THE FUZZY POTTS MODEL IN THE PLANE: SCALING LIMITS AND ARM EXPONENTS

LAURIN KÖHLER-SCHINDLER AND MATTHIS LEHMKUEHLER

Abstract. We study the fuzzy Potts model on a critical FK percolation in the plane, which is obtained by coloring the clusters of the percolation model independently at random. We show that under the assumption that this critical FK percolation model converges to a conformally invariant scaling limit (which is known to hold for the FK-Ising model), the obtained coloring converges to variants of Conformal Loop Ensembles constructed, described and studied by Miller, Sheffield and Werner. We also show, using discrete considerations that the arm exponents for this coloring in the discrete model are identical to the ones of the continuum model. Using the values of these arm exponents in the continuum, we determine the arm exponents for the fuzzy Potts model.

Contents

1. Introduction 2
  1.1. Coloring FK percolation clusters 2
  1.2. Phase diagram of the fuzzy Potts model 5
  1.3. Overview 6
2. Background 7
  2.1. FK percolation 7
  2.2. Fuzzy Potts model 10
  2.3. Loop (ensemble) topologies 12
  2.4. Encoding percolation models as loop configurations 12
  2.5. (Boundary) conformal loop ensembles 14
  2.6. CLE Percolations 16
  2.7. Imaginary geometry results 20
3. Fuzzy Potts model: Quasi-multiplicativity and other discrete ingredients 21
  3.1. Almost-arm events: Definitions and first results 22
  3.2. Quasi-multiplicativity 26
  3.3. Arm separation 30
  3.4. Quasi-multiplicativity and arm separation for the halfplane 34
  3.5. Bounding and comparing arm exponents 35
4. Convergence results 38
  4.1. Convergence of divide and color interfaces 41
  4.2. Limit of divide and color loops 42
5. Continuum exponents via exploration paths 44
6. Combining the discrete with the continuum results 50
Appendix A. A technical lemma on the loop topology 51
Appendix B. Distortion estimates up to the boundary 53
References 54

L.KS., M.L. – ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland

Date: September 26, 2022.
1. Introduction

1.1. Coloring FK percolation clusters

The subject of investigation of the present paper is the random coloring (with two colors) of a planar graph (more specifically, here $Z^2$ or $Z \times Z_+$) that is obtained when one colors independently the clusters of a FK percolation model as introduced in [FK72]. The resulting models are often referred to as fuzzy Potts models [Hä99, MVV95, CLM07, Hä01] (also known as fractional fuzzy Potts models) and belong to a wider class of random colorings known as ‘divide and color’ models. In this paper, we will study the case where we start with a critical FK percolation. Note that the construction involves two layers of randomness, first the sampling of a critical FK percolation with some cluster weight $q$ and then the coloring of each of the clusters in red or blue with respective probabilities $r$ and $1 - r$.

The name ‘fuzzy Potts model’ is due to the fact that if $r = i/q$ for $i \in \{1, \ldots, q - 1\}$ the fuzzy Potts model can also be obtained from a Potts model with spins $\{1, \ldots, q\}$ by coloring each vertex with spin in $\{1, \ldots, i\}$ in red and in blue otherwise. This is a consequence of the Edwards-Sokal coupling and it is exactly the construction of an Ising model from an FK-Ising percolation.

Before describing our results on these models, it is worth recalling how the scaling limit and arm exponents of critical models such as percolation or the Ising model have been derived mathematically: (i) A first crucial step is to show (via discrete considerations exhibiting conformal invariance features) that the interfaces in the discrete models converge to SLE type curves in the scaling limit. (ii) A second independent ingredient is the derivation of exponents related to this continuum scaling limit (this is performed via SLE computations and considerations) that in turn enable the computation of ‘arm exponents’ for this continuum scaling limit. (iii) A final step is to show that these continuum arm exponents are actually identical to the corresponding arm exponents for the discrete model (this step involves typically discrete ‘arm separation’ and ‘quasi-multiplicativity’ arguments for the discrete arm events). See for instance [Wer09] for an overview in the case of critical Bernoulli percolation (see also [SW01, Smi01, LSW01a, LSW01b, LSW02b, LSW02a]).

For the fuzzy Potts model mentioned above, the situation is more involved, due to this ‘two layer randomness’, so that this strategy needs to be adapted appropriately.

(i) We assume that the critical FK percolation model converges to its conjectured Conformal Loop Ensemble (CLE) scaling limit (we will state this conjecture more precisely as Conjecture 2.8 in this paper, and just refer to it as the FK conformal invariance conjecture). This conjecture is known to hold in the $q = 2$ case. Under this assumption, the first question is whether the scaling limit of the coloring is the coloring of the scaling limit. This is not obvious, because one has for instance to rule out the possibility that the coloring of the ‘small microscopic FK cluster’ that disappear in the scaling limit contribute to creating larger monochromatic islands. This will be the first type of results that we will derive in the present paper.

(ii) The conclusion of this first step is then that the scaling limit of these random colorings are described by the colorings of these continuous conformal loop ensembles. These continuum colorings have been introduced and studied in a series of works by Miller, Sheffield and Werner on the shoulder of which the present work will stand. It is worth recalling some aspects of this series of papers: In the seminal paper [MSW17], it is shown that when one starts with a non-simple CLE (i.e., a CLE$_{\kappa}$ for $\kappa' \in (4, 8)$ of the type that conjecturally appears as scaling limits of critical FK percolation models with cluster weight $q \in (0, 4)$) and one colors its clusters independently using some coloring parameter $r$, then the obtained coloring is described by a variant of CLE$_{\kappa}$ for $\kappa = 16/\kappa'$ that they describe in terms of Boundary Conformal Loop Ensembles (BCLE). The interfaces of these BCLEs are variants of SLE$_{\kappa}$ curves involving another parameter $\rho$.

In [MSW17], the explicit relation between $\rho$ and $r$ is not determined, but Miller, Sheffield and Werner have derived this relation in a subsequent recent paper [MSW21] using arguments based on Liouville Quantum Gravity. Altogether, it provides a full explicit description of these continuous fuzzy
Potts models in terms of $\kappa'$ and $r$. In [MSW17], a related setting is also considered when one starts with a simple CLE and the missing relation between two values is derived in [MSW22]. This led them to obtain the values of some continuum arm exponents. The formulas for these exponents are of a rather novel type, reflecting the fact that the relation between $r$ and $\rho$ seems to be connected directly to ‘Liouville Quantum Gravity’ considerations. In the present paper, we will use these facts to derive the (almost) complete set of continuum arm exponents for these ‘colored CLE’ models.

(iii) One has to show that the arm exponents for these continuum ‘colored CLE’ models do match those of the discrete fuzzy Potts models. This will require again some new inputs and results about these discrete models but is more subtle than the corresponding problem in the percolation setting since the fuzzy Potts model does not satisfy a Markov property which allows the existing proofs for arm separation and quasi-multiplicativity to be adapted. Instead, we will work via the FK percolation model and use its Markov property to reason about the fuzzy Potts model. In particular, the work [DCMT21] will be the key tool allowing us to perform the relevant constructions in the discrete setting.

The above ideas will then be combined into the main results of this paper that we now state. We consider $q \in [1, 4)$ and consider the critical FK percolation model with cluster weight $q$. Throughout this paper we will then define

$$\kappa' = 4\pi/\arccos(-\sqrt{q/2}) \in (4, 6) \quad \text{and} \quad \kappa = 16/\kappa' \in [8/3, 4).$$

The results are then conditional on the conformal invariance conjecture for this model, which says that the scaling limit of the cluster boundaries of this critical FK percolation with cluster weight $q$ is given by a CLE$_{\kappa'}$, which is a conformally invariant law on collections of loops. So far, this conjecture has only been proved in the case $q = 2$ (i.e., $\kappa' = 16/3$) in a series of works [Smi10, KS19, KS16] and so in this particular case, the theorems that we will now state are in fact unconditional.

