Numerical radius and zero pattern of matrices

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Abstract

Let $A$ be an $n \times n$ complex matrix and $r$ be the maximum size of its principal submatrices with no off-diagonal zero entries. Suppose $A$ has zero main diagonal and $x$ is a unit $n$-vector. Then, letting $\|A\|$ be the Frobenius norm of $A$, we show that

$$|\langle Ax, x \rangle|^2 \leq (1 - 1/2r - 1/2n) \|A\|^2.$$  

This inequality is tight within an additive term $O\left(n^{-2}\right)$.

If the matrix $A$ is Hermitian, then

$$|\langle Ax, x \rangle|^2 \leq (1 - 1/r) \|A\|^2.$$  

This inequality is sharp; moreover, it implies the Turán theorem for graphs.

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1 Introduction

Let $G$ be a simple graph, $\mu(G)$ be the spectral radius of its adjacency matrix, $\omega(G)$ be the maximum size of its complete subgraphs, and $e(G)$ be the number of its edges. In [7] it is shown that

$$\mu^2(G) \leq \left(2 - \frac{2}{\omega(G)}\right) e(G).$$  \hspace{0.5cm} (1)

The aim of this note is to extend this result to square matrices with zero main diagonal.

Let $\eta(A)$ be the numerical radius of a square matrix $A$, i.e.,

$$\eta(A) = \max_{\|x\|=1} |\langle Ax, x \rangle|.$$  

The value $\eta(A)$ has been extensively studied, see, e.g., [2]-[4], [6] and their references.
Given a complex matrix \( A = \{a_{ij}\} \), write \( \|A\| \) for its Frobenius's norm, i.e., \( \|A\| = \sqrt{\sum_{i,j} |a_{ij}|^2} \). We are interested in upper bounds on \( \eta(A) \) in terms of \( \|A\| \). It is easy to see that \( \eta(A) \leq \|A\| \) with equality holding, e.g., if \( A \) is a constant matrix. In this note we give conditions for the zero pattern of a square matrix \( A \) that imply \( \eta(A) \leq (1 - c) \|A\| \) for some \( c \in (0, 1) \) independent of the order of \( A \).

Given a square matrix \( A \), let \( \omega(A) \) be the maximum size of its principal submatrices with no off-diagonal zero entries.

Note that if \( A \) is the adjacency matrix of a graph \( G \), then \( \omega(A) = \omega(G) \), \( \mu(G) = \eta(A) \), and \( \|A\|^2 = 2e(G) \). Thus, the following theorem extends inequality (1).

**Theorem 1** For every Hermitian matrix \( A \) with zero main diagonal,

\[
\eta^2(A) \leq \left(1 - \frac{1}{\omega(A)}\right) \|A\|^2. \tag{2}
\]

Inequality (2) is sharp: for all \( n \geq r \geq 2 \), there exists an \( n \times n \) symmetric \((0, 1)\)-matrix \( A \) with zero main diagonal and \( \omega(A) = r \) such that equality holds in (2).

Note that inequality (2) implies a concise form of the fundamental theorem of Turán in extremal graph theory (see [1] for details). Indeed, if \( A \) is the adjacency matrix of a graph \( G \) with \( n \) vertices and \( m \) edges, then

\[
(2m/n)^2 \leq \eta^2(A) \leq (2 - 2/\omega(A)) m = 2 \left(1 - 1/\omega(G)\right) m,
\]

and so,

\[
m \leq \left(1 - \frac{1}{\omega(G)}\right) \frac{n^2}{2}. \tag{3}
\]

Moreover, inequality (2) follows from a result of Motzkin and Straus [5], following in turn from (3) (see [8] for details). The implications

\[ (2) \implies (3) \implies \text{MS} \implies (2) \]

justify regarding inequality (2) as a matrix form of Turán’s theorem.

We state without a proof a characterization of Hermitian matrices for which equality holds in (2).

