HS in flat spacetime. YM-like models

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Abstract: We introduce and analyze a few examples of massless higher spin theories in Minkowski spacetime. They are defined in terms of master fields, i.e. fields defined in the whole phase space. More specifically we introduce the HS YM-like theories in any dimension and HS CS-like ones in any odd dimension, in both Abelian and non-Abelian cases. These theories are invariant under gauge transformations that include ordinary gauge transformations, diffeomorphisms and HS gauge transformations. They are not at first sight invariant under local Lorentz transformations, but we show how this invariance can be recovered. We explicitly write down the actions, the eom’s as well as the (infinite many) conservation laws in both HS YM and HS CS cases. Then we focus in particular on the HS YM models, we illustrate their $L_\infty$ structure and perform their BRST quantization. We also introduce HS scalar and fermion master fields and show that the Higgs mechanism can be realized also in the case of HS YM theories. Next we start the discussion of the perturbative approach to quantization by means of Feynman diagrams. We show that the dependence on the conjugate momentum can be absorbed in a redefinition of the component fields, the coupling and the coordinates. The consequence is that a mass scale shows up. In such a new frozen momentum framework, we carry out a sample calculation. Finally we show that these theories do not respect a few basic hypotheses on which the no-go theorems on massless HS particles in flat background rely.

Keywords: []
1. Introduction

It is accepted nowadays that higher spin (HS) theories in dimension larger than 2, except for 3d examples, must involve an infinite number of (local) fields (for HS theories, see [1]). This characteristic, which was seen in the past as unattractive (to say the least), may actually be an inevitable feature of any theory with the ambition of unifying all the forces of nature. It is not yet clear why this is so, but there exist several hints in this direction. First and foremost (super)string theory, which is still the most authoritative example, has this feature. But also the AdS/CFT correspondence has shown that we may well limit ourselves to a (conformal or quasi-conformal) field theory on the boundary of AdS, but if we wish to resolve its singularities we had better consider the dual theory, which is a (super)string theory (and, so, has infinite many fields). Other arguments suggest that, when gravity is involved, infinite many local fields of increasing spins are necessary in order to avoid a conflict with causality, [2, 3]. On the other hand the infinite number of fields with increasing spins is related to the good UV behavior of string theory. Therefore HS theories are at the crossroad of many important themes: locality, causality, calculability.

The previous considerations are the general underlying motivation for our research on HS theories. However in this as well as in the previous paper, [4] referred to as I, we concentrate on a specific problem, for which there is no answer yet in the literature: can one formulate a sensible local massless HS theory in a Minkowski space-time (the issue of masslessness is fundamental here, being related to gauge invariance)? Actually the general attitude in the literature, for dimensions larger than 3, is skeptical. This is due to two reasons. The first is the so-called no-go theorems, which prevent the existence of such theories under rather general conditions. The second is experimental theory: the construction of fully interacting HS theories has been so far successful in AdS spaces (but see [5]), but

\footnote{For 3d models see [6].}
unsuccessful in flat spacetimes \[8, 13, 14\]. In this paper we present examples of theories defined in flat spacetime in any dimension, which are massless, HS gauge invariant, Poincaré invariant, classically consistent and fully interacting, and seem not to be unmanageable from the quantum point of view.

Let us be more specific. In I we have improved the analysis started in [12] of the effective action produced by integrating out the fermions in a theory of free fermions coupled to external potentials and quantized according to the worldline quantization [13, 14, 15, 16, 17, 18, 19, 20]. In particular we have developed methods to compute current correlators; we have clarified the relation with the analogous effective action obtained by integrating out the scalar field coupled to external sources; finally we have analysed the possible obstructions (anomalies) in the construction of the effective action. At this juncture we are faced with two possibilities. The first is to explicitly compute the above mentioned current correlators, much in the same way as in [21, 22, 23], and explicitly determine the effective action. We are guaranteed by the Ward identities (barring anomalies) that the resulting effective action will be HS gauge invariant and will lead to a realization of an \( L_\infty \) symmetry. However the experience made in [21, 22, 23] tells us that most likely the resulting effective action will be non-local. This in itself is not a negative feature, because we know that the completion of the Fronsdal program [24], at least at the linear level, requires non-local terms in the action [27]. The problem is the lack of an evident symmetry pattern in the two and three-point correlators (the most accessible ones), which makes it very difficult to reconstruct the effective action to all perturbative orders.

This first program may still be viable and worth pursuing, but there is perhaps a second possibility, a sort of shortcut. It consists in exploiting the (already remarked in I) analogy of the HS gauge transformations with the gauge transformations in ordinary non-Abelian gauge theories, to construct analogous HS invariants and covariant objects and in particular actions. As we show below this is rather elementary and allows us to directly ‘integrate the \( L_\infty \) algebra’, that is to find explicit equations of motion that satisfy the \( L_\infty \) axioms\(^3\). They are derived from HS gauge invariant \emph{primitive actions} represented by integrals of local polynomials of \emph{master fields} in the phase space. In this way we can define perturbatively local HS Chern-Simons in flat spacetimes in any odd dimensions, as well as HS Yang-Mills theories in any dimension, of which we can also easily carry out the BRST quantization. We can also define matter master field models and couple them to the previous theories and, for instance, reproduce the Higgs mechanism in the HS context. It is very likely possible to define other types of theory as well.

In this paper we focus in particular on the HS YM models. They are defined by means of a master field whose first two component can be identified with an ordinary gauge field and a vielbein fluctuation, respectively. They are characterized by a unique coupling constant, like the ordinary YM theories, and by invariance under HS gauge transformations.

\(^2\)Complete models, which however seem to be characterized by a trivial S matrix, have been formulated by means of chiral fields in a light-cone framework, see [24]. Partial attempts to construct consistent interacting vertices are numerous, see [13].

\(^3\)For a first introduction to \( L_\infty \), see [26], for the mathematical side see [27], for physical applications [28].
which include in particular ordinary gauge transformations and diffeomorphisms. However covariance is not attained by replacing ordinary derivatives by covariant ones like in many earlier attempts. In fact the way gravity appears in these models is different from the familiar Einstein-Hilbert theory, it is rather similar to teleparallel gravity. The formalism lends itself to interpret the first two component fields of the relevant master field as gauge and gravity fluctuations, an interpretation reinforced by the disclosure of a hidden local Lorentz covariance. This interpretation will be further analyzed in paper III.

A new feature appears when one tackles the problem of perturbative quantization by means of Feynman diagrams. We refer to the appearance of a mass scale. This comes down naturally from the momentum space integration and naturally follows from the worldline quantization. In this paper we only broach this problem, we make some sample computations, which are however enough to show that the perturbative approach is well defined, although it might not be technically easy when one comes to explicit computations. The final issue we consider here is the problem of no-go theorems for massless particles in flat background, (see also [31, 32, 33, 34]). In the last part of the paper we show that a few hypotheses on which the no-go theorems rely, notably the minimal coupling to gravity and the polynomial structure of the conserved charges, are not respected in HS YM theories.

The paper is organized as follows. In section 2 we review the effective action method, focusing in particular on the HS gauge transformations and their interpretation. In section 3 we define the HS YM and HS CS theories in both Abelian and non-Abelian case. Section 4 is devoted to the eom’s and the conservation laws in both HS YM and HS CS case. We discuss also the $L_{\infty}$ structure and the BRST quantization of the HS YM models. In section 5 we introduce HS scalar and fermion master fields and show that the Higgs mechanism can be reproduced also in the HS theories. Section 6 is devoted to a general discussion of the action principle in this kind of HS models. In section 7 we discuss the issue of local Lorentz covariance. In section 8 we start the discussion of the perturbative approach based on Feynman diagrams. We work out the example in which only the first two field are present (the gauge and vielbein fields). Finally we show that the dependence on the conjugate momentum $u_{\mu}$ can be absorbed in a redefinition of the component fields, the coupling and the coordinate $x^\mu$. The consequence is that a mass scale shows up. In section 9 we carry out a sample calculation (the 2-pt function at one loop) in such a new frozen momentum framework. Section 10 is devoted to a discussion of the above-mentioned no-go theorems and section 11 to some conclusions. Several cumbersome details and formulas are deferred to a few final appendices.

2. The method of effective action

The original matter model, analyzed in [12], is

$$S_{\text{matter}} = \int d^4x \bar{\psi}(i\gamma \cdot \partial - m)\psi + \sum_{s=1}^{\infty} \int d^4x \, J_{a_{1} \ldots a_{s-1}}^{(s)}(x) \, h_{a_{1} \ldots a_{s-1}}^{(s, \ldots, a_{s-1})}(x)$$

$$= S_0 + S_{\text{int}}$$

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The interaction part $S_{\text{int}}$ is

$$S_{\text{int}} = \langle J_a, h^a \rangle \equiv \int d^d x \frac{d^d u}{(2\pi)^d} J_a(x, u) h^a(x, u)$$  \hspace{1cm} (2.2)$$

The (external) gauge fields are collectively represented by \(^5\)

$$h^a(x, u) = \sum_{n=0}^{\infty} \frac{1}{n!} h^{\mu_1 \ldots \mu_n}_{(a)}(x) u_{\mu_1} \ldots u_{\mu_n},$$  \hspace{1cm} (2.3)$$

and

$$J_a(x, u) = \frac{\delta S_{\text{int}}}{\delta h^a(x, u)} = \int d^d z e^{ix \cdot z} \psi^\dagger(x + \frac{z}{2}) \gamma_{\mu} \psi(x - \frac{z}{2})$$  \hspace{1cm} (2.4)$$

which is obtained by expanding $e^{iu \cdot z}$. In order to extract $J_{\alpha_{\mu_1} \ldots \mu_{s-1}}^a(x)$ from $J_a(x, u)$ one must multiply it by $u_{\mu_1} \ldots u_{\mu_{s-1}}$, integrate over $u$ and divide by $(s - 1)!$. Also

$$J_{\alpha_{\mu_1} \ldots \mu_{s-1}}^a(x) = \frac{i^{s-1}}{(s - 1)!} \left[ \prod_{i=1}^{s-1} \frac{\partial}{\partial z_{\mu_i}} \right] \psi^\dagger(x + \frac{z}{2}) \gamma_{\mu} \psi(x - \frac{z}{2}) \bigg|_{z=0}. \hspace{1cm} (2.5)$$

A generic field, like $h_a(x, u)$, depending both on coordinates and momenta, will be called master field.

The gauge transformation of $h^a$ is

$$\delta_{\epsilon} h_a(x, u) = \partial_{\alpha} \epsilon(x, u) - i [h_a(x, u)^*, \epsilon(x, u)] \equiv D_{\alpha}^a \epsilon(x, u),$$  \hspace{1cm} (2.6)$$

where we introduced the covariant derivative

$$D_{\alpha}^a = \partial_{\alpha} - i [h_a(x, u)^*, ].$$

The effective action is denoted $W[h]$ and takes the form

$$W[h] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_i \frac{d^d u_i}{(2\pi)^d} \mathcal{W}^{(n)}_{a_1 \ldots a_n}(x_1, u_1, \ldots, x_n, u_n) h^{a_1}(x_1, u_1) \ldots h^{a_n}(x_n, u_n)$$  \hspace{1cm} (2.7)$$

\(^4\)There are in the literature also supersymmetric generalizations of $S_{\text{matter}}$, which we will not consider here. See for instance \cite{38} and references therein.

\(^5\)The position in the phase space are denoted by couples of letters $(x, u), (y, v), (z, t), (w, r)$, the first letter being for the space-time coordinate and the second for the momentum of the worldline particle. The letters $k, p, q$ will be reserved for the momenta of the (Fourier-transformed) physical amplitudes.
where

\[ W^{\left(n\right)}_{a_1 \ldots a_n}(x_1, u_1, \ldots, x_n, u_n) = \langle J_{a_1}(x_1, u_1) \ldots J_{a_n}(x_n, u_n) \rangle \] (2.8)

\[ = \left( \sum_{s_1=1}^{\infty} \frac{\partial}{\partial u_{\mu_1(1)}} \ldots \frac{\partial}{\partial u_{\mu_{s_1-1}(1)}} \delta(u_1) \right) \ldots \left( \sum_{s_n=1}^{\infty} \frac{\partial}{\partial u_{\mu_1(n)}} \ldots \frac{\partial}{\partial u_{\mu_{s_n-1}(n)}} \delta(u_n) \right) \]

\[ \times \langle J^{(s_1)}_{a_1 \mu_1(1) \ldots \mu_{s_1-1}(1)} \ldots J^{(s_n)}_{a_n \mu_1(n) \ldots \mu_{s_n-1}(n)}(x_1) \ldots (x_n) \rangle \]

The statement of invariance under (2.6) is the global Ward identity (WI)

\[ \delta_x W[h] = 0 \] (2.9)

Taking the variation with respect to \( \varepsilon(x, u) \) this becomes

\[ \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} d^d x_i \frac{d^d u_i}{(2\pi)^d} D^\mu_x W^{(n+1)}_{a_1 \ldots a_n}(x, u, x_1, u_1, \ldots, x_n, u_n) h_{a_1}(x_1, u_1) \ldots h_{a_n}(x_n, u_n) = 0 \] (2.10)

This must be true order by order in \( h \), i.e.

\[ 0 = \int \prod_{i=1}^{n} d^d x_i \frac{d^d u_i}{(2\pi)^d} \partial^\mu_x W^{(n+1)}_{a_1 \ldots a_n}(x, u, x_1, u_1, \ldots, x_n, u_n) h_{a_1}(x_1, u_1) \ldots h_{a_n}(x_n, u_n) \]

\[ -i n \int \prod_{i=1}^{n} d^d x_i \frac{d^d u_i}{(2\pi)^d} \left[ h^a(x, u) \delta^{(n)}_{a a_1, \ldots a_{n-1}}(x, u, x_1, u_1, \ldots, x_{n-1}, u_{n-1}) \right] \]

\[ \times h_{a_1}(x_1, u_1) \ldots h_{a_{n-1}}(x_{n-1}, u_{n-1}) \] (2.11)

### 2.1 The gauge transformation in the fermion model

In the fermion model the gauge transformation of the master field \( h_a(x, u) \) is,

\[ \delta_a h_a(x, u) = \partial^\mu_x \varepsilon(x, u) - i [h_a(x, u) \varepsilon(x, u)] = D^\mu_x \varepsilon(x, u) \] (2.12)

Now, the expansion of \( h_a(x, u) \) is

\[ h_a(x, u) = A_a(x) + \chi^\mu_a(x) u_\mu + \frac{1}{2} h_a^{\mu\nu}(x) u_\mu u_\nu + \frac{1}{6} h_a^{\mu\nu\lambda}(x) u_\mu u_\nu u_\lambda + \frac{1}{4!} h_a^{\mu\nu\lambda\rho}(x) u_\mu u_\nu u_\lambda u_\rho + \ldots \] (2.13)

Notice that in the expansion (2.13) the indices \( \mu_1, \ldots, \mu_n \) are upper (contravariant), as it should be, because in the Weyl quantization procedure the momentum has lower index, since it must satisfy \([x^\mu, p_\nu] = i \delta^\mu_\nu \). The index \( a \) is different in nature. As we will justify below, \( h_a \) will be referred to as a frame-like master field. Of course when the background metric is flat all indices are on the same footing, but writing in this way leads to the correct interpretation.
We also recall

\[ \varepsilon(x, u) = \varepsilon(x) + \xi^\mu(x) u_\mu + \frac{1}{2} \Lambda^{\mu\nu}(x) u_\mu u_\nu + \frac{1}{3!} \Sigma^{\mu\nu\lambda}(x) u_\mu u_\nu u_\lambda \\
+ \frac{1}{4!} P^{\mu\nu\lambda\rho}(x) u_\mu u_\nu u_\lambda u_\rho + \frac{1}{5!} \Omega^{\mu\nu\lambda\rho\sigma}(x) u_\mu u_\nu u_\lambda u_\rho u_\sigma + \ldots \]  

(2.14)

To the lowest order the transformation (2.12) reads

\[ \delta^{(0)} A_a = \partial_a \varepsilon \]
\[ \delta^{(0)} \chi_a^\nu = \partial_a \xi^\nu \]
\[ \delta^{(0)} b_a^{\nu \lambda} = \partial_a \Lambda^{\nu \lambda} \]  

(2.15)

To first order we have

\[ \delta^{(1)} A_a = \xi \cdot \partial A_a - \partial_\rho \varepsilon_\rho \chi_a^\rho \]
\[ \delta^{(1)} \chi_a^\nu = \xi \cdot \partial \chi_a^\nu - \partial_\rho \varepsilon_\rho \chi_a^\nu + \partial^\rho A_a \Lambda_\rho^{\nu} - \partial_\Lambda \varepsilon_\rho b_a^{\rho \nu} \]
\[ \delta^{(1)} b_a^{\nu \lambda} = \xi \cdot \partial b_a^{\nu \lambda} - \partial_\rho \varepsilon^\rho b_a^{\nu \lambda} - \partial_\mu \varepsilon^{\nu \lambda} b_a^{\rho \mu} + \partial_\rho \chi_a^{\nu \lambda} \Lambda_\rho + \partial_\Lambda \chi_a^{\nu \lambda} \Lambda_\rho - \chi_a^{\nu \lambda} \partial_\rho \Lambda_\rho \]  

(2.16)

The next orders contain three and higher derivatives. Let us denote by \( \tilde{A}_a \), \( \tilde{\varepsilon}_a^\mu = \delta_a^\mu - \tilde{\chi}_a^\mu \) the standard gauge and vielbein fields. The standard gauge and diff transformations, are

\[ \delta \tilde{A}_a \equiv \delta \left( \tilde{e}_a^\mu \tilde{A}_\mu \right) \equiv \delta \left( \delta_a^\mu - \tilde{\chi}_a^\mu \right) \tilde{A}_\mu \]
\[ = \left( -\xi \cdot \partial \tilde{\chi}_a^\mu + \partial_\rho \varepsilon^\rho \tilde{\chi}_a^{\rho \mu} \right) \tilde{A}_\mu + \left( \delta_a^\mu - \tilde{\chi}_a^\mu \right) \left( \partial_\mu \varepsilon + \xi \cdot \tilde{A}_\mu \right) \approx \partial_a \varepsilon + \xi \cdot \tilde{A}_a - \tilde{\chi}_a^\mu \partial_\mu \varepsilon \]  

and

\[ \delta \tilde{\varepsilon}_a^\mu \equiv \delta (\delta_a^\mu - \tilde{\chi}_a^\mu) = \xi \cdot \tilde{\varepsilon}_a^\mu - \partial_\rho \varepsilon^\rho \tilde{\varepsilon}_a^{\rho \mu} = -\xi \cdot \tilde{\chi}_a^\mu - \partial_\rho \varepsilon^\rho \tilde{\chi}_a^{\rho \mu} \]  

(2.17)

so that

\[ \delta \tilde{\chi}_a^\mu = \xi \cdot \tilde{\chi}_a^\mu + \partial_\rho \varepsilon^\rho \tilde{\chi}_a^{\rho \mu} - \partial_\mu \varepsilon^\rho \tilde{\chi}_a^{\rho \mu} \]  

(2.18)

where we have retained only the terms at most linear in the fields. From the above we see that we can make the identifications

\[ A_a = \tilde{A}_a, \quad \chi_a^\mu = \tilde{\chi}_a^\mu \]  

(2.19)

The transformations (2.13), (2.16) are consistent with Riemannian geometry, therefore the effective action may accommodate gravity. Concerning \( h_a(x, u) \), it contains more than symmetric tensors: beside the completely symmetric \( h^{(\mu_1 \ldots \mu_\alpha)} \) it includes also a Lorentz representation in which the index \( a \) and one of the other indices are antisymmetric.

