Primary Rate-Splitting Achieves Capacity for the Gaussian Cognitive Interference Channel

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Abstract—The cognitive interference channel models cognitive overlay radio systems, where cognitive radios overhear the transmission of neighboring nodes. Capacity for this channel is not known in general. For the Gaussian case capacity is known in three regimes, usually denoted as the “weak interference”, “very strong interference” and “primary decodes cognitive”. This paper provides a new capacity result, based on rate-splitting of the primary user’s message into a public and private part and that generalizes the capacity results in the "very strong interference" and “primary decodes cognitive” regimes. This result indicates that capacity of the cognitive interference channel not only depends on channel conditions but also the level of cooperation with the primary user.

Index Terms—cognitive interference channel, capacity, superposition coding, binning, rate-splitting, strong interference.

I. INTRODUCTION

A new generation of smart wireless devices is emerging that can sense and adapt to the surrounding radio environment and this technological development promises to drastically improve the efficiency in using the radio frequency spectrum. A model that captures the role of cooperation in overlay cognitive radio networks is the cognitive interference channel [1]. This channel is obtained from the classic interference channel by providing one of the transmitter, the cognitive transmitter, with the message of the other transmitter, the primary transmitter. The extra information at the cognitive transmitter models the ability of this node to acquire information about the primary user by exploiting the broadcast nature of the wireless medium.

The capacity of a cognitive interference channel for both the discrete memoryless case and the Gaussian case remains unknown in general. However, general outer bounds [2] as well as inner bounds [3] for this channel have been derived. Capacity is known for the discrete memoryless case in the “better cognitive decoding” regime, in which capacity is achieved using rate-splitting and superposition coding [3]. A larger set of capacity results is available for the Gaussian case: here capacity is known in three different regimes. In the “weak interference” regime [4] capacity is achieved by having the encoders cooperate in transmitting the primary message and by having the primary receiver treat the interference as noise while the cognitive transmitter pre-codes its message against the known interference. Capacity is known for channels in the “very strong interference” regime and is achieved by superimposing the cognitive message over the primary message and having both decoders decode both messages.

The last regime in which capacity is known for the Gaussian case is the “primary decodes cognitive” regime [5]. Here capacity is achieved by pre-coding the cognitive codeword against the interference created by the primary transmission and having the primary receiver decode both the primary and the cognitive codeword. The primary decoder gains insight over its own message by decoding the cognitive codeword, since the interference against which the cognitive codeword is pre-coded is indeed the primary codeword. Capacity for the Gaussian case is also known by within 1 bit/s/Hz and to within a factor of two [6], that is, a bounded difference between inner and outer bound has been established as well as a bounded ratio.

In the following we derive a new capacity result that generalizes the capacity results available for the “very strong interference” and the “primary decodes cognitive” regimes. This result is obtained by considering an achievable scheme that includes the capacity achieving schemes in the regimes above as a special cases. In this scheme the primary message is rate-split into a public and a private part and the private part is then superposed over the public one. The cognitive message is also superposed over the public primary message and binned against the private primary message. By determining the optimal rate-splitting between public and private primary message, we obtain capacity in a region that contains both the “very strong interference” and the “primary decodes cognitive” regimes. This result shows, in particular, that the optimal transmission strategy depends not only on the channel condition but also on the level of cooperation between the cognitive and the primary users. This is indeed a very interesting results since, in all the previously known capacity results for the Gaussian case, a single transmission scheme achieves capacity in the whole capacity region while, in this new result, capacity is achieved using two distinct transmission strategies.

Paper Organization:

The paper is organized as follows: In Sec. II we introduce the channel model, the Gaussian cognitive interference channel. In Sec. III we present some known results for this channel model while in Sec. IV we introduce the inner bound that we will use to prove capacity. In Sec. V we prove the new capacity result by showing the achievability of an outer bound presented in Sec. III with the inner bound in Sec. IV. In VI we show the new region in which capacity is derived using numerical simulations. Sec. VII concludes the paper.
III. Known Results for the G-CIFC

We begin by reviewing some known results for the G-CIFC that are relevant for the remainder of the paper.

Theorem III.1. “Weak Interference” capacity [4, Lem. 3.6]

If \(|b| < 1\), the capacity of the G-CIFC is the union over \(\alpha \in [0, 1]\) of the region

\[
R_1 \leq I(Y_1; X_1|X_2) = \mathcal{C}(\alpha P_1) \\
R_2 \leq I(Y_2; X_2) = \mathcal{C}(|b|^2 P_1 + P_2 + 2\sqrt{\alpha}|b|^2 P_1 P_2) - \mathcal{C}(\alpha P_1). 
\]

Capacity in the “weak interference” regime is achieved by pre-coding the cognitive codeword against the interference experienced at the cognitive decoder and treating the interference as noise at the primary decoder.

