FLOCKING AND LINE-SHAPED SPATIAL CONFIGURATION TO DELAYED CUCKER-SMALE MODELS

ZHISU LIU, YICHENG LI*, XIANG LI

Department of Mathematics, College of Liberal Arts and Sciences
National University of Defense Technology, Changsha, Hunan 410073, China

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Abstract. As we known, it is popular for a designed system to achieve a prescribed performance, which have remarkable capability to regulate the flow of information from distinct and independent components. Also, it is not well understand, in both theories and applications, how self propelled agents use only limited environmental information and simple rules to organize into an ordered motion. In this paper, we focus on analysis the flocking behaviour and the line-shaped pattern for collective motion involving time delay effects. Firstly, we work on a delayed Cucker-Smale-type system involving a general communication weight and a constant delay \( \tau > 0 \) for modelling collective motion. In a result, by constructing a new Lyapunov functional approach, combining with two delayed differential inequalities established by \( L^2 \)-analysis, we show that the flocking occurs for the general communication weight when \( \tau \) is sufficiently small. Furthermore, to achieve the prescribed performance, we introduce the line-shaped inner force term into the delayed collective system, and analytically show that there is a flocking pattern with an asymptotic flocking velocity and line-shaped pattern. All results are novel and can be illustrated by numerical simulations using some concrete influence functions. Also, our results significantly extend some known theorems in the literature.

1. Introduction. Self-organized systems appear very naturally in many scientific disciplines, such as physics, biology, economics and social science. As a kind of ordered collective behaviors, flocking can be regarded as the phenomena where autonomous agents reach a consensus based on limited environmental information and simple rules (see [36]). It is of particular significance in both theories and applications to understand how self-propelled individuals use limited environmental in formation and simple rules to organize into ordered motions. The celebrated Cucker-Smale model [10, 11] proposed in 2007 provides a framework to examine the emergent properties of flocks in order to explain self-organized behaviours in various complex systems. We say the system exhibits flocking behavior if there is asymptotic alignment of velocities, and the particle group stays uniformly bounded in time. The concept of unconditional flocking refers to the fact that the flocking behavior takes place for all initial conditions. The authors in [10, 11] stated the

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* Corresponding author: Yicheng Liu.
following system of equations

\[
\begin{align*}
\dot{x}_i &= v_i, \\
\dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i), & i = 1, 2, ..., N,
\end{align*}
\]

where \(x_i \in \mathbb{R}^d\) denotes the position of \(i\)-th particle and \(v_i \in \mathbb{R}^d\) stands for velocity. The communication rate \(\psi(\cdot) : [0, \infty) \to [0, \infty)\) quantifies the influence between \(i\)-th and \(j\)-th particles, and \(|\cdot|\) denotes the Euclidean norm in \(\mathbb{R}^d\). The communication rate is taken as

\[
\psi(s) = \frac{K}{(1 + s^2)\beta}, \quad K, \beta > 0.
\]  

Note that the unconditional flocking occurs if \(\beta < \frac{1}{2}\), and the conditional flocking occurs if \(\beta > \frac{1}{2}\). We refer the readers to \([7]\) for more detailed discussions. Moreover, unconditional flocking has also been obtained in the case \(\beta = \frac{1}{2}\) (see e.g. \([17, 18]\)). Recently, various modifications of the classical Cucker-Smale model have been considered. For examples, to avoid collision behaviour among particles, the singular communication functions are introduced, see \([3, 12, 27, 19, 30]\); to investigate the clustering emergence, the short-range communication functions are considered, see \([5, 20, 34]\). Moreover, the Cucker-Smale model with hierarchical leadership and kinetic Cucker-Smale model have been investigated, see \([35, 13, 25, 29, 18, 4, 26, 22, 23]\) and the references therein. The overview contributions can be referred to \([1, 2]\).

There has been some literatures on the flocking of Cucker-Smale models with time delays. More precisely, the existence of flocking solutions for the Motsch-Tadmor variant of the model with delay was obtained in \([28, 6]\). However, their analysis were only valid to some re-normalized systems. The authors \([7]\) considered heterogeneous delays only in the \(x_j\) and \(v_j\) terms and they proved asymptotic flocking for small delays with general weights. A system with time-varying delays was studied in \([31]\), where the strictly positive lower bound for \(\psi\) is assumed. For other flocking results for delay systems, see \([16, 14, 32, 37, 15, 33, 9]\) and the references therein. In particular, Erban, Haskovec and Sun \([16]\) studied the following Cucker-Smale type system with a fixed time delay \(\tau > 0\),

\[
\begin{align*}
\dot{x}_i &= v_i, \\
\dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} \psi(|\tilde{x}_j - \tilde{x}_i|)(\tilde{v}_j - \tilde{v}_i), & i = 1, 2, ..., N,
\end{align*}
\]  

(1.2)

where the parameter \(\lambda > 0\) measures the alignment strength, \(x_i \in \mathbb{R}^d\) and \(v_i \in \mathbb{R}^d\) are time-dependent position and velocity vectors of the \(i\)-th agent. Here, \(\tilde{x}_i(t) := x_i(t - \tau)\) and \(\tilde{v}_i(t) := v_i(t - \tau)\). For the above system (1.2), they considered prescribed initial position and velocity trajectories

\[
(x_i(s), v_i(s)) = (x_i^0(s), v_i^0(s)) \in C([-\tau, 0]; \mathbb{R}^d), \quad i = 1, 2, ..., N.
\]  

(1.3)

The authors in \([16]\) proved that there is a asymptotic alignment of velocities for systems (1.2)-(1.3) with or without noise, where delay \(\tau\) is less than some positive constant. However, the uniform bound of the particle group is not clear. Recently, Haskovec and Markou \([21]\) generalized the results of \([16]\), and established some stability estimates for the Cucker-Smale delayed flow to obtain the uniform bound of the particle group. We emphasize that the flocking results in \([21]\) for systems (1.2)-(1.3) are only valid for communication rate (1.1) and the constant initial datum. A
natural question is what about the flocking behaviour for systems (1.2)-(1.3) with more general communication weight. In the present paper we are attempt to prove the existence of flocking results for systems (1.2)-(1.3) with a general communication weight. Mathematically, we post the following assumption.

**Assumption on \( \psi \).** The communication weight \( \psi \) is bounded, positive, non increasing and Lipschitz continuous on \( \mathbb{R}^+ \).

**Remark 1.** In fact, such a general assumption on \( \psi \) is firstly introduced in [8] and the typical case for the general communication weight is \( \psi(r) = \frac{1}{(1+r^\gamma)^\alpha} \) for some \( \alpha, \beta > 0 \). When \( \alpha = 1 \), the communication weight is used in [5] recently. When \( \alpha = 2 \), the communication weight is widely adopted to the various modified Cucker-Smale model, see[10, 11, 21] and so on.

Based on above, we further prove that the particles finally come to a line-shaped formation by adding a prescribed driving force for system (1.2). The main novelty of this paper is threefold. First, we present a sufficient framework for asymptotic flocking behavior to the delayed Cucker-Smale system (1.2)-(1.3) with a general communication weight. Second, we present a sufficient framework for line-shaped spatial configuration to the delayed Cucker-Smale system (1.2)-(1.3) containing a prescribed driving force, which seems to be the first result on motion pattern of particles to delayed case. Third, some new tricks are used to obtain the uniform bounded of velocities of particles which is a key factor in proving flocking solutions.

To investigate the flocking condition for systems (1.2)-(1.3), we introduce the following macroscopic variables

\[
\bar{x}(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t), \quad \bar{v}(t) = \frac{1}{N} \sum_{i=1}^{N} v_i(t).
\]

Thus, from system (1.2) we deduce \( \frac{d \bar{x}(t)}{dt} = \bar{v}(t), \frac{d \bar{v}(t)}{dt} = 0 \). This in turn implies that \( \bar{x}(t) = \bar{x}(0) + \bar{v}(0)t \) for \( t \geq 0 \). Let \( \hat{x}_i(t) = x_i(t) - \bar{x}(t), \hat{v}_i(t) = v_i(t) - \bar{v}(t) \) be the fluctuations around the center of the mass system. It is not hard to check that

\[
\hat{x}_c(t) = \frac{1}{N} \sum_{i=1}^{N} \hat{x}_i(t) \equiv 0, \quad \hat{v}_c(t) = \frac{1}{N} \sum_{i=1}^{N} \hat{v}_i(t) \equiv 0.
\]

Moreover, the new variables \((\hat{x}_i(t), \hat{v}_i(t))\) satisfy systems (1.2)-(1.3) with \( \hat{x}_c(t) \equiv 0, \hat{v}_c(t) \equiv 0 \). For simplicity, we drop the hat notion in the microscopic variables and use \((x_i(t), v_i(t))\) instead of \((\hat{x}_i(t), \hat{v}_i(t))\). In order to obtain asymptotic flocking results, we mainly investigate systems (1.2)-(1.3) with the following conditions for \( t \geq -\tau \)

\[
\sum_{i=1}^{N} x_i(t) \equiv \sum_{i=1}^{N} v_i(t) \equiv 0. \tag{1.4}
\]

Define \( X(t) := \sum_{i=1}^{N} |x_i(t)|^2, V(t) := \sum_{i=1}^{N} |v_i(t)|^2 \). Then, by (1.4) we have

\[
\sum_{i,j=1}^{N} |x_i(t) - x_j(t)|^2 = 2NX(t), \quad \sum_{i,j=1}^{N} |v_i(t) - v_j(t)|^2 = 2NV(t). \tag{1.5}
\]

So we can state the following notation of flocking behavior.
Definition 1.1. We say that the system with particle positions $x_i(t)$ and velocities $v_i(t)$, $i = 1, ..., N$ and $t \geq 0$, exhibits asymptotic flocking if the spatial and velocity diameters satisfy

$$\sup_{t \geq 0} X(t) < \infty, \quad \lim_{t \to \infty} V(t) = 0.$$  

Without loss of generality, we assume that $\psi(0) = 1$. The remainder of this paper is organized as follows. In Section 2, the existence of flocking solutions to the delayed Cucker-Smale-type system is proved. Section 3 is devoted to proving the line-shaped motion pattern of the delayed Cucker-Smale-type system with a driving force. Some numerical simulations are finally stated in Section 4.

