On central-max-point tolerance graphs

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Abstract

Max-point-tolerance graphs (MPTG) were introduced by Catanzaro et al. in 2017 and the same class of graphs were studied in the name of $p$-box(1) graphs by Soto and Caro in 2015. This class has a wide application in genome studies as well as in telecommunication networks.

In our paper we consider central-max-point tolerance graphs (CMPTG) by taking the points of MPTG as center points of their corresponding intervals. In course of study on this class of graphs and we show that the class of CMPTG is same as the class of unit-max-tolerance graphs (UMTG). We prove the class of unit CMPTG is same as that of proper CMPTG and both of them are equivalent to the class of proper interval graphs.

Next we introduce 50% max-tolerance graphs and separate this class from UMTG which are same for min-tolerance graphs. We also show that proper interval graphs form a subclass of the class of 50% max-tolerance graphs. Moreover we show that every chordless cycle is a 50% max-tolerance graph.

We find a close relation between a CMPTG and a central interval catch digraph (CICD). We disprove a conjecture regarding CICD posed by Maehara in 1984. Finally we introduce the concepts of optimized digraphs, optimized graphs and prove that these classes of graphs merge with the class of CICD and that of proper interval graphs respectively.

Keywords: Interval graph, proper interval graph, tolerance graph, max-tolerance graph, max-point tolerance graph, interval catch digraph.

1 Introduction

The class of interval graphs was initially posed by Hajós in 1957 [10] as a study of intersection graphs of intervals on real line. In 1959, the molecular biological scientist Benzer [2] used the model of interval graphs to obtain a physical map from information on pairwise overlaps of the fragments of DNA. Interval graphs were well studied by many people in Computer Science and Discrete Mathematics for its wide application. Many combinatorial problems have been solved for interval graphs in linear time.

Due to their lot of applications in theories and practical situations the concept was generalized to several variations. In one direction it went in developing concepts of probe interval graphs, circular-arc graphs, interval digraphs [1], [9]. On the other hand in 1982, Golumbic and Monma introduced the concept of min-tolerance graphs (commonly known as tolerance graphs) [7]. A simple undirected graph $G = (V, E)$ is a min-tolerance graph if each vertex $u \in V$ corresponds to a real interval $I_u$ and a positive real number $t_u$ (known as tolerance), such that $uv$ is an edge of $G$ if and only if $|I_u \cap I_v| \leq \min \{t_u, t_v\}$.

In [8] Golumbic introduced max-tolerance graphs (in brief, MTG) where each vertex $u \in V$ corresponds to a real interval $I_u$ and a positive real number $t_u$ (known as tolerance) such that $uv$ is an edge of $G$ if and only if $|I_u \cap I_v| \geq \max \{t_u, t_v\}$. An MTG is a unit-max-tolerance graph (in brief, UMTG) if $|I_u| = |I_v|$ for all $u, v \in V$. Some combinatorial problems like finding maximal cliques were obtained in polynomial time whereas the recognition problem was proved to be NP-hard for max-tolerance graphs in 2006 [11]. Also they

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have given a geometrical connection of max-tolerance graphs to semi-squares. For further details of tolerance graphs see [8].

Recently max-point-tolerance graphs (in brief, MPTG) are introduced in [5], where each vertex $u \in V$ corresponds to a pointed interval $(I_u, p_u)$, where $I_u$ is an interval on the real line and $p_u \in I_u$, such that $uv$ is an edge of $G$ if and only if $\{p_u, p_v\} \subseteq I_u \cap I_v$. The graphs MPTG have a huge number of practical applications in human genome studies and modelling of telecommunication networks [5]. An MPTG $G = (V, E)$ is central if $p_u$ is the center point of $I_u$ for each $u \in V$. We call this graph by central-max-point-tolerance graph (in brief, CMPTG). This graph class actually matches with the graph defined as c-p-box(1) graph in [18]. They have shown that c-p-box(1) graphs form a subclass of max-tolerance graphs.

In our paper we prove that CMPTG is same as UMTG. Moreover we show that a unit CMPTG (in brief, UCMPTG) is same as a proper CMPTG (in brief, PCMPTG) and also is same as a proper interval graph. We introduce 50% max-tolerance graphs analogous to the similar concept for min-tolerance graphs. In case of min-tolerance graphs, unit and 50% are defining the same class of graphs [4]. In our paper we show that for max-tolerance graphs they are not same. Also we show that the classes CMPTG and 50% max-tolerance graphs are not comparable although they contain various classes of graphs (for example $C_n, n \geq 3$, proper intervals graphs) in common. Finally we find a close relation of CMPTG with central interval catch digraphs (in brief, CICD) where the points $p_v$ associated to the intervals $I_v$ are center points. For more details about interval catch digraph (in brief ICD) see Section 2. Further we introduce the concepts of optimized digraphs, optimized graphs and prove that these classes of graphs merge with the class of CICD and that of proper interval graphs respectively. Also we obtained the adjacency matrix characterization of a CICD which forms a tournament. In conclusion we show the relations between the subclasses of MTG and CMPTG and list major open problems in this area.

2 Preliminaries

The following characterization of MPTG is known:

**Theorem 2.1.** [5] Let $G = (V, E)$ be a simple undirected graph. Then $G$ is an MPTG if and only if there is an ordering of vertices of $G$ such that the following condition holds:

$$\text{For any } x < u < v < y, \text{ } xv, uy \in E \implies uv \in E. \quad (2.1)$$

**Theorem 2.2.** [18] The graph class CMPTG properly contains the class of interval graphs and the class of outer planar graphs.

A matrix whose entries are only zeros and ones is a binary matrix. A binary matrix is said to satisfy consecutive 1’s property for rows if its columns can be permuted in such a way that 1’s in each row occur consecutively [14]. For a simple undirected graph $G = (V, E)$, a matrix is known as the augmented adjacency matrix of $G$ if we replace all principal diagonal elements of the adjacency matrix of $G$ by 1 [6]. A simple directed graph $G = (V, E)$ is an interval catch digraph (in brief, ICD) if each vertex $u \in V$ corresponds to a pointed interval $(I_u, p_u)$, where $I_u$ is an interval on the real line and $p_u \in I_u$, such that $uv$ is an arc of $G$ if and only if $u \neq v$ and $p_v \in I_u$. [15] An interval catch digraph is central if $p_u$ is the center point of $I_u$ for each $u \in V$. We call this graph by central interval catch digraph (in brief, CICD) [2].
Theorem 2.3. [12] Let D be a simple directed graph. Then D is an interval catch digraph if and only if there exists an ordering of vertices such that the augmented adjacency matrix A(D) satisfies the consecutive 1’s property for rows.

Corollary 2.4. [13] Let D = (V, E) be a simple directed graph. Then D is an interval catch digraph if and only if there exists an ordering of vertices such that

\[
\text{for all } x < y < z \in V, \ (xz \in E \implies xy \in E) \quad \text{and} \quad (zx \in E \implies yz \in E). \tag{2.2}
\]

Among many characterizations of proper interval graphs we list the following which will serve our purpose. The reduced graph \( \tilde{G} \) is obtained from \( G \) by merging vertices having same closed neighborhood. \( G(n, r) \) is a graph with \( n \) vertices \( x_1, x_2, \ldots, x_n \) such that \( x_i \) is adjacent to \( x_j \) if and only if \( 0 < |i - j| \leq r \), where \( r \) is a positive integer.

Theorem 2.5. [12, 13] (pg 387, ex 11.17), [16, 17] Let \( G = (V, E) \) be an interval graph. then the following are equivalent:

1. \( G \) is a proper interval graph.
2. \( G \) is a unit interval graph.
3. For all \( v \in V \), elements of \( N[v] = \{u \in V \mid uv \in E\} \cup \{v\} \) are consecutive for some ordering of \( V \) (closed neighborhood condition).
4. \( \tilde{G} \) is an induced subgraph of \( G(n, r) \) for some positive integers \( n, r \) with \( n > r \).

3 Central max point tolerance graphs (CMPTG)

We begin with a trivial but important observation which will be used throughout rest of the paper. We denote the length of an interval \( I \) by \( |I| \).

Observation 3.1. Let \( \{I_u \mid u \in V\} \) be a collection of intervals, where \( I_u = [t_u, r_u] \), \( h_u = |I_u| = r_u - t_u \) and \( c_u = \frac{t_u + r_u}{2} \). Then \( \{c_u, c_v\} \subseteq I_u \cap I_v \iff |c_v - c_u| \leq \frac{1}{2} \min \{h_u, h_v\} \iff \ell_v \leq c_u \leq c_v \leq r_u \) (for \( c_u \leq c_v \)).

Proof. We have \( c_u \in I_v = [l_v, r_v] \iff |c_v - h_u|, c_v + \frac{h_u}{2} \iff c_v - \frac{h_u}{2} \leq c_u \leq c_v + \frac{h_u}{2} \iff -\frac{h_u}{2} \leq c_u - c_v \leq \frac{h_u}{2} \iff |c_u - c_v| \leq \frac{h_u}{2} \). Thus \( \{c_u, c_v\} \subseteq I_u \cap I_v \iff |c_v - c_u| \leq \frac{1}{2} \min \{h_u, h_v\} \). Also it is clear that \( c_u \in I_v = [l_v, r_v] \iff \ell_v \leq c_u \leq r_v \).

In the following theorem we show that the classes of CMPTG and UMTG are same.

Theorem 3.2. Let \( G \) be a simple undirected graph. Then \( G \) is a CMPTG if and only if \( G \) is a UMTG.

Proof. Let \( G = (V, E) \) be a CMPTG with a CMPTG representation \( \{I_u = [l_u, r_u] \mid u \in V\} \). Let \( c_u = \frac{l_u + r_u}{2} \) and \( h_u = r_u - l_u \) for all \( u \in V \). Choose \( h_0 > \max \{h_u \mid u \in V\} \). Define \( t_u = \frac{h_0 - h_u}{2} \), \( y_u = c_u + \frac{h_0}{2} \) and \( T_u = [c_u, y_u] \). Note that \( t_u > 0 \) and \( |T_u| = y_u - c_u = \frac{h_0}{2} \) which is a constant for all \( u \in V \).

