CHARACTERIZING MULTIGRADED REGULARITY ON PRODUCTS OF PROJECTIVE SPACES

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Abstract. We explore the relationship between multigraded Castelnuovo–Mumford regularity, truncations, Betti numbers, and virtual resolutions. We prove that on a product of projective spaces $X$, the multigraded regularity region of a module $M$ is determined by the minimal graded free resolutions of the truncations $M_{\geq d}$ for $d \in \text{Pic}(X)$. Further, by relating the minimal graded free resolutions of $M$ and $M_{\geq d}$ we provide a new bound on multigraded regularity of $M$ in terms of its Betti numbers. Using this characterization of regularity and this bound we also compute the multigraded Castelnuovo–Mumford regularity for a wide class of complete intersections.

1. Introduction

Let $S$ be the polynomial ring on $n + 1$ variables over an algebraically closed field $k$ and $m$ its maximal homogeneous ideal. A coherent sheaf $\mathcal{F}$ on the projective space $\mathbb{P}^n = \text{Proj} S$ is $d$-regular for $d \in \mathbb{Z}$ if

1. $H^i(\mathbb{P}^n, \mathcal{F}(b)) = 0$ for all $i > 0$ and all $b \geq d - i$.

The Castelnuovo–Mumford regularity of $\mathcal{F}$ is then the minimum $d$ such that $\mathcal{F}$ is $d$-regular. In [EG84], Eisenbud and Goto considered the analogous condition on the local cohomology of a finitely generated graded $S$-module $M$, proving the equivalence of the following:

2. $H^i_m(M)_b = 0$ for all $i \geq 0$ and all $b > d - i$;
3. the truncation $M_{\geq d}$ has a linear free resolution;
4. $\text{Tor}_i(M, k)_b = 0$ for all $i \geq 0$ and all $b > d + i$.

In particular, if $M = \bigoplus_p H^0(\mathbb{P}^n, \mathcal{F}(p))$ is the graded $S$-module corresponding to $\mathcal{F}$ (so that $H^0_m(M) = H^1_m(M) = 0$) then conditions (1) through (4) are equivalent (c.f. [Eis05, Prop. 4.16]).

In [MS04], Maclagan and Smith introduced the notion of multigraded Castelnuovo–Mumford regularity for finitely generated Pic($X$)-graded modules over the Cox ring of a smooth projective toric variety $X$. In essence their definition is a generalization of condition (2). In this setting the multigraded regularity of a module is a semigroup inside Pic($X$) rather than a single integer.

When $X = \mathbb{P}^n$ the minimum element of the multigraded regularity recovers the classical Castelnuovo–Mumford regularity. However, when $X$ has higher Picard rank, translating the geometric definition of Maclagan and Smith into algebraic conditions like (3) and (4) above is an open problem. In this article we focus on the case when $X$ is a product of projective spaces and explore the relationship between multigraded regularity, truncations, Betti numbers, and virtual resolutions.

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The obvious way one might hope to generalize Eisenbud and Goto’s result to products of projective spaces is false: the truncation \( M_{≥d} \) of a \( d \)-regular Pic(X)-graded module \( M \) can have nonlinear maps in its minimal free resolution (see Example 4.2). We show that under a mild saturation hypothesis, multigraded Castelnuovo–Mumford regularity is determined by a slightly weaker linearity condition, which we call quasilinearity (see Definition 4.3).

Let \( S \) be the \( \mathbb{Z}^r \)-graded Cox ring of \( \mathbb{P}^n := \mathbb{P}^n \times \cdots \times \mathbb{P}^n \) and \( B \) the corresponding irrelevant ideal. The following complex contains all allowed twists for a quasilinear resolution generated in degree zero on a product of 2 projective spaces:

\[
\begin{array}{ccc}
0 & \longrightarrow & S(-1, 0) \\
& \oplus & S(-1, -1) \\
& & S(0, -2)
\end{array}
\]

Within each term, the summands in the left column (green) are linear syzygies while those in the right column (pink) are nonlinear syzygies. In general, for twists \(-b\) appearing in the \( i\)-th step of a quasilinear resolution, the sum of the positive components of \( b - d - 1 \) is at most \( i - 1 \), where \( d \) is the degree of all generators.

Our main theorem characterizes multigraded regularity of modules on products of projective spaces in terms of the Betti numbers of their truncations.

**Theorem A.** Let \( M \) be a finitely generated \( \mathbb{Z}^r \)-graded \( S \)-module with \( H_B^0(M) = 0 \). Then \( M \) is \( d \)-regular if and only if \( M_{≥d} \) has a quasilinear resolution \( F_* \) with \( F_0 \) generated in degree \( d \).

In [BES20, Thm. 2.9] Berkesch, Erman, and Smith established a similar result characterizing multigraded regularity of modules on products of projective spaces in terms of the existence of short virtual resolutions of a certain shape. In contrast, Theorem A constructs specific, efficiently computable virtual resolutions that determine multigraded regularity (see Section 4.2). These resolutions still have length at most the dimension of the space.

The proof of Theorem A is based in part on a Čech–Koszul spectral sequence that relates the Betti numbers of \( M_{≥d} \) to the Fourier–Mukai transform of \( \tilde{M} \) with Beilinson’s resolution of the diagonal as the kernel. Precisely, if \( M \) is \( d \)-regular and \( H_B^0(M) = 0 \) we prove the equality

\[
\dim_k \ker \text{Tor}^S_j(M_{≥d}, k)_a = h^{|a|-j}(\mathbb{P}^n, \tilde{M} \otimes \Omega_{\mathbb{P}^n}(a)) \quad \text{for } |a| ≥ j ≥ 0, \tag{1.1}
\]

where the \( \Omega_{\mathbb{P}^n}^a \) are cotangent sheaves on \( \mathbb{P}^n \). The regularity of \( M \) implies certain cohomological vanishing for \( \tilde{M} \otimes \Omega_{\mathbb{P}^n}(a) \), which, using (1.1), implies quasilinearity of the resolution of \( M_{≥d} \). Conversely, building on [BES20, Thm. 2.9], a computation of \( H_B^1(S) \) shows that the cokernel of a quasilinear resolution generated in degree \( d \) is \( d \)-regular.

Since a linear resolution is necessarily quasilinear, Theorem A implies that having a linear truncation at \( d \) is strictly stronger than being \( d \)-regular. That is to say, when \( H_B^0(M) = 0 \):

\[
M_{≥d} \text{ has a linear resolution generated in degree } d \quad \Longrightarrow \quad M_{≥d} \text{ has a quasilinear resolution generated in degree } d \quad \iff \quad M \text{ is } d\text{-regular}.
\]

Using (1.1), we also get a cohomological characterization of when \( M_{≥d} \) has a linear resolution. In addition, Corollary 6.5 implies that when a \( d \)-regular module \( M \) has a linear presentation, then the virtual resolution used in the proof of [BES20, Thm. 2.9] is in fact isomorphic to the minimal free resolution of \( M_{≥d} \).
Unlike in the case of a single projective space, the multigraded Betti numbers of a module $M$ do not determine its multigraded regularity. For instance, in Example 5.1 we construct two modules with the same multigraded Betti numbers but different multigraded regularities. Hence the Betti numbers of $M$ also do not determine the Betti numbers of $M_{\geq d}$. Still, we can intersect combinatorially defined regions $L_i(b)$ and $Q_i(b)$ (see Figure 1) to specify a subset of the degrees $d \in \mathbb{Z}^r$ where $M_{\geq d}$ has a linear or quasilinear resolution generated in degree $d$.

**Theorem B.** Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $S$-module, and define the set $\beta_i(M) := \{b \in \mathbb{Z}^r \mid \text{Tor}_i^S(M, k)_b \neq 0\}$.

1. If $d \in \bigcap_{i \in \mathbb{N}} \bigcap_{b \in \beta_i(M)} Q_i(b)$ then $M_{\geq d}$ has a quasilinear resolution generated in degree $d$.
2. If $d \in \bigcap_{i \in \mathbb{N}} \bigcap_{b \in \beta_i(M)} L_i(b)$ then $M_{\geq d}$ has a linear resolution generated in degree $d$.

On a single projective space we recover condition (4) of Eisenbud–Goto. Our proof of Theorem B is based on the observation that we can construct a possibly nonminimal free resolution of $M_{\geq d}$ from the truncations of the terms in the minimal free resolution of $M$.

A number of inner\textsuperscript{1} bounds on the multigraded regularity of a module in terms of its Betti numbers exist in the literature. For example, [MS04, Cor. 7.3] used a local cohomology long exact sequence argument to deduce such a bound. These methods were extended in

\textsuperscript{1}We use the terms inner and outer bound since in general there is no total ordering on $\text{reg} X$ when $\text{Pic} X \neq \mathbb{Z}$. For a single projective space an inner bound corresponds to an upper bound and an outer bound to a lower bound.
BC17, Thm. 4.14] using a local cohomology spectral sequence. Our bound in Theorem B is generally larger and thus closer to the actual regularity than these results.

Moreover, Theorem B is sharp in a number of examples. For instance, we use Theorem A to show that the containment in (1) is equal to the regularity for all saturated ample complete intersections, meaning those determined by ample hypersurfaces.

**Theorem C.** Suppose \( \langle f_1, \ldots, f_c \rangle \subset B \) is a saturated complete intersection of codimension \( c \) in \( S \), so the affine subvariety defined by it contains the irrelevant locus \( V(B) \). Then

\[
\text{reg} \left( \frac{S}{\langle f_1, \ldots, f_c \rangle} \right) = Q_c \left( \sum_{i=1}^{c} \deg f_i \right).
\]

Note that on a product of projective spaces the intermediate cohomology of a complete intersection does not necessarily vanish. Even the local cohomology of a hypersurface in a product of projective spaces is not determined by its degree [BC17, Sec. 4.5]. Thus computing the multigraded regularity of complete intersections on products of projective spaces is more complicated than in the case of a single projective space.

This highlights a theme from the literature on multigraded regularity [MS04; HW04; SV04; SVW06; Hô7; CMR07; BC17; CN20], Tate resolutions [EES15; BE21], virtual resolutions [BES20; BKLY21; HNT21; Lop21; Yan21], and syzygies [HSS06; HS07; Her10; Bru19; Bru20], that algebraic and homological properties of modules on toric varieties are more nuanced than in the standard graded setting.

**Outline.** The organization of the paper is as follows: Section 2 gathers background results and fixes our notation. Section 3 defines minimal virtual resolutions and constructs one from the Beilinson spectral sequence. Readers not familiar with derived categories can skip Sections 3.2 and 3.3 except the statement of Proposition 3.7. Section 4 proves Theorem A, describing the relationship between multigraded regularity and quasilinear truncations. Section 5 proves Theorem B, describing the relationship between multigraded Betti numbers and resolutions of truncations, and Theorem C, computing regularity for a class of complete intersections. Section 6 sharpens our theorems in the case of linear truncations. Finally, Section 7 summarizes our results about the regions defined by truncations, Betti numbers, and multigraded regularity.

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2. Notation and Background

Throughout we denote the natural numbers by $\mathbb{N} = \{0, 1, 2, \ldots\}$. When referring to vectors in $\mathbb{Z}^r$ we use a bold font. Given a vector $v = (v_1, \ldots, v_r) \in \mathbb{Z}^r$ we denote the sum $v_1 + \cdots + v_r$ by $|v|$. For $v, w \in \mathbb{Z}^r$ we write $v \leq w$ when $v_i \leq w_i$ for all $i$, and use $\max\{v, w\}$ to denote the vector whose $i$-th component is $\max\{v_i, w_i\}$. We reserve $e_1, \ldots, e_r$ for the standard basis of $\mathbb{Z}^r$ and for brevity we write $1$ for $(1, 1, \ldots, 1) \in \mathbb{Z}^r$ and $0$ for $(0, 0, \ldots, 0) \in \mathbb{Z}^r$.

Fix a Picard rank $r \in \mathbb{N}$ and dimension vector $n = (n_1, \ldots, n_r) \in \mathbb{N}^r$. We denote by $\mathbb{P}^n$ the product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ of $r$ projective spaces over a field $k$. Given $b \in \mathbb{Z}^r$ we let

$$\mathcal{O}_{\mathbb{P}^n}(b) := \pi^*_i \mathcal{O}_{\mathbb{P}^{n_i}}(b_i) \otimes \cdots \otimes \pi^*_r \mathcal{O}_{\mathbb{P}^{n_r}}(b_r)$$

where $\pi_i$ is the projection of $\mathbb{P}^n$ to $\mathbb{P}^{n_i}$. This gives an isomorphism $\text{Pic} \mathbb{P}^n \cong \mathbb{Z}^r$, which we use implicitly throughout.

