The neutrino self-energy in a magnetized medium

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Abstract

In this work we calculate the neutrino self-energy in presence of a magnetized medium. The magnetized medium consists of electrons, positrons, neutrinos and a uniform classical magnetic field. The calculation is done assuming the background magnetic field is weak compared to the W-Boson mass squared, as a consequence of which only linear order corrections in the field are included in the W boson propagator. The electron propagator consists all order corrections in the background field. Although the neutrino self-energy in a magnetized medium in various limiting cases has been calculated previously in this article we produce the most general expression of the self-energy in absence of the Landau quantization of the charged gauge fields. We calculate the effect of the Landau quantization of the charged leptons on the neutrino self-energy in the general case. Our calculation is specifically suited for situations where the background plasma may be CP symmetric.

1 Introduction

The topic of neutrino self-energy in a thermal medium or a magnetized medium has attracted much attention in the last two decades [1, 2, 3]. Presence of magnetic field in active galactic nuclei as well as accretion disk of merging objects [4] and progenitors of Gamma Ray Bursts (GRBs) [5, 6] are obvious. So it is important to study the combined effect of both matter and magnetic field on neutrino propagation. As neutrinos are produced in elementary particle reactions, their presence in all of the above mentioned objects is a fact and once produced they propagate through the medium containing charged leptons, nucleons and/or neutrinos in presence of possible high/low magnetic fields.

In the present article we assume the magnetic field to be less than the critical field corresponding the \( W^\pm \) bosons and consequently we take only linear order, in the magnetic field strength, corrections to its propagator. The gauge bosons are assumed to be not in thermal equilibrium and

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so their thermal modifications are not used. In the present case we have worked in the unitary gauge and have not discussed about the gauge independence of the result, as it is noted that in such calculations the self-energy generally is dependent on the gauge choice but the dispersion relation is independent of the gauge [7, 8].

In this context we mention some differences from previous works namely those in [7, 8]. In most of the earlier works the authors have chosen a particular gauge as where the gauge parameter is unity, the Feynman gauge, and then tried to show that the dispersion relations are independent of the gauge choice. While in [1] the authors worked in unitary gauge to order \( G^2 \) but neither they nor the authors of Ref. [7, 8] did transparently exhibit the transverse and longitudinal decomposition of the neutrino self-energy which is expected in the presence of a magnetic field. In a related work [9] the authors decomposed the self-energy in the transverse and longitudinal parts and calculated the neutrino self-energy in a magnetized medium for high magnetic fields. In the last reference the authors approximated the electron propagator by its lowest Landau level value while used the most general \( W \)-boson propagator as they assumed that \( M^2_W \gg eB \gg m_e^2 \), where \( B \) is the magnitude of the magnetic field and \( m_e, M_W \) are the electron and the \( W \) boson masses. In ref. [10] the propagation of neutrino in an isotropic magnetized medium has been studied where they consider the magnetic field to be weak compared to the electron mass \( eB \ll m_e^2 \). In the present work we use the unitary gauge to calculate the self-energy of the neutrino to order \( G^2 \) and we find that the self-energy expression shows transverse and longitudinal neutrino momentum dependence as in [9] but we do not assume \( eB \gg m_e^2 \) and consequently we use the full electron propagator in presence of the magnetized medium while for the \( W \) boson propagator we only take linear order corrections in the magnetic field as in our case \( M^2_W \gg eB \). In the present article we only assume that \( T, \mu_\ell, \sqrt{B} < M_W \), where \( T \) and \( \mu \) are the temperature of the heat bath and the chemical potential of the charged leptons. We recover all the relevant result in the limit when \( B \to 0 \) and matches with the results in [11].

The paper is organized as follows. In section 2 we discuss about the general properties of the neutrino self-energy in a magnetized medium and its possible form. The various diagrams contributing to the neutrino self-energy and their individual contributions are also written down in the next section. In section 3 we calculate the contributions from the different diagrams and discuss about our scheme of computation. Section 4 summarizes the results of the previous section and there we write the dispersion relation of the neutrino in a magnetized medium. Finally we summarize our results in section 5.

2 General expression for the neutrino self-energy in a magnetized medium

The most general form of neutrino-self energy in presence of a magnetized medium can be written as:

\[
\Sigma(k) = R \left( a_\parallel k_\parallel^\mu + a_\perp k_\perp^\mu + bu^\mu + cb^\mu \right) \gamma_\mu L, \tag{1}
\]

where \( k_\parallel^\mu = (k^0, k^3) \) and \( k_\perp^\mu = (k^1, k^2) \). \( u^\mu \) stands for the 4-velocity of the centre-of-mass of the medium which looks like:

\[
u^\mu = (1, 0), \tag{2}
\]

in the rest frame of the medium. The \( u^\mu \) is normalized in such a way that,

\[
u^\mu u_\mu = 1. \tag{3}
\]
Likewise the effect of the magnetic field enters through the 4-vector $b^\mu$ which is defined in such a way that the frame in which the medium is at rest,

$$b^\mu = (0, \vec{b}),$$

where we denote the magnetic field vector by $\vec{B}\vec{b}$. The 4-vector $b^\mu$ is defined in such a way that,

$$b^\mu b_\mu = -1. \quad (5)$$

In this article we take the background classical magnetic field vector to be along the $z$-axis and consequently $b^\mu = (0, 0, 0, 1)$. The projection operators are conventionally defined as $R = \frac{1}{2}(1 + \gamma_5)$ and $L = \frac{1}{2}(1 - \gamma_5)$. Calculating the neutrino dispersion relation from Eq. (1) we get:

$$(1 - a_\parallel)\omega_\ell = \pm \left[ \left( (1 - a_\parallel)k_3 + c \right)^2 + (1 - a_\perp)^2 k_\perp^2 \right]^{1/2} + b. \quad (6)$$

In deriving the above dispersion relation Eq. (3) and Eq. (5) has been used and we have dropped terms like $b \cdot u$, $k_\perp \cdot u$, $k_\perp \cdot b$ which are zero in the rest frame of the medium, and $k_\perp^2 = k_1^2 + k_2^2$.

