SPECTRUM OF A LINEAR DIFFERENTIAL EQUATION OVER A FIELD OF
FORMAL POWER SERIES

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Abstract. In this paper we associate to a linear differential equation with coefficients in the field of
Laurent formal power series a new geometric object, a spectrum in the sense of Berkovich. We will
compute this spectrum and show that it contains interesting informations about the equation.

1. Introduction

Differential equations constitute an important tool for investigation of algebraic and analytic varieties.
Among ways to look for solutions of such equations, the study of formal Taylor solutions of the equation
around singular and regular points and the computation of their radii of convergence. This leads in
particular to the study of differential equations with coefficients in the field of Laurent formal power
series. In this special case, we have interesting classification results. Before discussing in more details
these results, we shall fix the setting of our context.

Let \((k, |.|)\) be a trivially valued field of characteristic 0. We set \(k((T))\) to be the field of Laurent
formal power series. Consider the differential field \((k((T)), \frac{dT}{dT})\). We mean by a differential equation with
coefficients in \(k((T))\) a differential module \((M, \nabla)\) over \((k((T)), \frac{dT}{dT})\). The main fundamental classification
results are the following:

1. Decomposition theorem according to the slopes, considering the \(T\)-adic valuation of coefficients of
the operator \(\nabla\) in a cyclic basis, we can associate a Newton polygon, called formal. The formal
slopes of \((M, \nabla)\) are the slopes of this polygon. The decomposition theorem is the following.

   **Theorem 1.1** ([DMR07, p. 97-107]). Let \(\gamma_1 < \cdots < \gamma_\mu\) be the slopes of the formal of \((M, \nabla)\),
   with multiplicity \(n_1, \cdots, n_\mu\) respectively. Then

   \[(M, \nabla) = \bigoplus_{i=1}^\mu (M_{\gamma_i}, \nabla_{\gamma_i}),\]

   where \(M_{\gamma_i}\) has dimension \(n_i\) and a unique slope \(\gamma_i\) with multiplicity \(n_i\).

2. The Turrittin-Levelt-Hukuhara decomposition theorem [Kat70], [Rob80], [Ked10]. It claims that
   for any differential module \((M, \nabla)\), there exists a suitable finite extension \(k'(\frac{dT}{dT})\) of \(k((T))\)
   for which the pull-back of \((M, \nabla)\) with respect to this extension is an extension of differential
   modules of rank one. Moreover, these differential modules are not isomorphic even after an
   algebraic extension.

In this paper we propose a new geometric invariant for differential modules over \((k((T)), \frac{dT}{dT})\), a
spectrum in the sense of Berkovich. We will compute this spectrum and show that it contains interesting
information about the equation. For this purpose we will use the classifications results listed above.
Before announcing the main result of the paper, we shall recall quickly the notion of the spectrum in the
sense of Berkovich.

Recall that for an element \(f\) of a non-zero \(k\)-algebra \(E\) with unit, the classical spectrum of \(f\) is the set
\(\{a \in k; f - a.1_E \text{ is not invertible in } E\}\).
This set may be empty, even if $E$ is a $k$-Banach algebra. To deal with that Berkovich propose to consider the spectrum not as a subset of $k$, but as a subset of the analytic affine line $\mathbb{A}^{1,\text{an}}_k$ (which is a bigger space than $k$)[Ber90, Chapter 7]. Let $E$ be a non-zero $k$-Banach algebra and $f \in E$. The spectrum $\Sigma_{f,k}(E)$ of $f$ in the sense of Berkovich is the set of points of $\mathbb{A}^{1,\text{an}}_k$ that correspond to a pair $(\Omega, c)$, where $\Omega$ is a complete extension of $k$ and $c \in \Omega$, for which $f \otimes 1 - 1 \otimes c$ is not invertible in $E \otimes_k \Omega$. This spectrum is compact, non-empty and satisfies other nice properties... (cf.[Ber90, Theorem 7.1.2]).

Let $(M, \nabla)$ be a differential module over $(k((T)), T \frac{d}{dT})$. Let $r > 0$ be a real positive number. From now on we endow $k((T))$ with the $T$-adic absolute value given by

$$(1.1) \quad | \sum_{i \geq N} a_i T^i | := r^N,$$

if $a_N \neq 0$. In this setting $k((T))$ is a complete valued field. Hence, $M$ can be endowed with $k$-Banach space for which $\nabla : M \to M$ is a bounded operator. As in our previous work [Azz18], the spectrum $\Sigma_{\nabla,k}(L_k(M))$ of $(M, \nabla)$ will be the spectrum of $\nabla$ as an element of the $k$-Banach algebra $L_k(M)$ of bounded endomorphism of $M$ with respect to operator norm.

Notice that we cannot use the classical index theorem of B. Malgrange [Mal74] to compute neither the spectrum in the sense of Berkovich nor the classical spectrum of $\nabla$. Indeed, it is relatively easy to show that any non trivial rank one connection on $k((T))$ is set theoretically bijective. However, the set theoretical inverse of the connection may not be automatically bounded. This is due to the fact that the base field $k$ is trivially valued and the Banach open mapping theorem does not hold in general.

In order to introduce our result, let $l$ be a real positive number, we set $x_{0,l}$ to be the point of $\mathbb{A}^{1,\text{an}}_k$ associated to $l$-Gauss norm on $k[T]$ (i.e. $\sum_i a_i T^i \mapsto \max_i |a_i|^l$). The main result of the paper is the following:

**Theorem 1.2.** Assume that $k$ is algebraically closed. Let $(M, \nabla)$ be a differential module over $(k((T)), T \frac{d}{dT})$. Let $\{\gamma_1, \cdots, \gamma_{\nu_1}\}$ be the set of the slopes of $(M, \nabla)$ and let $\{a_1, \cdots, a_{\nu_2}\}$ be the set of the exponents of the regular part of $(M, \nabla)$. Then the spectrum of $\nabla$ as an element of $L_k(M)$ is:

$$\Sigma_{\nabla,k}(L_k(M)) = \{x_{0,r^{-\gamma_1}}, \cdots, x_{0,r^{-\gamma_{\nu_1}}} \} \cup \bigcup_{i=1}^{\nu_2} (a_i + \mathbb{Z}).$$

This result clearly show the importance of the points of the spectrum that are not in $k$. On other hand, although differential modules over $(k((S)), S \frac{d}{dT})$ are algebraic objects, their spectra in the sense of Berkovich depends highly on the choice of the absolute value on $k((S))$.

The paper is organized as follows. Section 2 is devoted to providing setting and notation. It is divided onto three part, the first one is to recalling the notion of differential modules. The second one is to recalling the definition of the analytic affine line $\mathbb{A}^{1,\text{an}}_k$, and give, in the setting of the paper, a precise topological description for disks and annulus of $\mathbb{A}^{1,\text{an}}_k$. In the last part, we recall the definition of the spectrum in the sens of Berkovich and give some properties.

In Section 3, we introduce the spectrum of a differential module and recall some properties given in [Azz18]. We show also how the spectrum of a differential module behave after ramified ground field extension. In the end of the section we will recall the definition of the formal Newton polygon.

The section 4 is devoted to announcing and proving the main result of the paper. Using the decomposition theorem according to the slopes, we can reduce the probleme only to the computation of the spectrum of regular singular differential modules and differential modules without regular part. The spectrum of a regular singular module is mainly obtained by the computation of the spectrum of $T \frac{d}{dT}$ as an element of $L_k(k((T)))$. Regarding the spectrum of differential modules without regular part, in order to compute the spectrum, we reduce the computation to differential modules of rank one. This

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1 the completion of the tensor product with respect to tensor norm.
is possible by using Turrittin-Levelt-Hukura decomposition theorem and the behaviour of the spectrum after ramified ground field extension.

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2. Definitions and notations

All rings are with unit element. We will denote by $\mathbb{R}$ the field of real numbers, by $\mathbb{Z}$ the ring of integers and by $\mathbb{N}$ the set of natural numbers. We set

$$\mathbb{R}_+ := \{ r \in \mathbb{R}; r \geq 0 \}.$$

In all the paper we fix $(k, |.|)$ to be a trivially valued field and algebraically closed of characteristic 0. Let $E(k)$ be the category whose objects are $(\Omega, |.|_{\Omega})$, where $\Omega$ is a field extension of $k$, complete with respect to the valuation $|.|_{\Omega}$, and whose morphisms are isometric rings morphisms. For $(\Omega, |.|_{\Omega}) \in E(k)$, we set $\Omega^{alg}$ to be an algebraic closure of $\Omega$, the absolute value extends uniquely to $\Omega^{alg}$. We denote by $\hat{\Omega^{alg}}$ the completion of $\Omega^{alg}$ with respect to this absolute value. We set $k[T]$ to be the ring of polynomial with coefficients in $k$ and $k(T)$ its fractions field.

