Anomalous universality in the Anisotropic Ashkin–Teller model

A. Giuliani*, V. Mastropietro**

Abstract. The Ashkin–Teller (AT) model is a generalization of Ising 2-d to a four states spin model; it can be written in the form of two Ising layers (in general with different couplings) interacting via a four-spin interaction. It was conjectured long ago (by Kadanoff and Wegner, Wu and Lin, Baxter and others) that AT has in general two critical points, and that universality holds, in the sense that the critical exponents are the same as in the Ising model, except when the couplings of the two Ising layers are equal (isotropic case). We obtain an explicit expression for the specific heat from which we prove this conjecture in the weakly interacting case and we locate precisely the critical points. We find the somewhat unexpected feature that, despite universality holds for the specific heat, nevertheless nonuniversal critical indexes appear: for instance the distance between the critical points rescales with an anomalous exponent as we let the couplings of the two Ising layers coincide (isotropic limit); and so does the constant in front of the logarithm in the specific heat.

Our result also explains how the crossover from universal to nonuniversal behaviour is realized.

1. Introduction

1.1 Historical introduction. Ashkin and Teller [AT] introduced their model as a generalization of Ising 2-d to a four states spin model; in each site of a bidimensional lattice there is a spin which can take four values, and only nearest neighbor spins interact. The model can be also considered a generalization of the four state Potts model to which it reduces for a suitable choice of the parameters.

A very convenient representation of the Ashkin Teller model is in terms of Ising spins [F]; one associates with each site of the square lattice two spins variables, \(\sigma^{(1)}_x\) and \(\sigma^{(2)}_x\), the partition function is given by

\[
Z_{\Lambda_M} = \sum_{\sigma^{(1)} \in \mathbb{Z}^2} e^{-H_{\Lambda_M}},
\]

where

\[
H_{\Lambda_M}(\sigma^{(1)}, \sigma^{(2)}) = J^{(1)} H_I(\sigma^{(1)}) + J^{(2)} H_I(\sigma^{(2)}) + \lambda V(\sigma^{(1)}, \sigma^{(2)}) = \sum_{x \in \Lambda_M} H^{AT}_x,
\]

\[
H_I(\sigma^{(j)}) = - \sum_{x \in \Lambda_M} \left[ \sigma^{(j)}_x \sigma^{(j)}_{x+\hat{e}_1} + \sigma^{(j)}_x \sigma^{(j)}_{x+\hat{e}_0} \right],
\]

\[
V(\sigma^{(1)}, \sigma^{(2)}) = - \sum_{x \in \Lambda_M} \left[ \sigma^{(2)}_x \sigma^{(1)}_{x+\hat{e}_0} \sigma^{(1)}_{x+\hat{e}_1} + \sigma^{(2)}_x \sigma^{(1)}_{x+\hat{e}_1} \sigma^{(1)}_{x+\hat{e}_0} \right],
\]

where \(H_I\) is the Ising model hamiltonian, \(\hat{e}_1, \hat{e}_0\) are the unit vectors \(\hat{e}_1 = (1, 0), \hat{e}_0 = (0, 1)\) and \(\Lambda_M\) is a square subset of \(\mathbb{Z}^2\) of side \(M\). The free energy and the specific heat are given by

\[
f = \lim_{M \to \infty} \frac{1}{M^2} \log Z_{\Lambda_M}, \quad C_v = \lim_{M \to \infty} \frac{1}{M^2} \sum_{x,y \in \Lambda_M} <H^{AT}_x H^{AT}_y>_{\Lambda_M,T},
\]

where \(< \cdot >_{\Lambda_M,T}\) denotes the truncated expectation w.r.t. the Gibbs distribution with Hamiltonian (1.1). The case \(J^{(1)} = J^{(2)}\) is called isotropic. For \(\lambda = 0\) the model reduces to two independent Ising models and it has two critical points if \(J^{(1)} \neq J^{(2)}\); it was conjectured by Kadanoff and Wegner [K][KW] and later on by Wu and Lin [WL] that the AT model has in general two critical points also when \(\lambda \neq 0\), except when the model is isotropic.

* Partially supported by NSF Grant DMR 01–279–26; Dipartimento di Fisica, Università di Roma “La Sapienza”, P.zza A. Moro, 2, I-00185, Roma; and INFN, sezione di Roma1; e–mail: alessandro.giuliani@roma1.infn.it
** Dipartimento di Matematica, Università di Roma “Tor Vergata”, Via della Ricerca Scientifica, I-00133, Roma; e–mail: mastropi@mat.uniroma2.it
The isotropic case was studied by Kadanoff [K] who, by scaling theory, conjectured a relation between the critical exponents of isotropic AT and those of the Eight vertex model, which had been solved by Baxter and has nonuniversal indexes. Further evidence for the validity of Kadanoff’s prediction was given by [PB] (using second order renormalization group arguments) and by [LP][N] (by a heuristic mapping of both models into the massive Luttinger model describing one dimensional interacting fermions in the continuum). Indeed non universal critical behaviour in the specific heat in the isotropic AT model, for small $\lambda$, has been rigorously established in [M1].

The anisotropic case is much less understood. As we said, it is believed that there are two critical points, contrary to what happens in the isotropic case. Baxter [Ba] conjectured that ”presumably” universality holds at the critical points for $J^{(1)} \neq J^{(2)}$ (i.e. the critical indices are the same as in the Ising model), except when $J^{(1)} = J^{(2)}$ when the two critical points coincide and nonuniversal behaviour is found. Since the 1970’s, the anisotropic AT model was studied by various approximate or numerical methods: Migdal–Kadanoff Renormalization Group [DR], Monte Carlo Renormalization group [Be], finite size scaling [Bad]; such results give evidence of the fact that, far away from the isotropic point, AT has two critical points and belongs to the same universality class of Ising; however they do not give informations about the precise relative location of the critical points and the critical behaviour of the specific heat when $J^{(1)}$ is close to $J^{(2)}$. The problem of how the crossover from universal to nonuniversal behaviour is realized in the isotropic limit remained for years completely unsolved, even at a heuristic level.

We will study the anisotropic Ashkin–Teller model by writing the partition function and the specific heat as Grassmann integrals corresponding to a $d = 1 + 1$ interacting fermionic theory; this is possible because the Ising model can be reformulated as a free fermions model (see [SML][H][S] or [ID]). One can then take advantage from the theory of Grassmann integrals for weakly interacting $d = 1 + 1$ fermions, which is quite well developed, starting from [BG1] (see also [BG][GM] or [BM] for extensive reviews). Fermionic RG methods for classical spin models have been already applied in [PS] to the Ising model perturbed by a four spin interaction, proving a universality result for the specific heat; and in [M1] to prove a nonuniversality result for the 8 vertex or the isotropic AT model. By such techniques one can develop a perturbative expansion, convergent up to the critical points, uniformly in the parameters.

1.2 Main results. We find convenient to introduce the variables $t^{(j)} = \tanh J^{(j)}$, $j = 1, 2$ and

\[ t = \frac{t^{(1)} + t^{(2)}}{2}, \quad u = \frac{t^{(1)} - t^{(2)}}{2} \] (1.3).

The parameter $u$ measures the anisotropy of the system. We consider then the free energy or the specific heat as functions of $t, u, \lambda$.

If $\lambda = 0$, AT is exactly solvable, because the Hamiltonian (1.1) is the sum of two independent Ising model Hamiltonians. From the Ising model exact solution [O][SML][MW] one finds that $f$ is analytic for all $t$, $u$ except for

\[ t = t^\pm_c = \sqrt{2} - 1 \pm |u| \] (1.4)

and for $t$ close to $t^\pm_c$ the specific heat $C_v$ has a logarithmic divergence: $C_v \simeq -C \log |t - t^\pm_c|$, where $C > 0$ and $\simeq$ means that the ratio of both sides tends to 1 as $t \to t^\pm_c$.

We consider the case in which $\lambda$ is small with respect to $\sqrt{2} - 1$ and we distinguish two regimes.

1) If $u$ is much bigger than $\lambda$ (so that the unperturbed critical points are well separated) we find that the presence of $\lambda$ just changes by a small amount the location of the critical points, i.e. we find that the critical points have the form $t^\pm_c = \sqrt{2} - 1 + O(\lambda) \pm |u|(1 + O(\lambda));$ moreover the asymptotic behaviour of $C_v$ at criticality remains essentially unchanged: $C_v \simeq -C \log |t - t^\pm_c|$.

2) When $u$ is small compared to $\lambda$ the interaction has a more dramatic effect. We find that the system has still only two critical points $t^\pm_c(\lambda, u)$; their center $(t^+_c + t^-_c)/2$ is just shifted by $O(\lambda)$ from $\sqrt{2} - 1$, as in item (1); however their relative location scales, as $u \to 0$, with an “anomalous critical exponent” $\eta(\lambda)$, contiusly varying with $\lambda$: more precisely we find that $t^+_c - t^-_c = O(|u|^{1+\eta})$,
where \( \eta \) is analytic in \( \lambda \) near \( \lambda = 0 \) and \( \eta = -b\lambda + O(\lambda^2) \), \( b > 0 \). In particular the relative location of the critical points as a function of the anisotropy parameter \( u \) with \( \lambda \) fixed and small has a different qualitative behaviour, depending on the sign of \( \lambda \), see Fig 1.

![FIG 1. The qualitative behaviour of \( t_\pm^+(\lambda,u)-t_\pm^-\) as a function of \( u \) for two different values of \( \lambda \) (in arbitrary units). The graphs are (qualitative) plots of \( 2|u|^{1+\eta} \), with \( \eta \geq -b\lambda, b>0 \).](image)

For \( t \to t_\pm^+(\lambda,u) \) the specific heat \( C_v \) has still a logarithmic divergence but, for all \( u \neq 0 \), the constant in front of the log is \( O(|u|^{\eta_c}) \), where \( \eta_c \) is analytic in \( \lambda \) for small \( \lambda \) and \( \eta_c = a\lambda + O(\lambda^2) \), \( a \neq 0 \). The logarithmic behaviour is found only in an extremely small region around the critical points; outside this region, \( C_v \) varies as \( t \to t_\pm^\pm(\lambda,u) \) according to a power law behaviour with nonuniversal exponent. The conclusion is that, for all \( u \neq 0 \), there is universality for the specific heat (which diverges with the same exponent as in the Ising model); nevertheless nonuniversal critical indexes appear in the theory, in the difference between the critical points and in the constant in front of the logarithm in the specific heat. One can speak of anomalous universality as the specific heat diverges at criticality as in Ising, but the isotropic limit \( u \to 0 \) is reached with nonuniversal critical indices.

With the notations introduced above and calling \( D \) a sufficiently small \( O(1) \) interval (i.e. with amplitude independent of \( \lambda \)) centered around \( \sqrt{2} - 1 \), we can express our main result as follows.

**Main Theorem.** There exists \( \varepsilon_1 \) such that, for \( t \pm u \in D \), \( j = 1,2 \), and \( |\lambda| \leq \varepsilon_1 \), one can define two functions \( t_\pm^j(\lambda,u) \) with the following properties:

\[
t_\pm^j(\lambda,u) = \sqrt{2} - 1 + \nu^j(\lambda) \pm |u|^{1+\eta}(1 + F_\pm^j(\lambda,u)),
\]

where \( |\nu^j(\lambda)| \leq c|\lambda|, |F_\pm^j(\lambda,u)| \leq c|\lambda| \), for some positive constant \( c \) and \( \eta = \eta(\lambda) \) is an analytic function of \( \lambda \) s.t. \( \eta(\lambda) = -b\lambda + O(\lambda^2) \), \( b > 0 \), and:

1) the free energy \( f(t,u,\lambda) \) and the specific heat \( C_v(t,u,\lambda) \) in (1.2) are analytic in the region \( t \pm u \in D, |\lambda| \leq \varepsilon_1 \) and \( t \neq t_\pm^\pm(\lambda,u) \);

2) in the same region of parameters, the specific heat can be written as:

\[
C_v = -C_1 \Delta^{2\eta_c} \log \frac{|t-t_c^+||t-t_c^-|}{\Delta^2} + C_2 \frac{1-\Delta^{2\eta_c}}{\eta_c} + C_3
\]

where: \( \Delta^2 \equiv (t-t_c^+)^2 + (u^2)^{1+\eta} \) and \( \Delta^2 \equiv (t_c^- + t_c^+)/2 \); the exponent \( \eta_c = \eta_c(\lambda) = a\lambda + O(\lambda^2) \), \( a \neq 0 \), is analytic in \( \lambda \); the functions \( C_j = C_j(\lambda,t,u), j = 1,2,3 \), are bounded above and below by \( O(1) \) constants; finally \( C_1 - C_2 \) vanishes for \( \lambda = u = 0 \).

**Remarks**

1) The key hypothesis for the validity of Main Theorem is the smallness of \( \lambda \). When \( \lambda = 0 \) the
critical points correspond to \( t \pm u = \sqrt{2} - 1 \): hence for simplicity we restrict \( t \pm u \) in a sufficiently small \( O(1) \) interval around \( \sqrt{2} - 1 \). A possible explicit choice for \( D \), convenient for our proof, could be \( D = \left[ \frac{3(\sqrt{2} - 1)}{4}, \frac{5(\sqrt{2} - 1)}{4} \right] \). Our technique would allow us to prove the above theorem, at the cost of a lengthier discussion, for any \( t^{(1)}, t^{(2)} > 0 \): of course in that case we should distinguish different regions of parameters and treat in a different way the cases of low or high temperature or the case of big anisotropy (i.e. the cases \( t << \sqrt{2} - 1 \) or \( t >> \sqrt{2} - 1 \) or \( |u| >> 1 \)).

2) (1.6) shows how the crossover from universal to nonuniversal behaviour is realized. When \( u \neq 0 \) only the first term in (1.6) can be singular in correspondence of the two critical points; it has a logarithmic singularity (as in the Ising model) with a constant \( O(\Delta^{2\eta}) \) in front. However the logarithmic term dominates on the second one only if \( t \) varies inside an extremely small region \( O(|u|^{1+\eta}/\lambda) \), \( a > 0 \), around the critical points. Outside such region the power law behaviour corresponding to the second addend in (1.6) dominates. When \( u \to 0 \) one recovers the power law decay found in [M1] for the isotropic case. See Fig 2.

![FIG 2. The qualitative behaviour of \( C_v \) as a function of \( t-T_c \), where \( T_c = (t^+_c+t^-_c)/2 \). The three graphs are plots of (1.6), with \( C_1=C_2=1, C_3=0, u=0.01, \eta=\eta_c=0.1,0,-0.1 \) respectively; the central curve corresponds to \( \lambda=0 \), the upper one to \( \lambda>0 \) and the lower to \( \lambda<0 \).](image)

3) By the result of item (1) of Main Theorem, \( C_v \) is analytic in \( \lambda, t, u \) outside the critical line. This is not apparent from (1.6), because \( \Delta \) is non analytic in \( u \) at \( u = 0 \) (of course the bounded functions \( C_j \) are non analytic in \( u \) also, in a suitable way compensating the non analyticity of \( \Delta \)). We get to (1.6) by interpolating two different asymptotic behaviours of \( C_v \) in the regions \( |t - T_c| < 2|u|^{1+\eta} \) and \( |t - T_c| \geq 2|u|^{1+\eta} \) and the non analyticity of \( \Delta \) is introduced "by hands" by our estimates and it is not intrinsic for \( C_v \). (1.6) is simply a convenient way to describe the crossover between different critical behaviours of \( C_v \).

4) We do not study the free energy directly at \( t = t^\pm_c(\lambda, u) \), therefore in order to show that \( t = t^\pm_c(\lambda, u) \) is a critical point we must study some thermodynamic property like the specific heat by evaluating it at \( t \neq t^\pm_c(\lambda, u) \) and \( M = \infty \) and then verify that it has a singular behavior as \( t \to t^\pm_c \). The case \( t \) precisely equal to \( t^\pm_c \) cannot be discussed at the moment with our techniques, in spite of the uniformity of our bounds as \( t \to t^\pm_c \). The reason is that we write the AT partition function as a sum of 16 different partition functions, differing for boundary terms. Our estimates
on each single term are uniform up to the critical point; however, in order to show that the free energy computed with one of the 16 terms is the same as the complete free energy, we need to stay at $t \neq t_c^\pm$: in this case boundary terms are suppressed as $\sim e^{-\kappa M|t-t_c^\pm|}$, $\kappa > 0$, as $M \to \infty$.

If we stay exactly at the critical point cancellations between the 16 terms can be present (as it is well known already from the Ising model exact solution [MW]) and we do not have control on the behaviour of the free energy, as the infinite volume limit is approached.

1.3 Strategy of the proof. It is well known that the free energy and the specific heat of the Ising model can be expressed as a sum of Pfaffians [MW] which can be equivalently written, see [ID][S], as Grassmann functional integrals, see for instance App A of [M1] or §4 of [GM] for the basic definitions of Grassmann variables and Grassmann integration. The formal action of the Ising model in terms of Grassmann variables $\psi, \bar{\psi}$ has the form

$$
\sum_x \frac{t}{4} \left[ \psi_x (\partial_1 - i\partial_0) \psi_x + \bar{\psi}_x (\partial_1 + i\partial_0) \bar{\psi}_x - 2t \bar{\psi}_x (\partial_1 + \partial_0) \psi_x + i(\sqrt{2} - 1 - t) \bar{\psi}_x \psi_x \right],
$$

(1.7)

where $\partial_j$ are discrete derivatives. $\psi$ and $\bar{\psi}$ are called Majorana fields, see [ID], because of an analogy with relativistic Majorana fermions. They are massive, because of the presence of the last term in (1.7); criticality corresponds to the massless case ($t = \sqrt{2} - 1$). If $\lambda = 0$ the free energy and specific heat can be written as sum of Grassmann integrals describing two kinds of Majorana fields, with masses $m^{(1)} = t^{(1)} - \sqrt{2} + 1$ and $m^{(2)} = t^{(2)} - \sqrt{2} + 1$. The critical points are obtained by choosing one of the two fields massless (in the isotropic case $t^{(1)} = t^{(2)}$ and the two fields become massless together).

If $\lambda \neq 0$ again the free energy and the specific heat can be written as Grassmann integrals, but the Majorana fields are interacting with a short range potential. By performing a suitable change of variables, the partition function can be written, see §2 and §3, as a sum of terms $\Xi^{\gamma_1,\gamma_2}_{AT}$ ($\gamma_1, \gamma_2$ label different boundary conditions) of the form

$$
\Xi^{\gamma_1,\gamma_2}_{AT} = \int P(d\psi) e^{-V^{(1)}(\sqrt{Z_1} \psi)} \ , \quad P(d\psi) = D\psi e^{-Z_1(\psi^+, A\psi)},
$$

(1.8)

where: $\psi = \{ \psi^+_x, \psi^-_x \}_{\omega = \pm 1}$ are elements of a Grassmann algebra; $D\psi$ is a symbol for the Grassmann integration; $V^{(1)}$ is a short range interaction, sum of monomials in $\psi$ of any degree, whose quartic term is weighted by a constant $\lambda = O(\lambda)$; and $Z_1(\psi^+, A\psi)$ has the form:

$$
Z_1 \sum_{x, \omega} \psi^+_x (\partial_1 - i\omega \partial_0) \psi^-_x - i\sigma_1 \psi^+_x \psi^-_x \psi^-_x \psi^-_x + i\omega \mu_1 \psi^+_x \psi^-_x \psi^-_x \psi^-_x - \beta_1 \psi^+_x (\partial_1 - i\omega \partial_0) \psi^-_x \psi^-_x
$$

(1.9)

with $\sigma_1 = O(t - \sqrt{2} + 1) + O(\lambda)$, $\mu_1, \beta_1 = O(\mu)$ (in particular in the isotropic case the terms proportional to $\mu_1$ and $\beta_1$ are absent). If $\lambda = 0$, $\sigma_1 = (m^{(1)} + m^{(2)})/2$ and $\mu_1 = (m^{(2)} - m^{(1)})/2$. $\psi^\pm$ are called Dirac fields, because of an analogy with relativistic Dirac fermions; they are combinations of the Majorana variables $\psi^{(j)}, \bar{\psi}^{(j)}$, $j = 1, 2$, associated with the two Ising layers in (1.1); hence the description in terms of Dirac variables mixes intrinsically the two Ising models and will be useful in a range of momentum scale in which the two layers appear to be essentially equal.

One can compute $\Xi^{\gamma_1,\gamma_2}_{AT}$ by expanding $e^{-V^{(1)}(\sqrt{Z_1} \psi)}$ in Taylor series and integrating term by term the Grassmann monomials; since the propagators of $P(d\psi)$ (i.e. the elements of $A^{-1}$, see (1.8), (1.9)) diverge for $k = 0$ and $\sigma_1 \pm \mu_1 = 0$ in the infinite volume limit $M \to \infty$, the series can converge uniformly in $M$ only in a region outside $|\sigma_1 \pm \mu_1| \leq c$, for some $c$, i.e. in the thermodynamic limit it can converge only far from the critical points.

Since we are interested in the critical behaviour of the system, we set up a more complicated procedure to evaluate the partition function, based on (Wilsonian) Renormalization Group (RG). The first step is to decompose the integration $P(d\psi)$ as a product of independent integrations:
\[ P(d\psi) = \prod_{h=\infty}^{1} P(d\psi^{(h)}) \], where the momentum space propagator corresponding to \( P(d\psi^{(h)}) \) is not singular, but \( O(\gamma^{-h}) \), for \( M \to \infty \), \( \gamma \) being a fixed scaling parameter larger than 1. This decomposition is realized by slicing in a smooth way the momentum space, so that \( \psi^{(h)} \), if \( h \leq 0 \), depends only on the momenta between \( \gamma^{h-1} \) and \( \gamma^{h+1} \). We compute the Grassmann integrals defining the partition function by iteratively integrating the fields \( \psi^{(1)}, \psi^{(0)}, \ldots \), see §4. After each integration step we rewrite the partition function in a way similar to (1.8), with the quadratic form \( Z_{1}(\psi^{+}, A^{(h)} \psi) \) replaced by \( Z_{h}(\psi^{+}, A^{(h)} \psi) \), which has the same structure of (1.9), with \( Z_{h}, \sigma_{h}, \mu_{h} \) replacing \( Z_{1}, \sigma_{1}, \mu_{1} \); the structure of \( Z_{h}(\psi^{+}, A^{(h)} \psi) \) is preserved because of symmetry properties, guaranteeing that many other possible quadratic “local” terms are indeed vanishing, or irrelevant in a RG sense. The interaction \( \psi(1) \) is replaced by an effective action \( \psi^{(h)} \), \( h \leq 0 \), given by a sum of monomials of \( \psi \) of arbitrary order, with kernels decaying in real space on scale \( \gamma^{-h} \); in particular the quartic term is weighted by a coupling constant \( \lambda_{h} \) and the kernels of \( \psi^{(h)} \) are analytic functions of \( \{\lambda_{h}, \ldots, \lambda_{1}\} \), if \( \lambda_{h} \) are small enough, \( k \geq h \), and \( |\sigma_{k}\gamma^{-k}|, |\mu_{k}\gamma^{-k}| \leq 1 \) (say – the constant 1 could be replaced by any other constant \( O(1) \)).

In this way the problem of finding good bounds for \( \log \Xi_{\text{AT}} \) is reformulated into the problem of controlling the size of \( \lambda_{h}, \sigma_{h}, \mu_{h} \), \( h \leq 0 \), under the RG iterations. We use a crucial property, called vanishing of Beta function, to prove that actually, if \( \lambda \) is small enough, \( |\lambda_{h}| \leq 2|\lambda_{1}| \) (recall that \( \lambda_{1} = O(\lambda) \)). The possibility of controlling the flow of \( \lambda_{h} \) is the main reason for describing the system in terms of Dirac variables. For \( \sigma_{h}, \mu_{h}, Z_{h} \), we find that, under RG iterations, they evolve as: \( \sigma_{h} \simeq \sigma_{1}\gamma^{-h}, \mu_{h} \simeq \mu_{1}\gamma^{-h} \), \( Z_{h} \simeq \gamma^{-h}\lambda^{2h} \). Note in particular that \( Z_{h} \) grows exponentially with an exponent \( O(\lambda^{2}) \); this is connected with the presence of “critical indexes” in the correlation functions, which means that their long distance behaviour is qualitatively changed by the interaction.

We perform the iterative integration described above up to a scale \( h^{*} \) such that \( (|\sigma_{h^{*}}| + |\mu_{h^{*}}|)\gamma^{-h^{*}} = O(1) \), in such a way that \( (|\sigma_{1}| + |\mu_{1}|)\gamma^{-h} \leq O(1) \), for all \( h \geq h^{*} \) and convergence of the kernels of the effective potential can be guaranteed by our estimates. In the range of scales \( h \geq h^{*} \) the flow of the effective coupling constant \( \lambda_{h} \) is essentially the same as for the isotropic AT model [M1] (since \( |\mu_{h}|\gamma^{-h} \) is small the iteration “does not see” the anisotropy and the system seems to behave as if there was just one critical point) and nonuniversal critical indexes are generated (they appear in the flows of \( \sigma_{h}, \mu_{h} \) and \( Z_{h} \)), following the same mechanism of the isotropic case.

We note that after the integration of \( \psi^{(1)}, \ldots, \psi^{(h^{*}+1)} \), we can still reformulate the problem in terms of the original Majorana fermions \( \psi^{(1, \leq h^{*})}, \psi^{(2, \leq h^{*})} \), associated with the two Ising models in (1.1). On scale \( h^{*} \) their masses are deeply changed w.r.t. \( t^{(1)} - \sqrt{2} + 1 + t^{(2)} - \sqrt{2} + 1 \): they are given by \( m_{h^{*}}^{(1)} = |\sigma_{h^{*}}| + |\mu_{h^{*}}| \) and \( m_{h^{*}}^{(2)} = |\sigma_{h^{*}}| - |\mu_{h^{*}}| \). Note that the condition \( |\sigma_{h^{*}}| + |\mu_{h^{*}}| = O(\gamma^{h^{*}}) \) implies that the field \( \psi^{(1, \leq h^{*})} \) is massive on scale \( h^{*} \); (so that the Ising layer with \( j = 1 \) is “far from criticality” on the same scale). This implies that we can integrate (without any multiscale decomposition) the massive Majorana field \( \psi^{(1, \leq h^{*})} \), obtaining an effective theory of a single Majorana field with mass \( |\sigma_{h^{*}}| - |\mu_{h^{*}}| \), which can be arbitrarily small. The integration of the scales \( \leq h^{*} \), see §6, is done again by a multiscale decomposition similar to the one just described; an important feature is however that there are no more quartic marginal terms, because the anticommutativity of Grassmann variables forbids local quartic monomials of a single Majorana fermion. The problem is essentially equivalent to the study of a single perturbed Ising model with “upper” cutoff on momentum space \( O(\gamma^{h^{*}}) \) and mass \( |\sigma_{h^{*}}| - |\mu_{h^{*}}| \). The flow of the effective mass and of \( Z_{h} \) is non anomalous in this regime: in particular the mass of Majorana field is just shifted by \( O(\lambda\gamma^{h^{*}}) \) from \( |\sigma_{h^{*}}| - |\mu_{h^{*}}| \). Criticality is found when the effective mass on scale \( -\infty \) is vanishing; the values of \( t, u \) for which this happens are found by solving a non trivial implicit function problem.

Finally, see §7, we define a similar expansion for the specific heat and we compute its asymptotic behaviour arbitrarily near the critical points.

Technically it is an interesting feature of this problem that there are two regimes in which the system must be described in terms of different fields: a first one in which the natural variables
are Dirac Grassmann variables, and a second one in which they are Majorana; note that the scale separating the two regimes is dynamically generated by the RG iterations (and of course its precise location is not crucial and $h'_{1}$ can be modified in $h'_{1} + n$, $n \in \mathbb{Z}$, without qualitatively affecting the bounds).

