Framed link presentations of 3-manifolds by an $O(n^2)$ algorithm, II: colored complexes and boundings in their complexity

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May 5, 2014

Abstract

This is part 2 of a 3-part article where we provide an $O(n^2)$-algorithm to produce a surgery presentation of a 3-manifold induced by a gem with a resolution. In this part we produce a sequence of colored simplicial 2-complexes which are inverses and dual to the sequence of gems produced in the first part. The refinements of the PL2-faces that keep appearing are idempotent: the second refinement of a PL2-face is isomorphic to its first refinement. This fact inhibits exponentiability.

1 Colored 2-complexes, 2-skeleton of $\mathcal{H}^*$

This is the second of 3 closely related articles. References for the companion papers are [1] and [2].

We start with $\mathcal{H}_1^*$ which is easily obtained from $\mathcal{H}_1$. We get $\mathcal{H}_m^*$ from $\mathcal{H}_m$ and $\mathcal{H}_{m-1}^*$ by displaying the difference between $\mathcal{H}_m^*$ and its predecessor. The difference is precisely the balloon which becomes a pillow, encoded as $(c : u-r)$, where: $c \in \{0, 1\}$ and $u, r$ are the odd vertices in the $J^2$-gem defining the $c$-flip with $u, v = u + 1$ corresponding to the 2 PL3-faces of the balloon’s head.

1.1 The sequence of combinatorial 2-complexes: $\mathcal{H}_1^*, \mathcal{H}_2^*, \ldots, \mathcal{H}_n^*$

For $m \in \{1, 2, \ldots, n\}$ define $\mathcal{H}_m^*$ to be the 2-skeleton of the dual of the gem $\mathcal{H}_m \setminus 2n$. Removing vertex $2n$ corresponds to removing a tetrahedron from $S^3$ and so we know that $\mathcal{H}_m^*$ embeds into $\mathbb{R}^3$. Our goal is to obtain an explicit embedding of $(\mathcal{J}^2)^* = \mathcal{H}_n^*$ into $\mathbb{R}^3$. The embedding will be explicit in the sense that $\mathcal{H}_m^*$ will be given by a 2-dimensional PL simplicial complex where the 0-simplices are endowed with $\mathbb{R}^3$-coordinates.

It is a simple matter to obtain the required embedding for the bloboid $\mathcal{H}_1^*$. Our initial strategy is to proceed from a combinatorial 2-complex of $\mathcal{H}_m^*$ to generate a combinatorial description of the next, $\mathcal{H}_{m+1}^*$. In the process we give upper bounds for the number of 0-, 1- and 2-simplices arising. These are quadratic polynomials in $|V_{\mathcal{J}^2}|$, the number of vertices of the gem $\mathcal{J}^2$. Note that $|V_{\mathcal{J}^2}| = |V_{\mathcal{G}}|$, where $\mathcal{G}$ is the original resolvable gem.

The inverse of a 2-dipole thickening is a 3-dipole slimming. We need to consider the combinatorial duals of some objects. To a 2-dipole corresponds a pillow in the dual, namely two tetrahedra with two faces in common. To a 3-dipole (or blob) and a color involved in it corresponds a balloon in the dual. The balloon is formed by two tetrahedra with 3 faces in common together with a triangle sharing an edge with the two tetrahedra. The dual of a 3-dipole slimming is a balloon-pillow move, or bp-move. A sequence of these moves related to the $r_{5^24}$-example are depicted in Figs. [12] and [13].

There is a simple topological interpretation between primal and dual complexes, given in [3] pages 38, 39. Let’s take a look at this interpretation in our context. This will help to understand the PL-embedding $\mathcal{H}_m^*$. In what follows the $k$ in PL-$k$-face means the dimension $k \in \{0, 1, 2, 3\}$ of the PL-face.

i. a vertex $v$ in $G :=$ a solid PL-tetrahedron or PL3-face, denoted by $\nabla_v$ in the dual of the gem whose PL0-faces are labelled $z_{0v}, z_{1v}, z_{2v}, z_{3v}$; in this work is enough to work with the boundary of a PL3-face; this is topologically a sphere $S^2$ with four PL2-faces one of each color; the 3-simplices forming a PL3-face need not be explicitly specified;

*2010 Mathematics Subject Classification: 57M25 and 57Q15 (primary), 57M27 and 57M15 (secondary)
ii. an i colored edge $e_i$ in $G$ is a set of i-colored 2-simplices defining a PL2-face in the dual of the gem;

iii. a bigon $B_{ij}$ using the colors $i,j$ in $G$ is a set of 1-simplices $b_{ij}$ in $\mathcal{H}^*_m$ defining a PL1-face;

iv. an $i$-residue $V_i$ in $G$ is a 0-simplex in $\mathcal{H}^*_m$ defining a PL0-face.

