Leaking information to gain entanglement

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Entanglement lies at the root of quantum theory. It is a remarkable resource that is generally believed to diminish when entangled systems interact with their environment. On the contrary, we find that engaging a system with its environment increases its ability to retain entanglement. The maximum rate of retaining entanglement is given by the quantum channel capacity. We counter-intuitively boost the quantum capacity of a channel by leaking almost all quantum information to the channel’s environment. This boost exploits two-letter level non-additivity in the channel’s coherent information. The resulting non-additivity has a far larger magnitude and a qualitatively wider extent than previously known. Our findings have a surprising implication for quantum key distribution: maximum rates for key distribution can be boosted by allowing leakage of information to the eavesdropping environment.

I. INTRODUCTION

First recognized as a “spooky” aspect of quantum theory [1] and later measured [2–11] via violation of Bell type inequalities [12, 13], entanglement is a distinctly non-classical phenomenon. The phenomenon can give insights across disciplinary boundaries [14–20] and facilitate high precision experiments [21–27]. Entanglement is also a remarkable resource which plays a key role in quantum computing. When used for communication [28, 29], this resource can provide unconditional security [30–32] and reduce classical communication complexity [33–35]. As a result, strong efforts [36–40] are being made for intermediate and large scale [41–44] use of quantum entanglement. However, these efforts are impeded by the susceptibility of entanglement to noise. It is critical to devise protocols and find maximum rates for retaining entanglement in the presence of noise.

Any noise process is described by a completely positive trace preserving map, also known as a quantum channel. The ultimate limit for protecting entanglement from a noisy channel \( B \) is given by the channel’s quantum capacity \( Q(B) \). This fundamental capacity is also the highest rate for any quantum error correction code designed to recover quantum information affected by noise \( B \). Successful quantum error correction does not leak any information to the channel’s environment. Thus a channel’s private capacity \( P \), to send classical information hidden from the channel’s environment, is bounded from below by its quantum capacity. In turn, a channel’s capacity \( \chi \), to send regular classical information, is bounded from below by the channel’s private capacity. Due to their central importance, computing and understanding quantum capacities is a fundamental goal of quantum information theory.

The best way to understand quantum capacities is rooted in Shannon’s [45] original recipe for finding the capacity \( C(N) \) of a noisy classical channel \( N \). The recipe’s key ingredient is \( C(1)(N) \), the maximum mutual information between the channel input variable and its image under the channel. Shannon’s random coding argument shows that \( C(1)(N) \) is an achievable rate, i.e., \( C(1)(N) \leq C(N) \). Next, for \( k \) joint uses (also called parallel uses) of \( N \), \( \lim_{k \to \infty} C(1)(N^k)/k \) is shown to be a matching upper and lower bound on \( C(N) \). Finally, additivity, i.e, for any two channels \( N \) and \( N' \) used together \( C(1)(N' \times N) \) is \( C(1)(N') + C(1)(N) \), reduces these bounds to an elegant single-letter formula \( C(N) = C(1)(N) \). Given that \( C(N) \) is the maximum rate for sending classical data sent across asymptotically many uses of \( N \), this single letter formula is surprising.

Expressions for quantum capacities are found in a similar way. For quantum, private, and classical capacities the respective quantum analogs of the key classical ingredient \( C(1) \), are the coherent \( Q(1) \) [46], private \( P(1) \) [47], and Holevo information \( \chi(1) \) [48, 49]. Like \( C(1) \), each quantity \( Q(1), P(1), \) and \( \chi(1) \), is proved to be an achievable rate for its respective task using random coding arguments. However, unlike \( C(1) \), all of them are non-additive [50–56]. Such non-additivity (see (5) for a canonical definition) is remarkable. It presents an opportunity to find strategies for sending information across quantum channels at rates which surpass naïve random coding rates. Indeed a variety of strategies have been explored: structured quantum codes can sometimes improve upon \( Q(1) \) [57, 58], carefully crafted private codes can surpass \( P(1) \) in certain cases [59], and entangled inputs can in principle improve transmission of classical data beyond \( \chi(1) \) [55].

While non-additivity brings new opportunities, it also creates challenges. For instance the best known expressions for quantum capacities of general channels are multi-letter formulas (for example, see (4)).
Such formulas are hard to evaluate since they may require an intractable maximization of a non-convex function over an infinite number of variables.

This letter’s main contribution is to construct a simple noisy channel whose quantum and private capacity can be computed, but then show how a counter-intuitive scheme to leak almost all quantum information to the channel’s environment boosts the channel’s quantum and private capacities. This unexpected boost comes from leveraging non-additivity that has challenged our understanding of quantum capacities in the past. In the present case, non-additivity is used to show that engaging a system’s environment increases the system’s ability to retain entanglement and send private information.

Seemingly, our scheme doesn’t increase the channel’s coherent information $Q^{(1)}$. But it introduces a gap between channel parameter values where $Q^{(1)}$ is zero and threshold where the channel’s quantum capacity $Q$ can be shown to vanish. Filling such gaps [60–64] is crucial to understanding quantum capacities. In most cases, including the well studied case of a qubit depolarizing channel, such gaps remain unfilled. However, we completely fill ours by using an algebraic argument based on log-singularity [65]. This argument shows that the coherent information over two channel uses, and hence $Q$, remains positive until the threshold where $Q$ can be shown to vanish. In this way, we find two-letter level non-additivity whose extent is qualitatively wider than known before.

Despite its importance, the origin of non-additivity is obscure. A significant finding in this work is that leakage creates non-additivity and increases the channel’s capacity without increasing its output dimension. Since our constructions are simple, novel, and highly effective, we expect them to facilitate a deeper understanding of both non-additivity and quantum channel capacities.

II. QUANTUM CHANNEL CAPACITIES

An isolated quantum system becomes noisy by interacting with its environment. Such an interaction is captured by an isometry

$$J : \mathcal{H}_a \mapsto \mathcal{H}_b \otimes \mathcal{H}_c; \quad J^\dagger J = I_a$$

(1)

taking the system’s Hilbert space $\mathcal{H}_a$ to a subspace of the joint output-environment space $\mathcal{H}_b \otimes \mathcal{H}_c$. The interaction maps any system density operator $\rho_a$ to an output density operator $\rho_{ab} := B(\rho_a) := \text{Tr}_c(J \rho_a J^\dagger)$ and an environment density operator $\rho_e := C(\rho_a) := \text{Tr}_b(J \rho_a J^\dagger)$. These completely positive trace preserving maps, i.e., quantum channels $B : \mathcal{H}_a \mapsto \mathcal{H}_b$ and $C : \mathcal{H}_a \mapsto \mathcal{H}_c$ take the algebra of linear operators on $\mathcal{H}_a$ to the algebra of linear operator on $\mathcal{H}_b$ and $\mathcal{H}_c$ respectively. The quantum channel $C$ to the system’s environment is called the complement of $B$. If the system is originally entangled with some inaccessible reference, this noise $B$ can deplete the entanglement. The ultimate rate at which entanglement can be preserved is given by the channel’s quantum capacity $Q(B)$ (see Sec. 24.1 in [66] for an operational definition).

