Matter from $G_2$ Manifolds

Per Berglund

CIT-USC Center for Theoretical Physics
Department of Physics and Astronomy
University of Southern California
Los Angeles, CA 90089-0484

Andreas Brandhuber

Department of Physics
California Institute of Technology
Pasadena, CA 91125

and

CIT-USC Center for Theoretical Physics
University of Southern California
Los Angeles, CA 90089-0484

ABSTRACT

We describe how chiral matter charged under $SU(N)$ and $SO(2N)$ gauge groups arises from codimension seven singularities in compactifications of M-theory on manifolds with $G_2$ holonomy. The geometry of these spaces is that of a cone over a six-dimensional Einstein space which can be constructed by (multiple) unfolding of hyper-Kähler quotient spaces. In type IIA the corresponding picture is given by stacks of intersecting D6-branes and chiral matter arises from open strings stretching between them. Usually one obtains (bi)fundamental representations but by including orientifold six-planes in the type IIA picture we find more exotic representations like the anti-symmetric, which is important for the study of $SU(5)$ grand unification, and trifundamental representations. We also exhibit many cases where the $G_2$ metrics can be described explicitly, although in general the metrics on the spaces constructed via unfolding are not known.

1 e-mail: berglund@citusc.usc.edu
2 e-mail: andreas@theory.caltech.edu
1 Introduction

Compactifications of M-theory/string theory to $D = 4$ dimensions with $N = 1$ supersymmetry can be obtained in a number of ways. The historic approach, which is still very much viable, is via the heterotic $E_8 \times E_8$ theory \cite{1}. The compact manifold is a Calabi-Yau three-fold and in addition a choice of vector bundle has to be made, breaking the $E_8 \times E_8$ gauge symmetry \cite{4}. For example, the standard embedding of identifying the spin connection with the gauge connection gives rise to $h_{2,1}$ chiral multiplets in the $27$ and $h_{1,1}$ chiral multiplets in the $\overline{27}$ of $E_6$, where $h_{2,1}$ is the number of complex structure deformations and $h_{1,1}$ is the number of Kähler deformations of the Calabi-Yau three-fold.

An alternative approach, which is dual to another heterotic compactification \cite{3}, is to compactify M-theory on a seven dimensional real manifold, $X$, with $G_2$ holonomy. In this case the duality is inherited from fiberwise application of the duality between M-theory on a K3 manifold and the heterotic string on a $T^3$ with a choice of vector bundle \cite{4}. Contrary to the heterotic scenario, compactification of M-theory on a $G_2$ manifold does not in general lead to any charged chiral matter nor to non-abelian gauge symmetries as long as $X$ is smooth. Rather, one gets $b_2 \ U(1)$ vector multiplets and $b_3$ neutral chiral multiplets, where $b_q$ gives the number of $q$-cycles on $X$ \cite{5}. However, the situation changes drastically when $X$ admits singularities. In particular, codimension four and seven singularities lead to non-abelian gauge enhancement and charged chiral matter, respectively \cite{3, 5}. These compactifications also have an interesting interpretation in type IIA string theory, where they correspond to compactification on six-manifolds with RR two-form fluxes and/or D6-branes/O6 planes wrapped on supersymmetric three-cycles \cite{12, 13}. For examples of such type IIA compactifications which give rise to chiral fermions and which admit lifts to M-theory compactifications on $G_2$ manifolds, see \cite{14}.

In this paper, we continue the work of Acharya and Witten \cite{3}, who showed how chiral matter charged under unitary gauge groups appear at singularities of $G_2$ manifolds. These manifolds are constructed as circle quotients of conical hyper-Kähler (HK) eight-manifolds. In particular, we generalize their work to the antisymmetric representation of $SU(n)$ and the fundamental of $SO(2n)$. The former in particular, is essential in attempts of constructing realistic models of grand unified theories such as $SU(5)$ from string/M-theory. We describe how the corresponding M-theory backgrounds can be reduced to type IIA backgrounds that consist of intersecting D6-branes/O6 planes. This straight-

\footnote{Note that this is not a complete list of the possible singularities classified in terms of their codimension $p$. Other interesting cases are $p = 1, 5, 6$, where $p = 1$ corresponds to a boundary that localizes $E_8$ gauge symmetry \cite{10}, $p = 5$ corresponds to a singularity of the type studied in \cite{11}, and finally $p = 6$ are singularities typically encountered in Calabi-Yau three-folds which produce non-chiral matter.}
forwardly generalizes a collection of two and three stacks of intersecting D6-branes to $k$ groups of $n_i, i = 1, \ldots, k$ D6-branes (see also [13]). For $k = 3$ we compare our results with those of [3] which have an M-theory description as $\mathbb{R}^3$-bundles over $\mathbb{W} \mathbb{C} \mathbb{P}^2_{n_1,n_2,n_3}$ if at least two of the indices $n_i$ are equal. In the presence of an $\mathcal{O}_6$ plane the M-theory background is a $\mathbb{Z}_2$ orbifold thereof. We also describe how trifundamental matter charged under $SU(3) \times SU(3) \times SU(2)$ and $SU(4) \times SU(3) \times SU(2)$ are obtained from unfolding HK quotients [16] of $E_6$ and $E_7$ singularities, respectively.

The rest of the paper is organized as follows: in section 2 we generalize the construction of $G_2$ manifolds by multiple unfoldings of HK quotient singularities. In section 3 we show that the space arising from double unfoldings under certain conditions can be mapped to a different construction of $G_2$ manifolds as cones over twistor spaces, which makes it possible to find the metrics explicitly. In section 4 we include orientifolds in the type IIA picture which in M-theory lifts to $\mathbb{Z}_2$ orbifolds of manifolds constructed via multiple unfoldings. This gives rise to chiral multiplets in the anti-symmetric representation of $SU(n)$ gauge groups in addition to multiplets in the bifundamental representation. We also extend our work to exceptional gauge groups, which give rise to matter charged under more than two non-abelian gauge groups for which there is no perturbative type IIA description available. Finally, we end with some comments and discussions in section 5 while details of the map from M-theory to type IIA can be found in appendix A and the realizations of $D_n$ and $E_n$ singularities are given in appendices B and C, respectively.

2 Unfolding hyper-Kähler quotients

In this section we review and generalize a particular construction of $G_2$ manifolds introduced by Acharya and Witten in [3]. The manifolds are $U(1)$ quotients of HK manifolds where the $U(1)$ is chosen such that it commutes with the $SU(2)$ symmetry of the HK manifold which permutes the three complex structures, and can be obtained by unfolding four-dimensional HK quotient singularities. This unfolding, which will be described in more detail later, is nothing but a fibration of an ADE singularity over a three-dimensional base, $B$. If there are no singularities worse than the codimension four ADE singularity we would just obtain an $N = 1$ vector multiplet with ADE gauge group and $b_1(B)$ chiral multiplets in the adjoint. But in the construction of [3] the base manifold $B$ has one special point (or possibly several) over which the singularity is enhanced e.g. $A_{n-1} \to A_n$. In the dual heterotic string theory this worsening of the singularity is reflected by a jump in the rank of the gauge bundle at the same point(s) in the base $B$ as in M-theory. (We refer the reader to [3] for more details on the heterotic picture.) In the
case of a symmetry enhancement $A_{n-1} \rightarrow A_n$ the gauge theory is $SU(n) \times U(1)$ SYM with chiral matter in the $n_{-1}$ representation. The $SU(n)$ vector multiplet is produced by a co-dimension four $A_{n-1}$ singularity, whereas the $U(1)$ comes from conventional Kaluza-Klein reduction. As was shown in the anomaly from the chiral multiplet is canceled by an anomaly inflow mechanism from the bulk. This is the same mechanism that is at work to cancel the anomaly of chiral matter that arises from open strings stretched between configurations of intersecting D-branes which preserve four supercharges. Of course, this is more than a formal analogy since the $G_2$ singularities have an interpretation in type IIA string theory as sets of D6-branes intersecting over flat $\mathbb{R}^6$ as we will show in this section (with more details in appendix A).

Interestingly, there exists a closely related construction in Type IIA of Calabi-Yau threefold singularities which give rise to charged matter. The threefold is fibered by K3’s over a complex one-dimensional base $Q$ in a manner that over generic points of $Q$ the singularity is $G$ but over one point the symmetry is enhanced $G \rightarrow G'$. There exists a fruitful interplay between these two constructions and we will make this more explicit in examples presented later in this section.

We will start by reviewing Kronheimer’s HK quotient construction of ADE singularities which we will use to obtain charged chiral matter (see also [3]). The starting point is an $A_n$ singularity which is locally equivalent to $\mathbb{R}^4/Z_{n+1}$. The basic idea is that $\mathbb{R}^4/Z_{n+1}$ can be realized as the vacuum manifold of a particular linear sigma model (LSM) and will be denoted as $\mathbb{H}^{n+1}/K$, where $K$ is the LSM gauge group which is in this case is $U(1)^n$ and each factor of $\mathbb{H}$ denotes the four scalars of a hypermultiplet $\Phi_i$, $i = 0, \ldots, n$. The charges of the hypermultiplets follow from the affine Dynkin diagram of $A_n$, also denoted as the quiver diagram of the LSM. The hypermultiplet $\Phi_i$ has charge +1 and $\Phi_{i-1}$ has charge $-1$ under the $i^{th}$ $U(1)$ and all other charges are zero. The vacuum manifold, denoted in short by $\mathbb{H}^{n+1}/K$, of this theory is obtained by imposing the D/F-term constraints and dividing by the gauge group $K$. This manifold has real dimension $4(n+1) - 3n - n = 4$. The $3n$ D/F-term constraints form $n$ triplets under the $SU(2)_R$ symmetry of the LSM and can be expressed as linear combinations of the moment map $\mu : \mathbb{H} \rightarrow \mathbb{R}^3$:  

$$\Phi = (M, \bar{M}) \rightarrow \left( \begin{array}{c} \Re M \bar{M} \\ \Im M \bar{M} \\ M^* \bar{M} - \bar{M}^* M \end{array} \right) \equiv (\Phi, \sigma \Phi) , \quad (1)$$

where $M$ and $\bar{M}$ are two complex scalars contained in $\Phi$, and $\sigma_i$ are the standard her-
mitian Pauli matrices. Note that $\overline{M}$ has opposite charge of $M$ and together they form a doublet of $SU(2)_{\mathbf{R}}$. Therefore, and because of the charge assignments of the scalars, the D/F-term of the $l$-th $U(1)$ is

\[-(\Phi_{l-1}, \overline{\Phi}_{l-1}) + (\Phi_{l}, \overline{\Phi}_{l}) = \vec{t}_l , \tag{2}\]

where $\vec{t}_l$ is a triplet of FI-parameters.

