Effective temperatures of hot Brownian motion

G. Falasco,1 M. V. Gnann,2 D. Rings,3 and K. Kroy1

1Institut für Theoretische Physik, Universität Leipzig, Postfach 100 920, D-04009 Leipzig, Germany
2Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, 04103 Leipzig, Germany
3Department of Physics and Astronomy, University of Leeds, LS2 9JT Leeds, England

We derive generalized Langevin equations for the translational and rotational motion of a heated Brownian particle from the fluctuating hydrodynamics of its non-isothermal solvent. The temperature gradient around the particle couples to the hydrodynamic modes excited by the particle itself so that the resulting noise spectrum is governed by a frequency-dependent temperature. We show how the effective temperatures at which the particle coordinates and (angular) velocities appear to be thermalized emerge from this central quantity.

I. INTRODUCTION

Hot Brownian motion [1] is the diffusive dynamics of a colloidal particle persistently maintained at higher temperature than the surrounding fluid, so that the fluid temperature field \( T(r) = T(r) \) decays radially around the particle. It is of practical relevance, e.g. for laser-heated suspended nanoparticles involved in several experimental applications ranging from particle trapping and tracking to self-thermophoretic micro-swimmers [4, 5]. Besides, it is also of considerable theoretical interest, since it can be thought of as an archetypical example of a system in contact with a non-isothermal bath, hence far from thermal equilibrium. Nevertheless, for important conceptual and practical purposes, the hot particle can often be treated like an equivalent Brownian particle in equilibrium, with appropriate effective transport coefficients.

In particular, it has been shown analytically [6, 7] that free and confined diffusion of a hot spherical particle are, in the long-time limit, governed by effective “positional” temperatures, denoted by \( T^X \) and \( T^\Theta \) for translation in the \( X \)-direction and rotation along an angle \( \Theta \), respectively. The positional temperatures enter the effective Stokes–Einstein relations and Boltzmann factors for translation and rotation of the particle, respectively. Additionally, extensive numerical simulations [6, 8] have shown that the Maxwellian (angular) velocity distribution and the short-time response of the hot Brownian particle are characterized by yet other, somewhat higher effective temperatures, so-called kinetic temperatures \( T^V \) and \( T^n \). None of these effective temperatures is generally equal to the solvent temperature \( T_s = T(r \to R) \) at the particle surface or to the ambient temperature \( T_0 = T(r \to \infty) \). This complex behavior has led to the conclusion that an effective Langevin description of hot Brownian motion is restricted to the Markov limit [3].

Here, we show constructively how this limitation can be overcome, starting from the fluctuating hydrodynamics of a solvent maintained at local thermal equilibrium with a temperature field \( T(r) \). On this basis, we derive the generalized Langevin equation (GLE) for a heated spherical particle. Conceptually, non-spherical particles can be treated along the same lines, albeit with additional complications [7]. The most conspicuous feature of the theory is a frequency-dependent noise temperature \( T(\omega) \). It arises from the hydrodynamic coupling between the particle and distant solvent volume elements that are locally equilibrated at different temperatures \( T(r) \). From this central quantity analytical predictions for the mentioned kinetic and positional effective temperatures are derived.

The characteristic frequency scales that primarily select the dominant modes from the “temperature spectrum” \( T(\omega) \) are (for a translating sphere of mass \( M \), density \( \rho \), and radius \( R \))

\[
\omega_i \equiv \frac{2\nu}{R^2} \quad \text{and} \quad \omega_p \equiv \frac{6\pi\eta R}{M} = \frac{9\nu}{4\rho} \omega_i , \tag{1}
\]

namely, the inverse time scale for vorticity diffusion across the particle, and the inverse Stokes relaxation time of the particle momentum, respectively. The former characterizes how efficiently the particle momentum is spatially dispersed in a solvent of kinematic viscosity \( \nu = \eta/\rho \) and density \( \rho \), and the latter how, as a result, the motion of the particle adjusts to that of the fluid. The meaning of slow and fast processes, or low and high frequencies of the noise spectrum, is primarily provided by these rates. It should be clear, though, that any externally imposed additional time scale that interferes with these rates, can be expected to yield additional features.

The paper is structured as follows. In the next section we introduce the theoretical model of a Brownian particle in a non-isothermal solvent. We then sketch the contraction of the coupled solvent-particle system to the GLE for the particle motion, alone. Details of the calculation are given in Appendix A. In Sec. III we examine the frequency-dependent temperatures \( T(\omega) \) that govern the Langevin noise for the translational and rotational degrees of freedom of a heated sphere and give a qualitative physical interpretation of their functional form, while some technicalities are deferred to Appendix B. From this central quantity, we derive the effective rotational and translational kinetic temperatures of a free particle in Sec. IV. We analyze their explicit dependence on the characteristic time scales for the velocity relaxation of the particle and the solvent by varying their density ratio. Also we regain the known positional temperatures...
II. FLUCTUATING HYDRODYNAMICS OF A HEATED PARTICLE

We consider a spherical particle of radius $R$ immersed in an incompressible fluid of density $\rho$ described by the linearized fluctuating hydrodynamic equations [10, 11]

$$
\begin{align*}
\rho \partial_t v(r, t) - \nabla \cdot \sigma(r, t) &= \nabla \cdot \tau(r, t), \\
\nabla \cdot v(r, t) &= 0, \\
v(r, t) &= V(t) + \Omega(t) \times r \quad \text{on } S,
\end{align*}
$$

where the velocity field $v$ of the fluid is defined in the volume $V$ outside the particle and the no-slip boundary condition on the particle surface $S$ is imposed by Eq. (2a).