2. Fermionic representation

2.1 The partition function $\Xi_{I}^{(j)} = \sum_{\sigma^{(j)}} \exp \{-J^{(j)} H_{I}(\sigma^{(j)})\}$ of the Ising model can be written as a Grassmann integral; this is a classical result, mainly due to [LMS][Ka][H][MW][S]. In Appendix A1, starting from a formula obtained in [MW], we prove that

$$
\Xi_{I}^{(j)} = (-1)^{M^{2}} \frac{(2 \cosh J^{(j)})^{M^{2}}}{2} \sum_{\varepsilon, \varepsilon' = \pm} \int_{x \in \Lambda_{M}} \prod_{x} \left\{ dH_{x}^{(j)} d\overline{H}_{x}^{(j)} dV_{x}^{(j)} d\overline{V}_{x}^{(j)} (-1)^{\delta_{\varepsilon} e^{S_{\gamma}^{(j)}(x^{(j)})}} \right\}
$$

(2.1)

where $j = 1, 2$ denotes the lattice, $\gamma = (\varepsilon, \varepsilon')$ and $\delta_{\varepsilon}$ is $\delta_{\varepsilon, +} = 1$, $\delta_{\varepsilon, -} = 2$ and, if $t^{(j)} = \tanh J^{(j)},$

$$
S_{\gamma}^{(j)}(t^{(j)}) = t^{(j)} \sum_{x \in \Lambda_{M}} \left[ \overline{H}_{x}^{(j)} H_{x}^{(j)} + \overline{V}_{x}^{(j)} V_{x}^{(j)} \right] + \sum_{x \in \Lambda_{M}} \left[ \overline{H}_{x}^{(j)} + \overline{V}_{x}^{(j)} \right] V_{x}^{(j)} + V_{x}^{(j)} \overline{V}_{x}^{(j)} + H_{x}^{(j)} \overline{V}_{x}^{(j)} + V_{x}^{(j)} H_{x}^{(j)} \right],
$$

(2.2)

where $H_{x}^{(j)}, \overline{H}_{x}^{(j)}, V_{x}^{(j)}, \overline{V}_{x}^{(j)}$ are Grassmann variables verifying different boundary conditions depending on the label $\gamma = (\varepsilon, \varepsilon')$ which is not affixed explicitly, to simplify the notations, i.e.

$$
\overline{H}_{x}^{(j)} = \varepsilon H_{x}^{(j)}, \quad \overline{H}_{x}^{(j)} = \varepsilon' H_{x}^{(j)} \quad \varepsilon, \varepsilon' = \pm
$$

(2.3)

and identical definitions are set for the variables $V_{x}^{(j)}, \overline{V}_{x}^{(j)}$; we shall say that $\overline{H}_{x}^{(j)}, H_{x}^{(j)}, V_{x}^{(j)}, \overline{V}_{x}^{(j)}$ satisfy $\varepsilon$-periodic ($\varepsilon'$-periodic) boundary conditions in vertical (horizontal) direction.

2.2 By expanding in power series $\exp \{-\lambda V\}$, we see that the partition function of the model (1.1) is

$$
\Xi_{AT} = \sum_{\sigma^{(1)}, \sigma^{(2)}} e^{-J^{(1)} H_{I}(\sigma^{(1)})} e^{-J^{(2)} H_{I}(\sigma^{(2)})} e^{-\lambda V(\sigma^{(1)}, \sigma^{(2)})} =
$$

$$
= (\cosh \lambda)^{2M^{2}} \sum_{\sigma^{(1)}, \sigma^{(2)}} e^{-J^{(1)} H_{I}(\sigma^{(1)})} e^{-J^{(2)} H_{I}(\sigma^{(2)})},
$$

(2.4)

where $\lambda = \tanh \lambda$. The r.h.s. of (2.4) can be rewritten as:

$$
\Xi_{AT} = \prod_{x \in \Lambda_{M}} \left[ \left( 1 + \lambda \frac{\partial^{2}}{\partial J^{(1)} x^{(1)} x^{(2)} \partial J^{(2)} x^{(1)} x^{(2)}} \right) \Xi_{I}^{(1)} \left( \{J^{(1)}_{x, x'}\} \right) \Xi_{I}^{(2)} \left( \{J^{(2)}_{x, x'}\} \right) \right]_{\{J^{(1)}_{x, x'}\} = \{J^{(j)}_{x, x'}\}},
$$

(2.5)

where $\Xi_{I}^{(1)} \left( \{J^{(j)}_{x, x'}\} \right)$ is the partition function of an Ising model in which the couplings are allowed to depend on the bonds (the coupling associated to the n.n. bond $(x, x')$ on the lattice $j$ is called $J_{x, x'}^{(j)}$). Using for $\Xi_{I}^{(1)} \left( \{J^{(j)}_{x, x'}\} \right)$ an expression similar to (2.1), we find that we can express $\Xi_{AT}$ as a sum of sixteen partition functions labeled by $\gamma_{1}, \gamma_{2} = (\varepsilon_{1}, \varepsilon'_{1}), (\varepsilon_{2}, \varepsilon'_{2})$ (corresponding to choosing each $\varepsilon_{j}$ and $\varepsilon'_{j}$ as $\pm$):

$$
\Xi_{AT} = \frac{1}{4} (\cosh \lambda)^{2M^{2}} \sum_{\gamma_{1}, \gamma_{2}} (-1)^{\delta_{\varepsilon_{1}} + \delta_{\varepsilon_{2}}} Z_{AT}^{\gamma_{1}, \gamma_{2}},
$$

(2.6)
each of which is given by a functional integral

\[ \Xi_{\lambda T}^{\gamma_1, \gamma_2} = [4(1 + \lambda t(1 + 2))]^{M^2} \prod_{j=1}^{2} \left( \cosh J^{(j)} \right)^{M^2} (1)^{M^2}. \]

(2.7)

\[
\cdot \int \prod_{x \in \Lambda_M} dH^{(j)}_x d\bar{H}^{(j)}_x dV^{(j)}_x d\bar{V}^{(j)}_x e^{S^{(1)}(\phi^{\lambda_1}) + S^{(2)}(\phi^{\lambda_2}) + V_\lambda},
\]

where, if we define

\[
\chi^{(j)} = \lambda \left[ t(1 - t^2 + u^2) + (1)^j u(1 + t^2 - u^2) \right],
\]

(2.8)

we have that \( t^{(j)}_\lambda , j = 1, 2 \), is given by \( t^{(j)}_\lambda = t(j) + \lambda^{(j)} \) and \( V_\lambda \) by:

\[
V_\lambda = \sum_{x \in \Lambda_M} \lambda \left( \bar{V}^{(1)}_x H^{(1)}_x H^{(2)}_x + \bar{V}^{(1)}_x \bar{V}^{(2)}_x \bar{V}^{(2)}_x \right), \quad \lambda = \frac{\chi^{(1)}(\lambda)\chi^{(2)}(\lambda)}{\lambda(\lambda^2 - u^2)}.
\]

(2.9)

2.3 From now on, we shall study in detail only the partition function \( \Xi^{(-)}_{\lambda T} \equiv \Xi^{(-)}_{\lambda T}^{(-)} \), i.e. the partition function in which all Grassmanian variables verify antiperiodic boundary conditions (see (2.3)). We shall see in §5.5 below that, if \((\lambda, t, u)\) does not belong to the critical surface, which is a suitable 2-dimensional subset of \([-\varepsilon_1, \varepsilon_1] \times D \times [-|D|, |D|]\) which we will explicitly determine in §5.6, the partition function \( \Xi^{\gamma_1, \gamma_2}_{\lambda T} \) divided by \( \Xi^{(1)}_{\lambda T} \Xi^{(2)}_{\lambda T} \) is exponentially insensitive to boundary conditions as \( M \to \infty \).

As in [M1] we find convenient to perform the following change of variables, \( \alpha = \pm, \omega = \pm 1 \):

\[
\frac{1}{\sqrt{2}} \sum_{j=1,2} (-i\omega)^{(j)} \left( \bar{H}^{(j)}_x + i\omega H^{(j)}_x \right) = e^{i\omega x/4} \left( \psi^{\alpha}_{\omega, x} - \chi^{\alpha}_{\omega, x} \right)
\]

(2.10)

Let \( k \in D_{\omega,-} \), where \( D_{\omega,-} \) is the set of \( k \)'s such that \( k = 2\pi/M(n_1 + 1/2) \) and \( k_0 = 2\pi/M(n_0 + 1/2) \), where \([-M/2] \leq n_0, n_1 \leq [(M - 1)/2], n_0, n_1 \in \mathbb{Z} \). The Fourier transform of the Grassmanian fields \( \phi^{\alpha}_{\omega, x} \), \( \phi = \psi, \chi \), is given by \( \phi^{\alpha}_{\omega, k} \equiv \sum_{x \in \Lambda_M} e^{-i\omega k x} \phi^{\alpha}_{\omega, x} \).

With the above definitions, it is straightforward algebra to verify that the final expression is:

\[
\Xi^{\gamma_1, \gamma_2}_{\lambda T} = e^{-EM^2} \int P(d\psi) P(d\chi) e^{Q(\psi, \chi) + V(\psi, \chi)},
\]

(2.11)

where: \( E \) is a suitable constant; \( Q(\psi, \chi) \) collects the quadratic terms of the form \( \psi_{\omega_1, x_1} \chi_{\omega_2, x_2} \); \( V(\psi, \chi) \) is the quartic interaction (it is equal to \( V_\lambda \), see (2.9), in terms of the \( \psi^{\pm}_{\omega}, \chi^{\pm}_{\omega} \) variables); \( P(d\phi) \), \( \phi = \psi, \chi \), is:

\[
P(d\phi) = N^{-1}_\phi \prod_{k \in D_{\omega,-}} \prod_{\omega = \pm 1} d\phi^{+}_{\omega, k} d\phi^{-}_{\omega, k} \exp \left\{ -\frac{t_\chi}{4M^2} \sum_{k \in D_{\omega,-}} \Phi^{+T}_k A_\phi(k) \Phi_k \right\},
\]

\[
A_\phi(k) = \begin{pmatrix}
\sin k + \sin k_0 & -i\sigma_\phi(k) & -\frac{i}{2}(i \sin k + \sin k_0) & i\mu(k) \\
-i\sigma_\phi(k) & \sin k - \sin k_0 & -i\mu(k) & -\frac{i}{2}(i \sin k - \sin k_0) \\
-\frac{i}{2}(i \sin k + \sin k_0) & -i\mu(k) & \sin k + \sin k_0 & -i\sigma_\phi(k) \\
-i\mu(k) & \frac{i}{2}(i \sin k - \sin k_0) & i\sigma_\phi(k) & i\sin k - \sin k_0
\end{pmatrix}
\]

(2.12)
where
\[
\Phi^{±,T}_k = (\hat{\phi}_{{1,},k}^+, \hat{\phi}_{{1,k}}^-, \hat{\phi}_{{1,-k}}^-, \hat{\phi}_{{1,k}}^+), \quad \Phi^{T}_k = (\hat{\phi}_{{1,},k}^+, \hat{\phi}_{{1,k}}^-, \hat{\phi}_{{1,-k}}^+, \hat{\phi}_{{1,k}}^-),
\]
(2.13)

\( N_\phi \) is chosen in such a way that \( \int P(d\phi) = 1 \) and, if we define \( t_\lambda \) \( \equiv (t_\lambda^{(1)} + t_\lambda^{(2)})/2 \), \( u_\lambda \equiv (t_\lambda^{(1)} - t_\lambda^{(2)})/2 \), for \( \phi = \psi, \chi \) we have:
\[
\sigma_\phi(k) = 2\left(1 + \frac{\pm \sqrt{2} + 1}{t_k}\right) + \cos k_0 + \cos k - 2 , \quad \mu(k) = -(u_\lambda/t_\lambda)(\cos k + \cos k_0).
\]
(2.14)

In the first of (2.14) the \(- (+)\) sign corresponds to \( \phi = \psi (\phi = \chi) \). The parameter \( \mu \) in (2.12) is given by \( \mu = \mu(0) \).

It is convenient to split the \( \sqrt{2} - 1 \) appearing in the definition of \( \sigma_\psi(k) \) as:
\[
\sqrt{2} - 1 = (\sqrt{2} - 1 + \frac{\nu}{2}) - \frac{\nu}{2} - t_\psi - \frac{\nu}{2},
\]
(2.15)

where \( \nu \) is a parameter to be properly chosen later as a function of \( \lambda \), such that the average location of the critical points will be given by \( t_\lambda = t_\psi; \) in other words \( \nu \) has the role of a counterterm fixing the middle point of the critical temperatures. The splitting (2.15) induces the following splitting of \( P(d\psi) \):
\[
P(d\psi) = P_\sigma(d\psi)e^{-\nu F_\nu(\psi)}, \quad F_\nu(\psi) \equiv \frac{1}{2M^2} \sum_{k,\omega} (-i\omega)\hat{\psi}_m^{+}(k,\omega)\hat{\psi}_m^{−}(k,\omega),
\]
(2.16)

where \( P_\sigma(d\psi) \) is given by (2.12) with \( \phi = \psi \) and \( \sigma = 2(1 - t_\psi/t_\lambda) \) replacing \( \sigma_\psi(0) \).

2.4 Integration of the \( \chi \) variables. The propagators \( <\phi^{a}_{x,\omega,\phi^{a'}_{y,\omega'}}> \) of the fermionic integration \( P(d\phi) \) verify the following bound, for some \( A, \kappa > 0 \):
\[
| <\phi^{a}_{x,\omega,\phi^{a'}_{y,\omega'}}> | \leq Ae^{-\kappa m_\phi|x - y|},
\]
(2.17)

where \( m_\phi \) is the minimum between \( |m_\phi^{(1)}| \) and \( |m_\phi^{(2)}| \) and, for \( j = 1, 2, m_\phi^{(j)} \) is given by \( m_\phi^{(j)} = 2(t_\lambda^{(j)} - t_\psi)/t_\lambda \), \( j = 1, 2 \). Note that both \( m_\chi^{(1)} \) and \( m_\chi^{(2)} \) are \( O(1) \). This suggests to integrate first the \( \chi \) variables.

After the integration of the \( \chi \) variables we shall rewrite (2.11) as
\[
\sum_{\lambda \in \Lambda} = e^{-M^2 E_1} \int P_{Z_1, \sigma_1, \mu, C_1}(d\psi)e^{-\nu(1)(\sqrt{2} \psi)}, \quad \nu(1)(0) = 0,
\]
(2.18)

where \( C_1(k) \equiv 1, Z_1 = t_\psi, \sigma_1 = \sigma/(1 - \frac{\nu}{2}), \mu_1 = \mu/(1 - \frac{\nu}{2}) \) and \( P_{Z_1, \sigma_1, \mu, C_1}(d\psi) \) is the exponential of a quadratic form:
\[
P_{Z_1, \sigma_1, \mu, \psi, C_1}(d\psi) = N_1^{-1} \prod_{\omega = \pm 1} d\psi^+_{\omega, k} d\psi^-_{\omega, k} \exp \left[ -\frac{1}{4M^2} \sum_{k \in D_{\omega,-}} Z_1 C_1(k) \Psi^+_{k} A^{(1)}(k) \Psi_{k} \right],
\]
(2.19)
where $N_1$ is chosen in such a way that $\int P_{z_i,\sigma_i,\mu,1}(d\psi) = 1$. Moreover $V^{(1)}$ is the interaction, which can be expressed as a sum of monomials in $\psi$ of arbitrary order:

$$V^{(1)}(\psi) = \sum_{n=1}^{\infty} \sum_{k_1,\ldots,k_{2n-1}} \prod_{i=1}^{2n} \psi^{\alpha_i}(k_i) \hat{W}^{(1)}_{2n,\omega,\omega}(k_1,\ldots,k_{2n-1}) \delta(\sum_{i=1}^{2n} \alpha_i k_i)$$  

and $\delta(k) = \sum_{n \in \mathbb{Z}} \delta_{k,2\pi n}$. The constant $E_1$ in (2.18), the functions $a_1^\pm, b_1^\pm, c_1, d_1$ in (2.19) and the kernels $\hat{W}^{(1)}_{2n,\omega,\omega}$ in (2.20) have the properties described in the following Theorem, proved in Appendix A2. Note that from now on we will consider all functions appearing in the theory as functions of $\lambda, \sigma_1, \mu_1$ (of course $t$ and $u$ can be analytically and elementarily expressed in terms of $\lambda, \sigma_1, \mu_1$). We shall also assume $|\sigma_1, |\mu_1|$ bounded by some $O(1)$ constant. Note that if $t \pm u$ belong to a sufficiently small interval $D$ centered around $\sqrt{2}-1$, as assumed in the hypothesis of the Main Theorem in §1, then of course $|\sigma_1, |\mu_1| \leq c_1$ for a suitable constant $c_1$ (in particular, if $D$ is chosen as in Remark (1) following the Main Theorem, we find $|\sigma_1| \leq 1 + O(\varepsilon_1)$ and $|\mu_1| \leq 2 + O(\varepsilon_1)$).

**Theorem 2.1** Assume that $|\sigma_1, |\mu_1| \leq c_1$ for some constant $c_1 > 0$. There exist a constant $\varepsilon_1$ such that, if $|\lambda|, |\nu| \leq \varepsilon_1$, then $\mathbb{Z}_{\lambda,\mu}$ can be written as in (2.18), (2.19), (2.20), where:

1) $E_1$ is an $O(1)$ constant;

2) $a_1^\pm(k), b_1^\pm(k)$ are analytic odd functions of $k$ and $c_1(k), d_1(k)$ real analytic even functions of $k$;

3) in a neighborhood of $k = 0$, $a_1^\pm(k) = O(\lambda^1 k^1) + O(k^3)$, $b_1^\pm(k) = O(\mu_1 k^1) + O(k^3)$, $c_1(k) = O(k^2)$ and $d_1(k) = O(\mu_1 k^2)$;

4) $\hat{W}^{(1)}_{2n,\omega,\omega}(k_1,\ldots,k_{2n-1})$ are analytic functions of $k_1, \lambda, \nu, \sigma_1, \mu_1$, $i = 1,\ldots, 2n$ and, for some constant $C$,

$$|\hat{W}^{(1)}_{2n,\omega,\omega}(k_1,\ldots,k_{2n-1})| \leq M^2 C^n |\lambda|^{\max(1, n/2)}$$  

4-a) the terms in (2.21) with $n = 2$ can be written as

$$L_1 \sum_{k_1,\ldots,k_4} \psi^{\alpha_1}_{\omega,k_1} \psi^{\alpha_2}_{\omega,k_2} \psi^{\alpha_3}_{\omega,k_3} \psi^{\alpha_4}_{\omega,k_4} \delta(k_1 + k_2 - k_3 - k_4) +$$

$$+ \sum_{k_1,\ldots,k_4} \sum_{\omega,\nu} \hat{W}_{4,\omega,\nu}(k_1, k_2, k_3, k_4) \psi^{\alpha_1}_{\omega,k_1} \psi^{\alpha_2}_{\omega,k_2} \psi^{\alpha_3}_{\omega,k_3} \psi^{\alpha_4}_{\omega,k_4} \delta(\sum_{i=1}^{4} \alpha_i k_i) \tag{2.22}$$

where $L_1$ is real and $\hat{W}_{4,\omega,\nu}(k_1, k_2, k_3)$ vanishes at $k_1 = k_2 = k_3 = (\tilde{\omega}, \tilde{\nu})$;

4-b) the term in (2.21) with $n = 1$ can be written as:

$$\frac{1}{4} \sum_{\omega,\nu = \pm} \sum_k \left[ S_1 (-i \omega) \psi^{\alpha}_{\omega,k} \psi^{\alpha}_{\omega,-k} + M_1(i \omega) \psi^{\alpha}_{\omega,k} \psi^{\alpha}_{\omega,-k} + F_1(i \sin k + \omega \sin k_0) \psi^{\alpha}_{\omega,k} \psi^{\alpha}_{\omega,-k} + F_1(i \sin k + \omega \sin k_0) \psi^{\alpha}_{\omega,k} \psi^{\alpha}_{\omega,-k} \right] + \sum_{k} \sum_{\omega,\nu} \hat{W}_{2,\omega,\nu}(k) \psi^{\alpha}_{\omega,\nu,k} \psi^{\alpha}_{\omega,\nu,-k} \tag{2.23}$$

where: $\hat{W}_{2,\omega,\nu}(k)$ is $O(k^2)$ in a neighborhood of $k = 0$; $S_1, M_1, F_1, G_1$ are real analytic functions of $\lambda, \sigma_1, \mu_1, \nu$ s.t. $F_1 = O(\lambda^1)$ and

$$L_1 = l_1 + O(\lambda \sigma_1) + O(\lambda \mu_1), \quad S_1 = s_1 + \gamma n_1 + O(\lambda \sigma_1^3) + O(\lambda \mu_1^3)$$

$$M_1 = m_1 + O(\lambda \mu_1 \sigma_1) + O(\lambda \mu_1^3), \quad G_1 = z_1 + O(\lambda \sigma_1) + O(\lambda \mu_1)$$  

with $s_1 = \sigma_1 f_1$, $m_1 = \mu_1 f_2$, and $l_1, n_1, f_1, f_2, z_1$ independent of $\sigma_1, \mu_1$; moreover $l_1 = \lambda / Z_1 + O(\lambda^2), f_1, f_2 = O(\lambda), \gamma n_1 = \nu / Z_1 + c'_1 \lambda + O(\lambda^2)$, for some $c'_1$ independent of $\lambda$, and $z_1 = O(\lambda^2)$.

**Remark.** The meaning of Theorem 2.1 is that after the integration of the $\chi$ fields we are left with a fermionic integration similar to (2.12) up to corrections which are at least $O(k^2)$, and an effective interaction containing terms with any number of fields.
A priori many bilinear terms with kernel $O(1)$ or $O(k)$ with respect to $k$ near $k = 0$ could be generated by the $\chi$-integration besides the ones originally present in (2.12); however symmetry considerations restrict drastically the number of possible bilinear terms $O(1)$ or $O(k)$. Only one new term of the form $\sum_k (i \sin k + \omega \sin k_0) \hat{\psi}_\omega \hat{\psi}_{-\omega - k}$ appears, which is “dimensionally” marginal in a RG sense; however it is weighted by a constant $O(\lambda_{\mu})$ and this will improve its “dimension”, so that it will result to be irrelevant, see §3.2 below.

3. Integration of the $\psi$ variables: first regime

3.1 Multiscale analysis. From the bound on $\det A_\psi^{(3)}(k)$ described in Theorem 2.1, we see that the $\psi$ fields have a mass given by $\min\{\|\sigma_1 - \mu_1\|, |\sigma_1 + \mu_1|\}$, which can be arbitrarily small; their integration in the infrared region (small $k$) needs a multiscale analysis. We introduce a scaling parameter $\gamma > 1$ which will be used to define a geometrically growing sequence of length scales $1, \gamma, \gamma^2, \ldots$, i.e. of geometrically decreasing momentum scales $\gamma^h$, $h = 0, -1, -2, \ldots$. Correspondingly we introduce $C^\infty$ compact support functions $f_h(k)$ $h \leq 1$, with the following properties: if $|k| = \sqrt{\sin^2 k + \sin^2 k_0}$, when $h \leq 0$, $f_h(k) = 0$ for $|k| < \gamma^{h-2}$ or $|k| > \gamma^h$, and $f_h(k) = 1$, if $|k| = \gamma^{h-1}$; $f_1(k) = 0$ for $|k| \leq \gamma^{-1}$ and $f_1(k) = 1$ for $|k| \geq 1$; furthermore:

$$1 = \sum_{h=h_M}^1 f_h(k) \quad \text{where:} \quad h_M = \min\{h : \gamma^h > \sqrt{2 \sin \pi \frac{M}{E_h}}\},$$

and $\sqrt{2 \sin \pi / M}$ is the smallest momentum allowed by the antiperiodic boundary conditions, i.e. $\sqrt{2 \sin \pi / M} = \min_{k \in D_-} |k|$.

The purpose is to perform the integration of (2.19) over the fermion fields in an iterative way. After each iteration we shall be left with a “simpler” Grassmanni integration to perform: if $h = 1, 0, -1, \ldots, h_M$, we shall write

$$\Xi_{AT} = \int P_{Z_h, \sigma_h, \mu_h, C_h}(d\psi^{(\leq h)}) e^{-\mathcal{V}(h)(\sqrt{2 \sin \pi \frac{M}{E_h}} - M^2 E_h)} \quad \mathcal{V}(h)(0) = 0,$$

where the quantities $Z_h$, $\sigma_h$, $\mu_h$, $C_h$, $P_{Z_h, \sigma_h, \mu_h, C_h}(d\psi^{(\leq h)})$, $\mathcal{V}(h)$ and $E_h$ have to be defined recursively and the result of the last iteration will be $\Xi_{AT} = e^{-M^2 E_{1+h,M}}$, i.e. the value of the partition function.

$P_{Z_h, \sigma_h, \mu_h, C_h}(d\psi^{(\leq h)})$ is defined by (2.19) in which we replace $Z_1, \sigma_1, \mu_1, \sigma_1^\omega, b_1^\omega, c_1, d_1, C_1(k)$ with $Z_h, \sigma_h, \mu_h, \sigma_h^\omega, b_h^\omega, c_h, d_h, C_h(k)$, where $C_h(k)^{-1} = \sum_{j=1}^{h} f_j(k)$. Moreover

$$\mathcal{V}(h)(\psi) = \sum_{n=1}^{\infty} \frac{1}{M^{2n}} \prod_{i=1}^{2n} \psi_{\omega_i, k_i} \mathcal{W}_{2n, \omega_1, \omega_2, \cdots, \omega_{2n}}(x_1, \ldots, x_{2n}) \delta^d(\sum_{i=1}^{2n} \sigma_i k_i) \equiv \sum_{n=1}^{\infty} \prod_{i=1}^{2n} \theta_j^{\sigma_i} \psi_{\omega_i, \sigma_i k_i} \mathcal{W}_{2n, \sigma_i, \omega_1, \omega_2, \cdots, \omega_{2n}}(x_1, \ldots, x_{2n})$$

where in the last line $j_i = 0, 1$, $\sigma_i \geq 0$ and $\partial_j$ is the forward discrete derivative in the $\hat{\omega}_j$ direction.

Note that the field $\psi^{(\leq h)}$, whose propagator is given by the inverse of $Z_h C_h(k) A_\psi^{(h)}$, has the same support of $C_k^{-1}(k)$, that is on a strip of width $\gamma^h$ around the singularity $k = 0$. The field $\psi^{(\leq 1)}$ coincides with the field $\psi$ of previous section, so that (2.18) is the same as (3.2) with $h = 1$.

It is crucial for the following to think $\mathcal{W}_{2n, \sigma_i, \omega_1, \omega_2, \cdots, \omega_{2n}}$, $h \leq 1$, as functions of the variables $\sigma_i(k), \mu_k(k)$, $k = h, h + 1, \ldots, 0, 1, k \in D_-$. The iterative construction below will inductively imply that the
dependence on these variables is well defined (note that for \( h = 1 \) we can think the kernels of \( \mathcal{V}(1) \) as functions of \( \sigma_1, \mu_1, \) see Theorem 2.1).

### 3.2 The localization operator

We now begin to describe the iterative construction leading to (3.2). The first step consists in defining a localization operator \( \mathcal{L} \) acting on the kernels of \( \mathcal{V}(h) \), in terms of which we shall rewrite \( \mathcal{V}(h) = \mathcal{L}\mathcal{V}(h) + \mathcal{R}\mathcal{V}(h) \), where \( \mathcal{R} = 1 - \mathcal{L} \). The iterative integration procedure will use such splitting, see §3.3 below.

\( \mathcal{L} \) will be non zero only if acting on a kernel \( \widehat{W}^{(h)}_{2n,\omega} \) with \( n = 1, 2 \). In this case \( \mathcal{L} \) will be the combination of four different operators: \( \mathcal{L}_j, \ j = 0, 1 \), whose effect on a function of \( k \) will be essentially to extract the term of order \( j \) from its Taylor series in \( k \); and \( \mathcal{P}_j, \ j = 0, 1 \), whose effect on a functional of the sequence \( \sigma_h(k), \mu_h(k), \ldots, \sigma_1, \mu_1 \) will be essentially to extract the term of order \( j \) from its power series in \( \sigma_h(k), \mu_h(k), \ldots, \sigma_1, \mu_1 \).

The action of \( \mathcal{L}_j, \ j = 0, 1 \), on the kernels \( \widehat{W}^{(h)}_{2n,\omega}(k_1, \ldots, k_{2n}) \) is defined as follows.

1) If \( n = 1 \),

\[
\mathcal{L}_0 \widehat{W}^{(h)}_{2,\omega}(k, \alpha_1 \alpha_2 k) = \frac{1}{4} \sum_{\eta, \eta' = \pm 1} \widehat{W}^{(h)}_{2,\omega}(\bar{k}_{\eta \eta'}, \alpha_1 \alpha_2 \bar{k}_{\eta \eta'})
\]
\[
\mathcal{L}_1 \widehat{W}^{(h)}_{2,\omega}(k, \alpha_1 \alpha_2 k) = \frac{1}{4} \sum_{\eta, \eta' = \pm 1} \widehat{W}^{(h)}_{2,\omega}(\bar{k}_{\eta \eta'}, \alpha_1 \alpha_2 \bar{k}_{\eta \eta'}) \left[ \frac{\sin k}{\sin \frac{\pi}{M}} + \eta \frac{\sin k_0}{\sin \frac{\pi}{M}} \right],
\]

(3.4)

where \( \bar{k}_{\eta \eta'} = (\eta \frac{\pi}{M}, \eta' \frac{\pi}{M}) \) are the smallest momenta allowed by the antiperiodic boundary conditions.

