Exact solution of A-D Temperley-Lieb Models

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Abstract

We solve for the spectrum of quantum spin chains based on representations of the Temperley-Lieb algebra associated with the quantum groups $\mathcal{U}_q(X_n)$ for $X_n = A_1, B_n, C_n$ and $D_n$. We employ a generalization of the coordinate Bethe-Ansatz developed previously for the deformed biquadratic spin one chain. As expected, all these models have equivalent spectra, i.e. they differ only in the degeneracy of their eigenvalues. This is true for finite length and open boundary conditions. For periodic boundary conditions the spectra of the lower dimensional representations are contained entirely in the higher dimensional ones. The Bethe states are highest weight states of the quantum group, except for some states with energy zero.

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1 Introduction

The recent interplay between the field of solvable two-dimensional lattice (or quantum spin chain) models and quantum groups, has generated a lot of interesting results. One particular way of building models, which are quantum group invariant, uses the Temperley-Lieb (TL) algebra, satisfied by the Hamiltonian density $U_k$:

$$U_k^2 = \sqrt{Q}U_k, \quad U_k U_{k\pm 1} = U_k,$$

$$[U_k, U_l] = 0, \quad |k - l| \geq 2. \quad (1.1)$$

The Hamiltonian is now given by the following sum over $N$ sites:

$$H(q) = \sum_{k=1}^{N} U_k. \quad (1.2)$$

The paper is organized as follows. In section 2, we describe the representations of the TL algebra, constructed as projectors on total spin zero of two neighbouring spins. In section 3, we discuss a modified coordinate BA and show the reasons, why the techniques developed for the spin $1/2$ XXZ model don’t work here. Section 3 contains the algebraic details of the computation and section 4 is reserved for the conclusions.

2 Representations of the Temperley-Lieb algebra as spin zero projectors.

Representations of the TL algebra, commuting with quantum groups, can be constructed in the following way. Suppose $U_q(X_n)$ is the universal enveloping algebra of a finite dimensional Lie algebra $X_n$, equipped with the coproduct $\Delta : U_q \rightarrow U_q \otimes U_q$. If now $\pi : U_q \rightarrow EndV_\Lambda$ is a finite dimensional irreducible representation with highest weight $\Lambda$ and we assume that the decomposition $V_\Lambda \otimes V_\Lambda$ is multiplicity-free and includes one trivial representation on $V_0$, then the projector $P_0$ from
$V_\Lambda \otimes V_\Lambda$ onto $V_0$ is a representation of the TL algebra. The deformation parameter $q$, which plays the role of a coupling constant in the Hamiltonian, is related to $Q$ as:

$$\sqrt{Q} = Tr_V(q^{-2\rho}),$$  \hspace{1cm} (2.1)

where $\rho$ is half the sum of the positive roots.

By construction $P_0$ commutes with the quantum group $U_q(X_n)$.

Since we are not going to use any group-theoretical machinery, we will just lift the relevant formulas off Batchelor and Kuniba[4] in order to display explicitly the Hamiltonians to be diagonalized.

We will consider the following specific cases, $(V_\Lambda, U_q(X_n)) = (V_{2s\Lambda_1}, U_q(A_1))$ for spin $s$, $(V_{\Lambda_1}, U_q(B_n)), (V_{\Lambda_1}, U_q(C_n))$ and $(V_{\Lambda_1}, U_q(D_n))$. I.e, we treat the $q$-deformations of the spin-$s$ representation of $sl(2)$ and the vector representations of $so(2n + 1), sp(2n)$ and $so(2n)$. $V_\Lambda$ denotes the $U_q(X_n)$ module with highest weight $\Lambda$. $\Lambda_1$ is a highest weight of $X_n$.

Introduce the following notation. Let $e_i, i = 1, \ldots, n$ be orthonomal vectors and express the fundamental weight, $\Pi = \Lambda_1 + \ldots + \Lambda_n$, the set $\mathcal{A}$ of weights and the coupling constant $\sqrt{Q} \equiv -2\Delta$ as:

\begin{align*}
A_1 : \mathcal{A} &= \{s(e_1 - e_2), (s - 1)(e_1 - e_2), \ldots, -s(e_1 - e_2)\}, \\
\Lambda_1 &= (e_1 - e_2)/2, \\
\rho &= (e_1 - e_2)/2, \\
J &= \{s, s - 1, \ldots, -s\}, \\
\epsilon(\mu) &= (-1)^{\bar{\mu}}, \\
\sqrt{Q} &= [2s + 1];
\end{align*}

\begin{align*}
B_n (n \geq 2) : \mathcal{A} &= \{0, \pm e_1, \ldots, \pm e_n\}, \\
\Lambda_i &= e_1 + \ldots + e_i, \hspace{0.5cm} (1 \leq i < n),
\end{align*}
\[
\frac{e_1 + \ldots + e_i}{2}, \quad (i = n),
\]
\[
\rho = (n - 1/2)e_1 + \ldots + e_n/2,
\]
\[
J = \{0, \pm 1, \ldots, \pm n\},
\]
\[
\epsilon(\mu) = (-1)^\tilde{\mu},
\]
\[
\sqrt{Q} = [2n - 1][n + 1/2]/[n - 1/2];
\]
\[
C_n : A = \{\pm e_1, \ldots, \pm e_n\},
\]
\[
\Lambda_i = e_1 + \ldots + e_i, \quad (1 \leq i \leq n),
\]
\[
\rho = ne_1 + \ldots + e_n,
\]
\[
J = \{\pm 1, \ldots, \pm n\},
\]
\[
\epsilon(\mu) = \text{sign}(\mu),
\]
\[
\sqrt{Q} = [n][2n + 2]/[n + 1];
\]
\[
D_n : A = \{\pm e_1, \ldots, \pm e_n\},
\]
\[
\Lambda_i = e_1 + \ldots + e_i, \quad (1 \leq i < n - 1),
\]
\[
= \frac{e_1 + \ldots + e_{n-1} - e_n}{2}, \quad (i = n - 1),
\]
\[
= \frac{e_1 + \ldots + e_{n-1} + e_n}{2}, \quad (i = n),
\]
\[
\rho = (n - 1)e_1 + \ldots + e_{n-1},
\]
\[
J = \{0, \pm 1, \ldots, \pm n\},
\]
\[
\sqrt{Q} = [2n - 2][n]/[n - 1];
\]
\[
\epsilon(\mu) = 1.
\]