The $A_n$ singularity in the form $\mathbb{R}^4/\mathbb{Z}_{n+1}$ is recovered when we set all FI-parameters $\vec{t}_l = 0$. The quotient by $K$ is implemented by introducing gauge invariant meson and baryon like combinations of the complex scalars

\[x = \prod M_i = z_1^{n+1}, \quad y = \prod M_i = z_2^{n+1}, \quad z = M_0 \overline{M}_0 = \ldots = M_n \overline{M}_n = z_1 z_2 , \tag{3}\]

which obey the standard equation for an $A_n$ singularity $xy = z^{n+1}$. On the other hand when some or all of the moment maps are non-zero we get partial resolutions of the singularity in which case either the Kähler parameter of certain shrunk two-cycles is blown up and/or the complex structure equations is deformed to $xy = z^{n+1} - u_2 z^{n-1} - u_3 z^{n-2} - \ldots - u_{n+1}$.

This is the starting point for Katz and Vafa’s construction of charged matter from geometry \[19\]. The key idea is to deform a higher singularity into a lower by complex structure deformations which vary over space. This amounts to replacing the constant deformation parameters $u_i$ by functions over a base which we take to be $\mathbb{C}P^1$ and is parametrized by $t$. The breaking of an $A_r \rightarrow A_{r-1}$ singularity is given by the deformation $xy = z(z + t)^r$ which gives rise to matter hypermultiplets in the $r(-1)$ representation of $SU(r) \times U(1)$. This can be readily generalized in two ways. We can reduce the rank by one in more than one fashion, e.g. $xy = z^n (z + t)^m$ which corresponds to $A_{n+m-1} \rightarrow A_{n-1} \times A_{m-1}$ and gives matter in the $(n, \overline{m})$ of $SU(n) \times SU(m)$. The other possibility is to break the higher singularity into more than two smaller singularities by $xy = \prod_{i=1}^m (z + c_i t)^{n_i}$, where $c_i$ are arbitrary constants, which corresponds to $A_{N-1} \rightarrow \prod A_{n_i-1}$ with $N = \sum_{i=1}^m n_i$. In the context of $G_2$ manifolds we will soon employ a similar construction which we call $m$–1-fold unfolding of the singularity.

The construction shortly reviewed in the last two paragraphs \[19\] was mainly considered in the context of Calabi-Yau compactifications of type IIA string theory where it leads to charged hypermultiplets. In \[3\] a closely related construction of singularities of seven-dimensional spaces with $G_2$ holonomy was presented. These spaces are to be seen in the context of $M$-theory compactification to four dimensions with $N = 1$ supersymmetry and charged chiral matter. As before one is interested in fibering an $ADE$ singularity over a base manifold such that the singularity is maximal over a special point.
but smaller at generic points. However, the base manifold must be three-dimensional and the space cannot be written simply as a complex structure deformation of an $ADE$ singularity. As described in [3] this is achieved by unfolding a HK quotient singularity $\mathbb{H}^{n+1}/K$ by setting all moment maps to zero except for one

$$-(\Phi_{i-1}, \sigma \Phi_i) + (\Phi_i, \sigma \Phi_i) = 0$$

with $i = 1, \ldots, k - 1, k + 1, \ldots, n$ where $k$ has to be omitted. The corresponding unconstrained moment map is

$$-(\Phi_k, \sigma \Phi_k) + (\Phi_k, \sigma \Phi_k) = \vec{t}_1,$$

where the three-vector $\vec{t}_1$ should be thought of as parametrizing the three-dimensional base manifold. This clearly corresponds to a seven-manifold, which is conical and non-compact, and the symmetry breaking is of the form $A_{n-1} \rightarrow A_{k-1} \times A_{n-k-1}$. This is the prototype example of an unfolded $A_{n-1}$ singularity and in this case the manifold was shown to be a cone over $\mathbb{W} \mathbb{C} \mathbb{P}_{k,k,l,l}$ with $l = n - k$ [3]. The chiral matter localized at the singularity is in the representation $(k, n-k)$.

Let us explain in some more detail how the generalization to double unfoldings works. We start with an $A_{N-1}$ singularity realized as an HK quotient, where $N = n_1 + n_2 + n_3$. Let two of the moment maps be unrestricted

$$-(\Phi_{n_1}, \sigma \Phi_{n_1}) + (\Phi_{n_1}, \sigma \Phi_{n_1}) = \vec{t}_1,$$

$$-(\Phi_{n_1+n_2}, \sigma \Phi_{n_1+n_2}) + (\Phi_{n_1+n_2}, \sigma \Phi_{n_1+n_2}) = \vec{t}_2,$$

and otherwise

$$(\Phi_{i}, \sigma \Phi_i) = (\Phi_i, \sigma \Phi_i),$$

with $i = 1, \ldots, n_1, \ldots, n_1 + n_2, \ldots, N - 1$.

As it stands this corresponds to a ten-dimensional manifold since this is a fibration over a six-dimensional space spanned by $\vec{t}_{1,2}$. In a moment we will restrict to a three-dimensional subspace via a linear constraint between the two three-vectors. But let us first rephrase the D/F-term equations in a way that takes care of the gauge symmetry we have to divide by. As usual this is done by introducing gauge invariant combinations of the scalar fields $\Phi_i = (M_i, \overline{M}_i)$. We perform a double unfolding in which we leave the

$^5$It is straightforward to generalize our construction to multiple unfoldings, and we will return to the general case at the end of this section. However, in this paper we will be mainly concerned with double unfoldings.
D/F-terms corresponding to the $n_1$-th and $(n_1 + n_2)$-th $U(1)$ unconstrained,

\[
\bar{M}_i M_i = \bar{M}_{i+1} M_{i+1}, \quad |M_i|^2 - |\bar{M}_i|^2 = |M_{i+1}|^2 - |\bar{M}_{i+1}|^2,
\]

\[
\bar{M}_{n_1-1} M_{n_1-1} - \bar{M}_{n_1} M_{n_1} = t_1^{(1)} + it_1^{(2)} \equiv c_1,
\]

\[
|M_{n_1-1}|^2 - |\bar{M}_{n_1-1}|^2 - |M_{n_1}|^2 + |\bar{M}_{n_1}|^2 = t_1^{(3)} \equiv r_1,
\]

\[
\bar{M}_{n_1+n_2-1} M_{n_1+n_2-1} - \bar{M}_{n_1+n_2} M_{n_1+n_2} = t_2^{(1)} + it_2^{(2)} \equiv c_2,
\]

\[
|M_{n_1+n_2-1}|^2 - |\bar{M}_{n_1+n_2-1}|^2 - |M_{n_1+n_2}|^2 + |\bar{M}_{n_1+n_2}|^2 = t_2^{(3)} \equiv r_2,
\]

with $i = 0, \ldots, n_1 - 2, n_1, \ldots, n_1 + n_2 - 2, n_1 + n_2, \ldots, N - 2$. This corresponds to the breaking pattern $A_{N-1} \rightarrow A_{n_1-1} \times A_{n_2-1} \times A_{n_3-1}$. Eqs. (8) can be solved by introducing $K'$ invariant fields where $K = K' \times U(1)_{n_1} \times U(1)_{n_1+n_2}$. These baryonic fields are given by

\[
\prod_{i=0}^{n_1-1} M_i = z_1^{n_1}, \quad \prod_{i=0}^{n_1-1} \bar{M}_i = z_2^{n_1},
\]

\[
\prod_{i=n_1}^{n_1+n_2-1} M_i = z_3^{n_2}, \quad \prod_{i=n_1}^{n_1+n_2-1} \bar{M}_i = z_4^{n_2},
\]

\[
\prod_{i=n_1+n_2}^{N-1} M_i = z_5^{n_3}, \quad \prod_{i=n_1+n_2}^{N-1} \bar{M}_i = z_6^{n_3},
\]

where the $z_i$ are neutral under $K'$ and carry charges

\[
U(1)_{n_1} : (\pm 1/n_1, 1/n_1, 1/n_2, -1/n_2, 0, 0),
\]

\[
U(1)_{n_1+n_2} : (0, 0, 1/n_2, -1/n_2, -1/n_3, 1/n_3),
\]

under the remaining gauge symmetries.

In analogy with (3) we can introduce a different set of meson and baryon like variables.

\[
x = \prod_{i=0}^{N-1} M_i, \quad y = \prod_{i=0}^{N-1} \bar{M}_i,
\]

\[
\bar{z}^{n_1} = \prod_{i=0}^{n_1-1} M_i \bar{M}_i, \quad w^{n_2} = \prod_{i=n_1}^{n_1+n_2-1} M_i \bar{M}_i, \quad u^{n_3} = \prod_{i=n_1+n_2}^{N-1} M_i \bar{M}_i,
\]

with $N = n_1 + n_2 + n_3$. They obey the following equation

\[
xy = \bar{z}^{n_1} w^{n_2} u^{n_3} = \bar{z}^{n_1}(z - c_1)^{n_2}(z - c_1 - c_2)^{n_3}
\]

where in the last equality we have used the F-term equations from the $n_1$-th and $n_1 + n_2$-th $U(1)$ (see Eq. (8)). This allows us to make contact with the generalized construction of Katz and Vafa [13]. When $c_1 = c_2 = 0$ Eq. (12) describes an $A_{N-1}$ singularity. For $c_1 \neq 0$ and $c_1 + c_2 \neq c_1$ we have $A_{N-1} \rightarrow A_{n_1-1} \times A_{n_2-1} \times A_{n_3-1}$.
Let us study the geometry of the (singular) non-compact manifold given by the above HK quotient construction. The ambient space is $\mathbb{C}^6$ subject to the $U(1)^2$ actions given by (10). In order for this to describe a seven-dimensional manifold we impose a constraint of the type

$$\vec{t}_1 + R \cdot \vec{t}_2 = \vec{c}$$

(13)

where the $\vec{t}_i$ are the relevant moment maps from (8)

$$z_1 z_2 - z_3 z_4 = t_1^{(1)} + i t_2^{(2)} \equiv c_1,$$

$$|z_1|^2 - |z_2|^2 - (|z_3|^2 - |z_4|^2) = t_1^{(3)} \equiv r_1,$$

$$z_3 z_4 - z_5 z_6 = t_1^{(1)} + i t_2^{(2)} \equiv c_2,$$

$$|z_3|^2 - |z_4|^2 - (|z_5|^2 - |z_6|^2) = t_2^{(3)} \equiv r_2.$$ (14)

Non-vanishing $\vec{c}$ in Eq. (13) gives rise to interesting geometrical transitions, but we are mainly interested in conical $G_2$ holonomy spaces, that give rise to chiral fermions, and so we set $\vec{c} = 0$. It is then easy to see that the set of equations (14) is invariant under $z_i \to \lambda z_i$ with $\lambda \in \mathbb{R}^+$. Hence the space is a cone over a six dimensional base, $Y$. Furthermore, by enlarging the $U(1)^2$ actions on $\mathbb{C}^6$ to $(\mathbb{C}^*)^2$ we have a toric variety, $M$, of $\text{dim}_{\mathbb{C}} = 4$. Because of the linear constraint (13) the complex part of (14) gives rise to a quadratic relation between the $z_i$. Thus, the base $Y$ is obtained as a hypersurface in $M$. It is interesting to note that although the tools of algebraic geometry are not directly available to describe the geometry of the $G_2$ manifold itself, in the case of a conical geometry the base carries a Kähler structure and it often has singularities in codimension four (see appendix A). To be more precise, the $G_2$ holonomy condition implies that the base is a nearly Kähler Einstein manifold [20].

