The Jacobian Conjecture for the space of all the inner functions

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Abstract

We prove the Jacobian Conjecture for the space of all the inner functions in the unit disc.

1 Known facts

Definition 1.1. Let $B_F$ be the set of all the finite Blaschke products defined on the unit disc $\mathbb{D} = \{ z \in \mathbb{C} | |z| < 1 \}$.

Theorem A. $f(z) \in B_F \iff \exists n \in \mathbb{Z}^+ \text{ such that } \forall w \in \mathbb{D} \text{ the equation } f(z) = w \text{ has exactly } n \text{ solutions } z_1, \ldots, z_n \text{ in } \mathbb{D}, \text{ counting multiplicities.}$

That follows from [1] on the bottom of page 1.

Theorem B. $(B_F, \circ)$ is a semigroup under composition of mappings.

That follows by Theorem 1.7 on page 5 of [2].

Theorem C. If $f(z) \in B_F$ and if $f'(z) \neq 0 \forall z \in \mathbb{D}$ then

$$f(z) = \lambda \frac{z - \alpha}{1 - \overline{\alpha} z}$$

for some $\alpha \in \mathbb{D}$ and some unimodular $\lambda$, $|\lambda| = 1$, i.e. $f \in \text{Aut}(\mathbb{D})$.

For that we can look at Remark 1.2(b) on page 2, and remark 3.2 on page 14 of [2]. Also we can look at Theorem A on page 3 of [3].
2 introduction

We remark that the last theorem (Theorem C) could be thought of, as the (validity of) Jacobian Conjecture for $B_F$. This result is, perhaps, not surprising in view of the characterization in Theorem A above of members of $B_F$ (This is, in fact Theorem B on page 2 of [1]. This result is due to Fatou and to Rado). For in the classical Jacobian Conjecture one knows of a parallel result, namely:

If $F \in \text{et}(\mathbb{C}^2)$ and if $d_F(w) = |\{ z \in \mathbb{C}^2 | F(z) = w \}|$ is a constant $N$ (independent of $w \in \mathbb{C}^2$), then $F \in \text{Aut}(\mathbb{C}^2)$ (because $F$ is a proper mapping).

Thus we are led to the following,

**Definition 2.1.** Let $V_F$ be the set of all holomorphic $f : \mathbb{D} \to \mathbb{D}$, such that $\exists N = N_f \in \mathbb{Z}^+$ (depending on $f$) for which $d_f(w) = |\{ z \in \mathbb{D} | f(z) = w \}|$, $w \in \mathbb{D}$, satisfies $d_f(w) \leq N_f \forall w \in \mathbb{D}$.

We ask if the following is true:

$$f \in V_F, \ f'(z) \neq 0 \forall z \in \mathbb{D} \Rightarrow f \in \text{Aut}(\mathbb{D}).$$

The answer is negative. For example, we can take $f(z) = z/2$. So we modify the question:

$$f \in V_F, \ f'(z) \neq 0 \forall z \in \mathbb{D} \Rightarrow f(z) \text{ is injective}.$$

This could be written, alternatively as follows:

$$f \in V_F, \ f'(z) \neq 0 \forall z \in \mathbb{D} \Rightarrow \forall w \in \mathbb{D}, \ d_f(w) \leq 1.$$ 

Also the answer to this question is negative. For we can take $f(z) = 10^{-10}e^{10z}$ which will satisfy the condition $f(\mathbb{D}) \subset \mathbb{D}$ because of the tiny factor $10^{-10}$, while clearly $f \in V_F$ and $f'(z) \neq 0 \forall z \in \mathbb{D}$. But $d_f(w)$ can be as large as

$$\left[ \frac{2}{2\pi/10} \right] = \left[ \frac{10}{\pi} \right] = 3.$$

Thus we again need to modify the question (in order to get a more interesting result). It is not clear if the right assumption should include surjectivity or almost surjectivity. Say,

$$f \in V_F, \ f'(z) \neq 0 \forall z \in \mathbb{D}, \ \text{meas}(\mathbb{D} - f(\mathbb{D})) = 0 \Rightarrow f \in \text{Aut}(\mathbb{D}),$$

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where \( \text{meas}(A) \) is the Lebesgue measure of the Lebesgue measurable set \( A \).

Or maybe,
\[
f \in V_F, \ f'(z) \neq 0 \ \forall \ z \in \mathbb{D}, \ \lim_{r \to 1^-} |f(re^{i\theta})| = 1 \text{ a.e. in } \theta \Rightarrow f \in \text{Aut}(\mathbb{D}).
\]

This last question could be rephrased as follows:
\[
f \in V_F, \ f'(z) \neq 0 \ \forall \ z \in \mathbb{D}, \ f \text{ is an inner function } \Rightarrow f \in \text{Aut}(\mathbb{D}).
\]

3 The main result

We can answer the two last questions that were raised in the previous section. We start by answering affirmatively the last question.

