ATOMIC SEMIGROUP RINGS AND THE ASCENDING CHAIN CONDITION ON PRINCIPAL IDEALS

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Abstract. An integral domain is called atomic if every nonzero nonunit element factors into irreducibles. On the other hand, an integral domain is said to satisfy the ascending chain condition on principal ideals (ACCP) if every ascending chain of principal ideals terminates. It was asserted by Cohn back in the sixties that every atomic domain satisfies the ACCP, but such an assertion was refuted by Grams in the seventies with an explicit construction of a neat example. Still, atomic domains without the ACCP are notoriously elusive, and just a few classes have been found since Grams’ first construction. In the first part of this paper, we generalize Grams’ construction to provide new classes of atomic domains without the ACCP. In the second part of this paper, we construct what seems to be the first atomic semigroup ring without the ACCP in the existing literature.

1. Introduction

An integral domain is atomic if every nonzero nonunit factors into irreducibles, while an integral domain satisfies the ACCP if every ascending chain of principal ideals terminates. One can verify that every integral domain satisfying the ACCP is atomic. In particular, Noetherian domains are atomic. Further relevant classes of commutative rings, including Krull domains and Mori domains, satisfy the ACCP and are, therefore, atomic. Although the properties of being atomic and satisfying the ACCP are not equivalent in the context of integral domains, the distinction is subtle. In fact, the equivalence was asserted by P. Cohn in 1968. This wrong assertion was corrected by A. Grams in 1974, with a construction of the first atomic domain without the ACCP.

Since then, the interplay between atomicity and the ACCP has been the subject of several papers (see there recent paper [3] and references therein). Yet, producing atomic domains that do not satisfy the ACCP has been challenging, and only a few constructions have been provided since Grams constructed the first example five decades ago. The second construction of an atomic domain without the ACCP was given by A. Zaks in [18], where it is proved that certain quotient of a given polynomial ring in infinitely many variables is atomic (this construction was suggested by Cohn, who pointed out that such a quotient does not satisfy the ACCP). In 1993, M. Roitman [17] constructed the first atomic domain \( R \) whose ring of polynomials is not atomic, incidentally producing an atomic domain without the ACCP. More recently, J. Boynton and J. Coykendall [3] constructed a class of atomic domains without the ACCP using pullbacks of commutative rings.

In Section 2, we introduce the notation and remind the definitions and main results we will use throughout the paper. In Section 3, we briefly review Grams’ construction of the first atomic domain without the ACCP and introduce the notion of atomization. Then, in Theorem 3.3, we provide a generalization of Grams’ construction. The given generalization allows us to produce new atomic
domains without the ACCP by localizing monoid algebras, where the main ingredients are rank-one torsion-free atomic monoids, which are not that hard to come by. We illustrate this with some examples.

Our primary purpose in Section 4 is to construct a monoid algebra (i.e., a monoid domain over a field) that is atomic but does not satisfy the ACCP, and we do so in Theorem 4.4. We have mentioned before all the references of constructions of atomic domains without the ACCP that we have found in the literature, and it is worth noticing that each of them uses some algebraic construction on rings, namely, quotients, localizations, direct unions, or pullbacks. In particular, none of the existing examples of atomic domains without the ACCP is as elementary as the monoid domain we exhibit in Theorem 4.4.

2. Preliminary

2.1. General Notation. Following common notation, we let \( \mathbb{Z}, \mathbb{Q}, \) and \( \mathbb{R} \) denote the sets of integers, rational numbers, and real numbers, respectively. In addition, we let \( \mathbb{P}, \mathbb{N}, \) and \( \mathbb{N}_0 \) denote the sets of primes, positive integers, and nonnegative integers, respectively. For \( a, b \in \mathbb{Z} \), we let \([a, b]\) denote the discrete interval \( \{ n \in \mathbb{Z} \mid a \leq n \leq b \} \), allowing \([a, b]\) to be empty when \( a > b \). In addition, given \( S \subseteq \mathbb{R} \) and \( r \in \mathbb{R} \), we set \( S_{\geq r} = \{ s \in S \mid s \geq r \} \) and \( S_{> r} = \{ s \in S \mid s > r \} \). For \( q \in \mathbb{Q} \setminus \{0\} \), we let \( n(q) \) and \( d(q) \) denote, respectively, the unique \( n \in \mathbb{N} \) and \( d \in \mathbb{Z} \) such that \( q = n/d \) and \( \gcd(n, d) = 1 \). Accordingly, for any \( Q \subseteq \mathbb{Q} \setminus \{0\} \), we set

\[
\begin{align*}
n(Q) &= \{ n(q) \mid q \in Q \} \quad \text{and} \quad d(Q) = \{ d(q) \mid q \in Q \}.
\end{align*}
\]

Finally, for each \( p \in \mathbb{P} \) and \( n \in \mathbb{Z} \setminus \{0\} \), we let \( v_p(n) \) denote the maximum \( v \in \mathbb{N}_0 \) such that \( p^v \) divides \( n \), and for \( q \in \mathbb{Q} \setminus \{0\} \), we set \( v_p(q) = v_p(n(q)) - v_p(d(q)) \) (in other words, \( v_p \) is the \( p \)-adic valuation map of \( \mathbb{Q} \) restricted to nonzero rationals).

2.2. Monoids. In the scope of this paper, a monoid is a semigroup with identity that is both cancellative and commutative. Let \( M \) be an additively written monoid. We let \( M^\bullet \) denote the set of nonzero elements. In addition, we let \( \mathscr{U}(M) \) denote the group of invertible elements of \( M \), and we let \( M_{\text{red}} \) denote the quotient monoid \( M/\mathscr{U}(M) \). The monoid \( M \) is called reduced if \( \mathscr{U}(M) \) is the trivial group, in which case, \( M \) is naturally isomorphic to \( M_{\text{red}} \). The difference group of \( M \), denoted by \( \mathscr{D}(M) \), is the unique abelian group up to isomorphism satisfying that any abelian group containing a homomorphic image of \( M \) will also contain a homomorphic image of \( \mathscr{U}(M) \). The monoid \( M \) is torsion-free if \( \mathscr{D}(M) \) is a torsion-free group (or equivalently, if for all \( a, b \in M \), if \( na = nb \) for some \( n \in \mathbb{N} \), then \( a = b \)).

