COMPACT MANIFOLDS WITH POSITIVE $\Gamma_2$-CURVATURE

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Abstract. The Schouten tensor $A$ of a Riemannian manifold $(M, g)$ provides important scalar curvature invariants $\sigma_k$, that are the symmetric functions on the eigenvalues of $A$, where, in particular, $\sigma_1$ coincides with the standard scalar curvature $\text{Scal}(g)$. Our goal here is to study compact manifolds with positive $\Gamma_2$-curvature, i.e., when $\sigma_1(g) > 0$ and $\sigma_2(g) > 0$. In particular, we prove that a 3-connected non-string manifold $M$ admits a positive $\Gamma_2$-curvature metric if and only if it admits a positive scalar curvature metric. Also we show that any finitely presented group $\pi$ can always be realised as the fundamental group of a closed manifold of positive $\Gamma_2$-curvature and of arbitrary dimension greater than or equal to six.

1. Introduction and statement of the results

1.1. Motivation. Let $(M, g)$ be a Riemannian manifold of dimension $n \geq 3$. Recall that the Riemann curvature tensor $R$ of $(M, g)$ decomposes into

$$R = W + gA,$$

where $W$ is the trace-free part of $R$, that is, the Weyl tensor, and the product $gA$ is the Kulkarni-Nomizu product of the metric $g$ and the Schouten tensor $A$. The latter is defined by

$$A = \frac{1}{n-2} \left( \text{Ric} - \frac{\text{Scal}}{2(n-1)} g \right).$$

Here $\text{Ric}$ and $\text{Scal}$ are respectively the Ricci curvature tensor and the scalar curvature of $(M, g)$.

Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of the operator associated to $A$ via the metric $g$. For $1 \leq k \leq n$, the $\sigma_k$-curvature of $(M, g)$ is defined to be the scalar function

$$\sigma_k = \sigma_k(A) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

In this paper, we concentrate on the $\sigma_2$-curvature; more specifically, we study its positivity properties. First we note that

$$\sigma_1^2 = 2\sigma_2 + \|A\|^2.$$ 

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In particular, the positivity of $\sigma_2$ implies that $\sigma_2^2$ is never zero. This forces $\sigma_1$ to have a constant sign on $(M, g)$. In other words, the condition $\sigma_2 > 0$ has two possibilities: $\sigma_2 > 0$ and $\sigma_1 > 0$ or $\sigma_2 > 0$ and $\sigma_1 < 0$.

**Definition 1.1.** We say that $(M, g)$ has positive $\Gamma_2$-curvature if $\sigma_2 > 0$ and $\sigma_1 > 0$.

Here is the main question we would like to resolve:

**Question 1:** Which manifolds admit metrics with positive $\Gamma_2$-curvature?

1.2. **Main results on the existence of positive $\Gamma_2$-curvature.** It is important to emphasize that the positivity of $\Gamma_2$-curvature originally appeared as an ellipticity assumption that ensures that the $\sigma_2$-Yamabe equation is elliptic at any solution; see for instance [7, 16]. There are deep and interesting results concerning the $\sigma_2$-Yamabe problem, which inspired great interest from the mathematics community in the curvatures associated with the Shouten tensor.

Our first main result is an affirmative answer to Question 1 under certain topological restrictions on $M$. Let $p_k(M)$ stand for the $k$-th Pontryagin class of the manifold $M$. We recall that a smooth spin manifold $M$ is a string manifold if $\frac{1}{2}p_1(M) = 0$. Otherwise, we say that $M$ is not string.

**Theorem A.** Let $M$ be a compact 3-connected non-string manifold with dim $M = n \geq 9$ which admits a metric of positive scalar curvature. Then $M$ admits a Riemannian metric $g$ with positive $\Gamma_2$-curvature.

We notice that such a manifold is spin and determines a cobordism class $[M] \in \Omega_n^{\text{spin}}$. In particular, the index map $\alpha : \Omega_n^{\text{spin}} \to KO_n$ gives an element $\alpha([M]) \in KO_n$. Then, according to Gromov-Lawson [8] and Stolz [15], such a manifold $M$ admits a positive scalar curvature metric if and only if $\alpha([M]) = 0$. Thus, Theorem A provides an affirmative answer to the existence question as follows:

**Corollary A.** A compact 3-connected non-string manifold $M$ with dim $M = n \geq 9$ admits a metric of positive $\Gamma_2$-curvature if and only if $\alpha([M]) = 0$ in $KO_n$.

In the case when a manifold $M$ is string, we have the following result:

**Theorem B.** A compact 3-connected string manifold $M$ of dimension $n \geq 9$ that is string cobordant to a manifold of positive $\Gamma_2$-curvature admits a metric with positive $\Gamma_2$-curvature.

We recall that a string manifold $M$ determines a cobordism class $[M] \in \Omega_n^{\text{string}}$. We denote by $\phi_W : \Omega_*^{\text{string}} \to Z[[q]]$, the Witten genus; see [5,15]. We prove the following result which is analogous to [8, Corollary B].

**Corollary B.** Let $M$ be a 3-connected string manifold of dimension $n \geq 9$. Assume $\phi_W([M]) = 0$. Then some multiple $M\# \cdots \# M$ carries a metric of positive $\Gamma_2$-curvature.
It is well-known that some curvature conditions (such as positivity of Ricci curvature) impose severe restrictions on the fundamental group of a manifold. We show that the positivity of $\Gamma_2$-curvature does not require any restrictions. We prove the following:

**Theorem C.** Let $\pi$ be a finitely presented group. Then for every $n \geq 6$, there exists a compact $n$-manifold $M$ with positive $\Gamma_2$-curvature such that $\pi_1(M) = \pi$.

1.3. **Plan of the paper.** In Section 2, we describe close relationships between positivity of the Einstein tensor and other curvatures and positivity of the $\Gamma_2$-curvature, which provides relevant techniques and ideas to prove the above results. In Section 3, we examine the $\Gamma_2$-curvature on a Riemannian submersion. In particular, we show that a geometric $HP^2$-bundle carries a metric with positive $\Gamma_2$-curvature. Then, we prove a relevant surgery result in Section 4. We prove the main results in Section 5. In Section 6, we describe a general approach to the surgery developed by S. Hoelzel [9] and derive some applications. In particular, we show that any finitely presented group can be realized as a fundamental group of an $n$-manifold carrying a metric with positive $\Gamma_k$-curvature, provided that $2 \leq k < \frac{n+1}{2} - \frac{1}{2} \sqrt{n - \frac{1}{n-1}}$.

