Minmax Optimization: Stable Limit Points of Gradient Descent Ascent are Locally Optimal

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Abstract

Minmax optimization, especially in its general nonconvex-nonconcave formulation, has found extensive applications in modern machine learning frameworks such as generative adversarial networks (GAN), adversarial training and multi-agent reinforcement learning. Gradient-based algorithms, in particular gradient descent ascent (GDA), are widely used in practice to solve these problems. Despite the practical popularity of GDA, however, its theoretical behavior has been considered highly undesirable. Indeed, apart from possibility of non-convergence, recent results [Daskalakis and Panageas, 2018, Mazumdar and Ratliff, 2018, Adolphs et al., 2018] show that even when GDA converges, its stable limit points can be points that are not local Nash equilibria, thus not game-theoretically meaningful.

In this paper, we initiate a discussion on the proper optimality measures for minmax optimization, and introduce a new notion of local optimality—local minmax—as a more suitable alternative to the notion of local Nash equilibrium. We establish favorable properties of local minmax points, and show, most importantly, that as the ratio of the ascent step size to the descent step size goes to infinity, stable limit points of GDA are exactly local minmax points up to some degenerate points, demonstrating that all stable limit points of GDA have a game-theoretic meaning for minmax problems.

1 Introduction

Minmax optimization refers to problems of the form $\min_x \max_y f(x, y)$. Such problems arise in a number of fields, including mathematics, biology, social science, and particularly economics [Myerson 2013]. Due to the wide range of applications of these problems and their rich mathematical structure, they have been studied for several decades in the setting of zero-sum games. In the last few years, minmax optimization has also found significant applications in machine learning, in settings such as generative adversarial networks (GAN) [Goodfellow et al., 2014], adversarial training [Madry et al., 2017] and multi-agent reinforcement learning [Omidshafiei et al., 2017]. In practice, these minmax problems are often solved using gradient based algorithms, especially gradient descent ascent (GDA), an algorithm that alternates between a gradient descent step for $x$ and some number of gradient ascent steps for $y$. 
Such gradient-based algorithms have been well studied for convex-concave games, where \( f(\cdot, \cdot) \) is a convex function of \( x \) for any fixed \( y \) and a concave function of \( y \) for any fixed \( x \). In this case, it can be shown that the average of iterates of GDA converges to a Nash equilibrium; i.e., a point \((x^*, y^*)\) such that 

\[
    f(x^*, y^*) \leq f(x, y^*) \leq f(x, y) \quad \text{for every } x \text{ and } y
\]

[Bubeck, 2015, Hazan, 2016]. In the convex-concave setting, it turns out that Nash equilibria and global optima are equivalent: \((x^*, y^*)\) is a Nash equilibrium if and only if 

\[
    f(x^*, y^*) = \min_x \max_y f(x, y).
\]

Most of the minmax problems arising in modern machine learning applications do not, however, have this simple convex-concave structure.

Given the widespread usage of GDA in practice, it is natural to ask about its properties when applied to general nonconvex-nonconcave settings. It turns out that this question is extremely challenging—GDA dynamics do not monotonically decrease any known potential function and GDA may not converge in general [Daskalakis et al., 2017]. Worse still, even when GDA converges, recent results suggest that it has some undesirable properties. Specifically, [Daskalakis and Panageas, 2018], [Mazumdar and Ratliff, 2018], and [Adolphs et al., 2018] show that some of the stable limit points of GDA may not be Nash equilibria. This suggests that they may have nothing to do with the minmax problem being solved. This raises the following question:

**Is GDA an appropriate algorithm for solving general minmax problems?**

This paper provides a positive theoretical answer to this question in the general nonconvex-nonconcave setting. Critical to our perspective is a new notion of local optimality—local minmax, which we propose as a more useful alternative than local Nash equilibria for a range of problems. We show that, as the ratio of the ascent step size to the descent step size goes to infinity, the stable limit points of GDA are identical to local minmax points up to some degenerate points. Therefore, almost all stable limit points of GDA are game-theoretically meaningful for minmax problems.

### 1.1 Our contributions

The main contributions of the paper are as follows:

- We initiate a discussion on the proper optimality measures for minmax optimization, distinguishing among pure strategy Nash equilibria, global minmax points and mixed strategy Nash equilibria. We show that the latter two are well-defined and of practical relevance. We further show a reduction from the problem of finding mixed strategy Nash equilibria to the problem of finding global minmax points for Lipschitz games, demonstrating the central importance of finding global minmax points.

- We define a new notion of local optimality—local minmax—as a natural local surrogate for global minmaxity. We explain its relation to local Nash equilibria and global minmax points, and we establish its first- and second-order characterizations. It is worth noting that minmax optimization exhibits unique properties compared to nonconvex optimization in that global minmax points can be neither local minmax nor stationary (see Proposition 16).

- We analyze the asymptotic behavior of GDA, and show that as the ratio of the ascent step size to the descent step size goes to infinity, stable limit points of GDA are exactly local minmax points up to some degenerate points, demonstrating that almost all stable limit points of GDA have a game-theoretic meaning for minmax problems.

- We also consider the minmax problem with an approximate oracle for the maximization over \( y \). We show that gradient descent with inner maximization (over \( y \)) finds a point that is close to an approximate stationary point of \( \phi(x) := \max_y f(x, y) \).
Paper organization In Section 1.2, we review additional related work. Section 2 presents preliminaries. In Section 3, we discuss the right objective for general nonconvex-nonconcave minmax optimization. Section 4 presents our main results on a new notion of local optimality, the limit points of GDA and gradient descent with a maximization oracle. We conclude in Section 5. Due to space constraints, all proofs are presented in the appendix.

1.2 Related Work

GDA dynamics: There have been several lines of work studying GDA dynamics for minmax optimization. Cherukuri et al. [2017] investigate GDA dynamics under some strong conditions and show that it converges locally to Nash equilibria. Heusel et al. [2017] and Nagarajan and Kolter [2017] similarly impose strong assumptions in the setting of the training of GANs and show that under these conditions Nash equilibria are stable fixed points of GDA. Gidel et al. [2018] investigate the effect of simultaneous versus alternating gradient updates as well as the effect of momentum on the convergence in bilinear games. The most closely related analyses to ours are Mazumdar and Ratliff [2018] and Daskalakis and Panageas [2018]. While Daskalakis and Panageas [2018] study minmax optimization (or zero-sum games), Mazumdar and Ratliff [2018] study a much more general setting of non-zero-sum games and multi-player games. Both of these papers show that the stable limit points of GDA are not necessarily Nash equilibria. Adolphs et al. [2018] and Mazumdar et al. [2019] propose Hessian-based algorithms whose stable fixed points are exactly Nash equilibria. We note that all the papers in this setting use Nash equilibrium as the notion of goodness.

General minmax optimization in machine learning: There have also been several other recent works on minmax optimization that study algorithms other than GDA. Rafique et al. [2018] consider nonconvex but concave minmax problems where for any \( x \), \( f(x, \cdot) \) is a concave function. In this case, they analyze an algorithm combining approximate maximization over \( y \) and a proximal gradient method for \( x \) to show convergence to stationary points. Lin et al. [2018] consider a special case of the nonconvex-nonconcave minmax problem, where the function \( f(\cdot, \cdot) \) satisfies a variational inequality. In this setting, they consider a proximal algorithm that requires the solving of certain strong variational inequality problems in each step and show its convergence to stationary points. Hsieh et al. [2018] propose proximal methods that asymptotically converge to a mixed Nash equilibrium; i.e., a distribution rather than a point.

No regret dynamics for minmax optimization: Online learning/no regret dynamics have also been used to design algorithms for minmax optimization. All of these results require, however, access to oracles which solve the minimization and maximization problems separately, keeping the other variable fixed and outputting a mixed Nash equilibrium [see, e.g., Feige et al. [2015], Chen et al. [2017], Grnarova et al. [2017], Gonen and Hazan [2018]]. Finding the global minmax point even with access to these oracles is NP hard [Chen et al. [2017]].

Nonconvex optimization: Gradient-based methods are also widely used for solving nonconvex optimization problems in practice. There has been a significant amount of recent work on understanding simple gradient-based algorithms such as gradient descent in this setting. Since finding global minima is already NP hard, many papers focus on obtaining convergence to second-order stationary points. Lee et al. [2016] and Panageas and Piliouras [2016] show that gradient descent converges to only these points with probability one. Ge et al. [2015] and Jin et al. [2017] show that with a small amount of randomness gradient descent also converges to second-order stationary points and give nonasymptotic rates of convergence.
2 Preliminaries

In this section, we will first introduce our notation, and then present definitions and results for minmax optimization, zero-sum games, and general game-theoretic dynamics that are relevant to our work.

2.1 Notation

We use bold upper-case letters $\mathbf{A}, \mathbf{B}$ to denote matrices and bold lower-case letters $\mathbf{x}, \mathbf{y}$ to denote vectors. For vectors we use $\|\cdot\|$ to denote the $\ell_2$-norm, and for matrices we use $\|\cdot\|$ and $\rho(\cdot)$ to denote spectral (or operator) norm and spectral radius (largest absolute value of eigenvalues) respectively. Note that these two are in general different for asymmetric matrices. For a function $f : \mathbb{R}^d \to \mathbb{R}$, we use $\nabla f$ and $\nabla^2 f$ to denote its gradient and Hessian. For functions of two vector arguments, $f : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$, we use $\nabla_x f$, $\nabla_y f$ and $\nabla^2_{xx} f$, $\nabla^2_{xy} f$, $\nabla^2_{yy} f$ to denote its partial gradient and partial Hessian. We also use $O(\cdot)$ and $o(\cdot)$ notation as follows: $f(\delta) = O(\delta)$ means $\limsup_{\delta \to 0} |f(\delta)/\delta| \leq C$ for some large absolute constant $C$, and $g(\delta) = o(\delta)$ means $\lim_{\delta \to 0} |g(\delta)/\delta| = 0$. For complex numbers, we use $\Re(\cdot)$ to denote its real part, and $|\cdot|$ to denote its modulus. We also use $\mathcal{P}(\cdot)$, operating over a set, to denote the collection of all probability measures over the set.

