Comment on Mott Scattering in Strong Laser Field.

Y. Attaourti *, B. Manaut †

Laboratoire de Physique des Hautes Energies et d’Astrophysique,
Faculté des Sciences Semlalia, Université Cadi Ayyad, Marrakech, BP : 2390, Maroc.

Abstract

The first differential cross section for Mott scattering of a Dirac-Volkov electron is reviewed. The expression (26) derived by Szymanowski et al. [Physical Review A 56, 3846,(1997)] is corrected. In particular, we disagree with the expression of \( \frac{d\sigma}{d\Omega} \) they obtained and we give the exact coefficients multiplying the various Bessel functions appearing in the scattering differential cross section.

PACS number(s): 34.80.Qb, 12.20.Ds

1 Introduction

In a pioneering paper, Szymanowski et al. have studied the Mott scattering process in a strong laser field. The main purpose was to show that the modifications of the Mott scattering differential cross section for the scattering of an electron by the Coulomb potential of a nucleus in the presence of a strong laser field, can yield interesting physical insights concerning the importance and the signatures of the relativistic effects. Their spin dependent relativistic description of Mott scattering permits to distinguish between kinematics and spin-orbit coupling effects. They have compared the results of a calculation of the first Born differential cross section for the Coulomb scattering of the Dirac-Volkov electrons dressed by a circularly polarized laser field to the first Born cross section for the Coulomb scattering of spinless Klein-Gordon particles and also to the non relativistic Schrodinger-Volkov treatment. The aim of this comment is to provide the correct expression for the first-Born differential cross sections corresponding to the Coulomb scattering of the Dirac-Volkov electrons. One the one hand, We show that the terms proportional to \( \sin(2\phi_0) \) are missing in [1], where \( \phi_0 \) is the phase

* e-mail: attaourti@ucam.ac.ma
† e-mail: bmanaut@phfa.ucam.ac.ma
stemming from the expression of the circularly polarized electromagnetic field. The claim of [1] that they vanish is not true. These terms do not depend on the chosen description of the circular polarization in cartesian components. On the other hand, We perform the calculations with some details and throughout this work, we use atomic units ($\hbar = e = m = 1$) where $m$ denotes the electron mass. The abbreviation DCS stands for the differential cross section.

The organization of this paper is as follows: in Section 2, we establish the expression of the $S$-matrix transition amplitude as well as the formal expression of scattering DCS. In Section 3, we give a detailed account on the various trace calculations and show that indeed there is a missing term proportional to $\sin(2\phi_0)$ that is not equal to zero. This term as well as a term proportional to $\cos(2\phi_0)$ contribute to $\left(\frac{d\sigma}{d\Omega}\right)$ and multiply the product $J_{s+1}(z)J_{s-1}(z)$ where $J_s(z)$ is an ordinary Bessel function of argument $z$ and index $s$. The argument $z$ appearing in the above mentioned product will be defined later. Then, we carry out the derivation of the correct expression of the scattering DCS associated to the exchange of a given number of laser photons. We end by a brief a conclusion in Section 4.

2 The $S$-matrix element and the scattering differential cross section.

Exact solutions of relativistic wave equations [2] are very difficult to obtain. However, in seminal paper, Volkov [3] obtained the formal solution of The Dirac equation for the relativistic electron with 4-momentum $p^\mu$ inside a classical monochromatic electromagnetic field $A^\mu$. These solutions are called the relativistic Volkov states. The plane wave electromagnetic field $A^\mu$ of 4-momentum $k^\mu$ ($k_\mu k^\mu = k^2 = 0$) depends only on the argument $\phi = k.x = k_\mu x^\mu$ and therefore $A^\mu$ is such that:

$$A^\mu = A^\mu(k.x) = A^\mu(\phi)$$

The 4-vector $A^\mu$ satisfies the Lorentz gauge condition $\partial_\mu A^\mu = 0$ or equivalently $k_\mu A^\mu = 0$. The Dirac-Volkov equation in an external field $A_\mu$ is:

$$\left\{ \left(\hat{p} - \frac{i}{c} A\right)^2 - c^2 - \frac{i}{2c} F_{\mu\nu} \sigma^{\mu\nu} \right\} \psi(x) = 0$$

where $F_{\mu\nu}$ is the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$. The matrices $\gamma^\mu$ are the anticommuting Dirac matrices such that $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}1_4$ where $g^{\mu\nu}$ is the metric tensor $g^{\mu\nu} = diag(1, -1, -1, -1)$ and $1_4$ is the identity matrix in four dimensions.
The solutions of Eq. (2) are the relativistic Dirac-Volkov wave functions:

\[ \psi_p(x) = R(p) \frac{u(p,s)}{\sqrt{2p_0V}} e^{iS(x)} \]  

(3)

where:

\[ R(p) = \exp \left( \frac{k.A}{2c(k.p)} \right) = 1 + \frac{k.A}{2c(k.p)} \]  

(4)

and the function \( S(x) \) is given by:

\[ S(x) = -p.x - \int_0^{k.x} \frac{1}{c(k.p)} \left[ p.A(\xi) - \frac{1}{2c} A^2(\xi) \right] d\xi \]  

(5)

In Eq. (3), \( u(p,s) \) represents a Dirac bispinor which satisfies the free Dirac equation and is normalized according to

\[ u(p,s) u(p,s)^* == 2c^2. \]

We consider a circularly polarized field:

\[ A = a_1 \cos(\phi) + a_2 \sin(\phi) \]  

