Effects of dynamical noises on Majorana bound states

Roya Radgohar and Mehdi Kargarian
Department of Physics, Sharif University of Technology, Tehran 14588-89694, Iran

I. INTRODUCTION

Majorana bound states (MBSs) appear at the end of one-dimensional topological superconductors or in the vortex cores of two-dimensional chiral superconductors. Operationally a MBS is a fermionic quasiparticle that is its own antiparticle, i.e., $\gamma^\dagger = \gamma$. Therefore, the emergence of MBSs in a solid state system relies on equal superposition of electron and hole states, forming chargeless quasiparticles, and fermions with only one spin projection, e.g., the spinless fermions, are involved in the formation of Majorana states. The one-dimensional spinless Kitaev superconductor with $p$-wave pairing potential is topologically nontrivial, supporting MBSs at the ends of open chain in the weak coupling regime. However, any material design of a one-dimensional superconductor requires lifting the spin degeneracy.

Semiconductor heterostructures consisting of conventional materials such as nanowires with strong Rashba spin-orbit coupling in proximity to $s$-wave superconductors were proposed to exhibit nontrivial band topology, promising a hybrid structure supporting MBSs. The semiconductor heterostructure is shown schematically in the left panel of Fig. 1, where a nanowire of InSb (InAs) is grown on the surface of $s$-wave superconductor NbTiN (Al). Physically the strong Rashba coupling removes the spin degeneracy of electron states near the Fermi level, and a sizable Zeeman field can remove one of the energy bands. Hence, the single-particle states become effectively spinless giving rise to odd parity for pairing potential induced by the underneath superconductor. An observation of zero-bias conductance in hybrid structure signified the existence of MBSs at the ends of the nanowire. Another hybrid structure consists of a ferromagnetic chain of iron atoms deposited on the surface of a superconductor as shown in the right panel of Fig. 1. The intrinsically ordered magnetic moments break the time-reversal symmetry, eliminating the need for an external magnetic field. Moreover, the angle between adjacent moments induces inter-spin component of hopping terms that mimics the effects of spin-orbit coupling. An observation of zero-bias tunneling conductance has been associated to MBSs, though there are other explanations as well.

Besides the fundamental importance of MBSs in our understanding of exotic quantum states, the surge of recent interests on MBSs originates in possible use of them to build topologically protected qubits and perform fault-tolerant quantum computation. The degenerate subspace of multiple MBSs provides a topological memory to store quantum information and a proper set of braiding of non-Abelian quasiparticles serves as gate operations on quantum states, all immune to local errors.

Although, the topological qubits have some degree of robustness, especially against static disorder, they generically suffer from the time-dependent fluctuations of intrinsic properties of system as well as coupling to the environment. The latter coupling breaks the fermion parity– an important ingredient of existence of MBSs– through injection or removal of quasiparticles, giving rise to dynamic fluctuations that completely destroy coherence of Majorana qubits. Even the coupling to a parity-preserving reservoir such as finite-temperature bosonic bath can also destabilize MBSs, giving rise to an exponentially decay in correlation between MBSs and exposing braiding processes to errors. However, one can find a regime of parameters where there exists a long-lived quantum correlation between Majorana fermions in the presence of colored Markovian noise. For coupling to an Ohmic-like fermionic or bosonic bath with spectral density $\rho(\omega) \propto \omega^\alpha$, while the MBSs are robust in super-Ohmic regime $Q > 1$, the coherence of zero modes is strongly suppressed in the Ohmic and sub-Ohmic regimes with $Q \leq 1$. The non-equilibrium noise effects coming from trijunction setups, despite conserving parity, decrease the coherence time of Majorana qubits.

Since the noises are ubiquitous and indispensable in any physical system which could host MBSs, and hence, any successful protocol of quantum computation including initialization of qubits, implementation of gates, and readout is potentially subject to noises from various sources. Our paper is intended to investigate the effects of several noises on the robustness of MBSs and find the regime of parameters where the suffering effects of time-dependent noises are minimal. To this end, we focus on a class of noise sources relevant to the experimental setups inducing time-dependent fluctuations in chemical potential such as Lorentzian, thermal, and point con-
We identify the transition probability from zero-energy level to excited states as a measure for the fragility of MBSs against noises\(^{25}\) in three one-dimensional models: the \(p\)-wave Kitaev chain, Rashba nanowire, and magnetic chains, all in topological superconducting phases with MBSs at the ends of the chains. As discussed above, the last two models shown in Fig. 1 are relevant to the current experimentally designed heterostructures, calling for determination of regimes of parameters where the effects of noises are minimal. For the Kitaev chain it is shown that the repulsive electron-electron interactions between nearest-neighbor sites decrease the decoherence rate\(^{25}\), while the long-range many-body interactions between fermions reduce the lifetime of MBSs\(^{25,26}\). We instead consider the effects of long-range tunnelings and superconducting pairings on transition probability. For the nanowire proximized to the surface of an \(s\)-wave superconductor, the effects of strong Rashba coupling and Zeeman field on the robustness of bound states are studied. In particular we show that the stronger the former is, the more resilience against noises is achieved, a finding which could be important in looking for proper heterostructures with enhanced robust MBSs.

