Regularity of Minimizers for a General Class of
Constrained Energies in Two-Dimensional Domains
with Applications to Liquid Crystals

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Abstract

We investigate minimizers defined on a bounded domain $\Omega$ in $\mathbb{R}^2$ for singular constrained energy functionals that include Ball and Majumdar’s modification of the Landau-de Gennes Q-tensor model for nematic liquid crystals. We prove regularity of minimizers with finite energy and show that their range on compact subdomains of $\Omega$ does not intersect the boundary of the constraining set. We apply this result to prove that minimizers of the constrained Landau-de Gennes Q-tensor energy for liquid crystals composed of a singular Maier-Saupe bulk term and all elasticity terms with coefficients $L_1, \cdots, L_5$, are $C^2$ in $\Omega$; and their eigenvalues on compact subsets of $\Omega$ are contained in closed subintervals of the physical range $(-\frac{1}{3}, \frac{2}{3})$.

1 Introduction.

In this paper we consider minimizers to a singular constrained energy functional of the form

\begin{equation}
J[v] = \int_{\Omega} (F(v, Dv) + f(v))dx
\end{equation}

where $\Omega$ is a bounded $C^2$ domain in $\mathbb{R}^n$ and $n \geq 2$. We assume that $f$ is defined and real-valued on an open, bounded, convex set $\mathcal{K}$ in $\mathbb{R}^q$ with $q \geq 1$ and that $f(v) \to \infty$
as \( v \to \partial K \) for \( v \) in \( K \). We extend the definition of \( f \) to all of \( \mathbb{R}^q \) by setting \( f(v) = \infty \) for \( v \) in \( \mathbb{R}^q \setminus K \). Thus we assume throughout this paper that

\[
\begin{aligned}
&f : K \rightarrow \mathbb{R}, \quad f \in C^2(K), \quad D^2 f \geq -MI_q \text{ on } K, \\
&\lim_{v \to \partial K} f(v) = \infty, \quad \text{and } f(v) = \infty \text{ on } \mathbb{R}^q \setminus K
\end{aligned}
\]

where \( M \geq 0 \). We also assume the following structure conditions on \( F \):

\[
\begin{aligned}
F(v, P) &= A^\alpha_\beta(v)p^i_\alpha p^j_\beta + B^\alpha_\beta(v)p^i_\alpha \quad \text{for } v \in K, P \in M^{q \times n}, \\
A^\alpha_\beta(v)p^i_\alpha p^j_\beta &\geq \lambda |P|^2 \quad \text{for } v \in K, P \in M^{q \times n}, \\
\text{where } A^\alpha_\beta, B^\alpha_\beta &\in C^2(K) \quad \text{and } \lambda > 0.
\end{aligned}
\]

We use the convention in this paper that repeated indicies are summed. In this case \( i \) and \( j \) go from 1 to \( n \) and \( \alpha \) and \( \beta \) go from 1 to \( q \). Here \( M^{q \times n} \) denotes the set of \( q \times n \) real-valued matrices. We define

\[
M_1 = \max \left\{ \sup_{v \in K} |A^\alpha_\beta(v)|, \sup_{v \in K} |B^\alpha_\beta(v)| \right\}
\]

and \( M_2 \equiv \max \{ \| A^\alpha_\beta \|_{C^2(K)}, \| B^\alpha_\beta \|_{C^2(K)} \} \).

The energy functional \( J \) is defined for all \( v \) in

\[
H^1(\Omega, \overline{K}) = \{ v \in H^1(\Omega; \mathbb{R}^q) : v(x) \in \overline{K} \text{ almost everywhere in } \Omega \}.
\]

By assumption (1.2) on \( f \), if \( u \in H^1(\Omega, \overline{K}) \) and \( J[u] < \infty \), then \( u(x) \in K \) for almost every \( x \) in \( \Omega \).

Given \( u \) in \( H^1(\Omega; \overline{K}) \) such that \( J[u] < \infty \), it follows from direct methods in the calculus of variations (see [7]) that minimizers exist in the space \( A_u = \{ v \in H^1(\Omega; \overline{K}) : v - u \in H^1_0(\Omega; \mathbb{R}^q) \} \). In this paper we will refer to such minimizers as \textit{finite energy minimizers} of \( J \) in \( \Omega \).

A question of interest for applications is whether minimizers \( u \) of \( J[\cdot] \) are smooth and whether they satisfy \( u(x) \in K \) for all \( x \) in \( \Omega \). One of the difficulties in analyzing their regularity is to find finite energy variations in \( H^1(\Omega; \overline{K}) \) from which one can extract useful information about their properties.

Our results are for \( n = 2 \). We prove the following main theorem:
Theorem 1.1. Assume $\Omega$ is a bounded $C^2$ domain in $\mathbb{R}^n$, $n = 2$, $K$ is an open bounded, convex set in $\mathbb{R}^q$, $q \geq 1$, and (1.2) and (1.3) hold. If $u \in H^1(\Omega; \mathcal{K})$ is a finite energy minimizer for $J$ in $\Omega$, then $u$ is in $C^2(\Omega)$ for all $0 < \delta < 1$, $u(\Omega) \subset K$, and $u$ satisfies the equilibrium equation

$$\text{div} F_P(u, Du) - F_u(u, Du) = f_u(u) \quad \text{on} \ \Omega.$$ 

Moreover, if $\Omega'$ is an open set in $\Omega$ such that $\Omega' \subset \subset \Omega$, then dist $(u(\Omega'), \partial K) \geq c > 0$ where $c$ depends only on $J[u]$, $\Omega$, dist $(\Omega', \partial \Omega)$, $M_2$, $M$, and $\lambda$. Minimizers $u$ of (1.1) were investigated by Evans, Kneuss and Tran in [5]. Assuming that $n \geq 2$, $F = F(v, Dv)$ satisfies certain growth conditions and is uniformly strictly quasi-convex, and $f$ satisfies (1.2) with $M = 0$, they proved the following partial regularity result: there is an open subset $\Omega_0$ of $\Omega$ such that $|\Omega \setminus \Omega_0| = 0$, $u \in C^2(\Omega_0)$, and $u(\Omega_0) \subset K$. In particular it follows that $u$ satisfies the equilibrium equation (1.4) on $\Omega_0$ in this case.

They also proved in [5] that if, in addition, $F = F(Dv)$, $F(\cdot)$ is convex, and $f(v)$ is smooth and convex on $K$, then a finite energy minimizer $u$ is in $H^2_{\text{loc}}(\Omega)$. We use a similar approach here for part of our analysis. They considered variations $u + t\phi_k$ where $\phi_k$ is given by

$$\phi_k(x; h) = [\zeta^2(x)u(x + he_k) + \zeta^2(x - he_k)u(x - he_k) - (\zeta^2(x) + \zeta^2(x - he_k))u(x)]h^{-2} - \nabla^h_k(\zeta^2 \nabla^h_k u) \quad \text{for} \ 1 \leq k \leq n,$$

for $\zeta \in C^2_c(\Omega)$ and $\nabla^h_k w(x) = h^{-1}[w(x + he_k) - w(x)]$ for $h \neq 0$ and sufficiently small. They showed that for $0 \leq t < \bar{t}(h)$, $u(x) + t\phi_k(x) \in \mathcal{K}$. Using the definition of $\phi_k$ and the convexity of $f$ they proved that

$$\int_{\Omega} f(u + t\phi_k) dx \leq \int_{\Omega} f(u) dx \quad \text{for} \ 0 \leq t < \bar{t}(h).$$

and hence

$$0 \leq J(u + t\phi_k) - J(u) \leq \int_{\Omega} (F(D(u + t\phi_k)) - F(Du)) dx.$$ 

Dividing by $t$ and letting $t \to 0$ gives

$$0 \leq \int_{\Omega} (F_P(Du) : D\phi_k) dx \quad \text{for} \ 1 \leq k \leq n.$$
For $F = F(Dv)$ this inequality leads to $u \in H^2_{\text{loc}}(\Omega)$. (See \[5\], Thm 4.1.)

