Conserved non-linear quantities in cosmology

David Langlois$^{1,2}$, Filippo Vernizzi$^3$

$^1$APC (Astroparticules et Cosmologie),
UMR 7164 (CNRS, Université Paris 7, CEA, Observatoire de Paris)
11 Place Marcelin Berthelot, F-75005 Paris, France;
$^2$GReCO, Institut d’Astrophysique de Paris, CNRS,
98bis Boulevard Arago, 75014 Paris, France;
and
$^3$Helsinki Institute of Physics, P.O. Box 64,
FIN-00014 University of Helsinki - Finland

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Abstract

We give a detailed and improved presentation of our recently proposed formalism for non-linear perturbations in cosmology, based on a covariant and fully non-perturbative approach. We work, in particular, with a covector combining the gradients of the energy density and of the local number of e-folds to obtain a non-linear generalization of the familiar linear uniform density perturbation. We show that this covector obeys a remarkably simple conservation equation which is exact, fully non-linear and valid at all scales. We relate explicitly our approach to the coordinate-based formalisms for linear perturbations and for second-order perturbations. We also consider other quantities, which are conserved on sufficiently large scales for adiabatic perturbations, and discuss the issue of gauge invariance.
1 Introduction

The relativistic theory of cosmological perturbations is an essential tool to analyze cosmological data such as the Cosmic Microwave Background (CMB) anisotropies, and thus to connect the scenarios of the early universe, such as inflation, to cosmological observations. Because the temperature anisotropies of the CMB are so small ($\delta T/T \sim 10^{-5}$), considering only linear perturbations is an excellent approximation. This is why most of the efforts devoted to the theory of cosmological perturbations have dealt with linear perturbations [1, 2, 3, 4, 5].

There are however some issues where one must take into account the non-linear aspects of cosmological perturbations. A good example is the study of inhomogeneities on scales much larger than the Hubble radius, where the Universe could strongly deviate from a Friedmann-Lemaître-Robertson-Walker (FLRW) geometry. Another motivation, which has seen renewed interest recently, is to investigate the predictions of early universe models for primordial non-Gaussianity, with the hope to be able to detect this non-Gaussianity in the CMB data. This investigation requires the study of relativistic cosmological perturbations beyond linear order [6, 7, 8].

For small perturbations, as a first step beyond linear order, one can consider second-order relativistic perturbations. This is enough if one wants to compute the bispectrum of perturbations, an indicator of the presence of non-Gaussianities. In the context of general relativity, dealing with second-order perturbations is already an impressive task. The usual treatment introduces coordinates and the metric perturbations are then formally assumed to be the sum of a first-order quantity and of a second-order quantity [9, 10, 11, 12, 13, 14, 15]. One can treat similarly the fluid quantities and then write down Einstein’s equations up to second order. As one can imagine, this approach is rather cumbersome.

Other recent approaches of non-linear perturbations [16, 17, 18] (see also [19]) are based on the long wavelength approximation [20, 21, 22], which at lowest order is related to the so-called separate universe picture that represents our universe, on scales larger than the Hubble radius, as juxtaposed FLRW universes with slightly different scale factors [23, 24].

In the present work, we present the details of a different approach, which we proposed recently [25], based on a purely geometrical description of the perturbations. Our approach is inspired by the so-called covariant formalism for cosmological perturbations introduced by Ellis and Bruni [26]. Although the linearized version of this formalism is often used and it has also been used at second order (see e.g. [27]), our approach is fully non-perturbative and thus not limited to second-order perturbations. The equations and the
variables used in our approach encode the full non-linearity of cosmological perturbations on all scales. Although, at linear order, the covariant formalism is *computationally* equivalent to the much more used coordinate approach, the covariant approach turns out to be more efficient when one goes beyond linear order.

In particular, we show that it is possible to define, in a geometric way, the generalizations of quantities widely used in the linear theory because they are *conserved* on large scales. These quantities are covectors which obey conservation equations with respect to the directional derivation (Lie derivation) along the comoving worldlines. We give the full non-linear equations that govern these quantities. To emphasize the efficiency of our approach with respect to the more traditional coordinate-based approach, we show how gauge-invariant quantities, recently derived in the literature, can be obtained from our geometric quantities in a straightforward derivation. The particular case of scalar fields perturbations will be considered in a separate publication.

This paper is organized as follows. In the next section, we give a brief overview of the covariant formalism for cosmological perturbations due to Ellis and Bruni. In Sec. 3, we show how the energy conservation equation leads naturally to a covector which obeys a conservation equation. In Sec. 4, we show explicitly the connection between our approach and the familiar, coordinate-based, linear theory. In Sec. 5, we study second order perturbations. In Sec. 6, we consider other non-linear conserved quantities. In Sec. 7, we discuss the gauge invariance of our variables and finally, in Sec. 8, we summarize and conclude.

## 2 Covariant formalism

In this section, we briefly review the basic ideas of the covariant approach developed by Ellis, Bruni and collaborators [26, 28, 29], and based on earlier works by Hawking [30] and Ellis [31]. For simplicity, we consider the universe filled with a single perfect fluid, although most of what we will say applies to any independent (i.e., non-interacting with other fluids) perfect fluid, whether it is or not surrounded by other fluids. The case of an imperfect fluid, with non-vanishing heat flow and anisotropic stress, will be considered in a separate publication. The fluid we consider can be characterized by a comoving four-velocity $u^a = dx^a/d\tau$ ($u_a u^a = -1$), where $\tau$ is the proper time along the flow lines, a proper energy density $\rho$ and a pressure $P$.

The energy-momentum tensor associated to the perfect fluid is given by

$$T^a_b = (\rho + P) u^a u_b + P g^a_b. \quad (1)$$
To fully characterize the fluid, one needs an equation of state relating $P$ to $\rho$ and, possibly, to other physical quantities if the fluid is not barotropic.