To state the theorems, we first need to define discrete arm events $A^s_\tau(m,n) \subset \{R,B\}^{\mathbb{Z}^2}$ and $A^{s+}_\tau(m,n) \subset \{R,B\}^{\mathbb{Z}^2 \times \mathbb{Z}_+}$ for $1 \leq m \leq n$ where $\tau = \tau_1 \cdots \tau_k$ is a finite length word with letters $\tau_1, \ldots, \tau_k \in \{R,B\}$, called color sequence in the following. The superscript $s$ will be explained in Section 2.2; it should be ignored for the moment. We write $\Lambda_n = [-n,n]^2 \cap \mathbb{Z}^2$ for the box of size $n \geq 1$ with boundary $\partial \Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$ and $\Lambda_{m,n} = \Lambda_n \setminus \Lambda_{m-1}$ for the annulus from $m \geq 1$ to $n \geq m$. We also set $\Lambda^{m,n}_+ = \Lambda_{m,n} \cap (\mathbb{Z} \times \mathbb{Z}_+)$. Let $A^s_\tau(m,n)$ be the event which contains a configuration $\sigma \in \{R,B\}^{\mathbb{Z}^2}$ if and only if there are disjoint nearest-neighbor paths $\gamma_1, \ldots, \gamma_k$ in $\Lambda_{m,n}$ from $\partial \Lambda_m$ to $\partial \Lambda_n$ which are ordered counterclockwise and are such that $\sigma$ has color $\tau_i$ along $\gamma_i$ for all $1 \leq i \leq k$. Similarly, $A^{s+}_\tau(m,n)$ is the event in the upper halfplane which contains a color configuration $\sigma \in \{R,B\}^{\mathbb{Z} \times \mathbb{Z}_+}$ if and only if there are disjoint nearest-neighbor paths $\gamma_1, \ldots, \gamma_k$ in $\Lambda^{m,n}_+$ from $\partial \Lambda_m$ to $\partial \Lambda_n$ which are ordered counterclockwise, start with the rightmost arm, and are such that $\sigma$ has color $\tau_i$ along $\gamma_i$ for all $1 \leq i \leq k$.

We write $p_r(q,r,m,n)$ for the probability of the event $A^s_\tau(m,n)$ under the fuzzy Potts model on $\mathbb{Z}^2$ with coloring parameter $r$ on a critical FK percolation with weight $q$ and $p^{s+}_r(q,r,m,n)$ for the probability of the event $A^{s+}_\tau(m,n)$ under the fuzzy Potts model on $\mathbb{Z} \times \mathbb{Z}_+$ with coloring parameter $r$ on a critical FK percolation with weight $q$ and free boundary conditions.

Finally, we need to define the number of ‘interfaces’ that are associated to the arm events $A^s_\tau(m,n)$ and $A^{s+}_\tau(m,n)$; it will turn out that only this interface count affects the arm exponent. We let $\tau_{k+1} := \tau_1$ and define

$$I(\tau) = \#\{1 \leq i \leq k: \tau_i \neq \tau_{i+1}\} \quad \text{and} \quad I^+(\tau) = 1 + \#\{1 \leq i < k: \tau_i \neq \tau_{i+1}\}.$$

Note that $I(\tau)$ will always be an even number. One way of phrasing the fact that only the interface count is relevant in the theorems below is the statement that having one red arm yields any finite number of additional consecutive red arms with only a slightly smaller probability
that is insignificant for the arm exponent. The following theorem provides the values of all arm exponents (except the monochromatic ones) for the full plane.

**Theorem 1.1.** Let \( q \in [1, 4) \) and suppose that the conformal invariance conjecture holds for the critical FK percolation model with cluster weight \( q \) and write \( \kappa = 4 \arccos(-\sqrt{q}/2)/\pi \in [8/3, 4) \).

Let \( r \in (0, 1) \) and define for \( j \geq 1 \),

\[
\alpha_{2j}(r) = \frac{16j^2 - (\kappa - 4)^2}{8\kappa}.
\]

Then for any color sequence \( \tau = \tau_1 \cdots \tau_k \) which is not all red or all blue, i.e. \( I(\tau) > 0 \), we have

\[
p_\tau(q,r,m,n) = \left( \frac{m}{n} \right)^{\alpha_{I(\tau)}(r)+o(1)}
\]

as \( n/m \to \infty \) for \( k \leq m \leq n \), where the \( o(1) \) term may depend on \( \tau \).

For the upper halfplane, we obtain the values of all arm exponents.

**Theorem 1.2.** Let \( q \in [1, 4) \) and suppose that the conformal invariance conjecture holds for the critical FK percolation model with cluster weight \( q \) and write \( \kappa = 4 \arccos(-\sqrt{q}/2)/\pi \in [8/3, 4) \).

Let \( r \in (0, 1) \) and define for \( j \geq 1 \),

\[
\alpha_{2j+1}(r) = \frac{2j(2j + \kappa/2 - 2)}{\kappa},
\]

\[
\alpha_{2j-1}(r) = \frac{1}{\kappa} \left( 2j + \kappa - 4 - \frac{2}{\pi} \arctan \left( \frac{\sin(\pi\kappa/2)}{1 + \cos(\pi\kappa/2) - 1/r} \right) \right).
\]

Then for any color sequence \( \tau = \tau_1 \cdots \tau_k \) starting with \( R \) we have

\[
p_\tau^r(q,r,m,n) = \left( \frac{m}{n} \right)^{\alpha_{I(\tau)}^+(r)+o(1)}
\]

as \( n/m \to \infty \) for \( k \leq m \leq n \), where the \( o(1) \) term may depend on \( \tau \). The case where \( \tau \) starts with \( B \) is obtained by replacing \( r \) by \( 1 - r \).

In both cases, we stress the following feature: The ‘even’ exponents \( \alpha_{2j}(r) \) (resp. \( \alpha_{2j}^+(r) \)) do not depend on the actual value of \( r \in (0, 1) \). This is not at all clear (or even intuitive) from the definition of the discrete model (except in the special case \( \alpha_{2j}(r) = 1 \) which turns out to be a universal arm exponent) and it appears to be difficult to prove this result using only discrete tools (i.e., without relying on scaling limit conjectures).

It is worth emphasizing that in the special case of the FK-Ising model (i.e., \( q = 2 \)), the conformal invariance conjecture is known to hold by [Smi10, KS19, KS16] (see also [GW20]), so that the previous result is unconditional. In that case, we have \( \kappa = 3 \) and one can then summarize the previous theorems in the following (unconditional) formulation.

**Theorem 1.3.** If \( q = 2 \) then the critical exponents for the fuzzy Potts model with coloring parameter \( r \in (0, 1) \) are

\[
\alpha_{2j}(r) = \frac{(2j)^2 - 1/4}{6}, \quad \alpha_{2j}^+(r) = \frac{2j(2j - 1/2)}{3},
\]

\[
\alpha_{2j-1}(r) = \frac{1}{3} \left( 2j - 1 - \frac{2}{\pi} \arctan \left( \frac{r}{1-r} \right) \right) \left( 2j - \frac{1}{2} - \frac{2}{\pi} \arctan \left( \frac{r}{1-r} \right) \right)
\]

for \( j \geq 1 \) provided that the color sequence starts with \( R \). The case where it starts with \( B \) is obtained by replacing \( r \) by \( 1 - r \).
Remark 1.4. The case \( r = 1/q \) is special. In particular, when \( q = 2 \) (i.e. \( \kappa = 3 \)) or \( q = 3 \) (i.e. \( \kappa = 10/3 \)), it is related to the Potts models. We obtain the formulas
\[
\alpha^+_2(1/q) = \frac{1}{\kappa} \left( 2j + \frac{3\kappa}{2} - 6 \right) (2j + \kappa - 4),
\]
\[
\alpha^+_2(1 - 1/q) = \frac{1}{\kappa} \left( 2j - 2\kappa + 2 \right).
\]
The critical Ising model corresponds to cluster weight \( q = 2 \) (so \( \kappa = 3 \)) and coloring parameter \( r = 1/2 \). The exponents then agree with the results obtained for the Ising model in [Wu18a].

