) and \( \alpha \oplus \beta \) is defined. Then, necessarily, one of the following occurs:

1. \( (\alpha, \beta) = -1 \) and \( (\alpha, \beta) = S_X \);  
2. \( (\alpha, \beta) = -2 \) and \( \alpha \oplus \beta \simeq S^1 \).

In case (1), it follows from Corollary 5.8 that \( [x_\alpha^+, [x_\alpha^+, x_\beta^-]] = 0 \) (we assume \( \alpha \) is contractible) and therefore the Serre relation (6.2) with \( (\alpha, \beta) = -1 \) is satisfied. In case (2), it is easy to see that we are in the case described by Proposition 5.9 with \( k = 2 \). Therefore,

\[
[x_\alpha^+, [x_\alpha^+, x_\beta^-]] = 0,
\]

and the Serre relation (6.2) with \( (\alpha, \beta) = -2 \) is satisfied. The result follows.

**6.7. A presentation by generators and relations.** We now prove the main result of this section, providing a presentation by generators and relations of continuum Kac–Moody algebras. For simplicity, we set \( (\alpha, \beta) := (1_\alpha, 1_\beta) \) and \( (\alpha, \beta) := (1_\alpha, 1_\beta) \). Recall also that \( \alpha \perp \beta \) if \( \alpha \cap \beta = \emptyset \) and \( \alpha \oplus \beta \) is not defined.

**Theorem 6.19.** The continuum Kac–Moody algebra \( g_X \) is generated by the elements \( x_\alpha^+ \) and \( \zeta_\alpha \) for \( \alpha \in \text{Int}(X) \), subject to the following defining relations:
(1) for any $\alpha, \beta \in \text{Int}(X)$ such that $\alpha \oplus \beta$ is defined, 
$$\zeta_{\alpha \oplus \beta} = \zeta_\alpha + \zeta_\beta;$$

(2) for any $\alpha, \beta \in \text{Int}(X)$,
$$[\zeta_\alpha, \zeta_\beta] = 0,$$
$$[\zeta_\alpha, x^+_\beta] = \pm (\alpha, \beta) x^+_\beta,$$
$$[x^+_\alpha, x^-_\beta] = \delta_{\alpha, \beta} \zeta_\alpha + (\alpha, \beta) \left( x^+_{\alpha \oplus \beta} - x^-_{\beta \ominus \alpha} \right);$$

(3) if $(\alpha, \beta) \in \mathcal{S}_X$, then
$$[x^+_{\alpha}, x^-_\beta] = (-1)^{\beta, \alpha} x^+_{\alpha \oplus \beta},$$
$$[x^-_\alpha, x^+_\beta] = (-1)^{\alpha, \beta} x^-_{\alpha \oplus \beta}. \quad (6.4)$$

Proof. Recall that, by definition, $\mathfrak{g}_X := \mathfrak{g}_X / \tau_X$, where $\mathfrak{g}_X$ is the Lie algebra generated by $x^+_{\alpha}$ and $\zeta_\alpha, \alpha \in \text{Int}(X)$, with relations (1) and (2), and $\tau_X \subseteq \mathfrak{g}_X$ is the sum of all two-sided graded ideals with trivial intersection with the commutative subalgebra generated by $\zeta_\alpha$ with $\alpha \in \text{Int}(X)$. Let $\tau_X \subseteq \mathfrak{g}_X$ be the ideal generated by relations (6.4). By Corollary 6.15, we know that $\tau_X \subseteq \tau_X$. Thus we have to prove that $\tau_X = \tau_X$. We obtain this identity as a consequence of the Gabber–Kac theorem for Borcherds–Kac–Moody algebras [GK81, Bor88].

Set $\hat{\mathfrak{g}}(X) := \mathfrak{g}_X / \tau_X$. Let $\pi: \mathfrak{g}_X \to \hat{\mathfrak{g}}(X)$ be the natural projection and assume that there exists $v \in \pi(\tau_X)$ with $v \neq 0$. Let $\mathcal{J}(v)$ be any finite set of intervals such that $v$ belongs to $\hat{\mathfrak{g}}(\mathcal{J}(v))$, where $\hat{\mathfrak{g}}(\mathcal{J}(v)) \subseteq \hat{\mathfrak{g}}(X)$ is the Lie subalgebra generated by the elements $x^+_{\alpha}$ and $\zeta_\alpha$ with $\alpha \in \mathcal{J}(v)$. Using relations (6.4), we can always assume that $\mathcal{J}(v)$ is an irreducible set of intervals (cf. Section 6.6). Moreover, since the result of Proposition 6.18 relies exclusively on the relation (6.4), we can conclude that the homomorphism $\Phi_{\mathcal{J}(v)}$ factors through $\hat{\mathfrak{g}}(\mathcal{J}(v))$, i.e., there exists a surjective homomorphism $\hat{\Phi}_{\mathcal{J}(v)}: \mathfrak{g}_{\mathcal{J}(v)} \to \hat{\mathfrak{g}}(\mathcal{J}(v))$ such that $\Phi_{\mathcal{J}(v)} = \hat{\pi} \circ \hat{\Phi}_{\mathcal{J}(v)}$, where $\hat{\pi}: \hat{\mathfrak{g}}(\mathcal{J}(v)) \to \mathfrak{g}(\mathcal{J}(v))$ is the canonical projection. Since $v \in \pi(\tau_X)$ and $\hat{\Phi}_{\mathcal{J}(v)}$ is the identity on the Cartan subalgebras, $(\Phi_{\mathcal{J}(v)})^{-1}(v)$ generates a two-sided ideal which trivially intersect the Cartan subalgebra in $\mathfrak{g}_{\mathcal{J}(v)}$. By [Bor88, Corollary 2.6], this is necessarily trivial. Therefore, $v = 0$ and $\tau_X = \tau_X$. The result follows.

Remark 6.20. It follows from the proof above that the morphism $\Phi_{\mathcal{J}}: \mathfrak{g}_{\mathcal{J}} \to \mathfrak{g}(\mathcal{J})$ from Proposition 6.18 is an isomorphism. \hfill $\square$

6.8. Finite quivers and colimit structure. The proof of Theorem 6.19 crucially relies on the fact that $\mathfrak{g}_X$ can be covered by symmetric Borcherds–Kac–Moody algebras. We describe this relation in greater details.

Lemma 6.21. Let $\mathcal{J}, \mathcal{J}'$ be two irreducible finite sets of intervals in $X$.

(1) If $\mathcal{J}' \subseteq \mathcal{J}$, there is a canonical embedding $\phi_{\mathcal{J}' \to \mathcal{J}}: \mathfrak{g}(\mathcal{J}') \to \mathfrak{g}(\mathcal{J})$ sending $x^+_{\alpha} \mapsto x^+_{\alpha}$ and $\zeta_\alpha \mapsto \zeta_\alpha$ for $\alpha \in \mathcal{J}'$.

(2) If $\mathcal{J}$ is obtained from $\mathcal{J}'$ by replacing an element $\alpha_s \in \mathcal{J}'$ with two intervals $\alpha_1, \alpha_2$ such that $\alpha_s = \alpha_1 \oplus \alpha_2$, there is a canonical embedding $\phi_{\mathcal{J}' \to \mathcal{J}}: \mathfrak{g}(\mathcal{J}') \to \mathfrak{g}(\mathcal{J})$, which sends $x^+_{\alpha} \mapsto x^+_{\alpha}$ and $\zeta_\alpha \mapsto \zeta_\alpha$ for $\alpha \in \mathcal{J}' \setminus \{\alpha_s\}$, $\zeta_{\alpha_1} \to \zeta_{\alpha_1} + \zeta_{\alpha_2}$, $\zeta_{\alpha_2} \to (-1)^{\{\alpha_1, \alpha_2\}} [x^+_{\alpha_1}, x^+_{\alpha_2}]$, $x^-_{\alpha} \to (-1)^{\{\alpha_1, \alpha_2\}} [x^-_{\alpha_1}, x^-_{\alpha_2}]$.

