Anomalous Dimensions in the WF $O(N)$ Model with a Monodromy Line Defect

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Abstract: General ideas in the conformal bootstrap program are covered. Both numerical and analytical approaches to the bootstrap equation are reviewed to show how it can be manipulated in different ways. Further analytical approaches are studied for theories with defects. We consider the three-dimensional CFT at the corresponding WF fixed point in the $O(N) \phi^4$ model with a co-dimension two, monodromy defect. Anomalous dimensions for bulk- and defect-local fields as well as one of the OPE coefficients are found to the first loop order. Implications of inserting this defect and constraints that arises from symmetries of the theory are investigated.
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Chapter 1

Introduction and History

The idea behind bootstrap is to study the symmetries that exist in a quantum field theory (QFT) to determine the least amount of information we need in order to fully understand the correlators in the theory. Such an approach does not use perturbation theory nor an action, which could help us with problems where we cannot use perturbation theory. As it turns out, conformal bootstrap is a viable approach in conformal field theories (CFTs). A CFT is a QFT at a fixed point, i.e. where the $\beta$-function (that describes the running of a coupling constant) is zero. At fixed points, a QFT’s symmetry group is expanded beyond the Poincaré group (translations and rotations). In this expanded symmetry group: dilations (scalings) and a kind of transformations called special conformal transformations (SCTs) are added. Belavin, Polyakov and Zamolodchikov studied this expanded symmetry group for two dimensional theories (minimal models) in 1984 to find that all you need to classify a CFT are the scaling dimensions and spin of all primary fields (primaries) in the theory as well as all constants in every operators product expansion (OPE) of two primaries \[1\]. We call this information the CFT data. A non-primary field is called a descendant, and can always be written as derivatives of a primary. This means that once we know the CFT data for the primaries, we can find it for the descendants as well.

Finding all of the CFT data for a theory is not a simple task. Several minimal models were solved in the 80’s, e.g. the 2D Ising model was solved in \[1\]. It was not until 2008 that a higher dimensional theory was partially solved, namely the 3D Ising model \[2\]. It was studied numerically and not analytically. Global symmetries in the 3D Ising model does not restrict the theory as much as the two-dimensional case, hence bootstrapping the three-dimensional one is more complicated. It was not until 2012 that Fitzpatrick, Kaplan, Poland and Simmons-Duffin \[3\] as well as Komargodski and Zhiboedov \[4\] found a way to analytically study the bootstrap program for theories other than minimal models.

Since 2013 various approaches to analytically studying CFTs have been explored. One of these approaches is \[5\]. They studied the CFT at the Wilson-Fischer (WF) fixed point in $4 - \epsilon$ dimensional $\phi^4$-theory, where $\epsilon$ is a parameter we introduce when performing a dimensional regularization. The WF fixed point is the $\epsilon$-dependent fixed point in $\phi^4$ theory. We often let $\epsilon$ go to zero, but in this case we let it go to one. This yields the 3D Ising model. Gaiotto, Mazac and Paulos \[6\] found a way to analytically study this theory with the insertion of a defect. A defect is a subspace in the space of the theory where new fields and interactions between fields may occur. This subspace may not be invariant under the whole symmetry group of the theory, hence inserting a defect into the space of a theory will most likely break its global
symmetry. We call fields that only live on defect for defect-local fields and fields that live in the whole space of the theory for bulk-local fields. The main goal with this paper is to generalize the approach in [6] to a O(N) model by promoting the scalar fields into vector multiplets of O(N).

We start this paper with a brief review of the bootstrap program following the lecture notes [7]. In this review we use $d+2$ dimensional lightcone coordinates to study the generators of conformal transformations (CTs) and their algebra, primaries and descendants, correlators, radial quantization as well as OPEs. There will be a very important equation, called the bootstrap equation, found using the associativity of the correlators and the OPE. Solving this equation basically solves the bootstrap for a theory, i.e. yields the CFT data. The approach used in [2] and [3] to the bootstrap equation are studied to see how different various approaches to solving the bootstrap program can be.

Moving on to defects in CFT, we study [8]. This paper gives a good understanding how the conformal symmetry constraints can be studied when one or several defects are inserted into the theory. As previously mentioned, we generalize the approach in [6] to an O(N) model with a defect. We double check our answers by generalizing the non-perturbative approach of Yamaguchi [9] to O(N). This approach makes use of the framework for the WF O(N) model by Rychkov and Tan [5]. The WF O(N) model is the $\phi^4$ theory where the scalar fields have been promoted to vector multiplets of O(N).
2.1 Generators of Conformal Transformations

This review will follow [7]. We will study fixed point theories, i.e. theories where the \( \beta \)-function, which describes the running of the coupling constant, \( g \), is zero for some value(s) of \( g \). Only at a fixed point may a CFT appear. A conformal transformation (CT) is a composition of Weyl transformations. Hence, a CT has the generators [7]

\[
\begin{align*}
P_\mu &= i \partial_\mu, \quad \text{for translations.} \\
M_{\mu\nu} &= i x_\mu \partial_\nu, \quad \text{for rotations.} \\
D &= i x^\mu \partial_\mu, \quad \text{for dilations (scalings).} \\
K_\mu &= i \left( 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu \right), \quad \text{for special conformal transformations (SCT’s).}
\end{align*}
\]

(2.1.1)

There is a nice way of understanding these transformations using Euclidean lightcone coordinates

\[
X^\pm = X^{d+2} \pm X^{d+1} \quad \Rightarrow \quad ds^2 = dX^2 - dX^+ dX^-,
\]

(2.1.2)

\[
X^M Y_M = -\frac{1}{2} \left( X^+ Y^- + X^- Y^+ \right) + X^j Y_j, \quad M \in \{1, \ldots, d+2\}, \quad j \in \{1, \ldots, d\}.
\]

The antisymmetric \( SO(d+1, 1) \)-generators, \( J_{\mu\nu} = -J_{\nu\mu} \), identifies with the generators for a CT as

\[
\begin{align*}
J_{\mu\nu} &= M_{\mu\nu}, \\
J_{\mu+} &= P_\mu, \\
J_{\mu-} &= K_\mu, \\
J_{++} &= D.
\end{align*}
\]

(2.1.3)

Hence CTs are isomorphic to \( SO(d+1, 1) \)-transformations, whose generators satisfy the algebra

\[
\{J_{MN}, J_{RS}\} = -i \eta_{[M||R} J_{N||S]}.
\]

(2.1.4)

Here we have used an antisymmetric convention \( A_{[\mu} B_{\nu]} = A_{[\mu} B_{\nu]} = A_{\mu} B_{\nu} - A_{\nu} B_{\mu} \). This isomorphism lets us write a CT as a general \( SO(d+1, 1) \) transformation

\[
X'_M = \Lambda^N_M J_N, \quad \Lambda^N_M \in SO(d+1, 1).
\]

(2.1.5)

\(^1\)In Minkowski space, a CT is a general \( SO(d, 2) \) transformation.
2.2 Primaries and Descendants

The goal of conformal bootstrap is to understand the behavior of a CFT without knowing the action of the system. If we know this, we could possibly study non-perturbative problems that arise in CFTs. The scaling dimensions, \( \Delta \), and spins, \( l \), of all primaries in a CFT an important factor if we want to characterize the theory. An operator is primary if under (local) scaling
\[
\phi \rightarrow \tilde{\phi}(b(x)x) = \phi(b(x)x) = b(x)^{-\Delta} \phi(x), \quad (l = 0 \text{ in this case}),
\]
and an operator is a descendant if it can be written as derivatives of a primary, i.e. it has a homogeneous part (which only depend of \( z \)), and an anti-homogeneous part (which only depends on \( \bar{z} \)), e.g. \( \partial_\mu \phi \) is a descendant if \( \phi \) is a primary. If a primary has dimension \( \Delta_0 \) and spin \( l_0 \), then a descendant have dimension \( \Delta_0 + \#(\text{derivatives}) \) and spin \( l_0 + \#(\text{free indices}) \), where "\#" should be read as "number of". In a CFT, a field is either a primary or a descendant, e.g. symmetry currents and the stress-energy tensor, \( T_{\mu\nu} \), are both primaries. Generally there exists infinitely many primaries for each spin.

In every \( SO(D) \)-representation, there exist a lowest bound, \( \Delta_{\text{min}} \), for the dimensions called the unitary bound. For symmetric, traceless primaries, this lower bound equals
\[
\Delta \geq \Delta_{\text{min}}(d, l) = l + (d - 2) \left( 1 - \frac{\delta_{l0}}{2} \right).
\]

2.3 Projection onto Real Space

To project the lightcone coordinates onto physical space, we need to reduce the number of dimensions by two. We reduce one by restricting ourselves on the null space (lightcone)
\[
X^2 = 0,
\]
and another by restricting ourselves to a hypersurface of codimension one on this cone
\[
X^+ = f(X^\mu).
\]
In real space we have
\[
ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu.
\]
Compare this with (2.1.2) and we see that we achieve this if \( X^+ \) is an arbitrary constant. Since it is arbitrary, we may set it to one, i.e. \( f(X^\mu) = 1 \). This yields a projection from \( SO(d + 1, 1) \) onto real space, \( \mathbb{R}^{d-1,1} \)
\[
X^M = (X^+, X^-, X^\mu) = (1, x^2, x^\mu) \in SO(d + 1, 1), \quad x^\mu \in \mathbb{R}^{d-1,1}.
\]
Using this projection we find up to four-point correlators for scalar primaries with no spin
\[
\langle \phi_j(x_1)\phi_k(x_2) \rangle = \frac{\delta_{jk}}{|x_{12}|^{2\Delta_j}}, \quad x_{12} \equiv x_1 - x_2,
\]
\[
\langle \prod_{j=1}^3 \phi_j(x_j) \rangle = \frac{\lambda_{123}}{|x_{12}|^{\alpha_{123}}|x_{23}|^{\alpha_{231}}|x_{31}|^{\alpha_{312}}}, \quad \alpha_{jkl} = \Delta_j + \Delta_k - \Delta_l,
\]
\[
\langle \prod_{j=1}^4 \phi(x_j) \rangle = \frac{f(u, v)}{x_{12}^2 x_{34}^2 x_{13}^2 x_{24}^2}, \quad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = u|_{2 \rightarrow 4}.
\]
In these correlators \( f(u,v) \) is a function related to the three-point correlator, and \( u \) as well as \( v \) are conformally invariant coordinates. This means that we only need the dimensions, \( \Delta \), of the primaries and the constants \( \lambda_{123} \) to determine these correlators. In general, one need the spin, \( l \), of the primaries as well. The constants \( \lambda_{123} \) are real in an unitary theory.

**Note 1.** We have constructed \( \langle \phi_j(x)\phi_k(y) \rangle \) to be invariant under \( \text{SO}(d+1,1) \)-transformations and scalings \( \text{SO}_2 \). \( \text{SO} \)-transformations are rotations, so angles and lengths are left invariant. Therefore we want \( \langle \phi_j(X)\phi_k(Y) \rangle \) to depend on the angle \( XY \) and the lengths \( X^2 \) and \( Y^2 \), but since we are on the lightcone \( \text{lightcone} \) holds, and thus \( \langle \phi_j(X)\phi_k(Y) \rangle \) will only depend on \( XY \). If we let \( \langle \phi_j(X)\phi_k(Y) \rangle \propto (XY)^{-3} \), the correlator will be invariant under scalings as well. In \( \text{lightcone} \) we have normalized the correlators. Some details about this is in appendix section \( \text{Appendix A.1} \).

### 2.4 Radial Quantization

When we quantize the theory we foliate the \((d-1)\)-space as \( S^{d-2} \)-spheres centered at origo, each with their own Hilbert space. These foliations are called leaves. We let the radii, \( r \), of the spheres act as time-evolution operators. A dilation of a sphere changes it radius, hence it acts as a time-evolution operator in this quantization, and therefore also as the Hamiltonian. Thus if we were to classify the states with their dimensions (and spin)

\[
\begin{align*}
D|\Delta\rangle &= i\Delta|\Delta\rangle , \\
M_{\mu\nu}|\Delta, \ell\rangle &= (\Sigma_{\mu\nu})_{|[\ell]}|\Delta, \ell\rangle , \quad \Sigma_{\mu\nu} \neq 0 , \quad \text{only if } l > 0 .
\end{align*}
\]

If we study a conformal transformation from a statistical, non-quantized, point of view, we find how a conformal action, \( G \), acts on a scalar field

\[
G\phi(x) = \Delta \partial^\mu e^\mu(x)\phi(x) + e^\mu(x)\partial_\mu \phi(x) , \quad \delta b(x) = \partial^\mu e^\mu(x) , \quad \sum_{j=1}^n \langle \phi_1...\phi_{j-1}G\phi\phi_{j+1}...\phi_n \rangle = 0 , \quad \phi_j \equiv \phi(x_j) .
\]

Here \( b(x) \) is a local scaling, see \( \text{Appendix A.1} \). In a quantized theory, it is the same as above, but with the identifications

\[
G\phi \equiv |G,\phi\rangle , \\
\langle \phi_1...\phi_n | \equiv \langle 0|\phi_1...\phi_n |0\rangle .
\]

We define the vacuum as the state eliminated by \( G \), i.e. \( G|0\rangle = 0 \). The algebra of \( G \) is the following

\[
\begin{align*}
[P_\mu, \phi] &= -\partial_\mu \phi , \\
[M_{\mu\nu}, \phi] &= -i \{ \Sigma_{\mu\nu} + x_\mu \partial_\nu \} \phi , \\
[D, \phi] &= -i (\Delta + x^\mu \partial_\mu) \phi , \\
[K_\mu, \phi] &= -i \left( 2\Delta x_\mu + 2x^\nu \Sigma_{\nu\mu} + 2x_\mu x^i \partial_i - x^2 x_\mu \right) \phi .
\end{align*}
\]

There is a one-to-one correspondence between states and local operators \( \partial_\Delta \langle 0|0 \rangle = |\Delta\rangle \), with \( P_\mu/K_\mu \) acting as raising/lowering operators\(^2\).

\[
\begin{align*}
[D, P_\mu] = i P_\mu \Rightarrow |\Delta\rangle P_\mu |\Delta + 1\rangle P_\mu ... , \\
[D, K_\mu] = -i K_\mu \Rightarrow 0 K_\mu |\Delta\rangle K_\mu |\Delta + 1\rangle K_\mu ... .
\end{align*}
\]

\(^2\)In this way, the dimension of time is orthogonal to all of the spacial dimensions.

\(^3\)See appendix A.2 for why this is the case.
2.5 OPERATOR PRODUCT EXPANSION

The quantization yields the operator product expansion (OPE) between two operators:

\[ \langle \phi_1 \phi_2 \phi_3 \rangle = \sum_{\mathcal{O}} \lambda_{123} \mathcal{O} \langle \mathcal{O} \phi_3 \rangle \, , \quad C_{\mathcal{O}} = 1 + \frac{1}{2} \xi^\mu \partial_\mu + \mathcal{O}(q^2) \, . \] (2.5.1)

Here \( \lambda_{123} \) is the same constant as that in the three-point correlator \( \langle \mathcal{O}^{(n)} \rangle \), and the sum goes over all primaries \( \mathcal{O} \) in the theory. This OPE is associative, and only the first term in the differential operator \( C_{\mathcal{O}} \) may produce a primary. This means that if there only exist one primary with dimension \( \Delta \), and no descendant has the same dimension as any of the primaries, then the OPE above is non-zero only if \( \phi_1, \phi_2 \) and \( \phi_3 \) are all primaries (as can be seen from the two-point correlator \( \langle \mathcal{O} \phi_3 \rangle \)). However, this is not always the case.

