THE INERTIA GROUPS OF A TORIC DELIGNE-MUMFORD STACK

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ABSTRACT. This paper determines the isotropy groups (inertia groups) of the points of a toric Deligne-Mumford stack (over the category of smooth manifolds) that is realized from a quotient construction $[Z/G]$ arising from a stacky fan or stacky polytope, providing an explicit correspondence between certain geometric and combinatorial data. In the process, we express the connected component of the identity element $G_0 \subset G$ and the the component group $G/G_0$ in terms of the underlying stacky fan. We consequently obtain a characterization of those stacky polytopes that yield stacks equivalent to weighted projective spaces as well as, more generally, ‘fake’ weighted projective spaces. Our results are illustrated in the case of labelled sheared simplices—a class of labelled polytopes in the sense of Lerman-Tolman’s generalization of Delzant polytopes—where more detail can be obtained in terms of facet labels.

1. INTRODUCTION AND RESULTS

In their foundational paper [3], Borisov, Chen, and Smith introduce the notion of a stacky fan, the combinatorial data from which one constructs a toric Deligne-Mumford (DM) stack using a quotient construction in algebraic geometry. An abstract definition of toric DM stacks is given in [7], where it is shown to be compatible with the construction of Borisov, Chen, and Smith [3] (see also [13], [20] [9, 10] for related approaches). In this paper we determine the isotropy groups (also called inertia groups in the literature on stacks) of the points of a toric DM stack $[Z/G]$ constructed in [3] from a stacky fan (see below, and Section 2). These isotropy groups encode certain stacky information about the toric DM stack and play an important role in computing stacky invariants, such as (Chen-Ruan) orbifold cohomology.

Historically, toric varieties and toric DM stacks have been studied from the perspectives of both algebraic and symplectic geometry when the underlying fan is polytopal. This subfamily of toric DM stacks (whose fan is polytopal) admits a description in the language of symplectic geometry via Sakai’s stacky polytopes [22], where now the stacks are viewed over the category Diff of smooth manifolds. These stacks can be viewed as a generalization of Lerman and Tolman’s toric orbifolds associated to labelled polytopes [17] (cf. Section 2.2 for details), a generalization of the Delzant polytopes that classify symplectic toric manifolds. The authors of the present manuscript come from the symplectic-geometric viewpoint. As a consequence, in this manuscript we work in the category Diff of smooth manifolds; in particular, by a “stack” we shall mean a “stack over the category Diff.”

The main results of this paper explicitly compute the isotropy groups of toric DM stacks using basic algebra, without reference to much of the machinery of the theory of stacks. This simplicity is one of the selling features of our approach, particularly in the case of symplectic toric DM stacks constructed from labelled polytopes. In order to state our main result, Theorem 3.2 quoted below, recall that a stacky fan is a triple $(N, \Sigma, \beta)$ consisting of a finitely generated $\mathbb{Z}$-module $N$, a simplicial fan $\Sigma$ in $N \otimes \mathbb{R}$ with $n$ rays $\rho_1, \ldots, \rho_n$, and a homomorphism $\beta : \mathbb{Z}^n \to N$ satisfying certain conditions (see Definition 2.1). This data defines an action of an Abelian Lie group $G$ on a smooth manifold $Z_\Sigma \subset \mathbb{C}^n$, resulting in the toric DM stack $[Z_\Sigma/G]$.

Theorem 3.2. Let $(N, \Sigma, \beta)$ be a stacky fan and $[Z_\Sigma/G]$ its corresponding toric DM stack. For a point $z = (z_1, \ldots, z_n) \in Z_\Sigma \subset \mathbb{C}^n$, let $N_z \subset N$ denote the submodule generated by $\{\beta(e_i) \mid z_i = 0\}$. The isotropy group $\text{Stab}(z)$ is isomorphic to the torsion submodule $\text{Tor}(N/N_z)$.
Since Tor\((N/N_z)\) does not canonically appear as a subgroup of \(G\), we exhibit a concrete (non-canonical) homomorphism Tor\((N/N_z) \to G\) (see Equation (3.2) and Proposition 3.5). If \(N\) is free, however, this homomorphism is canonical and Proposition 3.16 recasts this map from the viewpoint of labelled polytopes, viewed as stacky polytopes. For a stacky polytope \((N, \Delta, \beta)\) (see Definition 2.2), one obtains a symplectic quotient stack \([\mu^{-1}(\tau)/K]\) of a certain Hamiltonian \(K\)-action on \(C^n\), where \(K\) is a compact Abelian Lie group, with moment map \(\mu : C^n \to \tau^*\) (see [22] for details). In Section 2.2, we see that for labelled polytopes, Sakai’s construction reproduces the extension of the Delzant construction in [17], so that in particular we may view \(K\) as a subgroup of \((S^1)^n\) with Lie algebra \(\mathfrak{a}\) a subspace of \(\mathbb{Z}^n \otimes \mathbb{R} \cong \text{Lie}(S^1)^n\).

Proposition 3.16. Let \((\Delta, \{m_i\}_{i=1}^n)\) be a labelled polytope in \((N \otimes \mathbb{R})^*\) with primitive inward pointing facet normals \(\nu_1 \otimes 1, \ldots, \nu_n \otimes 1\), and \([\mu^{-1}(\tau)/K]\) its corresponding toric DM stack. For \(z = (z_1, \ldots, z_n)\) in \(\mu^{-1}(\tau) \subset C^n\), let \(N_z \subset N\) denote the submodule generated by \(\{m_i \nu_i \mid z_i = 0\}\). Then the canonical map

\[
\text{Tor}(N/N_z) \to K, \quad x + N_z \mapsto \exp(y \otimes \frac{1}{m}),
\]

is an isomorphism onto its image \(\text{Stab}(z)\), where \(y\) is the unique element in \(\text{span}\{e_i \mid z_i = 0\} \subset \mathbb{Z}^n\) satisfying \(\beta(y) = m x\) for some smallest positive integer \(m\).

The analogous statement for a general stacky fan (not necessarily corresponding to a labelled polytope) requires some choices, and is stated in Proposition 3.9.

We also consider how the stacky fan data relates to the connectedness of the group \(G\). For example, we express the connected component of the identity element \(G_0 \subset G\) (see Proposition 3.12) as well as \(G/G_0\) (see Lemma 3.9) in terms of the underlying stacky fan data. In fact, one of our initial motivations was to determine the isotropy groups for the related toric DM stack \([Z/G_0]\); such a computation yields an answer to the question of whether \([Z/G]\) is equivalent to a global quotient of a smooth manifold by a finite group action. As shown in [12], \([Z/G]\) is equivalent as stacks to such a global quotient if and only if the isotropy groups for \([Z/G_0]\) are trivial; in Corollary 3.13 we provide an equivalent characterization in terms of the stacky fan.

Finally, in the special case the group \(G\) is 1-dimensional, we obtain more detailed results. In this case the underlying fan (polytope) is a simplex, and the resulting toric DM stacks yield stacky analogues of fake weighted projective spaces [14], i.e., (stacky) quotients \(W/\Lambda\) of a weighted projective space (viewed as a stack) equipped with a (stacky) action of a finite Abelian group \(\Lambda\) (see Section 5.1). Our results give a necessary and sufficient condition for \(G\) to be connected. Moreover, in this special case (when \(G\) is connected) we show that the toric DM stack is equivalent as stacks to a weighted projective space (Proposition 4.2). In the case of labelled sheared simplices—labelled simplices with all facets but one lying on coordinate hyperplanes—we identify the component group \(G/G_0\) explicitly (Lemma 4.3).

We close with some comments on possible further work. An understanding of isotropy groups is necessary for computing stacky invariants; for example, Borisov, Chen, and Smith compute the orbifold Chow ring of toric DM stacks in terms of the stacky fan [3]. It would be interesting to explore the analogous statement for Chen-Ruan cohomology in the symplectic setting using the inertial cohomology techniques in [11], which rely on isotropy data.

The paper is organized as follows. Section 2 recalls the construction of a toric DM stack from a stacky fan given in [3] and its relation with Lerman-Tolman’s generalization of the Delzant construction for labelled polytopes. In Section 3 we prove Theorem 3.2 stated above. Also in Section 3 we relate the connected component \(G_0\) of the identity of the group \(G\) to the underlying stacky fan data, showing in Proposition 3.12 that the quotient stack \([Z_G/G_0]\) is naturally a toric DM stack; we elaborate on this for the case of labelled polytopes in Section 3.3. In Sections 4 and 5 we characterize weighted projective spaces (Proposition 4.2) and more generally fake weighted projective spaces (Proposition 5.1), respectively, in terms of their underlying stacky fan data. Upon introducing labelled sheared simplices, we interpret the results on isotropy and connectedness of \(G\) in terms of the given labels and shearing data, with a complete treatment (Proposition 5.6) for labelled sheared simplices in the plane.

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2. Preliminaries

In this section we recall some background regarding stacky fans and polytopes and their associated toric Deligne-Mumford stacks. We assume some familiarity with stacks, in particular their use in modelling group actions on manifolds and orbifolds. We refer the reader to e.g. [5] and references therein for basic definitions and ideas in the theory of stacks. Within the field of algebraic geometry there is by now an extensive literature on (algebraic) stacks (see e.g. the informal guide [1]), but in other categories (e.g. Diff or Top) the literature continues to develop. The present authors learned a great deal from the unpublished (in-progress) notes [2] as well as [15] and [18].

The stacks appearing in this paper are quotient stacks over Diff associated to smooth, proper, locally-free Lie group actions on manifolds. Such quotient stacks are Deligne-Mumford (i.e. stacks that admit a presentation by a proper étale Lie groupoid, cf. [16, Theorem 2.4]) and therefore model smooth orbifolds. Often in the algebraic literature, the term orbifold is used for DM stacks with trivial generic stabilizer (e.g. as in [7]), what we shall refer to as an effective orbifold.

The class of DM stacks we work with are toric DM stacks, arising from the combinatorial data of a stacky fan [3]. Our original motivation was to work instead with stacky polytopes, the symplectic counterparts of stacky fans, which give rise to symplectic toric DM stacks [22]. These offer a modern perspective on the symplectic toric orbifolds of Lerman and Tolman [17] constructed from labelled polytopes (see Section 2.2 below). However, since our results do not depend on (or make use of) the symplectic structure from this perspective, we choose to work mainly with stacky fans.

2.1. Stacky fans and polytopes. Mainly to establish notation, we briefly recall some basic definitions of the combinatorial data appearing in the above discussion. We use $(-)^*$ to denote the functor $\text{Hom}_\mathbb{Z}(-, \mathbb{Z})$ or $\text{Hom}_\mathbb{R}(-, \mathbb{R})$; it should be clear from context which one is meant. Let $\mathbb{T}$ denote the group of units $\mathbb{C}^*$, and $\mu_k \subset \mathbb{T}$ the cyclic group of $k$-th roots of unity. Let $\{e_1, \ldots, e_n\}$ be the standard basis vectors in $\mathbb{Z}^n \subset \mathbb{R}^n$.

Definition 2.1. [3] A stacky fan is a triple $(N, \Sigma, \beta)$ consisting of a rank $d$ finitely generated Abelian group $N$, a rational simplicial fan $\Sigma$ in $N \otimes \mathbb{R}$ with rays $\rho_1, \ldots, \rho_n$ and a homomorphism $\beta : \mathbb{Z}^n \to N$ satisfying:

1. the rays $\rho_1, \ldots, \rho_n$ span $N \otimes \mathbb{R}$, and
2. for $1 \leq j \leq n$, $\beta(e_j) \otimes 1$ is on the ray $\rho_j$.

Given a polytope $\Delta \subseteq \mathbb{R}^d$, recall that the corresponding fan $\Sigma = \Sigma(\Delta)$ is obtained by setting the one dimensional cones $\Sigma^{(1)}$ to be the positive rays spanned by the inward-pointing normals to the facets of $\Delta$; a subset $\sigma$ of these rays is a cone in $\Sigma$ precisely when the corresponding facets intersect nontrivially in $\Delta$. Observe that under this correspondence, facets intersecting in a vertex of $\Delta$ yield maximal cones (with respect to inclusion) in $\Sigma(\Delta)$.

Definition 2.2. [22] A stacky polytope is a triple $(N, \Delta, \beta)$ consisting of a rank $d$ finitely generated Abelian group $N$, a simple polytope $\Delta$ in $(N \otimes \mathbb{R})^*$ with $n$ facets $F_1, \ldots, F_n$ and a homomorphism $\beta : \mathbb{Z}^n \to N$ satisfying:

1. the cokernel of $\beta$ is finite, and
2. for $1 \leq j \leq n$, $\beta(e_j) \otimes 1$ in $N \otimes \mathbb{R}$ is an inward pointing normal to the facet $F_j$.

