Elimination of loop-check for logic of idealized knowledge

Aida PLIUŠKEVIČIENĖ
Institute of Mathematics and Informatics
Akademijos 4, LT-08663 Vilnius, Lithuania
e-mail: aida@ktl.mii.lt

Abstract. In the paper loop-check-free sequent calculus for logic of idealized knowledge is presented. To obtain termination of derivation indices and marks are used instead of history.

Keywords: modal logics, sequent calculus, termination of derivation, loop-check, indexation.

1. Introduction

Traditional techniques used for test termination of backward proof search in modal (e.g., knowledge-based) sequent (and tableau) calculi are based on loop-check. Namely, before applying any rule it is checked if this rule was already applied to “essentially the same” sequent; if this is the case we block the application of the rule. In [2,4] efficient loop-check for modal logics KT, K4, S4, tense logic Kt, and a fragment of intuitionistic logic was presented using sequents extended by notion of history. In [5] a contraction-free (i.e., loop-check-free) calculus for mono-modal logic S5 is presented. In this paper instead of loop-check and history the method of indexation is used. With a view to construct a cut-free calculus for S5 Kanger [3] has introduced the index method. Later on Fitting [1] has developed the prefixed tableau method for various modal logics. Here multi-modal logic S5 is considered. This modal logic is used in artificial intelligence and computer science and is considered as the logic of idealized knowledge. The aim of this paper is to get a specialization of derivations for the multi-modal logic S5 that allows us to present loop-check-free decision procedure. To this aim we have used indexing as well and construct invertible loop-check-free sequent calculus. The result presented in the paper is an extension and revision of the results obtained in [6]. In the paper the multi-modal logic S5 when n = 2 (in notation S5^2) is considered.

2. Initial sequent calculus for S5^2

To get cut-free sequent calculus for mono-modal logic S5 Kanger proposed to use indexed propositional symbols along with usual propositional ones [3]. The indices in Kanger-style sequent calculus for S5 are arbitrary natural numbers. In Kanger-style sequent calculus KS5 for logic S5 a list (possibly empty) consisting of the ordered pairs (k, l) (where k ∈ {1, 2} and l is either zero or an arbitrary natural number) is
Elimination of loop-check for logic of idealized knowledge

used as index. Let $P$ be a propositional symbol, then $P^γ$ is an indexed propositional symbol. Let $γ$ be any index, then indexation procedure of formula in $S5_2$ is as follows:

1. $(P^γ)^{[i,r]} = P^{γ^{[i,r]}}$, where $i \in \{1, 2\}$;
2. $(A \odot B)^γ = A^γ \odot B^γ$, where $\odot \in \{\odot, \land, \lor\}$.
3. $(\neg A)^γ = \neg A^γ$;
4. $(\Box_j A)^γ = \Box_j A^γ$ ($j \in \{1, 2\}$).

In addition, for arbitrary indices $γ$ and $γ_1$ it is assumed that $P^{γ,\langle i,0 \rangle}, γ_1 = P^{γ,γ_1}$, and $P^{γ,\langle i,l \rangle}, γ_1 = P^{γ,\langle i,r \rangle}, γ_1$, where $i \in \{1, 2\}$.

A sequent is a formal expression $\Gamma \rightarrow \Delta$, where $\Gamma, \Delta$ are multisets of formulas.

Let $KS_{52}$ be a calculus obtained from invertible Kanger-style logical calculus [3] adding the following modal rules:

\[
\frac{\Gamma \rightarrow \Delta, A^{[i,r]}}{\Gamma \rightarrow \Delta, \Box_i A} (\rightarrow \Box_i) \quad \frac{\Box_i A, \Gamma \rightarrow \Delta}{\Box_i A, \Gamma \rightarrow \Delta} (\Box_i \rightarrow),
\]

where $i \in \{1, 2\}$; in the rule $(\rightarrow \Box_i)$ a natural number $r \in \{1, 2, \ldots\}$ is such that any index in the conclusion does not contain a pair $(i, r)$; in the rule $(\Box_i \rightarrow)$ value of metavariable $α$ is either zero or natural number defined as follows: if any index in the conclusion does not contain a pair of the shape $(i, b)$ where $b$ is an arbitrary natural number, then $α$ is zero, otherwise value of metavariable $α$ is natural number $l \in \{1, 2, \ldots\}$ such that $(i, l)$ enters in some index of the conclusion.

As in [6] the following theorems can be proved.

**Theorem 1.** The calculus $KS_{52}$ is a conservative extension of a traditional Hilbert-style calculus $HS_{52}$, i.e., the calculus $KS_{52}$ is sound and complete.

