LOOPS AND SEMIDIRECT PRODUCTS

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1. Introduction

A left loop $(B, \cdot)$ is a set $B$ together with a binary operation $\cdot$ such that (i) for each $a \in B$, the mapping $x \mapsto a \cdot x$ is a bijection, and (ii) there exists a two-sided identity $1 \in B$ satisfying $1 \cdot x = x \cdot 1 = x$ for every $x \in B$. A right loop is similarly defined, and a loop is both a right loop and a left loop [5] [6].

In this paper we study semidirect products of loops with groups. This is a generalization of the familiar semidirect product of groups. Recall that if $G$ is a group with subgroups $B$ and $H$ where $B$ is normal, $G = BH$, and $B \cap H = \{1\}$, then $G$ is said to be an internal semidirect product of $B$ with $H$. On the other hand, if $B$ and $H$ are groups and $\sigma : H \rightarrow \text{Aut}(B) : h \mapsto \sigma_h$ is a homomorphism, then the external semidirect product of $B$ with $H$ given by $\sigma$, denoted $B \rtimes_\sigma H$, is the set $B \times H$ with the multiplication

$$ (a, h)(b, k) = (a \cdot \sigma_h(b), hk).$$

A special case of this is the standard semidirect product where $H$ is a subgroup of the automorphism group of $B$, and $\sigma$ is the inclusion mapping. The relationship between internal, external and standard semidirect products is well known.

These considerations can be generalized to loops. We now describe the contents of the sequel.

In §2, we consider the natural embedding of a left loop $B$ into its permutation group $\text{Sym}(B)$. This leads to a factorization of $\text{Sym}(B)$ into a subset $L(B)$ consisting of the left translations of $B$ and a subgroup $\text{Sym}_1(B)$ consisting of permutations fixing the identity element $1 \in B$. We then discuss left inner mappings and deviations [33], and show how these characterize those permutations which are pseudo-automorphisms and automorphisms. We discuss how the aforementioned factorization of $\text{Sym}(B)$ is related to the group multiplication; here the left inner mappings and deviations play a role in decomposing the product of permutations. Finally, we give Sabinin’s definition of the standard semidirect product of a left loop $B$ with one of its transassociants [33]. This semidirect product has occasionally been rediscovered for various classes of loops. We conclude the section with an example.

In §3, we consider internal semidirect products of left loops and groups: given a group $G$, a subgroup $H < G$, and a transversal $B \subseteq G$ of $H$ which contains the identity, $B$ naturally has the structure of a left loop. This is equivalent to putting a loop structure on the set $G/H$ of cosets [2], but it is closer to the examples to work with transversals. We consider the relationship between the loop structure of $B$ and the multiplication in $G$, paralleling the discussion in §2. We follow Sabinin

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We then consider conditions under which subgroups and factor groups inherit the semidirect product structure. A particular case of the latter is obtained by modding out the core of \( H \) in \( G \).

In §4, we use transversal decompositions \( G = BH \) to study certain loop identities, especially those related to Bol loops and Bruck loops. This part of our study is related to work of Ungar [37] and Kreuzer and Wefelscheid [24], but we do not assume as much structure at the outset. We then give examples of internal semidirect products, illustrating some of the results.

In §5, we generalize the standard semidirect product of a left loop \( B \) with a particular subgroup of \( \text{Sym}_1(B) \) to an external semidirect product of a left loop \( B \) with a group \( H \). As for the usual semidirect product of groups, the main interest here is in the case where the defining homomorphism from \( H \) to \( \text{Sym}_1(B) \) is not injective. Our construction seems to be new, and we give examples. We conclude by discussing how the three semidirect products are related, generalizing the relationship between the usual standard, internal, and external semidirect products of groups.

There exist notions of semidirect products of loops which are different from that which we consider here. One definition is as follows: a loop \( R \) is an internal semidirect product of the normal subloop \( P \) by the subloop \( Q \) if \( R = PQ \) and \( P \cap Q = \{1\} \). This definition was given by Birkenmeier et al [3] and Birkenmeier and Xiao [4], who studied nonassociative loops which are internal semidirect products of groups. Goodaire and Robinson [12] defined an internal semidirect product similarly with additional conditions given in terms of associators. In contrast, our internal semidirect product follows Sabinin [33]: \( G = BH \) is a factorization of a group \( G \) into a subgroup \( H \) and a transversal \( B \). Even if \( B \) with its induced operation turns out to be a group, it is not necessarily a subgroup of \( G \). Also, \( H \) does not necessarily stabilize \( B \) by conjugation. Thus the two notions of semidirect product are quite distinct.

In group theory, the question of which groups have a semidirect product structure is answered by cohomology theory. Cohomology has been generalized to loops in at least two distinct ways; see Eilenberg and MacLane [7], and Johnson and Leedham-Green [16]. At present, we do not know if the semidirect product of the present paper has a suitable cohomological interpretation.

### 2. Standard Semidirect Products

For a set \( B \), let \( \text{Sym}(B) \) denote the group of permutations of \( B \). If \( (B, \cdot) \) is a left loop with identity element \( 1 \in B \), let \( \text{Sym}_1(B) \) denote the subgroup consisting of permutations fixing \( 1 \). For each \( x \in B \), define the left translation mapping \( L_x : B \to B \) by \( L_x y = x \cdot y \). Define the left division operation \( \backslash : B \times B \to B \) by \( x \backslash y = L_x^{-1} y \) for \( x, y \in B \). We denote the right inverse of \( x \in B \) by \( x^\rho = x \backslash 1 \). The right inversion mapping \( \rho : x \mapsto x^\rho \) fixes \( 1 \), and is a permutation if and only if each \( x \in B \) has a unique left inverse \( x^\lambda \).

Let \( L(B) = \{ L_x : x \in B \} \) be the set of left translations, and let \( \text{LMlt}(B, \cdot) = \langle L(B) \rangle \) be the left multiplication group, i.e., \( \text{LMlt}(B, \cdot) \) is the subgroup of \( \text{Sym}(B) \) generated by \( L(B) \). Let \( \text{LMlt}_1(B, \cdot) = \text{LMlt}(B, \cdot) \cap \text{Sym}_1(B) \). For \( x, y \in B \), the permutation \( L(x, y) \in \text{Sym}(B) \) defined by

\[
L(x, y) = L_{xy}^{-1} L_x L_y
\]
is called a left inner mapping. Clearly
\begin{equation}
L(1, x) = L(x, 1) = I
\end{equation}
for all \(x \in B\). We have \(L(x, y) \in \text{LMlt}_1(B, \cdot)\), and in fact,
\begin{equation}
\text{LMlt}_1(B, \cdot) = \langle L(x, y) : x, y \in B \rangle
\end{equation}
\cite[(2.6), p.61; (2.8), I.5.2]{a}. \text{LMlt}_1(B, \cdot) is called the left inner mapping group.

A permutation \(\phi \in \text{Sym}(B)\) is called a pseudo-automorphism with companion \(c \in B\) if \(c \cdot \phi(x \cdot y) = (c \cdot \phi(x)) \cdot \phi(y)\) for all \(x, y \in B\), or equivalently,
\begin{equation}
L_c \phi L_x \phi^{-1} = L_{c \cdot \phi(x)}
\end{equation}
for all \(x \in B\). The set \(\text{PsAut}(B)\) of pseudo-automorphisms of \(B\) is a group under composition of mappings. Since a pseudo-automorphism fixes 1 (take \(y = 1\) and cancel \(c \cdot \phi(x)\)), \(\text{PsAut}(B)\) is a subgroup of \(\text{Sym}_1(B)\). A left loop \((B, \cdot)\) is said to have the pseudo-
A_1 property if every left inner mapping \(L(x, y)\) is a pseudo-automorphism. By \cite[(2.3)]{a}, this is equivalent to the assertion that \(\text{LMlt}_1(B, \cdot) \subseteq \text{PsAut}(B)\).

A pseudo-automorphism with companion 1 is an automorphism of \(B\). Let \(\text{Aut}(B)\) denote the group of automorphisms of \(B\). A left loop \((B, \cdot)\) is said to have the A_1 property if every left inner mapping \(L(x, y)\) is an automorphism. By \cite[(2.3), p.61]{a}, this is equivalent to the assertion that \(\text{LMlt}_1(B, \cdot) \subseteq \text{Aut}(B)\).

For each \(x \in B\) and \(\phi \in \text{Sym}(B)\), the permutation \(\mu_x(\phi) \in \text{Sym}(B)\) defined by
\begin{equation}
\mu_x(\phi) = L_{\phi(x)}^{-1} \phi L_x \phi^{-1}
\end{equation}
is called the deviation of \(\phi\) at \(x\) \cite[(2.3), 2.C]{b}. Clearly, we have
\begin{align}
\mu_1(\phi) &= I \\
\mu_x(I) &= I
\end{align}
for all \(x \in B\), \(\phi \in \text{Sym}(B)\). As the next result shows, deviations measure how much arbitrary permutations “deviate” from being (pseudo-)automorphisms.

**Proposition 2.1.** Let \((B, \cdot)\) be a left loop, and let \(\phi \in \text{Sym}(B)\) be given.

1. \(\phi \in \text{Sym}_1(B)\) if and only if \(\mu_x(\phi) \in \text{Sym}_1(B)\) for all \(x \in B\).
2. \(\phi \in \text{PsAut}(B)\) if and only if there exists \(c \in B\) such that \(\mu_x(\phi) = L(c, \phi(x))^{-1}\) for all \(x \in B\).
3. \(\phi \in \text{Aut}(B)\) if and only if \(\mu_x(\phi) = I\) for all \(x \in B\).

**Proof.**
1. For \(x \in B\), \(\mu_x(\phi) = 1\) if and only if \(\phi(x) = \phi(x \cdot \phi^{-1}(1))\) if and only if \(x = x \cdot \phi^{-1}(1)\) if and only if \(1 = \phi^{-1}(1)\).
2. For all \(x, c \in B\), a computation using \(\mu_x(\phi)\) gives
\begin{equation}
L_c \phi L_x \phi^{-1} = L_{c \cdot \phi(x)} L(c, \phi(x)) \mu_x(\phi).
\end{equation}

From this, it is clear that \(\phi\) is a pseudo-automorphism with companion \(c\) if and only if \(L(c, \phi(x)) \mu_x(\phi) = I\).
3. This follows from taking \(c = 1\) in \(\mu_x(\phi)\) and using \(\mu_x(\phi)\).

For a left loop \((B, \cdot)\), the mapping \(B \to L(B) \colon x \mapsto L_x\) is bijective \((L_x = L_y\) implies \(x = L_x(1) = L_y(1) = y\)). Note that \(L(B)\) itself can be given the structure of a left loop isomorphic to \((B, \cdot)\) with the obvious definition:
\begin{equation}
L_x \cdot L_y = L_{x \cdot y}
\end{equation}
for \(x, y \in B\).

Let \(G\) be any group satisfying \(\text{LMlt}(B, \cdot) \leq G \leq \text{Sym}(B)\), and let \(H = G \cap \text{Sym}_1(B)\), so that \(\text{LMlt}_1(B, \cdot) \leq H \leq \text{Sym}_1(B)\). For any \(\phi \in G\), we have \(\phi = L_x \psi\) where \(x = \phi(1)\) and \(\psi = L_x^{-1} \phi\). Clearly \(\psi(1) = 1\), and thus \(\psi \in H\) since \(G\) contains \(\text{LMlt}(B, \cdot)\). The factorization of \(\phi\) into a left translation \(L_x\) in \(L(B)\) and a permutation \(\psi\) in \(H\) is unique. Indeed, if \(L_x \psi = L_y \varphi\) for \(x, y \in B, \psi, \varphi \in H\), then applying both sides to 1 gives \(x = y\), and thus \(L_x \psi = L_x \varphi\); cancelling \(L_x\) gives \(\psi = \varphi\).

