RANK TWO GLOBALLY GENERATED VECTOR BUNDLES
WITH $c_1 \leq 5$.

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Abstract. We classify globally generated rank two vector bundles on $\mathbb{P}^n$, $n \geq 3$, with $c_1 \leq 5$. The classification is complete but for one case ($n = 3$, $c_1 = 5$, $c_2 = 12$).

Introduction.

Vector bundles generated by global sections are basic objects in projective algebraic geometry. Globally generated line bundles correspond to morphisms to a projective space, more generally higher rank bundles correspond to morphism to (higher) Grassmann varieties. For this last point of view (that won’t be touched in this paper) see [11], [12], [13]. Also globally generated vector bundles appear in a variety of problems ([7] just to make a single, recent example).

In this paper we classify globally generated rank two vector bundles on $\mathbb{P}^n$ (projective space over $k$, $\kappa = \kappa$, $\text{ch}(\kappa) = 0$), $n \geq 3$, with $c_1 \leq 5$. The result is:

Theorem 0.1. Let $E$ be a rank two vector bundle on $\mathbb{P}^n$, $n \geq 3$, generated by global sections with Chern classes $c_1, c_2, c_1 \leq 5$.

1. If $n \geq 4$, then $E$ is the direct sum of two line bundles
2. If $n = 3$ and $E$ is indecomposable, then

$$(c_1, c_2) \in S = \{(2, 2), (4, 5), (4, 6), (4, 7), (4, 8), (5, 8), (5, 10), (5, 12)\}.$$ 

If $E$ exists there is an exact sequence:

$$0 \to \mathcal{O} \to E \to I_C(c_1) \to 0 \ (*)$$

where $C \subset \mathbb{P}^3$ is a smooth curve of degree $c_2$ with $\omega_C(4 - c_1) \simeq \mathcal{O}_C$. The curve $C$ is irreducible, except maybe if $(c_1, c_2) = (4, 8)$: in this case $C$ can be either irreducible or the disjoint union of two smooth conics.

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For every \((c_1, c_2) \in S, (c_1, c_2) \neq (5, 12)\), there exists a rank two vector bundle on \(\mathbb{P}^3\) with Chern classes \((c_1, c_2)\) which is globally generated (and with an exact sequence as in (2)).

The classification is complete, but for one case: we are unable to say if there exist or not globally generated rank two vector bundles with Chern classes \(c_1 = 5, c_2 = 12\) on \(\mathbb{P}^3\).

1. Rank two vector bundles on \(\mathbb{P}^3\).

1.1. General facts.

For completeness let’s recall the following well known results:

**Lemma 1.1.** Let \(E\) be a rank \(r\) vector bundle on \(\mathbb{P}^n, n \geq 3\). Assume \(E\) is generated by global sections.

1. If \(c_1(E) = 0\), then \(E \cong rO\).
2. If \(c_1(E) = 1\), then \(E \cong O(1) \oplus (r-1)O\) or \(E \cong T(-1) \oplus (r-n)O\).

**Proof.** If \(L \subset \mathbb{P}^n\) is a line then \(E|L \cong \bigoplus_{i=1}^r O_L(a_i)\) by a well known theorem and \(a_i \geq 0, \forall i\) since \(E\) is globally generated. It turns out that in both cases: \(E|L \cong O_L(c_1) \oplus (r-1)O_L\) for every line \(L\), i.e. \(E\) is uniform. Then (1) follows from a result of Van de Ven ([14]), while (2) follows from IV. Prop. 2.2 of [4]. \(\square\)

**Lemma 1.2.** Let \(E\) be a rank two vector bundle on \(\mathbb{P}^n, n \geq 3\). If \(E\) has a nowhere vanishing section then \(E\) splits. If \(E\) is generated by global sections and doesn’t split then \(h^0(E) \geq 3\) and a general section of \(E\) vanishes along a smooth curve, \(C\), of degree \(c_2(E)\) such that \(\omega_C(4-c_1) \cong O_C\). Moreover \(I_C(c_1)\) is generated by global sections.