Proof. (1) is clear. (2) is a straightforward consequence of Theorem 6.19. \hfill $\square$
Definition 6.22. Let $Q$ be a finite quiver. We say that $Q$ has shape $X$ if there exists an irreducible set of intervals $J$ such that $Q = Q_J$. We denote by $Sh(X)$ the set of all such pairs $(Q, J)$.

Thus, for any $(Q, J) ∈ Sh(X)$, there is a canonical embedding $ψ_J : g_Q → g_X$ which factors through the isomorphism $Φ_J : g_Q → g(J)$ given by Proposition 6.18. We also obtain analogous injective morphisms

$$ψ_{Q,Q'} : Φ_{Q,J}^{-1} ∘ Φ_{J,J'} ∘ Φ_{J',Q'}$$

and

$$ψ''_{Q,Q'} := Φ_{J,J'}^{-1} ∘ Φ_{Q,J} ∘ Φ_{J',Q'}.$$

Since the morphisms $ψ_{Q,Q'}, ψ''_{Q,Q'}$ clearly form a direct system and every element in $g_X$ is contained in the image of $ψ_J$ for some $J$, we obtain the following colimit realization of the continuum Kac–Moody algebra $g_X$.

Theorem 6.23. The morphisms $ψ_J$ induce a canonical isomorphism of Lie algebras

$$ψ_X : \text{colim}_{(Q,J) ∈ Sh(X)} g_Q → g_X.$$

Example 6.24. In the case of $X = R, S^1$, we compare our construction with the Lie algebras appearing in [SS19a], i.e., the Lie algebras $sl(R)$ and $sl(S^1)$ underlying the quantum group $U_q(sl(S^1))$.

If $X = R$, one checks easily that $g_R$ coincides with the Lie algebra $sl(R)$ introduced in Section 2.3. Consider now the case $X = S^1$. By Theorem 6.19, one checks immediately that the two Lie algebras do not coincide, but their difference is reduced to the elements $x^+_{S^1}$. More precisely, let $Ω_{S^1}$ be the subalgebra in $g_{S^1}$ generated by the elements $x^+_{S^1}$ and $ξ_{S^1} ∈ S^1$. Note that the elements $x^+_{S^1}$ and $ξ_{S^1}$ generate a Heisenberg Lie algebra his of order one (cf. [Kac90, Section 2.8]). It is then clear that $g_{S^1} = Ω_{S^1} ⊕ his$ and there is a canonical embedding $sl(S^1) → Ω_{S^1}$, whose image is $Ω_{S^1} ⊕ k · ξ_{S^1}$.

6.9. Continuum Kac–Moody algebras with marked support. We conclude this section by mentioning that Theorem 6.19 can be easily extended to the case of a vertex space with (possibly infinitely many) marked points. This shall be thought of as a generalization of the case of intervals with integral or rational boundaries.

Let $Y \subseteq X$ be a subset. We say that an interval $α$ is supported on $Y$ if $dα := π \setminus α^o \subseteq Y$. We denote by Int$_Y(X)$ the set of all intervals in $X$ supported on $Y$. It is clear that Int$_Y(X)$ is a subsemigroup of Int$(X)$ and the restrictions of $κ_X$ and $ξ_X$ to it gives rise to a good Cartan semigroup $Q_{X,Y} := (\text{Int}_Y(X), κ_X, ξ_X)$. Finally, the continuum Kac–Moody algebra of $X$ supported on $Y$ is the semigroup Lie algebra $g_{X,Y} := g(Q_{X,Y})$.

One checks easily that an analogue of Theorem 6.19 holds for $g_{X,Y}$, which can therefore be presented in terms of explicit Serre relations involving only intervals supported on $Y$. In particular, $g_{X,Y}$ can be thought of as a subalgebra of $g_X$ and, for $Y' \subseteq Y$, one has $g_{X,Y'} \subseteq g_{X,Y}$.

Example 6.25. For $X = R$ and $Y = Z, Q$, we obtain the sets of intervals with integral and rational boundaries defined in Remark 2.11. More generally, we can consider integral and rational points in an arbitrary vertex space $X$. Namely, let $K = Z, Q$. We say that a point $x ∈ X$ is a $K$-point if there exists a chart $(U, A, ϕ)$ around $x$ such that $ϕ_i(x) ∈ K$ holds for any $i$. We denote the subsets of integral and rational points by $X_Z$ and $X_Q$, respectively. In analogy with the case of $sl(Z)$ and $sl(Q)$, one can consider $g_{X,X_Z}$ and $g_{X,X_Q}$. Note that, in contrast with the case of $g_X$, these Lie algebras have countable dimensions.

Example 6.26. Let $x ∈ X$. Clearly, a loop centered at $x$ exists if and only if $x$ is contained in the image of a circle in $X$. In this case, it follows that $g_{X,\{x\}} ≃ his$. Conversely, if $x$ is not contained in a circle, $g_{X,\{x\}} = \{0\}$.

Example 6.27. The finite supports provide an alternative approach to the symmetric Borcherds–Kac–Moody subalgebras. Namely, let $(Q, J) ∈ Sh(X)$ such that $⋃_{α ∈ J} α$ is connected and set $\partial J := ⋃_{α ∈ J} \partial α$. Then, we have $g_Q ≃ g(J) = g_{X,α,J}$ and finally

$$g_X = \text{colim}_{Y < X, Y \subseteq X} g_{X,Y}.$$
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are instrumental to the notion of admissible pair from Definition 5.5.

A.1. Basic definitions.

Definition A.1. A partial semigroup is a tuple \((S, S^{(2)}_\sigma), \sigma\)
where \(S\) is a set, \(S^{(2)}_\sigma \subseteq S \times S\) a subset,
and \(\sigma: S^{(2)}_\sigma \to S\) a map such that, for any \(a, \beta, \gamma \in S\),
\[
\sigma(\sigma(a, \beta), \gamma) = \sigma(a, \sigma(\beta, \gamma))
\]
when both sides are defined, that is if the pairs \((a, \beta), (\sigma(a, \beta), \gamma), (\beta, \gamma), (a, \sigma(\beta, \gamma))\)
belong to \(S^{(2)}_\sigma\). We say that a partial semigroup is total
if \(S^{(2)}_\sigma = S \times S\).

A partial semigroup is commutative if \(S^{(2)}_\sigma\) is symmetric, i.e., \((a, \beta) \in S^{(2)}_\sigma\)
if and only if \((\beta, a) \in S^{(2)}_\sigma\), in which case \(\sigma(a, \beta) = \sigma(\beta, a)\).

Example A.2. Let
\[
Q_+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} a_i \subseteq \mathfrak{h}^*,
\]
be the root lattice of a Kac–Moody algebra \(g\) and \(R_+ := \{a \in Q_+ \setminus \{0\} \mid g_a \neq 0\}\)
the set of positive roots (cf. Section 2.1). Then, \(R_+\) is naturally endowed with a structure of partial semigroup \(\sigma\),
induced by \(Q_+\). That is, for any \(a, \beta \in R_+\), we set
\[
\sigma(a, \beta) = \begin{cases} 
\alpha + \beta & \text{if } \alpha + \beta \in R_+ \\
\text{n.d.} & \text{otherwise}
\end{cases}
\]

Remark A.3. It is common in the literature (see e.g., [Evs84]) to assume that the semigroup law \(\sigma\)
is strongly associative, i.e., for any \(a, \beta, \gamma \in S\),
\[
(a, \beta), (\sigma(a, \beta), \gamma) \in S^{(2)}_\sigma\quad \text{if and only if} \quad (\beta, \gamma), (a, \sigma(\beta, \gamma)) \in S^{(2)}_\sigma.
\]
This definition is stronger than the one given above, and is not suited for our purposes, since it
does not hold for root systems. For instance, in the root system of \(sl(4)\), \(a_2 + a_1\) and \((a_2 + a_1) + a_3\)
are defined, but clearly \(a_1 + a_3\) is not.

