Abstract

The three dimensional abelian fermionic determinant of a two component massive spinor in flat euclidean space-time is resetted to a pure Chern-Simons action through a nonlinear redefinition of the gauge field.
1 Introduction

Recently, it has been shown \cite{1,2} that any local Yang-Mills type action in the presence of the topological three dimensional Chern-Simons term can be reabsorbed into the pure Chern-Simons through a local covariant nonlinear gauge field redefinition. Choosing in fact as the Yang-Mills action the standard $\int F F$ term, we have \cite{1,2}

$$S_{CS}(A) + \frac{1}{4m} tr \int d^3 x F_{\mu \nu} F^{\mu \nu} = S_{CS}(\hat{A}) , \quad (1.1)$$

with

$$\hat{A}_\mu = A_\mu + \sum_{k=1}^\infty \frac{1}{m_k} \vartheta_k^\mu , \quad (1.2)$$

and

$$S_{CS}(A) = \frac{1}{2} tr \int d^3 x \varepsilon^{\mu \nu \rho} \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} g A_\mu A_\nu A_\rho \right) . \quad (1.3)$$

The two parameters $g, m$ in the expressions (1.1) – (1.3) identify respectively the gauge coupling and the so called topological mass \cite{6,7}. The coefficients $\vartheta_k^\mu$ in the eq.(1.2) turn out to be local and covariant, meaning that they are built only with the field strength $F_{\mu \nu}$

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g [A_\mu, A_\nu] , \quad (1.4)$$

and its covariant derivative $D_\mu$

$$D_\mu = \partial_\mu + g [A_\mu, ] . \quad (1.5)$$

For instance, the first four terms of the expansion (1.2) have been found to be \cite{1}.

\footnote{According to the BRST analysis of gauge theories \cite{3,4,5,6}, the name Yang-Mills type action is employed here to denote a generic integrated local invariant polynomial built only with the field strength $F$ and its covariant derivatives.}
The formulas (1.1), (1.2) can be generalized to any higher dimensional Yang-Mills term built with the field strength $F$ and its covariant derivatives, expressing therefore the classical equivalence, up to nonlinear field redefinitions, among the Yang-Mills actions in the presence of the Chern-Simons and the pure Chern-Simons term.

Although this equivalence has been rigorously proven \cite{1,2} only for the class of the local Yang-Mills type terms, it has been suggested that it could persist at the level of the 1PI effective quantum action. More precisely, it has been argued \cite{2} that the complete 1PI effective action obtained upon quantization in the Landau gauge of the massive Yang-Mills action (1.1) could be cast in the form of a pure Chern-Simons through a nonlinear and nonlocal gauge field redefinition. This hypothesis has been tested on a class of nonlocal gauge invariant terms expected to contribute to the 1PI effective action of topological massive Yang-Mills \cite{2}. As one can easily understand, the nonlocality of the field redefinition in the quantum case stems from the fact that the loop corrections to the effective 1PI action give rise to both local and nonlocal gauge invariant terms.

The aim of this work is to provide a further evidence in favour of this hypothesis by means of a direct example of a three dimensional system whose corresponding quantum effective action can be fully resetted to pure Chern-Simons, up to a nonlocal nonlinear redefinition of the gauge connection. The model we will refer to is the abelian fermionic determinant of a massive two component spinor interacting with an external gauge field in flat euclidean space-time \cite{6,8,9,10}. In particular, we shall be able to prove that the infinite number of one loop diagrams representing the perturbative expansion of the fermionic determinant can be reabsorbed into pure Chern-Simons, up to
field redefinitions. Remarkably in half and in spite of its nonlocal character, the redefined field turns out to transform still as a connection. This property will be of great relevance in order to give a geometrical interpretation of the final result.

It is worth recalling here that the fermionic determinant plays a rather important role in different areas of theoretical physics, going from pure solid state applications [11] to the three dimensional bosonization [12, 13, 14, 15, 16, 17, 18, 19].

The present work is organized as follows. In Sect.2 we present the strategy which will be adopted in order to reset the fermionic determinant to pure Chern-Simons. In Sect.3 we establish a very useful cohomological recursive formula. Sect.4 is devoted to the analysis of the fermionic determinant. Sect.5 deals with the generalization to a family of determinants with nonminimal gauge interaction. Finally, in Sect.6 we collect the concluding remarks and we outline possible further applications.

