CHARACTERIZING MODEL SPACES AMONG THE FINITE
DIMENSIONAL RKHS WITH PICK KERNELS

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ABSTRACT. We give several characterizations of those finite dimensional HSRK with complete Pick kernels which are model spaces. One characterization involves the size of the solution to a multiplier problem. Another involves having a conjugation operator which is compatible with the RKHS structure.

1. INTRODUCTION AND SUMMARY

We begin with an informal overview, detailed definitions and statements are in the later sections.

Let \( \mathcal{F} \) be the collection for finite dimensional reproducing kernel Hilbert spaces (RKHS) with irreducible complete Pick kernels. Those are exactly the spaces which are rescalings of spaces generated by finite sets of Dirichlet-Arveson reproducing kernels; the exact statement is Definition 2.5 below.

We interested in characterizing those \( H \in \mathcal{F} \) that arise from finite dimensional simple model spaces. That is, let \( H^2 \) be the the classical Hardy space and \( B \in H^2 \) a finite Blaschke product with simple zeros at the points \( X = \{x_i\}_{i=1}^n \) in the complex unit disk \( \mathbb{D} \). Let \( K_B \) be the finite dimensional Hilbert space

\[
K_B = H^2 \ominus BH^2.
\]

We regard \( K_B \) as an RKHS by declaring the functionals of evaluation at points of \( X \) to be the reproducing kernels; that is, the kernels are the \( k_i \in K_B, i = 1, \ldots, n \), which satisfy \( \langle f, k_i \rangle = f(x_i) \) for all \( f \in K_B \). We call those RKHS model spaces and call their rescalings \( r \)-model spaces. We denote the collection of all \( r \)-model spaces by \( \mathcal{M} \).

The \( r \)-model spaces are among the most easily described and thoroughly studied elements of \( \mathcal{F} \) and so it is interesting to know how typical they are as elements of \( \mathcal{F} \) and what distinguishes them from the general elements of \( \mathcal{F} \). Here we develop several criteria that characterize the \( H \in \mathcal{F} \) that are in \( \mathcal{M} \), including criteria based on the form of the associated set in complex hyperbolic space, on the values of certain extremal multipliers, on the existence of a conjugation operator taking the reproducing kernels to their dual basis, and on having a Gram matrix that is an orthogonal matrix.

The next section has preliminary notation, definitions and results. In Section 3 we state our main results, the proofs are in Section 4. A final section contains some comments.

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2. Background, Definitions, Notation

2.1. RKHS. For general background about spaces in $\mathcal{F}$ we refer to \textcite{AM} and \textcite{ARSW2}.

A finite dimensional RKHS is a finite dimensional Hilbert space $H$ together with a designated basis $\mathfrak{R} = \mathfrak{R}(H) = \{ k_i \}_{i=1}^n \subset H$ of vectors called reproducing kernels. We denote the dual basis by $\mathfrak{R}^\# = \{ k_i^\# \}_{i=1}^n$; that is, $k_i^\# \in H$ and $\langle k_i, k_j^\# \rangle = \delta_{ij}$ $1 \leq i, j \leq n$. Notice that the Hilbert space $H$ with vectors of $\mathfrak{R}^\#$ as its set of reproducing kernels is also a RKHS, we denote it by $H^\#$. Let $K = K(H)$ be the Gram matrix of $\mathfrak{R}$, the $n \times n$ matrix $(k_{ij})$ with $k_{ij} = \langle k_i, k_j \rangle$ $1 \leq i, j \leq n$. Similarly let $K^\#$ be the Gram matrix of $\mathfrak{R}^\#$. It is not hard to check that $K^\# = K^{-1}$.

We will use the metric $\delta = \delta_H$ defined on $\mathfrak{R}$ or, equivalently, the index set of $\mathfrak{R}$. For each $i$ let $P_i$ be the orthogonal projection onto the span of the kernel function $k_i$ and let $\| \| \Delta$ denote the operator norm. We set $\delta(k_i, k_j) = \|P_i - P_j\|$. There is also a formula for $\delta$ in terms of Gram matrix entries:

\[
\delta(k_i, k_j) = \sqrt{1 - \frac{|k_{ij}|^2}{k_{ii}k_{jj}}},
\]

More information about $\delta$ is in \textcite{ARSW1} and \textcite{R1}.

Two RKHS, $H$ and $\tilde{H}$, with reproducing kernels $\mathfrak{R} = \{ k_i \}_{i=1}^n$ and $\tilde{\mathfrak{R}} = \{ \tilde{k}_i \}_{i=1}^n$ are said to be rescalings of each other if there are nonzero scalars $\{ \lambda_i \}_{i=1}^n$ so that for $1 \leq i, j \leq n$ $\langle k_i, k_j \rangle = \lambda_i \lambda_j \langle \tilde{k}_i, \tilde{k}_j \rangle$. We denote this equivalence relation by $H \sim \tilde{H}$. Equivalently the Gram matrices of the two spaces are related through conjugation by a diagonal matrix with nonzero entries in which case we say that the two matrices are rescalings of each other. Most of the conditions we consider interact well with this equivalence relation; for instance if $H \sim \tilde{H}$ then $\delta_H(k_i, k_j) = \delta_{\tilde{H}}(\tilde{k}_i, \tilde{k}_j)$ and $H^\# \sim (\tilde{H})^\#$.

