Bound State Solution of the Klein-Fock-Gordon equation with the Hulthén plus a Ring-Shaped like potential within SUSY quantum mechanics

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Abstract

In this paper, the bound state solution of the modified Klein-Fock-Gordon equation is obtained for the Hulthén plus ring-shaped lake potential by using the developed scheme to overcome the centrifugal part. The energy eigenvalues and corresponding radial and azimuthal wave functions are defined for any $l \neq 0$ angular momentum case on the conditions that scalar potential is whether equal and nonequal to vector potential, the bound state solutions of the Klein-Fock-Gordon equation of the Hulthén plus ring-shaped like potential are obtained by Nikiforov-Uvarov (NU) and supersymmetric quantum mechanics (SUSYQM) methods. The equivalent expressions are obtained for the energy eigenvalues, and the expression of radial wave functions transformations to each other is revealed owing to both methods. The energy levels and the corresponding normalized eigenfunctions are represented in terms of the Jacobi polynomials for arbitrary $l$ states. A closed form of the normalization constant of the wave functions is also found. It is shown that the energy eigenvalues and eigenfunctions are sensitive to $n_r$ radial and $l$ orbital quantum numbers.

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I. INTRODUCTION

Since the early years of quantum mechanics (QM), the study of exactly solvable problems for some special potentials has aroused considerable interest in theoretical physics. In addition, since the wave function contains all necessary knowledge for the full description of a quantum system, so an analytical solution of the wave equations is of quite high significance in quantum mechanics [1, 2].

Since the exact solutions of the Klein-Fock-Gordon (KFG) equation with any potential play an important role in relativistic quantum mechanics [1, 2], there are many discussions about the KFG equation with physical potentials by using different methods. KFG equation is the well known relativistic wave equation that describes spin zero particles, as pseudoscalar pions. For example, the s-wave KFG equation with the vector Hulthén-type potential was treated by standard method [3], the same problem but with both vector and scalar Hulthén-type potentials was later discussed in [4, 5], the scattering state solutions of the s-wave KFG equation with vector and scalar Hulthén potentials are obtained for regular and irregular boundary conditions in [6]. Chetouani et al. successfully solved the Green function for the KFG operator with these two potentials by using the path-integral approach [7].

Many methods were developed and has been used successfully in solving the non-relativistic and relativistic wave equations in the presence of some well known potentials. Such as supersymmetry (SUSY) [8–10], factorization [11], Laplace transform approach [12] and the path integral method [13], shifted 1/N expansion approach [14, 15] for solving radial and azimuthal part of the wave equations exactly or quasi-exactly for \( l \neq 0 \) within different potentials. An other method known as the Nikiforov-Uvarov (NU) method [16] was proposed for solving the wave equations analytically.

In works [5, 17–26], the scalar potential is equal and non-equal to the vector potential have been assumed to obtain the bound states of the KFG equation with some typical potential by using the ordinary quantum mechanics. It is very significant to notice that KFG equation for the Ring-Shaped potential is fully studied in Ref. [23].

In order to give correction for non-relativistic quantum mechanics, the investigation of relativistic wave equations, which is invariant under Lorentz transformation, is required by the description of phenomena at high energies [2].
If we consider the case where the interaction potential is not enough to create particle-antiparticle pairs, we can apply the KFG equation to the treatment of a zero-spin particle and apply the Dirac equation to that of a 1/2-spin particle. When particle is in a strong field, then will interesting to consider the relativistic equations, so we can if possible extract the correction to non-relativistic quantum mechanics. Since it has been extensively used to describe the bound and continuum states of the interacting systems, it would be quite curious and significant investigation to the relativistic bound states of the arbitrary \( l \)-wave KFG equation with Hulthén potential plus a ring-shaped like potential.

The Hulthén potential is one of the important short-range potentials in physics, extensively using to describe the bound and continuum states of the interaction systems. It has been applied to the several research areas such as nuclear and particle physics, atomic physics, condensed matter and chemical physics, so the analyzing relativistic effects for a particle under this potential could become significant, especially for strong coupling. Therefore this problem has attracted a great deal of interests in solving the KFG equation with the Hulthén potential.

The Hulthén potential is defined by \[27, 28\]

\[
V(r) = -\frac{Ze^2\delta e^{-\delta r}}{(1 - e^{-\delta r})} \quad (1.1)
\]

At small values of the radial coordinate \( r \), the Hulthén potential behaves like a Coulomb potential, whereas for large values of \( r \) it decreases exponentially so that its influence for bound state is smaller than, that of Coulomb potential. In contrast to the Hulthén potential, the Coulomb potential is analytically solvable for any \( l \) angular momentum. Take into account of this point will be very interesting and important solving KFG equation for the Hulthén plus ring-shaped like potential for any \( l \) states within ordinary and supersymmetric quantum mechanics.

Unfortunately, for an arbitrary \( l \)-states \((l \neq 0)\), the KFG equation does not get an exact solution due to the centrifugal term. But many research are show the power and simplicity of NU method in solving central and noncentral potentials \[29, 37\] for arbitrary \( l \) states. This method is based on solving the second-order linear differential equation by reducing to a generalized equation of hypergeometric-type which is a second-order type homogeneous differential equation with polynomials coefficients of degree not exceeding the corresponding order of differentiation.
It should be noted that the nature of the radial function at the origin, especially for singular potentials was comprehensively studied by Khelashvili et al. [38–43]. While the Laplace operator is defined in spherical coordinates, the exact derivation of the radial wave equation displays the appearance of a delta function term. As a result, regardless of the behavior potential, the additional constraint is imposed on radial wave function in the form of a vanishing boundary condition at the origin.

The combined potential considering in this study is obtained by adding Hulthén potential term to Ring-Shaped potential as:

\[ V(r, \theta) = \frac{\hbar^2}{2M} \left[ -\frac{2M Ze^2 \delta e^{\delta r}}{\hbar^2 (1 - e^{\delta r})} + \frac{\beta'}{r^2 \sin^2 \theta} + \frac{\beta \cos \theta}{r^2 \sin^2 \theta} \right]. \] (1.2)

The non-central potentials are needed to obtain better results than central potentials about the dynamical properties of the molecular structures and interactions. Ring-shaped potentials can be used in quantum chemistry to describe the ring shaped organic molecules such as benzene and in nuclear physics to investigate the interaction between deformed pair of nucleus and spin orbit coupling for the motion of the particle in the potential fields.

This potential also is used as a mathematical model in the description of diatomic molecular vibrations and it constitutes a convenient model for other physical situations.

Therefore, it would be interesting and important to solve the relativistic radial and azimuthal KFG equation for Hulthén plus ring-shaped like potential for \( l \neq 0 \), since it has been extensively used to describe the bound and continuum states of the interacting systems.

Thus, the main purpose of our investigation is the analytical solution of modified KFG equation for the Hulthén plus ring-shaped potential within ordinary quantum mechanics using Nikiforov-Uvarov (NU) method [16] and in SUSY quantum mechanics the shape invariance concept that was introduced by Gendenshtein [44, 45] by using a novel improved scheme to overcome centrifugal term and found the energy eigenvalues and corresponding radial and azimuthal wave functions for any \( l \) orbital angular momentum case.