Condition 2 above implies that the polytope $\Delta$ in Definition 2.2 is a rational polytope. Also, from the preceding discussion it follows immediately that the data of a stacky polytope $(N, \Delta, \beta)$ specifies the data of a stacky fan by the correspondence $(N, \Delta, \beta) \leftrightarrow (N, \Sigma(\Delta), \beta)$. Indeed, $\Delta$ is simple if and only if $\Sigma(\Delta)$ is simplicial. Moreover, the fan $\Sigma(\Delta)$ is rational by condition 2.2.2. Finally, $(N, \Delta, \beta)$ satisfies conditions (1) and (2) of Definition 2.2 if and only if $(N, \Sigma(\Delta), \beta)$ satisfies conditions (1) and (2) of Definition 2.1.

The extra information encoded in a stacky polytope $(N, \Delta, \beta)$ (compared with the stacky fan $(N, \Sigma(\Delta), \beta)$) results in a symplectic structure on the associated toric DM stack. Given a presentation of a rational polytope $\Delta$ as the intersection of half-spaces

$$\Delta = \bigcap_{i=1}^n \{x \in (N \otimes \mathbb{R})^* \mid x(\beta(e_i) \otimes 1) \geq -c_i\}$$

for some $c_i \in \mathbb{R}$ and where each $\beta(e_i) \otimes 1 \in N \otimes \mathbb{R}$ is the inward pointing normal to the facet $F_i$, the fan $\Sigma(\Delta)$ only retains the data of the positive ray spanned by the normals, and not the parameters $c_i$, which encode the symplectic structure on the resulting DM stack (see [22] for details).
Recall (as in [3]) that given a stacky fan \((N, \Sigma, \beta)\), the corresponding DM stack may be constructed as a quotient stack \(\mathcal{Z}_\Sigma/G\) as follows. As with classical toric varieties, the fan \(\Sigma\) determines an ideal

\[
J(\Sigma) = \left\langle \prod_{\rho, \Sigma} z_{\rho} : \sigma \in \Sigma \right\rangle \subset \mathbb{C}[z_1, \ldots, z_n].
\]

Let \(\mathcal{Z}_\Sigma\) denote the complement \(\mathbb{C}^n \setminus V(J(\Sigma))\) of the vanishing locus of \(J(\Sigma)\). Next, we recall a certain group action on \(\mathcal{Z}_\Sigma\).

Choose a free resolution

\[
0 \to \mathbb{Z}^\ell \xrightarrow{Q} \mathbb{Z}^{d+\ell} \to N \to 0
\]

of the \(\mathbb{Z}\)-module \(N\), and let \(B : \mathbb{Z}^n \to \mathbb{Z}^{d+\ell}\) be a lift of \(\beta\). With these choices, define the dual group \(\text{DG}(\beta) = (\mathbb{Z}^{d+\ell})^*/\text{im}[B Q]^*\) where \([B Q] : \mathbb{Z}^{n+\ell} = \mathbb{Z}^n \oplus \mathbb{Z}^\ell \to \mathbb{Z}^{d+\ell}\) denotes the map whose restrictions to the first and second summands are \(B\) and \(Q\), respectively. Let \(\beta^\vee : (\mathbb{Z}^n)^* \to \text{DG}(\beta)\) be the composition of the inclusion \((\mathbb{Z}^n)^* \to (\mathbb{Z}^{d+\ell})^*\) (into the first \(n\) coordinates) and the quotient map \((\mathbb{Z}^{d+\ell})^* \to \text{DG}(\beta)\).

Applying the functor \(\text{Hom}(\mathbb{Z}, \mathbb{T})\) to \(\beta^\vee\) yields a homomorphism \(G := \text{Hom}(\text{DG}(\beta), \mathbb{T}) \to \mathbb{T}^n\), which defines a \(G\)-action on \(\mathbb{C}^n\) that leaves \(\mathcal{Z}_\Sigma \subset \mathbb{C}^n\) invariant. Define \(\mathcal{X}(N, \Sigma, \beta) = [\mathcal{Z}_\Sigma/G]\). By Proposition 3.2 in [3], \(\mathcal{X}(N, \Sigma, \beta)\) is a DM stack. At times, we shall simply use the notation \([\mathcal{Z}_\Sigma/G]\) to denote \(\mathcal{X}(N, \Sigma, \beta)\).

The above construction was adapted to stacky polytopes by Sakai in [22]. As the reader may verify, the DM stack \(\mathcal{X}(N, \Delta, \beta)\) obtained from a stacky polytope is a quotient stack obtained by symplectic reduction \([\mu^{-1}(\tau)/K]\) where \(\mu^{-1}(\tau) \subset Z_{\Sigma(\Delta)} \subset \mathbb{C}^n\) is a certain level set of a moment map \(\mu : \mathbb{C}^n \to \mathbb{R}^n\) for a Hamiltonian action of \(K := \text{Hom}(\text{DG}(\beta), \mathbb{T})\) on \(\mathbb{C}^n\). By [22] Theorem 24, the quotient stacks \([\mathcal{Z}_\Sigma/G]\) and \([\mu^{-1}(\tau)/K]\) are equivalent.

**Example 2.3.** Consider the stacky polytope \((N, \Delta, \beta)\), with \(N = \mathbb{Z}^2\), \(\Delta\) the simplex in \(\mathbb{R}^2 \cong (N \otimes \mathbb{R})^*\) given by the convex hull of \((0, 0), (0, 1)\) and \((1, 0)\), and \(\beta : \mathbb{Z}^3 \to N\) given by the matrix

\[
\beta = \begin{bmatrix}
-2 & 3 & 0 \\
-2 & 0 & 5
\end{bmatrix}.
\]

The corresponding stacky fan \((N, \Sigma, \beta)\) is then given by the same \(N\) and \(\beta\), and \(\Sigma = \Sigma(\Delta)\) the fan dual to \(\Delta\) (see Figure 2.1). A convenient way to represent the homomorphism \(\beta\) is to use ray or facet labels (see Section 2.2, as in Figure 2.1).

![Figure 2.1. A polytope \(\Delta\) and its dual fan \(\Sigma = \Sigma(\Delta)\). The labels on the facets of \(\Delta\) (resp. ray generators of \(\Sigma\)) encode the homomorphism \(\beta : \mathbb{Z}^3 \to N\).](image)

To compute the corresponding DM stack \([\mathcal{Z}_\Sigma/G]\), note that \(\mathcal{Z}_\Sigma = \mathbb{C}^3 \setminus \{0\}\). We find \(\text{DG}(\beta) = (\mathbb{Z}^3)^*/\text{im} \beta^* \cong \mathbb{Z}\), where the isomorphism may be chosen as \(f([a, b, c]) = 15a + 10b + 6c\). Therefore, \(G = \text{Hom}(\text{DG}(\beta), \mathbb{T}) \cong \mathbb{T}\). Since the map \(\beta^\vee\) is simply the projection \(f : (\mathbb{Z}^3)^* \to \text{DG}(\beta) \cong \mathbb{Z}\), where \(f(a, b, c) = 15a + 10b + 6c\), the homomorphism \(G \to \mathbb{T}^3\) induced by \(\beta^\vee\) is then \(t \mapsto (t^{15}, t^{10}, t^6)\). It follows that the corresponding stack is equivalent to a weighted projective space, \(\mathbb{P}(15, 10, 6)\).

We include next a modification of the above example to illustrates the construction for a \(\mathbb{Z}\)-module \(N\) with torsion.
Example 2.4. Let $N = \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$, and let $\Sigma$ be the fan in Figure 2.1. Set $\beta : \mathbb{Z}^3 \to N$ to be

$$
\beta(x, y, z) = (-2x + 3y, -2x + 5z, x + y + z \mod 2).
$$

As in Example 2.3, $Z_\Sigma = C^3 \setminus \{0\}$. To compute $G$, we choose the resolution $0 \to \mathbb{Z} \xrightarrow{Q} \mathbb{Z}^3 \to N \to 0$ with $Q = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 3 & 0 \\ 0 & 2 & 5 \\ -1 & 1 & 1 & 2 \end{bmatrix}$, and choose $B$ so that

$$
[BQ] = \begin{bmatrix}
-2 & 3 & 0 & 0 \\
-2 & 0 & 5 & 0 \\
-1 & 1 & 1 & 2
\end{bmatrix}.
$$

Therefore, $DG(\beta) = (\mathbb{Z}^4) / \text{im}[BQ]^* \cong \mathbb{Z}$, where the isomorphism can be chosen as $f([a, b, c, d]) = 30a + 20b + 12c - d$. Therefore, $G \cong \mathbb{T}$, and $\beta' : (\mathbb{Z}^3)^* \to DG(\beta) \cong \mathbb{Z}$ is given by $f(a, b, c) = 30a + 20b + 12c$. Hence the homomorphism $G \to \mathbb{T}$ describing the $G$-action on $Z_\Sigma$ is $t \mapsto (t^{30}, t^{20}, t^{12})$. It follows that the corresponding stack is equivalent to a weighted projective space $\mathbb{P}(30, 20, 12)$, which has global stabilizer $\mu_2$.

2.2. Relation with Delzant’s construction and labelled polytopes. The construction of the quotient stack $[\mu^{-1}(r)/K]$ from a stacky polytope may be viewed as a generalization of Lerman-Tolman’s generalization [17] of the Delzant construction, which we review next.

In its original form [17], a labelled polytope is a pair $(\Delta, \{m_i\}_{i=1}^n)$ consisting of a convex simple polytope $\Delta$ in $V^*$, where $V$ is a real vector space, with $n$ facets $F_1, \ldots, F_n$ whose relative interiors are labelled with positive integers $m_1, \ldots, m_n$. The polytope is assumed to be rational with respect to a chosen lattice $N \subset V$. Identifying $V \cong N \otimes \mathbb{R}$, we may denote the primitive inward pointing normals by $\nu_1 \otimes 1, \ldots, \nu_n \otimes 1$. Then defining $\beta : \mathbb{Z}^n \to N$ by the formula $\beta(e_i) = m_i \nu_i$ realizes $(N, \Delta, \beta)$ as a stacky polytope. Furthermore, any stacky polytope with $N$ free can be realized as a labelled polytope. Thus labelled polytopes are precisely those stacky polytopes for which the $\mathbb{Z}$-module $N$ is a free module.

Given a labelled polytope $(\Delta, \{m_i\}_{i=1}^n)$ in $(N \otimes \mathbb{R})^* \cong (\mathbb{R}^d)^*$, we may proceed with the Delzant construction to obtain a quotient stack $[\mu^{-1}(r)/K_D]$ as a symplectic reduction, where the group $K_D \subset (S^1)^n$ acts via the standard linear $(S^1)^n$-action on $\mathbb{C}^n$. As we shall see below $K_D$ is isomorphic to the group $K = \text{Hom}(DG(\beta), S^1)$ arising from Sakai’s construction. Furthermore, it is straightforward to verify that the isomorphism is compatible with the respective group actions of $K_D$ and $K$ on $\mathbb{C}^n$, and thus the symplectic quotient stacks $[\mu^{-1}(r)/K_D]$ and $[\mu^{-1}(r)/K]$ are equivalent.

The group $K_D$ in the Delzant construction is defined as follows. Let $\beta : \mathbb{Z}^n \to N$ be given by $\beta(e_i) = m_i \nu_i$, where $m_i \nu_i$ are the weighted normals to the facets of $\Delta$, and consider the resulting homomorphism $\tilde{\beta} : (S^1)^n \to (S^1)^d$ induced by $\beta_\mathbb{R} = \beta \otimes \mathbb{R}$ (where we have chosen identifications $N \cong \mathbb{Z}^d$, $S^1 \cong \mathbb{R}/\mathbb{Z}$). Define $K_D = \ker \tilde{\beta}$.

To compare the groups $K_D$ and $K$, we note that since $N$ is free it is easy to verify that $DG(\beta) = \text{coker} \beta^*$, and hence we have the short exact sequence

$$
0 \to N^* \to (\mathbb{Z}^n)^* \to DG(\beta) \to 0
$$

which yields the short exact sequence,

$$
1 \to K \to \text{Hom}((\mathbb{Z}^n)^*, S^1) \to \text{Hom}(N^*, S^1) \to 1.
$$

Using the natural isomorphism $\text{Hom}(M^*, S^1) \cong M \otimes S^1$ (for a free $\mathbb{Z}$-module $M$) and identifications $\mathbb{Z}^n \otimes S^1 \cong (S^1)^n$ and $N \otimes S^1 \cong (S^1)^d$, resulting from the chosen identification $N \cong \mathbb{Z}^d$, we may readily identify $K \cong K_D$.