Summarizing, for any group \(G\) satisfying \(\text{LMlt}(B, \cdot) \leq G \leq \text{Sym}(B)\), we have the following decomposition:

\[
\text{(2.10)} \quad G = L(B)H
\]

where \(H = G \cap \text{Sym}_1(B)\). The factorization of elements is unique, and we also have

\[
\text{(2.11)} \quad L(B) \cap H = \{1\}.
\]

It is natural to ask how the factorization \(\text{(2.10)}\) of a group \(G\) into a subset \(L(B)\) with a left loop structure given by \(\text{(2.9)}\) and a subgroup \(H\) interacts with the multiplication in \(G\). We first examine this question for \(G = \text{Sym}(B)\) and \(H = \text{Sym}_1(B)\). For permutations \(L_x \phi\) and \(L_y \psi\) with \(x, y \in B, \phi, \psi \in \text{Sym}_1(B)\), we have \((L_x \phi L_y \psi)(1) = x \cdot \phi(y)\). Also, \(L_x^{-1} L_y \phi L_y = L(x, \phi(y)) \mu_y(\phi)\). Put together, these observations give the following factorization of a product in \(\text{Sym}(B) = L(B) \text{ Sym}_1(B)\).

**Proposition 2.2.** Let \((B, \cdot)\) be a left loop. For all \(x, y \in B, \phi, \psi \in \text{Sym}_1(B)\),

\[
\text{(2.12)} \quad (L_x \phi)(L_y \psi) = L_{x \cdot \phi(y)} \left[ L(x, \phi(y)) \mu_y(\phi) \phi \psi \right].
\]

Using \(\text{(2.9)}, \text{(2.12)}\) can be rewritten as

\[
(L_x \phi)(L_y \psi) = (L_x \cdot L_{\phi(y)})(L(x, \phi(y)) \mu_y(\phi) \phi \psi).
\]

Thus we see that the operation \(L_x \cdot L_y\) defined by \(\text{(2.9)}\) is simply the projection of the composition \(L_x L_y\) onto \(L(B)\).

For the factorization \(\text{(2.12)}\) to hold in a subgroup \(G\) of \(\text{Sym}(B)\), it is clear that it is necessary that the part of \(\text{(2.12)}\) in square brackets be in the subgroup \(H = G \cap \text{Sym}_1(B)\). Thus assume \(L(x, \phi(y)) \mu_y(\phi) \in H\) for all \(x, y \in B, \phi \in H\). Taking \(\phi = I\), we have \(L(x, y) \in H\) for all \(x, y \in H\), and this implies \(\mu_y(\phi) \in H\) for all \(y \in B, \phi \in H\). This leads us to the following definition of Sabinin \(\text{[33]}\). A subgroup \(H \leq \text{Sym}_1(B)\) is said to be a transassociant of \(B\) if \(L(x, y) \in H\) for all \(x, y \in B\) and if \(\mu_x(\phi) \in H\) for all \(x \in B, \phi \in H\).

**Proposition 2.3.** Let \((B, \cdot)\) be a left loop.

1. \(\text{LMlt}_1(B, \cdot)\) is a transassociant.
2. If \((B, \cdot)\) has the pseudo-\(A_2\) property, and if \(\text{LMlt}_1(B, \cdot) \leq H \leq \text{PAut}(B)\), then \(H\) is a transassociant.
3. If \((B, \cdot)\) has the \(A_1\) property, and if \(\text{LMlt}_1(B, \cdot) \leq H \leq \text{Aut}(B)\), then \(H\) is a transassociant.

**Proof.** 1. If \(\phi \in \text{LMlt}_1(B, \cdot)\), then by \(\text{(2.3)}\) and Proposition \(\text{2.1}(1)\), \(\mu_x(\phi) \in \text{LMlt}_1(B, \cdot)\) for all \(x \in B\).
2. This follows from Proposition \(\text{2.1}(2)\).
3. This follows from Proposition \(\text{2.1}(3)\). \(\blacksquare\)
Consider again the case where $G = \text{Sym}(B) = L(B) \text{Sym}_1(B)$. The factorization (2.10) along with (2.11), gives a one-to-one correspondence between $B \times \text{Sym}_1(B)$ and $\text{Sym}(B)$ given by $(x, \phi) \mapsto L_x \phi$. Thus we may use Proposition 2.2 to define a binary operation on $B \times \text{Sym}_1(B)$:

$$(x, \phi) \cdot (y, \psi) = (x \cdot \phi(y), L(x, \phi(y))\mu_y(\phi)\psi)$$

for $x, y \in B$, $\phi, \psi \in \text{Sym}_1(B)$. By construction, $(B \times \text{Sym}_1(B), \cdot)$ is a group isomorphic to $\text{Sym}(B)$. We now present Sabinin’s definition [33] of the semidirect product of a left loop with one of its transassociants.

**Definition 2.4.** Let $B$ be a left loop and let $H \leq \text{Sym}_1(B)$ be a transassociant of $B$. Define a binary operation $\cdot$ on the set $B \times H$ as follows:

$$(x, \phi) \cdot (y, \psi) = (x \cdot \phi(y), L(x, \phi(y))\mu_y(\phi)\psi)$$

for all $a, b \in B$, $\phi, \psi \in H$. Then $(B \times H, \cdot)$ is called the **standard semidirect product** of $B$ with $H$, and is denoted $B \rtimes H$.

It is immediate from this definition that $B \rtimes H$ is a subgroup of the group $B \times \text{Sym}_1(B) \cong \text{Sym}(B)$. We have proven the following result.

**Proposition 2.5.** (33, Thm. 2) Let $B$ be a left loop and let $H \leq \text{Sym}_1(B)$ be a transassociant of $B$. Then $B \rtimes H$ is a group.

Incidentally, as Sabinin [33] has noted, in order for $B \rtimes H$ to be a group, it is only necessary for $B$ to have a **right** identity element $1$. In this case, (2.11) becomes $L(B) \cap H = \{L_1\}$ because $L_1 \neq I$.

We now consider some special cases.

**Remark 2.6.**

1. If $(B, \cdot)$ is a group, then the product in $B \rtimes H$ is given by

$$(x, \phi) \cdot (y, \psi) = (x \cdot \phi(y), \mu_y(\phi)\psi),$$

$x, y \in B$, $\phi, \psi \in H$. This generalized semidirect product of groups was rediscovered by Jajcay [15], who dubbed it the “rotary” product of groups. The semidirect product $\text{Sym}(B) \cong B \rtimes \text{Sym}_1(B)$ can be seen as a detailed description of the algebraic structure of the regular representation of $(B, \cdot)$.

2. Assume $(B, \cdot)$ is a pseudo-$A_l$ left loop and that $\text{LMlt}_1(B, \cdot) \leq H \leq \text{PsAut}(B)$. By Proposition 2.3(2), $H$ is a transassociant. In this case, the product in $B \rtimes H$ is given by

$$(x, \phi) \cdot (y, \psi) = (x \cdot \phi(y), L(x, \phi(y))L(c, \phi(y))^{-1}\phi\psi),$$

$x, y \in B$, $\phi, \psi \in H$, where $c$ is a companion of $\phi$, using Proposition 2.1(2). The semidirect product group $\text{PsAff}(B) \cong B \rtimes \text{PsAut}(B)$ is called the **pseudo-affine group** of $(B, \cdot)$.

3. Assume $(B, \cdot)$ is an $A_l$ left loop and that $\text{LMlt}_1(B, \cdot) \leq H \leq \text{Aut}(B)$. By Proposition 2.3(3), $H$ is a transassociant. In this case, the product in $B \rtimes H$ is given by

$$(x, \phi) \cdot (y, \psi) = (x \cdot \phi(y), L(x, \phi(y))\phi\psi),$$

$x, y \in B$, $\phi, \psi \in H$, using Proposition 2.1(3). The semidirect product group $\text{Aff}(B) \cong B \rtimes \text{Aut}(B)$ is called the **affine group** of $(B, \cdot)$. For $A_l$ left loops with the left inverse property (see §4), this semidirect product was rediscovered by Kikkawa [2] and later, using different terminology, by Ungar [37].
4. If $B$ is a group and $H$ is a subgroup of $\text{Aut}(B)$, then $B \rtimes H$ is the usual standard semidirect product of groups.

We now give an explicit example of a standard semidirect product.

**Example 2.7.** Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ be the complex unit disk. Note that the circle group $S^1 = \{ a \in \mathbb{C} : |a| = 1 \}$ acts on $\mathbb{D}$ by multiplication of complex numbers. For $x, y \in \mathbb{D}$, define
\[(2.14)\quad x \oplus y = \frac{x + y}{1 + \overline{xy}}.\]
($\mathbb{D}, \oplus$) turns out to be a $B$-loop [21] [29] (see §3 for the definition). The left inner mappings are given by unimodular complex numbers
\[(2.15)\quad L(x, y)z = \frac{1 + \overline{xy}z}{1 + \overline{xy}}\]
for $x, y, z \in \mathbb{D}$. If we identify $S^1$ with its natural image in $\text{Aut}(\mathbb{D})$, then $\text{Aut}(\mathbb{D})$ is generated by $S^1$ and the complex conjugation mapping $x \mapsto \overline{x}$. The semidirect product $\mathbb{D} \rtimes S^1$ turns out to be isomorphic to the orientation-preserving Möbius group of the disk $\mathbb{D}$. The semidirect product $\mathbb{D} \rtimes \text{Aut}(\mathbb{D})$ is isomorphic to the full Möbius group of $\mathbb{D}$.

3. **Internal Semidirect Products**

Let $G$ be a group with identity element $1$, let $H$ be a subgroup of $G$, and let $B$ be a left transversal of $H$ in $G$, i.e., for every $g \in G$, there exists a unique $a \in B$ and a unique $h \in H$ such that $g = ah$. (Equivalently, each element of $B$ is a representative of a unique left coset of $H$ in $G$.) We will call the factorization $G = BH$ a transversal decomposition. (Sabinin [33] calls $B$ a “quasi-reductant”. Here we adapt more standard group-theoretic terminology.)

Let $e \in B$ denote the representative of the coset $1H = H$. Then obviously $B \cap H = \{ e \}$ (since $eH = H$). We have $e^{-1} \in H$. Now let $\tilde{B} = B e^{-1} = \{ ae^{-1} : a \in B \}$. Then $\tilde{B}$ is a transversal of $H$ in $G$. Indeed, let $g \in G$ satisfy $g = ah$ where $a \in B$, $h \in H$. Then $g = (ae^{-1})(eh)$. On the other hand, if $g = bk$ for $b \in \tilde{B}$, $k \in H$, then $be \in B$ and $ae^{-1} = g(eh)^{-1} = (be)(e^{-1}kh^{-1}e^{-1})$. Thus $a = be$ and $k = eh$, which shows uniqueness of the decomposition. In addition, we clearly have $\tilde{B} \cap H = \{ 1 \}$. This discussion shows that there is no real loss in assuming that $e = 1$, i.e.,
\[(3.1)\quad B \cap H = \{ 1 \}.\]
In this case, $B$ is called a unital transversal and $G = BH$ is called a unital transversal decomposition. ($B$ is sometimes called a “uniform quasi-reductant” [33], and $G = BH$ is called an “exact” decomposition [21] [22] [37].) We will assume throughout that our transversal decompositions are unital without specifically mentioning it.