2.2 Minmax optimization and zero-sum games

In this paper, we consider general minmax optimization problems. Given a function $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, where $\mathcal{X} \subset \mathbb{R}^{d_1}$ and $\mathcal{Y} \subset \mathbb{R}^{d_2}$, the objective is to solve:

$$\min_{\mathbf{x} \in \mathcal{X}} \max_{\mathbf{y} \in \mathcal{Y}} f(\mathbf{x}, \mathbf{y}). \quad (1)$$

While classical theory mostly studied the convex-concave case where $f(\cdot, \mathbf{y})$ is convex for any fixed $\mathbf{y}$ and $f(\mathbf{x}, \cdot)$ is concave for any fixed $\mathbf{x}$, this paper considers the general case, where both $f(\mathbf{x}, \cdot)$ and $f(\cdot, \mathbf{y})$ can be nonconvex and nonconcave. Optimality in this setting is defined as follows:

**Definition 1.** $(\mathbf{x}^*, \mathbf{y}^*)$ is a **global minmax point**, if for any $(\mathbf{x}, \mathbf{y})$ in $\mathcal{X} \times \mathcal{Y}$ we have:

$$f(\mathbf{x}^*, \mathbf{y}) \leq f(\mathbf{x}^*, \mathbf{y}^*) \leq \max_{\mathbf{y}' \in \mathcal{Y}} f(\mathbf{x}, \mathbf{y}').$$

The minmax problem (1) has been extensively studied in the game theory literature under the name of “zero-sum game.” Here, two players play a competitive game with the first player playing $\mathbf{x} \in \mathcal{X}$, and then the second player playing $\mathbf{y} \in \mathcal{Y}$. $f(\mathbf{x}, \mathbf{y})$ is the payoff function which represents the value lost by the first player (which is in turn gained by the second player). In this setting the standard notion of equilibrium is the following:

**Definition 2.** $(\mathbf{x}^*, \mathbf{y}^*)$ is a **(pure strategy) Nash equilibrium** of $f$, if for any $(\mathbf{x}, \mathbf{y})$ in $\mathcal{X} \times \mathcal{Y}$:

$$f(\mathbf{x}^*, \mathbf{y}) \leq f(\mathbf{x}^*, \mathbf{y}^*) \leq f(\mathbf{x}, \mathbf{y}^*).$$

Pure strategy Nash equilibria play an essential role in convex-concave games since for those games, pure strategy Nash equilibria always exist, and are also global minmax points [Bubeck, 2015].

When we move to the nonconvex-nonconcave setting, these nice properties of pure strategy Nash equilibria no longer hold. Moreover, the problem of finding global solutions in this setting is NP hard in general. Therefore, previous work has consider local alternatives [see, e.g., Mazumdar and Ratliff [2018], Daskalakis and Panageas [2018].]
Definition 3. \((x^*, y^*)\) is a local (pure strategy) Nash equilibrium of \(f\), if there exists \(\delta > 0\) such that for any \((x, y)\) satisfying \(\|x - x^*\| \leq \delta\) and \(\|y - y^*\| \leq \delta\) we have:

\[
f(x^*, y) \leq f(x^*, y^*) \leq f(x, y^*).
\]

We can characterize local pure strategy Nash equilibria via first-order and second-order conditions.

**Proposition 4** (First-order Necessary Condition). Assuming \(f\) is differentiable, any local Nash equilibrium satisfies \(\nabla_x f(x, y) = 0\) and \(\nabla_y f(x, y) = 0\).

**Proposition 5** (Second-order Necessary Condition). Assuming \(f\) is twice-differentiable, any local Nash equilibrium satisfies \(\nabla_{xy}^2 f(x, y) \leq 0\), and \(\nabla_{xx}^2 f(x, y) \succeq 0\).

**Proposition 6** (Second-order Sufficient Condition). Assuming \(f\) is twice-differentiable, any stationary point (i.e., \(\nabla f = 0\)) satisfying the following condition is a local Nash equilibrium:

\[
\nabla_{xy}^2 f(x, y) < 0, \text{ and } \nabla_{xx}^2 f(x, y) > 0.
\]

We also call a stationary point satisfying (2) a strict local Nash equilibrium.

In contrast to pure strategies where each player plays a single action, game theorists have also considered mixed strategies where each player is allowed to play a randomized action sampled from a probability measure \(\mu \in \mathcal{P}(X)\) or \(\nu \in \mathcal{P}(Y)\). Then, the payoff function becomes an expected value \(\mathbb{E}_{x \sim \mu, y \sim \nu} f(x, y)\). This corresponds to the scenario where the second player knows the strategy (distribution) of the first player, but does not know the random action he plays. In this setting we define mixed strategy Nash equilibria:

**Definition 7.** A probability measure \((\mu^*, \nu^*)\) is a mixed strategy Nash equilibrium of \(f\), if for any measure \((\mu, \nu)\) in \(\mathcal{P}(X) \times \mathcal{P}(Y)\), we have

\[
\mathbb{E}_{x \sim \mu^*, y \sim \nu^*} f(x, y) \leq \mathbb{E}_{x \sim \mu, y \sim \nu} f(x, y) \leq \mathbb{E}_{x \sim \mu^*, y \sim \nu^*} f(x, y).
\]

2.3 **Dynamical systems**

One of the most popular algorithms for solving minmax problems is Gradient Descent Ascent (GDA). We outline the algorithm in Algorithm 1 with updates written in a general form \(z_{t+1} = w(z_t)\), where \(w : \mathbb{R}^d \to \mathbb{R}^d\) is a vector function. One notable distinction from standard gradient descent is that \(w(\cdot)\) may not be a gradient field (i.e., the gradient of a scalar function \(\phi(\cdot)\)), and so the Jacobian matrix \(J := \partial w / \partial z\) may be asymmetric. This results in the possibility of the dynamics \(z_{t+1} = w(z_t)\) converging to a limit cycle instead of a single point. Nevertheless, we can still define fixed points and stability for general dynamics.

**Definition 8.** \(z^*\) is a fixed point if \(z^* = w(z^*)\).

**Definition 9** (Linear Stability). For a differentiable dynamical system \(w\), a fixed point \(z^*\) is a linearly stable point of \(w\) if its Jacobian matrix \(J(z^*) := (\partial w / \partial z)(z^*)\) has spectral radius \(\rho(J(z^*)) \leq 1\). We also say that a fixed point \(z^*\) is a strict linearly stable point if \(\rho(J(z^*)) < 1\) and a strict linearly unstable point if \(\rho(J(z^*)) > 1\).

Intuitively, linear stability captures whether under the dynamics \(z_{t+1} = w(z_t)\) a flow that starts at point that is infinitesimally close to \(z^*\) will remain in a small neighborhood around \(z^*\).
3 What is the Right Objective?

We have introduced three notions of optimality in minmax games: global minmax points (Definition 1), pure strategy Nash equilibria (Definition 2) and mixed strategy Nash equilibria (Definition 7). For convex-concave games, these three notions are essentially identical. However, for nonconvex-nonconcave games, they are all different in general. So, what is the right objective to pursue in this general setting?

Pure strategy Nash equilibrium  First, we note that pure strategy Nash equilibria may not exist in nonconvex-nonconcave settings.

Proposition 10. There exists a twice-differentiable function \( f \), where pure strategy Nash equilibria (either local or global) do not exist.

Proof. Consider a two-dimensional function \( f(x,y) = \sin(x+y) \). We have \( \nabla f(x,y) = (\cos(x+y), \cos(x+y)) \). Assuming \( (x,y) \) is a local pure strategy Nash equilibrium, by Proposition 4 it must also be a stationary point; that is, \( x + y = (k + 1/2)\pi \) for \( k \in \mathbb{Z} \). It is easy to verify, for odd \( k \), \( \nabla^2_{xx} f(x,y) = \nabla^2_{yy} f(x,y) = 1 > 0 \); for even \( k \), \( \nabla^2_{xx} f(x,y) = \nabla^2_{yy} f(x,y) = -1 < 0 \). By Proposition 5 none of the stationary points is a local pure strategy Nash equilibrium. \( \square \)

Apart from the existence issue, the property that \( \mathbf{x}^* \) is optimal for \( f(\cdot, \mathbf{y}^*) \) is not meaningful in applications such as adversarial training, which translates to the property that the classifier needs to be optimal with respect to a fixed corruption.

Global minmax point  On the other hand, global minmax points, a simple but less mentioned notion of optimality, always exist.

Proposition 11. Assume that function \( f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R} \) is continuous, and assume that \( \mathcal{X} \subset \mathbb{R}^{d_1}, \mathcal{Y} \subset \mathbb{R}^{d_2} \) are compact. Then the global minmax point (Definition 1) of \( f \) always exists.

Proposition 11 is a simple consequence of the extreme-value theorem. Compared to pure strategy Nash equilibria, the notion of global minmax is typically important in the setting where our goal is to find the best \( \mathbf{x}^* \) subject to adversarial perturbation of \( \mathbf{y} \) rather than an \( \mathbf{x} \) which is optimal for a fixed \( \mathbf{y}^* \). Indeed, both GANs and adversarial training actually fall in this category, where our primary goal is to find the best generator subject to an adversarial discriminator, and to find the best robust classifier subject to adversarial corruption.

Mixed strategy Nash equilibrium  Finally, when each agent is allowed to play a random action according to some distribution, such as in the setting of multi-agent reinforcement learning, mixed strategy Nash equilibria are a valid notion of optimality. The existence of mixed strategy Nash equilibrium can be traced back to von Neumann [1928]. Here we cite a generalized version for continuous games.