For \(\mu \in J\) the symbol \(\tilde{\mu}\) is defined as \(\tilde{\mu} = \mu + (1 \pm 1)/4\) for \(A_1\) with \(s \in \mathbb{Z} + (1 \pm 1)/4\) and \(\tilde{\mu} = 0\) with the exception of \(\tilde{0} = 1\) for \(B_n\). The \(q\)-number notation is \([x] \equiv (q^x - q^{-x})/(q - q^{-1})\). For \(X_n = B_n, C_n, D_n\), we extend the suffix of \(e_\mu\) to \(-n \leq \mu \leq n\) by setting \(e_{-\mu} = -e_\mu\) (hence \(e_0 = 0\)). Using the index set \(J\), above, we can write \(A = \{\mu(e_1 - e_2)\}\) for \(A_1\) and \(A = \{e_\mu|\mu \in J\}\) for \(B_n, C_n, D_n\).

Denoting by \(E_{\mu\nu} \in \text{End } V_A\) the matrix unit, having all elements zero, except at
row $\mu$ and column $\nu$, the projector can be written as

$$P_0 = Q^{-1/2} \sum_{\mu,\nu \in J} \epsilon(\mu)\epsilon(\nu)q^{-<e_\mu+e_\nu,\rho>} E_{\mu\nu} \otimes E_{-\mu-\nu}. \quad (2.3)$$

In the following we will refer to all models generically as higher spin models for simplicity, even when not talking about $A_1$.

Consider a one-dimensional chain of length $N$ with a "spin" at each site. The spin variables range over the set of weight vectors $v_\mu |\mu \in J$ and our Hilbert space is an $N$-fold tensor product $V_\Lambda \otimes \ldots \otimes V_\Lambda$. For $A_1$, these are the $q$-analogues of the usual spin states.

The Hamiltonian densities acting on two neighboring sites are then given by:

$$\langle k, l | U | i, j \rangle = \epsilon(i)\epsilon(k)q^{-<e_i+e_k,\rho>} \delta_{i+j,0} \delta_{k+l,0}. \quad (2.4)$$

### 3 The coordinate Bethe - Ansatz

All the above Hamiltonians are $U(1)$ invariant and we can classify their spectra according to sectors. For $A_1(s = 1)$ the commuting operator is the total spin $S^z = \sum_{k=1}^N S^z_k$ and we set the conserved quantum number $r = N - S^z$. In general it equals $r = N * \omega - S^z$ for $A_1$ and $B_n$ and $r = N * (\omega - 1/2) - S^z$ for $C_n$ and $D_n$. We set $\omega = \max J$.

Therefore, there exists a reference state $|\Omega\rangle$, satisfying $H |\Omega\rangle = E_0 |\Omega\rangle$, with $E_0 = 0$. We take $|\Omega\rangle$ to be $|\Omega\rangle = \Pi_k^N |\omega, k\rangle$.

In every sector $r$ there are eigenstates degenerate with $|\Omega\rangle$. They contain a set of impurities. We call impurity any state obtained by lowering some of the $|\omega, k\rangle$'s, such that the sum of any two neighboring spins is non-zero. Since $H(q)$ is a projector on spin zero, all these states are annihilated by $H(q)$. In particular, they do not move under the action of $H(q)$, which is the reason for their name.

We will now start to diagonalize $H(q)$ in every sector. Nothing interesting happens in sector $r < 2\omega$. Sector $r = 2\omega$ is more interesting, although still trivial,
since it contains one free pseudoparticle. The main result of this paper is to show that \( H(q) \) can be diagonalized in a convenient basis, constructed from products of single pseudoparticle wavefunctions. The energy eigenvalues will be parametrized as a sum of single pseudoparticle contributions.

### 3.1 The sector \( r = 2\omega \), containing one pseudoparticle

Starting with \( r = 2\omega \), we encounter the situation, where the states \( |j, k\rangle \) and \( |-j, k \pm 1\rangle \), \( j \neq \omega \) occur in neighboring pairs. They do move under the action of \( H(q) \) and mix with states containing one \( |\omega, k\rangle \). Eigenstates are a superposition of \( |x[-\omega]\rangle = (\ldots \omega \omega \omega \omega \ldots)_x \) and \( |x[j, -j]\rangle = (\ldots \omega \omega \omega x \omega \omega \omega \omega \omega \ldots)_x \), i.e.

\[
|2\omega; \ldots \rangle = \sum_x \{a_\omega(x) |x[-\omega]\rangle + \sum_j b_j(x) |x[j, -j]\rangle\},
\]

where \( \sum_j \) means \( j \in J^* = J - \{\pm \omega\} \) and the ellipses stand for parameters the eigenvector is going to depend on. When \( H(q) \) now acts on \( |2\omega; \ldots \rangle \) it sees the reference configuration, except in the vicinity of \( x \) and we obtain the eigenvalue equations

\[
(E - q^2 <\epsilon_\omega, \rho> - q^{-2} <\epsilon_\omega, \rho>) a_\omega(x) = a_\omega(x + 1) + a_\omega(x - 1) + \\
\sum_l \epsilon(\omega)\epsilon(l)q^{-<\epsilon_\omega + \epsilon_l, \rho>} b_l(x - 1) + \sum_l \epsilon(-\omega)\epsilon(l)q^{-<\epsilon_\omega - \epsilon_l, \rho>} b_l(x)
\]

\[
Eb_j(x) = \epsilon(\omega)\epsilon(l)q^{<\epsilon_j + \epsilon_\omega, \rho>} a_\omega(x + 1) + \epsilon(-\omega)\epsilon(l)q^{<\epsilon_j - \epsilon_\omega, \rho>} a_\omega(x) + \\
\sum_l \epsilon(j)\epsilon(l)q^{-<\epsilon_j + \epsilon_l, \rho>} b_l(x), \quad j \in J^*.
\]