Let $G = BH$ be a transversal decomposition. Define a binary operation $\cdot : B \times B \to B$ as follows: for $x, y \in B$, $x \cdot y \in B$ is defined by
\[(3.2)\quad (x \cdot y)H = xyH.\]
Also define
\[(3.3)\quad l(x, y) = (x \cdot y)^{-1}xy.\]
We call \( l : B \times B \to H \) the \textit{transversal mapping}. Note that
\begin{align}
1 \cdot x &= x \cdot 1 = x \\
(3.4) \\
l(1,x) &= l(x,1) = 1
\end{align}
for all \( x \in B \). Finally we define \( \setminus : B \times B \to B \) as follows: for \( x,y \in B \), \( x \\setminus y \in B \) is defined by
\begin{equation}
(3.5)
(x \setminus y)H = x^{-1}yH
\end{equation}
for \( x,y \in B \).

The following result is well-known (e.g., [33], Thm. 7; [24], Thm. 3.2; [18], Thm. 2.7.).

\textbf{Proposition 3.1.} \((B, \cdot)\) is a left loop.

\textit{Proof.} From (3.4), 1 is a two-sided identity. For \( x,y \in B \), we have
\begin{align}
(x \cdot (x \setminus y))H &= x(x \setminus y)H = xx^{-1}yH = yH \\
\text{and}
(3.6)
x(x \setminus y)H &= x^{-1}(x \cdot y)H = x^{-1}xyH = yH.
\end{align}
Thus each left translation \( L_x : B \to B : y \mapsto x \cdot y \) has an inverse given by
\begin{equation}
(3.7)
l(x,x^{\rho}) = xx^{\rho}.
\end{equation}
In general, the mapping \( \rho : B \to B : x \mapsto x^{\rho} \) is not a permutation, i.e., not every element has a unique left inverse. The next result characterizes left loops for which this holds.

\textbf{Proposition 3.2.} Let \( G = BH \) be a transversal decomposition. The following are equivalent: (i) \( B \) is a right transversal of \( H \) in \( G \); (ii) \( B^{-1} \) is a left transversal of \( H \) in \( G \); (iii) each element of \( B \) has a unique left inverse in \((B, \cdot)\).

\textit{Proof.} The equivalence of (i) and (ii) is obvious. Since \( B \) is a transversal, \( B^{-1} \) is a transversal if and only if, for every \( x \in B \), there exists a unique \( x^\lambda \in B \) such that \( xH = (x^\lambda)^{-1}H \). This is equivalent to \( x^\lambda xH = H \), or \( x^\lambda \cdot x = 1 \), which establishes the equivalence of (ii) and (iii).

There is a natural action of \( G \) on \( B \) which is given by the action of \( G \) on the set of left cosets \( G/H \). For \( g \in G \), \( x \in B \), we define \( \sigma_g(x) \in B \) by
\begin{equation}
(3.8)
\sigma_g(x)H = gxH.
\end{equation}
for all \( x \in B, h \in H \). Recall that the \textit{core} of the subgroup \( H \) is defined by
\begin{equation}
(3.9)
\text{core}_G(H) = \bigcap_{g \in G} gHg^{-1},
\end{equation}
i.e., \( \text{core}_G(H) \) is largest subgroup of \( H \) which is normal in \( G \). For the next result, see also [18], Thm. 2.8.
**Theorem 3.3.** Let $G = BH$ be a transversal decomposition.

1. $\sigma : G \to \Sym(B)$ is a homomorphism.
2. $\sigma(H) \leq \Sym_1(B)$.
3. For all $x, y \in B$, $h \in H$,

\[
\sigma_x = L_x \\
\sigma_{l(x,y)} = L(x, y)
\]

4. $\ker(\sigma) = \operatorname{core}_G(H)$.

**Proof.**

1. For $g, g' \in G$, $x \in B$, we have by (3.8),

\[(\sigma_g \sigma_{g'})^H(x) = g \sigma_{g'}(x) H = g g' x H = \sigma_{gg'}(x) H.
\]

In particular, $\sigma_{g^{-1}} \sigma_g = \sigma_g \sigma_{g^{-1}} = I$, and thus $\sigma(G) \leq \Sym(B)$.

2. For $h \in H$, $x \in B$, we have by (3.8),

\[
\sigma_h(1) H = h H = H = 1 H,
\]

and thus $\sigma_h(1) = 1$ as claimed.

3. For $x, y \in B$, (3.8) and (3.2) imply $\sigma_x(y) = x \cdot y = L_x y$, which gives (3.10). For $x, y, z \in B$, we use (3.8), (3.3), (3.2), and (2.1) to compute

\[
\sigma_{l(x,y)}(z) H = l(x, y) z H = (x \cdot y)^{-1} x y z H \\
= [(x \cdot y)^{-1} \cdot (x \cdot y \cdot z)] H \\
= [(L_{x^{-1}} L_x L_y)(z)] H \\
= (L(x, y) z) H.
\]

This establishes (3.11).

4. If $h \in \ker(\sigma)$, then for all $x \in B$, $\sigma_h(x) H = h x H = x H$, i.e., for all $x \in B$, $h x \in x H$. This is equivalent to $hg \in g H$ for all $g \in G$, or $h \in \bigcap_{g \in G} g H g^{-1}$, as claimed.

For $x \in B$, $h \in H$, define $m(x, h) \in H$ by

\[
m(x, h) = \sigma_h(x)^{-1} h x h^{-1}.
\]

**Theorem 3.4.** Let $G = BH$ be a transversal decomposition. For all $x \in B$, $h \in H$,

\[
m(x, 1) = 1 \\
m(1, h) = 1 \\
\sigma_{m(x, h)} = \mu_x(\sigma_h)
\]
Proof. Since $\sigma_1 = I$, (3.13) follows. By Theorem 3.3(2), (3.14) follows. Finally, for $x, y \in B$, $h \in H$, we use (3.8), (3.12), (3.2), and (2.5) to compute

$$
(\sigma_{m(x,h)}(y))H = m(x,h)yH
$$

$$
= \sigma_h(x)^{-1}hx\sigma_h^{-1}(y)H
$$

$$
= \sigma_h(x)^{-1}h(x \cdot \sigma_h^{-1}(y))H
$$

$$
= \sigma_h(x)^{-1}\sigma_h(x \cdot \sigma_h^{-1}(y))H
$$

$$
= (\sigma_h(x)^{-1} \cdot \sigma_h(x \cdot \sigma_h^{-1}(y)))H
$$

$$
= \left[ (L_{\sigma_h(x)}^{-1}\sigma_h L_x \sigma_h^{-1}(y)) \right] H
$$

$$
= (\mu_x(\sigma_h(y)))H
$$

This establishes (3.15). □

Corollary 3.5. $\sigma(H)$ is a transassociant of $(B, \cdot)$.

Proof. This follows from Theorem 3.3(2), (3.11), and (3.15). □

We now consider the $B$- and $H$-components of a product of elements of $G$.

Proposition 3.6. For all $x, y \in B$, $h, k \in K$,

$$(3.16) \quad xhyk = (x \cdot \sigma_h(y))l(x, \sigma_h(y)m(y, h)hk.$$ 

Proof. This is a direct computation. □

Comparison of Proposition 3.6 with Proposition 2.2 suggests the following.

Definition 3.7. Let $(B, \cdot)$ be the left loop induced by a transversal decomposition $G = BH$. Then we say that $G$ is an internal semidirect product of $(B, \cdot)$ with $H$.

Remark 3.8. 1. If $l : B \times B \to H$ is trivial, but $m : B \times H \to H$ is nontrivial, then $(B, \cdot)$ is a subgroup of $G$, and $G = BH$ is the internal version of Jajcay’s “rotary product” \cite{15} of subgroups.

2. If $m : B \times H \to H$ is trivial, but $l : B \times B \to H$ is nontrivial, then the product of $xh, yk \in G$ simplifies to

$$(xh)(yk) = (x \cdot \sigma_h(y))l(x, y)hk.$$ 

As will be shown below, in this case $(B, \cdot)$ is an $A_1$ left loop, and $G = BH$ is the internal version of the semidirect product rediscovered (for $A_1$, LIP left loops) by Kikkawa \cite{20} and Ungar \cite{37}.

3. Both $l : B \times B \to H$ and $m : B \times H \to H$ are trivial if and only if $B$ is a normal subgroup of $G$. In this case, $G = BH$ is the usual internal semidirect product of subgroups.

4. Another case where $(B, \cdot)$ is a group is if $\sigma(H) = \{I\}$; this follows from (3.11). However, if the transversal mapping $l : B \times B \to H$ is nontrivial, then $B$ is not a subgroup of $G$, and if $m : B \times H \to H$ is nontrivial, then $H$ does not normalize $B$.
We now consider the inheritance of internal semidirect product structure by subgroups. Let $G = BH$ be a transversal decomposition giving an internal semidirect product of $(B, \cdot)$ by $H$. Assume that $G_1$ is a subgroup of $G$ and let $B_1 = B \cap G_1$ and $H_1 = H \cap G_1$. For $g \in G_1$, if $g = xh$ with $x \in B$, $h \in H$, then we clearly have $x \in B_1$ if and only if $h \in H_1$. When either of these conditions hold, we say that $G_1$ respects the internal semidirect product structure of $G$ and $H$ and say that $G_1$ respects the internal semidirect product structure of $G$. If $G_1$ respects $G = BH$, then $G_1 = B_1H_1$ is itself a transversal decomposition, which means that $G_1$ is an internal semidirect product of $B_1$ with $H_1$. In particular, the operation $\cdot$ on $B$ restricts to $B_1$, which shows that the left loop $(B_1, \cdot)$ is a subloop of $(B, \cdot)$. Finally, if $G_1$ and $G_2$ are both subgroups respecting $G = BH$, then clearly the intersection $G_1 \cap G_2$ satisfies this property as well.

Next we consider the inheritance of internal semidirect product structure by factor groups. Let $G = BH$ be a transversal decomposition and let $K < H$ be a normal subgroup of $G$. An arbitrary element $gK$ of $G/K$ factors as $gK = (xK)(hK)$ where $xK \in B_K = \{xK : x \in B\}$ and $hK \in H/K$. This factorization is clearly unique. Also, $B_K \cap H/K = \{K\}$. Thus

$$G/K = B_K (H/K)$$

is a transversal decomposition of the factor group $G/K$. Denote the induced binary operation (3.2) by $\cdot_K : B_K \times B_K \to B_K$ and the induced transversal mapping by $l_K : B_K \times B_K \to H/K$.

Since $B \cap K = \{1\}$, the set $B_K$ can be identified with $B$ itself. Thus we compare two factorizations of products. For $x, y \in B$, we have

$$(xK) (yK) = (xK \cdot_K yK) l_K(xK, yK),$$

and also

$$(xK) (yK) = (xy)K = (x \cdot y) l(x, y)K = ((x \cdot y) K) (l(x, y)K).$$

By uniqueness, we have

$$xK \cdot_K yK = (x \cdot y) K$$

$$l_K(xK, yK) = l(x, y)K$$

for all $x, y \in B$. It follows that under the mapping $x \mapsto xK$, the left loop $(B, \cdot)$ induced by the transversal decomposition $G = BH$ is isomorphic to the left loop $(B_K, \cdot_K)$ induced by the transversal decomposition $G/K = B_K (H/K)$. Making this identification, we may think of

$$G/K = B (H/K)$$

(3.17)

as being a transversal decomposition of $G/K$.

As a specific example of factor group inheritance, let $G = BH$ be an internal semidirect product of the left loop $(B, \cdot)$ with the subgroup $H$, and let $K = \text{core}_G(H)$. In addition to (3.17), we may make an additional observation. By Corollary 3.5, $\sigma(H)$ is a transassociant, and thus we may form the standard semidirect product $B \rtimes \sigma(H)$. Thus the exact sequence of groups

$$1 \to \text{core}_G(H) \to H \to \sigma(H) \to 1$$

(3.18)
induces an exact sequence of semidirect product groups
\[(3.19) \quad 1 \to \text{core}_G(H) \to G \to B \times \sigma(H) \to 1.\]
The exactness of (3.18) and (3.19) imply the isomorphisms \(H/\text{core}_G(H) \cong \sigma(H)\) and \(G/\text{core}_G(H) \cong B \times \sigma(H)\). Note also the obvious isomorphism of groups \(B \cdot H/\ker(\sigma) \to B \times \sigma(H)\) given by \(x(h\ker(\sigma)) \mapsto (x, \sigma_h)\).