2.1. Differential modules. Let $F \in E(k)$ and $d$ be a $k$-linear derivation on $F$. Consider the differential field $(F,d)$. A differential module $(M, \nabla)$ over $(F,d)$ of rank $n \in \mathbb{N}$ is an $F$-vector space $M$ of dimension $n$ together with a $k$-linear map $\nabla : M \rightarrow M$, called connection of $M$, satisfying $\nabla(fm) = df.m + f.\nabla(m)$ for all $f \in F$ and $m \in M$. If we fix a basis of $M$, then we get an isomorphism of $F$-vector spaces $M \simeq F^n$, and the operator $\nabla$ is given in the induced basis $\{e_1, \cdots, e_n\}$ by the formula:

$$\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} + G \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} $$

(2.1)

where $G \in M_n(F)$ is the matrix that its $i$ column is the vector $\nabla(e_i)$. Conversely the data of such matrix defines a differential module structure on $F^n$ by the formula (2.1).

A morphism between differential modules is a $k$-linear map $M \rightarrow N$ commuting with connections.

Notation 2.1. We denote by $d\text{-}\text{Mod}(F)$ the category of differential modules over $(F,d)$ whose arrows are morphisms of differential modules.

We set $\mathcal{D}_F := \bigoplus_{i \in \mathbb{N}} F.D^i$ to be the ring of differential polynomials on $D$ with coefficients in $F$, where the multiplication is non-commutative and defined as follows: $D.f = d(f) + f.D$ for all $f \in A$. Let $P(D) = g_0 + \cdots + g_n D^{n-1} + D^n$ be a monic differential polynomial. The quotient $\mathcal{D}_F/\mathcal{D}_F.P(D)$ is a $F$-vector space of dimension $n$. Equipped with the multiplication by $D$, it is a differential module over $(F,d)$. In the basis $\{1, D, \ldots, D^{n-1}\}$ the multiplication by $D$ satisfies:

$$D \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} -g_0 \\ -g_1 \\ -g_{n-1} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

(2.2)
Proposition 2.2 (The cyclic vector theorem). Let \((M, ∇)\) be a differential module over \((F, d)\). Then there exists \(P(D) \in \mathcal{D}_F\) such that \((M, ∇) \simeq (\mathcal{D}_F/\mathcal{D}_F P(D), D)\).

**Proof.** See [Ked10, Theorem 5.7.2]. 

Let \((M_1, ∇_1)\) and \((M_2, ∇_2)\) be two differential modules over \((F, d)\) of rank \(n_1\) and \(n_2\). The tensor product \(M_1 \otimes M_2\) equipped with the connection:

\[
∇ = ∇_1 \otimes 1 + 1 \otimes ∇_2
\]

is a differential module over \((F, d)\) of rank \(n_1 n_2\).

2.2. Analytic affine line. In this part we give basic notions related to the analytic affine line \(A_k^{1,\text{an}}\) in the sense of Berkovich, and describe some analytic domains of \(A_k^{1,\text{an}}\) that are helpful to describe spectra of differential modules later.

We will consider \(k\)-analytic spaces in the sense of Berkovich (see [Ber90]). We denote by \(A_k^{1,\text{an}}\) the affine analytic line over the ground field \(k\). Recall that a point \(x \in A_k^{1,\text{an}}\) corresponds to a multiplicative semi-norm \(|.|_x\) on \(k[T]\) (i.e. \(|0|_x = 0, |1|_x = 1, |P - Q|_x \leq \max(|P|_x, |Q|_x)\) and \(|P \cdot Q|_x = |P|_x \cdot |Q|_x\) for all \(P, Q \in k[T]\)) whose restriction coincides with the absolute value of \(k\). The set

\[
p_x := \{P \in k[T]; |f|_x = 0\}
\]

is a prime ideal of \(k[T]\). Therefore, the semi-norm extends to a multiplicative norm on the fraction field \(\text{Frac}(A/p_x)\).

**Notation 2.3.** Let \(c \in k\) and \(r \in \mathbb{R}_+\). Denote by \(x_{c,r}\) the point that corresponds to the semi-norm (norm if \(r > 0\))

\[
\sum_{i=0}^{n} a_i (T - c)^i \to \mathbb{R}_+ \to \max_i |a_i| r^i.
\]

**Remark 2.4.** Any point of \(A_k^{1,\text{an}}\) is a point of the form \(x_{c,r}\). In particular, the points of the form \(x_{c,0}\) coincides with the element of \(k\).

**Notation 2.5.** We denote by \(\mathscr{H}(x)\) the completion of \(\text{Frac}(A/p_x)\) with respect to \(|.|_x\), and by \(|.|\) the absolute value on \(\mathscr{H}(x)\) induced by \(|.|_x\).

Let \(Ω \in E(k)\). A nonzero \(k\)-algebra morphism \(χ : k[T] \to Ω\) is called a character of \(k[T]\). Let \(\sim\) be the equivalence relation defined as follows: let \(χ' : k[T] \to Ω'\) and \(χ'' : k[T] \to Ω''\) be two characters,

\[
χ' \sim χ'' \Leftrightarrow ∃χ : k[T] \to Ω\text{ such that:}
\]

\[
\begin{array}{c}
Ω' \\
\downarrow χ' \\
k[T] \xrightarrow{χ} Ω \\
\downarrow χ'' \\
Ω'' \\
\end{array}
\]

For any point \(x \in A_k^{1,\text{an}}\) we have a natural nonzero \(k\)-algebra morphism

\[
χ_x : k[T] \to \mathscr{H}(x)
\]

\[
P \mapsto P(x).
\]

This induces a bijection between \(A_k^{1,\text{an}}\) and the set of equivalence classes of characters of \(k[T]\).
For a points \( x_{c,r} \in \mathbb{A}^{1,\text{an}}_k \) with \( r > 0 \) the field \( \mathcal{H}(x_{c,r}) \) can be described more concretely. The case where \( r = 0 \) is trivial, indeed \( \mathcal{H}(x_{c,0}) \simeq k \).

In the case where \( r < 1 \), the field \( \mathcal{H}(x_{c,r}) \) coincides with the field of formal power series

\[
(2.5) \quad k[[T - c]] := \left\{ \sum_{i \geq N} a_i (T - c)^i ; a_i \in k, N \in \mathbb{Z} \right\}.
\]

equipped with \((T - c)\)-adic absolute value given by \(|\sum_{i \geq N} a_i (T - c)^i| := r^N\), if \( a_N \neq 0 \).

If \( r > 1 \), then \( \mathcal{H}(x_{c,r}) \) coincides with \( k[[((T - c)^{-1})]] \) equipped with \((T - c)^{-1}\)-adic absolute value given by \(|\sum_{i \geq N} a_i (T - c)^{-i}| := r^{-N}\), if \( a_N \neq 0 \).

Otherwise, \( \mathcal{H}(x_{c,1}) \) coincides with \( k(T) \) equipped with the trivial absolute value.

**Definition 2.6.** Let \( x \in \mathbb{A}^{1,\text{an}}_k \). We define the radius of \( x \) to be the value:

\[
(2.6) \quad r(x) := \inf_{c \in k} |T(x) - c|.
\]

Let \( \Omega \in E(k) \) and \( c \in \Omega \). For \( r \in \mathbb{R}_+ \setminus \{0\} \) we set

\[
D^+_\Omega(c, r) := \{ x \in \mathbb{A}^{1,\text{an}}_\Omega ; |T(x) - c| \leq r \}
\]

and

\[
D^-\Omega(c, r) := \{ x \in \mathbb{A}^{1,\text{an}}_\Omega ; |T(x) - c| < r \}.
\]

The point \( x_{c,r} \in \mathbb{A}^{1,\text{an}}_{\Omega} \) corresponds to the disk \( D^+_\Omega(c, r) \), more precisely it does not depend on the of the center \( c \) (cf. [Ber90, Section 1.4.4]).

Let \( c \in k \). The map

\[
(2.7) \quad [0, +\infty) \rightarrow \mathbb{A}^{1,\text{an}}_k, \quad r \mapsto x_{c,r}
\]

induces an homeomorphism between \([0, +\infty)\) and its image. Since the valuation is trivial on \( k \), we can describe disks of \( \mathbb{A}^{1,\text{an}}_k \) as follow.

