The quantum scattering time in a linear potential

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Abstract

In this paper, we calculate the quantum time delays for a neutral particle scattering off the Earth gravitational linear potential. The quantum time delays are obtained by subtracting the classical returning time (CRT) from the Wigner time, the dwell time and the Larmor time respectively. Different from the conventional definition, our Larmor time is defined by aligning the magnetic field along particle’s direction of motion, and this definition does give reasonable results for motions through a free region and a square barrier. It is worth noting that in the zero magnetic field limit, the Larmor time coincides well with the CRT, which is due to the special shape of linear barrier, and may have some relevance to the weak equivalence principle. It is also found that the classical forbidden region plays an essential role for the dwell time \(τ_{DW}\) to match with the CRT, and the difference between the dwell and the phase times, i.e. the self-interference time delay, is barrier shape sensitive and clearly shows the peculiarity of the linear barrier. All the time delays are on the order of sub-millisecond and exhibit oscillating behaviors, signaling the self-interference of the scattering particle, and the oscillations become evident only when the de Broglie wavelength \(λ_k\) is comparable to the characteristic length \(L_c = (2m^2 g/ℏ^2)^{-1/3}\). If the time delay measurement is experimentally realizable, it can probe the quantum nature for particles scattering off the gravitational potential in the temporal domain.

Keywords: tunneling time, Larmor time, linear barrier

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum tunneling is a pure quantum phenomenon without any classical analogy, and the relevant traversal time in tunneling is an intensively debated issue since the early days of quantum mechanics [1]. Many controversial definitions on tunneling time coexist along the debate. To name a few, the Wigner phase time [2, 3], dwell time [4] and Larmor time [5] are among the most widely discussed tunneling times, and they capture distinctive features of quantum tunneling by definition [6]. Aside from the lack of a unified definition of tunneling time, not all time definitions essentially describe traversal time [7, 8], and hence the interesting superluminal Hartman effect [9] is only an artifact of the misinterpretation [8, 10]. Nevertheless, a unified derivation of Larmor time, Büttiker–Landauer time [11], Wigner phase time and Pollak–Miller time [12], in terms of the Gell-Mann–Hartle decoherence functionals has been obtained recently [6], where various times have been neatly classified as the total time for particle sojourning in the barrier and the ultimate time difference for a particle traversal across the barrier. One can consult to several excellent review articles for further reading [7, 8, 13, 14].

In our opinion, there are at least two reasons for the issue being suspended more than 80 years. First, it is only very recently one seems to reach an acceptable answer to the question of whether traversal time is best viewed as a distribution, or as a single time scale [7]. From the path integral approach, reference [15] undeniably pointed out that...
at least for opaque barriers, there is no room for a definable unique tunneling time. Further, tunneling time does not possess a direct probability distribution, rather it can be assigned a somewhat vague distribution, a distribution where temporal interference occurs between different times corresponding to ’distinctive classical paths’ (in the sense of path integral) for a particle traveling beneath the barrier, just like spatial interference in a double-slit experiment for a particle traveling through alternative slits [15]. Besides, for different definitions of tunneling times, the exact meanings of temporal distributions maybe quite distinct [16, 17]. Second, the estimated time scales for most systems are far too small. For example, the possible ionization tunneling delay is about $10 \sim 100$ attoseconds ($1\times 10^{-18}$ s, approximately the time light orbiting around a circle with Bohr radius), so to pin down the time issue experimentally is extremely difficult. However, the situation is dramatically changed with the advent of attoclock [18], whose time resolution can already reach the level of attosecond [19, 20]. This ultra-precise temporal resolution revives a surge of interest in tunneling time [21–24], and also makes a satisfactory answer to the tunneling issue more urgent. Recently, evidence of finite tunneling time comes from the studies of strong field ionization in multi-electron atoms [25], and some theoretical calculations [21] also favor tunneling as a finite process, while other studies claim supports for instantaneous tunneling [20, 24]. Up to now, there is still no theoretical consensus on whether tunneling is an instantaneous or a finite process.

To shed new light on the issues mentioned above, we utilize a neutral particle [e.g. an ultracold neutron (UCN), and later we will use neutron for brevity] scattering off a linear gravitational potential as an illustrative model. Firstly, the linear potential provides an alternative analytical example to the study of tunneling time besides the extensively discussed square barrier. This may be used to explore the barrier-shape sensitive time definitions (e.g. dwell time [21]), which cannot be uncovered by a square barrier solely. Secondly, due to the tiny kinematic energy, the characteristic quantum time delay is on the order of sub-millisecond (see Larmor time in section 4), which is much easier to calibrate than the ultrafast ionization tunneling mentioned above. In fact, for slow-moving massive neutral particles such as atoms, the tunneling time measurements have successfully achieved the microseconds accuracy [26, 27]. Though compared to the precisely controllable cold atoms with laser field, UCN is neither easy to prepare nor to manipulate, precise measurement is still feasible, such as the precisely measured transition frequencies between different gravitational states, which has achieved 0.1 Hz accuracy already [28]. In a similar way, we believe precise time measurement with UCN is also likely to happen in the near future. Finally, this simple model can also evade the complication due to multi-time scales in ionization tunneling, where the temporal scales include not only the tunneling delay, but also the resonance lifetimes of bound states [21].

Though the tunneling delay for a linear potential has already been discussed by Davies [29] aiming to validate the weak equivalence principle (WEP), our study differs in several aspects from Davies’ approach, where a Peres’ clock [45] was utilized to measure the tunneling time delay. First, though Peres’ clock is an ideal quantum clock, it is more or less theoretical, without direct experimental realization, while we use the Larmor time approach, which is experimentally even feasible. Differ from the original definition of Larmor time [5], our new definition only needs weak magnetic field instead of zero magnetic field limit [5]. Furthermore, in our case, the magnetic field is parallel instead of orthogonal to the direction of motion. Meanwhile, we prove that our definition can reduce to the traversal time for a free particle and also give meaningful results for a square barrier. Similarly, we also show that the transmitted Larmor time through a square barrier approaches the Büttiker–Landauer time in the opaque limit. As our focus is the temporal behavior for a particle scattering off a linear barrier, we degrade the proof in the appendix. Second, our setting assumes an ideal injection region where gravity can be ignored, so the linear potential is truncated, while the linear potential in reference [29] is a full unscreened one. Though compared with Daves’ approach, this seems somehow unrealistic or a drawback, the conclusion remains the same, i.e. the classical returning time (CRT) equals to its quantum counterpart, which can be viewed as an evidence to the WEP, manifested in the quantum variant of the well-known Galileo Pisa tower experiment. This may indicate the initial setting, whether a free moving neutron beam or an upward moving neutron only seen as an asymptotic component of the wave function immersed in the linear potential all the time, does not affect the equality, a consequence of WEP. Moreover, our approach can be directly applied to other scenarios of linear barriers, not necessarily related to gravity (such as a charged particle in a uniform electric field, though will have no relevance to WEP in that cases). We hope with our extensive investigation on the various tunneling times and their relations, linear barrier as another test ground (besides square barrier) in resolving the tunneling time issue, can be appreciated and may even draw the interest of experimental physicists.

The remainder of the paper is organized as follows. In section 2, we review the preliminary knowledge in describing particle’s motion in a linear barrier, and we also provide rough estimates on the time scales involved in several tunneling processes based on the uncertainty principle and semi-classical approximation at the end of this section. In section 3, we discuss in detail about phase and dwell times, and their deviations from the CRT. In section 4, we illustrate our new definition of Larmor time and calculate its zero magnetic field limit for a linear barrier. Interestingly, the limit is just the CRT, which is also equals to the dwell time in the barrier region. At last, we summarize our main results in section 5.

2. Basic theory

2.1. A neutron in a linear potential

Before discussing tunneling time, we briefly review the quantum description of neutron’s motion in the linear potential $V_b = mgz$ [30]. The general solution to the Schrödinger
a beam of injected neutrons. The general solution is a linear equation

\[ i\hbar \frac{\partial \Phi(z, t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + mgz^2 \right] \Phi(z, t) \]  

(1)

is \( \Phi(z, t) = \int dE e^{-iE\hbar t/\hbar} \rho(E) \phi_E(z) \), where \( \phi_E(z) \) is the solution of the stationary Schrödinger equation

\[ \phi''_E(z) + \frac{2mE}{\hbar^2} \left( 1 - \frac{mg}{E} z \right) \phi_E(z) = 0, \]  

(2)

and \( \rho(E) \) is the weighting factor for the Fourier expansion of the wave-packet \( \Phi(z, t) \). Unless necessary to distinguish between them, we will assume the equality between the gravitational mass and inertial mass of neutron in the following, i.e. \( m = m_0 = m_g \). From the dimensional constants \( g, m, \) a characteristic length scale \( L_e \equiv (2m^2 g/\hbar^2)^{-1/3} \) (\( L_e = 5.866 \mu m \)) for neutron can be constructed. Associatively, the characteristic momentum \( p_c \equiv h/L_e \) and energy \( E_e \equiv [(mg)^2/2m]^{1/3} \) can be defined. With these dimensional constants, we can rewrite the equation (2) with the dimensionless variables \( z_0 = z/L_e \) and \( E_0 = E/E_e \). The general solution is

\[ \phi_E(z_0) = c_1 \text{Ai}[z_0 - E_0], \]  

(3)

where we have abandoned the \( \text{Bi}[z - E_0] \) branches as \( \lim_{z_0 \to \infty} \phi_E(z_0) = 0 \). To fix the undetermined constants \( c_1 \), a normalization condition is needed for bound states solutions and an additional ansatz and continuity condition are needed for scattering states solutions. According to different boundary conditions, the scattering and the bounded solutions can be obtained respectively. This paper is mainly to discuss the temporal behavior of the scattering states, which will be briefly reviewed in the following. For details on the gravitational bound states, see [31, 32].