The stress tensor $\sigma$ has components $\sigma_{ij} = -p \delta_{ij} + 2\eta \Gamma^{ij}$, where $p$ is the pressure and $\Gamma^{ij} = (\partial_i v_j + \partial_j v_i)/2$ the shear rate tensor, with the dynamic viscosity $\eta$. The incompressibility condition Eq. (2b) can be eliminated by expressing $p$ (and thus $\sigma$) as a functional of the flow field $v$. Finally, the thermal noise is represented by a zero-mean Gaussian random stress tensor $\tau$ that vanishes on the particle surface and otherwise obeys the fluctuation-dissipation relation

$$
\langle \tau_{ij}(r, t) \tau_{kl}(r', t') \rangle = 2\eta V(r, t) \delta(r-r') \times \delta(t-t') (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})
$$

corresponding to a local equilibrium with the prescribed temperature field $T(r, t)$ [12].

The vectors $V(t)$ and $\Omega(t)$, denoting the translational and rotational velocity of the Brownian particle, couple to the solvent dynamics via the boundary condition Eq. (2c) on the particle surface $S$. They evolve themselves according to Newton’s equations of motion

$$
\begin{align*}
M \dot{V}(t) &= F(t) + F_e(t), \\
I \dot{\Omega}(t) &= T(t) + T_e(t),
\end{align*}
$$

where $M$ is the mass of the particle, $I$ the moment of inertia, $F_e$ and $T_e$ are the external force and torque, and $F$ and $T$ are the hydrodynamic force and torque exerted by the fluid, defined by

$$
\begin{align*}
F(t) &= - \int_S \sigma(r, t) \cdot n \, d^2 r, \\
T(t) &= - \int_S r \times (\sigma(r, t) \cdot n) \, d^2 r,
\end{align*}
$$

with $n$ the inner radial unit vector. Note that we have suppressed the time dependence of $S$ in Eqs. (2d, e) in order to make the above set of equations linear not only in the flow field but also in the particle velocity. See [13, 14] for a discussion of the validity of linear hydrodynamics in relation to Brownian motion. We also suppress the corresponding time-dependent thermal advection, by requiring the temperature field to obey the stationary heat equation in the co-moving frame,

$$
\begin{align*}
\nabla^2 T(r) &= 0, \\
T(r) &= T_0 + \Delta T \text{ on } S, \\
T(r \to \infty) &= T_0.
\end{align*}
$$

This technical simplification and other implicit idealizations, such as taking the heat conductivity of the solvent to be constant, can be justified for common experimental conditions, such as those realized for laser-heated nanoparticles in water [1, 15]. Together with the prescription Eq. (6), the system Eq. (2a)–Eq. (3) then entirely describes the time evolution of the fluid and the heated Brownian particle. The solution of Eq. (6) is the radial field:

$$
T(r) = T_0 + \Delta T R/r.
$$

While the following derivation does not strictly depend on the specific form of $T(r)$ (as long as it does not depend on the particle velocity), and even an explicit externally imposed dependence on time could be included, we restrict the discussion in the following sections to this paradigmatic case.

We now proceed to contract the description of fluid plus particle into an equation for the particle alone. We rewrite the hydrodynamic forces introduced in Eq. (4) in the form

$$
\begin{align*}
F &= F_d + \xi^V, \\
T &= T_d + \xi^T
\end{align*}
$$

to account for contributions $\xi$ independent of the particle velocity that are expected to arise due to the inhomogeneity of Eq. (2a). By Eq. (2), $v(r, t)$ is a linear functional of $V(t')$ and $\Omega(t')$ with $-\infty < t' < t$, so in view of Eq. (6) this implies that the systematic components $F_d$ and $T_d$ are linear functionals of $V(t')$ and $\Omega(t')$ respectively, with $-\infty < t' < t$. Hence, we can write

$$
\begin{align*}
F_d(t) &= - \int_{-\infty}^t \zeta(t-t') \dot{V}(t') \, dt', \\
T_d(t) &= - \int_{-\infty}^t \gamma(t-t') \dot{\Omega}(t') \, dt',
\end{align*}
$$

where $\zeta(t)$ and $\gamma(t)$ are positive, time-symmetric memory kernels accounting for the time-dependent drag on the particle [11]. Equations (11) then take the GLE form

$$
M \dot{V}(t) = - \int_{-\infty}^t \zeta(t-t') V(t') \, dt' + \xi^V(t) + F_e(t).$$
\[ I\Omega(t) = -\int_{-\infty}^{t} \gamma(t - t') \Omega(t') dt' + \xi^R(t) + T_c(t). \] (12)

Once we identify \( \xi^{T, R} \) as the Langevin noise, whose statistical properties have to be derived from those of the random stress tensor \( \tau \).

For better readability, the actual calculation is detailed in Appendix A and only the main results and their physical interpretations are given in the main text. We focus mostly on the translational motion, but the rotational case is very analogous. It is moreover convenient to switch to the frequency representation defining, for a generic function \( g(\omega) \), the Fourier transform \( g(\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega t} g(t) dt \) and the half-Fourier transform \( g^{+}(\omega) \equiv \int_{0}^{\infty} e^{i\omega t} g(t) dt \).