Theorem III.2. “Strong interference” outer bound [9, Th. 4] If

\[|b| \geq 1,\]

the capacity of the G-CIFC is contained in the union over \(\alpha \in [0, 1]\) of the region

\[
R_1 \leq I(Y_1; X_1|X_2) = \mathcal{C}(\alpha P_1) \\
R_1 + R_2 \leq I(Y_2; X_1, X_2) = \mathcal{C}(|b|^2 P_1 + P_2 + 2\sqrt{\alpha}|b|^2 P_1 P_2). 
\]

The rate bound (8a) is a general bound that holds for any G-CIFC and suggests that the largest rate \(R_1\) can be achieved by either pre-canceling or decoding \(X_2\) at the cognitive receiver. The sum rate bound (8b) holds only under condition (7). The primary receiver, after having decoded its message, can reconstruct the channel output at the cognitive receiver. This consideration provides an intuitive interpretation of the sum rate bound in (8b) which suggest that the primary receiver can decode both messages without loss of optimality.

Theorem III.3. “Very strong interference” capacity [9, Th. 4] If condition (7) and

\[
(1 - |b|^2)P_1 + (|a|^2 - 1)P_2 \geq 0 \\
(1 - |b|^2)P_1 + (|a|^2 - 1)P_2 \geq 2(|b| - \Re\{a^*\})\sqrt{\pi P_1 P_2},
\]

hold, the region in (8) is the capacity region.

Proof: The proof for complex channel coefficients can be found in [7] App. B. This result can be improved by noticing that one can restrict \(\alpha \in [0, 1]\) in the inner bound to match the strong interference outer bound in Th. III.2.

In the “very strong interference” regime, capacity is achieved by having both decoders decode both messages and superimposing the cognitive message over the primary message.
Theorem III.4. “Primary decodes cognitive” capacity

\[ P_2|1 - a|b|^2(1 + P_1) \geq (|b|^2 - 1)(1 + P_1 + |a|^2P_2) \] (10a)

\[ P_2|1 - a|b|^2 \geq (|b|^2 - 1)(1 + P_1 + |a|^2P_2 - 2\text{Re}(a)\sqrt{P_1P_2}) \] (10b)

hold, the region in \([8]\) is the capacity region.

In Th. III.4 capacity is achieved by pre-coding the cognitive message against the interference and having the primary receiver decode this codeword as well.

IV. INNER BOUND

The largest known inner bound for a general CIFC is obtained in \([10]\) while a compact expression for this region is provided in \([7]\) Sec. IV. \([7]\) also provides a series of simpler transmission schemes that are special cases of the most general achievable scheme and can be expressed using a limited set of parameters. In the following we consider a transmission scheme that generalizes the capacity achieving schemes in the “very strong interference” and the “primary decodes cognitive” regimes.

Theorem IV.1. Achievable scheme (F) in \([7]\) Sec. IV.F The following region is achievable in a general CIFC

\[ R_1 \leq I(Y_1;U_{1c}|U_{2c}) - I(U_{1c}|U_{2c},X_2) \] (11a)

\[ R_1 \leq I(Y_2;U_{1c}|U_{2c},X_2) \] (11b)

\[ R_1 + R_2 \leq I(Y_2;X_1, X_2) \] (11c)

\[ R_1 + R_2 \leq I(Y_2;X_1, X_2) + I(Y_1;U_{1c}, U_{2c}) \] (11d)

\[ 2R_1 + R_2 \leq I(Y_2;U_{1c}, X_2 U_{2c}) + I(Y_1;U_{1c}, U_{2c}) \] (11e)

for some distribution

\[ P_{U_{2c}} P_{X_2|U_{2c}} P_{U_{1c}|U_{2c},X_2} P_{X_1|U_{1c},U_{2c},X_2}. \] (12)

The chain graph \([11]\) representation of the achievable scheme in Th. IV.1 can be found in Fig. 2. Each box represents a RV in \([11]\), a solid line represents superposition coding, a dashed line binning and a dotted line a deterministic dependence. Green, square boxes contain part of the message \(W_2\) while the blue diamond box the message \(W_1\).

For the G-CIFC in \([1]\) we consider the following assignment for the Random Variables (RVs) in \((12)\):

\[ X_1 \sim \mathcal{N}_C(0,1) \quad i \in \{1c,2c,2p\} \] (13a)

\[ X_2 = \sqrt{P_2}\beta X_{2c} + \sqrt{P_2}\beta X_{2p} \] (13b)

\[ X_1 = \sqrt{\alpha P_1} X_{1c} + \sqrt{\alpha P_1} X_{2c} \] (13c)

\[ U_{1c} = \sqrt{\alpha P_1} X_{1c} + \lambda_{\text{Costa}} \lambda_{1c} X_{2p} \] (13d)

\[ \lambda_{\text{Costa}} = \frac{\alpha P_1}{\alpha P_1 + 1} a \sqrt{P_2}, \] (13e)

We now show how the outer bound in Th. III.2 can be achieved using the inner bound of Th. IV.1 by optimally choosing the rate-splitting between the public and the private part. We begin by considering the case where only superposition coding is employed.

