Quantum Geometry In Action: Big Bang and Black Holes

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Abstract

Over the last three years, a number of fundamental issues in quantum gravity were addressed in the framework of quantum geometry, discussed extensively by John Baez in this conference. In particular, these include: A statistical mechanical derivation of the horizon entropy, encompassing astrophysically interesting black holes cosmological horizons, and a natural resolution of the big-bang singularity. The goal of this article is to communicate these advances in general terms.

1 Introduction

Major paradigm shifts in theoretical physics have required mathematical arenas that were, at the time, new to physics. Newton’s mechanics and theory of gravitation could not have been introduced without calculus; Maxwell’s electrodynamics required partial differential equations and analysis; Einstein had to learn differential geometry to develop general relativity; and quantum mechanics needed the theory of Hilbert spaces and operator algebras. It is widely believed that quantum gravity will lead to the next profound paradigm shift in physics. What would be the required mathematical arenas? The answers to this question vary. For example, Roger Penrose’s twistor theory posits that space-time would be a secondary, derived concept, arising from a 4-dimensional complex space; the fundamental theory would be based on complex manifolds, sheaf-cohomology and algebraic geometry. Alain Conne’s approach aims at describing fundamental physics through non-commutative geometry. In the loop quantum gravity approach I will discuss here, the basic tool is Riemannian quantum geometry. Just as differential geometry provides the mathematical language to formulate classical gravitational theories, such as general relativity, a specific quantum Riemannian geometry provides the required setting for quantum gravitational theories. Since this subject was covered in detail by Baez, I will only provide a semi-qualitative introduction and focus, rather, on applications of this quantum geometry. However, let me make a general remark in this Introduction. As Dennis Sullivan emphasized during discussions at the conference, from the perspective of graph theory, freedom in the construction of a (background independent) quantum theory of geometry is very limited. Thus, the mathematical structures, definitions and constructions we use are not only natural from this perspective, but essentially unique.

Let us now turn to physics. What are some of the central physical and conceptual questions of quantum gravity? I would like to outline a few of these to give a flavor of the subject. Since
This article is addressed to non-specialists, I will select questions that arise from what we already know about Nature, and what we expect based on physical theories which are firmly based on observations, avoiding issues—such as higher dimensions and supersymmetry—that are internal to specific quantum gravity programs and which have yet to receive observational support. Further discussion of the background material can be found in Section 3.

- **Big-Bang and other singularities:** It is widely believed that the prediction of a singularity, such as the big-bang of classical general relativity, is primarily a signal that the physical theory has been pushed beyond the domain of its validity. A key question to any quantum gravity theory, then, is: What replaces the big-bang? Is there a mathematically consistent description of the evolution of the quantum state of the universe which is singularity free? General relativity predicts that the space-time curvature must grow as we approach the big-bang but we expect the quantum effects, ignored by general relativity, to intervene, making quantum gravity indispensable before infinite curvatures are reached. If so, what is the upper bound on curvature? How close to the big-bang can we ‘trust’ classical general relativity? What can we say about the ‘initial conditions’, i.e., the quantum state of geometry and matter that correctly describes the big-bang? If they have to be imposed externally, is there a physical guiding principle?

- **Black holes:** In the early seventies, using imaginative thought experiments, Jacob Bekenstein argued that black holes must carry entropy proportional to their area. About the same time, Jim Bardeen, Brandon Carter and Stephen Hawking (BCH) showed that black holes in equilibrium obey two basic laws, which have the same form as the zeroth and the first laws of ordinary thermodynamics, provided one replaces the area of the black hole horizon $a_{\text{hor}}$ by a multiple of the entropy $S$ in thermodynamics and black hole surface gravity $\kappa$ by a corresponding multiple of the temperature $T$. However, at first this similarity was thought to be only a formal analogy because the BCH analysis was based on classical general relativity and simple dimensional considerations show that the proportionality factors must involve Planck’s constant $\hbar$. Two years later, using quantum field theory on a black hole background space-time, Hawking showed that black holes in fact radiate quantum mechanically as though they are black bodies at temperature $T = \hbar \kappa / 2\pi$. Using the analogy with the first law, one can then conclude that the black hole entropy should be given by $S_{\text{BH}} = a_{\text{hor}} / 4G\hbar$, where $G$ is Newton’s gravitational constant. This conclusion is striking and deep because it brings together the three pillars of fundamental physics—general relativity, quantum theory and statistical mechanics. However, the argument itself is a rather hodge-podge mixture of classical and semi-classical ideas, reminiscent of the Bohr theory of atom. A natural question then is: what is the analog of the more fundamental, Pauli-Schrödinger theory of the Hydrogen atom? More precisely, what is the statistical mechanical origin of black hole entropy? What is the nature of a quantum black hole and what is the interplay between the quantum degrees of freedom responsible for entropy and the exterior curved geometry? Can one derive the Hawking effect from first principles of quantum gravity?

- **Planck scale physics and the low energy world:** Perhaps the central lesson of general relativity is that gravity is geometry. There is no longer a background metric, no inert stage on which dynamics unfolds. Geometry itself is dynamical. Therefore, one expects that a fully satisfactory quantum gravity theory would also be free of a background space-time geometry. However, of necessity, a background independent description must use physical concepts and mathematical tools.

