Existence of quantum isometry group for a class of compact metric spaces

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Abstract
We formulate a definition of isometric action of a compact quantum group (CQG) on a compact metric space, generalizing Banica’s definition for finite metric spaces, and show that any CQG action on a compact Riemannian manifold which is isometric in the geometric sense of [12] automatically satisfies the isometry condition of the present article. We also prove for certain special class of metric measure spaces the existence of the universal object in the category of those compact quantum groups which act isometrically and in a measure-preserving way.

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1 Introduction
It is a very natural and interesting question to study quantum symmetries of classical spaces, particularly metric spaces. In fact, motivated by some suggestions of Alain Connes, S. Wang defined (and proved existence) of quantum group analogues of the classical symmetry or automorphism groups of various types of finite structures such as finite sets and finite dimensional matrix algebras (see [22], [21]), and then these quantum groups were investigated in depth by a number of mathematicians including Wang, Banica, Bichon and others (see, for example, [1], [2], [5] and the references therein). However, it is important to extend these ideas and construction to the ‘continuous’ or ‘geometric’ set-up. In a series of articles initiated by us in [12] and then followed up in [8], [9], [10] and other articles, we have formulated and studied quantum group analogues of the group of isometries (or orientation preserving isometries) of Riemannian manifolds, including in fact noncommutative geometric set-up in the sense of [13] as well. It remains to see whether such construction can be done in a metric space set-up, without
assuming any finer geometric (e.g. Riemannian or spin) structures. This aim is partially achieved in the present article, generalizing Banica’s formulation of quantum isometry groups of finite metric spaces. Indeed, in [10], we have proposed a natural definition of ‘isometric’ action of a (compact) quantum group on an arbitrary compact metric space (extending Banica’s definition which was given only for finite metric spaces), and showed in some explicit examples the existence of a universal object in the category of all such compact quantum groups acting isometrically on the given metric space. There is also an attempt in [19] to give such a formulation for even more general framework of compact quantum metric spaces à la Rieffel. However, the formulation of [19] does not seem to be very convenient for computations as it involves inequalities rather than equalities in the definition of ‘isometric action’.

In the present paper we slightly modify the definition proposed by us in [10] (for finite spaces it is still the same) and have been able to prove the existence of a universal object for some special class of metric measure spaces. In fact, the first part of the paper does not go into the isometry condition and concentrates on the general aspects of quantum group actions on compact metric spaces. In this context, we give a proof of the fact that only Kac algebras can act faithfully on compact spaces, which is perhaps well-known among experts (although we did not find it written up explicitly anywhere). After this, we assume that the compact space is metrizable and taking into account the metric $d$, give several equivalent formulations of the ‘quantum isometric action’. However, we could not prove existence of a ‘quantum isometry group’ in general, although for certain special class of metric measure spaces we could prove the existence of a universal object in the category of those compact quantum groups which act isometrically and in a measure-preserving way.

2 Quantum groups and their actions

A compact quantum group (CQG for short) is a unital $C^*$ algebra $S$ with a coassociative coproduct (see [23], [23]) $\Delta$ from $S$ to $S \otimes S$ (injective tensor product) such that each of the linear spans of $\Delta(S)(S \otimes 1)$ and that of $\Delta(S)(1 \otimes S)$ is norm-dense in $S \otimes S$. From this condition, one can obtain a canonical dense unital $*$-subalgebra $S_0$ of $S$ on which linear maps $\kappa$ and $\epsilon$ (called the antipode and the counit respectively) making the above subalgebra a Hopf $*$-algebra. In fact, we shall always choose this dense Hopf $*$-algebra to be the algebra generated by the ‘matrix coefficients’ of
the (finite dimensional) irreducible unitary representations (to be defined shortly) of the CQG. The antipode is an anti-homomorphism and also satisfies $\kappa(a^*) = (\kappa^{-1}(a))^*$ for $a \in S_0$.

It is known that there is a unique state $h$ on a CQG $S$ which is bi-invariant in the sense that $(\text{id} \otimes h) \circ \Delta(a) = (h \otimes \text{id}) \circ \Delta(a) = h(a)1$ for all $a$. The Haar state need not be faithful in general, though it is always faithful on $S_0$ at least. One also has $h(\kappa(a)) = h(a)$ for $a \in S_0$. We also recall from [23] that there exists a canonical one-parameter family $f_z$ indexed by $z \in \mathbb{C}$ of multiplicative linear functionals on $S_0$, with interesting properties listed in Theorem 1.4 of [23]. In particular, $f_z$ forms a one parameter group with respect to the ‘convolution’ $*$ (i.e. $f_{z+z'} = f_z * f_{z'} := (f_z \otimes f_{z'}) \circ \Delta$), and $\kappa^2(a) = f_{-1} * a * f_1$. The following fact, which is contained in Theorem 1.5 of [23], will be quite useful for us:

**Proposition 2.1** The Haar state $h$ is tracial, i.e. $h(ab) = h(ba)$ for all $a, b$, if and only if $\kappa^2 = \text{id}$, which is also equivalent to $f_z = \epsilon \ \forall z$. In such a case, $S$ is called a Kac algebra.

We also need the following:

**Lemma 2.2** If the tracial Haar state of a Kac algebra $S$ is faithful, then its antipode $\kappa$ admits a norm-bounded extension on $S$ satisfying $\kappa^2 = \text{id}$.

**Proof:**

It follows from Theorem 1.6, (4) of [23], by noting that $\tau_{i/2} = \text{id}$ for Kac algebras, since $f_z = \epsilon$ for all $z$ in this case. 

