ALMOST FULL ENTROPY SUBSHIFTS UNCORRELATED TO THE MÖBIUS FUNCTION

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Abstract. We show that if \( y = (y_n)_{n \geq 1} \) is a bounded sequence with zero average along every infinite arithmetic progression then for every \( N \geq 2 \) there exist (unilateral or bilateral) subshifts \( \Sigma \) over \( N \) symbols, with entropy arbitrarily close to \( \log N \), uncorrelated to \( y \). In particular, for \( y = \mu \) being the Möbius function, we get that there exist subshifts as above which satisfy the assertion of Sarnak’s conjecture. The existence of positive entropy systems uncorrelated to the Möbius function is claimed in Sarnak’s survey [S] (and attributed to Bourgain), however, to our knowledge no examples have ever been published. We fill in this gap and by the way we show that this has nothing to do with more advanced algebraic properties (for instance multiplicativity) of the considered sequence.

1. Introduction

Let \( y \) be a bounded, real-valued sequence with zero average along every infinite arithmetic progression, i.e., satisfying, for every \( t \geq 1 \) and \( l \geq 0 \), the condition

\[
\lim_{n} \frac{1}{n} \sum_{i=1}^{n} y_{i+t+l} = 0.
\]

Following the terminology used for multiplicative functions (see e.g. [FH]), we will call any such sequence aperiodic. Without loss of generality we will assume that \( |y_n| \leq 1 \) for all \( n \). For example, we can take the Möbius function \( y = \mu \), where

\[
\mu_n = \begin{cases} 
1 & \text{for } n = 1, \\
(-1)^r & \text{if } n \text{ is a product of } r \text{ distinct primes,} \\
0 & \text{otherwise (i.e., if } n \text{ has a repeated prime factor).}
\end{cases}
\]

It is known that this sequence satisfies the condition (1.1) (see e.g. [S]).

Once an aperiodic sequence \( y \) is fixed, we consider topological dynamical systems \((X,T)\) where \( X \) is a compact metric space and \( T : X \to X \) is a continuous transformation. Subshifts (in which the transformation is always the left shift) will be denoted using just one letter \( \Sigma \). Uncorrelation between a system and a sequence will be understood as follows:

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Definition 1.1. We say that $(X, T)$ is uncorrelated to $y$ if for each continuous function $f : X \to \mathbb{R}$ and every $x \in X$, we have
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(T^i x) y_i = 0.
\]

The celebrated Sarnak’s conjecture ([S]) asserts that any system with zero topological entropy is uncorrelated to the Möbius function. Most of the activity around this conjecture is aimed toward determining ever larger classes of zero entropy systems which obey the Sarnak’s uncorrelation condition (for a long list of references see the survey [AKLR], newer results are in [HWZ]). Much less (but not zero) effort is devoted to finding systems which correlate with $\mu$ (see e.g. [AKL], [DK], [K]). Clearly, all examples found so far have positive entropy. For these efforts to be meaningful, it becomes crucial to also consider the “inverse” of Sarnak’s problem i.e., the following question

Question 1.2. Are there positive entropy systems uncorrelated to $\mu$?

As a matter of fact, in Sarnak’s exposition [S] it is claimed that such systems do exist, and relevant example is attributed to Bourgain. However, no examples have ever been published and we failed to acquire any details, so we decided to consider the question as open.

Notice that if the answer to the above question was negative and if Sarnak’s conjecture held, one could view $\mu$ as a sequence able to precisely differentiate between positive and zero topological entropy systems. As we will show, it is not the case.

In this work we answer the above question in the positive, providing evidence for the claim in Sarnak’s survey. In fact we show a bit more: if $y$ is any aperiodic sequence as described at the beginning of this section, and $N \geq 2$ is an arbitrary integer, then there exist subshifts $\Sigma$ on $N$ symbols, with entropy arbitrarily close to $\log N$, uncorrelated to $y$. The proof relies on a complicated counting blocks argument.

As a byproduct, we can make the following remark. Weiss [W, Theorem 8.3] proved that any subshift $\Sigma$ on $N$ symbols, of entropy larger than $\log(N - 1)$, has a positive density independence set $A$ (i.e., a subset of $\mathbb{N}$ along which all combinations of symbols occur), with a lower bound on the density $\text{dens}(A)$ in $\mathbb{N}$ depending (obviously nondecreasingly) on the entropy $h(\Sigma)$. Since $h(\Sigma) = \log N$ holds only for the full shift (whose independence set is the whole $\mathbb{N}$), one might expect, that as $h(\Sigma)$ tends to $\log N$, these lower bounds tend to 1. Our examples show that the limit of these bounds cannot exceed $\frac{1}{2}$. Indeed, take any aperiodic sequence $y$ over $\{-1, 1\}$ (in fact almost every sequence is such, for the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure). In our example created for $y$, with $\Lambda = \{1, 2, \ldots, N\}$ and entropy arbitrarily close to $\log N$, consider the function $f(x) = (-1)^{x_0}$. There are points $x$ such that $f(x)$ matches $y$ along the independence set $A$. Then $(f(T^n x))_{n \geq 1}$ correlates with $y$ by at least $2 \text{dens}(A) - 1$, so this number cannot be positive.