Let $S$ be the $\mathbb{Z}^r$-graded Cox ring of $\mathbb{P}^n$, which is isomorphic to the polynomial ring $k[x_{i,j} \mid 1 \leq i \leq r, 0 \leq j \leq n_i]$ with $\deg(x_{i,j}) = e_i$. Further, let $B = \bigcap_{i=1}^r \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i} \rangle \subset S$ be the irrelevant ideal. For a description of the Cox ring and the relationship between coherent $\mathcal{O}_{\mathbb{P}^n}$-modules and $\mathbb{Z}^r$-graded $S$-modules, see [Cox95; CLS11]. In particular, the twisted global sections functor $\Gamma$ given by $\mathcal{F} \mapsto \bigoplus_{p \in \mathbb{Z}^r} H^0(\mathbb{P}^n, \mathcal{F}(p))$ takes coherent sheaves on $\mathbb{P}^n$ to $S$-modules. Given a $\mathbb{Z}^r$-graded $S$-module $M$, let $\beta_i(M) := \{b \in \mathbb{Z}^r \mid \text{Tor}_i^S(M, k)_b \neq 0\}$ denote the set of multidegrees of $i$-th syzygies of $M$.

2.1. Multigraded Regularity. In order to streamline our definitions of regions inside the Picard group of $\mathbb{P}^n$, we introduce the following subsets of $\mathbb{Z}^r$: for $d \in \mathbb{Z}^r$ and $i \in \mathbb{N}$ let

$$L_i(d) := \bigcup_{|\lambda| = i} (d - \lambda_1 e_1 - \cdots - \lambda_r e_r + \mathbb{N}^r) \quad \text{for } \lambda_1, \ldots, \lambda_r \in \mathbb{N}$$

$$Q_i(d) := L_{i-1}(d - 1) \quad \text{for } i > 0 \quad \text{and} \quad Q_0(d) = d + \mathbb{N}^r.$$

Note that for fixed $d \in \mathbb{Z}^r$ we have $L_i(d) \subseteq Q_i(d)$ for all $i$.

**Example 2.1.** When $r = 2$ the regions $L_i(d)$ and $Q_i(d)$ can be visualized as in Figure 1. For $i > 1$ they are shaped like staircases with $i + 1$ and $i$ “corners,” respectively; in other words, the semigroup $L_i(d)$ is generated by $i + 1$ elements and $Q_i(d)$ by $i$ elements.

**Remark 2.2.** An alternate description of $L_i(d)$ will also be useful: it is the set of $b \in \mathbb{Z}^r$ so that the sum of the positive components of $d - b$ is at most $i$. (This ensures that we can distribute the $\lambda_j$ so that $b + \sum_j \lambda_j e_j \geq d$.)

With this notation in hand we can recall the definition of multigraded regularity.

**Definition 2.3.** [MS04, Def. 1.1] Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $S$-module. We say $M$ is $d$-regular for $d \in \mathbb{Z}^r$ if the following hold:

1. $H^0_B(M)_p = 0$ for all $p \in \bigcup_{1 \leq j \leq r} (d + e_j + \mathbb{N}^r)$,
2. $H^i_B(M)_p = 0$ for all $i > 0$ and $p \in L_{i-1}(d)$.

The multigraded Castelnuovo–Mumford regularity of $M$ is then the set

$$\text{reg}(M) := \{d \in \mathbb{Z}^r \mid M \text{ is } d\text{-regular} \} \subset \text{Pic} \mathbb{P}^n \cong \mathbb{Z}^r.$$

It follows directly from the definition that if $M$ is $d$-regular, then $M$ is $d'$-regular for all $d' \geq d$. For other properties of multigraded regularity, such as $0$-regularity of $S$, see [MS04].
Remark 2.4. Several alternate notions of Castelnuovo–Mumford regularity for the multigraded setting exist in the literature. The initial extension was introduced by Hoffman and Wang for a product of two projective spaces [HW04]. Following Maclagan and Smith’s definition, Botbol and Chardin gave a more general definition working over an arbitrary base ring [BC17]. Recently, in their work on Tate resolutions on toric varieties, Brown and Erman introduced a modified notion of multigraded regularity for a weighted projective space, which they then extended to other toric varieties [BE21, §6.1].

2.2. Truncations and Local Cohomology. In this section we collect facts about truncations and local cohomology that will be used repeatedly. As in the case of a single projective space, the truncation of a graded module on a product of projective spaces at multidegree $d$ contains all elements of degree at least $d$.

Definition 2.5. For $d \in \mathbb{Z}^r$ and $M$ a $\mathbb{Z}^r$-graded $S$-module, the truncation of $M$ at $d$ is the $\mathbb{Z}^r$-graded $S$-submodule $M_{\geq d} := \bigoplus_{d' \geq d} M_{d'}$.

Immediate from the definition is the following lemma.

Lemma 2.6. The truncation map $M \mapsto M_{\geq d}$ is an exact functor of $\mathbb{Z}^r$-graded $S$-modules.

Remark 2.7. Since truncation is exact, if $F_\bullet$ is a graded free resolution of a module $M$ then the term by term truncation $(F_\bullet)_{\geq d}$ is a resolution of $M_{\geq d}$. However, in general the truncation of a free module is not free, so $(F_\bullet)_{\geq d}$ is generally not a free resolution of $M_{\geq d}$.

We denote by $H^p_B(M)$ the $p$-th local cohomology of $M$ supported at the irrelevant ideal $B$. For $p > 0$ and $a \in \mathbb{Z}^r$ there exist natural isomorphisms

$$H^p(\mathbb{P}^n, \tilde{M}(b)) \cong H^{p+1}_B(M)_b,$$

and for $p = 0$ there is a $\mathbb{Z}^r$-graded exact sequence

$$0 \longrightarrow H^0_B(M) \longrightarrow M \longrightarrow \Gamma_*(\tilde{M}) \longrightarrow H^1_B(M) \longrightarrow 0. \quad (2.1)$$

An important tool for computing local cohomology is the local Čech complex

$$\tilde{C}_\bullet(B, M) : 0 \longrightarrow M \longrightarrow \bigoplus M[g_i^{-1}] \longrightarrow \bigoplus M[g_i^{-1}, g_j^{-1}] \longrightarrow \cdots$$

where the $g_i$ range over the generators of $B$. We index the local Čech complex so that the summands of $\tilde{C}_p(B, M)$ are localizations of $M$ at $p$ distinct generators of $B$. Then we have

$$H^p_B(M) \cong H^p(\tilde{C}_\bullet(B, M)).$$

See [Iye+07] and [CLS11, §9] for more details.

Note that inverting a generator of $B$ inverts a variable from each factor of $\mathbb{P}^n$, so the distinguished open sets corresponding to the generators of $B$ form an affine cover $\mathcal{U}_B$ of $\mathbb{P}^n$. Denote by $\tilde{C}_\bullet(\mathcal{U}_B, \mathcal{F})$ the Čech complex of a sheaf $\mathcal{F}$ with respect to $\mathcal{U}_B$:

$$\tilde{C}_\bullet(\mathcal{U}_B, \mathcal{F}) : 0 \longrightarrow \bigoplus \mathcal{F}|_{\mathcal{U}_i} \longrightarrow \bigoplus \mathcal{F}|_{\mathcal{U}_i \cap \mathcal{U}_j} \longrightarrow \cdots .$$

Lemma 2.8. Given a complex of graded $S$-modules $L \rightarrow M \rightarrow N$ such that $\tilde{L} \rightarrow \tilde{M} \rightarrow \tilde{N}$ is exact, the complex $\tilde{C}_p(B, L) \rightarrow \tilde{C}_p(B, M) \rightarrow \tilde{C}_p(B, N)$ is exact for each $p \geq 0$. 
Proof. Fix $p$. Then $\check{C}^n(B, L) \to \check{C}^n(B, M) \to \check{C}^n(B, N)$ splits as a direct sum of complexes

$$L[g_1^{-1}, \ldots, g_p^{-1}] \to M[g_1^{-1}, \ldots, g_p^{-1}] \to N[g_1^{-1}, \ldots, g_p^{-1}]$$

each of which can be obtained by applying $\Gamma(U, -)$ to $\check{L} \to \check{M} \to \check{N}$, where $U$ is the complement of $V(g_1, \ldots, g_p)$. Since $U$ is affine they are exact. □

Since $M/M_{\geq d}$ is annihilated by a power of $B$, a module $M$ and its truncation define the same sheaf on $\mathbb{P}^n$. In particular $H^p_B(M) = H^p_B(M_{\geq d})$ for $p \geq 2$. The long exact sequence of local cohomology applied to $0 \to M_{\geq d} \to M \to M/M_{\geq d} \to 0$ gives

$$0 \longrightarrow H^0_B(M_{\geq d}) \longrightarrow H^0_B(M) \longrightarrow M/M_{\geq d} \longrightarrow H^1_B(M_{\geq d}) \longrightarrow H^1_B(M) \longrightarrow 0.$$ 

Hence $H^0_B(M) = 0$ implies $H^0_B(M_{\geq d}) = 0$. Since $M/M_{\geq d}$ is zero in degrees larger than $d$ we also have $H^1_B(M_{\geq d}) = H^1_B(M)_{\geq d}$. An immediate consequence is the following lemma, which we will use repeatedly to reduce to the case when $d = 0$.

**Lemma 2.9.** A $\mathbb{Z}^r$-graded $S$-module $M$ is $d$-regular if and only if $M_{\geq d}$ is $d$-regular.

2.3. Koszul Complexes and Cotangent Sheaves. For each factor $\mathbb{P}^n$ of $\mathbb{P}^n$, the Koszul complex on the variables of $S_i = \text{Cox} \mathbb{P}^n$ is a resolution of $k$:

$$K^i:\ 0 \leftarrow S_i \leftarrow S_i^{m_i + 1}(-1) \leftarrow \bigwedge^2 S_i^{m_i + 1}(-1) \leftarrow \cdots \leftarrow \bigwedge^{n_i + 1} S_i^{m_i + 1}(-1) \leftarrow 0. \quad (2.2)$$

The Koszul complex $K_i$ on the variables of $S$ is the tensor product of the complexes $\pi^*_i K^i$.

For $1 \leq a \leq n$ let $\Omega^a_{\mathbb{P}^n_i}$ be the kernel of $\bigwedge^{a-1} S_i^{m_i + 1} \leftarrow \bigwedge^a S_i^{m_i + 1}$ and let $\Omega^a_{\mathbb{P}^n_i}$ denote its sheafification. The minimal free resolution of $\Omega^a_{\mathbb{P}^n_i}$ then consists of the terms of $K^i$ with homological index greater than $a$. Write $\Omega^0_{\mathbb{P}^n_i}$ for the kernel of $k \leftarrow S_i$ (so that $\Omega^0_{\mathbb{P}^n_i} = \mathcal{O}_{\mathbb{P}^n_i}$) and take $\Omega^a_{\mathbb{P}^n_i}$ to be 0 otherwise. For $a \in \mathbb{Z}^r$ with $0 \leq a \leq n$ define

$$\Omega^a_{\mathbb{P}^n} := \pi_1^* \Omega^a_{\mathbb{P}^n_1} \otimes \cdots \otimes \pi_r^* \Omega^a_{\mathbb{P}^n_r}$$

and write $\hat{\Omega}^a_{\mathbb{P}^n}$ for the analogous tensor product of the modules $\hat{\Omega}^a_{\mathbb{P}^n_i}$.

Given a free complex $F_i$ and a multidegree $a \in \mathbb{Z}^r$, denote by $F^a_i$ the subcomplex of $F_i$ consisting of free summands generated in degrees at most $a$.

**Lemma 2.10.** Fix $a \in \mathbb{Z}^r$ and let $K_i$ be the Koszul complex on the variables of $S$. The subcomplex $K_i^a$ is equal to $K_i$ in degrees $\leq a$, and its sheafification is exact except at homological index $|a|$, where it has homology $\hat{\Omega}^a_{\mathbb{P}^n}$. 

**Proof.** The first statement follows from the fact that the terms appearing in $K_i$ but not $K^a_i$ have no elements in degrees $\leq a$.

Note that $K^a_i$ is a tensor product of pullbacks of subcomplexes of the $K^i$ in (2.2):

$$K^a_i = \pi_1^* (K_i)^{a_1} \otimes \cdots \otimes \pi_r^* (K_i)^{a_r}.$$ 

After sheafification, each complex $\pi_i^* (K_i)^{a_i}$ is exact away from its kernel $\pi_i^* \Omega^{a_i}_{\mathbb{P}^n_i}$, which appears at homological index $a_i$. Thus $K^a_i$ has homology $\hat{\Omega}^a_{\mathbb{P}^n}$, appearing in index $|a|$. □
3. Structure of Virtual Resolutions

Let \( X \) be a smooth projective toric variety with Pic\((X)\)-graded Cox ring \( S \) and irrelevant ideal \( B \). Consider a graded \( S \)-module \( M \). While a minimal free resolution \( F \) of \( M \) can be easily computed using Gröbner methods, it does not always provide a faithful reflection of the geometry of \( X \). For example, when the Picard rank of \( X \) is greater than one, the length of \( F \) may exceed \( \dim X \). To bridge this gap, Berkesch, Erman, and Smith introduced virtual resolutions in [BES20].

