GLOBAL WAVE FRONT SET
OF MODULATION SPACE TYPES
AND FOURIER INTEGRAL OPERATORS

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ABSTRACT. We continue our analysis of the global wave-front sets we introduced on modulation spaces, here in relation with the corresponding class of Fourier integral operators. We obtain propagation results in terms of canonical transformations of the phase space, without requiring that the involved phase functions and amplitude are classical.

CONTENTS

0. Introduction 2
1. Preliminaries 5
  1.1. Weight functions 5
  1.2. Modulation spaces 6
  1.3. Pseudo-differential operators and SG symbol classes 7
  1.4. Composition and further properties of SG classes of symbols, amplitudes, and functions 8
2. Symbolic calculus for generalised Fourier Integral Operators of SG type 12
   2.1. Phase functions of SG type 12
   2.2. Generalised Fourier integral operators of SG type 13
   2.3. Composition with SG pseudodifferential operators. 15
   2.4. Other composition theorems between SG FIOs and SG pseudodifferential operators 27
   2.5. Composition between SG FIOs of type I and type II 28
   2.6. Elliptic SG FIOs and parametrices 32
   2.7. Egorov theorem 33
3. Continuity on Lebesgue and modulation spaces 34
   3.1. Continuity on Lebesgue spaces 34
   3.2. Continuity on modulation spaces 35
4. Propagation results for global Wave-front Sets and Fourier Integral Operators of SG type 36
   4.1. Global Wave-front Sets 37

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4.2. Action of SG FIOs on global Wave-front Sets
4.3. Applications to SG-hyperbolic problems
Appendix A. Asymptotic expansions in the Weyl-Hörmander classes
A.1. Metrics
A.2. Asymptotic expansions
Appendix B. Direct proof of continuity on $L^2(\mathbb{R}^d)$ of regular generalized SG FIOs
References

0. Introduction

In [15], global wave-front sets with respect to convenient Banach or Frechet spaces were introduced, and global mapping properties of pseudo-differential operators of SG-type, see, e.g., [11,12,14,18,30,32], were established in terms of these wave-front sets. For any such Banach or Frechet space $\mathcal{B}$ and tempered distribution $f$, the global wave-front set $WF_{\mathcal{B}}(f)$ is the union of three components $WF_{\mathcal{B}}^m(f)$, $m = 1, 2, 3$. The first component (for $m = 1$) describes the local wave-front set which informs where $f$ locally fails to belong to $\mathcal{B}$, as well as the directions where the singularities (with respect to $\mathcal{B}$) propagates. The second and third components (for $m = 2$ or $m = 3$) informs where at infinity the growth and oscillations of $f$ are strong enough such that $f$ fails to belong to $\mathcal{B}$, as well as the directions where the singularities (with respect to $\mathcal{B}$) propagates. We remark that $WF_{\mathcal{B}}^1(f)$, $WF_{\mathcal{B}}^2(f)$ and $WF_{\mathcal{B}}^3(f)$ agree with $WF^\psi(f)$, $WF^{e}(f)$ and $WF^{\psi e}(f)$, respectively, in [17]. Note also that for admissible $\mathcal{B}$, these wave-front sets give suitable information for local and global behavior, since $f$ belongs to $\mathcal{B}$ globally (locally), if and only if $WF_{\mathcal{B}}(f) = \emptyset$ ($WF_{\mathcal{B}}^m(f) = \emptyset$).

It is convenient to formulate mapping properties for pseudo-differential operators of SG-type in terms of SG-ordered pairs $(\mathcal{B}, \mathcal{C})$, where $\mathcal{B}$ and $\mathcal{C}$ should be appropriate target and image spaces of the involved pseudo-differential operators. (Cf. [15].) More precisely, the pair $(\mathcal{B}, \mathcal{C})$ of spaces $\mathcal{B}$ and $\mathcal{C}$ containing $\mathcal{S}$ and contained in $\mathcal{S}'$, is called SG-ordered with respect to the weight $\omega$ if the mappings

$$
\text{Op}(a) : \mathcal{B} \to \mathcal{C}, \quad \text{Op}(b)^* : \mathcal{C} \to \mathcal{B},
$$

$$
\text{Op}(c) : \mathcal{B} \to \mathcal{B} \quad \text{and} \quad \text{Op}(c) : \mathcal{C} \to \mathcal{C}
$$

are continuous for every $a \in \text{SG}^{(\omega)}$, $b \in \text{SG}^{(1/\omega)}$ and $c \in \text{SG}^{0,0}$. If it is only required that the first mapping property in (0.1) holds, then the pair $(\mathcal{B}, \mathcal{C})$ is called weakly SG-ordered. Here $\text{SG}^{(\omega)}$, the set of all SG-symbols with respect to $\omega$, belongs to an extended family of symbol classes of SG-type. We refer to [17] for the definition of (also classical)
SG-symbols. We note that (0.1) is true also after $\text{Op}(b)^*$ replaces its adjoint $\text{Op}(b)^{\ast}$, because $\text{Op}(\text{SG}^{(\omega)})^* = \text{Op}(\text{SG}^{(\omega)})$.

Important examples on SG-ordered pairs are the Schwartz, tempered distributions and modulation spaces. More precisely, in [15] it is noticed that $(\mathcal{S}, \mathcal{S})$ and $(\mathcal{S}', \mathcal{S})$ are SG-ordered pairs, and for any weight $\omega$ and any modulation space $B$, there is a (unique) modulation space $C$ such that $(B, C)$ is an SG-ordered pair with respect to $\omega$. In particular, the family of SG-ordered pairs is broad in the sense that $B$ can be chosen as a Sobolev space, or, more general, as a Sobolev-Kato space, since such spaces are special cases of modulation spaces. Moreover, if $\text{SG}^{(\omega)}$ is a classical symbol class of SG-type and $B$ is a Sobolev-Kato space, then $C$ is also a Sobolev-Kato space.

For any SG-ordered pairs $(B, C)$ with respect to $\omega$, it is proved in [14, 15] that the wave-front sets with respect to $B$ and $C$ possess convenient mapping properties. For example, if $f \in \mathcal{S}'$ and $a \in \text{SG}^{(\omega)}$, then (0.1) is refined as

$$\text{WF}_C(\text{Op}(a)f) \subseteq \text{WF}_B(f),$$

(0.2)

and

$$\text{WF}_C^m(\text{Op}(a)f) \subseteq \text{WF}_B^m(f), \quad m = 1, 2, 3,$$

and that reversed inclusions are obtained by adding the set of characteristic points to the left-hand sides in (0.2). In particular, since the set of characteristic points is empty for elliptic operators, it follows that equalities are attained in (0.2) for such operators.

In this paper we establish similar properties for elliptic Fourier integral operators. More precisely, for any amplitude or symbol $a$ in $\text{SG}^{(\omega)}$ for some weight $\omega$, the Fourier integral operator $\text{Op}_\varphi(a)$ is given by

$$f \mapsto (\text{Op}_\varphi(a)f)(x) \equiv (2\pi)^{-d} \int e^{i\varphi(x,\xi)} a(x, \xi) \hat{f}(\xi) \, d\xi,$$

and its adjoint by

$$f \mapsto (\text{Op}_\varphi(a)^*f)(x) \equiv (2\pi)^{-d} \int \int e^{i\varphi(x,\xi) - \varphi(y,\xi)} a(y, \xi) f(y) \, dy \, d\xi.$$

The operator $\text{Op}_\varphi(a)^* = \text{Op}_\varphi(a)^{\ast}$ is sometimes called Fourier integral operator of type II, while $\text{Op}_\varphi(a)$ is called a Fourier integral operator of type I. Here the phase function $\varphi$ should be in $\text{SG}^{1,1}$ and satisfy

$$\langle \varphi_x'(x, \xi) \rangle < \langle \xi \rangle \quad \text{and} \quad \langle \varphi_\xi''(x, \xi) \rangle < \langle \xi \rangle.$$

(0.3)

In (0.3), $A \asymp B$ means that $A \lesssim B$ and $B \lesssim A$, where we write $A \lesssim B$ if there exists a constant $C > 0$ such that $A \leq C \cdot B$.

To achieve the desired continuity properties among the functional spaces we will deal with, $\varphi$ is also assumed to fulfill the usual (global) non-degeneracy condition

$$|\det(\varphi_{x\xi}''(x, \xi))| \geq c, \quad x, \xi \in \mathbb{R}^d,$$

(0.3)
for some constant $c > 0$.

In Section 3, the notion on SG-ordered pair from [15] is reformulate to include such Fourier integral operators, where the operators $\text{Op}(a)$ and $\text{Op}(b)^*$ in (0.1) are replaced by $\text{Op}_\varphi(a)$ and $\text{Op}_\varphi(b)^*$, respectively.

In order to establish wave-front results, similar to (0.2), it is also required that the phase functions fulfill some further natural conditions, namely, that they preserve the “shape” of the various kind of neighborhood of points in $T^*\mathbb{R}^d$ involved in the definition of our wave-front sets (see Section 4 below). In fact, the definitions of wave-front sets of appropriate distributions are based on the behavior in cones of corresponding Fourier transformations, after localizing the involved distributions near points or in certain directions.

In Section 4 we prove that if $\Phi$ is the canonical transformation of the phase space $T^*\mathbb{R}^d \simeq \mathbb{R}^{2d}$, $\text{Op}_\varphi(a)$ is an elliptic Fourier integral operator with amplitude in $\text{SG}^{(\omega)}$, and $(\mathcal{B}, \mathcal{C})$ being an SG-ordered pair, then

$$\text{WF}_C(\text{Op}_\varphi(a)f) = \Phi(\text{WF}_B(f)).$$

(0.4)

The result is based on comprehensive investigations of algebraic and continuity properties of the involved Fourier integral operators. A significant part of these investigations concern compositions between Fourier integral operators of type I or II, with pseudo-differential operators. This is performed in Section 2, where it is proved that for any Fourier integral operators $\text{Op}_\varphi(a)$ and $\text{Op}_\varphi^*(b)$ with $a, b \in \text{SG}^{(\omega_1)}$, and some $p \in \text{SG}^{(\omega_2)}$, then

$$\text{Op}(p) \circ \text{Op}_\varphi(a) = \text{Op}_\varphi(h_1) \mod \text{Op}(S),$$

$$\text{Op}(p) \circ \text{Op}_\varphi^*(b) = \text{Op}_\varphi^*(h_2) \mod \text{Op}(S),$$

$$\text{Op}_\varphi(a) \circ \text{Op}(p) = \text{Op}_\varphi(h_3) \mod \text{Op}(S)$$

$$\text{Op}_\varphi^*(b) \circ \text{Op}(p) = \text{Op}_\varphi^*(h_4) \mod \text{Op}(S),$$

for some $h_j \in \text{SG}^{(\omega_1 \omega_2)}$, $j = 1, \ldots, 4$. Here $\text{Op}(S)$ is a set of appropriate smoothing operators, depending on the hypotheses on the amplitudes $a, b$ and the symbol $p$. Furthermore, if $a \in \text{SG}^{(\omega_1)}$ and $b \in \text{SG}^{(\omega_2)}$, then it is also proved that $\text{Op}_\varphi^*(b) \circ \text{Op}_\varphi(a)$ is equal to a pseudo-differential operator $\text{Op}(q)$, for some $q \in \text{SG}^{(\omega_1 \omega_2)}$. Furthermore we present asymptotic formulae for $q$ and $h_1, h_2, h_3, h_4$, in terms of $a$ and $b$, respectively, or $a, b$ and $p$, modulus a smoothing term. The content of Section 2 generalizes the calculus of SG Fourier integral operators developed in [12] to the classes $\text{SG}^{(\omega, \rho)}$, introduced and systematically used in [14, 16].

The formula (0.4) also relies on certain asymptotic expansions in the framework of symbolic calculus of SG pseudo-differential operators, as well as on continuity properties for SG-ordered pairs. One of these
asymptotic expansions concerns making sense of expansions of the form

\[ a \sim \sum a_j, \]

in the framework of the generalized SG-classes \( \text{SG}_{r,\rho}^{(\omega_0)} \). The ideas are similar to corresponding properties in the usual Hörmander calculus in Section 18.1 in [28]. For this reason we have in Appendix A established properties on asymptotic expansions for symbols classes of the form \( S(m, g) \), parameterized by the weight function \( m \) and Riemannian metric \( g \) on the phase space (cf. Section 18.4 in [28]). Note here that any SG-class is equal to \( S(m, g) \) for some choice of \( m \) and \( g \), and that similar facts hold for the Hörmander classes \( S_{\rho,\delta}^{r,\rho} \). The results in Appendix A therefore cover several situations on asymptotic expansions for pseudo-differential operators.

Finally, we study in Section 3 some specific spaces which are SG-ordered or weakly SG-ordered. For example, we present necessary and sufficient conditions for the involved weight functions and parameters, in order for Sobolev-Kato spaces, Sobolev spaces and modulation spaces should be SG-ordered or weakly SG-ordered. Moreover, taking advantage of the calculus developed in Section 2 and relying on results in [27] and [10], we prove that our classes of SG Fourier integral operators are continuous between suitable couples of weighted modulation spaces \( (M_{(\omega_1)}^{p}(\mathbb{R}^d), M_{(\omega_2)}^{p}(\mathbb{R}^d)) \).

1. Preliminaries

We begin by fixing the notation and recalling some basic concepts which will be needed below. The material in Subsections 1.1-1.3 mainly summarizes part of the contents of Sections 2 in [15], and comes from [16]. In Subsection 1.4 we then state a few lemmas which will be useful in the subsequent Section 2. Some of these, compared with their original formulation in the SG context, appeared in [12], are here given in a slightly more general form, adapted to the definitions given in Subsection 1.3. In the sequel, we write \( A \lesssim B \) when there exists a constant \( C > 0 \) such that \( A \leq C \cdot B \). We also use the notation \( A \asymp B \) when \( A \lesssim B \lesssim A \).

1.1. Weight functions. Let \( \omega \) and \( v \) be positive measurable functions on \( \mathbb{R}^d \). Then \( \omega \) is called \( v \)-moderate if

\[ \omega(x + y) \lesssim \omega(x)v(y) \quad (1.1) \]

If \( v \) in (1.1) can be chosen as a polynomial, then \( \omega \) is called a function or weight of polynomial type. We let \( \mathcal{P}(\mathbb{R}^d) \) be the set of all polynomial type functions on \( \mathbb{R}^d \). If \( \omega(x, \xi) \in \mathcal{P}(\mathbb{R}^{2d}) \) is constant with respect to the \( x \)-variable or the \( \xi \)-variable, then we sometimes write \( \omega(\xi) \), respectively \( \omega(x) \), instead of \( \omega(x, \xi) \). In this case we consider \( \omega \) as an element in \( \mathcal{P}(\mathbb{R}^{2d}) \) or in \( \mathcal{P}(\mathbb{R}^d) \) depending on the situation. We say that \( v \) is
submultiplicative if (1.1) holds for \( \omega = v \). For convenience we assume that all submultiplicative weights are even, and we always let \( v \) and \( v_j \) stand for submultiplicative weights, if nothing else is stated.

Without loss of generality we may assume that every \( \omega \in \mathcal{P}(\mathbb{R}^d) \) is smooth and satisfies the ellipticity condition \( \partial^\omega \omega / \omega \in L^\infty \). In fact, by Lemma 1.2 in \cite{33} it follows that for each \( \omega \in \mathcal{P}(\mathbb{R}^d) \), there is a smooth and elliptic \( \omega_0 \in \mathcal{P}(\mathbb{R}^d) \) which is equivalent to \( \omega \) in the sense
\[
\omega \asymp \omega_0. \tag{1.2}
\]

The weights involved in the sequel have to satisfy additional conditions. More precisely let \( r, \rho \geq 0 \). Then \( \mathcal{P}_{r,\rho}(\mathbb{R}^{2d}) \) is the set of all \( \omega(x, \xi) \) in \( \mathcal{P}(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d}) \) such that
\[
\langle x \rangle^{r|\alpha|} \langle \xi \rangle^{\rho|\beta|} \frac{\partial^\alpha \partial^\beta \omega(x, \xi)}{\omega(x, \xi)} \in L^\infty(\mathbb{R}^{2d}),
\]
for every multi-indices \( \alpha \) and \( \beta \). Any weight \( \omega \in \mathcal{P}_{r,\rho}(\mathbb{R}^{2d}) \) is then called SG-moderate on \( \mathbb{R}^{2d} \), of order \( r \) and \( \rho \). Note that \( \mathcal{P}_{r,\rho} \) is different here compared to \cite{14}, and that there are elements in \( \mathcal{P}(\mathbb{R}^d) \) which have no equivalent elements in \( \mathcal{P}_{r,\rho}(\mathbb{R}^{2d}) \). On the other hand, if \( s, t \in \mathbb{R} \) and \( r, \rho \in [0, 1] \), then \( \mathcal{P}_{r,\rho}(\mathbb{R}^{2d}) \) contains all weights of the form \( \omega(x, \xi) = \langle x \rangle^s \langle \xi \rangle^t \), which are one of the most common type of weights in the applications.

It will also be useful to consider SG-moderate weights in one or three sets of variables. Let \( \omega \in \mathcal{P}(\mathbb{R}^{3d}) \cap C^\infty(\mathbb{R}^{3d}) \), and let \( r_1, r_2, \rho \geq 0 \). Then \( \omega \) is called SG moderate on \( \mathbb{R}^{3d} \), of order \( r_1, r_2 \) and \( \rho \), if it fulfills
\[
\langle x_1 \rangle^{r_1|\alpha_1|} \langle x_2 \rangle^{r_2|\alpha_2|} \langle \xi \rangle^{\rho|\beta|} \frac{\partial^\alpha_1 \partial^\alpha_2 \partial^\beta \omega(x_1, x_2, \xi)}{\omega(x_1, x_2, \xi)} \in L^\infty(\mathbb{R}^{3d}).
\]
The set of all SG-moderate weights on \( \mathbb{R}^{3d} \) of order \( r_1, r_2 \) and \( \rho \) is denoted by \( \mathcal{P}_{r_1,r_2,\rho}(\mathbb{R}^{3d}) \). Finally, we denote by \( \mathcal{P}_{r}(\mathbb{R}^d) \) the set of all SG-moderate weights of order \( r \geq 0 \) on \( \mathbb{R}^d \), which are defined in a completely similar fashion.

1.2. Modulation spaces. Let \( \phi \in \mathcal{S}(\mathbb{R}^d) \). Then the short-time Fourier transform of \( f \in \mathcal{S}(\mathbb{R}^d) \) with respect to (the window function) \( \phi \) is defined by
\[
V_\phi f(x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y) \overline{\phi(y - x)} e^{-i(y, \xi)} \, dy. \tag{1.3}
\]
More generally, the short-time Fourier transform of \( f \in \mathcal{S}'(\mathbb{R}^d) \) with respect to \( \phi \in \mathcal{S}'(\mathbb{R}^d) \) is defined by
\[
(V_\phi f)(x) = \mathcal{F}_2 F, \quad \text{where} \quad F(x, y) = (f \otimes \overline{\phi})(y, y - x), \tag{1.3'}
\]
where \( \mathcal{F}_2 F \) is the partial Fourier transform of \( F(x, y) \in \mathcal{S}'(\mathbb{R}^{2d}) \) with respect to the \( y \)-variable. We refer to \cite{25,26} for more facts about the short-time Fourier transform.
Let $B$ to $v$ such that the norm

\[ \|f\|_{M(\omega, B)} \equiv \|V\omega f\|_B \quad (1.4) \]

(cf. [23]).

**Remark 1.1.** Assume that $p, q \in [1, \infty]$, and let $L_p^q(\mathbb{R}^d)$ and $L_{2,p}^q(\mathbb{R}^d)$ be the sets of all $F \in L_{1,loc}^1(\mathbb{R}^d)$ such that

\[ \|F\|_{L_p^q} \equiv \left( \int \left( \int |F(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} < \infty \]

and

\[ \|F\|_{L_{2,p}^q} \equiv \left( \int \left( \int |F(x, \xi)|^q d\xi \right)^{p/q} dx \right)^{1/p} < \infty. \]

Then $M(\omega, L_1^{p,q}(\mathbb{R}^d))$ is equal to the classical modulation space $M_{p,q}(\mathbb{R}^d)$, and $M(\omega, L_{2,p}^{p,q}(\mathbb{R}^d))$ is equal to the space $W_{p,q}(\mathbb{R}^d)$, related to Wiener-amalgam spaces (cf. [22] [23] [26]). We will set $M_{p}(\omega) = M_{p,p}(\mathbb{R}^d) = W_{p,p}(\mathbb{R}^d)$. Furthermore, if $\omega = 1$, then we write $M_{p,q}^p, M_p^p$ and $W_{p,q}^p$ instead of $M_{p,q}^p(\omega), \ M_{p}(\omega)$ and $W_{p,q}(\omega)$ respectively.

**Remark 1.2.** Several important spaces agree with certain modulation spaces. In fact, let $s, \sigma \in \mathbb{R}$. If $\omega(x, \xi) = \vartheta_{s,\sigma}(x, \xi) = \langle x \rangle^s \langle \xi \rangle^\sigma$, then $M^2_{(\omega)}(\mathbb{R}^d)$ is equal to the weighted Sobolev space (or Sobolev-Kato space) $H^2_{s,\sigma}(\mathbb{R}^d)$ in [17] [30], the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\langle x \rangle^s \langle D \rangle^\sigma f \in L^2(\mathbb{R}^d)$. In particular, if $s = 0 (\sigma = 0)$, then $M^2_{(\omega)}(\mathbb{R}^d)$ equals to $H^2(\mathbb{R}^d)$ ($L^2(\mathbb{R}^d)$). Furthermore, if instead $\omega(x, \xi) = \langle x, \xi \rangle^s$, then $M_{(\omega)}^2(\mathbb{R}^d)$ is equal to the Shubin-Sobolev space of order $s$. (Cf. e.g. [29]).

### 1.3. Pseudo-differential operators and SG symbol classes.

Next we recall some facts in Chapter XVIII in [25] concerning pseudo-differential operators. Let $a \in \mathcal{S}(\mathbb{R}^d)$, and $t \in \mathbb{R}$ be fixed. Then the pseudo-differential operator $\text{Op}_t(a)$ is the linear and continuous operator on $\mathcal{S}(\mathbb{R}^d)$ defined by the formula

\[ (\text{Op}_t(a)f)(x) = (2\pi)^{-d} \int \left( \int e^{i(x-y, \xi)} a((1-t)x + ty, \xi) f(y) dy \right) d\xi. \]  

(1.5)

For general $a \in \mathcal{S}'(\mathbb{R}^d)$, the pseudo-differential operator $\text{Op}_t(a)$ is defined as the continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ with distribution kernel

\[ K_{t,a}(x, y) = (2\pi)^{-d/2}(\mathcal{F}^{-1} a((1-t)x + ty, x-y)). \]  

(1.6)

If $t = 0$, then $\text{Op}_t(a)$ is the Kohn-Nirenberg representation $\text{Op}(a) = a(x, D)$, and if $t = 1/2$, then $\text{Op}_t(a)$ is the Weyl quantization.
We now recall the definition of the generalized SG-symbol classes. Let \( m, \mu, r, \rho \in \mathbb{R} \) be fixed. Then \( \text{SG}^{m,\mu}_{r,\rho}(\mathbb{R}^{2d}) \) is the set of all \( a \in C^\infty(\mathbb{R}^{2d}) \) such that
\[
|D_x^a D_\xi^b a(x, \xi)| \lesssim \langle x \rangle^{m-r[a]} \langle \xi \rangle^{\mu-\rho[\beta]},
\]
for all multi-indices \( \alpha \) and \( \beta \). Usually we assume that \( r, \rho \geq 0 \) and \( \rho + r > 0 \).

More generally, assume that \( \omega \in \mathcal{P}_{r,\rho}(\mathbb{R}^{2d}) \). Then \( \text{SG}_r^{(\omega)}(\mathbb{R}^{2d}) \) consists of all \( a \in C^\infty(\mathbb{R}^{2d}) \) such that
\[
|D_x^a D_\xi^b a(x, \xi)| \lesssim \omega(x, \xi) \langle x \rangle^{-r[a]} \langle \xi \rangle^{-\rho[\beta]}, \quad x, \xi \in \mathbb{R}^d,
\]
for all multi-indices \( \alpha \) and \( \beta \). We note that
\[
\text{SG}_r^{(\omega)}(\mathbb{R}^{2d}) = S(\omega, g_{r,\rho}),
\]
when \( g = g_{r,\rho} \) is the Riemannian metric on \( \mathbb{R}^{2d} \), defined by the formula
\[
(g_{r,\rho})_{(y,\eta)}(x, \xi) = \langle y \rangle^{-2r}|y|^2 + \langle \eta \rangle^{-2\rho}|\eta|^2
\]
(cf. Section 18.4–18.6 in [28]). Furthermore, \( \text{SG}_r^{(\omega)} = \text{SG}^{m,\mu}_{r,\rho} \) when \( \omega(x, \xi) = \vartheta_{m,\mu}(x, \xi) = \langle x \rangle^m \langle \xi \rangle^\mu \).

In view of the properties of the Weyl-Hörmander calculus, the concept of asymptotic expansion extends to the classes \( \text{SG}_r^{(\omega)}(\mathbb{R}^{2d}) \), see Appendix A. We give below the explicit definition, focusing on the situations that we will meet in the sequel, and describing the corresponding type of remainders.