In the unitary gauge the three diagrams corresponding to the neutrino self-energy are as given in Fig. 1 and Fig. 2. The one-loop neutrino self-energy in a magnetized medium is comprised of

\begin{equation}
\Sigma(k) = \Sigma^W(k) + \Sigma^T(k) + \Sigma^Z(k). \quad (7)
\end{equation}

Figure 1: One-loop diagrams for neutrino self-energy in a magnetized medium. Diagram (a) is the $W$ exchange diagram and diagram (b) is the tadpole diagram. The heavy internal lines represent the $W$ and the charged lepton propagators in a magnetized medium.

Figure 2: The One-loop diagram for neutrino self-energy in a magnetized medium. This diagram is for the $Z$ exchange. The heavy dashed internal neutrino line corresponds to the neutrino propagator in a thermal medium.
Each of the individual terms appearing in the right-hand side of the above equation can be expressed as in Eq. (1). Expressing Eq. (7) as in Eq. (1) gives,

\[
a = a_W + a_T + a_Z, \quad (8)
\]
\[
b = b_W + b_T + b_Z, \quad (9)
\]
\[
c = c_W + c_T + c_Z. \quad (10)
\]

In Eq. (8) \(a_W, a_T\) and \(a_Z\) are composed of the parallel and perpendicular parts as shown in Eq. (1).

### 3 Contribution from the various diagrams

The individual terms on the right hand side of Eq. (7) can be explicitly written as:

\[
- i \Sigma^W(k) = \int \frac{d^4p}{(2\pi)^4} \left( -\frac{ig}{\sqrt{2}} \right) \gamma_\mu L i S_\ell(p) \left( -\frac{ig}{\sqrt{2}} \right) \gamma_\nu L i W^{\mu\nu}(q), \quad (11)
\]
\[
- i \Sigma^T(k) = - \left( \frac{g}{2 \cos \theta_W} \right)^2 R^{\mu\nu}(0) \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \gamma_\nu (c_V + c_A \gamma_5) i S_\ell(p) \right], \quad (12)
\]

and

\[
- i \Sigma^Z(k) = \int \frac{d^4p}{(2\pi)^4} \left( -\frac{ig}{\sqrt{2} \cos \theta_W} \right) \gamma_\mu L i S_\nu(p) \left( -\frac{ig}{\sqrt{2} \cos \theta_W} \right) \gamma_\nu L i Z^{\mu\nu}(q). \quad (13)
\]

In the above expressions \(g\) is the weak coupling constant and \(\theta_W\) is the Weinberg angle. The quantities \(c_V\) and \(c_A\) are the vector and axial-vector coupling constants which come in the neutral-current interaction of electrons, protons \((p)\), neutrons \((n)\) and neutrinos with the \(Z\) boson. Their forms are as follows,

\[
c_V = \begin{cases} 
-\frac{1}{2} + 2 \sin^2 \theta_W & e \\
\frac{1}{2} & \nu_e \\
\frac{1}{2} - 2 \sin^2 \theta_W & p \\
-\frac{1}{2} & \nu_p \\
\end{cases} 
\]

and

\[
c_A = \frac{1}{2} \begin{cases} 
-1 & \nu_p, \nu_e \\
1 & \mu, \tau \\
\end{cases} 
\]

Here \(W^{\mu\nu}(q)\) and \(S_\ell(p)\) stand for the \(W\)-boson propagator and charged lepton propagator respectively in presence of a magnetized plasma. The \(Z^{\mu\nu}(q)\) is the \(Z\)-boson propagator in vacuum and \(S_\nu(p)\) is the neutrino propagator in a thermal bath of neutrinos. The form of the charged lepton propagator in a magnetized medium is given by [12, 13]:

\[
S_\ell(p) = S^0_\ell(p) + S^3_\ell(p), \quad (16)
\]

where \(S^0_\ell(p)\) and \(S^3_\ell(p)\) are the charged lepton propagators in presence of an uniform background magnetic field and in a magnetized medium respectively. In this article we will always assume
that the magnetic field is directed towards the \( z \)-axis of the coordinate system. With this choice we have [14]:

\[
iS_\ell^0(p) = \int_0^\infty e^{\phi(p,s)}G(p, s) \, ds ,
\]

where,

\[
\phi(p, s) = is(p_0^2 - m_\ell^2 - \frac{\tan z}{z}p_\perp^2).
\]

In the above expression

\[
p_\parallel^2 = p_0^2 - p_3^2 ,
\]

\[
p_\perp^2 = p_1^2 + p_2^2 ,
\]

and \( z = eB\xi \) where \( e \) is the magnitude of the electron charge, \( B \) is the magnitude of the magnetic field and \( m_\ell \) is the mass of the charged lepton. In the above equation we have not written another contribution to the phase which is \( \epsilon|s| \) where \( \epsilon \) is an infinitesimal quantity. This term renders the \( s \) integration convergent. We do not explicitly write this term but implicitly we assume the existence of it and it will be written if required. The above equation can also be written as:

\[
\phi(p, s) = \psi(p_0) - is[p_3^2 + \frac{\tan z}{z}p_\perp^2],
\]

where,

\[
\psi(p_0) = is(p_0^2 - m_\ell^2) .
\]