\footnote{We are sure that this remark has a more direct proof.}
2. Preliminaries

Let \( B = (b_1, \ldots, b_n) \in \mathbb{R}^n \) be a finite sequence (block) of real numbers. We define its average as

\[
\overline{B} = \frac{1}{n} \sum_{i=1}^{n} b_i.
\]

If \( C = (c_1, \ldots, c_n) \) is another block (of the same length) we define

\[
corr(B, C) = [BC] \]

where \( BC = (b_1c_1, \ldots, b_nc_n) \). Further, if \( x = (x_i)_{i \geq 1} \) and \( y = (y_i)_{i \geq 1} \) are bounded sequences, the correlation between \( y \) and \( x \) is defined as

\[
corr(x, y) = \limsup_{n \to \infty} corr(x^n_1, y^n_1),
\]

where \( z^n_m \) stands for the block \((z_m, \ldots, z_n)\) \((m \leq n)\).

An elementary lemma concerns sequences with zero average:

**Lemma 2.1.** Let \((y_n)_{n \geq 1}\) be a bounded sequence with zero average. Then, for every \( \epsilon \in (0, 1) \) and every natural \( m \) there exists a natural \( L(\epsilon, m) \) such that for every \( L \geq L(\epsilon, m) \) the absolute value of the average of \( y \), over any interval \( I \subset [1, mL] \) of length at least \( L \), is less than \( \epsilon \).

**Proof.** Without loss of generality, we can assume that \(|y_n| \leq 1\) for all \( n \). Now simply define \( L(\epsilon, m) \) to be such that for every \( n \geq \frac{1}{2} L(\epsilon, m) \) the average of \( y \) over \([1, n]\) is less than \( \frac{\epsilon}{2m} \) in absolute value. Consider an interval \( I \) of some length \( i \geq L \geq L(\epsilon, m) \) as in the assertion of the lemma, and denote by \( J \) the interval extending from 1 to the left end of \( I \) and let \( j \) be its length. Note that \( \frac{j}{i} \leq m \). Denote by \( \alpha, \beta \) and \( \gamma \) the averages of \( y \) over \( I, J \) and \( I \cup J \), respectively. We have

\[
\gamma = \frac{i}{i+j} \alpha + \frac{j}{i+j} \beta,
\]

hence

\[
|\alpha| \leq \frac{i+j}{i} |\gamma| + \frac{j}{i} |\beta| \leq m|\gamma| + \frac{j}{i} |\beta|.
\]

Since \( i+j \geq L(\epsilon, m) > \frac{1}{2} L(\epsilon, m) \), we have \( |\gamma| < \frac{4}{2m} \). If \( \frac{j}{i} < \frac{1}{4} \), we are done (because \(|\beta| \leq 1\)). Otherwise \( j \geq \frac{1}{2} i \geq \frac{1}{2} L(\epsilon, m) \), hence \( |\beta| < \frac{1}{2m} \) while \( \frac{j}{i} \leq m \), and we are done as well.

\( \square \)

We will need a subtle version of Hoeffding’s inequality [1, Theorem 3], which we formulate in the form that suits us best:

**Theorem 2.2.** Let \( X_1, X_2, \ldots, X_m \) be independent (not necessarily identically distributed) random variables, each taking values in the interval \([-1, 1]\) and with a common bound \( v \) on the variance. Then for every \( \epsilon > 0 \) the following inequality holds:

\[
P\left(\overline{X} - E\overline{X} \geq \epsilon\right) \leq \left(1 + \frac{2v}{\epsilon^2} \right)^{-m/2} \cdot (1 - \frac{\epsilon}{2v})^{(1 - \frac{\epsilon}{2v})\frac{m}{2v}},
\]

where \( \overline{X} = \frac{1}{m} \sum_{i=1}^{m} X_i \) and \( E\overline{X} \) stands for the expectation of \( \overline{X} \).

