A symmetrization result for a class of anisotropic elliptic problems

A. Alberico∗– G. di Blasio†– F. Feo‡

Abstract

We prove estimates for weak solutions to a class of Dirichlet problems associated to anisotropic elliptic equations with a zero order term.

1 Introduction

We consider the class of Dirichlet problems for anisotropic elliptic equations, whose prototype has the form

\[
\begin{aligned}
\left\{ \begin{array}{ll}
- \sum_{i=1}^{N} \left( |u_{x_{i}}|^{p_{i} \cdot 2} u_{x_{i}} \right)_{x_{i}} + b(u) = f(x) & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega,
\end{array} \right.
\end{aligned}
\]  

where \( \Omega \) is a bounded open subset of \( \mathbb{R}^{N} \) with Lipschitz continuous boundary, \( N \geq 2, p_{i} \geq 1 \) for \( i = 1, \ldots, N \) such that their harmonic mean \( \overline{p} \) is greater than 1, the subscript \( x_{i} \) denotes partial derivative with respect to \( x_{i} \), \( b \) is a continuous, non-decreasing function such that \( b(0) = 0 \) and \( f \) is a nonnegative function with a suitable summability.

The anisotropy of problem (1.1) depends on differential operator whose growth with respect to the partial derivatives of \( u \) is governed by different powers. In the last years anisotropic problems have been extensively studied by many authors (see e.g. [AdBF2, AdBF3, ACh, BMS, DFG, DF, FGK, FGL, FS, G, Mar]).

The growing interest has led to an extensive investigation also for problems governed by fully anisotropic growth conditions (see e.g. [AC, A, AdBF1, CI, C3]) and problems related to different type of anisotropy (see e.g. [AFTL, BFK, DdB, DG]).

Our goal is to obtain an estimate of concentration of a weak solution to problem (1.1) via symmetrization methods. The use of the standard isoperimetric inequality in the study of isotropic elliptic Dirichlet problems was introduced in [Maz1, Maz2] and independently in [Ta1, Ta2]. Variants and extensions from these papers have been developed in a rich literature. We refer to Vazquez [V2] and Trombetti [T] for a quite comprehensive bibliography on this and related topics.

∗Istituto per le Applicazioni del Calcolo “M. Picone” (I.A.C.), Sez. Napoli, Consiglio Nazionale delle Ricerche (C.N.R.), Via P. Castellino 111, 80131 Napoli, Italy. E-mail:a.alberico@na.iac.cnr.it

†Dipartimento di Matematica e Fisica, Università degli Studi della Campania “Luigi Vanvitelli”, Via Vivaldi, 43 - 81100 Caserta, Italy. E-mail: giuseppina.diblasio@unicampania.it

‡Dipartimento di Ingegneria, Università degli Studi di Napoli “Parthenope”, Centro Direzionale Isola C4 80143 Napoli, Italy. E-mail: filomena.feo@uniparthenope.it

0 Mathematics Subject Classifications: 35B45, 35J25, 35J60

Key words: Anisotropic symmetrization rearrangements, Anisotropic Dirichlet problems, A priori estimate
It is well known that when isotropic elliptic Dirichlet problems with a zero order term are considered, the situation is quite different if we assume or not a sign condition (see, e.g., [D1, D2, Mad, V1, V2]). In the anisotropic setting there are two different cases as well. Indeed, when $b(u)u \geq 0$, it is showed (see, e.g., [C3]) that the symmetric rearrangement of a solution $u$ to anisotropic problem (1.1) is pointwise dominated by the radial solution to an isotropic problem, defined in a ball, with a radially symmetric decreasing data and with no zero order term. Otherwise, with no sign condition on $b(u)u$, we prove an integral comparison result between a solution $u$ to anisotropic problem (1.1) and the radial solution to a suitable isotropic problem defined in a ball, with a radially symmetric decreasing data again but, this time, which preserves a zero order term.

Just to give an idea of our results, let us consider problem (1.1) when the domain $\Omega$ is $B_R(0)$, the ball centered at the origin and with radius $R > 0$. We take into account two smooth strictly increasing functions $b$ and $\tilde{b}$ having the same domain such that $b(0) = \tilde{b}(0) = 0$, and two positive decreasing radial symmetric functions $f$ and $\tilde{f}$ defined in $B_R(0)$. Denote by $b^{-1}$ and $\tilde{b}^{-1}$ the inverse function of $b$ and $\tilde{b}$, respectively. Suppose that

$$((\tilde{b})^{-1})'(s) \leq (b^{-1})'(s) \quad \text{for every } s \in \mathbb{R}$$

and that the datum $f$ is less concentrated than the datum $\tilde{f}$, i.e.

