Peak Age of Information Distribution in Tandem Queue Systems

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Abstract—Age of Information is a critical metric for several Internet of Things (IoT) applications, where sensors keep track of the environment by sending updates that need to be as fresh as possible. Knowing the full distribution of the Peak Age of Information (PAoI) allows system designers to tune the protocols and dimension the network to provide reliability using the tail of the distribution as well as the average. In this letter we consider the most common relaying scenario in satellite communications, which consists of two subsequent links, model it as a tandem queue with two consecutive $M/M/1$ systems, and derive the complete peak age distribution.

Index Terms—Age of Information, Peak Age of Information, tandem queues

I. INTRODUCTION

Traditional communication networks consider packet delay as the one and only performance metric to capture the latency requirements of a transmission. However, numerous Internet of Things (IoT) applications require the transmission of real-time status updates of a process from a generating point to a remote destination. Sensor networks, vehicular networks and other tracking systems, and industrial control are examples of this kind of update process. For these cases, the Age of Information (AoI) is a novel concept that better represents the timeliness requirements by quantifying the freshness of the information at the receiver [1]. AoI measures the time elapsed since the latest received update was generated. Another age-related metric is the Peak Age of Information (PAoI), which is the maximum value of AoI for each update. As in other performance metrics of communication systems, the PAoI is more informative than the average age when the interest is in the worst case, e.g., when the system requirement is on the tail of the distribution.

AoI is a relatively new metric in networking, but it has gained widespread recognition thanks to its relevance to IoT applications. It is generally applied to queuing systems with a single node and First Come First Serve (FCFS) policy. However, a general result was proven for general queuing networks in [2], which shows that a preemptive Last Come First Serve (LCFS) policy minimizes the AoI: since queued packets increase the system delay, and updates are interchangeable, it is better to transmit the latest packet directly to send the freshest possible information. Preemption means that even the packet currently in service is blocked and queued after the new one. Similar results are shown for $M/M/k$ queues in [3]. The decision over whether to preempt or skip the subsequent update under different service time distributions is modeled as a Markov Decision Process (MDP) in [4]. A more realistic model considering a wireless channel with retransmissions was used to compute the PAoI distribution over a single-hop link in [5], and a recent live AoI measurement study on a public network generally confirmed that the theoretical models are realistic [6]. Multiple sources can also be considered, in which case the scheduling problem to maintain freshness for all sources becomes interesting [7].

A tandem queue models a system where the service is delivered in several successive stages. In communications, a relay network involves one or more intermediate nodes between transmitter and receiver to, e.g., overcome the physical distance between the two-end points. The case of a single relay corresponds to a 2-node tandem queue where the relay could be, for instance, a satellite. Another example is in some IoT scenarios, where the read from the sensor is first preprocessed, and then transmitted to the server [8]. This kind of model can capture the queuing dynamics of multi-hop links, which are much more complex than single-node models, as the combined effects of different service rates can be hard to gauge intuitively.

Given the wide range of relevant applications, we focus our attention in the study of the age in tandem queues. A recent work [8] models the AoI in tandem queues and derives a rate control algorithm for multiple sources with different priorities. In this case, each queue followed the FCFS discipline, but the authors derive only the average PAoI. An analysis of the effect of preemption on this kind of models on the average AoI is presented in [9], and [10] derives the average AoI for two queues in tandem with preemption and different arrival processes. Another recent work [11] uses the Chernoff bound to derive an upper bound of the quantile function of the AoI for two queues in tandem with deterministic arrivals. Finally, a general transport protocol to control the generation rate of status updates to minimize the AoI over the Internet is presented in [12].

In this letter, we analyze the distribution of the PAoI in a tandem queue with two systems and a single source, where each infinite queue follows the FCFS policy. This result will allow system designers to define reliability requirements using PAoI thresholds and derive the network specifications needed to meet those requirements. The structure of the letter is as follows. In Section [11] the system model is detailed, as well as
the procedure to calculate the AoI. Section III and Section IV present the calculations when the first system is busy and free, respectively. Numerical results are plotted in Section V and the paper is concluded in Section VI.

II. SYSTEM MODEL

We consider a tandem of two M/M/1 queues. Packets are generated by a Poisson process with rate \( \lambda \) and enter the first system, whose service time is exponentially distributed with rate \( \mu_1 \). When the packet exits the first system, it enters the second one, whose service time is an exponential random variable with rate \( \mu_2 \). In the following, we use the compact notation \( P_{X|Y}(x|y) \) for the conditioned probability \( P[X = x|Y = y] \). Probability Density Functions (PDFs) are denoted by a lower-case \( p \).