It is instructive to analyze what these manifolds correspond to when we reduce to Type IIA string theory along a circle that is generated by a $U(1)$ action with charges $(1, -1, 0, 0, 0, 0)$ on the variables $z_i$. We leave the derivation to appendix A and only state the results. In the case of double unfolding we find that the Type IIA background describes the intersection of three stacks of D6 branes at supersymmetric angles over $\mathbb{R}^6$, where the number of the D6-branes in the $i$-th stack is given by three integers $n_i$. This allows us to derive the expected chiral matter directly and we find multiplets in the representation $(n_1, n_2, 1) + (1, n_2, n_3) + (n_1, 1, n_3)$ of $SU(n_1) \times SU(n_2) \times SU(n_3)$. Hence, the physics at these singularities is quite straightforward to understand, unlike the

---

*Alternatively, we can use the equivalent formulation of a toric variety in terms of the symplectic quotient construction in which each $U(1)$ action is combined with the D-term component of the corresponding moment map, see e.g. [21], [22].*
case of $G_2$ singularities from twistor spaces [3] which are in general harder to analyze and probably contain new, interesting physics. (For recent work, see e.g. [23, 24]). However, in section 3 we show that certain singularities constructed by double unfolding can be related to a particular class of twistor space constructions, and, therefore, the $G_2$ metric is known explicitly in those examples.

It is natural to generalize this construction by considering multiple unfoldings of $A_N$ singularities by leaving $m$ D/F terms unconstrained. In order to obtain an appropriate seven-manifold we have to introduce $m-1$ linear relations of the type (13). By using a generalization of the discussion in appendix A we find that this space corresponds to an intersection of $m+1$ stacks of D6-branes with multiplicities $n_i$. In this case we obtain matter in the bifundamental representations of $SU(n_1) \times \ldots \times SU(n_{m+1})$ charged under all possible combinations of two gauge group factors which in total give $\frac{m(m+1)}{2}$ chiral multiplets.

3 Relation between unfolding and twistor space construction of $G_2$ singularities

In this section we will show how the procedure presented in section 2, specialized to a double unfolding of an $A_n$ singularity, can be related to the twistor space construction of $G_2$ holonomy cones considered in [3]. In particular we will find a connection with twistor spaces over weighted projective spaces $\mathbb{WCP}^2_{k_1,k_2,k_3}$. The advantage of the twistor space constructions is that they automatically provide a $G_2$ holonomy metric without extra work. However, the physics is harder to understand [3] and an interpretation in terms of sets of D6-branes intersecting over $\mathbb{R}^6$ is not always available [8]. As was explained in the previous section the unfolding procedure always gives singularities that can be viewed as intersecting D6-branes in Type IIA string theory and their physical interpretation is more accessible via duality with heterotic string theory. Unfortunately, it is very hard in general to find the $G_2$ holonomy metrics explicitly. But in order to understand the physics this is not really needed and duality with Type IIA strings guarantees that intersections of D6-branes at supersymmetric angles have lifts to $G_2$ holonomy metrics in M-theory.

Let us quickly review the example studied in [3] pertaining to a cone on the twistor space over $\mathbb{WCP}^2_{k_1,k_2,k_3}$. For this purpose we consider the vacuum manifold of a $U(1)$ gauge theory coupled to three hypermultiplets which contain complex scalars $a_i, \bar{b}_i, i = 1, 2, 3$ with $U(1)$ charges $(q_i, -q_i)$. The vacuum manifold is given by the D-term and F-term constraints modulo gauge transformations with the triplet of FI-terms set to zero. After
rescaling the scalars, \( x_i = \sqrt{q_i} a_i \) and \( y_i = \sqrt{q_i} b_i \), we find the equations

\[
\sum_{i=1}^{3} |x_i|^2 - |y_i|^2 = 0, \quad \sum_{i=1}^{3} x_i y_i = 0.
\] (15)

These equations describe a cone over \( SU(3) \) embedded in \( \mathbb{C}^6 = \mathbb{H}^3 \). The conical structure follows from the invariance of the equations (15) under the simultaneous rescaling \( x_i \rightarrow \lambda x_i \) and \( y_i \rightarrow \lambda y_i \). In order to find the base of the cone we fix this invariance by intersecting the cone with an eleven-sphere \( \sum |x_i|^2 + |y_i|^2 = 2 \subset \mathbb{C}^6 \). Together with (15) we find

\[
\sum_{i=1}^{3} |x_i|^2 = |y_i|^2 = 1, \quad \sum_{i=1}^{3} x_i y_i = 0,
\] (16)

which are nothing but the orthonormality conditions of two complex three-vectors \( \vec{x} \) and \( \vec{y} \). Introducing a third vector \( \vec{z} = \vec{x} \times \vec{y} \) we can package these three vectors into an \( SU(3) \) matrix \( M = \{ \vec{x}, \vec{y}, \vec{z} \} \). The charges of the components of \( M \) under the \( U(1) \) gauge symmetry can be summarized in a charge matrix

\[
\begin{pmatrix}
q_1 & q_1 & -(q_2 + q_3) \\
q_2 & q_2 & -(q_1 + q_3) \\
q_3 & q_3 & -(q_1 + q_2)
\end{pmatrix}.
\] (17)

Furthermore, the \( SU(3) \) manifold admits an \( SU(2) \) action that rotates \( \vec{x} \) and \( \vec{y} \) into each other but leaves \( \vec{z} \) fixed. It acts on \( M \) by right multiplication

\[
M \rightarrow M \cdot \begin{pmatrix} A_{SU(2)} & 0 \\ 0 & 1 \end{pmatrix}.
\] (18)

This \( SU(2) \) contains an abelian subgroup \( U(1)_H \) that acts by right multiplication with \( \text{diag}\{e^{i\alpha}, e^{-i\alpha}, 1\} \) and the corresponding charge table is

\[
\begin{pmatrix}
1 & -1 & 0 \\
1 & -1 & 0 \\
1 & -1 & 0
\end{pmatrix}.
\] (19)

The identification of the twistor space proceeds as follows. The base of the twistor space is given by \( SU(3)/(SU(2) \times U(1)) \). First note that the quotient \( SU(3)/SU(2) \) is simply a copy of \( S^5 \) since it is given by \( \vec{z} \) which is \( SU(2) \) invariant and obeys \( |\vec{z}|^2 = 1 \). We have to further divide by \( U(1) \) where the charges of the \( z_i \) can be read off from the third column of (17). By a redefinition of the \( U(1) \) generator we can remove the overall minus sign and find that \( S^5/U(1) \) is \( \mathbb{C}\mathbb{P}^2_{k_1,k_2,k_3} \) with \( k_i = \{q_2 + q_3, q_1 + q_3, q_1 + q_2\} \), or, if all \( q_i \) are odd \( k_i = \frac{1}{2}\{q_2 + q_3, q_1 + q_3, q_1 + q_2\} \).
Finally, the $S^2$ fiber of the twistor space is given by $SU(2)/U(1)_H$, and the twistor space is given by $SU(3)/(U(1) \times U(1)_H)$, where $U(1)_H$ is a subgroup of the $SU(2)$ group action \([18]\). The $G_2$ holonomy manifold constructed as the cone over the twistor space has a codimension seven singularity. Since in the case at hand the base of the cone is homogeneous $G/H$ a resolution of the singularity may be provided by a cohomogeneity one metric of the type $ds^2 = dr^2 + g_{G/H}(r)$, where $g_{G/H}(r)$ is an $r$ dependent metric on the so-called principal orbit $G/H$ and $r \geq r_0$. The necessary conditions for a smooth resolution \([28]\) are the existence of a singular orbit $G/K$ with $G \supset K \supset H$ which has finite volume at $r = r_0$ and that $K/H$ becomes a round sphere $S^n$. This means that the cone is resolved to a $\mathbb{R}^{n+1}$ bundle over $G/K$. In our case $K = SU(2) \times U(1)$ is the only possibility with $G/K = WCP^2_{k_1,k_2,k_3}$ and $K/H = S^2$. Note, that in general this only removes the codimension seven singularity but $G/K$ itself may have orbifold singularities, as is the case for weighted projective spaces \([29]\).

In general, a twistor space can be constructed over any four-manifold $M$ which is Einstein with positive curvature and has a self-dual Weyl tensor. In that case we can immediately write down a $G_2$ holonomy metric \([30,31]\)
\[
    ds^2 = \frac{dr^2}{1 - r_0^4/r^4} + \frac{r^2}{2} ds_M^2 + \frac{r^2}{4} (1 - r_0^4/r^4) |Dt|^2 , \tag{20}
\]
with $Dt = dt + \epsilon_{ijk}A^j t^k$, $\sum t_i^2 = 1$. The $A^i$ are three one-form gauge fields with anti-selfdual field strengths $F^i = dA^i + \frac{1}{2} \epsilon_{ijk} A^j \wedge A^k$. The parameter $r_0$ in the metric (20) is a blow-up parameter. For $r_0 = 0$ the metric has a conical singularity at $r = 0$, but for $r_0 > 0$ the four-manifold $M$ is blown-up to finite size in the interior $r = r_0$ and the total space is an $\mathbb{R}^3$ bundle over $M$. However, the manifold $M$ itself can have orbifold singularities as is the case for weighted projective spaces.