The rest of the present work is organized as follows. Bound-state solution of the radial KFG equation for Hulthén potential by NU method within ordinary quantum mechanics is provided in Section II. In Section III, we present the solution of angle-dependent part of the KFG equation. In Section IV we present the solution of KFG equation for Hulthén potential within SUSY quantum mechanics and the numerical results for energy levels and the cor-
responding normalized eigenfunctions are presented in Section [V]. Finally, some concluding remarks are stated in Section [VI].

II. BOUND STATE SOLUTION OF THE RADIAL KLEIN-FOCK-GORDON EQUATION

Two various type potentials can be introduced into this equation because KFG equation contains two objects; the four-vector linear momentum operator and the scalar rest mass. The first one is a vector potential \( V(r, \theta) \), introduced via minimal coupling and the second one is a scalar potential \( S(r, \theta) \) introduced via scalar coupling\([1]\). Hence, they allow one to introduce two types of potential coupling which are the four vector potential \( V \) and the space-time scalar potential \( S \).

In spherical coordinates, the KFG equation with scalar potential \( S(r, \theta) \) and vector potential \( V(r, \theta) \) can be written in the following form in natural units \((\hbar = c = 1)\)

\[
[-\nabla^2 + (M + S(r, \theta))^2] \psi(r, \theta, \phi) = [E - V(r, \theta)]^2 \psi(r, \theta, \phi), \quad (2.1)
\]

where \( E \) is the relativistic energy of the system and \( M \) denotes the rest mass of a scalar particle.

For separation of radial and angular parts of the wave function for the stationary KFG equation with Hulthén plus ring-shaped potential we use following wave function

\[
\psi(r, \theta, \phi) = \frac{\chi(r)}{r} \Theta(\theta) e^{im\phi}, \quad m = 0, \pm 1, \pm 2, \pm 3... \quad (2.2)
\]

and substituting this into Eq.\( (2.1) \) leads to the following second-order differential equations

\[
\chi''(r) + \left[ (E^2 - M^2) - 2(MS(r) + EV(r)) + (V^2(r) - S^2(r)) - \frac{l(l+1)}{r^2} \right] \chi(r) = 0, \quad (2.3)
\]

\[
\Theta''(\theta) + \cot(\theta) \Theta'(\theta) + \left[ -\frac{2}{\sin^2(\theta)} \left( (M + E)(\beta' + \beta cos(\theta)) + m^2 \right) + \lambda \right] \Theta(\theta) = 0. \quad (2.4)
\]

It should be noted that in Eq.\( (2.4) \), the scalar ring-shaped potential is taken to equal with vector potential \( V(r, \theta) = S(r, \theta) \).

If we take vector and scalar potentials as the general Hulthén potential in this form
\[
V(r) = -\frac{V_0 e^{-\delta r}}{(1 - e^{-\delta r})}, \quad S(r) = -\frac{S_0 e^{-\delta r}}{(1 - e^{-\delta r})}, \quad (2.5)
\]

then Eq. (2.3) becomes

\[
\chi''(r) + \left[ (E^2 - M^2) + 2 \left( \frac{(MS_0 + EV_0)e^{-2\delta r}}{1 - e^{-\delta r}} \right) + 2 \left( \frac{(S_0^2 - V_0^2)e^{-2\delta r}}{1 - e^{-\delta r}} \right) - \frac{l(l + 1)}{r^2} \right] \chi(r) = 0,
\quad (2.6)
\]

The effective Hulthén potential is defined in this form:

\[
V_{\text{eff}}(r) = -2 \left( \frac{(MS_0 + EV_0)e^{-\delta r}}{1 - e^{-\delta r}} \right) + 2 \left( \frac{(S_0^2 - V_0^2)e^{-2\delta r}}{1 - e^{-\delta r}} \right) + \frac{l(l + 1)}{r^2}, \quad (2.7)
\]

It is known that for this potential the KFG equation can be solved exactly using suitable approximation scheme to deal with the centrifugal term.

Therefore, in this research study, we attempt to use the following improved approximation scheme to deal with the centrifugal term. In order to solve Eq. (2.6) for \(l \neq 0\), we should make an approximation for the centrifugal term. When \(\delta r \ll 1\), we use an improved approximation scheme \([46–48]\) to deal with the centrifugal term,

\[
\frac{1}{r^2} \approx \delta^2 \left[ C_0 + \frac{e^{-\delta r}}{(1 - e^{-\delta r})^2} \right], \quad (2.8)
\]

where the parameter \(C_0 = \frac{1}{14}\) (Ref. \([49]\)) is a dimensionless constant. However, when \(C_0 = 0\), the approximation scheme becomes the convectional approximation scheme suggested by Greene and Aldrich \([50]\).

Now for applying NU method, Eq. (2.6) should be rewritten as the hypergeometric type equation form presenting below:

\[
\chi''(s) + \frac{\bar{\tau}}{\sigma} \chi'(s) + \frac{\bar{\sigma}}{\sigma^2} \chi(s) = 0,
\quad (2.9)
\]

The Eq. (2.6) can be further simplified using a new variable \(s = e^{-\delta r}\). Taking into account, that here \(r \in [0, \infty)\) and \(s \in [1, 0]\), then we obtain:

\[
\chi''(s) + \chi'(s) \frac{1}{s} + \left[ -\frac{s^2}{\varepsilon^2} + \frac{\alpha^2}{s(1 - s)} - \frac{\beta^2}{(1 - s)^2} - \frac{l(l + 1)}{s^2 \delta^2 r^2} \right] \chi(s) = 0, \quad (2.10)
\]
where we use the following notations for bound states

$$
\varepsilon = \frac{\sqrt{M^2 - E^2}}{\delta}, \quad \alpha = \frac{\sqrt{2E V_0 + 2MS_0}}{\delta}, \quad \beta = \frac{\sqrt{S_0^2 - V_0^2}}{\delta}.
$$

(2.11)

For the bound states, should \( E < M \), \( \varepsilon > 0 \). The boundary conditions for Eq.(2.3) are \( \chi(0) = 0 \) and \( \chi(\infty) = 0 \). Having inserting Eq.(2.8) in Eq.(2.10) and after such manipulations we obtain:

$$
\chi''(s) + \chi'(s) \frac{1-s}{s(1-s)} + \left[ \frac{1}{s(1-s)} \right]^2 \left[ -\varepsilon^2(1-s)^2 + \alpha^2 s(1-s) - \beta^2 s^2 
- l(l+1)C_0(1-s)^2 - l(l+1)s \right] \chi(s) = 0,
$$

(2.12)

Now, NU method can be successfully applied to define the eigenvalues of energy. By comparing Eq.(2.12) with Eq.(2.9) we can define the followings

$$
\tilde{\tau}(s) = 1 - s, \quad \sigma(s) = s(1-s),
$$

(2.13)

$$
\tilde{\sigma}(s) = -\varepsilon^2(1-s)^2 + \alpha^2 s(1-s) - \beta^2 s^2 - \lambda C_0(1-s)^2 - \lambda s.
$$

(2.14)