**2.2 Analogy with gauge transformations in gauge theories**

It should be remarked that in eqs. (2.12) and (2.15) the derivative \( \partial_a \) means \( \partial_a = \delta_a^\mu \partial_\mu \), not \( \partial_a = \varepsilon_a^\mu \partial_\mu = \left( \varepsilon_a^\mu - \chi_a^\mu + \ldots \right) \partial_\mu \). In fact the linear correction \( -\chi_a^\mu \partial_\mu \) is contained in the term \( -i [h_a(x, u), \varepsilon(x, u)] \), see for instance the second term in the RHS of the first equation.

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¿From this point of view the transformation (2.12) looks similar to the ordinary gauge transformation of a non-Abelian gauge field

\[ \delta \lambda A_a = \partial_a \lambda + [A_a, \lambda] \] (2.21)

where \( A_a = A_a^{\alpha} T^{\alpha} \), \( \lambda^{\alpha} T^{\alpha} \) being the Lie algebra generators.

In gauge theories it is useful to represent the gauge potential as a connection one form \( A = A_a dx^a \), so that (2.21) becomes

\[ \delta \lambda A = d \lambda + [A, \lambda] \] (2.22)

As was done in I, we can do the same for (2.12)

\[ \delta \varepsilon h(x, u) = d\varepsilon(x, u) - i[h(x, u) \varepsilon(x, u)] \equiv D\varepsilon(x, u) \] (2.23)

where \( d = \partial_a dx^a \), \( h = h_a dx^a \), \( x^a \) are coordinates in the tangent spacetime and it is understood that

\[ [h(x, u) \varepsilon(x, u)] = [h_a(x, u) \varepsilon(x, u)] dx^a \]

We will apply this formalism to the construction of CS- and YM-like actions

3. Integrating \( L_\infty \)

As we know from previous works \[22, \ 23\], the effective field theory method yields HS gauge invariant results, which in general are non-local. From them one may be able to eventually extract sensible, even though non-local actions. However the previously noticed analogy with the non-Abelian gauge theories formalism, suggests a shortcut: one can directly ‘integrate’ the \( L_\infty \) relations and find (perturbatively) local actions. Based on the analogy with ordinary local field theory it is not hard to construct HS gauge invariant action terms. Such terms are defined by means of an integral over the phase space. For this reason, we will call them at times, not simply actions, but \textit{primitive action functionals}.

This section is devoted to the construction of HS Chern-Simons (HS-CS) actions and HS Yang-Mills (HS-YM) theories, both Abelian and non-Abelian.

3.1 Preliminaries

In the sequel we will use for the curvature the notation, introduced in (I),

\[ G = dh - \frac{i}{2} [h \varepsilon, h] \] (3.1)

with the transformation property

\[ \delta \varepsilon G = -i [G \varepsilon] \] (3.2)

The functionals we will consider are integrated polynomials of \( G \) or of its components \( G_{ab} \).

In order to exploit the transformation property \( [3.2] \) in the construction we need the ‘trace

\[ ^* \text{In I we have used this analogy to construct HS anomalies.} \]
property’, analogous to the trace of polynomials of Lie algebra generators in ordinary non-Abelian gauge theories. The only object with trace properties we can define in the HS context is

\[ \langle \langle f * g \rangle \rangle \equiv \int d^d x \int \frac{d^d u}{(2\pi)^d} f(x, u) * g(x, u) = \int d^d x \int \frac{d^d u}{(2\pi)^d} f(x, u) g(x, u) = \langle \langle g * f \rangle \rangle \] (3.3)

¿From this, plus associativity, it follows that

\[ \langle \langle f_1 * f_2 * \ldots * f_n \rangle \rangle = \langle \langle f_1 * (f_2 * \ldots * f_n) \rangle \rangle \]

\[ = (-1)^{\epsilon_1(\epsilon_2 + \ldots + \epsilon_n)} \langle \langle f_2 * \ldots * f_n * f_1 \rangle \rangle = (-1)^{\epsilon_1(\epsilon_2 + \ldots + \epsilon_n)} \langle \langle f_2 * \ldots * f_n * f_1 \rangle \rangle \] (3.4)

where \( \epsilon_i \) is the Grassmann degree of \( f_i \). In particular

\[ \langle \langle [f_1 * f_2 * \ldots * f_n] \rangle \rangle = 0 \] (3.5)

where \([*,*]\) is the *-commutator or anti-commutator, as appropriate.

This property holds also when the \( f_i \) are valued in a Lie algebra, provided the symbol \( \langle \langle \rangle \rangle \) includes also the trace over the Lie algebra generators.

### 3.2 CS primitive action

In (I) we have shown that the action

\[ CS(h) = n \int_0^1 dt \langle \langle h * G_t * \ldots * G_t \rangle \rangle \] (3.6)

where

\[ G_t = dh_t - \frac{i}{2} [h_t, h_t], \quad h_t = t h, \] (3.7)

is HS gauge invariant in a space of odd dimension \( d = 2n - 1 \). We assume it as the HS gauge invariant CS action in such dimensions.

### 3.3 HS Yang-Mills action

The curvature form-components, see [3.3], are

\[ G_{ab} = \partial_a h_b - \partial_b h_a - i [h_a, h_b] \] (3.8)

Their transformation rule is

\[ \delta_\varepsilon G_{ab} = -i [G_{ab}, \varepsilon] \] (3.9)

Remembering that \( a \) and \( b \) are flat indices, it follows that

\[ \delta_\varepsilon \langle \langle G_{ab} * G_{ab} \rangle \rangle = -i \langle \langle G_{ab} * G_{ab} * \varepsilon - \varepsilon * G_{ab} * G_{ab} \rangle \rangle = 0 \] (3.10)

Therefore

\[ YM(h) = -\frac{1}{4g^2} \langle \langle G_{ab} * G_{ab} \rangle \rangle \] (3.11)
is invariant under HS gauge transformations and it is a well defined primitive functional in any dimension.

Remark 1. Observe that the dimensions of (3.6) and (3.11) are not the ones of an action. One should divide it by a factor \( V_u \) proportional to the integration volume over the momentum space. For the time being, for the sake of simplicity, we disregard this factor. We will resume it later on.

3.4 The non-Abelian case

All that has been done for the Abelian \( U(1) \) case up to now can be repeated for the non-Abelian case without significant changes. One has to consider a fermion field \( \psi \) belonging to some representation of a non-Abelian Lie algebra with generators \( T^\alpha \). Then one can introduce the non-Abelian sources

\[
\text{h}^\alpha = h_a^\alpha dx^a
\]

where summation over \( \alpha \) is understood. The interacting action is

\[
S_{\text{int},nA} = \int d^d x \frac{d^d u}{(2\pi)^d} \text{Tr} \left( J^\alpha(x,u) h_a(x,u) \right)
\]

where \( \text{Tr} \) is the trace over the Lie algebra generators. More explicitly

\[
h^\alpha_a(x,u) = \sum_{n=0}^{\infty} \frac{1}{n!} h^\alpha_{\mu_1...\mu_n}(x) u_{\mu_1}...u_{\mu_n},
\]

and

\[
J^\alpha_a(x,u) = \frac{\delta S_{\text{int}}}{\delta h^\alpha_a(x,u)} = \int d^d z e^{iz\cdot u} \left( x + \frac{z}{2} \right) \gamma_a T^\alpha \psi \left( x - \frac{z}{2} \right)
\]

Then the HS gauge parameter is

\[
e(x,u) = \varepsilon^\alpha(x,u) T^\alpha
\]

and the transformation of \( h(x,u) \) is

\[
\delta_e h(x,u) = \delta f e(x,u) - i [h(x,u) \, \varepsilon(x,u)],
\]

if the generators \( T^\alpha \) are anti-hermitean. In this case the curvature is

\[
G = dh - \frac{i}{2} [h \, \varepsilon]
\]

The \( \ast \)-commutator includes now also the Lie algebra commutator. Of course we have, in particular,

\[
\delta_e G(x,u) = -i [G(x,u) \, \varepsilon(x,u)]
\]

Everything works as before provided the symbol \( \langle \langle \rangle \rangle \) comprises also the trace over the Lie algebra generators. In particular

\[
\mathcal{Y}_{\mathcal{M}}(h) = -\frac{1}{4g^2} \langle \langle G^{ab} \ast G_{ab} \rangle \rangle
\]

is invariant under the HS non-Abelian gauge transformations and it is a well defined primitive functional in any dimension.
4. Covariant eom’s and conservation laws

The expressions (3.6) and (3.11) do not have the form of the usual field theory actions, because they are integrals over the phase space of the point particle with coordinate $x^a$. Nevertheless we can extract from them covariant eom’s by taking the variation with respect to $h$. In other words we assume that the action principle holds for fields defined in the phase space. We will justify this later on. For the time being we use this principle to extract from the primitive functional the relevant equations of motion.

4.1 Covariant YM-type eom’s

From (3.11) we get the following eom:

$$\partial_b G^{ab} - i[h_b, G^{ab}] \equiv \mathcal{D}_b G^{ab} = 0 \quad (4.1)$$

which is, by construction, covariant under the HS gauge transformation

$$\delta_x \left( \mathcal{D}_b G^{ab} \right) = -i[\mathcal{D}_b G^{ab}, \varepsilon] \quad (4.2)$$

This is analogous to eq.(2.58) of [12]. In components this equation splits into an infinite set according to the powers of $u$. Let us expand $G_{ab}$ in the notation of sec.2.1. We have

$$G_{ab} = F_{ab} + X^\mu_{ab} u_\mu + \frac{1}{2} B^{\mu\nu}_{ab} u_\mu u_\nu + \frac{1}{6} C^{\mu\nu\lambda}_{ab} u_\mu u_\nu u_\lambda + \frac{1}{4!} D^{\mu\nu\lambda\rho}_{ab} u_\mu u_\nu u_\lambda u_\rho + \ldots \quad (4.3)$$

An explicit expansion of $F_{ab}, X^\mu_{ab}, \ldots$ in terms of component fields is given in appendix A.1.

The eom’s from (4.1) are

$$0 = \partial_a F^{ab} + \partial_a A_a X^{ab\sigma} - \chi_\sigma \partial_b F^{ab} + \frac{1}{8} \left( \partial_{\sigma_a} \partial_{\sigma_2} \partial_{\sigma_3} A_a C^{\sigma_1 \sigma_2 \sigma_3}_{ab} + c^{\sigma_1 \sigma_2 \sigma_3} \partial_{\sigma_1} \partial_{\sigma_2} \partial_{\sigma_3} F_{ab} + 3 \partial_{\sigma_1} \partial_{\sigma_2} \chi^{b_{\sigma_3} \partial_{\sigma_3} \partial_{b_{\sigma_1}} \partial_{\sigma_2} X_{ab}^{\sigma_3} \right) + \ldots \quad (4.4)$$

$$0 = \partial_a X^{ab\mu} + \partial_a A_a B^{ab\mu}_{\sigma} - b^{\sigma\mu}_{\partial_b} F^{ab} + \partial_a \chi^\mu_{\sigma} F^{ab\sigma} - \chi^\mu_{\partial_b} X^{ab\sigma} + \chi^\mu_{\partial_a} X^{ab\mu} + \ldots \quad (4.5)$$

$$0 = \partial_a B^{ab\mu\nu} + \partial_a b^{ab\mu\nu}_{\sigma} X^{ab\sigma} + 2 \partial_a \chi^\mu_{\sigma} B^{ab\sigma\mu} + \partial_a A_a C^{ab\mu\nu}_{\sigma} - \partial_a F^{ab_{\sigma} \mu_{\partial_a}} - 2 \partial_a X^{ab_{\sigma} \mu_{\partial_a}} + \chi^\mu_{\partial_a} B^{ab\mu\nu} + \ldots \quad (4.6)$$

where the ellipses in the RHS refer to terms containing at three or more derivatives.

More explicitly, for instance the first eom is

$$0 = \Box A_b - \partial_b \partial_a A - (\partial_\sigma \partial_a A_{a^\sigma}) + \partial_a A_a A_{a^\sigma} + \partial_a A_{a^\sigma} \partial_\sigma A_a^\sigma - \partial_\sigma A_a^\sigma \partial_\sigma A_a - \partial_\sigma A_b^\sigma \partial_\sigma A_b$$

$$+ \partial a_A \left( \partial_a \partial_\sigma A_{a^\sigma} - \partial_\sigma A_a^\sigma + \frac{1}{2} \left( \partial_a A_a b_\delta^\sigma - \partial_\delta A_a b_a^\sigma + \partial_\delta A_a^\sigma \partial_\sigma A_a - \partial_\sigma A_{a^\sigma} \partial_\sigma A_a \right) \right) - \partial_\sigma A_{a^\sigma} \partial_\sigma A_b - \partial_\sigma \partial_b A_a + \frac{1}{2} \left( \partial_\sigma \partial_\delta A_{a^\delta} - \partial_\sigma A_a^\delta \partial_\delta A_a - \partial_\sigma A_{a^\delta} \partial_\delta A_a - \partial_\sigma A_a^\delta \partial_\delta A_a \right)$$

$$+ \ldots \quad (4.7)$$

- 11 -
\[ \square \chi^\mu_a - \partial_a \nabla^\mu_b \chi_b = \partial^b \left( \partial_a A^a_b b^\sigma_{\sigma} - \partial_a A^a_b b^\sigma_{\sigma} + \partial_a \chi^\mu_a \chi^\sigma - \partial_a \chi^\mu_a \chi^\sigma \right) \]  
(4.8)

\[ + \partial_a A^b \partial_a b^\mu_{\sigma} - \partial_a A^b \partial_a b^\mu_{\sigma} + \partial_a \chi^\mu b^\sigma_{\sigma} - \partial_a \chi^\mu b^\sigma_{\sigma} - \partial_b \chi \partial_a \chi^\sigma + \partial_b \chi \partial_a \chi^\sigma + \ldots \]  
(4.9)

Let us see a few elementary examples. Consider the case of a pure U(1) gauge field alone. The equation of motion is

\[ \partial_a F^{ab} = \square A^b - \partial_b \partial^a A = 0 \]  
(4.10)

In the Feynman gauge \( \partial^a A = 0 \) this reduces to \( \square A^b = 0 \).

Let us suppose next that only gravity is present. Eq. (4.5) becomes

\[ \partial_a X^{ab \mu} = \square \chi^\mu_b - \partial_b \partial^a \chi^\mu = 0 \]  
(4.11)

In the ‘Feynman gauge’ \( \partial^a \chi^\mu = 0 \), (4.11) reduces to \( \square \chi^\mu_b = 0 \).

Finally, keeping only the spin 3 field (4.6) becomes

\[ \partial_a B^{ab \mu \nu} = \square b^{\mu \nu}_b - \partial_b \partial^a b^{\mu \nu} = 0 \]  
(4.12)

Again in the ‘Feynman gauge’ \( \partial^a b^{\mu \nu}_a = 0 \) we get \( \square b^{\mu \nu} = 0 \).

In general we can impose for all the fields the Feynman gauge

\[ \partial^a h_a (x, u) = 0 \]  
(4.13)

and obtain the same massless Klein-Gordon equation.

**Remark 2.** As is clear from (4.7), for instance, the above eom’s are characterized by the fact that at each order, defined by the number of derivatives, there is a finite number of terms. We adopt the terminology of [34] and call a theory with this characteristic *perturbatively local*.

**Remark 3.** The above is an entirely new approach to covariance. The gauge transformation (2.0) reproduces both ordinary U(1) gauge transformations and diffeomorphisms, but the primitive action functional is defined in the phase space. It gives rise to local equations of motion that reproduce the ordinary YM eoms, but not completely the metric equations of motion: the linear eom coincide with the ordinary one after gauge fixing, but there is a huge difference with ordinary gravity because in the latter the interaction terms are infinite and include all powers in the fluctuating field, while in the (3.11) there are at most quartic interactions (at most cubic in the relevant eom). It should be noted, however, that the latter difference might be more apparent than real, the reason being the following: the ordinary gravity is formulated in terms of the fluctuating field \( h_{\mu \nu} \) where \( g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \), while the ‘HS gravity’ is formulated in terms of \( \chi^\mu_a \), where \( e^\mu_a = \delta^\mu_a - \chi^\mu_a \).

\[ \text{In ordinary gravity (} R_{\mu \nu} = 0 \text{) we have to impose the DeDonder gauge in order to obtain the same result.} \]
What is the relation between $\chi^\mu_a$ and $h_{\mu\nu}$? As it is well known one has to make a ‘gauge’ choice in order to find this relation. Choosing a symmetric ‘gauge’ for $\chi^\mu_a$ it is given by

$$
e^\mu_a = \delta^\mu_a - \chi^\mu_a$$

$$
e^a_\mu = \delta^a_\mu + \chi^a_\mu + \chi^b_\mu \chi^b_a + \ldots$$

$$
g^{\mu\nu} = \eta^{\mu\nu} - 2\chi^{\mu\nu} + \chi^a_\mu \chi^a_\nu + \ldots$$

$$
g_{\mu\nu} = \eta_{\mu\nu} + 2\chi_{\mu\nu} + 3\chi^a_\mu \chi^a_\nu + \ldots$$

(4.14)

So that

$$
h_{\mu\nu} = 2\chi_{\mu\nu} + 3\chi^a_\mu \chi^a_\nu + \ldots$$

(4.15)

$$
\chi^\mu_a = \frac{1}{2} h^\mu_a - \frac{3}{4} h^a_\mu h^a_\nu + \ldots$$

(4.16)

It follows that, expressed in terms of the fluctuation $h_{\mu\nu}$, the cubic and quartic powers in $\chi^\mu_a$ turn out to contain powers of any order.

This is not yet enough to clarify what kind of gravity is described by the equations (4.6). In a companion paper [29] we show that it resembles the so-called teleparallel gravity, [30].

4.2 CS covariant eom’s

For CS we start from the primitive functional

$$
CS(h) = n \int_0^1 dt \int d^dx \langle \langle h \ast G_t \ast \ldots \ast G_t \rangle \rangle, \quad d = 2n - 1,
$$

(4.17)

Taking a generic variation $\delta h$, with the usual manipulations, we get

$$
\delta CS(h) = n \int_0^1 dt \int d^dx \langle \langle \frac{d}{dt} (t \delta h \ast G_t \ast \ldots \ast G_t) \rangle \rangle
$$

$$
= n \langle \langle \delta h \ast G \ast \ldots \ast G \rangle \rangle
$$

(4.18)

It follows that the overall CS eom is

$$
G \ast \ldots \ast G = 0
$$

(4.19)

where an exterior product of $n-1 = \frac{d-1}{2}$ factors of $G$ is understood. Since $\delta G = -i[G \ast \varepsilon]$, it is evident that this equation is HS gauge covariant. For instance, in 3d in components this means

$$
F_{ab} = 0, \quad X_{ab}^\mu = 0, \quad B_{ab}^{\mu\nu} = 0, \quad \ldots
$$

(4.20)

These equations are covariant because a HS gauge transformation maps them to (infinite) linear combinations of themselves.
4.3 Conservation laws

The conservation laws of the HS models can be found following the analogy of a current in an ordinary gauge theory or the energy momentum tensor in gravity theories. For instance the latter is identified with the eom itself:

\[ T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}} \]

which, in absence of matter, vanishes on shell. It is singled out from the invariance relation of the action under diffeomorphisms

\[ 0 = \delta \xi S = \int d^4x \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} = -2 \int d^4x \xi_\mu \nabla_\nu \frac{\delta S}{\delta g_{\mu\nu}} \]

where \( \delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \).