For a subset \( S \) of \( M \), we let \( \langle S \rangle \) denote the submonoid of \( M \) generated by \( S \), that is, the smallest (under inclusion) submonoid of \( M \) containing \( S \). An ideal of \( M \) is a subset \( I \) of \( M \) such that \( I + M \subseteq I \) (or, equivalently, \( I + I = I \)). An ideal of \( M \) is principal if there exists \( b \in M \) satisfying \( I = b + M \). For \( b_1, b_2 \in M \), we say that \( b_2 \) divides \( b_1 \) in \( M \) if \( b_1 + M \subseteq b_2 + M \), in which case we write \( b_2 \mid_M b_1 \), and we say that \( b_1 \) and \( b_2 \) are associates if \( b_1 + M = b_2 + M \). The monoid \( M \) is a valuation monoid if for any \( b_1, b_2 \in M \) either \( b_1 \mid_M b_2 \) or \( b_2 \mid_M b_1 \). We say that \( M \) satisfies the ascending chain condition on principal ideals (ACCP) if every increasing sequence (under inclusion) of principal ideals eventually terminates. An element \( a \in M \setminus \mathscr{U}(M) \) is an atom (or an irreducible) if whenever \( a = u + v \) for some \( u, v \in M \), then either \( u \in \mathscr{U}(M) \) or \( v \in \mathscr{U}(M) \). We let \( \mathcal{A}(M) \) denote the set of atoms of \( M \). The monoid \( M \) is atomic if every non-invertible element factors into atoms. One can check that every monoid satisfying the ACCP is atomic.
2.3. **Factorizations.** Observe that the monoid $M$ is atomic if and only if $M_{\text{red}}$ is atomic. We let $Z(M)$ denote the free (commutative) monoid on $\mathcal{A}(M_{\text{red}})$, and we let $\pi: Z(M) \to M_{\text{red}}$ be the unique monoid homomorphism fixing the set $\mathcal{A}(M_{\text{red}})$. For every $b \in M$, we set

\[ Z(b) = Z_M(b) = \pi^{-1}(b + \mathcal{A}(M)). \]

Observe that $M$ is atomic if and only if $Z(b)$ is nonempty for any $b \in M$. The monoid $M$ is called a **finite factorization monoid (FFM)** if it is atomic and $|Z(b)| < \infty$ for every $b \in M$. In addition, $M$ is called a **unique factorization monoid (UFM)** if $|Z(b)| = 1$ for every $b \in M$. By definition, every UFM is an FFM. If $z = a_1 \cdots a_\ell \in Z(M)$ for some $a_1, \ldots, a_\ell \in \mathcal{A}(M_{\text{red}})$, then $\ell$ is called the **length** of $z$ and is denoted by $|z|$. For each $b \in M$, we set

\[ L(b) = L_M(b) = \{|z| \mid z \in Z(b)\}. \]

The monoid $M$ is called a **bounded factorization monoid (BFM)** if it is atomic and $|L(b)| < \infty$ for all $b \in M$. Observe that if $M$ is an FFM, then it is also a BFM. On the other hand, the reader can verify that every BFM satisfies the ACCP ([10, Corollary 1.4.4]).

The set consisting of all nonzero elements of an integral domain $R$ is a monoid, which is denoted by $R^{*}$ and called the *multiplicative monoid* of $R$. Every factorization property defined for monoids in the previous paragraph can be rephrased for integral domains. We say that $R$ is a **unique (resp., finite, bounded) factorization domain** provided that $R^{*}$ is a unique (resp., finite, bounded) factorization monoid. Accordingly, we use the acronyms UFD, FFD, and BFD. Observe that this new definition of a UFD coincides with the standard definition of a UFD. In order to simplify notation, we write $Z(R) = Z(R^{*})$, and for every $x \in R^{*}$, we write $Z(x) = Z_{R^{*}}(x)$ and $Z(x) = Z_{R^{*}}(x)$. As for monoids, we let $\mathcal{A}(R)$ denote the set of atoms/irreducibles of $R$.

Let $R$ be an integral domain, and let $M$ be a torsion-free monoid. Following R. Gilmer [11], we let $R[M]$ denote the monoid ring of $M$ over $R$, that is, the ring consisting of all polynomial expressions with exponents in $M$ and coefficients in $R$. It follows from [11, Theorem 8.1] that $R[M]$ is an integral domain. Accordingly, we often call $R[M]$ a monoid domain. In addition, it follows from [11, Theorem 11.1] that $R[M]^{*} = \{rx^{u} \mid r \in R^{*} \text{ and } u \in \mathcal{A}(M)\}$. In light of [11, Corollary 3.4], we can assume that $M$ is a totally ordered monoid. Let $f(x) = c_{n}x^{q_{n}} + \cdots + c_{1}x^{q_{1}}$ be a nonzero element in $R[M]$ for some coefficients $c_{1}, \ldots, c_{n} \in R^{*}$ and exponents $q_{1}, \ldots, q_{n} \in M$ satisfying $q_{a} > \cdots > q_{1}$. Then we call $\deg f = \deg_{R[M]} f := q_{n}$ and $\ord f = \ord_{R[M]} f := q_{1}$ the **degree** and the **order** of $f$, respectively. In addition, we call the set supp $f = \text{supp}_{R[M]}(f(x)) := \{q_{1}, \ldots, q_{n}\}$ the **support** of $f$.

