Bisets as categories, and tensor product of induced bimodules
Serge Bouc

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Abstract: Bisets can be considered as categories. This note uses this point of view to give a simple proof of a Mackey-like formula expressing the tensor product of two induced bimodules.

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1. Introduction

Let $R$ be a commutative ring, let $G$ and $H$ be finite groups, let $X$ be a subgroup of $H \times G$, and $M$ be an $RX$-module. If $m \in M$ and $(h, g) \in X$, set $h \cdot m \cdot g^{-1} = (h, g) \cdot m$: this is a slight extension of the usual correspondence between $R(H \times G)$-modules and $(RH, RG)$-bimodules.

The object of this note is to give a simple proof of the following result:

1.1. Theorem: Let $R$ be a commutative ring, let $G$, $H$, and $K$ be finite groups, let $X$ be a subgroup of $H \times G$ and $Y$ be a subgroup of $K \times H$. Let $M$ be an $RX$-module, and $N$ be an $RY$-module. Then there is an isomorphism of $(RK, RG)$-bimodules

$$(\text{Ind}_{Y}^{K \times H}N) \otimes_{RH} (\text{Ind}_{X}^{H \times G}M) \cong \bigoplus_{t \in [p_{2}(Y) \cap H/p_{1}(X)]} \text{Ind}_{Y \times \{t,1\}X}^{K \times G} (N \otimes k_{2}(Y) \cap k_{1}(X)) \quad (t,1) M,$$

where the notation is as follows (cf. [3]):

$p_{1}(X) = \{h \in H \mid \exists g \in G, (h, g) \in X\}$, \quad $k_{1}(X) = \{h \in H \mid (h, 1) \in X\}$

$p_{2}(Y) = \{h \in H \mid \exists k \in K, (k, h) \in Y\}$, \quad $k_{2}(Y) = \{h \in H \mid (1, h) \in Y\}$

$Y \star (t,1) X = \{(k, g) \in K \times G \mid \exists h \in H, (k, h) \in Y, (h^{t}, g) \in X\}$.

The action of $(k, g) \in Y \star (t,1) X$ on $N \otimes k_{2}(Y) \cap k_{1}(X) \quad (t,1) M$ is given by

$$k \cdot (n \otimes m) \cdot g^{-1} = (k \cdot n \cdot h^{-1}) \otimes (h^{t} \cdot m \cdot g^{-1}),$$

if $h \in H$ is chosen such that $(k, h) \in Y$ and $(h^{t}, g) \in X$. 


2. Functors over bisets

Recall that when $G$ and $H$ are groups, an $(H, G)$-biset $U$ is a set equipped with a left action of $H$ and a right action of $G$ which commute, i.e. such that $(hu)g = h(ug)$ for any $h \in H$, $u \in U$, and $g \in G$.

2.1. Notation: Let $G$ and $H$ be groups. When $U$ is an $(H, G)$-biset, let $\langle U \rangle$ denote the following category:

- The objects of $\langle U \rangle$ are the elements of $U$.
- If $u, v \in U$, then $\text{Hom}_{\langle U \rangle}(u, v) = \{(h, g) \in H \times G \mid hu = vg\}$.
- If $u, v, w \in U$, the composition of the morphisms $(h, g) : u \to v$ and $(h', g') : v \to w$ is the morphism $(h'h, g'g) : u \to w$.
- If $u \in U$, the identity morphism $\text{Id}_u : u \to u$ is the pair $(1, 1) \in G \times G$.

Note that the category $\langle U \rangle$ is a groupoid (any morphism is an isomorphism), and that for any $u \in U$, the group

$$A(u) = \text{Hom}_{\langle U \rangle}(u, u) = \{(h, g) \in H \times G \mid hu = ug\}$$

is a subgroup of $H \times G$.

A functor $M$ from $\langle U \rangle$ to a category $C$ consists of a collection of objects $M(u)$ of $C$, for $u \in U$, together with morphisms

$$M(h, g) : M(u) \to M(hug^{-1})$$

in the category $C$, for $(h, g) \in H \times G$, fulfilling the usual functorial conditions. In particular, for each $u \in U$, there is a group homomorphism $A(u) \to \text{Aut}_CM(u)$.

Functors from $\langle U \rangle$ to $C$ are the objects of a category $\text{Fun}(\langle U \rangle, C)$, in which the morphisms are natural transformation of functors.

2.2. Notation: When $C$ is a subcategory of the category Sets of sets, and $M$ is a functor $\langle U \rangle \to C$, the image of $m \in M(u)$ by the map $M(h, g) : M(u) \to M(hug^{-1})$, for $(h, g) \in H \times G$, will be denoted by $hmg^{-1}$.