Here we have $F = F(v, Dv)$ and $f$ is not convex. However, our assumption (1.2) implies that

$$f_0(v) \equiv f(v) + \frac{M}{2} |v|^2$$

is a convex function on $\mathcal{K}$ and by \[5\]

$$\int_\Omega f_0(u + t\phi_k) dx \leq \int_\Omega f_0(u) dx \quad \text{for } 0 \leq t < \bar{t}(h).$$

Using this we can argue just as above to show that a minimizer $u$ of $J$ satisfies

$$0 \leq \int_\Omega (F_F(u, Du) : D\phi_k + F_u(u, Du) \cdot \phi_k - Mu \cdot \phi_k) dx.$$  

From this and \[1.3\] it follows that

$$\frac{\lambda}{2} \int_\Omega |\nabla^h Du|^2 \zeta^2 \ dx \leq C_0 \int_\Omega |\nabla^h u|^2 |Du|^2 \zeta^2 \ dx + C_1$$

where $\lambda$ is the constant defined in our assumption \[1.3\].

To obtain an $H^2_{\text{loc}}$ estimate, we additionally need that $u$ is continuous and a second inequality. In Section 2 we show that when $n = 2$, our assumptions (1.2), (1.3), and a result in \[4\] imply that finite energy minimizers of $J$ are continuous in $\Omega$. We then construct additional variations $u + tw_l$ in $H^1(\Omega; \mathcal{K})$ so that $w_l$ satisfies \[1.5\] and \[1.6\], with $\phi_k$ replaced by $w_l$. We use this to prove a second inequality, from which we obtain the $H^2_{\text{loc}}$ regularity of minimizers. In Section 3 we use this result to prove Theorem 1.1.

Our formulation of the constrained energy (1.1) and the assumptions (1.2) and (1.3) is motivated by the constrained Landau-de Gennes Q-tensor energy for nematic liquid crystals. This energy is given by

$$I_{\text{LdG}}[Q] = \int_\Omega [G(Q, DQ) + \Psi_b(Q)] dx.$$  

where

$$G(Q, DQ) = L_1 |\nabla Q|^2 + L_2 \cdot D_{x_j} Q_{ij} \cdot D_{x_k} Q_{ik} + L_3 \cdot D_{x_j} Q_{ik} \cdot D_{x_k} Q_{ij} + L_4 \cdot Q_{ik} \cdot D_{x_j} Q_{ij} \cdot D_{x_k} Q_{ij} + L_5 \cdot \epsilon_{ijk} \cdot Q_{\ell i} \cdot D_{x_{\ell}} Q_{ki} \equiv L_1 I_1 + L_2 I_2 + L_3 I_3 + L_4 I_4 + L_5 I_5.$$
and $\Psi_b(Q) = Tf_{ms}(Q) - \kappa|Q|^2$. The constants, $L_1, L_2, L_3, L_4$, and $L_5$ are material-dependent elastic constants, $T$ and $\kappa$ are positive constants, and $\epsilon_{ijk}$ is the Levi-Civita tensor. The function $f_{ms}(Q)$ is a specific function (called the Maier-Saupe potential) defined on

$$
\mathcal{M} = \{ Q \in \mathbb{M}^{3 \times 3} : Q = Q^t, \text{tr} Q = 0, \text{ and } -\frac{1}{3} < \lambda(Q) < \frac{2}{3} \}
$$

for all eigenvalues $\lambda(Q)$ of $Q$.

It is defined abstractly using probability densities on a sphere that represent possible orientations of liquid crystal molecules. (See Section 4 for the definition of $f_{ms}$.) The bulk term $\Psi_b$ and the Maier-Saupe potential $f_{ms}$ were introduced and investigated in the papers [3] by Ball and Majumdar and [9] by Katriel, Kventsel, Luckhurst and Sluckin. It is known that $f_{ms}$ is convex. Moreover, $f_{ms}$ is bounded below and $f_{ms}(Q) \to \infty$ for $Q$ in $\mathcal{M}$ with $Q \to \partial \mathcal{M}$; hence the same is true for $\Psi_b$. As in (1.2), we set $\Psi_b(Q) = \infty$ for $Q \in S_0 \setminus \mathcal{M}$ where $S_0 = \{ Q \in \mathbb{M}^{3 \times 3} : Q = Q^t, \text{tr} Q = 0 \}$. Thus $\Psi_b$ blows up at $\partial \mathcal{M}$ as in (1.2.2), with $f$ replaced by $\Psi_b$ and $K$ replaced by $\mathcal{M}$. It follows that finite energy minimizers $Q$ of $I_{LdG}$ satisfy $Q(x) \in \mathcal{M}$ almost everywhere in $\Omega$, so that the eigenvalues of $Q(x)$ are in $(-\frac{1}{3}, \frac{2}{3})$ almost everywhere in $\Omega$. Conditions on $L_1, \cdots, L_4$ have been identified so that minimizers of $I_{LdG}$ exist in

$$
A_{Q_0} = \{ Q \in H^1(\Omega; \mathcal{M}) = \{ Q - Q_0 \in H_0^1(\Omega; \mathbb{M}^{3 \times 3}) \}
$$

provided that $Q_0 \in H^1(\Omega; \overline{\mathcal{M}})$ and that $I_{LdG}[Q_0] < \infty$. (See (4.7) and [10].) It was stated in [3] that for $\Omega$ in $\mathbb{R}^3$, if $Q_0(\overline{\Omega}) \subset \subset \mathcal{M}$, minimizers in $A_{Q_0}$ of the energy

$$
\int_{\Omega} [L_1|DQ|^2 + \Psi_b(Q)] dx
$$

with $L_1 > 0$ are smooth in $\Omega$ and valued in $\mathcal{M}$; thus their eigenvalues are in $(-\frac{1}{3}, \frac{2}{3})$ at all points in $\Omega$. A sketch of a proof of this statement is included in [11]. (See also [4].) Such minimizers are called "physically realistic." Additional features for minimizers, $Q$ of $I_{LdG}$ with $\Omega$ in $\mathbb{R}^3$ were obtained by Geng and Tong in [8]. In particular, assuming specific conditions on $G(Q, DQ)$ they proved higher integrability properties for $|DQ|$.

The elastic term with coefficient $L_4$ in $I_{LdG}$ is called the "cubic term." When $L_4 \neq 0$, the energy density is quasilinear. This makes it difficult to analyze the behavior of minimizers in this case.

Physicists have computed the elastic coefficients $L_1, \cdots, L_4$ in terms of the elastic coefficients $K_1, \cdots, K_4$ that account for the elastic energy of splay, twist and bend that occur in the well-known Frank energy density, which models liquid crystals in terms of
functions \( n = n(x) \) valued in \( \mathbb{S}^2 \). They found that \( L_4 = 0 \) if and only if \( K_1 = K_3 \), which is nonphysical for many applications. Thus it is desirable to consider the energy \( I_{\text{LandG}} \) with \( L_4 \neq 0 \). It is interesting to note that when \( L_4 \neq 0 \), the unconstrained Landau-de Gennes energy given by (1.8) with \( \Psi_b(Q) \) replaced by a polynomial is unbounded from below. Thus minimizers do not exist in general for boundary value problems with this energy. See [3].

For \( \Omega \) in \( \mathbb{R}^2 \), we proved Hölder continuity of finite energy minimizers in [4] under general conditions for energy functionals of the form
\[
F(Q) = \int_\Omega \left[ F_e(Q(x), \nabla Q(x)) + f_b(Q(x)) \right] dx
\]
by using harmonic and elliptic replacements to construct finite energy comparison functions. In particular we established that finite energy minimizers to the quasilinear constrained energy \( I_{\text{LandG}} \) in (1.8) under the coercivity condition (4.7) are Hölder continuous in \( \Omega \). We also proved under the additional assumption \( L_2 = L_3 = 0 \) that finite energy minimizers for (1.8) satisfy the "physicality condition", \( Q(x) \in \mathcal{M} \) for all \( x \in \Omega \). Here we establish this property without requiring the additional assumption.

