Strong Partially Greedy Bases and Lebesgue-Type Inequalities

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Received: 14 February 2020 / Revised: 19 August 2020 / Accepted: 10 September 2020 / Published online: 8 April 2021
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Abstract
In this paper, we continue the study of Lebesgue-type inequalities for greedy algorithms. We introduce the notion of strong partially greedy Markushevich bases and study the Lebesgue-type parameters associated with them. We prove that this property is equivalent to that of being conservative and quasi-greedy, extending a similar result given in Dilworth et al. (Constr Approx 19:575–597, 2003) for Schauder bases. We also give a characterization of 1-strong partial greediness, following the study started in Albiac and Ansorena (Rev Matem Compl 30(1):13–24, 2017), Albiac and Wojtaszczyk (J Approx Theory 138:65–86, 2006).

Keywords Nonlinear approximation · Lebesgue-type inequality · Greedy algorithm · Quasi-greedy bases · Partially greedy bases

Mathematics Subject Classification 41A65 · 41A46 · 41A17 · 46B15 · 46B45

1 Introduction and Background

Throughout this paper, $X$ is a separable infinite-dimensional Banach space over the field $F = \mathbb{R}$ or $\mathbb{C}$, with a semi-normalized Markushevich basis $B = (e_n)_{n=1}^\infty$ (M-basis...
for short). That is, denoting with $\mathbb{X}^*$ the dual space of $\mathbb{X}$, $(e_n)_{n=1}^\infty$ satisfies the following conditions:

(i) $\mathbb{X} = \{e_n : n \in \mathbb{N}\}$.
(ii) There is a (unique) sequence $(e_n^*)_{n=1}^\infty \subset \mathbb{X}^*$, called biorthogonal functionals, such that $e_n^*(e_k) = \delta_{k,n}$ for all $k, n \in \mathbb{N}$.
(iii) If $e_n^*(x) = 0$ for all $n$, then $x = 0$.
(iv) There exists a constant $c > 0$ such that $\sup_n \{\|e_n\|, \|e_n^*\|\} \leq c$.

Clearly, the semi-normalized condition of the basis is guaranteed by (ii) and (iv). Under the above assumptions, every $x \in \mathbb{X}$ is associated with a formal series $x \sim \sum_{n=1}^\infty e_n^*(x)e_n$, so that $\lim_n e_n^*(x) = 0$ and its coefficients $(e_n^*(x))_{n=1}^\infty$ are uniquely determined. As usual, we denote by $\text{supp}(x)$ the support of $x \in \mathbb{X}$, that is the set $\{n \in \mathbb{N} : e_n^*(x) \neq 0\}$. For $A$ and $B$ subsets of $\mathbb{N}$, we write $A < B$ to mean that $\max A < \min B$. If $m \in \mathbb{N}$, we write $m < A$ and $A < m$ for $\{m\} < A$ and $A < \{m\}$, respectively. Also, $A \cup B$ means the union of $A$ and $B$ with $A \cap B = \emptyset$.

We recall a few standard notions about greedy algorithms. For further information on the subject, we refer the reader to the monograph by Temlyakov [16] and to the articles [9,14,15,17] and the reference therein. For Lebesgue-type inequalities see, for instance, the more recent papers [4,6,10,12,13].

Given $x \in \mathbb{X}$, a greedy set for $x \in \mathbb{X}$ of order $m$ (or an $m$-greedy set for $x$) is a set of indices $A \subset \mathbb{N}$ such that $|A| = m$ and

$$\min_{n \in A} |e_n^*(x)| \geq \max_{n \notin A} |e_n^*(x)|.$$ 

A greedy operator of order $m$ is any mapping $G_m : \mathbb{X} \rightarrow \mathbb{X}$ such that

$$x \in \mathbb{X} \mapsto G_m(x) = G_m[B, \mathbb{X}] := \sum_{n \in A} e_n^*(x)e_n$$

with $A$ an $m$-greedy set for $x$, using the convention that $G_0 = 0$. We write $G_m$ for the set of all greedy operators of order $m$, and we consider $G = \cup_{m \geq 0} G_m$. Given $G$ and $G'$ in $G$, we write $G' < G$ whenever $G' \in G_m$ and $G \in G_n$ with $0 \leq m < n$ and the respective supporting sets satisfy $\text{supp}(G') \subset \text{supp}(G)$ (for all $x$).

Given a finite set $A \subset \mathbb{N}$, we denote by $P_A(x) = \sum_{n \in A} e_n^*(x)e_n$ the projection operator, with the convention that any sum over the empty set is zero.

For $A \subset \mathbb{N}$ finite, we denote by $\Psi_A$ the set of all collections of sequences $\varepsilon = (\varepsilon_n)_{n \in A} \subset \mathbb{F}$ such that $|\varepsilon_n| = 1$ and

$$1_{\varepsilon A} = 1_{\varepsilon A}[B, \mathbb{X}] := \sum_{n \in A} \varepsilon_n e_n.$$ 

If $\varepsilon \equiv 1$, we just write $1_A$. Also, every time we have index sets $A \subset B$ and $\varepsilon \in \Psi_B$, we write $1_{\varepsilon A}$ considering the natural restriction of $\varepsilon$ to $A$.

Given $x \in \mathbb{X}$, the error in the best $m$-term approximation with respect to the basis $B$ is

$$\sigma_m(x) = \sigma_m[B, \mathbb{X}](x) := \inf \{\|x - y\| : |\text{supp}(y)| \leq m\},$$

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and the error in the best $m$-term coordinate approximation is

$$\tilde{\sigma}_m(x) = \tilde{\sigma}_m[B, \mathcal{X}](x) := \inf \{\|x - P_B(x)\| : |B| \leq m\}.$$  

Greedy operators are frequently used for $m$-term approximations and, in order to study and quantify the performance of greedy operators one considers, for every $m = 1, 2, \ldots$, the smallest numbers $L_m = L_m[B, \mathcal{X}]$ and $\tilde{L}_m = \tilde{L}_m[B, \mathcal{X}]$ such that for all $x \in \mathcal{X}$ and all $G_m \in \mathcal{G}_m$,

$$\|x - G_m(x)\| \leq L_m \sigma_m(x), \quad (1.1)$$

and

$$\|x - G_m(x)\| \leq \tilde{L}_m \tilde{\sigma}_m(x). \quad (1.2)$$

The parameters $L_m$ and $\tilde{L}_m$ are called Lebesgue-type parameters and (1.1) and (1.2) their respective Lebesgue-type inequalities. When $L_m = O(1)$, the basis is called greedy [14], and if $\tilde{L}_m = O(1)$, the basis is called almost greedy [9]. Thanks to the main results of [9,14], we know that a basis $B$ is greedy if and only if $B$ is unconditional and democratic, and $B$ is almost greedy if and only if it is quasi-greedy (see Definition 1.2 below) and democratic. Recall that a basis $B$ is $K$-suppression-unconditional, with $K > 0$, if

$$K = K[B, \mathcal{X}] := \sup_{|A| < \infty} \|P_A\| < \infty,$$

and $B$ is $D$-democratic with $D > 0$, if for any pair of finite sets $A, B$ with $|A| \leq |B|$ we have

$$\|1_A\| \leq D \|1_B\|.$$

Different estimates for the parameters $L_m$ and $\tilde{L}_m$ have been studied in several papers: for example, in [10,12,13] the authors study estimates under the assumption of quasi-greediness, and in [4,5], for general Markushevich bases.

In order to study whether greedy approximations $(G_m(x))_m$ always perform better than the standard linear approximation $(P_m(x))_m$, the $m$th residual Lebesgue constant was introduced in [10] as follows: $L_r^e_m = L_r^e_m[B, \mathcal{X}]$ is the smallest number such that for all $x \in \mathcal{X}$ and $G_m \in \mathcal{G}_m$,

$$\|x - G_m(x)\| \leq L_r^e_m \|x - P_m(x)\|,$$

where $P_m$ is the $m$th partial sum, that is, $P_m(x) = P_A(x)$ for $A = \{1, \ldots, m\}$.