$$\int_{B_r(0)} f(x) \, dx \leq \int_{B_r(0)} \tilde{f}(x) \, dx \quad \text{for every } 0 \leq r \leq R.$$

Then, we are going to prove that

$$\int_{B_r(0)} b(u^*(x)) \, dx \leq \int_{B_r(0)} \tilde{b}(\tilde{u}(x)) \, dx \quad \text{for every } 0 \leq r \leq R,$$

where $u^*$ is the symmetric decreasing rearrangement of the solution $u$ to problem (1.1) and $\tilde{u}$ is the solution to the following problem

$$\begin{cases}
- \div (|\nabla \tilde{u}|^{\overline{p}-2} \nabla \tilde{u}) + \tilde{b}(\tilde{u}) = \tilde{f}(x) & \text{in } B_R(0) \\
\tilde{u} = 0 & \text{on } \partial B_R(0).
\end{cases}$$

The paper is organized as follows. In Section 2 we recall some backgrounds on the anisotropic spaces and on the properties of symmetrization. In Section 3 we state our main results, proved in Section 4.

## 2 Preliminaries

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, and let $1 \leq p_1, \ldots, p_N < \infty$ be $N$ real numbers. The anisotropic Sobolev space (see e.g. [D1])

$$W^{1,\overline{p}}(\Omega) = \{ u \in W^{1,1}(\Omega) : u_{x_i} \in L^{p_i}(\Omega), i = 1, \ldots, N \}$$

is a Banach space with respect to the norm

$$(2.1) \quad \| u \|_{W^{1,\overline{p}}(\Omega)} = \sum_{i=1}^N \| u_{x_i} \|_{L^{p_i}(\Omega)}.$$

The space $W^{1,\overline{p}}_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1) and we will denote by $\left(W^{1,\overline{p}}_0(\Omega)\right)'$ its dual.
A precise statement of our results requires the use of classical notions of rearrangement and of suitable symmetrization of a Young function, introduced by Klimov in [K]. Let \( u \) be a measurable function (continued by 0 outside its domain) fulfilling
\[
|\{ x \in \mathbb{R}^N : |u(x)| > t \}| < +\infty \quad \text{for every} \ t > 0.
\]
The symmetric decreasing rearrangement of \( u \) is the function \( u^\star : \mathbb{R}^N \to [0, +\infty[ \) satisfying\[
\{ x \in \mathbb{R}^N : u^\star(x) > t \} = \{ x \in \mathbb{R}^N : |u(x)| > t \}^\star \quad \text{for} \ t > 0.
\]
The decreasing rearrangement \( u^\ast \) of \( u \) is defined by
\[
u^\ast(s) = \sup \{ t > 0 : \mu_u(t) > s \} \quad \text{for} \ s \geq 0,
\]
where
\[
\mu_u(t) = |\{ x \in \Omega : |u(x)| > t \}| \quad \text{for} \ t \geq 0
\]
denotes the distribution function of \( u \).
Moreover,
\[
u^\star(x) = u^\ast(\omega_N |x|^N) \quad \text{for a.e.} \ x \in \mathbb{R}^N.
\]
Similarly, we define the symmetric increasing rearrangement \( u^\bullet \) on replacing “>” by “<” in the definitions of the sets in (2.2) and (2.3). We refer to [BS] for details on these topics.

In this paper we will consider an \( N \)-dimensional Young function \( \Phi : \mathbb{R}^n \to \mathbb{R} \) (namely, an even convex function such that \( \Phi(0) = 0 \) and \( \lim_{|\xi| \to +\infty} \Phi(\xi) = +\infty \)) of the following type:
\[
\Phi(\xi) = \sum_{i=1}^{N} \alpha_i |\xi_i|^{p_i} \quad \text{for} \ \xi \in \mathbb{R}^N \quad \text{with} \ \alpha_i > 0 \quad \text{for} \ i = 1, \ldots, N.
\]
We denote by \( \Phi^\bullet : \mathbb{R} \to [0, +\infty[ \) the symmetrization of \( \Phi \) introduced in [K]. It is the one-dimensional Young function fulfilling
\[
\Phi^\bullet(|\xi|) = \Phi^\bullet^\bullet(\xi) \quad \text{for} \ \xi \in \mathbb{R}^N,
\]
where \( \Phi^\bullet \) is the Young conjugate function of \( \Phi \) given by
\[
\Phi^\bullet(\xi') = \sup \{ \xi : \xi' - \Phi(\xi) : \xi \in \mathbb{R}^N \} \quad \text{for} \ \xi' \in \mathbb{R}^N.
\]
So \( \Phi^\bullet \) is the composition of Young conjugation, symmetric increasing rearrangement and Young conjugate again.