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[1] One can think of the horizon as the ‘surface’ of the black hole. In classical general relativity, one can not send causal signals from the region within the horizon to the region outside. Surface gravity $\kappa$ is, roughly, the black hole analog of the acceleration $g$ due to gravity on the surface of the earth.
that are quite different from those of the familiar, low energy physics formulated in flat space-time. A major challenge then is to show that this low energy description does arise from the pristine, Planckian world in an appropriate sense. In this ‘top-down’ approach, does the fundamental theory admit a ‘sufficient number’ of semi-classical states? Do these semi-classical sectors provide enough of a background geometry to anchor low energy physics? can one recover the familiar description? Furthermore, can one pin point why the standard ‘bottom-up’ perturbative approach fails? That is, what is the essential feature which makes the fundamental description mathematically coherent, but is absent in the standard perturbative quantum gravity?

Of course, this is by no means a complete list of challenges. There are many others. Since there is no background space-time metric, what does ‘time evolution’ mean? Without a fixed space-time at one’s disposal, what is one to make of quantum measurement theory and the associated questions of interpretation of quantum mechanics? What role does space-time topology play and can it change? . . . Recent advances within loop quantum gravity have led to illuminating answers to many of these questions and opened-up avenues to address others. As in my talk, in this report, I will focus on issues related to black holes and big-bang.

2 A bird’s eye view of loop quantum gravity

In this section, I will briefly summarize the salient features and current status of quantum geometry. The emphasis is on structural and conceptual issues; details can be found in references [1-9].

2.1 Viewpoint

In this approach, one takes the central lesson of general relativity seriously: gravity is geometry whence, in a fundamental theory, there should be no background metric. In quantum gravity, geometry and matter should both be ‘born quantum mechanically’. Thus, in contrast to approaches developed by particle physicists, one does not begin with quantum matter on a background geometry and use perturbation theory to incorporate quantum effects of gravity. There is a manifold but no metric, or indeed any other physical fields, in the background. In classical gravity, Riemannian geometry provides the appropriate mathematical language to formulate the physical, kinematical notions as well as the final dynamical equations. This role is now taken by quantum Riemannian geometry, discussed below. In the classical domain, general relativity stands out as the best available theory of gravity, some of whose predictions have been tested to an amazing accuracy, surpassing even the legendary tests of quantum electrodynamics. However, if one applies to general relativity the standard perturbative techniques of quantum field theory, one obtains a ‘non-renormalizable’ theory, i.e., a theory with uncontrollable infinities. Therefore, it is natural to ask: Does quantum general relativity, coupled to suitable matter (or supergravity, its supersymmetric generalizations) exist as a consistent theory non-perturbatively? There is no a priori implication that such a theory

\[2] The characteristic length scale in the Planck regime is \( \sim 10^{-33} \text{ cm} \) while the smallest distance we can probe with our highest energy accelerators today is \( \sim 10^{-17} \text{ cm} \).

\[3] In 2+1 dimensions, although one begins in a completely analogous fashion, in the final picture one can get rid of the background manifold as well. Thus, the fundamental theory can be formulated combinatorially [3, 1]. To achieve this goal in 3+1 dimensions, one needs a much better understanding of the theory of (intersecting) knots in 3 dimensions.
would be the final, complete description of Nature. Nonetheless, this is a fascinating open question at the level of mathematical physics.

In the particle physics circles, the answer is often assumed to be in the negative, not because there is concrete evidence against non-perturbative quantum gravity, but because of an analogy to the theory of weak interactions, where non-renormalizability of the initial ‘Fermi theory’ forced one to replace it by the renormalizable Glashow-Weinberg-Salam theory. However this analogy overlooks the crucial fact that, in the case of general relativity, there is a qualitatively new element. Perturbative treatments pre-suppose that the space-time can be assumed to be a continuum at all scales of interest to physics under consideration. Since this is a safe assumption for weak interactions, non-renormalizability was a genuine problem. However, in the gravitational case, the scale of interest is given by the Planck length $\ell_{\text{Pl}}$ and there is no physical basis to pre-suppose that the continuum picture should be valid down to that scale. The failure of the standard perturbative treatments may simply be due to this grossly incorrect assumption and a non-perturbative treatment which correctly incorporates the physical micro-structure of geometry may well be free of these inconsistencies.

As indicated above, even if quantum general relativity did exist as a mathematically consistent theory, there is no a priori reason to assume that it would be the ‘final’ theory of all known physics. In classical general relativity, while requirements of background independence and general covariance do restrict the form of interactions between gravity and matter fields and among matter fields themselves, they do not determine these interactions. Quantum general relativity would have the same limitation. Put differently, such a theory would not be a satisfactory candidate for unification of all known forces. However, just as general relativity has had powerful implications in spite of this limitation in the classical domain, quantum general relativity should have qualitatively new predictions, pushing further the existing frontiers of physics. Indeed, unification does not appear to be an essential criterion for usefulness of a theory even in other interactions. Quantum chromodynamics (QCD) for example, is a powerful theory of strong interactions even though it does not unify them with electro-weak ones. Furthermore, the fact that we do not yet have a viable candidate for the grand unified theory does not make QCD any less useful.

### 2.2 Quantum Geometry

Although there is no natural unification of dynamics of all interactions in loop quantum gravity, it does provide a kinematical unification. More precisely, in this approach one begins by formulating general relativity in the mathematical language of connections, the basic variables of gauge theories of electro-weak and strong interactions. Thus, now the configuration variables are not metrics (as in Wheeler’s geometrodynamics program), but certain *spin connections*; the emphasis is shifted from distances to holonomies. Consequently, the basic kinematical structures are the same as those used in gauge theories. A key difference, however, is that while a background space-time metric is available and crucially used in gauge theories, now there are no background fields whatsoever. This absence is forced on us by the requirement of diffeomorphism invariance.