To complete the contraction, we compare the energy dissipated by the fluid friction acting on the particle at a mean velocity \( \langle V(\omega) \rangle \)

\[ \zeta(\omega) \delta_{ij} \langle V_i(\omega) \rangle \langle V^*_j(\omega) \rangle = 2 \int_{V} \phi^{T}(r, \omega) d^3r, \] (13)

with the correlation function of the energy supplied by the random force at frequencies \( \omega \) and \( \omega^{'} \)

\[ \langle \xi^{T}_{i}(\omega) \xi^{T*}_{j}(\omega^{'}) \rangle \langle V_i(\omega) \rangle \langle V^*_j(\omega^{'}) \rangle = \\
= 2k_{B} \delta(\omega - \omega^{'}) \int_{V} \phi^{T}(r, \omega) T(r) d^3r. \] (14)

From Appendix A we have quoted the representation in terms of the dissipation function,

\[ \phi^{T}(r, \omega) \equiv \eta \left( \partial_{i} u_{j} \partial_{i} u_{j}^{*} + \partial_{i} u_{j} \partial_{j} u_{i}^{*} \right), \] (15)

which gives the energy dissipated by the fluid at position \( r \) and frequency \( \omega \) in terms of the flow field \( u(r, \omega) \).

From Eq. (13) and Eq. (12) we then find the relation

\[ \langle \xi^{T}_{i}(\omega) \xi^{T*}_{j}(\omega^{'}) \rangle \langle V_i(\omega) \rangle \langle V^*_j(\omega^{'}) \rangle = \\
= k_{B} T^{T}(\omega) \zeta(\omega) \delta_{ij} \delta(\omega - \omega^{'}) \langle V_i(\omega) \rangle \langle V^*_j(\omega^{'}) \rangle \] (16)

with

\[ T^{T}(\omega) \equiv \frac{\int_{V} \phi^{T}(r, \omega) T(r) d^3r}{\int_{V} \phi^{T}(r, \omega) d^3r}. \] (17)

Since \( \phi^{T}(r, \omega) \) is a quadratic function of \( \langle V(\omega) \rangle \) (see Appendix B), the ratio in Eq. (17) is independent of \( \langle V(\omega) \rangle \). Moreover, as the particle velocity \( \langle V(\omega) \rangle \) is arbitrary it can be deleted in Eq. (16), which renders Eq. (16) in the form of a generalized fluctuation-dissipation relation:

\[ \langle \xi^{T}_{i}(\omega) \xi^{T*}_{j}(\omega^{'}) \rangle = k_{B} T^{T}(\omega) \zeta(\omega) \delta_{ij} \delta(\omega - \omega^{'}). \] (18)

Analogous results hold for the rotational motion. They are obtained by substituting \( \zeta \to \gamma \) in Eq. (18) and \( \phi^{T} \to \phi^{R} \) in the definition Eq. (17).

### III. The Noise Temperature \( T(\omega) \)

Eq. (17) defines the frequency-dependent noise temperature that is the central quantity for the Brownian motion under non-isothermal conditions. Its nonlocal nature is manifest in the weighted average over the temperature field \( T(r) \), with the dissipation function determining how strongly the diverse local temperatures in the surroundings affect the Brownian motion of the particle at the origin.

Clearly, the noise autocorrelation can always be cast in such a form by defining a suitable function \( T(\omega) \) that measures the violation of the equilibrium fluctuation-dissipation relation. Here, the nontrivial statement is that \( T(\omega) \) is explicitly derived from an underlying hydrodynamic description. Moreover, in the next sections, we will show that \( T(\omega) \) plays the role of a frequency-dependent effective temperature, in the sense that dynamical isothermal relations can directly be extended to the non-isothermal case if the temperature \( T_0 \) is replaced by \( T(\omega) \).

Contending ourselves with explicit evaluations to leading order in the temperature heterogeneity \( T(r) - T_0 \), we can in the following neglect a possible temperature-dependence of the viscosity, which would affect our results to sub-leading order, only. Figure 1 shows the frequency-dependent temperatures \( T(\omega) \) for the translational and the rotational motion of a sphere, which are derived in Appendix B assuming constant heat conductivity and viscosity, i.e. Eq. (9) and \( \eta(r) = \eta \). As a consequence, \( \eta \) cancels in Eq. (17) and the obtained noise temperatures are universal functions independent of the solvent properties. All the subsequent results are derived under the latter approximation.

To gain a physical understanding of the functional form of \( T(\omega) \), consider its origin from the hydrodynamic coupling between the particle and distant solvent volume elements that are locally equilibrated at different temperatures \( T(r) \). In our low-Reynolds number approximation, the exchange of momentum is dominated by vorticity diffusion (7) with the diffusivity given by the kinematic viscosity \( \nu \equiv \eta / \rho \). This defines the inverse characteristic time scale \( \omega_{l} \equiv 2\nu / R^{2} \) for fluid transport over distances on the order of the particle radius, as introduced in Eq. (10).

**Low frequency fluctuations** are those with \( \omega \ll \omega_{l} \), during which the vorticity spreads out considerably from the particle. Since the translational field is more long-ranged than the rotational one \( (u^{T} \sim 1/r \text{ versus } u^{R} \sim 1/r^{2}) \), the translational noise is effectively cooler, as it involves an average over farther, i.e. cooler, regions of fluid. Ultimately, in the limit \( \omega \to 0 \), we find that the noise temperatures reduce to the effective temperatures \( T^{R, \text{HBM}}(0) \) known to characterize the overdamped hot Brownian motion of the positions and angles, respectively (8, 7), for which we employ the short-hand notation

\[ T^{X} \equiv T^{X}(0) = T_{0} + \frac{5}{12} \Delta T, \]
The added mass in Eq. (19), owing to the inertia of the displaced fluid, becomes relevant in the following. In the next sections we analyze some immediate implications of the above results for the dynamics of a hot Brownian sphere that is either freely diffusing or trapped in a confining potential.