In Section 4 of this paper we describe a connection between the constrained energies \( J[u] \) and \( I_{\text{LandG}}(Q) \). Using Theorem 1.1, we prove in Theorem 4.1 that under appropriate coercivity conditions on \( L_1, \cdots, L_4 \), finite energy minimizers of \( I_{\text{LandG}} \) with all elasticity terms are in \( C^2(\Omega) \); moreover, they satisfy a strong physicality condition: if \( \Omega' \) is an open set such that \( \Omega' \subset \subset \Omega \), then \( Q(\Omega') \subset \mathcal{M} \) and \( \text{dist}(Q(\Omega'), \partial \Omega) \geq c > 0 \). Thus in compact subsets of \( \Omega \), the eigenvalues of minimizers are contained in closed subintervals of the physical range \((-\frac{1}{3}; \frac{2}{3})\).

2 Continuity and \( H^2_{\text{loc}} \) estimates for minimizers in two-dimensional domains.

Assume that \( \Omega \) is a bounded \( C^2 \) domain in \( \mathbb{R}^2 \). Let \( \Lambda = \{ x \in \Omega : u(x) \in \partial \mathcal{K} \} \). In this section we will show that finite energy minimizers \( u \) of \( J \) are locally Hölder continuous in \( \Omega \), \( C^2 \) in \( \Omega \setminus \Lambda \), and globally Hölder continuous in \( \overline{\Omega} \) if their boundary values are sufficiently smooth. In addition, they are in \( H^2_{\text{loc}}(\Omega) \).

Our proof of the first statement is an application of (1.2), (1.3), and a result in [4] for two-dimensional domains. We will ultimately prove (in Section 3) that \( \Lambda = \emptyset \).

**Proposition 2.1.** Assume that \( \Omega \) is a bounded \( C^2 \) domain in \( \mathbb{R}^2 \). Assume \( u = u(x) \) is in \( H^1(\Omega; \overline{\mathcal{K}}) \) and \( u \) is a finite energy minimizer of \( J \) in \( \Omega \). Let \( \Lambda = \{ x \in \Omega : u(x) \in \partial \mathcal{K} \} \).
a) If \( \Omega' \) is a connected open set with \( \Omega' \subset \subset \Omega \), there exist constants \( 0 < \sigma < 1 \) and \( c_1 > 0 \) such that \( \omega(d) = c_1 d^\sigma \) is a modulus of continuity for \( u \) in \( \Omega' \). The constants \( \sigma \) and \( c_1 \) depend only on \( J(u), \Omega, \text{dist} (\Omega', \partial\Omega), M_1 \), and the constants \( M \) and \( \lambda \) in (1.2) and (1.3).

b) If \( u_0 \in H^1(\Omega; \mathbb{K}) \) and \( J(u_0) < \infty \) such that \( u_0 \in C^{0,1}(\partial\Omega; \mathbb{K}) \) and \( \int_{\partial\Omega} f(u_0)ds < \infty \) and if \( u \in H^1(\Omega; \mathbb{K}) \) is a minimizer of \( J \) in \( A_{u_0} = \{ v \in H^1(\Omega; \mathbb{K}) : v - u_0 \in H^1_0(\Omega; \mathbb{R}^q) \} \), then there exists a constant \( 0 < \beta < 1 \) such that \( u \in C^\beta(\Omega; \mathbb{K}) \). The modulus of continuity, \( \omega(d) = c_2 d^\beta \), has constants depending only on \( J(u), \Omega, M_1, M, \lambda \) and \( u_0 \).

c) The minimizer \( u \) is continuous in \( \Omega \) and \( C^{2,\delta} \) in the open set \( \Omega \setminus \Lambda \) for all \( 0 < \delta < 1 \).

Proof. In [4] we investigated finite energy minimizers \( \tilde{Q}(x) \) of a constrained energy of the form

\[
\tilde{J}(Q) = \int_\Omega (F_e(Q, DQ) + f_b(Q))dx
\]

over all \( Q \in H^1(\Omega; \mathcal{M}) \) such that \( Q - Q_0 \in H^1_0(\Omega; S_0) \) and \( \tilde{J}(Q_0) < \infty \), where \( \mathcal{M} \) is an open bounded convex subset of \( S_0 = \{ Q \in \mathbb{M}^{3 \times 3} : Q = Q' \text{ and } \text{tr} Q = 0 \} \) and \( \mathbb{M}^{3 \times 3} \) is the set of \( 3 \times 3 \) real-valued matrices. Note that \( S_0 \) is isometrically isomorphic to \( \mathbb{R}^5 \). Here \( f_b(Q) = b_0(Q) - \kappa |Q|^2 + b_0 \) for \( Q \in S_0 \) (and \( \infty \) otherwise) and it is assumed that \( g_b \) is a smooth convex function defined on \( \mathcal{M} \) such that \( g_b(Q) \to \infty \) as \( Q \to \partial \mathcal{M} \) with \( Q \in \mathcal{M} \). Since \( f(v) = f_0(v) - \frac{M}{2} |v|^2 \), our assumptions (1.2) and (1.3) on \( f(v) \) and \( F(v, Dv) \) correspond to the assumptions (1.2) and (1.4) on \( f_b(Q) \) and \( F_e(Q, DQ) \) in [4] that were used to prove the same Hölder continuity on \( \tilde{Q} = \tilde{Q}(x) \) that we wish to prove here for \( u = u(x) \). The change from energy densities that depend on the variable \( Q \in S_0 \) to those that depend on \( u \in \mathbb{R}^q \) is a trivial one, and the arguments in the proofs of Theorem 1 and 2 go through to prove a) and b).

To prove c), we first note that by a), \( u \) is continuous in \( \Omega \) and hence \( u^{-1}(\mathbb{K}) = \Omega \setminus \Lambda \) is an open set. To verify that \( u \) is \( C^{2,\delta} \) on this set, we argue as in [4], Corollary 2. Indeed, assume \( B_{4r}(x_0) \subset \subset \Omega \setminus \Lambda \). Note that \( f \) is bounded and \( C^2 \) on a neighborhood of \( u(B_{4r}(x_0)) \). We can then take smooth first variations for \( J \) about \( u \) supported in \( B_{4r}(x_0) \) to conclude that \( u \) is a weak solution of (1.4) on \( B_{4r}(x_0) \). We can apply the result from [7], Ch. VI, Proposition 1 asserting that in two space dimensions a continuous weak solution of (1.3)-(1.4) with \( f_b(u(x)) \) bounded is in \( W^{2,p}(B_{3r}(x_0)) \) for some \( p > 2 \) and thus its first derivatives are Hölder continuous on \( B_{2r}(x_0) \). Now we can apply techniques from linear elliptic theory in [7], Ch. III. Taking (1.3) into account these lead to \( u \in C^{2,\delta}(B_r(x_0)) \).
Our next objective is to show that \( u \in H^2_{\text{loc}}(\Omega) \). To define variations \( u + tw \) that will provide a proof, we will need several properties of the convex potential

\[
f_0(v) = f(v) + \frac{M}{2} |v|^2
\]

in a family of cones \( \mathcal{C} \) with vertices in the convex set \( \mathcal{K} \). For ease of notation, assume from now on without loss of generality that 0 is in \( \mathcal{K} \). Since \( \mathcal{K} \) is a bounded convex set in \( \mathbb{R}^q \), it is starlike with respect to 0. Let \( \mathbb{S}^{q-1} = \partial B_1(0) \) where \( B_1(0) \) is the open unit ball in \( \mathbb{R}^q \) centered at 0. Let \( g : \mathbb{S}^{q-1} \to \mathbb{R}^+ \) be in \( C^{0,1}(\mathbb{S}^{q-1}) \) such that the map

\[
\nu \in \mathbb{S}^{q-1} \to g(\nu)\nu \in \mathbb{R}^q
\]

is a parametrization of \( \partial \mathcal{K} \). Define 0 < \( m_1 < m_2 \) by

\[
(2.1) \quad m_1 = \inf \{ g(\nu) : \nu \in \mathbb{S}^{q-1} \} \quad \text{and} \quad m_2 = \sup \{ g(\nu) : \nu \in \mathbb{S}^{q-1} \}.
\]

Define \( G(x) : \overline{B_1(0)} \to \overline{\mathcal{K}} \) by

\[
(2.2) \quad G(x) = \begin{cases} 
  g(\frac{x}{|x|})x & \text{if } x \neq 0 \\
  0 & \text{if } x = 0.
\end{cases}
\]

Thus \( G \) is a bi-Lipschitz continuous map from \( \overline{B_1(0)} \) onto \( \overline{\mathcal{K}} \). Let \( \mathcal{Y}_\mu = G(B_\mu(0)) \) for 0 < \( \mu < 1 \). Then \( \mathcal{Y}_\mu \) is an open convex subset of \( \mathcal{K} \) and \( \mathcal{Y}_\mu \uparrow \mathcal{K} \) as \( \mu \uparrow 1 \). Fix \( r_0 > 0 \) and 0 < \( \mu_0 < 1 \) so that \( \overline{B_{r_0}(0)} \subset \mathcal{Y}_{\mu_0} \).