2. Flocking to the delayed Cucker-Smale-type system. Note that the uniform bound on particle velocities for the classical Cucker-Smale system without time delay can be easily proved by the dissipation of the kinetic energy. However, the presence of the delay in (1.3) destroys this dissipation property. In this section, we use the appropriate energy functional to derive the uniform bound on particle velocities. Motivated by [21], we define the following energy functional,

$$E(t) = V(t) + \frac{2\tau\lambda^3}{N} \sum_{i,j=1}^{N} \int_{t-\tau}^{t} \int_{\theta}^{\tau} \tilde{\psi}_{i,j}(s) \tilde{v}_j(s) - \tilde{v}_i(s))^2 dsd\theta,$$

where

$$\tilde{v}_i(t) := \psi(|\tilde{x}_i(t) - \tilde{x}_j(t)|), \quad \tilde{v}_i(t) := \psi(|x_j(t) - x_i(t)|),$$

and we define

$$W(t) := \sum_{i,j=1}^{N} \tilde{\psi}_{i,j}(t) \tilde{v}_j(t) - \tilde{v}_i(t))^2.$$  

We emphasize that $E(t)$ is not defined at $t \in [0, \tau)$ due to the integration term, and so $E(0)$ is meaningless. That is to say, we can not directly prove that the derivation of $E(t)$ is not less than zero for $t \geq 0$, which was used to obtain the uniform bounded of $V(t)$ in [21] whose upperbound is $E(0)$. So, we have to use some new tricks in our arguments to obtain the uniform bounded of $V(t)$.

2.1. Our main results. Now we state the following lemma whose conclusion will be of use in our main results.

Lemma 2.1. Systems (1.2)-(1.3) have a unique global solution $(x_i, v_i)_{i=1}^{N}$ for $\tau \in (0, \frac{1}{2\lambda})$, and $V(t) \leq E_{0}$ holds for all $t \geq 0$, Here,

$$E_{0} := 4\lambda^2 \tau \int_{-\tau}^{0} V(r) dr + 2V(0) + \frac{2\tau\lambda^3}{N} \sum_{i,j=1}^{N} \int_{0}^{\tau} \int_{\theta}^{\tau} \tilde{\psi}_{i,j}(s) \tilde{v}_j(s) - \tilde{v}_i(s))^2 dsd\theta.$$  

Moreover,

$$|v_i(t)| \leq \sqrt{E_{0}}, \quad |v_i(t) - v_j(t)| \leq 2\sqrt{E_{0}}, \quad \forall t \geq 0, i, j \in \{1, ..., N\}.$$  

Proof. It follows from the assumption on $\psi$ that the right-side of (1.2) is locally Lipschitz continuous with respect to a function of $(x_i, v_i)$. Then according to the Cauchy-Lipschitz theorem, system (1.2) has a unique local-in-time $C^{1}$-solution. Such a solution (denoted by $(x_i, v_i)_{i=1}^{N}$) is indeed global in time if we can prove the
Theorem 2.2. Suppose that there exist some \( \tau \) such that we have uniform-in-time boundedness of the velocity \( v_i \) since \( \psi \) is bounded and Lipschitz. Integrating two-sides of the second equation of system (1.2), we have for \( t \in (0, \tau) \)

\[
|v_i(t) - v_i(0)| \leq \int_0^t \frac{\lambda}{N} \sum_{j=1}^{N} \psi(|\tilde{x}_j(s) - \tilde{x}_i(s)|)|\tilde{v}_j(s) - \tilde{v}_i(s)|ds \\
\leq \int_0^t \frac{\lambda}{N} \sum_{j=1}^{N} |\tilde{v}_j(s) - \tilde{v}_i(s)|ds,
\]  

(2.4)

which implies that

\[
|v_i(t)| \leq \int_0^t \frac{\lambda}{N} \sum_{j=1}^{N} |\tilde{v}_j(s) - \tilde{v}_i(s)|ds + |v_i(0)|.
\]  

(2.5)

Thus, the following holds for \( t \in (0, \tau) \),

\[
V(t) \leq \sum_{i=1}^{N} |v_i(t)|^2 \\
\leq 2 \sum_{i=1}^{N} \left( \int_0^t \frac{\lambda}{N} \sum_{j=1}^{N} |\tilde{v}_j(s) - \tilde{v}_i(s)|ds \right)^2 + 2 \sum_{i=1}^{N} |v_i(0)|^2 \\
\leq 2 \frac{\lambda^2 \tau}{N^2} \sum_{i=1}^{N} \int_0^t \left( \sum_{j=1}^{N} |\tilde{v}_j(s) - \tilde{v}_i(s)| \right)^2 ds + 2 \sum_{i=1}^{N} |v_i(0)|^2 \\
\leq 2 \frac{\lambda^2 \tau}{N} \sum_{i=1}^{N} \int_0^\tau \sum_{j=1}^{N} |\tilde{v}_j(s) - \tilde{v}_i(s)|^2 ds + 2V(0) \\
\leq 4\lambda^2 \tau \int_{-\tau}^{0} V(r)dr + 2V(0),
\]

where we use Cauchy-Schwarz’s inequality and the definition of \( V(t) \) to get the second inequality and the fourth inequality, respectively. On the other hand, arguing as in Lemma 1 of [21], we obtain if \( t \geq \tau \) with \( \tau \in (0, \frac{1}{2\lambda}) \), \( E(t) \) is decreasing on \( t \). It is easy to see that \( V(t) \leq E(t) \leq E(\tau) \) for \( t \geq \tau \). Thus, from (2.6) and the definition of \( E \), we deduce that

\[
V(t) \leq 4\lambda^2 \tau \int_{-\tau}^{0} V(r)dr + 2V(0) + \frac{2\tau \lambda^3}{N} \sum_{i,j=1}^{N} \int_0^\tau \int_\theta \hat{v}_{i,j}(s)|\hat{v}_j(s) - \hat{v}_i(s)|^2 dsd\theta, \quad t \geq 0.
\]

Hence, systems (1.2)-(1.3) have a unique global solution \( (x_i, v_i)_{i=1}^{N} \). By the definition of \( V(t) \), the following holds immediately

\[
|v_i(t)| \leq \sqrt{E_0}, \quad |v_i(t) - v_j(t)| \leq 2\sqrt{E_0}, \quad \forall t \geq 0, i, j \in \{1, ..., N\}.
\]

Now we state the main results for systems (1.2)-(1.3).

**Theorem 2.2.** Let \( (x_i, v_i)_{i=1}^{N} \) be a unique global solution to systems (1.2)-(1.3). Suppose that there exist some \( \tau_0 \in (0, \frac{1}{2\lambda}) \) and \( \alpha > 0 \) such that

\[
G(\alpha) := \frac{4\sqrt{X(0)} + \alpha \cdot \max\{1, V(0)\}}{\alpha \lambda \cdot \psi(2\sqrt{X(0)} + \alpha + 2\sqrt{E_0}\tau_0)} < 1.
\]  

(2.7)
Then there exists $\tau^* \in (0, \tau_0]$ such that for all $\tau \in (0, \tau^*)$ we have
\[
\sup_{-\tau < t < +\infty} X(t) < +\infty \quad \text{and} \quad V(t) \leq C_0 e^{-2c_1 \lambda \psi(\sqrt{NX(0)+N\alpha+2\sqrt{E_0}})t}
\]
for all $t \geq 0$, where $C_0 \in (1, +\infty)$ and $c_1 \in (0, 1)$ are some positive constants.

The idea of the proof of Theorem 2.2 is as follows. We modify a Lyapunov functional constructed in [16, 21], together with some new tricks, to obtain the uniform bound of the velocities (see Lemma 2.1). And then we prove two new delay differential inequalities for $X(t)$ and $V(t)$ which are vital for us to obtain the existence of flocking solutions via the idea from [8]. However, in [8], the delay in the velocity equation for the $i$-th agent is present only in the $v_j(t)$ terms for $j \neq i$, which simplifies the arguments that the particle group stays uniformly bounded in time. That is to say, the technique in [8] can be used to deal with our problem. We also emphasize that the strictly positive lower bound in [16] for the weight function $\psi$ is not needed in our framework. Moreover, assumption on $\psi$ in our case is more general than the one in [21]. Indeed, the stability estimates for the particle flow in [21] strongly depends on the detail expression of the communication rate (1.1). So, in some sense, Theorem 2.2 extends some results in [16, 21, 8].