2) If \( n = 2, \ \mathcal{L}_0 \widehat{W}^{(h)}_{4,\omega} = 0 \) and

\[
\mathcal{L}_0 \widehat{W}^{(h)}_{4,\omega}(k_1, k_2, k_3, k_4) \overset{\text{def}}{=} \widehat{W}^{(h)}_{4,\omega}(\bar{k}_{++}, \bar{k}_{++}, \bar{k}_{++}, \bar{k}_{++}).
\]

(3.5)

3) If \( n > 2, \ \mathcal{L}_0 \widehat{W}^{(h)}_{2n,\omega} = \mathcal{L}_1 \widehat{W}^{(h)}_{2n,\omega} = 0. \)

The action of \( \mathcal{P}_j, \ j = 0, 1 \), on the kernels \( \widehat{W}^{(h)}_{2n,\omega} \), thought as functionals of the sequence \( \sigma_h(k), \mu_h(k), \ldots, \sigma_1, \mu_1 \) is defined as follows.

\[
\mathcal{P}_0 \widehat{W}^{(h)}_{2n,\omega} \overset{\text{def}}{=} \left. \mathcal{P}^{(h)}_0 \right|_{\omega^{(h)} = \mu^{(h)} = 0}
\]
\[
\mathcal{P}_1 \widehat{W}^{(h)}_{2n,\omega} \overset{\text{def}}{=} \sum_{k \geq n, k} \left[ \sigma(k) \frac{\partial \mathcal{W}^{(h)}_{2n,\omega}}{\partial \sigma(k)} \right|_{x^{(h)} = \mu^{(h)} = 0} + \mu(k) \frac{\partial \mathcal{W}^{(h)}_{2n,\omega}}{\partial \mu(k)} \right|_{x^{(h)} = \mu^{(h)} = 0}.
\]

(3.6)

Given \( \mathcal{L}_j, \mathcal{P}_j, \ j = 0, 1 \) as above, we define the action of \( \mathcal{L} \) on the kernels \( \widehat{W}^{(h)}_{2n,\omega} \) as follows.

1) If \( n = 1 \), then

\[
\mathcal{L} \widehat{W}^{(h)}_{2,\omega} \overset{\text{def}}{=} \begin{cases} \mathcal{L}_0 (\mathcal{P}_0 + \mathcal{P}_1) \widehat{W}^{(h)}_{2,\omega} & \text{if } \omega_1 + \omega_2 = 0 \text{ and } \alpha_1 + \alpha_2 = 0, \\ \mathcal{L}_0 \mathcal{P}_1 \widehat{W}^{(h)}_{2,\omega} & \text{if } \omega_1 + \omega_2 = 0 \text{ and } \alpha_1 + \alpha_2 \neq 0, \\ \mathcal{L}_1 \mathcal{P}_0 \widehat{W}^{(h)}_{2,\omega} & \text{if } \omega_1 + \omega_2 \neq 0 \text{ and } \alpha_1 + \alpha_2 = 0, \\ 0 & \text{if } \omega_1 + \omega_2 \neq 0 \text{ and } \alpha_1 + \alpha_2 \neq 0. \end{cases}
\]

2) If \( n = 2 \), then \( \mathcal{L} \widehat{W}^{(h)}_{4,\omega} \overset{\text{def}}{=} \mathcal{L}_0 \mathcal{P}_0 \widehat{W}^{(h)}_{4,\omega}. \)

3) If \( n > 2 \), then \( \mathcal{L} \widehat{W}^{(h)}_{2n,\omega} = 0. \)
Finally, the effect of $\mathcal{L}$ on $\mathcal{V}^{(h)}$ is, by definition, to replace on the r.h.s. of (3.3) $\hat{W}^{2n,\omega}_{\lambda,\omega}$ with $\mathcal{L}\hat{W}^{2n,\omega}_{\lambda,\omega}$. Note that $\mathcal{L}^2\mathcal{V}^{(h)} = \mathcal{V}^{(h)}$.

Using the previous definitions we get the following result, proven in Appendix A2.2. We use the notation $\sigma^{(h)} = \{\sigma_k^{(h)}\}_{k \in D_{-,-}}$ and $\mu^{(h)} = \{\mu_k^{(h)}\}_{k \in D_{-,-}}$.

**Lemma 3.1.** Let the action of $\mathcal{L}$ on $\mathcal{V}^{(h)}$ be defined as above. Then

\[
\mathcal{L}\mathcal{V}^{(h)}(\psi^{(h)}) = (s_h + \gamma^n \lambda_h)F^{(h)}_\sigma + m_h F^{(h)}_\mu + l_h F^{(h)}_\lambda + z_h F^{(h)}_\zeta ,
\]

where $s_h, \lambda_h, m_h, l_h$ and $z_h$ are real constants and: $s_h$ is linear in $\sigma^{(h)}$ and independent of $\mu^{(h)}$; $m_h$ is linear in $\mu^{(h)}$ and independent of $\sigma^{(h)}$; $n_h, l_h, z_h$ are independent of $\sigma^{(h)}, \mu^{(h)}$; moreover, if $D_h = D_{-, -} \cap \{k : C^{-1}_h(k) > 0\}$,

\[
\begin{align*}
F^{(h)}_\sigma(\psi^{(h)}) &= \frac{1}{2M^2} \sum_{k \in D_h} \sum_{\omega = \pm 1} (-i\omega)\hat{\psi}^{(h)}_{\omega,k} \hat{\psi}^{(-h)}_{-\omega,k} \overset{def}{=} \frac{1}{M^2} \sum_{k \in D_h} \hat{F}^{(h)}_\sigma(k), \\
F^{(h)}_\mu(\psi^{(h)}) &= \frac{1}{4M^2} \sum_{k \in D_h} \sum_{\omega, \alpha = \pm 1} i\omega \hat{\psi}^{\alpha(\omega)}_{\omega,k} \hat{\psi}^{\alpha(-\omega)}_{-\omega,k} \overset{def}{=} \frac{1}{M^2} \sum_{k \in D_h} \hat{F}^{(h)}_\mu(k), \\
F^{(h)}_\lambda(\psi^{(h)}) &= \frac{1}{M^2} \sum_{k_1, k_2 \in D_h} \hat{\psi}^{(h)}_{1,k_1} \hat{\psi}^{(-h)}_{-1,k_2} \hat{\psi}^{(-h)}_{-1} \delta(k_1 + k_2 - k_3 - k_4) \\
F^{(h)}_\zeta(\psi^{(h)}) &= \frac{1}{2M^2} \sum_{k \in D_h} \sum_{\omega = \pm 1} (i\sin k + \sin k_0)\hat{\psi}^{(h)}_{\omega,k} \hat{\psi}^{(-h)}_{-\omega,k} \overset{def}{=} \frac{1}{M^2} \sum_{k \in D_h} \hat{F}^{(h)}_\zeta(k).
\end{align*}
\]

where $\delta(k) = M^2 \sum_{n \in \mathbb{Z}^2} \delta_{k,2\pi n}$.

**Remark.** The application of $\mathcal{L}$ to the kernels of the effective potential generates the sum in (3.7), i.e. a linear combination of the Grassmannian monomials in (3.8) which, in the renormalization group language, are called “relevant” (the first two) or “marginal” operators (the two others).

We now consider the operator $\mathcal{R}^{def} = 1 - \mathcal{L}$. The following result holds, see Appendix A2 for the proof. We use the notation $\mathcal{R}_1 = 1 - \mathcal{L}_0, \mathcal{R}_2 = 1 - \mathcal{L}_0 - \mathcal{L}_1, \mathcal{S}_1 = 1 - \mathcal{P}_0, \mathcal{S}_2 = 1 - \mathcal{P}_0 - \mathcal{P}_1$.

**Lemma 3.2.** The action of $\mathcal{R}$ on $\hat{W}^{2n,\omega}_{\lambda,\omega}$ for $n = 1, 2$ is the following.

1) If $n = 1$, then

\[
\mathcal{R}\hat{W}^{2n,\omega}_{\lambda,\omega} = \begin{cases} [S_2 + \mathcal{R}_3(\mathcal{P}_0 + \mathcal{P}_1)]\hat{W}^{2n,\omega}_{\lambda,\omega} & \text{if } \omega_1 + \omega_2 = 0, \\
[S_1 + \mathcal{R}_2(\mathcal{P}_0)]\hat{W}^{2n,\omega}_{\lambda,\omega} & \text{if } \omega_1 + \omega_2 \neq 0 \text{ and } \alpha_1 + \alpha_2 = 0, \\
\mathcal{R}_1 S_1 \hat{W}^{2n,\omega}_{\lambda,\omega} & \text{if } \omega_1 + \omega_2 \neq 0 \text{ and } \alpha_1 + \alpha_2 \neq 0, \\
\end{cases}
\]

2) If $n = 2$, then $\mathcal{R}\hat{W}^{4,\omega}_{\lambda,\omega} = [S_1 + \mathcal{R}_1(\mathcal{P}_0)]\hat{W}^{4,\omega}_{\lambda,\omega}$.

**Remark.** The effect of $\mathcal{R}_j, j = 1, 2$ on $\hat{W}^{(h)}$ consists in extracting the rest of a Taylor series in $k$ of order $j$. The effect of $S_j, j = 1, 2$ on $\hat{W}^{(h)}$ consists in extracting the rest of a power series in $(\sigma^{(h)}, \mu^{(h)})$ of order $j$. The definitions are given in such a way that $\mathcal{R}\hat{W}^{2n,\omega}_{\lambda,\omega}$ is at least quadratic in $k, \sigma^{(h)}, \mu^{(h)}$ if $n = 1$ and at least linear in $k, \sigma^{(h)}, \mu^{(h)}$ when $n = 2$. This will give dimensional gain factors in the bounds for $\mathcal{R}\hat{W}^{(h)}_{2n,\omega}$ w.r.t. the bounds for $\hat{W}^{(h)}_{2n,\omega}$, $n = 1, 2$, as we shall see in details in Appendix A4.

### 3.3 Renormalization.

Once that the above definitions are given we can describe our integration
procedure for $h \leq 0$.

We start from (3.2) and we rewrite it as

$$\int P_{Z_h, \sigma_h, \mu_h, C_h} (d\psi^{(\leq h)} e^{-\mathcal{L}V^{(h)}}(\sqrt{Z_h}\psi^{(\leq h)}) - \mathcal{R}V^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) - M^2 E_h, \quad (3.9)$$

with $\mathcal{L}V^{(h)}$ as in (3.7). Then we include the quadratic part of $\mathcal{L}V^{(h)}$ (except the term proportional to $n_h$) in the fermionic integration, so obtaining

$$\int P_{\tilde{Z}_{h-1}, \sigma_{h-1}, \mu_{h-1}, C_h} (d\psi^{(\leq h)}) \int Z_h F_x (\sqrt{Z_h}\psi^{(\leq h)}) - \gamma_h n_h F_x (\sqrt{Z_h}\psi^{(\leq h)}) - \mathcal{R}V^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) - M^2 E_h, \quad (3.10)$$

where $\tilde{Z}_{h-1}(k) \overset{\text{def}}{=} Z_h(1 + z_h C_{h-1}(k))$ and

$$\begin{align*}
\sigma_{h-1}(k) &\overset{\text{def}}{=} \frac{Z_h}{Z_{h-1}(k)}(\sigma_h(k) + s_h C_{h-1}(k)), \quad \mu_{h-1}(k) \overset{\text{def}}{=} \frac{Z_h}{Z_{h-1}(k)}(\mu_h(k) + m_h C_{h-1}(k)) \\
a_{h-1}^w(k) &\overset{\text{def}}{=} \frac{Z_h}{Z_{h-1}(k)} a_{h}^w(k), \quad b_{h-1}^e(k) \overset{\text{def}}{=} \frac{Z_h}{Z_{h-1}(k)} b_{h}^e(k) \\
c_{h-1}(k) &\overset{\text{def}}{=} \frac{Z_h}{Z_{h-1}(k)} c_{h}(k), \quad d_{h-1}(k) \overset{\text{def}}{=} \frac{Z_h}{Z_{h-1}(k)} d_{h}(k).
\end{align*}\quad (3.11)$$

The integration in (3.10) differs from the one in (3.2) and (3.9): $P_{\tilde{Z}_{h-1}, \sigma_{h-1}, \mu_{h-1}, C_h}$ is defined by (2.19) with $Z_1$ and $A^{(1)}_{\psi}$ replaced by $\tilde{Z}_{h-1}(k)$ and $A^{(h-1)}_{\psi}$.

Now we can perform the integration of the $\psi^{(h)}$ field. It is convenient to rescale the fields:

$$\hat{\psi}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}) \overset{\text{def}}{=} \lambda_h F_x (\sqrt{Z_{h-1}}\psi^{(\leq h)}) + \gamma_h \psi^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}) + \mathcal{R}V^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}), \quad (3.12)$$

where $\lambda_h = (\frac{Z_h}{Z_{h-1}})^2 l_h, \nu_h = \frac{Z_h}{Z_{h-1}} n_h$ and $\mathcal{R}V^{(h)} = (1 - \mathcal{L})V^{(h)}$ is the irrelevant part of $V^{(h)}$, and rewrite (3.10) as

$$e^{-M^2(t_h + E_h) \int P_{\tilde{Z}_{h-1}, \sigma_{h-1}, \mu_{h-1}, C_h} (d\psi^{(\leq h-1)}) \int P_{\tilde{Z}_{h-1}, \sigma_{h-1}, \mu_{h-1}, \tilde{f}_{h-1}} (d\psi^{(h)}) \int P_{\sigma^{(h)}} (d\psi^{(h)}) \int P_{\tilde{Z}_{h-1}, \sigma_{h-1}, \mu_{h-1}, \tilde{f}_{h-1}} (d\psi^{(h)}) \int P_{\tilde{Z}_{h-1}, \sigma_{h-1}, \mu_{h-1}, \tilde{f}_{h-1}} (d\psi^{(h)}) e^{-\hat{\psi}^{(h)}}(\sqrt{Z_{h-1}}\psi^{(h)})} \quad (3.13)$$

where we used the decomposition $\psi^{(\leq h)} = \psi^{(\leq h-1)} + \psi^{(h)}$ (and $\psi^{(\leq h-1)}, \psi^{(h)}$ are independent) and $\tilde{f}_{h}(k)$ is defined by the relation $C_{h-1}(k)\tilde{f}_{h-1}(k) = C_{h-1}(k)\tilde{Z}_{h-1}^{-1}(k) + \tilde{f}_{h}(k)Z_{h-1}^{-1}$, namely:

$$\tilde{f}_{h}(k) \overset{\text{def}}{=} Z_{h-1}\left[ \frac{C_{h-1}(k)}{Z_{h-1}(k)} - \frac{C_{h-1}(k)}{Z_{h-1}(k)} \right] = f_{h}(k)\left[ 1 + z_h f_{h+1}(k) \right]. \quad (3.14)$$

Note that $\tilde{f}_{h}(k)$ has the same support as $f_{h}(k)$. Moreover $P_{\tilde{Z}_{h-1}, \sigma_{h-1}, \mu_{h-1}, \tilde{f}_{h-1}} (d\psi^{(h)})$ is defined in the same way as $P_{\tilde{Z}_{h-1}, \sigma_{h-1}, \mu_{h-1}, \tilde{f}_{h}} (d\psi^{(h)})$, with $\tilde{Z}_{h-1}(k)$ resp. $C_{h}$ replaced by $Z_{h-1}$ resp. $\tilde{f}_{h}^{-1}$. The single scale propagator is

$$\int P_{\tilde{Z}_{h-1}, \sigma_{h-1}, \mu_{h-1}, \tilde{f}_{h-1}} (d\psi^{(h)}) \psi_{\alpha}^{(h)} \psi_{\beta}^{(h)} = \frac{1}{Z_{h-1}} g_{\alpha \beta}^{(h)}(x - y) \quad (3.15)$$

$$g_{\alpha \beta}^{(h)}(x - y) = \frac{1}{2M^2} \sum_k e^{i\alpha \beta}(k \cdot x - y) f_{h}(k) A_{\psi}^{(h)}(k)[A_{\psi}^{(h)}(k)]^{-1} \quad (3.16)$$
with \( j(-,1) = j'(+,1) = 1, j(-,-1) = j'(+,1) = 2, j(+,1) = j'(-,1) = 3 \) and \( j(+,-1) = j'(-,-1) = 4 \). One finds that \( g^{(h)}_{a,a'}(x) = g^{(h)}_{a,a'}(x) - \alpha \lambda g^{(h)}_{a,a'}(x) \), where \( g^{(h)}_{a,a'}(x), j = 1,2 \) are defined in Appendix A3, see (A3.1).

The long distance behaviour of the propagator is given by the following Lemma, proved in Appendix A3.

**Lemma 3.3.** Let \( \sigma_h \overset{\text{def}}{=} \sigma_h(0) \) and \( \mu_h \overset{\text{def}}{=} \mu_h(0) \) and assume \( |\lambda| \leq \varepsilon_1 \) for a small constant \( \varepsilon_1 \).

Suppose that for \( h > \bar{h} \)

\[
|z_h| \leq \frac{1}{2}, \quad |s_h| \leq \frac{1}{2}|\sigma_h|, \quad |m_h| \leq \frac{1}{2}|\mu_h|, \quad (3.17)
\]

that there exists \( c \) s.t.

\[
e^{-c|\lambda|} \leq \left| \frac{\sigma_h}{\gamma_h} \right| \leq e^{c|\lambda|}, \quad e^{-c|\lambda|} \leq \left| \frac{\mu_h}{\mu_h-1} \right| \leq e^{c|\lambda|}, \quad e^{-c|\lambda|^2} \leq \left| \frac{Z_h}{Z_{h-1}} \right| \leq e^{c|\lambda|^2}, \quad (3.18)
\]

and that, for some constant \( C_1 \)

\[
\frac{|\sigma_h|}{\gamma_h} \leq C_1, \quad \frac{|\mu_h|}{\gamma_h} \leq C_1; \quad (3.19)
\]

then, for all \( h \geq \bar{h} \), given the positive integers \( N, n_0, n_1 \) and putting \( n = n_0 + n_1 \), there exists a constant \( C_{N,n} \) s.t.

\[
|\partial_x^n \partial_y^n g^{(h)}_{a,a'}(x-y)| \leq C_{N,n} \frac{\gamma^{(1+n)h}}{1+(\gamma h |d(x-y)|)^N}, \quad \text{where} \quad d(x) = \frac{M}{\pi} \left( \sin \frac{\pi x}{M} \sin \frac{\pi x_0}{M} \right). \quad (3.20)
\]

Furthermore, if \( P_0, P_1 \) are defined as in (3.6) and \( S_1, S_2 \) are defined as in Lemma 3.2, we have that \( P_j g^{(h)}_{a,a'}, j = 0,1 \) and \( S_j g^{(h)}_{a,a'}, j = 1,2 \), satisfy the same bound (3.20), times a factor \((\frac{|\sigma_h|+|\mu_h|}{\gamma_h})^j\).

The bounds for \( P_0 g^{(h)}_{a,a'} \) and \( P_1 g^{(h)}_{a,a'} \) hold even without hypothesis (3.19).

After the integration of the field on scale \( h \) we are left with an integral involving the fields \( \psi(\leq h-1) \) and the new effective interaction \( \mathcal{V}(\leq h-1) \), defined as

\[
e^{-\mathcal{V}(\leq h-1)}(\sqrt{Z_{h-1}}\psi(\leq h-1)-E_h)^M = \int P_{Z_{h-1},\sigma_{h-1},\mu_{h-1}} \mathcal{Z}_h(d\psi^{(h)}) e^{-\mathcal{V}(\leq h-1)}(\sqrt{Z_{h-1}}\psi(\leq h-1)). \quad (3.21)
\]

It is easy to see that \( \mathcal{V}(\leq h-1) \) is of the form (3.3) and that \( E_{h-1} = E_h + t_h + \tilde{E}_h \). It is sufficient to use the well known identity

\[
M^2 \tilde{E}_h + \mathcal{V}(\leq h-1)(\sqrt{Z_{h-1}}\psi(\leq h-1)) = \sum_{n \geq 1} \frac{1}{n!} (-1)^{n+1} \mathcal{E}_h^T (\sqrt{Z_{h-1}}\psi(\leq h)) n, \quad (3.22)
\]

where \( \mathcal{E}_h^T (X(\psi(h)); n) \) is the truncated expectation of order \( n \) w.r.t. the propagator \( Z_{h-1}^{-1} g^{(h)}_{a,a'} \), defined as

\[
\mathcal{E}_h^T (X(\psi(h)); n) = \frac{\partial}{\partial \lambda} \log \int P_{Z_{h-1},\sigma_{h-1},\mu_{h-1}} \mathcal{Z}_h(d\psi^{(h)}) e^{\lambda X(\psi(h))} \bigg|_{\lambda=0}. \quad (3.23)
\]

Note that the above procedure allow us to write the running coupling constants \( \tilde{\nu}_{h-1} = (\lambda_{h-1},\nu_{h-1}) \), \( h \leq 1 \), in terms of \( \tilde{\nu}_h, h \leq k \leq 1 \), namely \( \tilde{\nu}_h = \beta_h(\tilde{\nu}_h,\ldots,\tilde{\nu}_1) \), where \( \beta_h \) is the so-called Beta function.

### 3.4 Analyticity of the effective potential

We have expressed the effective potential \( \mathcal{V}(h) \) in terms of the running coupling constants \( \lambda_k, \nu_k, k \geq h \), and of the renormalization constants \( Z_k, \mu_k(k), \sigma_k(k) \), \( k \geq h \).
In Appendix A4 we will prove the following result.

**Theorem 3.1.** Let \( \sigma_h \equiv \sigma_h(0) \) and \( \mu_h \equiv \mu_h(0) \) and assume \( |\lambda| \leq \varepsilon_1 \) for a small constant \( \varepsilon_1 \). Suppose that for \( h > \bar{h} \) the hypothesis (3.17), (3.18) and (3.19) hold. If, for some constant \( c \),

\[
\max_{h > \bar{h}} \{ |\lambda_h|, |\nu_h| \} \leq c|\lambda| ,
\]  

(3.24)

then there exists \( C > 0 \) s.t. the kernels in (3.3) satisfy

\[
\int dx_1 \ldots dx_{2n}|W_{2n}^{(h)}(x_1, \ldots, x_{2n})| \leq M^2\gamma^{-\bar{h}D_k(n)}(C|\lambda|)^{\max(1,n-1)}
\]

(3.25)

where \( D_k(n) = -2 + n + k \) and \( k = \sum_{i=1}^{2n} \sigma_i \).

Moreover \( |\bar{E}_{\bar{h}+1}| + |t_{\bar{h}+1}| \leq c|\lambda|^{2\bar{h}} \) and the kernels of \( LV(\bar{h}) \) satisfy

\[
|s_{\bar{h}}| \leq C|\lambda||\sigma_{\bar{h}}| , \quad |m_{\bar{h}}| \leq C|\lambda||\mu_{\bar{h}}|
\]

(3.26)

and

\[
|n_{\bar{h}}| \leq C|\lambda| , \quad |z_{\bar{h}}| \leq C|\lambda|^2 , \quad |l_{\bar{h}}| \leq C|\lambda|^2 .
\]

(3.27)

The bounds (3.26) holds even if (3.19) does not hold. The bounds (3.27) holds even if (3.19) and the first two of (3.18) do not hold.

**Remarks.**

1) The above result immediately implies analyticity of the effective potential of scale \( h \) in the running coupling constants \( \lambda_k, \nu_k, k \geq h \), under the assumptions (3.17), (3.18), (3.19) and (3.24).

2) The assumptions (3.18) and (3.24) will be proved in §4 and Appendix A5 below, solving the flow equations for \( \bar{v}_h = (\lambda_h, \nu_h) \) and \( Z_h, \sigma_h, \mu_h \), given by \( \bar{v}_h = \beta_h(\bar{v}_h, \ldots, \bar{v}_1), Z_{h-1} = Z_h(1+z_h) \) and (3.11). They will be proved to be true up to \( h = -\infty \).

---

4. The flow of the running coupling constants.

The convergence of the expansion for the effective potential is proved by Theorem 3.1 under the hypothesis that the running coupling constants are small, see (3.24), and that the bounds (3.17), (3.18) and (3.19) are satisfied. We now want to show that, choosing \( \lambda \) small enough and \( \nu \) as a suitable function of \( \lambda \), such hypothesis are indeed verified. In order to prove this, we will solve the flow equations for the renormalization constants (following from (3.11) and preceding line):

\[
\frac{Z_{h-1}}{Z_h} = 1 + z_h , \quad \frac{\sigma_{h-1}}{\sigma_h} = 1 + \frac{s_h/\sigma_h - z_h}{1 + z_h} , \quad \frac{\mu_{h-1}}{\mu_h} = 1 + \frac{m_h/\mu_h - z_h}{1 + z_h} ,
\]

(4.1)

together with those for the running coupling constants:

\[
\lambda_{h-1} = \lambda_h + \beta_h^\lambda(\lambda_h, \nu_h; \ldots; \lambda_1, \nu_1) \\
\nu_{h-1} = \gamma \nu_h + \beta_h^\nu(\lambda_h, \nu_h; \ldots; \lambda_1, \nu_1).
\]

(4.2)

The functions \( \beta_h^\lambda, \beta_h^\nu \) are called the \( \lambda \) and \( \nu \) components of the Beta function, see the comment after (3.23), and, by construction, are independent of \( \sigma_k, \mu_k \), so that their convergence follow just from (3.24) and the last of (3.18), i.e. without assuming (3.19), see Theorem 3.1. While for a general kernel we will apply Theorem 3.1 just up to a finite scale \( h^*_1 \) (in order to insure the validity of (3.19) with \( \bar{h} = h^*_1 \), we will inductively study the flow generated by (4.2) up to scale \( -\infty \), and we shall prove that it is bounded for all scales. The main result on the flows of \( \lambda_h \) and \( \nu_h \) proven
in Appendix A5, is the following.

**Theorem 4.1.** If $\lambda$ is small enough, there exists an analytic function $v^*(\lambda)$ independent of $t, u$ such that the running coupling constants $\{\lambda_h, \nu_h\}_{h \leq 1}$ with $v_1 = v^*(\lambda)$ verify $|v_h| \leq c|\lambda|^\gamma (\theta / 2)^h$ and $|\lambda_h| \leq c|\lambda|$. Moreover the kernels $z_h, s_h$ and $m_h$ satisfy (3.17) and the solutions of the flow equations (4.1) satisfy (3.18).

Once that $v_1$ is conveniently chosen as in Theorem 4.1, one can study in more detail the flows of the renormalization constants. In Appendix A5 we prove the following.

**Lemma 4.1.** If $\lambda$ is small enough and $v_1$ is chosen as in Theorem 4.1, the solution of (4.1) can be written as:

$$Z_h = \gamma \eta (h-1) + F^h, \quad \mu_h = \mu_1 \gamma \eta (h-1) + F^h, \quad \sigma_h = \sigma_1 \gamma \eta (h-1) + F^h$$

where $\eta, \eta_1, \eta_2$ and $F^h, \zeta^h, \mu^h, \nu^h$ are $O(\lambda)$ functions, independent of $\sigma_1, \mu_1$.

Moreover $\eta - \eta_1 = -b\lambda + O(|\lambda|^2), b > 0$.

### 4.1 The scale $h^*_1$.

The integration described in §3 is iterated until a scale $h^*_1$ defined in the following way:

$$h^*_1 \overset{\text{def}}{=} \begin{cases} \min \left\{ 1, \left[ \log_\gamma |\sigma_1|^{-1/\eta} \right] \right\} & \text{if } |\sigma_1|^{-1/\eta} > 2|\mu_1|^{-1/\eta}, \\ \min \left\{ 1, \left[ \log_\gamma |u|^{-1/\eta} \right] \right\} & \text{if } |\sigma_1|^{-1/\eta} \leq 2|\mu_1|^{-1/\eta}. \end{cases}$$

(4.4)

From (4.4) it follows that

$$C_2 \gamma h^*_1 \leq |\sigma_1| + |\mu_1| \leq C_1 \gamma h^*_1,$$

with $C_1, C_2$ independent of $\lambda, \mu_1, \sigma_1$.

This is obvious in the case $h^*_1 = 1$. If $h^*_1 < 1$ and $|\sigma_1|^{-1/\eta} > 2|\mu_1|^{-1/\eta}$, then $\gamma h^*_1 - 1 = c_\eta |\sigma_1|^{-1/\eta}, \mu_1 |^{-1/\eta}$, with $1 \leq c_\eta < \gamma$, so that, using the third of (4.3), we see that $C_2 \gamma h^*_1 \leq |\sigma_1| \leq C'_1 \gamma h^*_1$, for some $C'_1, C_2 = O(1)$. Furthermore, using also the second of (4.3), we find

$$\left| \frac{\mu_1 h^*_1}{|\sigma_1 h^*_1|} \right| = c_{\eta} |\mu_1| |\sigma_1|^{-1/\eta} \gamma F_{\mu} h^*_1 - F_{\sigma} h^*_1 < 1$$

(4.6)

and (4.5) follows.

