STANDARD TRIPLES FOR ALGEBRAIC LINEARIZATIONS OF MATRIX POLYNOMIALS

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Abstract. Standard triples $X, C_1 - C_0, Y$ of a nonsingular matrix polynomials $P(z) \in \mathbb{C}^{r \times r}$ have the property $X(zC_1 - C_0)^{-1}Y = P^{-1}(z)$ for $z \notin \Lambda(P(z))$. They can be used in constructing algebraic linearizations; for example, for $h(z) = za(z)b(z) + c \in \mathbb{C}^{r \times r}$ from linearizations for $a(z)$ and $b(z)$. We tabulate standard triples for orthogonal polynomial bases, the monomial basis, and Newton interpolational bases; for the Bernstein basis; for Lagrange interpolational bases; and for Hermite interpolational bases. We account for the possibility of a transposed linearization, a flipped linearization, and a transposed-and-flipped linearization. We give proofs for the less familiar bases.

Key words. Standard triple, nonsingular matrix polynomial, polynomial bases, companion matrix, colleague matrix, comrade matrix, algebraic linearization, linearization.

AMS subject classifications. 65F15, 15A22, 65D05

1. Introduction. The paper is organized as follows. We tabulate the standard triples in Sections 2.1, 2.2, 2.3, and 2.4. We give proofs in Section 3. In the remainder of this first section, we establish notation and lemmas about transposition and about what we call flipping; transposition and flipping give altogether four common variations of companion matrix pencils.

For motivation of the study of linearizations of matrix polynomials consult the seminal book [12] or the masterful exposition [16]; some recent papers of interest include [4]. Linearizations using different polynomials bases were first systematically studied in [1]. An algebraic linearization is defined in [6]

1.1. Notation. We write a matrix polynomial $P(z) \in \mathbb{C}^{r \times r}$ as

(1.1) $P(z) = \sum_{k=0}^{n} p_k \phi_k(z)$

for some scalar polynomials $\{\phi_k(z)\}_{k=0}^{n}$ forming a basis for polynomials of degree at most $n$. (The phrase “of degree at most $n$” is sometimes shortened to “of grade $n$”.) The coefficient matrices $p_k \in \mathbb{C}^{r \times r}$ are assumed square. We mostly consider only nonsingular matrix polynomials, that is those with $\det(P(z))$ not identically zero. We say the matrix pencil $L(z) = zC_1 - C_0 \in \mathbb{C}^{N \times N}$ (usually $N = nr$ but not always) is a linearization of $P(z)$ if $\det(P(z)) = \det(L(z)) = \det(zC_1 - C_0)$. The polynomial eigenvalues of $P$ are thus computable from the generalized eigenvalues of $L$.

A standard triple for $P$ is a matrix $X \in \mathbb{C}^{r \times N}$, the pencil $L(z)$, and a matrix $Y \in \mathbb{C}^{N \times r}$ with

(1.2) $P^{-1}(z) = X(zC_1 - C_0)^{-1}Y$

for $z \notin \Lambda(P)$ (the set of polynomial eigenvalues of $P$).

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Lemma 1.1. If \( B_1 = H^{-1}C_1H \) and \( B_0 = H^{-1}C_0H \) so that the pencil \( zB_1 - B_0 \) has the same generalized eigenvalues as \( zC_1 - C_0 \), then another standard triple for \( P(z) \) is \( X, zB_1 - B_0, Y \) where \( X = XH \) and \( Y = H^{-1}Y \).

Proof.

\[
P^{-1}(z) = X(zC_1 - C_0)^{-1}Y
\]

\[
P^{-1}(z) = XHH^{-1}(zC_1 - C_0)^{-1}HH^{-1}Y
\]

\[
= (XH)(H^{-1}(zC_1 - C_0)H)^{-1}H^{-1}Y.
\]

Remark 1.2. The pencils \( zC_1H - C_0H \) and \( zH^{-1}C_1 - H^{-1}C_0 \) also have the same eigenvalues, but are not often used, principally because \( H^{-1}C_1 \) and \( C_1H \) are not “identity-like”.

Lemma 1.3. If \( X, L(z), Y \) is a standard triple for \( P(z) \), then \( Y^T, L^T(z) = (zC_1^T - C_0^T), X^T \) is a standard triple for \( P^T(z) \). \( P^T(z) \) has the same polynomial eigenvalues as \( P(z) \).

Proof. Immediate.

Lemma 1.4 (Flipping). Put \( J = \) the \( N \times N \) “anti-identity”, \( J_{i,j} = 0 \) unless \( i + j = N + 1 \) when \( J_{i,N+1-i} = 1 \). Then \( J^2 = I \) and the “flipped” linearization \( L_R(z) = J(zC_1 - C_0)J \) has the standard triple \( X_R = XJ \) and \( Y_R = JY \). The paper [17] calls this matrix “R”.

Proof. Immediate.

Remark 1.5. Flipping switches both the order of the equations and the order of the variables. It obviously does not change eigenvalues. Flipping, transposition, and flipping-with-transposition give four equivalent linearizations [19].

1.2. Companion matrices and linearizations. In the special case \( r = 1 \), a linearization is usually called a “companion pencil”; in the frequent monic case \( C_1 = I \), the generalized eigenproblem becomes a standard eigenproblem. For bases other than the monomial, the unfortunate nomenclature “colleague matrix” or “comrade matrix” is also used. This nomenclature hinders citation search and we prefer “generalized companion”, if a distinction is needed.

Construction of a linearization from a companion pencil is a simple matter of the Kronecker (tensor) product: given \( C_1, C_0 \in \mathbb{C}^{n \times n} \), take \( \tilde{C}_1 = C_1 \otimes I \), and then replace each block \( p_kI_r \) with the corresponding matrix coefficient \( \tilde{p}_k \in \mathbb{C}^{r \times r} \) (the first \( \tilde{p}_k \), in \( p_kI_r \), is the symbolic coefficient from \( p(z) = \sum_{k=0}^n p_k\phi_k(z) \); the matrix coefficient \( p_k \in \mathbb{C}^{r \times r} \) is from \( P(z) = \sum_{k=0}^n p_k\phi_k(z) \). This will be clearer by example.