If we take the following factorization

$$
\chi(s) = \phi(s)y(s),
$$

(2.15)

for the appropriate function \( \phi(s) \) the Eq.(2.12) takes the form of the well known hypergeometric-type equation,

$$
\sigma(s)y''(s) + \tau(s)y'(s) + \tilde{\lambda}y(s) = 0.
$$

(2.16)

The appropriate \( \phi(s) \) function must satisfy the following condition

$$
\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)},
$$

(2.17)

where \( \pi(s) \), the polynomial of degree at most one, is defined as

$$
\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left( \frac{\sigma' - \tilde{\tau}}{2} \right)^2 - \tilde{\sigma} + k\sigma}.
$$

(2.18)

Finally the equation, where \( y(s) \) is one of its solutions, takes the form known as hypergeometric-type if the polynomial \( \sigma(s) = \tilde{\sigma}(s) + \pi^2(s) + \pi(s)[\tilde{\tau}(s) - \sigma'(s)] + \pi'(s)\sigma(s) \) is
divisible by $\sigma(s)$, i.e., $\bar{\sigma} = \lambda \sigma(s)$. The constant $\bar{\lambda}$ and polynomial $\tau(s)$ in Eq. (2.16) defined as

$$\bar{\lambda} = k + \pi'(s) \quad (2.19)$$

and

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \quad (2.20)$$

respectively. For our problem, the $\pi(s)$ function is written as

$$\pi(s) = -s/2 \pm \sqrt{s^2[a - k] - s[b - k] + c} \quad (2.21)$$

where the values of the parameters are

$$a = \frac{1}{4} + \varepsilon^2 + \alpha^2 + \beta^2 + \lambda C_0,$$

$$b = 2\varepsilon^2 + 2\lambda C_0 + \alpha^2 - \lambda,$$

$$c = \varepsilon^2 + \lambda C_0.$$

The constant parameter $k$ can be found complying with the condition that the discriminant of the expression Eq. (2.21) under the square root is equal to zero. Hence we obtain

$$k_{1,2} = (b - 2c) \pm 2\sqrt{c^2 + c(a - b)}. \quad (2.22)$$

When the individual values of $k$ given in Eq. (2.21) are substituted into Eq. (2.20), the four possible forms of $\pi(s)$ are written as follows

$$\pi(s) = \frac{-s}{2} \pm \begin{cases} 
(\sqrt{c} - \sqrt{c + a - b})s - \sqrt{c} & \text{for} \ k = (b - 2c) + 2\sqrt{c^2 + c(a - b)}, \\
(\sqrt{c} + \sqrt{c + a - b})s - \sqrt{c} & \text{for} \ k = (b - 2c) - 2\sqrt{c^2 + c(a - b)}. 
\end{cases} \quad (2.23)$$

The polynomial $\pi(s)$ have four possible form according to NU method, but we select the one for which the function $\tau(s)$ has the negative derivative. Another forms are not suitable physically. Therefore, the appropriate function $\pi(s)$ and $\tau(s)$ are

$$\pi(s) = \sqrt{c} - s \left[ \frac{1}{2} + \sqrt{c + \sqrt{c + a - b}} \right], \quad (2.24)$$

$$\tau(s) = 1 + 2\sqrt{c} - 2s \left[ 1 + \sqrt{c + a - b} \right], \quad (2.25)$$
for

\[ k = (b - 2c) - 2\sqrt{c^2 + c(a - b)}. \]  
(2.26)

Also by Eq.\((2.18)\) we can define the constant \(\bar{\lambda}\) as

\[ \bar{\lambda} = b - 2c - 2\sqrt{c^2 + c(a - b)} - \left[ \frac{1}{2} + \sqrt{c + \sqrt{c + a - b}} \right]. \]  
(2.27)

Given a nonnegative integer \(n_r\), the hypergeometric-type equation has a unique polynomials solution of degree \(n_r\) if and only if

\[ \bar{\lambda} = \bar{\lambda}_{n_r} = -n_r\tau' - \frac{n_r(n_r - 1)}{2}\sigma'', \quad (n_r = 0, 1, 2...), \]  
(2.28)

and \(\bar{\lambda}_m \neq \bar{\lambda}_n\) for \(m = 0, 1, 2,..., n_r - 1\), then it follows that,

\[ \bar{\lambda}_{n_r} = b - 2c - 2\sqrt{c^2 + c(a - b)} \left[ 1 + \sqrt{c + \sqrt{c + a - b}} \right] + n_r(n_r - 1). \]  
(2.29)

We can solve Eq.\((2.29)\) explicitly for \(c\) by using the relation \(c = \epsilon^2 + \lambda C_0\) which brings

\[ \epsilon^2 = \left[ \frac{\alpha^2 - \lambda - 1/2 - n_r(n_r + 1) - (2n_r + 1)\sqrt{\frac{1}{4} + \beta^2 + \lambda}}{2n_r + 1 + 2\sqrt{\frac{1}{4} + \beta^2 + \lambda}} \right] - \lambda C_0, \]  
(2.30)

After inserting \(\epsilon^2\) into Eq.\((2.11)\) for energy levels in more common case \(V(r) \neq S(r)\) we find

\[ M^2 - E_{n_r,l}^2 = \left[ \frac{\alpha^2 - \lambda - 1/2 - n_r(n_r + 1) - (2n_r + 1)\sqrt{\frac{1}{4} + \beta^2 + \lambda}}{2n_r + 1 + 2\sqrt{\frac{1}{4} + \beta^2 + \lambda}} \cdot \delta \right]^2 - l(l + 1)C_0\delta^2. \]  
(2.31)

In case \(V_0 = S_0\), then for energy spectrum we obtain:

\[ M^2 - E_{n_r,l}^2 = \left[ \frac{\alpha^2 - \lambda - 1/2 - n_r(n_r + 1) - (2n_r + 1)\sqrt{\frac{1}{4} + \lambda}}{2n_r + 1 + 2\sqrt{\frac{1}{4} + \lambda}} \cdot \delta \right]^2 - l(l + 1)C_0\delta^2 = \left[ \frac{\alpha^2}{2(n_r + l + 1)} - \frac{(n_r + l + 1)}{2} \right]^2 \cdot \delta^2 - l(l + 1)C_0\delta^2. \]  
(2.32)

In this case \(\beta^2 = 0\), but \(\alpha^2 = \frac{2V_0(E + M)}{\delta^2}\). If we take \(C_0 = 0\) in Eq.\((2.31)\) and Eq.\((2.32)\) then we directly obtain results \([21, 22]\).
In case $V(r) = -S(r)$, then for energy spectrum we obtain:

$$M^2 - E_{n,r,l}^2 = \left[ \alpha'^2 - \lambda - 1/2 - n(n+1) - (2n+1)\sqrt{\frac{1}{4} + \frac{\lambda}{4}} \cdot \delta \right]^2 \cdot \frac{2n_r + 1 + 2\sqrt{\frac{1}{4} + \lambda}}{2(n + l + 1)} \cdot \frac{(n + l + 1)}{2} \cdot \delta^2 - l(l + 1)C_0 \delta^2 \cdot (2.33)$$

In this case also $\beta^2 = 0$, but $\alpha'^2 \neq \alpha^2$, here

$$\alpha'^2 = \frac{2V_0(E - M)}{\delta^2} \quad (2.34)$$

For fully investigation, we also studied non-relativistic limit of the formula for the energy spectrum. When $V(r) = S(r)$, then Eq. (2.3) reduces to a Schrödinger equation for the potential $2V(r)$. In this case from Eq. (2.3) we directly obtain result [34].