If $C_p := \sup_m L_r^e_m < \infty$, the basis is called $C_p$-partially greedy. Here, following the spirit of the inequalities (1.1) and (1.2), and using the definition of $w$-partially
greedy bases given in [7], we introduce the strong residual Lebesgue-type parameter as follows: given \( x \in X \) and \( m \in \mathbb{N} \), we define the \( m \)th strong residual error as

\[
\hat{\sigma}_m(x) = \hat{\sigma}_m[\mathcal{B}, X](x) := \inf_{0 \leq k \leq m} \| x - P_k(x) \|.
\]

Then, for each \( m = 1, 2, \ldots \), we define the strong residual Lebesgue-type parameter as the smallest number \( \hat{L}_m = \hat{L}_m[\mathcal{B}, X] \) such that for all \( x \in X \) and \( G_m \in \mathcal{G}_m \),

\[
\| x - G_m(x) \| \leq \hat{L}_m \hat{\sigma}_m(x).
\]

**Definition 1.1** An M-basis \( \mathcal{B} \) is \( C_{sp} \)-strong partially greedy if

\[
C_{sp} := \sup_m \hat{L}_m < \infty.
\]

Clearly, \( L_{re}^m \leq \hat{L}_m \) and, if the basis is Schauder, \( \hat{L}_m \approx L_{re}^m \). Consequently, if \( \mathcal{B} \) is Schauder, the basis is partially greedy if and only if it is strong partially greedy. For quasi-greedy bases, different bounds for \( L_{re}^m \) were studied in [10].

**Definition 1.2** [17] An M-basis \( \mathcal{B} \) is quasi-greedy if \( C_q := \sup_m g_m^c < \infty \), where

\[
g_m^c := \sup_{G \in \bigcup_{k \leq m} \mathcal{G}_k} \| I - G \|.
\]

Related to the parameter \( g_m^c \), we also consider (see [4])

\[
g_m := \sup_{G \in \bigcup_{k \leq m} \mathcal{G}_k} \| G \| \quad \text{and} \quad \hat{g}_m := \sup_{G \in \bigcup_{k \leq m} \mathcal{G}_k, G' < G} \| G - G' \|.
\]

Notice that as \( \mathcal{G}_0 = \{0\} \), we may define \( g_m^c \) (and also \( g_m \)) for \( m = 0 \) being \( g_0^c = 1 \) (and \( g_0 = 0 \)). This is the only parameter we will use with \( m = 0 \). On the other hand, in the definitions of the parameters that follow, we avoid without specifying it the undesirable situations in which the denominator could be zero.

**Definition 1.3** [4,17] An M-basis \( \mathcal{B} \) is \( C_u \)-unconditional for constant coefficients if \( C_u := \sup_m \gamma_m < \infty \), where

\[
\gamma_m := \sup \left\{ \frac{\| 1_{A} \|}{\| 1_{B} \|} : A \subset B, |B| \leq m, \epsilon \in \Psi_B \right\}.
\]

**Remark 1.4** Straightforward from the above definitions, for all \( m \in \mathbb{N} \), we have the following inequalities. The nontrivial calculation in (ii) can be found in [4, Lemma 2.1].

(i) \( |g_m - g_m^c| \leq 1 \),

1 We use the notation \( \| G \| = \sup_{x \neq 0} \| G(x) \| / \| x \| \) and \( \| I - G \| = \sup_{x \neq 0} \| x - G(x) \| / \| x \| \), even if \( G : X \to X \) is a nonlinear map.
(ii) $\tilde{g}_m \leq \min\{2 \min\{g_m, g_m^c\}, g_m g_m^c\}$,

(iii) $\gamma_m \leq \min\{g_m, g_m^c\}$.

**Definition 1.5** [2] An M-basis $B$ is $C_{ql}$-quasi-greedy for largest coefficients if $C_{ql} := \sup_m q_m < \infty$, where

$$q_m := \sup \left\{ \frac{\| x + 1_{eA} \|}{\| x + 1_{eA} \|} : |A| \leq m, A \cap \text{supp}(x) = \emptyset, \max_n |e_n^*(x)| \leq 1, \varepsilon \in \Psi_A \right\}.$$ 

**Definition 1.6** An M-basis $B$ is $C_{sc}$-superconservative if $C_{sc} := \sup_m sc_m < \infty$, where $sc_m$ is the $m$th superconservative parameter of a basis defined as follows:

$$sc_m := \sup \left\{ \frac{\| x + 1_{eA} \|}{\| x + 1_{eA} \|} : |A| \leq |B| \leq m, A \leq m, A < B, \varepsilon \in \Psi_A, \varepsilon' \in \Psi_B \right\}.$$ 

When $\varepsilon \equiv \varepsilon' \equiv 1$, we write $c_m$ instead of $sc_m$ and call this constant the $m$th conservative parameter. Also, we say that $B$ is $C_c$-conservative if $C_c := \sup_m c_m < \infty$.

**Remark 1.7** Conservative (Schauder) bases were introduced in [9]: a basis is conservative with constant $C$ if $\| 1_A \| \leq C \| 1_B \|$ whenever $A$ and $B$ are finite sets with $A < B$ and $|A| \leq |B|$. For such bases, our definition of a conservative basis is equivalent, with the same constant. Indeed, if $B$ is $C_c$-conservative, by Definition 1.6, given $A < B$ finite sets with $|A| \leq |B|$, for any $m \geq B$ we have

$$\| 1_A \| \leq c_m \| 1_B \| \leq \sup_m c_m \| 1_B \| = C_c \| 1_B \|,$$

so $B$ is also $C_c$-conservative as defined in [9]. On the other hand, if $B$ is $C_c$-conservative according to the definition in [9], it is immediate that $c_m \leq C$ for all $m$, so $C_c \leq C$.

**Remark 1.8** In [10], the authors define a more restrictive $m$th conservative parameters $c(m)$. The difference between our definition and that of [10] is that in the latter, the supremum is taken over all finite sets $A$ and $B$ such that $|A| = |B| \leq m$ and $A \leq m < B$ which entails the additional conditions of taking $m < B$ and $|A| = |B|$. For the purposes of this article, we find the definition of $c_m$ more convenient, not only because its definition is less restrictive than that of $c(m)$, but also because (by Remark 1.7) the sequence $(c_m)$ allows us to recover the original definition of conservative bases of [9].

Finally, we introduce the following concept, which will be used to study the behavior of $\tilde{1}_m$.

**Definition 1.9** An M-basis $B$ is $C_{pl}$-partially symmetric for largest coefficients ($C_{pl}$-PSLC for short) if $C_{pl} := \sup_m \omega_m < \infty$, where

$$\omega_m := \sup \left\{ \frac{\| x + t 1_{eA} \|}{\| x + t 1_{eB} \|} : |A| \leq |B| \leq m, A < \text{supp}(x) \cup B, A \leq m, \varepsilon \in \Psi_A, \varepsilon' \in \Psi_B, |t| \geq \max_n |e_n^*(x)| \right\}.$$
Notice that $\omega_m \geq 1$ for all $m$, which is clear from the definition if we take $A = B = \emptyset$.

**Remark 1.10** Note that a slight variant of the argument of Remark 1.7 shows that a basis is $C_{pl}$-PSLC if and only if there is $K > 0$ such that for any finite sets $A$ and $B$, any $x \in X$ with $|A| \leq |B|$, $A < \text{supp}(x) \cup B$, for $|t| \geq \max_n |e_n^*(x)|$, $\varepsilon \in \Psi_A$, $\varepsilon' \in \Psi_B$,

$$\|x + t1_{eA}\| \leq K\|x + t1_{e'B}\|,$$

and $C_{pl}$ is the minimum $K$ for which this inequality holds.