2. The standard triples. In this section, we tabulate the standard triples for four classes of linearizations. We do so by examples of companion pencils, leaving the reader to do the necessary tensor products to produce linearizations. This saves some space in the presentation. In contrast, in section 3 where we give proofs, we use the linearization notation, establishing generality.

2.1. Bases with three-term recurrence relations. The monomial basis, the shifted monomial basis, the Taylor basis, the Newton interpolational bases, and many common orthogonal polynomial bases all have three-term recurrence relations that can be written

\[
z\phi_n(z) = \alpha_n\phi_{n+1}(z) + \beta_n\phi_n(z) + \gamma_n\phi_{n-1}(z).
\]

We give a short table below, and refer the reader to the DLMF (dlmf.nist.gov) for more. See also [11].
| $\phi_n(z)$            | Name         | $\alpha_n$ | $\beta_n$ | $\gamma_n$ | $\phi_0$ | $\phi_1$ |
|-----------------------|--------------|------------|------------|------------|----------|----------|
| $z^n$                 | monomial     | 1          | 0          | 0          | 1 $z$    |          |
| $(z-a)^n$             | shifted monomial | 1          | $a$        | 0          | 1 $z-a$  |          |
| $(z-a)^n/n!$          | Taylor       | $n+1$      | $a$        | 0          | 1 $z-a$  |          |
| $\prod_{k=0}^{n-1}(z-\tau_k)$ | Newton interpolational | 1 | $\tau_n$   | 0          | 1 $z-\tau_0$ |        |
| $T_n(z) = \cos\left(n \cos^{-1}(z)\right)$ | Chebyshev | $\frac{n}{2}$ | 0 | $\frac{1}{2}$ | 1 | $z$ |        |
| $P_n(z)$              | Legendre     | $(n+1)/(2n+1)$ | 0 | $n/(2n+1)$ | 1 | $z$ |        |

| Table 1 |

A short list of three-term recurrence relations for some important polynomial bases. For a more comprehensive list, see *The Digital Library of Mathematical Functions*. These relations and others are coded in Walter Gautschi’s packages OPQ and SOPQ [11] and in the `MatrixPolynomialObject` implementation package in Maple (see [13, Chapter 27]).

For all such bases, we have the companion pencil

$$C_1 = \begin{bmatrix} p_5 \\ \alpha_4 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C_0 = \begin{bmatrix} -p_4 + \frac{\beta_4}{\alpha_4}p_5 & -p_3 + \frac{\gamma_4}{\alpha_4}p_5 & -p_2 & -p_1 & -p_0 \\ \alpha_3 & \beta_3 & \gamma_3 & \alpha_2 & \beta_2 \\ \beta_1 & \gamma_1 & \alpha_1 \\ \beta_0 & \gamma_0 \end{bmatrix}$$

and

$$X = [0 \ 0 \ 0 \ 0 \ 1]$$

$$Y = [1 \ 0 \ 0 \ 0 \ 0]^T$$

For instance, a flipped and transposed pencil of this class for the Chebyshev case is

$$L(z) = \begin{bmatrix} z & -\frac{1}{2} & p_0 \\ -1 & z & -\frac{1}{2} & p_1 \\ -\frac{1}{2} & z & -\frac{1}{2} & p_2 \\ -\frac{1}{2} & z & p_3 + p_5 \\ -\frac{1}{2} & 2zp_5 + p_4 \end{bmatrix}$$

has flipped and transposed $X = [0 \ 0 \ 0 \ 0 \ 1]$, $Y = [1 \ 0 \ 0 \ 0 \ 0]^T$. As another instance, a Newton

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1 Following Peter Lancaster’s dictum, namely that the $5 \times 5$ case almost always gives the idea.

2 For the matrix polynomial case, each $P_k$ would be transposed.
interpolational basis on the nodes \( \tau_0, \tau_1, \ldots, \tau_5 \) has a companion pencil

\[
\begin{pmatrix}
 p_5 \\
 1 \\
 1 \\
 1 \\
 \end{pmatrix}
\begin{pmatrix}
 \tau_0 \\
 \tau_1 \\
 \tau_2 \\
 \tau_3 \\
 \tau_4 \\
 \tau_5 \\
 \end{pmatrix}
\begin{pmatrix}
 -p_4 - \tau_4 p_5 \\
 -p_3 \\
 -p_2 \\
 -p_1 \\
 -p_0 \\
 1 \\
 \tau_3 \\
 1 \\
 1 \\
 1 \\
 1 \\
 \end{pmatrix}
\]

(2.12)

The corresponding linearization is

\[
\begin{pmatrix}
 p_5 \\
 I_r \\
 I_r \\
 I_r \\
 \end{pmatrix}
\begin{pmatrix}
 -p_4 - \tau_4 p_5 \\
 -p_3 \\
 -p_2 \\
 -p_1 \\
 -p_0 \\
 1 \\
 \tau_3 I_r \\
 \tau_2 I_r \\
 \tau_1 I_r \\
 \tau_0 I_r \\
 \end{pmatrix}
\]

(2.13)

2.2. The Bernstein basis. The set of polynomials \( \{B^k_n(z)\}_{k=0}^n \) is a set of \( n+1 \) polynomials each of exact degree \( n \) that forms a basis for polynomials of degree at most \( n \). They have many applications, for example in Computer Aided Geometric Design (CAGD), and many important properties including that of optimal condition number over all bases positive on \([0, 1]\). They do not satisfy a simple three term recurrence relation of the form discussed in section 2.1. See [10], [9], and [8] for more details of Bernstein bases.