If $h^*_1 < 1$ and $|\sigma_1|^{-1/\eta} \leq 2|\mu_1|^{-1/\eta}$, then $\gamma h^*_1 - 1 = c_\eta |u|^{-1/\eta}$, with $1 \leq c_\eta < \gamma$, so that, using the second of (4.3) and $|\sigma_1| = O(|u|)$, we see that $C_2 \gamma h^*_1 \leq |\mu_1| \leq C'_1 \gamma h^*_1$. Furthermore, using the third (4.3), we find

$$\left| \frac{\sigma_1 h^*_1}{|\mu_1 h^*_1|} \right| = c_{\eta} |\sigma_1| |u|^{-1/\eta} \gamma F_{\mu} h^*_1 - F_{\sigma} h^*_1 < C''_1,$$

(4.7)

for some $C''_1 = O(1)$, and (4.5) again follows.

**Remark.** The specific value of $h^*_1$ is not crucial: if we change $h^*_1$ in $h^*_1 + n, n \in Z$, the constants $C_1, C_2$ in (4.5) are replaced by different $O(1)$ constants and the estimates below are not qualitatively modified. Of course, the specific values of $C_1, C_2$ (then, the specific value of $h^*_1$) can affect the convergence radius of the perturbative series in $\lambda$. The optimal value of $h^*_1$ should be chosen by maximizing the corresponding convergence radius. Since here we are not interested in optimal estimates, we find the choice in (4.4) convenient.

Note also that $h^*_1$ is a non analytic function of $(\lambda, u)$. As a consequence, the asymptotic expression for the specific heat near the critical points (that we shall obtain in next section) will contain non analytic functions of $u$ (in fact it will contain terms depending on $h^*_1$). However, as explained in Remark (3) after the Main
Theorem, this does not imply that \( C_v \) is non analytic: it is clear that in this case the non analyticity is introduced “by hands” by our specific choice of \( h^*_1 \).

From the results of Theorem 4.1 and Lemma 4.1, together with (4.4) and (4.5), it follows that the assumptions of Theorem 3.1 are satisfied for any \( \tilde{h} \geq h^*_1 \). The integration of the scales \( \leq h^*_1 \) must be performed in a different way, as discussed in next section.

5. Integration of the \( \psi \) variables: second regime

5.1 Integration of the \( \psi^{(1)} \) field. If \( h^*_1 \) is fixed as in §4.1, we can use Theorem 3.1 up to the scale \( \tilde{h} = h^*_1 + 1 \).

Once that all the scales \( > h^*_1 \) are integrated out, it is more convenient to describe the system in terms of the fields \( \psi^{(1)}_\omega , \psi^{(2)}_\omega , \omega = \pm 1 \), defined through the following change of variables:

\[
\psi^{(1)}_{\omega,k,h^*_1} = \frac{1}{\sqrt{2}} (\psi^{(1)}_{\omega,-\alpha k} - i \alpha \psi^{(2)}_{\omega,-\alpha k}) , \quad \psi^{(2)}_{\omega,x} = \frac{1}{M^2} \sum_k e^{-ikx} \psi^{(j)}_{\omega,k} . \tag{5.1}
\]

If we perform this change of variables, we find \( P_{Z_{h^*_1}, \sigma_{h^*_1}, \mu_{h^*_1}, C_{h^*_1}} = \prod_{j=1}^2 P^{(j)}_{Z_{h^*_1}, m^{(j)}_{h^*_1}, C_{h^*_1}} \) where, if \( \psi^{(j)}_{(j \leq h^*_1) \cdot T} \defeq (\psi^{(j)}_{(j \leq h^*_1), 1}, \psi^{(j)}_{(j \leq h^*_1), -1} \),

\[
P^{(j)}_{Z_{h^*_1}, m^{(j)}_{h^*_1}, C_{h^*_1}} (d\psi_{(j \leq h^*_1)}) \defeq \\
\frac{1}{N(j)_{h^*_1}} \prod_{k,\omega} d\psi_{(j \leq h^*_1)} \exp \left\{ - \frac{Z_{h^*_1}}{4M^2} \sum_{k \in B_{h^*_1}} C_{h^*_1} (k) \psi^{(j \leq h^*_1), T}_k \right\} \}
\]

and \( a_{h^*_1}^{(j)}, m^{(j)}_{h^*_1}, C_{h^*_1} \) are given by (A3.2) with \( h = h^*_1 + 1 \).

The propagators \( g_{\omega_1, \omega_2}^{(j \leq h^*_1)} \) associated with the fermionic integration (5.2) are given by (A3.1) with \( h = h^*_1 + 1 \). Note that, by (4.5), \( \max \{ |m^{(1)}_{h^*_1}|, |m^{(2)}_{h^*_1}| \} = |\sigma_{h^*_1}| + |\mu_{h^*_1}| = O(\gamma h^*_1) \) (see (A3.2) for the definition of \( m^{(1)}_{h^*_1}, m^{(2)}_{h^*_1} \)). From now on, for definiteness we shall suppose that \( \max \{ |m^{(1)}_{h^*_1}|, |m^{(2)}_{h^*_1}| \} \approx |m^{(1)}_{h^*_1}| \). Then, it is easy to realize that the propagator \( g_{\omega_1, \omega_2}^{(1 \leq h^*_1)} \) is bounded as follows.

\[
|\partial^{n_{\omega_0}}_{x_0} \partial^{n_{\omega_2}}_{x_2} g_{\omega_1, \omega_2}^{(1 \leq h^*_1)} (x)| \leq C_{N,n} \frac{\gamma (1+n)_{h^*_1}}{1 + (\gamma h^*_1 |d(x)|)^N} , \quad n = n_0 + n_1 , \tag{5.3}
\]

namely \( g_{\omega_1, \omega_2}^{(1 \leq h^*_1)} \) satisfies the same bound as the single scale propagator on scale \( h = h^*_1 \). This suggests to integrate out \( \psi^{(1 \leq h^*_1)} \), without any other scale decomposition. We find the following result.

**Lemma 5.1** If \( |x| \leq \varepsilon_1 \), \( |\sigma_1|, |\mu_1| \leq c_1 (c_1, \varepsilon_1 \ being \ the \ same \ as \ in \ Theorem \ 2.1) \) and \( \nu_1 \) is fixed as in Theorem 4.1, we can rewrite the partition function as

\[
\Xi_{AT}^- = \int P^{(2)}_{Z_{h^*_1}, \nu_1, C_{h^*_1}} (d\psi_{(2 \leq h^*_1)} e^{-V_{h^*_1}^{(2 \leq h^*_1)}}) , \tag{5.4}
\]

where: \( \hat{m}^{(2)}_{h^*_1} (k) = m^{(2)}_{h^*_1} (k) - \gamma h^*_1 \pi_{h^*_1} C_{h^*_1}^{-1} (k) \), with \( \pi_{h^*_1} \) a free parameter, s.t. \( |\pi_{h^*_1}| \leq c |\lambda| ; |E_{h^*_1} - \]}
\[ E_{h_1} | \leq c|\lambda|^{2h_1}; \text{ and} \]
\[
\mathcal{V}^{(h_1)}(\psi^{(2)}) - \gamma^{h_1} \pi_{h_1} F_{\sigma}^{(2,\leq h_1^{\sigma})}(\psi^{(2,\leq h_1^{\sigma})}) = \sum_{n=1}^{\infty} \sum_{\omega} \prod_{i=1}^{2n} \psi^{(2)}_{\omega_i, k_i, \hat{W}^{(h_1^{\sigma})}_{2n, \omega}(k_1, \ldots, k_{2n-1})} \delta(\sum_{i=1}^{2n} k_i)
= \sum_{n=1}^{\infty} \sum_{\omega} \prod_{i=1}^{2n} \partial^\omega_i \psi^{(2)}_{\omega_i, x_i, \hat{W}^{(h_1^{\sigma})}_{2n, \omega}(x_1, \ldots, x_{2n})},
\]

with \(P^{(2,\leq h)}\) given by the first of (3.8) with \(\psi^{(2,\leq h)}_{\omega, k}\) replacing \(\psi^{(2,\leq h)}_{\omega, \omega, -k}\); and \(\hat{W}^{(h_1^{\sigma})}_{2n, \omega}\) satisfying the same bound (3.25) as \(W^{(h_1^{\sigma})}_{2n, \omega}\) with \(h = h_1^{\sigma}\).

In order to prove the Lemma it is sufficient to consider \(h = h_1^{\sigma}\) and rewrite \(P_{\eta_{h_1}, \eta_{h_1}^{\sigma}}^{(\sigma_1, \mu_{h_1}, \mu_{h_1}^{\sigma})}\) as the product \(\prod_{j=1}^{2} \hat{P}_{\eta_{h_1}^{(j)}, \eta_{h_1}^{(j)}},\) Then the integration over the \(\psi^{(1,\leq h_1^{\sigma})}\) field is done as the integration of the \(\chi_1^{(j)}\)'s in Appendix A2, recalling the bound (5.3).

Finally we rewrite \(m^{(2)}_{h_1^{\sigma}}(k)\) as \(\tilde{m}^{(2)}_{h_1^{\sigma}}(k) = \gamma^{h_1^{\sigma}} \pi_{h_1^{\sigma}} C_{h_1^{\sigma}}^{(2)}(k)\), where \(\pi_{h_1^{\sigma}}\) is a parameter to be suitably fixed below as a function of \(\lambda, \sigma_1, \mu_1\).

### 5.2 The localization operator

The integration of the r.h.s. of (5.4) is done in an iterative way similar to the one described in §3. now we shall perform an iterative integration of the field \(\psi^{(2)}\).

If \(h = h_1^{\sigma}, h_1^{\sigma} = 1, \ldots,\) we shall write:
\[
\Xi_{\mathcal{A}_T} = \int P^{(2)}_{\eta_{h_1^{(2)}, \eta_{h_1}^{(2)}}, C_{h_1}}, (d\psi^{(2,\leq h)}) e^{-\hat{V}^{(h)}(\sqrt{2 \bar{\pi}_{\psi^{(2,\leq h)}} - M^2 E_h)}, \]
where \(\hat{V}^{(h)}\) is given by an expansion similar to (5.5), with \(h\) replacing \(h_1^{\sigma}\) and \(Z_{h_1}, \tilde{m}^{(2)}_{h_1}\) are defined recursively in the following way. We first introduce a localization operator \(\mathcal{L}\). As in §3.2, we define \(\mathcal{L}\) as a combination of four operators \(\mathcal{L}_j, \hat{\mathcal{L}}_j, j = 0, 1, \mathcal{L}_j\) are defined as in (3.4) and (3.5), while \(\hat{\mathcal{L}}_0\) and \(\hat{\mathcal{L}}_1\), in analogy with (3.6), are defined as the operators extracting from a functional of \(\hat{m}^{(2)}_{h_1}(k), h < h_1^{\sigma}\), the contributions independent and linear in \(\hat{m}^{(2)}_{h_1}(k)\). Note that inductively the kernels \(\hat{W}^{(h)}_{2n, \omega}\) can be thought as functionals of \(\hat{m}^{(2)}_{h_1}(k), h < k \leq h_1^{\sigma}\). Given \(\mathcal{L}_j, \hat{\mathcal{L}}_j, j = 0, 1\) as above, we define the action of \(\mathcal{L}\) on the kernels \(\hat{W}^{(h)}_{2n, \omega}\) as follows.

1) If \(n = 1\), then
\[
\mathcal{L} \hat{W}^{(h)}_{2n, \omega} \text{def} \begin{cases} \mathcal{L}_0 (\hat{\mathcal{L}}_0 + \hat{\mathcal{L}}_1) \hat{W}^{(h)}_{2n, \omega} & \text{if } \omega_1 + \omega_2 = 0, \\ \mathcal{L}_1 \hat{\mathcal{L}}_0 \hat{W}^{(h)}_{2n, \omega} & \text{if } \omega_1 + \omega_2 \neq 0. \end{cases}
\]

2) If \(n > 2\), then \(\mathcal{L} \hat{W}^{(h)}_{2n, \omega} = 0\).

It is easy to prove the analogue of Lemma 3.1:
\[
\mathcal{L} \hat{V}^{(h)} = (s_h + \gamma^{h} p_h) F^{(2, \leq h)}_{\sigma} + z_h F^{(2, \leq h)}_{\xi},
\]

where \(s_h, p_h\) and \(z_h\) are real constants and: \(s_h\) is linear in \(\hat{m}^{(2)}_{h_1}(k), h \leq k \leq h_1^{\sigma}; p_h\) and \(z_h\) are independent of \(\hat{m}^{(2)}_{h_1}(k)\). Furthermore \(F^{(2, \leq h)}_{\sigma}\) and \(F^{(2, \leq h)}_{\xi}\) are given by the first and the last of (3.8) with \(\psi^{(2, \leq h)}_{\omega, k}\) replacing \(\psi^{(2, \leq h)}_{\omega, \omega, -k}\).

**Remark.** Note that the action of \(\mathcal{L}\) on the quartic terms is trivial. The reason of such a choice is that in the present case no quartic local term can appear, because of Pauli principle:
ψ_{1,0}^{(2,h)} \psi_{1,0}^{(2,h)} \psi_{1,0}^{(2,h)} \psi_{1,0}^{(2,h)} \equiv 0$, so that $\mathcal{L}_0 \overline{W}_{4,\varphi} = 0$.

Using the symmetry properties exposed in Appendix A2.2, we can prove the analogue of Lemma 3.2: if $n = 1$, then

$$\mathcal{R} \overline{W}_{2,\varphi} = \left\{ \begin{array}{ll} [S_2 + \mathcal{R}_2(\overline{P}_0 + \overline{P}_1)] \overline{W}_{2,\varphi} & \text{if } \omega_1 + \omega_2 = 0, \\
[S_1 + \mathcal{R}_2 \overline{P}_2] \overline{W}_{2,\varphi} & \text{if } \omega_1 + \omega_2 \neq 0, \end{array} \right. \quad (5.8)$$

where $S_1 = 1 - \overline{P}_0$ and $S_2 = 1 - \overline{P}_0 - \overline{P}_1$; if $n = 2$, then $\overline{W}_{4,\varphi} = \mathcal{R}_1 \overline{W}_{4,\varphi}$.

5.3 Renormalization for $h \leq h_1^*$. If $\mathcal{L}$ and $\mathcal{R} = 1 - \mathcal{L}$ are defined as in previous subsection, we can rewrite (5.6) as:

$$\int P_{Z_h, m_h}^{(2)}( \psi^{(2,\leq h)}) \psi^{(2,\leq h)} e^{-c V^{(h)}(\sqrt{Z_h} \psi^{(2,\leq h)})} - \mathcal{R} V^{(h)}(\sqrt{Z_h} \psi^{(2,\leq h)}) - M^2 E_h. \quad (5.9)$$

Furthermore, using (5.7) and defining:

$$\tilde{Z}_{h-1}(k) \stackrel{def}{=} Z_h(1 + C_0^{-1}(k) \omega), \quad \tilde{m}_{h-1}^{(2)}(k) \stackrel{def}{=} \frac{Z_h}{\tilde{Z}_{h-1}(k)} \left( \tilde{m}_h^{(2)}(k) + C_0^{-1}(k) s_h \right), \quad (5.10)$$

we see that (5.9) is equal to

$$\int P_{\tilde{Z}_{h-1}, \tilde{m}_{h-1}}^{(2)}( \psi^{(2,\leq h)}) \psi^{(2,\leq h)} e^{-c \pi_h F_{\tilde{Z}_{h-1}}^{(2,\leq h)}(\sqrt{\tilde{Z}_{h-1}} \psi^{(2,\leq h)}) - \mathcal{R} \tilde{V}^{(h)}(\sqrt{\tilde{Z}_{h-1}} \psi^{(2,\leq h)}) - M^2 (E_h + t_h) \quad (5.11)$$

Again, we rescale the potential:

$$\tilde{V}^{(h)}(\sqrt{\tilde{Z}_{h-1}} \psi^{(2,\leq h)}) \stackrel{def}{=} \gamma^h \pi_h F_{\tilde{Z}_{h-1}}^{(2,\leq h)}(\sqrt{\tilde{Z}_{h-1}} \psi^{(2,\leq h)}) + \mathcal{R} \tilde{V}^{(h)}(\sqrt{\tilde{Z}_{h-1}} \psi^{(2,\leq h)}) \quad (5.12)$$

where $Z_{h-1} = \tilde{Z}_{h-1}(0)$ and $\pi_h = (Z_h/Z_{h-1}) \pi_h$; we define $\tilde{f}_{h-1}^{-1}$ as in (3.14), we perform the single scale integration and we define the new effective potential as

$$e^{-\tilde{V}^{(h)}}(\sqrt{\tilde{Z}_{h-1}} \psi^{(2,\leq h-1)}) - M^2 E_h \stackrel{def}{=} \int P_{\tilde{Z}_{h-1}, \tilde{m}_{h-1}, \tilde{f}_{h-1}}^{(2)}( \psi^{(2,\leq h)}) \psi^{(2,\leq h)} e^{-\tilde{V}^{(h)}(\sqrt{\tilde{Z}_{h-1}} \psi^{(2,\leq h)}) \quad (5.13)}$$

Finally we pose $E_{h-1} = E_h + t_h + \tilde{E}_h$. Note that the above procedure allow us to write the $\pi_h$ in terms of $\pi_k$, $h \leq k \leq h_1^*$, namely $\pi_{h-1} = \gamma^h \pi_h + \beta_h^{(h)}(\pi_h, \ldots, \pi_{h-1})$, where $\beta_h^{(h)}$ is the Beta function.

Proceeding as in §3 we can inductively show that $\tilde{V}^{(h)}$ has the structure of (5.5), with $h$ replacing $h_1^*$ and that the kernels of $\tilde{V}^{(h)}$ are bounded as follows.

**Lemma 5.2.** Let the hypothesis of Lemma 5.1 be satisfied and suppose that, for $h < h_1^*$ and some constants $c, \vartheta > 0$

$$e^{-c|\lambda|} \lesssim \tilde{m}_h^{(2)} \leq e^{c|\lambda|}, \quad e^{-c|\lambda|^2} \lesssim \frac{Z_h}{\tilde{Z}_{h-1}} \lesssim e^{c|\lambda|^2}, \quad |\pi_h| \leq e|\lambda|, \quad |\tilde{m}_h^{(2)}| \leq \gamma^h. \quad (5.14)$$

Then the partition function can be rewritten as in (5.6) and there exists $C > 0$ s.t. the kernels of $\tilde{V}^{(h)}$ satisfy:

$$\int d\mathbf{x}_1 \cdots d\mathbf{x}_{2n} \overline{W}_{2n}^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_{2n}) \leq M^2 \gamma^{-h D_k(n)}(C |\lambda|)^{m_{(1,n-1)}} \quad (5.15)$$
where $D_k(n) = -2 + n + k$ and $k = \sum_{i=1}^{2n} \sigma_i$. Finally $|E_{h+1}| + |t_{h+1}| \leq c|\lambda|\gamma^{2h}$.

The proof of Lemma 5.2 is essentially identical to the proof of Theorem 3.1 and we do not repeat it here.

It is possible to fix $\pi_{h^*_1}$ so that the first three assumptions in (5.14) are valid for any $h \leq h^*_1$. More precisely, the following result holds, see Appendix A6.

**Lemma 5.3.** If $|\lambda| \leq \varepsilon_1$, $|\sigma_1|, |\mu_1| \leq c_1$ and $\nu_1$ is fixed as in Theorem 4.1, there exists $\pi_{h^*_1}(\lambda, \sigma_1, \mu_1)$ such that, if we fix $\pi_{h^*_1} = \pi_{h^*_1}(\lambda, \sigma_1, \mu_1)$, for $h \leq h^*_1$ we have:

$$|\pi_h| \leq c|\lambda|\gamma^{(a/2)(h-h^*_1)} \ , \quad \hat{m}_h^{(2)} = \hat{m}_{h^*_1}^{(2)} \gamma^{F_h} \ , \quad Z_h = Z_{h^*_1}^{\gamma \hat{F}_h} \ ,$$

where $F_h$ and $\hat{F}_h$ are $O(\lambda)$. Moreover:

$$|\pi_{h^*_1}(\lambda, \sigma_1, \mu_1) - \pi_{h^*_1}(\lambda, \sigma_1', \mu_1')| \leq c|\lambda| \left(\gamma^{(n_2-1)h^*_1}|\sigma_1 - \sigma_1'| + \gamma^{(n_2-1)h^*_1}|\mu_1 - \mu_1'|\right).$$

5.4 The integration of the scales $\leq h^*_2$. In order to insure that the last assumption in (5.14) holds, we iterate the preceding construction up to the scale $h^*_2$ defined as the scale s.t. $|\hat{m}_h^{(2)}| \leq \gamma^{h^*_2-1}$ for any $h^*_2 \leq k \leq h^*_1$ and $|\hat{m}_{h^*_1}^{(2)}| > \gamma^{h^*_2-2}$.

Once we have integrated all the fields $\psi(h^*_2)$, we can integrate $\psi(\leq h^*_2)$ without any further multiscale decomposition. Note in fact that by definition the propagator satisfies the same bound (5.3) with $h^*_2$ replacing $h^*_1$. Then, if we define

$$e^{-M^2 E_{\leq h^*_2}} = \int P_{h^*_2-1, \hat{m}_{h^*_2-1}^{(2)} C_{h^*_2}^{(2)}} e^{-\hat{V}^{(h^*_2)}(\sqrt{Z_{h^*_2-1}^{\leq h^*_2}})},$$

we find that $|E_{\leq h^*_2}| \leq c|\lambda|\gamma^{2h^*_2}$ (the proof is a repetition of the estimates on the single scale integration).

Combining this bound with the results of Theorem 3.1, Lemma 5.1, Lemma 5.2 and Lemma 5.3, together with the results of §4 we finally find that the free energy associated to $\Xi_{AT}$ is given by the following finite sum, uniformly convergent with the size of $\Lambda_M$:

$$\lim_{M \to \infty} \frac{1}{M^2} \log \Xi_{AT} = E_{\leq h^*_2} + (E_{h^*_1} - E_{h^*_1}) + \sum_{h=0}^{h^*_2+1} (E_{h} + t_h),$$

where $E_{\leq h^*_2} = \lim_{M \to \infty} E_{\leq h^*_2}$ and it is easy to see that $E_{\leq h^*_2}$, for any finite $h^*_2$, exists and satisfies the same bound of $E_{h^*_2}$.

5.5 Keeping $h^*_2$ finite. From the discussion of previous subsection, it follows that, for any finite $h^*_2$, (5.19) is an analytic function of $\lambda, t, u$, for $|\lambda|$ sufficiently small, uniformly in $h^*_2$ (this is an elementary consequence of Vitali’s convergence theorem). Moreover, repeating the discussion of Appendix G in [M1], it can be proved that, for any $h^*_2 > 0$ (here $h^*_2$ plays the role of $|t - t_c|$ in Appendix G of [M1]), the limit (5.19) coincides with $\lim_{M \to \infty} 1/M^2 \log \Xi_{AT}^{\gamma_1, \gamma_2}$ for any choice $\gamma_1, \gamma_2$ of boundary conditions; hence this limit coincides with $-2 \log \cosh \lambda$ plus the free energy in (1.2), see also (2.6). We can state the result as follows.

**Lemma 5.4.** There exists $\varepsilon_1 > 0$ such that, if $|\lambda| \leq \varepsilon_1$ and $t \pm u \in D$ (the same as in Main Theorem), the free energy $f$ defined in (1.2) is real analytic in $\lambda, t, u$, except possibly for the choices
of $\lambda,t,u$ such that $\gamma^{h^*_2} = 0$.

We shall see in §6 below that the specific heat is logarithmically divergent as $\gamma^{h^*_2} \to 0$. So the critical point is really given by the condition $\gamma^{h^*_2} = 0$. We shall explicitly solve the equation for the critical point in next subsection.

5.6 The critical points. In the present subsection we check that, if $t \pm u \in D$, $D$ being a suitable interval centered around $\sqrt{2} - 1$, see Main Theorem, there are precisely two critical points, of the form (1.5). More precisely, keeping in mind that the equation for the critical point is simply $\gamma^{h^*_2} = 0$ (see the end of previous subsection), we prove the following.

Lemma 5.5. Let $|\lambda| \leq \varepsilon_1$, $t \pm u \in D$ and $\pi_{h^*_1}$ be fixed as in Lemma 5.3. Then $\gamma^{h^*_2} = 0$ only if $(\lambda,t,u) = (\lambda, t^*_h(\lambda,u), u)$, where $t^*_h(\lambda,u)$ is given by (1.5).

Proof From the definition of $h^*_2$ given above, see §5.4, it follows that $h^*_2$ satisfies the following equation:

$$
\gamma^{h^*_2 - 1} = c_m \gamma^{F_{m}} \left| \sigma_{h^*_1} - |\mu_{h^*_1}| - \alpha_{\sigma} \gamma^{h^*_1} \pi_{h^*_1} \right|
$$

(5.20)

for some $1 \leq c_m < \gamma$ and $\alpha_{\sigma} = \text{sign} \gamma_1$. Then, the equation $\gamma^{h^*_2} = 0$ can be rewritten as:

$$
|\sigma_{h^*_1} - |\mu_{h^*_1}| - \alpha_{\sigma} \gamma^{h^*_1} \pi_{h^*_1} = 0.
$$

(5.21)

First note that the result of Lemma 5.5 is trivial when $h^*_1 = 1$. If $h^*_1 < 1$, (5.21) cannot be solved when $|\sigma_{1}|^{1-\eta_{\sigma}} > 2|\mu_{1}|^{1-\eta_{\sigma}}$. In fact,

$$
|\sigma_{1}|^{\eta_{\sigma}(h^*_1 - 1) + \nu_{\sigma} h^*_1} - |\mu_{1}|^{\eta_{\sigma}(h^*_1 - 1) + \nu_{\sigma} h^*_1} - \alpha_{\sigma} \gamma^{h^*_1} \pi_{h^*_1} = 0
$$

(5.22)

where $c_1, c'_1$ are constants $= 1 + O(\lambda)$, $\pi_{h^*_1} = O(\lambda)$ and $\gamma^{h^*_1 - 1} = c_{\sigma} |\sigma_{1}|^{1-\eta_{\sigma}}$, with $1 \leq c_{\sigma} < \gamma$. Now, if $|\mu_{1}| > 0$, the r.h.s. of (5.22) equation is strictly positive.

So, let us consider the case $h^*_1 < 1$ and $|\sigma_{1}|^{1-\eta_{\sigma}} \leq 2|\mu_{1}|^{1-\eta_{\sigma}}$ (s.t. $\gamma^{h^*_1} = c_{\sigma} \log_{\gamma} |u|^{1-\eta_{\sigma}}$, with $1 \leq c_{\sigma} \leq \gamma$). In this case (5.21) can be easily solved to find:

$$
|\sigma_{1}| = |\mu_{1}| |u|^{\frac{\nu_{\sigma}}{1-\eta_{\sigma}}} c_{u}^{-1-\eta_{\sigma}} \gamma^{h^*_1 - t} + |u|^{\frac{1-\eta_{\sigma}}{1-\eta_{\sigma}}} c_{u}^{-1-\eta_{\sigma}} \alpha_{\sigma} \gamma^{1-t} \pi_{h^*_1}.
$$

(5.23)

Note that $c_{u}^{\eta_{\sigma} - \eta_{\sigma}} \gamma^{h^*_1 - t} + \nu_{\sigma} h^*_1 = 1 + O(\lambda)$ is just a function of $u$, (it does not depend on $t$), because of our definition of $h^*_1$. Moreover $\pi_{h^*_1}$ is a smooth function of $t$: if we call $\pi_{h^*_1}(t,u)$ resp. $\pi_{h^*_1}(t',u)$ the correction corresponding to the initial data $\sigma_{1}(t,u), \mu_{1}(t,u)$ resp. $\sigma_{1}(t',u), \mu_{1}(t',u)$, we have

$$
|\pi_{h^*_1}(t,u) - \pi_{h^*_1}(t',u)| \leq c|\lambda||u|^{\frac{\nu_{\sigma}}{1-\eta_{\sigma}}}|t - t'|,
$$

(5.24)

where we used (5.17) and the bounds $|\sigma_{1} - \sigma_{1}'| \leq c|t - t'|$ and $|\mu_{1} - \mu_{1}'| \leq c|u||t - t'|$, following from the definitions of $(\sigma_{1}, \mu_{1})$ in terms of $(\sigma, \mu)$ and of $(t,u)$, see §2.