Let us proceed in an analogous way, for instance, for HS YM. If we express the invariance of the action under the HS gauge transformation we can write

\[ 0 = -\frac{1}{4} \delta \xi \langle \langle G_{ab} \ast G^{ab} \rangle \rangle = \langle \langle \delta \xi h_a \ast D^*_b G^{ab} \rangle \rangle = \langle \langle \delta \xi \ast D^*_a D^*_b G^{ab} \rangle \rangle, \]

which implies the off-shell relation or conservation law

\[ D^*_a D^*_b G^{ab} = 0 \]

from which we identify the conserved master current

\[ J_a = D^*_b G^{ab} \]

In other words the conserved currents are the first members of the eoms derived above. They vanish on shell and are conserved off-shell. Expanding in \( u \)

\[ J_a = \sum_{n=0}^{\infty} \frac{1}{n!} J^{\mu_1...\mu_n} u_{\mu_1}...u_{\mu_n} \]

we find the component generators.

4.4 The \( L_\infty \) structure

As was shown in [12] (see also I), the effective action obtained by integrating out a fermion field coupled to external sources hides an algebraic structure, which is revealed once we consider the relevant equations of motion. The basic relations in this game are the eoms

\[ \mathcal{F}_\mu(x, u) = 0 \]

where

\[ \mathcal{F}_\mu(x, u) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} d^4x_i \frac{d^4u_i}{(2\pi)^d} \mathcal{W}^{(n+1)}_{\mu, \mu_1...\mu_n}(x, u, x_1, u_1, ..., x_n, u_n) \times h^{\mu_1}(x_1, u_1) \cdots h^{\mu_n}(x_n, u_n) \]
where, of course,  

\[ F \]

all the other products, involving at least one  

\[ \varepsilon \]

D  

(which are at most three):  

identically 0. In order to define the products involving only  

h  

define  

X  

in the HS-YM case we have the eom (4.1) and the covariance relation (4.2), which we  

already been proven above, and this is all one needs in order t o prove (4.27). For instance,  

detail because it is enough to remark that the analog of the basic relation (4.26) have  
of this algebra, as one would expect. This is indeed so, and we do not need to prove it  

(\[ \sigma \])  

In this formula \( \sigma \) denotes a permutation of the entries so that \( \sigma(1) < \ldots \sigma(i) \) and \( \sigma(i+1) < \ldots \sigma(n) \), and \( \varepsilon(\sigma; x) \) is the Koszul sign.  

The obvious question is whether the HS-CS or HS-YM models eoms are representations  
of this algebra, as one would expect. This is indeed so, and we do not need to prove it  

\[ L \]

3

\[ W_{\mu_1,\mu_2,\ldots,\mu_n} \]

being the n-point correlators of the master fermion currents (2.3), and the  
covariance relation  

\[ \delta_{\varepsilon} F_\mu(x, u) = i[\varepsilon(x, u) \ast F_\mu(x, u)] \]  

(4.26)  

It was shown in section 3 of [2], that this allows us to define \( j \)-linear maps (products) \( L_j, j = 1, \ldots, \infty \) of degree \( d_j = j - 2 \) among vector spaces \( X_i \) of degree \( i \), defined by the assignments \( \varepsilon \in X_0, h^a \in X_{-1} \) and \( F_\mu \in X_{-2} \), which satisfy the relations  

\[ \sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} (-1)^{\sigma} \varepsilon(\sigma; x) L_j(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0 \]  

(4.27)  

In this formula \( \sigma \) denotes a permutation of the entries so that \( \sigma(1) < \ldots \sigma(i) \) and \( \sigma(i+1) < \ldots \sigma(n) \), and \( \varepsilon(\sigma; x) \) is the Koszul sign.  

The proof of the relations (4.27) is the same as in [12] and is based on (4.28). The only  
difference is that in [12] there is an infinite number of nonvanishing \( L_n(h_1, \ldots, h_n) \).  

A similar construction holds for HS-CS.
4.5 BRST quantization of HS Yang-Mills

To BRST quantize the action (3.11) we have to fix the gauge and apply the Faddeev-Popov approach. We impose the Lorentz gauge with parameter $\alpha$ and apply the standard approach, so the quantum action becomes

$$\mathcal{YM}(h_a, c, B) = \frac{1}{g^2} \left\langle -\frac{1}{4} G_{ab} * G^{ab} - h^a * \partial_a B - i \partial^a \overline{\varphi} \star D^*_a c + \frac{\alpha}{2} B * B \right\rangle$$ (4.34)

where $c, \overline{\varphi}$ and $B$ are the ghost, antighost and Nakanishi-Lautrup master fields, respectively. $c, \overline{\varphi}$ are anticommuting fields, while $B$ is commuting.

The action (4.34) is symmetric under the BRST transformations

$$s h_a = D^*_a c$$

$$s c = i c * c = \frac{i}{2} [c, c]$$

$$s \overline{\varphi} = i B$$

$$s B = 0$$

which are nilpotent. In particular

$$s(D^*_a c) = 0, \quad s(c * c) = 0$$

From the point of view of the $u$ dependence $c, \overline{\varphi}$ and $B$ are to be expanded like a scalar master field, see eq. (5.1) below.

Integrating out $B$ in (4.34) we obtain the standard gauge-fixed action.

$$\mathcal{YM}(h_a, c) = \frac{1}{g^2} \left\langle -\frac{1}{4} G_{ab} * G^{ab} - \frac{1}{2\alpha} \partial_a h^a * \partial_b h^b - i \partial^a \overline{\varphi} \star D^*_a c \right\rangle$$ (4.36)

5. Adding bosonic matter

So far we have treated only gauge fields of any spin. We can couple to the previous theories matter-type fields of any spin. Let us add, for instance, a complex multi-boson field

$$\Phi(x, u) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{\mu_1 \mu_2 ... \mu_n}(x) u_{\mu_1} u_{\mu_2} \ldots u_{\mu_n}$$ (5.1)

which transforms like

$$\delta_\varepsilon \Phi = i \varepsilon * \Phi, \quad \delta_\varepsilon \Phi^\dagger = -i \Phi^\dagger * \varepsilon,$$ (5.2)

Let us define the covariant derivative

$$D^*_a \Phi = \partial_a \Phi - i h_a * \Phi$$ (5.3)

and its hermitean conjugate

$$(D^*_a \Phi)^\dagger = \partial_a \Phi^\dagger + i \Phi^\dagger * h_a$$ (5.4)
They have the properties
\[ \delta_\varepsilon \mathcal{D}_a^* \Phi = i \varepsilon \ast \mathcal{D}_a^* \Phi, \]
\[ \delta_\varepsilon (\mathcal{D}_a^* \Phi) = -i (\mathcal{D}_a^* \Phi)^\dagger \ast \varepsilon \]  \hspace{1cm} (5.5)

As a consequence we have in particular
\[ \delta_\varepsilon (\mathcal{D}_a^* \mathcal{D}_b^* \Phi) = i \varepsilon \ast (\mathcal{D}_a^* \mathcal{D}_b^* \Phi) \]  \hspace{1cm} (5.6)

It follows that
\[ S(\Phi, h) = \frac{1}{2} \langle \langle (\mathcal{D}_a^0 \Phi)^\dagger \ast \mathcal{D}_a^0 \Phi \rangle \rangle + \sum_{n=1}^\infty \frac{\lambda_{2n}}{n!} \langle \langle (\Phi^\dagger \ast \Phi)_n \rangle \rangle \]  \hspace{1cm} (5.7)
is gauge invariant. We remark that this action is real because \( \Phi^\dagger \ast \Phi \) is. The generalized eom is
\[ \mathcal{D}_a^* \mathcal{D}_a^{*a} \Phi + \sum_{n=1}^\infty \frac{\lambda_{2n}}{(n-1)!} \Phi \ast (\Phi^\dagger \ast \Phi)^{n-1}_s = 0 \]  \hspace{1cm} (5.8)
which could be called interacting HS Klein-Gordon equation.

5.1 A Higgs mechanism

Let us expand \( \Phi \)
\[ \Phi(x, u) = \varphi_0(x) + \varphi_1^\mu(x) u_\mu + \frac{1}{2} \varphi_2^{\mu\nu}(x) u_\mu u_\nu + \frac{1}{6} \varphi_3^{\mu\nu\lambda} u_\mu u_\nu u_\lambda + \frac{1}{4!} \varphi_4^{\mu\nu\lambda\rho} u_\mu u_\nu u_\lambda u_\rho + \ldots \]  \hspace{1cm} (5.9)

Explicit formulas for the component transformations under (5.2) are given in Appendix A.2.

Let us consider the case in which in (5.7), \( \lambda_2 = \mu^2, \lambda_4 = -\lambda \), while the other couplings vanish, so that the potential is
\[ V(\Phi) = \frac{\mu^2}{2} \langle \langle \Phi^\dagger \ast \Phi - \frac{\lambda}{4} (\Phi^\dagger \ast \Phi)^2 \rangle \rangle \]  \hspace{1cm} (5.10)

Let us suppose that only \( \varphi_0 \) takes on a nonvanishing vacuum expectation value, say \( v \), so that
\[ \varphi_0(x) = v + \phi_0(x), \quad v = \frac{\mu}{\sqrt{\lambda}} \]  \hspace{1cm} (5.11)

Looking at (A.3), it is easy to see that this vev breaks the symmetry completely, for the HS gauge transformations on the vacuum take the form
\[ \delta_\varepsilon \langle \varphi_0 \rangle = i v \varepsilon \]  \hspace{1cm} (5.12)
\[ \delta_\varepsilon \langle \varphi_1^\mu \rangle = i v \varepsilon^\mu \]  
\[ \delta_\varepsilon \langle \varphi_2^{\mu\nu} \rangle = i v \Lambda^{\mu\nu} \]  
\[ \delta_\varepsilon \langle \varphi_3^{\mu\nu\lambda} \rangle = i v \Sigma^{\mu\nu\lambda} \]  
\[ \ldots = \ldots \]
Next it is convenient to use finite transformations:

\[ \Phi \rightarrow e^{ie} * \Phi, \quad \Phi^\dagger \rightarrow \Phi^\dagger * e^{-ie} \]

and

\[ h_a \rightarrow ie^{ie} * D_a^* e^{-ie} \]

Since \( e^{ie} * e^{-ie} = e^{-ie} * e^{ie} = 1 \), it follows, in particular, that

\[ D_a^* \Phi \rightarrow e^{ie} * D_a^* \Phi \]

So (5.13) is invariant under finite HS gauge transformations as well.

From the above (see (A.3)) it follows that we can parametrize a generic configuration of \( \Phi \) as

\[ \Phi = e^{i\omega} \varphi_0 \]

where

\[ \omega = \omega^\mu (x) u_\mu + \frac{1}{2} \omega^{\mu\nu} (x) u_\mu u_\nu + \ldots \]

Since the RHS of (5.16) is formally a HS gauge transformation, the terms of the action are form invariant. In particular the potential becomes

\[ V(\varphi_0) = \left\langle \left\langle \frac{\mu^2}{2} \varphi_0^2 - \frac{\lambda}{4} \varphi_0^4 \right\rangle \right\rangle \]

\[ = \left\langle \left\langle \frac{\mu^4}{4\lambda} - \mu^2 \varphi_0^2 - \mu \sqrt{\lambda} \varphi_0^3 - \frac{\lambda}{4} \varphi_0^4 \right\rangle \right\rangle \]

where \( \varphi_0 \) is given by (5.11). The term linear in \( \varphi_0 \) vanishes, while there is a constant term and a quadratic, cubic and quartic term in \( \varphi_0 \).

The kinetic term in (5.7) reduces to

\[ K(\varphi_0, h') = \frac{1}{2} \left\langle \left\langle (D_a^* \varphi_0) \dagger * D_a^* \varphi_0 \right\rangle \right\rangle \]

where \( D_a^* \) is the covariant \( * \)-derivative with respect to \( h'_a \)

\[ h'_a = e^{i\omega} * D_a^* e^{-i\omega} \]

In this way all the matter field components, except \( \varphi_0 \), are ‘eaten’ by the gauge fields. Moreover, since

\[ D_a^*(v + \varphi_0) = -ih'_a v + D_a^* \varphi_0 \]

the kinetic term becomes

\[ K(\varphi_0, h') = \frac{1}{2} \left\langle \left\langle (D_a^* \varphi_0) \dagger * D_a^* \varphi_0 + v^2 h'^a h'^a \right\rangle \right\rangle \]
The second term is a mass term for the gauge field components of $h^\prime_a$, whose kinetic term, obtained from (3.11) is

$$\mathcal{YM}(h') = \frac{1}{4g^2} \langle \langle G'_{ab} * G'_{ab} \rangle \rangle \quad (5.22)$$

Therefore the second term in (5.21) provides a mass term for the gauge fields, which become all massive by ‘eating’ the matter fields. The field $\phi_0$ survives and due to (5.18) it is massive.

**Warning.** The constant term in (5.18) is divergent due to the $x$ integration. This is a well-known fact in the ordinary Higgs mechanism. In addition the terms in (5.18) are infinite due to the momentum integration. This has to be seen in relation with the primitive functional and will be discussed later.

### 6. The action principle

As pointed out above the primitive functional (3.11) and (3.6) are integrals in the phase space. Our definition of effective action, (2.7), is also an integral in the phase space. The examples of equations of motion we obtain, (4.1), (4.19) are nevertheless space-time local (the leading term in the (4.1) equations is quadratic, that is a (pseudo)elliptic operator). The natural question is: does the action principle make sense in this case?

To answer this question let us recall that an ordinary field theory action is an abstract expression, a spacetime integral of a polynomial of the fields and its derivatives. For a generic field configuration one cannot say whether the integral is convergent or not: there are plenty of field configurations for which the integral converges and plenty of field configurations for which the integral is divergent. The action principle determines an extremum, which requires a calculus of variations, i.e. it requires a topology in the manifold of fields. Thus it is clear that the action principle is based on the assumption that the space of field configurations that give rise to a finite action integral is dense enough to define a topology in the space of fields.

Now, let us consider a primitive functional like (3.11) or (3.6). We can interpret the $u$ integrand as a series in $u^2$ (because of (global) Lorentz invariance, see below), the coefficient of each powers of $u^2$ being a spacetime integral. We suppose of course that the latter are convergent and small enough so that the series in $u^2$ is convergent and integrable. We suppose that the fields configurations that give rise to an overall convergent result are dense enough to define a topology in the space of master fields, so as to allow for a variational calculus.

Another point to be remarked is that in the primitive action $x$ and $u$ do not play the same role. While $x$ spans the dynamics of the fields (the dynamical derivatives are the spacetime ones), $u$ plays the role of auxiliary variable or bookkeeping device (much like a discrete summation over the fields would do in a theory with infinite many of them).

These considerations are at the basis of the discussion in the following subsection.
6.1 Primitive functionals

Let us denote a primitive functional by

\[ S = \langle \mathcal{L}(\Phi) \rangle = \int d^d x \frac{d^d u}{(2\pi)^d} \mathcal{L}(\Phi(x, u)) \]  

(6.1)

where \( \Phi(x, u) \) represents any master field (i.e. function of \( x \) and \( u \)). We assume that \( \mathcal{L} \) is an \(-\)-polynomial in \( \Phi \) and its space-time derivatives. As just said we interpret the \( u \) integration as a bookkeeping device: we could replace the functional with a sum of spacetime integrals over the component fields, provided the expansion in \( u \) is integrable; in this case the action can be written as an infinite series of spacetime integrals. Let us apply the action principle to the latter series. There is still a question to be answered: is the eom to be identified with the master field variation or with the variation of each component field separately? Let us derive the eom’s by taking the variation with respect to any component field and equating it to 0. Since the primitive functional is, so to speak, \(-\)-analytic in \( \Phi \) and its spacetime derivatives, this is equivalent to taking the variation with respect not to a single component field, but with respect to \( \delta \Phi \):

\[ \delta \Phi = \delta \phi_0(x) + \delta \phi^\mu_1(x) u_\mu + \frac{1}{2} \delta \phi^{\mu \nu}_2(x) u_\mu u_\nu + \frac{1}{3!} \delta \phi^{\mu \nu \lambda}_3(x) u_\mu u_\nu u_\lambda + \ldots \]

The action principle in this case takes the form

\[ 0 = \delta S = \langle \delta \Phi(x, u) \mathcal{F}(x, u) \rangle \]

(6.2)

where

\[ \mathcal{F}(x, u) = \mathcal{F}_0(x) + \mathcal{F}_1^\mu(x) u_\mu + \frac{1}{2} \mathcal{F}_2^{\mu \nu}(x) u_\mu u_\nu + \frac{1}{3!} \mathcal{F}_3^{\mu \nu \lambda}(x) u_\mu u_\nu u_\lambda + \ldots \]

So (6.2) has the explicit form

\[ 0 = \langle \delta \phi_0(x) \mathcal{F}_0(x) + (\delta \phi_0(x) \mathcal{F}_1^\mu(x) + \phi_1^\mu(x) \mathcal{F}_0(x)) u_\mu \]

\[ + \frac{1}{2} (\delta \phi^{\mu \nu}_2(x) \mathcal{F}_0 + 2 \delta \phi^\mu_1(x) \mathcal{F}_1^{\mu}(x) + \delta \phi_0(x) \mathcal{F}_2^{\mu \nu}(x)) u_\mu u_\nu \]

\[ + \frac{1}{6} (\delta \phi^{\mu \nu \lambda}_3(x) \mathcal{F}_0 + 3 \delta \phi^{\mu \nu}_2(x) \mathcal{F}_1^{\nu \lambda}(x) + 3 \delta \phi^\mu_1(x) \mathcal{F}_2^{\lambda}(x) + \delta \phi_0(x) \mathcal{F}_3^{\mu \nu \lambda}(x)) u_\mu u_\nu u_\lambda + \ldots \]

\[ + \ldots \rangle \]

\[ \mathcal{F}_{n}^{\mu_{1}\mu_{2}...\mu_{n}}(x) \] \( u_{\mu_{1}} u_{\mu_{2}} ... u_{\mu_{n}} \) \]

(6.3)

The integration over \( u \) simplifies due to (global) Lorentz covariance. So (6.3) becomes
\begin{align*}
0 &= \langle \delta \phi_0(x) F_0(x) + \frac{1}{2} (\delta \phi_0^\mu(x) F_0(x) + \delta \phi_0(x) F_0^\mu(x)) \rangle \eta_{\mu \nu} \frac{u^2}{d} \\
&\quad + \frac{1}{4!} (\delta \phi_4^{\mu \nu \lambda \rho}(x) F_0 + 4 \delta \phi_3^{\mu \nu \lambda}(x) F_0^\rho(x) + 6 \delta \phi_2^{\mu \nu}(x) F_0^{\lambda \rho}(x) + 4 \delta \phi_1^{\mu}(x) F_0^{\nu \lambda \rho}(x) \\
&\quad + \delta \phi_0(x) F_0^{\nu \lambda \rho \sigma}(x)) \eta_{\mu \nu} \eta_{\lambda \rho} + \eta_{\mu \lambda} \eta_{\nu \rho} + \eta_{\mu \rho} \eta_{\nu \lambda} \rangle \frac{u^4}{d(d+2)} + \ldots \ldots
\end{align*}

where 'perm' means all distinct permutations of $\mu_1, \ldots, \mu_2n$. Since the variations $\delta \phi_i$ are arbitrary it follows that this can only be true if all the terms vanish separately. So

$$F_0(x) = 0, \quad F_1^\mu(x) = 0, \quad F_2^{\mu \nu}(x) = 0, \ldots \tag{6.5}$$

i.e.

$$F(x, u) = 0 \tag{6.6}$$

which is the eom for the master field.