1.2. Phase diagram of the fuzzy Potts model

As explained in the previous section, this paper is concerned with the fuzzy Potts model on a well-suited to describe the phase diagram of FK percolation. Where
\[
E \subseteq \mathbb{Z}^2
\]
represents the nearest-neighbor edges on \( \mathbb{Z}^2 \). We sample the whole phase diagram of this model. We sample
\[
\{ 0 \leftrightarrow x \} \text{ for the event that } 0 \text{ and } x \text{ are in the same cluster.}
\]
The fuzzy Potts measure \( \mu_{\mathbb{Z}^2, p, q, r}^0 \) on \( \{ R, B \}^{\mathbb{Z}^2} \) is obtained by coloring the vertices of each cluster in \( C \) independently all with the color red (\( R \)) with probability \( r \in (0, 1) \) and all in blue (\( B \)) otherwise.

We would like to understand the geometry of the red and blue fuzzy Potts clusters as a function of the parameters \( (p, q, r) \) of the model. To this end, we first review some major results about the geometry of the underlying FK percolation model. It turns out that the event \( \{ 0 \leftrightarrow x \} \) is well-suited to describe the phase diagram of FK percolation.

- For every cluster weight \( q \geq 1 \), there is a sharp phase transition at the critical point \( p_c(q) = \sqrt{q}/(1 + \sqrt{q}) \). For subcritical \( p < p_c(q) \), the probability \( \phi_{\mathbb{Z}^2, p, q}^0(0 \leftrightarrow x) \) decays exponentially fast to 0 as \( \|x\| \to \infty \). For supercritical \( p > p_c(q) \), it is bounded away from 0 uniformly in the point \( x \). This has been established by [BDC12]; see also [DCRT19, DCM16, DCRT18].
- At the critical point \( p_c(q) \), the geometry depends on the cluster weight: For \( q > 4 \), the phase transition is discontinuous (see [DCGH+21, RS20, DCT20]) and in the subcritical phase, \( \phi_{\mathbb{Z}^2, p_c(q)}^0(0 \leftrightarrow x) \) decays exponentially fast to 0 as \( \|x\| \to \infty \). For \( q \in [1, 4] \), the phase transition is continuous as shown in [DCST17] and the probability \( \phi_{\mathbb{Z}^2, p, q}^0(0 \leftrightarrow x) \) decays polynomially fast to 0 as \( \|x\| \to \infty \). The key tool to study models at the point of a continuous phase transition are Russo-Seymour-Welsh (RSW) estimates on crossing probabilities. Recently, very powerful RSW estimates have been established in [DCMT21] for \( q \in [1, 4] \), and this allows for a good understanding of the critical and near-critical geometry. Much less is known in the \( q = 4 \) case, which is more subtle since the RSW estimates of [DCMT21] are expected to be wrong in this case.

We can now make some simple observations about the fuzzy Potts model with the parameters \( (p, q, r) \): Let \( R_\infty \) denote the event on \( \{ R, B \}^{\mathbb{Z}^2} \) that there is an infinite red fuzzy Potts cluster.

- When the clusters in \( C \) are exponentially small (i.e. for \( p < p_c(q) \)) and for \( p = p_c(q) \) when \( q > 4 \), the fuzzy Potts model with coloring parameter \( r \) should behave similarly to Bernoulli site percolation. In particular, one expects there to be a critical point \( r_c(p, q) \in (0, 1) \) such that
\[
\mu_{\mathbb{Z}^2, p, q, r}(R_\infty) > 0 \quad \text{if and only if} \quad r > r_c(p, q).
\]
In [BCM09], this has been established for \( q = 1 \) and in fact, there do not appear to be major obstacles to generalizing this result to \( q \geq 1 \) by making use of (more recent)
sharpness results for FK percolation. Further properties have been obtained for \( q = 1 \) in [BBT13b, BBT13a, Tas14].

- When there exists a unique infinite cluster in \( \mathcal{C} \) almost surely (i.e. for \( p > p_c(q) \)), we see that for any coloring parameter \( r \in (0,1) \),
  \[
  \mu_{\mathbb{Z}^2, p, q, r}(R_\infty) = r > 0 .
  \]
  In particular, there exists a unique infinite cluster in the fuzzy Potts model that is either red or blue.

- It remains to describe the geometry of the fuzzy Potts model for \( p = p_c(q) \) and \( q \in [1, 4] \).
  In this case, one can use weaker RSW estimates (see [DCST17]) to show that \( \mathcal{C} \) almost surely contains infinitely many disjoint clusters surrounding the origin. Since infinitely many of these clusters will be colored in blue almost surely, it follows that for any coloring parameter \( r \in (0,1) \),
  \[
  \mu_{\mathbb{Z}^2, p, q, r}(R_\infty) = 0 ,
  \]
  and analogously, there is also no infinite blue fuzzy Potts cluster almost surely. We emphasize that this behavior differs drastically from the two previous cases.

What is achieved in this paper is the derivation of more precise results on the geometry of the fuzzy Potts model for \( p = p_c(q) \) and \( q \in [1, 4] \) as described in the previous section. The case of \( p = p_c(q) \) and \( q = 4 \) is different since one expects there to be a sharp phase transition at \( r_c(p_c(4), 4) = 1/2 \). A construction of the conjectured near-critical scaling limits in this case appears in [MSW17, Leh21]. The key difference to the case \( q < 4 \) is that one expects the scaling limit of the coloring to be different from the coloring of the scaling limit when one considers \( q = 4 \) (since the scaling limit of the loop encoding of the model is conjectured to be a simple CLE).

### 1.3. Overview

Let us now provide an outline of the paper. As mentioned above, there will be three distinct steps which are needed to determine the exponents in Theorem 1.1 and 1.2 and which will be performed in this paper.

- **Section 2** contains all the relevant background information and references on the discrete objects (FK percolation and the fuzzy Potts model) as well as the continuum objects (CLE and BCLE). This is also the section within which we are precisely stating the conformal invariance conjecture as Conjecture 2.8.

- **In Section 3**, we develop arm separation and quasi-multiplicativity tools for the discrete fuzzy Potts model (which are instrumental for deducing the discrete from the continuum exponents). This section relies heavily on the recent paper [DCMT21] which is valid for \( q \in [1, 4] \). We however expect that quasi-multiplicativity also holds when \( q = 4 \) case, even though our proof techniques do not work in this case. Many techniques in this section are motivated by the seminal work [Kes87].

- **In Section 4**, we show that under the conformal invariance assumption for FK percolation with parameter \( q \in [1, 4] \), the scaling limit of the (discrete) fuzzy Potts cluster boundaries is given by the continuum fuzzy Potts cluster boundaries as constructed in the CLE percolation paper by Miller, Sheffield and Werner [MSW17]. We defer two technical lemmas on the topology involved to Appendix A. This is the step where the situation is significantly different when \( q = 4 \) instead of \( q \in [1, 4] \). We also obtain a new derivation of the scaling limit of the Ising model loops (see [BH19] for the existing proof).

- **In Section 5**, we work on the continuum side, and we compute the critical exponents in the setting of the continuum fuzzy Potts model. More specifically, now that the interfaces are described in terms of the SLE variants from the papers by Miller, Sheffield and Werner, we need to compute the corresponding exponents for these processes. This will bear many similarities (and will rely on) the SLE computations by Wu in [Wu18a]. Some technical results on conformal transformations relevant to this section are deferred to Appendix B.
Finally, we will combine the above ingredients in Section 6 to establish the main results, namely Theorem 1.1 and 1.2.

**Notation.** Throughout, if $f, g : X \to [0, \infty]$ are functions on some space $X$, we write $f(x) \lesssim g(x)$ if there is a constant $C \in (0, \infty)$ such that $f(x) \leq C g(x)$ for all $x \in X$. We write $f(x) \asymp g(x)$ if $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$ hold. We also let $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and write log for the logarithm to the base $2$. Finally, we write $B_r(z_0) = z_0 + r\mathbb{D}$ and $(a, b)$ for the open counterclockwise boundary arc from $a \in \partial \mathbb{D}$ to $b \in \partial \mathbb{D}$ along $\partial \mathbb{D}$.

**Acknowledgments.** L. KS. has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program (grant 851565). M. L. was supported by grant 175505 of the Swiss National Science Foundation. Both authors are part of NCCR SwissMAP. We would like to thank Wendelin Werner for proposing this project and we are grateful to Vincent Tassion and Wendelin Werner for many insightful inputs and discussions. We also thank Hugo Vanneuville for valuable comments on an earlier version of this paper.