Fig. 1 shows the evolution of the AoI over time: packet \( i \) is generated at time \( t_i \) and departs the system at time \( t_i \). We define the overall system time for the packet \( T_i = t_i - t_i \) as the difference between the departure and arrival times, and the interarrival time between two packets is denoted as \( Y_i \). The AoI \( \Delta_i \) is the AoI right before a packet reception, which corresponds to the maximum AoI in the cycle. It is given by \( T_i + Y_i \), as it is the difference between the departure time of a packet and the origin time of the last update received at the destination, as shown in Fig. 1. Since the arrival process is independent from the rest of the system, while the system time depends on it, we can compute the complete PDF of the AoI by conditioning on \( Y_i \) and using the law of total probability:

\[
p_{\Delta_i}(\tau_i) = \int_0^{\tau_i} p_{Y_i}(y_i)p_{T_i|Y_i}(\tau_i - y_i|y_i)dy_i \tag{1}
\]

We now need to compute the conditioned system time probability \( p_{T_i|Y_i}(t_i|y_i) \). For each system \( j \) in the tandem, we can define the system time \( T_{i,j} \), which is the sum of the waiting time \( W_{i,j} \) and the service time \( S_{i,j} \). We know from basic queuing theory that the system time for any of the two systems is exponentially distributed with rate \( \alpha_j = \mu_j - \lambda \), as long as the system is stable. We can also define the interarrival time for packet \( i \) in system \( j \), denoted as \( Y_{i,j} \), knowing that \( Y_{i,1} = Y_i \).

At each system, a packet can be queued for a certain time \( W_{i,j} \), or find that the system is free and go directly in service. We define the extended waiting time \( \Omega_{i,j} \) as the difference between the previous packet’s system time and the interarrival time at the system, i.e., \( \Omega_{i,j} = T_{i-1,j} - Y_{i,j} \). The reason we named \( \Omega_{i,j} \) the extended waiting time is that \( W_{i,j} = [\Omega_{i,j}]^+ \), where \([x]^+\) is equal to \( x \) if it is positive and 0 if \( x \) is negative. Knowing the PDF of the system time, we can derive the PDF of the extended waiting time, which will be useful in the next steps:

\[
p_{\Omega_{i,j}|Y_{i,j}}(\omega_{i,j}|y_{i,j}) = \alpha_j e^{-\alpha_j(\omega_{i,j}+y_{i,j})}u(\omega_{i,j} + y_{i,j}), \tag{2}
\]

where \( u(\cdot) \) is the step function. The interarrival time at the first relay \( Y_{i,1} \) is exponentially distributed with rate \( \lambda \), while in subsequent systems it is given by \( Y_{i,j+1} = S_{i,j} + [\Omega_{i,j}]^+ \).

We can combine (2) with the definition of \( Y_{i,j+1} \) to get:

\[
p_{\Omega_{i,j+1}|S_{i,j},\Omega_{i,j}}(\omega_{i,j+1}|s_{i,j},\omega_{i,j}) = \alpha_j e^{-\alpha_j(\omega_{i,j+1}+s_{i,j}+[-\omega_{i,j}]^+)}u(\omega_{i,j+1} + s_{i,j} + [-\omega_{i,j}]^+). \tag{3}
\]

III. CASE I: THE FIRST SYSTEM IS BUSY

We now consider the case in which the first system is busy, i.e., the \( i \)-th packet arrives before the departure of the \( (i-1) \)-th packet, and \( \Omega_1 > 0 \). In this case, we start from the conditioned distribution of the system time on \( \Omega_1, \Omega_2 \), and \( S_1 \), so \( S_2 \) is the only remaining random variable:

\[
P_{\Omega_1|\Omega_2, S_1}(t|\omega_1, \omega_2, s_1) = \mu_2 e^{-\mu_2(t-\omega_1-s_1-\omega_2)} u(t). \tag{5}
\]

We distinguish two sub-cases: one in which the second system is busy as well, and one in which the second system is free.