**Notation 2.7.** We denote by \([x_{c,r}, \infty) \) (resp. \((x_{c,r}, \infty)\)) the image of \([r, \infty) \) (resp. \((r, \infty)\)) by \([x_{c,r}, x_{c,r'}] \) (resp. \((x_{c,r}, x_{c,r'})\)) the image of \([r, r'] \) (resp. \((r, r')\), \((r, r')\)).

Let \( c \in k \) and \( r \in \mathbb{R}_+ \). In the case where \( r < 1 \), we have

\[
(2.8) \quad D^+_k(c, r) = [c, x_{c,r}] \quad D^-_k(c, r) = [c, x_{c,r}].
\]

Otherwise, recall that for all \( a \in k \), \( x_{a,1} = x_{0,1} \) and we have

\[
(2.9) \quad D^+_k(c, r) = \coprod_{a \in k} [a, x_{0,1}) \coprod [x_{0,1}, x_{0,r}].
\]

\[
(2.10) \quad D^-_k(c, r) = \begin{cases} [c, x_{0,1}) & \text{if } r = 1 \\ \coprod_{a \in k} [a, x_{0,1}) \coprod [x_{0,1}, x_{0,r}) & \text{otherwise}. \end{cases}
\]

For \( r_1, r_2 \in \mathbb{R}_+ \), such that \( 0 < r_1 \leq r_2 \) we set
and for $r_1 < r_2$ we set:

$$C^+_\Omega(c, r_1, r_2) := D^+_\Omega(c, r_2) \setminus D^+_\Omega(c, r_1)$$

We may suppress the index $\Omega$ when it is obvious from the context.

Let $X$ be an affinoid domain of $\mathbb{A}^1_{k^{an}}$, we denote by $\mathcal{O}(X)$ the $k$-Banach algebra of globale sections of $X$.

Let $X = D^+(c, r)$. Since $k$ is trivially valued, if $r < 1$ we have

$$\mathcal{O}(D^+(c, r)) = k[T - c] := \{ \sum_{i \in \mathbb{N}} a_i(T - c)^i; a_i \in k \}.$$ 

otherwise,

$$\mathcal{O}(D^+(c, r)) = k[T - c].$$

In the both cases, the multiplicative norm on $\mathcal{O}(D^+(c, r))$ is:

$$\| \sum_{i \in \mathbb{N}} a_i(T - c)^i \| = \max_{i \in \mathbb{N}} |a_i|r^i.$$ 

Let $X = C^+(c, r_1, r_2)$

$$\mathcal{O}(C^+(c, r_1, r_2)) = \left\{ \sum_{i \in \mathbb{N}} \frac{a_i}{(T - c)^i}; a_i \in k, |a_i|r_1^{-i} \to 0 \right\} \oplus \mathcal{O}(D^+(c, r_2)).$$

where $\| \sum_{i \in \mathbb{N} \setminus \{0\}} \frac{a_i}{(T - c)^i} \| = \max_{i \in \mathbb{N} \setminus \{0\}} |a_i|r_1^{-i}$ and the sum above is equipped with the maximum norm.

In the case where $r_2 < 1$ we have

$$\mathcal{O}(C^+(c, r_1, r_2)) = k [[T - c]].$$

**Notation 2.8.** Let $X$ be an analytic domain of $\mathbb{A}^1_{k^{an}}$, and $f \in \mathcal{O}(X)$. We can see $f$ as an analytic map $X \to \mathbb{A}^1_{k^{an}}$ that we still denote by $f$.

### 2.3. Berkovich spectrum

This part is devoted to recalling the definition of the spectrum in the sense of Berkovich and the definition of the sheaf of analytic function with value in a $k$-Banach space over analytic space, given by V. Berkovich in [Ber90, Chapter 7]. We recall here also the most important properties. We add in later sections some other properties that are necessary to compute the spectrum.

**Definition 2.9.** Let $M$ and $N$ be two $k$-Banach spaces. We defined a norm on the tensor product $M \otimes_k N$ as follows:

$$\| \cdot \| : M \otimes_k N \to \mathbb{R}_+ \quad \text{such that} \quad \inf \left\{ \max_i \left\{ \| m_i \|_M \| n_i \|_N \right\} \right\} \text{ for } f = \sum_i m_i \otimes n_i.$$ 

We denote by $M \hat{\otimes}_k N$ the completion of the tensor product with respect to this norm.

**Definition 2.10.** Let $E$ be a $k$-Banach algebra with unit and $f \in E$. The spectrum of $f$ is the set $\Sigma_{f,k}(E)$ of points $x \in \mathbb{A}^1_{k^{an}}$ such that the element $f \otimes 1 - 1 \otimes T(x)$ is not invertible in the $k$-Banach algebra $E \hat{\otimes}_k \mathcal{H}(x)$. The resolvent of $f$ is the function:

$$R_f : \mathbb{A}^1_{k^{an}} \setminus \Sigma_{f,k}(E) \to \prod_{x \in \mathbb{A}^1_{k^{an}} \setminus \Sigma_{f,k}(E)} E \hat{\otimes}_k \mathcal{H}(x) \quad x \mapsto (f \otimes 1 - 1 \otimes T(x))^{-1}.$$
Remark 2.11. If there is no confusion we denote the spectrum of \( f \), as an element of \( E \), just by \( \Sigma_f \).

Remark 2.12. The set \( \Sigma_f \cap k \) coincides with the classical spectrum, i.e.

\[
\Sigma_f \cap k = \{ a \in k; f - a \text{ is not invertible in } E \}.
\]

Definition 2.13. Let \( X \) be a \( k \)-affinoid space and \( B \) be a \( k \)-Banach space. We define the sheaf of analytic functions with values in \( B \) over \( X \) to be the sheaf:

\[
\mathcal{O}_X(B)(U) = \lim_{\mathcal{V} \subset U} B \hat{\otimes}_k A_V
\]

where \( U \) is an open subset of \( X \), \( V \) an affinoid domain and \( A_V \) the \( k \)-affinoid algebra associated to \( V \).

As each \( k \)-analytic space is obtained by gluing \( k \)-affinoid spaces (see [Ber90], [Ber93]), we can extend the definition to \( k \)-analytic spaces. Let \( U \) be an open subset of \( X \). Every element \( f \in \mathcal{O}_X(B)(U) \) induces a function: \( f : U \to \prod_{x \in U} B \hat{\otimes}_k \mathcal{H}(x) \), \( x \mapsto f(x) \), where \( f(x) \) is the image of \( f \) by the map \( \mathcal{O}_X(B)(U) \to B \hat{\otimes}_k \mathcal{H}(x) \). We will call an analytic function over \( U \) with value in \( B \), any function \( \psi : U \to \prod_{x \in U} B \hat{\otimes}_k \mathcal{H}(x) \) induced by an element \( f \in \mathcal{O}_X(B)(U) \).

Definition 2.14. Let \( E \) be a \( k \)-Banach algebra. The spectral semi-norm associated to the norm \( \| \cdot \| \) of \( E \) is the map:

\[
\| \cdot \|_{\text{sp}} : E \to \mathbb{R}_+
\]

\[
f \mapsto \lim_{n \to +\infty} \| f^n \|^{\frac{1}{n}}.
\]

If \( \| \cdot \|_{\text{sp}} = \| \cdot \| \), we say that \( E \) is a uniform algebra.

This notion of spectrum satisfies the same properties of the classical spectrum in the complex case.

Theorem 2.15 ([Ber90, Theorem 7.1.2]). Let \( E \) be a non-zero \( k \)-Banach algebra and \( f \in E \). Then:

1. The spectrum \( \Sigma_f \) is a non-empty compact subset of \( k^{1,\text{an}}_k \).
2. The radius of the smallest (inclusion) closed disk with center at zero which contains \( \Sigma_f \) is equal to \( \| f \|_{\text{sp}} \).
3. The resolvent \( R_f \) is an analytic function on \( k^{1,\text{an}}_k \setminus \Sigma_f \) which is equal to zero at infinity.

Lemma 2.16 ([Azz18, Lemma 2.20]). Let \( E \) be a non-zero \( k \)-Banach algebra and \( f \in E \). If \( a \in (k^{1,\text{an}}_k \setminus \Sigma_f) \cap k \), then the biggest open disk centred at \( a \) contained in \( k^{1,\text{an}}_k \setminus \Sigma_f \) has radius \( R = \| (f - a)^{-1} \|_{\text{sp}}^{-1} \).