IV. THE KINETIC TEMPERATURE

The GLE’s (1a), (1b) both contain a Gaussian noise satisfying a fluctuation-dissipation relation with constant effective temperatures in the high-frequency limit. Therefore, one may expect to find Maxwell-Boltzmann distributions of translational and angular velocities under stationary conditions, which is corroborated by molecular dynamics simulations [6, 7]. We thus define the kinetic temperatures such that the stationary averages of the velocities satisfy

\[ \frac{3}{2} k_B T^T = \frac{1}{2} M_{\text{eff}} \langle V^2 \rangle, \quad \frac{3}{2} k_B T^\alpha = \frac{1}{2} I(\Omega^2), \]

which reduce to the equipartition theorem with \( T^T = T^\alpha = T_0 \) in case of a constant fluid temperature \( T(r) \equiv T_0 \). For simplicity, we concentrate on the translational motion, in the following, but the same procedure applies also to the rotational motion.

From the Fourier transform of Eq. (11) in the absence of an external force,

\[ -i \omega M V(\omega) = -\zeta^+(\omega) V(\omega) + \xi^T(\omega), \]

we derive the velocity spectral density

\[ C_V(\omega) \equiv \langle V(\omega) \cdot V(-\omega) \rangle = |\mu(\omega)|^2 C_\xi^T(\omega). \]

Here

\[ C_\xi^T(\omega) = 3 k_B T^T(\omega) \zeta(\omega) \]

is the noise spectral density and \( \mu(\omega) \) is the particle mobility defined as

\[ \mu(\omega) = \frac{1}{\zeta(\omega) - i \omega M}. \]

The Wiener-Khinchine theorem then gives the velocity auto-correlation function

\[ \langle V(t) \cdot V(0) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mu(\omega)|^2 C_\xi^T(\omega) e^{-i \omega t} \, d\omega, \]

from which the translational kinetic temperature \( T^V \), defined in Eq. (20), follows as

\[ T^V = \frac{M_{\text{eff}}}{\pi} \int_0^{\infty} |\mu(\omega)|^2 T^T(\omega) \zeta(\omega) \, d\omega, \]

since the integrand is an even function of \( \omega \).

To further evaluate this result, we introduce into the mobility \( \mu \) the explicit expression for the memory kernel

\[ M_{\text{eff}} = M + M g/(2 \theta_B). \]
of a sphere translating in an incompressible fluid with no-slip boundary conditions [11]:

\[
\xi^+(\omega) = 6\pi\eta R \left[ 1 + (1 - i) \sqrt{\frac{R^2\omega}{2\nu} - iR^2\omega^2/9\nu} \right].
\]

The first term in the brackets is the usual Stokes friction \(\xi^+(\omega = 0) \equiv \xi\), the second describes the vorticity diffusion and gives rise to the long-time tails [18, 19]. The third term accounts for the mentioned mass renormalization, Eq. (19). With the notation \(x^2 = \omega/\omega_t\), Eq. (20) now reads

\[
T^\nu = \frac{1}{\pi} \int_0^{\infty} \frac{4\alpha x(x + 1)T^\nu(x)}{(1 + x)^2 + x^2(1 + \alpha x)^2} dx,
\]

which depends on the particle-to-fluid density ratio via the parameter \(\alpha \equiv 2(2\varrho_p/\varrho + 1)/9\). The same procedure gives the rotational kinetic temperature

\[
T^\Theta = \frac{1}{2} \int_0^{\infty} \frac{4\beta x(x + 1)T^\Theta(x)}{(1 + x)^2 + x^2(1 + \beta x)^2} dx
\]

with \(\beta \equiv 2\varrho_p/(15\varrho)\).

Equations (28), (29) can be integrated numerically using the translational and rotational noise temperatures \(T(\omega)\) introduced in Section III. The results are shown in Figure 2. The kinetic temperatures are seen to depend on the density ratio \(\varrho_p/\varrho\). To understand this, consider a translating sphere. In the Markov limit, its velocity relaxes within the Stokes time, corresponding to the relaxation rate

\[
\omega_p \equiv \frac{\zeta}{M} = \frac{6\pi\eta R}{M} = \frac{9\varrho}{4\varrho_p\omega_t},
\]

introduced in Eq. (1). The density ratio thus relates the characteristic time for the kinematic equilibration of the particle with the fluid — i.e. the time it takes to spread the particle momentum to a fluid mass comparable to the particle mass — to the time it takes to spread its momentum to a fluid volume comparable to the particle volume. Accordingly, the kinematic equilibration affects either a small or large fluid volume compared to the particle size, suggesting a kinetic temperature close to the temperature \(T_\alpha\) at the particle surface or close to the stationary effective temperature \(T^\infty\), respectively.

Indeed, if \(\varrho_p/\varrho \ll 1\), only the upper part of the spectrum \(T(\omega)\) contributes to the kinetic temperatures, as seen from Eqs. (21-26), where the integrand contributes significantly only for \(x \gg 1\). Hence, the rotational kinetic temperature \(T^\alpha\) approaches the surface temperature:

\[
T^\alpha \sim T^\infty(\infty) = T_0 + \Delta T = T_s \quad \text{for} \ \varrho_p/\varrho \to 0.
\]

Due to the mass renormalization, Eq. (19), the translational kinetic temperature \(T^\nu\) always remains somewhat below this limit, though. Although the noise temperature attempts to shake the particle with a strength proportional to the surface temperature \(T_s\), the particle cannot move without exciting a long ranged flow field that ultimately increases its own inertia. This effect limits the velocity fluctuations of the particle to a non-universal apparent “equipartition” temperature \(T^\nu\) that depends on the density ratio \(\varrho_p/\varrho\), and attains the limit

\[
T^\nu \simeq T_0 + 0.86 \Delta T < T_s \quad \text{for} \ \varrho_p/\varrho \to 0.
\]

As a consequence, the translational particle velocity never thermalizes to the fluid temperature at the particle surface.