The paper is organized as follows. In Sec. II, we introduce the noise models and the transition rate. We begin with a generalized version of the Kitaev chain with long-range hoppings and pairings in Sec. III and numerically calculate the transition rate for MBSs. In Sec. IV, the effects of noises on MBSs in semiconductor nanowires in the presence of strong spin-orbit coupling and magnetic field are presented, and in Sec. V the results for a magnetic atomic chain on the surface of superconductor are presented. We conclude in Sec. VI.

### II. NOISE MODELS

Before delving into the details of MBSs in one-dimensional systems and their resilience, in this section we introduce several noise models related to the hybrid structures and present a mathematical framework on how to calculate the transition probability of the MBSs to excited states.

One of the sources of noise which is intrinsic to the electronic materials is the charge noise resulting from the quantum fluctuations of occupation numbers. This noise manifests itself as time-dependent fluctuations in chemical potential. The fluctuations in the electron spin states caused by the nuclear spin fluctuations is also another source of noise. Here the chemical potential of each spin projection fluctuates.

Recent experiment shows that these two noise sources reveal a frequency spectrum\(^{25}\) that is described by a Lorentzian distribution function as

\[
S_{\text{Lorentz}}(\omega) = S_0(1 + (\omega - \omega_0)^2/(\delta\omega)^2)^{-1},
\]

where \(\omega_0\) is the central frequency, \(\delta\omega\) is the bandwidth, \(S_0\) is the amplitude of the spectrum. The limit of \(\delta\omega \to 0\) recovers quasi-monochromatic frequency spectrum and the limit of \(\delta\omega \to \infty\) corresponds to the quasi-white noise which contains equal contributions from all frequencies. We assume \(\delta\omega = 1\) throughout the paper. This kind of noise spectrum has been used to describe the effects of an externally random fluctuating noise on physical systems\(^{26}\).

Besides the intrinsic noise sources described above, the thermal fluctuations are another source of noise. At non-zero temperature, the thermal fluctuations give rise to fluctuations in the occupation number of energy states and consequently in the chemical potential. In thermal equilibrium, the frequency spectrum of thermal noise is given by\(^{25,27}\)

\[
S_{\text{Thermal}}(\omega) = S_0\exp(-\hbar\omega/k_B T),
\]

where \(k_B\) is the Boltzmann constant, \(T\) is the temperature, and \(\hbar\) is the reduced Planck constant.

The quantum transport across a quantum point contact (QPC) between a superconductor and a semiconductor or magnetic atomic chain in hybrid structures suffers from a non-equilibrium electrical current noise known as shot noises. The latter is a consequence of random transfer of quantized charged carriers through mesoscopic conductors. If the energy of an electron impinging on the surface of superconductor is smaller than the superconducting gap \((E < \Delta)\), it can be Andreev reflected, through which a hole is reflected back to the semiconductor and a Cooper pair with charge \(2e\) is injected to superconductor. The reverse process is also possible where a Cooper pair recombines with a hole in the semiconductor and produces an electron. In equilibrium, both processes occur with equal probability, leading to no net current flow. Hence, a bias voltage \((V)\) across the junction of semiconductor-superconductor is required to achieve a finite current flow. Using the scattering theory, the frequency spectrum of the shot noises for both cases have been computed in Ref.[28]. At zero temperature, they are as follows:

\[
S_{eq}(\omega) = \frac{2e^2}{\pi} \sum D_n n, \quad (3)
\]

\[
S(\omega) = \frac{2e^2}{\pi} \sum D_n^2 + \frac{4e^3V}{\pi \hbar} \sum D_n(1 - D_n), \quad (4)
\]

where \(S_{eq}\) is the shot noise in equilibrium \((V = 0)\), and \(S\) is the shot noise at finite applied voltage \((\hbar \omega < eV)\). Here \(D_n = T_n^2/(2 - T_n)^2\), where \(T_n\) is the \(n\)th transmission eigenvalue between the interface and semiconductor. The difference \(S(\omega) - S_{eq}\) is used to characterize the QPC noise (also called excess noise) as\(^{25,28}\).
\[ S_{QPC}(\omega) = S_0 \left( 1 - \frac{\hbar \omega}{eV} \right), \] (5)

where \( S_0 = (2e^3V/\pi \hbar) \sum_n D_n(1-D_n) \).