Example 2.5. Let $N = \mathbb{Z}$. Consider the labelled polytope $\Delta$ in $\mathbb{R} \cong (N \otimes \mathbb{R})^*$ consisting of a line segment with labels $r$ and $s$ at each endpoint (see Figure 2.2).

![Figure 2.2. A labelled polytope $\Delta$ in $\mathbb{R}$.](image-url)
The homomorphism $\beta : \mathbb{Z}^2 \to N$ in the corresponding stacky polytope (and stacky fan) is given by the matrix $\beta = [-s \, r]$. Let $g = \gcd(r, s)$. Then $DG(\beta) = (\mathbb{Z}^2)^* / \im \beta^* \cong \mathbb{Z} \oplus \mathbb{Z}/g\mathbb{Z}$, which may be realized by the isomorphism $f((a, b)) = (\frac{g}{\gcd(g, r)} a + \frac{g}{\gcd(g, s)} b, -ya + xb \mod g)$ where $x$ and $y$ are integers satisfying $g = xr + ys$. It follows that $G \cong \mathbb{T} \times \mu_g$, where $\mu_g \subset \mathbb{T}$ is the cyclic group of $g$-th roots of unity. Note that $G$ is connected if and only if $\gcd(r, s) = 1$.

Under the above identification, the homomorphism $\beta^* : (\mathbb{Z}^2)^* \to DG(\beta) \cong \mathbb{Z} \oplus \mathbb{Z}/g\mathbb{Z}$ is simply the projection $f(a, b) = (\frac{g}{\gcd(g, r)} a + \frac{g}{\gcd(g, s)} b, -ya + xb \mod g\mathbb{Z})$. Therefore, the homomorphism $\mathbb{T} \times \mu_g \cong G \to \mathbb{T}^2$ that determines the action on $Z_{\Sigma} = \mathbb{C}^2 \setminus \{0\}$ is then given by $(t, \xi_g^k) \mapsto (t^s \xi_g^{-ky}, t^r \xi_g^{kz})$ where $\xi_g \in \mathbb{C}$ denotes a primitive $g$-th root of unity.

3. Isotropy and Stacky Fans

Our goal in this section is to compute the local isotropy group of each point of a toric DM stack $\mathcal{X}(N, \Sigma, \beta) = [Z_{\Sigma}/G]$ by computing the subgroup $\text{Stab}(z) \subset G$ that fixes a given point $z \in Z_{\Sigma}$. The main result, Theorem 3.1 in Section 3.1, describes all possible isotropy groups that arise. A discussion of the connected component of $G$ and its role in detecting global quotient stacks appears in 3.2 along with further details in 3.3 for the case of labelled polytopes.

3.1. Isotropy and stacky fans. Recall that $Z_{\Sigma}$ is defined as the complement in $\mathbb{C}^n$ of the zero-set of the ideal $J(\Sigma)$, which is described in more detail next.

For $\sigma \in \Sigma$, let $I_\sigma = \{i : \rho_i \subset \sigma\}$, and $J_\sigma$ its complement. Then

$$V(J(\Sigma)) = \bigcap_{\sigma \in \Sigma} \{(z_1, \ldots, z_n) \mid \prod_{\rho_i \not\subset \sigma} z_i = 0\}$$

$$Z_{\Sigma} = \mathbb{C}^n \setminus V(J(\Sigma)) = \bigcup_{\sigma \in \Sigma} \{(z_1, \ldots, z_n) \mid z_i \neq 0 \text{ whenever } i \in J_\sigma\}$$

where $I_z = \{i \mid z_i = 0\}$.

There is a natural decomposition of $Z_{\Sigma}$ since an inclusion of cones $\sigma' \subset \sigma$ yields an inclusion $Z_\sigma \subset Z_{\sigma'}$, where $Z_\sigma := \{(z_1, \ldots, z_n) : I_z \subset I_{\sigma}\}$. Furthermore, for any $z = (z_1, \ldots, z_n) \in Z_{\sigma}$, there is a cone $\sigma_z \subset \sigma$ given by the span of the minimal generators of the rays $\rho_i$ with $i \in I_z$. This follows from our assumption that $\Sigma$ is simplicial. Since the number of rays $\rho_i$ with $i \in I_\sigma$ equals the dimension of $\sigma$, any subset of these rays spans a face of $\sigma$ and is thus in the fan $\Sigma$. It follows, then, that for every point $z \in Z_{\Sigma}$, we may write $z \in Z_{\sigma_z}$, where the cone $\sigma_z$ satisfies $\{i : z_i = 0\} = I_{\sigma_z}$. Moreover, $\sigma_z$ is minimal in the sense that $\sigma_z \subset \sigma$ for any $\sigma$ such that $z \in Z_{\sigma}$.

For a point $z \in \mathbb{C}^n$, the subgroup in $\mathbb{T}^n$ fixing $z$ is $\{(t_1, \ldots, t_n) : t_i = 1 \text{ if } z_i \neq 0\}$, which motivates the following definition. For any subset $I \subset \{1, \ldots, n\}$ and its complement $J_\sigma$, let

$$T^I = \{(t_1, \ldots, t_n) : i \in J \implies t_i = 1\} \subseteq \mathbb{T}^n.$$

Note that $T^I$ is the kernel of the map $\mathbb{T}^n \to \mathbb{T}^{|J_\sigma|}$ given by projection onto the coordinates indicated by $J$ with cardinality $|J|$.

Since $G$ acts on $Z_{\Sigma}$ via the homomorphism $G \to \mathbb{T}^n$ induced by $\beta^*$, then the isotropy $\text{Stab}(z)$ associated to a point $z \in Z_{\Sigma}$ is given by the kernel of the map

$$G \rightarrow \mathbb{T}^n \stackrel{\pi}{\rightarrow} \mathbb{T}^{|J_\sigma|}$$

where $J_z$ is the complement of $I_z$. At times it is useful to view the subset $I_z$ as $I_\sigma$ for the cone $\sigma = \sigma_z$ in $\Sigma$ described above, in which case we denote $\text{Stab}(z)$ by $\Gamma_z$.

Remark 3.1. Notice that an inclusion of cones $\sigma' \subset \sigma$ in $\Sigma$ induces an inclusion $J_\sigma \subset J_{\sigma'}$ and hence the following commutative diagram,
where the vertical map is the natural projection. Therefore, there is a natural inclusion of isotropy groups $\Gamma_{\sigma'} \subset \Gamma_{\sigma}$. It follows each such isotropy group is contained in $\Gamma_{\sigma}$ for some maximal cone $\sigma$. In particular, all isotropy groups are trivial if and only if $\Gamma_{\sigma}$ is trivial for maximal cones $\sigma$ in $\Sigma$.

**Theorem 3.2.** Let $(N, \Sigma, \beta)$ be a stacky fan and $[Z_\Sigma/G]$ its corresponding toric DM stack. For a point $z = (z_1, \ldots, z_n)$ in $Z_\Sigma \subset \mathbb{C}^n$, let $N_z \subset N$ denote the submodule generated by $\{\beta(e_i) \mid z_i = 0\}$. The isotropy group $\text{Stab}(z)$ is isomorphic to the torsion submodule $\text{Tor}(N/N_z)$.

**Proof.** Let $\sigma$ be a cone in $\Sigma$ with $I_\sigma = I_\sigma$. We noted above that the stabilizer of $z$ is given by $\Gamma_{\sigma}$, the kernel of the composition (3.1). This composition is realized by applying the functor $\text{Hom}(-, T)$ to the composition $f = \beta^\vee \circ \pi^*$

$$(\mathbb{Z}^{I_\sigma})^* \xrightarrow{\pi^*} (\mathbb{Z}^n)^* \xrightarrow{\beta^\vee} \text{DG}(\beta),$$

where $\pi^*$ is inclusion of the relevant factors. Moreover, since $T$ is injective as a $\mathbb{Z}$-module, the kernel of (3.1) is $\text{Hom}(\text{coker}(f, T)$). As we shall see, $\text{coker}(f)$ is finite; therefore, $\text{Hom}(\text{coker}(f, T)$ is isomorphic to $\text{coker}(f)$, which we compute next.

Let $N_{\sigma} = N_z \subset N$ denote the subgroup generated by $\{\beta(e_i) \mid i \in I_{\sigma}\}$, and let $\beta_\sigma : \mathbb{Z}^{I_{\sigma}} \rightarrow N_{\sigma}$ denote the restriction of $\beta$ to $\mathbb{Z}^{I_{\sigma}}$ together with its codomain.

We claim that $\beta_\sigma : \mathbb{Z}^{I_{\sigma}} \rightarrow N_{\sigma}$ is an isomorphism, and hence $N_{\sigma}$ is free. That $\Sigma$ is simplicial means that the $\{\beta(e_i) \otimes 1\}_{i \in I_{\sigma}}$ are linearly independent in $N \otimes \mathbb{R}$. Therefore, rank $N_{\sigma} = |I_{\sigma}|$. Since $\beta_{\sigma}$ is a surjective homomorphism of modules of the same rank, $\beta_{\sigma} \otimes \mathbb{R}$ is an isomorphism of vector spaces. But since the domain $\mathbb{Z}^{I_{\sigma}}$ of $\beta_{\sigma}$ is free, $\beta_{\sigma}$ must be injective as well. This verifies the claim.

In particular, this implies that $\text{DG}(\beta_{\sigma})$ is trivial. Any lift $B_{\sigma}$ of $\beta_{\sigma}$ is an isomorphism and $N_{\sigma}$ has no torsion, so $\text{DG}(\beta_{\sigma}) = \text{coker}(B_{\sigma})^* = \text{coker}e_{\sigma}$.

Consider the following diagram, whose rows are exact.

$$
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z}^{I_{\sigma}} & \xrightarrow{\beta_\sigma} & \mathbb{Z}^n & \xrightarrow{\beta} & \mathbb{Z}^{I_{\sigma}} & \rightarrow & 0 \\
\downarrow & & \downarrow \beta & & \downarrow \beta & & \downarrow \beta & & \\
0 & \rightarrow & N_{\sigma} & \xrightarrow{\beta} & N & \xrightarrow{\beta} & N/N_{\sigma} & \rightarrow & 0
\end{array}
$$

By Lemma 2.3 in [3], we get the following commutative diagram with exact rows, noting that $\text{DG}(\beta_{\sigma})$ is trivial.

$$
\begin{array}{ccccccc}
0 & \rightarrow & (\mathbb{Z}^{I_{\sigma}})^* & \xrightarrow{\beta^\vee} & (\mathbb{Z}^n)^* & \xrightarrow{\beta^\vee} & (\mathbb{Z}^{I_{\sigma}})^* & \rightarrow & 0 \\
\downarrow \beta^\vee & & \downarrow \beta^\vee & & \downarrow \beta^\vee & & \\
0 & \rightarrow & \text{DG}(\beta_\sigma) & \xrightarrow{\cong} & \text{DG}(\beta) & \rightarrow & 0
\end{array}
$$

We identify $f = \beta^\vee \circ \pi^*$ with the left vertical arrow.

Applying the exact sequence (2.0.3) from [3] to $\beta_\sigma : \mathbb{Z}^{I_{\sigma}} \rightarrow N/N_{\sigma}$, we get

$$
(N/N_{\sigma})^* \rightarrow (\mathbb{Z}^{I_{\sigma}})^* \rightarrow \text{DG}(\beta_\sigma) \rightarrow \text{Ext}^1_{\mathbb{Z}}(N/N_{\sigma}, \mathbb{Z}) \rightarrow 0
$$
whence $\text{coker}(f) \cong \text{Ext}^1_{\mathbb{Z}}(N/N_{\sigma}, \mathbb{Z}) \cong \text{Tor}(N/N_{\sigma})$, which completes the proof.

**Example 3.3.** Consider the toric DM stack from Example 2.3. We compute the isotropy for the points of the form $z = (0, a, 0)$ and $w = (0, a, b)$ in $Z_\Sigma$ with $a, b \neq 0$. Since $I_z = \{1, 3\}$, then $N_z \subset N$ is the subgroup generated by $(0, 5)$ and $(-2, -2)$. Therefore, $N/N_z \cong \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and $\text{Stab}(z) \cong \text{Tor}(N/N_z) \cong \mathbb{Z}/10$.