2 The strategy

The effective action \( \Gamma(A) \) generated by a two component massive spinor interacting with an abelian external gauge field can be written, via functional integral, as

\[
e^{\Gamma(A)} = \int D\psi \overline{D\psi} e^{\int d^3x \overline{\psi} (i\gamma^\mu \partial_\mu + \gamma^\mu A_\mu - m) \psi},
\]

\[
\Gamma(A) = \log \det (i\gamma^\mu \partial_\mu + \gamma^\mu A_\mu - m).
\]

Being the gauge field \( A_\mu \) an external field, the perturbative expansion of \( \Gamma(A) \) consists of an infinite series of one loop diagrams. The strategy which shall be adopted in order to reset the expression (2.7) to the Chern-Simons action relies on the analysis of the exact expression available for the two point function [1, 8, 9, 10, 20, 21, 22] and on two observations concerning the structure of a generic \( n \)-point, \( n \geq 2 \), contribution to the effective action \( \Gamma(A) \).

As it is well known, the one loop two point function (\( i.e. \) the spinor vacuum polarization) has been computed exactly. Although a detailed discussion will be given in Sect.4, it is worth emphasizing here that the complete contribution of the vacuum polarization contains several terms, among which the abelian Chern-Simons action.
\[ S_{CS}^{ab}(A) = \frac{1}{2} \int d^3x \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho . \] \quad (2.8)

Of course, the presence of this term will be crucial for the purposes of the present work.

Concerning now the aforementioned observations, the first one aims at showing that any action of the type

\[ S_\Omega(A) = \frac{1}{\xi} \int d^3x_1 \ldots d^3x_n F_{\mu_1\nu_1}(x_1) \ldots F_{\mu_n\nu_n}(x_n) \Omega^{\mu_1\nu_1 \ldots \mu_n\nu_n}(x_1, \ldots, x_n) , \] \quad (2.9)

with \( n \geq 2 \) and \( \Omega^{\mu_1\nu_1 \ldots \mu_n\nu_n}(x_1, \ldots, x_n) \) being a generic \( \text{(even nonlocal)} \) space-time dependent kernel\(^2\), can be reabsorbed into the pure Chern-Simons action \( (2.8) \) through a nonlinear redefinition, namely

\[ S_{CS}^{ab}(A) + S_\Omega(A) = S_{CS}(\hat{A}) , \] \quad (2.10)

with

\[ \hat{A}_\mu = A_\mu + \sum_{k=1}^{\infty} \frac{1}{\xi^k} \partial^k_\mu . \] \quad (2.11)

The parameter \( \xi \) in eq.(2.9) is an arbitrary coefficient. As it will be discussed in the next Section, this step will be handled by means of a useful recursive cohomological formula. In addition, it will be checked that the redefined field \( \hat{A}_\mu \) turns out to transform as a connection.

The second observation concerns the higher order terms of the effective action \( \Gamma(A) \). We shall be able to prove that the contributions with \( n \geq 2 \) external gauge fields can be cast in the form of the eq.(2.9), due to the absence of anomalies in three dimensions. This result combined with the knowledge of the spinor vacuum polarization will enable us to reset the fermionic determinant to pure Chern-Simons.

\(^2\text{Although the Lorentz structure of the kernel } \Omega^{\mu_1\nu_1 \ldots \mu_n\nu_n} \text{ can be specified by means of suitable combinations of the flat euclidean metric and of the } \varepsilon_{\mu\nu\rho} \text{ tensor, we leave it unspecified for the sake of generality.} \)
3 A recursive formula

The task of this section is to establish a simple recursive formula accounting for the eqs. (2.10), (2.11). For a better understanding of the mechanism which allows us to reabsorb the term $S_\Omega(A)$ into the pure Chern-Simons, let us first work out some of the coefficients $\vartheta^k_\mu$ of the expansion (2.11). Their computation is rather straightforward. After inserting the eq. (2.11) in the eq. (2.10) and identifying the terms with the same power in the inverse of the parameter $\xi$, for the coefficients $\vartheta^1_\mu, \vartheta^2_\mu, \vartheta^3_\mu, \vartheta^4_\mu$ we obtain

\[ \vartheta^1_\mu(x) = \epsilon_{\mu\nu\rho}\Xi^{\nu\rho}(x), \quad (3.12) \]

\[ \vartheta^2_\mu(x) = -\epsilon_{\mu\nu\rho} \int d^3y F_{3n}^{\sigma\tau\nu\rho}(y, x) \epsilon_{\sigma\tau\alpha} \partial_\beta^y \Xi^{\alpha\beta}(y) \]