We denote by \( \overline{p} \) the harmonic average of the exponents \( p_i \), i.e.
\[
\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}.
\]
The harmonic average \( \overline{p} \) plays a basic role in discussing anisotropic equations of the form (1.1). Let us assume that \( \overline{p} > 1 \) and set
\[
\Lambda = \frac{2\overline{p}(\overline{p} - 1)\overline{p}^{-1}}{\overline{p}^2} \left[ \prod_{i=1}^{N} \frac{1}{\omega_N \Gamma(1 + 1/p_i') \Gamma(1 + 1/p_i)} \left( \frac{N}{\alpha_i} \right)^{\frac{1}{p_i'}} \right]^{\frac{\overline{p}}{N}} \left( \prod_{i=1}^{N} \frac{\alpha_i^{1/p_i}}{\alpha_i} \right)^{\frac{\overline{p}}{N}}.
\]
with $\omega_N$ the measure of the $N-$dimensional unit ball, $\Gamma$ the Gamma function and $p_i' = \frac{p_i}{p_i-1}$, the Hölder conjugate of $p_i$ with the usual conventions if $p_i = 1$. We are now in position to evaluate $\Phi_\bullet(|\xi|)$. Easy calculations show (see e.g. [C3]) that

$$\Phi_\bullet(|\xi|) = \Lambda |\xi|^{\overline{p}}.$$  

In the anisotropic setting, we stress that $\overline{p}$ plays a role also in a Polya-Szegö principle which reads as follows (see [C3]). Let $u$ be a weakly differentiable function in $\mathbb{R}^N$ satisfying (2.2) and such that

$$\sum_{i=1}^{N} \alpha_i \int_{\mathbb{R}^N} |u_{x_i}|^{p_i} \, dx < +\infty.$$  

Then $u^\star$ is weakly differentiable in $\mathbb{R}^N$ and

$$\Lambda \int_{\mathbb{R}^N} |\nabla u^\star|^{\overline{p}} \, dx \leq \sum_{i=1}^{N} \alpha_i \int_{\mathbb{R}^N} |u_{x_i}|^{p_i} \, dx.$$  

### 3 Main results

In the present section, we focus our attention on the following class of anisotropic elliptic problems

$$\begin{cases} -\text{div}(a(x,u,\nabla u)) + g(x,u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$ with Lipschitz continuous boundary, $N \geq 2$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi, \xi' \in \mathbb{R}^N$  

(A1) $a(x,s,\xi) \cdot \xi \geq \sum_{i=1}^{N} \alpha_i |\xi_i|^{p_i}$ with $\alpha_i > 0$,  

(A2) $|a_j(x,s,\xi)| \leq \beta \left[ |s|^{\overline{p}_j} + |\xi_j|^{p_j-1} \right]$ with $\beta > 0$ \quad $\forall j = 1, \ldots, N$,  

(A3) $(a(x,s,\xi) - a(x,s,\xi')) \cdot (\xi - \xi') > 0$ for $\xi \neq \xi'$,  

where $1 \leq p_1, \ldots, p_N < \infty$ are real numbers and $\overline{p} > 1$. Moreover, we assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable, continuous and non-decreasing function in $s$ for fixed $x$, and bounded in $x$ uniformly for bounded $u$ such that

(A4) $g(x,s) s \geq b(s) s$ for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$, where $b$ is a continuous and strictly increasing function such that $b(0) = 0$.  

Finally, we assume that

(A5) $f : \Omega \to \mathbb{R}$ is a nonnegative function such that $f \in \left( W_0^{1,\overline{p}}(\Omega) \right)'$.  


In order to give a precise statement of our results, we need to precise what means to be less diffusive. Let \( b_1, b_2 \) be two continuous strictly increasing functions. We say that \( b_1 \) is \textit{weaker} than \( b_2 \) and we write

\[
(3.2) \quad b_1 \prec b_2,
\]

if they have the same domains and there exists a contraction \( \rho : \mathbb{R} \rightarrow \mathbb{R} \) such that \( b_1 = \rho \circ b_2 \).

We are interested in proving an integral estimate of a weak solution \( u \in W^{1,p}_0(\Omega) \) to problem (3.1) in terms of the weak solution \( w \in W^{1,p}_0(\Omega^\star) \) to the following problem

\[
(3.3) \quad \begin{cases}
- \operatorname{div}(\Lambda|\nabla w|^{p-2}\nabla w) + b(w) = f(x) & \text{in } \Omega^\star \\
w = 0 & \text{on } \partial\Omega^\star,
\end{cases}
\]

where \( \Omega^\star \) is the ball centered at the origin and having the same measure as \( \Omega \),

(A6) \( \tilde{b} \) is a continuous and strictly increasing function such that \( \tilde{b}(0) = 0 \),

(A7) \( (\tilde{b})^{-1} \prec b^{-1} \),

(A8) \( \tilde{f} : \Omega^\star \rightarrow \mathbb{R} \) is a nonnegative radially symmetric function and decreasing along the radii such that \( \tilde{f} \in (W^{1,p}_0(\Omega^\star))^\prime \).