**Lemma 1.3.** Let \(E\) be a non split rank two vector bundle on \(\mathbb{P}^3\) with \(c_1 = 2\). If \(E\) is generated by global sections then \(E\) is a null-correlation bundle.

**Proof.** We have an exact sequence: \(0 \to O \to E \to I_C(2) \to 0\), where \(C\) is a smooth curve with \(\omega_C(2) \cong O_C\). It follows that \(C\) is a disjoint union of lines. Since \(h^0(I_C(2)) \geq 2, d(C) \leq 2\). Finally \(d(C) = 2\) because \(E\) doesn’t split. \(\square\)

This settles the classification of rank two globally generated vector bundles with \(c_1(E) \leq 2\) on \(\mathbb{P}^3\).
1.2. Globally generated rank two vector bundles with $c_1 = 3$.

The following result has been proved in [3] (with a different and longer proof).

**Proposition 1.4.** Let $E$ be a rank two globally generated vector bundle on $\mathbb{P}^3$. If $c_1(E) = 3$ then $E$ splits.

**Proof.** Assume a general section vanishes in codimension two, then it vanishes along a smooth curve $C$ such that $\omega_C \simeq \mathcal{O}_C(-1)$. Moreover $\mathcal{I}_C(3)$ is generated by global sections. We have $C = \bigcup_{i=1}^{r} C_i$ (disjoint union) where each $C_i$ is smooth irreducible with $\omega_{C_i} \simeq \mathcal{O}_{C_i}(-1)$. It follows that each $C_i$ is a smooth conic. If $r \geq 2$ let $L = \langle C_1 \rangle \cap \langle C_2 \rangle$ (where $C_1$ is the plane spanned by $C_i$). Every cubic containing $C$ contains $L$ (because it contains the four points $C_1 \cap L$, $C_2 \cap L$). This contradicts the fact that $\mathcal{I}_C(3)$ is globally generated. Hence $r = 1$ and $E = \mathcal{O}(1) \oplus \mathcal{O}(2)$. \hfill $\Box$

1.3. Globally generated rank two vector bundles with $c_1 = 4$.

Let’s start with a general result:

**Lemma 1.5.** Let $E$ be a non split rank two vector bundle on $\mathbb{P}^3$ with Chern classes $c_1, c_2$. If $E$ is globally generated and if $c_1 \geq 4$ then:

$$c_2 \leq \frac{2c_1^3 - 4c_1^2 + 2}{3c_1 - 4}.$$ 

**Proof.** By our assumptions a general section of $E$ vanishes along a smooth curve, $C$, such that $\mathcal{I}_C(c_1)$ is generated by global sections. Let $U$ be the complete intersections of two general surfaces containing $C$. Then $U$ links $C$ to a smooth curve, $Y$. We have $Y \neq \emptyset$ since $E$ doesn’t split. The exact sequence of liaison: $0 \to \mathcal{I}_U(c_1) \to \mathcal{I}_C(c_1) \to \omega_Y(4 - c_1) \to 0$ shows that $\omega_Y(4 - c_1)$ is generated by global sections. Hence $\deg(\omega_Y(4 - c_1)) \geq 0$. We have $\deg(\omega_Y(4 - c_1)) = 2g' - 2 + d'(4 - c_1)$ ($g' = p_a(Y)$, $d' = \deg(Y)$). So $g' \geq \frac{d'(4 - c_1) + 2}{2} \geq 0$ (because $c_1 \geq 4$). On the other hand, always by liaison, we have: $g' - g = \frac{1}{2}(d' - d)(2c_1 - 4)$ ($g = p_a(C)$), $d = \deg(C)$). Since $d' = c_1^2 - d$ and $g = \frac{d(c_1 - 4)}{2} + 1$ (because $\omega_C(4 - c_1) \simeq \mathcal{O}_C$), we get: $g' = 1 + \frac{d(c_1 - 4)}{2} + \frac{1}{2}(c_1^2 - 2d)(2c_1 - 4) \geq 0$ and the result follows. \hfill $\Box$