**Remark 4.** It would seem that the eoms are not (6.5) but the quantities proportional to a component field variations equated to 0. For instance, looking in (6.4) at the term proportional to $\delta \phi_0$, the corresponding eom looks

$$0 = F_0(x) + \frac{1}{u^2} F_2^{\mu \nu} + \frac{3u^4}{d(d+2)} F_4^{\mu \nu} + \ldots \tag{6.7}$$

But this is not the case, because the vanishing must be true for any value of $u$, which is impossible unless each separate $x$-dependent coefficient vanishes. Thus we are back to (6.3).

### 7. Local Lorentz symmetry

As pointed out before the HS YM action is fully invariant in particular under diffeomorphism. This has prompted us to interpret the second component of $h_a(s, u)$ in the $u$ expansion, $\chi^a_s$, as a vielbein fluctuation, and $\delta_a^\mu - \chi^a_s$ as a vielbein or local frame. However this implies that $a$ is a flat index and must transform appropriately under local Lorentz transformations. But, at least at first sight, the local Lorentz invariance does not seem to be there. Consider simply the case in which only the field $A_a$ is non-vanishing, the form of the Lagrangian is

$$L_A \sim F_{a b} F^{a b}, \quad F_{a b} = \partial_a A_b - \partial_b A_a \tag{7.1}$$
This is not invariant under a Lorentz transformation, because, under \( A_a \rightarrow A_a + \Lambda_a^b A_b \), terms \((\partial_a \Lambda_c^e) A_e - (\partial_b \Lambda_a^e) A_c\) \( F^{ab} \) are generated, that do not vanish. This is a simple example of a general problem in HS YM. It is crucial to clarify it.

### 7.1 Inertial frames and connections

Let us start from the definition of trivial frame. A trivial (inverse) frame \( e^a_\mu(\mathbf{x}) \) is a frame that can be reduced to a Kronecker delta by means of a local Lorentz transformation (LLT), i.e. such that there exists a (pseudo)orthogonal transformation \( O_{ab}(\mathbf{x}) \) for which

\[
O_a^b(x)e_i^\mu(x) = \delta^\mu_a
\]  

(7.2)

As a consequence \( e^a_\mu(x) \) contains only inertial (non-dynamical) information. A full gravitational (dynamical) frame is the sum of a trivial frame and nontrivial piece

\[
\tilde{E}_a^\mu(x) = e_a^\mu(x) - \chi_a^\mu(x)
\]  

(7.3)

By means of a suitable LLT it can be cast in the form

\[
E_\mu^a(x) = \delta_a^\mu - \chi_a^\mu(x)
\]  

(7.4)

This is the form we have encountered above in HS theories. But it should not be forgotten that the Kronecker delta represents a trivial frame. If we want to recover local Lorentz covariance, instead of \( \partial_\mu = \delta_\mu^a \partial_a \) we must understand

\[
\partial_\mu = e_\mu^a(\mathbf{x})\partial_a,
\]  

(7.5)

where \( e_\mu^a(\mathbf{x}) \) is a trivial (or purely inertial) vielbein. In particular, under an infinitesimal LLT, it transforms according to

\[
\delta_\Lambda e_\mu^a(x) = \Lambda_a^b(x)e_b^\mu(x)
\]  

(7.6)

A trivial connection (or inertial spin connection) is defined by

\[
\mathcal{A}^{a}_{b\mu} = (O^{-1}(x)\partial_\mu O(x))^{a}_{b}
\]  

(7.7)

where \( O(x) \) is a generic local (pseudo)orthogonal transformation (finite local Lorentz transformation). As a consequence its curvature vanishes

\[
\mathcal{R}^{a}_{b\mu\nu} = \partial_\mu \mathcal{A}^a_{b\nu} - \partial_\nu \mathcal{A}^a_{b\mu} + \mathcal{A}^a_{c\mu} \mathcal{A}^c_{b\nu} - \mathcal{A}^a_{c\nu} \mathcal{A}^c_{b\mu} = 0
\]  

(7.8)

Let us recall that the space of connections is affine. We can obtain any connection from a fixed one by adding to it adjoint-covariant tensors, i.e. tensors that transform according to the adjoint representation. When the spacetime is topologically trivial we can choose as origin of the affine space the 0 connection. The latter is a particular member in the class of the trivial connections. This is done as follows. Suppose we start with the spin connection \( \mathcal{A}_\mu = \mathcal{A}_\mu^{ab}\Sigma_{ab} \) is

\[
\mathcal{A}_\mu(x) \rightarrow L(x)D_\mu L^{-1}(x) = L(x)(\partial_\mu + \mathcal{A}_\mu)L^{-1}(x)
\]  

(7.9)
where $L(x)$ is a (finite) LLT. If we choose $L = O$ we get
\[ A_\mu(x) \to 0 \] (7.10)

But at this point the LL symmetry gets completely concealed: choosing the zero spin connection amounts to fixing the local Lorentz gauge.

The connection $A_\mu$ contains only inertial and no gravitational information. It will be referred to as the **inertial connection**. It is a non-dynamical object (its content is pure gauge). It plays a role analogous to a trivial frame $e^\mu_a(x)$. The dynamical degrees of freedom will be contained in the adjoint tensor to be added to $A_\mu$ in order to form a fully dynamical spin connection\(^8\). $A_\mu$ is nevertheless a connection and it makes sense to define the inertial covariant derivative
\[ D_\mu = \partial_\mu - \frac{i}{2} A_\mu \] (7.11)

which is Lorentz covariant.

In ordinary Riemannian geometry the vielbein is annihilated by the covariant derivative provided we use it to build the metric and consequently the Christoffel symbols. A trivial frame and a trivial connection have an analogous relation provided the (pseudo)orthogonal transformation $O$ in (7.2) and (7.7) is the same in both cases. For we have
\[ D_\mu e^\nu_a = \left( \partial_\mu \delta^b_a + A_\mu^b \right) e^\nu_b = \partial_\mu e^\nu_a + \left( O^{-1} \partial_\mu O \right)^b_a O^{-1} e^\nu_c \delta^c_b \]
\[ = \partial_\mu O^{-1} e^\nu_a - \partial_\mu O^{-1} \delta^\nu_a = 0 \] (7.12)

\(\)From now on we assume that this is the case.

It is clear that the results ensuing from the effective action method as well as the HS YM and HS CS theories are all formulated in a trivial frame setting, eq.(7.4), with a trivial spin connection. In other words the local Lorentz gauge is completely fixed. However from this formalism it is not difficult to recover local Lorentz covariance.

### 7.2 How to recover local Lorentz symmetry

Let us restart from the definition of $J_\alpha(x, u)$
\[ J_\alpha(x, u) = \sum_{n, m=0}^{\infty} \frac{(-i)^n m^m}{2^n m! n!} \partial_{\mu_1} \ldots \partial_{\mu_n} \psi(x) \gamma_a \partial_{\nu_1} \ldots \partial_{\nu_n} \psi(x) \]
\[ \times \frac{\partial^{n+m}}{\partial u_{\mu_1} \ldots \partial u_{\mu_n} \partial u_{\nu_1} \ldots \partial u_{\nu_n}} \delta(u) \]
\[ = \sum_{s=1}^{\infty} (-1)^{s-1} J^{(s)}_{a_{\mu_1} \ldots \mu_{s-1}}(x) \frac{\partial^{s-1}}{\partial u_{\mu_1} \ldots \partial u_{\mu_{s-1}}} \delta(u) \] (7.13)

\(\)The splitting of vielbein and spin connection into an inertial and a dynamical part is characteristic of teleparallelism, see [30].
from which we derive

\[
J_{a_{\mu_1}...\mu_{s-1}}^s(x) = \sum_{n=0}^{s-1} \frac{(-1)^n}{2s-1(s-1)!} \partial_{(\mu_1}...\partial_{\mu_n} \bar{\psi}(x) \gamma_a \partial_{\mu_{n+1}}...\partial_{\mu_{s-1})} \psi(x) \tag{7.14}
\]

Assume now the following LLT

\[
\delta_\Lambda \psi = -\frac{i}{2} \Lambda \psi, \quad \Lambda = \Lambda^{ab} \Sigma_{ab}, \quad \Sigma_{ab} = \frac{i}{4} [\gamma_a, \gamma_b] \tag{7.15}
\]

and replace in (7.14) the ordinary derivative on \(\psi\) with the inertial covariant derivative

\[
\partial_{\mu} \psi \rightarrow D_{\mu} \psi = \left( \partial_{\mu} - \frac{i}{2} A_{\mu} \right) \psi \tag{7.16}
\]

and on \(\bar{\psi}\) with

\[
\partial_{\mu} \bar{\psi} \rightarrow D^\dagger_{\mu} \bar{\psi} = \partial_{\mu} \bar{\psi} + \frac{i}{2} \bar{\psi} A_{\mu} \tag{7.17}
\]

Eq. (7.14) becomes

\[
J_{a_{\mu_1}...\mu_{s-1}}^s(x) = \sum_{n=0}^{s-1} \frac{(-1)^n}{2s-1(s-1)!} D^\dagger_{(\mu_1}...D^\dagger_{\mu_n} \bar{\psi}(x) \gamma_a D_{\mu_{n+1}}...D_{\mu_{s-1})} \psi(x) \tag{7.18}
\]

Now, given

\[
\delta_\Lambda A_{\mu} = -\partial_{\mu} \Lambda + \frac{i}{2} [A_{\mu}, \Lambda] \tag{7.19}
\]

and (7.15), it is easy to prove that

\[
\delta_\Lambda (D_{\mu} \psi) = -\frac{i}{2} \Lambda (D_{\mu} \psi), \quad \delta (D^\dagger_{\mu} \bar{\psi}) = \frac{i}{2} (D^\dagger_{\mu} \bar{\psi}) \Lambda \tag{7.20}
\]

The same holds for multiple covariant derivatives

\[
\delta_\Lambda (D_{\mu_1}...D_{\mu_n} \psi) = -\frac{i}{2} \Lambda (D_{\mu_1}...D_{\mu_n} \psi), \quad \text{etc.}
\]

It follows that

\[
\delta_\Lambda J_{a_{\mu_1}...\mu_{s-1}}^s(x)
\]

\[
= -\frac{i}{2} \sum_{n=0}^{s-1} \frac{(-1)^n}{2s-1(s-1)!} D^\dagger_{(\mu_1}...D^\dagger_{\mu_n} \bar{\psi}(x) [\gamma_a, \Lambda] D_{\mu_{n+1}}...D_{\mu_{s-1})} \psi(x)
\]

\[
= \Lambda^b_a(x) J_{\mu_1...\mu_{s-1}}^s(x) \tag{7.21}
\]

Therefore the interaction term

\[
S_{\text{int}} = \sum_{s=1}^{\infty} \int d^d x \ J_{a_{\mu_1}...\mu_{s-1}}^s(x) h^{a_{\mu_1}...\mu_{s-1}} \tag{7.22}
\]
is invariant under (7.13) and (7.19) provided
\[ \delta_{\Lambda} h^{a\mu_1...\mu_n}(x) = \Lambda^a_b(x) h^{b\mu_1...\mu_n}(x) \] (7.23)

On the other hand, writing
\[ S_0 = \int d^4x \bar{\psi} \left( i\gamma^a \left( \partial_a - \frac{i}{2} A_a \right) \right) - m \] \psi \] (7.24)
also \( S_0 \) turns out to be invariant under LLT. So, provided we define LLT via (7.13) and (7.19), \( S = S_0 + S_{\text{int}} \) is invariant.

Replacing simple spacetime derivatives \( \partial_\mu \) with the inertial ones \( D_\mu \) everywhere is not enough. There is also another apparent inconsistency. Let us take the HS field strength \( G_{ab} \), (3.8). If we follow the above recipe we have to replace everywhere, also in the \( \ast \) product, the ordinary derivatives with covariant ones (covariant with respect to the spin connection \( A_a \))^9. This gives different transformation properties for the various pieces. \( D_a h_b \) transforms differently from
\[ \delta_{\Lambda}(h_a \ast h_b) = \Lambda^c_a (h_c \ast h_b) + \Lambda^c_b h_a \ast h_c \] (7.25)

The inertial frame fixes this inconsistency. Instead of writing \( \partial_\alpha = \delta_\alpha^\mu \partial_\mu \) we should write \( \partial_\alpha = e_\alpha^\mu(x) \partial_\mu \), where \( e_\alpha^\mu(x) \) is a purely inertial frame. In particular, under a LLT, it transforms according to
\[ \delta_{\Lambda} e_\alpha^\mu(x) = \Lambda^b_a e_\mu^b(x) \] (7.26)
Moreover, whenever a flat index \( O_\alpha \) is met we should rewrite it \( O_\alpha = e_\alpha^\mu O_\mu \).

Finally in spacetime integrated expression we must introduce in the integrand the factor \( e^{-1} \), where \( e = \det (e_\mu^a) \), the determinant of the inertial frame.

With this new recipes all inconsistencies disappear. For instance
\[ \delta_{\Lambda}(D_a J_b) = \Lambda^c_a (D_c J_b) + \Lambda^c_b (D_a J_c) \]
Therefore \( \delta_{\Lambda}(\eta^{ab} D_a J_b) = 0 \).

Likewise
\[ \delta_{\Lambda} G_{ab} = \Lambda^c_a G_{cb} + \Lambda^c_b G_{ac} \] (7.27)
which implies the local Lorentz invariance of \( G_{ab} \).

**Summary.** The HS effective action approach breaks completely the symmetry under the local Lorentz transformations. This is due the fact that in its formalism (and, in particular, in the HS YM and CS formalism) the choice \( e_\mu^a = \delta_\mu^a \) and \( A_a = 0 \) for the inertial frame and connection, is implicit. However the same formalism offers the possibility to recover the LL invariance by means of a simple recipe:

---

9Replacing \( \partial_\mu \) with \( D_\mu \) does not create any ordering problem because \( [D_\mu, D_\nu] = 0 \).
1. replace any spacetime derivative, even in the \(*\) product, with the inertial covariant derivative,

2. interpret any flat index \(a\) attached to any object \(O_a\) as \(e^\mu_a(x)O_\mu\),

3. in any spacetime integrand insert \(e^{-1}\).

In the process of quantization \(e^\mu_a(x)\) and \(A_a(x)\) should be treated as classical backgrounds. But in the rest of this paper, for simplicity, we stick to the gauge \(e^\mu_a = \delta^\mu_a\) and \(A_a = 0\).

### 7.3 Coupling to fermion master fields

A teleparallel framework allows us to introduce a coupling of the master field \(h_a(x, u)\) to fermion master fields. Let us start by defining the latter

\[
\Psi(x, u) = \sum_{n=0}^{\infty} \frac{1}{n!} \Psi^{\mu_1 \cdots \mu_n}(x) u_{\mu_1} \cdots u_{\mu_n},
\]

(7.28)

Under HS gauge transformations it transforms according to

\[
\delta_\varepsilon \Psi = i\varepsilon \ast \Psi, \quad \delta_\varepsilon \Psi^\dagger = -i\Psi^\dagger \ast \varepsilon,
\]

(7.29)

and let us define the covariant derivative

\[
{\mathcal{D}}^a_a \Psi = \partial_a \Psi - ih_a \ast \Psi
\]

(7.30)

together with its hermitean conjugate

\[
({\mathcal{D}}^a_a \Psi)^\dagger = \partial_a \Psi^\dagger + i\Psi^\dagger \ast h_a
\]

(7.31)

We get, in particular,

\[
\delta_\varepsilon (\mathcal{D}^a_a \Psi) = i\varepsilon \ast (\mathcal{D}^a_a \Psi)
\]

(7.32)

It is evident that the action

\[
S(\Psi, h) = \langle \langle \overline{\Psi} i\gamma^a {\mathcal{D}}_a \Psi \rangle \rangle = \langle \langle \overline{\Psi} \gamma^a (i\partial_a + h_a \ast) \Psi \rangle \rangle
\]

(7.33)

is invariant under the HS gauge transformations \((2.6)\). However, as it is, it is not invariant under the local Lorentz transformations

\[
\delta_\Lambda \Psi = -i\frac{\Lambda}{2} \Psi, \quad \delta_\Lambda \overline{\Psi} = i\frac{\Lambda}{2} \overline{\Psi}, \quad \delta_\Lambda h_a = \Lambda^b h_b
\]

(7.34)

But we know how to recover the LL invariance. We must replace in \((7.33)\) \(i\partial_a\) with \(e^\mu_a \left( i\partial_\mu + \frac{1}{2}A_\mu \right)\) and add \(e^{-1}\) in the spacetime integral. Then \((7.33)\) becomes

\[
S(\Psi, h, A) = S_1 + S_2
\]

(7.35)

\[
S_1 = \langle \langle \overline{\Psi} \gamma^a \left( i\partial_a + \frac{1}{2}A_\mu \right) \Psi \rangle \rangle = \langle \langle \overline{\Psi} \gamma^a e^\mu_a \left( i\partial_\mu + \frac{1}{2}A_\mu \right) \Psi \rangle \rangle
\]

(7.36)

\[
S_2 = \langle \langle \overline{\Psi} \gamma^a h_a \ast \Psi \rangle \rangle
\]

(7.37)
The result being contracted with the row vector \((A^1)\). One would think it is

\[
\text{The matrix operator (8.2) is acting on the column vector (}\not\alpha)
\]

becomes simply □. Let us consider the gauge-fixed HS YM

\[
\text{formations. But this is so. The proof is postponed to the Appendix B.}
\]

The rest of the proof is straightforward.

Having modified the form of the actions according to the rules contained in the summary above, it is not clear a priori that they remain invariant under the HS gauge transformations. But this is so. The proof is postponed to the Appendix [3].