(6)

where \( \phi = k.x \). We choose \( a_1^2 = a_2^2 = a^2 = A^2 \) and \( a_1.a_2 = a_2.a_1 = 0 \). The Lorentz condition \( k.A = 0 \) implies \( a_1.k = a_2.k = 0 \). If one assumes that \( A^\mu \) is quasi-periodic so that its time average is zero \( A^\mu = 0 \), then using the Gordon identity, the averaged 4-current is easily obtained:

\[ j^\mu = \frac{1}{p_0} \left\{ p^\mu - \frac{1}{2c^2(k.p)} A^2 k^\mu \right\} \]  

(7)

If one sets:

\[ q^\mu = p^\mu - \frac{1}{2c^2(k.p)} A^2 k^\mu \]  

(8)

this yields:

\[ q.q = q^\mu q_\mu = m_*^2 c^2 \]  

(9)

with:

\[ m_*^2 = 1 - \frac{A^2}{c^4} \]  

(10)

One often calls the averaged 4-momentum \( q^\mu \) a quasi-impulsion. Note that \( q^\mu = (Q/c,q) \).

The quantity \( m_* \) plays the role of an effective mass of the electron inside the electromagnetic field. For the study of the process of Mott scattering in presence of a laser field, we use the Dirac-Volkov wave functions \( \psi_q(x) \) normalized in the volume \( V \):

\[ \psi_q(x) = R(p) \frac{u(p,s)}{\sqrt{2QV}} e^{iS(q,x)} \]  

(11)

where:

\[ R(p) = R(q) = 1 + \frac{1}{2c(k.p)} k.A = 1 + \frac{1}{2c(k.p)} (k.\phi_1 \cos(\phi) + k.\phi_2 \sin(\phi)) \]  

(12)
and:

\[
S(q, x) = -q \cdot x - \frac{(a_1 \cdot p)}{c(k \cdot p)} \sin(\phi) + \frac{(a_2 \cdot p)}{c(k \cdot p)} \cos(\phi)
\]

\[
= -q \cdot x - \frac{(a_1 \cdot q)}{c(k \cdot q)} \sin(\phi) + \frac{(a_2 \cdot q)}{c(k \cdot q)} \cos(\phi)
\]

(13)

We turn now to the calculation of the transition amplitude. The interaction of the dressed electrons with the central Coulomb field:

\[
A^\mu = \left( \frac{Z}{|x|}, 0 \right)
\]

(14)

is considered as a first-order perturbation. This is well justified if \(Z\alpha \ll 1\), where \(Z\) is the nuclear charge of the nucleus considered and \(\alpha\) is the fine-structure constant. We evaluate the transition matrix element for the transition \((i \rightarrow f)\):

\[
S_{fi} = \frac{iZ}{c} \int d^4x \overline{\psi}_{qf}(x) \frac{\gamma^0}{|x|} \psi_{qi}(x)
\]

(15)

We first consider the quantity:

\[
\overline{\psi}_{qf}(x) \frac{\gamma^0}{|x|} \psi_{qi}(x) = \frac{1}{\sqrt{2Q_f V}} \frac{1}{\sqrt{2Q_i V}} \overline{\psi}(p_f, s_f) \overline{\gamma}^0 R(p_f) \gamma^0 R(p_i) u(p_i, s_i) e^{-i(S_{qf,x} - S_{qi,x})}
\]

(16)

We have:

\[
e^{-i(S_{qf,x} - S_{qi,x})} = \exp[i(q_f - q_i) \cdot x - iz \sin(\phi - \phi_0)]
\]

(17)

where \(z\) is such that:

\[
z = \sqrt{\alpha_1^2 + \alpha_2^2}
\]

(18)

whereas the quantities \(\alpha_1\) and \(\alpha_2\) are given by:

\[
\alpha_1 = \frac{(a_1 \cdot p_i)}{c(k \cdot p_i)} - \frac{(a_1 \cdot p_f)}{c(k \cdot p_f)} , \quad \alpha_2 = \frac{(a_2 \cdot p_i)}{c(k \cdot p_i)} - \frac{(a_2 \cdot p_f)}{c(k \cdot p_f)}
\]

(19)

and the phase \(\phi_0\) is such that \(\phi_0 = \arccos(\alpha_1 / z) = \arcsin(\alpha_2 / z) = \arctan(\alpha_2 / \alpha_1)\). It is important at this stage to perform intermediate calculations in order to reduce the numbers of \(\gamma\) matrices that will appear when one calculates the scattering DCS. After some algebraic manipulations, one gets:

\[
\overline{\psi}(p_f, s_f) \overline{\gamma}^0 R(p_f) \gamma^0 R(p_i) u(p_i, s_i) = [C_0 + C_1 \cos(\phi) + C_2 \sin(\phi)] u(p_i, s_i)
\]

(20)

where the three coefficients \(C_0\), \(C_1\) and \(C_2\) are respectively given by:

\[
C_0 = \gamma^0 - 2k_0 a^2 \gamma^0 c(p_i) c(p_f)
\]

\[
C_1 = c(p_i) \gamma^0 k \phi_1 + c(p_f) \phi_1 k \gamma^0
\]

\[
C_2 = c(p_i) \gamma^0 k \phi_2 + c(p_f) \phi_2 k \gamma^0
\]

(21)
with \( c(p) = \frac{1}{2c(k,p)} \) and \( k_0 = k^0 = \omega/c \). Therefore, the transition matrix element becomes:

\[
S_{fi} = \frac{iZ}{c} \int d^4x \frac{1}{\sqrt{2Q_fV}} \frac{1}{\sqrt{2Q_iV}} \bar{u}(p_f, s_f) [C_0 + C_1 \cos(\phi) + C_2 \sin(\phi)] u(p_i, s_i) \\
\times \exp[i(q_f - q_i).x - iz\sin(\phi - \phi_0)]
\]