The spatial projection tensor orthogonal to the fluid velocity $u^a$ is defined by

$$h_{ab} = g_{ab} + u_a u_b, \quad (h^a_{\ b} h^b_{\ c} = h^a_{\ c}, \ h_a^{\ b} u_b = 0). \quad (2)$$

It is also useful to introduce the familiar decomposition

$$\nabla_b u_a = \sigma_{ab} + \omega_{ab} + \frac{1}{3} \Theta h_{ab} - \dot{u}_a u_b, \quad (3)$$

with the (symmetric) shear tensor $\sigma_{ab}$, and the (antisymmetric) vorticity tensor $\omega_{ab}$; the volume expansion $\Theta$, is defined by

$$\Theta \equiv \nabla_a u^a, \quad (4)$$

while $\dot{u}^a$ is the acceleration, with the dot denoting the covariant derivative projected along $u^a$, i.e., $\dot{\equiv} u^a \nabla_a$.

The integration of $\Theta$ along the fluid world lines with respect to the proper time $\tau$ is

$$\alpha \equiv \frac{1}{3} \int d\tau \Theta, \quad (5)$$

and can be used to define, for each observer comoving with the fluid, a local scale factor $S = e^\alpha$. It follows that

$$\Theta = 3 \dot{\alpha} \equiv 3 u^a \nabla_a \alpha. \quad (6)$$

Note that $\alpha$ is defined up to an integration constant for each fluid world line. A convenient way to specify this integration constant is to introduce some reference hypersurface on which $\alpha = 0$. The arbitrariness in the choice of the integration constant along each world-line is thus translated in the arbitrariness in the choice of this reference hypersurface.

We now wish to define, in a geometrical way, quantities that can be interpreted as perturbations with respect to the FLRW configuration. Since we work directly with the clumpy spacetime, an obvious difficulty is how to separate homogeneous quantities from their perturbations. A way to get around this difficulty was first suggested by Ellis and Bruni [26]. They introduced spatial projections (defined as projections orthogonal to the four-velocity $u^a$) of the covariant derivative of various scalar quantities: of the energy density,

$$X_a = h_a^{\ \ b} \nabla_b \rho \equiv D_a \rho, \quad (7)$$

of the pressure

$$Y_a \equiv D_a P, \quad (8)$$

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and the volume expansion,

$$Z_a \equiv D_a \Theta.$$  \hfill (9)

Here, following [25], we also introduce the spatial gradient of the integrated expansion,

$$W_a \equiv D_a \alpha.$$  \hfill (10)

Note that these definitions are purely geometrical and depend only on the (physical) four-velocity $u^a$. All these quantities automatically vanish in a strictly FLRW spacetime: in this sense we call them perturbations. However, they are fully non-perturbative quantities, not restricted to linear order in a perturbation expansion. The last quantity, $W_a$, corresponds to the spatial gradient of the local number of e-folds $\alpha$. It was introduced in the non-perturbative approach in our previous work [25], because it plays a crucial role in our approach, replacing the familiar spatial curvature perturbation, similarly to the separate universe picture [24, 32]. Note that a quantity similar to $W_a$ has already been used in the linearized covariant theory in [33].

3 Generalized conserved quantities

Starting from the above definitions, we will now introduce a quantity that naturally satisfies a conservation equation. In [32], it was emphasized that the conservation, for adiabatic perturbations and on large scales, of the linear curvature perturbation on uniform-density hypersurfaces, usually denoted $\zeta$, introduced in [34], can be derived directly, without resorting to Einstein’s equations, from the conservation of the energy-momentum tensor,

$$\nabla_a T^a_{\ b} = 0.$$  \hfill (11)

Here, we will use the same starting point in order to define a non-linear generalization of $\zeta$, and we shall make use of the covariant approach described in the previous section.

Let us consider the projection along $u^a$ of the conservation equation (11). This yields

$$\dot{\rho} + \Theta (\rho + P) = 0,$$  \hfill (12)

where we remind the reader that the dot represents a covariant derivative projected along $u^a$, i.e. $\dot{\equiv} u^a \nabla_a$. If one takes the projected gradient of the previous expression one gets

$$D_a (\rho) + (\rho + P) Z_a + \Theta (X_a + Y_a) = 0,$$  \hfill (13)
where we have used the definitions (7), (8) and (9).

We now concentrate on the first term of Eq. (13) trying to invert the time derivative with the spatial gradient. In order to do so, it will be convenient to introduce the Lie derivative along $u^a$ of tensors. The corresponding definition for a covector, which will be useful below, is (see, e.g., [35] for the general definition)

$$L_u \chi_a \equiv u^c \nabla_c \chi_a + \chi_c \nabla_a u^c.$$  \hfill (14)

Note that, for scalars, the dot derivation is equivalent to the Lie derivation, i.e. $\dot{\rho} = L_u \rho$. Thus, we want to invert the Lie derivation along $u^a$ with the spatial gradient in the first term of Eq. (13). The two do not commute and we have

$$D_a (\dot{\rho}) = h^b_a \nabla_b (u^c \nabla_c \rho) = \left( h^b_a \nabla_b u^c \right) \nabla_c \rho + u^c h^b_a \nabla_b \nabla_c \rho$$

$$= L_u (D_a \rho) - \nabla_a u^c D_c \rho + \left( h^b_a \nabla_b u^c - u^b \nabla_b h^c_a \right) \nabla_c \rho. \quad \hfill (15)$$