The other term in Eq. (17) is given as:

\[
G(p, s) = \sec^2 z \left[ A + iB\gamma_5 + m_\ell (\cos^2 z - i\Sigma^3 \sin z \cos z) \right] ,
\]

where,

\[
A_\mu = p_\mu - \sin^2 z(p \cdot u \, u_\mu - p \cdot b \, b_\mu) ,
\]

\[
B_\mu = \sin z \cos z(p \cdot u \, b_\mu - p \cdot b \, u_\mu) ,
\]

and

\[
\Sigma^3 = \gamma_5\beta \gamma_\mu .
\]

The second term on the right-hand side of Eq. (16) denotes the medium contribution to the charged lepton propagator and its form is given by:

\[
S_\ell^\beta(p) = i\eta_F(p \cdot u) \int_{-\infty}^\infty e^{\phi(p,s)}G(p, s) \, ds ,
\]

where \( \eta_F(p \cdot u) \) contains the distribution functions of the particles in the medium and its form is:

\[
\eta_F(p \cdot u) = \frac{\theta(p \cdot u)}{e^{\beta(p \cdot u - \mu_\ell)} + 1} + \frac{\theta(-p \cdot u)}{e^{-\beta(p \cdot u - \mu_\ell)} + 1} ,
\]

where \( \beta \) and \( \mu_\ell \) are the inverse of the medium temperature and the chemical potential of the charged lepton.
The form of the $W$-propagator in presence of a uniform magnetic field along the $z$-direction is presented in [15] and in this article we only use the linearized (in the magnetic field) form of it. The reason we assume a linearized form of the $W$-propagator is because the magnitude of the magnetic field we consider is such that $eB \ll M_W^2$. In this limit the form of the propagator is:

$$ W^{\mu\nu}(q) = -\frac{1}{q^2 - M_W^2} \left[ g^{\mu\nu} - \frac{1}{M_W^2} \left( q^\mu q^\nu + i e F^{\mu\nu} \right) + \frac{2ie F^{\mu\nu}}{(q^2 - M_W^2)^2} \right]. $$

(29)

In the above equation $M_W$ is the $W$-boson mass and $\xi$ is the gauge-parameter. In the unitary gauge $\xi = 0$ and the last term on the right-hand side of the above equation drops out. Moreover in our work we assume $q^2 \ll M_W^2$ and keep terms up to $1/M_W^4$ in the $W$ propagator. The propagator form in the unitary gauge and in low momentum limit then looks like:

$$ W^{\mu\nu}(q) = \frac{g^{\mu\nu}}{M_W^2} \left( 1 + \frac{q^2}{M_W^2} \right) - \frac{q^\mu q^\nu}{M_W^4} + \frac{3ie}{2M_W^4} F^{\mu\nu}. $$

(30)

### 3.1 The $W$-exchange diagram

The $W$-exchange contribution to the neutrino self-energy is given in Eq. (11). In this subsection we will calculate the contribution to the neutrino self-energy coming from the magnetized plasma of electrons and positrons. With the propagator of the $W$ boson as given in Eq. (30) and the medium contribution from the lepton propagator in Eq. (10) we can write Eq. (11) in a magnetized medium. This form of the self-energy can be broken up into three parts as:

$$ -i\Sigma_1^W(k) = \left( \frac{g}{\sqrt{2}} \right)^2 \frac{1}{M_W^2} \int \frac{d^4p}{(2\pi)^4} R_{\gamma \mu} S_{\ell}^\beta(p) \gamma_\nu L g^{\mu\nu} \left( 1 + \frac{q^2}{M_W^2} \right), $$

(31)

$$ -i\Sigma_2^W(k) = -\left( \frac{g}{\sqrt{2}} \right)^2 \frac{1}{M_W^4} \int \frac{d^4p}{(2\pi)^4} R_{\gamma \mu} S_{\ell}^\beta(p) \gamma_\nu L q^\mu q^\nu, $$

(32)

$$ -i\Sigma_3^W(k) = \frac{3ie}{2M_W^4} F^{\mu\nu} \left( \frac{g}{\sqrt{2}} \right)^2 \int \frac{d^4p}{(2\pi)^4} R_{\gamma \mu} S_{\ell}^\beta(p) \gamma_\nu L, $$

(33)

in the above expressions $q = k - p$ as shown in Fig. 1. The above equations contain the contribution to the neutrino self-energy coming from the electrons and positrons in thermal equilibrium. In this article we do not consider the contribution to the neutrino self-energy in vacuum with a magnetic field.

After integrating out the 3-momenta in the loop using the integration results as presented in appendix A the form of Eq. (31) becomes:

$$ \Sigma_1^W(k) = \frac{g^2}{M_W^2} \int_{-\infty}^{\infty} \frac{dp_0}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\psi(p_0)} \tan z \eta_F(p_0) 
\left[ \mathcal{J}_0 \mathcal{R}_0 a_{30}(s)a_{20}(s')a_{10}(s') - \frac{\mathcal{R}_0 a_{30}(s)}{M_W^2} \left\{ a_{12}(s')a_{20}(s') + a_{22}(s')a_{10}(s') \right\} 
- \frac{\mathcal{R}_0 a_{32}(s)a_{20}(s')a_{10}(s')}{M_W^2 \sin z \cos z} \left\{ k_1 a_{12}(s')a_{20}(s') + k_2 a_{22}(s')a_{10}(s') \right\} 
- \frac{2k_3}{M_W^2} \left\{ \cot z \gamma + i \gamma \right\} a_{32}(s)a_{20}(s')a_{10}(s') \right] L. $$

