Revisiting the compatibility problem between the gauge principle and the observability of the canonical orbital angular momentum in the Landau problem

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Abstract

As is widely-known, the eigen-functions of the Landau problem in the symmetric gauge are specified by two quantum numbers. The first is the familiar Landau quantum number $n$, whereas the second is the magnetic quantum number $m$, which is the eigen-value of the canonical orbital angular momentum (OAM) operator of the electron. The eigen-energies of the system depend only on the first quantum number $n$, and the second quantum number $m$ does not correspond to any direct observables. This seems natural since the canonical OAM is generally believed to be a gauge-variant quantity, and observation of a gauge-variant quantity would contradict a fundamental principle of physics called the gauge principle. In recent researches, however, Bliokh et al. analyzed the motion of helical electron beam along the direction of a uniform magnetic field, which was mostly neglected in past analyses of the Landau states. Their analyses revealed highly non-trivial $m$-dependent rotational dynamics of the Landau electron, but the problem is that their papers give an impression that the quantum number $m$ in the Landau eigen-states corresponds to a genuine observable. This compatibility problem between the gauge principle and the observability of the quantum number $m$ in the Landau eigen-states was attacked in our previous letter paper. In the present paper, we try to give more convincing answer to this delicate problem of physics, especially by paying attention not only to the particle-like aspect but also to the wave-like aspect of the Landau electron.

Keywords: Landau problem, electron helical beam, canonical orbital angular momenta, gauge principle and observability, nucleon spin decomposition 71.70.Di, 03.65.-w, 11.15.-q, 11.30.-j

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1. Introduction

Under the presence of magnetic field background, the canonical momentum as well as the canonical orbital angular momentum (OAM) of a charged particle is believed to be gauge-variant quantities different from the mechanical (or kinetic) momentum and the mechanical OAM. Because of their gauge-variant nature, the canonical quantities are generally believed not to correspond to observables. In fact, an observation of the gauge-variant quantity would contradict the so-called gauge principle as one of the important principles of physics. To clearly understand the difference between the canonical OAM and the mechanical OAM is also a key issue for judging which of the two types of decomposition, i.e. the canonical type and of the mechanical type is favored from the physical perspective in the so-called nucleon spin decomposition problem \[1\] -\[9\]. (For review of the nucleon spin decomposition problem, see \[10\],\[11\].) We have already tried to clarify the meanings and the differences of these two different OAMs thorough the studies of the familiar Landau problem \[12\] as well as of the Aharonov-Bohm effect \[13\]. In the present paper, we extend these analyses to the physics of helical electron beam in a uniform magnetic field, with a particular intention of resolving the compatibility problem between the gauge principle and the observability of the canonical OAM. (A part of this investigation was already reported in a letter paper \[14\]. One of the interesting topics, which was discussed in this previous paper but is left out in the present paper, is an intuitive or physical explanation of the \(m\)-dependent splitting of the helical electron beam based on the novel concept of quantum guiding center in the Landau problem \[15\],\[16\].)

The existence of propagating wave carrying intrinsic orbital angular momentum (OAM) has been an object of intensive study for many years and it is firmly established by now not only for photon beams but also for electron beams \[17\] -\[21\]. These helical beams are characterized by an integer \(m\) sometimes called the topological index of the twisted beam. This integer is nothing but the eigen-value of the canonical OAM operator, or more precisely its component along the propagating direction of the photon or electron beam. Note that the canonical OAM is standardly believed to be a gauge-variant quantity. Nevertheless, it is somehow accepted that the observation of this topological index of the (free) helical beams does not contradict the gauge principle. This is probably because there actually exists no difference between the canonical OAM and the manifestly gauge-invariant mechanical (or kinetic) OAM in the case of free photon or electron beams. However, the problem becomes far more intricate, if one considers recently-investigated helical electron beam propagating along the direction of a uniform magnetic field \[22\],\[23\]. In the presence of non-zero magnetic field background, the two OAMs, the gauge-variant canonical OAM and the gauge-invariant mechanical OAM become absolutely different quantities. Hence, whether the canonical OAM ever corresponds to an observable quantity in such nontrivial setups is a fundamental question of physics with universal meaning.

The main purpose of the present paper is to elucidate the physical meanings of the canonical OAM and the mechanical OAM as transparently as possible and then to answer the compatibility question between the gauge principle and the observability of the gauge-variant canonical OAM. First, in sect.2, we briefly remind of the basics of the Landau problem and explain the motivation of our study. Next, in sect.3, we point out a novel symmetry of the Landau eigen-functions which contains the magnetic quantum number \(m\). With close attention to this symmetry, we analyze in detail the structures of the Landau electron's rotational dynamics which depends on the quantum number \(m\). We shall argue that the characteristic of the probability current distribution critically depends on the sign of the magnetic quantum number \(m\) and that it basically explains the three-fold splitting of the helical electron beam in a magnetic field observed by Bliokh et al.
In sect. 4, with a slight hope of possible observation of the magnetic quantum number \( m \) of the Landau electron, we analyze the interference phenomena of two helical electron beams with different magnetic quantum numbers \( m_1 \) and \( m_2 \). By considering various states of combination of \( m_1 \) and \( m_2 \), we realize that there appear rich structures in the rotational velocity of the superposition states. Next in sect. 5, we point out a delicate but critical difference between the nondiffractive Landau beams in a uniform magnetic field and the diffractive Laguerre-Gauss (LG) beams in free space. We argue that this difference is vitally important for resolving the compatibility problem between the observability of the canonical OAM and the gauge principle. Finally, in sect. 6, we summarize what we have learned from the present study.

2. Brief reminder of the Landau problem and research motives

The Landau Hamiltonian, which describes the motion of an electron with charge \(-e \) \((e > 0)\) and mass \( m_e \) in the uniform magnetic field, is given by

\[
H = \frac{1}{2m_e} \Pi^2 = \frac{1}{2m_e} (p + eA)^2 , \tag{1}
\]

where \( A \) is the gauge potential, which reproduces the uniform magnetic field pointing to the \( z \)-direction through the standard relation \( B = \nabla \times A \). (Since only the time-independent gauge symmetry is considered, only the vector potential \( A \) is involved in the following argument.) Here, \( p \) and \( \Pi = p + eA \) represent the canonical and mechanical momentum operators, respectively. (We shall use the natural unit \( \hbar = c = 1 \) throughout the paper.) Since the Landau Hamiltonian contains gauge-dependent vector potential \( A \), its eigen-functions depend on the choice of gauge. For our discussion below, it is convenient to work in the so-called symmetric gauge

\[
A = A^S = \frac{1}{2} B (y, x) . \tag{2}
\]

As is well-known, the eigen-functions of the Landau Hamiltonian in the symmetric gauge are specified by two quantum numbers \( n \) and \( m \) as

\[
\psi_{n,m}(r, \phi) = e^{im\phi} \sqrt{2\pi} R_{n,m}(r) , \tag{3}
\]

with \( R_{n,m}(r) \) being the normalization constant given by

\[
N_{n,m} = (-1)^n \frac{1}{l_B} \frac{1}{n + |m|} ! \sqrt{\frac{1}{(n - |m| + \frac{1}{2}) !}} . \tag{4}
\]

Here, \( l_B = 1/\sqrt{eB} \) is the so-called the magnetic length in the Landau problem, and \( L_n^m(x) \) is the associated (or generalized) Laguerre polynomials. The quantum number \( n \) takes any non-negative integers, i.e. \( n = 0, 1, 2, \cdots \), and is widely called the Landau quantum number, while another quantum number \( m \) takes any integers subject to the constraint \( m \leq n \) for a given \( n \). More
specifically, written above are the simultaneous eigen-functions of the Landau Hamiltonian $H$ and the canonical orbital angular momentum (OAM) operator $L^\text{can}_z \equiv (r \times p)_z$, which satisfy the following eigen-value equations:

$$
H \psi_{n,m}(r, \phi) = E_n \psi_{n,m}(r, \phi), \quad (5)
$$

$$
L^\text{can}_z \psi_{n,m}(r, \phi) = m \psi_{n,m}(r, \phi). \quad (6)
$$

Here, the eigen-energies $E_n$ are given by

$$
E_n = \left(n + \frac{1}{2}\right) \omega_c, \quad (7)
$$

with $\omega_c = \frac{eB}{mc}$ being the familiar cyclotron frequency. We recall that the expectation value of the mechanical OAM operator $L^\text{mech}_z \equiv (r \times \mathbf{\Pi})_z$ in the Landau eigen-states are given by

$$
\langle \psi_{n,m} | L^\text{mech}_z | \psi_{n,m} \rangle = 2n + 1, \quad (8)
$$

which is proportional to the observable Landau energies. In contrast, it is known that the expectation value of the canonical OAM operator, which is nothing but the quantum number $m$ specifying the Landau states, does not correspond to any direct observables. Undoubtedly, non-observable nature of the quantum number $m$ is inseparably connected with infinitely many degeneracy of the Landau levels on $m$, as exemplified by

$$
n = 0, \quad m = 0, -1, -2, -3, -4, \ldots \Rightarrow E_n = \frac{1}{2} \omega_c. \quad (9)
$$

$$
n = 1, \quad m = 1, 0, -1, -2, -3, \ldots \Rightarrow E_n = \frac{3}{2} \omega_c. \quad (10)
$$

$$
n = 2, \quad m = 2, 1, 0, -1, -2, \ldots \Rightarrow E_n = \frac{5}{2} \omega_c. \quad (11)
$$

However, in recent studies of the helical electron beam propagating along the direction of a uniform magnetic field, Bliokh et al. observed a peculiar splitting of the Landau levels depending on the sign of $m$ in the Landau states? It seems to us that this is a perplexing observation which needs clarification, considering that observation of gauge-variant canonical OAM appears to contradict a fundamental principle of physics called the gauge principle.