Eliminating the \( b_j \)'s, we get an equation very similar to the XXZ model:

\[
(E - \sum_j q^{2 <\epsilon_j, \rho>})a_\omega(x) = a_\omega(x + 1) + a_\omega(x - 1).
\]
We will treat periodic boundary conditions maintaining translational invariance in the following sections. They demand $a_{\omega}(x+N) = a_{\omega}(x)$ and $b_{j}(x+N) = b_{j}(x)$. We parametrize as: $a_{\omega}(x) = a_{\omega} \xi^{x}$ and $b_{l}(x) = b_{l} \xi^{x}$, $l \in J^{*}$. Substituting this into equ.(3.2) we get two eigenstates and their energies

$$a_{\omega} = \epsilon(-\omega) q^{-\langle \epsilon_{\omega}, \rho \rangle} + \epsilon(\omega) q^{-\langle \epsilon_{\omega}, \rho \rangle} \xi^{-1} \equiv \Gamma(\xi^{-1})$$

$$b_{l} = \epsilon(l) q^{-\langle \epsilon_{l}, \rho \rangle}, \; l \in J^{*}$$

$$E_{1} = \sum_{l} q^{-\langle \epsilon_{l}, \rho \rangle} + \Gamma(\xi) \Gamma(\xi^{-1}) = \sum_{l \in J} q^{-2\langle \epsilon_{l}, \rho \rangle} + \xi + \xi^{-1}$$

and a highly degenerate solution with $E_{2} = 0$, with the following constraint on the parameters:

$$\sum_{l} \epsilon(l) \epsilon(\omega) q^{-\langle \epsilon_{l+\omega}, \rho \rangle} b_{l} + \Gamma(\xi) \epsilon(\omega) q^{-\langle \epsilon_{\omega}, \rho \rangle} a_{\omega} = 0.$$

Here $\xi = e^{i\theta}$, $\theta$ being the momentum determined from the periodic condition to be: $\theta = 2\pi l/N$, with $l$ integer.

We describe this situation by saying that we have two types of pseudoparticles with energies $E_{1}$ and $E_{2}$. Whereas the pseudoparticle $|2\omega; \theta\rangle_{2}$ is degenerate with $|\Omega\rangle$, i.e. propagates with energy $E_{2} = 0$, the pseudoparticle $|2\omega; \theta\rangle_{1}$, propagates with energy

$$E_{1} = -2\Delta + 2 \cos \theta, \quad 2\Delta \equiv -\sum_{l \in J} q^{-2\langle \epsilon_{l}, \rho \rangle}.$$

As mentioned before, the energy eigenvalues are going to be parametrized as a sum of single pseudoparticle energies. Thus we write:

$$E = \sum_{n=1}^{p} \epsilon_{n}(\sum_{l} q^{-\langle \epsilon_{l}, \rho \rangle} + \Gamma(\xi_{n}) \Gamma(\xi_{n}^{-1})), $$

where $\epsilon_{n}$ depends on which pseudoparticle we use: $\epsilon_{n} = 1$ for $E = E_{1}$ and $E = E_{2} = 0$.

### 3.2 Two pseudoparticles and the XXZ Bethe - Ansatz

The next higher sector would be $r = 2\omega + 1$, but let us treat $r = 4\omega$ first, since then we can compare it with the first nontrivial sector in the XXZ model.
This sector contains states, which consist of two interacting pseudoparticles. We seek these eigenstates in the form:

\[ |4\omega; \ldots \rangle_{\epsilon_1\epsilon_2} = \sum_{x_1 < x_2} \{ a_{\omega\omega}(x_1, x_2) |x_1[-\omega], x_2[-\omega]\rangle + \sum_i' b_{\omega i}(x_1, x_2) |x_1[-\omega], x_2[i,-i]\rangle + \sum_j b_{j\omega}(x_1, x_2) |x_1[i,-j], x_2[-\omega]\rangle + \sum_i' \sum_j b_{ij}(x_1, x_2) |x_1[i,-j], x_2[j,-j]\rangle \}. \tag{3.9} \]

Translational invariance now specifies \( a_{\omega\omega}(x_1, x_2) = \xi^{x_1} a_{\omega\omega}(n) \) and similarly for the other wave functions, where \( n = x_2 - x_1 \). Periodic boundary conditions require that

\[
\begin{align*}
    a_{\omega\omega}(n) &= \xi^n a_{\omega\omega}(N-n), \\
    b_{\omega i}(n) &= \xi^n b_{\omega i}(N-n), \\
    b_{ij}(n) &= \xi^n b_{ji}(N-n), \\
\end{align*}
\tag{3.10}
\]

where \( \xi = \xi_1 \xi_2 \) \( (\xi_i = e^{i\theta_i}, i = 1, 2) \) and the total momentum is \( \theta_1 + \theta_2 = 2\pi l/N \), with \( l \) integer.

According to equ.(3.8), we will parametrize the energy as

\[
E = \sum_{n=1}^2 \epsilon_n \left[ \sum_l q^{-<e_l,\rho>} + \Gamma(\xi_n)\Gamma(\xi_n^{-1}) \right]. \tag{3.11}
\]

Let us take the block \( \epsilon_1 = \epsilon_2 = 1 \) first. We try to build 2-pseudoparticle eigenstates out of translationally invariant products of 1-pseudoparticle excitations at \( x_1 \) and \( x_2 \) with weight functions \( D_i(x_1, x_2), i = 1, 2 \):

\[
\begin{align*}
    |4\omega; \theta_1, \theta_2\rangle_{11} &= \sum_{x_1 < x_2} \{ D_1(x_1, x_2)[\Gamma(\xi_1^{-1})|x_1[-\omega]\rangle + \sum_i' \epsilon(i)q^{-<e_i,\rho>}|x_1[i,-i]\rangle ] \\
    &+ \left[ \Gamma(\xi_2^{-1})|x_2[-\omega]\rangle + \sum_j \epsilon(j)q^{-<e_j,\rho>}|x_2[j,-j]\rangle \right] + \\
    &+ D_2(x_1, x_2)[\Gamma(\xi_2^{-1})|x_1[-\omega]\rangle + \sum_j \epsilon(j)q^{-<e_j,\rho>}|x_1[j,-j]\rangle ] \\
    &+ \left[ \Gamma(\xi_1^{-1})|x_2[-\omega]\rangle + \sum_i \epsilon(i)q^{-<e_i,\rho>}|x_2[i,-i]\rangle \right] \}. \\
\end{align*}
\]
Comparing this with equ. (3.9) and using translational invariance, implying
\[ D(n) = D_1(n) + D_2(n). \]

Applying \( H(q) \) to the state of (3.9), we obtain a set of coupled equations for \( a_{\omega\omega}(n), b_{ij}(n) \). Following [3], we split the equations into far equations, when excitations do not meet and near equations, containing terms when they are neighbors.