In the opposite limit, \(\varrho_p/\varrho \gg 1\), the frequency-dependent terms in Eq. (27), which are proportional to \(R^2\omega_p/\nu = 2\omega_p/\omega_t \ll 1\), become small. In this limit, the kinetic temperature approaches the stationary values of the respective effective noise temperatures \(T(0)\), which coincide with the known temperatures for the configurational degrees of freedom, represented by the positional and orientational coordinates \(X\) and \(\Theta\) [6, 7] (see Sec. III). They determine the translational and rotational diffusion coefficient of the hot Brownian particle, e.g. for translation,

\[
D = \lim_{t \to \infty} \frac{1}{6} \frac{d}{dt} \langle (X(t) - X(0))^2 \rangle = \frac{1}{2} \int_{-\infty}^{\infty} \langle (V(t) - V(0))^2 \rangle dt = \frac{1}{2} C_V(\omega)|_{\omega=0}.
\]

Using Eqs. (22), (23), (24), and (27), we recover (to leading order in the temperature increment \(\Delta T\), i.e. not accounting for the temperature-induced spatial variations in the viscosity) the generalized Einstein relation [6]

\[
D_{\text{HBM}} = \frac{k_B T^\nu(\omega)}{\zeta^+(\omega)} \bigg|_{\omega=0} = \frac{k_B (T_0 + \frac{5}{18} \Delta T)}{6\pi\eta R}.
\]

The same reasoning applies to the orientation \(\Theta\). Hence, we see that for a hot Brownian particle that is much denser than the solvent, the kinetic temperatures reduce to the effective configurational temperatures,

\[
T^{\nu,\Theta} \sim T^{\infty,\Theta}(0) = T^{\infty,\Theta} \quad \text{for} \ \varrho_p/\varrho \to \infty.
\]

Moreover, in any case, both the translational and rotational velocities of a hot spherical particle can be sta-
tistically characterized by a (non-universal) Maxwell–Boltzmann distribution
\[
P(\mathbf{V}, \Omega) \propto \exp \left( -\frac{M_{\text{eff}} V^2}{2k_B T^v} - \frac{I\Omega^2}{2k_B T^u} \right) \quad (36)
\]
with effective temperatures that depend on the density ratio \( \varrho_p/\varrho \), in agreement with the fact that probability distributions of non-equilibrium ensembles explicitly depend on the dynamics of the system.

V. PARTICLE IN A HARMONIC POTENTIAL

The discussion of the previous section can be repeated for a particle trapped in a harmonic potential. While the kinetic temperature of a free particle is determined by the competition between the vorticity diffusion time \( \omega^{-1}_v \) and the Stokes relaxation time \( \omega^{-1}_s \) introduced in Eq. 11, a sufficiently narrow confining potential introduces an additional interfering time scale. In the following, we examine more closely the case of translational diffusion in confinement, but qualitatively similar results can be derived for the rotational case.

The parabolic confinement potential \( U(X) = KX^2/2 \) gives rise to the trap relaxation time
\[
\omega^{-1}_t = 6\pi\eta R/K = \omega_p/\omega_0^2, \quad (37)
\]
where \( \omega_0^2 = K/M \) is the undamped oscillation frequency. With \( F_c = -KX \), the Fourier transformed Eq. 11, \[-M\omega^2 X(\omega) = i\omega\zeta^{+}(\omega) X(\omega) - KX(\omega) + \xi(\omega), \]
yields the spectral density
\[
C_X(\omega) \equiv \langle X(\omega) \cdot X(-\omega) \rangle = |R(\omega)|^2 C_\zeta(\omega), \quad (38)
\]
where the (positional) response function is defined by
\[
R(\omega) = \frac{1}{M(\omega_0^2 - \omega^2) - i\omega\zeta^{+}(\omega)}. \quad (39)
\]
We use the relation
\[
\langle \mathbf{V}(t) \cdot \mathbf{V}(0) \rangle = -\frac{d^2}{dt^2} \langle \mathbf{X}(t) \cdot \mathbf{X}(0) \rangle,
\]
between the stationary correlation functions for position and velocity in frequency space, \( C_V(\omega) = \omega^2 C_X(\omega) \). The kinetic temperature, as defined in Eq. 20, follows as
\[
T^v = \frac{M_{\text{eff}}}{\pi} \int_0^{\infty} \omega^2 |R(\omega)|^2 T^v(\omega)\zeta^{+}(\omega) \, d\omega
= \frac{1}{\pi} \int_0^{\infty} 4\alpha x^3(x+1)T^v(x) \, dx \quad (40)
\]
The result is again integrated numerically and depicted in Fig. 3. Clearly, if \( \omega_t \ll \omega_v \), which means that the potential is not effective while the velocity is relaxing, we recover the result for free diffusion, Eq. 23. This should be the case for an optically trapped nano-particle in water under standard experimental conditions. Indeed for a gold particle with \( R \approx 100 \text{ nm} \), assuming a trap stiffness \( K = 10^{-6} \text{Nm}^{-1} \), we estimate \( \omega_t/\omega_v \approx 10^{-2} \). The velocity relaxation time decreases as we increase the ratio \( \omega_t/\omega_v \), resulting in a higher kinetic temperature. When \( \omega_t \gg \omega_v \) the narrow confinement eventually overrides the inertia of the particle motion due to its effective mass \( M_{\text{eff}} \), so that the kinetic temperature \( T^v \) approaches the surface temperature \( T_s \).