Now we wish to relate this approach to the kind of models constructed by double unfoldings of HK quotient spaces as introduced in section 2. The two non-zero moment maps correspond to the gauge factors $U(1)_{n_1} \equiv H_1$ and $U(1)_{n_1+n_2} \equiv H_2$. The double unfolding procedure produces hypersurface equations in $\mathbb{C}^6$ whose explicit form can be found in \([14]\) modulo the $U(1)$ actions \([10]\). This constitutes a ten-dimensional manifold and we have to impose linear relations \([13]\) between the constants $c_i$, $r_i$ in order to obtain a seven-dimensional space. The appropriate choice is $r_1 = r_2$, $c_1 = c_2$, so that \([14]\) becomes
\[
|z_1|^2 - |z_2|^2 - 2|z_3|^2 + 2|z_4|^2 + |z_5|^2 - |z_6| = 0 ,
\]
\[
z_1 z_2 - 2 z_3 z_4 + z_5 z_6 = 0 . \tag{21}
\]
where the $z_i$ have the following charges

| | $z_1$ | $z_2$ | $z_3$ | $z_4$ | $z_5$ | $z_6$ |
|---|---|---|---|---|---|---|
| $H_1$ | $1/n_1$ | $-1/n_1$ | $-1/n_2$ | $1/n_2$ | $0$ | $0$ |
| $H_2$ | $0$ | $0$ | $1/n_2$ | $-1/n_2$ | $-1/n_3$ | $1/n_3$ |

(22)

In order to bring (21) to a more familiar form, we make the replacements

$$(z_1, \ldots, z_6) \to (a_1, \bar{b}_1, \bar{b}_2/\sqrt{2}, -a_2/\sqrt{2}, a_3, \bar{b}_3)$$

(23)

which turns (21) into

$$3 \sum_{i=1}^3 |a_i|^2 - |b_i|^2 = 0 , \quad \sum_{i=1}^3 a_i \bar{b}_i = 0 .$$

(24)

At this point we have achieved part of our goal of identifying the unfolding and twistor space constructions. It remains to match the charges under $H_1$ and $H_2$ with the charges in (17) and (19) under $U(1)$ and $U(1)_H$. We are of course allowed to take arbitrary linear combinations of the generators. After rescaling the generators of $H_{1,2}$ we can turn the fractional charges of the $a_i$ and $b_i$ into integers

| | $a_1$ | $b_1$ | $a_2$ | $b_2$ | $a_3$ | $b_3$ |
|---|---|---|---|---|---|---|
| $H_1$ | $n_2$ | $n_2$ | $-n_1$ | $-n_1$ | $0$ | $0$ |
| $H_2$ | $0$ | $0$ | $n_3$ | $n_3$ | $-n_2$ | $-n_2$ |

(25)

As before we have a cone over $SU(3)$ modded out by $U(1)^2$ generated by $H_{1,2}$. By introducing the vector $\vec{c} = \vec{a} \times \vec{b}$ we get the $SU(3)$ matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} .$$

(26)

The charges of the fields in this matrix under $H_1$ and $H_2$ respectively are

$$Q_1 = \begin{pmatrix} n_2 & -n_1 & 0 \\ n_2 & -n_1 & 0 \\ n_1 & -n_2 & n_1 - n_2 \end{pmatrix}$$

and

$$Q_2 = \begin{pmatrix} 0 & n_3 & -n_2 \\ 0 & n_3 & -n_2 \\ n_2 - n_3 & n_2 & -n_3 \end{pmatrix}$$

(27)

In the simplest case we take $n_1 = n_2 = n_3 = 1$. We easily see that $Q_1$ and $Q_1 - Q_2$ agree with the charges of $U(1)$ and $U(1)_H$ respectively. This corresponds to the case of the cone over the twistor space of $\mathbb{CP}^2$.

Next we take two of the $n_i$ equal. Without loss of generality we can choose $n_1 = n_2 = p$ and $n_3 = q$. Note that the third column of $Q_1$ is zero which is necessary to make
contact with the twistor space construction. To complete the identification we introduce
\( \tilde{Q}_1 = \frac{1}{p} Q_1 \) and \( \tilde{Q}_2 = \frac{q}{2p} Q_1 + Q_2 \) which give
\[
\begin{pmatrix}
1 & -1 & 0 \\
1 & -1 & 0 \\
1 & -1 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\frac{q}{2} & \frac{q}{2} & -p \\
\frac{q}{2} & \frac{q}{2} & -p \\
q - \frac{p}{q} & p - q/2 & -q
\end{pmatrix}
\].
\quad (28)

From (19) and (17) this gives the twistor space over \( \mathbb{WCP}^2_{p,p,q} \). According to [3] this singularity has an interpretation in Type IIA as an intersection of three groups of D6-branes with multiplicities \( p, p \) and \( q \). The \( N = 1 \) supersymmetric gauge theory on these branes has gauge group \( SU(p) \times SU(p) \times SU(q) \) and is coupled to chiral multiplets in the \( (p, \bar{p}, 1) + (\bar{p}, 1, q) + (1, p, \bar{q}) \) representation.

Finally, when all \( n_i \) are distinct it is impossible to match the two constructions as can seen from the charge table (27). It turns out that this has an interesting physical interpretation. As we explained in section 2 the unfolding procedure naturally gives configurations of intersecting D6-branes in flat space, but it does not provide an easy way to construct the \( G_2 \) holonomy metric. On the other hand the manifolds obtained via the twistor space construction inherit a natural \( G_2 \) structure and the metric can be found explicitly [30, 31]. However, the physics of these singularities is in general much harder to understand because of the occurrence of codimension six singularities in addition to the much better understood codimension four singularities. In particular the cases based on twistor spaces of \( \mathbb{WCP}^2_{k_1,k_2,k_3} \) with all \( k_i \) different does not have a simple interpretation as an intersection of flat D6-branes over \( \mathbb{R}^6 \). Only if two or three of the \( k_i \) are equal is such an interpretation available which is precisely when we can make a connection with the unfolding construction.

4 \( D_n \) and \( E_n \) hyperkähler quotients

M-theory on ALE-spaces describing \( D_n \) and \( E_n \) singularities give rise to \( SO(2n) \) and \( E_n \) gauge groups, respectively. (We will focus on simply laced groups—for more details on the other, disconnected components of the space of seven-dimensional theories with sixteen supercharges, see [25].) The enhanced gauge symmetry comes from M2-branes wrapping vanishing 2-cycles which intersect according to the Cartan matrix for the corresponding Lie group [4]. In this section we will consider M-theory realizations of \( SO(2n) \) and \( E_n \) gauge symmetries, as well as charged chiral matter, in terms of the hyperkähler quotient construction described in section 2. We will concentrate on the HK quotient manifolds describing the unfolding of \( D_n \) singularities, and return to the exceptional groups at the end of the section.
The procedure is carried out in two steps. First, we have to describe the $D_n$ singularity in terms of a hyperkähler quotient construction. This is done by considering a $\mathbb{Z}_2$ orbifold of an $A_{n'}$-model. Second, the matter is described by unfolding the $D_n$ theory. Just as in the $N = 2$ compactification of type IIA theory on a K3-fibered Calabi-Yau manifold there are several possibilities for how this can be done [19], e.g. we get a $2(n - 1)$ of $SO(2(n - 1))$ when $D_n \to D_{n-1}$. Since we represent the $D_n$ model as an orbifold of an $A_{n'}$-model we have to consider the relevant unfolding in the covering space. Thus, we can use our general framework from section 2. The generalization from the $A$-series to the $D$-series gives us in addition the possibility of describing matter in the antisymmetric representation of $SU(n)$. This is of interest when studying grand unified models such as $SU(5)$ in which the $10$ plays an important role.

4.1 Orbifold of A-series gives D-series

We now use the hyperkähler quotient construction to describe a $D_n$ singularity. Because of the non-abelian nature of the corresponding $N = 1$ supersymmetric quiver theory, we will represent the $D_n$ singularity in terms of a $\mathbb{Z}_2$ orbifold of an $A_{n'}$ singularity. In particular, we will focus on the $\mathbb{Z}_2$ invariant untwisted sector. From the analysis in appendix B it is sufficient to consider an $A_{2n-1}$ singularity followed by a $\mathbb{Z}_2$ orbifold. This allows us to represent all of the unfoldings of the $D_n$ HK quotient in terms of $\mathbb{Z}_2$ invariant deformations of the $A_{2n-1}$ singularity.

To that effect we start by recalling the set-up for the hyperkähler quotient construction of an $A_{2n-1}$ singularity as discussed in section 2. The $A_{2n-1}$ singularity is locally realized as $\mathbb{C}^2/\mathbb{Z}_{2n}$, which in turn can be defined as a HK quotient $H^{2n}/U(1)^{2n-1}$. The complex scalars $(M_i, \overline{M}_i) \in \Phi_i$, $i = 0, \ldots, 2n - 1$ can be combined into gauge invariant baryons and mesons, which from Eq. (3) satisfy

$$xy = z^{2n}.$$  \hspace{1cm} (29)

In order to obtain a $D_n$ singularity we have to further divide by a $\mathbb{Z}_2$ generator. (For more detail, see appendix B.) The action of the $S$ generator translates into

$$S : \quad (z_1, z_2) \to (z_2, -z_1) \quad : \quad (x, y, z) \to (y, x, -z).$$  \hspace{1cm} (30)

i.e. $S$ acts non-trivially on the $SU(2n)$ gauge invariant variables. Following the discussion in appendix B new $SO(2n)$ invariant combinations can be defined (see Eq. (61)) which satisfies the relation of a $D_n$ singularity (62).
Next, let us consider a general unfolding of the hyperkähler quotient describing the $A_{2n-1}$ singularity along the lines of section 2. We consider a $\mathbb{Z}_2$ symmetric double unfolding given by (8), i.e. $A_{2n-1} \rightarrow A_{2(n-r)} \times A_{r-1}$. In terms of the invariant coordinates (11) we have the following relation

$$x \cdot y = z^{2(n-r)}w^ru^r,$$

(31)

where $z = z_1z_2$, $w = z_3z_4$ and $u = z_5z_6$.