A companion pencil for \( p_5(z) = \sum_{k=0}^5 p_k B^5_k(z) \) is

\[
\begin{pmatrix}
 -p_4 + \frac{1}{5} p_5 \\
 1 \\
 2 \\
 4 \\
 3 \\
 1 \\
 \end{pmatrix}
\begin{pmatrix}
 -p_0 \\
 -p_1 \\
 -p_2 \\
 -p_3 \\
 -p_4 \\
 -p_5 \\
 \end{pmatrix}
\]

(2.14)

\[
\begin{pmatrix}
 1 \\
 0 \\
 0 \\
 0 \\
 1 \\
 \end{pmatrix}
\begin{pmatrix}
 X \\
 Y \\
 \end{pmatrix}
= \begin{bmatrix}
 1 & 2 & 3 & 4 & 5 \\
 5 & 5 & 5 & 5 & 5 \\
 \end{bmatrix}
\]

(2.15)

We have \( p^{-1}(z) = X(zC_1 - C_0)^{-1}Y \) if \( p(z) \neq 0 \). We do not know who first established this pencil. One of the present authors implemented a version of this linearization in Maple (except for \( P^T(z) \), and reversed from the above form) in about 2004. This linearization has been independently rediscovered a few times, but the earliest publication seems to be [14]. For a review of Bernstein linearization, see [17]. We supply a proof in section 3. The standard triple is, we believe, new to this paper.
2.3. The Lagrange interpolational basis. There are by now several Lagrange basis pencils and linearizations. The use of barycentric forms means that Lagrange interpolation is efficient and numerically stable. For many sets of nodes (Chebyshev nodes on \([-1, 1] \), or roots of unity on the unit disk) the resulting interpolant is also well-conditioned, and can even be “better than optimal” [7], see also [5]. The linearization we use here is “too large” and has (harmless in our opinion) spurious roots at infinity; for alternative formulations see [20], [18]. Then pencil is \(zC_1 - C_0\) where

\[
C_1 = \begin{bmatrix}
0 & 1 \\
1 & 1 \\
\end{bmatrix}
\]

\[
C_0 = \begin{bmatrix}
0 & -\rho_0 & -\rho_1 & -\rho_2 & -\rho_3 & -\rho_4 \\
\beta_0 & \tau_0 & \tau_1 & \tau_2 & \tau_3 & \tau_4 \\
\beta_1 & \tau_0 & \tau_1 & \tau_2 & \tau_3 & \tau_4 \\
\beta_2 & \tau_0 & \tau_1 & \tau_2 & \tau_3 & \tau_4 \\
\beta_3 & \tau_0 & \tau_1 & \tau_2 & \tau_3 & \tau_4 \\
\beta_4 & \tau_0 & \tau_1 & \tau_2 & \tau_3 & \tau_4 \\
\end{bmatrix}
\]

Then \(\det(\tau_kC_1 - C_0) = \rho_k, 0 \leq k \leq 4\) and \(\deg(zC_1 - C_0) \leq 4\). Thus, \(p(z) = \det(zC_1 - C_0)\) interpolates the given data, assuming the \(\tau_k\) are distinct. Here the barycentric weights \(\beta_k\) are found by partial fraction expansion of \(\omega(z)^{-1}\) where

\[
\omega(z) = (z - \tau_0)(z - \tau_1)(z - \tau_2)(z - \tau_3)(z - \tau_4)
\]

is the node polynomial. Explicitly,

\[
\frac{1}{\omega(z)} = \sum_{k=0}^{5} \frac{\beta_k}{z - \tau_k}
\]

so

\[
\beta_k = \prod_{j=0}^{5} \frac{1}{(\tau_k - \tau_j)^{-1}}.
\]

The \(X\) and \(Y\) for the standard triple are

\[
X = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \end{bmatrix},
\]

\[
Y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T.
\]

Notice in this case that \(N = (n + 2)r\) while \(\deg p \leq n\), so there are at least \(2r\) eigenvalues at infinity. This can be inconvenient if \(r\) is at all large.

2.4. Hermite interpolational basis. The companion pencil of the previous section has been extended to Hermite interpolational bases, where some of the nodes have “flowed together,” collapsing to fewer distinct nodes. We suppose that at each node \(\tau_i\), there are now \(s_i \geq 1\) pieces of information known. The integer \(s_i\)
is called the confluency of the node. The known pieces of information are the local Taylor coefficients of the polynomial fitting the data:

\[(2.25)\]
\[\rho_{i,j} = \frac{f^{(j)}(\tau_i)}{j!}, \quad 0 \leq j \leq s_i - 1.\]

The companion pencil of the previous section changes to the following elegant form. The matrix \(C_1\) is unchanged,

\[(2.26)\]
\[C_1 = \begin{bmatrix}
0 & 1 & 1 \\
1 & 1 & \end{bmatrix},
\]
being \((d + 2) \times (d + 2)\) as before, although now

\[(2.27)\]
\[d = -1 + \sum_{i=0}^{n} s_i\]
is the grade of the resulting polynomial. The matrix \(C_0\) changes, picking up Jordan-like blocks for each distinct node. For instance, suppose we have two distinct nodes, \(\tau_0\) and \(\tau_1\). Suppose further that \(\tau_0\) has confluency \(s_0 = 3\) while \(\tau_1\) has confluency \(s_1 = 2\). This means that we know \(f(\tau_0), f'(\tau_0)/1!, f''(\tau_0)/2!, f(\tau_1)\) and \(f'(\tau_1)/1!\). Then,

\[(2.28)\]
\[C_0 = \begin{bmatrix}
0 & -f''(\tau_0)/2! & -f'(\tau_0)/1! & -f(\tau_0) & -f'(\tau_1)/1! & -f(\tau_1) \\
\beta_{02} & \tau_0 & -f''(\tau_0)/2! & -f'(\tau_0)/1! & -f(\tau_0) & -f'(\tau_1)/1! \\
\beta_{01} & -1 & \tau_0 & -f''(\tau_0)/2! & -f'(\tau_0)/1! & -f(\tau_0) \\
\beta_{11} & -1 & \tau_0 & \tau_1 & -f''(\tau_0)/2! & -f'(\tau_0)/1! \\
\beta_{10} & -1 & \tau_0 & \tau_1 & -1 & \tau_1
\end{bmatrix}
\]