We stress that, by standard arguments and thanks to the results contained in [BB] (see also [BCE] for the anisotropic setting), there exists a unique weak solution \( w \in W^{1,p}_0(\Omega^\star) \) to (3.3) such that

(i) \( \tilde{b}(w) \in L^1(\Omega^\star) \)

(ii) \( \tilde{b}(w) \in L^1(\Omega^\star) \)

(iii) \( \Lambda \int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla \phi \, dx + \int_\Omega \tilde{b}(w) \phi \, dx = \langle \tilde{f}, \phi \rangle_{(W^{1,p}_0(\Omega^\star))^\prime} \)

for every \( \phi \in W^{1,p}_0(\Omega^\star) \cap L^\infty(\Omega^\star) \) and \( \varphi = w \).

**Theorem 3.1** Assume that (A1)–(A8) hold. Let \( u \) be a weak solution to the problem (3.1) and \( w \) the weak solution to the problem (3.3). Then,

\[
(3.4) \quad \|(B - \tilde{B})_+\|_{L^\infty(0,|\Omega|)} \leq \|(F - \tilde{F})_+\|_{L^\infty(0,|\Omega|)},
\]

where

\[
(3.5) \quad B(s) = \int_0^s b(u^\star(t)) \, dt \quad \tilde{B}(s) = \int_0^s \tilde{b}(w^\star(t)) \, dt
\]

\[
(3.6) \quad F(s) = \int_0^s f^\star(t) \, dt \quad \tilde{F}(s) = \int_0^s \tilde{f}^\star(t) \, dt
\]

for \( s \in (0,|\Omega|] \).

If we assume that the datum of problem (3.1) dominates the datum of problem (3.3), then the following comparison result between concentrations holds as an easy consequence of Theorem 3.1.

---

1 By contraction we mean \( |\rho(a) - \rho(b)| \leq |a - b| \) for \( a, b \in \mathbb{R} \).
**Corollary 3.2** Under the same assumption of Theorem 3.1, if we suppose that

\[ F(s) \leq \tilde{F}(s) \quad \text{for any } s \in [0, |\Omega|], \]

then

\[ B(s) \leq \tilde{B}(s) \quad \text{for any } s \in [0, |\Omega|]. \]

In particular, we have

\[ \int_{\Omega} \Psi(b(u(x))) \, dx \leq \int_{\Omega} \Psi(\tilde{b}(w(x))) \, dx \]

for all convex and non-decreasing function \( \Psi : \mathbb{R} \to \mathbb{R} \).

An immediate consequence of Corollary 3.2 are norm estimate s of \( b(u) \) in terms of norm of \( \tilde{b}(w) \). An example of applications of (3.9) is the following one:

\[ \| b(u) \|_{L^p(\Omega)} \leq \| \tilde{b}(w) \|_{L^p(\Omega^*)} \quad \text{for } 1 \leq p \leq \infty. \]

We emphasize that in the spirit of [V2], Theorem 3.1 and Corollary 3.2 still hold if we do not require the strictly monotony of \( b \) and \( \tilde{b} \), but assume that \( b \) and \( \tilde{b} \) are non-decreasing functions or, more generally, maximal monotone graphs in \( \mathbb{R}^2 \) such that \( b(0) \ni 0 \) and \( \tilde{b}(0) \ni 0 \). Indeed, a maximal monotone graph is a natural generalization of the concept of monotone non-decreasing real function; moreover, the inverse of a maximal monotone graph is again a maximal monotone graph (see [V2] for more details).

### 4 Proof of Theorem 3.1

Let us consider the functions \( u_{\kappa,t} : \Omega \to \mathbb{R} \) defined by

\[
  u_{\kappa,t}(x) = \begin{cases} 
    0 & \text{if } |u(x)| \leq t, \\
    (|u(x)| - t) \text{sign}(u(x)) & \text{if } t < |u(x)| \leq t + \kappa \\
    \kappa \text{sign}(u(x)) & \text{if } t + \kappa < |u(x)|, 
  \end{cases}
\]

for any fixed \( t \) and \( \kappa > 0 \). This function can be chosen as a test function in (3.1). By (A1) and (A4),

\[ -\frac{d}{dt} \int_{\{u > t\}} \sum_{i=1}^{N} \alpha_i |u_{x_i}|^{p_i} \, dx \leq \int_{\{u > t\}} |f(x)| \, dx - \int_{\{u > t\}} b(u(x)) \text{sign } u \, dx \quad \text{for a.e. } t > 0. \]

