A NOTE ON GEOMETRY OF $\kappa$-MINKOWSKI SPACE

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Abstract. The infinitesimal action of $\kappa$-Poincaré group on $\kappa$-Minkowski space is computed both for generators of $\kappa$-Poincaré algebra and those of Woronowicz generalized Lie algebra. The notion of invariant operators is introduced and generalized Klein-Gordon equation is written out.

I. Introduction

In this short note we consider some simple properties of differential operators on $\kappa$-Minkowski space $\mathcal{M}_\kappa$ — a noncommutative deformation of Minkowski space-time which depends on dimensionful parameter $\kappa$ ([1]). We calculate the infinitesimal action of $\kappa$-Poincaré group $\mathcal{P}_\kappa$ ([1]) on $\mathcal{M}_\kappa$ both for the generators of $\kappa$-Poincaré algebra $\tilde{\mathcal{P}}_\kappa$ ([2]) (this is done using the duality $\tilde{\mathcal{P}}_\kappa \leftrightarrow \mathcal{P}_\kappa$ described in [3]) and for the elements of Woronowicz generalized Lie algebra ([4]) of $\kappa$-Poincaré group ([5]). The result supports the relation between both algebras found in [5]. We introduce also the notion of invariant differential operators on $\mathcal{M}_\kappa$ and write out the generalized Klein-Gordon equation.

Let us conclude this section by introducing the notions of $\kappa$-Poincaré group $\mathcal{P}_\kappa$ and algebra $\tilde{\mathcal{P}}_\kappa$. $\mathcal{P}_\kappa$ is defined by the following relations ([1])

\begin{align}
[x^\mu, x^\nu] &= \frac{i}{\kappa}(\delta^\mu_0 x^\nu - \delta^\nu_0 x^\mu), \\
[A^\mu, A^\alpha_\beta] &= 0, \\
[A^\mu, x^\nu] &= -\frac{i}{\kappa}((A^\mu_0 - \delta^\mu_0)A^\nu + (A^0_\nu - \delta^0_\nu)g^{\mu\rho}), \\
\Delta(A^\mu_\nu) &= A^\mu_\alpha \otimes A^\alpha_\nu, \\
\Delta(x^\mu) &= A^\mu_\alpha \otimes x^\alpha + x^\mu \otimes I, \\
S(A^\mu_\nu) &= A^\mu_\alpha, \\
S(x^\mu) &= -A^\mu_\nu x^\nu, \\
\varepsilon(A^\mu_\nu) &= \delta^\mu_\nu, \\
\varepsilon(x^\mu) &= 0.
\end{align}

(1)

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The dual structure, the $\kappa$-Poincaré algebra $\tilde{\mathcal{P}}_\kappa$, is, in turn, defined as follows ([6])

\[ [P_\mu, P_\nu] = 0, \]
\[ [M_i, M_j] = i\varepsilon_{ijk} M_k, \]
\[ [M_i, N_j] = i\varepsilon_{ijk} N_k, \]
\[ [N_i, N_j] = -i\varepsilon_{ijk} M_k, \]
\[ [M_i, P_0] = 0, \]
\[ [M_i, P_j] = i\varepsilon_{ijk} P_k, \]
\[ [N_i, P_0] = i P_i, \]
\[ [N_i, P_j] = i\delta_{ij} \left( \kappa^2 \left( 1 - e^{-2P_0/\kappa} \right) + \frac{1}{2\kappa} \vec{P}^2 \right) - \frac{i}{\kappa} P_i P_j, \]
\[ \Delta(M_i) = M_i \otimes I + I \otimes M_i, \]
\[ \Delta(N_i) = N_i \otimes e^{-P_0/\kappa} + I \otimes N_i - \frac{1}{\kappa} \varepsilon_{ijk} M_j \otimes P_k, \]
\[ \Delta(P_0) = P_0 \otimes I + I \otimes P_0, \]
\[ \Delta(P_i) = P_i \otimes e^{-P_0/\kappa} + I \otimes P_i, \]
\[ S(M_i) = -M_i, \]
\[ S(N_i) = -N_i + \frac{3i}{2\kappa} P_i, \]
\[ S(P_\mu) = -P_\mu, \]
\[ \varepsilon(P_\mu, M_i, N_i) = 0. \]

Structures (1), (2) are dual to each other, the duality being fully described in [3].