Now we have:

**Proposition 1.6.** Let $E$ be a rank two globally generated vector bundle on $\mathbb{P}^3$. If $c_1(E) = 4$ and if $E$ doesn’t split, then $5 \leq c_2 \leq 8$ and there is an exact sequence: $0 \to \mathcal{O} \to E \to \mathcal{I}_C(4) \to 0$, where $C$ is a smooth irreducible elliptic curve of degree $c_2$ or, if $c_2 = 8$, $C$ is the disjoint union of two smooth elliptic quartic curves.
Proof. A general section of $E$ vanishes along $C$ where $C$ is a smooth curve with $\omega_C = O_C$ and where $\mathcal{I}_C(4)$ is generated by global sections. Let $C = C_1 \cup \ldots \cup C_r$ be the decomposition into irreducible components: the union is disjoint, each $C_i$ is a smooth elliptic curve hence has degree at least three.

By Lemma 1.3, $d = \deg(C) \leq 8$. If $d \leq 4$ then $C$ is irreducible and is a complete intersection which is impossible since $E$ doesn’t split. If $d = 5, C$ is smooth irreducible.

Claim: If $8 \geq d \geq 6$, $C$ cannot contain a plane cubic curve.

Assume $C = P \cup X$ where $P$ is a plane cubic and where $X$ is a smooth elliptic curve of degree $d - 3$. If $d = 6$, $X$ is also a plane cubic and every quartic containing $C$ contains the line $\langle P \rangle \cap \langle X \rangle$. If $\deg(X) \geq 4$ then every quartic, $F$, containing $C$ contains the plane $\langle P \rangle$. Indeed $F|H$ vanishes on $P$ and on the $\deg(X) \geq 4$ points of $X \cap \langle P \rangle$, but these points are not on a line so $F|H = 0$. In both cases we get a contradiction with the fact that $\mathcal{I}_C(4)$ is generated by global sections. The claim is proved.

It follows that, if $8 \geq d \geq 6$, then $C$ is irreducible except if $C = X \cup Y$ is the disjoint union of two elliptic quartic curves.

Now let’s show that all possibilities of Proposition 1.6 do actually occur. For this we have to show the existence of a smooth irreducible elliptic curve of degree $d$, $5 \leq d \leq 8$ with $\mathcal{I}_C(4)$ generated by global sections (and also that the disjoint union of two elliptic quartic curves is cut off by quartics).

**Lemma 1.7.** There exist rank two vector bundles with $c_1 = 4, c_2 = 5$ which are globally generated. More precisely any such bundle is of the form $\mathcal{N}(2)$, where $\mathcal{N}$ is a null-correlation bundle (a stable bundle with $c_1 = 0, c_2 = 1$).

**Proof.** The existence is clear (if $\mathcal{N}$ is a null-correlation bundle then it is well known that $\mathcal{N}(k)$ is globally generated if $k \geq 1$). Conversely if $E$ has $c_1 = 4, c_2 = 5$ and is globally generated, then $E$ has a section vanishing along a smooth, irreducible quintic elliptic curve (cf.1.4). Since $h^0(\mathcal{I}_C(2)) = 0$, $E$ is stable, hence $E = \mathcal{N}(2)$.

**Lemma 1.8.** There exist smooth, irreducible elliptic curves, $C$, of degree 6 with $\mathcal{I}_C(4)$ generated by global sections.