### 3. Generalized Grams’ Construction

As we mentioned in the introduction, the first example of an atomic domain without the ACCP was constructed by Grams. The main purpose of this section is to generalize such construction. First, let us describe the integral domain given by Grams.

A torsion-free rank-one monoid that is not a group is called a Puiseux monoid. It follows from [9, Theorem 3.12.1] that nontrivial submonoids of $(\mathbb{Q}_{\geq 0}, +)$ account for all Puiseux monoids up to isomorphism, and their atomicity has been systematically studied recently (see [6] and references therein). Let $(p_{n})_{n \in \mathbb{N}}$ be the strictly increasing sequence consisting of odd primes, and consider the Puiseux monoid

\[ M := \left\langle \frac{1}{2^n}p_{n} \mid n \in \mathbb{N}\right\rangle. \]

Let $F$ be a field, and let $S$ denote the multiplicative set $\{f \in F[M] \mid \ord f = 0\}$ of the monoid domain $F[M]$. Then it follows from [14, Theorem 1.3] that the localization $F[M]_{S}$ of $F[M]$ at $S$ is an atomic domain, which does not satisfy the ACCP because the ascending chain of principal ideals.
$(x^{1/2^n} F[M]_S)_{n \in \mathbb{N}}$ does not terminate. Honoring Grams, we call $M$ the Grams monoid and $F[M]_S$ the Grams domain over $F$. The fact that $M$ contains the valuation monoid $N := \langle 1/2^n \mid n \in \mathbb{N} \rangle$ as a submonoid plays an important role. The second crucial property that makes Grams’ construction work is that every element of $M$ has a largest divisor in $N$.

To formalize and generalize the last two observations, let $M$ be a monoid, and let $N$ be a submonoid of $M$. For each $m \in M$, a greatest divisor of $m$ in $N$ is an element $d \in N$ satisfying the following two properties:

- $d \mid_M m$ and
- if $d' \mid_M m$ for some $d' \in N$, then $d' \not\mid_M d$.

Clearly, any two greatest divisors in $N$ of the same element of $M$ must be associates, and so if $M$ is reduced, then every element of $M$ has at most one greatest divisor in $N$. We say that $N$ is a greatest-divisor submonoid of $M$ provided that every element of $M$ has a greatest divisor in $N$. Assume now that $M$ is reduced, and also that $N$ is a greatest-divisor submonoid of $M$. We let $gd_N(m)$ denote the greatest divisor of $m$ in $N$. The following observations can be deduced directly from the definition of a greatest divisor:

- $\gcd_N(x - \gcd_N(x)) = 0$;
- if $x \mid_M y$, then $\gcd_N(x) \mid_M \gcd_N(y)$.

Let $N = \langle q_n \mid n \in \mathbb{N} \rangle$ be a Puiseux monoid generated by a sequence $(q_n)_{n \in \mathbb{N}}$ consisting of positive rationals, and let $(p_n)_{n \in \mathbb{N}}$ be a sequence of primes whose terms are pairwise distinct such that $\gcd(p_i, q(q_i)) = \gcd(p_i, d(q_i)) = 1$ for all $i, j \in \mathbb{N}$. We call the monoid $M := \langle \frac{q_n}{p_n} \mid n \in \mathbb{N} \rangle$ an atomization of $N$ at the sequence $(p_n)_{n \in \mathbb{N}}$. Observe that an atomization of $N$ not only depends on the sequence of primes $(p_n)_{n \in \mathbb{N}}$ but also on the chosen generating set of $N$.

**Proposition 3.1.** Let $N = \langle q_n \mid n \in \mathbb{N} \rangle$ be a Puiseux monoid with $q_n > 0$ for every $n \in \mathbb{N}$, and let $(p_n)_{n \in \mathbb{N}}$ be a sequence of pairwise distinct primes such that $\gcd(p_i, q(q_i)) = \gcd(p_i, d(q_i)) = 1$ for all $i, j \in \mathbb{N}$. Then the following statements hold.

1. The atomization $M := \langle \frac{q_n}{p_n} \mid n \in \mathbb{N} \rangle$ of $N$ is atomic with $\mathcal{A}(M) = \{ \frac{q_n}{p_n} \mid n \in \mathbb{N} \}$.

2. $N$ is a greatest-divisor submonoid of $M$.

**Proof.** (1) It suffices to verify that $\mathcal{A}(M) = \{ \frac{q_n}{p_n} \mid n \in \mathbb{N} \}$. This is indeed the case: observe that if $q_j/p_j = \sum_{i=1}^{n} c_i q_i/p_i$ for some $c_1, \ldots, c_n \in \mathbb{N}$, then after taking $p_i$-adic valuations (for all $i \in \mathbb{N}$) in both sides of this equality we obtain that $c_j = 1$ and $c_i = 0$ for every $i \in [1, n] \setminus \{ j \}$.

(2) We first observe that for each $b \in M$, there exist coefficients $c_n \in \{0, p_n - 1\}$ for all $n \in \mathbb{N}$ (only finitely many of them being different from 0) such that

$$b = \nu(b) + \sum_{n \in \mathbb{N}} c_n \frac{q_n}{p_n},$$

where $\nu(b) \in N$. We claim that $\nu(b)$ and the coefficients $c_n$ in the decomposition (3.2) are uniquely determined. To argue this, suppose that

$$\nu(b) + \sum_{n \in \mathbb{N}} c_n \frac{q_n}{p_n} = \mu(b) + \sum_{n \in \mathbb{N}} d_n \frac{q_n}{p_n}$$

for some $\mu(b) \in N$ and coefficients $d_n \in \{0, p_n - 1\}$, all but finitely many of them being zero. For each $n \in \mathbb{N}$, we can take $p_n$-adic valuation on both sides of (3.3) to see that $c_n \equiv d_n \pmod{p_n}$, which implies that $c_n = d_n$. Therefore $\nu(b) = \mu(b)$ and the claimed uniqueness follows.
We proceed to argue that $N$ is a greatest-divisor submonoid of $M$. For each $b \in M$, we verify that $\nu(b)$ is the greatest divisor of $b$ in $N$. Clearly, $\nu(b) \mid_M b$. Suppose now that $d \in N$ also satisfies $d \mid_M b$. Then after writing $b - d$ as in (3.2), the uniqueness of the decomposition will guarantee that $\nu(b) = \nu(b - d) + d$, which implies that $d \mid_M \nu(b)$. As a result, $\nu(b)$ is the greatest divisor of $b$ in $N$. Hence $N$ is a greatest divisor submonoid of $M$.