Using the OPE we may recursively reduce a \( n \)-point correlator to a sum over \( (n-1) \)-, \( (n-2) \)-, ..., and so on down to two-point correlators. This means that all correlators in a CFT, and therefore the CFT itself, are characterized by the dimensions, \( \Delta \), and the spin, \( l \), of the primaries as well as the coefficients, \( \lambda_{jkl} \), from the three-point correlators. We call the set of \( \{ \Delta, l, \lambda_{jkl} \} \) the CFT data.

Associativity of the OPE yields a relation, called the conformal bootstrap equation, between the \( \lambda_{jkl} \)s, which originates from the four-point correlators. We may write this relation in terms of the functions \( f(u, v) \) from the four-point correlator \( \langle \mathcal{O}^{(n)} \rangle \) and conformal blocks, \( G_{\mathcal{O}} \). By contracting \( \phi_1 \) with \( \phi_2 \) and \( \phi_3 \) with \( \phi_4 \) and comparing it with the case where we contract \( \phi_1 \) with \( \phi_4 \) and \( \phi_2 \) with \( \phi_3 \) yields this bootstrap equation

\[ f(u, v) = \sum_{\mathcal{O}} \lambda_{1234} \mathcal{O} G_{\mathcal{O}}(u, v) = \sum_{\mathcal{O}} \lambda_{1423} \mathcal{O} G_{\mathcal{O}}(u, v) \, . \] (2.5.2)

If we map the conformally invariant coordinates \( u \) and \( v \) to a cylinder, we find

\[ G_{\mathcal{O}}(u, v) = \sum_{n \geq 0} A_n(\alpha) r^{\Delta+n} \, , \quad A_0(0) = 1 \implies A_0(\pi) = (-1)^l \] (2.5.3)

Here \( A_n \) are some polynomials of the angle \( \alpha \), e.g. \( A_0 \) is a Gegenbauer polynomial for \( d \geq 5 \).

---

\[ \text{The two operators does not need to be primaries.} \]

\[ \text{Higher point correlators will not yield new relations} \]
There are numerous different approaches to the bootstrap equation. We will study two of the most famous approaches, one being used for numerical derivation of the scaling dimensions while the other is purely analytical. These two approaches are those used in [2] as well as [3]. It is worth mentioning that there exists other analytical approaches to the bootstrap program, see e.g. [4].

3.1 Geometrical Approach

In this section we review the approach of [2] to the bootstrap equation. Its end result may be used for numerical derivation of the lowest dimensions, see (2.2.2). Our starting point for this discussion will be the four-point correlator (2.3.5) with four identical scalars, $\phi$, with scaling dimension, $\Delta_\phi$. We note that the LHS is invariant under any exchange between two space-time coordinates, $x_j$, thus the RHS should be invariant under this exchange too. This yields crossing symmetries of the functions, $f(u,v)$

$$f(u,v) = f(u^{-1},v^{-1}), \quad \text{(follows from } x_1 \leftrightarrow x_2),$$

$$v^{\Delta_\phi} f(u,v) = u^{\Delta_\phi} f(v,u), \quad \text{(follows from } x_1 \leftrightarrow x_3).$$

(3.1.1)

Let us impose these symmetries on the conformal blocks, $G_\Theta$, through (2.5.2). We consider an unitary theory, making the constants, $\lambda_{\phi\phi\Theta}$, from the three-point correlator real-valued. Thus the coefficients, $\lambda^2_{\phi\phi\Theta}$, in the bootstrap equation will be non-negative. This yields that the spin of the operators $\Theta$ will all have even spin. The first crossing symmetry above will be satisfied automatically by $G_\Theta$, while the second one will bring the bootstrap equation to another form

$$G_\Theta (u,v) = (-1)^{l_\theta} G_\Theta (uv^{-1},v^{-1}),$$

$$\sum_{\Theta} p_\Theta F_{\Delta_\phi \Theta} (z,\bar{z}) = 1, \quad F_{\Delta_\phi \Theta} (z,\bar{z}) = \frac{\mu^{\Delta_\phi} G_\Theta (u,v) - u^{\Delta_\phi} G_\Theta (v,u)}{u^{\Delta_\phi} - v^{\Delta_\phi}}. \quad \text{(3.1.2)}$$

We call the second of these equations the sum rule, where we sum over all primaries $\Theta$. The "1" on the RHS of this equation has its origin from the unity operator. Here $z$ and $\bar{z}$ are space-like "diamond"
coordinates. In this diamond, $G_\theta$ is real, $F_{\Delta_\theta \phi}$ is smooth, vanishing on the boundary and converges fastest at its center, which is at

$$z = \bar{z} = 2^{-1}.$$  \hspace{1cm} (3.1.3)

We introduce coordinates $a$ and $b$ that are zero at the center of this diamond

$$\begin{cases} z = 2^{-1} + a + b, \\ \bar{z} = 2^{-1} + a - b. \end{cases}$$  \hspace{1cm} (3.1.4)

$F_{\Delta_\theta \phi}$ is even w.r.t. $a$ and $b$

$$F_{\Delta_\theta \phi}(\pm a, \pm b) = F_{\Delta_\theta \phi}(a, b).$$  \hspace{1cm} (3.1.5)

Comparing the sum rule with the equation for an elliptical cone with radius $r_j$ in the $x_j$-direction

$$r_1^{-2} x_1^2 + \ldots r_{n-1}^{-2} x_{n-1}^2 = r_n^{-2} x_n^2,$$  \hspace{1cm} (3.1.6)

and we see that the sum rule geometrically describes a cone in the function space $\{F_{\Delta_\theta \phi}(a, b)\}$. The sum rule is satisfied only if the "1" on the RHS of (3.1.2) is in the cone. Lowering the unitary boundary for scalars, $\Delta_{\min}(d, 0)$, will increase the number of primaries $O$ and thus make the cone wider and vice versa. This means that there exist a critical value, $\Delta_c$, for $\Delta_{\min}(d, 0)$ where this "1" lies on the boundary. Thus if $\Delta_{\min}(d, 0)$ is higher than $\Delta_c$, this "1" is outside the cone, which is not allowed. A consequence of this is that there must be fields with no spin in every CFT (since $\Delta_{\min}(d, 0) \to \infty$ is not allowed).

If we define vectors, $F_{\Delta_\theta \phi}^{(2m,2n)}$, as

$$F_{\Delta_\theta \phi}^{(2m,2n)} := \partial_a^{2m} \partial_b^{2n} F_{\Delta_\theta \phi}(a, b) \bigg|_{a=b=0},$$  \hspace{1cm} (3.1.7)

we can project the cone down to the $(F_{\Delta_\theta \phi}^{(2,0)}, F_{\Delta_\theta \phi}^{(0,2)})$-plane. For a vector in this plane, the sum rule yields a homogeneous equation\footnote{The sum rule for any $F_{\Delta_\theta \phi}^{(2m,2n)}$, $m, n \geq 1$, will be a homogeneous equation, while the sum rule for $F_{\Delta_\theta \phi}^{(0,0)}$ will be an inhomogeneous equation (the same as the original sum rule).}

$$\sum_{\theta} p_\theta (F_{\Delta_\theta \phi}^{(2,0)}, F_{\Delta_\theta \phi}^{(0,2)}) = 0.$$  \hspace{1cm} (3.1.8)

An important feature of the cone's projection onto the $(F_{\Delta_\theta \phi}^{(2,0)}, F_{\Delta_\theta \phi}^{(0,2)})$-plane is that the projection covers exactly half of this plane when $\Delta_{\min}(d, 0) = \Delta_c$, and the whole plane when $\Delta_{\min}(d, 0) < \Delta_c$. In \cite{2} they study the span of $\Delta_{\min}(d, l)$ for different spins in the $(F_{\Delta_\theta \phi}^{(2,0)}, F_{\Delta_\theta \phi}^{(0,2)})$-plane. By varying $\Delta_{\min}(d, 0)$, we find $\Delta_c$ when the set of $\Delta_{\min}(d, l)$ span exactly half of the plane.
3.2 Analytical Approach

In this section we review the purely analytical approach of [3] to the bootstrap equation. We first study a mean field theory (MFT) with arbitrary scaling dimensions, and then show that the same reasoning can be applied to a general CFT. A MFT is a free, i.e. non-interacting, CFT with only Gaussian fields. Its four-point correlator for four identical scalars equals

\[
\langle \prod_{j=1}^{4} \phi(x_j) \rangle = \frac{1 + u^{-\Delta} + v^{-\Delta}}{\lambda_1^2 \lambda_2^2}. \tag{3.2.1}
\]

Here the "1" comes from the s-channel, \( u^{-\Delta} \) comes from the t-channel and \( v^{-\Delta} \) comes from the u-channel. Comparing this with the bootstrap equation (2.5.2) yields

\[
1 + u^{-\Delta} + v^{-\Delta} = v^{-\Delta} \left( 1 + \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 \phi_{\mathcal{O}}(u,v) \right). \tag{3.2.2}
\]

Here the "1" originates from the unity operator. We define the twist, \( \tau \), of an operator \( \mathcal{O} \) as

\[
\tau := \Delta_{\mathcal{O}} - l_{\mathcal{O}}. \tag{3.2.3}
\]

In the regime where \( u \) and \( v \) are small

\[
|u| << |v| << 1, \tag{3.2.4}
\]

the LHS of (3.2.2) goes as \( u^{-\Delta} \) while the RHS goes as \( \log(u) \). A careful analysis of these divergences reveals that the logarithms can be resummed in a way that yields power-like divergences, provided that the spectrum of operators satisfies

\[
\tau \xrightarrow{l_{\mathcal{O}} \to \infty} 2(\Delta_{\phi} + n), \quad n \in \{0,1,...\}. \tag{3.2.5}
\]

Here \( n \) is an integer that we sum over in the RHS of the bootstrap equation (3.2.2). This result means that all of the operators, \( \mathcal{O} \), in this bootstrap equation will have the same twist in the large spin limit, i.e. we will have "towers" of operators whose spin approaches the above when \( l_{\mathcal{O}} \) becomes large.

Let us now apply the same reasoning to a general CFT. We write the conformal blocks as

\[
G_{\mathcal{O}}(u,v) = u^{\tau_{\mathcal{O}}/2} g_{\mathcal{O}}(u,v), \tag{3.2.6}
\]

which using crossing symmetry (3.1.1) brings the bootstrap equation to the form

\[
1 + \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 u^{\tau_{\mathcal{O}}/2} g_{\mathcal{O}}(u,v) = \left( \frac{u}{v} \right)^{\Delta_{\phi}} \left( 1 + \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 v^{\tau_{\mathcal{O}}/2} g_{\mathcal{O}}(v,u) \right). \tag{3.2.7}
\]

The RHS must reproduce the contribution from the unity operator on the LHS, i.e. "1" on LHS. An analysis, in the regime (3.2.4), similar to the one in MFT shows that \( \tau_{\mathcal{O}} \) approaches again (3.2.5) as \( l_{\mathcal{O}} \) becomes large. This means that for sufficiently large spin, any CFT behaves like a MFT.

Note 2. The unitary bound on the weights (2.2.2) yields a lower bound on the twists as well (seen from the definition (3.2.3))

\[
\tau_{\mathcal{O}} \geq (d - 2) \left( 1 - \frac{\delta l_{\mathcal{O}}}{2} \right). \tag{3.2.8}
\]
**Note 3.** In a general CFT, $\tau_\theta$ is given by

$$
\tau_\theta = 2(\Delta_\phi + n) + \gamma(n, l), \quad \gamma(n, l) \propto l_\theta^{-\tau_m}.
$$

(3.2.9)

Here $\gamma(n, l)$ is the anomalous dimension and $\tau_m$ is the lowest twist in the theory. As we can see is $\gamma(n, l)$ negligible if $l_\theta$ is large. This means that $\gamma(n, l)$ measures the rate at which the CFT differs from a MFT.

**Note 4.** If we consider two different scalars in the four-point correlator, giving us $\lambda^2_{\phi_1\phi_2\theta}$-terms in the OPE (3.2.7), then $\tau_\theta$ approaches $\Delta_{\phi_1} + \Delta_{\phi_2} + 2n$ as $l_\theta$ becomes big.
CHAPTER 4

DEFFECTS IN CFTs

In this section we review the approach of [8] to the bootstrap program when we consider a theory with planar or spherical defects, i.e. defects with a flat metric. We define a defect, \( D^{(m)} \), as a subspace in the space of the theory where new fields and new interactions between fields may occur. We call fields that only live on the defect for defect-local fields, \( \psi \), and fields that live in the entire space of the theory for bulk-local fields, \( \phi \). In action terms this may look like

\[
S = \int d^d x f_0(\phi) + \delta_{x \in D^{(m)}} \int d^{d-m} x f'(\phi, \psi)
\]

(4.0.1)

A hyperplane in \( \mathbb{R}^{D+1,1} \) is classified as time-like if it intersects the null-cone, space-like if it does not intersect the null-cone and light-like if it tangents the null-cone. We are interested in co-dimension \( m \)-defects whose support preserves \( SO(m) \times SO(d - m + 1,1) \) invariance. Only the support of time-like hyperplanes satisfy this. We parametrize these hyperplanes with \( m \) vectors, \( P_A^\alpha, A \in \{1,...,d+2\}, \alpha \in \{1,...,m\} \), which we normalize as

\[
P_\alpha P_\beta = \delta_{\alpha\beta}.
\]

(4.0.2)

The defect is characterized by \( m \)-dimensional orthonormal frames. This yields an \( O(m) \) gauge redundancy on \( P_a^M \). Thus we find the center, \( C \), and the radius, \( r \), of the defect by constructing \( O(m) \)-invariants out of \( P_a^M \) and a reference point \( \Omega^M \). We find

\[
C^M = \frac{\Omega^M - 2(P_a^\alpha \Omega) P_a^M}{4(P_b^\beta \Omega)^2}, \quad r^2 = \frac{1}{4(P_b^\beta \Omega)^2}.
\]

(4.0.3)

The question now is how to find \( P_a^M \). By definition \( P_a^M \) are orthogonal to the \( (d+2-m) \)-dimensional hyperplane in the embedding space, thus we need \( d+2-m \) vectors to fix this hyperplane. To fix a co-dimension \( m \) defect requires exactly the same amount of vectors. Thus we consider \( d+2-m \) vectors on the defect and project them onto the Poincaré section, i.e. the real space, \( P_a^M \) are then the solutions of

\[
X_j P_a = 0, \quad j \in \{1,...,d+2-m\}.
\]

(4.0.4)

Let us study \( P_a \) in some important examples.
Example 1. If the defect is spherical, has its center at the origin and is aligned so it lies in the \((d + 1 - m)\)-dimensional hyperplane spanned by an orthonormal basis, \(e_j, j \in \{1, \ldots, d + 1 - m\}\), then we pick

\[
X^M_j = (1, r^2, r e_j), \quad X^M_{d+2-m} = (1, r^2, -re_1).
\]

\(P^M_a\) could then be given by

\[
P^M_\beta = (0, 0, e_{d+1-m+\beta}), \quad P^M_\alpha = (r^{-1}, -r, 0), \quad \beta \in \{1, \ldots, m-1\}.
\]

Example 2. If the spherical defect from the previous example is shifted by \(l\) along \(e_1\), we pick

\[
X^M_j = (1, r^2, re_j + le_1), \quad X^M_{d+2-m} = (1, r^2, -re_1 + l e_1).
\]

\(P^M_a\) is then be given by

\[
P^M_\beta = (0, 0, e_{d+1-m+\beta}), \quad P^M_\alpha = (r^{-1}, -r + r^{-1}l^2, re_1), \quad \beta \in \{1, \ldots, m-1\}.
\]

Example 3. If the defect is planar and aligned in a plane spanned by \(e_j, j \in \{1, \ldots, d-m\}\), we pick

\[
X^M_j = (1, 1, e_j), \quad X^M_{d+1-m} = (1, 0, 0), \quad X^M_{d+2-m} \rightarrow \infty.
\]

\(P^M_a\) is then be given by

\[
P^M_\alpha = (0, 0, e_{d-m+a}), \quad \alpha \in \{1, \ldots, m\}.
\]

Example 4. If the planar defect from the previous example is tilted by an angle \(\theta\) in the \((e_1, e_d)\)-plane, \(P^M_a\) is given by

\[
P^M_\beta = (0, 0, e_{d-m+\beta}), \quad P^M_\alpha = (0, 0, \cos \theta e_d - \sin \theta e_1), \quad \beta \in \{1, \ldots, m-1\}.
\]

4.1 THE DEFECT/ BULK CORRELATOR

In this section we study the correlator function between a defect- and a bulk-local operator. We construct the correlators between a defect-local operator, \(D^{(m)}(P_a)\), a bulk-local operator, \(\mathcal{O}\), and a defect-local scalar, \(\sigma(Y)\), using the same reasoning as that in note 4.1.

\[
\langle D^{(m)}(P_a) \mathcal{O}(X) \rangle = \frac{C_{D^{(m)}}^{\mathcal{O} \mathcal{O}}}{\langle P_a X \rangle_{\mathcal{O} \mathcal{O}}^{\Delta \sigma/2}},
\]

\[
\langle D^{(m)}(P_a) \sigma(Y) \mathcal{O}(X) \rangle = \frac{C_{D^{(m)}}^{\mathcal{O} \mathcal{O}} \langle P_a X \rangle_{\mathcal{O} \mathcal{O}}^{\Delta \sigma - \Delta \sigma/2}}{(-2XY)^{\Delta \sigma/2}}.
\]

Note 5. \(X\) cannot be on the defect since the two-point correlator at \((4.1.1)\) will then diverge (follows from \((4.0.4)\)).