(34)
The above expression has $a_{im}$s which are not tensor components but Gaussian integrals defined as:

$$a_{im}(s) \equiv \int_{-\infty}^{\infty} dp_i e^{-isp_i^2} p_i^m ,$$  

(35)

where $i = 1, 2, 3$ and $m = 0, 2$ and $s' = s \tan \frac{z}{2}$. In the above equation $p_i^m$ is the $m$th power of the $i$th component of $p$. The properties of this integral are briefly discussed in appendix A. In the Eq. (34),

$$J_0 = 1 + \frac{p_0^2 - 2\omega \varepsilon p_0}{M_W^2} ,$$  

(36)

$$R_0 = p_0 (\cot z \dot{\phi} + i\dot{\theta}) .$$  

(37)

Using the form of the $a_{im}$s we can write Eq. (34) as:

$$\Sigma_1^W(k) = \pi^{3/2} e^{-3\pi i/4} \left( \frac{eB g^2}{M_W^2} \right) \int_{-\infty}^{\infty} \frac{dp_0}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{ds}{\sqrt{s}} e^{\psi(p_0)} \eta_F(p_0) W(p_0, s) ,$$  

(38)

where,

$$W(p_0, s) = J_0 p_0 (i\dot{\phi} + \dot{\theta} \cot z) + \frac{eB p_0}{M_W^2} \left( \frac{i\dot{\phi}}{\sin^2 z} - \dot{\theta} \cot z - i\dot{\theta} \right)$$  

$$+ \frac{k_3}{sM_W^2} (i\dot{\phi} \cot z - \dot{\phi}) + \frac{ieB}{M_W^2} \frac{(k_1 \gamma_1 + k_2 \gamma_2)}{\sin^2 z} \frac{p_0}{2sM_W^2} (i\dot{\phi} \cot z - \dot{\phi}) ,$$  

(39)

Factors $\cot z$ and $eB/\sin^2 z$ appearing in the above expression can be written as:

$$\cot z = i \sum_{n=0}^{\infty} \sum_{\lambda=\pm 1} e^{-is\mathcal{H}} ,$$  

(40)

$$\frac{eB}{\sin^2 z} = - \sum_{n=0}^{\infty} \sum_{\lambda=\pm 1} \mathcal{H} e^{-is\mathcal{H}} ,$$  

(41)

where,

$$\mathcal{H} = eB(2n + 1 - \lambda) .$$  

(42)

Here $n$ is a positive integer including zero and $\lambda$ can take only two values $\pm 1$ for $n \neq 0$ and for $n = 0$, $\lambda = 1$ only. The $n$ corresponds to the Landau level number occurring in the energy of the charged leptons in a magnetic field and $\lambda$ corresponds to the spin states of the leptons. Equations (40) and (41) are valid as long as $B \neq 0$ as they involve the landau quantization of the charged leptons in presence of a magnetic field. Using the above formulas for integrating the parameter $s$ in Eq. (38) and utilizing the fact that for leptons in the magnetized medium [16]:

$$p_0 \equiv E_{\ell,n} = \sqrt{m_{\ell}^2 + p_3^2 + \mathcal{H}} ,$$  

(43)

and shifting the integration variable from $p_0$ to $p_3$ we get,

$$\Sigma_1^W(k) = \frac{g^2}{4M_W^2} R \left[ \left( 1 + \frac{m_{\ell}^2}{M_W^2} \right) \left\{ (N_{\ell}^0 - \bar{N}_{\ell}^0) \dot{\theta} + (N_{\ell} - \bar{N}_{\ell}) \dot{\phi} \right\} \right.$$  

$$- \frac{eB}{M_W^2} \left\{ (N_{\ell} - \bar{N}_{\ell}) \dot{\theta} + (N_{\ell}^0 - \bar{N}_{\ell}^0) \dot{\phi} \right\} L$$  

$$- \frac{g^2 eB}{M_W^2} \int_{0}^{\infty} \frac{dp_3}{(2\pi)^2} \sum_{n=0}^{\infty} \sum_{\lambda=\pm 1} R \left[ \left( \omega_{\ell} E_{\ell,n} \delta_{\lambda,1}^{n,0} + \frac{k_3 p_3^2}{2} \right) \dot{\phi} + \frac{\mathcal{H}}{2E_{\ell,n}} \dot{\theta} \right] L(f_{\ell,n} + \bar{f}_{\ell,n}) ,$$  

(44)
where in the above expression,

\[ f_{\ell, n} = \frac{1}{e^{\beta(E_{\ell, n} - \mu_\ell)} + 1}, \quad \tilde{f}_{\ell, n} = \frac{1}{e^{\beta(E_{\ell, n} + \mu_\ell)} + 1}, \]

\[ N_\ell = \frac{e^B}{2\pi^2} \sum_{n=0}^{\infty} \sum_{\lambda=\pm 1} \int_0^\infty dp_3 f_{\ell, n}, \quad \tilde{N}_\ell = \frac{e^B}{2\pi^2} \sum_{n=0}^{\infty} \sum_{\lambda=\pm 1} \int_0^\infty dp_3 \tilde{f}_{\ell, n}, \]

and \( N_\ell^0 \) and \( \tilde{N}_\ell^0 \) corresponds to \( N_\ell \) and \( \tilde{N}_\ell \) with \( E_{\ell, n} \) in the distribution functions replaced by \( E_{\ell, 0} \), that is \( N_\ell^0 \) and \( \tilde{N}_\ell^0 \) are the particle and anti-particle number densities in the lowest Landau level. The symbol \( \delta_{\lambda, 1}^n = 1 \) only when \( n = 0 \) and \( \lambda = 1 \) and zero in other cases.

