The star formation rate density and the stochastic background of gravitational waves

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Abstract. There is in the literature a number of papers addressing the stochastic background of gravitational waves (GWs) generated by an ensemble of astrophysical sources. The main ingredient in such studies is the so called star formation rate density (SFRD), which gives the density of stars formed per unit time. Some authors argue, however, that there is, in the equation that determines the amplitude of the stochastic background of GWs, an additional \((1 + z)\) term dividing the SFRD, which would account for the effect of cosmic expansion onto the time variable. We argue here that the inclusion of this additional term is wrong. In order to clarify where the inclusion of the \((1 + z)\) term is really necessary, we briefly discuss the calculation of event rates in the study of GRBs (gamma ray bursts) from cosmological origin.

1. Stochastic background of gravitational waves
We have recently shown [1] that the dimensionless amplitude of the stochastic background of GWs produced by an ensemble of sources is given by

\[
h_{BG}^2 = \frac{1}{\nu_{\text{obs}}} \int h_{\text{single}}^2 dR, \tag{1}\]

where \(h_{\text{single}}^2\) is the dimensionless amplitude produced by an event that generates a signal with observed frequency \(\nu_{\text{obs}}\); and \(dR\) is the differential rate of production of GWs.

Eq.(1) is in fact a shortcut to the calculation of stochastic background of GWs. An interesting characteristic of this equation is that it is not necessary to know in detail the energy flux of the GWs produced at each frequency. If the characteristic values for the dimensionless amplitude and frequency are known and the event rate is given it is possible to calculate the stochastic background of GWs produced by an ensemble of sources.

In particular, for the case of a background produced by an ensemble of black holes, the differential rate reads

\[
dR_{BH} = \dot{\rho}_* (z) \frac{dV}{dz} \phi(m) dm dz, \tag{2}\]

where \(\dot{\rho}_* (z)\) is the star formation rate density (hereafter SFRD; in \(M_\odot \, \text{yr}^{-1} \, \text{Mpc}^{-3}\)), \(dV\) is the comoving volume element and \(\phi(m)\) is the initial mass function (IMF). In a few words, the \(\phi(m) \, dm\) represents the number of stars per unit mass in the interval \([m, m + dm]\).

Some authors argue that there is an additional \((1 + z)\) term in the equation for the differential rate dividing \(\dot{\rho}_* (z)\), which would take into account the effect of cosmic expansion onto the time variable.
In fact, there are two different points of view in the literature concerning the inclusion or not of the factor \((1 + z)\). The first group does not include this factor dividing the SFRD (see, e.g., Refs. [1, 2, 3]). On the other hand, the second group argues that it is necessary to include the factor \((1 + z)\) to account for the time dilation of the observed rate by cosmic expansion, converting a source-count equation to an event-rate equation (see, e.g., Refs. [4, 5, 6, 7]). We show here why such an additional term does not exist.

In order to address properly this issue, we should note firstly that the determination of the amplitude of the GWs is obtained from the flux received by a detector \((F_{GW})\). In this way, one divides the luminosity of the sources by \(4\pi d_L^2\), where \(d_L\) is the so called luminosity distance. The luminosity obviously refers to the source frame. Since the luminosity depends on the differential rate, given, for example, by Eq. (2), the latter also refers to the source frame, therefore it is not necessary to redshift it. It is worth stressing that any necessary redshifting is taken into account in the definition of \(d_L\).

Another way to be convinced that the \((1 + z)\) does not enter into the calculation of the differential rate of production of GWs, \(dR_{BH}\), is to derive Eq. (1) through a procedure completely different from that used in Ref. [1].

Let us write the specific flux received in GWs at the present epoch as (see, in particular, Eq. (15) in Ref. [8] and section 12.1 in Ref. [9])

\[
F_{\nu}(\nu_{\text{obs}}) = \int \frac{l_{\nu}}{4\pi d_L^2} \frac{d\nu}{d\nu_{\text{obs}}} dV,
\]

where

\[
l_{\nu} = \frac{dL_{\nu}}{dV}
\]

is the comoving specific luminosity density (given, e.g, in erg s\(^{-1}\) Hz\(^{-1}\) Mpc\(^{-3}\)), which obviously refers to the source frame.