Suppose \( uv \in E \) and \( c_u \leq c_v \). Then \( c_u - c_v \leq \frac{1}{2} \min \{h_u, h_v\} \leq \frac{h_0}{2} \). So \( c_u \leq c_v + \frac{h_0}{2} = y_v \). This implies \( c_v \leq c_u \leq y_v \). So \( T_u \cap T_v = [c_u, y_v] \neq \emptyset \) and \( |T_u \cap T_v| = y_v - c_u = c_v + \frac{h_0}{2} - c_u = \frac{h_0}{2} - (c_u - c_v) \geq \frac{h_0 - h_u}{2} + \frac{h_0 - h_v}{2} \). So \( y_v - c_u \geq t_u, t_v \), i.e.,

\[
|T_u \cap T_v| \geq \max \{t_u, t_v\}. \tag{3.1}
\]
On the other hand, $3.1$ implies $T_u \cap T_v \neq \emptyset$ and $c_u \leq y_v \leq y_u$. So $|T_u \cap T_v| = y_v - c_u$. Now $y_v - c_u \geq \max\{t_u, t_v\}$ implies $\frac{h_0}{2} - (c_u - c_v) \geq \max\{\frac{h_0 - h_u}{2}, \frac{h_0 - h_v}{2}\}$. Thus $c_u - c_v \leq \frac{1}{2}\min\{h_u, h_v\}$, i.e., $uv \in E$. Therefore $G$ is a UMTG with interval representation $\{T_u = [c_u, y_u] | u \in V\}$ and tolerances $\{t_u | u \in V\}$ as defined above.

Conversely, let $G = (V, E)$ be a UMTG with interval representation $\{T_u = [u, r_u] | u \in V\}$ and tolerances $\{t_u | u \in V\}$. Let $h = |T_u|$ for all $u \in V$. Define $I_u = [l_u - (h - t_u), l_u + (h - t_u)]$. Then $c_u$, the center of $I_u = l_u$ and $h_u = |I_u| = 2(h - t_u) < 2h$. Suppose $uv \in E$. Then $|T_u \cap T_v| \geq \max\{t_u, t_v\}$. Now for $l_u \leq l_v$, $|T_u \cap T_v| = r_u - l_v$. Then $r_u - l_v \geq \max\{t_u, t_v\}$, i.e., $h + l_u - l_v \geq \max\{t_u, t_v\}$. This implies $l_v - l_u \leq \min\{h - t_u, h - t_v\}$, i.e., $c_v - c_u \leq \frac{1}{2}\min\{h_u, h_v\}$. Finally, the condition that $0 < c_v - c_u \leq \frac{1}{2}\min\{h_u, h_v\}$ holds $l_v - l_u < h$ and $l_u \leq v \leq l_u + h = r_u$. So $T_u \cap T_v \neq \emptyset$ and $|T_u \cap T_v| = r_u - l_v$. Then $l_v - l_u = c_v - c_u \leq \frac{1}{2}\min\{h_u, h_v\}$ implies $|T_u \cap T_v| \geq \max\{t_u, t_v\}$, i.e., $uv \in E$. Thus $\{I_u | u \in V\}$ is a CMPTG representation of $G$.

**Definition 3.3.** A CMPTG $G = (V, E)$ is called proper if it has an interval representation with the required condition such that no interval contains another properly. We call it proper-central-max-point tolerance graph (in brief, PCMPTG). Similarly, a CMPTG $G = (V, E)$ is called unit if it has an interval representation with the required condition such that every interval has unit (or, same) length. We call it unit-central-max-point tolerance graph (in brief, UCMPTG).

**Theorem 3.4.** Let $G$ be a simple undirected graph. Then the following are equivalent.

1. $G$ is a PCMPTG.
2. $G$ is a UCMPTG.
3. $G$ is a proper interval graph.

**Proof.** (1 $\iff$ 2) : First from a PCMPTG representation of $G$, we obtain a UCMPTG representation. When no interval properly includes another, the right end points have the same order as the left end points and so as the center points as well. We process the representation from left to right, adjusting all intervals to length $l$ where $l$ is the length of first interval (i.e; $|I_1| = l$). At each step until all have been adjusted, let $I_x = [a, b]$ be the unadjusted interval that has the leftmost left endpoint among the unadjusted intervals. Now one of the following cases may happen.

1. $I_x$ does not contain centre point of any adjusted interval.
2. $I_x$ contains center points of some adjusted intervals and $c = \frac{a + b}{2}$ also belong to those intervals.
3. $I_x$ contains center points of some adjusted intervals but $c = \frac{a + b}{2}$ does not belong to any of them.

Let $I_i$ and $I_j$ be the adjusted intervals referred in (2) and (3) respectively. We note that $I_j$ must occur before $I_i$ (i.e; $a_j < a_i, b_j < b_i$ where $a_k, b_k$ denote the leftmost, rightmost endpoint of an adjusted interval $I_k$). Take $a = a_i$ if (1) occurs and $a = c_i$ if (2) occurs where $c_i$ is the leftmost center point which belong to $I_x$ such that $c_i \in I_i$. Note $c_i$ belong to an interval that has already been adjusted to length $l$. It is easy to observe $c_k \in I_x$ and $c \in I_k$ for all $k$ where $i < k < x$. Now when (1), (2) does not occur but (3) occurs then we take $a = b_i$ where $b_i$ is the rightmost endpoint for which $c_i \in I_x$ but $c \notin I_i$. We adjust the portion $[a, \infty)$ by shrinking or expanding $[a, b]$ to $[a, a + \ell]$ and scaling and shifting $[b, \infty)$ to $[a + \ell, \infty)$.

Note that in second case after adjustment centre point of $I_x$ becomes $c_x = a + \ell = c_i + \ell = b_i \in I_i$. Again $b_{i-1} < b_i < b_x = a + \ell, c_x \notin I_{i-1}$ but $c_x \in I_k$ for all $i \leq k \leq x$. In third case $c_x = b_i + \ell > b_i$ which
imply \( c_x \notin I_j \) for any \( j \leq i \). In all of three cases the order of endpoints does not change and the adjacency relation also remains intact with \( I_x \), intervals earlier than \( I_x \) still have length \( l \), and \( I_x \) also now has length \( l \). Iterating this operation produces the UCMPTG representation.

Conversely, if \( G \) is a UCMPTG then all intervals associated to the vertices of \( G \) must be of the same length. Thus none of them contains other properly and so \( G \) is a PCMTG with the same interval representation.

\[ (3 \Rightarrow 1) : \text{Let } G = (V, E) \text{ be a proper interval graph. So the reduced graph } \hat{G} - (V', \hat{E}) \text{ is an induced subgraph of } G(n, r) = (V_n, E') \text{ for some } n, r \in \mathbb{N} \text{ with } n > r, \text{ where } V_n = \{v_1, v_2, \ldots, v_n\} \text{ and } v_i \leftrightarrow v_j \text{ if and only if } |i - j| \leq r \text{ by Theorem 2.5} \text{[4]. Let } V = \{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\} \subseteq V_n. \text{ Now for each } u \in V, \text{ define } p_u = i_j \text{ if } u \text{ is a copy of } v_{i_j} \text{ and } I_u = [p_u - r, p_u + r]. \text{ Firstly all intervals } I_u \text{ are of same length } 2r \text{ and so none of them properly contains other.} \]

Next let \( u, v \in V \). Suppose \( p_u = i_j \) and \( p_v = i_k \). Then \( u \) is a copy of \( v_{i_j} \) and \( v \) is a copy of \( v_{i_k} \). If \( uv \in E \), then \( v_{i_j}v_{i_k} \in \hat{E} \subseteq E' \). Therefore \( |i_j - i_k| > r \Rightarrow |p_u - p_v| > r \Rightarrow p_v \in I_u \text{ and } p_u \in I_v \Rightarrow p_u, p_v \in I_u \cap I_v \).

Finally, let \( uv \notin E \). Then \( v_{i_j}v_{i_k} \notin \hat{E} \). Since \( \hat{G} \) is an induced subgraph of \( G(n, r) \), we have \( v_{i_j}v_{i_k} \notin E' \). Then \( |i_j - i_k| > r \Rightarrow |p_u - p_v| > r \Rightarrow p_v \notin I_u \text{ and } p_u \notin I_v \). Thus \( G \) is a PCMTG.

\[ (1 \Rightarrow 3) : \text{Let } G = (V, E) \text{ be a PCMTG with a PCMTG representation } \{I_u = [l_u, r_u]|u \in V\}. \text{ We arrange vertices according to the increasing order of center points, } V = \{v_1, v_2, \ldots, v_n\}. \text{ To prove that } G \text{ is a proper interval graph we show that neighbours of any vertex are consecutive in the above ordering. (see 2.5) [4].} \]

Denote \( I_u = [l_u, r_u] \) by \( [l_i, r_i] \) and \( c_i = \frac{l_i + r_i}{2} \) for \( i = 1, 2, \ldots, n \). Let \( i < j < k \text{ and } u_ju_k \in E \). Then \( c_i < c_j < c_k \). Now since \( G \) is a CMPTG, \( c_k - c_i \leq \min \{c_i - l_i, c_k - l_k\} \). Now \( c_j - c_i < c_k - c_i \leq c_i - l_i \). Now if \( l_j > c_i \), then \( l_k \leq c_i < l_j < c_j < c_k \) as \( c_i < c_k \leq l_k \). So \( [l_j, c_j] \subseteq [l_k, c_k] \). But this implies \( [l_j, r_j] \subseteq [l_k, r_k] \) which contradicts the fact that \( G \) is a CMPTG. Thus \( l_j \leq c_i \). So we have \( c_j - c_i \leq c_j - l_j \).

Hence \( c_j - c_i \leq \min \{c_i - l_i, c_j - l_j\} \). Therefore \( u_iu_j \in E \), as required. Similarly, it can be shown that \( u_ju_k \in E \). Thus \( G \) is a proper interval graph.

\[ \square \]

**Definition 3.5.** Let \( A = (a_{i,j}) \) and \( B = (b_{i,j}) \) be two \( n \times n \) binary matrices. We define \( A \cap B = (c_{i,j}) \) where \( c_{i,j} = a_{i,j} \cap b_{i,j} \) with the rules: \( 0 \wedge 0 = 0 \wedge 1 = 0 \) and \( 1 \wedge 1 = 1 \).

**Observation 3.6.** Let \( G \) be a simple undirected graph. Then following are equivalent:

1. \( G \) is a MPTG.
2. There is an ordering of vertices of \( G \) such that for any \( u < v, u, v \in V \),
   \[
   uv \notin E \iff uv \notin E \text{ for all } w > v \text{ or, } uv \notin E \text{ for all } w < u. \quad (3.2)
   \]
3. There exists an ordering of vertices such that every \( 0 \) above the principal diagonal of the augmented adjacency matrix \( A(G) \) has either all entries right to it are \( 0 \) or, all entries above it are \( 0 \).
4. There exists a binary matrix \( M \) with consecutive 1’s property for rows such that the augmented adjacency matrix \( A(G) = M \wedge M^T \).
5. There exists an interval catch digraph \( D \) such that \( G = D \cap D^T \) where \( D^T \) is the digraph obtained from \( D \) by reversing direction of every arc.