Proceeding in exactly the same way as done in the previous analysis the form of \( \Sigma_2^W(k) \) and \( \Sigma_3^W(k) \) are given as:

\[
\Sigma_2^W(k) = \frac{g^2}{8M_W^4} R \left[ (2k_3^2 - m_\ell^2)\hat{p} + eB\hat{p} \right] (N_\ell^0 - \tilde{N}_\ell^0) + (2\omega_{\ell, \hat{k}} + m_\ell^2)\hat{p} + eB\hat{p} \right] (N_\ell - \tilde{N}_\ell) L
\]

\[ + \frac{g^2eB}{2M_W^4} \int_0^\infty (2\pi)^2 \sum_{n=0}^{\infty} \sum_{\lambda=\pm 1} R \left[ \frac{\omega_{\ell, m}^2}{E_{\ell, n}} \delta_{\lambda, 1}^n \hat{p} - \frac{k_3m_\ell^2}{E_{\ell, n}} \delta_{\lambda, 1}^n \hat{p} - \frac{m_\ell^2}{E_{\ell, n}} \right] L(f_{\ell, n} + \tilde{f}_{\ell, n}) \]

and,

\[
\Sigma_3^W(k) = \frac{3g^2eB}{8M_W^4} R \left[ (N_\ell^0 - \tilde{N}_\ell^0)\hat{p} + (N_\ell - \tilde{N}_\ell)\hat{p} \right] L.
\]

### 3.2 The tadpole diagram

The lepton propagator contribution to the neutrino self-energy, up to one loop, is given by Eq. (12). Using the lepton propagator given by Eq. (27), which corresponds to the magnetized plasma contribution, and the vacuum Z boson propagator with zero momenta, the neutrino self-energy is:

\[
\Sigma^T(k) = \left( \frac{g}{2\cos\theta_W M_Z} \right)^2 R \gamma^\nu \int \frac{d^4p}{(2\pi)^4} \int_{-\infty}^{\infty} ds \ Tr \left[ \gamma_\nu (c_\nu + c_A\gamma_5) G(p, s) \right] e^{\phi(p, s)} \eta_F(p_0).
\]

In this case also we only write that part of the neutrino self-energy which arises from the electrons and positrons in the plasma and have not written the vacuum contribution to the self-energy.

Using Eq. (23) the trace is given by,

\[
Tr \left[ \gamma_\nu (c_\nu + c_A\gamma_5) G(p, s) \right] = 4 \sec^2 z (c_\nu A_\nu - iA_B) \]

where the 4-vectors \( A_\nu \) and \( B_\nu \) are as given in Eq. (24) and Eq. (25). Using the above equation the tadpole contribution to the self-energy comes out as:

\[
\Sigma^T(k) = 4 \left( \frac{g}{2\cos\theta_W M_Z} \right)^2 R \int \frac{d^4p}{(2\pi)^4} \int_{-\infty}^{\infty} ds \ e^{\phi(p, s)} \eta_F(p_0) \sec^2 z (c_\nu A - iA_B) \]

Doing the \( p_1, p_2 \) and \( p_3 \) integrals, by using the results in appendix [A] we obtain:

\[
\Sigma^T(k) = eB \pi^{3/2} e^{-3\pi i/4} \left( \frac{g}{\cos \theta_W M_Z} \right)^2 R \int_{-\infty}^{\infty} p_0 dp_0 \int_{-\infty}^{\infty} ds \sqrt{s} e^{\phi(p_0)} \eta_F(p_0) \times \left[ \hat{p} c_\nu \cot z - i\hat{p} c_A \right].
\]
Using Eq. (40) and doing some algebra the above equation can be written as:

\[
\Sigma^T(k) = eB\pi^{3/2}e^{-3\pi i/4} \left( \frac{g}{\cos \theta_W M_Z} \right)^2 \int_0^\infty \frac{p_0 dp_0}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{ds}{\sqrt{s}} e^{i\psi(p_0)}
R \left( ic_V \hat{\mu} \sum_{\lambda=\pm 1} e^{-i\mathcal{H}^\lambda} - ic_A \hat{\mathcal{B}} \right) L(f_{\ell,n} - \bar{f}_{\ell,n}),
\]

(53)

where \( \mathcal{H} \) is specified in Eq. (42). Now doing the \( s \) integral and re-introducing \( p_3 \) as the integration variable through Eq. (43) we obtain:

\[
\Sigma^T(k) = 2eB\pi^{2} \left( \frac{g}{\cos \theta_W M_Z^2} \right)^2 \int_0^{\infty} \frac{dp_3}{(2\pi)^4} \sum_{n=0}^{\infty} \sum_{\lambda=\pm 1} R \left( c_V \hat{\mu} - c_A \delta^n_\lambda \hat{\mathcal{B}} \right) L(f_{\ell,n} - \bar{f}_{\ell,n})
= \left( \frac{g^2}{4 \cos^2 \theta_W M_Z^2} \right) R \left[ c_V \hat{\mu} (\bar{N}_\ell - \bar{N}_\ell) - c_A \hat{\mathcal{B}} (N^0_\ell - \bar{N}^0_\ell) \right] L,
\]

(54)

where the symbols used in the above equation are explained in the last subsection.