Let \( M \), \( N \) two \( k \)-Banach spaces. Recall that a \( k \)-linear map \( \varphi : M \to N \) is said to be bounded if

\[
\exists C \in \mathbb{R}_+, \forall m \in M; \| \varphi(m) \| \leq C \| m \|.
\]

If moreover \( \varphi \) is an isomorphism such that \( \varphi^{-1} \) is a bounded we say that it is bi-bounded isomorphism.

Notation 2.17. Denote By \( \mathcal{L}_k(M, N) \) the \( k \)-algebra of bounded \( k \)-linear map \( M \to N \). We set \( \mathcal{L}_k(M, M) := \mathcal{L}_k(M) \).

The \( k \)-algebra \( \mathcal{L}_k(M, N) \) equipped with operator norm:

\[
\varphi \mapsto \| \varphi \|_{\text{op}} := \sup_{m \in M \setminus \{0\}} \frac{\| \varphi(m) \|}{\| m \|}
\]

is a \( k \)-Banach algebra.

Lemma 2.18. Let \( E \) and \( E' \) be two \( k \)-Banach algebras and \( \varphi : E \to E' \) be a bounded morphism of \( k \)-algebras. If \( f \in E \) then we have:

\[
\Sigma_{\varphi(f)}(E') \subset \Sigma_f(E).
\]

If moreover \( \varphi \) is a bi-bounded morphism then we have the equality.
Proof. Consequence of the definition.

The following Lemma allow us the computation spectrum of the translate of a connection $\nabla$ from the spectrum of $\nabla$. We will see that in Sections 3.2 and 4.1.

Let $P(T) \in k[T]$, let $E$ be a $k$-Banach algebra and let $f \in E$. We set $P(f)$ to be the image of $P(T)$ by the morphism $k[T] \to E, T \mapsto f$.

**Lemma 2.19.** We have the equality of sets:

$$\Sigma_{P(f)} = P(\Sigma_f)$$

**Proof.** The proof is analogous to the proof [Bou07, p. 2].

\[ \square \]

3. Differential module over $k[[S]]$ and spectrum

In this section we recall some properties of differential module $(M, \nabla)$ over $(F, d)$, where $(F, d)$ is a finite differential extension of $(k((S)), S \frac{d}{ds})$. We will recall also some general properties related to the spectrum associated to $(M, \nabla)$, and show how this last one behave under the ramification of the indeterminate $S$.

**Convention 3.1.** We fix $r \in (0, 1)$ and endow $k((S))$ with the $S$-adic absolute value given by $|\sum_{i \geq N} a_i S^i| := r^N$, if $a_N \neq 0$, where $|S| = r$. In this setting the pair $(k((S)), |.|)$ coincides with $\mathcal{H}(x_0, r, 1) = \mathcal{H}(x_0, r, 1), x_0, r \in k^{1, an}$.

**Remark 3.2.** The derivation $S \frac{d}{ds}$ is a bounded as an operator on $k((S))$.

3.1. Spectrum of differential modules. Recall that if $F$ is a finite extension of $k((S))$ of degree $m$, then we have $F \simeq k((S)^{\Delta})$ [VS12, Proposition 3.3]. The absolute value $|.|$ on $k((S))$ extends uniquely to an absolute value on $F$. The pair $(F, |.|)$ is an element of $E(k)$ and can be identified with $\mathcal{H}(x_0, r, 1)$. The derivation $S \frac{d}{ds}$ extends uniquely to a derivation $d$ on $F$, where $d(S^{\Delta}) = \frac{1}{m} S^{\Delta}$. Then $(F, d)$ is a finite differential extension of $(k((S)), S \frac{d}{ds})$. Hence, the derivation $d$ is also bounded as an operator on $F$. Conversely, any finite differential extension of $(k((S)), S \frac{d}{ds})$ is obtained by this way.

Let $(F, d)$ be a finite differential extension of $(k((S)), S \frac{d}{ds})$. Let $(M, \nabla)$ be a differential module over $(F, d)$. In order to associate to this differential module a spectrum we need to endow $M$ with a structure of $k$-Banach space. By fixing a basis on $M$ we can pull-back the structure of $F$-Banach space of $F^n$ equipped with the max norm to $M$, any other choice of basis induces an equivalent $F$-Banach structure. This is due to the following proposition.

**Proposition 3.3 ([Chr. Theorem 1.14]).** Let $(\Omega, |.|)$ be a complete valued field and $M$ a finite vector space over $\Omega$. Then all norms defined on $M$ are equivalent.

This induces a structure of $k$-Banach space on $M$. As $\nabla$ satisfies the formula (2.1) and $d \in \mathcal{L}_k(M)$, we have $\nabla \in \mathcal{L}_k(M)$. The spectrum of $(M, \nabla)$ will be the spectrum of $\nabla$ as an element of $\mathcal{L}_k(M)$, which we denote by $\Sigma_{\nabla, k}(\mathcal{L}_k(M))$ (or just by $\Sigma_{\nabla}$ if the dependence is obvious from the context). This spectrum does not depend on the choice of a basis on $M$. Indeed, since the norms are equivalent, by Lemma 2.18 we obtain the equality of spectra.

Let $\varphi : (M, \nabla) \to (N, \nabla')$ be a morphism of differential modules. If we endow $M$ and $N$ with a structures of $k$-Banach spaces then it induce a bounded $k$-linear map. In the case where $\varphi$ is an isomorphism, then it induce a bi-bounded $k$-linear isomorphism and according to Lemma 2.18 we have:

$$\Sigma_{\varphi}(\mathcal{L}_k(M)) = \Sigma_{\varphi}(\mathcal{L}_k(M))$$

This prove the following statement.

**Proposition 3.4.** The spectrum of a connection is an invariant by bi-bounded isomorphisms of differential modules.
Remark 3.5. We observe here that spectrum depends on the choice of the derivation. In this paper we chose in particular the derivation $S_{dS}$ to compute the spectrum.

This spectrum behave nicely under exact sequences and direct sum. This the aim of the following claims.

**Proposition 3.6.** Let $(M, \nabla), (M_1, \nabla_1)$ and $(M_2, \nabla_2)$ be three differential modules over $(F, d)$. If we have tow exact sequences of the form:

$$0 \to (M_1, \nabla_1) \to (M, \nabla) \to (M_2, \nabla_2) \to 0$$

$$0 \to (M_2, \nabla_2) \to (M, \nabla) \to (M_1, \nabla_1) \to 0.$$  

Then we have $\Sigma_\nabla(\mathcal{L}_k(M)) = \Sigma_\nabla_1(\mathcal{L}_k(M_1)) \cup \Sigma_\nabla_2(\mathcal{L}_k(M_2)).$

**Proof.** See [Azz18, Proposition 3.7] and [Azz18, Remark 3.8]. □

**Corollary 3.7.** Let $(M, \nabla), (M_1, \nabla_1)$ and $(M_2, \nabla_2)$ be three differential modules over $(F, d)$. If we suppose that $(M, \nabla) = (M_1, \nabla_1) \oplus (M_2, \nabla_2)$, then we have $\Sigma_\nabla(\mathcal{L}_k(M)) = \Sigma_\nabla_1(\mathcal{L}_k(M_1)) \cup \Sigma_\nabla_2(\mathcal{L}_k(M_2)).$

**Corollary 3.8.** Let $f \in F$. Let $(M, \nabla)$ be the differential modules over $(F, d)$ such that $(M, \nabla) \simeq (\mathcal{D}_F/\mathcal{D}_F.(D - f)^n, D)$. Then we have:

$$\Sigma_\nabla(\mathcal{L}_k(M)) = \Sigma_{S_{dS} + f}(\mathcal{L}_k(F)).$$

### 3.2. Spectrum of a differential module after ramified ground field extension

In this section we show how the spectrum of a differential module behave after ramified ground field extension.