**Definition 1.3.** Let \( a_j \in \text{SG}_r^{(\omega)}(\mathbb{R}^{2d}) \), \( j = 0, 1, \ldots \), be a sequence of generalized SG symbols such that \( r, \rho \geq 0 \), \( r + \rho > 0 \), \( \omega_j = \omega \cdot \vartheta_{s_j,\sigma_j} \), for a fixed \( \omega \in \mathcal{P}_{r,\rho}(\mathbb{R}^{2d}) \), and \( \vartheta_{s_j,\sigma_j}(x, \xi) = \langle x \rangle^{s_j} \langle \xi \rangle^{\sigma_j} \), with real sequences \( \{s_j\}, \{\sigma_j\} \), \( s_j, \sigma_j \leq 0 \). Assume that \( \{s_j\} \) is identically equal to 0 if \( r = 0 \), or diverging to \(-\infty\) if \( r > 0 \). Similarly, assume that \( \{\sigma_j\} \) is identically equal to 0 if \( \rho = 0 \), or diverging to \(-\infty\) if \( \rho > 0 \). We then say that \( \sum_{j \geq 0} a_j \) is an asymptotic expansion, and write \( a \sim \sum_{j \geq 0} a_j \) for \( a \in \text{SG}_r^{(\omega)}(\mathbb{R}^{2d}) \) if, for any integer \( N \geq 0 \) one has
\[
a - \sum_{j=0}^N a_j \in \text{SG}_r^{(\omega)N+1}(\mathbb{R}^{2d}).
\]  

It can be proved that to any asymptotic expansion \( \{a_j\} \) of the type described in Definition 1.3 there corresponds an asymptotic sum \( a \in \text{SG}_r^{(\omega)}(\mathbb{R}^{2d}) \) such that (1.10) holds. The asymptotic sum is uniquely determined modulo a remainder \( q \) such that, for suitable \( m, \mu \in \mathbb{R} \),
\[
q \in \text{SG}^{-\infty,\mu}_{r,\rho}(\mathbb{R}^{2d}) \text{ if } r > 0,
q \in \text{SG}^{m,-\infty}_{r,\rho}(\mathbb{R}^{2d}) \text{ if } \rho > 0,
q \in \text{SG}^{-\infty,-\infty}_{r,\rho}(\mathbb{R}^{2d}) \text{ if } r > 0, \rho > 0.
\]
The parameters $m, \mu$ in (1.11) depend only on the weight function, since $\omega$ is assumed to be polynomially moderate.

It is a well-known fact that SG-operators give rise to linear continuous mappings from $\mathcal{S}(\mathbb{R}^d)$ to itself, extendable as linear continuous mappings from $\mathcal{S}'(\mathbb{R}^d)$ to itself. They also act continuously between modulation spaces. Indeed, see [15], if $a \in \text{SG}_{r, \rho}(\mathbb{R}^{2d})$, then $\text{Op}_a(a)$ is continuous from $M(\omega, \mathcal{B})$ to $M(\omega/\omega_0, \mathcal{B})$. Moreover, there exist $a \in \text{SG}_{r, \rho}(\mathbb{R}^{2d})$ and $b \in \text{SG}_{(1/\omega), r, \rho}(\mathbb{R}^{2d})$ such that for every choice of $\omega \in \mathcal{P}(\mathbb{R}^{2d})$ and every translation invariant BF-space $\mathcal{B}$ on $\mathbb{R}^{2d}$, the mappings

$$\text{Op}_a(a) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d), \quad \text{Op}_b(a) : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$$

and

$$\text{Op}_a(a) : M(\omega, \mathcal{B}) \to M(\omega/\omega_0, \mathcal{B}).$$

are continuous bijections with inverses $\text{Op}_a(b)$.

1.4. Composition and further properties of SG classes of symbols, amplitudes, and functions. We define families of smooth functions with SG behaviour, depending on one, two or three sets of real variables (cfr. also [21]). We then introduce pseudodifferential operators defined by means of SG amplitudes. Subsequently, we recall sufficient conditions for maps of $\mathbb{R}^d$ into itself to keep the invariance of the SG classes.

**Definition 1.4.** Let $m_1, m_2, \mu, r_1, r_2, \rho \in \mathbb{R}$. We let $\text{SG}_{r_1, r_2, \rho}^{m_1, m_2, \mu}(\mathbb{R}^{3n})$ denote the set of all amplitude functions, that is, the set of all $a \in C^\infty(\mathbb{R}^{3n})$ which satisfy the condition

$$|\partial_1^{m_1} \partial_2^{m_2} \partial_3^{\beta} a(x_1, x_2, \xi)| \lesssim \langle x_1 \rangle^{m_1-r_1} \langle x_2 \rangle^{m_2-r_2} \langle \xi \rangle^{\mu-\rho|\beta|}, \quad (1.12)$$

for every multi-indices $\alpha_1, \alpha_2, \beta$. We will usually assume $r_1, r_2, \rho \geq 0$ and $r_1 + r_2 + \rho > 0$. More generally, let $\omega_0 \in \mathcal{P}_{r_1, r_2, \rho}(\mathbb{R}^{3n})$. Then $\text{SG}_{r_1, r_2, \rho}^{(\omega)}(\mathbb{R}^{3n})$ is the set of all $a \in C^\infty(\mathbb{R}^{3n})$ which satisfy the condition

$$|\partial_1^{m_1} \partial_2^{m_2} \partial_3^{\beta} a(x, y, \xi)| \lesssim \omega_0(x_1, x_2, \xi) \langle x_1 \rangle^{-r_1} \langle x_2 \rangle^{-r_2} \langle \xi \rangle^{-\rho|\beta|}, \quad (1.12)'$$

for every multi-indices $\alpha_1, \alpha_2, \beta$. The set $\text{SG}_{r_1, r_2, \rho}^{(\omega)}(\mathbb{R}^{3n})$ is equipped with the usual Fréchet topology based upon the seminorms implicit in (1.12)'.

As above, $\text{SG}_{r_1, r_2, \rho}^{(\omega)} = \text{SG}_{r_1, r_2, \rho}^{m_1, m_2, \mu}$ when $\omega(x_1, x_2, \xi) = \langle x_1 \rangle^{m_1} \langle x_2 \rangle^{m_2} \langle \xi \rangle^{\mu}$. Furthermore, we set

$$\text{SG}^\infty = \bigcup_{(m_1, m_2, \mu) \in \mathbb{R}^3} \text{SG}_{1, 1}^{m_1, m_2, \mu}, \quad \text{SG}^- = \bigcap_{(m_1, m_2, \mu) \in \mathbb{R}^3} \text{SG}_{1, 1}^{m_1, m_2, \mu} = \mathcal{S}.$$

**Definition 1.5.** Assume $\omega \in \mathcal{P}_{r_1, r_2, \rho}(\mathbb{R}^{3d})$, $r_1, r_2, \rho \geq 0$, $r_1 + r_2 + \rho > 0$, and let $a \in \text{SG}_{r_1, r_2, \rho}^{(\omega)}(\mathbb{R}^{3d})$. Then, the pseudo-differential operator $\text{Op}(a)$
is the linear and continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ with distribution kernel

$$K_a(x, y) = (2\pi)^{-d/2}(\mathcal{F}^{-1}_a)(x, y, x - y).$$

For $f \in \mathcal{S}(\mathbb{R}^d)$, we then have

$$(\text{Op}(a)f)(x) = (2\pi)^{-d}\iint e^{i(x-y, \xi)}a(x, y, \xi)f(y)\,dy\,d\xi.$$ 

The operators introduced in Definition 1.5 have properties analogous to the usual SG operator families described in [11]. They coincide with the operators defined in the previous subsection: the corresponding symbol is obtained by means of an asymptotic expansions, see Definition 1.3 modulo a remainder of the type given in (1.3). For the sake of brevity, we here omit the details. When neither the amplitude functions nor the corresponding weight $\omega$, depend on $x_2$, we of course reobtain the definition of SG-symbols and pseudo-differential operators, given in the previous subsection.

We next consider SG-functions, also called functions with SG behavior, that is, amplitudes which depend only on one set of variables in $\mathbb{R}^d$. We denote them by $\text{SG}_r(\mathbb{R}^d)$ and $\text{SG}_r^m(\mathbb{R}^d)$, $r > 0$, respectively, for a general weight $\omega \in \mathcal{P}_r(\mathbb{R}^d)$ and for $\omega(x) = \langle x \rangle^m$. When $\phi: \mathbb{R}^d \to \mathbb{R}^d$, and each component $\phi_j$, $j = 1, \ldots, d$, of $\phi$ belongs to $\text{SG}_r(\mathbb{R}^d)$, we will also write $\phi \in \text{SG}_r(\mathbb{R}^d)$. We use similar notation also for vector-valued SG symbols and amplitudes. The next two lemmas, dealing with compositions of SG amplitudes with functions with SG behavior, can easily be proved by induction.

**Lemma 1.6.** Let $f \in \text{SG}_{r_1, r_2, \rho}^{(\omega)}(\mathbb{R}^{3d})$ and $g$ be vector valued in $\mathbb{R}^d$ such that $g \in \text{SG}_{1, 1, 1}^{(0, 0, 0)}(\mathbb{R}^{3d})$ and $\langle g(x_1, x_2, \xi) \rangle \asymp \langle \xi \rangle$. Then $f(x_1, x_2, g(x_1, x_2, \xi)) \in \text{SG}_{r_1, r_2, \rho}^{(\omega)}(\mathbb{R}^{3d})$.

**Remark 1.7.** It is obvious that the requirements for $g$ in Lemma 1.6 need to be satisfied on supp $f$ only. By Lemma 1.6 it immediately follows

$$f \in \text{SG}_{r, s, \rho}^{(\omega)}(\mathbb{R}^{3d}), g_1 \in \text{SG}_{1, 1, 1}^{(0, 0, 0)}(\mathbb{R}^{3d}), g_2 \in \text{SG}_{1, 1, 1}^{(0, 0, 0)}(\mathbb{R}^{3d}), g_3 \in \text{SG}_{1, 1, 1}^{(0, 0, 0)}(\mathbb{R}^{3d}):
\langle g_1(x_1, x_2, \xi) \rangle \asymp \langle x_1 \rangle, \langle g_2(x_1, x_2, \xi) \rangle \asymp \langle x_2 \rangle, \langle g_3(x_1, x_2, \xi) \rangle \asymp \langle \xi \rangle
\Rightarrow f(g_1(x_1, x_2, \xi), g_2(x_1, x_2, \xi), g_3(x_1, x_2, \xi)) \in \text{SG}_{r_1, r_2, \rho}^{(\omega)}(\mathbb{R}^{3d}).$$

(1.13)

**Lemma 1.8.** Let $\phi: \mathbb{R}^d \to \mathbb{R}^d$ be smooth on $\mathbb{R}^d$ and such that $\langle \phi(x) \rangle \asymp \langle x \rangle$. Moreover, assume that, for any multiindex $\alpha$ such that $|\alpha| = 1$, $\phi^{(\alpha)}(x) = a_\alpha(\phi(x))$ with $a_\alpha(x) \in \text{SG}_{1}^{(0)}(\mathbb{R}^d)$. Then $\phi \in \text{SG}_{1}^{(1)}(\mathbb{R}^d)$.

**Lemma 1.9.** Let $\phi \in \text{SG}_{1}^{(1)}(\mathbb{R}^d)$ be vector-valued in $\mathbb{R}^d$, invertible and such that $|\det \phi'(x)| \geq \varepsilon > 0$. Then, the inverse $\phi^{-1}$ satisfies, for any
multiindex $\alpha$ such that $|\alpha| = 1$,

$$(\phi^{-1})'(y) = a_\alpha(\phi^{-1}(y))$$

with $a_\alpha(x) \in SG^0_1(R^d)$, and this also implies $\langle \phi(x) \rangle \asymp \langle x \rangle$.

**Remark 1.10.** The hypotheses of Lemma 1.9 imply that $\det \phi'(x)$ is an elliptic symbol belonging to $SG^0_1$, and this obviously implies, by the general properties of the calculus, that all the entries of $(\phi'(x))^{-1}$ belong to $SG^0_1$. The result on the first derivatives of $\phi^{-1}$ is then a consequence of the inverse function theorem. Lemma 1.6 and Lemma 1.8 give then immediately by writing $y = \phi(x)$.

Let $\phi = \phi(x_1, x_2, \xi) \in SG^{0,0,1}_{1,1,1}(R^{3d})$ be vector-valued in $R^d$. In several situations we consider $\phi$ as a function in $\xi$ with parameters $x_1, x_2$. If $\eta = \phi(x_1, x_2, \xi)$ can be globally solved with respect to $\xi$, then we write $\xi = \phi^{-1}(x_1, x_2, \eta)$. If $\phi$ satisfies the conditions in the subsequent Proposition 1.11, we say that $\phi$ and $\phi^{-1}$ are SG diffeomorphisms with $SG^0$ parameter dependence.

**Proposition 1.11.** Let $\phi = \phi(x_1, x_2, \xi) \in SG^{0,0,1}_{1,1,1}(R^{3d})$ be vector-valued in $R^d$ and satisfy

$$\langle \phi(x_1, x_2, \xi) \rangle \asymp \langle \xi \rangle \quad \text{and} \quad |\phi'_\xi(x_1, x_2, \xi)| \geq \varepsilon > 0.$$ 

If $\eta = \phi(x_1, x_2, \xi)$ can be globally solved with respect to $\xi$, then $\phi^{-1} \in SG^{0,0,1}_{1,1,1}(R^{3d})$.

**Proof.** $\phi^{-1}$ satisfies the required estimates with respect to $\eta$ in view of Lemmas 1.8 and 1.9. For what concerns the estimates with respect to $x_1$ and $x_2$, it is enough to use the well-known result about the derivatives of functions defined implicitly and an inductive process, completely analogous to that which can be used to prove Lemma 1.8.

**Definition 1.12.** The sets $\Xi^\Delta(k)$, $k > 0$, of the SG-compatible cut-off functions along the diagonal of $R^d \times R^d$, consist of all $\chi = \chi(x, y) \in SG^{0,0}_{1,1}(R^{2d})$ such that

$$|y - x| \leq k\langle x \rangle/2 \implies \chi(x, y) = 1,$$

$$|y - x| > k\langle x \rangle \implies \chi(x, y) = 0.$$ 

(1.14)

If not otherwise stated, we always assume $k \in (0, 1)$.

$\Xi(R)$ with $R > 0$ will instead denote the sets of all SG-compatible 0-excision functions, namely, the set of all $\phi = \phi(x, \xi) \in SG^{0,0}_{1,1}(R^{2d})$ such that

$$|x| + |\xi| \geq R \implies \phi(x, \xi) = 1,$$

$$|x| + |\xi| \leq R/2 \implies \phi(x, \xi) = 0.$$ 

(1.15)

By a standard construction, it is easy to prove that the sets $\Xi^\Delta(k)$ and $\Xi(R)$ are non-empty, for any $k, R > 0$. 
2. Symbolic calculus for generalised Fourier Integral Operators of SG type

2.1. Phase functions of SG type. We recall the definition of the class of admissible phase functions in the SG context, as it was given in [12]. We then observe that the subclass of regular phase functions generates (parameter-dependent) mappings of \( \mathbb{R}^d \) onto itself, which turn out to be SG diffeomorphisms with SG\(^0\) parameter-dependence. Finally, we define some regularizing operators, which are used to prove the properties of the SG Fourier integral operators introduced in the next subsection.

**Definition 2.1.** We will call a phase function or simply phase any real-valued \( \varphi \in \text{SG}^1_1(\mathbb{R}^{2d}) \) satisfying

\[
\langle \varphi'(x,\xi) \rangle \asymp \langle x \rangle \quad \text{and} \quad \langle \varphi''_x(x,\xi) \rangle \asymp \langle \xi \rangle.
\]

We denote by \( \mathfrak{F} \) the set of all such phases. Moreover, the phase \( \varphi \) is called regular, if \( \left| \det(\varphi''_{xx}(x,\xi)) \right| \geq c \) for some \( c > 0 \) and all \( x, \xi \in \mathbb{R}^d \).

The set of all regular phases is denoted by \( \mathfrak{F}^r \).

We observe that a regular phase function \( \varphi \) defines two globally invertible mappings, namely \( \xi \mapsto \varphi'_x(x,\xi) \) and \( x \mapsto \varphi'_\xi(x,\xi) \), in view of the following abstract results (see, e.g., [2], page 221).

**Theorem 2.2.** Assume that \( \phi \in C^1(X,Y) \) with \( X \) and \( Y \) Banach spaces. Then \( \phi \) is a diffeomorphism of \( X \) onto \( Y \) if and only if \( \phi \) is proper and \( \phi'(x) \) is a linear homeomorphism for each \( x \in X \).

**Theorem 2.3.** If \( X \) and \( Y \) are finite dimensional Banach spaces and \( \phi \in C^0(X,Y) \), then \( \phi \) is proper if and only if it is coercive, i.e., \( \lim_{\|x\| \to +\infty} \|f(x)\| = +\infty \).

As it is easy to see, the mappings generated by the first derivatives of the admissible regular phase functions satisfy the hypotheses of Theorems 2.2 and 2.3. In view of Lemmas 1.8 and 1.9 we finally conclude:

**Proposition 2.4.** Let \( \varphi \in \mathfrak{F}^r \). Then, \( \xi \mapsto \varphi'_x(x,\xi) \) and \( x \mapsto \varphi'_\xi(x,\xi) \) are globally invertible maps from \( \mathbb{R}^d \) to itself, which give rise to SG diffeomorphisms with SG\(^0\) parameter dependence.

In the following lemma we establish mapping properties for the operators \( R_1 \) and \( \mathcal{D} \), which, for \( \varphi \in \mathfrak{F} \), are defined by the formulas

\[
R_1 = \frac{1 - \Delta_{\xi}}{\langle \varphi'(x,\xi) \rangle^2 - i\Delta_{\xi} \varphi(x,\xi)},
\]

and

\[
\mathcal{D} q = \frac{q}{\langle \varphi'_\xi \rangle^2 - i\Delta_{\xi} \varphi}.
\]

**Lemma 2.5.** Let \( \varphi \in \mathfrak{F} \) and let \( R_1 \) and \( \mathcal{D} \) be defined by (2.2) and (2.3). Then the following is true:
Lemma 2.6. The next lemma is easily proved by induction.

(1) \( R_1 e^{i\varphi} = e^{i\varphi}; \)
(2) \( R_1 = \mathcal{D}(1 - \Delta_\varepsilon); \)
(3) for any positive integer \( l, \)
\[
\left( R_1 \right)^l = (1 - \Delta_\varepsilon) \mathcal{D} \ldots (1 - \Delta_\varepsilon) \mathcal{D} = \mathcal{D}^l + Q(\mathcal{D}, \Delta_\varepsilon),
\]
(2.4) \( l \) times

where \( Q \) is a suitable polynomial of total degree \( 2l \) in the variables \( \mathcal{D}, \Delta_\varepsilon \), whose terms contain exactly \( l \) factors equal to \( \mathcal{D} \) and at least one equal to \( \Delta_\varepsilon \). Then we have, for any admissible \( \omega \), the (continuous) maps
\[
\mathcal{D}^l : \text{SG}^{(\omega)}_{r,\rho}(\mathbb{R}^{2d}) \to \text{SG}^{(\omega_\varphi - 2_\alpha \theta)}_{r,\rho}(\mathbb{R}^{2d}),
\]
\[Q(\mathcal{D}, \Delta_\varepsilon) : \text{SG}^{(\omega)}_{r,\rho}(\mathbb{R}^{2d}) \to \text{SG}^{(\omega_\varphi - 2_\alpha \theta - 2)}_{r,\rho}(\mathbb{R}^{2d}).
\]

The next lemma is easily proved by induction.

Lemma 2.6. \( \varphi \in \text{SG}_{1,1}^{1,1}(\mathbb{R}^{2d}) \implies \partial_\xi^\alpha \partial_\beta e^{i\varphi(x,\xi)} = b_\alpha^\beta(x, \xi) e^{i\varphi(x, \xi)}, \) with \( b_\alpha^\beta \in \text{SG}_{1,1}^{1,1}(\mathbb{R}^{2d}). \)

2.2. Generalised Fourier integral operators of SG type. In analogy with the definition of generalized SG pseudo-differential operators, recalled in Subsection 1.1, we define the class of Fourier integral operators we are interested in terms of their distributional kernels. These belong to a class of tempered oscillatory integrals, studied in [21].

Definition 2.7. Let \( \omega \in \mathcal{D}_{r,\rho}(\mathbb{R}^{2d}), \ r, \rho \geq 0, \ r + \rho > 0, \ \varphi \in \mathfrak{F}, \ a, b \in \text{SG}_{r,\rho}^{(\omega)}(\mathbb{R}^{2d}). \) The generalized Fourier integral operators of SG type I (SG FIOs of type I) with phase \( \varphi \) and amplitude \( a \) are the linear continuous operators \( A = \text{Op}_\varphi(a) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d) \) with distribution kernel \( K_A \in \mathcal{S}'(\mathbb{R}^{2d}) \) given by
\[
K_A(x, y) = (2\pi)^{-d/2}(\mathcal{F}_2 \tilde{a})(x, y), \text{ with } \tilde{a}(x, \xi) = e^{i\varphi(x, \xi)}a(x, \xi).
\]

The generalized Fourier integral operators of SG of type II (SG FIOs of type II) with phase \( \varphi \) and amplitude \( b \) are the linear continuous operators \( B : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d) \) with distribution kernel \( K_B \in \mathcal{S}'(\mathbb{R}^{2d}) \) given by
\[
K_B(x, y) = (2\pi)^{-d/2}(\mathcal{F}_2^{-1} \tilde{b})(y, x), \text{ with } \tilde{b}(y, \xi) = e^{-i\varphi(y, \xi)}b(y, \xi).
\]

For functions \( u \in \mathcal{S}(\mathbb{R}^d) \) we then have, for SG FIOs of type I,
\[
Au(x) = \text{Op}_\varphi(a)u(x) = (2\pi)^{-d} \int e^{i\varphi(x, \xi)} a(x, \xi) (\mathcal{F}u)(\xi) \, d\xi;
\]
(2.5) and, respectively, for SG FIOs of type II,
\[
Bu(x) = \text{Op}_\varphi^*(b)u(x) = (2\pi)^{-d} \int \int e^{i((x, \xi) - \varphi(y, \xi))} \overline{b(y, \xi)} u(y) \, dy \, d\xi.
\]
(2.6)
Remark 2.8. Given a generalized SG pseudo-differential of Fourier integral operator $Q$, in the sequel we denote by $Q^*$ its formal $L^2(\mathbb{R}^d)$ adjoint. It is straightforward to observe that the SG FIOs of type II are the formal $L^2(\mathbb{R}^d)$-adjoints of the SG FIOs of type I. We will denote a SG FIO of type I with phase $\varphi$ and amplitude $a$ by $\text{Op}_{\varphi}(a)$. Then, a SG FIO of type II with phase $\varphi$ and amplitude $b$ will be denoted by $\text{Op}^*_{\varphi}(b) = \text{Op}_{\varphi}(b)^*$. For a function $c(x, \xi)$, define $(^tc)(x, \xi) = c(\xi, x)$, $(c^*)(x, \xi) = c(\xi, x)$. Clearly, for any $\omega \in \mathcal{S}_{\rho, r}(\mathbb{R}^{2d})$, $^t\omega = \omega^*$ is also an admissible weight, and, of course, $^t\omega \in \mathcal{S}_{\rho, r}(\mathbb{R}^{2d})$. Similarly, for arbitrary $\varphi \in \mathfrak{R}$ and $a \in \text{SG}^{(\omega)}_{\rho, r}(\mathbb{R}^{2d})$, we have $^t\varphi \in \mathfrak{R}$ and $^t a, a^* \in \text{SG}^{(\omega)}_{\rho, r}(\mathbb{R}^{2d})$. Definition 2.7 then also implies

$$B = (2\pi)^d \mathcal{F}^{-1} \circ A_{-^t\varphi, b} \circ \mathcal{F}^{-1} \iff A_{\varphi, a} = (2\pi)^{-d} \mathcal{F} \circ B_{-^t\varphi, a^*} \circ \mathcal{F},$$

(2.7)

Theorem 2.9. The generalized SG Fourier integral operators of type I and type II, introduced in Definition 2.7, are linear and continuous from $\mathcal{S}(\mathbb{R}^d)$ to itself, and extendable to linear continuous operators from $\mathcal{S}'(\mathbb{R}^d)$ to itself.

