Abstract. We examine the computational complexity of context-free languages, mainly concentrating on two well-known structural properties—immunity and pseudorandomness. An infinite language is REG-immune (resp., CFL-immune) if it contains no infinite subset that is a regular (resp., context-free) language. We prove that (i) there is a context-free REG-immune language outside REG/$n$ and (ii) there is a REG-bi-immune language that can be computed deterministically using logarithmic space. We also show that (iii) there is a CFL-simple set, where a CFL-simple language is an infinite context-free language whose complement is CFL-immune. Similar to the REG-immunity, a REG-prime-immune language has no polynomially dense subsets that are also regular. We further prove that (iv) there is a context-free language that is REG/$n$-bi-prime-immune but not even REG-immune. Concerning pseudorandomness of context-free languages, we show that (v) CFL contains REG/$n$-pseudorandom languages. Finally, we prove that (vi) against REG/$n$, there exists an almost 1-1 pseudorandom generator computable in nondeterministic pushdown automata equipped with a write-only output tape and (vii) against REG, there is no almost 1-1 weak pseudorandom generator computable deterministically in linear time by a single-tape Turing machine.

Keywords: regular language, context-free language, immune, simple, primeimmune, pseudorandom, pseudorandom generator

ACM Subject Classification: F.4.3, F.1.1, F.1.3

1 Motivations and a Quick Overview

The context-free language is one of the most fundamental concepts in formal language theory. Besides its theoretical interest, the context-freeness has drawn, since the 1960s, practical applications in key fields of computer science, including programming languages, compiler implementation, and markup languages, mainly attributed to unique traits of context-free grammars or phrase-structure grammars. Some of the traits can be highlighted by, for instance, pumping and swapping lemmas \cite{6,29}, normal form theorems \cite{9,14}, and undecidability theorems \cite{6,12}, all of which reveal certain substructures of context-free languages. The literature over half a century has successfully explored numerous fundamental properties (including operational closure, normal forms, and minimization) of the family CFL of context-free languages. The family CFL contains a number of non-regular languages, such as $L_{eq} = \{0^n1^n \mid n \geq 0\}$ and $Equal = \{w \in \{0,1\}^* \mid \#_0(w) = \#_1(w)\}$, where $\#_b(w)$ denotes the number of $b$'s in $w$. An effective use of a pumping lemma, for instance, easily separates them from the family REG of regular languages (see, e.g., \cite{18} for their proofs). Nonetheless, these two context-free languages look quite different in nature and in complexity. Is there any extremely “complex” context-free language? Since time-complexity might not bet a suitable complexity measure for context-free languages, another way to measure their complexity is to show “structural” differences among those languages.

Numerous structural properties have been proposed for polynomial-time complexity classes, including $P$ (deterministic polynomial-time class) and $NP$ (nondeterministic polynomial-time class), and have been studied to understand their characteristics. Many of those properties, which are important on their own light, have arisen naturally in a context of answering long-unsettled questions, such as the $P = \text{?NP}$ question (see, e.g., \cite{4} for these properties). To scale the complexity of context-free languages, we wish to target two well-known structural properties—immunity and pseudorandomness—-which have been studied since the 1940s in computational complexity theory and computational cryptography. These two properties are known to be closely related. In this paper, we shall spotlight them within a framework of formal language theory. Our approach may differ from standard ones in a setting of polynomial-time bounded computation.
In the first part of this paper (Sections 3–6), our special attention goes to languages that have only computationally “hard” non-trivial subsets. Those languages, known as immune languages and simple languages, naturally possess high complexity. Given a fixed family \( C \) of languages, an infinite language is called \( C \)-immune if it has no infinite subset in \( C \), and a \( C \)-simple language is an infinite language in \( C \) whose complement is \( C \)-immune. Significantly, the \( C \)-immunity satisfies a self-exclusion property: \( C \) cannot be \( C \)-immune. Notice that the notion of simplicity has played a key role in the theory of NP-completeness (see, e.g., [4]). In addition, a language is \( C \)-bi-immune if its complement and itself are both \( C \)-immune.

These notions of immunity and simplicity date back to the 1940s, in which they were first conceived by Post [24] for recursively enumerable languages. Their resource-bounded analogues were discussed later in the 1970s by Flajolet and Steyaert [11]. During the 1980s, Ko and Moore [19] intensively studied such limited immunity, whereas Homer and Maass [16] explored resource-bounded simplicity. The bi-immunity notion was introduced in mid-1980s by Balcazar and Schoning [5]. Since then, numerous variants of immunity and simplicity (for instance, strong immunity, almost immunity, balanced immunity, and hyperimmunity) have been proposed and studied extensively (see, e.g., [4, 28] for references therein).

Despite the past efforts, in a setting of polynomial-time bounded computation, the immunity notion has eluded from our full understandings; for instance, it has been open whether there exists a P-immune set in NP or even an NP-simple set since the existence of such a set immediately yields a class separation between NP and co-NP. While there is a large volume of work on the immunity of polynomial-time complexity classes, there has been little study done on the immunity of context-free languages in the past literature. An analysis of REG-immunity inside CFL could bring into new light a structural difference among various context-free languages. For instance, the aforementioned context-free language \( L_{eq} \) is REG-immune [11], whereas its accompanied language \( Equal \) is not REG-immune. The context-freeness provides tremendous advantages of proving immunity and non-immunity over polynomial-time complexity classes. Unlike NP-simplicity, we can demonstrate that CFL-simple languages actually exist.

There are, however, unsettled questions concerning the REG-immunity in CFL. One of those questions is related to REG-bi-immunity. It is unclear that REG-bi-immune languages actually exist inside CFL. At our best, we can prove that the language class \( L \) (deterministic logarithmic-space class) contains REG-bi-immune languages. Another unsolved question concerns with a density issue of immune languages. Notice that all known REG-immune languages \( L \) in CFL have exponentially small density rate \( |L \cap \Sigma^n|/|\Sigma^n| \). The REG-immune language \( L_{eq} \), for instance, has density rate \( |L_{eq} \cap \{0,1\}^n|/2^n \leq 1/2^n \) for any even length \( n \); in contrast, \( Equal \), which is not even REG-immune, has its density rate \( |Equal \cap \{0,1\}^n|/2^n \geq 1/n \) for any sufficiently large even number \( n \). Naturally, we can ask if there exists any context-free REG-immune language whose density \( |L \cap \Sigma^n| \) is lower-bounded by a “polynomial” fraction, i.e., \( 1/p(n) \) for a certain non-zero polynomial \( p \). Such a condition is referred to as polynomially dense or \( p \)-dense. In this paper, as the first step toward the open question, we shall show that there exists a \( p \)-dense REG-immune language in \( L \).

Our \( C \)-immunity requires the non-existence of an infinite subset in \( C \). Is there any language that lacks only \( p \)-dense subset in \( C \)? Such a natural question gives rise to a variant of \( C \)-immunity, referred to as \( C \)-primeimmunity. We turn our attention to this natural notion inside CFL. As an example, we shall prove that an “extended” language of \( Equal \), \( Equal_a = \{qw \mid a \in \{\lambda,0,1\}, w \in Equal\} \), is \( REG/n \)-primeimmune, where \( REG/n \) is obtained from \( REG \) by supplementing appropriate “advice” of size \( n \) [20]. In stark contrast with the REG-bi-immunity, we shall prove that REG-bi-primeimmune languages (even \( REG/n \)-bi-primeimmune languages) exist inside CFL.

The second part of this paper (Sections 7–8) is exclusively devoted to a property of computational randomness or pseudorandomness. An early computational approach to “randomness” began in the 1940s. Church’s [10] random 0-1 sequences, for instance, demand that every infinite subsequence contains asymptotically the same number of 0s and 1s. This line of study on computational randomness, also known as stochasticity, concerns with asymptotic behaviors of random sequences. It has been known a connection between stochasticity and bi-immunity.

To suit our study of context-free languages, we rather examine non-asymptotic behaviors of randomness inside languages. This paper discusses the following type of “random” languages. We say that a language \( L \) is \( C \)-pseudo-random if, for every language \( A \) in \( C \), the characteristic function \( \chi_A \) agrees with \( \chi_L \) on “nearly” 50% of strings of each length, where “nearly” means “with a negligible margin of error.” Our notion can be seen as a variant of Wilber’s [27] randomness, which dictates an asymptotic behavior of \( \chi_L \) and \( \chi_A \).

Similar in the case of primeimmunity, \( p \)-denseness requires our special attention. Targeting \( p \)-dense languages, we introduce another “randomness” notion, called weak \( C \)-pseudorandomness, as a non-asymptotic variant of Müller’s [23] balanced immunity, Loveland’s unbiasedness, or weak-stochasticity of Ambos-
Spies et al. [2]. Loosely speaking, a language $L$ is weak $C$-pseudorandom if the density rate $|L \cap A \cap \Sigma^n|/|A \cap \Sigma^n|$ is “nearly” a half for every $p$-dense language $A$ in $C$.

A typical example of $\text{REG}/n$-pseudorandom language is $\text{IP}_n$, whose strings are of the form $auv$ with $a \in \{0, 1\}$ and $|u| = |v|$ such that the binary inner product between $u^R$ and $v$ is odd. We show a close connection between pseudorandomness and primeimmunity. From this connection, we can conclude that $\text{IP}_n$ is also $\text{REG}/n$-bi-primeimmune. The aforementioned language $\text{Equal}_n$ can separate the notion of weak $\text{REG}$-pseudorandomness from the notion of $\text{REG}$-primeimmunity.

In early 1980s, Blum and Micali [7] studied pseudorandom generators, which produce unpredictable sequences. Our formulation of pseudorandom generators, attributed to Yao [31], use indistinguishability from uniform sequences. Loosely speaking, a pseudorandom generator is a function producing a string that looks random for any target adversary (in this case, we say that the generator fools it). In our language setting, we call a function from $\Sigma^*$ to $\Sigma^*$ with stretch factor $s(n)$ (that is, $|f(x)| = s(|x|)$) a pseudorandom generator against a language family $C$ if, for every language $A$ in $C$, $G$ fools every language in $C$. Our pseudorandom generator tries to fool languages in a sense that, over strings inputs of each length $n$, the outcome distribution of the generator is indistinguishable against the strings of length $s(n)$; that is, the function $\ell(n) = |\text{Prob}_x[\chi_A(x) = 1] - \text{Prob}_y[\chi_A(y) = 1]|$ is negligible, where $x$ and $y$ are chosen uniformly at random from $\Sigma^n$ and $\Sigma^{s(n)}$, respectively. We shall prove that, against $\text{REG}/n$, there exists an almost 1-1 pseudorandom generator computable by a nondeterministic pushdown automaton equipped with an output tape. As a limitation of generators, we can show that, even against $\text{REG}$, there is no almost 1-1 pseudorandom generator computable by a one-tape one-head linear-time deterministic Turing machine.

2 Foundations

The natural numbers are nonnegative integers and we write $\mathbb{N}$ to denote the set of all natural numbers. We set $\mathbb{N}^+ = \mathbb{N} - \{0\}$ for convenience. For any two integers $m, n$ with $m \leq n$, the notation $[m, n]_\mathbb{Z}$ stands for the integer interval $\{m, m + 1, m + 2, \ldots, n\}$. The symmetric difference between two sets $A$ and $B$, denoted $A \triangle B$, is the set $(A - B) \cup (B - A)$. In this paper, all logarithms are assumed to have base two unless otherwise stated. Let $\log^{(1)} n = \log n$ and $\log^{(i+1)} n = \log(\log^{(i)} n)$ for each number $i \in \mathbb{N}^+$. A function $\mu$ from $\mathbb{N}$ to $\mathbb{R}^{\geq 0}$ (all nonnegative reals) is called noticeable if there exists a non-zero polynomial $p$ such that $\mu(n) \geq 1/p(n)$ for all but finitely-many numbers $n \in \mathbb{N}$. By contrast, $\mu$ is called negligible if we have $\mu(n) \leq 1/p(n)$ for any non-zero polynomial $p$ and for all sufficiently large numbers $n \in \mathbb{N}$.

Our alphabet, often denoted $\Sigma$, is always a nonempty finite set. A string is a series of symbols taken from $\Sigma$, and the length of a string $x$ is the number of symbols in $x$ and is denoted $|x|$. For simplicity, the empty string is always denoted $\lambda$. For two strings $x$ and $y$, $xy$ denotes the concatenation of $x$ and $y$. In particular, $\lambda x$ coincides with $x$. The notation $\Sigma^n$ denotes the set of all strings of length $n$. For any string $x$ of length $n$ and for any index $i \in [0, n]_\mathbb{Z}$, $\text{pref}_i(x)$ is the substring of $x$, made up with the first $i$ symbols of $x$. For each string $w \in \Sigma^*$ and any symbol $a \in \Sigma$, the number of $a$’s appearing in $w$ is represented by $\#_a(w)$. A language over an alphabet $\Sigma$ is a subset of $\Sigma^*$, and the characteristic function $\chi_A$ of $A$ is defined as $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise for every string $x \in \Sigma^*$.

For any language $L$ over $\Sigma$, the complement $\Sigma^* - L$ of $L$ is often denoted $\overline{L}$ whenever $\Sigma$ is clear from the context. Furthermore, the complement of a family $\mathcal{C}$ of languages is the collection of all languages whose complements are in $\mathcal{C}$. We use the conventional notation $\overline{\mathcal{C}}$ to denote the complement of $\mathcal{C}$. The notation $\text{dense}(L)(n)$ expresses the cardinality of the set $L \cap \Sigma^n$; that is, $\text{dense}(L)(n) = |L \cap \Sigma^n|$. A language $L$ over $\Sigma$ is called tally if $L \subseteq \{a\}^*$ for a certain fixed symbol $a \in \Sigma$.

This paper mainly discusses regular languages and context-free languages. We assume the reader’s basic knowledge on fundamental mechanisms of one-tape one-head one-way finite automata, possibly equipped with pushdown (or first-in last-out) stacks. See, e.g., [7] [18] for the formal definitions of the aforementioned finite automata. Generally speaking, for a finite automaton $M$, the notation $L(M)$ represents the set of all strings “accepted” by $M$ under appropriate accepting criteria. Such criteria may significantly differ if we choose different machine types. Conventionally, we say that $M$ recognizes a language $L$ if $L = L(M)$. Languages recognized by deterministic finite automata (or dfa’s) and nondeterministic pushdown automata (or npda’s) are respectively called regular languages and context-free languages. In addition, deterministic pushdown automata (or dpda’s) recognize only deterministic context-free languages. For ease of notation, we denote by REG the family of regular languages and by CFL the family of context-free languages. As a proper subclass of CFL, DCFL denotes the family of all deterministic context-free languages.

It is known that the language family CFL is not closed under conjunction (see, e.g., [18]). This fact
Lemma 2.1

1. [pumping lemma for regular languages] Let $L$ be any infinite regular language. There exists a number $m > 0$ (referred to as a pumping-lemma constant) such that, for any string $w$ of length $\geq m$ in $L$, there is a decomposition $w = xyz$ for which (i) $|xy| \leq m$, (ii) $|y| \geq 1$, and (iii) $xy^iz \in L$ for any $i \in \mathbb{N}$.

2. [pumping lemma for context-free languages] Let $L$ be any infinite context-free language. There exists a positive number $m$ such that, for any $w \in L$ with $|w| \geq m$, $w$ can be decomposed as $w = uvxyz$ with the following three conditions: (i) $|uvx| \leq m$, (ii) $|vy| \geq 1$, and (iii) $uv^ixy^iz \in L$ for any $i \in \mathbb{N}$.

3. [swapping lemma for regular languages] Let $L$ be any infinite regular language on alphabet $\Sigma$ with $|\Sigma| \geq 2$. There exists a positive integer $m$ (called a swapping-lemma constant) such that, for any integer $n \geq 1$ and any subset $S$ of $L \cap \Sigma^n$ of cardinality at least $m$, the following condition holds: for any integer $i \in [0,n]$, there exist two strings $x = x_1x_2$ and $y = y_1y_2$ in $S$ with $|x_1| = |y_1| = i$ and $|x_2| = |y_2|$ satisfying that (i) $x \neq y$, (ii) $y_1x_2 \in L$, and (iii) $x_1y_2 \in L$.

