Stabilization of underactuated nonlinear systems to non-feasible curves

Victoria Grushkovskaya\textsuperscript{1,3,*} and Alexander Zuyev\textsuperscript{2,3}

1 Institute of Mathematics, Julius Maximilian University of Würzburg, Germany
2 Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, Germany
3 Institute of Applied Mathematics & Mechanics, National Academy of Science of Ukraine

This paper focuses on the development of stability conditions for systems of nonlinear non-autonomous ordinary differential equations and their applications to control problems. We present a novel approach for the study of asymptotic stability properties for nonlinear non-autonomous systems based on considering a parameterized family of sets. The proposed approach allows to state asymptotic stability conditions for a family of sets representing the level sets of a time-varying Lyapunov function and to estimate the rate of convergence of solutions to a prescribed neighbourhood of the given curve. The obtained stability results are applied to the trajectory tracking problem for a class of nonholonomic systems.

1 Introduction and Problem Statement

Consider a system of nonlinear non-autonomous ordinary differential equations

\[ \dot{x} = f(x, t), \]

where \( x \in \mathbb{R}^n, t \in \mathbb{R}^+ = [0, \infty), f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n \). Assume that there exists a function \( x^* \in C^1(\mathbb{R}^n; \mathbb{R}^n) \) such that \( f(x^*(t), t) = 0 \) for each \( t \in \mathbb{R}^+ \). Note that \( x^*(t) \) is not a solution of (1) in general. In this paper, we propose novel asymptotic stability conditions for system (1) in a neighborhood of the curve \( x^*(t) \). It should be noted that stability properties of such systems have already been studied in several publications, e.g., [5, 6, 8, 9]. Unlike the above-mentioned results, we consider a general class of systems without assuming that \( \|\dot{x}^*(t)\| \to 0 \) as \( t \to \infty \). Our approach is based on considering a parameterized family of sets: let \( V_t \subset \mathbb{R}^n, t \in \mathbb{R}^+ \), be a one-parameter family of non-empty sets. For a \( \delta > 0 \), we denote the \( \delta \)-neighborhood of the set \( V_t \) at time \( t \) as \( B_\delta(V_t) = \bigcup_{y \in V_t} \{ x \in \mathbb{R}^n : \| x - y \| < \delta \} \), and \( B_\delta(V_t) \) is the closure of \( B_\delta(V_t) \).

Definition 1.1 A family of sets \( V_t \) is said to be \textit{locally uniformly asymptotically stable} for (1) if:

– it is \textit{uniformly stable}, i.e., for each \( \varepsilon > 0 \) there are \( \delta > 0 \) such that, for all \( t_0 \in \mathbb{R}^+ \), if \( x^0 \in B_\delta(V_{t_0}) \) then the corresponding solution of (1) satisfies \( x(t) \in B_\varepsilon(V_t) \) for all \( t \geq t_0 \);

– \( \delta \)-\textit{uniformly attractive} with some \( \delta > 0 \), i.e., for each \( \varepsilon > 0 \) there exists a \( t_1 \in [0, \infty) \) such that, for all \( t_0 \in \mathbb{R}^+ \), if \( x^0 \in B_\delta(V_{t_0}) \) then the corresponding solution of (1) satisfies \( x(t) \in B_\varepsilon(V_t) \) for all \( t \geq t_0 + t_1 \);

If the attractivity property holds for every \( \delta > 0 \), then the family of sets \( V_t \) is called \textit{globally uniformly asymptotically stable} for (1).

In this paper, we consider a family of sets representing the level sets of a time-varying Lyapunov function [2, 7]. The obtained results extend the stability conditions obtained in [2, 3] for gradient-like systems to general nonlinear non-autonomous systems (1).

2 Stability Conditions

The basic stability result of this paper is as follows.

