Bounds for zeros of Collatz polynomials, with necessary and sufficient strictness conditions

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ABSTRACT

In a previous paper, we introduced the Collatz polynomials \( P_N(z) \), whose coefficients are the terms of the Collatz sequence of the positive integer \( N \). Our work in this paper expands on our previous results, using the Eneström-Kakeya Theorem to tighten our old bounds of the roots of \( P_N(z) \) and giving precise conditions under which these new bounds are sharp. In particular, we confirm an experimental result that zeros on the circle \( \{ z \in \mathbb{C} : |z| = 2 \} \) are rare: the set of \( N \) such that \( P_N(z) \) has a root of modulus 2 is sparse in the natural numbers. We close with some questions for further study.

1. Introduction

Let \( c(N) \) be the Collatz iterate of a number \( N \in \mathbb{N} \cup \{0\} \): i.e.

\[
c(N) := \begin{cases} 
    N/2, & \text{if } N \text{ even} \\
    (3N+1)/2, & \text{if } N \text{ odd and not 1} \\
    0, & \text{if } N = 1.
\end{cases}
\]  

(1)

Also, let \( c^j(N) \) have the standard meaning

\[
c^j(N) := \begin{cases} 
    N, & \text{if } j = 0 \\
    c(c^{j-1}(N)), & \text{if } j \geq 1,
\end{cases}
\]

(2)

and define \( n(N) \) (just \( n \) when \( N \) is clear from context) as

\[
n(N) := \min\{j \in \mathbb{N} : c^j(N) = 1\}.
\]

(3)

In this paper, we assume\(^1\) that \( n(N) < \infty \) for all \( N \), even though this remains an open question as of January 2022 (Harold Boas. Private communication.). As in [1], we define

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the \(N\)th Collatz polynomial to be

\[
P_N(z) := \sum_{j=0}^{n(N)} c_j^{(N)} \cdot z^j,
\]

i.e. the polynomial whose coefficients are the Collatz iterates of \(N\), or equivalently consecutive members of the Collatz trajectory/sequence of \(N\) (we will assume \(N \in \mathbb{N} - \{0, 1\}\) to avoid the trivial \(P_0(z) = 0\) and \(P_1(z) = 1\)). In our previous paper, we proved bounds on the zeros of \(P_N(z)\) and demonstrated that these zeros encode information about the Collatz sequence of \(N\). For example, if

\[
M_1, M_2, \ldots, M_\ell
\]

is the sequence of odd iterates of \(N\) and \(\{h_j\}_{j=1}^\ell\) are the non-negative integers such that \(2^{h_j}M_j\), but not \(2^{h_j+1}M_j\), is in the Collatz sequence of \(N\), then \(P_N(-2) = 0\) if and only if all the \(h_j\) are even. Furthermore, let \(t(N)\) be the natural number such that \(2^t(N)\), but not \(2^{t(N)+1}\), is in the Collatz sequence of \(N\); we proved that there exists an upper bound \(h(t^*) \geq 2\) on the zeros of \(P_N\) for \(t(N) = t^*\) such that

\[
\lim_{t^* \to \infty} h(t^*) = 2.
\]

Moreover, we showed that \(P_N(-1) = 0\) for \(N = 4^k \cdot \frac{2^{t(N)+1}-1}{3}, k = 0, 1, \ldots\), although these two statements are not equivalent. That is, if the Collatz sequence of \(N\) is “short” in a well-defined sense, then \(P_N(-1) = 0\).

Our results in this article not only greatly sharpen both the upper and lower bounds of the original article but also show that these new bounds cannot be improved when \(N\) is arbitrary. To wit, where \(z_N\) is an arbitrary root of \(P_N(z)\) and \([k] := [1, k] \cap \mathbb{N}\), we prove that

\[
\frac{2 \cdot M(N)}{3 \cdot M(N) + 1} \leq |z_N| \leq 2,
\]

where \(M(N)\) is the least odd iterate of \(N\) other than 1; these bounds are sharp when \(N\) is a power of two. In fact, we go somewhat further, proving that

\[
\frac{2 \cdot M(N)}{3 \cdot M(N) + 1} < |z_N| < 2
\]

for almost all \(N\) and giving precise conditions under which \(N\) belongs to the sparse set of exceptions: briefly, \(P_N(z)\) has zeros on \(|z| = 2\) if and only if the set of \(j\) such that \(c_j^{(N)}\) is odd form a subset of certain arithmetic sequences.