Let us take a second look at the Grams monoid from a different point of view.

Example 3.2. Consider the Puiseux monoid $N := \left\{ \frac{1}{2^n} \mid n \in \mathbb{N} \right\}$, and let $(p_n)_{n \in \mathbb{N}}$ be the strictly increasing sequence whose terms are the odd prime numbers. Then we can recover the Grams monoid as an atomization of $N$ (with respect to the defining generating set) at the sequence $(p_n)_{n \in \mathbb{N}}$. Therefore it follows from Proposition 3.1 that the Grams monoid is atomic with $\mathcal{A}(M) = \left\{ \frac{1}{p_n} \mid n \in \mathbb{N} \right\}$ and also that $M$ contains the valuation monoid $N$ as a greatest-divisor submonoid.

We proceed to establish the main result of this section.

Theorem 3.3. Let $F$ be a field, and let $M$ be an atomic reduced torsion-free monoid. Also, let $N$ be a submonoid of $M$ satisfying the following conditions:

(1) $N$ is a valuation greatest-divisor submonoid of $M$, and

(2) $L_M(m - \text{gd}_N(m))$ is finite for every $m \in M$.

Then $F[M]_S$ is atomic, where $S = \{ f \in F[M] \mid f(0) \neq 0 \}$.

Proof. We argue first that $X^a$ is irreducible in $F[M]_S$ for all $a \in \mathcal{A}(M)$. To do so, take $a \in \mathcal{A}(M)$, and suppose that $X^a = \frac{X^{a_1}}{s_1} \cdots \frac{X^{a_t}}{s_t}$ for some $f_1, f_2 \in F[M]$ and $s_1, s_2 \in S$. Then $X^a s_1 s_2 = f_1 f_2$ and, therefore, $\text{ord } f_1 + \text{ord } f_2 = \text{ord } f_1 f_2 = a$. Since $\text{ord } f_1$ and $\text{ord } f_2$ both belong to $M$ and $a \in \mathcal{A}(M)$, it follows that either $\text{ord } f_1 = 0$ or $\text{ord } f_2 = 0$, which implies that either $f_1$ or $f_2$ belongs to $S$. Hence $X^a \in \mathcal{A}(F[M]_S)$.

In order to prove that $F[M]_S$ is atomic, it suffices to show that every nonzero nonunit in $F[M]$ factors into irreducibles in $F[M]_S$. Take a nonzero nonunit $f \in F[M]$ and write $f = \sum_{j=1}^n r_j X^{m_j}$, assuming that $m_1 > \cdots > m_n$ and $r_1 \cdot \cdots \cdot r_n \neq 0$. Now set

$$q = \min\{ \text{gd}_N(m_j) \mid j \in [1, n] \} \quad \text{and} \quad k = \max\{ j \in [1, n] \mid \text{gd}_N(m_j) = q \}.$$ 

Since $N$ is a valuation monoid, we can write $f = X^q g$, where $g = \sum_{j=1}^n r_j X^{m_j-q} \in F[M]$. Let us argue that both $X^q$ and $g$ can be factored into irreducibles in $F[M]_S$. If $X^q$ does not belong to $F[M]^\times$, then $q \in M^\times$, and so the atomicity of $M$, in tandem with the fact that $X^a \in \mathcal{A}(F[M]_S)$ for all $a \in \mathcal{A}(M)$, ensures that $X^q$ factors into irreducibles in $F[M]_S$.

Let us prove that $g$ also factors into irreducibles in $F[M]_S$. To do this, write $g = \frac{g_1}{s_1} \cdots \frac{g_t}{s_t}$ for some nonunits $g_1, \ldots, g_t \in F[M]$ and $s_1, \ldots, s_t \in S$. Then $g s_1 \cdots s_t = g_1 \cdots g_t \in R[M]$. Since $\text{gd}_N(m_k - q) = 0$, for each $i \in [k+1, n]$ the fact that $\text{gd}_N(m_k - q) > 0$ implies that $m_k - q$ cannot divide $m_k - q$ in $M$. As a result, the coefficient of $X^{m_k - q}$ in the polynomial expression $g s_1 \cdots s_t$ is $r_k s_1(0) \cdots s_t(0)$, which is different from 0 because $s_1, \ldots, s_t \in S$. Therefore, there are $q_1, \ldots, q_t \in M$ with $q_i \in \text{supp } g_i$ for every $i \in [1, t]$ such that $m_k - q = q_1 + \cdots + q_t$. For each $i \in [1, t]$, the facts $q_i \in \text{supp } g_i$ and $g_i \notin S$ guarantee that $\ell \leq \max L_M(m_k - q)$, which is finite. Now, after assuming that $\ell$ was taken as large as it could possibly be, we find that $\frac{g_i}{s_i} \in \mathcal{A}(F[M]_S)$, whence $f$ factors into irreducibles in $F[M]_S$. Hence $F[M]_S$ is atomic.