This expression can be rewritten in a remarkably simple form as

$$D_a (\dot{\rho}) = L_u (D_a \rho) - \dot{u}_a \dot{\rho}. \quad \hfill (16)$$

Although we have derived it for the energy density $\rho$, this expression of course applies to any scalar quantity. In particular, it can be applied to the integrated expansion $\alpha$, thus allowing us to express $Z_a = D_a \Theta = 3D_a \dot{\alpha}$ in terms of $W_a = D_a \alpha$. This gives

$$Z_a = 3L_u W_a - 3\dot{u}_a \dot{\alpha}. \quad \hfill (17)$$

Substituting (16) and (17) in (13), one gets

$$L_u X_a + 3(\rho + P) L_u W_a + \Theta (X_a + Y_a) = 0, \quad \hfill (18)$$

where terms proportional to $\dot{u}_a$ have disappeared because of the continuity equation (12). Using the relation

$$\frac{L_u X_a}{\rho + P} = L_u \left( \frac{X_a}{\rho + P} \right) - \Theta (1 + c_s^2) \frac{X_a}{\rho + P}, \quad \hfill (19)$$

where we have introduced $c_s^2 \equiv \dot{P}/\dot{\rho}$, the non-linear generalization of the adiabatic speed of sound, Eq. (18) can be rewritten as an evolution equation for the covector

$$\zeta_a \equiv W_a + \frac{X_a}{3(\rho + P)} = D_a \alpha - \frac{\alpha}{\rho} D_a \rho, \quad \hfill (20)$$
which, as we will see later, can be seen as a generalization of the curvature perturbation on uniform density hypersurfaces of the linear theory, usually denoted by $\zeta$ (hence our name $\zeta_a$). Equation (18) then yields

$$L_u \zeta_a = -\frac{\Theta}{3(\rho + P)} \left( Y_a - c_s^2 X_a \right).$$

(21)

On the right hand side, the quantity

$$\Gamma_a \equiv Y_a - c_s^2 X_a = D_a P - \frac{\dot{P}}{\dot{\rho}} D_a \rho$$

(22)

can be interpreted as the projected gradient of the non-adiabatic pressure, and represents the non-linear generalization of the non-adiabatic pressure defined, e.g., in [32]. It vanishes for purely adiabatic perturbations, i.e., when the pressure $P$ is solely a function of the density $\rho$.

For practical purposes, it is useful to note that, although both $\zeta_a$ and $\Gamma_a$ are defined as linear combinations of spatially projected gradients, one can replace them by ordinary gradients, thus having

$$\zeta_a = \partial_a \alpha - \frac{\dot{\alpha}}{\dot{\rho}} \partial_a \rho, \quad \Gamma_a = \partial_a P - \frac{\dot{P}}{\dot{\rho}} \partial_a \rho.$$  

(23)

This is indeed possible for any linear combination of the form $D_a \chi - (\dot{\chi}/\dot{\eta}) D_a \eta$ since, for any scalar quantity $\chi$, we have

$$D_a \chi = \partial_a \chi + u^a \dot{\chi}.$$  

(24)

Substituting the definition of $\Gamma_a$ into Eq. (21) we finally obtain

$$L_u \zeta_a = -\frac{\Theta}{3(\rho + P)} \Gamma_a.$$  

(25)

This equation has a form very similar to the conservation equation for $\zeta$ of the linear theory, which will be rederived in the next section. However, in our case, the above equation is exact, fully non-perturbative and valid at all scales.

The time derivative that appears in the equation of the linear theory has been replaced here by a Lie derivative along $u^a$, which is consistent with the fact that we want to describe the conservation properties of a covector along the fluid worldline of a comoving observer. This makes Eq. (25) remarkably simple and improves our original formalism given in [25] where we did not introduce the Lie derivative. Moreover, the use of the Lie derivative simplifies the calculations in practice, because one does not need to compute the
covariant derivatives. Indeed, the left hand side of Eq. (25) can be rewritten, starting from the definition of Lie derivative (25), as
\[ \mathcal{L}_u \zeta_a = u^c \partial_c \zeta_a + \zeta_c \partial_a u^c, \] (26)
which will be very simple to use in the following.

A remark is in order here. In our approach, the covector \( \zeta_a \), defined as a linear combination of spatial gradients of \( \alpha \) and of \( \rho \), is the crucial quantity that enables us to write the conservation equation for perturbations in a remarkably simple form. It is interesting to note that the same linear combination appears in the work of Rigopoulos and Shellard [16]. Although their motivation is, like here, to generalize the conservation equation for the linear \( \zeta \) to the non-linear case, their approach differs from ours in two respects. First they do not use a covariant formalism but a coordinate based approach with an ADM decomposition of the metric. As a consequence, their correspondent equivalent of our \( \alpha \) is not defined as the integrated expansion along fluid worldlines but from the determinant of the spatial metric. In this, they follow an older work [20] which already emphasized the advantage of this quantity to generalize the linear \( \zeta \) to non-linear order. Second – and as a consequence of the first point – they restrict their analysis to super-Hubble scales as in [20].

In a similar spirit, i.e., using the ADM decomposition and invoking a long-wavelength approximation, the authors of [18] found, not exactly the linear combination appearing in [16], but its (spatially) integrated version. In contrast with these previous works, as already emphasized, our formulation is fully covariant and our non-linear generalization applies to all scales.

4 Linear theory

In this section, we compare our approach with the more familiar, coordinate-based, linear theory (see e.g., [36] or [37] for recent presentations of this topic). In order to do so, we first introduce conformal coordinates \( x^\mu = \{ \eta, x^1 \} \) to describe our almost-FLRW spacetime. A prime will denote a partial derivative with respect to conformal time \( \eta \), \( \eta' \equiv \partial/\partial \eta \). The background spacetime, i.e., at zeroth order, is a FLRW spacetime, endowed with the metric
\[ ds^2 = \bar{g}_{\mu \nu} dx^\mu dx^\nu = a^2 \left[ -d\eta^2 + \gamma_{ij} dx^i dx^j \right], \] (27)
where \( a = e^{\bar{\alpha}} \) is the background scale factor, and filled with a homogeneous perfect fluid characterized by the energy density \( \bar{\rho}(\eta) \), the pressure \( \bar{P}(\eta) \) and the four-velocity
\[ \bar{u}^\mu = \{ 1/a, 0, 0, 0 \}. \] (28)
The perturbed spacetime is described by the perturbed metric
\[ ds^2 = (\bar{g}_{\mu\nu} + \delta g_{\mu\nu}) \, dx^\mu dx^\nu, \]  
with
\[ \delta g_{00} = -2a^2 A, \quad \delta g_{0i} = a^2 B_i, \quad \delta g_{ij} = a^2 H_{ij}. \]

We decompose \( H_{ij} \) in the form
\[ H_{ij} = -2\psi \gamma_{ij} + 2\nabla_i \nabla_j E + 2\nabla_i (i E V_j) + 2E^T_{ij}, \]
with \( E^T_{ij} \) transverse and traceless, i.e., \( \nabla_i E^T_{ij} = 0 \) and \( \gamma^{ij} E^T_{ij} = 0 \), and \( E^V_{ij} \) transverse, i.e., \( \nabla_i E^V_{ij} = 0 \), where \( \nabla_i \) denotes the covariant derivative with respect to the homogeneous spatial metric \( \gamma_{ij} \) (which is also used to lower or raise the spatial indices).