Note the reverse ordering of the derivative values in this formulation. The barycentric weights \(\beta_{ij}\) again come from the partial fraction expansion of the reciprocal of the node polynomial

\[(2.29)\]
\[\omega(z) = \prod_{i=0}^{n} (z - \tau_i)^{s_i}.\]

That is,

\[(2.30)\]
\[\frac{1}{\omega(z)} = \sum_{i=0}^{n} \sum_{j=0}^{s_i-1} \frac{\beta_{ij}}{(z - \tau_i)^{j+1}}.\]

For the standard triple, take

\[(2.31)\]
\[Y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T\]
but for \(X\) take the coefficients of the expansion of the polynomial 1 in this Hermite interpolational basis:

\[(2.32)\]
\[
\begin{cases}
\rho_{ij} = 1 & \text{if } j = 0, \\
0 & \text{otherwise},
\end{cases}
\]
and sort them in order:

\[(2.33) \quad X = \begin{bmatrix} 0 & \rho_{0,s_0-1} & \rho_{0,s_0-2} & \cdots & \rho_{0,0} & \rho_{1,s_1-1} & \cdots & \rho_{n,0} \end{bmatrix} .\]

For the earlier instance (two nodes, of confluency 3 and 2, respectively,

\[(2.34) \quad X = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} .\]

Then

\[(2.35) \quad p^{-1}(z) = X(zC_1 - C_0)^{-1}Y .\]

**Remark 2.1.** We may re-order the nodes in any fashion we like, and each ordering generates its own companion pencil (both Hermite and Lagrange). We may also find a pencil where the confluent data is ordered \(p(\tau_i), \frac{p'(\tau_i)}{1!}, \frac{p''(\tau_i)}{2!}, \text{etc.},\) although we have not done so.

If there is just one node of confluency \(d + 1\), we recover the standard Frobenius companion form (plus two infinite roots):

\[(2.36) \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & \cdots & \cdots \\ 1 & 1 & \cdots & \cdots \\ \vdots & \vdots & \cdots & \cdots \end{bmatrix}, \quad \begin{bmatrix} 0 & -p_d & -p_{d-1} & \cdots & -p_1 & -p_0 \\ 1 & \tau_0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \tau_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \end{bmatrix}.
\]

Here \(p_k = \frac{p^{(k)}(\tau_0)}{k!}\) is the ordinary coefficient in the expansion \(p(z) = \sum_{k=0}^{d} p_k(z - \tau_0)^k\). The numerical stability of these Hermite interpolational companions has been studied briefly [15] but much remains unknown. We confine ourselves in this paper to the study of the standard triple.

Note that the Lagrange case \(X, X = [0 \ 1 \ \cdots \ 1]\), fits the Hermite pattern here also: the coefficients in the expansion of \(p(x) = 1\), namely \(\rho_i = 1\), appear in the vector. We will see why.

**Remark 2.2.** The modified linearizations of [20] also have standard triples that can be used for algebraic linearization, and arguably should be tabled here as well. They have the advantage of including fewer eigenvalues at infinity, or no spurious eigenvalues at infinity, which may lead to better algebraic linearizations. However, they are more involved, and we have less numerical experience with them. In particular we do not understand their dependence on the ordering of the nodes, and so we leave their analysis to a future study.

### 3. Proofs

We will use the Schur Complement, in the following form: assuming a matrix \(R\) is partitioned into

\[(3.37) \quad R = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \]

where \(A \in \mathbb{C}^{r \times r}, B \in \mathbb{C}^{r \times (N-r)}, C \in \mathbb{C}^{(N-r) \times r}\) and \(D \in \mathbb{C}^{(N-r) \times (N-r)}\) is assumed invertible, then

\[(3.38) \quad R = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & C \end{bmatrix} .\]
If further the Schur Complement $A - BD^{-1}C$ is invertible, then

$$R^{-1} = \begin{bmatrix}
  (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\
  -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1}
\end{bmatrix}$$

as can be verified by block multiplication of $R$ or by $R$. We will use $S$ for the Schur Complement $S = A - BD^{-1}C$. We will take $R = zC_1 - C_0$. We may already use this to establish for each of the four classes of linearizations that

$$\det R = \det(zC_1 - C_0) = \det(A - BD^{-1}C) \det D = \det P(z).$$

Notice that the coefficients of $P$ do not appear in the $D$ block (in any of our linearizations). Thus the Schur Complement carries all the information particular to $P(z)$. The computations verifying (3.40) are not obvious but in each case $D^{-1}$ plays an important role. We will see that generically $D^{-1}$ exists, except for isolated values of $z$, which we can safely ignore and recover later by continuity.

We take each case in turn.

**Theorem 3.1.** If $C_1 = \text{diag} \left[ \frac{1}{\alpha_{n-1}} p_n \ I_r \ I_r \ \cdots \ I_r \right]$ and

$$C_0 = \begin{bmatrix}
  \frac{\beta_{n-1}}{\alpha_{n-1}} p_n - p_{n-1} & \frac{\gamma_{n-1}}{\alpha_{n-1}} p_n - p_{n-2} & -p_{n-1} & \cdots & -p_0 \\
  \alpha_{n-2} I_r & \frac{\beta_{n-2}}{\alpha_{n-1}} I_r & \gamma_{n-2} I_r & \cdots & 0 \\
  \alpha_{n-3} I_r & \frac{\beta_{n-3}}{\alpha_{n-1}} I_r & \gamma_{n-3} I_r & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \gamma_1 I_r \\
  \alpha_0 I_r & \beta_0 I_r & & & \end{bmatrix}$$

and $X = [0 \ 0 \ \cdots \ 0 \ I_r]$ and $Y = [I_r \ 0 \ 0 \ \cdots \ 0]$ then $X(zC_1 - C_0)^{-1}Y = P^{-1}(z)$ where $P(z) = \sum_{k=0}^\infty p_k \phi_k(z)$ except for such $z$ that $\det P(z) = 0$. As in section 2.1 the polynomials $\phi_k(z)$ satisfy $z \phi_k = \alpha_k \phi_{k+1} + \beta_k \phi_k + \gamma_k \phi_{k-1}$, $\phi_{-1} = 0$, $\phi_0 = 1$, $\phi_1 = (z - \beta_0) / \alpha_0$. In this theorem, $n \geq 2$ and $N = nr$, and if $p_n \neq 0$, then degree $P = n$.