8. Perturbative approach

Let us consider the gauge-fixed HS YM

\[
\mathcal{YM}(h_{\alpha}, c) = \frac{1}{g^2} \left\{ -\frac{1}{4} G_{ab} \ast G^{ab} - \frac{1}{2\alpha} \partial_a h^a \ast \partial_b h^b - i\partial^a \bar{\Psi} \ast G^a c \right\}
\]

(8.1)
as an ordinary field theory and apply to it a perturbative quantization procedure.

8.1 The propagator

We use the results of the previous section replacing \(\Phi\) with \(h_\alpha\), the index \(\alpha\) being contracted with that of \(\not\alpha\). The linear part of the eom’s, see \([4.1, 1.34, 4.6]\), once the gauge is fixed\(^{10}\), becomes simply \(\Box h_{\alpha}^{\mu_1 \ldots} = 0\). If we wish to proceed to quantization we have to know the propagator. One would think it is \(\frac{1}{p^2}\), i.e. the inverse of \(\Box\). But, in fact, the situation is more complicated.

Let us specialize to the YM HS case and consider only the quadratic part in \([3.4]\). To start with we absorb the coupling \(g\) in \(h_{\alpha}\) and \(c, \bar{c}\). In the general Lorentz gauge (the Feynman gauge corresponds to \(\alpha = 1\)) of \([3.34]\) the kinetic operator takes the form\(^{11}\)

\[
K_{\mu \nu}^{(\mu \nu)} (x, u) = \left( \eta_{\mu \nu} \Box_x - \frac{\alpha - 1}{\alpha} \partial_\mu \partial_\nu \right) - \frac{\alpha}{\alpha} \delta_{\mu \nu} \partial_\alpha \partial_\alpha
\]

(8.2)

where \(\Pi_{\mu \nu}^{\lambda \rho} = \eta_{\mu \nu} \eta^{\lambda \rho} + \eta_{\lambda \rho} \eta_{\mu \nu} + \eta_{\mu \rho} \eta_{\lambda \nu}\). We will call \(N^{(\mu \nu)}(u)\) the matrix in the RHS. The matrix operator \(\left[\begin{array}{c}
\end{array}\right]\) is acting on the column vector \((A^0(x), \chi_{\mu \nu}(x), b_{\nu \nu \mu}(x), c_{\mu_1 \mu_2 \nu}(x), \ldots)^T\), the result being contracted with the row vector \((A_0(x), \chi_{\mu \nu}(x), b_{\mu \nu \mu}(x), c_{\mu_1 \mu_2 \nu}(x), \ldots)^T\).

\(^{10}\)We refer to the gauge fixed action \([3.3]\), but, as we shall see later, this gauge fixing is not sufficient.

\(^{11}\)From now on, for simplicity, we will disregard the ghosts.
If the matrix $N$ is invertible, the propagator is given by the inverse of \( \left(8.2\right) \). Let us denote it by $P_{ab}^{\{\mu\}\{\nu\}}(x, y, u, v)$. It has the structure

\[
P_{ab}^{\{\mu\}\{\nu\}}(x, y, u, v) = \langle h_a(x, u) h_b(y, v) \rangle_0 = \int \frac{d^d k}{\left(2\pi\right)^d} e^{i k \cdot (x - y)} \left( \eta_{ab} k^2 \right) \delta(u - v) M^{\{\mu\}\{\nu\}}(u) \tag{8.3}\]

where $M^{\{\mu\}\{\nu\}}(u)$ is an infinite matrix to be determined. It is the inverse of $N^{\{\mu\}\{\nu\}}(u)$, i.e.

\[
N^{\{\mu\}\{\nu\}}(u) M^{\{\nu\}\{\mu\}}(u) = \delta^{\{\mu\}}_{\{\lambda\}}
\]

One can guess the structure of $M$

\[
M^{\{\mu\}\{\nu\}}(u) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \eta^{\mu_1 \nu_1} \frac{a_1}{u^2} & 0 & 0 \\
0 & 0 & \mu^{\mu_1 \nu_1 \nu_2} \frac{a_2}{u^3} & 0 \\
0 & 0 & 0 & \mu^{\mu_1 \mu_2 \nu_3} \frac{a_4}{u^4} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

where $t^{\{\nu\}}$ are tensors constructed out of $\eta$, which are symmetric in $\{\mu\}$ and $\{\nu\}$ separately. $a_{i,j}$ are constants to be determined, with $a_{i,j} = a_{j,i}$.

Applying this to the column vector $(j^b_0(x), j^b_{1\nu_1}(x), j^b_{2\nu_1 \nu_2}(x), j^b_{3\nu_1 \nu_2 \nu_3}(x), \ldots)^T$ and contracting the result with the row vector $(j^a_0(x), j^a_{1\mu_1}(x), j^a_{2\mu_1 \mu_2}(x), j^a_{3\mu_1 \mu_2 \mu_3}(x), \ldots)$ we create the expression

\[
\langle j^a P_{ab} j^b \rangle = \int d^d x d^d y \frac{d^d u}{\left(2\pi\right)^d} \sum_{\{\mu\},\{\nu\}} j^a_{\{\mu\}}(x) P_{ab}^{\{\mu\}\{\nu\}}(x, y, u) j^b_{\{\nu\}}(y) \tag{8.5}\]

It is more convenient to go to the momentum representation:

\[
P_{ab}^{\{\mu\}\{\nu\}}(k, u, v) = \left( \eta_{ab} k^2 + (\alpha - 1) \frac{k_a k_b}{k^4} \right) \delta(u - v) M^{\{\mu\}\{\nu\}}(u) \tag{8.6}\]

Applying this to the column vector $(\hat{j}^b_0(-k), \hat{j}^b_{1\nu_1}(-k), \hat{j}^b_{2\nu_1 \nu_2}(-k), \ldots)^T$ and contracting the result with the row vector $(\hat{j}^a_0(k), \hat{j}^a_{1\mu_1}(k), \hat{j}^a_{2\mu_1 \mu_2}(k), \ldots)$ we create the expression

\[
\langle \hat{j}^a \tilde{P}_{ab} \hat{j}^b \rangle = \int \frac{d^d k}{\left(2\pi\right)^d} \int \frac{d^d u}{\left(2\pi\right)^d} \sum_{\{\mu\},\{\nu\}} \hat{j}^a_{\{\mu\}}(k) \tilde{P}_{ab}^{\{\mu\}\{\nu\}}(k, u) \hat{j}^b_{\{\nu\}}(-k) \tag{8.7}\]

This and \( \langle j^a P_{ab} j^b \rangle \) give the $\langle j^a \tilde{P}_{ab} j^b \rangle$ term, respectively, for each couple of local fields separately.

\footnote{We will see below that this may not be the case.}
It is evident that the crucial object to be determined is the matrix \( M^{(\mu)(\nu)}(u) \). In view of the \( u \) integration, the inverse powers of \( u^2 \) in it are hard if not impossible to deal with. To gain some insight about this obstacle let us consider, below, a simple example\(^\text{13}\).

### 8.2 An example: the \( A - \chi \) model, gauge field and vielbein

Let us suppose that the master field \( h^a(x,u) \) contains only two fields: \( A^a \) and \( \chi^\mu_a \). In this case the kinetic operator becomes

\[
K_{ab}^{(\mu)(\nu)}(x,u) = \left( \eta_{ab} \Box_x - \frac{\alpha - 1}{\alpha} \partial^x_a \partial^x_b \right) \begin{pmatrix} 1 & 0 \\ 0 & \eta^{\mu\nu} \frac{u^2}{d^2} \end{pmatrix}
\]  
(8.8)

Its inverse is

\[
P_{ab}^{(\mu)(\nu)}(x,y,u) = \hat{d} \hat{d} k (2\pi)^d \delta^d(x-y) \left( \eta_{ab} k^2 + (\alpha - 1) \frac{k_a k_b}{k^4} \right) \begin{pmatrix} 1 & 0 \\ 0 & \eta^{\mu\nu} \frac{d^2}{u^2} \end{pmatrix}
\]  
(8.9)

To gain some insight for the general case it is useful to develop the perturbative approach for this \( A - \chi \) model. Let us write down the interaction terms for these two fields. In Appendix \( \text{A.3} \) we have collected the \( u \) expansions for the interaction part of the action

\[
S = S_0 + S_{\text{int}}, \quad S_{\text{int}} = S_3 + S_4
\]  
(8.10)

where, in the \( A - \chi \) model, \( S_0 \) is the free part with kernel (8.8) and \( S_3, S_4 \) are the cubic, quartic interaction, respectively. The latter can be obtained from the equation (A.7) and (A.8) by suppressing all the other component fields.

The cubic term is

\[
S_3 = -g \langle \langle \partial^\rho A^b \partial_\rho A^a \chi^\sigma_b - \partial_\rho A^b \chi^\sigma_a \rangle + \frac{1}{d} \partial^\rho \chi^b \left( \partial_\rho \chi^\nu A^a \chi^\sigma_b - \partial_\rho \chi^\nu A^b \chi^\sigma_a \right) u^2 \rangle \]  
(8.11)

and the quartic is

\[
S_4 = -\frac{g^2}{2} \langle \langle \partial_\rho A^a \chi^{b\sigma} - \partial_\rho A^b \chi^{a\sigma} \rangle \partial_\tau A^a \chi^\tau_b + \frac{1}{d} \partial_\sigma \chi \chi^{a\nu} A^b \chi^\sigma - \partial_\sigma \chi \chi^{b\nu} A^a \chi^\sigma \rangle \partial_\tau \chi \chi^{\tau\nu} u^2 \rangle
\]  
(8.12)

exactly.

Let us define the Feynman rules in the usual way, by considering \( u^2 \) as a constant and ignoring, for the time being the integration over \( u \). The free \( AA \) propagator is of order \( u^0 \). Consider then the next order, i.e. the bubble diagram with two external \( A \)-legs. This can be formed with two cubic vertices (the first two terms in \( S_3 \)), one \( AA \) and one \( \chi \chi \) propagator. Looking at (8.9), we see that the result is of order \( \frac{1}{u^2} \). Another possibility is to create a seagull term by means of a quartic vertex (the first two terms in \( S_4 \)) and a \( \chi \chi \) propagator. This is also of order \( \frac{1}{u^2} \). Next let us consider the two-loop order, which is formed with three \( AA \chi \) and one \( \chi \chi \chi \) vertices, three internal \( \chi \chi \) plus two \( AA \) propagators. This is of order \( u^{-4} \).

\(^\text{13}\)For a more general discussion about the Feynman rules, see [1]

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Similarly, the vertex $AA\chi$ is of order $u^0$. The one-loop contribution to the 3-point $AA\chi$ is of order $u^{-2}$. And so on.

Consider next the diagrams with two external $\chi$-legs. The 0-th order $\chi\chi$ propagator is of order $u^2$. Using two cubic $AA\chi$ vertices and two $AA$ propagators we obtain a contribution of order $u^0$. We can obtain a bubble diagram with two external $\chi$-legs by means of two 3$\chi$ cubic vertices and two $\chi\chi$ propagators. The result is also of order $u^0$. We can also create two seagull terms using the quartic vertices $AA\chi\chi$ and one $AA$ propagator, and the vertex $\chi\chi\chi\chi$ with a $\chi\chi$ propagator. These results are also of order $u^0$.

Similarly the $3\chi$ vertex is of order $u^{-2}$. The one-loop $3\chi$ function is of order $u^0$.

Since different loop contributions have different powers of $u$ it is important to verify at least their dimensional correctness. This is done Appendix D.

We can continue by considering higher loop contributions or diagrams with more legs. The results are somewhat disconcerting, because we obtain a different at least their dimensional correctness. This is done Appendix D.

We see that $g$ has disappeared. But this is a bit too much, because now, since $u$ has the dimension of a mass, the field $\chi$ has the same dimension as $A$, i.e. 1. It cannot represent a gravity vielbein. What we have to do, instead, is split $\sqrt{u^2}$ = $m$ where $m$ is a fixed mass scale and $u$ is the dimensionless variable part, and absorb the latter in $\chi$ and $g$, by setting $\chi' = u\chi$ and $g' = \frac{g}{\sqrt{u^2}}$. We obtain in this way

\[ S_3 = -g\langle \partial^a A^b(\partial_\sigma A_a\chi^\sigma_b - \partial_\sigma A_b\chi^\sigma_a) + \frac{1}{d}\partial^a\chi^\sigma_b(\partial_\sigma \chi^\nu a\chi^\sigma_b - \partial_\sigma \chi^\nu b\chi^\sigma_a) \rangle \]  
\[ S_4 = -\frac{g^2}{2}\langle (\partial_\sigma A^a\chi^\nu b\sigma - \partial_\sigma A^b\chi^\nu a\sigma) \partial_\nu A_a\chi^\tau_b + \frac{1}{d}(\partial_\sigma \chi^\nu a\chi^\nu b\sigma - \partial_\sigma \chi^\nu b\chi^\nu a\sigma) \partial_\nu \chi^\nu a\chi^\nu b \rangle \]  

(8.13) \hspace{1cm} (8.14)

where, for simplicity, the new fields $\chi' = \sqrt{u^2}\chi$ and new coupling $g' = \frac{g}{\sqrt{u^2}}$ are denoted with the old symbols.

We see that $u$ has disappeared. But this is a bit too much, because now, since $u$ has the dimension of a mass, the field $\chi$ has the same dimension as $A$, i.e. 1. It cannot represent a gravity vielbein. What we have to do, instead, is split $\sqrt{u^2} = m$ where $m$ is a fixed mass scale and $u$ is the dimensionless variable part, and absorb the latter in $\chi$ and $g$, by setting $\chi' = u\chi$ and $g' = \frac{g}{u}$. We obtain in this way

\[ S_3 = -g\langle \partial^a A^b(\partial_\sigma A_a\chi^\sigma_b - \partial_\sigma A_b\chi^\sigma_a) + \frac{m^2}{d}\partial^a\chi^\sigma_b(\partial_\sigma \chi^\nu a\chi^\sigma_b - \partial_\sigma \chi^\nu b\chi^\sigma_a) \rangle \]  
\[ S_4 = -\frac{g^2}{2}\langle (\partial_\sigma A^a\chi^\nu b\sigma - \partial_\sigma A^b\chi^\nu a\sigma) \partial_\nu A_a\chi^\tau_b + \frac{m^2}{d}(\partial_\sigma \chi^\nu a\chi^\nu b\sigma - \partial_\sigma \chi^\nu b\chi^\nu a\sigma) \partial_\nu \chi^\nu a\chi^\nu b \rangle \]  

(8.15) \hspace{1cm} (8.16)

Now the integration over $u$ yields an overall dimensionful infinite factor which can be eliminated by dividing the primitive functional $\mathcal{J}\mathcal{M}(h)$ by the same factor, as suggested in Remark 1. Moreover in (8.9) $u^2$ is replaced by $m^2$, in this way disarming the risk triggered by the $u$ integration in the amplitudes.
8.3 Absorbing the $u$ dependence. A frozen momentum background

As in the previous example it is possible to absorb the $u$ dependence completely. However in the general case it is not enough to redefine $h_u^{\mu_1 \ldots \mu_s - 1} \rightarrow h_u^{\mu_1 \ldots \mu_s - 1} = u^{s-1} h_u^{\mu_1 \ldots \mu_s - 1}$, we must also redefine the coordinates as follows: $x^{\mu} \rightarrow u x^{\mu}$. The coupling is also redefined as before $g \rightarrow \frac{\alpha}{u}$. Under these redefinition, for instance $S_3$ and $S_4$ have the same form as in eqs. (A.7, A.8) with $u$ replaced by $m$ and $\langle \rangle$ replaced by $\langle \rangle'$. In the same way the kinetic term remains the same apart from

$$
\langle \langle h_a K^{ab} h_b \rangle \rangle' \rightarrow \langle \langle h_a K^{ab} h_b \rangle \rangle'
$$

(8.17)

The symbol $\langle \rangle'$ means that the integration measure has changed to

$$
\int d^d x d^d u \equiv m^d \int d^d x d^d u \rightarrow m^d \int d^d u \ u^{d-2}
$$

(8.18)

In other words, apart from this change of measure and the substitution of $u$ replaced by $m$, in the expressions $S_3, S_4$ and the kinetic term, nothing has changed. In particular the dependence on $u$ has disappeared from the integrand. Since now the integrand is $u$ independent we can factor out the quantity

$$
\mathcal{V}_d = m^d \int d^d u \ u^{d-2}
$$

(8.19)

and simplify it with the same factor coming from Remark 1.

So finally we are simply left with the spacetime action $S = S_2 + S_3 + S_4$:

$$
S_2 = \int d^d x \sum_{\{\mu\},\{\nu\}} h_a^{\mu}(x) K^{\rho(\nu)}(x, m) h_b^{\nu}(x)
$$

(8.20)

$$
S_3 = -g \langle \langle \partial^\rho A^b (\partial_\sigma A_a \chi_b^\sigma - \partial_\sigma A_b \chi_a^\sigma) \rangle \rangle
$$

(8.21)

$$
-\frac{1}{24} \left( \partial^\rho A^b - \partial^b A^\rho \right) (\partial_\sigma \partial_\tau \partial_\sigma A_a \chi_b^\sigma \partial_\tau \sigma_2 \sigma_3 + 3 \partial_\sigma A_a \chi^{\sigma_1 \sigma_2 \sigma_3} + \partial_\sigma A_a \chi^{\sigma_1 \sigma_2 \sigma_3} )
$$

$$
+ \frac{m^2}{2d} \left( \partial^\rho A^b \partial_\sigma b c_\mu \chi^\sigma - \partial^\rho A^b \partial_\sigma b c_\sigma \chi^\mu \right)
$$

+ 2 \partial^\rho A^b \partial_\sigma b c_\mu \chi^\sigma - 2 \partial^\rho A^b \partial_\sigma b c_\mu \chi^\sigma + \partial^\rho A^b \partial_\sigma A_a b c_\mu \chi^\sigma - \partial^\rho A^b \partial_\sigma A_a b c_\mu \chi^\sigma
$$

$$
+ \partial^\rho b c_\mu \partial_\sigma A_a b^\sigma - \partial^\rho b c_\mu \partial_\sigma A_a b^\sigma - 2 \partial^\rho b c_\mu \partial_\sigma A_a b^\sigma - \partial^\rho b c_\mu \partial_\sigma A_a b^\sigma
$$

(\partial_\sigma \chi_a \chi_b - \partial_\sigma \chi_a \chi_b)

$$
\frac{1}{24} \left( \left( \partial^\rho A^b - \partial_b A^\rho \right) (\partial_\sigma \partial_\tau \partial_\sigma A_a \chi_b^\sigma \partial_\tau \sigma_1 \sigma_2 \sigma_3 + \frac{1}{2} \partial_\sigma \partial_\tau \partial_\sigma A_a f_{\mu \nu \sigma_1 \sigma_2 \sigma_3}
$$

$$
+ \frac{1}{2} \partial_\sigma \partial_\tau \partial_\sigma A_a f_{\mu \nu \sigma_1 \sigma_2 \sigma_3} - \frac{3}{2} \partial_\sigma \partial_\tau \partial_\sigma \chi_a \partial_\tau \sigma_1 \sigma_2 \sigma_3 - 3 \partial_\sigma \partial_\tau \partial_\sigma \chi_a \partial_\tau \sigma_1 \sigma_2 \sigma_3
$$

$$
+ \frac{3}{2} \partial_\sigma \partial_\tau \partial_\sigma \chi_b \partial_\tau \sigma_1 \sigma_2 \sigma_3 + \left( \partial_\sigma b c^\mu - \partial_b c_\mu \right) (\partial_\sigma \partial_\tau \partial_\sigma A_a b_{\sigma_1 \sigma_2 \sigma_3}
$$

$$
+ 3 \partial_\sigma \partial_\tau \partial_\sigma \chi_a \partial_\tau \sigma_1 \sigma_2 \sigma_3
$$

$$
+ \left( \partial^\rho \chi_a \partial_\sigma b_{\mu} - \partial^\rho \chi_a \partial_\sigma b_{\mu} \right) \left( \partial_\sigma \partial_\tau \partial_\sigma A_a b_{\sigma_1 \sigma_2 \sigma_3}
$$

$$
+ 3 \partial_\sigma \partial_\tau \partial_\sigma \chi_a \partial_\tau \sigma_1 \sigma_2 \sigma_3
$$

$$
- 3 \partial_\sigma \partial_\tau \partial_\sigma \chi_b \partial_\tau \sigma_1 \sigma_2 \sigma_3 + 3 \partial_\sigma \partial_\tau \partial_\sigma \chi_a \partial_\tau \sigma_1 \sigma_2 \sigma_3
$$

$$
+ O(m^4, 6) \right)
$$

(8.21)
where \( \mathcal{O}(m^4, 6) \) means terms of order at least \( m^4 \) or containing at least six derivatives, and

\[
S_4 = -\frac{g^2}{2} \int d^4x \left\{ (\partial_\sigma A^a \chi^b - r_\sigma A^b \chi^a) \partial_\tau A_a \chi_b \right\}
\]

\[
+ \frac{m^2}{d} \left( (\partial_\sigma A^a \chi^b - \partial_\sigma A^b \chi^a) \left( \partial_{\tau} A_a c_{\nu \tau} + 2 \partial_{\nu} A_a b_{\nu \tau} + \partial_{\nu} b_{\nu \tau} \chi_b \right) + (\partial_\sigma A^a b_{\sigma \nu} - \partial_\sigma A^b b_{\sigma \nu} + \partial_\sigma b_{\sigma \nu} \chi^a - \partial_\sigma b_{\sigma \nu} \chi^a) \left( \partial_{\tau} A_a b_{\nu \tau} + \partial_{\tau} b_{\nu \tau} \chi_b \right) \right) + \mathcal{O}(m^4, 4)
\]

where \( \mathcal{O}(m^4, 4) \) means terms of order at least \( m^4 \) or containing at least four derivatives.