The far equations are:

\[
(E - 2q^{-2<\omega,\rho>} - 2q^{2<\omega,\rho>})a_{\omega\omega}(n) = (1 + \xi)^{-1}a_{\omega\omega}(n + 1) + (1 + \xi)a_{\omega\omega}(n - 1) + \\
\sum_l \epsilon(l)\epsilon(\omega)q^{-<e_l+e_{\omega,\rho}>}[\xi^{-1}b_{\omega\omega}(n + 1) + b_{\omega\omega}(n - 1)] + \\
\sum_l \epsilon(l)\epsilon(-\omega)q^{-<e_l-e_{\omega,\rho}>}[b_{\omega\omega}(n) + b_{\omega\omega}(n)], \quad 2 \leq n \leq N - 2,
\]

(3.13)

\[
(E - q^{-2<\omega,\rho>} - q^{2<\omega,\rho>})b_{ij}(n) = \xi^{-1}b_{ij}(n + 1) + \xi b_{ij}(n - 1) + \\
\epsilon(j)\epsilon(-\omega)q^{-<e_j-e_{\omega,\rho}>}a_{\omega\omega}(n) + \epsilon(j)\epsilon(\omega)q^{-<e_j+e_{\omega,\rho}>}a_{\omega\omega}(n + 1) + \\
\sum_l \epsilon(l)\epsilon^{-<e_{j,\rho}>}[\epsilon(\omega)q^{-<\omega,\rho>}\xi^{-1}b_{ij}(n + 1) + \epsilon(-\omega)q^{<\omega,\rho>}b_{ij}(n)] + \\
\epsilon(j)q^{-<e_{j,\rho}>}\xi^{-1}b_{\omega\omega}(n), \quad 2 \leq n \leq N - 2,
\]

(3.14)

\[
(E - q^{-2<\omega,\rho>} - q^{2<\omega,\rho>})b_{ij}(n) = b_{ij}(n - 1) + b_{ij}(n + 1) + \\
\epsilon(j)\epsilon(-\omega)q^{-<e_j-e_{\omega,\rho}>}a_{\omega\omega}(n) + \epsilon(j)\epsilon(\omega)q^{-<e_j+e_{\omega,\rho}>}\xi a_{\omega\omega}(n - 1) + \\
\sum_l \epsilon(l)\epsilon^{-<e_{j,\rho}>}[\epsilon(\omega)q^{-<\omega,\rho>}b_{ij}(n - 1) + \epsilon(-\omega)q^{<\omega,\rho>}b_{ij}(n)] + \epsilon(j)q^{-<e_{j,\rho}>}b_{\omega\omega}(n)] + \\
3 \leq n \leq N - 3,
\]

(3.15)

\[
E b_{ij}(n) = \epsilon(\omega)q^{-<\omega,\rho>}[\epsilon(i)q^{-<e_i,\rho>}\xi b_{\omega\omega}(n - 1) + \epsilon(j)q^{-<e_j,\rho>}b_{\omega\omega}(n + 1)] + \\
\epsilon(-\omega)q^{<\omega,\rho>}[\epsilon(i)q^{-<e_i,\rho>}b_{\omega\omega}(n) + \epsilon(j)q^{-<e_j,\rho>}b_{\omega\omega}(n)] +
\]

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\[
\sum_{l} \epsilon(l) q^{-\langle e_{l}, \rho \rangle} [\epsilon(i) q^{-\langle e_{i}, \rho \rangle} b_{ij}(n) + \epsilon(j) q^{-\langle e_{j}, \rho \rangle} b_{il}(n)],
\]
\[3 \leq n \leq N - 3. \tag{3.16}\]

We already know them to be satisfied, if we parametrize \(D_1(n)\) and \(D_2(n)\) by plane waves:
\[
D_1(n) = \xi_2^n, \quad D_2(n) = \xi_2^N \xi_1^n. \tag{3.17}\]

The real problem arises of course, when pseudoparticles are neighbors, so that they interact and we have no guarantee that the total energy is a sum of single pseudoparticle energies.

Let us now have a lightning review of the XXZ coordinate BA in order to be able to comment on the features, which are not going to survive generalizations to the present models.

The equations in the sector \(r_{XXZ} = 2\) are
\[
(E - 2q^{\frac{1}{2}} - 2q^{-\frac{1}{2}}) a(x_1, x_2) =
\]
\[a(x_1 + 1, x_2) + a(x_1 - 1, x_2) + a(x_1, x_2 + 1) + a(x_1, x_2 - 1), \tag{3.18}\]
if \(x_1\) and \(x_2\) are not neighbors. In case they are we get
\[
(E - q^{\frac{1}{2}} - q^{-\frac{1}{2}}) a(x_1, x_1 + 1) = a(x_1 - 1, x_1 + 1) + a(x_1, x_1 + 2). \tag{3.19}\]

One now supposes, that the parametrization equ.(3.9) for \(s = 1/2\) solves both the above equations. In this case we are allowed to set \(x_2 = x_1 + 1\) in equ.(3.18) and subtract it from equ.(3.20), yielding the following consistency condition:
\[
-(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) a(x_1, x_1 + 1) = a(x_1, x_1) + a(x_1 + 1, x_1 + 1). \tag{3.21}\]

This gives the \textit{BA equation} for the XXZ model, determining the 2-body phase shift:
\[
\xi_2^N = \frac{1 + \xi \xi_2(q^{\frac{1}{2}} + q^{-\frac{1}{2}})}{1 + \xi \xi_1(q^{\frac{1}{2}} + q^{-\frac{1}{2}})}. \tag{3.22}\]
This type of procedure only works for the spin 1/2 XXZ model, due to the following fact. When the two pseudoparticles come together in a configuration like \((\ldots+++-+++-\ldots)\) and when \(H\) is applied to the two down spins, it gives zero, since their total \(S_z\) equals \(-1\). But whenever two excitations approach each other becoming neighbors and the Hamiltonian applied to them yields a nonvanishing result, then the representation like equ.(3.12) cannot solve both the far and near equations. Yet this is exactly the situation arising for higher spins. As we shall see, in this case, the representation equ.(3.12) has to modified[11], the two-body wavefunction developing a ”discontinuity” at minimum separation. We call this the spin zero rule.