That this is a linearization is well-known; see e.g. [2]. We only prove $P^{-1}(z) = XR^{-1}Y$, here.

**Proof.** We use the first block column of Schur Complement inverse formula

$$R^{-1} = \begin{bmatrix}
  S^{-1} & * \\
  -D^{-1}CS^{-1} & * 
\end{bmatrix}.$$

Here

$$D = \begin{bmatrix}
  (z - \beta_{n-2}) I & -\gamma_{n-2} I \\
  -\alpha_{n-3} I & (z - \beta_{n-3}) I & -\gamma_{n-3} I \\
  \vdots & \vdots & \ddots & \gamma_1 I \\
  -\alpha_0 I & & & (z - \beta_0) I 
\end{bmatrix}$$

and

$$S^{-1} = \begin{bmatrix}
  \frac{1}{\alpha_{n-1}} p_n - p_{n-1} & \frac{\gamma_{n-1}}{\alpha_{n-1}} p_n - p_{n-2} & -p_{n-1} & \cdots & -p_0 \\
  \alpha_{n-2} I_r & \frac{\beta_{n-2}}{\alpha_{n-1}} I_r & \gamma_{n-2} I_r & \cdots & 0 \\
  \alpha_{n-3} I_r & \frac{\beta_{n-3}}{\alpha_{n-1}} I_r & \gamma_{n-3} I_r & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \gamma_1 I_r \\
  \alpha_0 I_r & \beta_0 I_r & & &
\end{bmatrix}.$$
is block tridiagonal, and

\[
C = \begin{bmatrix}
-\alpha_{n-2}I & 0 & 0 & \cdots & 0 \\
0 & -\alpha_{n-3}I & 0 & \cdots & 0 \\
0 & 0 & -\alpha_{n-4}I & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -\alpha_0I
\end{bmatrix}.
\]  

(3.44)

By inspection \( V = -D^{-1}C \) is

\[
V = q \begin{bmatrix}
\phi_{n-2}(z)I_r \\
\vdots \\
\phi_2(z)I_r \\
\phi_1(z)I_r \\
\phi_0(z)I_r
\end{bmatrix}
\]

(3.45)

for some constant \( q \), because

\[
-\alpha_k \phi_{k+1}(z) + (z - \beta_k) \phi_k(z) - \gamma_k \phi_{k-1}(z) = 0
\]

for \( k = 0, 1, \cdots, n - 3 \). The constant \( q \) is obtained from

\[
q \cdot (z - \beta_{n-2}) \phi_{n-2}(z) - q \cdot \gamma_{n-2} \phi_{n-3}(z) = +\alpha_{n-2}
\]

or

\[
q \cdot [\phi_{n-1}(z)] = +1
\]

(3.48)

So

\[
q = \frac{+1}{\phi_{n-1}(z)}.
\]

(3.49)

It follows that

\[
S = \frac{z - \beta_{n-1}}{\alpha_{n-1}} p_n + p_{n-1} + \begin{bmatrix}
-\gamma_{n-1}/\alpha_{n-1} & -\gamma_{n-2}/\alpha_{n-1} & \cdots & -\gamma_{n-3}/\alpha_{n-1} \\
\end{bmatrix} \begin{bmatrix}
\phi_{n-2}(z) \\
\phi_{n-3}(z) \\
\vdots \\
\phi_0(z)
\end{bmatrix} + \frac{1}{\phi_{n-1}(z)} \begin{bmatrix}
\phi_{n-2}(z)I_r \\
\vdots \\
\phi_0(z)I_r
\end{bmatrix}
\]

(3.50)

\[
= \frac{z - \beta_{n-1}}{\alpha_{n-1}} \phi_{n-1}(z)p_n + \phi_{n-1}(z)p_{n-1} - \frac{\gamma_{n-1}}{\alpha_{n-1}} \phi_{n-2}(z)p_n + \phi_{n-2}(z)p_{n-2} + \cdots + \phi_0(z)p_0
\]

(3.51)

\[
= \frac{\sum_{k=0}^{n} \phi_k(z)p_k}{\phi_{n-1}(z)} = \frac{P(z)}{\phi_{n-1}(z)}.
\]

(3.52)

Thus

\[
-D^{-1}CS^{-1} = \begin{bmatrix}
\phi_{n-2}(z)I_r \\
\vdots \\
\phi_0(z)I_r
\end{bmatrix} \frac{P^{-1}(z)}{\phi_{n-1}(z)}
\]

(3.53)
because \( \frac{1}{\phi_{n-1}(z)}S^{-1} = P^{-1}(z) \). Finally, \( \phi_{0}(z) = 1 \), so the bottom block is \( P^{-1}(z) \), establishing that

\[
X = \begin{bmatrix} 0 & 0 & \cdots & 0 & I_r \end{bmatrix}
\]

(3.54)

\[
Y = [I_r \ 0 \ \cdots \ 0]^T
\]

(3.55)

will produce \( XR^{-1}Y = P^{-1}(z) \).