Proof. Consider first the type I operators in (2.5). By differentiation under the integral sign and the properties of the symbol $a$, it is enough to show that, for any $u \in \mathcal{S}(\mathbb{R}^d)$, $|Au(x)| \lesssim p_k(u)$, where $p_k(u)$ is one of the seminorms which generate the topology of $\mathcal{S}(\mathbb{R}^d)$. By a regularization argument, using the operator $R_1$ defined in (2.4), in view of Lemma 2.5 we find, for arbitrary $l$ and $\mathcal{D} = (\varphi_\xi^2 - i\Delta_\xi)^l_{\varphi, a}$,

$$Au(x) = (2\pi)^{-d} \int e^{i\varphi(x, \xi)} (^t R_1)^l [a(x, \xi)(\mathcal{F} u)(\xi)] d\xi =$$

$$= (2\pi)^{-d} \int e^{i\varphi(x, \xi)} \left\{ \frac{a(x, \xi)}{(\mathcal{D}(x, \xi))^l} (\mathcal{F} u)(\xi) + Q(\mathcal{D}, \Delta_\xi) [a(x, \xi)(\mathcal{F} u)(\xi)] \right\} d\xi =$$

$$= (2\pi)^{-d} \int e^{i\varphi(x, \xi)} \left[ \frac{a(x, \xi)}{(\mathcal{D}(x, \xi))^l} (\mathcal{F} u)(\xi) + \sum_{|\gamma| \leq 2l} c_\gamma(x, \xi) D^\gamma (\mathcal{F} u)(\xi) \right] d\xi$$

with coefficients $c_\gamma \in \text{SG}_{\rho, r}^{(\omega - 2l - 2)}(\mathbb{R}^{2d})$ depending only on $a$ and $\mathcal{D}$, and $\frac{a(x, \xi)}{(\mathcal{D}(x, \xi))^l} \in \text{SG}_{\rho, r}^{(\omega - 2l - 2)}(\mathbb{R}^{2d})$. Since $\omega$ is polynomially bounded and $u \in \mathcal{S}(\mathbb{R}^d)$, it is then easily seen that, for any $l$ and a suitable $m \in \mathbb{R}$,

$$|Au(x)| \lesssim \langle x \rangle^{m-2l} p_k(u) \int \langle \xi \rangle^{-d-1} d\xi \lesssim p_k(u),$$

as desired, choosing $l$ and $k$ large enough. The continuity on $\mathcal{S}(\mathbb{R}^d)$ is then of course a consequence of the Closed Graph Theorem. The $\mathcal{S}'$-continuity of the operators of type II follows by a completely similar argument. The continuity on $\mathcal{S}'(\mathbb{R}^d)$ of operators of type I and type II then follows by duality, recalling Remark 2.8. \qed
2.3. Composition with SG pseudodifferential operators. The composition theorems presented in this and the subsequent subsections are variants of those originally appeared in [12]. We include anyway some of their proofs, focusing on the role of the parameters in the classes of the involved amplitudes and symbols, as well as on the different notion of asymptotic expansions needed here (cfr. Appendix A).

Theorem 2.10. Let \( \varphi \in \mathfrak{F} \), \( a \in \text{SG}_{r_1,\rho_1}(\mathbb{R}^{2d}) \), and \( p \in \text{SG}_{r_2,\rho_2}(\mathbb{R}^{2d}) \), \( \rho_2 \geq 1/2 \), \( r_1 + \rho_2 > 1/2 \). Then, the composed operator \( H = \text{Op}(p) \circ \text{Op}_a(\varphi) \) is, modulo smoothing remainders, a SG FIO of type I. In fact, \( H = \text{Op}_a(\varphi)(h) \) with amplitude \( h \in \text{SG}_{r,\rho}(\mathbb{R}^{2d}) \), \( \omega = \omega_1\omega_2 \), \( r = \min\{r_1, r_2, 1\} \), \( \rho = \min\{\rho_1, \rho_2, 1\} \), admitting the asymptotic expansion

\[
h(x, \xi) \sim \sum_{|\alpha|} \frac{j!}{\alpha!} (D_\xi^a p)(x, \varphi'_x(x, \xi)) D_y^a[e^{i\psi(x,y,\xi)} a(y, \xi)]_{y=x}, \tag{2.8}
\]

where

\[
\psi(x, y, \xi) = \varphi(y, \xi) - \varphi(x, \xi) - \langle y - x, \varphi'_x(x, \xi) \rangle. \tag{2.9}
\]

We split the proof of Theorem 2.10 into various steps. We begin considering the derivatives of the exponential functions appearing in (2.8). In fact, Lemma 2.11 is a special case of the so-called Fàa di Bruno formula, and can be proved by induction. For the proof of Lemma 2.13 see [12]. Then, in view of these two results, in Lemma 2.14 we can prove that the terms which appear in the right-hand side of (2.8) indeed give a generalized SG asymptotic expansion, in the sense described in Definition 1.3 and Appendix A.

Lemma 2.11. Let \( \psi(x, y, \xi) \) be defined as in (2.9). Then we have, for \( |\alpha| \geq 1 \),

\[
D_y^a e^{i\psi} = \tau_a e^{i\psi} = \left[ \sum_j c_j (\varphi'_y - \varphi'_x)^{\theta_j} \prod_{k=1}^{N_f} \partial_{y_k}^j \varphi \right] e^{i\psi} = \left[ (\varphi'_y - \varphi'_x)^{\alpha} + \sum_j c_j (\varphi'_y - \varphi'_x)^{\theta_j} \prod_{k=1}^{N_f} \partial_{y_k}^j \varphi + \sum_j c_j \prod_{k=1}^{N_f} \partial_{y_k}^j \varphi \right] e^{i\psi}, \tag{2.10}
\]

with suitable constants \( c_j \in \mathbb{R} \) and multindeces \( \theta_j, \beta_{jk} \) such that

\[
\theta_j + \sum_{k=1}^{N_f} \beta_{jk} = \alpha, \quad |\beta_{jk}| \geq 2. \tag{2.11}
\]

In (2.10), \( \varphi'_x = \varphi'_x(x, \xi) \), \( \varphi'_y = \varphi'_y(y, \xi) \) and \( \partial_y^j \varphi = \partial_y^j \varphi(y, \xi) \) is to be understood.
Remark 2.12. Note that, by (2.11), we have, in any term of (2.10),

\[ |\alpha| \geq \sum_{k=1}^{N} |\beta_{j_k}| \geq 2N_j \Rightarrow N_j \leq \frac{|\alpha|}{2}. \]  

(2.12)

Lemma 2.13. Let \( \psi \) be defined as in Lemma 2.11 with \( \varphi \in SG^{1,1}(\mathbb{R}^{2d}) \). Then we have, for any multi index \( \alpha \) such that \( |\alpha| \geq 1 \),

\[ \partial_{\alpha}^{d} e^{i\psi(x,y,\xi)} \big|_{y=x} \in SG^{[-|\alpha|/2,|\alpha|/2]}(\mathbb{R}^{2d}) \]

\[ \Rightarrow \partial_{\alpha}^{d} e^{i\psi(x,y,\xi)} \big|_{y=x} \lesssim \partial_{-|\alpha|/2,|\alpha|/2}(x,\xi). \]

Moreover, \( |y - x| \leq \varepsilon_1(x), \varepsilon_1 \in (0, 1) \), implies that each summand in the right-hand side of (2.11) can be estimated by the product of a suitable power \( |y - x|^\mu \) times a weight of the form \( \langle x \rangle^m \langle \xi \rangle^n \), with \( 0 \leq \mu \leq \mu \leq |\alpha|, m \leq \frac{|\alpha|}{2} \).

Lemma 2.14. If \( \varphi \in \mathfrak{F}, \psi \) is defined as in Lemma 2.11 \( a \in SG^{(\omega)}(\mathbb{R}^{2d}) \), and \( p \in SG^{(\omega_2)}(\mathbb{R}^{2d}), \rho_2 \geq 1/2, r_1 + r_2 > 1/2 \), the expression

\[ \sum_{\alpha} \frac{j_{|\alpha|}}{|\alpha|!} c_\alpha(x,\xi) = \sum_{\alpha} \frac{j_{|\alpha|}}{|\alpha|!} (D_\xi^\alpha p)(x, \varphi'_x(x,\xi)) D_y^\alpha [e^{i\psi(x,y,\xi)} a(y,\xi)] \big|_{y=x} \]

(2.13)

is a generalized SG asymptotic expansion which defines an amplitude \( h \in SG^{(\omega)}(\mathbb{R}^{2d}), \omega = \omega_1 \omega_2, r = \min\{r_1, r_2, 1\}, \rho = \min\{\rho_1, \rho_2, 1\} \), modulo a remainder of the type described in (1.12).

Proof. Using Lemma 2.13, the hypothesis \( a \in SG^{(\omega_1)} \), and the properties of the symbolic calculus, we see that

\[ D_y^\alpha [e^{i\psi(x,y,\xi)} a(y,\xi)] \big|_{y=x} = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} D_y^\beta e^{i\psi(x,y,\xi)} D_y^{\alpha-\beta} a(y,\xi) \big|_{y=x} \]

\[ \in \sum_{0 \leq \beta \leq \alpha} SG^{(\omega_{1} - |\beta|/2,|\beta|/2)} \cdot SG^{(\omega_{1} - r_1(|\alpha| - |\beta|),0)} \]

\[ = \sum_{0 \leq \beta \leq \alpha} SG^{(\omega_{1} - r_1(|\alpha| + |r_1 - 1/2| |\beta|),|\beta|/2)} \cdot SG^{(\omega_{1} - r_1(|\alpha| + |r_1 - 1/2| |\beta|),|\beta|/2)} \]

\[ \in SG^{(\omega_{1} - \min\{r_1, 1/2\} |\alpha|,|\alpha|/2)} \cdot SG^{(\omega_{1} - \min\{r_1, 1/2\} |\alpha|,|\alpha|/2)} \cdot \]

Using \( \varphi \in \mathfrak{F} \), in particular (2.11), we also easily have:

\( (D_\xi^\alpha p)(x, \varphi'_x(x,\xi)) \in SG^{(\omega_{2} - \rho_2 |\alpha|)} \).

Summing up, we obtain, for any multi index \( \alpha \),

\[ c_\alpha(x,\xi) \in SG^{(\omega_{2} - \rho_2 |\alpha|)} \cdot \]

which proves the lemma, by the hypotheses and the general properties of the symbolic calculus. \( \square \)
The next two lemmas are well-known, see, e.g., [11, 12], and can be proved by induction on \( l \).

**Lemma 2.15.** Let us consider the operator

\[
R_2 = \sum_{j=1}^{d} \frac{x_j - y_j}{|x - y|^2} D_{yj},
\]  
(2.14)

\( R_2 \) is well defined on \( \text{supp}(1 - \chi) \) for \( \chi \in \Xi^{\Delta}(k) \), \( k \in (0, 1) \). On that set it has the properties \( R_2 e^{i(x-y, \eta)} = e^{i(x-y, \eta)} \) and, for any positive integer \( l \),

\[
\left( t R_2 \right)^l = \sum_{|\theta|=l} c_\theta \frac{(x-y)^\theta}{|x-y|^{2l}} D_{\eta}^\theta,
\]  
(2.15)

for suitable coefficients \( c_\theta \). Moreover, it is possible to show that

\[
|y - x| \geq \varepsilon_1(x) \Rightarrow \exists \varepsilon'_1 > 0 : |y - x| \geq \varepsilon'_1(y)
\]  
[2.15]

\[
\Rightarrow |y - x| \gtrsim \langle x \rangle + \langle y \rangle \geq (\langle x \rangle \langle y \rangle)^{\frac{1}{2}}.
\]

**Lemma 2.16.** Let \( f = f(y) \) be a smooth function such that \(|f_y(y)| \neq 0\) and let us set

\[
R_3 = \frac{1}{|f_y(y)|^2} \sum_{k=1}^{d} f_{yk}(y) D_{yk},
\]  
(2.16)

so that \( R_3 e^{i\ell} = e^{i\ell} \). Then, for any positive integer \( l \),

\[
\left( t R_3 \right)^l = \frac{1}{|f_y(y)|^4} \sum_{|\alpha| \leq l} P_\alpha(y) D_\alpha,
\]  
(2.17)

with

\[
P_\alpha = \sum c_{\gamma_1 \ldots \gamma_l}^{\alpha} (f_y)^\gamma D_\gamma f \ldots D_\gamma f.
\]  
(2.18)

In (2.18) we have

\[
|\gamma| = 2l,
\]

\[
|\delta_j| \geq 1, \sum_{j=1}^{l} |\delta_j| + |\alpha| = 2l,
\]  
(2.19)

and \( c_{\gamma_1 \ldots \gamma_l}^{\alpha} \) are suitable constants.

**Lemma 2.17.** Let \( \varphi, a, p, r, \rho \) be as in Theorem 2.10 and let \( \chi \in \Xi^{\Delta}(\varepsilon_1) \). Then, the function \( h_2 = h_2(x, \xi) \) defined by

\[
h_2(x, \xi) = \int \int e^{i\varphi(y, \xi) - \varphi(x, \xi) - (y - x, \eta)} (1 - \chi(x, y)) a(y, \xi) p(x, \eta) dy d\eta
\]

is an element of \( \mathcal{S}(\mathbb{R}^{2d}) \).
Proof. We make use of the operators $\tilde{R}_1 = \frac{1-\Delta_y}{(\varphi_y(y,\xi) - \varphi(x,\xi) - (y-x,\eta))}$, which has properties similar to those of the operator $R_1$ defined in (2.2), and $R_2$, defined in (2.14). For any couple of positive integers $l_1, l_2$ we have

$$h_2(x, \xi) = \int e^{i(\varphi(y,\xi) - \varphi(x,\xi) - (y-x,\eta))} (1 - \chi(x,y)) a(y, \xi) \left[ (i R_2)^{l_2} p \right] (x, \eta) \, dy d\eta$$

$$= \int e^{i(\varphi(y,\xi) - \varphi(x,\xi) + (x,\eta))} (i \tilde{R}_1)^{l_1} e^{-i(y,\eta)} q(x, y, \xi, \eta) \, dy d\eta$$

(2.20)

where we have set

$$q(x, y, \xi, \eta) = (1 - \chi(x,y)) a(y, \xi) \left[ (i R_2)^{l_2} p \right] (x, \eta).$$

By Lemma (2.15) we find

$$\partial_y^\alpha q(x, y, \xi, \eta) =$$

$$= \partial_y^\alpha \left[ (1 - \chi(x,y)) a(y, \xi) \sum_{|\theta| = l_2} c_\theta \frac{(x-y)\theta}{|x-y|^{2l_2}} (D_\theta^\eta p)(x, \eta) \right]$$

$$= \sum_{|\theta| = l_2} (D_\theta^\eta p)(x, \eta) \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \alpha} \frac{\alpha_1!}{\alpha_1! \alpha_2! \alpha_3!} \left( \delta_{|\alpha_1|,0} - (\partial_y^{\alpha_1} \chi)(x,y) \right) \cdot (\partial_y^{\alpha_2} a)(y, \xi) \sum_{\beta_1 + \beta_2 = \alpha_3} \frac{\alpha_3!}{\beta_1! \beta_2!} c_{\beta_1} (x-y)^{\beta_1} \frac{P_{\beta_2}(x-y)}{|x-y|^{2(\rho+|\beta_2|)}}$$

with $P_{\beta_2}$ homogeneous polynomial of degree $|\beta_2|$, while $\delta_{|\alpha_1|,0} = 1$ for $\alpha_1 = 0$, $\delta_{|\alpha_1|,0} = 0$ otherwise. Then we obtain

$$\partial_y^\alpha q(x, y, \xi, \eta) \lesssim$$

$$\lesssim \sum_{|\theta| = l_2} \omega_2(x, \eta) \partial_{0, -\rho \vartheta} \theta_{|\alpha_1|,0} (x, \eta) \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \alpha} \langle y \rangle^{-|\alpha_1|} \omega_1(y, \xi) \partial_{-r_1, |\alpha_2|,0}$$

$$\lesssim \sum_{\beta_1 + \beta_2 = \alpha_3} \langle y \rangle^{-|\beta_1| + |\beta_2| - 2l_2 - 2|\beta_2|} \partial_{-\min(\nu, 1), |\alpha_1| + |\alpha_2|,0}$$

$$\lesssim \omega_1(y, \xi) \omega_2(x, \eta) \cdot \partial_{0, -\rho_1 l_2}(x, \eta) \cdot \partial_{-\min(\nu, 1), |\alpha_1| + |\alpha_2|,0}.$$

In view of the fact that $|y - x| \geq \frac{\xi_1}{2} \langle x \rangle$ on $\text{supp}(q)$, from (2.15) we also obtain

$$|y - x| \geq \frac{\xi_1}{2} \langle x \rangle \Rightarrow |y - x| \gtrsim \langle y \rangle \Rightarrow |y - x| \gtrsim \langle x \rangle + \langle y \rangle \gtrsim \left( \langle x \rangle \langle y \rangle \right)^{\frac{1}{3}},$$

and we can conclude

$$\partial_y^\alpha q(x, y, \xi, \eta) \lesssim \omega_1(y, \xi) \omega_2(x, \eta) \cdot$$

$$\partial_{0, -\rho_1 l_2}(x, \eta) \partial_{-l_2/2, -l_2/2}(x, y) \cdot \partial_{-\min(\nu, 1/2), |\alpha_1|,0} (y, \xi).$$

(2.21)
Finally, recalling that the admissible weight functions are polynomially moderate and \( p_2 \geq 1/2 \), we see that \( q \) “has order as low as we wish” with respect to \( x, y, \eta \), provided \( l_2 \) is chosen large enough. Moreover, when derivatives with respect to \( x, y, \eta \) with respect to \( \alpha, \beta \) \( S \), we find \( \omega \) moderate and \( \rho \).

We now estimate the integrand of \( (2.20) \). As shown in Lemma \( 2.5 \), we have:

\[
\left( l_1 R \right)^l \left[ e^{-i(y, \eta)} q(x, y, \xi, \eta) \right] = \]

\[
e^{-i(y, \eta)} \frac{q(x, y, \xi, \eta)}{(\langle \varphi'(y, \xi) \rangle^2 - i \Delta \varphi(y, \xi))^{l_1}} + Q(\mathcal{D}, \Delta_y) \left[ e^{-i(y, \eta)} q(x, y, \xi, \eta) \right],
\]

as in \( (2.24) \). Due to the presence of the exponential in the argument of \( Q(\mathcal{D}, \Delta_y) \), in the second term there are powers of \( \eta \) of height not greater than \( 2l_1 \). Owing to \( (2.21) \) we finally find

\[
x^{\alpha} \xi^\beta h_2(x, \xi) \lesssim \langle x \rangle^{-\frac{d}{2} + |\alpha|} \langle \xi \rangle^{-2l_1 + |\beta|} \int \omega_1(y, \xi) \langle y \rangle^{-\frac{d}{2}} dy \int \omega_2(x, \eta) \langle \eta \rangle^{-\frac{d}{2} + 2l_1} d\eta \lesssim 1,
\]

for all multiindices \( \alpha, \beta \), provided that \( l_1 \) and \( l_2 \) are large enough, since \( \omega_1 \) and \( \omega_2 \) are polynomially bounded (\( l_1 \) is chosen first, then \( l_2 \) fixed accordingly). Differentiating \( h_2 \) and multiplying it by powers of \( x \) and \( \xi \) would give a linear combination of expressions similar to \( (2.20) \), with different \( \omega_1, \omega_2 \) and parameters for the involved symbols, which could then be similarly estimated by constants. The proof is complete. \( \Box \)

**Proof of Theorem 2.10.** Writing explicitly \( \text{Op}(p) \circ \text{Op}_\varphi(a) u(x) \) with \( u \in \mathcal{F} \), we find

\[
\text{Op}(p) \circ \text{Op}_\varphi(a) u(x) =
\]

\[
= (2\pi)^{-d} \int e^{i(x, \xi)} p(x, \xi) \int e^{-i(y, \xi)} (2\pi)^{-d} \int e^{i\varphi(y, \eta)} a(y, \eta) (\mathcal{F} u)(\eta) d\eta dy d\xi
\]

\[
= (2\pi)^{-d} \int e^{i\varphi(x, \eta)} \left[ (2\pi)^{-d} \int e^{i\varphi(y, \eta) - \varphi(x, \xi) - (y - x, \xi)} a(y, \eta) p(x, \xi) dy d\xi \right] (\mathcal{F} u)(\eta) d\eta
\]

\[
= (2\pi)^{-d} \int e^{i\varphi(x, \xi)} \left[ (2\pi)^{-d} \int e^{i\varphi(y, \xi) - \varphi(x, \xi) - (y - x, \eta)} a(y, \xi) p(x, \eta) dy d\eta \right] (\mathcal{F} u)(\xi) d\xi
\]

\[
= (2\pi)^{-d} \int e^{i\varphi(x, \xi)} h(x, \xi)(\mathcal{F} u)(\xi) d\xi.
\]

We have to show

\[
h(x, \xi) = (2\pi)^{-d} \int e^{i\varphi(y, \xi) - \varphi(x, \xi) - (y - x, \eta)} a(y, \xi) p(x, \eta) dy d\eta \in \text{SG}_{r, \rho}^{(\omega)},
\]

with \( \omega, r, \rho \) given in the statement of the theorem. Choosing \( \chi \in \Xi^\Lambda(\varepsilon_1) \), with \( \varepsilon_1 \in (0, 1) \) fixed below (after equation (2.33)), we can
write
\[ h(x, \xi) = h_1(x, \xi) + h_2(x, \xi) \]
\[ = (2\pi)^{-d} \int e^{i(x(y, \xi) - \varphi(x, \xi) - (y-x, \eta))} \chi(x, y) a(y, \xi) p(x, \eta) \, dy \, d\eta \]
\[ + (2\pi)^{-d} \int e^{i(x(y, \xi) - \varphi(x, \xi) - (y-x, \eta))} (1 - \chi(x, y)) a(y, \xi) p(x, \eta) \, dy \, d\eta, \]

with \( h_2 \in \mathscr{S} \), by Lemma 2.17. We will prove \( h_1 \in \text{SG}^{(\omega)} \) by showing that it admits the asymptotic expansion already studied in Lemma 2.14.

In fact, setting \( \eta = \varphi'_x(x, \xi) + \theta \) in the expression of \( h_1 \) and using the Taylor expansion
\[ p(x, \eta) = \sum_{|\alpha| < M} \frac{\theta^\alpha}{\alpha!} (\partial^\alpha_x p)(x, \varphi'_x(x, \xi)) + \sum_{|\alpha| = M} \frac{M}{\alpha!} \theta^\alpha t_{\alpha}(x, \xi, \theta) \]
\[ t_{\alpha}(x, \xi, \theta) = \int_0^1 (1 - t)^{M-1} (\partial^\alpha_x p)(x, \varphi'_x(x, \xi) + t\theta) \, dt, \]
we have:
\[ h_1(x, \xi) = \]
\[ = \sum_{|\alpha| < M} \frac{(\partial^\alpha_x p)(x, d_x \varphi(x, \xi))}{\alpha!} \mathcal{F}_{\theta \rightarrow x}^{-1} [\theta^\alpha \mathcal{F}_{y \rightarrow \theta} (e^{i\psi(x, y, \xi)} \chi(x, y) a(y, \xi))] \]
\[ + \sum_{|\alpha| = M} \frac{M}{\alpha!} \mathcal{F}_{\theta \rightarrow x}^{-1} [\theta^\alpha t_{\alpha}(x, \xi, \theta) \mathcal{F}_{y \rightarrow \theta} (e^{i\psi(x, y, \xi)} \chi(x, y) a(y, \xi))] \]
\[ = \sum_{|\alpha| < M} \frac{(\partial^\alpha_x p)(x, \varphi'_x(x, \xi))}{\alpha!} D_y^{\alpha} [e^{i\psi(x, y, \xi)} \chi(x, y) a(y, \xi)]_{y=x} \]
\[ + \sum_{|\alpha| = M} \frac{M}{\alpha!} \int e^{i(x, \theta)} t_{\alpha}(x, \xi, \theta) \mathcal{F}_{y \rightarrow \theta} [D_y^{\alpha} (e^{i\psi(x, y, \xi)} \chi(x, y) a(y, \xi))] \, d\theta. \]

Now, since every derivative of \( \chi \) vanishes in a neighbourhood of the diagonal of \( \mathbb{R}^d \times \mathbb{R}^d \), and \( \chi(x, x) = 1 \), by an obvious use of the Leibniz rule in the last formula, we can write
\[ h_1(x, \xi) = \sum_{|\alpha| < M} \frac{i^{|\alpha|}}{\alpha!} c_{\alpha}(x, \xi) + \sum_{|\alpha| = M} \frac{M}{\alpha!} T_{\alpha}(x, \xi) \]
where the \( c_{\alpha} \) are the terms of the asymptotic expansion (2.13) and
\[ T_{\alpha}(x, \xi) = \int e^{i(x, \theta)} t_{\alpha}(x, \xi, \theta) \mathcal{F}_{y \rightarrow \theta} [D_y^{\alpha} (e^{i\psi(x, y, \xi)} \chi(x, y) a(y, \xi))] \, d\theta. \]

By the properties of the generalized SG asymptotic expansions, cfr. Appendix [A], to complete the proof we only have to estimate \( T_{\alpha}, |\alpha| = M \). To this aim, let us choose \( \tilde{\chi} \in \text{C}^{\omega}_r(\mathbb{R}^d) \) such that \( \tilde{\chi} \) is identically
equal to 1 in the ball $B_{\varepsilon_2}(0)$ and has support included in the ball $B_{\varepsilon_2}(0)$, $\varepsilon_2 \in (0, 1)$ to be fixed later (after equation (2.22)). Then, $\text{supp} \tilde{\chi}(\frac{\theta}{\langle \xi \rangle \rho_2}) \subset B_{\varepsilon_2(\langle \xi \rangle \rho_2)}(0)$. We split $T_\alpha$ into the sum of the two following integrals:

$$I_1 = \int e^{i(x, \theta) t_\alpha(x, \xi, \theta)} \tilde{\chi} \left( \frac{\theta}{\langle \xi \rangle \rho_2} \right) \mathcal{F}_{y \rightarrow \theta} \left[ D_\alpha (e^{i\psi(x, y, \xi)} \chi(x, y) a(y, \xi)) \right] d\theta;$$

$$I_2 = \int e^{i(x, \theta) t_\alpha(x, \xi, \theta)} \left[ 1 - \tilde{\chi} \left( \frac{\theta}{\langle \xi \rangle \rho_2} \right) \right] \cdot \mathcal{F}_{y \rightarrow \theta} \left[ D_\alpha (e^{i\psi(x, y, \xi)} \chi(x, y) a(y, \xi)) \right] d\theta.$$