Using the same definitions we also realize that (5.23) can be rewritten as

$$
t = \left[\sqrt{2} - 1 + \frac{\nu(\lambda)}{2} \pm |u|^{1+\eta}\left(1 + \lambda f(t,u)\right)\right]^{\frac{1+\lambda(t^2 - u^2)}{1+\lambda}},
$$

(5.25)

where

$$
1 + \eta \equiv \frac{1 - \eta_{\sigma}}{1 - \eta_{\mu}},
$$

(5.26)
and the crucial property is that $\eta = -b\lambda + O(\lambda^2), b > 0$, see Lemma 4.1 and Appendix A5. We also recall that both $\eta$ and $\nu$ are functions of $\lambda$ and are independent of $t, u$. Moreover $f(t, u)$ is a suitable bounded function s.t. $|f(t, u) - f(t', u)| \leq c|u|^{-(1+n)}|t - t'|$, as it follows from the Lipshitz property of $\pi_{\Lambda_t^0}$ (5.24). The r.h.s. of (5.25) is Lipshitz in $t$ with constant $O(\lambda)$, so that (5.25) can be inverted w.r.t. $t$ by contractions and, for both choices of the sign, we find a unique solution

$$t = t^\pm_c(\lambda, u) = \sqrt{2} - 1 + \nu^*(\lambda) \pm |u|^{1+\eta}(1 + F^\pm(\lambda, u)), \quad (5.27)$$

with $|F^\pm(\lambda, u)| \leq c|\lambda|$, for some $c$.

\subsection*{5.7 Computation of $h^2_2$}

Let us now solve (5.20) in the general case of $h^2_2 \geq 0$. Calling $\varepsilon \equiv h^2 - h^1 - F^h_m / c_m$, we find:

$$\varepsilon = \left| |\sigma_1| \gamma(\eta - \mu_1)(\eta - \mu_1) + F^h_m - \omega \gamma \pi h^1 \right| = \gamma(\eta - \mu_1)(\eta - \mu_1) + F^h_m - \omega \gamma \pi h^1.$$

If $|\sigma_1|^{1/\eta} \leq 2 |\mu_1|^{1/\eta}$, we use $\gamma h^1 = c_\sigma |u|^{1/\eta}$ and, from the second row of (5.27), we find:

$$\varepsilon = C(1 - |\sigma_1|^{1/\eta}) |u|^{1/\eta} - \omega \gamma \pi h^1 = C(1 - \gamma h^1 - \omega \gamma \pi h^1) \leq C,$$

where $C = C'(\lambda, t, u)$ and $C'' = C''(\lambda, t, u)$ are bounded above and below by $O(1)$ constants. In the opposite case ($|\sigma_1|^{1/\eta} > 2 |\mu_1|^{1/\eta}$), we use $\gamma h^1 = c_\sigma |\sigma_1|^{1/\eta}$ and, from the first row of (5.27), we find:

$$\varepsilon = C(1 - |\sigma_1|^{1/\eta}) |u|^{1/\eta} - \omega \gamma \pi h^1 \geq C,$$

where $C = C(\lambda, t, u)$ and $C'' = C''(\lambda, t, u)$ are bounded above and below by $O(1)$ constants. Since in this region of parameters $|t - t^\pm_c| \Delta^{-1}$ is also bounded above and below by $O(1)$ constants, we can in both cases write:

$$\varepsilon = C(\lambda, t, u) \frac{|t - t^+ | \cdot |t - t^- |}{\Delta^2}, \quad C_1 \leq C(\lambda, t, u) \leq C_2,$$

and $C_{j, \varepsilon}, j = 1, 2$, are suitable positive $O(1)$ constants.

\section{The specific heat}

Consider the specific heat defined in (1.2). The correlation function $< H^x \lambda Y^t >_{\Lambda_m, t}$ can be conveniently written as

$$< H^x \lambda Y^t >_{\Lambda, t} = \frac{\partial}{\partial \phi_x \partial \phi_y} \log \Xi(\phi) \bigg|_{\phi = 0} = \xi \sum_{\sigma^{(1)}, \sigma^{(2)}} \sum_{\gamma_1, \gamma_2} e^{-\sum_{\varepsilon \in \Lambda_1 (1 + \phi)} H^x_{\mu, \nu}} \Xi(\phi), \quad (6.1)$$

where $\phi_x$ is a real commuting auxiliary field (with periodic boundary conditions).

Repeating the construction of §2, we see that $\Xi(\phi)$ admit a Grassmanian representation similar to the one of $\Xi(\phi)$, and in particular, if $x \neq y$:

$$\frac{\partial^2}{\partial \phi_x \partial \phi_y} \log \Xi(\phi) \bigg|_{\phi = 0} = \frac{\partial^2}{\partial \phi_y \partial \phi_y} \log \sum_{\gamma_1, \gamma_2} (-1)^{\delta_{\gamma_1, \gamma_2}} \Xi^{\gamma_1, \gamma_2}(\phi) \bigg|_{\phi = 0}$$

$$\Xi^{\gamma_1, \gamma_2}(\phi) = \int \prod_{x \in \Lambda} dH^{x(j)} dH^{x*(j)} dV^{x(j)} dV^{x*(j)} e^{-\sum_{\varepsilon \in \Lambda} (1 + \phi)} \Xi^{\gamma_1, \gamma_2}(\phi)$$

$$= \int \prod_{x \in \Lambda_m} dH^{x(j)} dH^{x*(j)} dV^{x(j)} dV^{x*(j)} e^{-\sum_{\varepsilon \in \Lambda} (1 + \phi)} \Xi^{\gamma_1, \gamma_2}(\phi)$$

$$= \prod_{x \in \Lambda_m} dH^{x(j)} dH^{x*(j)} dV^{x(j)} dV^{x*(j)} e^{-\sum_{\varepsilon \in \Lambda} (1 + \phi)} \Xi^{\gamma_1, \gamma_2}(\phi)$$
where $\delta_x, S^{(j)}(t^{(j)})$ and $V_\lambda$ where defined in §2 (see (2.2) and previous lines, and (2.9)), the apex $\gamma_1, \gamma_2$ attached to $z_{AT}$ refers to the boundary conditions assigned to the Grassmanian fields, as in §2 and finally $B(\phi)$ is defined as:

\[
B(\phi) = \sum_{x \in A} \phi_x \left\{ a^{(1)}(H^{(1)} x H^{(1)} x) + \delta^{(1)} x V^{(1)} x + \delta^{(2)} x H^{(2)} x + \delta^{(2)} x V^{(2)} x \right\} +
\lambda \bar{\phi} (H^{(1)} x H^{(1)} x + \delta^{(1)} x V^{(1)} x + \delta^{(1)} x V^{(2)} x + \delta^{(2)} x V^{(2)} x) \}
\]

where $a^{(1)}, a^{(2)}$ and $\bar{\phi}$ are $O(1)$ constants, with $a^{(1)} - a^{(2)} = O(u)$. Using (6.2) and (6.3) we can rewrite:

\[
< H^{AT}_x H^{AT}_x >_{\Lambda,T} = \frac{1}{4} (\cosh J)^{2M^2} \sum_{\gamma_1, \gamma_2} \sum_{\delta_1, \delta_2} \bar{\Xi}_{AT}^{\gamma_1, \gamma_2} \Xi_{AT}^{\gamma_1, \gamma_2} < A_x A_y >_{\Lambda_1, \Lambda_2}^{\gamma_1, \gamma_2} ,
\]

where $< \cdot >_{\Lambda_1, \Lambda_2}$ is the average w.r.t. the boundary conditions $\gamma_1, \gamma_2$. Proceeding as in Appendix G of [M1] one can show that, if $\gamma^{h_2} > 0$, $< A_x A_y >_{\Lambda_1, \Lambda_2}$ is exponentially insensitive to boundary conditions and $\sum_{\gamma_1, \gamma_2} \sum_{\delta_1, \delta_2} \bar{\Xi}_{AT}^{\gamma_1, \gamma_2} \Xi_{AT}^{\gamma_1, \gamma_2}$ is an $O(1)$ constant. Then from now on we will study only $\bar{\Xi}_{AT}^{\gamma_1, \gamma_2}(\phi) = \Xi_{AT}^{\gamma_1, \gamma_2}(\phi)$ and $< A_x A_y >_{\Lambda_1, \Lambda_2}^{\gamma_1, \gamma_2}$.

As in §2 we integrate out the $\chi$ fields and, proceeding as in Appendix A, we find:

\[
\Xi_{AT}(\phi) = \int P_{z_1, \sigma_1, \mu_1, C_1}(d\psi)e^{\psi^{(1)} + B^{(1)}},
\]

where

\[
B^{(1)}(\psi, \phi) = \sum_{m,n=1}^{\infty} B^{(1)}_{m, 2n: \sigma_1, \omega_1}(x_1, \ldots, x_m; y_1, \ldots, y_{2n}) \left[ \prod_{i=1}^{m} \phi_{x_i} \right] \left[ \prod_{i=1}^{2n} \partial_{\psi_i}^{\sigma_1, \omega_1} \right] .
\]

We proceed as for the partition function, namely as described in §3 above. We introduce the scale decomposition described in §3 and we perform iteratively the integration of the single scale fields, starting from the field of scale 1. After the integration of the fields $\psi^{(1)}, \ldots, \psi^{(h + 1)}$, $h^*_1 < h \leq 0$, we are left with

\[
\Xi_{AT}(\phi) = e^{-M^2 E_n + S^{(h + 1)}(\phi)} \int P_{z_n, \sigma_n, \mu_n, C_n}(d\psi) e^{-\psi^{(h)}(\sqrt{\Xi_{AT}}^{\gamma_1, \gamma_2}) + B^{(h)}(\psi^{(h)}, \phi)} ,
\]

where $P_{z_n, \sigma_n, \mu_n, C_n}(d\psi)$ and $\psi^{(h)}$ are the same as in §3, $S^{(h + 1)}(\phi)$ denotes the sum of the contributions dependent on $\phi$ but independent of $\psi$, and finally $B^{(h)}(\psi^{(h)}, \phi)$ denotes the sum over all terms containing at least one $\phi$ field and two $\psi$ fields. $S^{(h + 1)}$ and $B^{(h)}$ can be represented as

\[
S^{(h + 1)}(\phi) = \sum_{m=1}^{\infty} \sum_{x_1, \ldots, x_m} S_m^{(h + 1)}(x_1, \ldots, x_m) \prod_{i=1}^{m} \phi_{x_i} ,
\]

\[
B^{(h)}(\psi^{(h)}, \phi) = \sum_{m,n=1}^{\infty} \sum_{x_1, \ldots, x_m} B_m^{(h)}_{2n: \sigma_1, \omega_1}(x_1, \ldots, x_m; y_1, \ldots, y_{2n}) \left[ \prod_{i=1}^{m} \phi_{x_i} \right] \left[ \prod_{i=1}^{2n} \partial_{\psi_i}^{\sigma_1, \omega_1} \right] .
\]

Since the field $\phi$ is equivalent, as regarding dimensional bounds, to two $\psi$ fields (see Theorem 6.1 below for a more precise statement), the only terms in the expansion for $B^{(h)}$ which are not irrelevant are those with $m = n = 1$, $\sigma_1 = \sigma_2 = 0$ and they are marginal. Hence we extend the definition of the localization operator $L$, so that its action on $B^{(h)}(\psi^{(h)}, \phi)$ is defined by its action
on the kernels $\hat{B}_{m,2n;\omega}(q_1, \ldots, q_m; k_1, \ldots, k_{2n})$:

1) if $m = n = 1$ and $\alpha_1 + \alpha_2 = \omega_1 + \omega_2 = 0$, then $L\hat{B}_{1,2;\omega}(q_1; k_1, k_2) \equiv P_0\hat{B}_{1,2;\omega}(k_+; k_+, k_+)$, where $P_0$ is defined as in (3.6);

2) in all other cases $L\hat{B}_{m,2n;\omega} = 0$.

Using the symmetry considerations of Appendix B together with the remark that $\phi_x$ is invariant under Complex conjugation, Hole–particle and (1) $\leftrightarrow$ (2), while under Parity $\phi_x \rightarrow \phi_{-x}$ and under Rotation $\phi(x, x_0) \rightarrow \phi(-x_0, -x)$, we easily realize that $LB^{(h)}$ has necessarily the following form:

$$LB^{(h)}(\psi^{(\leq h)}, \phi) = \frac{Z_h}{Z_1} \sum_{x, \omega} \frac{(-i\omega)}{2} \phi_x \psi_x^{(\leq h)} + \psi_{-x}^{(\leq h)} - \psi_{-x}^{(\leq h)}$$

(6.9)

where $Z_h$ is real and $Z_1 = a(1)\vert_{\sigma = \mu = 0} = a(2)\vert_{\sigma = \mu = 0}$.

Note that apriori a term $\sum_{x, \omega} \phi_x \psi_x^{(\leq h)} \psi_{-x}^{(\leq h)}$ is allowed by symmetry but, using (1) $\leftrightarrow$ (2) symmetry, one sees that its kernel is proportional to $\mu_k$, $k \geq h$. So, with our definition of localization, such term contributes to $RB^{(h)}$.

Now that the action of $L$ on $B$ is defined, we can describe the single scale integration, for $h > h_1^r$. The integral in the r.h.s. of (6.7) can be rewritten as:

$$e^{-M^2 t_h} \int P_{h-1, \sigma_{h-1}, \mu_{h-1}, C_{h-1}} (d\psi_{\leq h-1}),$$

$$\cdot \int P_{h-1, \sigma_{h-1}, \mu_{h-1}, J_h-1} (d\phi^{(h)}) e^{-\widehat{\psi}^{(h)}(\sqrt{Z_{h-1}} \psi^{(\leq h)}) + \widehat{\phi}^{(h)}(\sqrt{Z_{h-1}} \psi^{(\leq h)}, \phi)}$$

(6.10)

where $\widehat{\psi}^{(h)}$ was defined in (3.12) and

$$\widehat{B}^{(h)}(\sqrt{Z_{h-1}} \psi^{(\leq h)}, \phi) \overset{\text{def}}{=} B^{(h)}(\sqrt{Z_{h-1}} \psi^{(\leq h)}, \phi)$$

(6.11)

Finally we define

$$e^{-\overline{E}_h M^2 + \overline{S}^{(h)}(\phi)} - \overline{V}^{(h-1)}(\sqrt{Z_{h-1}} \psi^{(\leq h-1)}) + \overline{B}^{(h)}(\sqrt{Z_{h-1}} \psi^{(\leq h-1)}, \phi) \overset{\text{def}}{=}$$

$$\int P_{h-1, \sigma_{h-1}, \mu_{h-1}, J_h-1} (d\phi^{(h)}) e^{-\widehat{\psi}^{(h)}(\sqrt{Z_{h-1}} \psi^{(\leq h)}) + \widehat{\phi}^{(h)}(\sqrt{Z_{h-1}} \psi^{(\leq h)}, \phi)}$$

(6.12)

and

$$E^{(h)}(\phi) \overset{\text{def}}{=} \overline{E}_h + t_h + \overline{E}_h \quad \text{and} \quad S^{(h)}(\phi) \overset{\text{def}}{=} S^{(h+1)}(\phi) + \overline{S}^{(h)}(\phi).$$

(6.13)

With the definitions above, it is easy to verify that $\overline{Z}_{h-1}$ satisfies the equation $\overline{Z}_{h-1} = \overline{Z}_h (1 + \overline{\alpha}_h)$, where $\overline{\alpha}_h = \overline{\alpha} \mid \lambda_h + O(\lambda^2)$, for some $\overline{\alpha} \neq 0$. Then, for some $c > 0$, $\overline{Z}_1 e^{-c \mid \lambda_h} \leq \overline{Z}_h \leq \overline{Z}_1 e^{c \mid \lambda_h}$. The analogous of Theorem 3.1 for the kernels of $B^{(h)}$ holds:

**Theorem 6.1.** Suppose that the hypothesis of Lemma 5.1 are satisfied. Then, for $h^* \leq h \leq 1$ and a suitable constant $C$, the kernels of $B^{(h)}$ satisfy

$$\int d\mathbf{x}_1 \cdots d\mathbf{x}_{2n} |\overline{B}^{(h)}_{2n;\omega}(\mathbf{x}_1, \ldots, \mathbf{x}_m; \mathbf{y}_1, \ldots, \mathbf{y}_{2n})| \leq M^2 \gamma^{-h(D_k(n)+m)} (C \mid \lambda \mid)^{\max(1, n-1)},$$

(6.14)

where $D_k(n) = -2 + n + k$ and $k = \sum_{i=1}^{2n} \sigma_i$.

Note that, consistently with our definition of localization, the dimension of $B^{(h)}_{2,1;\omega}$ is $D_0(1) + 1 = 0$.  

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Again, proceeding as in §4, we can study the flow of \( \mathcal{Z}_h \) up to \( h = -\infty \) and prove that \( \mathcal{Z}_h = \mathcal{Z}_1 \gamma^h \), where \( \gamma \) is a non-trivial analytic function of \( \lambda \) (its linear part is non vanishing) and \( F^h \) is a suitable \( O(\lambda) \) function (independent of \( \sigma_1, \mu_1 \)). We recall that \( \mathcal{Z}_1 = O(1) \).

We proceed as above up to the scale \( h_1^* \). Once that the scale \( h_1^* \) is reached we pass to the \( \psi^{(1)}, \psi^{(2)} \) variables, we integrate out (say) the \( \eta \) variables, with \( \eta \) trivial, i.e. over \( \sum \eta \psi^{(2)}(\pm h_1^*) \). Moreover, if \( h_2^* \leq h \leq h_1^* \), \( \mathcal{Z}_h = \mathcal{Z}_1 \gamma^h \), with \( F^h = O(\lambda) \).

The conclusion is that the correlation function \( \langle H^T_x H^T_y \rangle \mathcal{Z}_{\Lambda M} \) is given by a convergent power series in \( \lambda \), uniformly in \( \Lambda M \). Then, the leading behaviour of the specific heat is given by the sum of the \( \gamma^{(1)} \) fields and we get

\[
\int \frac{d\psi^{(2)}(\pm h_1^*)}{Z_{h_1^*}^2} e^{-\mathcal{B}^{h_1^*}(\sqrt{Z_{h_1^*}^2} \psi^{(2)}(\pm h_1^*) + \mathcal{B}^{h_1^*}(\sqrt{Z_{h_1^*}^2} \psi^{(2)}(\pm h_1^*))},
\]

with \( \mathcal{B}^{h_1^*}(\sqrt{Z_{h_1^*}^2} \psi^{(2)}(\pm h_1^*)) = \mathcal{Z}_{h_1^*} \sum x_i \phi_{\psi^1, x} \psi^{(2)}(\pm h_1^*) \).

The scales \( h_2^* \leq h \leq h_1^* \) are integrated as in §5 and one finds that the flow of \( \mathcal{Z}_h \) in this regime is trivial, i.e. if \( h_2^* \leq h \leq h_1^* \), \( \mathcal{Z}_h = \mathcal{Z}_1 \gamma^h \), with \( F^h = O(\lambda) \).

The result is that the correlation function \( \langle H^T_x H^T_y \rangle \mathcal{Z}_{\Lambda M} > \Lambda M, T \) is given by a convergent power series in \( \lambda \), uniformly in \( \Lambda M \). Then, the leading behaviour of the specific heat is given by the sum over \( x \) and \( y \) of the lowest order contributions to \( \langle H^T_x H^T_y \rangle \mathcal{Z}_{\Lambda M} > \Lambda M, T \), namely by the diagrams in Fig 3. Absolute convergence of the power series of \( \langle H^T_x H^T_y \rangle \mathcal{Z}_{\Lambda M} > \Lambda M, T \) implies that the rest is a small correction.

The conclusion is that \( C_v \), for \( \lambda \) small and \( |t - \sqrt{2}| + |u| \leq \left( \sqrt{2} - 1 \right) / 4 \), is given by:

\[
C_v = \frac{1}{|A|} \sum_{x, y \in \Lambda M} \sum_{\omega_1, \omega_2 = \pm 1} \sum_{h, h' = h_2^*} \frac{(Z_{h_2^*}^2)^2}{Z_{h-1}^2 Z_{h'-1}^2} \left[ G^{(h)}_{(+, \omega_2, +, \omega_1)}(x - y) G^{(h')}_{(-, -, \omega_2, -, -, \omega_1)}(y - x) + G^{(h)}_{(+, \omega_1, -)}(x - y) G^{(h')}_{(-, -, \omega_2, +, \omega_1)}(x - y) \right] + \frac{1}{|A|} \sum_{x, y \in \Lambda M} \sum_{h, h'} \frac{Z_{h_2^*}^2 \Omega^{(h)}_{\Lambda M}(x - y) \Omega^{(h')}_{\Lambda M}(x - y)}{Z_{h_2^*}^2} \Omega^{(h)}_{\Lambda M}(x - y),
\]

where \( h \vee h' = \max\{h, h'\} \) and \( G^{(h)}_{(\omega_1, \omega_2)}(x) \) must be interpreted as

\[
G^{(h)}_{(\omega_1, \omega_2)}(x) = \begin{cases} g^{(h)}_{(\omega_1, \omega_2)}(x) & \text{if } h > h_1^*, \\ g^{(1, \omega_1, \omega_2)}(x) + g^{(2, \omega_1, \omega_2)}(x) & \text{if } h = h_1^*, \\ g^{(2, \omega_1, \omega_2)}(x) & \text{if } h_2^* < h < h_1^*, \\ g^{(2, \omega_1, \omega_2)}(x) & \text{if } h = h_2^*. \end{cases}
\]

Moreover, if \( N, n_0, n_1 \geq 0 \) and \( n = n_0 + n_1 \), \( |\partial_{x_1}^n \partial_{x_0}^n \Omega^{(h)}_{\Lambda M}(x) | \leq C_{N, n} \left| \langle H^T_x H^T_y \rangle \right| \left( n + m \right) ! \sqrt{n_{\Lambda M}} \). Now, calling \( \eta_c \) the exponent associated to \( \mathcal{Z}_h / Z_h \), from (6.16) we find:

\[
C_v = -C_1 \gamma^{2\eta_c h_1^*} \log \gamma^{h_1^* - h_2^*} (1 + \Omega^{(1)}_{h_1^*}(h_2^*) + C_2 \frac{1 - \gamma^{2\eta_c (h_1^* - 1)}}{2\eta_c} (1 + \Omega^{(2)}_{h_1^*}(\lambda)),
\]
where \(|\Omega^{(1)}_{h_1,h_2}\lambda|, |\Omega^{(2)}_{h_1}\lambda| \leq c|\lambda|\), for some \(c\). Note that, defining \(\Delta\) as in (1.6), \(\gamma^{(1-n_\sigma)}h^i_1\Delta^{-1}\) is bounded above and below by \(O(1)\) constants. Then, using (5.30), (1.6) follows.

Appendix A1. Proof of (2.1)

We start from eq. (V.2.12) in [MW], expressing the partition function of the Ising model with periodic boundary condition on a lattice with an even number of sites as a combination of the Pfaffians of four matrices with different boundary conditions, defined by (V.2.10) and (V.2.11) in [MW]. In the general case (i.e. \(M^2\) not necessarily even), the (V.2.12) of [MW] becomes:

\[
Z_I = \sum_{\sigma} e^{-\beta J H_I(\sigma)} = (-1)^{M^2} \frac{1}{2} (2 \cosh \beta J)^{M^2} \left( -\text{Pf} \mathcal{A}_1 + \text{Pf} \mathcal{A}_2 + \text{Pf} \mathcal{A}_3 + \text{Pf} \mathcal{A}_4 \right),
\]

(1A1)

where \(\mathcal{A}_i\) are matrices with elements \((\mathcal{A}_i)_{x,j:y,k}\), with \(x,y \in \Lambda_M, j,k = 1, \ldots, 6\), given by:

\[
(\mathcal{A}_i)_{x,x} = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0 & -1 & 0
\end{pmatrix}
\]

(1A2)

and \((\mathcal{A}_i)_{x,x+\hat{e}_i}, \mathcal{A}_i)_{x,x+\hat{e}_j}\) \(\gamma\), with \(x,y \in \Lambda_M, j,k = 1, \ldots, 6\), given by:

\[
(\mathcal{A}_i)_{x,x+\hat{e}_i} = t\delta_{i,1}\delta_{j,2},
(\mathcal{A}_i)_{x,x+\hat{e}_j} = t\delta_{i,2}\delta_{j,1},
(\mathcal{A}_i)_{x,x+\hat{e}_i} = -\mathcal{A}_i, \mathcal{A}_i)_{x,x+\hat{e}_0} = -\mathcal{A}_i,
\]

(1A3)

and \((\mathcal{A}_i)_{x,x+\hat{e}_i}\) \(\gamma\), with \(x,y \in \Lambda_M, j,k = 1, \ldots, 6\), given by:

\[
(\mathcal{A}_i)_{x,x+\hat{e}_i} = t\delta_{i,1}\delta_{j,2},
(\mathcal{A}_i)_{x,x+\hat{e}_j} = t\delta_{i,2}\delta_{j,1},
(\mathcal{A}_i)_{x,x+\hat{e}_i} = -\mathcal{A}_i, \mathcal{A}_i)_{x,x+\hat{e}_0} = -\mathcal{A}_i,
\]

(1A4)

where \((\mathcal{A}_i)_{x,y}\) are identical zero.

Given a \((2n) \times (2n)\) antisymmetric matrix \(A\), it is well-known that \(\text{Pf} A = (-1)^n \int d\psi_1 \cdots d\psi_{2n}, \cdot \exp \{ \frac{1}{2} \sum_{i,j} \psi_i A_{ij} \psi_j \}\), where \(\psi_1, \ldots, \psi_{2n}\) are Grassmanian variables. Then, we can rewrite (1A1) as:

\[
\frac{1}{2} (2 \cosh \beta J)^{M^2} \sum_{\gamma} (-1)^{\gamma} \int \prod_{x \in \Lambda_M} d\mathcal{T}_x d\mathcal{H}_x d\mathcal{V}_x d\mathcal{\gamma}_x e^{S^{\gamma}(t;H,V,T)},
\]

(1A5)

where: \(\gamma = (\varepsilon, \varepsilon'); \varepsilon, \varepsilon' = \pm 1\); \(\delta_\gamma\) is defined after (2.1); \(\mathcal{T}_x, \mathcal{H}_x, \mathcal{V}_x, \mathcal{\gamma}_x\) are Grassmanian variables with \(\varepsilon\)-periodic resp. \(\varepsilon'\)-periodic boundary conditions in vertical resp. horizontal direction, see (2.3) and following lines. Furthermore:

\[
S^{\gamma}(t;H,V,T) = t \sum_x \left[ \mathcal{H}_x \mathcal{V}_x + \mathcal{V}_x \mathcal{\gamma}_x \right] + \sum_x \left[ \mathcal{T}_x \mathcal{H}_x + \mathcal{H}_x \mathcal{T}_x + \mathcal{V}_x \mathcal{H}_x + \mathcal{H}_x \mathcal{V}_x \right].
\]

(1A6)

The \(T\)-fields appear only in the diagonal elements and they can be easily integrated out:

\[
\prod_{x \in \Lambda_M} \int d\mathcal{H}_x d\mathcal{T}_x \exp \left\{ \mathcal{T}_x \mathcal{T}_x + \mathcal{H}_x \mathcal{H}_x + \mathcal{V}_x \mathcal{V}_x \right\} =
\]

(1A7)

\[
\Pi_{x \in \Lambda_M} \left\{ (-1\mathcal{T}_x \mathcal{H}_x - \mathcal{V}_x \mathcal{H}_x - \mathcal{H}_x \mathcal{H}_x - \mathcal{V}_x \mathcal{H}_x) \right\} = (-1)^M \exp \sum_{x \in \Lambda_M} \left[ \mathcal{H}_x \mathcal{H}_x + \mathcal{V}_x \mathcal{V}_x + \mathcal{V}_x \mathcal{H}_x + \mathcal{H}_x \mathcal{V}_x \right],
\]
where in the last identity we used that \( \left[ \Pi_x H_x^2 + \nabla_x V_x + V_x \Pi_x + H_x \nabla_x \right]^2 = 0 \). Substituting (A1.6) into (A1.4) we find (2.1).