**Theorem 2.** The structural rules of weakening, contraction, and cut are admissible in $KS_{52}$.

From the admissibility of the weakening rules it follows the invertibility of the rule $(\Box_i \rightarrow)$.

A derivation in a calculus $I$ is called an atomic one if the main formula of an axiom is a propositional symbol. It is obvious that backward applying rules of $KS_{52}$ each derivation in $KS_{52}$ can be reduced to an atomic one with the same end-sequent.

By induction on the height of derivation $V$ denoted by $h(V)$ we can prove

**Lemma 1.** Let $S (S_1)$ be a conclusion (premise, correspondingly) of a logical rule or the rule $(\rightarrow \Box_i)$. Let $KS_{52} \vdash^V S$ where $V$ is an atomic derivation of $S$ in $KS_{52}$ and $h(V)$ is a height of this derivation. Then $KS_{52} \vdash^{V^*} S_1$ and $h(V^*) < h(V)$.

A backward proof search in the calculus $KS_{52}$ is not terminative, in general. Indeed, let $S$ be a sequent $\Box_i (P \lor Q) \rightarrow P$. Then the backward proof search contains an infinitive branch because we repeatedly get almost the same sequents $S_m = \overbrace{Q, \ldots, Q}^{\text{m times}}, \Box_i (P \lor Q) \rightarrow P$, $m \in \{1, 2, \ldots\}$.
To prune the infinite branch the method of loop-check [1] is used. Since the sequents
$S_1$ and $S_2$ are almost the same we block applications of the rules ($\Box_i \to$) and ($\lor \to$) and conclude that $KS_{S_2} \not\vdash S$.

3. Specialization of modal rules

With the aim to construct loop-check-free sequent calculus for $S_{S_2}$ let us introduce
some specialization of the modal rules ($\to \Box_i$) and ($\Box_i \to$). At first, a notion of
primary sequent is defined.

**Definition 1.** A sequent $S$ is a primary one if $S = \Sigma_1, \Box_1 \Gamma_1, \Box_2 \Gamma_2 \to \Sigma_2, \Box_1 \Delta_1, \Box_2 \Delta_2$, where $\Sigma_i$ ($i \in \{1, 2\}$) is empty or consists of propositional symbols, $\Box_i \Gamma_j$ and $\Box_j \Delta_i$ ($i \in \{1, 2\}$) is empty or consists of the formulas of the shape $\Box_i A$.

Using invertibility of logical rules we can prove

**Lemma 2** (reduction to primary sequents). It is possible automatically construct
a reduction of a sequent $S$ to a set $\{S_1, \ldots, S_m\}$, where $S_j$ ($1 \leq j \leq m$) is a primary sequent. Moreover, if $K_1 S_{S_2} \vdash V$, where $V$ is an atomic derivation, then $K_1 S_{S_2} \vdash V_j (j \in \{1, \ldots, m\})$ and $h(V_j) < h(V)$.

Let $K_1 S_{S_2}$ be a calculus obtained from the initial calculus $KS_{S_2}$ replacing the
modal rules ($\to \Box_i$) and ($\Box_i \to$) by the following ones:

$$
\frac{\Sigma_1, \Box_1 \Gamma_1, \Box_2 \Gamma_2 \to \Sigma_2, \Box_1 \Delta_1, \Box_2 \Delta_2, A^{(i,r)}}{\Sigma_1, \Box_1 \Gamma_1, \Box_2 \Gamma_2 \to \Sigma_2, \Box_1 \Delta_1, \Box_2 \Delta_2, \Box_i A (\to \Box_i \to)}
$$

$$
\frac{\Sigma_1, A^{(i,\alpha)}, \Box_1 \Gamma_1, \Box_2 \Gamma_2 \to \Sigma_2, \Box_1 \Delta_1, \Box_2 \Delta_2}{\Sigma_1, \Box_1 \Gamma_1, \Box_2 \Gamma_2 \to \Sigma_2, \Box_1 \Delta_1, \Box_2 \Delta_2 (\Box_i \to)}
$$

where conclusion of the rule is a primary sequent and $\Sigma_j$ ($j \in \{1, 2\}$) is empty or consists of propositional symbols, moreover, $\Sigma_1 \cap \Sigma_2$ is empty. In the premises of these rules the indices $(i, r)$ and $(i, \alpha)$ are defined as in the rules ($\to \Box_i$) and ($\Box_i \to$), correspondingly.