2.2. Delayed differential inequalities. In this part of our proof we use $L^2$ analysis to derive two delayed differential inequalities for $X(t)$ and $V(t)$, which play a key role in proving that the flocking occurs.

**Lemma 2.3.** Let $(x_i, v_i)_{i=1}^N$ be a global solution to systems (1.2)-(1.3). Then $X(t)$ and $V(t)$ satisfy
\[
\begin{align*}
\frac{dX(t)}{dt} & \leq 2\sqrt{X(t)}\sqrt{V(t)}, \\
\frac{dV(t)}{dt} & \leq -2\lambda \psi(2\sqrt{X(t)}+2\sqrt{E_0}\tau)V(t)+4\lambda\Delta_N^\tau, 
\end{align*}
\]
for all $t > 0$, where $\Delta_N^\tau = \frac{1}{4N} \sum_{i,j=1, i \neq j}^N \sum_{1}^N |v_j(t) - v_i(t)| \cdot (|\dot{x}_j(t) - v_j(t)| + |\dot{x}_i(t) - v_i(t)|)$. Moreover,
\[
\Delta_N^\tau(t) \leq \lambda \sqrt{2E_0}\tau. \tag{2.9}
\]

**Proof.** We apply the Cauchy-Schwartz inequality to obtain
\[
\frac{dX(t)}{dt} = \frac{d}{dt} \sum_{i=1}^N |x_i(t)|^2 = 2 \sum_{i=1}^N \langle x_i(t), v_i(t) \rangle \\
\leq 2 \sqrt{\sum_{i=1}^N |x_i(t)|^2} \sqrt{\sum_{i=1}^N |v_i(t)|^2} \\
\leq 2\sqrt{X(t)}\sqrt{V(t)}.
\]

It remains to derive the differential inequality for $V(t)$. Note that for any $i \in \{1, \ldots, N\}$,
\[
\begin{align*}
|\dot{x}_j(t) - \dot{x}_i(t)| & = |x_j(t) - x_i(t) + (\dot{x}_j(t) - \dot{x}_i(t) - x_j(t) + x_i(t))| \\
& \leq |x_j(t) - x_i(t)| + |\dot{x}_j(t) - \dot{x}_i(t) - x_j(t) + x_i(t)| \\
& \leq 2\sqrt{X(t)} + \int_{t-\tau}^t |v_i(s)| ds + \int_{t-\tau}^t |v_j(s)| ds \\
& \leq 2\sqrt{X(t)} + 2\sqrt{E_0}\tau. \tag{2.10}
\end{align*}
\]
We next estimate the term $\Delta^*_N(t)$. Indeed, it follows from Lemma 2.1 that

$$\Delta^*_N(t) = \frac{1}{4N} \sum_{i,j=1}^{N} |v_j(t) - v_i(t)| \cdot (|\tilde{v}_j(t) - v_j(t)| + |\tilde{v}_i(t) - v_i(t)|) \leq \sqrt{E_0} \sum_{i=1}^{N} |\tilde{v}_i(t) - v_i(t)|. $$

(2.12)

Moreover, the following holds for any $i \in \{1, \ldots, N\}$,

$$|\tilde{v}_i(t) - v_i(t)| = \left| \int_{t-\tau}^{t} \frac{dv_i(s)}{ds} \, ds \right| \leq \int_{t-\tau}^{t} \left| \frac{dv_i(s)}{ds} \right| \, ds \leq \frac{\lambda}{N} \int_{t-\tau}^{t} \sum_{j=1}^{N} \tilde{\psi}(s) |\tilde{v}_j(s) - \tilde{v}_i(s)| \, ds \leq \frac{\lambda\sqrt{\tau}}{N} \left( \int_{t-\tau}^{t} \left( \sum_{j=1}^{N} |\tilde{v}_j(s) - \tilde{v}_i(s)| \right)^2 \, ds \right)^{\frac{1}{2}} \leq \frac{\lambda\sqrt{\tau}}{N} \left( \int_{t-\tau}^{t} N \sum_{j=1}^{N} |\tilde{v}_j(s) - \tilde{v}_i(s)|^2 \, ds \right)^{\frac{1}{2}} \leq \frac{\lambda\sqrt{\tau}}{N} \left( \int_{t-\tau}^{t} 2N^2 V(s-\tau) ds \right)^{\frac{1}{2}} \leq \lambda\sqrt{2E_0 \tau},$$

(2.13)

from which (2.9) follows immediately. This completes the proof. □
2.3. **Proof of Theorem 2.2.** It is easy to see from Lemma 2.3 that \( \sup_{t \geq 0} \triangle_N(t) \to 0 \) as \( \tau \to 0 \). Set

\[
\mathcal{N} := \left\{ t \in [0, \infty) : X(s) < X(0) + \alpha \quad \text{for} \quad s \in [0, t) \right\}.
\]

From the continuity of function \( X(t) \), we deduce that \( 0 \in \mathcal{N} \). Now we set \( \overline{\mathcal{N}} := \sup \mathcal{N} \).

We divide two steps to complete the proofs. **Step 1.** Choose some positive constants \( \beta > 0 \) and \( 0 < c < 1 \) such that

\[
\left( \frac{4V(0)}{c \lambda \psi(2\sqrt{X(0) + \alpha} + 2\sqrt{E_\alpha \tau_0})} + \frac{8\beta}{c(1 - c)\lambda} \right) < \frac{\alpha}{\sqrt{X(0) + \alpha}}.
\]

(2.14)

Then we define

\[
\mathcal{M} := \left\{ t \in [0, \overline{\mathcal{N}}) : V(s) < \left( V(0) + \frac{2\beta \psi(\infty)}{1 - c} \right) e^{-2c\lambda \psi(\infty)s} \right\}.
\]

and \( \triangle_{\mathcal{N}}(s) < \beta(\psi(\infty))^2 e^{-2c\lambda \psi(\infty)s} \) for \( s \in [0, t) \).

Here, for simplicity, we denote \( \psi(\infty) := \psi(\sqrt{NX(0) + N\alpha} + 2\sqrt{E_\alpha \tau_0}) \). According to the definition of \( \mathcal{M} \) and Lemma 2.3, for \( \tau > 0 \) small sufficiently, we have

\[
V(0) < V(0) + \frac{2\beta \psi(\infty)}{1 - c}, \quad \sup_{t \geq 0} \triangle_{\mathcal{N}}(t) < \beta(\psi(\infty))^2 e^{-2c\lambda \psi(\infty)t}.
\]

(2.15)

So, \( \mathcal{M} \neq \emptyset \). From the continuity of the functions \( V(t) \) and \( \triangle_{\mathcal{N}}(t) \), we deduce that there exists \( \tau_1 \in [0, \tau_0] \) such that \( 0 < \underline{\mathcal{M}} := \sup \mathcal{M} \) for any \( \tau \in (0, \tau_1) \). We now show \( \overline{\mathcal{M}} = \overline{\mathcal{N}} \). If \( \overline{\mathcal{M}} < \sup \mathcal{N} \), then one of the two conclusions holds

(i) \( \lim_{t \to \underline{\mathcal{M}}} V(t) = \left( V(0) + \frac{2\beta \psi(\infty)}{1 - c} \right) e^{-2c\lambda \psi(\infty)\underline{\mathcal{M}}} \); (ii) \( \lim_{t \to \underline{\mathcal{M}}} \triangle_{\mathcal{N}}(t) = \beta(\psi(\infty))^2 e^{-2c\lambda \psi(\infty)\underline{\mathcal{M}}} \).

Assume (i) holds. It follows from Lemma 2.3 that

\[
\dot{V}(t) \leq -2\lambda \psi(t) V(t) + 4\lambda \beta(\psi(\infty))^2 e^{-2c\lambda \psi(\infty)t}
\]

(2.16)

for all \( t \in (0, \underline{\mathcal{M}}) \). It follows from Gronwall’s inequality that

\[
V(t) \leq V(0)e^{-2\lambda \psi(\infty)t} + \frac{2\beta \psi(\infty)}{1 - c} \left( e^{-2c\lambda \psi(\infty)t} - e^{-2\lambda \psi(\infty)t} \right)
\]

