A Completion of the Spectrum of 3-Way \((v, k, 2)\) Steiner Trades

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Abstract
A 3-way \((v,k,t)\) trade \(T\) of volume \(m\) consists of three pairwise disjoint collections \(T_1, T_2\) and \(T_3\), each of \(m\) blocks of size \(k\), such that for every \(t\)-subset of \(v\)-set \(V\), the number of blocks containing this \(t\)-subset is the same in each \(T_i\) for \(1 \leq i \leq 3\). If any \(t\)-subset of \(\text{found}(T)\) occurs at most once in each \(T_i\) for \(1 \leq i \leq 3\), then \(T\) is called 3-way \((v,k,t)\) Steiner trade. We attempt to complete the spectrum \(S_{3s}(v,k)\), the set of all possible volume sizes, for 3-way \((v,k,2)\) Steiner trades, by applying some block designs, such as BIBDs, RBs, GDDs, RGDDs, and \(r \times s\) packing grid blocks. Previously, we obtained some results about the existence some 3-way \((v,k,2)\) Steiner trades. In particular, we proved that there exists a 3-way \((v,k,2)\) Steiner trade of volume \(m\) when \(12(k-1) \leq m\) for \(15 \leq k\) (Rashidi and Soltankhah in Discrete Math. 339(12): 2955–2963, 2016). Now, we show that the claim is correct also for \(k \leq 14\).

Keywords 3-Way \((v,k,2)\) Steiner trade · 1-Solely balanced set · Resolvable block design · Resolvable GDD · Resolvable \(r \times c\) grid-block packing

Mathematics Subject Classification 05B30 · 05B05

1 Introduction

Let \(k\), \(t\) and \(\lambda\) be positive integers such that \(v > k > t\). A \(t - (v,k,\lambda)\) design \((V, B)\) is a collection of blocks such that each \(t\)-subset of \(v\)-set \(V\) is contained in \(\lambda\) blocks of collection \(B\). A \((v,k,t)\) trade \(T = \{T_1, T_2\}\) of volume \(m\) consists of two disjoint collections \(T_1\) and \(T_2\), each containing \(m\) blocks. Each block is a \(k\)-subset of \(v\)-set \(V\). Every \(t\)-subset of \(v\)-set \(V\) is contained in the same number of blocks in \(T_1\) and \(T_2\). In a \((v,k,t)\) trade, both collections of
blocks must cover the same set of elements. This set of elements is foundation of \((v, k, t)\) trade and is denoted by found \((T)\). A \((v, k, t)\) trade is called \((v, k, t)\) Steiner trade if any \(t\)-subset of found \((T)\) occurs at most once in \(T_1(T_2)\).

Recently, a generalization for the concept of trade as has been defined in [10] by the name of \(\mu\)-way \((v, k, t)\) trade as follows:

A \(\mu\)-way \((v, k, t)\) trade of volume \(m\) consists of \(\mu\) pairwise disjoint collections \(T_1, \ldots, T_\mu\) each of \(m\) blocks, such that for every \(t\)-subset of \(v\)-set \(V\), the number of blocks containing this \(t\)-subset is the same in each \(T_i\) for \(1 \leq i \leq \mu\). In the other words any set \(T = \{T_i, T_j\}\) for \(i \neq j\) is a \((v, k, t)\) trade of volume \(m\). A \(\mu\)-way \((v, k, t)\) trade is \(\mu\)-way \((v, k, t)\) Steiner trade if any \(t\)-subset of found \((T)\) occurs at most once in every \(T_j\) for \(j \geq 1\).

The concept of \(\mu\)-way trade was defined under a different name as \(N\)-legged trade, before, see [2]. Each \(T_i\) contains \(m\) blocks \(B_{ij}\), \(B_{ab}\), \(B_{mi}\), where \(B_{ij}\) denote \(i\)th block of \(T_j\). Therefore, a \(\mu\)-way trade \(T\) of volume \(m\) has the following form:

\[
\begin{array}{cccccc}
T_1 & T_2 & \cdots & T_i & \cdots & T_\mu \\
T_{11} & T_{12} & \cdots & T_{1i} & \cdots & T_{1\mu} \\
T_{21} & T_{22} & \cdots & T_{21} & \cdots & T_{1\mu} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
T_{m1} & T_{m2} & \cdots & T_{mi} & \cdots & T_{m\mu} \\
\end{array}
\]

A type of \(\mu\)-way \((v, k, t)\) Steiner trade with additional property is named \(\mu\)-way \(t\)-solely balanced set where those sets have the important role in constructing the Steiner trade with other parameters.

**Definition 1.1** Let \(T = \{T_1, \ldots, T_\mu\}\) be a \(\mu\)-way \((v, k, t)\) Steiner trade. It is \(\mu\)-way \(t\)-solely balanced set if there exist no blocks \(B_{ij}\) and \(B_{ab}\) such that \(|B_{ij} \cap B_{ab}| > t\) for \(1 \leq j < b \leq \mu\) and \(1 \leq i, a \leq m\). In the other words \(B_{ij}\) and \(B_{ab}\) contain no common \((t + 1)\)-subset.

For \(t = 1\) there exists a \(\mu\)-way 1-soley balanced set for \(k = 2\) and \(v = 2m\), \(\mu \leq 2m - 1\) of volume \(m\). It is a one factorization of complete graph \(K_{2m}\). For \(t \geq 2\) each large set of a super simple design or a resolvable super simple design is a \(\mu\)-way \(t\)-solely balanced set. Hamilton and Khodkar [6] named 2-soley balanced set with strong Steiner trades that is a \((v, k, 2)\) Steiner trade \(T = \{T_1, T_2\}\) so that any block of \(T_1\) intersects any block of \(T_2\) in at most two elements. Gray and Ramsay [3] applied 2-way 1-soley balanced set for constructing 2-way \((v, k, t)\) Steiner trades. These sets are named solely sets by them.

There exists the following main construction, which uses the \(\mu\)-way 1-soley balanced set \(T\) and \(\mu\) new elements \(\{x_1, \ldots, x_\mu\}\) for constructing the \(\mu\)-way \((v + \mu, k + 1, t + 1)\) Steiner trade \(T^*\):

\[
\begin{array}{cccccc}
T_1^* & T_2^* & \cdots & T_\mu^* \\
x_1T_1 & x_1T_2 & \cdots & x_1T_\mu \\
x_2T_2 & x_2T_3 & \cdots & x_2T_1 \\
x_3T_3 & x_3T_4 & \cdots & x_3T_2 \\
\cdots & \cdots & \cdots & \cdots \\
x_\mu T_\mu & x_\mu T_1 & \cdots & x_\mu T_{\mu-1} \\
\end{array}
\]

This construction is mentioned in the following theorem.
Theorem 1.2 [10] (i) Let $T = \{T_1, \ldots, T_\mu\}$ be a $\mu$-way $(v, k, t)$ trade of volume $m$. Next based on $T$, a $\mu$-way $(v + \mu, k + 1, t + 1)$ trade $T^*$ of volume $\mu m$ can be constructed.
(ii) If $T$ is a $\mu$-way $t$-solely balanced, then a $\mu$-way $(v + \mu, k + 1, t + 1)$ Steiner trade $T^*$ can be constructed.

We need some notation. Let $T$ and $T^*$ be two $\mu$-way $(v, k, t)$ trades of volume $m$.

We consider $T + T^* = \{T_1 \cup T^*_1, \ldots, T_\mu \cup T^*_\mu\}$. It is easy to see that $T + T^*$ is a $\mu$-way $(v, k, t)$ trade. If $T$ and $T^*$ are Steiner trades and $\text{found}(T) \cap \text{found}(T^*) = \emptyset$, then $T + T^*$ is also a Steiner trade.