A. Case 1A: both systems are busy

We now consider the case in which \( \omega_2 > 0 \). In this case, (5) becomes:

\[
P_{\Omega_1|\Omega_2, S_1}(t|\omega_1, \omega_2, s_1) = \mu_2 e^{-\mu_2(t-\omega_1-s_1-\omega_2)} u(t). \tag{6}
\]

We can uncondition the PDF on \( \Omega_2 \) using the law of total probability:

\[
P_{\Omega_1|S_1}(t|\omega_1, s_1) = \int_0^{\omega_1} P_{\Omega_1|\Omega_2, S_1}(t|\omega_1, \omega_2, s_1)P_{\Omega_2|Y_2}(\omega_2|y_2)dy_2 \tag{7}
\]

\[
= \frac{\alpha_2 \mu_2}{\lambda} \left( e^{-\alpha_2(t-\omega_1)} - e^{-\mu_2(t-\omega_1)+\lambda s_1} \right).
\]
We repeat the operation to uncondition on $S_1$:

$$P^{(1A)}_{T|\Omega_1}(t|\omega_1) = \int_0^{t-\omega_1} P^{(1A)}_{T|\Omega_1,S_1}(t|\omega_1,s_1)p_{S_1}(s_1)ds_1$$

$$= \alpha_2\mu_2e^{-\alpha_2(t-\omega_1)}\left(\alpha_1 + \lambda e^{-\mu_1(t-\omega_1)} - \mu_1e^{-\lambda(t-\omega_1)}\right).$$

(8)

We then condition on $Y_1$ and uncondition on $\Omega_1$:

$$P^{(1A)}_{T|Y_1}(t|y_1) = \int_0^{t} P^{(1A)}_{T|\Omega_1}(t|\omega_1)p_{\Omega_1|Y_1}(\omega_1|y_1)d\omega_1$$

$$= \mu_2\alpha_2e^{-\alpha_2\omega_1}\frac{\alpha_1(e^{-\alpha_1t} - e^{-\alpha_2t})}{(\mu_2 - \mu_1)}$$

$$+ \frac{\lambda e^{-\alpha_1t}(1 - e^{-\mu_1t}) - \mu_1(e^{-\alpha_1t} - e^{-\mu_1t})}{\mu_2 - \alpha_1}. \quad (9)$$

We can now derive the PDF of the system time $T$:

$$P^{(1A)}_T(t) = \int_0^\infty p_Y(y_1)p_{T|Y_1}(t|y_1)dy_1$$

$$= \mu_2\alpha_2\frac{\alpha_1(e^{-\alpha_1t} - e^{-\alpha_2t})}{(\mu_2 - \mu_1)}$$

$$+ \frac{\lambda e^{-\alpha_1t}(1 - e^{-\mu_1t}) - \mu_1(e^{-\alpha_1t} - e^{-\mu_1t})}{\mu_2 - \alpha_1}. \quad (10)$$

Finally, we get the PDF of the PAoI, given by $T + Y_1$:

$$P^{(1A)}_{T+Y_1}(\tau) = \int_0^\tau P^{(1A)}_T(t|\tau - t)p_{Y_1}(\tau - t)dt$$

$$= \frac{\alpha_1\alpha_2\mu_2(e^{-\mu_1\tau} - e^{-\alpha_2\tau})}{(\mu_2 - \mu_1)(\mu_1 - \alpha_2)}$$

$$+ \frac{\alpha_2\mu_2\mu_1(e^{-\mu_1\tau} - e^{-\mu_2\tau})}{(\mu_2 - \mu_1)(\mu_2 - \alpha_1)}$$

$$+ \frac{\alpha_1\mu_2\mu_2(e^{-\mu_1\tau} - e^{-\mu_1\tau})}{(\mu_2 - \mu_1)(\mu_2 - \alpha_1)}$$

$$- \lambda e^{-\mu_1\tau}(1 - e^{-\alpha_2\tau}). \quad (11)$$

### IV. Case 2: The First System is Free

We now consider the case in which the first system is free. The $i$-th packet arrives and does not experience any queuing, i.e., $\Omega_1 \leq 0$. The system time distribution, conditioned on $S_1$, $\Omega_1$, and $\Omega_2$, is:

$$P^{(2)}_{T|\Omega_1,\Omega_2,S_1}(t|\omega_1,\omega_2,s_1) = \mu_2 e^{-\mu_2(t-s_1-\omega_2)}u(t). \quad (12)$$

As for case 1, we now further divide the derivation into two sub-cases.