We can now use the F-term equations in (8) to express $w$ and $u$ in terms of $z$,

$$w = z + c_1, \quad u = w + c_2 = z - c_1 \quad \text{with} \quad c_2 = -2c_1$$

(32)

In particular, the relation between the moment maps (13) is chosen to reflect the $\mathbb{Z}_2$ symmetric nature of the configuration. Thus, we can rewrite (31) as

$$x \cdot y = z^{2(n-r)}(z^2 - c_1^2)r.$$

(33)

The $\mathbb{Z}_2$ action in (30) now implies that

$$S: (x, y, z, w, u) \rightarrow (y, x, -z, -u, -w).$$

(34)

Equivalently, in terms of the $z_i$ we have

$$S: (z_1, z_2, z_3, z_4, z_5, z_6) \rightarrow (z_2, -z_1, z_6, -z_5, z_4, -z_3)$$

(35)

which is a symmetry of the D/F term equations (14) if $c_2 = -2c_1$ and $r_2 = -2r_1$ and can be viewed as the M theory lift of the orientifold action. When applied to (33) this gives the following unfolding of the $D_n$ singularity

$$Y^2 = ZX^2 - Z^{-1}Z^{n-r}(Z + t^2)r, \quad t^2 = -c_1^2.$$

(36)

Following Katz and Vafa’s analysis for localized matter hypermultiplets in type IIA [19] we will now show that (36) describes

$$SO(2n) \rightarrow SO(2(n-r)) \times SU(r), \quad (2(n-r), r) + (1, r(r-1)/2).$$

(37)

We first note that for $t = 0$ Eq. (36) describes a $D_n$ singularity. On the other hand, when $t \neq 0$ there is a $D_{n-r}$ singularity at $X = Y = Z = 0$ while at $X = Y = Z + t^2 = 0$ there is an $A_{r-1}$ singularity. The matter, as we will explain in more detail below, is obtained by decomposing the adjoint of $SO(2n)$ in terms of representations of $SO(2(n-r)) \times SU(r)$. Note that we keep the chiral multiplets that survive the breaking of $N = 4$ supersymmetry.
to $N = 1$ by the non-trivial fibration of the ALE space describing the $D_n$ singularity over the base parametrized by the moment maps.

The fact that the covering space of these spaces can be obtained by the double unfolding procedure of section 2 allows us to determine their metric in most cases. The covering space is of the type which can be related to the twistor space construction, as explained in section 3, and hence its metric is given by the metric on the cone over the twistor space of $\mathbb{CP}^2_{2(n-r-2),r,r}$ for $n - r > 2$. For $n - r = 2$ the double unfolding Eqs. (8)-(14) degenerates to a simple unfolding $[8, 3]$ and the metric on the covering space is a cone over $\mathbb{CP}^3/Z_r$. Hence, the metric we are interested in is just a $Z_2$ orbifold (33) of a known metric. However, this requires $n - r \geq 2$ for the following reason: Although we can represent all $D_n$ singularities as manifolds in terms of orbifolds of $A_n'$ singularities, this is not true for their metrics. In particular in [32] it was shown that the M theory lift of an $O6^-$ plane with $r$ D6-branes plus mirror branes is given by a $Z_2$ orbifold of a multi-center Taub-NUT metric if at least two D6-branes sit on top of the orientifold plane. If all D6-branes and the orientifold are together the metric is an orbifold of a single center Taub-NUT with charge $2r - 4$ where the $-4$ comes from the charge of the orientifold plane which is non-singular for $r \geq 2$. On the other hand a $O6^-$ plane or $O6^-$ plane with a single D6-brane is described by the Atiyah-Hitchin metric or a $Z_2$ orbifold of it, respectively, which are both smooth. In particular this means that only the cases where at least two D6-branes lie on top of the $O6^-$ plane can be related to orbifolds of $G_2$ spaces from the twistor space construction. These correspond to the breaking pattern $D_n \to D_{n-r\geq 2} \times A_{r-1}$.

4.2 Map from M-theory to IIA

In section 2 (see also appendix A) we discussed how compactifying on an $S^1$ maps the KK-monopoles in M-theory to D6-branes in type IIA theory. In particular, the fixed points under the $U(1)$ action give the locations of the D6-branes. The enhanced $SU(n)$ gauge symmetry and charged matter arises from M2-branes wrapped on 2-cycles in M-theory while in the type IIA picture open strings stretch between intersecting D6-branes.

This picture gets modified when we consider a $D_n$ singularity. Following our earlier discussion we represent the $D_n$ singularity in terms of a $Z_2$ orbifold of an $A_n'$ singularity. In a beautiful paper [26] Sen showed that the $Z_2$ parity symmetry of the Taub-Nut space, representing the KK-monopoles, gets mapped to $(-1)^{F_L} \Omega I$ where $I$ is the parity

---

8This is also the reason why the covering space is the cone over the twistor space of $\mathbb{CP}^2_{2(n-r-2),r,r}$ and not $\mathbb{CP}^2_{2(n-r),r,r}$. 

15
symmetry in $\mathbb{R}^3$ transverse to the D6-branes. Indeed, the parity transformation in M-theory is exactly the $\mathbb{Z}_2$ given in (30). We thus have M-theory compactified on an Atiyah-Hitchin space $[27]$ with overlapping KK monopoles at the origin. Furthermore, this is in agreement with the $D_n$ singularity described by (58). In type IIA the KK-monopoles get mapped to D6-branes which are paired by the $\mathbb{Z}_2$ action, while the Atiyah-Hitchin space corresponds to an $O6^-$ plane.

It is appropriate to make the following remarks here. For an $A_{2n-5}$ singularity we can formally write

$$xy = z^{-4}z^{2n}.$$  

This is a semiclassical description of $n$ pairs of D6-branes located on top of an $O6^-$ plane $[26]$. Second, to describe the lift of the $O6^-$ plane to M-theory we need to include the $S$-generator of the binary dihedral group $D_{n-2}$, corresponding to the above $\mathbb{Z}_2$ action. This leads indeed to the correct description of the Atiyah-Hitchin space. With the $n$ pairs of D6-branes taken into account one obtains a $D_n$ singularity, (58).

The above picture can be generalized to the unfolding of a $D_n$ singularity. As in section 2 we start by considering the covering space $A_{2n-1}$ and its deformations. For definiteness let $A_{2n-1} \rightarrow A_{2(n-r)-1} \times A_r^2$, i.e. following section 2 we have an enhanced gauge symmetry $SU(2(n-r)) \times SU(r)^2$ with bifundamental matter $(2(n-r), \mathbf{r}, \mathbf{1}) + (2(n-r), \mathbf{1}, r) + (\mathbf{1}, r, \mathbf{f})$. It is now straightforward to read off the gauge symmetry and matter after the $\mathbb{Z}_2$ transformation (34). It is clear that the $\mathbb{Z}_2$ acts such as $SU(2(n-r)) \rightarrow SO(2(n-r))$. Since the two $SU(r)$ are exchanged, see (34), this results in a single $SU(r)$. The matter is read off by decomposing the adjoint of $SO(2n)$ in terms of $SO(2(n-r)) \times SU(r)$.

### 4.3 Exceptional Groups

We now use the hyperkähler quotient construction to describe an $E_n$ singularity. In particular, we will focus on $E_6$ and $E_7$. From our discussion in appendix C it is sufficient to consider a $D_4$ singularity followed by a $Z_3$ ($Z_3 \times Z_2$) orbifold to construct an $E_6$ ($E_7$) singularity. Let us start by considering the case of $E_6$. In order to correctly describe the deformations of the $E_6$ singularity that are inherited as $U$-invariant deformations of the $D_4$ we find that the correct covering space is the $A_3$ singularity (see Appendix C). We consider the single unfolding, $A_3 \rightarrow A_1 \times A_1$. Following the analysis outlined in section 2, identifying the appropriate coordinates in terms of the chiral components $M_i, \overline{M}_i$ of the hypermultiplet $\Phi_i$ we find

$$xy = w^2u^2, \quad u = w + c.$$  

(39)
By a change of variables, $z = w - e^{i\pi/4}3^{1/4}t$, $c = -e^{i\pi/4}23^{1/4}t$, Eq. (39) becomes $xy = (z^2 + i\sqrt{3}t)^2$, where $z = z_1z_2$, $w = z_3z_4$ and $u = z_5z_6$, $z_i \in \mathbb{C}^6$ (see section 2). We show in appendix C that this is a deformation appropriate for studying the $E_6$ singularity (74).

The $U(1)$ action on the $z_i$, $i = 3, \ldots, 6$ follows straightforwardly from section 2 Eq. (10),

$$U(1) : (1/2, -1/2, -1/2, 1/2) . \quad (40)$$

Following the arguments in [3], by rescaling the charges this corresponds to a $\mathbb{Z}_2$ orbifold of $\mathbb{CP}^3$. In terms of the projective coordinates $w_i$ of $\mathbb{CP}^3$, where we identify $w_1 = z_3$, $w_2 = \bar{z}_4$, $w_3 = z_5$, $w_4 = \bar{z}_6$, the $\mathbb{Z}_2$ action is given by

$$\mathbb{Z}_2 : (w_1, w_2, w_3, w_4) \rightarrow (-w_1, -w_2, w_3, w_4) . \quad (41)$$

To obtain the unfolding of the $E_6$ singularity we have to further divide by $S$ and $U$,

$$S : (w_1, w_2, w_3, w_4) \rightarrow (w_4, -w_3, w_2, -w_1) \quad (42)$$

$$U : (w_1, w_2, w_3, w_4) \rightarrow$$

$$\left(\frac{\epsilon}{\sqrt{2}}(w_1 - w_4), \frac{\epsilon^7}{\sqrt{2}}(w_2 + w_3), \frac{\epsilon}{\sqrt{2}}(-w_2 + w_3), \frac{\epsilon^7}{\sqrt{2}}(w_1 + w_4)\right) , \quad (43)$$

with $\epsilon^8 = 1$. The action on the projective coordinates is deduced as in the case of the unfolding of the $D_n$ singularity as a symmetry of the D/F-term equations [35].

Finally, the matter localized at the singularity is obtained by decomposing the adjoint of $E_6$, 78, in terms of representations of $SU(3)^2 \times SU(2) \times U(1)$,

$$\left(3, \bar{3}, 1\right)_2 + \left(\bar{3}, 3, 2\right)_1 + \left(1, 1, 2\right)_{-3} . \quad (44)$$

Notice the matter charged with respect to all three non-abelian factors. This is an example in which a perturbative description in terms of open strings stretching between D6-branes cannot be used to explain this particular matter content.

For $E_7$ the situation is very similar. The $A_3$ singularity is once again the covering space, since the deformation given in (74) is invariant under the $V$ transformation. Hence the unfolding of the hyperkähler quotient proceeds exactly as for the $E_6$ case above. Thus, the base of the cone is $\mathbb{CP}^3/(\mathbb{Z}_2 \cdot S \cdot U \cdot V)$, where $\mathbb{Z}_2$, $S$ and $U$ are given in Eqs. (11), (12) and (43), respectively, and $V$ is given by

$$(w_1, w_2, w_3, w_4) \rightarrow (\alpha w_1, \alpha w_2, \alpha w_3, \alpha w_4) , \quad \alpha^8 = 1 . \quad (45)$$

This action is consistent with the D/F-term equations of a single unfolding [3].