**Proof.** The condition 2 is equivalent to (2.1) in the other way. Condition 3 is a matrix version of condition 2. Condition 4 and 5 follow from definition of MPTG and ICD.

**Definition 3.7.** (C-order) Let \( G = (V, E) \) be a CMPTG with (distinct) center points \( \{c_u | u \in V\} \) of intervals \( \{I_u | u \in V\} \) in its CMPTG interval representation. Define \( u <_C v \iff c_u < c_v \). Then this linear order of vertices is called a C-order of \( V \).
In the following we present a necessary condition for CMPTG.

**Theorem 3.8.** Let $G = (V, E)$ be a CMPTG. Then there is an ordering of vertices of $G$ such that the following condition holds:

$$
\text{For any } x < u < v < y, \ xv, uy \in E \implies uv \in E \text{ and } (xu \in E \text{ or } vy \in E \text{ or } xu, vy \in E).\quad (3.3)
$$

*Proof.* Let $G = (V, E)$ be a CMPTG with a CMPTG representation $\{I_u\} u \in V$. We arrange vertices according to the increasing order of center points (i.e., in $\triangleleft_C$ order) of representing intervals. Suppose in this ordering we have $x < u < v < y$ and $xv, uy \in E$. Then $c_x, c_y \in I_v \cap I_u$ and $c_u, c_y \in I_u \cap I_v$. Also we have $c_x < c_u < c_v < c_y$. Now $c_x, c_y \in I_v \Rightarrow c_u \in I_v$ and $c_u, c_y \in I_u \Rightarrow c_v \in I_u$. Therefore $uv \in E$. Again $c_x, c_y \in I_v \Rightarrow c_u \in I_v$ and $c_u, c_y \in I_y \Rightarrow c_v \in I_y$. Thus if $xu, vy \notin E$, then $c_x \notin I_u$ and $c_y \notin I_v$. But then $c_u - c_x > c_y - c_u$ as $c_x \notin I_u$ but $c_y \in I_v$, and $c_y - c_v > c_v - c_x$ as $c_y \notin I_u$ but $c_x \in I_v$. Combining these inequalities we have $c_u < \frac{c_x + c_y}{2} < c_u$ which is a contradiction. Therefore $xu \in E$ or, $vy \in E$ or, $xu, vy \in E$.

**Definition 3.9.** Let $G$ be a CMPTG and $C_n$ $(n \geq 4)$ be an induced cycle of $G$. Let $A = (a_1, \ldots, a_n)$ be the list of all vertices of $C_n$, $n \geq 4$ arranged in a $C$-order. Then $a_x, a_y$ are said to be circularly consecutive in $A$ if they are in consecutive places of the list, or if $x = n$ and $y = 1$ or $x = 1$ and $y = n$. $C_n, n \geq 4$ is said to be circularly consecutive ordered if starting from a fixed vertex (say $u$) we can order all of its vertices in a circularly consecutive way in clockwise (or anticlockwise) direction until $u$ is reached.

**Definition 3.10.** An ending edge in a path $P_n$ is an edge that contains a pendant vertex. Let $P_n$ $(n \geq 4)$ be an induced path in a CMPTG $G$. Then $P_n$ is said to have vertex consecutive ending edges if vertices corresponding to ending edges of $P_n$ occur consecutively in a $C$-order (up to permutations between them) at least in one end.

**Corollary 3.11.** All induced $C_4$ in CMPTG are circularly consecutive ordered and all induced $P_4$ in CMPTG have vertex consecutive ending edges.

*Proof.* In all other cases it will violate condition $(3.3)$ follows from the proof of Theorem 3.8.

The following theorem is a sufficient condition that an MPTG to be a CMPTG.

**Theorem 3.12.** Let $G = (V, E)$ be a MPTG with $n$ vertices. Let the ordering $\{v_1, v_2, \ldots, v_n\}$ of vertices of $G$ that satisfies $[2.1]$ and each $v_i$ corresponds to a natural number $x_i$ such that $x_1 < x_2 < \cdots < x_n$ and the following conditions hold for all $i = 1, 2, \ldots, n$:

$$
\begin{align*}
  x_{i_2 + 1} - x_i &> x_i - x_{i_1} \quad \text{when } i_2 < n \quad (3.4) \\
  x_i - x_{i_1 - 1} &> x_{i_2} - x_i \quad \text{when } i_1 > 1 \quad (3.5)
\end{align*}
$$

where $i_1$ and $i_2$ be the least and the highest indices such that $i_1 = i$ or, $v_iv_{i_1} \in E$ and $i_2 = i$ or, $v_iv_{i_2} \in E$. Then $G$ is a CMPTG.

*Proof.* Suppose the conditions hold. Define $r_i = \max\{x_i - x_{i_1}, x_{i_2} - x_i\}$ and $I_i = [x_i - r_i, x_i + r_i]$ for $i = 1, 2, \ldots, n$. We show that $G = (V, E)$ is a CMPTG with a CMPTG representation $\{I_{v_i}| i = 1, 2, \ldots, n\}$ where $V = \{v_1, v_2, \ldots, v_n\}$ and this ordering of vertices satisfies $[2.1]$. Suppose $v_i v_j \in E$. Then by definition of $i_1$ and $i_2$, we have $x_i \leq x_j \leq x_{i_1}$ and $x_j \leq x_{i_1} \leq x_{i_1} x_{i_2}$. Then $x_i - x_{i_1} \geq x_j - x_i$ and $x_{i_2} - x_i \geq x_{i_1} - x_i$ which imply $|x_i - x_j| \leq r_i$ and so $x_j \in I_{v_i}$. Similarly $x_i \in I_{v_j}$. Hence $\{x_i, x_j\} \subseteq I_{v_i} \cap I_{v_j}$. Now let $v_i v_j \notin E$. Without loss of generality we assume $i < j$. Suppose $j_1 < i < j < i_2$ and $v_{j_1} v_j, v_i v_{i_2} \in E$. Then by $[2.1]$, $v_i v_j \in E$, which is a contradiction. Thus either $i < j_1$ or $j > i_2$. Then
\[ i \leq j_1 - 1 \text{ or } j \geq i_2 + 1. \] For the first inequality by (3.4), we have \( x_j - x_i \geq x_j - x_{j_1 - 1} > x_{j_2} - x_j. \) Also \( x_j - x_i > x_j - x_{j_1}, \) as \( x_i < x_{j_1}. \) Thus \( x_j - x_i > r_j \) which implies \( x_i \notin I_{v_j}. \) Similarly \( j \geq i_2 + 1 \) implies \( x_j \notin I_{v_i}. \) Therefore \( G \) is a CMPTG.

Next, following Observation 3.6, we have the following in a similar way:

**Proposition 3.13.** Let \( G \) be a simple undirected graph. Then \( G \) is a CMPTG if and only if there exists a central interval catch digraph \( D \) such that \( G = D \cap D^T \) where \( D^T \) is the digraph obtained from \( D \) by reversing direction of every arc.

**Definition 3.14.** Let \( G = (V, E) \) be a simple undirected graph and \( u, v \in V \). Then \( N(u) = \{ x \in V \mid ux \in E \} \) is the set of (open) neighbors of \( u \) and \( G[N(u) \cap N(v)] \) is the graph induced by the vertices of \( N(u) \cap N(v). \)

It is proved in [5] that if \( G \) is an MPTG with non-adjacent vertices \( u \) and \( v \), then \( G[N(u) \cap N(v)] \) is an interval graph. We found the following analogous result for CMPTG.

**Proposition 3.15.** If \( G \) is a CMPTG with non-adjacent vertices \( u \) and \( v \), then \( G[N(u) \cap N(v)] \) is a proper interval graph.

**Proof.** Let \( G = (V, E) \) be a CMPTG with a CMPTG representation \( \{ I_u = [a_u, b_u] \mid u \in V \} \) where vertices are arranged according to the increasing order of center points (i.e., in \( C \)-order \( \langle C \rangle \)). Define \( c_u = \frac{a_u + b_u}{2} \) for all \( u \in V \). Suppose \( u < v \). We observe the followings.

- Vertices belong to \( N(u) \cap N(v) \), occurs between \( u, v \) form a clique. Let \( x, y \) be two such vertices such that \( u < x < y < v \). By (3.3) one can conclude that \( xy \in E \).

- If there exist a vertex belongs to \( N(u) \cap N(v) \), occurs before \( u \) then no vertex can occur after \( v \) which belongs to \( N(u) \cap N(v) \) and conversely. On contrary let \( u' < u < v < v' \) such that \( u, v' \in N(u) \cap N(v) \) then from (3.3) one can find \( uv \in E \) which is a contradiction.

- Vertices belong to \( N(u) \cap N(v) \), occurs before \( u \) forms a clique. Let \( u' < u'' < u < v \) such that \( u', u'' \in N(u) \cap N(v) \) then \( c_{u'} < c_{u''} < c_u < c_v < b_u \). This imply \( c_{u''} \in [c_{u'}, b_u] \subset I_{u'} \). Now if \( a_u > c_{u'} \), then \( a_u \leq c_{u'} < a_u < c_u < c_v \leq b_u \) which implies \( |I_{u''}| < |I_u| \). Again as \( a_u, c_u \) from above we can conclude that \( b_{u''} \leq b_u \). But this is a contradiction as \( u' \in E \) implies \( a_{u'} < c_{u''} < c_u < c_v \leq b_u \) which imply \( c_{u'} \in [a_{u'}, c_u] \subset I_{u'} \), but as \( u'' \in v \), \( c_v > b_u \), hence from above we get \( b_u < c_v \leq b_{u''} \). Therefore \( a_{u''} \leq c_{u'}. \) So \( c_{u''} \in [a_{u''}, c_{u''}] \subset I_{u''}. \) Hence \( u' u'' \in E \).

Similarly one can show vertices belong to \( N(u) \cap N(v) \), occurs after \( v \) forms a clique.