**1. Estimate of $I_1$.**

Let us set

$$f_\alpha(x, \xi, \cdot) = \mathcal{F}_{\theta \rightarrow \cdot}^{-1} \left[ t_\alpha(x, \xi, \theta) \tilde{\chi} \left( \frac{\theta}{\langle \xi \rangle \rho_2} \right) \right]$$

and rewrite

$$I_1 = \iint e^{i(x, \theta) t_\alpha(x, \xi, \theta)} \tilde{\chi} \left( \frac{\theta}{\langle \xi \rangle \rho_2} \right) \cdot e^{-i(y, \theta)} D_\alpha (e^{i\psi(x, y, \xi)} \chi(x, y) a(y, \xi)) dy d\theta$$

$$= \int \left[ \int e^{i(x - y, \theta) t_\alpha(x, \xi, \theta)} \tilde{\chi} \left( \frac{\theta}{\langle \xi \rangle \rho_2} \right) d\theta \right] \cdot D_\alpha (e^{i\psi(x, y, \xi)} \chi(x, y) a(y, \xi)) dy$$

$$= \int f_\alpha(x, \xi, x - y) D_\alpha (e^{i\psi(x, y, \xi)} \chi(x, y) a(y, \xi)) dy.$$

Remembering our choice of $\tilde{\chi}$ and $\psi \in \mathfrak{G}$, we have, for any multiindex $\beta$,

$$\partial_\theta^\beta t_\alpha(x, \xi, \theta) \lesssim$$

$$\lesssim \int_0^1 \omega_2(x, \varphi'_x(x, \xi) + t\theta) (\varphi'_x(x, \xi) + t\theta)^{-\rho_2(\langle \alpha \rangle + |\beta|)} dt$$

$$\lesssim \omega_2(x, \xi) (\xi)^{-\rho_2(\langle \alpha \rangle + |\beta|)}.$$

In fact, the presence of $\tilde{\chi}$ in the integrand of $I_1$ and $t \in [0, 1]$ imply $|\theta| \leq \varepsilon_2(\langle \xi \rangle)$ imply $|t\theta| \leq \varepsilon_2(\langle \xi \rangle)$ imply $(\varphi'_x(x, \xi) + t\theta) \approx (\xi).$ We have also, for any multiindices $\alpha, \beta$,

$$|u^\beta f_\alpha(x, \xi, u)| = \left| \mathcal{F}_{\theta \rightarrow u}^{-1} \left[ D_\theta^\beta \left( t_\alpha(x, \xi, \theta) \tilde{\chi} \left( \frac{\theta}{\langle \xi \rangle \rho_2} \right) \right) \right] \right|$$

$$\lesssim \mu(B_{\varepsilon_2(\langle \xi \rangle \rho_2)}(0)) \sup_{\theta \in B_{\varepsilon_2(\langle \xi \rangle \rho_2)}(0)} \left| D_\theta^\beta \left( t_\alpha(x, \xi, \theta) \tilde{\chi} \left( \frac{\theta}{\langle \xi \rangle \rho_2} \right) \right) \right|.$$

\[ (2.23) \]
In view of (2.22),

\[ \partial_\theta ^\beta \left( t_\alpha (x, \xi, \theta) \tilde{\chi} \left( \frac{\theta}{\langle \xi \rangle^{\rho_2}} \right) \right) \lesssim \sum_{\gamma \leq \beta} \left| \partial_\gamma ^\beta \partial_\alpha t_\alpha (x, \xi, \theta) \right| \left| \partial_\theta ^{\beta-\gamma} \tilde{\chi} \left( \frac{\theta}{\langle \xi \rangle^{\rho_2}} \right) \right| \]

\[ \lesssim \sum_{\gamma \leq \beta} \omega_2 (x, \xi) \langle \xi \rangle^{-\rho_2 (|\alpha|+|\gamma|)} \langle \xi \rangle^{\rho_2 (|\gamma|-|\beta|)} \]

\[ \lesssim \omega_2 (x, \xi) \langle \xi \rangle^{-\rho_2 (|\alpha|+|\beta|)}, \]

while, of course, \( \text{vol}(B_{\varepsilon_2 \langle \xi \rangle^{\rho_2}(0)}) \lesssim \langle \xi \rangle^{d\rho_2} \), uniformly with respect to \( \varepsilon_2 \in (0, 1) \). Then (2.22), (2.23), and (2.24) imply, for arbitrary multiindices \( \alpha, \beta \) and integer \( j \),

\[ |u^\beta f_\alpha (x, \xi, u)| \lesssim \omega_2 (x, \xi) \langle \xi \rangle^{\rho_2 (d-|\alpha|-|\beta|)} \iff \]

\[ |(u \langle \xi \rangle^{\rho_2})^\beta| f_\alpha (x, \xi, u) | \lesssim \omega_2 (x, \xi) \langle \xi \rangle^{\rho_2 (d-|\alpha|)} \Rightarrow \]

\[ |u| \langle \xi \rangle^{\rho_2} f_\alpha (x, \xi, u) | \lesssim \omega_2 (x, \xi) \langle \xi \rangle^{\rho_2 (d-|\alpha|)} \]

which finally gives, for any multiindex \( \alpha \) and integer \( L \),

\[ (1 + |u| \langle \xi \rangle^{\rho_2})^L |f_\alpha (x, \xi, u)| \lesssim \omega_2 (x, \xi) \langle \xi \rangle^{\rho_2 (d-|\alpha|)} \Rightarrow \]

\[ |f_\alpha (x, \xi, u)| \lesssim \omega_2 (x, \xi) \langle \xi \rangle^{\rho_2 (d-|\alpha|)} (1 + |u| \langle \xi \rangle^{\rho_2})^{-L}. \]

So, setting \( L = L_1 + L_2 \) with \( L_1, L_2 \) arbitrary positive integers, we can say that, for any multiindex \( \alpha \),

\[ I_1 \lesssim \omega_2 (x, \xi) \langle \xi \rangle^{\rho_2 (d-|\alpha|)} \cdot \sup_y \left[ \left| D_\alpha ^y \left( e^{iy(x,y,\xi)} \chi (x, y)a (y, \xi) \right) \right| (1 + |y - x| \langle \xi \rangle^{\rho_2})^{-L_1} \right] \cdot \int (1 + |y - x| \langle \xi \rangle^{\rho_2})^{-L_2} dy. \]  

(2.25)

For what concerns the integral in (2.25), by the change of variable \( y = x + z \langle \xi \rangle^{-\rho_2} \), it turns out to be estimated by a constant times \( \langle \xi \rangle^{-d\rho_2} \), by choosing \( L_2 \) large enough. It remains to estimate the factor containing the supremum with respect to \( y \):
this can be achieved by observing that
\[
\partial_y (e^{iy(x,y,\xi)} \chi(x, y) a(y, \xi)) = \\
= \sum_{\beta + \gamma + \delta = \alpha} \frac{\alpha!}{\beta! \gamma! \delta!} \tau_{\beta}(x, y, \xi) e^{iy(x,y,\xi)} \partial_y^\beta \chi(x, y) \partial_y^\gamma a(y, \xi) \\
\lesssim \sum_{\beta + \gamma + \delta = \alpha} |\tau_{\beta}(x, y, \xi)| \langle y \rangle^{-|\gamma|} \omega_1(y, \xi) \langle y \rangle^{-r_1|\delta|} \\
\lesssim \omega_1(x + (y - x), \xi + 0) \cdot \sum_{\beta + \gamma + \delta = \alpha} |y - x|^{|\theta_j|} \langle x \rangle^{N_j - |\beta|} \langle \xi \rangle^{N_j + |\theta_j|} \langle y \rangle^{-\min\{r_1, 1\}(|\gamma| + |\delta|)} \\
\lesssim \omega_1(x, \xi) |v(x - y)| \cdot \sum_{\beta + \gamma + \delta = \alpha} |y - x|^{|\theta_j|} \langle x \rangle^{-\min\{r_1, 1\}(|\gamma| + |\delta|)} \\
\lesssim \omega_1(x, \xi) \langle x \rangle^{-\min\{r_1, 1\}/2\} \langle \xi \rangle^{\rho_2} \sum_{\beta + \gamma + \delta = \alpha} \langle \xi \rangle^{N_j + (1 - \rho_2)|\theta_j|} \sum_{j} \langle |x - y| \rangle^{\rho_2} \langle \xi \rangle^{\rho_2} \sum_{\beta + \gamma + \delta = \alpha} \langle \xi \rangle^{N_j + (1 - \rho_2)|\theta_j|},
\]
where we used Lemma 2.13 \( \langle x \rangle \lesssim \langle y \rangle \) (owing to the presence of \( \chi \)), the fact that \( \omega_1 \) is polynomially moderate, and suitable polynomials \( \tilde{v}, \tilde{\tilde{v}} \). Note that, when \( \rho_2 > 1 \), all terms in the sum in the last line of the above estimate, are never greater than \( \langle \xi \rangle^{\frac{\rho_1}{2}} \lesssim \langle \xi \rangle^{\alpha/2} \), in view of (2.12). The same holds for \( 1 \geq \rho_2 \geq 1/2 \), since (2.11) implies
\[
N_j + (1 - \rho_2)|\theta_j| \leq N_j + \frac{1}{2}|\theta_j| = \frac{1}{2}(2N_j + |\theta_j|) \leq \frac{1}{2} (|\theta_j| + \sum_{k=1}^{N_j} |\beta_{jk}|) = \frac{\beta}{2} \leq \frac{\alpha}{2}.
\]

We conclude that, for \( L_1 \) large enough,
\[
I_1 \lesssim \omega_1(x, \xi) \langle x \rangle^{-\min\{r_1, 1/2\}/2\} \langle \xi \rangle^{-(\rho_2 - 1/2)/2\}. \\
\cdot \sup_{y \in \mathbb{R}^d} \tilde{v}(|x - y|\langle \xi \rangle^{\rho_2})(1 + |y - x|\langle \xi \rangle^{\rho_2})^{-L_1} \\
\lesssim \omega_1(x, \xi) \langle x \rangle^{-\min\{r_1, 1/2\}/2\} \langle \xi \rangle^{-(\rho_2 - 1/2)/2\}.
\]

(2) Estimate of \( I_2 \).
Let us set
\[
f(x, y, \xi, \theta) = \langle y, \theta \rangle - \psi(x, y, \xi) \\
= \langle y, \theta \rangle - (\varphi(y, \xi) - \varphi(x, \xi) - \langle y - x, \varphi'(x, \xi) \rangle), \quad (2.26)
\]
which implies
\[
f'_y(x, y, \xi, \theta) = \theta - (\varphi'_y(y, \xi) - \varphi'_x(x, \xi)) \lesssim \langle \theta \rangle + \langle \xi \rangle.
\]

We begin by using the operator \( R_4 \) for the sake of brevity. Note that under the conditions (2.28), (2.29) and (2.30). We will write
\[
\text{This can be proved by induction on } l_1, \text{ for any arbitrary integer } l_1,
\]
\[
\begin{align*}
I_2 &= \int e^{i(x, \theta)} R_4^t \left\{ t_\alpha(x, \xi, \theta) \left[ 1 - \bar{\chi} \left( \frac{\theta}{\langle \xi \rangle^{\rho_2}} \right) \right] \cdot \mathcal{F}_{y \to \theta} \left[ D_y^\alpha \left( e^{i\psi(x, y, \xi) \chi(x, y) a(y, \xi)} \right) \right] d\theta \\
&= \sum_j \int e^{i(x, \theta)} t_\alpha^j(x, \xi, \theta) \bar{\chi}_j \left( \frac{\theta}{\langle \xi \rangle^{\rho_2}} \right) \cdot \mathcal{F}_{y \to \theta} \left[ y^\beta D_y^\alpha \left( e^{i\psi(x, y, \xi) \chi(x, y) a(y, \xi)} \right) \right] d\theta,
\end{align*}
\]
with, for any \( j \),
\[
\bar{\chi}_j \lesssim 1, \text{ supp } \bar{\chi}_j \left( \frac{\cdot}{\langle \xi \rangle} \right) \subseteq \mathbb{R}^d \setminus B_{\varepsilon_2}(\xi^{\rho_2}(0)),
\]
\[
t_\alpha^j(x, \xi, \theta) \lesssim \omega_2(x, \xi) \theta^{-2l_1, -\rho_2 |\alpha|} (x, \xi),
\]
\[
|\beta_j| \leq 2l_1.
\]

This can be proved by induction on \( l_1 \). From now on, we will consider only one of the integrals in the sum (2.27), since all the estimates we will find are independent of \( j \). Writing explicitly the Fourier transform and the derivative with respect to \( y \) in one of such integrals, and using (2.26) and the notation in Lemma 2.11, we have to estimate
\[
I_2 = \sum_{j, \beta + \gamma + \delta = \alpha} \frac{\alpha!}{\beta! \gamma! \delta!} \int e^{i(x, \theta)} t_\alpha^j(x, \xi, \theta) \bar{\chi}_j \left( \frac{\theta}{\langle \xi \rangle} \right) \cdot e^{-i f(x, y, \xi, \theta)} \tau_\beta(x, y, \xi) \partial_y^\beta \chi(x, y) y^\gamma \partial_y^\gamma a(y, \xi) dy d\theta,
\]
under the conditions (2.28), (2.29) and (2.30). We will write
\[
g_{\beta \gamma \delta}^j(x, y, \xi) = \tau_\beta(x, y, \xi) \partial_y^\beta \chi(x, y) y^\gamma \partial_y^\gamma a(y, \xi)
\]
for the sake of brevity. Note that
\[
g_{\beta \gamma \delta}^j \in \text{SG}^{(\omega)}_{1, \min \{r_1, 1\}, \min \{\rho_1, 1\}},
\]
\[
\bar{\omega}(x, y, \xi) = \omega_1(y, \xi) \partial_{2l_1, |\alpha|} (y, \xi),
\]
owing to
- \( \tau_\beta \in \text{SG}^{0, 0, |\beta|}_{1, 1, 1} \subseteq \text{SG}^{0, 0, |\alpha|}_{1, 1, 1} \), by (2.11);
- \( \chi \in \text{SG}^{0, 0, 0}_{1, 1, 1} \Rightarrow \partial_y^\beta \chi \in \text{SG}^{0, -|\gamma|, 0}_{1, 1, 1} \subseteq \text{SG}^{0, 0, 0}_{1, 1, 1};
\]
- \( a \in \text{SG}^{(\omega)}_{r_1, \rho_1} \) and (2.30) which imply \( y^\beta a(y, \xi) \in \text{SG}^{(\omega_1, \partial_{2l_1, \alpha})}_{r_1, \rho_1} \).
Let us now use the operator $R_3$ defined in (2.16). This is admissible, since, for a suitable $C > 0$,

$$
|f'_y(x, y, \xi, \theta)| = |\theta - (\varphi'_x(y, \xi) - \varphi'_x(x, \xi))| \geq |\theta| - |\varphi'_x(y, \xi) - \varphi'_x(x, \xi)| \geq C(\langle \theta \rangle + \langle \xi \rangle^{\min(p_2, 1)}) \gtrsim \langle \theta \rangle \langle \xi \rangle^{\min(p_2, 1)} + \varepsilon_1 \langle \xi \rangle \min\{\rho_2, 1\},
$$

provided $\varepsilon_1 \in (0, 1)$ in the definition of $\chi$ is chosen small enough. In fact, owing to the presence of $\tilde{\chi}_j$, we have here $|\theta| \geq \varepsilon_1^{1/2} \langle \xi \rangle^{\rho_2}$. Moreover, recall that, owing to the presence of $\chi$, which implies $|y - x| \leq \varepsilon_1 \langle x \rangle$,

$$
\varphi'_x(y, \xi) - \varphi'_x(x, \xi) = \sum_{k=1}^d \int_0^1 \varphi''_{x,x_k}(x + t(y - x), \xi)(y_k - x_k) \, dt \lesssim \varepsilon_1 \langle x \rangle \langle \xi \rangle \int_0^1 \langle x + t(y - x) \rangle^{-1} dt \lesssim \varepsilon_1 \langle \xi \rangle \langle x \rangle^{-1} = \varepsilon_1 \langle \xi \rangle.
$$

This implies

$$
|f'_y(x, y, \xi, \theta)| \geq |\theta| - |\varphi'_x(y, \xi) - \varphi'_x(x, \xi)| \gtrsim |\theta| - \varepsilon_1 \langle \xi \rangle \gtrsim \langle \theta \rangle + \langle \xi \rangle^{\rho_2} - \varepsilon_1 \langle \xi \rangle \gtrsim \langle \theta \rangle + (1 - \varepsilon_1) \langle \xi \rangle^{\min(p_2, 1)},
$$

and (2.33) follows, with an appropriate choice of $\varepsilon_1$. Note also that $^4R_4$ acts only on $g_{\beta\gamma\delta}^j$, leaving $e^{i(x, \theta)}$, $t_{\alpha}^j(x, \xi, \theta)$ and $\tilde{\chi}_j$ unchanged, so that we can use the estimates (2.28) and (2.29) for them. By applying formulae (2.17), (2.18), (2.31) and (2.32) we find, for any integer $l_2$,

$$
\int e^{-if(x,y,\xi,\theta)} g_{\beta\gamma\delta}^j(x, y, \xi, \theta) \, dy = \int e^{-if(x,y,\xi,\theta)} (t R_3)^{l_2} g_{\beta\gamma\delta}^j(x, y, \xi, \theta) \, dy = \int e^{-if(x,y,\xi,\theta)} \left| f'_y(x, y, \xi, \theta) \right|^{4l_2} \sum_{|\kappa| \leq l_2} P_{n,l_2} \partial_{\kappa}^y g_{\beta\gamma\delta}^j(x, y, \xi, \theta) \, dy.
$$
Inserting this into (2.31), it follows that

\[
I_2 \lesssim \omega_2(x, \xi) v_{2l_1, \rho_2|\alpha|}(x, \xi) \sum_{j, \beta+\gamma+\delta=\alpha} \int (\langle \theta \rangle + \langle \xi \rangle^{\min\{\rho_2, 1\}})^{-4l_2}. \]

\[
\cdot \sum_{|\upsilon| \leq l_2} \omega_1(y, \xi) v_{2l_1-|\upsilon|, |\upsilon|}(y, \xi). \]

\[
\cdot \sum (\langle \theta \rangle + \langle \xi \rangle^{\min\{\rho_2, 1\}})^{2l_2} (\langle y \rangle^{l_2} (\langle \theta \rangle + \langle \xi \rangle^{\min\{\rho_2, 1\}})^{l_2} d\theta dy
\]

\[
\lesssim \omega_1(x, \xi) \omega_2(x, \xi) (x)^{-2l_1} (\xi)^{(1-\rho_2)|\alpha|}. \]

\[
\cdot (\xi)^{-\min\{\rho_2, 1\}} \frac{l_2}{l_2} \int \bar{v}(|x-y|) (\langle y \rangle^{2l_1} d\theta dy \int (\theta)^{-\frac{l_2}{2}} d\theta
\]

\[
\lesssim \omega_1(x, \xi) \omega_2(x, \xi) (x)^{-2l_1} (\xi)^{m_1+(1-\rho_2)|\alpha|-\min\{\rho_2, 1\}} \frac{l_2}{l_2},
\]

for a certain fixed \( m_1 \geq 0 \) depending only on \( \omega_1 \), provided that

\[
l_2 > \max\{2d, 2l_1 + d\}.
\]

By all what we showed above, recalling once more that, for suitable \( m, \mu \geq 0 \),

\[
\omega_1(x, \xi) \omega_2(x, \xi) \lesssim (x)^m \langle \xi \rangle^\mu,
\]

for all \( x, \xi \in \mathbb{R}^d \), it is now possible to conclude as follows. For any arbitrary integer \( l \), fix the integer \( M \) such that

\[
r_1 l + m - \min\left\{ r_1, \frac{1}{2} \right\} M \leq 0 \quad \text{and/or} \quad \left( \rho_2 - \frac{1}{2} \right) l + \mu - \left( \rho - \frac{1}{2} \right) M \leq 0.
\]

In view of the hypotheses, at least one of the two above conditions can be fulfilled. Then, with \( \varepsilon_1, \varepsilon_2 \in (0, 1) \) chosen as in the above discussion about the estimate of \( I_2 \), fix \( l_1 \) such that

\[
r_1 l + m - 2l_1 \leq 0
\]

and \( l_2 \) such that

\[
l_2 > 2d, \quad l_2 > m + d + 2l_1 \geq d + 2l_1,
\]

\[
\left( \rho_2 - \frac{1}{2} \right) l + \mu + m_1 + (1-\rho_2)M - \min\{\rho_2, 1\} \frac{l_2}{2} \leq 0.
\]

We conclude that for any arbitrary integer \( l \) it is possible to find an integer \( M \) such that
\[(\langle x \rangle^{r_1} \langle \xi \rangle^{\rho_2 - 1/2}) \left( h_1(x, \xi) - \sum_{|\alpha|<M} \frac{1}{\alpha!} c_\alpha(x, \xi) \right) = \langle x \rangle^{r_1} \langle \xi \rangle^{\rho_2 - 1/2} \sum_{|\alpha|=M} \frac{M}{\alpha!} T_\alpha(x, \xi) \lesssim w(x, \xi), \]

where

\[w(x, \xi) = \langle \xi \rangle^m \text{ if } r > 0, \rho = 1/2; \]
\[w(x, \xi) = \langle x \rangle^m \text{ if } r = 0, \rho > 1/2; \]
\[w(x, \xi) = 1 \text{ if } r > 0, \rho > 1/2.\]

Recalling Definition 1.3 and (1.11), this gives the desired result, by the general properties of the symbolic calculus (see Appendix A). \(\square\)

2.4. Other composition theorems between SG FIOs and SG pseudodifferential operators. The next three theorems are immediate consequences of the Composition Theorem 2.10. In fact, they can easily be proved by considering the transposed and the formal-adjoints of the involved operators, in view of the symmetry in the roles of variable and covariable in the SG setting, as explained in Remark 2.8.

**Theorem 2.18.** Let \(\varphi \in \mathfrak{F}, a \in \text{SG}_{r_1,\rho_1}(\mathbb{R}^{d_2}),\) and \(p \in \text{SG}_{r_2,\rho_2}(\mathbb{R}^{d_2}),\) \(r_2 \geq 1/2, \rho_1 + r_2 > 1/2.\) Then, the composed operator \(\tilde{H} = \text{Op}_\varphi(a) \circ \text{Op}(p)\) is, modulo smoothing operators, a FIO of type I. In fact, \(\tilde{H} = \text{Op}_\varphi(h)\) where \(\varphi\) is the same phase function and the transpose \(\tilde{h}\) of the amplitude \(h \in \text{SG}_{r,\rho}(\mathbb{R}^{d_d}),\) \(\omega = \omega_1 \omega_2, r = \min\{r_1, r_2, 1\}, \rho = \min\{\rho_1, \rho_2, 1\}\) admits the asymptotic expansion (2.8), (2.9), with \(p\) changed into \(\tilde{t}p, a\) changed into \(\tilde{t}a\) and \(\varphi\) changed into \(\tilde{t} \varphi.\)

**Theorem 2.19.** Let \(\varphi \in \mathfrak{F}, b \in \text{SG}_{r_1,\rho_1}(\mathbb{R}^{d_2}),\) and \(p \in \text{SG}_{r_2,\rho_2}(\mathbb{R}^{d_2}),\) \(r_2 \geq 1/2, r_1 + r_2 > 1/2.\) Then, the composed operator \(W = \text{Op}_\varphi^*(b) \circ \text{Op}(p)\) is, modulo smoothing remainders, a SG FIO of type II. In fact, \(W = \text{Op}_\varphi^*(w)\) with the amplitude \(w \in \text{SG}_{r,\rho}(\mathbb{R}^{d_d}),\) \(\omega = \omega_1 \omega_2, r = \min\{r_1, r_2, 1\}, \rho = \min\{\rho_1, \rho_2, 1\}\) admitting the asymptotic expansion

\[w(x, \xi) \sim \sum_\alpha \frac{|\alpha|}{\alpha!} (D^\varphi \xi q)(x, \varphi'_x(x, \xi)) D_y^\alpha [e^{i\psi(x,y,\xi)} b(y, \xi)]_{y=x}, \quad (2.34)\]

where

\[\psi(x, y, \xi) = \varphi(y, \xi) - \varphi(x, \xi) - (y - x, \varphi'_x(x, \xi)) \quad (2.35)\]

and \(q\) is given by

\[q(x, \xi) \sim \sum_\alpha \frac{|\alpha|}{\alpha!} D_x^\alpha D_y^\alpha p(x, \xi). \quad (2.36)\]
where we have set \( f_b \). That, under the same condition, \( r_p \), write \( \omega \). Of course, \( q_2 \) is a smoothing operator; then we will show that, under the same condition, \( C_1 \) can be rewritten as a SG pseudodifferential operator defined by an amplitude \( p \in \text{SG}^{(\omega)}_{r,s} \), provided \( \varepsilon \in (0,1) \) is chosen suitably small.