**Corollary 2.3** Let $S$ be a CQG with faithful Haar state $h$ and assume that the mutiplicative functionals $f_z$ are identically equal to the counit $\epsilon$ on a norm-dense $*$-subalgebra $S_1$ of $S_0$ ($S_1$ may be strictly smaller than $S_0$). Then $S$ must be a Kac algebra.

**Proof:**

In the notation of [23], we observe from the proof of Theorem 1.6 of [23] that $\sigma_t(a) = a$ for all $a \in S_1$, where $\sigma_t$ denotes the $*$-automorphism defined on the whole of $S$ mentioned in (3) of Theorem 1.6 of [23]. By the boundedness of $\sigma_t$ and the norm-density of $S_1$, we conclude that $\sigma_t = \text{id}$ on the whole of $S$, hence $h$ must be tracial. 

We say that a CQG $S$ (with a coproduct $\Delta$) (co)acts on a unital $C^*$ algebra $C$ if there is a unital $C^*$-homomorphism $\beta : C \to C \otimes S$ such that $\text{Span}\{\beta(C)(1 \otimes S)\}$ is norm-dense in $C \otimes S$, and it satisfies the coassociativity condition, i.e. $(\beta \otimes \text{id}) \circ \beta = (\text{id} \otimes \Delta) \circ \beta$. It has been shown in [18]
that there is a unital dense $\ast$-subalgebra $C_0$ of $C$ such that $\beta$ maps $C_0$ into $C_0 \otimes_{\text{alg}} S_0$ (where $S_0$ is the dense Hopf $\ast$-algebra mentioned before) and we also have $(\text{id} \otimes \epsilon) \circ \beta = \text{id}$ on $C_0$. In fact, this subalgebra $C_0$ comes from the canonical decomposition of $C$ into subspaces on each of which the action $\beta$ is equivalent to an irreducible representation. More precisely, $C_0$ is the algebraic direct sum of finite dimensional vector spaces $C_{\pi}^i$, say, where $i$ runs over some index set $J_i$, and $\pi$ runs over some subset (say $T$) of the set of (inequivalent) irreducible unitary representations of $S$, and the restriction of $\beta$ to $C_{\pi}^i$ is equivalent to the representation $\pi$. Let $\{a_{j}^{(\pi,i)}; j = 1, \ldots, d_{\pi}\}$ (where $d_{\pi}$ is the dimension of the representation $\pi$) be a basis of $C_{\pi}^i$ such that $\beta(a_{j}^{(\pi,i)}) = \sum_{k} a_{k}^{(\pi,i)} \otimes t_{jk}^{\pi}$, for elements $t_{jk}^{\pi}$ of $S_0$. The elements $\{t_{jk}^{\pi}; \pi \in T; j, k = 1, \ldots, d_{\pi}\}$ are called the ‘matrix coefficients’ of the action $\beta$.

We say that the action $\beta$ is faithful if the $\ast$-subalgebra of $S$ generated by elements of the form $(\omega \otimes \text{id})(\beta(a))$, where $a \in C_0$, and $\omega$ being a bounded linear functional on $C_0$, is norm-dense in $S$. Since every bounded linear functional on a $C^\ast$ algebra is a linear combination of states, it is clear that in the definition of faithfulness, we can replace ‘bounded linear functional’ by ‘state’.

**Lemma 2.4** Given an action $\beta$ of a CQG $S$ on $C$, with $C_0$, $S_0$ etc. as above, the following are equivalent:

(i) The action $\beta$ is faithful.

(ii) The $\ast$-algebra generated by the matrix coefficients is norm-dense in $S$.

(iii) The $\ast$-algebra generated by elements of the form $(\omega \otimes \text{id})(\beta(a))$, where $a \in C_0$, and $\omega$ is a (not necessarily bounded) linear functional on $C_0$, is norm-dense in $S$.

**Proof:**
The equivalence of (ii) and (iii) is quite obvious, and so is the implication (i) $\Rightarrow$ (iii). To prove (ii) $\Rightarrow$ (i), fix any $\pi, i, j, k$ and consider (by Hahn-Banach theorem) a bounded linear functional $\omega$ on $C$ such that $\omega(a_{l}^{(\pi,i)}) = 1$, $\omega(a_{l}^{(\pi,i)}) = 0$ for $l \neq k$. Clearly, $(\omega \otimes \text{id})(\beta(a_{j}^{(\pi,i)})) = t_{jk}^{\pi}$. $\square$

For a Hilbert $C^\ast$ module $E$ over a $C^\ast$ algebra $C$, we shall denote by $L(E)$ the $C^\ast$ algebra of adjointable $C$-linear maps from $E$ to $E$. We shall typically consider the Hilbert $C^\ast$ modules of the form $H \otimes S$, where $S$ is a $C^\ast$ algebra and the Hilbert module is the completion of $H \otimes_{\text{alg}} S$ w.r.t. the weakest topology which makes $H \otimes_{\text{alg}} S \ni X \mapsto \langle X, X \rangle^{\frac{1}{2}} \in S$ continuous in norm. We shall use two kinds of ‘leg-numbering’ notation: for $T \in L(H \otimes S)$, we denote by $T_{23}$ and $T_{13}$ the elements of $L(H \otimes H \otimes S)$ given by $T_{23} = I_H \otimes T$, 

\[ 4 \]
\[ T_{13} = \sigma_{12} \circ T_{23} \circ \sigma_{12}, \] where \( \sigma_{12} \) flips two copies of \( \mathcal{H} \). On the other hand, we shall denote by \( T^{12} \) and \( T^{13} \) the elements \( T \otimes \text{id}_S \) and \( \sigma_{23} \circ T^{12} \circ \sigma_{23} \) respectively, of \( \mathcal{L}(\mathcal{H} \otimes S \otimes S) \), where \( \sigma_{23} \) flips two copies of \( S \).