(22)

We now invoke the well-known identities involving ordinary Bessel functions \( J_s(z) \):

\[
\left\{ \begin{array}{c} \cos(\phi) \\ \sin(\phi) \end{array} \right\} e^{-iz\sin(\phi - \phi_0)} = \sum_{s=-\infty}^{\infty} \left\{ \begin{array}{c} B_s \\ B_{1s} \\ B_{2s} \end{array} \right\} e^{-is\phi}
\]

(23)

with:

\[
\left\{ \begin{array}{c} B_s \\ B_{1s} \\ B_{2s} \end{array} \right\} = \left\{ \begin{array}{c} J_s(z) e^{is\phi_0} \\ (J_{s+1}(z) e^{i(s+1)\phi_0} + J_{s-1}(z) e^{i(s-1)\phi_0})/2 \\ (J_{s+1}(z) e^{i(s+1)\phi_0} - J_{s-1}(z) e^{i(s-1)\phi_0})/2i \end{array} \right\}
\]

(24)

Evaluating the integrals over \( x_0 \) and \( x \) yields for \( S_{fi} \):

\[
S_{fi} = \frac{i4\pi Z}{\sqrt{2Q_iV} \sqrt{2Q_fV}} \sum_{s=-\infty}^{\infty} \frac{2\pi\delta(Q_f - Q_i - s\omega)}{|\mathbf{q}_f - \mathbf{q}_i - sk|^2} M_{fi}^{(s)}
\]

(25)

where the quantity \( M_{fi}^{(s)} \) is defined by:

\[
M_{fi}^{(s)} = \pi(p_f, s_f) [C_0 B_s + C_1 B_{1s} + C_2 B_{2s}] u(p_i, s_i)
\]

(26)

To evaluate the DCS, we first evaluate the transition probability per particle into final states within the range of momentum \( d\mathbf{q}_f \):

\[
dW_{fi} = |S_{fi}|^2 \frac{V d\mathbf{q}_f}{(2\pi)^3}
\]

\[
= \frac{(4\pi)^2 Z^2}{2Q_iV 2Q_fV} \sum_{s=-\infty}^{\infty} \frac{T2\pi\delta(Q_f - Q_i - s\omega)}{|\mathbf{q}_f - \mathbf{q}_i - sk|^4} |M_{fi}^{(s)}|^2 \frac{V d\mathbf{q}_f}{(2\pi)^3}
\]

(27)

where we have used the rule of replacement:

\[
[2\pi\delta(Q_f - Q_i - sw)]^2 \rightarrow 2\pi\delta(0)2\pi\delta(Q_f - Q_i - sw) = T2\pi\delta(Q_f - Q_i - sw)
\]

(28)

Next, we have for the transition probability per unit time:

\[
dR_{fi} = \frac{dW_{fi}}{T} = \frac{(4\pi)^2 Z^2}{2Q_iV 2Q_fV} \sum_{s=-\infty}^{\infty} \frac{2\pi\delta(Q_f - Q_i - s\omega)}{|\mathbf{q}_f - \mathbf{q}_i - sk|^4} |M_{fi}^{(s)}|^2 \frac{V d\mathbf{q}_f}{(2\pi)^3}
\]

(29)
Dividing $dR_{fi}$ by the flux of incoming particles:

$$|J^{inc}| = \frac{|q_i|c^2}{Q_iV}$$

(30)

then using the relation $|q_f|d|q_f| = \frac{1}{c^2}Q_f dQ_f$ and integrating over the final energy, we get for the scattering DCS:

$$\frac{d\sigma}{d\Omega_f} = \frac{Z^2 |q_f|}{c^4 |q_i|} \sum_{s=\infty}^{\infty} \frac{|M^{(s)}_{fi}|^2}{|q_f - q_i - sk|^4} \bigg|_{Q_f=Q_i+sw} = \sum_{s=-\infty}^{\infty} \frac{d\sigma^{(s)}}{d\Omega_f} \bigg|_{Q_f=Q_i+sw}$$

(31)

where:

$$\frac{d\sigma^{(s)}}{d\Omega_f} \bigg|_{Q_f=Q_i+sw} = \frac{Z^2 |q_f|}{c^4 |q_i|} \frac{|M^{(s)}_{fi}|^2}{|q_f - q_i - sk|^4}$$

(32)

The calculation is now reduced to the computation of traces of $\gamma$ matrices. This is routinely done using Reduce [4]. We consider the unpolarized DCS. Therefore, the various polarization states have the same probability and the actually measured DCS is given by summing over the final polarization $s_f$ and averaging over the initial polarization $s_i$. Therefore, the unpolarized DCS is formally given by:

$$\frac{d\sigma}{d\Omega_f} = \sum_{s=-\infty}^{\infty} \frac{d\sigma^{(s)}}{d\Omega_f} \bigg|_{Q_f=Q_i+sw}$$

(33)

where:

$$\frac{d\sigma^{(s)}}{d\Omega_f} \bigg|_{Q_f=Q_i+sw} = \frac{Z^2 |q_f|}{c^4 |q_i|} \frac{1}{|q_f - q_i - sk|^4} \frac{1}{2} \sum_{s_i} \sum_{s_f} |M^{(s)}_{fi}|^2$$

(34)