We define the combinatorial 2-dimensional PL complex $\mathcal{H}^*_1$ as follows.

The 0-simplices $z_0$, $z_1$ and $z_2$ are positioned in clockwise order as the vertices of an equilateral triangle of side $\psi$ in the $xy$-plane so that $z_0 z_1$ is parallel to the x-axis and the center of the triangle coincides with the origin of an $\mathbb{R}^2$-cartesian system. The 0-simplex $a_1 := \frac{\psi}{2} x + \frac{\psi}{2} y$. The 0-simplex $b_1 := \frac{\psi}{2} x - \frac{\psi}{2} y$. Let the 0-simplices $z_3^j$ be defined as $z_3^j = (0, 0, (2n - j)\psi)$, $1 \leq j \leq 2n$, where $\psi$, as $\varphi$, is a positive constant, see Fig. 1. It is convenient, to leave $\varphi$ and $\psi$ as independent arbitrary positive constants, for adjusting the visual aspect of the embedded $\mathcal{H}^*_m$.

Suppose $u$ is an odd vertex of the $J^2$-gem, $u' = u - 1$, $v = u + 1$ and $v' = v + 1$. The dual of a $3$-residue is $z_3^j$, where $j$ is even. When $j$ is odd, then $z_3^j$ is a 0-simplex in the middle of a PL2-face, incident to five 2-simplices of color 3. The dual of the 03-gon is the PL1-face formed by the pair of 1-simplices $z_1 b_1$ and $b_1 z_2$. The dual of the 12-gon is the PL1-face formed by the pair of 1-simplices $z_2 a_1$ and $a_1 z_2$. The dual of the 23-gon is the PL1-face formed by the 1-simplex $z_0 z_1$. The dual of the 01-gon relative to vertices $u$ and $v$ is the 1-simplex $z_2 z_1^\dagger$. The dual of the 02-gon relative to vertices $u$ and $v$ is the 1-simplex $z_1 z_2^\dagger$. The dual of the 12-gon relative to vertices $u$ and $v$ is the 1-simplex $z_0 z_3^\dagger$. The dual of a 3-colored edge $u'v'$ is the image of PL2-face with odd index $u$ in the vertices. The dual of an i-colored edge $uv$ with $i \in \{0, 1, 2\}$ is the PL2-face with even index $v$.

Before presenting $\mathcal{H}^*_m$, $1 \leq m \leq n$, and its embeddings, we need to understand the dual of the $(pb)^*$-move and its inverse. In the primal, to apply a $(pb)^*$-move, we need a blob and a 0- or 1-colored edge. The dual of this pair is the balloon: the balloon’s head is the dual of the blob; the balloon’s tail is the dual of the i-edge. To make it easier to understand, the $(pb)^*$-move can be factorable into a 3-dipole move followed by a 2-dipole move, so in the dual, it is a smashing of the head of the balloon followed by the pillow move described in the book [3], page 39. This composite move is the balloon-pillow move or bp-move. Considering it as a single move is easier to implement and the code is faster. Restricting our basic change in the colored 2-complex to bp-moves we have nice theoretical properties which are responsible for avoiding an exponential process. In what follows we describe the bp-move assuming that the balloon’s tail is 0-colored using a generic balloon’s tail, of which we just draw the contour. The other case, color 1, is similar.

i. if the image of $v_5^b$ and $v_5^b$ is $b_1$, create two 0-simplices $b_1^\prime$ and $b_1^\prime\prime$, define the images of $v_5^b$ and $v_5^b$ as $b_1^\prime$ and $b_1^\prime\prime$ and change the label of the image of $v_5^b$ from $b_4$ to $B_4$;

ii. make two copies of the PL0-face, if necessary, refine each, from the middle vertex of the segment $z_2 z_1$ to the third vertex $z_3^j$, where $\dagger = j$, for an adequate height $j$;

iii. change the colors of the medial layer of the pillow as specified by the dual structure, namely by the current $J^2 B$-gem, see Fig. 2

The choice of the letters $P, B, R, G$ in the set types of next proposition comes from the colors $0 = (P)ink$, $1 = (B)lue$, $2 = (R)ed$ and $3 = (G)reen$. In the next proposition $R_{2k-1}^p$ is a PL2-face which is inside the pillow neighboring a PL2-face. Similarly $R_{2k-1}^p$ is a PL2-face which is inside the pillow neighboring a PL2-face.