17
The matter localized at the singularity is obtained by decomposing the adjoint of $E_7$, 133 in terms of representations of $SU(4) \times SU(3) \times SU(2) \times U(1)$\textsuperscript{[1]}

$$(4, 1, 2)_3 + (6, 3, 1)_2 + (4, \bar{3}, 2)_1 + (1, 3, 1)_4 . \quad (46)$$

As in the case of $E_6$ there is matter charged with respect to all three non-abelian factors a reflection of the non-perturbative nature of the existence of this particular matter content.

There is however no map from M-theory to type IIA in this case. Recall that for the D-series it was essential that the parity transformation was accompanied by $\Omega$. However, $U$ acts like a $Z_3$ which does not have a well-defined description in type IIA. Furthermore, the exotic matter in (44) and (46) is another reflection of the non-perturbative nature of these models. A similar situation occurs in the construction of $E_n$ singularities in F-theory, in which $\tau$, the axion-dilaton, has to take a particular value corresponding to a strongly coupled type IIB theory [3].

5 Discussion and Conclusion

In this paper we studied a large class of singular $G_2$ holonomy manifolds that give rise to chiral matter. Our construction is based on a generalization of the simple unfolding of HK quotient singularities [3] to multiple unfoldings. We showed that in general these manifolds, after a reduction along a suitable circle to type IIA, correspond to configurations of stacks of D6-branes intersecting over flat $\mathbb{R}^6$. This gives rise to chiral matter in bifundamental representations under the gauge group $SU(n_1) \times \ldots \times SU(n_m)$ which can be understood from the open string spectrum in type IIA. In general the explicit $G_2$ metrics on these spaces are not known but since the angles between the branes can be chosen to be supersymmetric this guarantees that a lift of the type IIA background to a $G_2$ holonomy metric exists. One might wonder where the angles between the brane enter in our construction. As long as we do not commit to a metric we only describe the space as a manifold in Eqs. (13) and (14) but there is still a lot of freedom in rescaling the coordinates which leads to the same manifold. So at this stage, without a metric, it does not make sense to talk about angles and trying to attribute them to the numbers appearing in Eq. (13). These will eventually be fixed by imposing $G_2$ holonomy but to find these metrics explicitly will be very hard, except for some of the examples discussed in section 3 and 4.

\textsuperscript{9}The relative $U(1)$ charge assignment is not clear from the hyperkähler quotient construction. A more detailed study of the anomaly cancellation condition will hopefully resolve this issue.
In particular in section 3 we were able to map manifolds obtained via double unfolding to cones over twistor spaces over weighted projective spaces $\mathbb{WP}^2_{k_1,k_2,k_3}$ if at least two of the $k_i$ are equal, say $k_2 = k_3$. In this case the metric and $G_2$ structure can be constructed explicitly $\cite{30,31}$. In type IIA this describes the intersection of three sets of D6-branes with $k_1$, $k_2$ and $k_2$ branes in every set.

Furthermore, we extended the possible representations that appear at the singularities from bifundamental representations of unitary groups to anti-symmetric representations of unitary groups and (bi)fundamental representations of orthogonal groups. This was achieved by introducing $\mathcal{O}6^-$ planes into the picture. In principle this can be done by unfolding $D_n$ singularities, which is technically harder because the orbifold action is non-abelian. But as we show in section 4 in many cases we can construct these spaces as $\mathbb{Z}_2$ orbifolds of double unfolded $A_{2n-5}$ singularities. For this to work the $\mathcal{O}6^-$ plane has to be superimposed with at least two D6-branes (and their mirrors, if we work in the covering space). In these cases we can do even more. The $G_2$ metric can be constructed explicitly since it is simply a $\mathbb{Z}_2$ orbifold of spaces arising from double unfoldings which can be related to the twistor space construction.

We are also able to describe unfoldings of $E_6$ and $E_7$ where the group is broken to three or more non-abelian factors. (For these constructions, the metric is also known to have $G_2$ holonomy since the unfolding is given by a cone over a non-abelian orbifold of $\mathbb{CP}^3$.) The corresponding chiral matter fields appear in representations charged simultaneously under three gauge factors. Obviously, this has no analogue in weakly coupled string theory and cannot be explained by open strings stretching between D6-branes. It would be very interesting to generalize to other breaking patterns discussed in $\cite{19}$, e.g. $E_6 \rightarrow SO(10)$ and $E_7 \rightarrow E_6$ which give rise to the 16 of $SO(10)$ and the 27 of $E_6$, respectively.

It is definitely interesting and important to get more examples of singular $G_2$ spaces that produce chiral matter, e.g. recently a large class of conical $G_2$ spaces was constructed from a generalization of the twistor space construction, which automatically gives the corresponding $G_2$ metric $\cite{23}$ (see also $\cite{24}$). Another challenging task is to find the explicit metrics on the $G_2$ cones constructed via unfolding of HK quotient singularities. In the case of simple unfolding $\cite{3}$ the base of the cone is a weighted projective space $\mathbb{WP}^3_{n,n,m,m}$ but the corresponding nearly Kähler metric is not known. Similarly, for the general double unfolding with distinct $k_i$ or multiple unfoldings the metrics are not known. A possible route to finding them is to exploit the construction of $G_2$ holonomy metrics from harmonic three-forms $\Phi$. In this case one starts from an ansatz for the three-form $\Phi$ and imposing $G_2$ holonomy amounts to solving $d\Phi = d(*_g \Phi) = 0$, which is a highly non-linear differential equation for $\Phi$. The metric itself is a non-linear function...
of $\Phi$. This approach is very efficient and has been employed successfully in \cite{30, 36} to construct complete, non-compact $G_2$ holonomy spaces. See also \cite{34, 37} for a closely related method and \cite{38} where an effective Lagrangian approach is used.

Our understanding of honest M theory compactification on $G_2$ manifolds is hampered by the lack of a microscopic description of M theory and the scarcity of examples of compact $G_2$ manifolds. Dualities with string theory imply the existence of huge classes of compact $G_2$ manifolds, see e.g. \cite{14, 3}, but up to now only three methods exist, that can be used to construct them (See \cite{38} for a short review and more references). In particular most of the $G_2$ singularities that give rise to chiral matter, which were described in \cite{3} and this paper, do not arise in the known examples of compact $G_2$ manifolds. But since chiral matter is necessary for physically interesting compactifications more general constructions of compact $G_2$ manifolds are needed. A related problem is the absence of a simple topological condition for the existence of a $G_2$ structure on a given seven-manifold. This should be contrasted with the situation for Calabi-Yau manifolds where one only has to show the vanishing of the first Chern class.

**Acknowledgments**

We would like to thank C. I. Lazaroiu and E. Witten for discussions. P. B. would like to thank LBL and in particular the CIG, Berkeley for their hospitality. The work of P. B. was supported in part by the US Department of Energy under grant number DE-FG03-84ER40168. The work of A. B. is supported in part by the DOE under grant No. DE-FG03-92ER40701. A. B. would like to thank Therapy West for hospitality during the final stages of this research.

**A Reduction to Type IIA**

The singular $G_2$ holonomy manifolds $X$ we constructed via double (and multiple) unfolding of HK quotient spaces are all cones over some six-dimensional Einstein manifolds $Y$. Hence the metric can be written as

$$ds^2 = dr^2 + r^2 d\Omega_Y^2.$$  \hspace{1cm} (47)

We want to show that $M$-theory compactifications on these manifolds are the uplift of type IIA backgrounds of D6-brane and $O6^-$ plane intersecting over flat $\mathbb{R}^6$. This means that we have to identify a $U(1)$ isometry with the following properties \cite{8}: The coset
$X/U(1)$ is topologically $\mathbb{R}^6$, however, in general the metric will differ from flat space, since the dilaton, which is proportional to the size of the $U(1)$, varies over $\mathbb{R}^6$. In particular this means that the base of the cone $Y$ modded by this $U(1)$ action gives $S^5$ since $\mathbb{R}^6$ is a cone over $S^5$. Furthermore, the $U(1)$ may have fixed points in $\mathbb{R}^6$ which have the interpretation as the location of D6-branes/O6$^-\,$ planes only if they occur in codimension four of the seven-dimensional $G_2$ manifold. Since we claim that our manifolds correspond to the intersection of flat D6-branes/O6$^-\,$ planes we expect the fixed point sets to be copies of $\mathbb{R}^3$ inside $\mathbb{R}^6$ and, because of the conical structure of $X$, they correspond to copies of two-spheres, $S^2 = \mathbb{R}^3/U(1)$ inside $S^5 = Y/U(1)$.

The hyper-Kähler moment map plays an important role in what follows. Let us represent $\mathbb{R}^4 = \mathbb{C}^2$ by a complex two-component vector $x = (x_1, x_2)^t$ and introduce the $U(1)$ action $x \to e^{i\theta}x$. Then the quotient $\mathbb{R}^4 \to \mathbb{R}^4/U(1) = \mathbb{R}^3$ is given in terms of $U(1)$ invariant moment maps

$$\vec{r} = x^\dagger \vec{\sigma} x \equiv (x, \vec{\sigma} x) ,$$

where $\vec{\sigma}$ are the standard hermitian Pauli matrices. This map was already introduced in Eq. (1) in component form.

To make contact with our constructions via double (and multiple) unfoldings Eq. (14) we reorganize the coordinates of $\mathbb{C}^6$ in three groups of two:

$$x = (z_1, \bar{z}_2) , \quad y = (z_3, \bar{z}_4) , \quad z = (z_5, \bar{z}_6)$$

(49)

It is then easy to see that the equations in (48) are just linear combinations of the moment maps of $x, y$ and $z$, and are invariant under the two $U(1)$ actions (10). The $U(1)$ actions on the new coordinates take the form

$$(x, y, z) \to (xe^{-i\theta_1/n_1}, ye^{i(\theta_1+\theta_2)/n_2}, ze^{-i\theta_2/n_3}) .$$

(50)

In the remaining part of this section we want to show that the $U(1)$ action

$$(x, y, z) \to (e^{i\theta} x, y, z)$$

(51)

provides the reduction to type IIA with the desired features listed above. First we note that the space indeed has a conical structure since we can rescale $x, y, z$ by a non-zero real number without changing the F and D term equations. To construct the base of the cone we just have to intersect this hypersurface with the eleven-sphere$^{10}$

$$(x, x)^2 + (y, y)^2 + (z, z)^2 = 1 ,$$

(52)

$^{10}$Note, that this describes an eleven-sphere although the condition is quartic in the coordinates $z_i$. It is however advantageous for the rest of the section to use this convention. Since we are only interested in topology here, we could actually use arbitrary positive exponents in this equation.
which avoids the origin in $\mathbb{C}^6$. Dividing out the three $U(1)$ actions leads to an eight-sphere embedded in $\mathbb{R}^9$. The eight-sphere can be parametrized by the nine-vector

\[
(\vec{r}_x, \vec{r}_y, \vec{r}_z) = \left( s \vec{e}_x, t \vec{e}_y, \sqrt{1 - s^2 - t^2} \vec{e}_z \right),
\]

with $s \in [0, 1]$, $t \in [0, 1]$ and $\vec{e}_x, \vec{e}_y, \vec{e}_z \in \mathbb{S}^2$. If we define $\vec{e}_x = (x, \vec{\sigma}_x)/(x, x)$, $\vec{e}_y = (y, \vec{\sigma}_y)/(y, y)$, $\vec{e}_z = (z, \vec{\sigma}_z)/(z, z)$, $s = (x, x)$, $t = (y, y)$, then this gives an isomorphism from $\mathbb{S}^{11}/U(1)^3$ to $\mathbb{S}^8$.