**Remark 3.9.** Consider again the special case of Remark 3.8(4) where \(\sigma(H) = \{1\}\), i.e., \(H \triangleleft G\). Then making the usual identifications, (3.19) simplifies to
\[(3.20) \quad 1 \to H \to G \to B \to 1.\]
As noted, if \(l\) is nontrivial, then \(B\) is a group, but not a subgroup. Instead, we see from (3.20) that \(B\) is an isomorphic copy of the factor group \(G/H\).

Let \(G = BH\) be a transversal decomposition. Let \(G_0 = \langle B \rangle\), and let \(H_0 = G_0 \cap H\). It is straightforward to show that \(H_0 = \langle l(B, B) \rangle\) where \(l(B, B) = \{(x, y) : x, y \in B\}\). If \(H\) is corefree, i.e., \(\text{cor}_G(H) = \{1\}\), then the restriction of \(\sigma\) to \(l(B, B)\) is a bijection onto the set \(\{L(x, y) : x, y \in B\}\) of left inner mappings. Thus \(\sigma|_{H_0}\) is an isomorphism onto the left inner mapping group \(\text{LMI}_1(B, \cdot)\).

Putting these considerations together, we have the following result.

**Proposition 3.10.** Let \(G = BH\) be a transversal decomposition. Then \(G \cong \text{LMI}(B, \cdot)\) if and only if \(H\) is corefree and \(G = \langle B \rangle\).

There is a similar characterization of the multiplication group of a loop; see Niemennaa and Kepka [27]. In the more general case where the core is not required to be trivial, Phillips has shown that a group \(G = BH\) with \(G = \langle B \rangle\) can be viewed as a left relative multiplication group of \((B, \cdot)\) as a subloop in some larger left loop [30].

We will consider some examples of internal semidirect products in the next section, after using transversal decompositions to study certain varieties of left loops.

### 4. Varieties of Left Loops

We begin by reviewing the definitions and properties of the varieties we will consider. Let \((B, \cdot)\) be a left loop. \((B, \cdot)\) is said to satisfy the left inverse property (LIP) if
\[(4.1) \quad L_x^{-1} = L_{x^p}\]
for all \(x \in B\). This implies \(x^p\) is a (unique) two-sided inverse of \(x \in B\). \((B, \cdot)\) is said to satisfy the left alternative property (LAP) if
\[(4.2) \quad L_x L_x = L_{x \cdot x}\]
for all \(x \in B\). \((B, \cdot)\) is said to be a (left) Bol loop if
\[(4.3) \quad L_x L_y L_x = L_{x \cdot (y \cdot x)}\]
for \(x, y \in B\). A Bol loop satisfies LIP (take \(x = y^p\) in (4.3)), LAP (take \(y = 1\) in (4.3)), and is also a right loop [33]. If \(B\) is a Bol loop, then for all \(x, y \in B\), \(L(x, y)\) is a pseudo-automorphism with companion \((x \cdot y) \cdot (x^p \cdot y^p)\) [12]. Thus every Bol loop has the pseudo-\(A_1\) property. An \(A_1\) Bol loop is sometimes called a “gyrogroup”, as defined by Ungar in [40]. The equivalence of gyrogroups and \(A_1\) Bol loops was noted by Ròzga [22].
There is an interesting intermediate variety of left loops which is defined by the following identity: for all \(x, y \in B\),
\[
L_x L_{y(y)} L_x = L_{x(\langle y \cdot y \rangle \cdot x)}.
\]
(4.4)

Every such left loop has LAP (take \(x = 1\)) and every Bol loop clearly satisfies (4.4). If \((B, \cdot)\) satisfies (4.4) and the squaring mapping \(x \mapsto x^2\) is surjective, then \((B, \cdot)\) is a Bol loop.

A left loop \((B, \cdot)\) is said to satisfy the automorphic inverse property (AIP) if
\[
(x \cdot y)^\rho = x^\rho \cdot y^\rho
\]
for all \(x, y \in B\), or simply \(\rho L_x = L_{x^\rho} \rho\), where \(\rho : x \mapsto x^\rho\). If \(\rho\) is a permutation on \(B\) with inverse \(\lambda : x \mapsto x^\lambda\), then AIP is characterized by \(\rho L_x \lambda \in L(B)\) for all \(x \in B\). (Indeed, if \(\rho L_x \lambda\) is a translation, then applying it to 1, we see that \(\rho L_x \lambda = L_{x^\lambda}\).) An \(A_l\) LIP, AIP left loop is called a Kikkawa left loop [8].

An identity closely related to AIP is
\[
L_x L_y L_{y(y)} L_x = L_{(x,y)} L_{(x,y)}
\]
(4.6)
for all \(x, y \in B\). Applying both sides to 1, we have
\[
x \cdot (y \cdot (y \cdot x)) = (x \cdot y) \cdot (x \cdot y)
\]
(4.7)
for all \(x, y \in B\). Taking \(y = x^\rho\), we obtain \(x^\rho = x^\rho \cdot (x^\rho \cdot x)\). Cancelling, we have that every element of \(B\) has a two-sided inverse.

The relationship between AIP, (4.6) and our other identities is summarized in the following result.

**Theorem 4.1.** Let \((B, \cdot)\) be a left loop.

1. If \((B, \cdot)\) satisfies (4.6), then LIP and AIP are equivalent.
2. If \((B, \cdot)\) is a Kikkawa left loop, then (4.4) holds.
3. If \((B, \cdot)\) satisfies LAP and (4.4), then (4.6) and (4.4) are equivalent.
4. If \((B, \cdot)\) is a Bol loop, then AIP and (4.4) are equivalent.

**Proof.** 1. Applying both sides of (4.6) to \(x \cdot y^\rho\), we have \(x \cdot y = (x \cdot y) \cdot ((x \cdot y) \cdot (x \cdot y^\rho))\). Cancelling, we obtain \((x \cdot y)^\rho = x \cdot y^\rho\) for all \(x, y \in B\), or equivalently, \(\rho L_x = L_{x^{-1}}^\rho\) for all \(x \in B\). If LIP holds, then \(\rho L_x = L_{x^\rho} \rho\) for all \(x \in B\), which is AIP. Conversely, if AIP holds, then \(L_{x^\rho} \rho = \rho L_x = L_{x^\rho}^{-1}\). Cancelling \(\rho\), we obtain LIP.

2. See [20], Prop. 1.13.
3. Using LAP, (4.7), and LAP again, we have
\[
L_x y L_{x(y)} = L_{x(y)} (x \cdot y) = L_{x(y)} (y \cdot x) = L_{x((y \cdot y) \cdot x)}
\]
for all \(x, y \in B\). From this the equivalence is clear.

4. From the remarks following (4.3), we have that in a Bol loop satisfying AIP, each left inner mapping \(L(x, y)\) is an automorphism, i.e., the \(A_l\) property holds. By (2), (4.6) holds. The converse follows from (1), since every Bol loop has LIP.

For a related discussion, see also [8], especially pp. 37-38 and 58-59. The implication “AIP implies LIP” in Theorem 4.1(1) seems to be new, as is the connection with the variety of left loops satisfying (4.4).

A Bol loop satisfying AIP or (4.6) is called a Bruck loop. This was a term coined by Robinson in his dissertation [31] under the additional assumption that the mapping \(x \mapsto x \cdot x\) is a permutation. The term acquired its generally accepted present...
meaning due to a remark of Glauberman ([11], p.376), who also coined the term B-loop to describe finite, odd order, Bruck loops. The term “B-loop” is now used for the general (not necessarily finite) case in which squaring is a permutation. Bruck loops are also known as “K-loops” [17] [18] [19] [23] [37] and as “(gyrocommutative) gyrogroups” [23] [30] [31]. The direct equivalence between Bruck loops and K-loops was shown by Kreuzer [23]. The direct equivalence between Bruck loops and gyrocommutative gyrogroups was shown by Sabinin et al [34], and was also noted by Rózga [32]. (The direct equivalence of K-loops and gyrocommutative gyrogroups was a well-known folk result.)

Let \( G = BH \) be a transversal decomposition. In the following discussion we will abbreviate the core of \( H \) in \( G \) by \( N = \text{core}_G(H) \). For \( B \subseteq G \), let \( B^2 = \{ x^2 : x \in B \} \) and let \( B^{-1} = \{ x^{-1} : x \in B \} \). We introduce the following conditions on the transversal \( B \).

\[
\begin{align*}
(G-LIP) & \quad B^{-1} \subseteq BN. \\
(G-LAP) & \quad B^2 \subseteq BN. \\
(G-Bol) & \quad \text{For all } x \in B, \ xBx \subseteq BN. \\
(G-W) & \quad \text{For all } x \in B, \ xB^2x \subseteq BN. \\
(G-Br) & \quad \text{For all } x, y \in B, \ xy^2x \in (x \cdot y)^2N.
\end{align*}
\]

In the following result, we show that these transversal conditions characterize the aforementioned left loop varieties. In this respect, our approach is sharper than that of other authors [33] [24]. For instance, it is well-known that the property \( B^{-1} \subseteq B \) implies that the left loop \((B, \cdot)\) has LIP. However, this characterizes LIP only in decompositions \( G = BH \) in which \( H \) is corefree.

Recall the notation \( B_N = \{ xN : x \in B \} \) and that the left loop \((B_N, \cdot_N)\) is an isomorphic copy of \((B, \cdot)\).

**Theorem 4.2.** Let \( G = BH \) be a transversal decomposition. In each of the following assertions \((i), \ (ii), \text{ and } (iii)\) are equivalent.

1. \((i) \ (G-LIP); \ (ii) \ (G/N-LIP); \ (iii) \ (B, \cdot) \text{ satisfies LIP.}\)
2. \((i) \ (G-LAP); \ (ii) \ (G/N-LAP); \ (iii) \ (B, \cdot) \text{ satisfies LAP.}\)
3. \((i) \ (G-Bol); \ (ii) \ (G/N-Bol); \ (iii) \ (B, \cdot) \text{ is a Bol loop.}\)
4. \((i) \ (G-W); \ (ii) \ (G/N-W); \ (iii) \ (B, \cdot) \text{ satisfies (4.4).}\)
5. \((i) \ (G-Br); \ (ii) \ (G/N-Br); \ (iii) \ (B, \cdot) \text{ satisfies (4.4).}\)

**Proof.** 1. For all \( x \in B \), we have \( L_xL_{x^\rho} = \sigma_{x\rho}x^\rho = \sigma_{x\rho} \cdot x \). By (4.1), \((B, \cdot)\) has LIP if and only if, for each \( x \in B \), \( \sigma_{x\rho} = I \), or equivalently, \( x_\rho \in N \). This is equivalent to \( x^{-1} \in x^\rho N \). Since \( x^{-1}H = x^\rho H \), LIP holds if and only if, for each \( x \in B \), \( x^{-1} \in BN \), which is (G-LIP), or equivalently, \( x^{-1}N \in B_N \), which is (G-N-LIP).

2. For all \( x \in B \), we have \( L_{(x^{-1})}L_xL_x = \sigma_{(x^{-1})}x^{-1} \). By (4.2), \((B, \cdot)\) has LAP if and only if, for each \( x \in B \), \( \sigma_{(x^{-1})} = I \), or equivalently, \( (x \cdot x)^{-1}x^{-1}N \in N \), or \( x^2 \in (x \cdot x)N \). Since \( x^{-1}H = (x \cdot x)H \), LAP holds if and only if, for each \( x \in B \), \( x^2 \in BN \), which is (G-LAP), or equivalently, \( x^2N \in B_N \), which is (G-N-LAP).