Now let \( \{ u_1', \ldots, u_k' \} \) be the vertices of \( G[N(u) \cap N(v)] \) arranged in \( \langle C \rangle \) order, occurs before \( u \in G \) and \( \{ x_1, \ldots, x_m \} \) be the vertices of \( G[N(u) \cap N(v)] \) arranged according to increasing order of left end points \( \langle L \rangle \) order say), occurs between \( u \) and \( v \) in \( G \). Now we will show \( G[N(u) \cap N(v)] \) forms a proper interval graph with respect to the ordering \( \{ u_1', \ldots, u_k', x_1, \ldots, x_m \} \). We show the neighbours of any vertex are consecutive in this ordering (see 2.5 [4]). As \( x_i (u_j') \)'s form clique for \( 1 \leq i \leq m (1 \leq j \leq k) \), it is sufficient to prove the followings.

- Let \( u_i' < x_i < x_j \) where \( 1 \leq l \leq k, 1 \leq i, j \leq m \) such that \( u_i' x_j \in E \). Then \( c_{u_i'} < c_u < c_{x_i}, c_{x_j} < c_v < b_{u_i'}. \) This implies \( c_{x_i} \in [c_{u_i'}, b_{u_i'}] \subset I_{u_i'} \). Now as \( u_i' x_j \in E, a_{x_j} \leq c_{u_i'} < c_u < c_{x_j} \) implies \( a_{x_i} < a_{x_j} \leq c_{u_i'} < c_{x_i} \) (as \( x_i \in L \iff a_{x_i} < a_{x_j} \)) which imply \( c_{u_i'} \in [a_{x_i}, c_{x_i}] \subset I_{x_i}. \) Hence \( u_i' x_i \in E \).

- Let \( u_i' < u_j' < x_i \) where \( 1 \leq i, j \leq k, 1 \leq l \leq m \) such that \( u_i' x_i \in E. \) Then \( u_i' < u_j' < x_i < v \) clearly. Now from (3.3) one can conclude \( u_i' x_i \in E \).
Similarly one can show if there exists vertices of $G[N(u) \cap N(v)]$, occurs after $v$ in $G$ then with respect to the ordering \( \{x_1, \ldots, x_m, v'_1, \ldots, v'_k\} \), $G[N(u) \cap N(v)]$ forms a proper interval graph where $\{x_1, \ldots, x_m\}$ are vertices of $G[N(u) \cap N(v)]$ arranged according to increasing order of right end points ($<_{N'}$ say), occurs between $u$ and $v$ in $G$ and $\{v'_1, \ldots, v'_k\}$ are vertices of $G[N(u) \cap N(v)]$ arranged in $<_{C'}$, occurs after $v$ in $G$.

The above proposition leads to a construction of the following forbidden graph for the class of CMPTG.

**Example 3.16.** By Proposition 3.15 we see that the graph $G$ (see Figure 1) formed by taking $K_{1, 3}$ together with two non-adjacent vertices (say, $u, v$) which are adjacent to each vertex of $K_{1, 3}$ is not a CMPTG whereas $G \setminus \{u, v\}, G \setminus \{u\}$ are CMPTG. It follows from Lemma 4.3 that $G \setminus \{u, v\} = K_{1, 3}$ is a CMPTG. Now if we assign the intervals $[-13, 15], [3, 15], [11, 15]$ to the pendants of $K_{1, 3}$ and $[1, 21], [1, 29]$ to $v$ and the central vertex of $K_{1, 3}$ respectively then it is easy to check $G \setminus \{u\}$ becomes a CMPTG with this representation.

![Figure 1: The graph $G$ in Example 3.16](image)

### 4 50% max-tolerance graphs

**Definition 4.1.** A max-tolerance graph (MTG) $G = (V, E)$ is a 50% max-tolerance graph if $t_u = \frac{|I_u|}{2}$ for all $u \in V$ where $t_u$ denotes the tolerance associated with the vertex $u \in V$.

We know that classes of unit (min) tolerance graphs and 50% (min) tolerance graphs are same [5]. But the following theorem shows that this is not the case for max-tolerance graphs

**Theorem 4.2.** CMPTG (i.e., UMTG) and the class of 50% max-tolerance graphs are not comparable.

**Proof.** We prove this theorem with the help of following Lemmas.

**Lemma 4.3.** $K_{1, n}$ is a CMPTG for any natural number $n$.

**Proof.** Let $u$ be the central vertex and $\{v_1, v_2, \ldots, v_n\}$ be pendant vertices of $K_{1, n}$. We will give its CMPTG representation by introducing the following interval representation. Let $I_u = [10, 20], I_k = [15 - (3.2^{k-1} - 2)\alpha, 15 + \alpha]$ where $k = 1, 2, \ldots, n$ and $0 < \alpha < \frac{5}{3.2^n - 1 - 2}$. First of all note that $c_k = 15 \in I_k$ for all $k \in \{1, 2, \ldots, n\}$. Moreover $c_k \in I_u$ for all $k$ trivially since each $I_k \subset I_u$ as $10 < 15 - (3.2^{k-1} - 2)\alpha < 15 < 15 + \alpha < 20$. So for each $k \in \{1, 2, \ldots, n\}, \{c_u, c_k\} \subset I_u \cap I_k$. Thus $u \leftrightarrow v_k$ for each $k$. Now we will show that $c_{k'} \notin I_k$ for any $k' > k$ that will imply $v_k \leftrightarrow v_{k'}$ for any $k, k' \in \{1, 2, \ldots, n\}$. Let $k' > k$. Here $a_k = 15 - (3.2^{k-1} - 2)\alpha, c_{k'} = \frac{15 - (3.2^{k'-1} - 2)\alpha + 15 + \alpha}{2} = \frac{30 + 3\alpha - 3.2^{k'-1}\alpha}{2}$. Sufficient to prove $c_{k'} < a_k$ i.e., $a_k - c_{k'} > 0$. Now $a_k - c_{k'} = 3\alpha.2^{k-1}(2^{k'-k-1} - 1) + \frac{\alpha}{2} > 0$ clearly as $k' > k \geq 1$ and $\alpha > 0$.

**Lemma 4.4.** $K_{1, n}$ where $n \geq 8$ is a CMPTG but is not a 50% max-tolerance graph.
Proof. It is sufficient to show that $K_{1,n}$, $n \geq 8$ is not a 50\% max-tolerance graph. Then the result will follow using Lemma 4.3.

We will prove that $K_{1,n}$ where $n \geq 8$ cannot have any 50\% max-tolerance representation. On the contrary, let assume that it has a 50\% max-tolerance representation. By suitable scaling and shifting origin without loss of generality, we may assume that $I_u = [0, 1]$, interval corresponding to central vertex $u$ and $I_i = [a_i, b_i]$, intervals corresponding to the pendant vertices $v_i$ where $i \in \{1, \ldots, n\}$.

We consider the intervals corresponding to any two pendants must be distinct as they are non-adjacent. We note the following observations which will lead us to the proof.

- **Claim 1: Number of pendant vertices whose intervals satisfy $a_i \leq 0 < 1 \leq b_i$ is at most 1.**

  Let $[a_1, b_1], [a_2, b_2]$ be the two intervals associated to two such pendant vertices (say $v_1, v_2$). Then the $[a_i, b_i] \supseteq [0, 1]$ for $i = \{1, 2\}$. Now $[a_i, b_i] \cap [0, 1] = [0, 1]$ clearly. As every pendant vertex is adjacent to $u$ and $I_u$ is of unit length, so each pendant vertex has tolerance at most 1 and therefore has length at most 2. But $[a_1, b_1] \cap [a_2, b_2] \supseteq [0, 1] \Rightarrow ||a_1, b_1] \cap [a_2, b_2]|| \geq 1$. Hence $v_1 \leftrightarrow v_2$ which contradicts the fact that they are pendant vertices.

- **Claim 2: Number of pendant vertices whose intervals satisfy $a_i < 0, 0 < b_i < 1$ is at most 2.**

  Let $[a_1, b_1], [a_2, b_2]$ and $[a_3, b_3]$ be three such intervals corresponding to the vertices (say $v_1, v_2, v_3$). First we will show that for any two of the three intervals, one must contain the other. On contrary, without loss of generality assume that $[a_1, b_1], [a_2, b_2]$ are two such intervals which are not containing each other and let $a_1 < a_2$, $b_1 < b_2$. It is sufficient to show for one case. Now let $[a_1, b_1] \cap [a_2, b_2] = [a_2, b_1]$. Now as $v_1 \leftrightarrow u$, $|[a_1, b_1] \cap [0, 1]| \geq \max\{\frac{1}{2}, \frac{b_1 - a_1}{2}\} \Rightarrow b_1 \geq \frac{1}{2} \frac{b_1 - a_1}{2}$. This implies

  $$b_1 \geq -a_1$$  \hspace{1cm} (4.1)

  Similarly $b_2 \geq -a_2$.

  Now $-a_1 \leq b_1 \Rightarrow -a_1 + b_1 \leq 2b_1 \Rightarrow -a_1 + b_1 < 2b_1 - 2a_2 \Rightarrow \frac{-a_1 + b_1}{2} < b_1 - a_2$. But $v_1 \leftrightarrow v_2$. Hence $b_1 - a_2 < \frac{b_2 - a_2}{2} \Rightarrow 2b_1 - 2a_2 < b_2 - a_2 \Rightarrow 2b_1 - 2a_2 < b_2$. Hence $b_2 > 2b_1 - 2a_2 > 2b_1 \geq 1$ (Since $b_1 \geq \frac{1}{2}$ and $a_2 < 0$) which contradicts the fact $b_2 < 1$. Hence the three intervals form a well ordered set with inclusion as the ordering.

  From the statement above without loss of generality we can assume that $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3]$. Now $[a_1, b_1] \cap [a_2, b_2] = [a_2, b_2]$. Therefore $2(b_2 - a_2) < b_1 - a_1$. Similarly $b_2 - a_2 > 2(b_3 - a_3)$. Hence $b_1 - a_1 > 4(b_3 - a_3) > 4b_3 \geq 2$ (since $b_3 \geq \frac{1}{2}$ as $v_3 \leftrightarrow u$). Now $b_1 - a_1 \leq 2b_1 < 2$ from (4.1) and the fact $b_1 < 1$. Hence we are through.

- **Claim 3: Number of pendant vertices whose intervals satisfy $0 < a_i < 1, b_i > 1$ is at most 2.**

  This proof is same as in Claim 2.

- **Claim 4: Number of pendant vertices whose intervals satisfy $0 \leq a_i < b_i \leq 1$ is at most 2.**

  On the contrary we assume that there exists three such vertices with representations $[a_i, b_i] \subseteq [0, 1]$ for $i \in \{1, 2, 3\}$.