(1) \( C_2 \) is smoothing.

First of all, note that we have \( |x - y| \geq \frac{\varepsilon}{2} \) on \( \text{supp} \, q_2 \). Then, in the integral defining \( C_2 u(x) \), we can use the operator

\[
R_3 = \frac{1}{|f'_\xi(x,y,\xi)|^2} \sum_{k=1}^{\infty} f_\xi'(x,y,\xi) D_\xi,
\]

2.5. Composition between SG FIOs of type I and type II. The subsequent Theorems 2.21 and 2.22 deal with the composition of a type I operator with a type II operator.

Theorem 2.21. Let \( \varphi \in \mathfrak{F}_r \), \( a \in \text{SG}^{(\omega)}_{r_1,p_1} (\mathbb{R}^{2d}) \), \( b \in \text{SG}^{(\omega)}_{r_2,p_2} (\mathbb{R}^{2d}) \), \( \rho_1, \rho_2 > 0 \). Then, the operator \( P = \text{Op}_\varphi^* (a) \circ \text{Op}_\varphi^* (b) \) is, modulo smoothing remainders, a SG FIO of type II. In fact, \( \widetilde{W} = \text{Op}_\varphi^* (w) \) with the transpose \( ^t \tilde{w} \) of the amplitude \( \tilde{w} \in \text{SG}^{(\omega)}_{r,p} (\mathbb{R}^{2d}) \), \( \omega = \omega_1, \omega_2, \, r = \min \{ r_1, r_2, 1 \} \), \( \rho = \min \{ \rho_1, \rho_2, 1 \} \), admitting the asymptotic expansion expansion (2.34), (2.35), (2.36), with \( q \) changed in \( ^t q \), \( b \) changed in \( ^t b \) and \( \varphi \) changed in \( ^t \varphi \).

Proof. Let us write explicitly the composition for \( u \in \mathfrak{S} \). We find

\[
\text{Op}_\varphi^* (a) \circ \text{Op}_\varphi^* (b) u(x) = \int e^{i\varphi(x,y,\xi)} a(x,\xi) \left[ \int e^{-i\varphi(y,\xi)} \overline{b(y,\xi)} u(y) dy \right] d\xi
\]

where we have set \( f(x,y,\xi) = \varphi(x,\xi) - \varphi(y,\xi) \) and \( c(x,y,\xi) = a(x,\xi) \cdot \overline{b(y,\xi)} \). Let us choose \( \chi \in \Xi^\Delta (\varepsilon), \varepsilon \in (0,1) \), and write

\[
\text{Op}_\varphi^* (a) \circ \text{Op}_\varphi^* (b) u(x) = \int e^{i\varphi(x,y,\xi)} q_1(x,y,\xi) u(y) dy d\xi
\]

with \( q_1(x,y,\xi) = \chi(x,y) c(x,y,\xi) \) and \( q_2(x,y,\xi) = (1-\chi(x,y)) c(x,y,\xi) \).

Of course, \( q_1, q_2 \in \text{SG}^{(\omega)}_{r,s} \). We begin by proving that, since \( \rho = \min \{ \rho_1, \rho_2, 1 \} > 0 \), \( C_2 \) is a smoothing operator; then we will show that, under the same condition, \( C_1 \) can be rewritten as a SG pseudodifferential operator defined by an amplitude \( p \in \text{SG}^{(\omega)}_{r,s} \), provided \( \varepsilon \in (0,1) \) is chosen suitably small.

Theorem 2.22. Let \( \varphi \in \mathfrak{F}_r \), \( b \in \text{SG}^{(\omega)}_{r_1,p_1} (\mathbb{R}^{2d}) \), \( p \in \text{SG}^{(\omega)}_{r_2,p_2} (\mathbb{R}^{2d}) \), \( r_2 \geq 1/2, r_1 + r_2 > 1/2 \). Then, the composed operator \( \tilde{W} = \text{Op}(p) \circ \text{Op}_\varphi^* (b) \) is, modulo smoothing remainders, a SG FIO of type II. In fact, \( \tilde{W} = \text{Op}_\varphi^* (w) \) with the transpose \( ^t \tilde{w} \) of the amplitude \( \tilde{w} \in \text{SG}^{(\omega)}_{r,p} (\mathbb{R}^{2d}) \), \( \omega = \omega_1, \omega_2, \, r = \min \{ r_1, r_2, 1 \} \), \( \rho = \min \{ \rho_1, \rho_2, 1 \} \), admitting the asymptotic expansion expansion (2.34), (2.35), (2.36), with \( q \) changed in \( ^t q \), \( b \) changed in \( ^t b \) and \( \varphi \) changed in \( ^t \varphi \).
analogous to that defined in \((2.10)\). In fact, let us set \(v = \varphi'_\xi(x, \xi)\) and \(w = \varphi'_\xi(y, \xi)\). By making use of Proposition \(2.4\) and in view of \(\varphi \in SG^{1,1}_{r,s,\rho}\), we can write, for a suitable constant \(M > 0\),
\[
|x - y| = |(\varphi'_\xi)^{-1}(v, \xi) - (\varphi'_\xi)^{-1}(w, \xi)|
\leq |v - w| \sup_{\mathbb{R}^d \times \mathbb{R}^d} \|d_x(\varphi'_\xi)^{-1}(z, \xi)\|
\leq M|\varphi'_\xi(x, \xi) - \varphi'_\xi(y, \xi)|
= M|f'_\xi(x, y, \xi)|,
\]
which implies
\[
|f'_\xi(x, y, \xi)| \gtrsim |x - y| \gtrsim \langle x \rangle + \langle y \rangle
\]
on \text{supp} q_2. Then, using \(R_3e^{if} = e^{if}, \ (2.17), \ (2.18), \ (2.19)\) and again \(\varphi \in SG^{1,1}_{r,s,\rho}\), for any integer \(l\),
\[
C_2u(x) = \int e^{if(x, y, \xi)} ((t^lR_3)^l q_2)(x, y, \xi) u(y) dy d\xi
\]
and
\[
((t^lR_3)^l q_2)(x, y, \xi) = \frac{1}{|f'_\xi(x, y, \xi)|^d l^d} \sum_{|\alpha| \leq l} P_{t\alpha} \partial^\alpha \omega_1 q_2(x, y, \xi)
\lesssim \sum_{|\alpha| \leq l} \omega_1(x, \xi) \omega_2(y, \xi) \frac{\langle \xi \rangle^{l} \min\{\rho_1, \rho_2, 1\}|\alpha|}{\langle (x) + \langle y \rangle \rangle^{d l}}
\lesssim \omega_1(x, \xi) \omega_2(y, \xi) \langle \xi \rangle^{l} \min\{\rho_1, \rho_2, 1\}|\alpha|.
\]
Then, we can rewrite \(C_2\) as
\[
C_2u(x) = \int \left[ \int e^{if(x, y, \xi)} \langle t^lR_3 \rangle^l q_2(x, y, \xi) \right] u(y) dy d\xi
= \int c_2(x, y) u(y) dy,
\]
with an arbitrarily chosen large integer \(l\). Recalling that \(\rho > 0\) and \(\langle x \rangle + \langle y \rangle \geq (\langle x \rangle \langle y \rangle)^{\frac{1}{2}}\), it follows \(c_2(x, y) \lesssim (\langle x \rangle \langle y \rangle)^{-N}\) for any integer \(N\). The estimates for the derivatives of \(D_x^a D_y^b c_2(x, y)\) follow similarly by differentiation under the integral sign, since then we just have to start with some other \(\tilde{q}_2 \in SG^{(G)}_{r,s,\rho}\).

(2) \textbf{\(C_1\) is a SG pseudodifferential operator.}\nOn \text{supp} \(\tilde{q}_1\) we have \(|x - y| \leq \varepsilon \langle x \rangle \Rightarrow \langle x \rangle \sim \langle y \rangle\). Let us define
\[
d_x \varphi(x, y, \xi) = \int_0^1 \varphi'_\xi(y + t(x - y), \xi) dt,
\]
and observe
\[
d\varphi(x, y, \xi) = \varphi'(y, \xi) + w(x, y, \xi),
\]
\[
w(x, y, \xi) = \int_0^1 \int_0^1 (x - y) \cdot H(y + t_1 t_2 (x - y), \xi) dt_1 dt_2,
\]
\[
H(x, \xi) = \varphi''_{xx}(x, \xi),
\]
\[
\Rightarrow d\xi d\varphi(x, y; \xi) = \varphi''_{xx}(y, \xi)
\]
\[
+ \int_0^1 \int_0^1 t_1 (x - y) \cdot H'_x(y + t_1 t_2 (x - y), \xi) dt_1 dt_2.
\]

Provided \( \varepsilon \in (0, 1) \) is small enough, the integrand in (2.37) can be estimated on \( \text{supp } q_1 \) as follows:
\[
\sum_{k=1}^d (x_k - y_k) \partial_{\xi_k} \varphi''_{xx}(y + t_1 t_2 (x - y), \xi)
\]
\[
\lesssim |x - y| \sup_{t \in [0,1]} \langle y + t(x - y) \rangle^{-1}
\]
\[
\lesssim \varepsilon \langle x \rangle \langle y \rangle^{-1} \lesssim \varepsilon,
\]
so that the Jacobian of \( d\varphi(x, y, \xi) \) is a small perturbation of the one of \( \varphi'_x(y, \xi) \). Then, possibly taking a smaller value of \( \varepsilon \) and recalling \( \varphi \in \mathcal{F} \), on \( \text{supp } q_1 \) we can assume
\[
\left| \det d\xi d\varphi(x, y, \xi) \right| \geq \frac{\kappa}{2} > 0.
\]
Moreover, it is easy to see that, on \( \text{supp } q_1 \), the components of \( d\varphi(x, y, \xi) \) satisfy \( \text{SG}^{0,0,1}_{1,1,1} \) estimates, since
\[
\partial^a_x \partial^b_y \partial^c_\xi d\varphi(x, y, \xi) \lesssim \langle y \rangle^{-|a|+|b|} \langle \xi \rangle^{1-|c|} = \langle x \rangle^{-|a|} \langle y \rangle^{-|b|} \langle \xi \rangle^{1-|c|}. \tag{2.38}
\]

We now prove that, on \( \text{supp } q_1 \),
\[
\langle d\varphi(x, y, \xi) \rangle \ll \langle \xi \rangle.
\]
In fact, the upper bound is immediate, and we also have
\[
|w(x, y, \xi)| \leq |x - y| \cdot \sup_{t \in [0,1]} \|H(y + t(x - y), \xi)\| \lesssim \varepsilon \langle x \rangle \langle y \rangle^{-1} \langle \xi \rangle
\]
\[
\lesssim \varepsilon \langle \xi \rangle \ll \varepsilon \langle \varphi'_x(y, \xi) \rangle
\]
\[
\langle d\varphi(x, y, \xi) \rangle = \langle \varphi'_x(y, \xi) + w(x, y, \xi) \rangle \approx \langle \varphi'_x(y, \xi) \rangle \ll \langle \xi \rangle.
\]
Then, with a suitable choice of \( \varepsilon \in (0, 1) \), \( d\varphi(x, y; \xi) \) satisfies all the requirements of Proposition 1.11 and, on \( \text{supp } q_1 \),
\( \tilde{d}_x \varphi(x, y, \xi) \) is an SG diffeomorphism with SG\(^0\) parameter dependence. With this in mind, we can rewrite \( C_1 u(x) \) as

\[
C_1 u(x) = \int e^{i(\varphi(x, \xi) - \varphi(y, \xi))} q_1(x, y, \xi) u(y) \, d\xi \, dy
\]

\[
= \int e^{i(x-y, \tilde{d}_x \varphi(x, y, \eta))} q_1(x, y, \eta) u(y) \, d\eta \, dy.
\]

By the above arguments, it follows that we can make the substitution

\[
\xi = \tilde{d}_x \varphi(x, y; \eta) \Leftrightarrow \eta = (\tilde{d}_x \varphi)^{-1}(x, y; \xi),
\]

so that we can conclude

\[
C_1 u(x) = \int e^{i(x-y, \xi)} q_1(x, y, (\tilde{d}_x \varphi)^{-1}(x, y, \xi)) \cdot
\]

\[
\cdot \det d_\xi (\tilde{d}_x \varphi)^{-1}(x, y, \xi) \big| u(y) \, dy \, d\xi,
\]

setting

\[
p(x, y, \xi) = q_1(x, y, (\tilde{d}_x \varphi)^{-1}(x, y, \xi)) \big| \det d_\xi (\tilde{d}_x \varphi)^{-1}(x, y, \xi) \big|.
\]  

(2.39)

By (2.38), Lemma 1.6 and Proposition 1.11 we find \( p \in \text{SG}_{r,s,\rho} \), as claimed.

We omit the proof of the next theorem, which can be obtained along the same lines of the one of Theorem 2.21.

**Theorem 2.22.** Let \( \varphi \in \mathfrak{F}^r, \ a \in \text{SG}^{(1)}_{r_1, \rho_1} (\mathbb{R}^{2d}), \ b \in \text{SG}^{(2)}_{r_2, \rho_2} (\mathbb{R}^{2d}), \) \( r_1, r_2 > 0 \). Then, the operator \( P = \text{Op}_\phi^r(b) \circ \text{Op}_\varphi(a) \) is, modulo smoothing operators, a SG pseudodiffferential operator with symbol \( p \in \text{SG}^{(\omega)}_{r,\rho} (\mathbb{R}^{2d}) \) with \( \omega(x, \xi) = \omega_1(x, \xi) \omega_2(x, \xi), \ r = \min\{r_1, r_2, 1\}, \rho = \min\{\rho_1, \rho_2, 1\} \), given by

\[
p(x, \xi) \sim \sum_{\alpha} \frac{j^{|\alpha|}}{\alpha !} (D^\alpha_\eta D^\alpha_\eta \tilde{q}_1)(\xi, \eta, x)|_{\eta = \xi},
\]

where

\[
\tilde{q}_1(\xi, \eta, x) = q_1(\xi, \eta, (\tilde{d}_x \varphi)^{-1}(\xi, \eta, x)) \cdot \big| \det d_x (\tilde{d}_x \varphi)^{-1}(\xi, \eta, x) \big|,
\]

\[
q_1(\xi, \eta, y) = \alpha(y, \xi) \bar{b}(y, \eta) \chi(\xi, \eta), \quad \chi \in \Xi_\Delta(\varepsilon),
\]

\[
\tilde{d}_x \varphi(\xi, \eta, x) = \int_0^1 \varphi'_x(x, \eta + t(\xi - \eta)) \, dt,
\]

and \( \varepsilon \in (0, 1) \) is chosen suitably small.
2.6. Elliptic SFIOs and parametrices.

Definition 2.23. A type I or a type II SFIO, $\text{Op}_\varphi(a)$ or $\text{Op}_\varphi^+(b)$, respectively, is said elliptic if $\varphi \in \mathcal{S}'$ and the amplitude $a$, respectively $b$, is SF-elliptic.

Lemma 2.24. If a type I SFIO $\text{Op}_\varphi(a)$ is elliptic, $a \in \text{SG}_{r,\rho}^{(\omega)}(\mathbb{R}^{2d})$, $r, \rho > 0$, then the two pseudodifferential operators $\text{Op}_\varphi(a) \circ \text{Op}_\varphi^+(a)$ and $\text{Op}_\varphi^+(a) \circ \text{Op}_\varphi(a)$ are SF-elliptic.

Proof. It is of course enough to prove that the first term of the asymptotic expansion of the two symbols is SF-elliptic. From Theorem 2.21 cf. (2.39), the symbol of $\text{Op}_\varphi(a) \circ \text{Op}_\varphi^+(a)$ is

$$p(x, \xi) = |a(x, \eta)|_{\eta = (\varphi'_x)^{-1}(x, \xi)}^2 \cdot |\det d\varphi_x'(x, \xi)| \mod \text{l.o.t.}$$

$|a(x, \eta)|_{\eta = (\varphi'_x)^{-1}(x, \xi)}^2$ is SF-elliptic, in view of the hypothesis on the amplitude $a$, the properties of the SF-diffeomorphism with SF$^0$ parameter dependence given by $\xi \mapsto \varphi'_x(x, \xi)$, and the composition properties in the SF symbol classes. $|\det d\varphi_x'(x, \xi)| \in \text{SG}_{1,1}^{0,0}$ is of course SF-elliptic, since it is the Jacobian determinant of an SF-diffeomorphism. By a completely similar argument, it turns out that also $\text{Op}_\varphi^+(a) \circ \text{Op}_\varphi(a)$ is SF-elliptic. \[\square\]

Theorem 2.25. Let $\varphi \in \mathcal{S}'$, $a \in \text{SG}_{r,\rho}^{(\omega)}(\mathbb{R}^{2d})$, $r, \rho \geq 1/2$, with a SF-elliptic. Then, the elliptic SF IOs $\text{Op}_\varphi(a)$ and $\text{Op}_\varphi^+(a)$ admit a parametrix. These are elliptic SF IOs of type II and type I, respectively.

Proof. The proof is standard. For the sake of completeness, we repeat here the argument in the case of SF IOs of type I. Let us denote $\text{Op}_\varphi(a)$ by $A$, the parametrix of $P = A \circ A^*$ by $P^{-1}$, and the parametrix of $Q = A^* \circ A$ by $Q^{-1}$. Both $P^{-1}$ and $Q^{-1}$ exist, owing to Lemma 2.23.

We have

$$P \circ P^{-1} = I - K_1, \quad P^{-1} \circ P = I - K_2,$$

$$Q \circ Q^{-1} = I - K_3, \quad Q^{-1} \circ Q = I - K_4,$$

with $K_1, K_2, K_3, K_4$ smoothing operators. Let us set $F_l = Q^{-1} \circ A^*$ and $F_r = A^* \circ P^{-1}$. We then have

$$F_l \circ A = (A^* \circ A)^{-1} \circ (A^* \circ A) = I - K_4,$$

$$A \circ F_r = (A \circ A^*) \circ (A \circ A^*)^{-1} = I - K_1,$$

$$F_l \circ A \circ F_r = (I - K_4) \circ F_r \Rightarrow F_l - F_l \circ R_1 = F_r - K_4 \circ F_r$$

$$\Leftrightarrow F_l = F_r \mod \text{smoothing operators},$$

so that $F_r$ or $F_l$ can be chosen as parametrices of $A$. With similar arguments it is possible to find a parametrix for $A^*$, namely setting $G_r = A \circ Q^{-1}$ and $G_l = P^{-1} \circ A$. The second part of the statement follows from the composition Theorems 2.10, 2.18, 2.19 and 2.20. \[\square\]
2.7. Egorov theorem. The versions of the Egorov’s theorem adapted to the present situation follow from the composition results of the previous subsections.

**Theorem 2.26.** Let $A = \text{Op}_\varphi (a)$ be a SG FIO of type I with $a \in \text{SG}^{(\omega_1)}_{r_1,p_1} (\mathbb{R}^{2d})$ and $P = \text{Op}(p)$ a pseudo-differential operator with $p \in \text{SG}^{(\omega_2)}_{r_2,p_2} (\mathbb{R}^{2d})$, $r_1, r_2, p_1, p_2 \geq 1/2$. Then, setting $\eta = (\varphi'_x)^{-1}(x, \xi)$ we have

$$\text{Sym} (A \circ P \circ A^*) (x, \xi) = p(\varphi'_x(x, \eta), \eta) |a(x, \eta)|^2 |\det \varphi''_{x\xi}(x, \eta)|^{-1} \mod \text{l.o.t.},$$

(2.40)

which is an element of $\text{SG}^{(\omega)}_{r,\rho} (\mathbb{R}^{2d})$ with $\omega = \omega_1 \cdot \omega_2^2$, $r = \min \{r_1, r_2, 1\}$, $\rho = \min \{p_1, p_2, 1\}$.

**Proof.** By Theorem 2.20 for the amplitude $b$ of $B = \text{Op}_{\varphi^*} (b) = P \circ A^*$ we have

$$b(x, \xi) = \langle (\text{Sym} (P^*)) \rangle (x, (\text{Sym} (P^*)) (x, \xi)) \langle a \rangle (x, \xi) \mod \text{l.o.t.}$$

$$= p(\varphi'_x(x, \eta), \eta) |a(x, \eta)|^2 \mod \text{l.o.t.} \Rightarrow$$

$$\Rightarrow b(x, \xi) = p(\varphi'_x(x, \xi), \eta) a(x, \xi) \mod \text{l.o.t.}$$

Then, setting $\eta = (\varphi'_x)^{-1}(x, \xi)$, owing to Theorem 2.21 we find

$$\text{Sym} (A \circ P \circ A^*) (x, \xi) = \text{Sym} (A \circ B) (x, \xi)$$

$$= p(\varphi'_x(x, \eta), \eta) |a(x, \eta)|^2 |\det d_{\xi}(\varphi'_x)^{-1}(x, \xi)| \mod \text{l.o.t.},$$

(2.41)

which is (2.40), recalling that, of course,

$$|\det d_{\xi}(\varphi'_x)^{-1}(x, \xi)| = |\det (\varphi''_{x\xi}(x, \eta))|^{-1}.$$  

We also observe that, as in the standard theory (cf. Section 25.3 in [28]),

$$\text{Sym} (A \circ A^*) (x, \xi) = |a(x, \eta)|^2 |\det (\varphi''_{x\xi}(x, \eta))|^{-1} \mod \text{l.o.t.}$$

That $\text{Sym} (A \circ P \circ A^*) (x, \xi)$ is indeed a SG symbol belonging to the indicated class follows again by the mentioned composition theorems, and by the properties of the SG calculus recalled in the previous subsection. \qed

The next result can be proved by means of a similar argument, using Theorem 2.25.

**Theorem 2.27.** Let $A = \text{Op}_\varphi (a)$ be an elliptic SG FIO of type I with $a \in \text{SG}^{(\omega_1)}_{r_1,p_1} (\mathbb{R}^{2d})$ and $P = \text{Op}(p)$ a pseudo-differential operator with $p \in \text{SG}^{(\omega_2)}_{r_2,p_2} (\mathbb{R}^{2d})$, $r_1, r_2, p_1, p_2 \geq 1/2$. Then, setting $\eta = (\varphi'_x)^{-1}(x, \xi)$ we have

$$\text{Sym} (A \circ P \circ A^{-1}) (x, \xi) = p(\varphi'_x(x, \eta), \eta) \mod \text{l.o.t.}$$

(2.42)
In this section we recall some basic facts about continuity properties for Fourier integral operators when acting on Lebesgue and modulation spaces. We also use the analysis in previous sections in combination with certain lifting properties for modulation spaces in order to establish weighted versions of continuity results for Fourier integral operators on modulation spaces.

3. Continuity on Lebesgue and modulation spaces

In this section we recall some basic facts about continuity properties for Fourier integral operators when acting on Lebesgue and modulation spaces. We also use the analysis in previous sections in combination with certain lifting properties for modulation spaces in order to establish weighted versions of continuity results for Fourier integral operators on modulation spaces.

3.1. Continuity on Lebesgue spaces. We start by considering the following result, which, for trivial Sobolev parameters, is related to Theorem 2.6 in [20]. Here $B_r(a)$ is the open ball with center at $a \in \mathbb{R}^d$ and radius $r$.

**Theorem 3.1.** Let $\sigma_1, \sigma_2 \in \mathbb{R}$, $p \in (1, \infty)$ and $m, \mu \in \mathbb{R}$ be such that

$$m \leq -(d-1) \left| \frac{1}{p} - \frac{1}{2} \right|, \quad \mu \leq -(d-1) \left| \frac{1}{p} - \frac{1}{2} \right| + \sigma_1 - \sigma_2.$$  

Also let $\varphi \in SG_{1,1}^1(\mathbb{R}^{2d})$ be such that for some constants $c > 0$ and $R > 0$ and every multi-index $\alpha$ it holds

$$|\det \varphi''_{x,\xi}(x,\xi)| \geq c, \quad |\partial_x^\alpha \varphi(x,\xi)| \lesssim (x)^{1-|\alpha|} \langle \xi \rangle \langle \varphi_x'(x,\xi) \rangle \asymp \langle x \rangle,$$

and

$$\varphi(x, t\xi) = t\varphi(x, \xi), \quad x, \xi \in \mathbb{R}^d, \ |\xi| \geq R, \ t \geq 1.$$  

If $a \in SG_{1,1}^{m,\mu}(\mathbb{R}^{2d})$ is supported outside $\mathbb{R}^d \times B_r(0)$ for some $r > 0$, then $\text{Op}_\varphi(a)$ extends to a continuous operator from $H^p_{\sigma_1}(\mathbb{R}^d)$ to $H^p_{\sigma_2}(\mathbb{R}^d)$.

**Proof.** Let $T = \langle D \rangle^{\sigma_2} \circ \text{Op}_\varphi(a) \circ \langle D \rangle^{-\sigma_1}$. Since

$$\langle D \rangle^{\sigma_2} : H^p_{\sigma_2} \rightarrow L^p \quad \text{and} \quad \langle D \rangle^{-\sigma_1} : L^p \rightarrow H^p_{\sigma_1}$$

are continuous bijections, the result follows if we prove that $T$ is continuous on $L^p$.