Since \(\mathcal{O}\) is normalized through its two-point correlator \((2.3.3)\), and \(D^{(m)}(P_a)\) is normalized through \((4.0.2)\), the constants \(C_{D^{(m)}}^{\mathcal{O} \mathcal{O}}\) and \(C_{D^{(m)}}^{\mathcal{O} \sigma}\) will hold physical meaning. It is important to remember that \(C_{D^{(m)}}^{\mathcal{O} \sigma}\) is from an OPE unlike \(C_{D^{(m)}}^{\mathcal{O} \mathcal{O}}\). If we let \(\sigma(Y)\) be the unit operator, \(\mathcal{I}\), we find

\[
C_{\mathcal{I} \sigma}^{\mathcal{O} \mathcal{O}} = C_{\mathcal{O} \mathcal{O}}^{\mathcal{O} \mathcal{O}}.
\]
If we consider a defect as that in example[1], then
\[ \langle D^{(m)}(P \alpha) \Theta(X) \rangle = C_{\Theta}^{D^{(m)}} \left( \frac{l_{\min} l_{\max}}{2r} \right)^{-\Delta_\Theta/2} \],
\[ \langle D^{(m)}(P \alpha) o(Y) \Theta(X) \rangle = C_{\Theta_o}^{D^{(m)}} \cdot \text{other terms} \] (4.1.3)

Here \( l_{\min}/l_{\max} \) is the minimum/maximum distance from \( P \alpha \) in the defect to the point \( X \) outside the defect, and \( l \) is the distance form the center of the defect to the point \( X \).

### 4.2 The Defect Expansion

The OPE between a bulk- and a defect-local operator is a sum over scalars on the defect, and the OPE between a defect-local operator and a scalar on the defect is a sum over bulk-local operators \[8].

\[ \langle D^{(m)}(P \alpha) o(Y) \rangle = \sum_{\Theta} C_{\Theta}^{D^{(m)}} f_{\Delta_\Theta}(P \alpha, X, \partial X) \Theta(X) \] (4.2.1)

If we let \( o(Y) \) be the unit operator, we find through (4.1.2) that we can express a defect-local operator in terms of bulk-local operators. We call this expansion the defect expansion

\[ D^{(m)}(P \alpha) = \sum_{\Theta} C_{\Theta}^{D^{(m)}} f_{\Delta_\Theta}(P \alpha, X, \partial X) \Theta(X) \] (4.2.2)

Using this expansion we find that the correlator between two defect-local operators is given in terms of conformal blocks

\[ \langle D^{(m)}(P \alpha) D^{(k)}(Q \rho) \rangle = \sum_{\Theta} C_{\Theta}^{D^{(m)}} C_{\Theta}^{D^{(k)}} G_{\Theta}(\eta_a) \] (4.2.3)

Here \( \eta_a \) is the cross-ratio of the matrices

\[ M_{a\beta} = \langle P \alpha Q \rho \rangle (Q^P P_\beta) \] (4.2.4)

Using SO(\( m \)) and SO(\( k \)) transformations, we find that the only gauge invariant data in \( M_{a\beta} \) is its diagonal \[8]. Thus we find the cross-ratios through

\[ \eta_a = \text{Tr}(M_{a\beta})^a \] (4.2.5)

**Example 5.** \( \eta_1 = M_{a^a}, \eta_2 = M_{a\beta}M^{\beta\alpha} \).

The number of physical cross-ratios will be \( \min(m, k, d + 2 - m, d + 2 - k) \). The conformal blocks satisfies the Casimir eigenvalue equation

\[ (L^2 + C_{\Theta}) G_{\Theta}(\eta) = 0 \] (4.2.6)

Leibniz rule yields

\[ \left( \frac{1}{2} L^{AB} \eta_a L_{AB} \eta_b \frac{\partial^2}{\partial \eta_a \partial \eta_b} + L^2 \eta_a \frac{\partial}{\partial \eta_a} + C_{\Theta} \right) G_{\Theta}(\eta) = 0 \] (4.2.7)

---

8. References or sources for the equations and concepts are typically found at the end of the text. In the context of this document, the number 8 might refer to a specific reference or source. For a complete understanding, one would need to refer to the indicated source. If this is a question-based or theoretical physics context, the number 8 could point to a related theorem, equation, or concept in the referenced material. In a more practical or technological context, the number 8 might refer to a version, iteration, or release of a software, system, or methodology. The number screwed up for understanding. The correct understanding is crucial for effective use or application. If this is a question-based or theoretical physics context, the number 8 could point to a related theorem, equation, or concept in the referenced material. In a more practical or technological context, the number 8 might refer to a version, iteration, or release of a software, system, or methodology.
Here we sum over all physical cross-ratios. The components in the above equation are given by \cite{8}

\[ C_{\theta} = \Delta_{\theta} (\Delta_{\theta} - d) + l_{\theta} (l_{\theta} + d - 2) , \]

\[ \frac{1}{2} L^{AB} \eta_{a} L_{AB} \eta_{b} = 4 a b (\eta_{a+b-1} - \eta_{a+b}) , \]

\[ L^{2} \eta_{a} = \begin{cases} 2 \{ m k - (d + 2) \eta_{1} \}, & \text{if } a = 1 . \\ 4 \{ (1 + m + k) \eta_{1} - (d + 3) \eta_{2} - \eta_{1}^{2} \}, & \text{if } a = 2 . \\ 2 a \{(a + m + k - 1) \eta_{a-1} - (a + d + 1) \eta_{a} + \sum_{b=1}^{a-2} \eta_{b} \eta_{a-b-1} - \sum_{b=1}^{a-1} \eta_{b} \eta_{a-b} \}, & \text{if } a \geq 3 . \end{cases} \]

**Note 6.** Unphysical cross-ratios, i.e. \( \eta_{a} \) with \( a > \min(m, k, d + 2 - m, d + 2 - k) \), may appear in the expressions above. Using trace relations, we may write those in terms of physical ones.

**Example 6.** Let us review this chapter with an example of how we can find the conformal blocks in a theory with two spherical defects. One is a co-dimension \( m \) defect, \( D^{(m)}(P_{a}) \), and the other is a co-dimension one defect, \( D^{(1)}(Q) \). Let \( D^{(m)}(P_{a}) \) have radius \( r_{1} \), \( D^{(1)}(Q) \) have radius \( r_{2} \) and let the distance between their centers be \( l \). If we place \( D^{(m)}(P_{a}) \) at the origin, \( P_{a} \) will be given by that in example 1 and \( Q \) will be given by that in example 2. There will be one physical cross-ratio\(^1\)

\[ \eta_{1} = (P_{\beta} Q)(Q P^{\beta}) + (P_{m} Q)(Q P^{m}) = \left( \frac{1}{r_{2}} \delta_{\beta(d-m)} \right)^{2} + \left( \frac{r_{2}^{2} + r_{1}^{2} - l^{2}}{2 r_{1} r_{2}} \right)^{2} = \frac{l^{2}}{r_{2}^{2}} + \frac{r_{2}^{2} + r_{1}^{2} - l^{2}^{2}}{4 r_{1}^{2} r_{2}^{2}} . \] (4.2.8)

We find the conformal blocks from the Casimir equation (4.2.7)

\[ \left( 4 \{ \eta_{1} - \eta_{2} \} \frac{\partial^{2}}{\partial \eta_{1}^{2}} + 2 \{ m - (d + 2) \eta_{1} \} \frac{\partial}{\partial \eta_{1}} + \Delta_{\theta} (\Delta_{\theta} - d) \right) G_{\theta}(\eta_{1}) = 0 . \] (4.2.9)

Since the vectors \( P_{a} \) and \( Q \) commute, the unphysical cross-ratio \( \eta_{2} \) is given by

\[ \eta_{2} = (P_{a} Q)(Q P_{\beta})(Q P^{\beta}) = \eta_{1}^{2} . \] (4.2.10)

Thus

\[ \left( \eta_{1} (1 - \eta_{1}) \frac{\partial^{2}}{\partial \eta_{1}^{2}} + \frac{m - (d + 2) \eta_{1}}{2} \frac{\partial}{\partial \eta_{1}} + \frac{\Delta_{\theta} (\Delta_{\theta} - d)}{4} \right) G_{\theta}(\eta_{1}) = 0 , \] (4.2.11)

which is solved by hypergeometric functions \cite{8}

\[ G_{\theta}(\eta_{1}) = \eta_{1}^{-\Delta_{\theta}/2} F_{1} \left( \frac{\Delta_{\theta}}{2}, \frac{\Delta_{\theta} - m}{2} + 1, \Delta_{\theta} - \frac{d}{2} + 1, \eta_{1}^{-1} \right) . \] (4.2.12)

\(^{1}\)Remember that we are using lightcone coordinates, see 2.1.2.
Recently there has been a lot of development in the conformal bootstrap program for the 3D Ising model with a co-dimension two, monodromy defect, both numerically [10], and analytically [6]. A monodromy defect is defined with the condition

\[ \phi(r, \theta + 2\pi, y) = g \phi(r, \theta, y), \quad r = |\vec{r}|, \quad g \in \mathcal{G}. \]  

(5.0.1)

Here \( r \) is the shortest distance from the bulk-local fields, \( \phi^j \), to the defect, \( \theta \) is an angle between \( \vec{r} \) and a specified vector transverse to the line defect, \( y \) is the coordinate along the axis parallel to the defect and \( g \) is an element of the global symmetry group \( \mathcal{G} \) of the theory. This condition means that if we transport \( \phi^j \) around the defect, we get back a transformed field. The choice of \( g \) will define the defect.

**Example 7.** In the 3D Ising model, the global symmetry group is \( \mathbb{Z}_2 \). Thus the monodromy defect in this theory can be defined with either \( g = \pm 1 \).

We expect that insertion of a defect will break \( \mathcal{G} \) since acting with an arbitrary element from \( \mathcal{G} \) will change the defect. Only the subgroups of \( \mathcal{G} \) leaving \( g \) invariant will survive as a symmetry of the theory.

In order to get consistency, the result from the above definition should be the same as that from flipping the defect

\[ \phi(r, \theta - 2\pi, y) = g^{-1} \phi(r, \theta, y), \quad r = |\vec{r}|, \quad g \in \mathcal{G}. \]  

(5.0.2)

The approach to the bootstrap program in [6] starts with the WF fixed point in \( \phi^4 \)-theory. As it turns out, we get the 3D Ising model if we let the dimension go to three when we perform a dimensional regularization [10]. In [6] they compare an expansion of bulk-local operators in terms of defect-local operators (similar to (4.2.2), but for bulk-local operators instead) with correlators found from Feynman diagrams to find the scaling dimensions of the primaries, and some of the OPE coefficients to the first order in \( \epsilon \). In this expansion they assume that the constants in the \( \epsilon \)-expansion gets smaller and smaller at higher powers of \( \epsilon \), since \( \epsilon \) itself is not small. A framework was later created in [5] that makes it possible to find the scaling dimensions and the OPE coefficients in the 3D Ising Models (without a defect) at the WF fixed point without using perturbation theory. This frameworks constrains the theory by defining three axioms that contain information about the dynamic of the Ising model. A generalization of this framework was made in [9] to the 3D Ising model with a monodromy line defect. The same results of the scaling
dimensions and OPE coefficients in [6] was derived using this generalized framework.

For the rest of this paper, we generalize the method in [6] to the WF O(N) model. We will double check the results we find from this generalization using the O(N) framework in [5] in chapter [9]. This double check will be very similar to the method in [9]. The WF O(N) model is governed by the Lagrangian [11]

\[ \mathcal{L} = \frac{1}{2} (\partial \mu \phi^j)^2 + \frac{\lambda}{4!} (\phi^j)^4, \quad j \in \{1, \ldots, N\}. \]  

(5.0.3)

We renormalize it using dimensional regularization, i.e. we shift the dimension from four dimensions into \(4 - \epsilon\) dimensions\(^1\). The \(\beta\)-function is given by [12]

\[ \beta(\lambda) = \frac{\lambda}{3!} \left( -\epsilon + \frac{N + 8}{3! 8 \pi^2} \lambda \right) + O(\epsilon^3), \]  

(5.0.4)

which have fixed points at

\[ \lambda = 0 \quad \text{and} \quad \lambda = \frac{3! 8 \pi^2 \epsilon}{N + 8} + O(\epsilon^2). \]  

(5.0.5)

We align the line defect in the three dimensional theory to be parallel the basis space vector \(y\) as well as rescale the bulk-local fields as

\[ \Phi^j \rightarrow \frac{1}{2\pi} \Phi^j. \]  

(5.0.6)

The bulk-defect expansion for the rescaled \(\Phi^j\) presented in [6] is generalized into

\[ \Phi^j(r, \theta, y) = \sum_s \sum_{l \geq 0} C^j_{k_1 \ldots k_l} (s) \frac{e^{-i \theta}}{r^{\Delta - \Delta\psi}} B_{\Delta\psi}(r, \partial y) \psi^s_{k_1 \ldots k_l} (y), \quad k_1, \ldots, k_l \in \{1, \ldots, N\}, \]  

\[ B_{\Delta\psi}(r, \partial y) = \sum_{m \geq 0} \frac{(-1)^m (\Delta\psi)_m}{m! (2\Delta\psi)_{2m}} r^{2m} \partial_y^{2m}, \quad C^j_{k_1 \ldots k_l} (s) \equiv C^j_{\psi^s_{k_1 \ldots k_l}}, \quad (x)_m \equiv \frac{\Gamma(x + 1)}{\Gamma(x - m + 1)}. \]  

(5.0.7)

Here \(s\) is the spin of the defect-local operator \(\psi^s_{k_1 \ldots k_l} (y)\), \(C^j_{k_1 \ldots k_l} (s)\) is a OPE coefficient that we have promoted to a tensor and \((x)_m\) is the Pochhammer symbol. Summations over the indices \(k_1, \ldots, k_l\) are explicit. The first thing we need to ask ourselves is what kinds of defect-local operators may appear in this expansion. We may be able to constrain the theory using the definition of a monodromy line defect [5.0.1] and the global O(N)-symmetry.