This is a key difference and it causes a host of conceptual as well as technical difficulties in the passage to quantum theory. For, most of the techniques used in the familiar, Minkowskian quantum theories are deeply rooted in the availability of a flat background metric. It is this structure that enables one to single out the vacuum state, perform Fourier transforms to decompose fields canonically in to creation and annihilation operators, define masses and spins of particles and carry...
out regularizations of products of operators. Already when one passes to quantum field theory in curved space-times, extra work is needed to construct mathematical structures that are adequate for physics. In our case, the situation is much more drastic: there is no background metric whatsoever. Therefore new physical ideas and mathematical tools are now necessary. Fortunately, they were constructed by a number of researchers in the mid-nineties and have given rise to a detailed quantum theory of geometry [4, 5, 6, 7, 8].

Because the situation is conceptually so novel and because there are no direct experiments to guide us, reliable results require mathematical precision to ensure that there are no hidden infinities. Achieving this precision has been a high priority in the program. Thus, while one is inevitably motivated by heuristic, physical ideas and formal manipulations, the final results are mathematically rigorous. In particular, due care is taken in constructing function spaces, defining measures and functional integrals, regularizing products of field operator, and calculating eigenvectors and eigenvalues of geometric operators. The final results are all free of divergences, well-defined, and respect the background independence and diffeomorphism invariance.

Let me now turn to specifics. It is perhaps simplest to begin with a Hamiltonian or symplectic description of general relativity. The phase space is the cotangent bundle. The configuration variable is a connection, $A$ on a fixed 3-manifold $\Sigma$ representing ‘space’ and (as in gauge theories) the momenta are the ‘electric field’ 2-forms $E$, both of which take values in the Lie-algebra of SU(2). In the present gravitational context, the momenta acquire a geometrical significance: their Hodge-duals $\star E$ can be naturally interpreted as orthonormal triads (with density weight 1) and determine the dynamical, Riemannian geometry of $\Sigma$. Thus, (in contrast to Wheeler’s geometrodynamics) the Riemannian structures on $\Sigma$ are now built from momentum variables. The basic kinematic objects are holonomies of $A$, which dictate how spinors are parallel transported along curves, and the 2-forms $E$, which determine the Riemannian metric of $\Sigma$. (Matter couplings to gravity have also been studied extensively [2, 1].)

In the quantum theory, the fundamental excitations of geometry are most conveniently expressed in terms of holonomies [3, 4]. They are thus one-dimensional, polymer-like and, in analogy with gauge theories, can be thought of as ‘flux lines of the electric field’. More precisely, they turn out to be flux lines of areas: an elementary flux line deposits a quantum of area on any 2-surface $S$ it intersects. Thus, if quantum geometry were to be excited along just a few flux lines, most surfaces would have zero area and the quantum state would not at all resemble a classical geometry. Semi-classical geometries can result only if a huge number of these elementary excitations are superposed in suitably dense configurations [13, 14]. The state of quantum geometry around you, for example, must have so many elementary excitations that $\sim 10^{68}$ of them intersect the sheet of paper you are reading, to endow it an area of $\sim 100\text{cm}^2$. Even in such states, the geometry is still distributional, concentrated on the underlying elementary flux lines; but if suitably coarse-grained, it can be approximated by a smooth metric. Thus, the continuum picture is only an approximation that arises from coarse graining of semi-classical states.

These quantum states span a specific Hilbert space $\mathcal{H} = L^2(\bar{\mathcal{A}}, d\mu_o)$, consisting of functions on the space of (suitably generalized) connections which are square integrable with respect to a natural, diffeomorphism invariant (regular, Borel) measure $\mu_o$. This space is very large. However, it can be conveniently decomposed in to a family of orthonormal, finite dimensional subspaces $\mathcal{H} = \bigoplus_{\gamma, \vec{j}} \mathcal{H}_{\gamma, \vec{j}}$, labelled by finite graphs $\gamma$ each edge of which itself is labelled by a non-trivial irreducible representation of SU(2) (or, a half-integer, or a spin $j$). $\mathcal{H}_{\gamma, \vec{j}}$ can be regarded as the Hilbert space of a ‘spin-system’. These spaces are extremely simple to work with; this is why very
explicit calculations are feasible. Elements of $\mathcal{H}_{\gamma,\vec{j}}$ are referred to as spin-network states [5].

As one would expect from the structure of the classical theory, the basic quantum operators are the holonomies $\hat{h}_p$ along paths $p$ in $\Sigma$ and the triads $\hat{\triangledown} E$ [6]. Both sets of operators are densely defined and self-adjoint on $\mathcal{H}$. Furthermore, a striking result is that all eigenvalues of the triad operators are discrete. This key property is, in essence, the origin of the fundamental discreteness of quantum geometry. For, just as the classical Riemannian geometry of $\Sigma$ is determined by the triads $\hat{\triangledown} E$, all Riemannian geometry operators —such as the area operator $\hat{A}_S$ associated with a 2-surface $S$ or the volume operator $\hat{V}_R$ associated with a region $R$— are constructed from $\hat{\triangledown} E$. However, since even the classical quantities $A_S$ and $V_R$ are non-polynomial functionals of the triads, the construction of the corresponding $\hat{A}_S$ and $\hat{V}_R$ is quite subtle and requires a great deal of care. But their final expressions are rather simple [6].