To compute the quantum capacity, one follows Shannon’s footsteps. First, define the appropriate quantum analog of the classical mutual information: the coherent information or entropy bias of $B$ at $\rho_a$,

$$\Delta(B, \rho_a) = S(\rho_b) - S(\rho_c),$$

(2)

where $S(\rho) := -\text{Tr}(\rho \log \rho)$ is the von-Neumann entropy of $\rho$ (all log functions use base 2). Next, maximize the entropy bias to obtain an achievable rate called the channel coherent information (also called the one-letter coherent information),

$$Q^{(1)}(B) = \max_{\rho_a} \Delta(B, \rho_a).$$

(3)

Finally, the quantum capacity of $B$ is a multi-letter formula [47, 67–69],

$$Q(B) = \lim_{n \to \infty} \frac{1}{n} Q^{(1)}(B^\otimes n),$$

(4)

where $B^\otimes n$ represents $n \in \mathbb{N}$ parallel uses of the channel $B$.

The limit in (4) is intractable to compute due to non-additivity of $Q^{(1)}$: for two channels $B$ and $B'$ used in parallel, the inequality

$$Q^{(1)}(B \otimes B') \geq Q^{(1)}(B) + Q^{(1)}(B'),$$

(5)

can be strict. There are stunning examples of channels which display non-additivity: for any integer $m > 1$ there is a channel $B$ for which $Q^{(1)}(B^\otimes m) = 0$ but $Q^{(1)}(B^\otimes k) > 0$ for some $k > m$ [52]. Such examples show that using (4) to check positivity of $Q$ is hard. A simpler example where checking positivity of $Q$ has been difficult [61, 70] is the qubit depolarizing channel: $D_q(\rho) = (1 - q)\rho + qI/2$, $0 \leq q \leq 4/3$. Here, $Q^{(1)}(D_q) = 0$ for $q \geq q_0 := .2524$ [50], but the quantum capacity $Q(D_q)$ is known to vanish only for $q_1 := 1/3 > q_0$ [71–73]. This creates a gap $q_0 < q < q_1$ where the capacity can be zero or positive. This gap has been partially filled using an “extreme form” of non-additivity: $Q^{(1)}(D_q) = 0$ but $Q^{(1)}(D_q^\otimes n) > 0$, found at $n \geq 3$ for $q_0 \leq q \leq .255 < q_1$ [50, 57, 60].

Perhaps even more appealing than extreme non-additivity are examples of two-letter level non-additivity where $\delta_n(B) := Q^{(1)}(B^\otimes n)/n - Q^{(1)}(B)$ is positive for $n \geq 2$. This type of non-additivity is embedded in high dimensional constructions of unbounded non-additivity [53]. However, simpler qubit constructions found recently [74] have not
only given more insight but also made it possible to test non-additivity experimentally [75]. These simpler constructions can be viewed as generalized erasure channels \( B_q \) [76]. While the non-additivity in such channels is not known to be extreme, its amount \( \delta_2(B_q) \approx O(10^{-3}) \) is comparatively larger than \( \delta_4(D_q) \). In this letter, we construct a new and simple channel which displays two-letter level non-additivity in its extreme form. The non-additivity is an order of magnitude larger and has a qualitatively wider extent that the non-additivity found in qubit depolarizing and generalized erasure channels.

The private capacity \( P(\mathcal{B}) \) is the ultimate limit for sending private information over \( \mathcal{B} \). It also equals the maximum rate for exchanging a secret key by sending quantum states over a channel whose environment is monitored by an eavesdropper (see [47] or Ch. 23 in [66]). Like the quantum capacity, \( P \) is given by a multi-letter formula of the form (4), where \( Q^{(1)} \) is now replaced with the private information,

\[
P^{(1)}(\mathcal{B}) = \max_{\{p(x),\rho_a(x)\}} \Delta(B,\rho_a) - \sum_x p(x)\Delta(B,\rho_a(x)),
\]

where \( \rho_a = \sum_x p(x)\rho_a(x) \). The private information is non-additive in exactly the same way as \( Q^{(1)} \) in eq. (5). From eqns. (3) and (6) we see that \( Q^{(1)}(\mathcal{B}) \leq P^{(1)}(\mathcal{B}) \), thus \( Q(\mathcal{B}) \leq P(\mathcal{B}) \). Both these inequalities can be strict [74,77]. The strictness can be extreme in the sense that a channel with zero quantum capacity can have strictly positive private capacity [78–80].

Equality \( Q(\mathcal{B}) = P(\mathcal{B}) \), is known to hold when \( \mathcal{B} \)'s complement \( \mathcal{C} \) has zero quantum capacity, i.e., \( Q(\mathcal{C}) = 0 \). When \( P(\mathcal{C}) \) is also zero, additivity holds in the sense that [81],

\[
Q^{(1)}(\mathcal{B}) = Q(\mathcal{B}) = P(\mathcal{B}) = P^{(1)}(\mathcal{B}).
\]

This additivity greatly simplifies the discussion of the quantum and private capacities of \( \mathcal{B} \). However, this simplicity comes with the headache of showing that the channel’s complement \( \mathcal{C} \) has no quantum or private capacity. For a general channel this headache has no known cure due to non-additivity. Under special circumstances when \( \mathcal{B} \) is degradable and thus its complement \( \mathcal{C} \) antidegradable [82], i.e., \( D \circ \mathcal{B} = \mathcal{C} \) for some quantum channel \( D \), data processing ensures that the antidegradable channel \( \mathcal{C} \) has \( Q(\mathcal{C}) = P(\mathcal{C}) = 0 \). Thus, degradability implies (7), but in addition it enforces a more general additivity: for any two channels \( \mathcal{B} \) and \( \mathcal{B}' \), either degradable or antidegradable, equality holds in eq. (5). Due to their pleasant properties, (anti)degradable channels have also been widely studied [83–85]. Such studies show that (anti)degradability can be checked using a semi-definite program [86] and used to find computable bounds on quantum capacities [86–90].

