Significant non-classical paths with atoms and cavities in the double-slit experiment

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In the double-slit experiment, non-classical paths are Feynman paths that go through both slits. Prior work with atom cavities as which-way detectors in the double-slit experiment, has shown these paths to be experimentally inaccessible. In this paper, we show how such a setup can indeed detect non-classical paths with 1% probability, if one considers a different type of non-classical path than previously investigated. We also show how this setup can be used to erase and restore coherence of the non-classical paths. Finally, we also show how atom cavities may be used to implement a Born-rule violation measure (the Quach parameter \cite{1}), which up until now has only been a formal construct.

I. INTRODUCTION

Quantum mechanics is undoubtedly one of the most successful theories of the last century. Recent phenomenological developments have led to a plethora of applications in high precision and sensing tasks \cite{1,2}. However, to continue increasing the sensitivity of precision measurements requires a better understanding of the fundamental aspects of the quantum theory \cite{3}. Interference and coherence effects are some of the most useful measures in studying quantum mechanical effects. In this work, we will investigate the contributions of non-classical paths \cite{4,7} in the precise measurement of interference effects.

The double-slit experiment is the foundation of studies in interference effects \cite{8,10}, as well as revealing the wave-nature of matter \cite{11-14}. Typically, the nodes (or anti-nodes) are calculated as the result of the path difference arising out of the distance from the slits to the detection screen. This, however, is only an approximation, as first pointed out by Yabuki \cite{4}. In the Feynman path integral formulation of quantum mechanics \cite{15}, all possible paths between points contribute to the wave function. The direct paths from the slits to the detection screen are just one set of an infinite number of possible paths. Higher-order or non-classical paths include paths which enter both slits before reaching the detector, as shown in Fig. 1. Typically, these non-classical paths are much less probable than the direct or classical paths; nevertheless, it has been shown that in regimes, where the wavelength is large compared to the slit-spacing, these non-classical paths can be significant \cite{16}. The non-classical path contributions to the interference pattern is not uniquely a quantum mechanical effect. Such contributions to the interference pattern arise also out of Maxwell’s equations, as shown with finite-difference time-domain simulations \cite{17,19}.

In the double-slit experiment, the particle nature of matter is revealed if one knows which slit the particle went through \cite{20}. In 1991, Scully \textit{et al.} \cite{21} introduced cavities into the slits as a means to mark which slit the particle went through, thereby acting as which-way detectors. They showed how the setup could implement the delayed-choice quantum erasure experiment \cite{22-27}. More recently, de Oliveira Jr. \textit{et al.} \cite{28} showed how the setup proposed by Scully can be used to isolate non-classical paths. Their work consisted of modeling looped trajectories of Rubidium Rydberg atoms in the double-slit experiment, will the result that the probability of detection of these non-classical paths were too small to be feasible.

The Born rule states that if a quantum object is represented by a wave function \( \psi(\mathbf{r}, t) \), then the probability density of detecting it at position \( \mathbf{r} \) and time \( t \) is given by the absolute square of the wave function \( \psi^*(\mathbf{r}, t) \psi(\mathbf{r}, t) \) (1).

Despite being a cornerstone of quantum mechanics, a direct test of the Born rule was not attempted until 2010 by Sinha \textit{et al.} \cite{30}. The test was a measure of the Sorkin parameter \cite{31}, which quantifies non-pairwise interference, in a triple-slit experiment \cite{32}. Since the exponent of the Born rule only allows for pairwise interference, a non-zero Sorkin parameter would suggest violation of the Born rule. Shortly after this experiment, it was pointed out that a non-zero Sorkin parameter would not necessarily indicate Born-rule violation \cite{33}; instead it could be a signature of non-classical paths. Most recently, Quach \cite{34} proposed an alternative parameter, using the double-slit experiment with which-way detectors, as a more accurate measure of Born-rule violation. However, the Quach parameter up until now, has only been a formal construct.

The goals of this paper are to propose a practical set up to detect non-classical paths and test the validity of the Born-rule. In the first part of this paper (Sec. \ref{I} and \ref{III}), we show how a double-slit experiment with an
atom-cavity setup, can detect non-classical paths with 1% probability. Further, we show the delay-choice quantum erasure in the context of non-classical paths. In the last part of the paper (Sec. IV), we extend the set up to show its applicability in implementing the Quach parameter.

II. CLASSICAL AND NON-CLASSICAL PATHS

Let us consider the double-slit experiment as depicted in Fig. 1. We make the usual assumption that the slits run infinite in the y direction (perpendicular to the figure plane), and the slit-plane extends infinitely in the x direction. This allows us to reduce the system to a one-dimensional problem in the x-direction. The source is an atom described by the wave packet

$$\psi_0(x, t = 0) = \frac{1}{\sqrt{\sigma_0 \sqrt{\pi}}} \exp \left[ -\frac{x^2}{2\sigma_0^2} \right],$$

(2)

where $\sigma_0$ is the effective width of the atomic wave packet.