The analysis given below was suggested to two of the authors (P. Kosiński and P. Maślanka) by J. Lukierski.

II. $\kappa$-MINKOWSKI SPACE

The $\kappa$-Minkowski space $\mathcal{M}_\kappa$ ([1]) is a universal $*$-algebra with unity generated by four selfadjoint elements $x^\mu$ subject to the following conditions

\[ [x^\mu, x^\nu] = \frac{i}{\kappa} \left( \delta_\mu^\nu x^\sigma - \delta_\nu^\mu x^\sigma \right). \] (3a)

Equipped with the standard coproduct

\[ \Delta x^\mu = x^\mu \otimes I + I \otimes x^\mu, \] (3b)

antipode $S(x^\mu) = -x^\mu$ and counit $\varepsilon(x^\mu) = 0$ it becomes a quantum group.

On $\mathcal{M}_\kappa$ one can construct a bicovariant five-dimensional calculus which is defined
by the following relations ([5])

\[ \tau^\mu \equiv dx^\mu, \quad \tau \equiv d \left( x^2 + \frac{3i}{\kappa}x^0 \right) - 2x_\mu dx^\mu, \]

\[ [\tau^\mu, x^\nu] = \frac{i}{\kappa} g^{0\mu} \tau^\nu - \frac{i}{\kappa} g^{\mu\nu} x^0 + \frac{1}{4} g^{\mu\nu} \tau, \]

\[ [\tau, x^\mu] = -\frac{4}{\kappa^2} \tau^\mu, \quad \tau^\mu \wedge \tau^\nu = -\tau^\nu \wedge \tau^\mu, \quad \tau \wedge \tau^\mu = -\tau^\mu \wedge \tau, \]

\[ (\tau^\mu)^* = \tau^\mu, \quad \tau^* = -\tau, \]

\[ d\tau^\mu = 0, \quad d\tau = -2\tau_\mu \wedge \tau^\mu. \]  

The \( \kappa \)-Minkowski space carries a left-covariant action of \( \kappa \)-Poincaré group \( P_\kappa ([1]) \), \( \rho_L : M_\kappa \rightarrow P_\kappa \otimes M_\kappa \), given by

\[ \rho_L(x^\mu) = \Lambda^\mu_\nu \otimes x^\nu + a^\mu \otimes I. \]  

The calculus defined by (4) is covariant under the action of \( P_\kappa \) which reads

\[ \tilde{\rho}_L(\tau^\mu) = \Lambda^\mu_\nu \otimes \tau^\nu, \quad \tilde{\rho}_L(\tau) = I \otimes \tau. \]  

III. Derivatives, infinitesimal actions and invariant operators

The product of generators \( x^\mu \) will be called normally ordered if all \( x^0 \) factors stand leftmost. This definition can be used to ascribe a unique element : \( f(x) : \) of \( M_\kappa \) to any polynomial function of four variables \( f \). Formally, it can be extended to any analytic function \( f \).