In this and future works, we wish to uncover further relationships between the behaviour of the Collatz sequence of \(N\) and the location of zeros of \(P_N(z)\). Given our assumption that the Collatz conjecture is true (equivalently, that \(P_N(z)\) is a polynomial), a material contradiction of our findings could be sufficient to disprove the conjecture, which has been open for decades and is widely believed to be true.

Section 2 of this paper contains general bounds on the zeros of \(P_N(z)\). Sections 3 and 4 address the strictness of, respectively, our new lower and upper bounds. Finally, Section 5 closes the paper with some suggestions for further research.
2. General bounds

**Lemma 2.1 (Eneström-Kakeya, Theorem A of [2]):** Let \( f(z) = a_kz^k + \cdots + a_1z + a_0 \) have all strictly positive coefficients and set

\[
\alpha[f] := \min_{j=0, \ldots, k-1} \frac{a_j}{a_{j+1}} \tag{7}
\]

\[
\beta[f] := \max_{j=0, \ldots, k-1} \frac{a_j}{a_{j+1}}. \tag{8}
\]

Then

\[
\alpha[f] \leq |w| \leq \beta[f] \tag{9}
\]

for all roots \( w \) of \( f(z) \).

With Lemma 2.1, we prove the following general bound for \( |z_N| \):

**Theorem I:**

\[
\frac{2 \cdot M(N)}{3 \cdot M(N) + 1} \leq |z_N| \leq 2, \tag{10}
\]

where \( M(N) := \max((-1/2) \cup \{\ell > 1 : \ell \text{ an odd Collatz iterate of } N\}) \).

**Proof:** If \( a_j \) is odd and not 1, then

\[
\frac{a_j}{a_{j+1}} = \frac{2a_j}{3a_j + 1}, \tag{11}
\]

and if \( a_j \) is even, then

\[
\frac{a_j}{a_{j+1}} = \frac{2a_j}{a_j} = 2. \tag{12}
\]

If \( N \) has an odd iterate other than 1, then the conclusion follows by Lemma 2.1. Otherwise, \( P_N(z) \) is the partial sum

\[
N \cdot \left(1 + \frac{z}{2} + \cdots + \frac{z^n}{2^n}\right), \tag{13}
\]

whose roots all lie on \( \{z \in \mathbb{C} : |z| = 2\} \). \( \blacksquare \)

3. Strictness of the lower bound

**Lemma 3.1 (Theorem B of [2]):** For a polynomial \( f(z) \) of degree \( k \) with strictly positive coefficients, let \( S[f] \) be the set of all \( j \in [k+1] \) such that

\[
\beta[f] > \frac{a_{k-j}}{a_{k+1-j}}. \tag{14}
\]

Then the upper bound of Equation (9) is an equality if and only if \( 1 < d := \gcd(j \in S[f]) \), in which case
all the zeros on $|z| = \beta[f]$ are simple and given by \( \{ \beta[f] \cdot \exp\left(\frac{2\pi ij}{d}\right), \ j = 1, \ldots, d - 1 \} \) and

\[
\hat{f}(\beta[f] \cdot z) = (1 + z + \cdots + z^{d-1}) \cdot q_m(z^d)
\]

for a degree $m$ polynomial $q_m$ with strictly positive coefficients.

Moreover, if $m > 0$, then all zeros of $q_m$ belong to $\mathbb{D}$ and $\beta[q_m] \leq 1$.