A unitary representation of a CQG \( (S, \Delta) \) in a Hilbert space \( \mathcal{H} \) is given by a complex linear map \( U \) from the Hilbert space \( \mathcal{H} \) to the Hilbert \( S \)-module \( \mathcal{H} \otimes S \), which is isometric in the sense that \( \langle U\xi, U\eta \rangle_S = \langle \xi, \eta \rangle \) for all \( \xi, \eta \in \mathcal{H} \) (where \( \langle \cdot, \cdot \rangle_S \) denotes the \( S \)-valued inner product) and the \( S \)-linear span of the range of \( U \) is dense in \( \mathcal{H} \in S \). There is an equivalent description of the unitary representation given by the \( S \)-linear unitary \( \tilde{U} \) in \( L(\mathcal{H} \otimes S) \) defined by \( \tilde{U}(\xi \otimes b) = U(\xi)b \), for \( \xi \in \mathcal{H}, b \in S \), satisfying \( (\text{id} \otimes \Delta)(\tilde{U}) = \tilde{U}^{12}\tilde{U}^{13} \). We denote by \( \text{ad}_U \) the map \( B(\mathcal{H}) \ni X \mapsto \tilde{U}(X \otimes 1)\tilde{U}^* \), for \( X \in B(\mathcal{H}) \). We use similar notation for an action \( \beta \) of \( S \) on some \( C^\ast \)-algebra \( \mathcal{C} \) implemented by \( U \), i.e. take \( \beta^{(2)} \equiv \beta^{(2)}_U \) (this may in general depend on \( U \) to be the restriction of \( \text{ad}_U^{(2)} \) to \( \mathcal{C} \otimes \mathcal{C} \). This will be referred to as the ‘diagonal action’, since in the commutative case, i.e. when \( \mathcal{C} = C(X) \), \( S = C(G) \), with \( G \) acting on \( X \), the action \( \beta^{(2)} \) does indeed correspond to the diagonal action of \( G \) on \( X \times X \). However, we warn the reader that when \( S \) is no longer commutative as a \( C^\ast \)-algebra, i.e. not of the form \( C(G) \) for some group \( G \), \( \beta^{(2)} \) may not leave \( \mathcal{C} \otimes \mathcal{C} \) (or even its weak closure) invariant, so may not be an action of \( S \) on \( \mathcal{C} \otimes \mathcal{C} \).

We shall denote by \( \beta^{(2)}_U \) the \( * \)-homomorphism \( \text{ad}_W \), where \( W = \tilde{U}_{23}\tilde{U}_{13} \).

**Remark 2.5** The ‘diagonal action’ \( \beta^{(2)}_U \) is not same as the one considered in \[10\]; in fact, the diagonal map of \[10\] is actually (at least for finite spaces) the unitary \( U^{(2)} \) considered in the present paper, so is not an algebra homomorphism in general.

### 3 Necessity of the Kac algebra condition

The aim of this section is to prove that if a CQG with faithful Haar state acts faithfully on \( C(X) \), then the CQG must be Kac algebra.

**Remark 3.1** As we have already mentioned in the introduction, this result is probably well-known among experts, although we could not find any ex-
plicit reference for this fact and so decided to prove it. The main tool is the results known about ergodic actions of CQG, which basically relate the modularity property of the Haar state of the compact quantum group acting with the modularity of some canonical functional on the C* algebra on which the quantum group is acting ergodically. However, if one drops the assumption of ergodicity, this is no longer true, as the example of Cuntz algebra $\mathcal{O}_n$ shows. In fact, any CQG with an n-dimensional fundamental representation acts faithfully (but not ergodically) on $\mathcal{O}_n$, hence in particular many Kac algebras can have faithful actions on $\mathcal{O}_n$. But $\mathcal{O}_n$ does not admit any faithful trace. Fortunately, commutative C* algebras have a unique special feature: they have a separating family of *-homomorphic pure states (point evaluations). This allows us to determine the modularity of a general (possibly non-ergodic) faithful CQG action $\alpha$ on $C(X)$ by a family of ergodic actions $\{\alpha_x : x \in X\}$, which are essentially restrictions of the original action to the ‘orbits’. Thus we can apply the results about ergodic actions to prove similar results for more general actions on commutative C* algebras. But this strategy would fail for a more general (noncommutative) C* algebra.

Let is recall that an action $\beta$ of a CQG $S$ on a C* algebra $C$ is called ergodic if the fixed point subalgebra is trivial, i.e. $\beta(x) = x \otimes 1$ for some $x \in A$ implies that $x$ is a scalar multiple of the identity element of $A$. We refer the reader to [11] for a detailed analysis of such ergodic actions. The following is essentially contained in [11] (see also [6]), but we state and prove it in a form suitable to us:

**Theorem 3.2** Let $C$ be a commutative unital C* algebra on which a CQG $S$ has an ergodic action $\beta$. Let $C_0$, $S_0$ be the dense *-subalgebras mentioned before, with the counit $\epsilon$ and antipode $\kappa$, and let $f_z$ be the one-parameter family of multiplicative maps discussed in Section 2. Then $f_z(x) = \epsilon(x)$ for all $x$ belonging to the subalgebra of $S_0$ spanned by elements of the form $(\theta \otimes \text{id})(\beta(a))$, for $a \in C_0$ and $\theta$ being a linear functional on $C_0$.