The energy levels $E_{n,r,l}$ is determined by the energy equation Eqs. (2.31-2.33), which is rather complicated transcendental equation.

Now, applying the NU-method we can obtain the radial eigenfunctions. After substituting $\pi(s)$ and $\sigma(s)$ into Eq. (2.17) and solving first order differential equation, it is easy to obtain

$$\phi(s) = s^{\sqrt{c}}(1 - s)^K, \quad (2.35)$$

where $K = 1/2 + \sqrt{\frac{1}{4} + \beta^2 + l(l + 1)}$.

Furthermore, the other part of the wave function $y_n(s)$ is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s)\rho(s)], \quad (2.36)$$

where $C_n$ is a normalizing constant and $\rho(s)$ is the weight function which is the solutions of the Pearson differential equation. The Pearson differential equation and $\rho(s)$ for this problem have the form,

$$(\sigma \rho)' = \tau \rho, \quad (2.37)$$

$$\rho(s) = (1 - s)^{2K-1}s^{2\sqrt{c}}, \quad (2.38)$$

respectively.
Substituting Eq. (2.38) into Eq. (2.36) then we get
\[ y_{n_r}(s) = C_{n_r}(1 - s)^{1 - 2K} s^{2\sqrt{c} \frac{d^{n_r}}{ds^{n_r}}} \left[ s^{2\sqrt{c} + n_r}(1 - s)^{2K - 1 + n_r} \right]. \] (2.39)

Then by using the following definition of the Jacobi polynomials [51]
\[ P_n^{(a,b)}(s) = \frac{(-1)^n}{n!2^n(1 - s)^a(1 + s)^b} \frac{d^n}{ds^n} \left[ (1 - s)^{a+n}(1 + s)^{b+n} \right], \] (2.40)
we can write
\[ P_n^{(a,b)}(1 - 2s) = \frac{C_n}{s^a(1 - s)^b} \frac{d^n}{ds^n} \left[ s^{a+n}(1 - s)^{b+n} \right], \] (2.41)
and
\[ \frac{d^n}{ds^n} \left[ s^{a+n}(1 - s)^{b+n} \right] = C_n s^a(1 - s)^b P_n^{(a,b)}(1 - 2s). \] (2.42)

If we use the last equality in Eq. (2.39), we can write
\[ y_{n_r}(s) = C_{n_r} P_n^{(2\sqrt{c},2K-1)}(s). \] (2.43)

Substituting \( \phi(s) \) and \( y_{n_r}(s) \) into Eq. (2.17), we obtain
\[ \chi_{n_r}(s) = C_{n_r} s^{2\sqrt{c}}(1 - s)^K P_n^{(2\sqrt{c},2K-1)}(s). \] (2.44)

Using the following definition of the Jacobi polynomials [51]
\[ P_n^{(a,b)}(s) = \frac{\Gamma(n + a + 1)}{n!\Gamma(a + 1)} \frac{\Gamma(n_r + 2\sqrt{c} + 1)}{n_r!\Gamma(2\sqrt{c} + 1)} F \left( \frac{\Gamma(n_r + 2\sqrt{c} + 1)}{n_r!\Gamma(2\sqrt{c} + 1)} \right) \] (2.45)
we are able to write Eq. (2.44) in terms of hypergeometric polynomials as
\[ \chi_{n_r}(s) = C_{n_r} s^{2\sqrt{c}}(1 - s)^K \frac{\Gamma(n_r + 2\sqrt{c} + 1)}{n_r!\Gamma(2\sqrt{c} + 1)} F \left( \frac{-n_r, 2\sqrt{c} + 2K + n_r, 1 + 2\sqrt{c}, 1}{s} \right). \] (2.46)

The normalization constant \( C_{n_r} \) can be found from normalization condition
\[ \int_0^\infty |R(r)|^2 r^2 dr = \int_0^\infty |\chi(r)|^2 dr = \frac{1}{\delta} \int_0^1 \frac{1}{s} |\chi(s)|^2 ds = 1, \] (2.47)
by using the following integral formula [51]
\[ \int_0^1 (1 - z)^{-2(\delta+1)} z^{2\lambda - 1} \left\{ F \left( \frac{-n_r, 2(\delta + \lambda + 1) + n_r, 2\lambda + 1; z} {21} \right) \right\}^2 dz \]
\[ = \frac{(n_r + \delta + 1)n_r!\Gamma(n_r + 2\delta + 2)\Gamma(2\lambda)\Gamma(2\lambda + 1)}{(n_r + \delta + \lambda + 1)\Gamma(n_r + 2\lambda + 1)\Gamma(2(\delta + \lambda + 1) + n_r)}, \] (2.48)
here \( \delta > \frac{3}{2} \) and \( \lambda > 0 \). After simple calculations, we obtain normalization constant as
\[ C_{n_r} = \sqrt{\frac{n_r!2\sqrt{c}(n_r + K + \sqrt{c})\Gamma(2(K + \sqrt{c}) + n_r)}{b(n_r + K)\Gamma(n_r + 2\sqrt{c} + 1)\Gamma(n_r + 2K)}}. \] (2.49)
III. SOLUTION OF AZIMUTHAL ANGLE-DEPENDENT PART OF THE KLEIN-FOCK-GORDON EQUATION

We may also derive the eigenvalues and eigenvectors of the azimuthal angle dependent part of the KFG equation in Eq. (2.4) by using NU method. Introducing a new variable 
\[ z = \cos \theta \] 
and \[ \gamma = 2(E + M) \] and inserting these into Eq. (2.4) yield

\[ \Theta''(z) - \frac{2z}{1 - z^2} \Theta'(z) + \frac{1}{(1 - z^2)^2} \left[ \lambda(1 - z^2) - m^2 - \gamma(\beta' + \beta z) \right] \Theta(z) = 0. \] (3.1)

After the comparison of Eq. (3.1) with Eq. (2.9) we obtain

\[ \tilde{\tau}(z) = -2z, \sigma(z) = 1 - z^2, \tilde{\sigma}(z) = -\lambda z^2 - \gamma\beta z + (\lambda - m^2 - \gamma\beta'). \] (3.2)

In the NU method the new function \( \pi(z) \) is calculated for angle-dependent part as

\[ \pi(z) = \pm \sqrt{z^2(\lambda - k) + \gamma\beta z - (\lambda - \gamma\beta' - m^2 - k)}. \] (3.3)

The constant parameter \( k \) can be determined as

\[ k_{1,2} = \frac{2\lambda - m^2 - \gamma\beta'}{2} \pm \frac{u}{2}, \] (3.4)

where \( u = \sqrt{(m^2 + \gamma\beta')^2 - \gamma^2\beta^2} \). The appropriate function \( \pi(z) \) and parameter \( k \) are

\[ \pi(x) = - \left[ x \sqrt{\frac{m^2 + \gamma\beta' + u}{2}} + \sqrt{\frac{m^2 + \gamma\beta' - u}{2}} \right], \] (3.5)

\[ k = \frac{2\lambda - m^2 - \gamma\beta'}{2} - \frac{u}{2}. \] (3.6)