Since $I_w = \{1\}$, then $N_w$ is the subgroup generated by $(-2, -2)$, and $N/N_w \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Therefore, $\text{Stab}(w) \cong \text{Tor}(N/N_w) \cong \mathbb{Z}/2$. Note that the isotropy for $w$ can simply be read off from the corresponding facet label (see Figure 2.1) in this case. For higher dimensional cones, a more detailed analysis is required—e.g. see Section 5.3.

**Remark 3.4.** The proof of Theorem 3.2 does not show explicitly how $\text{Tor}(N/N_z)$ may be viewed as a subgroup of $G$. Below we define a map $\text{Tor}(N/N_z) \rightarrow G$ (3.2) that models the inclusion of the isotropy group $\text{Stab}(z)$. See Proposition 3.16 for a more direct approach in the case that $N$ is free.
Remark 3.6. In practice, the isotropy groups in Theorem 3.2 may be computed using the Smith Normal Form of a matrix with non-zero entries $a_1, a_2, \ldots, a_{\min(d+\ell,|I_\sigma|+\ell)}$ appearing on the diagonal, satisfying the divisibility relations.
a_j |a_{j+1}. The entries a_j \neq 1 give the orders of the cyclic subgroups appearing in the invariant factor decomposition of \( \Gamma_\sigma \).

**Example 3.7.** To illustrate Remark 3.6, we consider the following example. Let \( N = \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \), and let \( \Sigma \) be the fan in Figure 3.1 with ray generators \((1, 0), (0, 1), (0, -1)\) and \((-1, -2)\). Set \( \beta : \mathbb{Z}^4 \rightarrow N \) to be \( \beta(x, y, z, w) = (-2x + 3z, -4x + 6y - 2w, x + y + z + w \mod 2) \).

Fix the resolution \( 0 \rightarrow \mathbb{Z} \xrightarrow{Q} \mathbb{Z}^3 \rightarrow N \rightarrow 0 \) with \( Q = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}^T \), and choose \( B : \mathbb{Z}^4 \rightarrow \mathbb{Z}^3 \) so that \( [BQ] = \begin{bmatrix} -2 & 0 & 3 & 0 & 0 \\ -4 & 6 & 0 & -2 & 0 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix} \).

Let \( \sigma \) be the cone in \( \Sigma \) generated by \( \rho_1 \) and \( \rho_2 \). Then since

\[
[B_\sigma Q] = \begin{bmatrix} -2 & 0 & 0 \\ -4 & 6 & 0 \\ 1 & 1 & 2 \end{bmatrix}
\]

we see that the isotropy group \( \Gamma_\sigma \cong \mu_2 \times \mu_{12} \).

Notice that if \( \sigma \) is maximal, then \( N_\sigma \) has the same rank as \( N \). The following Corollary is immediate. (Compare with [3, Prop 4.3].)

**Corollary 3.8.** Let \( (N, \Sigma, \beta) \) be a stacky fan, and \( [Z_\Sigma/G] \) its corresponding toric DM stack. If \( z = (z_1, \ldots, z_n) \in Z_\Sigma \) has \( d = \text{rank } N \) vanishing coordinates, then \( \sigma = \sigma_z \) is maximal and \( \text{Stab}(z) \cong N/N_z \).

**3.2. On the connected component of \( G \).** Let \( G_0 \subset G \) denote the connected component of the identity element. By construction, the short exact sequence \( 0 \rightarrow \text{Tor}(DG(\beta)) \rightarrow DG(\beta) \rightarrow DG(\beta)/\text{Tor}(DG(\beta)) \rightarrow 0 \) dualizes to give the short exact sequence \( 1 \rightarrow G_0 \rightarrow G \rightarrow G/G_0 \rightarrow 1 \). Below, we identify \( G_0 \) and the quotient \( G/G_0 \) in terms of stacky fan data.

**Lemma 3.9.** Let \( (\Sigma, N, \beta) \) be a stacky fan, and \( G = \text{Hom}(DG(\beta), \mathbb{T}) \). If \( G_0 \subset G \) denotes the connected component of the identity element, then \( G/G_0 \cong \text{coker } \beta \). In particular, \( G \) is connected if and only if \( \beta \) is surjective.

**Proof.** As \( G = \text{Hom}(DG(\beta), \mathbb{T}) \), the group of connected components \( G/G_0 \) is \( \text{Hom}(\text{Tor}(DG(\beta)), \mathbb{T}) \), where \( \text{Tor}(DG(\beta)) \) denotes torsion submodule of \( DG(\beta) \). This torsion submodule may be identified by computing \( \text{Ext}_1^Z(DG(\beta), \mathbb{Z}) \), which we show next is isomorphic to \( \text{coker } \beta \).

Note that \( \text{coker } \beta \cong \text{coker } [BQ] \), where \( [BQ] : \mathbb{Z}^{n+\ell} \rightarrow \mathbb{Z}^{d+\ell} \) is the homomorphism described in Section 2. This can be seen by verifying that the surjective composition \( \mathbb{Z}^{d+\ell} \rightarrow N \rightarrow \text{coker } \beta \) has kernel \( \text{im } [BQ] \). Therefore, it suffices to show that \( \text{Ext}_1^Z(DG(\beta), \mathbb{Z}) \cong \text{coker } [BQ] \).

To that end, consider the free resolution that defines \( DG(\beta) \):

\[
0 \longrightarrow (\mathbb{Z}^{d+\ell})^* \xrightarrow{[BQ]^*} (\mathbb{Z}^{n+\ell})^* \longrightarrow DG(\beta) \longrightarrow 0
\]

(Note that since \( \text{coker } [BQ] \cong \text{coker } \beta \) is assumed to be finite, then \( [BQ]^* \) is injective.) Applying \( \text{Hom}(-, \mathbb{Z}) \) and taking homology shows that \( \text{Ext}_1^Z(DG(\beta), \mathbb{Z}) \cong \text{coker } [BQ] \), as required. \( \square \)
Remark 3.10. By [12, Cor. 3.5], Lemma 3.9 shows that the (stacky) fundamental group of \( \mathcal{X}(N, \Sigma, \beta) \) is therefore isomorphic to \( \text{coker}\beta \). (Cf. [3, Section 3.2])

Remark 3.11. As in the proof of the preceding Lemma, \( \text{coker}\beta \cong \text{coker}[BQ] \) and thus the invariant factor decomposition of \( G/G_0 \) may be determined immediately from the Smith Normal Form of the matrix \([BQ]\). For example, the reader may verify that \( G/G_0 \cong \mathbb{Z}/2\mathbb{Z} \) in Example 3.7.

Given a stacky fan \((N, \Sigma, \beta)\), we may model the quotient stack \([Z_\Sigma/G_0]\) as a toric DM stack of a related stacky fan \((N_0, \Sigma_0, \beta_0)\), defined as follows. Consider the submodule \( N_0 = \text{im}(\beta) \subset N \), and let \( \beta_0 : \mathbb{Z}^n \to N_0 \) be given by \( \beta \) with its restricted codomain. Finally, we let \( \Sigma_0 \) be the fan in \( N_0 \otimes \mathbb{R} \) corresponding to \( \Sigma \) defined by the natural isomorphism \( N_0 \otimes \mathbb{R} \cong N \otimes \mathbb{R} \) induced by the inclusion \( N_0 \subset N \) (where we have used the fact that \( \beta \) has finite cokernel).

Proposition 3.12. Let \((N, \Sigma, \beta)\) be a stacky fan and \( \mathcal{X}(N, \Sigma, \beta) = [Z_\Sigma/G] \) its corresponding toric DM stack. The triple \((N_0, \Sigma_0, \beta_0)\) defined above as a stacky fan whose corresponding toric DM stack is \( \mathcal{X}(N_0, \Sigma_0, \beta_0) = [Z_\Sigma/G_0] \).

Proof. It is straightforward to verify that \((N_0, \Sigma_0, \beta_0)\) defines a stacky fan. Since \( \Sigma_0 \) and \( \Sigma \) contain the same combinatorial information, \( J(\Sigma) = J(\Sigma_0) \), and thus \( Z_{\Sigma_0} = Z_\Sigma \).

It remains to verify that the group action on \( Z_\Sigma \) determined by the stacky fan \((N_0, \Sigma_0, \beta_0)\) is the same as that obtained by the restriction of the action of \( G \) to the connected component of the identity \( G_0 \) on \( Z_\Sigma \).

To see this, we apply Lemma 2.3 of [3] to the following diagram of short exact sequences.

\[
\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}^n & \to & \mathbb{Z}^n & \to & 0 & \to & 0 \\
\downarrow{\beta_0} & & \downarrow{\beta} & & \downarrow & & \downarrow & & \\
0 & \to & N_0 & \to & N & \to & \text{coker}\beta & \to & 0
\end{array}
\]

Since \( \text{DG}([0] \to \text{coker}(\beta)) \) can be naturally identified with \( \text{coker}(\beta) \), we obtain the diagram below with exact rows.

\[
\begin{array}{ccccccccc}
0 & \to & 0 & \to & (\mathbb{Z}^n)^* & \to & (\mathbb{Z}^n)^* & \to & 0 \\
\downarrow{\beta} & & \downarrow & & \downarrow{\beta(0)} & & \downarrow{\beta(0)} & & \\
0 & \to & \text{coker}\beta & \to & \text{DG}(\beta) & \to & \text{DG}(\beta_0) & \to & 0
\end{array}
\]

This shows that \( \text{DG}(\beta_0) \) and \( \text{DG}(\beta) \) have the same rank and thus \( \text{Hom}(\text{DG}(\beta_0), \mathbb{T}) \) and \( G \) have the same dimension. To show that \( \text{Hom}(\text{DG}(\beta_0), \mathbb{T}) \) is connected, it suffices to verify that \( \text{DG}(\beta_0) \) is torsion free, which follows from Lemma 3.9.

Thus \( \text{Hom}(\text{DG}(\beta_0), \mathbb{T}) = G_0 \), the connected component of the identity in \( G \). Lastly, note that the \( G_0 \) action on \( Z_\Sigma \) is induced by the composition \( (\mathbb{Z}^n)^* \to \text{DG}(\beta) \to \text{DG}(\beta_0) \) so that \( G_0 \) acts via its inclusion into \( G \), as desired. \( \square \)

Identifying the connected component of \( G \) is useful in detecting whether or not a toric DM stack \( \mathcal{X}(N, \Sigma, \beta) \) is equivalent to a global quotient of a finite group action — that is, equivalent to a quotient stack \([M/\Lambda]\) where \( \Lambda \) is a finite group acting on a manifold \( M \). Indeed, global quotients among toric DM stacks are completely characterized by the requirement that the restriction of the \( G \)-action on \( Z_\Sigma \) to the connected component of the identity \( G_0 \subset G \) is a free action (see [12]). Moreover, in this case, one may choose the finite group to be \( \Lambda = G/G_0 \), acting on the quotient \( M = Z_\Sigma/G_0 \), which is indeed a manifold, provided \( G_0 \) acts freely on \( Z_\Sigma \).

This motivates the following Corollary, which follows readily from Proposition 3.12 and Theorem 3.2 (cf. Remark 3.1).

Corollary 3.13. Let \((N, \Sigma, \beta)\) be a stacky fan, and let \( N_0 = \text{im}(\beta) \).

1. \( N_\sigma = N_0 \) for all maximal cones \( \sigma \in \Sigma \) if and only if \( G_0 \) acts freely on \( Z_\Sigma \).
2. \( N_\sigma = N \) for all maximal cones \( \sigma \in \Sigma \) if and only if \( G \) is connected and acts freely on \( Z_\Sigma \).

As mentioned above, the first item in Corollary 3.13 gives a criterion for detecting global quotients among toric DM stacks. In particular, it shows that if \( N_\sigma = N_0 \) for all maximal cones, the toric DM stack \( \mathcal{X}(N_0, \Sigma_0, \beta_0) = [Z_\Sigma/G_0] \) is in fact a smooth manifold. As shown in [12], this exhibits the toric DM stack \( \mathcal{X}(N, \Sigma, \beta) \) as a global quotient in this case — more precisely, when \( N_\sigma = N_0 \) for all maximal cones \( \sigma \in \Sigma \), there is a natural equivalence of stacks \( \mathcal{X}(N, \Sigma, \beta) \cong [(Z_\Sigma/G_0)/\Lambda] \) where \( \Lambda = G/G_0 \).
Example 3.14. Consider the toric DM stack \([Z_S/G]\) from Example 2.5 the labelled line segment with labels \(r\) and \(s\). Since \(N_0 = g\mathbb{Z}\) where \(g = \gcd(r, s)\), \([Z_S/G]\) is a global quotient if and only if \(r = g = s\). In that case, the \(G \cong T \times \mu_r\)-action on \(Z_S\) is induced by the homomorphism \(G \to T^2\), \((t, \xi^r_1) \mapsto (t, t\xi^r_2)\). Therefore, \(Z_S/G = P_1\), and the residual \(\Lambda = G/G_0 \cong \mu_r\)-action may be written in homogeneous coordinates as \(\xi^k_r \cdot [z_0 : z_1] = [z_0 : \xi^k_r z_1]\), and \([Z_S/G] \cong \mathbb{P}^1/\mu_r\).