We briefly recall the straightforward notions of a morphism of partial semigroups and of a
partial subsemigroup. Let \((S, \sigma), (T, \tau)\) be partial semigroups. A morphism \(\phi: (S, \sigma) \to (T, \tau)\) is a
map such that \((a, \beta) \in S^{(2)}_\sigma\) if and only if \((\phi(a), \phi(\beta)) \in T^{(2)}_\tau\), and \(\phi(\sigma(a, \beta)) = \tau(\phi(a), \phi(\beta))\)
for any \((a, \beta) \in S^{(2)}_\sigma\).

Any subset \(S' \subseteq S\) inherits a partial semigroup structure. Namely, we denote by \(t(S')\) the
semigroup with underlying set \(S'\),
\[
t(S')^{(2)} := \{(a, \beta) \in S' \times S' \mid (a, \beta) \in S^{(2)}_\sigma\ \text{and} \ \sigma(a, \beta) \in S'\},
\]
and semigroup law induced by that of \(S\). Note that a priori \(t(S')^{(2)}_\sigma \subseteq (S' \times S') \cap S^{(2)}_\sigma\). The
corresponding embedding \(t(S') \to S\) is a morphism of semigroups if and only if \(S'\) is a subsemigroup of \(S\), i.e.,
if \((a, \beta) \in (S' \times S') \cap S^{(2)}_\sigma\) implies \(\sigma(a, \beta) \in S'\) (which means exactly \(t(S')^{(2)}_\sigma = (S' \times S') \cap S^{(2)}_\sigma\)).
Finally, for any \( a \in S \), set

\[
S^{(2)}_{\nu,a} := \{ (\beta, \gamma) \in S^{(2)} | \sigma(\beta, \gamma) = a \}.
\]

**Definition A.4.** We say that a subset \( S' \subseteq S \) is **saturated** if \( S^{(2)}_{\nu,a} \subseteq S' \times S' \) for any \( a \in S' \).

**A.2. Partial semigroups with cancellation laws.**

**Definition A.5.** A partial semigroup \((S, \sigma)\) is **(right) cancellative** if, for any two pairs \((a, \beta), (a', \beta) \in S^{(2)}_\sigma\),

\[
\sigma(a, \beta) = \sigma(a', \beta) \implies a = a'.
\]

Then, a **(right) partial cancellation law** on \( S \) is a partial map \( \nu : S \times S \to S \) defined on a (possibly empty) subset \( S^{(2)}_\nu \subseteq S \times S \) such that

- for any \((a, \beta) \in S^{(2)}_\nu\), then, whenever defined, \( \nu(\sigma(a, \beta), \gamma) = a \), and \( \nu(a, \nu(\sigma(a, \beta))) = \beta \);
- for any \( a, \beta, \gamma \in S \), then

\[
\begin{align*}
\nu(\sigma(a, \beta), \gamma) &= \sigma(a, \nu(\beta, \gamma)), \\
\sigma(\nu(a, \beta), \gamma) &= \nu(a, \nu(\beta, \gamma)), \\
\nu(a, \nu(\beta, \gamma)) &= \nu(\nu(a, \beta), \gamma),
\end{align*}
\]

(A.1)

when both sides are defined.

We say that \( \nu \) is **strong** if, for any \( a, \beta, \gamma \in S \), \( \sigma(a, \beta) = \gamma \) if and only if \( a = \nu(\gamma, \beta) \).

In particular, if \( \nu \) is strong, one has \( \nu(\sigma(a, \beta), \beta) = a \) for any \( a, \beta \in S \). Left partial cancellation laws are similarly defined.

**Remark A.6.** Note that every cancellative partial semigroup is endowed with a standard cancellation law. Namely, let \((S, \sigma)\) be a right cancellative partial semigroup. Then, for any \( a, \beta \in S \), there is at most one element \( \gamma \in S \) such that \((\gamma, \beta) \in S^{(2)}_\sigma\) and \( \sigma(\gamma, \beta) = a \), i.e., \(|S^{(2)}_\sigma \cap S \times \{\beta\}| \leq 1 \). Set

\[
S^{(2)}_\nu := \{ (a, \beta) \in S \times S \mid |S^{(2)}_\nu \cap S \times \{\beta\}| = 1 \}.
\]

Then the partial map \( \nu : S^{(2)}_\nu \subseteq S \times S \to S \) defined by \( \nu(\beta, \gamma) = \gamma \), where \( \gamma \) is the only element in \( S^{(2)}_\nu \cap S \times \{\beta\} \), is a right partial cancellation law.

**A.3. Commutative partial semigroups.** Let \((S, \oplus)\) be a commutative partial semigroup with a maximal cancellation law \( \ominus : S \times S \to S \). To alleviate the notation, for any \( a, \beta \in S \), we write \( a \oplus \beta \) and \( a \ominus \beta \) in place of \( \ominus(\alpha, \beta) \) and \( \ominus(\alpha, \beta) \).

**Definition A.7.** An element \( 0 \in S \) is a **partial zero** if, whenever defined, \( a \oplus 0 = a = 0 \oplus a \) for any \( a \in S \). We denote by \( Z(S) \) the set of partial zeros in \( S \). We say that \((S, \oplus, \ominus)\) is **positive** if the following holds:

1. \( Z(S) = \emptyset \);
2. the elements \( a \ominus a \) and \( (a \ominus \beta) \ominus a \) are never defined;
3. the elements \( a \ominus \beta \) and \( \beta \ominus a \) are never simultaneously defined;
4. up to commutation, \( \ominus \) is strongly associative, i.e., \( (a \ominus \beta) \ominus \gamma \) is defined if and only if either \( a \ominus (\beta \ominus \gamma) \) or \( \beta \ominus (a \ominus \gamma) \) is defined.

We shall make use of the following elementary lemma.

**Lemma A.8.** Assume that \((S, \oplus, \ominus)\) is positive. Then

1. the elements \( (\gamma \ominus \beta) \ominus a, \gamma \ominus (a \ominus \beta), (\gamma \ominus a) \ominus \beta \) are either all of them not defined or at least two of them are simultaneously defined;
(2) at most three of the elements \( a \oplus (\gamma \ominus \beta), \beta \ominus (\gamma \ominus a), (a \ominus \gamma) \oplus \beta, a \ominus (\beta \ominus \gamma), (a \ominus \beta) \ominus \gamma \) are simultaneously defined.

**Proof.** By strong associativity and thanks to Formula (A.1), the existence of an element implies that of at least another one. Namely, set \( x = (\gamma \ominus \beta) \ominus a \). Then, \( \gamma = (x \ominus a) \ominus \beta \) and therefore either \( \gamma = x \ominus (a \ominus \beta) \) or \( \gamma = (x \oplus \beta) \ominus a \). In the first case, we get \( \gamma \ominus (a \ominus \beta) \) and in the second one we get \( (\gamma \ominus a) \ominus \beta \). Similarly for the other cases.

Note that, by Definition A.7–(3), the elements \( a \ominus (\gamma \ominus \beta) \) and \( a \ominus (\beta \ominus \gamma) \) (resp. \( \beta \ominus (\gamma \ominus a) \) and \( (a \ominus \gamma) \ominus \beta \)) are never simultaneously defined. This proves the assertion. \( \square \)

By using the same arguments as in the proof of the above lemma, we get the following.