Having introduced several dynamical noise spectra in equations (1), (2), and (3), we now discuss how the effects of latter noises on the MBSs are taken into account, which is the main subject of this work. We also ignore other sources which could lead to fluctuations in spin-orbit interactions\(^{25-31}\) and superconducting pairings\(^{25-31}\). Following Ref. [21], we assume that the dynamical noises perturb the chemical potential as \( \mu(t) = \mu + \zeta f(t) \), where \( \zeta \) is the unperturbed chemical potential, \( \zeta \) is the coupling constant, and \( f(t) \) is the interacting potential amplitude encoding the information about the type of noise under consideration. The latter term perturbs the Hamiltonian as \( H = H_0 + \zeta f(t)M \), where the unperturbed Hamiltonian is given by \( H_0 \), and \( M \) is a density operator associated with the change in the chemical potential which will be specified for our models in next sections. Let us denote the zero-energy state by \( |0\rangle \) and the excited states by \( |q\rangle \). The transition probability out of \( |0\rangle \) is given by

\[ P(t) = \sum_q |\langle q|U(t)|0\rangle|^2, \] (6)

where \( U(t) \) is the time-evolution operator which will be specified shortly. Assuming the coupling \( \zeta \) is small, we may apply the first order time-dependent perturbation theory to obtain the following expression for the time-evolution operator:

\[ U(t) \approx U_0(t) + \frac{\zeta}{\hbar} \int_0^t U_0^\dagger(\tau) f(\tau)MU_0(\tau)d\tau, \] (7)

where \( U_0(t) = e^{-iH_0t/\hbar} \). Since we are interested in averaged time evolution of the system, we obtain the average probability \( \bar{P} \)

\[ \bar{P}(t) = \frac{\zeta^2}{\hbar^2} \sum_q \int_0^t \int_0^t d\tau d\tau' \langle f(\tau)f(\tau') \rangle \langle q|U_0^\dagger(\tau)MU_0(\tau)|0\rangle \times (0|U_0(\tau')MU_0^\dagger(\tau')q) \] (8)

where the noise correlation function \( \langle f(\tau)f(\tau') \rangle \) is related to the frequency spectrum of noise \( S(\omega) \) in equations (1), (2), and (5), as\(^{32}\)

\[ \langle f(\tau)f(\tau') \rangle = \int \frac{d\omega}{2\pi} e^{i\omega(\tau'-\tau)} S(\omega). \] (9)

Finally, the probability rate is given by the time-derivative of \( \bar{P} \). It reads as

\[ \Gamma \equiv \frac{d\bar{P}}{dt} = \frac{\zeta^2}{\hbar^2} \sum_q |\langle q|M|0\rangle|^2 \int d\omega S(\omega) \delta (\omega - \epsilon_q/\hbar), \] (10)

where \( \epsilon_q \) is the eigenenergy of \( q \)-th excited state. For simplicity, we take \( \zeta^2S_0/\hbar^2 = 1 \) and define \( \omega_D = \Delta/\hbar \) as a frequency associated to superconducting gap. In the following sections we use Eq. (10) to evaluate the effect of various noise sources on the MBSs.

### III. KITAEV \( p \)-WAVE CHAIN WITH LONG-RANGE HOPPINGS AND PAIRINGS

The simple theoretical model satisfying both conditions of equal superposition of electron and hole states and having only one spin species is the Kitaev chain introduced in Ref. [2]. The model is composed of spinless fermions with nearest-neighbor tunnelings and superconducting pairings. In the weak coupling regime the bulk states are topological and MBSs appear at the ends of an open chain. While the original model has short-range hopping and pairing amplitudes, the recent theoretical and experimental works have generalized the Kitaev chain to include long-range interactions\(^{13-36}\). Our main objection in this section is to study the influence of long-range interactions in Kitaev chain on the sensitivity of MBSs when subjected to noise sources introduced in preceding section.