To explain the notion of advice, we first adapt a “track” notation $[\sigma \tau]$ from [26]. For any pair of symbols $\sigma \in \Sigma_1$ and $\tau \in \Sigma_2$, the notation $[\sigma \tau]$ denotes a new symbol made from $\sigma$ and $\tau$. For two strings $x = x_1x_2 \cdots x_n$ and $y = y_1y_2 \cdots y_n$ of the same length $n$, the notation $[\sigma \tau \nu \gamma]$ is shorthand for the string $[\sigma_1 \tau_1 \nu_1 \gamma_1] [\sigma_2 \tau_2 \nu_2 \gamma_2] \cdots [\sigma_n \tau_n \nu_n \gamma_n]$. An advice function is a map $h$ from $\mathbb{N}$ to $\Gamma^*$, where $\Gamma$ is an appropriate alphabet. For any family $C$ of languages, the advised class $C/n$ denotes the collection of languages $L$ over an alphabet $\Sigma$ for which there exist another alphabet $\Gamma$, an advice function $h : \mathbb{N} \rightarrow \Gamma^*$, and a language $A \subseteq C$ such that, for every string $x \in \Sigma^*$, (i) $|h(|x|)| = |x|$ (i.e., length preserving) and (ii) $x \in L$ iff $[h(|x|)] \in A$ [26, 29].

As an additional computation model, we introduce the notion of one-tape one-head off-line Turing machines whose tape heads move in all directions. All tape cells of an infinite input/work tape are indexed with integers and an input string of length $n$ is given in the cells indexed between 1 and $n$ surrounded by two designated endmarkers. We take a notation 1-DTIME($t(n)$) from [26] to denote the collection of all languages that are recognized within time $t(n)$ by those machines. As a special case, write 1-DLIN for 1-DTIME($O(n)$). It is well-known that REG = 1-DLIN = 1-DTIME($o(n \log n)$) [15, 20].

To handle (multi-valued partial) functions, we further consider Turing machines that produce output strings. Conventionally, whenever a single-tape machine halts along the tape that contains only a block of non-blank symbols beginning at the left endmarker and surrounded only by blanks, we treat the string given in this block as an outcome of the machine. A (partial) function $f$ from $\Sigma^*$ to $\Gamma^*$, where $\Sigma$ and $\Gamma$ are alphabets, is called length preserving if $|f(x)| = |x|$ for any string $x$ in the domain of $f$.

Let us introduce several function classes, which are natural extensions of our language families REG and CFL. The function class 1-FLIN is the set of all single-valued total functions computable in time $O(n)$ by one-tape one-head off-line deterministic Turing machines whose tape heads move in all directions. Similarly, the notation 1-FLIN(partial) expresses the set of all single-valued partial functions $f$ such that there exists a one-tape one-head off-line deterministic Turing machine $M$ that starts with input $x$ and halts with output $f(x)$ by entering an accepting state whenever $f(x)$ is defined and $M$ enters a rejecting state when $f(x)$ is not defined.

In a similar fashion, we define 1-NLINMV as the class of all multi-valued partial functions $f$ for which there exists a one-tape one-head off-line nondeterministic Turing machine $M$ provided that all computation (both accepting and rejecting) paths terminates with output values in time $O(n)$, with the condition that $f(x)$ consists of all output values produced along accepting paths. Notice that, when $f(x) = \emptyset$, there should be no accepting path. See [26] for their fundamental properties.

The original npda model was introduced to recognize languages. Let us expand this model to compute (partial) functions. For this purpose, we equip an npda with an additional output tape and its associated tape head. Now, our npda has two tapes: a read-only input tape and a write-only output tape. This new npda acts as a standard npda with a single stack except for moves of an output-tape head. In the write-only output tape, its tape head always moves to the right whenever it writes a symbol in its tape cell. We also allow each tape head to stay still while it scans a blank symbol but does not write any non-blank symbol. Since the head moves only to a new blank cell, it cannot read any symbol that have already written in the output tape. Along each computation path, we define an output as follows. When the npda enters
3 Resource-Bounded Immunity and Simplicity

Intuitively, an immune language contains only finite subsets and infinite subsets that are “hard” to compute; in other words, it lacks any non-trivial “easy” subset. In contrast, a simple language inherits the immunity only for its complement. Such languages turn out to possess quite high complexity. The original notions of immunity and simplicity are rooted in the 1940s and later adapted to computational complexity theory in the 1970s with various restrictions on their computational resources.

The notion of resource-bounded immunity for an arbitrary family $C$ of languages can be introduced in the following abstract way. A language $L$ is said to be $C$-immune if (i) $L$ is infinite and (ii) no infinite subset of $L$ exists in $C$. When a language family $D$ contains a $C$-immune language, we briefly say that $D$ is $C$-immune. Since $C$ cannot be $C$-immune, if $D$ is $C$-immune then it immediately follows that $D \not\subseteq C$. On the contrary, the separation $D \not\subseteq C$ cannot, in general, guarantee the existence of $C$-immune languages inside $D$. By this reason, a separation between two language families by immune languages is sometimes referred to as a strong separation. In a polynomial-time setting, for instance, even if assuming that $P \neq \text{NP}$, it is not known whether there is a $P$-immune language in NP or equivalently NP is $P$-immune.

Within a framework of formal language theory, we extensively discuss the immunity of two well-known families of languages: REG and CFL. Earlier, Flajolet and Steyaert [11] presented two apparent examples: a REG-immune language $L_{eq} = \{0^n1^n \mid n \in \mathbb{N}\}$ and a CFL-immune language $L_{3eq} = \{a^n b^p c^n \mid n \in \mathbb{N}\}$. Notice that, in contrast, similar non-regular languages $\text{Equal} = \{x \in \{0,1\}^* \mid \#_0(x) = \#_1(x)\}$ and $3\text{Equal} = \{x \in \{0,1,2\}^* \mid \#_0(x) = \#_1(x) = \#_2(x)\}$ are not REG-immune, because two regular languages $\{(01)^n \mid n \in \mathbb{N}\}$ and $\{(012)^n \mid n \in \mathbb{N}\}$ are respectively infinite subsets of $\text{Equal}$ and of $3\text{Equal}$. This clear contrast signifies a structural difference among those languages.

Unlike the well-studied $P$-immunity, we can take quite different approaches toward REG-immune and CFL-immune languages because these immunity notions embody unique characteristics. For example, in many cases, as we shall see later, a diagonalization technique—a standard technique of constructing immune languages in a polynomial setting—is no longer necessary.

Since $\text{REG} \subseteq \text{CFL}$, the CFL-immunity clearly implies the REG-immunity but the converse does not hold because, for instance, $L_{3eq}$ is REG-immune and also belongs to CFL. Since $L_{eq}$ and $L_{3eq}$ are tally languages (because, e.g., dense$(L_{eq})(n) = 1$ for all even lengths $n \in \mathbb{N}$), they belong to the advised class $\text{REG}/n$. In particular, since $L_{eq}$ is in DCFL, we can conclude that $\text{DCFL} \cap \text{REG}/n$ is REG-immune. Similarly, since $L_{3eq} \in \text{CFL}(2)$, the language family $\text{CFL}(2) \cap \text{REG}/n$ is CFL-immune.

As the first nontrivial example of REG-immunity, we want to show that the language family $\text{DCFL} - \text{REG}/n$ is REG-immune, complementing the aforementioned REG-immunity of $\text{DCFL} \cap \text{REG}/n$.

Proposition 3.1 The language family $\text{DCFL} - \text{REG}/n$ is REG-immune.

Our REG-immune language is a “marked” language $\text{Pal}_\# = \{w\#w^R \mid w \in \{0,1\}^*\}$ over the ternary alphabet $\{0,1,\#\}$, where $\#$ is used as a separator. Notice that a use of this separator is crucial because a corresponding unmarked version $\text{Pal} = \{ww^R \mid w \in \{0,1\}^*\}$ (even-length palindromes) is no longer REG-immune. Although the proof of Proposition 3.1 is relatively easy, we include it for completeness.

Proof of Proposition 3.1 We shall show that $\text{Pal}_\#$ is REG-immune and is also located outside of $\text{REG}/n$. First, we shall prove the REG-immunity of $\text{Pal}_\#$. Assume on the contrary that $\text{Pal}_\#$ has an infinite regular subset $A$. Let $m$ be a pumping-lemma constant (in Lemma 2.1(1)) and, since $A$ is infinite, choose a string $w = w\#w^R$ in $A$ with $w \in \{0,1\}^*$ and $|w| \geq m$. Let us consider any decomposition of the form $w = xyz$ with $|xy| \leq m$ and $|y| \geq 1$. Because $xy$ is a substring of $w$, the string $xy^2z$ cannot be of the form $v\#v^R$ for any string $v \in \{0,1\}^*$. This contradicts the conclusion of the pumping lemma, and therefore $A$ does not exist. As a result, $\text{Pal}_\#$ is indeed REG-immune.
Second, we shall prove that $\text{Pal}_{\#}$ is not in $\text{REG}/n$. Instead of applying the swapping lemma (i.e., Lemma 23.1(3)) directly, we want to prove our claim by applying a known result of $\text{Pal} \notin \text{REG}/n$ 29. In what follows, we want to show that if $\text{Pal}_{\#} \in \text{REG}/n$ then $\text{Pal} \in \text{REG}/n$, which contradicts the fact that $\text{Pal} \notin \text{CFL}/n$. Therefore, it immediately follows that $\text{Pal}_{\#}$ does not belong to $\text{REG}/n$. Write $\Sigma$ for the ternary alphabet $\{0, 1, \#\}$. Now, let us assume that $\text{Pal}_{\#} \in \text{REG}/n$. There are an alphabet $\Gamma$, an advice function $h$ from $\mathbb{N}$ to $\Gamma^*$, and a language $L \in \text{REG}$ such that, for every string $x \in \Sigma^*$, (i) $|h(|x|)| = |x|$ and (ii) $x \in \text{Pal}_{\#}$ iff $[h(|x|)] \in L$. For later convenience, assuming that $\{0, 1\} \subseteq \Gamma$, we define $h'(n) = \left[\begin{array}{c} \text{odd}\ h(n) \\
 \end{array}\right]$ if $n = 2k + 1$, and $h'(n) = \left[\begin{array}{c} \text{odd}\ n \\
 \end{array}\right]$ otherwise, and moreover, we define $L' = \{[\#] \mid \exists u, v \in \Gamma^{|w|} \left[ w = [\#] \text{ and } [\#] \in L \text{ and } \exists m, m'[v = 0^m10^m]\right] \}$. Clearly, $L'$ is regular and, for every string $x, x \in \text{Pal}_{\#}$ iff $[g(|x|)] \in L'$. Our plan is to remove the separator $\#$ out of $\text{Pal}_{\#}$, resulting in $\text{Pal}$. To carry out this plan, we introduce a new Turing machine $M$ and a new advice function $g$ for $\text{Pal}$. If $h'(n+1)$ is of the form $[\sigma \#] [\gamma] [\alpha]$, then we define $g(n) = \left[\begin{array}{c} \sigma \# \\\\\\\\\\\\gamma \\\\\\\\\\\\alpha \end{array}\right]$. On the contrary, if $h'(n+1)$ is of the form $[\sigma \#]$, then let $g(n) = [\sigma \#]$. Our two-way off-line Turing machine $M$, which is equipped with a single input/work tape, behaves as follows.

On input $[\#]$, $M$ checks if $|x|$ is even, $w$ is of the form $[\#]$ with $|u| = |v|$, and $v$ is of the form $[\sigma u] [\gamma] [\alpha]$ for certain elements $u, w \in \Sigma^*$ and $\sigma, \gamma, \alpha \in \Sigma$. If not, $M$ rejects the input. Otherwise, from the input string $[\#]$, $M$ generates $[\# w \#]$, where $w' = \left[\begin{array}{c} \sigma \# \\\\\\\\\\\\gamma \\\\\\\\\\\\alpha \end{array}\right]$, on the single tape, and $M$ then checks if $[\# w \#] \in L$.

It is not difficult to check that $M$ halts in time $O(n)$. Since $1\text{-DLIN} = \text{REG}$, the set of all strings accepted by $M$ belongs to $\text{REG}$. Moreover, we can show that, for every input $x \in \{0, 1\}^*$, $x \in \text{Pal}$ iff $M$ accepts $[g(|x|)]$. From this equivalence, we conclude that $\text{Pal}$ is in $\text{REG}/n$, as we have planned.

Away from the $\text{REG}$-immunity, we next discuss $\text{CFL}$-immune languages. As noted before, the language family $\text{CFL}(2) \cap \text{REG}/n$ (and thus $\text{CFL}(2) \cap \text{CFL}/n$) is $\text{CFL}$-immune; however, it is not known that $\text{CFL}(2) - \text{CFL}/n$ is also $\text{CFL}$-immune. Instead, we demonstrate in the following proposition that $L - \text{CFL}/n$ is $\text{CFL}$-immune, where $L$ consists of all languages recognized by deterministic Turing machines with a single read-only input tape and a logarithmic-space bounded work tape.

**Proposition 3.2** The language family $L - \text{CFL}/n$ is $\text{CFL}$-immune.

As a $\text{CFL}$-immune language outside of $\text{CFL}/n$, we plan to consider a marked version of the language $\text{Dup} = \{ww \mid w \in \{0, 1\}^*\}$ (duplicating strings), which is denoted $\text{Dup}_{\#}$; namely, $\text{Dup}_{\#} = \{w\#w \mid w \in \{0, 1\}^*\}$. The major reason for using this marked language is that, similar to the case of $\text{Pal}$, $\text{Dup}$ is not even $\text{REG}$-immune.

**Proof of Proposition 3.2** Letting $\Sigma = \{0, 1, \#\}$, we shall show that $\text{Dup}_{\#}$ is $\text{CFL}$-immune but not in $\text{CFL}/n$. Our first claim is the $\text{CFL}$-immunity of $\text{Dup}_{\#}$. Toward a contradiction, we assume that $\text{Dup}_{\#}$ has an infinite context-free subset $A$. Let $m$ be a pumping-lemma constant (in Lemma 23.1(2)), take any string $v = w\#w$ with $|w| \geq m$, and let $v = xyz$ be any decomposition satisfying that $|xy| \leq m$ and $|y| \geq 1$. Since $|xy| \leq m$, the string $xy^2z$ should be of the form $w'\#w$ ($w' \neq w$) and thus it cannot belong to $\text{Dup}_{\#}$, a contradiction against the pumping lemma. Therefore, the conclusion that $\text{Pal}_{\#}$ is $\text{CFL}$-immune follows immediately.

Our next claim is that $\text{Dup}_{\#} \notin \text{CFL}/n$. Now, assuming otherwise that $\text{Dup}_{\#} \in \text{CFL}/n$, we take an advice function $h$ and a context-free language $L$ such that, for every string $x \in \Sigma^*$, $x \in \text{Dup}_{\#}$ iff $[h(|x|)] \in L$. Similar to the proof of Proposition 3.1, we can show the existence of another language $L' \in \text{CFL}$ and another advice function $g$ such that, for any string $x \in \{0, 1\}^*$, $x \in \text{Dup}$ iff $[g(|x|)] \in L'$. This equivalence concludes that $\text{Dup}$ belongs to $\text{CFL}/n$, contradicting the fact that $\text{Dup} \notin \text{CFL}/n$. As an immediate consequence, we obtain the desired result that $\text{Dup}_{\#} \notin \text{CFL}/n$.}

The immunity notion has given rise to the notion of *simplicity*. In general, a language $L$ is called $C$-simple if (i) $L$ is infinite, (ii) $L$ is in $C$, and (iii) $\overline{T}$ is $C$-immune. The existence of such a $C$-simple language clearly leads to a class separation $C \neq \text{co-}C$. Because of this implication, we do not know whether $\text{NP}$-simple languages exist (since, otherwise, $\text{NP} \neq \text{co-NP}$ follows). It is therefore natural to ask if $\text{CFL}$-simple languages actually exist. In what follows, we shall prove the existence of such $\text{CFL}$-simple languages.
Proposition 3.3 There exist CFL-simple languages. Moreover, the complements of some of those languages belong to CFL(2) \(\cap\) REG/\(n\).

Our example of CFL-simplicity is the complement of a language \(L_{keq}\) \((k \geq 3)\), which is a natural generalization of \(L_{3eq}\). Let \(k \geq 3\) be fixed. We define \(L_{keq} = \{a_1^m a_2^n \cdots a_k^n \mid n \in \mathbb{N}\}\) over the \(k\)-letter alphabet \(\Sigma_k = \{a_1, a_2, \ldots, a_k\}\). We shall show that the complement of \(L_{keq}\), where \(k \geq 3\), is indeed CFL-simple. This gives a clear contrast with the fact that an associated language \(3\text{Equal} = \{w \in \{a, b, c\}^* \mid \#_a(w) = \#_b(w) = \#_c(w)\}\) is not even REG-immune.

Proof of Proposition 3.3 We intend to show that, for each index \(k \geq 3\), (1) \(\overline{L_{keq}}\) is in CFL, (2) \(L_{keq}\) is in CFL(2) \(\cap\) REG/\(n\), and (3) \(L_{keq}\) is CFL-immune.