By Theorems 2.18 and 2.19 it follows that

$$T = \text{Op}_\varphi(a_1) \mod \text{Op}(\mathcal{F}),$$

where $a_1 \in SG_{1,1}^{m,\mu_0}(\mathbb{R}^{2d})$ with

$$\mu_0 \leq -(d-1) \left| \frac{1}{p} - \frac{1}{2} \right|.$$  

Furthermore, by the symbolic calculus and the fact that $a$ is supported outside $\mathbb{R}^d \times B_r(0)$ we get

$$\text{Op}_\varphi(a_1) = \text{Op}_\varphi(a_2) \mod \text{Op}(\mathcal{F}),$$

where $a_2 \in SG_{1,1}^{m,\mu_0}(\mathbb{R}^{2d})$ is supported outside $\mathbb{R}^d \times B_r(0)$. Hence

$$T = \text{Op}_\varphi(a_2) + \text{Op}(c),$$

where $c$ is a constant.
where \( c \in \mathcal{S} \), giving that \( \text{Op}(c) \) is continuous on \( L^p \).

Since \( \text{Op}_\varphi(a_2) \) is continuous on \( L^p \), by [20, Theorem 2.6] and its proof, the result follows.

\[ \square \]

3.2. **Continuity on modulation spaces.** Next we consider continuity properties on modulation spaces. The following result extends Theorem 1.2 in [10]. Here we let \( M^\infty_0(\mathbb{R}^d) \) be the completion of \( \mathcal{S}(\mathbb{R}^d) \) under the norm \( \| \cdot \|_{M^\infty_0} \). We also say that a (complex valued) Gauss function \( \phi \) is non-degenerate, if \( |\phi| \) tends to zero at infinity.

**Theorem 3.2.** Let \( m, \mu \in \mathbb{R} \) and \( 1 \leq p < \infty \) be such that

\[
m \leq -d \left( \frac{1}{2} - \frac{1}{p} \right), \quad \mu \leq -d \left( \frac{1}{2} - \frac{1}{p} \right),
\]

and let \( \omega_j \in \mathcal{P}_{1,1}(\mathbb{R}^{2d}), \ j = 0, 1, 2 \), be such that

\[
\omega_0(x, \xi) \lesssim \frac{\omega_1(x, \xi)}{\omega_2(x, \xi)} |x|^m \langle \xi \rangle^\mu.
\]

Also let \( a \in \text{SG}_{1,1}^{(\omega_0)}(\mathbb{R}^{2d}) \) and \( \varphi \in \mathcal{F}^\circ \). Then \( \text{Op}_\varphi(a) \) is uniquely extendable to a continuous map from \( M^p_0(\mathbb{R}^d) \) to \( M^p(\mathbb{R}^d) \) and from \( M^\infty_0(\mathbb{R}^d) \) to \( M^\infty(\mathbb{R}^d) \).

**Proof.** Let \( \phi \) be a Gaussian, and let \( T_1 \) and \( T_2 \) be the operators, defined by the formulas

\[
(T_1 f, g) = (\omega^{-1}_1 V_\phi f, V_\phi g) \quad \text{and} \quad (T_2 f, g) = (\omega_2 V_\phi f, V_\phi g).
\]

Then it follows from Theorem 1.1 in [27] that \( T_1 \) and \( T_2 \) on \( \mathcal{S} \) are uniquely extendable to continuous bijections between \( M^p \) to \( M^p_0(\mathbb{R}^d) \), and from \( M^\infty_0(\mathbb{R}^d) \) to \( M^\infty(\mathbb{R}^d) \). Since \( \mathcal{S} \) is dense in \( M^p_0(\mathbb{R}^d) \) and in \( M^\infty_0(\mathbb{R}^d) \), the result follows if we prove

\[
\|(T_2 \circ \text{Op}_\varphi(a) \circ T_1)f\|_{M^p} \lesssim \|f\|_{M^p}, \quad f \in \mathcal{S}.
\]

For some non-degenerate Gauss function \( \Phi \) which depends on \( \phi \) we have

\[
T_j = \text{Op}(a_j), \ j = 1, 2, \quad \text{where} \quad a_1 = ((\omega_1)^{-1}) \ast \Phi \quad \text{and} \quad a_2 = \omega_2 \ast \Phi.
\]

Furthermore, using the fact that \( \omega_j \in \mathcal{P}_{1,1} \), it follows by straightforward computations that \( a_1 \in \text{SG}_{1,1}^{(1/\omega_1)} \) and \( a_2 \in \text{SG}_{1,1}^{(\omega_2)} \).

By using these facts in combination with Theorems 2.18 and 2.19 we get

\[
T_2 \circ \text{Op}_\varphi(a) \circ T_1 = T_2 \circ (\text{Op}_\varphi(h_1) + S_1) = \text{Op}_\varphi(h_2) + S_2 + T_2 \circ S_1,
\]

for some operators \( S_j \in \text{Op}(\mathcal{S}) \), \( j = 1, 2 \), where

\[
h_1 \in \text{SG}_{1,1}^{(\omega_0/\omega_1)} \quad \text{and} \quad h_2 \in \text{SG}_{1,1}^{(\omega_0 \omega_2/\omega_1)} \subseteq \text{SG}_{1,1}^{m,\mu}.
\]

Since

\[
T_2 \circ S_1 \in \text{Op}(\text{SG}_{1,1}^{\omega_2}) \circ \text{Op}(\mathcal{S}) \subseteq \text{Op}(\mathcal{S}),
\]

35
it follows that
\[ T_2 \circ \text{Op}_\varphi(a) \circ T_1 = \text{Op}_\varphi(h_2) + S_0, \]
where \( S_0 \in \text{Op}(\mathcal{S}) \), giving that \( S_0 \) is continuous on \( M^p \). Furthermore, the fact that \( h_2 \in \text{SG}^{m,p}_{1,1} \) and Theorem 1.2 in \([10]\) imply that
\[ \| \text{Op}_\varphi(h_1)f \|_{M^p} \lesssim \| f \|_{M^p}, \quad f \in \mathcal{S}. \]
This gives the result. \( \square \)

The following definition is justified by Theorems 3.1 and 3.2.

**Definition 3.3.** Let \( r, \rho \in [0, 1], t \in \mathbb{R}, B \) be a topological vector space of distributions on \( \mathbb{R}^d \) such that
\[ \mathcal{S}(\mathbb{R}^d) \subseteq B \subseteq \mathcal{S}'(\mathbb{R}^d) \]
with continuous embeddings. Then \( B \) is called SG-admissible (with respect to \( r, \rho \) and \( d \)) when \( \text{Op}_t(a) \) maps \( B \) continuously into itself, for every \( a \in \text{SG}^{0,0}_{r,\rho}(\mathbb{R}^d) \).

Let \( \varphi \in \text{SG}^{1,1}_{1,1}(\mathbb{R}^{2d}) \) be a regular phase, \( B \) and \( C \) be SG-admissible with respect to \( r, \rho \) and \( d, \omega_0 \in \mathcal{S}'_{r,\rho}(\mathbb{R}^{2d}) \), and let \( \Omega \subseteq \mathbb{R}^d \) be open. Then the pair \((B, C)\) is called weakly SG-ordered (with respect to \( r, \rho, \omega_0, \varphi \) and \( \Omega \)), when the mapping
\[ \text{Op}_\varphi(a) : B \to C \]
is continuous for every \( a \in \text{SG}^{(\omega_0)}_{r,\rho}(\mathbb{R}^{2d}) \) which is supported outside \( \mathbb{R}^d \times \Omega \). Furthermore, \((B, C)\) is called SG-ordered (with respect to \( r, \rho, \omega_0, \varphi \) and \( \Omega \)), when the mappings
\[ \text{Op}_\varphi(a) : B \to C \quad \text{and} \quad \text{Op}_\varphi(b) : C \to B \]
are continuous for every \( a \in \text{SG}^{(\omega_0)}_{r,\rho}(\mathbb{R}^{2d}) \) and \( b \in \text{SG}^{(1/\omega_0)}_{r,\rho}(\mathbb{R}^{2d}) \).

**Remark 3.4.** Let \( \sigma_1, \sigma_2, p, m \) and \( \mu \) be the same as in Theorem 3.1. Then it follows that \( (H^p_{\sigma_1,\sigma_2}, H^p_{\sigma_1-\mu,\sigma_2-m}) \) are weakly SG-ordered with respect to
\[ 1, 1, \omega_0(x, \xi) = \langle x \rangle^m \langle \xi \rangle^\mu, \varphi \] and \( \Omega = B_r(0) \).

Furthermore, if instead \( p, m, \mu \) and \( \omega_j \), \( j = 0, 1, 2 \) are the same as in Theorem 3.2, then it follows that \( (M^p_{(\omega_1)}, M^p_{(\omega_2)}) \) are weakly SG-ordered with respect to
\[ 1, 1, \omega_0(x, \xi), \varphi \] and \( \Omega = \emptyset \).

4. **Propagation results for global Wave-front Sets and Fourier Integral Operators of SG type**

We first recall the definition of the global wave-front set with respect to modulation spaces, given in \([15]\). The content of Subsection 4.1 again comes from \([16]\). In Subsection 4.2, we prove our main results about the propagation of singularities in the SG context, under the action of the Fourier integral operators described above.
4.1. Global Wave-front Sets. Here we recall the definition given in [15] of global wave-front sets for temperate distributions with respect to Banach or Fréchet spaces and state some of their properties (see also [16]). First of all, we recall the definitions of characteristic sets to Banach or Fréchet spaces and state some of their properties (see in [15] of global wave-front sets for temperate distributions with respect to $\mathbb{R}^d$).

**4.1. Global Wave-front Sets.**

On the other hand, $a$ is invertible, in the sense that $1/a$ is a symbol in $\text{SG}^{(1/\omega_0)}(\mathbb{R}^{2d})$, if and only if

$$|a(x, \xi)| \lesssim \omega_0(x, \xi).$$

We need to deal with the situations where (4.1) holds only in certain (conic-shaped) subset of $\mathbb{R}^d \times \mathbb{R}^d$.

**Definition 4.1.** Let $r, \rho \geq 0$, $\omega_0 \in \mathcal{P}_{r, \rho}(\mathbb{R}^{2d})$ and let $a \in \text{SG}^{(\omega_0)}(\mathbb{R}^{2d})$.

1. $a$ is called *locally or type-1 invertible* with respect to $\omega_0$ at the point $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$, if there exist a neighbourhood $X$ of $x_0$, an open conical neighbourhood $\Gamma$ of $\xi_0$ and a positive constant $R$ such that (4.1) holds for $x \in X$, $\xi \in \Gamma$ and $|\xi| \geq R$.

2. $a$ is called *Fourier-locally or type-2 invertible* with respect to $\omega_0$ at the point $(x_0, \xi_0) \in (\mathbb{R}^d \setminus 0) \times \mathbb{R}^d$, if there exist an open conical neighbourhood $\Gamma$ of $x_0$, a neighbourhood $X$ of $\xi_0$ and a positive constant $R$ such that (4.1) holds for $x \in \Gamma$, $|x| \geq R$ and $\xi \in X$.

3. $a$ is called *oscillating or type-3 invertible* with respect to $\omega_0$ at the point $(x_0, \xi_0) \in (\mathbb{R}^d \setminus 0) \times (\mathbb{R}^d \setminus 0)$, if there exist open conical neighbourhoods $\Gamma_1$ of $x_0$ and $\Gamma_2$ of $\xi_0$, and a positive constant $R$ such that (4.1) holds for $x \in \Gamma_1$, $|x| \geq R$, $\xi \in \Gamma_2$ and $|\xi| \geq R$.

If $m \in \{1, 2, 3\}$ and $a$ is not type-$m$ invertible with respect to $\omega_0$ at $(x_0, \xi_0)$, then $(x_0, \xi_0)$ is called *type-$m$ characteristic* for $a$ with respect to $\omega_0$. The set of type-$m$ characteristic points for $a$ with respect to $\omega_0$ is denoted by $\text{Char}^m_{(\omega_0)}(a)$.

The *global set of characteristic points* (the characteristic set), for a symbol $a \in \text{SG}^{(\omega_0)}(\mathbb{R}^{2d})$ with respect to $\omega_0$, is

$$\text{Char}(a) = \text{Char}^1_{(\omega_0)}(a) \cup \text{Char}^2_{(\omega_0)}(a) \cup \text{Char}^3_{(\omega_0)}(a).$$

**Remark 4.2.** In the case $\omega_0 = 1$ we exclude the phrase “with respect to $\omega_0$” in Definition 4.1. For example, $a \in \text{SG}^{0, 0}_{r, \rho}(\mathbb{R}^{2d})$ is type-1 invertible at $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ if $(x_0, \xi_0) \notin \text{Char}^1_{(\omega_0)}(a)$ with $\omega_0 = 1$. This means that there exist a neighbourhood $X$ of $x_0$, an open conical neighbourhood $\Gamma$ of $\xi_0$ and $R > 0$ such that (4.1) holds for $\omega_0 = 1$, $x \in X$ and $\xi \in \Gamma$ satisfies $|\xi| \geq R$.

In the next definition we introduce different classes of cutoff functions (see also Definition 1.9 in [14]).
**Definition 4.3.** Let $X \subseteq \mathbb{R}^d$ be open, $\Gamma \subseteq \mathbb{R}^d \setminus 0$ be an open cone, $x_0 \in X$ and let $\xi_0 \in \Gamma$.

1. A smooth function $\varphi$ on $\mathbb{R}^d$ is called a cutoff (function) with respect to $x_0$ and $X$, if $0 \leq \varphi \leq 1$, $\varphi \in C^\infty_0(X)$ and $\varphi = 1$ in an open neighbourhood of $x_0$. The set of cutoffs with respect to $x_0$ and $X$ is denoted by $\mathscr{C}_{x_0}(X)$ or $\mathscr{C}_{x_0}$.

2. A smooth function $\psi$ on $\mathbb{R}^d$ is called a directional cutoff (function) with respect to $\xi_0$ and $\Gamma$, if there is a constant $R > 0$ and open conical neighbourhood $\Gamma_1 \subseteq \Gamma$ of $\xi_0$ such that the following is true:
   - $0 \leq \psi \leq 1$ and $\text{supp} \psi \subseteq \Gamma$;
   - $\psi(t\xi) = \psi(\xi)$ when $t \geq 1$ and $|\xi| \geq R$;
   - $\psi(\xi) = 1$ when $\xi \in \Gamma_1$ and $|\xi| \geq R$.

The set of directional cutoffs with respect to $\xi_0$ and $\Gamma$ is denoted by $\mathscr{C}^\text{dir}_{\xi_0}(\Gamma)$ or $\mathscr{C}^\text{dir}_{\xi_0}$.

The next proposition shows that $\text{Op}_t(a)$ for $t \in \mathbb{R}$ satisfies convenient invertibility properties of the form
\[
\text{Op}_t(a) \text{ Op}_t(b) = \text{Op}_t(c) + \text{Op}_t(h),
\]
(4.2) outside the set of characteristic points for a symbol $a$. Here $\text{Op}_t(b)$, $\text{Op}_t(c)$ and $\text{Op}_t(h)$ have the roles of “local inverse”, “local identity” and smoothing operators respectively. From these statements it also follows that our set of characteristic points in Definition 4.1 are related to those in [17, 28]. We let $\mathbb{I}_m$ and $\Omega_m$, $m = 1, 2, 3$, be the sets
\[
\mathbb{I}_1 \equiv [0, 1] \times (0, 1], \quad \mathbb{I}_2 \equiv (0, 1] \times [0, 1], \quad \mathbb{I}_3 \equiv (0, 1] \times (0, 1] = \mathbb{I}_1 \cap \mathbb{I}_2,
\]
(4.3) and
\[
\Omega_1 = \mathbb{R}^d \times (\mathbb{R}^d \setminus 0), \quad \Omega_2 = (\mathbb{R}^d \setminus 0) \times \mathbb{R}^d,
\]
\[
\Omega_3 = (\mathbb{R}^d \setminus 0) \times (\mathbb{R}^d \setminus 0),
\]
(4.4)
which will be useful in the sequel.

**Proposition 4.4.** Let $m \in \{1, 2, 3\}$, $(r, \rho) \in \mathbb{I}_m$, $\omega_0 \in \mathscr{P}_{r, \rho}(\mathbb{R}^{2d})$ and let $a \in \text{SG}_{r, \rho}^{(\omega_0)}(\mathbb{R}^{2d})$. Also let $\Omega_m$ be as in (4.4), $(x_0, \xi_0) \in \Omega_m$, and let $(r_0, \rho_0)$ be equal to $(r, 0)$, $(0, \rho)$ and $(r, \rho)$ when $m$ is equal to 1, 2 and 3, respectively. Then the following conditions are equivalent:

1. $(x_0, \xi_0) \notin \text{Char}^m_{(\omega_0)}(a)$;

2. there is an element $c \in \text{SG}_{r, \rho}^{0, 0}$ which is type-$m$ invertible at $(x_0, \xi_0)$, and an element $b \in \text{SG}_{r, \rho}^{1/(\omega_0)}$ such that $ab = c$;

3. (4.2) holds for some $c \in \text{SG}_{r, \rho}^{0, 0}$ which is type-$m$ invertible at $(x_0, \xi_0)$, and some elements $h \in \text{SG}_{r, \rho}^{-\tau_0, -\rho_0}$ and $b \in \text{SG}_{r, \rho}^{1/(\omega_0)}$.
(4) (4.2) holds for some $c_m \in \text{SG}^{0,0}_{r,\rho}$ in Remark 4.5 which is type-$m$ invertible at $(x_0, \xi_0)$, and some elements $h$ and $b \in \text{SG}^{(1,\omega)}_{r,\rho}$, where $h \in \mathcal{S}$ when $m \in \{1, 3\}$ and $h \in \text{SG}^{-\infty,0}$ when $m = 2$.

Furthermore, if $t = 0$, then the supports of $b$ and $h$ can be chosen to be contained in $X \times \mathbb{R}^d$ when $m = 1$, in $\Gamma \times \mathbb{R}^d$ when $m = 2$, and in $\Gamma_1 \times \mathbb{R}^d$ when $m = 3$.

Remark 4.5. Let $X \subseteq \mathbb{R}^d$ be open and $\Gamma, \Gamma_1, \Gamma_2 \subseteq \mathbb{R}^d \setminus \{0\}$ be open cones. Then the following is true.

1. if $x_0 \in X$, $\xi_0 \in \Gamma$, $\varphi \in \mathcal{C}_{x_0}(X)$ and $\psi \in \mathcal{C}_{\xi_0}^{\text{dir}}(\Gamma)$, then $c_1 = \varphi \otimes \psi$ belongs to $\text{SG}^{1,1}_{1,1}(\mathbb{R}^{2d})$, and is type-$1$ invertible at $(x_0, \xi_0)$;
2. if $x_0 \in \Gamma$, $\xi_0 \in X$, $\psi \in \mathcal{C}_{x_0}^{\text{dir}}(\Gamma)$ and $\varphi \in \mathcal{C}_{\xi_0}(X)$, then $c_2 = \varphi \otimes \psi$ belongs to $\text{SG}^{1,1}_{1,1}(\mathbb{R}^{2d})$, and is type-$2$ invertible at $(x_0, \xi_0)$;
3. if $x_0 \in \Gamma_1$, $\xi_0 \in \Gamma_2$, $\psi_1 \in \mathcal{C}_{x_0}^{\text{dir}}(\Gamma_1)$ and $\psi_2 \in \mathcal{C}_{\xi_0}^{\text{dir}}(\Gamma_2)$, then $c_3 = \psi_1 \otimes \psi_2$ belongs to $\text{SG}^{0,0}_{1,1}(\mathbb{R}^{2d})$, and is type-$3$ invertible at $(x_0, \xi_0)$.

We can now introduce the complements of the wave-front sets. More precisely, let $\Omega_m$, $m \in \{1, 2, 3\}$, be given by (4.4), $\mathcal{B}$ be a Banach or Fréchet space such that $\mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{B} \subseteq \mathcal{S}'(\mathbb{R}^d)$, and let $f \in \mathcal{S}'(\mathbb{R}^d)$. Then the point $(x_0, \xi_0) \in \Omega_m$ is called type-$m$ regular for $f$ with respect to $\mathcal{B}$, if

$$\text{Op}(c_m)f \in \mathcal{B},$$

(4.5) for some $c_m$ in Remark 4.5. The set of all type-$m$ regular points for $f$ with respect to $\mathcal{B}$, is denoted by $\Theta^m_{\mathcal{B}}(f)$.

Definition 4.6. Let $m \in \{1, 2, 3\}$, $\Omega_m$ be as in (4.4), and let $\mathcal{B}$ be a Banach or Fréchet space such that $\mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{B} \subseteq \mathcal{S}'(\mathbb{R}^d)$.

1. the type-$m$ wave-front set of $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to $\mathcal{B}$ is the complement of $\Theta^m_{\mathcal{B}}(f)$ in $\Omega_m$, and is denoted by $\text{WF}^m_{\mathcal{B}}(f)$;
2. the global wave-front set $\text{WF}^m_{\mathcal{B}}(f) \subseteq \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$ is the set

$$\text{WF}^m_{\mathcal{B}}(f) \equiv \text{WF}^1_{\mathcal{B}}(f) \cup \text{WF}^2_{\mathcal{B}}(f) \cup \text{WF}^3_{\mathcal{B}}(f).$$

The sets $\text{WF}^1_{\mathcal{B}}(f)$, $\text{WF}^2_{\mathcal{B}}(f)$ and $\text{WF}^3_{\mathcal{B}}(f)$ in Definition 4.6 are also called the local, Fourier-local and oscillating wave-front set of $f$ with respect to $\mathcal{B}$.

From now on we assume that $\mathcal{B}$ in Definition 4.6 is SG-admissible, and recall that Sobolev-Kato spaces and, more generally, modulation spaces, and $\mathcal{S}(\mathbb{R}^d)$ are SG-admissible, see [15][16].

The next result describes the relation between “regularity with respect to $\mathcal{B}$” of temperate distributions and global wave-front sets.

Theorem 4.7. Let $\mathcal{B}$ be SG-admissible, and let $f \in \mathcal{S}'(\mathbb{R}^d)$. Then

$$f \in \mathcal{B} \iff \text{WF}^m_{\mathcal{B}}(f) = \emptyset.$$
For the sake of completeness, we recall that microlocality and microellipticity hold for these global wave-front sets and pseudo-differential operators in $\text{Op}(\mathcal{S}G_{r,\rho}^{(\omega_0)})$, see [15]. This implies that operators which are elliptic with respect to $\omega_0 \in \mathcal{P}_{r,\rho}(\mathbb{R}^{2d})$ when $0 < r, \rho \leq 1$ preserve the global wave-front set of temperate distributions. The next result is an immediate corollary of microlocality and microellipticity for operators in $\text{Op}(\mathcal{S}G_{r,\rho}^{(\omega_0)})$:

**Theorem 4.8.** Let $m \in \{1, 2, 3\}$, $(r, \rho) \in \mathbb{I}_m$, $t \in \mathbb{R}$, $\omega_0 \in \mathcal{P}_{r,\rho}(\mathbb{R}^{2d})$, $a \in \mathcal{S}G_{r,\rho}^{(\omega_0)}(\mathbb{R}^{2d})$ be SG-elliptic with respect to $\omega_0$ and let $f \in \mathcal{S}'(\mathbb{R}^d)$. Moreover, let $(\mathcal{B}, \mathcal{C})$ be a SG-ordered pair with respect to $\omega_0$. Then

$$\text{WF}_{\mathcal{E}}^m(\text{Op}_t(a)f) = \text{WF}_{\mathcal{E}}^m(f).$$

4.2. **Action of SG FIOs on global Wave-front Sets.** Let us denote by $\Phi$ the canonical transformation of $T^*\mathbb{R}^d$ into itself generated by $\varphi \in \mathcal{F}^r$, that is, $\Phi : (x, \xi) \mapsto (y, \eta) = (\Phi_1(x, \xi), \Phi_2(x, \xi))$ is given by

$$\begin{align*}
y &= \varphi'_x(x, \eta) \\
\xi &= \varphi'_\xi(x, \eta).
\end{align*} \quad (4.6)$$

As we have seen in Subsection 2.7, such transformations appear in the Egorov Theorem, through which we will prove our main result below. This justifies the following definition of admissibility of phase functions. Namely, the latter are required to generate transformations of the type (4.6) which “preserve the shape” of the different kinds of neighborhoods appearing in the Definition 4.1 of characteristic set.

**Definition 4.9.** Let $\varphi \in \mathcal{F}^r$ and let $\Phi$ be the canonical transformation (4.6), generated by $\varphi$. Let $m \in \{1, 2, 3\}$, $\Omega_m$ be as in (4.4), and $(x_0, \xi_0) \in \Omega_m$.