The generalized Kitaev chain is obtained by letting hopping and pairing amplitudes to extend to \( r \)-th and \( s \)-th neighbors, respectively. The Hamiltonian reads as

\[ H_0 = -\sum_{j=1}^N \mu \left( a_j^\dagger a_j - \frac{1}{2} \right) - \sum_{j=1}^r \sum_{l=1}^{N-l} \left( J a_j^\dagger a_{j+l} + h.c. \right) + \sum_{s=1}^N \sum_{j=1}^{N-l} \left( \Delta a_j a_{j+l} + h.c. \right), \] (11)

where \( \mu \) is chemical potential, \( N \) denotes total number of sites, and \( a_j(a_j^\dagger) \) is a fermionic annihilation (creation) operator. Moreover, the strength of long-range hoppings and pairings decreases with the distance between sites as power law functions \( J_l = J_0 l^{-\nu_r} \) and \( \Delta_l = \Delta_0 l^{-\nu_s} \), respectively, where \( J_0 \) and \( \Delta_0 \) are the corresponding nearest-neighbor values and the exponents of \( \nu_r \) and \( \nu_s \) control the strength of amplitudes so that \( \nu_r, \nu_s < 1(\nu_r, \nu_s > 1) \) correspond to long (short)-range interactions. Taking the limit \( \nu_r, \nu_s \to \infty \), the original Kitaev model is recovered.

The model (11) can be simulated in cold atomic gases interacting through tunable Feshbach resonance\(^{37-39}\) or in a setup of planar Josephson junctions in proximity to a 2D electron gas where long-range pairings and hoppings are controlled experimentally\(^{40}\). The phase diagram of the Hamiltonian (11) contains topological superconducting phases with MBSs\(^{36}\).

The model is much easier to analyze in Majorana representation of fermion operators. The transformation reads as

\[ a_j = \frac{1}{2}(c_{2j-1} + ic_{2j}) \quad a_j^\dagger = \frac{1}{2}(c_{2j-1} - ic_{2j}), \] (12)

where the Majorana operators satisfy the Clifford algebra \( \{c_i, c_j\} = 2 \delta_{i,j} \) for \( i, j = 1, \ldots, N \). The Hamiltonian becomes

\[ H_0 = -\sum_{j=1}^N \frac{\mu}{2} c_{2j-1} c_{2j} + \frac{\mu}{2} \sum_{j=1}^r \sum_{l=1}^{N-l} \frac{1}{\epsilon_j} \left( c_{2j} c_{2(j+l)-1} - c_{2j-1} c_{2(j+l)} \right) + \frac{i\Delta_0}{2} \sum_{s=1}^N \sum_{j=1}^{N-l-1} \frac{1}{\epsilon_j} \left( c_{2j-1} c_{2(j+l)} + c_{2j} c_{2(j+l)-1} \right). \] (13)
We consider a finite open chain with odd number of sites $N$ and $r = s = (N - 1)/2$, and work in a regime of parameters where the model is in a topological superconducting phase. For the sake of simplicity and arguments we consider two cases separately: 1) the case of nearest-neighbor pairing and long-range hoppings is studied in Sec. III A, and 2) the case of nearest-neighbor hopping and long-range pairings is discussed in Sec. III B. To connect to our discussions of noises in the preceding section Sec. II, a dynamical shift in chemical potential $\mu \rightarrow \mu + \zeta f(t)$ in Eq. (13) yields $M = \sigma y$, where $\sigma y$ is the Pauli matrix. Using Eq. (10) and eigenvectors and eigenvalues of the Hamiltonian, we evaluate the transition rate $\Gamma$ of MBSs.

### A. Kitaev chain with long-range hopping

For this case the hopping terms are long-ranged, i.e., $\nu_r$ in Eq. (13) is finite, but the nearest-neighbor pairing is obtained by taking the limit $\nu_s \rightarrow \infty$. This model has a rich phase diagram studied in Ref.[36]. The topological phase is characterized by the nontrivial winding numbers $w = \pm 1$ in the following regime of parameters:

$$-2J_0 \sum_{l=1}^{N-1} \frac{1}{p_r} < \mu < 2J_0 \sum_{l=1}^{N-1} \frac{(-1)^{l+1}}{p_r}.$$  \hspace{1cm} (14)

In this regime the chain hosts Majorana modes localized at the ends. The transition probabilities of MBSs affected by distinct types of noises are shown in Fig. 2. In all panels the black solid curve corresponds to the behavior of $\Gamma$ in the original Kitaev model obtained by $\nu_r \rightarrow \infty$. Therefore, the plots provide insights on how the range of hopping affects the transition.