3. For all \( x, y \in B \), we have \( L_{(x \cdot y \cdot x)}L_xL_yL_x = \sigma_{(x \cdot y \cdot x)}x^{-1}y \). By (4.3), \((B, \cdot)\) is a Bol loop if and only if, for every \( x, y \in B \), \( \sigma_{(x \cdot y \cdot x)} = I \), or equivalently, \( (x \cdot (y \cdot x))^{-1}x^{-1}y \in N \), or \( xy \in (x \cdot (y \cdot x))N \). Since \( xyH = (x \cdot (y \cdot x))H \), \((B, \cdot)\) is a Bol loop if and only if, for every \( x, y \in B \), \( xy \in BN \), which is (G-Bol), or equivalently, for every \( x, y \in B \), \((xN)(yN)(xN) = x(yN)x \in B_N \), which is (G/N-Bol).

4. The proof is similar to that of (3), \textit{mutatis mutandis}. 

5. For all \( x, y \in B \), we have \( L^{-1}_{(x\cdot y)}L^{-1}_{(x\cdot y)}L_xL_yL_x = \sigma_{(x\cdot y)}^{-1}x^2y^2x \). Thus \((B, \cdot)\) satisfies (4.6) if and only if, for every \( x, y \in B \), \((x\cdot y)^{-1}x^2y^2x \in N \), or \( xy^2x \in (x\cdot y)^2N \), which is \((G\cdot Br)\), or equivalently, \((xN)(yN)^2(xN) = (xN \cdot N \cdot yN)^2 \), which is \((G/N\cdot Br)\).

**Corollary 4.3.** Let \((B, \cdot)\) be a left loop.

1. \((B, \cdot)\) satisfies LIP if and only if, for all \( x \in B \), \( L_x^{-1} \in L(B) \).
2. \((B, \cdot)\) satisfies LAP if and only if, for all \( x \in B \), \( L^2 \in L(B) \).
3. \((B, \cdot)\) is a Bol loop if and only if, for all \( x, b \in B \), \( L_xL_y L_x \in L(B) \).
4. \((B, \cdot)\) is satisfies (4.4) if and only if, for all \( x, y \in B \), \( L_xL_y L_x \in L(B) \).

**Remark 4.4.** 1. Other varieties of left loops may be characterized in terms of internal semidirect product structure. For instance, a left loop \((B, \cdot)\) said to have the LC property if \( L_xL_y L_y = L_y(x\cdot y) \) for all \( x, y \in B \). If \( G = BH \) is a transversal decomposition and \( N = \text{core}_G(H) \), then \((B, \cdot)\) is an LC left loop if and only if \( x^2B \subseteq BN \) for all \( x \in B \). Similarly, a left loop \((B, \cdot)\) is said to be left conjugacy closed if \( L_xL_y L_x^{-1} \in L(B) \) for all \( x, y \in B \) [26]. If \( G = BH \) is a transversal decomposition, then \((B, \cdot)\) is left conjugacy closed if and only if \( xBx^{-1} \subseteq BN \) for all \( x \in B \). The proofs of both of these assertions are similar to that of Theorem 4.2(3), mutatis mutandis.

2. Let \( G \) be a group. A subset \( B \subseteq G \) is said to be a twisted subgroup of \( G \) if \( 1 \in B \) and if \( xBx^{-1} \subseteq B \) for all \( x \in B \) [26]. In this jargon, we can restate Theorem 4.2(3) as follows: If \( G = BH \) is a transversal decomposition with \( H \) corefree, then \( B \) is a twisted subgroup if and only if \((B, \cdot)\) is a Bol loop.

3. Let \( G = BH \) with \( H \) corefree. Assume \( G \) satisfies \((G\cdot Bol)\) and \((G\cdot Br)\). By Theorem 4.2(3),(5), \((B, \cdot)\) is a Bruck loop. If the squaring mapping \( B \to B : x \mapsto x^2 \) is a permutation, then \((B, \cdot)\) is a B-loop, and the loop operation in \( B \) is given by

\[
(4.8) \quad x \cdot y = (xy^2x)^{1/2},
\]

\( x, y \in B \). It is interesting to compare the operation (4.8) in \( B \) with the operation discussed by Glauberman [11] and Kikkawa [20] (Ex. 1.5), namely, \( x \ast y = x^{1/2}yx^{1/2} \). Observe that \( x \cdot y = (x^2 \ast y^2)^{1/2} \). The operation \( \cdot \) of the present paper is more natural with respect to the group structure \( G = BH \).

See also [18], pp. 60-62.

The significance of Theorem 4.2 is that in the study of these particular varieties of left loops via internal semidirect products, there is no loss in assuming that the subgroup \( H \) is corefree. In particular, for simply proving facts about abstract left loops, one can always work with the transversal \( L(B) \) in the left multiplication group \( \text{LMLt}(B, \cdot) \). On the other hand, for examples it is frequently the case that the group \( G \) and the transversal decomposition \( G = BH \) are given. In this instance it is preferable to work directly with the group \( G \).

Let \( G = BH \) be a transversal decomposition, and again let \( N = \text{core}_G(H) \). We now consider conditions on the interaction between \( B \) and \( H \).

- \((G\cdot PsA_i)\) For each \( h \in H \), \( chBh^{-1} \subseteq BN \) for some \( c \in B \).
- \((G\cdot A_i)\) For all \( h \in H \), \( hBh^{-1} \subseteq BN \).
Lemma 4.5. Let $G = BH$ be a transversal decomposition, and let $h \in H$ be given. In each of the following, (i), (ii), and (iii) are equivalent.

1. (i) $\sigma_h \in \text{PsAut}(B, \cdot)$; (ii) $chBh^{-1} \subseteq BN$ for some $c \in B$; (iii) for some $c \in B$, $l(c, \sigma_h(x))m(x, h) \in N$ for all $x \in B$.

2. (i) $\sigma_h \in \text{Aut}(B, \cdot)$; (ii) $hBh^{-1} \subseteq BN$; (iii) $m(x, h) \in N$ for all $x \in B$.

Proof. 1. For $x, c \in B$, we have

$$chxh^{-1} = c\sigma_h(x)m(x, h) = (c \cdot \sigma_h(x))l(c, \sigma_h(x))m(x, h).$$

Thus the equivalence of (ii) and (iii) is clear. For $x, y, c \in B$, we have

$$(c \cdot \sigma_h(x \cdot y))H = chyH = chx^{-1}hyH = chx^{-1}\sigma_h(y)H,$$

and

$$(c \cdot \sigma_h(x)) \cdot \sigma_h(y)H = (c \cdot \sigma_h(x))\sigma_h(y)H.$$
Proof. 1. Assume $\rho$ is injective and $\tau(xh) = \tau(yk)$ for some $x, y \in B$, $h, k \in H$. Then $x^\rho l(x, x^\rho)^{-1}h = y^\rho l(y, y^\rho)^{-1}k$, using (3.7). Matching $B$ and $H$ components, this is equivalent to $x^\rho = y^\rho$ and $l(x, x^\rho)^{-1}h = l(y, y^\rho)^{-1}k$. Now $x^\rho = y^\rho$ implies $x = y$, and cancelling $l(x, x^\rho)^{-1} = l(y, y^\rho)^{-1}$, we have $h = k$. Thus $\tau$ is injective. Conversely, assume $\tau$ is injective and $x^\rho = y^\rho$. Then using (3.7) again,

$$\tau(xl(x, x^\rho)) = x^{-1}l(x, x^\rho) = x^\rho = y^\rho = y^{-1}l(y, y^\rho) = \tau(yl(y, y^\rho)).$$

Thus $xl(x, x^\rho) = yl(y, y^\rho)$. Matching components, $x = y$.

2. $\tau$ is surjective if and only if, for each $a \in B$, there exists $x \in B$ and $h \in H$ such that $a = \tau(xh) = x^{-1}h$, i.e., $xa = h \in H$. This implies $x \cdot a = e$, and conversely, if $x \cdot a = e$, then $xa = h$ for some $h \in H$. This establishes the desired equivalence.

3. This follows from (1) and (2).

4. Using (3.7), we have for all $x \in B$,

$$\tau^2(x) = \tau(x^{-1}) = \tau(x^\rho l(x, x^\rho)^{-1}) = x^\rho l(x^\rho, x^\rho)^{-1}l(x, x^\rho)^{-1}.$$

Now

$$l(x^\rho, x^\rho)^{-1}l(x, x^\rho)^{-1} = (l((x, x^\rho)l(x^\rho, x^\rho)^{-1} = (x^\rho x^\rho x^\rho)l^{-1} = x^\rho (x^\rho x^\rho)^{-1} = x.$$

If $\tau^2(x) = x$, then matching components gives $x = x^\rho$ (i.e., $x^\rho$ is a two-sided inverse of $x$), and $xx^\rho x^\rho x = x$. Conversely, if $xx^\rho x^\rho x = e$ for all $x \in B$, then $\tau^2(x) = x^\rho (x^\rho x^\rho)^{-1} = x^\rho (x^\rho x^\rho)^{-1} = x$.

Remark 4.8. If $G = BH$ is a transversal decomposition, then obviously (G-LIP) holds if and only if $\tau(B) \subseteq BN$. In previous studies of the mapping $\tau$ (e.g., [17], [20]), (G-LIP) was assumed at the outset. Theorem 4.7 shows that $\tau$ contains information about $(B, \cdot)$ even under much weaker conditions.

Now assume that $\tau : G \to G$ is a semi-automorphism, i.e.,

$$\tau(g_1 g_2 g_1) = \tau(g_1) \tau(g_2) \tau(g_1)$$

for all $g_1, g_2 \in G$. For $x \in B$, we have

$$x^{-1} = \tau(x) = \tau((xx^{-1}x) = \tau(x)\tau(x^{-1})\tau(x) = x^{-1}(x^{-1})x^{-1}.$$

Cancelling, we obtain $\tau^2(x) = \tau(x^{-1}) = x$. By Theorem 4.7(4), each element of $(B, \cdot)$ has a two-sided inverse. Now for $x, y \in B$, we compute

$$x^{-1}y^{-1}x^{-1} = \tau(x)\tau(y)\tau(x) = \tau((xy)(x^{-1}y^{-1})l(x, y \cdot x)l(y, x)) = (x \cdot (y \cdot x))^{-1}l(x, y \cdot x)l(y, x) = (x \cdot (y \cdot x))^{-2}xy.$$

Thus

$$\tau(xy)^2 = (x \cdot (y \cdot x))^2.$$

Proposition 4.9. Let $G$ be such that the squaring mapping $g \dashv g^2$ is injective. If $\tau : G \to G$ is a semi-automorphism, then $(B, \cdot)$ is a Bol loop.

Proof. From (1.1), we have that $xy = x \cdot (y \cdot x) \in B$ for all $x, y \in B$. This implies (G-Bol), and the result follows from Theorem 4.2(3).

Remark 4.10. Let $G = BH$ be a transversal decomposition with $N = \text{core}_{G}(H)$. Clearly (G-Bol) and (G-A1) hold if and only if $gB\tau(g)^{-1} \subseteq BN$ for all $g \in G$.

Finally, we characterize those groups for which $\tau$ is an automorphism.
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Theorem 4.11. Let $G = BH$ be a transversal decomposition.