**Proof:**
Let us recall the vector space decomposition of $C_0$ into subspaces $C_0^\pi_i$ discussed before, and denote by $F^\pi_i$ the operator (as in [11], and also [23], where the symbol $F^\pi$ was used) the operator $(\text{id} \otimes f_1) \circ \beta$ restricted to the span of $C_0^\pi_i$'s, for any fixed $\pi, i$. Note that the operator essentially depends on $\pi$ only, as it is given on $a_j^{(\pi, i)}$ by $\sum_k a_k^{(\pi, i)} f_1(t_j^{\pi, k})$, so that we did not keep $i$ in the suffix. It is clear that it suffices for us to prove that $F^\pi_i = I$ for each $\pi$. It is in fact a positive invertible finite-dimensional operator satisfying $\text{Tr}(F^\pi_i) = \text{Tr}(F^{-1}_{\pi})$ (see [23]). Now by Proposition 18 of [11], there is a unique faithful $S$-invariant state $\omega$.
on $C$ and multiplicative linear map $\Theta : C_0 \to C_0$ satisfying $\omega(x\Theta(y)) = \omega(yx)$ for all $x, y \in C_0$. But $C$ being commutative, we have $\omega(x\Theta(y)) = \omega(xy)$, and by the faithfulness of $\omega$, we must have $\Theta(y) = y$ for all $y \in C_0$. Moreover, by [11], for each $\pi$ and $i$, the restriction of $\Theta$ to $C_i^\pi$ coincides with a scalar multiple of $F_\pi$, and hence (noting also that $\text{Tr}(F_\pi) = \text{Tr}(F_\pi^{-1})$) we get $F_\pi = I$. □

We now prove the main result of this section.

**Theorem 3.3** Let $(S, \Delta)$ be a CQG with faithful Haar state which acts faithfully on $C(X)$ for a compact space $X$. Then $S$ must be a Kac algebra.

**Proof:**
Let $\kappa, \epsilon$ be the antipode and counit of $S$ (defined at least on $S_0$) respectively, and let $f_z$ be the one-parameter group of multiplicative functionals discussed before. For any $x \in X$, let $\beta_x$ denote the $\ast$-homomorphism $(\text{ev}_x \otimes \text{id}) \circ \beta$ from $C(X)$ to $S$, and let the range of $\beta_x$ be denoted by $S^x$. Clearly, $S^\ast$ is a unital commutative $C^\ast$-algebra and by the coassociativity of $\beta$, it follows that $\Delta(S^\ast) \subseteq S^\ast \otimes S$. Thus, $\Delta|_{S^\ast}$ gives an action of $S$ on $S^\ast$, and we claim that this is ergodic. Indeed, if $\Delta(a) = a \otimes 1$ for $a \in S^\ast$, by applying the Haar state $h$ (say) of $S$ on the second copy of the tensor product we get $h(a)1 = a$, i.e. $a$ is a scalar multiple of 1.

Now, let $C_0$ be the dense $\ast$-subalgebra of $C(X)$ on which $\beta$ is algebraic, and by the assumption of faithfulness, the $\ast$-algebra generated by $S_0^\beta := \beta_x(C_0)$, with $x$ varying in $X$, is dense in $S$. Since the action $\Delta|_{S^\ast}$ is ergodic, we conclude by Theorem 3.2 that $f_z(b) = \epsilon(b) \forall z$, for all elements $b$ of the form $b = (\omega \otimes \text{id})(\Delta(a))$, for any linear functional $\omega$ on $S_0^\ast$, and $a \in S_0^\beta$. Taking $\omega = \epsilon$, (which is defined on $S_0^\ast \subseteq S_0$), we see that $f_z = \epsilon$ on $S_0^\ast$ for every $x$, and using the facts that $f_z(ab) = f_z(a)f_z(b)$ and $f_z(a^*) = f_{-z}(a)$ for all $a, b$, and $\epsilon$ is $\ast$-homomorphism, we get that $f_z = \epsilon$ on the $\ast$-subalgebra (say $S_1^\ast$) of $S_0$ generated by $S_0^\beta$'s, which is norm-dense in $S$ by faithfulness. The theorem now follows from Corollary 2.3 □

4 Quantum group of isometries of $(X, d)$

4.1 Definition of isometric action of compact quantum groups

We have already noted that for any $C^\ast$ action $\beta$ of a CQG $S$ with faithful Haar state on $C(X)$, the antipode, say $\kappa$ of $S$ is defined and bounded on the $C^\ast$ subalgebra generated by $\beta(f)(x) \equiv (\text{ev}_x \otimes \text{id}) \circ \beta(f)$, $f \in C(X)$, $x \in X$. So $(\text{id} \otimes \kappa) \circ \beta$ is a well-defined and norm-bounded map on $C(X)$.

In view of this, it is natural to make the following definition:
Definition 4.1  Given an action $\beta$ of a CQG $S$ (with faithful Haar state) on $C = C(X)$ (where $(X,d)$ is a compact metric space), we say that $\beta$ is ‘isometric’ if the metric $d \in C(X) \otimes C(X)$ satisfies the following:

$$
(id_C \otimes \beta)(d) = \sigma_{23} \circ ((id_C \otimes \kappa) \circ \beta \otimes id_C)(d),
$$

where $\sigma_{23}$ denotes the flip of the second and third tensor copies.