3 Trace calculations.

Since the controversy is very acute and precise about the results of the sum over the polarization $\frac{1}{2} \sum_{s_i} \sum_{s_f} |M^{(s)}_{fi}|^2$, we devote a whole section to the calculations of the various traces that intervene in the formal expression of the unpolarized DCS given by Eq. (34). We have to calculate:

$$\frac{1}{2} \sum_{s_i} \sum_{s_f} |M^{(s)}_{fi}|^2 = \frac{1}{2} \sum_{s_i} \sum_{s_f} |\overline{u}(p_f, s_f)| \left[ C_0 B_s + C_1 B_{1s} + C_2 B_{2s} \right] u(p_i, s_i)^2$$

$$= \frac{1}{2} \sum_{s_i} \sum_{s_f} |\overline{u}(p_f, s_f)\Lambda^{(s)} u(p_i, s_i)|^2$$

(35)
with:

\[
\Lambda^{(s)} = [\gamma^0 - 2k_0a^2k(c(p_i)c(p_f))]B_s^* + [c(p_i)\gamma_0^{0}k\phi_i + c(p_f)c(p_f)\phi_0]B_{1s}^* + [c(p_i)\gamma^0_0k\phi_2]B_{2s}^* \tag{36}
\]

using standard techniques of the $\gamma$ matrix algebra, one has:

\[
\frac{1}{2} \sum_{s_i} \sum_{s_f} |M_{f_i}^{(s)}|^2 = \frac{1}{2} Tr\{(\dot{\phi}f + c^2)\Lambda^{(s)}(\dot{\phi}c + c^2)\Lambda^{(s)}\} \tag{37}
\]

with:

\[
\Lambda^{(s)} = \gamma^0\Lambda^{(s)t}\gamma^0 = [\gamma^0 - 2k_0a^2k(c(p_i)c(p_f))]B_s^* + [c(p_i)\phi_1\gamma^0 + c(p_f)c(p_f)\phi_1]B_{1s}^* + [c(p_i)\phi_2\gamma^0 + c(p_f)c(p_f)\phi_2]B_{2s}^* \tag{38}
\]

There are nine main traces to be calculated. We write them explicitly:

\[
\begin{align*}
M_1 &= Tr\{(\dot{\phi}f + c^2)C_0(\dot{\phi}c + c^2)\overline{C_0}\}|B_s|^2 \\
M_2 &= Tr\{(\dot{\phi}f + c^2)C_0(\dot{\phi}c + c^2)\overline{C_1}\}B_sB_{1s}^* \\
M_3 &= Tr\{(\dot{\phi}f + c^2)C_0(\dot{\phi}c + c^2)\overline{C_2}\}B_sB_{2s}^* \\
M_4 &= Tr\{(\dot{\phi}f + c^2)C_1(\dot{\phi}c + c^2)\overline{C_0}\}B_{1s}B_{1s}^* \\
M_5 &= Tr\{(\dot{\phi}f + c^2)C_1(\dot{\phi}c + c^2)\overline{C_1}\}|B_{1s}|^2 \\
M_6 &= Tr\{(\dot{\phi}f + c^2)C_1(\dot{\phi}c + c^2)\overline{C_2}\}B_{1s}B_{2s}^* \\
M_7 &= Tr\{(\dot{\phi}f + c^2)C_2(\dot{\phi}c + c^2)\overline{C_0}\}B_{2s}B_{2s}^* \\
M_8 &= Tr\{(\dot{\phi}f + c^2)C_2(\dot{\phi}c + c^2)\overline{C_1}\}B_{1s}B_{2s} \\
M_9 &= Tr\{(\dot{\phi}f + c^2)C_2(\dot{\phi}c + c^2)\overline{C_2}\}|B_{2s}|^2
\end{align*} \tag{39}
\]

To simplify the notations, we will drop the argument of the various ordinary Bessel functions that appear. The diagonal terms give rise to:

\[
\begin{align*}
M_1 &\propto |B_s|^2 = J_s^2 \\
M_5 &\propto |B_{1s}|^2 = \frac{1}{2}(J_{s+1}^2 + 2J_{s+1}J_{s-1}\cos(2\phi_0) + J_{s-1}^2) \\
M_9 &\propto |B_{2s}|^2 = \frac{1}{2}(J_{s+1}^2 - 2J_{s+1}J_{s-1}\cos(2\phi_0) + J_{s-1}^2) \tag{40}
\end{align*}
\]

So, taking into account the fact that the traces multiplying $|B_s|^2$, $|B_{1s}|^2$ and $|B_{2s}|^2$ are not zero, one expects that terms proportional to $J_{s+1}J_{s-1}\cos(2\phi_0)$ will be present in the expression
of the scattering DCS. The first controversy between our work and the result of Szymanowski et al. [1] concerns the traces \( \mathcal{M}_6 \) and \( \mathcal{M}_8 \). Since:

\[
\mathcal{M}_6 \propto B_{1s}B^*_{2s} = \frac{i}{4}(J_{s+1}^2 - 2iJ_{s+1}J_{s-1}\sin(2\phi_0) - J_{s-1}^2) \\
\mathcal{M}_8 \propto B^*_{1s}B_{2s} = \frac{-i}{4}(J_{s+1}^2 + 2iJ_{s+1}J_{s-1}\sin(2\phi_0) - J_{s-1}^2)
\]

(41)

and with little familiarity with the \( \gamma \) matrix algebra, one can see at once that if the corresponding traces are not zero then the net contribution of \( \mathcal{M}_6 + \mathcal{M}_8 \) will contain a term proportional to \( J_{s+1}J_{s-1}\sin(2\phi_0) \). We shall demonstrate that in what follows. We have:

\[
\mathcal{M}_6 = Tr\{(\hat{p}_f c + c^2)C_1(\hat{p}_ic + c^2)\mathcal{C}_2\}B_{1s}B^*_{2s} \\
= Tr\{(\hat{p}_f c + c^2)[c(\hat{p}_i)\gamma^0\hat{k}\phi_1 + c(\hat{p}_f)\phi_1\gamma^0](\hat{p}_i c + c^2) \\
[c(\hat{p}_i)\phi_2\gamma^0 + c(\hat{p}_f)\gamma^0\hat{k}\phi_2]\}B_{1s}B^*_{2s}
\]