2.2. Model Spaces. The main facts we use about model spaces are in \textcite{GP}; a general reference is \textcite{CMR}.

Suppose $B$ is a Blaschke product with simple zeros at the points $\{ z_i \}_{i=1}^n \subset \mathbb{B}^1$. Thus $B = \Pi_i B_i$ where, for $x_i \neq 0$

\[
B_i(z) = \frac{|x_i|}{x_i} \frac{x_i - z}{1 - \overline{x_i}z}
\]

and $B_i(z) = z$ if $x_i = 0$. The reproducing kernels for the space $K_B$ are the functions $\{ k_i \} \subset K_B$ defined by

\[
k_i(w) = \frac{1}{1 - \langle w, x_i \rangle}.
\]

To see this recall from Hardy space theory that for each $i$ and each $f \in H^2$ we have $\langle f, k_i \rangle = f(x_i)$. With this in hand it follows that $k_i \in \left( BH^2 \right)^\perp = K_B$. These two facts together insure that the $k_i$ are the reproducing kernels. It then follows with a bit of computation that

\[
\delta(k_{x_i}, k_{x_j}) = \delta(x_i, x_j) = \left| \frac{x_i - x_j}{1 - \overline{x_i}x_j} \right| = |B_i(x_j)| = |B_j(x_i)|.
\]
A conjugation operator on a Hilbert space $H$ is an isometric conjugate linear map $J$ of $H$ to itself that is an involutive automorphism; that is $\|Jh\| = \|h\|$ for all $h \in H$ and $J^2 = I$. Each $K_B$ carries a conjugation operator $J = J_B$ given by

$$J_B f = Bz^T. \tag{2.4}$$

In (2.4) functions in $H^2$ are identified with their boundary values and (2.4) is an equation involving functions on $T^1$, the boundary circle of $B^1$. Using (2.4) and the fact that each of the factors $B_i$ of (2.2) is unimodular on $T^1$ it is not hard to check that the action of $J_B$ is on the kernel functions is given by

$$Jk_i(x_j) = \begin{cases} B(x_j)/(x_j - x_i) & \text{if } i \neq j \\ B'(x_i) & \text{if } i = j \end{cases}, \tag{2.5}$$

[GP], [GMR]. Hence for $1 \leq i, j \leq n$

$$\langle k_i, Jk_j \rangle = B'(x_i)\delta_{ij}. \tag{2.5}$$

Because $B$ has only simple zeros none of the $B'(x_i)$ are zero. Thus it is almost true that $J$ maps the basis of reproducing kernels of $K_B$ to its dual basis. By rescaling we can make that exactly true. We describe the general pattern.

**Proposition 2.1.** Suppose $H$ is a RKHS with kernel functions $\{k_i\}_{i=1}^n$. Suppose $J$ is a conjugation operator on $H$ and that for nonzero $\{c_i\}$ we have

$$\langle k_i, Jk_j \rangle = c_i\delta_{ij} \quad 1 \leq i, j \leq n.$$

Let $\tilde{H}$ be the RKHS obtained by using the same Hilbert space as $H$ and the reproducing kernels $\{\tilde{k}_i\}$ defined by $\tilde{k}_i = c_i^{-1/2}k_i$, $1 \leq i \leq n$. Then $\tilde{H} \sim H$ and $J$ is a conjugation operator on $\tilde{H}$ which maps the reproducing kernels of $\tilde{H}$ to their dual basis; that is

$$\langle \tilde{k}_i, J\tilde{k}_j \rangle = \delta_{ij} \quad 1 \leq i, j \leq n.$$

**Proof.** The first two statements are clear. To check the third we compute, for $1 \leq i, j \leq n$

$$\langle \tilde{k}_i, J\tilde{k}_j \rangle = \langle c_i^{-1/2}k_i, Jc_j^{-1/2}k_j \rangle = \langle c_i^{-1/2}k_i, c_j^{-1/2}Jk_j \rangle = c_i^{-1/2}c_j^{-1/2}\langle k_i, Jk_j \rangle = c_i^{-1/2}c_j^{-1/2}c_i\delta_{ij} = \delta_{ij}. \quad \square$$

We will call such an $\tilde{H}$ orthogonal and say its rescaling $H$ is $r$–orthogonal. The name is based on the following simple proposition. Recall that a matrix is called orthogonal if its inverse equals its transpose.

**Proposition 2.2** ([H1 Prop. 8]). $H$, a finite dimensional RKHS, is orthogonal if and only if its Gram matrix is an orthogonal matrix.

Suppose $H$ is a finite dimensional RKHS with kernel functions $\{k_i\}$ and Gram matrix $K = (k_{ij})$. Let $\{k_{ij}^\#\}$ be the dual basis and hence the set of kernel functions of $H^\#$ and write its Gram matrix as $K^\# = (k_{ij}^\#)$. It is always true that the transpose of $K$, $K^\dagger$ is equal to the entrywise conjugate matrix $\tilde{K}$. It is also always true that
\( K^\# = K^{-1} \). Thus \( H \) is orthogonal exactly if \( KK^t = K\bar{K} = KK^\# = I \). Formulated in terms of matrix entries this is, for \( 1 \leq i, j \leq n \),

\begin{equation}
\sum_{s=1}^{n} k_{is}k_{js} = \sum_{s=1}^{n} k_{is}k_{sj} = \sum_{s=1}^{n} k_{is}k_{sj}^\# = \delta_{ij}.
\end{equation}

**Corollary 2.3.** If \( H \in \mathcal{M} \) then \( H^\# \in \mathcal{M} \). Specifically, if \( H \sim K_B \) where \( B \) is the Blaschke product with zeros \( \{x_i\} \) then \( H^\# \sim K_{B^\#} \) where \( B^\# \) is the Blaschke product with zeros \( \{t_i\} \).