**Definition 3.1.** A Pic\((X)\)-graded complex of free \( S \)-modules \( F \) is a virtual resolution of \( M \) if the complex \( \tilde{F} \) of locally free sheaves on \( X \) is a resolution of the sheaf \( \tilde{M} \).

Despite more faithfully capturing the geometry of \( X \), virtual resolutions are often less rigid than minimal free resolutions. For example, a module \( M \) generally has many non-isomorphic virtual resolutions. In this section we consider virtual resolutions containing no degree 0 maps, which we show in certain situations are subcomplexes of minimal free resolutions. This additional structure is necessary to our work in Section 6.

Virtual resolutions will also appear in our proof of Theorem A. Inspired by the work of Berkesch, Erman, and Smith, we use a Fourier-Mukai construction to give a virtual resolution of \( M \) whose Betti numbers are computable in terms of certain cohomology groups. In Section 4.1 we then relate this virtual resolution to the minimal free resolution of \( M_{\geq d} \).

**Remark 3.2.** When \( X = \mathbb{P}^n \), Berkesch, Erman, and Smith constructed virtual resolutions of length at most \( \dim \mathbb{P}^n \). Note that since \( \tilde{M} = \tilde{M}_{\geq d} \), a free resolution of \( M_{\geq d} \) is automatically a virtual resolution of \( M \). In Section 4.1 we use this fact to give an alternative construction of short virtual resolutions on \( \mathbb{P}^n \).

3.1. Subcomplexes of Minimal Free Resolutions. A complex of \( S \)-modules is trivial if it is a direct sum of complexes of the form

\[
\cdots \longrightarrow 0 \longrightarrow S \underset{1}{\longrightarrow} S \underset{0}{\longrightarrow} \cdots.
\]

A free resolution of a finitely generated Pic\((X)\)-graded \( S \)-module \( M \) is isomorphic to the direct sum of the Pic\((X)\)-graded minimal free resolution of \( M \) and a trivial complex. With this in mind we introduce the following notion of a minimal virtual resolution.

**Definition 3.3.** A virtual resolution \( F \) is minimal if it is not isomorphic to a Pic\((X)\)-graded chain complex of the form \( F' \oplus F'' \) where \( F'' \) is a trivial complex.

Note that, unlike in the case of ordinary free resolutions, minimal virtual resolutions are not unique, even up to isomorphism. Further, minimal virtual resolutions need not have the same length. That said, analogous to the case of minimal free resolutions, minimal virtual resolutions are characterized by having no constant entries in their differentials.

**Lemma 3.4.** A virtual resolution of \( M \) is minimal if and only if its differentials have no degree 0 components.

**Proof.** Since \( S \) is positively graded, a graded version of Nakayama’s Lemma holds (see [MS05, pp. 155-156]). The statement follows immediately from an argument similar to those in [Eis95, Thm. 20.2, Exc. 20.1].
The following structure result shows that a minimal virtual resolution \( F \) of a module \( M \) satisfying certain conditions on the Betti numbers arises as a subcomplex of the minimal free resolution of \( H_0(F_\bullet) \). We use this in the proof of Corollary 6.5 to show that the virtual resolution which we construct in Section 3.3 is exact. Here we denote by \( \text{Eff}(X) \) the cone generated by the degrees of the variables of \( S \) in \( \text{Pic} X \).

**Proposition 3.5.** Let \((F_\bullet, \varphi_\bullet)\) be a finite minimal virtual resolution and let \( N = H_0(F_\bullet) \). Suppose that

1. \( \dim_k \text{Tor}_i(F_\bullet, k)_d \leq \dim_k \text{Tor}_i(N, k)_d \) for all \( d \) and all \( i \);
2. whenever \( c - d \in \text{Eff}(X) \) and \( \text{Tor}_i(F_\bullet, k)_c \neq 0 \) equality holds in (1).

Then \( F_\bullet \) is a subcomplex of the minimal free resolution of \( N \).

**Proof.** First, we will inductively construct a resolution \((G_\bullet, \psi_\bullet)\) of \( N \) which contains \((F_\bullet, \varphi_\bullet)\) as a subcomplex. Let \( G_0 = F_0, G_1 = F_1, \) and \( \psi_1 = \varphi_1, \) so that \( H_0(G_\bullet) = N \).

Suppose \( G_i \) has been defined for \( 0 \leq i \leq n - 1 \) so that \( F_\bullet \) is a summand and \( G_\bullet \) is exact for \( 0 < i < n - 1 \). Consider \( \varphi_n \) as a map \( F_n \to G_{n-1} \) by composing with the inclusion \( F_{n-1} \hookrightarrow G_{n-1} \). Choose \( z_1, \ldots, z_s \in \ker \psi_{n-1} \) such that their images generate \( \ker \psi_{n-1} / \text{im} \varphi_n \).

Let \( G_n = F_n \oplus S(-a_1) \oplus \cdots \oplus S(-a_s) \) where \( \deg z_j = a_j \). Define \( \psi_n \) by \( \psi_n |_{F_n} = \varphi_n \) and \( \psi_n(g_j) = z_j \), where \( g_j \) is the generator of \( S(-a_j) \). Then \( \ker \psi_n = \text{ker} \psi_{n-1} \), so that \( G_\bullet \) is a complex and exact at \( n - 1 \).

We will now show by induction that it is possible to prune \( G_\bullet \) to a minimal free resolution of \( N \) that contains \( F_\bullet \) as a subcomplex. At each step, take a nonminimal homogeneous relation among the images of generators of some \( G_i \). Write it as

\[
\psi_i \left( \sum a_j f_j + \sum b_j g_j \right) = 0,
\]

where \( f_j \in F_i, g_j \in G_i \setminus F_i \), and \( a_j, b_j \neq 0 \) for all \( j \). As \( F_\bullet \) is minimal, at least one \( g_j \) does appear. Since each \( G_i \) has only finitely many generators, it is possible to choose a relation whose degree \( c \) satisfies \( c - d \notin \text{Eff}(X) \) for all degrees \( d \neq c \) of other available relations.

Assume by induction that no generator of \( F_\bullet \) has been removed in a previous step. Since the chosen relation is nonminimal, at least one of its coefficients is a unit. If some \( b_j \) is a unit then we may remove the corresponding \( g_j \) and continue pruning.

Suppose instead that all unit coefficients appear among the \( a_j \). In this case we must prune some \( f_k \) in order to remove the relation. Note that by homogeneity

\[
\deg f_k = \deg a_k f_k = c = \deg b_j g_j = \deg b_j + \deg g_j
\]

for all \( j \). Thus \( c - \deg g_j = \deg b_j \in \text{Eff}(X) \), so equality holds in (1) for \( d = \deg g_j \) by hypothesis. By choice of \( c \) we cannot remove anything of degree \( \deg g_j \) in a subsequent step. Hence \( g_j \) appears in the minimal free resolution of \( N \), so by the equality in (1) some generator \( f \) of \( F_i \) with degree \( d \) must be removed. However, it cannot have been removed before \( f_k \) by the induction hypothesis, and it cannot be removed after \( f_k \) by choice of \( c \). This is a contradiction, so we are never required to prune a generator of \( F_\bullet \), completing the proof. \( \square \)

In the language of [BES20], this proposition implies that a virtual resolution that appears to be a *virtual resolution of a pair* based only on its Betti numbers can indeed be produced by that construction. Note that the proposition is not true without conditions on the Betti numbers. For instance, [BPC21, Ex. 1.2] gives a minimal virtual resolution which is not a subcomplex of the minimal free resolution of its cokernel.
3.2. **Fourier–Mukai Transforms.** The sheafification of a virtual resolution of \( M \) is a resolution of \( \tilde{M} \) by direct sums of line bundles. More generally, following [EES15, §8], we define a **free monad** of a coherent sheaf \( \mathcal{F} \) to be a finite complex

\[
\mathcal{L} : 0 \leftarrow L_{-s} \leftarrow \cdots \leftarrow L_{-1} \leftarrow L_0 \leftarrow \cdots \leftarrow L_t \leftarrow 0
\]

whose terms are direct sums of line bundles and whose homology is \( H_s(\mathcal{L}) = H_0(\mathcal{L}) \cong \mathcal{F} \).

In this section we introduce a type of geometric functor between derived categories known as a Fourier–Mukai transform. We will use a particular instance in Section 3.3 to prove that a complex constructed from the Beilinson spectral sequence is a free monad. See [Huy06, §5] for background and further details.

Let \( X \) and \( Y \) be smooth projective varieties and consider the two projections

\[
X \xleftarrow{q} X \times Y \xrightarrow{p} Y.
\]

A **Fourier–Mukai transform** is a functor

\[
\Phi_K : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)
\]

between the derived categories of bounded complexes of coherent sheaves. It is represented by an object \( K \in \mathcal{D}^b(X \times Y) \) and constructed as a composition of derived functors

\[
\mathcal{F} \mapsto Rp_*(Lq^* \mathcal{F} \otimes K).
\]

Here \( Lq^* \), \( Rp_* \), and \(-\otimes L \mathcal{K}\) are the derived functors induced by \( q^* \), \( p_* \), and \(-\otimes K\), respectively. Moreover, since \( q \) is flat \( Lq^* \) is the usual pull-back, and if \( \mathcal{K} \) is a complex of locally free sheaves \(-\otimes L \mathcal{K}\) is the usual tensor product. In fact, all equivalences between \( \mathcal{D}^b(X) \) and \( \mathcal{D}^b(Y) \) arise in this way.

A special case of the Fourier–Mukai transform occurs when \( Y = X \) and \( K \in \mathcal{D}^b(X \times X) \) is a resolution of the structure sheaf \( \mathcal{O}_\Delta \) of the diagonal subscheme \( \iota : \Delta \to X \times X \). Such \( K \) is referred to as a **resolution of the diagonal**.

Using the projection formula, one can see that the Fourier–Mukai transform \( \Phi_{\mathcal{O}_\Delta} \) is simply the identity in the derived category; that is to say, it produces quasi-isomorphisms. We will use this fact in the proof of Proposition 3.7.

3.3. **The Beilinson Spectral Sequence.** Returning to the case of products of projective spaces, we consider coherent sheaves on \( X = \mathbb{P}^n \). We construct a free monad for \( \tilde{M} \) from the Beilinson spectral sequence on \( \mathbb{P}^n \times \mathbb{P}^n \) and describe its Betti numbers. When \( M \) is 0-regular it is a minimal virtual resolution, which we will use in Sections 4 and 6. See [OSS80, §3.1] for a geometric exposition and [Huy06, §8.3] or [AO89, §3] for an algebraic exposition on a single projective space.

For sheaves \( \mathcal{F} \) and \( \mathcal{G} \) on \( \mathbb{P}^n \), denote \( p^* \mathcal{F} \otimes q^* \mathcal{G} \) by \( \mathcal{F} \boxtimes \mathcal{G} \). Consider the vector bundle

\[
\mathcal{W} = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(e_i) \boxtimes \mathcal{T}_{\mathbb{P}^n}^e_i(-e_i),
\]

where \( \mathcal{T}_{\mathbb{P}^n}^e_i \) is the pullback of the tangent bundle, as in the Euler sequence on the factor \( \mathbb{P}^{n_i} \):

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n_i}} \longrightarrow \mathcal{O}_{\mathbb{P}^{n_i+1}}^{e_i}(e_i) \longrightarrow \mathcal{T}_{\mathbb{P}^{n_i}} \longrightarrow 0. \quad (3.1)
\]
There is a canonical section \( s \in H^0(\mathbb{P}^n \times \mathbb{P}^n, \mathcal{M}) \) whose vanishing cuts out the diagonal subscheme \( \Delta \subset \mathbb{P}^n \times \mathbb{P}^n \) (see [BES20, Lem. 2.1]), giving a Koszul resolution of \( \mathcal{O}_\Delta \):

\[
\mathcal{K}: 0 \leftarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \leftarrow \mathcal{W}^\vee \leftarrow \bigwedge^2 \mathcal{W}^\vee \leftarrow \cdots \leftarrow \bigwedge^n \mathcal{W}^\vee \leftarrow 0. \tag{3.2}
\]

The terms of \( \mathcal{K} \) can be written as

\[
\mathcal{K}_j = \bigwedge^j \left( \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(-e_i) \boxtimes \Omega^{n}_{\mathcal{O}_{\mathbb{P}^n}}(e_i) \right) = \bigoplus_{|a|=j} \mathcal{O}_{\mathbb{P}^n}(-a) \boxtimes \Omega^{n}_{\mathcal{O}_{\mathbb{P}^n}}(a), \quad \text{for } 0 \leq j \leq |n|. \tag{3.3}
\]

As in Section 3.2, we are interested in the derived pushforward of \( q^*\widetilde{M} \otimes \mathcal{K} \), which we will compute by resolving the second term of each box product with a Čech complex to obtain a spectral sequence. Since \( \mathcal{K} \) is a resolution of the diagonal, the pushforward will be quasi-isomorphic to \( \widetilde{M} \).

