On $W_0$ and $W_2 \phi$-Symmetric Contact Manifold Admitting Quarter-Symmetric Metric Connection

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Abstract. The paper deals locally $W_0$ and $W_2$ curvature tensor of $\phi$-symmetric $K$-contact manifolds with quarter-symmetric metric connection and some results are obtained.

1. Introduction
In 1970, K. Yano [23] studied conditions of curvatures for semi-symmetric connections in Riemannian manifolds. S. Golab [6] in 1975, defined and studied quarter-symmetric connection in a differentiable manifold with affine connection. In 1977, T. Takahashi [17], has introduced the notion of locally $\phi$-symmetry on Sasakian manifolds.

A linear connection $\tilde{\nabla}$ in an $n$-dimensional differentiable manifold is said to be a quarter-symmetric connection [6] if its torsion tensor $T$ is of the form

$$
T(\Gamma, \Lambda) = \tilde{\nabla}_\Gamma \Lambda - \tilde{\nabla}_\Lambda \Gamma - [\Gamma, \Lambda] = \eta(\Lambda)\phi \Gamma - \eta(\Gamma)\phi \Lambda,
$$

If $\phi \Gamma = \Gamma$, the quarter-symmetric connection reduces to the semi-symmetric connection [5].

2. Preliminaries
A differentiable manifold $M$ of dimensional-$n$ is said to have an almost contact structure $(\phi, \xi, \eta)$ if it carries a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ on manifold $M$ respectively such that,

$$
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi \xi = 0.
$$

Thus with this structure a manifold $M$ equipped is called an almost contact manifold and is denoted by $(M, \phi, \xi, \eta)$. A Riemannian metric $g$ on an almost contact manifold $M$ such that,

$$
g(\phi \Gamma, \phi \Lambda) = g(\Gamma, \Lambda) - \eta(\Gamma)\eta(\Lambda),
g(\Gamma, \xi) = \eta(\Gamma),
g(\Gamma, \phi \Lambda) = -g(\phi \Gamma, \Lambda),
$$
Here, $\Gamma, \Lambda$ are vector fields defined on manifold $M$. If $\xi$ is a killing vector field, then manifold is called a $K$-contact Riemannian manifold ([1], [14]). In a $K$-contact manifold $M$, the following relations hold:

$$\nabla_\Gamma \xi = -\phi \Gamma,$$

$$g(R(\xi, \Gamma)\Lambda, \xi) = g(\Gamma, \Lambda) - \eta(\Gamma)\eta(\Lambda),$$

$$R(\xi, \Gamma)\xi = -\Gamma + \eta(\Gamma)\xi,$$

$$S(\Gamma, \xi) = (n - 1)\eta(\Gamma),$$

for any vector fields $\Gamma, \Lambda$ and $\Upsilon$. Where $R$ is the Riemannian curvature tensor and $S$ is the Ricci tensor of manifold $M$.

A manifold $M$ is said to be locally $\phi$-symmetric if

$$\phi^2((\nabla_\Omega R)(\Gamma, \Lambda)\Upsilon) = 0,$$

for vector fields $\Gamma, \Lambda, \Upsilon$ and $\Omega$ orthogonal to $\xi$. The notion was first introduced by Takahashi [17].

A manifold $M$ is said to be $\phi$-symmetric if

$$\phi^2((\nabla_\Omega R)(\Gamma, \Lambda)\Upsilon) = 0,$$

A $K$-contact manifold $M$ is said to be locally $W_0$ and $W_2$ $\phi$-symmetric if

$$\phi^2((\nabla_\Omega W_0)(\Gamma, \Lambda)\Upsilon) = 0,$$

$$\phi^2((\nabla_\Omega W_2)(\Gamma, \Lambda)\Upsilon) = 0,$$

for $\Gamma, \Lambda, \Upsilon$ and $\Omega$ orthogonal to $\xi$, where $W_0$ and $W_2$ curvature tensors respectively given by [18]

$$W_0(\Gamma, \Lambda)\Upsilon = R(\Gamma, \Lambda)\Upsilon - \frac{1}{n - 1}[S(\Lambda, \Upsilon)\Gamma - g(\Gamma, \Upsilon)Q\Lambda],$$

$$W_2(\Gamma, \Lambda)\Upsilon = R(\Gamma, \Lambda)\Upsilon - \frac{1}{n - 1}[g(\Lambda, \Upsilon)Q\Gamma - g(\Gamma, \Upsilon)Q\Lambda].$$