**Lemma A.9.** Under the same hypothesis of Lemma A.8, the elements in (1) (resp. in (2)) pairwise coincide (whenever defined).\(^8\)

**Example A.10.**

(1) Every root system is a semigroup with respect to the operations \( \oplus, \ominus \) induced by the inclusion \( \mathbb{Q}_+ \subseteq (\mathbb{Q}, +, -) \).

(2) Let \( \mathbb{K} = \mathbb{Z}, \mathbb{Q}, \mathbb{R} \). In Section 2.3.1, we introduced the set \( \text{Int}_k(\mathbb{R}) \) together with the partial operations \( \oplus \) and \( \ominus \) (cf. Formulas (2.7) and (2.8)). Then \( \text{S}(\mathbb{K}) := (\text{Int}_k(\mathbb{R}), \oplus, \ominus) \) is a semigroup.

\( \triangle \)

**APPENDIX B. PROOF OF PROPOSITION 2.9**

In this section, we complete the proof of Proposition 2.9, providing a new presentation for the Lie algebra of the line. Specifically, we shall prove that the relations (2.16) and (2.17) hold in \( \mathfrak{sl}(\mathbb{R}) \).

**Relation (2.17).** We first observe that, if \( J_1, J_2 \) are nested, \( J_1 \oplus J_2 \) is not defined and (2.17) coincides with (2.14). If \( J_1 \rightarrow J_2 \) or \( J_2 \rightarrow J_1 \), then (2.17) coincides with (2.12) and (2.13). It remains to prove that, if \( J'_1 \vdash J'_2 \), then

\[
[e_{J'_1}, e_{J'_2}] = 0 = [f_{J'_1}, f_{J'_2}].
\]

In this case, there exists an interval \( J_2 \) such that \( J'_1 = J_1 \ominus J_2 \), with \( J_1 = J'_1 \ominus J_2 \) and \( J_1 \rightarrow J_2 \), and \( J'_2 = J_2 \ominus J_3 \), with \( J_3 = J'_2 \ominus J_2 \) and \( J_2 \rightarrow J_3 \). It is therefore equivalent to show that

\[
[[e_{J_1}, e_{J_2}], [e_{J_2}, e_{J_3}] = 0 = [[f_{J_1}, f_{J_2}], [f_{J_2}, f_{J_3}]].
\]

By Jacobi identity,

\[
[[e_{J_1}, e_{J_2}], [e_{J_2}, e_{J_3}]] = -[[e_{J_2}, e_{J_2 \oplus J_3}], e_{J_1}] - [[e_{J_2 \oplus J_3}, e_{J_1}], e_{J_2}] = 0.
\]

Now, \( [e_{J_2}, e_{J_2 \oplus J_3}] = 0 \) by (2.14) since \( J_2 \vdash J_2 \oplus J_3 \), hence the first quantity on the RHS is zero. Since \( J_1 \rightarrow J_2 \oplus J_3 \), we have \( [e_{J_2 \oplus J_3}, e_{J_1}] = e_{J_2 \oplus J_3 \oplus J_1} \), hence by (2.14) the second quantity on the RHS is zero as well. The computation for \( f \) is similar. Therefore, (2.17) holds.

**Relation (2.16).** We shall now prove that relation (2.16) holds. We proceed case–by–case, according to the relative position of two arbitrary intervals. If \( J_1 = J_2, J_1 \rightarrow J_2, J_2 \rightarrow J_1, \) or \( J_1 \perp J_2, \) this is clear. We shall analyze the other cases separately.

---

\(^8\)For example, set \( y = (\gamma \ominus a) \ominus \beta, z = (\gamma \ominus (a \ominus \beta)). \) Then

\[
y = (((x \ominus a) \ominus \beta) \ominus a) \ominus \beta = (\beta \ominus ((x \ominus a) \ominus a)) \ominus \beta = (\beta \ominus x) \ominus \beta = x
\]

where the second equality is (A.1). Similarly for \( x = z = y. \)
Case $J_1 \vdash J_2$. In this case, $\langle 1_{J_1}, 1_{J_2} \rangle = 0$, $\langle 1_{J_2}, 1_{J_1} \rangle = 1$, $\langle 1_{J_1}, 1_{J_2} \rangle = 1$, $J_2 \ominus J_1$ is defined, and $J_1 \odot J_2, J_1 \odot J_2$ are not. We have to show that
\[ [e_{1_j}, f_{1_j}] = -f_{1_j} \ominus 1_j. \]

By definition, $J_2 = J_1 \oplus J_3$, where $J_3 = J_2 \ominus J_1$ and $J_1 \rightarrow J_3$. Therefore, by (2.13), it is equivalent to show that
\[ [e_{1_j}, [f_{1_j}, f_{1_j}]] = f_{1_j}. \]

By Jacobi identity, we have
\[ [e_{1_j}, [f_{1_j}, f_{1_j}]] = -[f_{1_j}, [e_{1_j}, f_{1_j}]] - [f_{1_j}, [e_{1_j}, f_{1_j}]] = [h_{1_j}, f_{1_j}] = - (1_{J_1}, 1_{J_2}) f_{1_j}, \]
where the second identity follows from (2.10). It remains to observe that $(1_{J_1}, 1_{J_2}) = -1$.

Case $J_2 \vdash J_1$. This case is identical to the previous one, but we will prove it for completeness. In this case, $\langle 1_{J_1}, 1_{J_2} \rangle = 1$, $\langle 1_{J_2}, 1_{J_1} \rangle = 0$, $\langle 1_{J_1}, 1_{J_2} \rangle = 1$, $J_1 \ominus J_2$ is defined, and $J_1 \odot J_2, J_2 \odot J_1$ are not. We have to show that
\[ [e_{1_j}, f_{1_j}] = -e_{1_j} \oplus 1_j. \]

By definition, $J_1 = J_2 \oplus J_3$, where $J_3 = J_1 \ominus J_2$ and $J_2 \rightarrow J_3$. Therefore, by (2.12), it is equivalent to show that
\[ [[e_{1_j}, e_{1_j}], f_{1_j}] = -e_{1_j}. \]

By Jacobi identity, we have
\[ [[e_{1_j}, e_{1_j}], f_{1_j}] = -[[e_{1_j}, f_{1_j}], e_{1_j}] - [[f_{1_j}, e_{1_j}], e_{1_j}] = [h_{1_j}, e_{1_j}] = (1_{J_1}, 1_{J_2}) e_{1_j}. \]

It remains to observe that $(1_{J_1}, 1_{J_2}) = -1$, since $J_2 \rightarrow J_3$.

Case $J_1 \vdash J_2$. In this case, $\langle 1_{J_1}, 1_{J_2} \rangle = 1$, $\langle 1_{J_2}, 1_{J_1} \rangle = 0$, $\langle 1_{J_1}, 1_{J_2} \rangle = 1$, $J_2 \ominus J_1$ is defined, and $J_1 \odot J_2, J_1 \odot J_2$ are not. We have to show that
\[ [e_{1_j}, f_{1_j}] = f_{1_j} \ominus 1_j. \]

By definition, $J_2 = J_1 \ominus J_3$, where $J_3 = J_2 \ominus J_1$ and in this case $J_3 \rightarrow J_1$. Therefore, by (2.13), it is equivalent to show that
\[ [e_{1_j}, [f_{1_j}, f_{1_j}]] = -f_{1_j}. \]

By Jacobi identity, we have
\[ [e_{1_j}, [f_{1_j}, f_{1_j}]] = -[f_{1_j}, [f_{1_j}, e_{1_j}]] - [f_{1_j}, [e_{1_j}, f_{1_j}]] = -[h_{1_j}, f_{1_j}] = (1_{J_1}, 1_{J_2}) f_{1_j}. \]

It remains to observe that $(1_{J_1}, 1_{J_2}) = -1$. The case $J_2 \vdash J_1$ is identical.