Remark 3.15. Analogous to the map constructed in Proposition 3.5 modelling the inclusion of the isotropy groups, we may also model the quotient map \(G \to G/G_0\) (cf. Lemma 3.9) more concretely as in the discussion that follows. See Proposition 3.18 for a more direct approach in the case that \(N\) is free.

Recall that the quotient \(G \to G/G_0\) is the map \(\text{Hom}(DG(\beta), \mathbb{T}) \to \text{Hom}(\text{Tor}(DG(\beta)), \mathbb{T})\), induced by the inclusion of the torsion submodule \(\text{Tor}(DG(\beta)) \to DG(\beta)\). We shall describe an explicit isomorphism \(\text{Hom}(\text{Tor}(DG(\beta)), \mathbb{T}) \cong \text{coker}(BQ)\).

Consider the diagram of short exact sequences,

\[
\begin{array}{ccc}
0 & \longrightarrow & (\mathbb{Z}^{d+1})^* \\
\| & & \| \\
0 & \longrightarrow & (\mathbb{Z}^{d+1})^* \rightarrow \text{Tor}(DG(\beta)) \rightarrow P \rightarrow \text{DG}(\beta) \rightarrow 0 \\
\| & & | \\
& & |\beta \downarrow \\
& & | \\
& & |(\mathbb{Z}^{d+1})^* \\
& & | \\
& & |(\mathbb{Z}^{d+1})^* \rightarrow \text{Tor}(DG(\beta)) \rightarrow P \rightarrow \text{DG}(\beta) \rightarrow 0 \\
\| & & | \\
& & |BQ \\
& & | \\
& & |BQ \\
& & | \\
& & |BQ \\
\end{array}
\]

obtained by restriction (pullback) to \(\text{Tor}(DG(\beta))\). Given a homomorphism \(\varphi : \text{Tor}(DG(\beta)) \to \mathbb{T} = \mathbb{C}/\mathbb{Z}\), choose a homomorphism \(\tilde{\varphi} : P \to \mathbb{C}\) covering \(\varphi\). The restriction \(\tilde{\varphi}|_{(\mathbb{Z}^{d+1})^*}\) is integer-valued, and hence defines a vector \(v_\varphi \in \mathbb{Z}^{d+1}\) by duality (i.e. \(\tilde{\varphi}|_{(\mathbb{Z}^{d+1})^*}(u) = u v_\varphi\) for all row vectors \(u \in (\mathbb{Z}^{d+1})^*\).

Any two covers \(\tilde{\varphi}_1, \tilde{\varphi}_2\) of \(\varphi\) differ by a homomorphism \(\alpha : P \to \mathbb{Z}\), which by restriction to \((\mathbb{Z}^{d+1})^*\) defines a vector \(v_\alpha \in \mathbb{Z}^{d+1}\) that is the difference between \(v_{\tilde{\varphi}_1}\) and \(v_{\tilde{\varphi}_2}\). We check that \(v_\alpha\) is in the image of \([BQ] : \mathbb{Z}^{d+1} \to \mathbb{Z}^{d+1}\), and hence the correspondence \(\varphi \mapsto v_\varphi\) descends to a well-defined homomorphism \(\text{Hom}(\text{Tor}(DG(\beta)), \mathbb{T}) \to \text{coker}(BQ)\). Applying the Snake Lemma to the diagram of short exact sequences above shows that the quotient \((\mathbb{Z}^{d+1})^*/P \cong \text{DG}(\beta)/\text{Tor}(DG(\beta))\) is free; therefore, \(\alpha : P \to \mathbb{Z}\) may be extended to a homomorphism \(\hat{\alpha} : (\mathbb{Z}^{d+1})^* \to \mathbb{Z}\), which by duality defines a vector \(w_{\hat{\alpha}}\in \mathbb{Z}^{d+1}\). Then for any row vector \(u \in (\mathbb{Z}^{d+1})^*\), we have

\[
\alpha(u) = \hat{\alpha}(u[BQ]) = u[BQ]w_{\hat{\alpha}},
\]

and hence \(v_\alpha = [BQ]w_{\hat{\alpha}}\), as required.

An argument similar to the above shows that \(\text{Hom}(\text{Tor}(DG(\beta)), \mathbb{T}) \to \text{coker}(BQ)\) is injective, which implies it is then also surjective since these groups are already known to be abstractly isomorphic finite groups.

3.3. Isotropy and labelled polytopes. Let \(N\) be a free \(\mathbb{Z}\)-module of rank \(d\). Viewing stacky polytopes as labelled polytopes, as in Section 2.2 we now give a more direct description of the maps modelling the inclusion of isotropy groups \(\text{Stab}(z) \hookrightarrow G\) (see Theorem 3.2 and Proposition 3.5), and the quotient map \(G \to G/G_0\) (see Lemma 3.9 and the discussion following Remark 3.15).

Let \((\Delta, \{m_i\}_{i=1}^n)\) in \((N \otimes \mathbb{R})^* \cong (\mathbb{R}^d)^*\) be a labelled polytope. Let \(\beta : \mathbb{Z}^n \to N\) be given by \(\beta(e_i) = m_i\nu_i\), where \(m_i\nu_i\) are the weighted normals to the facets of \(\Delta\), and consider the resulting homomorphism \(\beta : (S^1)^n \to (S^1)^d\) with kernel \(K_D = \ker \beta\). Let \(\exp : \mathbb{Z}^n \otimes \mathbb{R} \to (S^1)^n\) denote the exponential map.

We begin with an explicit description of the stabilizers of Theorem 3.2

Proposition 3.16. Let \((\Delta, \{m_i\}_{i=1}^n)\) be a labelled polytope in \((N \otimes \mathbb{R})^*\) with primitive inward pointing facet normals \(\nu_1 \otimes 1, \ldots, \nu_n \otimes 1\) and \([\mu^{-1}(\tau)/K_D]\) its corresponding toric DM stack. For \(z = (z_1, \ldots, z_n)\) in \(\mu^{-1}(\tau) \subset \mathbb{C}^n\), let \(N_z \subset N\) denote the submodule generated by \(\{m_i\nu_i \ | \ z_i = 0\}\). Then the canonical map

\[
\text{Tor}(N/N_z) \to K_D, \quad x + N_z \mapsto \exp(y \otimes \frac{1}{m}),
\]

is an isomorphism onto its image \(\text{Stab}(z)\), where \(y\) is the unique element in \(\text{span}\{e_i \ | \ z_i = 0\} \subset \mathbb{Z}^n\) satisfying \(\beta(y) = mx\) for some smallest positive integer \(m\).
Proof. Let $\sigma$ be a cone in the normal fan $\Sigma(\Delta)$ with $I_\sigma = I_z$. Analogous to Section 3.1 we see that $\text{Stab}(z)$ is given by the kernel $\Gamma_\sigma$ of the composition (Cf. (3.1))

$$K_D \to (S^1)^n \to (S^1)^{|I_\sigma|}.$$ 

We exhibit an isomorphism $\psi : \text{Tor}(N/N_\sigma) \to \Gamma_\sigma$, for any cone $\sigma$ of the normal fan $\Sigma(\Delta)$.

Define $\psi : \text{Tor}(N/N_\sigma) \to \Gamma_\sigma$ as follows. Given $x + N_\sigma$ of order $m$, we may find a unique (see the Claim in Theorem 4.2) $y \in \mathbb{Z}^{|I_\sigma|} \subset \mathbb{Z}^n$ with $\beta(y) = mx$. Consider $y \otimes \frac{1}{m} \in \mathbb{Z}^n \otimes \mathbb{R}$. Since $\beta(\exp(y \otimes \frac{1}{m})) = \exp(x \otimes 1) = 1$, $\exp(y \otimes \frac{1}{m}) \in K_D$. And it is straightforward to see that $q_\sigma(\exp(y \otimes \frac{1}{m})) = 1$, where $q_\sigma : (S^1)^n \to (S^1)^{|I_\sigma|}$ is the projection onto the “non-trivial” components; therefore, $\exp(y \otimes \frac{1}{m}) \in \Gamma_\sigma$ and we may set $\psi(x + N_\sigma) = \exp(y \otimes \frac{1}{m})$.

The map $\psi$ is well defined. Indeed, suppose we choose a different representative $x' + N_\sigma$ for $x + N_\sigma$, and let $y'$ denote the corresponding element in $\mathbb{Z}^{|I_\sigma|}$ with $\beta(y') = mx'$. Then there exists a (unique) $\eta \in \mathbb{Z}^{|I_\sigma|}$ satisfying $\beta(\eta) = x - x'$ and hence $\beta_\sigma(y - y') = \beta_\sigma(m\eta)$, which shows $y - y' = m\eta$. Therefore, $\exp((y - y') \otimes \frac{1}{m}) = \exp(\eta \otimes 1) = 1$.

We check that $\psi$ is an isomorphism. To check injectivity, suppose $\psi(x + N_\sigma) = \exp(y \otimes \frac{1}{m}) = 1$, where $y$ and $m$ are as above. Then $y \otimes \frac{1}{m}$ lies in the image of $\mathbb{Z}^{|I_\sigma|} \to \mathbb{Z}^n \otimes \mathbb{R}$, which implies that $m = 1$ so that $x \in N_\sigma$. To check surjectivity, suppose $\gamma \in \Gamma_\sigma \subset (S^1)^{|I_\sigma|}$. Choose an element $v \in \mathbb{Z}^{|I_\sigma|} \otimes \mathbb{R}$ with $\exp(v) = \gamma$ and consider $(\beta_\sigma \otimes 1)(v) \in N_{\mathbb{R}}$. Since $\exp((\beta_\sigma \otimes 1)(v)) = 1$, there is an element $x \in N$ such that $x \otimes 1 = \beta_\sigma(v)$. We now check that $\psi(x + N_\sigma) = \gamma$. Let $m$ be the order of $\gamma$. Then $\exp(mv) = 1$ and hence $mv = \xi \otimes 1$ for some $\xi \in \mathbb{Z}^{|I_\sigma|}$, and $\beta_\sigma(\xi) \otimes 1 = \beta_\sigma(mv) = mx \otimes 1$ Therefore, $\beta_\sigma(\xi) = mx$, and $\exp(\xi \otimes \frac{1}{m}) = \exp(v) = \gamma$, as required.

Remark 3.17. The isomorphism given in Proposition 3.16 is compatible with the one given in Proposition 3.5 for general $\mathbb{Z}$-modules $N$. Indeed, if $N$ is assumed to be free, then we may choose an isomorphism $\tilde{q} : \mathbb{Z}^d \to N$, and take $R = \tilde{q}^{-1} \circ \beta_\sigma : \mathbb{Z}^d \to \mathbb{Z}^d$ in (3.3). With this choice, it can be shown that the following diagram commutes.

$$\begin{array}{ccc}
K_D & \xrightarrow{\beta_\sigma} & (S^1)^n \cong \mathbb{Z}^n \otimes S^1 \\
\downarrow_{\psi} & & \downarrow_{\exp} \\
\text{Tor}(N/N_\sigma) & \xrightarrow{\gamma_\sigma} & \text{Hom}((\mathbb{Z}^n)^*, S^1) \\
\end{array}$$

Next, we give an explicit description of the component group $K_D/(K_D)_0$, which models Lemma 3.9 which shows that $\text{coker} \beta \cong K_D/(K_D)_0$.

Define a homomorphism $\varphi : N \to K_D/(K_D)_0$ as follows. For $x \in N$, choose $y \in \mathbb{Z}^n \otimes \mathbb{R}$ with $\beta_\sigma(y) = x \otimes 1$ and consider $\exp(y) \in (S^1)^n$. Then $\beta(\exp(y)) = 1$, and thus $\exp(y) \in K_D$. Set $\varphi(x) = \exp(y)(K_D)_0$. The map $\varphi$ is well-defined since $\ker \beta_\sigma$ is the Lie algebra of $K_D$, which exponentiates onto $(K_D)_0$.