Lemma V.1. Partial achievability of the “strong interference” outer bound with superposition coding \([2]\) The “strong interference” outer point for \(\alpha = \alpha' \in [0,1]\) is achievable if \(|b| \geq 1\) and

\[ (1 - |b|^2)P_1 + (a^2 - 1)P_2 \geq \frac{2(a - \text{Re}(a^*)\sqrt{P_1P_2})}{1 + a^2 P_2}. \] (14)

Proof: When fixing the rate of the private primary message to zero in \((11)\), which is equivalent to setting \(\beta = 1\) in \((13)\), and for \(|b| \geq 1\), the rate bounds \((11b)\) and \((11e)\) can be dropped. With this choice, the outer bound point in Th. III.2 for \(\alpha = \alpha' \in [0,1]\) is achievable when

\[ I(Y_1;U_{1c}, U_{2c}) \geq I(Y_2;X_1 X_2) \quad \iff \quad \beta X_1, X_2 \geq I(Y_2;X_1 X_2) \] (15a)

which corresponds to the condition in \((14)\) for the assignment in \((13)\) with \(\beta = 1\).

The capacity result in Th. III.3 is obtained by imposing condition \((14)\) for all \(\alpha' \in [0,1]\).

We now consider the case when only binning is employed in the achievable scheme of Th. IV.1.

Theorem V.2. Partial achievability of the “strong interference” outer bound with binning \([5]\) The “strong interference” outer point for \(\alpha = \alpha' \in [0,1]\) is achievable if \(|b| \geq 1\) and

\[ P_2 (1 - a|b|) (\alpha P_1 + 1) \] (16)

\[ - (|b|^2 - 1)(P_1 + |a|^2 P_2 + 2a \sqrt{P_1 P_2} + 1) \geq 0. \]

Proof: When fixing the rate of the common primary message to zero in \((11)\), which is equivalent to setting \(\beta = 0\) in \((13)\), and for \(|b| \geq 1\), the rate bounds \((11b)\) and \((11e)\) can
be dropped. With this choice, the outer bound point in Th. [IV.2] for \( \alpha = \alpha' \in [0, 1] \) is achievable when

\[
I(Y_2; X_2|U_{1c}) + I(Y_1; U_{1c}) \geq I(Y_2; X_1, X_2) \quad (17a)
\]

\[
I(Y_1; U_{1c}) \geq I(Y_2; U_{1c}) \quad (17b)
\]

which corresponds to the condition in (16) for the assignment in (13) with \( \beta = 0 \).

With the aid of Lem. V.1 and Lem. V.2 we now show the achievability of the “strong interference” outer bound for \( |b| \geq 1 \) using the inner bound in Th. IV.1.

**Theorem V.3. A new capacity result**

Let \((i)_{\alpha' = \gamma}\) indicates that condition \((i)\) holds for the assignment \(\alpha' = \gamma\) and define

\[
\tilde{\alpha} = \max \left\{ 0, \min \left\{ 1, \frac{(|a|^2 - 1)P_2 + (1 - |b|^2)P_1}{2|\text{Re}\{a^*\} - |b|\sqrt{P_1P_2}} \right\} \right\}.
\]

(18)

If

\[
(16)_{\alpha' = 0} \quad \text{or} \quad (14)_{\alpha' = 1} \quad \text{or} \quad (16)_{\alpha' = \tilde{\alpha}} \quad (19)
\]

or

\[
(16)_{\alpha' = 1} \quad \text{or} \quad (14)_{\alpha' = 0} \quad \text{or} \quad (16)_{\alpha' = \tilde{\alpha}} \quad (20)
\]

the region in (8) is the capacity region.

**Proof:** We now seek to extend the results of Lem. V.1 and Lem. V.2 to all the \( \alpha \in [0, 1] \). Since we can show achievability under condition (14) and (16), we only need to focus on the range of \( \alpha \) for which neither of these conditions hold. In particular, (14) is linear in \( \alpha \), so if it holds for \( \alpha_1 \) and \( \alpha_2 \), then it holds for the whole interval \([\alpha_1, \alpha_2]\). Similarly, (16) is quadratic and concave in \( \sqrt{\alpha} \), so if it holds for \( \alpha_1 \) and \( \alpha_2 \), then it holds for the whole interval \([\alpha_1, \alpha_2]\).