The first panel exhibits the behavior of $\Gamma$ versus the central frequency $\omega_0/\omega_D$ of the Lorentzian noise in Eq. (1). For all range of hoppings $\nu_r$, a peak appears for $\omega_0/\omega_D < 1$, which is attributed to the resonance with superconducting gap. It is seen that the strength of the long-range hopping can significantly affect the transition probability. In the regime of short-range hopping interactions ($\nu_r > 1$), the probability rate surpasses the corresponding values of the original Kitaev model (black curve), and by further increase of $\nu_r$ the curves approach the latter model. In the long-range hopping regime, where $\nu_r < 1$, the probability rate shows a totally different behavior. For values around $\nu_r = 0.2$, the $\Gamma$ is quite large, while for $\nu_r = 0.4$ is exceedingly small. An inspection of Eq. (10) shows that two factors conspire to determine the probability rate: the transition matrix element $\langle q|\mathbf{M}|0\rangle^2$ and the accumulation of states whose energies $\epsilon_q$ are close to $\omega_0$. The latter makes $S_{\text{Lorentz}}$ quite appreciable for many states.

We found that for example for $\nu_r = 0.2$ the matrix element is rather large for states near the energy gap. Also, the gap in the energy spectrum is small, and therefore many states $|q\rangle$ contribute to the noise spectrum which is detrimental in having small values of $\Gamma$. For other values of long-rang hopping, say $\nu_r = 0.4, 0.6$, the gap in the spectrum pushes many states away from MBSs, suppressing the transition probability rate. On the other hand, for $\nu_r > 1$ the superconducting energy gap is relatively large, so less states are involved in the noise spectrum, and since the $\langle |q|\mathbf{M}|0\rangle^2$s become rather large for states near the gap, a relatively large value of $\Gamma$ arises. And, in the limit of $\nu_r \rightarrow \infty$ the original Kitaev model is reached out.

The results for thermal and QPC noises are shown in Fig. 2(b) and (c), respectively. The transition rate increases at high temperatures, since the thermal weight in Eq. (2) is relatively large for many states and they contribute in $\Gamma$. However, it turns out at least for some ranges of small values of long-range hopping strength $\nu_r$ the transition rate can be significantly suppressed. For the QPC noise the values of transition rate do not change with gate voltage, however, again it’s seen that there exists a window of $\nu_r$ where the transition rate is decreased substantially. We note that the general behavior of probability rate with $\nu_r$ is similar for all three noise sources.

FIG. 2. The probability rate to excite MBSs in an extended Kitaev chain with long-range hopping in the presence of (a) Lorentzian, (b) thermal, and (c) QPC noises. The parameters used are $N = 101$, $J_0 = 1$, $\mu = -1$ and $\Delta_0 = 0.1$. The value of $\nu_r$ determines the strength of long-range hoppings; $\nu_r \rightarrow \infty$ corresponds to the original Kitaev model.
B. Kitaev chain with long-range pairing

Next we move to the second case of non-local superconducting pairing amplitudes given by finite value of \( \nu_s \) in Eq. (13), while the hopping amplitudes are restricted to nearest-neighbor sites. Again note that the limit \( \nu_s \to \infty \) recovers the original Kitaev model. The model with finite \( \nu_s \) exhibits a nontrivial topological phase in a parameter range of \(-2 < \mu/J_0 < 2^{36}\).

The results of transition rate for three noise sources are shown in Fig. 3. Again the black curves in all panels show the variation of \( \Gamma \) for the original Kitaev model. It is clearly seen that for all types of sources the transition rate for the latter model lies at the upper limit of curves. A striking feature of these plots is that as the strength of the long-range pairing is increasing by decreasing \( \nu_s \), the transition probability rate is reduced. The reason can be traced back to the energy gap, the number of energy states close to the zero-energy state, and the matrix elements as discussed above. Indeed, for this case the energy gap is increasing smoothly with increasing \( \nu_s \), while the values of matrix elements remains rather small. Both effects then cooperate in yielding comparatively small values for \( \Gamma \). Having established such a unique behavior, the results show that the harmful effects of noises on MBSs can be reduced in systems with long-range superconducting pairing amplitudes.