**Proposition 2** Let \( A = \{a_{ij}\} \) be an \( n \times n \) Hermitian matrix with zero main diagonal with \( \omega(A) = r \geq 2 \). Then the equality \( \eta^2(A) = (1 - 1/r) \|A\|^2 \) holds if and only if there exist a complex number \( c \neq 0 \), a partition \([n] = \bigcup_{i=0}^{r} N_i\), and a unit vector \( \mathbf{x} = (x_1, \ldots, x_n) \) such that:

(i) \( x_i = 0 \) for all \( i \in N_0 \).
(ii) \( \sum_{i \in N_i} |x_i|^2 = 1/r \) for all \( 1 \leq i \leq r \).
(iii) \( a_{ij} = cx_i x_j \) for all \( 1 \leq i < j \leq n \).

It turns out that Theorem 1 has analogues for non-Hermitian matrices as well.
Theorem 3 For every complex $n \times n$ matrix $A$ with zero main diagonal,

$$
\eta^2(A) \leq \left( 1 - \frac{1}{2\omega(A)} - \frac{1}{2n} \right) \|A\|^2.
$$

Inequality (4) is tight: for all $n \geq r \geq 2$, there exists an $n \times n$ matrix $A$ with zero main diagonal and $\omega(A) = r$ such that

$$
\eta^2(A) \geq \left( 1 - \frac{1}{2\omega(A)} - \frac{1}{2n} + O\left(n^{-2}\right) \right) \|A\|^2.
$$

Let $P_n$ be the set of vectors $(x_1, \ldots, x_n)$ with $x_1 \geq 0, \ldots, x_n \geq 0$, and $x_1 + \cdots + x_n = 1$. Recall a result of Motzkin and Straus [5]: if $A$ is the adjacency matrix of a graph $G$ of order $n$, and $x \in P_n$, then

$$
\langle Ax, x \rangle \leq 1 - 1/\omega(G).
$$

We shall need the following extension of this result.

Lemma 4 For every square $(0,1)$-matrix $A$ of size $n$ with zero main diagonal and every $x \in P_n$,

$$
\langle Ax, x \rangle \leq 1 - \frac{1}{2\omega(A)} - \frac{1}{2n}.
$$

Inequality (6) is tight: for all $n \geq r \geq 2$, there exists a square $(0,1)$-matrix $A$ of size $n$ with zero main diagonal and $\omega(A) = r$ such that,

$$
\langle Ax, x \rangle = 1 - \frac{1}{2r} - \frac{1}{2n} + O\left(n^{-2}\right)
$$

for some $x \in P_n$.

2 Proofs

Proof of Lemma 4 Define the $n \times n$ matrix $B = \{b_{ij}\}$ setting $b_{ij} = a_{ij}a_{ji}$ for all $i, j \in [n]$; let $C = A - B$. Note that for every two distinct $i, j \in [n]$, we have

$$
c_{ij} + c_{ji} = a_{ij} + a_{ij} - 2a_{ij}a_{ji} \leq 1.
$$

We may and shall assume that $c_{ij} + c_{ji} = 1$ for all distinct $i, j \in [n]$ with $b_{ij} = 0$, since otherwise some off-diagonal zero entry of $A$ can be changed to 1 so that $\omega(A)$ remains the same and the left-hand side of (4) does not decrease. Hence, for every $x = (x_1, \ldots, x_n)$,

$$
\langle Bx, x \rangle + 2 \langle Cx, x \rangle = 1 - \|x\|^2.
$$

Since $B$ is a symmetric $(0,1)$-matrix with zero main diagonal, the result of Motzkin and Straus implies that

$$
\langle Bx, x \rangle \leq 1 - 1/\omega(B)
$$

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for every $x \in P_n$. Since $\omega(B) = \omega(A)$, we find that

$$
\langle Ax, x \rangle = \langle Bx, x \rangle + \langle Cx, x \rangle = \frac{1}{2} \left( 1 - \|x\|^2 \right) + \frac{1}{2} \langle Bx, x \rangle \leq 1 - \frac{1}{2\omega(A)} - \frac{1}{2n}.
$$

completing the proof of (6).