Proposition 12 (Glicksberg [1952]). Assume that the function \( f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R} \) is continuous and that \( \mathcal{X} \subset \mathbb{R}^{d_1}, \mathcal{Y} \subset \mathbb{R}^{d_2} \) are compact. Then

\[
\min_{\mu \in \mathcal{P}(\mathcal{X})} \max_{\nu \in \mathcal{P}(\mathcal{Y})} \mathbb{E}_{(\mu,\nu)} f(\mathbf{x}, \mathbf{y}) = \max_{\nu \in \mathcal{P}(\mathcal{Y})} \min_{\mu \in \mathcal{P}(\mathcal{X})} \mathbb{E}_{(\mu,\nu)} f(\mathbf{x}, \mathbf{y}).
\]

Let \( \mu^* \) be the minimum for the minmax problem, and let \( \nu^* \) be the maximum for the maxmin problem. Then \( (\mu^*, \nu^*) \) is a mixed strategy Nash equilibrium.
In conclusion, both global minmax points and mixed strategy Nash equilibria are well-defined objectives, and of practical interest. For a specific application, which notion is more suitable depends on whether randomized actions are allowed or of interest.

3.1 Reduction from mixed strategy Nash equilibria to minmax points

We concluded in the last section that both global minmax points and mixed strategy Nash equilibria (or mixed strategies, for short) are of practical interest. However, finding mixed strategy equilibria requires optimizing over a space of probability measures, which is infinite dimensional, making the problem computationally infeasible in general. In this section, we show instead how to find approximate mixed strategy Nash equilibria for Lipschitz games. We show that it is sufficient to find a global minmax point of a problem with polynomially large dimension.

Definition 13. Let \((\mu^*, \nu^*)\) be a mixed strategy Nash equilibrium. A probability measure \((\mu^\dagger, \nu^\dagger)\) is an \(\epsilon\)-approximate mixed strategy Nash equilibrium if:

\[
\forall \nu' \in \mathcal{P}(Y), \quad E_{(\mu^\dagger, \nu')} f(x, y) \leq E_{(\mu^*, \nu^*)} f(x, y) + \epsilon
\]

\[
\forall \mu' \in \mathcal{P}(X), \quad E_{(\mu', \nu^\dagger)} f(x, y) \geq E_{(\mu^*, \nu^*)} f(x, y) - \epsilon.
\]

Theorem 14. Assume that function \(f\) is \(L\)-Lipschitz, and the diameters of \(X\) and \(Y\) are at most \(D\). Let \((\mu^*, \nu^*)\) be a mixed strategy Nash equilibrium. Then there exists an absolute constant \(c\), for any \(\epsilon > 0\), such that if \(N \geq c \cdot d_2^2(\text{LD}/\epsilon)^2 \log(\text{LD}/\epsilon)\), we have:

\[
\min_{(x_1, \ldots, x_N) \in X^N} \max_{y \in Y} \frac{1}{N} \sum_{i=1}^{N} f(x_i, y) \leq E_{(\mu^*, \nu^*)} f(x, y) + \epsilon.
\]

Intuitively, Theorem 14 holds because function \(f\) is Lipschitz, \(Y\) is a bounded domain, and thus we can establish uniform convergence of the expectation of \(f(\cdot, y)\) to its average over \(N\) samples for all \(y \in Y\) simultaneously. A similar argument was made in Arora et al. [2017].

Theorem 14 implies that in order to find an approximate mixed strategy Nash equilibrium, we can solve a large minmax problem with objective \(\tilde{F}(X, y) := \sum_{i=1}^{N} f(x_i, y)/N\). The global minmax solution \(X^* = (x_1^*, \ldots, x_n^*)\) gives an empirical distribution \(\bar{\mu}^* = \sum_{i=1}^{N} \delta(x - x_i^*)/N\), where \(\delta(\cdot)\) is the Dirac delta function. By symmetry, we can also solve the maxmin problem to find \(\bar{\nu}^*\). Since optimal pure strategies are always as good as optimal mixed strategies for the second player, we know \((\bar{\mu}^*, \bar{\nu}^*)\) is an \(\epsilon\)-approximate mixed strategy Nash equilibrium. That is, approximate mixed strategy Nash can be found by finding two global minmax points.

4 Main Results

In the previous section, we concluded that the central question in minmax optimization is to find a global minmax point. However, the problem of finding global minmax points is in general NP hard. In this section, we present our main results, suggesting possible ways of circumventing this NP-hardness challenge. In Section 4.1 we develop a new notion of local surrogacy for global minmax points which we refer to as local minmax points, and we study their properties. In Section 4.2 we establish relations between stable fixed points of GDA and local minmax points. In Section 4.3 we study the behavior of gradient descent with an approximate maximization oracle for \(y\) and show that it converges to approximately stationary points of \(\max_y f(\cdot, y)\).
4.1 Local minmax points

While most previous work [Daskalakis and Panageas, 2018, Mazumdar and Ratliff, 2018] has focused on local Nash equilibria (Definition 3), which are local surrogates for pure strategy Nash equilibria, we propose a new notion—local minmax—as a natural local surrogate for global minmaxity. To the best of our knowledge, this notion has not been considered before.

Definition 15. A point $(x^\star, y^\star)$ is said to be a local minmax point of $f$, if there exists $\delta_0 > 0$ and a continuous function $h$ satisfying $h(\delta) \to 0$ as $\delta \to 0$, such that for any $\delta \leq \delta_0$, and any $(x, y)$ satisfying $\|x - x^\star\| \leq \delta$ and $\|y - y^\star\| \leq \delta$, we have

\[
    f(x^\star, y) \leq f(x^\star, y^\star) \leq \max_{y': \|y' - y^\star\| \leq h(\delta)} f(x, y').
\]

A notion of local maxmin point can be defined similarly. Local minmax points are different from local Nash equilibria since local minmax points only require $x^\star$ to be the minimum of a local max function $\max_{y': \|y' - y^\star\| \leq h(\delta)} f(\cdot, y')$, while local Nash equilibria require $x^\star$ to be the local minimum after fixing $y^\star$ (see Figure 1). The local radius $h(\delta)$ over which the maximum is taken needs to decrease to zero as $\delta$ approaches zero. We note that our definition does not control the relative rate at which $h(\delta)$ and $\delta$ go to zero; indeed, it is allowed that $\lim_{\delta \to 0} h(\delta)/\delta = \infty$.

We would like to highlight an interesting fact: in minmax optimization, global minmax can be neither local minmax nor stationary points (and thus not local Nash equilibria). This is in contrast to the well-known fact in nonconvex optimization where global minima are always local minima.

Proposition 16. The global minmax point can be neither local minmax nor a stationary point.

See Figure 2 for an illustration and Appendix B for the proof. The proposition is a natural consequence of the definitions where global minmax points are obtained as a minimum of a global maximum function while local minmax points are the minimum of a local maximum function. This also illustrates that minmax optimization is a challenging task, and worthy of independent study, beyond nonconvex optimization.

Nevertheless, global minmax points can be guaranteed to be local minmax if the problem has some structure. For instance, this is true when $f$ is strongly-concave in $y$, or more generally when $f$ satisfies the following properties that have been established to hold in several popular machine learning problems [Ge et al., 2017, Boumal et al., 2016].
Figure 2: Left: \( f(x, y) = 0.2xy - \cos(y) \), the global minmax points \((0, -\pi)\) and \((0, \pi)\) are not stationary. Right: The relations among local Nash equilibria, local minmax points, local maxmin points and linearly stable points of \(\gamma\)-GDA, and \(\infty\)-GDA (up to degenerate points).

**Theorem 17.** Assume that \( f \) is twice differentiable, and for any fixed \( x \), the function \( f(x, \cdot) \) is strongly concave in the neighborhood of local maxima and satisfies the assumption that all local maxima are global maxima. Then the global minmax point of \( f(\cdot, \cdot) \) is also a local minmax point.

We consider local minmax as a more suitable notion of local optimality than local Nash equilibrium for minmax optimization. First, local minmaxity is a strictly relaxed notion of local Nash equilibrium, and it alleviates the non-existence issue for local Nash equilibria.

**Proposition 18.** Any local pure strategy Nash equilibrium is a local minmax point.

Second, local minmax points enjoy simple first-order and second-order characterizations.

**Proposition 19 (First-order Necessary Condition).** Assuming that \( f \) is continuously differentiable, then any local minmax point \((x, y)\) satisfies \( \nabla_x f(x, y) = 0 \) and \( \nabla_y f(x, y) = 0 \).

**Proposition 20 (Second-order Necessary Condition).** Assuming that \( f \) is twice differentiable, then \((x, y)\) is a local minmax point implies that \( \nabla_{yy}^2 f(x, y) \preceq 0 \) and for any \( v \) satisfying \( \nabla_{yx}^2 f(x, y) \cdot v \in \text{col-span}(\nabla_{yy}^2 f(x, y)) \) that \( v^\top [\nabla_{xx}^2 f - \nabla_{xy}^2 f(\nabla_{yy}^2 f)^{-1} \nabla_{yx}^2 f] (x, y) \cdot v \geq 0 \). (Here \( ^\dagger \) denotes Moore-Penrose inverse.)

**Proposition 21 (Second-order Sufficient Condition).** Assume that \( f \) is twice differentiable. Any stationary point \((x, y)\) satisfying \( \nabla_{yy}^2 f(x, y) \prec 0 \) and

\[
[\nabla_{xx}^2 f - \nabla_{xy}^2 f(\nabla_{yy}^2 f)^{-1} \nabla_{yx}^2 f](x, y) \succ 0
\]

(4)

is a local minmax point. We call stationary points satisfying (4) strict local minmax points.