1. $\tau : G \rightarrow G$ is an automorphism, then for all $x, y \in B$,

\begin{equation}
xy^2x = (x \cdot y)^2
\end{equation}

and for all $y \in B$, $h \in H$,

\begin{equation}
\sigma_h(y)^2 = h y^2 h^{-1}.
\end{equation}

2. If $H$ is corefree and $(G-A_l)$ holds, then $\tau$ is an automorphism if and only if \( \frac{\sigma}{\mu} \) holds.

3. If $G = (B)$, then $\tau$ is an automorphism if and only if \( \frac{\sigma}{\mu} \) holds.

Proof. 1. For $x, y \in B$, $h \in H$, we compute

$$
\tau(xhyk) = \tau((x \cdot \sigma_h(y))[(x \cdot \sigma_h(y))^{-1}xhyk]) = (x \cdot \sigma_h(y))^{-2}xhyk
$$

and

$$
\tau(xh)\tau(yk) = x^{-1}hy^{-1}k.
$$

Thus $\tau$ is an automorphism if and only if

\begin{equation}
(x \cdot \sigma_h(y))^2 = x y h^2 h^{-1} x
\end{equation}

for all $x, y \in B$, $h \in H$. Taking $h = 1$ gives \( \frac{\sigma}{\mu} \). Taking $x = 1$ gives \( \frac{\sigma}{\mu} \).

Conversely, \( \frac{\sigma}{\mu} \) and \( \frac{\sigma}{\mu} \) clearly imply \( \frac{\sigma}{\mu} \).

2. If $H$ is corefree and $(G-A_l)$ holds, then \( \frac{\sigma}{\mu} \) holds, so the result follows from (1).

3. Assume $G = (B)$. We will show that \( \frac{\sigma}{\mu} \) implies \( \frac{\sigma}{\mu} \). First note that \( \frac{\sigma}{\mu} \) implies $y^2 = (x \cdot (x \backslash y)^2) = x(x \backslash y)^2 x$, or

\begin{equation}
x^{-1} y^2 x^{-1} = (x \backslash y)^2
\end{equation}

for all $x, y \in B$. Fix $h \in H$. Then $h = a_1 \cdot a_2 \cdot \cdots \cdot a_n \cdot \sigma_n \cdot \cdots \cdot \sigma_2 \cdot \sigma_1$ for some $a_i \in B$, where $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$. Using \( \frac{\sigma}{\mu} \) and \( \frac{\sigma}{\mu} \), we have for $y \in B$,

$$
hy^2h^{-1} = a_{\varepsilon_1}^1 \cdot a_{\varepsilon_2}^2 \cdot y \cdot a_{\varepsilon_n}^n \cdots a_{\varepsilon_1}^1 = (a_{\varepsilon_1} \cdot a_{\varepsilon_2} \cdots a_{\varepsilon_n} \cdot y) \cdots
$$

where $\varepsilon_i = \varepsilon$ if $\varepsilon_i = 1$ and $\varepsilon_i = -1$. Thus

$$
hy^2h^{-1} = \sigma_{a_1}^1 \cdot \sigma_{a_2}^2 \cdots \sigma_{a_n}^n(y^2) = \sigma_{a_1}^1 \cdot \cdots \sigma_{a_n}^n(y^2) = \sigma_h(y^2).
$$

This completes the proof.

Note that \( \frac{\sigma}{\mu} \) implies $(G-Br)$, and reduces to $(G-Br)$ if $H$ is corefree.

Corollary 4.12. Let $(B, \cdot)$ be a left loop, let $G = \text{LMlt}(B, \cdot)$, and define $\tau : G \rightarrow G$ by $\tau(L_x \phi) = L_x^{-1} \phi$ (for $x \in B, \phi \in \text{LMlt}(B, \cdot)$).

1. $(B, \cdot)$ satisfies \( \frac{\sigma}{\mu} \) if and only $\tau$ is an automorphism.

2. If the squaring mapping $G \rightarrow G : g \mapsto g^2$ is injective, then $(B, \cdot)$ is a Bruck loop if and only if $\tau$ is an automorphism.

Proof. 1. This follows from Proposition 3.10, Theorem 4.11(2), and Theorem 4.2(5).

2. This follows from (1) and Proposition 4.9.

Remark 4.13. Automorphisms (or, equivalently, anti-automorphisms of order 2) have been used by various authors to study Bruck loops. See, for instance, [9], [11], [14], [17], [20]. Theorem 4.11 and its corollary clarify the exact relationship between $\tau$ and \( \frac{\sigma}{\mu} \).
We now consider some examples of transversal decompositions giving internal semidirect products.

**Example 4.14. (Polar Decomposition)** Let \( GL(n, \mathbb{C}) \) denote the general linear group of \( n \times n \) complex invertible matrices, let \( \mathcal{P}(n) \) denote the subset of all \( n \times n \) positive definite Hermitian matrices, and let \( U(n) \) denote the subgroup of unitary \( n \times n \) complex matrices. The polar decomposition asserts that every \( M \in GL(n, \mathbb{C}) \) can be uniquely factored as \( M = AU \) for a unique \( A = (MM^*)^{1/2} \in \mathcal{P}(n) \) and \( U = (MM^*)^{-1/2}A \in U(n) \), where \( M^* \) is the conjugate transpose of \( M \) and where the unique positive definite square root of \( MM^* \) is intended. Thus the polar decomposition is a transversal decomposition

\[
GL(n, \mathbb{C}) = \mathcal{P}(n) \cdot U(n).
\]

The induced binary operation \((\text{3.3})\), denoted here by \( \odot \), is given by

\[
A \odot B = (AB(AB)^*)^{1/2} = (AB^2A)^{1/2}
\]

for \( A, B \in \mathcal{P}(n) \); compare with \((4.17)\). The transversal mapping \( l : \mathcal{P}(n) \times \mathcal{P}(n) \to U(n) \) is given by

\[
l(A, B) = (AB^2A)^{-1/2}AB
\]

for \( A, B \in \mathcal{P}(n) \). We have \( \mathcal{P}(n)^{-1} \subseteq \mathcal{P}(n) \), i.e., (G-LIP) holds. The involution \( \tau : GL(n, \mathbb{C}) \to GL(n, \mathbb{C}) : AU \mapsto A^{-1}U \) is given explicitly by

\[
\tau(M) = \tau((MM^*)^{1/2}(MM^*)^{-1/2}M)
= (MM^*)^{-1/2}(MM^*)^{-1/2}M
= (MM^*)^{-1}M
= (M^*)^{-1}
\]

for \( M \in GL(n, \mathbb{C}) \). For \( A \in \mathcal{P}(n) \), \( M \in GL(n, \mathbb{C}) \), we have \( MA\tau(M)^{-1} = MAM^* \in \mathcal{P}(n) \). By Remark \((4.10)\), (G-Bol) and (G-A) hold. Since \( \tau \) is the composition of two involutory anti-automorphisms (conjugate transposition and inversion), \( \tau \) is an automorphism. By Theorem \((4.11)\), (G-Br) holds. In addition, the squaring mapping is a permutation of \( \mathcal{P}(n) \). Putting all these facts together, we have that \( (\mathcal{P}(n), \oplus) \) is a B-loop.

**Example 4.15. (Subgroups of \( GL(n, \mathbb{C}) \))** Subgroups of \( GL(n, \mathbb{C}) \) respecting the polar decomposition include the special linear group \( SL(n, \mathbb{C}) \), the real general linear group \( GL(n, \mathbb{R}) \), the group \( U(m, n) \), and the complex symplectic group \( Sp(n, \mathbb{C}) \). Taking intersections of these yields more such groups. Here we limit ourselves to one specific example: The group \( SU(1, 1) \) consists of those \( 2 \times 2 \) complex matrices preserving the form \( |z_1|^2 - |z_2|^2 \) on \( \mathbb{C}^2 \) which also have determinant 1. The polar decomposition of this group is

\[
SU(1, 1) = \mathcal{P}U(1, 1) \cdot S(U(1) \times U(1)).
\]

where \( S(U(1) \times U(1)) \) is the subgroup of matrices of the form \( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \), \( a \in S^1 \), and \( \mathcal{P}U(1, 1) = \mathcal{P}(n) \cap U(1, 1) \) is the set of positive definite Hermitian matrices in \( U(1, 1) \) (such matrices necessarily have determinant 1). Thus \( (\mathcal{P}U(1, 1), \oplus) \) is a subloop of the B-loop \( (\mathcal{P}(2), \oplus) \) of Example \((4.14)\). Note that the mapping \( Q : \)
$S^1 \to S(U(1) \times U(1))$ defined by $Q(a) = \begin{pmatrix} \bar{a} & 0 \\ 0 & a \end{pmatrix}$ is an isomorphism of groups. A matrix $L$ in $PU(1, 1)$ can be parametrized by a number $z \in \mathbb{D}$ as follows:

$$L = L(z) = \gamma_z \begin{pmatrix} 1 & z \\ z & 1 \end{pmatrix}$$

(4.19)

where $\gamma_z = (1 - |z|^2)^{-1/2}$. The mapping $L : \mathbb{D} \to PU(1, 1)$ turns out to be an isomorphism from the $B$-loop $(\mathbb{D}, \oplus)$ of Example 2.7 to $(PU(1, 1), \oplus)$.

Remark 4.16. The polar decompositions of Examples 4.14 and 4.15 are special cases of the global Cartan decomposition of a Lie group associated with a Riemannian symmetric space of noncompact type [13]. Any such space, realized as a subset of the Lie group, can be given the structure of a $B$-loop; this actually follows quite easily from the results herein. For the Hermitian case (bounded symmetric domains), the result was shown in [10], while the general Riemannian case was worked out in [22] and [32]. For related work over Pythagorean fields, see [19].

Example 4.17. We will use the polar decomposition of the group $U(1, 1)$ to construct a different transversal decomposition, leading to a different internal semidirect product structure. The polar decomposition of a given $A \in U(1, 1)$ is $A = L(z)V(a, b)$, with $L(z) \in PU(1, 1)$ given by (4.19) for some $z \in \mathbb{D}$ and $V(a, b) = \begin{pmatrix} \bar{a} & 0 \\ 0 & b \end{pmatrix}$ for some $a, b \in S^1$. Let

$$R(a, z) = \bar{a}L(z)$$

$$T(ab) = aV(a, b).$$

Then for each $A \in U(1, 1)$, there exists $z \in \mathbb{D}$, $a, c \in S^1$ such that

$$A = R(a, z)T(c) = \bar{a}\gamma_z \begin{pmatrix} 1 & \bar{c} \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}.$$ 

Set

$$S^1 \cdot PU(1, 1) = \{R(a, z) : a \in S^1, z \in \mathbb{D}\}$$

and note that $T(c) \in \{1\} \times U(1)$. Then we have shown that

$$U(1, 1) = (S^1 \cdot PU(1, 1)) (\{1\} \times U(1))$$

is a transversal decomposition; its uniqueness follows from the uniqueness of the polar decomposition. It is easy to show that the induced binary operation and corresponding transversal mapping on $S^1 \cdot PU(1, 1)$ are given by

$$R(a, z) \circ R(b, w) = R \left( ab \begin{pmatrix} 1 + \bar{z}w \\ 1 + z\bar{w} \end{pmatrix}, z \oplus w \right)$$

and

$$l(R(a, z), R(b, w)) = T \left( \begin{pmatrix} 1 + \bar{z}w \\ 1 + z\bar{w} \end{pmatrix}, z \oplus w \right),$$

respectively, where $\oplus$ is given by (2.14). Straightforward computations give

$$T(a)R(b, z)T(a)^{-1} = R(b, a^2 z)$$

and

$$R(a, z)R(b, w)R(a, z) = R(ab^2 c/|c|, (z \oplus w) \oplus z),$$

where $\oplus$ is given by (2.14).
where \( c = 1 + \bar{z}w + z\bar{w} + |z|^2 \). Thus (G-AI) and (G-Bol) hold, from which it follows that \( (S^1 \cdot \mathcal{P}U(1,1), \circ) \) is an \( A_I \) Bol loop. It is easy to show directly that \( S^1 \cdot \mathcal{P}U(1,1) \) does not satisfy AIP and hence is not a Bruck loop.

