MODULI OF ÉTALE SUBALGEBRAS IN AN AZUMAYA ALGEBRA

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Abstract. Let \( A \) be a sheaf of Azumaya algebras over a Noetherian base \( S \). In this paper we describe using generalized Severi-Brauer varieties, a quasi-projective moduli space parametrizing sheaves of étale subalgebras of \( A \).

In the case that \( S \) is the spectrum of a field, we study the geometry of this moduli space, and show that in certain cases it is a rational variety.

1. Preliminaries

Let \( S \) be a Noetherian scheme, and let \( A \) be a sheaf of Azumaya algebras over \( S \). Our goal is to study the functor \( \text{ét}(A) \), which associates to every \( S \)-scheme \( X \), the set of sheaves of commutative étale subalgebras of \( A_X \). We will show that this functor is representable by a scheme which may be described in terms of the generalized Severi-Brauer variety of \( A \).

Unless said otherwise, all products are fiber products over \( S \). If \( X \) is an \( S \)-scheme with structure morphism \( f : X \to S \), then we write \( A_X \) for the sheaf of \( \mathcal{O}_X \)-algebras \( f^*(A) \). For a \( S \)-scheme \( Y \), we occasionally write \( Y_X \) for \( Y \times X \), thought of as an \( X \)-scheme.

Every sheaf of étale subalgebras may be assigned a discrete invariant, which we call its type, and therefore our moduli scheme is actually a disjoint union of other moduli spaces.

To begin, let us define the notion of type.

**Definition 1.1.** Let \( R \) be a local ring, and \( B/R \) an Azumaya algebra. If \( e \in B \) is an idempotent, we define the rank of \( e \), denoted \( r(e) \) to be the reduced rank of the right ideal \( eB \). Recall that the reduced rank of a \( B \)-module \( M \) is the rank of \( M \) as an \( R \)-module divided by the degree of \( B \) (\textbf{KMRT98}).

Let \( E \) be a sheaf of étale subalgebras of \( A/S \), and let \( p \in S \). Let \( R \) be the local ring of \( p \) in the étale topology (so that \( R \) is a strictly Henselian local ring). Then taking étale stalks, we see that \( E_p \) is an
étale subalgebra of $A_p/R$, and it follows that

$$E_p = \bigoplus_{i=1}^k Re_i,$$

for a uniquely defined collection of idempotents $e_i$, which are each minimal idempotents in $S_p$.

**Definition 1.2.** The type of $E$ at the point $p$ is the unordered collection of positive integers $[r(e_1), \ldots, r(e_m)]$.

**Definition 1.3.** We say that $E$ has type $[n_1, \ldots, n_m]$ if it has this type for each point $p \in S$.

**Remark 1.** Since $1 = \sum e_i$, the ideals $I_i = e_iA$ span $A$. Further it is easy to see that the ideals $I_i$ are linearly independent since $e_i a = e_j b$ implies $e_i a = e_i e_i a = e_i e_j b = 0$. We therefore know that the numbers making up the type of $E$ give a partition of $\deg(A_p)$.

Some additional notation for partitions will be useful. Let $\rho = [n_1, \ldots, n_m]$. For a positive integer $i$, let $\rho(i)$ be the number of occurrences of $i$ in $\rho$. Let $S(\rho)$ be the set of distinct integers $n_i$ occurring in $\rho$, and let $N(\rho) = |S(\rho)|$. Let

$$\ell(\rho) = \sum_{i \in S(\rho)} \rho(i) = m$$

be the length of the partition.

2. **Moduli spaces of étale subalgebras**

Suppose $A/S$ is an sheaf of Azumaya algebras, and suppose $S$ is a connected, Noetherian scheme. Let $\rho = [n_1, \ldots, n_m]$ be a partition of $n = \deg(A)$. Let $\text{ét}_\rho(A)$ be the functor which associates to every $S$ scheme $X$ the set of étale subalgebras of $A_X$ of type $\rho$. That is, if $X$ has structure map $f : X \to S$,

$$\text{ét}_\rho(A)(X) = \left\{ \begin{array}{l}
\text{sub-}\mathcal{O}_X\text{-modules } E \subset f^* A \\
\text{E is a sheaf of commutative étale subalgebras of } f^* A \text{ of type } \rho
\end{array} \right\}$$

Our first goal will be to describe the scheme which represents this functor. We use the following notation:

$$V_\rho(A) = \prod_{i \in S(\rho)} V_i(A)^{\rho(i)},$$

where $V_i(A)$ is the $i$'th generalized Severi-Brauer variety of $A$ ([Bla91]), which parametrizes right ideals of $A$ which are locally direct summands of reduced rank $i$.

