The Einstein static universe with torsion and the sign problem of the cosmological constant

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Abstract

In the field equations of Einstein-Cartan theory with cosmological constant a static spherically symmetric perfect fluid with spin density satisfying the Weyssenhoff restriction is considered. This serves as a rough model of space filled with (fermionic) dark matter. From this the Einstein static universe with constant torsion is constructed, generalising the Einstein Cosmos to Einstein-Cartan theory.

The interplay between torsion and the cosmological constant is discussed. A possible way out of the cosmological constant’s sign problem is suggested.

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1 Introduction

Cosmological observations [15, 19] give strong indications of the presence of a positive cosmological constant, which would mean that the universe is of de Sitter type. On the other hand the low energy limit of supersymmetry theories prefers a negative cosmological constant, implying an anti-de Sitter cosmos [23]. Existing solutions to this problem necessitate the inclusion of a further field ( quintessence) [2].

It is shown that this may not be necessary. By considering the Einstein-Cartan theory with cosmological constant a model is constructed which indicates that a relatively simple solution to the problem may be possible.

2 Field equations in Einstein-Cartan theory

In the Cartan formalism the metric is expressed in terms of vielbein 1-forms $e^i$

$$\text{d}s^2 = \eta_{ij} e^i \otimes e^j,$$  \hspace{1cm} (1)

indicating anholonomic indices. In standard notation torsion and curvature 2-forms are given by

$$T^i = (D e)^i = d e^i + \omega^i_j \wedge e^j = \frac{1}{2} T_{jk}^i e^j \wedge e^k,$$ \hspace{1cm} (2)

$$R^i_j = (D \omega)^i_j = d \omega^i_j + \omega^i_k \wedge \omega^k_j = \frac{1}{2} R_{jkl}^i e^k \wedge e^l,$$ \hspace{1cm} (3)

respectively, where $\omega^i_j$ is the connection 1-form. The field equations of the Einstein-Cartan theory are obtained from the action functional [4, 8, 9, 11, 20]

$$S = \int (L + 2\Lambda \epsilon + \kappa L_m),$$ \hspace{1cm} (4)

where $L = \frac{1}{4} R e$ and $L_m = L_m(e^i, \omega^i_j)$ is the matter Lagrangian. $\epsilon$ is the volume four form, $R = \eta^{lm} \delta^m_k R_{lmn}$ is the Ricci scalar and $\kappa$ the gravitational coupling constant.

Variation of (4) with respect to $e^i$ and $\omega^i_j$ together with (2) and (3) yields the field equations in Einstein-Cartan theory [10, 11, 20]

$$R^i_j - \frac{1}{2} R \delta^i_j + \Lambda \delta^i_j = -\kappa t^i_j,$$ \hspace{1cm} (5)

$$T^i_{jk} - \delta^i_j T^i_{lk} - \delta^i_k T^i_{jl} = -\kappa s^i_{jk},$$ \hspace{1cm} (6)

where $t^i_{jk}$ is the canonical energy-momentum tensor and $s^i_{jk}$ is the tensor of spin.
A static and spherically symmetric spacetime is described by a line element of the form
\[ ds^2 = A^2 dt^2 - B^2 dr^2 - r^2 d\Omega^2, \]
with \( A, B \) functions of the radial coordinate \( r \) only. If one assumes that the spacetime is filled with a fermionic fluid, a classical description of the spin contribution in (6) is
\[ s^{i j} = u^i S_{j}, \quad u^i S_{ji} = 0, \]
where \( u^i \) is the four velocity of the fluid and \( S_{ij} \) is the intrinsic angular momentum tensor. Further restricting to an isotropic perfect fluid yields
\[ t_{ij} = h_i u_j - P \eta_{ij}, \]
\[ h_i = (\rho + P) u_i - u^j \nabla_k (u^k S_{ji}) \]
\[ = (\rho + P) u_i + a^j S_{ji}, \]
where \( \rho \) is the matter density and \( P \) the isotropic pressure, \( a^j = (u^k \nabla_k) u^j \) is the fluid’s acceleration. Thus the canonical energy-momentum tensor is symmetric if the acceleration of the fluid is zero.