2.2. Scattering solution

We assume initially a beam of neutrons with a given energy is injected from the \( z < 0 \) region, and the effect of gravity can be ignored in the negative-\( z \) zone to simplify our discussion. Consequently, the potential can be approximated as \( V_z = mgz \theta(z) \). Note since nearly all forms of matter gravitators (dark energy antigravitators), this is somehow only a toy model facilitating our calculations, and at the same time provide a simpler check of the peculiarity of linear potential without decomposing the wave function (Airy function) into the sum of incoming and reflecting partial waves [29] at the asymptotically negative infinite region. However, we may view the linear potential in a broader way as well, where \( mg \) in \( V_z \) is replaced by some proportional constant, such as \( qE \) for a charged particle in a uniform electric field. In fact, even for a neutron which has nonzero magnetic moment, the potential \( V_z = -\vec{\mu} \cdot \vec{B} \) with \( \vec{B} = B(z) \hat{n} \) can be made linear provided \( B(z) = b \zeta(z) \), where \( b \) is a constant. This is entirely feasible as unlike gravity, magnetic field can be screened. For the moment, we may just regard \( V_z \) as a simple model to the neutron-gravity interaction for a beam of injected neutrons. The general solution is a linear superposition

\[ \Phi(z, t) = \int_0^\infty dk \, \rho(k) e^{-iE_k t/\hbar} \phi(k, z), \]  

(4)

where the energy \( E_k = (k \hbar)^2/2m, \rho(k) \) is the weighting factor and \( \phi(k, z) \) is the plane wave component

\[ \phi(k, z) = \begin{cases} e^{ikz} + R e^{-ikz}, & z < 0, \\ c_1 \text{Ai}[z/L_e - E_0], & z > 0. \end{cases} \]  

(5)

From the continuity equation

\[ 1 + R = c_1 \text{Ai}[z - E_0], \quad (1 - R) i k = \frac{c_1}{L_e} \text{Ai}'[z - E_0] \]  

(6)

we get the coefficients

\[ c_1 = \frac{2ikL_e}{\text{Ai}'[-E_0] + ikL_e \text{Ai}[-E_0]}, \]  

(7)

\[ R = \frac{\text{Ai}[-E_0] + \frac{i}{L_e} \text{Ai}'[-E_0]}{\text{Ai}[-E_0] - \frac{i}{L_e} \text{Ai}'[-E_0]} \]  

(8)

From (8), we can readily obtain \( R = |R|^2 = 1 \), since for any real argument \( x, \text{Ai}[x] \in \mathbb{R} \) and \( \text{Ai}'[x] \in \mathbb{R} \). From the asymptotic form of Airy function at sufficiently (positive) large \( z \),

\[ \text{Ai}[z] \sim \frac{e^{-z^{1/3} \pi}}{2\sqrt{\pi}z^{2/3}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n + \frac{1}{3}) \Gamma(n + \frac{2}{3}) \Gamma \left( \frac{1}{3} \right)^n}{2^n n! z^{2n}}, \]  

(9)

we get the expected asymptotic decay behavior. From the asymptotic behavior, we can define the barrier penetration length as \( L_p \equiv L_e(1 + k^2 L_e^2) \), which is the depth where the argument in the Airy function in equation (5), \( z/L_e - E_0 = L_p/L_e = 1 \). We plot the relative probability density \( |\phi(k, z)|^2 \) with the linear potential \( V_z \) in figure 1, where we can see the small tails of \( |\phi(k_i, z)|^2 \) penetrating into the classical forbidden region (CFR), and the amplitudes of \( |\phi(k, z)|^2 \) decrease at much lower \( z \) than the classical turning points \( z_c = E_k/mg \) for each \( k_i \) (\( i = 1, 2, 3 \)), indicated by the cross points of the dashed horizontal lines (representing the corresponding eigen-energies) and the oblique solid blue line (representing the linear potential). The small penetrating tails indicate that the incoming neutron has a small probability tunneling into the CFR. Later we will see the advanced decreasing of wave amplitude at position lower than \( z_c \) and the penetration of wave amplitude into the CFR conspire to match the total dwell time with the CRT.

Before diving into any specific definition of tunneling time, we give simple estimates on the time scales involved in three particular tunneling processes based on:

- Uncertainty principle estimate, \( \tau_{uc} \sim \frac{\hbar}{E} \), where \( E, m \) is the characteristic energy involved in a specific process;
- Semi-classical estimate, \( \tau_{sm} \sim \frac{L}{\sqrt{2V - E} \sqrt{m}} \), where \( L \) is the characteristic length, and \( V - E \) are the mass and the negative kinetic energy of the tunneling particle.
3. Reflective phase time and dwell time

Let’s take the electron tunneling ionization in a hydrogen atom as an example, where the ionization potential is \( I_0 = 13.6 \text{ eV} \). Given that the laser peak intensity is \( 2.9 \times 10^{14} \text{ W cm}^{-2} \), the corresponding electric field strength is around \( E = 4.67 \times 10^{10} \text{ V m}^{-1} \). There are two characteristic lengths, the effective barrier width \( d_1 = I_0/(eE) = 2.91 \text{ Å} \) [34] and the Bohr radius \( a_0 = h/(m_e c) \approx 0.53 \text{ Å} \) (for the natural field strength \( E = \alpha c h/(e a_0^2) = 5.14 \times 10^{11} \text{ V m}^{-1} \), the effective barrier width \( d_2 \approx a_0/2 \). Here \( e, m_e \) are the charge and mass of the electron, and \( \alpha \) is the fine structure constant. The uncertainty principle estimate gives \( \tau_{\text{unc}} \approx 48 \text{ as} \), and the semiclassical estimate gives \( \tau_{\text{sm}} = d_1/\sqrt{2 I_0/m_e} \approx 134 \text{ as} \) (for \( d_2, \tau_{\text{sm}} \approx 12 \text{ as} \), roughly the same order as \( \tau_{\text{unc}} \)). The estimations give roughly 10 \( \sim 100 \text{ as} \) for the duration of tunneling ionization, comparable to the measurement performed on Helium [33]. For other tunneling processes, we summarize the simple estimates in table 1, where in the last row, ‘?’ means that no corresponding experimental result is known up to now.

3. Reflective phase time and dwell time

3.1. Wigner phase time

The Wigner phase time (or Eisenbud–Wigner time, or group time delay) [2, 4] is to follow the peak evolution of the wave packet and can be determined by the stationary phase method. As there is no obvious causal connection between the peaks of incoming packet and the transmitted one [7], and a simple barrier behaves like a high-energy components filter [35], phase time cannot directly correspond to the true time duration in tunneling [8]. However, it can still characterize the time scales in a quantum process [10], and clearly demonstrate the self-interference between the incoming and reflective partial waves. By definition, a stringent and satisfactory treatment should start with a wave packet with finite width \( \Delta x \sim d \), and \( d \) must be much smaller than the characteristic width of the finite barrier. However, as mentioned above, since phase time is utilized only as a rough estimate, and a linear potential is an infinite barrier, \( d \ll L \) cannot be directly imposed here as a requirement for the wave packet. Instead, we choose the monochromatic approximation \( \rho(k) \approx \delta(k - k_0) \) to see what happens for a highly non-local wave packet, which is both for calculational simplicity and to explore the more quantum nature of neutron scattering. A natural requirement of non-locality is that the position spreading of a wave packet, \( \Delta x \approx \frac{1}{2 \kappa \rho(\rho)} \gg L_c \), which leads to the de Broglie wavelength spreading \( \delta \lambda_k \ll \frac{\lambda^2}{2 \kappa L_c} \), where \( \lambda_k = 2\pi/k \). As for UCN with \( \lambda_k \approx L_c \), it reduces to \( \delta \lambda_k \ll \frac{\lambda_k}{2} \). In the following, we stick to this assumption and drop the lower index 0 in \( k_0 \).

5.433peV
5.622peV
8.245peV

Figure 1. Potential barrier \( V_S(z) \) and the sample scattering wavefunctions. The red, green and purple curves correspond to the \( |\phi_{d,k}|^2 \) with different \( E_D = (\hbar k)^2/2m_e \), which are also shown in the corresponding colors with dashed horizontal lines. The three points are the cross points for each \( E_D \) at the corresponding classical turning heights.

\[ \tau_{\text{phase}} = \frac{2m \partial \theta}{\hbar k} \delta k = \frac{2L_c}{v(k)} \left( 2E_D (A_i^[y])^2 - A_i[y] A_i'[y] - A_i'[y] A_i[y] \right) A_i[y]^2 + E_D A_i[y]^2 \]
by the solid blue curve in figure 2. Clearly, with the increase of neutron wave number \( k \), the oscillating amplitude of \( \Delta \tau_p \) decreases quickly.

We can interpret the oscillating behavior as a manifestation of the self-interference between incoming and reflective partial waves [36]. The de Broglie wave length of a neutron with wave number \( k \) is \( \lambda_k = 2\pi / k \). Only two partial waves with comparable de Broglie wave lengths can interfere coherently, so a rough estimate of the time scale is \( \tau_{coh} = \alpha \lambda_k / 1(k) = \alpha \frac{\Delta \tau_p}{\lambda_k} \), where \( \alpha \) is a \( O(1) \) numerical fitting fitting factor, and we find \( \alpha = 1/(2\pi) \) fits the envelope of the quantum deviation \( \Delta \tau_p \) very well, see the dashed red and dashed green fitting curves in figure 2. Therefore the excellent fitting supports our self-interference interpretation [36]. The excellent fitting of the envelope of \( \Delta \tau_p \) by \( \tau_{coh} \) can also be viewed through the energy-time uncertainty relation \( \Delta E \Delta \tau \geq \frac{\hbar}{4} \). The deviation of \( \Delta \tau_p \) can be regarded as the quantum time fluctuation, so its oscillation amplitude must satisfy the energy-time uncertainty relation, which exactly gives \( \lambda_k = \tau_{coh} \). In fact, the perfect fitting is unaffected by varying the slope of the linear potential, such as replacing \( g \) by \( 2g \) or \( 5g \), which may reflect the fact that the energy-time uncertainty relation is a more general feature beyond this particular model [10].

