Graphic statics: projective funicular polygon

Athanasios A. Markou *, Gengmu Ruan

Department of Civil Engineering, Aalto University, Rakentajanaukio 4A, Espoo FI-00076, Finland

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Abstract

Since its birth, around the mid-19th century, graphic statics has been through several stages, with the most recent one being its “renaissance” phase, due to mainly the development of digital graphical tools, after almost being abandoned during the last two thirds of the 20th century. The power of graphic statics rises from the reciprocity between its two main pillars, namely the force polygon and the funicular polygon. These two pillars, tend to represent in the modern era two distinct disciplines: architectural (form) and structural engineering (force). To this end, graphic statics can act as the common ground, as we move, potentially, towards the direction of the multidisciplinary master-builder approach. The majority of graphic statics applications, in its “renaissance” phase, reported in the literature is related to form-finding of axially-loaded statically determinate structures. In the current work, we suggest a so-called projective funicular polygon, that is necessary for the structural analysis of statically indeterminate beams with inclined loads that impose an axial force component to the beam. In addition, the construction of the projective funicular polygon explicitly defines the bending moments, shear forces and axial forces in the beam and therefore provides information regarding an efficient cross-sectional variation of the beam element, while the classical funicular polygon provides information regarding an efficient alternative structural form. Finally, apart from highlighting that graphic statics is a powerful tool not only for form-finding, but for structural analysis as well, it is worth emphasizing that the information provided by the polygons (force, funicular, projective) can help us generate insights in the light of geometry towards more efficient and sustainable structures.

1. Introduction

Even though the roots of graphic statics can be traced back to the studies of (i) composition of forces by Leonardo da Vinci and Galileo Galilei [1], (ii) parallelogram law by Simon Stevin [2] and (iii) force and motion by Isaac Newton [3], the force polygon and the funicular polygon, graphic statics’ two main pillars, were presented for the first time in 1725 by Pierre Varignon in his Nouvelle Mecanique ou Statique [4]. Few decades later in 1748, Giovanni Poleni published his graphical analysis of the dome of St. Peter’s Basilica in Rome [5]. Almost a century after Nouvelle Mecanique ou Statique, in 1823 Lame and Clapeyron used the force and the funicular polygon [6] to analyze the dome of St. Isaac’s Cathedral in Saint Petersburg [7]. Surprisingly enough, it was the work on earth pressure theory [8] by Jean-Victor Poncelet, through the development of projective geometry, the geometric statics by August F. Möbius [9] and the graphical analysis by Barthélémy-Édouard Cousinery [10], that led Karl Culmann establish the field of graphic statics with his monograph Die Graphische Statik in 1864 [11].

It is around this time that the preparatory phase of graphic statics ends and we enter in its development phase. During this phase, James Clerk Maxwell published his work on the reciprocity between form and force diagram [12] and on the graphic analysis of trusses [13], while Macquorn Rankine proposed a force diagram comprising polyhedral forms [14]. In 1868, Otto Mohr developed a method for the calculation of the deflection curve for beams [15] and this opened the way for solving statically indeterminate systems. In 1872, Luigi Cremona published his Le figure reciproche nella statica grafica, a work on graphic statics for trusses [16]. One year later, Robert H. Bow published his extensive work on the graphic analysis of trusses [17]. Additionally, Bow introduced the notation system that is used nowadays.

In the years to come, graphic statics enters its spread phase, during which various reference textbooks have been published in different countries, some of which are: in France by Maurice Levy [18], in UK by James B. Chalmers [19], in Germany by August Föppl [20] and by Heinrich Müller-Breslau [21], in Switzerland by Wilhelm Ritter [22], in U.S.A. by William Wolfe [23], in India by R. Nanjundaya [24]. The last
two thirds of the 20th century graphic statics enters its abandoned phase, due to mainly the development of frame-type bearing systems with reinforced concrete that could be numerically solved with the use of linear algebra [7].

At the beginning of the 21st century the “renaissance” phase of graphic statics starts. Due to the advancement in computer-aided design, graphic statics has been mainly used as a design tool, particularly for architects, because of its ability of visualizing form and force in real time. Besides standard textbooks by Allen, Zalewski and Saliklis [1,25], the current progress on graphic statics appears in the design of mainly two types of structural systems: funicular and truss. Within the funicular group of structures, graphic statics is mainly used as a tool for equilibrium (form-finding) and stability. More specifically, Block et al. [26] studied arch behavior with graphical means, while Block and Ochendorf [27] presented a methodology for equilibrium of three-dimensional shell structures. Beghini et al. [28] studied form-finding of cable nets for a canopy application. Within the truss group of structures, Beghini et al. [29] studied structural topology optimization of trusses with the use of graphic statics. McRobie et al. [30] presented a graphical method to determine rigidity and stability of prestressed trusses. Akbarzadeh et al. [31] presented an extension of graphic statics into the third dimension by using polyhedral form and force diagrams for trusses. Finally, Schwartz [32] used strut-and-tie models for the analysis of concrete structures for digital design fabrication.