Theorem 4.2  Given a $C^*$-action $\beta$ of a CQG $S$ (with faithful Haar state) on $C(X)$, the following are equivalent:

(i) The action is isometric.

(ii) $\forall x, y \in X$, one has $\beta(d_x)(y) = \kappa(\beta(d_y)(x))$, where $d_x(z) := d(x,z)$.

(iii) For some (hence all) unitary representation $U$ of $S$ on $H_U$ which implements $\beta$, we have $\beta^{(2)}((\pi_U \otimes \pi_U)(d)) = (\pi_U \otimes \pi_U)(d) \otimes 1$, where $\pi_U : C(X) \to B(H_U)$ denotes the imbedding of $S$ into $B(H_U)$.

(iv) For some (hence all) unitary representation $U$ of $S$ on $H_U$ which implements $\beta$, we have $\beta^{(2),U}(\pi_U \otimes \pi_U)(d)) = (\pi_U \otimes \pi_U)(d) \otimes 1$.

Proof: The equivalence of (i) and (ii) is a consequence of the continuity of the map $x \mapsto d_x \in C(X)$, and hence (by the norm-contractivity of $\beta$), the continuity of $x \mapsto \beta(d_x) \in C(X) \otimes S$.

Next, to prove the equivalence of (i) and (iii), we need to first note that in (i), i.e. the condition (1), $\beta$ can be replaced by $ad_U$ for any unitary representation $U$ of the CQG $S$ in some Hilbert space, which implements $\beta$. Then, using the observation that $(\tilde{U})^{-1}(\pi_U(f) \otimes 1)\tilde{U} = (\pi_U \otimes \kappa)(\beta(f))$, we see that (i) is equivalent to the following:

$$
\tilde{U}_{23}((\pi_U \otimes \pi_U)(d) \otimes 1))\tilde{U}_{23}^{-1} = \tilde{U}_{13}^{-1}((\pi_U \otimes \pi_U)(d) \otimes 1)\tilde{U}_{13},
$$

which is clearly nothing but (iii).

Finally, the equivalence of (i) and (iv) follows from the symmetry of $d$, i.e. $d\sigma = d$, where $\sigma \in C(X \times X)$ is the map $\sigma(x,y) = (y,x)$. To elaborate this calculation little more, let us denote by $\sigma_{12}$ the map $\sigma \otimes id_S$, and observe that $\sigma_{12} \circ \text{ad}_{U_{13}} \circ \sigma_{12} = \text{ad}_{U_{23}}$, and $\sigma_{12} \circ \text{ad}_{U_{23}}^{-1} \circ \sigma_{12} = \text{ad}_{U_{13}}^{-1}$. Now, it is clear that (iv) is obtained from (i) by replacing $d$ by $\sigma(d)$ in (1) and then also applying $\sigma_{12}$ on both sides of it. □

Remark 4.3  In case $S = C(G)$ and $\beta$ corresponds to a topological action of a compact group $G$ on $X$, it is clear that the condition (ii) above is nothing
but the requirement \(d(x, gy) = d(y, g^{-1}x) = d(g^{-1}x, y)\), which is obviously the usual definition of isometric group action.

**Remark 4.4** For a finite metric space \((X, d)\), the present definition does coincide with Banica’s definition in [11] as well as the one proposed in [10]. Indeed, for such a space, \(C(X) = l^2(X)\), and we can be choose \(\mathcal{H}_U = l^2(X)\) and \(U\) to be the natural representation of \(S\) coming from the action, with \(d\) viewed both as an element of \(C(X \times X)\) as well as of \(l^2(X \times X)\). There is also the identically 1 function, say \(1\), in \(l^2(X \times X)\), which is cyclic and separating for \(C(X \times X)\). Thus, the requirement \(\beta_U^{(2)}(d) = d \otimes 1\) is clearly equivalent to \(U(d) = d \otimes 1\), since \(U^{-1}1 = 1 \otimes 1\). This is precisely the proposed condition of [10] (and equivalent to the definition of Banica, as observed in [10]).

**Remark 4.5** In the more general situation, consider any faithful state \(\phi\) on \(C(X)\) (given by integration w.r.t. some Borel probability measure \(\mu\), say), and by averaging w.r.t. the Haar state, we get another faithful (as the Haar state is assumed to be faithful) and \(S\)-invariant state, say \(\bar{\phi}\), with the corresponding measure being \(\bar{\mu}\). Clearly, the action \(\beta\) extends to a unitary representation, say \(U\), on \(\mathcal{H} := L^2(X, \bar{\mu})\) which implements \(\beta\). Moreover, since \(\bar{\mu}\) is a Borel probability measure, we have \(C(X) \subseteq L^2(X, \bar{\mu})\). and \(1\) is a cyclic separating vector for \(C(X \times X)\) in \(L^2(X \times X)\) as before, such that \(U^{(2)^{-1}}(1) = 1 \otimes 1\). Thus, \(\beta\) is isometric in our sense if and only if \(U^{(2)}(d) = d \otimes 1\), \(d\) being viewed as a vector in \(L^2(X \times X)\). Similarly, using condition (iv) of Theorem 4.2, we get \(U_{23}U_{13}(d) = d \otimes 1\). More generally, for any function \(\phi(d)\), where \(\phi\) is a continuous real-valued function on \(\mathbb{R}^+\), we have \(\beta^{(2)}(\phi(d)) = \phi(d) \otimes 1 = \beta^{(2)}(\phi(d))\), and hence also \(U_{13}U_{23}(\phi(d)) = U_{23}U_{13}(\phi(d)) = \phi(d) \otimes 1\).