### 3.3 The Z-exchange diagram

When the energy of the neutrinos is much greater than their chemical potentials, the result of the Z exchange process is known [17]:

\[
\Sigma^Z(k) = \left( \frac{g}{2 \cos \theta_W M_Z^2} \right)^2 R \left\{ \omega_\ell (N_{\nu_\ell} - \bar{N}_{\nu_\ell}) + \frac{2}{3} (\omega^B_\ell N_{\nu_\ell} + \omega^B_\ell \bar{N}_{\nu_\ell}) \right\} \hat{\mathcal{B}}
+ \left( \frac{g}{2 \cos \theta_W M_Z^2} \right)^2 \left\{ (N_{\nu_\ell} - \bar{N}_{\nu_\ell}) - \frac{8 \omega_\ell}{3 M_Z^2} (\omega^B_\ell N_{\nu_\ell} + \omega^B_\ell \bar{N}_{\nu_\ell}) \right\} \hat{\mu} L,
\]

(55)

where,

\[
N_{\nu_\ell} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\beta(\bar{\omega}^B_\ell - \mu_{\nu_\ell})} + 1}, \quad \bar{N}_{\nu_\ell} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\beta(\omega^B_\ell + \mu_{\nu_\ell})} + 1},
\]

(56)

are the number densities of the neutrinos and antineutrinos. Here \( \omega^B_\ell \) is the energy of the background neutrinos and \( \langle \omega^B_\ell \rangle \) (\( \langle \bar{\omega}^B_\ell \rangle \)) is the average neutrino (anti-neutrino) energy per unit volume per neutrino (anti-neutrino).

### 4 The dispersion relation

We can write the forms of \( b \) and \( c \) from the discussions on the last section and from Eqs. (8), (9) and (10). The forms of \( b \) and \( c \) are:

\[
b = \frac{g^2 (N_{\ell} - \bar{N}_{\ell})}{4M_W^2} \left( 1 + c_V + \frac{3m_\ell^2}{2M_W^2} \right) + \frac{g^2 eB}{4M_W^4} (N^0_\ell - \bar{N}^0_\ell)
+ \frac{g^2}{4M_W^2} \left( (N_{\nu_\ell} - \bar{N}_{\nu_\ell}) - \frac{8 \omega_\ell}{3 M_Z^2} (\omega^B_\ell N_{\nu_\ell} + \omega^B_\ell \bar{N}_{\nu_\ell}) \right)
- \frac{eB\pi^2}{M_W^4} \int_0^\infty \frac{dp_3}{(2\pi)^2} \sum_{n=0}^{\infty} \sum_{\lambda=\pm 1} \left[ \frac{k_3}{p_3 + m_\ell^2} \right] \delta^n_\lambda \left. \left( f_{\ell,n} + \bar{f}_{\ell,n} \right) \right|_{n=0}.
\]

(57)
\[ c = \frac{g^2(N^0_\ell - \bar{N}^0_\ell)}{4M_W^2} \left( 1 - c_A + \frac{m_\ell^2}{2M_W^2} \right) + \frac{g^2eB}{4M_W^4}(N_\ell - \bar{N}_\ell) \]

\[ - \frac{eBg^2}{M_W^4} \int_0^\infty \frac{dp_3}{(2\pi)^2} \sum_{n=0}^\infty \sum_{\lambda=\pm 1} \left[ \frac{\omega_\ell}{E_{\ell,n}} \left( E_{\ell,n} \right) \delta_{\lambda,1}^n \right] f_\ell,n + \bar{f}_\ell,n. \]  

(58) 

In our calculation we see that in general we cannot say a-priori that the magnitude of \( a_\parallel \) or \( a_\perp \) is smaller than the other coefficients in the self-energy expression. Consequently we specify the forms of \( a_\parallel \) and \( a_\perp \):

\[ a_\perp = -\frac{g^2eB}{M_W^4} \int_0^\infty \frac{dp_3}{(2\pi)^2} \sum_{n=0}^\infty \sum_{\lambda=\pm 1} \left( \frac{\mathcal{H}}{2E_{\ell,n}} + \frac{m_\ell^2}{E_{\ell,n}} \right) f_\ell,n + \bar{f}_\ell,n \]

\[ + \frac{g^2}{4M_W^4} \left[ k_3(N^0_\ell - \bar{N}^0_\ell) + \omega_\ell(N_\ell - \bar{N}_\ell + N_{\nu_\ell} - \bar{N}_{\nu_\ell}) + \frac{2}{3} \langle (\omega_\ell^B)N_{\nu_\ell} + \langle \bar{\omega}_\ell^B \rangle \bar{N}_{\nu_\ell} \rangle \right]. \]  

(59) 

\[ a_\parallel = -\frac{g^2eB}{M_W^4} \int_0^\infty \frac{dp_3}{(2\pi)^2} \sum_{n=0}^\infty \sum_{\lambda=\pm 1} \frac{m_\ell^2}{E_{\ell,n}} f_\ell,n + \bar{f}_\ell,n \]

\[ + \frac{g^2}{4M_W^4} \left[ k_3(N^0_\ell - \bar{N}^0_\ell) + \omega_\ell(N_\ell - \bar{N}_\ell + N_{\nu_\ell} - \bar{N}_{\nu_\ell}) + \frac{2}{3} \langle (\omega_\ell^B)N_{\nu_\ell} + \langle \bar{\omega}_\ell^B \rangle \bar{N}_{\nu_\ell} \rangle \right]. \]  

(60) 