**Proof.** Let $X$ be the union of three skew lines. The curve $X$ lies on a smooth quadric surface, $Q$, and has $\mathcal{I}_X(3)$ globally generated (indeed the exact sequence
0 \to \mathcal{I}_Q \to \mathcal{I}_X \to \mathcal{I}_{X,Q} \to 0 \text{ twisted by } \mathcal{O}(3) \text{ reads like: } 0 \to \mathcal{O}(1) \to \mathcal{I}_C(3) \to \mathcal{O}_Q(3,0) \to 0). \text{ The complete intersection, } U, \text{ of two general cubics containing } X \text{ links } X \text{ to a smooth curve, } C, \text{ of degree 6 and arithmetic genus 1. Since, by liaison, } h^1(\mathcal{I}_C) = h^1(\mathcal{I}_X(-2)) = 0, \text{ } C \text{ is irreducible. The exact sequence of liaison: } 0 \to \mathcal{I}_U(4) \to \mathcal{I}_C(4) \to \omega_X(2) \to 0 \text{ shows that } \mathcal{I}_C(4) \text{ is globally generated.} \ □

In order to prove the existence of smooth, irreducible elliptic curves, \( C \), of degree \( d = 7, 8 \), with \( \mathcal{I}_C(4) \text{ globally generated} \), we have to recall some results due to Mori ([10]).

According to [10] Remark 4, Prop. 6, there exists a smooth quartic surface \( S \subset \mathbb{P}^3 \) such that \( \text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}X \) where \( X \) is a smooth elliptic curve of degree \( 7 \leq d \leq 8 \). The intersection pairing is given by:

\[ H^2 = 4, \quad X^2 = 0, \quad H \cdot X = d. \]

Such a surface doesn’t contain any smooth rational curve ([10] p.130). In particular:

\( \ast \) every integral curve, \( Z \), on \( S \) has degree \( \geq 4 \) with equality if and only if \( Z \) is a planar quartic curve or an elliptic quartic curve.

**Lemma 1.9.** With notations as above, \( h^0(\mathcal{I}_X(3)) = 0. \)

*Proof.* A curve \( Z \in |3H - X| \) has invariants \( (d_Z, g_Z) = (5, -2) \) (if \( d = 7 \)) or \( (4, -5) \) (if \( d = 8 \)), so \( Z \) is not integral. It follows that \( Z \) must contain an integral curve of degree \( < 4 \), but this is impossible. \ □

**Lemma 1.10.** With notations as above \( |4H - X| \) is base point free, hence there exist smooth, irreducible elliptic curves, \( X \), of degree \( d, 7 \leq d \leq 8 \), such that \( \mathcal{I}_X(4) \) is globally generated.

*Proof.* Let’s first prove the following: **Claim:** Every curve in \( |4H - X| \) is integral.

If \( Y \in |4H - X| \) is not integral then \( Y = Y_1 + Y_2 \) where \( Y_1 \) is integral with \( \deg(Y_1) = 4 \) (observe that \( \deg(Y) = 9 \) or 8).

If \( Y_1 \) is planar then \( Y_1 \sim H \), so \( 4H - X \sim H + Y_2 \) and it follows that \( 3H \sim X + Y_2 \), in contradiction with \( h^0(\mathcal{I}_X(3)) = 0 \) (cf [1.9]).

So we may assume that \( Y_1 \) is a quartic elliptic curve, i.e. (i) \( Y_1^2 = 0 \) and (ii) \( Y_1 \cdot H = 4 \). Setting \( Y_1 = aH + bX \), we get from (i): \( 2a(2a + bd) = 0 \). Hence (\( \alpha \)) \( a = 0 \), or (\( \beta \)) \( 2a + bd = 0. \)

(\( \alpha \)) In this case \( Y_1 = bX \), hence (for degree reasons and since \( S \) doesn’t contain curves of degree \( < 4 \)), \( Y_2 = \emptyset \) and \( Y = X \), which is integral.

(\( \beta \)) Since \( Y_1 \cdot H = 4 \), we get \( 2a + (2a + bd) = 2a = 4 \), hence \( a = 2 \) and \( bd = -4 \) which is impossible (\( d = 7 \) or 8 and \( b \in \mathbb{Z} \)).
This concludes the proof of the claim.