2. Background

2.1. FK percolation

In this subsection, we formally introduce critical FK percolation and we state all preliminary results that will be used throughout the paper. We direct the reader to [Gri06] for a broader introduction and to [DC20] for an exposition of recent results. Besides standard properties of FK percolation, our work mostly relies on the strong crossing estimates which were recently established in [DCMT21].

We consider the square lattice $(\mathbb{Z}^2, E(\mathbb{Z}^2))$ which is the graph with vertex set $\mathbb{Z}^2$ and edges between nearest neighbors. Abusing notation slightly, we often refer to the graph itself as $\mathbb{Z}^2$.

Let $G = (V, E)$ be a finite subgraph of $\mathbb{Z}^2$. We define its vertex boundary $\partial V = \{v \in V : \deg_G(v) < \deg_{\mathbb{Z}^2}(v) = 4\}$, where the $\deg_G(v) = \{|\{w \in V : vw \in E\}\}$ is the number of neighbors of $v$ in $G$. An element $\omega \in \{0, 1\}^E$ encodes a subgraph of $G$ with vertex set $V$ and edge set $o(\omega) := \{e \in E : \omega_e = 1\}$, where we recall that an edge $e$ is called open (resp. closed) if $\omega_e = 1$ (resp. $\omega_e = 0$). We now consider a partition $\xi$ of $\partial V$, called boundary condition, and we denote by $\omega^\xi$ the graph obtained from $\omega$ by adding additional edges between vertices belonging to the same partition element. Let $\mathcal{C}(\omega^\xi)$ denote the set of clusters of this graph. FK percolation on $G = (V, E)$ with boundary condition $\xi$, edge weight $p \in [0, 1]$ and cluster weight $q > 0$ is the measure

$$\phi^\xi_{G,p,q}(\omega) := \frac{1}{Z^\xi_{G,p,q}} p^{o(\omega)}(1 - p)^{E \setminus o(\omega)} q^{\mathcal{C}(\omega^\xi)}$$

on $\{0, 1\}^E$, where $Z^\xi_{G,p,q}$ is a normalizing constant, called partition function.

The product ordering provides a partial order on $\{0, 1\}^E$. A function $f : \{0, 1\}^E \to \mathbb{R}$ is said to be increasing if $\omega \leq \omega'$ implies $f(\omega) \leq f(\omega')$, and an event $A$ is increasing if its indicator function $1_A$ is increasing. For cluster weights $q \geq 1$, FK percolation is positively associated, i.e. for any two increasing functions $f, g : \{0, 1\}^E \to \mathbb{R}$,

$$\phi^\xi_{G,p,q}(f \cdot g) \geq \phi^\xi_{G,p,q}(f) \cdot \phi^\xi_{G,p,q}(g).$$

This property is also referred to as the FKG inequality. Actually, it does not hold for $q \in (0, 1)$ and much less is known about the model in this case. Throughout the paper, we will therefore only consider cluster weights $q \geq 1$.

We will now discuss boundary conditions in more detail. For two partitions $\xi, \xi'$ of $\partial V$, we write $\xi \leq \xi'$ if $\xi$ is a finer partition than $\xi'$, i.e. every element in $\xi$ is a subset of an element in $\xi'$. We refer to the finest partition, where $\partial V$ is partitioned into singletons, as free boundary conditions, denoted by $\phi_{G,p,q}^0$, and to the coarsest partition, which consists only of the element $\partial V$,
as *wired* boundary conditions, denoted by $\phi_{G,p,q}^\xi$. Boundary conditions can easily be compared for increasing events $A \subseteq \{0,1\}^E$: For any two boundary conditions $\xi \leq \xi'$,
\[
\phi_{G,p,q}^\xi(A) \leq \phi_{G,p,q}^{\xi'}(A)
\]

The domain Markov property is another fundamental tool to study FK percolation. Let $G' = (V',E')$ be a subgraph of the finite graph $G = (V,E)$ and $\xi$ be a boundary condition on $G$. Given an event $A$ that is measurable with respect to the status of edges in $E'$, it says that for any $\psi \in \{0,1\}^{E\setminus E'}$,
\[
\phi_{G,p,q}^\xi(A | \omega_e = \psi_e, \forall e \in E \setminus E') = \phi_{G,p,q}^{\xi'}(A)
\]

where $\xi' = \xi'(\psi,\xi)$ denotes the partition of $\partial V'$ that is induced by $\xi$ and $\psi$, i.e. two vertices $x',y' \in \partial V'$ belong to the same element of $\xi'$ if they can be connected to the same element of $\xi$ or directly to each other using open edges in $\psi$.

Throughout the paper, we naturally embed the square lattice $(\mathbb{Z}^2, E(\mathbb{Z}^2))$ in the plane $\mathbb{R}^2$ by viewing edges as closed line segments. Whenever $S \subseteq \mathbb{R}^2$ is closed and bounded, by definition, the finite subgraph induced by $S$ is the subgraph of $\mathbb{Z}^2$ with vertex set $S \cap \mathbb{Z}^2$ and edge set $E(S)$ containing those edges that are completely contained in $S$. We often refer to the subgraph itself by $S$ when no confusion can arise, and define its vertex boundary $\partial S$ as before. If the subgraph induced by $S$ is contained in the upper halfplane, we also define its vertex boundary $\partial_+ S$ with respect to $\mathbb{Z} \times \mathbb{Z}_+$, consisting of those vertices $v \in S$ with incomplete degree, i.e. $\deg_+(v) < \deg_{\mathbb{Z} \times \mathbb{Z}_+}(v) \in \{3,4\}$.

For $r,s > 0$, we define the box $\Lambda_r = [-r,r]^2$ and the annulus $\Lambda_{r,s} = [-s,s]^2 \setminus (-r,r)^2$. We also define their intersections with the upper halfplane as $\Lambda^+_r = [-r,r] \times [0,r]$ and $\Lambda^+_{r,s} = [-s,s] \times [0,s] \setminus (-r,r) \times [0,r)$. As mentioned before, we will often use the same notation when referring to their induced subgraphs. In particular, our notation is consistent with the definitions given in Section 1.

To define FK percolation on the infinite graph $\mathbb{Z}^2$, one can consider the sequence $(\Lambda_n)_{n \geq 1}$ and then take the weak limit of the measures $\phi_{\Lambda_n,p,q}^\xi$ along this sequence for $\xi = 0$ and $\xi = 1$, respectively. While the limiting measures $\phi_{\mathbb{Z}^2,p,q}^0$ and $\phi_{\mathbb{Z}^2,p,q}^1$ might a priori be different, it is straightforward to see that they also satisfy the FKG inequality. In the same way, one can define the FK percolation measure $\phi_{\mathbb{Z} \times \mathbb{Z}_+,p,q}^0$ on the upper halfplane $\mathbb{Z} \times \mathbb{Z}_+$ by considering the sequence $(\Lambda^+_n)_{n \geq 1}$.

As explained before, this paper is concerned with critical FK percolation. Therefore, we fix the edge weight to be $p_c(q) = \sqrt{q}/(1 + \sqrt{q})$ from now on and we drop it from our notation. It was proven in [DCST17] that for $q \in [1,4]$, the two extremal measures $\phi_{\mathbb{Z}^2,p,q}^0$ and $\phi_{\mathbb{Z}^2,p,q}^1$ are in fact the same, and so we will also drop the dependence on boundary conditions and simply write $\phi_{\mathbb{Z}^2,p,q}$ for the (critical) FK percolation measure on $\mathbb{Z}^2$ with cluster weight $q \in [1,4]$.

A consequence of the crossing estimates in rectangles that were established in [DCST17] is the following mixing statement which says that the boundary conditions do not affect the values of the probabilities significantly (see [DCST17, Section 1.3.1]).

**Corollary 2.1.** Let $q \in [1,4]$. For all $c > 0$, there exists a constant $C > 0$ such that for any $n \geq 1$ with $\Lambda_{(1+c)n} \subseteq G$ and for any event $A$ measurable with respect to the edges in $\Lambda_n$,
\[
C^{-1} \cdot \phi_{\mathbb{Z}^2,p,q}^\xi(A) \leq \phi_{\mathbb{Z}^2,q}^\xi(A) \leq C \cdot \phi_{\mathbb{Z}^2,p,q}^{\xi'}(A)
\]
uniformly in the boundary condition $\xi$.