There exist many questions concerning $\mu$-way trades. Some of the most important questions are about the minimum volume and minimum foundation size and the set of all possible volume sizes of $\mu$-way trades. Not much is known for the mentioned questions about $\mu$-way $(v, k, t)$ trades for $\mu \geq 3$ and most of the papers have been focused on the case $\mu = 2$. Some questions have been answered about the existence and non-existence of 3-way $(v, k, t)$ trades for some special values of $k$ and $t$, see [1, 5, 10]. Let $S_{3\mu}(t, k)$ denote the set of all possible volume sizes of a 3-way $(v, k, t)$ Steiner trade. The set of all possible volume sizes of a 2-way $(v, k, 2)$ Steiner trade has been answered completely in [3, 4, 7].

The aim of this paper is to complete the lower bound for the spectrum of 3-way $(v, k, 2)$ Steiner trades. Previously in [11], we obtained some results about the existence some 3-way $(v, k, 2)$ Steiner trades. In particular, we proved that there exists a 3-way $(v, k, 2)$ Steiner trade of volume $m$ when $12(k - 1) \leq m$ for $15 \leq k$. Here, we show it is correct also for $k \leq 14$. Also we improve the lower bound to $11(k - 1)$ for even $k$. The obtained results in [11] are as follows:

Theorem 1.3 [11] $m \in S_{3\mu}(2, k)$ for all $m \geq 12(k - 1)$ when $k \geq 15$.

Also there are some other results.

Theorem 1.4 [11] There exists a 3-way $(v, k, 2)$ Steiner trade of volume $9(k - 1) - r$ for $r \in \{1, \ldots, k - 1\}$ with block size $k$.

Theorem 1.5 [11] (1) There exists a $\mu$-way $(q^2 + \mu, q + 1, 2)$ Steiner trade of volume $m = q\mu$ for $\mu \in \{2, \ldots, q + 1\}$ when $q$ is a prime power.
(2) $S_{3\mu}(2, k) \subseteq \mathbb{N} \setminus \{1, \ldots, 3k - 4\}$ for $k \neq 4$.
(3) There exists a 3-way $(v, k, 2)$ Steiner trade of volume:
(a) $m = 3l$ for $l \geq k - 1$;
(b) $m = 4l$ for $l = k - 1$ or $l \geq 2(k - 1)$ when $l$ is odd;
(c) $m = 4l$ for all $l \geq 4(k - 1)$.

But we obtain three new constructions that will be introduced in the next section for this aim.

2 New Constructions

In this section, we propose two constructive methods for 3-way $(v, k, t)$ Steiner trades. In these constructions, we first have an 1-solely balanced set and then apply the Theorem 1.2 for constructing the Steiner trades.
2.1 Construction 1 (GDD and RB)

The main combinatorial objects in this section are GDDs and RBs.

**Definition 2.1** Suppose \((V, B)\) is a \(2 - (v, k, \lambda)\) Design. A parallel class in \((V, B)\) is a subset of pairwise disjoint blocks from \(B\) whose union is \(V\). A partition of \(B\) into \(r\) parallel classes is called a resolution, and \((V, B)\) is said to be a resolvable if \(B\) has at least one resolution. A resolvable \(2 - (v, k, \lambda)\) Design is named RB or \(RB(v, k, \lambda)\).

A parallel class contains \(\frac{v}{r}\) blocks. Therefore, a \(2 - (v, k, \lambda)\) Design can have a parallel class only if \(v \equiv 0 \pmod{k}\). Also a RB\((v, k, \lambda)\) has \(r = \frac{\lambda(v-1)}{k-1}\) parallel classes.

**Definition 2.2** Let \(K\) be a set of positive integers. A group divisible design \(K\)-GDD (as GDD for short) is a triple \((X, \mathcal{G}, \mathcal{A})\) satisfying the following properties:

1. \(\mathcal{G}\) is a partition of a finite set \(X\) into subsets (called groups);
2. \(\mathcal{A}\) is a set of subsets of \(X\) (called blocks), each of cardinality from \(K\), such that a group and a block contain at most one common element;
3. every pair of elements from distinct groups occurs in exactly one block.

If \(\mathcal{G}\) contains \(u_i\) groups of size \(g_i\), for \(i \in \{1, \ldots, s\}\), then we denote by \(g_1^{u_1}g_2^{u_2}\cdots g_s^{u_s}\) the group type (or type) of the GDD. If \(K = \{k\}\), we write \(\{k\}\)-GDD as \(k\)-GDD. This is exponential notation for the group type.

A GDD is resolvable if the blocks of it can be partitioned into parallel classes. A resolvable GDD is denoted by RGDD. Let \(r^*\) be the number of parallel classes of the blocks of RGDD.

**Example 2.3** The collection \(D\) is a \(RB(9, 3, 1)\). The blocks are written as columns. The blocks of one parallel class can be the groups. Hens the collection \(D\) is a 3-GDD of type \(3^3\) with \(r^* = 3\).

\[
D: 1 \ 4 \ 7 \ 1 \ 2 \ 3 \ 1 \ 2 \ 3 \\
2 \ 5 \ 8 \ 4 \ 5 \ 6 \ 5 \ 6 \ 4 \ 6 \ 4 \ 5 \\
3 \ 6 \ 9 \ 7 \ 8 \ 9 \ 9 \ 7 \ 8 \ 9 \ 7
\]

Now, we construct new 1-soley balanced sets and related Steiner trades using some RBs or GDDs.

**Theorem 2.4** [8] The necessary conditions for the existence of a \(4\)-RGDD of type \(g^u\), namely, \(u \geq 4, gu \equiv 0 \pmod{4}\) and \(g(u - 1) \equiv 0 \pmod{3}\) except \((g, u) \in \{(2, 4), (3, 4), (6, 4), (2, 10)\}\) and possibly excepting; \(g = 2\) and \(u \in \{34, 46, 52, 70, 82, 94, 100, 118, 130, 142, 178, 184, 202, 214, 238, 250, 334, 346\}\); \(g = 10\) and \(u \in \{4, 34, 52, 94\}\); \(g \in \{14, 454\} \cup \{478, 502, 514, 526, 614, 626, 686\}\) and \(u \in \{10, 70, 82\}\); \(g = 6\) and \(u \in \{6, 54, 68\}\); \(g = 18\) and \(u \in \{18, 38, 62\}\); \(g = 9\) and \(u = 44\); \(g = 12\) and \(u = 27\); \(g = 24\) and \(u = 23\); and \(g = 36\) and \(u \in \{11, 14, 15, 18, 23\}\).

**Remark 2.5** The number of blocks of \(k\)-RGDD, denoted \(b\), is obtained from the following equations.

\[v = (k - 1)r^* + g\]
\[(r^* + 1)v = kb + gu\]

We explain this equation for the Example 2.3. Consider the group 147. The elements of this block appear in the three blocks of each parallel class of GDD. For example, consider the first parallel class. The element 1 appears with the elements 2 and 3. The element 4 appears with the elements 5 and 6. The element 7 appears with the elements 8 and 9. Hence 
\[v = 3 + 2 + 2 + 3 = 2 \times 3 = g + r^*(k - 1).\]

The second equation: Each element occur exactly once in each parallel class and exactly once in the groups. Therefore, \(v(r^* + 1) = gu + bk\).

**Theorem 2.6** There exists a 3-way \((23, 5, 2)\) Steiner trade of volume 25.

**Proof** By Theorem 2.4 and Remark 2.5, there exists a 4-RGDD of type 54 and
\[20 = 3r^* + 5\]

\[(r^* + 1)(3r^* + 5) = 4b + 20.\]

Therefore, \(b = 25, m = \frac{v}{k} = \frac{20}{4} = 5\) and \(\mu = \frac{b}{m} = \frac{25}{5}\). Therefore, we have a 5-way 1-soley balanced set with volume \(m = 5\), foundation \(v = 20\) and block size \(k = 4\). By, Theorem 1.2, we can construct a 3-way \((23, 5, 2)\) Steiner trade of volume 15, 20, and 25. □

The necessary conditions for the existence of a \(RB(v, k, \lambda)\) is:
1. \(\lambda(v - 1) \equiv 0 \pmod{k - 1}\),
2. \(v \equiv 0 \pmod{k}\).