**Definition.** We define a family of cones \( \mathcal{C} \) as follows: For each \( v \) in \( \mathcal{K} \setminus \overline{B_{r_0}(0)} \), we define the cone \( C_v^- \) to be the closed half-cone with vertex \( v \), axis containing the ray from \( v \) to 0, and aperture \( \alpha = \alpha(v) \) in \( (0, \frac{\pi}{2}) \) determined by \( \sin \alpha = \frac{r_0}{|v|} \). (See Figure 1.) The cone \( C_v^+ \) is the reflection of \( C_v^- \) about the point \( v \), i.e.

\[
C_v^+ = \{ w = v + \xi : \xi \in \mathbb{R}^q \text{ and } v - \xi \in C_v^- \}.
\]

We define \( \mathcal{C} \) to be the family of all cones, \( C_v^- \) and \( C_v^+ \), with \( v \) in \( \mathcal{K} \setminus \overline{B_{r_0}(0)} \).

For \( v \) as above, the ball \( \overline{B_{r_0}(0)} \) is contained in \( C_v^- \) and is tangent to its boundary. Thus each ray in \( C_v^- \) with initial point \( v \) intersects \( \partial B_{r_0}(0) \) at least once. Also \( r_0 < |v| \leq m_2 \) and thus there exists \( \alpha_0 \in (0, \frac{\pi}{2}) \) such that

\[
(2.3) \quad 1 > \sin \alpha \geq \frac{r_0}{m_2} \equiv \sin \alpha_0 > 0 \quad \text{for all } v \in \mathcal{K} \setminus \overline{B_{r_0}(0)}.
\]

The result below follows from (2.3) and the symmetry of \( C_v^- \) and \( C_v^+ \).
Figure 1: The cones $C_{v}^{-}$ and $C_{v}^{+}$.

**Proposition 2.2.** Assume $v \in \mathcal{K} \setminus \overline{B_{r_{0}}(0)}$, $z$ is a point on the axis of $C_{v}^{+}$ with $z \neq v$, and $\gamma$ satisfies $|z - v| \geq \gamma > 0$. If $r > 0$ satisfies $r \leq \gamma \sin \alpha_0$, then

$$B_{r}(z) \subset C_{v}^{+}.$$  

**Proof.** The hypotheses ensure that

$$r \leq \gamma \sin \alpha \leq |z - v| \sin \alpha$$

for $\alpha = \alpha(v)$. It follows from this and the symmetry of $C_{v}^{-}$ and $C_{v}^{+}$ that $B_{r}(z) \subset C_{v}^{+}$. \qed

**Proposition 2.3.** If $v \in \mathcal{K} \setminus \mathcal{Y}_{\mu}$ and $\mu > \mu_0$, then $C_{v}^{+} \cap \mathcal{K} \subset \mathcal{K} \setminus \mathcal{Y}_{\mu}$.

**Proof.** If not, there exists $w \in C_{v}^{+} \cap \mathcal{K}$ such that $w \notin \mathcal{K} \setminus \mathcal{Y}_{\mu}$ and hence $w \in \mathcal{Y}_{\mu}$. Thus $w \neq v$ and $w = v + \xi$ for some $\xi \neq 0$. The ray with initial point $v$ that passes through $v - \xi$ is in $C_{v}^{-}$ and hence contains a point $y$ in $\partial B_{r_{0}}(0) \cap C_{v}^{-}$. Since $\mu > \mu_0$, $\partial B_{r_{0}}(0) \subset \mathcal{Y}_{\mu}$. 9
By the convexity of $\mathcal{Y}_\mu$, the segment $yw$ is contained in $\mathcal{Y}_\mu$. But $v \in \overline{yw} \subset \mathcal{Y}_\mu$, which contradicts the fact that $v \in K \setminus \mathcal{Y}_\mu$.

Define $s_0 = \max\{f_0(v) : v \in \partial B_{r_0}(0)\}$. Since $f_0$ is continuous on $K$ and $f_0(v) \to \infty$ as $v \to \partial K$ with $v$ in $K$, there exists a constant $\mu_1$ in $(\mu_0, 1)$ such that

(2.5) $f_0(v) \geq 1 + s_0$ for all $v$ in $K \setminus \mathcal{Y}_{\mu_1}$.

From now on, we fix $0, r_0, \alpha_0, m_1, m_2, 0 < \mu_0 < \mu_1 < 1$, and $s_0$ as above. We then have the following:

**Lemma 2.4.** For any $v$ in $K \setminus \mathcal{Y}_{\mu_1}$ and $w$ in $C_v^+ \cap K$ such that $w \neq v$, we have $\nabla f_0(w) \cdot (w - v) > 0$.

**Proof.** If this is false, there exists $v$ in $K \setminus \mathcal{Y}_{\mu_1}$ and $w$ in $C_v^+ \cap K$ such that $w \neq v$ and $\nabla f_0(w) \cdot (w - v) \leq 0$. By Proposition 2.3, $w \in K \setminus \mathcal{Y}_{\mu_1}$. Consider a linear path given by $p(t) = v + t \frac{w - v}{|w - v|}$ for $t \leq t \leq \tilde{t}$ where $p(t) \in \partial B_{r_0}(0)$ and $\tilde{t} = |v - w|$. Setting $h(t) = f_0(p(t))$ we have $h'(\tilde{t}) \leq 0$. By definition of $m_2$ (see (2.1)), we have

$$0 < \tilde{t} - t = |p(\tilde{t}) - p(t)| = |w - p(t)| \leq |w| + |p(t)| \leq 2m_2.$$

Since $h(t)$ is convex, we then have

$$0 \geq h'(\tilde{t}) \geq \frac{h(\tilde{t}) - h(t)}{\tilde{t} - t} \geq \frac{h(0) - h(t)}{2m_2}.$$

This is impossible since $m_2 > 0$ and by definition of $s_0$,

$$h(\tilde{t}) - h(t) = f_0(w) - f_0(p(t)) \geq (1 + s_0) - s_0 = 1.$$

**Corollary 2.5.** If $v \in K \setminus \mathcal{Y}_{\mu_1}$ and $\tilde{l} = wv$ is a ray in $C_v^+ \cap K$ with initial point $w$ and final point $v$, then $f_0$ decreases along $\tilde{l}$.

We can now prove $H^2_{loc}$ estimates on minimizers using appropriate variations $u + t\phi$ and $u + tw$.

**Lemma 2.6.** Assume that $\Omega'$ is an open connected set and $\Omega' \subset \subset \Omega$. If $u \in H^1_{loc}(\Omega; K)$ is a finite energy local minimizer of $J$ in $\Omega$, then $u \in H^2(\Omega')$ and

$$\|u\|_{H^2(\Omega')} \leq C_o,$$

where $C_o$ depends only on $J(u)$, $\Omega$, $\text{dist}(\Omega', \partial \Omega)$, $M_2$, $M$ and $\lambda$. 