The homomorphism $\varphi$ has kernel $\text{im}(\beta)$. Indeed, if $\varphi(x) = \exp(y) \in (K_D)_0$, then there is an element $\xi \in \ker \beta_\sigma$ with $\exp(y - \xi) = 1$, and thus an integer vector $a \in \mathbb{Z}^n$ with $y - \xi = a \otimes 1$. Since $N \to N \otimes \mathbb{R}$ is injective ($N$ is free!), $x \otimes 1 = \beta_\sigma(y) = \beta_\sigma(a \otimes 1) = \beta(a) \otimes 1$ implies $\beta(a) = x$, and hence $\ker \varphi \subset \text{im}(\beta)$. Finally, if $x = \beta(a)$ for some $a \in \mathbb{Z}^n$, then we may choose $y = a \otimes 1$ to compute $\varphi(x) = \exp(y)(K_D)_0$ to see that $\exp(y) = 1$, showing $\ker \beta \subset \ker \varphi$. Hence we obtain the following:

Proposition 3.18. Suppose $N$ is torsion free. The map $\varphi : N \to K_D/(K_D)_0$ defined above descends to an isomorphism $\tilde{\varphi} : \text{coker} \beta \cong K_D/(K_D)_0$.

Remark 3.19. The above arguments are essentially applications of the Snake Lemma (and parts of its proof). For example, Proposition 3.18 above follows part of the proof of the Snake Lemma for the diagram of short exact sequences,

$$\begin{array}{cccc}
0 & \xrightarrow{} & \ker \beta_\sigma & \xrightarrow{} & \mathbb{Z}^n \otimes \mathbb{R} \xrightarrow{\beta_\sigma} N \otimes \mathbb{R} & \xrightarrow{} & 0 \\
1 & \xrightarrow{} & K_D & \xrightarrow{} & (S^1)^n \xrightarrow{} & (S^1)^d & \xrightarrow{} & 1 \\
\end{array}$$

and the map $\varphi$ is the connecting homomorphism.
4. WEIGHTED PROJECTIVE SPACES

In this section we interpret the results in Section 3 for an important class of toric DM stacks known as weighted projective spaces. In Section 4.1 we identify those toric DM stacks that are equivalent to weighted projective spaces in terms of their stacky fan data. Labelled sheared simplices are introduced in Section 4.2 as a natural family of stacky polytopes to study, and those labelled sheared simplices that give rise to weighted projective spaces are determined.

4.1. Which stacky polytopes correspond to weighted projective spaces? Here we identify those stacky polytopes corresponding to weighted projective spaces.

Definition 4.1. For positive integers \(b_0, \ldots, b_d\), let \(T\) act on \(\mathbb{C}^{d+1} \setminus \{0\}\) by \(t \cdot (z_0, \ldots, z_d) = (t^{b_0} z_0, \ldots, t^{b_d} z_d)\). The resulting quotient stack \([\mathbb{C}^{d+1} \setminus \{0\}] / T\) is called a weighted projective space, denoted \(\mathbb{P}(b_0, \ldots, b_d)\).

In particular, a weighted projective space is a quotient by a connected one-dimensional Abelian Lie group action, but the action need not be effective.

Recall that to each \(\mathbb{P}(b_0, \ldots, b_d)\) there is associated stacky polytope \((N, \Delta, \beta)\) with \(\text{DG}(\beta) \cong \mathbb{Z}\) (see Example 21) whose associated toric DM stack is equivalent to \(\mathbb{P}(b_0, \ldots, b_d)\). Proposition 4.2 below shows that the toric DM stack corresponding to any stacky polytope satisfying \(\text{DG}(\beta) \cong \mathbb{Z}\) results in a weighted projective space.

Proposition 4.2. Let \((N, \Delta, \beta)\) be a stacky polytope, and let \(\Sigma(\Delta)\) be the dual fan to \(\Delta\). The associated toric DM stack \(X(N, \Sigma(\Delta), \beta)\) is a weighted projective space \(\mathbb{P}(b_0, \ldots, b_d)\) if and only if \(\text{DG}(\beta) \cong \mathbb{Z}\). In this case, the polytope \(\Delta\) is a simplex, and the weights are determined by the condition that \((b_0, \ldots, b_d)\) generates \(\mathbb{Z}^{d+1}\).

Remark 4.3. Let \((N, \Delta, \beta)\) be a stacky polytope satisfying the condition \(\text{DG}(\beta) \cong \mathbb{Z}\). By Lemma 3.9 it follows that \(\beta\) must be surjective. Additionally, the torsion submodule \(\text{Tor}(N)\) of \(N\) must be cyclic, and the proof of Proposition 4.2 shows that the order of \(\text{Tor}(N)\) is \(g = \gcd(b_0, \ldots, b_d)\).

Proof of Proposition 4.2. Since \(\text{DG}(\beta) \cong \mathbb{Z}\), the homomorphism \(\beta\) of the stacky polytope \((N, \Delta, \beta)\) must have domain \(\mathbb{Z}^{d+1}\) where (as usual) \(\text{rank } N = d\). That is the polytope \(\Delta\) has \(d+1\) facets, and is therefore a simplex. Hence \(V(J(\Sigma(\Delta))) = \{0\}\) and \(Z_{\Sigma(\Delta)} = \mathbb{C}^{d+1} \setminus \{0\}\).

We determine the \(G\)-action on \(\mathbb{C}^{d+1} \setminus \{0\}\). Recall that the action is determined by applying \(\text{Hom}(-, T)\) to the map \(\beta^* : (\mathbb{Z}^{d+1})^* \rightarrow \text{DG}(\beta)\), obtaining a homomorphism \(G \rightarrow \mathbb{T}^{d+1}\). We set out to determine \(\beta^*\).

We first show that \(\text{DG}(\beta) \cong (\ker \beta)^*\). Let \(0 \rightarrow \mathbb{Z}^t \xrightarrow{Q} \mathbb{Z}^{d+\ell} \rightarrow N \rightarrow 0\) be a free resolution of \(N\), and let \(B : \mathbb{Z}^{d+1} \rightarrow \mathbb{Z}^{d+\ell}\) denote a lift of \(\beta\). Then \(\text{DG}(\beta) = \text{coker}(BQ)^*\). Similar to the proof of Proposition 2.2 in [3], an application of the snake lemma to the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \mathbb{Z}^t \\
\downarrow & & \downarrow \\
\mathbb{Z}^t \rightarrow \mathbb{Z}^{d+\ell} & \rightarrow & \mathbb{Z}^{d+1} \\
\downarrow & \downarrow & \downarrow \\
0 & \rightarrow & \mathbb{Z}^{d+\ell} \rightarrow [BQ] \rightarrow N \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

shows that \([BQ]\) and \(\beta\) have isomorphic cokernels, which are trivial by assumption, as well as isomorphic kernels. It follows that \(\text{coker}(BQ)^* \cong (\ker \beta)^*\).

Since \(\beta : \mathbb{Z}^{d+1} \rightarrow N\) has finite cokernel, it follows there is an exact sequence

\[
0 \rightarrow (N^*) \xrightarrow{\beta^*} (\mathbb{Z}^{d+1})^* \xrightarrow{\beta^*} (\ker \beta)^* \rightarrow \text{Ext}^1(N, \mathbb{Z}) = 0
\]

obtained by the identification \(\text{DG}(\beta) \cong (\ker \beta)^*\), where exactness at \(\text{Ext}^1(N, \mathbb{Z})\) follows from the fact that \(\text{Ext}^1(N, \mathbb{Z})\) is the first right derived functor of \(\text{Hom}(-, \mathbb{Z})\) (and \(N\) is finitely generated).

Since \(\Delta\) is a simplex, we can find positive integers \(b_0, \ldots, b_d\) such that \(\sum_{j=0}^d b_j(e_j) \otimes 1 = 0\), where \(\{e_0, \ldots, e_d\}\) is the standard basis for \(\mathbb{Z}^{d+1}\). Without loss of generality, assume that \(b = (b_0, \ldots, b_d)\) generates \(\ker \beta\), which we now identify with \(\mathbb{Z}\) according to \(b \mapsto 1\). The resulting identification \((\ker \beta)^* \cong \mathbb{Z}\) in (4.1) gives \(\beta^*\) the matrix representation \([b_0 \cdots b_d]\). It follows that \(\beta^*\) induces the homomorphism \(T \rightarrow \mathbb{T}^{d+1}\), \(t \mapsto (t^{b_0}, \ldots, t^{b_d})\), which completes the proof. \(\square\)
4.2. Labelled sheared simplices and weighted projective spaces. We now introduce labelled sheared simplices. Applying the results of the previous section, an analysis of the isotropy groups arising in toric DM stacks constructed from labelled sheared simplices follows.

**Definition 4.4.** Let \( a = (a_1, \ldots, a_d) \in \mathbb{Z}^d \) be a primitive vector in the positive orthant and \( \{\epsilon_i\} \) the standard basis of \( \mathbb{R}^d \). The **sheared simplex** \( \Delta(a) \) is the convex hull of the origin together with the points \( \frac{\text{lcm}(a)}{a_j}\epsilon_j \) \( (j = 1, \ldots, d) \) in \( \mathbb{R}^d \). Given positive integers \( m_0, \ldots, m_d \), we define a **labelled sheared simplex** as the stacky polytope \( (\mathbb{Z}^d, \Delta(a), \beta) \), where the homomorphism \( \beta : \mathbb{Z}^{d+1} \to \mathbb{Z}^d \) is given by \( \beta(e_0) = -m_0 a \) and \( \beta(e_j) = m_j \epsilon_j \) \( (j = 1, \ldots, d) \), where \( e_0, \ldots, e_d \) denote the standard basis vectors for \( \mathbb{Z}^{d+1} \).

The results in this section depend only on the dual fan \( \Sigma(\Delta) \), which consists of all cones \( \sigma \) generated by any subset of the ray generators \( \{a, \epsilon_1, \ldots, \epsilon_d\} \). Proposition 4.7 below characterizes those labelled sheared simplices whose corresponding toric DM stacks are weighted projective spaces. By Proposition 4.2, this involves translating the condition \( \text{DG}(\beta) \cong \mathbb{Z} \) in terms of the labels \( \{m_0, \ldots, m_d\} \) and the vector \( a \). Since \( \text{rank} \text{DG}(\beta) = 1 \), \( \text{DG}(\beta) \cong \mathbb{Z} \) if and only if \( G \) is connected; therefore, we begin with the following lemma.

**Lemma 4.5.** Let \( (\mathbb{Z}^d, \Delta(a), \beta) \) be a labelled sheared simplex with labels \( \{m_0, \ldots, m_d\} \), and let \( [Z_\Sigma/G] \) be its associated toric DM stack. If \( G_0 \subset G \) denotes the connected component of the identity element, then

\[
G/G_0 \cong \left[ \bigoplus_{i=0}^d \mathbb{Z}/m_i\mathbb{Z} \right] / \langle \{1 \text{ mod } m_0, a_1 \text{ mod } m_1, \ldots, a_d \text{ mod } m_d\} \rangle
\]

**Example 4.6.** Consider the toric DM stack from Example 4.5, the labelled line segment with labels \( r \) and \( s \), where it was seen directly that \( G \cong \mathbb{T} \times \mu_g \), where \( g = \gcd(r, s) \) and \( \mu_g \subset \mathbb{T} \) is the cyclic group of \( g \)-th roots of unity. Hence \( G/G_0 \cong \mu_g \). This can be seen from the Lemma above as well, since \( G/G_0 \cong (\mathbb{Z}/r\mathbb{Z} \oplus \mathbb{Z}/s\mathbb{Z})/H \) where \( H \) is the subgroup generated by \( (1, 1) \).

**Proof of Lemma 4.5.** By Lemma 3.3, \( G/G_0 \cong \ker \beta \), which we now compute. Consider the commutative diagram of short exact sequences

\[
0 \to \mathbb{Z}^d \to \mathbb{Z}^{d+1} \xrightarrow{\times (m_0, \ldots, m_d)} \mathbb{Z}^{d+1} \to \bigoplus_{i=0}^d \mathbb{Z}/m_i\mathbb{Z} \to 0
\]

where \( \beta' \) may be written as the matrix

\[
\begin{bmatrix}
-a_1 & 1 & 0 \\
\vdots & \ddots & \vdots \\
-a_d & 0 & 1
\end{bmatrix}
\]

Applying the Snake Lemma gives the exact sequence

\[
0 \to \ker \beta \to \ker \beta' \to \bigoplus_{i=0}^d \mathbb{Z}/m_i\mathbb{Z} \to \ker \beta \to 0.
\]

Since \( \ker \beta' \cong \mathbb{Z} \) is generated by \( (1, a_1, \ldots, a_d) \in \mathbb{Z}^{d+1} \), the second map in the sequence above sends the generator to \( (1 \text{ mod } m_0, a_1 \text{ mod } m_1, \ldots, a_d \text{ mod } m_d) \). The result follows.