The necessary conditions are sufficient for any \(k\) and \(\lambda\) if \(v\) is large enough. There exist four related theorems, for \(\lambda = 1\), which are as follows.

**Theorem 2.7** [12] There exists a \(RB(v, 4, 1)\) if and only if \(v \equiv 4 \pmod{12}\).

**Theorem 2.8** [12] If \(v\) and \(k\) are both powers of the same prime, then the necessary conditions for the existence of an \(RB(v, k, \lambda)\) are sufficient.

**Theorem 2.9** [12] Let \(q = k(k - 1) + 1\) be a prime power, odd number and \(q > \left(\frac{k(k-1)}{2}\right)^{k(k+1)}\), there exists a \(RB(kq, k, 1)\).

**Theorem 2.10** [12] For \(k \geq 3\) and \(q \equiv k \pmod{k(k - 1)}\) and \(v > \exp(e^{12k^2})\), there exists a \(RB(v, k, 1)\).

**Theorem 2.11** The existence of a \(RB(v, k, 1)\) design is equivalent to a \(\mu\)-way 1-soley balanced set with block size \(k\), volume \(m = \frac{v}{k}\) and \(\mu = \frac{v-1}{k-1}\).

**Proof** Each \(RB(v, k, 1)\) has \(\mu = \frac{v-1}{k-1}\) parallel classes, such as \(P_1, \ldots, P_\mu\). Also each class contains \(m = \frac{v}{k}\) blocks. Let \(T_i = P_i\) for \(1 \leq i \leq \mu\). Now \(T = \{T_1, \ldots, T_\mu\}\) is a \(\mu\)-way 1-soley balanced set with block size \(k\), volume \(m = \frac{v}{k}\) and \(\mu = \frac{v-1}{k-1}\). □

Now, we apply Theorem 1.2 and 2.11, then there exists a \(\mu\)-way \((v + \mu, k + 1, 2)\) Steiner trade for large \(v\) and volume \(\mu m\). This confirms our results in [11].

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Theorem 2.12 If there exists a \( \mu \)-way 1-soley balanced set with block size \( k \), volume \( m \) and foundation size \( v \), such that \( k = \frac{v}{m} \), then \( \mu \leq \frac{v-1}{k-1} \). The equation holds when there exists a \( RB(v, k, 1) \).

Proof The number of pairs which appear in the \( \mu \)-way 1-solely balanced set with block size \( k \) and volume \( m \), is \( C(k, 2).m.\mu \). By the property of 1-solely balanced sets (Each pair appears in at most one block.):

\[
C(k, 2).v^k.\mu \leq C(v, 2) = v(v-1) \implies \mu \leq \frac{v-1}{k-1}
\]

Now, if \( \mu = \frac{v-1}{k-1} \), then the \( \mu \)-way 1-solely balanced set is a \( RB(v, k, 1) \).

Theorem 2.13 There exists a 3-way \((v + 2, 5, 2)\) Steiner trade of volume \( \mu(v - 1) \frac{4}{3} \), for \( 3 \leq \mu \leq \frac{v-2}{3} \), \( v \geq 11 \) and \( v \equiv 5 \) (mod 12).

Proof By Theorem 2.7, there exist \( RB(v - 1, 4, 1) \) for \( v - 1 \equiv 4 \) (mod 12). Therefore, by Theorem 2.11 there exists a \( \mu \)-way 1-solely balanced set with block size \( k = 4 \), volume \( m = \frac{v-1}{4} \) and \( 3 \leq \mu \leq \frac{v-2}{3} \). Now, we apply Theorem 1.2 and obtain a 3-way \((v + 2, 5, 2)\) Steiner trade of volume \( \mu m \) for \( m = \frac{v-1}{4} \) and \( 3 \leq \mu \leq \frac{v-2}{3} \).

2.2 Construction 2 (Resolvable \( r \times c \) Grid-Block Packing)

In this subsection, we obtain some \( \mu \)-way 1-solely balanced sets by resolvable \( r \times c \) grid-block packings and vice versa.

Definition 2.14 [9] For a \( v \)-set \( V \), let \( A \) be a collection of \( r \times c \) arrays with elements in \( V \). A pair \((V, A)\) is called an \( r \times c \) grid-block packing (an \( r \times c \) grid-block design) if any two distinct points \( i \) and \( j \) in \( V \) occur together at most once (exactly once) in the same row or in the same column of arrays in \( A \). A \( r \times c \) grid-block packing is denoted by \( P_{r \times c}(K_v) \) (\( D_{r \times c}(K_v) \)).

Remark 2.15 Notice that by this definition the following table cannot be a grid-block of an \( r \times c \) grid-block packing. Since the pair \( \{1, 4\} \) occurs in the first column and the second row. Therefore, the grid-block cannot contain the repetitive element.

| 1 | 2 | 3 |
|---|---|---|
| 4 | 5 | 1 |
| 6 | 1 | 7 |

Definition 2.16 [9] An \( r \times c \) grid-block packing \((V, A)\) is said to be resolvable if the collection of arrays \( A \) can be partitioned into sub-classes \( R_1, \ldots, R_t \) such that every point of \( V \) is contained in precisely one array of each class.

If a packing is resolvable, then \( v \) is divisible by \( rc \) and the number of grid-block is \( b = t \frac{v}{rc} \leq \lfloor \frac{v-1}{r+c-2} \rfloor \), where \( t \) is the number of resolution class. A resolvable packing \( P_{r \times c}(K_v) \) attaining this bound is said to be optimal.

There exist some existence results about \( P_{q \times q}(K_q^n) \) and \( D_{q \times q}(K_q^n) \). We apply the following theorems in this section.
Theorem 2.17 [9] An optimal resolvable grid-block packing $P_{q \times q}(K_{q^n})$ exists for a prime power $q$ and an integer $n$. Moreover, when $n$ is even and $q$ is odd, the optimal resolvable grid-block packing is a resolvable grid-block design $D_{q \times q}(K_{q^n})$.

The aim of $PA(v, c, 1)$ is a resolvable packing however there does not exist any definition of it in [9].

Theorem 2.18 [9] Assume $r \leq c$. If there exists a resolvable $PA(v, c, 1)$ with $t$ resolution classes and a resolvable $P_{r \times c}(K_{rc})$ with $s + 1$ grid-blocks, then there exists a resolvable $P_{r \times c}(K_{rv})$ with $st + 1$ resolution classes.

Theorem 2.19 [9] Assume $r \leq c$. If there exists a resolvable $PA(v, c, 1)$ with $t$ resolution classes, then there exists a resolvable $P_{r \times c}(K_{rv})$ with $t$ resolution classes.

The third construction is explained as follows.