(2.17)

for all \( t \in (0, \underline{\mathcal{M}}) \). Taking the limit \( t \to \underline{\mathcal{M}}^- \) in (2.17) yields

\[
\lim_{t \to \underline{\mathcal{M}}} V(t) < \left( V(0) + \frac{2\beta \psi(\infty)}{1 - c} \right) e^{-2c\lambda \psi(\infty)\underline{\mathcal{M}}},
\]

which contradicts with (i) due to \( \beta > 0 \) and \( c \in (0, 1) \). Assume (ii) holds, we claim that there exists \( \tau_2 \in (0, \tau_1] \) such that the following inequality holds strictly for \( \tau \in (0, \tau_2] \)

\[
\lim_{t \to \underline{\mathcal{M}}^-} \triangle_{\mathcal{N}}(t) < \beta(\psi(\infty))^2 e^{-2c\lambda \psi(\infty)\underline{\mathcal{M}}}.
\]
Indeed, by the definitions of $\Delta_N(t)$ and $V(t)$, we have

$$\lim_{t \to \overline{M}} \Delta_N^\tau(t) = \lim_{t \to \overline{M}} \frac{1}{4N} \sum_{i,j=1}^{N} |v_j(t) - v_i(t)| \cdot (|\check{v}_j(t) - v_j(t)| + |\check{v}_i(t) - v_i(t)|)$$

$$\leq \lim_{t \to \overline{M}} \frac{\lambda \sqrt{V(t)}}{N} \int_{t-\tau}^{t} \sum_{i,j=1}^{N} \tilde{\psi}(s) |\check{v}_j(s) - \check{v}_i(s)| ds$$

$$\leq \lim_{t \to \overline{M}} \frac{\lambda \sqrt{V(t)}}{N} \sqrt{\left( \int_{t-\tau}^{t} \left( \sum_{i,j=1}^{N} |\check{v}_j(s) - \check{v}_i(s)| \right)^2 ds \right)^{\frac{1}{2}}}$$

$$\leq \lim_{t \to \overline{M}} \frac{\lambda \sqrt{V(t)}}{N} \sqrt{\left( \int_{t-\tau}^{t} 2N^2V(s-\tau) ds \right)^{\frac{1}{2}}}$$

And then, by the definition of $\overline{M}$, we have

$$\lim_{t \to \overline{M}} \Delta_N^\tau(t)$$

$$\leq \sqrt{\lambda} \lambda \sqrt{\left( V(0) + \frac{2\beta \psi_{\infty}}{1 - c} e^{-2c\lambda \psi_{\infty} t} \right)} \sqrt{\left( V(0) + \frac{2\beta \psi_{\infty}}{1 - c} e^{-2c\lambda \psi_{\infty} (\overline{M} - 2\tau)} \right)}$$

$$= \sqrt{\lambda} \lambda \sqrt{\left( V(0) + \frac{2\beta \psi_{\infty}}{1 - c} e^{4c\lambda \psi_{\infty} t} \right)} \cdot \left( V(0) + \frac{2\beta \psi_{\infty}}{1 - c} e^{-2c\lambda \psi_{\infty} t} \right)$$

from which the claim follows for $\tau$ small enough. It contradicts with (ii). Therefore, $\overline{M} = \overline{N}$.

**Step 2.** We are attempt to show $\overline{N} = \infty$ when $\tau > 0$ is small enough. Note that for $t \in [0, \overline{N})$ and $\tau \in (0, \tau_2]$, the following holds

$$X(t) < X(0) + \alpha, \quad V(t) < \left( V(0) + \frac{2\beta \psi_{\infty}}{1 - c} e^{-2c\lambda \psi_{\infty} t} \right) e^{-2c\lambda \psi_{\infty} t}, \quad \Delta_N^\tau(t) < 1 \beta(\psi)_{\infty}^2 e^{-2c\lambda \psi_{\infty} t}.$$
which implies a contradiction. Otherwise, if $V(0) + \frac{2\beta \psi^\infty}{1-c} < \frac{1}{4}$, then by (2.7) we have

$$\lim_{t \to M^-} X(t) = \lim_{t \to M^-} \left( X(0) + \int_0^t X(s) ds \right) \leq \lim_{t \to M^-} \left( X(0) + 2 \int_0^t \sqrt{X(s)}\sqrt{V(s)} ds \right) \leq X(0) + \sqrt{X(0)} + \alpha \sqrt{\frac{8\beta \psi^\infty}{1-c} \int_0^t e^{-c\lambda \psi^\infty} s ds} \leq X(0) + \alpha$$

for $t \in [0, \overline{M})$, which implies also a contradiction and yields $\overline{M} = \infty$. Based on the above discussion, we know $\overline{N} = \overline{M} = \infty$. This means that the following inequalities hold for $t \geq 0$:

$$x(t) \leq X(0) + \alpha \quad \text{and} \quad V(t) \leq \left( V(0) + \frac{2\beta \psi^\infty}{1-c} \right) e^{-2c\lambda \psi^\infty t},$$

where $\psi^\infty$, $\beta$ and $c$ are given in Step 1. This completes the proof. \hfill \Box

3. Pattern motion to the modified Cucker-Smale-type system. To the best of our knowledge, there are very few works on the pattern motion to Cucker-Smale-type model, which occurs very naturally in many physical and biological phenomena. Recently, the authors in [24] modified the classical Cucker-Smale model by taking the target motion pattern driving forces into consideration, and proved that there is a line-shaped motion pattern with asymptotic flocking velocity. And soon, the authors in [27] considered non-collision flocking and line-shaped spatial configuration for a modified singular Cucker-Smale model. It can be viewed as a reasonable explanation for the wild geese flying in a line-shape in the sky. A natural question is whether the motion pattern with an asymptotic flocking velocity can occur in the delayed Cucker-Smale model by taking a reasonable target motion pattern driving forces. Our aim in this section is to show that the $N$-agent self-organizing system with delay converges to the line-shaped flocking pattern by designing properly the driving force.

In order to carry the information of the final motion patterns, we slightly modify the classic delay C-S model by adding the final motion pattern driving force function. Precisely, we introduce a suitable $F$ so that all the agents in the self-organizing system (1.2) converge to a line-shaped flock. The modified C-S model is stated as follows

$$\begin{align*}
\dot{x}_i &= v_i, \\
\dot{v}_i &= \frac{\lambda}{N} \sum_{j=1}^{N} \psi(|\tilde{x}_j - \tilde{x}_i|)(\tilde{v}_j - \tilde{v}_i) + F(x_i), \quad i = 1, 2, ..., N,
\end{align*} \tag{3.1}$$

where the driving force $F$ is introduced to determine the final motion pattern of the self-organized systems. Motivated by [24], the function $F$ takes the form

$$F(x_i(t)) = \gamma \langle x_i(t), 1 \rangle \cdot 1 - x_i(t), \quad i = 1, 2, ..., N,$$  \tag{3.2}$$

where $1$ is a unit constant vector denoting the desired final motion direction, $\gamma$ is a positive constant measuring the force strength, $\langle \cdot, \cdot \rangle$ denotes the inner product in
$\mathbb{R}^d$. Particularly, $F$ has the properties:

$$\langle F(x_i(t)), 1 \rangle = 0 \quad \text{for all } t \text{ and } i,$$

$$\sum_{i=1}^{N} F(x_i(t)) = \gamma \sum_{i=1}^{N} [(x_i(t), 1) \cdot 1 - x_i(t)] = 0.$$

### 3.1. Our main results

Define

$$\tilde{E}_0 := (1+2\gamma \tau)C(1+e^{8\tau^4 \gamma}) + 2\gamma X(0) + 2\gamma X(0) + 2\tau \lambda^3 \sum_{i,j} \int_{-\tau}^{\tau} \int_{0}^{T} \tilde{v}_{i,j}(s)|\tilde{v}_i(s) - \tilde{v}_i(s)|^2 ds d\theta,$$

where

$$C := 4\lambda^2 \int_{-\tau}^{\tau} V(s) ds + 8\tau \gamma X(0) + 4V(0).$$

We now show that the solutions of the modified system (3.1) which is given by (3.2) converge to a flock, and the final flocking position pattern is the prescribed line $1$.

**Theorem 3.1.** Suppose that there exist some $\tau_0 \in (0, 1/\kappa \lambda^2)$ with $\kappa > 2$ and $\alpha > 0$ such that

$$4\sqrt{X(0)} + \alpha \cdot \max\{1, V(0)\} \leq \alpha \lambda \cdot \psi(\sqrt{\alpha N X(0) + N \alpha + 2\sqrt{E_0 \tau_0}}) < 1,$$

(3.3)

then there exists $\tau^* \in (0, \tau_0]$ such that for all $\tau \in (0, \tau^*], \text{ the solution } (x_i, v_i)^N_{i=1} \text{ of the self-organizing system (3.1) converges to a flock. In particular, the final flocking pattern is a line paralleled to } 1.$

**Remark 2.** It worthy to pointing out that the function $\psi$ in [24] takes the following detail form

$$\psi(r) = \frac{1}{(1 + r^2)^\beta} \text{ for } r \geq 0,$$

where $\beta$ is a constant satisfying $\int_0^\infty \psi(r) dr = \infty$. But we only acquire that the weight function $\psi$ in this paper is bounded, positive, non increasing and Lipschitz continuous on $\mathbb{R}^+$. Therefore, we generalize and improve the ones in [24]. Moreover, our arguments is different from those in [24].

### 3.2. The energy functional

In this subsection, we construct a suitable energy functional to derive the uniform bound on particle velocities. Indeed, we define the following energy functional,

$$\tilde{E}(t) = V(t) + \zeta \int_{t-\tau}^{t} W(s) ds d\theta + \gamma \left( X(t) - \sum_{i=1}^{N} [\langle x_i, 1 \rangle]^2 \right), \quad t \geq \tau,$$

(3.4)

where $\zeta > 0$, $V(t)$ and $W(t)$ have been defined in Section 2. Now we state a lemma which will be of use in our arguments.