Their investigations start from noticing close resemblance between the Landau states of the electron represented as

$$
\psi^L_{p,m}(r, \phi, z) \propto \left( \frac{r}{R} \right)^{|m|/2} L^{|m|}_p \left( \frac{r^2}{2R^2} \right) e^{-\frac{z^2}{4R^2}} e^{i(m\phi + k z)}, \quad (12)
$$

with $p \equiv n - \frac{|m|+m}{2}$ being the number of nodes in the radial part of the wave function, and the (free) electron Laguerre-Gauss (LG) beam propagating along the $z$-direction

$$
\psi^{LG}_{p,m}(r, \phi, z) \propto \left( \frac{r^2}{w^2(z)} \right)^{|m|/2} L^{|m|}_p \left( \frac{2r^2}{w^2(z)} \right) e^{-\frac{z^2}{w^2(z)}} + ik \frac{\hat{z}^2}{w^2(z)}
$$

$$
\times e^{i(m\phi + k z)} e^{-(2p + |m| + 1) \arctan \left( \frac{z}{a} \right) / 4}, \quad (13)
$$
with the identification \(2 l_B \leftrightarrow w(z)\). Here, \(l_B\) represents the previously-mentioned magnetic length of the Landau problem, while \(w(z)\) stands for the (weakly) \(z\)-dependent transverse width of the LG beam. The other quantities in the expression of the LG beam can be found in standard literatures of laser physics or helical electron beam physics [17]-[20].

A remarkable observation by Bliokh et al. is that the rotation of electrons in a uniform magnetic field is drastically different from the classical cyclotron motion [22]. According to them, instead of rotation with a single cyclotron frequency \(\omega_c = e B / m_e\), the Landau electrons, while propagating along the direction of the magnetic fields, receive characteristic rotation with three different angular velocities, depending on the eigen-value \(m\) of the canonical OAM operator 

\[
L_{\text{can}}^z = (r \times p)_z ,
\]

\[
\langle \omega \rangle = \begin{cases} 
0 & (m < 0), \\
\omega_L & (m = 0), \\
\omega_c & (m > 0),
\end{cases}
\]

where \(\omega_c\) is the cyclotron frequency, while \(\omega_L = \omega_c / 2\) is the Larmor frequency. In the following analyses, we shall look for a clear physical explanation on the above peculiar splitting of the Landau levels with a special intention of testing the validity of the gauge principle.

3. Landau electron’s probability and probability current distributions

To answer the questions raised in the previous section, we need a thorough understanding of the \(m\)-dependent rotational dynamics of the Landau electron. As argued in our previous letter paper [14], the following way of looking at the Landau problem is particularly helpful. That is, as first pointed out by Johnson and Lippmann a long time ago [29], the Landau Hamiltonian 

\[
H = \frac{1}{2 m_e} (p + e A)^2
\]

in the symmetric gauge can be expressed as a sum of the two pieces, i.e. the Hamiltonian of the familiar 2-dimensional harmonic oscillator and the Zeeman term as

\[
H = H_{\text{osc}} + H_{\text{Zeeman}},
\]

where

\[
H_{\text{osc}} = \frac{1}{2 m_e} (p_x^2 + p_y^2) + \frac{1}{2} m_e \omega_L^2 (x^2 + y^2),
\]

\[
H_{\text{Zeeman}} = \omega_L L_{\text{can}}^z .
\]

Here, \(\omega_L\) is the Larmor frequency, while \(L_{\text{can}}^z\) is just the canonical OAM operator defined by \(L_{\text{can}}^z = (r \times p)_z\). The eigen-functions and the associated eigen-energies of the 2-dimensional harmonic oscillator are textbook material and they are given as

\[
H_{\text{osc}} \hat{\Psi}_{p,m}(r, \phi) = (2 p + |m| + 1) \omega_L \hat{\Psi}_{p,m}(r, \phi),
\]

where

\[
\hat{\Psi}_{p,m}(r, \phi) = e^{i m \phi} \sqrt{\frac{1}{2 \pi}} \hat{R}_{p,m}(r),
\]

with

\[
\hat{R}_{p,m}(r) = (-1)^p \frac{1}{b} \frac{1}{(p + |m|)!} e^{- \frac{r^2}{b^2}} \left( \frac{r^2}{b^2} \right)^{|m|/2} L_p^{|m|} \left( \frac{r^2}{b^2} \right),
\]
and with \( b^2 = 1 / (m_e \omega_L) = 2 / (eB) = 2 \ell_r^2 \). In the above equations, \( p \) stands for the number of radial nodes, which takes zero or any positive integers. On the other hand, \( m \) represents the azimuthal or magnetic quantum number, which is the eigen-value of the canonical OAM operator \( L_{can}^z = -i \frac{\partial}{\partial \phi} \):

\[
L_{can}^z \tilde{\psi}_{p,m}(r, \phi) = m \tilde{\psi}_{p,m}(r, \phi),
\]

with \( m \) taking any (positive, zero, or negative) integers. Since \( \tilde{\psi}_{p,m} \) are simultaneous eigen-functions of \( H_{osc} \) and \( H_{Zeeman} \), it immediately follows that they are also the eigen-functions of the whole Landau Hamiltonian:

\[
H \tilde{\psi}_{p,m}(r, \phi) = E \tilde{\psi}_{p,m}(r, \phi),
\]

with the corresponding eigen-energies,

\[
E = E_{osc} + E_{Zeeman} = \left(2p + |m| + 1\right) \omega_L.
\]

Here, \( E_{osc} \) and \( E_{Zeeman} \) respectively stand for the energy of the 2-dimensional harmonic oscillator and that of the Zeeman term.

Already at this stage, we notice a remarkable fact. If the sign of the magnetic quantum number \( m \) is negative, the energy of the Landau state becomes

\[
E(m < 0) = \left. (2p + 1) \omega_L \right|_{m < 0},
\]

irrespectively of the absolute magnitude of \( m \). This explains the reason why the Landau states with \( m < 0 \) are infinitely degenerate as illustrated by Eqs.(7)-(11). What is worthy of special mention here is how the cancellation of these \( m \)-dependent terms happens. Note that the \( m \)-dependent part of the energy of the 2-dimensional harmonic oscillator depends only on the absolute value of \( m \). This is related to the time-reversal symmetry of the 2-dimensional harmonic oscillator Hamiltonian. Physically, it means that the system energy is independent of the direction of the electron’s rotational motion specified by the sign of \( m \). On the other hand, since the action of the magnetic field breaks the time-reversal symmetry, the Zeeman energy depends on the sign of \( m \) or on the direction of rotation. The infinite degeneracy of the Landau states with \( m < 0 \) can therefore be understood as an interplay of the two different rotational dynamics of the Landau electron. In any case, what is disclosed by the above consideration is a very strange property of the Landau states with negative \( m \) and a simple explanation about its origin. (Although it is left out in the present paper, the singular nature of the Landau state with \( m < 0 \) has another more intuitive explanation based on the quantum guiding center concept in the Landau problem [14].)

In the standard representation of the Landau eigen-states, it is customary to introduce a new quantum number \( n \) defined by

\[
n \equiv p + \frac{|m| + m}{2}.
\]

This quantum number takes zero or any positive integer and is called the Landau quantum number. (Note that, from the above relation between \( n \), \( p \), and \( m \), the quantum number \( n \) must satisfy the inequality \( m \leq n \), that is, the maximum value of \( m \) is \( n \) for a given \( n \), as already exemplified by eqs.(7)-(8).) Accordingly, the eigen-functions of the Landau Hamiltonian are standardly expressed in terms of \( n \) and \( m \) instead of \( p \) and \( m \), which motivates to define new wave functions \( \psi_{n,m}(r) \) by \( \phi_{n,m}(r, \phi) \equiv \tilde{\psi}_{p,m}(r, \phi) \). As a consequence, the eigen-energies of the Landau Hamiltonian depend only on the quantum number \( n \) as

\[
H \psi_{n,m}(r, \phi) = \left(2n + 1\right) \omega_L \psi_{n,m}(r, \phi).
\]
These are basically known stories, but the fact that the Landau eigen-states are also the eigen-states of the 2-dimensional harmonic oscillator makes us aware of an remarkable symmetry of the Landau eigen-functions as first pointed out in [14]. First, as can be explicitly convinced from Eq. (20), the radial wave functions $\tilde{R}_{p,m}(r)$ of the 2-dimensional harmonic oscillator have a very simple symmetry:

$$\tilde{R}_{p,-m}(r) = \tilde{R}_{p,m}(r),$$

(26) i.e. the symmetry under the reverse of the magnetic quantum number $m$. It is clear that this symmetry also originates from the time-reversal invariance of the Hamiltonian of the 2-dimensional harmonic oscillator. Very interestingly, if this symmetry of $\tilde{R}_{p,m}(r)$ is translated into the symmetry of the standard form of radial wave functions in the Landau problem, defined by $R_{n,m}(r) \equiv \tilde{R}_{p,m}(r)$ with $b = \sqrt{2}l_B$, we are led to a highly nontrivial relation represented as

$$R_{n-m,-m}(r) = R_{n,m}(r).$$

(27)

To understand how surprising this symmetry relation is, let us, as an example, consider a special case where $n = m = 20$. In this particular case, one has the relation:

$$R_{0,-20}(r) = R_{20,20}(r).$$

(28)

In view of Eq. (19), this relation dictates that the probability density $|\psi_{0,-20}(r,\phi)|^2$ of the state with $n = 0$ and $m = -20$ is exactly the same as the probability density $|\psi_{20,20}(r,\phi)|^2$ of the state with $n = 20$ and $m = 20$. Note however that the eigen-energies of these two states are totally different as seen from

$$E(n = 0, m = -20) = (2 \times 0 + 1) \omega_L = \omega_L,$$

$$E(n = 20, m = 20) = (2 \times 20 + 1) \omega_L = 41 \omega_L.$$  

(29)  

(30)

We thus observe that, while these two states have exactly the same probability densities, they have totally different eigen-energies. As we shall see below, the cause of this peculiar observation can be traced back to the fact that, although the probability densities of these two states are exactly the same, they have totally different probability current distributions. The point is that, under the presence of the external magnetic field, the internal electric current carried by the electron interacts with this magnetic field, and this interaction also contributes to the energy of the system. This means that, if the two states in question have different probability current distributions, they generally have different energies even if they have the identical probability distributions.