For a wide class of channels called \textit{pcubed}, degradability can be checked without solving a semi-definite program [91]. The simplest channel of this type has a qubit input, output, and environment. These simple qubit channels have two real parameters. Their quantum and private capacities are easy to find because these channels are either degradable or antidegradable [92]. This leads us to consider another simple channel \( \mathcal{B}_1 \) whose quantum and private capacities are easy to find because the channel is either degradable or antidegradable. It has a single real parameter, a qubit input and a qutrit output and environment.

The channel \( \mathcal{B}_1 \) is defined using an isometry \( J_1 : \mathcal{H}_{a1} \rightarrow \mathcal{H}_{b1} \otimes \mathcal{H}_{c1}, \)

\[
J_1(0) = \sqrt{\lambda} |00\rangle + \sqrt{1/2 - \lambda} (|11\rangle + |22\rangle),
\]

\[
J_1(1) = |01\rangle,
\]

where the parameter \( 0 \leq \lambda \leq 1 \), \(|i\rangle \) denotes an element of the standard orthonormal basis, and \(|ij\rangle \) denotes \(|i\rangle \otimes |j\rangle \in \mathcal{H}_{a1} \otimes \mathcal{H}_{c1} \). This channel is degradable for \( 0 \leq \lambda \leq \lambda_0 = 1/3 \) and antidegradable for \( \lambda_0 \leq \lambda \leq 1 \) (see App. A1). As a result eq. (7) holds for any \( \lambda \), and simply computing \( Q^{(1)}(\mathcal{B}_1) \) gives the quantum and private capacities of \( \mathcal{B}_1 \). One can always compute \( Q^{(1)}(\mathcal{B}_1) \) via a maximization of the form (3), where the input density operators are diagonal in the standard basis (see App. A2). The resulting maximum (see plot in Fig. 1), shows that \( Q^{(1)}(\mathcal{B}_1) \) decreases monotonically with \( \lambda \),

\[
Q^{(1)}(\mathcal{B}_1) > 0,
\]

for \( 0 \leq \lambda < \lambda_0 \), and zero for \( \lambda_0 \leq \lambda \leq 1 \).

III. NEAR PERFECT LEAKAGE

Transform \( \mathcal{B}_1 \) to a channel \( \mathcal{B} \) defined via an isometry \( J : \mathcal{H}_a \rightarrow \mathcal{H}_b \otimes \mathcal{H}_c, \)

\[
J|0\rangle = J_1|0\rangle, \quad J|1\rangle = J_1|1\rangle, \quad J|i\rangle = |0i\rangle,
\]

where \( \mathcal{H}_a \) and \( \mathcal{H}_c \) are obtained by appending kets \(|i\rangle \), \( 2 \leq i \leq n - 1 \) to \( \mathcal{H}_{a1} \) and \( \mathcal{H}_{c1} \) respectively, and \( \mathcal{H}_b = \mathcal{H}_{b1} \). Unlike \( \mathcal{B}_1 \), the \( \mathcal{B} \) channel defined this way is not degradable for any \( \lambda \), however, \( \mathcal{B} \) is antidegradable i.e.,

\[
Q(\mathcal{B}) = 0,
\]

for \( \lambda_1 := 1/2 \leq \lambda \leq 1 \) (see App. B2).

The transformation (10) adds quantum states to the input of \( \mathcal{B}_1 \) and maps them perfectly to its environment. To analyze how this changes the flow of
information to the environment, consider the channel \( C_1 \) to the environment of \( B_1 \). This channel \( C_1 \) gets transformed to \( C \), the complement of \( B \). Perfect transmission to the environment in (10) is expected to make \( C \) much better than \( C_1 \). This exception is met: unlike \( C_1 \), whose quantum capacity is non-zero only when \( \lambda > \lambda_0 \) (see App. A2), \( C \) has near perfect quantum capacity for all \( \lambda > 0 \) (see App. B1), i.e.,

\[
\log(n-1) \leq Q(C) \leq \log n, \quad (12)
\]

where the upper bound is the maximum quantum capacity achieved by a channel identically mapping \( \mathcal{H}_a \) to \( \mathcal{H}_c \).

IV. FILLING GAPS IN QUANTUM CAPACITY

Transforming \( B_1 \) to \( B \) has introduced a near perfect flow of quantum information to the channel’s environment. Such a modification is not expected to improve the ability of \( B_1 \) to send quantum information. This expectation is not entirely misplaced, indeed we find no (see App. B1) change in the channel’s coherent information, i.e.,

\[
Q^{(1)}(B) = Q^{(1)}(B_1). \quad (13)
\]

Combining the equation above with eqns. (9) and (11) reveals a gap,

\[
G = [\lambda_0, \lambda_1], \quad (14)
\]

of channel parameter values where \( Q^{(1)}(B) = 0 \) but \( Q(B) \) is not known vanish. To fill this gap, one must either show that \( Q(B) = 0 \) or show \( Q(B) > 0 \) for all \( \lambda \in G \). In general, there is no straightforward way to fill such a gap. However, using a log-singularity based argument (see App. B4) we show,

\[
Q^{(1)}(B \otimes^2) > 0, \quad (15)
\]

for all \( \lambda \in G \). The above equation, together with eqns. (9), (11), and (13), implies that \( Q(B) \) is strictly positive for \( 0 \leq \lambda < \lambda_1 \), and exactly zero for \( \lambda_1 \leq \lambda \leq 1 \).

V. NON-ADDITIVITY BOOST IN QUANTUM AND PRIVATE CAPACITIES

The channel \( B \) displays an “extreme form” of non-additivity at the two-letter level, i.e., \( Q^{(1)}(B) = 0 \) but \( Q^{(1)}(B \otimes^2) > 0 \) for \( \lambda_0 \leq \lambda < \lambda_1 \). This non-additivity follows from eqns. (9), (13), and (15). To help understand it, fix \( \lambda = \lambda_0 \), and consider a density operator,

\[
\tau_{aa} = \frac{1}{2} |n_0\rangle\langle n_0| + |n_1\rangle\langle n_1|, \quad (16)
\]

on \( \mathcal{H}_a \otimes \mathcal{H}_a \), where

\[
|n_0\rangle = |10\rangle, \quad |n_1\rangle = \sqrt{p}|11\rangle + \sqrt{1-p}|02\rangle, \quad (17)
\]