With the notation as in Theorem 3.3, the following corollary can be used as a tool to construct atomic integral domains that do not satisfy the ACCP.
Corollary 3.4. Let \( F \) be a field, and let \( M \) be a monoid satisfying the conditions in Theorem 4.4. If \( M \) does not satisfy the ACCP, then \( F[M]_S \) is an atomic integral domain that does not satisfy the ACCP.

Now we use Corollary 3.4 to exhibit new examples of atomic domains without the ACCP.

Example 3.5. Let \( N \) be a Puiseux monoid that is also a valuation monoid (that is, a seminormal Puiseux monoid by [9, Proposition 3.1]), and assume that \( N \) admits an atomization \( M \). It follows from Proposition 3.1 that \( N \) is a greatest-divisor submonoid of \( M \). In addition, from the uniqueness of the decomposition (3.2), we can infer that for any \( b \in M \) the element \( b - \text{gd}_N(b) \) has a unique factorization in \( M \), and so \( |L_M(b - \text{gd}_N(b))| = 1 \). Therefore if \( F \) is a field and \( S = \{ f \in F[M] \mid f(0) \neq 0 \} \), then it follows from Theorem 4.4 that \( F[M]_S \) is an atomic domain. Now if we choose \( N \) to be a non-finitely generated valuation monoid (for instance, \( N = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \) for some \( d \in \mathbb{N}_{\geq 2} \)), then neither \( N \) nor \( M \) satisfy the ACCP, and so Corollary 3.4 guarantees that \( F[M]_S \) is an atomic domain that does not satisfy the ACCP. In particular, we obtain that the Grams domain is an atomic domain without the ACCP.

To obtain further examples of atomic domains without the ACCP, we can use Theorem 4.4 on monoids that cannot be produced using atomization. The following example illustrates this.

Example 3.6. Take \( a, b \in \mathbb{N}_{\geq 2} \) such that \( a < b \) and \( \gcd(a, b) = 1 \). Now consider the Puiseux monoid \( M = \langle q^n \mid n \in \mathbb{N}_0 \rangle \), where \( q = a/b \). It is not hard to verify that \( M \) is atomic. In addition, observe that \( \mathbb{N}_0 \) is a submonoid of \( M \). We will argue that \( \mathbb{N}_0 \) is indeed a greatest-divisor submonoid of \( M \). To do this, fix \( b \in M \). By virtue of [5, Lemma 3.1], we can uniquely write \( b = \nu(b) + \sum_{n \in \mathbb{N}} c_n q^n \) under the constrains \( \nu(b) \in \mathbb{N}_0 \) and \( c_n \in [0, b - 1] \) for every \( n \in \mathbb{N} \), where all but finitely many of the terms in \( \sum_{n \in \mathbb{N}} c_n q^n \) equal zero. Mimicking the last two paragraphs in the proof of Proposition 3.1, one can verify the uniqueness of the decomposition \( b = \nu(b) + \sum_{n \in \mathbb{N}} c_n q^n \) and, as a consequence, the equality \( \nu(b) = \text{gd}(b) \). Hence \( \mathbb{N}_0 \) is a greatest-divisor submonoid of \( M \). Thus, if \( F \) is a field and \( S := \{ f \in F[M] \mid f(0) \neq 0 \} \), then \( F[M]_S \) is an atomic domain by virtue of Theorem 4.4. Since \( bq^n = (b - a)q^n + bq^{n+1} \) for every \( n \in \mathbb{N}_0 \), the sequence \( (X^{bq^n} F[M]_S)_{n \in \mathbb{N}} \) is an ascending chain of principal ideals of \( F[M]_S \) that does not terminate. Hence \( F[M]_S \) does not satisfy the ACCP.

4. Atomic Semigroup Rings without the ACCP

The primary purpose of this section is to construct a new class of atomic monoid algebras that do not satisfy the ACCP. In order to do so, we consider monoid domains with coefficients in a field and exponents in the nonnegative ray of \( \mathbb{R} \). Several classes of atomic monoid domains with coefficients in a field and exponents in the nonnegative ray of \( \mathbb{Q} \) were recently considered by the first author in [13]. However, every atomic monoid domain considered in the mentioned paper satisfies the ACCP.

In what follows, we shall assume that for every sequence \( (r_n)_{n \in \mathbb{N}_0} \) of real numbers, \( \sum_{n=k}^{n=k} r_n = 0 \) provided that \( k > \ell \). Let \( (\alpha_n)_{n \in \mathbb{N}} \) be a sequence of pairwise distinct positive irrational numbers such that the set \( \{1, \alpha_n \mid n \in \mathbb{N} \} \) is linearly independent over \( \mathbb{Q} \) and \( \sum_{n=1}^{\infty} \alpha_n < \frac{1}{2} \). Now consider the set

\[
A := \left\{ \alpha_{j_k} + \sum_{i=1}^{\ell} \alpha_{j_i} \mid \ell, j_1, \ldots, j_\ell \in \mathbb{N}, k \in [1, \ell], \text{ and } j_1 < \cdots < j_\ell \right\},
\]

that is, \( A \) is the set consisting of all possible finite summations of the terms of \( (\alpha_n)_{n \in \mathbb{N}} \) where exactly one of the terms appears twice while the rest appear at most once. In addition, take \( \beta_0 = 1 \) and set

\[
B := \{ \beta_0 \} \cup \left\{ \beta_\ell := 1 - \sum_{i=1}^{\ell} \alpha_i \mid \ell \in \mathbb{N} \right\}.
\]
Proposition 4.1. The monoid $M$ is atomic with $\mathcal{A}(M) = A \cup B$. In addition, $M$ does not satisfy the ACCP.