We define $V_\rho(A)^o$ to be the open subscheme parametrizing ideals which are linearly independent. That is to say, for a $S$-scheme $X$, if
$I_1, \ldots, I_{\ell_\rho}$ is a collection of sheaves of ideals in $A_X$, representing a point in $V^\rho(A)(X)$, then by definition, this point lies in $V^\rho(A)^\circ$ if and only if $\oplus I_i = A$. Let $S_\rho$ be the subgroup $\prod_{i \in S(\rho)} S_{\rho(i)}$ of the symmetric group $S_n$. For each $i$, we have an action of $S_{\rho(i)}$ on $V_i(A)^{\rho(i)}$ by permuting the factors. This induces an action of $S_\rho$ on $V^\rho(A)$, and on $V^\rho(A)^\circ$. Denote the quotients of these actions by $S^\rho V(A)$ and $S^\rho V(A)^\circ$ respectively. We note that since the action on $V^\rho(A)^\circ$ is free, the quotient morphism

$$V^\rho(A)^\circ \to S^\rho V(A)^\circ$$

is an étale morphism which is a Galois covering with group $S_\rho$.

**Theorem 2.1.** Let $\rho = [n_1, \ldots, n_m]$ be a partition of $n$. Then the functor $\text{ét}(A)_\rho$ is represented by the scheme $S^\rho V(A)$.

**Proof.** To begin, we first note that both $\text{ét}(A)_\rho$ and the functor represented by $S^\rho V(A)$ are sheaves in the étale topology. Therefore, to show that these functors are naturally isomorphic, it suffices to construct a natural transformation $\psi : S^\rho V(A)^\circ \to \text{ét}(A)_\rho$, and then show that this morphism induces isomorphisms on the level of stalks.

Let $X$ be an $S$-scheme, and let $p : X \to S^\rho V(A)^\circ$. To define $\psi(X)(p)$, since both functors are étale sheaves, it suffices to define it on an étale cover of $X$. Let $\tilde{X}$ be the pullback in the diagram

$$\begin{array}{ccc}
\tilde{X} & \longrightarrow & V^\rho(A)^\circ \\
\downarrow & & \downarrow \pi \\
X & \longrightarrow & S^\rho V(A)^\circ \\
\end{array}$$

Since the quotient morphism $\pi$ is étale, so is the morphism $\tilde{X} \to X$. Therefore we see that after passing to an étale cover, and replacing $X$ by $\tilde{X}$, we may assume that $p = \pi(q)$ for some $q \in V^\rho A^\circ(X)$. Passing to another cover, we may also assume that $X = \text{Spec}(R)$.

Since $p = \pi(q)$, we may find right ideals $I_1, \ldots, I_{\ell(\rho)}$ of $A_R$ such that $\oplus I_i = A_R$, which represent $q$. Writing

$$1 = \sum e_i, \quad e_i \in I_i,$$

we define $E_p = \oplus e_i R$. This is a split étale extension of $R$, which is a subalgebra of $A$, and we set $\psi(p) = E_p$. One may check that this defines a morphism of sheaves. Note that this definition with
respect to an étale cover gives a general definition since the association 
\((I_1, \ldots, I_\ell) \mapsto E_p\) is \(S_\rho\) invariant.

To see that \(\psi\) is an isomorphism, it suffices to check that it is an isomorphism on étale stalks. In other words, we may restrict to the case that \(X = \text{Spec}(R)\), where \(R\) is a strictly Henselian local ring.

We first show that \(\psi\) is injective. Suppose \(E\) is an étale subalgebra of \(A_R\) of type \(\rho\). Since \(R\) is strictly Henselian, we have

\[ E = \bigoplus_{i \in S(\rho)} \bigoplus_{j=1}^{\rho(i)} e_{i,j} R. \]

By definition, since the type of \(E\) is \(\rho\), if we let \(I_{i,j} = e_{i,j} A\), then we the tuple of ideals \((I_{i,j})\) defines a point \(q \in V_\rho(A)\circ(R)\). Further, since \(\sum e_{i,j} = 1\), we actually have \(q \in V_\rho(A)\circ(R)\). If we let \(p = \pi(q)\), then tracing through the above map yields \(\psi(R)(p) = E\). Therefore \(\psi\) is surjective.