To convince the statements above in more concrete manner, we first recall that the total probability current density of the electron is given by the following familiar expression:

$$j = \frac{1}{m_e} \text{Im} (\psi^* D \psi),$$

(31)

with $D = \nabla + i e A$ being the so-called covariant derivative. Note that the net current shown above is gauge-invariant. (Here and hereafter, we use the word gauge-invariant following the custom of prevailing literatures. More precisely, however, we should use the word gauge-covariant. In fact, what is gauge-invariant is the expectation value of the above current operator.) Obviously, this total current consists of two pieces as

$$j = j^{\text{can}} + j^{\text{gauge}},$$

(32)
with
\[ j^{\text{can}} = \frac{1}{m_e} \text{Im} \left( \psi^* \nabla \psi \right), \quad j^{\text{gauge}} = \frac{1}{m_e} \psi^* e A \psi, \]
(33)
which we hereafter call the canonical current and the gauge current, respectively. (They are sometimes called the paramagnetic current and the diamagnetic current in the field of electronic physical properties [39].) Widely-accepted viewpoint is that the canonical current and the gauge current are not separately gauge-invariant. Only the sum of them is gauge-invariant.

In the Landau states described by the eigen-functions \((19)\) and \((20)\), both parts of current have only the azimuthal components as\( j^{\text{can}} = j^{\text{can}}_\phi e_\phi \) and \( j^{\text{gauge}} = j^{\text{gauge}}_\phi e_\phi \), where
\[ j^{\text{can}}_\phi = \frac{1}{m_e} \frac{m}{r} \rho(r), \quad j^{\text{gauge}}_\phi = \frac{1}{m_e} \frac{r}{2 l_B^2} \rho(r), \]
(34)
with \( \rho(r) = |\psi_{n,m}|^2 \) being the electron probability density. Here, the probability density is normalized as
\[ \int_0^{2\pi} d\phi \int_0^{\infty} dr \, r \, \rho(r) = 2 \pi \int_0^{\infty} dr \, r \, \rho(r) = 1. \]
(35)
Note that, due to the axial symmetry of the Landau eigen-states in the symmetric gauge, \( \rho \) is a function of the radial coordinate \( r \) alone. For convenience, we also introduce the dimensionless probability density \( \tilde{\rho}(R) \) and the dimensionless current density \( \tilde{j}_\phi(R) \) by \( \tilde{\rho}(R) \equiv l_B^2 \rho(r) \) and \( \tilde{j}_\phi(R) \equiv m_e l_B^3 j\phi(r) \) with \( R \equiv r/l_B \) being the dimensionless radial coordinate.

Figure 1: The grey scale images on the left and right panels of this figure respectively show the 2-dimensional plots of the probability distributions corresponding to the Landau state with \( p = 0, m = 20 \) and that with \( p = 0, m = -20 \) in units of dimensionless coordinates \( X = x/l_B \) and \( Y = y/l_B \). Shown by arrows (red in color) in both panels are the dimensionless (total) probability current distributions corresponding to the Landau state with \( p = 0, m = 20 \) and that with \( p = 0, m = -20 \).

In Fig[1] we compare the probability distributions as well as the probability current distribution of the Landau state with \( p = 0, m = 20 \) (or equivalently \( n = 20, m = 20 \)) and those with \( p = 0, m = -20 \) (or \( n = 0, m = -20 \)). The greyscale images shown in the left and right panels of this figure are respectively the 2-dimensional density plots of the probability distributions corresponding to the Landau state with \( p = 0, m = 20 \) and that with \( p = 0, m = -20 \) in the units of dimensionless coordinates \( X = x/l_B \) and \( Y = y/l_B \). One can confirm that the probability distribution of the \( p = 0, m = 20 \) state and that of the \( p = 0, m = -20 \) state perfectly coincide with each other. Shown by arrows (red in color) in the same figures are the dimensionless probability current densities corresponding to the Landau state with \( p = 0, m = 20 \) and that
with \( p = 0, m = -20 \). Taking a closer look at these figures, one sees that the current distributions of these two states are drastically different. For the state with \( p = 0, m = 20 \), the flow of probability current is counter-clock wise in the whole region in which the magnitude of the probability density is appreciable (the brighter region in color). On the other hand, for the state with \( p = 0, m = -20 \), the flow of current is counter-clock-wise in the outer part of the high probability density region, while it is clock-wise in the inner part.

The above remarkable difference in the behavior of the probability currents for the \( m > 0 \) state and for the \( m < 0 \) state would even more clearly be seen if one looks into the Landau states with one radial node. Shown by gray scale images in the left and right panels of Fig. 2 are the 2-dimensional plots of the probability densities corresponding to the state with \( p = 1, m = 20 \) and with \( p = 1, m = -20 \), respectively. Due to the existence of a node of the radial wave function, the probability densities of these two states show double-ring structure. Still, the probability densities of these two states are confirmed to be exactly the same. On the other hand, shown by arrows (red in color) in the same figures are the (net) probability current distributions corresponding to the two Landau states with \( p = 1, m = 20 \) and \( p = 1, m = -20 \). For the state with \( p = 1, m = 20 \), the flow of probability current shown by arrows (red in color) is counter-clock wise in the outer ring as well as in the inner ring. In contrast, for the state with \( p = 1, m = -20 \), the flow of current is counter-clock-wise in the outer ring, while it is clock-wise in the inner ring.

A natural question is then how we can understand the above-mentioned characteristic difference in the structures of the current distributions for the \( m > 0 \) state and for the \( m < 0 \) state. As discussed in the paper [22] by Bliokh et al., it can be understood as an interplay of the canonical and gauge parts of the probability current. In Fig. 3 we compare the probability densities and the probability current densities in the 2-dimensional \((X, Y)\) plane for the Landau state with \( p = 0, m = 20 \) (upper panel) and that with \( p = 0, m = -20 \) (lower panel). The upper panel corresponds to the state with \( p = 0, m = 20 \) or equivalently to the state with \( n = 20, m = 20 \), while the lower panel to the state with \( p = 0, m = -20 \) or equivalently that with \( n = 0, m = -20 \). One can again confirm that the probability density of the state with \( n = 20, m = 20 \) shown in the upper panel and that with \( n = 0, m = -20 \) shown in the lower panel perfectly coincide in spite of the fact that their eigen-energies are totally different as convinced from their different Landau quantum numbers. From the behavior of the canonical and the gauge current distributions, one can easily understand the reason why the behavior of the probability net currents for these two states are remarkably different. Since \( m > 0 \) for the state with \( p = 0, m = 20 \), both

![Figure 2: The same as Fig.1 except that the left panel correspond to the Landau state with \( p = 1, m = 20 \), while the right panel to the Landau state with \( p = 1, m = -20 \).](image-url)
Figure 3: Shown in the three figures on the upper and lower panels are the probability density and the current densities in unit of $X$ and $Y$. The upper panel corresponds to the Landau state with $p = 0$ and $m = +20$, while the lower panel to that with $p = 0$ and $m = -20$. The arrows (red in color) in the left, middle, and right figures in both panels represents the canonical current, the gauge current, and the total current, respectively.

of the canonical current $j_{\phi}^{\text{can}}$ and the gauge current $j_{\phi}^{\text{gauge}}$ are positive, which means that both of the canonical current and the gauge current are circulating in a counter-clock-wise direction. Accordingly, total current is also flowing counter-clock-wise. On the other hand, since $m < 0$ for the state with $p = 0$, $m = -20$, the canonical current is flowing clock-wise, whereas the gauge current is flowing counter-clock-wise. An important fact here is that, because of different radial dependencies of the canonical and gauge currents given as $j_{\phi}^{\text{can}}(r) \propto \frac{1}{r} \rho(r)$ and $j_{\phi}^{\text{gauge}}(r) \propto r \rho(r)$ (see Eq. (34)), the flow of the total current shows highly nontrivial behavior as illustrated in the rightmost figure of the lower panel of this figure. Namely, the flow of the net current for the state with $p = 0, m = -20$ is counter-clock-wise in the outer part of the high probability density region, whereas it is clock-wise in the inner part of the high probability density region.

For the sake of completeness, we show in Fig. 4 the probability densities and the separate contributions of the canonical and gauge currents to the probability current densities for the state with $p = 1, m = 20$ and that with $p = 1, m = -20$. What we can learn from these figures are basically the same as learned from Fig. 3 for the $p = 0$ case, but the double-ring structure coming from a node of the radial wave functions makes it easier to see the interplay between the canonical and gauge currents.

So far, we have seen that the interplay of the canonical current and the gauge current causes a drastic difference in the behavior of the probability current of the electron, which critically depends on the sign of the quantum number $m$ in the Landau states. In fact, it is the origin of the highly nontrivial rotational dynamics of the Landau electron pointed out by Bliokh et al. To understand this, it is convenient to introduce electron’s angular velocity distribution $\omega(r)$ related
Figure 4: The same as Fig. 3 except that all the figures correspond to the Landau state with \( p = 1 \), \( m = \pm 20 \), which has one node in the radial wave function.

\[
\omega(r) = \frac{j_\phi(r)}{r}. \tag{36}
\]

Naturally, this quantity is also expressed as a sum of the contributions of the canonical current and the gauge current as

\[
\omega(r) = \frac{1}{r} \left( J_\phi^{\text{can}}(r) + J_\phi^{\text{gauge}}(r) \right) = \frac{1}{m_e} \frac{m}{r^2} \rho(r) + \frac{1}{m_e} \frac{1}{2 \ell_B^2} \rho(r). \tag{37}
\]

The average angular velocity \( \bar{\omega} \) is obtained by integrating \( \omega(r) \) over the whole \( xy \)-plane as

\[
\bar{\omega} = 2 \pi \int_0^\infty \omega(r) r \, dr. \tag{38}
\]