  1. First we will show that for any two intervals, one must not contain the other.

     If not, we assume $[a_i, b_i] \subseteq [a_j, b_j]$ for some $i, j \in \{1, 2, 3\}$. Then $v_i \leftrightarrow u \Rightarrow b_i - a_i \geq \frac{1}{2}$. Hence $|[a_i, b_i] \cap [a_j, b_j]| = |[a_i, b_i]| \geq \frac{1}{2}$. Again $v_i \leftrightarrow v_j \Rightarrow b_i - a_i < \frac{b_j - a_j}{2} \leq \frac{1}{2}$ which is a contradiction.
2. Next we will show that no two intervals are disjoint.

If possible let \([a_1, b_1] \) and \([a_2, b_2] \) be disjoint. Without loss of generality we can assume \(a_1 < a_2 \).

Hence since they are disjoint \(b_1 < a_2 \). But \(b_1 \geq \frac{1}{2} \) as \(v_1 \leftrightarrow u \). Hence \(a_2 > \frac{1}{2} \). This implies \(b_2 - a_2 < \frac{1}{2} \) which contradicts \(v_2 \leftrightarrow u \).

Now without loss of generality we can assume \(a_1 < a_2 < a_3 \). Hence from 1 and 2 we conclude \(a_1 < a_2 < a_3 \leq b_1 < b_2 < b_3 \). Under this situation we will show that there exist no choice of three such intervals. For this we first show \(b_2 > \frac{3}{4} \). If \(a_2 \leq \frac{a_1 + b_1}{2} \) then \(b_1 - a_2 \geq b_1 - \frac{a_1 + b_1}{2} = \frac{b_1 - a_1}{2} \). Then \(b_1 - a_2 < \frac{b_2 - a_2}{2} \) since \(v_1 \leftrightarrow v_2 \). Hence \(b_2 > 2b_1 - a_2 \geq 2b_1 - \frac{a_1 + b_1}{2} = b_1 + \frac{b_1 - a_1}{2} \geq \frac{1}{2} + \frac{1}{2} = \frac{3}{4} \).

(Since \(v_1 \leftrightarrow u \), \(b_1 - a_1 \geq \frac{1}{2} \) and hence \(b_1 \geq \frac{1}{2} \). For other case, i.e., \(a_2 > \frac{a_1 + b_1}{2} \) then \(b_2 = \frac{b_1 - a_1}{2} + a_1 \geq \frac{1}{2} \).

Now \(b_2 - a_2 \geq \frac{1}{2} \Rightarrow b_2 \geq a_2 + \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \Rightarrow b_2 > \frac{3}{4} \).

For the remaining of the proof we split into two cases.

1. If \(a_3 \leq \frac{a_2 + b_2}{2} \) then \(b_2 - a_3 \geq b_2 - \frac{a_2 + b_2}{2} = b_2 - a_2 \). Then \(b_2 - a_3 < \frac{b_1 - a_3}{2} \) as \(v_2 \leftrightarrow v_3 \).

This implies \(b_3 > 2b_2 - a_3 \geq 2b_2 - \frac{a_2 + b_2}{2} = b_2 + \frac{b_2 - a_2}{2} > \frac{3}{4} + \frac{1}{2} = \frac{1}{2} \). Hence we arrive at a contradiction.

2. If \(a_3 > \frac{a_2 + b_2}{2} \) then since \(b_3 - a_3 \geq \frac{1}{2} \) (as \(v_3 \leftrightarrow u \)), we have

\[
b_3 > \frac{a_2 + b_2}{2} + \frac{1}{2} \tag{4.2}
\]

Now if \(a_2 \leq \frac{a_1 + b_1}{2} \) then \(b_2 > 2b_1 - a_2 \) as before. Hence \(b_2 + a_2 > b_1 \). Hence using (4.2) we get \(b_3 > b_1 + \frac{1}{2} \geq \frac{1}{2} + \frac{1}{2} = 1 \) contradicting \(b_3 \leq 1 \).

Now if \(a_2 > \frac{a_1 + b_1}{2} \) then \(b_2 > \frac{a_1 + b_1}{2} + \frac{1}{2} \) since \(b_2 - a_2 \geq \frac{1}{2} \). Hence \(a_2 + b_2 > a_1 + b_1 + \frac{1}{2} \). Again using (4.2) we get \(b_3 > \frac{a_1 + b_1}{2} + \frac{1}{4} + \frac{1}{2} \geq \frac{1}{2} + \frac{1}{2} = 1 \) (as \(b_1 \geq \frac{1}{2}, a_1 \geq 0 \)). Hence we again arrive at a contradiction.

Hence we have established our claim.

Now using the above results we will show that \(K_{1,n} \) where \(n \geq 8 \) is not a 50% max-tolerance graph. Let \([a, b] \) be an interval corresponding to a pendant vertex \(I_v \). Then \(I_v \) must belong to one of the four sets, \(S_1 = \{I_v|a \leq 0 < 1 \leq b\} \), \(S_2 = \{I_v|a < 0 < 0 < b \leq 1\} \), \(S_3 = \{I_v|0 < a < 1, b > 1\} \), \(S_4 = \{I_v|0 \leq a < b \leq 1\} \).

We note that any pendant vertex cannot have \([0, 1]\) as its interval representation. Hence the above sets are mutually exclusive. As every pendant vertex is adjacent to the central vertex \(u \) hence it follows that the four sets are also exhaustive.

From above, we can see that \(|S_1| \leq 1 \) and \(|S_i| \leq 2 \) for \(i \in \{2, 3, 4\} \). Hence the maximum number of possible pendant vertices is \(1 + 2 + 2 + 2 = 7 \). So we are done if we take at least 8 pendant vertices, i.e., \(n \geq 8 \).

**Lemma 4.5.** \(C_6\) is a 50% max-tolerance graph but is not a CMPTG.

**Proof.** Let \(\{v_1, \ldots, v_6\} \) are the vertices occurred in circularly consecutive way (clockwise or anticlockwise order) in \(C_6\). We assign the following intervals and tolerances for all the vertices so that they satisfy 50%
max-tolerance representation in $\overline{C_6}$. $I_{v_1} = [0,20], t_{v_1} = 10, I_{v_2} = [12,24], t_{v_2} = 6, I_{v_3} = [0,22], t_{v_3} = 11, I_{v_4} = [9.5,19.5], t_{v_4} = 5, I_{v_5} = [7.5,30.5], t_{v_5} = 11.5, I_{v_6} = [10.5,21.5], t_{v_6} = 5.5$. It is easy to verify the adjacency relations from Definition 3.1.

Now we will show that $\overline{C_6}$ is not a CMPTG. On contrary suppose $\overline{C_6}$ is a CMPTG with interval representation \(\{I_v = [a_v, b_v] \mid v \in V\}\) and the corresponding center points \(c_i = \frac{a_i + b_i}{2}\). It is easy to check that the subgraph induced by deleting the vertices \(\{v_2, v_3\}\) is a $C_4$. Now from Corollary 3.11 we can conclude that the vertices in $C_4 = \{v_1, v_4, v_6, v_3\}$ are circularly consecutive ordered. So without loss of generality we can assume that in its CMPTG representation $c_1 < c_4 < c_6 < c_3$. Now we observe the following,

1. $v_3 \leftrightarrow v_6 \Rightarrow c_3 < I_6 = c_3 < b_6$. Again $v_1 \leftrightarrow v_3 \Rightarrow [c_1, c_3] \subseteq I_3$. Note that $c_4 \in [c_1, c_3] \subseteq I_3$. But as $c_4 < c_3, v_3 \leftrightarrow v_4 \Rightarrow b_4 < c_3$. Combining we get $b_4 < c_3 < b_6$.

2. $v_4 \leftrightarrow v_6 \Rightarrow a_6 < c_4$. Again $v_1 \leftrightarrow v_3 \Rightarrow [c_1, c_3] \subseteq I_1$. Note that $c_6 \in [c_1, c_3] \subseteq I_1$. But as $c_1 < c_6, v_1 \leftrightarrow v_6 \Rightarrow v_6 \Rightarrow c_1 < a_6$. Combining we get $c_1 < a_6 < c_4$.

3. $v_1 \leftrightarrow v_4 \Rightarrow a_4 < c_1$. Again $v_4 \leftrightarrow v_5 \Rightarrow c_6 < b_4$. Combining these with the inequalities of 1 and 2 we get $a_4 < c_1 < a_6 < c_4 < c_6 < b_4 < c_3 < b_6$.

4. $v_2 \leftrightarrow v_4, v_6 \Rightarrow c_2 \in I_4 \cap I_6 = [a_6, b_4] \subseteq [c_1, c_3] \subseteq I_1$ (from 2 and 3) which imply $c_2 \in I_1, c_1 < c_2$ (from 3). Now as $v_2 \leftrightarrow v_1 \Rightarrow a_2 > c_1$. Again $v_1 \leftrightarrow v_3 \Rightarrow a_3 < c_1$. Combining we get $a_3 < c_1 < a_2 < c_2 < b_4 < c_3$. Thus imply $c_2 \in [c_1, c_3] \subseteq I_3$. But as $v_2 \leftrightarrow v_3 \Rightarrow b_2 < c_3$. Thus we get $c_1 < a_2 < c_2 < b_2 < c_3$. Using 3 and 4 which imply $c_5 \notin [a_2, b_2] = I_2$ which is a contradiction as $v_5 \leftrightarrow v_2$.

5. $v_5 \leftrightarrow v_1, v_3 \Rightarrow c_5 \in [c_1, c_3] \subseteq [a_4, b_3] \Rightarrow c_5 \in I_4$ or $I_6$ (from 3). (If $c_5 < c_1$ then as $v_5 \leftrightarrow v_3 \Rightarrow a_3 < c_5 < c_1$. From 4 we get $a_3 < c_5 < c_1 < a_2 < c_2 < b_2 < c_3$ which imply $c_5 \notin [a_2, b_2] = I_2$ which is a contradiction as $v_5 \leftrightarrow v_2$. Again if $c_5 > c_3$ as $v_5 \leftrightarrow v_1 \Rightarrow a_5 \leq c_1 < c_4 < c_6 < c_3 < c_3$ which imply $c_6 \in [a_5, c_3] \subseteq I_5$. Now as $v_5 \leftrightarrow v_6 \Rightarrow b_6 < c_5$. Again $a_2 < b_2 < c_3 < b_6 < c_5$ using 3 and 4 which imply $c_5 \notin [a_2, b_2] = I_2$ which is a contradiction as $v_5 \leftrightarrow v_2$.)