(42)

From now on, we define a 4-vector:

\[
\eta^\mu = (1, 0, 0, 0)
\]

(43)

We can therefore write:

\[
\gamma^0 = \eta
\]

(44)

Then, Eq. (41) becomes:

\[
\mathcal{M}_6 = Tr\{(\hat{p}_f c + c^2)C_1(\hat{p}_i c + c^2)\mathcal{C}_2\}B_{1s}B^*_{2s} \\
= Tr\{(\hat{p}_f c + c^2)[c(\hat{p}_i)\eta\hat{k}\phi_1 + c(\hat{p}_f)\phi_1\eta\gamma^0](\hat{p}_i c + c^2) \\
[c(\hat{p}_i)\phi_2\eta\gamma^0 + c(\hat{p}_f)\gamma^0\hat{k}\phi_2]\}B_{1s}B^*_{2s}
\]

(45)

In [1], the authors claim that the controversial \( \sin(2\phi_0) \) term disappear because it is proportional to terms like \( Tr\{(\hat{p}_f c + c^2)\gamma^0\hat{k}\phi_1(\hat{p}_i c + c^2)\phi_2\gamma^0\} \). This term as well as \( Tr\{(\hat{p}_f c + c^2)\phi_1\gamma^0(\hat{p}_i c + c^2)\gamma^0\hat{k}\phi_2\} \) are indeed zero but for \( Tr\{(\hat{p}_f c + c^2)\gamma^0\hat{k}\phi_1(\hat{p}_i c + c^2)\gamma^0\phi_2\} \) and \( Tr\{(\hat{p}_f c + c^2)\phi_1\gamma^0(\hat{p}_i c + c^2)\phi_2\gamma^0\} \) this is no longer true. These terms are not zero and we give explicitly their values:

\[
Tr\{(\hat{p}_f c + c^2)\gamma^0\hat{k}\phi_1(\hat{p}_i c + c^2)\gamma^0\phi_2\} = Tr\{(\hat{p}_f c + c^2)\phi_1\gamma^0(\hat{p}_i c + c^2)\phi_2\gamma^0\} \\
= 8w^2\{(a_1, p_f)(a_2, p_i) + (a_1, p_i)(a_2, p_f)\}
\]

(46)

In most case, the various traces are zero except when the cyclic process of taking scalar products of pairs comes to products such that:

\[
(k, \eta)(k, \eta)(a_1, p_i)(a_2, p_f) \\
(k, \eta)(k, \eta)(a_1, p_f)(a_2, p_i)
\]

(47)
in which case, one has contributions proportional to \(w^2(a_1.p_i)(a_2.p_f)\) and \(w^2(a_1.p_f)(a_2.p_i)\) respectively. Explicitly, we give the result for \(\mathcal{M}_6\) and \(\mathcal{M}_8\). One has:

\[
\mathcal{M}_6 = \frac{w^2}{c^2} (2 \sin(2\phi_0)) \left[ \frac{(a_1.p_i)(a_2.p_f)}{(k.p_i)(k.p_f)} + \frac{(a_2.p_i)(a_1.p_f)}{(k.p_i)(k.p_f)} \right] J_{s+1}J_{s-1} \\
+ i\{-[(a_1.p_i)(a_2.p_f) + (a_1.p_f)(a_2.p_i)]J_{s-1}^2 \}
+ \{(a_1.p_i)(a_2.p_f) + (a_1.p_f)(a_2.p_i)\}J_{s+1}^2) \] (48)

while \(\mathcal{M}_8\) is given by:

\[
\mathcal{M}_8 = \frac{w^2}{c^2} (2 \sin(2\phi_0)) \left[ \frac{(a_1.p_i)(a_2.p_f)}{(k.p_i)(k.p_f)} + \frac{(a_2.p_i)(a_1.p_f)}{(k.p_i)(k.p_f)} \right] J_{s+1}J_{s-1} \\
- i\{-[(a_1.p_i)(a_2.p_f) + (a_1.p_f)(a_2.p_i)]J_{s-1}^2 \}
+ \{(a_1.p_i)(a_2.p_f) + (a_1.p_f)(a_2.p_i)\}J_{s+1}^2) \] (49)

The fact that complex numbers appear in the expressions of \(\mathcal{M}_6\) and \(\mathcal{M}_8\) is not surprising since the former is the complex conjugate of the latter and their real sum is such that:

\[
\mathcal{M}_6 + \mathcal{M}_8 = \frac{4w^2}{c^2} \sin(2\phi_0) \left[ \frac{(a_1.p_i)(a_2.p_f)}{(k.p_i)(k.p_f)} + \frac{(a_2.p_i)(a_1.p_f)}{(k.p_i)(k.p_f)} \right] J_{s+1}J_{s-1} \] (50)