**Remark 4.6** In the recent article [19], the authors have generalized the notion of isometric action of CQG to the framework of Rieffel’s compact quantum metric spaces. However, it is not yet clear whether their definition of isometric CQG-action is the same as the one given by us for a general compact metric space, although for finite spaces their equivalence has been proved by them. It will be also interesting to see whether analogues of the equivalent conditions (i)-(iv) proved above can be generalized too.
4.2 Comparison with the geometric definition of quantum isometry

In [12], a definition of isometric action of compact quantum groups was given in a geometric set-up, generalizing the notion of isometric action of groups on Riemannian manifolds. We refer [12] for the details of this formulation, but only recall that the meaning of isometry in [12] is that the action leaves invariant the subalgebra of smooth functions and commutes with the Hodge-Laplacian on this subalgebra. It is also proved in [12] that for a classical compact connected Riemannian manifold $X$, there is indeed a universal object (say $QISO_L^C(X)$) in the category of all compact quantum groups acting isometrically on it in the (geometric) sense of [12]. However, the manifold $X$ also has a canonical metric $d$ coming from the geodesic distance, so it is natural to consider isometric action in the sense of Subsection 4.1, and it is quite interesting to compare these two notions. In the classical situation, i.e. when we consider action by groups only, the two definitions of isometry do coincide. We prove below that even in the quantum case, the geometric definition of [12] is at least stronger than the definition of the present article. We are not yet able to decide whether the two definitions are actually equivalent in the quantum case, but conjecture that they are indeed so.

**Theorem 4.7** Let $X$ be a compact connected Riemannian manifold with $d$ the corresponding geodesic distance and let the quantum isometry group in the Laplacian sense be denoted by $QISO_L^C(X)$, and let $Q = (QISO_L^C(X))_r$ be the corresponding reduced quantum group having faithful Haar state. Then the action of $Q$ is also isometric in the sense of metric space (as defined in the Subsection 4.1) for $(X, d)$.

**Proof:**
Let $\alpha$ be the action of $Q$. It is known that $\alpha$ preserves the state $\tau$ coming from the Riemannian volume measure ([12]). Thus, the action $\alpha$ naturally extends to a unitary representation on the Hilbert space $\mathcal{H} = L^2(\mu)$, which we denote by $U$, and $\tilde{U}$ denotes the corresponding unitary on the Hilbert module $\mathcal{H} \otimes Q$. Let $T_t$ be the heat semigroup generated by $\mathcal{L}$, $k_t$ its heat kernel, i.e $T_t f = (\text{id} \otimes \tau)(k_t(1 \otimes f))$ for $f \in C(X) \subseteq L^2(X, \mu)$.

We have for all $t \geq 0$

$$U \circ T_t = (T_t \otimes \text{id}) \circ U. \quad (2)$$

We claim that

$$U_{23}U_{13}(k_t) = k_t \otimes 1_Q = U_{13}U_{23}(k_t). \quad (3)$$

10
This will complete the proof of the theorem from the well-known asymptotic result (see, e.g., [15]):

\[ d^2(x, y) = -4 \lim_{t \to 0^+} t \log(k_t(x, y)). \]

If \( f, g \in C(X) \), \( q \in \mathcal{Q} \), we have the following (where \( \langle \cdot, \cdot \rangle_{\mathcal{Q}} \) will denote the \( \mathcal{Q} \)-valued inner product of the Hilbert modules \( \mathcal{H} \otimes \mathcal{Q} \) or \( \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{Q} \)):

\[
q^* \langle g \otimes 1_{\mathcal{Q}} U(T_t f) \rangle_{\mathcal{Q}} = q^* \langle g \otimes 1 \otimes 1_{\mathcal{Q}}, U_{13}((1 \otimes f)k_t \otimes 1_{\mathcal{Q}}) \rangle_{\mathcal{Q}} \\
= q^* \langle g \otimes 1 \otimes 1_{\mathcal{Q}}, (1 \otimes f \otimes 1_{\mathcal{Q}})U_{13}(k_t \otimes 1_{\mathcal{Q}}) \rangle_{\mathcal{Q}} \\
= q^* \langle \tilde{U}_{23}(g \otimes 1 \otimes 1_{\mathcal{Q}}), \tilde{U}_{23}(1 \otimes f \otimes 1_{\mathcal{Q}})U_{13}^{-1}\tilde{U}_{23}\tilde{U}_{13}(k_t \otimes 1_{\mathcal{Q}}) \rangle_{\mathcal{Q}} \\
= q^* \langle g \otimes 1 \otimes 1_{\mathcal{Q}}, (1 \otimes \alpha(f))\tilde{U}_{23}\tilde{U}_{13}(k_t \otimes 1_{\mathcal{Q}}) \rangle_{\mathcal{Q}} \\
= (g \otimes \alpha(\tilde{f})(1 \otimes q)) \langle \tilde{U}_{23}\tilde{U}_{13}(k \otimes 1_{\mathcal{Q}}) \rangle_{\mathcal{Q}} \\
= q^* \langle g \otimes 1 \otimes 1_{\mathcal{Q}}, (T_t \otimes \text{id})(\alpha(f)) \rangle_{\mathcal{Q}} \\
= q^* \langle g \otimes 1 \otimes 1_{\mathcal{Q}}, (T_t \otimes \text{id})(Uf) \rangle_{\mathcal{Q}}.
\]