### A. Case 2A: The Second System is Busy

We now consider the case in which $\omega_2 > 0$. In this case, we becomes:

$$P^{(2A)}_{T|\Omega_1,\Omega_2,S_1}(t|\omega_1,\omega_2,s_1) = \mu_2 e^{-\mu_2(t-s_1-\omega_2)}u(t). \quad (19)$$

As in case 1A, we can uncondition the PDF on $\Omega_2$ using the law of total probability:

$$P^{(2A)}_{T|\Omega_1,S_1}(t|\omega_1,s_1) = \int_0^{t-s_1} P^{(2A)}_{T|\Omega_1,S_1}(t|\omega_1,s_1)p_{S_1}(s_1)ds_1$$

$$= \mu_2 e^{-\mu_2(t-s_1-\omega_2)}u(t). \quad (20)$$

$$= \frac{\mu_2\alpha_2(e^{-\alpha_2(t-s_1-\omega_2)} - e^{-\mu_2(t-s_1-\omega_2)})}{\alpha_1(\mu_2 - \mu_1)}$$

$$+ \frac{\mu_2\alpha_2(e^{-\alpha_2(t-s_1-\omega_2)} - e^{-\mu_2(t-s_1-\omega_2)})}{\alpha_1(\mu_2 - \mu_1)}$$

$$+ \frac{\mu_2\alpha_2(e^{-\alpha_2(t-s_1-\omega_2)} - e^{-\mu_2(t-s_1-\omega_2)})}{\alpha_1(\mu_2 - \mu_1)}$$

$$- \frac{\mu_2\alpha_2(e^{-\alpha_2(t-s_1-\omega_2)} - e^{-\mu_2(t-s_1-\omega_2)})}{\alpha_1(\mu_2 - \mu_1)}.$$
We now uncondition on $S_1$:

$$P_{T|Y_1}^{(2A)}(t|\omega_1) = \int_0^t P_{T|Y_1, S_1}^{(2A)}(t|\omega_1, s_1) p_{S_1}(s_1) ds_1$$

$$= \frac{\mu_2 \alpha_2 e^{-\alpha_2(t-\omega_1)}}{\lambda \alpha_1} \left( \alpha_1 - \lambda e^{-\mu_1 t} + \mu_1 e^{-\lambda t} \right).$$

We then condition on $Y_1$ and uncondition on $\Omega_1$:

$$P_{T|Y_1}^{(2A)}(t|y_1) = \int_{\omega_1}^\infty P_{T|Y_1, \Omega_1}^{(2A)}(t|\omega_1, y_1) \omega_1 d\omega_1$$

$$= \frac{\mu_2 \alpha_2 e^{-\alpha_2(t-\omega_1)}}{\lambda (\mu_2 - \mu_1)} \left( \alpha_1 - \mu_1 e^{-\lambda t} + \lambda e^{-\mu_1 t} \right).$$

We can now find the PDF of the system delay:

$$P_T^{(2A)}(t) = \int_0^\infty P_{T|Y_1}^{(2A)}(t|y_1) p_{Y_1}(y_1) dy_1$$

$$= \alpha_2 e^{-\alpha_2 t} \left( \alpha_1 - \mu_1 e^{-\lambda t} + \lambda e^{-\mu_1 t} \right).$$

The PDF of the PAoI is then:

$$P_{T+Y_1}^{(2A)}(\tau) = \int_0^\tau P_{T|Y_1}^{(2A)}(t|\tau) p_{Y_1}(\tau) (\tau-t) dt$$

$$= \frac{\mu_2}{\mu_2 - \mu_1} \left( \frac{\alpha_1 \alpha_2 (e^{-\mu_1 \tau} - e^{-\alpha_2 \tau})}{\alpha_2 - \mu_1} + \alpha_2 \mu_1 \alpha_2 e^{-\mu_2 \tau} \right)$$

$$- \frac{\mu_1 \alpha_2 (e^{-\mu_1 \tau} - e^{-\mu_2 \tau})}{\mu_2 - \mu_1} + \lambda e^{-\mu_1 \tau} (1 - e^{-\alpha_2 \tau})$$

$$- \frac{\alpha_1 \alpha_2 (e^{-\alpha_2 \tau} - e^{-\mu_2 \tau})}{\lambda} - \frac{\alpha_2 \mu_2 e^{-\mu_2 \tau} (1 - e^{-\alpha_1 \tau})}{\alpha_1}.$$  