Since \((4H - X)^2 \geq 0\), the claim implies that \(4H - X\) is numerically effective. Now we conclude by a result of Saint-Donat (cf [10], Theorem 5) that \(|4H - X|\) is base point free, i.e. \(\mathcal{I}_{X,S}(4)\) is globally generated. By the exact sequence: \(0 \to \mathcal{O} \to \mathcal{I}_X(4) \to \mathcal{I}_{X,S}(4) \to 0\) we get that \(\mathcal{I}_X(4)\) is globally generated. \(\square\)

**Remark 1.11.** If \(d = 8\), a general element \(Y \in |4H - X|\) is a smooth elliptic curve of degree 8. By the way \(Y \neq X\) (see [1]). The exact sequence of liaison: \(0 \to \mathcal{I}_U(4) \to \mathcal{I}_X(4) \to \omega_Y \to 0\) shows that \(h^0(\mathcal{I}_X(4)) = 3\) (i.e. \(X\) is of maximal rank). In case \(d = 8\) Lemma 1.10 is stated in [3], however the proof there is incomplete, indeed in order to apply the enumerative formula of [8] one has to know that \(X\) is a connected component of \(\bigcap_{i=1}^{3} F_i\); this amounts to say that the base locus of \(|4H - X|\) on \(F_1\) has dimension \(\leq 0\).

To conclude we have:

**Lemma 1.12.** Let \(X\) be the disjoint union of two smooth, irreducible quartic elliptic curves, then \(\mathcal{I}_X(4)\) is generated by global sections.

**Proof.** Let \(X = C_1 \sqcup C_2\). We have: \(0 \to \mathcal{O}(-4) \to 2\mathcal{O}(-2) \to \mathcal{I}_{C_1} \to 0\), twisting by \(\mathcal{I}_{C_2}\), since \(C_1 \cap C_2 = \emptyset\), we get:

\[
0 \to \mathcal{I}_{C_2}(-4) \to 2\mathcal{I}_{C_2}(-2) \to \mathcal{I}_X \to 0
\]

and the result follows. \(\square\)

Summarizing:

**Proposition 1.13.** There exists an indecomposable rank two vector bundle, \(E\), on \(\mathbb{P}^3\), generated by global sections and with \(c_1(E) = 4\) if and only if \(5 \leq c_2(E) \leq 8\) and in these cases there is an exact sequence:

\[
0 \to \mathcal{O} \to E \to \mathcal{I}_C(4) \to 0
\]

where \(C\) is a smooth irreducible elliptic curve of degree \(c_2(E)\) or, if \(c_2(E) = 8\), the disjoint union of two smooth elliptic quartic curves.

1.4. Globally generated rank two vector bundles with \(c_1 = 5\).

We start by listing the possible cases:

**Proposition 1.14.** If \(E\) is an indecomposable, globally generated, rank two vector bundle on \(\mathbb{P}^3\) with \(c_1(E) = 5\), then \(c_2(E) \in \{8, 10, 12\}\) and there is an exact
sequence:

\[ 0 \to \mathcal{O} \to E \to \mathcal{I}_C(5) \to 0 \]

where \( C \) is a smooth, irreducible curve of degree \( d = c_2(E) \), with \( \omega_C \simeq \mathcal{O}_C(1) \).

In any case \( E \) is stable.

Proof. A general section of \( E \) vanishes along a smooth curve, \( C \), of degree \( d = c_2(E) \) with \( \omega_C \simeq \mathcal{O}_C(1) \). Hence every irreducible component, \( Y \), of \( C \) is a smooth, irreducible curve with \( \omega_Y \simeq \mathcal{O}_Y(1) \). In particular \( \deg(Y) = 2g(Y) - 2 \) is even and \( \deg(Y) \geq 4 \).