**Proof.** The Gram matrix of \( H^\# \) is the inverse matrix of the Gram matrix of \( H \) and hence is a rescaling of the inverse of \( K_B \), the Gram matrix of \( K_B \). From the previous proposition \( K_B \) is \( r \)-orthogonal hence \( K_B \) is a rescaling of an orthogonal matrix and thus \( (K_B)^{-1} \) is a rescaling of \( K_B \), the matrix of complex conjugates of entries of \( K_B \). It is a direct consequence of the definitions that \( K_B = K_{B^\#} \). In sum, the Gram matrix of \( H^\# \) is a rescaling of the Gram matrix of \( K_{B^\#} \). Hence \( H^\# \sim K_{B^\#} \).

This corollary shows that the Gram matrix of \( K_B \) is a rescalings of the complex conjugate of the Gram matrix of \( K_B^\# \). A similar result holds for some infinite Blaschke products \( B \) and that fact has been used in the study of \( H^2 \) interpolating sequences [AM, Exercise 9.54] [ARSW2, Sec. 7.3.1].

Because \( H^{\#\#} = H \) the converse of the corollary also holds and thus we have an necessary and sufficient condition for \( H^\# \in \mathcal{M} \). On the other hand it is not clear what conditions insure \( H^\# \in \mathcal{F} \). We discuss that question briefly in Section 5.2.

### 2.3. Multiplier Algebras.

If \( H \in \mathcal{F} \) then there is a particularly close relationship between \( H \) and \( M(H) \). In particular suppose \( X \) is the index set of \( \mathcal{H}(H), Y \subset X \) and \( x \in X \setminus Y \). Let \( m(Y, x) \in M(H) \) be the multiplier of norm one which vanishes on \( Y \) and maximizes \( \text{Re} m(Y, x) \). Let \( h(Y, x) \) be the vector in \( H \) of norm one which on \( Y \) and maximizes \( \text{Re} h(Y, x) \).

**Proposition 2.4 ([ARSW2] Prop. 6.27).** In the situation just described and with \( k_x \) the reproducing kernel for \( x \)

\begin{equation}
m(Y, x) \frac{k_x}{\|k_x\|} = h(Y, x).
\end{equation}

With this as a starting point it is not hard to show that for \( H \in \mathcal{F} \) the metric \( \delta \) on \( X \), the index set of \( \mathcal{H}(H), \) is the same as the Gleason metric on \( X \) induced by \( M(H) \), that is

\begin{equation}
\delta(x, y) = \max \{\text{Re} m(x) : m \in M(H), m(y) = 0, \|m\| = 1\},
\end{equation}

[ARSW2] Remrk 7.2]. The equality of the term on the left \( 2.3 \) and the last two terms on the right is an instance of this general fact.

### 2.4. Drury Arveson Spaces.

For \( m = 1, 2, \ldots \), the Drury-Arveson space \( DA_m \) is the Hilbert space of functions defined on the complex \( m \)-ball \( \mathbb{B}^m \) which is the closure of the span of the reproducing kernels \( \{k_z(w) = \frac{1}{1-\langle w, z \rangle} : z \in \mathbb{B}^m\} \). Here \( \langle w, z \rangle \) is the standard Hermitian inner product on \( \mathbb{C}^n \). References for this space and its properties include [AM], [ARSW2], and [Sh].

For \( X \) a finite subset of some \( \mathbb{B}^m \) we define the associated space \( DA_m(X) \) to be the subspace of \( DA_m \) spanned by \( \{k_x : x \in X\} \) and having those functions as
kernel functions. In particular if \( m = 1 \) then the kernel functions of \( DA_1(X) \) are the same as those of the model space \( K_{B_X} \) for \( B_X \) the Blaschke product with zeros at the points of \( X \); thus those two spaces are the same (or, pedantically, are trivial rescalings of each other). When \( m > 1 \) then the details of our discussion are effectively independent of \( m \) and we will not keep track of that index; this is discussed in [R1].

**Definition 2.5.** We say \( H \in \mathcal{F} \) if there is an integer \( m \) and finite \( X(H) \subset \mathbb{B}^m \) such that \( H \sim DA_m(X(H)) \).

Although this definition suits our purposes it is very different from the traditional definition of finite dimensional RKHS with complete Pick kernels. The equivalence of our definition with the traditional one is a basic theorem in the subject [AM].

2.5. **Hyperbolic Geometry.** The unit ball \( \mathbb{B}^m \subset \mathbb{C}^m \) is a model for complex hyperbolic \( m \)-space \( \mathbb{CH}^m \). This is discussed in detail in [Go]; here we will just recall a few pieces of information that we need.

The space \( \mathbb{CH}^m \) carries a transitive set of orientation preserving automorphisms. In the ball model these are realized by the group of conformal automorphisms of \( \mathbb{B}^m \).

The space \( \mathbb{CH}^m \) carries several natural metrics, of particular interest to us is the pseudohyperbolic metric \( \Delta \) which can be defined by setting, for \( z \in \mathbb{B}^m \), \( \Delta(0, z) = |z| \) and requiring \( \Delta \) to be invariant under the automorphism group. (We note for context, but will not use the fact, that the length metric generated by \( \Delta \) is the classical Bergman metric on the ball.) In particular, if \( z, w \in \mathbb{B}^1 \) then

\[
\Delta(z, w) = \left| \frac{z - w}{1 - \overline{z}w} \right|.
\]

The disk \( \mathbb{B}^1 \) sits inside \( \mathbb{B}^m = \mathbb{CH}^m \) as a totally geodesically embedded manifold of complex dimension one, called a complex geodesic. All the other totally geodesically embedded complex one manifolds, the other complex geodesics, are the images of that unit disk under the group of conformal automorphisms of the ball. In particular the group of automorphisms acts transitively on the set of complex geodesics.