The following track in this selection is to achieve the condition \( \tau' < 0 \). Therefore \( \tau(z) \) becomes

\[ \tau(z) = -2z \left[ 1 + \sqrt{\frac{m^2 + \gamma\beta' + u}{2}} \right] - 2\sqrt{\frac{m^2 + \gamma\beta' - u}{2}}. \] (3.7)

We can also write the values \( \tilde{\lambda} = k + \pi'(s) \) as

\[ \tilde{\lambda} = \frac{2\lambda - \gamma\beta' - m^2}{2} - \frac{u}{2} - \sqrt{\frac{m^2 + \gamma\beta' + u}{2}}, \] (3.8)

and using Eq. (2.28), then from the Eq. (3.8) we can obtain

\[ \tilde{\lambda}_N = \frac{2\lambda - \gamma\beta' - m^2}{2} - \frac{u}{2} - \sqrt{\frac{m^2 + \gamma\beta' + u}{2}} = 2N \left[ 1 + \sqrt{\frac{m^2 + \gamma\beta' + u}{2}} \right] + N(N - 1), \] (3.9)
In order to obtain unknown $\lambda$ we can solve Eq.\((3.9)\) explicitly for $\lambda = l(l + 1)$

$$\lambda - \zeta^2 - \zeta = 2N(1 + \zeta) + N(N - 1),$$

(3.10)

where $\zeta = \sqrt{\frac{m^2 + \gamma \beta + u}{2}}$, and

$$\lambda = \zeta^2 + \zeta + 2N\zeta + N(N + 1) = (N + \zeta)(N + \zeta + 1) = l(l + 1),$$

(3.11)

then

$$l = N + \zeta.$$  

(3.12)

Substitution of this result Eq.\((3.12)\) in Eqs.\((2.30-2.32)\) yields the desired energy spectrum, in terms of $n_r$ and $l$ quantum numbers. Similarly, the wave function of polar angle dependent part of KFG equation can be formally derived by a process to the derivation of radial part of KFG equation. Thus using Eq.\((2.16)\), we obtain

$$\phi(z) = (1 - z)^{(B+C)/2},$$

(3.13)

where $B = \sqrt{\frac{m^2 + \gamma \beta + u}{2}}$, $C = \sqrt{\frac{m^2 + \gamma \beta - u}{2}}$.

On the other hand, to find a solution for $y_N(s)$ we should first obtain the weight function $\rho(s)$. From Pearson equation, we find weight function as

$$\rho(z) = (1 - z)^{B+C}(1 + z)^{B-C}.$$  

(3.14)

Substituting Eq.\((3.14)\) into Eq.\((2.36)\) allows us to obtain the polynomial $y_N(s)$ as follows

$$y_N(z) = B_N(1 - z)^{-(B+C)}(1 + z)^{C-B}\frac{d^N}{dz^N}[(1 - z)^{B+C+N}(1 + z)^{B-C+N}].$$

(3.15)

From the definition of Jacobi polynomials, we can write

$$\frac{d^N}{dz^N}[(1 - z)^{B+C+N}(1 + z)^{B-C+N}] = (-1)^N2^N(1 - z)^{B+C}(1 + z)^{B-C}P_N^{(B+C,B-C)}(z).$$

(3.16)

Having inserted Eq.\((3.16)\) into Eq.\((3.15)\) and after long but straightforward calculations we obtain the following result,

$$\Theta_N(z) = C_N(1 - z)^{(B+C)/2}(1 + z)^{(B-C)/2}P_N^{(B+C,B-C)}(z),$$

(3.17)

where $C_N$ is the normalization constant. Using orthogonality relation of the Jacobi polynomials \([51]\) the normalization constant can be found as

$$C_N = \sqrt{\frac{(2N + 2B + 1)\Gamma(N + 1)\Gamma(N + 2B + 1)}{2^{2B+1}\Gamma(N + B + C + 1)\Gamma(N + B - C + 1)}}.$$  

(3.18)
Thus after inserting Eq. (3.12) in Eqs. (2.30-2.32) then we directly obtain energy spectrum for combined potential, so Hulthén plus Ring-Shaped potential in this form:

In case \( V(r) \neq S(r) \) we find

\[
M^2 - E_{n_r,l}^2 = \left[ \frac{\alpha^2 - \eta - 1/2 - n_r(n_r + 1) - (2n_r + 1)\sqrt{\frac{1}{4} + \beta^2 + \eta}}{2n_r + 1 + 2\sqrt{\frac{1}{4} + \beta^2 + \eta}} \cdot \delta \right]^2 - C_0 \eta \delta^2 \quad (3.19)
\]

In case \( V_0 = S_0 \), then for energy spectrum we obtain:

\[
M^2 - E_{n_r,l}^2 = \left[ \frac{\alpha^2}{2(n_r + N + \zeta + 1)} - \frac{(n_r + N + \zeta + 1)}{2} \right]^2 \cdot \delta^2 - C_0 \eta \delta^2. \quad (3.20)
\]

In this case \( \beta^2 = 0 \), but \( \alpha^2 = \frac{2V_0(E + M)}{\delta^2} \).

In case \( V(r) = -S(r) \), then for energy spectrum we obtain:

\[
M^2 - E_{n_r,l}^2 = \left[ \frac{\alpha^2}{2(n_r + N + \zeta + 1)} - \frac{(n_r + N + \zeta + 1)}{2} \right]^2 \cdot \delta^2 - C_0 \eta \delta^2. \quad (3.21)
\]

In this case also \( \beta^2 = 0 \), but \( \alpha^2 \neq \alpha^2 \), here

\[
\alpha^2 = \frac{2V_0(E - M)}{\delta^2} \quad (3.22)
\]

In the equations Eqs. (3.19, 3.21) we used notation \( \eta = (N + \zeta)(N + \zeta + 1) \)

IV. THE SOLUTION OF KLEIN-FOCK-GORDON EQUATION FOR HULTHÉN POTENTIAL WITHIN SUSY QUANTUM MECHANICS

The eigenfunction of ground state \( \chi_0(r) \) in Eq. (2.3) according to supersymmetric quantum mechanics, is a form as

\[
\chi_0(r) = N exp \left( - \int W(r)dr \right). \quad (4.1)
\]

where \( N \) and \( W(r) \) are normalized constant and superpotential, respectively. The connection between the supersymmetric partner potentials \( V_1(r) \) and \( V_2(r) \) of the superpotential \( W(r) \) is as follows [8, 9]:

\[
V_1(r) = W^2(r) - W'(r) + E, \quad V_2(r) = W^2(r) + W'(r) + E. \quad (4.2)
\]
The particular solution of the Riccati equation Eq.(4.2) searches the following form:

\[ W(r) = -\left( C + \frac{D e^{-\delta r}}{1 - e^{-\delta r}} \right), \]