Similar as for the particle velocity, we define the positional temperature of a hot Brownian particle in a har-
monic potential via the generalized equipartition theorem,

\[ \frac{3}{2} k_B T^X = \frac{1}{2} \omega_0^2 M \langle X^2 \rangle, \quad (41) \]

where the average is taken with respect to the stationary distribution. Using Eqs. \ref{eq:equipartition} and \ref{eq:generalized equipartition} we straightforwardly obtain

\[ T^X = \frac{\omega_0^2 M}{\pi} \int_0^\infty |R(\omega)|^2 T^X(\omega) \langle \xi(\omega) \rangle \, d\omega \]

\[ = \frac{1}{\pi} \int_0^\infty \frac{4(\omega_t/\omega_0) x(x + 1) T^X(x)}{x^4(1 + x)^2 + (x^3 + \alpha x^4 - \omega_t/\omega_0)^2} \, dx. \quad (42) \]

This result is integrated numerically and plotted in Fig. \ref{fig:position_temperature}. Again, if \( \omega_t \ll \omega_0 \), we recover the configurational temperature of a free particle, since the integral in Eq. \ref{eq:generalized equipartition} is sharply peaked at \( x \ll 1 \), corresponding to \( \omega \ll \omega_0 \). Physically, the relaxation in the potential takes place quasi-statically with respect to the free hot Brownian motion, which can then be represented in the Markov approximation, in perfect analogy to the equilibrium case. The corresponding Langevin equation is

\[ \zeta \dot{X} = -\nabla \mathcal{U} + \xi, \quad \langle \xi(t)\xi(t') \rangle = 2 \mathcal{D}_{HBM} \delta(t - t') \delta_{ij}, \]

and its stationary solution is the generalized Boltzmann distribution

\[ P(X) \propto \exp \left( \frac{\mathcal{U}(X)}{k_B T^X} \right). \]

with the effective temperature

\[ T^X = \zeta D_{HBM} = T_0 + \frac{5}{12} \Delta T \quad (43) \]

of free hot Brownian motion \ref{eq:free HBM} (originally denoted by \( T_{HBM} \)).

In contrast, if \( \omega_t \approx \omega_p \) the potential interferes with the relaxation of the particle, resulting in a higher \( T^X \) than in the free case. Eventually, in the extreme limit \( \omega_t \gg \omega_p \), the integral peaks near \( \omega_t \) and \( T^X \) approaches the kinetic temperature \( T^K \) (non-uniformly in \( \theta_0/\theta \)). It is moreover worth noting that the stationary probability distribution can in any case still be written in the form of Eq. \ref{eq:generalizedequipartition}, albeit with non-universal temperatures \( T^V \) and \( T^X \) that generally depend on the density ratio \( \theta_0/\theta \) and on the stiffness \( K \) of the potential. Analogous conclusions hold for the rotational degrees of freedom.

VI. CONCLUSION

Starting from the fluctuating hydrodynamic description of the solvent, which we required to be in local thermal equilibrium with an inhomogeneous temperature field \( T(r) \), we have derived a generalized Langevin equation for the motion of a suspended particle. While the discussion was limited to the important case of hot Brownian motion, where \( T(r) \) decays radially around the particle, essentially the same reasoning applies to more general temperature profiles \ref{fig:temperature_profile}. As a consequence of the non-isothermal conditions, the noise temperature \( T(\omega) \) characterizing the strength of the stochastic Langevin forces becomes frequency dependent and differs for different degrees of freedom, which couple to different hydrodynamic modes. From the noise temperature, we derived approximate expressions for the effective temperatures at which the rotational and translational degrees of freedom of a spherical particle appear to thermalize. Explicit numerical results have been limited to first order in the temperature increment \( \Delta T \), so that the temperature-dependence of the fluid viscosity could be neglected. We found the (angular) velocities to be Maxwell–Boltzmann distributed with non-universal, but explicitly known, effective temperatures. In the long-time limit we regained previous results for the configurational temperatures governing free and weakly confined hot Brownian motion.
Appendix A: Derivation of the GLE’s noise autocorrelation function

Extending the calculation presented in [21] to a non-isothermal solvent, we derive the expressions Eq. (A3a) and Eq. (A4b) of Section II for the translational motion. The same procedure can be applied separately to rotational motion bearing in mind that, tanks to linearity and spherical symmetry, the flow field \( \mathbf{v} \) can be divided into the two independent fields \( \mathbf{v}^T \) and \( \mathbf{v}^R \) generated, respectively, by the particle translation and rotation, and satisfying the boundary conditions:

\[
\begin{align*}
\mathbf{v}^T(r,t) &= \mathbf{V}(t) \text{ on } \mathcal{S}, \\
\mathbf{v}^R(r,t) &= \Omega(t) \times r \text{ on } \mathcal{S}.
\end{align*}
\]

Since we focus on the translational motion only, we omit the superscript \( T \). Using Eq. (A3b), the Fourier transform of the generalized Langevin Eq. (II) reads

\[
-i\omega M \mathbf{V}(\omega) = -\zeta^+(\omega)\mathbf{V}(\omega) + \xi(\omega) + \mathbf{F}_e(\omega),
\]

and may be rewritten as

\[
-i\omega M \mathbf{V}(\omega) = \mathbf{f}(\omega) + \tilde{\mathbf{f}}(\omega) + \mathbf{F}_e(\omega), \tag{A1}
\]

where we have divided the force exerted by the fluid into deterministic \( \mathbf{f}(\omega) \) and random \( \tilde{\mathbf{f}}(\omega) \) components:

\[
\begin{align*}
\mathbf{f}(\omega) &\equiv -\zeta^+(\omega)\langle \mathbf{V}(\omega) \rangle, \tag{A2a} \\
\tilde{\mathbf{f}}(\omega) &\equiv -\zeta^+(\omega)\tilde{\mathbf{V}}(\omega) + \xi(\omega), \tag{A2b}
\end{align*}
\]

with \( \mathbf{V} \equiv \langle \mathbf{V} \rangle + \tilde{\mathbf{V}} \). It is easy to see that \( \mathbf{f}(\omega) \) is the force exerted by the deterministic flow field \( \mathbf{u} \equiv \langle \mathbf{v} \rangle \), the solution of

\[
\begin{align*}
i\omega \rho \mathbf{u}(r,\omega) + \nabla \cdot \mathbf{\sigma}(r,\omega) &= 0, \tag{A3a} \\
\nabla \cdot \mathbf{u}(r,\omega) &= 0, \tag{A3b} \\
\mathbf{u}(r,\omega) &= \langle \mathbf{V}(\omega) \rangle \text{ on } \mathcal{S}. \tag{A3c}
\end{align*}
\]

while \( \tilde{\mathbf{f}}(\omega) \) is the force exerted by the stochastic flow field \( \tilde{\mathbf{u}} \equiv \mathbf{v} - \langle \mathbf{v} \rangle \), the solution of

\[
\begin{align*}
i\omega \rho \tilde{\mathbf{u}}(r,\omega) + \nabla \cdot \tilde{\mathbf{\sigma}}(r,\omega) &= -\nabla \cdot \mathbf{\tau}(r,\omega), \tag{A4a} \\
\nabla \cdot \tilde{\mathbf{u}}(r,\omega) &= 0, \tag{A4b} \\
\tilde{\mathbf{u}}(r,\omega) &= \mathbf{\tilde{V}}(\omega) \text{ on } \mathcal{S}. \tag{A4c}
\end{align*}
\]

This splitting of equations and boundary conditions is again allowed by the linearity of the problem. In the following calculation, in order to ease the notation, we omit the arguments \( r \) and \( \omega \) of the hydrodynamic fields where there is no possibility of confusion.

We start by calculating twice the energy dissipated by the particle moving at velocity \( \langle \mathbf{V}(\omega) \rangle \):

\[
\begin{align*}
\langle V_i(\omega) \rangle (\zeta^+(\omega) + \zeta^{+*}(\omega)) \langle V_i^*(\omega) \rangle &= \\
\equiv -(f_i(\omega)\langle V_i^*(\omega) \rangle + f_i^*(\omega)\langle V_i(\omega) \rangle) \\
\equiv \langle V_i^*(\omega) \rangle \int_{\mathcal{S}} \sigma_{ij} n_j \, d^2r + \langle V_i(\omega) \rangle \int_{\mathcal{S}} \sigma_{ij}^* n_j \, d^2r \\
\equiv \int_{\mathcal{V}} \partial_j (u_i^* \sigma_{ij}) \, d^3r + \int_{\mathcal{V}} \partial_j (u_i \sigma_{ij}^*) \, d^3r \\
\equiv \int_{\mathcal{V}} (\sigma_{ij} \partial_j u_i^* + \sigma_{ij}^* \partial_j u_i) \, d^3r \\
\equiv 2\eta (\Gamma_{ij} \partial_j u_i + \Gamma_{ij}^* \partial_j u_i) \, d^3r \\
&= 2 \int_{\mathcal{V}} \phi(r,\omega) \, d^3r. \tag{A5}
\end{align*}
\]
where in Eq. \( A5 \) we employed the divergence theorem and in Eq. \( A6 \) we defined the dissipation function:

\[
\phi(r, \omega) = \eta(r) \left( \partial_i u_i \partial_i u_j^* + \partial_i u_j \partial_j u_i^* \right)(r, \omega).
\]

Since \( \zeta(\omega) = 2 \text{Re} \zeta^+(\omega) = \zeta^+(\omega) + \zeta^{+\ast}(\omega) \), being \( \zeta(t) \) real and time-symmetric, we can rewrite Eq. \( A6 \) in the following form:

\[
\zeta(\omega) \delta_{ij} \langle V_i(\omega) \rangle \langle V_j^*(\omega) \rangle = 2 \int_V \phi(r, \omega) \, d^3r
\]

(A7)

This proves Eq. \( 13 \). We proceed with the evaluation of the energy supplied by the random force \( \mathbf{\xi}(\omega) \):

\[
\xi_i(\omega) \langle V_i(\omega) \rangle = \int_V \tau_{ij}(r, \omega) \partial_i u_j(r, \omega) \, d^3r,
\]

(A9)

which asserts that \( \mathbf{\xi}(\omega) \) is Gaussian with vanishing mean, being the integral of the deterministic quantity \( \partial_j u_i \) times the zero-mean Gaussian field \( \tau_{ij} \). Using Eq. \( A9 \) we evaluate the noise correlation function:

\[
\langle \xi_i(\omega) \xi_j^*(\omega') \rangle \langle V_i(\omega) \rangle \langle V_j^*(\omega') \rangle = 2 k_B \delta(\omega - \omega') \int_V \phi(r, \omega) T(r) \, d^3r.
\]

(A10)

In Eq. \( A10 \) we used the Fourier transform of Eq. \( 3 \) together with \( \tau_{kl}^\ast(r', \omega') = \tau_{kl}(r', -\omega') \), since \( \mathbf{\tau} \) is real. This proves Eq. \( 14 \).