Using the relation \( 2 \pi \int \frac{\rho(r)}{\rho} r \, dr = 1 / (2 \ell_B^2 |m|) \), one thus finds that

\[
\bar{\omega} = \omega_L \left( \frac{m}{|m|} + 1 \right). \tag{39}
\]

which confirms the 3-fold splitting of the Landau electron’s rotational velocity depending on the sign of the magnetic quantum number \( m \) already shown in Eq. (14). We point out that this relation was already written down in the paper by Li and Wang [26], although its practical importance became clear only after the proposal of using the helical electron beams by Bliokh et al. [22], [23].
Since an important aim of our present study is to clarify the physical meanings and the differences of the canonical and mechanical OAMs of the electron or of any charged particles especially in relation with their observability or nonobservability, we are also interested in the angular momentum density \( l \), which is related to the probability current density \( j \) by

\[
I = m_e r \times j.
\]  

(40)

Since the probability current density is made up of the canonical part and the gauge part, the total angular momentum density also consists of the two pieces as

\[
I = I^{\text{can}} + I^{\text{gauge}},
\]

(41)

with

\[
I^{\text{can}} = m_e r \times j^{\text{can}}, \quad I^{\text{gauge}} = m_e r \times j^{\text{gauge}}.
\]

(42)

The total orbital angular momentum (OAM) density \( l \) is called by various names such as the kinetic OAM, mechanical OAM, or the dynamical OAM [31]. In the present paper, we shall mostly use the terminology, the mechanical OAM. As is widely believed, this mechanical OAM is a gauge-invariant quantity, whereas the canonical OAM as well as the gauge part of the OAM are not separately gauge-invariant. We recall that the Landau eigen-states in the symmetric gauge is the simultaneous eigen-states of the Landau Hamiltonian and the canonical OAM operator. However, one should keep in mind the fact that the Landau eigen-states in other gauges like the so-called Landau gauges are not the eigen-states of the canonical OAM operator [13].

For the Landau eigen-states in the symmetric gauge, both of the canonical OAM and the gauge part of the OAM have only the \( z \)-component and take the following form

\[
l^{\text{mech}}_z = l^{\text{can}}_z + l^{\text{gauge}}_z,
\]

(43)

with

\[
l^{\text{can}}_z = m \rho (r), \quad l^{\text{gauge}}_z = \frac{r^2}{2 l_B^2} \rho (r).
\]

(44)

The spatial integrals of these densities coincide with the expectation values of the relevant OAM operators \( L^{\text{mech}}_z, L^{\text{can}}_z \) and \( L^{\text{gauge}}_z \) in the Landau eigen-states. It is easy to show that the expectation values of these OAM operators are given by

\[
\langle L^{\text{can}}_z \rangle = m, \quad \langle L^{\text{gauge}}_z \rangle = 2n + 1 - m,
\]

(45)

which in turn gives

\[
\langle L^{\text{mech}}_z \rangle = \langle L^{\text{can}}_z \rangle + \langle L^{\text{gauge}}_z \rangle = 2n + 1.
\]

(46)

One observes that, in the expectation value of the mechanical OAM operator, the quantum number \( m \) appearing in the expectation value of the canonical OAM is exactly canceled by the \( m \)-dependent part appearing in the gauge part of the OAM. As a consequence, the expectation value of the mechanical OAM depends only on the Landau quantum number \( n \) and is proportional to the observable Landau energy, which is a well-known fact. (See, for example, [12].)

Before ending this section, we recall that Johnson and Lippmann also pointed out that the magnetic moment \( \mu_z \) of the Landau system is obtained from the relation [32]

\[
\mu_z = \frac{\partial H}{12 B^2}.
\]

(47)
where $H = \sum e_m (\Pi_x^2 + \Pi_y^2)$ is the Landau Hamiltonian in the symmetric gauge. Using the relation
\begin{align*}
\frac{\partial \Pi_x}{\partial B} &= \frac{\partial}{\partial B} \left( -\frac{1}{2} e B y \right) = -\frac{1}{2} e y, \\
\frac{\partial \Pi_y}{\partial B} &= \frac{\partial}{\partial B} \left( +\frac{1}{2} e B x \right) = +\frac{1}{2} e x,
\end{align*}
(48)
together with the commutation relation
\begin{align*}
[x, \Pi_y] &= 0, \\
[y, \Pi_x] &= 0,
\end{align*}
(50)
we are led to a simple relation
\begin{align*}
\mu_z &= \frac{1}{m_e} \left( \Pi_x \frac{\partial \Pi_x}{\partial B} + \Pi_y \frac{\partial \Pi_y}{\partial B} \right) = \frac{e}{2 m_e} \left( x \Pi_y - y \Pi_x \right) = \frac{e}{2 m_e} L_{\text{mech}}^z.
\end{align*}
(51)
This relation just reconfirms that the magnetic moment of the Landau system originates from the cyclotron motion of the electron. Taking the expectation value in the Landau state, it gives
\begin{align*}
\langle B \cdot \mu \rangle &= \omega_L \langle L_{\text{mech}}^z \rangle,
\end{align*}
(52)
where we have used the relation $\omega_L = e B / (2 m_e)$. Remember here the fact that the Landau Hamiltonian is given as a sum of the 2-dimensional harmonic oscillator Hamiltonian $H_{\text{osc}}$ and the Zeeman term $H_{\text{Zeeman}} = \omega_L L_c^z$. Furthermore, there is a simple relation resulting from the Virial theorem that relates the expectation value of the kinetic term and that of the potential term in the 2-dimensional harmonic oscillator Hamiltonian:
\begin{align*}
\left\langle \frac{1}{2} m_e (p_x^2 + p_y^2) \right\rangle &= \left\langle \frac{1}{2} m_e \omega_L^2 (x^2 + y^2) \right\rangle.
\end{align*}
(53)
By using this relation, the expectation value of the Landau Hamiltonian can be written as
\begin{align*}
\langle H \rangle &= \omega_L \langle L_c^z + m_e \omega_L r^2 \rangle.
\end{align*}
(54)
Here, for the eigen-states in the symmetric gauge, it holds that
\begin{align*}
\langle m_e \omega_L r^2 \rangle &= \langle L_{\text{gauge}}^z \rangle.
\end{align*}
(55)
This therefore gives the relation
\begin{align*}
\langle H \rangle &= \omega_L \langle L_c^z + L_{\text{gauge}}^z \rangle = \omega_L \langle L_{\text{mech}}^z \rangle.
\end{align*}
(56)
Comparing (52) and (56), we are then led to a remarkable relation
\begin{align*}
\langle B \cdot \mu \rangle &= \langle H \rangle,
\end{align*}
(57)
with
\begin{align*}
\mu &= \frac{e}{2 m_e} L_c^z + \frac{e}{2 m_e} L_{\text{gauge}}^z = \mu_{\text{can}} + \mu_{\text{gauge}}.
\end{align*}
(58)
This is also an expected relation, since the Landau energy can be thought to come from the interaction between the external magnetic field and the magnetic moment originating from the
cyclotron motion of the electron. (Note that the spin degrees of freedom of the electron is totally neglected in the present paper.) One might call the above energy a generalized Zeeman energy in the sense that the Zeeman term $\omega L^R$ is only a part of this total energy of Zeeman type. An important fact learned from the above consideration is that the interaction between the external magnetic field and the electron’s current always appears in a single combination of the canonical and gauge currents. This implies that each part of the total current cannot be separately measured. According to common wisdom, one would say that this is not unrelated to the fact that only the net current or the mechanical OAM is gauge-invariant, while the canonical part and the gauge part are not separately gauge-invariant.

Still, we should be ready for the following objection. Namely, someone might say that the arguments above is primarily based on the particle aspect of the electron even though its theoretical foundation is quantum mechanics. In fact, in a recent paper [33], Greenshields et al. argue that the canonical and mechanical momenta (and also the corresponding OAMs) are associated with wavelike and particlelike properties, respectively. This appears to suggest a possibility that the canonical momentum and/or the canonical OAM might be observed by making use of a wavelike property of the electron beam, say, through some interference phenomenon of electron wave functions. In the subsequent sections, we shall discuss this very delicate issue with minute attention.

4. Interference of helical electron beams with two different orbital angular momenta

In the typical Landau problem, the motion of the electron along the $z$-direction is neglected or simply treated as a free plane-wave state, so that the problem essentially reduces to a two-dimensional one. As argued in the previous section, the 3-fold splitting of the electron’s average angular velocity $\langle \omega(r) \rangle$ can essentially be understood as an interplay of the canonical and gauge-field parts of the electron’s probability current in the 2-dimensional transverse plane. In the simple 2-dimensional setup, however, there is no way to observe this splitting of the angular velocity of the electron. The wisdom of Blokh et al. for observing this novel splitting is to consider the Landau vortex mode represented by [12] and their superpositions generated in a system with a fixed electron energy $E$ and free propagation along the $z$-direction [22], [23]. Such beams are prepared in the following way. Let us first introduce a longitudinal Larmor length $z_m$ determined by the Larmor frequency $\omega_L$ and the electron velocity $v = \sqrt{2E/m_e}$ as

$$ z_m = \frac{v}{\omega_L}. \quad (59) $$

Note that, by using the transverse magnetic length $w_m = 2/\sqrt{eB} = 2l_B$, this longitudinal scale can also be expressed as

$$ z_m = \frac{2 \sqrt{2E/m_e}}{eB} = \frac{E}{\omega_L w_m}. \quad (60) $$

Here, different modes will have different wave number $k_z$ that satisfies the dispersion relation

$$ E = E_{||} + E_{\perp}, \quad (61) $$

where $E_{||} = k_z^2/(2 m_e)$ represents the energy corresponding to the electron’s motion along the $z$-direction, whereas

$$ E_{\perp} = E_Z + E_G, \quad (62) $$
stands for the energy corresponding to the electron’s motion in the transverse plane, i.e. the plane perpendicular to the z-axis. Here, the first part \( E_Z = m \omega_L \) corresponds to the Zeeman energy, while the second part \( E_G \) corresponds to the eigen-energy of the 2-dimensional harmonic oscillator discussed in sect.3. In \[22\], this second part was called the Gouy term because it can be related to the Gouy phase appearing in the diffractive LG beam represented by \( \ell_\parallel \). Under the condition that the paraxial conditions \( E_\perp \ll E \) and \( w_m \ll z_m \) are satisfied, the wave number \( k_z \) can approximately be written as
\[
k_z \approx k + \Delta k_z,
\]
where \( k = \sqrt{2 \epsilon E m} \), and
\[
\Delta k_z(p, m) = - \left[ m + (2p + |m| + 1) \right] / z_m.
\]
This gives
\[
e^{i k_z z} \approx e^{i k z} e^{i \Delta k_z z} = e^{i k_z z} e^{i \Phi_{LZG}},
\]
with
\[
\Phi_{LZG} = - \left[ m + (2p + |m| + 1) \right] z / z_m.
\]
This additional phase \( \Phi_{LZG} \) was called the Landau-Zeeman-Gouy phase in \[22\]. We emphasize the fact that this extra phase depends on the two quantum numbers \( p \) and \( m \) only through the combination \( m + (2p + |m| + 1) \), which is equal to \( 2n + 1 \) if we use the Landau quantum number defined by \( n \equiv p + (|m| + m) / 2 \).