### 3.2. Dwell time

Next we turn to the dwell time, which was first introduced in reference [4]. Unlike phase time, dwell time is a positively defined quantity, averaged over all scattering channels. The indistinguishability between reflected and transmitted channels means dwell time is better viewed as a lifetime or a storage time rather than a traversal time. However, for a barrier with zero transmission rate (\(|R| = 1\)), such as the linear barrier, as all particles including the ones penetrated into the barrier are finally reflected, dwell time does encode the tunneling time delay. According to reference [37], dwell time can be defined as

\[
\tau_{DW}[z_L, z_R] \equiv \frac{m}{\hbar k} \int_{-z_L}^{z_R} \mathrm{d}z |\phi(k, z)|^2, \quad (11)
\]

where the positive \( z_L, z_R \) can be chosen as the characteristic length \( L_c \) and the penetration depth \( L_p \), respectively. Note this choice is somehow arbitrary, as we can also choose \( z_R = z_C + 2\lambda_k \) as well, since \(|\phi(k, z)|^2 \) decays rapidly when \( z > z_C = \frac{E}{m_g} \). Choosing a finite \( z_R \) can facilitate our numerical integral as the contribution of \( z > z_R \) is negligibly small, while choosing \( z_L = L_c \) instead of \( z_L = 0 \) is to include the contribution of wave function ahead of the linear barrier, as our neutron state is highly nonlocal and we do not want to lose the information encoded in the wave function just before it interacting with the linear barrier, though we may choose \( z_L = \lambda_k \) as well. The integral in (11) is given by

\[
\int_{-L_c}^{L_p} \mathrm{d}z |\phi(k, z)|^2 = \int_{-L_c}^{0} \mathrm{d}z \left[ 2 + (Re^{-2ikz} + c.c) \right] + |c_1|^2 \int_{0}^{L_p} \mathrm{d}z \frac{\mathrm{Ai}^2}{\Delta \tau_p} \frac{|z/L_c - E_D|}{\Delta \tau_p} = L_{eff} + \frac{4}{\mathrm{Ai}^2 + \alpha' \Delta \tau_p} \int_{0}^{L_p} \mathrm{d}z \frac{\mathrm{Ai}^2}{\Delta \tau_p} \frac{|z/L_c - E_D|}{\Delta \tau_p} \right.
\]

\[
\left. \times \mathrm{Ai}^2 \left| z/L_c - E_D \right|, \quad (12) \right.
\]

where \( \mathrm{Ai} \) and \( \mathrm{Ai}' \) represent \( \mathrm{Ai}[-E_D], \mathrm{Ai}'[-E_D] \) respectively,

\[
L_{eff} = \left[ \frac{\mathrm{Ai}^2}{\Delta \tau_p} \left( \frac{\alpha'}{\Delta \tau_p} \right)^2 \right] \left( \frac{\sin(2\lambda_k)}{2\lambda_k} + \frac{\mathrm{Ai}'(\alpha')}{\Delta \tau_p} \cos(2L_c) \right) - 1 \right]
\]

\[
+ 2L_c. \quad (13)
\]

As for \( \int_{0}^{L_p} \mathrm{d}z \frac{\mathrm{Ai}^2}{\Delta \tau_p} \frac{|z/L_c - E_D|}{\Delta \tau_p} \), it is given in (A1). Just as mentioned, we can also choose \( z_L = \lambda_k N \), where \( N \in \mathbb{Z} \), then \( L_{eff} = 2\lambda_k \) and the first oscillating term in (13) vanishes. Substituting all these terms into (11), we obtain

\[
\tau_{DW}[L_c, L_p] = \tau_{DW}^{+\infty} \left\{ 1 + \frac{\mathrm{Ai}^2}{E_D \mathrm{Ai}'^2 + (\alpha')^2} \right\} + \frac{mL_{eff}}{\hbar k}, \quad (14)
\]

where \( \tau_{DW}^{+\infty} = \tau_{DW}[0, +\infty] \) is the dwell time in the region of linear barrier. It is interesting to note that \( \tau_{DW}^{+\infty} = \tau_{CE} \), and the

| Table 1. Simple estimations on time scales in various tunneling processes. In this table, \( \tau_{coh} \) is the estimate from uncertainty principle, \( \tau_{un} \) is the semi-classical estimate, and \( \tau_{exp} \) comes from experimental measurements. To know we don’t know any measurement on neutron’s tunneling delay in the surface gravity of the Earth. For briefness, ETI, NTLG and ATL refer to electron tunneling ionization in helium, neutron tunneling in linear gravity, and atom tunneling through blue detuned laser field, respectively. |
|-----------------|-----------------|-----------------|
| Tunneling       | ETI [33]        | NTLG            | ATL [26]        |
| \( L \)         | 0.591 nm(\( \frac{\hbar}{2m} \)) | 5.866 \( \mu m(L_c) \) | 1.3 \( \mu m(waist) \) |
| \( m \)         | 510.99 keV(\( m \)) | 939.57 MeV(\( m \)) | 81.69 GeV(\( m \)) |
| \( E \)         | 24.39 eV        | 0.602 peV(\( E_c \)) | 10.47 peV(\( 122 nK \)) |
| \( V \)         | 24.39 eV        | 6.624 peV(\( m g L_p \)) | 15.51 peV(\( 180 nK \)) |
| \( \tau_{uc} \) | 48 as           | 1.09 ms         | 0.06 ms         |
| \( \tau_{un} \) | 284 as          | 0.55 ms         | 0.39 ms         |
| \( \tau_{exp} \) | 34 as [33]      | ?               | 0.62 ms [26]    |
envelope fitting curves are given by $\pm \tau_{\text{qph}}$, represented by the dashed red and green curves, respectively. The inset on the upper right corner shows the reflective phase time $\tau_{\text{phase}}$ and the CRT $\tau_{\text{CE}}$, represented by solid blue and dashed red line, respectively.

$$\int_{0}^{+\infty} dz \frac{d}{d\tau} (\phi(k,z))^{2} = 2 \int_{0}^{\pi/2} dz \frac{v(k,z)}{v(k,z)}$$

where $v_{k} \equiv \hbar k/m$ and $v(k,z) \equiv \sqrt{v_{k}^{2} - 2gz}$. The left-hand side is the time expressed in terms of quantum probability density $|\phi(k,z)|^{2}$ and the integral includes the CFR $(z_{C}, +\infty)$, while the right-hand side is purely a classical expression of the time a vertically injected particle with initial velocity $v_{k}$ takes to get back to the starting point in the Earth’s gravitational field. This can be viewed as a signal in supporting the WEP holding true in the quantum domain, at least for a non-relativistic particle in weak gravity. WEP states that any two test particles with negligible self-gravity will follow the same geodesic determined by the nearby metric, provided they share the same initial conditions. It is also termed as universality of free fall. In fact, if we distinguish inertial mass $m_{I}$ with gravitational mass $m_{G}$ from the beginning of our calculation, we will find $\tau_{\text{DW}} \approx \frac{2\hbar}{m_{G}g}$ and $\tau_{\text{CE}} \approx \frac{2\hbar}{m_{G}}$. So the equality holds only when $m_{I} = m_{G}$ for any massive neutral particle (since our calculation doesn’t specify particle species and only requires charge neutrality for the injected particle), and thus provides a clear manifestation of WEP (at least for neutral particles). A through exploration of the WEP in quantum domain is interesting, but is out of the scope of this paper. We plot the dwell time with respect to the CRT $\tau_{\text{CE}}$ in figure 3(a), where the dashed blue line corresponds to $\tau_{\text{CE}}$. To show the influence of the choice of $z_{C}$ to the behavior of dwell time, we plot $\tau_{\text{DW}}[-L_{c}, L_{p}]$ (briefly referred as $\tau_{\text{DW}}$ in the following) and $\tau_{\text{DW}}[-\lambda_{c}/2, L_{p}]$, represented by the solid red and solid cyan curves, respectively, in the same figure. To have a detailed comparison with $\tau_{\text{CE}}$, we also plot the first part of $\tau_{\text{DW}}$ in (14), $\tau_{\text{DW}}[0, L_{p}]$, and the dwell time in the classical allowed region $\tau_{\text{DW}}[0, z_{C}]$ in figure 3(a). These are represented by the dashed green and solid purple curves, respectively. The solid red and cyan curves ($\tau_{\text{DW}}[-\lambda_{c}/2, L_{p}]$) behaves like $ak + \frac{b}{\tau}$, where $a, b$ are two positive constants) in figure 3(a) clearly demonstrate that the oscillatory behavior is due to the fact $L_{c} \propto \frac{1}{\tau}$. Different from classical naive expectation, $\tau_{\text{DW}}[0, L_{p}]$ gets closer to $\tau_{\text{CE}}$ than $\tau_{\text{DW}}[0, z_{C}]$. In fact, the dashed blue line and the dashed green curve nearly overlap, while an obvious gap can be found between the solid purple curve and the dashed blue line. To show the importance of the contribution of CFR, $(z_{C}, L_{p})$, to $\tau_{\text{DW}}$, we plot $\tau_{\text{DW}}[z_{C}, L_{p}]$ in figure 3(b), see the solid blue curve. To facilitate the comparison, we also plot the constantly shifted deviations of $\tau_{\text{DW}}[0, z_{C}] - \tau_{\text{CE}}$ and $\tau_{\text{DW}}[0, L_{p}] - \tau_{\text{CE}}$ in figure 3(b), represented by the solid green and dashed red curves, respectively. The shifted time constants are shown in the legends below the figure. By comparing the oscillating amplitudes of the solid green and dashed red curves in figure 3(b) (the constant time shifts are not very relevant here), we see $\tau_{\text{DW}}[0, L_{p}]$ is indeed much closer to $\tau_{\text{CE}}$ than $\tau_{\text{DW}}[0, z_{C}]$, which has already shown transparently by the gap between the dash green and the solid purple curves in figure 3(a). So without the contribution from the CFR, represented by the solid blue curve in figure 3(b), the intuitively more ‘classical’ returning time $\tau_{\text{DW}}[0, z_{C}]$ deviates from $\tau_{\text{CE}}$ by an indisputable discrepancy, which is a constant $-4mL_{c}^{2}/(h3/2\Gamma[1/3])$ when $k \rightarrow +\infty$ [shown by the rapidly decreasing oscillating amplitude of $\tau_{\text{DW}}[z_{C}, L_{p}]$, see the solid blue curve in figure 3(b)].