2. **Positivity of $\Gamma_2$ and related curvatures**

2.1. **Positive $\Gamma_2$-curvature and positive Einstein curvature.** Here, we will explain that the ellipticity condition for the $\sigma_2$-Yamabe equation is closely related to the positive definiteness of the Einstein tensor

$$S = \frac{\text{Scal}}{2} g - \text{Ric}.$$

We notice first that from an algebraic view point, the Einstein tensor is the first Newton transformation of Schouten tensor $A$ as follows:

**Lemma 2.1.** The Einstein tensor $S$ is the image of the Schouten tensor $A$ under the first Newton transformation:

$$S = (n-2) (\sigma_1(A) g - A).$$

**Proof.** Indeed, we have:

$$\sigma_1(A) g - A = \frac{\text{Scal}}{2(n-1)} g - \frac{1}{n-2} \left( \text{Ric} - \frac{\text{Scal}}{2(n-1)} g \right) = \frac{1}{n-2} \left( \frac{\text{Scal}}{2} g - \text{Ric} \right).$$

This proves it. $\square$

The next proposition shows that positivity of $\Gamma_2$-curvature implies positive definiteness of the Einstein tensor:

**Proposition 2.2.** Let $n \geq 3$. Assume that a Riemannian $n$-manifold $(M, g)$ has positive $\Gamma_2$-curvature. Then its Einstein tensor $S$ is positive definite. In particular, if $n = 3$, positive $\Gamma_2$-curvature implies positive sectional curvature.
Remark. The previous proposition is a special case of a general result, see Lemma 2.7 below and [2, Proposition 1.1]. Below, we provide a direct proof of Proposition 2.2. In particular, we obtain a lower bound for the Einstein tensor. This proof is inspired by the proof of a similar result in dimension 4 in [3].

Proof of Proposition 2.2. We show that if \( \sigma_1(A) > 0 \) and \( \sigma_2(A) > 0 \), then the Einstein tensor is positive definite. Assume that \( \sigma_1(A) > 0 \) and write \( A = A_1 + \frac{\sigma_1(A)}{n}g \) where \( A_1 = A - \frac{\sigma_1(A)}{n}g \) is a trace-free tensor. Then the first Newton transformation is given as

\[
 t_1(A) = \sigma_1(A)g - A = \frac{n-1}{n} \sigma_1 g - A_1.
\]

Now let \( X \) be a unit vector. Then we have:

\[
 t_1(A)(X, X) = \frac{n-1}{n} \sigma_1 - A_1(X, X) \geq \frac{n-1}{n} \sigma_1 - |A_1(X, X)|.
\]

Next, since \( A_1 \) is trace free, a simple Lagrange multiplier argument shows that

\[
 |A_1(X, X)| \leq \frac{\sqrt{n-1}}{\sqrt{n}} ||A_1||.
\]

Therefore, we obtain

\[
 t_1(A)(X, X) \geq \frac{n-1}{n} \sigma_1 - \frac{\sqrt{n-1}}{\sqrt{n}} ||A_1||
\]

\[
 \geq \frac{n-1}{n} \sigma_1 - 2 \left( \frac{||A_1||}{\sqrt{2} \sqrt{\sigma_1}} \frac{\sqrt{n-1}}{\sqrt{2} \sqrt{n}} \right)
\]

\[
 \geq \frac{n-1}{n} \sigma_1 - \frac{||A_1||^2}{2 \sigma_1} - \frac{\sigma(n-1)}{2n}
\]

\[
 = \frac{n-1}{2n} \sigma_1 - \frac{||A_1||^2}{2 \sigma_1}
\]

\[
 = \left( \frac{n-1}{n} \sigma_1^2 - ||A_1||^2 \right) \frac{1}{2 \sigma_1} = \frac{2 \sigma_2}{2 \sigma_1} = \frac{\sigma_2}{\sigma_1}.
\]

Thus, \( t_1(A) \geq \frac{\sigma_2}{\sigma_1} \). \( \square \)

As a direct consequence of the previous proof, we have:

**Proposition 2.3.** Let \((M, g)\) be a Riemannian manifold of dimension \( n \geq 3 \) and positive scalar curvature. Then the first eigenvalue of the Einstein tensor is bounded from below by \((n - 2) \frac{\sigma_2}{\sigma_1}\).

On the other hand, positivity of the Einstein tensor does not imply positivity of the \( \Gamma_2 \)-curvature. Moreover, there are examples of metrics with positive Einstein tensor that cannot be conformally deformed to a metric with positive \( \Gamma_2 \)-curvature.

Let \( n \geq 2 \), \((S^{n+2}, g_0)\) be the standard sphere of curvature +1, and \((M^n, g_1)\) be a compact space form of curvature −1. We define \( g \) to be the Riemannian product of \( g_0 \) and \( g_1 \) on \( S^{n+2} \times M^n \).
Proposition 2.4. Let \((S^{n+2} \times M^n, g)\) be as above with \(n \geq 2\). Then

1. \(\text{Scal}(g) > 0\) and the Einstein tensor \(S(g)\) is positive definite;
2. there is no metric \(\bar{g}\) in the conformal class \([g]\) with positive \(\Gamma_2\)-curvature.

Proof. It is not difficult to check that \(\text{Scal}(g) > 0\) and the Einstein tensor \(S(g)\) is positive definite. The metric \(g\) is conformally flat, and therefore, any metric in the conformal class of \(g\) is conformally flat as well. Then [6, Theorem 1.1] asserts that the existence of a conformally flat metric with positive \(\Gamma_2\)-curvature on a compact manifold of dimension \(2n + 2\) implies the vanishing of its \(n\)-th Betti number. However, the \(n\)-th Betti number of \(S^{n+2} \times M^n\) is the same one as for \(M^n\), which cannot be equal to zero. □

Thus we see that positivity of the \(\Gamma_2\)-curvature of a Riemannian metric is a rather strong condition since it implies positivity of the Einstein tensor in all dimensions at least three. In dimension 3, it guarantees positivity of sectional curvature, and in dimension 4, it implies positivity of Ricci curvature.

On the other hand, as we shall show in the sequel to this paper, there are very simple examples, such as the product of spheres \(S^3 \times S^q\), with \(q \geq 2\), which admit a Riemannian metric that has non-negative sectional curvature, positive Ricci curvature and positive definite Einstein tensor, but its \(\sigma_2\)-curvature is negative. We address here the following natural question:

Question 2. Which curvature conditions imply positive \(\Gamma_2\)-curvature?

Clearly, positivity of the Schouten tensor implies positivity of \(\Gamma_2\)-curvature, however, this condition is too strong. In fact, positivity of the Schouten tensor is equivalent to positivity of all \(\sigma_k\)-curvatures for \(1 \leq k \leq n\). In the next proposition, we give much weaker conditions that imply positivity of \(\Gamma_2\)-curvature:

Proposition 2.5. Let \(n = 2k + 2\) be even and \((M, g)\) be a Riemannian \(n\)-manifold such that the sum of the smallest \(k + 1\) eigenvalues of the Schouten tensor is positive. Then the metric \(g\) has positive \(\Gamma_2\)-curvature.

Proof. One can easily prove the previous result using the exterior product of double forms as in [11, 10]. Using the notations of these two references, we get

\[
\sigma_2 = \frac{1}{(n-2)!} g^{n-2} A^2 = \frac{1}{(n-2)!} \left( g^k A g^k A \right) = \frac{1}{(n-2)!} \langle *g^k A, g^k A \rangle.
\]

The last inner product can be easily seen as a sum whose terms are products of two factors. One factor is the sum of \((k + 1)\) eigenvalues of \(A\) and the other factor is the sum of the remaining \((k + 1)\) eigenvalues of \(A\). This shows that \(\sigma_2\) is positive. To be more explicit, let \(e_1, \ldots, e_n\) be an orthonormal basis that diagonalises \(A\). Then

\[
\langle *g^k A, g^k A \rangle = \sum_{1 \leq i_1 < \ldots < i_{k+1} \leq n} \left( \lambda_{i_1} + \ldots + \lambda_{i_{k+1}} \right) \left( \sigma_1 - \lambda_{i_1} - \ldots - \lambda_{i_{k+1}} \right).
\]

Clearly, \(\sigma_1 > 0\). This completes the proof. □
2.2. **Positive $\sigma_2$-curvature and positive second Gauss-Bonnet curvature.** We recall that the second Gauss-Bonnet curvature $h_4$ is defined by

\[
h_4 = \|R\|^2 - \|\text{Ric}\|^2 + \frac{1}{4}\text{Scal}^2.
\]

The positivity properties of $h_4$ were studied in [13, 1]. It turns out that the $\sigma_2$-curvature coincides (up to a constant factor) with $h_4$ for conformally-flat manifolds. Precisely, the $\sigma_2$-curvature is the non-Weyl part of $h_4$ as follows:

**Proposition 2.6 ([14]).** For an arbitrary Riemannian manifold of dimension $n \geq 4$, we have

\[
h_4 = |W|^2 + 2(n-2)(n-3)\sigma_2.
\]

In particular, positive (resp. nonnegative) $\sigma_2$-curvature implies $h_4 > 0$ (resp. $h_4 \geq 0$).