**Lemma 3.2.** Let the parameters $\tau, \lambda > 0$, and $\zeta = \frac{2\gamma \lambda^3}{N}$, systems (3.1) and (1.3) have a unique global solution $(x_i, v_i)^N_{i=1}$ for $\tau \in (0, \tau_0]$ with some $\tau_0 \in (0, 1/\kappa \lambda)$, and $V(t) \leq \tilde{E}_0$ holds for all $t \geq 0$. Moreover,

$$|v_i(t)| \leq \sqrt{\tilde{E}_0}, \quad |v_i(t) - v_j(t)| \leq 2\sqrt{\tilde{E}_0}, \quad \forall t \geq 0, \quad i, j \in \{1, \ldots, N\}.$$
Similarly to Lemma 2.1, systems (3.1) and (1.3) have a unique local-in-time $C^1$-solution (denoted by $(x_i, v_i)_{i=1}^N$), which is indeed global in time if we can prove the uniform-in-time boundedness of the velocity $v_i$. Integrating two-sides of the first equation of system (3.1), we have for $t \in (0, \tau]$

$$|x_i(t) - x_i(0)| \leq \int_0^t |v_i(s)| ds. \quad (3.5)$$

Integrating two-sides of the second equation of system (3.1), by Cauchy-Schwarz’s inequality, we have for $t \in (0, \tau]$

$$|v_i(t)| \leq |v_i(t) - v_i(0)| + |v_i(0)|$$

$$= \frac{\lambda}{N} \sum_{j=1}^N \int_0^t \left|(\tilde{v}_j(s) - \tilde{v}_i(s))\right| ds + \int_0^t |F(x_i(s))| ds + |v_i(0)|$$

$$= \frac{\lambda \sqrt{T}}{N} \sum_{j=1}^N \left( \int_0^t \left|\tilde{v}_j(s) - \tilde{v}_i(s)\right|^2 ds \right)^{1/2} + \int_0^t |F(x_i(s))| ds + |v_i(0)|, \quad (3.6)$$

which, together with Cauchy-Schwarz’s inequality, the definition of $F(x_i)$ and (3.5), yields that

$$|v_i(t)|^2 \leq (|v_i(t) - v_i(0)| + |v_i(0)|)^2$$

$$\leq \frac{2\lambda^2 \tau}{N^2} \sum_{j=1}^N \left( \int_0^t \left|\tilde{v}_j(s) - \tilde{v}_i(s)\right|^2 ds \right)^{1/2} + 4 \left( \int_0^t |F(x_i(s))| ds \right)^2 + 4|v_i(0)|^2$$

$$\leq \frac{2\lambda^2 \tau}{N} \sum_{j=1}^N \int_0^t \left|\tilde{v}_j(s) - \tilde{v}_i(s)\right|^2 ds + 4\tau \int_0^t |F(x_i(s))|^2 ds + 4|v_i(0)|^2$$

$$\leq \frac{2\lambda^2 \tau}{N} \sum_{j=1}^N \int_0^\tau \left|\tilde{v}_j(s) - \tilde{v}_i(s)\right|^2 ds + 4\tau \gamma \int_0^t \left|\tilde{v}_i(s)\right|^2 ds + 4|v_i(0)|^2$$

$$\leq \frac{2\lambda^2 \tau}{N} \sum_{j=1}^N \int_0^\tau \left|\tilde{v}_j(s) - \tilde{v}_i(s)\right|^2 ds + 8\tau^3 \gamma \int_0^t \left|\tilde{v}_i(r)\right|^2 dr + 4|v_i(0)|^2$$

$$\leq \frac{2\lambda^2 \tau}{N} \sum_{j=1}^N \int_0^\tau \left|\tilde{v}_j(s) - \tilde{v}_i(s)\right|^2 ds + 8\tau^3 \gamma \int_0^t \left|\tilde{v}_i(r)\right|^2 dr + 8\tau^2 \gamma |x_i(0)|^2 + 4|v_i(0)|^2. \quad (3.7)$$

Thus, it follows from (3.7) and the definition of $V(t)$ that for $t \in (0, \tau]$

$$V(t) = \sum_{i=1}^N |v_i(t)|^2 \leq \frac{2\lambda^2 \tau}{N} \sum_{j=1}^N \int_0^\tau |\tilde{v}_j(s) - \tilde{v}_i(s)|^2 ds + 8\tau^3 \gamma \sum_{i=1}^N \int_0^t |v_i(r)|^2 dr + 8\tau^2 \gamma \sum_{i=1}^N |x_i(0)|^2 + 4 \sum_{i=1}^N |v_i(0)|^2$$

$$= 4\lambda^2 \tau \int_{-\tau}^0 V(s) ds + 8\tau^3 \gamma \int_0^t V(s) ds + 8\tau^2 \gamma X(0) + 4V(0). \quad (3.8)$$

Using Gronwall’s inequality, we have for $t \in (0, \tau]$

$$V(t) \leq \overline{C} + \overline{C} \frac{8\tau^3 \gamma e^{8\tau^3 \gamma (t-s)}}{1 - e^{8\tau^4 \gamma}} ds \leq \overline{C} e^{8\tau^4 \gamma}. \quad (3.9)$$
On the other hand, it follows from Cauchy-Schwarz’s inequality, the definition of \( V(t) \), (3.5) and (3.9) that

\[
X(t) = \sum_{i=1}^{N} |x_i(t)|^2 \leq 2 \sum_{i=1}^{N} \left( \int_{0}^{t} |v_i(s)| ds \right)^2 + 2 \sum_{i=1}^{N} |x_i(0)|^2 \\
\leq 2\tau \sum_{i=1}^{N} \int_{0}^{t} |v_i(s)|^2 ds + 2 \sum_{i=1}^{N} |x_i(0)|^2 \tag{3.10}
\]

Hence, systems (3.1) and (1.3) have a unique global solution \((x_i, v_i) \) on \((0, \infty) \). From the definition of \( V(t) \), we can easily obtain

\[
|v_i(t)| \leq \sqrt{E_0}, \quad |v_i(t) - v_j(t)| \leq 2\sqrt{E_0}, \quad \forall t \geq 0, \quad i, j \in \{1, \ldots, N\}.
\]

It remains to prove that \( E(t) \) is decreasing on \( t \geq \tau \). In fact, by differentiating the velocity fluctuation \( V(t) \) with respect to \( t \), using the conservation of total momentum, we have

\[
\frac{dV(t)}{dt} = 2 \sum_{i=1}^{N} \langle v_i(t), \dot{v}_i(t) \rangle = 2 \sum_{i=1}^{N} \left( v_i(t), \frac{\lambda}{N} \sum_{j=1}^{N} \bar{v}_{i,j}(t)(\bar{v}_j(t) - \bar{v}_i(t)) + F(x_i) \right) \\
= 2\lambda \sum_{i,j=1}^{N} \langle \bar{v}_i(t) - \bar{v}_i(t) + v_i(t), \bar{v}_{i,j}(t)(\bar{v}_j(t) - \bar{v}_i(t)) \rangle + 2 \sum_{i=1}^{N} \langle v_i(t), F(x_i) \rangle \\
= 2\lambda \sum_{i,j=1}^{N} \langle \bar{v}_i(t), \bar{v}_{i,j}(t)(\bar{v}_j(t) - \bar{v}_i(t)) \rangle - 2\lambda \sum_{i,j=1}^{N} \langle \bar{v}_i(t) - v_i(t), \bar{v}_{i,j}(t)(\bar{v}_j(t) - \bar{v}_i(t)) \rangle \\
+ 2 \sum_{i=1}^{N} \langle v_i(t), F(x_i) \rangle \\
\leq \frac{\lambda}{N} \sum_{i,j=1}^{N} \bar{v}_{i,j}(t)(\bar{v}_j(t) - \bar{v}_i(t))^2 - 2\lambda \sum_{i,j=1}^{N} \langle \bar{v}_i(t) - v_i(t), \bar{v}_{i,j}(t)(\bar{v}_j(t) - \bar{v}_i(t)) \rangle \\
+ 2 \sum_{i=1}^{N} \langle v_i(t), F(x_i) \rangle. \tag{3.11}
\]

Using the Young inequality with some \( \delta > 0 \) to be specified later, we have

\[
|2\langle \bar{v}_i(t) - v_i(t), \bar{v}_{i,j}(t)(\bar{v}_j(t) - \bar{v}_i(t)) \rangle| \leq \delta \bar{v}_{i,j}^2(t)|\bar{v}_j(t) - \bar{v}_i(t)|^2 + \frac{1}{\delta} |\bar{v}_i(t) - v_i(t)|^2.
\]

Hence,

\[
\frac{dV(t)}{dt} \leq \frac{(\delta-1)\lambda}{N} \sum_{i,j=1}^{N} \bar{v}_{i,j}(t)|\bar{v}_j(t) - \bar{v}_i(t)|^2 + \frac{\lambda}{\delta} \sum_{i=1}^{N} |\bar{v}_i(t) - v_i(t)|^2 + 2 \sum_{i=1}^{N} \langle v_i(t), F(x_i) \rangle. \tag{3.12}
\]

One hand, it follows from Cauchy-Schwarz’s inequality that
On the other hand,
\[
\frac{dX(t)}{dt} - \frac{d}{dt}\sum_{i=1}^{N}|x_i| = 2\sum_{i=1}^{N}(x_i - \langle x_i, 1 \rangle \cdot 1, v_i(t)) = -2\gamma \sum_{i=1}^{N} \langle F(x_i), v_i(t) \rangle.
\]
(3.14)