Proof. Since $\{1, \alpha_n \mid n \in \mathbb{N}\}$ is linearly independent over $\mathbb{Q}$, none of the elements in $B$ can divide any element of $A$ in $M$. Now it follows from the linearity independence of $(\alpha_n)_{n \in \mathbb{N}}$ that $\alpha \notin (A \setminus \{\alpha\})$ for any $\alpha \in A$. Thus, $A \subseteq \mathcal{A}(M)$. Because $B \subset (\frac{1}{2}, 1]$, if we express $\beta_k \in B$ as the addition of elements of $A \cup B$, then at most one element $\beta_k \in B$ can appear in such an expression and, therefore, $\ell \geq k$. In this case, we see that $\sum_{i=k}^{n} \alpha_i \in (A)$, which can only happens if $k = \ell$. Hence $B \subseteq \mathcal{A}(M)$ and, as a result, we obtain that $M$ is an atomic monoid with $\mathcal{A}(M) = A \cup B$. To argue the second statement, it suffices to observe that $(2\beta_n + M)_{n \in \mathbb{N}}$ is an ascending chain of principal ideals of $M$ that does not stabilize: this is because $2\beta_n = 2\beta_{n+1} + 2\alpha_{n+1}$ for every $n \in \mathbb{N}$. $\square$

In order to establish the main result of this section, we need the next two lemmas.

Lemma 4.2. Suppose that $x := \sum_{i=1}^{n} c_i \alpha_i$ for some $c_1, \ldots, c_n \in \mathbb{N}_0$. If $\min\{c_j, c_k\} \geq 2$ for different $j, k \in [1, n]$, then $x \in (A)$.

Proof. Let $S$ be the set of elements $\sum_{i=1}^{n} c_i \alpha_i$ with $c_1, \ldots, c_n \in \mathbb{N}_0$ such that $\min\{c_j, c_k\} \geq 2$ for some $j, k \in [1, n]$ with $j \neq k$. For each $x := \sum_{i=1}^{n} c_i \alpha_i \in S$, set $\omega(x) = \sum_{i=1}^{n} \max\{c_i - 1, 0\}$. We will show that every $S \subseteq (A)$ by induction on $\omega(x)$. If $\omega(x) = 2$, then exactly two of the coefficients $c_1, \ldots, c_n$ equal 2 and the rest are 1. Thus, if $\alpha_j = 2$, then $x = 2\alpha_j + (x - 2\alpha_j) \in A + A \subseteq (A)$. If $\omega(x) = 3$, then either exactly three of the coefficients $c_1, \ldots, c_n$ equal 2 and the rest are zero. In the former case, if $c_j = 2$ and $c_k = 2$ for $j \neq k$, then $x = 2\alpha_j + 2\alpha_k + (x - 2\alpha_j - 2\alpha_k) \in A + A \subseteq (A)$. In the latter case, if $c_j = 2$ and $c_k = 3$, then $x = (2\alpha_j + \alpha_k) + (x - 2\alpha_j - \alpha_k) \in A + A \subseteq (A)$.

Now suppose that $s \in (A)$ for every $s \in S$ with $\omega(s) < n$, and take $x \in S$ with $\omega(x) = n \geq 4$. We split the rest of the proof into the following two cases.

Case 1: There exist pairwise different indices $j, k, \ell \in [1, n]$, with $\min\{c_j, c_k, c_\ell\} \geq 2$. In this case, we see that $x - 2\alpha_j \in S$ and $\omega(x - 2\alpha_j) \geq 2$. Thus, $x - 2\alpha_j \in (A)$ by induction hypothesis, and so $x = 2\alpha_j + (x - 2\alpha_j) \in A + (A) \subseteq (A)$.

Case 2: There exist exactly two distinct indices $j, k \in [1, n]$ such that $c_j \geq 2$ and $c_k \geq 2$. For $\omega(x) \geq 4$, one deduces that $c_j + c_k \geq 6$. If $c_j = c_k = 3$, then $2\alpha_j + \alpha_k$ and $x - (2\alpha_j + \alpha_k)$ both belong to $A$, whence $x \in A + A \subseteq (A)$. Otherwise, we can assume that $c_j \geq 4$. Since $\omega(x - 2\alpha_j) = \omega(x) - 2 < n$, it follows by induction that $x - 2\alpha_j \in (A)$, and so $x = 2\alpha_j + (x - 2\alpha_j) \in A + (A) \subseteq (A)$. $\square$

From the definition of the sequence $(\alpha_n)_{n \in \mathbb{N}}$, we deduce that the set $\{1, \alpha_n \mid n \in \mathbb{N}\}$ is a $\mathbb{Z}$-module basis for the abelian group $G := \mathbb{Z} + \sum_{n \in \mathbb{N}} \mathbb{Z}\alpha_n$ they generate. Thus, the map $\psi: G \to \mathbb{Z}$ given by $\psi(c_0 + \sum_{i=1}^{k} c_i \alpha_i) = \sum_{i=1}^{k} \max\{c_i, 0\}$, where $c_0, \ldots, c_k \in \mathbb{Z}$, is well defined.

Lemma 4.3. The following statements hold.

1. $B + (A) = (1 + \sum_{n \in \mathbb{N}} (\mathbb{Z}_{\geq 1}) \alpha_n) \cap M$.
2. If $x \in B + (A)$, then $\psi(x + a) \geq \psi(x) + 1$ for every $a \in A$. 