Most of the recent research topics related to graphic statics, are linked with axially-loaded statically determinate structures, while the most recent research related to combined bending, shear and axial response of statically indeterminate structures dates back almost a century [23]. To this end, the purpose of the current work is related to beams with inclined loads with any boundary conditions and is three-fold: (i) presentation of projective funicular polygon, (ii) axial force and shear force diagrams extracted from force polygon and bending moment diagrams extracted from the projective funicular polygon and (iii) analysis of statically indeterminate beams. In the next section application examples of the projective funicular polygon are presented for statically determinate and indeterminate beams. Through the lenses of sustainability the projective funicular polygon provides information about a variable cross-section beam element. The funicular polygon demonstrates the flow of forces in the beam element and therefore suggests an alternative more efficient structural system. Finally, the force polygon, can replace the traditional axial force, shear force and bending moment diagrams.

2. Inclined loads/beams
2.1. Statically determinate beams

In this section, the process of deriving the projective funicular polygon for statically determinate beams is described along with three examples. All terminology required has been defined in Fig. 1. The general process can be divided into the following steps:

Step 1: Derive the reaction forces.
Step 2: Construct the force polygon, which represents equilibrium of forces, \( \Sigma F_x = \Sigma F_y = 0 \).
Step 3: Construct the funicular polygon, which represents equilibrium of moments, \( \Sigma M = 0 \).
Step 4: Construct the projective funicular polygon as follows:

(i) Select the position of the pole, so that the closing string is parallel to the neutral axis of the beam. Note that in Fig. 1 the closing string is selected non parallel to the neutral axis of the beam for demonstration purposes.
(ii) From the endpoint of the ray of the closing string (not the pole) in the force polygon draw a perpendicular line to the closing string (which is parallel to the neutral axis of the beam).

(iii) Project the rest of the points of the force polygon to the perpendicular line drawn in step (ii).

(iv) Join the points determined in step (iii) with the pole of the force polygon.

(v) Draw the projective funicular polygon in the form diagram by starting with the rays between the projection points and the pole (defined in step (iv)), and the action lines of the components of the external loads that act perpendicular to the neutral axis of the beam.

All examples presented here have been solved using a parametric software called Geogebra, [33]. The first example for the statically determinate structures, deals with the case of a beam loaded with the inclined load $F$, shown in Fig. 2. Firstly, the reactions are calculated by recalling that when three forces are in equilibrium, they need to be concurrent and therefore all action lines of forces (two reactions and one external load) will intersect at the same point. Additionally, due to the fact that the action line of the roller support is known, the intersection between the action line of the roller and the external load determines uniquely the action line of the hinge support.

In this way, the force polygon is drawn and the magnitude and direction of the support reactions are specified. The next step is to select the pole $p$, so that the closing string coincides with the neutral axis of the beam. The main funicular polygon $1 \rightarrow 2 \rightarrow 3$ is constructed parallel to rays $ap, bp$ and $cp$ in the form diagram.

From point $c$ in the force polygon, we draw a line perpendicular to the ray of the closing string and then we project all points on it. In this way point $a'$ is defined, while points $b'$ and $c'$ coincide with points $b$ and $c$. The rays of the projective funicular polygon are drawn by joining points $a', b'$ and $c'$ to the pole $p$. The next step is to draw the projective funicular polygon $1 \rightarrow 2 \rightarrow 3$ in the form diagram. During this process, only the action line of the component of the external load that acts perpendicular to its neutral axis of the beam, namely $F_p$, is used.