Case $J_2 \vdash J_1$. In this case, $\langle 1_{J_1}, 1_{J_2} \rangle = 0$, $\langle 1_{J_2}, 1_{J_1} \rangle = 0$, $\langle 1_{J_1}, 1_{J_2} \rangle = 0$, and $J_1 \odot J_2, J_2 \odot J_1, J_1 \odot J_2$ are not defined. We have to show that
\[ [e_{1_j}, f_{1_j}] = 0. \]

By definition, $J_2 = J_3 \oplus J_1 \oplus J_4$ with $J_3 \rightarrow J_1 \rightarrow J_4$, where $J_3, J_4$ are the two connected components of $J_2 \setminus J_1$. It is therefore equivalent to show that
\[ [e_{1_j}, [f_{1_j}, f_{1_j \oplus J_4}]] = 0. \]

By Jacobi identity,
\[ [e_{1_j}, [f_{1_j}, f_{1_j \oplus J_4}]] = -[f_{1_j}, [f_{1_j \oplus J_4}, e_{1_j}]] - [f_{1_j \oplus J_4}, e_{1_j}, f_{1_j}] = 0, \]

since $J_1 \vdash J_1 \oplus J_4$ and $J_3 \rightarrow J_1$. The case $J_2 < J_1$ is identical.
Case $J_1 \oplus J_2$. In this case, $\langle 1_{J_1}, 1_{J_2} \rangle = -1$, $\langle 1_{J_2}, 1_{J_1} \rangle = 1$, $\langle 1_{J_1}, 1_{J_2} \rangle = 0$, and $J_1 \oplus J_2, J_2 \ominus J_1, J_1 \ominus J_2$ are not defined. We have to show that
\[ [e_{J_1}, f_{J_2}] = 0 . \]
In this case, there exists an interval $J'_2$ such that $J_1 = J'_1 \oplus J'_2$, with $J'_1 = J_1 \ominus J'_2$ and $J'_2 = J_2 \ominus J'_1$, with $J'_3 = J_2 \ominus J'_2$ and $J'_2 \rightarrow J'_3$. It is therefore equivalent to show that
\[ [e_{J'_1 \oplus J'_2}, [f_{J'_2}, f_{J'_1}]] = 0 . \]
By Jacobi identity,
\[ [e_{J'_1 \oplus J'_2}, [f_{J'_2}, f_{J'_1}]] = -[f_{J'_2}, [f_{J'_1}, e_{J'_1 \oplus J'_2}]] - [f_{J'_1}, [e_{J'_1 \oplus J'_2}, f_{J'_2}]] = 0 , \]
since $J'_1 \oplus J'_2 \rightarrow J'_3$ and $J'_2 \rightarrow J'_1 \oplus J'_2$. The case $J_2 \oplus J_1$ is identical.

APPENDIX C. PROOF OF PROPOSITION 4.1

In this section, we complete the proof of Proposition 4.1, providing necessary and sufficient conditions for the Serre relations to hold in a semigroup Lie algebra.

For any $a, b \in S$ set
\[ X_{\pm, a, b} := [x^\pm_a, x^\pm_b] - \mu_\pm (a, b) x^\pm_{a \oplus b} . \]
We denote by $S_{\pm, a, b}$ the subspace spanned by the elements $X_{\pm, a, b}$ with $a \in S^\pm_{c \cap \alpha}, b \in S^\pm_{\beta \cap \alpha}$. We shall prove that, for any $c \in S$, $[S_{\pm, a, b}, x^\pm_a] \subseteq S_{\pm, a, b}$.

By the Jacobi identity, the commutation relations (3.7) and (3.8), the commutator of $X_{\pm, a, b}$ and $x^\pm_a$ is an element in $S_{\pm, a, b}$ if and only if the following identities hold in $L_0$ and $L_-$:
\[ \mu_\pm (a, b) \delta_{a \oplus b, c} \xi_e = \delta_{b, s \ominus a} \xi_e (c, a) \xi_b - \delta_{a, s \ominus b} \xi_e (c, b) \xi_a , \quad (C.1) \]
\[ \mu_\pm (a, b) \xi_e (c, a \oplus b) x^\pm_{c \ominus (a \ominus b)} = \xi_e (c, a) \xi^\pm (c \ominus a, b) x^\pm_{c \ominus (a \ominus b)} - \xi_e (c, b) \xi^\pm (c \ominus b, a) x^\pm_{c \ominus (a \ominus b)} , \quad (C.2) \]
and the element
\[ \delta_{a, c} \kappa (a, b) x^\pm_a + \xi^\pm (a, c) [x^\pm_a, x^\pm_{b \ominus c}] + \xi^\pm (a, c) \xi^\pm (b, c \ominus a) x^\pm_{b \ominus (a \ominus c)} - \delta_{b, c} \kappa (b, a) x^\pm_a + \xi^\pm (a, c) [x^\pm_a, x^\pm_{b \ominus c}] - \xi^\pm (a, c \ominus b) \xi^\pm (a, c \ominus b) x^\pm_{a \ominus (b \ominus c)} - \mu_\pm (a, b) \xi^\pm (a, c \ominus b, c) x^\pm_{a \ominus (b \ominus c)} \quad (C.3) \]
belongs to $S_{\pm, a, b}$.

The identity (C.1). The identity is non–trivial only if $c = a \oplus b$, in which case $a = c \ominus b, b = c \ominus a,$ and it reduces to relation (4.2), i.e., one has
\[ \xi^\pm (a \oplus b, a) \xi_b - \xi^\pm (a \ominus b, b) \xi_a = \mu_\pm (a, b) (a, \xi_a + \xi_b) . \]
and therefore,
\[ \xi^\pm (a \oplus b, a) = \mu_\pm (a, b) = -\xi^\pm (a \ominus b, b) . \]

The identity (C.2). By Lemma A.9, it is enough to observe that the elements $c \ominus (a \oplus b), (a \ominus b) \ominus a$, and $(c \ominus a) \ominus b$ coinde whenever defined. Therefore, (C.2) reduces to (4.4).
The identity (C.3). We first consider the cases $c = a, b$.

Assume that $a \oplus b$ is not defined.

- If $(b, a) \in S_{(b)}$ and $(a, b) \not\in S_{(a)}$, then for $c = a$, we get
  \[ \kappa(a, b) x_\pm^\pm + \xi_\pm(b, a)[x_\pm^a, x_\pm^b], \]
  which gives
  \[ \kappa(a, b) = -\xi_\pm(b, a) \xi_\pm(b, a). \]
  For $c = b$, we get
  \[ -\kappa(b, a) x_\pm^a + \xi_\pm(b, a) \xi_\pm(b, a) x_\pm^b, \]
  which gives
  \[ \kappa(b, a) = \xi_\pm(b, a) \xi_\pm(b, a), \]
  where the second equality follows from (4.2).

- The case $(b, a) \not\in S_{(b)}$ and $(a, b) \in S_{(a)}$ is identical to the previous one.

- If $(a, b), (b, a) \not\in S_{(a)}, (C.3)$ reduces to the condition $\kappa(a, b) = 0 = \kappa(b, a)$.

If $a \oplus b$ is defined, it is enough to add to the previous identities the summand $\xi_\pm(a \oplus b, a) \xi_\pm(a \oplus b, b)$. This proves (4.3).

We move to the case $c \neq a, b$. By Lemma A.9, the elements $a \oplus (b \odot c), b \odot (c \odot a), (a \odot c) \oplus b, a \odot (c \odot b), (a \odot b) \ominus c$ coincide whenever defined. In particular, it follows that the element (C.3) belongs to the subspace $S_{\pm,a,b}$ if and only if (4.5) holds.