**Proposition 4.7.** Let \( (\mathbb{Z}^d, \Delta(a), \beta) \) be a labelled sheared simplex with labels \( \{m_0, \ldots, m_d\} \) and \( a = (a_1, \ldots, a_d) \). Let \( M = \text{mom}_1 \cdots m_d \). The following conditions are equivalent:

1. \( \text{DG}(\beta) \cong \mathbb{Z} \)
2. \( \gcd\left( \frac{M}{m_0 m_1 \cdots m_d} \right) = 1 \)
3. \( \gcd(m_i, m_j) = 1 \) for all \( i \neq j \) in \( \{0, \ldots, d\} \), and \( \gcd(a_i, m_i) = 1 \) for all \( i \) in \( \{1, \ldots, d\} \).

In addition, if one of the above holds, then \( \mathcal{X}(\mathbb{Z}^d, \Delta(a), \beta) = \mathbb{P}\left( \frac{M}{m_0}, \frac{M}{m_1}, \ldots, \frac{M}{m_d} \right) \).
Proof. The equivalence of conditions (0) and (1) is part of Proposition 4.2 which also gives the identification of the weights.

Let \((Z^d, \Delta(a), \beta)\) be a labelled sheared simplex, with labels \(\{m_0, \ldots, m_d\}\) and \(a = (a_1, \ldots, a_d)\). Since the rank of \(\text{DG}(\beta)\) is 1, by Lemma 3.9 condition (0) is equivalent to \(\text{coker } \beta = 0\). By Lemma 4.5 \(\text{coker } \beta = 0\) if and only if

\[
H = \langle (1 \mod m_0, a_1 \mod m_1, \ldots, a_d \mod m_d) \rangle
\]

is equal to \(\oplus_i \mathbb{Z}/m_i \mathbb{Z}\), or equivalently if \(H\) is cyclic of order \(M = m_0 m_1 \cdots m_d\). If (3) holds, this is immediate.

Conversely, suppose \(H\) is cyclic of order \(M\), and hence equals \(\oplus_i \mathbb{Z}/m_i \mathbb{Z}\). Therefore, \(\gcd(m_i, m_j) = 1\) for \(i \neq j\). Consider the isomorphism \(\oplus_i \mathbb{Z}/m_i \mathbb{Z} \to \mathbb{Z}/M\mathbb{Z}\) defined by sending \(f_i \mapsto M/m_i\), where \(f_i\) denotes the element whose only non-zero component is 1 in the \(i\)th component. Under this isomorphism, the generator of \(H\) is sent to \(b = (M/m_0 + Ma_1/m_1 + \cdots + Ma_d/m_d) \mod M\), which has order \(M/\gcd(M, b) = M\) since \(H\) has order \(M\). Therefore, \(\gcd(M, b) = 1\) and hence \(\gcd(m_j, a_j) = 1\), which shows (3).

The equivalence of conditions (2) and (3) is straightforward to verify. \(\square\)

Proposition 4.8 below describes the isotropy groups for weighted projective spaces that correspond to labelled sheared simplices. For a weighted projective space \(\mathbb{P}(b_0, \ldots, b_d)\), the resulting isotropy groups are straightforward to compute directly from the defining action of \(T\). Namely, the isotropy of a point \(z \in \mathbb{C}^{d+1} \setminus \{0\}\) is easily seen to be cyclic of order \(\gcd(b_j : z_j \neq 0)\). For those weighted projective spaces arising from labelled sheared simplices, we may use Proposition 4.2 to express this in terms of the labels \(\{m_0, \ldots, m_d\}\) and \(a\).

Proposition 4.8. Suppose that \((Z^d, \Delta(a), \beta)\) is a labelled sheared simplex with labels \(\{m_0, m_1, \ldots, m_d\}\) that corresponds to a weighted projective space. Let \(z = (z_0, \ldots, z_d) \in \mathbb{C}^{d+1} \setminus \{0\}\), and set \(d_z = \gcd(a_j : z_j \neq 0)\), where \(a_0\) is set to 1. If \(z_0 \neq 0\), then \(\text{Stab}(z) \cong \bigoplus_{i \in I_z} \mathbb{Z}/m_i \mathbb{Z}\). If \(z_0 = 0\), then \(\text{Stab}(z) \cong \mathbb{Z}/(m_z d_z) \mathbb{Z}\), where \(m_z = \prod_{i \in I_z} m_i\).

5. FAKE WEIGHTED PROJECTIVE SPACES

Analogous to fake weighted projective spaces (e.g. considered in [9][14]), in this section we consider a fake weighted projective space as a stack quotient \(\mathcal{W}/\Lambda\) where \(\Lambda\) is a finite Abelian group acting (in the sense of group actions on stacks [21], [16]) on a weighted projective space \(\mathcal{W}\). In Section 5.1 we characterize fake weighted projective spaces in terms of their associated combinatorial data, while in Section 5.2 we interpret the results of Section 3 for all labelled sheared simplices. A more complete treatment for labelled sheared simplices in dimension 2 appears in Section 5.3.

5.1. Which stacky polytopes correspond to fake weighted projective spaces? As verified by Proposition 4.2, the toric DM stack associated to a stacky fan \((N, \Sigma, \beta)\) satisfying \(\text{DG}(\beta) \cong \mathbb{Z}\) is a weighted projective space. If we require only that \(\text{rank } \text{DG}(\beta) = 1\) (i.e. allowing \(\text{DG}(\beta)\) with torsion), the next Proposition shows that the resulting toric DM stack is as a fake weighted projective space. Note that the proof of the ‘only if’ direction of the proposition relies more heavily on the language of stacks, which in the interest of brevity will not be reviewed. The reader may wish to consult the indicated references.

Proposition 5.1. Let \((N, \Delta, \beta)\) be a stacky polytope, and let \(\Sigma(\Delta)\) be the dual fan to \(\Delta\). The toric DM stack \(\mathcal{X}(N, \Sigma(\Delta), \beta)\) is a fake weighted projective space \(\mathbb{P}(b_0, \ldots, b_d)/\Lambda\) if and only if \(\text{rank } \text{DG}(\beta) = 1\). In this case, the polytope \(\Delta\) is a simplex, and the weights are determined by the condition that \((b_0, \ldots, b_d)\) generates \(\ker \beta \subset \mathbb{Z}^{d+1}\).

Proof. Suppose \((N, \Delta, \beta)\) is a stacky polytope with \(\text{rank } \text{DG}(\beta) = 1\). By [16], \(\mathcal{X}(N, \Sigma(\Delta), \beta) = [\mathbb{Z}_\Sigma/G] \cong [\mathbb{Z}_\Sigma/G_0]/\Lambda\), where \(\Lambda = G/G_0\) and \(G_0\) denotes the connected component of the identity element in \(G\). By Proposition 3.12, \(\mathbb{Z}_\Sigma/G_0\) is the toric DM stack associated to the stacky fan \((N_0, \Sigma(\Delta), \beta_0)\), which satisfies \(\text{DG}(\beta_0) \cong \mathbb{Z}\) and is thus a weighted projective space, by Proposition 4.2.

Conversely, suppose \(\mathcal{X}(N, \Sigma(\Delta), \beta)\) is equivalent to a fake weighted projective space \(\mathcal{W}/\Lambda\) with \(\mathcal{W} = \mathbb{P}(b_0, \ldots, b_d)\). We shall verify below that the quotient map of stacks \(\mathcal{W} \to \mathcal{W}/\Lambda\) is a covering projection—i.e. a representable map of stacks such that every representative is a covering projection (see [19]). Since \(\mathcal{W}\) has trivial (stacky) fundamental group, \(\mathcal{W}\) is then the universal cover, which by [12], coincides with \(\mathcal{X}(N_0, \Sigma_0, \beta_0)\). Therefore \(\text{rank } \text{DG}(\beta) = \text{rank } \text{DG}(\beta_0) = 1\) by Proposition 4.2. The statement about the weights is also a direct consequence of Proposition 4.2.

To see that \(p : \mathcal{W} \to \mathcal{W}/\Lambda\) is a covering projection, it is enough to show that the base extension of \(p\) along a presentation (also called chart or atlas) is a covering projection of topological spaces (since coverings are...
invariant under base change and local on the target—see [19 Example 4.6]). Choose a $\Lambda$-atlas $X_1 \Rightarrow X_0$ for $\mathcal{W}$ with $\Lambda$ acting freely on $X_1$ and $X_0$ (see [16]). Then $X_1/\Lambda \Rightarrow X_0/\Lambda$ is an atlas for the quotient $\mathcal{W}/\Lambda$. It is straightforward to check that $X_0/\Lambda \times _{\mathcal{W}/\Lambda} \mathcal{W} \cong X_0$ (e.g. see the discussion of 2-fiber products in [19 Section 9]), and hence the base extension of $p$ is $X_0 \to X_0/\Lambda$, which is a covering projection.  

Notice that the condition $\text{rank } \text{DG}(\beta) = 1$ is equivalent to the polytope $\Delta$ being a simplex. Therefore, the following is immediate.

**Proposition 5.1.** Let $(N, \Delta, \beta)$ be a stacky polytope and let $\Sigma(\Delta)$ be the dual fan to $\Delta$. The polytope $\Delta$ is a simplex if and only if the associated toric DM stack $X(N, \Delta, \beta)$ is a fake weighted projective space.

### 5.2. Labelled sheared simplices and fake weighted projective spaces

In this section we consider fake weighted projective spaces, specializing to a family of toric DM stacks constructed from the labelled sheared simplices of Definition 4.4.

The following is immediate from Proposition 5.1.

**Remark 5.2.** For a labelled sheared simplex $(\mathbb{Z}^d, \Delta(a), \beta)$, the associated toric DM stack $[Z_{\Sigma(\Delta)}/G]$ is a fake weighted projective space.

Note that a labelled sheared simplex has a smooth vertex (at the origin); therefore, not all fake weighted projective spaces arise from a labelled sheared simplex.

For general labelled sheared simplices, we have the following result realizing the various isotropy groups.

**Proposition 5.3.** Let $(\mathbb{Z}^d, \Delta(a), \beta)$ be a labelled sheared simplex with labels $\{m_0, \ldots, m_d\}$ and set $a_0 = 1$. Let $z = (z_0, \ldots, z_d) \in \mathbb{C}^d \setminus \{0\}$, and set $d_z = \gcd(a_j : z_j \neq 0)$. The isotropy group $\text{Stab}(z)$ is an extension

$$0 \to \bigoplus_{i \in I_z} \mathbb{Z}/m_i \mathbb{Z} \to \text{Stab}(z) \to \mathbb{Z}/d_z \mathbb{Z} \to 0.$$  

**Proof.** Recall that by Theorem 3.2, the isotropy group $\text{Stab}(z)$ is isomorphic to the torsion submodule of $N/N_z$. Form the matrix $B_z$ by deleting the $i$-th column of $\beta$ whenever $z_i \neq 0$. Viewing $B_z$ as a homomorphism $\mathbb{Z}^{1|I_z} \to \mathbb{Z}^d$ realizes $N/N_z = \text{coker } B_z$. We compute the torsion submodule of $\text{coker } B_z$ next.

If $z_0 \neq 0$, then one may readily see that $\text{Tor}(N/N_z) \cong \bigoplus_{i \in I_z} \mathbb{Z}/m_i \mathbb{Z}$. It remains to consider the case where $z_0 = 0$ (i.e. with corresponding matrix $B_z$ containing the first column of $\beta$).

To begin, observe that $\beta : \mathbb{Z}^{d+1} \to N = \mathbb{Z}^d$ factors as the composition $\mathbb{Z}^{d+1} \xrightarrow{L} \mathbb{Z}^{d+1} \xrightarrow{\beta'} \mathbb{Z}^d$, where

$$L = \begin{bmatrix} m_0 & 0 & \cdots & 0 \\ 0 & m_1 & \cdots & m_d \end{bmatrix}, \quad \text{and} \quad \beta' = \begin{bmatrix} -a_1 & 1 & 0 \\ \vdots & \ddots & \vdots \\ -a_d & 0 & 1 \end{bmatrix}.$$  

Accordingly, we may factor $B_z$ as a composition $\mathbb{Z}^{1|I_z} \xrightarrow{L_z} \mathbb{Z}^{1|I_z} \xrightarrow{B_z'} \mathbb{Z}^d$, where $B_z'$ is a matrix whose first column is the first column of $\beta'$ and whose other columns are standard basis vectors $e_i \in \mathbb{Z}^d$ for $i \neq 0$ in $I_z$.