The situation is similar in the case of the halfplane measure $\phi_{\mathbb{Z} \times \mathbb{Z}_+,q}^0$. The following corollary can be proven analogously to the full plane case. This makes use of stronger crossing estimates as obtained in [DCMT21] and we therefore exclude the case $q = 4$. 


Corollary 2.2. Let \( q \in [1, 4) \). For all \( c > 0 \), there exists \( C > 0 \) such that for all \( n \geq 1 \) with \( \Lambda_n^{(1+c)n} \subset G \subset \mathbb{Z} \times \mathbb{Z}^+ \) and for any event \( A \) measurable with respect to the edges in \( \Lambda_n^1 \),
\[
C^{-1} \cdot \phi^0_{\mathbb{Z} \times \mathbb{Z}_+, q}(A) \leq \phi^c_{G, q}(A) \leq C \cdot \phi^0_{\mathbb{Z} \times \mathbb{Z}_+, q}(A),
\]
uniformly in boundary conditions \( \xi \) for which \( \{(i, 0)\} \in \xi \) when \( |i| \leq n \).

Let us now present the stronger crossing estimates from [DCMT21] which allow for a precise understanding of the behavior at criticality. We introduce a few notions which will also appear throughout the paper.

A path \( \gamma = (\gamma_i)^{n}_{i=0} \) is a finite sequence of nearest-neighbor vertices and it is called simple if the vertices are distinct. A loop is a path \( \lambda \) with \( \lambda_n = \lambda_0 \) and such that \((\lambda_i)^{n}_{i=1} \) is simple. A path \( \gamma = (\gamma_i)^{n}_{i=0} \) is said to be open in a percolation configuration \( \omega \) if \( \omega_{\gamma_{i-1} \gamma_i} = 1 \) for all \( 1 \leq i \leq n \). A discrete domain is a finite subgraph \( D = (V, E) \) of \( \mathbb{Z}^2 \) which is enclosed by a loop (all vertices and edges on the loop belong to the domain, see also [DCMT21]) and we write \( \partial D \) for the vertices on the boundary loop. If \( a, b, c, d \in \partial D \) are distinct and ordered counterclockwise, we call \( (D, a, b, c, d) \) a discrete quad. The boundary points define four boundary arcs \((ab), (bc), (cd)\) and \((da)\) (which are subsets of \( \partial D \) and contain their endpoints by convention).

Later, we will allow for a slightly more general type of domain, which we refer to as an approximate discrete domain. D is defined in terms of a loop \( \lambda \) (as above), we can view \( \lambda \) as a piecewise linear Jordan curve surrounding a domain \( \lambda^o \) in \( \mathbb{R}^2 \). By the Riemann mapping theorem, there exists \( \ell > 0 \) and a unique conformal transformation from \( \lambda^o \) to \( (0, 1) \times (0, \ell) \) such that its continuous extension maps \( a, b, c, d \) to the corners of \([0, 1] \times [0, \ell]\) in counterclockwise order with \( a \) being mapped to \( (0, 0) \). This unique value \( \ell \), denoted by \( \ell_D((ab), (cd)) \), is referred to as the extremal distance from \((ab)\) to \((cd)\) in \( \mathcal{D} \).

Theorem 2.3 ([DCMT21]). Let \( q \in [1, 4) \). For every \( L > 0 \), there exists \( \epsilon = \epsilon(L) \in (0, 1) \) such that if \( (\mathcal{D}, a, b, c, d) \) and \( \mathcal{D}' \) are as above, then for any boundary condition \( \xi \),
\[
\begin{align*}
\bullet & \quad \text{if } \ell_D((ab), (cd)) \leq L, \text{ then } \phi^c_{\mathcal{D}', q}(\mathcal{D}) \leq (ab) \xrightarrow{\mathcal{D}'} (cd) \geq \epsilon, \\
\bullet & \quad \text{if } \ell_D((ab), (cd)) \geq L^{-1}, \text{ then } \phi^c_{\mathcal{D}', q}(\mathcal{D}) \xrightarrow{\mathcal{D}'} (cd) \leq 1 - \epsilon.
\end{align*}
\]

These crossing estimates are the main result of [DCMT21] and they will serve as the key tool to study the critical fuzzy Potts model in Sections 3 and 4. Note that in [DCMT21] they are only stated in the case of discrete domains but deducing the case for approximate discrete domains is straightforward. While boundary conditions are irrelevant for Bernoulli percolation \( (q = 1) \), it is difficult to control the effect of boundary conditions for \( q \in (1, 4) \) and these estimates are a major improvement compared with [DCST17] since they include crossings that touch unfavorable boundary conditions in arbitrary domains.

We will now state some applications of Theorem 2.3 that were presented (among others) in Sections 6 and 7 of [DCMT21]. Many of the arguments go back to [Kes87] (see also [Nol08]) in the case of Bernoulli percolation \( (q = 1) \) and have been extended to the case of FK-Ising \( (q = 2) \) in [CDCH16].

We begin with the universal arm exponents for FK percolation. We consider the dual lattice \((\mathbb{Z}^2)^* = (1/2, 1/2) + \mathbb{Z}^2\) with edge set \( E^*((\mathbb{Z}^2)^*) \). It can naturally be embedded in the plane \( \mathbb{R}^2 \) together with the (primal) lattice \( \mathbb{Z}^2 \) by drawing its edges as straight line segments. Every
dual edge $e^* \in E^*((\mathbb{Z}^2)^*)$ intersects a unique (primal) edge $e \in E(\mathbb{Z}^2)$, and we define the dual percolation configuration $\omega^*$ by $\omega^*_e = 1 - \omega_e$. We say that an edge $e^* \in E^*((\mathbb{Z}^2)^*)$ is dual open if $\omega^*_e = 1$. A type sequence $\tau = \tau_1 \cdots \tau_k$ is a finite length word with letters $\tau_1, \ldots, \tau_k \in \{0, 1\}$. An open simple path in the annulus $\Lambda_{m,n}$ from the inner boundary $\partial \Lambda_m$ to the outer boundary $\partial \Lambda_n$ is called an arm of type 1. Similarly, an arm of type 0 from $\partial \Lambda_m$ to $\partial \Lambda_n$ in $\Lambda_{m,n}$ denotes a dual open simple path on $\Lambda^*_n$ (the subgraph of $(\mathbb{Z}^2)^*$ induced by the edges dual to the edges of $\Lambda_{m,n}$) from the inner boundary to the outer boundary.

**Definition 2.4.** Let $1 \leq m \leq n$ and let $\tau$ be a type sequence. The arm event $A_\tau(m, n)$ denotes the existence of $|\tau|$ counterclockwise-ordered, disjoint arms $\gamma_1, \ldots, \gamma_{|\tau|}$ from $\partial \Lambda_m$ to $\partial \Lambda_n$ in $\Lambda_{m,n}$ such that $\gamma^i$ has type $\tau_i$ for all $1 \leq i \leq |\tau|$. Similarly, we define the arm event $A_\tau^+(m, n)$ by requiring that the arms stay in the upper halfplane $\mathbb{Z} \times \mathbb{Z}_+$ and that $\gamma_1, \ldots, \gamma_{|\tau|}$ starts with the rightmost arm.

Note that the condition $m \geq |\tau|$ is sufficient to guarantee that $A_\tau(m, n)$ and $A_\tau^+(m, n)$ are non-empty events.

We will make use of the following results on critical exponents for FK percolation. They appear in [DCMT21, Proposition 6.6, Corollary 6.7] and are stated for the measure $\phi_{22,q}$ there. However, using the inclusions $A_{010}^+(m, n) \subset A_{010}^+(m, n/2)$ and $A_{100100}^+(m, n) \subset A_{100100}^+(m, n/2)$ for $m \leq n/2$ together with the mixing property, we obtain the result as stated below.