The corresponding matter content is a perfect fluid with perturbed energy density and pressure,
\[ \rho(\eta, x^i) = \bar{\rho}(\eta) + \delta \rho(\eta, x^i), \quad P(\eta, x^i) = \bar{P}(\eta) + \delta P(\eta, x^i) \]
and four-velocity
\[ u^\mu = \bar{u}^\mu + \delta u^\mu, \quad \delta u^\mu = \{-A/a, v^i/a\}, \quad v_k = \nabla_k v + v^V_k, \]
where \( v^V_i \) is transverse, \( \nabla_i v^V_i = 0 \).

We now wish to write down explicitly the components of our vector \( \zeta_a \) in this generic coordinate system. We will start from the expression for \( \zeta_a \) given in (23) rather than from its definition (20). At zeroth order, i.e., in the unperturbed FLRW spacetime, \( \zeta_a \) automatically vanishes. At linear order, the spatial components are simply
\[ \zeta^{(1)}_i = \partial_i \zeta^{(1)}, \quad \zeta^{(1)} = \alpha^{(1)} - \frac{\ddot{a}}{a} \alpha^{(1)}, \]
where we recall that a prime denotes a partial derivative with respect to conformal time.

To compute the component \( \zeta_0 \), it is useful to note that, for any function \( \chi \), one can write
\[ D_0 \chi = u_0 u^i \partial_i \chi - u^i u_i \partial_0 \chi, \]
where we have used the normalization of \( u^a \). Since \( u^i \) is first order, this implies that \( \zeta^{(1)}_0 = 0 \) (although this will not be the case at second order).

In order to make the link with the usual quantities of the linear theory, one needs to reexpress \( \alpha^{(1)} \) in terms of the metric and matter perturbations.
The detailed calculations, up to second order in the perturbations, are given in the appendix. Retaining only the first order terms, we obtain

\[
\alpha^{(1)} = \frac{1}{6} \gamma^{ij} H_{ij} + \frac{1}{3} \int d\eta \nabla_k v^k = -\psi + \frac{1}{3} \nabla^2 E + \frac{1}{3} \int d\eta \nabla^2 v,
\]

where the decompositions (31) and (33) have been used to obtain the second line.

The components of the non-adiabatic term \( \Gamma^a = \partial^a P - (\dot{P}/\dot{\rho})\partial^a \rho \) can be deduced directly from the components of \( \zeta^a \) by substituting \( P \) to \( \alpha \). Therefore, one finds

\[
\Gamma^{(1)}_i = \partial_i \Gamma^{(1)}_i, \quad \Gamma^{(1)} = P_{(1)} - \frac{\dot{P}}{\dot{\rho}} \rho_{(1)},
\]

while the time component vanishes.

Let us now specialize our equation (25) to first order in perturbation theory. One can first notice that, at first order, \( L_u \zeta^{(1)}_i = \zeta^{(1)}_i' / a \). We then get

\[
\zeta^{(1)}_i' = -H \rho + P \delta P_{nad} - \frac{1}{3} \nabla^2 (E' + v),
\]

where \( \zeta \) denotes the conformal Hubble rate, \( \zeta = a' / a = \alpha' \). We then note that, from Eq. (36),

\[
\alpha^{(1)}' = -\psi' + \frac{1}{3} \nabla^2 (E' + v).
\]

Consequently, making use of Eqs. (31) and (37) and getting rid of the \( \partial_i \) common to both sides of (38), one finds

\[
\left( -\psi - \frac{\dot{H}}{\dot{\rho}} \right) + \frac{1}{3} \nabla^2 (E' + v) = -\frac{\dot{H}}{\dot{\rho} + P} \left( P_{(1)} - \frac{\dot{P}}{\dot{\rho}} \rho_{(1)} \right).
\]

One recognizes in the parenthesis of the left hand side the familiar quantity

\[
\zeta \equiv -\psi - \frac{\dot{\rho}_{(1)}}{\dot{\rho}}.
\]

Our linearized equation for \( \zeta_a \) is thus shown to be completely equivalent to the analogous equation obtained directly in the linear theory

\[
\zeta' = -\frac{\dot{H}}{\rho + P \delta P_{nad}} - \frac{1}{3} \nabla^2 (E' + v),
\]
with

\[ \delta P_{\text{nad}} \equiv P_1 - \frac{\bar{\rho}'}{\rho'} \rho_1. \]  

(43)

As one can see, at the linear level, our quantity \( \zeta_i \) is conserved at all scales because our definition of \( \zeta_i \) automatically includes the Laplacian terms that appear on the right hand side of (42). In other words, whereas our \( \zeta_\alpha \) coincides with the usual \( \zeta \) on long wavelengths when the spatial gradients can be neglected, the two quantities will differ on small scales, our quantity being more adapted to the underlying conservation law.

5 Second order perturbations

In the following, we decompose any function \( X \) in the form

\[ X(\eta, x^i) = \bar{X}(\eta) + X_1(\eta, x^i) + \frac{1}{2} X_2(\eta, x^i) + \ldots, \]  

(44)

where \( X_1 \) and \( X_2 \) represent, respectively, the first and second order perturbations.