Example 2.20 There exists the following resolvable $3 \times 3$ grid-block packing in [9]. Now, we construct some Steiner trades from it.

\[
\begin{array}{ccc}
0, 4, 8 & 0, 9, d & a, h, 8 \\
3, 4, 5 & 5, 6, 9 & c, 7, 3 \\
6, 7, 8 & 7, a, e & \\
\end{array}
\]

If we consider the rows of arrays of this resolvable $P_{3 \times 3}(K_{18})$, then it is a 3-way $1$-solely balanced set for $k = 3$ of volume 6. If we consider the rows and columns of arrays of this resolvable $P_{3 \times 3}(K_{18})$, then it is a 6-way $1$-solely balanced set of volume 6 with same parameter. We can do this because this $P_{3 \times 3}(K_{18})$ is resolvable and all elements of $V(K_{18})$ appeared in the columns of $S_1$. Notice that the columns are written respectively from up to down. For example, in the following 6-way $1$-solely balanced set of volume 6. The blocks of $S_i$ are the columns of $S_{i-3}$ ($4 \leq i$).

\[
\begin{array}{ccccccc}
S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \\
012 & 048 & 09d & 036 & 057 & 0ac \\
345 & 569 & ah8 & 147 & 46a & 9hc \\
678 & 7ae & c73 & 258 & 89e & d83 \\
9ab & 13f & 15b & 9cf & 1ch & 16g \\
cde & cgb & 6f2 & adg & 3g2 & 5fe \\
fg \ h2d & ge4 & beh & fbd & b24 \\
\end{array}
\]

Now, we apply Proposition 1.2 and obtain a $\mu$-way $(18 + \mu, 4, 2)$ Steiner trade of volume $6\mu$ for $\mu \in \{2, 3, 4, 5, 6\}$.

By the method of this example, we have the following theorem.
Theorem 2.21  i) If there exists a resolvable \( P_{r \times c}(K_n) \) with \( t \) resolution classes, then there exists a \( \mu \)-way \((n + \mu, c + 1, 2)\) Steiner trade of volume \( \frac{n}{\mu} \mu \) for \( \mu \leq 2t \).

ii) If there exists a resolvable \( P_{r \times c}(K_n) \) with \( t \) resolution classes, then there exists a \( \mu \)-way \((n + \mu, c + 1, 2)\) Steiner trade of volume \( \frac{n}{\mu} \mu \) for \( \mu \leq t \).

\[ \text{Proof} \] The proof is similar the previous example. \( \Box \)

Corollary 2.22 There exists a \( \mu \)-way \((q^n + \mu, q + 1, 2)\) Steiner trade of volume \( q^{n-1} \mu \) for \( \mu \leq 2\left\lceil \frac{q^n-1}{2q-2} \right\rceil \).

\[ \text{Proof} \] The result is obtained by Theorem 2.17 and Theorem 2.21. \( \Box \)

In Theorem 2.19, we can replace a \( \mu \)-way 1-soley balanced set by resolvable packing \( PA(v, c, 1) \) and obtained the new resolvable \( P_{r \times c}(K_{rv}) \) with \( \mu \) resolution classes. The proof of Theorem 2.19 (It is proved in [9]) is mentioned again in the following example. But, we apply the 1-soley balanced set instead of the resolvable packing \( PA(v, c, 1) \).

Example 2.23 Consider the 1-soley balanced set that it is obtained in the previous example. Corresponding to each block of the one-soley balanced set an array is constructed as follows:

Each block \( a, b, c \) change to the this block.

\[
\begin{array}{ccc}
    a_0 & b_0 & c_0 \\
    b_1 & c_1 & a_1 \\
    c_2 & a_2 & b_2 \\
\end{array}
\]

Therefore, we have resolvable \( P_{3 \times 3}(K_{54}) \) grid packing with six resolution classes. The following arrays are the resolution class \( R_1 \) of resolvable \( P_{3 \times 3}(K_{54}) \) which are constructed from the \( S_1 \).

\[
\begin{array}{ccc}
    0_0 & 1_0 & 2_0 \\
    1_1 & 2_1 & 0_1 \\
    2_2 & 0_2 & 1_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
    3_0 & 4_0 & 5_0 \\
    4_1 & 5_1 & 3_1 \\
    5_2 & 3_2 & 4_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
    9_0 & a_0 & b_0 \\
    a_1 & b_1 & 9_1 \\
    b_2 & 9_2 & a_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
    c_0 & d_0 & e_0 \\
    d_1 & e_1 & c_1 \\
    e_2 & c_2 & d_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
    f_0 & g_0 & h_0 \\
    g_1 & h_1 & f_1 \\
    h_2 & f_2 & g_2 \\
\end{array}
\]

Now, we can apply the method of example 2.20 and construct the 6-way 1-soley balanced set of volume 18 and 36 from this resolvable \( P_{3 \times 3}(K_{54}) \). However, the new 6-way 1-soley balanced set can be obtain by adding three 6-way 1-soley balanced sets of volume 6 on disjoint foundations. The 6-way 1-soley balanced set of volume 6 is obtained in Example 2.20.

According to this example, we have the following theorem.

Theorem 2.24 Assume \( r \leq c \). If there exists a \( \mu \)-way 1-soley balanced set with block size \( c \) and foundation \( v \), then there exists a resolvable \( P_{r \times c}(K_{rv}) \) with \( \mu \) resolution classes.
The characterization of \( \mu \)-way 1-solely balanced set with different parameters can be useful for constructing the new resolvable \( r \times c \) grid-block packing. For example, in the [9] it is proved that, there exists optimal \( P_{2 \times 2}(K_n) \) for any \( v \equiv 0 \) (mod 4) in three pages. But, we can obtain this results by the 1-solely balanced set and Theorem 2.21 as follows.

**Theorem 2.25** There exists optimal resolvable \( P_{2 \times 2}(K_v) \) for any \( v \equiv 0 \) (mod 4)

**Proof** Each complete graph \( K_{2m} \) has \( 2m - 1 \) one-factors. We consider this one factorization of \( K_{2m} \) as a \((2m - 1)\)-way 1-solely balanced set of volume \( m \) and block size 2. Now apply Theorem 2.19. We use this \((2m - 1)\)-way 1-solely balanced set instead of a \( PA(2v, 2, 1) \). Therefore, we have a resolvable \( P_{2 \times 2}(K_{4m}) \). This resolvable \( P_{2 \times 2}(K_{4m}) \) has \( 2m - 1 = \lfloor \frac{4m - 1}{2} \rfloor - \lfloor \frac{4m - 2}{2} \rfloor \) resolution classes. This number is optimal. Since \( 2m - 1 = \lfloor \frac{v - 1}{r + c - 2} \rfloor = \lfloor \frac{4m - 2}{2} \rfloor \).

The method of Theorem 2.25 is explained in the following example for complete graph \( K_4 \).

**Example 2.26** The one-factorization of \( K_4 \) is a 3-way 1-solely balanced set.

\[
\begin{array}{ccc}
S_1 & S_2 & S_3 \\
12 & 13 & 14 \\
34 & 24 & 23
\end{array}
\]

Now, we construct the following optimal resolvable \( P_{2 \times 2}(K_8) \) by the method of Theorem 2.19.

\[
R_1: \begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}, \quad R_2: \begin{bmatrix}
1 & 3 \\
4 & 2
\end{bmatrix}, \quad R_3: \begin{bmatrix}
1 & 4 \\
2 & 3
\end{bmatrix}
\]

Also we can apply the \( \mu \)-way 1-solely balanced set and Theorem 2.18. Next we obtain some new \( \mu' \)-way 1-solely balanced set for \( \mu' \geq \mu \). This result is interesting. Since, we can have new \( \mu' \)-way steiner trades. It is explained in the next example and Theorem.

**Theorem 2.27** If there exists a \( \mu \)-way 1-solely balanced set with block size \( c \) and foundation \( v \) and a resolvable \( P_{r \times c}(K_{rc}) \) with \( s + 1 \) grid blocks, then there exists a \( \mu' \)-way \( (rv + \mu', c + 1, 2) \) Steiner trade for \( \mu' \leq s\mu + 1 \) of volume \( v\mu' \).

**Proof** The proof has resulted from Theorems 2.18 and 2.21.

This theorem can be applied for constructing the Steiner trades. For example, we know that There exists a \((2v - 1)\)-way 1-solely balanced set with block size 2 and foundation \( v \) and an optimal resolvable \( P_{2 \times 2}(K_8) \) with \( 5 + 1 \) grid blocks, then there exists a \( \mu' \)-way \((2v + \mu', 2 + 1, 2) \) Steiner trade for \( \mu' \leq 5(2v - 1) + 1 = 10v - 4 \) of volume \( v\mu' \). Also if we apply \( \mu \)-way 1-solely balanced set for block size \( k \geq 3 \), then we obtain the other Steiner trades for block size \( k \geq 4 \).