2.6. **Hyperbolic Geometry and \( \mathcal{F} \).** The structure of the spaces \( DA_m(X) \) is closely related to the geometry of \( X \) regarded as a subset of \( \mathbb{CH}^m \). This theme is developed in [ARSW1], [R1], and [R2].

Comparing (2.9) with (2.3) we see that if \( B \) is a Blaschke product which vanishes at \( z \) and \( w \) and possibly other points, and with \( \delta \) denoting the metric on \( \mathbb{B}(B) \) then we have \( \Delta(z, w) = \delta(z, w) \). In fact this is the general pattern.

**Proposition 2.6 ([R1, 4.1]).** Suppose \( X \) is a finite set in some \( \mathbb{B}^m \) with \( z, w \in X \). Let \( \delta \) be the metric on \( X \) which is the index set of \( \mathbb{B}(DA_m(X)) \) and let \( \Delta \) be the pseudohyperbolic metric on \( \mathbb{CH}^m \) restricted to \( X \), then \( \Delta(z, w) = \delta(z, w) \).

There is another fundamental relation between elements of \( \mathcal{F} \) and hyperbolic geometry. Given \( H \in \mathcal{F} \) we know from Definition 2.5 that \( H \sim DA_m(X) \) for some finite \( X \). In fact we can describe the possible choices for \( X \).

**Theorem 2.7 ([R1 Thm 7]).** Suppose \( X, Y \) are finite sets in some \( \mathbb{CH}^m \), \( DA(X) \sim DA(Y) \) if and only if there is an automorphism \( \Phi \) of \( \mathbb{CH}^m \) with \( \Phi(X) = Y \).

In sum, questions of equivalence of elements of \( \mathcal{F} \) under rescaling are equivalent to questions about congruence of finite sets in complex hyperbolic space.
2.7. Subspaces. Suppose $H$ is a finite dimensional RKHS with kernel functions $\{k_\alpha\}_{\alpha \in X}$. We say $J$ is a regular subspace of $H$ if there is a $Y \subset X$ and $J$ is the Hilbert space spanned by $\{k_\beta\}_{\beta \in Y}$ and is regarded as a RKHS with the $\{k_\beta\}_{\beta \in Y}$ as its its kernel functions. It follows from the definitions that if $H \in \mathcal{F}$ then $J \in \mathcal{F}$, and similarly for membership in $\mathcal{M}$.

Statements which can be formulated using Gram matrix entries can be passed between a space $H$ and a regular subspace $J$. For instance, given $H, J, X, Y$ as above and $y, y' \in Y$ then the value $\delta(k_y, k_{y'})$ does not depend on whether we regard the kernel functions as in $J$ or in $H$.

Another instance of this, one we use in the proof below in showing Statement (2) implies Statement (1), is that if $J$ is a regular subspace of a $DA(X)$, and hence is $DA(Y)$ for some $Y \subset X$ then knowing that three points of $Y$ are in a complex geodesic implies that the "same" points, regarded as inside of $X$, and hence as index points for kernel functions in $DA(X)$, are also in a complex geodesic.

Another fact about regular subspaces which we will use is that if $H \in \mathcal{F}$ and $J$ is a regular subspace of $H$ then it is a consequence of the Pick property of $H$ than given a multiplier $m_J \in \mathcal{M}(J)$ there is an extension to a multiplier $m_H \in \mathcal{M}(H)$, defined on all of $H$, with the same norm, $\|m_J\| = \|m_H\|$.

At times we will use these facts without mention.

3. The Results

Our main result is the following

**Theorem 3.1.** Suppose $H \in \mathcal{F}$, that is for some $m$, and $X = \{x_i\}_{i=1}^n \subset \mathbb{B}^m$, we have $H \sim DA_m(X)$. The following are equivalent:

1. $X$ lies in a single complex geodesic in $\mathbb{B}^m = \mathbb{C}H^m$.
2. For each $1 < i < j \leq n$, $\{x_1, x_i, x_j\}$ lies in a single complex geodesic in $\mathbb{B}^m = \mathbb{C}H^m$.
3. There is a renumbering of $X$ after which, with $m_1 \in \mathcal{M}(H)$ the multiplier of norm one which satisfies $m_1(x_j) = 0$, $j = 2, \ldots, n$ and which maximizes $\Re m_1(x_1)$ we have

   $m_1(x_1) = \prod_{j=2}^n \delta(x_1, x_j)$.

4. $H$ is $r-$orthogonal. That is $H \sim \tilde{H}$ for some $\tilde{H}$ which carries a conjugation operator taking the basis of kernels of $H$ to the dual basis.
5. $H \sim \tilde{H}$ for an $\tilde{H}$ which has a Gram matrix which is an orthogonal matrix.
6. $H$ is an $r-$model space; that is there is a finite Blaschke product with simple zeros $B$ such that $H \sim K_B$.

4. The Proofs

Some parts of the proof follow from our earlier discussion and we begin with those. First we show (1) and (6) are equivalent. Suppose (6) holds and let $X(B)$ be the zero set of the Blaschke product $B$. As we noted earlier the spaces $K_B$ and $DA_1(X(B))$ have the same kernel functions and hence are the same space. Furthermore $DA_1(X(B))$ is in the form described in (1), that is $X(B)$ is in the unit disk which is a complex geodesic, both on its own and as a subset of any larger $\mathbb{B}^m = \mathbb{C}H^m$. On the other hand, if (1) holds then we can use the fact that
the group of automorphisms acts transitively on the set of complex geodesics to select an automorphism \( \Phi \) mapping the geodesic containing \( X \) to the unit disk. By Theorem 2.7 \( DA_m(X) \sim DA_1(\Phi(X)) \) and as before \( DA_1(\Phi(X)) \sim K_B \) where \( B \) is now the Blaschke product with zeros at \( \Phi(X) \). Thus we have (6) as required.