The parameter $\omega_m$ is a weaker version of the parameter $\nu_m$ used to define the constant associated to Property (A) that appears in [3, 4, 11]: a basis has the $C_a$-Property (A) (or is $C_a$-symmetric for largest coefficients) if $C_a := \sup_m \nu_m < \infty$, where

$$v_m := \sup\left\{\|x + t1_{eA}\| : |A| \leq |B| \leq m, \ A \cap B = \emptyset, \ \text{supp}(x) \cap (A \cup B) = \emptyset\right\}.$$

Among the definitions given above, the parameters $g_m, g_m^c, \tilde{g}_m, q_m, \gamma_m, L_m, \tilde{L}_m$ and $L_{re}^m$ are well known in the literature and were studied, for instance, in [2, 4, 6, 9, 10, 17].

Our main results for the strong residual parameters $(\tilde{L}_m)_m$ are the following.

**Proposition 1.11** For every $m = 1, 2, \ldots,$

$$\tilde{L}_m \leq 1 + 2\kappa m,$$

where $\kappa = \sup_{m,n} \|e_m\|\|e_n^*\|$.

**Theorem 1.12** For each $m = 1, 2, \ldots,$

$$\tilde{L}_m \leq g_m^c + \tilde{g}_m sc_m.$$

**Proposition 1.13** For each $m = 1, 2, \ldots,$

$$\omega_m \leq \max_{1 \leq k \leq m} \tilde{L}_k.$$

**Theorem 1.14** For each $m = 1, 2, \ldots,$

$$g_m^c \leq \tilde{L}_m \leq g_{m-1}^c \omega_m.$$

Moreover, $\tilde{L}_1 = \omega_1$.

On the other hand, we characterize strong partially greedy bases in terms of partial greediness and quasi-greediness and 1-strong partially greedy bases in terms of the 1-PSLC property. Also, we prove that 1-partially greedy bases are strong partially greedy.
Theorem 1.15  An M-basis $\mathcal{B}$ is 1-strong partially greedy if and only if $\mathcal{B}$ is 1-partially symmetric for largest coefficients.

The above result is an improvement with respect to Theorem 1.14, since it allows us to deduce that PSLC with $C_{pl} = 1$ implies quasi-greediness with $C_q = 1$.

The paper is structured as follows: in Sect. 2, we give some basic results that will be used later. In Sect. 3, we prove Theorem 1.12 and the respective corollaries, whereas in Sect. 4, we prove Theorem 1.15 and give a characterization of 1-PSLC bases. In Sect. 5, we discuss the relation between the concepts of partial greediness and strong partial greediness and extend [9, Theorem 3.2] to the context of Markushevich bases.

2 Preliminary Results

This section is devoted to giving some different estimates of the parameters $\omega_m$ and $sc_m$. In order to do so, we recall the definition of the truncation operator, which was first considered in [8]. We use some of its properties that connect this operator with the quasi-greedy parameter. Also, we appeal to several lemmas given in [4] that we list below for the sake of the reader.

2.1 Truncation Operator

For each $t > 0$, we define the $t$-truncation of $x \in \mathbb{F}$ by

$$T_t(x) = \begin{cases} \sign(x) & \text{if } |x| > t, \\ x & \text{if } |x| \leq t. \end{cases}$$

We can extend $T_t$ to an operator in $\mathbb{X}$ - which we still call $T_t$ - by formally assigning $T_t(x) \sim \sum_{n=1}^{\infty} T_t(e_n^*(x))e_n$, that is,

$$T_t(x) := t1_{\Lambda_t(x)} + (I - P_{\Lambda_t(x)})(x),$$

where $\epsilon = \{\sign(e_n^*(x))\}$ and $\Lambda_t(x) := \{n \in \mathbb{N} : |e_n^*(x)| > t\}$ is the $t$-index set for $x$ associated to $T_t$. Since $\Lambda_t(x)$ is finite, $T_t : \mathbb{X} \to \mathbb{X}$ is well defined.

Lemma 2.1  [4, Lemma 2.3] Let $x \in \mathbb{X}$ and $\epsilon = \{\sign(e_n^*(x))\}$. For each $m$-greedy set $A$ of $x$,

$$\min_{n \in A} |e_n^*(x)| \|1_{e_A}\| \leq \tilde{g}_m \|x\|.$$

With the notation above, we also have:

Lemma 2.2  [4, Lemma 2.5] For all $t > 0$ and $x \in \mathbb{X}$,

$$\|T_t(x)\| \leq g_{|\Lambda_t(x)|} \|x\|.$$
Moreover, if \( \Lambda_t(x) = \emptyset \), the equality is attained as \( T_t(x) = x \), \( |\Lambda_t(x)| = 0 \) and \( g^c_0 = 1 \).

Given a subset \( E \subset \mathbb{X} \), its convex hull is denoted by \( \text{co}(E) \).

**Lemma 2.3** [4, Lemma 2.7] For every finite set \( A \subset \mathbb{N} \), we have

\[
\text{co}(\{1_{\varepsilon A} : \varepsilon \in \Psi_A\}) = \left\{ \sum_{n \in A} z_n \varepsilon \varepsilon_n : |z_n| \leq 1 \right\}.
\]

Now we focus on the task of giving estimates of the parameters \( s_{c_m} \) and \( \omega_m \) in terms of known parameters appearing in [2–4, 11]. Our estimates allow us to characterize bases which are PSLC (see Corollary 2.8).

**Proposition 2.4** For each \( m = 1, 2, \ldots, \)

\[
c_m \leq s_{c_m} \leq 4 \nu^2 \gamma_m c_m,
\]

where \( \nu = 1 \) or 2 if \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \), respectively.

**Proof** The lower bound for \( s_{c_m} \) is trivial by definition. To prove the upper bound, take \( A < B, A \leq m, |A| \leq |B| \leq m \) and \( \varepsilon \in \Psi_A, \varepsilon' \in \Psi_B \). Assume first that \( \varepsilon \in \{ \pm 1 \}^{|A|} \). Then, if \( A^+ = \{ n \in A : \varepsilon_n = 1 \} \) and \( A^- = A \setminus A^+ \),

\[
\|1_{\varepsilon A}\| \leq \|1_{A^+}\| + \|1_{A^-}\| \leq c_m \|1_B\| + c_m \|1_B\| = 2c_m \|1_B\| \leq 4c_m \gamma_m \nu \|1_{\varepsilon' B}\|,
\]

where the last inequality follows by [4, Lemma 3.3]. If \( \mathbb{F} = \mathbb{R} \), the proof is complete. For \( \varepsilon \notin \{ \pm 1 \}^{|A|} \), by convexity we have

\[
\max \left\{ \left\| \sum_{n \in A} \Re(\varepsilon \varepsilon_n) \varepsilon_n \right\|, \left\| \sum_{n \in A} \Im(\varepsilon \varepsilon_n) \varepsilon_n \right\| \right\} \leq \max_{\eta \in \{ \pm 1 \}^{|A|}} \|1_{\eta A}\|.
\]

Thus, by the triangle inequality we get \( \|1_{\varepsilon A}\| \leq 2\|1_{\eta A}\| \) for some \( \eta \in \{ \pm 1 \}^{|A|} \), and the result follows by (2.1).

**Remark 2.5** Proposition 2.4 shows that the superconservative parameter can be controlled by the parameter \( \gamma_m \) of Definition 1.3 and the conservative parameter. This implies that a basis that is conservative and unconditional for constant coefficients is superconservative. We do not know whether every superconservative basis is unconditional for constant coefficients, but as Example 2.6 shows, the parameter \( \gamma_m \) cannot be controlled by \( s_{c_m} \). As usual, \( c_{00} \) denotes the space of sequences of finite support.