The loop \( (S^1 \cdot \mathcal{P}U(1,1), \circ) \) is isomorphic to a loop structure on the set \( H_C = \{ (x_0, x_1) \in \mathbb{C}^2 : |x_0|^2 - |x_1|^2 = 1 \} \). For \( (x_0, x_1)^t, (y_0, y_1)^t \in H_C \), define

\[
\left( \begin{array}{c} x_0 \\ x_1 \end{array} \right) \circ \left( \begin{array}{c} y_0 \\ y_1 \end{array} \right) = \left( \begin{array}{c} x_0 \bar{x}_0 y_0 + \bar{x}_1 y_1 \\ x_0 y_1 + x_1 y_0 \end{array} \right).
\]

Then \( (H_C, \circ) \) is a loop. (This is a simplified version of an example in \([36]\).)

The following sequence of loop homomorphisms is exact:

\[
1 \to S^1 \xrightarrow{\alpha} H_C \xrightarrow{\pi} \mathbb{D} \to 0,
\]

where \( \alpha(z) = (z, 0)^t \) for \( z \in S^1 \) and \( \pi((x_0, x_1)^t) = x_1/x_0 \). In fact, \( H_C \) is a central, invariant extension of \( \mathbb{D} \) by \( S^1 \) \( [32] \). The mapping \( (H_C, \circ) \to (S^1 \cdot \mathcal{P}U(1,1), \circ) : (x_0, x_1) \mapsto R(x_0/|x_0|, x_1/x_0) \) is an isomorphism of loops.

**Example 4.18.** (Projective Groups) Let \( G \leq SL(n, \mathbb{C}) \) be a subgroup respecting the polar decomposition. Let \( B = G \cap \mathcal{P}(n) \) and \( H = G \cap U(n) \). Then \( G = BH \) is a transversal decomposition. The kernel of the conjugation homomorphism \( U \to \sigma_U \), where \( \sigma_U(A) = UAU^* \) for \( A \in B, U \in H \), is the group \( \ker(\sigma) = G \cap \{ eI : c \in \mathbb{C} \} \) of scalar matrices in \( G \). Thus \( PG = G/\ker(\sigma) \) is the projective group associated to \( G \), and similarly define \( PH \). Applying (3.17) to the present setting, we have the (corefree) transversal decomposition

\[
PG = B \cdot PH.
\]

We will refer to this as a **projective polar decomposition**.

As a specific example, consider the Möbius group \( PSU(1,1) \). The projective polar decomposition of this group is

\[
PSU(1,1) = \mathcal{P}U(1,1) \cdot PSU(1) \times U(1).
\]

The only scalar matrices in \( S(U(1) \times U(1)) \) are \( \pm I \), and thus \( PSU(1) \times U(1) = S(U(1) \times U(1))/\{ \pm I \} \). In terms of (3.18) and (3.19), the exact sequence of groups

\[
1 \to \{ \pm I \} \to S(U(1) \times U(1)) \to PSU(1) \times U(1) \to 1
\]

induces the exact sequence of semidirect product groups

\[
1 \to \{ \pm I \} \to SU(1,1) \to PSU(1,1) \to 1.
\]

**Example 4.19.** (Upper Triangular Matrices) Let \( \mathbb{F} \) be a field containing \( 1/2 \). For \( x, y, z \in \mathbb{F} \), let

\[
T(x, y, z) = \begin{pmatrix}
1 & x & y \\
0 & 1 & z \\
0 & 0 & 1
\end{pmatrix},
\]

and let \( \mathcal{T}(3, \mathbb{F}) = \{ T(x, y, z) : x, y, z \in \mathbb{F} \} \) be the group of \( 3 \times 3 \) strictly upper triangular matrices over \( \mathbb{F} \). For \( x_1, x_2 \in \mathbb{F} \), let

\[
A(x_1, x_2) = \begin{pmatrix}
1 & x_1 & \frac{1}{2}x_1x_2 \\
0 & 1 & x_2 \\
0 & 0 & 1
\end{pmatrix},
\]
and let $A(3,\mathbb{F}) = \{A(x_1, x_2) : x_1, x_2 \in \mathbb{F}\}$. For $c \in \mathbb{F}$, let
\[
M(c) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
and let $M(3,\mathbb{F}) = \{M(c) : c \in \mathbb{F}\}$. Then $M(3,\mathbb{F})$ is a subgroup of $\mathcal{T}(3,\mathbb{F})$. An arbitrary matrix $T(x, y, z) \in \mathcal{T}(3,\mathbb{F})$ factors as follows:
\[
T(x, y, z) = A(x, z)M(y - xz/2).
\]
It is easy to show by direct computation that this factorization is unique. In addition, $A(x, z) = M(y - xz/2)$ if and only if $x = y = z = 0$, which implies $A(3,\mathbb{F}) \cap M(3,\mathbb{F}) = \{I\}$. Thus
\[
\mathcal{T}(3,\mathbb{F}) = A(3,\mathbb{F}) \cdot M(3,\mathbb{F})
\]
is a transversal decomposition. Denote the induced binary operation (3.2) on $A(3,\mathbb{F})$ by $\oplus$. Then $\oplus$ is given by
(4.21)
\[A(x_1, x_2) \oplus A(y_1, y_2) = A(x_1 + y_1, x_2 + y_2).\]

It is immediate that $A(3,\mathbb{F})$ is an abelian group isomorphic to $\mathbb{F}^2$. The transversal mapping $l : A(3,\mathbb{F}) \times A(3,\mathbb{F}) \rightarrow M(3,\mathbb{F})$ is given by
(4.22)
\[l(A(x_1, x_2), A(y_1, y_2)) = M((x_1 y_2 - x_2 y_1)/2).
\]

Since this is nontrivial, $A(3,\mathbb{F})$ is not a subgroup of $\mathcal{T}(3,\mathbb{F})$. The subgroup $M(3,\mathbb{F})$ is the center of $\mathcal{T}(3,\mathbb{F})$, and thus trivially normalizes $A(3,\mathbb{F})$. Thus $\mathcal{T}(3,\mathbb{F})$ is an internal semidirect product of the abelian group $(A(3,\mathbb{F}), \oplus)$ with the abelian group $M(3,\mathbb{F})$. However, as noted in Remark 3.8(4), this is not the usual internal semidirect product of groups.

This example turns out to generalize to $\mathcal{T}(n,\mathbb{F})$ for any $n \geq 3$. For $n > 3$, $(A(n,\mathbb{F}), \oplus)$ is a loop but not a group, and the homomorphism $\sigma : M(n,\mathbb{F}) \rightarrow \text{Aut}(A(n,\mathbb{F}))$ is nontrivial.

**Remark 4.20.** Examples 4.18 and 4.19 are counterexamples to a claim of Sabinin (3.8, Thm. 8) that (in the notation and terminology of the present paper) $\sigma : H \rightarrow \text{Sym}_1(B)$ is always injective.

## 5. External Semidirect Products

In this section we generalize the standard semidirect product $B \rtimes H$ to the case where $H$ is not necessarily a transassociant of $\text{Sym}_1(B)$, but rather there is a homomorphism $\sigma : H \rightarrow \text{Sym}_1(B)$. Our discussion will show that, in a certain sense, our definition of external semidirect product is the optimal one.

Let $B$ be a set with a distinguished element $1$ and let $H$ be a group with identity element $e$. Assume that $B \times H$ has a binary operation which makes it a group satisfying the following properties:

\begin{enumerate}[(E1)]
\item $\{1\} \times H$ is a subgroup isomorphic to $H$.
\item $(x, h) = (x, 1)(1, h)$ for all $x \in B$, $h \in H$.
\end{enumerate}

Then $B \times H = (B \times \{1\}) \{(1) \times H\}$ is a transversal decomposition. Indeed, by (E2), $(x, e)(\{1\} \times H) = \{(x, h) : h \in H\}$ for all $x \in B$, which implies that $B \times \{e\}$ is a transversal. Making the usual identifications $B \cong B \times \{1\}$ and $H \cong \{1\} \times H$, we have
an induced operation $\cdot$ on $B$ and an induced transversal mapping $l : B \times B \to H$, both defined by

$$(x, 1)(y, 1) = (x \cdot y, l(x, y))$$

for $x, y \in B$. We also have a mapping $m : B \times H \to H$ and (what turns out to be) a homomorphism $\sigma : H \to \text{Sym}_1(B)$, both defined by

$$(1, h)(x, e)(1, h^{-1}) = (\sigma_h(x), m(x, h))$$

for $x \in B$, $h \in H$. Thus we see that $\hat{\sigma}$ and $\hat{m}$:

(S1) For all $x, y \in B$,

$$\sigma_{l(x,y)} = L(x, y).$$

(S2) For all $x \in B$, $h \in H$,

$$\sigma_{m(x,h)} = \mu_x(h).$$

Define a binary operation on $B \times H$ by

$$(5.1) \quad (x, h)(y, k) = (x \cdot \sigma_h(y), l(x, \sigma_h(y))m(y, h)hk).$$

The question is thus: what are the minimal additional assumptions necessary for $B \times H$ with the product given by (5.1) to be a group?

If $B \times H$ is a group and both (E1) and (E2) hold, then $B \times H = (B \times \{e\}) \times \{1\} \times H)$ is a transversal decomposition. Thus there is an induced product on $B$, which we will denote by $\cdot$, an induced transversal mapping $\hat{l} : B \times B \to H$, an induced mapping $\hat{m} : B \times H \to H$, and an induced homomorphism $\hat{\sigma} : H \to \text{Sym}_1(B)$. Using (5.1), (E1) and (E2), we compute

$$(x \cdot y, \hat{l}(x, y)) = (x, e)(y, e)$$

and

$$(\hat{\sigma}_h(x), \hat{m}(x, h)) = (1, h)(x, e)(1, h^{-1})$$

and

$$(5.2) \quad \hat{l}(x, y) = l(x, y)m(y, e)$$

for all $x, y \in B$, $h \in H$. If we assume that $\hat{l} = l$ and $\hat{m} = m$, then we have the following necessary requirements:
(S3) For all \(x \in B\),
\[ l(1, x) = e. \]

(S4) For all \(x \in B\),
\[ m(x, e) = e. \]

Taking \(y = 1\) in (5.2), applying (3.5) to \(\hat{l}(x, 1)\), and using (S4), we obtain
(S5) For all \(x \in B\),
\[ l(x, e) = e. \]

Taking \(x = 1\) in (5.3), applying (3.14) to \(\hat{m}(1, h)\), and using (S3), we obtain
(S6) For all \(h \in H\),
\[ m(1, h) = e. \]

Next we consider the group axioms which must be satisfied by \(B \times H\). We have
\[(x, h)(1, e) = (x \cdot \sigma_h(1), l(x, \sigma_h(1))m(1, h)h) = (x, h)\]
by (S5) and (S6), and
\[(1, e)(x, h) = (1 \cdot \sigma_e(x), l(1, \sigma_e(x))m(x, e)h) = (x, h)\]
by (S3) and (S4). Therefore, the hypotheses we have so far give us that \((1, e)\) is the identity element of \(B \times H\).

Next, we impose associativity on \(B \times H\). By computing an arbitrary product \((x, h)(y, k)(z, t)\) in two different ways, matching \(H\)-components (matching \(B\)-components gives no new information), and simplifying, we obtain the following technical condition which must be satisfied.