Let $(F, d)$ be a finite differential extension of $(k(S), S_{dS})$. Let $m \in \mathbb{N}$ such that $F \simeq k(\frac{d}{m})$. Now, if we set $Z = S_{\frac{1}{m}}$, then we have $F = k(Z)$ and $d = \frac{d}{m}$. Note that we can see $(F, \frac{d}{m})$ as a differential module over $(k(S), S_{dS})$. In the basis $\{1, Z, \cdots, Z^{m-1}\}$ we have:

$$Z \frac{d}{m} \frac{d}{m} \frac{d}{m} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} S_{dS} f_1 \\ \vdots \\ S_{dS} f_m \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{m}{d} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \frac{m-1}{d} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$$

We have a functor:

$$I_F^*: S_{dS} - \text{Mod}(k(S)) \to \frac{Z}{m} \frac{d}{dZ} - \text{Mod}(F)$$

where $I_F^* M = M \otimes_{k(S)} F$ and the connection $I_F^* \nabla$ is defined as follows:

$$I_F^* \nabla = \nabla \otimes 1 + 1 \otimes \frac{Z}{m} \frac{d}{dZ}$$

Let $(M, \nabla)$ be an object of $S_{dS} - \text{Mod}(k(S))$ of rank $n$. If $\{e_1, \ldots, e_n\}$ is a basis of $M$ such that we have:

$$\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} S_{dS} f_1 \\ \vdots \\ S_{dS} f_n \end{pmatrix} + G \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},$$
with \( G \in \mathcal{M}(k((S))) \), then \((I_F^*M, I_F^*\nabla)\) is of rank \( n \) and in the basis \( \{e_1 \otimes 1, \ldots, e_n \otimes 1\} \) we have:

\[
(I_F^* \nabla \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} \frac{Z}{m} \frac{d}{dz} f_1 \\ \vdots \\ \frac{Z}{m} \frac{d}{dz} f_n \end{pmatrix} + G \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.
\]

We have also the natural functor:

\[
(I_F^*: Z \frac{d}{dz} - \text{Mod}(F) \rightarrow S \frac{d}{dz} - \text{Mod}(k((S)))
\]

where \( I_{F^*}M \) is the restriction of scalars of \( M \) via \( k((S)) \rightarrow F \), and \( \nabla = I_{F^*} \nabla \) are equal as \( k \)-linear maps. If \((M, \nabla)\) has rank equal to \( n \), the rank of \((I_{F^*}M, I_{F^*} \nabla)\) is equal to \( nm \).

Let \((M, \nabla)\) be an object of \( S \frac{d}{dz} - \text{Mod}(k((S))) \) of rank \( n \). The differential module \((I_{F^*}I_F^*M, I_{F^*}I_F^* \nabla)\) has rank \( nm \). Let \( \{e_1, \ldots, e_n\} \) be a basis of \((M, \nabla)\) and let \( G \) be the associated matrix in this basis.

Then the associated matrix of \((I_{F^*}I_F^*M, I_{F^*}I_F^* \nabla)\) in the basis \( \{e_1 \otimes 1, \ldots, e_n \otimes 1, e_1 \otimes Z, \ldots, e_n \otimes Z^{m-1} \} \) is:

\[
\begin{pmatrix}
G & 0 & \cdots & 0 \\
0 & G + \frac{1}{m} \cdot I_n & \cdots & 0 \\
0 & \cdots & 0 & G + \frac{m-1}{m} \cdot I_n
\end{pmatrix}
\]

Therefore we have the following isomorphism:

\[
(I_{F^*}I_F^*M, I_{F^*}I_F^* \nabla) \cong \bigoplus_{i=0}^{m-1} (M, \nabla + \frac{i}{m})
\]

As \( k \)-Banach spaces \( I_{F^*}I_F^*M \) and \( I_{F^*}M \) are the same, and \( I_{F^*}I_F^* \nabla \) as a \( k \)-linear map coincides with \( I_{F^*} \nabla \). Therefore, we have

\[
\Sigma_{I_F^* \nabla, k}(L_k(I_F^*M)) = \Sigma_{I_F^*I_F^* \nabla, k}(L_k(I_{F^*}I_F^*M)).
\]

By Lemma 2.19 and Corollary 3.7 we have:

\[
\Sigma_{I_{F^*} \nabla, k}(L_k(I_{F^*}M)) = \bigcup_{i=0}^{m-1} \frac{i}{m} + \Sigma_{\nabla, k}(L_k(M))^2.
\]

### 3.3. Newton polygon and the decomposition according to the slopes

Let \( v: k((S^\frac{1}{m})) \rightarrow \mathbb{Z} \cup \{\infty\} \) be the valuation map associated to the absolute value of \( k((S^\frac{1}{m})) \), that satisfies \( v(S^\frac{1}{m}) = \frac{1}{m} \). Let \( P = \sum_{i=0}^{n} g_i D_i \) be an element of \( \mathcal{D}_k((S)) \). Let \( L_P \) to be the convex hull in \( \mathbb{R}^2 \) of the set of points

\[
\{(i, v(g_i)) | 0 \leq i \leq n\} \cup \{(0, \min_{0 \leq i \leq n} v(g_i))\}
\]

**Definition 3.9** ([VS12, Definition 3.44]). The Newton polygon \( NP(P) \) of \( P \) is the boundary of \( L_P \). The finite slopes \( \gamma_i \) of \( P \) are called the slopes of \( NP(P) \). The horizontal width of the segment of \( NP(P) \) of slope \( \gamma_i \) is called the multiplicity of \( \gamma_i \).

\footnote{This is the image of the spectrum by the polynomial function \( \frac{1}{m} + T \).}
Definition 3.10. A differential module \((M, \nabla)\) over \((k(S), S \frac{d}{dS})\) is said to be regular singular if there exists a basis for which \(G\) (cf. (2.1)) has constant entries (i.e \(G \in \mathcal{M}_n(k)\)). We will call the eigenvalues of such \(G\) the exponents of \((M, \nabla)\).

Proposition 3.11. Let \((M, \nabla)\) be a differential module over \((k(S), S \frac{d}{dS})\). The following properties are equivalent:

\begin{itemize}
  \item \((M, \nabla)\) is regular singular;
  \item There exists differential polynomial \(P(D)\) with only one slope equal to 0 such that \((M, \nabla) \simeq (\mathcal{G}_{k(S)}/\mathcal{G}_{k(S)} \cdot P(D))\);
  \item There exists \(P(D) = g_0 + g_1 D + \cdots + g_{n-1} D^{n-1} + D^n\) with \(g_i \in k[S]\), such that \((M, \nabla) \simeq (\mathcal{G}_{k(S)}/\mathcal{G}_{k(S)} \cdot P(D), D)\);
  \item There exists \(P(D) = g_0 + g_1 D + \cdots + g_{n-1} D^{n-1} + D^n\) with \(g_i \in k\), such that \((M, \nabla) \simeq (\mathcal{G}_{k(S)}/\mathcal{G}_{k(S)} \cdot P(D), D)\);
\end{itemize}

Proof. See [Ked10, Corollary 7.1.3] and [Chr, Proposition 10.1].

Proposition 3.12 ([Ked10, Proposition 7.3.6]). Let \(P(D) = g_0 + g_1 D + \cdots + g_{n-1} D^{n-1} + D^n\) such that \(g_i \in k[S]\). Then we have the isomorphism in \(S \frac{d}{dS} - \text{Mod}(k(S))\):

\[(\mathcal{G}_{k(S)}/\mathcal{G}_{k(S)} \cdot P(D), D) \simeq (\mathcal{G}_{k(S)}/\mathcal{G}_{k(S)} \cdot P_0(D)),\]

where \(P_0(D) = g_0(0) + g_1(0) D + \cdots + g_{n-1}(0) D^{n-1} + D^n\).

Remark 3.13. This Proposition means in particular that for all \(f \in k[S]\) there exists \(g \in k(S)\) \(\backslash \{0\}\) such that \(f = \frac{S \frac{d}{dS}}{g} g\). Indeed, by Proposition 3.12 we have \((k(S), S \frac{d}{dS} + f) \simeq (k(S), S \frac{d}{dS} + f(0))\). This is equivalent (to say that there exists \(g \in k(S)\) \(\backslash \{0\}\) such that

\[g^{-1} \circ S \frac{d}{dS} \circ g + f = S \frac{d}{dS} + f(0)\].

Definition 3.14. Let \((M, \nabla)\) be a differential module of \(S \frac{d}{dS} - \text{Mod}(k(S))\), and let \(P(D) \in \mathcal{G}_{k(S)}\) such that \((M, \nabla) \simeq (\mathcal{G}_{k(S)}/\mathcal{G}_{k(S)} \cdot P(D), D)\). If all the slopes of \(P(D)\) are different from 0, then we say that \((M, \nabla)\) is without regular part.