**Corollary 2.5 ([DCMT21]).** Let $q \in [1, 4)$. There exists a constant $\beta_1 = \beta_1(q) > 0$ such that for all $n \geq m \geq 1$,

$$
\phi_{\Lambda_n,q}^\xi(A_{010}^+(m, n)) \lesssim \left( \frac{m}{n} \right)^{1+\beta_1},
$$

$$
\phi_{\Lambda_n,q}^\xi(A_{100100}^+(m, n)) \lesssim \left( \frac{m}{n} \right)^{2+\beta_1},
$$

where the bounds are uniform in $\xi$ and in $m, n$.

Another application of Theorem 2.3 will be needed in Section 4 to prove the convergence of the (discrete) fuzzy Potts cluster boundaries. A sequence of distinct clusters $C_1, \ldots, C_k$ such that for every $1 \leq i < k$, the clusters $C_i$ and $C_{i+1}$ have graph distance 1, is called a cluster chain of length $k \geq 1$. For $\ell > 0$ and $n \geq 1$, denote by $\Lambda_n^\ell$ the subgraph induced by $[-\ell n, \ell n] \times [-n, n]$. The following theorem (see [DCMT21, Theorem 7.5]) states that with probability close to 1, the box $\Lambda_n^\ell$ is crossed from left to right by a cluster chain consisting of macroscopic clusters. The result in [DCMT21] is stated with respect to $\phi_{22,q}$ but the proof goes through with more general boundary conditions (away from the box $\Lambda_n^\ell$ under consideration).

**Theorem 2.6 ([DCMT21]).** Let $q \in [1, 4)$. For $\alpha > 0$ let $S(\alpha, n, \ell)$ be the event that there exists a cluster chain with each cluster having diameter at least $\alpha n$, that crosses $\Lambda_n^\ell$ from the left side to the right side. Then, for every $\epsilon, \ell > 0$ and $c > 0$, there exists $\alpha > 0$ such that for every $n \geq 1$,

$$
\phi_{\Lambda_n^\ell(1+c)n,q}^\xi(S(\alpha, n, \ell)) \geq 1 - \epsilon.
$$

### 2.2. Fuzzy Potts model

Having introduced critical FK percolation in the previous subsection, we are now in the position to define the fuzzy Potts model. Let $G = (V, E)$ be a finite subgraph of $\mathbb{Z}^2$. We consider the space $\{0, 1\}^E \times \{R, B\}^V$ consisting of elements $(\omega, \sigma)$ with $\omega$ representing a (bond) percolation configuration and $\sigma$ representing a (site) coloring. We say that a vertex $v \in V$ is red if $\sigma_v = R$ and blue if $\sigma_v = B$. Fix a cluster weight $q \in [1, 4]$, a coloring parameter $r \in [0, 1]$, and a boundary condition $\xi$. The probability measure $P_{G,q,r}^\xi$ on $\{0, 1\}^E \times \{R, B\}^V$ is constructed in two steps:

(i) Sample $\omega \in \{0, 1\}^E$ according to the critical FK percolation measure $\phi_{G,q}^\xi$. 

(ii) Color every cluster \( C \in \mathcal{C}(\omega^\xi) \) in red with probability \( r \) and in blue with probability \( 1 - r \), independently of the other clusters. Here, coloring a cluster \( C \) in red (resp. blue) means to set \( \sigma_v = R \) (resp. \( \sigma_v = B \)) for all vertices \( v \in C \).

In the above, recall that \( \mathcal{C}(\omega^\xi) \) denotes the set of clusters of the graph \( \omega^\xi \) that is obtained from \( \omega \) by adding additional edges between vertices belonging to the same partition element of \( \xi \).

The second marginal of \( P_{\xi,q,r}^\mu \), denoted by \( \mu_{\xi,q,r}^\mu \), is called the fuzzy Potts measure. Similarly, we obtain \( P_{\mathcal{Z}^2,q,r} \) (resp. \( P_{\mathcal{Z}^2,q,r}^0 \)) and its second marginal \( \mu_{\mathcal{Z}^2,q,r}^\mu \) (resp. \( \mu_{\mathcal{Z}^2,q,r}^0 \)) from \( \phi_{\mathcal{Z}^2,q} \) (resp. \( \phi_{\mathcal{Z}^2,q}^0 \)) using the analogous construction.

To describe the geometry of the fuzzy Potts model, we again want to define the notions of red (resp. blue) arms and clusters. Compared with FK percolation, which is a measure on the status of edges of \( \mathbb{Z}^2 \), the fuzzy Potts model is a measure on the color of vertices of \( \mathbb{Z}^2 \) and therefore, it is a priori not clear which adjacency relation should be chosen to define arms and clusters. Recall that we have naturally embedded \( \mathbb{Z}^2 \) in the plane \( \mathbb{R}^2 \) and denote by the \( d_1 \) (resp. \( d_\infty \)) metric induced by the 1-norm (resp. by the \( \infty \)-norm) on \( \mathbb{R}^2 \). We call a finite sequence of vertices \( \gamma = (\gamma_i)_{i=0}^n \) a strong path (resp. weak path) if for all \( 1 \leq i \leq n \), \( d_1(\gamma_i, \gamma_{i-1}) = 1 \) or \( d_\infty(\gamma_i, \gamma_{i-1}) = 1 \). Note that the notion of strong path coincides with a nearest-neighbor path on the square lattice, whereas a weak path is allowed to move along diagonals.

The reason why we are also introducing the notion of weak paths is that to prove the alternating property of arm events we need to specify whether arms correspond to strong or weak paths in the definition below. In the definition below, a color sequence \( \tau = \tau_1 \cdots \tau_k \) is a finite length word with letters \( \tau_1, \ldots, \tau_k \in \{ R, B \} \).

**Definition 2.7.** Let \( 1 \leq m \leq n \) and let \( \tau \) be a color sequence. The arm event \( A_\tau^\mu(m,n) \) denotes the existence of \( |\tau| \) counterclockwise-ordered, disjoint strong paths \( \gamma^1, \ldots, \gamma^{|\tau|} \) from \( \partial \Lambda_m \) to \( \partial \Lambda_n \) in \( \Lambda_{m,n} \) such that \( \gamma^i \) has color \( \tau_i \) for all \( 1 \leq i \leq |\tau| \). The arm event \( A_\tau(m,n) \) is defined in the same way except that \( \gamma^i \) is allowed to be a weak path if \( \tau_i = B \).

Similarly, we define the arm events \( A_{\tau}^+(m,n) \) and \( A_{\tau}^+(m,n) \) by requiring that the arms stay in the upper halfplane \( \mathbb{Z} \times \mathbb{Z}_+ \) and that \( \gamma^1, \ldots, \gamma^{|\tau|} \) starts with the rightmost arm.

Hence, the superscript \( s \) indicates that all arms correspond to strong paths. We do not introduce the arm event in which all arms correspond to weak paths since in that case red and blue weak paths can cross each other. As in the case of arm events for FK percolation, we remark that assuming \( m \geq |\tau| \) is sufficient to guarantee that \( A_\tau(m,n) \), \( A_\tau^+(m,n) \), \( A_{\tau}^+(m,n) \) and \( A_{\tau}^+(m,n) \) are non-empty.

Let us recall the definition of the number of ‘interfaces’ associated to arm events: Setting \( \tau_{k+1} := \tau_k \), it was defined by

\[
I(\tau) = \#\{1 \leq i \leq k : \tau_i \neq \tau_{i+1} \} \quad \text{and} \quad I^+(\tau) = 1 + \#\{1 \leq i < k : \tau_i \neq \tau_{i+1} \}.
\]

A color sequence \( \tau \) is called alternating if \( |\tau| = 1 \) or \( |\tau| = I(\tau) \), and alternating for the halfplane if \( I^+(\tau) = |\tau| \). In both cases, it means that no color occurs twice subsequently (with the only difference that the last and the first letter of \( \tau \) are not viewed as subsequent in the halfplane). For example, the color sequence \( \tau = RBR \) is not alternating but it is alternating for the halfplane.
2.3. Loop (ensemble) topologies

Since we will investigate the convergence of collections of loops in planar domains, we will need to define the sense of convergence carefully and we use the setup from [BH19] (see this work for more details).