\( S \) does not have the elegant form of (8.14), but it is a good starting point for quantization.

### 8.4 Back to the propagator

The kinetic operator in (8.24) is

\[
K_{ab}^{\{\mu\} \{\nu\}}(x, m) = \left( \eta_{ab} \Box_x - \frac{\alpha - 1}{\alpha} \partial_a \partial_b \right) N^{\{\mu\} \{\nu\}}(m) \equiv \mathcal{K}_{ab} N^{\{\mu\} \{\nu\}}(m)
\]

where

\[
N^{\{\mu\} \{\nu\}}(m) = \begin{pmatrix}
1 & 0 & \eta_{\mu_1 \nu_2} \frac{m^2}{d} & \Pi_{\mu_1 \nu_2 \nu_3 \nu_4} \frac{m^4}{d(d+2)} & 0 \\
0 & 1 & 0 & 0 & \Pi_{\mu_1 \nu_2 \nu_3 \nu_4} \frac{m^4}{d(d+2)} \\
\eta_{\mu_1 \nu_2} \frac{m^2}{d} & 0 & 1 & 0 & 0 \\
\Pi_{\mu_1 \nu_2 \nu_3 \nu_4} \frac{m^4}{d(d+2)} & 0 & 0 & 1 & \cdots \\
0 & \cdots & \cdots & \cdots & 1
\end{pmatrix}
\]

If the inverse of this matrix exists the propagator in momentum space is

\[
\tilde{F}_{ab}^{\{\mu\} \{\nu\}}(k, u, v) = \left( \eta_{ab} \frac{k^2}{k^2} + (\alpha - 1) \frac{k_a k_b}{k^4} \right) M^{\{\mu\} \{\nu\}}(u) \Rightarrow \delta_{ab}^{\{\mu\} \{\nu\}}
\]

where \( M \) is the inverse of \( N \), i.e.

\[
N^{\{\mu\} \{\nu\}}(m) M^{\{\nu\} \{\lambda\}}(m) = \delta_{\{\nu\} \{\lambda\}}^{\{\mu\}}
\]

\( M \) must have the structure

\[
M^{\{\mu\} \{\nu\}}(m) = \begin{pmatrix}
1 & 0 & \eta_{\mu_1 \nu_2} \frac{\alpha_1 \alpha_2}{m^2} & 0 & \mu_{\mu_1 \nu_2 \nu_3 \nu_4} \frac{\alpha_1 \alpha_2}{m^4} \\
0 & 1 & 0 & \mu_{\mu_1 \mu_2 \nu_2 \nu_3} \frac{\alpha_1 \alpha_3}{m^4} & 0 \\
\eta_{\mu_1 \nu_2} \frac{\alpha_1 \alpha_2}{m^2} & 0 & 1 & 0 & \mu_{\mu_1 \mu_2 \nu_2 \nu_3} \frac{\alpha_1 \alpha_3}{m^4} \\
\mu_{\mu_1 \mu_2 \nu_2 \nu_3} \frac{\alpha_1 \alpha_3}{m^4} & 0 & 0 & 1 & \cdots \\
0 & \cdots & \cdots & \cdots & 1
\end{pmatrix}
\]
where $t^{(\mu)(\nu)}$ are tensors constructed out of $\eta$, which are symmetric in $\{\mu\}$ and $\{\nu\}$ separately. $a_{i,j}$ are constants to be determined, with $a_{i,j} = a_{j,i}$.

**Comment.** As noted in the previous subsection, the propagators in the frozen momentum background do not suffer from the pathological feature of higher and higher inverse powers of $u^2$. In this sense we view the move to the frozen momentum background as a part of the gauge fixing procedure, because it does not leave the HS gauge symmetry unaffected. In turn this raises a new question: what are the new eom’s and what becomes of the HS symmetry in the frozen momentum background? If the momentum $u$ is frozen to $m$ we cannot anymore partially integrate over it, as it is necessary to do in order to guarantee the property (3.5), which is basic in order to prove that

$$\langle \langle [G_{ab} * G^{ab}, \varepsilon] \rangle \rangle = 0$$  \hspace{1cm} (8.27)

Do we have to conclude that in this phase the dynamics is different and the HS gauge symmetry is completely lost? We can actually convince ourselves that some symmetry does survive. Let us return to (6.5). The rescaling to the frozen momentum background affects this equation simply by the change of $u^2$ to $m^2$. In this regard it may be useful to notice that the frozen momentum expressions (8.21),(8.22) can be obtained by carrying out the integration over $u$ for the modified HS Yang-Mills action

$$\tilde{\mathcal{Y}M}(h) = -\frac{1}{4g^2} \langle \langle \delta(u^2 - m^2) * G^{ab} * G_{ab} \rangle \rangle,$$  \hspace{1cm} (8.28)

where the delta distribution may help understanding the fate of HS gauge symmetry in the frozen momentum framework. The latter affects the eom’s, because we cannot anymore use the argument that the master eom has to vanish for any $u$. For instance the equation $F_0(x) = 0$ is modified to an infinite series

$$0 = F_0(x) + \frac{m^2}{d} F_{2\mu} + 3 \frac{m^4}{d(d + 2)} F_{4\mu \nu \mu \nu} + \ldots$$  \hspace{1cm} (8.29)

The second and following terms in the RHS are nothing but traces of the eom’s (6.5). Therefore the solutions of (6.5) are also solutions of the new eom’s such as (8.29). Moreover we know that the latter are covariant under HS gauge transformations. Therefore we expect that in some reshuffled form some kind of symmetry should appear also in the frozen momentum background.

The problem of residual symmetry in the frozen momentum framework deserves further investigation, but leaving it aside for the time being, the frozen momentum background offers a viable procedure for perturbative quantization.

\footnote{Needless to say the frozen momentum background is reminiscent of a spontaneously broken symmetry phase in ordinary field theory.}
9. Feynman diagrams in the frozen momentum background

Let us return to (C.13) in Appendix and rewrite it in terms of component fields (for two point functions, for the sake of simplicity)

\[
\langle \tilde{h}_{a}^{\mu_{1}...\mu_{m}}(q_{1})\tilde{h}_{b}^{\nu_{1}...\nu_{n}}(q_{2}) \rangle = \frac{\delta}{\delta j_{\mu_{1}...\mu_{m}}(q_{1})} \frac{\delta}{\delta j_{\nu_{1}...\nu_{n}}(q_{2})} e^{iS_{\text{int}}(\frac{\delta}{\delta j})} e^{-i\langle \tilde{P}_{ab} \rangle} \bigg|_{\tilde{j}=0} \tag{9.1}
\]

The crucial objects are \( S_{3} \), \( S_{4} \) and \( \langle \tilde{P}_{ab} \rangle \). We have to rewrite them in the frozen momentum language. Let us start from the latter. The re-definitions needed are:

\[
u^{2} = m^{2} u^{2}, \quad x^{\mu} \rightarrow x^{\mu} = x^{\mu} u, \quad k^{\mu} \rightarrow k^{\mu} \frac{k^{\mu}}{u}, \quad \tilde{h}_{a}^{\mu_{1}...\mu_{s}} \rightarrow \tilde{h}_{a}^{\prime \mu_{1}...\mu_{s}} = u^{s} h_{a}^{\mu_{1}...\mu_{s}} \tag{9.2}
\]

This implies, in particular,

\[
\tilde{j}_{\mu_{1}...\mu_{s}} \rightarrow \tilde{j}_{\mu_{1}...\mu_{s}} = u^{-s} j_{\mu_{1}...\mu_{s}} \tag{9.3}
\]

Taking into account the explicit form of the propagator, see (8.23), we can decompose the integrand of \( \langle \tilde{P}_{ab} \rangle \) into a \( u \)-independent factor and a \( m^{d} u^{d-2} \) factor. Finally, as before, we can factor out \( V_{d} = m^{d} \int d^{d}u u^{d-2} \) and simplify it to get

\[
\langle \tilde{P}_{ab} \rangle = \sum_{n, m} \frac{1}{m^{n+m}} \int \frac{d^{d}k}{(2\pi)^d} \tilde{j}_{\mu_{1}...\mu_{m}}(k) \tilde{P}_{ab}^{\mu_{1}...\mu_{m} \nu_{1}...\nu_{n}}(k) j_{\nu_{1}...\nu_{n}}(-k) \tag{9.4}
\]

where

\[
\tilde{P}_{ab}^{\mu_{1}...\mu_{m} \nu_{1}...\nu_{n}}(k) = a_{m, n} t^{\mu_{1}...\mu_{m} \nu_{1}...\nu_{n}} \left( \frac{\eta_{ab}}{k^{2}} + (\alpha - 1) \frac{k_{a}k_{b}}{k^{4}} \right) \tag{9.5}
\]

Next we have to rewrite \( S_{3} \). Here a more subtle representation of \( u_{\mu} \) must be used:

\( u_{\mu} = m u_{\mu} \), where \( n_{\mu} \) represents the normal unit vector to the sphere \( S^{d-1} \) of radius 1, such that

\[
\int_{S^{d-1}} dn \ n^{\mu_{1}} ... n^{\mu_{2n}} = \frac{\Pi^{\mu_{1}...\mu_{2n}}}{d(d+2)...(d+2n-2)} \tag{9.6}
\]

Using (C.14), rewriting it in components, rescaling as above, then factoring out \( V_{d} \) and simplifying it as before, we get

\[
S_{3} = -g \int \frac{d^{d}k_{1}}{(2\pi)^d} \frac{d^{d}k_{2}}{(2\pi)^d} \delta(k_{1} + k_{2} + k_{3}) k_{1a} \int_{S^{d-1}} dn \ \sum_{l=0}^{\infty} \frac{m^{l}}{l!} \frac{\delta}{\delta j_{\mu_{1}...\mu_{l}}^{\alpha}(k_{1})} n_{\lambda_{1}} ... n_{\lambda_{l}} \times \sum_{n, m=0}^{\infty} \frac{m^{n+m}}{n!m!} \left[ \frac{\delta}{\delta j_{\mu_{1}...\mu_{m}}(k_{2})} \left( n - \frac{k_{3}}{2} \right)_{\mu_{1}} ... \left( n - \frac{k_{3}}{2} \right)_{\mu_{m}} \frac{\delta}{\delta j_{\nu_{1}...\nu_{n}}(k_{3})} \left( n + \frac{k_{2}}{2} \right)_{\nu_{1}} ... \left( n + \frac{k_{2}}{2} \right)_{\nu_{n}} \right] \tag{9.7}
\]
and a similar expression for $S_4$.

As a sample let us compute the two-point function from (9.1): Let us recall that formula (9.1) gives the two-point function multiplied on the left by the propagator $P_{\hat{h}_b^{(m)}\hat{h}_b^{(n)}}$ and on the right by $P_{\hat{h}_b^{(n)}\hat{h}_b^{(n)}}$, a piece that contributes to the self-energy. In order to find the genuine two-point function we have to truncate the two external legs by multiplying by the respective inverse propagators. The calculation proceeds in the usual way and the result is $(n + m = \text{even},$ otherwise 0)

$$\langle \hat{h}_a^{\mu_1 \ldots \mu_m}(q_1)\hat{h}_b^{\nu_1 \ldots \nu_n}(q_2) \rangle_0 = g^2 q_1^b q_2^d \delta(q_1 + q_2) \int \frac{d^4p}{(2\pi)^d} \frac{1}{p^2(p - q_1)^2} \frac{m^{n+m}}{n!m!}$$  \hspace{1cm} (9.8)

$$\times \int_{S^d-1} dn_1 \int_{S^d-1} dn_2 n_1 \ldots n_1 n_2 \ldots n_2 \sum_{l, r, j, s = 0} \frac{1}{l!r!j!s!} a_{l,s} a_{r,j} \delta^{\lambda_1 \ldots \rho_l \sigma_1 \ldots \tau_s} \delta^{\rho_1 \ldots \rho_l \tau_1 \ldots \tau_s}$$

$$\times \left[ \left( n_1 + \frac{p - q_1}{2m} \right) \lambda_1 \ldots \left( n_1 + \frac{p - q_1}{2m} \right) \lambda_1 \left( n_1 + \frac{p}{2m} \right) \rho_1 \ldots \left( n_1 + \frac{p}{2m} \right) \rho_1 \right]$$

$$- \left[ \left( n_1 - \frac{p - q_1}{2m} \right) \lambda_1 \ldots \left( n_1 - \frac{p - q_1}{2m} \right) \lambda_1 \left( n_1 - \frac{p}{2m} \right) \rho_1 \ldots \left( n_1 - \frac{p}{2m} \right) \rho_1 \right]$$

$$\times \left[ \left( n_2 + \frac{p - q_1}{2m} \right) \sigma_1 \ldots \left( n_2 + \frac{p - q_1}{2m} \right) \sigma_1 \left( n_2 + \frac{p}{2m} \right) \tau_1 \ldots \left( n_2 + \frac{p}{2m} \right) \tau_1 \right]$$

$$- \left[ \left( n_2 - \frac{p - q_1}{2m} \right) \sigma_1 \ldots \left( n_2 - \frac{p - q_1}{2m} \right) \sigma_1 \left( n_2 - \frac{p}{2m} \right) \tau_1 \ldots \left( n_2 - \frac{p}{2m} \right) \tau_1 \right]$$

It is easy to check that the dimension of this 2-pt function is $2 + m + n$, as it should be because it must have the same dimension as the propagator with opposite sign. The full result is a series in $\frac{1}{m}$. Thus $\frac{1}{m}$ plays a role similar to $\sqrt{\alpha'}$ in the field theory limit of string theory.

10. Diagonalizing the propagator (further gauge fixing)

It is clear that the difficulties met in the previous section could be handled more effectively were we able to diagonalize the kinetic operator (8.2). We would like to show in this section that this can be done provided we fix a surviving gauge freedom. As it is not hard to guess this corresponds to go to a traceless basis for the component fields. Since this can in fact be formulated on a general ground we will use a general notation rather than the specific one of the previous sections. Let us consider a generic master field

$$\Phi(x,u) = \sum_{n=0}^{\infty} \frac{1}{n!} \phi^{\mu_1 \ldots \mu_n}(x) u_{\mu_1} \ldots u_{\mu_n}$$ \hspace{1cm} (10.1)
Φ may have additional indices, like for instance \( h_a \), but they will be understood in the sequel. Its components can be reshuffled as follows

\[
\Phi(x, u) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(2n)!} \left( \frac{2n}{2k} \right) (2k - 1)!! \frac{\mu^{2k}}{(d + 4(n - 1) - 2(k - 1)) \ldots (d + 4(n - 1) - 4(k - 1))} \]

\[
\times \tilde{\phi}^{(k) \mu_{2k+1} \ldots \mu_{2n}} u_{\mu_{2k+1}} \ldots u_{\mu_{2n}}
\]

\[
+ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(2n + 1)!} \left( \frac{2n + 1}{2k} \right) (2k - 1)!! \frac{\mu^{2k}}{(d + 4(n - 1) - 2(k - 1) + 2) \ldots (d + 4(n - 1) - 4(k - 1) + 2)}
\]

\[
\times \tilde{\phi}^{(k) \mu_{2k+1} \ldots \mu_{2n+1}} u_{\mu_{2k+1}} \ldots u_{\mu_{2n+1}}
\]

(10.2)

where \( \tilde{\phi}_0 = \phi_0, \tilde{\phi}^\mu = \phi^\mu \) and \( \tilde{\phi}^{(k) \mu_{1} \ldots \mu_{p}} \) are traceless fields obtained from \( \phi^{[k] \mu_{1} \ldots \mu_{p}} \) by subtracting traces. Here \( [k] \) denotes a \( k \)-fold trace.