6. $v_1, v_3 \leftrightarrow v_5 \Rightarrow [c_1, c_3] \subseteq I_5$. Now $c_6 \in [c_1, c_3]$ (from 3) $\Rightarrow c_6 \in I_5$ which imply $c_5 \notin I_6$ since $v_5 \leftrightarrow v_6$. Now $a_6 \leq c_5 \Rightarrow a_6 \leq c_5 < c_3 < b_6$ (from 3, 5) which implies $c_5 \in I_6$ which is a contradiction. Hence $c_5 < a_6$. Hence from 5 we get $c_5 \in I_4$. But as $v_5 \leftrightarrow v_4 \Rightarrow c_4 \notin I_5$. This imply $b_5 < c_4$ as if $b_5 \geq c_4$ then $c_3 < a_6 < c_4 \leq b_5$ (from 3) $\Rightarrow c_4 \in I_5$ which is a contradiction. So we can conclude $b_5 < c_4 < a_6$ (from 3). But as $c_6 \in I_5 \Rightarrow c_6 \leq b_5$. So contradiction arises. Hence we are done.

Thus the proof is complete.

Now from Lemma 4.4 and 4.5 we can easily conclude that CMPTG and 50% max-tolerance graphs are incomparable.

The following theorem shows that the class of proper interval graphs is subclass of the class of 50% max-tolerance graphs.

**Theorem 4.6.** A proper interval graph is a 50% max-tolerance graph.

**Proof.** Let $G = (V, E)$ be a proper interval graph. Now as we know from Theorem 3.4 that proper interval graphs are same as PCMTG. We can consider a PCMTG representation \(\{I_u = [a_u, b_u] \mid u \in V\}\) of it. We will show that $G$ is a 50% max-tolerance graph with this same interval representation.

Now \(u \leftrightarrow v\) in its PCMTG representation implies $I_u \cap I_v \supseteq \{c_u, c_v\}$ where $c_u, c_v$ are center points of $I_u, I_v$ respectively. This implies $|I_u \cap I_v| \geq \frac{|I_u| + |I_v|}{2}$ as none of them contains other properly. Hence $|I_u \cap I_v| \geq \frac{1}{2} \max\{|I_u|, |I_v|\} \Rightarrow u \leftrightarrow v$ in its 50% max-tolerance representation.
Now, let \( u \leftrightarrow v \) in its PCMPTG representation implies \( c_u \notin I_v \) or \( c_v \notin I_u \). Let \( c_u \leq c_v \). Now \( c_u \notin I_v \Rightarrow a_v > c_u \Rightarrow |I_u \cap I_v| < \frac{|I_u|}{2} \). Also \( c_v \notin I_u \Rightarrow c_v > b_u \Rightarrow |I_u \cap I_v| < \frac{|I_v|}{2} \). From this we can conclude that 
\[ |I_u \cap I_v| < \frac{1}{2} \max\{|I_u|, |I_v|\} \]. So \( u \leftrightarrow v \) in its 50% max-tolerance representation.

**Theorem 4.7.** Any cycle \( C_n \) \((n \geq 3)\) is a 50% max-tolerance graph.

**Proof.** We prove that \( C_n \) belongs to the class of 50% max-tolerance graphs by constructing a realization 
\( \{I_v = [a_i, b_i] | v \in V \} \). We label the vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) clockwise starting from an arbitrary vertex in \( C_n \). \( c_i = \frac{a_i + b_i}{2} \) be the centre point of interval \( I_v \) for each \( i \in \{1, \ldots, n\} \). Consider tolerance (say \( t_u \)) for each vertex (say \( u \)) as \( \frac{|I_u|}{2} \).

- **Let** \( n = 3 \). We associate intervals \( I_{v_1} = I_{v_2} = I_{v_3} = [1, 2] \).
- **Let** \( n = 4 \). We associate intervals \( I_{v_1} = [1, 4.6], I_{v_2} = [2, 4], I_{v_3} = [2.9, 4.9], I_{v_4} = [2.7, 6.3] \).
- **Let** \( n = 5 \). We associate intervals \( I_{v_1} = [10, 30], I_{v_2} = [16, 28], I_{v_3} = [18, 24], I_{v_4} = [15, 21], I_{v_5} = [9, 21] \).
- **Let** \( n \geq 6 \). We prove now in two cases considering \( n \) even and odd. We define

\[ k = \frac{n}{2} \text{ when } n \text{ is even, and } k = \frac{n+1}{2} \text{ when } n \text{ is odd.} \]

**Claim 1: We will show that \( v_1 \) is adjacent to only \( v_2, v_n \).**

We note \( |I_{v_1} \cap I_{v_2}| = |I_{v_2}| = n - 1 = \frac{|I_{v_1}|}{2} \). This implies \( v_1 \leftrightarrow v_2 \). We note \( a_n = 5 - 2n \). Also since \( n \geq 6, 5 - 2n < 2 - n \). Hence \( a_n < a_1 \). Also \( b_n = 1 < n = b_1 \). So \( |I_{v_1} \cap I_{v_n}| = ||a_1, b_n|| = |[2-n, 1]| = n - 1 = \frac{|I_{v_1}|}{2} > n - 2 = \frac{|I_{v_n}|}{2} \). This implies \( v_1 \leftrightarrow v_n \).

Next we show \( v_1 \leftrightarrow v_i \) for \( i \in \{3, \ldots, k\} \). Clearly \( 2 - n < 1 \) implies \( a_1 < a_i \). Also \( n - 2 < (n-1)2^{i-3} \) for \( i \geq 3 \).

Hence

\[ 1 + \frac{k-1}{2^{i-4}} < n \quad (4.3) \]

Therefore \( b_i < b_1 \). Hence \( |I_{v_1} \cap I_{v_i}| = |I_{v_i}| = \frac{k-1}{2^{i-4}} < n - 1 \) from (4.3) = \( \frac{|I_{v_1}|}{2} \). Hence \( v_1 \leftrightarrow v_i \).

Analagously \( v_1 \leftrightarrow v_j \) for \( j \in \{k+2, \ldots, n-1\} \).

Again \( n - 2 < (n-1)2^{k-2} \) implies

\[ 1 + \frac{k-1}{2^{k-3}} < n \quad (4.4) \]

Hence \( a_{k+1} = 1 + \frac{k-1}{2^{k-3}} > 2 - n = a_1 \). Similarly, \( b_{k+1} = 1 + \frac{k-1}{2^{k-3}} < n = b_1 \).

Hence \( |I_{v_1} \cap I_{v_{k+1}}| = \frac{k-1}{2^{k-4}} = n - 2 < \frac{2k-3}{2} < n - 1 = \frac{|I_{v_1}|}{2} \). Hence \( v_1 \leftrightarrow v_{k+1} \).

**Claim 2: We will show that \( v_2 \) is adjacent to only \( v_1, v_3 \).**

From previous case \( v_2 \leftrightarrow v_1 \). Note that \( |I_{v_2} \cap I_{v_3}| = |[1, n-1]| = n - 2 > \frac{n - 1}{2} = \frac{|I_{v_3}|}{2} \). Hence \( v_2 \leftrightarrow v_3 \).

Next to show \( v_2 \leftrightarrow v_i \) for \( i \in \{4, \ldots, k\} \) we note, \( a_2 = a_1 + 1 = 1 \) and \( b_i < b_1 \) [from (4.3)] = \( b_2 \) for \( i \in \{4, \ldots, k\} \). Hence \( |I_{v_2} \cap I_{v_i}| = |I_{v_i}| = \frac{k-1}{2^{i-4}} = n - 2 < \frac{2k-3}{2} < \frac{n - 1}{2} = \frac{|I_{v_2}|}{2} \). This implies
\(v_2 \leftrightarrow v_i\) for \(i \in \{4, \ldots, k\}\). Again \(|V_{v_2} \cap V_{v_j}| = \{1\} < \frac{n-1}{2} = \frac{|I_{v_2}|}{2}\). Hence \(v_2 \leftrightarrow v_j\) for \(j \in \{k+2, \ldots, n\}\).

Now clearly \(a_{k+1} < 1 = a_2\) and \(b_{k+1} < n = b_2\) from \([4.4]\). Hence, \(|I_{v_2} \cap I_{v_{k+1}}| = |\{a_2, b_{k+1}\}| = \frac{k-1 - \frac{n-2}{2^{k-3}}}{2^{k-3}} < \frac{n-1 - \frac{2}{2}}{2}\). Hence \(v_2 \leftrightarrow v_{k+1}\).

**Claim 3:** Vertex \(v_i\) where \(i \in \{3, \ldots, k\}\) is adjacent to only \(v_{i-1}, v_{i+1}\).

From previous case \(v_3 \leftrightarrow v_2\). Let \(v_i, v_{i'}\) be two vertices where \(i, i' \in \{3, \ldots, k\}\). Let \(i < i'\). Then clearly \(b_i > b_i'\). Hence \(|I_i \cap I_{i'}| = |I_{i'}| = \frac{k-1}{2^{i'-3}}\). Now if \(i' = i+1\), then \(\frac{k-1}{2^{i'-3}} = \frac{k-1}{2^{i-3}} = \frac{|I_{v_i}|}{2}\). Hence \(v_i \leftrightarrow v_{i'}\). Also one can check trivially that \(a_{k+1} < 1 = c_{k+1} = a_i\) where \(i \in \{3, \ldots, k\}\). Moreover \(|I_{v_k} \cap I_{v_{k+1}}| = |\{c_{k+1}, b_{k+1}\}| = |\{1, 1+\frac{k-1}{2^{k-3}}\}| = \frac{k-1}{2^{k-3}} = \frac{|I_{v_k}|}{2}\). Hence \(v_k \leftrightarrow v_{k+1}\).

**Claim 4:** Vertex \(v_{k+1}\) is adjacent to only \(v_k, v_{k+2}\).

**Claim 5:** Vertex \(v_i\) where \(i \in \{k+2, \ldots, n\}\) is adjacent to only \(v_{i-1}, v_{i+1}\).

Claim 4, 5 can be shown similarly to the previous cases.

- \(n\) is odd

\[\begin{align*}
I_{v_1} &= [2-n, n], I_{v_2} = [1, n], I_{v_i} = [1, 1 + \frac{n-2}{2^{i-3}}] \quad \text{for} \quad i \in \{3, \ldots, k\}, \quad I_{v_{k+1}} = [1 - \frac{3n-6}{2^{k-3}}, 1 + \frac{3n-6}{2^{k-3}}], \\
I_{v_j} &= [1 - \frac{3n-6}{2^{j+1}}, 1] \quad \text{for} \quad j \in \{k+2, \ldots, n\}.
\end{align*}\]

The proof is analogous to the case when \(n\) is even.