(4.3)

where \( C \) and \( D \) unknown constants. Since \( V_1(r) = V_{\text{eff}}(r) \), having inserted the relations Eq.(2.7) and Eq.(4.3) into the expression Eq.(4.2), and from comparison of compatible quantities in the left and right sides of the equation, we find the following relations for \( C \) and \( D \) constants:

\[ C^2 = \varepsilon^2 \delta^2 + \delta^2 C_0 l(l + 1), \]

(4.4)

\[ 2CD - \delta D = \delta^2 l(l + 1) - \alpha^2 \delta^2, \]

(4.5)

\[ D^2 - \delta D = \delta^2 l(l + 1) + \delta^2 \beta^2. \]

(4.6)

Considering extremity conditions to wave functions, we obtain \( D > 0 \) and \( C < 0 \). Solving Eq.(4.6) yields

\[ D = \frac{\delta \pm \sqrt{\delta^2 + 4\delta^2 l(l + 1) + \beta^2 \delta^2}}{2} = \frac{\delta \pm 2\delta \sqrt{l^2 l + l(l + 1) + \beta^2}}{2}, \]

(4.7)

and considering \( B > 0 \) from Eqs.(4.5) and (4.6), we find

\[ 2CD - D^2 = -\alpha^2 \delta^2 - \beta^2 \delta^2, \]

(4.8)

or

\[ C = \frac{D}{2} - \frac{(\alpha^2 + \beta^2)\delta^2}{2D}, \]

(4.9)

From Eq.(4.4) and Eq.(4.9), we find

\[ \varepsilon^2 \delta^2 + l(l + 1)\delta^2 C_0 = \frac{1}{\delta^2} \left[ \frac{D}{2} - \frac{(\alpha^2 + \beta^2)\delta^2}{2D} \right]^2, \]

(4.10)

After inserting (4.10) into (2.11) for energy eigenvalue, we obtain

\[ M^2 - E^2 = \left[ \frac{D}{2} - \frac{(\alpha^2 + \beta^2)\delta^2}{2D} \right]^2 - l(l + 1)C_0 \delta^2 \]

(4.11)

In Eq. (4.8) insert \( D > 0 \) from Eq.(4.4) finally, for energy eigenvalue, we obtain

\[ M^2 - E^2 = \delta^2 \left[ \frac{1 + 2\sqrt{l^2 + \beta^2 + l(l + 1)}}{4} - \frac{\alpha^2 + \beta^2}{\sqrt{l^2 + \beta^2 + l(l + 1)}} \right]^2 - l(l + 1)C_0 \delta^2 \]

(4.12)
When \( r \to \infty \), the chosen superpotential \( W(r) \) is \( W(r) \to -\frac{\hbar A}{\sqrt{2} \mu} \).

Having inserted the Eq. (4.3) into Eq. (4.2), then we can find supersymmetric partner potentials \( V_1(r) \) and \( V_2(r) \) in the form

\[
V_1(r) = W^2(r) - W'(r) + E = \left[ C^2 + \frac{(2CD - \delta D)e^{-\delta r}}{1 - e^{-\delta r}} + \frac{(D^2 - \delta D)e^{-2\delta r}}{(1 - e^{-\delta r})^2} \right]. \tag{4.13}
\]

\[
V_2(r) = W^2(r) + W'(r) + E = \left[ C^2 + \frac{(2CD + \delta D)e^{-\delta r}}{1 - e^{-\delta r}} + \frac{(D^2 + \delta D)e^{-2\delta r}}{(1 - e^{-\delta r})^2} \right]. \tag{4.14}
\]

By using superpotential \( W(r) \) from Eq.(4.1) we can find \( \chi_0(r) \) radial eigenfunction in this form:

\[
\chi_0(r) = N_0 e^{\int W(r) \, dr} = N_0 e^{\int \left( C + \frac{D e^{-\delta r}}{1 - e^{-\delta r}} \right) \, dr} = N_0 e^{C r} e^{\int \frac{D(1 - e^{-\delta r})}{1 - e^{-\delta r}} \, dr} = N_0 e^{C r (1 - e^{-\delta r}) D}, \tag{4.15}
\]

here \( r \to 0; \chi_0(r) \to 0, D > 0 \) and \( r \to \infty; \chi_0(r) \to 0, C < 0 \).

Two partner potentials \( V_1(r) \) and \( V_2(r) \) which differ from each other with additive constants and have the same functional form are called invariant potentials [44, 45]. Thus, for the partner potentials \( V_1(r) \) and \( V_2(r) \) given with Eq.(4.13) and Eq.(4.14), the invariant forms are:

\[
R(D_i) = V_2[D + (i - 1)\delta, r] - V_1[D + i\delta, r] = \left[ \left( \frac{D + (i - 1)\delta}{2} \right)^2 - \left( \frac{D + i\delta}{2(D + \delta)} \right)^2 \right]. \tag{4.16}
\]

\[
R(D_i) = V_2[D + (i - 1)\delta, r] - V_1[D + i\delta, r] = \left[ \left( \frac{D + (i - 1)\delta}{2} \right)^2 - \left( \frac{D + i\delta}{2(D + (i - 1)\delta)} \right)^2 \right]. \tag{4.17}
\]

where the remainder \( R(D_i) \) is independent of \( r \).

If we continue this procedure and make the substitution \( D_{n_r} = D_{n_r-1} + \delta = D + n_r\delta \) at every step until \( D_n \geq 0 \), the whole discrete spectrum of Hamiltonian \( H_-(D) \):
\[ E_{n,r,l} = E_0 + \sum_{i=0}^{n} R(D_i) = \]
\[ = \lambda \delta^2 C_0 - \left( \frac{D}{2} - \frac{\delta^2(\alpha^2 + \beta^2)}{2D} \right)^2 \] 
\[ - \left( \frac{D}{2} - \frac{\delta^2(\alpha^2 + \beta^2)}{2D} \right)^2 + \left( \frac{D + 2\delta}{2} - \frac{\delta^2(\alpha^2 + \beta^2)}{2(D + 2\delta)} \right)^2 \] 
\[ + \left( \frac{D + (n_r - 1)\delta}{2} - \frac{\delta^2(\alpha^2 + \beta^2)}{2(D + (n_r - 1)\delta)} \right)^2 \] 
\[ = \lambda \delta^2 C_0 - \left( \frac{D + n_r\delta}{2} - \frac{\delta^2(\alpha^2 + \beta^2)}{2(D + n_r\delta)} \right)^2, \quad (4.18) \]

and we obtain

\[ E_{n,r} = \lambda \delta^2 C_0 - \frac{\hbar^2}{2\mu} \left( \frac{D + n_r\delta}{2} - \frac{\delta^2(\alpha^2 + \beta^2)}{2(D + n_r\delta)} \right)^2, \quad (4.19) \]

Finally, for energy eigenvalues we found

\[ M^2 - E^2_{n,r,l} = \left[ \frac{\alpha^2 - \lambda - 1/2 - n(n + 1) - (2n + 1)\sqrt{\frac{1}{4} + \beta^2 + \lambda}}{2n_r + 1 + 2\sqrt{\frac{1}{4} + \beta^2 + \lambda}} \cdot \delta \right]^2 \] 
\[ - l(l + 1)C_0\delta^2. \quad (4.20) \]