In sum, the above consideration reveals that, while propagating along the \( z \)-direction, the magnetic field correction to the longitudinal wave number \( k_z \) yields an additional phase to the wave functions. After this correction is taken into account, the superposed nondiffractive Landau beam becomes
\[
\psi_{p, m}(r, \phi, z) \approx \frac{N_{p, m}}{\sqrt{2 \pi}} \left( \frac{2 \pi^2}{w_m} \right)^{|m|/2} L_p^{(|m|)} \left( \frac{2 \pi^2}{w_m^2} r^2 \right) e^{-\frac{r^2}{w_m^2}} e^{i k_z z} e^{i \ell_\parallel(p, m) z},
\]
with \( N_{p, m} \) being the normalization constant of the Landau state given by
\[
N_{p, m} = \frac{2}{w_m} \sqrt{\frac{p!}{(p + |m|)!}}.
\]
This means that the electron wave function acquires an extra phase \( \Phi_{LZG} = \Delta k_z(p, m) z \), which changes as a function of the propagation distance \( z \) along the directions of the magnetic field. However, this \( z \)-dependent phase change of the wave function cannot be detected straightforwardly by observing the beam intensity. In fact, the probability density of the electron beam is given by
\[
\rho = |\psi_{p, m}|^2 = \frac{N_{p, m}^2}{2 \pi} \left( \frac{2 \pi^2}{w_m^2} \right)^{|m|} L_p^{(|m|)} \left( \frac{2 \pi^2}{w_m^2} \right) e^{-\frac{r^2}{w_m^2}} e^{-2 \frac{z^2}{w_m^2}},
\]
which is axially-symmetric with respect to the \( z \)-axis and is independent of the coordinate \( z \). This is the very reason why the authors of \[23\] had to invent a clever experiment in which half of the beam is obstructed to stop with an opaque knife edge stop and the spatial rotation of the visible part of the beam is traced by moving the knife edge along the beam direction, in order to verify the \( m \)-dependent splitting of the electron helical beam.
Observation of the phase rotation of the wave function would however be possible if one constructs a superposition state of two Landau beams as given by
\[
\psi = \frac{1}{\sqrt{a^2 + b^2}} \left( a \psi_{p_1,m_1} + b \psi_{p_2,m_2} \right),
\] (70)
with two different values of the quantum numbers \( p \) and \( m \). For simplicity, we assume here that the mixing coefficients \( a \) and \( b \) are both real numbers. By using the relation \( \Delta k_z(p,m) = -(2n + 1)/z_m \) with \( n = p + (|m| + m)/2 \) together with the relation \( z/z_m = \omega \mu_{L}(z/v) \), the probability density of the above superposition state is easily calculated and given by
\[
\rho = |\psi|^2 = \frac{1}{2\pi(a^2 + b^2)} \left\{ a^2 N_{p_1,m_1}^2 \left( \frac{2r^2}{w_m^4} \right) e^{-\frac{2r^2}{w_m^2}} \left[ L_{p_1}^{|m_1|} \left( \frac{2r^2}{w_m^2} \right) \right]^2 \\
+ b^2 N_{p_2,m_2}^2 \left( \frac{2r^2}{w_m^4} \right) e^{-\frac{2r^2}{w_m^2}} \left[ L_{p_2}^{|m_2|} \left( \frac{2r^2}{w_m^2} \right) \right]^2 \\
+ 2ab N_{p_1,m_1} N_{p_2,m_2} \cos \left( (m_1 - m_2) \left( \phi - \frac{2(n_1 - n_2)}{m_1 - m_2} \omega \mu_{L}(z/v) \right) \right) \\
\times \left( \frac{2r^2}{w_m^2} \right) \left[ L_{p_1}^{|m_1|} \left( \frac{2r^2}{w_m^2} \right) L_{p_2}^{|m_2|} \left( \frac{2r^2}{w_m^2} \right) \right],
\] (71)
where
\[
N_{p_1,m_1} = \frac{2}{w_m} \left( \frac{p_1}{(p_1 + |m_1|)!} \right), \quad N_{p_2,m_2} = \frac{2}{w_m} \left( \frac{p_2}{(p_2 + |m_2|)!} \right),
\] (72)
and
\[
n_1 = p_1 + \frac{|m_1| + m_1}{2}, \quad n_2 = p_2 + \frac{|m_2| + m_2}{2}.
\] (73)
Noticeable here is the appearance of the interference term, which depends on the azimuthal angle \( \phi \) as well as on the propagation distance \( z \). In consideration of the relation \( z = vt \), the above expression dictates that the interference term in the probability density rotates around the \( z \)-axis with the angular velocity
\[
\omega = \frac{2(n_1 - n_2)}{m_1 - m_2} \omega \mu_{L}.
\] (74)
Here, we emphasize again the fact that, aside from these two quantum numbers \( m_1 \) and \( m_2 \) specifying the incident beam, \( \omega \) depends on \( p_1, m_1 \) and \( p_2, m_2 \) only through the two quantum number \( n_1 \) and \( n_2 \) related to the Landau quantum number \[34].

In the following analyses, we confine ourselves to simple cases in which the numbers of node of the radial wave functions are both zero, i.e. we limit to the cases in which \( p_1 = p_2 = 0 \) with the arbitrary magnetic quantum numbers \( m_1 \) and \( m_2 \). For the mixing coefficients \( a \) and \( b \), we shall use the values given by \( a = 1.0 \times \left( \sqrt{|m_1|} / 2|m_1|/2 \right) \) and \( b = 2.0 \times \left( \sqrt{|m_2|} / 2|m_2|/2 \right) \) throughout all the analyses below. After setting \( p_1 = p_2 = 0 \), the above rotational velocity \( \omega \) reduces to the form:
\[
\omega = \left( 1 + \frac{|m_1| - |m_2|}{m_1 - m_2} \right) \omega \mu_{L}.
\] (75)
Note that this expression is symmetric under the exchange of \( m_1 \) and \( m_2 \). With use of the dimensionless radial coordinate \( \xi = 2 r^2/w_m^2 \), the probability density of the superposition state is then given by

\[
\rho = \frac{1}{2 \pi} \frac{1}{a^2 + b^2} e^{-\xi} \left\{ a^2 N_{0,m_1}^2 \xi^{m_1} + b^2 N_{0,m_2}^2 \xi^{m_2} \right\} + 2 a b N_{0,m_1} N_{0,m_2} \xi^{m_1+m_2} \cos \left( (m_1 - m_2) \left( \phi - \frac{|m_1| + m_1 - |m_2| - m_2}{m_1 - m_2} \frac{z}{z_m} \right) \right). \tag{76}
\]

We can also calculate the probability current density for the same superposition state. For this state, the canonical part of the current contains both of radial and azimuthal components as

\[
j_{\text{can}} = j_{\phi} e_r + j_{\theta} e_\theta.
\]

whereas the gauge part contains only the azimuthal component as

\[
j_{\text{gauge}} = j_\phi e_\theta.
\]

By using the dimensionless radial coordinate \( \xi = 2 r^2/w_m^2 \), they are expressed as

\[
j_{r,\text{can}} = \frac{1}{2 \pi} \frac{a b}{a^2 + b^2} N_{0,m_1} N_{0,m_2} \sqrt{2} \frac{r}{m_e w_m} \xi^{m_1+m_2} \left\{ a^2 N_{0,m_1}^2 \xi^{m_1} + b^2 N_{0,m_2}^2 \xi^{m_2} \right\} e^{-\xi} \times \sin \left( (m_1 - m_2) \left( \phi - \frac{|m_1| + m_1 - |m_2| - m_2}{m_1 - m_2} \frac{z}{z_m} \right) \right), \tag{79}
\]

\[
j_{\phi,\text{can}} = \frac{1}{2 \pi} \frac{1}{a^2 + b^2} \frac{1}{m_e r} e^{-\xi} \left\{ m_1 a^2 N_{0,m_1}^2 \xi^{m_1} + m_2 b^2 N_{0,m_2}^2 \xi^{m_2} \right\} \cos \left( (m_1 - m_2) \left( \phi - \frac{|m_1| + m_1 - |m_2| - m_2}{m_1 - m_2} \frac{z}{z_m} \right) \right), \tag{80}
\]

and

\[
j_{\phi,\text{gauge}} = \frac{1}{2 \pi} \frac{1}{a^2 + b^2} \frac{e B}{2 m_e} r e^{-\xi} \left\{ a^2 N_{0,m_1}^2 \xi^{m_1} + b^2 N_{0,m_2}^2 \xi^{m_2} \right\} \cos \left( (m_1 - m_2) \left( \phi - \frac{|m_1| + m_1 - |m_2| - m_2}{m_1 - m_2} \frac{z}{z_m} \right) \right). \tag{81}
\]

Following \[22\], we consider some typical cases in order to see the effect of phase rotation in the superposed electron beam. The first is the OAM-balanced superposition specified by \( m_1 = -m_2 \equiv m \) with \( m \) being arbitrary positive integer. Shown in Fig.5 are the probability densities as well as the probability current densities projected on the \( xy \)-plane as functions of the dimensionless propagation distance \( Z = z/z_m \). They are shown for four representative values of \( Z \), i.e. \( Z = 0, 0.4, 0.8, \) and \( 1.2 \). The upper, middle and lower panels respectively correspond to the superposition states with \( (m_1, m_2) = (-1, 1) \), \( (m_1, m_2) = (-2, 2) \), and \( (m_1, m_2) = (-3, 3) \).
probability and current densities: $m_1 = -1, m_2 = +1$

Figure 5: The three (upper, middle, and lower) panels represent the probability densities and the probability current densities in the $xy$-plane as functions of the dimensionless propagation distance $Z = z/z_m$ for the OAM-balanced superposition state with $m_1 = -m_2$. Here and in the subsequent figures, the dimensionless coordinate $X$ and $Y$ are defined by $X \equiv x/l_B$ and $Y \equiv y/l_B$ with $l_B = w_m/2$. The upper, middle, and lower panels respectively correspond to the states with $(m_1, m_2) = (-1, +1), (-2, +2),$ and $(-3, +3)$.