and \( 0 < p < 1 \). For now, fix \( p = 1/2 \). The input density operator \( \tau_{aa} \) is mapped to an output state \( \tau_{bb} = B^{\otimes 2}(\tau_{aa}) \) and environment state \( \tau_{cc} = C^{\otimes 2}(\tau_{aa}) \). A simple PPT [93] test shows that \( \tau_{aa} \) and \( \tau_{cc} \) are entangled across \( \mathcal{H}_a \otimes \mathcal{H}_a \) and \( \mathcal{H}_c \otimes \mathcal{H}_c \) respectively, however \( \tau_{bb} \) is separable across \( \mathcal{H}_b \otimes \mathcal{H}_b \) (see App. B6). As a result, one may suspect that using \( \tau_{aa} \) to send quantum information across \( B^{\otimes 2} \) is not a good idea. However this is not the case, the entropy bias \( \Delta(B^{\otimes 2}, \tau_{aa}) \approx .067 \) is positive. Thus noiseless quantum information can be sent at a positive achievable rate by encoding quantum states in the typical subspace of \( \tau_{aa} \). Intuitively, this positive entropy bias arises because \( B^{\otimes 2}(|n_0\rangle\langle n_0|) \) and \( B^{\otimes 2}(|n_1\rangle\langle n_1|) \) are well separated (for instance in \( l_1 \) distance [94] in comparison to \( C^{\otimes 2}(|n_0\rangle\langle n_0|) \) and \( C^{\otimes 2}(|n_1\rangle\langle n_1|) \)). In the present case, this separation is perhaps of greater value than entanglement of \( \tau_{bb} \) and \( \tau_{cc} \).

\[\text{FIG. 1. A plot of } Q^{(1)}(B_1), Q^{(1)}(B), \text{ and } \Delta^*/2 \text{ versus } \lambda. \text{ Here } Q^{(1)}(B_1) = Q^{(1)}(B) \text{ decreases monotonically with } \lambda \text{ and becomes zero at } \lambda_0. \text{ The difference } \delta^* = \Delta^*/2 - Q^{(1)}(B) \text{ is positive for } \lambda_0 < \lambda < \lambda_1 = 1/2. \text{ The inset shows } \Delta^*/2 \text{ on a logarithmic scale for } \lambda \text{ close to } \lambda_1. \]

Non-additivity,

\[
Q^{(1)}(B^{\otimes 2}) > 2Q^{(1)}(B), \quad (18)
\]

also occurs when \( Q^{(1)}(B) > 0 \) (see Fig. 1), it is found by optimizing the entropy bias \( \Delta(B^{\otimes 2}, \Lambda_{aa}) \) over

\[
\Lambda_{aa} = \sum_{i=0}^{2} r_i |n_i\rangle\langle n_i|, \quad (19)
\]

where \( |n_2\rangle = |00\rangle \), and \( r_i \) are real positive numbers that sum to one. The maximum entropy bias \( \Delta^* \), found this way, reveals an amount of non-additivity \( \delta^* := \Delta^*/2 - Q^{(1)}(B) \). This amount is positive for
an interval of values $\lambda_s < \lambda < \lambda_1$, where $\lambda_s \simeq 0.23$ is found numerically. The maximum value of $\delta^*$, $\simeq 4.4 \times 10^{-2}$, occurs at $\lambda = \lambda_0$ (see App. B5 for details).

In the interval of values $\lambda_s < \lambda < \lambda_1$, where $\delta^*$ is positive, the quantum and private capacities of $B$ are strictly larger than the quantum and private capacities of $B_1$. In light of discussions containing eqns. (9), (13), and (18), the amount by which $Q(B) \leq P(B)$ is larger than $Q^{(1)}(B_1) = Q(B_1) = P(B_1)$ is at least $\delta^*$. Thus $\delta^*$ indicates a magnitude of boost in capacities $Q$ and $P$, found by transforming $B_1$ to $B$. Notice the boost occurs over an interval which includes $\lambda_0 \leq \lambda < \lambda_1$, where $Q(B_1) = P(B_1) = 0$. Thus, the boost not only increases the magnitude of the quantum and private capacities of $B_1$, but it also extends the parameter interval over which these capacities are non-zero. This boost comes from transforming $B_1$, whose coherent information is additive, to $B$, whose coherent information is non-additive.

Transformations which take a channel with additive coherent information to a channel with non-additive coherent information have been proposed in the past. One transformation, used to great effect in superactivation [95–99], is to place channels in parallel. Another transformation is to place channels in series [74, 76]. Clarifying when and why such transformations give rise to non-additivity is a significant challenge in quantum information. It is clear that placing channels in series decreases their capacity. Parallel use can non-additively increase capacity, however it also increases the channel’s output dimension. This work provides a key insight that a counter-intuitive scheme to leak information to a channel’s environment is an entirely new way to obtain non-additivity. Unlike previous methods, the non-additivity arising from this scheme increases a channel’s quantum and private capacity without increasing the channel’s output dimension. This increase in a channel’s quantum capacity improves the maximum rate for protecting entanglement from noise created by the channel. Additionally, the increase in private capacity signifies an uptick in the maximum rate for quantum key distribution across the channel. Both increases arise from leaking information to a channel’s environment. Because the environment is generally believed to destroy entanglement and reduce rates for quantum key distribution, these increases are counter-intuitive.

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Appendix A: Properties of the qubit input channel pair

Focus on the isometry \( J_1 : \mathcal{H}_{a1} \rightarrow \mathcal{H}_{b1} \otimes \mathcal{H}_{c1} \) in eq. (8) of the main text. It has a qubit input \( \mathcal{H}_{a1} \) and two qutrit outputs \( \mathcal{H}_{b1} \) and \( \mathcal{H}_{c1} \). This isometry generates a pair of noisy quantum channels

\[
B_1(A) = \text{Tr}_c(J_1 A J_1^\dagger), \quad C_1(A) = \text{Tr}_b(J_1 A J_1^\dagger),
\]

where \( A \) is any linear operator on \( \mathcal{H}_{a1} \). Here \( B_1 : \mathcal{H}_{a1} \rightarrow \mathcal{H}_{b1} \) and \( C_1 : \mathcal{H}_{a1} \rightarrow \mathcal{H}_{c1} \). In what follows, we discuss various properties of these two channels.