That (4) and (5) are equivalent is Proposition 2.2.

It is immediate that (1) implies (2). To see that (2) implies (1) recall that any two points in \( \mathbb{C} \mathbb{H}^n \) are contained in a unique complex geodesic. Hence, by the same argument we would use to study colinear points in Euclidian space we see that if (2) holds then so does (1).

The demonstrations that (6) implies (4) and that (6) implies (3) both use the function theory of the Hardy space applied to \( K_B \subset H^2 \). That (6) implies (4) follows from Proposition 2.1 and the discussion proceeding it. To see that (6) implies (3) note that from classical function theory on the Hardy space, in particular Pick’s theorem, the extreme value \( m_1(x_1) \) for the multiplier \( m_1 \in M(K_B) \) is the same as the extreme value for the \( H^\infty \) interpolation problem of finding.

\[
\max \left\{ \operatorname{Re} g(x_1) : g(x_2) = ... = g(x_n) = 0, \sup_{z \in \mathbb{B}^1} |g(z)| \leq 1 \right\}.
\]

That problem is solved using Blaschke products and we find that the maximum is \( |D(x_1)| \) where \( D \) is a Blaschke product with zeros \( \{x_2,...,x_n\} \). The relation between Blaschke factors and \( \delta \) given in (2.3) completes the argument.

It remains to show that (3) implies (6) and that (4) implies (6). If \( H \) is three dimensional then both implications are known. We recall those results and then use them to pass to the general cases.

**Proposition 4.1.** Suppose \( H \in \mathcal{F} \) is three dimensional with kernel functions \( \{k_a, k_b, k_c\} \). Let \( s \in M(H) \) the multiplier of norm one which vanishes at \( k_b \) and \( k_c \) and maximizes \( \operatorname{Re} s(k_a) \). If

\[
\tag{4.1}
s(k_a) = \delta(k_a, k_b)\delta(k_a, k_c).
\]

then \( H \in \mathcal{M} \).

This is Proposition 27 of [R1]. The proof is based on a detailed analysis of \( DA(X) \) and its multipliers for three point sets \( X \).

The next result is from [R1] where it is stated with an oversight. We will say a three dimensional RKHS is non-degenerate if no entry of its Gram matrix is zero, equivalently no two kernel functions are orthogonal. The property is preserved under rescaling and under passage to regular subspaces.

**Proposition 4.2.** Suppose \( H \) is a non-degenerate three dimensional RKHS which is \( r \)-orthogonal, then \( H \in \mathcal{M} \).

If \( H \in \mathcal{F} \) then \( H \) is automatically non-degenerate. However the proposition does not have the assumption that \( H \in \mathcal{F} \).

This is Theorem 28 of [R1] where the requirement of non-degeneracy was omitted. The shape of the proof is that finding rescaling parameters that will transform the Gram matrix of \( H \) into an orthogonal matrix involves solving a system of equations. For those equations to have a solution a determinant involving functions of the Gram matrix entries must vanish. That vanishing gives an equation for the Gram matrix entries which is equivalent to knowing \( H \sim DA(X) \) for \( X \) in a single complex geodesic. (This is the place where the proof in [R1] fails. Without the
non-degeneracy assumption the equation involving the Gram matrix entries may trivialize.)

Suppose now that (3) holds and we want to establish (6). After rescaling we may suppose $H = DA(X_H)$ for some $X_H \subset \mathbb{B}^n$. Let $J$ be the regular subspace of $H$ with kernel functions $\{k_\alpha, k_\beta, k_\gamma\}$. If we have (4.1), then by Proposition 4.1 $J \in \mathcal{M}$ and thus $J \sim DA(X_J)$ for some $X_J \subset \mathbb{B}^1$; in particular $X_J$ is in a single complex geodesic. As discussed in Section 2.7, this implies that the points of $X$ corresponding to those three kernel functions also lie in a single geodesic. The choice of which three kernel functions we considered was arbitrary and hence we would have (2) for $H$. We have already seen that (2) implies (1) which implies (6).

Thus we are done if we can show that for any choice of three kernel functions (4.1) holds. From (2.5), we know that given any kernel functions $k_\alpha, k_\beta$ in an $H \in \mathcal{F}$ there will be a multiplier $m_{\alpha\beta} \in M(H)$ of norm one which satisfies

$$(4.2) \quad m_{\alpha\beta}(k_\alpha) = 0, \quad m_{\alpha\beta}(k_\beta) = \delta(k_\alpha, k_\beta).$$

We now proceed by contradiction. Suppose we have found three kernel functions, $\{k_i\}_{i=1,2,3}$, for which (4.1) fails and denote their span by $J$. Using the notation of (4.2) form the multiplier $r = m_{12}m_{13} \in M(J)$. This multiplier is a candidate for the extremal problem defining the multiplier $s$ in (4.1) and by evaluating if we see that the left hand side in (4.1) will never be smaller than the right. Hence by our construction of $J$ (4.1) fails because the left hand side is larger. Thus we have a multiplier $t \in M(J)$ of norm one with

$$(4.3) \quad t(k_1) > \delta(k_1, k_2)\delta(k_1, k_3).$$

As noted in Section 2.7 the multipliers $m_{\alpha\beta}$ and $t$ all have norm preserving extensions to elements in $M(H)$. We will regard those extensions as having been made and use the same notation for the extended multipliers. Staying with the notation in (4.2) consider the multiplier in $q \in M(H)$ given by

$$q = t \prod_{j=4}^n m_{1j}.$$  

By the Banach algebra property of $M(H)$ $q$ has norm at most one and by construction it vanishes at $k_2, \ldots, k_n$; hence $q$ is a competitor in the extremal problem defining $m_1$. Furthermore, comparing the definition of $q$, (4.2), and (4.3) we see that $q(k_1)$ is larger than the right hand side of (3.1). This inequality contradicts the extremal value suggested by (3.1) and hence contradicts the assumption that (3) holds. This completes the proof that (3) implies (6).