We note that if \( \nabla_{yy}^2 f(x, y) \) is non-degenerate, then the second-order necessary condition (Proposition 20) becomes \( \nabla_{yy}^2 f(x, y) \prec 0 \) and \( [\nabla_{xx}^2 f - \nabla_{xy}^2 f(\nabla_{yy}^2 f)^{-1} \nabla_{yx}^2 f](x, y) \succeq 0 \), which is identical to the sufficient condition Eq.(4) up to an equals sign.
Algorithm 1: Gradient Descent Ascent (γ-GDA)

Input: \((x_0, y_0)\), step size \(\eta\), ratio \(\gamma\).

for \(t = 0, 1, \ldots, \) do

\[x_{t+1} \leftarrow x_t - \left(\frac{\eta}{\gamma}\right) \nabla_x f(x_t, y_t).\]

\[y_{t+1} \leftarrow y_t + \eta \nabla_y f(x_t, y_t).\]

Comparing Eq. (4) to the second-order sufficient condition for local Nash equilibrium in Eq. (2), we see that local minmax requires the Shur complement to be positive definite instead of requiring \(\nabla^2_{xx} f(x, y)\) to be positive definite. Contrary to local Nash equilibria, this characterization of local minmax not only takes into account the interaction term \(\nabla^2_{xy} f\) between \(x\) and \(y\), but also reflects the order of minmax vs maxmin.

4.2 Limit points of gradient descent ascent

In this section, we consider the asymptotic behavior of Gradient Descent Ascent (GDA). As shown in the pseudo-code in Algorithm 1, GDA simultaneously performs gradient descent on \(x\) and gradient ascent on \(y\). We consider the general form where the step size for \(x\) can be different from the step size for \(y\) by a ratio \(\gamma\), and denoted this algorithm by \(\gamma\)-GDA. When the step size \(\eta\) is small, this is essentially equivalent to gradient descent with multiple steps of gradient ascent where \(\gamma\) indicates how many gradient ascent steps are performed for one gradient descent step.

To study the limiting behavior, we primarily focus on linearly stable points of \(\gamma\)-GDA, since with random initialization, \(\gamma\)-GDA will almost surely escape strict linearly unstable points.

Theorem 22 (Daskalakis and Panageas [2018]). For any \(\gamma > 1\), assuming the function \(f\) is \(\ell\)-gradient Lipschitz, and the step size \(\eta \leq 1/\ell\), then the set of initial points \(x_0\) so that \(\gamma\)-GDA converges to its strict linear unstable point is of Lebesgue measure zero.

We further simplify the problem by considering the limiting case where the step size \(\eta \to 0\), which corresponds to \(\gamma\)-GDA flow

\[
\frac{dx}{dt} = -\frac{1}{\gamma} \nabla_x f(x, y) \quad \frac{dy}{dt} = \nabla_y f(x, y).
\]

The strict linearly stable points of the \(\gamma\)-GDA flow have a very simple second-order characterization.

Proposition 23. \((x, y)\) is a strict linearly stable point of \(\gamma\)-GDA if and only if for all the eigenvalues \(\{\lambda_i\}\) of following Jacobian matrix,

\[
J_\gamma = \begin{pmatrix} -(1/\gamma) \nabla^2_{xx} f(x, y) & -(1/\gamma) \nabla^2_{xy} f(x, y) \\ \nabla^2_{yx} f(x, y) & \nabla^2_{yy} f(x, y) \end{pmatrix},
\]

their real part \(\text{Re}(\lambda_i) < 0\) for any \(i\).

In the remainder of this section, we assume that \(f\) is a twice-differentiable function, and we use Local Nash to represent the set of strict local Nash equilibria, Local Minmax for the set of strict local minmax points, Local Maxmin for the set of strict local maxmin points, and \(\gamma\)-GDA for the set of strict linearly stable points of the \(\gamma\)-GDA flow. Our goal is to understand the relationships between these sets. Daskalakis and Panageas [2018] and Mazumdar and Ratliff [2018] provided a relation between Local Nash and 1-GDA which can be generalized to \(\gamma\)-GDA as follows.
Algorithm 2 Gradient Descent with Max-oracle

Input: \(x_0\), step size \(\eta\).

for \(t = 0, 1, \ldots, T\) do

find \(y_t\) so that \(f(x_t, y_t) \geq \max_y f(x_t, y) - \epsilon\).

\(x_{t+1} \leftarrow x_t - \eta \nabla_x f(x_t, y_t)\).

Pick \(t\) uniformly at random from \(\{0, \ldots, T\}\).

return \(\bar{x} \leftarrow x_t\).

Proposition 24 (Daskalakis and Panageas [2018]). For any fixed \(\gamma\), for any twice-differentiable \(f\), \(\text{Local Nash} \subset \gamma - \text{GDA}\), but there exist twice-differentiable \(f\) such that \(\gamma - \text{GDA} \not\subset \text{Local Nash}\).

That is, if \(\gamma\)-GDA converges, it may converge to points not in \(\text{Local Nash}\). This raises a basic question as to what those additional stable limit points of \(\gamma\)-GDA are. Are they meaningful? This paper answers this question through the lens of \(\text{Local Minmax}\). Although for fixed \(\gamma\), the set \(\gamma - \text{GDA}\) does not have a simple relation with \(\text{Local Minmax}\), it turns out that an important relationship arises when \(\gamma\) goes to \(\infty\). To describe the limit behavior of the set \(\gamma - \text{GDA}\) when \(\gamma \to \infty\) we define two set-theoretic limits:

\[
\infty - \text{GDA} := \limsup_{\gamma \to \infty} \gamma - \text{GDA} = \bigcap_{\gamma > 0} \gamma - \text{GDA}
\]

\[
\infty - \text{GDA} := \liminf_{\gamma \to \infty} \gamma - \text{GDA} = \bigcup_{\gamma > 0} \bigcap_{\gamma > 0} \gamma - \text{GDA}.
\]

The relations between \(\gamma - \text{GDA}\) and \(\text{Local Minmax}\) are given as follows:

Proposition 25. For any fixed \(\gamma\), there exists twice-differentiable \(f\) such that \(\text{Local Minmax} \not\subset \gamma - \text{GDA}\); there also exists twice-differentiable \(f\) such that \(\gamma - \text{GDA} \not\subset \text{Local Minmax} \cup \text{Local Maxmin}\).

Theorem 26 (Main Theorem). For any twice-differentiable \(f\), \(\text{Local Minmax} \subset \infty - \text{GDA} \subset \infty - \text{GDA} \subset \text{Local Minmax} \cup \{(x, y) | (x, y)\text{ is stationary and }\nabla^2_{yy} f(x, y)\text{ is degenerate}\}.

That is, \(\infty - \text{GDA} = \text{Local Minmax}\) up to some degenerate points. Intuitively, when \(\gamma\) is large, \(\gamma\)-GDA can move a long distance in \(y\) while only making very small changes in \(x\). As \(\gamma \to \infty\), \(\gamma\)-GDA can approximately find the local maximum of \(f(x + \delta_x, \cdot)\), subject to any small change in \(\delta_x\); therefore, stable limit points are indeed local minmax.

Algorithmically, one can view \(\infty - \text{GDA}\) as a set that describes the strict linear stable limit points for GDA with \(\gamma\) very slowly increasing with respect to \(t\), and eventually going to \(\infty\). To the best of our knowledge, this is the first result showing that all stable limit points of GDA are meaningful and locally optimal up to some degenerate points.

4.3 Gradient descent with max-oracle

In this section, we consider solving the minmax problem when we have access to an oracle for approximate inner maximization; i.e., for any \(x\), we have access to an oracle that outputs a \(\hat{y}\) such that \(f(x, \hat{y}) \geq \max_y f(x, y) - \epsilon\). A natural algorithm to consider in this setting is to alternate between gradient descent on \(x\) and a (approximate) maximization step on \(y\). The pseudocode is presented in Algorithm 2.

It can be shown that Algorithm 2 indeed converges (in contrast with GDA which can converge to limit cycles). Moreover, the limit points of Algorithm 2 satisfy a nice property—they turn out to be approximately
stationary points of \( \phi(x) := \max_y f(x, y) \). For a smooth function, “approximately stationary point” means the norm of gradient is small. However, even when \( f(\cdot, \cdot) \) is smooth (up to whatever order), \( \phi(\cdot) \) as defined above need not be differentiable. The norm of subgradient can be a discontinuous function which is an undesirable measure for closeness to stationarity. Fortunately, however, and \( \ell \)-gradient Lipschitz of \( f(\cdot, \cdot) \) imply that \( \phi(\cdot) \) is \( \ell \)-weakly convex \( \text{Rafique et al., 2018} \); i.e., \( \phi(x) + (\ell/2)\|x\|^2 \) is convex. In such settings, the approximate stationarity of \( \phi(\cdot) \) can be measured by the norm of gradient of its Moreau envelope \( \phi_\lambda(\cdot) \).

\[
\phi_\lambda(x) := \min_{x'} \phi(x') + \frac{1}{2\lambda} \|x - x'\|^2. \tag{5}
\]

Here \( \lambda < 1/\ell \). The Moreau envelope satisfies the following two important properties if \( \lambda < 1/\ell \). Let \( \hat{x} = \arg\min_{x'} \phi(x') + (1/2\lambda)\|x - x'\|^2 \), then:

\[
\|\hat{x} - x\| = \lambda \|\nabla \phi_\lambda(x)\|, \quad \text{and} \quad \min_{g \in \partial \phi(\hat{x})} \|g\| \leq \|\nabla \phi_\lambda(x)\|,
\]

where \( \partial \) denotes the subdifferential of a weakly convex function. A proof of this fact can be found in \( \text{Rockafellar 2015} \). Therefore, \( \|\nabla \phi_\lambda(x)\| \) being small means that \( x \) is close to a point \( \hat{x} \) that is approximately stationary. We now present the convergence guarantee for Algorithm 2.

**Theorem 27.** Suppose \( f \) is \( \ell \)-smooth and \( L \)-Lipschitz and define \( \phi(\cdot) := \max_y f(\cdot, y) \). Then the output \( \bar{x} \) of GD with Max-oracle (Algorithm 2) with step size \( \eta = \gamma/\sqrt{T + 1} \) will satisfy

\[
\mathbb{E} [\|\nabla \phi_{1/2\ell}(\bar{x})\|^2] \leq 2 \cdot \left( \phi_{1/2\ell}(x_0) - \min_x \phi(x) \right) + \ell L^2 \gamma^2 / (\gamma \sqrt{T + 1}) + 4\ell \epsilon,
\]

where \( \phi_{1/2\ell} \) is the Moreau envelope (5) of \( \phi \).