Eq. (10.2) is simply a rewriting and we cannot in general assign any meaning to the coefficients of \( u_{\mu_{2k+1}} \ldots u_{\mu_{2n}} \) and \( u_{\mu_{2k+1}} \ldots u_{\mu_{2n+1}} \) because they contain powers of \( u^2 \). However in the frozen momentum background things change and, after the rescalings of section 3, we can define new component fields

\[
\tilde{\phi}^{\mu_{1} \ldots \mu_{2p}} = \phi^{\mu_{1} \ldots \mu_{2p}} + \sum_{n=1}^{\infty} \frac{1}{(2p)!(2n)!} \frac{(2n - 1)!! \mu^{2n}}{(d + 2(n - 1) + 2p + 2) \ldots (d + 4p + 2)} \tilde{\phi}^{[n] \mu_{1} \ldots \mu_{2p}}
\]

(10.3)

and

\[
\tilde{\phi}^{\mu_{1} \ldots \mu_{2p+1}} = \phi^{\mu_{1} \ldots \mu_{2p+1}} + \sum_{n=1}^{\infty} \frac{1}{(2p + 1)!(2n)!} \frac{(2n - 1)!! \mu^{2n}}{(d + 2(n - 1) + 2p + 2) \ldots (d + 4p + 2)} \tilde{\phi}^{[n] \mu_{1} \ldots \mu_{2p+1}}
\]

(10.4)

Here are some examples:

\[
\tilde{\phi}_0 = \phi_0 + \frac{m^2}{2d} \phi_2' + \frac{m^4}{8d(d + 2)} \phi_4'' + \frac{m^6}{48d(d + 2)(d + 4)} \phi_6''' + \ldots
\]

(10.5)

\[
\tilde{\phi}_1^\mu = \phi_1^\mu + \frac{m^2}{2(d + 2)} \phi_3^{\mu'} + \frac{m^4}{8(d + 2)(d + 4)} \phi_5^{\mu''} + \ldots
\]

(10.6)

\[
\tilde{\phi}_2^{\mu \nu} = \phi_2^{\mu \nu} + \frac{m^2}{2(d + 4)} \phi_4^{\mu \nu'} + \frac{m^4}{8(d + 4)(d + 6)} \phi_6^{\mu \nu''} + \ldots
\]

(10.7)

etc.

Next let us consider our original kinetic term in the frozen momentum background

\[
h^{\alpha T}_{\{\mu\}} K_{ab}^{\{\nu\}} (x, m) h^b_{\{\nu\}}
\]

(10.8)

where

\[
K_{ab}^{\{\nu\}} (x, m) = \left( \eta_{ab} - \frac{\alpha - 1}{\alpha} \partial_a \partial_b \right)
\]

(10.9)
and $h^{Ta}_{\{\mu\}} = (A^a, \chi^a_{\mu}, b^{a}_{\mu_1\mu_2}, e^{a}_{\mu_1\mu_2\mu_3}, \ldots)$. The claim is that (10.3) is equal to

\[ \hat{h}^{Ta}_{\{\mu\}} \tilde{K}^{\{\nu\}}(x, m) \hat{h}^b_{\{\nu\}} \]

where \( \hat{h}^{Ta}_{\{\mu\}} = (\hat{A}^a, \hat{\chi}_{\mu}, \hat{b}^{a}_{\mu_1\mu_2}, \hat{e}^{a}_{\mu_1\mu_2\mu_3}, \ldots) \), the component fields being given by (10.3) and (10.4), and

\[ \tilde{K}^{\{\mu\}}(x, m) = \left( \eta_{ab} \Box - \frac{\alpha - 1}{\alpha} \partial_a \partial_b \right) \]

\[ \times \left( \begin{array}{cccccc}
1 & 0 & 0 & \ldots \\
0 & \eta^{\mu_1\nu_1} & 0 & \ldots \\
0 & 0 & \eta^{\mu_2\nu_2} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{array} \right)
\]

The tensors $\tilde{H}_{\{\mu\}}^{\{\nu\}}$ are symmetric and traceless in the indices $\{\mu\}$ and $\{\nu\}$, separately. For instance

\[ \tilde{H}_{\mu_1\mu_2\nu_1\nu_2}^{\mu_3} = \eta^{\mu_1\nu_1} \eta^{\mu_2\nu_2} + \eta^{\mu_1\nu_2} \eta^{\mu_2\nu_1} - \frac{2}{d} \eta^{\mu_1\mu_2} \eta^{\nu_1\nu_2} \]

\[ \tilde{H}_{\mu_1\mu_2\mu_3}^{\nu_1\nu_2} = \delta_{\nu_1}^{\mu_1} \delta_{\nu_2}^{\mu_2} \delta_{\nu_3}^{\mu_3} + \text{perm}(\nu_1, \nu_2, \nu_3)
- \frac{2}{d} \left( \eta^{\mu_1\mu_2} \eta^{\nu_1\nu_2} + \text{perm}(\mu_1, \mu_2, \mu_3)(\nu_1, \nu_2, \nu_3) \right) \]

The terms at the RHS in the second line are 6, while the terms in the third line are 9.

When replacing (10.3) and (10.4) in (10.10), one should notice that the fields $\tilde{H}_{\{\mu\}}^{\{\nu\}}$ and $\tilde{H}_{\{\mu\}}^{\{\nu\}}$ can be replaced by $\tilde{H}_{\{\mu\}}^{\{\nu\}}$ and $\tilde{H}_{\{\mu\}}^{\{\nu\}}$, respectively, because the difference is made of trace parts (i.e. they contain at least one $\eta$), saturated with some traceless tensor $\tilde{H}_{\{\mu\}}^{\{\nu\}}$. The equivalence between (10.8) and (10.10) has been explicitly verified up for the 4x4 matrix in (10.11). We believe it can be verified in general, but, as will be seen in a moment, this is not necessary.

Now comes the surprise. These $\tilde{H}$ are projectors

\[ \tilde{H}_{\mu_1\mu_2\nu_1\nu_2}^{\mu_3\nu_3} \Pi_{\lambda_1\lambda_2}^{\nu_1\nu_2} = 2\Pi^{\mu_1\mu_2}_{\lambda_1\lambda_2}, \quad \tilde{H}_{\mu_1\mu_2\mu_3}^{\nu_1\nu_2\nu_3} \Pi^{\nu_1\nu_2}_{\lambda_1\lambda_2} = 6\Pi^{\mu_1\mu_2\mu_3}_{\lambda_1\lambda_2\lambda_3}, \ldots \]

Therefore the inverse of the matrix in (10.11) does not exist. This means that we have not yet properly fixed the gauge. It is also easy to guess what the right gauge fixing is: we have to set to zero all the component field traces. More precisely

\[ b'_a = 0, \quad c'_a = 0, \quad d'_a = 0, \ldots \]

i.e. all the component fields of $h_a$ must be traceless on the $\mu$ indices. We can now return to the kinetic operator (8.3) and realize that, in this new gauge all the non-diagonal terms in the matrix $N_{\{\mu\}}^{\{\nu\}}$ are absent and only the diagonal ones survive with the replacement

\[ \Pi^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n} \rightarrow \Delta^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n} = \delta_{\mu_1}^{\nu_1} \ldots \delta_{\mu_n}^{\nu_n} + \text{perm}(\mu_1, \ldots, \mu_n) \]
The RHS has in total $n!$ different products of Kronecker delta’s. Let us call the new matrix $\tilde{N}^{(\mu)(\nu)}$. Its inverse $\tilde{M}^{(\mu)(\nu)}$ is very easy to determine

$$
\tilde{M}^{(\mu)(\nu)}(u) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{\delta^{\mu_1}_{\nu_1}}{m^2} & 0 & 0 & 0 & \cdots \\
0 & 0 & \frac{\Delta^{\mu_1\mu_2}_{\nu_1\nu_2}}{2m^2} & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{\Delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}}{3m^2} & 0 & \cdots \\
0 & 0 & 0 & 0 & \frac{\Delta^{\mu_1\mu_2\mu_3\mu_4}_{\nu_1\nu_2\nu_3\nu_4}}{4m^2} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
$$

from which one can extract the values of the coefficients $a_{l,r}$: $a_{l,l} = \frac{1}{l!}$ and $= 0$ otherwise.

The tensors $t^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}$ are given by $\Delta^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}_{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}$. Replacing these in (9.8) one obtains the formula for the 2-pt correlators of any two (traceless) components of $h^a$.

We remark that, generally, many contributions like (9.8) are UV divergent, the degree of divergence increasing with the power of $\frac{1}{m}$. However these integrals can be dealt with, they are similar to the ones explicitly calculated in [22, 23] via dimensional regularization. The main problem here is not as much the integration over the internal momentum, but the fact that if, for instance, we wish to compute the self-energy for a given spin $s$ field we have to sum infinite many contributions. This is because a given spin $s$ field is connected to infinite many other fields via the 3-vertices, see (8.21). Computing these summations is a necessary step to successfully complete the perturbative quantization outlined in the last two sections.

11. The no-go theorems

Let us pause to consider the results obtained so far for HS YM theories. The eom’s are well defined in a Minkowski background in any dimensions! They are perturbatively local, i.e. the number of terms with a fixed number of derivatives is finite. They encompass a Maxwell or YM eom, gravity, etc. They are interacting eom’s, which include up to third order (fourth order in the action) interactions (infinite order in the metric fluctuation). They are characterized by a unique coupling $g$. We have also introduced a perturbative quantization, which may be complicated to deal with but is well defined.

As anticipated in the introduction, this at first cannot but be surprising, for there exist in the literature no-go theorems forbidding massless HS particles in flat spacetime. Let us briefly review this issue, relying on the nice review [34].

The argument goes as follows (in a 4d Minkowski spacetime). Particles with spin $s \leq 2$ are known to couple minimally to gravity. Weinberg equivalence principle (based on an S-matrix argument), [24], states that all particles of whatever spin must as well couple minimally to gravity at low energy (if we want a non-vanishing emission of such particles, Weinberg’s soft emission theorem), but the Weinberg-Witten theorem, [22], and its generalizations say that HS particles cannot couple minimally to gravity. As a consequence HS particles decouple from low spin ones at low energies, which means that an action containing LS and HS particles split into two non-interacting pieces (at low energy).
These theorems on a general ground are based on the existence of the S-matrix, which requires in particular the existence of asymptotic states in the full range of energy; more specifically they are based on a lemma, which in Lagrangian language, can be formulated as follows: any local polynomial which is at least quadratic in a spin *s* massless field, non-trivial on-shell and gauge invariant, contains at least 2*s* derivatives.

The consequence of this lemma is that any perturbatively local theory with a Lorentz covariant and gauge invariant energy-momentum tensor cannot have spin higher than 2. The reason is that the energy-momentum tensor is assumed to contain two derivatives. Now since the em tensor is quadratic in an HS field *h*(*s*) (the coupling of *h*(*s*) to gravity is in accord with the scheme $g - h^{(s)} - h^{(s)}$), according to the lemma, it must contain at least 2*s* derivatives, which is impossible.

In [34], from the above no-go theorems the following conclusions are drawn for local cubic vertices in flat space including at least one massless particle:

1. the number of derivatives in any consistent local cubic vertex is at least equal to the highest spin in the vertex;
2. a local cubic vertex containing at least one massless field with spin higher than 2 contains at least three derivatives;
3. massless higher spin particle couple non-minimally to low-spin particles.

None of these is true for HS YM models. Looking at (8.21) we see that in the third line the coupling $A\chi b$ contains two derivatives (not three, like the spin of *b*). This disagrees with 1 and 2. As for 3, comparing the first and third line, one sees that the coupling $AA\chi$, which is minimal, has the same structure as the coupling $A\chi b$, where *b* has spin 3.

So the question is: where is the bug? Let us observe that there are several ways to evade the hypotheses on which no-go theorems rely:

- The em tensor is in general not gauge-invariant. But this is not a compelling counter-argument: the theorem has been extended to the case of a non-covariant em tensors [33].
- The em tensor (like in the HS YM case) may not be a polynomial but an infinite series, like in [1,24].
- The coupling to gravity via the em tensor is non-minimal, that is it contains more than two derivatives (for instance, the coupling $\chi bb$ in (A.7) has four derivatives).
- No-go theorems always understand Einstein-Hilbert gravity: the coupling to gravity is implemented by replacing simple derivatives with covariant ones. In HS YM models, for instance, this is not the case, while covariance is nevertheless implemented.
- The no-go theorems are based on the existence of asymptotic free particle state. The question is: do these states always exist? or, at least, do they exist in the full range of energy? For instance, the escape for Vasilev’s models is that they hold in AdS spaces where asymptotic states and S-matrix do not exist.
Summarizing: in the HS YM models there are at least two important differences with the hypotheses of the no-go theorems:

- the coupling of HS fields to gravity is non-minimal;
- the em tensor and the other conserved currents are non-polynomial.

But also other issues may play a role. First, the gravity formalism underlying the HS YM models is not the familiar one of EH gravity, but a different one, similar to teleparallelism, in which formalism universality is not a manifest, but an indirect feature (see paper III). Second, quantization of HS YM models conjures up a massive constant which, while HS fields remain massless, might play the same role as the cosmological constant does in the Fradkin-Vasiliev’s treatment of HS theories.

There seem to be enough reasons to explain the failure of no-go theorems for HS YM models.

12. Conclusions

In this paper we have shown that massless HS theories exist in a flat spacetime. In fact, inspired by the effective action method, by which integrating out fermion matter fields one can derive HS models, we have constructed HS YM-type theories in any dimension and CS-like theories in any odd dimension. We have defined their actions and found their equations of motion, as well as their conserved currents. These theories are perturbatively local. They are characterized by a HS gauge symmetry which includes in particular ordinary gauge transformations and diffeomorphisms. On the same footing we may also introduce HS scalar type theories. Focusing in particular on HS YM theories we have shown that, with the addition of ghosts and auxiliary fields, they can be easily BRST quantized. It is possible to reproduce the Higgs phenomenon, by which the HS potentials acquire a mass. Finally we have shown how to recover local Lorentz covariance in all of these HS models. Then we have broached the problem of perturbative quantization of HS YM-like models (with unbroken gauge symmetry) by means of Feynman diagrams. We have seen that this leads to the appearance of a mass parameter, although the HS particles remain massless. A perturbative series can indeed be defined and used for calculations (although we believe here we have only scratched the surface). Finally we have argued that these theories evade a few basic hypotheses of the no-go theorems on massless HS particles in flat background, which seem to be thus ineffective in their case.

And now a few words of caution. The fact that the no-go theorems do not apply to the theories we have introduced is not enough to guarantee that they are well defined, and even less that they have any physical relevance. To this we should add that we have introduced new techniques, such as defining a theory by means of an integral over the phase space and quantizing in the frozen momentum framework, which, needless to say, deserve further tests. In particular, the problem of the residual gauge symmetry, after gauge fixing, remains open. Nevertheless we think we have derived enough new and unexpected results to conclude that they deserve attention.
In the third paper of this series we will discuss the interpretation of these theories for what concerns in particular gravity. We have already remarked that in HS YM theories gravity does not have the traditional EH form, but is rather akin to teleparallel gravity. This is an interesting new aspect of our HS models as compared to previous attempts. It is also interesting to compare a theory like HS YM with string theory/theories. Since it is massless it makes sense to compare it with tensionless (open) string field theory \[39\], or, better, with a subsector thereof. On the other hand it is an interacting theory and it contains a mass scale, so HS YM seems rather to be related with an intermediate stage between tensionless and tensile string theory. In any case it may represent a new tool also to understand string theory.

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A. Master field expansions

In this Appendix we collect some \( u \) expansions of various master fields referred to in the text.

A.1 Curvature components

Here are the expansions referred to in section \[4.1\]:

\[ F_{ab} = \partial_a A_b - \partial_b A_a + \partial_a A_b \chi_a^\sigma - \partial_a A_b \chi_b^\sigma - \frac{1}{24}(\partial_{\sigma_1} \partial_{\sigma_2} \partial_{\sigma_3} A_b c_b^{\sigma_1 \sigma_2 \sigma_3}) \] \hspace{1cm} (A.1)

\[ X_{\mu a}^\mu = \partial_a \chi_b^\mu - \partial_b \chi_a^\mu + \partial_a A_b b_{\alpha \beta}^\mu - \partial_a A_b b_{\beta \alpha}^\mu - \chi_b^\mu \partial_a \chi_a^\mu + \chi_b^\mu \partial_a \chi_a^\mu \] \hspace{1cm} (A.2)

\[ B_{ab}^{\mu \nu} = \partial_a b_{b}^{\mu \nu} - \partial_b b_{a}^{\mu \nu} + 2 \partial_a \chi_b^{(\mu \alpha \beta)} - 2 \partial_a \chi_b^{(\mu \beta \alpha)} \] \hspace{1cm} (A.3)
$$C_{ab}^{\mu\nu\lambda} = \partial_a c_b^{\mu\nu\lambda} - \partial_b c_a^{\mu\nu\lambda} + \partial_a A_b^{\mu\nu\lambda} - \partial_b A_a^{\mu\nu\lambda} + 3\partial_\sigma \chi_a^{(\mu\nu)}c_b^{\lambda\sigma} - 3\partial_\sigma \chi_b^{(\mu\nu)}c_a^{\lambda\sigma}$$

\[+3\partial_\sigma b_a^{(\mu\nu\lambda)} - 3\partial_\sigma b_b^{(\mu\nu\lambda)} + \partial_\sigma d_b^{\mu\nu\lambda} - \partial_\sigma d_a^{\mu\nu\lambda} \quad (A.4)\]

\[-\frac{1}{24}\left( \partial_1 \partial_2 \partial_3 \partial_4 A_b^{\mu\nu\lambda} - \partial_1 \partial_2 \partial_3 b_a^{(\mu\nu\lambda)} + 3\partial_1 \partial_2 \partial_3 \chi_b^{(\mu\nu\lambda)} + 3\partial_1 \partial_2 \partial_3 \chi_a^{(\mu\nu\lambda)} \right) + \ldots \]

where the ellipses in the RHS refer to terms containing at least five derivatives.