In case $V_0 = S_0$, then for energy spectrum we obtain:

\[ M^2 - E^2_{n,r,l} = \left[ \frac{\alpha^2 - \lambda - 1/2 - n(n + 1) - (2n + 1)\sqrt{\frac{1}{4} + \lambda}}{2n_r + 1 + 2\sqrt{\frac{1}{4} + \lambda}} \cdot \delta \right]^2 - l(l + 1)C_0\delta^2 = \]
\[ \left[ \frac{\alpha^2}{2(n + l + 1)} - \frac{(n + l + 1)}{2} \right]^2 \cdot \delta^2 - l(l + 1)C_0\delta^2. \quad (4.21) \]

In this case $\beta^2 = 0$
In case $V(r) = -S(r)$, then for energy spectrum we obtain:

$$M^2 - E_{n,r,l}^2 = \left[ \frac{\alpha^2 - \lambda - 1/2 - n(n + 1) - (2n + 1)\sqrt{\frac{1}{4} + \lambda}}{2n_r + 1 + 2\sqrt{\frac{1}{4} + \lambda}} \cdot \delta \right]^2 - l(l + 1)C_0\delta^2 =$$

$$\left[ \frac{\alpha^2}{2(n + l + 1)} - \frac{(n + l + 1)}{2} \right]^2 \cdot \delta^2 - l(l + 1)C_0\delta^2.(4.22)$$

In this case also $\beta^2 = 0$, but $\alpha^2 \neq \alpha^2$, here

$$\alpha^2 = \frac{2V_0(E - M)}{\delta^2}$$

(4.23)

Based on Eqs.(A.14) and (A.17), the obtained result of radial KFG equation by using the Eq.(4.1) of the ground state eigenfunction is exactly same with the result obtained by using NU method.

This indisputable opens a new window for determining of the properties of the interactions in quantum system.

V. NUMERICAL RESULTS AND DISCUSSION

Solutions of the modified Klein-Fock-Gordon equation for the Hulthén plus ring-shaped like potential are obtained respectively within ordinary quantum mechanics by applying the Nikiforov-Uvarov method and within SUSY QM by applying the shape invariance concept. Both ordinary and SUSY quantum mechanical energy eigenvalues and corresponding eigenfunctions are obtained for arbitrary $l$ quantum numbers.

After analytically solving the bound states of $l$-wave KFG equation with vector and scalar Hulthén plus ring-shaped potentials, we should make next important remarks. First, when $l = 0$, the approximation centrifugal term $l(l + 1)\delta^2(C_0 + \frac{e^{-\delta r}}{(1 - e^{-\delta r})^2}) = 0$, too. Thus letting $l = 0$ in Eq.(2.31) and Eq.(2.46), they reduce to the exact energy spectrum formula and the unnormalized radial wave functions for the bound states of s-wave KFG equation with vector and scalar Hulthén potentials [4, 5, 7], respectively. Second, in the case scalar potential is equal to the vector potential, as $S_0 = V_0$, and $\zeta = 0$ in Eq.(3.20), then formula for energy spectrum reduces to Eq.(2.31)

$$M^2 - E_{n,r,l}^2 = \left[ \frac{\alpha^2}{2(n_r + l + 1)} - \frac{(n_r + l + 1)}{2} \right]^2 \cdot \delta^2 - C_0l(l + 1)\delta^2.$$
In this case \( \beta^2 = 0 \).

Third, in case \( V(r) = -S(r) \), and \( \zeta = 0 \) in Eq. (3.21) then for energy spectrum we obtain Eq. (2.33):

\[
M^2 - E_{n,r,l}^2 = \left[ \frac{\alpha^2}{2(n_r + l + 1)} - \frac{(n_r + l + 1)}{2} \right]^2 \cdot \delta^2 - C_0 l(l + 1)\delta^2.
\]

In this case also.

Fourth, we also discussed non-relativistic limit of the formula for the energy spectrum. When \( V(r) = S(r) \), then Eq. (2.3) will be transformed a Schrödinger equation for the potential \( 2V(r) \). If we take \( C_0 = 0 \) in Eq. (2.31), and also \( C_0 = 0 \) and \( \zeta = 0 \) in Eq. (3.19) then we obtain results [21, 22].

If \( \zeta = 0 \), then Eqs. (3.19, 3.21) reduces to energy spectrum for the Hulthén potential.

The energy eigenvalues and corresponding eigenfunctions are obtained for arbitrary \( l \) quantum numbers. Two important cases must be emphasized in the results of this study. In the first case which \( \beta = \beta' = 0 \) the potentials turn to central Hulthén potential. For this case, by using \( u = m^2 \), \( \zeta = |m| \) and \( l = N + |m| \) (\( N = 0, 1, 2 \ldots \)) then \( l \geq |m| \) by substituting this \( l \) values in Eq. (3.19) we obtain energy spectrum for Hulthén potential.

Finally, we want to deal with some restrictions about bound state solutions of KFG for Hulthén plus ring shaped like potential. First, it is seen from Eq. (3.4) and expression from \( u \) that in order to obtain real energy values the condition \((m^2 + \beta')^2 \geq \beta^2 \) must be hold. Since the parameters \( \beta \) and \( \beta' \) are real and positive, we can write

\[
m^2 \geq \gamma(\beta - \beta'). \tag{5.1}
\]

If \( \beta \leq \beta' \) the inequality in Eq. (5.1) is provided automatically. But if \( \beta \geq \beta' \) then \( m \) becomes bounded. Secondly, in Eq. (2.31) if

\[
l(l + 1)C_0 > \left[ \frac{\alpha^2 - \lambda - 1/2 - n_r(n_r + 1) - (2n_r + 1)\sqrt{1/4 + \beta^2 + \lambda}}{2n_r + 1 + 2\sqrt{1/4 + \beta^2 + \lambda}} \right]^2 \tag{5.2}
\]

then energy eigenvalues take non-negative values, this means there is no bound states.

If both conditions in Eqs. (5.1, 5.2) are satisfied simultaneously, the bound states exist.
VI. CONCLUSION

We used alternative two methods to obtain the energy eigenvalues and corresponding eigenfunctions of the Klein-Fock-Gordon equation for the Hulthén plus ring-shaped lake potential.

The energy eigenvalues of the bound states and corresponding eigenfunctions are analytically found via both of NU and SUSY quantum mechanics. The same expressions were obtained for the energy eigenvalues, and the expression of radial and azimuthal wave functions transformed each other is shown by using these methods. A closed form of the normalization constant of the wave functions is also found. The energy eigenvalues and corresponding eigenfunctions are obtained for arbitrary $l$ angular momentum and $n_r$ radial quantum numbers. It is shown that the energy eigenvalues and eigenfunctions are sensitive to $n_r$ radial and $l$ orbital quantum numbers.

It is worth to mention that the Hulthén plus ring-shaped lake potential is one of the important exponential potential, and it is a subject of interest in many fields of physics and chemistry. The main results of this paper are the explicit and closed form expressions for the energy eigenvalues and the normalized wave functions. The method presented in this study is a systematic one and in many cases it is one of the most concrete works in this area.

Consequently, studying of analytical solution of the modified KFG equation is obtained for the Hulthén plus ring-shaped lake potential within the framework ordinary and SUSY QM could provide valuable information on the QM dynamics at nuclear, atomic and molecule physics and opens new window.