First, by looking at the leftmost three figures with $Z = 0$, one clearly sees that the above three states respectively have two, four and six brighter domains which corresponds to the higher probability density regions. One can also see that, for all these three states, the probability current densities shown by arrows (red in color) are circulating around the origin and that they have no components along the radial direction. This can be analytically convinced if one looks at the expression for radial component of the current $j_{r}^{\text{im}}$ given in (79). The radial component is clearly seen to vanish in the OAM-balanced superposition with $|m_1| = |m_2|$. As $Z$ increases, i.e. as the beam propagates along the $z$-direction, one clearly sees the rotation of the high density regions in the interference patterns. One can readily check that the rotational velocity of these three superposition states with $(m_1, m_2) = (-1, 1), (-2, 2),$ and $(-3, 3)$ are all the same and it is given by $\omega_L = \frac{1 + (|m_1| - |m_2|)(m_1 - m_2)}{m_1 - m_2} \omega_L = \omega_L$, i.e. the Larmor
frequency. This confirms the observation in the paper [22].

Next, we consider the superposition in which one of the magnetic quantum number, say $m_1$, is zero, while another magnetic number $m_2$ is arbitrary nonzero integer. Shown in Fig.6 are the probability densities and the probability current densities projected on the $xy$-plane as functions of the dimensionless propagation distance $Z = z/z_m$. Similar densities shown in the lower panel are those of the superposition state with $(m_1, m_2) = (0, -1)$.

Figure 6: The upper panel shows the probability densities and the current densities projected on the $xy$-plane for the superposition state with $(m_1, m_2) = (0, +1)$ as functions of the dimensionless propagation distance $Z = z/z_m$. Similar densities shown in the lower panel are those of the superposition state with $(m_1, m_2) = (0, -1)$.

One clearly observes that, as $Z$ increases, the high density region for the state with $(m_1, m_2) = (0, +1)$ state rotates with a constant angular velocity. One can verify that this angular velocity is given by

$$\omega_L = \left[ 1 + (|m_1| - |m_2|)/(m_1 - m_2) \right] \omega_c,$$

where $\omega_c$ is the cyclotron frequency. In sharp contrast, the interference pattern for the superposition state with $(m_1, m_2) = (0, -1)$ does not show any rotational behavior as $Z$ increases. This is just consistent with the fact that the average rotational velocity of this state is given by

$$\omega_L = \left[ 1 + (|m_1| - |m_2|)/(m_1 - m_2) \right] \omega_c = 0.$$
We also show in Fig. 7 one more example of superposed beam in which the first magnetic quantum number \( m_1 \) is zero but the second magnetic quantum \( m_2 \) is nonzero. The upper panel of Fig. 7 corresponds to the case with \((m_1, m_2) = (0, +2)\), while the lower panel corresponds to the superposition state with \((m_1, m_2) = (0, -2)\). Again, by looking at the leftmost two figures with \( Z = 0 \), one confirms that the probability densities of these two states have two brighter regions and that probability densities of these two states coincide perfectly with each other. However, the current densities of these two states shown by arrows (red in color) are drastically different, and this also generates totally different rotational behavior of the interference patterns. One confirms that the interference pattern for the state with \((m_1, m_2) = (0, +2)\) rotates with the angular velocity \( \omega_c \) when propagating along the \( z \)-direction. On the other hand, the rotational velocity of the interference pattern is zero for the superposition state with \((m_1, m_2) = (0, -2)\).

Aside from the number of peaks in the probability densities, all the features of the states with \((m_1, m_2) = (0, \pm 2)\) appear to be nothing different from those of the states with \((m_1, m_2) = (0, \pm 1)\). However, there is one profound difference between the states with \((m_1, m_2) = (0, \pm 2)\) and the states with \((m_1, m_2) = (0, \pm 1)\). In fact, making a careful inspection of Fig. 7, one realizes that the centroid of the former state with \((m_1, m_2) = (0, \pm 2)\) is always located at the coordinate center, but this is not the case with the \((m_1, m_2) = (0, \pm 1)\) states shown in Fig. 7. At \( Z = 0 \), the centroid of the \((m_1, m_2) = (0, \pm 1)\) states lies on the positive \( x \)-axis. When \( Z \) is increased, the centroid of the \((m_1, m_2) = (0, +1)\) state rotates with the angular velocity \( \omega_c \) around the \( z \)-axis, which means that the centroid of this superposition state makes a spiral motion while propagating along the \( z \)-axis. On the other hand, the centroid of the \((m_1, m_2) = (0, -1)\) state makes no rotation in the
xy-plane, because it remains on the same position along X-axis. In other words, the centroid of this states makes a rectilinear motion along the z-axis.

The helical motion of the centroid of the superposition state with \( m_1 = 0 \) and \( m_2 = 1 \) was already pointed out and called the “spiralling of the Bohmian trajectory” in the paper by Bliokh et al. [22],[23]. However, the description in their paper is a little misleading in that it fails to emphasize the fact that \( m_2 = \pm 1 \) mode is exceptional. As a consequence, there might be a danger of giving a wrong impression that the superposition state with \( m_1 = 0 \) and arbitrary \( m_2 (>0) \) also makes a spiral motion. Actually, as already pointed out above, the centroid of the \((m_1, m_2) = (0, \pm 2)\) state is always located at the coordinate origin, so that it does not make a helical motion. One can check that this is also the case for all the state with \( m_1 = 0 \) and \(|m_2| = 2, 3, 4, \ldots\). The reason why the \( m_2 = \pm 1 \) modes are special can be understood if one remembers the theory of nuclear deformation or that of the electric dipole giant resonance in nuclei. (See, for example, [24].) For example, with use of the time-dependent shape parameter \( \alpha_{\lambda \mu}(t) \), the nuclear deformation is standardly parametrized in the following form

\[
R(\theta, \phi, t) = R_0 \left( 1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda \mu}^*(t) Y_{\lambda \mu}(\theta, \phi) \right).
\]

Here, \( R(\theta, \phi, t) \) stands for the nuclear radius in the direction \((\theta, \phi)\) at time \( t \), while \( R_0 \) is a radius of spherical nucleus. It is well-known that the \( \lambda = 1 \) mode is special among others. In fact, the dipole deformations \( \lambda = 1 \) really do not correspond to a deformation of the nucleus but rather to the shift of the center of mass of the nucleus. Similarly, in the present problem of electron beams, one should notice that the wave function of the \( m_2 = \pm 1 \) mode is proportional to \( r e^{i\theta} \approx r Y_{1+1}(\theta = \frac{\pi}{2}, \phi) \). This means that the mixture of the \( m_2 = \pm 1 \) mode into the \( m_1 = 0 \) mode works to shift the centroid of the superposition beam toward the perpendicular direction to the beam axis.

Now we summarize in Table 1 the rotational velocity of the interference patterns of the typical superposition states investigated so far. They are the OAM-balanced superposition with \( m_1 = -m_2 \) and the OAM-unbalanced superposition in which one of the magnetic quantum number is zero. As already pointed out in [22], three different rotational velocities appear in these cases. They are the Larmor frequency \( \omega_L \), the cyclotron frequency \( \omega_c \), and the zero frequency. Table 1 summarizes these situations.

| \( m_1 \) | \( m_2 \) | \( \omega \) |
|---------|---------|--------|
| \( +|m| \) | \( -|m| \) | \( \omega_L \) |
| \( 0 \) | \( \text{positive integer} \) | \( \omega_c \) |
| \( 0 \) | \( \text{negative integer} \) | \( 0 \) |

Table 1: The rotational velocities \( \omega \) of the OAM-balance superposition state with \((m_1 = +|m|, m_2 = -|m|)\) and also the OAM-unbalanced superposition states with \((m_1 = 0, m_2 = \text{any positive integer})\) and with \((m_1 = 0, m_2 = \text{any negative integer})\).

For the sake of completeness, let us consider some other types of OAM-unbalanced superpositions, which were not touched upon in [22]. Shown in Table 2 are the cases in which both of \( m_1 \) and \( m_2 \) are positive integers or both of \( m_1 \) and \( m_2 \) are negative integers. One sees that the superposition state with \( m_1 > 0 \) and \( m_2 > 0 \) rotates with the cyclotron frequency, while the superposition state with \( m_1 < 0 \) and \( m_2 < 0 \) state shows no rotational behavior. In any case, for the superposition state considered so far, only three types of rotation frequency appear. They are the cyclotron frequency \( \omega_c \), the Larmor frequency \( \omega_L \), and zero frequencies.