The atom wave function at a later time is the weighted sum of all possible paths,

$$\psi(x, t) = \int_{x_i} K(x_f, t_f; x_i, t_i) \psi_0(x_i, t_i),$$

(3)

where $K(x_f, t_f; x_i, t_i)$ is the free propagator for a particle with mass $m > 0$ from point $(x_i, t_i)$ to $(x_f, t_f)$:

$$K(x_f, t_f; x_i, t_i) = \sqrt{\frac{m}{2\pi i \hbar (t_f - t_i)}} \exp \left[ \frac{i m (x_f - x_i)^2}{2\hbar (t_f - t_i)} \right].$$

(4)

The presence of the slit-plane reduces the number of possible paths between the source and the detection screen. Following the literature we categorize the two types of allowed paths: classical paths that go through only one slit, and non-classical paths which go through both slits [28]. Classical and non-classical paths are also known as non-exotic and exotic [28] or higher-order paths [34].

A. Classical paths

The classical paths incorporate all possible paths connecting the source and the detection screen, whenever a single slit, $A$ or $B$, is open. The wave function resulting from the summation of all paths that go through slit $A$ only is

$$\psi_A(x, t) = \int_{x', x_0} K(x, t; x', t') T(x' + d/2) K(x', t'; x_0, 0) \psi_0(x_0),$$

(5)

where

$$T(x) = \exp \left[ -\frac{(x)^2}{2\beta^2} \right].$$

(6)

In Eq. (5), $K(x', t'; x_0, 0)$ is the free propagator from the source to the slit plane, and $K(x, t; x', t')$ is the free propagator from the slit plane to the detection screen. $T(x)$ is the slit transmission function, which we take to be a Gaussian function of slit-width $\beta$ [28]. Performing the integral, and taking the limits of integration to infinity yields the following form,

$$\psi_A(x) = \Gamma_e \exp \left[ c_2 x^2 + c_1 x + c_0 \right],$$

(7)

where the explicit expression for the constants ($\Gamma_e, c_2, c_1, c_0$) are given in Appendix A. The wave function resulting from the summation of classical paths that go through slit $B$ is similarly calculated,

$$\psi_B(x, t) = \int_{x', x_0} K(x, t; x', t') T(x' - d/2) K(x', t'; x_0, 0) \psi_0(x_0)$$

$$= \Gamma_e \exp \left[ c_2 x^2 - c_1 x + c_0 \right].$$

(8)

In the next subsection, we will use this formalism to calculate the non-classical path that goes through two slits before reaching the detection screen.

B. Non-classical paths

There are an infinite number of non-classical paths that enter both slits. Non-classical paths that loop through both slits were considered in the literature [1, 28]. An example of such a path is depicted by the green-dotted line in Fig. 1. the particle enters slits A, then slit B, then slit A again, before traveling to the detection screen. In this paper, we will focus on non-classical paths entering each slit only once. An example of such a path is depicted by the red-solid line in Fig. 1. the particle enters slit A, then slit B, then travels to the detection screen. The wave function resulting from the summation of such non-
classical paths is
\[
\psi_{AB}(x,t,\tau) = \int_{x_1',x_2',x_0} K(x, i + \tau; x_2', t) T(x_2' + \frac{d}{2}) K(x_2', i + \tau; x_2', t) T(x_2' + \frac{d}{2}) K(x_2', i + \tau; x_0) \psi_0(x_0)
\]
\[
= \Gamma_{nc} \exp[c_i^2 x^2 + c_{i'}^2 x + c_0^2] ,
\]
(9)

with constants given in Appendix A. Similarly, the wave function resulting from the summation of non-classical paths that go through slit B then A is
\[
\psi_{BA}(x,t,\tau) = \int_{x_1',x_2',x_0} K(x, i + \tau; x_2', t) T(x_2' + \frac{d}{2}) K(x_2', i + \tau; x_2', t) T(x_2' + \frac{d}{2}) K(x_2', i + \tau; x_0) \psi_0(x_0)
\]
\[
= \Gamma_{nc} \exp[c_i^2 x^2 - c_{i'}^2 x + c_0^2] ,
\]
(10)
The difference between \(\psi_{AB}(x,t,\tau)\) and \(\psi_{BA}(x,t,\tau)\) lies in the sign of \(\pm c_1'\).

Using the same formalism, other non-classical paths can also be calculated. For example, the looped trajectory (green-dotted line in Fig. 1), requires an additional transmission through the slits, and therefore has the wave function
\[
\psi_{ABAB}(x,t,\tau) = \int_{x_1',x_2',x_0} K(x, i + \tau; x_2', t) T(x_2' + \frac{d}{2}) K(x_2', i + \tau; x_2', t) T(x_2' + \frac{d}{2}) K(x_2', i + \tau; x_0) \psi_0(x_0)
\]
\[
= \Gamma_{nc} \exp[c_i^2 x^2 - c_1' x + c_0^2] .
\]
(11)

In general, each additional slit transmission attenuates the wave function, such that \(|\psi(x,i)|^2 \approx |\alpha^m \psi_0(x,0)|^2\), where \(\alpha\) is the attenuation factor and \(m\) is the number of time the path traverses a slit. For classical paths \(m = 1\), minimal non-classical paths \(m = 2\), and single looped paths \(m = 3\). For the values show in Appendix A, \(\alpha \approx 0.1\); as such, the probability of detecting minimal non-classical paths is 1% and loop paths is 0.01%, relative to the classical paths. We will use these facts in the next section to show one may indeed detect minimal non-classical paths.