As elegantly displayed by the manipulation of the stationary Schrödinger equation in reference [39], there is a relation between dwell time and phase time. The direct application of the relation in [39] does not work, as there is no asymmetrically transmitted region in the far right side for a linear barrier. However, with a small alteration, the relation becomes

$$\tau_{\text{DW}}[0, +\infty] = \frac{m}{\hbar k^{2}} \text{Im}[R] + \tau_{\text{phase}}$$

The formal simplicity of this relation strongly depends on the specifically chosen integration interval $[0, +\infty]$, where the wave function and its derivatives simply vanishes at $+\infty$. Other choice of integration interval may generate a more
Figure 3. (a) Dwell times with τCE = −Lc and zL = −L are represented by the solid red and solid cyan curves, respectively. As a comparison, the CRT τCE is plotted in dashed blue lines, and the first part of τDW in (14), τDW[0, Lp], and the dwell time in the classical allowed region, τDW[0, 2L], are represented by the dashed green and solid purple curves, respectively. (b) Dwell time in the CFR τCE[2L, Lp], and deviations of various dwell time from τCE. The deviations τDW[2L, Lp] − τCE and τDW[0, 2L] − τCE are plotted with dashed red and solid green curves, respectively. For the convenience of comparison, they are constantly shifted by 0.45 ms and +0.87 ms, respectively.

In general, equation (16) is also applicable to other barriers with |R| = 1, and the term τIF ≡ −τCE Im[R] is called self-interference delay [8], originated from the overlap between incoming and reflected partial waves, and is very sensitive to barrier shape. As an illustration, we plot the corresponding −τIF in figure 4 for three different semi-infinite barriers: the step barrier V = Aθ(x), the tanh-like barrier V = 1/2[1 + tanh(x/a)] [40] and the linear barrier V = mgθ(x). To facilitate the comparison, we set A = 27 mgLc for step barrier and tanh-like barriers [40], and we choose two different length parameters, a = 6.6 μm and a = 0.9 μm, for tanh-like barriers. In comparison, the barrier with a = 0.9 μm is much steeper than a = 6.6 μm and resembles more to the step barrier.

From figure 4, we see that the self-interference effect is more significant for low energy particles, and becomes more evident for smooth and gentle barriers (linear barrier and a = 6.6 μm tanh-like barrier, see the solid red and blue curves, respectively) than for steep barriers (step barrier and a = 0.9 μm tanh-like barrier, see the solid green and dashed purple curves, respectively). This is not unexpected, since particle with lower energy has longer De Broglie wavelength, thus easier to interfere, and smooth and gentle barrier provides more chance of self-interference between incoming and reflected partial waves. Interestingly, as τIF∞ ≡ τCE = 4mkLc/h for linear barrier, τIF is exactly the phase time delay shown in figure 2, and confirms our assertion that the oscillating phase time delay is due to the self-interference. Since τIF is sensitive to barrier shape, it also provides a clue of why dwell time is barrier shape sensitive [21] from the relation (16). Note dwell time can be defined as the expectation value of the Hermitian operator τD = ℏ ∫|x⟩⟨x| dx |x⟩ (where {x, x'} is the region of interest) [41], and thus corresponds to an measurable operator at least in principle. It was also proved that the real part of the conditional expectation value of τD equals the unconditional one ⟨τD⟩ for either transmitted or reflected channels in reference.
However, the equality shall work only for a symmetric barrier, not necessarily hold true for asymmetric ones. For e.g. \( \text{Re}(\tilde{\tau}_D(1)) = \text{Re}(\tilde{\tau}_D(\alpha)) \) may not be right for a triangular barrier, and will be interesting to check. Hope our discussion of linear barrier may provide a valuable warm-up exercise for that more challenging task.

At last, we also note that the time estimate from the WKB approximation due to the CFR, \( \tau_{WKB} \equiv \int_0^{\phi_1} d\phi x_0 \sqrt{2}(mgz - E) = 2\sqrt{2L_0}/\bar{g} \approx 2.19 \text{ ms} \), is roughly the same order as \( \delta\gamma_{NW} \) for small \( k \), and is consistent with the rough estimate in table 1.

4. Larmor time and neutron in an external magnetic field

In analogy with the attoclock, where a highly circularly polarized electric field rotating on the plane orthogonal to its direction of motion plays the role of a hand on the face of a clock, the neutron spin can also act as a clock pointer. The picture is that, as neutron carries non-zero magnetic moment, its spin precesses in an external magnetic field. This internal degree of freedom acts as a ‘pointer’ and the precession angle with respect to the initial spin measures the time elapsed during which the neutron is in the region covered by magnetic field. The spin precession is the well-known Larmor precession, and the measured time is called the Larmor time [5]. Larmor time has been discussed extensively in the literature [5, 6, 21, 37], and has an intimate connection with the complex time obtained by the path integral approach [42]. Different from the conventional definition of Larmor time, in our approach the external magnetic field is defined to be parallel instead of orthogonal to the direction of motion, whereas the initial spin polarization is still orthogonal to the magnetic field to allow spin precession. Clearly, our configuration provides an alternative realization of ‘Larmor clock’ for measuring traversal time. To illustrate the distinction, the initial spin, momentum and magnetic field for both definitions are shown in figure 5. We find that for a free particle, our definition reduces to the classical traversal time \( L/v \), where \( L \) is the length of the space interval and \( v \) is the speed of the incoming particle. For a rectangular barrier, the definition also gives a meaningful transmitted and reflected Larmor time. As our goal is to discuss the tunneling time of a linear barrier, we degrade the Larmor time for the motion through the free space and a rectangular barrier to the appendix C.

For a linear potential, to implement a Larmor clock, we introduce a magnetic field in the region \( z \in [0, z_a] \), where \( z_a \gg z_c = E/mg^2 \). The Hamiltonian is given by

\[
\hat{H}_D = \frac{\tilde{p}_0^2}{2m} + mgz\theta(z) - \tilde{\mu}_s \cdot \vec{B}(z)\theta(z - z_a),
\]

where \( \tilde{\mu}_s \equiv h\frac{\vec{\sigma}}{2} \) is the magnetic moment, \( \bar{g} = -3.826085 \) is the neutron Landé g-factor and \( \mu_s \) is the nuclear magneton. For simplicity, suppose the magnetic field is homogeneous such that \( \vec{B} = B_0\vec{n} \), and the non-relativistic neutron wave-function is described by a two-component Pauli-spinor \( \phi \equiv (\chi, \eta)^T \). Defining \( \tilde{\mu}_N \equiv \mu_N m_v \), \( b_0 \equiv B_0 L_0^2/c \), we can recast the static eigen-equation in the region \( 0 < z < z_a \) into the dimensionless form

\[
\tilde{\phi}''(z_D) + \frac{E_D (1 - z_D) - \tilde{\mu}_N b_0 \vec{n}}{E_D} \cdot \tilde{\phi}(z_D) = 0.
\]

Assume the upward going neutron is initially polarized in a horizontal direction, e.g. being initially prepared in the eigenstate of \( \sigma_z \), and the magnetic field is along the direction \( \vec{n} \equiv (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \). We keep a general \( (\theta, \phi) \) temporarily in order to check the results by finally setting \( \theta = 0 \). Imposing the continuity condition at \( z = 0 \) with the ansatz

\[
\phi(z) = \begin{cases} 
\frac{1}{\sqrt{2}} \left[ e^{-i\alpha_z} e^{ikz} + \left( \frac{R^+}{R^-} \right) e^{-ikz} \right], & z < 0, \\
U \begin{pmatrix} c_+ Ai \left( \frac{z}{L_v} \right) - E_D - \tilde{\mu}_N b_0 \end{pmatrix}, & z \geq 0,
\end{cases}
\]

where \( U \equiv \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2)e^{-i\phi} \\
\sin(\theta/2)e^{i\phi} & -\cos(\theta/2) \end{pmatrix} \), we get the coefficients below

\[
c_+ = \frac{\sqrt{2} \left[ e^{-i\alpha} \cos \phi + e^{-i\phi} \sin \phi \right]}{Ai[-E_D(1 + r)] - \frac{1}{i\phi}Ai[-E_D(1 + r)]},
\]
the spin state when the magnetic field strength $A_i$.

When a reflected neutron, we can read out the elapsed time through the magnetic field dependent Larmor time can be defined as well.

$L_{\text{Lar}}(\omega) = \frac{2}{\gamma N B_0} \left[ G \right]$, (22)

where we have defined $r = \frac{\delta \tau_{\text{Lar}}}{\delta \omega} = (\tilde{g})_{\mu B}(B_0)/(k^2 \hbar)$, $A_i \equiv A_i[E_D(1 \pm r)]$ and $A'_i \equiv A_i[-E_D(1 \pm r)]$.

At this stage, we set $\theta = 0$. Since the incoming neutron is in the spin state $\langle e^{i \omega t} \rangle$, comparing the spin state of the reflected neutron, we can read out the elapsed time through the spin precession angle

$\Theta_R = 2 \tan^{-1} \left[ \frac{(A_i^i A_+ - A_+^i A_i)}{(kD A_i^i + A_+^i A_i)} \right] \mod \pi$, (25)

for calculational details, see appendix B. Note the modulus of $\pi$ in (25) is only for mathematical rigour and not necessary when the magnetic field strength $B_0$ is sufficiently small. The magnetic field dependent Larmor time can be defined as

$\tau_{\text{Lar}} \equiv \frac{\Theta_R}{\omega_{\text{Lar}}} = \frac{2m}{k^2 \hbar R} \left[ \frac{G}{F} \right] = \frac{2m}{\gamma N B_0} \left[ \frac{G}{F} \right]$, (26)

where $F \equiv [A_i^i A_+ - A_+^i A_i]$, $G \equiv [A_i^i A_+ - A_+^i A_i]kD$, $\gamma N \equiv \tilde{g}^{\gamma N} = 1.832 \times 10^3 \text{rad}/(\text{sT})$ is the gyromagnetic ratio of neutron and $\omega_{\text{Lar}} = \gamma N B_0 = \hbar k^2/m$ is the Larmor frequency. Clearly, our definition can also be applied to other potentials. For the case of a finite barrier such as a rectangular barrier or a free region, the transmitted Larmor time can be defined as well. In that case, $\Theta_R$ in (26) has to be replaced by the spin precession angle of the corresponding transmitted partial wave with respect to the incoming partial wave, see appendix C.