**Example 2.6** Let \( \mathbb{X} \) be the completion of \( c_{00} \) under the norm

\[
\|(a_n)_n\| := \sup_{D \in \mathcal{S}} \left| \sum_{n \in D} a_n \right|,
\]
where

\[ S := \left\{ D \subset \mathbb{N} : 0 < |D| \leq \sqrt{\min D}, \text{ and } i < j < k, i \in D, k \in D \implies j \in D \right\}. \]

There is no constant \( K \) such that

\[ \gamma_m \leq K \text{sc}_m, \quad \forall m \in \mathbb{N}. \]

**Proof** It is clear that the canonical basis \((e_n)_n\) is a normalized Schauder basis for \( X \).

Note that if \( D \in S \), then \( D \) is a nonempty interval in \( \mathbb{N} \), that is, either \( D = \{ m \} \) for some \( m \in \mathbb{N} \), or \( D = \{ m, m+1, \ldots, m+n \} \) for some \( m, n \in \mathbb{N} \). Moreover, any nonempty interval \( D' \subset D \) is also an element of \( S \). This guarantees that for every \( x \) with finite support,

\[
\|x\| = \max_{D \in S} \left\{ \left\| \sum_{n=\min D}^{\max D} e_n^*(x) \right\| \right\},
\]

where \((e_n^*)_n\) are the biorthogonal functionals corresponding to \((e_n)_n\). In particular, the basis is monotone. Note also that, since \( \{n\} \in S \) for all \( n \in \mathbb{N} \), for all \( x \in X \) we have

\[
\|x\| \geq \sup_{n \in \mathbb{N}} |e_n^*(x)| = \|x\|_\infty.
\]

Now fix \( m \in \mathbb{N} \), a finite set \( A \) with \( A \leq m \), and \( \varepsilon \in \Psi_A \). Then, for any \( D \in S \) such that \( D \leq \max A \), we have

\[
|D| \leq \sqrt{\min D} \leq \sqrt{\max A} \leq \sqrt{m}.
\]

Thus,

\[
\|1_{\varepsilon A}\| \leq \sup_{D \in S, D \leq m} \left\{ \sum_{n \in D} |e_n^*(1_{\varepsilon A})| \right\} \leq \sqrt{m}.
\]

Given that for any finite set \( B \) and any \( \varepsilon' \in \Psi_B \),

\[
1 = \|1_{\varepsilon'B}\|_\infty \leq \|1_{\varepsilon'B}\|,
\]

it follows that

\[
\text{sc}_m \leq \sqrt{m}, \quad \forall m \in \mathbb{N}.
\]

Now fix \( m \in \mathbb{N} \), and let

\[
B := \{4m^2 + 1, \ldots, 4m^2 + 2m\}, \quad A := \{n \in B : n \text{ is even}\}.
\]
Clearly, \(|B| = 2m, B \in S, \) and \(|A| = m\). Now define \(\varepsilon \in \Psi_B\) as follows

\[
\varepsilon_n := (-1)^n \forall n \in B.
\]

Since \(B\) is an interval, for any \(D \in S\) with \(4m^2 + 1 \leq D \leq 4m^2 + 2m\) we have

\[
\max_D \left| \sum_{n=\min D}^{\max D} e_n^*(1_{\varepsilon B}) \right| = \max_D \left| \sum_{n=\min D}^{\max D} (-1)^n \right| \leq 1 = \|1_{\varepsilon B}\|_{\infty}.
\]

Hence,

\[
\|1_{\varepsilon B}\| = 1.
\]

On the other hand, since \(B \in S\) and \(A \subset B\),

\[
\|1_{\varepsilon A}\| = \sup_{D \in S} \left\{ \left| \sum_{n \in D} e_n^*(1_{\varepsilon A}) \right| \right\} \geq \left| \sum_{n \in B} e_n^*(1_{\varepsilon A}) \right| = \left| \sum_{n \in A} e_n^*(1_{\varepsilon A}) \right| = \sum_{n \in A} 1 = m.
\]

Thus,

\[
\gamma_{2m} \geq \frac{\|1_{\varepsilon A}\|}{\|1_{\varepsilon B}\|} = m,
\]

and the assertion is proved. \(\square\)

**Proposition 2.7** For each \(m = 1, 2, \ldots\),

\[
\max \left\{ \frac{q_m}{2}, sc_m \right\} \leq \omega_m \leq 1 + q_m(1 + sc_m).
\]

**Proof** First, we prove the lower bound. It is clear that \(sc_m \leq \omega_m\). To prove that \(q_m \leq 2\omega_m\), take \(A, x\) and \(\varepsilon\) as in the definition of \(q_m\). Then,

\[
\|1_{\varepsilon A}\| \leq \|x + 1_{\varepsilon A}\| + \|x\| \leq \|x + 1_{\varepsilon A}\| + \omega_m\|x + 1_{\varepsilon A}\|
\]

\[
\leq 2\omega_m\|x + 1_{\varepsilon A}\|.
\]

To prove the upper bound, let \(A, B, x, t, \varepsilon\) and \(\varepsilon'\) be as in the definition of \(\omega_m\), that is, \(A < \text{supp}(x) \cup B, \ |A| \leq |B| \leq m, A \leq m, t \geq \max_n |e_n^*(x)|\) and \(\varepsilon \in \Psi_A, \varepsilon' \in \Psi_B\). We have

\[
\|x + t1_{\varepsilon A}\| \leq \|x + t1_{\varepsilon' B}\| + \|t1_{\varepsilon A}\| + \|t1_{\varepsilon' B}\|
\]

\[
\leq \|x + t1_{\varepsilon' B}\| + sc_m \|t1_{\varepsilon' B}\| + q_m\|x + t1_{\varepsilon' B}\|
\]

\[
\leq (1 + q_m + q_msc_m)\|x + t1_{\varepsilon' B}\|,
\]

which concludes the proof. \(\square\)
**Corollary 2.8** An $M$-basis $\mathcal{B}$ in a Banach space $X$ is partially symmetric for largest coefficients if and only if $\mathcal{B}$ is superconservative and quasi-greedy for largest coefficients.

**Proof** The lower bound of Proposition 2.7 shows that a basis that is $C_{pl}$-PSLC is also superconservative with constant $C_{sc} \leq C_{pl}$ and quasi-greedy for largest coefficients with constant $C_{ql} \leq 2C_{pl}$. On the other hand, if the basis is $C_{sc}$-superconservative and $C_{ql}$-quasi-greedy for largest coefficients, applying the upper bound of Proposition 2.7 we obtain

$$C_{pl} = \sup_m \omega_m \leq 1 + C_{ql}(1 + C_{sc}).$$

Hence, the basis is PSLC. □

### 3 Main Results for $\hat{L}_m$

In this section, we prove the main results concerning $\hat{L}_m$.