The bending moment can be calculated with three different ways. The first way is to use the main funicular polygon $1 \rightarrow 2 \rightarrow 3$ and the vertical distance $v$. The vertical distance $v$ for the segment $1 \rightarrow 2$ of the beam is defined as the distance between closing string $1 \rightarrow 3$ and line $1 \rightarrow 2'$, while for the segment $2 \rightarrow 3$ of the beam, the distance $v$ is measured from the closing string $1 \rightarrow 3$ to line $2' \rightarrow 3$. It is important to realize that $F_{ap}$ and $F_{cp}$ are the two components of force $R_l$, while $F_{cp}$ and $F_{bp}$ are the components of force $R_r$. Due to the fact that components of the forces acting on the beam vary in the section $1 \rightarrow 2$ compared to $2 \rightarrow 3$, the vertical distance $v_i$ between the closing string and the funicular, needs to be multiplied by different force. More specifically, for the section $1 \rightarrow 2$ the vertical distance $v_i$ needs to be multiplied by the horizontal component of $F_{ap}$, namely $H_{ap}$, while for the section $2 \rightarrow 3$ the vertical distance $v_i$ needs to be multiplied with the horizontal component of $F_{bp}$, namely $H_{bp}$, in order to find the bending moments. Note that the distance $v_i$ in the case of section $2 \rightarrow 3$ will require that the ray $bp$ of the funicular will be extended in order to provide the moment close to point 2.

The second way to define the bending moments, is to use the main funicular polygon $1 \rightarrow 2 \rightarrow 3$ and their perpendicular (to the funicular segments) distance $v_p$ to the closing string (i.e. the beam itself in the case of Fig. 2. Projective, force and funicular polygons, along with bending moments, shear and axial forces of a beam with inclined load.
where the closing string lies on top of it). The perpendicular distance \( v_p \) for the segment \( 1-2 \) of the beam is taken from the closing string \( 1-3 \) and the segment \( 1-2' \), while for the segment \( 2-3 \) of the beam is the distance between the closing string \( 1-3 \) and the segment \( 2'-3 \). In this case, the bending moment will be for segment \( 1-2' \): \( F_{ap} \cdot v_p \) and for the segment \( 2-3 \): \( F_{bp} \cdot v_p \). Note once again that the distance \( v_p \) in the case of section \( 2-3 \) might require that the segment \( 2'-3 \) of the funicular be extended in order to provide the moment close to point 2.

The last way to define the bending moments, may be the easiest one, is to use the projective funicular \( 1-2' \). In this case, only the components of forces perpendicular to the beam are taken into account multiplied by the distance \( v \), which is a distance perpendicular to the beam. Note that in order to construct the projective funicular, the action line of the perpendicular-to-the-beam component of external force \( F \) is used. The force component that generates the moment is the force \( F_{cp} \).

Fig. 3. Projective, force and funicular polygons, along with bending moments, shear and axial forces of an inclined beam with two inclined loads.

Fig. 4. Projective, force and funicular polygons, along with bending moments, shear and axial forces of an inclined cantilevering beam with two inclined loads.
multiplied with distance \( v \). In the force polygon, the force \( F_{ap} \) is analyzed with two components \( a \) and \( a'p \). Note that in this case some more work is required to construct the projective funicular, but after that the moment along the whole span is calculated with a single operation: \( F_{cp} \cdot v \).

In addition to that, from the projective funicular in the force polygon, information about the shear and axial forces can be extracted. From the force polygon the shear force can be read as follows: from the left side of the beam shear force is equal to \( c' \), after point 2 the shear force changes direction and becomes equal to \( b' \) and finally at point 3 the shear force is balanced by the reaction \( R \). Along the same lines, the axial force can be read from the force polygon as follows: from the left side the axial force is equal to \( a' \) and drops to zero at point 2 and remains zero for the span 2–3.

In the next example, an inclined beam with two inclined loads will be investigated, Fig. 3. The difference with the previous case is that the complexity has increased due to more-than-one loads in the inclined beam. Nevertheless the process we follow is the similar as before. The force and funicular polygons are constructed in the same way as in previous example, see Fig. 2.

Once again, the location of the pole \( p \) of the force polygon is selected, so that the ray of the closing string \((\phi p)\) is parallel to the neutral axis of the beam. The point of the ray of the closing string \( d \) is selected so that a line perpendicular to it is drawn. Then all points in the force polygon are projected to this perpendicular line. More specifically, points \( a, b, c \) are projected to points \( a', b', c' \), while point \( d \) coincides with point \( d' \). Then the rays of the projective funicular polygon are constructed and the projective funicular polygon is constructed in the form diagram.

As in the previous example, the simplest way to read the bending moments, is from the projective funicular polygon \( 1 - 2 - 3 - 4 \) and the distance \( v \). Then the bending moment can be calculated by using force \( F_{dp} \) times the perpendicular distance to the beam \( v \). The shear forces can be calculated as well, as follows: starting from left side of the beam the shear force is equal to \( d' \), at point 2 becomes equal to \( b' \), at point 3 becomes \( c' \) and at point 4 vanishes. The axial forces from the left side start with a force equal to \( a' \), at point 2, the force reduces to \( b' \) at point 3, becomes \( c' \) at segment 3–4 and at point 4 the axial force vanishes.