Hence the proof is complete. \(\square\)

**Remark 4.8.** Combining Theorems 4.6 and 4.7 along with results of [18] we can easily conclude that proper intervals graphs and \(C_n\) both belong to the intersection class of CMPTG and 50% max-tolerance graphs.

### 5 Optimized Digraph

In the following we define a class of directed graphs, namely optimized digraphs. Let \(\mathbb{R}^+\) be the set of all positive real numbers.

**Definition 5.1.** Let \(D = (V, E)\) be a directed graph with a distinct labeling \(f : V \rightarrow \mathbb{R}^+\) of vertices. Let \(d(i, j) = |f(v_i) - f(v_j)|\) for all \(i, j \in \{1, 2, \ldots, n\}\). Then \(D\) is called an **optimized digraph** if

\[d(i, j) < d(i, k)\] for all \(i, j, k \in \{1, 2, \ldots, n\}\) such that \(v_i v_j \in E\) and \(v_i v_k \notin E\) \quad (5.1)

(i.e., every out-neighbor distance is less than every non-out-neighbor distance from a vertex).

**Example 5.2.** Consider the directed graph \(G = (V, E)\) in Figure 3. It is easy to check that with respect to the labeling of vertices (written at the top of every vertex in the table), \(d(i, j) < d(i, k)\) for all \(i, j, k \in \{1, 2, \ldots, n\}\) such that \(v_i v_j \in E\) and \(v_i v_k \notin E\). So from our above definition it follows that \(G\) is an optimized digraph.
Case I: $i < j < k$ or, $k < j < i$. Then $d(i, k) = d(i, j) + d(j, k) > d(i, j)$ for $d(j, k) > 0$ as $p_i$’s are distinct.

Case II: $i < k < j$ or, $j < k < i$. These cases are not possible by (2.2) as $D$ is an ICD.

Case III: $j < i < k$ or, $k < i < j$. Now $v_iv_j \in E$ and $v_iv_k \notin E$ imply $p_j \in I_i$ but $p_k \notin I_i$. Let $r = \frac{|L_i|}{2}$. Since $p_i$ is the central point of $I_i$, we have $I_i = [p_i - r, p_i + r]$. Then $d(i, j) = |p_i - p_j| < r < |p_i - p_k| = d(i, k)$.

Conversely, suppose $D$ is an optimized digraph with a labeling $f$. Let us arrange vertices according to increasing order of their labels. For each $i = 1, 2, \ldots, n$, define $p_i = f(v_i)$. Let $i_1$ and $i_2$ be the least and the highest numbers such that $i_1 = i$ or, $v_iv_{i_1} \in E$ and $i_2 = i$ or, $v_iv_{i_2} \in E$. Note that $i_1 \leq i \leq i_2$. Define $r_i = \max \{d(i, i_1), d(i, i_2)\}$ and $I_i = [p_i - r_i, p_i + r_i]$. We show that $\{(I_i, p_i) \mid i = 1, 2, \ldots, n\}$ is a central point-interval representation of $D$, i.e., $D$ is a CICD. As $p_i$ is the center point of $I_i$ for each $i \in \{1, \ldots, n\}$ for this it is sufficient to prove that $D$ is an ICD.

We verify (2.2) to show that $D$ is an ICD. Let $i < j < k$ and $v_iv_k \in E$. Now $d(i, k) = d(i, j) + d(j, k) > d(i, j)$. So $v_iv_j \in E$. Let $v_kv_j \in E$. Again $d(k, i) = d(k, j) + d(j, i) > d(k, j)$. So $v_kv_j \in E$ as required.

Note that the digraph $G$ in Example 5.2 is indeed a CICD as it is evident from the matrix in Figure 2, where intervals are shown in each row at the right most positions and their center points are given at the top of each column.

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3Given a positive real number labeling, one can easily obtain a positive rational number labeling with slight adjustment and again those can be changed to positive integers by scaling as required. Thus we note that natural number labeling will produce the same class of digraphs.
**Proposition 5.4.** Let \( D = (V, E) \) be a CICD. Then there is an ordering of vertices \( V = \{v_1, v_2, \ldots, v_n\} \) which satisfies (2.2) and the following condition:

\[
\text{for any } i < j, \text{ either } i_1 \leq j_1 \text{ or } i_2 \leq j_2, \tag{5.2}
\]

where \( i_1 \) and \( i_2 \) be the least and the highest numbers such that \( i_1 = i \) or, \( v_iv_i \in E \) and \( i_2 = i \) or, \( v_iv_i \in E \) for each \( i = 1, 2, \ldots, n \).

**Proof.** We first note that it follows from the proof of Theorem 5.3 that \( D \) is an optimized digraph with a labeling for each vertex the center point of the corresponding interval in a CICD representation of \( D \). Thus arrangement of vertices according to the increasing sequence center points is same as the arrangement of vertices according to the increasing labeling of vertices when \( D \) is considered as an optimized digraph. Let \( V = \{v_1, v_2, \ldots, v_n\} \) is one such ordering. Let \( i < j < k \) and \( v_iv_k \in E \). Then \( d(i, j) < d(i, k) \) which implies \( v_iv_j \in E \) by (5.1). Similarly, let \( v_kv_j \in E \). Then \( d(k, j) < d(k, i) \) and so \( v_kv_j \in E \). Therefore the ordering satisfies (2.2).

Next suppose there are \( i < j \) such that \( i_1 > j_1 \) and \( i_2 > j_2 \). Then \( j_1 < i_1 \leq i < j_2 < i_2 \). Now by (5.1), \( d(i, j_1) > d(i, i_2) > d(j, i_2) > d(j, j_1) = d(j, i) + d(i, j_1) \) So we have \( d(j, i) < 0 \) which is a contradiction. \( \square \)

This necessary condition gives rise to many forbidden digraphs for the class of optimized digraphs. We describe one of them in the following example.

**Example 5.5.** In the digraph \( G_1 \) of Figure 3 one can easily verify that \( \{v_1, v_2, v_3, v_4\} \) is the unique ICD ordering of vertices, i.e., it is the only ordering of vertices such that the augmented adjacency matrix satisfies consecutive 1s property for rows. But it follows from the above proposition that \( G_1 \) is not a CICD as \( (2)_1 = 2 > 1 = (3)_1 \) and \( (2)_2 = 4 > 3 = (3)_2 \), though \( 2 < 3 \).

**Remark 6.6.** In [12], Maehara conjectured that if an ICD does not contain any of the digraphs in Figure 4 as an induced subgraph, then it is a CICD. Clearly the graph \( G_1 \) in Example 5.5 disproves the conjecture. Note that the digraph \( G_1 \) can be obtained from the digraph (a) of Figure 4 by adding the arc 4 to 3.

![Figure 3: The digraph \( G_1 \) in Example 5.5 and its augmented adjacency matrix](image)

![Figure 4: Maehara’s forbidden digraphs for CICD](image)

Interestingly the undirected version of the Definition 5.1 leads to a well known class of graphs, namely, proper interval graphs.
**Definition 5.7.** Let $G = (V, E)$ be an undirected graph. Let $V = \{v_1, v_2, \ldots, v_n\}$ be an ordering of vertices and $f : V \to \mathbb{R}^+$ be a distinct labeling of vertices such that $d(u, v) < d(u, w)$ for all $u, v, w \in V$ such that $uv \in E$ but $uw \notin E$. Then $G$ is called an optimized graph.

**Theorem 5.8.** Let $G = (V, E)$ be an undirected graph. Then $G$ is an optimized graph if and only if $G$ is a proper interval graph.

**Proof.** Let $G = (V, E)$ be a proper interval graph. By Theorem 2.5, the reduced graph $\tilde{G} = (\tilde{V}, \tilde{E})$ of $G$ is an induced subgraph of $G(n, r) = (V_n, E')$ for some $n, r \in N$ with $n > r$. Let $V_n = \{x_1, x_2, \ldots, x_n\}$ such that $x_i \leftrightarrow x_j$ in $G(n, r)$ if and only if $0 < |i - j| \leq r$. Let $\tilde{V} = \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ such that $i_1 < i_2 < \ldots < i_m$.

For convenience, we write $y_j = x_i$, for $j = 1, 2, \ldots, m$. Define $f : V \to \mathbb{R}^+$ by $f(u) = i_j + \frac{k}{z + 1}$ if $u$ is a $k$th copy among $z$ copies of $y_j$ (for any but a fixed permutation of them). We arrange the vertices of $V$ according to the increasing order of vertices in $\tilde{V}$ keeping copies of same vertices together. Let $u, v, w \in V$ such that $u \leftrightarrow v$ and $u \leftrightarrow w$. Let $f(u) = i_p, f(v) = i_q$ and $f(w) = i_t$ for some $p, q, t \in \{1, 2, \ldots, n\}$. Then $|i_p - i_q| \leq r$ and $|i_p - i_t| > r$. So $d(u, v) = |f(u) - f(v)| < |f(u) - f(w)| = d(u, w)$. Therefore $d(u, v) < d(u, w)$ for all $u, v, w \in V$ such that $u \leftrightarrow v$ and $u \leftrightarrow w$. Then $G$ is an optimized graph.

Conversely, let $G = (V, E)$ be an optimized graph with a labeling $f$. We arrange the vertices of $G$ according to the increasing order of their labels. Let $i < j < k$, where $i, j, k \in \{1, 2, \ldots, n\}$ and $v_i \leftrightarrow v_k$. Then $d(v_i, v_j) = d(v_i, v_k) - d(v_j, v_k) < d(v_i, v_k)$. Also $d(v_j, v_k) = d(v_i, v_k) - d(v_i, v_j) < d(v_i, v_k)$. Then by Definition 5.7, $v_i \leftrightarrow v_j$ and $v_j \leftrightarrow v_k$. Thus in the given ordering of vertices, (closed) neighbors of any vertex is consecutively ordered. Therefore by Theorem 2.5, $G$ is a proper interval graph.

A Ferrers digraph $D = (V, E)$ is a directed graph whose successor sets are linearly ordered by inclusion, where the successor set of $v \in V$ is its set of out-neighbors $\{u \in V \mid vu \in E\}$. A binary matrix $M$ is a Ferrers matrix if 1’s are clustered in a corner of $M$. A digraph $D$ is Ferrers digraph if and only if there exists a permutation of vertices of $D$ such that its adjacency matrix is a Ferrers matrix. For a binary matrix $B$, $\overline{B}$ denotes the matrix obtained from $B$ by interchanging 0’s and 1’s.