Consider the double complex

\[
C^{-s,t} = \bigoplus_{|a|=s} \mathcal{O}_{\mathbb{P}^n}(-a) \boxtimes \hat{C}^t \left( \mathfrak{U}_B, \widetilde{M} \otimes \Omega^{n}_{\mathcal{O}_{\mathbb{P}^n}}(a) \right),
\]

with vertical maps from the Čech complexes and horizontal maps from \( \mathcal{K} \). Since taking Čech complexes is functorial and exact we have \( \mathrm{Tot}(C) \sim q^*\widetilde{M} \otimes \mathcal{K} \), which is a resolution of \( q^*\widetilde{M} \otimes \mathcal{O}_\Delta \) because \( \mathcal{K} \) is locally free. Moreover, since the first term of each box product in \( q^*\widetilde{M} \otimes \mathcal{K} \) is locally free, the columns of \( C \) are \( p_* \)-acyclic (c.f. [Har66, Prop. 3.2], [AO89, Lem. 3.2]). Hence the pushforward

\[
E_0^{-s,t} = p_*(C^{-s,t}) = \bigoplus_{|a|=s} \mathcal{O}_{\mathbb{P}^n}(-a) \otimes \Gamma \left( \mathfrak{P}^n, \hat{C}^t \left( \mathfrak{U}_B, \widetilde{M} \otimes \Omega^{n}_{\mathcal{O}_{\mathbb{P}^n}}(a) \right) \right) \tag{3.4}
\]

satisfies \( \mathrm{Tot}(E_0) = \Phi_{\mathcal{K}}(\widetilde{M}) \sim \widetilde{M} \). With this notation, the Beilinson spectral sequence is the spectral sequence of the double complex \( E_0 \), whose (vertical) first page has terms

\[
E_1^{-s,t} = \bigoplus_{|a|=s} \mathcal{O}_{\mathbb{P}^n}(-a) \otimes H^t \left( \mathfrak{P}^n, \widetilde{M} \otimes \Omega^{n}_{\mathcal{O}_{\mathbb{P}^n}}(a) \right) = R^t p_*(q^*\widetilde{M} \otimes \mathcal{K}_s). \tag{3.5}
\]

Beilinson’s resolution of the diagonal and the associated spectral sequence are crucial ingredients in constructions of Beilinson monads, Tate resolutions, and virtual resolutions [EFS03; EES15; BES20]. Recently, Brown and Erman [BE21] expanded these constructions to toric varieties using a noncommutative analogue of a Fourier–Mukai transform. More generally, Costa and Miró-Roig [CMR07] have introduced a Beilinson type spectral sequence for a smooth projective variety under certain conditions on its derived category.

The main result of this section is the next proposition, which describes the Betti numbers of a free monad constructed from the Beilinson spectral sequence (c.f. [BES20, Thm. 2.9]). A key component of the construction is the following lemma.

**Lemma 3.6.** Let \((C_\bullet, d_\bullet)\) be a bounded above complex of free \(S\)-modules and let \(B_i = \mathrm{im} d_{i-1}\) and \(Z_i = \ker d_i\). If every homology module \(Z_i/B_i\) of \(C_\bullet\) is free then there is a splitting \(f_i: C_i \to B_i \oplus Z_i/B_i \oplus C_i/Z_i\) such that \(f\) and \(d\) commute on each summand.

**Proof.** Since \(C_\bullet\) is bounded above, there is some \(k\) such that \(B_i = 0\) for all \(i > k\), so in particular \(B_{k+1} \cong C_k/Z_k\) is free. Since \(C_k\) is free, the exact sequence \(0 \to Z_k \to C_k \to \)
$C_k/Z_k \to 0$ implies that $Z_k$ is free and $C_k \cong Z_k \oplus C_k/Z_k$. Since $Z_k/B_k$ is free by assumption, the exact sequence $0 \to B_k \to Z_k \to Z_k/B_k \to 0$ implies that $B_k$ is free and $Z_k \cong B_k \oplus Z_k/B_k$. Together, we get $C_k \cong B_k \oplus Z_k/B_k \oplus C_k/Z_k$, and the freeness of $B_k$ means that we can induct backwards on the whole complex.

**Proposition 3.7.** Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $S$-module. There is a free monad $L$ for $M$ with terms

$$L_k = \bigoplus_{|a|=k} \mathcal{O}_{\mathbb{P}^n}(-a) \otimes H^{|a|-k}(\mathbb{P}^n, \widetilde{M} \otimes \Omega^n_{\mathbb{P}^n}(a))$$

so that

1. the free complex $F_\bullet = \Gamma_*(L)$ has Betti numbers $\beta_{k,a}(F_\bullet) = h^{|a|-k}(\mathbb{P}^n, \widetilde{M} \otimes \Omega^n_{\mathbb{P}^n}(a))$;
2. if $H^i(\mathbb{P}^n, \widetilde{M} \otimes \Omega^n_{\mathbb{P}^n}(a)) = 0$ for $i > |a|$ then $F_\bullet$ is a minimal virtual resolution for $M$.

**Proof.** Let $K$ be the resolution of the diagonal from (3.3) and let $\Phi_K$ be the corresponding Fourier–Mukai transform. The Beilinson spectral sequence has (vertical) first page $E_1^{s,t}$:

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
R^2p_*(q^*\widetilde{M} \otimes K_0) & \leftarrow & R^2p_*(q^*\widetilde{M} \otimes K_1) & \leftarrow R^2p_*(q^*\widetilde{M} \otimes K_2) & \leftarrow \cdots \\
R^1p_*(q^*\widetilde{M} \otimes K_0) & \leftarrow & R^1p_*(q^*\widetilde{M} \otimes K_1) & \leftarrow R^1p_*(q^*\widetilde{M} \otimes K_2) & \leftarrow \cdots \\
p_*(q^*\widetilde{M} \otimes K_0) & \leftarrow & p_*(q^*\widetilde{M} \otimes K_1) & \leftarrow p_*(q^*\widetilde{M} \otimes K_2) & \leftarrow \cdots \\
\end{array}
\]

Since both (3.4) and (3.5) have locally free terms, by Lemma 3.6 the vertical differential of $E_0$ satisfies the splitting hypotheses of [EFS03, Lem. 3.5], which implies that the total complex of $E_0$ is homotopy equivalent to a complex $L$ with terms $L_k = \bigoplus_{s-t=k} E_1^{s,t}$. Hence

$$L \sim \text{Tot}(E_0) = \Phi_K(\widetilde{M}) \sim \widetilde{M}.$$  

Since the terms of $E_1$ are direct sums of line bundles, the complex $L$ is a free monad for $\widetilde{M}$.

Observe that the only terms with twist $a$ appear in $K_s$ for $s = |a|$ and that the Betti numbers in homological index $k$ come from the higher direct images $E_1^{s,t}$ on diagonals with $s-t = k$. Hence $\beta_{k,a}(F_\bullet)$ is the rank of $\mathcal{O}_{\mathbb{P}^n}(-a)$ in $E_1^{s-t,k}$ which is $h^{|a|-k}(\mathbb{P}^n, \widetilde{M} \otimes \Omega^n(\mathbb{P}^n)(a))$.

Lastly, note that the hypothesis of part (2) implies that the terms of (3.6) on diagonals with $k < 0$ vanish; hence the free monad $L$ is a locally free resolution. Since each map in the construction from [EFS03, Lem. 3.5] increases the index $-s$, the differentials in $F_\bullet$ are minimal, so $F_\bullet$ is a minimal virtual resolution.

**Remark 3.8.** In the proof of [BES20, Prop. 1.2], Berkesch, Erman, and Smith show that if $M$ is sufficiently twisted so that all higher direct images of $\widetilde{M} \otimes \Omega^n_{\mathbb{P}^n}(a)$ vanish, then the $E_1$ page will be concentrated in one row, which results in a linear virtual resolution. Similarly in [EES15, Prop. 1.7], Eisenbud, Erman, and Schreyer prove that for sufficiently positive twists, the truncation of $M$ has a linear free resolution. However, in both cases
the positivity condition is stronger than 0-regularity for \( M \), as illustrated by the following example.

**Example 3.9.** Write \( S = \mathbb{k}[x_0, x_1, y_0, y_1, y_2] \) for the Cox ring of \( \mathbb{P}^1 \times \mathbb{P}^2 \) and consider the ideal \( I = (y_0 + y_1 + y_2, x_0y_0 + x_0y_1 + x_0y_2 + x_1y_0 + x_1y_1) \). Then \( \tilde{M} = S/I \) is a bigraded, \((0, 0)\)-regular \( S \)-module. The global sections of the Beilinson spectral sequence for \( \tilde{M} \) has first page

\[
0 \quad 0 \quad S(-1, -1) \quad S(-1, -2) \quad 0
\]

where the dotted diagonal maps are lifts of maps from the second page of the spectral sequence, which agree with the maps from [EFS03, Lem. 3.5].

In the next section we state and prove Theorem A by illustrating the restrictions on the virtual resolution above that follow from the regularity of \( \tilde{M} \) and using them to bound the shape of the minimal free resolution of a truncation of \( M \). Later, in Corollary 6.5, we examine the maps in the first row in order to compute the Beilinson spectral sequence when \( M \) has a linear presentation.

### 4. A Criterion for Multigraded Regularity

To investigate the relationship between multigraded regularity and resolutions of truncations we first need to establish a definition of linearity for a multigraded resolution. We would like the differentials to be given by matrices with entries of total degree at most 1. However, we will examine only the twists in the resolution, requiring that they lie in the \( L \) regions from Section 2.1. In particular, we will identify a complex with a map of degree \( > 1 \) as nonlinear even if that map is zero.

**Definition 4.1.** Let \( F_\bullet \) be a \( \mathbb{Z}^r \)-graded free resolution. We say \( F_\bullet \) is **linear** if \( F_0 \) is generated in a single multidegree \( d \) and the twists appearing in \( F_j \) lie in \( L_j(-d) \).

We require \( F_0 \) to be generated in a single degree so that the truncation of a module with a linear resolution also has a linear resolution (see Proposition 4.5). Otherwise, for instance, the minimal resolution of \( M \) in the following example would be considered linear, yet the resolution of its truncation \( M_{\geq(1,0)} \) would not.

**Example 4.2.** Write \( S = \mathbb{k}[x_0, x_1, y_0, y_1] \) for the Cox ring of \( \mathbb{P}^1 \times \mathbb{P}^1 \) and let \( M \) be the module with resolution \( S(-1, 0)^2 \oplus S(0, -1)^2 \leftarrow S(-1, -1)^4 \leftarrow 0 \) given by the presentation matrix

\[
\begin{bmatrix}
x_0 & x_1 & 0 & 0 \\
0 & 0 & x_1 & x_0 \\
-y_0 & 0 & -y_0 & 0 \\
0 & -y_1 & 0 & -y_1
\end{bmatrix}.
\]

A Macaulay2 computation shows that \( M \) is \((1, 0)\)-regular. However, the minimal graded free resolution of the truncation \( M_{\geq(1,0)} \) is

\[
0 \leftarrow S(-1, 0)^2 \leftarrow S(-2, -1)^2 \leftarrow 0
\]
which is not linear because \((-2, -1) \notin L_1(-1, 0)\).

This example shows that a module can be \(d\)-regular yet have a nonlinear resolution for \(M_{\geq d}\). Thus in order to characterize regularity in terms of truncations we need to weaken the definition of linear. We will use the larger \(Q\) regions from Section 2.1 in order to allow some maps of higher degree.

**Definition 4.3.** Let \(F_\bullet\) be a \(\mathbb{Z}^r\)-graded free resolution. We say \(F_\bullet\) is quasilinear if \(F_0\) is generated in a single multidegree \(d\) and for each \(j\) the twists appearing in \(F_j\) lie in \(Q_j(-d)\).