In this regularization, the underlying background independence turns out to be a blessing. For, diffeomorphism invariance constrains the possible forms of the final expressions severely and the detailed calculations then serve essentially to fix numerical coefficients and other details. Let us illustrate this point with the example of the area operators $\hat{A}_S$. Since they are associated with 2-surfaces $S$ while the states have 1-dimensional support, the diffeomorphism covariance requires that the action of $\hat{A}_S$ on a state $\Psi_{\gamma,\vec{j}}$ must be concentrated at the intersections of $S$ with $\gamma$. The detailed expression bears out this fact: the action of $\hat{A}_S$ on $\Psi_{\gamma,\vec{j}}$ is dictated simply by the spin labels $j_I$ attached to those edges of $\gamma$ which intersect $S$. For all surfaces $S$ and 3-dimensional regions $R$ in $\Sigma$, $\hat{A}_S$ and $\hat{V}_R$ are densely defined, self-adjoint operators. All their eigenvalues are discrete [6]. Naively, one might expect that the eigenvalues would be uniformly spaced, given by, e.g., integral multiples of the Planck area or volume. This turns out not to be the case; the distribution of eigenvalues is quite subtle. In particular, the eigenvalues crowd rapidly as areas and volumes increase. In the case of area operators, the complete spectrum is known in a closed form, and the first several hundred eigenvalues have been explicitly computed numerically. For a large eigenvalue $a_n$, the separation $\Delta a_n = a_{n+1} - a_n$ between consecutive eigenvalues decreases exponentially: $\Delta a_n \leq \ell_{Pl}^2 \exp\left(-\sqrt{a_n}/\ell_{Pl}\right)!$ Because of such strong crowding, the continuum approximation becomes excellent quite rapidly just a few orders of magnitude above the Planck scale. At the Planck scale, however, there is a precise and very specific replacement. This is the arena of quantum geometry. The premise is that the standard perturbation theory fails because it ignores this fundamental discreteness (see Section 2.1).

There is however a key mathematical subtlety [2, 7]. This non-perturbative quantization has a one parameter family of ambiguities labelled by $\gamma > 0$. This $\gamma$ is called the Barbero-Immirzi parameter (and is rather similar to the well-known $\theta$-parameter of QCD). In the classical theory, $\gamma$ is irrelevant but in quantum theory different values of $\gamma$ correspond to unitarily inequivalent representations of the algebra of geometric operators. The overall mathematical structure of all these sectors is very similar; the only difference is that the eigenvalues of all geometric operators scale with $\gamma$. For example, the simplest eigenvalues of the area operator $\hat{A}_S$ in the $\gamma$ quantum sector is given by

$$a_{\{j\}} = 8\pi\gamma\ell_{Pl}^2 \sum_I \sqrt{j_I(j_I + 1)}$$  \hspace{1cm} (1)

where $I = 1, \ldots N$ for some integer $N$ and each $j_I$ is a half-integer. Since the representations are unitarily inequivalent, as usual, one must rely on Nature to resolve this ambiguity: Just as Nature must select a specific value of $\theta$ on QCD, it must select a specific value of $\gamma$ in loop quantum gravity.
With one judicious experiment —e.g., measurement of the lowest eigenvalue of the area operator $A_S$ for a 2-surface $S$ of any given topology— we could determine the value of $\gamma$ and fix the theory. Unfortunately, such experiments are hard to perform! However, we will see in Section 3.2 that the Bekenstein-Hawking formula of black hole entropy provides an indirect measurement of this lowest eigenvalue of area for the 2-sphere topology and can therefore be used to fix the value of $\gamma$.

3 Applications of quantum geometry

In this section, I will summarize two recent developments that answer several of the questions raised under first two bullets in the Introduction.

3.1 Big bang

Let us first recall how the big-bang singularity arises in classical general relativity. Observations have shown that the universe is spatially homogeneous and isotropic on a cosmological scale. Therefore, to model the large scale behavior of the universe, one begins by assuming that the 4-manifold representing space-time is foliated by 3-dimensional spatial manifolds, each equipped with a metric of constant curvature. Thus there are three possibilities: the leaves of the preferred foliation are either metric 3-spheres, or flat, or metric 3-hyperboloids, depending on whether the scalar curvature (which is constant on each leaf) is positive, zero or negative. To be specific, let me assume the first case. Then, although the scalar curvature is constant on any one spatial slice, it changes in time, giving rise to an overall expansion or contraction. The radius $a$ of the 3-sphere encodes the full information of the 3-metric at that instant of time and is called the scale factor. One then suitably models matter-sources —galaxies and the observed radiation fields— and seeks solutions of Einstein’s equation with these symmetries. The equation implies that the universe must have ‘originated from a big-bang’: if we evolve the solution backwards in time, the scale factor $a$ must eventually go to zero and the curvature must diverge as $1/a^2$. At this ‘initial instant’, Einstein’s equation breaks down; classical physics stops. As discussed in the Introduction, the general belief is that this singular behavior is an artifact of our insistence of applying general relativity beyond the domain of its validity. Quantum effects are thought to intervene and dominate the ‘real physics’ in the high curvature regions. The question then is: what replaces the big-bang in this new, more accurate theory?

This question has been discussed for over thirty years in a framework called ‘quantum cosmology’. Traditionally, one has proceeded by first imposing spatial symmetries —such as homogeneity and isotropy— to freeze out all but a finite number of degrees of freedom already at the classical level and then quantizing the reduced system. In the simplest case, the basic variables of the reduced classical system are the scale factor $a$ and matter fields $\phi$. One then asks: in the theory so quantized, do the singularities of the classical general relativity disappear? Unfortunately, without an additional input, they do not: typically, to resolve the singularity one either had to introduce matter with unphysical properties or introduce boundary conditions by invoking new principles.