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Proof. Let \( \eta_0 > 0 \) satisfy \( 3\sqrt{2}\eta_0 < \frac{1}{4} \) \( \text{dist} (\Omega', \partial\Omega) \). Define

\[
(2.6) \quad \Omega'' = \{ x \in \Omega : \text{dist} (x, \Omega') < \frac{1}{2} \text{dist} (\Omega', \partial\Omega) \}
\]

and let \( \omega(d) = C d^n \) be a modulus of continuity for \( u \) on \( \Omega'' \). Given \( \eta > 0 \) with \( \eta < \eta_0 \), let \( \mathcal{D}_0 = \{ D_m : m \in \mathbb{N} \} \) be a tiling of \( \mathbb{R}^2 \) by closed squares of side length \( \eta \), so that \( \mathbb{R}^2 = \cup_{m=1}^\infty D_m \) and distinct squares in this tiling have at most one edge in common. For each \( m \in \mathbb{N} \) let \( E_m \) be the union of \( D_m \) and its eight neighbors. Thus \( E_m \) and \( D_m \) have the same centers and \( E_m \) has side length \( 3\eta \). Set

\[
\mathcal{D}_1 = \{ D_m \in \mathcal{D}_0 : D_m \cap \Omega' \neq \emptyset \}.
\]

Then \( \mathcal{D}_1 = \{ D_{m_l} : 1 \leq l \leq L \} \). For ease of notation, let \( \tilde{D}_l \) and \( \tilde{E}_l \) denote \( D_{m_l} \) and \( E_{m_l} \), respectively, for \( 1 \leq l \leq L \). Note that by our definition of \( \eta_0 \), \( \tilde{E}_l \subset \Omega'' \) for \( 1 \leq l \leq L \).

We first work through the case \( q \geq 2 \). Let \( \mu \in [<1, 1) \). Thus \( \mu_1 \leq \mu \leq \frac{\mu_1 + 3}{4} < 1 \). (Later we will also require that \( (1 - \mu) \) is sufficiently small.) We partition \( \mathcal{D}_1 = \mathcal{D}_2 \cup \mathcal{D}_3 \) as follows:

\[
\tilde{D}_l \in \mathcal{D}_2 \text{ if } u(\tilde{E}_l) \cap (\tilde{K} \setminus \mathcal{Y}_{\mu+2}) \neq \emptyset,
\]

\[
\tilde{D}_l \in \mathcal{D}_3 \text{ if } u(\tilde{E}_l) \subset \mathcal{Y}_{\mu+2}.
\]

We first assume \( \tilde{D}_l \in \mathcal{D}_2 \) and show that \( u \in H^2(\tilde{D}_l) \). Note that in this case, \( u(\tilde{E}_l) \) is not a subset of \( \partial K \) because if so, \( \tilde{E}_l \subset \Lambda \) (by (1.1) and (1.2) since \( J(u) < \infty \)) and this would imply that \( |\tilde{E}_l| = 0 \), a contradiction. From this and the definition of \( \mathcal{D}_2 \) it follows that there exists \( \tilde{y} \in \tilde{E}_l \) such that \( u(\tilde{y}) \in K \setminus \mathcal{Y}_{\mu+2} \). Thus

\[
u(\tilde{y}) = \tilde{\mu} g(\tilde{\nu}) \tilde{\nu} \text{ for } \tilde{\nu} = \frac{u(\tilde{y})}{|u(\tilde{y})|} \in S^{q-1},
\]

and some constant \( \tilde{\mu} \) depending on \( \tilde{y} \) such that \( \frac{\mu_1 + 3}{4} < \tilde{\mu} < 1 \). Set \( v_l = \mu g(\tilde{\nu}) \tilde{\nu} \). Recall that \( \mu_0 < \mu_1 \leq \mu < \frac{\mu_1 + 3}{4} \), hence \( v_l \) is on the segment between 0 and \( u(\tilde{y}) \), and is contained in \( K \) by convexity. Also \( v_l \in K \setminus \mathcal{Y}_{\mu_1} \subset K \setminus \mathcal{B}_{v_l}(0) \). Thus the cone \( C_{v_l}^+ \) is in our family of cones \( \mathcal{C} \) and \( u(\tilde{y}) \) is on the axis of \( C_{v_l}^+ \). Since \( m_2 \geq g(\tilde{\nu}) \geq m_1 \) and \( 1 - \mu > \tilde{\mu} - \mu > \frac{\mu_1 + 3}{4} - \mu = \frac{3}{4}(1 - \mu) \), using the definition of \( u(\tilde{y}) \) and \( v_l \), we have

\[
m_2(1 - \mu) \geq g(\tilde{\nu})(\tilde{\mu} - \mu) = |u(\tilde{y}) - v_l| > m_1(\frac{\mu + 3}{4} - \mu) = m_1 \frac{3m_1}{4}(1 - \mu) \equiv \gamma_1 > 0.
\]
Next assume further that $\eta$ satisfies

$$\omega(3\sqrt{2}\eta) < \frac{3m_1}{4}(1-\mu)\sin \alpha_0 \equiv r_1 = \gamma_1 \sin \alpha_0.$$  \hfill (2.8)

Since $\bar{y} \in \bar{E}_l$ and the diameter of $\bar{E}_l$ is $3\sqrt{2}\eta$, $u(\bar{E}_l) \subset B_{r_1}(u(\bar{y}))$. By (2.7) and (2.8),

$$0 < r_1 = \gamma_1 \sin \alpha_0 < |u(\bar{y}) - v_l| \sin \alpha_0.$$

From this and Proposition 2.2, we have $B_{r_1}(u(\bar{y})) \subset C^+_{v_l}$. Thus

$$u(\bar{E}_l) \subset B_{r_1}(u(\bar{y})) \subset C^+_{v_l}. \hfill (2.9)$$

Let $\zeta_l \in C^2_c((\bar{E}_l)^o)$ such that $\zeta_l = 1$ on $\bar{D}_l$ and $0 \leq \zeta_l \leq 1$. Define $\zeta_l$ to be zero in $\Omega \setminus (\bar{E}_l)^o$. (For other values $1 \leq j \leq L$, we shall assume that the definition of $\zeta_j$ in $\bar{E}_j$ differs only by a rigid translation that maps $\bar{E}_j$ onto $\bar{E}_l$.) For $x \in \Omega$, and $h \neq 0$ sufficiently small, define

$$w_l(x) = \zeta_l^2(x)(v_l - u(x))|\nabla^h u(x)|^2 \hfill (2.10)$$

for $1 \leq l \leq L$ where $\nabla^h u = (\nabla^h u_1, \nabla^h u_2)$ with $\nabla^h u_l(x) \equiv h^{-1}[u(x + he_k) - u(x)]$ for $k = 1, 2$. By (2.6) and Proposition 2.1, we can choose $\bar{t}(h) > 0$ depending only on $h, J(u)$, and $\text{dist}(\Omega', \partial \Omega)$ so that

$$r \equiv t\zeta_l^2(x)|\nabla^h u(x)|^2 \leq 1 \quad \text{for } x \in \Omega \quad \text{and } 0 \leq t \leq \bar{t}(h).$$

Note that if $w_l(x) \neq 0$, then $x \in \bar{E}_l \subset \Omega'$ and by (2.9), $u(x) \in \bar{K} \cap B_{r_1}(u(\bar{y})) \subset \bar{K} \cap C^+_{v_l}$. Hence

$$u(x) + tw_l(x) = u(x) + \tau(v_l - u(x))$$

$$= (1 - \tau)u(x) + \tau v_l \in \bar{K} \cap C^+_{v_l} \hfill (2.11)$$

by convexity, and it is located on the ray from $u(x)$ to $v_l$ in $\bar{K} \cap C^+_{v_l}$. By Corollary 2.5, we have

$$f_0(u(x) + tw_l(x)) \leq f_0(u(x)) \quad \text{for all } x \in \Omega \text{ and } 0 \leq t \leq \bar{t}(h).$$

Thus

$$\int_\Omega f_0(u + tw_l) dx \leq \int_\Omega f_0(u) dx \quad \text{for } 0 \leq t \leq \bar{t}(h)$$

and since $u$ is a minimizer and $f(u) = f_0(u) - \frac{M}{2}|u|^2$,

$$0 \leq J(u + tw_l) - J(u) \leq \int_\Omega \left( (F(u + tw_l, Du + tw_l)) - F(u, Du) + \frac{M}{2}(|u|^2 - |u + tw_l|^2) \right) dx.$$
Dividing by $t$ and letting $t \to 0$, we obtain
\begin{equation}
0 \leq \int_\Omega (F_P(u, Du) : Dw_l + [F_u(u, Du) - Mu] \cdot w_l) dx
= \int_{\tilde{E}_l} (F_P(u, Du) : Dw_l + [F_u(u, Du) - Mu] \cdot w_l) dx.
\end{equation}