2.3. **Higher $\sigma_k$-curvatures and higher Gauss-Bonnet curvatures.** Even though this paper is mainly about the $\Gamma_2$-curvature (i.e., scalar and $\sigma_2$-curvature), we briefly discuss here the higher $\sigma_k$-curvatures and corresponding $\Gamma_k$-curvatures. In this subsection we shall use the formalism of double forms as in [11, 10]. Recall that the $\sigma_k$-curvature and the Newton transformation $t_k(A)$ of a Riemannian manifold $(M, g)$ are given by:

\[
\begin{align*}
\sigma_k &= \sigma_k(A) = *\frac{g^{n-k}A^k}{(n-k)!k!} = \frac{c^kA^k}{(k!)^2} & \text{for } 0 \leq k \leq n, \\
t_k &= t_k(A) = *\frac{g^{n-k-1}A^k}{(n-k-1)!k!} = \sigma_k(A)g - \frac{c^{k-1}A^k}{(k-1)!k!} & \text{for } 1 \leq k \leq n.
\end{align*}
\]

**Remark.** Here we considered the Schouten tensor $A$ as a $(1,1)$-double form.

The following algebraic fact is a classical result, see [2, Proposition 1.1]:

**Lemma 2.7.** If for all $r$ such that $1 \leq r \leq k+1 \leq n$ we have $\sigma_r(A) > 0$ then the tensor $t_k(A)$ is positive definite.

We are now going to apply Lemma 2.7 to the higher Gauss-Bonnet curvatures and Einstein-Lovelock tensors. First, recall that for $2 \leq 2k \leq n$, the $2k$-th Gauss-Bonnet curvature $h_{2k}$ of $(M, g)$ is the function defined on $M$ by

\[
h_{2k} = \frac{1}{(n-2k)!} * \left(g^{n-2k}R^k\right).
\]

The $2k$-th Einstein-Lovelock tensor of $(M, g)$, denoted $T_{2k}$, is defined by

\[
T_{2k} = *\frac{1}{(n-2k-1)!}g^{n-2k-1}R^k.
\]

If $2k = n$, we set $T_n = 0$. For $k = 0$, we set $h_0 = 1$ and $T_0 = g$. 
Suppose now that \((M, g)\) is a conformally flat manifold. Then its Riemann curvature tensor is determined by the Schouten tensor \(A\) as follows:

\[ R = gA. \]

Consequently, for \(n - 2k - 1 \geq 0\), we have:

\[ t_k(A) = \frac{g^{n-k} A^k}{(n-k)!} = \frac{g^{n-2k} (gA)^k}{(n-k)!} = \frac{g^{n-2k} (R)^k}{(n-k)!} = \frac{(n-2k)!}{k! (n-k-1)!} T_{2k}. \]

Similarly, we have for \(n \geq 2k\):

\[ \sigma_k(A) = \frac{g^{n-k} A^k}{(n-k)!} = \frac{g^{n-2k} (gA)^k}{(n-k)!} = \frac{g^{n-2k} (R)^k}{(n-k)!} = \frac{(n-2k)!}{(n-k)!} h_{2k}. \]

We have therefore proved the following result:

**Proposition 2.8.** For a conformally flat \(n\)-manifold \((M, g)\), the following are true

1. For \(2 \leq 2k \leq n\), the \(\sigma_k\)-curvature is positive if and only if the \(h_{2k}\)-Gauss-Bonnet curvature is positive.
2. If the Gauss-Bonnet curvatures \(h_2, h_4, ..., h_{2k+2}\) of \((M, g)\) are all positive then the Einstein-Lovelock tensors \(T_2, ..., T_{2k}\) of \((M, g)\) are all positive definite.

### 3. Riemannian submersions and \(\sigma_2\)-curvature

#### 3.1. General observations

Let \((M, g)\) be a total space of a Riemannian submersion \(p : M \to B\). We denote by \(\hat{g}\) the restriction of \(g\) to the fiber \(F = p^{-1}(x), \ x \in B\). We denote by \(\text{Scal}(\hat{g})\) and \(\sigma_2(\hat{g})\) the scalar curvature and \(\sigma_2\)-curvature of the metric \(\hat{g}\), respectively.

For a Riemannian submersion \(p : M \to B\) as above, there is a canonical variation \(g_t\) of the original metric \(g\), which is a fiberwise scaling by \(t^2\). Then, if the fiber metrics \(\hat{g}\) satisfies some bounds, this construction delivers a metric on a total space with positive (negative) curvature.

**Theorem 3.1 (See [14]).** Let \(p : M \to B\) be a Riemannian submersion where the total space \((M, g)\) is a compact manifold with \(\dim M = n\) and fibers \(F\), where \(\dim F = p\). Let \(\hat{g}\) be the induced metric on fiber \(F\). Assume that the inequality

\[ 8(n-1)(p-1)(p-2)^2 \sigma_2(\hat{g}) > (n-p) \text{Scal}^2(\hat{g}) \]

holds for every fiber \((F, \hat{g})\). Then there exists \(t_0 > 0\) such that the canonical variation metric \(g_t\) on \(M\) has \(\sigma_2(g_t) > 0\) (resp. \(\sigma_2(g_t) < 0\)) for all \(0 < t \leq t_0\).

Furthermore, if \(\text{Scal}(\hat{g}) > 0\) (resp. \(\text{Scal}(\hat{g}) < 0\)), then \(\text{Scal}(g_t) > 0\) (resp. \(\text{Scal}(g_t) < 0\)) for all \(0 < t \leq t_0\).
Proof. We modify the proof of [13, Theorem B] a bit. Using O’Neill’s formulas for Riemannian submersions, we get an estimate of $\text{Scal}^2(g_t)$ and the norm of the Ricci curvature $||\text{Ric}(g_t)||_{g_t}$ of the canonical variation metric $g_t$ as follows:

\[
||\text{Ric}(g_t)||_{g_t}^2 = \frac{1}{t^4}||\text{Ric}(\hat{g})||^2 + O(\frac{1}{t^2}), \quad \text{and} \quad \text{Scal}^2(g_t) = \frac{1}{t^4}\text{Scal}^2(\hat{g}) + O(\frac{1}{t^2});
\]

see [13] for details. Consequently, the $\sigma_2$-curvature of the metric $g_t$ is given by

\[
2(n - 2)^2\sigma_2(g_t) = -||\text{Ric}(g_t)||_{g_t}^2 + \frac{n}{4(n-1)}\text{Scal}^2(g_t)
\]

\[
= \frac{1}{t^4}\left\{ (2(p - 2)^2\sigma_2(\hat{g}) - \frac{n-p}{4(n-1)p(n-1)}\text{Scal}^2(\hat{g}) \right\} + O(\frac{1}{t^2}).
\]

To prove the second part of the theorem, it is enough to recall that $\text{Scal}(g_t) = \frac{1}{t^2}\text{Scal}(\hat{g}) + O(1)$.