1. Our first claim is that \(\overline{L_{keq}}\) belongs to CFL. To simplify our proof, we shall argue only on the case \(L_{3eq}\). Let us introduce two additional languages \(L_{3eq} = \{a_k b^l c^m \mid k \neq l, l \neq m, or k \neq m\}\) and \(L_3 = \{a_k b^l c^m \mid k, l, m \in \mathbb{N}\}\). Note that \(L_{3eq}\) equals the union of the following three sets: \(\{a_k b^l c^m \mid k \neq l, m \geq 0\}\), \(\{a_k b^l c^m \mid l \neq m, k \geq 0\}\), and \(\{a_k b^l c^m \mid m \neq k, l \geq 0\}\), all of which are apparently context-free. Since CFL is closed under union, \(L_{3eq}\) belongs to CFL. Moreover, since \(\overline{L_{3eq}} = L_{3eq} \cup \overline{L_3}\) and \(\overline{L_3} \in \text{REG} \subseteq \text{CFL}\), the language \(\overline{L_{3eq}}\) is also in CFL.

2. To show that \(L_{keq} \in \text{REG}/\(n\)\), choose an advice function \(h\) defined as \(h(n) = a_1^{n/k} a_2^{n/k} \cdots a_k^{n/k}\) for all numbers \(n \equiv 0\pmod{k}\) and \(h(n) = 0^n\) for all the other \(n\)'s. If we define \(S = \{\[w^{m}\] \mid w \in \Sigma_k\}\), then \([h(n)] \in S\) holds exactly when \(w = h([w])\), which means \(w \in L_{keq}\). Thus, it follows that \(L_{keq} \in \text{REG}/\(n\)\), as requested. To show that \(L_{keq} \in \text{CFL}(2)\), let us deal only with the case where \(k = 2m\) and \(m = 2j + 1\) for a certain number \(j \in \mathbb{N}_+\), since the other cases are similar. We introduce two useful languages \(L_1\) and \(L_2\) defined as follows.

- \(L_1\) consists of all strings of the form \(a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}\) such that \(n_i = n_{i+1} - 1\), for all indices \(i \in [1, m]\).
- \(L_2\) consists of all strings of the form \(a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}\) such that \(n_{2i+1} = n_{2i+2}\) and \(n_{2i+2m+1} = n_{2i+1} + 3\) for all \(i \in [0, j]\).

Clearly, \(L_1\) and \(L_2\) are both context-free. Since the target language \(L_{keq}\) can be expressed as \(L_1 \cap L_2\), \(L_{keq}\) belongs to CFL(2).

3. Finally, we shall check the CFL-immunity of \(L_{keq}\). Assume that there exists an infinite subset \(A \in \text{CFL of} \ L_{keq}\). Let \(m\) be a pumping-lemma constant (in Lemma 2.1(2)). Choose \(w = a_1^{a_2^n} \cdots a_k^n\) in \(A\) with \(n \geq m\). Take a decomposition \(w = uvxyz\) with \(|vxy| \leq m\) and \(|vy| \geq 1\) such that \(w_j = uv^jxy^jz\) is in \(A\) for every index \(i\). Since \(|vxy| \leq m \leq n\), there is an index \(i\) such that \(vyx\) is a substring of either \(a_i^n\) or \(a_i^n a_{i+1}^n\). Thus, there are only two cases: (i) \(v\) and \(y\) are both substrings of \(a_i^n\) or (ii) \(v\) is a substring of \(a_i^n\) and \(y\) is a substring of \(a_{i+1}^n\). In either case, the string \(w_3 = uv^2xy^2z\) cannot belong to \(A\). This is absurd and therefore \(A\) does not exist. We thus reach a conclusion of the CFL-immunity of \(L_{keq}\). \(\square\)

Notice that our CFL-simple languages \(\overline{L_{keq}}\) is not even REG-immune because, for instance, \(L_3\) is an infinite regular subset of \(\overline{L_{3eq}}\). It is still open whether there is a REG-immune CFL-simple language.

In the remaining portion of this section, we briefly discuss a density issue of REG-immune languages. Note that all context-free REG-immune languages \(L\) shown in this section satisfy the following density property: its density rate \(\text{density}(\text{L})(n)/|\Sigma^n|\) is “exponentially small” in terms of a length parameter \(n\). The language \(\text{Pal}_\#\), for example, satisfies, for densities \(\text{density}(\text{Pal}_\#)(n)/|\Sigma^n| \leq 2^{\lfloor n/2 \rfloor}/3^n\) (thus \(\text{density}(\text{Pal}_\#)(n) \leq |\Sigma^n|/(2^n)\)) for every odd length \(n \geq 1\). Naturally, we can question if there exists a context-free REG-immune language whose density rate is “polynomially large.” To be more precise, we call a language \(L\) over an alphabet \(\Sigma\) **polynomially dense** (or \(p\)-dense, in short) exactly when there exist a number \(n_0 \in \mathbb{N}\) and a non-zero polynomial \(p\) such that \(\text{density}(A)(n) \geq |\Sigma^n|/p(n)\) for all numbers \(n \geq n_0\). Our previous question is now rephrased as: is there any \(p\)-dense REG-immune language in CFL, or is CFL \(p\)-dense REG-immune? Unfortunately, we are unable to settle this question at present; however, we can show that \(L \cap \text{CFL}/n\) is \(p\)-dense REG-immune.

Proposition 3.4 The language family \(L \cap \text{CFL}/n\) is \(p\)-dense REG-immune.

Let us consider the language \(\text{LCenter} = \{au0^m10^mv \mid a \in \{\lambda, 0, 1\}, 2^m \leq |u| = |v| < 2^{m+1}\}\) over the alphabet \(\{0, 1\}\). We claim that this language is REG-immune and also \(p\)-dense. Notice that \(\text{LCenter}\) is in CFL/\(n\).

Proof of Proposition 3.4 We want to show that \(\text{LCenter}\) is \(p\)-dense REG-immune. We first show that \(\text{LCenter}\) is \(p\)-dense. Let \(w = au0^m10^mv\) in \(\text{LCenter}\) with \(2^m \leq |u| = |v| < 2^{m+1}\). Let \(n = |w|\). Consider the
case where \( a = \lambda \). In this case, we have \( 2^m \leq |u| = (n - 2m - 1)/2 < 2^{m+1} \), which implies \( 2^{m+1} + 2m + 1 \leq n \). Thus, \( n^2 \geq 2^{2m+1} \). Since \( \text{dense}(\text{LCenter})(n) = 2^{n-2m-1} \), we obtain
\[
\frac{\text{dense}(\text{LCenter})(n)}{|\Sigma^n|} = \frac{2^{n-2m-1}}{2^n} = \frac{1}{2^{2m+1}} \geq \frac{1}{n^2}.
\]
The other cases where \( a \in \{0, 1\} \) are similar. Therefore, \( \text{LCenter} \) is \( p \)-dense.

Next, we show that \( \text{LCenter} \) is REG-immune. Assuming otherwise, we choose an infinite subset \( A \) of \( \text{LCenter} \) in REG. We use Lemma 2.1. Take a pumping-lemma constant \( m > 0 \). Let \( w = au0^k1^kv \) be any string in \( A \) with \( k > m \) and \( 2^k \leq |u| = |v| < 2^{k+1} \). Consider the case where \( a = \lambda \). The other cases are similar.

Consider any decomposition \( w = xyz \) with \( |xy| \leq m \) and \( |y| \geq 1 \). Consider \( w_0 = xz \). Since \( |y| \leq m \), \( y \) is a substring of \( u \). Since \( k > m \), the center symbol of \( w_0 \) should be \( 0 \). Thus, \( w_0 \) cannot belong to \( \text{LCenter} \). This is a contradiction against the conclusion of the pumping lemma. Therefore, \( \text{LCenter} \) must be REG-immune.

\( \Box \)

4 Properties of Immune Languages

Immune languages lack infinite subsets of certain complexity and therefore they are of relatively high complexity. We have presented a few REG-immune languages in the previous section. For a much better understanding of REG-immunity, we intend to examine the fundamental nature of the REG-immunity by studying its relationships to other notions, such as quasireduction, hardcore, and levelability. The first example concerns with non-regularity measure, which gives another characterization of the REG-immunity. The nonregularity \( N_L(n) \) of a language \( L \) at \( n \) is the minimal number of equivalence classes \( \Sigma^n / \equiv_L \), where the relation \( \equiv_L \) is defined as: \( x \equiv_L y \) if and only if \( \forall z \in \Sigma^* \{ xz \in L \iff yz \in L \} \).

**Proposition 4.1** A language \( L \) is REG-immune iff \( L \) is infinite and, for every infinite subset \( A \) of \( L \) and for every constant \( c > 0 \), \( N_A(n) > c \) holds for an infinite number of indices \( n \in \mathbb{N} \).

This proposition is a natural extension of the so-called Myhill-Nerode Theorem [17], which bridges between the nonregularity and REG. We include its proof only for completeness.

**Proof of Proposition 4.1** (If-part) We prove a contrapositive. Assume that \( L \) has an infinite subset \( A \) in REG. Since \( A \in \text{REG} \), by the Myhill-Nerode Theorem [17], the cardinality of the set \( \Sigma^* / \equiv_A \) is constant, say, \( c \) (not depending on \( n \)). In other words, \( N_A(n) \) is upper-bounded by \( c \).

(Only If-part) Let \( \{ A_1, A_2, \ldots \} \) be a set of equivalence classes in \( \Sigma^* / \equiv_L \). Take the lexicographically minimal string, say, \( a_i \) from each set \( A_i \). Consider a dfa \( M \) with its transition function \( \delta \) defined by: \( \delta(i, \sigma) = j \) if \( a_i \sigma \equiv_L a_j \). The set of final states is \( F = \{ i \mid a_i \in L \} \). It is not difficult to check that \( M \) indeed recognizes \( A \). This implies that \( A \) is regular, a contradiction. \( \Box \)

Our notion of 1-DLIN-m-quasireduction gives another characterization of the REG-immunity. Recall from Section 2 the partial function class 1-FLIN(partial). A 1-DLIN-m-quasireduction from \( L \) to \( A \) is a single-valued partial function \( f \) that satisfies the following two conditions: for every string \( x \), (i) when \( f(x) \) is defined, \( x \in L \) iff \( f(x) \in A \) and (ii) \( f \) is in 1-FLIN(partial).

**Lemma 4.2** The language \( L \) is REG-immune iff \( L \) is infinite and for any set \( B \) and for any 1-DLIN-m-quasireduction \( f : L \rightarrow B \) and for any \( u \in B \), \( f^{-1}(u) \) is finite.

**Proof.** (\( \Rightarrow \)) Assume that \( L \) is not REG-immune. Take an infinite regular subset \( A \subseteq L \). Choose an element \( u_0 \in A \) and, for every string \( x \), define \( f(x) = u_0 \) if \( x \in A \) and undefined otherwise. Clearly, \( f \) belongs to 1-FLIN(partial).

(\( \Leftarrow \)) Assume that we have an infinite set \( L \), another set \( A \), a 1-DLIN-m-quasireduction \( f : L \rightarrow A \), and an element \( u_0 \in A \) such that \( f^{-1}(u_0) \) is infinite. Consider the set \( B = f^{-1}(u_0) \). Obviously, \( B \) is infinite. Note that \( x \in B \) iff \( M(x) \) halts in an accepting state and outputs \( u_0 \). Hence, \( B \) is in REG. Therefore, \( L \) has an infinite regular subset. \( \Box \)

Next, we give a “hardcore” characterization; however, our definition of “hardcore” is quite different from a standard definition of a (polynomial) hardcore for polynomial-time bounded computation (see, e.g., [4] for
its definition). With a use of an npda, we rather place a space restriction on the size of a stack used by an npda. More accurately, for any npda $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$, any constant $k \in \mathbb{N}$, and any input string $x \in \Sigma^*$, we introduce the notation $M(x)_k$ as follows: (1) $M(x)_k = 1$ if there is an accepting path of $M$ on the input $x$ with stack size at most $k$; (2) $M(x)_k = 0$ if all computation paths of $M$ on $x$ are rejecting paths with stack size at most $k$; and (3) $M(x)_k$ is undefined otherwise. A context-free language $A$ is called a REG-hardcore for a language $S$ if, for any constant $k \in \mathbb{N}$ and any npda $M$ recognizing $A$, there exists a finite set $B \subseteq S$ such that $M(x)_k$ is undefined for all strings $x \in S - B$.

**Proposition 4.3** The following two statements are equivalent. Let $S$ be any infinite context-free language.

1. The language $S$ is REG-immune.
2. The language $S$ is a REG-hardcore for $S$.

**Proof.** (1 implies 2) We shall prove a contrapositive. Let $S$ be any context-free language. Assuming that $S$ is not a REG-hardcore for $S$, we plan to prove that $S$ is not REG-immune. There exist a constant $k \in \mathbb{N}$ and an npda $M$ with $L(M) = S$ such that, for every finite set $B \subseteq S$, $M(x)_k$ is defined (i.e., $M(x)_k \in \{0, 1\}$) for a certain input $x \in S - B$. Now, let us introduce a new npda $N$ as follows: on input $x$, we simulate $M$ on $x$ nondeterministically and, along each computation path, whenever its stack size exceeds $k$, we immediately reject $x$. Consider the set $L(N)$ of all strings accepted by $N$. By the definition of $N$, it follows that $L(N) \subseteq S$.

First, we claim that $L(N)$ is regular. Since $k$ is a fixed constant, we can express the entire content of the stack as a certain new internal state. Tracking down this state, we can simulate $N$ using a certain nondeterministic finite automaton (or nfa). This implies that $L(N)$ is regular.

Next, we claim that $L(N)$ is infinite. Notice that $S$ is infinite. For every finite subset $B$ of $S$, there is a string $x \in S - B$ satisfying $x \in L(N)$. From this property, we can conclude that $L(N)$ is infinite. Since $L(N)$ is an infinite regular subset of $S$, $S$ is not REG-immune.

(2 implies 1) Similarly, we first assume that a context-free language $S$ is not REG-immune. This means that there exists a dfa $M$ for which $L(M) \subseteq S$ and $L(M)$ is infinite. Since $S$ is context-free, take an npda $N$ that recognizes $S$. Now, let us define a new npda $M'$ as follows: on input $x$, $M'$ splits its computation into two nondeterministic computation paths and then simulates $M$ and $N$ along these paths separately. Clearly, $L(M') = L(M) \cup L(N) = S$. Choose $k = 1$ and consider $M'(x)_k$. For every string $x \in L(M)$, $M'(x)_k = 1$ follows since $M$ is a dfa. Let $B$ be any finite subset of $S$. Because $L(M) - B$ is infinite within $S$, there exists a string $x \in S - B$ for which $M'(x)_k = 1$. This implies that $S$ cannot be a REG-hardcore for $S$. \qed

At the end of this section, we shall discuss a slightly weak immunity notion, known as almost immunity. A language $L$ is said to be almost $C$-immune if $L$ is the union of a $C$-immune set and a set in $C$. Since $C$-immune languages are almost $C$-immune, CFL naturally contains almost $C$-immune languages. Let us consider a simple example $L = \{0^n x \mid x \in \{0^n, 1^n\}, n \in \mathbb{N}\}$. This language $L$ is almost REG-immune (because $L = \{0^n \mid n \in \mathbb{N}\} \cup L_{eq}$) but obviously not REG-immune. When an infinite language $L$ is not almost $C$-immune, it is said to be $C$-levelable. Since every $C$-levelable language is not $C$-immune, the levelability of a language strengthens its non-immunity. A language family $D$ is $C$-levelable if $D$ contains a $C$-levelable language. Concerning NP-levelability, all “known” NP-complete languages are NP-levelable.

Let us demonstrate two examples of REG-levelable languages. We have already seen in Section 3 that Equal and Pal are not REG-immune. We shall strengthen this fact by showing that Equal and Pal are both REG-levelable. Note that Equal is in $\text{CFL} \cap \text{REG}/n$ and Pal is in $\text{CFL} - \text{REG}/n$ [29].

**Proposition 4.4** The languages Equal and Pal are both REG-levelable.

To show this proposition, we need a general statement on a necessary condition for a language to be REG-levelable. In our REG setting, we need to require a slightly different conditions (in comparison to a polynomial-time setting, see [28]). We say that a language $L$ is 1-DLIN-m-autoreducible if there exist a function $f$ (called an autoreduction) and a linear-time one-tape one-head Turing machine $M$ such that, for every string $x$, (1) $M$ on the input $x$ outputs $f(x)$ and (2) $x \in L$ iff $f(x) \in L$. We say that a function $f$ is length increasing if $|f(x)| > |x|$ for every string $x$. We say that a function $f$ is 1-DLIN-invertible if there exists a one-tape one-head off-line linear-time deterministic Turing machine $M$ such that $M(f(x))$ outputs $x$ for every string $x$.

The proof of the following lemma is a simple modification of [28] Lemma 5.4], which is based on an argument in [25]. We include the proof only for completeness.