Note however that, if one considers the superposition of LG beams with \( m_1 > 0 \) and \( m_2 < 0 \) or with \( m_1 < 0 \) and \( m_2 > 0 \), one finds that the rotational dynamics of the superposition state
Table 2: The rotational velocities $\omega$ for the superposition state with ($m_1 = \text{any positive integer}, m_2 = \text{any positive integer}$) and the state with ($m_1 = \text{any negative integer}, m_2 = \text{any negative integer}$).

| $m_1$ | positive integer | negative integer |
|-------|------------------|------------------|
| $m_2$ | positive integer | negative integer |
| $\omega$ | $\omega_c$ | 0 |

Table 3: The rotational velocities $\omega$ for the superposed states with ($m_1 = 1, m_2 = \text{any negative integer}$) and the states with ($m_1 = -1, m_2 = \text{any positive integer}$).

| $m_1$ | +1 | +1 | +1 | +1 | +1 | $\cdots$ | +1 |
|-------|----|----|----|----|----|--------|----|
| $m_2$ | -1 | -2 | -3 | -4 | -5 | $\cdots$ | $-\infty$ |
| $\omega$ | $\omega_L$ | $\frac{2}{3} \omega_L$ | $\frac{4}{5} \omega_L$ | $\frac{6}{7} \omega_L$ | $\frac{8}{9} \omega_L$ | $\cdots$ | 0 |

| $m_1$ | -1 | -1 | -1 | -1 | -1 | $\cdots$ | -1 |
|-------|----|----|----|----|----|--------|----|
| $m_2$ | +1 | +2 | +3 | +4 | +5 | $\cdots$ | $+\infty$ |
| $\omega$ | $\omega_L$ | $\frac{2}{3} \omega_L$ | $\frac{4}{5} \omega_L$ | $\frac{6}{7} \omega_L$ | $\frac{8}{9} \omega_L$ | $\cdots$ | $2 \omega_L$ |

becomes far richer. To see it, we investigate here two simple cases. The first is the superposition state with $m_1 = 1$ and $m_2$ being any negative integer, while the second is the one with $m_1 = -1$ and $m_2$ being any positive integer. As seen from Table 3, if one changes the value of $m_2$ from $-1$ to $-\infty$ with maintaining the value of $m_1$ to be 1, the average rotational velocity $\bar{\omega}$ gradually decreases and vanishes in the $m_2 \to -\infty$ limit. On the other hand, Table 3 also shows that, if one changes the value of $m_2$ from $+1$ to $+\infty$ with maintaining the value of $m_1$ to be $-1$, $\bar{\omega}$ gradually increases and approaches the cyclotron frequency $\omega_c$ in the $m_2 \to +\infty$ limit. Obviously, if one extends this analysis to more general cases with arbitrary value of $m_1$ and $m_2$ with $m_1 m_2 < 0$, wider variety of $\omega$ in the form of rational number times $\omega_L$ will appear.

To explicitly convince the above-explained structure of the rotational dynamics of the superposition states, we show in Fig. 8 the probability densities as well as the probability current densities projected on the $xy$-plane as functions of $Z = z/z_m$. The upper, middle, and the lower panels respectively correspond to the superposition states with $(m_1, m_2) = (+1, -2)$, $(m_1, m_2) = (+1, -4)$, and $(m_1, m_2) = (+1, -8)$. (In the last case with $(m_1, m_2) = (+1, -8)$, the density distribution contains actually 9 blobs in the azimuthal direction, but they are so close to each other that it looks like a ring.)

One clearly sees the rotations of the interference patterns as functions of the dimensionless propagation distance $Z$. One can also convince that the rotational frequencies of these three states are respectively given by $\frac{2}{3} \omega_L$, $\frac{4}{5} \omega_L$, and $\frac{6}{7} \omega_L$. Clearly, $\bar{\omega}$ is approaching zero as $|m_2|$ increases.

Next, in Fig. 9 we show the probability densities as well as the probability current densities of the states with $m_1 = -1$ and $m_2 = +2, +4, +8$, projected on the $xy$-plane as functions of $Z = z/z_m$. The upper, middle, and the lower panels respectively correspond to the superposition
states with \((m_1, m_2) = (-1, +2)\), and \((m_1, m_2) = (-1, +4)\), \((m_1, m_2) = (-1, +8)\). (Just like the third figure in Fig.8, the density distribution in the third figure here actually contains 9 blobs in the azimuthal direction.) One again observes the rotations of the interference patterns as \(Z\) increases. One can also convince that the rotational frequencies of these three states are respectively given by \(\frac{1}{3} \omega_L\), \(\frac{2}{5} \omega_L\), and \(\frac{4}{9} \omega_L\). Clearly seen here is the tendency that \(\omega\) approaches \(2 \omega_L = \omega_c\) as \(|m_2|\) increases.

![Probability and current densities](image)

In this way, we have seen that, if we consider the interference of the Landau beams with two different magnetic quantum numbers \(m_1\) and \(m_2\), there appear rich structures in the rotational dynamics of the superposition state. The emerging rotational frequency of the superposed probability density is not limited to the cyclotron, Larmor, and zero frequencies. Arbitrary rotational frequency in the form of rational number times \(\omega_L\) can appear in general. These rotational velocities, which depend on the magnetic quantum number or the topological index of the beam, are direct observables, since the rotation of the probability density of the superposition state must be
observed without any difficulty. Our question here is the relation with observables in the Landau state, especially the observability of the magnetic quantum number $m$ as the eigenvalue of the canonical OAM. From the three-fold splitting of the rotational dynamics of the Landau states as well as from the rotation of the interference pattern of the superposition of the two beams, the authors of [22], [23] gave an impression that the Larmor and the Gouy term of the total energy can separately be observed, which amounts to claiming that the eigenvalue of the canonical OAM is an observable. On the other hand, however, it appears to contradict the fact that the canonical OAM in the Landau problem is a gauge-variant quantity. The next section is devoted to the discussion of this very delicate and confusing problem.

5. Discussions

Throughout the investigations so far, motivated by the works of Bliokh et al. [22], [23], attention has been mostly paid to the similarity between the wave functions of the Landau states

![Figure 9](image_url)
and those of the helical LG beams of the electron. Although the wave functions of these two systems are both described by the Laguerre-Gauss type functions, one should not overlook a critical difference between them. To understand the importance of this difference, let us first recall that, leaving aside the trivial free motion in the $z$-direction, the Landau Hamiltonian is expressed as a sum of the Hamiltonian of the 2-dimensional harmonic oscillator and the Zeeman term. Naturally, the Hamiltonian of the 2-dimensional harmonic oscillator contains the restoring force potential, which is proportional to the square of the transverse distance from the coordinate origin. The cause of this restoring force term is of course the vector potential in the symmetric gauge, which reproduces the uniform magnetic field along the $z$-direction. To put it another way, what generates the cyclotron motion of the Landau electron is the Lorentz force caused by the external magnetic field. In sharp contrast, the helical LG beam of the electron is a solution of the free Schrödinger equation although in the paraxial approximation. It may be a little perplexing that the wave function of the free electron beam is expressed with the same functions as those of the Landau problem with nonzero magnetic field. The magic is the diffractive nature of the LG beam, which is embedded in the $z$-dependence of the beam width. This $z$-dependence of the beam width as well as that of the curvature radius of the wave front are the key factors that the Laguerre-Gauss types of wave functions are solutions of the free Schrödinger equation in the paraxial approximation. (This has been a long known fact in the field of twisted or helical photon beams in optics [17],[18],[19].)

Let us now minutely compare the two OAMs, i.e. the canonical and mechanical OAMs, with close attention to the above difference between the non-diffractive Landau beams and the diffractive LG beams. We first recall that the magnetic quantum number $m$ (also called the topological index) of the LG beam of the electron is widely believed to be a direct observable. An immediate question is whether the observation of the canonical OAM does not contradict the gauge principle or not, since we know that the canonical OAM is believed to be a gauge-variant quantity. A way out of this seeming conflict may be found out as follows. First, note that, for free electrons, there is no substantial difference between the canonical OAM and the mechanical OAM because the gauge potential term is absent in a free space. Next, in the most general context, what appears in the dynamical equation of motion of a charged particle is the mechanical (or total) momentum or the mechanical (or total) OAM not the canonical momentum or canonical OAM. In a free space, however, the canonical OAM simply coincides with the total OAM, so that it is a quantity which appears in the equation of motion of the electron. Then, it can in principle be extracted, for example, by measuring the torque exerted on the external atoms, etc. Undoubtedly, this would be the reason why canonical OAM of the free LG beam can be a direct observable without conflicting with the gauge principle. Note, however, that the situation is a little different for the canonical OAM in the Landau problem. Under the presence of the nonzero magnetic field, the canonical current and the gauge current and also the canonical OAM and the gauge part of the OAM are both nonzero, and besides they always appear as a single combination. As a consequence, the canonical part of the current or the OAM cannot be separately extracted, unless there are some external probes, which couples to the canonical current and the gauge current in a different manner.

Now we recall that, based on the decomposition of the transverse part of the energy given by Eq.(62), Bliokh et al. argued that the Landau energy of an electron in a magnetic field is given as the sum of the Zeeman and Gouy contributions as $E_{\perp} = E_Z + E_G$ (22). They further claim that these two contributions are separately observable. This statement is a little misleading, however. The statement would be certainly true for the diffractive LG beams, but it is not necessarily true for the electron energy in the original Landau problem or for the transverse energy in the
nondiffractive Landau beam. First, in the original Landau problem, the Zeeman part of the energy is proportional to the magnetic quantum number \( m \) which is the eigen-value of the gauge-variant canonical OAM. Besides, as is clear from the discussion in sect.3, what-they-call the Gouy term of the energy (which is nothing but the eigen-energy of the 2-dimensional harmonic oscillator) is also dependent on the choice of gauge. By this reason, it is usually believed that these two parts of energy are not separately observable at least in the original Landau problem. The situation is basically the same also for the case of nondiffractive Landau beam. As can be seen from (66), the Zeeman phase part proportional to \( m \) and the Gouy phase part proportional to \( 2p + |m| + 1 \) always appear in a single combination, so that they are be separately measurable.