We now show (4) implies (6). We are given $H \in \mathcal{F}$ that is $n$ dimensional and is $r-$orthogonal. Let $\{k_i\}_{i=1}^n$ be an arbitrary numbering of the kernel functions of $H$ and let $H_- \in \mathcal{F}$ be the $n - 1$ dimensional regular subspace of $H$ spanned by the kernel functions $\{k_i\}_{i=1}^{n-1}$. We will show that $H_-$ is $r-$orthogonal. Repeating this shows that for any three kernel functions $\{k_\alpha, k_\beta, k_\gamma\}$ their span, $H_{\alpha\beta\gamma}$, is $r-$orthogonal. By Proposition 1.2 this insures $H_{\alpha\beta\gamma} \in \mathcal{M}$. Hence after rescaling $H_{\alpha\beta\gamma} = DA(X_{\alpha\beta\gamma})$ for a three point set $X_{\alpha\beta\gamma}$ in the unit disk. As we noted in Section 2.7 this insures that when $H$ is rescaled as $DA(X)$ for some $X$ in some $\mathbb{B}^n$ then the points of $X$ corresponding to the kernel functions $\{k_\alpha, k_\beta, k_\gamma\}$ will lie in a single complex geodesic in $\mathbb{B}^n$. The numbering of the kernel functions of $H$ was arbitrary and hence this establishes (2) for $H$ from which (1) follows and then (6).
To prove the reduction we start with \( n \)-dimensional \( H \in \mathcal{F} \) with kernel functions \( \{k_j\}_{j=1}^n \) which, by rescaling, we can assume has an orthogonal Gram matrix \( \mathbf{K} = (k_{ij})_{i,j=1}^n \). We denote the dual basis, the kernel functions of \( H^\# \), by \( \{k^\#_i\}_{i=1}^n \) and write its Gram matrix as \( \mathbf{K}^\# = (k^\#_{ij})_{i,j=1}^n \). Because \( H \) is orthogonal we have \( \mathbf{K}^{-1} = \mathbf{K}^\# = \mathbf{K}^t \) and hence the condition that \( H \), and thus \( \mathbf{K} \), is orthogonal is expressed by the equations, for \( 1 \leq i, j \leq n \).

(4.4) \[
\sum_{s=1}^n k_{i,s} k^\#_{s,j} = \delta_{ij}.
\]

Let \( H_- \) be the regular subspace of \( H \) spanned by the kernel functions \( \{k_i\}_{i=1}^{n-1} \). The inner product of those \( k_j \) is the same whether they are regarded as vectors in \( H \) or \( H_- \) and hence the Gram matrix of \( H_- \) is the upper left \((n-1) \times (n-1)\) block of the Gram matrix of \( H \): \( \mathbf{K}_- = (k_{ij})_{i,j=1}^{n-1} \). Let \( \{k_i\}_{i=1}^{n-1} \) be the dual basis of the basis \( \{k^\#_i\}_{i=1}^{n-1} \) and let \( \mathbf{K}^\#_- = (k^\#_{ij})_{i,j=1}^{n-1} \) be its Gram matrix. For ease of reading we set \( g_j = k^\#_j \) and \( g_{ij} = k^\#_{ij} \).

The vectors \( \{k^\#_j\}_{j=1}^{n-1} \) are nearly but not quite the \( \{g_j\}_{j=1}^{n-1} \). They give the correct inner products; for \( 1 \leq i, j \leq n-1 \)

(4.5) \[
\langle k_i, k^\#_j \rangle = \delta_{ij},
\]

but they are not in \( H \). To obtain vectors in \( H \) which satisfy the analog of (4.5) we apply \( P \), the orthogonal projection from \( H \) to \( H_- \). That gives

\[
\langle k_i, P(k^\#_j) \rangle = \delta_{ij};
\]

and hence the \( P(k^\#_j) \) are our desired \( g_j, j = 1, ..., n-1 \).

We know \( P(k^\#_j) = k^\#_j - Q(k^\#_j) \) where \( Q \) is the projection complimentary to \( P \) and we use that to compute \( g_j \). \( Q \) is the projection onto the orthocomplement of \( H_- \) in \( H \) and by the definition of the dual basis that is the subspace of \( H \) spanned by \( k^\#_n \). Hence for any \( b \)

\[
P(b) = b - \left( b, \frac{k^\#_n}{\|k^\#_n\|} \right) \frac{k^\#_n}{\|k^\#_n\|}.
\]

In particular, for \( 1 \leq j \leq n-1 \)

\[
g_j = k^\#_j - \left( k^\#_j, \frac{k^\#_n}{\|k^\#_n\|} \right) \frac{k^\#_n}{\|k^\#_n\|}
= k^\#_j - \frac{k^\#_n}{k^\#_n \cdot k^\#_n}.
\]