(1.1) Proposition. Each PL2-face of the combinatorial simplicial complex $\mathcal{H}^*_m$, $1 \leq m \leq n$, is isomorphic to one in the set of types of triangulations

$$\{ G, P_{2k-1}, P'_{2k-1}, B_{2k-1}, B'_{2k-1}, R^p_{2k-1}, R^p_{2k-1} \mid k \in \mathbb{N} \}.$$
we use arithmetic mod 2n
u is odd, u' = u - 1
v = u + 1, v' = u + 2

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Primal and dual bp-moves.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{All kinds of PL2-faces.}
\end{figure}

\textbf{Proof.} We need to fix a notation for the head of the balloon, instead of drawing all the PL2-faces of the head, we just draw one PL2-face and put a label u'-v'. If the balloon’s tail, is of type \( P_1 \), by applying a bp-move we can see at Fig. \[\text{4}\] that we get a PL2-face of type \( B_3 \) and a PL2-face of type \( R_3^b \). The others PL2-faces are already known. If the balloon’s tail, is of type \( B_3 \), by applying a bp-move, we need to refine the tail and the
copies, otherwise we would not be able to build a pillow because some 2-simplices would be collapsed, so we get two PL2\(_1\)-faces of type \(B'_3\), one PL2\(_0\)-face of type \(P_5\) and a PL2\(_2\)-face \(R^p_{b2k-1}\). The others PL2-faces are already known.

In what follows given \(X \in \{P'_{2k-1}, B'_{2k-1}\}\) denote by \(\hat{X}\) the copy of \(X\) which is a PL2-face of the PL-tetrahedra whose PL2\(_3\)-face is below the similar PL2\(_3\)-face of the other PL-tetrahedra which completes the pillow in focus. In face of these conventions, if balloon's tail is of type \(P_{2k-1}\), then by applying a \(bp\)-move, we get types \(P'_{2k-1}, \hat{P}'_{2k-1}, B_{2k+1}, R^b_{2k-1}\)

- \(P'_{2k-1}\), then by applying a \(bp\)-move, we get types \(P_{2k-1}, B_{2k+1}, R^b_{2k-1}\)
- \(B_{2k-1}\), then by applying a \(bp\)-move, we get types \(B'_{2k-1}, \hat{B}'_{2k-1}, P_{2k+1}, R^p_{2k-1}\)
- \(B'_{2k-1}\), then by applying a \(bp\)-move, we get types \(B_{2k-1}, P_{2k+1}, R^p_{2k-1}\)

In the sequel we will see that with this combinatorics attached to the PL2-faces the combinatorial \(H^*\)'s can be PL-embedded into \(\mathbb{R}^3\). It is worthwhile to mention, in view of the above proof, that each PL2-face is refined at most one time. So, if \(X\) is a type of PL2-face, \(X'\) is its refinement, then \(X'' = X'\). This idempotency is a crucial property inhibiting the exponentiality of our algorithm.

### 1.2 Upper bounds for the number of simplices of the complex \(H^*_n\)

Now we give quadratic upper bounds for the number of \(i\)-simplices, \(i \in \{0, 1, 2\}\) of \(H^*_n\).

#### (1.2) Lemma. The quadratic expressions

\[
3n^2 - 5n + 9, \quad 11n^2 - 17n + 21, \quad 8n^2 - 10n + 12
\]

are upper bounds for the numbers of 0-simplices, 1-simplices and 2-simplices of the colored 2-complex \(H^*_n\) induced by a resolvable gem \(G\) with \(2n\) vertices.

**Proof.**
**Case $i = 0$**

We know that $\mathcal{H}_i^*$ has exactly $z_0, z_1, z_2, a_1, b_1$ and $z_j^i, j \in \{1, \ldots, 2n\}$ as 0-simplices, which is $2n + 5$ 0-simplices. In first step balloon’s tail has to be of type $P_1$ or $B_1$, so by applying $bp$-moves, we get two new 0-simplices, which inverse image is a black and white disk in Fig. 6 (first part).