\footnote{Formally, the correlation should be the distance between the average of the product and the product of the averages. But since our aperiodic sequence has zero average, we can skip the latter term.}
Of course, in order to obtain a more convenient upper bound on the above probability, we can replace the expression in square brackets (henceforth denoted by $W$) by a smaller (yet positive) one. First, since in our case it only makes sense to consider $\epsilon < 2$, we can use the facts that $\frac{\epsilon^2}{\epsilon + 4} \geq \frac{\epsilon}{\epsilon}$ and that $x^2 > \frac{1}{2}$ for $x \in (0, 1)$, and write

$$W \geq (1 + \frac{2\epsilon}{\epsilon + 4}) \cdot \frac{\epsilon}{\epsilon + 4} = W_1.$$  

Now we can simply skip the first $1$ and the second (smaller than $1$) exponent:

$$W_1 \geq (\frac{2\epsilon}{\epsilon + 4}) \cdot \frac{\epsilon}{\epsilon + 4} = \frac{1}{2} (2\epsilon)^{-\frac{1}{\epsilon}} = W_2.$$  

Finally, we note that $(2x)^{-\frac{1}{\epsilon}} > \frac{1}{2}$ for $x \in (0, 1)$, hence

$$W_2 > \frac{1}{2} v^{-\frac{1}{\epsilon}} = W_3.$$  

Replacing $W$ by $W_3$ in Hoeffding’s inequality and combining with a symmetric estimate for $-\overline{X}$, we obtain

$$P(\{|X - E[X]| \geq \epsilon\} < 2 \cdot 4^m v^{-\frac{1}{\epsilon}}.$$  

3. The main result

**Theorem 3.1.** Let $y$ be an aperiodic sequence and $N \geq 2$ be a fixed integer. There exists a subshift $\Sigma$ over $N$ symbols of entropy arbitrarily close to $\log N$, uncorrelated to $y$.

**Proof.** We need to guarantee uncorrelation to $y$ of any sequence obtained using a function $f \in C(\Sigma)$ (and a starting point $x$). Since uncorrelation to $y$ is preserved under linear combinations and uniform limits of sequences, it suffices to consider functions $f$ from a family linearly dense in $C(\Sigma)$. We can choose in this role the family consisting of $\{-1, 1\}$-valued functions depending on finitely many coordinates. Indeed, the collection of linear combinations of such functions is an algebra which contains constants and separates points of $\Sigma$, and thus the Stone–Weierstrass Theorem applies. Further, even for bilateral subshifts, it suffices to consider functions which depend on finitely many nonnegative coordinates. Indeed, if a function depends also on some negative coordinates, composing it with an appropriate iterate of the shift we obtain a function depending only on nonnegative coordinates, and which yields the same set of values of the correlation with $y$.

Any $\{-1, 1\}$-valued function $f$ depending on finitely many nonnegative coordinates will be called a code and the horizon of $f$ (denoted by $r_f$) is defined as the minimal $r \geq 1$ such that $f$ does not depend on the coordinates $r + 1, r + 2, \ldots$ (for $f$ constant we have $r_f = 1$). The name “code” is justified by the fact that every such $f$ determines a sliding block code which can be applied to any block $B$ appearing in $\Sigma$ of any length $n$ larger than or equal to $r_f$, producing a block $f(B)$ over $\{-1, 1\}$, of length $n - r_f + 1$. The rule is

$$f(B)_i = f(b_i, b_{i+1}, \ldots, b_{i+r_f-1}).$$

Since there are countably many such codes, we enumerate them by natural numbers (to be used later).

Our goal is to build a special subshift $\Sigma$. This subshift will be the intersection of a nested sequence of subshifts $\Sigma_k$, where each $\Sigma_k$ consists of all (unilateral or bilateral, depending of the preferred type of the subshift) infinite concatenations (and their shifts) of blocks belonging to some family $G_k \subset \Lambda^N_k$, where $\Lambda$ is an alphabet of
cardinality $N$ (i.e., $\mathcal{G}_k$ is a subfamily of blocks of some common length $N_k$). It is an elementary exercise to show that the topological entropy of the intersection of a nested sequence of subshifts equals the limit of their entropies, thus

$$h(\Sigma) = \lim_{k \to \infty} h(\Sigma_k) = \lim_{k \to \infty} \frac{1}{N_k} \log(\#\mathcal{G}_k).$$

We begin the construction by setting $N_0 = 1$ and $\mathcal{G}_0 = \Lambda$. Now $\Sigma_0$ is simply the full shift on $N$ symbols. In each following step $k \geq 1$ of the construction we will refer to several parameters, for which we are about to fix the notation consistently used throughout the remainder of this paper. In the description of the inductive step $k$, in the notation of most of these parameters we will skip the subscript $k$. And so:

- The multiplier $m = m_k$ will play the role of the ratio $\frac{N_k}{N_{k-1}}$; the family $\mathcal{G}_k$ will consist of concatenations of $m$ blocks from $\mathcal{G}_{k-1}$. In the first step $m$ is equal to some large $M \geq 81$, then it tends nondecreasingly to infinity, but very slowly (it is constant on long intervals, rarely increasing by 1). The dependence $k \mapsto m$ will be specified more precisely later.