To compute the component \( \zeta_0 \) at second order we use again Eq. (35). Since \( u^i \) is first order, this implies \( \zeta_0^{(2)} = u^i \zeta_i^{(1)} \). Expanding now \( \zeta_i = \partial_i \alpha - (\dot{\alpha}/\dot{\rho}) \partial_i \rho \) up to second order, one finds

\[ \zeta_i^{(2)} = \partial_i \left( \alpha_2 - \bar{\alpha}' \bar{\rho}_2 \right) - \frac{2}{\bar{\rho}} \left( \alpha_1' - \bar{\alpha}' \bar{\rho}_1' \right) \partial_i \rho_1. \]  

(45)

In contrast with the first order expression, the coefficient in front of the gradient in the second term on the right hand side cannot be directly absorbed in the gradient because it depends on space. One can however do the following manipulations:

\[
\begin{align*}
\zeta_i^{(2)} &= \partial_i \left( \alpha_2 - \bar{\alpha}' \bar{\rho}_2 \right) - \frac{2}{\bar{\rho}} \left( \alpha_1' - \bar{\alpha}' \bar{\rho}_1' \right) \partial_i \rho_1 \\
&= \partial_i \left( \alpha_2 - \bar{\alpha}' \bar{\rho}_2 \right) - 2\alpha_1' \frac{\rho_1(1)}{\bar{\rho}} + 2\bar{\alpha}' \frac{\rho_1(1)}{\bar{\rho}} \frac{\rho_1'}{\bar{\rho'}}
+ 2\partial_i \left( \alpha_1' - \bar{\alpha}' \bar{\rho}_1' \right) \frac{\rho_1(1)}{\bar{\rho}} \\
&= \partial_i \left[ \alpha_2 - \bar{\alpha}' \bar{\rho}_2 - 2\alpha_1' \frac{\rho_1(1)}{\bar{\rho}} + 2\bar{\alpha}' \frac{\rho_1(1)}{\bar{\rho}} \frac{\rho_1'}{\bar{\rho'}} \right]
+ \left( \bar{\alpha}'' - \bar{\alpha}' \frac{\rho''(1)}{\bar{\rho}'} \right) \frac{\rho_1(1)}{\bar{\rho}'^2} + 2\frac{\rho_1(1)}{\bar{\rho}} \partial_i \zeta_i^{(1)}',
\end{align*}
\]  

(46)
which can be written in the form
\[ \zeta^{(2)}_i = \partial_i \zeta^{(2)} + \frac{2}{\bar{\rho}} \rho^{(1)} \partial_i \zeta^{(1)'}, \]  
(47)
with
\[ \zeta^{(2)} \equiv \alpha^{(2)} - \frac{\dot{\alpha}'}{\rho} \rho^{(2)} - \frac{2}{\bar{\rho}} \alpha^{(1)' \rho^{(1)}} + 2 \frac{\ddot{\alpha}' \rho^{(1)}}{\bar{\rho}^2 \rho^{(1)}} + \frac{1}{\bar{\rho}} \left( \frac{\dot{\alpha}'}{\rho} \right)' \rho^{(1)}. \]  
(48)

If one substitutes in the above equation the expression for \( \alpha \) up to second order derived in the appendix, keeping only the scalar terms without gradients, i.e., which are not negligible on large scales,
\[ \alpha \simeq \ln a - \psi - \psi^2, \]  
(49)
one finds
\[ \zeta^{(2)} \simeq -\psi^{(2)} - 2\psi^{(1)} \]  
(50)

The right hand side can be easily related to the conserved second order quantity defined by Malik and Wands in [12], and thus,
\[ \zeta^{(2)} \simeq \zeta^{(2)}_{MW} - \zeta^{(1)2}_{MW}. \]  
(51)
(See also the discussion in [38].)

Now that we have identified the second order perturbation variable \( \zeta^{(2)} \), we can expand Eq. (25) to second order. On making use of Eq. (26) to reexpress the Lie derivative along \( u^a \) in terms of coordinate time derivative, and retaining only second order terms, we have
\[ L_{u^a} \zeta^{(2)}_i = \frac{1}{a} \left[ \zeta^{(2)'}_i - 2 A \zeta^{(1)'}_i + 2 \left( \psi_j \partial_j \zeta^{(1)} + \zeta^{(1)}_j \partial_i \psi_j \right) \right]. \]  
(52)
Finally, on making use of Eqs. (54), (47), and (48), and that \( \Theta = 3(1-A)\alpha'/a \) up to first order, we can explicitly write the conservation equation (25) at second order and on all scales:
\[ \zeta^{(2)'} = -\frac{\mathcal{H}}{\bar{\rho} + P} \Gamma^{(2)} - \frac{2}{\bar{\rho} + P} \Gamma^{(1)} \zeta^{(1)'} - 2 \psi_j \partial_j \zeta^{(1)}. \]  
(53)
The definition of \( \Gamma^{(2)} \) can be read from the expansion of \( \zeta^{(2)} \) by substituting \( P \) to \( \alpha \). For adiabatic perturbations, we find that the second order scalar variable \( \zeta^{(2)} \) is conserved only on large scales, when the last term on the right hand side of Eq. (53) can be neglected. In principle, one can extend straightforwardly the procedure presented here to higher orders in the perturbation expansion.
6 Other conserved quantities

6.1 Comoving curvature perturbation

We have seen in the previous sections that our $\zeta_a$ can be interpreted as the non-linear generalization of the curvature perturbation on uniform energy density hypersurfaces of the linear theory. Another useful quantity in the linear theory is the so-called curvature perturbation on comoving hypersurfaces, usually denoted by $\mathcal{R}$ and one can wonder if a non-linear generalization can be constructed within our formalism. As we have seen before, what plays the rôle of the spatial curvature in our formalism is the integrated expansion $\alpha$ and since our spatial gradients are defined with respect to the comoving observers, a candidate that generalizes $\mathcal{R}$ is simply

$$\mathcal{R}_a \equiv -D_a \alpha = -W_a. \quad (54)$$

We can explicitly check the connection between this quantity and the linear comoving curvature perturbation $\mathcal{R}$ by noticing that, in the coordinate system defined in the appendix, one finds for the spatial components of $\mathcal{R}_a$, at first order in the perturbations,

$$\mathcal{R}^{(1)}_i = -\partial_i \alpha^{(1)} - \ddot{\alpha} u_i = -\partial_i \alpha^{(1)} - \ddot{\alpha}' (v_i + B_i). \quad (55)$$