Also, there exist two theorems for achieving a new \( \mu \)-way 1-solely balanced set (resolvable \( r \times c \) grid-block packing) from another by applying the resolvable \( r \times c \) grid-block packings (\( \mu \)-way 1-solely balanced sets).
Theorem 2.28 If there exists a $\mu$-way 1-solely balanced set with foundation size $v$ and block size $c$, then there exists a $\mu'$-way 1-solely balanced set with foundation size $c \times v$, block size $c$ and volume $v$ for $\mu' \leq 2\mu$.

Proof By Theorem 2.24, there exists a resolvable $P_{r \times c}(K_{rv})$ with $\mu$ resolution classes. Now, apply Theorem 2.21 and obtain the result. \(\Box\)

Example 2.29 In Example 2.26 a $P_{2 \times 2}(K_8)$ is constructed from a 3-way 1-solely balanced set with foundation size 4 and block size 2. Now, we can construct a 6-way 1-solely balanced set from this $P_{2 \times 2}(K_8)$ of volume four as follows.

\[
\begin{array}{ccccccc}
S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \\
1_02_0 & 1_03_0 & 1_04_0 & 1_02_1 & 1_03_1 & 1_04_1 \\
2_11_1 & 3_11_1 & 4_11_1 & 2_01_1 & 3_01_1 & 4_01_1 \\
3_04_0 & 2_04_0 & 2_03_0 & 3_04_1 & 2_04_1 & 2_03_1 \\
4_13_1 & 4_12_1 & 3_12_1 & 4_03_1 & 4_02_1 & 3_02_1 \\
\end{array}
\]

The second theorem handles with a resolvable $P_{r \times c}(K_v)$.

Theorem 2.30 If there exists a resolvable $P_{c \times c}(K_v)$ with $t$ resolution classes, then there exists a resolvable $P_{r \times c}(K_{rv})$ with $2t$ resolution classes for $r \leq c$.

Proof By Theorem 2.21, there exists a $\mu$-way 1-solely balanced set with block size $c$, foundation size $v$ and volume $v^c$ for $\mu \leq 2t$. Now apply Theorem 2.24 and obtain the result. \(\Box\)

3 Completion of Spectrum

In [11], we prove that there exists a 3-way $(v, k, 2)$ Steiner trade of volume $m$ when $m \geq 12(k - 1)$ for $k \geq 15$. In this section, we show it is correct also for $k \leq 14$. For $k = 1$, there does not exist any Steiner trade and for $k = 2$, we have the trivial case. For $k = 3$ and $k = 4$, it is proved in [10].

3.1 Block Size 5

For $k = 5$, we have five parts as follows:
1- By Theorem 1.5, $S_{3c}(2, k) \subseteq N \setminus \{1, \ldots, 3k - 4\}$ for $k \neq 4$. Therefore, $S_{3c}(2, 5) \subseteq N \setminus \{1, \ldots, 11\}$ and there does not exist any 3-way $(v, 5, 2)$ Steiner trade of volume $m$ for $m \leq 11$.
2- By the third part of Theorem 1.5, there exists 3-way $(v, 5, 2)$ Steiner trade of volume $\{12, 15, 18, 21, 24, 27, 30, 33, 36, 39, \ldots\} = \{m : m = 3l, l \geq 4\}$.
3- By Theorem 1.4, there exists 3-way $(v, 5, 2)$ Steiner trade of volume $m \in \{32, 33, 34, 35\} = \{m : m = 9 \times 4 - r, 0 \leq r \leq 4\}$.
4- By the fourth part of Theorem 1.5, there exists 3-way $(v, 5, 2)$ Steiner trade of volume $m$ for $m \in \{16, 20, 24, 28, 40, \ldots\} = \{m : m = 4l, l \geq 4\}$.
5- There exist 3-way $(v, 5, 2)$ Steiner trades of volume $m \in \{15, 20, 25\}$ by Construction 1.

Theorem 3.1 There exists a 3-way $(v, 5, 2)$ Steiner trade of volume $m \geq 12$ except possibly when $m \in \{13, 14, 17, 19, 22, 23, 26, 29\}$. 

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Proof If there exist two 3-way Steiner trades $T$ and $T^*$ of volume $m_1$ and $m_2$ with disjoint foundations and same block size, then there exists a 3-way Steiner trade $T + T^*$ of volume $m_1 + m_2$. By this method, we have the existence of a 3-way $(v, 5, 2)$ Steiner trade of the volumes $\{31, 41, 38, 46, 43, 37, 47, 48, 49\}$ and volumes $m = 20 + m'$, for $m' \geq 30$.

By previous description the remainder volumes are $\{13, 14, 17, 19, 22, 23, 26, 29\}$. These volumes are prime numbers or products of pairs of prime numbers less than 30.

3.2 Block Size 6

For $k = 6$, we have five parts as follows:
1- By Theorem 1.5, $S_{3,2}(2, 6) \subseteq \mathcal{N} \setminus \{1, \ldots, 14\}$ and there does not exist any 3-way $(v, 6, 2)$ Steiner trade of volume $m$ for $m \leq 14$.
2- By the second part of Theorem 1.5, there exists a 3-way $(v, 6, 2)$ Steiner trade of volume $m \in \{15, 18, 21, \ldots\} = \{m : m = 3l, \ l \geq 5\}$.
3- By Theorem 1.4, there exists a 3-way $(v, 6, 2)$ Steiner trade of volume $m \in \{40, 41, 42, 43, 44\} = \{m : m = 9 \times 5 - r, \ 1 \leq r \leq 5\}$.
4- By the fourth part of Theorem 1.5, there exists a 3-way $(v, 6, 2)$ Steiner trade of volume $m \in \{40, 44, 48, \ldots\} = \{m : m = 4l, \ l \geq 10\}$.
5- By the fifth part of Theorem 1.5, there exists a $(v, 6, 2)$ 3-way Steiner trade of volume $m \in \{15, 20, 25, 30\}$.

Now, we can state the following theorem.

**Theorem 3.2** There exists a 3-way $(v, 6, 2)$ Steiner trade of volume $m \geq 15$ except possibly when $m \in \{16, 17, 19, 22, 23, 26, 28, 29, 31, 32, 34, 37\}$.

Proof Some volumes are multiples of three or four with the conditions of Theorem 1.5. Other volumes can be written as $m_1 + m_2$ from previous parts. By this method, we have the existence of a 3-way $(v, 6, 2)$ Steiner trade of volumes $\{35, 38, 46, 47, 49, 50, 53, 55, 56, 58, 59, 61, 62, 65, 67, 68, 70, 71, 73, 74\}$ and $m = 15 + m'$, for $m' \geq 60$. There does not exist any 3-way $(v, 6, 2)$ Steiner trade of volume $m \leq 14$. Therefore, the remainder volumes are $\{16, 17, 19, 22, 23, 26, 28, 29, 31, 32, 34, 37\}$.

Some values of volumes can be obtained by other ways such as by using Construction 1.

**Example 3.3** There exists a $RB(45, 5, 1)$. Therefore, by Construction 1, there exist $\mu$-way $(45 + \mu, 6, 2)$ Steiner trades of volumes $\mu \frac{45}{5} = 9\mu$, for $3 \leq \mu \leq 11$.

Steiner trade of volume 36 – 6 = 30.

There exists a $RB(25, 5, 1)$. By Construction 1, there exist $\mu$-way $(25 + \mu, 6, 2)$ Steiner trades of volumes $\mu \frac{25}{5} = 5\mu$, for $3 \leq \mu \leq 6$. Therefore, there exists a 3-way $(v, 6, 2)$ Steiner trade of volume $m \in \{15, 20, 25, 30\}$.