Let us now one define the (left) partial derivatives: for any \( f \in M_\kappa \) we write

\[ df = \partial_\mu f \tau^\mu + \partial f \tau. \]  

It is a matter of some boring calculations (using the commutation rules (3a)) to find the following formula

\[ d : f : =: \left( \kappa \sin \left( \frac{\partial_0}{\kappa} \right) + \frac{i}{2\kappa} e^{i \frac{\partial_0}{\kappa} \Delta} \right) f : \tau^0 + e^{i \frac{\partial_0}{\kappa} \Delta} \frac{\partial f}{\partial x^i} : \tau^i \]

\[ + : \left( \frac{\kappa^2}{4} \left( 1 - \cos \left( \frac{\partial_0}{\kappa} \right) \right) - \frac{1}{8} e^{i \frac{\partial_0}{\kappa} \Delta} \right) f : \tau \]  

or

\[ \partial_0 : f : =: \left( \kappa \sin \left( \frac{\partial_0}{\kappa} \right) + \frac{i}{2\kappa} e^{i \frac{\partial_0}{\kappa} \Delta} \right) f : \]

\[ \partial_i : f : =: e^{i \frac{\partial_0}{\kappa} \Delta} \frac{\partial f}{\partial x^i} : \]

\[ \partial : f : =: \left( \frac{\kappa^2}{4} \left( 1 - \cos \left( \frac{\partial_0}{\kappa} \right) \right) - \frac{1}{8} e^{i \frac{\partial_0}{\kappa} \Delta} \right) f : \]
Let us now define the infinitesimal action of $P_{\kappa}$ on $M_{\kappa}$. Let $X$ be any element of the Hopf algebra dual to $P_{\kappa}$ — the $\kappa$-Poincaré algebra $\tilde{P}_{\kappa}$ (cf. [3] for the proof of duality). The corresponding infinitesimal action

$$\hat{X} : M_{\kappa} \to M_{\kappa}$$

is defined as follows: for any $f \in M_{\kappa}$,

$$\hat{X}f = (X \otimes \text{id}) \circ \rho_{L}(f).$$

(10)

Using the standard duality rules ([3]), we conclude that

$$\hat{P}_\mu : x^\alpha = i\delta^\alpha_\mu;$$

$$\hat{P}_\mu : x^\alpha x^\beta = i\delta^\beta_\mu x^\alpha + i\delta^\alpha_\mu x^\beta$$

(11)

etc. One can show that, in general, 

$$\hat{P}_\mu : f := i \frac{\partial f}{\partial x^\mu};$$

(12)

Also, using the fact that $\tilde{\rho}_{L}$ is a left action of $P_{\kappa}$ on $M_{\kappa}$ together with the duality $P_{\kappa} \to \tilde{P}_{\kappa}$, we conclude that

$$F(\hat{P}_\mu) : f := F \left( i \frac{\partial f}{\partial x^\mu} \right) f :$$

(13)

Formulae (11)–(13) have the following interpretation. In [5] the fifteen-dimensional bicovariant calculus on $P_{\kappa}$ has been constructed using the methods developed by Woronowicz ([4]). The resulting generalized Lie algebra is also fifteen-dimensional, the additional generators being the generalized mass square operator and the components of generalized Pauli-Lubanski fourvector. All generators of this Lie algebra can be expresses in terms of the generators $P_\mu, M_{\alpha\beta}$ of $\tilde{P}_{\kappa}$ ([5]). In particular, the translation generators $\chi_\mu$ as well as the mass squared operator $\chi$ are expressible in terms of $P_\mu$ only. The relevant expressions are given by formulae (20) of [5]. Comparing them with (9), (13) above, we conclude that

$$\hat{\chi}_\mu = \partial_\mu;$$

$$\hat{\chi} = \partial.$$

(14)

These relations, obtained here by explicit computations, follow also from (7) if one takes into account that $M_{\kappa}$ is a quantum subgroup of $P_{\kappa}$.