**Lemma 4.5** Let $L$ be any non-regular language. If $L$ is 1-DLIN-m-autoreducible by an autoreduction $f$
that is length-increasing and 1-DLIN-invertible, then $L$ and $\overline{L}$ are both REG-levelable.

**Proof.** Assume that $L$ is almost REG-immune and 1-DLIN-autoreducible by an autoreduction $f$ that is length-increasing and 1-DLIN-invertible. Take $B \in \text{REG}$ and $C$, which is REG-immune such that $L = B \cup C$. Define $D = \{x \mid x \notin B, f(x) \in B\}$. Clearly, $D \in \text{REG}$. We want to show that $D$ is infinite, leading to a contradiction against the immunity of $C$. If $D$ is finite, then $C - (B \cup C)$ is infinite. Take $z_0 \in D$, which is the lexicographically largest element. Let $x \in C - (B \cup C)$, which is minimal such that $|x| > |z_0|$. Define $H = \{f^{(i)}(x) \mid i \in \mathbb{N}\}$, where $f^{(i)}(x)$ denotes the $i$-fold composition of $f$ on $x$ (in particular, $f^{(0)}(x) = x$). Since $f$ is 1-DLIN-invertible, $H$ is in REG. We claim that $H \cap B \neq \emptyset$, because, otherwise, $F$ is an infinite subset of $C$, a contradiction. Thus, $f^{(k)}(x) \in D$ for a certain number $k$. This implies that $|f^{(k)}(x)| \leq |z_0| < |x|$, a contradiction. 

Proposition 4.4 is now easily proven by Lemma 4.5.

**Proof of Proposition 4.4.** Following Lemma 4.5, it suffices to show that $\text{Equal}$ and $\text{Pal}$ are both 1-DLIN-m-autoreducible by certain autoreductions that are length-increasing and 1-DLIN-invertible. First, we consider the case $\text{Equal}$. Define our desired autoreduction $f$ as $f(x) = x01$. It is easy to see that $x \in \text{Equal}$ iff $f(x) \in \text{Equal}$. Moreover, $f$ is length-increasing and 1-DLIN-invertible. Next, we show that $\text{Pal}$ is length-increasing 1-DLIN-m-autoreducible. In this case, define our autoreduction $f$ as $f(x) = 0x0$. Obviously, it holds that $x \in \text{Pal}$ iff $f(x) \in \text{Pal}$. Obviously, $f$ is length-increasing and 1-DLIN-invertible.

## 5 Existence of Bi-Immune Languages

The existence of natural REG-immune languages within CFL encourages us to search for much stronger “immune” languages in CFL. One such candidate is another variant of $C$-immunity, known as $C$-bi-immunity in \[5\], where a language $L$ is $C$-bi-immune if $L$ and its complement $\overline{L}$ are both $C$-immune. For brevity, a language family $D$ is said to be $C$-bi-immune if there is a $C$-bi-immune language in $D$. Time-bounded bi-immunity has been known to be related to the notion of genericity, which corresponds to certain finite-extension diagonalization arguments (see, e.g., \[1\] [25] for its connection).

Is there any REG-bi-immune language in CFL? When we look at all the examples of context-free REG-immune languages shown in Section 3, they appear to lack the REG-bi-immunity property. Concerning the existence of REG-immune CFL-simple languages discussed in Section 3, they are both REG-immune and 1-DLIN-invertible. Although we are unable to answer the question at this point, we instead prove that $L \cap \text{REG}/n$ is REG-bi-immune.

**Proposition 5.1.** The languages family $L \cap \text{REG}/n$ is REG-bi-immune.

How can we prove this proposition? Balcázar and Schöning \[5\] employed a diagonalization technique to construct a $P$-bi-immune language inside EXP (deterministic exponential-time class). A disadvantage of such a construction is that the constructed $P$-bi-immune language depends on how to enumerate all languages in $P$. In our proof, we rather present two REG-bi-immune languages explicitly. Our desired REG-bi-immune languages are $L_{\text{even}}$ and $L_{\text{odd}}$ given as follows:

- $L_{\text{even}} = \{w \in \{0, 1\}^* \mid \exists k \in \mathbb{N} [2k < \log^2|w| \leq 2k + 1] \} \cup \{\lambda\}$, and
- $L_{\text{odd}} = \{w \in \{0, 1\}^* \mid \exists k \in \mathbb{N} [2k + 1 < \log^2|w| \leq 2k + 2] \} \cup \{0, 1\}$.

Notice that these two languages form a partition of $\{0, 1\}^*$; namely, $L_{\text{even}} \cup L_{\text{odd}} = \{0, 1\}^*$ and $L_{\text{even}} \cap L_{\text{odd}} = \emptyset$.

**Proof of Proposition 5.1.** It suffices to show that $L_{\text{even}}$ and $L_{\text{odd}}$ are both REG-immune because each of them is the complement of the other. For brevity, let $\Sigma$ represent the binary alphabet $\{0, 1\}$. We begin with proving the REG-immunity of $L_{\text{even}}$ by contradiction. Assume now that there exists an infinite regular subset $A$ of $L_{\text{even}}$. Take a pumping-lemma constant $m > 0$, given in Lemma 2.1(1), and choose a string $w$ in $A \cap \Sigma^*$ for a certain length $n$ with $n \geq m + 1$. Such $n$ satisfies that $2k < \log^2 n \leq 2k + 1$ for a certain number $k \in \mathbb{N}$. The pumping lemma (i.e., Lemma 2.1(1)) provides a decomposition $w = xyz$ with $|xy| \leq m$ and $|y| \geq 1$ for which $w_i =_{\text{def}} xy^iz$ belongs to $A$ for an arbitrary number $i \in \mathbb{N}$. Write $\ell$ for the length of $y$. Toward a contradiction, there are two cases to consider.
Case 1: Consider the case where $\log^{(2)} n = 2k + 1$. In this case, we choose $i = n + 1$. Since $1 \leq \ell \leq m$, the length $|w_i|$ is sandwiched by two terms as

$$2^{2k+1} = n < |w_i| = n + (i-1)\ell \leq n + n\ell \leq n(m+1) \leq n^2 = 2^{2k+2}.$$ 

In short, it holds that $2k + 1 < \log^{(2)} |w_i| \leq 2k + 2$, implying that $w_i$ is in $L_{odd}$. Since $A \cap L_{odd} = \emptyset$, it immediately follows that $w_i \notin A$, a contradiction.

Case 2: Consider the case where $\log^{(2)} n < 2k + 1$. This means that $2^{2k} < n \leq 2^{2k+1} - 1$. When we choose $i = \lceil n(n-1)/\ell \rceil + 1$, the length $|w_i|$ can be lower-bounded by

$$|w_i| \geq n + \frac{n(n-1)}{\ell} \cdot \ell = n + n(n-1) = n^2 > 2^{2k+1}.$$ 

In contrast, since $n \geq m/2$, we can upper-bound $|w_i|$ as

$$|w_i| < n + \left(\frac{n(n-1)}{\ell} + 1\right) \cdot \ell = n^2 + \ell \leq n^2 + m < (n+1)^2 \leq 2^{2k+2}.$$ 

The above two bounds together imply that $2k + 1 < \log^{(2)} |w_i| < 2k + 2$, concluding that $w_i \in L_{odd}$, a contradiction against the fact that $w_i \in A$.

From the above two cases, we can conclude that $A$ does not exist; in other words, $L_{even}$ is REG-immune, as requested. Similarly, we can show that $L_{odd}$ is REG-immune. Since $L_{even} = L_{odd}$, the REG-bi-immunity of $L_{even}$ and $L_{odd}$ follows immediately.

We still need to argue that $L_{even}$ and $L_{odd}$ are both in $L \cap \text{REG}/n$. Since $L \cap \text{REG}/n$ is closed under complementation, it suffices to show that $L_{even}$ belongs to $L \cap \text{REG}/n$. We prove that $L_{even} \in \text{REG}/n$. Consider the following advice function $h(n) = 10^{n-1}$ if $L_{even} \cap \Sigma^n \neq \emptyset$ and $h(n) = 0^n$ if $L_{odd} \cap \Sigma^n \neq \emptyset$. Define a set $A$ as $A = \{[x \mapsto y] \mid |x| = |y| + 1, y \in \{0,1\}^n\}$. It is obvious that, for every $x, y \in L_{even}$ iff $[h(x)] \in A$. Since $A \in \text{REG}$, $L_{even}$ belongs to $\text{REG}/n$.

To show that $L_{even} \in L$, we consider the following algorithm for $L_{even}$.

On input $x$, if $x = \lambda$ then accept it. Assume that $|x| \geq 1$. With access to $w$ on a read-only input tape, compute $\lceil \log^{(2)} |w| \rceil$ on its log-space work tape. If $\lceil \log^{(2)} |w| \rceil$ is odd, then accept the input; otherwise, reject it.

It is not difficult to show that this algorithm recognizes $L_{even}$ using only logarithmic space. This completes our proof of the proposition.

6 P-Denseness and Primeimmunity

Non-immunity of a language guarantees the existence of a certain infinite subset that is computationally “easy.” In practice, many non-REG-immune languages have infinite regular subsets of low density. In typical examples, there are infinite tally subsets $\{(01)^n \mid n \in \mathbb{N}\}$ and $\{(012)^n \mid n \in \mathbb{N}\}$ inside $\text{Equal}$ and $\text{3Equal}$, respectively. These subsets are not even close to be polynomially dense or p-dense. Moreover, as discussed in Section 5 it is unknown whether there exists a p-dense REG-immune language in CFL. This situation also signifies the importance of p-denseness.

Apart from the standard $C$-immunity, we turn our attention to p-dense languages that lack only p-dense regular subsets. Such languages are referred to as $C$-primeimmune. More generally, for a language family $C$, we say a language $L$ over $\Sigma$ is $C$-primeimmune if (1) $L$ is p-dense and (2) $L$ has no p-dense subset in $C$, and a language family $D$ is $C$-primeimmune if there exists a $C$-primeimmune language in $D$. This definition immediately yields the following self-exclusion property: $C$ cannot be $C$-primeimmune.

An obvious relationship holds between p-dense REG-immunity and REG-primeimmunity. If $L$ is p-dense but not REG-primeimmune, then $L$ contains a p-dense regular subset $A$. By the definition of p-denseness, $A$ should be infinite and thus $L$ must not be REG-immune. The next lemma therefore follows.

Lemma 6.1 Let $L$ be any language over an alphabet $\Sigma$ with $|\Sigma| \geq 2$. If $L$ is p-dense REG-immune, then $L$ is REG-primeimmune.
An apparent example of REG-primeimmunity is the language \( L_{Center} \) given in Section 5. We have already shown that \( L_{Center} \) is p-dense REG-immune; hence, Lemma 6.1 shows the REG-primeimmunity of \( L_{Center} \). Unfortunately, \( L_{Center} \) is not context-free, and therefore this example is not sufficient to conclude that CFL is REG-primeimmune. Rather, we shall take a more direct approach to the REG-primeimmunity of CFL.

Let us recall the context-free language \( Equal \) over the binary alphabet \( \{0, 1\} \). Since \( Equal \) is technically not p-dense, we need to extend it slightly and define its “extended” language \( Equal_* = \{aw \mid a \in \{\lambda, 0, 1\}, w \in Equal\} \). Despite \( Equal_* \)’s non-REG-immunity, we can prove that \( Equal_* \) is REG-primeimmune.

In the next proposition, however, we shall prove a slightly stronger statement: \( Equal_* \) is REG\(/n\)-primeimmune. Such REG\(/n\)-primeimmunity signifies a stark difference from REG\(/n\)-immunity, since there exists no REG\(/n\)-immune language (because every infinite language \( L \) over an alphabet \( \Sigma \) has an infinite subset of the form \( \{x \in L \mid h(|x|) = \sigma x\} \) in REG\(/n\), where \( \sigma = [\_\_\_] \) and \( h \) is an advice function defined as \( h(n+1) = \sigma u \) if \( \sigma u \) is the minimal string in \( L \cap \Sigma^n \) and \( h(n+1) = 0^{n+1} \) otherwise). The REG\(/n\)-primeimmunity of \( Equal_* \) also draws an obvious conclusion that \( Equal_* \notin \text{REG}/n \), because REG\(/n\) is not REG\(/n\)-primeimmune.

**Proposition 6.2** The language \( Equal_* \) is REG\(/n\)-primeimmune.  

**Proof.** We start our proof with an easy claim that \( Equal_* \) is p-dense. For any sufficiently large even number \( n \), by Stirling’s approximation formula, the density of \( Equal_* \) can be estimated as

\[
\text{dense}(Equal_*)(n) = \left(\frac{n}{n/2}\right)^2 = \frac{2^n \sqrt{\pi n}}{\sqrt{\pi n}} \left(1 + O\left(\frac{1}{n}\right)\right) \geq \frac{2^n}{n}.
\]

When \( n \) is odd, on the contrary, we obtain

\[
\text{dense}(Equal_*)(n) = 2 \cdot \text{dense}(Equal_*)(n-1) \geq \frac{2 \cdot 2^{n-1}}{n-1} > \frac{2^n}{n}.
\]

The above two lower bounds clearly yield the desired p-denseness of \( Equal_* \).

Our next target is to prove the non-existence of p-dense subset of \( Equal_* \) in REG\(/n\). Assume otherwise that there is a p-dense set \( A \subseteq Equal_* \) in REG\(/n\). Since \( A \) is p-dense, a certain constant \( d \geq 1 \) satisfies \( \text{dense}(A)(n) \geq 2^n/n^d \) for all but finitely-many numbers \( n \). Let \( m \) be a swapping-lemma constant for \( A \) (from Lemma 2.1.3) and let \( n \) be any sufficiently large number in \( \mathbb{N} \). We consider only the case that \( m \) is odd. The other case where \( m \) is even is similar and omitted. For each pair \( i, k \in \{0, n\} \), let \( A_{i,k} \) denote the set \( \{x \in A \cap \Sigma^n \mid \#_0(\text{pref}_1(x)) = k\} \) so that \( A \cap \Sigma^n \) can be expressed as \( A \cap \Sigma^n = \bigcap_{i=1}^{n} \bigcup_{k=0}^{m} A_{i,k} \). Here, we claim the following key statement of \( \{A_{i,k}\}_{i,k} \).

**Claim 1** There are an index \( i \in [1,n] \) and at least \( m \) distinct indices \( (k_1, k_2, \ldots, k_m) \) such that \( A_{i,k} \neq \emptyset \) for every index \( j \in [1,m] \).

Assuming that Claim 1 is true, let us choose \( m \) distinct indices \( (k_1, \ldots, k_m) \) and an index \( i \) that satisfies the claim. We then choose one string \( x_j \) from each set \( A_{i,k_j} \) and define \( S = \{x_1, x_2, \ldots, x_m\} \). Clearly, \( |S| \geq m \). By the swapping lemma (Lemma 2.1.3), there are two distinct strings \( x = x_1 x_2 \) and \( y = y_1 y_2 \) in \( S \) with \( |x_1| = |y_1| = i \) and \( |x_2| = |y_2| \) such that the swapped strings \( x_1 y_2 \) and \( y_1 x_2 \) belong to \( A \). This leads to a contradiction because the choice of \( S \) makes \( x_1 y_2 \) satisfy \( \#_0(x_1 y_2) \neq \#_1(x_1 y_2) \). This contradiction leads to conclude that \( A \) does not exist, and therefore we finish the proof of Proposition 6.2.

Now, our remaining task is to prove Claim 1. Assume that this claim is false. Since \( m \) is fixed, we omit “\( m \)” in the rest of the proof. We abbreviate \( |m/2| \) as \( m_0 \) for brevity.

To simplify our description, we introduce new terminology: an \( m \)-index series \( E \) is \( \{E_{m-1}, E_m, E_{m+1}, \ldots, E_n\} \) with \( E_i \subseteq [0,i] \) and \( |E_i| = m \) for every index \( i \in [m-1,n] \). For each \( m \)-index series \( E \) and for any index \( \ell \in [m-1,n] \), let \( T_{E,\ell} = \{w \in \Sigma^n \mid \forall i \in [m-1,\ell], \#_0(\text{pref}_i(w)) \in E_i\} \). Our assumption yields the existence of an appropriate \( m \)-series \( E = \{E_{m-1}, E_m, \ldots, E_n\} \) for which \( A \cap \Sigma^n = \bigcap_{i=m-1}^{n} \bigcup_{k \in E_i} A_{i,k} \).

To estimate the cardinality \( |A \cap \Sigma^n| \), we further define a special \( m \)-series \( D = \{D_{m-1}, D_m, \ldots, D_n\} \); for each index \( i \in [m-1,n] \), let \( D_i = (\lfloor i/2 \rfloor - m_0, \lfloor i/2 \rfloor + m_0] \). The corresponding value \( T_{D,\ell} \) is abbreviated as \( S_\ell \). In what follows, we claim that \( |S_n| \) upper-bounds \( |A \cap \Sigma^n| \).