- The length $N_k$ equals the product $m_1 m_2 \cdots m_k$. Clearly, $M^k \leq N_k \leq m^k$. Since in most formulas this parameter appears several times with different indices, exceptionally, we will never skip the subscript (besides, $N$ is already reserved to denote the cardinality of the alphabet).

All of the following parameters depend on $k$ indirectly, via the multiplier $m$.

- We fix a decreasing to zero sequence of parameters $\epsilon = \epsilon_m$ starting with $\epsilon_M = 1$ and assuming the values $\epsilon_m = \frac{2}{m}$ for $m > M$. Clearly, $\epsilon$ tends to zero with $k$, but very slowly, remaining constant throughout many steps.

- The number $p = m - M$ (which is always strictly less than $k$) will serve as the index of some previous step (called the reference step); we will view the elements of $\mathcal{G}_k$ (and also of $\mathcal{G}_{k-1}$) as concatenations of the blocks from $\mathcal{G}_p$. In the initial step we have $m = M$, so that $p = 0$ and we imagine the elements of $\mathcal{G}_k$ decomposed into elements of $\mathcal{G}_0$ (single symbols).

- We will also refer to the multiplier that was used in step number $p + 1$. According to our notation it is $m_{p+1}$. The number $2^{-m_{p+1}}$ will be denoted by $\delta$. This parameter tends to zero with $k$ (but very very slowly).

- Our estimates in step $k$ will involve some finite collection $\mathcal{F} = \mathcal{F}_k$ of codes. Two conditions must be fulfilled to include a code $f$ in this collection: its index in the ordering of all codes must not exceed $m$, and its horizon $r_f$ must not exceed $\delta N_p$. It is clear that $\#\mathcal{F} \leq m$. The numbers $\delta N_p = N_p 2^{-m_{p+1}} \geq N_p 2^{-p-M} \geq 2^{-M} \left( \frac{M}{2} \right)^p$ tend nondecreasingly to infinity, hence the collections ascend, and every code will eventually be included.

We can now define $\mathcal{G}_k$ more precisely: Suppose that for some $k \geq 1$ the family $\mathcal{G}_{k-1} \subset \Lambda^{N_{k-1}}$ has been established. We define $\mathcal{G}_k$ as the family of all concatenations $B$ of $m$ blocks from $\mathcal{G}_{k-1}$ which satisfy the following requirement:

(R) for every $1 \leq j \leq (m^2 - 1)N_k$ and every $f \in \mathcal{F}$, letting $C = y_j^{j+N_k-1}$ we have $|f(B)|C| < 2(\epsilon + \delta)$.

(by convention, we denote by $\overline{f(B)f'}$ what should formally be $\overline{f(B)f'}$, where $f'$ is $C$ trimmed by $r_f - 1$ terminal symbols, to match the length of $f(B)$). In words, we

\footnote{Attention, this is not true for more general topological dynamical systems.}
require that all images of $B$ under the codes from $\mathcal{F}$ have small correlations with every block of $y$ of length $N_k$, ending before the position $m^2N_k$.

We can identify the family of all concatenations of $m$ blocks from $\mathcal{G}_{k-1}$ with the product space $(\mathcal{G}_{k-1})^m$. Notice that if $\mathcal{G}_{k-1}$ is equipped with the normalized counting measure, then the product measure on $(\mathcal{G}_{k-1})^m$ coincides with the normalized counting measure. Similarly, this measure conditioned on $\mathcal{G}_k$ is the normalized counting measure. In order to estimate (from below) the cardinality of $\mathcal{G}_k$, we need to estimate the probability $\gamma_k$ that a block $B \in (\mathcal{G}_{k-1})^m$ satisfies (R). Then

$$\#\mathcal{G}_k = (\#\mathcal{G}_{k-1})^m \gamma_k.$$  

Recursive application of the above dependence (in which we replace the varying parameter $m$ by $\frac{N_k}{N_{k-1}}$) yields

$$\#\mathcal{G}_k = N^{N_k} \cdot \gamma_1 N^2 \cdot \gamma_2 N^2 \cdot \gamma_3 N^2 \cdots \gamma_{k-1} N^2 \cdot \gamma_k N^2.$$  