Now back to our problem to solve the near equations. They are:

\[
(E - q^{-2}\langle e_{\omega,\rho} \rangle - q^2\langle e_{\omega,\rho} \rangle) a_{\omega\omega}(1) = (1 + \xi)^{-1} a_{\omega\omega}(2) + \sum_l \epsilon(l)q^{-\langle e_l,\rho \rangle} \left[ \epsilon(\omega)q^{-\langle e_{\omega,\rho} \rangle} \xi^{-1} b_{l\omega}(2) + \epsilon(-\omega)q^{\langle e_{\omega,\rho} \rangle} b_{wl}(1) \right];
\]

\[
(E - q^{-2}\langle e_{\omega,\rho} \rangle) b_{\omega j}(1) = \xi^{-1} b_{\omega j}(2) + \epsilon(j)\epsilon(-\omega)q^{-\langle e_j - e_{\omega,\rho} \rangle} a_{\omega\omega}(1) + \epsilon(j)\epsilon(\omega)q^{-\langle e_j + e_{\omega,\rho} \rangle} a_{\omega\omega}(2) + \sum_l \epsilon(l)q^{-\langle e_l,\rho \rangle} \left[ \epsilon(\omega)q^{-\langle e_{\omega,\rho} \rangle} \xi^{-1} b_{lj}(2) + \epsilon(j)q^{-\langle e_j,\rho \rangle} b_{wl}(1) \right];
\]

\[
(E - q^2\langle e_{\omega,\rho} \rangle) b_{j\omega}(2) = b_{j\omega}(3) + \epsilon(j)\epsilon(-\omega)q^{-\langle e_j - e_{\omega,\rho} \rangle} a_{\omega\omega}(2) + \epsilon(j)\epsilon(\omega)q^{-\langle e_j + e_{\omega,\rho} \rangle} \xi a_{\omega\omega}(1) + \sum_l \epsilon(l)q^{-\langle e_l,\rho \rangle} \left[ \epsilon(-\omega)q^{\langle e_{\omega,\rho} \rangle} b_{jl}(2) + \epsilon(j)q^{-\langle e_j,\rho \rangle} b_{wl}(2) \right];
\]

\[
Eb_{ij}(2) = \epsilon(\omega)q^{-\langle e_{\omega,\rho} \rangle} \left[ \epsilon(i)q^{-\langle e_i,\rho \rangle} \xi b_{\omega j}(1) + \epsilon(j)q^{-\langle e_j,\rho \rangle} b_{\omega i}(3) \right] + \epsilon(-\omega)q^{\langle e_{\omega,\rho} \rangle} \left[ \epsilon(i)q^{-\langle e_i,\rho \rangle} b_{\omega j}(2) + \epsilon(j)q^{-\langle e_j,\rho \rangle} b_{\omega i}(2) \right] + \sum_l \epsilon(l)q^{-\langle e_l,\rho \rangle} \left[ \epsilon(i)q^{-\langle e_i,\rho \rangle} b_{lj}(2) + \epsilon(j)q^{-\langle e_j,\rho \rangle} b_{li}(2) \right],
\]

\(i \neq j;\)
\[ Eb_{li}(2) = \epsilon(i)\epsilon(\omega)q^{-<\epsilon_i+\epsilon_\omega,\rho>}[\xi b_{\omega i}(1) + b_{\omega i}(3)] + \]
\[ \epsilon(i)\epsilon(-\omega)q^{-<\epsilon_i-\epsilon_\omega,\rho>}[b_{\omega i}(2) + b_{\omega i}(2)] + \sum' \epsilon(l)q^{-<\epsilon_l+\epsilon_\rho,\rho>}[b_{li}(2) + b_{li}(2)] + \]
\[ \sum' \epsilon(-i)\epsilon(l)q^{-<\epsilon_l-\epsilon_i,\rho>}B_i^{(l)}. \quad (3.23) \]

Here some new states are showing up. \( B_i^{(l)} \) are the wavefunctions of the states of the type \( \ldots \omega_l^i l - l \omega_l \ldots \), \( l \neq i \). Applying \( H(q) \) to them we obtain the system:

\[ (E - q^2<\epsilon_i,\rho>)B_i^{(l)} = \sum_{j \in \mathcal{J}, j \neq -i} \epsilon(l)\epsilon(j)q^{<\epsilon_l+\epsilon_j,\rho>}B_i^{(l)} + \epsilon(l)\epsilon(-i)q^{-<\epsilon_l-\epsilon_i,\rho>}b_{li}(2), \quad (3.24) \]

yielding

\[ B_i^{(l)} = \frac{\epsilon(l)\epsilon(-i)q^{-<\epsilon_l-\epsilon_i,\rho>}}{(E - \sum_{j \in \mathcal{J}, j \neq -i} q^{-2<\epsilon_l,\rho>})}b_{li}(2). \quad (3.25) \]

Eliminating \( B_i^{(l)} \) from equation (3.23), we get

\[ \frac{E(E + 2\Delta)}{E + 2\Delta + q^{<\epsilon_i,\rho>}} b_{li}(2) = \epsilon(i)\epsilon(\omega)q^{-<\epsilon_i+\epsilon_\omega,\rho>}[\xi b_{\omega i}(1) + b_{\omega i}(3)] + \]
\[ \epsilon(i)\epsilon(-\omega)q^{-<\epsilon_i-\epsilon_\omega,\rho>}[b_{\omega i}(2) + b_{\omega i}(2)] + \]
\[ \sum' \epsilon(l)\epsilon(i)q^{-<\epsilon_l+\epsilon_i,\rho>}[b_{li}(2) + b_{li}(2)]. \quad (3.26) \]