On the other hand,

\[
\langle (g \otimes \alpha(\tilde{f})(1 \otimes q)), (k_t \otimes 1_{\mathcal{Q}}) \rangle_{\mathcal{Q}} = q^* \langle \tilde{U}_{23}(g \otimes 1 \otimes 1_{\mathcal{Q}}), \tilde{U}_{23}(1 \otimes f \otimes 1_{\mathcal{Q}})U_{13}^{-1}\tilde{U}_{23}\tilde{U}_{13}(k_t \otimes 1_{\mathcal{Q}}) \rangle_{\mathcal{Q}} \\
= q^* \langle g \otimes 1 \otimes 1_{\mathcal{Q}}, (1 \otimes \alpha(f))\tilde{U}_{23}\tilde{U}_{13}(k_t \otimes 1_{\mathcal{Q}}) \rangle_{\mathcal{Q}} \\
= (g \otimes \alpha(\tilde{f})(1 \otimes q)) \langle \tilde{U}_{23}\tilde{U}_{13}(k \otimes 1_{\mathcal{Q}}) \rangle_{\mathcal{Q}} \\
= q^* \langle g \otimes 1 \otimes 1_{\mathcal{Q}}, (T_t \otimes \text{id})(Uf) \rangle_{\mathcal{Q}}.
\]

This proves

\[
\langle (g \otimes \alpha(\tilde{f})(1 \otimes q)), \tilde{U}_{23}\tilde{U}_{13}(k \otimes 1_{\mathcal{Q}}) \rangle_{\mathcal{Q}} = \langle (g \otimes \alpha(\tilde{f})(1 \otimes q)), (k \otimes 1_{\mathcal{Q}}) \rangle_{\mathcal{Q}},
\]

and this implies [3], by the density of the linear span of elements of the form \( \alpha(\tilde{f})(1 \otimes q) \) in \( C(X) \otimes \mathcal{Q} \).

4.3 Existence of a universal isometric action

It is a natural question to ask: does there exist a universal object in the category (say \( \mathcal{Q}_{X,d} \)) of all CQG (with faithful Haar state) acting isometrically (in our sense) on \( (X, d) \)? For finite metric spaces, the answer is clearly affirmative, and the universal object is the quantum isometry group defined by Banica. We are not yet able to settle this question in full generality. However, we shall give an affirmative answer for a slightly modified question for a class of metric measure spaces.
Lemma 4.8 Let \( \mathcal{C} \) be a unital \( C^* \) algebra. Define \(< < \cdot , \cdot > > \) on \( \mathbb{R}^m \otimes \mathcal{C}^{s.a.} \) by \(< < Z, W > > := \frac{1}{2} \sum_i (Z_i W_i + W_i Z_i) \). Let \( F \) be a function from \( \mathbb{R}^m \) to \( \mathbb{R}^m \otimes \mathcal{C}^{s.a.} \) which satisfies \(< < F(x), F(y) > > = < x, y > \) for all \( x, y \), where \(< \cdot , \cdot > \) is the Euclidean inner product of \( \mathbb{R}^n \). Then \( F \) must be linear.

Proof:
Let \( \| A \|_C^2 := < < A, A > > \). It is easy to see that \( \| A \|_C = 0 \) if and only if \( A = 0 \) for \( A \in \mathbb{R}^m \otimes \mathcal{C}^{s.a.} \). We now observe that \( \| F(x + y) - F(x) - F(y) \|_C^2 = 0 \) and \( \| F(cx) - cF(x) \|_C^2 = 0 \) by direct computation using the condition \(< < F(x), F(y) > > = < x, y > \).

Remark 4.9 The bilinear form \(< < \cdot , \cdot > > \) is not a \( \mathcal{C}^{s.a.} \) valued inner product in the sense of Hilbert module. It is only bilinear w.r.t. scalars, but not w.r.t. \( \mathcal{C}^{s.a.} \).

Lemma 4.10 Let \( \mathcal{C}, < < \cdot , \cdot > > \) and \( \| \cdot \|_C^2 \) be as in the statement and proof of the previous lemma, but let \( F \) be a function from a nonempty subset \( X \) of \( \mathbb{R}^m \) to \( \mathbb{R}^n \otimes \mathcal{C}^{s.a.} \) which satisfies

\[
\| F(x) - F(y) \|_C^2 = \| x - y \|^2 1
\]

for all \( x, y \in X \). Then \( F \) is affine in the sense that there are elements \( a_{ij}, i = 1, \ldots, n; j = 1, \ldots, m \) of \( \mathcal{C}^{s.a.} \) and \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \otimes \mathcal{C}^{s.a.} \) such that

\[
F(x) = Ax + \xi, \quad A := ((a_{ij})).
\]

Proof:
Consider \( x_0 \) in \( X \) and the function \( G : Y := X - x_0 \equiv \{ x - x_0 : x \in X \} \rightarrow \mathbb{R}^n \otimes \mathcal{C}^{s.a.} \) by \( G(y) = F(y + x_0) - F(x_0) \) for \( y \in Y \). Then \( \| G(y) - G(y') \|_C^2 = \| y - y' \|^2 1 \) by hypothesis, and also \( \| G(y) \|_C^2 = \| y \|^2 1 \). Observe that for any \( Z, W \in \mathbb{R}^n \otimes \mathcal{C}^{s.a.} \) we have \( \| Z + W \|_C^2 = \| Z \|_C^2 + \| W \|_C^2 + 2 < < Z, W > > \). Using this with \( Z = G(y), W = G(y') \), we get that

\[
< < G(y), G(y') > > = < y, y' > 1 \quad \forall y, y' \in Y.
\]