III. CAVITY WHICH-WAY DETECTORS

Our set up consists of placing a cavity into each of the slits as depicted in Fig. 2. The source is a two-level Rydberg atom with ground and excited states \(|g\rangle\) or \(|e\rangle\). The transition frequency between the two states is resonant with the cavity mode \(\Omega\). The initial configuration is such that the atom is in the excited state, and there is one photon in each of the cavities,
\[
|\psi_0\rangle = |e\rangle |1\rangle_A |1\rangle_B .
\]
(12)
The speed of the atom is tuned so that the interaction time with the cavity is
\[
\tau = \frac{\pi}{\sqrt{n + \frac{1}{\Omega}}} ,
\]
(13)
where \(n + 1\) is the number of excitation in the cavity. This interaction time affects a \(\pi\) pulse on the atom \([37]\).

Here we are interested in the case where \(n = 1\), to ensure that the transition between \(|e\rangle |1\rangle_i\) and \(|g\rangle |2\rangle_i\) \((i = A,B)\) occurs with unit probability. Therefore our interaction time is set to \(\tau = \frac{\pi}{\sqrt{2\Omega}}\).

FIG. 2. Scheme of atom and double-slit with photonic cavities in each slit. The blue and green box in between the double slit contains the shutters and photodetection scheme. The inset (right bottom) shows a magnified view of the process of photodetection, it describes in detail one possible implementation to detect the cavity photons. In this example, with the opening of the shutters, the cavity photons go through a 50:50 beam splitter before its detection.

A. Atom-cavity interaction

Initially the atom is in the excited state and there is a single photon in each cavity. If the atom follows the classical path, it enters a single cavity once; in this case the transition \(|e\rangle |1\rangle_i \rightarrow |g\rangle |2\rangle_i\) will occur. If the atom follows the non-classical path it will enter both cavities. Upon leaving the first cavity the atom emits a photon \(|e\rangle |1\rangle_{i'} \rightarrow |e\rangle |0\rangle_{i'}\) \((i \neq i')\). As such, the system state just before the detection screen is
\[
|\psi\rangle = \frac{1}{\sqrt{N_i}} \left( |g\rangle |2\rangle_A |1\rangle_B |\psi_A\rangle + |1\rangle_A |2\rangle_B |\psi_B\rangle \right) + |e\rangle |2\rangle_A |0\rangle_B |\psi_{AB}\rangle + |0\rangle_A |2\rangle_B |\psi_{BA}\rangle \right) ,
\]
(14)
where \(N_i\) is the overall normalization factor, which in general will be dependent on the number of slits \((i)\) present. The first term represents the state of the system when the atom’s (classical) path traverses though slit-A only: here the atom emits a photon into cavity A.
Similarly, the second term represents the state of the system when the atom’s (classical) path traverses through slit-B only. The third term represents the state of the system when the atom’s (non-classical) path traverses first through slit-A then slit-B: here the atom emits a photon into cavity A and absorbs a photon in cavity B. Similarly, the fourth term represents the state of the system when the atom’s (non-classical) path traverses first through slit-B then slit-A.

Defining the following symmetric and anti-symmetric basis states,

\[
|\psi_{c}^{\pm}\rangle = \frac{1}{\sqrt{2}} (|\psi_{A}\rangle \pm |\psi_{B}\rangle) ,
\]

\[
|\psi_{nc}^{\pm}\rangle = \frac{1}{\sqrt{2}} (|\psi_{AB}\rangle \pm |\psi_{BA}\rangle) ,
\]

\[
|\mu^{\pm}\rangle = \frac{1}{\sqrt{2}} (|2\rangle_{A}|1\rangle_{B} \pm |1\rangle_{A}|2\rangle_{B}) ,
\]

\[
|\nu^{\pm}\rangle = \frac{1}{\sqrt{2}} (|2\rangle_{A}|0\rangle_{B} \pm |0\rangle_{A}|2\rangle_{B}) ,
\]

we can re-write state of the system before the detection screen [Eq. (14)] as

\[
|\psi_{f}^{\prime}\rangle = \frac{1}{\sqrt{N_{2}}} [ (|g\rangle (|\psi_{c}^{\prime}\rangle |\mu^{+}\rangle + |\psi_{nc}^{-}\rangle |\mu^{-}\rangle) + \\
+ |e\rangle (|\psi_{nc}^{+}\rangle |\nu^{+}\rangle + |\psi_{nc}^{-}\rangle |\nu^{-}\rangle) ] .
\]

Eq. (19) shows that by measuring the state of the atom, we can isolate the classical and non-classical paths. Keeping count only when an excited atom is detected, gives the probability distribution of the non-classical paths,

\[
P_{\text{c}}(x) = \frac{1}{N_{2}} (|\psi_{nc}^{+}(x)|^{2} + |\psi_{nc}^{-}(x)|^{2})
\]

\[
= \frac{1}{N_{2}} (|\psi_{AB}(x)|^{2} + |\psi_{BA}(x)|^{2}) .
\]