Proposition 3.15. Let \(P \in \mathcal{G}_{k(S)}\) and \(\gamma\) a slope of \(P\). Let \(\nu\) be the multiplicity of \(\gamma\). Then there exist differential polynomials \(R, R', Q\) and \(Q'\) which satisfy the following properties:

\begin{itemize}
  \item \(P = RQ = Q'R'\);
  \item The degree of \(R\) and \(R'\) is equal to \(\nu\), and their only slopes are \(\gamma\) with multiplicity equal to \(\nu\).
  \item All the slopes of \(Q\) and \(Q'\) are different from \(\gamma\).
  \item \(\mathcal{G}_{k(S)}/\mathcal{G}_{k(S)} \cdot P = \mathcal{G}_{k(S)}/\mathcal{G}_{k(S)} \cdot R \oplus \mathcal{G}_{k(S)}/\mathcal{G}_{k(S)} \cdot Q = \mathcal{G}_{k(S)}/\mathcal{G}_{k(S)} \cdot Q' \oplus \mathcal{G}_{k(S)}/\mathcal{G}_{k(S)} \cdot R'\).
\end{itemize}

Proof. See [Chr, Proposition 12.1] and [VS12, Theorem 3.48].

Corollary 3.16. Let \((M, \nabla)\) be a differential module over \((k(S), S \frac{d}{dS})\). Then we have the decomposition in \(S \frac{d}{dS} - \text{Mod}(k(S))\):

\[(M, \nabla) = (M_{\text{reg}}, \nabla_{\text{reg}}) \oplus (M_{\text{irr}}, \nabla_{\text{irr}})\]

where \((M_{\text{reg}}, \nabla_{\text{reg}})\) is a regular singular differential module and \((M_{\text{irr}}, \nabla_{\text{irr}})\) is a differential module without regular part.
4. Spectrum of a differential module

In this section we compute the spectrum of a differential module \((M, \nabla)\) over \((k((S)), S_{dS})\), which is the aim of this paper. The main statement of the paper is the following.

**Theorem 4.1.** Let \((M, \nabla)\) be a differential module over \((k((S)), S_{dS})\). Let \(\{\gamma_1, \cdots, \gamma_n\}\) be the set of the slopes of \((M, \nabla)\) and let \(\{a_1, \cdots, a_n\}\) be the set of the exponents of the regular part of \((M, \nabla)\). Then the spectrum of \(\nabla\) as an element of \(L_k(M)\) is:

\[
\Sigma_{\nabla, k}(L_k(M)) = \{x_{0, r-\gamma_1}, \cdots, x_{0, r-\gamma_n}\} \cup \bigcup_{i=1}^{r_2}(a_i + \mathbb{Z})
\]

**Remark 4.2.** We observe that although differential modules over \((k((S)), S_{dS})\) are algebraic objects, their spectra in the sense of Berkovich depends highly on the choice of the absolute value on \(k((S))\).

According to Corollary 3.16 we have the decomposition:

\[
(M, \nabla) = (M_{\text{reg}}, \nabla_{\text{reg}}) \oplus (M_{\text{irr}}, \nabla_{\text{irr}})
\]

We know that \(\Sigma_{\nabla} = \Sigma_{\nabla_{\text{reg}}} \cup \Sigma_{\nabla_{\text{irr}}}\) (cf. Corollary 3.7). Therefore, in order to obtain the main statement, it is enough to know the spectrum of a regular singular differential module and the spectrum of a pure irregular singular differential module.

4.1. Spectrum of regular singular differential module. Let \((M, \nabla)\) be a regular singular differential module. The behaviour of the spectrum of \(\nabla\) is recapitulated in the following Theorem.

**Theorem 4.3.** Let \((M, \nabla)\) be a regular singular differential module over \((k((S)), S_{dS})\). Let \(G\) the matrix associated to \(\nabla\) with constant entries (i.e. \(G \in M_n(k)\)), and let \(\{a_1, \cdots, a_N\}\) be the set of eigenvalues of \(G\). The spectrum of \(\nabla\) is

\[
\Sigma_{\nabla, k}(L_k(M)) = \bigcup_{i=1}^{N}(a_i + \mathbb{Z}) \cup \{x_{0, 1}\}.
\]

**Lemma 4.4 ([VS12, Proposition 3.12]).** We can assume that the set of the eigenvalues \(\{a_1, \cdots, a_N\}\) satisfies \(a_i - a_j \notin \mathbb{Z}\) for each \(i \neq j\).

In order to prove this Theorem, we use the following Proposition to reduce the computation of the spectrum of \(S_{dS}\) as an element of \((k((S)), \nabla)\).

**Proposition 4.5 ([Azz18, Proposition 3.15]).** Let \((M, \nabla)\) be a differential module over \((k((S)), S_{dS})\) such that:

\[
\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} + G \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},
\]

with \(G \in M_n(k)\). The spectrum of \(\nabla\) is \(\Sigma_{\nabla, k}(L_k(M)) = \bigcup_{i=1}^{N}(a_i + \Sigma_{d, k}(L_k(k((S)))))\), where \(\{a_1, \cdots, a_N\}\) are the eigenvalues of \(G\).

We now compute the spectrum of \(S_{dS}\).

**Lemma 4.6.** The norm and spectral semi-norm of \(S_{dS}\) as an element of \(L_k(k((S)))\) satisfy:

\[
\|S_{dS}\| = 1, \quad \|S_{dS}\|_{sp} = 1.
\]
Proof. Since $\|S\| = |S| = r$ and $\|\frac{d}{ds}\| = \frac{1}{r}$ (cf. [Pul15, Lemma 4.4.1]), we have $\|S \frac{d}{ds}\| \leq 1$. Hence also, $\|S \frac{d}{ds}\|_{Sp} \leq 1$. The map

$$L_k(\mathcal{H}(x)) \mapsto L_k(\mathcal{H}(x))$$

$$\varphi \mapsto S^{-1} \circ \varphi \circ S$$

is bi-bounded and induces change of basis. Therefore, as $S^{-1} \circ (S \frac{d}{ds}) \circ S = S \frac{d}{ds} + 1$, we have $\|S \frac{d}{ds}\|_{Sp} = \|S \frac{d}{ds}\| + 1\|_{Sp}$. Since $1$ commutes with $S \frac{d}{ds}$, we have:

$$1 = \|1\|_{Sp} = \|S \frac{d}{ds}\| + 1\|S \frac{d}{ds}\|_{Sp} \leq \max(\|S \frac{d}{ds}\| + 1\|S\|, \|S \frac{d}{ds}\|_{Sp}).$$

Consequently, we obtain

$$\|S \frac{d}{ds}\| = \|S \frac{d}{ds}\|_{Sp} = 1.$$

\[\square\]

**Proposition 4.7.** The spectrum of $S \frac{d}{ds}$ as an element of $L_k(k(\langle S \rangle))$ is equal to:

$$\Sigma_{S \frac{d}{ds}}(L_k(k(\langle S \rangle))) = \mathbb{Z} \cup \{0\}$$

We need the following Lemma to prove to compute the spectrum.

**Lemma 4.8 (Azz18, Lemma 2.29).** Let $M_1$ and $M_2$ be $k$-Banach spaces and let $M = M_1 \oplus M_2$ endowed with the max norm. Let $p_1, p_2$ be the respective projections associated to $M_1$ and $M_2$ and $i_1, i_2$ be the respective inclusions. Let $\varphi \in L_k(M)$ and set $\varphi_1 = p_1 \varphi i_1 \in L_k(M_1)$ and $\varphi_2 = p_2 \varphi i_2 \in L_k(M_2)$. If $\varphi(M_1) \subseteq M_1$, then we have:

i) $\Sigma_{\varphi_1}(L_k(M_1)) \subseteq \Sigma_{\varphi}(L_k(M)) \cup \Sigma_{\varphi_2}(L_k(M_2))$, where $i, j \in \{1, 2\}$ and $i \neq j$.

ii) $\Sigma_{\varphi}(L_k(M)) \subseteq \Sigma_{\varphi_1}(L_k(M_1)) \cup \Sigma_{\varphi_2}(L_k(M_2))$. Furthermore, if $\varphi(M_2) \subseteq M_2$, then we have the equality.

iii) If $\Sigma_{\varphi_1}(L_k(M_1)) \cap \Sigma_{\varphi_2}(L_k(M_2)) = \emptyset$, then $\Sigma_{\varphi}(L_k(M)) = \Sigma_{\varphi_1}(L_k(M_1)) \cup \Sigma_{\varphi_2}(L_k(M_2))$.