IV. NANOWIRES IN PROXIMITY TO AN s-WAVE SUPERCONDUCTOR

In this section we present the results for noise on the MBSs in one of the most realistic and experimentally realized platforms. A schematic of the model is shown in the left panel of Fig. 1(a). The system is a heterostructure of a semiconductor nanowire in proximity to an s-wave superconductor. The role of latter superconducting substrate is to induce pairing potential into the nanowire. The main microscopic ingredients to have a topologically nontrivial pairing gap in the nanowire are strong spin-orbit coupling and a moderate Zeeman field, which is provided by a magnetic field\(^{15} \). The heterostructure has been designed experimentally with strong evidence of the existence of MBSs appearing at the open ends of the nanowire\(^5 \). Our objection is to investigate the effects of noises on the MBSs and determine the range of parameters where the latter states remain less influenced by noises.

The continuum model Hamiltonian capturing the main physics of topological superconductor in this heterostructure is\(^{15,41} \)

\[
H = \sum_{j,\lambda} \int_0^L dx \psi_\lambda^\dagger(x) \left( -\frac{\hbar^2 \partial_x^2}{2m^*} - \mu + i\alpha \sigma_\lambda \sigma_x + h\sigma_x \right) \psi_\lambda(x) + \Delta \int_0^L dx \left( \psi_\lambda^\dagger(x) \psi_{\lambda'}(x) + h.c. \right),
\]

where \( m^* \) and \( \mu \) are the effective mass and chemical potential, respectively. The third term describes the Rashba spin orbit coupling (RSOC) in semiconductor nanowire which lifts the spin degeneracy. The Zeeman energy \( h = g_\mu B \), where \( g \) is the Lande g-factor and \( \mu_B \) is the Bohr magneton, opens a gap in the energy spectrum. A strong enough magnetic field can push one of the bands above the Fermi level, and therefore creates single-degenerate electron state near the Fermi level.

Since we are interested in full spectrum of an open chain, in the following we use the corresponding Hamiltonian on a lattice. The Hamiltonian reads as\(^{12} \)

\[
H = -\mu \sum_{j,\lambda} a_\lambda^\dagger a_\lambda - J \sum_{j=1}^{N-1} \left( a_{\lambda,j}^\dagger a_{\lambda,j+1} + h.c. \right) + \alpha \sum_{j,\lambda,j'=1}^{N-1} \left[ i\sigma_\lambda \left( a_{\lambda,j}^\dagger a_{\lambda+1,j'} - a_{\lambda+1,j}^\dagger a_{\lambda,j} \right) \right] - h \sum_{j,\lambda,j'=1}^{N} \left( a_\lambda^\dagger a_{\lambda,j} + h.c. \right). 
\]

Representation of this Hamiltonian in terms of the Majorana fermions is given in Appendix A. When \( h >
The probability rate of MBSs in a Rashba nanowire with \( N = 401 \). Different solid colored curves in top (bottom) row are for different values of magnetic field \( h \) (spin-orbit coupling \( \alpha \)). The panels are for (a,d) Lorentzian, (b,e) thermal, and (c,f) QPC noises. The parameters are \( J = 1, \mu = -2 \) and \( \Delta = 0.1 \). In top panels \( \alpha = 0.1 \) and in bottom panels \( h = 1 \).

Another very important parameter as discussed above is the spin-orbit coupling \( \alpha \). The results are shown in the second row of panels in Fig.4. For the Lorentzian noise, the transition rates for several values of \( \alpha \) are shown in Fig.4(d). The results indicate that for small values of \( \alpha \) the transition rate becomes large for lower part of spectrum. Our detailed analysis show that for small \( \alpha \), despite having a small gap, the matrix element is large for excited states near the gap giving rise to a large transition rate. It starts diminishing by increasing \( \alpha \) within the low-frequency window of noise spectrum. For larger frequencies, however, the rise of matrix elements leads to increment of transition rate. The results for thermal and QPC noises are shown in Fig.4(e) and (f), respectively. Now we see that the noise effects are substantially diminished by increasing \( \alpha \), and thus the transition rate is decreased. These results show that choosing nanowires with large spin-orbit interaction will make the MBSs more immune to noises.