This factorization yields the following diagram of short exact sequences.

\[
\begin{array}{cccccc}
0 & \to & \mathbb{Z}^{1|I_z} & \xrightarrow{L_z} & \mathbb{Z}^{1|I_z} & \xrightarrow{B_z} \bigoplus_{i \in I_z} \mathbb{Z}/m_i \mathbb{Z} & \to & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{Z}^d & \xrightarrow{B_z'} & \mathbb{Z}^d & \to & 0 \\
& & & \downarrow & \text{coker } B_z & & \downarrow & \text{coker } B_z' & \to & 0
\end{array}
\]

Applying the Snake Lemma (and observing that $B_z'$ is injective) yields the exact sequence,

$$0 \to \bigoplus_{i \in I_z} \mathbb{Z}/m_i \mathbb{Z} \to \text{coker } B_z \to \text{coker } B_z' \to 0. \quad (5.1)$$

Let $d_z = \gcd(a_i : i \notin I_z)$. Then $B_z'$ is row equivalent to the matrix $C_z'$ obtained from $B_z'$ by replacing all $a_i$ with $i \notin I_z$ with 0’s except for one which is replaced with $d_z$. It follows that $\text{coker } B_z' \cong \mathbb{Z}^{d-|I_z|} \oplus \mathbb{Z}/d_z \mathbb{Z}$. Moreover, we may describe the generators of the free summand as follows. Let $E_z : \mathbb{Z}^d \to \mathbb{Z}^d$ denote the invertible homomorphism defined by $C_z' = E_z B_z'$. Then the free summand is generated by the images of the $E^{-1}(e_i)$ ($i \notin I_z$) in $\mathbb{Z}^d/\text{im } B_z'$. It follows that the $E^{-1}(e_i)$ ($i \notin I_z$) must generate a free submodule in
coker $B_z$, which maps isomorphically onto the free summand of coker $B'_z$. Thus, we may pass to torsion submodules in (5.1):

$$0 \rightarrow \bigoplus_{i \in I_0} \mathbb{Z}/m_i \mathbb{Z} \rightarrow \text{Tor}(\text{coker } B_z) \rightarrow \mathbb{Z}/d_2 \mathbb{Z} \rightarrow 0.$$ 

□

Using Corollary 3.13 we may also characterize those labelled sheared simplices yielding global quotient stacks. Note that since the toric DM stack constructed from a labelled sheared simplex is a fake weighted projective space, it is immediate that a global quotient in this case must then be a quotient of smooth projective space $\mathbb{P}^d$.

**Proposition 5.4.** Let $(\mathbb{Z}^d, \Delta(a), \beta)$ be a labelled sheared simplex with labels $\{m_0, \ldots, m_d\}$. The toric DM stack $\mathcal{X}(\mathbb{Z}^d, \Delta(a), \beta)$ is equivalent to a global quotient if and only if $m_i = m_0 a_i$ for all $i = 1, \ldots, d$.

**Proof.** We apply Corollary 3.13 and [12, Theorem 4.4]. Let $\sigma_j$ be the maximal cone generated by the rays $\{\rho_0, \ldots, \rho_j, \ldots, \rho_d\}$, where $\rho_j$ signifies omission from the list. Then

$$N'_{\sigma_0} = \text{span}\{m_1 e_1, \ldots, m_d e_d\}, \quad \text{and} \quad N'_{\sigma_j} = \text{span}\{m_0 \sum_{i=1}^d a_i e_i, m_1 e_1, \ldots, m_j e_j, \ldots, m_d e_d\}.$$ 

Observe that if $m_0 a_i = m_i$ for all $i$, then it is clear that $N' = N'_{\sigma_j}$ for all $j = 0, \ldots, d$.

To prove the converse, suppose $N' = N'_{\sigma_j}$ for all $j = 0, \ldots, d$. For $j = 0$, this implies that there exist $\alpha_1, \ldots, \alpha_d \in \mathbb{Z}$ such that

$$\sum_{i=1}^d (m_0 a_i - \alpha_i m_i) e_i = 0,$$

and thus $m_i | m_0 a_i$ for $i = 1 \ldots d$. Similarly, for $j = 1, \ldots, d$, we see that there exist $\gamma_0, \ldots, \gamma_j, \ldots, \gamma_d \in \mathbb{Z}$ such that

$$m_j e_j = \gamma_0 m_0 \sum_{i=1}^d a_i e_i + \sum_{i \neq j} \gamma_i m_i e_i,$$

or equivalently,

$$\sum_{i \neq j} (\gamma_0 m_0 a_i + \gamma_i m_i) e_i + (\gamma_0 m_0 a_j - m_j) e_j = 0.$$ 

Therefore, $m_0 a_j | m_j$ for $j = 1, \ldots, d$ whence $m_0 a_i = m_i$ for all $i$. □

5.3. 2-dimensional labelled sheared simplices. As a consequence of Theorem 3.2, we can now determine the isotropy groups corresponding to a labelled sheared simplex in the plane (see also Remark 3.6).

Let $a = (a_1, a_2)$ be a primitive vector in the positive quadrant, and suppose $(\mathbb{Z}^2, \Delta(a), \beta)$ is a labelled sheared simplex with labels $\{m_0, m_1, m_2\}$. Explicitly, $\Delta(a)$ is the convex hull of the origin together with $(a_2, 0)$ and $(0, a_1)$, with assigned labels $m_1$ to the edge along the $y$-axis, $m_2$ to the edge along the $x$-axis, and $m_0$ to the remaining edge (see Figure 5.1).

![Figure 5.1. A labelled sheared simplex in the plane.](image)
The isotropy for points \((z_0, 0, 0) \in Z_\Sigma = \mathbb{C}^3 \setminus \{0\}\) with \(z_0 \neq 0\) is easily seen to be \(\mathbb{Z}/m_1 \mathbb{Z} \oplus \mathbb{Z}/m_2 \mathbb{Z}\) (by Proposition 5.3). For points of the form \(z = (0, 0, z_2)\) with \(z_2 \neq 0\)—i.e. for points corresponding to the vertex \((0, a_1)\)—we shall describe the isotropy \(\text{Stab}(z)\) below. (The isotropy for points of the form \(z = (0, z_1, 0)\) with \(z_1 \neq 0\) can be obtained by exchanging the indices 1 and 2.)

We begin with a Lemma describing the Smith normal form of an integer matrix with exactly one zero entry.

**Lemma 5.5.** For non-zero \(a, b, c \in \mathbb{Z}\), the Smith Normal Form of \(\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}\) is \(\begin{bmatrix} g & 0 \\ 0 & bc/g \end{bmatrix}\), where \(g = \gcd(a, b, c)\).

**Proof.** Suppose that \(\gcd(a, b) = d\). Then, we claim that there exist \(x, y \in \mathbb{Z}\) such that \(xa + yb = d\), and \(\gcd(x, y) = 1\).

To prove this claim, we first note that it is equivalent to the following: Suppose \(u, v \in \mathbb{Z}\) are relatively prime. Consider the set of solutions \(X = \{x \mid xu + yv = 1\}\) for some \(y \in \mathbb{Z}\). For any given integer \(d\), there is some \(x \in X\) so that \(\gcd(x, d) = 1\).

In this latter formulation, let \(d \in \mathbb{Z}\) be given and suppose that \(x_0\) is any solution to \(x_0u + yv = 1\). Recall that all solutions are then of the form \(x = x_0 + tv\) with \(t \in \mathbb{Z}\). Moreover, \(x_0u + yv = 1\) implies that \(\gcd(x_0, v) = 1\). Then, showing that there is \(x \in X\) such that \(\gcd(x, d) = 1\) is equivalent to showing that there is \(t \in \mathbb{Z}\) such that \(\gcd(x_0 + tv, d) = 1\). We will construct such an integer, \(t\).

Suppose that \(d = p_1^{a_1} \cdots p_s^{a_s}\) is the prime factorization of \(d\). Let \(t = \prod p_i\) such that \(p_i\) does not appear in the prime factorization of either \(x_0\) or \(v\). Because \(x_0\) and \(v\) are relatively prime, it follows that in the sum \(x_0 + tv\) each prime in the factorization of \(d\) appears exactly once. That is, each \(p_i\) divides exactly one of \(x_0\) or \(tv\). Thus, \(d\) cannot divide the sum and \(\gcd(x, d) = 1\).

Therefore, we may find \(x\) and \(y\) such that \(ax + by = d\) and \(\gcd(x, d) = 1\). Moreover, since \(\gcd(cx, d) = \gcd(c, d) = \gcd(a, b, c) = g\), there exist \(p, q \in \mathbb{Z}\) such that \(p(cx) + qd = g\). Hence,

\[
\begin{bmatrix} x & -b/d \\ y & a/d \end{bmatrix}, \quad \begin{bmatrix} q & p \\ -cx/g & d/g \end{bmatrix} \in SL_2(\mathbb{Z}),
\]

and

\[
\begin{bmatrix} q & p \\ -cx/g & d/g \end{bmatrix} \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \begin{bmatrix} x & -b/d \\ y & a/d \end{bmatrix} = \begin{bmatrix} g & -cbp/d \\ 0 & -bc/g \end{bmatrix}.
\]

Recall that \(d \mid b\) and that \(g \mid c\), so \(g \mid (bcp/d)\). Thus, by elementary column operations

\[
\begin{bmatrix} g & -cbp/d \\ 0 & -bc/g \end{bmatrix} \hookrightarrow \begin{bmatrix} g & 0 \\ 0 & -bc/g \end{bmatrix}.
\]

Finally, note that \(g \mid (bc/g)\), so the above is the desired Smith normal form. \(\square\)

We now can give the explicit form of the isotropy groups of a labelled sheared simplex in the plane.

**Proposition 5.6.** Let \((\mathbb{Z}^2, \Delta(a), \beta)\) be a labelled sheared simplex with labels \(\{m_0, m_1, m_2\}\) and \([\mathbb{Z}_\Sigma/G]\) its corresponding toric DM stack. The isotropy of \(z = (0, 0, z_2) \in Z_\Sigma\) with \(z_2 \neq 0\) is \(\text{Stab}(z) \cong \mathbb{Z}/g\mathbb{Z} \oplus \mathbb{Z}/((m_0m_1a_1)/g)\mathbb{Z}\), where \(g = \gcd(m_0, m_1)\).

**Proof.** We consider the map \(\beta : \mathbb{Z}^3 \rightarrow \mathbb{Z}^2\) given by the matrix

\[
\beta = \begin{bmatrix} -m_0a_1 & m_1 & 0 \\ -m_0a_2 & 0 & m_2 \end{bmatrix}.
\]

By Lemma 5.5, the Smith normal form of \(B_z = \begin{bmatrix} -m_0a_1 & m_1 \\ -m_0a_2 & 0 \end{bmatrix}\) is \(\begin{bmatrix} g & 0 \\ 0 & m_1a_2/g \end{bmatrix}\) since \(g = \gcd(m_0, m_1) = \gcd(m_0a_1, m_1, m_0a_2)\), which by Theorem 3.2 gives the result. \(\square\)

Though Proposition 5.6 gives the general form of the isotropy group of points corresponding to the vertex \((0, a_1)\) of a sheared simplex, it can be instructive to consider several special cases to illustrate the interplay of the facet labels and the geometry of the sheared simplex—see Table 5.1.
| Labels | Lengths | \( \text{Stab}(z) \) |
|--------|---------|------------------|
| \( m_0 = m_1 = m_2 = 1 \) | \( a_1 = a_2 = 1 \) | \( \mathbb{Z}/a_1 \mathbb{Z} \) |
| \( m_0, m_1, m_2 \) arbitrary | \( a_1 = a_2 = 1 \) | \( \mathbb{Z}/m_0 \mathbb{Z} \oplus \mathbb{Z}/m_1 \mathbb{Z} \) |
| \( a_1, a_2 \) arbitrary | \( \mathbb{Z}/g \mathbb{Z} \oplus \mathbb{Z}/(m_0 m_1 a_1/g ) \mathbb{Z} \) |

Table 5.1. The isotropy group \( \text{Stab}(z) \) corresponding to points of the form \((0, 0, z_2)\) with \( z_2 \neq 0 \) (i.e. corresponding to the vertex \((0, a_1)\) of \( \Delta \)) for a toric DM stack corresponding to a labelled sheared simplex \((\mathbb{Z}^2, \Delta(\alpha), \beta)\). Here, \( g = \text{gcd}(m_0, m_1) \).

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