1. $\varphi$ is type-1 admissible if, for any $(x_0, \xi_0) \in \Omega_1$, $\tilde{\delta} > 0$, $\tilde{R} > 0$, there exist $\delta > 0$, $R > 0$, such that

$$|x - x_0| \leq \delta, \ |\xi| \geq R, \ \text{and} \ \frac{\xi \cdot \xi_0}{|\xi| \cdot |\xi_0|} \geq 1 - \delta,$$

$$\Rightarrow |\Phi_2(x, \xi)| \geq \tilde{R} \ \text{and} \ \frac{\Phi_2(x, \xi) \cdot \Phi_2(x_0, \xi_0)}{|\Phi_2(x, \xi)| \cdot |\Phi_2(x_0, \xi_0)|} \geq 1 - \tilde{\delta}.$$

2. $\varphi$ is type-2 admissible if, for any $(x_0, \xi_0) \in \Omega_2$, $\tilde{\delta} > 0$, $\tilde{R} > 0$, there exist $\delta > 0$, $R > 0$, such that

$$|\xi - \xi_0| \leq \delta, \ |x| \geq R, \ \text{and} \ \frac{x \cdot x_0}{|x| \cdot |x_0|} \geq 1 - \delta,$$

$$\Rightarrow |\Phi_1(x, \xi)| \geq \tilde{R} \ \text{and} \ \frac{\Phi_1(x, \xi) \cdot \Phi_1(x_0, \xi_0)}{|\Phi_1(x, \xi)| \cdot |\Phi_1(x_0, \xi_0)|} \geq 1 - \tilde{\delta}.$$
(3) \( \varphi \) is type-3 admissible if, for any \((x_0, \xi_0) \in \Omega_3, \tilde{\delta} > 0, \tilde{R} > 0, \) there exist \( R > 0, \delta > 0 \), such that

\[
|x|, |\xi| \geq R, \quad \frac{x \cdot x_0}{|x| \cdot |x_0|} \geq 1 - \delta \quad \text{and} \quad \frac{\xi \cdot \xi_0}{|\xi| \cdot |\xi_0|} \geq 1 - \delta
\]

\[
\Rightarrow |\Phi_1(x, \xi)|, |\Phi_2(x, \xi)| \geq \tilde{R},
\]

\[
\frac{\Phi_1(x, \xi) \cdot \Phi_1(x_0, \xi_0)}{|\Phi_1(x, \xi)| \cdot |\Phi_1(x_0, \xi_0)|} \geq 1 - \tilde{\delta},
\]

and

\[
\frac{\Phi_2(x, \xi) \cdot \Phi_2(x_0, \xi_0)}{|\Phi_2(x, \xi)| \cdot |\Phi_2(x_0, \xi_0)|} \geq 1 - \tilde{\delta}.
\]

We say that \( \varphi \) is admissible if it is type-\( m \) admissible for all \( m = 1, 2, 3 \).

**Remark 4.10.** Since both the maps \( \mathbb{R}^d \to \mathbb{R}^d \) appearing in the two lines of (4.6) are SG-diffeomorphisms with SG\(^0\) parameter dependence, we immediately see that, for any weight function \( \omega_0 \in \mathcal{P}_{r, \rho}(\mathbb{R}^{2d}), r, \rho \geq 0, \omega_0 \circ \Phi \in \mathcal{P}_{r, \rho}(\mathbb{R}^{2d}) \), and \( \omega_0 \asymp \omega_0 \circ \Phi \). Then, in view of Definition 4.1, if \( \varphi \) is type-\( m \) admissible, \( m = 1, 2, 3 \), with \( \mathbb{I}_m \) as in (4.3), \((r, \rho) \in \mathbb{I}_m, \omega_0 \in \mathcal{P}_{r, \rho}(\mathbb{R}^{2d}) \), for any \( a \in \text{SG}^{(\omega_0)}(\mathbb{R}^{2d}) \) we find

\[
(x_0, \xi_0) \in \text{Char}^m_{(\omega_0)}(a \circ \Phi) \Leftrightarrow (y_0, \eta_0) = \Phi(x_0, \xi_0) \in \text{Char}^m_{(\omega_0)}(a).
\]

**Remark 4.11.** It is easy to see that the “standard” phase functions \( \varphi(x, \xi) \), 1-homogeneous with respect to \( \xi \) for large \(|\xi|\), are type-1 admissible. Examples of type-2 and type-3 admissible phase functions can be obtained by assuming similar homogeneity properties with respect to \( x \) or to both variables. A main example of admissible phase functions, which is not necessarily homogeneous, is given by the SG-classical phase functions. Families of such objects, smoothly depending on a parameter \( t \in [-T, T], T > 0, \) are obtained solving Cauchy problems associated with classical SG-hyperbolic systems with diagonal principal part, or, similarly, for classical SG-hyperbolic linear operators of Levi type and order \( N \) (see [13,17,18] for the detailed definitions, and Subsection 4.3 below). In short, omitting the dependence on the time variable \( t \), a SG-classical phase functions \( \varphi \) admits expansions in terms which are homogeneous with respect to \( x \), respectively \( \xi \), satisfying suitable compatibility relations, see, e.g., [17,18]. Then, in particular, \( \varphi \) admits a principal symbol, given by a triple \((\varphi_1, \varphi_2, \varphi_3)\), that is, it can be written as

\[
\varphi(x, \xi) = \chi(\xi) \varphi_1(x, \xi) + \chi(x)(\varphi_2(x, \xi) - \chi(\xi) \varphi_3(x, \xi)) + l.o.t. \quad (4.7)
\]

In (4.7), \( \chi \) is a 0-excision function, while \( \chi(\xi) \varphi_1(x, \xi), \chi(x) \varphi_2(x, \xi), \chi(\xi) \chi(x) \varphi_3(x, \xi) \in \text{SG}^{1,1}(\mathbb{R}^{2d}), \) where \( \varphi_1 \) is 1-homogeneous with respect to the variable \( \xi \), \( \varphi_2 \) is 1-homogeneous with respect to the variable \( x \), and \( \varphi_3 \) is 1-homogeneous with respect to each one of the variables \( x, \xi \). Observe
that then
\[ \varphi(x, \xi) = \chi(\xi) \varphi_1(x, \xi) \mod \text{SG}_{1,1}^0(\mathbb{R}^{2d}), \]
\[ \varphi(x, \xi) = \chi(x) \varphi_2(x, \xi) \mod \text{SG}_{1,1}^0(\mathbb{R}^{2d}), \] (4.8)
\[ \varphi(x, \xi) = \chi(x) \chi(\xi) \varphi_3(x, \xi) \mod \text{SG}_{1,1}^0(\mathbb{R}^{2d}) + \text{SG}_{1,1}^0(\mathbb{R}^{2d}). \]

The homogeneity of the leading terms in (4.8) implies, in particular,
\[ \varphi'_x(x, \xi) = |\xi| \left[ \varphi'_{1,x} \left( x, \frac{\xi}{|\xi|} \right) + |\xi|^{-1} r_1(x, \xi) \right] \quad \text{for } |\xi| > R, \]
\[ \varphi'_\xi(x, \xi) = |x| \left[ \varphi'_{2,\xi} \left( x, \frac{\xi}{|x|} \right) + |x|^{-1} r_2(x, \xi) \right] \quad \text{for } |x| > R, \]
\[ \varphi'_x(x, \xi) = |\xi| \left[ \varphi'_{3,x} \left( \frac{x}{|x|}, \frac{\xi}{|\xi|} \right) + |\xi|^{-1} (r_{31}(x, \xi) + s_{31}(x, \xi)) \right] \]
(4.9)
for \( |x|, |\xi| > R, \)
\[ \varphi'_\xi(x, \xi) = |x| \left[ \varphi'_{3,\xi} \left( \frac{x}{|x|}, \frac{\xi}{|\xi|} \right) + |x|^{-1} (r_{32}(x, \xi) + s_{32}(x, \xi)) \right] \]
for \( |x|, |\xi| > R, \)

with \( r_1, r_2, r_{31}, r_{32} \in \text{SG}_{1,1}^0(\mathbb{R}^{2d}), s_{31} \in \text{SG}_{1,1}^{-1,1}(\mathbb{R}^{2d}), s_{32} \in \text{SG}_{1,1}^{1,-1}(\mathbb{R}^{2d}). \)

By the properties of generalized SG symbols and (4.9) it is possible to prove that all the conditions in Definition 4.9 are fulfilled.

We can now state our main results concerning the propagation of (global) singularities under the action of the generalized SG FIOs.

**Theorem 4.12.** Let \( \varphi \in \mathcal{F}_m^g \) be type \( m \)-admissible, \( m \in \{1, 2, 3\} \), and \( \omega_0 \in \mathcal{P}_{r, \rho}(\mathbb{R}^{2d}), a \in \text{SG}_{r, \rho}(\mathbb{R}^{2d}), r, \rho \geq 1/2 \). Assume that \( a \) is SG-elliptic and \( (\mathcal{B}, \mathcal{C}) \) are weakly SG-ordered with respect to
\[ 1, 1, \omega_0(x, \xi), \varphi \text{ and } \Omega = \emptyset. \]

Then, for any \( u \in \mathcal{B} \), we have
\[ \text{WF}_m^m(\mathcal{C})(Au) = \Phi(\text{WF}_m^m(u)), \] (4.10)
where \( \Phi \) is the canonical transformation generated by \( \varphi \), defined in (4.6).

**Proof.** Let \( (x_0, \xi_0) \in \Theta_0^m(u) \) and let \( c_m \in \text{SG}_{1,1}^{0,0} \), be a symbol as in (4.5) and Remark 4.5 and set \( C_m = \text{Op}(c_m) \), so that \( C_m u \in \mathcal{B} \). Let us set \( A = \text{Op}_\varphi(a) \) and \( Q_m = A c_m A^{-1} \equiv Q_m \circ A \equiv A c_m \), that is, the two sides of the last formula coincide modulo type-\( m \) smoothing operators. By Theorem 2.27 and (4.6), we have \( \text{Sym}(Q_m) = c_m \circ \Phi \mod l.o.t. \), which implies \( \text{Sym}(Q_m) \in \text{SG}_{r, \rho}^{0,0} \). Then, \( \Phi(x_0, \xi_0) \in \Theta_\varphi^m(Au) \), since, by the hypotheses on \( (\mathcal{B}, \mathcal{C}) \), \( Q_m(Au) \equiv A(Pu) \in \mathcal{C} \). This means that
\[ \Omega_m \setminus \text{WF}_m^m(Au) \supseteq \Phi(\Omega_m \setminus \text{WF}_m^m(u)). \] (4.11)

Complementing (4.11) with respect to \( \Omega_m \), repeating a similar argument starting from \( Au \), recalling Remark 4.10 and that \( \Phi \) is a diffeomorphism, we finally obtain (4.10). \( \square \)
4.3. Applications to SG-hyperbolic problems. In this subsection we apply the results obtained above to the SG-hyperbolic problems considered in [17, 18], to which we refer for the details omitted here. We prove that, under natural conditions, the singularities described by the generalized wave-front sets \( \text{WF}^m_\nu(U_0) \), \( m = 1, 2, 3 \), for a scalar or vector-valued initial data \( U_0 \in \mathcal{B} \), propagate to the solution \( U(t) \), \( t \in [-T, T] \), in the sense that the points of \( \text{WF}^m_\nu(U(t)) \), \( m = 1, 2, 3 \) lie on bicharacteristics curves determined by the phase functions of the Fourier operators \( A(t) \) such that, modulo smooth remainders, \( U(t) = A(t)U_0 \). Here we choose, for simplicity, \( \omega_0(x, \xi) = \langle x \rangle \langle \xi \rangle \).

In this subsection, the subscript “cl” denotes the subclasses of SG symbols which are classical, see [18]. We first need to recall some definitions and results, mainly taken from [12, 13, 18].

**Definition 4.13.** Let \( J = [-T, T] \subset \mathbb{R}, T > 0 \), and consider the linear operator

\[
L = D_t^\nu + P_1(t) D_t^{\nu-1} + \cdots + P_\nu(t),
\]

(4.12)

with \( P_j(t) = \text{Op}(p_j(t)) \), \( p_j = p_j(t, x, \xi) \in C^\infty(J, \text{SG}_{1,1,1,1,1}(\mathbb{R}^{2d})) \). Let

\[
l(x, \xi, t, \tau) = \tau^\nu + q_1(t, x, \xi) \tau^{\nu-1} + \cdots + q_\nu(t, x, \xi)
\]

be the principal symbol of \( L \), with \( q_j \in C^\infty(J, \text{SG}_{1,1,1,1,1}(\mathbb{R}^{2d})) \) such that \( q_j(t, \ldots) \) is the principal symbol of \( p_j(t, \ldots) \), in the sense of (4.7). \( L \) is called SG-classical hyperbolic with constant multiplicities if the characteristic equation

\[
\tau^\nu + q_1(t, x, \xi) \tau^{\nu-1} + \cdots + q_\nu(t, x, \xi) = 0
\]

(4.13)

has \( \mu \leq \nu \) distinct real roots \( \tau_j = \tau_j(t, x, \xi) \in C^\infty(J, \text{SG}_{1,1,1,1,1}(\mathbb{R}^{2d})) \) with multiplicities \( l_j, 1 \leq l_j \leq \nu, j = 1, \ldots, \mu \), which satisfy, for a suitable \( C > 0 \) and all \( t \in J, x, \xi \in \mathbb{R}^d \),

\[
\tau_{j+1}(t, x, \xi) - \tau_j(t, x, \xi) \geq C \langle \xi \rangle \langle x \rangle, j = 1, \ldots, \mu - 1.
\]

(4.14)

\( L \) is called strictly hyperbolic if it is hyperbolic with constant multiplicities and the multiplicity of all the \( \tau_j, j = 1, \ldots, \nu = \mu \), is equal to 1.

A standard strategy to solve the Cauchy problem

\[
\begin{cases}
L u(t) = 0, & t \in J, \\
D^k_t u(0) = u_0^k, & k = 0, \ldots, \nu - 1,
\end{cases}
\]

(4.15)

for \( L \) hyperbolic with constant multiplicities and initial data \( u_0^k, k = 1, \ldots, \nu - 1 \), chosen in appropriate functional spaces, is to show that this is equivalent to solving, modulo smooth elements, a Cauchy problem for a first order system

\[
\begin{cases}
\frac{\partial U}{\partial t}(t) - iK(t) U(t) = 0, & t \in J, \\
U(0) = U_0,
\end{cases}
\]

43
with a coefficient matrix $K$ of special form. In our case, one obtains that $K = \text{Op}((k_{ij}(t,x,D)_{i,j}))$, is a $\mu\nu \times \mu\nu$ matrix of SG pseudo-differential operators with symbols $k_{ij} \in C^\infty(J,SG_{1,1,cl}^{k,1})$. Under suitable assumptions, see [17-18], the principal part $k_1$ of $k = k_1 + k_0$, $k_j \in C^\infty(J,SG_{1,1,cl}^{k,1})$, $j = 0,1$, turns out to be diagonal, so that the system will be symmetric, cfr. [11-13]. This implies that the corresponding Cauchy problem is well-posed. One of the main advantages for using this algorithm is the following Proposition 4.14 which is an adapted version of the Mizohata Lemma of Perfect Factorization, proved in [19] for the general SG symbols (see also the references quoted therein).

**Proposition 4.14.** Let $L$ be a SG-classical hyperbolic linear operator with constant multiplicities $l_j$, $j = 1,\ldots,\mu \leq \nu$, as in Definition 4.13 Then, it is possible to factor $L$ as

$$L = L_\mu \cdots L_1 + \sum_{s=1}^{\nu} \text{Op}(r_s(t)) D_t^{\nu-s}$$

with $L_j = (D_t - \text{Op}(\tau_j(t)))^{l_j} + \sum_{k=1}^{l_j} \text{Op}(s_{jk}(t)) (D_t - \text{Op}(\tau_j(t)))^{l_j-k}$ and

$$s_{jk} \in C^\infty(J,SG_{1,1,cl}^{k-1,0}(\mathbb{R}^{2d})), r_s \in C^\infty(J,\mathcal{S}(\mathbb{R}^{2d})),

j = 1,\ldots,\mu, k = 1,\ldots,l_j, s = 1,\ldots,\nu.$$

The following corollary, also obtained in [19], follows by means of a reordering of the roots $\tau_j$ of the principle symbol of $L$.

**Corollary 4.15.** Let $c_j$, $j = 1,\ldots,\mu$, denote the reorderings of the $\mu$-tuple $(1,\ldots,\mu)$ given by

$$c_j(i) = \begin{cases} j + i & \text{for } j + i \leq \mu \\
 j + i - \mu & \text{for } j + i > \mu, \end{cases}, \quad i, j = 1,\ldots,\mu,$$

that is, $c_1 = (2,\ldots,\mu,1), \ldots, c_\mu = (1,\ldots,\mu).$ Then, under the same hypotheses of Proposition 4.14 we have

$$L = L_{c_\mu(1)}^{(m)} \cdots L_{c_1(1)}^{(m)} + \sum_{s=1}^{\nu} \text{Op}(r_s^{(m)}(t)) D_t^{\nu-s}$$

with $L_j^{(m)} = (D_t - \text{Op}(\tau_j(t)))^{l_j} + \sum_{k=1}^{l_j} \text{Op}(s_{jk}^{(m)}(t)) (D_t - \text{Op}(\tau_j(t)))^{l_j-k}$ and

$$s_{jk}^{(m)} \in C^\infty(J,SG_{1,1,cl}^{k-1,0}(\mathbb{R}^{2d})), r_s^{(m)} \in C^\infty(J,\mathcal{S}(\mathbb{R}^{2d})),

m, j = 1,\ldots,\mu, k = 1,\ldots,l_j, s = 1,\ldots,\nu.$$

**Definition 4.16.** We say that a SG-classical hyperbolic operator $L$ is of Levi type if it satisfies the SG-Levi condition\[3\]

$$s_{jk}^{(m)} \in C^\infty(J,SG_{1,1,cl}^{0,0}(\mathbb{R}^{2d})), m, j = 1,\ldots,\mu, k = 1,\ldots,l_j. \quad (4.16)$$

\[3\]Let us observe that (4.13) needs to be fulfilled only for a single value of $m$. 


The following Theorem 4.17 gives the well-posedness for the Cauchy problem (4.15) and the propagation results of the global wave-front sets of modulation space type $WF^m_C(u(t))$, $m = 1, 2, 3$, for the corresponding solution $u(t)$, under natural assumptions on the initial data and the modulation space $C$. It immediately follows by the analysis of SG-classical hyperbolic Cauchy problems in [18], by Section 3 and by Theorem 4.12.

**Theorem 4.17.** Let $L$ be a SG-classical hyperbolic operator with constant multiplicities and of Levi type, and denote by $l = \max\{l_1, \ldots, l_\mu\}$ the maximum multiplicity of the distinct real roots $\tau_j$, $j = 1, \ldots, \mu$, of the characteristic equation (4.13). Then, for any choice of initial data $u_0^j \in S'(\mathbb{R}^d)$, $j = 0, \ldots, \nu - 1$, the Cauchy problem (4.15) admits a unique solution $u \in C(J', S'(\mathbb{R}^d))$, $J' = [-T', T']$, $0 < T' \leq T$. Collecting the initial conditions in the vector

$$c_0 = \begin{pmatrix}
  u_0^0 \\
  u_0^1 \\
  \vdots \\
  u_0^{\nu - 1}
\end{pmatrix},$$

the solution $u$ is given by

$$u(t) = (A_1(t) + \cdots + A_\mu(t))c_0,$$

where each $A_j(t) = Op_{\varphi_j(t)}(a_j(t))$ is a type I FIO with regular phase function $\varphi_j \in C^\infty(J', S') \cap C^\infty(J', SG_{1,1,c}(\mathbb{R}^{2d}))$, solution of the eikonal equation associated with $\tau_j$, and vector-valued amplitude functions $a_j = (a_{j0}, \ldots, a_{j\nu - 1})$ with $a_{jk} \in C^\infty(J', SG_{1,1,c}^{l-k-1,l-k-1}(\mathbb{R}^{2d}))$, $j = 1, \ldots, \mu$, $k = 0, \ldots, \nu - 1$. Moreover, let $u_0^k \in B_k$, and assume that $C$ is such that $(B_k, C)$, $k = 0, \ldots, \nu - 1$, are weakly SG-ordered with respect to

$$1, 1, \langle x \rangle^{l-k-1} \langle \xi \rangle^{l-k-1}, \varphi_k(t) \text{ and } \emptyset.$$

Then, we have well-posedness with respect to the $\nu$-tuple of generalized modulation spaces $(B_0, \ldots, B_{\nu - 1})$ and $C$, $u \in C(J', C)$, and

$$WF^m_C(u(t)) \subseteq \bigcup_{j=1}^\mu \bigcup_{k=0}^{\nu-1} \Phi_j^{-1}(t)(WF_{B_k}^m(u_0^k)),$$

where $\Phi_j(t)$ is the canonical transformation of the form (4.6) associated with the phase function $\varphi_j(t)$.

**Appendix A. Asymptotic Expansions in the Weyl-Hörmander Classes**

**A.1. Metrics.** We recall the definition of symbol classes which are considered. (See Sections 18.4–18.6 in [28].) Assume that $a \in C^N(W)$, $g$ is
an arbitrary Riemannian metric on $W$, and that $m > 0$ is a measurable function on $W$. For each $k = 0, \ldots, N$, let
\[ |a|^g_k(X) = \sup_{Y_1, \ldots, Y_k \in W} |a^{(k)}(X; Y_1, \ldots, Y_k)|, \]
where the supremum is taken over all $Y_1, \ldots, Y_k \in W$ such that $g_X(Y_j) \leq 1$ for $j = 1, \ldots, k$. Also set
\[ \|a\|^g_{m,N} \equiv \sum_{k=0}^{N} \sup_{X \in W} \left( |a|^g_k(X)/m(X) \right), \]
let $S_N(m, g)$ be the set of all $a \in C^N(W)$ such that $\|a\|^g_{m,N} < \infty$, and let
\[ S(m, g) \equiv \bigcap_{N \geq 0} S_N(m, g). \]

Next we recall some properties for the metric $g$ on $W$ (cf. [9, 28, 34]). It follows from Section 18.6 in [28] that for each $X \in W$, there are symplectic coordinates $Z = \sum_{j=1}^{n} (z_j e_j + \zeta_j \varepsilon_j)$ which diagonalize $g_X$, i.e. $g_X$ takes the form
\[ g_X(Z) = \sum_{j=1}^{n} \lambda_j(X)(z_j^2 + \zeta_j^2), \]
where
\[ \lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X) > 0, \]
only depend on $g_X$ and are independent of the choice of symplectic coordinates which diagonalize $g_X$.

The dual metric $g^\sigma$ and Planck’s function $h_g$ with respect to $g$ and the symplectic form $\sigma$ are defined by
\[ g^\sigma_X(Z) \equiv \sup_{Y \neq 0} g_X(Y) \quad \text{and} \quad h_g(X) = \sup_{Z \neq 0} \left( g_X(Z) \right)^{1/2}, \]
respectively. It follows that if (A.3) and (A.4) are fulfilled, then $h_g(X) = \lambda_1(X)$ and
\[ g^\sigma_X(Z) = \sum_{j=1}^{n} \lambda_j(X)^{-1}(z_j^2 + \zeta_j^2). \]

In most of the applications we have that $h_g(X) \leq 1$ everywhere, i.e. the uncertainty principle holds.

The metric $g$ is called symplectic if $g_X = g^\sigma_X$ for every $X \in W$. It follows that $g$ is symplectic if and only if $\lambda_1(X) = \cdots = \lambda_n(X) = 1$ in (A.3).

We recall that parallel to $g$ and $g^\sigma$, there is also a canonical way to assign a corresponding symplectic metric $g^0$. (See e.g. [34].) More precisely, let $Mg = (g + g^\sigma)/2$ and define
\[ g^0_X = \lim_{k \to \infty} M^k g. \]
Then $g^0$ is a symplectic metric, defined in a symplectically invariant way and if $Z = \sum_{j=1}^n (z_j e_i + \zeta_j \epsilon_j)$ are symplectic coordinates such that (A.3) is fulfilled, then

$$g_X^0(Z) = \sum_{j=1}^n (z_j^2 + \zeta_j^2).$$

More generally, in [34], it is proved that if (A.3) holds and $g^t$ is the Riemannian metric, given by

$$g_X^t(Z) = \sum_{j=1}^n \lambda_j(X)^t(z_j^2 + \zeta_j^2),$$

then $g^t$ is symplectically invariant defined, for every $t \in \mathbb{R}$. We note that $g^t = g$, $g^t = g^0$ and $g^t = g^0$, when $t = 0$ and $t = 1$, $t = -1$ and $t = 0$, respectively. We also note that the dual metric for $g^t$ is given by $g_t^\ast$.

The Riemannian metric $g$ on $W$ is called slowly varying if there are positive constants $c$ and $C$ such that

$$g_X(Y - X) \leq c \implies C^{-1}g_Y \leq g_X \leq Cg_Y. \quad \text{(A.5)}$$

More generally, assume that $g$ and $G$ are Riemannian metrics on $W$. Then $G$ is called $g$-continuous, if there are positive constants $c$ and $C$ such that

$$g_X(Y - X) \leq c \implies C^{-1}G_Y \leq G_X \leq CG_Y. \quad \text{(A.5)'}$$

By duality it follows that $g$ is slowly varying if and only if $g^\sigma$ is $g$-continuous, and that (A.5) is equivalent to (A.5)', when $G = g^\sigma$. More generally, in [34] it is proved that if $g$ is slowly varying and $t_1, t_2 \in [-1, 1]$, then $g^{t_1}$ is $g^{t_2}$-continuous. In particular, $g^{t_1}$ is $g$-continuous.

A positive function $m$ on $W$ is called $g$-continuous if there are constants $c$ and $C$ such that

$$g_X(Y - X) \leq c \implies C^{-1}m(Y) \leq m(X) \leq Cm(Y). \quad \text{(A.6)}$$

We observe that if $g$ is slowly varying, $N \geq 0$ is an integer and $m$ is $g$-continuous, then $S_N(m, g)$ is a Banach space when the topology is defined by the norm (A.2). Moreover, $S(m, g)$ is a Frechét space under the topology defined by the norms (A.2) for all $N \geq 0$.