In view of (3.12)-(3.14), we derive
\[
\frac{d\tilde{E}(t)}{dt} = \frac{d}{dt} \left( V(t) + \zeta \int_{t-\tau}^{t} W(s)ds + \gamma (X(t) - \sum_{i=1}^{N}|x_i|) \right)
\leq \frac{(\delta - 1)\lambda}{N} \sum_{i,j=1}^{N} \tilde{\psi}_{i,j}(s) |\tilde{v}_j(t) - \tilde{v}_j(t)|^2 + \left( \frac{\tau\lambda^3}{\delta N} - \zeta \right) \int_{t-\tau}^{t} W(s)ds + \tau \zeta W(t)
= \left( \frac{(\delta - 1)\lambda}{N} + \tau \zeta \right) W(t) + \left( \frac{\tau\lambda^3}{\delta N} - \zeta \right) \int_{t-\tau}^{t} W(s)ds.
\]
(3.15)

Take \( \zeta = \frac{\tau\lambda^3}{\delta N} \), then it follows from (3.15) that
\[
\frac{d\tilde{E}(t)}{dt} \leq \left( \frac{(\delta - 1)\lambda}{N} + \frac{\tau^2\lambda^3}{\delta N} \right) W(t)
\leq \frac{\lambda}{N} \left( \frac{(\delta - 1) + \frac{1}{4\delta}}{N} \right) W(t).
\]
(3.16)

Take \( \delta = \frac{1}{2} \) then \( \frac{d\tilde{E}(t)}{dt} < 0 \) for any \( t > 0 \), which implies that the conclusions follows easily.

Now we simplify some formulations by denoting \( h_i^2 := |x_i|^2 - \langle x_i, 1 \rangle |^2 \) with \( h_i \geq 0 \), and \( h^2 := \gamma \sum_{i=1}^{N} h_i^2 \) with \( h \geq 0 \). In view of Lemma 3.2, we have
\[
\frac{dV(t)}{dt} + \frac{d}{dt} \left( \frac{\zeta}{N} \int_{t-\tau}^{t} W(s)ds + \delta \right) + \frac{dh^2(t)}{dt} \leq 0,
\]
from which we have
\[
V(t) + h^2(t) \leq \tilde{E}_0.
\]
(3.17)
Since \( V(t) \geq 0 \) for all \( t \in [0, +\infty) \), we can easily obtain \( h^2(t) \leq \tilde{E}_0 \) for all \( t \in [0, +\infty) \). Furthermore, there exists \( M > 0 \) such that \( h_i(t) \leq M \) for all \( i \) and \( t \in [0, +\infty) \). From the definition of \( h_i \), we know that if we project the modified system (3.1) into the plane which is perpendicular to \( \mathbf{1} \), then the projective system is bounded. It remains to consider the one-dimension space which is parallel with \( \mathbf{1} \). The modified system (3.1) is bounded only when the projective system is bounded in the one-dimension space. Now we introduce a new system by making inner-product both sides of system (3.1) by \( \mathbf{1} \),

\[
\begin{aligned}
\dot{x}_i &= v_i, \\
\dot{v}_i &= \lambda \sum_{j=1}^{N} \psi(|\tilde{x}_j - \tilde{x}_i|)(\tilde{v}_j - \tilde{v}_i), \quad i = 1, 2, ..., N, 
\end{aligned}
\tag{3.18}
\]

where \( \tilde{x}_i = \langle x_i, \mathbf{1} \rangle \), \( \tilde{v}_i = \langle v_i, \mathbf{1} \rangle \), \( \tilde{v}_i = \langle \tilde{v}_i, \mathbf{1} \rangle \), \( i = 1, 2, ..., N \). For the new system, we introduce similarly

\[
\begin{align*}
X(t) := & \sum_{i=1}^{N} x_i^2(t), \\
V(t) := & \sum_{i=1}^{N} v_i^2(t).
\end{align*}
\]

By the definition of \( h(t) \) and the fact that \( h^2(t) \leq \tilde{E}_0 \) for \( t \in [0, +\infty) \), we have

\[
X(t) \leq X(t) + \frac{\tilde{E}_0}{\gamma}. \tag{3.19}
\]

Using the same conditions and similar arguments as Theorem 2.2, we obtain the following lemma to the new system (3.18).

**Lemma 3.3.** Let \( (\tilde{x}_i, \tilde{v}_i)_{i=1}^{N} \) be a global solution to system (3.18). Suppose that there exist some \( \tau_0 \in (0, \frac{1}{2\lambda N}) \) and \( \alpha > 0 \) such that (2.7) holds, then there exists \( \tau^* \in (0, \tau_0] \) such that for all \( \tau \in (0, \tau^*) \) we have

\[
\sup_{-\tau < t < +\infty} X(t) < +\infty \quad \text{and} \quad \lim_{t \to +\infty} \sum_{i=1}^{N} V(t) = 0.
\]

The proof is very similar to that of Theorem 2.2, so we omit it. \( \square \)

### 3.3. The proof of Theorem 3.1.

We divide three steps to complete the proofs.

**Step 1.** We show that \( \lim_{t \to +\infty} v_i(t) = 0 \) for all \( i \).

Indeed, it follows from Proposition 3.3 that there exists \( C^* > 0 \) such that

\[
\sup_{-\tau < t < +\infty} X(t) < C^*.
\]

By (3.19), we have

\[
X(t) \leq C^* + \frac{\tilde{E}_0}{\gamma} \tag{3.20}
\]

for all \( t \in [-\tau, +\infty) \). Take \( \delta = \frac{1}{2} \) in (3.16), then we have for \( t \geq \tau \)

\[
\begin{align*}
\frac{d\tilde{E}(t)}{dt} & \leq \left( \frac{(\delta - 1)\lambda}{N} + \frac{\tau^2\lambda^3}{\delta N} \right) W(t) \\
& \leq \frac{\lambda \kappa^2 \delta^2 - \kappa^2 \delta + 1}{N} W(t) \\
& \leq \frac{\lambda 4 - \kappa^2}{N} W(t). \tag{3.21}
\end{align*}
\]
We deduce from the definition of $W$ and (3.21) that for $t \geq \tau$,
\[
\frac{dE(t)}{dt} \leq \frac{\lambda}{N} 4 - \kappa^2 \sum_{i,j=1}^{N} \bar{\psi}_{i,j}(t)|\bar{v}_j(t) - \bar{v}_i(t)|^2
\]
\[
\leq \frac{\lambda}{N} 4 - \kappa^2 \psi(\sqrt{2NX(t-\tau)}) \sum_{i,j=1}^{N} |\bar{v}_j(t) - \bar{v}_i(t)|^2
\]
\[
\leq \frac{\lambda}{N} 4 - \kappa^2 \psi(\sqrt{2N(C^* + \gamma^{-1}E_0)}) \sum_{i,j=1}^{N} |\bar{v}_j(t) - \bar{v}_i(t)|^2
\]
\[
= \lambda \frac{4 - \kappa^2}{\kappa^2} \psi(\sqrt{2N(C^* + \gamma^{-1}E_0)})V(t-\tau).
\]
Integrating both sides of formula (3.22) from $\tau$ to $+\infty$, we have
\[
\bar{E}(t) - \bar{E}(\tau) \leq \lambda \frac{4 - \kappa^2}{\kappa^2} \psi(\sqrt{2N(C^* + \gamma^{-1}E_0)}) \int_{\tau}^{+\infty} V(s-\tau)ds
\]
\[
= \lambda \frac{4 - \kappa^2}{\kappa^2} \psi(\sqrt{2N(C^* + \gamma^{-1}E_0)}) \int_{0}^{+\infty} V(t)dt.
\]
In view of Lemma 3.2, we know that the left-hand side of (3.23) is bounded. So the right-hand side $\int_{0}^{+\infty} V(t)dt$ must be bounded as well. Moreover, combining (3.12), (3.17) and (3.19), we deduce that $\frac{dV(t)}{dt}$ is bounded for any $t \in [0, +\infty)$. So, it is easy to see that
\[
\lim_{t \to +\infty} V(t) = 0.
\]
That is, $\lim_{t \to +\infty} v_i(t) = 0$ for all $i$.

**Step 2.** We prove that $\lim_{t \to +\infty} \int_{\tau}^{t} \int_{0}^{t} W(s)dsd\theta = 0$.
In fact, using integral mean value theorem, (1.5) and (2.3), we deduce that
\[
\int_{\tau}^{t} \int_{0}^{t} W(s)dsd\theta = \tau \int_{\tau}^{t} W(s)ds
\]
\[
= W(t')\tau(t-t')
\]
\[
= \tau(t-t') \sum_{i,j=1}^{N} \bar{\psi}_{i,j}(t'')|\bar{v}_j(t'') - \bar{v}_i(t'')|^2
\]
\[
\leq 2\tau^2 NV(t'' - \tau),
\]
where $t' \in (t-\tau, t)$ and $t'' \in (t', t)$. By the conclusion of Step 1, we can easily obtain $\lim_{t \to +\infty} V(t'' - \tau) = 0$. So we also get $\lim_{t \to +\infty} \int_{\tau}^{t} \int_{0}^{t} W(s)dsd\theta = 0$.