3.3 Block Size 7

For $k = 7$, we have four parts as follows:
1- By Theorem 1.5, $S_{3,2}(2, 7) \subseteq \mathcal{N} \setminus \{1, \ldots, 17\}$ and there does not exist any 3-way $(v, 7, 2)$ Steiner trade of volume $m$ for $m \leq 17$.
2- By the second part of Theorem 1.5, there exists a 3-way $(v, 7, 2)$ Steiner trade of volume $m \in \{18, 21, \ldots\} = \{m : m = 3l, \ l \geq 6\}$.
3- By Theorem 1.4, there exists a 3-way $(v, 7, 2)$ Steiner trade of volume $m \in \{47, 48, 49, 50, 51, 52, 53\} = \{m : m = 9 \times 6 - r, \ 1 \leq r \leq 6\}$. 

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4- By the fourth part of Theorem 1.5, there exists a 3-way \((v, 7, 2)\) Steiner trade of volume \(m \in \{48, \ldots \} = \{m : m = 4l, \ l \geq 12\}.

Now, we can state the following theorem.

**Theorem 3.4** There exists a 3-way \((v, 7, 2)\) Steiner trade of volume \(m \geq 18\) except possibly when \(m \in \{19, 20, 22, 23, 25, 26, 28, 29, 31, 32, 34, 35, 37, 38, 40, 41, 43, 44, 46, 47, 55, 58, 59, 61, 62, 65\}.

**Proof** Some volumes are multiples of three or four with the conditions of Theorem 1.5. Other volumes can be written as \(m_1 + m_2\) from previous parts. By this method, we have the existence of a 3-way \((v, 7, 2)\) Steiner trade of volumes \(\{57, 63, 67, 70, 71, 73, 74, 75, 76, 77, 79, 80, 82, 83, 85, 86, 88, 89, 91\}\) and \(m = 18 + m', \ for \(m' \geq 72\). There does not exist any 3-way \((v, 7, 2)\) Steiner trade of volume \(m \leq 17\). Therefore, the reminder volumes are \(\{19, 20, 22, 23, 25, 26, 28, 29, 31, 32, 34, 35, 37, 38, 40, 41, 43, 44, 46, 47, 55, 58, 59, 61, 62, 65\}.

\(\square\)

**Remark 3.5** There exists a \(RB(36, 6, 1)\). Therefore, by Construction 1 there exist \(\mu\)-way \((36 + \mu, 7, 2)\) Steiner trades of volumes \(\mu \frac{36}{6} = 6\mu\), for \(3 \leq \mu \leq \frac{35}{5} = 7\).

### 3.4 Block Size 8

For \(k = 8\), we have five parts as follows:

1- By Theorem 1.5, \(S_3(2, 8) \subseteq N \setminus \{1, \ldots, 20\}\) and there does not exist any 3-way \((v, 8, 2)\) Steiner trade of volume \(m \ for \ m \leq 20\).

2- By from the second part of Theorem 1.5, there exists a 3-way \((v, 8, 2)\) Steiner trade of volume \(m \in \{21, 24, \ldots \} = \{m : m = 3l, \ l \geq 7\}\).

3- By Theorem 1.4, there exists a 3-way \((v, 8, 2)\) Steiner trade of volume \(m \in \{56, \ldots, 62\} = \{m : m = 9 \times 7 - r, \ 1 \leq r \leq 7\}\).

4- By the fourth part of Theorem 1.5, there exists a 3-way \((v, 8, 2)\) Steiner trade of volume \(m \in \{56, 60, 68, \ldots \} = \{m : m = 4l, \ l \geq 14\} \cup \{28 = 4 \times 7\}\).

5- By the fifth part of Theorem 1.5, there exists a 3-way \((v, 8, 2)\) Steiner trade of volume \(m \in \{39, 52, 65, 78, 91, 104, 117, 130, 143, 156, 169, 182\}\).

Now, we can state the following theorem.

**Theorem 3.6** There exists a 3-way \((v, 8, 2)\) Steiner trade of volume \(m \geq 21\) except possibly when \(m \in \{22, 23, 25, 26, 29, 30, 31, 34, 37, 38, 40, 41, 43, 44, 46, 47, 50, 53\}\).

**Proof** Some volumes are multiples of three or four with the conditions of Theorem 1.5. Other volumes can be written as \(m_1 + m_2\) from previous parts. By this method, we have the existence of a 3-way \((v, 8, 2)\) Steiner trade of volumes \(\{52, 55, 64, 67, 68, 70, 71, 73, 74, 76, 77, 79, 80, 82, 83, 85, 86, 89, 91, 94, 95, 97, 98, 101, 103\}\) and \(m = 21 + m', \ for \(m' \geq 84\). There does not exist any 3-way \((v, 8, 2)\) Steiner trade of volume \(m \leq 20\). Therefore, the reminder volumes that we don’t know the existence or non-existence of them are \(\{22, 23, 25, 26, 29, 30, 31, 3437, 38, 40, 41, 43, 44, 46, 47, 50, 53\}\).

\(\square\)

**Remark 3.7** There exists a \(RB(49, 7, 1)\). Therefore, by Construction 1 there exist 3-way \((v, 8, 2)\) Steiner trades of volumes \(\frac{49}{7}\mu = 7\mu\) for \(3 \leq \mu \leq \frac{48}{6} = 8\). Therefore, there exists a 3-way \((v, 8, 2)\) Steiner trade of volume \(m \in \{21, 28, 35, 42, 49, 56\}\).
3.5 Block Size 9

For \( k = 9 \), we have four parts as follows:
1- By Theorem 1.5, \( S_3(2, 9) \subseteq \mathcal{N} \setminus \{1, \ldots, 23\} \) and there does not exist any 3-way \((v, 9, 2)\) Steiner trade of volume \( m \) for \( m \leq 23 \).
2- By the second part of Theorem 1.5, there exists a 3-way \((v, 9, 2)\) Steiner trade of volume \( m \in \{24, 27, 30, \ldots\} = \{m : m = 3l, \ l \geq 8\} \).
3- By Theorem 1.4, there exists a 3-way \((v, 9, 2)\) Steiner trade of volume \( m \in \{65, 64\} = \{m : m = 9 \times 8 - r, \ 1 \leq r \leq 8\} \).
4- By the fourth part of Theorem 1.5, there exists a 3-way \((v, 9, 2)\) Steiner trade of volume \( m \in \{64, 68, 72, \ldots\} = \{m : m = 4l, \ l \geq 16\} \).

Now, we can state the following theorem.

**Theorem 3.8** There exists a 3-way \((v, 9, 2)\) Steiner trade of volume \( m \geq 24 \) except possibly when \( m \in \{25, 26, 28, 29, 31, 32, 34, 35, 37, 38, 40, 41, 42, 43, 44, 46, 47, 49, 50, 52, 53, 55, 56, 58, 59, 61, 62, 73, 74, 77, 79, 82, 83, 85, 86, \} \)

**Proof** Some volumes are multiples of three or four with the conditions of Theorem 1.5. Other volumes can be written as \( m_1 + m_2 \) from previous parts. By this method, we have the existence of a 3-way \((v, 9, 2)\) Steiner trade of volumes \{88, 89, 94, 95, 97, 98, 101, 103, 106, 107, 109, 110, 113, 115, 118, 119\} and \( m = 24 + m' \), for \( m' \geq 96 \). There does not exist any 3-way \((v, 9, 2)\) Steiner trade of volume \( m \leq 38 \). Therefore, the remainder volumes that we don’t know the existence or non-existence of them are \{25, 26, 28, 29, 31, 32, 34, 35, 37, 38, 40, 41, 42, 43, 44, 46, 47, 49, 50, 52, 53, 55, 56, 58, 59, 61, 62, 73, 74, 77, 79, 82, 83, 85, 86, \}.