In a series of seminal papers [10], Martin Bojowald has shown that the situation in loop quantum cosmology is quite different: the underlying quantum geometry makes a qualitative difference very near the big-bang and naturally resolves the singularity. In the standard procedure summarized above, the reduction is carried out at the classical level and this removes all traces of the fundamental discreteness. Therefore, the key idea in Bojowald’s analysis is to retain the essential
features of quantum geometry by first quantizing the kinematics of the full theory as in Section 2.2 and then restricting oneself to quantum states which are spatially homogeneous and isotropic. As a result, the scale factor operator $\hat{a}$ has discrete eigenvalues. The continuum limit is reached rapidly. For example, the gap between an eigenvalue of $\hat{a}$ of $\sim 1\text{cm}$ and the next one is less than $\sim 10^{-30}\ell_{\text{Pl}}$. Nonetheless, near $a \sim \ell_{\text{Pl}}$ there are surprises. Predictions of loop quantum cosmology are very different from those of traditional quantum cosmology.

The first surprise occurs already at the kinematical level. Recall that, in the classical theory, curvature is essentially given by $1/a^2$, and blows up at the big-bang. What is the situation in quantum theory? Denote the Hilbert space of spatially homogeneous, isotropic kinematical quantum states by $\mathcal{H}_{\text{HI}}$. A self-adjoint operator $\hat{\text{curv}}$ corresponding to curvature can be constructed on $\mathcal{H}_{\text{HI}}$ and turns out to be bounded from above. This is very surprising because $\mathcal{H}_{\text{HI}}$ admits an eigenstate of the scale factor operator $\hat{a}$ with a discrete, zero eigenvalue! At first, it may appear that this could happen only by an artificial trick in the construction of $\hat{\text{curv}}$ and that this quantization cannot possibly be right because it seems to represent a huge departure from the classical relation $(\text{curv})a^2 = 1$. However, these concerns turn out to be misplaced. The procedure for constructing $\hat{\text{curv}}$ is natural and, furthermore, descends from full quantum theory.

Let us examine the properties of $\hat{\text{curv}}$. Its upper bound $u_{\text{curv}}$ is finite but absolutely huge:

$$u_{\text{curv}} \sim \frac{256}{81} \frac{1}{\ell_{\text{Pl}}^2} = \frac{256}{81} \frac{1}{G\hbar}$$

or, about $10^{77}$ times the curvature at the horizon of a solar mass black hole. The functional form of the upper bound is also illuminating. Recall first the Pauli-Schrödinger treatment of the hydrogen atom in non-relativistic quantum mechanics. Because the Coulomb potential between the proton (nucleus of the atom) and the electron diverges as $-1/r$, in the classical theory the energy is unbounded from below. However, thank to the Planck’s constant $\hbar$, in the quantum theory, we obtain a finite value, $E_0 = -(me^4/\hbar^2)$. Similarly, $u_{\text{curv}}$ is finite because $\hbar$ is non-zero and tends to the classical answer as $\hbar$ tends to zero.

At curvatures as large as $u_{\text{curv}}$, it is natural to expect large departures from classical relations such as $(\text{curv})a^2 = 1$. But is this relation recovered in the semi-classical regime? The answer is in the affirmative. In fact it is somewhat surprising how quickly this happens. As one would expect, one can simultaneously diagonalize $\hat{a}$ and $\hat{\text{curv}}$. If we denote their eigenvalues by $a_n$ and $b_n$ respectively, then $a_n \cdot b_n - 1$ is of the order $10^{-4}$ at $n = 100$ and decreases rapidly as $n$ increases. These properties show that, in spite of the initial surprise, the quantization procedure is viable. Furthermore, one can apply it also to more familiar systems such as a particle moving on a circle and obtain results which at first seem surprising but are in complete agreement with the standard quantum theory of these systems.

Since the curvature is bounded above in the entire Hilbert space, one might hope that the quantum evolution may be well-defined right through the big-bang singularity. Is this in fact the case? The second surprise is that although the quantum evolution is close to that of the so-called Wheeler-DeWitt equation of standard quantum cosmology for large $a$, there are dramatic differences near the big-bang which makes it well defined even at the big-bang, without any additional input.

To solve the quantum Einstein equation, Bojowald again follows, step by step, the procedure introduced (by Thomas Thiemann) in the full theory. Let us expand the full quantum state as $|\Psi> = \sum_n \psi_n(\phi) |n>$ where $|n>$ are the eigenstates of the scale factor operator and $\phi$ denotes
Figure 1: The product $a_n \cdot b_n$ as a function of $n$. The corresponding classical product $a \cdot \sqrt{\text{curv}}$ equals 1.

matter fields. Then, the quantum Einstein equation takes the form:

$$c_n \psi_{n+8}(\phi) + d_n \psi_{n+4}(\phi) + e_n \psi_{n}(\phi) + f_n \psi_{n-4}(\phi) + g_n \psi_{n-8}(\phi) = \gamma \ell_P^2 \tilde{H}_\phi \psi_n(\phi)$$  \hspace{1cm} (3)

where $c_n, \ldots, g_n$ are fixed numerical coefficients, $\gamma$ the Barbero-Immirzi parameter and $\tilde{H}_\phi$ is the matter Hamiltonian. (Again, using the Thiemann regularization, one can show that the matter Hamiltonian is a well-defined operator.)