Let \( C^*(\partial \mathbb{D}, \mathbb{C}) \) be the closure in the uniform topology of all injective elements of \( C(\partial \mathbb{D}, \mathbb{C}) \), i.e. closed curves that may touch but not cross themselves. When we talk of loops in \( \mathbb{C} \) in this work, we refer to elements of \( \mathcal{C} := C^*(\partial \mathbb{D}, \mathbb{C})/ \sim \), where the equivalence relation \( \sim \) is defined by \( \eta \sim \eta \circ \phi \) whenever \( \phi \) is a homeomorphism from \( \partial \mathbb{D} \) to itself. We define a metric \( d_C \) on \( \mathcal{C} \) by

\[
d_C([\eta], [\eta']) = \inf_{\phi} \| \eta - \eta' \circ \phi \|_{\infty}
\]

where \([\eta]\) denotes the equivalence class associated to \( \eta \) and the infimum is taken over all homeomorphisms \( \phi \) from \( \partial \mathbb{D} \) to itself. This turns \( (\mathcal{C}, d_C) \) into a Polish space. By a small abuse of notation, we will write \( \eta \) instead of \([\eta]\) throughout.

We write \( \text{diam} (\eta) \) for the diameter of the image of \( \eta \) and we also let \( \eta' \) denote all the points in \( \mathbb{C} \) which are surrounded by \( \eta \) or more precisely all the points around which \( \eta \) has winding \( \pm 1 \). We say that \( \eta \) surrounds \( \eta' \) if \( \eta' \) is contained in the closure of \( \eta \).

We will also want to consider collections of loops: Let \( \mathcal{L} \) be the set of countable subsets \( \Gamma \) of \( \mathcal{C} \) with the property that \( \Gamma_\varepsilon := \{ \eta \in \Gamma : \text{diam} (\eta) > \varepsilon \} \) is finite for all \( \varepsilon > 0 \) and the property that \( \Gamma = \bigcup_{\varepsilon > 0} \Gamma_\varepsilon \) (this is called local finiteness). We define

\[
d_{\mathcal{L}}(\Gamma, \Gamma') = \inf_{G, G', \pi \in \mathcal{G}} \sup_{\eta \in \Gamma \setminus G} \text{diam} (\eta) \cup \sup_{\eta' \in \Gamma' \cap G'} \text{diam} (\eta')
\]

where the infimum is taken over all \( G \subset \Gamma, G' \subset \Gamma' \) and all bijections \( \pi : G \rightarrow G' \). Again, this definition turns \( \mathcal{L} \) with the topology induced by \( d_{\mathcal{L}} \) into a Polish space.

When \( \Gamma \in \mathcal{L} \) has the property that for any two loops \( \eta, \eta' \in \Gamma \) with \( \eta' \cap (\eta')^o \neq \emptyset \), \( \eta \) surrounds \( \eta' \) or \( \eta' \) surrounds \( \eta \), we can associate a nesting level \( N_{\Gamma, \eta} \) to each loop \( \eta \in \Gamma \). Indeed, we let \( N_{\Gamma, \eta} = n + 1 \) where \( n \) is the number of distinct loops surrounding \( \eta \) (not counting \( \eta \) itself). In particular, the outermost loops have nesting level 1.

We will also introduce the space \( \mathcal{C}' \) of curves. We let \( C^*([0, 1], \mathbb{C}) \) be the closure in the uniform topology of all injective elements of \( C([0, 1], \mathbb{C}) \). We call the space \( \mathcal{C}' = C^*([0, 1], \mathbb{C})/ \sim \) the space of curves where we identify \( \gamma \sim \gamma \circ \phi \) whenever \( \phi : [0, 1] \rightarrow [0, 1] \) is an increasing homeomorphism. We let

\[
d_{\mathcal{C}'}([\gamma], [\gamma']) = \inf_{\phi} \| \gamma - \gamma' \circ \phi \|_{\infty}
\]

where the infimum is taken over all increasing homeomorphisms \( \phi \) from \([0, 1]\) to itself. Also in this case we obtain a Polish space \( (\mathcal{C}', d_{\mathcal{C}'}) \). As in the case of loops we will write \( \gamma \) instead of \([\gamma]\) to avoid visual clutter.

When working with curves and collections of loops, it will always be understood that we fix some arbitrary parametrization for them.

2.4. Encoding percolation models as loop configurations

In this section, we explain how both a bond and a site percolation can be encoded as a collection of loops (in the sense of the previous section). Since we will use these notions to talk about scaling limits in this paper, we will consider subgraphs of \( \varepsilon \mathbb{Z}^2 \) for \( \varepsilon > 0 \). Let \( \mathcal{D} = (V, E) \) be a discrete domain in \( \varepsilon \mathbb{Z}^2 \), so \( \mathcal{D} \) is therefore a finite subgraph of \( \varepsilon \mathbb{Z}^2 \). We can view the boundary of \( \mathcal{D} \) as a piecewise linear simple loop \( \lambda \) and write \( \lambda^o \) for the open set of points it surrounds.

To each bond percolation \( \omega \in \{0, 1\}^E \) and a cluster \( C \in \mathcal{C}(\omega) \) we can associate the set

\[
O^C_\omega = \bigcup_{v, w \in C : \omega_{vw} = 1} R_{vw} \cup \bigcup_{v \in V} [\varepsilon / 4, \varepsilon / 4]^2
\]
Figure 1. Left. The set $\cup_{C \in C(\omega)} O^C_\omega$ associated to a bond percolation configuration $\omega$ is shaded in blue and the clusters $C(\omega)$ are colored to obtain a configuration $\sigma$. Edges which are part of the boundary of the domain but are not open with respect to $\omega$ are drawn as a dashed line. Center. We consider a site percolation configuration $\sigma$ and draw a tile $X^+$ (as shown at the bottom) over each blue vertex and take the union to obtain the area shaded in blue. Right. The set in the center figure is intersected with the closure of the domain to obtain $O^{+}_\sigma$. This step corresponds to the definition of blue clusters in terms of weak paths which stay within the discrete domain. The curve in green shows the curve $\gamma^+_{\sigma,a,b}$ from the boundary point $a$ (at the bottom) to $b$ (at the top).

where $R_{vw} = \{tv + (1-t)w: t \in [0,1]\} + [-\epsilon/4,\epsilon/4]^2$ is a rectangle with width $\epsilon/2$ and length $3\epsilon/2$ centered around the edge $vw$.

We write $\eta^C_\omega$ for the boundary of the unbounded connected component of $C \setminus O^C_\omega$ and $\Gamma^C_\omega$ for the collection of boundaries of the bounded connected component of $C \setminus O^C_\omega$. We call the loop $\eta^C_\omega$ the outer boundary and view it as an element of $C$ and $\Gamma^C_\omega$ the collection of inner boundaries of $C$, viewed as an element of $\mathcal{L}$ (see Figure 1).

We now define the loop encoding of the bond percolation configuration $\omega$ by $\Gamma_\omega := \Gamma^O_\omega \cup \Gamma^I_\omega$ where

$$\Gamma^O_\omega = \{\eta^C_\omega: C \in C(\omega)\} \quad \text{and} \quad \Gamma^I_\omega = \bigcup_{C \in \mathcal{C}(\omega)} \Gamma^C_\omega$$

are the sets of outer and inner boundaries respectively. Note that $\Gamma^O_\omega$ (resp. $\Gamma^I_\omega$) is the set of loops in $\Gamma_\omega$ with odd (resp. even) nesting level in $\Gamma_\omega$.

The constant $1/4$ in the definition here is completely arbitrary and any constant $<1/2$ would work. In much of the literature, the loop encoding is defined slightly differently but for statements about scaling limit results, the differing definitions of the loop encodings do not matter.

We write $\Gamma^O_\omega$ for the loops in $\Gamma_\omega$ which do not stay within $\lambda^o$ (these are the outer boundaries of clusters of $\omega$ intersecting $\partial D$).