A quarter-symmetric metric connection $\tilde{\nabla}$ in a $K$-contact manifold is given by [12]

$$\tilde{\nabla}_\Gamma \Lambda = \nabla_\Gamma \Lambda - \eta(\Gamma)\phi \Lambda.$$

A relation between the curvature tensor of manifold with quarter-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection $\nabla$ is given by

$$\tilde{R}(\Gamma, \Lambda)\Upsilon = R(\Gamma, \Lambda)\Upsilon - 2g(\Gamma, \phi \Lambda)\phi \Upsilon + [\eta(\Gamma)g(\Lambda, \Upsilon)$$

$$- \eta(\Lambda)g(\Gamma, \Upsilon)]\xi + [\eta(\Lambda)\Gamma - \eta(\Gamma)\Lambda]$$

$$\eta(\Upsilon),$$

where $\tilde{R}$ and $R$ are curvature of $\tilde{\nabla}$ and $\nabla$ connections respectively.

From (12), we have

$$\tilde{S}(\Lambda, \Upsilon) = S(\Lambda, \Upsilon) - g(\Lambda, \Upsilon) + n \eta(\Lambda)\eta(\Upsilon),$$

where $\tilde{S}$ and $S$ are the Ricci tensors of $\tilde{\nabla}$ and $\nabla$ connections respectively.

On contraction of (13), it follows that

$$\tilde{r} = r,$$

here $\tilde{r}$ and $r$ are the scalar curvatures of $\tilde{\nabla}$ and $\nabla$ connections respectively.
3. Locally $\phi$-symmetric quarter-symmetric metric connection

A locally $\phi$-symmetric $K$-contact manifold with quarter-symmetric metric connection is given by

$$\phi^2((\tilde{\nabla}_\Omega \tilde{R})(\Gamma, \Lambda)\Upsilon) = 0,$$

for vector fields $\Gamma, \Lambda, \Upsilon$ and $\Omega$ orthogonal to $\xi$.

From (11) we have

$$\tilde{\nabla}_\Omega \tilde{R}(\Gamma, \Lambda)\Upsilon = (\nabla_\Omega R)(\Gamma, \Lambda)\Upsilon - \eta(\Omega)\phi \tilde{R}(\Gamma, \Lambda)\Upsilon$$

$$+ \eta(\Omega)\{\tilde{R}(\phi \Gamma, \Lambda)\Upsilon + \tilde{R}(\Gamma, \phi \Lambda)\Upsilon + \tilde{R}(\Gamma, \Lambda)\phi \Upsilon\}.$$  

Differentiating (12), we obtain

$$\nabla_\Omega \tilde{R}(\Gamma, \Lambda)\Upsilon = (\nabla_\Omega R)(\Gamma, \Lambda)\Upsilon + 2\eta(\Lambda)\phi \tilde{R}(\Gamma, \Lambda)\Upsilon$$

$$+ \phi \tilde{R}(\Gamma, \Lambda)\phi \Upsilon + (\eta(\Lambda)g(\Omega, \phi \Upsilon) - \eta(\Gamma)g(\Omega, \phi \Upsilon))\phi \Upsilon + (\eta(\Lambda)g(\Omega, \Upsilon) - \eta(\Gamma)g(\Omega, \Upsilon))\phi \Upsilon.$$  

Now applying $\phi^2$ and considering $\Gamma, \Lambda, \Upsilon$ and $\Omega$ orthogonal to $\xi$ in (16), we obtain

$$\phi^2(\tilde{\nabla}_\Omega \tilde{R}(\Gamma, \Lambda)\Upsilon) = \phi^2((\nabla_\Omega R)(\Gamma, \Lambda)\Upsilon).$$

Hence we state a $K$-Contact manifold is locally $\phi$-symmetric with quarter-symmetric metric connection $\nabla$ if and only if it is so with Levi-Civita connection $\nabla$.