**Proof of Proposition 1.11:** The proof of this result is immediate using that by definition

$$\hat{L}_m \leq \tilde{L}_m,$$

and also that $\tilde{L}_m \leq 1 + 2\kappa m$, by [4, Theorem 1.8]. □

**Proof of Theorem 1.12:** Take $x \in X$, fix $m$ and take a greedy operator of order $m$, $G_m$. Set $A := \text{supp}(G_m(x))$, take $k \leq m$ and define $D := \{1, \ldots, k\}$. We have the following possibly trivial decomposition:

$$x - G_m(x) = P_{(A \cup D)^c}(x) + P_{D \setminus A}(x).$$

On the one hand, since

$$P_{(A \cup D)^c}(x) = (I - P_{A \setminus D})(x - P_k(x)),$$

we have

$$\|P_{(A \cup D)^c}(x)\| \leq g_m \|x - P_k(x)\|. \quad (3.1)$$

On the other hand, notice that $D \setminus A \leq k < A \setminus D$ and $A \setminus D$ is a greedy set for $x - P_k(x)$. Thus, applying first Lemma 2.3 and then Lemma 2.1 with $e = (\text{sign}(e_n^*(x)))_n$, we obtain
\[ \|P_{D\setminus A}(x)\| \leq \text{sc}_m \max_{n \in D \setminus A} |e_n^*(x)| \|1_{\epsilon(A \setminus D)}\| \]
\[ \leq \text{sc}_m \min_{n \in A \setminus D} |e_n^*(x - P_k(x))| \|1_{\epsilon(A \setminus D)}\| \]
\[ \leq \tilde{g}_{|A \setminus D|}\text{sc}_m \|x - P_k(x)\| \]
\[ \leq \tilde{g}_m \text{sc}_m \|x - P_k(x)\|. \quad (3.2) \]

As (3.1) and (3.2) hold for any \( k \leq m \), a direct combination of both inequalities shows that \( \|x - G_m(x)\| \leq (g_m^c + \tilde{g}_m \text{sc}_m) \hat{\sigma}_m(x) \), and therefore, the result follows. \( \square \)

**Corollary 3.1** If an M-basis is \( C_q \)-quasi-greedy, then for each \( m = 1, 2, \ldots \),
\[ \hat{\Lambda}_m \leq C_q + 2C_q \text{sc}_m. \]

**Proof** Just apply Theorem 1.12 and use the estimate \( \tilde{g}_m \leq 2C_q \) for all \( m \in \mathbb{N} \). \( \square \)

**Proof of Proposition 1.13:** Take \( A, B, x, \epsilon, \epsilon' \) as in the definition of \( \omega_m \). A careful look at this definition allows us to only consider \( t > 0 \). Now, let \( m_1 := \max A \). Since \( |A| \leq |B| \leq m, A \leq m_1 \leq m \) and \( A < B \), there exists a possibly empty set \( D \) such that \( D \subset \{1, \ldots, m_1\} \setminus A \) and \( m_1 \leq |D \cup B| \leq m \). Let \( m_2 := |D \cup B| \) and define, for \( \eta > 0 \), the element
\[ y := x + t1_{\epsilon A} + (t + \eta)(1_{\epsilon'B} + 1_D). \]

As in particular \( A \cap \text{supp}(x) \) is the empty set and \( t \geq \max_n |e_n^*(x)| \), we see that \( G_{m_2}(y) = (t + \eta)(1_{\epsilon'B} + 1_D) \) and \( P_{m_1}(y) = t1_{\epsilon A} + (t + \eta)1_D \), then
\[ \|x + t1_{\epsilon A}\| = \|y - G_{m_2}(y)\| \leq \hat{\Lambda}_{m_2} \|y - P_{m_1}(y)\| \leq \max_{1 \leq k \leq m} \hat{\Lambda}_k \|x + (t + \eta)1_{\epsilon'B}\|. \]

Since the inequality holds for any \( \eta > 0 \), we conclude that \( \omega_m \leq \max_{1 \leq k \leq m} \hat{\Lambda}_k. \) \( \square \)

The following lemma will be used to prove Theorem 1.14. It is based on [4, Lemma 2.8], but stated for the parameter \( \omega_m \) instead of \( \nu_m \). We give a proof for the sake of completeness.

**Lemma 3.2** Let \( x \in X \) and \( |t| \geq \max_n |e_n^*(x)| \). Then,
\[ \|x + z\| \leq \omega_m \|x + t1_{\epsilon B}\|, \]
for any finite set \( B, |B| \leq m, \) any \( \epsilon \in \Psi_B, \) and any \( z \) such that \( \text{supp}(z) \leq m, \) \( \text{supp}(z) \subset B \cup \text{supp}(x) \) and \( |t| \geq \max_n |e_n^*(z)|. \)

\( \square \) Springer
Proof Notice that by the definition of $t \mathbf{1}_{\varepsilon B}$ with $\varepsilon \in \Psi_B$, it is enough to give a proof for $t > 0$. By the definition of the parameter $\omega_m$, the result is true if we take $z = \mathbf{1}_{\tilde{\varepsilon} A}$, for any $\tilde{\varepsilon} \in \Psi_{A}$, $A < \text{supp}(x) \cup B$, $A \leq m$, and $|A| \leq |B| \leq m$. Thanks to the convexity of the norm, it continues to be true for any element $z \in \text{co} \left( \{ \mathbf{1}_{\tilde{\varepsilon} A} : \tilde{\varepsilon} \in \Psi_{A} \} \right)$. Then, the general case follows from Lemma 2.3.

Proof of Theorem 1.14: Taking $k = 0$ in the definition of $\tilde{\sigma}_m(x)$, we see that $\tilde{\sigma}_m(x) \leq \|x\|$ for all $x$. Then, the inequality $g_{m}^{c} \leq \tilde{L}_m$ is immediate for all $m \in \mathbb{N}$.

To prove the upper bound, fix $m \in \mathbb{N}, x \in \mathbb{K}$ and a greedy operator of order $m$, $G_m$.

We will show that $\|x - G_m(x)\| \leq g_{m-1}^{c} \omega_m \|x - P_k(x)\|$ for all $k \leq m$.

Set $A := \text{supp}(G_m(x))$, fix $k \leq m$ and take $D := \{1, \ldots, k\}$. If $A = D$, then $k = m$ and $G_m(x) = P_m(x)$; as $g_{m-1}^{c}, \omega_m \geq 1$ there is nothing to prove. If $A \neq D$, consider the following decomposition:

$$x - G_m(x) = P_{(A \cup D)^{c}}(x - P_k(x)) + P_D \setminus A(x). \quad (3.3)$$

Applying Lemma 3.2 with $t = \min_{n \in A} |\varepsilon_n^{*}(x)|$, $z = P_D \setminus A(x)$, and $\varepsilon \in \Psi_{A \setminus D}$ such that $\varepsilon_n = \text{sign}(\varepsilon_n^{*}(x))$, we have

$$\|P_{(A \cup D)^{c}}(x - P_k(x)) + P_D \setminus A(x)\| \leq \omega_m \|P_{(A \cup D)^{c}}(x - P_k(x)) + t \mathbf{1}_{\varepsilon A \setminus D}\|. \quad (3.4)$$

Let $T_t$ be the $t$-truncation operator and $\Lambda_t$ be its associated $t$-index set. Notice that $|\Lambda_t(x - P_k(x))| \leq m - 1$ and

$$P_{(A \cup D)^{c}}(x - P_k(x)) + t \mathbf{1}_{\varepsilon A \setminus D} = T_t(x - P_k(x)).$$

Thus, an application of Lemma 2.2 yields

$$\|P_{(A \cup D)^{c}}(x - P_k(x)) + t \mathbf{1}_{\varepsilon A \setminus D}\| \leq g_{|\Lambda_t(x - P_k(x))|}^{c} \|x - P_k(x)\| \leq g_{m-1}^{c} \|x - P_k(x)\|. \quad (3.5)$$

Hence, the upper bound in the statement follows combining (3.3), (3.4) and (3.5). Finally, by Proposition 1.13 we have $\omega_1 \leq \tilde{L}_1$, which completes the proof.