This, and the convergence of entropies, allows us to write the entropy of $\Sigma$ as

$$h(\Sigma) = \lim_{N \to \infty} \frac{1}{N} \log(\#\mathcal{G}_k) = \log N + \sum_{k=1}^{\infty} \log(\gamma_k) \cdot \frac{N_k}{N}.$$  

If we arrange (which we will) that all $\gamma_k$ are larger than or equal to $\frac{1}{2}$, then we shall have

$$h(\Sigma) \geq \log N - \log 2\sum_{k=1}^{\infty} \frac{1}{M^k} = \log N - \frac{1}{M-1} \log 2.$$  

So, solely by the choice of the initial multiplier $M$, we will be able make $h(\Sigma)$ as close to $\log N$ as we wish.

Now we specify the assignment $k \mapsto m$. For each $m \geq M$ let the jump step $K_m$ be defined as the first index $k$ such that $m_k = m$. Clearly, $K_M = 1$, so there is no choice, but for $m > M$ we are free to choose the jump steps arbitrarily large. We choose them so that they satisfy two requirements:

(a) In step $K_m$ the reference index $p$ will be increased from $m-1-M$ to $m-M$. Recall that $N_p$ is the length of the blocks built in step $p$. This parameter will not change regardless of how we choose $K_m$. We require that the ratio $\frac{N_p}{N_{p+1}}$ be as large as the maximum of the parameters $L(\epsilon, m^2)$ evaluated for $y$ along all arithmetic progressions of the form $(IN_p + l)_{l \geq 1}$ with $0 \leq l < N_p$. Recall that $y$ has zero average along every arithmetic progression, so Lemma 2.1 applies.

(b) Let $\alpha(m) = m^4 \cdot 2 \cdot 4^m (2N_p)^{\frac{m}{2}}$. We require $K_m$ to be so large that

$$9 \cdot \alpha(m) \cdot (\frac{8}{9})^{K_m-1} < \frac{1}{2m^{1+\epsilon}}.$$  

This concludes the construction of the subshift. Now we need to prove its properties.

In step $k$ let us fix a block $C$ and a code $f$, as they appear in the condition (R). On the probability space $(\mathcal{G}_{k-1})^m$ let us denote by $X$ the random variable $B \mapsto f(B)C$.

Lemma 3.2. With the above notation, for every $k$, $f$ and $C$, we have:

(A) $\sum_{n=p+1}^{N_p} (1 - \gamma_n) < \frac{\delta}{2}$.
(B) $|EX| < \epsilon + 2\delta$. 

(C) \( \gamma_k > 1 - \alpha(m)(\frac{8}{9})^{k-1} \), (which, by (b) and since \( k \geq K_m \), is much larger than \( \frac{1}{2} \)).

Proof. In steps 1, 2, \ldots, \( K_{M+1} - 1 \), the multiplier equals \( M \), the reference index is 0, hence \( \epsilon = 1 \) implying that \( \gamma_k = 1 \) and all three conditions hold trivially.

Fix any \( k \) such that the corresponding multiplier \( m \) is larger than \( M \). We have \( k \geq K_m \). Suppose we have proved the lemma for all indices smaller than \( k \). Then (A) holds. Indeed, using (C) for indices smaller than \( k \), and (b), we can compute as follows:

\[
\sum_{s=p+1}^{k-1} (1 - \gamma_s) \leq \sum_{n \geq m_{p+1}} \sum_{s=K_n}^{K_{n+1}} \alpha(n) \left( \frac{8}{9} \right)^{s-1} \leq \\
\sum_{n \geq m_{p+1}} \alpha(n) \cdot 9 \cdot \left( \frac{8}{9} \right)^{K_n-1} \leq \sum_{n \geq m_{p+1}} \frac{1}{2^{n+2}} = \frac{1}{2^{m_{p+1}+1}} = \frac{\delta}{2}.
\]