\[ F_{3n}^{\sigma\tau\nu\rho} = \int \left( \prod_{j=3}^n d^3x_j F_{\mu_j\nu_j}(x_j) \right) \Omega^{\sigma\tau\nu_3\nu_3...\nu_n\nu_n}(y, x, x_3, .., x_n), \quad (3.13) \]

\[ \frac{1}{2} \vartheta^3_\mu(x) = \epsilon_{\mu\nu\rho} \int d^3yd^3z F_{4n}^{\sigma\tau\lambda\nu\rho}(y, z, x) \epsilon_{\sigma\tau\alpha} \partial_\beta^y \Xi^{\alpha\beta}(y) \epsilon_{\lambda\delta\omega} \partial_\tau^z \Xi^{\omega\tau}(z) \]

\[ F_{4n}^{\sigma\tau\lambda\nu\rho} = \int \left( \prod_{j=4}^n d^3x_j F_{\mu_j\nu_j}(x_j) \right) \Omega^{\sigma\tau\lambda\rho\mu_4\nu_4...\nu_n\nu_n}(y, z, x_4, .., x_n), \quad (3.14) \]

\[ -\frac{1}{4} \vartheta^4_\mu(x) = \epsilon_{\mu\nu\rho} \int d^3td^3yd^3z F_{5n}^{\sigma\tau\lambda\alpha\beta\nu\rho}(t, y, z) \epsilon_{\sigma\tau\gamma} \partial_\delta^z \Xi^{\gamma\delta}(z) \]

\[ -\frac{1}{8} \epsilon_{\mu\nu\rho} \int d^3yd^3z F_{4n}^{\alpha\beta\chi\nu\rho}(z) \epsilon_{\alpha\beta\gamma} \partial_\tau^y \Xi^{\gamma\tau}(y) \epsilon_{\lambda\delta\omega \tau\xi}(z), \]

\[ F_{5n}^{\sigma\tau\lambda\alpha\beta\nu\rho} = \int \left( \prod_{j=5}^n d^3x_j F_{\mu_j\nu_j}(x_j) \right) \Omega^{\sigma\tau\lambda\alpha\beta\rho\mu_5...\mu_n\nu_n}(t, y, z, x_5, .., x_n), \quad (3.15) \]

with
\[ \Xi^{\nu \rho}(x) = \int d^3x_2 \ldots d^3x_n F_{\mu_2 \nu_2}(x_2) \ldots F_{\mu_n \nu_n}(x_n) \Omega^{\nu \rho \mu_2 \nu_2 \ldots \mu_n \nu_n}(x, x_2, \ldots, x_n) . \] (3.16)

Observe that, as already mentioned, the expressions in eqs. (3.12)-(3.15) are gauge invariant. As a consequence, the redefined field \( \hat{A}_\mu \) behaves as a connection under gauge transformations

\[ \delta A_\mu(x) = -\partial_\mu \alpha(x) , \quad \delta \hat{A}_\mu(x) = -\partial_\mu \alpha(x) . \] (3.17)

It follows then that the resulting Chern-Simons action \( S_{CS}^{ab}(\hat{A}) \) is gauge invariant as well, \( i.e. \)

\[ \delta S_{CS}^{ab}(\hat{A}) = 0 . \] (3.18)

Although the higher order coefficients of the expansion (2.11) can be easily obtained, let us present here a cohomological recursive argument for the equation (2.10). To this purpose, we introduce a ghost field \( c \) and a set of antifields \( A^*, c^* \) in order to implement in cohomology the equations of motion stemming from the action

\[ S_{CS}^{ab}(A) + S_\Omega(A) . \] (3.19)

For the BRST differential we have

\[ sA_\mu = -\partial_\mu c , \]

\[ sc = 0 , \]

\[ sA^*_\mu = \frac{\delta(S_{CS}^{ab} + S_\Omega)}{\delta A^\mu} = \frac{1}{2} \varepsilon^{\mu \nu \rho} F^\nu_{\mu \rho} + \frac{\delta S_\Omega(A)}{\delta A^\mu} , \]

\[ sc^* = -\partial^\mu A^* , \] (3.20)

where

\[ \frac{\delta S_\Omega(A)}{\delta A^\mu} = \frac{2}{\xi} \int \left( \prod_{j=2}^n d^3x_j F_{\mu_j \nu_j}(x_j) \right) \partial_\nu \Omega^{\nu \mu_2 \nu_2 \ldots \mu_n \nu_n}(x, x_2, \ldots, x_n) \] (3.21)