So, the first controversy is settled and there is indeed a term containing \(\sin(2\phi_0)\) in the expression of the scattering cross section. To put an end to any further criticism, we give in the Appendix the Reduce program we have written with the necessary commentaries and observations so that anyone in the scientific community having some knowledge of this powerful symbolic computational software can easily try it and check our results. Before writing our Reduce program, we have extensively studied the textbook by A. G. Grozin [5] which is full of worked examples in various fields of physics particularly in QED. We give the final result for the unpolarized DCS for the Mott scattering of a Dirac-Volkov electron:

\[
\frac{\sigma^{(s)}}{\sigma^{(f)}} = \frac{Z^2}{c^2} q_f \frac{1}{|q_f - q_i - s|} \times \frac{1}{c^2} \{J_s^2 A + (J_{s+1}^2 + J_{s-1}^2) B + (J_{s+1} J_{s-1} + J_s J_{s+1}) C + J_s (J_{s-1} + J_{s+1}) D \} \] (51)

where for notational simplicity we have dropped the argument \(z\) in the various ordinary Bessel functions. The coefficients \(A, B, C\) and \(D\) are respectively given by:

\[
A = c^4 - (q_f q_i)c^2 + 2Q_f Q_i - \frac{a^2}{2} \left( \frac{1}{k.q_f} - \frac{1}{k.q_i} \right) + \frac{a^2 \omega^2}{c^2(k.q_f)(k.q_i)}((q_f q_i) - c^2) + \frac{(a^2)^2 \omega^2}{c^4(k.q_f)(k.q_i)} + \frac{a^2 \omega^2}{c^2(Q_f - Q_i)} \left( \frac{1}{k.q_i} - \frac{1}{k.q_f} \right) \] (52)
\[
B = -\frac{(a^2)^2 \omega^2}{2c^2(k.q_f)(k.q_i)} + \frac{\omega^2}{2c^2} \left( \frac{(a_1.q_f)(a_1.q_i)}{(k.q_f)(k.q_i)} + \frac{(a_2.q_f)(a_2.q_i)}{(k.q_f)(k.q_i)} \right) - \frac{a^2}{2} + \frac{a^2}{4} \left( \frac{k.q_f}{(k.q_i)} + \frac{k.q_i}{(k.q_f)} \right) - \frac{a^2 \omega^2}{2c^2(k.q_f)(k.q_i)} \left( (q_f.q_i) - \epsilon^2 \right)
\]
\[
C = \frac{\omega^2}{c^2(k.q_f)(k.q_i)} \left( \cos(2\phi_0) \left\{ (a_1.q_f)(a_1.q_i) - (a_2.q_f)(a_2.q_i) \right\} + \sin(2\phi_0) \left\{ (a_1.q_f)(a_2.q_i) + (a_1.q_i)(a_2.q_f) \right\} \right)
\]
\[
D = \frac{c}{2} \left( (\dot{A}.q_i) + (\dot{A}.q_f) \right) - \frac{c}{2} \left( \frac{k.q_f}{(k.q_i)}(\dot{A}.q_i) + \frac{k.q_i}{(k.q_f)}(\dot{A}.q_f) \right) + \frac{\omega}{c} \left( \frac{Q_i(\dot{A}.q_f)}{(k.q_f)} + \frac{Q_f(\dot{A}.q_i)}{(k.q_i)} \right)
\]

where \( \dot{A} = a_1 \cos(\phi_0) + a_2 \sin(\phi_0) \).

3.1 Comparison of the coefficients.

The argument about the missing term proportional to \( \sin(\phi_0) \) having been given a convincing explanation, we now turn to other remarks along the same lines since there are indeed other differences between our result and the result of \([1]\). We discuss now the difference occurring in our expression of the coefficient \( A \) and the corresponding one of \([1]\). To make the comparison easier we give explicitly the simple relations between our coefficients and the corresponding coefficients of \([1]\). One has: \( A([1]) = A/c^2 \), \( B([1]) = 2B/c^2 \), \( C([1]) = 2C/c^2 \) and \( D([1]) = 2D/c^2 \). In their expression multiplying the product \( 2J^2_n(\xi) \), the single term \( \frac{(a^6)^2 \omega^2}{c^2(k.q)(k.q_f)} \) should come with a coefficient \( \frac{1}{2} \). In the appendix, we give a second Reduce program that allows the comparison between the coefficient \( A \) of \([1]\) and the coefficient \( A \) of this work. There are so many differences between our result and the result they found for the coefficient \( B \) that we refer the reader to our main Reduce program. The coefficient \( C \) has already been discussed. As for the coefficient \( D \), we have found an expression that is linear in the electromagnetic potential. In the appendix, we give a third Reduce program. It is shown explicitly that if we ignore the first term in the coefficient multiplying \( J_s(J_{s-1} + J_{s+1}) \) given in \([1]\), one easily gets the result we have obtained. This term does not come from the passage from the variables \( (p, \tilde{p}) \) to the variable \( (q, \tilde{q}) \). The introduction of such 4-vector \( \tilde{q} \) is not useful, makes the calculations rather lengthy and gives rise to complicated expressions. As a supplementary consistency check of
our procedure used in writing the main Reduce program, we have reproduced the result of the DCS corresponding to the Compton scattering in an intense electromagnetic field given by Berestetzkii, Lifshitz and Pitaevskii \[6\].

4 Conclusion.

In this comment, we derived the correct expression of the first Born differential cross section for the scattering of the Dirac-Volkov electron by a Coulomb potential of a nucleus in the presence of a strong laser field. We have given the correct relativistic generalization of the Bunkin and Fedorov treatment \[7\] that is valid for an arbitrary geometry. To prove that our results are correct, we give the Reduce program to let the scientific community judge their accuracy. We are adamant that the core of the whole controversy stems from the fact that in \[1\], the vector $\eta^\mu$ introduced in Eq. (43) of our work has not been properly dealt with while it is the common method to use when a trace contains a $\gamma^0$ matrix. Any standard QED textbook introduces this very elementary method.

Appendix

We give the main Reduce program that calculates the traces in Eq. (37). For this program to be readable, before every line, we give an explanation of the different instructions. Some are obvious, other are less straightforward and we also give the number of the equation to which it refers in the text wherever it is possible. In a Reduce program, a commentary is preceded by the symbol \%. A Reduce instruction is not preceded by any symbol.