Taking into account (2.4), (2.8) and (2.9), analogous arguments as in [C3] yield

\[ -\frac{d}{dt} \int_{\{u > t\}} \Lambda |\nabla u|^p \, dx \leq -\frac{d}{dt} \int_{\{u > t\}} \sum_{i=1}^{N} \alpha_i |u_{x_i}|^{p_i} \, dx \quad \text{for a.e. } t > 0. \]

By the Coarea formula and the H"older inequality,

\[ \left( -\frac{d}{dt} \int_{\{u > t\}} |\nabla u|^p \, dx \right)^{\frac{1}{p}} \geq N \omega_N \frac{1}{\mu(t)^{\frac{1}{p'}}} \mu(t)^{\frac{1}{p'}} \left( -\mu(t) \right)^{-\frac{1}{p'}} \quad \text{for a.e. } t > 0. \]
Since $f$ is nonnegative, the maximum principle assures that $u \geq 0$. Since $b$ is monotone, we obtain

$$
(4.4) \quad \int_{\{u > t\}} b(u(x)) \text{sign } u \, dx = \int_0^{\mu_u(t)} b(u^*(s)) \, ds \quad \text{for a.e. } t > 0.
$$

Thus, as a consequence of (4.1), (4.2), (4.3) and (4.4), it follows that

$$
(4.5) \quad \Lambda \left( N \omega_N^\frac{1}{N} \mu_u(t) \frac{1}{\alpha^p} \right) \left( -\mu_u(t)^{\frac{1}{p}} \right) \leq \int_0^{\mu_u(t)} f^*(s) \, ds - \int_0^{\mu_u(t)} b(u^*(s)) \, ds \quad \text{for a.e. } t > 0.
$$

The relation (4.5) implies that

$$
(4.6) \quad 1 \leq \frac{-\mu_u(t) \Lambda^{-\frac{1}{p-1}}}{\left( N \omega_N^{\frac{1}{N}} \right)^{\frac{1}{p-1}} (\mu_u(t))^{-\frac{1}{p}}} \left[ \mathcal{F}(\mu_u(t)) - \mathcal{B}(\mu_u(t)) \right]^{-\frac{1}{p-1}} \quad \text{for a.e. } t > 0,
$$

where $\mathcal{F}$ and $\mathcal{B}$ are defined as in (3.5) and (3.6), respectively.

By standard arguments (see, e.g., [Ta1]), it follows that

$$
(4.7) \quad (-u^*(s))' \leq \left( N \omega_N^\frac{1}{N} \right)^{-\frac{1}{p}} \Lambda^{-\frac{1}{p-1}} s^{-\frac{1}{pN}} [\mathcal{F}(s) - \mathcal{B}(s)]^{-\frac{1}{p-1}} \quad \text{for a.e. } s \in (0, |\Omega|).
$$

By (3.5),

$$
(4.8) \quad \mathcal{B}'(s) = b(u^*(s)) \quad \text{for a.e. } s \in (0, |\Omega|).
$$

Relations (3.5), (4.7) and (4.8) imply that

$$
(4.9) \quad \left\{ \begin{array} {l}
\Lambda \left( N \omega_N^\frac{1}{N} \right)^{-\frac{1}{p}} s^{\frac{p-1}{Np}} \left[ -\frac{d}{ds} \left( \gamma \left( \mathcal{B}'(s) \right) \right) \right]^{-\frac{1}{p}} + \mathcal{B}(s) \leq \mathcal{F}(s) \quad \text{for a.e. } s \in (0, |\Omega|) \\
\mathcal{B}(0) = 0, \quad \mathcal{B}'(|\Omega|) = 0,
\end{array} \right.
$$

where $\gamma$ is the inverse function of $b$, i.e. $\gamma = b^{-1}$.

Let us consider problem (3.3). A weak solution $w$ to problem (3.3) is unique and the symmetry of data assures that $w(x) = w(|x|)$, i.e. $w$ is positive and radially symmetric. Moreover, setting $s = \omega_N |x|^N$ and $\tilde{w}(s) = w((s/\omega_N)^{1/N})$, we get that for all $s \in [0, |\Omega|]$}

$$
-\Lambda |\tilde{w}'(s)|^{p-2} \tilde{w}'(s) = \frac{s^{-p/N'}}{(N \omega_N^{1/N})^p} \int_0^s \left( f^*(\sigma) - b(\tilde{w}(\sigma)) \right) d\sigma \quad \text{for a.e. } s \in (0, |\Omega|).
$$