**Theorem 5.9.** Let $D = (V, E)$ be a tournament. Then $D$ is an ICD if and only if there is an ordering of vertices of $D$ with respect to which the augmented adjacency matrix $A(D)$ of $D$ takes one of the following forms:

$$A(D) = \begin{bmatrix} \frac{M}{F^T} & \frac{F}{N} \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_i & v_{i+1} & \cdots & v_k & v_n \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

or

$$A(D) = \begin{bmatrix} v_1 & \cdots & v_n \\ 1 & 0 & 0 & 0 \\ \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = N$$

where $M$ is an upper triangular matrix with all entries above diagonal are one, $N$ is a lower triangular matrix with all entries below diagonal are one and $F$ is a Ferrers matrix.

**Proof.** Let $D = (V, E)$ be a tournament which is also an ICD. Let the vertices of $V$ be ordered according to the increasing value of the associated points with their corresponding intervals. Let $\{v_1, \ldots, v_n\}$ be such an ordering. It is easy to check $\{v_1, \ldots, v_n\}$ satisfies 2.2. Now it follows from the proof of Theorem 2.3.
that $A(D)$ satisfies consecutive 1’s property along its rows with respect to this ordering. For each $1 \leq i \leq n$, $(v_i)_1, (v_i)_2$ be the columns where first and last one in the row corresponding to the vertex $v_i$ occurred. As $A(D)$ is augmented we consider $v_iv_i = 1$ for all $1 \leq i \leq n$.

**Case I:** First we assume $(v_1)_2 = v_k > v_1$ where $k \geq 2$. As $D$ is a tournament $v_2v_1 = 0$ clearly. So $v_1 = (v_1)_1 < (v_2)_1 = v_2$. Now the following cases may happen.

- If $(v_2)_2 = v_2$ then $v_2 = (v_2)_2 = (v_2)_3$ which imply $v_2$ does not have any out degree vertex. Rename $v_l = v_2$.

- If $(v_2)_2 > v_2$, then we can show that $(v_2)_2 \leq (v_1)_2$. If not let $(v_2)_2 > (v_1)_2$. Then $v_1(v_2)_2 = 0$ implies $(v_2)_2v_1 = 1$ as $D$ is a tournament. But as $(v_2)_2v_2 = 0$ and $(v_2)_2(v_2)_2 = 1$ in $A(D)$, this contradicts the consecutive 1’s property for the row of $(v_2)_2$. Hence $(v_2)_2 \leq (v_1)_2$. Since $(v_2)_2 > v_2$, $v_2v_3 = 1$ and hence $v_3v_2 = 0$. Thus $(v_3)_1 = v_3$. Let

$$S = \{ i \mid (v_i)_1 = v_i, (v_i)_2 \leq (v_{i-1})_2 \}.$$

Then clearly $2 \in S$. This imply $S \neq \emptyset$. As the vertex set is finite $S$ must have a maximum $l$ (say).

First we will show $\{ j : 2 \leq j \leq l \} \subseteq S$. As $2 \in S$, applying induction we can assume $2, \ldots, j \in S$. If $j + 1 \leq l$ then we will show $j + 1$ also in $S$. If $(v_j)_2 = v_j$ then $(v_j)_2 \geq (v_{j+1})_1 = v_1 > v_j = (v_j)_2$ which is a contradiction as $l > j$. So $(v_j)_2 > v_j$. This imply $(v_{j+1})_1 = v_{j+1}$. Now we will show $(v_{j+1})_2 \leq (v_j)_2$.

On contrary let $(v_{j+1})_2 > (v_j)_2$. This implies $(v_{j+1})_2 \geq v_{j+1}$. Now if $(v_{j+1})_2 = v_{j+1}$. Then $(v_{j+1})_2 \leq (v_j)_2$ clearly as $(v_j)_2 > v_j$. So consider $(v_{j+1})_2 > v_{j+1}$ then $(v_j)_2(v_{j+1})_2 = 0$ as $(v_j)_2 < (v_{j+1})_2$ which imply $(v_{j+1})_2v_j = 1$ as $D$ is a tournament. Now $v_{j+1}(v_{j+1})_2 = 1$ implies $(v_{j+1})_2v_{j+1} = 0$.

Again $(v_{j+1})_2v_{j+1} = 1$ in $A(D)$ contradicts consecutive 1’s property along the row $(v_{j+1})_2$ (as $(v_{j+1})_2 > v_{j+1} > v_j$). So $(v_{j+1})_2 \leq (v_j)_2$. This imply $j + 1 \in S$.

Now we will show now $l \leq k$ and $(v_2)_2 = v_l$. On contrary let $l > k$. Then $v_1v_2 = 0$ clearly. This imply $v_2v_1 = 1$ from which we can conclude $(v_1)_1 = v_1 < v_l$ which contradicts $l \in S$. So $l \leq k$. Now if $v_l < (v_l)_2$ then $v_lv_{l+1} = 1$ clearly which implies $v_{l+1}v_l = 0$, i.e., $(v_{l+1})_1 = v_{l+1}$. Again $l + 1 \notin S$ implies $(v_{l+1})_2 > (v_l)_2$.

So $v_1(v_{l+1})_2 = 0$ which imply $(v_{l+1})_2v_l = 1$. But as $(v_{l+1})_2v_{l+1} = 0$ and $(v_{l+1})_2(v_{l+1})_2 = 1$, this contradicts consecutive 1’s property for the row $(v_{l+1})_2$. So $(v_l)_2 = v_l$. Thus row of $v_l$ does not contain any out degree vertex.

Thus the vertices $\{ v_1, \ldots, v_l \}$ form upper triangular matrix $M$ with all entries above diagonal as 1 and $F$ is basically a Ferrers matrix formed by the rows $\{ v_1, \ldots, v_l \}$ and columns $\{ v_{l+1}, \ldots, v_n \}$ as $(v_l)_2 = v_l \leq (v_{l-1})_2 \leq \ldots \leq (v_1)_2 = v_k$. As $D$ forms a tournament, matrix formed by rows $\{ v_{l+1}, \ldots, v_n \}$ and columns $\{ v_1, \ldots, v_l \}$ is $^TF^T$.

Now the only remaining thing is to show $(v_i)_2 = v_i$ for all $i > l$. On contrary let there exist some $v_i, l < i \leq n$ for which $(v_i)_2 > v_i$. From this it follows $v_iv_{i+1} = 1$ which implies $v_{i+1}v_i = 0$ i.e., $(v_{i+1})_1 = v_{i+1}$ which is a contradiction as $(v_i)_2 = v_i$ we get $v_iv_i = 0$ for all $i > l$. So $v_iv_i = 1$ which implies $(v_i)_1 \leq v_i < v_l$ for all $l < i \leq n$. Thus we get the required lower triangular matrix $N$ formed by the vertices $\{ v_{l+1}, \ldots, v_n \}$.

**Case II:** If $(v_1)_2 = v_1$, then $v_jv_j = 0$ for all $j$, $1 < j \leq n$. As $D$ is oriented this imply $v_jv_1 = 1$. Again as $v_jv_j = 1$ in $A(D)$ and $A(D)$ satisfies consecutive 1’s property along its rows we can conclude now $v_jv_1 = 1$ for all $i, 1 \leq i < j$. Hence $v_jv_j = 0$ for all $i, j$ where $1 \leq i < j, 1 < j \leq n$. This induces the lower triangular matrix $N$ with all entries one below diagonal formed by the vertices $\{ v_1, \ldots, v_n \}$. \□
6 Conclusion

We have established the following relations between some graph classes related to this paper.

- Proper Interval Graphs = PCMPTG = UCMPTG ⊂ Interval Graphs ⊂ CMPTG = UMTG ⊂ MTG.
- Proper Interval Graphs ⊂ 50% Max-Tolerance Graphs.
- $C_4 \in UMTG \setminus Interval \ Graphs$. $C_4$ has a unit-max-tolerance representation having intervals $[1, 5], [2, 6], [3, 7], [4, 8]$ and corresponding tolerances 1, 3, 3, 1 for its consecutive vertices (clockwise or anti-clockwise). But $C_4$ is not an interval graph.
- $K_{1,3} \in 50\% \ Max\text{-}Tolerance \ Graphs \setminus Proper \ Interval \ Graphs$. $K_{1,3}$ is an example of a graph which has a 50% max-tolerance representation having interval $[1.9, 6.1]$ and tolerance 2.1 for its central vertex and the intervals $[0, 8], [1.8, 4.3], [3.6, 5.9]$ and corresponding tolerances 4, 1.25, 1.15 for pendant vertices. But it is not a proper interval graph.
- $K_{1,n}, n \geq 8 \in CMPTG \setminus 50\% \ Max\text{-}Tolerance \ Graphs$ by Lemma 4.4.
- $\overline{C_6} \in 50\% \ Max\text{-}Tolerance \ Graphs \setminus CMPTG$ by Lemma 4.5.

The following examples lead us to conclude that interval graphs and 50% max-tolerance graphs are incomparable.

- $C_4 \in 50\% \ Max\text{-}Tolerance \ Graphs \setminus Interval \ Graphs$. $C_4$ is an example of a graph which has 50% max-tolerance representation having intervals $[1, 4.6], [2, 4], [2.9, 4.9], [2.7, 6.3]$ and corresponding tolerances 1.8, 1, 1.8 for its consecutive vertices (clockwise or anticlockwise). But it is not an interval graph.
- $K_{1,n}, n \geq 8 \in Interval \ Graphs \setminus 50\% \ Max\text{-}Tolerance \ Graphs$. $K_{1,n}$ is an interval graph having interval $[1, 2n]$ for central vertex and intervals $[2i - 1, 2i]$ for pendant vertices where $i \in \{1, \ldots, n\}$. But it has no 50% max-tolerance representation follows from Lemma 4.4.

Finally we note the major unsolved problems in this area.

1. Recognition algorithm and forbidden subgraph characterization of max-point tolerance graphs.
2. Combinatorial characterization, adjacency matrix characterization, recognition algorithm and forbidden subgraph characterization of central max-point tolerance graphs.

3. Forbidden subgraph characterization of interval catch digraphs.

4. Adjacency matrix characterization, recognition algorithm and forbidden subgraph characterization of central interval catch digraphs.

Acknowledgements:

This research is supported by UGC (University Grants Commission) NET fellowship (21/12/2014(ii)EU-V) of the author Sanchita Paul.

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