To see that it is injective, we suppose that we have points \(p, p' \in S^\rho V(A)^\circ(R)\). By forming the pullbacks as in equation 1 since \(R\) is strictly Henselian, we immediately find that in each case, because \(\tilde{X}\) is an étale cover of \(X\), it is a split étale extension, and hence we have sections. This means we may write

\[ p = \pi(I_1, \ldots, I_{\ell(\rho)}), \quad p' = \pi(I'_1, \ldots, I'_{\ell(\rho)}). \]

Note that in order to show that \(p = p'\) suffices to prove that the ideals are equal after reordering. Now, if \(E_{p} = E_{p'}\), then both rings have the same minimal idempotents. However, by remark 1 the ideals are generated by these idempotents. Therefore, the ideals concide after reordering, and we are done.

Since we now know that the functor \(\text{ét}_{\rho}(A)\) is representable, we will abuse notation slightly and refer to it and the representing variety by the same name.

**Definition 2.2.** \(\text{ét}(A)\) is the disjoint union of the schemes \(\text{ét}_{\rho}(A)\) as \(\rho\) ranges over all the partitions of \(n = \text{deg}(A)\).

**Corollary 2.3.** The functor which associates to any \(S\)-scheme \(X\) the set of étale subalgebras of \(A_X\) is representable by \(\text{ét}(A)\).

**Remark 2.** By associating to an étale subalgebra \(E \subset A_X\) its underlying module, we obtain a natural transformation to the Grassmannian functor.
3. Subfields of central simple algebras

In this section we specialize to the case where $S = \text{Spec}(k)$ for a field $k$, and $A$ is a central simple $k$-algebra. We use the notation in this section that tensor products are always taken over $k$, unless specified otherwise. If $E$ is an étale subalgebra of $A$, then taking the étale stalk at $\text{Spec}(k)$ amounts to extending scalars to the separable closure $k^{\text{sep}}$ of $k$. Let $G$ be the absolute Galois group of $k^{\text{sep}}$ over $k$. Writing $E \otimes k^{\text{sep}} \cong \bigoplus_{\rho \in S(\rho)} e_{i,j} k^{\text{sep}}$, we have an action of $G$ on the idempotents $e_{i,j}$. One may check that the idempotents $e_{i,j}$ are permuted by $G$, and there is a correspondence between the orbits of this action and the idempotents of $E$. In particular we have

**Lemma 3.1.** In the notation above, if $E$ is a subfield of $A$, then $|S(\rho)| = 1$.

*Proof.* $E$ is a field if and only if $G$ acts transitively on the set of idempotents. On the other hand, this action must also preserve the rank of an idempotent, which implies that all the idempotents have the same rank. □

Therefore, if we are interested in studying the subfields of a central simple algebra, we may restrict attention to partitions of the above type. If $m|n = \text{deg}(A)$, we write

$$\hat{\text{et}}_m(A) = \hat{\text{et}}_{\frac{n}{m}, \frac{n}{m}, \ldots, \frac{n}{m}}.$$

Note that every subfield of dimension $m$ is represented by a $k$-point of $\hat{\text{et}}_m(A)$, and in the case that $A$ is a division algebra, this gives a 1-1 correspondence. In particular, elements of $\hat{\text{et}}_n(A)(F)$ are in natural bijection with the maximal subfields of $A$. We will now show that this variety is rational and R-trivial. This argument will be a geometric analog of one in [KS04].

If $a \in A$ is an element whose characteristic polynomial has distinct roots, then the field $k(a)$ is a maximal subfield of $A$.

**Theorem 3.2.** Let $U \subset A$ be the Zariski open subset of elements of $A$ whose characteristic polynomials have distinct roots. Then there is a dominant rational map $U \to \hat{\text{et}}_n(A)$ which is surjective on $F$-points.

*Proof.* Let $\tilde{U}$ be the degree $n!$ étale cover of $U$ whose $k^{\text{sep}}$-points consists of pairs $(r, a)$, where $a \in U$, and $r = (r_1, \ldots, r_n)$ is an ordered $n$-tuple of distinct roots of the characteristic polynomial $\xi(a)$. One may check that
the indecomposable idempotents in the étale extension $k(a)$ are given by the elements:

$$e_i = \prod_{j \neq i} \frac{a - r_j}{r_i - r_j}.$$ 

We may then define a morphism $\tilde{U} \to V_n(A)^d$ by taking $(a, r)$ to $(e_1 A, \ldots, e_n A)$. We compose this with the quotient map and obtain a morphism $\tilde{U} \to \acute{e}t_n(A)$. Since this morphism is constant on the fibers of $\tilde{U} \to U$, this map descends to a map $U \to \acute{e}t_n(A)$.

By the description above, this is the morphism which takes an element of $A$ to the subfield which it generates. Since every étale subfield of $A$ can be generated by a single element, this morphism is surjective on $F$-points. Since this also holds after fibering with the algebraic closure, it follows also that this morphism is surjective at the algebraic closure and hence dominant.