---

\(^1\) The defect is always of co-dimension two.
5.1 Monodromy Action Constraint

We start studying constraints that arises from the definition (5.0.1). A general $O(N)$-matrix satisfies
\[ O(N) = \{ M^T M = I, \ det[M^T M] = 0, \ M \in \mathbb{R}^{N \times N} \} \Rightarrow \ det M = \pm 1. \tag{5.1.1} \]

By conjugation, an $O(N)$-matrix is then given by\(^2\)
\[ \gamma(\theta) = \begin{bmatrix} Y_{a_1 \times a_1} & 0 & 0 & 0 \\ 0 & \pm \cos \theta & 0 & \mp \sin \theta \\ 0 & 0 & Y_{a_2 \times a_2} & 0 \\ 0 & \sin \theta & 0 & \cos \theta \end{bmatrix}, \tag{5.1.2} \]
\[
\sum_{t=1}^{3} a_t = N - 2, \quad a_t \in \{0, 1, ..., N - 2\} \quad \forall t \in \{1, 2, 3\}. \tag{5.1.3}
\]

Here the matrices $Y_{a_t \times a_t}, t \in \{1, 2, 3\}$ only have non-zero elements along their diagonal. These elements are either plus or minus one. Let us assume that all of the $Y_{a_t \times a_t}$s together contain $\chi \in \{0, 1, ..., N - 2\}$ number of ones. Using $\gamma(\theta)$ when defining our defect yields the same constraints as using
\[ g(\theta) = \begin{bmatrix} R_\theta & 0 & 0 \\ 0 & 1_{\chi \times \chi} & 0 \\ 0 & 0 & -1_{(N-\chi-2) \times (N-\chi-2)} \end{bmatrix}, \quad R_\theta = \begin{bmatrix} \pm \cos \theta & \mp \sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad \chi \in \{0, 1, ..., N - 2\}. \tag{5.1.4} \]

Monodromy of the defect (5.0.1) together with the bulk-defect expansion (5.0.7) yields\(^3\)
\[
\begin{align*}
e^{-2\pi i s C^1_{k_1...k_1}} &= \pm \cos \theta C^1_{k_1...k_1} \mp \sin \theta C^2_{k_1...k_1}, \\
e^{-2\pi i s C^2_{k_1...k_1}} &= \sin \theta C^1_{k_1...k_1} + \cos \theta C^2_{k_1...k_1}, \\
e^{-2\pi i s C^q_{k_1...k_1}} &= C^q_{k_1...k_1}, \quad q \in \{3, ..., \chi + 2\}, \\
e^{-2\pi i s C^r_{k_1...k_1}} &= -C^r_{k_1...k_1}, \quad r \in \{\chi + 3, ..., N\}.
\end{align*} \tag{5.1.5}
\]

There are two important special cases for the above equation system. These special cases are when we cannot write $C^1_{k_1...k_1}$ in terms of $C^2_{k_1...k_1}$, i.e. when
\[ \sin \theta = 0 \quad \Leftrightarrow \quad \theta = 0 \text{ or } \pi \quad \text{if } \theta \in (-\pi, \pi]. \tag{5.1.6} \]

We will get two different sets of solutions depending on whether $R_\theta$ describes an improper ($\det R_\theta = -1$) or proper ($\det R_\theta = 1$) rotation.

Before we study these solutions, it would be good to check the consistency of them, i.e. to check that (5.0.2) yields the same solutions as (5.1.5). The matrix $g$ at (5.1.4) have the inverse
\[ g(\theta)^{-1} = \begin{bmatrix} R_\theta^{-1} & 0 & 0 \\ 0 & 1_{\chi \times \chi} & 0 \\ 0 & 0 & -1_{(N-\chi-2) \times (N-\chi-2)} \end{bmatrix}, \quad \chi \in \{0, 1, ..., N - 2\}, \tag{5.1.7} \]
\[ R_\theta^{-1} = \begin{bmatrix} \cos \theta & \pm \sin \theta \\ \pm \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & \mp \sin \theta \\ \mp \sin \theta & \cos \theta \end{bmatrix} = R^T. \]

\(^2\)The (im)proper rotation elements are studied in appendix chapter A.3.
\(^3\)The only differences between using $\gamma(\theta)$ and $g(\theta)$ when defining our defect are the upper indices on the constrained tensors, $C^j_{k_1...k_1}$.\]
Using (5.0.2) yields
\[
\begin{align*}
\begin{cases} 
    e^{2\pi i s} C^{1}_{k_1 \ldots k_l} & = \pm \cos \theta C^{1}_{k_1 \ldots k_l} + \sin \theta C^{2}_{k_1 \ldots k_l}, \\ 
    e^{2\pi i s} C^{2}_{k_1 \ldots k_l} & = \mp \sin \theta C^{1}_{k_1 \ldots k_l} + \cos \theta C^{2}_{k_1 \ldots k_l}, \\ 
    e^{2\pi i s} C^{q}_{k_1 \ldots k_l} & = C^{q}_{k_1 \ldots k_l}, \quad q \in \{3, \ldots, \chi + 2\}, \\ 
    e^{2\pi i s} C^{r}_{k_1 \ldots k_l} & = -C^{r}_{k_1 \ldots k_l}, \quad r \in \{\chi + 3, \ldots, N\}.
\end{cases}
\end{align*}
\] (5.1.8)

In the next two sections we study the solutions to (5.1.5) and the above system of equations (in the two different cases of the plus and minus signs). We will see that it does not matter which one of these system of equations we use, meaning that the theory is consistent.

### 5.1.1 Proper Rotation, \( \det R_\theta = 1 \)

In this section we study the solutions to (5.1.5) as well as (5.1.8) when the determinant of \( R_\theta \) is one. We consider first the two special cases when \( C^{1}_{k_1 \ldots k_l} \) can not be written in terms of \( C^{2}_{k_1 \ldots k_l} \) and vice versa, see (5.0.6). If \( \theta \) equals zero, the equation system (5.1.5) reduces to
\[
\begin{align*}
\begin{cases} 
    e^{-2\pi i s} C^{v}_{k_1 \ldots k_l} & = C^{v}_{k_1 \ldots k_l}, \quad v \in \{1, \ldots, \chi + 2\}, \\ 
    e^{-2\pi i s} C^{r}_{k_1 \ldots k_l} & = -C^{r}_{k_1 \ldots k_l}, \quad r \in \{\chi + 3, \ldots, N\}.
\end{cases}
\end{align*}
\] (5.1.9)

Which have two solutions. Either
\[
\begin{align*}
C^{v}_{k_1 \ldots k_l} = 0, \quad C^{v}_{k_1 \ldots k_l} \text{ is non-zero}, \quad s \in \mathbb{Z},
\end{align*}
\] (5.1.10)
or
\[
\begin{align*}
C^{v}_{k_1 \ldots k_l} = 0, \quad C^{r}_{k_1 \ldots k_l} \text{ is non-zero}, \quad s = n + \frac{1}{2}, \quad n \in \mathbb{Z}.
\end{align*}
\] (5.1.11)

These solutions tells us that the global symmetry group, \( \text{O}(N) \), has been broken into \( \text{O}(\chi + 2) \oplus \text{O}(N - \chi - 2) \). The branching rule yields that the bulk-local fields \( \Phi^j \) can then be separated into fields that transforms in \( \text{O}(\chi + 2) \) and fields that transforms in \( \text{O}(N - \chi - 2) \), i.e.
\[
\Phi^j = \Phi^a_{\chi + 2} \oplus \Phi^b_{N - \chi - 2}, \quad a \in \{1, \ldots, \chi + 2\}, \quad b \in \{1, \ldots, N - \chi - 2\}. \] (5.1.12)

Both \( \Phi^a_{\chi + 2} \) and \( \Phi^b_{N - \chi - 2} \) will have bulk-defect expansions similar to that of \( \Phi^j \), see (5.0.7). The defect-local operators in these expansions will transform under the same orthogonal symmetry group as the bulk-local fields, e.g. the defect-local operators in the bulk-defect expansion of \( \Phi^a_{\chi + 2} \) will transform under \( \text{O}(\chi + 2) \). Defect-local operators, \( \psi^{a_1 \ldots a_l}_{\chi + 2}, a_1, \ldots, a_l \in \{1, \ldots, \chi + 2\} \), that transform in \( \text{O}(\chi + 2) \) have integer spin, and defect-local operators, \( \psi^{b_1 \ldots b_l}_{N - \chi - 2}, b_1, \ldots, b_l \in \{1, \ldots, N - \chi - 2\} \), that transform in \( \text{O}(N - \chi - 2) \) have half-integer spin, i.e.
\[
\begin{align*}
s_{\chi + 2} = n, \quad s_{N - \chi - 2} = n + \frac{1}{2}, \quad n \in \mathbb{Z}.
\end{align*}
\] (5.1.13)

It is a similar story when \( \theta = \pi \). The \( \text{O}(N) \) symmetry is then broken into \( \text{O}(\chi) \oplus \text{O}(N - \chi) \), and fields that transforms in \( \text{O}(\chi) \) have defect-local operators with integer spin in their bulk-defect expansions, while fields that transforms in \( \text{O}(N - \chi) \) have defect-local operators with half-integer spin in their bulk-defect expansions.
Solving the equations [5.1.8] when [5.1.6] is satisfied yields the same results as those previously discussed in this section.

A more interesting case is when we consider $\theta$ to be real-valued, i.e. $\theta = \pm \Theta, \Theta \in (0, \pi)$. Then [5.1.5] yields the following equation system:

$$
\theta = \pm \Theta \Rightarrow \begin{cases}
C^1_{k_1...k_l} = \pm iC^2_{k_1...k_l}, & s = n + \frac{\Theta}{2\pi}, \ n \in \mathbb{Z}, \\
e^{-2\pi i C^q_{k_1...k_l}} = C^q_{k_1...k_l} \forall q \in [3, ..., 3 + \chi], & s = n', \ n' \in \mathbb{Z}, \\
e^{-2\pi i C^r_{k_1...k_l}} = -C^r_{k_1...k_l} \forall r \in [4 + \chi, ..., N], & s = n'' + \frac{1}{2}, \ n'' \in \mathbb{Z}.
\end{cases}
$$

These constraints are on the dynamics of the theory coming from the monodromy action. We see that the first two components of the OPE tensor $C^I_{k_1...k_l}$ relate to each other, and does not mix with other components of the tensor. The above system of equations has three solutions:

$$
\theta = \pm \Theta \Rightarrow C^1_{k_1...k_l} = \pm iC^2_{k_1...k_l}, \ C^2_{k_1...k_l} = 0 \forall \nu \in \{3, ..., N\}, \ s = n + \frac{\Theta}{2\pi}, \ n \in \mathbb{Z},
$$

or

$$
C^{\nu'}_{k_1...k_l} = 0 \forall \nu' \in \{1, 2, 3 + \chi, ..., N\}, \ s \in \mathbb{Z},
$$

or

$$
C^{\nu''}_{k_1...k_l} = 0 \forall \nu'' \in \{1, ..., \chi + 2\}, \ s = n + \frac{1}{2}, \ n \in \mathbb{Z}.
$$

Thus the $O(N)$ symmetry has been broken into $O(2) \otimes O(\chi) \otimes O(N - \chi - 2)$, with fields $\phi_a^b, a \in \{1, 2\}$ that transforms under $O(2)$ having bulk-defect expansions with defect-local operators, $\psi^{a_1...a_l}_{\chi+2}, a_1, ..., a_l \in \{1, 2\}$, that have fractional spin, $\phi^b_a, b \in \{1, ..., \chi\}$ that transforms under $O(\chi)$ having bulk-defect expansions with defect-local operators, $\psi^{b_1...b_l}_{\chi+2}, b_1, ..., b_l \in \{1, ..., \chi\}$, that have integer spin and $\phi^c_{N-\chi-2}, c \in \{1, ..., N - \chi - 2\}$ that transforms under $O(N - \chi - 2)$ having bulk-defect expansions with defect-local operators, $\psi^{c_1...c_l}_{N-\chi-2}, c_1, ..., c_l \in \{1, ..., N - \chi - 2\}$, that have half-integer spin.

We get the same results using [5.1.8] instead of [5.1.5], meaning the theory is consistent for proper rotations.

### 5.1.2 Improper Rotation, $\det R_\theta = -1$

The solutions to [5.1.5] considering the special cases when $\theta$ equals zero or $\pi$ will yield similar solutions as those in the proper case. In both of these cases the global $O(N)$ symmetry is broken, leaving a $O(\chi + 1) \otimes O(N - \chi - 1)$ symmetry. Defect-local operators that transforms in $O(\chi + 1)$ will have integer spin, while defect-local operators that transforms in $O(N - \chi - 1)$ will have half-integer spins. The procedure of finding this is exactly the same as that discussed in the previous section.

---

4 See the "Proper Rotation" section of appendix chapter for details on this.

5 The solutions are easily read off by matching the spin required for the equations to hold.
If we consider a real-valued $\theta = \pm \Theta$, $\Theta \in (0, \pi)$, (5.1.5) yields the following equation system\(^6\)

$$\theta = \pm \Theta \Rightarrow \left\{ \begin{array}{l}
C^1_{k_1...k_l} = \frac{\sin(\pm \Theta)}{1 + \cos \Theta} C^2_{k_1...k_l}, \quad C^q_{k_1...k_l} = \frac{\sin(\pm \Theta)}{1 - \cos \Theta} C^q_{k_1...k_l}, \quad s = \frac{n}{2}, \quad n \in \mathbb{Z}, \\
\end{array} \right. (5.1.18)$$

As in the proper case, these are constraints on the OPE coefficients coming from the monodromy action. Above system of equations has two solutions. Either

$$C^q_{k_1...k_l} \neq 0 \quad \forall q \in \{1, 2, ..., \chi + \chi + 2\}, \quad s = n', \quad n' \in \mathbb{Z},$$

or

$$C^q_{k_1...k_l} = 0 \quad \forall q \in \{1, 2, ..., \chi + \chi + 2\}, \quad s = n + \frac{1}{2}, \quad n \in \mathbb{Z}. \quad (5.1.20)$$

These solutions tell us that the symmetry group has again been broken into $O(\chi + 1) \oplus O(N - \chi - 1)$, where two of the fields, one that transforms in $O(\chi + 1)$, and the other transforms in $O(N - \chi - 1)$, have OPE tensors $C^1_{k_1...k_l}$ and $C^2_{k_1...k_l}$ in their bulk-defect expansions that are compositions of the two tensors $C^1_{k_1...k_l}$ and $C^2_{k_1...k_l}$, which both transforms in the unbroken symmetry group $O(N)$. Tensors that transforms in $O(\chi + 1)$, i.e. corresponds to defect-local fields with integer spin, should not mix with tensors that transforms in $O(N - \chi - 1)$, i.e. corresponds to defect-local fields with half-integer spin. Thus the tensor $C^1_{k_1...k_l}$ that transforms in $O(\chi + 1)$ should be zero when we are considering half-integer spin, see (5.1.20), and the tensor $C^2_{k_1...k_l}$ that transforms in $O(N - \chi - 1)$ should be zero when we are considering integer spin, see (5.1.19). We find

$$\theta = \pm \Theta \Rightarrow \bar{C}^1_{k_1...k_l} = C^1_{k_1...k_l} \pm \frac{\sin \Theta}{1 - \cos \Theta} C^2_{k_1...k_l}, \quad \bar{C}^q_{k_1...k_l} = C^q_{k_1...k_l} \mp \frac{\sin \Theta}{1 + \cos \Theta} C^q_{k_1...k_l}. \quad (5.1.21)$$

We can check that this result is correct by representing the OPE tensors that transforms in $O(\chi + 1)$ and $O(N - \chi - 1)$ as vectors, $\sigma_{\chi + 1}$ and $\sigma_{N - \chi - 1}$, both containing $N$ elements. These elements are the coefficients in front of $C_{k_1...k_l}, ..., C_{k_1...k_l}$, i.e.

$$\sigma_{\chi + 1} = (1, \mp(1 - \cos \Theta)^{-1} \sin \Theta, 1, ..., 1, 0, ..., 0), \quad \sigma_{N - \chi - 1} = (1, \mp(1 + \cos \Theta)^{-1} \sin \Theta, 0, ..., 0, 1, ..., 1). \quad (5.1.22)$$

The "±" sign corresponds to $\theta = \pm \Theta$, $\Theta \in (0, \pi)$. Since the OPE tensors that transforms in $O(\chi + 1)$ should not mix with OPE tensors that transforms in $O(N - \chi - 1)$, the two vectors $\sigma_{\chi + 1}$ and $\sigma_{N - \chi - 1}$ should be orthogonal to each other. Indeed, using the trigonometric identity we find

$$\sigma_{\chi + 1}^T \sigma_{N - \chi - 1} = 1 - \frac{\sin^2 \Theta}{(1 - \cos \Theta)(1 + \cos \Theta)} = 0. \quad (5.1.23)$$

Which is a sign that our construction seems to be correct.