This concludes the proof. \qed

If the fiber $(F, \hat{g})$ is Einstein with $\dim F = p$, then its $\sigma_2$-curvature is determined by the scalar curvature as follows (see [14, Proposition 3.2]):

\[
\sigma_2(\hat{g}) = \frac{1}{8p(p - 1)}\text{Scal}^2(\hat{g}).
\]

Consequently, the $\sigma_2$-curvature of the metric $g_t$ of the total space is given by

\[
2(n - 2)^2\sigma_2(g_t) = \frac{\text{Scal}^2(\hat{g})}{t^4} \left( \frac{n(p-4)+4}{p(n-1)} \right) + O\left( \frac{1}{t^2} \right).
\]

Therefore, the first inequality in (8) holds for such a metric $\hat{g}$, provided that the dimension $\dim F = p \geq 4$ and $\hat{g}$ is not Ricci flat. Thus, we have proved the following result:

**Corollary 3.2.** Let $\hat{g}$ be the induced metric on fiber $F$. Assume that the fiber metrics $\hat{g}$ are non Ricci-flat Einstein metrics. Then there exists $t_0 > 0$ such that the canonical variation metric $g_t$ on $M$ has $\sigma_2(g_t) > 0$ for all $0 < t \leq t_0$.

Furthermore, the scalar curvature $\text{Scal}(g_t)$ of the metric $g_t$ has the same sign as the scalar curvature of the fiber metrics $\hat{g}$ for all $0 < t \leq t_0$.

Let $\mathbf{HP}^2$ be a quaternionic projective plane equipped with a standard metric $\hat{g}_0$. It is well-known that the metric $\hat{g}_0$ is Einstein with positive sectional curvature and that it has isometry group $PSp(3)$.

Then a smooth fiber bundle $\pi : M \to B$ is called a **geometric $\mathbf{HP}^2$-bundle** if its fiber is $\mathbf{HP}^2$ and the structure group is $PSp(3)$. Given any Riemannian metric on the base $B$, there is a canonical metric $g_0$ on the total space $M$ inducing the metric $\hat{g}_0$ along every fiber, and the projection $\pi : M \to B$ becomes a Riemannian submersion. Thus, we have the following:
Corollary 3.3. Let \( \pi : M \to B \) be a geometric \( \text{HP}^2 \)-bundle. Then \( M \) admits a metric with positive \( \Gamma^2 \)-curvature.

3.2. Negative \( \sigma_2 \)-curvature for low-dimensional fibers. If the fibers have lower dimensions, the above machinery helps to construct metrics with negative \( \sigma_2 \)-curvature as follows.

Corollary 3.4. Let \( p : M \to B \) be a Riemannian submersion where the total space \( (M, g) \) is a compact manifold with \( \dim M = n \) and fibers \( F \), where \( \dim F = 2 \). Let \( \hat{g} \) be the induced metric on fiber \( F \). Assume that the Gaussian curvature of the fiber metrics \( \hat{g} \) does not vanish and \( n \geq 3 \). Then there exists \( t_0 > 0 \) such that the canonical variation metric \( g_t \) on \( M \) has \( \sigma_2(g_t) < 0 \) for all \( 0 < t \leq t_0 \).

Furthermore, the scalar curvature \( \text{Scal}(g_t) \) of the metric \( g_t \) has the same sign as the Gaussian curvature of the fiber metrics \( \hat{g} \) for all \( 0 < t \leq t_0 \).

In the case of three-dimensional fibres we have:

Corollary 3.5. Let \( p : M \to B \) be a Riemannian submersion where the total space \( (M, g) \) is a compact manifold with \( \dim M = n \geq 5 \) and fibers \( F \), where \( \dim F = 3 \). Let \( \hat{g} \) be the induced metric on fiber \( F \). Assume that the fiber metrics \( \hat{g} \) are non Ricci-flat Einstein metrics. Then there exists \( t_0 > 0 \) such that the canonical variation metric \( g_t \) on \( M \) has \( \sigma_2(g_t) < 0 \) for all \( 0 < t \leq t_0 \).

Furthermore, the scalar curvature \( \text{Scal}(g_t) \) of the metric \( g_t \) has the same sign as the scalar curvature of the fiber metrics \( \hat{g} \) for all \( 0 < t \leq t_0 \).

Next, we specify the previous results to products with the standard spheres. Let \( \hat{g}(r) \) be a standard round metric on the sphere \( S^p \) of radius \( r \).

Corollary 3.6. Let \( (M, g(r)) = (S^p, \hat{g}(r)) \times (B, g_B) \), where \( (B, g_B) \) is an arbitrary compact Riemannian manifold.

- If \( p = 2 \) and \( \dim B \geq 1 \), then \( \text{Scal}(g(r)) > 0 \) and \( \sigma_2(g(r)) < 0 \) for all \( r \) sufficiently small.
- If \( p = 3 \) and \( \dim B \geq 2 \), then \( \text{Scal}(g(r)) > 0 \) and \( \sigma_2(g(r)) < 0 \) for all \( r \) sufficiently small.
- If \( p \geq 4 \), then \( \text{Scal}(g(r)) > 0 \) and \( \sigma_2(g(r)) > 0 \) for all \( r \) sufficiently small.

Remark 3.1. We notice that the Riemannian product
\[
(M, g(r)) = (S^p(r), \hat{g}(r)) \times (S^q, ds^2)
\]
of the standard spheres (where \( \hat{g}(r) \) is a round metric of radius \( r \) and \( q \geq 2 \)) is such that for small enough \( r \), the sectional curvature of \( g(r) \) is nonnegative, Ricci curvature, the Einstein tensor and the \( h_4 \)-curvature are all positive, but its \( \sigma_2 \)-curvature is negative.
Remark 3.2. Recall that any finitely presented group can be realised as the fundamental
group of a compact manifold of an arbitrary dimension $n \geq 4$. Consequently, the above
examples show that any finitely presented group can be realized as the fundamental
group of a compact $n$-manifold of positive scalar curvature and, at the same time, of
negative $\sigma_2$-curvature for any arbitrary $n \geq 6$.

The same is true for compact $n$-manifolds of positive $\Gamma_2$-curvature for $n \geq 8$. In Section
3.1 we will show that this is still true for $n \geq 6$.

4. A surgery theorem for metrics with positive $\Gamma_2$-curvature

Let $X$ be a closed manifold, $\dim X = n$, and $S^p \subset X$ be an embedded sphere in $X$
with trivial normal bundle. We assume that it is embedded together with its tubular
neighbourhood $S^p \times D^q \subset X$. Here, $p + q = n$. Then we define $X'$ to be the manifold
resulting from the surgery along the sphere $S^p$:

$$X' = (X \setminus (S^p \times D^q)) \cup_{S^p \times S^q - 1} (D^{p+1} \times S^{q-1}).$$

The codimension of the sphere $S^p \subset X$ is called a codimension of the surgery. In the
above terms, the codimension of the above surgery is $q$.