1. Degradability and Antideградability

We prove a statement, made below eq. (8) in the main text, that \( B_1 \) is degradable for \( 0 \leq \lambda \leq \lambda_0 := 1/3 \) and antidegradable for \( \lambda_0 \leq \lambda \leq 1 \). To show that \( B_1 \) is degradable, we construct an isometry \( K_1 : \mathcal{H}_{b1} \rightarrow \mathcal{H}_{c1} \otimes \mathcal{H}_{d1} \),

\[
K_1 |0\rangle = |10\rangle, \quad K_1 |1\rangle = \sqrt{\delta} |00\rangle + \sqrt{1 - \delta} |11\rangle, \quad K_1 |2\rangle = |22\rangle,
\]
where $\mathcal{H}_{d1}$ has dimension three and $0 \leq \delta \leq 1$. For $0 \leq \lambda \leq \lambda_0$, set $\delta = 2\lambda/(1 - \lambda)$ in the above equation, then for any operator $O \in \mathcal{H}_{b1}$, the channel

$$D_1(O) = \text{Tr}_{d1}(K_1OK_1^\dagger),$$

(A3)

satisfies

$$D_1 \circ B_1 = C_1,$$

(A4)
i.e., $B_1$ is degradable. To show $B_1$ is antidegradable we use an isometry $L_1 : \mathcal{H}_{c1} \mapsto \mathcal{H}_{b1} \otimes \mathcal{H}_{c1}$,

$$L_1|0\rangle = \sqrt{1 - \eta}|00\rangle + \sqrt{\eta}|11\rangle, \quad L_1|1\rangle = |01\rangle, \quad L_1|2\rangle = |22\rangle,$$

(A5)

where $\mathcal{H}_{c1}$ is three dimensional and $0 \leq \eta \leq 1$. In the parameter interval $\lambda_0 \leq \lambda \leq 1$, we set $\eta = (1 - \lambda)/2\lambda$. In this setting, the isometry $L$ defines (using an equation analogous to (A3)) a channel $E_1 : \mathcal{H}_{c1} \mapsto \mathcal{H}_{b1}$ which satisfies

$$E_1 \circ C_1 = B_1,$$

(A6)
i.e., $B_1$ is antidegradable.

By definition, if $B_1$ is antidegradable then $C_1$ is degradable, and if $B_1$ is degradable then $C_1$ is antidegradable. From the discussion above, it follows that $C_1$ is antidegradable for $0 \leq \lambda \leq \lambda_0$ and degradable for $\lambda_0 \leq \lambda \leq 1$.

2. Channel Coherent Information

A statement, made above eq. (9) in the main text, states that one can always compute $Q^{(1)}(B_1)$ by maximizing the entropy bias $\Delta(B_1, \sigma_{a1})$ over density operators $\sigma_{a1}$ that are diagonal in the standard basis. To prove this statement, consider unitary operators

$$U_{a1} = [0] - [1], \quad U_{b1} = [0] - [1] + [2], \quad \text{and} \quad U_{c1} = [0] - [1] + [2],$$

(A7)
on $\mathcal{H}_{a1}$, $\mathcal{H}_{b1}$, and $\mathcal{H}_{c1}$ respectively, where $|\psi\rangle$ is our notation for $|\psi\rangle\langle\psi|$. The isometry $J_1$ in eq. (8) has a symmetry in the sense that

$$J_1 U_{a1} = (U_{b1} \otimes U_{c1}) J_1.$$

(A8)

In light of the above symmetry, the superoperators of the two channels $B_1$ and $C_1$ defined by this isometry $J_1$ satisfy

$$B_1(U_{a1}AU_{a1}^\dagger) = U_{b1}AU_{b1}^\dagger, \quad C_1(U_{a1}AU_{a1}^\dagger) = U_{c1}AU_{c1}^\dagger,$$

(A9)

where $A$ is any linear operator on $\mathcal{H}_{a1}$. As a consequence of eq. (A9) the entropy bias of $B_1$ at any input density operator $\rho_{a1}$ satisfies

$$\Delta(B_1, \rho_{a1}) = \Delta(B_1, U_{a1}\rho_{a1}U_{a1}^\dagger).$$

(A10)

For $0 \leq \lambda \leq \lambda_0$, the $B_1$ channel is degradable and thus $\Delta(B_1, \rho_{a1})$ is a concave function of $\rho_{a1}$ [100]. Maximizing this concave function over all possible density operators $\rho_{a1}$ gives $Q^{(1)}(B_1)$. Given any $\rho_{a1}$, construct

$$\sigma_{a1} = \frac{1}{2}(\rho_{a1} + U_{a1}\rho_{a1}U_{a1}^\dagger),$$

(A11)

and observe

$$\Delta(B_1, \sigma_{a1}) \geq \frac{1}{2}(\Delta(B_1, \rho_{a1}) + \Delta(B_1, U_{a1}\rho_{a1}U_{a1}^\dagger)) = \Delta(B_1, \rho_{a1}),$$

(A12)

where the inequality above follows from concavity of $\Delta(B_1, \rho_{a1})$, and the equality comes from eq. (A10). From eq. (A11) it follows that $\sigma_{a1}$ is diagonal in the standard basis, i.e.,

$$\sigma_{a1} = \hat{u}|0\rangle + (1 - \hat{u})|1\rangle,$$

(A13)
where \( u \) is some real number between zero and one. In the interval \( 0 \leq \lambda \leq \lambda_0 \), eq. (A12) holds and thus

\[
Q^{(1)}(B_1) = \max_{0 \leq u \leq 1} \Delta(B_1, \sigma_{a_1}).
\]  

(A14)

For the parameter values \( \lambda_0 \leq \lambda \leq 1 \), \( B_1 \) is antidegradable and \( Q^{(1)}(B_1) = 0 \). Since \( \Delta(B_1, \sigma_{a_1}) = 0 \) at \( u = 0 \), the above equation holds for all \( 0 \leq \lambda \leq 1 \). Since \( B_1 \) is either degradable or antidegradable, \( Q^{(1)}(B_1) = Q(B_1) \). Thus, by performing the maximization in the above equation we obtain values of both \( Q^{(1)}(B_1) \) and \( Q(B_1) \), these values are plotted in Fig. 2.

As mentioned at the end of Sec. A1, \( C_1 \) is antidegradable for \( 0 \leq \lambda \leq \lambda_0 \) and degradable for \( \lambda_0 \leq \lambda \leq 1 \). As a result \( Q^{(1)}(C_1) = Q(C_1) \) for all \( 0 \leq \lambda \leq 1 \). To find \( Q^{(1)}(C_1) \) we must maximize the entropy bias \( \Delta(C_1, \rho_{a_1}) \) over all qubit density operators \( \rho_{a_1} \). However, using arguments analogous to those given in the above discussion, we can restrict our maximization to density operators of the form \( \sigma_{a_1} \) in eq. (A13). Results from this maximization, plotted in Fig. 2, show that \( Q^{(1)}(C_1) \) increases monotonically with \( \lambda \), it is zero for \( 0 \leq \lambda \leq \lambda_0 \) and positive for \( \lambda_0 < \lambda \leq 1 \). This final statement supports the claim, made above eq. (12) in the main text, that the quantum capacity of \( C_1 \) is non-zero only when \( \lambda > \lambda_0 \).