\(^6\)See the "Improper Rotation" section of appendix chapter\(^3\) for details on this.
To summarize this section, inserting a monodromy defect using a proper O(2) rotation, i.e. \( \det R_\theta = 1 \), possibly (depending on the angle \( \theta \)) breaks the global O(N) symmetry into three parts O(2) \( \otimes \) O(\( \chi \)) \( \otimes \) O(N – \( \chi \) – 2), where fields that transform in one of these subgroups does not mix with fields from the other subgroups. Each of these bulk-local fields will have a bulk-defect expansion with defect-local operators that transform under the same unbroken subgroup as their corresponding bulk-local field. These defect-local operators have different spin depending on what subgroup they transform under. The situation is very similar when considering an improper O(2) rotation, i.e. \( \det R_\theta = -1 \), when defining the defect. In this case however, the global O(N) symmetry (independently of the angle \( \theta \)) breaks into O(\( \chi + 1 \)) \( \otimes \) O(N – \( \chi \) – 1), meaning that in general, using \( \det R_\theta = -1 \) does not break the symmetry as much as when using \( \det R_\theta = 1 \).

### 5.2 Symmetry Constraints

In this section we study constraints from the broken O(N) symmetry. The transformed bulk-local field, \( \Phi^j \), is to be the same as when we transform the defect-local fields, \( \Psi^k \), inside the bulk-defect expansion [5.0.7]. Let \( \Omega^j_k \in \text{O}(X) \) be a transformation matrix from one of the subgroups that is preserved after the global O(N)-symmetry has been broken (see the previous section). Then the transformation of bulk-local fields under \( \Omega \) must be compatible with the transformation of defect-local fields under the same \( \Omega \)

\[
\Omega^j_j \Phi^j = \sum_s \sum_{l \geq 0} C^j_{k'_1...k'_l}(s) \frac{e^{-i s \theta}}{r^{3s} \Delta \psi} B_{\Delta \psi}(r, \partial \gamma) \prod_{n=1}^l \Omega^k_n \Psi^k \psi^{k_1...k_l} (y). \tag{5.2.1}
\]

Comparing the two sides constrains the OPE tensors

\[
\Omega^j_j C^j_{k_1...k_l} = C^j_{k'_1...k'_l} \prod_{n=1}^l \Omega^k_n \Leftrightarrow C^j_{k_1...k_l} = (\Omega^{-1})^j_j C^j_{k'_1...k'_l} \prod_{n=1}^l \Omega^k_n. \tag{5.2.2}
\]

This tells us that \( C^j_{k_1...k_l} \) is a tensor invariant of O(X). In the upcoming chapter we study these tensors.

**Note 7.** In the case of scalars on the defect, i.e. when \( l = 0 \), we have

\[
\Omega^j_j C^j = C^j, \tag{5.2.3}
\]

which means that any O(X)-transformation is to leave \( C^j \) invariant. However, this is impossible, since only a subgroup of O(X) leaves \( C^j \) invariant, namely (im)proper rotations around the vector \( C^j \) in the O(X)-space. This means that scalars are not allowed on the defect.

**Note 8.** Matrices, i.e. when \( l = 1 \), on the defect commute with every element from O(X), i.e.

\[
\left[ \Omega^j_j, C^j_{k} \right] = 0. \tag{5.2.4}
\]
CHAPTER 6

TENSOR INVARIANTS OF $O(X)$

We know that the OPE tensors are to be invariants of their corresponding $O(X)$ group that they transform under, where $O(X)$ is one of the subgroups that are preserved after the symmetry breaking that concurs when we insert a co-dimension two, monodromy defect in the theory. Let us study these tensor invariants of $O(X)$ to see if they follow a certain pattern.

We find tensor invariants of $O(X)$ by making an arbitrary tensor invariant under all of its basis matrices corresponding to its generators. Once we have found a tensor invariant this way, we need to check if the final tensor is indeed invariant under an arbitrary $O(X)$ matrix.

6.1 BASIS MATRICES OF $O(X)$

From the $SO(X)$ group, we know that its generators, $\Lambda_j$, satisfy

\[
\begin{align*}
\lambda_j^T &= -\lambda_j \\
\text{Tr} \lambda_j &= 0 \quad \forall j \in \{1,\ldots,X^2\}.
\end{align*}
\]

(6.1.1)

The first of these conditions follows from orthogonality of $SO(X)$, and the second one follows from its determinant being equal to one. When we have found basis matrices from these generators, we can change the sign on one of the rows to find all of the desired $O(X)$ basis matrices. The above constraints yields the generators

\[
\begin{align*}
\lambda_1 &= \begin{bmatrix}
0 & -1 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix}, \quad \ldots, \quad \lambda_{X-1} &= \begin{bmatrix}
0 & 0 & \ldots & 0 & -1 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0
\end{bmatrix},
\end{align*}
\]
\[ \Lambda_X = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & -1 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \ldots, \quad \Lambda_2^{(X-1)} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & 0 \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix}, \ldots, \quad (6.1.2) \]

\[ \lambda_{(X-1)^2} = \begin{bmatrix} 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & -1 \end{bmatrix}. \quad (6.1.3) \]

We note that we can write all of these generators which do not have elements in the off-diagonal in terms of the generators which have elements in the off-diagonal, e.g.

\[ \lambda_2 = [\Lambda X, \Lambda_1], \quad \lambda_3 = [\lambda_{2X-1}, \lambda_2] = [\lambda_{2X-1}, [\Lambda X, \Lambda_1]], \quad \lambda_{2X} = [\Lambda_1, \lambda_3] = [\Lambda_1, [\lambda_2X - 1, [\Lambda X, \Lambda_1]]]. \]

This means that we only need to consider the \( N - 1 \) generators which only have elements in its off-diagonal. We denote these generators as \( \Lambda_j, j \in \{1, \ldots, X-1\} \).

\[ \Lambda_1 = \begin{bmatrix} 0 & -1 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \ldots, \Lambda_{X-1} = \begin{bmatrix} 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 1 \end{bmatrix}. \quad (6.1.4) \]

We find basis matrices, \( \xi_j, j \in \{1, \ldots, X-1\} \), in \( \text{SO}(N) \) from these generators using

\[ \xi_j = e^{a_j \Lambda_j} = \sum_{n \geq 0} \frac{1}{n!} (a_j \Lambda_j)^n = \sum_{n \geq 0} \frac{1}{(2n)!} (a_j \Lambda_j)^{2n} + \sum_{n \geq 0} \frac{1}{(2n+1)!} (a_j \Lambda_j)^{2n+1}, \quad \text{(no sum over } j), \quad (6.1.5) \]

The generators satisfy

\[ \Lambda_j^2 = -M_j, \quad M_1 = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \ldots, \quad M_{X-1} = \begin{bmatrix} 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 1 \end{bmatrix}. \quad (6.1.6) \]

Thus

\[ \begin{cases} \Lambda_j^{2n} = (-1)^n M_j, \\ \Lambda_j^{2n+1} = (-1)^n \Lambda_j. \end{cases} \quad (6.1.7) \]
If we insert this back into (6.1.5) we find

\[
\zeta_j = 1 - M_j + M_j \sum_{n \geq 0} \frac{(-1)^n}{(2n)!} \left( a_j \right)^{2n} + \Lambda_j \sum_{n \geq 0} \frac{(-1)^n}{(2n + 1)!} \left( a_j \right)^{2n+1} = 1 - M_j + \cos \alpha M_j + \sin \alpha \Lambda_j. \tag{6.1.8}
\]

Changing the sign of all elements in one of the rows in \( \zeta_j \) yields the basis matrices in \( \text{O}(N) \)

\[
\Xi_1 = \begin{bmatrix}
\pm \cos \alpha & \mp \sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \ldots, \quad \Xi_{X-1} = \begin{bmatrix}
1_{X-2} & 0 & 0 \\
0 & \mp \cos \alpha & \pm \sin \alpha \\
0 & \sin \alpha & \cos \alpha
\end{bmatrix} \tag{6.1.9}
\]

**Note 9.** If \( X = 2 \) we get the result from appendix chapter [A.3]

### 6.2 Matrix Invariants

We use the basis matrices from the previous section to calculate matrix invariants of \( \text{O}(X) \). Let us consider an arbitrary \( X \times X \) complex-valued matrix \( C \in \mathbb{C}^{X \times X} \) and then force it to be a matrix invariant of \( \text{O}(X) \)

\[
\Xi_j^T C \Xi_j = C \quad \forall \; j \in \{1, \ldots, X-1\}, \quad \text{(no sum over } j). \tag{6.2.1}
\]

If we consider \( \Xi_1 \) we get

\[
\Xi_1^T C \Xi_1 = \begin{bmatrix}
\cos^2 \alpha c_{11} + \sin^2 \alpha c_{22} \pm \cos \alpha \sin \alpha \left( c_{12} + c_{21} \right) & - \cos \alpha \sin \alpha \left( c_{11} - c_{22} \right) \pm \left( \cos^2 \alpha c_{12} - \sin^2 \alpha c_{21} \right) & \pm \cos \alpha c_{13} + \sin \alpha c_{23} & \pm \cos \alpha c_{1X} + \sin \alpha c_{2X} \\
- \cos \alpha \sin \alpha \left( c_{11} - c_{22} \right) \mp \left( \sin^2 \alpha c_{12} - \cos^2 \alpha c_{21} \right) & \sin^2 \alpha c_{11} + \cos^2 \alpha c_{22} \mp \cos \alpha \sin \alpha \left( c_{12} + c_{21} \right) & \mp \sin \alpha c_{13} + \cos \alpha c_{23} & \mp \sin \alpha c_{1X} + \cos \alpha c_{2X} \\
\pm \cos \alpha c_{13} + \sin \alpha c_{23} & \mp \sin \alpha c_{13} + \cos \alpha c_{23} & c_{33} & \cdots & \cdots \\
\pm \sin \alpha c_{1X} + \cos \alpha c_{2X} & \mp \sin \alpha c_{1X} + \cos \alpha c_{2X} & c_{X3} & \cdots & c_{XX} \\
\end{bmatrix}.
\]

Here \( c_{jk} \) is the element on the \( j \)th row and \( k \)th column in \( C \). The condition (6.2.1) yields

\[
\begin{align*}
c_{11} &= \cos^2 \alpha c_{11} + \sin^2 \alpha c_{22} \pm \cos \alpha \sin \alpha \left( c_{12} + c_{21} \right), \\
c_{12} &= - \cos \alpha \sin \alpha \left( c_{11} - c_{22} \right) \pm \left( \cos^2 \alpha c_{12} - \sin^2 \alpha c_{21} \right), \\
c_{21} &= - \cos \alpha \sin \alpha \left( c_{11} - c_{22} \right) \mp \left( \sin^2 \alpha c_{12} - \cos^2 \alpha c_{21} \right), \\
c_{22} &= \sin^2 \alpha c_{11} + \cos^2 \alpha c_{22} \mp \cos \alpha \sin \alpha \left( c_{12} + c_{21} \right), \\
c_{33} &= \pm \cos \alpha c_{33} + \sin \alpha c_{33}, \\
c_{X3} &= \pm \sin \alpha c_{X3} + \cos \alpha c_{X3}, \\
c_{12} &= c_{12} \quad \forall \; (\pi, \rho) \in \{ (13, 23), (1X, 2X), (31, 32), \ldots, (X1, X2) \}.
\end{align*}
\tag{6.2.2}
\]

Here \( \pi, \rho \in \{ (13, 23), (1X, 2X), (31, 32), \ldots, (X1, X2) \} \). Let us start by studying the first three equations in the above system\footnote{The fourth equation is actually the first equation, but with an overall minus sign on both sides.}.

Using the trigonometric identity, we get from the first equation

\[
\sin \alpha \left( c_{11} - c_{22} \right) = \pm \cos \alpha \left( c_{12} + c_{21} \right) \quad \Rightarrow \quad c_{21} = \mp c_{12}. \tag{6.2.3}
\]

To understand this we compare the proper case with the improper one. Since we are interested in \( \text{O}(X) \) invariants, we want \( C \) to satisfy both. The proper case corresponds to \( c_{21} = -c_{12} \), and the improper case
corresponds to $c_{21} = c_{12}$. We present the proper case with the following upper two relations, and the improper case with the lower two relations

$$\begin{cases} c_{22} = c_{11}, & c_{21} = -c_{12} \\ c_{22} = c_{11} + 2 \tan \alpha c_{12}, & c_{21} = c_{12} \end{cases} \Rightarrow c_{12} = c_{21} = 0, \quad c_{22} = c_{11}.$$  \hfill (6.2.4)

We move on to study the equations with $c_\pi$ and $c_\rho$ in (6.2.2)

$$c_\pi = \frac{\sin \alpha}{1 + \cos \alpha} c_\rho \Rightarrow [(1 - \cos \alpha)(1 + \cos \alpha) \pm \sin^2 \alpha] c_\rho = 0.$$  \hfill (6.2.5)

This means that $c_\rho$ (and therefore also $c_\pi$) are non-trivial only if the angular part in the above expression is zero for all values of $\alpha$. However, the proper case, $(1 - \cos \alpha)^2 - \sin^2 \alpha = 2 \cos \alpha(1 + \cos \alpha)$, is not zero for all values of $\alpha$, thus

$$c_\pi = c_\rho = 0 \forall \pi, \rho.$$  \hfill (6.2.6)

Putting it all together, yields the matrix invariant of $\Xi_1$

$$C = \begin{pmatrix} c_{11} & 0 & \ldots & 0 \\ 0 & c_{33} & \ldots & c_{3X} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{X3} & \ldots & c_{XX} \end{pmatrix}.$$  \hfill (6.2.7)

If we use the above matrix, and make it invariant under $\Xi_2$, and then make the resulting matrix invariant under $\Xi_3$, and so on, until we get a matrix that is invariant under every $\Xi_j$, $j \in \{1, \ldots, X - 1\}$, we end up with a constant times the unitary matrix. In component form

$$C^l_k = c_{11} \delta^l_k, \quad c_{11} \in \mathbb{C}.$$  \hfill (6.2.8)

This is indeed an $O(X)$ invariant, i.e.

$$(\Omega^{-1})^l_j C^{l'}_{k'} \Omega^{k'}_k = c_{11} (\Omega^{-1})^l_j \Omega^{l'}_k = c_{11} \delta^l_k = C^l_k.$$  \hfill (6.2.9)

We have double checked this result using Mathematica.