Conversely, keeping count only when a grounded atom is detected, gives the probability distribution of the classical paths,

\[
P_{\text{g}}(x) = \frac{1}{N_{2}} (|\psi_{c}^{+}(x)|^{2} + |\psi_{c}^{-}(x)|^{2})
\]

\[
= \frac{1}{N_{2}} (|\psi_{A}(x)|^{2} + |\psi_{B}(x)|^{2}) .
\]

From Eqs. (20) and (21) and the wave functions calculated in previous section, we plot in Fig. 3 the non-classical path probability distribution, normalised to the central maximum of the double-slit classical probability distribution, i.e. $P_{\text{c}}(x)/P_{\text{g}}(0)$.

Fig. 3 shows that non-classical paths accounts for about 1% of the classical paths detection events. This result indicates this set up as a good candidate to measure the non-classical paths.

**FIG. 3.** Probability of detecting non-classical paths normalized by the classical paths. The parameters are defined in Appendix A.

### B. Erasing which-way information with cavity photodetection

An interesting feature of the atom-cavity implementation of the which-way detectors is that one can partially erase the which-way information and restore coherent interference, even after the atom has been detected. To partially erase the which-way information, we add a beam splitter and photodetectors between the two cavities (Fig. 2). Shutters are positioned in each cavity. When the shutters are open, the photons are mixed in a beam splitter device and photodetectors are placed at each output port; the photodetectors act as a reservoir and in the limit of long detection time, all photons present in the cavities are absorbed. This procedure allows to mix the photons from both cavities losing the which-slit information, retrieving interference.

The shutters opening and photon detection occurs after the passage of each single atom. The statistics is obtained in the limit of infinite repetitions of this procedure. The beam splitter action on the intra-cavity photons, corresponds to the following transformation of the $A,B$ input modes:

\[
\hat{a}_{\pm}^\dagger |0\rangle = \frac{(a_{A} \pm a_{B})^\dagger}{\sqrt{2}} |0\rangle ,
\]

for example in the new basis, the state $|\nu^{-}\rangle = |1\rangle_{+} |1\rangle_{-}$. At each output port $+$, $-$, there is a photodetector, we shall refer to their probability distributions as $P^{+}$ and $P^{-}$, respectively.

We modeled the detection statistics, following the Markovian view of photon absorption [39]. This strategy predicts photon counts of one, two and three photons in a time interval. The density matrix time evolution contains the detection probabilities at each photon absorption. At time $t = 0$ the density matrix is:

\[
\rho(0) = |\psi_{f}^{\prime}\rangle \langle \psi_{f}^{\prime}| .
\]
\( |\psi_f^j\rangle \) was defined in Eq. (19), with photon number in the \(+/−\) basis [Eq. (22)]. The procedure to calculate \( \rho(t) \) and the probability distributions are described in Appendix B [Eq. (31) and Table II 39]. Here we will discuss the most relevant results, and analyse the probabilities in the limits of zero and infinite detection time.

At zero interaction time, no photons are absorbed, the statistics recover the results of Sec. IIIA. Moreover, for infinite detection time, all photons are absorbed and the number of photons in the cavities is conditioned to the atomic state, as we can see in Eq. (19). If the atom is detected in the ground state, there are three photons in the cavities, whereas if detected in the excited state, only two photons are in the cavities.

The probability of measuring the atom in the excited state with two photon counts in the same output photodetector, \( P_{e(kk)} \), or one photon in each output, \( P_{e(kj)} \), \( k = +/−, j = +/− \) with \( k \neq j \). In the regime of long detection time is:

\[
P_{e(kk/kj)}(x) = \frac{1}{2N_2} \left( \left| \psi_{nc}(+)\right|^2 \right); \tag{24}
\]

it represents the retrieval of interference, between non-classical paths \( AB \) and \( BA \), due to detection of the cavity photons. This is implemented keeping count of the excited atoms, only when two photons trigger the same \( (P_{e(kk)}) \) or different \( (P_{e(kj)}) \) detectors.

Similarly, the probability of measuring the atom in the ground state and detecting three photons in the same detector, \( P_{g(kk)} \), and two photons in one detector and one in the other, \( P_{g(kjk)} \) (including all permutations of \( kjk \)). In the regime of long detection time is:

\[
P_{g(kk)}(x) = \frac{1}{N_2} \left( \frac{3}{4} \right) \left| \psi_{e(k)}(x) \right|^2, \tag{25}
\]

\[
P_{g(kjk)}(x) = \frac{1}{N_2} \left( \frac{1}{12} \right) \left| \psi_{e(j)}(x) \right|^2, \tag{26}
\]

with total probability of atoms in the ground state given by:

\[
P_g(x) = \sum_{k,j=+,-} P_{g(kk)}(x) + 3 P_{g(kjk)}(x), \tag{27}
\]

this result recovers Eq. (20), as expected. This is implemented keeping count of the grounded atoms, only when three photons trigger the same or different detectors, respectively.