The proof of Theorem 27 is similar to the convergence analysis for nonsmooth weakly-convex functions \( \text{Davis and Drusvyatskiy 2018} \), except here the max-oracle has error \( \epsilon \). Theorem 27 claims, other than an additive error \( 4\ell \epsilon \) as a result of the oracle solving the maximum approximately, that the remaining term decreases at a rate of \( 1/\sqrt{T} \).

## 5 Conclusion

In this paper, we consider general nonconvex-nonconcave minmax optimization. While gradient descent ascent (GDA) is widely used in practice for such problems, previous results suggest that GDA has undesirable limiting behavior, questioning GDA’s relevance for this problem. We formulate a new notion of local optimum for minmax problems, which we refer to as \( \text{local minmax} \), and show that it is more suitable for many learning problems than standard notions such as local Nash equilibrium. We establish that as the ratio of the ascent step size to the descent step size in GDA goes to infinity, all strict stable limit points are equivalent to local minmax points except for degenerate points. This yields a game-theoretic meaning for all stable limit points of GDA. We also consider the minmax problem when we have access to an approximate inner maximization. In this setting, we analyze gradient descent with maximization and show that it finds a point close to an approximate stationary point.
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A Proofs for Reduction from Mixed Strategy Nash to Minmax Points

In this section we prove Theorem 14 in Section 3.1.

**Theorem 14.** Assume that function $f$ is $L$-Lipschitz, and the diameters of $X$ and $Y$ are at most $D$. Let $(\mu^*, \nu^*)$ be a mixed strategy Nash equilibrium. Then there exists an absolute constant $c$, for any $\epsilon > 0$, such that if $N \geq c \cdot d_2(LD/\epsilon)^2 \log(LD/\epsilon)$, we have:

$$\min_{(x_1, \ldots, x_N) \in X^N} \max_{y \in Y} \frac{1}{N} \sum_{i=1}^{N} f(x_i, y) \leq \mathbb{E}_{(\mu^*, \nu^*)} f(x, y) + \epsilon.$$  

**Proof.** Note that WLOG, the second player can always play pure strategy. That is, assume that function $f$ be a minimal $\mu^* \in \mathcal{P}(X)$ and $\nu^* \in \mathcal{P}(Y)$.

Therefore, we only need to solve the problem of RHS. Suppose the minimum over $\mathcal{P}(X)$ is achieved at $\mu^*$. First, sample $(x_1, \ldots, x_N)$ i.i.d. from $\mu^*$, and note that $\max_{x_1, x_2 \in X} |f(x_1, y) - f(x_2, y)| \leq LD$ for any fixed $y$. Therefore by Hoeffding inequality, for any fixed $y$:

$$\Pr \left( \frac{1}{N} \sum_{i=1}^{N} f(x_i, y) - \mathbb{E}_{x \sim \mu^*} f(x, y) \geq t \right) \leq e^{-\frac{Nt^2}{(LD)^2}}.$$

Let $\tilde{Y}$ be a minimal $\epsilon/(2L)$-covering over $Y$. We know the covering number $|\tilde{Y}| \leq (2DL/\epsilon)^d$. Thus by union bound:

$$\Pr \left( \forall y \in \tilde{Y}, \frac{1}{N} \sum_{i=1}^{N} f(x_i, y) - \mathbb{E}_{x \sim \mu^*} f(x, y) \geq t \right) \leq e^{d \log \frac{2DL}{\epsilon} - \frac{Nt^2}{(LD)^2}}.$$

Pick $t = \epsilon/2$ and $N \geq c \cdot d(LD/\epsilon)^2 \log(LD/\epsilon)$ for some large absolute constant $c$, we have:

$$\Pr \left( \forall y \in \tilde{Y}, \frac{1}{N} \sum_{i=1}^{N} f(x_i, y) - \mathbb{E}_{x \sim \mu^*} f(x, y) \geq \frac{\epsilon}{2} \right) \leq \frac{1}{2}.$$

Let $y^* = \arg \max_{y} \frac{1}{N} \sum_{i=1}^{N} f(x_i, y)$, by definition of covering, we can always find a $y' \in \tilde{Y}$ so that $\|y^* - y'\| \leq \epsilon/(4L)$. Thus, with probability at least 1/2:

$$\max_y \frac{1}{N} \sum_{i=1}^{N} f(x_i, y) - \max_y \mathbb{E}_{x \sim \mu^*} f(x, y) = \frac{1}{N} \sum_{i=1}^{N} f(x_i, y^*) - \max_y \mathbb{E}_{x \sim \mu^*} f(x, y) \leq \left[ \frac{1}{N} \sum_{i=1}^{N} f(x_i, y^*) - \frac{1}{N} \sum_{i=1}^{N} f(x_i, y') \right] + \left[ \frac{1}{N} \sum_{i=1}^{N} f(x_i, y') - \mathbb{E}_{x \sim \mu^*} f(x, y') \right] + [\mathbb{E}_{x \sim \mu^*} f(x, y') - \max_y \mathbb{E}_{x \sim \mu^*} f(x, y)] \leq \epsilon/2 + \epsilon/2 + 0 \leq \epsilon$$

That is, with probability at least 1/2:

$$\max_{y \in Y} \frac{1}{N} \sum_{i=1}^{N} f(x_i, y) \leq \min_{\mu \in \mathcal{P}(X)} \max_{y \in Y} \mathbb{E}_{x \sim \mu} f(x, y) + \epsilon.$$
This implies:

\[
\min_{(x_1, \ldots, x_N) \in X^N} \max_{y \in Y} \frac{1}{N} \sum_{i=1}^{N} f(x_i, y) \leq \min_{\mu \in P(X)} \max_{y \in Y} \mathbb{E}_{x \sim \mu} f(x, y) + \epsilon
\]

Combine with Proposition 12 we finish the proof.

B Proofs for Properties of Local Minmax Points

In this section, we prove the propositions and theorems presented in Section 4.1.

Proposition 16. The global minmax point can be neither local minmax nor a stationary point.

Proof. Consider function \( f(x, y) = 0.2xy - \cos(y) \) in region \([-1, 1] \times [-2\pi, 2\pi] \) as shown in Figure 2. Clearly, the gradient is equal to \((0.2y, 0.2x + \sin(y))\). And for any fixed \( x \), there are only two maxima \( y^*(x) \) satisfying \( 0.2x + \sin(y^*) = 0 \) where \( y_1^*(x) \in (-3\pi/2, -\pi/2) \) and \( y_2^*(x) \in (\pi/2, 3\pi/2) \). On the other hand, \( f(x, y_1^*(x)) \) is monotonically decreasing with respect to \( x \), while \( f(x, y_2^*(x)) \) is monotonically increasing, with \( f(0, y_1^*(0)) = f(0, y_2^*(0)) \) by symmetry. It is not hard to check \( y_1^*(0) = -\pi \) and \( y_2^*(0) = \pi \). Therefore, \((0, -\pi)\) and \((0, \pi)\) are two global solutions of minmax problem. However, the gradients at both points are not 0, thus they are not stationary points. By Proposition 19, they are also not local minmax points.

Theorem 17. Assume that \( f \) is twice differentiable, and for any fixed \( x \), the function \( f(x, \cdot) \) is strongly concave in the neighborhood of local maxima and satisfies the assumption that all local maxima are global maxima. Then the global minmax point of \( f(\cdot, \cdot) \) is also a local minmax point.

Proof. Denote \( A := \nabla^2_x f(x, y) \), \( B := \nabla^2_{yx} f(x, y) \), \( C := \nabla^2_{xy} f(x, y) \), \( g_x := \nabla_x f(x, y) \) and \( g_y := \nabla_y f(x, y) \). Let \((x, y)\) be a global minmax point. Since \( y \) is the global argmax of \( f(x, \cdot) \) and locally strongly concave, we know \( g_y = 0 \) and \( B < 0 \). Let us now consider a second-order Taylor approximation of \( f \) around \((x, y)\).

\[
f(x + \delta_x, y + \delta_y) = f(x, y) + g_x^\top \delta_x + \frac{1}{2} \delta_x^\top A \delta_x + \delta_x^\top C \delta_y + \frac{1}{2} \delta_y^\top B \delta_y + o(\|\delta_x\|^2 + \|\delta_y\|^2)
\]

Since by hypothesis, \( B < 0 \), we see that when \( \|\delta_x\| \) is sufficiently small, there is a unique \( y^*(\delta_x) \) so that \( y + y^*(\delta_x) \) is a local maximum of \( f(x + \delta_x, \cdot) \), where \( y^*(\delta_x) = -B^{-1}C^\top \delta_x + o(\|\delta_x\|) \). It is clear that \( \|y^*(\delta_x)\| \leq (\|B^{-1}C^\top\| + 1)\|\delta_x\| \) for sufficiently small \( \|\delta_x\| \). Let \( h(\delta) = (\|B^{-1}C^\top\| + 1)\delta \), we know for small enough \( \delta \):

\[
f(x + \delta_x, y + y^*(\delta_x)) = \max_{\|\delta_y\| \leq h(\delta)} f(x + \delta_x, y + \delta_y)
\]

Finally, since by assumption for any \( f(x, \cdot) \) all local maxima are global maxima and \( x \) is the global min of \( \max_y f(x, y) \), we know:

\[
f(x, y) \leq \max_{y'} f(x + \delta_x, y') = f(x + \delta_x, y + y^*(\delta_x)) = \max_{\|\delta_y\| \leq h(\delta)} f(x + \delta_x, y + \delta_y)
\]

which finishes the proof.