### 6.3 Three Tensor Invariants

We now perform the same procedure as that in the previous section, but for an arbitrary three tensor, $C^j_{kl}$. We portray this tensor as a matrix where each element is promoted to a vector. Let $C^j_{kl}$ describe the $l^\text{th}$ element in the vector at row $j$ and column $k$. A three tensor invariant of $O(X)$ satisfies

$$\Xi^T_a j^l C^{l'}_{k'l'} \Xi_a^l k'^l = C^{j'}_{kl} \forall a \in \{1, \ldots, X - 1\}, \quad (\text{no sum over} \ a).$$  \hfill (6.3.1)

Since we want this equation to hold for any values of the angle $\alpha$ (this angle resides in $\Xi_a$), we may first consider the cases when $\cos \alpha$ or $\sin \alpha$ is zero before we move on to study the general case. We start with
Ξ₁ and let \( \cos \alpha \) be one. Since the above equation needs to hold for both proper and improper \( \Xi_1 \), we end up with

\[
(C^j_{kl}) = \begin{pmatrix}
(0, c_{112}, \ldots, c_{11X}) & (c_{121}, 0, \ldots, 0) & \ldots & (c_{1XX}, 0, \ldots, 0) \\
(c_{211}, 0, \ldots, 0) & (0, c_{222}, \ldots, c_{22X}) & \ldots & (0, c_{2XX}, \ldots, c_{2XX}) \\
\vdots & \vdots & \ddots & \vdots \\
(c_{X11}, 0, 0, 0) & (0, c_{XX2}, \ldots, c_{XX2}) & \ldots & (0, c_{XX2}, \ldots, c_{XXX})
\end{pmatrix}.
\]  

(6.3.2)

Using this tensor and consider the case when \( \sin \alpha \) is one, then \[6.3.1\] tells us that

\[
(C^j_{kl}) = \begin{pmatrix}
(0, 0, c_{113}, \ldots, c_{11X}) & 0 & \ldots & (c_{1XX}, 0, \ldots, 0) \\
0 & (0, 0, c_{113}, \ldots, c_{11X}) & \ldots & (0, c_{1XX}, 0, \ldots, 0) \\
(c_{311}, 0, \ldots, 0) & (0, c_{311}, 0, \ldots, 0) & \ldots & (0, 0, c_{33X}, \ldots, c_{33X}) \\
\vdots & \vdots & \ddots & \vdots \\
(c_{X11}, 0, \ldots, 0) & (0, c_{X11}, 0, \ldots, 0) & \ldots & (0, 0, c_{XX3}, \ldots, c_{XX3})
\end{pmatrix}.
\]

One can check that this tensor is indeed a tensor invariant of \( \Xi_1 \) for any angle. Following the same procedure, i.e. by first considering \( \cos \alpha = 1 \) and then \( \sin \alpha = 1 \) before considering the general case, we make this tensor invariant under every \( \Xi_a, a \in \{1, \ldots, X-1\} \) using the same procedure as we that for matrix invariants. We find that there does not exist any non-trivial three tensor invariants of \( O(X) \), i.e.

\[
C^j_{kl} = 0 .
\]

(6.3.3)

Just as the matrix invariants of \( O(X) \), we have double checked this result using Mathematica.

**Note 10.** There may still exist three tensor invariants of \( SO(X) \). The plus-minus signs in \( \Xi_a, a \in \{1, \ldots, X-1\} \) makes tensor invariants of \( O(X) \) more constricted. One could also argue that \( O(X) \) is a bigger group than \( SO(X) \), thus tensor invariants of \( O(X) \) needs to stay invariant under more transformations than those of \( SO(X) \).

### 6.4 Higher Order Tensor Invariants

From this point, when calculating tensor invariants of \( O(X) \), we only use Mathematica. The procedure is the same as that for the three tensor. We find the four tensor invariants of \( O(X) \) to be

\[
C^j_{klm} = c_1 \delta^j_l \delta^k_m + c_2 \delta^j_l \delta^k_m + c_3 \delta^j_l \delta^k_m, \quad c_j \in \mathbb{C} \ \forall \ j \in \{1, 2, 3\} .
\]

(6.4.1)

This is indeed an \( O(X) \) invariant:\footnote{The elements in an \( O(X) \) matrix are numbers, hence they commute in component form.}

\[
(\Omega^{-1}){j^l} {k^l} [\Omega^k]_{i^l} \Omega^m = c_1 (\Omega^{-1}){j^l} [\Omega^k]_{i^l} \Omega^m + c_2 (\Omega^{-1}){j^l} (\Omega^k)_{i^l} \Omega^m + c_3 (\Omega^{-1}){j^l} (\Omega^k)_{i^l} \Omega^m
\]

(6.4.2)

Using the same procedure we find that there are no five tensor invariants of \( O(2) \). It is worth mentioning that in all of the previous cases, the tensor invariants of \( O(X) \) have all been on a similar form as those of
O(2), thus we believe that there does not exist any tensor invariants of O(X).

From the tensor invariants that we have calculated, it seems like the \((n + 1)\) tensor invariant of O(X) is on the form

\[
C^j_{k_1, \ldots, k_n} = \begin{cases} 
\sum_\pi c_\pi \delta^j_{k_{\pi(1)}} \prod_{j=2}^{n-1} \delta_{k_{\pi(j)} k_{\pi(j+1)}} & \text{if } n \text{ is odd.} \\
0 & \text{if } n \text{ is even.}
\end{cases}
\] (6.4.3)

Here we sum over all unique pair configurations. This sum contains \((n - 1)!!\) terms, e.g. the six tensor will contain 15 constants. It is important to remember that it is not proven that this is the most general O(X) tensor invariant. The above formula correctly reproduces the previous results we have studied when \(n \in \{1, \ldots, 5\}\).

We have generated the six tensor invariant of O(2) using the above formula and it is indeed invariant under O(2). So the above formula seems to hold, but there are several questions we may ask ourselves here. Does it hold for five and six tensor invariants of O(X) when \(X > 2\)? Does it hold for \(n > 6\)? For this project we will be satisfied with this formula, but it is important to keep in mind that we have not proved that it is the most general one nor that it holds for an arbitrary amount of indices.
CHAPTER 7

GREEN’S FUNCTION

In this section we follow the steps in [6], but for the WF O(N) model instead of \( \phi^4 \)-theory. Our starting point for this discussion is Green’s function, i.e. the correlator, for two bulk-local fields. We proceed to find this Green’s function from both the bulk-defect expansion of bulk-local fields and Feynman diagrams, and then compare the two with each other in order to find some of the CFT data.

7.1 GREEN’S FUNCTION FROM THE OPE

From the bulk-defect expansion (5.0.7) we get the full two-point correlator

\[
G^{\phi_{\theta}} = \langle 0|\phi(r_1, \theta_1, y_1)\phi^{\prime}(r_2, \theta_2, y_2)|0\rangle = \left(\phi(r_1, \theta_1, y_1)|0\rangle\right)^{\dagger}\phi^{\prime}(r_2, \theta_2, y_2)|0\rangle
\]

\[
= \sum_{s_1, s_2, \ell, \ell^\prime \geq 1} (C^\ell)^{k_1...k_i} (C^\ell^\prime)^{k_1'...k_i'} e^{i (s_1 \theta_1 \psi k_1...k_i)} \frac{1}{r_{1\Delta_{\theta} \Delta_{\psi}}^2 r_{2\Delta_{\theta} \Delta_{\psi}}^2} \left[ 1 + \mathcal{O}(r_1^2 \Delta_{\psi}) + \mathcal{O}(r_2^2 \Delta_{\psi}) \right] \times
\]

\[
\langle 0|\psi_{r,s}^{k_1...k_i}(y_1)\psi^{k_1'...k_i'}(y_2)|0\rangle.
\]

The defect-local fields, \( \psi_{r,s}^{k_1...k_i} \), are normalized through its two-point correlator

\[
\langle 0|\psi_{r,s}^{k_1...k_i}(y)\psi^{k_1'...k_i'}(y')|0\rangle = \frac{\delta_{s_1 s_2}}{|y_{12}|^2} \prod_{m=1}^{l} \delta_{k_m k_m'}, \quad y_{12} \equiv y_1 - y_2.
\]

We place \( \phi^{\prime}(r, \theta, y) \) and \( \phi^{\prime}(r', \theta', y') \) on the same distance from the defect

\[
r = r_1 = r_2 \Rightarrow G^{\phi_{\theta}} = (C^\ell)^{k_1...k_i} (C^\ell^\prime)^{k_1'...k_i'} \frac{e^{i (s_1 \theta_1 \psi k_1...k_i)}}{r_{2\Delta_{\psi}}^2} \rho^{2\Delta_{\psi}} \left[ 1 + \mathcal{O}(\rho^2) \right], \quad \theta_{12} \equiv \theta_1 - \theta_2, \quad \rho \equiv \frac{r}{|y_{12}|}.
\]

Here \( G^{\phi_{\theta}} \) is the summand of (7.1.1). By comparing this OPE with the result that we will calculate from diagrams at tree-level we find the zeroth loop order correction to \( \Delta_{\phi} \), \( \Delta_{\psi} \) and \( (C^\ell)^{k_1...k_i} (C^\ell^\prime)^{k_1'...k_i'} \). The logarithm of \( G^{\phi_{\theta}} \) will be useful when finding correction from one-loop diagrams

\[
\log G^{\phi_{\theta}} = \log \left[ (C^\ell)^{k_1...k_i} (C^\ell^\prime)^{k_1'...k_i'} \right] + i s_{12} \delta^{\phi_{\theta}} - 2 \Delta_{\phi} \log r \delta^{\phi_{\theta}} + 2 \Delta_{\psi} \log \rho \delta^{\phi_{\theta}} + \mathcal{O}(\rho^2).
\]
7.2 Green's Function from Feynman Rules

When calculating diagrams using Feynman rules, we calculate one loop order at a time, hence we write Green's function as a sum over loop order corrections where \( G_n \) represents the correction from the \( n \)th loop order

\[
G^{jj'} = \sum_{n \geq 0} G_n^{jj'}.
\]  

(7.2.1)

The logarithm of (7.2.1) will be useful when finding first loop order corrections to the CFT data. We Taylor expand the logarithm of the above sum so we can compare it later on with (7.1.4)

\[
\log G^{jj'} = \log G_0^{jj'} + \log \left( G_0^{jj'} \right)^{-1} G_1^{jj'} + \log \left( G_0^{jj'} \right)^{-1} G_1^{jj'} + \mathcal{O}(\varepsilon^2).
\]  

(7.2.2)

### 7.2.1 Tree-Level Diagram

We calculate the two-point Feynman diagrams for bulk-local fields on the defect. These calculations of diagrams will be very similar to those in [6]. We start from the propagator that we normalize in the same way as [6].

\[
G_0^{jj'}(x_1, x_2) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \delta^{jj'} \sum_{k} \int \frac{d^D k}{(2\pi)^D} e^{i[s\theta_1 + k\theta_2]} I_{tt}(k r_\pm) K_{tt}(r_\pm, z) \equiv \sum_s G_s^{jj'}(x_1, x_2),
\]

\[
x = (r, \theta, y), \quad D = 2 - \epsilon, \quad r_\pm = \min(r_1, r_2), \quad r_\pm = \max(r_1, r_2),
\]

\[
I_{tt}(Z) = \sum_{m \geq 0} \frac{1}{m!\Gamma(m + \alpha + 1)} \left( \frac{z}{2} \right)^{2m + \alpha}, \quad K_{tt}(Z) = \frac{\pi}{2} \frac{I_{tt}(Z) - I_{tt}(Z)}{\sin(\alpha \pi)}.
\]  

(7.2.3)

Here \( D \) is the dimension of the monodromy defect, \( r_\pm \) is the largest/shortest distance from \( \phi^j \) or \( \phi^{j'} \) to the defect and \( I_{tt} \) as well as \( K_{tt} \) are modified Bessel functions.

**Note 11.** We have constructed the propagator such that it satisfies

\[
-\nabla^2 G_0^{jj'}(x_1, x_2) = \frac{4\pi^{D/2 + 1}}{\Gamma(D/2)} \delta^{jj'} \delta^{D+2}(x_1 - x_2),
\]

\[
\lim_{x_2 \to x_1} G_0^{jj'}(x_1, x_2) = \frac{\delta^{jj'}}{|x_1 - x_2|^{2\alpha}} + \mathcal{O}(|x_1 - x_2|^{0}).
\]  

(7.2.4)

We rewrite the summand \( G_0^{jj'} \) in (7.2.3) using modified Bessel relations

\[
G_0^{jj'}(x_1, x_2) = \frac{\Gamma(|s| + D/2)}{\Gamma(|s| + 1)} \frac{e^{i\phi_{12}}}{(r_1 r_2)^{D/2}} \alpha^{-|s|} \beta^{D/2} 2F_1 \left[ |s| + D/2, |s| + 1/2, |s| + 1, -4\alpha^{-1} \right],
\]

\[
2F_1(a, b, c, z) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \mathcal{O}(z), \quad \alpha = \frac{\gamma^2}{12} + \frac{r_1^2}{r_1 r_2}, \quad r_{12} = r_1 - r_2.
\]  

(7.2.5)
Here \( _2F_1 \) is a hyper geometric function. We place the bulk-local fields on the same distance from the defect so we can compare it with the result from the OPE

\[
r \equiv r_1 = r_2 \implies G^j_{0s}(x_1, x_2) = \frac{\Gamma(|s| + D/2)}{\Gamma(D/2)\Gamma(|s| + 1)} \frac{e^{i\theta_{12}}}{r^D} \rho^{2|s| + D} \delta^{jj'} \left[ 1 + O(\rho^2) \right]. \tag{7.2.6}
\]

Comparing (7.2.6) with the result (7.1.3) from the OPE yields\(^1\)

\[
(C^j C)^m = \delta^{jj'} - \frac{\psi(|s| + 1) - \psi(1)}{2} \delta^{jj'} \epsilon + O(\epsilon^2), \quad \psi(x) \equiv \frac{\Gamma'(x)}{\Gamma(x)}, \tag{7.2.7}
\]

Here \( \psi(x) \) is the digamma function, \((C^j C)^m\) is the \( m \)-loop correction to \((C^j)^{k_1 \ldots k_l} C^{j_1 \ldots j_k}\), and \((C^j)^m \cdot (\Delta \psi)_m\) is the \( m \)-loop correction to \( \Delta \phi \). Let us find the one-loop correction before we study the constraint \((C^j C)^m\).

\[\textbf{Note 12. It is important to remember that in all of the } \epsilon \text{-expansions in this section, } \epsilon \text{ is not small, but one. This means that many times we are assuming that the constants in front of the } \epsilon \text{ gets smaller at higher power of } \epsilon. \text{ In the } \phi^4 \text{-theory case, this seems to hold by comparing it with numerical data [6].}\]

### 7.2.2 One-Loop Diagram

The two-point, one-loop diagram (not in momentum space) is given by\(^1\)

\[
\begin{align*}
G^{jj'}_{1s}(x_1, x_2) &= \frac{2i^2 \lambda}{(2\pi)^4 S} \int_{\mathbb{R}^4} d^4 x_0 \left\{ G^{j_0}_{0s}(x_1, x_0) G^{j'}_{k_0}(x_0, x_2) + G^{j_0}_{0s}(x_1, x_0) G^{j'}_{0k}(x_0, x_2) + G^{j_0}_{0s}(x_1, x_0) G^{j'}_{0j}(x_0, x_2) + G^{j_0}_{0s}(x_1, x_0) G^{j'}_{0j}(x_0, x_2) \right\}, \tag{7.2.8}
\end{align*}
\]

\[
S = 3!.
\]

Here \( \lambda \) is the coupling constant at the \( \epsilon \)-dependent fixed point, see (5.0.5), and \( S \) is the symmetry factor. The "2" in \( S \) comes from the intertwining two of the lines at the vertex and the "3!" comes from perturbation of the indices. Please note that one of the Green’s functions, \( G^j_{0s}\), is the whole sum and not only the summand, \( G^{jj'}_{0s}\), of (7.2.3). In appendix section C.1 we rewrite \( G^j_{0s}\) using hypergeometric function relations

\[
G^{jj'}_{0s}(x_k, x_l) = e^{is_{kl}} \frac{(4r_k r_l)^{|s|}}{d_k^+ d_k^- (d_k^+ + d_k^-)^2} r_{kl}^{jj'}, \quad d_{kl}^\pm = \sqrt{y_{kl}^2 + (r_k^\pm)^2 + z_{kl}^2}, \quad r_{kl}^\pm = r_k \pm r_l. \tag{7.2.9}
\]

The sum \( G^j_{0s}\) is the propagator for the theory. Renormalization yields that we only need to care about the finite piece of this propagator when we perform the resummation\(^1\)

\[
G^j_{0s}(x_0, x_0) = \frac{v(v - 1)}{2r_0^2} \delta^{jj'}. \tag{7.2.10}
\]

---

\(^1\)Details on the Taylor expansion of \((C^j)^{k_1 \ldots k_l} C^{j_1 \ldots j_k}\) is in appendix section A.4.