It is also not difficult to obtain the action of Lorentz generators. Combining (1) and (3a) with the duality $P_{\kappa} \to \tilde{P}_{\kappa}$ described in detail in [5], we conclude first that the action of $M_i$ and $N_i$ coincides with the proposal of Majid and Ruegg ([6]); the actual computation is then easy and gives

$$\hat{M}_i : f(x^\mu) := -i\varepsilon_{ijl}x^j \frac{\partial f(x^\mu)}{\partial x^l};$$

$$\hat{N}_i : f(x^\mu) := \left( i x^0 \frac{\partial}{\partial x^i} + x^i \frac{\kappa}{2} \left( 1 - e^{-\frac{\kappa}{2}} \frac{\partial}{\partial x^0} \right) - \frac{1}{2\kappa} \Delta \right)$$

$$+ \frac{1}{\kappa} x^k \frac{\partial^2}{\partial x^k \partial x^i} \right) f(x^\mu);$$

(15)
Let us now pass to the notion of invariant operator; \( \hat{C} \) is an invariant operator on \( \mathcal{M}_\kappa \) if
\[
\rho_L \circ \hat{C} = (\text{id} \otimes \hat{C}) \circ \rho_L. \tag{16}
\]

We shall show that if \( C \) is a central element of \( \tilde{\mathcal{P}}_\kappa \), then
\[
\hat{C}f = (C \otimes \text{id}) \circ \rho_L(f) \tag{17}
\]
is an invariant operator. To prove this let us take any \( Y \in \tilde{\mathcal{P}}_\kappa \), then
\[
YC = CY \tag{18}
\]
or, in other words, for any \( a \in \mathcal{P}_\kappa \),
\[
Y(a(1))C(a(2)) = C(a(1))Y(a(2)) \tag{19}
\]
where \( \Delta a = a(1) \otimes a(2) \). Let us fix \( a \) and write (19) as
\[
Y(a(1))C(a(2)) = C(a(1))Y(a(2)) \tag{20}
\]
As (20) holds for any \( Y \in \tilde{\mathcal{P}}_\kappa \) we conclude that for any \( a \in \mathcal{P}_\kappa \)
\[
C(a(1))a(1)(2) = C(a(2))a(1) \tag{21}
\]
Now let
\[
\rho_L(x) = a(1) \otimes x(1), \quad \rho_L(x(1)) = a(1) \otimes x(2), \quad \Delta a(1) = a(1) \otimes a(1). \tag{22}
\]
The identity
\[
(id \otimes \rho_L) \circ \rho_L = (\Delta \otimes \text{id}) \circ \rho_L \tag{23}
\]
implies
\[
a(1) \otimes a(1)(2) \otimes x(2) = a(1) \otimes a(1) \otimes x(1). \tag{24}
\]
Applying to both sides \( id \otimes C \otimes \text{id} \) and \( C \otimes \text{id} \otimes \text{id} \), we get
\[
C(a(1))a(1) \otimes x(2) = C(a(1))a(1) \otimes x(1), \tag{25}
\]
\[
C(a(1))a(1) \otimes x(2) = C(a(1))a(1) \otimes x(1).
\]
It follows from (21) applied to \( a(1) \) that the right-hand sides of (25) are equal. So,
\[
C(a(1))a(1) \otimes x(2) = C(a(1))a(1)(2) \otimes x(2) \tag{26}
\]
i.e.
\[
(id \otimes \hat{C}) \circ \rho_L(x) = \rho_L \circ \hat{C}(x). \tag{27}
\]
Using the above result we can easily construct the deformed Klein-Gordon equation. Namely, we take as a central element the counterpart of mass squared Casimir operator \( \chi \) ([5]). Due to (14) the generalized Klein-Gordon equation reads
\[
\left( \theta + \frac{m^2}{8} \right)f = 0; \tag{28}
\]
the coefficient \( \frac{1}{8} \) is dictated by the correspondence with standard Klein-Gordon equation in the limit \( \kappa \to \infty \). Let us note that (28) can be written, due to (9), in the form
\[
\left[ \partial_0^2 - \partial_i^2 + m^2 \left( 1 + \frac{m^2}{4 \kappa^2} \right) \right]f = 0; \tag{29}
\]
here \( \partial_0, \partial_i \) are the operators given by (9). It seems therefore that the Woronowicz operators \( \chi_\mu \) are better candidates for translation generators than \( P_\mu \)'s. Note that the operators \( \chi_\mu \) already appeared in [7], [8].
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