The magnitude of the non-classical paths is presented in Fig. 4; the interference pattern is recovered, due to the opening of the shutters, which allows for the photons interference and detection. The red solid curve illustrates the non-classical paths fringes pattern, showing it contributes with up to 1% of the total probability distribution; it is the total probability of interfering and measuring in a single detector the intra-cavity photons, \( P_{e(kk)}(x) \). The blue dashed curve, shows the anti-fringes pattern, it indicates probability distribution of interfering and measuring the photons in both detectors, \( P_{e(kj)}(x) \).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{Fig_4.png}
\caption{Probability of detecting non-classical paths in comparison with the classical paths. The red solid curve illustrates the non-classical paths fringes pattern, it is the total probability of interfering and measuring in a single detector the intra-cavity photons after measuring the atom in the excited state, \( P_e(kk) \). The blue dashed curve, shows the anti-fringes pattern, it indicates probability distribution of interfering and measuring the photons in both detectors, \( P_e(kj) \).}
\end{figure}

In the next subsection, we will discuss how cavities which-way detectors can be used to implement the Quach parameter and test the Born rule.

IV. THE QUACH PARAMETER

The Sorkin parameter for the triple-slit experiment is defined as

\[
\mathcal{I}_{ABC} \equiv \mathcal{P}_{ABC} - \mathcal{P}_{AB} - \mathcal{P}_{AC} - \mathcal{P}_{BC} + \mathcal{P}_A + \mathcal{P}_B + \mathcal{P}_C, \tag{28}
\]

where \( \mathcal{P}_{ABC} \) is the probability of detection when all 3 slits \( (A,B,C) \) are open, \( \mathcal{P}_{AB} \) is the probability of detection when 2 slits \( (A,B) \) are open, and so on. If one assumes that the probabilities are simply given by the linear superposition of the individual wave functions of the constituent single-slit set ups \( \mathcal{P}_{ABC} = |\psi_A + \psi_B + \psi_C|^2, \mathcal{P}_{AB} = |\psi_A + \psi_B|^2 \), and so on, then by rewriting probabilities in Eq. (28) in terms of wave functions, it can be shown that \( \mathcal{I}_{ABC} = 0 \) if the Born rule is correct. The Sorkin parameter can be generalised to systems with 3 and more slits, but not 2 slits. The reason for this is that \( \mathcal{I}_{AB} \equiv \mathcal{P}_{AB} - \mathcal{P}_A - \mathcal{P}_B \neq 0 \).

If one accounts for non-classical paths, the probability of detection must be corrected to \( \mathcal{P}_{ABC} = |\psi_A + \psi_B +
\( \psi_C + \psi_{ABC} \), where \( \psi_{ABC} \) is the wave function made up of non-classical paths when slits \( A, B, C \) are open, which are not accounted for by single-slit wave functions \( \psi_A, \psi_B, \psi_C \). The inclusion of these corrections mean that \( I_{ABC} \neq 0 \). To overcome these shortcomings, Quach \textsuperscript{34} proposed an alternative parameter using which-way detectors in a double-slit set up:

\[
I_{AB} = P_{AB} - P_{DA} - P_{DB} - P_{DAB} + 2P_{DADB},
\]

where

\[
P_{DA}(x) = |\psi_A(x) + \psi_{AB}(x) + \psi_{BA}(x)|^2 + |\psi_B(x)|^2,
\]

\[
P_{DB}(x) = |\psi_A(x)|^2 + |\psi_{AB}(x) + \psi_{BA}(x)|^2 + |\psi_B(x)|^2,
\]

\[
P_{DAB}(x) = |\psi_A(x) + \psi_{AB}(x) + \psi_{BA}(x)|^2,
\]

\[
P_{DADB}(x) = |\psi_A(x)|^2 + |\psi_B(x)|^2 + |\psi_{AB}(x) + \psi_{BA}(x)|^2.
\]

\( P_{DA}(x) \) and \( P_{DB}(x) \) are the probability distributions when there is a which-way detector in slit \( A \) or \( B \), respectively. \( P_{DAB}(x) \) and \( P_{DADB}(x) \) are the probability distributions of distinguishable and indistinguishable which-way detectors in both slits, respectively. Distinguishable which-way detectors identify whether a particle went through slit \( A \) or \( B \), indistinguishable which-way detectors knows that a particle went through the slits, but does not know which one. \( \psi_{AB}(x) \) consists of Feynman paths that go through slit \( A \) first then slit \( B \), and vice versa for \( \psi_{BA}(x) \).

The Quach parameter has the advantage that \( I_{AB} = 0 \) if the Born is rule is not violated, even in the presence of high-order paths and it applies to the double-slit set up. However, up until now the Quach parameter has only been a formal construct. Here we propose how the Quach parameter could be implemented using atom-cavities.