Relying on Lemma 2 we get

**Lemma 3.** $KS_{S_2} \vdash S$ if and only if $K_1 S_{S_2} \vdash S$.

4. Loop-check-free sequent calculus for $S_{S_2}$

Along with usual modality $\Box_i$ let us introduce a marked modality $\Box_i^+$ which has the
same semantical meaning as non-marked modality $\Box_i$ and serves as a stopping device
for a backward application of the rules ($\to \Box_i$) and ($\Box_i \to$).

Let us introduce operation $*$ defined in the following way:

1. $(P^\gamma)^* = P^\gamma$, where index $\gamma$ can be empty.
2. \((A \oplus B)^* = A^* \oplus B^*,\) where \(\oplus \in \{\cap, \land, \lor\}.

3. \((\neg A)^* = \neg A^*.

4. \((\Box_i A)^* = \Box_i^* A^* (i \in \{1, 2\}).

5. \((\Box_i^* A = \Box_i^* A (i \in \{1, 2\}).

Let \(K_1^i S_52\) be a calculus obtained from the calculus \(K_1 S_52\) replacing the rules
\((\rightarrow \Box_i)\) and \((\rightarrow \Box_i^*)\) by the following marked rules:

\[
\begin{align*}
\Sigma_1, \Box_i^{\alpha} \Gamma_1, \Box_i \Theta_1, \Box_2^{\alpha} \Gamma_2, \Box_2 \Theta_2 & \rightarrow \Sigma_2, \Box_i^{\alpha} \Pi_1, \Box_1 \Delta_1, \Box_2^{\alpha} \Pi_2, \Box_2 \Delta_2, A^{i, r} \\
\Sigma_1, \Box_i^{\alpha} \Gamma_1, \Box_i \Theta_1, \Box_2^{\alpha} \Gamma_2, \Box_2 \Theta_2 & \rightarrow \Sigma_2, \Box_i^{\alpha} \Pi_1, \Box_1 \Delta_1, \Box_2^{\alpha} \Pi_2, \Box_2 \Delta_2, \Box_i A
\end{align*}
\]

\[
\begin{align*}
\Sigma_1, \Box_i^{\alpha} A^*, \Box_i^{\alpha} \Gamma_1, \Box_i \Theta_1, \Box_2^{\alpha} \Gamma_2, \Box_2 \Theta_2 & \rightarrow \Sigma_2, \Box_i^{\alpha} \Pi_1, \Box_1 \Delta_1, \Box_2^{\alpha} \Pi_2, \Box_2 \Delta_2, \Box_i A
\end{align*}
\]

where \(\Sigma_j (j \in \{1, 2\})\) is empty or consists of propositional symbols, moreover, \(\Sigma_1 \cap \Sigma_2\) is empty; in the conclusion of the rules the modality \(\Box_i\) in the main formula \(\Box_i A\) is not marked; in the premises of the rules the indices \((i, r)\) and \((i, \alpha)\) are defined as in the rules \((\rightarrow \Box_i)\) and \((\rightarrow \Box_i^*)\), correspondingly; in the premise of the rule \((\rightarrow \Box_i^*)\),

If \(i = j (j \in \{1, 2\})\) then \(\Box_j^{\alpha} \Gamma_j = \Box_j^* \Gamma_j\) otherwise \(\Box_j^{\alpha} \Gamma_j = \Box_j^* \Gamma_j\).

Lemma 4. If \(K_1^i S_52 \vdash \Sigma\) then \(K_1 S_52 \vdash \Sigma\) where \(\Sigma\) does not contain occurrences of marked modality.

Proof. Let us replace the marked modalities \(\Box_i^* (i \in \{1, 2\})\) in the derivation \(\Sigma\) by the non-marked ones. As a result we get that each application of marked modal rule transforms into application of non-marked modal rule.

To prove the inverse implication let us introduce marked rules \((\rightarrow \Box_i^{\alpha \rightarrow})\) and \((\rightarrow \Box_i^{\alpha \rightarrow})\) which are obtained from the rules \((\rightarrow \Box_i^*)\) and \((\rightarrow \Box_i^*)\) correspondingly by replacing the non-marked main formula \(\Box_i A\) by the marked formula \(\Box_i^* A\), i.e., the main formula \(\Box_i^* A\) is marked.

Let \(K_1^{\alpha \rightarrow} S_52\) be an auxiliary calculus obtained from the calculus \(K_1^i S_52\) adding the rules \((\rightarrow \Box_i^{\alpha \rightarrow}), (\rightarrow \Box_i^{\alpha \rightarrow})\).

We can prove the following

**Lemma 5.** If \(K_1^{\alpha \rightarrow} S_52 \vdash \Sigma\) then \(K_1 S_52 \vdash \Sigma\), where \(\Sigma\) does not contain marked modality.