The main program

% This program calculates the trace appearing in Eq. (37) of the text.
% The result must be multiplied by 4. Reduce calculates the quarter of any trace.
% This is well explained in the manual \[4\].
% We first define the vector $p_f$, $p_i$, $a$, $a_1$, $a_2$, $k$, $\eta$ (stands for nu), $q_i$, $q_f$.
vector pfin, pin, aa, a1, a2, k, nu, qin, qfin;
% The command mass associates the relevant scalar variable as a mass with.
% The corresponding vector. In the next instruction, cv stands for the velocity of light.
mass k=0, pfin=cv, pin=cv;
% The command mshell put a particle 'on the mass shell'.
% A substitution <vector variable> = <mass>**2 is set up.

mshell k, pfin, pin;

% The results of the above instruction are: \( k^2 = 0 \), \( p_f^2 = c^2 \) and \( p_i^2 = c^2 \).

on div;

% We define properties of the vector \( \eta \) introduced in Eq. (43).

let nu.nu=1, nu.k=w/cv, k.nu=w/cv;

let nu.pfin=efin/cv, pfin.nu=efin/cv, nu.pin=ein/cv, pin.nu=ein/cv;

let nu.a1=0, a1.nu=0, nu.a2=0, a2.nu=0;

% We define the Maxwell gauge condition.

let a1.k=0, k.a1=0, a2.k=0, k.a2=0;

% We define the properties of the electromagnetic field potential.

% We cannot use the variable \( a \) in our program.

% Reduce interprets it as the matrix \( \gamma^5 \).

let a1.a1=aa.aa, a2.a2=aa.aa, a1.a2=0, a2.a1=0;

% We define \( \langle p c + c^2 \rangle \). The variable 'l' denote the fermionic line.

for all p let gp(p)=cv*g(l,p)+cv**2;

% We define the properties of the various quantities stemming from Eq. (24).

% impart denotes the imaginary part and repart denotes the real part.

% js, jsp1 and jsm1 denote respectively \( J_s, J_{s+1} \) and \( J_{s-1} \).

let impart(js)=0, impart(jsp1)=0, impart(jsm1)=0;

let repart(js)=js, repart(jsp1)=jsp1, repart(jsm1)=jsm1;

let impart(s)=0, impart(phi0)=0;

let repart(s)=s, repart(phi0)=phi0;

% We define the quantities of Eq. (24).

bs:=js*exp(i*s*phi0);

b1s:=(jsp1*exp(i*(s+1)*phi0)+jsm1*exp(i*(s-1)*phi0))/2;

b2s:=(jsp1*exp(i*(s+1)*phi0)-jsm1*exp(i*(s-1)*phi0))/(2*i);

repart(bs):=js*cos(s*phi0);

impart(bs):=js*sin(s*phi0);

repart(b1s):=(jsp1*cos((s+1)*phi0)+jsm1*cos((s-1)*phi0))/2;

impart(b1s):=(jsp1*sin((s+1)*phi0)+jsm1*sin((s-1)*phi0))/2;

repart(b2s):=(jsp1*sin((s+1)*phi0)-jsm1*sin((s-1)*phi0))/2;

impart(b2s):=(jsm1*cos((s-1)*phi0)-jsp1*cos((s+1)*phi0))/2;

% We ask Reduce not to perform the various traces for the time being.
% We define the various products appearing in Eq. (37).
\[ t_1 := \text{gp}(p_{\text{fin}}); \]
\[ t_2 := (g(l, \nu) - 2c \cdot cp \cdot (aa \cdot aa) \cdot (k \cdot \nu) \cdot g(l, k)) \cdot b_s; \]
\[ t_3 := (c \cdot g(l, \nu) \cdot g(l, k) \cdot g(l, a_1) + cp \cdot g(l, a_1) \cdot g(l, k) \cdot g(l, \nu)) \cdot b_{1s}; \]
\[ t_4 := (c \cdot g(l, \nu) \cdot g(l, k) \cdot g(l, a_2) + cp \cdot g(l, a_2) \cdot g(l, k) \cdot g(l, \nu)) \cdot b_{2s}; \]
\[ t_5 := \text{gp}(p_{\text{in}}); \]
\[ t_6 := (g(l, \nu) - 2c \cdot cp \cdot (aa \cdot aa) \cdot (k \cdot \nu) \cdot g(l, k)) \cdot \text{conj}(b_s); \]
\[ t_7 := (cp \cdot g(l, \nu) \cdot g(l, k) \cdot g(l, a_1) + c \cdot g(l, a_1) \cdot g(l, k) \cdot g(l, \nu)) \cdot \text{conj}(b_{1s}); \]
\[ t_8 := (cp \cdot g(l, \nu) \cdot g(l, k) \cdot g(l, a_2) + c \cdot g(l, a_2) \cdot g(l, k) \cdot g(l, \nu)) \cdot \text{conj}(b_{2s}); \]
% To obtain compact expressions, we define:
for all \( fi \), let \( \cos(fi) + i \cdot \sin(fi) = \exp(i \cdot fi) \);
for all \( fi \), let \( \cos(fi) - i \cdot \sin(fi) = \exp(-i \cdot fi) \);
% We explicitly define the coefficients \( c(p_i) = c(q_i) \) and \( c(p_f) = c(q_f) \).
\[ c := 1/(2 \cdot cv \cdot (k \cdot q_{\text{in}})); \]
\[ cp := 1/(2 \cdot cv \cdot (k \cdot q_{\text{fin}})); \]
% We define the product \( J_s(J_{s-1} + J_{s+1}) \) of Bessel functions.
let \( js \cdot j_{s-1} + js \cdot j_{s+1} = js_{\text{oim}}; \)
% We define the relation between the 4-vector \( p \) and \( q \).
let \( p_{\text{in}} = q_{\text{in}} + (aa \cdot aa) \cdot c \cdot k / cv \);
let \( p_{\text{fin}} = q_{\text{fin}} + (aa \cdot aa) \cdot cp \cdot k / cv \);
% Same definition for the energy.
let \( e_{\text{in}} = g_{qi} + (aa \cdot aa) \cdot c \cdot w / cv \);
let \( e_{\text{fin}} = g_{qf} + (aa \cdot aa) \cdot cp \cdot w / cv \);
% We load the package ASSIST [4] that simplifies the calculations.
load_package assist;
% We now ask Reduce to calculate the traces related to the fermionic line 'l'.
spur l;
% Each trace \( \text{res}_i \) is calculated separately corresponds to the trace \( \mathcal{M}_i \) of Eq. (39).
\[ \text{res}_1 := t_1 \cdot t_2 \cdot t_5 \cdot t_6; \]
\[ \text{trigreduce \ res}_1; \]
\[ \text{res}_2 := t_1 \cdot t_2 \cdot t_5 \cdot t_7; \]
\[ \text{trigreduce \ res}_2; \]
\[ \text{res}_3 := t_1 \cdot t_2 \cdot t_5 \cdot t_8; \]
trigreduce res3;
res4:=t1*t3*t5*t6;
trigreduce res4;
res5:=t1*t3*t5*t7;
trigreduce res5;
res6:=t1*t3*t5*t8;
trigreduce res6;
res7:=t1*t4*t5*t6;
trigreduce res7;
res8:=t1*t4*t5*t7;
trigreduce res8;
res9:=t1*t4*t5*t8;
trigreduce res9;