Assuming spherical symmetry for the spin implies that \( S_{ij} \) has only one non-vanishing component, \( S_{23} = K \), where \( K \) is a function of \( r \). Since a static configuration is also assumed \( u^i = \delta^i_0 \) and hence \( s^0_{23} = K \). Therefore, one can solve (6) and one finds
\[ T^0_{23} = -T^0_{32} = -\kappa K, \]
all other components vanish. Equation (2) implies that \( T^0 = -\kappa K e^2 \wedge e^3 \), and \( T^1 = T^2 = T^3 = 0 \). Taking (8) into account simplifies to \( t^0_0 = \rho \) and \( t^1_1 = t^2_2 = t^3_3 = -P \).

The remaining field equations (5) are three independent equations which imply energy-momentum (plus spin) conservation. For convenience we use the first two field equations and the conservation equation. With \( \kappa = -8\pi \) this yields
\[ \frac{1}{r^2} \frac{d}{dr} \left( r - \frac{r}{B^2} \right) - \Lambda + 16\pi^2 K^2 = 8\pi \rho, \]
\[ - \frac{1}{r^2} + \frac{1}{B^2} \left( \frac{2A'}{Ar} + \frac{1}{r^2} \right) + \Lambda + 16\pi^2 K^2 = 8\pi P, \]
\[ P' + (\rho + P) \frac{A'}{A} - 4\pi K \left( K' + K \frac{A'}{A} \right) = 0, \]
where the prime denotes differentiation with respect to \( r \). Furthermore if we assume that the conservation equation
\[ P' + (\rho + P) \frac{A'}{A} = 0, \]
of general relativity holds then (13) implies
\[ K' + K \frac{A'}{A} = 0, \quad K \propto A^{-1}, \quad \text{for } K \neq 0. \] (15)

One may redefine the pressure and the energy density by
\[ \rho_{\text{eff}} = \rho - 2\pi K^2 + \frac{\Lambda}{8\pi}, \] (16)
\[ P_{\text{eff}} = P - 2\pi K^2 - \frac{\Lambda}{8\pi}, \] (17)
and rewrite equations (11)-(13). This leads to the usual field and conservation equations with vanishing torsion and vanishing cosmological constant.

The effect of the cosmological constant can be seen as a special type of an energy-momentum tensor. It acts as an unusual fluid with \( P^\Lambda = -\Lambda/8\pi \) and \( P^\Lambda = -\rho^\Lambda \) as an equation of state. On the other hand, the torsion contribution \( K \) acts as a fluid with \( P^K = -2\pi K^2 \) and equation of state \( P^K = \rho^K \). For simplicity from now on \( \rho, P \) and \( K \) are assumed to be constant.

It is instructive to have a closer look at the effective quantities (16) and (17). The cosmological solution consisting of an incoherent dust \( P = 0 \) with vanishing torsion \( K = 0 \) and de Sitter type universe implies \( \rho_{\text{eff}} > 0 \) and \( P_{\text{eff}} < 0 \).

Thus one can try to reconcile this with a negative cosmological constant as required from supergravity. Two assumptions are needed. (i) Cosmological observations measure the effective energy density and pressure and from this the sign of the cosmological constant is defined. Note that this is a crucial assumption for what is argued in the following. (ii) The output of supersymmetry theories is correct in the sense that the cosmological constant is negative.

Then one can check whether the three conditions \( |\Lambda| = -\Lambda > 0, \rho_{\text{eff}} > 0 \) and \( P_{\text{eff}} < 0 \) can be satisfied simultaneously.

\[ \rho_{\text{eff}} > 0 \Rightarrow \rho - \frac{|\Lambda|}{8\pi} > 2\pi K^2, \] (18)
\[ P_{\text{eff}} < 0 \Rightarrow P + \frac{|\Lambda|}{8\pi} < 2\pi K^2. \] (19)

Assuming as before an incoherent dust \( P = 0 \), the above leads to the inequality
\[ \frac{|\Lambda|}{8\pi} < 2\pi K^2 < \rho - \frac{|\Lambda|}{8\pi}. \] (20)

If the (AdS) cosmological constant has an upper bound given by torsion and if the cosmic energy density is sufficiently large then the three conditions can simultaneously be satisfied. From this one can conclude that under the above assumptions it is possible to reconcile observational data leading to a positive cosmological constant, and the supersymmetry requirement yielding a negative sign. The argument also works for vanishing cosmological constant.
For easier comparison with observational data the inequality (20) is divided by the critical density \( \rho_c \) and rewritten in terms of density parameters \( \Omega \). This gives
\[
\Omega_{\text{susy}}^{\Lambda} < \frac{1}{\rho_c} \left( \frac{K}{2M_{\text{pl}}} \right)^2 < \Omega - \Omega_{\text{susy}}^{\Lambda},
\]
where the notation of [16] was used. The present value of the dark matter contribution is denoted by \( \Omega_0 \), recent observations suggest [15] that \( \Omega_0 = 0.3 \pm 0.1 \). Thus for small enough \( \Omega_{\text{susy}}^{\Lambda} \) one gets an upper and a lower bound for the torsion contribution.