\[ + \frac{2}{\xi} \int \left( \prod_{j \neq 2}^n d^3x_j F_{\mu_j \nu_j}(x_j) \right) \partial_\nu \Omega^{\nu \mu_1 \nu_1 \mu \ldots \mu_n \nu_n}(x_1, x, x_3 \ldots, x_n) \]
The fields and antifields $A_\mu, c, A^*_\mu, e^*$ possess respectively ghost number 0, 1, $-1$, $-2$. Following now refs. [1, 2], it is easily established that the third equation of (3.20) can be cast in the form of a recursive formula. In fact, contracting both sides with $\varepsilon_{\mu
u\rho}$ and using

$$\varepsilon_{\mu
u\rho}\varepsilon^{\rho\sigma\tau} = \delta_\mu^\sigma\delta_\nu^\tau - \delta_\nu^\sigma\delta_\mu^\tau,$$

we get

$$F_{\mu\nu} = s(\varepsilon_{\mu\nu\rho}A^{*\rho}) - \varepsilon_{\mu\nu\rho}\frac{\delta S_{\Omega}(A)}{\delta A_\rho}$$

$$= s(\varepsilon_{\mu\nu\rho}A^{*\rho}) - \frac{2\varepsilon_{\mu\nu\rho}}{\xi} \int \left( \prod_{j=2}^{n} d^3 x_j F_{\mu_j\nu_j}(x_j) \right) \partial_\tau^\rho \Omega^{\rho \mu_2 \cdots \mu_n \nu_n}(x, x_2, \ldots, x_n)$$

$$- \frac{2\varepsilon_{\mu\nu\rho}}{\xi} \int \left( \prod_{j=2}^{n} d^3 x_j F_{\mu_j\nu_j}(x_j) \right) \partial_\tau^\rho \Omega^{\mu_1 \mu_2 \cdots \mu_n \nu_n}(x_1, x, x_3, \ldots, x_n)$$

$$- \cdots$$

$$- \frac{2\varepsilon_{\mu\nu\rho}}{\xi} \int \left( \prod_{j=1}^{n-1} d^3 x_j F_{\mu_j\nu_j}(x_j) \right) \partial_\tau^\rho \Omega^{\mu_1 \cdots \mu_{n-1} \nu_{n-1} \rho}(x_1, \ldots, x_{n-1}, x).$$

This equation has the meaning of an iterative formula since the field strength $F_{\mu\nu}$ appears on both sides. At each step of the iteration the $F_{\mu\nu}$’s contained in the term $\delta S_{\Omega}/\delta A_\rho$ can be replaced by the exact BRST variation $s(\varepsilon_{\mu\nu\rho}A^{*\rho})$ with the addition of terms of higher order in $1/\xi$. Obviously, the whole iteration procedure will result in a BRST exact power series of the kind

$$F_{\mu\nu} = s\left(\varepsilon_{\mu\nu\rho}A^{*\rho} + \sum_{k=1}^{\infty} \frac{1}{\xi^k} \mathcal{M}^k_{\mu\nu}(\Omega, A, A^*)\right),$$

where the coefficients $\mathcal{M}^k_{\mu\nu}(\Omega, A, A^*)$ depend on the space-time kernel $\Omega$, the gauge field $A$, the antifield $A^*$ and their space-time derivatives. The formula (3.24) expresses the exactness of the field strength. Therefore we can write the action $S_{\Omega}(A)$ in the form of an exact cocycle, i.e.
\begin{equation}
S_{\Omega} = \int \frac{1}{\xi} \prod_{j=2}^{n} d^3 x_j F_{\mu_j \nu_j} (x_j) \left( \varepsilon_{\mu_1 \nu_1 \rho} A^{* \rho} + \sum_{k=1}^{\infty} \frac{M_{\mu_1 \nu_1}^{k}}{\xi_k} \right) \Omega_{\mu_1 \nu_1 \ldots \mu_n \nu_n}.
\end{equation}