Appendix B: Hydrodynamics of a translating and rotating sphere

1. Translational motion

The Fourier transform of the flow field generated by a sphere translating with velocity \( V(\omega) \mathbf{e}_z \) reads in polar coordinates \( (r, \varphi, \theta) \) [22, p. 623]:

\[
\mathbf{u}^\tau(r, \theta, \omega) = \frac{1}{r} \sin \theta \left( g + \frac{dg}{dr} \right) \mathbf{e}_\theta - 2 g \cos \theta \mathbf{e}_r,
\]

with

\[
g(r, \omega) = \frac{3 \eta V(\omega) R}{2(k r)^2} \left[ (i k - 1) e^{ik(r-R)} - \left( 1 + i k R - \frac{1}{3} (kR)^2 \right) \right],
\]

where in Eq. \( A5 \) we employed the divergence theorem and in Eq. \( A6 \) we defined the dissipation function:

\[
\phi(r, \omega) = \eta(r) \left( \partial_i u_j \partial_i u_j^* + \partial_i u_j \partial_i u_j^* \right)(r, \omega).
\]

Since \( \zeta(\omega) = 2 \text{Re} \zeta^+(\omega) = \zeta^+(\omega) + \zeta^{+\ast}(\omega) \), being \( \zeta(t) \) real and time-symmetric, we can rewrite Eq. \( A6 \) in the following form:

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\langle \xi_i(\omega) \xi_j^*(\omega') \rangle \langle V_i(\omega) \rangle \langle V_j^*(\omega') \rangle = 2 k_B \delta(\omega - \omega') \int_V \phi(r, \omega) T(r) \, d^3r.
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Appendix B: Hydrodynamics of a translating and rotating sphere

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\]

with

\[
g(r, \omega) = \frac{3 \eta V(\omega) R}{2(k r)^2} \left[ (i k - 1) e^{ik(r-R)} - \left( 1 + i k R - \frac{1}{3} (kR)^2 \right) \right],
\]
where $k = (1 + i)k_0$, and $k_0 = \sqrt{\nu/2\pi}$ is the inverse of the characteristic fluid diffusion length. The associated dissipation function is:

$$\phi^T = \eta \left( \frac{12}{r^4} \cos^2 \theta \left| g - r \frac{dg}{dr} \right|^2 + \sin^2 \theta \left| \frac{d^2 g}{dr^2} \right|^2 \right),$$

that becomes after integration over $\theta$:

$$\int_0^{\pi} \phi^T(r, \theta, \omega) \sin \theta \, d\theta = \frac{3\eta |V(\omega)|^2 R^2}{2k_0^6 r^6} \times \left\{ \begin{array}{c} 5[9 + 2k_0 R(9 + k_0 R(9 + 2k_0 R(3 + k_0 R)))] + e^{-2k_0(r-R)}[45 + 2k_0 r(45 + k_0 r(45 + k_0 r(30 + k_0 r(15 + 2k_0 r(3 + k_0 r))))))] \\
-2e^{-k_0(r-R)} [(45 + 45k_0 R + 15k_0(3 + 2k_0 R(3 + k_0 R)))r + 12k_0^3 R(3 + 2k_0 R)r^2 + 2k_0^3 (-3 + 2k_0^2 R^2) r^3 \cos[k_0(R-r)] \\
-k_0 (15R(3 + 2k_0 R) + 15 (3 - 2k_0^2 R^2) r + 36k_0(1 + k_0 R)r^2 + 2k_0^2 (3 + 2k_0 R(3 + k_0 R))r^3) \sin[k_0(R - r)] \end{array} \right\}. \quad (B1)$$

Notice that Eq. (B1) displays a term which does not decay with an exponential cutoff but only algebraically as $1/r^6$. But its contribution to $\mathcal{T}(\omega)$ actually diminishes at high frequencies, $k_0 \to \infty$. In order to obtain $\mathcal{T}(\omega)$ we numerically integrate Eq. (17) together with Eq. (B1). The result is shown in Figure 1.

### 2. Rotational motion

The Fourier transform of the flow field generated by a sphere rotating with angular velocity $\Omega(\omega) e_z$ reads [11, p. 91]:

$$u^R(r, \theta, \omega) = \frac{\Omega(\omega)R^3}{r^2} \sin \vartheta \frac{1 - ikr}{1 - ikR} e^{ik(r-R)} e_\varphi \equiv f(r, \vartheta, \omega)e_\varphi. \quad (B2)$$

The associated dissipation function $\phi^R(r, \omega)$ is:

$$\phi^R = \frac{\eta}{r^2} \left( |r \partial_r f - f|^2 + |\partial_\theta f - \cot \vartheta f|^2 \right) = \frac{\eta |\Omega(\omega)|^2 R^6 \left[ 9 + 18k_0 R + 18(k_0 R)^2 + 12(k_0 R)^3 + 4(k_0 R)^4 \right] \sin^2 \vartheta e^{-2k_0(r-R)}}{r^6 \left[ 1 + 2k_0 R + 2(k_0 R)^2 \right]} \quad. \quad (B2)$$

Notice in Eq. (B2) the exponential cutoff where the fluid’s diffusion characteristic length $k_0^{-1}$ appears. Using Eq. (17) and Eq. (B2) we obtain the first-order approximation in $\Delta T$ for the noise temperature of rotational motion:

$$\frac{T^R(\omega) - T_0}{\Delta T} = \frac{9 + 18k_0 R + 18(k_0 R)^2 + 12(k_0 R)^3 - 8(k_0 R)^4 E_1(2k_0 R)e^{2k_0 R}}{4 (3 + 6k_0 R + 6(k_0 R)^2 + 2(k_0 R)^3)}, \quad (B3)$$

where $E_1(x) = \int_1^\infty dy e^{-xy}/y$ is the exponential integral. The result is plotted in Figure 1.

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For rotation, Eq. (A3a) reduces to the diffusion equation
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