We now proceed as in the proof of [7], Ch. II, Thm 1.2. Let $C_i$ denote constants that are independent of $h$. Consider $\tilde{D}_l \in D_2$ and $\zeta \equiv \zeta_l$ as above. The first step is to consider the test function $\phi \equiv \phi_l = \phi_{l,k}$ defined by
\[
\phi_{l,k}(x) = \left[ \zeta^2(x)u(x + he_k) + \zeta^2(x - he_k)u(x - he_k) - (\zeta^2(x) + \zeta^2(x - he_k))u(x) \right]h^{-2}
= \nabla^{-h}_k(\zeta^2 \nabla^h_k u) \quad \text{for } 1 \leq k \leq 2,
\]
where $\nabla^h_k u(x) = h^{-1}[u(x+he_k) - u(x)]$ for $h \neq 0$ sufficiently small so that $\phi$ is compactly supported in $(\tilde{E}_l)^\circ$ for $k = 1, 2$. Recall that $\tilde{E}_l \subset \Omega''$ and $u(\tilde{E}_l) \subset B_{r_1}(u(y)) \subset \overline{K}$. Thus by (2.9), $u(\tilde{E}_l) \subset K \cap C^1_{v_l}$. Since $f_0(v)$ is convex on $K$, it was proved in [3] that for $h$ sufficiently small and $0 < t < t(h)$ sufficiently small, we have
\[
\int_{\tilde{E}_l} (f_0(u + t\phi_{l,k}) - f_0(u)) dx \leq 0.
\]
As in (2.12), since $u$ is a minimizer of $J$ we have
\begin{equation}
0 \leq \int_\Omega (F_P(u, Du) : D\phi_{l,k} + [F_u(u, Du) - Mu] \cdot \phi_{l,k}) dx
= \int_{\tilde{E}_l} (F_P(u, Du) : D\phi_{l,k} + [F_u(u, Du) - Mu] \cdot \phi_{l,k}) dx.
\end{equation}
Here

\[
\int_{\tilde{E}_t} (F_p(u, Du) : D\phi_{l,k}) \, dx = \int_{\tilde{E}_t} (F_p(u, Du) \cdot \frac{\partial}{\partial x_\alpha} (\nabla_k^{-h} [\zeta^2 \nabla_k^h u]) ) \, dx
\]

\[
= \int_{\tilde{E}_t} [A_{ij}^{\alpha\beta}(u) \frac{\partial u^j}{\partial x_\beta} + B_i^\alpha(u)] \cdot \frac{\partial}{\partial x_\alpha} (\nabla_k^{-h} [\zeta^2 \nabla_k^h u]) \, dx
\]

\[
= -\int_{\tilde{E}_t} \{\nabla_k^h [A_{ij}^{\alpha\beta}(u) \frac{\partial u^j}{\partial x_\beta} + B_i^\alpha(u)] \} \cdot \frac{\partial}{\partial x_\alpha} (\zeta^2 \nabla_k^h u) \, dx
\]

\[
- \int_{\tilde{E}_t} \{\nabla_k^h [A_{ij}^{\alpha\beta}(u) \frac{\partial u^j}{\partial x_\beta} + B_i^\alpha(u)] \} \cdot \{\zeta^2 \nabla_k^h (\frac{\partial u_i}{\partial x_\alpha}) + 2\zeta (\frac{\partial \zeta}{\partial x_\alpha})(\nabla_k^h u_i)\} \, dx
\]

and

\[
\int_{\tilde{E}_t} ([F_u(u, Du) - Mu] \cdot \phi_{l,k}) \, dx
\]

\[
= \int_{\tilde{E}_t} [D_u(A_{ij}^{\alpha\beta}(u)) \cdot \frac{\partial u^i}{\partial x_\alpha} \cdot \frac{\partial u^j}{\partial x_\beta} + D_u(B_i^\alpha(u)) \cdot \frac{\partial u^i}{\partial x_\alpha} - Mu] \cdot [\nabla_k^{-h} (\zeta^2 \nabla_k^h u)] \, dx
\]

\[
= -\int_{\tilde{E}_t} \nabla_k^h \{[D_u(A_{ij}^{\alpha\beta}(u)) \cdot \frac{\partial u^i}{\partial x_\alpha} \cdot \frac{\partial u^j}{\partial x_\beta} + D_u(B_i^\alpha(u)) \cdot \frac{\partial u^i}{\partial x_\alpha} - Mu]\} \cdot (\zeta^2 \nabla_k^h u) \, dx
\]

for \(1 \leq k \leq 2\). It follows that

(2.14) \quad \frac{\lambda}{2} \int_{\tilde{E}_t} |\nabla^h Du|^2 \zeta^2 \, dx \leq C_0 \int_{\tilde{E}_t} |\nabla^h u|^2 |Du|^2 \zeta^2 \, dx + C_1

for all \(\eta\) sufficiently small, where \(\lambda\) is the constant defined in our assumption (1.3). Next we use (2.12) and our definition of \(w_l\) to prove a second inequality that will provide an upper bound on the second derivatives of \(u\) in \(L^2(\tilde{E}_t)\).
By (2.12) and our definition of $w_l$, we have

\begin{equation}
0 \leq \int_{\tilde{E}_l} (F_P(u, Du) : Dw_l + [F_u(u, Du) - Mu] \cdot w_l) dx
\end{equation}

\begin{align*}
&= \int_{\tilde{E}_l} [A_{ij}^{\alpha\beta}(u) \cdot \frac{\partial w^i}{\partial x_\beta} + B_i^{\alpha}(u)] \cdot \frac{\partial}{\partial x_\alpha} (\zeta^2 (v_l^i - u^i(x)) |\nabla^h u|^2) dx \\
&\quad + \int_{\tilde{E}_l} [D_u(A_{ij}^{\alpha\beta}(u)) \cdot \frac{\partial u^i}{\partial x_\alpha} \cdot \frac{\partial u^j}{\partial x_\beta} + (D_u(B_i^{\alpha}(u)) \cdot \frac{\partial u^i}{\partial x_\alpha} - Mu] \cdot \zeta^2 (v_l - u(x)) |\nabla^h u|^2 dx.
\end{align*}

Recall that $u(\tilde{E}_l) \subset B_{r_1}(u(\bar{y}))$ and by (2.7) and (2.8),

\[
|v_l - u(x)| \leq |v_l - u(\bar{y})| + |u(\bar{y}) - u(x)| \leq |v_l - u(\bar{y})| + r_1
\]

\[
\leq m_2(1 - \mu) + \frac{3m_1}{4} \sin \alpha_0 (1 - \mu) \equiv C_2(1 - \mu).
\]

for all $x$ in $\tilde{E}_l$. Using this we see from (2.15) that

\begin{equation}
\frac{\lambda}{2} \int_{\tilde{E}_l} |\nabla^h u|^2 |Du|^2 \zeta^2 \, dx \leq C_3(1 - \mu) \int_{\tilde{E}_l} |\nabla^h Du|^2 \zeta^2 \, dx + C_4
\end{equation}

Note that $C_0$ and $C_3$ depend only on $\lambda$ and $M_2$. Taking $(1 - \mu)$ sufficiently small so that $C_3(1 - \mu) \leq \frac{\lambda^2}{8C_5}$, it follows from (2.14) and (2.16) that

\begin{equation}
\int_{\bar{D}_l} (|\nabla^h Du|^2 + |\nabla^h u|^2 |Du|^2) dx \leq C_5.
\end{equation}

The constants $C_j$ are uniform in $h$ for $|h| \leq h_0$. Letting $h \to 0$ we get

\begin{equation}
\int_{\bar{D}_l} (|D^2 u|^2 + |Du|^4) dx \leq C_5.
\end{equation}
This inequality holds for all \( \tilde{D}_l \in D_2 \). If \( \tilde{D}_l \in D_3 \) then \( f_u \) is bounded on a neighborhood of the union of all such squares. It follows as in the proof from \([7]\), Ch. II referred to above that we have (2.18) (with a possibly larger value of \( C_5 \)) in this case as well.