As one would expect from the phase space-formulation of classical general relativity, primarily, Eq(3) serves to constrain the coefficients $\psi_n(\phi)$ of physically permissible quantum states. However, if we choose to interpret the scale factor (more precisely, the square of the scale factor times the determinant of the triad) as a time variable, Eq(3) can be interpreted as an ‘evolution equation’ which evolves the state through discrete time steps. In a (large) neighborhood of the big-bang singularity, this notion of time is viable. For the choice of factor ordering used in the Thiemann regularization, one can evolve in the past through $n = 0$, i.e. right through the classical singularity. Thus, the infinities predicted by the classical theory at the big-bang are indeed artifacts of assuming that the classical, continuum space-time approximation is valid right up to the big-bang. In the quantum theory, the state can be evolved through the big-bang without any difficulty. However, the classical space-time description fails near the big-bang; quantum evolution is well-defined but the classical space-time ‘dissolves’.

The ‘evolution’ equation (3) has other interesting features. To begin with, the space of solutions is 16 dimensional. Can we single out a preferred solution by imposing a physical condition? One possibility is to impose a pre-classicality condition, i.e., to require that the quantum state not oscillate rapidly from one step to the next at late times when we know our universe behaves classically. Although this is an extra input, it is not a theoretical prejudice about what should happen at (or near) the big-bang but an observationally motivated condition that is clearly satisfied.
by our universe. The coefficients $c_n, \ldots, g_n$ of (3) are such that this condition singles out a solution uniquely. One can ask what this state does at negative times, i.e., before the big-bang. (Time becomes negative because triads flip orientation on the ‘other side’.) Preliminary indications are that the state does not become pre-classical there. If this is borne out by detailed calculations, then the ‘big-bang’ separates two regimes; on ‘our’ side, classical geometry is both meaningful and useful at late times while on the ‘other’ side, it is not. Another interesting feature is that the standard Wheeler-DeWitt equation is recovered if we take the limit $\gamma \rightarrow 0$ and $n \rightarrow \infty$ such that the eigenvalues of $\hat{a}$ take on continuous values. This is completely parallel to the limit we often take to coarse grain the quantum description of a rigidly spinning rotor to ‘wash out’ discreteness in angular momentum eigenvalues and arrive at the classically allowed continuous angular momenta. From this perspective, then, one is led to say that the most striking of the consequences of loop quantum gravity are not seen in standard quantum cosmology because it ‘washes out’ the fundamental discreteness of quantum geometry.

Finally, the detailed calculations have revealed another surprising feature. The fact that the quantum effects become prominent near the big bang, completely invalidating the classical predictions, is pleasing but not unexpected. However, prior to these calculations, it was not clear how soon after the big-bang one can start trusting semi-classical notions and calculations. It would not have been surprising if we had to wait till the radius of the universe became, say, a few million times the Planck length. These calculations strongly suggest that few hundred Planck lengths should suffice. This is fortunate because it is now feasible to develop quantum numerical relativity; with computational resources commonly available, grids with $(10^6)^3$ points are hopelessly large but one with $(100)^3$ points are readily available.

### 3.2 Black-holes

Loop quantum cosmology illuminates dynamical ramifications of quantum geometry but within the context of mini-superspaces where all but a finite number of degrees of freedom are frozen. In this sub-section, I will discuss a complementary application where one considers the full theory but probes consequences of quantum geometry which are not sensitive to full quantum dynamics —the application of the framework to the problem of black hole entropy. This discussion is based on joint work with Baez, Corichi and Krasnov [11] which itself was motivated by earlier work of Krasnov, Rovelli and others.

Let us begin with classical general relativity. Consider first the simplest solution of Einstein’s equation: a manifold which is topologically $\mathbb{R}^4$, equipped with a flat metric (of signature $-+++$). It has the property that an observer near infinity can receive a causal signal from any point in the interior, sometime along its infinite world-line. Thus, no part of space-time is permanently hidden from infinity. However, Einstein’s equation admits solutions which do not share this property. In such solutions, the portion of space-time which is hidden from all asymptotic observers is called a black hole region. Physically, since gravity is attractive, the space-time metric around dense, compact astrophysical objects is such that the light cones in their vicinity are ‘bent towards the object’. Since causal signals propagate with a speed less than or equal to that of light, it is ‘harder

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4There is thus a qualitative similarity to the phenomenon of phase transitions in magnets. ‘Our side’ of the big-bang is analogous to the ferro-magnetic phase (the role of the ‘magnetization’ mean field —the vector pointing from the south to the north pole of a ferro-magnet— being played by the classical geometry) and the ‘other side’ is analogous to the para-magnetic phase (where ‘magnetization’ is no longer a useful concept."

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for the information to leak out’ from their vicinity. A black hole region results if light cones are bent so much that they are ‘tilted completely inwards’; i.e., no causal signal can leave the region of space-time in question. A space-like surface representing an ‘instant of time’ intersects the boundary of the black-hole region in a 2-sphere. We will refer to this 2-sphere boundary as the black hole horizon. (This terminology is not the standard one but is more convenient for our purposes here.)

As explained in the Introduction, since mid-seventies, a key question in the subject has been: What is the statistical mechanical origin of the black hole entropy $S_{BH} = a_{hor}/4\ell_{Pl}^2$? What are the microscopic degrees of freedom that account for this entropy? This relation implies that a solar mass black hole must have $\sim (\exp 10^{77})$ quantum states, a number that is huge even by the standards of statistical mechanics. Where do all these states reside? To answer these questions, in the early nineties John Wheeler suggested the following heuristic picture, which he christened ‘It from Bit’. Divide the black hole horizon into elementary cells, each with one Planck unit, $\ell_{Pl}^2$, of area and assign to each cell two microstates, or one ‘bit’. Then the total number of states $N$ is given by $N = 2^n$ where $n = (a_{hor}/\ell_{Pl}^2)$ is the number of elementary cells, whence entropy is given by $S = \ln N \sim a_{hor}$. Thus, apart from a numerical coefficient, the entropy (‘It’) is accounted for by assigning two states (‘Bit’) to each elementary cell. This qualitative picture is simple and attractive. Therefore it is natural to ask if it can be made precise. Can these heuristic ideas be supported by a systematic analysis from first principles? What is the rationale behind dividing the black hole horizon into elementary cells of unit Planck area? Why are there exactly two quantum states associated with each cell? It turned out that quantum geometry could supply answers to these questions through a detailed analysis.