V. CHAIN OF MAGNETIC ATOMS ON A SUPERCONDUCTOR

The last system we study is a linear chain of magnetic atoms deposited on the surface of an s-wave superconductor, as schematically shown in the right panel of Fig. 1. Using the state-of-art spin-polarized scanning tunneling microscopy, it is observed that magnetic chains with more than eight atoms exhibit stable Néel states which is described by the classical spin model aligned along a local axis. The magnetic ordering naturally breaks the time-reversal symmetry, and therefore the need for an applied external field is lifted. The magnetic texture induces the effective spin-orbit interaction as electrons move along the chain.
A model Hamiltonian describing the above observation is as follows:

\[
H = J \sum_{j,N} a_j^{\dagger} a_{j+1,0} + \sum_{j,N} \left[ (B_j \cdot \sigma)_{j,j'} - (\mu \phi_{j,j'}) \right] a_j^{\dagger} a_{j,j'} + \sum_{j} \Delta (a_j^{\dagger} c_j + c_j^{\dagger} a_j)
\]

where the magnetic field \( B = B \hat{r} \) with \( \hat{r} = (\sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}) \) and \( \lambda, \lambda' \) are for up and down spin projections. To diagonalize the Hamiltonian we rotate the spins in a local basis with the quantization axis directed along the unit vector \( \hat{r} \) and without loss of generality assume that \( \phi = 0 \), the details of this transformation and subsequent Majorana representation are relegated to Appendix B.

For Zeeman fields satisfying

\[
\sqrt{\Delta^2 + (|\mu| - 2|J f|)^2} < |B| < \sqrt{\Delta^2 + (|\mu| + 2|J f|)^2},
\]

where \( f = \cos \theta / 2 \) (see Appendix B), the superconducting model (17) becomes topologically nontrivial.

When exposed to dynamical noises, the results of transition rate \( \Gamma \) are shown in Fig. 5. All panels show a qualitatively similar results to nanowire model discussed in preceding section. As seen in Fig. 5(a) by decreasing the Zeeman field the transition rate is reduced at the lower part of the spectrum. The gap in the quasiparticle spectrum deceases with the rise of the Zeeman field, and the matrix element of MBSs and low lying states increases simultaneously. The cooperation of these two effects gives rise to the enhancement of the transition rate.

### VI. CONCLUSIONS

Before summarizing the main findings of this work, let us recapitulate the main idea and outlines of what we have done. We started by posing an important question of how resilient the MBSs appearing at the open ends of one-dimensional topological superconductors are against dynamical noise sources. We studied the effects of three experimentally relevant time-dependent noises such as Lorentzian, thermal and QPC on MBSs in the Kitaev \( p \)-wave model, Rashba nanowires, and magnetic atomic chains.

We showed that in a topological phase the response of MBSs to noise sources depends on the microscopic parameters, and provide a pathway in selecting material combinations where the effects of noises are least. Our findings show that long-range pairings in the Kitaev chain, which can be tuned experimentally, reduce the destructive effects of noises and enhance the robustness of MBSs. For the experimentally realized Rashba nanowire and magnetic chain in proximity to an \( s \)-wave superconductor, which are the most promising proposals for realizing MBSs, we have shown that smaller magnetic fields yield more resilience MBSs. In the former case, we showed that the materials with strong Rashba-spin orbit coupling support highly robust MBSs in a noisy environment.

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### Appendix A: Majorana representation of superconducting nanowire Hamiltonian

In the basis of the \( 4N \)-components Nambu spinor \( \psi = [\ldots, a_j^{\dagger} a_j, a_{j+1}^{\dagger}, a_j, \ldots] \), the matrix representation of the Hamiltonian (16) can be obtained as:

\[
H_{4N \times 4N} = \begin{pmatrix}
H_1 & H_2 & 0 & 0 \\
H_2^T & H_1 & H_2 & 0 \\
0 & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & H_2^T & H_1 & H_2 \\
0 & 0 & \cdots & 0 & H_2^T & H_1
\end{pmatrix},
\]

where

\[
H_1 = \begin{pmatrix}
-\mu & -h & 0 & \Delta \\
-h & -\mu & -\Delta & 0 \\
0 & -\Delta & \mu & h \\
\Delta & 0 & h & \mu
\end{pmatrix},
H_2 = \begin{pmatrix}
-J & -\alpha & 0 & 0 \\
-\alpha & J & 0 & 0 \\
0 & 0 & J & \alpha \\
0 & 0 & -\alpha & J
\end{pmatrix}.
\]
In the Majorana basis, we use the following unitary transformation:
\[
\begin{pmatrix}
c_{j-1,1} \\
c_{j-1,1}^\dagger \\
i c_{j,1} \\
i c_{j,1}^\dagger
\end{pmatrix} = U \begin{pmatrix}
a_{j,1} \\
a_{j,1}^\dagger \\
\theta_{j,1} \\
\theta_{j,1}^\dagger
\end{pmatrix}
\]
where
\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1
\end{pmatrix}
\]
and rewrite the Hamiltonian in the following form:
\[
H = U_{4N\times 4N} H_{4N\times 4N} U_{4N\times 4N}^T.
\]
that yields \( M = I_N \otimes (\sigma^x \otimes I_2) \).