Theorem 4.1. Let $X$ be a compact manifold with $\dim X \geq 5$ and $X'$ be a manifold
obtained from $X$ by a surgery of codimension at least 5. Assume that $X$ has a
Riemannian metric $g$ with positive $\Gamma_2$-curvature. Then there exists a metric $g'$ on $X'$
with positive $\Gamma_2$-curvature.

Proof. We adapt the proof of the similar result about positive Gauss-Bonnet curvature
in [13] and we use similar notations.

Let $(X, g)$ be a compact $n$-dimensional Riemannian manifold with positive $\Gamma_2$-
curvature, i.e., $\text{Scal}(g) > 0$ and $\sigma_2(g) > 0$. Let $S^p \subset X$ be an embedded sphere of
codimension $q$ with a trivial normal bundle $N$, which we identify with the product
$N \cong S^p \times \mathbb{R}^q$.

Let $\exp : N \to X$ be the exponential map. For each $r > 0$, we denote by $D^q(r) \subset \mathbb{R}^q$
the standard closed Euclidian disk of radius $r$. Then there exists $r_0 > 0$ such that
the restriction $\exp : S^p \times D^q(r_0) \to X$ of the exponential map is an embedding. We
define $\exp^* g$ to be the pull-back metric to the total space $S^p \times D^q(r_0)$ of the normal
disk bundle

$$S^p \times D^q(r_0) \subset S^p \times \mathbb{R}^q.$$

There is another natural metric $g^\nabla$ on the normal bundle, which is defined by the normal
connection $\nabla$. The metric $g^\nabla$ is compatible with the normal connection and is such
that the natural projection $\pi : S^p \times D^q(r) \to S^p$ is a Riemannian submersion. We shall
also denote by $g^\nabla$ its restriction to the subbundles

$$S^p \times D^q(r) \to S^p \quad \text{and} \quad \partial(S^p \times D^q(r)) = S^p \times S^{q-1}(r) \to S^p.$$
It turns out that these two metrics are tangent up to the order two in the directions tangent to $D^q$. More precisely, for tangent vectors $u_1, u_2 \in T_{(x,v)}(S^p \times D^q(r))$, we have
\begin{equation}
\exp^* g(u_1, u_2) = g^\nabla (u_1, u_2) + \Pi_n(\pi_* u_1, \pi_* u_2) r + O(r^2),
\end{equation}
see [13]. Here $\Pi_n$ denotes the second fundamental form of the submanifold $S^p \subset X$ at the point $x$ in the direction of the unit normal vector $n = \frac{u}{||u||}$. Notice that if $u_1$ or $u_2$ is tangent to $D^q$, then the second term in the previous expansion vanishes as the map $\pi_*$ sends $D^q$ to zero.

Next, we define a hypersurface $M$ in the product $S^p \times D^q(r_0) \times \mathbb{R}$ endowed with the product metric $\exp^* g \times \mathbb{R}$ by the relation
\[ M = \{ (x, v, t) \in (S^p \times D^q(r_0)) \times \mathbb{R} \mid (||v||, t) \in \gamma \}, \]
where $\gamma$ is a curve in the $(r, t)$-plane that is curved as in [13]. In the following, $k$ denotes the curvature of $\gamma$ and $\theta$ the angle between the normal vector to the hypersurface $M$ and the $t$-axis at the corresponding point.

Now we are going to evaluate the $\sigma_2$-curvature of the induced metric on the hypersurface $M$. First, recall the following two formulas for the curvatures of $M$, see [13]:
\begin{align*}
\|\text{Ric}^M\|^2 &= \|\text{Ric}^{S^p \times D^q}\|^2 + \frac{(q-1)(q-2)^2}{r^4} \sin^4 \theta + q(q-1) \frac{k^2}{r^2} \sin^2 \theta \\
&\quad - \frac{(q-1)(q-2)^2k}{r^3} \sin^2 \theta + O(\frac{1}{r^2}) \sin \theta,
\end{align*}
\begin{align*}
\|\text{scal}^M\|^2 &= \|\text{scal}^{S^p \times D^q}\|^2 + \frac{(q-1)^2(q-2)^2}{r^4} \sin^4 \theta + 4(q-1) \frac{k^2}{r^2} \sin^2 \theta \\
&\quad - 2\frac{(q-1)^2(q-2)^2k}{r^3} \sin^3 \theta + O(\frac{1}{r^2}) \sin \theta.
\end{align*}
Here, we assumed that the curvature $k$ of the plane curve $\gamma$ satisfies $k = O(\frac{1}{r})$.

Consequently, one can easily evaluate the $\sigma_2$-curvature of the induced metric on $M$ as follows:
\begin{align*}
2(n - 2)^2 \sigma_2^M &= 2(n - 2)^2 \sigma_2^{S^p \times D^q} + \frac{(q-1)(q-2)^2[n(q-5)+4]}{4(n-1)r^4} \sin^4 \theta \\
&\quad + \frac{(q-1)(q-2)[(n-2)(q-3)-2]}{2(n-1)r^3} k \sin^3 \theta - \frac{(q-1)(q-2)^2k^2}{(n-1)r^2} \sin^2 \theta + O(\frac{1}{r^2}) \sin \theta.
\end{align*}

Next, we shall show that it is possible to choose the curve $\gamma$ so that the metric induced on $M$ has $\sigma_2^M > 0$ and $\text{Scal}^M > 0$ everywhere on $M$.

Formula [12] shows that for $\theta = 0$, we have $\sigma_2^M = \sigma_2^{S^p \times D^q}$ is positive and there exists an angle $\theta_0 > 0$, such that for all $0 < \theta \leq \theta_0$ we still have $\sigma_2^M > 0$.

We continue then with a straight line ($k = 0$) of angle $\theta_0$, say $\gamma_1$, until the term
\[ \frac{(q-1)(q-2)^2[n(q-5)+4]}{4(n-1)r^4} \sin^4 \theta_0 \]
is strongly dominating. On the other hand, when \( \theta = \pi/2 \), then \( k = 0 \) and \( r = \epsilon \), and we have

\[
2(n-2)^2\sigma^M_2 = \frac{(q-1)(q-2)^2[n(q-5)+4]}{4(n-1)^2r^4} + O\left(\frac{1}{r^4}\right),
\]

which is positive as \( \epsilon \) is small enough and \( q \geq 5 \). We now choose \( r_1 > 0 \) and small.

Consider the point \((r_1, t_1)\) \( \in \gamma_1 \). We bend the straight line \( \gamma_1 \), beginning at this point, with a curvature \( k(s) \) of the following form:

\[
k(s) = 2A \sin \theta_0 \frac{k}{r_1}
\]

\[r_1/2 \]
\[s\]

**Figure 1.** The curvature function

The variable \( s \) denotes the arc length along the curve.