\(^2\)Details about this resummation is in appendix section C.2.
We consider fields with fractional spin, i.e. $s = Z + \nu, \nu \in [0, 1)$. Inserting $C_{0g}^{ij}$ and $G_{0g}^{ij}$ back into (7.2.8) yields

$$G_{1s}^{ij}(x_1, x_2) = -\frac{v(u-1)\lambda}{(2\pi)^4 S} e^{i\theta_{12}} \int_{0}^{2\pi} d\theta_0 \int_{0}^{\infty} r_0^2 \int_{0}^{\infty} r_0^2 (d_{10}^2 + d_{10}^2) (d_{02}^2 + d_{02}^2) \frac{(4\pi r_0)^2}{2|s|} \cdot$$

Here we are using cylindrical coordinates and the positions $x_1$ and $x_2$ are at the same distance from the defect, i.e., $r_1 = r_2 \equiv r$, as well as $z_1 = z_2 = 0$. Let the bulk-local fields in this correlator transform under one of the unbroken subgroups, $O(X)$, after the symmetry breaking that occurs when we insert a defect, see section 5.1. We rewrite this integral using the variable change

$$y' = y_0 + \frac{y}{2}, \quad y \equiv y_{12}. \quad (7.2.11)$$

$$d_{10} = \sqrt{\left(y'_0 - \frac{y}{2}\right)^2 + (r_0 \pm r)^2 + z_0^2} \equiv e^z, \quad (7.2.12)$$

$$d_{02} = \sqrt{\left(y'_0 + \frac{y}{2}\right)^2 + (r_0 \pm r)^2 + z_0^2} \equiv e^z. \quad (7.2.11)$$

$$G_{1s}^{ij}(x_1, x_2) = -\frac{v(u-1)(X+2)\lambda}{(2\pi)^4 S} e^{i\theta_{12}} \delta^{ij} H_s(r, y), \quad (7.2.13)$$

The asymptotics of the integral $H_s(r, y)$ is carefully studied in [6]

$$H_s(r, y) = \frac{2\pi}{|s|} r^{2(\nu+1)} \log \rho + \Theta(\rho^0) \Rightarrow G_{1s}^{ij} (x_1, x_2) = -\frac{v(u-1)(X+2)\lambda}{2(X+8)|s|} e^{i\theta_{12}} \delta^{ij} \frac{(4\pi r_0)^2}{2|s|} \log \rho + \Theta(\rho^0) + \Theta(\epsilon^2). \quad (7.2.14)$$

Comparing this with the result from the OPE (7.1.4) and we find that only $\Delta_\nu$ receives corrections from the one-loop diagram. This correction is given by

$$\Delta_\nu = \frac{v(u-1)(X+2)e}{2(X+8)|s|}. \quad (7.2.15)$$

To summarize this section, up to one-loop corrections (or up to order $\epsilon$), we have

$$(C^l C)^l = \delta^{ij} - \frac{\psi(|s|+1) - \bar{\psi}(1)}{2} \delta^{ij} \epsilon + \Theta(\epsilon^2), \quad (7.2.16)$$

$$\Delta_\nu = \frac{\epsilon}{2} + \Theta(\epsilon^2), \quad \Delta_\nu = |s| + 1 + \left(\frac{v(u-1)(X+2)}{2(X+8)|s|} - 1\right) \frac{\epsilon}{2} + \Theta(\epsilon^2). \quad (7.2.17)$$

Note 13. This reduces to the results in [6] when $X = 1$ and $v = 2^{-1}$, which is a sign that this is correct, e.g.

$$X = 1, \quad v = \frac{1}{2} \Rightarrow \Delta_\nu = |s| + 1 + \left(\frac{1}{12|s|} + 1\right) \frac{\epsilon}{2} + \Theta(\epsilon^2).$$
7.3 Using Tensor Invariants of O(X)

In this section we study the constraint (7.2.16) for the tensor invariants of O(X) found in chapter 6. We start with the matrix invariants

\[ C_{j k} = c \delta_{j k}, \quad |c| = 1 - \frac{\bar{\psi}(|s| + 1) - \bar{\psi}(1)}{4} \epsilon + O(\epsilon^2). \] (7.3.1)

Here we have Taylor expanded a square root that appears in the RHS of (7.2.16). This is essentially the same as the result in [6], but times a unitary matrix.

The next non-trivial tensor invariant of O(X) is the four tensor

\[ C_{j k l m} = c_1 \delta_{j k} \delta_{l m} + c_2 \delta_{j l} \delta_{k m} + c_3 \delta_{j m} \delta_{k l}, \]

\[ \sum_{j=1}^{3} X|c_j|^2 + \sum_{k=1, k \neq j}^{3} c_j^* c_k \right) = 1 - \frac{\bar{\psi}(|s| + 1) - \bar{\psi}(1)}{2} \epsilon + O(\epsilon^2). \] (7.3.2)

This condition tells us that only certain values on the constants \((c_1, c_2, c_3) \in \mathbb{C}^3\) are allowed, meaning these constants will only take values on the subspace of \(\mathbb{C}^3\) that satisfy the above relation.

The six tensor invariant invariant of O(X) is the next non-trivial one. Since it is described by 15 constants, see (6.4.3), the constants \((c_1, \ldots, c_{15})\) will only take values from a specific subgroup of \(\mathbb{C}^{15}\) according to (7.2.16).
Chapter 8

Non-Perturbative Approach

In this chapter we generalize the $O(N)$ framework created in [5] to the WF $O(N)$ model with a co-dimension two, monodromy defect. This approach is very similar to that in [9]. We define three axioms for the theory that contains information about its dynamics.

**Axiom 1.** The $\epsilon$-dependent fixed point in the WF $O(N)$ model, see [5.0.5], is conformally invariant, hence the theory at this point is a CFT.

**Axiom 2.** Correlators in the $\epsilon$-dependent fixed point, approach free theory correlators (when the coupling constant is zero) in the limit

$$\epsilon \to 0 . \quad (8.0.1)$$

This can easily be seen from the $\epsilon$-dependent fixed point [5.0.5] since it is proportional to $\epsilon$. It yields that every operator in the $4-\epsilon$ dimensional theory tends to operators in the free theory in the above limit.

**Axiom 3.** The operators

$$T_{2p} = (\phi^k \phi^k)^p, \quad T_{2p+1}^j = \phi^j (\phi^k \phi^k)^p, \quad j, k \in \{1, \ldots, X\} , \quad (8.0.2)$$

are all primary except $T_3^j$. The equations of motion from [5.0.3] with the rescaling of bulk-local fields [5.0.6] tells us that it is a descendant of $T_1$

$$a T_3^j = \delta^2 \delta^1_\mu T_1^1, \quad \alpha = \frac{\lambda}{3!(2\pi)^2} = \frac{2e}{X+8} + O(\epsilon^2) . \quad (8.0.3)$$

**Note 14.** In chapter [5] we learned that the global $O(N)$ symmetry is broken into either $O(\chi + a) \otimes O(N - \chi - a), \ a \in \{0, 1, 2\}$ or $O(2) \otimes O(\chi) \otimes O(N - \chi - 2)$ depending on which group element from $O(N)$ we use for the monodromy action [5.1.4]. This means that the primaries discussed above must belong to one of the unbroken subgroups after the insertion of the defect. In this chapter we assume that $\phi^j$ transform under $O(X), X \in \{2, \chi + a, N - \chi - a\}$.

We will find $T_3^j$ using first [8.0.2] and then compare it with the $T_3^j$ that we find from [8.0.3]. In this way we do not need to use perturbation theory to find the scaling dimensions of defect-local operators. We find $T_3^j$ from [8.0.2] using Wick’s theorem

$$T_3^j = : \phi_j \phi_k \phi^k : + 2 \phi^j \phi^k \phi^k + \phi^j \phi^j \phi^j . \quad (8.0.4)$$
Derivation of the anomalous dimension, $\gamma$

As we can see, this approach reproduces the same $S$-matrix $\mathcal{S}$, which yields

$$S = \mathcal{O}(r^0) + \mathcal{O}(r^\Delta) + \mathcal{O}(r^\gamma)$$

Using the bulk-defect expansion (5.0.7) of $\phi^j$

$$T^j_3 = \frac{v(v-1)(X+2)}{2r^2} \phi^j + \mathcal{O}(r^3).$$

We move on to find $T^j_3$ using axiom 3. In cylindrical coordinates

$$\partial^2 \beta(r, \theta, y) = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \beta}{\partial r} \right) + \frac{\partial^2 \beta}{\partial \theta^2} + \frac{\sigma^2 \beta}{\partial y^2},$$

which yields

$$T^j_3 = \alpha^{-1} \sum_{s, n} \left( C^j_{s, k_1...k_n} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( -r (\Delta_\phi - \Delta_\psi) \frac{e^{-is\theta}}{r^{\Delta_\phi - \Delta_\psi + 2}} \right) + (-is)^2 \frac{e^{-is\theta}}{r^{\Delta_\phi - \Delta_\psi + 2}} \right) \psi^j_{s, k_1...k_n} + \mathcal{O}(r^{\Delta_\phi - \Delta_\psi}) \right).$$

Compare the $r^{\Delta_\phi - \Delta_\psi + 2}$ terms above with those in (8.0.6) to get the relation

$$\frac{v(v-1)(X+2)}{2} = (\Delta_\phi - \Delta_\psi)^2 - s^2.$$  

The scaling dimension for bulk-local fields is found using the framework for $O(N)$ models from [5]  

$$\Delta_\phi = 1 + \frac{\epsilon}{2} + \gamma = 1 + \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2).$$  

Derivation of the anomalous dimension, $\gamma$ in [5]. If we write $\Delta_\psi$ as a power series in $\epsilon$

$$\Delta_\psi = x + ye + \mathcal{O}(\epsilon^2) \Rightarrow \frac{v(v-1)(X+2)}{X+8} \epsilon = 1 - \epsilon - 2 \left( x + ye - \frac{x}{2} \right) + x^2 + 2xye + \mathcal{O}(\epsilon^2) - s^2.$$  

Comparing $\epsilon^0$- and $\epsilon$-terms yields the system of equations

$$\begin{cases} 1 - 2x + x^2 - s^2 = 0, \\ -1 - 2y + x + 2xy = \frac{v(v-1)(X+2)}{X+8}. \end{cases}$$

Solving these equations

$$\Delta_\psi \geq 0 \Rightarrow x = 1 + |s| \Rightarrow |s|(1 + 2y) = \frac{v(v-1)(X+2)}{X+8} \Rightarrow y = \frac{1}{2} \left( \frac{v(v-1)(X+2)}{(X+8)|s|} - 1 \right).$$

As we can see, this approach reproduces the same $\Delta_\phi$ and $\Delta_\psi$ as in chapter 7, see (7.2.16), which is a sign that our construction is correct.

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Chapter 9

Discussion and Conclusion

Conformal bootstrap is a powerful, non-perturbative tool which can be used in CFTs. We may reduce any $n$-point correlator down to a two-point correlator, with coefficients from the three-point correlator, using conformal invariance. These coefficients also appear in the OPE between two primaries, from which all other operators descend from. As we have seen, the goal of the bootstrap is to find the scaling dimensions and all of the spins of the primary operators in addition to the previously mentioned OPE coefficients. We conveniently call this information the CFT data. The physiology of the bootstrap program is that all of the CFT data can be found through solving the bootstrap equation which originates from associativity in using the OPE on the four-point correlator. It is not a simple task considering there may exist infinitely many primaries in a theory.

There are many different ways to study the bootstrap equation. In chapter 3 we reviewed two different approaches. One of these was a numerical one that made the subject popular to study again [2]. This approach interpreted the bootstrap equation as a cone equation in a function space of the conformal blocks. Carefully studying this cone equation makes it possible to numerically generate the unitary bound for the scaling dimensions of the primaries. The other approach was among the first to successfully bootstrap a CFT analytically [3]. The result of this approach tells us that any CFT has a sector which can be treated as a MFT which is the set of operators with large spin. The anomalous dimensions for the primaries acts as the growth rate for which the CFT becomes a MFT.

Recently there has been a lot of development on studies of CFTs with one (or several) defects, see e.g. [8]. It is important to further study such theories to investigate the possibilities of the conformal bootstrap. In the last chapters of this thesis the WF $O(N)$ model with a co-dimension two, monodromy defect was studied. In an $O(N)$ model, the OPE coefficients are promoted to tensors that transform in $O(N)$. We studied the effects of insertion of a co-dimension two, monodromy defect as well as constraints that follow from the global symmetry of the theory. As expected, the global symmetry group is broken after a defect is inserted into the theory. Depending on which group element from the global symmetry group we use in the definition of the monodromy action, the global symmetry is broken into two or three of subgroups. In each of these subgroups there will be primaries, with corresponding OPE tensors and different spin, that does not mix with primaries in the other subgroups. In order to make the theory invariant under each of the unbroken subgroups we need the OPE tensors to be $O(X)$ invariants, where $O(X)$ is one of the unbroken subgroups.
We studied up to six tensor invariants in chapter 6 by letting an arbitrary, complex-valued matrix be invariant under each of the basis tensors of $O(X)$, $X \in \{2, \ldots, 6\}$. We found that a general tensor invariant of $O(X)$ seem to be a sum over all different compositions of tensor products of unitary operators. It is important to remember that this general formula has not been proven. It may exist some literature on tensor invariants of $O(X)$ that we are not aware of at the moment.