In order to solve these equations, we follow [11] and now leave the value of the wavefunctions for nearest separation as arbitrary parameters:

\[ a_{\omega i}(1) = \Gamma(\xi)\Gamma(\xi^{-1})D(1) + \mathcal{F}_{a_{\omega i}}(1), \]
\[ b_{\omega i}(1) = \epsilon(i)q^{-<\epsilon_i,\rho>}[\Gamma(\xi^{-1})D_1(1) + \Gamma(\xi^{-1})D_2(1)] + \mathcal{F}_{b_{\omega i}}(1), \]
\[ b_{\omega i}(2) = \epsilon(i)q^{-<\epsilon_i,\rho>}[\Gamma(\xi^{-1})D_1(2) + \Gamma(\xi^{-1})D_2(2)] + \mathcal{F}_{b_{\omega i}}(2), \]
\[ b_{ij}(2) = \epsilon(i)\epsilon(j)q^{-<\epsilon_l+\epsilon_\rho,\rho>}D(2) + \mathcal{F}_{b_{ij}}(2). \quad (3.27) \]

In order for this modification to leave the far equations still satisfied, the following conditions have to hold:

\[ (1 + \xi^{-1})\mathcal{F}_{a_{\omega l}}(1) + \sum' \epsilon(l)\epsilon(\omega)q^{-<\epsilon_l+\epsilon_\omega,\rho>}\mathcal{F}_{b_{\omega l}}(1) \]
\[
\sum_l \epsilon(l)\epsilon(-\omega)q^{-\langle e_l-e_\omega,\rho \rangle} F_{b_l}(2) = 0, \\
\xi F_{b_j}(1) + \sum_l \epsilon(l)\epsilon(-\omega)q^{-\langle e_l-e_\omega,\rho \rangle} F_{b_l}(2) = 0, \\
F_{b_j}(2) + \sum_l \epsilon(l)\epsilon(\omega)q^{-\langle e_l+e_\omega,\rho \rangle}, F_{b_l}(2) = 0. \quad (3.28)
\]

Now using equs. (3.12) and equ. (3.27) in equ. (3.26), we get the following equation for \( F_{b_i}(2) \):

\[
F_{b_i}(2) = \frac{D(2)}{E+2\Delta}, \quad i \in J^*. \quad (3.29)
\]

Doing the same with equ.(3.23), we get:

\[
F_{b_{ij}}(2) = 0, \quad i \neq j. \quad (3.30)
\]

These results for \( F_{b_i}(2), F_{b_{ij}}(2) \) are reasonable. In the first case the colliding excitations satisfy the zero spin rule and we get a non-zero result, whereas in the second case the rule is not satisfied and we get zero. Using this in equ.(3.28), we obtain for the remaining constants:

\[
F_{a_\omega}(1) = -(q^{-2<e_\omega,\rho>} + q^{2<e_\omega,\rho>} + 2\Delta) F_{b_i}(2), \\
F_{b_j}(1) = -\epsilon(j)\epsilon(-\omega)q^{-\langle e_j-e_\omega,\rho \rangle} F_{b_i}(2), \\
F_{b_{j\omega}}(2) = -\epsilon(j)\epsilon(\omega)q^{-\langle e_j+e_\omega,\rho \rangle} F_{b_i}(2), \quad j \in J^*. \quad (3.31)
\]

Substituting finally the complete parametrization into the remaining near equations, get the following Bethe-Ansatz equation:

\[
\frac{D(2)}{E+2\Delta} = \frac{\xi}{1+\xi} D(1), \quad (3.32)
\]

which can also be rewritten as

\[
\xi^N_2 = -\frac{\xi_2[(1+\xi^{-1})\xi_2 - 2\Delta - E]}{\xi_1[(1+\xi^{-1})\xi_1 - 2\Delta - E]}, \quad (3.33)
\]

Using the explicit form of the energy, the set of equations determining the spectrum are:

\[
\xi^N_2 = -\frac{1+\xi\xi_2 - 2\Delta\xi_2}{1+\xi\xi_1 - 2\Delta\xi_1}, \quad \xi^N = 1. \quad (3.34)
\]
Notice that this equation is independent of \( n \) or any other representation specific quantities. All the models considered show therefore an equivalent spectrum, when parametrized in terms of \( \Delta \).

In particular, this is the same consistency condition one finds for the XXZ model, showing that for \( \epsilon_1 = \epsilon_2 = 1 \), even for periodic boundary conditions, the spectra of all our models are equivalent to the spectrum of the XXZ model, if expressed in terms of \( \Delta \).

We will refrain from discussing the other two blocks: \( \epsilon_1 = 1, \epsilon_2 = 0 \) and \( \epsilon_1 = \epsilon_2 = 0 \), since the calculations are analogous to the ones presented above. For details the reader might consult reference[11]. Anyhow, the first of the two cases doesn’t show up for the more interesting situation of free boundary conditions. It is too asymmetric to satisfy free boundary conditions. This is the reason, why for free boundary conditions, the spectrum of all of our models is equivalent to the XXZ spectrum. The block \( \epsilon_1 = \epsilon_2 = 0 \) has \( E = 0 \) and the BA equation reduces to \( \xi^N = 1 \), being highly degenerate. The eigenvalue \( E = 0 \) also occurs in the XXZ spectrum, albeit with different degeneracy.

### 3.3 One pseudoparticle and impurities

Since the setup with pseudoparticles and impurities is a little different from the case of two pseudoparticles, we will dedicate some space to it.