APPENDIX D. PROOF OF THEOREM 5.7

In this section, we complete the proof of Theorem 5.7. Given a good Cartan semigroup $S$, for any $(a, b) \in \text{Serre}(S)$, we have to show that the relation $[x_\pm^a, x_\pm^b] = \xi_\pm(a \oplus b, a) \cdot x_\pm^a b^{b} \in g(S)$. By Proposition 4.1, it is enough to show that the relations (4.2), (4.3), (4.4), and (4.5) hold for any $a \in S_{\pm,a}, b \in S_{\pm,b}, c \in S$.

The first two follow, respectively, from the properties (5.1), (5.2), and (5.5) of good Cartan semigroups. In order to prove the remaining two relations, we shall proceed by analyzing different cases. Recall that the two relations we are interested in are the following:

(4.4) For any $a \in S_{\pm,a}, b \in S_{\pm,b}, c \in S$,
\[ \delta_{c \ominus (a \oplus b)} [x_\pm^a, x_\pm^b](a \oplus b, a) = \delta_{c \ominus (a \oplus b)} x_\pm^c (a \ominus b) \delta_{c \ominus (a \oplus b)} x_\pm^c (a \ominus b, a). \]

(4.5) For any $a \in S_{\pm,a}, b \in S_{\pm,b}, c \in S, c \neq a, b$,
\[ \xi_\pm(a, c) x_\pm^c [(a \ominus c) \ominus b, b] - \xi_\pm(c, b) x_\pm^c (a \ominus (b \ominus c), a) = \delta_{b \ominus (c \ominus a)} x_\pm^c (c, a) x_\pm^c (b, c) \delta_{b \ominus (c \ominus a)} x_\pm^c (c, b) - \delta_{a \ominus (c \ominus b)} x_\pm^c (c, a) x_\pm^c (a, c \ominus b)
- \delta_{a \ominus (c \ominus b)} x_\pm^c (a \ominus b, a) x_\pm^c (a \ominus b, c). \]

Note that, as we show in Corollary 4.5, the relations (4.4) and (4.5) are satisfied whenever $a \perp b$. Therefore, we can assume $a \neq b$. 

Proof of relation (4.4). First, note that, since $a \not\perp b$, the relation (4.4) is trivial whenever $a \oplus b$ is not defined, since $c \ominus (a \oplus b)$, $(c \ominus a) \ominus b$, and $(c \ominus b) \ominus a$ cannot exist in this case. Indeed, the first element does not exist by definition. By Lemma A.8, the second and third element either do not exist or they both exist. However, the latter situation cannot occur because of condition (L1). Therefore we can assume that $a \oplus b$ is defined and thus real by Remark 5.6.

Next, (4.4) is trivial whenever $c \ominus (a \oplus b)$ is not defined. Indeed, in this case it follows by strong associativity (cf. Definition A.7–(4)) that the elements $(c \ominus a) \ominus b$ and $(c \ominus b) \ominus a$ are also not defined. Therefore we can assume that $c \ominus (a \oplus b)$ is defined.

Thus, we are left considering only the case in which both $a \oplus b$ and $c \ominus (a \oplus b)$ are defined.

Case $a, b \in S^{re}$. We claim that, if $a, b \in S^{re}$, we can assume $c \in S^{re}$. Indeed, in this case $a \oplus b$ is real and, by the reality condition, if $c \not\in S^{re}$, there exists a real element $c'$ such that $c \ominus c'$ is defined and orthogonal to $a \oplus b$. It follows that $c \ominus c'$ is orthogonal to $a$ and $b$ and, thus, by condition (L1), $c' \ominus a$ (resp. $c' \ominus b$) is defined whenever $c \ominus a$ (resp. $c \ominus b$) is. Therefore, by orthogonality, we have

$$
\xi_\pm(x, a \oplus b) = \xi_\pm(c', a \ominus b), \quad \xi_\pm(x, a) = \xi_\pm(c', a), \quad \xi_\pm(x, b) = \xi_\pm(c', b).
$$

It is also clear that

$$
\delta_{(c \ominus a) \ominus b} = \delta_{(c' \ominus a) \ominus b} \quad \text{and} \quad \delta_{(c \ominus b) \ominus a} = \delta_{(c' \ominus b) \ominus a}
$$

since both $a$ and $b$ are orthogonal to $c \ominus c'$. This proves that the relation (4.4) holds for the triple $(a, b, c)$ if and only if it holds for the triple $(a, b, c')$. Therefore we can assume $c \in S^{re}$.

Case $a \in S^{re}, b \in S^{im}$. Note that, in this case, $c$ is necessarily an element in $S^{im}$.

Assume therefore that either $(a, b, c) \in S^{re} \times S^{re} \times S^{re}$ or $(a, b, c) \in S^{re} \times S^{im} \times S^{im}$, with $a \not\perp b$ and $c \ominus (a \oplus b)$ defined. We first observe that in any good semigroup the conditions (4.4) can be simplified. Namely, it follows from the admissibility condition (1a) from Definition 5.5 and Lemma A.8 that either none or exactly two elements among $c \ominus (a \ominus b)$, $(c \ominus a) \ominus b$, and $(c \ominus b) \ominus a$ can be simultaneously defined. Recall that, by definition, $\xi_\pm(x, y) = \xi_\pm(y, x)$ and $\xi := \xi_+$. Then, one checks easily that (4.4) reduces to the condition

$$
\xi_\pm(x \ominus y, x)\xi_\pm((x \ominus y) \oplus z, x \ominus y) = \xi_\pm((x \ominus y) \ominus z, x)\xi_\pm(y \ominus z, y),
$$

where either $(x, y, z) = (a, b, (c \ominus a) \ominus b))$ or $(x, y, z) = (b, a, (c \ominus b) \ominus a)$. It is easy to check that this holds. Indeed,

$$
\xi_\pm(x \ominus y, x)\xi_\pm((x \ominus y) \oplus z, x \ominus y) = \xi_\pm(x \ominus y, x)\xi_\pm(x \ominus y) = \xi_\pm(x \ominus y, x)\xi_\pm(x \ominus y) = \xi_\pm(x \ominus y, y)\xi_\pm(y \ominus z, y) = \xi_\pm(x \ominus y, y)\xi_\pm(y \ominus z, y) = \xi_\pm(x \ominus y, y)\xi_\pm(y \ominus z, y) = \xi_\pm(x \ominus y, y)\xi_\pm(y \ominus z, y),
$$

where the second and fourth identities rely on (5.3), while the third and sixth ones rely on (5.1) and (5.2), and the first and fifth ones follow by associativity. Note that we are allowed to use (5.3) since the elements $c, a \oplus b$, and $c \ominus a$ (resp. $c \ominus b$) can never be locally degenerate.

Proof of relation (4.5). First, we claim that, if $a, b \in S^{re}$, we can assume $c \in S^{re}$, since (4.5) is trivial otherwise. Indeed, if $c \not\in S^{re}$, the elements $a \ominus c, b \ominus c$, and $(a \oplus b) \ominus c$ are certainly not defined, since $S_{<a}$ and $S_{<b}$ are contained in $S^{re}$. Now assume that $a \ominus (c \ominus b)$ is defined and $\xi_\pm(a, c \ominus b) \neq 0$. Then, $c \ominus b$ is defined, is real (since $a$ is), and belongs to $S_{<a}$. It follows that $c = (c \ominus b) \ominus b$ is necessarily real, because the pair $(c \ominus b, b)$ belongs to $S_{<a}^{im}$ by Remark 5.6.

---

Therefore, for example, $(a \ominus c) \ominus b = ((c \ominus c') \ominus (c' \ominus a)) \ominus b = (c \ominus c') \ominus ((c' \ominus a) \ominus b)$. 

Therefore, either \( a \oplus (c \ominus b) \) is not defined or \( \zeta_{\pm}(a, c \ominus b) = 0 \), and similarly for \( b \ominus (c \oplus a) \). It follows that (4.5) is trivial if \( c \not\in S^e \).