**Claim 2** For any length \( n \in \mathbb{N} \), \( |A \cap \Sigma^n| \leq |S_n| \).

As an immediate consequence of Claim 2, since \( A \) is p-dense, we obtain a lower bound \( |S_n| \geq 2^n/n^d \) for all but finitely-many numbers \( n \). In contrast, the following statement gives an upper bound of \( |S_n| \).
Claim 3 There exists a constant $c$ with $1 < c < 2$ such that $|S_n| < c^n$ for all sufficiently large numbers $n$.

Together with the $p$-denseness of $A$, Claim 3 yields a relation $2^n/n^d \leq |S_n| < c^n$, from which we obtain $c > 2n^{-d/n}$. Since $\lim_{n \to \infty} n^{-d/n} = 1$, we reach a conclusion $c \geq 2$, which clearly contradicts the choice of $c$ in Claim 3. Therefore, Claim 1 holds.

To complete the proof of our proposition, we need to prove Claims 2 and 3, which are a core of our proof. We begin with the proof of Claim 3 by making a direct estimation of the target value $|S_n|$.

Proof of Claim 3 Remember that $m$ is an odd constant and, for simplicity, we assume that $m \geq 5$. Now, we introduce the notation $a^{(k)}_i$ as follows. For any two indices $k \in [1, m]_Z$ and $i \in [m-1, n]_Z$, let $a^{(k)}_i$ denote the cardinality of the set $\{w \in \Sigma^i \mid w \in S_k \text{ and } b \in \{0, 1\} \text{ and } \#_0(wb) + k = i \geq 2m_0 + 1\}$. In particular, it always holds that $a^{(1)}_{m-1} = a^{(m)}_{m-1} = 1$. We see a quick example. When $m = 5$ and $i = 6$, we obtain $a^{(1)}_6 = 6$, $a^{(2)}_6 = a^{(4)}_6 = 15$, $a^{(3)}_6 = 20$, and $a^{(6)}_6 = 5$. For each index $j \in [1, n]_Z$, $|S_j|$ equals $\sum_{k=1}^{m} a^{(k)}_j$.

A simple observation yields the following relations among $a^{(k)}_i$: for each index $k \in [1, m - 2]_Z$ and $i \in [m_0 + 1, (n - 1)/2]_Z$,

$$a^{(1)}_{2i+1} = a^{(1)}_{2i-1} + a^{(2)}_{2i+1}, \quad a^{(k+1)}_{2i+1} = a^{(k)}_{2i-1} + 2a^{(k+1)}_{2i-1} + a^{(k+2)}_{2i-1}, \quad a^{(m)}_{2i+1} = a^{(m-1)}_{2i-1} + 2a^{(m)}_{2i-1}. \tag{1}$$

We want to show that $\sum_{k=1}^{m-1} a^{(k)}_{2i+1} \leq \gamma \sum_{k=1}^{m} a^{(k)}_{2i+1}$ for any index $i \in [m_0, (n - 1)/2]_Z$, where $\gamma$ is a constant less than 1 independent of $i$. Assuming $\gamma$, since

$$|S_{2i+1}| = \sum_{k=1}^{m} a^{(k)}_{2i+1} = 2a^{(1)}_{2i-1} + 4 \sum_{k=2}^{m-1} a^{(k)}_{2i-1} + 3a^{(m)}_{2i-1},$$

we obtain a recurrence:

$$|S_{2i+1}| \leq 3 \sum_{k=1}^{m} a^{(k)}_{2i-1} + \sum_{k=2}^{m-1} a^{(k)}_{2i-1} \leq (3 + \gamma m) \sum_{k=1}^{m} a^{(k)}_{2i-1} = (3 + \gamma)|S_{2i-1}|.$$

This recurrence has a solution $|S_n| \leq (3 + \gamma)^{n/2}|S_m|$. Since $|S_m|$ is a constant and $1 < \sqrt{3 + \gamma} < 2$, Claim 3 immediately follows.

Hereafter, we show $\gamma$. To show this, it suffices to show that $\gamma$ exists a constant $\delta \geq 1$ for which $\sum_{k=2}^{m-1} a^{(k)}_{2i-1} \leq \delta(a^{(1)}_{2i-1} + a^{(m)}_{2i-1})$ for any index $i \in [m_0, (n - 1)/2]_Z$, because, from this inequality $\gamma$, the following holds:

$$m \sum_{k=1}^{m} a^{(k)}_{2i+1} = (a^{(1)}_{2i+1} + a^{(m)}_{2i+1}) + \sum_{k=2}^{m-1} a^{(k)}_{2i+1} \geq \left(1 + \frac{1}{\delta}\right) \sum_{k=2}^{m-1} a^{(k)}_{2i+1}.$$

Hence, the desired constant $\gamma$ should be defined as $1/(1/\delta + 1)$, which is clearly less than 1. Our goal is therefore to prove $\gamma$, which follows as a special case of the next claim.

Claim 4 For every index $j \in [0, m_0 - 1]_Z$, $\sum_{k=m_0+1-j}^{m_0} a^{(k)}_{2i+1} \leq \delta_j (a^{(m_0-j)}_{2i+1} + a^{(m_0+j+2)}_{2i+1})$, where $\delta_j = 2^{2j+1} - 1$.

When $j = m_0 - 1$, this claim implies that $\sum_{k=2}^{m-1} a^{(k)}_{2i-1} \leq \delta_{m_0-1}(a^{(1)}_{2i-1} + a^{(m)}_{2i-1})$. By setting $\delta = \delta_{m_0-1}$, we obtain $\gamma$.

To end the proof of Claim 3 we need to prove Claim 4 by induction on $j$. For the basis case $j = 0$, from [1] we can estimate the sum $a^{(m_0)}_{2i+1} + a^{(m_0+2)}_{2i+1}$ as

$$a^{(m_0)}_{2i+1} + a^{(m_0+2)}_{2i+1} = a^{(m_0+1)}_{2i-1} + 2a^{(m_0+1)}_{2i-1} + 2a^{(m_0+2)}_{2i-1} + a^{(m_0+3)}_{2i-1} \geq a^{(m_0)}_{2i-1} + 2a^{(m_0+1)}_{2i-1} + a^{(m_0+2)}_{2i-1} = 2a^{(m_0+1)}_{2i-1},$$

which yields the desired relation $a^{(m_0+1)}_{2i+1} \leq \delta_0(a^{(m_0)}_{2i+1} + a^{(m_0+2)}_{2i+1})$ since $\delta_0 = 1$.

Next, let us consider the induction step with $0 \leq j \leq m_0 - 2$. As our induction hypothesis, we assume that $\sum_{k=m_0+1-j}^{m_0} a^{(k)}_{2i+1} \leq \delta_j (a^{(m_0-j)}_{2i+1} + a^{(m_0+j+2)}_{2i+1})$. We want to show that $\sum_{k=m_0+1-j}^{m_0} a^{(k)}_{2i+1} \leq \delta_{j+1}(a^{(m_0-j-1)}_{2i+1} + a^{(m_0+j+3)}_{2i+1})$. Let $Z_j = \sum_{k=m_0+1-j}^{m_0} a^{(k)}_{2i+1}$, then

$$Z_j \leq \delta_j (a^{(m_0-j)}_{2i+1} + a^{(m_0+j+2)}_{2i+1}),$$

and

$$Z_{j+1} = \sum_{k=m_0+1-(j+1)}^{m_0} a^{(k)}_{2i+1} \leq \delta_{j+1}(a^{(m_0-j-1)}_{2i+1} + a^{(m_0+j+3)}_{2i+1}).$$

Therefore, $Z_{j+1} \leq \delta_{j+1}(a^{(m_0-j-1)}_{2i+1} + a^{(m_0+j+3)}_{2i+1})$.
Each block \(k\) following.

By choosing \(0 < s \leq 1\), we place \(j\) holds.

\[
\sum_{k=m_0-j}^{m_0+j+2} a_{2i+1}^{(j)} = a_{2i-1}^{(m_0-j)} + 3a_{2i-1}^{(m_0-j)} + 4 \sum_{k=m_0-j}^{m_0+j+2} a_{2i-1}^{(k)} + 3a_{2i-1}^{(m_0+j+2)} + a_{2i-1}^{(m_0+j+3)}
\]

\[
\leq a_{2i-1}^{(m_0-j)} + 3a_{2i-1}^{(m_0-j)} + 4\delta_j(a_{2i-1}^{(m_0-j)} + a_{2i-1}^{(m_0+j+2)}) + 3a_{2i-1}^{(m_0+j+2)} + a_{2i-1}^{(m_0+j+3)}
\]

\[
= a_{2i-1}^{(m_0-j)} + (4\delta_j + 3)(a_{2i-1}^{(m_0-j)} + a_{2i-1}^{(m_0+j+2)}) + a_{2i-1}^{(m_0+j+3)}
\]

Since \(\delta_j+1\) satisfies that \(\delta_j+1 = 4\delta_j + 3\), the above sum is further bounded as

\[
\sum_{k=m_0-j}^{m_0+j+2} a_{2i+1}^{(j)} \leq (4\delta_j + 3)(a_{2i-1}^{(m_0-j)} + 2a_{2i-1}^{(m_0-j)} + a_{2i-1}^{(m_0-j)} + a_{2i-1}^{(m_0+j+2)} + 2a_{2i-1}^{(m_0+j+3)} + a_{2i-1}^{(m_0+j+4)})
\]

\[
= \delta_j+1(a_{2i+1}^{(m_0-j)} + a_{2i+1}^{(m_0+j+3)}),
\]

from which we obtain the desired inequality for \(j = \ell + 1\). By applying the induction, we obtain Claim \(\square\) holds.

What follows is the proof of Claim \(\square\) which also proceeds by induction.

**Proof of Claim \(\square\)** Recall the notation \(a_{\ell}^{(i)}\) and let \(c_{\ell}^{(1)} \geq c_{\ell}^{(2)} \geq \cdots \geq c_{\ell}^{(m)}\) be an enumeration of all the elements in \(\{a_{\ell}^{(i)}\}_{i \leq m}\) in a non-increasing order. For later convenience, whenever \(c_{\ell}^{(i)} = a_{\ell}^{(j)}\) with \(i < j\), we place \(a_{\ell}^{(i)}\) ahead of \(a_{\ell}^{(j)}\). As a quick example, when \(m = 5\) and \(\ell = 5\), we obtain \(c_6^{(1)} = a_6^{(3)}\), \(c_6^{(2)} = a_6^{(2)}\), \(c_6^{(3)} = a_6^{(4)}\), \(c_6^{(4)} = a_6^{(5)}\), and \(c_6^{(5)} = a_6^{(1)}\). This enumeration admits the following recurrence:

\[
\sum_{i=1}^{k} c_{\ell}^{(i)} = \sum_{i=1}^{k-1} c_{\ell}^{(i)} + \sum_{i=1}^{k+1} c_{\ell}^{(i)} \text{ for any index } k \in [1, m]_2, \text{ provided that } c_{\ell}^{(m+1)} = 0. \text{ In particular, } c_{\ell}^{(1)} = c_{\ell}^{(2)}.
\]

Recall the notation \(T_{E,\ell}\) and write \(T_{E,\ell}[j]\) for the set \(\{w \in T_{E,\ell} \mid \#_0(\text{pref}_{\ell+1}(w)) = j\}\). By our choice of the \(m\)-index series \(E\), we have \(A \cap \Sigma^n \subseteq T_{E,n}\). Similar to the enumeration of \(\{a_{\ell}^{(i)}\}_i\), we also enumerate all the elements in \(\{T_{E,\ell}[j]\}_{j \in E_{\ell}}\) as \(e_{E,\ell}^{(1)} \geq e_{E,\ell}^{(2)} \geq \cdots \geq e_{E,\ell}^{(m)}\). Note that \(|T_{E,n}| = \sum_{i=1}^{m} e_{E,n}^{(i)}\). We claim the following.

**Claim 5** For any pair \(\ell \in [m-1, n]_2\) and \(k \in [1, m]_2\), it holds that \((*)\) \(\sum_{i=1}^{k} c_{\ell}^{(i)} \geq \sum_{i=1}^{k} c_{E,\ell}^{(i)}\).

By choosing \(\ell = n\) and \(k = m\) in Claim \(\square\) we obtain \(|S_n| = \sum_{i=1}^{m} c_{n}^{(i)} \geq \sum_{i=1}^{m} c_{E,n}^{(i)} = |T_{E,n}| \geq |A \cap \Sigma^n|\), as requested. The remaining task is to prove Claim \(\square\).

This claim can be proven by double induction on \(\ell\) and \(k\). When \(\ell = m\), the inequality \((*)\) is true for any index \(k \in [1, m]_2\), because \(c_{E,m}^{(i)}\) coincides with \(c_{m}^{(i)}\). Assume that \(m \leq \ell < n\) and we target the case \(\ell + 1\). If \((**)\) \(\sum_{i=1}^{k} e_{E,\ell+1}^{(i)} \leq \sum_{i=1}^{k-1} e_{E,\ell}^{(i)} + \sum_{i=1}^{k+1} e_{E,\ell}^{(i)}\) where \(e_{E,\ell}^{(m+1)} = 0\), then we have

\[
\sum_{i=1}^{k} e_{E,\ell+1}^{(i)} \leq \sum_{i=1}^{k-1} e_{E,\ell}^{(i)} + \sum_{i=1}^{k+1} e_{E,\ell}^{(i)} \leq \sum_{i=1}^{k-1} c_{\ell}^{(i)} + \sum_{i=1}^{k+1} c_{\ell}^{(i)} = \sum_{i=1}^{k} c_{\ell}^{(i)},
\]

where the second inequality follows from the induction hypothesis that \(\sum_{i=1}^{k-1} c_{\ell}^{(i)} \geq \sum_{i=1}^{k-1} e_{E,\ell}^{(i)}\) and \(\sum_{i=1}^{k+1} c_{\ell}^{(i)} \geq \sum_{i=1}^{k+1} e_{E,\ell}^{(i)}\). Therefore, we have \(\sum_{i=1}^{k} c_{\ell}^{(i)} \geq \sum_{i=1}^{k} c_{E,\ell+1}^{(i)}\), as requested.

Finally, we show \((**)\). We proceed our proof by observing how to compute \(T_{E,\ell+1}\) from \(T_{E,\ell}\). Consider a partition of \(E_{\ell+1}\) into a number of blocks, say, \(E_{\ell+1}^{(1)}, E_{\ell+1}^{(2)}\), \ldots, each of which has a form \([p, q]_2\) with \(p \leq q\). Each block \(E_{\ell+1}^{(i)} = [p_i, q_i]_2\) defines \(T_{E,\ell+1}^{(i)} = \bigcup_{j \in E_{\ell+1}^{(i)}} T_{E,\ell+1}[j]\), whose cardinality \(|T_{E,\ell+1}^{(i)}|\) is bounded from above, similar to Equation \(\square\). By \(|T_{E,\ell}[p_i]| + 2 \sum_{p_i < j < q_i} |T_{E,\ell}[j]| + |T_{E,\ell}[q_i]|\). By summing up such \(|T_{E,\ell}^{(i)}|\)'s, \(|T_{E,\ell+1}^{(i)}|\) can be upper-bounded by the sum of at least two terms, say, \(|T_{E,\ell}[p']|\) and \(\sum_{|T_{E,\ell}[q']| + 2 \times \sum_{p' < j < q'} |T_{E,\ell}[j]|\) plus two times the sum of at most \(k - 2\) remaining terms \(|T_{E,\ell}[j]|\)’s. In other words, since \(|T_{E,\ell+1}^{(k)}| = \sum_{i=1}^{k} c_{E,\ell+1}^{(i)}\), we obtain
\[ \sum_{i=1}^{k} e_{E,t}^{(i)} \leq \sum_{i=1}^{k-1} e_{E,t}^{(i)} + \sum_{i=1}^{k+1} e_{E,t}^{(i)}. \] Claim 5 immediately follows. \(\square\)

This completes the proof of Proposition 6.2. \(\square\)

Unlike the REG-bi-immunity, it is possible to prove the existence of context-free REG/n-bi-primeimmune languages. Later, in Section 7 we shall show that a context-free language, called IP*, is indeed REG/n-bi-primeimmune.