The eigenstates sought for, will be like:

\[
|2\omega + 1; \ldots\rangle = \sum_{x_1 < x_2} \{ a_{\omega k}(x_1, x_2) |x_1[-\omega], x_2[k]\rangle + \sum_i^i b_{jk}(x_1, x_2) |x_1[j, -j], x_2[k]\rangle + \\
 a_{k\omega}(x_1, x_2) |x_1[-k], x_2[-\omega]\rangle + \sum_j^j b_{kj}(x_1, x_2) |x_1[k], x_2[j, -j]\rangle \}.
\]

Translational invariance and periodic boundary conditions impose:

\[
a_{\omega k}(x_1, x_2) = \xi^{x_1} a_{\omega k}(n), \quad b_{jk}(x_1, x_2) = \xi^{x_1} b_{jk}(n),
\]

(3.35)
\[ a_{\omega k}(n) = \xi^n a_{k\omega}(N-n), \quad b_{jk}(n) = \xi^n b_{kj}(N-n), \] (3.37)

where \( n = x_2 - x_1, \) \( \xi = \xi_1 \xi_2 \) \((\xi_i = e^{i\theta_i}, \ i = 1, 2)\) and the total momentum is \( \theta_1 + \theta_2 = 2\pi l/N, \) with \( l \) integer.

Let us take the block \( \epsilon_1 = 1 \), building eigenstates out of translationally invariant products of 1-pseudoparticle excitations at \( x_1 \) and an impurity at \( x_2 \) with weight functions \( D_i(x_1, x_2), \ i = 1, 2 \) as in the previous section. This yields the parametrizations:

\[
\begin{align*}
a_{\omega k}(n) &= \Gamma(\xi_1^{-1}) D_1(n), \\
a_{k\omega}(n) &= \Gamma(\xi_1^{-1}) D_2(n), \\
b_{jk}(n) &= \epsilon(j) q^{-<\epsilon_j, \rho>} D_1(n), \\
b_{kj}(n) &= \epsilon(j) q^{-<\epsilon_j, \rho>} D_2(n).
\end{align*}
\] (3.38)

The far equations for the impurity at the right are now:

\[
\begin{align*}
(E - 2q^{-2<\epsilon_\omega, \rho>} - 2q^{2<\epsilon_\omega, \rho>}) a_{\omega k}(x_1, x_2) &= a_{\omega k}(x_1 - 1, x_2) + a_{\omega k}(x_1 + 1, x_2) + \\
\sum_l \epsilon(l) \epsilon(\omega) q^{-<\epsilon_l + \epsilon_\omega, \rho>} b_{lk}(x_1 - 1, x_2) + \sum_l \epsilon(l) \epsilon(-\omega) q^{-<\epsilon_l - \epsilon_\omega, \rho>} b_{lk}(x_1, x_2), \\
x_1 + 2 \leq x_2 \leq N - x_1 - 2, \\
E b_{jk}(x_1, x_2) &= \epsilon(j) \epsilon(\omega) q^{-<\epsilon_j + \epsilon_\omega, \rho>} a_{\omega k}(x_1 + 1, x_2) + \\
\epsilon(j) \epsilon(-\omega) q^{-<\epsilon_j - \epsilon_\omega, \rho>} a_{\omega k}(x_1, x_2) + \\
\sum_l \epsilon(j) \epsilon(l) q^{-<\epsilon_j + \epsilon_l, \rho>} [\epsilon(i) q^{-<\epsilon_i, \rho>} b_{lk}(x_1, x_2), \\
x_1 + 3 \leq x_2 \leq N - x_1 - 3,
\end{align*}
\] (3.39)

and analogous equations for the impurity at the left. Eliminating the \( b \)-functions, we get:

\[
\begin{align*}
(E - \sum_{l \in J} q^{-2<\epsilon_l, \rho>}) a_{\omega k}(x_1, x_2) &= a_{\omega k}(x_1 - 1, x_2) + a_{\omega k}(x_1 + 1, x_2), \\
(E - \sum_{l \in J} q^{-2<\epsilon_l, \rho>}) a_{k\omega}(x_1, x_2) &= a_{k\omega}(x_1, x_2 - 1) + a_{k\omega}(x_1, x_2 + 1), \\
x_1 + 3 \leq x_2 \leq N - x_1 - 3.
\end{align*}
\] (3.41)
We know them to be satisfied, if the energy is given by equ.(3.11). The near equations require of course the by now costumary treatment of modifying the ansatz of the wavefunctions at nearest separations.

The near equations for the impurity at the right are:

\[
(E - 2q^{-2<\epsilon_\omega,\rho>})a_{\omega k}(x, x + 1) = \\
a_{\omega k}(x - 1, x + 1) + \sum_l \epsilon(l)\epsilon(\omega)q^{-<\epsilon_l + \epsilon_\omega,\rho>}b_{lk}(x - 1, x + 1),
\]

\[
Eb_{jk}(x, x + 1) = \epsilon(j)\epsilon(\omega)q^{-<\epsilon_j + \epsilon_\omega,\rho>}a_{\omega k}(x + 1, x + 2) + \\
\epsilon(j)\epsilon(-\omega)q^{-<\epsilon_j - \epsilon_\omega,\rho>}a_{\omega j}(x, x + 2) + \sum_l \epsilon(j)\epsilon(l)q^{-<\epsilon_l + \epsilon_\omega,\rho>}b_{lk}(x, x + 2),
\]

\[
Eb_{jj}(x, x + 2) = \\
\epsilon(j)\epsilon(\omega)q^{-<\epsilon_j + \epsilon_\omega,\rho>}a_{\omega j}(x + 1, x + 2) + \epsilon(-j)\epsilon(\omega)q^{-<\epsilon_j - \epsilon_\omega,\rho>}a_{\omega j}(x, x + 2) + \\
\sum_l \epsilon(j)\epsilon(l)q^{-<\epsilon_l + \epsilon_\omega,\rho>}b_{lj}(x, x + 2) + \sum_l \epsilon(-j)\epsilon(l)q^{-<\epsilon_l - \epsilon_\omega,\rho>}b_{lj}(x, x + 2),
\]