**Theorem 3.2.** Put

\[
C_1 = \begin{bmatrix}
\frac{1}{n}p_n - p_{n-1} & -p_{n-2} & \cdots & -p_1 & -p_0 \\
I_r & 2 & -p_{n-1} & \cdots & 0 \\
I_r & 2 & 3 & \cdots & 0 \\
& & & & \ddots \\
I_r & & & & n I_r
\end{bmatrix}
\]

(3.56)

and

\[
C_0 = \begin{bmatrix}
-p_{n-1} & -p_{n-2} & \cdots & -p_1 & -p_0 \\
I_r & 0 & \cdots & 0 \\
I_r & 0 & \cdots & 0 \\
& & & \ddots \\
I_r & & & & 0
\end{bmatrix}
\]

(3.57)

and \( Y = [I_r \ 0 \ \cdots \ 0]^T \) with \( X = \begin{bmatrix} \frac{1}{n}I_r & \frac{2}{n}I_r & \frac{3}{n}I_r & \cdots & \frac{n}{n}I_r \end{bmatrix} \). Then \( X(zC_1 - C_0)^{-1}Y = P^{-1}(z) \), unless \( z \in \Lambda(P) \), and \( \det P(z) = \det R(z) = \det(zC_1 - C_0) \).

**Proof.** This linearization in proved e.g. in [17], but for convenience we supply one here as well. The Schur factoring is

\[
R = \begin{bmatrix} I_r & BD^{-1} \\ 0 & I_{N-r} \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & D \end{bmatrix}
\]

(3.58)

where \( S = A - BD^{-1}C \) is the Schur Complement. Here

\[
A = \frac{z}{n}p_n + (1 - z)p_{n-1}
\]

(3.59)

\[
B = [(1 - z)p_{n-2} \ (1 - z)p_{n-3} \ \cdots \ (1 - z)p_0]
\]

(3.60)

\[
C = \begin{bmatrix}
(z - 1)I_r \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

(3.61)
and

\[
D = \begin{bmatrix}
\frac{2}{n-1}zI_r \\
(z-1)I_r \\
\frac{3}{n-2}zI_r \\
(z-1)I_r \\
\vdots \\
\vdots \\
(z-1)I_r \\
\frac{n}{1}zI_r
\end{bmatrix}
\]  

(3.62)

Therefore  \( V = D^{-1}C \) satisfies

\[
\begin{bmatrix}
\frac{2}{n-1}zI_r \\
(z-1)I_r \\
\frac{3}{n-2}zI_r \\
(z-1)I_r \\
\vdots \\
\vdots \\
(z-1)I_r \\
\frac{n}{1}zI_r
\end{bmatrix}
\begin{bmatrix}
\mathbf{v}_1 \\
\mathbf{v}_2 \\
\mathbf{v}_3 \\
\vdots \\
\mathbf{v}_{n-1}
\end{bmatrix}
= \begin{bmatrix}
(z-1)I_r \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

(3.63)

So

\[
\mathbf{v}_1 = \frac{n-1}{2} \left( \frac{z-1}{z} \right) I_r = -\frac{n-1}{2} \left( \frac{1-z}{z} \right) I_r
\]

(3.64)

\[
\mathbf{v}_2 = -\frac{n-2}{3} \cdot \mathbf{v}_1 = -\frac{n-2}{3} \cdot \frac{n-1}{2} \left( \frac{1-z}{z} \right)^2 I_r
\]

(3.65)

\[
\mathbf{v}_3 = -\frac{n-3}{4} \cdot \frac{n-2}{3} \cdot \frac{n-1}{2} \left( \frac{1-z}{z} \right)^3 I_r
\]

(3.66)

and so in; by inspection, confirmed by a formal induction not given here,

\[
\mathbf{v}_k = -\frac{(n-1)!}{(n-k+1)!} \left( \frac{1-z}{z} \right)^k I_r = -\frac{1}{n} \frac{n}{k+1} \left( \frac{n}{k} \right) \left( \frac{1-z}{z} \right)^k I_r
\]

for  \( k = 1, \ldots, n-1 \). Thus

\[
S = \frac{z}{n} \mathbf{p}_n + (1-z) \mathbf{p}_{n-1} + (1-z) \left[ \mathbf{p}_{n-2} \quad \mathbf{p}_{n-3} \quad \cdots \quad \mathbf{p}_0 \right]
\]

\[
= \frac{1}{nz^{n-1}} \left[ z^n \mathbf{p}_n + n z^{n-1}(1-z) \mathbf{p}_{n-1} + \binom{n}{2} z^{n-2}(1-z)^2 \mathbf{p}_{n-2} + \cdots + \binom{n}{n} (1-z)^n \mathbf{p}_0 \right]
\]

(3.68)

\[
= \frac{P(z)}{nz^{n-1}}
\]

(3.70)
Hence

\begin{equation}
\det \mathbf{R} = \det \mathbf{S} \det \mathbf{D} = \frac{\det \mathbf{P}(z)}{(nz^{n-1})^r} \left( \frac{2}{n-1} \cdot \frac{3}{n-2} \cdots \frac{n-1}{2} \cdot n \cdot z \right)^r
\end{equation}

This establishes the linearization. Moreover,

\begin{equation}
S^{-1} = nz^{n-1}P^{-1}(z)
\end{equation}

and the first column of \( \mathbf{R}^{-1} \) is

\begin{equation}
\begin{bmatrix}
S^{-1} \\
-D^{-1}CS^{-1}
\end{bmatrix} = 
\begin{bmatrix}
\frac{nz^{n-1}}{1} \left( \frac{1}{n} \right) \left( \frac{1-z}{z} \right) P^{-1} \\
\frac{nz^{n-1}}{2} \left( \frac{1}{n} \right) \left( \frac{1-z}{z} \right)^2 P^{-1} \\
\frac{nz^{n-1}}{3} \left( \frac{1}{n} \right) \left( \frac{1-z}{z} \right)^3 P^{-1} \\
\vdots \\
\frac{nz^{n-1}}{n} \left( \frac{1}{n} \right) \left( \frac{1-z}{z} \right)^{n-1} P^{-1}
\end{bmatrix} = 
\begin{bmatrix}
\frac{nz^{n-1}}{1} \left( \frac{1}{n} \right) \left( \frac{1-z}{z} \right) P^{-1} \\
\frac{n}{2} \left( \frac{1-z}{z} \right)^2 P^{-1} \\
\frac{n}{3} \left( \frac{1-z}{z} \right)^3 P^{-1} \\
\vdots \\
\frac{n}{n} \left( \frac{1-z}{z} \right)^{n-1} P^{-1}
\end{bmatrix}
\end{equation}