**Proof of Proposition 4.7.** We set $d := S \frac{d}{ds}$ and $\Sigma_{d-n} := \Sigma_{d-n,k}(L_k(k(\langle S \rangle)))$. As $\|d\|_{Sp} = 1$ (cf. Lemma 4.6), we have $\Sigma_d \subseteq D^+(0, 1)$.

Let $a \in k \cap D^+(0, 1)$. If $a \in \mathbb{Z}$, then we have $(d - a)(S^n) = 0$. Hence, $d - a$ is not injective and $\mathbb{Z} \subseteq \Sigma_d$. As the spectrum is compact, we have $\mathbb{Z} \cup \{a, 0\} \subseteq \Sigma_d$. If $a \notin \mathbb{Z}$, then $d - a$ is invertible in $L_k(k(\langle S \rangle))$.

Indeed, let $g(S) = \sum_{i \in \mathbb{Z}} b_i S^i \in k(\langle S \rangle)$, if there exists $f = \sum_{i \in \mathbb{Z}} a_i S^i \in k(\langle S \rangle)$ such that $(d - a)f = g$, then for each $i \in \mathbb{Z}$ we have

$$a_i = \frac{b_i}{(i - a)}.$$

For each $i \in \mathbb{Z}$ we have $|a_i| = |b_i|$. This means that $f$ is unique and converges in $k(\langle S \rangle)$. We obtain also $|f| = |g|$. Consequently, the set theoretical inverse $(d - a)^{-1}$ is bounded and $\|(d - a)^{-1}\| = 1$. Hence, we have $\|\Sigma_{d-n,K}^{-1}\|_{Sp} = 1$. According to Lemma 2.16, we have $D^+(a, 1) \subseteq K_{k,\mathbb{Z}}^{1,\text{an}} \setminus \Sigma_d$.

In order to end the proof, since $D^+(0, 1) = \bigcup_{a \in k} [a, x_{0,1}]$ (cf. (2.9)), it is enough to show that $(n, x_{0,1}) \subseteq K_{k,\mathbb{Z}}^{1,\text{an}} \setminus \Sigma_d$ for all $n \in \mathbb{Z}$. Let $n \in \mathbb{Z}$. Then we have

$$k(\langle S \rangle) = k.S^n \oplus \bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i.$$

The operator $(d - n)$ stabilises both $k.S^n$ and $\bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i$. We set $(d - n)|_{k.S^n} = \nabla_1$ and $(d - n)|_{\bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i} = \nabla_2$. We set $\Sigma_{\nabla_1} := \Sigma_{\nabla_1,k}(L_k(k.S^n))$ and $\Sigma_{\nabla_2} := \Sigma_{\nabla_2,k}(L_k(\bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i))$. We have $\nabla_1 = 0$. By Lemma 4.8, we have:

$$\Sigma_{d-n} = \Sigma_{\nabla_1} \cup \Sigma_{\nabla_2} = \{0\} \cup \Sigma_{\nabla_2}.$$
We now prove that
\[ D^-(0, 1) \cap \Sigma_{\mathcal{V}_2} = \emptyset. \]

The operator \( \nabla_2 \) is invertible in \( \mathcal{L}_k(\bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i) \). Indeed, let \( g(S) = \sum_{i \in \mathbb{Z} \setminus \{n\}} b_i S^i \in \bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i \). If there exists \( f = \sum_{i \in \mathbb{Z} \setminus \{n\}} a_i S^i \in \bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i \) such that \( \nabla_2(f) = g \), then for each \( i \in \mathbb{Z} \setminus \{n\} \) we have
\[ a_i = \frac{b_i}{(i - n)}. \]

Since \( |a_i| = |b_i| \), the element \( f \) exists and it is unique, moreover \( |f| = |g| \). Hence, \( \nabla_2 \) is invertible in \( \mathcal{L}_k(\bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i) \) and as \( k \)-linear map it is isometric. Therefore, we have \( \|\nabla_2^{-1}\|_{sp} = 1 \). Hence, by Lemma 2.16 \( D^-(0, 1) \subset A^{1,an}_k \setminus \Sigma_{\mathcal{V}_2} \). Consequently, \( D^-(0, 1) \cap \Sigma_{d-n} = \{0\} \). As \( \Sigma_d = \Sigma_{d-n} + n \) (cf. Lemma 2.19), we have \( D^-(n, 1) \cap \Sigma_d = \{n\} \). Therefore, for all \( n \in \mathbb{Z} \) we have \( (n, x_0, 1) \subset A^{1,an}_k \setminus \Sigma_d \) and the claim follows. \( \square \)

### 4.2. Spectrum of a differential module without regular part

Recall the following Theorem, which is the celebrated theorem of Turrittin. It ensures that any differential module becomes extension of one differential modules after pull-back by a convenient ramification ramified extension.

**Theorem 4.9 ([Tur55]).** Let \( (M, \nabla) \) be a differential module over \( (k((S)), S \frac{d}{dx}) \). There exists a finite extension \( F = k((S^{\pi})) \) such that we have:

\[ (I_F^* M, I_F^* \nabla) = \bigoplus_{i=1}^{N} (\mathcal{D}_F / \mathcal{D}_F.(D - f_i)^{\alpha_i}, D) \]

where \( f_i \in k[[S^{\pi}]] \) and \( \alpha_i \in \mathbb{N} \).

**Proof.** See [VS12, Theorem 3.1]. \( \square \)

Now, in order to compute the spectrum, we need the following Theorem.

**Proposition 4.10.** Let \( f = \sum_{i \in \mathbb{Z}} a_i S^{\pi} \) an element of \( F := k((S^{\pi})) \) and let \( (F, \nabla) \) be the differential module of rank one such that \( \nabla = S \frac{d}{dx} + f \). If \( v(f) < 1 \), then the spectrum of \( \nabla \) as an element of \( \mathcal{L}_k(F) \) is:

\[ \Sigma_{\nabla,k}(\mathcal{L}_k(F)) = \{x_{a,r^{\pi}(t)}\}. \]

The following results are necessary to prove this Proposition.

**Lemma 4.11.** Let \( \Omega \in E(k) \). Consider the isometric embedding of \( k \)-algebras

\[ \begin{align*}
\Omega & \rightarrow \mathcal{L}_k(\Omega) \\
\alpha & \mapsto b \mapsto a.b.
\end{align*} \]

With respect to this embedding, \( \Omega \) is a maximal commutative subalgebra of \( \mathcal{L}_k(\Omega) \).

**Proof.** Let \( A \) be a commutative subalgebra of \( \mathcal{L}_k(\Omega) \) such that \( \Omega \subset A \). Then each element of \( A \) is an endomorphism of \( \Omega \) that commutes with the elements of \( \Omega \). Therefore, \( A \subset \mathcal{L}_\Omega(\Omega) = \Omega \). Hence, we have \( A = \Omega \). \( \square \)

**Lemma 4.12.** Let \( \Omega \in E(k) \) and \( \pi_{\Omega/k} : A^{1,an}_k \rightarrow A^{1,an}_k \) be the canonical projection. Let \( \alpha \in \Omega \). The spectrum of \( \alpha \) as an element of \( \mathcal{L}_k(\Omega) \) is \( \Sigma_{\alpha,k}(\mathcal{L}_k(\Omega)) = \{ \pi_{\Omega/k}(\alpha) \} \).

**Proof.** By [Ber90, Proposition 7.1.4, i)], the spectrum of \( \alpha \) as an element of \( \Omega \) is the point which corresponds to the character \( k[T] \rightarrow \Omega, T \mapsto \alpha \). Hence, \( \Sigma_{\alpha,k}(\Omega) = \{ \pi_{\Omega/k}(\alpha) \} \). By Lemma 4.11 and [Ber90, Proposition 7.2.4] we conclude. \( \square \)
Lemma 4.13 ([Azz18, Lemma 2.3]). Let \( \Omega \in E(k) \) and let \( M \) be a \( k \)-Banach space. Then, the inclusion \( M \hookrightarrow M \otimes_k \Omega \) is an isometry. In particular, for all \( v \in M \) and \( c \in K \) we have \( \| v \otimes c \| = |c| \cdot \| v \| \).