In the last example, different boundary conditions are considered, by examining a cantilever beam, Fig. 4. The different support condition, compared to the previous examples, can be represented by two forces, and the pole of the force polygon coincides with the representative space between them.

2.2. Statically indeterminate beams

In this section, the statically indeterminate cases are presented. Compared to the statically determinate cases, the process is reversed, due to the fact that the reaction forces cannot be calculated before
constructing the projective funicular polygon. More specifically, the following process need to be followed:

Step 1: Construct the projective funicular polygon following the steps:

(i) Select randomly a point in the force polygon that represents the area between the supports. Then select the pole of the force polygon so that the ray of the closing string will be parallel to the neutral axis of the beam.

(ii) From the endpoint of the ray of the closing string (not the pole) in the force polygon draw a perpendicular line to the closing string.

(iii) Project the rest of the points of the force polygon to the perpendicular line drawn in step (ii) by keeping representative names.

(iv) Join the points determined in step (iii) with the pole of the force polygon.

(v) Draw the projective funicular polygon in the form diagram by using the rays between the projection points and the pole (defined in step (iv)) and the action lines of the perpendicular to the neutral axis of the beam, external load components. Note that in the case of a clamped-clamped beam, the location of the point in the force polygon between the supports needs to be adjusted in addition to the location of the pole.

Step 2. The projective funicular polygon, is discretised, see [25], and its areas are used to construct the elastic curve of the beam.

Step 3. The randomly selected points in steps 1(i) and (v), are adjusted so that the compatibility of the deflections are fulfilled.

Step 4. Draw the funicular polygon.

As a first example the beam shown in Fig. 5 is considered. After constructing the force line a−b of the force polygon, point c is selected randomly. The pole p of the force polygon is selected randomly on a line that is parallel to the neutral axis of the beam through point c (ray of closing string). For the case of clamped beams the pole coincides with the point the two reaction forces in the fixed end, in this case point d. Then a line perpendicular to the ray of the closing string cp is drawn. Points a, b are projected to this perpendicular line and points a′, b′ are determined. Note that points c′ and d′ coincide. Points a′, b′, c′, d′ are connected to the pole to generate the rays of the projective funicular polygon. In the form diagram the projective funicular polygon (1⊥−2⊥−3⊥−2⊥−1) is drawn from its rays in the force polygon. The projective funicular polygon is discretised in different areas, which are used to construct the elastic curve of the beam. Adjust point c in the force polygon so that the deflections of the elastic curve of the beam are compatible with its boundary conditions. As a last step the funicular polygon 1′−2′−3′−2′−1 can be drawn. (See Fig. 5). The bending moments are defined from the distance between the projective funicular polygon 1⊥−2⊥−3⊥−2⊥−1 and its closing string 1−2−3, times the force H from the force polygon.

A more general example of statically indeterminate structures is shown in Fig. 6, namely a clamped-clamped beam. The difference with the previous example is that apart from the adjustment of point d in the force polygon, the location of the pole p needs to be adjusted as well so that the elastic curve produced by the projective funicular polygon can
satisfy the boundary conditions. After the location of point d is defined, the reactions and the funicular polygon can be determined. The bending moments are defined as in the previous example, from the distance between the projective funicular polygon $1⊥2⊥3⊥$ and its closing string $1–2–3$, times the force H from the force polygon.

### 3. Conclusions

In the current work the projective funicular polygon is presented. For the cases of statically determinate beam elements with external loads that introduce axial forces (inclined beams/loads), the projective funicular polygon can be used to derive bending moments, shear forces and axial forces. In addition to that, for the case of the statically indeterminate beams the projective funicular polygon is necessary to determine the reactions of the element, by fulfilling the compatibility of deflections due to bending. From the design perspective, the projective funicular polygon offers beneficial information regarding the bending rigidity required along the beam element, in contrast to the funicular polygon, which informs about the flow of forces in the structural deflections due to bending. From the design perspective, the projective funicular polygon offers beneficial information regarding the bending rigidity required along the beam element, in contrast to the funicular polygon, which informs about the flow of forces in the structural deflections due to bending. From the design perspective, the projective funicular polygon offers beneficial information regarding the bending rigidity required along the beam element, in contrast to the funicular polygon, which informs about the flow of forces in the structural deflections due to bending.

### Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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