We want to evaluate \( \mathbf{K}^\#_- = (g_{ij}) \);
\[ g_{ij} = \langle g_i, g_j \rangle = \left\langle k_i^\# - \frac{k_i^\#}{k_{nn}^\#} k_n^\#, k^\#_j - \frac{k_j^\#}{k_{nn}^\#} k_n^\# \right\rangle = \left( k_i^\#, k_j^\# \right) - \left\langle \frac{k_i^\#}{k_{nn}^\#} k_n^\#, k_j^\# \right\rangle + 0 = k_{ij}^\# - \frac{k_{in}^\# k_{nj}^\#}{k_{nn}^\#} \]

To show that \( K^- \) is an orthogonal matrix we will show
\[ \sum_{s=1}^{n-1} k_{-is} k_{-sj}^\# = \sum_{s=1}^{n-1} k_{is} g_{sj} = \delta_{ij}. \]

We compute
\[ \sum_{s=1}^{n-1} k_{is} g_{sj} = \sum_{s=1}^{n-1} k_{is} \left( k_{sj}^\# - \frac{k_{is}^\# k_{nj}^\#}{k_{nn}^\#} \right) = \sum_{s=1}^{n-1} k_{is} k_{sj}^\# - \sum_{s=1}^{n-1} \frac{k_{is}^\# k_{nj}^\#}{k_{nn}^\#} = (\delta_{ij} - k_{in} k_{nj}^\#) + (\delta_{in} - k_{in} k_{nn}^\#) k_{nj}^\# k_{nn}^\# = \delta_{ij}. \]

In the passage from the second line to the third we used (4.4) for the index pair \((i, j)\) and for the index pair \((i, n)\). The passage to the final line used the fact that \(i < n\) and hence \(\delta_{in} = 0\).

5. Comments and Variations

5.1. Reformulations of \( m_1(x_1) = \prod_{j=2}^{n} \delta(x_1, x_j) \). Suppose \( m_1 \in M(H) \) is the multiplier described in (3) of Theorem 3.1. Taking note of Proposition 2.4 we can write (2.7) for this multiplier and obtain
\[ m_1 \frac{k_1}{\|k_1\|} = h. \]

Here \( h \) is the function of norm one which vanishes at \( x_2, \ldots, x_n \) and maximizes \( \text{Re} \ h(x_1) \). The space of competitors for \( h \) is one dimensional and hence \( h = k_1^\#/ \|k_1^\#\| \).

Using this and taking the inner product of both sides of the previous equation with \( k_1 \) we find
\[ m_1(k_1) \frac{k_{11}}{\|k_1\|} = \left\langle \frac{k_1^\#, k_1}{k_1^\#} \right\rangle \]

which simplifies as
\[ m_1(k_1) = (\|k_1\| \|k_1^\#\|)^{-1}. \]

We obtain another expression for \( m_1(k_1) \) if we consider the idempotent multiplier in \( M(H) \) which takes the value 1 at \( k_1 \) and is zero elsewhere. Denote that multiplier
by $\text{Idem}_1$. It is in the one dimensional space spanned by $m_1$ and hence $\text{Idem}_1 = m_1/m_1(k_1)$. Thus

$$m_1(k_1) = (\|\text{Idem}_1\|)^{-1}.$$ 

Either of these evaluations of $m_1(k_1)$ could be used on the left hand side of (3.1). We had already noted that the $\delta$'s on the right hand side can be evaluated using the Hilbert space structure $H$ or the using the structure of $M(H)$. Hence we can write versions of (3.1) based entirely on data from $H$ or entirely on using data from $M(H)$.

5.2. When is $H^\# \in \mathcal{F}$? We noted that Corollary 2.3 leads to the fact that $H \in \mathcal{M}$ is a necessary and sufficient condition for $H^\# \in \mathcal{M}$. On the other hand it is not clear what conditions on a RKHS $H$, even one in $\mathcal{F}$, will insure $H^\# \in \mathcal{F}$. Here we indicate that the condition $H \in \mathcal{M}$, which is sufficient because $H^\# \in \mathcal{M} \subset \mathcal{F}$, is not necessary, and the condition $H \in \mathcal{F}$ is not sufficient.

Suppose $X \subset \mathbb{B}^1 \subset \mathbb{B}^2$ and $H = DA(X)$. Arbitrarily slight modification of $X$ can produce $\tilde{X} \subset \mathbb{B}^2$ which are not contained in a single complex geodesic. For those $\tilde{X}$ by Theorem 3.1 $H = DA(\tilde{X}) \notin \mathcal{F} \setminus \mathcal{M}$. If $\tilde{X}$ is sufficiently close to $X$ then the Gram matrix of $\tilde{H}$ is arbitrarily close to the Gram matrix of $H$. Furthermore, passing to inverse matrices, the Gram matrix of $\tilde{H}^\#$ is arbitrarily close to the Gram matrix of $H^\#$. We know $H^\# \in \mathcal{M} \subset \mathcal{F}$ and we know that being in $\mathcal{F}$ is an open condition on the Gram matrix of a HSRK [AM] (this is in contrast to the condition for being in $\mathcal{M}$) hence, if our perturbation was sufficiently small then $H^\# \in \mathcal{F}$.

On the other hand $H \in \mathcal{F}$ is not itself sufficient to insure $H^\# \in \mathcal{F}$. Consider, for $0 < a, b < 1$, $X = \{(0,0), (a,0), (0,b)\} = \mathbb{B}^2$, in some sense the extreme opposite of $X$ being in a single geodesic. The Gram matrix of $H = DA(X)$ has the form

$$G = \begin{pmatrix}
1 & 1 & 1 \\
1 & * & 1 \\
1 & * & *
\end{pmatrix}$$

The Gram matrix of $H^\#$ is $G^{-1}$ and by explicit computation the $(2,3)$ entry of $G^{-1}$ is 0 which is impossible for a RKHS in $\mathcal{F}$.