In second step the worse case is when balloon’s tail is of type $P_3$ or $B_3$, generated by last $bp$-move, so we add $6 \times 1 + 2 = 8$ to the number of 0-simplices in the upper bound. See Fig. 6 (second part).

![Figure 6: Upper bound for the number of the 0-simplices, first and second steps.](image)

In step $k$ we note that worst case is when we use the greatest ranked PL2-face generated by last $bp$-move, so it means that balloon’s tail has to be of type $P_{2k-1}$ or $B_{2k-1}$, (in Fig. 7 $j = 2k - 1$) which we add $6 \cdot (k - 1) + 2$ 0-simplices. By adding the number of 0-simplices created by $bp$-moves from step 1 until step $k$ we get $3k^2 - k$

![Figure 7: Upper bound for the number of the 0-simplices, $j$-th step.](image)

0-simplices. As the number of steps is $n - 1$, and we have at the beginning $2n + 5$, we have that $3n^2 - 5n + 9$ is an upper bound for the number of 0-simplices.

**Case $i = 1$**

We know that $\mathcal{H}_i^*$ has exactly $z_0z_1, z_0a_1, z_1b_1, z_2b_1, z_2a_1, z_0z_j^1, z_1z_j^1, b_1z_j^1, z_2z_j^1, a_1z_j^1, j \in \{1, \ldots, 2n\}$ as 1-simplices which give us $10n + 5$ 1-simplices. In first step balloon’s tail has to be of type $P_1$ or $B_1$, so by applying $bp$-moves, we add $2 \times 3 = 6$ to the number of 1-simplices in the upper bound.

In second step we know that worst case is when balloon’s tail is of type $P_3$ or $B_3$, see Fig. 8 (second part) and we add $8 + 2 \times 11 - 2 = 28$ to the number of 1-simplices in the upper bound.

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In step \( k \), as in the 0-simplex case, the worst case is when we use the greatest ranked PL2-face generated by last \( bp \)-move, so it means that balloon’s tail has to be of type \( P_{2k-1} \) or \( B_{2k-1} \). (in Fig. 8 \( j = 2k - 1 \)) and we add \( 2(3 + 8(k - 1)) + 8(k - 1) - 2(k - 1) = 22k - 16 \) to the number of 1-simplices in the upper bound. An upper bound for the number of 1-simplices created by \( bp \)-moves from step 1 until step \( k \) is \( 11k^2 - 5k \). As the number of steps is \( n - 1 \) and we have at the beginning \( 10n + 5 \), it follows that, by adding the arithmetical progression, \( 11n^2 - 17n + 12 \) is an upper bound for the number of 1-simplices.

**Case \( i = 2 \)**

We know that \( H_1^\prime \) has exactly \( z_0z_1^i, z_1z_3b_1, z_2b_1z_3^j, z_2z_3^ja_1, z_0a_1z_3^j, j \in \{1, \ldots, 2n\} \) as 2-simplices which give us \( 10n \) 2-simplices.

In first step balloon’s tail has to be of type \( P_1 \) or \( B_1 \), so by applying \( bp \)-moves, we add \( 2 \times 2 \) 2-simplices. In second step we add \( 3 \times 8 \) and subtract 4.

As we know, the worst case in step \( k \) is when balloon’s tail is of type \( P_{2k-1} \), or \( B_{2k-1} \). By apply \( bp \)-move we add \( 3 \cdot (6 \cdot (k - 4)) \) and subtract \( 2k \), for \( k \geq 2 \).

By adding the number of 2-simplices created by \( bp \)-moves from step 1 until step \( k \) we get \( 8k^2 - 4k \) 2-simplices. As the number of steps is \( n - 1 \), and we have at the beginning \( 10n \) 2-simplices, \( 8n^2 - 10n + 12 \) is an upper bound for the number of 2-simplices.

\( \square \)
Figure 10: Upper bound for the number of the 2-simplices, first and second steps.

Figure 11: Upper bound for the number of the 2-simplices, $j$-th step.

References

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1.3 Appendix A:
sequence of $bp$-move corresponding to $r_{S^5}^{24}$

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Figure 12: sequence of \( bp \)-moves, \( m = 1, \ldots, 6 \) (\( r_{24}^q \)-example).
Figure 13: sequence of $bp$-moves, $m = 7, \ldots, 11$ ($r_5^{24}$-example).