**Appendix B: Properties of the extended channel pair**

In eq. (10) of the main text we constructed an isometry \( J : \mathcal{H}_a \mapsto \mathcal{H}_b \otimes \mathcal{H}_c \), where \( \mathcal{H}_a \) and \( \mathcal{H}_c \) have dimension \( n \geq 3 \), and \( \mathcal{H}_b \) has dimension three. This isometry has a single real parameter \( 0 \leq \lambda \leq 1 \), and it defines a pair of quantum channels \( B : \mathcal{H}_a \mapsto \mathcal{H}_b \) and \( C : \mathcal{H}_a \mapsto \mathcal{H}_c \). Several claims from the main text about this channel pair are discussed here.
1. Subchannels

Let $\mathcal{H}_{a2}, \mathcal{H}_{b2},$ and $\mathcal{H}_{c2}$ denote subspaces of $\mathcal{H}_a, \mathcal{H}_b,$ and $\mathcal{H}_c$ respectively, where each subspace is the support of

$$P_{a2} = \sum_{i=1}^{n-1} [i], \quad P_{b2} = [0], \quad \text{and} \quad P_{c2} = \sum_{i=1}^{n-1} [i], \quad (B1)$$

respectively. Clearly, both $\mathcal{H}_{a2}$ and $\mathcal{H}_{c2}$ have dimension $n - 1$ while $\mathcal{H}_{b2}$ is one dimensional. Restricting the input of the isometry $J : \mathcal{H}_a \mapsto \mathcal{H}_b \otimes \mathcal{H}_c$ to the subspace $\mathcal{H}_{a2}$ yields

$$J_2 := JP_{a2}, \quad \text{where} \quad J_2^2J_2 = P_{a2}. \quad (B2)$$

Since $P_{a2}$ is the identity on $\mathcal{H}_{a2}, J_2$ is an isometry with input $\mathcal{H}_{a2}$. In general, $J_2 = JP_{a2}$ maps its input to some subspace of $\mathcal{H}_b \otimes \mathcal{H}_c$. In the present case, this subspace is exactly $\mathcal{H}_{b2} \otimes \mathcal{H}_{c2}$. Thus, we write

$$J_2 : \mathcal{H}_{a2} \mapsto \mathcal{H}_{b2} \otimes \mathcal{H}_{c2}. \quad (B3)$$

The above isometry generates a pair of quantum channels $\mathcal{B}_2 : \mathcal{H}_{a2} \mapsto \mathcal{H}_{b2}$ and $\mathcal{C}_2 : \mathcal{H}_{a2} \mapsto \mathcal{H}_{c2}$. Since $\mathcal{B}_2$ is obtained by restricting the input space of $\mathcal{B}$ to a subspace, one may call $\mathcal{B}_2$ a subchannel of $\mathcal{B}$. Similarly, one may call $\mathcal{C}_2$ a subchannel of $\mathcal{C}$.

We shall be interested in two properties of subchannels. First, the quantum capacity of a channel is bounded from below by the quantum capacity of any of its subchannels. Second, if a channel is degradable then all its subchannels are also degradable. The first property follows from the definition of the quantum capacity in eq. (4) of the main text. The second property follows from the definition of degradability mentioned below eq. (7) of the main text. Next, we employ these properties of subchannels to prove two claims made in the main text.

Our first claim, stated below (10) in the main text, is

$$\log(n - 1) \leq Q(\mathcal{C}), \quad (B4)$$

for all $0 \leq \lambda \leq 1$. The above inequality is obtained by noticing that $Q(\mathcal{C})$ is bounded from below by $Q(\mathcal{C}_2), \quad \text{the quantum capacity of the subchannel } \mathcal{C}_2. \quad \text{Since this subchannel } \mathcal{C}_2 \text{ perfectly maps its } n - 1 \text{ dimensional input } \mathcal{H}_{a2} \text{ to its output } \mathcal{H}_{c2}, \quad Q(\mathcal{C}_2) = \log(n - 1).

The second claim, stated below (13) in the main text, states that $\mathcal{B}$ is not degradable for any value of the parameter $0 \leq \lambda \leq 1$. This absence of degradability of $\mathcal{B}$ follows from an absence of degradability of the subchannel $\mathcal{B}_2$. In fact one can show that $\mathcal{B}_2$ is antidegradable, i.e., $T_2 \circ \mathcal{C}_2 = \mathcal{B}_2$ where $T_2(A) = \text{Tr}(A)|0\rangle$.

2. Antidegradability

In the main text below eq. (13), it was claimed that $\mathcal{B}$ is antidegradable for $1/2 \leq \lambda \leq 1$, i.e.,

$$\mathcal{B} = \mathcal{E} \circ \mathcal{C}, \quad (B5)$$

for some quantum channel $\mathcal{E} : \mathcal{H}_c \mapsto \mathcal{H}_b$. To construct $\mathcal{E}$, consider an isometry $L : \mathcal{H}_c \mapsto \mathcal{H}_b \otimes \mathcal{H}_c$,

$$L|0\rangle = \sqrt{1 - \zeta}|0\rangle + \sqrt{\zeta/2}(|11\rangle + |22\rangle), \quad L|i\rangle = |0i\rangle, \quad (B6)$$

where $1 \leq i \leq n - 1$, $\mathcal{H}_c$ has dimension $n$, and $0 \leq \zeta \leq 1$. Let $\mathcal{E}$ be the channel generated by $L$ with channel input $\mathcal{H}_c$ and output $\mathcal{H}_b$. Then for $1/2 \leq \lambda \leq 1$, $\mathcal{E}$ satisfies eq. (B5) with $\zeta = (1 - \lambda)/\lambda$.

3. Channel coherent information

In this sub-section we provide details supporting eq. (13) in the main text:

$$Q^{(1)}(\mathcal{B}) = Q^{(1)}(\mathcal{B}_1). \quad (B7)$$

The channel coherent information $Q^{(1)}(\mathcal{B})$ is found by maximizing the entropy bias $\Delta(\mathcal{B}, \rho)$ over all density operators $\rho$ on $\mathcal{H}_a$. The $\mathcal{H}_a$ space has a special $n - 1$ dimensional subspace $\mathcal{H}_{a2}$. All states in this subspace
$H_a$ and $H_c$ are mapped perfectly to the channel environment $H_c$. Thus, a density operator $\rho$ with support in $H_a$ cannot help maximize $\Delta(B, \rho)$. However, if the support of $\rho$ does not intersect with $H_a$, then $\rho = |0\rangle$ and $\Delta(B, \rho) = 0$. A large, non-negative $\Delta(B, \rho)$ may be obtained only when the support of $\rho$ is two dimensional, and this two dimensional support has a one dimensional intersection with $H_a$. Such a two dimensional support must contain $|0\rangle$ and some $|\psi\rangle$ in $H_a$. To maximize $\Delta(B, \rho)$, choose $|\psi\rangle$ such that $B(|0\rangle)$ and $B(|\psi\rangle)$ have small overlap $\text{Tr}(B(|0\rangle)B(|\psi\rangle))$, while $C(|0\rangle)$ and $C(|\psi\rangle)$ have large overlap $\text{Tr}(C(|0\rangle)C(|\psi\rangle))$. This can be arranged by letting $|\psi\rangle$ be any linear combination of $|0\rangle$ and $|1\rangle$. All such linear combinations are treated identically by $J$, thus a choice $|\psi\rangle = |1\rangle$ can be made without loss of generality. Thus $\Delta(B, \rho)$ is maximum when $\rho$ has support $S = H_a$. Input $\rho$ with support $H_a$ gives a channel output $B(\rho)$ which is identical to $B_1(\rho)$, thus the maximum entropy bias $\Delta(B, \rho)$ and $\Delta(B_1, \rho)$ are equal. As a result, both channels $B$ and $B_1$ have the same $Q^{(1)}$.

Numerics confirm the above heuristic. In these numerics $\Delta(B, \rho)$ is maximized by setting $\rho = AA^\dagger/\text{Tr}(AA^\dagger)$ where $A$ is an upper triangular matrix. The matrix has $n$ real parameters on the diagonal and $n^2 - n$ real parameters representing the real and imaginary parts of the off diagonal elements. To optimize the parameters, we use standard numerical techniques (SciPy) [101].

4. Positivity of the two-letter coherent information

We briefly discuss $\epsilon \log$-singularity methods from [65]. These have been used to prove eq. (15) of the main text. Let $\rho(\epsilon)$ be a density operator which depends on a real parameter $\epsilon$ and $S(\epsilon) := -\text{Tr}(\rho(\epsilon) \log \rho(\epsilon))$ denote its von-Neumann entropy. If one or several eigenvalues of $\rho(\epsilon)$ increase linearly from zero to leading order in $\epsilon$, then $S(\epsilon)$ is said to have an $\epsilon \log$ singularity. As a result of this singularity, $S(\epsilon) \simeq x |\epsilon| \log |\epsilon|$ for small $\epsilon$, where $x > 0$ is called the rate of the $\epsilon \log$-singularity.

A channel $B$ with complement $C$ maps an input density operator $\rho_a(\epsilon)$ to an output $\rho_b(\epsilon) := B(\rho_a(\epsilon))$ and complementary output $\rho_c(\epsilon) := C(\rho_a(\epsilon))$. Denote the entropy bias of $B$ at $\rho_a(\epsilon)$ by

$$\Delta(\epsilon) := S_b(\epsilon) - S_c(\epsilon),$$

where $S_b(\epsilon) := S(\rho_b(\epsilon))$ and $S_c(\epsilon) := S(\rho_c(\epsilon))$. If an $\epsilon \log$ singularity is present in either one or both $S_b(\epsilon)$ and $S_c(\epsilon)$, then the singularity with larger rate is said to be stronger.

A $\log$-singularity based argument to show $Q^{(1)}(B) > 0$ proceeds as follows: choose $\rho(\epsilon)$ such that at $\epsilon = 0$, $\Delta(0) = 0$ and $S_b(\epsilon)$ has a stronger $\epsilon \log$ singularity than $S_c(\epsilon)$. Then for small $\epsilon$, $\Delta(\epsilon) \simeq |O(\epsilon \log \epsilon)| > 0$. Since $\Delta(\epsilon) \leq Q^{(1)}(B)$, we have a positive $Q^{(1)}(B)$.

In the main text below eq. (15), we state that $Q^{(1)}(B \otimes B) > 0$ for all values $\lambda_0 \leq \lambda < \lambda_1$, where $\lambda_0 = 1/3$ is the threshold at and above which $Q^{(1)}(B)$ is zero, and $\lambda_1 = 1/2$ is the threshold at and above which $B$ is antidegradable and thus $Q(B) = 0$. To prove $Q^{(1)}(B \otimes B) > 0$ we use a $\log$-singularity based argument which utilizes a density operator,

$$\rho_{aa}(\epsilon) := \epsilon|n_0\rangle + (1 - \epsilon)|n_1\rangle,$$

where $0 \leq \epsilon \leq 1$, $|n_0\rangle$ is a fixed state, and $|n_1\rangle$ depends on a parameter $0 < p < 1$ (see eq. (17) in the main text for definitions). Concisely denote the entropy of $\rho_{bb}(\epsilon)$ and $\rho_{cc}(\epsilon)$ by $S_{bb}(\epsilon)$ and $S_{cc}(\epsilon)$ respectively. Let $\Delta(\epsilon)$ denote and the entropy bias of $B \otimes B$ at $\rho_{aa}(\epsilon)$. We are interested in the parameter region $\lambda_0 \leq \lambda < \lambda_1$ where $Q^{(1)}(B) = \Delta(0) = 0$. In this parameter region, $S_{bb}(\epsilon)$ and $S_{cc}(\epsilon)$ have $\epsilon \log$-singularities with rates

$$x_{bb} = 1 - \lambda \quad \text{and} \quad x_{cc} = \lambda \left(1 + \frac{p(1 - \lambda)}{2p(1 - \lambda)} - 1\right),$$

respectively. The singularity in $S_{bb}(\epsilon)$ is stronger, i.e., $x_{bb} > x_{cc}$ for all $\lambda_0 \leq \lambda < \lambda_1$ when $p$ is strictly less than

$$p_{\text{max}} := \frac{2(1 - 2\lambda)}{(1 - \lambda)(2 - 3\lambda)},$$

where $p_{\text{max}} > 0$ for any $\lambda_0 \leq \lambda < \lambda_1$. This stronger singularity implies that $\Delta(\epsilon) > 0$ and thus $0 < \Delta(\epsilon) \leq Q^{(1)}(B \otimes B)$ for $\lambda_0 \leq \lambda < \lambda_1$. 