Proposition 18. Any local pure strategy Nash equilibrium is a local minmax point.
Proof. Let $h$ be the constant function $h(\delta) = 0$ for any $\delta$. Suppose $(x^*, y^*)$ is a local pure strategy Nash equilibrium, by definition it implies the existence of $\delta_0$, so that for any $\delta \leq \delta_0$, and any $(x, y)$ satisfying $\|x - x^*\| \leq \delta$ and $\|y - y^*\| \leq \delta$:

$$f_2(x^*, y) \leq f_2(x^*, y^*) \leq f(x, y) \leq \max_{y': \|y' - y^*\| \leq h(\delta)} f_2(x, y').$$

which finishes the proof. \Box

**Proposition 19** (First-order Necessary Condition). Assuming that $f$ is continuously differentiable, then any local minmax point $(x, y)$ satisfies $\nabla_x f(x, y) = 0$ and $\nabla_y f(x, y) = 0$.

Proof. Since $y$ is the local maximum of $f(x, \cdot)$, it implies $\nabla_y f(x, y) = 0$. Denote local optima $\delta^*_y(\delta_x) := \arg\max_{\|y\| \leq h(\delta)} f(x + \delta_x, y + \delta_y)$. By definition we know, $\|\delta^*_y(\delta_x)\| \leq h(\delta) \to 0$ as $\delta \to 0$. Thus

$$0 \leq f(x + \delta_x, y + \delta^*_y(\delta_x)) - f(x, y) = f(x + \delta_x, y + \delta^*_y(\delta_x)) - f(x, y + \delta^*_y(\delta_x)) + f(x, y + \delta^*_y(\delta_x)) - f(x, y)$$

$$\leq f(x + \delta_x, y + \delta^*_y(\delta_x)) - f(x, y + \delta^*_y(\delta_x))$$

$$= \nabla_x f(x, y + \delta^*_y(\delta_x)) + o(\|\delta_x\|) = \nabla_x f(x, y) + o(\|\delta_x\|)$$

holds for any small $\delta_x$, which implies $\nabla_x f(x, y) = 0$. \Box

**Proposition 20** (Second-order Necessary Condition). Assuming that $f$ is twice differentiable, then $(x, y)$ is a local minmax point implies that $\nabla^2_{xy} f(x, y) \preceq 0$, and for any $v$ satisfying $\nabla^2_{xy} f(x, y) \cdot v \in \text{column span}(\nabla^2_{xy} f(x, y))$ that $v^T [\nabla^2_{xx} f - \nabla^2_{xy} f (\nabla^2_{yy} f)^\dagger \nabla^2_{xy} f](x, y) \cdot v \geq 0$. (Here $\dagger$ denotes Moore-Penrose inverse.)

Proof. Denote $A := \nabla^2_{xx} f(x, y)$, $B := \nabla^2_{xy} f(x, y)$ and $C := \nabla^2_{yy} f(x, y)$. Since $y$ is the local maximum of $f(x, \cdot)$, it implies $B \preceq 0$. On the other hand,

$$f(x + \delta_x, y + \delta_y) = f(x, y) + \frac{1}{2} \delta_x^{\top} A \delta_x + \delta_y \delta_y + \frac{1}{2} \delta_y \delta_x + o(\|\delta_x\|^2 + \|\delta_y\|^2).$$

Since $(x, y)$ is a local minmax point, by definition, there exists a function $h$ such that Eq. holds. Denote $h'(\delta) = 2\|B^{-1} C^\top\| \delta$. We note both $h(\delta)$ and $h'(\delta) \to 0$ as $\delta \to 0$. For any $\delta_x$ satisfying $C^\top \delta_x \in \text{column span}(B)$, it is not hard to verify that argmax$_{\|\delta_y\| \leq \max(1, h(\delta), h'(\delta))} f(x + \delta_x, y + \delta_y) = -B^{-1} C^\top \delta_x + o(\|\delta_x\|^2)$. Since $(x, y)$ is a local minmax point, we have

$$0 \leq \max_{\|\delta_y\| \leq h(\delta)} f(x + \delta_x, y + \delta_y) - f(x, y) \leq \max_{\|\delta_y\| \leq \max(h(\delta), h'(\delta))} f(x + \delta_x, y + \delta_y) - f(x, y)$$

$$= \frac{1}{2} \delta_x^{\top} (A - CB^{-1} C^\top) \delta_x + o(\|\delta_x\|^2).$$

Above equation holds for any $\delta_x$ satisfying $C^\top \delta_x \in \text{column span}(B)$, which finishes the proof. \Box

**Proposition 21** (Second-order Sufficient Condition). Assume that $f$ is twice differentiable. Any stationary point $(x, y)$ satisfying $\nabla^2_{yy} f(x, y) \prec 0$ and

$$[\nabla^2_{xx} f - \nabla^2_{xy} f (\nabla^2_{yy} f)^{-1} \nabla^2_{xy} f](x, y) \prec 0$$

is a local minmax point. We call stationary points satisfying $\ddagger$ strict local minmax points.
Proof. Again denote $A := \nabla_{xx}^2 f(x, y)$, $B := \nabla_{xy}^2 f(x, y)$ and $C := \nabla_{yx}^2 f(x, y)$. Since $(x, y)$ is a stationary point, and $B \prec 0$, it is clear that $y$ is the local maximum of $f(x, \cdot)$. On the other hand, pick $\delta^\top = B^{-1} C^\top \delta_x$ and let $h(\delta) = \|B^{-1} C^\top\| \delta$, we know when $\|\delta_x\| \leq \delta, \|\delta^\top_y\| \leq h(\delta)$, thus

$$\max_{\|\delta_y\| \leq h(\delta)} f(x + \delta_x, y + \delta^\top) - f(x, y) \geq f(x + \delta_x, y + \delta^\top_y) - f(x, y)$$

$$= \frac{1}{2} \delta^\top_x (A - CB^{-1} C^\top) \delta_x + o(\|\delta_x\|^2) > 0$$

which finishes the proof.

C Proofs for Limit Points of Gradient Descent Ascent

In this section, we provides proofs for propositions and theorems presented in Section 4.2.

Proposition 23. $(x, y)$ is a strict linearly stable point of $\gamma$-GDA if and only if for all the eigenvalues $\{\lambda_i\}$ of following Jacobian matrix,

$$J_{\gamma} = \left(\begin{array}{cc}
-(1/\gamma) \nabla_{xx}^2 f(x, y) & -(1/\gamma) \nabla_{xy}^2 f(x, y) \\
\nabla_{yx}^2 f(x, y) & \nabla_{yy}^2 f(x, y)
\end{array}\right)$$

their real part $\text{Re}(\lambda_i) < 0$ for any $i$.

Proof. Consider GDA dynamics with step size $\eta$, then the Jacobian matrix of this dynamic system is $I + \eta J_{\gamma}$ whose eigenvalues are $\{1 + \eta \lambda_i\}$. Therefore, $(x, y)$ is a strict linearly stable point if and only if $\rho(I + \eta J_{\gamma}) < 1$, that is $|1 + \eta \lambda_i| < 1$ for all $i$. When taking $\eta \rightarrow 0$, this is equivalent to $\text{Re}(\lambda_i) < 0$ for all $i$.

Proposition 24 [Daskalakis and Panageas 2018]. For any fixed $\gamma$, for any twice-differentiable $f$, $\text{Local } \text{Nash} \subset \gamma-GDA$, but there exist twice-differentiable $f$ such that $\gamma-GDA \not\subset \text{Local } \text{Nash}$.

Proof. [Daskalakis and Panageas 2018] showed the proposition holds for 1-GDA. For completeness, here we show how similar proof goes through for $\gamma$-GDA for general $\gamma$. Let $\epsilon = 1/\gamma$, and denote $A := \nabla_{xx}^2 f(x, y)$, $B := \nabla_{xy}^2 f(x, y)$ and $C := \nabla_{yx}^2 f(x, y)$.

To prove the statement $\text{localNash} \subset \gamma-GDA$, we note by definition, $(x, y)$ is a strict linear stable point of $1/\epsilon$-GDA if the real part of the eigenvalues of Jacobian matrix

$$J_{\epsilon} := \left(\begin{array}{cc}
-\epsilon A & -\epsilon C \\
C^\top & B
\end{array}\right)$$

satisfy that $\text{Re}(\lambda_i) < 0$ for all $1 \leq i \leq d_1 + d_2$. We first note that:

$$\tilde{J}_{\epsilon} := \left(\begin{array}{cc}
B & \sqrt{\epsilon} C^\top \\
-\sqrt{\epsilon} C & -\epsilon A
\end{array}\right) = U J_{\epsilon} U^{-1}, \text{ where } U = \left(\begin{array}{cc}
0 & \sqrt{\epsilon} I \\
I & 0
\end{array}\right)$$

Thus, the eigenvalues of $\tilde{J}_{\epsilon}$ and $J_{\epsilon}$ are the same. We can also decompose:

$$J_{\epsilon} = P + Q, \text{ where } P := \left(\begin{array}{cc}
B & 0 \\
-\epsilon A & 0
\end{array}\right), \text{ and } Q := \left(\begin{array}{cc}
0 & \sqrt{\epsilon} C^\top \\
-\sqrt{\epsilon} C & 0
\end{array}\right)$$

If $(x, y)$ is a strict local pure strategy Nash equilibrium, then $A \succ 0$, $B \prec 0$, then $P$ is a negative definite symmetric matrix, and $Q$ is anti-symmetric matrix, i.e. $Q = -Q^\top$. For any eigenvalue $\lambda$ if $\tilde{J}_{\epsilon}$, assume $w$
is the associated eigenvector. That is, \( \tilde{J}_\epsilon w = \lambda w \), also let \( w = x + iy \) where \( x \) and \( y \) are real vectors, and \( w \) be the complex conjugate of vector \( w \). Then:

\[
\text{Re}(\lambda) = \frac{[w^\top \tilde{J}_\epsilon w + w^\top \tilde{J}_\epsilon w] / 2 = [(x - iy)^\top \tilde{J}_\epsilon (x + iy) + (x + iy)^\top \tilde{J}_\epsilon (x - iy)] / 2}
\]

\[
= x^\top \tilde{J}_\epsilon x + y^\top \tilde{J}_\epsilon y = x^\top Px + y^\top Py + x^\top Qx + y^\top Qy
\]

Since \( P \) is negative definite, that is \( x^\top Px \) and \( y^\top Py < 0 \) (\( x \) and \( y \) are fixed points),\n
\[
\text{Re}(\lambda) = \frac{[x^\top \tilde{J}_\epsilon x + y^\top \tilde{J}_\epsilon y] / 2 =} {x^\top \tilde{J}_\epsilon x + y^\top \tilde{J}_\epsilon y = x^\top Px + y^\top Py + x^\top Qx + y^\top Qy}
\]

Since \( P \) is negative definite, that is \( x^\top Px \) and \( y^\top Py < 0 \). Meanwhile, since \( Q \) is antisymmmetric \( x^\top Qx = x^\top Q^\top x = 0 \) and \( y^\top Qy = y^\top Q^\top y = 0 \). This proves \( \text{Re}(\lambda) < 0 \), that is \( (x, y) \) is a strict linear stable point of \( 1/\epsilon\text{-GDA} \).