Proof. (1) Take $x \in B + \langle A \rangle$ and write $x = \beta + \sum_{i=1}^{m} a_i$ for some $\beta \in B$ and $a_1, \ldots, a_m \in A$. After taking $i \in \mathbb{N}$ with $\beta = 1 - \sum_{i=1}^{\ell} \alpha_i$, we see that $x = 1 + \sum_{i=1}^{\ell} \alpha_i - \sum_{i=1}^{\ell} \alpha_i = 1 + \sum_{n \in \mathbb{N}} (\mathbb{Z}_{\geq 1}) \alpha_n$. Thus, $B + \langle A \rangle \subseteq (1 + \sum_{n \in \mathbb{N}} (\mathbb{Z}_{\geq 1}) \alpha_n) \cap M$. Conversely, suppose that $x = 1 + \sum_{n \in \mathbb{N}} \alpha_n \in M$ for some $c_1, \ldots, c_k \in \mathbb{Z}_{\geq 1}$. Since $M$ is atomic with $\mathcal{A}(M) = A \cup B$ by Proposition 4.1, the fact that $\{1, \alpha_n \mid n \in \mathbb{N}\}$ is linearly independent over $\mathbb{Q}$ guarantees that when we write $x$ as a sum of atoms in $M$ exactly one atom of $B$ will show as a summand, which implies that $x \in B + \langle A \rangle$. Hence $(1 + \sum_{n \in \mathbb{N}} (\mathbb{Z}_{\geq 1}) \alpha_n) \cap M \subseteq B + \langle A \rangle$.

(2) Take $x \in B + \langle A \rangle$ and $a \in A$. In light of part (1), we can write $x := 1 + \sum_{n \in \mathbb{N}} \alpha_n$ for some $c_1, \ldots, c_k \in \mathbb{Z}_{\geq 1}$. Let $\alpha_j$ be the term in $(\alpha_n)_{n \in \mathbb{N}}$ that appears twice in the linear combination defining $a$, and assume that $k \geq j$ by using zero coefficients $c_{k+1}, \ldots, c_j$ if necessary. Since $c_j \geq 1$,

$$\psi(x + a) \geq \max\{2 + c_j, 0\} + \sum_{i \in \{1, \ldots, j\}} \max\{c_i, 0\} \geq 1 + \sum_{i=1}^{k} \max\{c_i, 0\} = 1 + \psi(x).$$

We are in a position to exhibit a class of atomic monoid domains that do not satisfy the ACCP.

**Theorem 4.4.** For any field $F$, the monoid domain $F[M]$ is atomic but does not satisfy the ACCP.

**Proof.** Let $G$ be the smallest subgroup of $(\mathbb{R}, +)$ containing the sequence $(\alpha_n)_{n \in \mathbb{N}}$. Observe that every $b \in M$ can be uniquely expressed as $b = m_0 + \sum_{n=1}^{n} m_n \alpha_n$ for some $m_0 \in \mathbb{N}$ and $m_1, \ldots, m_n \in \mathbb{Z}$. Therefore $M$ can be embedded into the monoid $\mathbb{N}_0 \times G$ by the assignment $b \mapsto (m_0, b - m_0)$, and so we can identify $F[M]$ with a subring of the polynomial ring $R[x]$ over the group algebra

$$R := F[G] \cong F[\prod_{n \in \mathbb{N}} X_n^\pm 1 \mid n \in \mathbb{N}],$$

where the identification is the canonical isomorphism given by the assignments $Y \mapsto x$ and $Y^\alpha \mapsto X_n$ for every $n \in \mathbb{N}$ (here $Y$ is the indeterminate of $F[M]$). For every $F[G]$-monomial $\prod_{n \in \mathbb{N}} X_n^{m_n}$, where $m_n = 0$ for all but finitely many $n \in \mathbb{N}$, we say that $\deg_{F[G]} \prod_{n \in \mathbb{N}} X_n^{m_n} := \sum_{n \in \mathbb{N}} m_n \alpha_n$ is the total degree of $\prod_{n \in \mathbb{N}} X_n^{m_n}$. Accordingly, for every $f \in F[M]$, the notation $\deg f$ (resp., $\ord f$) will refer to the degree (resp., order) of $f$ as a polynomial in $R[x]$. Since $R$ is a Laurent polynomial ring over a field, it is a UFD.

**Claim 1:** For each $f \in F[M]$ with $\ord f = 0$, there exists $N \in \mathbb{N}$ such that $f$ cannot be written as a product of more than $N$ elements of $F[M] \setminus F$.

**Proof of Claim 1:** Let $M_{\alpha}$ be the submonoid of $G$ generated by $(\alpha_n)_{n \in \mathbb{N}}$. Note that $F[M_{\alpha}]$ is a subring of $R$ and also that $F[M_{\alpha}]^\times = F^\times$. For each $g \in F[M] \subseteq R[x]$, we observe that $g(0)$ is a sum of finitely many monomials $Y^k \in F[M]$ with $b \in M$ not divisible by any element of $B$ in $M$, which implies that $b \in M_{\alpha}$. Thus, $g(0) \in F[M_{\alpha}]$ for every $f \in F[M]$. Since $M_{\alpha}$ is a free commutative monoid, it follows from [15, Proposition 3.14] that $F[M_{\alpha}]$ is a BFD and, therefore, $\sup \{f(0) \mid f(0) \in F[M_{\alpha}]\} < N - \deg f$ for some $N \in \mathbb{N}$. Write $f = g_1 \cdots g_n$ for some $g_1, \ldots, g_n \in F[M] \setminus F$, assuming that for some $k \in [1, n]$ the inequality $\deg g_i \geq 1$ holds if and only if $i \in [k, n]$. Clearly, $n - k < \deg f$. If $k = 1$, then $n \leq \deg f < N$. Suppose, on the other hand, that $k > 1$. Since $g_i = g_i(0) \in F[M_{\alpha}] \setminus F$ for every $i \in [1, k-1]$, the fact that $g_1(0) \cdots g_n(0)$ divides $f(0)$ in $F[M_{\alpha}]$, which is a BFD, ensures that $k < N - \deg f$. Hence $n = k + (n - k) \leq (N - \deg f) + \deg f = N$.

**Claim 2:** For each $f \in F[M]$ with $\ord f = 1$, there exists $N \in \mathbb{N}$ such that $f$ cannot be written as a product of more than $N$ elements of $F[M] \setminus F$.