We pass to proving (B). We have the following descending sets:

\[
(G_{p+1})_{N_{p+1}}^N \supset (G_{p+1})_{N_{p+1}+1}^N \supset \cdots \supset (G_{k-2})_{N_{k-2}}^N \supset (G_{k-1})_1^N,
\]

each containing blocks of length \( N_k \) concatenated of blocks from some previous step \( s \) of the construction, with \( s \) ranging from the reference index \( p \) to the index \( k - 1 \) of the preceding step. We will denote \( G^{(s)} = (G_{p+1})_{N_{p+1}^N} \) treated as a probability space with the normalized counting measure. Any block \( B \) from any of these spaces decomposes as a concatenation \( Q_1 Q_2 \ldots Q_t \) of blocks from \( G_{p+1} \), with \( q = \frac{N_{p+1}}{N_p} \). Fix a code \( f \in \mathcal{F} \) (we have \( r_f - 1 \ll N_p \)) and we fix a block \( C \) of length \( N_k \) appearing in \( y \), ending before the position \( m^2 N_k \) (only such blocks appear in (R)). This block can be represented as the concatenation

\[
C = U_1 I_1 U_2 I_2 \ldots U_q I_q,
\]

where each \( U_i \) has length \( N_p - r_f + 1 \) and each \( I_i \) has length \( r_f - 1 \). On each of the spaces \( G^{(s)} \) we define the \( p \)-approximate correlation function

\[
(B) \mapsto \frac{1}{q} \sum_{i=1}^{q} f(Q_i) U_i =: \frac{f(B)C}{C}.
\]

It is obvious that \( \frac{f(B)C}{C} \) differs from \( \frac{f(B)C}{C} \) by at most \( \frac{r_f-1}{N_p} \) (less than \( \delta \)), because this is the contribution of \( \{ -1,0,1 \} \)-valued terms in the evaluation of \( \frac{f(B)C}{C} \) not included in the evaluation of \( \frac{f(B)C}{C} \). We have

\[
\frac{f(Q_i) U_i}{C} = \frac{1}{N_p - r_f + 1} \sum_{l=1}^{N_p - r_f + 1} (f(Q_i))_l (U_i)_l,
\]

where \( f(Q_i)_l \) and \( U_i)_l \) are single symbols in \( f(Q_i) \) and \( U_i \), respectively. Changing the order of summation, we get

\[
\frac{f(B)C}{C} = \frac{1}{N_p - r_f + 1} \sum_{l=1}^{N_p - r_f + 1} \frac{1}{q} \sum_{i=1}^{q} (f(Q_i))_l (U_i)_l.
\]
In evaluating the expected value, which will be denoted by $E^{(s)}(\overline{f(\cdot)C})$, over any of the above spaces $G^{(s)}$, the terms $(U_i)_i$ are constant, so we can write

$$E^{(s)}(\overline{f(\cdot)C}) = \frac{1}{N_p - r_f + 1} \sum_{i=1}^{N_p - r_f + 1} \frac{1}{q} \sum_{i=1}^{q} (U_i)_i E^{(s)}(f_{i,l}),$$

where $f_{i,l}$ is the $\{-1, 1\}$-valued variable $B \mapsto (f(Q_i))_l$ selecting one symbol of $f(B)$ (precisely, the symbol at the position $iN_p + l$).

Now, on the largest space $G^{(p)}$, i.e., on all blocks of length $N_k$ which are concatenations of blocks from $G_p$, the variables $f_{i,l}$ with a common index $l$ have the same distribution for all indices $i$, and hence a common expected value denoted $E^{(p)}_l$ (note that $|E^{(p)}_l| \leq 1$). Then

$$E^{(p)}(\overline{f(\cdot)C}) = \frac{1}{N_p - r_f + 1} \sum_{i=1}^{N_p - r_f + 1} E^{(p)}_l \frac{1}{q} \sum_{i=1}^{q} (U_i)_l.$$

The last average is the average of $y$ along an arithmetic progression with step $N_p$ and consisting of $q = \frac{N_k}{N_p}$ terms, contained in the first $m^2 \frac{N_k}{N_p}$ terms. Since $k \geq K_m$, and hence $\frac{N_k}{N_p} \geq \frac{N_k}{N_p} \geq L(\epsilon, m^2)$, the condition (a) implies that the last average (for every $l$) is less than $\epsilon$, and thus so is the double average.

Having proved that $E^{(p)}(\overline{f(\cdot)C}) < \epsilon$, we need to control how this expected value changes as we pass to smaller spaces $G^{(s)}$, till we reach $G^{(k-1)}$. Decomposing each $B \in G^{(s-1)}$ into $B \in N_k$ subblocks $B_i \in G_s$ and also decomposing $C$ into corresponding subblocks $C_i$ of length $N_s$, we can write

$$\overline{f(B)C} = \frac{N_s}{N_k} \sum_{i=1}^{N_s} \overline{f(B_i)C_i},$$

Since for $\tau \in \{s-1, s\}$ the blocks $B_i$ range over a set $G_{(\tau)}$ independent of $i$ (more specifically, $G_{(s-1)} = (G_{s-1})^{N_k}$ and $G_{(s)} = G_s$), we have