Using this lemma we get

**Lemma 6.** If \(K_1 S_52 \vdash \Sigma\) then \(K_1 S_52 \vdash \Sigma\).

From Lemmas 4 and 6 we get

**Lemma 7.** \(K_1 S_52 \vdash \Sigma\) if and only if \(K_1 S_52 \vdash \Sigma\).
Relying on Lemmas 3, 7 and Theorem 1 we get

**Theorem 3.** The calculus $K^*_1 S 52$ is sound and complete.

Using traditional technique we can prove

**Lemma 8.** Let $S$ be a conclusion (premise, correspondingly) of any rule of $K^*_1 S 52$. Let $K^*_1 S 52 \vdash V S$ where $V$ is an atomic derivation of $S$. Then $K^*_1 S 52 \vdash S$.

A primary sequent $S$ of the shape $\Sigma_1, \Box_1 \Gamma_1, \Box_2 \Gamma_2 \rightarrow \Sigma_2, \Box_1 \Delta_1, \Box_2 \Delta_2$ is a critical one if $\Sigma_1 \cap \Sigma_2$ is empty. A derivation $V$ of a sequent $S$ in $K^*_1 S 52$ is successful if each branch of $V$ ends with an axiom. In this case $K^*_1 S 52 \vdash S$. A derivation $V$ of a sequent $S$ in $K^*_1 S 52$ is unsuccessful if $V$ contains a branch ending with a critical sequent. In this case $K^*_1 S 52 \nvdash S$.

From Lemma 8 on invertibility of the rules of $K^*_1 S 52$ and shape of these rules we get

**Theorem 4.** $K^*_1 S 52$ is a loop-check-free decidable calculus, i.e., for any sequent $S$ there exists a successful or unsuccessful derivation without loops of the sequent $S$ in $K^*_1 S 52$.

**Example 1.** Let us construct a derivation of the sequent $S = \Box_1 P \rightarrow \Box_1 \neg \Box_2 \neg P$ in $K^*_1 S 52$.

$$S^* = P(1, \beta | \beta \in \{1\}), \Box_1 P, \Box_2 \neg P(1, 1) \rightarrow P(1, 1), (2, \alpha | \alpha \in \{0\}) (\Box_1 \rightarrow)$$

$$S' = \Box_1 P, \Box_2 \neg P(1, 1) \rightarrow P(1, 1), (2, \alpha | \alpha \in \{0\}) (\Box_2 \rightarrow), (\neg \rightarrow)$$

$$\Box_1 P, \Box_2 \neg P(1, 1) \rightarrow (\neg \neg)$$

$$S = \Box_1 P \rightarrow \Box_1 \neg \Box_2 \neg P$$

Choosing $\beta = 1$ and $\alpha = 0$ and applying indexation procedure, we get that the sequent $S^*$ is an axiom of the shape $P(1, 1), \Box_1 P, \Box_2 \neg P(1, 1) \rightarrow P(1, 1)$.

**References**

1. M. Fitting. *Proof Methods for Modal and Intuitionistic Logics.* D.Reidel, Dordrecht, Holland, 1983.
2. A. Heuerding, M. Seyfried, H. Zimmermann. Efficient loop-check for backward proof search in some non-classical propositional logics. *Lecture Notes in Computer Science*, 1071:210–225, 1996.
3. S. Kanger. *Provability in Logic.* Almgvist&Wiksell, Stockholm, 1957.
4. M. Mouri. Constructing counter-models for modal logic K4 from refutation trees. *Bulletin of the Section of Logic*, 31(2):81–90, 2002.
5. F. Poggiolesi. A cut-free simple sequent calculus for modal logic S5. *The Review of Symbolic Logic*, 1(1):3–15, 2008.
6. A. Plisškevičienė. Specialization of derivations in modal logic S5. *Liet. matem. rink.*, 46(spec. nr.):242–246, 2006.
REZIUME

A. Plisučienė. Ciklų tikrinimo eliminavimas idealaus žinojimo logikai

Pateiktas korektiškas ir pilnas sekvencinis skaičiavimas idealaus žinojimo logikai. Naudojant sukonsuota skaičiavima gaunama neturinti ciklų išpregrąžiamojo procedūra. Vietoje istorijų sąvokos išvedimų baigtiniu naudojami indeksai ir žymės.

Raktiniai žodžiai: modalumo logikos, sekvencinis skaičiavimas, išvedimo baigtinumas, ciklų tikrinimas, indekswagenimas.