Remark 3.3 Note that the sequence $(g_{m}^{c})_m$ is increasing, so from Theorem 1.14 we also get, for all $m \in \mathbb{N}$,

$$g_{m}^{c} \leq \tilde{L}_m \leq g_{m}^{c} \omega_m.$$

We end this section presenting two examples that allow us to study the optimality of the inequalities of the main results about $\tilde{L}_m$. First, note that if $B$ is the unit vector basis of $c_0$ or $\ell_p$ with $1 \leq p < \infty$, we have $g_{m}^{c} = 1$ and $\omega_m = 1$ for all $m \in \mathbb{N}$. Hence, equality holds throughout in Theorem 1.14. With the next example, we show that equality holds in Propositions 1.13 and 1.11.
Example 3.4 Let $\mathbb{X}$ be the completion of $c_{00}$ under the norm

$$
\|(a_n)_n\| := \sup_{n \geq 1} \left| \sum_{j=1}^{n} a_j \right|.
$$

The canonical basis of $\mathbb{X}$, $(e_n)_n$, is monotone, and the following hold for all $m \in \mathbb{N}$:

(i) $g_m = \tilde{g}_m = 2m$ and $g_m^\ast = 2m + 1$.
(ii) $\mathbf{s}c_m = m$ and $c_m = 1$.
(iii) $\omega_m = \hat{L}_m = \max_{1 \leq k \leq m} \hat{L}_k = 1 + 4m$.

Then, equality holds in Propositions 1.11 and 1.13.

Proof A proof of (i) can be found in [4, Proposition 5.1]. To see (ii), notice that for every finite set $A$,

$$
\|1_A\| = |A|.
$$

Hence, $c_m = 1$. Also, notice that for $\mathbf{e} \in \Psi_A$ we have

$$
1 \leq \|1_{\mathbf{e}A}\| \leq |A|.
$$

Then, for $A, B, \mathbf{e}$ and $\mathbf{e}'$ as in the definition of $\mathbf{s}c_m$, we have

$$
\frac{\|1_{\mathbf{e}A}\|}{\|1_{\mathbf{e}'B}\|} \leq |A| \leq m.
$$

Finally, with $A := \{1, \ldots, m\}, B := \{m + 1, \ldots, 2m\}, \mathbf{e} \equiv 1$ and $\mathbf{e}' = ((-1)^n)_n$, and noting that $\|1_{\mathbf{e}'B}\| = 1$, we see that the bound above is attained and $\mathbf{s}c_m = m$.

Next, let us show that (iii) holds. Taking into account that $\|e_n\| = 1$ for all $n \in \mathbb{N}$, $\|e_1\| = 1$ and $\|e_n^\ast\| = 2$ for all $n \geq 2$, the constant in Proposition 1.11 is $\kappa = 2$, so we obtain that $\hat{L}_k \leq 1 + 4k$ for any $k$. Then, clearly, $\max_{1 \leq k \leq m} \hat{L}_k \leq 1 + 4m$.

To show the lower bound, consider the sets $A$ and $B$ with $|A| = |B| = m$ so that

$$
1_A := \left(1, \ldots, 1, 0, 0, \ldots \right)_{\text{m}},
$$

$$
1_B := \left(0, \ldots, 0, 0, 1, 0, \ldots, 0, 1, 0, 0, \ldots \right)_{\text{m}},
$$

and define $x \in \mathbb{X}$ with $|\text{supp}(x)| = 2m + 1$ as follows

$$
x := \left(0, \ldots, 0, \frac{1}{2}, 0, \frac{1}{2}, \ldots, 0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \ldots \right)_{\text{supp}(x)}.
$$
Then \( \|x + 1_A\| = 2m + \frac{1}{2} \) and \( \|x - 1_B\| = \frac{1}{2} \). Now, using Proposition 1.13 we get

\[
\max_{1 \leq k \leq m} \hat{L}_k \geq \omega_m \geq \frac{\|x + 1_A\|}{\|x - 1_B\|} = 1 + 4m,
\]

and the proof is complete. \(\Box\)

Next, we prove the optimality of the estimate of Theorem 1.12 and the right-hand side of the inequality in Theorem 1.14. Additionally, we give another example in which equality in Proposition 1.13 holds.

**Example 3.5** Let \((e_n)_n\) and \((f_n)_n\) be the unit vector bases of \(\ell_1\) and \(c_0\), respectively, and let \(X\) be the space \(\ell_1 \times c_0\) with the norm

\[
\|(x, y)\| := \max\{\|x\|_1, \|y\|_\infty\}.
\]

For each \(m \in \mathbb{N}\), define

\[
x_{2m-1} := (e_m, 0), \quad x_{2m} := (0, f_m).
\]

Then, \((x_n)_n\) is a Schauder basis for \(X\), and the following hold for all \(m \in \mathbb{N}\):

(i) \(g_m = \tilde{g}_m = g^c_m = 1\).
(ii) \(s_{c2m} = c_{2m} = sc_{2m-1} = c_{2m-1} = m\).
(iii) \(\omega_{2m} = \omega_{2m-1} = \hat{L}_{2m-1} = \hat{L}_{2m} = m + 1\).

Thus, equalities in Theorem 1.12 and Proposition 1.13 hold, and also the right-hand side of the inequality in Theorem 1.14.

**Proof** It is easy to check that \((x_n)_n\) is a 1-suppression-unconditional basis for \(X\), so (i) holds trivially. To prove (ii), fix \(m \in \mathbb{N}\) and take any nonempty set \(A\), with \(A \leq 2m\) and \(e \in \Psi_A\). Define \(A_e := \{j \in A : j \text{ is even}\}\) and \(A_o := A \setminus A_e\). We have

\[
\|1_{eA}\| = \max\{|1_{eA_e}|, |1_{eA_o}|\} \leq \max\{|A_o|, 1\} \leq m.
\]

It follows that

\[
s_{c2m} \leq m. \quad (3.6)
\]

Now, define the sets

\[
A_m := \{2j - 1 : 1 \leq j \leq m\}, \quad B_m := \{2m + 2j : 1 \leq j \leq m\}.
\]

Note that for all \(m\),

\[
|A_m| = |B_m| = m, \quad A_m \leq 2m - 1 < 2m + 2 \leq B_m, \quad \|1_{A_m}\| = m, \quad \|1_{B_m}\| = 1.
\]

In particular, this immediately gives

\[
c_{2m-1} \geq m. \quad (3.7)
\]
Since $c_n \leq \min\{c_{n+1}, sc_n\} \leq sc_{n+1}$ for all $n \in \mathbb{N}$, combining (3.6) and (3.7) we obtain (ii). Finally, to prove (iii) first notice that by (i), (ii) and Theorem 1.12, for all $m$ it follows that

$$
\hat{L}_{2m-1} \leq m + 1, \quad \hat{L}_{2m} \leq m + 1.
$$

Given that $(\omega_n)_n$ is an increasing sequence, (3.8) and Proposition 1.13 yield for all $m \in \mathbb{N}$,

$$
\omega_{2m-1} \leq \omega_{2m} \leq \max_{1 \leq k \leq 2m} \hat{L}_k \leq m + 1.
$$

(3.9)

Considering $A_m$ and $B_m$ as before, with $\epsilon' \in \Psi_{B_m}$ we have

$$
\|1_{A_m} + x_{2m+1}\| = \|\sum_{j=1}^{m+1} (e_j, 0)\| = m + 1,
$$

(3.10)

whereas

$$
\|1_{\epsilon'B_m} + x_{2m+1}\| = \max\{\|e_{m+1}\|_1, \|\sum_{j=1}^{m} \epsilon'_{2m+2j} f_{m+j}\|_\infty\} = 1.
$$

Thus, by the choice of $A_m$ and $B_m$, it follows that for all $m \in \mathbb{N}$,

$$
\omega_{2m-1} \geq m + 1.
$$

(3.11)

Now take $\eta > 0$, and let

$$
y_m := 1_{A_m} + x_{2m+1} + (1 + \eta)1_{B_{2m}}
$$

By (3.10), we have

$$
\|y_m - G_{2m}(y_m)\| = \|1_{A_m} + x_{2m+1}\| = m + 1,
$$

and by the 1-suppression-unconditionality of the basis,

$$
\|y_m - G_{2m-1}(y_m)\| \geq \|y_m - G_{2m}(y_m)\| = m + 1.
$$

Since

$$
\|y_m - P_{2m-1}(y_m)\| = \|x_{2m+1} + (1 + \eta)1_{B_{2m}}\| = (1 + \eta)
$$

and $\eta$ is arbitrary, it follows that

$$
\hat{L}_{2m} \geq m + 1, \quad \hat{L}_{2m-1} \geq m + 1.
$$

Combining these inequalities with (3.8), (3.9) and (3.11), we obtain (iii).  

\[\square\]
4 Characterization of 1-Strong Partially Greedy Bases

Since 2006, some authors have studied bases that have their respective greedy constants attaining the least possible value, that is bases that have constant one. For instance, Albiac and Wojtaszczyk [3] gave a characterization of 1-greedy bases and Albiac and Ansorena [1] characterized 1-almost-greediness in terms of 1-symmetry for largest coefficients (see the paragraph below Remark 1.10 for its definition). Following this spirit, in this section we prove a similar result for 1-strong partially greedy and 1-PSLC bases showing that this last condition is stronger than quasi-greediness.