A systematic approach requires that we first specify the class of black holes of interest. Since the entropy formula is expected to hold unambiguously for black holes in equilibrium, most analyses were confined to space-times with ‘eternal’ black holes admitting a global time-translation isometry, rather than the astrophysical ones which result from a gravitational collapse. From a physical viewpoint however, this assumption seems overly restrictive. After all, in statistical mechanical calculations of entropy of ordinary systems, one only has to assume that the given system is in equilibrium, not the whole world. Therefore, it should suffice to assume that the black hole itself is in equilibrium; the exterior geometry should not be forced to be time-independent. Finally, it has been known since the mid-seventies that the thermodynamical considerations apply not only to black holes but also to so-called ‘cosmological horizons’. A natural question is: Can these diverse situations be treated in a single stroke? Within the quantum geometry approach, the answer is in the affirmative. The idea that the black hole (or the cosmological horizon) is itself in equilibrium is captured by certain boundary conditions which ensure that the horizon itself is isolated, allowing time-dependent space-time geometry and matter fields in the exterior region. Entropy associated with an isolated horizon refers to the family of observers in the exterior region for whom the isolated horizon is a physical boundary that separates the region which is accessible to them from the one which is not. (This point is especially important for cosmological horizons where, without reference to observers, one can not even define these horizons.) States which contribute to this entropy are the ones which can interact with the states in the exterior; in this sense, they ‘reside’ on the horizon.

In the detailed analysis, one considers space-times admitting an isolated horizon as inner boundary and carries out a systematic quantization. The quantum geometry framework can be naturally

\footnote{I should add, however, that this account does not follow chronology. Black hole entropy was computed in quantum geometry quite independently and the realization that the ‘It from Bit’ picture works so well was somewhat of a surprise.}
extended to this case. The isolated horizon boundary conditions imply that the intrinsic geometry of the quantum horizon is described by the so called U(1) Chern-Simons theory on the horizon. This is a well-developed, topological quantum field theory. A deeply satisfying feature of the analysis is that there is a seamless matching of three otherwise independent structures: the isolated horizon boundary conditions which come from classical general relativity; the quantum geometry in the bulk; and the Chern-Simons theory on the horizon. In particular, one can calculate eigenvalues of certain physically interesting operators using purely bulk quantum geometry without any knowledge of the Chern-Simons theory, or using the Chern-Simons theory without any knowledge of the bulk quantum geometry. The two theories have never heard of each other. Yet, thanks to the isolated horizon boundary conditions, the two infinite sets of numbers match exactly, providing a coherent description of the quantum horizon.

In this description, the polymer excitations of the bulk geometry, each labelled by a spin $j_I$, pierce the horizon, endowing it an elementary area $a_{j_I}$ given by $a_{j_I}$. The sum $\sum_I a_{j_I}$ adds up to the total horizon area $a_{\text{hor}}$. The intrinsic geometry of the horizon is flat except at these puncture, but at each puncture there is a quantized deficit angle. These add up to endow the horizon with a 2-sphere topology, as required by a quantum analog of the Gauss-Bonnet theorem. For a solar mass black hole, a typical horizon state would have $10^{77}$ punctures, each contributing a tiny deficit angle. So, although the quantum geometry is distributional, it can be well approximated by a smooth metric.

The counting of states can be carried out as follows. First one constructs a micro-canonical ensemble by restricting oneself only to those states for which the total area, angular momentum, and charges lie in small intervals around fixed values $a_{\text{hor}}, J_{\text{hor}}, Q^i_{\text{hor}}$. (As is usual in statistical mechanics, the leading contribution to the entropy is independent of the precise choice of these small intervals.) For each set of punctures, one can compute the dimension of the surface Hilbert space, consisting of Chern-Simons states compatible with that set. One allows all possible sets of punctures (by varying both the spin labels and the number of punctures), subject to the constraint that the total area $a_{\text{hor}}$ be fixed, and adds up the dimensions of the corresponding surface Hilbert spaces to obtain the number $N$ of permissible surface states. One finds that the horizon entropy
$S_{\text{hor}}$ is given by

$$S_{\text{hor}} := \ln \mathcal{N} = \frac{\gamma_o}{\gamma} \frac{a_{\text{hor}}}{\ell_{\text{Pl}}} + \mathcal{O}(\ell_{\text{Pl}}^2), \quad \text{where} \quad \gamma_o = \frac{\ln 2}{\sqrt{3\pi}}$$

(4)

Thus, for large black holes, entropy is indeed proportional to the horizon area. This is a non-trivial result; for examples, early calculations often led to proportionality to the square-root of the area. However, even for large black holes, one obtains agreement with the Hawking-Bekenstein formula only in the sector of quantum geometry in which the Barbero-Immirzi parameter $\gamma$ takes the value $\gamma = \gamma_o$. Thus, while all $\gamma$ sectors are equivalent classically, the standard quantum field theory in curved space-times is recovered in the semi-classical theory only in the $\gamma_o$ sector of quantum geometry. It is quite remarkable that thermodynamic considerations involving large black holes can be used to fix the quantization ambiguity which dictates such Planck scale properties as eigenvalues of geometric operators. Note however that the value of $\gamma$ can be fixed by demanding agreement with the semi-classical result just in one case —e.g., a spherical horizon with zero charge, or a cosmological horizon in the de Sitter space-time, or, . . . . Once the value of $\gamma$ is fixed, the theory is completely fixed and we can ask: Does this theory yield the Hawking-Bekenstein value of entropy of all isolated horizons, irrespective of the values of charges, angular momentum, and cosmological constant, the amount of distortion, or hair. The answer is in the affirmative. Thus, the agreement with quantum field theory in curved space-times holds in all these diverse cases.