(1) If \( d = 4 \), then \( C \) is a planar curve and \( E \) splits.
(2) If \( d = 6 \), \( C \) is necessarily irreducible (of genus 4). It is well known that any such curve is a complete intersection \((2, 3)\), hence \( E \) splits.
(3) If \( d = 8 \) and \( C \) is not irreducible, then \( C = P_1 \sqcup P_2 \), the disjoint union of two planar quartic curves. If \( L = \langle P_1 \rangle \cap \langle P_2 \rangle \), then every quintic containing \( C \) contains \( L \) in contradiction with the fact that \( \mathcal{I}_C(5) \) is generated by global sections. Hence \( C \) is irreducible.
(4) If \( d = 10 \) and \( C \) is not irreducible, then \( C = P \sqcup X \), where \( P \) is a planar curve of degree 4 and where \( X \) is a degree 6 curve \((X \text{ is a complete intersection (2, 3)})\). Every quintic containing \( C \) vanishes on \( P \) and on the 8 points of \( X \cap \langle P \rangle \), since these 8 points are not on a line, the quintic vanishes on the plane \( \langle P \rangle \). This contradicts the fact that \( \mathcal{I}_C(5) \) is globally generated.
(5) If \( d = 12 \) and \( C \) is not irreducible we have three possibilities:
   (a) \( C = P_1 \sqcup P_2 \sqcup P_3 \), \( P_i \) planar quartic curves
   (b) \( C = X_1 \sqcup X_2 \), \( X_i \) complete intersection curves of types \((2, 3)\)
   (c) \( C = Y \sqcup P \), \( Y \) a canonical curve of degree 8, \( P \) a planar curve of degree 4.
(a) This case is impossible (consider the line \( \langle P_1 \rangle \cap \langle P_2 \rangle \)).
(b) We have \( X_i = Q_i \cap F_i \). Let \( Z \) be the quartic curve \( Q_1 \cap Q_2 \). Then \( X_i \cap Z = F_i \cap Z \), i.e. \( X_i \) meets \( Z \) in 12 points. It follows that every quintic containing \( C \) meets \( Z \) in 24 points, hence such a quintic contains \( Z \). Again this contradicts the fact that \( \mathcal{I}_C(5) \) is globally generated.
(c) This case too is impossible: every quintic containing \( C \) vanishes on \( P \) and on the points \( \langle P \rangle \cap Y \), hence on \( \langle P \rangle \).

We conclude that if \( d = 12 \), \( C \) is irreducible.
The normalized bundle is $E(-3)$, since in any case $h^0(I_C(2)) = 0$ (every smooth irreducible subcanonical curve on a quadric surface is a complete intersection), $E$ is stable. □

Now we turn to the existence part.

**Lemma 1.15.** There exist indecomposable rank two vector bundles on $\mathbb{P}^3$ with Chern classes $c_1 = 5$ and $c_2 \in \{8, 10\}$ which are globally generated.

**Proof.** Let $R = \bigsqcup_{i=1}^s L_i$ be the union of $s$ disjoint lines, $2 \leq s \leq 3$. We may perform a liaison $(s, 3)$ and link $R$ to $K = \bigsqcup_{i=1}^s K_i$, the union of $s$ disjoint conics. The exact sequence of liaison: $0 \to I_U(4) \to I_K(4) \to \omega_R(5 - s) \to 0$ shows that $I_K(4)$ is globally generated (n.b. $5 - s \geq 2$).