To prove the statement \( \gamma\text{-GDA} \not\subset \text{localNash} \), since \( \epsilon \) is also fixed, we consider function \( f(x, y) = x^2 + 2\sqrt{\epsilon}xy + (\epsilon/2)y^2 \). It is easy to see \( (0, 0) \) is a fixed point of \( 1/\epsilon\text{-GDA} \), and Hessian \( A = 2, B = \epsilon, C = 2\sqrt{\epsilon} \). Thus the Jacobian matrix

\[
J_\epsilon := \begin{pmatrix}
-2\epsilon & -2\epsilon^{3/2} \\
2\epsilon^{1/2} & -\epsilon
\end{pmatrix}
\]

has two eigenvalues \( \epsilon(-1 \pm i\sqrt{7})/2 \). Therefore, \( \text{Re}(\lambda_1) = \text{Re}(\lambda_2) < 0 \), which implies \( (0, 0) \) is a strict linear stable point. However \( B = \epsilon > 0 \), thus it is not a strict local pure strategy Nash equilibrium.

\[\square\]

**Proposition 25.** For any fixed \( \gamma \), there exists twice-differentiable \( f \) such that \( \text{Local Minmax} \not\subset \gamma\text{-GDA} \); there also exists twice-differentiable \( f \) such that \( \gamma\text{-GDA} \not\subset \text{Local Minmax} \).

**Proof:** Let \( \epsilon = 1/\gamma \), and denote \( A := \nabla_{xx}^2 f(x, y), B := \nabla_{xy}^2 f(x, y), \) and \( C := \nabla_{yy}^2 f(x, y) \).

To prove the first statement \( \text{localminmax} \not\subset \gamma\text{-GDA} \), since \( \epsilon \) is also fixed, we consider function \( f(x, y) = x^2 + 2\sqrt{\epsilon}xy + (\epsilon/2)y^2 \). It is easy to see \( (0, 0) \) is a fixed point of \( 1/\epsilon\text{-GDA} \), and Hessian \( A = -2, B = \epsilon, C = 2\sqrt{\epsilon} \). It is easy to verify that \( B < 0 \) and \( A - CB^{-1}C = 2 > 0 \), thus \( (0, 0) \) is a local minmax point. However, inspect the Jacobian matrix of \( 1/\epsilon\text{-GDA} \):

\[
J_\epsilon := \begin{pmatrix}
-2\epsilon & -2\epsilon^{3/2} \\
2\epsilon^{1/2} & -\epsilon
\end{pmatrix}
\]

We know the two eigenvalues are \( \epsilon(1 \pm i\sqrt{7})/2 \). Therefore, \( \text{Re}(\lambda_1) = \text{Re}(\lambda_2) > 0 \), which implies \( (0, 0) \) is not a strict linear stable point.

To prove the second statement \( \gamma\text{-GDA} \not\subset \text{localminmax} \cup \text{localmaxmin} \), since \( \epsilon \) is also fixed, we consider function \( f(x, y) = x^2 + 2\sqrt{\epsilon}x_1y_1 + (\epsilon/2)y_1^2 - x_2^2/2 + 2\sqrt{\epsilon}x_2y_2 - ey_2^2 \). It is easy to see \( (0, 0) \) is a fixed point of \( 1/\epsilon\text{-GDA} \), and Hessian \( A = \text{diag}(2, -1), B = \text{diag}(\epsilon, -2\epsilon), C = 2\sqrt{\epsilon} \cdot \text{diag}(1, 1) \). Thus the Jacobian matrix

\[
J_\epsilon := \begin{pmatrix}
-2\epsilon & 0 & -2\epsilon^{3/2} & 0 \\
0 & \epsilon & 0 & -2\epsilon^{3/2} \\
2\epsilon^{1/2} & 0 & \epsilon & 0 \\
0 & 2\epsilon^{1/2} & 0 & -2\epsilon
\end{pmatrix}
\]

has four eigenvalues \( \epsilon(-1 \pm i\sqrt{7})/2 \) (each with multiplicity of 2). Therefore, \( \text{Re}(\lambda_i) < 0 \) for \( 1 \leq i \leq 4 \), which implies \( (0, 0) \) is a strict linear stable point. However, \( B \) is not negative definite, thus \( (0, 0) \) is not a strict local maxmin point; similarly, \( A \) is also not positive definite, thus \( (0, 0) \) is not a strict local maxmin point.

\[\square\]

**Theorem 26 (Main Theorem).** For any twice-differentiable \( f \), \( \text{Local Minmax} \subset \infty\text{-GDA} \subset \infty\text{-GDA} \subset \text{Local Minmax} \cup \{(x, y) \mid (x, y) \text{ is stationary and } \nabla_{xy}^2 f(x, y) \text{ is degenerate.} \)
Proof. For simplicity, denote $A := \nabla^2_{xx} f(x, y)$, $B := \nabla^2_{xy} f(x, y)$ and $C := \nabla^2_{yy} f(x, y)$. Let $\epsilon = 1/\gamma$. Consider sufficiently small $\epsilon$ (i.e. sufficiently large $\gamma$), we know the Jacobian $J$ of $1/\epsilon$-GDA at $(x, y)$ is:

$$J_\epsilon := \begin{pmatrix} -\epsilon A & -\epsilon C \\ C^T & B \end{pmatrix}$$

According to Lemma 28, for sufficient $\epsilon$, $J_\epsilon$ has $d_1 + d_2$ complex eigenvalues $\{\lambda_i\}_{i=1}^{d_1+d_2}$ with following form for sufficient small $\epsilon$:

$$|\lambda_i + \epsilon \mu_i| = o(\epsilon) \quad 1 \leq i \leq d_1$$
$$|\lambda_i + d_1 - \nu_i| = o(1), \quad 1 \leq i \leq d_2$$

(6)

where $\{\mu_i\}_{i=1}^{d_1}$ and $\{\nu_i\}_{i=1}^{d_2}$ are the eigenvalues of matrices $A - CB^{-1}C^T$ and $B$ respectively. Now we are ready to prove the three inclusion statement in Theorem 26 seperately.

First, for $\infty-GDA \subset \infty-GDA$ always holds by their definitions.

Second, for $Local_{\text{Minmax}} \subset \infty-GDA$ statement, if $(x, y)$ is strict local minmax point, then by its definition:

$$B < 0, \quad \text{and} \quad A - CB^{-1}C^T \succ 0$$

By Eq.(6) the eigenvalue structure of $J_\epsilon$, we know there exists sufficiently small $\epsilon_0$, so that for any $\epsilon < \epsilon_0$, the real part $\text{Re}(\lambda_i) < 0$, i.e. $(x, y)$ is a strict linear stable point of $1/\epsilon$-GDA.

Finally, for $\infty-GDA \subset Local_{\text{Minmax}} \cup \{(x, y)|(x, y)\}$ is stationary and $B$ is degenerate statement, if $(x, y)$ is strict linear stable point of $1/\epsilon$-GDA for a sufficiently small $\epsilon$, then for any $i$, the real part of eigenvalue of $J_\epsilon$: $\text{Re}(\lambda_i) < 0$. By Eq.(6), if $B$ is invertible, this implies:

$$B < 0, \quad \text{and} \quad A - CB^{-1}C^T \succeq 0$$

Finally, suppose matrix $A - CB^{-1}C^T$ has an eigenvalue 0. This means the existence of unit vector $w$ so that $(A - CB^{-1}C^T)w = 0$. It is not hard to verify then $J_\epsilon \cdot (w, -B^{-1}C^T w)^T = 0$. This implies $J_\epsilon$ has a 0 eigen-value, which contradicts the fact that $\text{Re}(\lambda_i) < 0$ for any $i$. Therefore, we can conclude $A - CB^{-1}C^T \succ 0$, and $(x, y)$ is a strict local minmax point.

□

Lemma 28. For any symmetric matrix $A \in \mathbb{R}^{d_1 \times d_1}$, $B \in \mathbb{R}^{d_2 \times d_2}$, and any rectangular matrix $C \in \mathbb{R}^{d_1 \times d_2}$, assume $B$ is nondegenerate. Then, matrix

$$\begin{pmatrix} -\epsilon A & -\epsilon C \\ C^T & B \end{pmatrix}$$

has $d_1 + d_2$ complex eigenvalues $\{\lambda_i\}_{i=1}^{d_1+d_2}$ with following form for sufficient small $\epsilon$:

$$|\lambda_i + \epsilon \mu_i| = o(\epsilon) \quad 1 \leq i \leq d_1$$
$$|\lambda_i + d_1 - \nu_i| = o(1), \quad 1 \leq i \leq d_2$$

where $\{\mu_i\}_{i=1}^{d_1}$ and $\{\nu_i\}_{i=1}^{d_2}$ are the eigenvalues of matrices $A - CB^{-1}C^T$ and $B$ respectively.