### A. Implementation of the Quach parameter

To implement the Quach parameter we follow the reasoning of Sec. III using the cavity as which-way detectors. We also write the parameter in terms of the normalised probability distributions \( (P_i) \), as this is what is actually measured:

\[
I_{AB} = N_0 P_{AB} - N_1 (P_{DA} + P_{DB}) - N_2 (P_{DAB} - 2P_{DADB}),
\]

where \( N_i \) are normalisation factors that satisfy

\[
\int_{-\infty}^{\infty} \frac{1}{N_i} P_{AB}(x) dx = \frac{1}{N_i} \int_{-\infty}^{\infty} P_{DA}(x) dx = \frac{1}{N_i} \int_{-\infty}^{\infty} P_{DAB}(x) dx = 1.
\]

To calculate each of these probability a different initial set up is required, as it is summarized in Table I.

| Probability distribution | Setup |
|--------------------------|-------|
| \( P_{AB} \) | Atom Slit A Slit B |
| \( P_{DA} \) | \( |g\rangle \) | 1 | - |
| \( P_{DB} \) | - | 1 | - |
| \( P_{DAB} \) | \( |e\rangle \) | 1 | 1 |

TABLE I. Initial set up of the system, to obtain the respective probability distributions. The atom is either initialised in the ground \( |g\rangle \) or excited \( |e\rangle \) state. \( |1\rangle \) represent a slit-cavity initialised with a single photon.

empty. To obtain, for example, \( P_{DA}(x) \) the set up has a cavity in slit-\( A \) only, and the atom is initially in the ground state \( |g\rangle |1\rangle_A |0\rangle_B \). The evolved state of the system before the detection screen is

\[
|\psi_{DA} \rangle = \frac{1}{\sqrt{N_1}} (|g\rangle |1\rangle_A |0\rangle_B |\psi_B \rangle +
+ |e\rangle |0\rangle_A |0\rangle_B (|\psi_A \rangle + |\psi_{AB} \rangle + |\psi_{BA} \rangle)) \).
\]

The first term represents the state of the system when the atom traverses slit-\( B \) only. The second term represents the state of the system whenever the atom traversed slit-\( A \). In all three cases, the atom absorbed the intra-cavity photon and transitioned to the excited atomic state \( |e\rangle \).

Tracing out the cavity states \( (\mathcal{T}_c) \) and projecting on to the position basis, one can retrieve \( P_{DA}(x) \):

\[
P_{DA}(x) = \langle \mathcal{T}_c (|e\rangle \langle e| + |g\rangle \langle g|) |\psi_{DA} \rangle |\psi_{DA} \rangle |x\rangle
= \frac{1}{N_1} (|\psi_A(x) + \psi_{AB}(x) + \psi_{BA}(x)|^2 + |\psi_B(x)|^2).
\]

In other words, selecting the atoms in the ground and excited state at the detection screen, allows one to obtain the probability of adding a which-way detector in a single slit \( P_{DA}(x) \). \( P_{DB}(x) \) is similarly obtained with the initial state \( |g\rangle |0\rangle_A |1\rangle_B \). We plot \( P_{DA}(x) \) in Fig. 3(c) using the wave functions analytically calculated in Sec. III \( P_{DB}(x) \) has a similar pattern.

To obtain \( P_{DAB}(x) \) and \( P_{DADB}(x) \) we use the analysis developed in Sec. III i.e. the system is initially \( |e\rangle |1\rangle_A |1\rangle_B \), with the cavity photodetector shutters open. The probability distribution of distinguishable which-way detectors, \( P_{DADB}(x) \) is

\[
P_{DADB}(x) = P_g + 2P_e^{(kk)}
= \frac{1}{N_2} (|\psi_A(x)|^2 + |\psi_B(x)|^2 + |\psi_{AB}(x) + \psi_{BA}(x)|^2).
\]

where \( P_e^{(kk)} \) is defined in Eq. (24): the factor of 2 accounts for \( k = + \) and \( k = - \). \( P_g \) is the sum defined in Eq. (24).

We count all atoms in the ground state, while the ones in the excited state are kept only when two photons trigger the same cavity photodetector. The grounded atoms
at each $x$-position give the first term, and the excited atoms give the second term of the probability distribution. We plot $P_{D_A D_B}(x)$ in Fig. 5(b).

To calculate the probability distribution of indistinguishable which-way detectors, $P_{D_A D_B}(x)$, we first review table 1 in Appendix C based on which we can define the required detection strategy. The implementation requires us to keep count of the atoms in the ground state whenever three photons arrive in detector “$P^+$”, and also when one photon arrives in “$P^+$” and two in “$P^-$”. It also requires one to keep count of atoms in the excited state, only when two photons reach the same detector,

$$P_{D_A D_B}(x) = P_{g^+++} + P_{g^-} + P_{g^+-} + P_{g^-} + P_{c^+} + P_{c^-}$$

$$= \frac{1}{N_2} \left( \frac{3}{4} \psi_c^+(x)^2 + 3 \left( \frac{1}{12} \psi_c^+(x)^2 \right) + |\psi_{nc}(x)|^2 \right)$$

$$= \frac{1}{N_2} \left( |\psi_A(x) + \psi_B(x)|^2 + |\psi_{AB}(x) + \psi_{BA}(x)|^2 \right).$$

(38)

We plot $P_{D_A D_B}(x)$ in Fig. 5(d).