Interestingly, we find in the $B_0 \to 0$ limit, where $r \to 0$ as well,

$\tau_{\text{Lar}} \equiv \frac{2m}{k^2 \hbar R} \left[ \tan^{-1}(-G/F) \right]$,

$= \lim_{r \to 0} \frac{2m k^2}{k^2 \hbar} \left\{ 2 + r \left[ \frac{G \tilde{G}}{F^2 + G^2} \right] \right\}$

$= \frac{4m k \tilde{n}^2}{\hbar} = \tau_{\text{CE}}$. (27)

where $\tilde{G} = \frac{1}{2} \left( A_i^i + A_+^i \right) kD$. This is clearly demonstrated in figure 6, where we plot the deviation of $\tau_{\text{Lar}}$ from the $\tau_{\text{CE}}$, $\delta \tau_{\text{Lar}} \equiv \tau_{\text{Lar}} - \tau_{\text{CE}}$. In figure 6, we see that as the weak magnetic field decreases toward zero, $\tau_{\text{Lar}}$ approaches the $\tau_{\text{CE}}$ as close as possible, and the deviation oscillates much more frequently with the increase of the wave number $k$. Compared with those deviations of the phase and the dwell times from CRT, the oscillating amplitude decreases more gently with increasing $k$.

Note that $r \to 0 \tau_{\text{Lar}} = \tau_{\text{CE}}$ is not a common feature for general potentials, such as a rectangular barrier, where the reflective Larmor time does not reduce to the CRT in the zero magnetic field limit. This provides a further support of WEP just as $\tau_{\text{Lar}} = \tau_{\text{CE}}$, since different definitions of quantum sojourning time agree well with the CRT $\tau_{\text{CE}} = 2\hbar k/(mg)$ for the linear barrier. On the other hand, the peculiarity of various tunneling times for linear barrier also proves that this potential profile is virtue of study for shape sensitive tunneling delays.

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Similar to conventional definition of Larmor times, we can also define

$\tau_{\text{Lar}} = \langle S_x \rangle / (\hbar \omega_{\text{Lar}})$, $\tau_{\text{CE}} = \langle S_x \rangle / (\hbar \omega_{\text{CE}})$ (28)

associated with spin precession for an incoming spin-$\frac{1}{2}$ particle initially polarized in the $x$-direction. The superscript or subscript $s = R$, $T$, denote the reflective and transmitted scattering channels, respectively, for a general potential. However, by intentionally aligning the magnetic field parallel instead of orthogonal to the momentum of an incoming neutron, the magnetic interaction in the potential region or the free region of interest gives rise to the asymmetry between opposite helicity components instead of ‘spin-$c$’ components (should be the spin-$x$ components in the coordinate frame defined here) for the spin state of the scattered particle. We observe that, when the barrier interaction dominates, such as the case of a particle tunneling through an opaque barrier, the helicity asymmetry is large, and correspondingly, $\tau_{\text{Lar}}^T$ gets much closer to $\tau_{\text{Lar}}$. Whereas for the case of a transversely polarized particle reflected from an opaque barrier, or transmitted through a free
Interestingly, by appropriately choosing region or a low barrier with \( A < E \) (\( A \) is the maximum height of the barrier), i.e. when magnetic interaction dominates, the resultant helicity asymmetry is tiny and the spin evolution is largely confined in the transverse plane, then our definition of \( \tau_{	ext{Lar}} \) is dominated by \( \tau_Y \).

In fact, resembling the Büttiker–Landauer (BL) time \( \tau_{	ext{BL}} \equiv \sqrt{\langle \tau^2 \rangle} \)\(^2 \) \( \tau^2 \) \[37, 38\] defined as the modulus of the complex time \( \tau_0 \equiv \tau_{	ext{Lar}} - i \tau_{	ext{CE}} \) \[42, 43\], where \( \tau^2_{	ext{Lar}} = \tau_{	ext{Lar}} \), \( \tau_{	ext{LM}} \) are the conventional Larmor times, we can also define the modulus \( \tau_L \equiv \sqrt{\langle \tau^2 \rangle} \equiv \sqrt{\langle \tau^2 \rangle + \langle \tau^2 \rangle} \). Note we have omitted the superscripts \( R \) and \( T \) unless the distinction is necessary. Interestingly, for the barriers we considered, \( \tau_L \equiv \tau_{	ext{Lar}} \), just as expected. Also it is not surprising that for opaque barriers, \( \tau_{	ext{Lar}} \) gets close to \( \tau_{	ext{BL}} \), and hence to the semi-classical time \( \tau_{	ext{SC}} \equiv \int \sqrt{\frac{m}{2(V(x) - E)}} \), since \( \tau_{	ext{BL}} \approx \tau_{	ext{SC}} \) for opaque barriers.

For the linear barrier, we find

\[
\tau_{\text{Lar}}^2 = \langle S_J \rangle \left( \frac{h}{2 \omega_{	ext{Lar}}} \right) = -\frac{1}{\gamma N B_0} \frac{2 F G}{F^2 + G^2},
\]

where we have set \( \alpha = 0 \) for calculational simplicity. Also we have omitted the superscript \( R \) in \( \tau_{\text{Lar}}^2 \), since there is no transmitted flux for a linear barrier. As mentioned above, for the reflective particle off a linear barrier, \( \tau_{	ext{Lar}} \) is dominated by \( \tau_L \); thus \( \tau_{\text{Lar}}^2 \) also gets closer to \( \tau_{	ext{CE}} \), with decreasing \( B_0 \) as \( \tau_{	ext{Lar}} \). However, unlike the case of \( \tau_{	ext{Lar}} \), \( \tau_{\text{Lar}}^2 \) monotonically deviates away from the CRT \( \tau_{	ext{CE}} \) with increasing \( k \), rather than oscillating around it.

Another interesting observation for linear barrier is that, if we formally follow the standard definition \( \tau_{	ext{LM}} \equiv -h \frac{\partial \phi_R}{\partial k} \) \[6, 21\], where \( \phi_R \) is the phase of the reflective amplitude \( \mathcal{R} \equiv |R|e^{i\phi_R} \) and \( V \) is the height of the barrier, and define the reflective Larmor time as

\[
\tau_{	ext{LM}}^R \equiv -h \frac{\partial \phi_R}{\partial k} = -2 \hbar \left[ \frac{\partial \theta_R}{\partial g} \frac{\partial g}{\partial V_g} + \frac{\partial \theta_R}{\partial m} \frac{\partial m}{\partial V_g} \right],
\]

where \( V_g = mgz \) and \( \theta_R = 2\theta_R \) is independent of \( z \), see the statements above equation (10). Clearly, \( \tau_{	ext{LM}}^R \) is \( z \)-dependent. Interestingly, by appropriately choosing \( z = (\hbar k)^2 / (m^2 g) \) and substituting it into (30), a direct calculation shows that \( \tau_{	ext{LM}}^R = \tau_{\text{phase}}^R \), where \( \tau_{\text{phase}}^R \) is given by equation (10). We think this ‘coincident equality’ is due to the special choice of \( z = 2\epsilon_c \) and a particular feature of linear barrier, since for other semi-infinite barriers, such as the step barrier \( V = A \theta(x) \), \( \tau_{	ext{LM}}^R = \frac{2m}{\kappa} \neq \tau_{\text{phase}}^R = \frac{2m}{\kappa} \), where \( \kappa \equiv \sqrt{2m(A - E)/\hbar^2} \).

To avoid confusion due to various symbols in the definitions of Larmor time, we summarize the symbols and the corresponding definitions of Larmor time appeared previously in table 2.

### Table 2.

For notational convenience, ‘Larmor Time’, ‘relevant tunneling time’ and ‘definition’ are briefly denoted as LT, RT and Def, respectively. Note for any Larmor time definition, there are reflective and transmitted channels for a general potential, which is denoted as the superscript ‘s’ in equation (28), but is omitted here for simplicity.

| LT        | Traditional LT | Def          | Redefined LT | Def          |
|-----------|----------------|--------------|--------------|--------------|
| \( \tau_{	ext{LM}} \) | \( \tau_{	ext{LM}} \) \[37\] | \( \tau_{\text{Lar}} \) \[37\] | \( \tau_{\text{Lar}} \) \[37\] | \( \tau_{\text{Lar}} \) \[37\] |
| \( \tau_{	ext{LM}} \) | \( \tau_{	ext{LM}} \) \[37\] | \( \tau_{\text{Lar}} \) \[37\] | \( \tau_{\text{Lar}} \) \[37\] | \( \tau_{\text{Lar}} \) \[37\] |
| \( \tau_{	ext{BL}} \) | \( \tau_{	ext{BL}} \) \[38\] | \( \tau_{\text{Lar}} \) \[38\] | \( \tau_{\text{Lar}} \) \[38\] | \( \tau_{\text{Lar}} \) \[38\] |

5. Summary

In this paper, we utilize phase time, dwell time and Larmor time to calculate the time delays for a neutral particle (taking neutron as an example) scattering off the linear gravitational potential. As the Earth gravitational field is very gentle due to the extreme weakness of gravity, the time scale for a vertically injected UCN climbing the potential is on the order of sub-millisecond, which is indicated by the naïve estimates in table 1 and the following calculations. As far as we know, the tunneling particles in most time measurements include photon \[44\] and electron \[18, 20\], whose relevant time scales are very short, say, femosecond. A fraction of experiments use atoms \[26, 27\], and the time scales are 10 ~ 100 microseconds. So sub-millisecond shall be experimentally realizable, though UCN may be not easy to manipulate. However, if the time measurement of UCN in the gravitational potential is experimentally feasible, it may probe the quantum nature of scattering states of the gravitational linear potential in the temporal domain, in complementary to the spatial domain quantum test of the discrete turning heights of gravitational bound states \[46\].