Two letter level non-additivity, mentioned in eq. (18) of the main text, is obtained by maximizing the entropy bias \( \Delta(B^{\otimes 2}, \Lambda_{aa}) \) over

\[
\Lambda_{aa} = r_0|n_0\rangle\langle n_0| + r_1|n_1\rangle\langle n_1| + r_2|n_2\rangle\langle n_2|,
\]

(B12)
defined using eqns. (17) and (19) of the main text. This maximization has three positive parameters, \( r_0, r_1, \) and \( r_2, \) which some to one, and a fourth parameter \( 0 \leq p \leq 1 \) coming from \( |n_1\rangle = \sqrt{p}|11\rangle + \sqrt{1-p}|02\rangle. \) The maximum entropy bias \( \Delta(B^{\otimes 2}, \Lambda_{aa}) \), over all four parameters gives \( \Delta^* \). Using \( \Delta^* \) and pre-computed values of \( Q^{(1)}(B) \) (see comments in App. B3) we obtain an amount of non-additivity \( \delta^* = \Delta^*/2 - Q^{(1)}(B). \)

The four parameter maximization to obtain \( \Delta^* \) (and hence \( \delta^* \)) is first performed using standard maximization techniques (SciPy). The maximum obtained this way shows (see Fig. 3) that as \( \lambda \) in increased from zero, \( \delta^* \) remains zero until \( \lambda \) reaches \( \lambda_* \approx 0.23 \), at which point \( \delta^* \) becomes positive. The value of \( \delta^* \) continues to increase and reaches a maximum at \( \lambda_0 = 1/3 \), where \( \delta^* \) has a sharp peak. The peak is genuine and reflects the fact that \( Q^{(1)}(B) \) goes to zero at \( \lambda_0 \) with finite slope while \( \Delta^*/2 \) is positive and strictly decreasing. As \( \lambda \) is increased beyond \( \lambda_0 \), the value of \( r_2 \) which maximizes \( \Delta(B^{\otimes 2}, \Lambda_{aa}) \) becomes zero. For these larger \( \lambda \) values, one essentially maximizes \( \Delta(B^{\otimes 2}, \Lambda_{aa}) \) over two parameters \( 0 < \epsilon < 1 \) and \( 0 < p < 1 \), where \( r_0 = \epsilon, r_1 = 1 - \epsilon, \) and \( r_2 \) is set to zero. This two parameter maximization shows that \( \delta^* \) decreases as \( \lambda \) is increased from \( \lambda_0 \). As \( \lambda \) approaches \( \lambda_1 = 1/2 \), i.e., for small positive \( \delta \lambda := \lambda_1 - \lambda \), we find that the maximum value of \( \Delta(B^{\otimes 2}, \Lambda_{aa}) \) occurs at \( \epsilon \) tending to zero. In this small \( \delta \lambda \) limit, standard numerical maximization becomes unstable because one is trying to maximize a function of the form,

\[
f(p, \epsilon) = -\alpha \ln \epsilon + \beta \epsilon,
\]

(B13)
where \( \ln \) is the natural logarithm, \( \alpha = (x_{bb} - x_{cc})/\ln 2 \) is the difference of \( \epsilon \log \) singularity rates (B10) divided by \( \ln 2 \), and \( \beta \) depends on \( p \) and \( \lambda \). If \( \alpha \) is positive, then \( f \) is positive for small enough \( \epsilon \). From comments below eq. (B11), it follows that as long as \( 0 < p < p_{\text{max}} \), \( \alpha \) is positive for any small positive \( \delta \lambda \). One may express \( \alpha \) and \( \beta \) as follows

\[
\alpha = \alpha_0 \delta \lambda + \alpha_1 (\delta \lambda)^2 + \ldots,
\beta = \beta_0 + \beta_1 (\ln \delta \lambda) + \beta_2 (\delta \lambda \ln \delta \lambda) + \beta_3 (\delta \lambda) + \ldots,
\]

(B14)
where \( \alpha_0 = 2(1 - r)/\ln 2, \alpha_1 = 8r \alpha_0, \beta_0 = 1 + \ln r/(4 \ln 2), \beta_1 = 1/(4 \ln 2), \beta_2 = (2r + 1/2)/\ln 2, \)

\[
\beta_3 = \frac{1}{\ln 2} \left[ -3/2 - 9r \ln 2 + 8r \ln(16r) + \ln(256r)/2 \right],
\]

(B15)
and $r := p/p_{\text{max}}$. For small $\delta \lambda$, we use the above expressions in (B13) and maximize $f(p, \epsilon)$ to obtain $\Delta^*$. For this maximization it is convenient to replace $p$ with $r$ where $r$ lies between zero and one. The replacement gives a function $g(r, \epsilon)$ which equals $f(rp_{\text{max}}, \epsilon)$. For small $\delta \lambda$, the values of $g$ are also small, hence one maximizes $\ln g$. To obtain this maximum, we set to zero the derivative of $g(r, \epsilon)$ with respect to $\epsilon$. This occurs at

$$\epsilon = \epsilon^* := \exp[-(1 - \beta/\alpha)].$$

Next, we numerically maximize

$$\ln g(\epsilon^*, r) = \ln(\alpha) - (1 - \beta/\alpha),$$

over $0 < r < 1$ to obtain $\ln \Delta^*$. The values of $\ln \Delta^*$ obtained this way are in good agreement with standard numerical calculations for moderate values of $\delta \lambda$. In the inset of Fig. 1 in the main text, we plot $\ln \Delta^*$ for small $\delta \lambda$. Since $Q^{(1)}(B) = 0$ for small $\delta \lambda$, $\ln \Delta^*$ readily gives

$$\ln \delta^* = \ln[\Delta^*/2 - Q^{(1)}(B_1)] = \ln[\Delta^*/2].$$

In the inset of Fig. 3 we plot $\ln \delta^*$ for small $\delta \lambda$.

6. Entanglement in input and outputs

A bipartite density operator $\rho_{aa}$ on $\mathcal{H}_a \otimes \mathcal{H}_a$ is separable if it can be written as

$$\rho_{aa} = \sum_i p_i |\alpha_i\rangle \otimes |\alpha'_i\rangle,$$

where $p_i$ are positive numbers that sum to one, $|\alpha_i\rangle$ and $|\alpha'_i\rangle$ are pure states in the first and second $\mathcal{H}_a$ space respectively. The density operator $\rho_{aa}$ is entangled across $\mathcal{H}_a \otimes \mathcal{H}_a$ if it is not separable. The PPT condition [93] for entanglement states that $\rho_{aa}$ is entangled if its partial transpose with respect to the second $\mathcal{H}_a$ space gives an operator with negative eigenvalues.

Under (17) in the main text, we mentioned that $\tau_{aa}$ and $\tau_{cc}$ are entangled while $\tau_{bb}$ is not. The partial transpose of $\tau_{aa}$ with respect to the second $\mathcal{H}_a$ space gives an operator with one negative eigenvalue $-1/4$, and the partial transpose of $\tau_{cc}$ with respect to the second $\mathcal{H}_c$ space gives an operator with one negative eigenvalue $(3 - \sqrt{21})/24$. Thus the PPT test shows that both $\tau_{aa}$ and $\tau_{cc}$ are entangled. It is easy to check that $\tau_{bb}$ is diagonal in the standard basis $|i\rangle \otimes |j\rangle$, thus it is separable.