5.3. The Use of Hardy Space Theory and the Pick Condition. In our proof of Theorem 3.1 we used the function theory of the Hardy space to study $K_B \subset H^2$, in particular to prove that (6) implies (3) and that (6) implies (4). It would be interesting to have a proof of either of these implication, or of the equivalence of (3) and (4) without involving (6), that was inside the the theory of RKHS and did not use function theory. In this context we note the work of Cole, Lewis, and Wermer in [CLW]. They study conditions on $M(H)$, the multiplier algebra of a RKHS $H$, which suffice to insure that $H \in \mathcal{M}$. They have two approaches, one using operator theory and a second which uses their hypotheses on the multiplier algebras to, in effect, reconstruct and thus reintroduce parts of Hardy space function theory.

Many of the steps in the proof of Theorem 3.1 did not require the Pick property and hence can be used for any finite dimensional RKHS. However there were places where we did use the Pick property and do not know the extent to which it could be avoided. In particular we used Proposition 2.4 to connect multiplier algebra statements to Hilbert space statements. This fact, which is a characteristic property of spaces in $\mathcal{F}$, was used in showing $\delta$ satisfies (2.8) and in our analysis of condition (3) in Theorem 3.1.
Also, the hypothesis $H \in \mathcal{F}$ was used in passing from condition (2) to condition (1). For instance it is not clear that given a general four dimensional RKHS $H$, and knowing that all of its regular three dimensional subspaces are in $\mathcal{M}$, is enough to insure $H \in \mathcal{F}$. If we could show that, that $H \in \mathcal{F}$, then by the implication (2) implies (1) in Theorem 3.1 we would also know $H \in \mathcal{M}$. In this context it is interesting to note that there are examples due to Quiggen in which all the regular subspaces of a four dimensional $H$ are in $\mathcal{F}$ but $H$ is not in $\mathcal{F}$. Those are discussed in [12].

To see this issue in context, suppose we wanted to show that an orthogonal $H$, not necessarily in $\mathcal{F}$, satisfied Statement (1), and hence Statement (6), in Theorem 3.1. The reduction to the case of three dimensional regular subspaces is a linear algebra argument and so continues to hold. If we assume $H$ is non-degenerate we can then use Proposition [12] to obtain a version of Statement (2). However without knowing $H \in \mathcal{F}$ it is not clear how to proceed from Statement (2) to Statement (1).

5.4. Replacing RKHS by Point Sets in Projective Space. It is possible to recast much of the previous discussion in the language of point sets in complex projective space. The viewpoint is intriguing but it is not clear where it leads. We will be brief and informal.

Suppose we start with the finite dimensional Hilbert space $\mathbb{C}^n$ and regard it as a RKHS, $H$, by selecting a set of basis vectors $\{k_i\}$ and declaring those vectors to be reproducing kernels. In fact any finite dimensional RKHS is a rescaling of such an $H$.

Let $\mathbb{P}\mathbb{C}^{n-1}$ be the complex projective space of lines in $\mathbb{C}^n$; for each nonzero vector $v \in \mathbb{C}^n$ we denote the line containing $v$ by $[v]$; thus $[v] \in \mathbb{P}\mathbb{C}^{n-1}$. Hence we can associate to $H$ the set $[H] = \{[k_i]\}_{i=1}^n \subset \mathbb{P}\mathbb{C}^{n-1}$. Note that if we rescale $H$ to $\tilde{H}$ by selecting scalars $\{\lambda_i\}$ declaring the $\{\lambda_i k_i\}$ to be the kernels for $\tilde{H}$, then $[\tilde{H}] = [H]$. The rescaled space $\tilde{H}$, and in some sense $\tilde{H}$ is the generic rescaling of $H$, produces the same set in $\mathbb{P}\mathbb{C}^{n-1}$. From our point of view, our interest is in properties invariant under rescaling, this is an attractive feature.

Next note that there set $[H^\#]$ associated to the dual RKHS $H^\#$ can also be described in the language of $\mathbb{P}\mathbb{C}^{n-1}$. To $n-1$ generic points in $\mathbb{P}\mathbb{C}^{n-1}$ there correspond $n-1$ linearly independent lines in $\mathbb{C}^n$. There is a unique line in $\mathbb{C}^n$ orthogonal to those lines and that orthogonal line corresponds to a point in $\mathbb{P}\mathbb{C}^{n-1}$ which is ”orthogonal” to each of the $n-1$ points in the starting set. This process applied to each of the $n-1$ point subsets of $[H]$ gives a set $[H^\#]$. Tracking the definitions we see that $[H^\#] = [H^\#]$.

If $H$ is orthogonal then $\{k_i\}$ and $\{k_i^\#\}$ are related through a period two conjugate linear isometry of the Hilbert space. This isometry descends to an isometry of $\mathbb{P}\mathbb{C}^{n-1}$ which interchanges $[H]$ and $[H^\#]$. On the other hand if there is such an isometry connecting $[H]$ and $[H^\#]$ then one can show by analysis in $\mathbb{P}\mathbb{C}^{n-1}$ that the isometry extends to a global isometry of $\mathbb{P}\mathbb{C}^{n-1}$ which, by Wigner’s theorem, must come from a unitary or antiunitary map of the original $\mathbb{C}^n$.

In sum, many of the ideas we have considered can be comfortably reformulated as statements about certain types of symmetric point sets in projective space.
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