Since it is possible to show (see [D1] Lemma 1.31) that the above integral is positive, we deduce that $w(x) = w^*(x)$. By the properties of $w$ we can repeat arguments used to prove (4.7) replacing all the inequalities by equalities and obtaining

$$
(4.10) \quad (-w^*(s))' = \left( N \omega_N^\frac{1}{N} \right)^{-\frac{1}{p}} \Lambda^{-\frac{1}{p-1}} s^{-\frac{p}{pN'}} \left[ \tilde{\mathcal{F}}(s) - \tilde{\mathcal{B}}(s) \right]^{-\frac{1}{p-1}} \quad \text{for a.e. } s \in (0, |\Omega|).
$$

Moreover, we have

$$
(4.11) \quad \left\{ \begin{array} {l}
\Lambda \left( N \omega_N^\frac{1}{N} \right)^{-\frac{1}{p}} s^{\frac{p-1}{Np}} \left[ -\frac{d}{ds} \left( \tilde{\gamma} \left( \tilde{\mathcal{B}}'(s) \right) \right) \right]^{-\frac{1}{p}} + \tilde{\mathcal{B}}(s) = \tilde{\mathcal{F}}(s) \quad \text{for a.e. } s \in (0, |\Omega|) \\
\tilde{\mathcal{B}}(0) = 0, \quad \tilde{\mathcal{B}}'(|\Omega|) = 0,
\end{array} \right.
$$

7
where \( \tilde{\gamma} \) is the inverse function of \( \tilde{b} \), i.e. \( \tilde{\gamma} = (\tilde{b})^{-1} \).

Since \( B, \tilde{B} \in C([0, |\Omega|]) \), there exists \( s_0 \in (0, |\Omega|) \) such that

\[
\| (B - \tilde{B})_+ \|_{L^\infty(0, |\Omega|)} = (B - \tilde{B})(s_0).
\]

In order to prove (4.11), we argue by contradiction. Assume that

\[
(B - \tilde{B})(s_0) > \| (\mathcal{F} - \tilde{\mathcal{F}})_+ \|_{L^\infty(0, |\Omega|)}.
\]

We distinguish two cases: \( s_0 < |\Omega| \) and \( s_0 = |\Omega| \).

Case \( s_0 < |\Omega| \). Combining (4.9) and (4.11) yields

\[
\Lambda \left( N \omega^\frac{1}{N} \right) \frac{s}{\sqrt{s}} \left[ \left( -\frac{d}{ds} \left( \gamma \left( B'(s) \right) \right) \right)^{\frac{\sqrt{s}}{\sqrt{s}}} - \left( -\frac{d}{ds} \left( \tilde{\gamma} \left( \tilde{B}'(s) \right) \right) \right)^{\frac{\sqrt{s}}{\sqrt{s}}} \right] \leq \mathcal{F}(s) - \tilde{\mathcal{F}}(s) + \tilde{B}(s) - B(s) \quad \text{for a.e. } s \in (0, |\Omega|)
\]

By (4.13),

\[
\mathcal{F}(s) - \tilde{\mathcal{F}}(s) + \tilde{B}(s) - B(s) \leq \| (\mathcal{F} - \tilde{\mathcal{F}})_+ \|_{L^\infty(0, |\Omega|)} - (B - \tilde{B})(s) < 0
\]

for \( s \in (s_0 - \varepsilon, s_0 + \varepsilon) \). As a consequence of (4.14) and (4.15) we obtain

\[
\Lambda \left( N \omega^\frac{1}{N} \right) \frac{s}{\sqrt{s}} \left[ \left( -\frac{d}{ds} \left( \gamma \left( B'(s) \right) \right) \right)^{\frac{\sqrt{s}}{\sqrt{s}}} - \left( -\frac{d}{ds} \left( \tilde{\gamma} \left( \tilde{B}'(s) \right) \right) \right)^{\frac{\sqrt{s}}{\sqrt{s}}} \right] = \Lambda \left( N \omega^\frac{1}{N} \right) \frac{s}{\sqrt{s}} \omega(s) \left[ \frac{d}{ds} \left( \gamma \left( B'(s) \right) - \tilde{\gamma} \left( \tilde{B}'(s) \right) \right) \right] < 0,
\]

where

\[
\omega(s) = (\sqrt{s} - 1) \int_0^1 \left\{ \left[ \tau \left( -\frac{d}{ds} \left( \gamma \left( B'(s) \right) \right) \right) + (1 - \tau) \left( -\frac{d}{ds} \left( \tilde{\gamma} \left( \tilde{B}'(s) \right) \right) \right) \right]^{\sqrt{s}} \right\} d\tau > 0.
\]