With all these probabilities in hand, we calculate $I_{AB}(x) = 0$. Obviously, as our theoretical description assumed the Born rule, this result is expected. However, we propose a detailed practical description to test the Quach parameter. In an experiment, an $I_{AB}(x) \neq 0$, would implicate a Born-rule violation.

V. CONCLUSIONS

In this paper, we have shown using Feynman path integrals a class of of non-classical paths, that can be detected with 1% probability. We used an atomic double-slit setup with cavities in each slit to achieve this. Our proposal operates in the microwave regime, where the non-classical paths are significant, as the wavelength is large compared to the slits spacing. We also show how our setup may be used to implement a delayed-choice quantum eraser for the non-classical paths.

As this setup explicitly utilises quantum mechanical behaviour to isolate and detect the non-classical paths, it offers a possible implementation of the Quach parameter. This is in contrast to other non-classical path proposals which utilise only the classical nature of light [19, 30, 33].

As a future prospect one could use the include the non-classical paths in studying the Aharonov-Bohm effect [40] in the double-slit [11]. Considering the solenoid is strong enough to move the electrons between the slits could be a method to isolate the non-classical paths.

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### Appendix A: Non-classical paths wave function

In this appendix section we define the constants used in Eq. (4):

\[
\Gamma_c = -\frac{im\beta}{\pi^{1/4}\sqrt{-im\sigma_0 + \frac{d^2}{\sigma_0}}} \times \frac{1}{\sqrt{-im^2\beta^2\sigma_0^2 + m(t^2 + (\beta^2 + \sigma_0^2)t)\hbar + it\sigma_0^2}} \quad (A1)
\]

\[
\Gamma_{nc} = -\frac{m^{3/2}\beta^2 \left(\frac{1}{2}\right)^{1/4}}{\sqrt{-im\sigma_0 + \frac{d^0}{\sigma_0}}} \times \frac{1}{\sqrt{-im^2\beta^2\sigma_0^2 + m(t^2 + (\beta^2 + \sigma_0^2)t)\hbar + it\sigma_0^2}} \quad (A5)
\]

\[
c_0' = -\frac{m(2d^2m^2\beta^2\sigma_0^2 + id^2m(2t^2\beta^2 + 2\beta^2(e + \tau) + \sigma_0^2(e + 4\tau))h - d^2(\tau + t(e + 4\tau))h^2)}{8(m^3\beta^4\sigma_0^2 + im^2\beta^2(t^2\beta^2 + \beta^2(e + \tau) + \sigma_0^2(e + 2\tau))h - m(\beta^2 + \sigma_0^2)\tau + t\beta^2(2\tau)h^2 - it\sigma_0^2h^2)} \quad (A6)
\]

\[
c_1' = -\frac{m(4id\sigma_0^2 + \beta^2h - 4d\tau h^2)}{8(m^3\beta^4\sigma_0^2 + im^2\beta^2(t^2\beta^2 + \beta^2(e + \tau) + \sigma_0^2(e + 2\tau))h - m(\beta^2 + \sigma_0^2)\tau + t\beta^2(2\tau)h^2 - it\sigma_0^2h^2)} \quad (A7)
\]

\[
c_2' = -\frac{m(4m^2\beta^2(2\beta^2 + \sigma_0^2) + 4im(2t^2\beta^2 + \beta^2(2\tau))h - 4t\kappa h^2)}{8(m^3\beta^4\sigma_0^2 + im^2\beta^2(t^2\beta^2 + \beta^2(e + \tau) + \sigma_0^2(e + 2\tau))h - m(\beta^2 + \sigma_0^2)\tau + t\beta^2(2\tau)h^2 - it\sigma_0^2h^2)} \quad (A8)
\]

The parameter constants used in this letter are $m = 1.44.10^{-25} \text{kg}$, $d = 5\mu m$, $\sigma_0 = \beta = 0.3\mu m$, $t = \tau = 5\text{ms}$, $\epsilon = 2.9\text{ms}$.

### Appendix B: Conditional probability distribution including intra-cavity photodetection

The time evolution of the density matrix [state in Eq. (19)], with cavities in both slits including the beam splitter and intra-cavity photodetectors $P^+$ and $P^-$. Each photon is detected at individual time interval, such that the density matrix is

\[
\rho(t) = e^{\mathcal{N}t} \rho(0) e^{-\mathcal{N}t} + 2\Gamma \sum_{x=x-rac{1}{2}}^{x=+rac{1}{2}} \int_0^1 dt' e^{-\mathcal{N}(t-t')} a_x e^{\mathcal{N}t'} \rho(0) e^{-\mathcal{N}t'} a_x^\dagger e^{-\mathcal{N}(t-t')} +
\]

\[
+ (2\Gamma)^2 \sum_{x,x'=x-rac{1}{2}}^{x=x+rac{1}{2}} \int_0^1 dt' \int_0^1 dt'' e^{-\mathcal{N}(t-t')} a_x e^{-\mathcal{N}(t''-t')} a_{x'} e^{-\mathcal{N}t''} \rho(0) e^{-\mathcal{N}t''} a_{x'}^\dagger e^{-\mathcal{N}(t''-t')} a_x^\dagger e^{-\mathcal{N}(t-t')} +
\]