7 Pseudorandomness of Languages

From this section to the next section, we shall discuss “computational randomness” of context-free languages. Although there are numerous ways to describe the intuitive notion of computational randomness, we choose the following notion, which we prefer to call \(C\)-pseudorandomness to distinguish another notion of “\(C\)-randomness” used in the past literature. Let \(\Sigma\) denote our alphabet with \(|\Sigma| \geq 2\) and let \(C\) be any language family. Roughly speaking, a language \(L\) over \(\Sigma\) is \(C\)-pseudorandom when the characteristic function \(\chi_L\) of any language \(A\) in \(C\) agrees with \(\chi_L\) over “nearly” 50% of strings of each length, where the word “nearly” is meant for “negligibly small margin.” In other words, since \(L \Delta A = \{ x \in \Sigma^* | \chi_L(x) \neq \chi_A(x) \}\), the density \(\text{dense}(L \Delta A)(n)\) “nearly” halves the total size \(|\Sigma^n|\). This new notion can be seen as a non-asymptotic variant of Wilber’s randomness [27] (which is also referred to as Wilber-stochasticity in [2]) and Meyer-McCreight’s randomness [22].

Let us formalize our intuitive notion. We say that a language \(L\) over \(\Sigma\) is \(C\)-pseudorandom if, for any language \(A\) over \(\Sigma\) in \(C\), the function \(\ell(n) = \left| \text{dense}(L \Delta A)(n) - \frac{1}{2} \right|\) is negligible; namely, for any non-zero polynomial \(p\), there is a number \(n_0 \geq 1\) such that (\ast) \(\left| \frac{\text{dense}(L \Delta A)(n)}{|\Sigma^n|} - \frac{1}{2} \right| \leq 1/p(n)\) for all numbers \(n \geq n_0\).

Assuming that \(\emptyset \in C\), we note that, by setting \(A = \emptyset\) in (\ast), every \(C\)-pseudorandom language \(L\) should satisfy
\[
\left( \frac{1}{2} - \frac{1}{p(n)} \right) |\Sigma^n| \leq \text{dense}(L)(n) \leq \left( \frac{1}{2} + \frac{1}{p(n)} \right) |\Sigma^n| \tag{2}
\]
for any non-zero polynomial \(p\) and for all but finitely-many lengths \(n \in \mathbb{N}\).

Similar in spirit to the previous \(C\)-primeimmunity, we can naturally restrict our attention to p-dense languages in \(C\). As a non-asymptotic variant of the notions of Müller’s balanced immunity [23] and weak-stochasticity of Ambos-Spies et al. [2], we introduce another notion, called weak \(C\)-pseudorandomness, which refers to a language that “nearly” splits every p-dense set in \(C\) by half. Let \(C\) be any language family containing the set \(\Sigma^*\). Formally, a language \(L\) over \(\Sigma\) is called weak \(C\)-pseudorandom if, for every p-dense language in \(C\), the function \(\ell'(n) = \left| \frac{\text{dense}(L \triangle A)(n)}{\text{dense}(A)(n)} - \frac{1}{2} \right|\) is negligible. By choosing \(A = \Sigma^*\), provided that \(\Sigma^* \in C\), we can show that \(L\) satisfies Equation (\ast\ast), and thus \(L\) cannot belong to \(C\).

For any language family \(D\), we say that \(D\) is \(C\)-pseudorandom (resp., weak \(C\)-pseudorandom) if \(D\) contains a \(C\)-pseudorandom (resp., weak \(C\)-pseudorandom) language. In fact, as we shall show later, CFL is REG-pseudorandom.

Meanwhile, we want to explore useful characteristics of (weak) \(C\)-pseudorandom languages. The following lemma gives other characterizations of weak \(C\)-pseudorandomness.

**Lemma 7.1** Assume that \(|\Sigma| \geq 2\). Let \(C\) be any language family that is closed under complementation. For every set \(S \subseteq \Sigma^*\), the following three statements are equivalent.

1. \(S\) is weak \(C\)-pseudorandom.
2. The function \(\ell(n) = \left| \frac{\text{dense}(S \triangle A)(n)}{|\Sigma^n|} - \frac{1}{2} \right|\) is negligible for every p-dense language \(A \in C\) over \(\Sigma\).
3. The function \(\ell''(n) = \left| \frac{\text{dense}(S \cap A)(n)}{|\Sigma^n|} - \frac{\text{dense}(\overline{S} \cap A)(n)}{|\Sigma^n|} \right|\) is negligible for every p-dense language \(A \in C\) over \(\Sigma\).

Notice that the statements (2) and (3) are still equivalent although we remove a requirement of the p-denseness of \(A\). Hence, with an appropriate change, a similar characterization of \(C\)-pseudorandomness follows. For a later reference, we call this fact a “pseudorandom” version of Lemma 7.1.

**Proof of Lemma 7.1** Let \(\Sigma\) be our alphabet with \(|\Sigma| \geq 2\) and let \(S\) be any language over \(\Sigma\). We use the following abbreviation: write \(S_n\) for \(S \cap \Sigma^n\) and \(\overline{S}_n\) for \(\overline{S} \cap \Sigma^n\). A language family \(C\) satisfies \(C = \text{co-}C\).


(1 ⇒ 2) Assume (1). Choose an arbitrary non-zero polynomial \( p \) and also any \( p \)-dense language \( A \) in \( C \). By the \( p \)-denseness of \( A \), there exists another non-zero polynomial \( q \) satisfying that \( |A_n| \geq |\Sigma^*|/q(n) \) for all but finitely many numbers \( n \). Hereafter, we assume that \( n \) is a sufficiently large number. From (1) follows the inequality 
\[
\frac{\text{dense}(S \setminus A)(n)}{\text{dense}(A)(n)} - \frac{1}{2} \leq 1/2p(n),
\]
which is equivalent to \( |A_n \cap S_n| - |A_n \cap \overline{S}_n| \leq |A_n|/p(n) \).

The closure property of \( C \) under complementation implies that \( \overline{A} \) is also in \( C \). Hence, similar to the case of \( A \), we obtain another inequality \( |\overline{A} \cap S_n| - |\overline{A} \cap \overline{S}_n| \leq |\overline{A}|/p(n) \).

Since \( |S_n \triangle A_n| = |A_n \cap S_n| + |\overline{A} \cap S_n| \) and \( |S_n \triangle \overline{A}_n| = |A_n \cap S_n| + |\overline{A} \cap \overline{S}_n| \), it follows that
\[
2|S_n \triangle A_n| - |\Sigma^n| = |S_n \triangle A_n| - |S_n \triangle \overline{A}_n|\
\leq |A_n \cap S_n| - |\overline{A} \cap S_n| + |\overline{A} \cap \overline{S}_n| - |A_n \cap \overline{S}_n|\
\leq \frac{|A_n|}{p(n)} + \frac{|\overline{A}_n|}{p(n)} = \frac{|\Sigma^n|}{p(n)}.
\]

Using this inequality, we obtain
\[
\ell(n) = \frac{\text{dense}(S \Delta A)(n)}{\text{dense}(A)(n)} - \frac{1}{2} = \frac{2|S_n \triangle A_n| - |\Sigma^n|}{|\Sigma^n|} \leq \frac{1}{p(n)}.
\]

Since \( p \) is arbitrary, the above bound of \( \ell(n) \) clearly implies (2).

(2 ⇒ 3) Assume (2). Let \( p \) be any non-zero polynomial and let \( A \) be any \( p \)-dense language in \( C \). From (2), we can assume that \( \ell(n) = \frac{|S_n \triangle A_n|}{|\Sigma^n|} - \frac{1}{2} \leq 1/2p(n) \) for any sufficiently large number \( n \). Since \( \Sigma^* \subset C \), it also holds that \( \left| \frac{|S_n|}{|\Sigma^n|} - \frac{1}{2} \right| \leq 1/2p(n) \). Hence, since \( |S_n \cap A_n| - |\overline{S}_n \cap A_n| = |S_n \triangle A_n| - |S_n| \), we can bound the term \( \ell'(n) \) as
\[
\ell'(n) = \frac{|S_n \cap A_n| - |\overline{S}_n \cap A_n|}{|\Sigma^n|} \leq \frac{|S_n \triangle A_n|}{|\Sigma^n|} - \frac{1}{2} + \frac{|S_n|}{|\Sigma^n|} - \frac{1}{2} \leq \frac{1}{2p(n)} + \frac{1}{2p(n)} = \frac{1}{p(n)}.
\]

Therefore, (3) holds.

(3 ⇒ 1) Assume (3). For any non-zero polynomial \( p \) and any \( p \)-dense language \( A \) in \( C \), take a certain non-zero polynomial \( q \) such that \( |A_n| \geq |\Sigma^n|/q(n) \) for any sufficiently large number \( n \). We then have
\[
\ell(n) = \frac{|S_n \cap A_n|}{|A_n|} - \frac{1}{2} = \frac{|S_n \cap A_n| - |\overline{S}_n \cap A_n|}{|A_n|} \leq q(n) \cdot \frac{|S_n \cap A_n| - |\overline{S}_n \cap A_n|}{|\Sigma^n|}.
\]

Since \( \left| \frac{|S_n \cap A_n| - |\overline{S}_n \cap A_n|}{|A_n|} \right| \leq \frac{1}{p(n)q(n)} \) from (3), the above inequality implies that \( \ell'(n) \leq \frac{1}{p(n)} \). The arbitrariness of \( p \) leads to a conclusion that \( \ell'(n) \) is negligible, or equivalently (1) holds.

Notice that the two implications (2 ⇒ 3) and (3 ⇒ 1) in the proof of Lemma 7.1 require no extra closure property of \( C \). As an immediate consequence, we can draw the following relationship for any language family \( C \).

**Corollary 7.2** Every \( C \)-pseudorandom language is weak \( C \)-pseudorandom.

We further argue that weak \( C \)-pseudorandomness implies \( C \)-primeimmunity. This implication bridges between primeimmunity and pseudorandomness.

**Lemma 7.3** Let \( C \) be any language family, which is closed under complementation. Every weak \( C \)-pseudorandom language is \( C \)-bi-primeimmune.

**Proof.** Let \( S \) be any weak \( C \)-pseudorandom language. Assuming that \( S \) is not \( C \)-primeimmune, we take a \( p \)-dense subset \( A \) of \( S \) in \( C \). From the \( p \)-denseness of \( A \), there exist a non-zero polynomial \( p \) and a constant \( n_0 \in \mathbb{N}^+ \) satisfying that \( |A_n| \geq 2^n/p(n) \) for all numbers \( n \geq n_0 \). Since \( A \subseteq S \), it follows that \( \ell'(n) = \frac{|S_n \cap A_n|}{|A_n|} - \frac{1}{2} = \frac{|A_n|}{|A_n|} - \frac{1}{2} \geq |1 - 1/2| = 1/2 \), which is clearly not negligible. Hence, \( S \) is not weak \( C \)-pseudorandom. A similar argument can be carried out under the assumption that \( S \) is not \( C \)-primeimmune. As a consequence, \( S \) is \( C \)-bi-primeimmune.

The converse of Lemma 7.3, however, does not hold in general. As a counterexample, we present a context-free language that is REG/n-primeimmune but not weak REG/n-pseudorandom. Our example is the language Equal*, defined in Section 8.
Proposition 7.4  The language family CFL contains a REG/n-primeimmune language that is not weak REG/n-pseudorandom.

Proof. In Proposition 6.2 the context-free language Equal is shown to be REG/n-primeimmune. Hence, our remaining task is to show that Equal is not weak REG/n-pseudorandom. Choose \( A = \Sigma^* \) in REG/n and consider the function \( \ell(n) = \frac{2^n - \text{dense}(\text{Equal}_n)(n)}{2^n} - \frac{1}{2} \). Obviously, \( \ell(n) \) is bounded by

\[
\ell(n) = \frac{2^n - \text{dense}(\text{Equal}_n)(n)}{2^n} - \frac{1}{2} = \frac{1}{2} - \frac{\text{dense}(\text{Equal}_n)(n)}{2^n}
\]

for any sufficiently large number \( n \), because \( \left( \frac{n}{2^n} \right) \leq \frac{2^n + 1}{2^n} \). Since \( \ell(n) \geq 1/4 \), Equal is not weak REG-pseudorandom. \( \square \)

Proposition 6.2 makes CFL to be REG/n-primeimmune. We shall strengthen CFL’s REG/n-primeimmunity by proving that CFL is actually REG/n-pseudorandom. Since the REG/n-pseudorandomness implies the REG/n-bi-primeimmunity by Corollary 7.2 and Lemma 7.3 we can conclude that CFL is also REG/n-bi-primeimmune, as stated in Section 4.

Proposition 7.5  The language family CFL is REG/n-pseudorandom.

To prove Proposition 7.5 we introduce a context-free language \( \text{IP}_n \). First, let us define the \textit{(binary) inner product} of \( x \) and \( y \) as \( x \odot y = \sum_{i=1}^{n} x_i \cdot y_i \), where \( x = x_1x_2\cdots x_n \) and \( y = y_1y_2\cdots y_n \) are \( n \)-bit strings. The language \( \text{IP}_n \) is defined as \( \text{IP}_n = \{ auv \mid |a| = |v|, u^R \odot v \equiv 1 \pmod{2} \} \). Now, we demonstrate that \( \text{IP}_n \) is context-free. Let us consider the following npda \( M \). On input \( auv \), we nondeterministically check two possibilities. Along one computation path, we assume that \( a = \lambda \), and we nondeterministically check if \( |u| = |v| \) and \( u^R \odot v \equiv 1 \pmod{2} \). On the other path, we assume that \( a \neq \lambda \), and we ignore the first bit \( a \) and check if \( |u| = |v| \) and \( u^R \odot v \equiv 1 \pmod{2} \). The latter condition \( u^R \odot v \equiv 1 \pmod{2} \) can be checked by storing \( u \) in a (first-in last-out) stack and then computing each \( u_{n/2-i} \cdot v_i \) while reading \( v_i \), where \( i = 1, 2, \ldots, n/2 \).

The reader may heed an attention to the fact that \( \text{IP}_n \) is REG-levelable, because, by Lemma 4.3 \( f(auv) = a0uv0 \) is a length-increasing 1-DLIN-invertible 1-DLIN-m-autoreduction for \( \text{IP}_n \).

Our proof of Proposition 7.5 requires a certain unique property of REG/n, called a \textit{swapping property}, which has a loose similarity with the swapping lemma for regular languages.

Lemma 7.6  \textit{[swapping property lemma]}  Let \( S \) be any language over an alphabet \( \Sigma \). If \( S \subseteq \Sigma^*/n \), then there exists a positive integer \( m \) that satisfies the following property. For any three numbers \( n, \ell_1(n), \ell_2(n) \in \mathbb{N} \) with \( \ell_1(n) + \ell_2(n) = n \), there is a group of disjoint sets, say, \( S_1^{(n)}, S_2^{(n)}, \ldots, S_m^{(n)} \) such that \( i) \ S \cap \Sigma^n = \bigcup_{i=1}^{m} S_i^{(n)} \) and \( ii) \) (swapping property) for any pair \( x, y \in A_1^{(n)}, \) if \( x = x_1x_2 \) and \( y = y_1y_2 \) with \( |x_j| = |y_j| = \ell_j(n) \) for each index \( j \in \{1, 2\} \), then a swapped strings \( x_1y_2 \) and \( y_1x_2 \) are in \( A_1^{(n)} \).

Proof. From our assumption \( S \subseteq \Sigma^*/n \), we choose a dfa \( M \) with a set \( Q \) of inner states, and an advice function \( h : \mathbb{N} \rightarrow \Gamma^* \) with \( |h(n)| = n \) satisfying that, for every string \( x \in \Sigma^*, x \in A \) iff \( M \) accepts \( [_{h(x)}^x] \). Assume that \( Q = \{ q_1, q_2, \ldots, q_m \} \) with \( m \geq 1 \). For any numbers \( n, \ell_1(n), \ell_2(n) \in \mathbb{N} \) with \( \ell_1(n) + \ell_2(n) = n \), we define \( S_j^{(n)} \) as the set \( \{ x_1x_2 \in S \cap \Sigma^n \mid |x_1| = \ell_1(n), |x_2| = \ell_2(n) \} \), where \( h_1 \) satisfies \( h(n) = h_1h_2 \) and \( h_1 = \ell_1(n) \). It is clear that \( S \cap \Sigma^n = \bigcup_{i=1}^{m} S_i^{(n)} \). If \( x_1x_2 \) and \( y_1y_2 \) are in \( S_i^{(n)} \), then \( M \)'s inner state after reading either \( [_{h_1}^{x_1}] \) or \( [_{h_2}^{y_2}] \) are the same state \( q_i \). Since \( M \) accepts both \( [_{h_1}^{x_1}] \) and \( [_{h_2}^{y_2}] \), \( M \) also accepts both \( [_{h_1}^{x_2}] \) and \( [_{h_2}^{y_1}] \). This implies that \( x_1y_2 \) and \( y_1x_2 \) belong to \( S_i^{(n)} \). \( \square \)

Now, we are ready to give the proof of Proposition 7.5. In the proof, we utilize a well-known discrepancy upper bound of the inner-product-modulo-two function.