In turn, this implies that the action $S_{\Omega}(A)$ can be reabsorbed into pure Chern-Simons through a nonlinear field redefinition, accounting then for the eq. (2.10). In fact, from the eqs. (3.20) it follows that the BRST variation of antifield dependent expressions, as for instance (3.25), gives rise to terms which are proportional to the equations of motion, thereby corresponding to field redefinitions. It is worth underlining here that the possibility of reabsorbing the term $S_{\Omega}(A)$ depends crucially on the presence of the Chern-Simons in the starting action. The recursive formula (3.23) works in fact due to the presence of the field strength $F_{\mu \nu}$ in the left hand side. Needless to say, this term follows from the field variation of the Chern-Simons action $S_{CS}^{ab}$. Let us conclude this Section by remarking that the use of the antifields and of the BRST differential do not have here the meaning of a quantization procedure. We are not attempting to quantize the action (3.19). This would be a very hard task, due to the highly nonlocal character of $S_{\Omega}(A)$. The gauge field $A_\mu$ is always meant to be an external classical field. The introduction of the antifields has to be seen as a useful device in order to exploit from a cohomological point of view the consequences following from the classical equations of motion, as for instance the recursive formula (3.23). This role of the BRST differential is well known, being related to the so called characteristic cohomology [23] and to the Koszul-Tate differential [1, 23]. Notice also that the formulas (3.24), (3.25) have been derived by means of direct straightforward manipulations, without relying on any particular property of the BRST differential (3.20) or on the explicit knowledge of its cohomology.

4 The fermionic determinant

We are now ready to analyse the perturbative expansion of the fermionic determinant. To this purpose we recall that because of charge conjugation invariance the Green functions with an odd number of external gauge fields
vanish and that, although superficially divergent, the spinor vacuum polarization turns out to be finite \[6, 8, 9, 10, 20, 21, 22\]. The higher $n$-point Green functions are finite by power counting. It is also worth mentioning that in the infinite mass limit $m \to \infty$ the fermionic determinant reduces, modulo the well known regularization ambiguity \[6, 8, 9, 10, 20, 21, 22\], to the pure Chern-Simons action.

Let us begin by showing that, due to the absence of gauge anomalies in three dimensions, the generic $n$-point contribution to $\Gamma(A)$ can be cast in the form of the eq.(2.9).

### 4.1 Absence of anomalies and structure of the perturbation theory

It is a well established fact that in three dimensions there are no gauge anomalies, so that the functional $\Gamma(A)$ of eq.(2.7) is gauge invariant

$$\partial_\mu \frac{\delta \Gamma(A)}{\delta A_\mu} = 0 . \tag{4.26}$$

This equation has useful consequences. Acting indeed on (4.26) with the test operator

$$\delta \frac{\delta}{\delta A_\mu_1(x_1)} \cdots \frac{\delta}{\delta A_{n-1}(x_{n-1})} , \tag{4.27}$$

and setting $A_\mu$ to zero we get

$$\partial_{\mu_1} \frac{\delta}{\delta A_\mu_1(x_1)} \cdots \frac{\delta}{\delta A_{n-1}(x_{n-1})} \Gamma(A) \bigg|_{A=0} = 0 , \tag{4.28}$$

expressing the conservation law for the fermionic current insertions

$$\partial_{x_1}^{\mu_1} < j_{\mu_1}(x_1) \ldots j_{\mu_n}(x_n) > = 0 , \tag{4.29}$$

$j_\mu(x)$ denoting the spinor current

$$j_\mu(x) = \bar{\psi} \gamma_\mu \psi . \tag{4.30}$$

The eq.(4.29) implies that the functional $\Gamma(A)$ depends only on the gauge invariant variable $A_\mu^T$ (see also App.A)
\[ A^\nu_\mu = (g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}) A^\nu = A_\mu(x) - \partial_\mu^\nu \partial^\nu \int d^3 y \, G(x - y) A^\nu(y) , \quad (4.31) \]

where

\[ G(x - y) = -\frac{1}{4\pi} \frac{1}{|x - y|} , \quad (4.32) \]

is the inverse of the three dimensional laplacian

\[ \partial^2_3 G(x - y) = \delta^3(x - y) . \quad (4.33) \]

The nonlocal variable \( A^T_\mu \) corresponds to the pure transverse part of the gauge field \( A_\mu \) according to the decomposition

\[ A_\mu = A^T_\mu + A^L_\mu , \quad (4.34) \]

\[ A^L_\mu = \frac{\partial_\mu \partial_\nu A^\nu}{\partial^2} . \quad (4.35) \]

Of course,

\[ \partial^\mu A^T_\mu = 0 , \quad (4.36) \]

and

\[ \delta A^T_\mu = -\partial_\mu \alpha + \partial_\mu \alpha = 0 . \quad (4.37) \]