The D/F-term equations impose a linear relation between the three terms in (53), which for general D and F-terms takes the form

\[
\vec{r}_x + (R - I_{3 \times 3}) \cdot \vec{r}_y - R \cdot \vec{r}_z = 0
\]

where the three by three matrix $R$ was defined in (13). These are the equations of three hyperplanes intersecting the $\mathbb{S}^8$ defined by $|\vec{r}_x|^2 + |\vec{r}_y|^2 + |\vec{r}_z|^2 = 1$. The three equations in (54) are linearly independent, hence, this gives an $\mathbb{S}^5$ as desired. The same reasoning can be repeated for multiple unfoldings with the same result $Y = X/U(1) = \mathbb{S}^5$.

Finally, we want to find the fixpoint set of the $U(1)$ action (51), which, if it appears at codimension four, corresponds to the location of D6-branes and/or $\mathcal{O}6^-$ planes. For this we assume that the $\mathcal{O}6^-$ plane has at least two D6-branes on top of it. Otherwise the discussion becomes more involved since $D_{0,1}$ singularities which correspond to $\mathcal{O}6^-$ planes with zero or one D6-branes on top are represented by the Atiyah-Hitchin space or a $\mathbb{Z}_2$ orbifold of it, whereas $D_{n \geq 2}$ singularities are represented by much simpler $\mathbb{Z}_2$ orbifolds of ALE spaces with $A_{2n-5}$ singularities [32]. The corresponding $D_n$ ALF metric, if at least two D6-branes coincide with the $\mathcal{O}6^-$ plane, is just the multicentered Taub-NUT metric with $2n - 4$ KK-monopoles arranged in a $\mathbb{Z}_2$ symmetric fashion [32]. The $\mathbb{Z}_2$ is the M-theory lift of the orientifold action and the multi-Taub-NUT metric describes the double covering space before modding by the $\mathbb{Z}_2$.

From the M-theory circle action (51) and the $U(1)^2$ action given by (20) we see that (51) gives a fixed point set consisting of three components: $x = 0$, $y = 0$ and $z = 0$. On any of these components Eq. (52) reduces to the equation for an $\mathbb{S}^7$. Out of the three $U(1)$ actions one particular linear combination acts trivially, so that dividing out the $U(1)$ symmetries gives an $\mathbb{S}^5$. Finally, we have to impose the three linear relations (54) which yield an $\mathbb{S}^2$ as expected. The fixed point set is the union of three $\mathbb{S}^2$’s, which do not intersect except at the tip of the cone where the whole base shrinks to a point. Hence, in type IIA this corresponds to three sets of intersecting D6-branes. To find the multiplicities of the D6-branes we have to identify the type of singularity at the fixed points. Inspecting the $U(1)$ actions we find an $A_{n_1-1}$ singularity at $x = 0$, an $A_{n_2-1}$
singularity at $y = 0$ and an $A_{n_3-1}$ singularity at $z = 0$, which means that we have three stacks of D6-branes with multiplicities $n_i$, $i = 1, 2, 3$.

It is important to note that the codimension four singularities all coincide with the $U(1)$ fixed point set. Only for this reason do we have a clean interpretation in terms of D6-brane configurations. In the twistor construction (see also [23]) this is not the case in general and the interpretation is more complicated which makes it hard to identify the correct spectrum of chiral fields at the singularity [3, 23]. Also, most of the arguments go through for multiple unfoldings under relatively mild assumptions for the generalization of Eqs. (14) and (54). Concretely, this means for an $m$-fold unfolding that there should be $3m - 3$ linearly independent equations. In this case the corresponding $G_2$ singularity corresponds to the intersection of $m + 1$ stacks of D6-branes.

**B Construction of $D_n$ singularities**

$D_n$ singularities can be constructed by dividing the complex two-plane $\mathbb{C}^2$ by a finite non-abelian group, the binary dihedral group $D_{n-2}$ of rank $4(n-2)$, generated by $T = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ with $\xi^{n-2} = 1$, and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acting on $(z_1, z_2) \in \mathbb{C}^2$ [39]. The well-known form of the singularities as equations in $\mathbb{C}^3$ can be found by introducing invariant monomials in terms of the coordinates $z_1, z_2$.

We will present this in a two step process where we first divide by the $\mathbb{Z}_{2(n-2)}$ subgroup generated by $T$, followed by the $\mathbb{Z}_2$ generated by $S$. A $T$-invariant set of variables is given by

$$x = z_1^{2(n-2)}, \quad y = z_2^{2(n-2)}, \quad z = z_1 z_2$$

yielding the $A_{2n-5}$ singularity

$$xy = z^{2(n-2)}.$$  

The fully invariant set of variables is

$$
Y &= \frac{1}{2}(x + y) = \frac{1}{2}(z_1^{2n-4} + z_2^{2n-4}) \\
X &= \frac{1}{2}z(x - y) = \frac{1}{2}z_1 z_2 (z_1^{2n-4} - z_2^{2n-4}) \\
Z &= z^2 = z_1^2 z_2^2
$$

which yields the equation for the $D_n$ singularity

$$X^2 = YZ^2 - ZZ^{n-2}.$$  

Note that in deriving (58) we have in the last term used the $A_{2n-5}$ relation (56) above.
It is clear from (58) that there are only \(n - 2\ \mathbb{Z}_2\) \((S)\)-invariant deformations of (56), while a \(D_n\) singularity has \(n\) deformations. However, this problem can be circumvented. We start by considering \(\mathbb{Z}_{2n}\) generated by \(T\), for which

\[
x = z_1^{2n}, \quad y = z_2^{2n}, \quad z = z_1 z_2
\]

yield the \(A_{2n-1}\) singularity

\[
xy = z^{2n}.
\]

Then, introduce a new set of invariant variables (under the \(T\) and \(S\) generators)\footnote{This map is not valid at \(x, y \neq 0, z = 0\). As we argue below, there are certain limits when \(x, y, z\) all vanish in which \((59)\) is still well-defined.}

\[
Y = \frac{1}{2} z^{-2}(x + y) = \frac{1}{2} (z_1 z_2)^{-2}(z_1^{2n} + z_2^{2n})
\]

\[
X = \frac{1}{2} z^{-1}(x - y) = \frac{1}{2} (z_1 z_2)^{-1}(z_1^{2n} - z_2^{2n})
\]

\[
Z = z^2 = z_1^2 z_2^2
\]

which also yields the equation for a \(D_n\) singularity

\[
X^2 = ZY^2 - Z^{-1} z^n.
\]

However, the difference is that all of the deformations of (62) can be accounted for when deforming (60) with \(S\)-invariant deformations,

\[
xy = \prod_{i=1}^{n} (z^2 - z_i^2) \rightarrow (63)
\]

\[
X^2 = ZY^2 - Z^{-1} \left( \prod_{i=1}^{n} (Z + Z_i) - \prod_{i=1}^{n} Z_i \right) + 2Y \prod_{i=1}^{n} Z_i \quad (64)
\]

and the map (59) becomes

\[
Y = \frac{1}{2} z^{-2}(x + y + 2 \prod_{i=1}^{n} z_i)
\]

\[
X = \frac{1}{2} z^{-1}(x - y)
\]

\[
Z = z^2.
\]

This argument agrees with the general idea that when the D6-branes are located away from the \(O6^-\) plane they do not feel its effect, and the theory is perfectly described by \(n\) pairs of D6-branes [29]. However, we have to be careful when considering the limit \(Z_i \rightarrow 0\), \(i.e.\) is the map between the covering space of deformations of the \(A_{2n-1}\) singularity and the space of deformations of the \(D_n\) singularity well-defined in this limit? For our purpose it is enough to analyze limits of deformations corresponding to

\[
D_n \rightarrow D_{n-r} \times A_{r-1}, \quad (66)
\]
as they are the ones relevant for studying the localized matter. Hence, let $z_i^2 = -t^2$, $i = 1, \ldots, r$. In the $A_{2n-1}$ covering space we get from (63)

$$xy = z^{2(n-r)}(z^2 + t^2)^r$$

while the $D_n$ deformation from (64) is similarly given by

$$X^2 = ZY^2 - Z^{-1}(Z^{n-r}(Z + t^2)^r - \delta_{n,r}t^{2n}) + 2\delta_{n,r}t^{n}y.$$ (68)

Clearly, when $t \neq 0$ we have a $D_{n-r}$ singularity at $X = Y = Z = 0$ while at $X = Y = Z - t^2 = 0$ we have an $A_{r-1}$ singularity. In the covering space this corresponds to an $A_{2(n-r)-1}$ singularity at $x = y = z = 0$ while at $x = y = z \pm it = 0$ we have two $A_{r-1}$ singularities respectively. Thus, for $t \neq 0$ we have a well-defined map from the $\mathbb{Z}_2$ invariant deformations of $A_{2n-1}$ to the deformations of $D_n$ given by (65).

Let us now consider the limit $t \to 0$. From (68) we find a $D_n$ singularity at $X = Y = Z = 0$. In the covering space, both $x$ and $y$ vanish at the location of the singularities when $t \neq 0$. Thus, in spite of the apparent singular map due to the negative powers of $z$ in (63), the limit $t \to 0$ is well-defined also in the covering space, and

$$A_{2(n-r)-1} \times (A_{r-1})^2 \to A_{2n-1}.\quad (69)$$

We can therefore extend the map between the deformation spaces to also include the origin in the limit $t \to 0$.\[25]

Finally, note that for $r = n$ the $t \neq 0$ deformation corresponds to $D_0 \times A_{n-1}$, where $D_0$, the Atiyah-Hitchin space, is located at $X = Y = Z = 0$ and the $A_{n-1}$ singularity is at $X = Y - t^{2(n-2)} = Z + t^2 = 0$. Just as for $r < n$ the limit $t \to 0$ is well-defined; $D_0 \times A_{n-1} \to D_n$ and in the covering space $A_{n-1}^2 \to A_{2n-1}$.