4. Locally $W_0$ $\phi$-symmetric $K$-contact manifold

A $K$-contact manifold $M$ is said to be a locally $W_0$ $\phi$-symmetric with quarter-symmetric metric connection if

$$\phi^2((\tilde{\nabla}_\Omega \tilde{W}_0)(\Gamma, \Lambda)\Upsilon) = 0,$$

for $\Gamma, \Lambda, \Upsilon$ and $\Omega$ orthogonal to $\xi$, where $\tilde{W}_0$ is the $W_0$ curvature tensor with quarter-symmetric metric connection given by

$$\tilde{W}_0(\Gamma, \Lambda)\Upsilon = \tilde{R}(\Gamma, \Lambda)\Upsilon - \frac{1}{(n-1)}[\tilde{S}(\Lambda, \Upsilon)\Gamma - g(\Gamma, \Upsilon)\Omega \Lambda],$$

From (11) we can write

$$\tilde{\nabla}_\Omega \tilde{W}_0(\Gamma, \Lambda)\Upsilon = (\nabla_\Omega \tilde{W}_0)(\Gamma, \Lambda)\Upsilon - \eta(\Omega)\phi \tilde{W}_0(\Gamma, \Lambda)\Upsilon$$

$$+ \eta(\Omega)\{\tilde{W}_0(\phi \Gamma, \Lambda)\Upsilon + \tilde{W}_0(\Gamma, \phi \Lambda)\Upsilon + \tilde{W}_0(\Gamma, \Lambda)\phi \Upsilon\}.$$  

Differentiating (20) with respect to $\Omega$, we obtain

$$(\nabla_\Omega \tilde{W}_0)(\Gamma, \Lambda)\Upsilon = (\nabla_\Omega \tilde{R})(\Gamma, \Lambda)\Upsilon$$

$$- \frac{1}{(n-1)}[(\nabla_\Omega \tilde{S})(\Lambda, \Upsilon)\Gamma].$$
Use of (17) and (13) in (22), we have
\[
(\nabla_{\Omega}W_0)(\Gamma, \Lambda)\phi = (\nabla_{\Omega}R)(\Gamma, \Lambda)\phi + 2[\eta(\Lambda)g(\Gamma, \Omega) - \eta(\Gamma)g(\Omega, \Lambda)]\phi \phi + [g(\Omega, \phi \Lambda)g(\Lambda, \phi \Lambda) - g(\Omega, \phi \Lambda)g(\Lambda, \phi \Lambda)]\phi \phi + [g(\Gamma, \phi \Lambda)g(\Gamma, \phi \Lambda) + g(\Omega, \phi \Lambda)]\phi \phi - g(\Omega, \phi \Lambda)\eta(\phi) + [\eta(\Lambda)g(\Omega, \phi \phi)\phi \phi + g(\Omega, \phi \Lambda)\Lambda] - \frac{1}{(n-1)}[n\eta(\Lambda)g(\Omega, \phi \phi)\phi \phi + n\eta(\Gamma)g(\Gamma, \phi \phi)\phi \phi + n\eta(\Lambda)g(\Lambda, \phi \phi)\phi \phi + n\eta(\Gamma)g(\Gamma, \phi \phi)\phi \phi].
\]

Taking account of (9), we write (23) as
\[
(\nabla_{\Omega}W_0)(\Gamma, \Lambda)\phi = (\nabla_{\Omega}W_0)(\Gamma, \Lambda)\phi + 2[\eta(\Lambda)g(\Gamma, \Omega) - \eta(\Gamma)g(\Omega, \Lambda)]\phi \phi + [g(\Omega, \phi \Lambda)g(\Lambda, \phi \Lambda) - g(\Omega, \phi \Lambda)g(\Lambda, \phi \Lambda)]\phi \phi + [g(\Gamma, \phi \Lambda)g(\Gamma, \phi \Lambda) + g(\Omega, \phi \Lambda)]\phi \phi - g(\Omega, \phi \Lambda)\eta(\phi) + [\eta(\Lambda)g(\Omega, \phi \phi)\phi \phi + g(\Omega, \phi \Lambda)\Lambda] - \frac{1}{(n-1)}[n\eta(\Lambda)g(\Omega, \phi \phi)\phi \phi + n\eta(\Gamma)g(\Gamma, \phi \phi)\phi \phi + n\eta(\Lambda)g(\Lambda, \phi \phi)\phi \phi + n\eta(\Gamma)g(\Gamma, \phi \phi)\phi \phi].
\]