The situation is quite different for the diffractive LG beam, however. In fact, in the case of the diffractive LG beam, the Gouy term of the transverse energy is connected with the Gouy phase of the diffractive LG beam. Here, the Gouy phase of the diffractive LG beam takes the following form with \( z_R \) being the Rayleigh diffraction length,

\[
\Phi_G = - (2p + |m| + 1) \arctan \left( \frac{z}{z_R} \right),
\]

which shows that the Gouy phase makes drastic jump before and behind the beam waist where \( z = 0 \). This enables us to observe it directly and separately with the Zeeman term. This feature is of course a specific property of the diffractive LG beam of the electron not possessed by the nondiffractive Landau beam. This after all means that the separate observation of the Zeeman term and the what-they-call the Gouy term of the electron’s energy is a property of the diffracting LG beam but not a property of the Landau electron. For that reason, it would still be legitimate to say that the gauge-principle is not violated in the original Landau problem as well as in the physics of (free) helical electron beams.

Finally, the difference between the non-diffractive Landau electron beam and the diffractive LG vortex electron beam was also discussed in a recent paper [36]. (See also the related paper [37], which treats electron vortex beam in non-uniform magnetic field.) In that paper, a general quantum-mechanical solution was obtained for the twisted electrons in a uniform magnetic field. This solution is remarkably different from the nondiffractive Landau beam of Blokh et al. and also from the free vortex beam of the electron. A prominent feature of this solution is that, as a reflection of its non-plane wave character along the magnetic field direction, it correctly describes the \( z \)-dependence of the beam width \( w(z) \) for vortex electrons in a uniform magnetic field. As a consequence, the angular velocity has no longer a simple relation with the Larmor frequency \( \omega_L \), as given by (74), but it is generally \( z \)-coordinate dependent. Another prominent feature of the above general vortex solution is that it coincides with Landau solution when condition \( w_0 = w_m \) is fulfilled for the beam waist of the free vortex electron \( w_0 \) and the beam width parameter \( w_m \). If this condition is fulfilled, the beam radius becomes a constant given by \( w(z) = w_m = 2l_B \). Under this circumstance, the electron angular velocity becomes \( z \)-independent and the conclusion obtained with Eq. (74) would be justified. We also recall that, in [23], the electron beam was tuned such that the beam radius \( w(z) \) comes close to magnetic radius \( w_m \), i.e. such that the relation \( w(z) \approx w_m \) holds. This might explain why the experimental data in [23] shows a good linear relation between the angular velocity and the Larmor frequency. As explained above, since the quantum-mechanical solutions of vortex electron beam and approximate solutions for the nondiffractive Landau beam are fairly different, a more careful analysis based on the exact solution would be highly desirable. It will be reported in a separated paper.
6. Summary and conclusion

Motivated by the observation of the formal resemblance between the eigen-functions of the Landau problem and the wave functions of the diffractive LG beam, Bliokh et al. [22] and also Schattschneider et al. [23] tried to disentangle the internal rotational dynamics of the Landau states, which were not fully resolved in past studies of this old but popular problem. Among others, they verified that the Landau modes with different magnetic (or azimuthal) quantum numbers \( m \) belong to three classes, which are characterized by rotations with zero, Larmor and cyclotron frequencies, respectively. At first glance, in view of the fact that the magnetic quantum number \( m \) of the Landau eigen-state is the eigen-value of the gauge-dependent canonical OAM operator, this \( m \)-dependent splitting of the Landau levels appears to expose a breakdown of the widely-accepted gauge principle. Motivated by this naive suspicion, we have carried out a careful analysis of the \( m \)-dependent rotational dynamics of the Landau eigen-states \( |n, m\rangle \) in the symmetric gauge and confirmed that unexpectedly rich structure is hidden in the \( m \)-dependencies of the Landau states. First, we pointed out a novel symmetry of the electron’s probability densities of the two Landau states \( |n - m, - m\rangle \) and \( |n, m\rangle \). It was shown that these two states have exactly the same probability densities, in spite that they have totally different eigen-energies. The cause of this peculiar observation was verified to be traced back to the difference between the probability current distributions of these two states, which is generated by the interplay of the canonical and gauge-potential parts of the total (or mechanical) current, which critically depends on the sign of the quantum number \( m \). In any case, since this 3-fold splitting of the electron’s rotational velocity is a prediction based on the gauge-invariant total current (or equivalently the mechanical current), our conclusion is that it does not contradict the gauge principle.

The conclusions up to this point were drawn chiefly based on the similarities between the Landau states and the wave functions of the single LG beams. If one considers superposition of the two Landau beams with different magnetic quantum numbers \( m_1 \) and \( m_2 \), far richer structure turns out to appear in the rotational dynamics of such superposition states. We begin with some simple cases already investigated in the paper by Bliokh et al. [22]. They are the OAM-balanced superposition with \( m_1 = -m_2 \), the OAM-unbalanced superposition with \( m_1 = 0 \) and \( m_2 \) being some positive or negative integers. In these cases, we confirmed their observation that the rotational velocity of the superposed probability densities are limited to the three values, i.e. zero, Larmor and cyclotron frequencies. However, if we consider more general superpositions with arbitrary integers \( m_1 \) and \( m_2 \) with \( m_1 m_2 < 0 \), it turns out that wider variety of rotational velocity in the form of rational number times \( \omega_L \) appear. A question is whether all these facts mean observations of the magnetic quantum number \( m \) in the Landau states. The answer is probably no. To convince it, we have emphasized a delicate but important difference between the nondiffractive Landau beam and the diffractive LG beam of the electron. The former is a solution of a Schrödinger equation under the presence of the nonzero magnetic field, while the latter is a solution of a free Schrödinger equation in a paraxial approximation.

Suppose now that, by following the experiments carried out by Schattschneider [23], the (free and diffractive) LG beam of the electron is incident into a nonzero magnetic field region. As explained in the paper by Karlovets [38], which takes account of realistic experimental setup for preparing for magnetic fields, the free LG beam turns into nondiffractive Landau beam around the boundary of the free space and the nonzero magnetic field region. This transition happens in a finite transition period, i.e. in the cyclotron period. Once entering the nonzero magnetic field region, the phase rotation of the nondiffractive Landau beam is determined by \( m + (2p + |m| + 1) = 2n + 1 \), so that the Larmor and Gouy part of the phase rotation cannot be separated. However,
after going out of the nonzero magnetic field region, the nondiffractive Landau beam again turns into the free LG beam. After propagating a long distance, the $z$-dependence of the width of this free LG beam becomes appreciable and the beam eventually reaches the focal point where the drastic change of the Gouy phase happens. This would make it possible to separate the Zeeman and Gouy part of the phase rotation. We therefore think that the separation of the corresponding Zeeman and Gouy parts of the electron’s transverse energy should be thought of as the property of the diffractive LG beam, and it should not be interpreted as $m$-dependent splitting of the Landau state. If we think this way, we can say that any of the observations made in the papers [22, 23] would not contradict the widely-accepted gauge principle.

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[1] R.L. Jaffe and A. Manohar, The $g_1$ problem : Deep inelastic electron scattering and the spin of the nucleon, Nucl. Phys. B337 (1990) 509.
[2] X. Ji, Gauge-Invariant Decomposition of Nucleon Spin, Phys. Rev. Lett. 78 (1997) 610.
[3] S.V. Bashinsky, R.L. Jaffe, Quark and gluon orbital angular momentum and spin in hard processes, Nucl. Phys. B536 (1998) 303.
[4] M. Wakamatsu, Gauge-invariant decomposition of nucleon spin, Phys. Rev. D81 (2010) 114010.
[5] M. Wakamatsu, Gauge- and frame-independent decomposition of nucleon spin, Phys. Rev. D83 (2011) 014012.
[6] M. Burkardt, Parton orbital angular momentum and final state interactions, Phys. Rev. D88 (2013) 014014.
[7] X. Ji, J.-H. Zhang, Y. Zao, Justifying the naive partonic sum rule for proton spin, Phys. Lett. B743 (2015) 180.
[8] M. Wakamatsu, On the two remaining issues in the gauge-invariant decomposition problem of the nucleon spin, Eur. Phys. J. A51 (2015) 52.
[9] M. Wakamatsu, Unravelling the physical meaning of the Jaffe-Manohar decomposition of the nucleon spin, Phys. Rev. D95 (2016) 056004.
[10] E. Leader and C. Lorcé, The angular momentum controversy : What’s it all about and does it matter? Phys. Rep. 541 (2014) 163.
[11] M. Wakamatsu, Is gauge-invariant complete decomposition of the nucleon spin possible? Int. J. Mod. Phys. A29 (2014) 1430012.
[12] M. Wakamatsu, Y. Kitadono, P.-M. Zhang, The issue of gauge choice in the Landau problem and the physics of canonical and mechanical orbital angular momenta, Ann. Phys. 392 (2018) 287–322.
[13] M. Wakamatsu, Y. Kitadono, L.-P. Zou, P.-M. Zhang, The role of electron orbital angular momentum in the Aharonov-Bohm effect revisited, Ann. Phys. 397 (2018) 259–277.
[14] M. Wakamatsu, Y. Kitadono, L.-P. Zou, P.-M. Zhang, The physics of helical electron beams in a uniform magnetic field as a testing ground of gauge principle, Phys. Lett. A384 (2020) 126415.
[15] Y. Kitadono, M. Wakamatsu, L.-P. Zou, P.-M. Zhang, Role of guiding center in Landau level system and mechanical and pseudo orbital angular momenta, Int. J. Mod. Phys. A35 (2020) 2050096.
[16] S.J. van Enk, Angular momentum in the fractional quantum Hall effect, American Journal of Physics 88 (2020) 286.
[17] L. Allen, M. Beijersbergen, R. Spreeuw, J. Woerdman, Orbital angular momentum of light and the transformation of laguerre-gaussian laser modes, Phys. Rev. A45 (1992) 8185–8189.
[18] L. Allen, M. Padgett, M. Babiker, The orbital angular momentum of light, Prog. Opt. 39 (1999) 291–371.
[19] I. Torres, L. Torner, Twisted Photons : Applications of Light with Orbital Angular Momentum, Wiley-VCH, New York, 2011.
[20] K. Bliokh, Y. Bliokh, S. Savel’ev, F. Nori, Semiclassical dynamics of electron wave packet states with phase vorteces, Phys. Rev. Lett. 99 (2007) 190404/1-4.