By comparing these times, we obtain the relation between Wigner phase time and dwell time, shown in equation (16). In the end of last section, we also find that the conventionally defined reflective Larmor time, equation (30) coincides with the phase time for the linear barrier. Interestingly, our new definition of Larmor time reduces to the CRT \( \tau_{	ext{CE}} = 2\epsilon_c / g \) in the zero magnetic field limit, and the dwell time in the barrier region, \( \tau_{\text{DW}} \), also equals to \( \tau_{	ext{CE}} \). This may not be an accident, but rather a temporal manifestation of the WEP, as first proposed in \[29\]. Our results seem support this proposal and will be interesting to explore with its own right.

By subtracting off the \( \tau_{	ext{CE}} \) from these times, we obtain the corresponding time delays. All these time delays tend to be vanishingly small with increasing neutron wave number \( k \), and the amplitude of Larmor time delay decreases more gen-
tly with increasing $k$, compared with other time delays, see figure 6. The excellent fit of the envelope of phase time delay, $\tau_{\text{coherence}}$, can be viewed as an evidence of the self-interference between incoming and reflective partial waves [36]. To further reveal the self-interference effects, we plot the reflective self-interference delay $\tau_B$ for linear, step and tanh-like barriers in figure 4, where the peculiar self-interference delay of linear barrier is demonstrated transparently. This can be attributed to the very gentleness and particular shape of the linear barrier. This peculiarity may be one of the reasons that WEP holds true even in quantum domain [47], where the advance decrease of the wave amplitude previous to the classical turning height cancels exactly with the tunneling induced quantum lag in the CFR. In fact, we have shown that the CFR contributes a small but indispensable part for the dwell time to match with the CRT.

It is also interesting to note that the Larmor time defined here does have operational meanings since no need for zero magnetic field limit. The magnetic field in our calculation is on the order of 0.1 mG, which is not very stringent for current technologies. Further, the configuration of magnetic field, initial spin and momentum is quite distinct from conventional definitions, and thus provides an alternative for experimental realization.

Further, the formalism can be directly applied to neutral atoms such as lithium. Lithium is of comparable mass with neutron, say, $^7\text{Li} \sim 7m_e$, and the combined Laser and the rf-induced forced evaporative coolings have already been able to reach $T \simeq 300$ nK for $^7\text{Li}$ [48, 49]. Since alkali atom is much easier to manipulate [50] and the sub-millisecond timing accuracy is not technically very stringent, ultracold $^7\text{Li}$ ($^6\text{Li}$) may be a good candidate to probe the quantum temporal behavior in the Earth gravitational field. For example, for a $^7\text{Li}$ atom with effective $T \simeq 823$ nK (corresponds to $v \simeq 54$ mm s$^{-1}$ and classical turning height $\simeq 148$ μm), the dwell time delay is on the order of $20 \sim 35$ μs.

At last, we note that a more rigorous and complete treatment is to start with a wave packet. In that case, all the time delays and the relevant processes we have calculated before, such as finding the spin expectation value of the reflective wave packet, have to be averaged over the weighting factor (or the distribution in $k$-space) $\rho(k)$ of the wave packet, see equation (4). That involves very tedious calculations, and will be left to future work.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Integral of Airy function

The integral of Airy function used in the main context is

$$\int_0^{\infty} dz \text{Ai}^2[z/L_c - E_D] = L_c \left\{ E_D \text{Ai}^2 + \text{Ai}^2 \right\} + \left( \frac{z}{L_c} - E_D \right) \frac{\text{Ai}\left[ \frac{z}{L_c} - E_D \right]^2}{\text{Ai}^2} - \frac{\text{Ai}\left[ \frac{z}{L_c} - E_D \right]^2}{\text{Ai}^2}. \quad (A1)$$

Appendix B. Spin-precession angle

For a spin-$\frac{1}{2}$ particle carrying non-zero magnetic moment, the spin precession equation is

$$\frac{dS_i}{dt} = \frac{i}{\hbar} [-\vec{\mu} \cdot \vec{B}, S_i] = \vec{g}_N \nabla \times \vec{B}. \quad (B1)$$

From (B1) we get the spin precession frequency, $\omega_0 = \vec{g}_N \nabla \times \vec{B}$, suppose the magnetic field $|\vec{B}| = B_0$.

Now assume initially the spin is polarized along the polar and azimuthal angles ($\beta, \gamma$), i.e. the initial spin state is $\frac{1}{\sqrt{2}} (\cos[\beta/2]e^{-i\gamma}, \sin[\beta/2])$. For later comparison, we can write it in the standard way up to a normalization constant

$$|\beta, \gamma\rangle = \frac{\text{Ai}e^{-i\gamma}}{\sqrt{2}}. \quad (B2)$$

For example, for the neutron polarized in the antipodal direction, $\beta \rightarrow \beta + \pi$, the spin state is $\frac{1}{\sqrt{2}} (-\tan[\beta/2]e^{-i\gamma}, 1)^T$. Assume the magnetic field is in the $z$-direction ($\theta = 0$ in the main context) and the initial state is polarized horizontally with the azimuthal angle $\alpha$, in other words,

$$|\pi/2, \alpha\rangle \langle z| = \frac{1}{\sqrt{2}} (e^{-i\alpha}, 1) e^{ik}. \quad (B3)$$

The reflected partial wave is in the spin state

$$|\pi/2 + \delta \theta, \alpha + \delta \phi\rangle \langle z| - k \rangle = \frac{1}{\sqrt{2}} (R^+, R_-) e^{-ik}. \quad (B3)$$

where $\delta \theta, \delta \phi$ are the change of polarization angles. From the standard form of spin state (B2), we can immediately read out

$$\text{cot}[\frac{1}{2}(\pi/2 + \delta \theta)] = |\frac{R^+}{R_-}| = 1 \Rightarrow \delta \theta = 0. \quad (B4)$$
so the spin evolves on the horizontal large circle on the Bloch sphere, which is dictated by the spin evolution equation (B1).

The rotated angle is then purely $\delta \phi$, and can be determined by the following two methods.

**B.1. Method A**

First note that

$$\mathcal{R}_+ = e^{-i\alpha} \left( \frac{A_i^+ A_i^+ + \frac{1}{2} \tilde{A}_i^+ \tilde{A}_i^+}{A_i^+ A_i^- + \frac{1}{2} \tilde{A}_i^+ \tilde{A}_i^-} \right) + \frac{i}{2} \left( \frac{A_i^+ A_i^- - \tilde{A}_i^+ \tilde{A}_i^-}{A_i^+ A_i^- + \frac{1}{2} \tilde{A}_i^+ \tilde{A}_i^-} \right),$$

where $-\delta \phi$ is the rotated angle of the spin on the horizontal large circle. From (B5), we find

$$-\tan[\delta \phi/2] = \frac{k_0 (A_i^+ A_i^- - A_i^+ A_i^-)}{(k_0^2 A_i^+ A_i^- + A_i^+ A_i^-)} \Rightarrow \delta \phi = -2 \tan^{-1} \left( \frac{k_0 (A_i^+ A_i^- - A_i^+ A_i^-)}{(k_0^2 A_i^+ A_i^- + A_i^+ A_i^-)} \right).$$

(B6)

So the time measured by the Larmor clock is

$$\tau_{L} = \frac{\delta \phi}{\omega_{L}} = \frac{2m_t}{k^2 \hbar} \tan^{-1} \left( \frac{k_0 (A_i^+ A_i^- - A_i^+ A_i^-)}{(k_0^2 A_i^+ A_i^- + A_i^+ A_i^-)} \right).$$

**B.2. Method B**

The other method is to calculate the expectation value of spin vector $\vec{S} = \frac{1}{2} \sigma$. Substituting the reflective spin state (B3), we can get

$$\langle S_z \rangle = \frac{1}{4} \left[ |R_+|^2 - |R_-|^2 \right],$$

$$\langle S_y \rangle = \frac{1}{4i} \left[ |R_+|^2 - |R_-|^2 \right],$$

$$\langle S_x \rangle = \frac{1}{4} \left[ |R_+|^2 + |R_-|^2 \right].$$

(B8)

In the case $\theta = 0$, where the magnetic field is along $z$-direction, direct calculation gives ($\langle S_z \rangle = 0$), and

$$\langle S_y \rangle = \frac{2GF \cos \alpha + (F + G)(F - G) \sin \alpha}{2[AI_i^+ + k_0 AI_i^-][AI_i^+ + k_0 AI_i^-]},$$

$$\langle S_x \rangle = \frac{(F + G)(F - G) \cos \alpha - 2GF \sin \alpha}{2[AI_i^+ + k_0 AI_i^-][AI_i^+ + k_0 AI_i^-]},$$

(B9)

where $F \equiv [A_i^+ A_i^- + k_0^2 A_i^+ A_i^-], G \equiv [A_i^+ A_i^- - A_i^+ A_i^-], k_0$, and the denominator in (B9) is $2(F^2 + G^2)$. Note $\langle S_z \rangle = 0$ means the reflected spin state is on the horizontal plane, i.e. $\delta \theta = 0$, so the azimuthal angle of the new spin state can be obtained from

$$\tan[\alpha + \delta \phi] = \frac{\langle S_y \rangle}{\langle S_x \rangle} = \frac{2GF/(F^2 - G^2) + \tan \alpha}{1 - 2GF/(F^2 - G^2) \tan \alpha}.$$  

(B10)

Finally we can read out the rotated angle

$$\delta \phi = \arctan[2GF/(F^2 - G^2)] = 2 \arctan[G/F].$$

(B11)

It is easy to check that (B11) is exactly the same as (B6), confirming our calculations.

**Appendix C. Larmor time for free motion and square barrier**

In this section, we discuss two special cases with our definition of Larmor time, the free motion and rectangular barrier. For the former case, we will show that it indeed reduces to the classical traversal time as expected, while for the latter case, the definition also generates reasonable results. To show that our Larmor time definition doesn’t rely on special coordinate configurations, different from the main text, we choose instead the $x$-coordinate as the moving direction, while the particle is initially polarized in the spin-$z$ eigenstate.