Assume therefore that either \((a, b, c) \in S^e \times S^e \times S^e \) or \((a, b, c) \in S^e \times S^{im} \times S \), with \( a \not\perp b \). By Lemma A.8 and the admissibility condition (1b) in Definition 5.5, either none or exactly two elements among \((a \oplus b) \ominus c, (a \ominus c) \oplus b, a \ominus (b \ominus c), a \ominus (c \ominus b), b \ominus (c \ominus a)\) are simultaneously defined. Then, one checks easily that (4.5), similarly to the case of (4.4), reduces to the conditions
\[
\zeta(x \oplus z, (x \oplus z) \ominus y, y \oplus z) = \zeta(y, y \ominus z) \zeta(x \oplus z, x), \tag{D.1}
\]
\[
\zeta(x \oplus (y \ominus z), y \ominus z) \zeta(y, x \oplus y) = \zeta(z, y \ominus z) \zeta(x \oplus (y \ominus z), y), \tag{D.2}
\]
\[
\zeta(x, (x \oplus y) \ominus z) \zeta(y, y \ominus z) = \zeta(y, (x \oplus y) \ominus z) \zeta(x, x \oplus z), \tag{D.3}
\]
whenever all terms are defined. Therefore, we are left to prove the identities (D.1), (D.2), and (D.3). One checks by direct inspection that, as before, these follow directly from properties (5.1), (5.2), and (5.3).

### Appendix E. Proof of Proposition 6.11

In this section, we complete the proof of Proposition 6.11, showing the continuum quiver \( \mathbb{Q}_X \) is a good Cartan semigroup. We shall show that \( \mathbb{Q}_X \) satisfies the conditions (1)–(5) from Definition 5.3. We proved that (1), (2), and (3) hold. It remains to prove that the functions \( \zeta_X \) and \( \kappa_X \) satisfy (4) and (5). Note that \( \kappa_X \) is symmetric and satisfies the condition (5.4) by definition. Below, we prove the conditions (5.1), (5.2), (5.3), and (5.5).

**Proof of conditions (5.1) and (5.2).**

**Case 1: \( a, \beta \) are elementary intervals.** Note that, whenever \( a \oplus \beta \) is defined, \( \langle 1_{a \oplus \beta}, 1_a \rangle + \langle 1_{a \oplus \beta}, 1_\beta \rangle = 1 \) and \( \langle 1_{a \oplus \beta}, 1_a \rangle = 1 = \langle 1_{a \oplus \beta}, 1_\beta \rangle \). Therefore, \( \zeta_X \) satisfies (5.1) and (5.2), i.e.,
\[
\zeta_X(a \oplus \beta, a) = (-1)^{\|1_{a \oplus \beta} 1_a \|} \langle 1_{a \oplus \beta}, 1_a \rangle = (-1)^{\|1_{a \oplus \beta} 1_\beta \|} \langle 1_{a \oplus \beta}, 1_\beta \rangle = -\zeta_X(a \oplus \beta, \beta),
\]
\[
\zeta_X(a, a \oplus \beta) = (-1)^{\|1_{a \oplus \beta} 1_a \|} \langle 1_{a \oplus \beta}, 1_a \rangle = (-1)^{\|1_{|a \oplus \beta|} 1_\beta \|} \langle 1_{a \oplus \beta}, 1_\beta \rangle = -\zeta_X(\beta, a \oplus \beta).
\]

Note that, if either \( a \oplus \beta \), \( a \ominus \beta \) or \( \beta \ominus a \) is defined, \( \zeta_X(a, \beta) = -\zeta_X(\beta, a) \).

**Case 2: \( a, \beta \) are contractible intervals.** If \( a \oplus \beta \) is contractible or undefined, the proof is essentially identical to the case of elementary intervals. If \( a \oplus \beta \) is non–contractible, then the conditions holds by a direct computation.

**Case 3: \( a \) is a contractible interval, \( \beta \) is homeomorphic to \( S^1 \).** By a direct computation, one sees that, if \( a \oplus \beta \) is defined, then both sides of (5.1) (resp. (5.2)) equals \(-1 \) (resp. 1).

**Case 4: \( a \) is a contractible interval, \( \beta \neq S^1 \) is a non–contractible interval.** First, by Lemma 6.5, \( \beta \) is of the form \( S^1 \oplus \bigoplus_k T_k \) for some pairwise disjoint contractible intervals \( T_k \). Now, \( a \ominus \beta \) exists if and only if \( S^1 \rightarrow a \) or \( T_h \rightarrow a \) for some \( h \). In the first case, \( a \) is perpendicular to all \( T_k \), so the check of conditions (5.1) and (5.2) reduces to the third case above. On the other hand, in the second case, \( a \) is perpendicular to \( \beta \ominus T_h \), so the check of conditions (5.1) and (5.2) reduces to the second case above.

**Case 5: \( a, \beta \) are non–contractible intervals.** Since \( a \oplus \beta \) is never defined, (5.1) and (5.2) are automatically satisfied.

**Proof of the condition (5.5).**
Case 1: \(a, \beta\) are elementary intervals. If \(a \oplus \beta\) is defined, then
\[
\xi_X(a \oplus \beta, a)\xi_X(a, a \oplus \beta) = -\xi_X(a \oplus \beta, a)^2 = -1 = \kappa_X(a, \beta).
\]
If either \(a \ominus \beta\) or \(\beta \ominus a\) is defined, then
\[
-\xi_X(a, \beta)\xi_X(\beta, a) = \xi_X(a, \beta)^2 = 1 = \kappa_X(a, \beta).
\]
Finally, if \(a \oplus \beta\), \(a \ominus \beta\) and \(\beta \ominus a\) are not defined, then \(\kappa_X(a, \beta) = 0\).

Case 2: \(a, \beta\) are contractible intervals. Since we need to verify (5.5) only for those \(a, \beta\) such that \(a \oplus \beta\) is real (cf. condition (\(\ast\)) in (5.5)) and if \(a \ominus \beta\) or \(\beta \ominus a\) are defined, they are real, this case reduces to the previous one.

Case 3: \(a\) is a contractible interval, \(\beta\) is homeomorphic to \(S^1\). If \(a \oplus \beta\) is defined, in particular, it is not locally degenerate. Thus, by using similar arguments as in the third case of Section E, both sides of (5.5) equals \(-1\). Next, \(a \ominus \beta\) is never defined, while if \(\beta \ominus a\) is defined, this implies \(a \subset \beta\) and therefore (5.5) is trivial. Finally, if neither \(a \oplus \beta\) or \(\beta \ominus a\) are defined, then \(a \cap \beta = \emptyset\) and the result follows.

Case 4: \(a\) is a contractible interval, \(\beta \neq S^1\) is a non–contractible interval. It follows by the same arguments as in the previous section.

Proof of the condition (5.3).

Case 1: \(a, \beta, \gamma\) are elementary intervals. First, \(a \oplus \beta\) is defined if and only if either \(\beta \vdash a\) or \(\beta \triangleright a\). Note also that, if \(a \oplus \gamma\) and \(\beta \oplus \gamma\) are both defined, then one of the following holds (we use the notation from Section 2.3):
- \(\beta \vdash a, \gamma \to a, \gamma \to \beta\);
- \(\beta \vdash a, a \to \gamma, \beta \to \gamma\).