Finally, we generalized the approach of [6] to find the scaling dimensions of bulk-local and defect-local fields as well as a relation for the OPE tensors in the WF $O(N)$ model. The scaling dimension for bulk-local fields is the same as in the case without the defect. It also coincides with the corresponding operator in the 3D Ising model when $N$ goes to one, just like the result from [5] predicts. We were able to double check the scaling dimension of defect-local fields through a generalization of the approach in [9]. This approach uses the framework for a $O(N)$ model from [5]. If the $n$ tensor invariants of $O(N)$ are a sum over tensor products of unitary matrices, then the relation we found for OPE tensors tells us that the $(n - 1)!!$ complex-valued constants in these tensor invariants only takes values in a subspace in $\mathbb{C}^{(n - 1)!!}$. 
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Appendices
APPENDIX A

MINOR CALCULATIONS

A.1 SCALING INVARIANCE

We want to show that a two-point correlator must be proportional to \((XY)^{-\Delta}\) for it to be invariant under scalings. To get a Lorentz invariant correlator it must have the form

\[ \langle \phi_j(X)\phi_k(Y) \rangle = B_{jk}(XY)^a. \] (A.1.1)

Here \(B_{jk}\) is a normalization factor and \(a\) is a constant that we determine by forcing this correlator to be invariant under scaling \([2.2.1]\)

\[ \langle \phi_j(bX)\phi_k(bY) \rangle = b^{-2\Delta} \langle \phi_j(X)\phi_k(Y) \rangle \quad \Rightarrow \quad B_{jk}b^{2a}(XY)^a = b^{-2\Delta}B_{jk}(XY)^a \quad \Rightarrow \quad a = -\Delta. \] (A.1.2)

If we write \(X\) and \(Y\) in terms of \(x\) and \(y\) using \([2.1.2]\) and \([2.3.4]\)

\[ XY = -\frac{x^2 + y^2}{2} + xy = -\frac{(x-y)^2}{2} \Rightarrow \langle \phi_j(x)\phi_k(y) \rangle = B_{jk} \frac{(-2)^\Delta}{(x-y)^{2\Delta}}. \] (A.1.3)

We normalize this correlator as

\[ B_{jk} = \frac{\delta_{jk}}{(-2)^\Delta} \Rightarrow \langle \phi_j\phi_k \rangle = \frac{\delta_{jk}}{(-2XY)^\Delta} = \frac{\delta_{jk}}{(x-y)^2}. \] (A.1.4)

A.2 RAISING AND LOWERING OPERATORS

The eigenvalues of the dilation operator, \(D\), acting on a primary, \(\phi\), are the scaling dimensions, \(\Delta\), i.e.

\[ D\phi = i\Delta\phi. \] (A.2.1)

From the algebra of the generators we have

\[ [D, P_\mu] = iP_\mu, \quad [D, K_\mu] = -iK_\mu. \] (A.2.2)

Using these two we can find the dimensions of the operators \(P_\mu\phi\) and \(K_\mu\phi\).

\[ D\{P_\mu\phi\} = [D, P_\mu]\phi + P_\mu\{D\phi\} = iP_\mu\phi + i\Delta P_\mu\phi = i(\Delta + 1)\{P_\mu\phi\}, \]
\[ D\{K_\mu\phi\} = [D, K_\mu]\phi + P_\mu\{D\phi\} = -iK_\mu\phi + i\Delta K_\mu\phi = i(\Delta - 1)\{K_\mu\phi\}. \] (A.2.3)
A.3 General O(2) - Matrices

Consider an arbitrary $2 \times 2$ matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}. \tag{A.3.1}$$

If $M \in O(N)$, then from the definition (5.1.1) its elements has to satisfy

$$\begin{cases} 
M_{11}M_{22} - M_{12}M_{21} = \pm 1, \\
M_{21}^2 + M_{22}^2 = 1, \\
M_{11}M_{12} + M_{21}M_{22} = 0, \\
M_{21} + M_{22}^2 = 1. 
\end{cases} \tag{A.3.2}$$

Depending on the value of $\det M$, the above will have different solutions. Let us denote $M_{\pm}$ such that 

$$\det M_{\pm} = \pm 1. \tag{A.3.3}$$

Then from the equation system (A.3.2) we get

$$M_{\pm} = \begin{bmatrix} \pm \sqrt{1 - M_{21}^2} & -M_{21} \\ M_{21} & \pm \sqrt{1 - M_{21}^2} \end{bmatrix}, \quad M_{\mp} = \begin{bmatrix} \mp \sqrt{1 - M_{21}^2} & M_{21} \\ M_{21} & \mp \sqrt{1 - M_{21}^2} \end{bmatrix}. \tag{A.3.4}$$

We parametrize the elements as

$$\begin{cases} 
M_{22} = \sqrt{1 - M_{21}^2} \\
M_{21} = \sin \theta
\end{cases} \Rightarrow M_{\pm} = \begin{bmatrix} \pm \cos \theta & \mp \sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}. \tag{A.3.5}$$

Note 15. $M_+ \in SO(2) \subset O(2)$ is a proper rotation by an angle $\theta$, and $M_- \in O(2)$ is an improper rotation.

A.4 Taylor Expansion of Tensor Product

In this appendix we Taylor expand $(C^i C^j)_{0 \epsilon}$ around $\epsilon = 0$. It is important to remember that $\epsilon \rightarrow 1$ even though we expand around $\epsilon = 0$.

$$(C^i C_j^i|_k_{\ldots} k_j^\prime)_{k_{\ldots} k_j^\prime} = \Gamma(s + D/2) / \Gamma(s + 1) \delta^{i j^\prime}$$

$$= \frac{\Gamma(s + 1 - \epsilon/2)}{\Gamma(1 - \epsilon/2) \Gamma(s + 1)} \delta^{i j^\prime} + \frac{1}{\Gamma(s + 1)} \left( \frac{\Gamma(s + 1 - \epsilon/2)}{\Gamma(1 - \epsilon/2)^2} \right) \left( \frac{\Gamma(s + 1 - \epsilon/2)}{\Gamma(s + 1)} - \frac{\Gamma(s + 1 - \epsilon/2)}{\Gamma(1 - \epsilon/2)} \right) \delta^{i j^\prime} \epsilon + O(\epsilon^2) \tag{A.4.1}$$

$$= \delta^{i j^\prime} - \frac{1}{2 \Gamma(1 - \epsilon/2)} \left( \frac{\Gamma(s + 1 - \epsilon/2)}{\Gamma(s + 1)} - \frac{\Gamma(s + 1 - \epsilon/2)}{\Gamma(1 - \epsilon/2)} \right) \delta^{i j^\prime} \epsilon + O(\epsilon^2),$$

$$\psi(x) \equiv \Gamma(x) / \Gamma(1).$$
APPENDIX B

PROPER AND IMPROPER O(2) SOLUTIONS

In this appendix we solve the first two equations from (5.1.5) when \( \sin \theta \neq 0 \)

\[
\begin{align*}
\frac{e^{-2\pi i s} C_{1}^{1}_{k_1 \ldots k_l}}{e^{-2\pi i s} C_{2}^{2}_{k_1 \ldots k_l}} &= \pm \cos \theta C_{1}^{1}_{k_1 \ldots k_l} + \mp \sin \theta C_{2}^{2}_{k_1 \ldots k_l}, \\
\frac{e^{-2\pi i s} C_{2}^{1}_{k_1 \ldots k_l}}{e^{-2\pi i s} C_{2}^{2}_{k_1 \ldots k_l}} &= \sin \theta C_{1}^{1}_{k_1 \ldots k_l} + \cos \theta C_{2}^{2}_{k_1 \ldots k_l}.
\end{align*}
\] (B.0.1)

The first of these equations yields

\[ C_{1}^{1}_{k_1 \ldots k_l} = \pm \frac{\sin \theta}{\mp \cos \theta} C_{2}^{2}_{k_1 \ldots k_l}. \] (B.0.2)

Inserting this into the second equation in (B.0.1) gives us

\[ (e^{-2\pi i s} - \cos \theta)(e^{-2\pi i s} \mp \cos \theta) C_{2}^{2}_{k_1 \ldots k_l} = \mp \sin^2 \theta C_{2}^{2}_{k_1 \ldots k_l}. \] (B.0.3)

Which is the same as

\[ (e^{-2\pi i s} - \cos \theta)(e^{-2\pi i s} \mp \cos \theta) = \mp \sin^2 \theta. \] (B.0.4)

This will yield different results depending on whether \( R_\theta \) in (5.1.4) has determinant one or minus one.

Let us also solve the first two equations in (5.1.8) when \( \sin \theta \neq 0 \), and compare its solutions with those from (B.0.1). Consistency states that solutions from (5.1.8) are to be the same as solutions from (B.0.1).

We have

\[
\begin{align*}
\frac{e^{2\pi i s} C_{1}^{1}_{k_1 \ldots k_l}}{e^{2\pi i s} C_{2}^{2}_{k_1 \ldots k_l}} &= \pm \cos \theta C_{1}^{1}_{k_1 \ldots k_l} + \sin \theta C_{2}^{2}_{k_1 \ldots k_l}, \\
\frac{e^{2\pi i s} C_{2}^{1}_{k_1 \ldots k_l}}{e^{2\pi i s} C_{2}^{2}_{k_1 \ldots k_l}} &= \mp \sin \theta C_{1}^{1}_{k_1 \ldots k_l} + \cos \theta C_{2}^{2}_{k_1 \ldots k_l}.
\end{align*}
\] (B.0.5)

In a similar way as how we found (B.0.2) and (B.0.4) from (B.0.1), we find from the above equations

\[ C_{1}^{1}_{k_1 \ldots k_l} = \frac{\sin \theta}{e^{2\pi i s} \mp \cos \theta} C_{2}^{2}_{k_1 \ldots k_l}, \] (B.0.6)

\[ (e^{2\pi i s} - \mp \cos \theta)(e^{2\pi i s} \mp \cos \theta) = \mp \sin^2 \theta. \]
B.1 Proper Rotation

A proper $R_\theta$, i.e. $\det R_\theta = 1$, yields
\[ (e^{-2s} - \cos \theta)^2 = -\sin^2 \theta . \] (B.1.1)

Solving for $s$
\[ e^{-2\pi s} = \cos \theta \pm i \sin \theta = e^{\pm i(\theta + 2\pi n)}, \quad n \in \mathbb{Z} \Leftrightarrow s = n + \frac{\theta}{2\pi}. \] (B.1.2)

Insert this back into (B.0.2)
\[ C^1_{k_1...k_l} = \pm i C^2_{k_1...k_l}. \] (B.1.3)

In the same way, we find the same spin and relation between $C^1_{k_1...k_l}$ and $C^2_{k_1...k_l}$ when studying (B.0.6), meaning that the theory is consistent for proper rotations.

B.2 Improper Rotation

An improper $R_\theta$, i.e. $\det R_\theta = -1$, yields
\[ (e^{-2s} - \cos \theta)(e^{-2s} + \cos \theta) = \sin^2 \theta . \] (B.2.1)

Solving for $s$
\[ e^{-4\pi s} = 1 \Leftrightarrow s = \frac{n}{2}, \quad n \in \mathbb{Z}. \] (B.2.2)

We find the same spin starting from (B.0.6). Since the spin is an integer or half-integer
\[ e^{2\pi s} = e^{-2\pi s}, \] (B.2.3)

which yields the same relation between $C^1_{k_1...k_l}$ and $C^2_{k_1...k_l}$, see (B.0.2). Thus (B.0.1) is consistent with (B.0.6) for improper rotations as well.
If we study the components of (7.2.8), we can solve it by carefully studying its asymptotic expansion. Hence we start by studying its components which are two summands on the same form and one sum that we may resum after we have massaged the expression for the previously mentioned summands. The asymptotic behavior of (7.2.8) will not be studied here. The interested reader may find details on its asymptotics in [6].

C.1 Summand

We start with the summand $G^{jj'}_{0s}$. We cannot consider $r_1 = r_2 = r_0$, which corresponds to (7.2.6), since we are integrating over one of the coordinates. Thus we need to massage (7.2.5) using hypergeometric function relations. Taylor expanding around $\epsilon = 0$ yields

$$G^{jj'}_{0s}(x_k, x_l) = \frac{\Gamma(|s|+1)}{\Gamma(1)\Gamma(|s|+1)} \frac{e^{i\delta_{kl}}}{r_k r_l} \alpha^{-|s|+1} \frac{\delta^{jj'}}{2} F_1 \left(\frac{|s|+1}{2}, \frac{|s|+1}{2}; \frac{|s|+1}{2}; -4\alpha^{-1}\right) + \mathcal{O}(\epsilon)$$

$$= \frac{e^{i\delta_{kl}}}{r_k r_l} \frac{4^s}{\sqrt{\alpha\sqrt{4+\alpha}} \left(\sqrt{\alpha\sqrt{4+\alpha}}\right)^{|s|}} + \mathcal{O}(\epsilon)$$

$$= \frac{e^{i\delta_{kl}}}{r_k r_l} \frac{4^s}{(r_k r_l)^{-1} \sqrt{y_{kl}^2 + r_{kl}^2 + z_{kl}^2} \sqrt{4r_k r_l + y_{kl}^2 + r_{kl}^2 + z_{kl}^2} \times}$$

$$\times \frac{1}{(r_k r_l)^{-1} \sqrt{y_{kl}^2 + r_{kl}^2 + z_{kl}^2} \sqrt{4r_k r_l + y_{kl}^2 + r_{kl}^2 + z_{kl}^2} \times}$$

$$= e^{i\delta_{kl}} \frac{(4r_k r_l)^{|s|}}{d_k^+ d_k^- (d_k^- + d_k^+)^{|s|}} \delta^{jj'} + \mathcal{O}(\epsilon), \quad d_k^\pm = \sqrt{y_{kl}^2 + (r_{kl}^\pm)^2 + z_{kl}^2}, \quad r_{kl}^\pm = r_k \pm r_l.$$  \hspace{1cm} (C.1.1)

Note 16. The $z$-components are zero unless it is one the integration variables in (7.2.8), i.e. $z_k = 0$ if $k \neq 0$. 

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C.2 Resummation

The next component in (7.2.8) that we need to study is the sum \( G_{j j'}^0(x_0, x_0) \). This component will be divergent, but we renormalize the theory so that we only care about its finite part. Let us denote

\[
x \equiv \sqrt{y_{00}^2 + z_{00}^2} \quad \Rightarrow \quad d_{00}^- = \lim_{x \to 0} x, \quad d_{00}^+ = \lim_{x \to 0} \sqrt{(2r_0^2 + x^2)}.
\]  

(C.2.1)

We consider the defect-local operators in the bulk-defect expansion to have fractional spin, \( s \in \mathbb{Z} + \nu, \nu \in [0, 1) \), since it reproduces the result of integer spin (\( \nu = 0 \)) and half-integer spin (\( \nu = 1/2 \)). Using (C.1.1)

\[
G_{j j'}^0(x_0, x_0) = \lim_{x \to 0} \delta_{j j'} x \sqrt{(2r_0^2 + x^2)} \sum_{s \in \mathbb{Z} + \nu} \left( \frac{2r_0}{x + \sqrt{(2r_0^2 + x^2)}} \right)^{2|s|}.
\]  

(C.2.2)

Resumming a geometric sum on the form

\[
\sum_{s \in \mathbb{Z} + \nu} \eta^{s} \delta_{00} - \delta_{00} = \sum_{s \geq 0} \eta^{s} - \delta_{00} = 2 \sum_{s \geq 0} \eta^{s + \nu} - \delta_{00} = \frac{2\eta^{\nu}}{1 - \eta} - \delta_{00},
\]  

(C.2.3)

and using the following Taylor expansions

\[
\frac{1}{\sqrt{(2r_0^2 + x^2)}} \approx \frac{1}{2r_0} + \mathcal{O}(x^2),
\]

\[
\frac{1}{\sqrt{(2r_0^2 + x^2)}}^{2\nu} \approx \frac{1}{(2r_0)^{2\nu}} \left( \frac{1}{x} + \frac{1 - 2\nu}{2(2r_0^2)} x \right) + \mathcal{O}(x^2),
\]

(C.2.4)

yields

\[
G_{j j'}^0(x_0, x_0) = \lim_{x \to 0} \delta_{j j'} \left( \frac{2(2r_0)^{2\nu}}{(2r_0)^{2\nu}} \left( \frac{1}{x} + \frac{1 - 2\nu}{2(2r_0^2)} x \right) - \delta_{00} + \mathcal{O}(x^2) \right)
\]

\[
= \delta_{j j'} \left( \lim_{x \to 0} \frac{1}{x} \left( \frac{1}{x} + \frac{1 - 2\nu - \delta_{00}}{2r_0} + \frac{\nu(\nu - 1)}{2r_0^2} \right) \right).
\]  

(C.2.5)

We renormalize the theory such that we can ignore the divergent part (\( x^{-2} \) and \( x^{-1} \) terms) in the above propagator. This propagator seems to be correct since in the half-integer (\( \nu = 1/2 \)) case we reproduce the result from [6] with an overall factor of \( \delta_{j j'} \), i.e.

\[
G_{j j'}^0(x_0, x_0) = \delta_{j j'} \left( \lim_{x \to 0} \frac{1}{x^2} - \frac{1}{8r_0^2} \right).
\]  

(C.2.6)
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