Therefore, for the scalar part of $\mathcal{R}^{(1)}_i$, we recover the usual definition of the comoving curvature perturbation,

$$\mathcal{R}^{S(1)}_i = \partial_i \mathcal{R}^{(1)}, \quad \mathcal{R}^{(1)} = -\alpha^{(1)} - \mathcal{H} (v + B), \quad (56)$$

provided one can replace $-\alpha^{(1)}$ by $\psi$.

It is relatively easy to derive an evolution equation for $\mathcal{R}_a$. As a first step, one can rewrite Eq. (17) as

$$\mathcal{L}_u \mathcal{R}_a = -\frac{1}{3} \left( \Theta u_a + Z_a \right). \quad (57)$$

If one wishes to write the right hand side of this equation only in terms of quantities characterizing the matter, there is some extra work. This consists in replacing $Z_a$ with the use of the field equations, i.e., Einstein’s equations. Contracting the identity

$$\nabla_c \nabla_d u_a - \nabla_d \nabla_c u_a = R_{abcd} u^b, \quad (58)$$

where $R_{abcd}$ is the Riemann curvature tensor, by $g^{ac} h^c_b$, one gets, upon using the decomposition (3) and Einstein’s field equations $[R_{ab} = \kappa \left( T_{ab} - \frac{1}{2} T g_{ab} \right)$, with $\kappa = 8 \pi G$], the so-called momentum constraint equation (see, e.g., [31]),

$$\frac{2}{3} Z_a - h^b_a \nabla_c (\sigma^c_b + \omega^c_b) \geq \left( \sigma^b_a + \omega^b_a \right) v^b = -\kappa h^b_a T^a_b u^c = 0, \quad (59)$$
where the right-hand side vanishes for the energy-momentum tensor (1).

Moreover, one can write \( \dot{u}_a \) as

\[
\dot{u}_a = -\frac{Y_a}{\rho + P}.
\]

(60)

This is simply the Euler equation, which follows from the projection, orthogonally to \( u^a \), of the energy-momentum tensor conservation (11). The covector \( Y_a \) can be reexpressed as

\[
Y_a = \Gamma_a + c_s^2 X_a.
\]

(61)

It is also worth noticing the identity that relates our two covectors \( \zeta_a \) and \( R_a \):

\[
\zeta_a + R_a = \frac{X_a}{3(\rho + P)}.
\]

(62)

Finally, substituting the relations (59-61) into Eq. (57), we obtain

\[
\mathcal{L}_u R_a = \left[ \frac{\Theta \delta_a^b}{3(\rho + P)} - \frac{\sigma_a^b + \omega_a^b}{2(\rho + P)} \right] \left( \Gamma_b + c_s^2 X_b \right) - \frac{1}{2} h_a^b \nabla_c (\sigma^c_b + \omega^c_b).
\]

(63)

This gives a fully non-linear evolution equation for \( R_a \). As one can see, it is much more complicated than the evolution equation for \( \zeta_a \).

At linear order, considering only scalar perturbations, one finds, from the definition of \( X_a \),

\[
X^{(1)} = \rho^{(1)} - 3\mathcal{H}(\bar{\rho} + \bar{P})(v + B),
\]

(64)

where \( X^{(s)} = \partial_i X^{(s)} \). The quantity \( X^{(1)} \) corresponds to the so-called comoving density perturbation [2], known to vanish on large scales because of the relativistic Poisson equation. Therefore, since \( \zeta^{(1)} + R^{(1)} = X^{(1)}/[3(\bar{\rho} + \bar{P})] \), \(-R^{(1)}\) coincides with \( \zeta^{(1)} \) on large scales, and is thus conserved on large scales. Note that this also holds at second order, as shown in [14].

6.2 The conserved variable \( C_a \)

In the previous subsection, we introduced the variable \( R_a \) and showed that, in the linear limit and on large scales, it reduces to the familiar comoving curvature perturbation. This is however not the only non-linear quantity that satisfies this property. As shown in [28], this is also the case for the projected gradient of the spatial scalar curvature,

\[
C_a = S^a D_a K.
\]

(65)
To be more precise, the quantity $\mathcal{K}$ is defined as

$$\mathcal{K} = 2\left(\frac{-1}{3}\Theta^2 + \sigma^2 - \omega^2 + \kappa \rho\right),$$

(66)

with $\sigma^2 \equiv \frac{1}{2} \sigma_{ab} \sigma^{ab}$ and $\omega^2 \equiv \frac{1}{2} \omega_{ab} \omega^{ab}$. When $\omega_{ab} = 0$, $\mathcal{K}$ corresponds to the Ricci scalar curvature $^{(3)}R$ of the three-dimensional hypersurfaces orthogonal to $u^a$. However, in the general case when $\omega_{ab} \neq 0$, such hypersurfaces cannot be defined, according to Frobenius' theorem (see, e.g., [35]).

As shown in [28], $C_a$ is conserved on large scales in a linearly perturbed flat FLRW universe, for adiabatic perturbations. More generally, it is convenient to define the quantity [39]

$$\tilde{C}_a = C_a - \frac{4aK}{\rho + P} X_a,$$

(67)

where $K = 0, \pm 1$ characterizes the spatial curvature of the unperturbed FLRW spacetime. The variable $\tilde{C}_a$ reduces to $C_a$ for a spatially flat background and turns out to be conserved on large scales for adiabatic linear perturbations for all values of $K$.