**Step 3.** We prove that $\lim_{t \to +\infty} F(x_i(t)) = 0$ for all $i$.
Since $\bar{E}(t)$ is monotonically decreasing for $t \geq \tau$, we see that $\lim_{t \to +\infty} \bar{E}(t)$ exists. Furthermore, by steps 1 and 2, we have
\[
\lim_{t \to +\infty} \left( V(t) + \int_{\tau}^{t} \int_{0}^{t} W(s)dsd\theta + h^2(t) \right) = \lim_{t \to +\infty} h^2(t).
\]
Now we claim $\lim_{t \to +\infty} h^2(t) = 0$. Assume on the contrary that there exists $c > 0$ such that $\lim_{t \to +\infty} h^2(t) = c$, then according to the definition of $h(t)$ and $h_i(t)$, there exists
It follows from the definition of \( h_i(t) \) satisfying
\[
\limsup_{t \to +\infty} h_i(t) = c_i^* \quad \text{for some } c_i^* > 0.
\]
(3.25)

This yields two cases on \( h_i \): one is \( \liminf_{t \to +\infty} h_i(t) = 0 \), the other is \( \liminf_{t \to +\infty} h_i(t) = c_i \) for some \( c_i > 0 \). For the first case, we take \( \tilde{c}_i \in (0, c_i^*) \) such that
\[
\frac{(c_i^* - \tilde{c}_i)^2}{2\gamma \tilde{c}_i \sqrt{C^* + \gamma^{-1} E_0}} < N^2.
\]
(3.26)

From the conclusion of Step 1, we deduce that there exists \( t_1 > 0 \) such that for \( t \in (t_1, +\infty) \),
\[
|v_i(t)| < \frac{(c_i^* - \tilde{c}_i)^2}{8N \sqrt{C^* + \gamma^{-1} E_0}}, \quad \left| \frac{\lambda N}{\sqrt{N}} \sum_{j=1}^{N} \psi(|\tilde{x}_j - \tilde{x}_i|)(\bar{v}_j(t) - \bar{v}_i(t)) \right| < \frac{\gamma \tilde{c}_i}{2}.
\]
(3.27)

According to (3.25), we can take \( t_2 \in (t_1, +\infty) \) such that
\[
c_i^* - h_i(t_2) < \frac{c_i^* - \tilde{c}_i}{2}.
\]
(3.28)

It follows from (3.25), \( \liminf_{t \to +\infty} h_i(t) = 0 \) and the continuity of \( h_i \), there exists \( t_3 \in (t_2, +\infty) \) satisfying
\[
h_i(t_3) = \tilde{c}_i \quad \text{and} \quad h_i(t) \neq \tilde{c}_i \quad \text{for } t \in (t_2, t_3).
\]
(3.29)

Combining (3.28) and (3.29), we have
\[
h_i(t_2) - h_i(t_3) \geq \frac{c_i^* - \tilde{c}_i}{2}.
\]
(3.30)

It follows from the definition of \( h_i(t) \), the mean value theorem, (3.20) and (3.27) that
\[
h_i(t_2) - h_i(t_3) = \frac{h_i^2(t_2) - h_i^2(t_3)}{h_i(t_2) + h_i(t_3)} \leq \frac{\langle |x_i(t_2)|^2 - |x_i(t_3)|^2 \rangle - (\langle |x_i(t_2)|, 1 \rangle^2 - |\langle x_i(t_3), 1 \rangle|^2)}{h_i(t_2) + h_i(t_3)} \leq \frac{2(|v_i(\xi_1)| |x_i(\xi_1)| + |x_i(\xi_2)||v_i(\xi_2)|)(t_3 - t_2)}{h_i(t_2) + h_i(t_3)} \leq \frac{2 \sqrt{C^* + \gamma^{-1} E_0}(|v_i(\xi_1)| + |v_i(\xi_2)|)(t_3 - t_2)}{c_i^* - \tilde{c}_i} \leq \frac{(c_i^* - \tilde{c}_i)(t_3 - t_2)}{2N}.
\]
(3.31)
where $\xi_1, \xi_2 \in (t_2, t_3)$. Combining (3.30) and (3.31), we have $t_3 - t_2 > N$. Then, from (3.27)-(3.29) we deduce that

$$v_i(t_3) - v_i(t_2) = \int_{t_2}^{t_3} \left( \frac{\lambda}{N} \sum_{j=1}^{N} \psi(|\tilde{x}_j(t) - \tilde{x}_i(t)|)(\tilde{v}_j(t) - \tilde{v}_i(t)) + F(x_i) \right) dt$$

$$\geq \int_{t_2}^{t_3} \left( \frac{\lambda}{N} \sum_{j=1}^{N} \psi(|\tilde{x}_j(t) - \tilde{x}_i(t)|)(\tilde{v}_j(t) - \tilde{v}_i(t)) \right) dt$$

$$\geq \int_{t_2}^{t_3} \frac{\gamma h_i(t)}{2} dt > \frac{N\gamma \tilde{c}_i}{2} > \frac{(c_i^* - \tilde{c}_i)^2}{4N \sqrt{C^* + \gamma^{-1} E_0}}.$$

which contradicts (3.27). For the second case, we can use similar analysis to obtain that $\liminf_{t \to +\infty} h_i(t) = c_i > 0$ is also impossible. Summarizing from the above two cases, we have $\lim_{t \to +\infty} h^2(t) = 0$. It is easy to see $\lim_{t \to +\infty} F(x_i(t)) = 0$ for all $i$. Thus there exists constant $k_i$ such that

$$\lim_{t \to +\infty} x_i(t) = k_i 1 \quad \text{for all } i.$$

Based on the above three steps, we see that the modified system (3.1) converges to a flock if (2.7) holds. Particularly, the final flocking pattern is a line parallelled to 1. This completes the proof. □

**Remark 3.** In view of Theorem 3.1, we see that the driving force $F$ does not change the final velocity of the flocking, but changes the final flocking pattern.

4. **Numerical experiments.** In this section we present several numerical experiments with regard to the dynamic behavior of the position and velocity for the particle system (1.2)-(1.3) with condition (1.4). We aim to illustrate analytical results on flocking and pattern motion in Sections 2 and 3, and demonstrate the evolution of $\lambda, \tau$ with respect to some initial conditions. This led to the conclusion that the analytically derived sufficient conditions for the asymptotic flocking and pattern motion are qualitatively right. More precisely, we first provide the numerical examples(Examples 4.1 and 4.2) to see whether there exist a exponential flocking for diameters and time differences. Then, we also provide several numerical examples (Examples 4.3 and 4.4) to see whether the information of the final motion patterns can be obtained for the modified delay C-S model with the final motion pattern driving force function, and to see whether the smallness of the delay time is necessary or not. For the sake of numerical simulations we will assume that $d = 1$ for Examples 4.1 and 4.2, and $d = 2$ for Examples 4.3 and 4.4. We use the standard explicit Euler scheme for the discretization in time and the method of steps to treat the delayed terms. The differential system in there examples to be integrated is discretized with $\Delta t = 0.01$.

**Example 4.1.** Take $N = 2$, $\lambda = 2$, $\tau = \tau_0 = \frac{1}{2}$, $\psi(t) \equiv 1$, $x_1(t) \equiv -0.5, x_2(t) \equiv 0.5$ for $t \in [-\tau, 0]$, $v_1(t) \equiv 2, v_2(t) \equiv -2$ for $t \in [-\tau, 0]$, then systems (1.2)-(1.3) reduced as

$$\begin{cases}
\dot{x}_1 = v_1, \\
\dot{x}_2 = v_2, \\
\dot{v}_1 = v_2(t - \tau) - v_1(t - \tau), \\
\dot{v}_2 = v_1(t - \tau) - v_2(t - \tau).
\end{cases}$$
By a direct calculation, we obtain $X(t) \equiv \frac{1}{2}, V(t) \equiv 8$ for $t \in [-\tau, 0]$, and

$$E_0 = 4\lambda^2 \int_{-\tau}^{0} V(r)dr + 2V(0) + \frac{2\tau \lambda^3}{N} \sum_{i,j=1}^{N} \int_{0}^{\tau} \int_{\theta}^{\tau} \tilde{\psi}_{i,j}(s)|\tilde{v}_j(s) - \tilde{v}_i(s)|^2dsd\theta$$

$$= \frac{128}{25} + 16 + 8\tau \sum_{i,j=1}^{2} \int_{0}^{\tau} \int_{\theta}^{\tau} \tilde{\psi}_{i,j}(s)|\tilde{v}_j(s) - \tilde{v}_i(s)|^2dsd\theta$$

$$= \frac{2768}{125}.$$ 

Moreover,

$$G(\alpha) = \frac{4\sqrt{X(0) + \alpha \cdot \max\{1, V(0)\}}}{\alpha \cdot \psi(2\sqrt{X(0) + \alpha + 2E_0\tau_0})}$$

$$= \frac{16\sqrt{1 + 2\alpha}}{\sqrt{2\alpha}}.$$ 

Based on above, we take $\alpha = 1000$, then $G(\alpha) < 1$ holds immediately. Thus, there exists some $\alpha > 0$ such that condition (2.7) in Thoerem 2.2 holds and implies the flocking occurs. Fig.1 shows the corresponding numerical simulation with initial value $x_1(t) \equiv -0.5, x_2(t) \equiv 0.5$ for $t \in [-0.2, 0], v_1(t) \equiv 2, v_2(t) \equiv -2$ for $t \in [-0.2, 0]$.