Proof of Proposition 7.5 We shall show that \( \text{IP}_n \) is REG/n-pseudorandom. Assume on the contrary that, by a “pseudorandom” version of Lemma 7.1 there is a set \( S \) in REG/n, a polynomial \( p \), and an infinite set \( I \subseteq \mathbb{N} \) such that \( \ell^*(n) = \frac{\log |\text{dense}(\text{IP}_n \cap S)(n)| - \log |\text{IP}_n \cap S(n)|}{\log(n/p(n))} \geq 1/p(n) \) for all numbers \( n \in I \). Take a constant \( m \) given in Lemma 7.6. Let \( n \) be any sufficiently large number in \( I \) satisfying \( m < 2^{n/8} \) and
$p(n) < 2^{n/8}$, and consider any input $a uv$ of length $n$. It is sufficient to check the case where $n$ is even (that is, $a = \lambda$), because, when $n$ is odd, we can ignore the first bit $a$ and follow a similar argument. Abbreviate $S \cap IP_+ \cap S^n$ and $S \cap \overline{IP}_+ \cap S^n$ by $U_1$ and $U_0$, respectively. From our assumption, it then follows that $|U_1| - |U_0| = t^{(n)}(n) S^n \geq 2^n/p(n)$.

By setting $\ell_1(n) = \ell_2(n) = n/2$, we take $s_1^{(n)}, \ldots, s_m^{(n)}$ given by Lemma 7.4. Let us consider two partitions $U_0 = \bigcup_{i=1, m}^{m} U_0(i)$ and $U_1 = \bigcup_{i=1, m}^{m} U_1(i)$, where $U_1(i) = IP_+ \cap s_i^{(n)}$ and $U_0(i) = \overline{IP}_+ \cap s_i^{(n)}$. Toward our desired contradiction, we aim at proving the inequality $|U_1| - |U_0| < 2^n/p(n)$. For our purpose, we claim the following.

Claim 6. For all indices $i \in [1, m]$, $|U_1(i)| - |U_0(i)| \leq 2^{5n/4}$.

From this claim, since $m < 2^{n/8}$, it follows that

$$|U_1| - |U_0| \leq \sum_{i \in [1, m]} |U_1(i)| - |U_0(i)| \leq m \cdot 2^{5n/4} < 2^{7n/8} < 2^n/p(n).$$

This consequence obviously contradicts our assumption that $|U_1| - |U_0| \geq 2^n/p(n)$. Hence, the proposition follows immediately.

Now, we give the proof of Claim 6. For this proof, we need a discrepancy upper bound of the inner-product-modulo-two function. Let $M$ be a $\Sigma/n/2$-by-$\Sigma/n/2$ matrix whose $(x, y)$-entry has a value $x \circ y \mod 2$. The discrepancy of a rectangle $A \times B$ in $M$ is $\text{Disc}_M(A \times B) = 2^{-n} \left| \#_1^{(M)}(A \times B) - \#_0^{(M)}(A \times B) \right|$, where $\#_b^{(M)}(A \times B)$ means the total number of 1 entries in $M$ when $M$’s entries are limited to $A \times B$. It is known that $\text{Disc}_M(A \times B) \leq 2^{-3n/4} \sqrt{|A||B|} \leq 2^{-n/4}$ (see, e.g., [3, Example 12.14]). Although it is not quite tight, this loose bound still serves well for our purpose.

For each index $i \in [1, m]$, we define two sets $A_i = \{ u \in \Sigma/n/2 | \exists v \in \Sigma/n/2 [uv \in s_i^{(n)}] \}$ and $B_i = \{ v \in \Sigma/n/2 | \forall u \in \Sigma/n/2 [uv \in s_i^{(n)}] \}$, and we claim the following.

Claim 7. For each bit $b$, $\#_b^{(M)}(A_i \times B_i) = |U_b(i)|$.

It is clear from this claim that $2^{-n}|U_1(i)| - |U_0(i)| = \text{Disc}_M(A_i \times B_i) \leq 2^{-n/4}$. This inequality implies that $|U_1(i)| - |U_0(i)| \leq 2^{3n/4}$ as in Claim 6.

To end our proof, we shall prove Claim 7. Consider the case $b = 0$. The other case is similar and omitted here. First, let $N$ be another $\Sigma/n/2$-by-$\Sigma/n/2$ matrix in which the value of each $(x, y)$-entry is $x \circ y \mod 2$. Obviously, we have $\#_0^{(M)}(A_i \times B_i) = \#_0^{(N)}(A_i^R \times B_i)$, where $A_i^R = \{ w^R | w \in A_i \}$. Second, we show that $A_i^R \times B_i = S_i^{(n)}$ by identifying $(u, v)$ with $uv$ whenever $|u| = |v|$. This is shown as follows. Assume that $uv \in S_i^{(n)}$. By the definitions of $A_i$ and $B_i$, it follows that $w^R \in A_i$ and $v \in B_i$; hence, $(u, v) \in A_i^R \times B_i$. Conversely, assume that $(u, v) \in A_i^R \times B_i$. Take $\hat{u}, \hat{v} \in \Sigma/n/2$ such that $\hat{u} \hat{v} \in S_i^{(n)}$ and $\hat{u} \hat{v} \in S_i^{(n)}$. The swapping property of $S_i^{(n)}$ given in Lemma 7.6 implies that $uv \in S_i^{(n)}$. Therefore, it holds that $A_i^R \times B_i = S_i^{(n)}$.

From the above two equations, it follows that $\#_0^{(M)}(A_i \times B_i) = \#_0^{(N)}(A_i^R \times B_i) = |S_i^{(n)} \cap IP_+| = |U_0(i)|$. From this equation, Claim 7 follows.

To close this section, we exhibit a closure property of the family of $C$-pseudorandom languages under a certain relation between two languages. Two languages $A$ and $B$ over the same alphabet $\Sigma$ are said to be almost equal if the function $\delta(n) = \frac{\text{dense}(A \triangle B)(n)}{|\Sigma^n|}$ is negligible. Note that this binary relation is actually an equivalence relation (satisfying reflexivity, symmetry, and transitivity).

Lemma 7.7. Let $C$ be any language family and let $A$ and $B$ be any two languages over an alphabet $\Sigma$. If $A$ and $B$ are almost equal and $A$ is $C$-pseudorandom, then $B$ is also $C$-pseudorandom.

Proof. Let $A$ and $B$ be any two languages over an alphabet $\Sigma$. We assume that $A$ is $C$-pseudorandom for a language family $C$ and that $A$ and $B$ are almost equal. As before, we use the following abbreviation: for each integer $n \in \mathbb{N}$ and a language $D$, write $D_n$ and $\overline{D}_n$ for $D \cap \Sigma^n$ and $\overline{D} \cap \Sigma^n$, respectively. To show the $C$-pseudorandomness of $B$, let $p$ be any non-zero polynomial and let $n$ be any number, which is sufficiently large to withstand our argument that proceeds in the rest of this proof.

It suffices to show that (*) $\left| \frac{B_n \triangle C_n}{|\Sigma^n|} - \frac{1}{2} \right| \leq \frac{1}{p(n)}$. Since $A$ is $C$-pseudorandom, it holds that $\left| \frac{A_n \triangle C_n}{|\Sigma^n|} - \frac{1}{2} \right| \leq \frac{1}{p(n)}$. Therefore, it follows that $\left| \frac{B_n \triangle C_n}{|\Sigma^n|} - \frac{1}{2} \right| \leq \frac{1}{p(n)}$.
Moreover, since $A$ and $B$ are almost equal, we have $\frac{|A\triangle B|}{|\Sigma^n|} \leq \frac{1}{4p(n)}$. It is not difficult to show that $\overline{A}$ and $\overline{B}$ are also almost equal; thus, it follows that $\frac{|\overline{A}\triangle \overline{B}|}{|\Sigma^n|} \leq \frac{1}{4p(n)}$.

We first give an upper bound of $||B_n\triangle C_n|| - |A_n\triangle C_n||$. Note that

$$||B_n\triangle C_n|| - |A_n\triangle C_n|| \leq ||B_n \cap C_n|| - |A_n \cap C_n|| + ||B_n \cap C_n|| - |A_n \cap C_n||.$$  

The first term of the right side of the above formula is bounded by

$$||B_n \cap C_n|| - |A_n \cap C_n|| \leq |A_n \cap B_n|| + |A_n \triangle B_n||.$$  

A similar bound is given for $||B_n \cap C_n|| - |A_n \cap C_n||$. Combining these two bounds leads to

$$||B_n\triangle C_n|| - |A_n\triangle C_n|| \leq |A_n\triangle B_n|| + |\overline{A_n}\triangle B_n|| \leq \frac{1}{4p(n)} + \frac{1}{4p(n)} = \frac{1}{2p(n)}.$$  

From this bound, our desired inequality (*) follows as

$$\frac{|B_n\triangle C_n|}{|\Sigma^n|} - \frac{1}{2} \leq \frac{|A_n\triangle C_n|}{|\Sigma^n|} - \frac{1}{2} + \frac{|B_n\triangle C_n| - |A_n\triangle C_n||}{|\Sigma^n|} \leq \frac{1}{2p(n)} + \frac{1}{2p(n)} = \frac{1}{p(n)}.$$  

Since $C$ is arbitrary, we conclude the $C$-pseudorandomness of $B$, as requested. \qed

8 Pseudorandom Generators

Rather than determining the pseudorandomness of strings, we intend to produce pseudorandom strings. A function that generates such strings, known as a pseudorandom generator, is an important cryptographic primitive, and a large volume of work has been dedicated to its theoretical and practical applications. In accordance with this paper's main theme of formal language theory, we define our pseudorandom generator so that it fools “languages” rather than “probabilistic algorithms” as in its conventional definition in, e.g., [13].

A similar treatment also appears in designing of generators that fool “Boolean circuits.” For ease of notation, we always denote the binary alphabet $\{0,1\}$ by $\Sigma$. Let us recall the notation $\chi_A$, which expresses the characteristic function of $A$. In cryptography, we often restrict our interest to a function $G$ that maps $\Sigma^*$ to $\Sigma^*$ with a stretch factor $\ell(n)$: namely, $|G(x)| = s(|x|)$ holds for all strings $x \in \Sigma^*$. Such a function $G$ is said to fool a language $A$ over $\Sigma$ if the function $\ell(n) = |\text{Prob}_x[\chi_A(G(x)) = 1] - \text{Prob}_y[\chi_A(y) = 1]|$ is negligible, where $x$ and $y$ are random variables over $\Sigma^n$ and $\Sigma^*(n)$, respectively. We often call an input $x$ fed to $G$ a seed. A function $G$ is called a pseudorandom generator against a language family $C$ if $G$ fools every language $A$ over $\Sigma$ in $C$. Taking the significance of p-denseness into our consideration, we also introduce a weaker form of pseudorandom generator, which fools only p-dense languages. Formally, a weak pseudorandom generator against $C$ is a function that fools every p-dense language over $\Sigma$ in $C$. Obviously, every pseudorandom generator is a weak pseudorandom generator. As shown later, C-pseudorandomness has a close connection to pseudorandom generators against $C$.

In particular, this paper draws our attention to “almost one-to-one” pseudorandom generators. A generator $G$ with the stretch factor $n+1$ is called almost 1-1 if there is a negligible function $\tau(n) \geq 0$ such that $|\{G(x) \mid x \in \Sigma^n\}| = |\Sigma^n|(1 - \tau(n))$ for all numbers $n \in \mathbb{N}$.

Recall from Section 2 the single-valued total function class CFLSV_t, which includes 1-FLIN as a proper subclass (because REG = CFL if 1-FLIN = CFLSV_t). In the rest of this section, we aim at proving that CFLSV_t contains an almost 1-1 pseudorandom generator against REG/n.

**Proposition 8.1** There exists an almost 1-1 pseudorandom generator in CFLSV_t against REG/n.

To prove this proposition, let us discuss a close relation between two notions: $C$-pseudorandomness and pseudorandom generators against $C$. Our key lemma below states that any almost 1-1 (weak) pseudorandom generator against $C$ can be characterized by the notion of (weak) $C$-pseudorandomness.

**Lemma 8.2** Let $\Sigma = \{0,1\}$. Let $C$ be any language family that is closed under complementation. Let $G$ be any almost 1-1 function from $\Sigma^*$ to $\Sigma^*$ with the stretch factor $n+1$.

*This factor is also called an expansion factor in, e.g., [13].
1. G is a pseudorandom generator against C iff the range \( S = \{ G(x) \mid x \in \Sigma^* \} \) of G is an \( \mathbb{C} \)-pseudorandom set.

2. G is a weak pseudorandom generator against C iff the range \( S = \{ G(x) \mid x \in \Sigma^* \} \) of G is a weak \( \mathbb{C} \)-pseudorandom set.

**Proof.** Let \( \mathbb{C} \) be any language family such that \( \mathbb{C} = \text{co-} \mathbb{C} \). Assume that G is an almost 1-1 function stretching \( n \)-bit seeds to \( (n + 1) \)-bit strings. Consider G’s range \( S = \{ G(x) \mid x \in \Sigma^* \} \). For any language \( \mathcal{B} \) over \( \Sigma \) and for each length \( n \in \mathbb{N} \), \( B_{n+1} \) denotes \( B \cap \Sigma^{n+1} \) and \( \overline{B}_{n+1} \) denotes \( \overline{B} \cap \Sigma^{n+1} \). In particular, \( S_{n+1} \) equals \( \{ G(x) \mid x \in \Sigma^n \} \). Since G is almost 1-1, it holds that \( |S_{n+1}| = |\Sigma^n|(1 - \tau(n)) \) for a certain negligible function \( \tau(n) \). In other words, \( |\Sigma^n| - |S_{n+1}| = |\Sigma^n|\tau(n) \). We write \( \ell_B(n) \) for \( |\text{Prob}_{x \in \Sigma^n}[\chi_B(G(x)) = 1] - \text{Prob}_{y \in \Sigma^{n+1}}[\chi_B(y) = 1]| \). Henceforth, we want to show only (1) since (2) can be proven similarly.

(Only If – part) Assume that G is a pseudorandom generator against \( \mathbb{C} \). Let \( \mathcal{B} \) be any language in \( \mathbb{C} \). Since G fools \( \mathcal{B} \), the function \( \ell_B(n) \) is negligible. Let \( p \) be any non-zero polynomial. Assume that \( n \) is sufficiently large so that \( \ell_B(n) \leq 1/2p(n) \) and \( \tau(n) \leq 1/2p(n) \). It thus follows that \( |\Sigma^n| - |S_{n+1}| = |\Sigma^n|\tau(n) \leq |\Sigma^n|/2p(n) \). We set \( \delta_n \) and \( \epsilon_n \) to satisfy that \( \sum_{y \in S_{n+1} \cap B_{n+1}} |G^{-1}(y)| = \delta_n |S_{n+1} \cap B_{n+1}| \) and \( \sum_{y \in S_{n+1} \cap \overline{B}_{n+1}} |G^{-1}(y)| = \epsilon_n |S_{n+1} \cap \overline{B}_{n+1}| \). Obviously, \( \delta_n, \epsilon_n \geq 1 \). Since \( |\Sigma^n| = \sum_{y \in S_{n+1}} |G^{-1}(y)| \) and

\[
\sum_{y \in S_{n+1}} |G^{-1}(y)| = \sum_{y \in S_{n+1} \cap B_{n+1}} |G^{-1}(y)| + \sum_{y \in S_{n+1} \cap \overline{B}_{n+1}} |G^{-1}(y)|,
\]

we have

\[
|\Sigma^n| = \delta_n |S_{n+1} \cap B_{n+1}| + \epsilon_n |S_{n+1} \cap \overline{B}_{n+1}|.
\]

Using this relation, since \( \epsilon_n \geq 1 \), we have

\[
(\delta_n - 1) |S_{n+1} \cap B_{n+1}| \leq (\delta_n - 1) |S_{n+1} \cap B_{n+1}| + (\epsilon_n - 1) |S_{n+1} \cap \overline{B}_{n+1}|
\]

\[
= (\delta_n |S_{n+1} \cap B_{n+1}| + \epsilon_n |S_{n+1} \cap \overline{B}_{n+1}|) - (|S_{n+1} \cap B_{n+1}| + |S_{n+1} \cap \overline{B}_{n+1}|)
\]

\[
= |\Sigma^n| - |S_{n+1}| \leq \frac{|\Sigma^n|}{2p(n)}.
\]

Therefore, it holds that \( (\delta_n - 1) |S_{n+1} \cap B_{n+1}| \leq |\Sigma^n|/2p(n) \). We will use this inequality later.