To check the relation between $C_a$ and our quantity $\mathcal{R}_a$ at linear order, we now work with the perturbed metric of Sec. 4. The first order perturbation of $\mathcal{K}$ is given by [29]

$$\mathcal{K}^{(1)} = \frac{4}{a^2} \left[ (\nabla^2 + 3K) \psi - \mathcal{H} \nabla_k (v^k + B^k) \right].$$

(68)

It follows from the definition (65) that $C_{a}^{(1)} = 0$ and, using $\tilde{\mathcal{K}} = 6K/a^2$,

$$C_i^{(1)} = a^3 \left[ (v_i + B_i) \partial_0 \tilde{\mathcal{K}} + \nabla_i \bar{\mathcal{K}}^{(1)} \right]$$

$$= 4a \left\{ \nabla_i \left[ (\nabla^2 + 3K) (\psi - \mathcal{H}(v + B)) \right] - 3 \mathcal{H} K (v_i^V + B_i^V) \right\}. \quad (69)$$

On making use of Eq. (67) one finds $\tilde{C}_{0}^{(1)} = 0$ and

$$\tilde{C}_i^{(1)} = 4a \nabla_i \left\{ \nabla^2 [\psi - \mathcal{H}(v + B)] + 3K \left[ \psi - \frac{\rho^{(1)}}{3(\rho + P)} \right] \right\}$$

$$= 4a \partial_i \left( \nabla^2 \mathcal{R} - 3K \zeta \right), \quad (70)$$

where $\zeta$ and $\mathcal{R}$ are the standard first order perturbation variables.

### 6.3 Conserved number densities

We can define other types of conserved quantities starting from any scalar quantity which obeys a continuity equation. In analogy with the analysis of
Lyth and Wands in the context of the linear theory \cite{10}, one can, for example, consider the particle number density in the physical contexts where the particle number is conserved. The corresponding non-perturbative continuity equation is simply
\[ \nabla_a (n u^a) = 0, \tag{71} \]
which yields
\[ \dot{n} + \Theta n = 0. \tag{72} \]

Via a derivation similar to that of Sec. 3 for the energy density, it is not difficult to show that the spatial projection of (72) can be rewritten as
\[ L^a \zeta_a^{(n)} = 0, \tag{73} \]
where we have defined
\[ \zeta_a^{(n)} \equiv W_a + D_a n + \frac{1}{3n} D_a \alpha - \frac{\dot{\alpha}}{n} D_a n. \tag{74} \]

We thus see that, for any quantity that satisfies a standard local conservation equation, one can construct a covector that embodies a non-linear perturbation for this quantity, defined as a combination of the spatial gradient of the quantity and that of \( \alpha \). This covector then satisfies an exact fully non-linear conservation equation.

7 Non-perturbative approach and gauge invariance

The approach proposed in this work is completely non-perturbative: it does not rely on a perturbative expansion of the variables describing cosmological perturbations. This is why the equations derived here encode the non-linear evolution of these variables. In this section, we would like to comment about the notion of gauge-invariance in the context of this approach.

Gauge-invariance has become ubiquitous in the studies of linear cosmological perturbations. Indeed, using gauge-invariant quantities helps avoiding ambiguities by getting rid of unphysical degrees of freedom and makes easier the comparison between calculations done in different gauges. However, completely fixing a gauge, either physically (i.e., with reference to some specific matter) or geometrically, is equivalently acceptable.

In our case we do not really need to care about gauge-invariance because we use tensor quantities that are physically well specified independently of any coordinate system. Our approach, however, contains one possible source
of ambiguity: our quantity $\alpha$ is defined up to a constant of integration along each fluid worldline. Assuming that $\alpha$ is continuous implies that there is some arbitrariness in the choice of the $\alpha = 0$ hypersurface. A similar arbitrariness can be found in the separate universe approach [24].

This being said, it is nevertheless instructive to examine our approach in the light of the treatment of gauge-invariance beyond linear perturbations which has been discussed thoroughly in [10]. In the perturbative approach, one considers a collection of space-times $\mathcal{M}_\lambda$ that interpolates continuously, as the parameter $\lambda$ goes from 0 to 1, between the background spacetime $\mathcal{M}_0$ and the “real” universe $\mathcal{M}_1$. A choice of gauge corresponds to a choice of a continuous family of diffeomorphisms $\varphi_\lambda : \mathcal{M}_0 \to \mathcal{M}_\lambda$. The perturbation of a tensor field $T$, which is associated to a continuous family of $T_\lambda$ defined on each $\mathcal{M}_\lambda$, is defined as

$$\Delta T_\lambda = \varphi_\lambda^* T - T_0,$$

where $\varphi_\lambda^* T$ is the pull-back of $T_\lambda$. The perturbation $\Delta T_\lambda$ is thus a tensor field defined on $\mathcal{M}_0$. One can then do a Taylor expansion of the perturbation with respect to the parameter $\lambda$,

$$\Delta T_\lambda = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \mathcal{L}_\xi^k T,$$

where $\xi$ is the tangent vector associated with the flow $\varphi_\lambda$, and one can thus define, order by order, the notion of gauge-invariance [10].

In a non-perturbative approach, such as ours, one can define an alternative notion of gauge-invariance. Indeed one needs to consider only the background spacetime $\mathcal{M}_0$ and the real spacetime $\mathcal{M}$. A correspondence between $\mathcal{M}_0$ and $\mathcal{M}$ can be specified via a diffeomorphisms $\varphi : \mathcal{M}_0 \to \mathcal{M}$. The gauge ambiguity results from the arbitrariness in the choice of $\varphi$. However, now, instead of defining the perturbation of a tensor $T$ on the background spacetime $\mathcal{M}_0$, one can do it on the perturbed spacetime:

$$\Delta T = T - \varphi_* T_0,$$

where $\varphi_*$ is the push-forward of $T$. With this alternative definition for the perturbation, it is immediate to see that $\Delta T$ is gauge-invariant, i.e., independent of the choice of $\varphi$, if $T_0$ vanishes. In this sense, which does not coincide with the perturbative gauge-invariance defined in [10] but which agrees implicitly with the discussion of [26], the quantities defined in our approach are gauge-invariant.
8 Conclusions

In this work, we have discussed in detail a new approach to the theory of non-linear cosmological perturbations, which was presented briefly in [25]. Our starting point is the covariant formalism for cosmological perturbations and in particular its use of a spatially projected gradient, defined orthogonally to the fluid four-velocity. Our new formalism is then based on the following additional ingredients:

- We have introduced $\alpha$, the local number of e-folds, or integrated expansion, along each worldline, and its spatially projected gradient.