In both cases, we have \(a \oplus \gamma = (a \ominus \beta) \oplus (\beta \oplus \gamma)\), hence
\[
\langle 1_{a \oplus \gamma}, 1_{\beta \oplus \gamma} \rangle = \langle 1_{\beta \oplus \gamma} + 1_{a \ominus \beta}, 1_{\beta \oplus \gamma} \rangle = \langle 1_{\beta \oplus \gamma}, 1_{\beta \oplus \gamma} \rangle + \langle 1_{a \ominus \beta}, 1_{\beta} + 1_{\gamma} \rangle = \langle 1_{\gamma}, 1_{\beta} \rangle + \langle 1_{a \ominus \beta}, 1_{\beta} \rangle = \langle 1_{a}, 1_{\beta} \rangle.
\]

Here, we have applied Formula (2.9) and Remark 6.6 (since \(\gamma \perp a \ominus \beta\)). One can show similarly that \(\langle 1_{\beta \oplus \gamma}, 1_{a \ominus \beta} \rangle = \langle 1_{\beta}, 1_{a} \rangle\). Thus, condition (5.3) holds.

Case 2: \(a, \beta, \gamma\) are contractible intervals. Assume that there exists \(a \ominus \beta\). If \(a \oplus \gamma\) and \(\beta \oplus \gamma\) are contractible, we can reduce to the case of elementary intervals. It remains to be checked when
\begin{enumerate}
\item \(\beta \oplus \gamma\) is non–contractible and it is not homeomorphic to \(S^1\);
\item \(a \oplus \gamma\) is non–contractible and it is not homeomorphic to \(S^1\), and \(\beta \oplus \gamma\) is contractible but not contained in the locally degenerate part of \(a \ominus \gamma\).
\end{enumerate}

In the first case, we can decompose \(\gamma\) as \(a''_1 \oplus a''_2\) such that both \(a \oplus a''_1\) and \(\beta \oplus a''_1\) exist and are contractible and \(a''_2\) is perpendicular to both \(a\) and \(\beta\). Thus, we can reduce the the situation described in the above paragraph. On the other hand, in the second case one needs to do a direct computation to show that (5.3) holds.

Case 3: \(a, \beta\) are contractible elements, \(\gamma\) is homeomorphic to \(S^1\). Assume that \(a \ominus \beta\), \(a \oplus \gamma\) and \(\beta \oplus \gamma\) are defined. In this case, one can explicitly verify that
\[
\xi_X(a \oplus \gamma, a \ominus \gamma) = (-1)^{-1}(-1) = 1 = \xi_X(a, \beta).
\]
Case 4: \( \alpha, \beta \) are contractible elements, \( \gamma \neq S^1 \) is non-contractible. First, by Lemma 6.5, \( \gamma \) is of the form \( S^1 \oplus \bigoplus T_k \) for some pairwise disjoint contractible intervals \( T_k \). Now, \( \alpha \oplus \beta, \alpha \oplus \gamma \) and \( \beta \oplus \gamma \) are defined. Thus, we have to consider only the following two situations: \( S^1 \rightarrow \alpha, S^1 \rightarrow \beta \) or \( T_k \rightarrow \alpha, T_k \rightarrow \beta \) for some \( k \). In the first situation, we reduce to the third case above, while in the second one to the second case above.

Case 5: \( \alpha \) is a contractible interval, \( \beta \) is homeomorphic to \( S^1 \), \( \gamma \) is an interval. \( \alpha \oplus \beta \) is never defined, while \( \beta \oplus \alpha \) is defined, this implies \( \alpha \subset \beta \). Thus, \( \xi_X(\beta, \alpha) = 0 \). Let \( \gamma \) be an interval such that both \( \alpha \oplus \gamma \) and \( \beta \oplus \gamma \) exist and are not homeomorphic to \( S^1 \). First note that \( \gamma \) can be only contractible since sums of non-contractible intervals never exist. In addition, \( \beta \oplus \gamma = (\beta \oplus \alpha) \oplus (\beta \oplus \gamma) \) and \( \gamma \perp \beta \oplus \alpha \). Thus,

\[
\langle 1_{\beta \oplus \gamma}, 1_{\alpha \oplus \gamma} \rangle = \langle 1_{\alpha \oplus \gamma} + 1_{\beta \oplus \gamma}, 1_{\alpha \oplus \gamma} \rangle = \langle 1_{\beta \oplus \gamma}, 1_{\beta \oplus \gamma} \rangle + \langle 1_{\beta \oplus \gamma}, 1_{\alpha} + 1_{\gamma} \rangle
\]

\[
= \langle 1_{\beta}, 1_{\beta} \rangle + \langle 1_{\beta \oplus \alpha}, 1_{\alpha} \rangle = \langle 1_{\beta}, 1_{\beta} \rangle + 0 = 0.
\]

Similarly, \( \langle 1_{\alpha \oplus \gamma}, 1_{\beta \oplus \gamma} \rangle = 0 \).

Case 6: \( \alpha \) is a contractible interval, \( \beta \neq S^1 \) is a non-contractible interval, \( \gamma \) is an interval. \( \alpha \oplus \beta \) is never defined, while \( \beta \oplus \alpha \) is defined if and only if \( \alpha \subset \beta \). By Lemma 6.5, \( \beta \) is of the form \( S^1 \oplus \bigoplus T_k \) for some pairwise disjoint contractible intervals \( T_k \). Let \( \gamma \) be an interval such that \( \alpha \oplus \gamma \) and \( \beta \oplus \gamma \) are defined. We have that \( \gamma \) is real, since sums of imaginary elements are never defined. We have two mutually exclusive cases:

- \( \gamma \) is perpendicular to \( T_k \) for all \( k \), so to verify that (5.3) holds, one can suitably reduce to the fifth case above;
- there exists \( T_k \) such that \( T_k \rightarrow \gamma \), so to verify that (5.3) holds, one can suitably reduce to the second case above.

Case 7: \( \alpha, \beta \) are non-contractible intervals, \( \gamma \) is an interval. Assume that \( \beta \oplus \alpha \) is defined, hence it is necessarily real. As before, we decompose \( \beta \) as \( \beta = S \oplus T \), with \( S \in S^{im} \) and \( T \in S^{re} \), such that \( \alpha \oplus \beta \) is perpendicular to \( S \). Let \( \gamma \) be an interval such that \( \alpha \oplus \gamma \) and \( \beta \oplus \gamma \) are defined. As seen before, \( \gamma \) is necessarily a real element. If \( \gamma \) sums the imaginary part \( S \), then set \( S' := S \oplus \gamma \). Therefore,

\[
\langle 1_{\alpha \oplus \gamma}, 1_{\beta \oplus \gamma} \rangle = \langle 1_{\alpha \oplus \beta} + 1_{T} + 1_{S'}, 1_{T} + 1_{S'} \rangle = \langle 1_{\alpha \oplus \beta}, 1_{T} \rangle + \langle 1_{T \oplus S'}, 1_{T \oplus S'} \rangle = \langle 1_{\alpha \oplus \beta}, 1_{T} \rangle = \langle 1_{\alpha}, 1_{\beta} \rangle.
\]

Similarly, one can prove \( \langle 1_{\beta \oplus \gamma}, 1_{\alpha \oplus \gamma} \rangle = \langle 1_{\beta}, 1_{\alpha} \rangle \). If \( \gamma \) sums the real part \( T \), denote by \( T' \) the sum between them. We have \( T' \rightarrow \alpha \oplus \beta \). In this case, we get:

\[
\langle 1_{\alpha \oplus \gamma}, 1_{\beta \oplus \gamma} \rangle = \langle 1_{\alpha \oplus \beta} + 1_{T'} + 1_{S'}, 1_{T'} + 1_{S'} \rangle = \langle 1_{\alpha \oplus \beta}, 1_{T'} \rangle + \langle 1_{T' \oplus S'}, 1_{T' \oplus S'} \rangle = \langle 1_{\alpha \oplus \beta}, 1_{T'} \rangle = 0 = \langle 1_{\alpha}, 1_{\beta} \rangle.
\]

Similarly, one can show \( \langle 1_{\beta \oplus \gamma}, 1_{\alpha \oplus \gamma} \rangle = -1 = \langle 1_{\beta}, 1_{\alpha} \rangle \).

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