Next, we want to estimate \( \ell_B''(n) = \frac{|S_{n+1} \cap B_{n+1}| - |S_{n+1} \cap \overline{B}_{n+1}|}{|\Sigma^n|} \). With \( \delta_n \), we obtain

\[
\text{Prob}_{x \in \Sigma^n}[\chi_B(G(x)) = 1] = \frac{\sum_{y \in S_{n+1} \cap B_{n+1}} |G^{-1}(y)|}{|\Sigma^n|} = \frac{\delta_n |S_{n+1} \cap B_{n+1}|}{|\Sigma^n|}.
\]

Since \( \text{Prob}_{y \in \Sigma^{n+1}}[\chi_B(y) = 1] = |B_{n+1}|/|\Sigma^n| \), \( \ell_B(n) \) equals

\[
\ell_B(n) = \frac{\delta_n |S_{n+1} \cap B_{n+1}|}{|\Sigma^n|} - \frac{|B_{n+1}|}{|\Sigma^n|}
\]

\[
= \frac{\delta_n |S_{n+1} \cap B_{n+1}|}{|\Sigma^n|} - \frac{|S_{n+1} \cap B_{n+1}|}{|\Sigma^n|} - \frac{|S_{n+1} \cap \overline{B}_{n+1}|}{|\Sigma^n|},
\]

which is lower-bounded by

\[
\ell_B(n) \geq \frac{|S_{n+1} \cap B_{n+1}| - |S_{n+1} \cap B_{n+1}|}{|\Sigma^n|} - 2(\delta_n - 1) \frac{|S_{n+1} \cap B_{n+1}|}{|\Sigma^n|}.
\]

From our assumption \( \ell_B(n) \leq 1/2p(n) \), we can conclude that

\[
\ell_B''(n) = \frac{|S_{n+1} \cap B_{n+1}| - |S_{n+1} \cap \overline{B}_{n+1}|}{|\Sigma^n|} \leq \frac{1}{2p(n)} + 2(\delta_n - 1) \frac{|S_{n+1} \cap B_{n+1}|}{|\Sigma^n|} \leq \frac{1}{p(n)},
\]

where the last inequality follows from the previous bound \( (\delta_n - 1) |S_{n+1} \cap B_{n+1}| \leq |\Sigma^n|/2p(n) \).

Apply a “\( \mathbb{C} \)-pseudorandom” version of Lemma 7.1 and we obtain the \( \mathbb{C} \)-pseudorandomness of S.
(If - part) Assume that the set \( S = \{ G(x) \mid x \in \Sigma^* \} \) is \( C \)-pseudorandom. To show that \( G \) is a pseudorandom generator against \( C \), we want to show that the function \( \ell_B(n) \) is negligible for any language \( B \) in \( C \). Let \( p \) be any non-zero polynomial and let \( B \) be any language in \( C \). Since \( S \) is \( C \)-pseudorandom, by a “\( C \)-pseudorandom” version of Lemma \( \ref{lem:lower_bound} \) the following term

\[
\ell''(n) = \frac{|S_{n+1} \cap B_{n+1}|}{|\Sigma^{n+1}|} - \frac{|S_{n+1} \cap B_{n+1}|}{|\Sigma^{n+1}|}
\]

is upper-bounded by \( 1/2p(n) \) for all but finitely-many numbers \( n \).

We choose two numbers \( \delta_n \) and \( \epsilon_n \) so that \( \text{Prob}_{x \in \Sigma^*} [X_B(G(x)) = 1] = \delta_n |S_{n+1} \cap B_{n+1}| \) and \( \text{Prob}_{x \in \Sigma^*} [X_B(G(y)) = 1] = \epsilon_n |S_{n+1} \cap \overline{B_{n+1}}| \). Since \( \delta_n \geq 1 \), it follows that

\[
\ell_B(n) = \frac{\delta_n |S_{n+1} \cap B_{n+1}|}{|\Sigma^{n+1}|} - \frac{|B_{n+1}|}{|\Sigma^{n+1}|} = \left( \frac{\delta_n - 1}{\delta_n} \right) \frac{|S_{n+1} \cap B_{n+1}|}{|\Sigma^{n+1}|} + \left( \frac{\delta_n - 1}{\delta_n} \right) \frac{|B_{n+1}|}{|\Sigma^{n+1}|} + \frac{1}{2p(n)}.
\]

Since \( \epsilon_n \geq 1 \), we further estimate \( \ell_B(n) \) as

\[
\ell_B(n) \leq \left( \frac{\delta_n - 1}{\delta_n} \right) \frac{|S_{n+1} \cap B_{n+1}|}{|\Sigma^{n+1}|} + \frac{1}{2p(n)} \leq \frac{1}{2p(n)}.
\]

Therefore, we obtain \( \ell_B(n) \leq 1/p(n) \). From the arbitrariness of \( B \) in \( C \), we can conclude that \( G \) is a pseudorandom generator against \( C \).

Let us describe the proof of Proposition \( \ref{prop:lower_bound} \). First, recall the context-free language \( IP_r \) given in Section \( \ref{sec:context_free_language} \). We want to build our desired pseudorandom generator based on the \( \text{REG}/n \)-pseudorandomness of \( IP_r \).

**Proof of Proposition \( \ref{prop:lower_bound} \)** The desired generator \( G \) is defined as follows. Let \( n \) be an arbitrary number \( \geq 3 \) and let \( w = axy \) be any input of length \( n \) satisfying \( a \in \{ \lambda, 0, 1 \} \) and \( |x| = |y| + 1 \). We consider the first case where \( n \) is odd (i.e., \( a = \lambda \)), assuming further that \( x = bz \) for a certain bit \( b \). Since \( n \) is odd, let \( k = (n-1)/2 \). As described below, our generator \( G \) outputs a string of the form \( x'y'e \) of length \( n+1 \), where \( |x'| = |x|, |y'| = |y|, \) and \( e \in \{0, 1\} \).

1. If \( w = bzy \) for a certain bit \( b \) and \( z^R \odot y \equiv 1 \) (mod 2), then let \( G(w) = bzyb \).
2. If \( w = 1zy \) and \( z^R \odot y \equiv 0 \) (mod 2), then let \( G(w) = 1zy1 \).
3. If \( w = 0zy \) and \( z^R \odot y \equiv 0 \) (mod 2), then check if there is the minimal index \( i \) such that \( z_{k-i+1} = 1 \).
   
   (3a) Consider the case where such \( i \) exists. In this case, let \( G(w) = 0zy0 \), where \( \hat{y} \) is obtained from \( y \) by flipping the \( ith \) bit; that is, \( \hat{y} = yyi_2 \cdots y_{i-1}y_{i+1} \cdots y_k \).
   
   (3b) Consider the other case where \( i \) does not exist; in other words, \( z = 0^k \). In this case, we define \( G(w) = 1zy1 \).

In the remaining case where \( n \) is even (i.e., \( a \in \{0, 1\} \)), we simply define \( G(w) = au \), where \( u = G(xy) \).

Our next goal is set to show that \( G \) is a pseudorandom generator in CFLSV \( t \) against \( \text{REG}/n \). We begin with the claim that \( G \) is an almost 1-1 function.

**Claim 8** \( G \) is almost 1-1.

**Proof.** Consider the case where \( n \) is odd, and set \( k = (n-1)/2 \) as before. In the above definition of \( G \), it is not difficult to check that all the cases except Case (3b) make \( G \) one-to-one. It is thus sufficient to deal with Case (3b). In this case, for each fixed string \( y \in \Sigma^k \), only inputs taken from the set \( \{00^k y, 10^k y\} \)
are mapped by $G$ into the string $10^k y 1$. Now, we define $\tau(n) = 1/2^k \ (= 1/2^{(n-1)/2})$. Letting $A_k$ denote $igcup_{y \in \Sigma^n} \{00^k y, 10^k y\}$, we note that $G$ is one-to-one on the domain $\Sigma^n - A_k$ and 2-to-1 on the domain $A_k$. Hence, since $|A_k| = 2^{k+1}$, it thus follows that

$$|\{G(w) \mid w \in \Sigma^n\}| = |\Sigma^n - A_k| + \frac{|A_k|}{2} = 2^n - 2^k = |\Sigma^n| \left( 1 - \frac{1}{2^n} \right).$$

The other case where $n$ is even follows from the previous case and we can define $\tau$ accordingly. Clearly, $\tau$ is negligible, and therefore $G$ is almost 1-1.

**Claim 9** The range $S = \{G(w) \mid w \in \Sigma^n\}$ of $G$ coincides with $IP_\ast$.

**Proof.** The containment $S \subseteq IP_\ast$ can be shown as follows. Let $w \in \Sigma^n$ be any input string and we want to show that $G(w) \in IP_\ast$. Now, assume that $n$ is odd. Let us consider Case (1) with $w = b z y$ and $z_R \circ y \equiv 1 \pmod{2}$. In this case, $G(w) = b z y \overline{b}$. Since $(b z)^R \circ (y \overline{b}) \equiv z_R \circ y + b \circ \overline{b} \equiv 1 \pmod{2}$, it follows that $G(w) \in IP_\ast$. Next, we consider Case (3a) with $w = 0 z y$ and $z_R \circ y \equiv 0 \pmod{2}$. Let $j = \min\{i \mid z_{k-i+1} = 1\}$. Notice that $z_{k-j+1} \cdot y_j \neq z_{k-j+1} \cdot \overline{y_j}$. We have

$$z^R \circ y = \sum_{i:j \neq j} z_{k-i+1} y_i + z_{k-j+1} \cdot y_j \neq \sum_{i:j \neq j} z_{k-i+1} y_i + z_{k-j+1} \cdot \overline{y_j} = z^R \circ \overline{y}.$$ 

Thus, we then show the other containment $IP_\ast \subseteq S$. Choose an arbitrary string $u \in IP_\ast \cap \Sigma^n$ and assume that $n$ is even. Let $k = (n - 2)/2$. Consider the case where $u = b z y \overline{b}$ with $b \in \{0, 1\}$ and $|z| = |y| = k$. Since $u \in IP_\ast$, we have $(b z)^R \circ (y \overline{b}) \equiv z^R \circ y \equiv 1 \pmod{2}$. Hence, $G$ maps $w = b z y$ to $u$. This means that $u$ is in $S$. Next, we consider the case where $u = 0 z y 0$ with $|z| = |y|$. Let $i = \min\{i \mid z_{k-i+1} = 1\}$. As before, we define $\tilde{y}$ from $y$ by flipping the $i$th bit of $y$. Hence, $G(0 z \tilde{y})$ equals $0 z y 0$, which obviously equals $u$. Thus, $u \in S$. The other cases are similarly proven.

Since $IP_\ast$ is REG/n-pseudorandom, by Claim 9, $S$ is also REG/n-pseudorandom. From $G$’s almost one-oneness, Lemma 8.2 guarantees that $G$ is a pseudorandom generator against REG/n. What remains unproven is that $G$ actually belongs to CFLSV since $I_F$ is in CFLSV.

**Claim 10** $G$ is in CFLSV.

**Proof.** Here we give a npda with a write-only output tape, which computes $G$. Our npda $N$ works as follows. On input $w = a x y$, guess nondeterministically whether $a = \lambda$ or not. Along a nondeterministic branch associated with a guess “$a = \lambda$,” check nondeterministically whether $|x| = |y| + 1$ using a stack as storage space. During this checking process, $N$ also computes $z^R \circ y$ and finds the minimal index $i_0$ such that $z_{k-i_0+1} = 1$ (if any). While reading input bits, for each nondeterministic computation, $N$ produces three types of additional computation paths. Along the first one of such paths, $N$ writes $10^k y 1$ on its output tape; on the second path, $N$ writes $b x y$ on the output tape; on the third path, $N$ writes $0 z \tilde{y} 0$, provided that $i_0$ exists. At the end of scanning the input, if Case (3b) does not hold, $N$ enters a rejecting state on the first path to invalidate its output $10^k y 1$. If Case (3a) does not hold, $N$ also invalidate its output $0 z \tilde{y} 0$ on the third path. In Cases (1)-(2), assume that $N$ has written $b x y$ on the second path. Now, $N$ writes down $b$ or 1, respectively, on the output tape following $b x y$ if Case (1) or Case (2) holds. It is not difficult to show that, for each input string $w$, $N$’s valid output is unique and matches $G(w)$. This npda $N$ therefore places $G$ into CFLSV.

To this end, we have already completed our proof of Proposition 8.3.

**Proposition 8.3** There is no almost 1-1 weak pseudorandom generator in 1-FLIN with the stretch factor $n + 1$ against $REG$.

Our proof of this proposition demands new terminology. For any two multi-valued partial functions $f$ and $g$ mapping $\Sigma^*$ to $\Gamma^*$, where $\Gamma$ could be another alphabet, $f$ is called a refinement of $g$ if, for any string
Proof of Proposition 8.3. Let $G$ be any almost 1-1 weak pseudorandom generator against REG stretching $n$-bit seeds to $(n + 1)$-bit long strings. Toward a contradiction, we assume that $G$ belongs to 1-FLIN. By Lemma 8.2, the range $S = \{G(x) \mid x \in \Sigma^*\}$ is weak REG-pseudorandom. If $S$ is regular, then REG is weak REG-pseudorandom; however, this contradicts the self-exclusion property. REG cannot be weak REG-pseudorandom. To obtain this contradiction, it remains to prove that $S$ is a regular language.

To make $G$ length-preserving, we slightly expand $G$ and define $\hat{G}(xb) = G(x)$ for each string $x$ and each bit $b$. This new function $\hat{G}$ is also in 1-FLIN. Let us consider its inverse function $\hat{G}^{-1}(y) = \{x \mid \hat{G}(x) = y\}$. Obviously, the inverse function $\hat{G}^{-1}$ belongs to 1-NLINMV (by guessing $x$ and then checking if $\hat{G}(x) = y$). Since every length-preserving function in 1-NLINMV has a refinement in 1-FLIN(partial) \cite{23}, there exists a refinement $g \in$ 1-FLIN(partial) of $\hat{G}^{-1}$, and we denote by $N$ a one-tape one-head linear-time deterministic Turing machine that computes $g$.

Claim 11. For every string $y$, $y \in S$ if and only if $N$ on the input $y$ terminates with an accepting state.

As a consequence of Claim 11, $S$ is in 1-DTIME($O(n)$), which equals REG \cite{15}. We thus obtain the regularity of $S$, as we have planned.

Finally, we want to prove Claim 11. Assume that $y$ is in $S$, meaning that $\hat{G}^{-1}(y) \neq \emptyset$. Since $g$ is a refinement of $\hat{G}^{-1}$, we have $g(y) \neq \emptyset$, which indicates that $N$ terminates with an accepting state. Conversely, assume that $N$ on $y$ terminates with an accepting state. In other words, $g(y) \neq \emptyset$. Since $g(y) \subseteq \hat{G}^{-1}(y)$, we obtain $\hat{G}^{-1}(y) \neq \emptyset$. This implies that $y = \hat{G}(x)$ for a certain string $x$. Since $S = \{\hat{G}(x) \mid x \in \Sigma^*\}$, it immediately follows that $y \in S$. Therefore, Claim 11 holds. 

9 Discussion and Open Problems

We have discussed two fundamental notions—immunity and pseudorandomness—in a framework of formal language theory. Our main target of this paper is the context-free language. Our initial study in this paper has revealed a quite rich structure that lies inside CFL. For instance, CFL contains complex languages, which are REG-immune, CFL-simple, and REG/n-pseudorandom. Moreover, its function class CFLSV$_k$ contains a pseudorandom generator against REG/n.

There remain several key questions that we have not answered throughout this paper. To direct future research, we generate a short list of those questions for the interested reader.

1. As shown in Section 8, $L \cap$ REG/n is REG-bi-immune. Determine whether CFL is also REG-bi-immune. More strongly, is CFL $-$ REG/n REG-bi-immune?
2. Prove or disprove that CFL(2) $-$ CFL/n is CFL-immune.
3. Is there any context-free language that is p-dense REG-immune? Is one of such languages located outside of REG/n?
4. The languages $L_{k\text{eq}}$, where $k \geq 3$, are shown to be CFL-simple; however, they cannot be REG-immune. Is there any REG-immune CFL-simple language?
5. We can define the notion of “CFL-primesimplicity” analogous to “CFL-simplicity.” Find natural CFL-primesimple languages.
6. Is DCFL REG/n-pseudorandom? An affirmative answer implies the REG/n-bi-primeimmunity of DCFL.
7. As noted in Section 8, the language $L_{3\text{eq}}$ belongs to CFL(2) and it is also CFL(1)-immune. In short, CFL(2) is CFL(1)-immune. Naturally, we can ask if, for each index $k \geq 2$, CFL($k + 1$) is CFL($k$)-immune.
8. Our pseudorandom generator $G$ given in Section 8 is almost 1-1. Find a natural 1-1 pseudorandom generator against REG/n.
9. Find a natural and easy-to-compute pseudorandom generator against CFL/n.

The answers to the above questions will surely enrich our knowledge on context-free languages.
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