The above forms of the coefficients determine the dispersion relation of the neutrino in a magnetized plasma containing charged leptons. In writing the above equations we have used \( M_W = M_Z \cos \theta_W \). Equations (57), (58), (59) and (60) are the central results of this article. The expressions of \( a_\parallel \) and \( a_\perp \) only has the leptonic contributions as we did not take into account the Landau quantization of the charged gauge fields. If we used the full \( W \) propagator instead of the one in Eq. (30), containing only the linear order correction \( eB/M_W^4 \) to the \( W \) propagator, then \( a_\parallel \) and \( a_\perp \) would get contributions from the gauge sector also. In the absence of the Landau quantization of the charged gauge fields it is noticed that \( a_\parallel - a_\perp \) diminishes as the magnetic field increases and the temperature of the system remains constant. The coefficient \( b \) closely resembles the corresponding coefficient in [3] in the four-Fermi limit. Both \( b \) and \( c \) match exactly with the corresponding forms in the four-Fermi limit with the results obtained in [15]. But the expressions of \( a_\perp \) and \( a_\parallel \) in Equations (59) and (60) has not been calculated before, in general. Previously \( a_\parallel - a_\perp \) has only been calculated in the CP symmetric plasma but it must be noted that \( a_\parallel - a_\perp \) exists in the CP asymmetric plasma also.

Of special interest is the charge symmetric plasma, which perhaps existed in the early universe. In this situation where the chemical potentials of the particles are negligible, the coefficients become:

\[ b = -\frac{4g^2\omega_\ell}{3M_W^2M_Z^2} \langle (\omega_\ell^B)N_{\nu_\ell} \rangle \]

\[ - \frac{2eBg^2}{M_W^4} \int_0^\infty \frac{dp_3}{(2\pi)^2} \sum_{n=0}^\infty \sum_{\lambda=\pm 1} \left[ \frac{k_3}{E_{\ell,n}} \left( \frac{p_3^2 + m_\ell^2}{2} \right) \delta_{\lambda,1}^n + \omega_\ell E_{\ell,n} \right] f_\ell,n. \]  

(61) 

\[ c = -\frac{2eBg^2}{M_W^4} \int_0^\infty \frac{dp_3}{(2\pi)^2} \sum_{n=0}^\infty \sum_{\lambda=\pm 1} \left[ \omega_\ell \left( E_{\ell,n} - \frac{m_\ell^2}{E_{\ell,n}} \right) \delta_{\lambda,1}^n + \frac{k_3p_3^2}{E_{\ell,n}} \right] f_\ell,n. \]  

(62) 

\[ a_\perp = -\frac{2g^2eB}{M_W^4} \int_0^\infty \frac{dp_3}{(2\pi)^2} \sum_{n=0}^\infty \sum_{\lambda=\pm 1} \left( \frac{\mathcal{H}}{2E_{\ell,n}} + \frac{m_\ell^2}{E_{\ell,n}} \right) f_\ell,n + \frac{g^2}{3M_W^4} \langle (\omega_\ell^B)N_{\nu_\ell} \rangle, \]  

(63) 

\[ a_\parallel = -\frac{2g^2eB}{M_W^4} \int_0^\infty \frac{dp_3}{(2\pi)^2} \sum_{n=0}^\infty \sum_{\lambda=\pm 1} \frac{m_\ell^2}{E_{\ell,n}} f_\ell,n + \frac{g^2}{3M_W^4} \langle (\omega_\ell^B)N_{\nu_\ell} \rangle. \]  

(64)
From the above expressions we immediately notice that all the contributions in the charge symmetric case are proportional to $M_W^4$ or $G_F^2$. From the above equations if we neglect all the terms explicitly containing the mass of the charged leptons then we reproduce closely the form of the neutrino-self-energy calculated in [15] in a CP symmetric plasma.

In Eq. (61) the most general form of the neutrino dispersion relation was written. To order of $g^2$ the dispersion relation becomes,

$$\omega_{\ell} = \left| |\mathbf{k}| - c \mathbf{k} \cdot \mathbf{b} + 2(a_\parallel - a_\perp)k_\perp^2 \right|^{1/2} + b,$$

where we have taken the positive sign of the square root in Eq. (61). The above dispersion relation can be simplified by binomially expanding the square root and neglecting terms of order more than $g^2$. The expansion gives,

$$\omega_{\ell} = |\mathbf{k}| - c \mathbf{k} \cdot \mathbf{b} + (a_\parallel - a_\perp)k_\parallel^2 + b,$$

where $k^3 = k_\parallel = |\mathbf{k}| \cos \theta$. The above equation implies that in presence of a magnetized medium the effective-potential acting on the neutrinos is of the form,

$$V_{\text{eff}} = b - c \cos \theta + (a_\parallel - a_\perp)|\mathbf{k}| \sin^2 \theta. \tag{67}$$

From the expressions of $a_{\parallel}$ and $a_\perp$ in the CP symmetric case we see that in the lowest Landau level $a_\parallel - a_\perp$ is zero and in that case the effective potential is independent of $a$. With the form of the effective potential in Eq. (67) the problem of neutrino oscillations in the CP symmetric magnetized plasma in the early universe can be tackled.