Now applying (2) and (24) in (21), we have
\[
\phi^2(\nabla_{\Omega}W_0)(\Gamma, \Lambda)\phi = \phi^2(\nabla_{\Omega}W_0)(\Gamma, \Lambda)\phi + 2[\eta(\Lambda)g(\Gamma, \Omega) - \eta(\Gamma)g(\Omega, \Lambda)]\phi \phi + [g(\Omega, \phi \Lambda)g(\Lambda, \phi \Lambda) - g(\Omega, \phi \Lambda)g(\Lambda, \phi \Lambda)]\phi \phi + [g(\Gamma, \phi \Lambda)g(\Gamma, \phi \Lambda) + g(\Omega, \phi \Lambda)]\phi \phi - g(\Omega, \phi \Lambda)\eta(\phi) + [\eta(\Lambda)g(\Omega, \phi \phi)\phi \phi + g(\Omega, \phi \Lambda)\Lambda] - \frac{1}{(n-1)}[n\eta(\Lambda)g(\Omega, \phi \phi)\phi \phi + n\eta(\Gamma)g(\Gamma, \phi \phi)\phi \phi + n\eta(\Lambda)g(\Lambda, \phi \phi)\phi \phi + n\eta(\Gamma)g(\Gamma, \phi \phi)\phi \phi].
\]

If we consider \( \Gamma, \Lambda, \phi \phi \phi \) orthogonal to \( \xi \), (25) reduces to
\[
\phi^2(\nabla_{\Omega}W_0)(\Gamma, \Lambda)\phi = \phi^2(\nabla_{\Omega}W_0)(\Gamma, \Lambda)\phi.
\]

Hence A K-contact manifold is locally \( W_0 \) \phi-symmetric with \( \tilde{\nabla} \) if and only if it is so with Levi-Civita connection \( \nabla \).

Next from (2) and (23) in (21) then we get
\[
\phi^2(\nabla_{\Omega}W_0)(\Gamma, \Lambda)\phi = \phi^2(\nabla_{\Omega}R)(\Gamma, \Lambda)\phi + 2[\eta(\Lambda)g(\Gamma, \Omega) - \eta(\Gamma)g(\Omega, \Lambda)]\phi \phi + [g(\Omega, \phi \Lambda)g(\Lambda, \phi \Lambda) - g(\Omega, \phi \Lambda)g(\Lambda, \phi \Lambda)]\phi \phi + [g(\Gamma, \phi \Lambda)g(\Gamma, \phi \Lambda) + g(\Omega, \phi \Lambda)]\phi \phi - g(\Omega, \phi \Lambda)\eta(\phi) + [\eta(\Lambda)g(\Omega, \phi \phi)\phi \phi + g(\Omega, \phi \Lambda)\Lambda] - \frac{1}{(n-1)}[n\eta(\Lambda)g(\Omega, \phi \phi)\phi \phi + n\eta(\Gamma)g(\Gamma, \phi \phi)\phi \phi + n\eta(\Lambda)g(\Lambda, \phi \phi)\phi \phi + n\eta(\Gamma)g(\Gamma, \phi \phi)\phi \phi].
\]

Taking \( \Gamma, \Lambda, \phi \phi \phi \) orthogonal to \( \xi \) in (27) followed by a simplification we get
\[
\phi^2(\nabla_{\Omega}W_0)(\Gamma, \Lambda)\phi = \phi^2(\nabla_{\Omega}R)(\Gamma, \Lambda)\phi.
\]

Thus we can state that if \( M \) is \phi-symmetric with quarter-symmetric metric connection then a K-contact manifold is locally \( W_0 \) \phi-symmetric with quarter-symmetric metric connection \( \tilde{\nabla} \) if and only if it is locally \phi-symmetric with Levi-Civita connection \( \nabla \).
5. $\xi$-$W_0$ flat $K$-contact manifold

A $K$-contact manifold $M$ with the quarter-symmetric metric connection is said to be $\xi$-$W_0$ flat if $\bar{W}_0(\Gamma, \Lambda)\xi = 0$, for $\Gamma, \Lambda$ on $M$. If this expression holds for $\Gamma, \Lambda$ orthogonal to $\xi$, then a manifold is a horizontal $\xi$-$W_0$ flat manifold.