In this case, a functor $M : \langle U \rangle \to C$ is a collection of objects $M(u)$ of $C$, for $u \in U$, together with morphisms $m \mapsto hmg^{-1} : M(u) \to M(hug^{-1})$ in $C$. 

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for \((h, g) \in H \times G\), such that \(h'(hmg^{-1})g'^{-1} = (h'h)m(g'g)^{-1}\) and \(1m1 = m\),
for any \((h, g), (h', g') \in H \times G\), any \(u \in U\), and any \(m \in M(u)\).

2.3. Example : Suppose that \(C = \text{Sets}\). Then the disjoint union \(\bigsqcup M = \bigsqcup_{u \in U} M(u)\) becomes an \((H, G)\)-biset, and the map \(\bigsqcup M \to U\) sending elements of \(M(u)\) to \(u\), for \(u \in U\), is a map of \((H, G)\)-bisets. Conversely, if \(\pi : S \to U\) is a map of \((H, G)\)-bisets, then the assignment \(u \mapsto \pi^{-1}(u)\) is a functor from \(\langle U \rangle\) to \(\text{Sets}\).

In other words, a functor \(\langle U \rangle \to \text{Sets}\) is just an \((H, G)\)-biset over \(U\).

2.4. Example : Let \(R\) be a commutative ring. In the remainder of this note, the category \(C\) will be the category \(R\)-\text{Mod} of (left) \(R\)-modules. If \(M\) is functor from \(\langle U \rangle\) to \(R\)-\text{Mod}, then for each \(u \in U\), the \(R\)-module \(M(u)\) has a natural structure of \(\mathbb{A}(u)\)-module.

Conversely, let \([H \backslash U/G]\) be a set of representatives of \((H, G)\)-orbits on \(U\). Equivalently \([H \backslash U/G]\) is a set of representatives of isomorphism classes in the category \(\langle U \rangle\). Since \(\langle U \rangle\) is a groupoid, it is equivalent to its full subcategory \([H \backslash U/G]\). In particular, this yields an equivalence of categories

\[
\text{Fun}(\langle U \rangle, R\text{-Mod}) \cong \prod_{u \in [H \backslash U/G]} \mathbb{A}(u)\text{-Mod}.
\]

2.6. Remark : In the situation of Example 2.4, the direct sum

\[
\Sigma(M) = \bigoplus_{u \in U} M(u)
\]

has a natural structure of \((RH, RG)\)-bimodule, i.e. using the usual group isomorphism \((h, g) \mapsto (h, g^{-1})\) from \(H \times G^{\text{op}}\) to \(H \times G\), of left \(R(H \times G)\)-module.

Moreover, is easy to see that there is an isomorphism of \((RH, RG)\)-bimodules

\[
\Sigma(M) \cong \bigoplus_{u \in [H \backslash U/G]} \text{Ind}^{H \times G}_{\mathbb{A}(u)} M(u).
\]

3. Product of bisets, and product of functors

Let \(G, H\) and \(K\) be groups. If \(U\) is an \((H, G)\)-biset and \(V\) is a \((K, H)\)-biset, recall that the product (or composition) of \(V\) and \(U\) is the set

\[
V \times_H U = (V \times U)/H,
\]
where the right action of $H$ on $(V \times U)$ is defined by $(v, u) \cdot h = (vh, h^{-1}u)$, for $v \in V$, $u \in U$, and $h \in H$. The set $V \times H U$ is a $(K, G)$-biset for the following action

$$\forall z \in K, \forall x \in G, \forall v \in V, \forall u \in U, \quad z \cdot (v, H u) \cdot x = (zv, H ux) ,$$

where $(v, H u)$ denotes the $H$-orbit of $(v, u)$.

### 3.1. Definition

Let $G$, $H$, and $K$ be finite groups. Let $U$ be a finite $(H, G)$-biset, and $V$ be a finite $(K, H)$-biset. If $M$ is a functor $\langle U \rangle \rightarrow R\text{-Mod}$ and $N$ is a functor $\langle V \rangle \rightarrow R\text{-Mod}$, the tensor product $N \otimes H M$ is the functor $\langle V \times H U \rangle \rightarrow R\text{-Mod}$ defined by

$$(N \otimes H M)(v, H u) = \left( \bigoplus_{h \in H} N(vh) \otimes_R M(h^{-1}u) \right) / I_{v,u} ,$$

where $I_{v,u}$ is the $R$-submodule generated by the elements of the form

$$[ny \otimes y^{-1}m]_h - [n \otimes m]_h ,$$

where $y \in H$, and where $[n \otimes m]_h$ denotes the element $n \otimes m$ of the component indexed by $h \in H$ in the direct sum, for $n \in N(vh)$, and $m \in M(h^{-1}u)$.