**Theorem 3.1.** If \( f \in V_F, \ f'(z) \neq 0 \ \forall \ z \in \mathbb{D}, \ f \text{ is an inner function, then } f \in \text{Aut}(\mathbb{D}).
\)

**Proof.**
We recall the following result,

**Theorem.** Every inner function is a uniform limit of Blaschke products.

We refer to the theorem on page 175 of [4]. Let \( \{B_n\}_{n=1}^{\infty} \) be a sequence of Blaschke products that uniformly converge to \( f \). Since \( f \in V_F \) there exist a natural number \( N_f \) such that \( d_f(w) \leq N_f \ \forall \ w \in \mathbb{D}. \) By Hurwitz Theorem we have \( \lim_{n \to \infty} d_{B_n}(w) = d_f(w) \ \forall \ w \in \mathbb{D} \) and hence \( d_{B_n}(w) = d_f(w) \) for \( n \geq n_w. \) We should note that \( n_w \) depends on \( w \) but it is a constant in a neighborhood of the point \( w. \) Hence the Blaschke products in the tail subsequence \( \{B_n\}_{n \geq n_w} \) all have a finite valence which is bounded from above by \( d_f(w) \) at the point \( w. \) In particular, the valence of these finite Blaschke products are bounded from above by the number \( N_f \) in definition 2.1 (of the set \( V_F \)). Hence we can extract a subsequence of these Blaschke products that have one and the same number of zeroes. Again by the Hurwitz Theorem it follows that \( d_f(w) = N \) is a constant, independent of \( w, \) and so by Theorem A in section 1 we conclude that \( f(z) \) is a finite Blaschke product with exactly \( N \) zeroes. By Theorem C in section 1 we conclude (using the assumption that \( f'(z) \neq 0, \ \forall \ z \in \mathbb{D} \)) that \( f(z) \in \text{Aut}(\mathbb{D}). \)

Next, we answer negatively the one before the last question.

**Theorem 3.2.** There exist functions \( f \in V_F \) that satisfy \( f'(z) \neq 0 \ \forall \ z \in \mathbb{D} \) and also \( \text{meas}(\mathbb{D} - f(\mathbb{D})) = 0 \) such that \( f \notin \text{Aut}(\mathbb{D}). \) In fact, we can construct such functions that will not be surjective and not injective.
Proof.
Consider the domain $\Omega = \mathbb{D} - \{x \in \mathbb{R} | 0 \leq x < 1\}$. Then $\Omega$ is the unit disc with a slit along the non-negative $x$-axis. It is a simply connected domain. Let $g : \mathbb{D} \to \Omega$ be a Riemann mapping (i.e. it is holomorphic and conformal. Finally, let $k \geq 2$ any natural integer and define $f = g^k$. This gives the desired function. □

Can the result in Theorem 3.1 be generalized to higher complex dimensions?
We make the obvious:

Definition 3.3. Let $V_F(n) (n \in \mathbb{Z}^+) \text{ be the set of all the holomorphic } f : \mathbb{D}^n \to \mathbb{D}^n, \text{ such that } \exists N = N_f (\text{depending on } f) \text{ for which } d_f(w) = \left| \{z \in \mathbb{D}^n | f(z) = w\} \right|, \text{ } w \in \mathbb{D}^n \text{ satisfies } d_f(w) \leq N_f \forall a \in \mathbb{D}^n.

We ask if the following assertion holds true:

If $f \in V_F(n)$ satisfies $\det J_f(z) \neq 0 \forall z \in \mathbb{D}^n$ and also $\lim_{r \to 1^-} |f(re^{i\theta_1}, . . . , re^{i\theta_n})| = 1$ a.e. in $(\theta_1, . . . , \theta_n)$ then $f \in \text{Aut}(\mathbb{D}^n)$.

References

[1] Emmanuel Fricain, Javad Mashreghi, On a characterization of finite Blaschke products.

[2] Daniela Kraus and Oliver Roth, Maximal Blaschke products, 2013.

[3] Daniela Kraus and Oliver Roth, Critical points of inner functions, non-linear partial differential equations, and an extension of Liouville’s Theorem.

[4] Banach Spaces of Analytic Functions, by Kenneth Hoffman, Prentice-Hall, inc. Englewood Cliffs, N.J., 1962.

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