Define the reciprocal polynomial $\tilde{f}$ of the degree $k$ polynomial $f(z) = a_k z^k + \cdots + a_j z^j + \cdots + a_0$, for which we assume $a_0 a_k \neq 0$, to be

\[
\tilde{f}(z) := z^k \cdot f\left(\frac{1}{z}\right)
\]

\[
= a_0 z^k + \cdots + a_{k-j} z^j + \cdots + a_k.
\]

Note that

\[
\beta[\tilde{f}] = \max_{j=0,\ldots,k-1} \frac{a_{k-j}}{a_{k-j-1}}
\]

\[
= \frac{1}{\min_{j=0,\ldots,k-1} \frac{a_j}{a_{j+1}}}
\]

\[
= \frac{1}{\alpha[f]},
\]

and that, more generally, an upper bound for the zeros of $\tilde{f}$ is the reciprocal of a lower bound for the zeros of $f$.

**Theorem II:** Equality holds for the leftmost inequality of Equation (10) if and only if $N$ is a power of 2.

**Proof:** We have observed the ‘if’ direction in the proof of Theorem I. For the ‘only if’ direction, suppose that $N$ is not a power of 2 and let $M(N)$ again signify the minimum odd iterate of $N$ not equal to 1. Then

\[
\beta[\tilde{P}_N] = \frac{1}{\alpha[P_N]} = \frac{3}{2} + \frac{1}{2 \cdot M(N)}.
\]

Now, for $j \in [n(N) + 1]$,

\[
\frac{a_{n-(n-j)}}{a_{n-(n+1-j)}} = \frac{a_j}{a_{j-1}} = \begin{cases} 
1, & a_{j-1} \text{ even and } j \neq n + 1 \\
\frac{1}{2}, & a_{j-1} = M \\
\frac{3}{2} + \frac{1}{2a_{j-1}}, & a_{j-1} \text{ odd} \\
0, & j = n + 1,
\end{cases}
\]

so that $\beta[\tilde{P}_N] > \frac{a_j}{a_{j-1}}$ for all but the unique $j$ for which $a_{j-1} = M$. Thus, there exist at least two consecutive values of $j$ such that $\beta[\tilde{P}_N] > \frac{a_j}{a_{j-1}}$, implying that $\gcd(j \in S[\tilde{P}_N]) = 1$. Therefore, by Lemma 3.1, $\beta[\tilde{P}_N]$ is a non-sharp upper bound for the zeros of $\tilde{P}_N$, whence $\alpha[P_n]$ is a non-sharp lower bound for the zeros of $P_N$. ■
4. Strictness of the upper bound

For a given $N$, define the set $T_N := \{n(N) + 1\} \cup \{j \in [n(N)] : c^{n-j}(N) \text{ is odd}\}$. Then Lemma 3.1 implies the following sharpness result:

**Theorem III:** Let $d_N := \gcd(j \in T_N)$. Then the upper bound of Equation (10) is an equality if and only if $d_N > 1$, in which case

1. the zeros of $P_N$ on $\{|z| = 2\}$ are simple and given precisely by $\{2\omega : \omega^{d_N} = 1 \land \omega \neq 1\}$
2. $P_N$ factors as

   $P_N(2z) = \left(1 + z + \cdots + z^{d_N-1}\right) \cdot Q_N(z^{d_N}), \tag{22}$

   where $Q_N$ is a polynomial with positive coefficients. Moreover, if $Q_N$ is non-constant then all its zeros lie in $\mathbb{D}$ and $\beta[Q_N] \leq 1$.

**Proof:** By the proof of Theorem I (and the fact that $a_{n-1} = 0$ because $P_N$ is a polynomial), the set $T_N$ is precisely the set of $j \in [n + 1]$ such that $\frac{a_{n-j}}{a_{n+1-j}} < \beta[P_N] = 2$. Thus the result follows from applying Lemma 3.1.

The exacting conditions of Theorem III suggest that zeros on $|z| = 2$ are rare, and indeed we prove that this is so. First, we need two lemmas.

**Lemma 4.1:** The number of length $k$ binary strings with no consecutive 1s is $F_{k+1}$, the $(k+1)^{th}$ Fibonacci number. In particular, let $p_k$ be the probability that a length $k$ binary string selected uniformly at random contains no consecutive 1s. Then $\lim_{k \to \infty} p_k = 0$.