### Appendix A2. Integration of the heavy fermions. Symmetry properties

#### A2.1 Integration of the \( \chi \) fields

Calling \( \mathbf{V}(\psi, \chi) = Q(\psi, \chi) - \nu F_\sigma(\psi) + V(\psi, \chi) \), we obtain

\[
-\vec{E}_1 M^2 - Q^{(1)}(\psi) - V^{(1)}(\psi) = \log \int P(d\chi) e^{\mathbf{V}(\psi, \chi)} = \sum_{n=0}^\infty \frac{(-1)^n}{n!} E_x^T(\mathbf{V}(\psi, \chi); n), \tag{A2.1}
\]

where \( \vec{E}_1 \) is a constant and \( V^{(1)} \) is at least quadratic in \( \psi \) and vanishing when \( \lambda = \nu = 0 \). \( Q^{(1)} \) is the rest (quadratic in \( \psi \)). Given \( s \) set of labels \( P_{v_i}, i = 1, \ldots, s \) and \( \tilde{\chi}(P_{v_i}) \) the truncated expectation \( E_x^T(\tilde{\chi}(P_{v_i}), \ldots, \tilde{\chi}(P_{v_i})) \) can be written as

\[
E_x^T(\tilde{\chi}(P_{v_i}), \ldots, \tilde{\chi}(P_{v_i})) = \sum T \alpha_T \prod_{i \in T} g_x(f^T_i, f^T_i) \int dP_T(t) Pf G^T(t) \tag{A2.2}
\]

where \( T \) is a set of lines forming an anchored tree between the cluster of points \( P_{v_i}, \ldots, P_{v_s} \) i.e. \( T \) is a set of lines which becomes a tree if one identifies all the points in the same clusters; \( t = \{ t_{i,i'} \in [0,1], 1 \leq i, i' \leq s \} \), \( dP_T(t) \) is a probability measure with support on a set of \( t \) such that \( t_{i,i'} = u_i \cdot u_{i'} \) for some family of vectors \( u_i \in \mathbb{R}^s \) of unit norm; \( \alpha_T \) is a sign (irrelevant for the subsequent bounds); \( f^T_i, f^T_i \) are the field labels associated to the points connected by \( t \); if \( g(f) = (\alpha(f), \omega(f)) \), the propagator \( g_x(f, f') \) is equal to

\[
g_x(f, f') = g_x^\alpha(f) g_x^{\alpha(f')}(\chi(f) - \chi(f')) \tag{A2.3}
\]

if \( 2n = \sum_{i=1}^s |P_{v_i}| \), then \( G^T(t) \) is a \((2n - 2s + 2) \times (2n - 2s + 2) \) antisymmetric matrix, whose elements are given by \( G^T_{f, f'} = t_{i(f), i(f')} g_x(f, f') \), where: \( f, f' \notin F_T \) and \( F_T \) is the set of points \( \ell \) such that \( f \in P(f) \); finally \( Pf G^T \) is the Pfaffian of \( G^T \). If \( s = 1 \) the sum over \( T \) is empty, but we can still use the above equation by interpreting the r.h.s. as \( 1 \) if \( P_{v_i} \) is empty, and det \( G^T \) otherwise.

#### Sketch of the proof of (A2.2)

Equation (A2.2) is a trivial generalization of the well–known formula expressing truncated fermionic expectations in terms of sums of determinants [Le]. The only difference here is that the propagators \( \chi_{\omega_1, \omega_2, \omega_3}^{\alpha_1, \alpha_2, \alpha_3} \) are not vanishing, so that Pfaffians appear instead of determinants. The proof can be done along the same lines of Appendix A3 of [GM]. The only difference here is that the identity known as the Berezin integral, see (A3.15) of [GM], that is the starting point to get to (A2.2), must be replaced by the (more general) identity:

\[
E_x \left( \prod_{j=1}^S \tilde{\chi}(P_{v_j}) \right) = Pf G = (-1)^n \int D\chi \exp \left[ \frac{1}{2} (\chi, G\chi) \right], \tag{A2.4}
\]

where: the expectation \( E_x \) is w.r.t. \( P(d\chi) \); if \( 2m = \sum_{j=1}^s |P_{v_j}| \), \( G \) is the \( 2m \times 2m \) antisymmetric matrix with entries \( G_{f, f'} = g_{x(f), x(f')}^{\alpha(f), \alpha(f')} (\chi(f) - \chi(f')) \); and

\[
D\chi = \prod_{j=1}^n \prod_{f \in P_{v_j}} d\chi_{x(f), \omega(f)}^{\alpha(f)} (\chi, G\chi) = \sum_{f, f' \in \cup_i P_{v_i}} \chi_{x(f), \omega(f)}^{\alpha(f)} G_{f, f'} G_{x(f), \omega(f)}^{\alpha(f)} \tag{A2.5}
\]

Starting from (A2.4), the proof in Appendix A3 of [GM] can be repeated step by step in the present case, to find finally the analogue of (A3.55) of [GM]. Then, using again that \( \int D\chi \exp(\chi, G\chi)/2 \)
is, unless for a sign, the Pfaffian of \( G \), we find (A2.2).

We now use the well–known property \( \text{Pf} G^T = \sqrt{\det G} \) and we can bound \( \det G^T \) by Gram–Hadamard (GH) inequality. Let \( \mathcal{H}^d = \mathbb{R}^s \otimes \mathcal{H}_0 \), where \( \mathcal{H}_0 \) is the Hilbert space of complex four dimensional vectors \( F(k) = (F_1(k), \ldots, F_4(k)) \), \( F_i(k) \) being a function on the set \( \mathcal{D}_{-,-} \), with scalar product \( \langle F,G \rangle = \sum_{i=1}^4 1/M^2 \sum_k F_i^*(k)G_i(k) \). We can write the elements of \( G^T \) as inner products of vectors of \( \mathcal{H} \):

\[
G_{f,f'} = t_{i(f),i(f')}g_\chi(f,f') = <u_{i(f)} \otimes A_f, u_{i(f')} \otimes B_{f'}>,
\]

where \( u_i \in \mathbb{R}^s \), \( i = 1, \ldots, s \), are vectors such that \( t_{i,i'} = u_i \cdot u_{i'} \), and, if \( g_\chi^{x,y}(k) \) is the Fourier transform of \( g_\chi^{x,y}(x − y), A_f(k) \) and \( B_{f'}(k) \) are given by

\[
A_f(k) = e^{-ikx(f)}
\]

\[
B_{f'}(k) = e^{-ikx(f')}
\]

(42.7)

With these definitions and remembering (2.17), it is now clear that \( |\text{Pf} G^T| \leq C^{n-s+1} \), for some constant \( C \). Then, applying (A2.2) and the previous bound we find the second of (2.21).

We now turn to the construction of \( P_{Z_1,\sigma_1,\mu_1}C_1 \), in order to prove (2.19).

We define \( e^{-t_1M^2}P_{\sigma_1,\sigma_1',\mu_1}C_1(\psi) \) \( \stackrel{df}{=} \) \( P_\sigma(\psi)e^{-Q^{(1)}}(\psi) \), where \( t_1 \) is a normalization constant. In order to write \( P_{Z_1,\sigma_1,\mu_1}C_1(\psi) \) as an exponential of a quadratic form, it is sufficient to calculate the correlations

\[
<\psi^{\alpha_1}_{\omega_1,k}\psi^{\alpha_2}_{\omega_2,-\alpha_1\omega_2 k}> \stackrel{df}{=} \int P_{\sigma_1,\sigma_1',\mu_1}(\psi)\psi^{\alpha_1}_{\omega_1,k}\psi^{\alpha_2}_{\omega_2,-\alpha_1\omega_2 k} =
\]

\[
e^{-t_1M^2} \int P_\sigma(\psi)P(\chi)e^{Q(\chi,\psi)}\psi^{\alpha_1}_{\omega_1,k}\psi^{\alpha_2}_{\omega_2,-\alpha_1\omega_2 k}.
\]

(42.8)

It is easy to realize that the measure \( \sim P_\sigma(\psi)P(\chi)e^{Q(\chi,\psi)} \) factorizes into the product of two measures generated by the fields \( \psi_j^{(1)}, j = 1, 2 \), defined by \( \psi_{\omega,x}^{(a)} = (\psi_{\omega,x}^{(1)} + i(-1)^a\psi_{\omega,x}^{(2)})/\sqrt{2} \). In fact, using this change of variables, one finds that

\[
P_\sigma(\psi)P(\chi)e^{Q(\chi,\psi)} = \prod_{j=1,2} P_j(\psi_{(j)}, d\chi_{(j)}) = \prod_{j=1,2} \frac{1}{N^{(j)}} \exp\left\{-\frac{t^{(j)}_{\Lambda}}{4M^2} \sum_k c_{k}^{(j)} T C_{k}^{(j)} \xi_{k}^{(j)} \right\},
\]

(42.9)

for two suitable matrices \( C_{k}^{(j)} \), whose determinants \( B_{(j)}(k) \) \( \stackrel{df}{=} \) \( \det c_{k}^{(j)} \) are equal to

\[
B_{(j)}(k) = \frac{16}{(t^{(j)}_{\Lambda})^4} \{2t^{(j)}_{\Lambda} [1 - (t^{(j)}_{\Lambda})^2] (2 - \cos k - \cos k_0) + (t^{(j)}_{\Lambda} - t_\psi)^2 (t^{(j)}_{\Lambda} - t_\chi)^2 \}
\]

(42.10)

From the explicit expression of \( C_{k}^{(j)} \) one finds

\[
<\psi_{-k}^{(j)}\psi_{k}^{(j)}> > 1 = \frac{4M^2 c_{-1}^{(j)}(k)}{t^{(j)}_{\Lambda} B^{(j)}(k)}, \quad <\psi_{-k}^{(j)}\psi_{k}^{(j)}> = \frac{4M^2 c_{-1}^{(j)}(k)}{t^{(j)}_{\Lambda} B^{(j)}(k)},
\]

(42.11)

\[
<\psi_{-k}^{(j)}\psi_{k}^{(j)}> > 1 = \frac{4M^2 c_{1}^{(j)}(k)}{t^{(j)}_{\Lambda} B^{(j)}(k)}.
\]
where, if $\omega = \pm 1$, recalling that $t_\psi = \sqrt{2} - 1 + \nu/2$ and defining $t_\chi = -\sqrt{2} - 1$,

\[
c_{(j,\omega)}(k) \overset{\text{def}}{=} \frac{4}{(i\chi_j)^2} \left\{ 2t_{\chi}^j t_{\psi}^{j\omega}(1 - i \sin k \cos k_0 + \omega \sin k_0 \cos k) + [(t_{\chi}^j)^2 + t_{\psi}^2]((i \sin k - \omega \sin k_0) - \omega)
\right. \\
- \left. (t_{\chi}^j (t_{\psi} + t_{\chi}) + 2t_{\psi}^j t_{\chi}^2) \right\} .
\]  

(A2.12)

It is clear that, for any monomial $F(\psi^{(j)})$, \( P(d\psi^{(j)}, d\chi^{(j)}) F(\psi^{(j)}) = \int P(\psi^{(j)}) F(\psi^{(j)}) \), with

\[
P^{(j)}(d\psi^{(j)}) = \frac{1}{N^j_1} \prod_{k,\omega} d\psi^{(j)}_k \psi^{(j)}_k \exp \left\{ - \frac{1}{4M^2} \frac{B^{(j)}(j)}{\det c_{(j,\omega)}^j} \left( \begin{array}{cc}
\psi^{(j)}_{1,1} & -c_{1,1}^{(j,\omega)} \\
-c_{1,1}^{(j,\omega)} & \psi^{(j)}_{1,1}
\end{array} \right) \right\},
\]

(A2.13)

where \( \det c_{(j,\omega)}^j = c_{1,1}^{(j,\omega)}(j) c_{1,1}^{(j,\omega)} - c_{1,1}^{(j,\omega)} c_{1,1}^{(j,\omega)} \). If we now use the identity \( t_{\chi}^j = t_\psi (2 + (-1)^j \mu)/2 - \sigma \) and rewrite the measure \( P^{(1)}(d\psi^{(1)}) P^{(2)}(d\psi^{(2)}) \) in terms of $\psi^{(j)}_{\omega, k}$ we find:

\[
P^{(1)}(d\psi^{(1)}) P^{(2)}(d\psi^{(2)}) = \frac{1}{N^{(1)}_1} \prod_{k,\omega} d\psi^{(1)}_k d\psi^{(2)}_k \exp \left\{ - \frac{Z_1 C_1(k)}{4M^2} \psi^{(1)}_k A^{(1)}_\psi \psi^{(2)}_k \right\} = P_{Z_1, \sigma_1, \mu_1, c_1}(d\psi),
\]

(A2.14)

with $C_1(k)$, $Z_1$, $\sigma_1$ and $\mu_1$ defined as after (2.18), and $A^{(1)}_\psi(k)$ as in (2.19), with

\[
M^{(1)}(k) = \frac{2}{2 - \sigma} \left( \begin{array}{cc}
-c_{1,1}^{(1,1)} & c_{1,1}^{(1,1)} \\
-c_{1,1}^{(1,1)} & c_{1,1}^{(1,1)}
\end{array} \right), \quad N^{(1)}(k) = \frac{2}{2 - \sigma} \left( \begin{array}{cc}
-c_{1,1}^{(1,1)} & c_{1,1}^{(1,1)} \\
-c_{1,1}^{(1,1)} & c_{1,1}^{(1,1)}
\end{array} \right),
\]

(A2.15)

where $c_{(1,\omega_1,\omega_2)}^{(1)}(k) \overset{\text{def}}{=} [(1 - \mu/2)B^{(1)}(1)k_{\omega_1,\omega_2}(k)/ \det c_{(1,\omega)}^j + \alpha(1 + \mu/2)B^{(2)}(k)k_{\omega_1,\omega_2}(k)/ \det c_{(2)}^j]/2$. It is easy to verify that $A^{(1)}_\psi(k)$ has the form (2.19). In fact, computing the functions in (A2.15), one finds that, for $k$, $\sigma_1$ and $\mu_1$ small,

\[
M^{(1)}(k) = \left( \begin{array}{cc}
(1 + \frac{a_1}{2})(i \sin k + \sin k_0) & O(k^3) \\
i0_1 + O(k^2) & (1 + \frac{a_1}{2})(i \sin k - \sin k_0) + O(k^3)
\end{array} \right), \quad N^{(1)}(k) = \left( \begin{array}{cc}
(-\frac{a_1}{2})(i \sin k + \sin k_0) + O(k^3) & i0_1 + O(\mu_1 k^2) \\
i0_1 + O(\mu_1 k^2) & -\frac{a_1}{2}(i \sin k - \sin k_0) + O(k^3)
\end{array} \right),
\]

(A2.16)

where the higher order terms in $k$, $\sigma_1$ and $\mu_1$ contribute to the corrections $a_1^{(1)}(k)$, $b_1^{(1)}(k)$, $c_1(k)$ and $d_1(k)$. They have the reality and parity properties described after (2.19) and it is apparent that $a_1^{(1)}(k) = O(\sigma_1 k) + O(k^3)$, $b_1^{(1)}(k) = O(\mu_1 k) + O(k^3)$, $c_1(k) = O(k^2)$ and $d_1(k) = O(\mu_1 k^2)$.

A2.2 Symmetry properties. In this section we identify some symmetries of model (2.7) and we prove that the quadratic and quartic terms in $V^{(1)}$ have the structure described in (2.22), (2.23) and (2.24).

The formal action appearing in (2.7) (see also (2.2) and (2.9) for an explicit form) is invariant under the following transformations.

1) Parity. $H^{(j)} \rightarrow \overline{H}^{(j)}$, $\overline{H}^{(j)} \rightarrow -H^{(j)}$ (the same for $V$ and $V^{(1)}$). In terms of the variables $\psi_{\omega, k}$, this transformation is equivalent to $\psi_{\omega, k} \rightarrow \pm \psi_{\omega, k}$ (the same for $\psi$) and we shall call it parity.
2) Complex conjugation: \( \hat{\psi}^\alpha_{\omega,k} \rightarrow \hat{\psi}^{-\alpha}_{-\omega,-k} \) (the same for \( \chi \)) and \( c \rightarrow c^* \), where \( c \) is a generic constant appearing in the formal action and \( c^* \) is its complex conjugate. Note that (2.10) is left invariant by this transformation, that we shall call complex conjugation.

3) Hole-particle: \( H^{(j)}_x \rightarrow (-1)^{j+1} H^{(j)}_x \) (the same for \( \overline{H}, V, \overline{V} \)). This transformation is equivalent to \( \hat{\psi}^\alpha_{\omega,k} \rightarrow \hat{\psi}^{-\alpha}_{-\omega,-k} \) (the same for \( \chi \)) and we shall call it hole-particle.

4) Rotation: \( H^{(j)}_{x,x_0} \rightarrow i\overline{V}^{(j)}_{-x_0,-x}, \overline{V}^{(j)}_{x,x_0} \rightarrow iV^{(j)}_{-x_0,-x}, \psi^{(j)}_{x,x_0} \rightarrow -i\overline{V}^{(j)}_{-x_0,-x}, \overline{V}^{(j)}_{x,x_0} \rightarrow iV^{(j)}_{-x_0,-x} \). This transformation is equivalent to
\[
\hat{\psi}^\alpha_{\omega,(k,k_0)} \rightarrow -\omega e^{-i\omega\pi/4}\hat{\psi}^{-\alpha}_{-\omega,(-k_0,-k)}, \quad \hat{\chi}^\alpha_{\omega,(k,k_0)} \rightarrow \omega e^{-i\omega\pi/4}\hat{\chi}^{-\alpha}_{-\omega,(-k_0,-k)}
\]
and we shall call it rotation.

5) Reflection: \( H^{(j)}_{x,x_0} \rightarrow i\overline{H}^{(j)}_{-x_0,x}, \overline{H}^{(j)}_{x,x_0} \rightarrow iH^{(j)}_{-x_0,x}, V^{(j)}_{x,x_0} \rightarrow -i\overline{V}^{(j)}_{-x_0,x}, \overline{V}^{(j)}_{x,x_0} \rightarrow iV^{(j)}_{-x_0,x} \). This transformation is equivalent to \( \hat{\psi}^\alpha_{\omega,(k,k_0)} \rightarrow i\hat{\psi}^{-\alpha}_{-\omega,(-k_0,k)} \) (the same for \( \chi \)) and we shall call it reflection.

6) The (1) \( \leftrightarrow \) (2) symmetry: \( H^{(1)}_x \leftrightarrow H^{(2)}_x, \overline{H}^{(1)}_x \leftrightarrow \overline{H}^{(2)}_x, V^{(1)}_x \leftrightarrow V^{(2)}_x, \overline{V}^{(1)}_x \leftrightarrow \overline{V}^{(2)}_x, u \rightarrow -u \). This transformation is equivalent to \( \hat{\psi}^\alpha_{\omega,k} \rightarrow -i\alpha \hat{\psi}^{-\alpha}_{-\omega,-k} \) (the same for \( \chi \)) together with \( u \rightarrow -u \) and we shall call it (1) \( \leftrightarrow \) (2) symmetry.

It is easy to verify that the quadratic forms \( P(\delta x) \), \( P(\delta \psi) \) and \( P_{Z_1,\tau_1,\mu_1,C_1}(\delta \psi) \) are separately invariant under the symmetries above. Then the effective action \( V^{(1)}(\psi) \) is still invariant under the same symmetries. Using the invariance of \( V^{(1)} \) under transformations (1)–(6), we now prove that the structure of its quadratic and quartic terms is the one described in Theorem 2.1, see in particular (2.22), (2.23) and (2.24).

Quartic term. The term \( \sum_{j=1}^6 W(k_1,k_2,k_3,k_4)\hat{\psi}^\dagger_{-1,k_1}\hat{\psi}^\dagger_{-1,k_2}\hat{\psi}^-_{1,k_3}\hat{\psi}^-_{1,k_4}\delta(k_1+k_2-k_3-k_4) \) under complex conjugation becomes equal to \( \sum_{j=1}^6 W^*(k_1,k_2,k_3,k_4)\hat{\psi}^-_{1,k_3}\hat{\psi}^-_{1,k_4}\hat{\psi}^\dagger_{-1,k_1}\hat{\psi}^\dagger_{-1,k_2}\delta(k_1+k_4-k_1-k_2) \), so that \( W(k_1,k_2,k_3,k_4) = W^*(k_3,k_4,k_1,k_2) \). Then, defining \( L_1 = W(k_1_+\hat{+}k_2_+\hat{+}k_3_+\hat{+}k_4_+), \) where \( k_1_+ = (\pi/M, \pi/M) \), and \( l_1 = P_0L_1 \), we see that \( L_1 \) and \( l_1 \) are real. From the explicit computation of the lower order term we find \( l_1 = \lambda + O(\lambda^2) \).

Quadratic terms. We distinguish 4 cases (items (a)–(d) below).

a) Let \( \alpha_1 = -\alpha_2 = + \) and \( \omega_1 = -\omega_2 = \omega \) and consider the expression \( \sum_{\omega,k} W_0(k;\mu_1)\hat{\psi}^\dagger_{\omega,k}\hat{\psi}^-_{-\omega,-k} \). Under parity it becomes \( \sum_{\omega,k} W_0(k;\mu_1)(i\omega)^\dagger\hat{\psi}^\dagger_{-\omega,-k} \hat{\psi}^-_{\omega,k} \), so that \( W_0(k;\mu_1) \) is even in \( k \).

Under complex conjugation it becomes \( \sum_{\omega,k} W_0(k;\mu_1)^*\hat{\psi}^-_{-\omega,-k}\hat{\psi}^\dagger_{\omega,k} = -\sum_{\omega,k} W_0(k;\mu_1)\hat{\psi}^-_{-\omega,-k}\hat{\psi}^\dagger_{-\omega,-k} \), so that \( W_0(k;\mu_1) \) is purely imaginary.

Under hole-particle it becomes \( \sum_{\omega,k} W_0(k;\mu_1)^*\hat{\psi}^-_{-\omega,-k}\hat{\psi}^\dagger_{-\omega,-k} = -\sum_{\omega,k} W_0(k;\mu_1)^*\hat{\psi}^-_{\omega,k}\hat{\psi}^\dagger_{-\omega,-k} \), so that \( W_0(k;\mu_1) \) is odd in \( \omega \).

Under (1) \( \leftrightarrow \) (2) it becomes \( \sum_{\omega,k} W_0(k;\mu_1)^*(-\omega)^\dagger\hat{\psi}^-_{\omega,k} \hat{\psi}^\dagger_{-\omega,-k} = -\sum_{\omega,k} W_0(k;\mu_1)^*\hat{\psi}^-_{\omega,k}\hat{\psi}^\dagger_{-\omega,-k} \), so that \( W_0(k;\mu_1) \) is even in \( \omega \).

Let us define \( S_1 = i\omega/2 \sum_{n\eta} W_0(k_{n\eta}) \), where \( k_{n\eta} = (n\pi/M, \eta\pi/M) \), and \( \gamma_{n1} = P_0S_1, \gamma_1 = P_0S_1 = \sigma_1\partial_{\sigma_1}, \gamma_1 |_{\sigma_1=0} = 0 \) and \( \gamma_1 \) is independent of \( \mu_1 \). From the previous discussion we see that \( S_1 \) and \( n_1 \) are real and \( n_1 \) is independent of \( \mu_1 \). From the exploitation of the lower order terms we find \( s_1 = O(\lambda\sigma_1) \) and \( \gamma_{n1} = \nu + c_n^\dagger\lambda + O(\lambda^2) \), for some constant \( c_n^\dagger \) independent of \( \lambda \). Note that, since \( W_0(k;\mu_1) \) is even in \( k \) (so that in particular no linear terms in \( k \) appear) in real space no terms of the form \( \hat{\psi}^\dagger_{\omega,k}\partial\hat{\psi}^-_{-\omega,-k} \) can appear.

b) Let \( \alpha_1 = \alpha_2 = \alpha \) and \( \omega_1 = -\omega_2 = \omega \) and consider the expression \( \sum_{\omega,\alpha,k} W_0^\alpha(k;\mu_1)\hat{\psi}^\alpha_{\omega,k}\hat{\psi}^{-\alpha}_{-\omega,-k} \).
We proceed as in item (a) and, by using parity, we see that $W_\omega^\alpha(k;\mu_1)$ is even in $k$ and odd in $\omega$.

By using complex conjugation, we see that $W_\omega^\alpha(k;\mu_1) = -W_\omega^{-\alpha}(k;\mu_1)^*$. By using hole-particle, we see that $W_\omega^\alpha(k;\mu_1)$ is even in $\alpha$ and $W_\omega^\alpha(k;\mu_1) = -W_\omega^{-\alpha}(k;\mu_1)^*$ implies that $W_\omega^\alpha(k;\mu_1)$ is purely imaginary.

By using (1)→(2) we see that $W_\omega^\alpha(k;\mu_1)$ is odd in $\mu_1$.

If we define $M_1 = -i\omega/2\sum_{\eta,\eta'}W_\omega^\alpha(\mathbf{k}\eta\eta';\mu_1)$ and $m_1 = \mathcal{P}_1M_1$, from the previous properties follows that $M_1$ and $m_1$ are real, $m_1$ is independent of $\sigma_1$ and, from the computation of its lower order, $m_1 = O(\lambda\mu_1)$. Note that, since $W_\omega^\alpha(k;\mu_1)$ is even in $k$ (so that in particular no linear terms in $k$ appear) in real space no terms of the form $\psi_{\omega,k}^\alpha\partial\psi_{-\omega,k}^\alpha$ can appear.

c) Let $\alpha_1 = -\alpha_2 = +$, $\omega_1 = \omega_2 = \omega$ and consider the expression $\sum_{\omega,k}W_\omega^\alpha(k;\mu_1)\psi_{\omega,k}^\alpha\psi_{-\omega,k}^\alpha$. By using parity we see that $W_\omega^\alpha(k;\mu_1)$ is odd in $k$.

By using reflection we see that $W_\omega(k,k_0;\mu_1) = W_{-\omega}(k,-k_0;\mu_1)$.

By using complex conjugation we see that $W_\omega(k,k_0;\mu_1) = W_\omega^*(k,k_0;\mu_1)$.

By using rotation we find $W_\omega(k,k_0;\mu_1) = -i\omega W_\omega(k_0,-k;\mu_1)$.

By using (1)→(2) we see that $W_\omega(k;\mu_1)$ is even in $\mu_1$.

If we define

$$G_1(k) = \frac{1}{4}\sum_{\eta,\eta'}W_\omega^\alpha(\mathbf{k}\eta\eta';\mu_1)(\eta\frac{\sin k}{\sin \pi/M} + \eta'\frac{\sin k_0}{\sin \pi/M}) \equiv a_\omega \sin k + b_\omega \sin k_0,$$  \hspace{2cm} (A2.18)

it can be easily verified that the previous properties imply that

$$a_\omega = a_{-\omega} = -a_\omega^\dagger = i\omega b_\omega \overset{def}{=} ia, \quad b_\omega = -b_{-\omega} = b_\omega^\dagger = -i\omega a_\omega \overset{def}{=} \omega b = -i\omega ia$$ \hspace{2cm} (A2.19)

with $a = b$ real and independent of $\omega$. As a consequence, $G_1(k) = G_1(i\sin k + \omega \sin k_0)$ for some real constant $G_1$. If $z_1 \overset{def}{=} \mathcal{P}_0G_1$ and we compute the lowest order contribution to $z_1$, we find $z_1 = O(\lambda^2)$.

d) Let $\alpha_1 = \alpha_2 = \alpha$, $\omega_1 = \omega_2 = \omega$ and consider the expression $\sum_{\omega,k}W_\omega^\alpha(k;\mu_1)\hat{\psi}_{\omega,k}^\alpha\hat{\psi}_{-\omega,k}^\alpha$. Repeating the proof in item (c) we see that $W_\omega^\alpha(k;\mu_1)$ is odd in $k$ and in $\mu_1$ and, if we define $F_1(k) = \frac{1}{4}\sum_{\eta,\eta'}W_\omega^\alpha(\mathbf{k}\eta\eta';\mu_1)(\eta\frac{\sin k}{\sin \pi/M} + \eta'\frac{\sin k_0}{\sin \pi/M})$, we can rewrite $F_1(k) = F_1(i\sin k + \omega \sin k_0)$. Since $W_\omega^\alpha(k;\mu_1)$ is odd in $\mu_1$, we find $F_1 = O(\lambda\mu_1)$.

Note that, with the definition of $\mathcal{L}$ introduced in §3.2, the result of the previous discussion is the following:

$$\mathcal{L}\mathcal{V}^{(1)}(\psi) = (s_1 + \gamma n_1)F_\sigma^{(\leq 1)} + m_1 F_\mu^{(\leq 1)} + \lambda F_\lambda^{(\leq 1)} + z_1 F_\zeta^{(\leq 1)},$$ \hspace{2cm} (A2.20)

where $s_1, n_1, m_1, l_1$ and $z_1$ are real constants and: $s_1$ is linear in $\sigma_1$ and independent of $\mu_1$; $m_1$ is linear in $\mu_1$ and independent of $\sigma_1$; $n_1, l_1, z_1$ are independent of $\sigma_1, \mu_1$; moreover $F_\sigma^{(\leq 1)}$, $F_\mu^{(\leq 1)}$, $F_\lambda^{(\leq 1)}$, $F_\zeta^{(\leq 1)}$ are defined by (3.8) with $h = 1$.

**Proof of Lemma 3.1.** The symmetries (1)−(6) discussed above are preserved by the iterative integration procedure. In fact it is easy to verify that $\mathcal{L}\mathcal{V}^{(h)}$, $\mathcal{R}\mathcal{V}^{(h)}$ and $P_{Z_{h-1}}\mathcal{R}_{\sigma_{h-1}}\mathcal{P}_{\mu_{h-1}}\mathcal{T}_h(\psi^{(h)})$ are, step by step, separately invariant under the transformations (1)−(6). Then Lemma 3.1 can be proven exactly in the same way (A2.20) was proven above.