- We have introduced $\zeta_a$, a linear combination of the spatial gradients of $\alpha$ and of the energy density $\rho$. For linear perturbations and on large scales, our definition reduces to the well-known curvature perturbation on uniform energy density hypersurfaces (more precisely, its gradient).

- We have shown that $\zeta_a$ obeys a remarkably simple conservation equation, Eq. (25), which is fully non-linear, exact and valid at all scales. Here, improving the formulation of our previous work [25], we describe the evolution of $\zeta_a$ by using the Lie derivative with respect to the four-velocity $u^a$, instead of the covariant derivative with respect to $u^a$. This explicitly shows that $\zeta_a$ is conserved, at all scales and fully non-perturbatively, for adiabatic perturbations. Indeed, the only source of evolution of $\zeta_a$ is the non-linear generalization of the non-adiabatic pressure, which describes entropy perturbations and is defined as a linear combination of the spatial gradients of $\rho$ and $P$.

We have shown explicitly the connection between our approach and previous works, in particular on linear perturbations and on second order perturbations. As an example, our equation (25) is so simple that it is straightforward to write down the conservation equation for the components of $\zeta_a$ at second order and at all scales. One of the conclusions of this work is that the local number of e-folds and the related combination $\zeta_a$ appear to be the most natural quantities to express the conservation law for adiabatic cosmological perturbations, in its fully non-linear and all-scale form.

As a final remark, let us stress that we did not use Einstein’s equations to derive the conservation equation for $\zeta_a$ but only the conservation of the energy-momentum tensor. Therefore, our equation (25) applies to any gravity theory, as long as the energy-momentum tensor is conserved.
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A Appendix: Perturbed quantities up to second order

We consider a perturbed metric with

\[ g_{00} = -a^2 (1 + 2A), \quad g_{0i} = a^2 B_i, \quad g_{ij} = a^2 (\gamma_{ij} + H_{ij}), \quad (78) \]

where \( A, B_i \) and \( H_{ij} \) are the perturbations. We decompose \( H_{ij} \) in the form

\[ H_{ij} = -2\psi \gamma_{ij} + 2\nabla_i \nabla_j E + 2\nabla^2 (\nu^i E^V_j) + 2E^T_{ij}, \quad (79) \]

with \( E^T_{ij} \) transverse and traceless (TT), i.e. \( \nabla_i E^T_{ij} = 0 \) and \( \gamma^{ij} E^T_{ij} = 0 \), and \( E^V_i \) is transverse, i.e. \( \nabla_i E^V_i = 0 \). The components of the inverse metric \( g^{ab} \) are given, up to second order, by the following expressions:

\[ g^{00} = \frac{1}{a^2} \left( -1 + 2A - 4A^2 + B_i B^i \right), \quad g^{0i} = \frac{1}{a^2} \left( B^i - 2AB^i - H^{ik} B_k \right), \]
\[ g^{ij} = \frac{1}{a^2} \left( \gamma^{ij} - H^{ij} - B^i B^j + H^{ik} H^{kj} \right), \quad (80) \]

where the spatial indices are raised or lowered via the background comoving spatial metric \( \gamma_{ij} \). One has to imagine that each perturbation quantity can be expanded up to second order, e.g., \( A = A^{(1)} + \frac{1}{2} A^{(2)} \).

The time component \( u^0 \) of the four-velocity can be derived from the normalization condition \( g_{ab} u^a u^b = -1 \), and one finds at second order

\[ u^0 = \frac{1}{a} \left( 1 - A + \frac{3}{2} A^2 + \frac{1}{2} v_k v^k + B_k v^k \right), \quad (81) \]

where we have introduced \( v^k \equiv a u^k \).

We now have all the ingredients to compute the expansion \( \Theta = \nabla_a u^a \) up to second order. One finds

\[ \Theta = \frac{1}{a} \left\{ (1 - A) \left( 3H + \frac{1}{2} H^i i^i \right) - \frac{1}{2} H_{ij} H^{ij} + \frac{1}{2} \left[ (v^k + B^k) (v_k + B_k) \right]' \right. \]
\[ + 3H \left( \frac{3}{2} A^2 + \frac{1}{2} v_k v^k + B_k v^k \right) + \nabla_k v^k + v^k \nabla_k \left( A + \frac{1}{2} H^i i \right) \}. \quad (82) \]
where \( H^i_i \equiv \gamma^{ij} H_{ij} = -6\psi + 2\nabla^2 E \). This formula must be compared to the expression of \( \Theta \) as a function of the integrated expansion \( \alpha \), \( \Theta = 3\hat{\alpha} = u^a \nabla_a \alpha \), which reads up to second order:

\[
\Theta = \frac{3}{a} \left[ \left( 1 - A + \frac{3}{2} A^2 + \frac{1}{2} v_k v^k + B_k v^k \right) \alpha' + v^k \nabla_k \alpha \right].
\] (83)

The order by order comparison finally yields

\[
\alpha' = H + \frac{1}{3} \left\{ \frac{1}{2} H^i_i' - \frac{1}{2} H_{ij} H^{ij'} + \frac{1}{2} \left[ \left( v^k + B^k \right) \left( v_k + B_k \right) \right]' \right\}
+ (1 + A) \nabla_k v^k + v^k \nabla_k \left( A + \frac{1}{2} H^i_i \right) \right\} - v^k \nabla_k \alpha^{(1)}.
\] (84)

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