As discussed in Refs. [8, 9], the above equations are valid to estimate a stochastic background radiation received on Earth independently of its origin. In the present paper \(l_{\nu}\) can be written as follows

\[
l_{\nu} = \int \frac{dE_{GW}}{d\nu} \frac{d\nu}{d\nu_{\text{obs}}} \rho_*(z)\phi(m) dm,
\]

where \(dE_{GW}/d\nu\) is the specific energy of the source. Note that in the above equation \(\rho_*(z)\) refers to the source frame, therefore, there is not the putative \((1 + z)\) factor responsible to the time dilation.

Thus, the flux \(F_{\nu}(\nu_{\text{obs}})\) received on Earth reads

\[
F_{\nu}(\nu_{\text{obs}}) = \int \frac{1}{4\pi d_L^2} \frac{dE_{GW}}{d\nu} \frac{d\nu}{d\nu_{\text{obs}}} \rho_*(z)\phi(m) dm dV.
\]

Using Eq. (2) it follows that

\[
F_{\nu}(\nu_{\text{obs}}) = \int \frac{1}{4\pi d_L^2} \frac{dE_{GW}}{d\nu} \frac{d\nu}{d\nu_{\text{obs}}} dR_{BH}.
\]

Note that in the above equation, what multiplies \(dR_{BH}\) is nothing but the specific energy flux per unity frequency (in, e.g., erg cm\(^{-2}\) Hz\(^{-1}\)), i.e.,

\[
f_{\nu}(\nu_{\text{obs}}) = \frac{1}{4\pi d_L^2} \frac{dE_{GW}}{d\nu} \frac{d\nu}{d\nu_{\text{obs}}}.
\]
(see, e.g., Ref.[10]).

On the other hand, the specific energy flux per unit frequency for GWs is given by Ref.[11]

$$f_\nu(\nu_{\text{obs}}) = \frac{\pi c^3}{2G} h_{\text{BH}}^2.$$  \hspace{1cm} (9)

Also, the spectral energy density, the flux of GWs, received on Earth, $F_\nu$, in erg cm$^{-2}$ s$^{-1}$ Hz$^{-1}$ can be written from Refs.[12, 13] as

$$F_\nu(\nu_{\text{obs}}) = \frac{\pi c^3}{2G} h_{\text{BG}}^2 \nu_{\text{obs}}.$$  \hspace{1cm} (10)

From the above equations one obtains

$$h_{\text{BG}}^2 = \frac{1}{\nu_{\text{obs}}} \int h_{\text{BH}}^2 dR_{\text{BH}},$$  \hspace{1cm} (11)

which is nothing but the Eq. (1).

Thus, to calculate the amplitude of a stochastic background of GWs, it is not necessary to include the extra $(1+z)$ term. Since the time dilation is already incorporated into the definition of the luminosity distance, Eq. (1) is just a different way to write the relation between luminosity of an ensemble of sources and the flux received on Earth by a detector.

One could argue that what we derive here is nothing but a rehash of a well known equation for the calculation of electromagnetic backgrounds, found in textbooks on cosmology, applied to gravitational radiation backgrounds. But, since some authors in the GW community are erroneously introducing extra redshift factors in their calculations, it is worth presenting clearly how to calculate GW backgrounds of cosmological origin.

Also, these authors do not base their calculations either on the formulation of electromagnetic backgrounds presented in textbooks on cosmology or on an equation such as the one derived here. They inappropriately base their calculations on event rate equations, in which the time dilation needs to be considered. As a result, they consider that in other calculations involving the SFRD the redshifting should be taken into account.

One could argue when it is necessary to take into account the redshifting related to the time dilation. We now discuss a case where we must include the factor $(1+z)$ to take into account the time dilation of the observed rate.

The case is related to the Gamma Ray Bursts (hereafter GRBs), which are short and intense pulses of $\gamma$-rays, which last from a fraction of a second to several hundred seconds. It is worth mentioning that there are in the literature many papers dealing with the GRB event rates of cosmological origin, namely, Refs. [14, 15, 16, 17], among others.

The event rate for the GRBs of cosmological origin is related to the SFRD. The main reason to include the $(1+z)$ factor in this case is that we observe GRB events over a fixed time window $\Delta t_{\text{obs}}$, which corresponds to $\Delta t_{\text{obs}}/(1+z)$ in the source frame. Thus, the total number of GRBs can be written as (see, in particular, Ref.[14])

$$N(>z) = \int_{z}^{\infty} \psi_{\text{GRB}}(z') \frac{\Delta t_{\text{obs}}}{(1+z')} \frac{dV}{dz} dz',$$  \hspace{1cm} (12)

where $dV/dz$ is the comoving volume element and $\psi_{\text{GRB}}(z)$ is the number of GRB events per comoving volume per unit time, which is proportional to $\dot{\rho}_s(z)$.