**Proof of Lemma 3.2.** It is sufficient to note that the symmetry properties discussed above imply that: $\mathcal{L}_1W_{2,\omega} = 0$ if $\omega_1 + \omega_2 = 0$; $\mathcal{L}_0W_{2,\omega} = 0$ if $\omega_1 + \omega_2 \neq 0$; $\mathcal{P}_0W_{2,\omega} = 0$ if $\omega_1 + \omega_2 \neq 0$; and use the definitions of $\mathcal{R}_i$, $\mathcal{S}_i$, $i = 1, 2$.

**Appendix A3. Proof of Lemma 3.3.**

The propagators $g_{\omega}^{(b)}(x)$ can be written in terms of the propagators $g_{\omega}^{(j,h)}(x)$, $j = 1, 2$, see (3.16)
and following lines; $g_{(j,h)}(x)$ are given by

$$
\begin{align*}
g_{(j,h)}(x,\omega)(x-y) &= \frac{2}{M^2} \sum_k e^{-ik(x-y)} \bar{f}_h(k) \frac{-i \sin k + \omega \sin k_0 + a_{h-1}^{-1}(k)}{\sin^2 k + \sin^2 k_0 + (m_{h-1}^{(j)}(k))^2 + \delta B_{h-1}^{(j)}(k)} \\
g_{(j,h)-\omega}(x-y) &= \frac{2}{M^2} \sum_k e^{-ik(x-y)} \bar{f}_h(k) \frac{-i \omega m_{h-1}^{(j)}(k)}{\sin^2 k + \sin^2 k_0 + (m_{h-1}^{(j)}(k))^2 + \delta B_{h-1}^{(j)}(k)},
\end{align*}
$$

(A3.1)

where

$$
\begin{align*}
a_{h-1}^{(j)}(k) &\overset{\text{def}}{=} a_{h-1}^\omega(k) + (-1)^j b_{h-1}^\omega(k) , \quad c_{h-1}^{(j)}(k) \overset{\text{def}}{=} c_{h-1}(k) + (-1)^j d_{h-1}(k) \\
m_{h-1}^{(j)}(k) &\overset{\text{def}}{=} \sigma_{h-1}(k) + (-1)^j \mu_{h-1}(k) , \quad m_{h-1}^{(j)}(k) \overset{\text{def}}{=} m_{h-1}(k) + e^{(j)}(k) \\
\delta B_{h-1}^{(j)}(k) &\overset{\text{def}}{=} \sum_\omega \left[ a_{h-1}^{(j)}(k)(i \sin k - \omega \sin k_0) + a_{h-1}^{(j)}(k)a_{h-1}^{(j)}(k)/2 \right].
\end{align*}
$$

(A3.2)

In order to bound the propagators defined above, we need estimates on $\sigma_h(k), \mu_h(k)$ and on the “corrections” $a_{h-1}^\omega(k), b_{h-1}^\omega(k), c_{h-1}(k), d_{h-1}(k)$. As regarding $\sigma_h(k)$ and $\mu_h(k)$, in [BM] is proved (see Proof of Lemma 2.6) that, on the support of $f_h(k)$, for some $c, c^{-1}|\sigma_h| \leq |\sigma_{h-1}(k)| \leq c|\sigma_h|$ and $c^{-1}|\mu_h| \leq |\mu_{h-1}(k)| \leq c|\mu_h|$. Note also that, if $h \geq \tilde{h}$, using the first two of (3.18), we have $\frac{|\sigma_h|+|\mu_h|}{\gamma^h} \leq 2C_1$. As regarding the corrections, using their iterative definition (3.11), the asymptotic estimates near $k = 0$ of the corrections on scale $h = 1$ (see lines after (2.19)) and the hypothesis (3.18), we easily find that, on the support of $f_h(k)$:

$$
\begin{align*}
a_{h-1}^\omega(k) &= O(\sigma_h \gamma^{(1-2c|\lambda|^2)}h) + O(\gamma^{(3-c|\lambda|^2)}h) , \quad b_{h}^\omega(k) = O(\mu_h \gamma^{(1-2c|\lambda|^2)}h) + O(\gamma^{(3-c|\lambda|^2)}h) \\
c_{h}(k) &= O(\gamma^{(2-c|\lambda|^2)}h) , \quad d_{h}(k) = O(\mu_h \gamma^{(2-2c|\lambda|^2)}h).
\end{align*}
$$

(A3.3)

The bounds on the propagators follow from the remark that, as a consequence of the estimates discussed above, the denominators in (A3.1) are $O(\gamma^{2h})$ on the support of $f_h$.

**Appendix A4. Analyticity of the effective potentials**

It is possible to write $V^{(h)}$ (3.3) in terms of Gallavotti-Nicolo’ trees.

*FIG 4. A tree with its scale labels.*
We need some definitions and notations.

1) Let us consider the family of all trees which can be constructed by joining a point \( r \), the root, with an ordered set of \( n \geq 1 \) points, the endpoints of the unlabeled tree, so that \( r \) is not a branching point. \( n \) will be called the order of the unlabeled tree and the branching points will be called the non trivial vertices. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. Then the number of unlabeled trees with \( n \) end-points is bounded by \( 4^n \).

2) We associate a label \( h \leq 0 \) with the root and we denote \( T_{h,n} \) the corresponding set of labeled trees with \( n \) endpoints. Moreover, we introduce a family of vertical lines, labeled by an integer taking values in \([h,2]\), and we represent any tree \( \tau \in T_{h,n} \) so that, if \( v \) is an endpoint or a non trivial vertex, it is contained in a vertical line with index \( h_v > h \), to be called the scale of \( v \), while the root is on the line with index \( h \). There is the constraint that, if \( v \) is an endpoint, \( h_v > h + 1 \); if there is only one end-point its scale must be equal to \( h + 2 \), for \( h \leq 0 \). Moreover, there is only one vertex immediately following the root, which will be denoted \( v_0 \) and can not be an endpoint; its scale is \( h + 1 \).

3) With each endpoint \( v \) of scale \( h_v = +2 \) we associate one of the contributions to \( V^{(1)} \) given by (2.21); with each endpoint \( v \) of scale \( h_v \leq 1 \) one of the terms in \( \mathcal{L}V^{(h_v - 1)} \) defined in (3.7). Moreover, we impose the constraint that, if \( v \) is an endpoint and \( h_v \leq 1 \), \( h_v = h_v + 1 \), if \( v' \) is the non trivial vertex immediately preceding \( v \).

4) We introduce a field label \( f \) to distinguish the field variables appearing in the terms associated with the endpoints as in item 3); the set of field labels associated with the endpoint \( v \) will be called \( I_v \). Analogously, if \( v \) is not an endpoint, we shall call \( I_v \) the set of field labels associated with the endpoints following the vertex \( v \); \( x(f) \), \( \sigma(f) \) and \( \omega(f) \) will denote the space-time point, the \( \sigma \) index and the \( \omega \) index, respectively, of the field variable with label \( f \).

5) We associate with any vertex \( v \) of the tree a subset \( P_v \) of \( I_v \), the external fields of \( v \). These subsets must satisfy various constraints. First of all, if \( v \) is not an endpoint and \( v_1, \ldots, v_s \) are the \( s \) vertices immediately following it, then \( P_v \subseteq \cup_i P_{v_i} \); if \( v \) is an endpoint, \( P_v = I_v \). We shall denote \( Q_v \) the intersection of \( P_v \) and \( P_{v_i} \); this definition implies that \( P_v = \cup_i Q_{v_i} \). The subsets \( P_v \setminus Q_v \), whose union will be made, by definition, of the internal fields of \( v \), have to be non empty, if \( s_v > 1 \), that is if \( v \) is a non trivial vertex. Given \( \tau \in T_{h,n} \), there are many possible choices of the subsets \( P_v \), \( v \in \tau \), compatible with the previous constraints; let us call \( P \) one of this choices. Given \( P \), we consider the family \( \mathcal{G}_P \) of all connected Feynman graphs, such that, for any \( v \in \tau \), the internal fields of \( v \) are paired by propagators of scale \( h_v \), so that the following condition is satisfied: for any \( v \in \tau \), the subgraph built by the propagators associated with all vertices \( v' \geq v \) is connected. The sets \( P_v \) have, in this picture, the role of the external legs of the subgraph associated with the graphs belonging to \( \mathcal{G}_P \) will be called compatible with \( P \) and we shall denote \( \mathcal{P}_{\tau} \) the family of all choices of \( P \) such that \( \mathcal{G}_P \) is not empty.

6) we associate with any vertex \( v \) an index \( \rho_v \in \{s,p\} \) and correspondingly an operator \( \mathcal{R}_{\rho_v} \), where \( \mathcal{R}_s \) or \( \mathcal{R}_p \) are defined as

\[
\mathcal{R}_s \overset{df}{=} \begin{cases} S_2 & \text{if } n = 1 \text{ and } \omega_1 + \omega_2 = 0, \\ S_1 \cup \mathcal{R}_1 S_1 & \text{if } n = 1 \text{ and } \omega_1 + \omega_2 \neq 0, \\ S_1 & \text{if } n = 2, \\ 1 & \text{if } n > 2; \end{cases}
\]  

and

\[
\mathcal{R}_p \overset{df}{=} \begin{cases} \mathcal{R}_2 (P_0 + P_1) & \text{if } n = 1 \text{ and } \omega_1 + \omega_2 = 0, \\ \mathcal{R}_2 P_0 & \text{if } n = 1, \omega_1 + \omega_2 \neq 0 \text{ and } \alpha_1 + \alpha_2 = 0, \\ 0 & \text{if } n = 1, \omega_1 + \omega_2 \neq 0 \text{ and } \alpha_1 + \alpha_2 \neq 0, \\ \mathcal{R}_1 P_0 & \text{if } n = 2, \\ 0 & \text{if } n > 2. \end{cases}
\]
Note that $\mathcal{R}_s + \mathcal{R}_p = \mathcal{R}$, see Lemma 3.1.

The effective potential can be written in the following way:

$$\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) + M^2 \hat{E}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in T_{h,n}} \mathcal{V}^{(h)}(\tau, \sqrt{Z_h}\psi^{(\leq h)});,$$  \hspace{1cm} (A4.3)

where, if $v_0$ is the first vertex of $\tau$ and $\tau_1, \ldots, \tau_s$ are the subtrees of $\tau$ with root $v_0$, $\mathcal{V}^{(h)}(\tau, \sqrt{Z_h}\psi^{(\leq h)})$ is defined inductively by the relation

$$\mathcal{V}^{(h)}(\tau, \sqrt{Z_h}\psi^{(\leq h)}) = \frac{(-1)^{s+1}}{s!} \mathcal{E}^{T_{h+1}}_h[\hat{V}^{(h+1)}(\tau_1, \sqrt{Z_h}\psi^{(\leq h+1)}); \ldots; \hat{V}^{(h+1)}(\tau_s, \sqrt{Z_h}\psi^{(\leq h+1)})],$$  \hspace{1cm} (A4.4)

and $\hat{V}^{(h+1)}(\tau_i, \sqrt{Z_h}\psi^{(\leq h+1)})$;

a) is equal to $\mathcal{R}_{P_{v_i}} \hat{V}^{(h+1)}(\tau_{v_i}, \sqrt{Z_h}\psi^{(\leq h+1)})$ if the subtree $\tau_i$ with first vertex $v_i$ is not trivial (see (3.12) for the definition of $\hat{V}^{(h+1)}$);  

b) if $\tau_i$ is trivial and $h \leq -1$, it is equal to one of the terms in $\mathcal{L}\hat{\mathcal{V}}^{(h+1)}$, see (3.12), or, if $h = 0$, to one of the terms contributing to $\mathcal{V}^{(1)}(\sqrt{Z_1}\psi^{(\leq 1)})$.

**A4.1** The explicit expression for the kernels of $\mathcal{V}^{(h)}$ can be found from (A4.3) and (A4.4) by writing the truncated expectations of monomials of $\psi$ fields using the analogue of (A2.2): if $\tilde{\psi}(P_{v_i}) = \prod_{f \in P_{v_i}} \psi^{(h)}(f_{v_i})$, the following identity holds:

$$\mathcal{E}^{T_{h_v}}_v(\tilde{\psi}(P_{v_1}), \ldots, \tilde{\psi}(P_{v_s})) = \left(\frac{1}{Z_{h_v-1}}\right)^n \sum_{T_v} \prod_{v \in P_{v_0}} g^{(h_v)}(f_{i_t}, f^2_{i_t}) \int dP_{T_v}(t) \text{Pf} G^{T_v}(t)$$  \hspace{1cm} (A4.5)

where $g^{(h_v)}(f, f') = g^{(h_v)}(\Delta(f') - \Delta(f))$ and the other symbols in (A4.5) have the same meaning as those in (A2.2).

Using iteratively (A4.5) we can express the kernels of $\mathcal{V}^{(h)}$ as sums of products of propagators of the fields (the ones associated to the anchored trees $T_v$) and Pfaffians of matrices $G^{T_v}$.

**A4.2** If the $\mathcal{R}$ operator were not applied to the vertices $v \in \tau$ then the result of the iteration would lead to the following relation:

$$\mathcal{V}^{(h)}_h(\tau, \sqrt{Z_h}\psi^{(\leq h)}) = \sqrt{Z_h} P_{v_0} \sum_{P \in P_{v_0}} \sum_{T \in T_v} \int dx_{v_0} W^{*}_{\tau, P, T}(x_{v_0}) \left\{ \prod_{f \in P_{v_0}} \psi^{(h_v)}(f_{v_i}) \right\},$$  \hspace{1cm} (A4.6)

where $x_{v_0}$ is the set of integration variables associated to $\tau$ and $T = \bigcup_v T_v$; $W^{*}_{\tau, P, T}$ is given by

$$W^{*}_{\tau, P, T}(x_{v_0}) = \left[ \prod_{v \not\in \text{e.p.}} \left( \frac{Z_{h_v}}{Z_{h_v-1}} \right)^{\frac{|P_v|}{2}} \right] \left[ \prod_{i=1}^{n} K_{f_1}^{(h_v)}(x_{v_i}) \right] \left[ \prod_{v \not\in \text{e.p.}} \frac{1}{s!} \int dP_{T_v}(t_v) \right] \cdot \text{Pf} G^{h_v, T_v}(t_v) \left[ \prod_{i \in T_v} g^{(h_v)}(f^1_{i_t}, f^2_{i_t}) \right],$$  \hspace{1cm} (A4.7)

where: e.p. is an abbreviation of “end points”; $v_1^*, \ldots, v_n^*$ are the endpoints of $\tau$, $h_i \equiv h_{v_i}$ and $K^{(h_v)}(x_{v_i})$ are the corresponding kernels (equal to $\lambda_{h_v-1}(x_{v_i})$ or $\nu_{h_v-1}(x_{v_i})$ if $v_i$ is an endpoint of type $\lambda$ or $\nu$ on scale $h_v \leq 1$; or equal to one of the kernels of $\mathcal{V}^{(1)}$ if $h_v = 2$).

We can bound (A4.7) using (3.20) and the Gram–Hadamard inequality, see Appendix A2, we would find:

$$\int dx_{v_0} |W^{*}_{\tau, P, T}(x_{v_0})| \leq C^n M^2 |\lambda|^n \gamma^{-\frac{2+2\varepsilon}{2}} \prod_{v \not\in \text{e.p.}} \left\{ \frac{1}{s!} \left( \frac{Z_{h_v}}{Z_{h_v-1}} \right)^{\frac{|P_v|}{2}} \gamma^{-\frac{2+2\varepsilon}{2}} \right\},$$  \hspace{1cm} (A4.8)
We call $D_v = -2 + \frac{P_1}{2}$ the *dimension of $v$, depending on the number of the external fields of $v$. If $D_v < 0$ for any $v$ one can sum over $\tau, \mathbf{P}, T$ obtaining convergence for $\lambda$ small enough; however $D_v \leq 0$ when there are two or four external lines. We will take now into account the effect of the $R$ operator and we will see how the bound (A4.8) is improved.

**A4.3** The effect of application of $P_j$ and $S_j$ is to replace a kernel $W^{(h)}_{2n, \mathbf{j}, \mathbf{f}, \omega}$ with $P_j W^{(h)}_{2n, \mathbf{j}, \mathbf{f}, \omega}$ and $S_j W^{(h)}_{2n, \mathbf{j}, \mathbf{f}, \omega}$. If inductively, starting from the end–points, we write the kernels $W^{(h)}_{2n, \mathbf{j}, \mathbf{f}, \omega}$ in a form similar to (A4.7), we easily realize that, eventually, $P_j$ or $S_j$ will act on some propagator of an anchored tree or on some Pfaffian Pf $G^T_v$, for some $v$. It is easy to realize that $P_j$ and $S_j$, when applied to Pfaffians, do not break the Pfaffian structure. In fact the effect of $P_j$ on the Pfaffian of an antisymmetric matrix $G$ with elements $G_{f,f'}, f, f' \in J; |J| = 2k$, is the following (the proof is trivial):

$$\mathcal{P}_0 Pf G = Pf G^0, \quad \mathcal{P}_1 Pf G = \frac{1}{2} \sum_{f_1, f_2 \in J} \mathcal{P}_1 G_{f_1, f_2} (1) \mathcal{P}_1 Pf G_1,$$

(A4.9)

where $G^0$ is the matrix with elements $\mathcal{P}_0 G_{f,f'}$, $f, f' \in J$; $G^0$ is the matrix with elements $\mathcal{P}_0 G_{f,f'}$, $f, f' \in J$ when $i = 1, 2$; $(-1)^{\sigma}$ is the sign of the permutation leading from the ordering $J$ of the labels $f$ in the l.h.s. to the ordering $f_1, f_2, J_1$ in the r.h.s. The effect of $S_j$ is the following, see Appendix A7 for a proof:

$$S_1 Pf G = \frac{1}{2} \cdot \sum_{f_1, f_2 \in J} S_1 G_{f_1, f_2} \sum_{J_1, J_2 = J \setminus \{f_1, f_2\}} \mathcal{P}_1 Pf G_1,$$

(A4.10)

where: the * on the sum means that $J_1 \cap J_2 = \emptyset$; $|J_i| = 2k_i$, $i = 1, 2$; $(-1)^{\sigma}$ is the sign of the permutation leading from the ordering $J$ of the fields labels on the l.h.s. to the ordering $f_1, f_2, J_1$ on the r.h.s.; $G^0$ is the matrix with elements $\mathcal{P}_0 G_{f,f'}$, $f, f' \in J_1$; $G_2$ is the matrix with elements $G_{f,f'}$, $f, f' \in J_2$. The effect of $S_2$ on Pf $G^T$ is given by a formula similar to (A4.10). Note that the number of terms in the sums appearing in (A4.9), (A4.10) (and in the analogous equation for $S_2 Pf G^T$), is bounded by $c^k$ for some constant $c$.

**A4.4** It is possible to show that the $R_j$ operators produce derivatives applied to the propagators of the anchored trees and on the elements of $G^T_v$; and a product of “zeros” of the form $d_\beta^f (x(f) - x(f'))$, $j = 0, 1, b = 0, 1, 2$, associated to the lines $\ell \in T_v$. This is a well known result, and a very detailed discussion can be found in §3 of [BM]. By such analysis, and using (A4.9),(A4.10), we get the following expression for $\mathcal{R}V^{(h)}(\tau, \sqrt{Z_h}\psi^{(\leq h)})$:

$$\mathcal{R}V^{(h)}(\tau, \sqrt{Z_h}\psi^{(\leq h)}) = \sqrt{Z_h}^{\mathcal{P}_0} \mathcal{P}_0 \mathcal{P}_1 Pf G = \frac{1}{2} \sum_{f_1, f_2 \in J} S_1 G_{f_1, f_2} \sum_{J_1, J_2 = J \setminus \{f_1, f_2\}} \mathcal{P}_1 Pf G_1,$$

(A4.11)

where: $B_T$ is a set of indices which allows to distinguish the different terms produced by the non trivial $R$ operations; $x_\beta(f)$ is a coordinate obtained by interpolating two points in $x_{v_0}$, in a suitable way depending on $\beta$; $q_\beta(f)$ is a nonnegative integer $\leq 2$; $j_\beta(f) = 0, 1$ and $\partial_{j_\beta(f)}$ is a suitable differential operator, dimensionally equivalent to $\partial_{j_\beta}$ [see [BM] for a precise definition]; $W_{\tau, \mathbf{P}, T, \beta}(x_{v_0})$ is given by:

$$W_{\tau, \mathbf{P}, T, \beta}(x_{v_0}) = \left[ \prod_{v \text{ not e.p.}} \left( \frac{Z_{h_v}}{Z_{h_v - 1}} \right) \right] \left[ \prod_{i = 1}^n d_{j_\beta(v_i)}(x_{x_i}^\gamma, y_\beta^\gamma) \mathcal{P}_i C_{j_\beta(v_i)}(x_{x_i}^\gamma) S_{j_\beta(v_i)}(x_{x_i}^\gamma) K_{h_i}^\gamma(x_{x_i}^\gamma) \right] \cdot \left[ \prod_{v \text{ not e.p.}} \frac{1}{v_0} \int d\mathcal{P}_0 \mathcal{P}_1 Pf C_{j_\beta(v_0)} S_{j_\beta(v_0)} Pf K_{h_0}^\gamma(x_{v_0}^\gamma) \right] \cdot \left[ \prod_{v \text{ not e.p.}} \frac{1}{v_0} \int d\mathcal{P}_0 \mathcal{P}_1 Pf C_{j_\beta(v_0)} S_{j_\beta(v_0)} Pf K_{h_0}^\gamma(x_{v_0}^\gamma) \right],$$

(A4.12)
where: $v_1^*, \ldots, v_n^*$ are the endpoints of $\tau$; $b_\beta(v)$, $b_\beta(l)$, $q_\beta(f_1^j)$ and $q_\beta(f_2^j)$ are nonnegative integers $\leq 2$; $j_\beta(v)$, $j_\beta(f_1^j)$, $j_\beta(f_2^j)$ and $j_\beta(l)$ can be 0 or 1; $i_\beta(v)$ and $i_\beta(l)$ can be 0 or 2; $I_\beta(v)$ and $I_\beta(l)$ can be 0 or 1; $C_\beta(v)$, $c_\beta(v)$, $C_\beta(l)$ and $c_\beta(l)$ can be 0, 1 and $\max\{C_\beta(v) + c_\beta(v), C_\beta(l) + c_\beta(l)\} \leq 1$; $G_{\beta,T}^{h,v}(t_v)$ is obtained from $G_{h,v}^{h,v}(t_v)$ by substituting the element $\tau_{1(f),i(j)} g^{(h,v)}(f,f')$ with $\tau_{1(f),i(j)} r_{1(f),i(j)} g^{(h,v)}(f,f')$.

It would be very difficult to give a precise description of the various contributions to the sum over $B_T$, but fortunately we only need to know some very general properties, which easily follows from the construction in §3.

1) There is a constant $C$ such that, $\forall T \in T_T$, $\vert B_T \vert \leq C^n$; for any $\beta \in B_T$, the following inequality is satisfied

$$\left( \prod_{f \in \cup v \gamma h(f) q_\beta(f)} \prod_{l \in T} \gamma^{-h(l) b_\beta(l)} \right) \leq \prod_{v \notin \text{e.p.}} \gamma^{-z(P_v)} , \quad (A4.13)$$

where: $h(f) = h_{v_0} - 1$ if $f \in P_{v_0}$, otherwise it is the scale of the vertex where the field with label $f$ is contracted; $h(l) = h_{v_0}$, if $l \in T_v$ and

$$z(P_v) = \begin{cases} 1 & \text{if } \vert P_v \vert = 4 \text{ and } \rho_v = p, \\ 2 & \text{if } \vert P_v \vert = 2 \text{ and } \rho_v = p, \\ 1 & \text{if } \vert P_v \vert = 2, \rho_v = s \text{ and } \sum_{f \in P_v} \omega(f) = 0, \\ 0 & \text{otherwise}. \end{cases} \quad (A4.14)$$

2) If we define

$$\prod_{v \in T} \left( \frac{\sigma h_v}{\gamma h_v} + \frac{\mu h_v}{\gamma h_v} \right) \prod_{l \in T_v} \left( \frac{\sigma h_v}{\gamma h_v} + \frac{\mu h_v}{\gamma h_v} \right) = \prod_{v \in T_v} \left( \frac{\sigma h_v}{\gamma h_v} + \frac{\mu h_v}{\gamma h_v} \right) \left( i(v, \beta) \right) . \quad (A4.15)$$

the indices $i(v, \beta)$ satisfy, for any $B_T$, the following property:

$$\sum_{w \geq v} i(v, \beta) \geq z'(P_v) , \quad (A4.16)$$

where

$$z'(P_v) = \begin{cases} 1 & \text{if } \vert P_v \vert = 4 \text{ and } \rho_v = s, \\ 2 & \text{if } \vert P_v \vert = 2 \text{ and } \rho_v = s \text{ and } \sum_{f \in P_v} \omega(f) = 0, \\ 1 & \text{if } \vert P_v \vert = 2, \rho_v = s \text{ and } \sum_{f \in P_v} \omega(f) = 0, \\ 0 & \text{otherwise}. \end{cases} \quad (A4.17)$$

A4.5 We can bound any $\mid P_{-\beta, v_0}^{C_\beta(v)} S_{\beta(v)}^{C_\beta(v)} \mid$ in (A4.12), with $C_\beta(v) + c_\beta(v) = 0, 1$, by using (A4.9), (A4.10) and Gram inequality, as illustrated in Appendix A2 for the case of the integration of the $\chi$ fields. Using that the elements of $G$ are all propagators on scale $h_v$, dimensionally bounded as in Lemma 3.3, we find:

$$\mid P_{-\beta, v_0}^{C_\beta(v)} S_{\beta(v)}^{C_\beta(v)} \mid \leq C \sum_{i=1}^{n} \vert P_{v_i} \vert - \vert P_{v_i} \vert - 2(\sigma_v - 1) . \quad (A4.18)$$

where $J_v = \cup_{i=1}^{n} P_{v_i} \setminus Q_{v_i}$. We will bound the factors $\left( \frac{\sigma h_v}{\gamma h_v} + \frac{\mu h_v}{\gamma h_v} \right) C_\beta(v) I_\beta(v)$ using (3.19) by a constant.

If we call

$$J_{\tau, P, T, \beta} = \int d\mathbf{x}_{v_0} \left[ \prod_{i=1}^{n} \frac{d_{\beta, v_0}(x_{v_i}^*)}{\gamma h_v} \mathbf{P}_{-\beta, v_0}^{C_\beta(v_i)} S_{\beta(v_i)}^{C_\beta(v_i)} K_{\varepsilon_i}^{h_v}(x_{v_i}^*) \right] \cdot \left\{ \prod_{v \notin \text{e.p.}} \frac{1}{\gamma h_v} \left[ \prod_{l \in T_v} \gamma h(l) \right] \right\} , \quad (A4.19)$$

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we have, under the hypothesis (3.24),

\[
J_{\tau,P,T,\alpha} \leq C^n M^2 |\lambda|^n \left[ \prod_{i=1}^{n} \left( \frac{|\sigma_{h_i^*}| + |\mu_{h_i}|}{\gamma_{h_i^*}} \right) e_{\beta}(v_i^*) i_{\beta}(v_i^*) \right].
\]

\[
\left\{ \sum_{v \notin \text{e.p.}} \frac{1}{8!} C^{2(s_v-1)} \gamma^{h_v} n_v(v) \gamma^{-h_v} \sum_{\ell \in T} b_{\beta}(\ell) \gamma^{-h_v} \sum_{i=1}^{n} b_{\beta}(v_i^*) \gamma^{-h_v(s_v-1)}. \right. 
\]

\[
\left. \cdot \gamma^{h_v} \sum_{\ell \in T} \left[ q_{\alpha}(f_1^*) + q_{\beta}(f_2^*) \right] \prod_{\ell \in T} \left( \frac{|\sigma_{h_{\ell}}| + |\mu_{h_{\ell}}|}{\gamma_{h_{\ell}}} \right) e_{\alpha}(\ell) i_{\beta}(\ell) \right\} 
\]

where \( n_v(v) \) is the number of vertices of type \( \nu \) with scale \( h_v + 1 \).