Since \( q \geq 5 \), the formula (12) shows that

\[
2(n-2)^2\sigma^M_2 - 2(n-2)^2\sigma^S_{p=D^n}
\]

\[
\geq \frac{36\sin^4 \theta}{(n-1)r^4} + \frac{12(n-3)\sin^3 \theta}{(n-1)r^3k} - \frac{4(n-5)\sin^2 \theta}{(n-1)r^2k^2} + O\left(\frac{1}{r^3}\right) \sin \theta
\]

\[
= \frac{18\sin^4 \theta}{(n-1)r^4} + \frac{18\sin^4 \theta}{(n-1)r^4} + \frac{12(n-3)\sin^3 \theta}{(n-1)r^3k} - \frac{4(n-5)\sin^2 \theta}{(n-1)r^2k^2} + O\left(\frac{1}{r^3}\right) \sin \theta
\]

\[
= \frac{18\sin^4 \theta}{(n-1)r^4} + O\left(\frac{1}{r^4}\right) \sin \theta + \frac{2\sin^2 \theta}{(n-1)r^4}\left\{-2(n-5)(kr)^2 + 6(n-3)\sin \theta \sin k \epsilon - 9 \sin^2 \theta\right\}.
\]

The expression between brackets in the previous last term is quadratic in \( kr \) and can be easily seen to be positive for

\[
0 \leq kr < \left(\frac{n-3+\sqrt{(n-3)^2+2(n-5)}}{n-5}\right)\frac{3\sin \theta}{2} =: 2A \sin \theta.
\]

Note that if \( n = 5 \) then that expression is always positive. Consequently the hypersurface \( M \) will continue to have \( \sigma^M_2 > 0 \), if the curvature \( k \) is chosen such that \( k < \frac{2A \sin \theta_0}{r_1} \); see Fig. 1 above. After this first bending, we have \( \Delta r \leq \Delta s = r_1/2 \), and \( r \geq r_1 - \Delta r \geq r_1 - r_1/2 > 0 \). Consequently the curve will not cross the \( t \)-axis.

On the other hand, \( \Delta \theta = \int kds \approx A \sin \theta_0 \) is independent of \( r_1 \). Clearly, by scaling down the curvature \( k \), we can produce any \( \Delta \theta \) such that \( 0 < \Delta \theta \leq A \sin \theta_0 \).

Our curve now continues with a new straight line \( \gamma_2 \) with angle \( \theta_1 = \theta_0 + \Delta \theta \). By repeating this process finitely many times, we can achieve a total bend of \( \frac{\pi}{2} \).
Let \( g_\epsilon \) denote the induced metric from \( \exp^* g \) on \( \partial(S^p \times D^q(\epsilon)) = S^p \times S^{q-1}(\epsilon) \). We have seen that the new metric defined on \( M \) has positive \( \Gamma_2 \)-curvature (it has positive scalar curvature which could be easily checked as in [8]). Recall that the new metric on \( M \) coincides with the old metric when \( t = 0 \), and finishes with the product metric \( g_\epsilon \times \mathbb{R} \).

In the following we shall deform the product metric \( g_\epsilon \times dt^2 \) on \( S^p \times S^{q-1}(\epsilon) \times \mathbb{R} \) to the standard product metric through metrics with \( \Gamma_2 \)-curvature. This will be done in two steps.

**Step 1:** We deform the metric \( g_\epsilon \) on \( S^p \times S^{q-1}(\epsilon) \) to the standard product metric \( S^p(1) \times S^{q-1}(\epsilon) \) through metrics with \( \Gamma_2 \)-curvature as follows:

First, the metric \( g_\epsilon \) can be homotoped through metrics with positive \( \sigma_2 \)-curvature and positive scalar curvature to the normal metric \( g^\nabla \) since their \( \sigma_2 \)-curvature and scalar curvatures are respectively high and close enough, see formulas (13) and (10).

Then, for \( \epsilon \) small enough, we can deform the normal metric \( g^\nabla \) on \( S^p \times S^{q-1}(\epsilon) \) through Riemannian submersions to a new metric where \( S^p \) is the standard sphere \( S^p(1) \), keeping the horizontal distribution fixed. This deformation keeps \( \sigma_2 > 0 \) and \( \text{Scal} > 0 \) as long as \( \epsilon \) is small enough, see formula (10).

Finally, we deform the horizontal distribution to the standard one and again by the same formula (10) this can be done keeping \( \sigma_2 > 0 \) and \( \text{Scal} > 0 \).

**Step 2:** Let us denote the previous family of deformations on \( S^p \times S^{q-1}(\epsilon) \) by \( ds^2_t, 0 \leq t \leq 1 \). They all have positive \( \Gamma_2 \)-curvature. Here \( ds_0 = g_\epsilon \) and \( ds_1 \) is the standard product metric of round spheres.

It is clear that the metric \( ds^2_{t/a} + dt^2, 0 \leq t \leq a \), glues together the two metrics \( ds_0 \times \mathbb{R} \) and \( ds_1 \times \mathbb{R} \). Furthermore, there exists \( a_0 > 0 \) such that for all \( a \geq a_0 \) the metric \( ds^2_{t/a} + dt^2 \) on the product

\[ S^m \times S^{q-1}(\epsilon) \times [0, a] \]

has positive \( \Gamma_2 \)-curvature. In fact, via a change of variable, the previous statement is equivalent to the existence of \( \lambda_0 > 0 \) such that the metric \( \lambda^2 ds^2_t + dt^2 \) has positive \( \Gamma_2 \)-curvature for all \( 0 < \lambda \leq \lambda_0 \). To complete the proof one can check that our family of deformation metrics \( ds^2_t, 0 \leq t \leq 1 \), on \( S^p \times S^{q-1}(\epsilon) \) satisfies the assumptions of Theorem 3.1 when \( \epsilon \) is sufficiently small. This completes the proof of the surgery theorem. \( \square \)

### 5. Proofs of main theorems

5.1. **Fundamental groups of manifolds with positive \( \Gamma_2 \)-curvature.** We already mentioned that Corollary 3.6 implies that there are no restrictions on the fundamental group of a compact manifold of positive scalar curvature and positive \( \sigma_2 \)-curvature in dimensions at least eight. We use surgery Theorem 4.1 to prove that the same holds in dimensions 6 and 7:
Theorem C. Let $\pi$ be a finitely presented group. Then for every $n \geq 6$, there exists a compact $n$-manifold $M$ with positive $\Gamma_2$-curvature such that $\pi_1(M) = \pi$.

Remark. We emphasize that the previous result is no longer true in dimensions 4 and 3. In dimension 4, the positivity of the $\Gamma_2$-curvature implies the positivity of the Ricci curvature; consequently, the fundamental group must be finite. The same holds for the fundamental group of a 3-dimensional compact manifold of positive $\Gamma_2$-curvature as in this case, the sectional curvature must be positive. However, it remains an open question whether there exist any restriction on the fundamental group of 5-dimensional compact manifold of positive $\Gamma_2$-curvature.

Proof of Theorem C. Let $n \geq 6$ and $\pi$ be a group which has a presentation consisting of $k$ generators $x_1, x_2, ..., x_k$ and $\ell$ relations $r_1, r_2, ..., r_\ell$. Let the manifold $S^1 \times S^{n-1}$ be given a standard product metric which has positive $\Gamma_2$-curvature. Since $\pi_1(S^1 \times S^{n-1}) \cong \mathbb{Z}$, the Van-Kampen theorem implies that the fundamental group of the connected sum $N := \#k(S^1 \times S^{n-1})$ is a free group on $k$ generators, which we denote by $x_1, x_2, ..., x_k$. By the surgery Theorem 4.1, $N$ admits a metric with positive $\Gamma_2$-curvature.