Let us consider then the perturbative expansion of \( \Gamma(A) \)

\[ \Gamma(A) = \sum_{n=2}^{\infty} \Gamma^n(A) , \quad (4.38) \]

\[ \Gamma^n(A) = \int d^3 x_1 ... d^3 x_n A^{\mu_1}(x_1) ... A^{\mu_n}(x_n) < j_{\mu_1}(x_1) ... j_{\mu_n}(x_n) > . \]

It is almost immediate now to verify that we can replace \( A_\mu \) by \( A^T_\mu \) in the eq.(4.38). In fact, due to the conservation law (4.29), the longitudinal components \( A^L_\mu \) do not couple to the spinor current, i.e.
\[ 0 = - \int d^3x_1 \ldots d^3x_n \left( \frac{\partial}{\partial^2} A^\nu(x_1) \right) \ldots A^{\mu_n}(x_n) \partial_{x_1}^{\mu_1} \langle j_{\mu_1}(x_1) \ldots j_{\mu_n}(x_n) \rangle > \\
= \int d^3x_1 \ldots d^3x_n A^{L^{\mu_1}(x_1) \ldots A^{\mu_n}(x_n)} \langle j_{\mu_1}(x_1) \ldots j_{\mu_n}(x_n) \rangle > . \quad (4.39) \]

Thus

\[ \Gamma^n(A) = \Gamma^n(A^T) . \quad (4.40) \]

Recalling now that

\[ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} = \partial_{\mu} A^T_{\nu} - \partial_{\nu} A^T_{\mu} , \quad (4.41) \]
\[ \partial^\mu F_{\mu\nu} = \partial^2 A^T_{\nu} , \quad (4.42) \]

it follows

\[ A^T_{\nu} = \frac{1}{\partial^2} \partial^\mu F_{\mu\nu} = \frac{1}{4\pi} \int d^3y \frac{(x-y)^\mu}{|x-y|^3} F_{\mu\nu}(y) . \quad (4.43) \]

Finally, for the \( n \)-point contribution \( \Gamma^n(A) \) we get

\[ \Gamma^n(A) = \int d^3y_1 \ldots d^3y_n F_{\mu_1 \nu_1}(y_1) \ldots F_{\mu_n \nu_n}(y_n) \Omega^{\mu_1 \nu_1 \ldots \mu_n \nu_n}(y_1, \ldots, y_n) , \quad (4.44) \]

with the space-time kernel \( \Omega^{\mu_1 \nu_1 \ldots \mu_n \nu_n}(y_1, \ldots, y_n) \) given by

\[ \Omega^{\mu_1 \nu_1 \ldots \mu_n \nu_n} = \frac{1}{(4\pi)^n} \int \left( \prod_{j=1}^n d^3x_j \frac{(x_j - y_j)^{\mu_j}}{|x_j - y_j|^3} \right) \langle j^{\nu_1}(x_1) \ldots j^{\nu_n}(x_n) \rangle > , \quad (4.45) \]

where a suitable antisymmetrization in the Lorentz indices \((\mu_j, \nu_j)\) has to be understood. We see then that, as announced, the \( n \)-point contribution to the effective action \( \Gamma^n(A) \) can be cast in the form of the eq.(2.9). It remains now to analyse the two point function. This will be the task of the next Subsection.
4.2 The spinor vacuum polarization

The conclusions of the previous Subsection can be generalized to the fermionic
determinant in higher space-time dimensions, provided one is able to guar-
ante the absence of anomalies. However, the three dimensional case is pecu-
liar with respect to the higher dimensional ones. Of course, the peculiarity
lies in the appearance of the Chern-Simons action $\mathcal{S}_{CS}^{ab}(A)$ in the two point
function, as it has been established by the various exact computations of the
one loop spinor vacuum polarization done till now [6, 8, 9, 10, 20, 21, 22].

Although higher dimensional generalizations of the Chern-Simons are
known, it is only in three dimensions that the field variation of the Chern-
Simons action yields the field strength $F_{\mu\nu}$. As already underlined, it is this
property which allows us to reabsorb gauge invariant $F$-dependent actions
through nonlinear redefinitions.