□

**Corollary 3.3.** Suppose $k$ is an infinite field. Then $\acute{e}t_n(A)$ is a rational variety.

**Proof.** Suppose $L \subset A$ is a linear affine subvariety of $A$. Then from remark 2 the $n$-dimensional étale subfields of $\acute{e}t_n(A)$ which intersect $L$ transversely form an open subvariety $V$ of $\acute{e}t_n(A)$.

Choose a maximal étale subalgebra $E \subset A$, and let $L$ be a complementary subspace of dimension $n^2 - n = \text{dim}(A) - \text{dim}(E)$. Choose $a \in E$ with distinct eigenvalues generating $E$. Note that by construction, $(L + a) \cap E = a$.

Let $U \subset L + a$ be the dense open set of elements with distinct characteristic roots, and $\psi$ be the restriction of the morphism from theorem 3.2 to $U$. I claim that this is a birational isomorphism from $U$ to $\acute{e}t_n(A)$. To see this note that for $E' \in V$, there is a unique element $b \in E' \cap (L + a)$, if we define $\phi(E') = b$, we obtain a morphism $V \xrightarrow{\phi} U$, which is a birational inverse to $\psi$. □

**Lemma 3.4.** Let $A$ be a degree 4 central simple $k$-algebra. Then $V_2(A)$ is isomorphic to an involution variety $V(B, \sigma)$ of a degree 6 algebra with orthogonal involution $\sigma$.

**Proof.** Consider the map

$$Gr(2, 4) \to \mathbb{P}^5$$

, given by the Plücker embedding. Fixing $V$ a 4 dimensional vector space, we may consider this as the map which takes a 2 dimensional subspace $W \subset V$ to the 1 dimensional subspace $\wedge^2 W \subset \wedge^2 V$. This morphism gives an isomorphism of $Gr(2, 4)$ with a quadric hypersurface.
This quadric hypersurface may be thought of as the quadric associated to the bilinear form on \( \wedge^2 V \) defined by \( \langle \omega_1, \omega_2 \rangle = \omega_1 \wedge \omega_2 \in \wedge^4 V \cong F \). Note that one must choose an isomorphism \( \wedge^4 V \cong F \) to obtain a bilinear form, and so it is only defined up to similarity. Nevertheless, the quadric hypersurface and associated adjoint (orthogonal) involution depend only on the similarity class and are hence canonically defined.

Since the Plücker embedding defined above is clearly \( PGL(V) \) invariant, using [Art82], for any degree 4 algebra \( A \) given by a cocycle \( \alpha \in H^1(k, PGL_4) \), we obtain a morphism:

\[
V_2(A) \to V(B),
\]

where \( B \) is given by composition of \( \alpha \) with the standard representation \( PGL(V) \to PGL(V \wedge V) \). By [Art82] this implies that \( B \) is similar to \( A^{\otimes 2} \) in \( Br(k) \). Also, it is easy to see that the quadric hypersurface and hence the involution is \( PGL_4 \) invariant, and hence descends to an involution \( \sigma \) on \( B \). We therefore obtain an isomorphism \( V_2(A) \cong V(B, \sigma) \) as claimed.

**Lemma 3.5.** Suppose \( B \) is a central simple \( k \) algebra with orthogonal involution \( \sigma \). Then there is a natural birational morphism \( V_2(B) \to S^2V(B, \sigma) \) which is surjective on \( k \) points. In particular, \( S^2V(B, \sigma) \) is a rational variety.

**Proof.** The proof is along similar lines to the previous proof. Let \( V \) be a vector space with a quadratic form \( q \), and let \( Q \subset \mathbb{P}(V) \) be the associated quadric hypersurface. Consider the map

\[
Gr(2, V) \to S^2Q,
\]

defined by taking a projective line in \( \mathbb{P}(V) \) to its intersection points with \( Q \). Note that this morphism is a birational morphism. If we let \( O_q(V) \) be the orthogonal group for the quadratic form \( q \), it is easy to see that the map described above is \( O_q(V) \) invariant.

To say that an algebra \( B \) has an orthogonal involution is to say that we have chosen a representing cocycle \( \alpha \in H^1(k, PO_q(V)) \). Therefore, the above map will be invariant under any Galois action defined by \( \alpha \), and we will get a morphism

\[
V_2(B) \to S^2V(B, \sigma).
\]

Putting these lemmas together yeilds the following corollary:

**Corollary 3.6.** Suppose \( A \) is a degree 4 central simple \( k \) algebra. Then \( \text{ét}_2(A) \) is a rational variety.
References

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