**Proof:** The first statement holds for $k = 0$ and 1. Consider a binary string $x$ of length $k \geq 2$ and let $\text{substring}(x, j)$ be the substring of $x$ comprising its first $j$ digits. If the last digit of $x$ is 0, then $x$ has two consecutive 1s if and only if $\text{substring}(x, k-1)$ does. Otherwise, its last two digits are either 11, in which case it has consecutive 1s, or 01, in which case it has consecutive 1s if and only if $\text{substring}(x, k-2)$ does. Therefore, the number of length $k$ binary strings with no consecutive 1s satisfies the same recurrence relation as the Fibonacci numbers.

Since $F_k \sim \frac{\phi^k}{\sqrt{5}}$, $p_k \sim \frac{\phi}{\sqrt{5}} \left(\frac{\phi}{2}\right)^k$, and this quantity has limit 0 as $k \to \infty$.

**Lemma 4.2:** Let $X$ be the set of natural numbers $N$ with the property that

$e^j(N) \text{ odd } \Rightarrow e^{j+1}(N) \text{ even}$

for all integers $j \geq 0$. The set $X$ has density zero in the natural numbers.

**Proof:** Let $x_j(N) := e^j(N) \pmod{2}$. By [3, Theorem B], the function $\mathbb{Z} \to \mathbb{Z}/2^{k+1}\mathbb{Z}$ defined by

$N \to (x_0(N), \ldots, x_k(N)) \tag{23}$

is periodic with period $2^{k+1}$. For a fixed $k$, the set $X$ is a subset of the preimage of the set of length $k+1$ strings with no consecutive 1s under this function. By Lemma 4.1, as $k \to \infty$, this preimage becomes sparse as a subset of the natural numbers.
Lemmas 4.1 and 4.2 culminate in the following theorem.

**Theorem IV:** The set of $N$ such that $P_N$ has a root on $|z| = 2$ has density 0 in the natural numbers.

**Proof:** By Lemma 4.2, $T_N$ contains consecutive natural numbers for almost all $N$, so that $\gcd(j \in T_N) = 1$ for almost all $N$. The conclusion follows from Theorem III. ■

5. Open problems

We conclude with a few open problems.

1. Strengthen Theorem II by finding an explicit, closed-form lower bound for $|z_N|$ in terms of $N$.
2. Strengthen Theorems III and IV by finding an explicit, closed-form upper bound for $|z_N|$ in terms of $N$.
3. Figure 1 suggests that certain subsets of $\{z \in \mathbb{C} : |z| \leq 2\}$ contain large clusters of Collatz zeros while others are zero-free. Give an algorithm for proving that a subset of $\{z \in \mathbb{C} : |z| \leq 2\}$ is free of Collatz zeros.
4. While Descartes’ Rule of Signs implies that no Collatz zero is positive real, there appears to be a sequence of zeros that approaches $z = 1$ arbitrarily closely, bounded
by a parabola with vertex at $z = 1$. Prove or disprove the existence of such a convergent sequence and parabola.

(5) Under some modest assumptions, the zeros of a polynomial with random coefficients cluster near the unit circle with uniform angular distribution [4]. Prove or disprove that these assumptions hold for the Collatz polynomials.

(6) Find the Galois groups of $P_N$ for general classes of $P_N$.

(7) Expanding on Theorem III, find conditions under which $P_N$ has zeros at real multiples (other than 2) of roots of unity.

(8) Except for Theorem II, all the theorems in this paper are functions of the Collatz sequence of $N$ rather than of $N$ itself. Prove theorems on $z_N$ which can be applied without calculating the Collatz sequence of $N$ (e.g. if $N \equiv 3 \mod 4$, then the zeros of $P_N$ lie in $\ldots$).

(9) Our results rest on the assumption, equivalent to the Collatz conjecture itself, that $P_N$ is a polynomial for all $N$. Using other methods of proof, find a property of Collatz zeros that contradicts a theorem in this paper, thereby disproving the Collatz conjecture.

**Note**

1. cf. item 9 of Section 5

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