If the second assumption is dropped (no supersymmetry) and if one considers a positive cosmological constant it is found that observations are compatible with the presence of torsion. In this case from (16) and (17) one can deduce
\[
\Omega + \Omega_\Lambda > \frac{1}{\rho_c} \left( \frac{K}{2M_{\text{pl}}} \right)^2 \geq 0,
\]
that the density parameter of torsion is heavily constrained. Observations are, of course, also compatible with vanishing torsion.

4 Einstein universe with constant torsion

Finally the Einstein universe with constant torsion is constructed. The first field equation (11) can easily be integrated and yields
\[
\frac{1}{B^2} = 1 - \frac{2M(r)}{r} - \frac{\Lambda}{3} r^2 + \frac{\mathcal{K}(r)}{r},
\]
where the constant of integration was set to zero because of regularity at the centre and
\[
\mathcal{M}(r) = \int_0^r 4\pi \rho s^2 ds, \quad \mathcal{K}(r) = \int_0^r 16\pi K^2 s^2 ds.
\]
From equations (12) and (13) one can eliminate \( A'/A \) and gets the Tolman-Oppenheimer-Volkoff equation [18, 22] with cosmological constant and spin contribution
\[
P' = -r \frac{(12\pi P + 4\pi \rho_0 - \Lambda - 32\pi^2 K_0^2)(P + \rho_0 - 4\pi K_0^2)}{3 - (8\pi \rho_0 + \Lambda - 16\pi^2 K_0^2)r^2},
\]
where we assumed positive constant density \( \rho = \rho_0 = \text{const.} \) and positive constant torsion \( K = K_0 = \text{const.} \).

If \( P' = 0 \) for all \( r \) the differential equation (25) of the pressure implies
\[
\Lambda = 4\pi (3P_0 + \rho_0 - 8\pi K_0^2),
\]
provided the modified energy condition

\[ P_0 + \rho_0 - 4\pi K_0^2 > 0, \quad (27) \]

holds. Moreover, under these assumptions \((23)\) implies that \(A = \text{const.}\) which we can re-scale to one.

The metric function \(B\) can now be read from \((23)\). We also insert the value of the cosmological constant \((26)\) and arrive at

\[ \frac{1}{B^2} = 1 - 4\pi \left( \rho_0 + P_0 - 4\pi K_0^2 \right) r^2. \quad (28) \]

From this one can read off the radius of the Einstein static universe with torsion

\[ R_E^2 = \frac{1}{4\pi \left( \rho_0 + P_0 - 4\pi K_0^2 \right)}, \quad (29) \]

which for vanishing \(K_0\) coincides with the radius of the standard Einstein static universe.

The above solution is a modified Einstein static universe. The modification is due to the additional spin contribution \(K\). Of course for \(K = 0\) the cosmological constant of the usual Einstein static universe \([3]\) is reproduced by \((26)\). Torsion free generalisations of the Einstein static universe have been published earlier in Ref. \([12]\) and also in Ref. \([1]\).

5 Conclusions and Outlook

Under the assumptions (i) and (ii) it was possible to construct a model in which the different observational (dS) and supersymmetry (AdS) requirement concerning the cosmological constant’s sign could be incorporated.

It would be interesting to find a model in which one could clearly define an effective cosmological constant and therefore solving the full problem. Other sources of torsion might be a good starting point.

A detailed investigation of observational data along the lines of the recent report \([6]\) is beyond the scope of the present work.

The static Einstein Cosmos was generalised to Einstein-Cartan theory. A possible generalisation of the present approach is to weaken to strong Weyssenhoff condition and consider instead a hyperfluid \([7,17]\). The qualitative result that the Einstein Cosmos can be generalised will be unaffected by that.

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