**Proof of Claim 2:** Let $f'$ be the formal derivative of $f$ when considered as a polynomial in $R[x]$, and set $d := \deg_{F[G]} f'(0)$. From the equality $\ord f = 1$, we obtain that $1 + d$ has the form $\beta + \sum_{i=1}^{m} a_i$,
for some $\beta \in B$ and $a_1, \ldots, a_m \in A$, and so we can take $N \in \mathbb{N}$ such that $\psi(1 + d) + \deg f < N$. Write $f = g_0 \cdots g_n$ for some $g_0, \ldots, g_n \in F[M] \setminus F$, and let us show that $n \leq N$. Because $\ord f = 1$, we can assume, without loss of generality, that $\ord g_0 = 1$ and $g_1(0) \cdots g_n(0) \neq 0$. After relabeling $g_1, \ldots, g_n$ if necessary, we can further assume the existence of $k \in [2, n]$ such that $\deg g_i \geq 1$ if and only if $i \in [k + 1, n]$, and so $g_i = g_i(0) \in F[M_n]$ for every $i \in [1, k]$. Since $\ord g_0 = 1$, we see that $n - k < \deg f$. Considering the coefficients of the monomials of degree 1 in both sides of $f = g_0 \cdots g_n$, we see that $f'(0) = g'_0(0)g_1(0) \cdots g_n(0)$ in $F[G]$. Set $d_0 := \deg_{F[G]} g_0(0)$ and $d_i := \deg_{F[G]} g_i(0)$ for every $i \in [1, n]$. As $\ord g_0 = 1$, we can write $1 + d_0 = \beta' + \sum_{i=1}^{\ell} a'_i$ for some $\beta' \in B$ and $a'_1, \ldots, a'_\ell \in A$. On the other hand, for every $i \in [1, k]$, the fact that $g_i(0) \in F[M_n] \setminus F$ implies that $d_i \in M^*_n$. Thus, in light of Lemma 4.3, we obtain

$$\psi(1 + d) \geq \psi\left(1 + d_0 + \sum_{i=1}^{k} d_i\right) = \psi\left(\beta' + \sum_{i=1}^{\ell} a'_i + \sum_{i=1}^{k} d_i\right) \geq \psi(\beta') + k = k.$$ 

As a result, $n = k + (n - k) < \psi(1 + d) + \deg f < N$, which completes the proof of our claim.

Now we are in a position to prove that $F[M]$ is an atomic domain. To do so, we proceed by induction on the order of elements of $F[M] \setminus F$ as polynomials in $R[x]$. Take $f \in F[M] \setminus F$. If $\ord f \in \{0, 1\}$, then it immediately follows from Claim 1 and Claim 2 that $f$ can be factored into irreducibles. Therefore suppose that $\ord f = n \geq 2$, assuming that every element of $F[M] \setminus F$ whose order in $R[x]$ is less than $n$ can be factored into irreducibles.

Write $f = \sum_{i=1}^\ell c_i Y^{\theta_i}$ for some $c_1, \ldots, c_\ell \in F$ and $\theta_1, \ldots, \theta_\ell \in M$. Now, for each $i \in [1, \ell]$, write $\theta_i = a_i + \sum_{j=1}^{n_i} \beta_{ij}$, for some $a_i \in \langle A \rangle$ and $\beta_{1i}, \ldots, \beta_{n_i} \in B$. Let $m$ be the largest index such that $\beta_m$ appears in the right-hand side of one of the equalities $\theta_i = a_i + \sum_{j=1}^{n_i} \beta_{ij}$ (for every $i \in [1, \ell]$). Now fix $i \in [1, \ell]$ and then write

$$a_i + \sum_{j=1}^{n_i} \beta_{ij} = 2\beta_{m+2} + (n_i - 2)\beta_m + a'_i$$

for some $a'_i \in R$. If we express both $a_i + \sum_{j=1}^{n_i} \beta_{ij}$ and $2\beta_{m+2} + (n_i - 2)\beta_m$ as linear combinations of the elements in the linearly independent set $\{\alpha_n \mid n \in \mathbb{N}\}$, then the coefficients of 1 in both linear combinations are the same, namely, $n_i$. Therefore $a'_i \in G$. Furthermore,

$$a'_i = a_i + 2(\beta_m - \beta_{m+2}) + \sum_{j=1}^{n_i} (\beta_{ij} - \beta_m) = a_i + 2\alpha_{m+1} + 2\alpha_{m+2} + \sum_{j=1}^{n_i} \sum_{k=i}^{m} \alpha_k,$$

and so it follows from Lemma 4.2 that $a'_i \in \langle A \rangle$. This implies that $Y^{2\beta_{m+2}}$ divides $f$ in $F[M]$, whence we can factor $f$ as $f = Y^{2\beta_{m+2}}(f/Y^{2\beta_{m+2}})$ in $F[M]$. Since $Y^\beta$ is an irreducible of $F[M]$ for every $\beta \in \mathcal{M}(M)$, the monomial $Y^{2\beta_{m+2}}$ factors into irreducibles in $F[M]$, namely, $Y^{2\beta_{m+2}} = (Y^{\beta_{m+2}})^2$. On the other hand, observe that $\ord (f/Y^{2\beta_{m+2}}) = (\ord f) - 2 < n$, and so it follows from the induction hypothesis that $f/Y^{2\beta_{m+2}}$ also factors into irreducibles in $F[M]$. Hence every $f \in F[M] \setminus F$ factors into irreducibles in $F[M]$, which means that $F[M]$ is an atomic domain.

Finally, we observe that $F[M]$ cannot satisfy the ACCP as, otherwise, the monoid $M$ would also satisfy the ACCP by [16, Proposition 1.4], which is not the case, as we have already seen in Proposition 4.1. \qed
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