Using (12) in (20), we get
\[
\bar{W}_0(\Gamma, \Lambda)\Upsilon = R(\Gamma, \Lambda)\Upsilon - 2g(\Gamma, \phi\Lambda)\phi\Upsilon + [\eta(\Gamma)g(\Lambda, \Upsilon) - \eta(\Lambda)g(\Gamma, \Upsilon)]\xi + \frac{1}{(n-1)}[\tilde{S}(\Lambda, \Upsilon)\Gamma - g(\Upsilon, \Lambda)\phi\Gamma].
\] (29)

Putting $\Upsilon = \xi$ and using (2), (5) and (13) in (29), we get
\[
\bar{W}_0(\Gamma, \Lambda)\xi = 2[\eta(\Lambda)\Gamma - \eta(\Gamma)\Lambda] = \frac{1}{(n-1)}[2(n-1)\eta(\Lambda)\Gamma - \eta(\Lambda)\phi\Lambda].
\] (30)

If we consider $\Gamma, \Lambda$ orthogonal to $\xi$ then (30), implies that
\[
\bar{W}_0(\Gamma, \Lambda)\xi = 0.
\] (31)

Hence we state that a $K$-contact manifold is horizontal $\xi$-$W_0$ flat with quarter-symmetric metric connection.

Again using (13) in (29), we have
\[
\bar{W}_0(\Gamma, \Lambda)\Upsilon = W_0(\Gamma, \Lambda)\Upsilon - 2g(\Gamma, \phi\Lambda)\phi\Upsilon + [\eta(\Gamma)g(\Lambda, \Upsilon) - \eta(\Lambda)g(\Gamma, \Upsilon)]\xi + \frac{1}{(n-1)}[g(\Lambda, \Upsilon)\Gamma - n\eta(\Lambda)\eta(\Upsilon)\Gamma].
\] (32)

Putting $\Upsilon = \xi$ and using (2) in (32), it follows that
\[
\bar{W}_0(\Gamma, \Lambda)\xi = W_0(\Gamma, \Lambda)\xi - \eta(\Gamma)\Lambda.
\] (33)

From (33), it implies that
\[
\bar{W}_0(\Gamma, \Lambda)\xi = W_0(\Gamma, \Lambda)\xi.
\] (34)

Hence a $K$-contact manifold is horizontal $\xi$-$W_0$ flat with quarter-symmetric metric connection if and only if the manifold is $\xi$-$W_0$ flat with Levi-Civita connection.

6. Locally $W_2$ $\phi$-symmetric $K$-contact manifold

A $K$-contact manifold $M$ is said to be a locally $W_2$ $\phi$-symmetric with quarter-symmetric metric connection if
\[
\phi^2((\tilde{\nabla}_{\Omega}\bar{W}_2)(\Gamma, \Lambda)\Upsilon) = 0,
\] (35)

for all $\Gamma, \Lambda, \Upsilon$ and $\Omega$ orthogonal to $\xi$, where $\bar{W}_2$ is the $W_2$ curvature tensor with quarter-symmetric metric connection given by
\[
\bar{W}_2(\Gamma, \Lambda)\Upsilon = \bar{R}(\Gamma, \Lambda)\Upsilon - \frac{1}{(n-1)}[g(\Lambda, \Upsilon)\phi\Gamma - g(\Upsilon, \Lambda)\phi\Gamma],
\] (36)

Using (11) we can write
\[
(\tilde{\nabla}_{\Omega}\bar{W}_2)(\Gamma, \Lambda)\Upsilon = (\nabla_{\Omega}\bar{W}_2)(\Gamma, \Lambda)\Upsilon - \eta(\Omega)\phi\bar{W}_2(\Gamma, \Lambda)\Upsilon + \eta(\Omega)\{\bar{W}_2(\phi\Gamma, \Lambda)\Upsilon + \tilde{\Omega}_2(\Gamma, \phi\Lambda)\Upsilon + \bar{W}_2(\Gamma, \Lambda)\phi\Upsilon\}. 
\] (37)
Differentiating (36), we obtain
\[
(\nabla_\Omega \tilde{W}_2)(\Gamma, \Lambda) \Upsilon = (\nabla_\Omega \tilde{R})(\Gamma, \Lambda) \Upsilon.
\] (38)