In conclusion, we first fix \( \mu \in (\mu_1, 1) \) sufficiently close to 1 so that (2.17) follows from (2.14) and (2.16). We then choose \( \eta \in (0, \eta_0) \) sufficiently small so that (2.8), (2.14), and (2.16) hold. This fixes the covering \( D_1 \) such that (2.18) holds for a fixed constant \( C_5 \) for all \( 1 \leq l \leq L \). Summing on \( l \) we have \( \int_{\Omega} |D^2 u| dx < \infty \).

We now comment on the case \( q = 1 \). In this instance \( \mathcal{K} \) is a bounded interval \((a, b)\) such that \( f\delta_0(v) \) is a convex function satisfying \( \lim_{v \uparrow b} f\delta_0(v) = \infty = \lim_{v \downarrow a} f\delta_0(v) \). It follows that there exists \( \delta > 0 \) so that \( f\delta_0'(v) > 0 \) for \( b - \delta < v < b \). In the same way we have that \( f\delta_0'(v) < 0 \) for \( a < v < a + \delta \). With this information we can carry out the argument as above using half lines \( R^-_v (R^+_v) \) in place of the cones \( C^-_v (C^+_v) \). \( \square \)

3 Proof of Theorem 1.1.

Recall that \( u \in C^2(\Omega_0) \) and \( \Omega_0 \subset \Omega \setminus \Lambda \). The two facts that \( u \in H^2_{\text{loc}}(\Omega) \) and \( u \) satisfies equation (1.4) on \( \Omega_0 \) imply that \( u \) is a strong solution to the equilibrium equations (1.4) throughout \( \Omega \) and that each term appearing in this equation is in \( L^2_{\text{loc}}(\Omega) \). We use this to prove Theorem 1.1.

Proof. Suppose that \( B_{2R}(x_0) \subset \subset \Omega \). Let \( (\rho, \theta) \) be polar coordinates centered at \( x_0 \) and consider the field of pure second derivatives \( D^2_{\psi \psi} u(x) \) where \( \psi = \psi(x) = \psi_{\theta} \) for \( x \in B_R(x_0) \). We have \( D^2_{\psi \psi} u(x) = \frac{u_x}{\rho} \frac{u_{\rho \rho}}{\rho} + \frac{u_{\psi \psi}}{\rho^2} \) is in \( L^2(B_R(x_0)) \). Let \( \zeta = \zeta(\rho) \in C^2_c(B_R(x_0)) \) such that \( \zeta = 1 \) on \( B_{R/2}(x_0) \). Fix \( 0 < r \leq \frac{R}{2} \). Multiplying the equation (1.4) by \( \zeta^2 D^2_{\psi \psi} u \) and integrating over \( B_R(x_0) \setminus B_r(x_0) \), we obtain

\[
(3.1) \quad \int_{B_R(x_0) \setminus B_r(x_0)} (\text{div} F_P - F_u) \cdot \zeta^2 D^2_{\psi \psi} u \, dx = \int_{B_R(x_0) \setminus B_r(x_0)} f_u(u) \cdot \zeta^2 \left( \frac{u_x}{\rho} + \frac{u_{\theta \theta}}{\rho^2} \right) dx
\]

Since \( \text{div} F_P - F_u \) and the test function \( D^2_{\psi \psi} u \) are in \( L^2(B_R(x_0)) \) we get

\[
(3.2) \quad \left| \int_{B_R(x_0) \setminus B_r(x_0)} (\text{div} F_P - F_u) \cdot \zeta^2 D^2_{\psi \psi} u \, dx \right| \leq C_1
\]
where $C_1$ is independent of $r$ for $r$ in $(0, \frac{R}{2}]$. By (1.4), $f_u$ is also in $L^2(B_R(x_0))$. Consider

$$\int_{B_R(x_0) \setminus B_r(x_0)} f_u(u) \cdot \zeta^2 \frac{u_{\theta\theta}}{\rho^2} \, dx = \int_r^R \int_0^{2\pi} f_u(u) \cdot \zeta^2 \frac{u_{\theta\theta}}{\rho} \, d\theta d\rho.$$ 

Since $u$ is a finite energy minimizer of $J$ in $\Omega$, $f(u)$ is in $L^1(\Omega)$ and for almost every $\rho$ satisfying $r \leq \rho \leq R$ we have

$$\int_0^{2\pi} (|f(u(\rho, \theta))| + |f_u(u(\rho, \theta))|^2 + |u_{\theta}(\rho, \theta)|^2) d\theta < \infty. \quad (3.3)$$

For such a value of $\rho$, if there is an interval $(\alpha, \beta) \subset [0, 2\pi)$ so that $u(\rho, \beta) \in \Lambda$ and $u(\rho, \theta) \notin \Lambda$ for all $\theta$ in $[\alpha, \beta)$, then

$$\int_{\alpha}^{\theta} \partial_{\phi} f(u(\rho, \phi)) d\phi = \int_{\alpha}^{\theta} f_u(u(\rho, \phi)) \cdot u_{\phi} d\phi \leq C_1 < \infty.$$

Thus $f(u(\rho, \theta)) - f(u(\rho, \alpha)) \leq C_1 < \infty$. However $\lim_{\theta \to \beta} f(u(\rho, \theta)) = \infty$ and this is not possible. It follows that for each $\rho$ for which $(3.3)$ holds, we have $\partial B_\rho(x_0) \cap \Lambda = \emptyset$. Since $u \in C^2(\Omega \setminus \Lambda)$ it follows that the functions $u(\rho, \cdot)$ and $f(u(\rho, \cdot))$ are smooth. This allows us to integrate by parts and obtain

$$\frac{\zeta^2(\rho)}{\rho} \int_0^{2\pi} f_u(u) \cdot u_{\theta\theta} d\theta = -\frac{\zeta^2(\rho)}{\rho} \int_0^{2\pi} u_{\theta} \cdot D^2 f(u) \cdot u_{\theta} d\theta \leq M \int_{\partial B_\rho(x_0)} |Du|^2 ds.$$

We conclude then that

$$\int_{B_R(x_0) \setminus B_r(x_0)} f_u(u) \cdot \zeta^2 \frac{u_{\theta\theta}}{\rho^2} \, dx \leq M \int_{B_R(x_0)} |Du|^2 \, dx \leq C_2. \quad (3.4)$$

Next using the estimate

$$\int_{B_R(x_0)} (|f_u(u(x))|^2 + |Du(x)|^2) \, dx < \infty$$

and the same argument as above on almost every line parallel to one of the coordinate axes it follows that $f(u(x)) \in W^{1,1}(B_R(x_0))$ and $Df(u) = f_u \cdot Du$ almost everywhere.
on $B_R(x_0)$. With this fact we see that
\[
\int_{B_R(x_0) \setminus B_r(x_0)} \zeta^2 f_u(u(x)) \cdot \frac{u_\rho}{\rho} \, dx
\]
\[= \int_0^{2\pi} \int_r^R \zeta^2(\rho)(f(u(\rho, \theta)))_\rho d\rho d\theta \]
\[= -r^{-1} \int_{\partial B_r(x_0)} f(u) ds - \int_{B_R(x_0) \setminus B_{R/2}(x_0)} f(u) \frac{(\zeta^2)_\rho}{\rho} \, dx \]
where for the last term we used the fact that $\zeta^2 = 1$ on $B_{R/2}(x_0)$.