To match the inner bound in Th. IV.1 with the assignment in (13) and the outer bound in Th. IV.2 for \( |b| \geq 1 \) we need equations (11b), (11d) and (11e) to be redundant, that is

\[
I(Y_2; U_{1c}, X_2|U_{2c}) \geq I(Y_1; U_{1c}, X_2, U_{2c}) \quad (21a)
\]

\[
I(Y_1; U_{1c}, U_{2c}) \geq I(Y_2; U_{1c}, U_{2c}) \quad (21b)
\]

\[
I(Y_1; U_{2c}) \geq I(Y_2; U_{2c}). \quad (21c)
\]

Condition (21c) can be rewritten as

\[
\frac{|a|^2P_2 + P_1 + 2|\text{Re}\{a^*\}\sqrt{\alpha P_1P_2}}{\alpha P_1 + \beta \sqrt{\alpha P_1 + \beta P_2}^2} + 1 \geq \frac{|b|^2P_1 + P_2 + 2|b|\sqrt{\alpha P_1P_2} + 1}{|b|^2\alpha P_1 + \beta \sqrt{|b|^2\alpha P_1 + \beta P_2}^2 + 1} \quad (22)
\]

Note that condition (22) holds only for \( \beta = 0 \) in the “strong interference” but outside the “very strong interference” regime this means that when condition (14) does not hold, one can hope to achieve the outer bound only with the choice \( \beta = 0 \). With this observation we conclude that we can achieve capacity using \( \beta = 0 \) for a subset of the \( \alpha \) while using \( \beta = 1 \) for the remaining subset, that is either condition (14) or (16) must hold for any \( \alpha \in [0, 1] \). Since both conditions have at most two zeros in \( \alpha' \), the above condition is satisfied when either

- one condition holds in both zero and one, or
- one condition holds in zero and is in \( \alpha' = \tilde{\alpha} \), the other holds in one and in \( \alpha' = \tilde{\alpha} \).

For simplicity, one can chose \( \tilde{\alpha} \) to be the \( \alpha' \), for which condition (14) holds with equality as in (18).

In the above proof, the optimal transmission strategy is obtained by either a private public message on a private one, depending on the cooperation level between the transmitters. This is somewhat surprising as one would expect some rate advantage from primary message private and a part public. The key intuition here is provided by (22), outside the “very strong interference” regime there is a rate penalty in decoding the primary message at the cognitive decoder at some rates. When such penalty exists, the best thing to do is to set the rate of the private cognitive message to zero. Note that this may not be the case when considering an assignment different from (13) in (5) it is shown that partial interference cancelation, i.e. \( \lambda \neq \lambda_{\text{Costa}} \) in (22), can yield large achievable regions then full interference cancelation.

**VI. Numerical Results**

In this section we present some numerical results to illustrate the capacity region associated with Th. V.3.

We begin by illustrating the result in (19) with Fig. 3, here we plot the region where (16) holds for \( \alpha' = 0 \), (14) holds for \( \alpha' = 1 \) and finally where (19) holds. Both conditions (16) for \( \alpha' = 0 \) and (14) for \( \alpha' = 1 \) are sufficient conditions for (19) to holds.

In Fig. 4 we present the improvement on the known capacity region that is provided by condition (21c). In this figure we represent the “very strong interference” regime of Th. III.3 and
Fig. 4. The primary decoder “very strong interference” capacity region (blue hatched) holds, where (14) for $\alpha' = 1$ (cross hatched) holds and where (19) holds (solid color), for $P_1 = 10$, $P_2 = 1$ and $a \times |b| \in [-5, 5] \times [1, 5]$.

Fig. 5. The region where (16) for $\alpha' = 1$ (left hatched) holds, where (14) for $\alpha' = 0$ (right hatched) holds and where (19) holds (solid color), for $P_1 = 10^{-3}$, $P_2 = 1$ and $a \times |b| \in [-1.1, -1] \times [3, 3.1]$.

The “primary decodes cognitive” regime of Th. III.4 together with the new capacity result in Th. V.3.

Fig. 6. The primary decoder “very strong interference” capacity region (blue hatched) holds, where (14) for $\alpha' = 1$ (cross hatched) holds and where (19) holds (solid color), for $P_1 = 10^{-3}$, $P_2 = 1$ and $a \times |b| \in [-1.1, -1] \times [3, 3.1]$.

Fig. 5 and Fig. 6 are the analogous of Fig. 3 and Fig. 4 for condition (20).

VII. CONCLUSION

In this paper we derive a new capacity result for the cognitive interference channel, a classic interference channel where the first transmitter is additionally provided with the message of the second user. This new capacity result is obtained by generalizing the capacity proof for the “very strong interference” regime, where superposition coding achieves capacity, and for the “primary decodes cognitive” regime, where binning is optimal. Although this result improves on the class of channels for which capacity is known, the complete characterization of the capacity of this channel is still an open problem.

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