Generally speaking, in any event rate of cosmological sources, which involves a formation rate as a function of redshift or the like, the time dilation must be applied.

We see that the definition presented in Eq.(12) is completely different from the definition presented in Eq.(1). The misuse of the time dilation of the SFRD in the calculation of the
energy flux is due to the fact that this involves a time rate, as the event rate of cosmological
sources does. But, one has to bear in mind that the luminosity distance, that links the luminosity
(source frame) and the energy flux (observer frame), is defined in such a way that any redshifting,
including the time dilation, is implicitly in its definition.

2. Concluding remarks
The \((1 + z)\) dividing the SFRD, which some authors claim would account for the time dilation
of the observed rate by cosmic expansion, converting a source-count equation to an event-rate
equation (see, e.g., \([4, 5, 6, 7]\)), is nothing but the arrival rates, which is implicit in the luminosity
distance. In particular, Refs.\([2, 3]\) implicitly corroborate with the argumentation given here.

It is important to stress that for the cosmological background of GW models we studied in
Ref.\([18]\), the inclusion of the \((1 + z)\) term would reduce by a factor of \(\sim 3 - 5\) the signal-to-noise
ratio (SNR) predicted for the LIGO observatories. For some models, this would mean to predict
a non detection of such a stochastic background of GWs.

We hope the present work contributes to clarify the reader why in the calculation of the
amplitude of a stochastic background of GWs, from cosmological sources, it is not necessary
to redshift the SFRD as some authors are arguing for; and also why the GRB community, or
any one who calculates cosmological event rates, should keep using the time dilation in their
calculations. For further details we refer the reader to Ref. \([19]\).

Acknowledgments
JCNA would like to thank the Brazilian agency CNPq for partial support (grant 303868/2004-0).

References
[1] J.C.N. de Araujo, O.D. Miranda and O.D. Aguiar, Phys. Rev. D \textbf{61}, 124015 (2000).
[2] V. Ferrari, S. Matarrese and R. Schneider, Mon. Not. R. Astron. Soc. \textbf{303}, 247 (1999).
[3] T. Regimbau and J.A. de Freitas Pacheco, Astron. Astrophys. \textbf{376}, 381 (2001).
[4] R. Schneider, A. Ferrara, B. Ciardi, V. Ferrari and S. Matarrese, Mon. Not. R. Astron. Soc. \textbf{317}, 385 (2000).
[5] D.M. Coward, R.R. Burman and D.G. Blair, Mon. Not. R. Astron. Soc. \textbf{324}, 1015 (2001).
[6] D.M. Coward, M.H.P.M. van Putten and R.R. Burman, Astrophys. J. \textbf{580}, 1024 (2002).
[7] D.M. Coward, R.R. Burman and D.G. Blair Mon. Not. R. Astron. Soc. \textbf{329}, 411 (2002).
[8] A.J. Farmer and E.S. Phinney, Mon. Not. R. Astron. Soc. \textbf{346}, 1197 (2003).
[9] J.A. Peacock, in \textit{Cosmological Physics} (Cambridge University Press, Cambridge, England, 1999).
[10] V. Ferrari, R. Schneider and S. Matarrese, Mon. Not. R. Astron. Soc. \textbf{303}, 258 (1999).
[11] B.J. Carr, Astron. Astrophys. \textbf{89}, 6 (1980).
[12] D.H. Douglass and V.G. Braginsky, in General Relativity: An Einstein Centenary Survey, edited by S. W.
Hawking and W. Israel (Cambridge University Press Cambridge, England, 1979), p. 90.
[13] D. Hils, P.L. Bender, and R.F. Webbink, Astrophys. J. 360, 75 (1990).
[14] V. Bromm and A. Loeb, Astrophys. J. \textbf{575}, 111 (2002).
[15] T. Totani and A. Panaitescu, Astrophys. J. \textbf{576}, 129 (2002).
[16] C. Firmani, V. Avila-Reese, G. Ghisellini and A.V. Tutukov, Astrophys. J. \textbf{611}, 1033 (2004).
[17] C. Porciani and P. Madau, Astrophys. J. \textbf{548}, 522 (1991).
[18] J.C.N. de Araujo, O.D. Miranda and O.D. Aguiar, Class. Quantum Grav. \textbf{21}, S545 (2004).
[19] J.C.N. de Araujo and O.D. Miranda, Phys. Rev. D \textbf{71}, 127503 (2005).