**Figure 1.** Time-domain behaviors of the state variables $x_1(t)$, $x_2(t)$, $v_1(t)$, $v_2(t)$ in Example 4.1.

**Example 4.2.** Take $N = 2, \lambda = 20, \psi(r) = \frac{1}{(1+r)^\beta}$, $\tau = \tau_0 = \frac{1}{25}$, $x_1(t) \equiv -1, x_2(t) \equiv 1$ for $t \in [-\tau, 0], v_1(t) \equiv 1, v_2(t) \equiv -1$ for $t \in [-\tau, 0]$, then systems (1.2)-(1.3) can be reduced as

\[
\begin{align*}
\dot{x}_1 &= v_1, \\
\dot{x}_2 &= v_2, \\
\dot{v}_1 &= \frac{10}{(1+|x_2(t-\tau)-x_1(t-\tau)|)^\beta}[v_2(t-\tau) - v_1(t-\tau)], \\
\dot{v}_2 &= \frac{10}{(1+|x_1(t-\tau)-x_2(t-\tau)|)^\beta}[v_1(t-\tau) - v_2(t-\tau)].
\end{align*}
\]
By a direct calculation, we obtain $X(t) = V(t) \equiv 2$ for $t \in [-\tau, 0]$, and
\[
E_0 = 4\lambda^2 \tau \int_{-\tau}^{0} V(r)dr + 2V(0) + \frac{2\tau\lambda^3}{N} \sum_{i,j=1}^{N} \int_{0}^{\tau} \int_{\theta}^{\tau} \tilde{\psi}_{i,j}(s)\tilde{v}_j(s) - \tilde{v}_i(s) |^2 dsd\theta
\]
\[
= \frac{132}{25} + 20^3\tau \sum_{i,j=1}^{2} \int_{0}^{\tau} \int_{\theta}^{\tau} \tilde{\psi}_{i,j}(s)\tilde{v}_j(s) - \tilde{v}_i(s) |^2 dsd\theta
\]
\[
= \frac{132}{25} + 20^3 \cdot 4 \cdot 3^{-\beta^2}.3.
\]
Moreover, if $\beta = \frac{1}{2}$ and $\alpha = 100$, then
\[
G(\alpha) = \frac{4\sqrt{X(0)} + \alpha \cdot \max\{1, V(0)\}}{\alpha \cdot \psi(2\sqrt{X(0)} + \alpha + 2\sqrt{E_0}\tau_0)}
\]
\[
= \frac{4\sqrt{X(0)} + \alpha \cdot \max\{1, V(0)\}}{\alpha \cdot \psi(2\sqrt{X(0)} + \alpha + 2\sqrt{E_0}\tau_0)}
\]
\[
= 0.1864 < 1,
\]
which implies that condition (2.7) in Theorem 2.2 holds. So it follows from Theorem 2.2 that the flocking occurs. Consider the initial value $x_1(t) \equiv 1, x_2(t) \equiv 1$ for $t \in [-\tau, 0], v_1(t) \equiv 0, v_2(t) \equiv 1$ for $t \in [-\tau, 0]$. As shown in Fig. 2, numerical simulation also confirms that the position diameter is uniformly bounded and velocity diameters tend to zero exponentially fast.

If $\beta = 1$ and $\alpha = 100$, then
\[
G(\alpha) = \frac{4\sqrt{X(0)} + \alpha \cdot \max\{1, V(0)\}}{\alpha \cdot \psi(2\sqrt{X(0)} + \alpha + 2\sqrt{E_0}\tau_0)}
\]
\[
= \frac{4\sqrt{X(0)} + \alpha \cdot \max\{1, V(0)\}}{\alpha \cdot \psi(2\sqrt{X(0)} + \alpha + 2\sqrt{E_0}\tau_0)}
\]
\[
= 0.8601 < 1.
\]

It is easy to see that condition (2.7) in Theorem 2.2 holds. The numerical simulation is shown in Fig. 3 which also confirms that the position diameter is uniformly bounded and velocity diameter tends to zero exponentially fast. That is, the system exhibits flocking behavior. It is worthy of pointing out that the systems in [21] for the case $\beta = 1$ only exhibit velocity diameter tends to zero exponentially fast, but the uniform bound of the particle group is not clear. Besides, our arguments are totally different from those in [21]. Based on the above facts, some flocking results of systems (1.2)-(1.3) in [21] are generalized by our theorem.

Observe that Examples 4.1 and 4.2 describe the case of two agents but with different interaction functions. In these examples, $E_0$ and $G$ in Theorem 2.2 can be obtained directly. We can take some parameters so that $G < 1$ which implies flocking can attain eventually, which is underpinned by the simulations.

**Example 4.3.** To gain a further understanding of the interesting phenomenon of asymptotic flocking behavior, we would like to simulate the case where pattern motion can be achieved under specific conditions as illustrated in the second part of the article. For computational convenience and demonstration purpose, we take some classical communication rate functions, i.e. $\psi(r) = \frac{1}{(1+r^2)^{1.2}}$. Moreover, we fix $N = 100, \lambda = 2, \tau = \tau_0 = 0.2$, and take $x_i(t)$ and $v_i(t)$ ($i = 1, 2, \ldots, N$) for $t \in [-\tau, 0]$ are generated randomly in the region $[0, 10] \times [0, 10]$. Similar to the analysis of Example 2.1, we know such $\alpha$ satisfying inequality (3.3) always exists to the modified C-S model (3.1) for all $\tau_0 < \frac{1}{2\lambda}$, and all conditions in Theorem 2.2 hold. Based on the previous analytical results, convergence to line-shaped flocking pattern can be obtained for any value of the delay satisfying $\tau_0 < \frac{1}{2\lambda}$. As expected,
Figure 2. Time-domain behaviors of the state variables \(x_1(t), x_2(t), v_1(t), v_2(t)\) for the case \(\beta = \frac{1}{2}\) in Example 4.2.

Figure 3. Time-domain behaviors of the state variables \(x_1(t), x_2(t), v_1(t), v_2(t)\) for the case \(\beta = 1\) in Example 4.2.

The simulation reveals in this case, flocking indeed occurs though we choose the extreme case where \(\tau = \tau_0\). See Figure 4. We also observe that while for \(\tau = \tau_0\) the agents show tendency to converge to a common velocity during the indicated time interval. Thus, the sufficient conditions in Theorem 3.1 for the asymptotic line-shaped flocking pattern motion are qualitatively right. Let us note that the parameter \(\lambda\) measures the influence strength among \(N\) agents. The larger \(\lambda\) is, the stronger influence the agents places on each other. Hence, our theoretical result with simulation reveals: the asymptotic line-shaped flocking pattern motion occurs for the larger \(\lambda\), and then for the smaller \(\tau\).

**Example 4.4.** We are now trying to simulate the modified C-S model (3.1) when the delay term is larger than the influence strength among agents, i.e., \(\tau > \frac{1}{\sqrt{N}}\). We also take the classical communication rate function \(\psi(r) = \frac{1}{(1+r^2)^{1/3}}\) as Example 4.3, Moreover, take \(N = 100, \lambda = 20, \tau = \tau_0 = 0.2\), and \(x_i(t)\) and \(v_i(t)\) \((i = 1, 2, \cdots, N)\) for \(t \in [-\tau, 0]\) are generated randomly in the region \([0, 10] \times [0, 10]\). A direct
Figure 4. The first is the initial position of each particle in the system; the second is the population distribution after iteration 200 (2s); the third is the population distribution after iteration 2000 (20s). The value of each parameter is given as: $N = 100$, $\lambda = 2$, $\tau = 0.2$, $\psi = \frac{1}{(1+r^2)^{1/3}}$, $\gamma = 1$.

calculation yields that the time-delay $\tau_0$ is larger than the threshold $\frac{1}{2\lambda}$, that is to say, not all conditions of Theorem 3.1 are satisfied. One can indeed observe that we have found in this case some particles in Figure 5 always oscillate in the neighborhood of the preset line, i.e., periodic solutions emerge. Convergence to the prescribed line-shaped pattern motion does not hold, which is different from dynamical behaviors stated in Example 4.3. Let us note that the results presented in Figure 5 do not contradict our analytical results. This simulation, as well as others that are not reported here, illustrates that the delay threshold $\frac{1}{2\lambda}$ is almost sharp.

We numerically explore the relationships among the flocking patterns, the pairwise influence function and the target motion pattern driving force function in Examples 4.3 and 4.4. In these examples, we find if the delay is smaller than the threshold $\tau_0$, then emergent flocking behavior with line-pattern can always be obtained, as expected, while if the delay is greater than the threshold $\frac{1}{2\lambda}$, flocking cannot be assured, in which case the system will oscillate in some range of the parameters.
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E-mail address: liuzhisu183@sina.com
E-mail address: liuyc2001@hotmail.com
E-mail address: mmxl@leeds.ac.uk