Let $G$ be a complete $r$-partite graph whose vertex classes differ in size by at most 1. Let $T = \{t_{ij}\}$ be the adjacency matrix of $G$; set $t_{ij} = 1$ for $i < j$ and write $A$ for the resulting matrix. We have

$$
\|A\|^2 = \binom{n}{2} + \frac{1}{2} \|T\|^2 = \binom{n}{2} + \frac{\binom{r}{2} n^2 - \nu^2}{r^2} + \binom{\nu}{2},
$$

Letting $x$ to be the $n$-vector $(1/n, \ldots, 1/n) \in P_n$, we find that

$$
\langle Ax, x \rangle = \frac{1}{n^2} \|A\|^2 = \frac{1}{n^2} \left( \binom{n}{2} + \frac{\binom{r}{2} n^2 - \nu^2}{r^2} + \binom{\nu}{2} \right) = 1 - \frac{1}{2r} - \frac{1}{2n} + \frac{1}{r^2} \binom{\nu^2}{2} - \frac{1}{2} \binom{\nu^2}{2} - \frac{1}{2} \binom{\nu^2}{2} - \frac{1}{8n^2},
$$

completing the proof of the lemma. \qed

**Proof of Theorem 1** Select $y = (y_1, \ldots, y_n)$ with $\|y\| = 1$ and $\eta(A) = |\langle Ay, y \rangle|$. We have, by the Cauchy-Schwarz inequality,

$$
\eta^2(A) = \left| \sum_{i,j} a_{ij} y_i y_j \right|^2 \leq \sum_{i,j} |a_{ij}|^2 \sum_{i,j} |y_i|^2 |y_j|^2 = \|A\|^2 \sum_{a_{ij} \neq 0} |y_i|^2 |y_j|^2.
$$

Define a graph $G$ with $V(G) = [n]$, joining $i$ and $j$ if $a_{ij} \neq 0$. Obviously, $\omega(G) = \omega(A)$. Since $\|y\| = 1$, the result of Motzkin and Straus implies that

$$
\sum_{a_{ij} \neq 0} |y_i|^2 |y_j|^2 = \sum_{ij \in E(G)} |y_i|^2 |y_j|^2 \leq 1 - \frac{1}{\omega(A)},
$$

completing the proof of (2).

Let $A$ be the adjacency matrix of the union of a complete graph on $r$ vertices and $n - r$ isolated vertices. Since $\omega(A) = r$, $\eta(A) = r - 1$, and $\|A\|^2 = r (r - 1)$, we see that

$$
\eta^2(A) = \mu^2(A) = (1 - 1/\omega(A)) \|A\|^2,
$$

completing the proof of the theorem. \qed

**Proof of Theorem 3** Select $y = (y_1, \ldots, y_n)$ with $\|y\| = 1$ and $\eta(A) = |\langle Ay, y \rangle|$. Lemma 4 implies that

$$
\eta^2(A) = \left| \sum_{i,j} a_{ij} y_i y_j \right|^2 \leq \sum_{i,j} |a_{ij}|^2 \sum_{a_{ij} \neq 0} |y_i|^2 |y_j|^2 = \|A\|^2 \sum_{a_{ij} \neq 0} |y_i|^2 |y_j|^2 \leq \left( 1 - \frac{1}{2\omega(A)} - \frac{1}{2n} \right) \|A\|^2,
$$

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proving (4).

To complete the proof, select $A$ as in the proof of Lemma 4. Hence, letting $\nu$ be the remainder of $n$ modulo $r$, we have

$$\|A\|^2 = \sum_{i,j} a_{ij} = \left(\binom{n}{2} + \binom{r}{2} \frac{n^2 - \nu^2}{r^2} + \binom{\nu}{2}\right).$$

Selecting $x$ to be the $n$-vector $(n^{-1/2}, \ldots, n^{-1/2})$, as in the proof of Lemma 4, we find that

$$\eta^2(A) \geq \frac{1}{n^2} \|A\|^2 = 1 - \frac{1}{2r} - \frac{1}{2n} + \left(\frac{\nu^2}{2r} - \frac{\nu}{2}\right) \frac{1}{n^2} \geq 1 - \frac{1}{2r} - \frac{1}{2n} - \frac{r}{8n^2},$$

completing the proof of the theorem. \qed

Concluding remarks

- The example constructed in the proof of Lemma 4 shows that equality may hold in (4) and (6) whenever $n$ is a multiple of $r$.
- It would be interesting to drop the requirement for zero main diagonal in Theorem 1 and 3. Note that inequalities (5) and (6) are no longer valid if ones are present on the main diagonal of $A$.
- Since the spectral radius of a square matrix does not exceed its numerical radius, Theorem 1 and 3 provide upper bounds on the spectral radius as well.

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