Theorem 4.1 An M-basis $B$ is 1-strong partially greedy if and only if $B$ is 1-PSLC.

Proof If $\hat{L}_m = 1$ for all $m \in \mathbb{N}$, by Proposition 1.13 it follows that $\omega_m = 1$ for all $m \in \mathbb{N}$. Reciprocally, if $\omega_m = 1$ for all $m \in \mathbb{N}$, from the fact that $g_m^* \geq 1$ and Theorem 1.14 we obtain

$$\hat{L}_1 = 1, \quad 1 \leq g_m^* \leq \hat{L}_m \leq g_{m-1}^* \quad \text{for all } m \geq 2.$$ 

Thus, inductively it follows that $g_m^* = \hat{L}_m = 1$ for all $m \in \mathbb{N}$. $\square$

We also have the following characterization of 1-PSLC bases.

Proposition 4.2 Let $B = (e_n)_n$ be an M-basis for a Banach space $X$. The following are equivalent:

(i) For any finite sets $A$ and $B$ such that $|A| \leq |B|$, $x \in X$, $A < \text{supp}(x) \cup B$, $\varepsilon \in \Psi_A$, $\varepsilon' \in \Psi_B$, $t \in F$ with $|t| \geq \max_n |e^*_n(x)|$, 

$$\|x + t1_{\varepsilon A}\| \leq \|x + t1_{\varepsilon' B}\|. \quad (4.1)$$

(ii) $B$ has the 1-PSLC property.

(iii) The following conditions hold:

(I) For all $x \in X$, $t \in F$ such that $|t| \geq \max_n |e^*_n(x)|$, and for all $k \notin \text{supp}(x)$, 

$$\|x\| \leq \|x + te_k\|.$$ 

(II) For all $x \in X$, $s$, $t \in F$ such that $|t| = |s| \geq \max_n |e^*_n(x)|$, and for all $j \not\in \{k\} \cup \text{supp}(x)$, 

$$\|x + se_j\| \leq \|x + te_k\|.$$ 

Proof The equivalence between (i) and (ii) is Remark 1.10 when $C_{pl} = 1$, whereas the implication from (i) to (iii) is immediate.

Suppose now that (iii) holds and let us see that (i) is satisfied. For $A$, $B$, $x$, $t$, $\varepsilon$ and $\varepsilon'$ as in (i), we prove by induction on $|B|$ that (4.1) holds. For $|B| = 0$, there is nothing to prove, and for $|B| = 1$, (4.1) follows at once from (I) when taking $A = \emptyset$, and from (II) when taking $|A| = 1$. Suppose now that (4.1) holds for $|B| \leq n$. For $|B| = n + 1$,
take \( k \) any element of \( B \) and \( B_0 := B \setminus \{k\} \). In the case \( A = \emptyset \) applying the inductive hypothesis and then (I), we get
\[
\|x\| \leq \|x + t_1 e'B_0\| \leq \|x + t_1 e'_0 + t e'_k\| = \|x + t_1 e'_B\|.
\]
In the case \( A \neq \emptyset \), define
\[
j := \max A, \quad A_0 := A \setminus \{j\}, \quad y := t e_j, \quad z := x + t_1 e'_B.
\]
Since \( A_0 \subset \text{supp}(y) \cup B_0, |A_0| \leq |B_0| = n, \) and \( j \subset \{k\} \cup \text{supp}(z) \), applying the inductive hypothesis first and then (II) it follows that
\[
\|x + t_1 e_A\| = \|y + t_1 e_{A_0}\| \leq \|y + t_1 e'B_0\| = \|z + t e_j\| \leq \|z + t e'_k\| = \|x + t_1 e'_B\|.
\]
This completes the inductive step and thus the proof. \( \square \)

In [1], the authors ask a difficult—and still open—question about the relation between 1-almost greediness and greediness; the central issue is whether 1-almost greediness implies unconditionality and thus greediness. A similar question can be asked for strong partial greediness and almost greediness, as follows:

If \( B \) is 1-strong partially greedy, is \( B \) \( C \)-almost greedy for some \( C > 0 \)?

As almost greediness is equivalent to quasi-greediness and democracy, in order to give a negative answer it suffices to show that 1-strong partial greediness does not imply democracy. To that end, we use an example from the family given in [7, Proposition 6.10].

**Example 4.3** Let \( \mathbb{X} \) be the completion of \( c_{00} \) under the norm
\[
\|(a_n)_n\| := \sup_{A \in \mathcal{S}} \sum_{n \in A} |a_n|,
\]
where \( \mathcal{S} \subset \mathbb{N} \) is the set
\[
\mathcal{S} := \left\{ A \subset \mathbb{N} : |A| \leq \sqrt{\min A} \right\}.
\]
The canonical basis \((e_n)_n\) of \( \mathbb{X} \) is 1-PSLC, and it is not democratic.

**Proof** First, notice that \((e_n)_n\) is a normalized 1-suppression-unconditional Schauder basis. Thus, Condition (iii)(I) of Proposition 4.2 holds. Now choose \( x \in \mathbb{X}, s, t \in \mathbb{F} \) so that \( |t| = |s| \geq \max_n |e^*_n(x)| \), and \( j, k \in \mathbb{N} \) so that \( j \subset \{k\} \cup \text{supp}(x) \). For \( A \in \mathcal{S} \) with \( j \notin A \), we have
\[
\sum_{n \in A} |e^*_n(x + se_j)| = \sum_{n \in A} |e^*_n(x)| \leq \sum_{n \in A} |e^*_n(x + te_k)| \leq \|x + te_k\|.
\]
On the other hand, given \( A \in \mathcal{S} \) with \( j \in A \), let \( B := \{k\} \cup (A \setminus \{j\}) \). Since \( |B| \leq |A| \) and \( \min(B) \geq \min(A) \), it follows that \( B \in \mathcal{S} \). Hence,

\[
\sum_{n \in A} |e_n^* (x + s e_j)| = |s| + \sum_{n \in A \setminus \{j\}} |e_n^*(x)| = \sum_{n \in B} |e_n^* (x + t e_k)| \leq \|x + t e_k\|.
\]

Taking supremum over all \( A \in \mathcal{S} \), we obtain \( \|x + s e_j\| \leq \|x + t e_k\| \). Hence, Condition (iii)(II) of Lemma 4.2 holds as well, and therefore, \((e_n)_n\) has the 1-PSLC.

The fact that the basis is not democratic was proven in [7]. Here, we include a proof for the sake of completeness. Let \( A := \{m^2 + 1, \ldots, m^2 + m\} \) and \( B := \{1, \ldots, m\} \). Then, since \( A \in \mathcal{S} \), \( \|1_A\| = m \). We claim that \( \|1_B\| \leq \sqrt{m} \), hence the basis is not democratic. Indeed, to prove this upper estimate, take a set \( A_1 \in \mathcal{S} \) such that \( \|1_{B_0}\| = |A_1| \). Then, \( \min A_1 \leq m \), so \( |A_1| \leq \sqrt{m} \). Thus, \( \|1_B\| \leq \sqrt{m} \).