**Appendix B: Majorana representation of superconducting magnetic chain Hamiltonian**

Following the strategy used in \(^7\), we start with the Hamiltonian (17) and align the spin basis with the unit vector of \( \hat{n} \) by the following transformation\(^8\):
\[
\begin{pmatrix}
a_{j,1} \\
a_{j,1}^\dagger
\end{pmatrix} = U \begin{pmatrix}
b_{j,1} \\
b_{j,1}^\dagger
\end{pmatrix}
\]
where \( b_{j,\lambda} \) satisfies the same anti-commutation relation as \( a_{j,\lambda} \). The Hamiltonian in the new basis reads:
\[
H = J \sum_{j,j',\lambda} (\Omega_{j,j',\lambda} b_{j,\lambda}^\dagger b_{j',\lambda} + \Omega_{j,j',\lambda}^\dagger b_{j,\lambda} b_{j',\lambda}^\dagger) + \Delta \sum_{j} (a_{j,1}^\dagger a_{j,1} + a_{j,1} a_{j,1}^\dagger),
\]
where
\[
\Omega_{j} = U_{j}^\dagger U_{j+1} = \begin{pmatrix} f_{j} & -g_{j}^* \\
g_{j} & f_{j}^*
\end{pmatrix}
\]
and
\[
f_{j} = \cos(\theta_{j}/2) \cos(\theta_{j+1}/2) + \sin(\theta_{j}/2) \sin(\theta_{j+1}/2) e^{i(\theta_{j}-\theta_{j+1})}
\]
\[
g_{j} = \cos(\theta_{j}/2) \sin(\theta_{j+1}/2) e^{i\phi_{j+1}} - \sin(\theta_{j}/2) \cos(\theta_{j+1}/2) e^{i\phi_{j}}.
\]

To write the Hamiltonian in the Majorana basis, we use the following definitions
\[
b_{j,1} = \frac{1}{2} (c_{j-1,1} + ic_{j,1}), \quad b_{j,1}^\dagger = \frac{1}{2} (c_{j-1,1} - ic_{j,1}),
\]
as well as the assumptions of \( \phi_{j} = 0 \) and the constant angle \( \theta \) between nearest-neighbor moments. We define \( f_{j} := f = \cos(\theta/2) \) and \( g_{j} := g = \sin(\theta/2) \) and rewrite the Hamiltonian (B2) as:
\[
H = \frac{i}{2} f (c_{j-1,1} c_{j,1} + c_{j-1,1} c_{j,1}^\dagger + c_{j-1,1} c_{j,1} - c_{j-1,1} c_{j,1}^\dagger - c_{j-1,1} c_{j,1}^\dagger - c_{j-1,1} c_{j,1}^\dagger + c_{j-1,1} c_{j,1}^\dagger - c_{j-1,1} c_{j,1}^\dagger + c_{j-1,1} c_{j,1}^\dagger) + \frac{iB}{2} (c_{j-1,1} c_{j,1} + c_{j-1,1} c_{j,1}^\dagger - c_{j-1,1} c_{j,1} - c_{j-1,1} c_{j,1}^\dagger).
\]
Introducing the following Nambu spinor:
\[
\psi^j = (\ldots, c_{j-1,1}, c_{j-1,1}, ic_{j,1}, ic_{j,1}, \ldots),
\]
the matrix representation of the Hamiltonian reads as (A1) where
\[
H_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & -\mu + B & -\Delta \\
-\mu + B & 0 & \Delta & 0 \\
-\Delta & \Delta & 0 & 0 \\
-\mu - B & 0 & 0 & 0
\end{pmatrix}
\]
and
\[
H_2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & ft & -g^t \\
0 & 0 & gt & ft \\
ft & -gt & 0 & 0 \\
gt & ft & 0 & 0
\end{pmatrix}
\]
leading to \( M = I_N \otimes (\sigma^x \otimes I_2) \).

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* kargarian@physics.sharif.edu

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