We now notice that 1, expressed as a linear combination of

\begin{equation}
\left( \frac{n}{1} \right) z^{n-1}, \left( \frac{n}{2} \right) z^{n-2}(1-z), \ldots, \left( \frac{n}{n} \right) z^0(1-z)^{n-1}
\end{equation}

is

\begin{equation}
1 = \frac{1}{n} \left( \frac{n}{1} \right) z^{n-1} + \frac{2}{n} \left( \frac{n}{2} \right) z^{n-2}(1-z) + \cdots + \frac{n}{n} \left( \frac{n}{n} \right) z^0(1-z)^{n-1}
\end{equation}

\begin{equation}
= \left( \frac{n-1}{0} \right) z^{n-1}(1-z)^0 + \left( \frac{n-1}{1} \right) z^{n-2}(1-z)^1 + \cdots + \left( \frac{n-1}{n-1} \right) z^0(1-z)^{n-1}
\end{equation}

\begin{equation}
= (z + 1 - z)^{n-1}.
\end{equation}

Indeed we use a degree-reduced Bernstein bases here, \( \binom{n-1}{k} z^k (1-z)^{n-1-k} \), to express 1. In any case, the coefficients of 1 give us our \( \mathbf{X} \) vector: \( \mathbf{X} \mathbf{R}^{-1} \mathbf{Y} = \mathbf{P}^{-1}(z) \).

\textbf{Theorem 3.3 (Lagrange Basis).} If \( \mathbf{P}(z) \in \mathbb{C}^{r \times r} \) is of degree at most \( d \), and takes the values \( \mathbf{p}_k \in \mathbb{C}^{r \times r} \) at the \( d+1 \) distinct nodes \( z = \tau_k, 0 \leq k \leq d \), i.e. \( \mathbf{P}(\tau_k) = \mathbf{p}_k \in \mathbb{C}^{r \times r} \), and the reciprocal of the node polynomial \( \omega(z) = \prod_{k=0}^{d} (z - \tau_k) \) has partial fraction expansion

\begin{equation}
\frac{1}{\omega(z)} = \sum_{k=0}^{d} \frac{\beta_k}{z - \tau_k}
\end{equation}
then a linearization for \( P(z) \) is \( zC_1 - C_0 \) where 
\[
C_1 = \text{diag}(0, I_r, I_r, \cdots, I_r)
\]
with \( d + 2 \) diagonal blocks, so 
\[
N = (d + 2)r,
\]
and
\[
C_0 = \begin{bmatrix}
0 & -\rho_0 & -\rho_1 & -\rho_2 & \cdots & -\rho_d \\
\beta_0 I & \tau_0 I & & & & \\
\beta_1 I & \tau_1 I & \tau_2 I & & & \\
\vdots & & & \ddots & & \\
\beta_d I & & & & \tau_d I
\end{bmatrix}.
\]

Moreover, if 
\[
Y = \begin{bmatrix} I_r & 0 & 0 & \cdots & 0 \end{bmatrix}^T \quad \text{and} \quad X = \begin{bmatrix} 0_r & I_r & I_r & \cdots & I_r \end{bmatrix}
\]
then 
\[
X(zC_1 - C_0)^{-1}Y = P^{-1}(z)
\]
where \( z \in \Lambda(P) \).

**Proof.** Again we use the Schur complement: 
\[
S = A - BD^{-1}C
\]
where here
\[
A = 0_r,
\]
\[
B = -[\rho_0 \ \rho_1 \ \cdots \ \rho_d]
\]
\[
D^{-1} = \text{diag} \left( \frac{1}{z - \tau_0}I_r, \frac{1}{z - \tau_1}I_r, \cdots, \frac{1}{z - \tau_d}I_r \right)
\]
\[
C = \begin{bmatrix}
\beta_0 I_r \\
\beta_1 I_r \\
\beta_2 I_r \\
\vdots \\
\beta_d I_r
\end{bmatrix}
\]

So
\[
S = \sum_{k=0}^{d} \frac{\beta_k}{z - \tau_k} \rho_k = \omega(z)^{-1} P(z)
\]

from the first barycentric formula [3].

Note the first column of 
\[
R^{-1}(z) = \begin{bmatrix} S^{-1} \\ -CD^{-1}S^{-1} \end{bmatrix}
\]
or
\[
\begin{bmatrix}
\omega(z)P^{-1}(z) \\
\left(\frac{z - \tau_0}{\beta_0}\right) \omega(z)P^{-1}(z) \\
\left(\frac{z - \tau_1}{\beta_1}\right) \omega(z)P^{-1}(z) \\
\vdots \\
\left(\frac{z - \tau_d}{\beta_d}\right) \omega(z)P^{-1}(z)
\end{bmatrix}
\]

Note that 
\[
\sum_{k=0}^{d} \frac{\beta_k}{z - \tau_k} = \frac{1}{\omega(z)},
\]

so
\[
\begin{bmatrix} 0 & I_r & I_r & \cdots & I_r \end{bmatrix} \cdot R^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \left( \sum_{k=0}^{d} \frac{\beta_k}{z - \tau_k} \right) \omega(z)P^{-1}(z) = P^{-1}(z)
\]
Theorem 3.4. In the Hermite interpolational bases on \( n + 1 \) nodes each with coefficient \( s_i \), so the degree \( d \) is at most \(-1 + \sum_{k=0}^{n} s_k\), the barycentric weights are

\[
\frac{1}{\omega(z)} = \sum_{i=0}^{n} \sum_{j=0}^{s_i-1} \frac{\beta_{ij}}{(z - \tau_i)^{j+1}}
\]

As in the Lagrange case, \( C_1 = \text{diag}(0, I_r, \cdots, I_r) \). \( C_0 \) is as below:

\[
C_0 = \begin{bmatrix}
0 & -\hat{\rho}_0 & -\hat{\rho}_1 & \cdots & -\hat{\rho}_n \\
\beta_{0,s_0-1}I & J_0^T & & & \\
\beta_{0,s_0-2}I & J_1^T & J_1 & & \\
& & & \ddots & \\
& & & & \beta_{n,s_n-1}I & J_n^T
\end{bmatrix}
\]

where

\[
\hat{\rho}_i = \begin{bmatrix} \rho_{i,s_i-1} & \rho_{i,s_i-2} & \cdots & \rho_{i,0} \end{bmatrix}
\]

and

\[
J_i = \begin{bmatrix}
\tau_i I_r & I_r & \cdots & 1 \\
I_r & \tau_i I_r & \cdots & 1 \\
\cdots & \cdots & \ddots & \cdots \\
1 & \cdots & \cdots & \tau_i I_r
\end{bmatrix}
\]

with Jordan-like blocks for each node. This form arises naturally on flowing distinct Lagrange nodes together.