The $D_0$ singularity is a smooth configuration obtained from the $D_1$ singularity by a $\mathbb{Z}_2$ orbifold $(X, Y, Z) \to (-X, -Y, Z)$, where the $X, Y, Z$ are given in terms of the $A_1$ covering space coordinates $x, y, z$ in (61) for $n = 1$. Introducing new variables $\tilde{Y} = Y^2, \tilde{X} = XY$ we find

$$\tilde{X}^2 = Z\tilde{Y}^2 - \tilde{Y}.$$ (70)

Indeed, (70) is non-singular.

\[25\]There are other limits in which the origin can be included but this suffices for our discussion.
C Construction of $E_n$ singularities

The $E_n$ singularities can be constructed by extending the binary dihedral group action, $D_2$, on the complex two-plane, $C^2$ \[^{[39]}\]. Consider the following actions on $(z_1, z_2) \in C^2$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} e^7 & e^7 \\ e^5 & e \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} e & 0 \\ 0 & e^7 \end{pmatrix},$$

(71)

where $e^8 = 1$. The binary tetrahedral group, $T$ of order 24 corresponding to $E_6$ is generated by adding $U$ to $D_2$. The $E_7$ singularity is obtained by adding $V$ to $T$ which gives the binary octahedral group of order 48.

Let us first consider the $E_6$ singularity. In terms of the $D_4$ invariant variables $X, Y, Z$ we have the following set of variables invariant under $T$, $S$ and $U$,

$$\tilde{X} = Y(Y^2 - 9Z^2)$$
$$\tilde{Y} = 3Z^2 + Y^2$$
$$\tilde{Z} = (-108)^{1/4}X$$

(72)

which gives the equation for an $E_6$ singularity

$$\tilde{X}^2 = \tilde{Y}^3 + \frac{\tilde{Z}^4}{4}.$$  

(73)

We are interested in the deformation space associated to the $E_6$ singularity. In analogy with the $D_n$ singularities we want to describe this parameter space in terms of the covering space of deformations of the $A_3$ singularity. However, it is not possible to find an auxiliary representation in which all the deformations of $E_6$ can be represented. Since the $U$-invariant deformation also has to be invariant under the $S$ transformations of the $A_3$ singularity we are left with only one invariant deformation,

$$xy = (z^2 + i\sqrt{3}t^2)^2$$

(74)

which is mapped to the following deformation of the $E_6$ singularity

$$\tilde{X}^2 = \tilde{Y}^3 + \frac{\tilde{Z}^4}{4} - 3t^2\tilde{Y} \tilde{Z}^2 + 9t^4\tilde{Y}^2 - 4t^6\tilde{Z}^2 + 24t^8\tilde{Y} + 16t^{12}.$$  

(75)

Clearly, when $t = 0$ (75) describes an $E_6$ singularity. When $t \neq 0$ it is straightforward to show that there are three different singularities located at $(\tilde{X}, \tilde{Y}, \tilde{Z}) = (0, 0, \pm \sqrt{2}t^3)$ and $(0, -4t^4, 0)$ corresponding to $A_2$ and $A_1$ type singularities respectively. Thus, the deformations in (75) correspond to the following breaking of $E_6$

$$E_6 \to A_2^2 \times A_1.$$  

(76)
Let us now turn to the $E_7$ singularity. We can express the $D_4$ invariant variables in combinations that are invariant under $T$, $S$, $U$ and $V$,

\[
\begin{align*}
\hat{X} &= XY(Y^2 - 9Z^2) \\
\hat{Y} &= 48^{1/3}(3Z^2 + Y^2) \\
\hat{Z} &= 3X^2
\end{align*}
\]  

which gives the equation for an $E_7$ singularity

\[
\hat{X}^2 = \hat{Z}\hat{Y}^3 + 16\hat{Z}^3.
\]

As for $E_6$ we are interested in the deformation space associated to the $E_7$ singularity. The $U$ and $S$-invariant deformation of the $A_3$ singularity is also invariant under the $V$-transformation, and gets mapped to the following $E_7$ deformation

\[
\hat{X}^2 = \frac{\hat{Y}^3\hat{Z}}{16} - Z^3 + 12t^2\hat{Y}\hat{Z}^2 - 36t^4\hat{Y}^2\hat{Z} - 4t^6\hat{Z}^2 + 24t^8\hat{Y}\hat{Z} - 4t^{12}\hat{Z}.
\]

Clearly, when $t = 0$ (79) describes an $E_7$ singularity. When $t \neq 0$ it is straightforward to show that there are three different singularities located at $(\hat{X}, \hat{Y}, \hat{Z}) = (0, t^4, 0)$, $(0, t^4/4, 0)$ and $(0, 0, -2t^6)$ corresponding to $D_3 \equiv A_3$, $A_2$ and $A_1$ type singularities respectively. Thus, the deformations in (79) correspond to the following breaking of $E_7$

\[
E_7 \rightarrow (D_3 \equiv A_3) \times A_2 \times A_1.
\]

References

[1] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, Phys. Rev. Lett. 54 (1985) 502.

[2] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B258 (1985) 46.

[3] B. Acharya and E. Witten, “Chiral fermions from manifolds of G(2) holonomy”, hep-th/0109152.

[4] E. Witten, Nucl. Phys. B443 (1995) 85, hep-th/9503124.

[5] G. Papadopoulos and P. K. Townsend, Phys. Lett. B357 (1995) 300, hep-th/9506150.

[6] B. S. Acharya, “On realising N = 1 super Yang-Mills in M theory”, hep-th/0011089.
[7] M. Atiyah, J. Maldacena and C. Vafa, J. Math. Phys. 42 (2001) 3209, hep-th/0011256.

[8] M. Atiyah and E. Witten, “M-theory dynamics on a manifold of G(2) holonomy”, hep-th/0107177.

[9] E. Witten, “Anomaly cancellation on G(2)manifolds”, hep-th/0108168.

[10] P. Horava and E. Witten, Nucl. Phys. B460 (1996) 506, hep-th/9510209; Nucl. Phys. B475 (1996) 94, hep-th/9603142.

[11] N. Seiberg, Phys. Lett. B408 (1997) 98, hep-th/9705221.

[12] J. Gomis, Nucl. Phys. B606, 3 (2001), hep-th/0103115.

[13] S. Kachru and J. McGreevy, JHEP 0106, 027 (2001), hep-th/0103223.

[14] M. Cvetic, G. Shiu and A. M. Uranga, Phys. Rev. Lett. 87, 201801 (2001), hep-th/0107143; Nucl. Phys. B615 (2001) 3, hep-th/0107160; “Chiral type II orientifold constructions as M theory on G(2) holonomy spaces”, hep-th/0111179.

[15] S. Gukov and D. Tong, JHEP 0204, 050 (2002), hep-th/0202126.

[16] P. Kronheimer, “The Construction of ALE Spaces As Hyper-Kahler Quotients”, J. Diff. Geom. 28 (1989) 665; “A Torelli-Type Theorem For Gravitational Instantons”, J. Diff. Geom. 29 (1989) 685.

[17] M. B. Green, J. A. Harvey and G. W. Moore, Class. Quant. Grav. 14 (1997) 47, hep-th/9605033.

[18] M. Berkooz, M. R. Douglas and R. G. Leigh, Nucl. Phys. B480 (1996) 265, hep-th/9606139.

[19] S. Katz and C. Vafa, Nucl. Phys. B497 (1997) 146, hep-th/9606086.

[20] A. Gray, “Weak holonomy groups”, Math. Z. 123 (1971) 290.

[21] E. Witten, Nucl. Phys. B403, (1993) 159, hep-th/9301042.

[22] D. R. Morrison and M. Ronen Plesser, toric varieties,” Nucl. Phys. B440, (1995) 279, hep-th/9412239.

[23] C. I. Lazaroiu and L. Anguelova, “M-theory compactifications on certain ‘toric’ cones of G2 holonomy”, hep-th/0204249; L. Anguelova and C. I. Lazaroiu, “M-theory on ‘toric’ G2 cones and its type II reduction”, hep-th/0205070.
[24] K. Behrndt, “Singular 7-manifolds with G(2) holonomy and intersecting 6-branes”, hep-th/0204061.

[25] J. de Boer, R. Dijkgraaf, K. Hori, A. Keurentjes, J. Morgan, D. R. Morrison and S. Sethi, Adv. Theor. Math. Phys. 4 (2002) 995, hep-th/0103170.

[26] A. Sen, JHEP 9709 (1997) 001, hep-th/9707123.

[27] M. F. Atiyah and N. J. Hitchin, Phys. Lett. A107 (1985) 21.

[28] R. Cleyton and A. Swann, “Cohomogeneity-one G2-structures”, math.DG/0111056.

[29] K. Galicki and H. B. Lawson, “Quaternionic Reduction And Quaternionic Orbifolds”, Math. Ann. 282 (1988) 1.

[30] R.L. Bryant and S. Salamon, “On the construction of some complete metrics with exceptional holonomy”, Duke Math. J. 58 (1989) 829.

[31] G. W. Gibbons, D. N. Page and C. N. Pope, Commun. Math. Phys. 127 (1990) 529.

[32] G. Chalmers, M. Rocek and S. Wiles, JHEP 9901 (1999) 009, hep-th/9812212.

[33] J. A. Minahan and D. Nemeschansky, Nucl. Phys. B489 (1997) 24, hep-th/9610076.

[34] A. Brandhuber, J. Gomis, S. S. Gubser and S. Gukov, Nucl. Phys. B611 (2001) 179, hep-th/0106034.

[35] M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, Nucl. Phys. B620, 3 (2002), hep-th/0106026.

[36] A. Brandhuber, Nucl. Phys. B629 (2002) 393, hep-th/0112113.

[37] M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, Phys. Rev. Lett. 88, 121602 (2002) hep-th/0112098, “A G(2) unification of the deformed and resolved conifolds”, hep-th/0112138.

[38] D. Joyce, “Constructing compact manifolds with exceptional holonomy”, math.DG/0203158.

[39] For a nice review, see P. Slodowy, “Simple Singularities and Simple Algebraic Groups”, Lecture Notes in Math., Vol. 815, Springer Verlag, Berlin, 1980.