Setting

\[
Z = B - \tilde{B} \in W^{2, \infty}(s_0 - \varepsilon, s_0 + \varepsilon),
\]

we get

\[
-\frac{d}{ds} \left( \tilde{\gamma} \left( \tilde{B}'(s) \right) - \gamma \left( B'(s) \right) \right) = -\frac{d}{ds} \left( Z'(s) \eta(s) \right),
\]

where

\[
\eta(s) = \int_0^1 \tau \left( \gamma \left( B'(s) \right) + (1 - \tau) \tilde{B}'(s) \right) d\tau > 0.
\]

By (A7), we can conclude that

\[
-\frac{d}{ds} \left( \gamma \left( B'(s) \right) - \tilde{\gamma} \left( \tilde{B}'(s) \right) \right) \geq 0 \quad \text{for a.e. } s \in (0, |\Omega|).
\]

Then, by (4.18) and (4.20),

\[
-\frac{d}{ds} \left( Z'(s) \eta(s) \right) \leq -\frac{d}{ds} \left( \gamma \left( B'(s) \right) - \tilde{\gamma} \left( \tilde{B}'(s) \right) \right) \quad \text{for a.e. } s \in (0, |\Omega|).
\]
Finally, thanks to (4.16) and (4.21), we have
\begin{align}
\Lambda \left( N \omega_\gamma \frac{1}{\beta} \right)^p \int_\Omega \omega(s) \left( \frac{d}{ds} (\eta(s)Z'(s)) \right) \leq \\
\leq \Lambda \left( N \omega_\gamma \frac{1}{\beta} \right)^p \int_\Omega \omega(s) \left[ -\frac{d}{ds} \left( \gamma \left( B'(s) \right) - \gamma \left( \bar{B}'(s) \right) \right) \right] < 0 \quad \text{for a.e. } s \in (0, |\Omega|).
\end{align}

We can conclude that
\begin{align}
-d \left( \eta(s)Z'(s) \right) < 0 \quad \text{for } s \in (s_0 - \epsilon, s_0 + \epsilon),
\end{align}

which is in contradiction with the assumption (4.12), i.e. \( Z \) has a maximum in \( s_0 \).

Case \( s_0 = |\Omega| \). In this case, the inequality (4.23) holds for \( s \in (|\Omega| - \epsilon, |\Omega|) \). So \( Z'(|\Omega|) > 0 \), but this is not true since \( Z'(|\Omega|) = 0 \).

Acknowledgements  This work has been partially supported by GNAMPA of the Italian INdAM (National Institute of High Mathematics) and “Programma triennale della Ricerca dell’Università degli Studi di Napoli “Parthenope” - Sostegno alla ricerca individuale 2015-2017”.

References

[A] A. Alberico, *Boundedness of solutions to anisotropic variational problems*, Comm. Part. Diff. Eq. 36 (2011), 470–486; Corrigendum, ibid, 41, No. 5, 877–878 (2016).

[AC] A. Alberico, A. Cianchi, *Comparison estimates in anisotropic variational problems*, Manuscripta Math. 126 (2008), 481–503.

[AdBF1] A. Alberico, G. di Blasio, F. Feo, *A priori estimates for solutions to anisotropic elliptic problems via symmetrization*, Math. Nachr., Version of Record online : 25 OCT 2016, DOI: 10.1002/mana.201500282.

[AdBF2] A. Alberico, G. di Blasio, F. Feo, *Estimates for solutions to anisotropic elliptic equations with zero order term*, Geometric Properties for Parabolic and Elliptic PDEs. Contributions of the 4th Italian-Japanese Workshop, GPPEPDEs, Palinuro, Italy, May 25–29, 2015, pp. 1–15, Springer (2016).

[AdBF3] A. Alberico, G. di Blasio, F. Feo, *Comparison results for nonlinear anisotropic parabolic problems*, in press on Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl..

[ALT] A. Alvino, G. Trombetti, P. L. Lions, *On optimization problems with prescribed rearrangements*, Nonlinear Anal. 13 (1989), 185–220.

[AFTL] A. Alvino, V. Ferone, G. Trombetti, P. L. Lions, *Convex symmetrization and applications*, Ann. Inst. H. Poincaré Anal. Non Linéaire 14 (1997), 275–293.

[ACh] S. Antontsev, M. Chipot, *Anisotropic equations: uniqueness and existence results*, Diff. Int. Eq. 21 (2008), 401–419.

[BFK] M. Belloni, V. Ferone, B. Kawohl, *Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic equations*, Zeit. Angew. Math. Phys. 54 (2003), 771–789.
[BCE] M. Bendahmane, M. Chrif, S. El Manouni, An approximation result in generalized anisotropic Sobolev spaces and applications. Z. Anal. Anwend. 30 (2011), 341–353.