(TC) For all \(x, y, z \in B\), \(h, k \in H\),
\[ l(x \cdot \sigma_h(y), L(x, \sigma_h(y))\mu_y(h)\sigma_{hk}(z))m(z, l(x, \sigma_h(y))m(y, h)h\cdot l(x, \sigma_h(y))m(y, h) = l(x, \sigma_h(y) \cdot \mu_y(h)\sigma_{hk}(z))m(y \cdot \sigma_h(z), h)\cdot l(y, \sigma_h(z))m(z, k)h^{-1}. \]

Fortunately, (TC) can be replaced by three simpler conditions to which it is equivalent. First, taking \(h = k = e\) in (TC) and using (S4), we obtain
(S7) For all \(x, y, z \in B\),
\[ l(x \cdot y, L(x, y)z)m(z, l(x, y))l(x, y) = l(x, y \cdot z)l(y, z). \]

Second, taking \(x = y = 1\) in (TC) and using (S3) and (S6) (and writing \(x\) for \(z\)) we obtain
(S8) For all \(x, y \in B\), \(h, k \in H\),
\[ m(x, hk) = m(\sigma_k(x), h)hm(x, k)h^{-1}. \]

Finally, taking \(x = 1\), \(k = e\) in (TC) and using (S3) and (S4) (and making the replacements \(y \to x\), \(z \to y\)), we obtain
(S9) For all \(x, y \in B\), \(h \in H\),
\[ l(\sigma_h(x), (\mu_x(h)\sigma_h)(y))m(y, m(x, h)h)m(x, h) = m(x \cdot y, h)h\cdot l(x, y)h^{-1}. \]

Thus we have shown one direction of the following.

**Lemma 5.1.** Condition (TC) is equivalent to conditions (S7), (S8) and (S9).
We omit the tedious proof that (S7), (S8) and (S9) imply (TC), except to say that starting with the left hand side of (TC), one can obtain the right hand side by two applications of (S8), then one application of (S7), and then one application of (S9).

Next we consider inverses in \( B \times H \). Consider first an element of the form \((x, e)\). If \((y, h)\) is to be the right inverse of \((a, e)\), then

\[
(1, e) = (x, e)(y, h)
\]

This implies \( y = x^o \), and using (S4), \( h = l(x, x^o)^{-1} \), so that

\[
(5.4) \quad (x, e)^{-1} = (x^o, l(x, x^o)^{-1}).
\]

Now we consider a general element \((x, h) \in B \times H\) and use (E2), (E1), \((5.4)\) and (S3) to derive the inverse:

\[
(x, h)^{-1} = ((x, e)(1, h))^{-1} = (1, h^{-1})(x^o, l(x, x^o)^{-1}) = (\sigma_{h^{-1}}(x^o), l(1, \sigma_{h^{-1}}(x^o))m(x^o, h^{-1})h^{-1}l(x, x^o)^{-1}) = (\sigma_{h^{-1}}(x^o), m(x^o, h^{-1})h^{-1}l(x, x^o)^{-1}).
\]

We check that this candidate is indeed a right inverse:

\[
(x, h)(\sigma_{h^{-1}}(x^o), m(x^o, h^{-1})h^{-1}l(x, x^o)^{-1}) = (x \cdot x^o, l(x, x^o)m(\sigma_{h^{-1}}(x^o), h)m(x^o, h^{-1})h^{-1}l(x, x^o)^{-1}) = (1, e).
\]

By (S8), the \( H \)-component of the last step of this calculation indeed simplifies to 1. Similar computations show that (5.5) is a left inverse, although it is not necessary to check this; a two-sided identity, right inverses and associativity are sufficient for \( B \times H \) to be a group.

**Definition 5.2.** Let \( B \) be a left loop and let \( H \) be a group. Assume there exist a mapping \( l : B \times B \to H \), a mapping \( m : B \times H \to H \), and a homomorphism \( \sigma : H \to \text{Sym}_1(B) \) such that conditions (S1) through (S9) are satisfied. Define a binary operation \( \cdot \) on the set \( B \times H \) by

\[
(x, h) \cdot (y, k) = (x \cdot \sigma_h(y), l(x, \sigma_h(y))m(y, h)hk)
\]

for \( x, y \in B, h, k \in H \). Then \((B \times H, \cdot)\) is called the **external semidirect product** of \( B \) with \( H \) given by \((\sigma, l, m)\), and is denoted \( B \rtimes_{(\sigma, l, m)} H \).

Of course, the whole discussion leading up to this result was a sketch of the proof of the following.

**Theorem 5.3.** \((B \rtimes_{(\sigma, l, m)} H, \cdot)\) is a group.

**Remark 5.4.** As with the internal product, there are various special cases of the external product which are of interest.

1. If the transversal mapping \( l : B \times B \to H \) can be chosen to be trivial, i.e., \( l(x, y) = e \) for all \( x, y \in B \), then \( B \) is a group (by (S1)). In this case, we have an external version of Jajcay’s “rotary product” of groups [13].
2. If the mapping \( m : B \times B \to H \) can be chosen to be trivial, i.e., \( m(x, h) = e \) for all \( x \in B, h \in H \), then \( B \) is an \( A_l \)-loop (by (S2)).
3. If both \( l \) and \( m \) can be chosen to be trivial, then \( B \rtimes_{(\sigma, l, m)} H = B \rtimes_{\sigma} H \) is the usual external semidirect product of groups.

4. If \( \sigma(H) = \{ I \} \), then \( B \) is a group, but if \( l : B \times B \rightarrow H \) is nontrivial, then \( B \) is not (isomorphic to) a subgroup of \( B \rtimes_{(\sigma, l, m)} H \).

Our examples are \( A_l \) Bol loops. Thus we will choose \( m \) to be trivial, so that \((S2), (S4), (S6) \) and \((S8)\) are trivial. \((S7)\) simplifies to

\[(S7') \text{ For all } x, y, z \in B, \ h \in H,\]

\[l(x \cdot y, L(x, y)z)l(x, y) = l(x, y \cdot z)l(y, z).\]

and \((S9)\) simplifies to

\[(S9') \text{ For all } x, y \in B, \ h \in H,\]

\[l(\sigma_h(x), \sigma_h(y)) = h\overline{l(x, y)h^{-1}}.\]

**Example 5.5.** The mapping \( \phi : S^1 \rightarrow S^1 \) defined by \( \phi(a) = a^2 \) is a homomorphism of groups, and we consider the target copy of \( S^1 \) to be a subgroup of \( \text{Aut}(\mathbb{D}, \oplus) \). Define \( l : \mathbb{D} \times \mathbb{D} \rightarrow S^1 \) by

\[l(x, y) = \frac{1 + xy}{|1 + xy|}\]

for \( x, y \in \mathbb{D} \). Then \( \phi(l(x, y)) = L(x, y) = (1 + x\bar{y})/(1 + \bar{x}y) \), as in Example 2.7.

Define \( m : \mathbb{D} \times S^1 \rightarrow S^1 \) trivially. It is easy to check that \((S1), (S3), (S6), (S7')\) and \((S9')\) are satisfied. Thus we have an external semidirect product \( \mathbb{D} \rtimes_{(\phi, l)} S^1 \).

The exact sequence of groups

\[1 \rightarrow \{ \pm 1 \} \rightarrow S^1 \rightarrow \mathbb{D} \rightarrow S^1 \rightarrow 1\]

induces an exact sequence of semidirect product groups

\[1 \rightarrow \{ \pm 1 \} \rightarrow \mathbb{D} \rtimes_{(\phi, l)} S^1 \rightarrow \mathbb{D} \times S^1 \rightarrow 1\]

where \( \hat{\phi}(z, a) = (z, \phi(a)) = (z, a^2) \). We now show the relationship between this construction and Example 4.18. Recall the isomorphism \( Q : S^1 \rightarrow S(U(1) \times U(1)) \) given by \( Q(a) = \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right) \). As the following diagram indicates, there exists an isomorphism \( \hat{Q} : S^1 \rightarrow PS(U(1) \times U(1)) \).

\[
\begin{array}{ccccc}
1 & \rightarrow & \{ \pm 1 \} & \rightarrow & S^1 \\
\downarrow & & \downarrow Q & & \downarrow \hat{Q} \\
1 & \rightarrow & S(U(1) \times U(1)) & \rightarrow & PS(U(1) \times U(1)) & \rightarrow & 1
\end{array}
\]

This is explicitly given by

\[
\hat{Q}(a) = \left\{ \begin{array}{ll}
\left[ Q \left( \frac{1 + a}{|1 + a|} \right) \right] & \text{if } a \neq -1 \\
\left[ Q(i) \right] & \text{if } a = -1
\end{array} \right.
\]

where the square brackets denote equivalence classes. (Note that \((1 + a)/|1 + a|)^2 = a\). On the other hand, the mapping \((z, a) \mapsto P(z)Q(a)\) gives an isomorphism from
$D \times_{\phi} S^1$ to $SU(1,1) = PU(1,1) \cdot S(U(1) \times U(1))$. This and the following diagram yield an isomorphism between $D \times S^1$ and $PSU(1,1)$:

\[
\begin{array}{ccc}
1 & \rightarrow & \{(0, \pm 1)\} \\
\downarrow & & \downarrow_{4pq} \\
1 & \rightarrow & \{\pm I\} \\
\end{array}
\]

The isomorphism is explicitly given by $(z, a) \mapsto P(z)\hat{Q}(a)$.

**Example 5.6.** Let $F$ be a field containing $1/2$. Let $P = F^2$ with the operation of vector addition, and let $H = F$ with the operation of addition. Define $l : F^2 \times F^2 \rightarrow F$

\[
l((x_1, x_2), (y_1, y_2)) = \frac{1}{2}(x_1y_2 - x_2y_1).
\]

Define $\phi : F \rightarrow \text{Aut}(F^2)$ trivially: $\phi(c) = I$. Finally, define $m : F^2 \times \text{Aut}(F^2) \rightarrow \text{Aut}(F^2)$ trivially. Then $(S1)$, $(S3)$, $(S6)$, $(S7')$ and $(S9')$ are satisfied. We thus have an external semidirect product $F^3 = F^2 \times_{(\phi, l)} F$.

The operation is given by

\[
(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1y_2 - x_2y_1)).
\]

As in Remark 5.4(4), this external semidirect product of groups does not reduce to the usual one. Clearly this external semidirect product is isomorphic to the internal semidirect product of Example 4.10.

We conclude with a few remarks about the relationship of the external semidirect product with the standard and internal semidirect products. These naturally generalize the usual relationships between these products of groups. First, if $G = BH$ is an internal semidirect product of a left loop $(B, \cdot)$ with the subgroup $H$, then the mapping $g = xh \mapsto (x, h)$ is clearly an isomorphism of $BH$ with $B \rtimes_{(\sigma, l, m)} H$. On the other hand, by factoring a given external semidirect product as $B \rtimes_{(\sigma, d, m)} H \cong (B \times \{e\})(\{1\} \times H)$, it is easy to see that it is isomorphic to an internal semidirect product; this was essentially our starting point for deriving the definition of external semidirect product. The standard semidirect product is, of course, a special case of the external semidirect product. On the other hand, if $B \rtimes_{(\sigma, l, m)} H$ is an external semidirect product, then the natural mapping $\hat{\sigma} : B \rtimes_{(\sigma, d, m)} H \rightarrow B \rtimes \sigma(H) : (x, h) \mapsto (x, \sigma_h)$ is an epimorphism. We have $\ker(\hat{\sigma}) = \{1\} \times \ker(\sigma)$, and thus the exact sequence of groups

\[
1 \rightarrow \ker(\sigma) \rightarrow H \rightarrow \sigma(H) \rightarrow 1
\]

induces an exact sequence of semidirect product groups

\[
1 \rightarrow \ker(\hat{\sigma}) \rightarrow B \rtimes_{(\sigma, l, m)} H \rightarrow B \rtimes \sigma(H) \rightarrow 1.
\]

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