% The total trace is the sum of all these traces.
restot:=res1+res2+res3+res4+res5+res6+res7+res8+res9;
trigreduce restot;

% The program is complete and Reduce takes seconds to give the answer.

The most controversial terms correspond to res6 and res8. It is very easy to check that our claim is well founded and that there is indeed a term proportional to \( \sin(2\phi_0) \) that is missing in [1]. We now turn to the Reduce program that shows that the single term \( \frac{(a^2)^2w^2}{c^2(k.q)(k.q')} \) should come with a coefficient \( \frac{1}{2} \).

The coefficient A.

This coefficient comes from the calculation of \( \mathcal{M}_1 \) in Eq. (33). The coefficient A of [1] is transformed so that no vector \( \tilde{q} \) appears anymore. The symbols qit and qft stand for \( \tilde{q}_i \) and \( \tilde{q}_f \) respectively. The instruction f1:=c1 is the correct instruction that allows to find exactly the same result as that given in Eq. (52).

on div;

vector qi, qf, qit, qft, k;
a1:=c**2-qi.qf+2*gqi*gqf/c**2;
b1:=aa**2*(k.qft/k.qi+k.qit/k.qf)/(2*c**2);
c1:=aa**4*w**2/(2*c**4*(k.qi)*(k.qf));
d1:=1-aa**2*w**2/(c**4*(k.qi)*(k.qf));
e1:=-aa**2*(1-k.qft*k.qit/(k.qi*k.qf))/(2*c**2);
% Here, we use the fact that there is a factor 1/2 in the above mentioned single term.
fl:=c1;
g1:=aa**4*w**2*(k.qf*k.qft+k.qi*k.qit+aa**2*w**2/c**4);
h1:=g1/(2*c**6*(k.qi)**2*(k.qf)**2);
k.qit:=-k.qi+2*w*gqi/c**2;
k.qft:=-k.qf+2*w*gqf/c**2;
res:=(a1+b1+c1)*d1+e1+f1+h1;

This program gives exactly our coefficient $A$ given in Eq. (52)

The coefficient $D$.

For the coefficient $D$ of [1], we give the Reduce program omitting the first term that contains a factor that is quadratic in $a$. Doing so, we find exactly the same expression as that given in Eq. (55).

on div;
vector qi, qf, qit, qft, k, aron;
term1:=0;
term2:=aron.qi*k.qit/(c*k.qf)+aron.qi*k.qft/(c*k.qi);
term3:=aron.qi/c+aron.qf/c;
k.qit:=-k.qi+2*w*gqi/c**2;
k.qft:=-k.qf+2*w*gqf/c**2;
resd:=term1+term2+term3;

We have given convincing arguments to support our results. We also gave programs that will allow anyone to check every stage of our reasoning and to reach the same conclusion: there are indeed mistakes in [1].

References

[1] C. Zsymanowski, V. Véniard, R. Taïeb, A. Maquet and C.H. Keitel, Physical Review A, 56, 3846, 1997.

[2] V. G. Bagrov and D. M. Gitman, Exact solution of relativistic Wave Equations (Kluwer Academic Publishers, Dordrecht, 1990)

[3] D. M. Volkov, Z. Phys, 94, 250, (1935).
[4] A. C. Hearn, Reduce User’s and Contributed Packages Manual, Version 3.7 (Konrad-Zuse-Zentrum für Informationstechnik, Berlin, 1999).

[5] V. Berestetzkii, E. M. Lifshitz and L. P. Pitaevskii, Quantum Electrodynamics, 2nd ed. (Pergamon Press, Oxford, 1982).

[6] A. G. Grozin, Using Reduce in Hight Energy Physics (Cambridge University Press, 1997).

[7] F.V. Bunkin and M.V. Fedrov, Zh Eksp. teor. Fiz. 49, 1215, (1965)[Sov. Phys. JETP 22, 844 (1966)]