**C.1. Free motion**

The relevant Hamiltonian for free particle is

$$\hat{H}_{\text{free}} = \frac{\hat{p}^2}{2m} - \frac{\hbar \mu_B \tilde{g} \hat{B}_0}{2} \cdot \hat{n}(\theta(x)\theta(a - x),$$

(C1)

where $[0, a]$ is the space interval we are interested, and is covered by a homogeneous weak magnetic field $\hat{B} = \hat{B}_0 \tilde{n}$. As mentioned in the main text, $\tilde{n}$ is chosen to be parallel to the $x$-direction, i.e. $\tilde{n} = \hat{x}$. For simplicity, consider the stationary solution with a monochromatic neutron beam. The ansatz of the corresponding wave function is

$$\phi(x) = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ix} + \begin{pmatrix} R'_+ \\ R'_- \end{pmatrix} e^{-ix}, & x < 0, \\ U \begin{pmatrix} c_+ e^{ik x} + d_+ e^{-ik x} \\ c_- e^{-ik x} + d_- e^{ik x} \end{pmatrix}, & 0 < x < a, \\ \frac{1}{\sqrt{2}} \begin{pmatrix} T'_+ \\ T'_- \end{pmatrix} e^{ix}, & x > a, \end{cases}$$

(C2)

where $U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, k^2 \equiv \frac{2mE}{\hbar^2}$ and $k_0^2 \equiv 2m(E + \hbar \mu_B \tilde{g} B_0)/\hbar^2$. Imposing the continuity conditions at $x = 0$ and $x = a$, we get the following solutions for the transmitted and reflective coefficients

$$T'_+ = \frac{1}{2} (T_k + T_k), \quad T'_- = \frac{1}{2} (T_k - T_k),$$

(C3)

$$R'_+ = \frac{1}{2} (R_k + R_k), \quad R'_- = \frac{1}{2} (R_k - R_k),$$

(C4)

where $T_0 \equiv (k^2 - r^2)(\sin(\theta a) + 2k_0 \cos(\theta a))$ and $R_0 \equiv (k^2 - r^2)(\sin(\theta a) + 2k_0 \cos(\theta a))$. The transmitted and reflective amplitudes for a rectangular barrier (or well), respectively, and $\rho$ takes values of $k_+ , k_-$. The spin of the incoming particle is along the positive $z$-direction and is denoted as
\[ |\theta_1, \phi_i \rangle = |0, \phi_i \rangle = (1, 0)^T, \] while the spin for the transmitted particle is denoted as \(|\theta_f, \phi_f \rangle\). Since in the zero magnetic field limit, \( T^I \to 0 \), we’d better normalize the spin state with the polar and azimuthal angles \((\beta, \gamma)\) by
\[
|\beta, \gamma \rangle = \frac{1}{\sqrt{2}} \left( e^{i\beta} |\gamma \rangle + e^{-i\beta} |\gamma \rangle \right).
\]

According to this normalization, the polarization of the transmitted partial wave can be written as
\[
|\theta_f, \phi_f \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ T^I \end{pmatrix} = \frac{1}{\sqrt{2}} \left( \frac{T_{k_+} - T_{k_-}}{T_{k_+} + T_{k_-}} \right).
\]

Comparing (C5) with (C6), we can readily get
\[
\tan[\theta_f/2]e^{i\phi_f} = \frac{T_{k_-} - T_{k_+}}{T_{k_-} + T_{k_+}}.
\]

Since at the north pole \( \phi_i \) can be arbitrary chosen, for convenience, we chose \( \phi_i = \phi_f \), then we only need the modulus to read out the rotation angle
\[
\theta_f = 2 \arctan \left( \frac{T_{k_-} - T_{k_+}}{T_{k_-} + T_{k_+}} \right),
\]
and the Larmor time is given by
\[
\tau_{Lar} = \frac{\theta_f}{\omega_{Lar}} = \frac{2}{\gammaNB_0} \arctan \left( \frac{T_{k_-} - T_{k_+}}{T_{k_-} + T_{k_+}} \right),
\]
where \( \omega_{Lar} \equiv 3\gamma B_0 \) is the Larmor frequency and \( \gamma_N = 1.832 \times 10^9 \text{rad}/(\text{sT}) \) is the gyromagnetic ratio of neutron. In figure 7, we plot the Larmor time with respect to classical traversal time \( \tau_{CT} \equiv am/(\hbar k) \) and the relative time error \( \delta\tau \equiv |\tau_{Lar} - \tau_{CT}|/\tau_{CT} \). From figure 7, we see the Larmor time given by (C9) fits well with the classical traversal time for two arbitrarily chosen space interval \( a = 6 \mu m \) and \( a = 10.56 \mu m \). In figure 7(b), we also see the relative time error grows large only around relatively small \( k \) and large \( B_0 \), which is as expected because the magnetic interaction becomes more relevant when \( \hbar \mu g B_0 \) grows large, and this also drives the neutron away from free motion.

### C.2. Rectangular barrier

For a rectangular barrier, the relevant Hamiltonian in calculating Larmor time is
\[
H_{SB} = \frac{\hat{p}^2}{2m} + A \theta(x) \hat{\sigma} (L - x) - \frac{\hbar \mu g B_0}{2} \hat{\sigma} \cdot \hat{n} \theta(x + a) \hat{\sigma} (L - x),
\]
where \( a > 0 \) is to allow for comparison of reflective Larmor time with CRT. Though it seems more reliable and complete to start with a wave packet, and in that case, for a wave packet with finite width \( \delta k \) in \( k \)-space, it is suitable to choose \( a > 1/\delta k \), we still work in the monochromatic limit, since the wave packet formalism evolves integration over wave number \( k \), and hence is not easy or even impossible to obtain simple analytical formulas. For the wave packet formalism, we leave it to future work. Then the ansatz of the stationary wave function is
\[
\phi(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left( e^{ikx} + \frac{R^+}{R^-} e^{-ikx} \right), \quad x < -a,
\]
\[
U \begin{pmatrix} f_+ e^{ikx} + g_+ e^{-ikx} \\ f_- e^{-ikx} + g_- e^{ikx} \end{pmatrix}, \quad -a < x < 0,
\]
\[
U \begin{pmatrix} c_+ e^{ikx} + d_+ e^{-ikx} \\ c_- e^{-ikx} + d_- e^{ikx} \end{pmatrix}, \quad 0 < x < L,
\]
where \( k, k_\pm \) have already been defined in the subsection of free motion, and \( \kappa_0^2 = 2m(A - E \pm \frac{\hbar \mu g B_0}{2})/\hbar^2 \). Imposing the continuity conditions at \( x = -a, 0, L \), we can get the reflec-
Figure 8. (a) Reflective Larmor time $\tau^R_{Lar}$ and CRT $\tau^R_{CRT}$ with respect to the free space interval $a$ in front of the barrier. For comparison, we also plot $y$-component Larmor time $\tau^R_{y,L}$. These times are divided into two classes with different incoming neutron energies, which are labeled in the square bracket in the legends. The height and width of the barrier are 48 peV and $L = 2.7 \mu m$, respectively. (b) Transmitted Larmor time $\tau^T_{Lar}$ and Büttiker–Landauer (BL) time $\tau^T_{BL}$ with respect to the barrier width $L$. For comparison, we also plot component Larmor times $\tau^T_{x,L}$, $\tau^T_{y,L}$, free traversal time $\tau^T_{Free}$ and Wigner phase time $\tau^T_{EW}$. The incoming neutron energy is 20.3 peV, the barrier height is 21 peV and the free space interval is $a = 0.2 \mu m$. In both subfigures, the external magnetic field is chosen as $B_0 = 0.1 \text{ mG}$.

Figure 9. (a) The wave number $k$-spectrum of reflective Larmor times $\tau^R_{Lar}$ and the CRT. For comparison, we also show the Wigner phase time $\tau^T_{EW}$, BL time $\tau^T_{BL}$ and component Larmor times $\tau^R_{x,L}$, $\tau^R_{y,L}$ together in the subfigure. The barrier parameters are: height $A = 90$ peV and height $L = 1.2 \mu m$, and the free space interval $a = 9.6 \mu m$. (b) The wave number $k$-spectrum of transmitted Larmor time $\tau^T_{Lar}$ with transmitted BL time $\tau^T_{BL}$. In comparison, we also plot the component barrier times $\tau^T_{x,L}$, $\tau^T_{y,L}$, together with semi-classical time $\tau^T_{SC}$ and Wigner phase time $\tau^T_{EW}$. The barrier parameters are $A = 36$ peV and $L = 9 \mu m$, and $a = 0.2 \mu m$. In both subfigures, the external magnetic field is chosen as $B_0 = 0.1 \text{ mG}$.

tive and transmitted coefficients as before. Interestingly, these coefficients take the similar form as in (C3), in other words,

$$ R^0_\pm = \frac{1}{2} [R^0_+ \pm R^0_-], \quad T^0_\pm = \frac{1}{2} [T^0_+ \pm T^0_-], \quad (C12) $$

where $R^0_\pm$, $T^0_\pm$ are just the reflective and transmitted amplitudes for a spinless particle scattering off the barrier $A \theta(x)\theta(L-x) + V_0 \theta(x+a)\theta(L-x)$, where $V_0 = \frac{\hbar \mu \omega B}{2}$. For example,

$$ \tau^0_\mu = \frac{e^{-(A+L)}}{\cos(k_\rho a)A - i \sin(k_\rho a)B}. \quad (C13) $$

where $A \equiv \text{ch}(\kappa_\rho L) + \frac{1}{4}(k_\rho^2 + \kappa_\rho^2)\text{sh}(\kappa_\rho L)$ and $B \equiv \frac{1}{2}(k_\rho^2 \text{ch}(\kappa_\rho L) + \frac{1}{4}(k_\rho^2 + \kappa_\rho^2)\text{sh}(\kappa_\rho L)$, and $\rho$ represents $\pm$. Since the full expression is lengthy, we do not show explicitly the
reflective amplitude here. From the discussion of the free motion Larmor time, we can get the reflective and transmitted Larmor times as

\[
\tau_{\text{R,Lar}}^R = \frac{2}{\gamma\hbar B_0} \tan^{-1} \left[ \frac{\mathcal{R}_s^-}{\mathcal{R}_s^+} \right],
\]

(C14)

\[
\tau_{\text{L,Lar}}^R = \frac{2}{\gamma\hbar B_0} \tan^{-1} \left[ \frac{\mathcal{T}_s^-}{\mathcal{T}_s^+} \right].
\]

(C15)