Proof. By definition of eigenvalues, $\{\lambda_i\}_{i=1}^{d_1+d_2}$ are the roots of characteristic polynomial:

$$p_\epsilon(\lambda) := \text{det} \begin{pmatrix} \lambda I + \epsilon A & \epsilon C \\ -C^T & \lambda I - B \end{pmatrix}$$
We can expand this polynomial as:
\[
p_{\epsilon}(\lambda) = p_0(\lambda) + \sum_{i=1}^{d_1 + d_2} \epsilon^i p_i(\lambda), \quad p_0(\lambda) = \lambda^{d_1} \cdot \det(\lambda I - B).
\]

Here, \( p_i \) are polynomials of order at most \( d_1 + d_2 \). It is clear that the roots of \( p_0 \) are 0 (with multiplicity \( d_1 \)) and \( \{\nu_i\}_{i=1}^{d_2} \). According to Lemma 29, we know the roots of \( p_\epsilon \) satisfy:
\[
|\lambda_i| = o(1) \quad 1 \leq i \leq d_1 \\
|\lambda_{i+d_1} - \nu_i| = o(1), \quad 1 \leq i \leq d_2
\]

Since \( B \) is non-degenerate, we know when \( \epsilon \) is small enough, \( \lambda_1 \ldots \lambda_{d_1} \) are very close to 0 while \( \lambda_{d_1+1} \ldots \lambda_{d_1+d_2} \) have modulus at least \( \Omega(1) \). To provide the sign information of the first \( d_1 \) roots, we proceed to lower order characterization.

On the other hand, reparametrize \( \lambda = \epsilon \theta \), we have:
\[
p_{\epsilon}(\epsilon \theta) = \det \left( \epsilon \theta I + \epsilon A \begin{array}{c} C \\ -C^\top \end{array} \right) = \epsilon^{d_1} \det \left( \theta I + \begin{array}{c} C \\ -C^\top \end{array} \right)
\]

Therefore, we know \( q_{\epsilon}(\theta) := p_{\epsilon}(\epsilon \theta)/\epsilon^{d_1} \) is still a polynomial, and has polynomial expansion:
\[
q_{\epsilon}(\theta) = q_0(\theta) + \sum_{i=1}^{d_2} \epsilon^i q_i(\lambda), \quad q_0(\theta) = \det \left( \theta I + \begin{array}{c} C \\ -C^\top \end{array} \right)
\]

It is also clear polynomial \( q_{\epsilon} \) and \( p_{\epsilon} \) have same roots up to \( \epsilon \) scaling. Furthermore, we have following factorization:
\[
\left( \begin{array}{c} \theta I + A \\ -C^\top \end{array} \right) = \left( \begin{array}{c} \theta I + A - CB^{-1}C^\top \\ 0 \end{array} \right) \left( \begin{array}{c} I \\ -B \\ \left( B^{-1}C^\top \right) \end{array} \right)
\]

Since \( B \) is non-degenerate, we have \( \det(B) \neq 0 \), and
\[
q_0(\theta) = (-1)^{d_2} \det(B) \det(\theta I + A - CB^{-1}C^\top)
\]

\( q_0 \) is \( d_1 \)-order polynomial having roots \( \{\mu_i\}_{i=1}^{d_1} \), which are the eigenvalues of matrices \( A - CB^{-1}C^\top \).

According to Lemma 29, we know \( q_{\epsilon} \) has at least \( d_1 \) roots so that \( |\theta_i + \mu_i| \leq o(1) \). This implies \( d_1 \) roots of \( p_{\epsilon} \) so that:
\[
|\lambda_i + \epsilon \mu_i| = o(\epsilon) \quad 1 \leq i \leq d_1
\]

By Eq.(7), we know \( p_{\epsilon} \) has exactly \( d_1 \) roots which are of \( o(1) \) scaling. This finishes the proof.

**Lemma 29** (Continuity of roots of polynomials [Zedeck 1965]). *Given a polynomial \( p_n(z) := \sum_{k=0}^{n} a_k z^k \), \( a_n \neq 0 \), an integer \( m \geq n \) and a number \( \epsilon > 0 \), there exists a number \( \delta > 0 \) such that whenever the \( m+1 \) complex numbers \( b_k, 0 \leq k \leq m \), satisfy the inequalities:
\[
|b_k - a_k| < \delta \quad 0 \leq k \leq n, \quad \text{and} \quad |b_k| < \delta \quad n+1 \leq k \leq m
\]

then the roots \( \beta_k, 1 \leq k \leq m \) of the polynomial \( q_m(z) := \sum_{k=0}^{m} b_k z^k \) can be labeled in such a way as to satisfy with respect to the zeros \( \alpha_k, 1 \leq k \leq n \) of \( p_n(z) \) the inequalities:
\[
|\beta_k - \alpha_k| < \epsilon \quad 1 \leq k \leq n, \quad \text{and} \quad |\beta_k| > 1/\epsilon \quad n+1 \leq k \leq m
\]
D Proofs for Gradient Descent with Max-oracle

In this section, we present the proof for Theorem 27 presented in Section 4.3.

**Theorem 27.** Suppose $f$ is $\ell$-smooth and $L$-Lipschitz and define $\phi(\cdot) := \max_y f(\cdot, y)$. Then the output $\tilde{x}$ of GD with Max-oracle (Algorithm 2) with step size $\eta = \gamma/\sqrt{T} + 1$ will satisfy

\[
\mathbb{E} \left[ \|\nabla \phi_{1/2\ell}(\tilde{x})\|^2 \right] \leq 2 \cdot \frac{\phi_{1/2\ell}(x_0) - \min_x \phi(x) + \ell L^2 \gamma^2}{\gamma \sqrt{T} + 1} + 4 \epsilon,
\]

where $\phi_{1/2\ell}$ is the Moreau envelope \eqref{eq:moreau} of $\phi$.

**Proof.** The proof of this theorem mostly follows the proof of Theorem 2.1 from Davis and Drusvyatskiy \cite{Davis18}. The only difference is that $y_t$ in Algorithm 2 is only an approximate maximizer and not exact maximizer. However, the proof goes through fairly easily with an additional error term.

We first note an important equation for the gradient of Moreau envelope.

\[
\nabla \phi(\lambda)(x) = \lambda^{-1} \left( x - \arg\min_{\tilde{x}} \left( \phi(\tilde{x}) + \frac{1}{2\lambda} \|x - \tilde{x}\|^2 \right) \right).
\]

We also observe that since $f(\cdot)$ is $\ell$-smooth and $y_t$ is an approximate maximizer for $x_t$, we have that any $x_t$ from Algorithm 2 and $\tilde{x}$ satisfy

\[
\phi(\tilde{x}) \geq f(\tilde{x}, y_t) \geq f(x_t, y_t) + \langle \nabla_x f(x_t, y_t), \tilde{x} - x_t \rangle - \frac{\ell}{2} \|\tilde{x} - x_t\|^2
\]

\[
\geq \phi(x_t) - \epsilon + \langle \nabla_x f(x_t, y_t), \tilde{x} - x_t \rangle - \frac{\ell}{2} \|\tilde{x} - x_t\|^2.
\]

Let $\tilde{x}_t := \arg\min_x \phi(x) + \ell \|x - x_t\|^2$. We have:

\[
\phi_{1/2\ell}(x_{t+1}) \leq \phi(\tilde{x}_t) + \ell \|x_{t+1} - \tilde{x}_t\|^2
\]

\[
\leq \phi(\tilde{x}_t) + \ell \|x_t - \eta \nabla_x f(x_t, y_t) - \tilde{x}_t\|^2
\]

\[
\leq \phi(\tilde{x}_t) + \ell \|x_t - \tilde{x}_t\|^2 + 2 \eta \ell \langle \nabla_x f(x_t, y_t), \tilde{x}_t - x_t \rangle + \eta^2 \ell \|\nabla_x f(x_t, y_t)\|^2
\]

\[
\leq \phi_{1/2\ell}(x_t) + 2 \eta \ell \langle \nabla_x f(x_t, y_t), \tilde{x}_t - x_t \rangle + \eta^2 \ell \|\nabla_x f(x_t, y_t)\|^2
\]

\[
\leq \phi_{1/2\ell}(x_t) + 2 \eta \ell \left( \phi(\tilde{x}_t) - \phi(x_t) + \epsilon + \frac{\ell}{2} \|x_t - \tilde{x}_t\|^2 \right) + \eta^2 \ell L^2,
\]

where the last line follows from \eqref{eq:ineq2}. Taking a telescopic sum over $t$, we obtain

\[
\phi_{1/2\ell}(x_T) \leq \phi_{1/2\ell}(x_0) + 2 \eta \ell \sum_{t=0}^{T} \left( \phi(\tilde{x}_t) - \phi(x_t) + \epsilon + \frac{\ell}{2} \|x_t - \tilde{x}_t\|^2 \right) + \eta^2 \ell L^2 T
\]

Rearranging this, we obtain

\[
\frac{1}{T+1} \sum_{t=0}^{T} \left( \phi(x_t) - \phi(\tilde{x}_t) - \frac{\ell}{2} \|x_t - \tilde{x}_t\|^2 \right) \leq \epsilon + \frac{\phi_{1/2\ell}(x_0) - \min_x \phi(x)}{2 \eta \ell T} + \frac{\eta L^2}{2}.
\]
Since $\phi(x) + \ell \| x - x_t \|^2$ is $\ell$-strongly convex, we have

$$\begin{align*}
\phi(x_t) - \phi(\tilde{x}_t) - \frac{\ell}{2} \| x_t - \tilde{x}_t \|^2 &\geq \phi(x_t) + \ell \| x_t - x_t \|^2 - \phi(\tilde{x}_t) - \ell \| \tilde{x}_t - x_t \|^2 + \frac{\ell}{2} \| x_t - \tilde{x}_t \|^2 \\
&= \left( \phi(x_t) + \ell \| x_t - x_t \|^2 - \min_x \phi(x) + \ell \| x - x_t \|^2 \right) + \frac{\ell}{2} \| x_t - \tilde{x}_t \|^2 \\
&\geq \ell \| x_t - \tilde{x}_t \|^2 = \frac{1}{4\ell} \| \nabla \phi_{1/2\ell}(x_t) \|^2,
\end{align*}$$

where we used (8) in the last step. Plugging this in (10) proves the result. \qed