3.6 Block Size 10

For \( k = 10 \), we have five parts as follows:
1- By Theorem 1.5, \( S_3(2, 10) \subseteq \mathcal{N} \setminus \{1, \ldots, 26\} \) and there does not exist any 3-way \((v, 10, 2)\) Steiner trade of volume \( m \) for \( m \leq 26 \).
2- By the second part of Theorem 1.5, there exists a 3-way \((v, 10, 2)\) Steiner trade of volume \( m \in \{27, 30, 33, \ldots\} = \{m : m = 3l, \ l \geq 9\} \).
3- By Theorem 1.4, there exists a 3-way \((v, 10, 2)\) Steiner trade of volume \( m \in \{80, 79, \ldots, 72\} \).
4- By the fourth part of Theorem 1.5, there exists a 3-way \((v, 10, 2)\) Steiner trade of volume \( m \in \{72, 76, \ldots\} = \{m : m = 4l, \ l \geq 18\} \cup \{36\} \).
5- There exists a \( RB(153, 9, 1) \). Therefore, by Construction 1, there exist 3-way \((v, 10, 2)\) Steiner trades of volumes \( \frac{153}{9} \mu = 17 \mu \) for \( 3 \leq \mu \leq \frac{152}{8} = 19 \). Therefore, there exists a 3-way \((v, 10, 2)\) Steiner trade of volume \( m \in \{68, 85, 102, 119, \ldots\} \). The existence of a 3-way \((v, 10, 2)\) Steiner trade of volumes 68 and 85 is new.

Now, we can state the following theorem.

**Theorem 3.9** There exists a 3-way \((v, 10, 2)\) Steiner trade of volume \( m \geq 27 \) except possibly when \( m \in \{28, 29, 31, 32, 34, 35, 37, 38, 40, 41, 43, 44, 46, 47, 49, 50, 52, 53, 55, 56, 58, 59, 61, 62, 64, 65, 67, 70, 71, 82, 83, 86, 91, 89, 94, 97\} \).

**Proof** Some volumes are multiples of three or four with the conditions of Theorem 1.5. Other volumes can be written as \( m_1 + m_2 \) from previous parts. By this method, we have the
existence of a 3-way \((v, 10, 2)\) Steiner trade of volumes \(\{95, 98, 100, 101, 103, 104, 106, 107, 109, 110, 113, 115, 118, 119, 121, 122, 125, 127, 130, 131, 133, 134, 135\}\) and \(m = 39 + m'\), for \(m' \geq 156\). There does not exist any 3-way \((v, 10, 2)\) Steiner trade of volume \(m \leq 26\). Therefore, the remainder volumes that we don’t know the existence or non-existence of them are \(\{28, 29, 31, 32, 34, 35, 37, 38, 40, 41, 43, 44, 46, 47, 49, 50, 52, 53, 55, 56, 58, 59, 61, 62, 64, 65, 67, 70, 71, 82, 83, 86, 91, 89, 94, 97\}\).

### 3.7 Block Size 11

For \(k = 11\), we have four parts as follows:

1- By Theorem 1.5, \(S_3(2, 11) \subseteq N \setminus \{1, \ldots, 29\}\) and there does not exist any 3-way \((v, 11, 2)\) Steiner trade of volume \(m\) for \(m \leq 29\).

2- By the second part of Theorem 1.5, there exists a 3-way \((v, 11, 2)\) Steiner trade of volume \(m \in \{30, 33, 36, \ldots\}\) = \{\(m : m = 3l, \ l \geq 10\)\}.

3- By Theorem 1.4, there exists a 3-way \((v, 11, 2)\) Steiner trade of volume \(m \in \{89, 92, 94, 97, 98, 101, 103, 106, 107, 109, 113, 115, 118, 119, 121, 122, 125, 127, 130, 131, 133, 134, 137, 139, 142, 143, 145, 146, 149\}\) and \(m = 30 + m'\), for \(m' \geq 120\). There does not exist any 3-way \((v, 11, 2)\) Steiner trade of volume \(m \leq 29\). Therefore, the remainder volumes that we don’t know the existence or non-existence of them are \{31, 32, 34, 35, 37, 38, 41, 43, 44, 46, 47, 49, 52, 53, 55, 56, 58, 59, 61, 62, 64, 65, 67, 68, 71, 74, 77\}.

**Theorem 3.10** There exists a 3-way \((v, 11, 2)\) Steiner trade of volume \(m \geq 30\) except possibly when \(m \in \{31, 32, 34, 35, 37, 38, 41, 43, 44, 46, 47, 49, 52, 53, 55, 56, 58, 59, 61, 62, 64, 65, 67, 68, 71, 74, 77\}\).

**Proof** Some volumes are multiples of three or four with the conditions of Theorem 1.5. Other volumes can be written as \(m_1 + m_2\) from previous parts. By this method, we have the existence of a 3-way \((v, 11, 2)\) Steiner trade of volumes \(\{73, 79, 91, 92, 94, 97, 98, 101, 103, 106, 107, 109, 113, 115, 118, 119, 121, 122, 125, 127, 130, 131, 133, 134, 137, 139, 142, 143, 145, 146, 149\}\) and \(m = 30 + m'\), for \(m' \geq 120\). Therefore, there does not exist any 3-way \((v, 11, 2)\) Steiner trade of volume \(m \leq 29\). Therefore, the remainder volumes that we don’t know the existence or non-existence of them are \{31, 32, 34, 35, 37, 38, 41, 43, 44, 46, 47, 49, 52, 53, 55, 56, 58, 59, 61, 62, 64, 65, 67, 68, 71, 74, 77\}.

**Remark 3.11** There exists a \(RB(100, 10, 1)\) and \(RB(190, 10, 1)\). By Construction 1, there exists 3-way \((v, 11, 2)\) Steiner trades of volumes \(\frac{100}{10} \mu = 10 \mu\) for \(3 \leq \mu \leq \frac{90}{9} = 11\) and \(\frac{190}{10} \mu = 19 \mu\) for \(3 \leq \mu \leq \frac{189}{9} = 21\). Therefore, there exists a 3-way \((v, 11, 2)\) Steiner trade of volume \(m \in \{76, 95, 133, 30, 40, 50, 60, 70, 80, 90, 100, 110\}\). The existence of some of 3-way \((v, 11, 2)\) Steiner trades of these volumes are known from Theorem 1.5 too.

### 3.8 Block Size 12

For \(k = 12\), we have five parts as follows:

1- By Theorem 1.5, \(S_3(2, 12) \subseteq N \setminus \{1, \ldots, 32\}\) and there does not exist any 3-way \((v, 12, 2)\) Steiner trade of volume \(m\) for \(m \leq 32\).

2- By the second part of Theorem 1.5, there exists a 3-way \((v, 12, 2)\) Steiner trade of volume \(m \in \{33, 36, 39, 42, 45, \ldots\}\) = \{\(m : m = 3l, \ l \geq 11\)\}.

3- By Theorem 1.4, there exists a 3-way \((v, 12, 2)\) Steiner trade of volume \(m \in \{98, 97, 96, \ldots, 89, 88\}\) = \{\(m : m = 9 \times 11 - r, \ 1 \leq r \leq 11\)\}.
4- By the fourth part of Theorem 1.5, there exists a 3-way \((v, 12, 2)\) Steiner trade of volume 
\[ m \in \{88, 92, 96, \ldots \} = \{m : m = 4l, \ l \geq 26\} \cup \{44\} (44 = 4 \times 11) .\]
5- By the fifth part of Theorem 1.5, there exists a 3-way \((v, 12, 2)\) Steiner trade of volume 
\[ m \in \{33, 44, 55, 66, 77, 88, 99, 110, 132\} .\]

Now, we can state the following theorem.