We now perform surgery $\ell$-times on the manifold $N$ such that each surgery is of codimension $n - 1 \geq 5$, killing in succession the elements $r_1, r_2, ..., r_\ell$. Again, according to Theorem 4.1, the resulting manifold $M$ has fundamental group $\pi_1(M) \cong \pi$ and admits a metric with positive $\Gamma_2$-curvature, as desired. $\square$

5.2. Existence of metrics with positive $\Gamma_2$-curvature. Let $M$ be a 3-connected manifold. In particular, $M$ has a canonical spin-structure. We use a standard notation $p_i(M)$ for the Pontryagin classes of $M$. There are two cases to consider here:

(1) The manifold $M$ is not string, i.e., $\frac{1}{2}p_1(M) \neq 0$.
(2) The manifold $M$ is string, i.e., $\frac{1}{2}p_1(M) = 0$.

In case (1), the manifold $M$ is spin and it determines a cobordims class $[M] \in \Omega^\text{spin}_n$, where $\Omega^\text{spin}_n$ is the spin-cobordims group. We recall also that there is the homomorphism $\alpha : \Omega^\text{spin}_n \to KO_n$ evaluating the index of the Dirac operator. It is well-known that a simply connected spin-manifold $M$ of dimension at least five admits a metric with positive scalar curvature if and only if $\alpha([M]) = 0$ in $KO_n$.

Proof of Theorem A. Let $M$ be 3-connected, non-string (i.e. case (1) above) such that $M$ admits a metric of positive scalar curvature. In particular, this means that $\alpha([M]) = 0$. Then, according to [15, Theorem B], there exists a spin cobordism between $M$ and $M'$, where $M'$ is a total space of a geometric $\mathbb{HP}^2$-bundle and has a metric with positive $\Gamma_2$-curvature by Corollary 3.3.

We recall the following result [1, Proposition 3.7]:

\[ N := \#k(S^1 \times S^{n-1}) \]
Lemma 5.1. Let $M$ be a 3-connected, non-string manifold with $\dim M \geq 9$. Assume $M$ is spin cobordant to a manifold $M'$. Then $M$ can be obtained from $M'$ by surgeries of codimension at least five.

Thus, we can use the surgery Theorem 4.1 to “push” a metric with positive $\Gamma_2$-curvature from $M'$ to $M$. □

Proof of Theorem B. The proof is completely analogous to the arguments given to prove [11, Theorem B]. □

If $M$ is string cobordant to zero, then the conclusion of the theorem holds for $M$. It is known that $\Omega_n^{\text{string}} = 0$ for $n = 11$ or $n = 13$; therefore any compact 3-connected string manifold of dimension 11 or 13 always has a metric with positive scalar curvature and positive $\Gamma_2$-curvature.

6. A general surgery theorem and applications

6.1. A general surgery theorem. After we finished writing the proof of the surgery Theorem 4.1, Hoelzel sent us his paper [9] where he proved an interesting general surgery theorem, which can be used to easily prove our surgery result.

His approach is as follows. Let $C_1(\mathbb{R}^n)$ denote the vector space of curvature structures on the Euclidean space $\mathbb{R}^n$ that satisfy the first Bianchi identity. Recall that the orthogonal group $O(n)$ acts in a natural way on $C_1(\mathbb{R}^n)$ and that the latter space is endowed with a canonical Euclidean inner product.

We shall say that a subset $C$ of $C_1(\mathbb{R}^n)$ is a curvature condition if $C$ is an open convex $O(n)$-invariant cone in $C_1(\mathbb{R}^n)$.

We say that a Riemannian manifold $(M, g)$ satisfies the above (pointwise) curvature condition $C$ if for every $p \in M$ and for any linear isometry $i : \mathbb{R}^n \rightarrow T_p M$, the pull back by $i$ of the Riemann curvature tensor $R$ of $(M, g)$ belongs to the subset $C$.

Theorem 6.1. (Hoelzel, [9, Theorem A]) Let $C \subset C_1(\mathbb{R}^n)$ be a curvature condition that is satisfied by the standard Riemannian product metric on $S^{c-1} \times \mathbb{R}^{n-c+1}$ for some $c$, $3 \leq c \leq n$.

If a Riemannian manifold $(M^n, g)$ satisfies the curvature condition $C$, then so does any manifold obtained from $M^n$ by surgeries of codimension at least $c$.

Remark. It would be great to see Theorem 6.1 generalized for families of metrics in the spirit of papers [17, 18]. This would allow us to understand much more about the topology of the space of “metrics satisfying the curvature condition $C$.”

Recall that a curvature structure $R \in C_1(\mathbb{R}^n)$ decomposes into $R = W + gA$ where $g$ denotes the Euclidean metric on $\mathbb{R}^n$, $W$ is trace free, $A$ is a symmetric bilinear form called the Schouten tensor, and the product $gA$ is the Kulkarni-Nomizu product. We
shall say that the curvature structure $R$ has positive $\Gamma_r$-curvature if the $k$-th elementary symmetric function $\sigma_k(A)$ in the eigenvalues of $A$ is positive for all $k$ with $1 \leq k \leq r$.

Let $C(\Gamma_2)$ be the subset of $C_1(\mathbb{R}^n)$ consisting of curvature structures with positive $\Gamma_2$-curvature. It is not difficult to check that $C(\Gamma_2)$ is a curvature condition in the above sense. Furthermore, one can check without difficulties that the $\sigma_2$-curvature of the standard product $S^{c-1} \times \mathbb{R}^{n-c+1}$ is equal to

$$\sigma_2 = \frac{(c - 2)^2(c - 1)}{8((n - 2)^2(n - 1))(n(c - 5) + 4)}.$$

This value is clearly positive for $c \geq 5$. The previous theorem asserts the stability of positive $\Gamma_2$-curvature under surgeries of codimension at least five. Therefore, we recover the surgery result of Theorem 4.1.

6.2. Some applications. Next, we are going to show that the previous theorem applies to Gauss-Bonnet curvatures $h_{2r}$. We recall that $h_2$ coincides with one half of the scalar curvature.

Corollary 6.2. Let $M$ be a compact manifold with $\dim M \geq 5$ and $M'$ be a manifold obtained from $M$ by a surgery of codimension at least 5. If the manifold $M$ admits a Riemannian metric with $h_4 > 0$ and $h_2 > 0$, then so does the manifold $M'$.

Proof. Let $C(h_4^+)$ be the subset of the space $C_1(\mathbb{R}^n)$ consisting of curvature structures with $h_4 > 0$ and $h_2 > 0$. We notice that the $h_{2r}$-curvature of the standard Riemannian product $S^{c-1} \times \mathbb{R}^{n-c+1}$ is equal to the $h_{2r}$-curvature of the standard unit sphere $S^{c-1}$, which is equal to $\frac{(c-1)!}{2^{(c-1)-2r}}/(c-1-2r)$ for $c-1 \geq 2r$; see [13, Examples 2.1 and 2.2]. Consequently, the $h_{2r}$-curvatures of the standard Riemannian product $S^{c-1} \times \mathbb{R}^{n-c+1}$ are positive for $1 \leq r \leq k$, provided $c - 1 - 2k \geq 0$.

Let us emphasize here that the set $C(h_4^+)$ is not a convex subset of $C_1(\mathbb{R}^n)$, therefore Theorem 6.1 does not apply directly. We use instead a more general version, namely [9, Theorem B], where the convexity condition is replaced by an inner cone condition with respect to the standard Riemannian curvature structure, say $S_4$, of the standard product $S^4 \times \mathbb{R}^{n-4}$, see [9].