They can be solved modifying the parametrization for nearest neighbors in the usual way. The result is:

\[
a_{\omega k}(1) = \Gamma(\xi^{-1})\xi_2 + \mathcal{F}_{a_{\omega k}}, \quad b_{k}(2) = \epsilon(j)q^{-<\epsilon_j,\rho>}\xi_2^2 + \mathcal{F}_{b_{k}},
\]

\[
a_{k\omega}(1) = \Gamma(\xi^{-1})\xi_2^N\xi_1 + \mathcal{F}_{a_{k\omega}}, \quad b_{kj}(1) = \epsilon(j)q^{-<\epsilon_j,\rho>}\xi_2^N\xi_1 + \mathcal{F}_{b_{kj}},
\]

where

\[
\mathcal{F}_{a_{k\omega}} = -\epsilon(\omega)q^{-<\epsilon_\omega,\rho>}\xi_2^N, \quad \mathcal{F}_{b_{kj}} = \epsilon(k)q^{-<\epsilon_k,\rho>}\xi_2^N\delta_{k+j,0},
\]

\[
\mathcal{F}_{a_{\omega k}} = -\epsilon(-\omega)q^{-<\epsilon_\omega,\rho>}\xi_2, \quad \mathcal{F}_{b_{kj}} = \epsilon(-k)q^{-<\epsilon_k,\rho>}\xi_2\delta_{k-j,0},
\]

with $\xi^N = 1, \xi_1^{N-2}\xi_2 = 1$ resulting from periodic boundary conditions.
4 Free boundary conditions

It is for free boundary conditions, that the Hamiltonian $H(Q)$ commutes with the quantum group $\mathcal{U}_q(X_n)$. As expected, the Bethe states are highest weight states of $\mathcal{U}_q(X_n)$, except some $E = 0$ states. Since the extension of the BA-procedure from the periodic boundary conditions to the free case, follows exactly the lines of ref. [11], we will only state the results for the sector $r = 4$.

Take the block $\epsilon_1 = \epsilon_2 = 1$. The nearest approach constants to be added to the now standing waves are the same as in the periodic case, namely equs. (3.30), (3.31), only $F_{bii}(2)$ is different [11]). The BA equations are now:

$$\xi^2_{2N} = \prod_{b=1, b\neq a}^{r} \frac{b(\xi^{-1}_a, \xi_b)}{b(\xi_a, \xi_b)}, \quad a = 1, 2,$$

where

$$b(\xi_a, \xi_b) = \frac{\xi_b}{\xi_a} [\xi_b + \xi^{-1}_a - 2\Delta - E_{ab}] [\xi^{-1}_b + \xi^{-1}_a - 2\Delta - E_{ab}].$$

and

$$E_{ab} = 2 \sum_{l} q^{-x_l} + \Gamma(\xi_a)\Gamma(\xi^{-1}_a) + \Gamma(\xi_b)\Gamma(\xi^{-1}_b).$$

The only other block is $\epsilon_1 = \epsilon_2 = 0, E = 0$. It is again highly degenerate with $\mathcal{F}_{bii}(2), \xi_1, \xi_2$ as free parameters[1].

Thus all models have \textit{spectra equivalent to the one of the XXZ model}.

5 Conclusion

We obtained the spectra of quantum spin chain models, arising as representations of the Temperley-Lieb algebra associated with quantum groups. The tool is a modified version of the coordinate Bethe Ansatz, since the simpler algebraic Bethe Ansatz is not immediately available for these models. We find that all models have equivalent

\[\text{\footnote{Actually there is one more free parameter, called $\alpha_5$ in ref. [1].}}\]
spectra, i.e. they differ at most in their degeneracies. The energy eigenvalues are given by

\[ E = \sum_{n=1}^{p} (\epsilon_n (-2 \Delta + \Gamma(\xi_n)\Gamma(\xi_n^{-1}))), \]  

(5.1)

where \(-2\Delta = \sum_i q^{-<\epsilon_i,\rho>}\) and the rapidities \(\xi_n\) are solutions of the BA equations.

In the sector \(r\) we may have \(p\) pseudoparticles \(N_{\omega^*-1}, N_{\omega^*-1}, \ldots, N_{-\omega^*+1}\) impurities of the type \((\omega^*-1), (\omega^*-2), \ldots, (-\omega^*+1)\), respectively, such that

\[ N_{\omega^*-1} + 2N_{\omega^*-1} + \cdots + (2\omega^*-1)N_{-\omega^*+1} = r - 2\omega^*p. \]  

(5.2)

Here \(\omega^* = \omega\) for \(A_1\) and \(B_n\) and \(\omega^* = \omega - 1/2\) for \(C_n\) and \(D_n\).

For example, for periodic boundary conditions\(^\dagger\), the total rapidity \(\xi = \xi_1\xi_2\cdots\xi_p\xi_{imp}\), \(\xi_{imp} = \xi_{p+1}\xi_{p+2}\cdots\xi_{r-\sigma}\) obeys \(\xi^N = 1\) and the BA equations for \(E \neq 0\) are:

\[ \xi_a^N = \prod_{b=1, b\neq a}^{r-\sigma} \frac{\xi_a (1 + (\xi_a\xi_b)^{-1}\xi_a - \epsilon_{ab})}{\xi_b (1 + (\xi_a\xi_b)^{-1}\xi_b - \epsilon_{ab})}, \]  

(5.3)

where \(\epsilon_{ab} = E_{ab} + 2\Delta\) and \(\sigma\) can be fixed for each allowed case from the equations (5.2). The Bethe eigenstates are highest weight states of the quantum group, except for the states with energy \(E = 0\), for which this is not always true. In the sector \(r\) with \(p\) impurities and \(E \neq 0\), the total rapidity \(\xi = \xi_1\xi_2\cdots\xi_p\xi_{imp}\), \(\xi_{imp} = \xi_{p+1}\xi_{p+2}\cdots\xi_{r-p}\) obeys \(\xi^N = 1\) and the BA equations are:

\[ \xi_a^N \xi_{imp}^2 = \prod_{b=1, b\neq a} \frac{\xi_a (1 + (\xi_a\xi_b)^{-1}\xi_a - \epsilon)}{\xi_b (1 + (\xi_a\xi_b)^{-1}\xi_b - \epsilon)}, \]  

(5.4)

where \(\epsilon = E + 2\Delta\). The Bethe eigenstates are highest weight states of the quantum group, except for the states with energy \(E = 0\), for which this is not always true. These results are expected, but as far as we know, unproven using only Temperley-Lieb algebraic statements as input.

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\(^\dagger\)We don’t list the equations for free boundary conditions, since they are identical to those of ref.\(^{[1]}\).
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