**Theorem 3.12** There exists a 3-way \((v, 12, 2)\) Steiner trade of volume 
\[ m \geq 33 \] except possibly when 
\[ m \in \{34, 35, 37, 38, 40, 41, 43, 46, 47, 49, 50, 52, 53, 56, 58, 59, 61, 62, 64, 67, 68, 70, 71, 73, 74, 76, 85\} .\]

**Proof** Some volumes are multiples of three or four with the conditions of Theorem 1.5. 
Other volumes can be written as \(m_1 + m_2\) from previous parts. By this method, we have the 
existence of a 3-way \((v, 12, 2)\) Steiner trade of volumes 
\(\{86, 101, 103, 118, 119, 121, 122, 124, 125, 127, 128, 130, 131, 133, 134, 137, 139, 142, 143, 145, 146, 149, 151, 154, 155, 157, 158, 161, 163\} \) and 
\[ m = 33 + m', \] for \(m' \geq 132\). There does not exist any 3-way 
\((v, 12, 2)\) Steiner trade of volume \(m \leq 32\). Therefore, the remainder volumes that we don’t 
know the existence or non-existence of them are 
\(\{34, 35, 37, 38, 40, 41, 43, 46, 47, 49, 50, 52, 53, 56, 58, 59, 61, 62, 64, 67, 68, 70, 71, 73, 74, 76, 85\} .\]

3.9 **Block Size 13**

For \(k = 13\), we have four parts as follows:
1- By Theorem 1.5, 
\[ S_3(2, 13) \subseteq N \setminus \{1, \ldots, 35\} \] and there does not exist any 3-way 
\((v, 13, 2)\) Steiner trade of volume \(m\) for 
\[ m \leq 35 .\]
2- By the second part of Theorem 1.5, there exists a 3-way \((v, 13, 2)\) Steiner trade of volume 
\[ m \in \{36, 39, 42, \ldots \} = \{m : m = 3l, \ l \geq 12\} .\]
3- By Theorem 1.4, there exists a 3-way \((v, 13, 2)\) Steiner trade of volume 
\[ m \in \{107, 106, 105, \ldots, 97, 96\} = \{m : m = 9 \times 12 - r, \ 1 \leq r \leq 12\} .\]
4- By the fourth part of Theorem 1.5, there exists a 3-way \((v, 13, 2)\) Steiner trade of volume 
\[ m \in \{96, 100, \ldots \} = \{m : m = 4l, \ l \geq 24\} .\]

Now, we can state the following theorem.

**Theorem 3.13** There exists a 3-way \((v, 13, 2)\) Steiner trade of volume 
\[ m \geq 36 \] except possibly 
\[ m \in \{37, 38, 40, 41, 43, 44, 46, 47, 49, 50, 53, 55, 56, 58, 59, 61, 62, 64, 67, 68, 70, 71, 73, 74, 76, 77, 79, 80, 82, 83, 85, 86, 89, 92, 95\} .\]

**Proof** Some volumes are multiples of three or four with the conditions of Theorem 1.5. 
Other volumes can be written as \(m_1 + m_2\) from previous parts. By this method, there exists a 3-way 
\((v, 13, 2)\) Steiner trade of volumes 
\(\{88, 94, 118, 119, 122, 125, 127, 131, 133, 134, 137, 139, 142, 142, 143, 145, 146, 149, 151, 154, 155, 157, 158, 161, 163, 166, 167, 170, 171, 173, 175\} \) and 
\[ m = 36 + m', \] for \(m' \geq 144\). There does not exist any 3-way \((v, 13, 2)\) Steiner trade of volume 
\[ m \leq 35 .\] Therefore, the remainder volumes are 
\(\{37, 38, 40, 41, 43, 44, 46, 47, 49, 50, 53, 55, 56, 58, 59, 61, 62, 64, 67, 68, 70, 71, 73, 74, 76, 77, 79, 80, 82, 83, 85, 86, 89, 92, 95\} .\]

**Remark 3.14** There exists a \(RB(144, 12, 1)\). By Construction 1, there exists a 3-way 
\((v, 13, 2)\) Steiner trade of volume 
\[ \frac{144}{12} \mu = 12 \mu \] for 
\[ 3 \leq \mu \leq \frac{143}{17} = 13 .\] The existence of 3-way 
\((v, 13, 2)\) Steiner trades of these volumes are known from Theorem 1.5 too.
3.10 Block Size 14

For $k = 14$, we have five parts as follows:

1- By Theorem 1.5, $S_3(2, 14) \subseteq \mathcal{N} \setminus \{1, \ldots, 38\}$ and there does not exist any 3-way $(v, 14, 2)$ Steiner trade of volume $m$ for $m \leq 38$.

2- By the second part of Theorem 1.5, there exists a 3-way $(v, 14, 2)$ Steiner trade of volume $m \in \{39, 42, 45, \ldots\} = \{m : m = 3l, l \geq 13\}$.

3- By Theorem 1.4, there exists a 3-way $(v, 14, 2)$ Steiner trade of volume $m \in \{116, 115, 114, \ldots, 105, 104\} = \{m : m = 9 \times 13 - r, 1 \leq r \leq 13\}$.

4- By the fourth part of Theorem 1.5, there exists a 3-way $(v, 14, 2)$ Steiner trade of volume $m \in \{104, 108, \ldots\} = \{m : m = 4l, l \geq 26\} \cup 52 = 4 \times 13$.

5- By the fifth part of Theorem 1.5, there exists a 3-way $(v, 14, 2)$ Steiner trade of volume $m \in \{39, 52, 65, 78, 91, 104, 117, 130, 143, 156, 169, 182\}$

Now, we can state the following theorem.

**Theorem 3.15** There exists a 3-way $(v, 14, 2)$ Steiner trade of volume $m \geq 39$ except possibly when $m \in \{40, 41, 43, 44, 46, 47, 49, 50, 53, 55, 56, 58, 59, 61, 62, 64, 67, 68, 70, 71, 73, 74, 76, 77, 79, 80, 82, 83, 85, 86, 88, 89, 92, 95, 98, 101, 131\}$.

**Proof** Some volumes are multiples of three or four with the conditions of Theorem 1.5. Other volumes can be written as $m_1 + m_2$ from previous parts. By this method, there exists a 3-way $(v, 14, 2)$ Steiner trade of volumes $\{94, 97, 100, 103, 118, 119, 122, 125, 127, 133, 134, 137, 139, 142, 143, 145, 146, 148, 149, 151, 152, 154, 155, 157, 158, 161, 163, 166, 167, 169, 170, 173, 175, 178, 179, 181, 182, 183, 185, 187, 190, 191, 193, 194\}$ and volumes $m = 39 + m'$, for $m' \geq 156$. There does not exist any 3-way $(v, 14, 2)$ Steiner trade of volume $m \leq 38$. Therefore, the remainder volumes that we don’t know the existence or non-existence of them are $\{40, 41, 43, 44, 46, 47, 49, 50, 53, 55, 56, 58, 59, 61, 62, 64, 67, 68, 70, 71, 73, 74, 76, 77, 79, 80, 82, 83, 85, 86, 88, 89, 92, 95, 98, 101, 131\}$. □

**Remark 3.16** There exists a $RB(169, 13, 1)$. By Construction 1 there exists a 3-way $(v, 14, 2)$ Steiner trade of volume $m = 169 \mu = 13 \mu$ for $3 \leq \mu \leq \frac{168}{12} = 14$. Therefore, there exists a 3-way $(v, 14, 2)$ Steiner trade of volume $m \in \{52, 65, 91, 104, 130, 143, 169, 182\}$. The existence of a 3-way $(v, 14, 2)$ Steiner trade of these volumes is known from Theorem 1.5 too.

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