Thus
\[
(3.5) \quad \int_{B_R(x_0) \setminus B_r(x_0)} \zeta^2 f_u(u(x)) \cdot \frac{u_\rho}{\rho} \, dx \leq -r^{-1} \int_{\partial B_r(x_0)} f ds + C_3. \]

Taking (3.2), (3.4), and (3.5) together with (3.1) we see that
\[r^{-1} \int_{\partial B_r(x_0)} f(u) ds \leq C_4 \quad \text{for } 0 < r < \frac{R}{2} \]
where $C_4$ is independent of $r$. Since $u(x)$ is continuous on $\Omega$ and $\lim_{u \to u_0} f(u) = f(u_0) \in (-\infty, \infty]$ for each $u_0 \in K$, we have
\[2\pi f(u(x_0)) \leq C_4, \]
where $C_4$ depends on $R$, $J(u)$, and $\|u\|_{H^2(B_R(x_0))}$. In particular $\Lambda = \emptyset$. Furthermore, applying Lemma 2.6 we see that $\text{dist}(u(x_0), \partial K) \geq c > 0$ where $c$ depends on $\text{dist}(x_0, \partial \Omega)$ and $J(u)$.

4 Applications to Liquid crystals.

We briefly describe the liquid crystal model that motivates our constrained problem and state our result as it applies to this case. A more detailed overview of energies for liquid crystals is given in [2]. Let $\Omega \subset \mathbb{R}^3$ be a region filled with rod–like liquid crystal molecules. For $x \in \Omega$ and $p \in S^2$ denote by $\rho(x, p)$ the probability distribution for the long axes of the molecules near $x$ aligned with the direction $p$. We have
\[
(4.1) \quad \rho(x, p) \geq 0, \quad \int_{S^2} \rho(x, p) dp = 1. \]
If the directions are random so that no direction is preferred then \( \rho(x, p) = \rho_0 = \frac{1}{4\pi} \) and the liquid crystal is in the isotropic state at \( x \). The de Gennes \( Q \) tensor is introduced as a macroscopic order parameter

\[
Q(x) = \int_{S^2} (p \otimes p - \frac{1}{3} I) \rho(x, p) dp
\]

representing the second moments of \( \rho \) normalized so that \( Q = 0 \) if \( \rho = \rho_0 \). Set \( S_0 = \{ A \in M^{3 \times 3} : A = A^t, tr A = 0 \} \). Then from (4.1) and (4.2) we see that \( Q \) takes on values in the open, bounded and convex set

\[
\mathcal{M} = \{ A \in S_0 : -\frac{1}{3} < \lambda_{\min}(A) \leq \lambda_{\max}(A) < \frac{2}{3} \}
\]

where \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) are the minimum and maximum eigenvalues of \( A \) respectively. This is the set of physically attainable states in \( S_0 \). A free energy for constant nematic liquid crystal states identified with \( Q \in \mathcal{M} \) was developed by Katriel, Kventsel, Luckhurst and Sluckin [9] and Ball and Majumdar [3]. Assuming Maier–Saupe molecular interactions it takes the form

\[
\begin{align*}
\psi_b(Q) &= Tf_{ms}(Q) - \kappa |Q|^2 & \text{for } Q \in \mathcal{M}, \\
&= \infty & \text{for } Q \in S_0 \setminus \mathcal{M}, \\
f_b(Q) &= \inf_{\rho \in A_Q} \left( \int_{S^2} \rho \log \rho dp \right)
\end{align*}
\]

where

\[
A_Q = \{ \rho \in L^1(S^2) : \rho \geq 0, \int_{S^2} \rho(p) dp = 1, \quad Q = \int_{S^2} (p \otimes p - \frac{1}{3} I) \rho(p) dp \}
\]

and \( T, \kappa > 0 \). It is shown in [3] that \( f_{ms} \) is convex on \( \mathcal{M} \) with \( \lim_{Q \to \partial \mathcal{M}} f(Q) = \infty \) and it is shown in [6] that \( f \in C^{\infty}(\mathcal{M}) \). Ball and Majumdar then used \( \psi_b \) to define an energy functional to characterize stable spatially varying liquid crystal configurations. They considered local minimizers \( Q \in H^1(\Omega, \mathcal{M}) \) to

\[
I_{\text{LdG}}[Q] = \int_{\Omega} (G(Q, DQ) + \psi_b(Q)) dx
\]

where following [10] the elastic energy density takes the form

\[
G(Q, DQ) = \sum_{i=1}^{5} L_i I_i(Q, DQ),
\]
such that

\[ I_1 = D_{x_k} Q_{ij} D_{x_k} Q_{ij} \quad I_2 = D_{x_j} Q_{ij} D_{x_k} Q_{ik}, \]
\[ I_3 = D_{x_j} Q_{ik} D_{x_k} Q_{ij} \quad I_4 = Q_{\ell k} D_{x_\ell} Q_{ij} D_{x_k} Q_{ij}, \]
\[ I_5 = \epsilon_{\ell j k} Q_{\ell i} D_{x_j} Q_{ki}. \]

These are polynomial expressions in terms of \( Q \) and \( DQ \) that satisfy the principle of frame indifference and material symmetry. The expressions \( I_1, I_2, I_3 \) and \( I_4 \) satisfy both properties. Analytically this means that \( I_1, \ldots, I_4 \) are invariant under transformations over \( O(3) \) while \( I_5 \) satisfies only frame indifference and is invariant under transformations over \( SO(3) \). The first four terms are quadratic in \( DQ \) while the fifth is linear and is included in the elastic energy density when modeling chiral liquid crystals. Here \( \epsilon_{\ell j k} \) is the Levi–Civita tensor.

Let \( D := \{ D = [D_{ijk}] \mid 1 \leq i, j, k \leq 3 : D_{ijk} = D_{jik} \text{ and } \sum_{i=1}^{3} D_{\ell i k} = 0 \text{ for each } i, j \text{ and } k \} \). The elasticity constants are to be chosen so that \( \sum_{\ell=1}^{4} L_\ell I_\ell(Q, D) \geq c_0 |D|^2 \) for some \( c_0 > 0 \) for all \( Q \in \mathcal{M} \) and \( D \in D \). Inequalities that ensure the coercivity condition are

\[ L'_1 + \frac{5}{3} L_2 + \frac{1}{6} L_3 > 0, \quad L'_1 - \frac{1}{2} L_3 > 0, \quad L'_1 + L_3 > 0 \]

where

\[ L'_1 = \begin{cases} L_1 - \frac{1}{3} L_4 & \text{if } L_4 \geq 0 \\ L_1 + \frac{2}{3} L_4 & \text{if } L_4 \leq 0. \end{cases} \]

(See [2]).

The space \( S_0 \) has dimension five. If we take an orthonormal basis \( \{E_1, \ldots, E_5\} \) we can parameterize the space with the isometry

\[ Q(v) : \mathbb{R}^5 \rightarrow S_0, \quad Q(v) = \sum_{j=1}^{5} v_j E_j. \]

Set \( \mathcal{K} = \{ v \in \mathbb{R}^5 : Q(v) \in \mathcal{M} \} \), \( f(v) = \psi_b(Q(v)) \) and \( F(v, Dv) = G(Q(v), DQ(v)) \). Then \( \mathcal{K} \) is an open, bounded and convex region in \( \mathbb{R}^5 \), \( f \) satisfies (1.2) and \( F \) satisfies (1.3). For the case \( n = 2 \) we view the liquid crystal body on the infinite cylinder \( \Omega \times \mathbb{R} \) where \( \Omega \subset \mathbb{R}^2 \) is the cylinder’s cross–section and the order parameter is \( Q = Q(x_1, x_2) \). We can then apply our results from Theorem 1.1 to \( J[v] = I_{LdG}[Q(v)] \) to obtain:
Theorem 4.1. Let $\Omega \subset \mathbb{R}^2$ and let $Q \in H^1(\Omega; \mathcal{M})$ be a finite energy local minimizer for $I_{LdG}$ satisfying (4.4)-(4.7). Then $Q \in C^2(\Omega)$. If $E \subset \subset \Omega$ then $Q(E) \subset \subset \mathcal{M}$ and $Q$ satisfies the equilibrium equation

$$[\text{div}G_D(G, DQ) - G_Q(Q, DQ) - \psi_{b,Q}(Q)]^{\text{st}} = 0 \text{ in } \Omega.$$ 

where $[A]^{\text{st}}$ is the symmetric and traceless part of $A \in \mathcal{M}^{3 \times 3}$.

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