To show this definition is reasonable, we compare the reflective Larmor time \(\tau_{\text{R,Lar}}^R\) with the CRT \(\tau_{\text{CRT}}^R \equiv 2am/(\hbar k)\), and also compare the transmitted Larmor time with the Wigner phase time \(\tau_{\text{EW}}^T = \partial |T|/\partial A\). To get an intuition, we plot them in figures 8 and 9. Further, we also plot the component Larmor times \(\tau_{\text{Lar}}^x\), \(\tau_{\text{Lar}}^y\) defined in (28), only here the particle motion direction, i.e. the \(z\)-coordinate defined in the main context has been changed into \(x\)-coordinate. In figure 8(a), the barrier height and width are 48 peV and 2.7 \(\mu\)m, respectively. Comparing the distances of the solid blue curve to the dashed red line, and the solid yellow curve to the dashed purple line, we find that the oscillating Larmor time \(\tau_{\text{Lar}}^R\) gets more closer to the CRT \(\tau_{\text{CRT}}\) for less energetic particles. In other words, the more opaque the barrier is, the more it resembles a classical wall. We can also see that \(\tau_{\text{Lar}}^R\) nearly overlaps with \(\tau_{\text{Lar}}^R\) from the two pairs curves. This indicates that the spin precession for reflected partial wave is nearly confined in the transversal plane, i.e. \(y = z\) plane in this coordinates frame. This fact can also be confirmed from the nearly overlapped curves of \(\tau_{\text{R,Lar}}^R\) and \(\tau_{\text{Lar}}^R\) in figure 9(a), where \(\tau_{\text{R,Lar}}^R\) is very close to the BL time \(\tau_{\text{BL}}^R\) and Wigner phase time \(\tau_{\text{EW}}^R\), but far smaller than \(\tau_{\text{R,Lar}}^R\). Note it is not easy to find \(\tau_{\text{R,Lar}}^R\) in figure 9(b), since they nearly overlap with the horizontal \(k\) axis. We can also see in figure 9(a) that, \(\tau_{\text{Lar}}^R\) oscillates around the curve \(\tau_{\text{CRT}}^R\) which means that for thick barriers, disregard quantum fluctuations, \(\tau_{\text{Lar}}^R\) can give a good measure of the returning time.

For the transmitted Larmor time, we also plot the classical transversal time \(\tau_{\text{Free}} = \frac{2mL_{\text{Free}}}{\hbar k}\) in figure 8(b) as a comparison, see the dotted red line. However, even for energetic particles (the particle’s energy in figure 8(b) is 20.3 peV compared to the barrier height 21 peV), \(\tau_{\text{Lar}}^T\) gets close to the free traversal time only for very thin barriers, while for thick barrier, our transmitted Larmor time resembles more closely to the BL time \(\tau_{\text{BL}}^R\) as presented by the dashed green curve. The \(\tau_{\text{SC}} = \frac{L_{\text{Free}}}{\hbar k} \frac{L_{\text{Free}}}{\sqrt{2m(\hbar k)^2}}\) matches \(\tau_{\text{Lar}}^T\) (the solid blue curve) only for partial intruder, see the solid purple curve. We can also see that \(\tau_{\text{Lar}}^T\) nearly coincides with \(\tau_{\text{BL}}^R\) and is very close to \(\tau_{\text{Lar}}^T\), this can be further confirmed from figure 9(b), while \(\tau_{\text{Lar}}^T\) is much smaller than \(\tau_{\text{Lar}}^T\) and it is very close to Wigner phase time \(\tau_{\text{EW}}^T\) for energetic neutrals, as can be also seen in figure 9(b). So for opaque barriers, \(\tau_{\text{Lar}}^T\) is dominated by \(\tau_{\text{Lar}}^T\). This means that the helicity asymmetry caused by barrier interactions during tunneling is large, and the spin precession cannot be confined in the transversal plane. As a complementary observation, the good match of \(\tau_{\text{EW}}^T\) with the sub-dominate \(\tau_{\text{LY}}^T\) for energetic tunneling particles confirms that Wigner phase time delay is not a meaningful measure of particle tunneling time.

In conclusion, we see our Larmor time definition is free of the coordinate choice and is rational, as verified by the comparison of traversal time and the other frequently used time definitions (such as the Böttiker–Landauer time) in free motion and square barrier cases. A further clarification of the relation between the component Larmor times \(\tau_{\text{Lar}}^x\), \(\tau_{\text{Lar}}^y\) with our Larmor time definition \(\tau_{\text{Lar}}^R\) and various discussed tunneling times in the literature will be interesting, but is out of the scope of this work.

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**References**

[1] Condon E U 1931 Rev. Mod. Phys. 3 43

[2] Maccoll L A 1932 Phys. Rev. 40 621–6

[3] Eisenbud L 1948 PhD Thesis Princeton

[4] Wigner E P 1955 Phys. Rev. 98 145

[5] Bohm D 1951 Quantum Theory (Englewood Cliffs, NJ: Prentice-Hall)

[6] Smith F T 1960 Phys. Rev. 118 349

[7] Baz’ A I 1967 Sov. J. Nucl. Phys. 1 182

[8] Baz’ A I 1967 Sov. J. Nucl. Phys. 5 161

[9] Yamada N 2004 Phys. Rev. Lett. 93 170401

[10] Landauer R and Martin T 1994 Rev. Mod. Phys. 66 217

[11] Winful H G 2006 Phys. Rep. 436 1–69

[12] Hartman T E 1962 J. Appl. Phys. 33 3427

[13] Xiao Z, Huang H and Lu X-X 2015 Int. J. Mod. Phys. B 29 1550052

[14] Büttiker M and Landauer R 1982 Phys. Rev. Lett. 49 1739

[15] Pollak E and Miller W H 1984 Phys. Rev. Lett. 53 115

[16] Hauge E H and Støvneng J A 1989 Rev. Mod. Phys. 61 917

[17] Landsman A S and Keller U 2015 Phys. Rev. Lett. 115 24

[18] Yamada N 1999 Phys. Rev. Lett. 83 3350

[19] Nussenzveig H M 2000 Phys. Rev. A 62 042107

[20] Haro K and Obha I 2003 Phys. Rev. A 67 052105

[21] Eckle P, Smolarski M, Schlup P, Biegert J, Staudte A, Schöffler M, Muller H G, Dörner R and Keller U 2008 Nat. Phys. 4 565

[22] Schulzke M et al 2010 Science 328 1658

[23] Sainadh N S et al 2019 Nature 568 75

[24] Zimmermann T, Mishra S, Doran B R, Gordon D F and Landsman A S 2016 Phys. Rev. Lett. 116 233603

[25] Ni H, Saalmann U and Rost J M 2016 Phys. Rev. Lett. 117 023002

[26] Sokolovski D 2017 Phys. Rev. A 96 022120

[27] Trilina L et al 2015 Nat. Phys. 11 503

[28] Camus N et al 2017 Phys. Rev. Lett. 119 032301

[29] Ramos R, Spierings D, Racicott I and Steinberg A M 2020 Nature 583 529

[30] Okulov D 2017 Phys. Rev. A 96 022120

[31] Fortun A, Cabrera-Gutiérrez C, Condon G, Michon E, Billy J and Gury-Oedlin D 2016 Phys. Rev. Lett. 117 010401

[32] Jenke T, Gelentenbort P, Lemmel H and Abele H 2011 Nat. Phys. 7 468

[33] Cronenberg G, Brax P, Filter H, Gelentenbort P, Jenke T, Pignol G, Pittschmann M, Thalhammer M and Abele H 2018 Nat. Phys. 14 1022
[29] Davies P C W 1986 J. Phys. A: Math. Gen. 19 2115
Davies P C W 2004 Class. Quantum Grav. 21 2761
Davies P C W 2005 Am. J. Phys. 73 23

[30] Ballentine L E 1998 Quantum Mechanics: A Modern Development (Singapore: World Scientific) https://doi.org/10.1142/3142
Zeng J 2003 Quantum Mechanics Course (Beijing: Science Press)

[31] Nesvizhevsky V V et al 2005 J. Res. Natl Inst. Stand. Technol. 110 263

[32] Xiao Z and Shao L 2020 J. Phys. G: Nucl. Part. Phys. 47 085002

[33] Eckle P, Pfeiffer A N, Cirrelli C, Staudte A, Dorner R, Muller H G, Büttiker M and Keller U 2008 Science 322 1525

[34] Keldysh L V 1965 J. Exp. Theor. Phys. 20 1307

[35] Hofmann C, Landsman A S and Keller U 2019 J. Mod. Opt. 66 1052

[36] Fertig H A 1990 Phys. Rev. Lett. 65 2321

[37] Büttiker M 1983 Phys. Rev. B 27 6178

[38] Büttiker M and Landauer R 1988 J. Phys. C: Solid State Phys. 21 6207

[39] Winful H G 2003 Phys. Rev. Lett. 91 260401

[40] Winful H G, Ngom M and Litchinitser N M 2004 Phys. Rev. A 70 052112

[41] Steinberg A M 1995 Phys. Rev. Lett. 74 2405

[42] Sokolovski D and Baskin L M 1987 Phys. Rev. A 36 4604

[43] Leavens C R and Aers G C 1987 Solid State Commun. 63 1101

[44] Steinberg A M, Kwiat P G and Chiao R Y 1993 Phys. Rev. Lett. 71 708

Borjemscaia N, Polyakov S V, Lett P D and Migdall A 2010 Opt. Express 18 2279

[45] Peres A 1980 J. Phys. A 48 552

[46] Nesvizhevsky V V et al 2002 Nat. Phys. 415 297

[47] Giulini D 2011 arXiv:1105.0749[gr-qc]

[48] Bradley C C, Sackett C A and Hulet R G 1997 Phys. Rev. Lett. 78 985

[49] Grier A T, Ferrierre-Barbut I, Rem B S, Delehaye M, Khaykovich L, Chevy F and Salomon C 2013 Phys. Rev. A 87 063411

Hamilton P, Kim G, Joshi T, Mukherjee B, Tiarks D and Müller H 2014 Phys. Rev. A 89 023409

[50] Hulet R G, Nguyen J H V and Senaratne R 2020 Rev. Sci. Instrum. 91 011101