Now, substituting (A4.18), (A4.20) into (A4.12), using (A4.13), we find that:

\[
\int dx_v |W_{\tau,P,T,\beta}(x_v)| \leq C^n M^2 |\lambda|^n \gamma^{-h D_h(|P_v|)} \prod_{v \in V_{\beta}} \left( \frac{|\sigma_{h_{\ell}}| + |\mu_{h_{\ell}}|}{\gamma_{h_{\ell}}} \right) i(v,\beta).
\]

\[\prod_{v \notin \text{e.p.}} \left\{ \frac{1}{8!} C \sum_{i=1}^{s_v} |P_v|-|P_v| \left( \frac{Z_{h_v}}{Z_{h_v-1}} \right) \left( \frac{|P_v|}{z(P_v)} \right) \gamma^{-|-2+|P_v|+z(P_v)+(1-c|\lambda|)|} \right\}
\]

where, if \( k = \sum_{f \in P_v} q_{\beta}(f) \), \( D_h(p) = -2 + p + k \) and we have used (A4.15). Note that, given \( v \in \tau \) and \( \tau \in T_{h,n} \) and using (3.19) together with the first two of (3.18),

\[
\frac{|\sigma_{h_{\ell}}|}{\gamma_{h_{\ell}}} = \frac{|\sigma_{h_{\ell}}|}{\gamma_{h_{\ell}}}, \frac{|\mu_{h_{\ell}}|}{\gamma_{h_{\ell}}} \leq \frac{|\sigma_{h_{\ell}}|}{\gamma_{h_{\ell}}} \gamma^{(h-h_{\ell})(1-c|\lambda|)} \leq C_1 \gamma^{(h-h_{\ell})(1-c|\lambda|)}
\]

Moreover the indices \( i(v,\beta) \) satisfy, for any \( B_T \), (A4.17) so that, using (A4.22) and (A4.16), we find

\[
\prod_{v \notin \text{e.p.}} \left( \frac{|\sigma_{h_{\ell}}| + |\mu_{h_{\ell}}|}{\gamma_{h_{\ell}}} \right) i(v,\beta) \leq C_1^n \prod_{v \notin \text{e.p.}} \gamma^{-z(P_v)}.
\]

Substituting (A4.22) into (A4.21) and using (A4.16), we find:

\[
\int dx_v |W_{\tau,P,T,\beta}(x_v)| \leq C^n M^2 |\lambda|^n \gamma^{-h D_h(|P_v|)}.
\]

\[
\prod_{v \notin \text{e.p.}} \left\{ \frac{1}{8!} C \sum_{i=1}^{s_v} |P_v|-|P_v| \left( \frac{Z_{h_v}}{Z_{h_v-1}} \right) \left( \frac{|P_v|}{z(P_v)} \right) \gamma^{-|-2+|P_v|+z(P_v)+(1-c|\lambda|)|}} \right\}.
\]

where

\[
D_v \equiv -2 + \frac{|P_v|}{2} + z(P_v) + (1 - c|\lambda|)|z'(P_v) \geq \frac{|P_v|}{6}.
\]

Then (3.25) in Theorem 3.1 follows from the previous bounds and the remark that

\[
\sum_{\tau \in T_{h,n}} \sum_{P_v} \sum_{T} \sum_{\beta \in B_T} \prod_{v \notin \text{e.p.}} \frac{1}{8!} \gamma^{-\frac{|P_v|}{6}} \leq c^n,
\]

for some constant \( c \), see [BM] or [GM] for further details.

The bound on \( \tilde{E}_h, t_h \), (3.26) and (3.27) follow from a similar analysis. The remarks following (3.26) and (3.27) follows from noticing that in the expansion for \( \mathcal{L}^{\tilde{\gamma}(h)} \) appear only propagators of type \( P_0 g_{a,a}(h_v) \) or \( P_1 g_{a,a}(h_v) \) (in order to bound these propagators we do not need (3.19), see
the last statement in Lemma 3.3). Furthermore, by construction \( l_k, n_h \) and \( z_h \) are independent of \( \sigma_k, \mu_k \), so that, in order to prove (3.27) we do not even need the first two inequalities in (3.18).

**A4.6** The sum over all the trees with root scale \( h \) and with at least \( v \) with \( h_v = k \) is \( O(|\lambda|^2 \gamma^h (h-k)) \); this follows from the fact that the bound (A4.26) holds, for some \( c = O(1) \), even if \( \gamma^{-|P_v|/6} \) is replaced by \( \gamma^{-k|P_v|} \), for any constant \( \kappa > 0 \) independent of \( \lambda \); and that \( D_v \), instead of using (A4.25), can also be bounded as \( D_v \geq 1/2 + |P_v|/12 \). This property is called short memory property.

**Appendix A5. Proof of Theorem 4.1 and Lemma 4.1**

We consider the space \( M_\varrho \) of sequences \( \varrho = \{\nu_h\}_{h \leq 1} \) such that \( |\nu_h| \leq c|\lambda|\gamma^{(\vartheta/2)k} \); we shall think of \( M_\varrho \) as a Banach space with norm \( \|\cdot\|_\varrho \), where \( \|\varrho\|_\varrho = \sup_{k \leq 1} |\nu_k|\gamma^{-k(\vartheta/2)k} \). We will proceed as follows: we first show that, for any sequence \( \varrho \in M_\varrho \), the flow equation for \( \nu_h \), the hypothesis (3.17), (3.18) and the property \( |\lambda_h(\varrho)| \leq c|\lambda| \) are verified, uniformly in \( \varrho \). Then we fix \( \varrho \in M_\varrho \) via an exponentially convergent iterative procedure, in such a way that the flow equation for \( \nu_h \) is satisfied.

**A5.1 Proof of Theorem 4.1.** Given \( \varrho \in M_\varrho \), let us suppose inductively that (3.17), (3.18) and that, for \( k > h + 1 \),

\[
|\lambda_{k-1}(\varrho) - \lambda_k(\varrho)| \leq c_0|\lambda|^2 \gamma^{(\vartheta/2)k},
\]

(A5.1)

for some \( c_0 > 0 \). Note that (A5.1) is certainly true for \( h = 1 \) (in that case the r.h.s. of (A5.1) is just the bound on \( \beta_1^h \)). Note also that (A5.1) implies that \( |\lambda_k| \leq c|\lambda| \), for any \( k > h \).

Using (3.26), the second of (3.27) and (4.1) we find that (3.17), (3.18) are true with \( h \) replaced by \( h - 1 \).

We now consider the equation \( \lambda_{h-1} = \lambda_h + \beta_h^h(\lambda_h, \nu_h; \ldots; \lambda_1, \nu_1) \), \( h > h \). The function \( \beta_h^h \) can be expressed as a convergent sum over tree diagrams, as described in Appendix A4; note that it depends on \( \lambda_h(\varrho), \nu_h; \ldots; \lambda_1, \nu_1 \) directly through the end–points of the trees and indirectly through the factors \( Z_h/Z_{h-1} \).

We can write \( g(\varrho)_{(+\omega),(-\omega)}(x-y) = g_{L,\omega}(x-y) + r^{(h)}_{\omega}(x-y), \) where

\[
g^{(h)}_{L,\omega}(x-y) \overset{\text{def}}{=} \frac{4}{M^2} \sum_k e^{-ik(x-y)} f_{\omega}(k) \frac{1}{1 + \omega k}
\]

(A5.2)

and \( r^{(h)}_{\omega} \) is the rest, satisfying the same bound as \( g^{(h)}_{(+\omega),(-\omega)} \), times a factor \( \gamma^h \). This decomposition induces the following decomposition for \( \beta_h^h \):

\[
\beta_h^h(\lambda_h, \nu_h; \ldots; \lambda_1, \nu_1) = \beta_{h,L}(\lambda_h, \ldots, \lambda_h) + \sum_{k=h+1}^{1} D_{\lambda,k}^h + r_{\omega}(\lambda_h, \ldots, \lambda_1) + \sum_{k \geq h} \nu_k \beta_{\lambda,k}^h(\lambda_k, \nu_k; \ldots; \lambda_1, \nu_1),
\]

(A5.3)

with

\[
|\beta_{h,L}^h| \leq c|\lambda|^2 \gamma^{(\vartheta/2)h}, \quad |D_{\lambda,k}^h| \leq c|\lambda|\gamma^{(h-k)}|\lambda_k - \lambda_h|,
\]

\[
|r_{\omega}^{(k)}| \leq c|\lambda|^2 \gamma^{(\vartheta/2)h}, \quad |\beta_{\lambda,k}^h| \leq c|\lambda|\gamma^{(h-k)}.
\]

(A5.4)

The first two terms in (A5.3) \( \beta_{h,L}^h \) collect the contributions obtained by posing \( r_{\omega}^{(k)} = 0, k \geq h \) and substituting the discrete \( \delta \) function defined after (3.8) with \( M^2 \delta_{k,0} \). The first of (A5.4) is called the vanishing of the Luttinger model Beta function property, see [BGPS][GS][BM1] (or [BeM1] for a simplified proof), and it is a crucial property of interacting fermionic systems in \( d = 1 \).

Using the decomposition (A5.3) and the bounds (A5.4) we prove the following bounds for \( \lambda_h(\varrho), \varrho \in M_\varrho \):

\[
|\lambda_h(\varrho) - \lambda_1(\varrho)| \leq c_0|\lambda|^2, \quad |\lambda_h(\varrho) - \lambda_{h+1}(\varrho)| \leq c_0|\lambda|^2 \gamma^{(\vartheta/2)h},
\]

(A5.5)
for some $c_0 > 0$. Moreover, given $\nu, \nu' \in \mathcal{M}_\delta$, we show that:

$$|\lambda_h(\nu) - \lambda_h(\nu')| \leq c|\lambda||\nu - \nu'||_0, \quad (A5.6)$$

where $||\nu - \nu'||_0 \overset{def}{=} \sup_{h \leq 1} |\nu_h - \nu'_h|$. 

**Proof of (A5.5).** We decompose $\lambda_h - \lambda_{h+1} = \beta(\lambda)$ as in (A5.3). Using the bounds (A5.4) and the inductive hypothesis (A5.1), we find:

$$|\lambda_h(\nu) - \lambda_{h+1}(\nu')| \leq c|\lambda|^2 \gamma^{(h+1)} + \sum_{k \geq h+2} c|\lambda| \gamma^{(h+1-k)} \sum_{k' = h+2} c_0 |\lambda|^2 \gamma^{(k')h+1} + c|\lambda|^2 \gamma^{(h+1)} + \sum_{k \geq h+1} c^2|\lambda|^2 \gamma^{(k')h+1}, \quad (A5.7)$$

which, for $c_0$ big enough, immediately implies the second of (A5.5) with $h \to h-1$; from this bound and the hypothesis (A5.1) follows the first of (A5.5). 

**Proof of (A5.6).** If we take two sequences $\nu, \nu' \in \mathcal{M}_\delta$, we easily find that the beta function for $\lambda_h(\nu) - \lambda_h(\nu')$ can be represented by a tree expansion similar to the one for $\beta_h$, with the property that the trees giving a non vanishing contribution have necessarily one end-point on scale $k \geq h$ associated to a coupling constant $\lambda_k(\nu) - \lambda_k(\nu')$ or $\nu_k - \nu'_k$. Then we find:

$$\lambda_h(\nu) - \lambda_h(\nu') = \lambda_1(\nu) - \lambda_1(\nu') + \sum_{h+1 \leq k \leq h} [\beta_h(\lambda_k(\nu); \nu_k; \ldots; \lambda_1, \nu_1) - \beta_h(\lambda_k(\nu'); \nu_k; \ldots; \lambda_1, \nu_1)]. \quad (A5.8)$$

Note that $|\lambda_1(\nu) - \lambda_1(\nu')| \leq c_0 |\lambda||\nu_k - \nu'_k|$, because $\lambda_1 = \lambda/Z^2 + O(\lambda^2/Z^4)$ and $Z \approx \sqrt{2} - 1 + \nu/2$. 

If we inductively suppose that, for any $k > h$, $|\lambda_k(\nu) - \lambda_k(\nu')| \leq 2c_0 |\lambda||\nu - \nu'||_0$, we find, by using the decomposition (A5.3):

$$|\lambda_h(\nu) - \lambda_h(\nu')| \leq c_0 |\lambda||\nu_1 - \nu'_1| + c|\lambda| \sum_{h+1 \leq k \leq h} \gamma^{(h+1)} \sum_{k' \geq k} \gamma^{(k-k')} [2c_0 |\lambda||\nu - \nu'||_0 + |\nu_k - \nu'_k|]. \quad (A5.9)$$

Choosing $c_0$ big enough, (A5.6) follows. 

We are now left with fixing the sequence $\nu$ in such a way that the flow equation for $\nu$ is satisfied. Since we want to fix $\nu$ in such a way that $\nu_{-\infty} = 0$, we must have:

$$\nu_1 = -\sum_{k=-\infty}^1 \gamma^{k-2} \beta_k(\lambda_k, \nu_k; \ldots; \lambda_1, \nu_1). \quad (A5.10)$$

If we manage to fix $\nu_1$ as in (A5.10), we also get:

$$\nu_k = -\sum_{k \leq h} \gamma^{k-h-1} \beta_k(\lambda_k, \nu_k; \ldots; \lambda_1, \nu_1). \quad (A5.11)$$

We look for a fixed point of the operator $T : \mathcal{M}_\delta \to \mathcal{M}_\delta$ defined as:

$$(T \nu)_h = -\sum_{k \leq h} \gamma^{k-h-1} \beta_k(\lambda_k(\nu), \nu_k; \ldots; \lambda_1, \nu_1). \quad (A5.12)$$

where $\lambda_k(\nu)$ is the solution of the first line of (4.2), obtained as a function of the parameter $\nu$ as described above.
If we find a fixed point $\mu^*$ of (A5.12), the first two lines in (4.2) will be simultaneously solved by $\lambda(z^*)$ and $z^*$ respectively, and the solution will have the desired smallness properties for $\lambda_h$ and $\nu_h$.

First note that, if $|\lambda|$ is sufficiently small, then $T$ leaves $\mathfrak{M}_\theta$ invariant: in fact, as a consequence of parity cancellations, the $\nu$–component of the Beta function satisfies:

$$\beta_h^\nu(\lambda_h, \nu_h; \ldots; \lambda_1, \nu_1) = \sum_k \nu_k \tilde{\beta}_{\nu,k}^h(\lambda_h, \nu_h; \ldots; \lambda_1, \nu_1, \ldots; \lambda_1, \nu_1)$$  \hspace{1cm} (A5.13)

where, if $c_1, c_2$ are suitable constants

$$|\beta_{\nu,1}| \leq c_1 |\lambda|^{\alpha_h} \quad |\tilde{\beta}_{\nu,k}^h| \leq c_2 |\lambda|^{\alpha(h-k)}.$$  \hspace{1cm} (A5.14)

by using (A5.13) and choosing $c = 2c_1$ we obtain

$$|(T\nu)_h| \leq \sum_{k \leq h} 2c_1 |\lambda|^{\alpha(\nu/2)k - h} \leq c |\lambda|^{\alpha(\nu/2)h},$$  \hspace{1cm} (A5.15)

Furthermore, using (A5.13) and (A5.6), we find that $T$ is a contraction on $\mathfrak{M}_\theta$:

$$|(T\nu) - (T\nu')| \leq c \sum_{k \leq h} |\gamma^k \lambda^k - \gamma^k \lambda' k - h| \leq c' \sum_{k \leq h} |\gamma^k \lambda^k - \gamma^k \lambda' k - h| \leq c' \gamma^{k-h} \sum_{k \leq h} \nu_k - \nu'_k,$$

$$\sum_{k \leq h} \gamma^k \lambda^k \nu_k - \nu'_k \leq c'' |\lambda|^{\alpha(\nu/2)h} |\nu - \nu'| = O(\nu^k),$$

hence $||(T\nu) - (T\nu')||_\theta \leq c'' |\lambda|^{\alpha(\nu/2)h}$. Then, a unique fixed point $\mu^*$ for $T$ exists on $\mathfrak{M}_\theta$. Proof of Theorem 4.1 is concluded by noticing that $T$ is analytic (in fact $\beta_h^\nu$ and $\lambda$ are analytic in $\nu$ in the domain $\mathfrak{M}_\theta$). \hfill $\blacksquare$

**A5.2 Proof of Lemma 4.1** From now on we shall think $\lambda_h$ and $\nu_h$ fixed, with $\nu_1$ conveniently chosen as above ($\nu_1 = \nu^*_1(\lambda)$). Then we have $|\lambda_h| \leq c |\lambda|$ and $|\nu_h| \leq c |\lambda|^{\alpha(\nu/2)h}$, for some $c, \vartheta > 0$.

Having fixed $\nu_1$ as a convenient function of $\lambda$, we can also think $\lambda_h$ and $\nu_h$ as functions of $\lambda$.

The flow of $Z_h$. The flow of $Z_h$ is given by the first of (4.1) with $z_h$ independent of $\sigma, k, k \geq h$. By Theorem 3.1 we have that $|z_h| \leq c |\lambda|^2$, uniformly in $h$. Again, as for $\lambda_h$ and $\nu_h$, we can formally study this equation up to $h = -\infty$. We define $\eta = \lim_{h \to -\infty} 1 + z_h$, so that

$$\log(1 + z_h) = \eta_z(h - 1) + \sum_{k \geq h+1} r_z^k,$$  \hspace{1cm} (A5.17)

Using the fact that $z_{k-1} - z_k$ is necessarily proportional to $\lambda_{k-1} - \lambda_k$ or to $\nu_{k-1} - \nu_k$ and that $\lambda_{k-1} - \lambda_k$ is bounded as in (A5.1), we easily find: $|r_z^k| \leq c |z_k - z_{k-1}| \leq c |\lambda|^{\alpha(\nu/2)k}$. So, if $F^{(h)} \equiv \sum_{k \geq h+1} r_z^k$ and $F_z^k = 0$, then $F^{(h)} = O(\lambda)$ and $z_h = \gamma^{h-1} + F_z^h$. Clearly, by definition, $\eta_z$ and $F_z^h$ only depend on $\lambda_h, \nu_h, k \leq 1$, so they are independent of $t$ and $u$.

The flow of $\mu_h$. The flow of $\mu_h$ is given by the last of (4.1). One can easily show inductively that $\mu_h(k)/\mu_h, k \geq h$, is independent of $\mu_1$, so that one can think $\mu_{h-1}/\mu_h$ is just a function
of \( \lambda_h, \nu_h \). Then, again we can study the flow equation for \( \mu_h \) up to \( h \to -\infty \). We define 
\[
\gamma^{-\eta} = \lim_{h \to -\infty} 1 + (m_h/\mu_h - z_h)/(1 + z_h),
\]
so that, proceeding as for \( Z_h \), we see that
\[
\mu_h = \mu_1 \gamma^{-\eta}, \tag{A5.18}
\]
for a suitable \( F^h_\mu = O(\lambda) \). Of course \( \eta_\mu \) and \( F^h_\mu \) are independent of \( t \) and \( u \).

The flow of \( \sigma_h \). The flow of \( \sigma_h \) can be studied as the one of \( \mu_h \). If we define 
\[
\gamma^{-\eta} = \lim_{h \to -\infty} 1 + (\eta_h/\sigma_h - z_h)/(1 + z_h),
\]
we find that
\[
\sigma_h = \sigma_1 \gamma^{-\eta} + F^h_\sigma, \tag{A5.19}
\]
for a suitable \( F^h_\sigma = O(\lambda) \). Again, \( \eta_\sigma \) and \( F^h_\sigma \) are independent of \( t, u \).

We are left with proving that \( \eta_\sigma - \eta_\mu \neq 0 \). It is sufficient to note that, by direct computation of the lowest order terms, for some \( \theta > 0 \), (4.1) can be written as:
\[
z_h = b_1 \lambda_h^2 + O(|\lambda|^2 \gamma^{\theta} h) + O(|\lambda|^3), \quad b_1 > 0
\]
\[
\eta_h/\sigma_h = - b_2 \lambda_h + O(|\lambda| \gamma^{\theta} h) + O(|\lambda|^2), \quad b_2 > 0
\]
\[
m_h/\mu_h = b_2 \lambda_h + O(|\lambda|^2 \gamma^{\theta} h) + O(|\lambda|^2), \quad b_2 > 0,
\]
where \( b_1, b_2 \) are constants independent of \( \lambda \) and \( h \). Using (A5.20) and the definitions of \( \eta_\mu \) and \( \eta_\sigma \) we find: \( \eta_\sigma - \eta_\mu = (2b_2/\log \gamma) \lambda + O(\lambda^2) \).

### Appendix A6. Proof of Lemma 5.3

Proceeding as in §4 and Appendix A5, we first solve the equations for \( Z_h \) and \( m^{(2)}_h \) parametrically in \( \Sigma = \{ \pi_h \}_{h \leq h_1^*} \). If \( |\pi_h| \leq c|\lambda|^{\gamma(h-h_1^*)} \), the first two assumptions of (5.14) easily follow. Now we will construct a sequence \( \Sigma \) such that \( |\pi_h| \leq c|\lambda|^{\gamma(h-h_1^*)} \) and satisfying the flow equation \( \pi_{h-1} = \gamma^h \pi_h + b^h(\pi_h, \ldots, \pi_{h_1^*}) \).

**A6.1 Tree expansion for \( \beta^h_\pi \).** \( \beta^h_\pi \) can be expressed as sum over tree diagrams, similar to those used in Appendix A4. The main difference is that they have vertices on scales \( k \) between \( h \) and \( +2 \). The vertices on scales \( h_v \geq h_1^* + 1 \) are associated to the truncated expectations \( A4.4 \); the vertices on scale \( h_v = h_1^* \) are associated to truncated expectations w.r.t. the propagators \( g^{(1,h_1^*)}_{\omega_1,\omega_2} \); the vertices on scale \( h_v < h_1^* \) are associated to truncated expectations w.r.t. the propagators \( g^{(2,h_1^*+1)}_{\omega_1,\omega_2} \). Moreover the end–points on scale \( \geq h_1^* + 1 \) are associated to the couplings \( \lambda_h \) or \( \nu_h \), as in Appendix A4; the end–points on scales \( h \leq h_1^* \) are necessarily associated to the couplings \( \pi_h \).

**A6.2 Bounds on \( \beta^h_\pi \).** The non vanishing trees contributing to \( \beta^h_\pi \) must have at least one vertex on scale \( \geq h_1^* \): in fact the diagrams depending only on the vertices of type \( \pi \) are vanishing (they are chains, so they are vanishing, because of the compact support property of the propagator). This means that, by the short memory property, see the Remark at the end of Appendix A4: 
\[
|\beta^h_\pi| \leq c|\lambda|^{\gamma(h-h_1^*)}.
\]

**A6.3 Fixing the counterterm.** We now proceed as in Appendix A5 but the analysis here is easier, because no \( \lambda \) end–points can appear and the bound \( |\beta^h_\pi| \leq c|\lambda|^{\gamma(h-h_1^*)} \) holds. As in Appendix A5, we can formally consider the flow equation up to \( h = -\infty \), even if \( h_1^* \) is a finite integer. This is because the beta function is independent of \( m^{(2)}_k \), \( k \leq h_1^* \) and admits bounds uniform in \( h \). If we want to fix the counterterm \( \pi_{h_1^*} \) in such a way that \( \pi^-\infty = 0 \), we must have, for any \( h \leq h_1^* \):
\[
\pi_h = - \sum_{k \leq h} \gamma^{k-h-1} \beta^k_\pi(\pi_k, \ldots, \pi_{h_1^*}). \tag{A6.1}
\]
Let $\mathfrak{M}$ be the space of sequences $\pi = \{\pi_{-\infty}, \ldots, \pi_{h^*}\}$ such that $|\pi_h| \leq c|\lambda|\gamma^{-(\theta/2)(h-h^*)}$. We look for a fixed point of the operator $T : \mathfrak{M} \to \mathfrak{M}$ defined as:

$$
(T\pi)_h = - \sum_{k \leq h} \gamma^{k-h-1} \beta^k_{\pi}(\pi_k; \ldots; \pi_{h^*}).
$$

(A6.2)

Using that $\beta^k_{\pi}$ is independent from $\hat{m}_{h^*}^{(2)}$ and the bound on the beta function, choosing $\lambda$ small enough and proceeding as in the proof of Theorem 4.1, we find that $T$ is a contraction on $\mathfrak{M}$, so that we find a unique fixed point, and the first of (5.16) follows.

**A6.4 The flows of $Z_h$ and $\hat{m}_h^{(2)}$.** Once that $\pi_{h^*}$ is fixed via the iterative procedure of §A6.3, we can study in more detail the flows of $Z_h$ and $\hat{m}_h^{(2)}$ given by (5.10). Note that $z_h$ and $s_h$ can be again expressed as a sum over tree diagrams and, as discussed for $\beta^k_{\pi}$, see §A6.2, any non-vanishing diagram must have at least one vertex on scale $\geq h^*_1$. Then, by the short memory property, see §A4.6, we have $z_h = O(\lambda^2 \gamma^{(h-h^*_1)})$ and $s_h = O(\lambda^2 m_h^{(2)} \gamma^{(h-h^*_1)})$ and, repeating the proof of Lemma 4.1, we find the second and third of (5.16).

**A6.5 The Lipschitz property (5.17).** Clearly, $\pi_{h^*_1}(\lambda, \sigma_1, \mu_1) - \pi_{h^*_1}(\lambda, \sigma'_1, \mu'_1)$ can be expressed via a tree expansion similar to the one discussed above; in the trees with non-vanishing value, there is either a difference of propagators at scale $h \geq h^*_1$ with couplings $\sigma_h, \mu_h$ and $\sigma'_h, \mu'_h$, giving in the dimensional bounds an extra factor $O(|\sigma_h - \sigma'_h| \gamma^{-h})$ or $O(|\mu_h - \mu'_h| \gamma^{-h})$; or a difference of propagators at scale $h \leq h^*_1$ (computed by definition at $\hat{m}_h^{(2)} = 0$) with the “corrections” $a_{\nu}^{h_1}, c_h$ associated to $\sigma_1, \mu_1$ or $\sigma'_1, \mu'_1$, giving in the dimensional bounds an extra factor $O(|\sigma_1 - \sigma'_1|)$. Then,

$$
\pi_{h^*_1}(\lambda, \sigma_1, \mu_1) - \pi_{h^*_1}(\lambda, \sigma_1', \mu_1') \leq c|\lambda| \sum_{k \leq h^*_1} \gamma^{k-h^*_1-1} \sum_{h \geq h^*_1} \left( \frac{|\sigma_h - \sigma'_h|}{\gamma_h} + \frac{|\mu_h - \mu'_h|}{\gamma_h} \right) + \sum_{k \leq h \leq h^*_1} (|\sigma_1 - \sigma_1'| + |\mu_1 - \mu_1'|),
$$

(A6.3)

from which, using (A5.18) and (A5.19), we easily get (5.17).

---

**Appendix A7. Proof of (A4.10)**

We have, by definition $\text{Pf} G = (2k!)^{-1} \sum_p (-1)^p G_{p(1)p(2)} \cdots G_{p(2k-1)p(2k)}$, where $p = (p(1), \ldots, \ldots, p(|J|))$ is a permutation of the indices $f \in J$ (we suppose $|J| = 2k$) and $(-1)^p$ its sign.

If we apply $S_1 = 1 - \text{P}_f G$ and we call $G^0_{f,f'} = \text{P}_f G_{f,f'}$, we find that $S_1 \text{Pf} G$ is equal to

$$
\frac{1}{2k!} \sum_p (-1)^p \left[ G_{p(1)p(2)} \cdots G_{p(2k-1)p(2k)} - G^0_{p(1)p(2)} \cdots G^0_{p(2k-1)p(2k)} \right] = \frac{1}{2k!} \sum_p (-1)^p \sum_{j=1}^k \left[ G^0_{p(1)p(2)} \cdots G^0_{p(2k-1)p(2k)} S_1 G_{p(1)p(2)} \cdots G_{p(2j-1)p(2j-2)} \right],
$$

(A7.1)

where in the last sum the meaningless factors must be put equal to 1. We rewrite the two sums over $p$ and $j$ in the following way:

$$
\sum_p \sum_{j=1}^k \sum_{f_1, f_2} \sum_{J_1, J_2} \sum_p \sum_{j_1, j_2}^{**} = \sum_{j=1}^k \sum_{f_1, f_2} \sum_{J_1, J_2} \sum_p \sum_{j_1, j_2}^{**},
$$

(A7.2)

where: the ** on the second sum means that the sets $J_1$ and $J_2$ are s.t. $(f_1, f_2, J_1, J_2)$ is a partition of $J$; the * on the second sum means that $p(1), \ldots, p(2j - 2)$ belong to $J_1$, $(p(2j - 1), p(2j)) = (f_1, f_2)$.
and \(p(2j + 1), \ldots, p(2k)\) belong to \(J_2\). Using (A7.2) we can rewrite (A7.1) as

\[
S_1 \text{Pf } G = \frac{1}{2^k k!} \sum_{j_1, j_2} (-1)^{\pi} S_1 G_{f_1, f_2} \sum_{J_1, J_2} \left( (-1)^{p_1 + p_2} \left( G_{p_1(1)}^{f_1(1)} \cdots G_{p_1(2k_1-1)}^{f_1(2k_1)} \right) \left( G_{p_2(1)}^{f_2(1)} \cdots G_{p_2(2k_2-1)}^{f_2(2k_2)} \right) \right),
\]

(A7.3)

where: \((-1)^{\pi}\) is the sign of the permutation leading from the ordering \(J\) to the ordering \((f_1, f_2, J_1, J_2)\); \(p_i, i = 1, 2\) is a permutation of the labels in \(J_i\) (we suppose \(|J_i| = 2k_i\)) and \((-1)^{p_i}\) is its sign. It is clear that (A7.3) is equivalent to (A4.10).

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