Lemma 2.1 Let the following conditions be satisfied for system (1) with \( f \in C^1(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^n) \):

1. there exists a function \( x^* \in C^1(\mathbb{R}^n; \mathbb{R}^n) \) such that \( f(x^*(t), t) = 0 \) for all \( t \in \mathbb{R}^+ \);

2. there exist a bounded domain \( \Gamma \subset \mathbb{R}^n \) and a constant \( \nu \geq 0 \) such that \( x^*(t) \in \Gamma \) and \( \|\dot{x}^*(t)\| \leq \nu \) for all \( t \in \mathbb{R}^+ \);

3. there exist constants \( \Delta > 0, L, H \geq 0 \), and functions \( V \in C^2(B_\Delta(0); \mathbb{R}), w_{11}, w_{12}, w_{21}, w_{22}, w_3 \in K_{[0, \Delta]} \) such that, for all \( x \in B_\Delta(0), t \geq 0 \),

\[
\begin{align*}
(a) & \quad w_{11}(\|x\|) \leq V(x) \leq w_{12}(\|x\|), \\
(b) & \quad \|\frac{\partial V(x)}{\partial x}\| \leq w_{21}(\|x\|), \quad \|\frac{\partial^2 V(x)}{\partial x \partial x}\| \leq H, \\
(c) & \quad \frac{\partial V(x)}{\partial x}(x + x^*(t), t) \leq -w_3(x),
\end{align*}
\]

Given a \( \lambda \in [0, w_{21}^0(\Delta)], \) if \( \nu < \lambda \) then \( V_t^\lambda = \{ x \in B_\Delta(x^*(t)) : V(x - x^*(t)) \leq \lambda, t \in \mathbb{R} \} \) is locally uniformly asymptotically stable for system (1).

\footnote{* Corresponding author: e-mail viktoriia.grushkovska@mathematik.uni-wuerzburg.de}

\footnote{1 Let us recall that a function \( W(x) : [0, \rho) \to [0, \infty) (\rho > 0) \) is called to be of class \( K_{[0, \rho]} \) if it is continuous, strictly increasing, and \( W(0) = 0 \).

This is an open access article under the terms of the Creative Commons Attribution License 4.0, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.
The proof goes along the same line as the proof of [2, Theorem 3], and thus we omit it due to space limitations. The proof techniques rely on a careful analysis of the behavior of the time-varying function \( V(x - x^*(t)) \) along the trajectories of system (1).

Note that the proposed result serves as a basis for obtaining various stability conditions, depending on properties of \( f \), which we plan to present in future works. In the next section, we demonstrate an application of the obtained stability conditions to several control problems. In particular, we present a constructive solution to the trajectory tracking problem for a class of nonholonomic systems.

3 Stabilizing Control Strategies

Consider a class of driftless control systems of the form

\[
\dot{x} = \sum_{i=1}^{m} u_i f_i(x), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad m < n, \quad f_i \in C^2(\mathbb{R}^n).
\]  

(2)

The vector fields \( f_i(x) \) are assumed to be linearly independent in a neighborhood of a given curve \( x^*(t) \) and satisfying the condition

\[
\text{span} \{ f_{i_1}(x), f_{i_j}(x) \} = \{ f_{i_1}, f_{i_2}, f_{i_3} \}(x) : i \in S_1, (j_1,j_2) \in S_2, (\ell_1, \ell_2, \ell_3) \in S_3 \} = \mathbb{R}^n
\]

for all \( x \in D \), with \( f_{i_1}(x) = \frac{\partial f_{i_2}(x)}{\partial x} f_{i_1}(x) - \frac{\partial f_{i_3}(x)}{\partial x} f_{i_2}(x) \), and with some sets of indices \( S_k \subseteq \{1,2,\ldots,m\}^k \) such that \( |S_1| + |S_2| + |S_3| = n \). It is well-known that any equilibrium of (2) cannot be stabilized by a regular time-invariant feedback law [1]. To stabilize (2) in a neighborhood of \( x^*(t) \), we define control functions in the following way:

\[
\begin{align*}
    u'_i(t, x, x^*) &= \sum_{j \in S_1} \delta_{ij} a_j(x, x^*) \\
    &+ \sqrt{\frac{4\pi}{\epsilon}} \sum_{(j_1,j_2) \in S_2} \sqrt{\kappa_{j_1,j_2}} |a_{j_1,j_2}(x,x^*)| \left( \delta_{j_1} \cos \frac{2\pi \kappa_{j_1,j_2}\ell t + \delta_{j_2} \text{sign}(a_{j_1,j_2}(x,x^*)) \sin \frac{2\pi \kappa_{j_1,j_2}\ell t}{\epsilon} \right) \\
    &+ \sqrt{\frac{16\pi^2}{\epsilon^2}} \sum_{(\ell_1,\ell_2,\ell_3) \in S_3} \sqrt{\kappa_{3,\ell_1,\ell_2,\ell_3}} \left( \delta_{\ell_1} \cos \frac{2\pi \kappa_{3,\ell_1,\ell_2,\ell_3}\ell t}{\epsilon} + \delta_{\ell_2} \sin \frac{2\pi \kappa_{2,\ell_1,\ell_2,\ell_3}\ell t}{\epsilon} \right), \quad i = 1,2,\ldots,m.
\end{align*}
\]

(3)

Here \( \delta_{ij} \) is the Kronecker delta, \( \kappa_{j_1,j_2} \in \mathbb{N} \) are pairwise distinct, and \((a_j(x,x^*))_{j \in S_1}, (a_{j_1,j_2}(x,x^*))_{(j_1,j_2) \in S_2} \) and \((f_{i_1}, f_{i_2}, f_{i_3})_{(i_1,i_2,i_3) \in S_3} \) are locally asymptotically stable for system (2).

Theorem 3.1 Let \( x^* \in C^1(\mathbb{R}^+; \mathbb{R}^n) \) satisfy condition 2 of Lemma 2.1, and let a function \( V \in C^2(\mathbb{R}^n; \mathbb{R}^+) \) satisfy conditions 3(a)–3(b) of Lemma 2.1. Assume also that the matrix \( F(x) \) is nonsingular in \( D = \cup_{t \geq 0} B_{\Delta}(x^*(t)) \), and there exist \( \mu > 0, \nu \geq 0 \) such that \( \|F^{-1}(x)\| \leq \mu \) for all \( x \in D \), and \( \|\dot{x}\| \leq \nu \) for all \( t \geq 0 \).

Take \( P(x,t) = V(x - x^*(t)) \), then, for any \( \lambda \in (0, w_{11}(\Delta)) \), there exist \( \epsilon \geq 0 \) such that, for any \( \epsilon < (0, \bar{\epsilon}) \), the family of sets \( V_{\lambda} = \{ x \in B_{\Delta}(x^*(t)) : V(x - x^*(t)) \leq \lambda \} \subseteq \mathbb{R}^n \) is locally asymptotically stable for system (2)–(3) in the sense of \( \pi_\epsilon \)-solutions.

Note that Theorem 3.1 has been proved in [4] for the particular case \( S_3 = \emptyset \), \( V = \frac{1}{2}\|x(t) - x^*(t)\|^2 \).

Acknowledgements This work was supported in part by the German Research Foundation (GR 5293/1-1) and NAS of Ukraine (budget program KPKBK 6541230)

References

[1] R. W. Brockett, in: Differential Geometric Control Theory, edited by R. W. Brockett, R. S. Hillman, H. J. Sussmann (Birkhäuser, 1983).
[2] V. Grushkovskaya, H.-B. Dürr, C. Ebenbauer, and A. Zuyev, IFAC-PapersOnLine 50, 5522–5528 (2017).
[3] V. Grushkovskaya, M. Kelemen, A. Zuyev, M. May, and C. Ebenbauer, Proc. 2018 European Control Conf., 912–917 (2018).
[4] V. Grushkovskaya, and A. Zuyev, to appear in Proc. 2019 European Control Conf., arXiv preprint arXiv:1904.10219 (2019).
[5] M. Kelemen, IEEE Trans. Autom. Control 31, 766—768 (1986).
[6] H. K. Khalil, Nonlinear Systems 3rd Edition (Prentice Hall, 2002).
[7] J. A. Lang, J. C. Robinson, and A. Sáez, Nonlinearity 15, 887–903 (2002).
[8] D. A. Lawrence, and W. J. Rugh, IEEE Trans. Autom. Control 35, 860—864 (1990).
[9] I. G. Malkin, Theory of Stability of Motion (United States Atomic Energy Commission, 1952).