Proof of Lemma 4.10. We set \( d := S \frac{d}{dx} \). We can assume that \( f = \sum_{i \in \mathbb{N}} a_i S^{\frac{d}{dx}} \). Indeed, since \( f = f_- + f_+ \) with \( f_- := \sum_{i < 0} a_i S^{\frac{d}{dx}} \) and \( f_+ := \sum_{i \geq 0} a_i S^{\frac{d}{dx}} \), according to Remark 3.13 there exists \( g \in k((S^{\frac{d}{dx}})) \) such that \( f_+ - a_0 = \frac{S^{\frac{d}{dx}}(g)}{g} \). Therefore, we have \( \langle k((S^{\frac{d}{dx}})), \nabla \rangle \simeq \langle k((S^{\frac{d}{dx}})), S \frac{d}{dx} \rangle + f_- + a_0 \). Since the point \( \pi_{F/k}(f) \) corresponds to the character \( k[T] \twoheadrightarrow F, T \mapsto f \) and \( F \simeq \mathcal{H}(x_{0,r \frac{d}{dx}}) \), it coincides with \( f(x_{0,r \frac{d}{dx}}) \) (cf. Notation 2.8). Moreover, we have \( f(x_{0,r \frac{d}{dx}}) = x_{0,r}, f_0 = x_{0,r}(f) \). By Lemma 4.12 \( \Sigma_{f,k}(L_k(F)) = \{ x_{0,r}(f) \} \). Let us prove now that \( \Sigma_v,k(L_k(F)) = \{ x_{0,r}(f) \} \). Let \( y \in k_{1,\text{an}} \setminus \{ x_{0,r}(f) \} \). We know that \( f \otimes 1 - \mathcal{T}(y) \) is invertible in \( F \otimes_k \mathcal{H}(y) \), hence invertible in \( L_k(F) \otimes_k \mathcal{H}(y) \). Since \( d \otimes 1 = (\nabla \otimes 1 - \mathcal{T}(y)) \), in order to prove that \( \nabla \otimes 1 - \mathcal{T}(y) \) is invertible, it is enough to show that

\[
\| (f \otimes 1 - \mathcal{T}(y))^{-1} \| = \| f \otimes 1 - \mathcal{T}(y) \|^{-1}.
\]

In order to do so, since \( \| d \| = \| d \otimes 1 \| = 1 \) (cf. Lemmas 4.13 and 4.6), it is enough to show that

\[
1 < \| (f \otimes 1 - \mathcal{T}(y))^{-1} \| = \max(|f - a_0|, |\mathcal{T}(y) - a_0|).
\]

Consequently, we obtain \( 1 < \| (f \otimes 1 - \mathcal{T}(y))^{-1} \| = \max(|f - a_0|, |\mathcal{T}(y) - a_0|) \). Since the spectrum \( \Sigma_v,k(L_k(F)) \) is not empty, we conclude that \( \Sigma_v,k(L_k(F)) = \{ x_{0,r}(f) \} \).

Proposition 4.14. Let \((M, \nabla)\) be a differential module over \((k((S)), S \frac{d}{dx})\) without regular part. The spectrum of \( \nabla \) as an element of \( L_k(M) \) is:

\[
\Sigma_v,k(L_k(M)) = \{ x_{0,r}(f_1), \ldots, x_{0,r}(f_N) \}
\]

where the \( f_i \) are as in the formula (4.1).

Proof. We set \( \Sigma_v := \Sigma_v,k(L_k(M)) \). By Theorem 4.9, there exists \( F = k((S^{\frac{d}{dx}})) \) such that

\[
(I_F \ast M, I_F \ast \nabla) = \bigoplus_{i=1}^N \mathcal{D}_F / \mathcal{D}_F(D - f_i)^{s_i}
\]

where \( f_i \in k[S^{\frac{d}{dx}}] \). We set \( \Sigma_{I_F} := \Sigma_{I_F \ast \nabla,k}(L_k(I_F \ast M)) \). Since \((M, \nabla)\) is without regular part, we have \( f_i \in k[S^{\frac{d}{dx}}] \setminus k \). By Corollaries 3.7 and 3.8, we have:

\[
\Sigma_{I_F} = \bigcup_{i=1}^N \Sigma_{S^{\frac{d}{dx}} + f_i}(L_k(F)).
\]

By Proposition 4.10, we have \( \Sigma_{S^{\frac{d}{dx}} + f_i}(L_k(F)) = \{ x_{0,r}(f_i) \} \). Hence,

\[
\Sigma_{I_F} = \{ x_{0,r}(f_1), \ldots, x_{0,r}(f_N) \}.
\]

By the formula (3.8), we have:

\[
\Sigma_{I_F} = \bigcup_{i=0}^{m-1} \frac{i}{m} + \Sigma_v.
\]

Since \( r^{\nu(f_i)} > 1 \) for all \( 1 \leq i \leq N \), then each element of \( \Sigma_{I_F} \) is invariant by translation by \( \frac{1}{m} \) where \( 1 \leq j \leq m \). This means that \( \Sigma_v = \Sigma_v + \frac{j}{m} \). Therefore, we have \( \Sigma_{I_F} = \Sigma_v \).

\[\Box\]
Remark 4.15. Note that, it is not easy to compute the $f_i$ of the formula (4.1). However, the values $-v(f_i)$ coincide with the slopes of the differential module (cf. [Kat87] and [VS12, Remarks 3.55]).

We now prove the main statement of the paper that summarizes all the previous results.

Proof of Theorem 4.1. According to Theorem 4.3, Proposition 4.14 and Remark 4.15 we obtain the result. □

REFERENCES

[Azz18] T. A. Azzouz. “Spectrum of a linear differential equation with constant coefficients”. In: (Feb. 2018). eprint: 1802.07306. url: https://arxiv.org/abs/1802.07306.

[Ber90] V. G. Berkovich. Spectral Theory and Analytic Geometry Over non-Archimedean Fields. AMS Mathematical Surveys and Monographs 33. AMS, 1990.

[Ber93] V. G. Berkovich. “Étale cohomology for non-Archimedean analytic spaces.” English. In: Publ. Math., Inst. Hautes Étud. Sci. 78 (1993), pp. 5–161. issn: 0073-8301; 1618-1913/e. DOI: 10.1007/BF02712916.

[Bou07] N. Bourbaki. Théories spectrales: Chapitres 1 et 2. Bourbaki, Nicolas. Springer Berlin Heidelberg, 2007. isbn: 9783540353317.

[Chr] G. Christol. “Le théorème de turritin p-adique (version du 11/06/2011)”.

[DMR07] P. Deligne, B. Malgrange, and J. Ramis. Singularités irrégulières. Correspondance et documents. French. Vol. 5. Paris: Société Mathématique de France, 2007, pp. x + 188. isbn: 978-2-85629-241-9/hbk.

[Kat70] N. M. Katz. “Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin.” English. In: Publ. Math., Inst. Hautes Étud. Sci. 39 (1970), pp. 175–232. issn: 0073-8301; 1618-1913/e. DOI: 10.1007/BF02684688.

[Kat87] N. M. Katz. “On the calculation of some differential galois groups”. In: Inventiones mathematicae 87.1 (Feb. 1987), pp. 13–61. issn: 1432-1297. DOI: 10.1007/BF01389152. url: https://doi.org/10.1007/BF01389152.

[Ked10] K. S. Kedlaya. p-adic differential equations. English. Cambridge: Cambridge University Press, 2010, pp. xvii + 380. isbn: 978-0-521-76879-5/hbk.

[Mal74] B. Malgrange. “Sur les points singuliers des équations différentielles”. In: Enseignement Math. (2) 20 (1974), pp. 147–176. issn: 0013-8584.

[Poi13] J. Poineau. “Les espaces de Berkovich sont angéliques”. In: Bull. soc. Math. France (2013).

[Pul15] A. Pulita. “The convergence Newton polygon of a p-adic differential equation. I: Affinoid domains of the Berkovich affine line.” English. In: Acta Math. 214.2 (2015), pp. 307–355. issn: 0001-5962; 1871-2509/e. DOI: 10.1007/s11511-015-0126-9.

[Rob80] P. Robba. “Lemmes de Hensel pour les opérateurs différentiels. Application à la reduction formelle des équations différentielles.” French. In: Enseign. Math. (2) 26 (1980), pp. 279–311. issn: 0013-8584; 2309-4672/e.

[Tur55] H. L. Turrittin. “Convergent solutions of ordinary linear homogeneous differential equations in the neighborhood of an irregular singular point”. In: Acta Math. 93 (1955), pp. 27–66. DOI: 10.1007/BF02392519. url: https://doi.org/10.1007/BF02392519.

[VS12] M. Van der Put and M. F. Singer. Galois Theory of Linear Differential Equations. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2012. isbn: 9783642557507.

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