**Example 4.4.** Unlike on a single projective space, the resolution of \(S/B\) for the irrelevant ideal \(B\) on a product of projective spaces is not linear. However it is quasilinear. On \(\mathbb{P}^1 \times \mathbb{P}^2\), for instance, \(S/B\) has resolution

\[
0 \leftarrow S \leftarrow S(-1, -1)^6 \leftarrow \begin{array}{c} S(-1, -2)^6 \\
S(-2, -1)^3 \\
S(-2, -2)^3 \\
S(-2, -3) \leftarrow 0,
\end{array}
\]

which has generators in degree \((0, 0)\) and relations in degree \((1, 1)\). Thus the resolution is not linear, since \((-1, -1) \notin L_1(0, 0)\). However \((-1, -1) \in Q_1(0, 0)\) is compatible with quasilinearity.

This condition is inspired by [BES20, Thm. 2.9], which characterized regularity in terms of the existence of virtual resolutions with Betti numbers similar to those of \(S/B\)—see Section 4.2 for a more complete discussion. Note that both linear and quasilinear reduce to the standard definition of linear on a single projective space. As one might expect from that setting, they satisfy the property below, which will follow from Theorems 5.4 and 5.5.

**Proposition 4.5.** Let \(M\) be a \(\mathbb{Z}^r\)-graded \(S\)-module. If \(M_{\geq d}\) has a linear (respectively quasilinear) resolution and \(d' \geq d\) then \(M_{\geq d'}\) has a linear (respectively quasilinear) resolution.

A linear resolution for \(M_{\geq d}\) implies that \(M\) is \(d\)-regular when \(H_B^0(M) = 0\). To obtain a converse that generalizes Eisenbud–Goto’s result one should instead check that the resolution is quasilinear. This gives a criterion for regularity that does not require computing cohomology.

**Theorem 4.6.** Let \(M\) be a finitely generated \(\mathbb{Z}^r\)-graded \(S\)-module such that \(H_B^0(M) = 0\). Then \(M\) is \(d\)-regular if and only if \(M_{\geq d}\) has a quasilinear resolution \(F_\bullet\) such that \(F_0\) is generated in degree \(d\).

We prove one direction of Theorem 4.6 in Section 4.1 (Theorem 4.8) and the other in Section 4.2 (Theorem 4.14).

### 4.1. Regularity Implies Quasilinearity.

In Proposition 3.7 we constructed a virtual resolution with Betti numbers determined by the sheaf cohomology of \(\tilde{M} \otimes \Omega^n_{\mathbb{P}^n}(a)\). By resolving the \(\Omega^n_{\mathbb{P}^n}(a)\) in terms of line bundles and tensoring with \(\tilde{M}\), we can relate the cohomological vanishing in the definition of multigraded regularity to the shape of this virtual resolution. The following lemma implies that when \(M\) is \(d\)-regular the virtual resolution is quasilinear, i.e., the coefficients of twists outside of \(Q_i(-d)\) are zero. The lemma is a variant of [BES20, Lem. 2.13] (see Section 4.2).
Lemma 4.7. If a \( \mathbb{Z}^r \)-graded \( S \)-module \( M \) is \( 0 \)-regular then \( H^{[a]-i}(\mathbb{P}_n^r, \tilde{M} \otimes \Omega^a_{\mathbb{P}_n}(a)) = 0 \) for all \( -a \notin Q_i(0) \) and all \( i > 0 \).

Proof. Fix \( i \) and \( a \in \mathbb{Z}^r \) with \( -a \notin Q_i(0) \), and suppose that \( H^{[a]-i}(\mathbb{P}_n^r, \tilde{M} \otimes \Omega^a_{\mathbb{P}_n}(a)) \neq 0 \). We will show that \( M \) is not \( 0 \)-regular. We must have \( 0 \leq a \leq n \), else \( \Omega^a_{\mathbb{P}_n}(a) = 0 \). Let \( \ell \) be the number of nonzero coordinates in \( a \).

A tensor product of locally free resolutions for the factors \( \pi_i^*(\Omega^a_{\mathbb{P}_n}(a)) \) gives a locally free resolution for \( \Omega^a_{\mathbb{P}_n}(a) \). Since \( \Omega^a_{\mathbb{P}_n} = \mathcal{O}_{\mathbb{P}_n} \), we can use \( r - \ell \) copies of \( \mathcal{O}_{\mathbb{P}_n} \) and \( \ell \) linear resolutions, each generated in total degree 1, to obtain such a resolution \( F \) (see Section 2.3).

Thus the twists in \( F \) have nonpositive coordinates and total degree \( -j - \ell \), so they are in \( L_{j+\ell}(0) \).

Since \( F \) is locally free the cokernel of \( \tilde{M} \otimes F \) is isomorphic to \( \tilde{M} \otimes \Omega^a_{\mathbb{P}_n}(a) \). By a standard spectral sequence argument, explained in the proof of Theorem 4.14, the nonvanishing of \( H^{[a]-i}(\mathbb{P}_n^r, \tilde{M} \otimes \Omega^a_{\mathbb{P}_n}(a)) \) implies the existence of some \( j \) such that \( H^{[a]-i+j}(\mathbb{P}_n^r, \tilde{M} \otimes \mathcal{F}_j) \neq 0 \).

If \( i = 0 \) then

\[
|a| - i + j \geq \ell - i + j = j + \ell.
\]

If \( i > 0 \) then \( a - 1 \) has \( \ell \) nonnegative coordinates that sum to \( |a| - \ell \). Thus \( |a| - \ell > i - 1 \), since \( -a \notin Q_i(0) = L_{i-1}(-1) \) (see Remark 2.2). This also gives

\[
|a| - i + j \geq (\ell + i) - i + j = j + \ell.
\]

so in either case \( L_{j+\ell}(0) \subseteq L_{|a|-i+j}(0) \). Therefore \( H^{[a]-i+j}(\mathbb{P}_n^r, \tilde{M} \otimes \mathcal{F}_j) \neq 0 \) for \( \mathcal{F}_j \) with twists in \( L_{j+\ell}(0) \) implies that \( M \) is not \( 0 \)-regular.

See [CMR07, Thm. 5.5] for a similar result relating Hoffman and Wang’s definition of regularity [HW04] to a different cohomology vanishing for \( \tilde{M} \otimes \Omega^a_{\mathbb{P}_n}(a) \).

Motivated by the quasilinearity of the virtual resolution in Proposition 3.7, we will prove that the \( d \)-regularity of \( M \) implies that the minimal free resolution of \( M_{\geq d} \) is quasilinear. Let \( K \) be the Koszul complex from Section 2.3 and \( \check{C}^n(B, \cdot) \) the Čech complex as in Section 2.2. We will use the spectral sequence of a double complex with rows from subcomplexes of \( K \) and columns given by Čech complexes in order to relate the Betti numbers of \( M_{\geq d} \) to the sheaf cohomology of \( \tilde{M} \otimes \Omega^a_{\mathbb{P}_n}(a) \).

Theorem 4.8. Let \( M \) be a finitely generated \( \mathbb{Z}^r \)-graded \( S \)-module such that \( H^0_B(M)_d = 0 \). If \( M \) is \( d \)-regular then \( M_{\geq d} \) has a quasilinear resolution \( F \), with \( F_0 \) generated in degree \( d \).

Proof. Without loss of generality we may assume that \( d = 0 \) and \( M = M_{\geq 0} \) (see Lemma 2.9).

By Proposition 3.7 there exists a free monad \( G \) of \( M \) with \( j \)-th Betti number given by \( h^{[a]-j}(\tilde{M} \otimes \Omega^a_{\mathbb{P}_n}(a)) \). Since \( M \) is \( 0 \)-regular the vanishing of these cohomology groups results in a quasilinear virtual resolution by Lemma 4.7 and (2) from Proposition 3.7. Let \( F \) be the minimal free resolution of \( M \). We will show that the Betti numbers of \( F \) are equal to those of \( G \), so that \( F \) is also quasilinear and \( F_0 = G_0 \) is generated in degree \( d \).

Fix a degree \( a \in \mathbb{Z}^r \). Construct a double complex \( E^{s,t} \) by taking the Čech complex of each term in \( M \otimes K^{\leq a} \) and including the Čech complex of \( M \otimes \Omega^a_{\mathbb{P}_n} \) as an additional column. Index \( E^{s,t} \) so that

\[
E^{s,t} = \begin{cases} 
\check{C}^t(B, M \otimes K^{\leq a}_{[a]+1-s}) & \text{if } s > 0, \\
\check{C}^t(B, M \otimes \Omega^a_{\mathbb{P}_n}) & \text{if } s = 0.
\end{cases}
\]
We will compare the vertical and horizontal spectral sequences of $E^{••}$ in degree $a$. By Lemma 2.10 and the fact that $K^{≤a}$ is locally free, the sheafification of the 0-th row $E^{•0}$ is exact. Thus by Lemma 2.8 the rows of $E^{••}$ are exact for $t ≠ 0$.

Since the elements of $M$ have degrees $≥0$, the elements of degree $a$ in $M ⊗ K^∗$ come from elements of degree $≤a$ in $K^∗$. Thus by Lemma 2.10 the homology of $M ⊗ K^{≤a}$ in degree $a$ is the same as that of $M ⊗ K^∗$. Hence the cohomology of the 0-th row $E^{•0}$ in degree $a$ computes the degree $a$ Betti numbers of $F_j$ for $0 ≤ j ≤ |a|$, i.e., for $s > 0$,

$$H^s(E^{•0})_a = \text{Tor}_{|a|+1-s}(M, k)_a.$$ (4.1)

The vertical cohomology of $E^{••}$ gives the local cohomology of the terms of $M ⊗ K^{≤a}$ along with $M ⊗ \hat{\Omega}^a_{\mathbb{P}^n}$. Consider the degree $a$ part of this double complex. The cohomology coming from $M ⊗ K^{≤a}$ has summands of the form $H^s_B(M(−b))_a = H^s_B(M)_{a−b}$ where $b ≤ a$. These vanish because $M$ is 0-regular, except possibly $H^0_B(M)_0$ which vanishes by hypothesis, so the only nonzero terms come from $M ⊗ \hat{\Omega}^a_{\mathbb{P}^n}$.

Since $K^{≤a}$ is a resolution of $k$ in degrees $≤a$, there are no elements of degree $a$ in $M ⊗ \hat{\Omega}^a_{\mathbb{P}^n}$. Hence, using (2.1),

$$H^1_B\left(M ⊗ \hat{\Omega}^a_{\mathbb{P}^n}\right)_a = H^0\left(\mathbb{P}^n, \widetilde{M} ⊗ \hat{\Omega}^a_{\mathbb{P}^n}(a)\right).$$

Therefore the cohomology of the 0-th column $E^{0•}$ in degree $a$ is

$$H^t(E^{0•})_a = H^t_B(M ⊗ \hat{\Omega}^a_{\mathbb{P}^n})_a = H^{t−1}(_{\mathbb{P}^n, \widetilde{M} ⊗ \hat{\Omega}^a_{\mathbb{P}^n}(a)})$$ (4.2)

for $t > 0$, i.e., the Betti numbers of $G^a_∗$ indexed differently.

Since both spectral sequences of the double complex $E^{••}$ converge after the first page, their total complexes agree in degree $a$, so by equating the dimensions of (4.1) and (4.2) in total degree $|a| + 1 - j$ we get

$$\dim_k \text{Tor}_j(M, k)_a = \dim_k H^{|a|−j}(\mathbb{P}^n, \widetilde{M} ⊗ \hat{\Omega}^a_{\mathbb{P}^n}(a))$$ (4.3)

for $|a| ≥ j ≥ 0$. When $j > |a|$, neither $F^a_∗$ nor $G^a_∗$ has a nonzero Betti number for degree reasons, and when $a$ has $\hat{\Omega}^a_{\mathbb{P}^n} = 0$ the argument above still holds. Hence the Betti numbers of $G^a_∗$ and $F^a_∗$ are equal in degree $a$.

**Example 4.9.** A smooth hyperelliptic curve of genus 4 can be embedded into $\mathbb{P}^1 × \mathbb{P}^2$ as a curve of degree $(2, 8)$. An example of such a curve is given explicitly in [BES20, Ex. 1.4] as
the $B$-saturation $I$ of the ideal
\[ \langle x_0^2y_0^2 + x_1y_1^2 + x_0x_1y_2^2, x_0y_0^2 + x_1(y_0 + y_1) \rangle. \]

Using Theorem 4.8 it is relatively easy to check that $S/I$ is not $(2,1)$-regular: the minimal, graded, free resolution of $(S/I)_{\geq(2,1)}$ is
\[
\begin{align*}
0 & \longleftarrow S(-3, -1)^7 \oplus S(-3, -2)^6 \\
& \longleftarrow S(-2, -1)^9 \oplus S(-2, -2)^{10} \oplus S(-2, -3)^3 \oplus S(-3, -3)^2 \longleftarrow 0
\end{align*}
\]
which is not quasilinear because $(-2, -3) \notin Q_1(-2, -1)$.