Express 1 as a polynomial in this basis. Then \( 1 \leftrightarrow \rho_{n0} = 1, \rho_{10} = 1, \cdots, \rho_{n0} = 1 \) and all other components are zero. Put

\[
X = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1
\end{bmatrix} \otimes I_r
\]

and \( Y = \begin{bmatrix} I_r & 0 & 0 & \cdots & 0 \end{bmatrix} \).

A similar but more involved computation than in theorem 3.3 gives

\[
S = \frac{1}{\omega(z)} P(z) = \sum_{i=0}^{d} \sum_{j=0}^{s_i-1} \sum_{k=0}^{j} \beta_{ij} \rho_{ik} (z - \tau_i)^{k-j-1}
\]

and \( D^{-1}C \) contains just the correct powers of \((z - \tau_i)\) divided into \( \beta_{ij} \) to make the sums come out right; the inverse of the block

\[
\begin{bmatrix}
(z - \tau_0)I_r & & \\
-I_r & (z - \tau_0)I_r & \\
& -I_r & \ddots \\
& & \ddots & \ddots
\end{bmatrix}
\]

(3.95)
is

\[
\begin{bmatrix}
\frac{1}{z - \tau_1} & I_r \\
\frac{1}{(z - \tau_0)^2} & \frac{1}{z - \tau_1} & I_r \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{(z - \tau_0)^n} & \frac{1}{(z - \tau_0)^{n-1}} & \ldots & \frac{1}{z - \tau_0} & I_r
\end{bmatrix}.
\]

(3.96)

and thus each block is reminiscent of theorem 3.1, in fact.

Remark 3.5. In every case \(X = [0, \text{coefficients of } 1 \otimes I, \ Y = [1, 0, \ldots , 0] \otimes I\). This suggests that there is a simpler proof, a universal proof which we see in the next section.

4. Concluding remarks. Putting

\[
1 = \sum_{k=0}^{n-1} \hat{e}_k \phi_k(z)
\]

defines the coefficients \(\hat{e}_k\) uniquely because the \(\phi_k\) are a basis. Putting

\[
X = [\hat{e}_{n-1} \ \hat{e}_{n-2} \ \ldots \ \hat{e}_1 \ \hat{e}_0] \otimes I
\]

always gives our standard triple \(P^{-1}(z) = X(zC_1 - C_0)^{-1}Y\) with \(Y = [I \ 0 \ 0 \ \cdots \ 0]^T\). The proof is simple and universal: Denote the change-of-bases matrix by \(\Phi\) (in the cases 3.1 and 3.2)

\[
\begin{bmatrix}
\phi_{n-1}(z) \\
\phi_{n-2}(z) \\
\vdots \\
\phi_1(z) \\
\phi_0(z)
\end{bmatrix} = \Phi \begin{bmatrix} z^{n-1} \\ \vdots \\ z^2 \\ z \\ 1 \end{bmatrix}.
\]

(4.98)

and for the Lagrange case (and similarly for the Hermite case)

\[
\begin{bmatrix}
\omega(z) \\
\ell_0(z) \\
\vdots \\
\ell_n(z)
\end{bmatrix} = \Phi \begin{bmatrix} z^{n+1} \\ \vdots \\ z \\ 1 \end{bmatrix}
\]

(4.99)

Then if \(zC_1 - C_0\) is a companion pencil for \(P(z)\) in the basis \(\phi_k(z)\), we have

\[
C_0 \begin{bmatrix}
\phi_{n-1}(z) \\
\phi_{n-2}(z) \\
\vdots \\
\phi_1(z) \\
\phi_0(z)
\end{bmatrix} = zC_1 \begin{bmatrix}
\phi_{n-1}(z) \\
\phi_{n-2}(z) \\
\vdots \\
\phi_1(z) \\
\phi_0(z)
\end{bmatrix}
\]

(4.100)

by construction, so

\[
zC_1 \Phi \begin{bmatrix} z^{n-1} \\ \vdots \\ z \\ 1 \end{bmatrix} = C_0 \Phi \begin{bmatrix} z^{n-1} \\ \vdots \\ z \\ 1 \end{bmatrix}
\]

(4.101)
so $z(C_1 \Phi) - (C_0 \Phi)$ is the companion pencil

$$
\begin{bmatrix}
z p_n + p_{n-1} & p_{n-2} & p_{n-3} & \cdots & p_0 \\
-I & zI & & & \\
-I & zI & & \\
& & & \ddots & \ddots \\
& & & & -I & zI
\end{bmatrix}
$$

(4.102)

for the monomial basis, which has $X_n = [0 \cdots 0 1] \otimes I$ and $Y_m = [I 0 0 \cdots 0]^T$. Since $(zC_1 \Phi - C_0 \Phi)^{-1} = \Phi^{-1} (zC_1 - C_0)^{-1}$, $X_d = X_m \Phi^{-1}$ is the 1st entry of the standard triple for $zC_1 - C_0$. But this is exactly $[\hat{e}_{n-1} \hat{e}_{n-2} \cdots \hat{e}_0] \otimes I$.

The new contributions of this paper are the explicit expressions for the standard triples and the proof that the standard triples are in this sense universal.

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