Use of (17) in (38), we have

\[
(\nabla_\Omega \tilde{W}_2)(\Gamma, \Lambda) \Upsilon = (\nabla_\Omega R)(\Gamma, \Lambda) \Upsilon + 2[\eta(\Lambda)g(\Gamma, \Omega) - \eta(\Gamma)g(\Omega, \Lambda)](\phi \Upsilon) \\
+ [g(\Omega, \phi \Gamma)g(\Lambda, \Upsilon) - 2g(\Gamma, \phi \Lambda)g(\Omega, \Upsilon) - g(\Omega, \phi \Lambda)g(\Gamma, \Upsilon)]\xi \\
+ [\eta(\Lambda)g(\Gamma, \Upsilon) - \eta(\Gamma)g(\Lambda, \Upsilon)](\phi \Upsilon) + 2g(\Gamma, \phi \Lambda)\Omega + g(\Omega, \phi \Lambda)\Gamma
\] (39)

Taking account of (10), we write (39) as

\[
(\nabla_\Omega \tilde{W}_2)(\Gamma, \Lambda) \Upsilon = (\nabla_\Omega W_2)(\Gamma, \Lambda) \Upsilon + 2[\eta(\Lambda)g(\Gamma, \Omega) - \eta(\Gamma)g(\Omega, \Lambda)](\phi \Upsilon) \\
+ [g(\Omega, \phi \Gamma)g(\Lambda, \Upsilon) - 2g(\Gamma, \phi \Lambda)g(\Omega, \Upsilon) - g(\Omega, \phi \Lambda)g(\Gamma, \Upsilon)]\xi \\
+ [\eta(\Lambda)g(\Gamma, \Upsilon) - \eta(\Gamma)g(\Lambda, \Upsilon)](\phi \Upsilon) + 2g(\Gamma, \phi \Lambda)\Omega + g(\Omega, \phi \Lambda)\Gamma
\] (40)

Now applying (2) and (40) in (37), we have

\[

\phi^2(\nabla_\Omega \tilde{W}_2)(\Gamma, \Lambda) \Upsilon = \phi^2(\nabla_\Omega W_2)(\Gamma, \Lambda) \Upsilon + 2[\eta(\Lambda)g(\Gamma, \Omega) - \eta(\Gamma)g(\Omega, \Lambda)](\phi \Upsilon) \\
+ [\eta(\Lambda)g(\Gamma, \Upsilon) - \eta(\Gamma)g(\Lambda, \Upsilon)](\phi \Upsilon) + 2g(\Gamma, \phi \Lambda)(\phi \Upsilon) \phi^2 \Omega \\
+ g(\Omega, \phi \Lambda)(\phi \Upsilon) - g(\Omega, \phi \Gamma)\phi^2 \Lambda(\phi \Upsilon) + \phi(\Upsilon) + [\eta(\Lambda)g(\Omega, \phi \Upsilon)](\phi \Upsilon) - \eta(\Omega)(\phi \Upsilon) - \eta(\Omega)(\phi \Upsilon)(\phi \Upsilon)
\] (41)

Considering \(\Gamma, \Lambda, \Upsilon\) and \(\Omega\) orthogonal to \(\xi\), (41) reduces to

\[
\phi^2(\nabla_\Omega \tilde{W}_2)(\Gamma, \Lambda) \Upsilon = \phi^2(\nabla_\Omega W_2)(\Gamma, \Lambda) \Upsilon.
\] (42)

Hence we can state a \(K\)-contact manifold is locally \(W_2\) \(\phi\)-symmetric with \(\tilde{\nabla}\) if and only if it is so with Levi-Civita connection \(\nabla\).