5 Discussions on the Relation Between Partially Greedy, Strong Partially Greedy and Quasi-Greedy Bases

As it has been mentioned in the introduction, in [9] the authors introduced the notion of partially greedy Schauder bases and characterized them as those which are quasi-greedy and conservative [9, Theorem 3.4]. Recall that for Schauder bases, being partially greedy and strong partially greedy are equivalent notions. Thus, the following theorem extends [9, Theorem 3.4] to the context of Markushevich bases and also shows a relationship between partially and strong partially greedy Markushevich bases.

**Theorem 5.1** Let \( B \) be an M-basis in a Banach space \( X \). The following are equivalent.

(i) \( B \) is strong partially greedy.
(ii) \( B \) is quasi-greedy and partially greedy.
(iii) \( B \) is quasi-greedy and superconservative.
(iv) \( B \) is quasi-greedy and conservative.

**Proof** The implications (i) \( \implies \) (ii) and (iii) \( \implies \) (iv) are immediate. To prove that (ii) \( \implies \) (iii), fix \( m \in \mathbb{N} \) and let \( A, B, \varepsilon, \varepsilon' \) as in Definition 1.6. Choose \( B_0 \subset B \) with \( |B_0| = |A| \), and let

\[
m_1 := \max A, \quad m_2 := |B \setminus B_0|, \quad D := \{1, \ldots, m_1\} \setminus A.
\]

Choose \( \eta > 0 \) and define

\[
y := 1_{eA} + (1 + \eta)(1_{e'B_0} + 1_D), \quad z := (1 + \eta)1_{e'B_0} + (1 + 2\eta)1_{e'B \setminus B_0}.
\]
We have
\[
\|1_{\mathcal{A}}\| = \|y - G_{m_1}(y)\| \leq C_p \|y - P_{m_1}(y)\| = C_p \|(1 + \eta)1_{\mathcal{B}_0}\| = C_p \|z - G_{m_2}(z)\| \\
\leq C_p C_q \|z\|.
\]

Since \(\eta\) is arbitrary, it follows that \(\|1_{\mathcal{A}}\| \leq C_p C_q \|1_{\mathcal{B}}\|\) and \(\mathbf{s c}_m \leq C_p C_q\). Then, taking supremum over \(m\) we obtain \(C_{\mathbf{s c}} \leq C_p C_q\), so the basis is superconservative.

We do not know whether strong partially and partially greedy Markushevich bases are equivalent notions. In light of Theorem 5.1, the question is to find out whether or when every partially greedy Markushevich basis is quasi-greedy. In the case of 1-partially greedy bases, we give a positive answer.

**Proposition 5.2** Let \(\mathcal{B} = (e_n)_{n=1}^{\infty}\) be 1-partially greedy Markushevich basis for \(\mathcal{X}\) and let \(c := \sup_n \{\|e_n\|, \|e_n^*\|\}\). The following hold:

(i) For all \(x \in \mathcal{X}, m \in \mathbb{N}\), any greedy sum \(G_m\) and \(1 \leq k \leq m\),
\[
\|x - G_m(x)\| \leq \|x - P_k(x)\|.
\]

(ii) \(\mathcal{B}\) is \(C_{\mathbf{sp}}\)-strong partially greedy with \(C_{\mathbf{sp}} \leq 1 + c^2\).

(iii) The sequence \((e_n)_{n \geq 2}\) is a 1-strong partially greedy basis for \(\mathcal{Y} := [e_n : n \geq 2]\).

(iv) \(\mathcal{B}\) is 1-superconservative.

**Proof** We prove (5.1) by induction on \(m\). For \(m = 1\), that is just the 1-partially greedy condition. Suppose (5.1) holds for \(1 \leq m \leq m_0 + 1\) and let us prove that it holds for \(m_0 + 1\). Take \(x \in \mathcal{X}\) and fix \(\mathcal{A}\) an \((m_0 + 1)\)-greedy set for \(x\) with greedy operator \(G_{m_0+1}\). For \(k = m_0 + 1\), (5.1) holds because \(\mathcal{B}\) is 1-partially greedy. Fix \(1 \leq k_0 \leq m_0\), and consider first the case that there is \(1 \leq j \leq k_0\) such that \(j \in \mathcal{A}\). Then
\[
G_{m_0+1}(x) = e_j^*(x)e_j + G_{m_0}(x - e_j^*(x)e_j).
\]

Thus, by inductive hypothesis,
\[
\|x - G_{m_0+1}(x)\| = \|x - e_j^*(x)e_j - G_{m_0}(x - e_j^*(x)e_j)\| \leq \|x - e_j^*(x)e_j - P_{k_0}(x - e_j^*(x)e_j)\| = \|x - P_{k_0}(x)\|.
\]

On the other hand, if \(k_0 < \mathcal{A}\), then for all \(1 \leq j \leq k_0\) and all \(1 \leq l \leq m_0 + 1\),
\[
G_l(x) = G_l(x - P_j(x)); \quad P_j(x - G_l(x)) = P_j(x).
\]

Hence, by inductive hypothesis and the 1-partially greedy condition,
\[ \| x - G_{m+1}(x) \| = \| x - G_1(x) - G_{m_0}(x - G_1(x)) \| \leq \| x - G_1(x) - P_{k_0}(x - G_1(x)) \| \]
\[ = \| x - P_{k_0}(x) - G_1(x - P_{k_0}(x)) \| \leq \| x - P_{k_0}(x) - P_1(x - P_{k_0}(x)) \| \]
\[ = \| x - P_{k_0}(x) \|. \]

This completes the inductive step and thus the proof of (5.1).

To prove (ii), we use (5.1) for \( k = 1 \). For each \( x \in X \) and \( m \in \mathbb{N} \), we have
\[ \| x - G_m(x) \| \leq \| x - P_1(x) \| \leq \| x \| + \| P_1(x) \| \leq (1 + c^2)\| x \|. \]

Therefore, \( B \) is strong partially greedy with constant \( C_{sp} \leq 1 + c^2 \).

Now, we show (iii). It is clear that \( (e_n)_{n \geq 2} \) is a basis for \( Y \). Denote by \( (\overline{P}_m)_{m} \) and \( (G_m)_{m} \) the projections and greedy sums with respect to \( (e_n)_{n \geq 2} \). Given \( y \in Y, m \in \mathbb{N} \), and \( 0 \leq k \leq m \), choose \( a > |e_j^* (y)| \) for all \( j \).
\[ \| y - G_m(y) \| = \| ae_1 + y - G_{m+1}(ae_1 + y) \| \leq \| ae_1 + y - P_{k+1}(ae_1 + y) \| = \| y - \overline{P}_k(y) \|. \]

To prove (iv), fix \( A \neq \emptyset, B, \varepsilon, \varepsilon' \) as in Definition 1.6, choose \( \eta > 0 \) and define
\[ k := \max A, \quad D := \{1, \ldots, k\} \setminus A, \quad y := 1_{\varepsilon A} + (1 + \eta)(1_{\varepsilon'B} + 1_D), \quad m := |D \cup B|. \]

Given that \( m \geq k \geq 1 \), it follows by (i) that
\[ \| 1_{\varepsilon A} \| = \| y - G_m(y) \| \leq \| y - P_k(y) \| = (1 + \eta)\| 1_{\varepsilon'B} \|. \]

Letting \( \eta \to 0 \), we obtain the desired inequality \( \| 1_{\varepsilon A} \| \leq \| 1_{\varepsilon'B} \| \), which shows that \( B \) is 1-superconservative. Now the proof is complete. \( \square \)

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