The calculation of the coefficients $c$, $(a_\parallel - a_\perp)$, which are directly related to the presence of a non-zero magnetic field, do not allow us to take the $B \to 0$ limit because of the non-perturbative nature of the Landau quantization of the charged fermions. Interestingly it turns out that in the zero external magnetic field limit the form of $b$ exactly matches the form in an unmagnetized plasma. We can write Eq. (61) as:

$$b = -\frac{4g^2\omega_{\ell}}{3M_W^2M_Z^2}(\omega_{\ell}^B)N_{\nu_{\ell}} - \frac{eB^2g^2\omega_{\ell}}{M_W^4} \int_0^\infty \frac{dp_3}{2\pi^2} E_{\ell,n} f_{\ell,n}$$

$$- \frac{2eB^2g^2}{M_W^4} \int_0^\infty \frac{dp_3}{2\pi^2} \sum_{n=0}^{\infty} \sum_{\lambda=\pm 1} \left[ \frac{k_3}{E_{\ell,n}} \left( p_3^2 + \frac{m_\ell^2}{2} \right) \delta_{n,0}^{\lambda,1} \right] f_{\ell,n}. \tag{68}$$

Now if we substitute the lepton energy in the loop integral by the thermal average of the lepton energy and use the relation $\tilde{N}_{\ell} = \frac{eB}{2\pi^2} \sum_{n=0}^{\infty} \sum_{\lambda=\pm 1} \int_0^\infty dp_3 f_{\ell,n}$, the above equation becomes,

$$b = -\frac{16\sqrt{2}G_F\omega_{\ell}}{3M_Z^2}(\omega_{\ell}^B)N_{\nu_{\ell}} - \frac{16\sqrt{2}G_F\omega_{\ell}}{M_W^4}(E_{\ell})N_{\ell}$$

$$- \frac{2eB^2g^2}{M_W^4} \int_0^\infty \frac{dp_3}{2\pi^2} \sum_{n=0}^{\infty} \sum_{\lambda=\pm 1} \left[ \frac{k_3}{E_{\ell,n}} \left( p_3^2 + \frac{m_\ell^2}{2} \right) \delta_{n,0}^{\lambda,1} \right] f_{\ell,n}, \tag{69}$$

where we have utilized $\sqrt{2}G_F = \frac{g^2}{4M_W^2}$ and replaced $E_{\ell,n}$ by $4/3(E_{\ell})$ because there are three flavours and two spin states for the electron and the positron. In the $B \to 0$ limit the last term on the right hand side of the above equation vanishes and we have,

$$b_{B \to 0} = -\frac{16\sqrt{2}G_F\omega_{\ell}}{3M_Z^2} \left( (E_{\nu_{\ell}}^B)N_{\nu_{\ell}} - (E_{\ell})N_{\ell} \right). \tag{70}$$

The Eq. (70) resembles the results found in [11]. The signs are different as in [11] the authors used the opposite sign for $b$. 

11
5 Conclusion

In the present work we calculated the self-energy of the neutrino in a medium seeded with a uniform classical magnetic field. The calculations were carried out in the unitary gauge where the unphysical Higgs contribution does not appear. The background is supposed to be comprised of the charged leptons and neutrinos equilibrated at the same temperature. The magnitude of the magnetic field is such that only linear contributions of the field appear in the charged $W^\pm$ boson propagators but all orders of the field are present in the charged lepton propagators. Equations (57), (58), (59) and (60) contain the most general results of the self-energy calculation in the absence of Landau quantization of the charged gauge fields. Earlier calculations have been done in this regard but the authors have used separate assumptions to calculate the expression of the neutrino self-energy in various cases. The expression of the neutrino self-energy in various specific cases can be calculated from the general results given in the previous section. The correspondence of our result with results from [15] confirms the gauge invariance of the calculations as in the last reference the authors used a different gauge to calculate the neutrino-self energy. We point out the special role played by the Landau quantization of the charged leptons in the medium in modifying the neutrino self-energy through the coefficient $a_\parallel - a_\perp$. The zero field limit of the coefficient $b$ matches perfectly with the form predicted in presence of an unmagnetized medium.

The main application of our results are expected to be in the early universe, specifically before the period of neutrino decoupling which was expected to happen around an energy scale of 1 MeV when the age of the universe was about 1 second. After that period the neutrinos decoupled from the thermal equilibrium and its temperature started to red-shift with time. Other possible application of our calculation can be in GRB fireballs [18] where neutrinos propagate in the presence of an electron, positron plasma with a very small baryon contamination.

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Appendix

A Gaussian Integral results

The integral in Eq. (35) is:

$$a_{im}(s) = \int_{-\infty}^{\infty} dp_i e^{-isp_i^2} p_i^m,$$

where $i = 1, 2, 3$ and $m = 0, 2$. In the actual calculations the $p_\perp$ components, which stands for $(p_1, p_2)$, and the $p_3$ component are integrated in different manner as for the perpendicular components we have

$$a_{im}(s') = \int_{-\infty}^{\infty} dp_i e^{-is'p_i^2} p_i^m,$$

where $s' = s \tan z$. In practice $a_{1m}$ and $a_{2m}$ are different from $a_{3m}$. In general we will have,

$$a_{10}(s') = a_{20}(s') = \sqrt{\frac{\pi}{s}} \exp[-\frac{i\pi}{4}] = \sqrt{\frac{\pi}{s\tan z}} \exp[-\frac{i\pi}{4}],$$

$$a_{12}(s') = a_{22}(s') = \sqrt{\frac{\pi}{2s\tan z}} \exp[-\frac{3\pi i}{4}] = \frac{\sqrt{\pi}}{2} (\frac{z}{s\tan z})^{3/2} \exp[-\frac{3\pi i}{4}],$$
and,
\[
a_{30}(s) = \sqrt{\frac{\pi}{s}} \exp\left[-\frac{i\pi}{4}\right], \quad a_{32}(s) = \sqrt{\frac{\pi}{2s^{3/2}}} \exp\left[-\frac{3\pi i}{4}\right].
\]  
(75)

From the above results it can be shown,
\[
a_{32}(s) = -\frac{i}{2s} a_{30}(s), \quad a_{12}(s') = -\frac{iz}{2s \tan z} a_{10}, \quad a_{22}(s') = -\frac{iz}{2s \tan z} a_{20}.
\]  
(76)

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