$$E^{(\tau)}(\overline{f(\cdot)C}) = \frac{N_s}{N_k} \sum_{i=1}^{N_s} E^{(\tau)}(\overline{f(\cdot)C_i}),$$

where the latter expectation is over $G_{(\tau)}$ (the dot represents the varying block $B_i$). Now, in the passage from $\tau = s - 1$ to $\tau = s$ we must renormalize the measure from $(G_{s-1})^{N_k}$ to $G_s$. The expected value of any function with values in $[-1, 1]$ may change by at most $2(1 - \gamma_s)$. After averaging over $i$ we get

$$|E^{(s)}(\overline{f(\cdot)C}) - E^{(s-1)}(\overline{f(\cdot)C})| \leq 2(1 - \gamma_s).$$

Composing over $s = p + 1, \ldots, k - 1$ and using (A), we arrive at

$$|E^{(k-1)}(\overline{f(\cdot)C})| < \epsilon + 2 \sum_{s=p+1}^{k-1} (1 - \gamma_s) < \epsilon + \delta.$$
Recall that $X$ appearing in the condition (B) equals $\overline{f(B)C}$ restricted to $G^{(k-1)}$, and that $\overline{f(B)C}$ differs from $\overline{f(B)C}$ by less than $\delta$. Thus we can conclude the proof of (B):

$$|EX| = |E^{(k-1)}(\overline{fB})| < \epsilon + 2\delta.$$  

We pass to the proof of (C). We will need once again to refer to the $p$-approximate correlations $\overline{f(B)C}$, but now on the spaces $G_s$ ($s = p, \ldots, k - 1$), (the blocks $B$ will now have lengths $N_s$) and on the space $(G_{k-1})^m$ (on which we have already been working). The code $f$ remains fixed, and $C$ is any block of the appropriate length ($N_s$ or $N_k$) appearing in $y$, ending before the position $m^2 N_k$.

Suppose $v_{s-1}$ is a bound on the variance of all $p$-approximate correlations on $G_s$. Then the variance on the space $(G_{s-1})^m$ is at most $\frac{N_s - 1}{N_s} v_{s-1}$, because we are averaging independent random variables. Now, $G_s$ is a subset of the above product space, where it has measure $\gamma_s$, which, by the inductive assumption, is larger than $\frac{1}{2}$. Conditioning on such a subset can enlarge the variance at most 4 times. Thus we obtain the estimate

$$v_s \leq 4 \frac{N_s - 1}{N_s} v_{s-1}.$$  

By recursive application of the above, we can estimate $v_{k-1}$ referring to the step $p$ and safely estimating the variances on $G_p$ by $v_p = 2$ (our variables take values in $[-1, 1]$):

$$v_{k-1} \leq 4^{k-1-p} \frac{N_p}{N_{k-1}} \cdot 2 \leq 2 N_p 4^{k-1}. $$

Denote by $\overline{X}$ (to match the notation in (2.1)), the $p$-approximate correlation regarded on the product space $(G_{k-1})^m$ (on which it is indeed the average of $m$ independent random variables). Now (2.1) applies and reads:

$$P\{|\overline{X} - E\overline{X}| \geq \epsilon\} \leq 2 \cdot 4^m v_{k-1} \gamma^m \leq 2 \cdot 4^m (2N_p)^{\frac{\epsilon}{v_{k-1}}} \gamma.$$  

(we have also used the equality $\epsilon = \frac{\alpha}{m}$). The expectation $E\overline{X}$ coincides with what was previously denoted by $E^{(k-1)}(\overline{f(B)C})$, so, by the already proved inequality (3.2), we have $|E\overline{X}| < \epsilon + \delta$. We can thus continue:

$$P\{|\overline{X} - E\overline{X}| \geq \epsilon\} \geq P\{|\overline{X} \geq 2\epsilon + \delta\}.$$  

Recall that $\overline{X}$ differs from the corresponding correlation function $B \mapsto \overline{f(B)C}$ by less than $\delta$. This implies that

$$P\{|\overline{f(B)C}| \geq 2(\epsilon + \delta)\} \leq 2 \cdot 4^m (2N_p)^{\frac{\epsilon}{v_{k-1}}} \gamma.$$  