To check that a module $M$ is $d$-regular directly from Definition 2.3, condition (2) requires one to show that $H^i_B(M)_p$ vanishes for all $i > 0$ and all $p \in \bigcup_{|\lambda|=1} (d - \lambda_1 e_1 - \cdots - \lambda_r e_r + \mathbb{N}^r)$ with $\lambda \in \mathbb{N}^r$. The proof of Theorem 4.8, when combined with Theorem 4.6 and Lemma 4.7, shows that on a product of projective spaces the full strength of this condition is unnecessary. In particular, one only needs to consider $\lambda_j$ with $\lambda_j \leq n_j + 1$.

**Proposition 4.10.** Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $S$-module. If
\begin{enumerate}
\item $H^0_B(M)_p = 0$ for all $p \geq d$
\item $H^i_B(M)_p = 0$ for all $i > 0$ and all $p \in \bigcup_{|\lambda|=1} (d - \sum_{i=1}^r \lambda_j e_j + \mathbb{N}^r)$ where $0 \leq \lambda_j \leq n_j + 1$
\end{enumerate}
then $M$ is $d$-regular.

**Proof.** The only difference between (2) above and condition (2) in Definition 2.3 is the restriction to $\lambda_j \leq n_j + 1$. By the proof of Theorem 4.8, if $H^0_B(M)_p = 0$ and $M$ satisfies the hypotheses of Proposition 3.7 and Lemma 4.7 then $M$ has a quasilinear resolution generated in degree $d$ and is thus $d$-regular by Theorem 4.6. In the proof of Lemma 4.7 it is sufficient for the cohomology of $M(d)$ to vanish in degrees appearing in the resolution of some $\Omega_{\mathbb{P}^d}^n(a)$, which excludes those with coordinates not $\leq n + 1$.

**Example 4.11.** On $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, to show that a module $M$ is $0$-regular using Definition 2.3 one must check that $H^3_B(M)_p = 0$ for $p$ in the region with minimal elements
\[ (-3, 0, 0), (-2, -1, 0), (-2, 0, -1), \ldots, (0, -3, 0), \ldots, (0, 0, -3). \]
However, Proposition 4.10 implies that a smaller region is sufficient. For instance, we need not check that $H^3_B(M)_p = 0$ for $p$ equal to each of $(0, 0, 0)$, $(0, -3, 0)$, and $(0, 0, -3)$.

**Remark 4.12.** One may also deduce Proposition 4.10 from the proofs in [BES20] without the hypothesis that $H^0_B(M)_d = 0$. See Section 4.2 for further discussion.

4.2. **Quasilinearity Implies Regularity.** We will now prove the reverse implication of Theorem 4.6, namely that a quasilinear resolution generated in degree $d$ for $M_{\geq d}$ implies that $M$ is $d$-regular. We use a hypercohomology spectral sequence argument, which relates the local cohomology of $M$ to that of the terms in a resolution for $M_{\geq d}$.

The following lemma will show that entire diagonals in our spectral sequence vanish when the resolution is quasilinear. Thus the local cohomology modules $H^i_B(M)$ to which the diagonals converge also vanish in the same degrees.
Lemma 4.13. If \( i, j \in \mathbb{N} \) then \( H^{i+j+1}_B(S)_{a+b} = 0 \) for all \( a \in L_i(0) \) and all \( b \in Q_j(0) \).

Proof. Note that \( L_i(0) + Q_j(0) = L_i(0) + L_{j-1}(-1) = L_{i+j-1}(-1) \) as sets. We also have \( H^0_B(S) = H^1_B(S) = 0 \), so it suffices to show that \( H^{k+1}_B(S)_c = H^k(P^n, O_{P^n}(c)) = 0 \) for \( k \geq 1 \) and \( c \in L_{k-1}(-1) \).

The cohomology of \( O_{P^n} \) is given by the Künneth formula. Fix a nonempty set of indices \( J \subseteq \{1, \ldots, r\} \) and consider the term

\[
\left( \bigotimes_{j \in J} H^{n_j}(P^n_i, O_{P^n}(d_j)) \right) \otimes \left( \bigotimes_{j \notin J} H^0(P^n_i, O_{P^n}(d_j)) \right),
\]

which contributes to \( H^k(P^n, O_{P^n}(c)) \) for \( k = \sum_{j \in J} n_j \). It will be nonzero if and only if \( d_j \leq -n_j - 1 \) for \( j \in J \) and \( d_j \geq 0 \) for \( j \notin J \). If \( c \in L_{k-1}(-1) \) then

\[ c \geq -1 - \lambda_1 e_1 - \cdots - \lambda_r e_r \]

for some \( \lambda_i \) with \( \sum \lambda_i = k - 1 = -1 + \sum_{j \in J} n_j \). It is not possible for the right side to have components \( \leq -n_j - 1 \) for all \( j \in J \). Since all cohomology of \( O_{P^n} \) arises in this way, the lemma follows. \( \square \)

In [BES20, Thm. 2.9] Berkesch, Erman, and Smith show for \( M \) with \( H^0_B(M) = H^1_B(M) = 0 \) that \( M \) is \( d \)-regular if and only if \( M \) has a virtual resolution \( F_\bullet \) so that the degrees of the generators of \( F(d)_\bullet \) are at most those appearing in the minimal free resolution of \( S/B \). This Betti number condition is stronger than quasilinearity, but the additional strength is not used in their proof, so the existence of such a virtual resolution is equivalent to the existence of a quasilinear one.

Since a resolution of \( M_{\geq d} \) is a type of virtual resolution, the reverse implication of Theorem 4.6 mostly reduces to this result. We present a modified proof for completeness. In particular, we do not need to require \( H^1_B(M) = 0 \) because we have more information about the cokernel of our resolution.

From this perspective Theorem 4.6 says that the regularity of \( M \) is determined not only by the Betti numbers of its virtual resolutions, but by the Betti numbers of only those virtual resolutions that are actually minimal free resolutions of truncations of \( M \). Thus we provide an explicit method for checking whether \( M \) is \( d \)-regular.

Theorem 4.14. Let \( M \) be a finitely generated \( \mathbb{Z}^r \)-graded \( S \)-module such that \( H^0_B(M) = 0 \). If \( M_{\geq d} \) has a quasilinear resolution \( F_\bullet \) with \( F_0 \) generated in degree \( d \), then \( M \) is \( d \)-regular.

Proof. Without loss of generality we may assume that \( d = 0 \) and \( M = M_{\geq 0} \) (see Lemma 2.9).

Let \( F_\bullet \) be a quasilinear resolution of \( M \), so that the twists of \( F_j \) are in \( Q_j(0) \). Then the spectral sequence of the double complex \( E^{\bullet, \bullet} \) with terms

\[ E^{s,t}_i = C^i(B, F_{-s}) \]

converges to the cohomology \( H^i_B(M) \) of \( M \) in total degree \( i \). The first page of the vertical spectral sequence has terms \( H^i_B(F_{-s}) \), so \( H^{i+j}_B(F_j)_a = 0 \) for all \( j \) (i.e., for all \( (s, t) = (-j, i+j) \)) implies \( H^i_B(M)_a = 0 \).

Therefore it suffices to show that \( H^{i+j}_B(S(b))_a = 0 \) for \( i \geq 1 \) and all \( a \in L_{i-1}(0) \) and \( b \in Q_j(0) \), as is done in Lemma 4.13. \( \square \)
5. Multigraded Regularity and Betti Numbers

Unlike in the single graded setting, it is possible for two modules on a product of projective spaces to have the same multigraded Betti numbers but different multigraded regularities.

**Example 5.1.** Let $M$ be the module on $\mathbb{P}^1 \times \mathbb{P}^1$ with resolution

$$S(-1,0)^2 \oplus S(0,-1)^2 \leftarrow S(-1,-1)^4 \leftarrow 0$$

given in Example 4.2. Computation shows that $M$ is $(1,0)$-regular but not $(0,1)$-regular.

Notice that all of the twists appearing in the minimal resolution of $M$ are symmetric with respect to the factors of $\mathbb{P}^1 \times \mathbb{P}^1$. Hence the cokernel $N$ given by exchanging $x$ and $y$ in the presentation matrix has the same multigraded Betti numbers as $M$. However $N$ is not $(1,0)$-regular because $M$ was not $(0,1)$-regular.

**Remark 5.2.** Example 5.1 answers a question of Botbol and Chardin [BC17, Ques. 1.2].

5.1. Inner Bound from Betti Numbers. While the multigraded Betti numbers of a module do not determine its regularity, in this section we show that they do determine a subset of the regularity. In particular, the following lemma restricts the possible Betti numbers of a truncation of $M$ given the Betti numbers of $M$.

**Lemma 5.3.** Let $M$ be a $\mathbb{Z}^r$-graded $S$-module. If $M_{\geq d}$ has $\text{Tor}^S_{m'}(M_{\geq d},k)_{b'} \neq 0$ for some $b' \in \mathbb{Z}^k$ then there exist $b \leq b'$ and $m \leq m'$ such that $\text{Tor}^S_m(M,k)_b \neq 0$ and $|b' - c| \leq m' - m$ where $c = \max\{b, d\}$.

**Proof.** Let $0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$ be the minimal free resolution of $M$. Then the termwise truncation $0 \leftarrow M_{\geq d} \leftarrow (F_0)_{\geq d} \leftarrow (F_1)_{\geq d} \leftarrow \cdots$ is also exact by Lemma 2.6. For each $i$, let $G^i_\bullet$ be a minimal free resolution of $(F_i)_{\geq d}$.

We will see in Corollary 6.3 that $S(-b)_{\geq d}$ has a linear resolution for all $b \in \mathbb{Z}^k$. Thus the $G^i_\bullet$ are linear. By taking iterated mapping cones we can construct a free resolution of $M_{\geq d}$ with terms

$$0 \leftarrow G^0_0 \leftarrow G^0_1 \oplus G^1_0 \leftarrow G^0_2 \oplus G^1_1 \oplus G^2_0 \leftarrow \cdots \quad (5.1)$$

Then $b'$ corresponds to the degree of a generator of some $G^i_j$ with $i + j = m'$. Since $G^i_\bullet$ is linear, there is a minimal generator of $(F_i)_{\geq d}$ with degree $c$ such that $|b' - c| = j$. 


However the generators of \((F_i)_{i \geq 4}\) have degrees equal to \(\max\{b, d\}\) for degrees \(b\) of generators of \(F_i\). These correspond to \(b \in \mathbb{Z}^k\) such that \(\text{Tor}_i^S(M, k)_b \neq 0\). Thus the lemma holds for \(m = i\), so that \(m' - m = j = |b' - c|\) as desired.

Lemma 5.3 shows that each Betti number of \(M_{\geq d}\) comes from a Betti number of \(M\) in a predictable way. Note that the process cannot be reversed—not all Betti numbers of \(M\) produce minimal Betti numbers of \(M_{\geq d}\). However, the Betti numbers of \(M\) limit the degrees where a nonlinear truncation could exist. The following theorem identifies such degrees.

**Theorem 5.4.** Let \(M\) be a \(\mathbb{Z}^r\)-graded \(S\)-module. For all \(d \in \bigcap L_m(b)\), the truncation \(M_{\geq d}\) has a linear resolution generated in degree \(d\), where the intersection is over all \(m\) and all \(b\) with \(\text{Tor}_m^S(M, k)_b \neq 0\).

**Proof.** We may assume that \(d = 0\). Suppose instead that \(M_{\geq 0}\) does not have a linear resolution generated in degree \(0\). Then there exist \(b' \in \mathbb{N}^k\) and \(m' \in \mathbb{Z}\) such that \(\text{Tor}_m^S(M_{\geq 0}, k)_{b'} \neq 0\) and \(|b'| > m'\).

By Lemma 5.3 there exist \(b\) and \(m\) so that \(\text{Tor}_m^S(M, k)_b \neq 0\) and \(|b' - c| \leq m' - m\) where \(c = \max\{b, 0\}\). The sum of the positive components of \(b\) is

\[
|c'| = |b'| - |b' - c| > m' - (m' - m) = m
\]

so \(0 \notin L_m(b)\) (see Remark 2.2).

An analogous statement to Theorem 5.4 exists for truncations with quasilinear resolutions. By Theorem 4.6 it also gives a subset of the multigraded regularity. We will see in Section 5.2 that this inner bound is sharp.