Owing to the results [6, 8, 9, 10, 20, 21, 22], the contribution of the spinor
vacuum polarization to the effective action can be written as

$$
\Gamma^2(A) = \Gamma^2(A^T) = \eta \mathcal{S}_{CS}^{ab}(A) + \Gamma_T^2(A) ,
$$

where $\eta$ is the well known regularization ambiguity [6, 8, 9, 10, 20, 21, 22]
and where $\Gamma_T^2(A)$ has the general form (4.44). In particular it turns out that,
when expanded in the inverse of the mass parameter $m$, $\Gamma_T^2(A)$ leads to an
infinite sum of terms of the type

$$
\int d^3x F_{\mu\nu}(\partial^2)^n F^{\mu\nu} , \quad \int d^3x \varepsilon^{\mu\nu\rho} A_{\mu} \partial_{\nu} (\partial^2)^n A_{\rho} , \quad n \geq 1 .
$$

Observe that

$$
\int d^3x \varepsilon^{\mu\nu\rho} A_{\mu} \partial_{\nu} (\partial^2)^n A_{\rho} = - \int d^3x \varepsilon^{\mu\nu\rho} F_{\mu\sigma} \partial_{\nu} (\partial^2)^{n-1} F_{\sigma} , \quad (4.48)
$$

has indeed the form of the eq.(4.44).

We also remark that the presence of the Chern-Simons term in the eq.(4.46)
is in complete agreement with the general fact that the effective action $\Gamma$ de-
pends only on the transverse component $A^T_{\mu}$. It is almost immediate to check
that the Chern-Simons term is in fact already purely transverse,

$$
\int d^3x \varepsilon^{\mu\nu\rho} A_{\mu} \partial_{\nu} A_{\rho} = \int d^3x \varepsilon^{\mu\nu\rho} A^T_{\mu} \partial_{\nu} A^T_{\rho} . \quad (4.49)
$$
4.3 Resetting the fermionic determinant to pure Chern-Simons

To summarize, we have been able to show that the perturbative expansion of the fermionic determinant can be written as

$$\Gamma(A^T) = \eta S^{ab}_{CS}(A) + \sum_{n \geq 2} \int d^3 y_1 \ldots d^3 y_n F_{\mu_1 \nu_1} \ldots F_{\mu_n \nu_n} \Omega^{\mu_1 \nu_1 \ldots \mu_n \nu_n},$$

(4.50)

for a suitable space-time dependent kernel $\Omega^{\mu_1 \nu_1 \ldots \mu_n \nu_n}$. Therefore, owing to the results of Sect. 3, we can write for the whole quantum action $\Gamma$ the following formula

$$\Gamma(A) = \log \det(i \gamma^\mu \partial_\mu + \gamma^\mu A_\mu - m) = \eta S^{ab}_{CS}(\hat{A}),$$

(4.51)

up to a nonlinear redefinition $\hat{A}$ of the gauge field of the kind of eq. (2.11).

This formula is the essence of the present work, expressing the fact that the quantum effects can be reabsorbed in the pure Chern-Simons action, up to nonlinear field redefinitions.

Although being out of the aim of this work, let us emphasize that the equation (4.51) calls for a deeper understanding of the regularization ambiguity coefficient $\eta$ [6, 8, 9, 10, 20, 21, 22].

5 Generalization

It is not difficult now to prove that a result similar to (4.51) holds in the case in which the spinor fields interact in a nonminimal way with the gauge field $A_\mu$. For instance, the inclusion of an interaction term of the type

$$\varepsilon^{\mu \nu \rho} \bar{\psi} \gamma_\mu \psi F_{\nu \rho},$$

(5.52)

will not alter the gauge invariance of the resulting fermionic determinant. As a consequence, the $n$-point contribution to the effective action will be always of the form (4.44).

Therefore, up to nonlinear field redefinitions, we get

$$\log \det(i \gamma^\mu \partial_\mu + \gamma^\mu A_\mu + \frac{q}{2} \varepsilon^{\mu \nu \rho} \gamma_\mu F_{\nu \rho} - m) = \eta S^{ab}_{CS}(\hat{A}).$$

(5.53)
The above formula generalizes to any higher dimensional $F$-dependent non-minimal interaction, implying that a whole family of quantum effective actions can be actually resetted to pure Chern-Simons, up to nonlinear field redefinitions. The suggestive picture which emerges from these results is that the introduction of a nonminimal gauge coupling in the fermionic determinant corresponds to a change of the redefined connection $\hat{A}_\mu$ for the resulting Chern-Simons. In other words, it seems rather natural to interpret the Chern-Simons as a gauge invariant functional defined on the space of the connections of the type (2.11). One moves from a given determinant to another one by an appropriate change of the connection $\hat{A}_\mu$. This point will lead to rather interesting conclusions.