From (2) and (39) in (37) then we get

\[
\phi^2(\nabla_\Omega \tilde{W}_2)(\Gamma, \Lambda) \Upsilon = \phi^2(\nabla_\Omega R)(\Gamma, \Lambda) \Upsilon + 2[\eta(\Lambda)g(\Gamma, \Omega) - \eta(\Gamma)g(\Omega, \Lambda)](\phi \Upsilon) \\
+ [\eta(\Lambda)g(\Gamma, \Upsilon) - \eta(\Gamma)g(\Lambda, \Upsilon)](\phi \Upsilon) + 2g(\Gamma, \phi \Lambda)(\phi \Upsilon) \phi^2 \Omega \\
+ g(\Omega, \phi \Lambda)(\phi \Upsilon) - g(\Omega, \phi \Gamma)(\phi \Upsilon) \phi^2 \Lambda(\phi \Upsilon) + \phi(\Upsilon) + [\eta(\Lambda)g(\Omega, \phi \Upsilon)](\phi \Upsilon) - \eta(\Omega)(\phi \Upsilon) - \eta(\Omega)(\phi \Upsilon)(\phi \Upsilon)
\] (43)

Taking \(\Gamma, \Lambda, \Upsilon\) and \(\Omega\) orthogonal to \(\xi\) in (43) followed by a simplification we get

\[
\phi^2(\nabla_\Omega \tilde{W}_2)(\Gamma, \Lambda) \Upsilon = \phi^2(\nabla_\Omega R)(\Gamma, \Lambda) \Upsilon.
\] (44)

Thus we can state that if \(M\) is \(\phi\)-symmetric with quarter-symmetric metric connection then a \(K\)-contact manifold is locally \(W_2\) \(\phi\)-symmetric with quarter-symmetric metric connection \(\tilde{\nabla}\).
7. $\xi$-$W_2$ flat $K$-contact manifold

A $K$-contact manifold $M$ with quarter-symmetric metric connection is said to be $\xi$-$W_2$ flat if $\tilde{W}(\Gamma,\Lambda)\xi = 0$, for $\Gamma, \Lambda$ on $M$. If this expression holds for $\Gamma, \Lambda$ orthogonal to $\xi$, then a manifold is a horizontal $\xi$-$W_2$ flat manifold.

Using (12) in (36), we get

$$\tilde{W}(\Gamma,\Lambda)\Upsilon = R(\Gamma,\Lambda)\Upsilon - 2g(\Gamma, \phi\Lambda)\phi\Upsilon + [\eta(\Gamma)g(\Lambda, \Upsilon) - \eta(\Lambda)g(\Gamma, \Upsilon)]\xi$$

$$+ [\eta(\Lambda)\Gamma - \eta(\Gamma)\Lambda]g(\Upsilon, \Upsilon) - \frac{1}{(n-1)}[g(\Lambda, \Upsilon)Q\Gamma - g(\Gamma, \Upsilon)Q\Lambda]. \quad (45)$$

Putting $\Upsilon = \xi$ and using (2), (5) and (13) in (45), we get

$$\tilde{W}(\Gamma,\Lambda)\xi = 2[\eta(\Lambda)\Gamma - \eta(\Gamma)\Lambda] - \frac{1}{(n-1)}[\eta(\Lambda)Q\Gamma - \eta(\Gamma)Q\Lambda]. \quad (46)$$

If $\Gamma, \Lambda$ orthogonal to $\xi$ then (46), implies that $\tilde{W}(\Gamma,\Lambda)\xi = 0, \quad (47)$

Hence we state a $K$-contact manifold is horizontal $\xi$-$W_2$ flat with quarter-symmetric metric connection.

Again using (13) in (45), we have

$$\tilde{W}(\Gamma,\Lambda)\Upsilon = W_2(\Gamma,\Lambda)\Upsilon - 2g(\Gamma, \phi\Lambda)\phi\Upsilon + [\eta(\Lambda)\Gamma - \eta(\Gamma)\Lambda]g(\Upsilon, \Upsilon)$$

$$+ [\eta(\Gamma)g(\Lambda, \Upsilon) - \eta(\Lambda)g(\Gamma, \Upsilon)]\xi. \quad (48)$$

Putting $\Upsilon = \xi$ and using (2) in (48), it follows that

$$\tilde{W}(\Gamma,\Lambda)\xi = W_2(\Gamma,\Lambda)\xi + \eta(\Lambda)\Gamma - \eta(\Gamma)\Lambda. \quad (49)$$

From (49), it implies that

$$\tilde{W}(\Gamma,\Lambda)\xi = W_2(\Gamma,\Lambda)\xi. \quad (50)$$

Hence we have the following:

**Theorem:** A $K$-contact manifold is horizontal $\xi$-$W_2$ flat with quarter-symmetric metric connection if and only if the manifold is $\xi$-$W_2$ flat with Levi-Civita connection.

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