We can conclude that our analytical results of this study are expected to enable new possibilities for pure theoretical and experimental physicist, because the results are exact and more general.

Appendix A: Supersymmetric Quantum Mechanics

Supersymmetric Quantum Mechanics (SUSYQM) for $N = 2$, we have two nilpotent operators namely, $Q$ and $Q^+$, satisfying the following algebra:

$$\{Q, Q^+\} = H, \quad \{Q, Q\} = \{Q^+, Q^+\} = 0,$$  \hspace{1cm} (A1)
where $H$ is the supersymmetric Hamiltonian, $Q = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}$ and $Q^+ = \begin{pmatrix} 0 & A^0 \\ 0 & 0 \end{pmatrix}$ are the operators of supercharges, $A^-$ is bosonic operators and $A^0$ is its adjoint. The supersymmetric $H$ Hamiltonian in terms of these operators defined in the form \[8, 9\]:

\[
H = \begin{pmatrix} A^+ A^- & 0 \\ 0 & A^- A^+ \end{pmatrix} = \begin{pmatrix} H^- & 0 \\ 0 & H^+ \end{pmatrix},
\]

where $H^-$ and $H^+$ are called supersymmetric partner Hamiltonians. The supercharges $Q$ and $Q^+$ commute with SUSY $H$ Hamiltonian: \([H, Q] = [H, Q^+] = 0\).

If the ground state energy of a Hamiltonian $H$ is zero (i.e. $E_0 = 0$), it can always be written in a factorable form as a product of a pair of linear differential operators. That is why, one has from the Schrödinger equation that the ground state wave function $\psi_0(x)$ obeys

\[
H \psi_0(x) = -\frac{\hbar^2}{2m} \frac{d^2 \psi_0}{dx^2} + V(x) \psi_0(x) = 0, \tag{A3}
\]

so that

\[
V(x) = \frac{\hbar^2}{2m} \frac{\psi_0''(x)}{\psi_0(x)}. \tag{A4}
\]

This allows a global reconstruction of the potential $V(x)$ from the knowledge of its ground state wave function which possesses no nodes. Once we realize this, factorizing the Hamiltonian is now quite simple by using the following ansatz \[8, 9\]:

\[
H^- = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) = A^+ A^- \tag{A5}
\]

here

\[
A^- = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \quad A^+ = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x). \tag{A6}
\]

By factorizing procedure of the Hamiltonian, the Riccati equation for Superpotential is obtained:

\[
V^-(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x). \tag{A7}
\]

The solution for superpotential $W(x)$ in terms of the ground state wave function is

\[
W(x) = -\frac{\hbar^2}{2m} \frac{\psi_0'(x)}{\psi_0(x)}. \tag{A8}
\]
This solution is obtained by recognizing that once we satisfy $A^-\psi_0(x) = 0$, we automatically have a solution to $H\psi_0 = A^+A^-\psi_0 = 0$.

The next step in constructing the SUSY theory related to the original Hamiltonian $H_-$ is to define the operator $H_+ = A^-A^+$ obtained by reversing the order of $A^-$ and $A^+$. A little simplification shows that the operator $H_+$ is in fact a Hamiltonian corresponding to a new potential $V_+(x)$.

$$H_+ = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_+(x), \quad V_+(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}} W'(x). \quad (A9)$$

The potentials $V_-(x)$ and $V_+(x)$ are known as supersymmetric partner potentials. It is then clear that if the ground state energy of a Hamiltonian $H_1$ is $E_1^0$ with eigenfunction $\psi_1^0$ then in view of Eq.(A.5), it can always be written in the form below as,

$$H_1 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x) = A^+A^- + E_1^0, \quad (A10)$$

here

$$A^-_1 = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W_1(x), \quad A^+_1 = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W_1(x),$$
$$V_1(x) = W^2_1(x) - \frac{\hbar}{\sqrt{2m}} W'_1(x) + E_1^0, \quad W_1(x) = -\frac{\hbar^2}{2m} \frac{d\ln \psi_1^0}{dx}. \quad (A11)$$

The SUSY partner Hamiltonian is then given by

$$H_2 = A^-_1 A^+_1 + E_0^1 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_2(x), \quad (A12)$$

where

$$V_2(x) = W^2_1(x) + \frac{\hbar}{\sqrt{2m}} W'_1(x) + E_0^1 =$$
$$= V_1(x) + \frac{2\hbar}{\sqrt{2m}} W'_1(x) = V_1(x) - \frac{\hbar}{m} \frac{d^2}{dx^2}(\ln \psi_1^0). \quad (A13)$$

From Eq.(A.12), the energy eigenvalues and eigenfunctions of the two Hamiltonians $H_1$ and $H_2$ are related by

$$E_n^2 = E_{n+1}^1, \quad \psi_n^2 = [E_{n+1}^1 - E_0^1]^{-\frac{1}{2}} A^-_n \psi_{n+1}^1, \quad \psi_{n+1}^1 = [E_n^2 - E_0^2]^{-\frac{1}{2}} A^+_1 \psi_n^2. \quad (A14)$$

Here $E_n^m$ is the energy level, where $n$ denotes the energy level and $(m)$ refers to the $m$’th Hamiltonian $H_m$. 

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Thus, it is clear that if the original Hamiltonian \( H_1 \) has \( p \geq 1 \) bound states with eigenvalues \( E_n^1 \), and eigenfunctions \( \psi_n^1 \) with \( 0 < n < p \), then we can always generate a hierarchy of \((p - 1)\) Hamiltonians \( H_2, H_3, ..., H_p \) such that the \( m \)’th member of the hierarchy of Hamiltonians \((H_m)\) has the same eigenvalue spectrum as \( H_1 \) except that the first \((m - 1)\) eigenvalues of \( H \) are missing in \( H_1 \): 

\[
H_m = A_m^+ A_m^- + E_{m-1}^1 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_m(x),
\]  

(A15)  

where  

\[
A_m^\pm = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W_m(x), \quad W_m(x) = -\frac{\hbar}{\sqrt{2m}} \frac{d\ln(\psi_0^{(m)})}{dx}, \quad m = 2 3 4, \cdots p.
\]  

(A16)  

One also has  

\[
E_n^{(m)} = E_{n+1}^{(m-1)} = \cdots = E_{n+m-1}^1 ; \quad \psi_n^{(m)} = [E_{n+m-1}^1 - E_{n+m-2}^1]^{-\frac{1}{2}} \cdots [E_{n+1}^1 - E_0^1]^{-\frac{1}{2}} A_{m-1}^- A_1^- \psi_{n+m-1}^1 ,
\]  

(A17)  

\[
V_m(x) = V_1(x) - \frac{\hbar}{m} \frac{d^2}{dx^2} \ln(\psi_0^{(1)} \cdots \psi_0^{(m-1)}) .
\]

i.e., knowing all the eigenvalues and eigenfunctions of \( H_1 \) we immediately know all the energy eigenvalues \( E_n^1 \) and eigenfunctions \( \psi_n^1 \) of the hierarchy of \((p - 1)\) Hamiltonians \( H_2, H_3, ..., H_p \).

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