## 6 Conclusion

Several remarks follow from the previous considerations.

- The gauge invariance of the coefficients $\vartheta^k_\mu$ entering the nonlinear re-definition of the gauge field (see eqs. (3.12)-(3.15)) implies that the re-defined field $\hat{A}_\mu$ is still a connection. As already underlined in ref. 2, this feature allows us to interpret the resulting Chern-Simons action $S_{CS}^{ab}(\hat{A})$ as a gauge invariant functional defined on the space of the connections of the type (2.11). This provides a simple geometrical set up for the fermionic determinant for an arbitrary finite nonvanishing value of the mass parameter $m$.

- The possibility of resetting a whole family of fermionic determinants (see for instance eq. (5.53)) to pure Chern-Simons suggests the existence of a kind of universal behaviour for the corresponding effective quantum actions. The universality factor is precisely the Chern-Simons functional. The effective action of a given determinant is obtained thus by evaluating the Chern-Simons functional at a suitably chosen gauge connection $\hat{A}_\mu$.

- This universality character should persist for any model which can be related in some way to the fermionic determinant, as it is the case for instance of the three dimensional Thirring model [16, 19]. This point could be of great relevance for the three dimensional bosonization program [12, 13, 14, 15, 16, 17, 18, 19].
• Although the form of the coefficients \( \vartheta^k_\mu \) entering the nonlinear redefinition (2.11) relies on the explicit computation of the space-time kernel \( \Omega \) in eq.(4.50), we believe that the knowledge of the fact that the quantum effects can be reabsorbed into pure Chern-Simons and that the resulting field \( \hat{A}_\mu \) is a connection represents a nonempty information. Perhaps, the pure geometrical interpretation of the final result could help us in finding a kind of recursive procedure for the \( \vartheta^k_\mu \)'s. This would allow us to obtain exact bosonized formulas [24]. We observe also that the geometrical interpretation of the nonlinear redefinition of the gauge field naturally reminds us the well known normal coordinates expansion of the general relativity and of the nonlinear sigma model.

• Finally, it is worth underlining that the present results yield a further evidence in favour of the conjectured quantum equivalence [2], up to nonlinear field redefinitions, between the pure nonabelian Chern-Simons and the fully 1PI effective action of topological massive Yang-Mills.

Acknowledgements
The Conselho Nacional de Pesquisa e Desenvolvimento, CNPq Brazil, the Faperj, Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro and the SR2-UERJ are gratefully acknowledged for financial support.

A Appendix
A.1 A gauge invariant perturbative expansion
The dependence of the effective action \( \Gamma(A) \) in eq.(2.7) from the transverse component \( A^T_\mu \) can also be seen as a consequence of the invariance of the functional measure under phase transformations of the type
\[
\psi' = e^{-i\tau(x)}\psi, \quad \bar{\psi}' = e^{i\tau(x)}\bar{\psi},
\]
(A.54)
namely
\[
\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = \mathcal{D}\psi\mathcal{D}\bar{\psi}.
\]
(A.55)
As it is well known, this property is related to the absence of anomalies in three dimensions. In particular, this implies that the longitudinal components $A_{\mu}^L$ can be completely gauged away. Moving in fact from $(\psi, \overline{\psi})$ to a set of gauge invariant spinor variables $(\chi, \overline{\chi})$

$$\chi(x) = e^{-i \int d^3y \mathcal{G}(x-y) \partial A(y)} \psi(x), \quad (A.56)$$

$$\overline{\chi}(x) = e^{+i \int d^3y \mathcal{G}(x-y) \partial A(y)} \overline{\psi}(x),$$

with $\mathcal{G}(x - y)$ given in eq.(4.32), we get

$$D\psi D\overline{\psi} = D\chi D\overline{\chi}, \quad (A.57)$$

so that

$$\int D\psi D\overline{\psi} e^{\int d^3x \psi (i\gamma^\mu \partial_\mu + \gamma^\mu A_\mu - m)\psi} = \int D\chi D\overline{\chi} e^{\int d^3x \overline{\chi} (i\gamma^\mu \partial_\mu + \gamma^\mu A_\mu^T - m)\chi} . \quad (A.58)$$

This equation implies thus

$$\Gamma(A) = \Gamma(A^T) . \quad (A.59)$$

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