A.2 \(\delta\Phi\)

Here we consider the transformation of the complex scalar field \(\Phi\) introduced in sec 5.4. Under (13-2), its components transform as

$$\delta \varphi_0 = i \varepsilon \varphi_0 - \frac{1}{2} \varepsilon \partial \varphi_0 + i \left[ \frac{1}{2} \mu^{\prime \nu} \partial_\mu \varepsilon + \frac{i}{8} \left( \varphi_0^\mu \partial_\mu \varepsilon + \varphi_0^\nu \partial_\nu \varepsilon - \varphi_0^\lambda \partial_\lambda \varepsilon \right) \right]$$

$$\delta \varphi_1^\lambda = i \varepsilon \varphi_1^\lambda + i \varphi_0 \varepsilon \varphi_1^\lambda - \frac{1}{2} \left( \varphi_0^\mu \partial_\mu \varphi_1^\lambda + \varphi_0^\nu \partial_\nu \varphi_1^\lambda - \varphi_0^\lambda \partial_\lambda \varphi_1^\lambda \right)$$

$$\delta \varphi_2^\nu = i \varepsilon \varphi_2^\nu + i \varphi_0 (\varepsilon \varphi_2^\nu) + i \varphi_0 \Lambda^\nu - \frac{1}{2} \left( \varphi_0^\mu \partial_\mu \varphi_2^\nu + 2\partial_\mu \varepsilon \varphi_2^\nu + \frac{1}{2} \partial_\mu \Lambda^\nu \right)$$

$$\delta \varphi_3^{\mu\nu\lambda} = i \varepsilon \varphi_3^{\mu\nu\lambda} + \frac{3}{2} i \varepsilon \varphi_3^{\mu\nu\lambda} + \frac{3}{2} i \Lambda^a \varphi_3^{\mu\nu\lambda}$$

$$\quad + \frac{1}{2} \left( \partial_\sigma \partial_\varphi_3^{\mu\nu\lambda} + 3\partial_\sigma \Lambda^\nu \varphi_3^{\mu\lambda\sigma} + 3\partial_\sigma \varphi_3^{(\mu\nu\lambda)} + \partial_\sigma \partial_\varphi_3^{(\mu\nu\lambda)} \right)$$

$$\quad + \frac{1}{2} \left( \xi \partial_\varphi_3^{\mu\nu\lambda} + 3\partial_\sigma \varphi_3^{(\mu\nu\lambda)} + 3\partial_\sigma \partial_\varphi_3^{(\mu\nu\lambda)} + \partial_\sigma \partial_\varphi_3^{(\mu\nu\lambda)} \right)$$

$$\quad + \frac{1}{2} \left( \partial_\sigma \partial_\varphi_3^{(\mu\nu\lambda)} + 3\partial_\sigma \Lambda^\nu \varphi_3^{(\mu\nu\lambda)} + 3\partial_\sigma \varphi_3^{(\mu\nu\lambda)} + \partial_\sigma \partial_\varphi_3^{(\mu\nu\lambda)} \right)$$

$$\quad + \frac{1}{2} \left( \xi \partial_\varphi_3^{(\mu\nu\lambda)} + 3\partial_\sigma \varphi_3^{(\mu\nu\lambda)} + 3\partial_\sigma \partial_\varphi_3^{(\mu\nu\lambda)} + \partial_\sigma \partial_\varphi_3^{(\mu\nu\lambda)} \right)$$

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A.3 $S_3$ and $S_4$

These are the explicit expression for the lowest order terms of $S_3$ and $S_4$:

$$S_3 = -ig \langle \partial_\mu h_b * [h^a, h^b] \rangle$$

$$= -\frac{g}{2} \langle (\partial_\mu A^b - \partial_\mu A^a) \partial_\sigma \partial_\tau \partial_\sigma_3 A^a \epsilon^{\sigma_1 \sigma_2 \sigma_3} + 3 \partial_\sigma_3 f^{\sigma_1 \sigma_2 \sigma_3} \partial_\sigma \partial_\tau \partial_\sigma_3 A^a \rangle$$

$$+ \frac{1}{2d} \langle \partial_\mu A^b \partial_\sigma b_{\mu \lambda} \chi^\lambda - \partial_\mu A^b \partial_\sigma b_{\mu \lambda} \chi^\lambda \rangle$$

$$+ 2 \partial_\mu A^b \partial_\sigma_3 c_{\mu \lambda} b_{\mu \lambda} - 2 \partial_\mu A^b \partial_\sigma_3 c_{\mu \lambda} b_{\mu \lambda} - \partial_\mu A^b \partial_\sigma_3 A^a c_{\mu \lambda} + 2 \partial_\mu A^b (\partial_\sigma A^a b_{\mu \lambda} + \partial_\sigma A^a b_{\mu \lambda})$$

$$+ \partial_\mu A^b (\partial_\sigma A^a b_{\mu \lambda} - \partial_\sigma A^a b_{\mu \lambda})$$

$$+ \langle \partial_\mu A^b \partial_\sigma_3 c_{\mu \lambda} b_{\mu \lambda} \rangle + \langle \partial_\sigma A^a b_{\mu \lambda} - \partial_\sigma A^a b_{\mu \lambda} \rangle$$

$$+ \langle \partial_\sigma A^a b_{\mu \lambda} - \partial_\sigma A^a b_{\mu \lambda} \rangle$$

where $\mathcal{O}(u^4, 6)$ means terms of order at least $u^4$ or containing at least six derivatives, and

$$S_4 = \frac{g_4}{4} \langle [[h^a, h^b], [h^a, h^b]] \rangle$$

$$= -\frac{g_4}{2} \langle (\partial_\mu A^a \chi^\lambda - \partial_\mu A^b \chi^\lambda) \partial_\tau A^a \chi^\tau \rangle$$

$$+ \frac{1}{d} \langle (\partial_\mu A^a \chi^\lambda - \partial_\mu A^b \chi^\lambda) (\partial_\tau A^a c_{\mu \lambda} \chi^\tau + 2 \partial_\tau \chi^\lambda b_{\mu \lambda} + \partial_\tau b_{\mu \lambda} \chi^\tau) \rangle$$

$$+ \langle (\partial_\mu A^b \chi^\lambda - \partial_\mu A^b \chi^\lambda + \partial_\tau \chi^\lambda \chi^\lambda - \partial_\tau \chi^\lambda \chi^\lambda) (\partial_\tau A^a b_{\mu \lambda} + \partial_\tau A^a b_{\mu \lambda}) \rangle$$

$$+ \mathcal{O}(u^4, 4)$$

B. Compatibility of LL and HS gauge transformations

In this Appendix we answer the question: after the introduction of the inertial frame $e^\mu_a$ and connection $A_{\mu a}$, does the HS gauge symmetry still hold? For instance, is (7.3) still invariant under (7.3)? Let us consider first

$$\delta_c S_2 = \delta_c \langle [\bar{\Psi} \gamma^a h_a * \Psi] \rangle = \delta_c \langle [\bar{\Psi} * \gamma^a h_a * \Psi] \rangle$$

$$= i \langle [\bar{\Psi} * \gamma^a h_a * \psi] - i \langle [\bar{\Psi} \gamma^a h_a * \psi] \rangle + \langle [\bar{\Psi} * \gamma^a (\partial_\mu \varepsilon - i [h_a, \varepsilon]) * \psi] \rangle$$

$$= \langle [\bar{\Psi} * \gamma^a \partial_\mu \varepsilon * \psi] \rangle = \langle [\bar{\Psi} * \gamma^a D_\mu \varepsilon * \psi] \rangle$$

(B.1)
where it is understood that \( \partial_a = e^\mu_a \partial_\mu = e^\mu_a D_\mu \) and in the \(*\) products the ordinary spacetime derivatives are replaced by inertial covariant ones. This is possible because, not only \([D_\mu, D_\nu] = 0\), but also \([D_a, D_b] = 0\) due of (7.12).

Next

\[
\delta_v S_1 = \delta_v \langle \Psi^\gamma a e^\mu_a \left(i \partial_\mu + \frac{1}{2} A_\mu\right) \Psi \rangle = \delta_v \langle \Psi^\gamma \gamma^a D_a \Psi \rangle - \langle \Psi^\gamma \gamma^a D_a (\varepsilon * \Psi) \rangle - \langle \Psi^\gamma \gamma^a D_a \varepsilon * \Psi \rangle
\]

(B.2)

where we have used \(D_a (\varepsilon * \Psi) = D_a \varepsilon * \Psi + \varepsilon * D_a \Psi\), which is possible because we have inserted the inertial covariant derivative in the \(*\) product, and because \([D_a, D_b] = 0\), as already pointed out.

In conclusion the HS gauge invariance of \(S(\Psi, h, A)\) still holds. Let us remark that, in order to achieve invariance, the inertial frame and connection must not transform under HS gauge transformations.

As another example let us consider the transformation of

\[
G_{ab} = D_a h_b - D_b h_a - i[h_a *, h_b]
\]

(B.3)

It follows immediately

\[
\delta G_{ab} = -i[G_{ab}, \varepsilon]
\]

provided one remarks that, once we replace ordinary spacetime derivatives with inertial covariant ones in the \(*\) product, the inertial covariant derivative commutes with the \(*\) product. For instance:

\[
D_a(h_b * \varepsilon) = D_a h_b * \varepsilon + h_b * D_a \varepsilon
\]

and \(D_a \varepsilon = \partial_a \varepsilon = e^\mu_a \partial_\mu \varepsilon\).

**C. Functional calculus**

In the functional integral manipulations of the perturbative approach the conjugate variables are

\[
h_a(x, u) = \sum_{n=0}^\infty \frac{1}{n!} h_a^{\mu_1...\mu_n}(x) u_{\mu_1} \cdots u_{\mu_n},
\]

(C.1)

and

\[
j^a(x, u) = \sum_{n=0}^\infty (-1)^n j^a_{\mu_1...\mu_n}(x) \frac{\partial^n}{\partial u_{\mu_1} \cdots \partial u_{\mu_n}} \delta(u)
\]

(C.2)

so that

\[
\langle j^a h^a \rangle = \int d^4 x j^a_{\mu_1...\mu_n}(x) h_a^{\mu_1...\mu_n}(x)
\]

(C.3)
In the presence of the factor $e^{i\langle j_a h^a \rangle}$ we can represent

$$h_a(x, u) = \frac{\delta}{\delta j^a(x, u)} \langle j_a h^a \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} u_{\mu_1} \cdots u_{\mu_n} \frac{\delta}{\delta j^a_{\mu_1 \cdots \mu_n}(x)} \langle j_a h^a \rangle$$  \hspace{1cm} (C.4)

It follows that

$$\frac{\delta}{\delta j^a(x, u)} j^b(y, v) = \sum_{n=0}^{\infty} \frac{1}{n!} u_{\mu_1} \cdots u_{\mu_n} \frac{\delta}{\delta j^a_{\mu_1 \cdots \mu_n}(x)} \frac{(-1)^l}{l!} j^b_{\nu_1 \cdots \nu_l}(y) \frac{\partial^l}{\partial v_{\nu_1} \cdots \partial v_{\nu_l}} \delta(v)$$

$$= \delta_a^b \delta(x - y) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} u_{\mu_1} \cdots u_{\mu_n} \frac{\partial^n}{\partial v_{\mu_1} \cdots \partial v_{\mu_n}} \delta(v)$$

$$= \delta_a^b \delta(x - y) \delta(u - v)$$  \hspace{1cm} (C.5)

since

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} u_{\mu_1} \cdots u_{\mu_n} \frac{\partial^n}{\partial v_{\mu_1} \cdots \partial v_{\mu_n}} \delta(v) = \delta(u - v)$$  \hspace{1cm} (C.6)

for one can show, integrating by parts, that

$$\int \frac{d^d v}{(2\pi)^d} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} u_{\mu_1} \cdots u_{\mu_n} \frac{\partial^n}{\partial v_{\mu_1} \cdots \partial v_{\mu_n}} \delta(v) f(v) = f(u)$$  \hspace{1cm} (C.7)

Similarly one can prove

$$\frac{\delta}{\delta h_a(x, u)} h_b(y, v) = \delta_a^b \delta(x - y) \delta(u - v)$$  \hspace{1cm} (C.8)

In momentum representation

$$h_a(x, u) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \tilde{h}_a(k, u), \quad j^a(x, u) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \tilde{j}^a(k, u)$$

$$h_{a_{\mu_1 \cdots \mu_n}}(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \tilde{h}_{a_{\mu_1 \cdots \mu_n}}(k), \quad j_{a_{\mu_1 \cdots \mu_n}}(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \tilde{j}_{a_{\mu_1 \cdots \mu_n}}(k)$$

and

$$\langle j_a(x, u) h^a(x, u) \rangle = \int d^d x j_{\mu_1 \cdots \mu_n}^a(x) h^{\mu_1 \cdots \mu_n}_a(x) = \int \frac{d^d k}{(2\pi)^d} \tilde{z}_{a_{\mu_1 \cdots \mu_n}}(k) \tilde{h}_{a_{\mu_1 \cdots \mu_n}}(-k)$$

C.1 Perturbative series for master fields

Let us try to develop a perturbative treatment for master fields (instead of simple space-time fields). First let us introduce external currents $j_a(x, u)$ and define the coupling

$$\langle h^a(x, u) \rangle = \int d^d x \int \frac{d^d u}{(2\pi)^d} h^a(x, u) \ast j_a(x, u).$$  \hspace{1cm} (C.9)
Then we define the generating functional
\[ Z[h_a; j_a] = \int \mathcal{D}h_a e^{i \left( S_0 + \langle h^a(x,u) * j_a(x,u) \rangle \right)} e^{i S_{int}} \] (C.10)
where
\[ S_0 = -\frac{1}{2} \langle \langle h^a(x,u) * K_{ab}(x) h^b(x,u) \rangle \rangle \] (C.11)
and
\[ S_{int} = S_3 + S_4 \]
see Appendix \[ \text{[8.10]} \].

The (unnormalized) n-point function of the master field \( h_a \) is defined by
\[ \langle h_{a_1}(x_1, u_1) \ldots h_{a_n}(x_n, u_n) \rangle = \int \mathcal{D}h_a h_{a_1}(x_1, u_1) \ldots h_{a_n}(x_n, u_n) e^{i \left( S_0 + \langle h^a(x,u) * j_a(x,u) \rangle \right)} e^{i S_{int}} \] (C.12)

Going through the usual process (completing the square and integrating over \( h_a \)) one gets
\[ \langle h_{a_1}(q_1, u_1) \ldots h_{a_n}(x_n, u_n) \rangle = \frac{\delta}{\delta j_{a_1}(x_1, u_1)} \ldots \frac{\delta}{\delta j_{a_n}(x_n, u_n)} e^{i S_{int}} \left( \frac{\delta}{\delta j_a(x,u)} e^{-i \langle j_a P_{ab} j^b \rangle} \right) \] (C.13)
where \( P_{ab} \) is the propagator \([8.4]\). It is convenient to express everything in terms of Fourier transforms:
\[ \tilde{h}_a(k, u) = \int d^d x e^{i k \cdot x} h_a(x,u), \quad \tilde{j}_a(k, u) = \int d^d x e^{i k \cdot x} j_a(x,u) \]

For instance \( S_3 \) can be rewritten
\[ S_3 = -g \int \frac{d^d k_1}{(2 \pi)^d} \frac{d^d k_2}{(2 \pi)^d} \frac{d^d k_3}{(2 \pi)^d} \delta(k_1 + k_2 + k_3) \int \frac{d^d u}{(2 \pi)^d} \]
\[ \times k_{1a} \tilde{h}_b(k_1, u) \left[ \tilde{h}_a \left( k_2, u - \frac{k_3}{2} \right) \tilde{h}_b \left( k_3, u + \frac{k_2}{2} \right) - \tilde{h}_a \left( k_2, u + \frac{k_3}{2} \right) \tilde{h}_b \left( k_3, u - \frac{k_2}{2} \right) \right] \] (C.14)

Next
\[ \langle h_{a_1}(x_1, u_1) \ldots h_{a_n}(x_n, u_n) \rangle = \int \frac{d^d q_1}{(2 \pi)^d} \ldots \frac{d^d q_n}{(2 \pi)^d} e^{i (q_1 \cdot x_1 + \ldots + q_n \cdot x_n)} \langle \tilde{h}_{a_1}(q_1, u_1) \ldots \tilde{h}_{a_n}(q_n, u_n) \rangle \]

Since
\[ \tilde{h}_a(k, u) = \frac{\delta}{\delta j^a(-k, u)} = \int d^d x e^{-i k \cdot x} \frac{\delta}{\delta j^a(x,u)} \]
we can write
\[ \langle \tilde{h}_{a_1}(q_1, u_1) \ldots \tilde{h}_{a_n}(q_n, u_n) \rangle = \frac{\delta}{\delta j_{a_1}(-q_1, u_1)} \ldots \frac{\delta}{\delta j_{a_n}(-q_n, u_n)} e^{i S_{int}} \left( \frac{\delta}{\delta j^a(x,u)} e^{-i \langle j_a P_{ab} j^b \rangle} \right) \] (C.15)
Now one could apply the machinery of Feynman diagrams. But there are some unanswered questions. First, we do not have an explicit expression for the propagator in $\langle \tilde{\gamma}^a P_{ab} \tilde{\gamma}^b \rangle$, see (8.25). The second problem is that so far we have formulated everything in terms of master fields, but, unfortunately, the just mentioned term (8.25) is not expressed in terms of master fields but only in terms of component fields. For the time being, at least, as far as the perturbative approach is concerned, we can rely only in a component field formulation.

D. A dimensional check in the $A - \chi$ model

In this Appendix we make the dimensional check anticipated in subsection 8.3. Since the different loop contributions have different powers of $u$ it is important to verify that their dimensions are correct. Let us recall that the self-energy of the particle $A$ in momentum space is given by the series

$$P_{AA} + P_{AA} \Sigma P_{AA} + P_{AA} \Sigma P_{AA} \Sigma P_{AA} + \ldots$$  \hspace{1cm} (D.1)

where $P_{AA}$ is the $AA$ propagator and $\Sigma$ is the perturbative one-particle irreducible contribution to the two-point function. Therefore $\Sigma$ has dimension opposite to $P_{AA}$. Let us check that this is true. Let us start with $d = 4$, in which case $g$ is dimensionless ($A$ and $u$ have dimension 1). The 2-pt function

$$\langle \tilde{A}(k)\tilde{A}(-k) \rangle = \int d^d x \ e^{ik \cdot x} \langle A(x)A(0) \rangle$$  \hspace{1cm} (D.2)

has dimension $-2$. Let us consider the first contribution to $\Sigma$, which comes from the just mentioned one-loop diagram

$$\frac{P_{\chi\chi}}{P_{AA}}$$  \hspace{1cm} (D.3)

Looking at (8.11) and (8.9), in momentum space the vertex $AA\chi$ has dimension 2, $P_{AA}$ has dimension -2 and $P_{\chi\chi}$ has dimension -4. Taking into account the integration over the internal momentum $p$, the overall dimension is 2, which is the right dimension for $\Sigma$.

If $d \neq 4$ to start with $g$ has dimension $2 - \frac{d}{2}$. In order for the kinetic term of $A$ to have the canonical form we must redefine $A \rightarrow A' = A/g$, so $\text{dim}(A') = \frac{d}{2} - 1$. Using (D.2) with $A$ replaced by $A'$ and $d^d x$ by $d^d x$ one finds that

$$\text{dim} \left( \langle \tilde{A}'(k)\tilde{A}'(-k) \rangle \right) = -d + 2 \left( \frac{d}{2} - 1 \right) = -2$$  \hspace{1cm} (D.4)

As for $\chi\chi$, let us also redefine $\chi' = \chi/g$. Using the formula (D.2) adapted to $\chi$ we find

$$\text{dim} \left( \langle \tilde{\chi}'(k)\tilde{\chi}'(-k) \rangle \right) = -d + 2 \left( \frac{d}{2} - 2 \right) = -4$$  \hspace{1cm} (D.5)

which corresponds to what one reads off (8.9). Therefore concerning the diagram (D.3), taking into account these results (which are the same as in $d = 4$), the integration $\int d^d p$
over the internal momentum and the factor $g^2$ in front of the diagram, one gets $d + 4 - d + 2 + 2 - 4 - 2 = 2$, i.e. the same result as in $d = 4$.

As for the seagull diagram, one has to take into account that the vertex $AA\chi\chi$ has dimension 2 and is proportional to $g^2$. Using the previous results one gets that also the seagull diagram has dimension 2, as expected. As for the two-loop diagram which is formed with three $AA\chi$ and one $\chi\chi\chi$ vertex, three internal $\chi\chi$ plus two $AA$ propagators, one has to take into account the double internal momentum integration and the fact that $\chi\chi\chi$ vertex has dimension 4: the result is again of dimension 2 as it should.

Proceeding in the same way as with the fields $A$ and $\chi$, with the spin 3 field $b$ one finds
\[
\dim\left(\langle \tilde{b}'(k)\tilde{b}'(-k)\rangle\right) = -6 \quad (D.6)
\]
and in general for a field $h^\mu_\alpha^{\mu_1...\mu_{s-1}}$ one finds
\[
\dim\left(\langle \tilde{h}'(k)\tilde{h}'(-k)\rangle\right) = -2s \quad (D.7)
\]

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