B. Case 2B: both systems are free

In this case, the packet experiences no queuing, and (18) becomes:

$$P_{T|\Omega_2, S_1}^{(2B)}(t|\omega_1, \omega_2, s_1) = \mu_2 e^{-\mu_2(t-s_1)} u(t).$$

Since the system time probability is independent of $\Omega_2$, we can just give the unconditioned probability as:

$$P_{T|\Omega_1, S_1}^{(2B)}(t|\omega_1, s_1) = \mu_2 e^{-\mu_2(t-s_1)} (1 - e^{-\alpha_2(s_1-\omega_1)}) u(t).$$

We can then uncondition on $S_1$:

$$P_{T|\Omega_1}^{(2B)}(t|\omega_1) = \int_0^t P_{T|\Omega_1, S_1}^{(2B)}(t|\omega_1, s_1) p_{S_1}(s_1) ds_1$$

$$= \mu_1 \mu_2 \left( e^{-\mu_1 t} - e^{-\mu_2 t} \right) \left( e^{-\alpha_2 t} - e^{-\mu_2 t} \right)$$

$$+ \frac{(e^{-\mu_2 t} - e^{-(\alpha_1 + \mu_2) t}) e^{\alpha_2 \omega_1}}{\alpha_1}.$$  

V. Simulation results

We compared the results of our analysis with a Monte Carlo simulation, transmitting 10 million packets and computing the system delay and PAoI for each. The initial stages of each simulation were discarded, removing enough packets to ensure that the system had reached a steady state.

Fig. 2 shows the PAoI CDF in the four subcases for $\lambda = 0.5$, $\mu_1 = 1$ and $\mu_2 = 1.2$. It is easy to note that the PAoI does not have the same behavior as the system time, which is the highest when the first system is busy, i.e., when there is queuing at the bottleneck, and lowest in case 2B, in which both systems are free. The PAoI is the lowest in case 2A, and almost identical in cases 1B and 2B. This difference is due to the effect of the interarrival times on the PAoI, as case 2B usually means that the instantaneous load of the system is low and packets are far apart, increasing the PAoI. In all cases, the simulation results fit the analytically derived curve with minimal error, for both the system time and the PAoI.
In case 2A, the faster system is busy and the bottleneck is empty. Intuitively, this can reduce age, as the second system will probably be able to serve packets fast enough, but at the same time the instantaneous load will be high enough to avoid having a strong impact on the age.

We can now examine the PAoI CDFs for different values of $\lambda$: the system time is always higher for higher values of $\lambda$, as it depends on the traffic. The same is not true for the PAoI, as Fig. 3 shows: the PAoI is lowest for $\lambda = 0.5$, as the high interarrival time becomes the dominant factor for $\lambda = 0.25$. As for the subcase analysis, the system time and PAoI from the Monte Carlo simulation follow the analytical curve perfectly. On the other hand, the values of $\mu_1$ and $\mu_2$ also have an important effect, as Fig. 4 shows: while the bottleneck always has a service rate 1, changing the service rate of the other link and even switching the two can have an impact on the PAoI. Naturally, increasing the rate of the other link from 1.2 to 1.6 slightly reduces the PAoI, but we note that for both values, having the first system as the bottleneck reduces performance, particularly when the systems are similar.

Finally, Fig. 5 shows how the worst-case PAoI, measured using the 95th, 99th and 99.9th percentiles, changes as a function of $\lambda$: if the traffic is very high, the queuing time is the dominant factor, causing the worst-case PAoI to diverge. The same happens if the traffic is too low, as the interarrival times can be very large: in this case, the system will almost always be empty, but updates will be very rare. The best performance in terms of PAoI is close to the middle.

VI. CONCLUSIONS AND FUTURE WORK

In this letter, we derived the PDF of the PAoI for a tandem of two $M/M/1$ queues. This result can give more flexibility in the design of bounded AoI systems, both for IoT and other relay applications. The results are derived for two nodes, but the procedure is generic for $K$ nodes.

A first possible avenue of future work is the introduction of multiple independent sources in the system, possibly with different priorities. The extension of the system to longer line networks is also a possibility, but the complexity of the derivation might make the results unwieldy. Other potential directions are the inclusion of error probabilities in the links and preemption-based policies. Finally, the extension to tandem $M/D/1$ or $D/M/1$ systems might be very interesting, as these systems are often used to model real update applications.

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