If $(k, g) \in K \times G$, then by definition

$$k [n \otimes m]_h g = [kn \otimes mg]_h .$$

### 3.2. Remark

It follows from this definition that

$$(N \otimes H M)((v, H u)) \cong N(v) \otimes_{RH_{v,u}} M(u) ,$$

where $H_{v,u}$ is the set of elements $h \in H$ such that $vh = v$ and $hu = u$.

### 3.3. Lemma

There is an isomorphism of $(RK, RG)$-bimodules

$$\Sigma(N) \otimes_R \Sigma(M) \cong \Sigma(N \otimes H M) ,$$

sending (from right to left) the element $[n \otimes m]_h$ to $n \otimes_{RH} m$.

**Proof:** To be more precise, the map $\alpha$ from

$$\Sigma(N \otimes H M) = \bigoplus_{(v, H u) \in V \times H U} \left( \bigoplus_{h \in H} N(vh) \otimes_R M(h^{-1}u) \right) / I_{v,u}$$

sending the element \([n \otimes m]_h\) in the component indexed by \((v, u)\) to the element \(n \otimes m\) of the tensor product

\[
\Sigma(N) \otimes_{RH} \Sigma(M) = \left( \bigoplus_{v \in V} N(v) \right) \otimes_{RH} \left( \bigoplus_{u \in U} M(u) \right)
\]
is well defined. To show that it is an isomorphism, define a map

\[
\beta : \Sigma(N) \otimes_{RH} \Sigma(M) \to \Sigma(N \otimes_H M)
\]
in the following way: choose a set \(S\) of representatives of the classes \((v, u)\).

Now map the element \(n \otimes RH m \in N(v) \otimes M(u) \subseteq \Sigma(N) \otimes_{RH} \Sigma(M)\) to \([n \otimes m]_h\), where \(h \in H\) is chosen such that \((vh^{-1}, hu) \in S\). Again, it is easy to see that this map is well defined, and that the maps \(\alpha\) and \(\beta\) are mutual inverse isomorphisms of \((RK, RG)\)-bimodules.

\[\square\]

3.4. Corollary : Let \(G, H,\) and \(K\) be finite groups. Let \(X\) be a subgroup of \(H \times G\) and \(Y\) be a subgroup of \(K \times H\). Let \(M\) be an \(RX\)-module, and \(N\) be an \(RY\)-module. Then there is an isomorphism of \((RK, RG)\)-bimodules

\[
(\text{Ind}_{Y \times H}^K N) \otimes_{RH} (\text{Ind}_{X \times G}^H M) \cong \bigoplus_{t \in [p_2(Y) \backslash H/p_1(X)]} \text{Ind}_{Y \times (t, 1) \times X}^{K \times G} (N \otimes k_{[2]}(Y)^{(t, 1)} \otimes k_{1}(X)) M.
\]

**Proof :** Set \(U = (H \times G)/X\). Then \(U\) is an \((H, G)\)-biset by \(h \cdot (t, s)X \cdot g = (ht, g^{-1}s)X\), and this biset is transitive. If \(u\) is the point \(X\) of \(U\), then \(A(u) = X\), and the equivalence of categories \[\square\] reads

\[
\text{Fun}(U, R\text{-Mod}) \cong RX\text{-Mod}.
\]

More precisely, for an \(RX\)-module \(M\), this equivalence yields a functor \(\tilde{M} : (U) \to R\text{-Mod}\) in the following way: for any \((h, g) \in H \times G\), set

\[
\tilde{M}((h, g)X) = M.
\]

Next, fix a set \(S\) of representatives of elements of \(U\), i.e. \(X\)-cosets in \(H \times G\). For \((t, s) \in S\), and \((h, g) \in H \times G\), define a map

\[
\tilde{M}(h, g) : \tilde{M}((t, s)X) = M \to \tilde{M}((ht, gs)X) = M
\]

by \(\tilde{M}(h, g)(m) = (y, x)m\), where \((y, x)\) is the unique element of \(X\) such that \((ht, gs)(y, x)^{-1} \in S\).

Then it is easy to check that \(\tilde{M}\) is indeed a functor, and that there is an isomorphism of \((RH, RG)\)-bimodules

\[
\Sigma(\tilde{M}) \cong \text{Ind}_{X \times G}^H M.
\]
Similarly, set $V = (K \times H)/Y$, and define a functor $\tilde{N}(V) \to R\text{-Mod}$, using the $RY$-module $N$. Then the corollary is a straightforward consequence of the lemma, applied to the functors $\tilde{M}$ and $\tilde{N}$, using Remark 2.6 and Remark 3.2.

**References**

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Serge Bouc
CNRS-LAMFA
Université de Picardie
33 rue St Leu
80039 - Amiens Cedex 1
FRANCE
email : serge.bouc@u-picardie.fr