[BB] H. Brezis, F. E. Browder, Some properties of higher order Sobolev spaces, J. Math. Pures Appl. 61 (1982), 245–259.

[BS] C. Bennett, R. Sharpley, Interpolation of operators, Pure and Applied Mathematics, 129, Academic Press, Inc., Boston, MA, 1988.

[BMS] L. Boccardo, P. Marcellini, C. Sbordone, $L^\infty$-regularity for variational problems with sharp nonstandard growth conditions, Boll. Un. Mat. Ital. A 4 (1990), 219–225.

[C1] A. Cianchi, Local boundedness of minimizers of anisotropic functionals, Ann. Inst. Henri Poincaré, Analyse non linéaire 17 (2000), 147–168.

[C3] A. Cianchi, Symmetrization in anisotropic elliptic problems, Comm. Part. Diff. Eq. 32 (2007), 693–717.

[DdB] F. Della Pietra, G. di Blasio, Blow-up solutions for some nonlinear elliptic equations involving a Finsler-Laplacian, Publ. Mat. 61, No. 1, 213–238 (2017).

[DG] F. Della Pietra, N. Gavitone, Anisotropic elliptic equations with general growth in the gradient and Hardy-type potentials, J. Differential Equations 255 (2013), 3788–3810.

[DFG] R. Di Nardo, F. Feo, O. Guibé, Uniqueness result for nonlinear anisotropic elliptic equations. Adv. Differential Equations 18 (2013), 433–458.

[DF] R. Di Nardo, F. Feo, Existence and uniqueness for nonlinear anisotropic elliptic equations, Arch. Math. (Basel) 102 (2014), 141–153.

[D1] J. I. Díaz, Nonlinear partial differential equations and free boundaries. Vol. I. Elliptic equations, Research Notes in Mathematics, 106. Pitman, Boston, MA, 1985.

[D2] J. I. Díaz, Inequalities of isoperimetric type for the Plateau problem and the capillarity problem, Rev. Acad. Canaria Cienc. 3 (1991), 127–166.

[FGK] I. Fragalà, F. Gazzola, B. Kawohl, Existence and nonexistence results for anisotropic quasilinear elliptic equations, Ann. Inst. Henri Poincaré, Analyse non linéaire 21 (2004), 715–734.

[FGL] I. Fragalà, F. Gazzola, G. Lieberman, Regularity and nonexistence results for anisotropic quasilinear elliptic equations in convex domains, Disc. Cont. Dynam. Syst. (2005), 280–286.

[FS] N. Fusco, C. Sbordone, Some remarks on the regularity of minima of anisotropic integrals, Comm. Part. Diff. Equat. 18 (1993), 153-167.

[G] M. Giaquinta, Growth conditions and regularity, a counterexample, Manus. Math. 59 (1987), 245–248.

[Mad] C. Maderna, Optimal problems for a certain class of nonlinear Dirichlet problems, Boll. Un. Mat. Ital. Suppl. 1 (1980), 31–43.

[Mar] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions, Arch. Rat. Mech. Anal. 105 (1989), 267–284.

[Maz1] V. G. Maz’ya, Some estimates of solutions of second-order elliptic equations, Dokl. Akad. Nauk. SSSR 137 (1961), 1057–1059 (Russian); English translation: Soviet Math. Dokl. 2 (1961), 413–415.
[Maz2] V. G. Maz’ya, *On weak solutions of the Dirichlet and Neumann problems*, Trusdy Moskov. Mat. Obšč. 20 (1969), 137–172 (Russian); English translation: Trans. Moscow Math. Soc. 20 (1969), 135–172.

[K] V. S. Klimov, *Isoperimetric inequalities and imbedding theorems*, (Russian) Dokl. Akad. Nuak SSSR 217 (1974), 272–275.

[Ta1] G. Talenti, *Elliptic equations and rearrangements*, Ann. Sc. Norm. Sup. Pisa IV 3 (1976), 697–718.

[Ta2] G. Talenti, *Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces*, Ann. Mat. Pura Appl. 120 (1979), 160–184.

[Tr] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ricerche Mat. 18 (1969), 3–24.

[T] G. Trombetti, *Symmetrization methods for partial differential equations*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. 3 (2000), 601–634.

[V1] J. L. Vazquez, *Symétrisation pour ut = \Delta \varphi(u) et applications*, C. R. Acad. Sci. Paris Sér. I Math. 295 (1982), 71–74.

[V2] J. L. Vazquez, *Symmetrization and Mass Comparison for Degenerate Nonlinear Parabolic and related Elliptic Equations*, Advances in Nonlinear Studies, 5 (2005), 87–131.