**Theorem 5.5.** Let \(M\) be a \(\mathbb{Z}^r\)-graded \(S\)-module. For all \(d \in \bigcap Q_m(b)\), the truncation \(M_{\geq d}\) has a quasilinear resolution generated in degree \(d\), where the intersection is over all \(m\) and all \(b\) with \(\text{Tor}_m^S(M, k)_b \neq 0\).

**Proof.** Assume \(d = 0\) and suppose instead that \(M_{\geq 0}\) does not have a quasilinear resolution generated in degree \(0\). If \(M_{\geq 0}\) is not generated in degree \(0\) then some generator of \(M\) has a degree \(b\) with a positive coordinate, so that \(0 \notin b + \mathbb{N}^r = Q_0(b)\).

Otherwise there exist \(b' \in \mathbb{N}^k\) and \(m' \in \mathbb{Z}\) such that \(\text{Tor}_{m'}^S(M_{\geq 0}, k)_{b'} \neq 0\) and \(|b'| > m' + \ell' - 1\) where \(\ell'\) is the number of nonzero coordinates in \(b'\). Thus by Lemma 5.3 there exist \(b\) and \(m\) so that \(\text{Tor}_m^S(M, k)_b \neq 0\) and \(|b' - c| \leq m' - m\) for \(c = \max\{b, 0\}\).

Let \(\ell\) be the number of coordinates for which \(c\) differs from \(c' = \max\{b, 1\}\). Then \(|c'| = |c| + \ell\), so the sum of the positive components of \(b - 1\) is

\[
|c' - 1| = |c| + \ell - r
\]

\[
= |b'| - |b' - c| - r + \ell
\]

\[
> (m' + \ell' - 1) - (m' - m) - r + \ell
\]

\[
= m - 1 + \ell' - (r - \ell).
\]

Note that \(r - \ell\) is the number of nonzero coordinates in \(c\). Since \(b' \geq 0\) and \(b' \geq b\) we have \(b' \geq c \geq 0\), so \(\ell' \geq r - \ell\). Hence the right side of the inequality is \(\geq m - 1\), so \(0 \notin L_{m-1}(b - 1) = Q_m(b)\) (see Remark 2.2).
Corollary 5.6. Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $S$-module. If $H^0_B(M) = 0$, then

$$\bigcap_{i \in \mathbb{N}} \bigcap_{b \in \beta_i(M)} Q_i(b) \subseteq \text{reg}(M).$$

We can now prove Proposition 4.5.

Proof of Proposition 4.5. Suppose that $M_{\geq d}$ has a linear resolution. We will apply Theorem 5.4 to $M_{\geq d}$ to show that $M_{\geq d'}$ has a linear resolution for $d' \geq d$ as desired. We may assume that the intersection contains all possible terms that could arise from a linear resolution:

$$\bigcap_{i \in \mathbb{N}} \bigcap_{b \in L_i(-d)} L_i(b)$$

Note that $-b \in L_i(-d)$ if and only if $d \in L_i(b)$. Thus $d \in L_i(b)$ for all $b$, so $d'$ is in the intersection as well. For quasilinear resolutions replace $L$ with $Q$. □

Other bounds on the multigraded regularity of a module in terms of its Betti numbers exist in the literature. For example, Maclagan and Smith use a long exact sequence argument to bound regularity in [MS04, Thm. 1.5, Cor 7.2]. While our theorem has the added hypothesis that $H^0_B(M) = 0$, it is often sharper than Maclagan and Smith’s.

Example 5.7. In [MS04, Ex. 7.6] Maclagan and Smith consider the $B$-saturated ideal $I = \langle x_1 - x_1, x_2 - x_2, x_3 - x_3 \rangle \cap \langle x_1 - 2x_1, x_2 - 2x_2, x_3 - 2x_3 \rangle$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. They show that the regularity of $S/I$ is

$$\text{reg}(S/I) = ((1, 0, 0) + \mathbb{N}^3) \cup ((0, 1, 0) + \mathbb{N}^3) \cup ((0, 0, 1) + \mathbb{N}^3)$$

and their bound from the Betti numbers of $S/I$ is

$$((2, 2, 1) + \mathbb{N}^3) \cup ((2, 1, 2) + \mathbb{N}^3) \cup ((1, 2, 2) + \mathbb{N}^3) \subseteq \text{reg}(S/I).$$

However, Corollary 5.6 implies that $(1, 1, 1) + \mathbb{N}^r \subseteq \text{reg}(S/I)$, giving a larger inner bound.

5.2. Regularity of Complete Intersections. As an application of Theorems A and B, in this section we compute the multigraded regularity of a saturated complete intersection satisfying minor hypotheses on its generators. To do this we make the bound from Corollary 5.6 explicit in the case of complete intersections. We then use our characterization of regularity to prove that the resulting bound is sharp by explicitly constructing truncations outside this region that do not have quasilinear resolutions.

Lemma 5.8. If $b, c \in \mathbb{N}^r$ with $b_j, c_j > 0$ for all $j$ then $Q_{i+1}(b + c) \subseteq Q_i(b)$ for all $i > 0$.

Proof. By definition the semigroup generators of $Q_{i+1}(b + c)$ are of the form $b + c - 1 - v$ where $v \in \mathbb{N}^r$ and $|v| = i$. Thus it is enough to show that each $b + c - 1 - v$ is in $Q_i(b)$. Since $|v| = i$ it has at least one nonzero coordinate, say $v_j$. From this we have

$$b + c - 1 - v = (b - 1 - (v - e_j)) + (c - e_j).$$

The desired containment follows from the above equality given that $|v - e_j| = i - 1$ and that by assumption $c - e_j$ is in $\mathbb{N}^r$. □
Theorem 5.9. Let \( I = (f_1, \ldots, f_c) \subset B \) be a saturated complete intersection of codimension \( c \) in \( S \), meaning that the \( f_i \) form a regular sequence of elements from \( B \) and \( H_B^0(S/I) = 0 \). Then

\[
\text{reg}(S/I) = Q_c \left( \sum_{i=1}^c \deg f_i \right).
\]

Proof. Write \( a = \sum_{i=1}^c \deg f_i \). By Theorem 4.6 it suffices to show that \( (S/I)_{\geq a} \) has a quasilinear resolution generated in degree \( d \) if and only if \( d \in Q_c(a) \). We will prove one direction by showing that \( Q_c(a) \) is the bound from Corollary 5.6, i.e., that

\[
\bigcap_{j \in \mathbb{N}} \bigcap_{b \in \beta_j(S/I)} Q_j(b) = Q_c(a)
\]

By hypothesis the minimal free resolution \( F_\bullet \) of \( S/I \) is a Koszul complex, so the elements of \( \beta_j(S/I) \) are sums of \( j \) choices of \( \deg f_i \). In particular \( \beta_0(S/I) = \{0\} \) and \( \beta_c(S/I) = \{a\} \). We have \( Q_c(a) \subset \mathbb{N}^r = Q_0(0) \), so it suffices to show that

\[
Q_{j+1}(\deg f_{i_1} + \cdots + \deg f_{i_j}, \deg f_{i_{j+1}}) \subseteq Q_j(\deg f_{i_1} + \cdots + \deg f_{i_j})
\]

for all \( 0 < j < c \) and all \( 1 \leq i_1 < \cdots < i_{j+1} \leq c \), since each of the other sets in the intersection can be obtained from \( Q_c(a) \) in this way. Note that since \( I \subset B \), all coordinates of each \( \deg f_i \) are positive; therefore the inclusion follows from Lemma 5.8.

Now we need that \( (S/I)_{\geq a} \) does not have a quasilinear resolution if \( d \notin Q_c(a) \). Specifically, we will show that the resolution of \( (S/I)_{\geq a} \) has a \( c \)-th syzygy in degree \( a' = \max\{d, a\} \). If \( d \notin Q_c(a) \) then \( d \notin Q_c(a') \) and thus \( -a' \notin Q_c(-d) \), so this will complete our argument.

The proof of Lemma 5.3 constructs a possibly nonminimal free resolution \( (S/I)_{\geq a} \) from resolutions of truncations of the \( F_j \). Since \( (F_c)_{\geq a} \) has a generator of degree \( a' \), the minimal resolution of \( (S/I)_{\geq a} \) will contain a \( c \)-th syzygy of degree \( a' \) unless there is a nonminimal map from the generators \( G^c_i \) of \( (F_c)_{\geq a} \) to \( G^{-1} \oplus \cdots \oplus G_{c-1}^0 \). Suppose for contradiction that this is true.

The degrees of the summands in \( G^{c-1}_{i-i} \) have the form \( \max\{d, b\} + v \) where \( b \) is the sum of the degrees of \( c - 1 - i \) choices of the generators \( f_j \) and some \( v \in \mathbb{N}^r \) with \( |v| = i \). In order to have a degree 0 map we need \( \max\{d, b\} + v = a' = \max\{d, a\} \) for some \( b \) and \( v \). Since all coordinates of each \( \deg f_j \) are positive \( b_j + i + 1 \leq a_j \) for each \( j \), so \( b_j + v_j \neq a_j \). Thus \( d \geq b \), so \( d + v = a' \), contradicting the fact that \( d \notin Q_c(a') \).

\[
\square
\]

Note the assumption that \( H_B^0(S/I) = 0 \) is automatically satisfied if \( \text{codim}(P) \neq \text{codim}(I) \) for all minimal primes \( P \) over \( B \). However, based on a number of examples it seems that a weaker saturation hypothesis may be sufficient.

Example 5.10. Write \( S = \mathbb{k}[x_0, x_1, x_2, y_0, y_1, y_2] \) and consider the saturated complete intersection ideal \( I = (x_0y_0, x_1y_1^2) \) that defines a surface in \( \mathbb{P}^2 \times \mathbb{P}^2 \). Then Theorem 5.9 implies

\[
\text{reg}(S/I) = Q_2((2, 3)) = ((0, 2) + \mathbb{N}^2) \cup ((1, 1) + \mathbb{N}^2).
\]

6. Linear Truncations

As demonstrated by Example 4.2, in general \( d \)-regularity is a stronger condition than having a linear resolution for \( M_{\geq d} \). Still, linear truncations have been independently studied in the literature [EES15; BES20].
Our main result in this section is a cohomological vanishing condition that specifies when $M_{\geq d}$ has a linear resolution. Our arguments largely mimic those for the analogous statements about quasilinear resolutions by switching the roles of $L$ and $Q$. However, in this case we can identify not only the terms but also the maps in the first page of the Beilinson spectral sequence with those in the resolution of $M_{\geq d}$.

Lemma 6.1. Let $M$ be a $\mathbb{Z}^r$-graded $S$-module. If $H^i(\mathbb{P}^n, \widetilde{M}(b)) = 0$ for all $i > 0$ and all $b \in Q_i(0)$, then $H^{[a]-i}(\mathbb{P}^n, \widetilde{M} \otimes \Omega^n_{\mathbb{P}^n}(a)) = 0$ for all $i \geq 0$ and all $-a \notin L_i(0)$.

Proof. We will modify the argument from Lemma 4.7.

Suppose that $-a \notin L_i(0)$ and $H^{[a]-i}(\mathbb{P}^n, \widetilde{M} \otimes \Omega^n_{\mathbb{P}^n}(a)) \neq 0$. Since $a \geq 0$ we have $|a| > i$. There must exist $j$ such that $H^{[a]-i+j}(\mathbb{P}^n, \widetilde{M} \otimes \mathcal{F}_j) \neq 0$, where the twists $b$ in $F_j$ have total degree $-j - \ell$ for $\ell$ the number of nonzero coordinates in $a$. Each twist has $\ell$ negative coordinates, so that the positive coordinates of $-1 - b$ sum to $j + \ell - \ell = j$. Hence $H^{[a]-i+j}(\mathbb{P}^n, \widetilde{M}(b)) \neq 0$ for some $b \in L_j(-1) = Q_{j+1}(0) \subseteq Q_{|a|-i+j}(0)$ with $|a|-i+j > 0$. $\square$

As in our main theorem, the conclusion of this lemma ensures the vanishing of certain Betti numbers of $M_{\geq d}$.

Theorem 6.2. Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $S$-module with $H^0_B(M) = 0$. Then $M_{\geq d}$ has a linear resolution $F_\bullet$ with $F_0$ generated in degree $d$ if and only if $H^j_B(M)_b = 0$ for all $i > 0$ and all $b \in Q_{i-1}(d)$.