This allows us to extend \( G \) linearly to the span of \( Y \) in \( \mathbb{R}^m \). Indeed, the extension \( G(\sum_i c_i y_i) = \sum_i c_i G(y_i) \), for \( c_i \in \mathbb{R}, y_i \in Y \) is well-defined because \( \| \cdot \|_C^2 \) of the right hand side equals the Euclidean norm square of \( \sum_i c_i y_i \) by (4). We can then extend \( G \) further on the whole of \( \mathbb{R}^m \) as a linear map denoted again by \( G \). Thus, we get \( A = ((a_{ij})) \), say, such that \( G(y) = Ay \) for all \( y \in \mathbb{R}^m \). This implies, \( F(x) = G(x - x_0) + F(x_0) = Ax + \xi \), where \( \xi = F(x_0) - Ax_0 \). \( \square \)
Definition 4.11 We say that an action \( \beta : C \to C \otimes Q \) of a CQG \( Q \) on a unital \( C^* \) algebra \( C \) preserves a bounded functional \( \phi \) on \( C \) if \( (\phi \otimes \text{id})\beta(a) = \phi(a)1_Q \) for all \( a \in C \). In case \( C = C(X) \), \( X \) compact Hausdorff space, and \( \phi \) corresponds to integral w.r.t. some regular Borel Hausdorff measure \( \mu \) on \( X \), we say that the action \( \beta \) preserves \( \mu \), or, it is \( \mu \)-preserving.

Theorem 4.12 Let \( (X,d) \) be a compact metric space which can be isometrically embedded in some Euclidean space say \( \mathbb{R}^n \) with the usual metric. Assume furthermore that there exists a faithful, regular, Borel probability measure \( \mu \) on \( X \). Then the subcategory \( Q_{(X,d,\mu)} \) of \( Q_{X,d} \) consisting of those isometric CQG actions which also preserve \( \mu \), has a universal object, to be denoted by \( QISO(X,d,\mu) \).

Proof:
Note that w.l.g. we can assume (replacing \( X \) by its isometric image as a subset of \( \mathbb{R}^n \)) that \( X \) is a subset of \( \mathbb{R}^n \) and \( d^2(x,y) = \sum_{i=1}^n (x_i - y_i)^2 \).

Let \( X_1, \ldots, X_n \) denote the restrictions of coordinate functions of \( \mathbb{R}^n \) to \( X \). \( C(X) \) is clearly generated as a unital \( C^* \) algebra by the self-adjoint elements \( \{X_1, \ldots, X_n\} \). Let \( Q \) be a CQG with a faithful isometric and \( \mu \)-preserving action \( \alpha \) on \( C(X) \). Applying Lemma 4.10 with \( C = Q \), \( F(x) := (F_1(x), \ldots, F_n(x)) \) for \( x \in X \), where \( F_i(x) := \alpha(X_i)(x) \), we conclude that \( \alpha \) is affine, say, \( \alpha(X_i) = 1 \otimes \xi_i + \sum_j X_j \otimes a_{ij} \) for some \( \xi_i, a_{ij} \in Q^{*,a} \). Consider the subspaces \( \mathcal{G}_m, m = 0, 1, 2, \ldots \), with \( \mathcal{G}_0 = C1, \mathcal{G}_1 = \text{Span}\{1, X_1, \ldots, X_n\}, \mathcal{G}_m = \text{Span}\{X_i X_j \ldots X_m, \ i_k \in \{0\} \cup \mathbb{N}\} \), where \( X_0 := 1 \). As \( \alpha \) is \( * \)-homomorphpic and \( X_i \)'s commute among themselves, it is clear that \( \alpha \) maps each of the finite-dimensional subspaces \( \mathcal{G}_m \) into \( \mathcal{G}_m \otimes Q \). Furthermore, as \( \alpha \) is \( \mu \)-preserving, it extends as a unitary representation on each \( \mathcal{G}_m \), and thus also leaves invariant the subspaces \( \mathcal{F}_m \) given by \( \mathcal{F}_m = \mathcal{G}_m \otimes \mathcal{G}_{m-1}, m = 1, 2, \ldots \), \( \mathcal{F}_0 = \mathcal{G}_0, \) where \( K_1 \otimes K_2 \) means the orthogonal complement of \( K_2 \) in \( K_1 \). But \( \mathcal{F}_m \)'s are mutually orthogonal w.r.t. \( \mu \) and thus is an orthogonal filtration in the terminology of \( [3] \), which is preserved by \( \alpha \). In other words, the category \( Q_{(X,\mu,d)} \) coincides with the category of CQG actions which preserve the orthogonal filtration \( \{\mathcal{F}_m, m \geq 0\} \) and hence the existence of the universal object on this category follows from Theorem 2.7 of \( [3] \). \( \Box \)

Remark 4.13 For any finite subset \( X \) of \( \mathbb{R}^n \), any CQG \( Q \) faithfully acting on \( C(X) \) is a quantum subgroup of the quantum permutation group of \( X \) in the sense of Wang, which preserves the counting measure (say \( \mu \)) on \( X \).
Thus, the action of $Q$ will preserve this measure as well. It follows that the category $Q_{(X,d)}$ coincides with $Q_{(X,\mu,d)}$, where $d$ is the Euclidean metric restricted to $X$. This implies that for the finite metric space $(X,d)$, the universal object obtained by us in Theorem 4.12 is nothing but the quantum isometry group of Banica. However, a word of caution here is that there are finite metric spaces which cannot be embedded isometrically in some Euclidean space, and our result does not apply to such spaces.

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