\[
+ (2\Gamma)^3 \sum_{x,x',x''=x-rac{1}{2}}^{x=x+rac{1}{2}} \int_0^1 dt' \int_0^1 dt'' \int_0^1 dt''' e^{-\mathcal{N}(t-t')} a_x e^{-\mathcal{N}(t''-t')} a_{x'} e^{-\mathcal{N}(t'''-t'')} a_{x''} e^{-\mathcal{N}t'''} \rho(0) e^{-\mathcal{N}t'''} a_{x''}^\dagger e^{-\mathcal{N}(t'''-t'')} a_{x'}^\dagger e^{-\mathcal{N}(t''-t')} a_x^\dagger e^{-\mathcal{N}(t-t')} (B1)
\]

where $N$ is the number of photons operator and $\Gamma$ is the cavity width. Eq. (B1) is obtained under the assumption photodetection is a stochastic jump process [14, 15, 16]. The first term corresponds to the probability of zero photon absorption, the second term corresponds to the probability of a single photon absorption, the third is the two photon absorption and the last is three photon absorption.

In the limit of long interaction time, $\Gamma t \to \infty$, both $\frac{1}{2} (e^{2\Gamma t} - 1)^3$ and $\frac{1}{2} (e^{2\Gamma t} - 1)^2$ tend to 1.
| Number of photons | Photodetector | Atom - ground state \( \times \frac{1}{N}e^{-6\Gamma t} \) | Atom - excited state \( \times \frac{1}{N}e^{-4\Gamma t} \) |
|------------------|---------------|----------------------|----------------------|
| 0                | ---           | \( \left( |\psi_c^+|^2 + |\psi_c^-|^2 \right) \) | \( \left( |\psi_{nc}^+|^2 + |\psi_{nc}^-|^2 \right) \) |
| 1                | +             | \( (e^{2\Gamma t} - 1) \left( \frac{1}{2} |\psi_c^+|^2 + \frac{1}{2} |\psi_c^-|^2 \right) \) | \( (e^{2\Gamma t} - 1) \left( |\psi_{nc}^+|^2 + |\psi_{nc}^-|^2 \right) \) |
|                  | -             | \( (e^{2\Gamma t} - 1) \left( \frac{1}{2} |\psi_c^+|^2 + \frac{1}{2} |\psi_c^-|^2 \right) \) | \( (e^{2\Gamma t} - 1) \left( |\psi_{nc}^+|^2 + |\psi_{nc}^-|^2 \right) \) |
| 2                | ++            | \( \frac{1}{2} (e^{2\Gamma t} - 1)^2 \left( \frac{1}{4} |\psi_c^+|^2 + \frac{1}{2} |\psi_c^-|^2 \right) \) | \( \frac{1}{2} (e^{2\Gamma t} - 1)^2 \left( |\psi_{nc}^+|^2 \right) \) |
|                  | --            | \( \frac{1}{2} (e^{2\Gamma t} - 1)^2 \left( \frac{1}{4} |\psi_c^+|^2 + \frac{1}{2} |\psi_c^-|^2 \right) \) | \( \frac{1}{2} (e^{2\Gamma t} - 1)^2 \left( |\psi_{nc}^+|^2 \right) \) |
|                  | +- or --      | \( \frac{1}{2} (e^{2\Gamma t} - 1)^2 \left( \frac{1}{4} |\psi_c^+|^2 + \frac{1}{2} |\psi_c^-|^2 \right) \) | \( \frac{1}{2} (e^{2\Gamma t} - 1)^2 \left( |\psi_{nc}^+|^2 \right) \) |
| 3                | ++ +          | \( \frac{1}{5} (e^{2\Gamma t} - 1)^3 \left( \frac{1}{4} |\psi_c^+|^2 \right) \) | \( \frac{1}{5} (e^{2\Gamma t} - 1)^3 \left( |\psi_{nc}^+|^2 \right) \) |
|                  | -- --         | \( \frac{1}{5} (e^{2\Gamma t} - 1)^3 \left( \frac{1}{4} |\psi_c^+|^2 \right) \) | \( \frac{1}{5} (e^{2\Gamma t} - 1)^3 \left( |\psi_{nc}^+|^2 \right) \) |
|                  | + - or - + - or - - + | \( \frac{1}{5} (e^{2\Gamma t} - 1)^3 \left( \frac{1}{4} |\psi_c^+|^2 \right) \) | \( \frac{1}{5} (e^{2\Gamma t} - 1)^3 \left( |\psi_{nc}^+|^2 \right) \) |
|                  | ++ - or - + + or ++ | \( \frac{1}{5} (e^{2\Gamma t} - 1)^3 \left( \frac{1}{4} |\psi_c^+|^2 \right) \) | \( \frac{1}{5} (e^{2\Gamma t} - 1)^3 \left( |\psi_{nc}^+|^2 \right) \) |

**TABLE II.** Probability distribution at each single photodetection, the order of + and – signs correspond to the temporal order of the photodetection.