The Riemannian metric $g$ on $W$ is called $\sigma$-temperate, if there is a constant $C > 0$ and an integer $N \geq 0$ such that

$$g_Y(Z) \leq Cg_X(Z)(1 + g^\sigma_X(X - Y))^N, \quad \text{for all } X, Y, Z \in W. \quad \text{(A.7)}$$

We observe that if (A.7) holds, then (A.7) still holds after the term $g^\sigma_X(X - Y)$ is replaced by $g^0_X(X - Y)$, provided the constants $C$ and $N$ have been replaced by larger ones if necessary. (See also [28].)
More generally, if $g$ and $G$ are Riemannian metrics on $W$, then $G$ is called $(\sigma, g)$-temperate, if there is a constant $C$ and an integer $N \geq 0$ such that

$$\begin{align*}
G_X(Z) &\leq CG_Y(Z)(1 + g^g_X(X - Y))^N, \\
G_X(Z) &\leq CG_Y(Z)(1 + g^\sigma_X(X - Y))^N, \quad \text{for all } X, Y, Z \in W.
\end{align*}$$

By duality it follows that $G$ is $(\sigma, g)$-temperate, if and only if $G^\sigma$ is $(\sigma, g)$-temperate. In particular, $g$ is $\sigma$-temperate, if and only if $g^\sigma$ is $(\sigma, g)$-temperate. We also note that if $g$ is $\sigma$-temperate and one of the inequalities in \((A.7)^{'}\) holds, then $G$ is $(\sigma, g)$-temperate.

The weight function $m$ is called $(\sigma, g)$-temperate if \((A.7)^{'}\) holds after $G_X(Z)$ and $G_Y(Z)$ have been replaced by $m(X)$ and $m(Y)$ respectively.

In the following proposition we give examples on important functions related to the slowly varying metric $g$ and which are symplectically invariantly defined. Here we set

$$\Lambda_g(X) = \lambda_1(X) \cdots \lambda_n(X),$$

when $g_X$ is given by \((A.3)\).

**Proposition A.1.** Assume that $g$ is a Riemannian metric on $W$, and that $X \in W$ is fixed. Also assume that the symplectic coordinates are chosen such that \((A.3)\) holds. Then the following are true:

1. $\lambda_j$ for $1 \leq j \leq n$ and $\Lambda_g$ are symplectically invariantly defined;
2. if in addition $g$ is slowly varying, then $\lambda_j$ for $1 \leq j \leq n$ and $\Lambda_g$ are $g$-continuous;
3. if in addition $g$ is $\sigma$-temperate, then $\lambda_j$ for $1 \leq j \leq n$ and $\Lambda_g$ are $(\sigma, g)$-temperate.

**Proof.** The assertion follows immediately from the fact that

$$\lambda_j(X) = \inf_{W_j} \left( \sup_{Y \in W_j \setminus 0} \left( \frac{g^g_X(Y)}{g^\sigma_X(Y)} \right)^{1/2} \right),$$

where the infimum is taken over all symplectic subspaces $W_j$ of $W$ of dimension $2(n - j + 1)$.

We note that an alternative proof of (1) in Proposition A.1 can be found in Section 18.5 in [28].

The following definition is motivated by the general theory of Weyl calculus. (See Section 18.4–18.6 in [28].)

**Definition A.2.** Assume that $g$ is a Riemannian metric on $W$. Then $g$ is called

1. *feasible* if $g$ is slowly varying and $h_g \leq 1$ everywhere;
2. *strongly feasible* if $g$ is feasible and $\sigma$-temperate.
If $g$ is feasible and $m$ is $g$-continuous, then $S(h^r_g m, g)$ decreases with respect to $r$. For conveniency we set

$$S(h^\infty_g m, g) \equiv \bigcap_{r \geq 0} S(h^r_g m, g),$$

in this situation.

Note that feasible and strongly feasible metrics are not standard terminology. In the literature it is common to use the term “Hörmander metric” or “admissible metric” instead of “strongly feasible” for metrics which satisfy (ii) in Definition A.2. (See [4–8].) An important reason for us to follow [9, 34] concerning this terminology is that we permit metrics which are not admissible in the sense of [4–8], and that we prefer similar names for metrics which satisfy (i) or (ii) in Definition A.2.

It is obvious that $g^{t_1} \leq g^{t_2}$ when $t_1 \leq t_2$ and $h_g \leq 1$. In particular, $g \leq g^{t_1} \leq g^\sigma$ when $-1 \leq t \leq 1$ and $h_g \leq 1$. In the following proposition we list some important properties for strongly feasible metrics. The proof is omitted since the result can be found in [34].

**Proposition A.3.** Let $g$ be a strongly feasible metric on $W$, $G$ be a Riemannian metric on $W$, and let $t_1, t_2 \in [-1, 1]$ be such that $t_2 > -1$. If $G$ is $(\sigma, g)$-temperate, then $G^{t_1}$ is $(\sigma, g^{t_2})$-temperate.

In particular, $g^{t_1}$ is $(\sigma, g^{t_2})$-temperate, and if $t \in [0, 1]$, then $g^t$ is strongly feasible.

**Remark A.4.** Assume that $g$ is slowly varying on $W$ and let $c$ be the same as in [A.5]. Then it follows from Theorem 1.4.10 in [28] that there is a constant $\varepsilon_0 > 0$, an integer $N_0 \geq 0$ and a sequence $\{X_j\}_{j \in \mathbb{N}}$ in $W$ such that the following is true:

1. there is a positive number $\varepsilon$ such that $g_{X_j}(X_j - X_k) \geq \varepsilon_0$ for every $j, k \in \mathbb{N}$ such that $j \neq k$;
2. $W = \bigcup_{j \in \mathbb{N}} U_j$, where $U_j$ is the $g_{X_j}$-ball $\{X : g_{X_j}(X - X_j) < c\}$;
3. the intersection of more than $N_0$ balls $U_j$ is empty.

**Remark A.5.** It follows from Section 1.4 and Section 18.4 in [28] that if $g$ is a slowly varying metric on $W$, and (1)–(3) in Remark A.4 holds, then there is a sequence $\{\varphi_j\}_{j \in \mathbb{N}}$ in $C_0^\infty(W)$ such that the following is true:

1. $0 \leq \varphi_j \in C_0^\infty(U_j)$ for every $j \in \mathbb{N}$;
2. $\sup_{j \in \mathbb{N}} \|g^{X_j}_{1, N} \varphi_j\|_{1, N} < \infty$ for every integer $N \geq 0$ (i.e. $\{\varphi_j\}_{j \in \mathbb{N}}$ is a bounded sequence in $S(1, g)$);
3. $\sum_{j \in \mathbb{N}} \varphi_j = 1$ on $W$.

**A.2. Asymptotic expansions.** We shall next consider properties on asymptotic expansions for elements in the symbol class $S(m, g)$. Let $g$ be a feasible metric, and let $m$ be $g$-continuous. Then the following
proposition shows that $S(m, g)$ fulfills convenient asymptotic expansion properties.

**Proposition A.6.** Let $g$ be feasible and let $m$ be $g$-continuous. If $r_j \geq 0$, $j \geq 1$, strictly increases to infinity when $j \to \infty$, and and $a_j \in S(h_g^{r_j} m, g)$, then there is an element $a \in S(h_g^{r_j} m, g)$ such that

$$a - \sum_{k<j} a_k \in S(h_g^{r_j} m, g), \quad \text{for every } j \geq 1. \quad (A.9)$$

**Proof.** Let $U_j$ and $\varphi_j$ be the same as in Remarks A.4 and A.5. For any integer $k \geq 1$, let $J_k$ be the set of all $l \geq 1$ such that

$$|a_k \varphi_l|_{n}^{\varphi}(X) \leq 2^{-k} h_g^{r_{k-1}}(X)m(X), \quad n \leq k, \ X \in W, \quad (A.10)$$

and let $\psi_k = \sum_{i \in J_k} \varphi_i$. We also let $b_1 = a_1$, and define inductively

$$b_{k+1} = b_k + \psi_{k+1}a_{k+1}, \quad k \geq 1.$$

We claim that

1. $b = \lim_{k \to \infty} b_k$ exists and defines an element in $S(m, g)$. Furthermore, $\lim_{k \to \infty} \|b - b_k\|_{N, h_g^r}^\varphi = 0$, for any $r > 0$ and $N \geq 0$;

2. $b - \sum_{j<k} a_j \in S(h_g^{r_j} m, g)$.

In fact, by the definitions and Weierstrass theorem it follows that $b$ exists in $S(m, g)$, and that for $N$ fixed, then $\lim_{k \to \infty} \|b - b_k\|_{N, h_g^r}^\varphi = 0$, for every fixed $r > 0$. This gives (1) in the claim.

Next we prove (2). We have

$$b - \sum_{j<k} a_j = u_1 + u_2,$$

where

$$u_1 = \sum_{j=2}^{k-1} (\psi_j - 1)a_j \quad \text{and} \quad u_2 = \sum_{j=k}^{\infty} \psi_j a_j.$$

The result follows if we prove that $u_1 \in S(h_g^{r_j} m, g)$ and $u_2 \in S(h_g^{r_j} m, g)$ for every $r > 0$.

Let

$$v_j = \sum_{i \notin J_j} \varphi_i a_j.$$

Then

$$u_1 = \sum_{j=2}^{k-1} v_j,$$

and we shall investigate the terms $v_j$ separately. First let $c_0 > 0$ be fixed and let $J_{1,j}$ be the set of all $i \in J_j$ such that $h_g(X) \geq c_0$ when $X \in U_i$. 
Alo let \( J_{2,j} = J_j \setminus J_{1,j} \), and let \( \Omega_{k,j} = \bigcup_{i \in J_{k,j}} U_i \). Then \( v_j = v_{1,j} + v_{2,j} \), where
\[
v_{k,j} = \sum_{i \in J_{k,j}} \varphi_i a_j.
\]
Since \( h_g \geq c_0 \) on \( \Omega_{1,j} \), it follows that if \( r \geq 0 \) and \( d \) is smooth on \( W \) with support in \( \Omega_{1,j} \), then
\[
v \in S(h^*_g m, g) \iff v \in S(m, g).
\]
In particular, \( v_{1,j} \in S(h^*_g m, g) \).

Next we consider \( v_{2,j} \). From the fact that (A.10) is violated for some \( n \in [0, j] \), \( a_j \in S(h^{r_j}_g m, g) \), \( r_{j-1} < r_j \) and \( h_g(X) < Cc_0 \) when \( X \in U_{i \in J_{2,j}} U_i \), it follows that \( J_{2,j} \) is a finite set, provided \( c_0 \) was chosen small enough. Here \( C \) is the same as in Definition A.5. This implies that
\[
v_{2,j} \in C_0^\infty \subseteq S(h^{r_k}_g m, g).
\]
Consequently, \( v_{2,j} \), and thereby \( u_1 \) belong to \( S(h^*_g m, g) \).

It remains to consider \( u_2 \). We have that \( u_2 = b - b_{k-1} \). Since \( \psi_j a_j \in S(h^{r_j}_g m, g) \) when \( j \geq k \), the fact that \( \|b - b_{k-1}\|_{N,h^*_g m} \to 0 \) when \( k \to \infty \), for every fixed \( r > 0 \), it follows that \( u_2 \in S(h^*_g m, g) \). This gives the result.

We also have

**Proposition A.7.** Let \( g \) be strongly feasible and \( m \) be \( g \)-continuous such that \( S(m, g) \) posses the asymptotic expansion property. If \( r_j, R_j \in \mathbb{R}, j \geq 1 \), satisfy
\[
\lim_{j \to \infty} r_j = \infty \quad \text{and} \quad R_j = \min_{k \geq j} r_k,
\]
and \( a_j \in S(h^{r_j}_g m, g) \), then there is an element \( a \in S(h^{R_1}_g m, g) \) such that
\[
a - \sum_{k < j} a_k \in S(h^{R_j}_g m, g), \quad \text{for every} \quad j \geq 1.
\]
(A.11)
The element \( a \) is uniquely determined modulo \( S(h^*_g m, g) \).

For any feasible metric \( g \) and weight \( m \) and \( r_j, R_j \) being the same as in Proposition A.7, we write
\[
a \sim \sum a_j \quad \text{with respect to} \quad m \quad \text{and the metric} \quad G,
\]
when
\[
a - \sum_{k < j} a_k \in S(h^{R_j}_g m, G), \quad \text{for every} \quad j \geq 1.
\]
(A.11)'

If \( G = g \), then we omit the last part in (A.11)', and write only \( a \sim \sum a_j \).

**Proof.** Let \( n \) be the largest number such that \( r_n < 0 \). By replacing \( a \) with
\[
a - \sum_{k \leq n} a_j,
\]
we also have
it follows that we may assume that $R_1 = 0$. Since $r_j \geq R_j$, it follows that $S(h_g^{r_j} m, g) \subseteq S(h_g^{R_j} m, g)$. Hence it is no restriction to assume that $r_j = R_j$, which in particular implies that $r_j$ increases with $j$. Finally, by letting

$$b_k = \sum_{r_j=r_k} a_j,$$

and considering the sequence $\{b_k\}$ instead of $\{a_j\}$, we reduce ourself to the case that $r_j \geq 0$ are strictly increasing. The expansion (A.11) now follows from Proposition A.6.

If $b \in S(h_g^{r_j} m, g)$ satisfies $b \sim \sum a_j$, then it follows from (A.11) that $a - b \in S(h_g^{R_j} m, g)$. This gives the result. \qed

We have now the following proposition.

**Proposition A.8.** Let $g$ be strongly feasible on $W$ and let $m$ be $g$-continuous. Also let $r_j, R_j$ be the same as in Proposition A.7 and let $a \in S(m, g^0)$ and $a_j \in S(h_g^{r_j}, g)$ be such that

$$|a - \sum_{k<j} a_k| \prec h_g^{R_j} m, \quad \text{for all } j \geq 1. \quad (A.12)$$

Then $a \in S(h_g^{R_1} m, g)$, and (A.11) holds with $G = g$.

In certain steps of the proof of Proposition A.7 we shall use similar arguments as in the proof of Proposition 18.1.4 in [28].

**Proof.** We shall use the same ideas as in the proof of proposition 18.1.4 in [28].

By Proposition A.7 there is an element $b \in S(h_g^{R_1} m, g)$ such that $b \sim \sum a_k$. Then it follows from the assumptions that $u \in S(h_g^{R_1} m, g^0)$ and

$$|u| \prec h_g^N m, \quad \text{for every } N \geq 0.$$

Let $c$ is the same as in (A.5), $N \geq 0$, $\varepsilon \in (0, 1)$, $X \in W$ and $Y \in W$ be fixed such that $g_X(Y) < c$. Then Taylor’s formula gives

$$|u(X + \varepsilon Y) - u(X) - \varepsilon(\partial_Y u)(X)| \leq 2^{-1}\varepsilon^2|\partial^2_{Y,Y} u)(X + \theta Y)|,$$

for some $\theta \in [0, 1]$. This gives

$$|(\partial_Y u)(X)| \leq 2^{-1}|u(X + Y)| + |u(X)| + 2^{-1}\varepsilon|\partial^2_{Y,Y} u)(X + \theta Y)|.
\leq C_1(\varepsilon^{-1}(h_g(X + Y)^{2N} m(X + Y) + h_g(X)^{2N} m(X)) + \varepsilon m(X + \theta Y))
\leq C_2((\varepsilon^{-1} h_g(X)^{2N} + \varepsilon) m(X),$$

for some constants $C_1$ and $C_2$ which only depend on the constants in (A.5) and (A.6), and the semi-norms of $a$ in $S(m, g^0)$. In the last step we have used the fact that $m$ is $g$-continuous, $g_X(Y) < c$ and $g_X(\theta Y) < c$. 52
By taking the supremum of the left-hand side over all possible $Y$ and choosing $\varepsilon = h_y^N(X)$, we obtain
\[ \sqrt{\varepsilon} \| u \|_2^2(X) \leq C_3 h_y(X)^N m(X), \]
which gives $\| u \|_{h_y^N,m_1}^q < \infty$. By induction, using similar arguments after $u$ has been replaced by $(\partial_{x_1} \cdots \partial_{x_N})u$, we get $\| u \|_{h_y^N,m_k}^q < \infty$ for all $k \geq 0$. This proves that $u \in S(h_y^\infty,m,g)$, and the proof is complete. \( \square \)

**APPENDIX B. DIRECT PROOF OF CONTINUITY ON $L^2(\mathbb{R}^d)$ OF REGULAR GENERALIZED SG FIOS**

**Theorem B.1.** Let $A = Op_{\varphi}(a)$ be a type I SG Fourier integral operator with $\varphi \in \mathcal{F}$ and $a \in SG_{0,0}^{r,0}(\mathbb{R}^{2d})$, $r, \rho > 0$. Then, $A \in L(L^2(\mathbb{R}^d))$.

We give here a proof of Theorem B.1 as an adapted version of a general $L^2$-boundedness result by Asada and Fujiwara \[1\]. This is a slight modification of the argument originally given in \[12\] for the case $a \in SG_{1,1}^{0,0}$ (see also, e.g., \[3\] and \[31\], and the references quoted therein). We will need the following classical Schur’s lemma.

**Lemma B.2.** If $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$ and
\[ \sup_y \int dx \left| H(x,y) \right| \leq T, \quad \sup_x \int dy \left| H(x,y) \right| \leq T, \quad (B.1) \]
then the integral operator with kernel $H$ has norm $\leq T$ in $L(L^2)$.

**Proof of Theorem B.1.** Let us choose a non-increasing $\psi \in C^\infty(\mathbb{R})$ such that $\psi(t) = 1$ for $t < \frac{1}{2}$ and $\psi(t) = 0$ for $t > 1$. Then, set, for $w = (s, \sigma) \in \mathbb{R}^n \times \mathbb{R}^n$,
\[ \psi_w(x, \xi) = \frac{\psi(|x-s|)\psi(|\xi-\sigma|)}{\int \psi(|x-s|)\psi(|\xi-\sigma|) ds d\sigma}, \quad (B.2) \]
so that
\[ \text{supp } \psi_w \subseteq U_w = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x-s| \leq 1, |\xi-\sigma| \leq 1\}, \quad (B.3) \]
\[ \max_{|\alpha+\beta| \leq m} \sup_{(x, \xi) \in \mathbb{R}^n} \left| \partial_\xi^\alpha \partial_x^\beta \psi_w(x, \xi) \right| \leq C_m, \quad (B.4) \]
\[ \forall x, \xi \int \psi_w(x, \xi) ds d\sigma = 1, \quad (B.5) \]
where the constants $C_m$ do not depend on $w$. For fixed $w$, let us set
\[ a_w(x, \xi) = \psi_w(x, \xi)a(x, \xi), \quad (B.6) \]
\[ A_w = Op_{\varphi}(a_w), \quad (B.7) \]
(B.3), (B.4) and (B.6) imply $A_w$ is linear from $C_0^\infty$ to itself, and $\|A_w u\|_{L^2} \leq C\|u\|_{L^2}$ with constant $C$ independent of $w$. In fact, $a_w$ has compact support and (B.4) holds. Moreover,
\[ \psi_w \in C_0^\infty \Rightarrow \psi_w \in \mathcal{F} \Rightarrow a_w \in SG_{\min\{r,1\},\min\{\rho,1\}} \quad (B.8) \]
and

\[ Au(x) = \lim_{N \to \infty} \int_{|w| \leq N} A_w u(x) \, dw, \quad (B.9) \]

where the limit exists pointwise for all \( x \in \mathbb{R}^n \) and with respect to the strong topology of \( L^2 \). We will prove the theorem if we can show that for all compact sets \( K \subset \mathbb{R}^n \times \mathbb{R}^n \)

\[ \| \int_K A_w u(\cdot) \, dw \|_{L^2} \leq M \| u \|_{L^2}, \quad u \in C_0^\infty \quad (B.10) \]

with constant \( M \) independent of \( u \) and \( K \). To this aim, we will use Cotlar’s lemma, which, adapted to our operators \( A_w \), can be stated in the following form.

**Lemma B.3.** Let \( h(w, w') \) and \( k(w, w') \) be two positive functions on \( \mathbb{R}^{2n} \times \mathbb{R}^{2n} \) such that

\[ \| A_w A_w^* \| \leq h(w, w')^2, \quad \| A_w^* A_w \| \leq k(w, w')^2. \quad (B.11) \]

If \( h \) and \( k \) satisfy

\[ \int h(w, w') \, dw \leq M, \quad \int k(w, w') \, dw \leq M, \quad (B.12) \]

then \( (B.10) \) holds for the same value \( M \).

Here we shall not use Theorem 2.21, but observe that the kernel \( H_{w,w'}(x, y) \) of \( A_w A_w^* \) can be written in the form

\[ H_{w,w'}(x, y) = \int e^{i(\varphi(x, \xi) - \varphi(y, \xi))} q_{w,w'}(x, y, \xi) \, d\xi \quad (B.13) \]

with

\[ q_{w,w'}(x, y, \xi) = a_w(x, \xi) a_{w'}(y, \xi). \quad (B.14) \]

We now want to show that \( H_{w,w'} \) in \( (B.13) \) satisfies the hypotheses of Lemma [B.2] for a suitable \( T \). Let us introduce the operator

\[ \mathcal{L} = d^{-1}(1 - L) \quad (B.15) \]

where

\[ L = i \sum_{j=1}^n \partial_\xi^j (\varphi(x, \xi) - \varphi(y, \xi)) \partial_\xi^j, \quad (B.16) \]

\[ d = 1 + |\nabla_\xi (\varphi(x, \xi) - \varphi(y, \xi))|^2, \quad (B.17) \]

so that

\[ \mathcal{L} e^{i(\varphi(x, \xi) - \varphi(y, \xi))} = e^{i(\varphi(x, \xi) - \varphi(y, \xi))}. \]

Take note that

\[ |\nabla_\xi (\varphi(x, \xi) - \varphi(y, \xi))| \gtrsim |x - y| \Rightarrow d \gtrsim (x - y)^2. \]

\[ ^2 \text{See the first part of the proof of Theorem 2.21.} \]
and also, setting $Df = f/d$,
\[
D : \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n),
\]
\[
L : \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n),
\]
\[
\text{supp } q_{w,w'} \subseteq \{ (x, y, \xi) \mid |x - s| \leq 1, |y - s'| \leq 1, |\xi - \sigma| \leq 1, |\xi - \sigma'| \leq 1 \} \\
\Rightarrow q_{w,w'} \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n).
\]

Since for $(L^m)$ a formula analogous to (2.4) holds, by the hypotheses and the above observations we have, for arbitrary $m \in \mathbb{N}$ and a suitable polynomial $Q_m$ in the variables $D, L$,
\[
H_{w,w'}(x,y) = \\
= \int L^m e^{i(x\cdot\xi - y\cdot\xi)} q_{w,w'}(x, y, \xi) \, d\xi \\
= \int e^{i(x\cdot\xi - y\cdot\xi)} (L^m q_{w,w'}(x, y, \xi) \, d\xi \\
= \int e^{i(x\cdot\xi - y\cdot\xi)} (D^m + Q_m(D, L)) q_{w,w'}(x, y, \xi) \, d\xi \\
\Rightarrow H_{w,w'}(x,y) \lesssim \tau \left( \frac{\sigma - \sigma'}{2} \right) \tau(x - s) \tau(y - s') (1 + |x - y|^2) (B.18)
\]
where $\tau = \chi_{B_1(0)}$ is the characteristic function of the unit ball in $\mathbb{R}^n$. Then
\[
\sup_y \int |H_{w,w'}(x,y)| \, dx \lesssim \\
\lesssim \tau \left( \frac{\sigma - \sigma'}{2} \right) \sup_{y \in B_1(s')} \int_{u \in B_1(0)} (1 + |u + (s - y)|^2)^{-m} \, du \\
\lesssim \tau \left( \frac{\sigma - \sigma'}{2} \right) \sup_{y \in B(s', 1)} (1 + |s - y|^2)^{-m} \\
\lesssim \tau \left( \frac{\sigma - \sigma'}{2} \right) (1 + |s - s'|^2)^{-m}
\]
and analogously for $\sup_x \int |H_{w,w'}(x,y)| \, dy$, owing to the symmetry in the estimate (B.18). So, all requirements of Lemma [B.2] are satisfied

\footnote{The minimum of $|x - y| = |z + (s - y)|$ for $x \in B_1(s) \Leftrightarrow z \in B_1(0)$ is achieved for $z = \text{vers}(s - y)$, as it is easily verified (the result is, in fact, an obvious consequence of the theorem of projections). We also have
\[
|\text{vers}(s - y) + (s - y)|^2 = 1 + 2|s - y| + |s - y|^2 > |s - y|^2
\]
and, obviously,
\[
\min_{y \in B_1(s')} |s - y| = \min_{z \in B_1(0)} |(s - s') + z| > |s - s'|.\]}

55
and summing up, we have:

\[
|\sigma - \sigma'| \geq 2 \Rightarrow A_w A^*_w = 0 \\
|\sigma - \sigma'| \leq 2 \Rightarrow \|A_w A^*_w\| \lesssim (1 + |s - s'|^2)^{-m}.
\]

An analogous estimate can be obtained for \(A^*_w A_{w'}\), in view of the symmetry in the role of variables and covariables in \(\text{SG}\) phases and amplitudes. Then, also the requirements (B.11) and (B.12) of Lemma B.3 are satisfied, and the theorem is proved. □

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