Proof. The proof of the forward implication is analogous to the proof of Theorem 4.14, switching the roles of $L$ and $Q$. For the reverse, notice that the proof of Theorem 4.8 shows that the virtual resolution of $M$ from Proposition 3.7 has the same Betti numbers as the minimal free resolution of $M_{\geq d}$, i.e.,

$$\dim_k \text{Tor}_j(M_{\geq d}, k)_a = \dim_k H^{[a]-j}(\mathbb{P}^n, \widetilde{M} \otimes \Omega^n_{\mathbb{P}^n}(a))$$

for $|a| \geq j \geq 0$ and both are 0 otherwise. The vanishing of the right hand side for $-a \notin L_j(0)$, given by Lemma 6.1, then implies that the minimal free resolution of $M_{\geq d}$ is linear. $\square$

Corollary 6.3. The minimal free resolution of $S(-b)_{\geq d}$ is linear for all $b, d \in \mathbb{Z}^r$.

Proof. By adjusting $d$ we may assume that $b = 0$. Note that $S_{\geq d} = S_{\geq d'}$ for $d' = \max\{d, 0\} \in 0 + \mathbb{N}^r$. Thus by Theorem 6.2 and Proposition 4.5 it suffices to show that $H^j_B(S)_b = 0$ for all $i > 0$ and all $b \in Q_{i-1}(0)$, which follows from Lemma 4.13. $\square$

A key part in the proofs of Theorems 4.8 and 6.2 is that if $M$ is $d$-regular and $H^0_B(M) = 0$, then the Betti numbers of $M_{\geq d}(d)$ and the virtual resolution $F_\bullet$ of $M(d)$ constructed in Proposition 3.7 are equal. In the case when $H^0_B(M) = H^1_B(M) = 0$ and $F_\bullet$ has a linear presentation, however, we can prove an even stronger result: in fact $F_\bullet$ is the minimal free resolution of $M_{\geq d}(d)$. We do this by using the following lemma to show that the cokernel of $F_\bullet$ is $M_{\geq d}(d)$ and then using Proposition 3.5.

Lemma 6.4. If $H^0_B(M)_0 = H^1_B(M)_0 = 0$ then the first map in the virtual resolution from Proposition 3.7:

$$S \otimes H^0(\mathbb{P}^n, \widetilde{M}) \leftarrow \bigoplus_{i=1}^r S(-e_i) \otimes H^0(\mathbb{P}^n, \widetilde{M} \otimes \Omega^n_{\mathbb{P}^n}(e_i))$$

is the linear part of a presentation of $M_{\geq 0}$.
Proof. Recall from Section 3.3 that the first map in the virtual resolution comes from the first map in the resolution of the diagonal (3.2), which is the dual of a section

\[ s: O_{P^n} \boxtimes O_{P^n} \to W = \bigoplus_{i=0}^r O_{P^n}(e_i) \boxtimes T_{P^n}^e(-e_i). \]

We can describe \( s \) on the \( i \)-th summand of \( W \) by a map to its presentation, which we obtain by pulling back the Euler sequence (3.1) along \( \pi_i \) and taking the box product with \( O_{P^n}(e_i) \).

\[
\begin{array}{c}
0 \to O_{P^n}(e_i) \boxtimes O_{P^n}(-e_i) \xrightarrow{[1 \boxtimes x_{i,0}] \ldots [1 \boxtimes x_{i,n_i}]} O_{P^n}(e_i) \boxtimes O_{P^n}^{e_i+1} \xrightarrow{O_{P^n} \boxtimes O_{P^n}} O_{P^n}(e_i) \boxtimes T_{P^n}^e(-e_i) \to 0
\end{array}
\]

Summing over \( i \), dualizing, and tensoring with \( q^*\tilde{M} \) gives a diagram of sheaves on the product \( P^n \times P^n \). In particular, the map \( s \) gives the presentation

\[
0 \longleftarrow q^*\tilde{M} \otimes O_\Delta \longleftarrow O_{P^n} \otimes \tilde{M} \longleftarrow \bigoplus_{i=1}^r O_{P^n}(-e_i) \otimes (\tilde{M} \otimes O_{P^n}^e(e_i))
\]

Pushing forward along \( p_* \) and taking global sections, we get the commutative diagram

\[
\begin{array}{c}
M \xleftarrow{g} S \otimes H^0(P^n, \tilde{M}) \xleftarrow{\bigoplus_{i=1}^r S(-e_i) \otimes H^0(P^n, \tilde{M})} M
\end{array}
\]

where the vertical maps come from the Euler sequence and the horizontal maps from the resolution of the diagonal. Note that \( g: S \otimes H^0(P^n, \tilde{M}) \to M \) is surjective onto \( M_0 \) by hypothesis, so it suffices to show that in total degree 1 the kernel of \( g \) is the image of \( f \).

Take an element \( v \in S \otimes H^0(P^n, \tilde{M}) \) of total degree 1 such that \( g(v) = 0 \). Since the diagonal map is surjective in total degree 1, we get a lift \( w \) of \( v \). Then \( w \) maps to 0 vertically because \( M \) and \( \bigoplus S(-e_i) \otimes H^0(P^n, \tilde{M}(e_i)) \) are isomorphic in total degree 1 and this isomorphism commutes with the diagram above. Since the vertical sequence is exact \( w \) is in the image of the map below and \( v \) is in the image of \( f \).

\[ \square \]

**Corollary 6.5.** If \( M \) is \( d \)-regular with \( H_B^0(M) = H_B^1(M) = 0 \) and the virtual resolution \( G_* \) of \( M(d) \) from Proposition 3.7 has a linear presentation, then \( G_* \) is the minimal free resolution of \( M_{\geq d}(d) \).
Proof. Since \( M \) is \( \mathbf{d} \)-regular, by the proof of Theorem 4.8 the minimal free resolution \( F_\bullet \) of \( M_{\geq \mathbf{d}}(\mathbf{d}) \) and the virtual resolution \( G_\bullet \) have the same Betti numbers. Hence \( M_{\geq \mathbf{d}}(\mathbf{d}) \) has a linear presentation, so by Lemma 6.4 it is the cokernel of \( G_\bullet \). Since \( F_\bullet \) and \( G_\bullet \) have the same Betti numbers they must be isomorphic by Proposition 3.5. \( \square \)

The hypotheses of Corollary 6.5 do not require the truncation to have a linear resolution.

Example 6.6. Consider the ideal \( I = (x_0y_0 + x_0y_1 + x_0y_2, x_1y_0y_1 + x_1y_0y_2 + x_1y_1y_2) \) inside \( S = \text{Cox} \mathbb{P}^1 \times \mathbb{P}^2 \). Then \( N = S/I \) is a bigraded \((1,1)\)-regular \( S \)-module. The virtual resolution \( G_\bullet \) of \( M = N_{\geq (1,1)}(1,1) \) comes from the spectral sequence with first page

\[
\begin{array}{cccc}
0 & 0 & 0 & S'(-1,-2) \\
S^5 & S(-1,0)^3 \oplus S(0,-1)^2 & S(-1,-1)^3 \oplus S(0,-2)^2 & S(-1,-2) \\
\end{array}
\]

where the diagonal map is from the virtual resolution. Observe that \( G_\bullet \) has a linear presentation; hence it is a resolution by Corollary 6.5.

In light of the isomorphism in Corollary 6.5 and the more general equality of Betti numbers in the proof of Theorem 4.8, it seems reasonable to guess that the hypothesis of the corollary on the presentation of \( M_{\geq \mathbf{d}} \) may be unnecessary.

Conjecture 6.7. If \( M \) is \( \mathbf{d} \)-regular with \( H_B^0(M) = H_B^1(M) = 0 \), then the virtual resolution of \( M(\mathbf{d}) \) from Proposition 3.7 is the minimal free resolution of \( M_{\geq \mathbf{d}}(\mathbf{d}) \).

7. Generalizing Eisenbud–Goto

Recall Eisenbud–Goto’s conditions (2) through (4) from the introduction. As we have seen, these conditions diverge substantially for products of projective spaces. However, they can each be generalized to give interesting, albeit different, regions inside \( \text{Pic} \mathbb{P}^n \).

If \( M \) is a finitely generated \( \mathbb{Z}^r \)-graded \( S \)-module, then (2) defines the multigraded regularity region \( \text{reg}(M) \subset \text{Pic} \mathbb{P}^n \) of Maclagan and Smith. On the other hand condition (3) naturally generalizes to two truncation regions. First, the obvious generalization gives the linear truncation region:

\[
\text{trunc}^L(M) := \{ \mathbf{d} \in \mathbb{Z}^r \mid M|_{\geq \mathbf{d}} \text{ has a linear resolution generated in degree } \mathbf{d} \}.
\]

Second, our characterization of regularity gives the quasilinear truncation region:

\[
\text{trunc}^Q(M) := \{ \mathbf{d} \in \mathbb{Z}^r \mid M|_{\geq \mathbf{d}} \text{ has a quasilinear resolution generated in degree } \mathbf{d} \}.
\]

Finally, condition (4) on the Betti numbers of \( M \) also naturally generalizes to two Betti regions; the \( L \)-Betti region as in Theorem 5.4 and the \( Q \)-Betti region as in Theorem 5.5:

\[
\text{betti}^L(M) := \bigcap_{i \in \mathbb{N}} \bigcap_{\mathbf{d} \in B_i(M)} L_i(\mathbf{d}), \quad \text{betti}^Q(M) := \bigcap_{i \in \mathbb{N}} \bigcap_{\mathbf{d} \in B_i(M)} Q_i(\mathbf{d}).
\]

Theorem 4.6 now states that \( \text{reg}(M) = \text{trunc}^Q(M) \) when \( H_B^0(M) = 0 \). Moreover, since all linear resolutions are quasilinear we get \( \text{trunc}^L(M) \subseteq \text{trunc}^Q(M) \). Similarly, since \( L_i(\mathbf{d}) \subseteq Q_i(\mathbf{d}) \), by definition \( \text{betti}^L(M) \subseteq \text{betti}^Q(M) \).

Theorem 5.4 shows that the \( L \)-Betti region \( \text{betti}^L(M) \) is a subset of the linear truncation region \( \text{trunc}^L(M) \). Similarly, Theorem 5.5 shows that the \( Q \)-Betti region \( \text{betti}^Q(M) \) is a
subset of the quasilinear truncation region \( \text{trunc}^Q(M) \). We can summarize all of the above relations in the following highly non-commutative diagram:

\[
\begin{array}{c}
\text{betti}^L(M) \\ \downarrow \\
\text{betti}^Q(M)
\end{array} \xrightarrow{5.4} \begin{array}{c}
\text{trunc}^L(M) \\ \downarrow \\
\text{trunc}^Q(M)
\end{array} \xrightarrow{5.5} \begin{array}{c}
\text{reg}(M)
\end{array}
\]

We saw in Section 6 that we can switch the roles of \( Q \) and \( L \) in the proof of Theorem 4.6 to complete the upper right corner of this diagram. The resulting cohomological characterization of \( \text{trunc}^L(M) \) in Theorem 6.2 is related to the positivity conditions described in Remark 3.8. We suspect that the reversal of \( Q \) and \( L \) between the Betti number and cohomological conditions has a deeper explanation in terms of the BGG correspondence.

We illustrate the four regions above in the following example.

**Example 7.1.** Let \( I \) be the \( B \)-saturated ideal in Example 4.9, defining a smooth hyperelliptic curve of genus 4 embedded into \( \mathbb{P}^1 \times \mathbb{P}^2 \) as a curve of degree \((2, 8)\). As noted in [BES20, Ex. 1.4], using Macaulay2 one finds that the minimal graded free resolution of \( I \) is:

\[
\begin{array}{c}
S(-3, -1) \\ S(-2, -2) \\ S(-2, -3)^2 \\ S(-1, -5)^3 \\ S(0, -8)
\end{array} \xleftarrow{\oplus} \begin{array}{c}
S(-3, -3)^3 \\ S(-2, -5)^6 \\ S(-1, -7) \\ S(-1, -8)^2 \\ S(-3, -7) \oplus S(-2, -7)^2 \oplus S(-2, -8) \oplus 0.
\end{array}
\]

From this we can calculate that \( \text{betti}^L(S/I) \) and \( \text{betti}^Q(S/I) \) are both equal to \((2, 7) + \mathbb{N}^2\). These regions, depicted in Figure 2, can also be computed using linearTruncationsBound and regularityBound from the Macaulay2 package LinearTruncations, which implement Theorems 5.4 and 5.5, respectively [CHN21].

Further, using the functions linearTruncations and multigradedRegularity from the package VirtualResolutions [ABLS20], we can compute where \( S/I \) has a linear or quasilinear truncation inside the box \([0, 9]^2\). We see that the minimal elements of \( \text{trunc}^L(S/I) \) are \((1, 5)\), \((2, 2)\), and \((5, 1)\). On the other hand the minimal elements of \( \text{trunc}^Q(S/I) \)—which equals \( \text{reg}(S/I) \) as \( I \) is saturated—are \((1, 5)\), \((2, 2)\), and \((4, 1)\).

![Figure 2. The four regions for Example 4.9](image-url)
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