We recall that a non-empty open and $O(n)$-invariant subset $C$ of $C_1(\mathbb{R}^n)$ is said to satisfy an inner cone condition with respect to $S$ if for any $R$ in $C$, there exists a positive real number $\rho(R)$, that depends continuously on $R$, such that

$$R + C_\rho := \{R + T : T \in C_\rho\} \subset C,$$

where $C_\rho$ is an open convex $O(n)$-invariant cone that contains the ball $B_\rho(S_k)$ of radius $\rho$ in $C_1(\mathbb{R}^n)$. From another side, we recall that we have the following standard orthogonal decomposition of $C_1(\mathbb{R}^n)$ into irreducible subspaces

$$C_1(\mathbb{R}^n) = W \oplus gA_1 \oplus g^2A_0,$$
Figure 2. The subsets $h_4 = 0$, $h_4 > 0$ and $h_4 < 0$.

where $W$ is the subspace consisting of trace free curvature structures, $g$ denotes the Euclidean inner product of the Euclidean space $\mathbb{R}^n$, $A_1$ is the space of trace free bilinear forms on $\mathbb{R}^n$, $gA_1 = \{ ga : a \in A_1 \}$, the products $ga$ and $g^2 = gg$ being the Kulkarni-Nomizu product of bilinear forms, and $g^2A_0 = \mathbb{R}g^2 = \{ \lambda g^2 : \lambda \in \mathbb{R} \}$.

Figure 3. The cone $C(h_4^+)$ satisfies an inner cone condition.

The map $h_4$ is then a quadratic function defined on $C_1(\mathbb{R}^n)$. With respect to the previous splitting, it sends a curvature structure $R = \omega_2 + g\omega_1 + g^2\omega_0$ to (see Lemma 2.6 and formula 6 of [13])

\[
(17) \quad h_4(R) = h_4(\omega_2 + g\omega_1 + g^2\omega_0) = ||\omega_2||^2 - (n-3)||g\omega_1||^2 + \frac{(n-2)(n-3)}{2}||g^2\omega_0||^2.
\]

Here the norms are the induced norms from the natural Euclidean product on $C_1(\mathbb{R}^n)$. In particular, the curvature structures with null $h_4$ form a cone in $C_1(\mathbb{R}^n)$ as in Fig. 2.
Note that the curvature structures with positive $h_2$ (that is positive scalar curvature) form the upper half of the space, that is $\{R = \omega_2 + g\omega_1 + \lambda g^2 : \lambda > 0\}$. It is clear from the previous discussion that $C(h^+_4)$ is not a convex cone. However one can check without difficulties that it satisfies an inner cone condition with respect to the standard curvature structure $S_4$ of the Riemannian product $S^4 \times \mathbb{R}^{n-4}$ as illustrated in Fig. 3. This completes the proof.

The previous result was first proved in [13].

Now we return to the stability question of higher $\sigma_k$-curvatures under surgeries. We say that a metric $g$ has positive $\Gamma_k$-curvatures if $\sigma_i > 0$ for all $1 \leq i \leq k$.

**Corollary 6.3.** Let $M$ be a compact manifold with $\dim M = n \geq 2k + 1$ and $M'$ be a manifold obtained from $M$ by a surgery of codimension $n$. If the manifold $M$ admits a Riemannian metric with positive $\Gamma_k$-curvature, then so does the manifold $M'$.

**Remark.** Corollary 6.3 is equivalent to the stability of the $\Gamma_k$-curvatures under connected sums. This was first proved by Guan-Lin-Wang in [6].

**Proof of Corollary 6.3.** The standard Riemannian product $S^{n-1} \times \mathbb{R}$ is conformally flat, and therefore, its $\sigma_k$-curvature coincides with the $h_{2k}$-curvature; see Proposition 2.8. Since $h_{2k}$-curvature is positive for $n - 1 \geq 2k$, the result follows from Corollary 6.2.

Our last result concerns 1-dimensional surgeries and positive $\Gamma_k$-curvature.

**Corollary 6.4.** Let $M$ be a compact manifold with $\dim M = n$, and let $k \geq 2$ be a positive integer such that $2k < n + 1 - \sqrt{n - 1}$. Let $M'$ be a manifold obtained from $M$ by a surgery of codimension $n - 1$ or $n$. If the manifold $M$ admits a Riemannian metric with positive $\Gamma_k$-curvature, then so does the manifold $M'$.

**Proof.** Let $C(\sigma_k^+)$ be the subset of $C_1(\mathbb{R}^n)$ consisting of curvature structures with positive $\Gamma_k$-curvature. We are now going to show that the $\Gamma_k$-curvature of the standard product $S^{n-2} \times \mathbb{R}^2$ are positive for $k$ as in the corollary. Therefore, the result follows immediately from the above theorem of Hoelzel.

Let $\bar{A} = \frac{2(n-1)(n-2)}{n-3}A$. Then the operator $\bar{A}$ has only two distinct eigenvalues: $\lambda_1 = n$ with multiplicity $n - 2$, and $\lambda_2 = 2 - n$ with multiplicity 2. A straightforward computation shows that the $\sigma_k$-curvature is given by

$$\sigma_k(\bar{A}) = \frac{(n-2)!n^{k-2}}{(n-k-2)!(k-2)!} \left\{ \frac{n^2}{k(k-1)} - \frac{2n(n-2)}{(k-1)(n-k-1)} + \frac{(n-2)^2}{(n-k)(n-k-1)} \right\}.$$

It is easy to see that the sign of $\sigma_k(\bar{A})$ is determined by the expression:

$$(n-1)(n^3 - 4kn^2 + 4k^2n + 4k - 4k^2) = (n-1)(4(n-1)k^2 + 4(1-n^2)k + n^3).$$
The second factor in the previous product is quadratic in \( k \) and can easily be seen positive for \( k \) and \( n \) as in the corollary.

As a consequence of Corollary 6.4, we have:

**Corollary 6.5.** Let \( \pi \) be a finitely presented group, and \( k, n \) are arbitrary positive integers satisfying \( 2 \leq k < \frac{n+1}{2} - \frac{1}{2} \sqrt{n - \frac{1}{n-1}} \). Then there exists a compact \( n \)-manifold \( M \) with positive \( \Gamma_k \)-curvature such that \( \pi_1(M) = \pi \).

In particular, there are no restrictions on the fundamental group of a compact \( n \)-manifold of positive \( \Gamma_k \)-curvature in the following cases:

- \( n \geq 6 \) and \( k = 2 \).
- \( n \geq 8 \) and \( k = 3 \).
- \( n \geq 11 \) and \( k = 4 \).

On the other hand, a result of Guan-Viaclovsky-Wang [7] asserts that positive \( \Gamma_k \)-curvature on an \( n \)-manifold implies positive Ricci curvature if \( k \geq n/2 \). In particular, the fundamental group of a compact \( n \)-manifold of positive \( \Gamma_k \)-curvature is finite provided that \( k \geq n/2 \).

**Open Question.** Are there any restrictions on the fundamental group of a compact \( n \)-manifold of positive \( \Gamma_k \)-curvature if \( k \geq 2 \) belongs to the following gap:

\[
\frac{n+1}{2} - \frac{1}{2} \sqrt{n - \frac{1}{n-1}} < k < \frac{n}{2} ?
\]

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