Why does $\gamma_o$ not depend on other horizon parameters such as the charges $Q_{\text{hor}}$? This important property can be traced back to a key consequence of the isolated horizon boundary conditions: detailed calculations show that only the gravitational part of the symplectic structure has a surface term at the horizon; the matter symplectic structures have only volume terms. (Furthermore, the gravitational surface term is insensitive to the value of the cosmological constant.) Consequently, in the geometric quantization procedure used in this analysis, there are no independent surface quantum states associated with matter. This provides a natural explanation of the fact that the Hawking-Bekenstein entropy depends only on the horizon geometry and is independent of electromagnetic (or other) charges.

Finally, let us return to Wheeler’s ‘It from Bit’. One can ask: what are the states that dominate the counting? Perhaps not surprisingly, they turn out to be the ones which assign to each puncture the smallest quantum of area (i.e., spin value $j = \frac{1}{2}$), thereby maximizing the number of punctures. In these states, each puncture defines one of Wheeler’s ‘elementary cell’ and his two states correspond to the $j_z = \pm 1/2$ states, i.e. to whether the deficit angle is positive or negative. However, in the complete theory, all values of $j$ (and hence of $j_z$) must be allowed to obtain a complete description of the geometry of the quantum horizon. If one is only interested in counting states for large black holes, however, the leading contribution comes from the $j = 1/2$ states.

To summarize, quantum geometry naturally provides the micro-states responsible for the huge entropy associated with horizons. In this analysis, all black holes —including the ones of direct astrophysical interest— and cosmological horizons are treated in an unified fashion. The sub-leading term has also been calculated and shown to be proportional to $\ln a_{\text{hor}}$ [11].

4 Conclusion

In this brief report, I have summarized two of the recent advances which have answered some of the long standing questions of quantum gravity raised in the Introduction. There have been two
other notable advances: i) the development of ‘spin-foam models’ discussed briefly by Baez at the conference which provide a new, non-perturbative path integral approach to quantum gravity and have led to a variety of interesting and intriguing mathematical results on state sum models, extending in certain ways the very interesting work on Turiev and Viro in 3-dimensions; and, ii) the introduction of new measures on the space of connections relating the quantum geometry framework to the standard Fock description of photons and gravitons, which is paving the way to relate the Planck scale calculations of quantum geometry to the more familiar world of ‘low energy physics’. The vitality of the program is reflected in the fact that many of the key ideas in all these developments came from young researchers —from Bojowald in quantum cosmology, Krasnov in the understanding of quantum black holes, Perez in spin foams and Varadarajan in the relation to low energy physics.

Throughout the development of loop quantum gravity, unforeseen simplifications have arisen regularly, leading to surprising solutions to seemingly impossible difficulties. Progress could occur because some of the obstinate problems which had slowed developments in background independent approaches, sometimes for decades, evaporated when ‘right’ perspectives were found. I will conclude with a few examples.

- Up until the early nineties, it was widely believed that spaces of connections do not admit non-trivial diffeomorphism invariant measures. This would have made it impossible to develop our background independent approach. Quite surprisingly, such a measure could be found by looking at connections in a slightly more general perspective. It is simple, natural, and has just the right structure to support quantum geometry. This geometry, in turn, supplied some missing links, e.g., by providing just the right expressions that Ponzano-Regge had to postulate without justification in their celebrated, early work on 3-dimensional gravity.

- Fundamental discreteness first appeared in a startling fashion in the construction of the so-called ‘weaves’, quantum states which approximate given classical 3-geometries. In this construction, quantum states based on finite graphs were introduced as a starting point, with the goal of taking a ‘continuum limit’ as in lattice gauge theories. It came as a major surprise that, if one wants to recover a given classical geometry on large scales, one can not take this limit, i.e., one can not refine the underlying graph arbitrarily; there is an in-built discreteness.

- Traces of holonomies of a suitably defined connection around a smooth loop define a natural set of functions of connections. At a heuristic level, it was found that they automatically solve the most difficult part of the quantum Einstein’s equation. No one expected to find such simple and natural solutions even heuristically. This calculation suggested that the action of this part of the quantum Einstein equation is concentrated at vertices of graphs, which in turn led to strategies for its regularization.

- As I indicated in some detail, unforeseen insights arose in the well-studied subject of quantum cosmology essentially by taking an adequate account of the quantum nature of geometry, i.e., by respecting the fundamental discreteness of the eigenvalues of the scale factor operator. Similarly, in the case of black holes, three quite distinct structures —the isolated horizon boundary conditions, the bulk quantum geometry and the surface Chern-Simons theory— blended together unexpectedly to provide a coherent theory of quantum horizons.

Repeated occurrence of such ‘unreasonable’ simplifications suggest that the ideas underlying loop quantum gravity may have captured an essential germ of truth.

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