Since $\omega_K(1) \simeq O_K$ we have an exact sequence: $0 \to O \to E(2) \to I_K(3) \to 0$, where $E$ is a rank two vector bundle with Chern classes $c_1 = -1, c_2 = 2s - 2$. Twisting by $O(1)$ we get: $0 \to O(1) \to E(3) \to I_K(4) \to 0$ (∗). The Chern classes of $E(3)$ are $c_1 = 5, c_2 = 2s + 4$ (i.e. $c_2 = 8, 10$). Since $I_K(4)$ is globally generated, it follows from (∗) that $E(3)$ too, is generated by global sections. □

**Remark 1.16.**

(1) If $E$ is as in the proof of Lemma 1.15 a general section of $E(3)$ vanishes along a smooth, irreducible (because $h^1(E(-2)) = 0$) canonical curve, $C$, of genus $1 + c_2/2$ ($g = 5, 6$) such that $I_C(5)$ is globally generated. By construction these curves are not of maximal rank ($h^0(I_C(3)) = 1$ if $g = 5$, $h^0(I_C(4)) = 2$ if $g = 6$). As explained in 3 §4 this is a general fact: no canonical curve of genus $g, 5 \leq g \leq 6$ in $\mathbb{P}^3$ is of maximal rank. We don’t know if this is still true for $g = 7$.

(2) According to 3 the general canonical curve of genus 6 lies on a unique quartic surface.

(3) The proof of 1.15 breaks down with four conics: $I_K(4)$ is no longer globally generated, every quartic containing $K$ vanishes along the lines $L_i (5 - s = 1)$. Observe also that four disjoint lines always have a quadrisection and hence are an exception to the normal generation conjecture (the homogeneous ideal is not generated in degree three as it should be).

**Remark 1.17.** The case $(c_1, c_2) = (5, 12)$ remains open. It can be shown that if $E$ exists, a general section of $E$ is linked, by a complete intersections of two
quintics, to a smooth, irreducible curve, $X$, of degree 13, genus 10 having $\omega_X(-1)$ as a base point free $g^5_1$. One can prove the existence of curves $X \subset \mathbb{P}^3$, smooth, irreducible, of degree 13, genus 10, with $\omega_X(-1)$ a base point free pencil and lying on one quintic surface. But we are unable to show the existence of such a curve with $h^0(I_X(5)) \geq 3$ (or even with $h^0(I_X(5)) \geq 2$). We believe that such bundles do not exist.

2. Globally generated rank two vector bundles on $\mathbb{P}^n$, $n \geq 4$.

For $n \geq 4$ and $c_1 \leq 5$ there is no surprise:

**Proposition 2.1.** Let $E$ be a globally generated rank two vector bundle on $\mathbb{P}^n$, $n \geq 4$. If $c_1(E) \leq 5$, then $E$ splits.

**Proof.** It is enough to treat the case $n = 4$. A general section of $E$ vanishes along a smooth (irreducible) subcanonical surface, $S$: $0 \to \mathcal{O} \to E \to I_S(c_1) \to 0$. By [5], if $c_1 \leq 4$, then $S$ is a complete intersection and $E$ splits. Assume now $c_1 = 5$. Consider the restriction of $E$ to a general hyperplane $H$. If $E$ doesn’t split, by [1.14] we get that the normalized Chern classes of $E$ are: $c_1 = -1$, $c_2 \in \{2, 4, 6\}$. By Schwarzenberger condition: $c_2(c_2 + 2) \equiv 0 \pmod{12}$. The only possibilities are $c_2 = 4$ or $c_2 = 6$. If $c_2 = 4$, since $E$ is stable (because $E|H$ is, see [1.14]), we have (3) that $E$ is a Horrocks-Mumford bundle. But the Horrocks-Mumford bundle (with $c_1 = 5$) is not globally generated.

The case $c_2 = 6$ is impossible: such a bundle would yield a smooth surface $S \subset \mathbb{P}^4$, of degree 12 with $\omega_S \simeq \mathcal{O}_S$, but the only smooth surface with $\omega_S \simeq \mathcal{O}_S$ in $\mathbb{P}^4$ is the abelian surface of degree 10 of Horrocks-Mumford. □

**Remark 2.2.** For $n > 4$ the results in [3] give stronger and stronger (as $n$ increases) conditions for the existence of indecomposable rank two vector bundles generated by global sections.

Putting everything together, the proof of Theorem [1.1] is complete.

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