The above concerns the probability on $(G_{k-1})^m$, a fixed block $C$ in $y$ and a fixed code $f \in F$. The condition (R) requires the inequality $|\overline{f(B)C}| \geq 2(\epsilon + \delta)$ to be satisfied for $(m^2 - 1) N_k \leq m^3 N_{k-1}$ blocks $C$ and all codes $f \in F$. Since $\# F \leq m$, the overall probability $1 - \gamma_k$ of a block $B$ failing (R) is estimated by

$$m^4 N_{k-1} \cdot 2 \cdot 4^m (2N_p)^{\frac{\epsilon}{v_{k-1}}} \gamma \leq \alpha(m) \frac{4^{k-1}}{N_{k-1}} \gamma \leq \alpha(m) \frac{4^{k-1}}{\sqrt{N_{k-1}}} \gamma \leq \alpha(m) \frac{4^{k-1}}{\frac{1}{8}} \gamma \leq \alpha(m) \frac{4^{k-1}}{\frac{1}{8}} \gamma.$$  

(recall that $M \geq 81$). This ends the proof of (C) and thus of the lemma. 

\footnote{To see this, write the variance as $\iint_{\mathbb{R}^2} (x - y)^2 \, dx \, dy$ where $dx$ and $dy$ stand for the distribution on $\mathbb{R}$ of the random variable. The set on which we condition in $\mathbb{R}^2$ has measure larger than $\frac{1}{4}$.}
Since we have proved, in particular, that all $\gamma_k$ are larger than $\frac{1}{2}$, it now becomes certain that the entropy of $\Sigma$ can be made arbitrarily close to $\log N$. It remains to prove lack of correlation between $\Sigma$ and $y$. Let $f$ be any $\{-1, 1\}$-valued function depending on finitely many nonnegative coordinates. Fix some point $x \in \Sigma$ and pick $n \in \mathbb{N}$. Let $k$ be the smallest integer such that $n < m^2 N_k$. If $f$ is not in $\mathcal{F} = \mathcal{F}_k$ then we simply must pick a larger $n$. So, we can assume that $f \in \mathcal{F}$. Now, $x \in \Sigma_k$, which means that $x^n_1$ is a concatenation of the blocks from $\mathcal{G}_k$, except that the first and last component blocks may be incomplete. The contribution of these parts in the length is at most $\frac{2N_k}{n}$, and since $n \geq m^2 N_k - 1 \geq (m-1)^2 N_k - 1 > (m-2) N_k$, this contribution is less than $\frac{2(m-2)}{m^2 k}$, and such is also the maximal contribution of these parts in the evaluation of the correlation between $x^n_1$ and $y^n_1$. The rest of the correlation is the average of the correlations of the complete component blocks from $\mathcal{G}_k$ with their respective subblocks of length $N_k$ of $y$. Since all these subblocks end before the position $m^2 N_k$, by (R), each of these correlations is less than $2(\epsilon + \delta)$ in absolute value. Jointly, the absolute value of the correlation of $x^n_1$ with $y^n_1$ does not exceed

$$\frac{2(m-2)}{m^2} \cdot 1 + \frac{m-4}{m^2} \cdot 2(\epsilon + \delta).$$

Obviously, as $n$ grows, so does $k$, and so does $m$, while both $\epsilon$ and $\delta$ tend to zero. This proves the desired uncorrelation condition concluding the entire proof of the main result. \(\Box\)

References

[AKLR] H. El Abdalaoui, J. Kuś, M. Lemańczyk, T. de la Rue, The Chowla and the Sarnak conjectures from ergodic theory point of view (extended version), preprint, https://arxiv.org/pdf/1410.1673.pdf

[AKL] H. El Abdalaoui, S. Kasjan, M. Lemańczyk, $0$-$1$ sequences of the Thue-Morse type and Sarnak’s conjecture, Proceedings of the American Mathematical Society. 144 (2016), 161–176

[DK] T. Downarowicz, S. Kasjan, Odometers and Toeplitz systems revisited in the context of Sarnak’s conjecture, Studia Mathematica 229 (2015), 45–72

[FH] N. Frantzikinakis and B. Host, Higher order Fourier analysis of multiplicative functions and applications, J. Amer. Math. Soc. 30 (2017), 67–157

[H] W. Hoeffding, Probability inequalities for sums of bounded random variables, Journal of the American Statistical Association 58 (1963), 13–30

[HWZ] W. Huang, Z. Wang and G. Zhang, M"{o}bius disjointness for topological models of ergodic systems with discrete spectrum, preprint, https://arxiv.org/pdf/1608.08289.pdf

[K] D. Karagulyan, On M"{o}bius orthogonality for subshifts of finite type with positive topological entropy, preprint

[S] P. Sarnak, Three lectures on the M"{o}bius function randomness and dynamics, http://publications.ias.edu/sites/default/files/MobiusFunctionsLectures2.pdf.

[W] B. Weiss, Single Orbit Dynamics, CBMS Regional Conference Series in Mathematics (2000)