Let us now suppose that $\omega \in \{0,1\}^E$ and that $\sigma \in \{R,B\}^V$ is obtained from $\omega$ by assigning colors to its clusters $C(\omega)$ (we are therefore working implicitly with free boundary conditions). Let us first make the following definitions:

$$\Gamma^{OR}_{\omega,\sigma} = \{\eta^C_\omega: C \in C(\omega), \sigma_v = R, \forall v \in C\},$$

$$\Gamma^{IR}_{\omega,\sigma} = \bigcup_{C \in \mathcal{C}(\omega): \sigma_v = R \forall v \in C} \Gamma^C_\omega.$$

So $\Gamma^{OR}_{\omega,\sigma}$ (resp. $\Gamma^{IR}_{\omega,\sigma}$) is the set of outer (resp. inner) boundaries of red clusters. We analogously define $\Gamma^{OB}_{\omega,\sigma}$ and $\Gamma^{IB}_{\omega,\sigma}$ as the outer and inner boundaries of blue clusters. Now we encode the site percolation configuration $\sigma$ in terms of a collection as loops as well. To this end, let

$$O^+_\sigma = \left(\bigcup_{v \in V: \sigma_v = B} (v + X^\pm)\right) \cap \overline{\lambda^o}$$
where $X^+$ (resp. $X^-$) is a square with some boxes superimposed on (resp. removed from) its corners; more precisely, we define

$$X^+ = [-\varepsilon/2, \varepsilon/2]^2 \cup \{ (\pm \varepsilon/2, \pm \varepsilon/2) \} + [-\varepsilon/4, \varepsilon/4]^2 \},$$

$$X^- = [-\varepsilon/2, \varepsilon/2]^2 \setminus \{ (\pm \varepsilon/2, \pm \varepsilon/2) \} + (-\varepsilon/4, \varepsilon/4]^2 \}.$$

The boundary of $O^\pm_\sigma$ can be written as the disjoint union of simple loops and we denote this set of simple loops by $\Sigma^\pm_\sigma$.

The convention in the definition of $\Sigma^\pm_\sigma$ corresponds to the convention of defining blue fuzzy Potts clusters in terms of weak paths and red fuzzy Potts clusters in terms of strong paths while the definition of $\Sigma_\sigma$ arises from considering strong blue and weak red paths. It will turn out that this choice does not matter when we are determining the scaling limit of the critical fuzzy Potts measure with $q \in [1, 4]$ since the loops in the scaling limit will all be disjoint.

If $a, b \in \partial D$ are distinct we also want to associate an interface from $a$ to $b$ which is blue on its right and red on its left. We make the following definition: The interface $\gamma^n_{\sigma,a,b}$ is given by the unique simple curve from $a$ to $b$ which traces the counterclockwise boundary arc of $\lambda$ from $a$ to $b$ except that it follows the boundary of $O^\pm_\sigma$ in clockwise order whenever it hits $O^\pm_\sigma$. This definition is illustrated in Figure 1.

Let us now quickly explain what we mean by the convergence of discrete domains to continuum ones. This will appear in the formulation of the scaling limit conjectures and results. Consider a Jordan domain $D \subset \mathbb{C}$ and write $\lambda_\infty \in \mathcal{C}$ for its boundary. If $\epsilon_n \to 0$ we consider discrete domains $D_n = (V_n, E_n)$ in $\epsilon_n \mathbb{Z}^2$ and associate to $D_n$ its boundary curve $\lambda_n$ which we view as an element of $\mathcal{C}$. We say that $D_n$ converges to $D$ as $n \to \infty$ if $d_C(\lambda_n, \lambda_\infty) \to 0$ as $n \to \infty$.

2.5. (Boundary) Conformal Loop Ensembles

We assume familiarity with Schramm-Loewner evolutions (SLE), see [Sch00, Wer04, Law05], conformal loop ensembles (CLE), see [She09, SW12, MSW17], and boundary conformal loop ensembles (BCLE), see [MSW17]. However, we will briefly review some of the key elements of the theory of CLEs and BCLEs for the reader’s convenience.

Let us begin by recalling the definition of $\text{SLE}_\kappa(\rho_-, \rho_+)$ started from $\xi_0$ with initial force points at $O_0^- \leq \xi_0$ (on the left) and $O_0^+ \geq \xi_0$ (on the right) when $\kappa > 0$ and $\rho_\pm > -2$.

It is the curve generated by the chordal Loewner chain with driving function $\xi$ where $\xi$ is the unique weak solution to the SDE

$$d\xi_t = \sqrt{\kappa} dB_t + \frac{\rho_-}{\xi_t - O_t^-} dt + \frac{\rho_+}{\xi_t - O_t^+} dt,$$

$$dO_t^\pm = \frac{2 dt}{O_t^\pm - \xi_t},$$

with initial values $\xi_0$, $O_0^\pm$ and $O^- \leq \xi \leq O^+$ where $B$ is a standard Brownian motion. The fact that a solution exists, uniqueness in law holds and that the Loewner chain is generated by a continuous curve from $\xi_0$ to $\infty$ is not trivial, see [MS16].

If $\xi_0 = O_0^+$ or $O_0^- = 0$ we obtain a scale invariant law on curves from 0 to $\infty$ which we simply call $\text{SLE}_\kappa(\rho_-, \rho_+)$ without reference to force or initial points. By applying a conformal transformation from $\mathbb{H}$ to any Jordan domain, we can define $\text{SLE}_\kappa(\rho_-, \rho_+)$ in any Jordan domain. This is also well-defined in the case when the force points and the starting point are all the same (and so the conformal transformation is not unique) by the scale-invariance of the curve in this case.

Boundary conformal loop ensembles are conformally invariant laws on $\mathcal{C}$ in Jordan domains such that each loop intersects the boundary of the domain. BCLEs consist either of simple or non-simple curves:

- BCLE$_\kappa(\rho)$ is defined for $\kappa \in (2, 4]$ and $\rho \in (-2, \kappa - 4)$ and is a conformally invariant law on collections of simple loops in the unit disk $\mathbb{D}$. 

Figure 2. This figure illustrates the relation between $\Xi \sim \text{BCLE}_\kappa(\rho)$ (in red), its false loops $\Xi^* \sim \text{BCLE}_\kappa(\kappa - 6 - \rho)$ (the boundaries of the regions shaded in light blue) and the curve $\gamma \sim \text{SLE}_\kappa(\rho, \kappa - 6 - \rho)$ (in green).

- $\text{BCLE}_{\kappa'}(\rho')$ is defined for $\kappa' \in (4, 8)$ and $\rho' \in [\kappa'/2 - 4, \kappa'/2 - 2]$ and is a conformally invariant law on collections of non-simple loops in $\mathbb{D}$.

The conformal invariance property implies that by applying conformal transformations we obtain a well-defined notion of BCLE in any Jordan domain; we restrict to Jordan domains here since this ensures that the BCLE loops in the new domain are again continuous curves in the Euclidean topology (this fails for domains with more pathological boundaries).

The definition of BCLEs further extends to domains which are disjoint unions of Jordan domains by sampling independent BCLEs within each connected component and the notion of false loops introduced in the paragraph below extends to this setting as well.

If $\Xi \sim \text{BCLE}_\kappa(\rho)$ then we can consider all the ‘boundary to boundary’ segments of the loops (i.e. loop segments which touch the boundary at exactly two points) and observe that these loop segments in fact encode another collection of loops which we will denote by $\Xi^*$ with the same ‘boundary to boundary’ loop segments (see Figure 2). The collection $\Xi^*$ is called the collection of false loops of $\Xi$ and $\Xi^* \sim \text{BCLE}_\kappa(\kappa - 6 - \rho)$. The analogous construction can be performed in the case of non-simple BCLE.

We will momentarily want to sample BCLE within the complementary components of the loops of BCLE so we need to mention that the complementary components of BCLE loops are again Jordan domains. More precisely, suppose that $\eta$ is a loop within a BCLE. Then $\mathbb{C} \setminus \eta(\partial \mathbb{D})$ can be decomposed into its connected components. It turns out that by SLE duality (as established in the required generality in [MS16]) all the bounded connected components are Jordan domains. Moreover, the boundary of the unbounded component of $\mathbb{C} \setminus \eta(\partial \mathbb{D})$ is a simple curve which we call the boundary of the filling of $\eta$. 

\[\text{BCLE}_\kappa(\rho) \quad \text{BCLE}_\kappa(\kappa - 6 - \rho) \quad \text{SLE}_\kappa(\rho, \kappa - 6 - \rho)\]
The object $\text{CLE}_{\kappa'}$ (which is a law on $\mathcal{L}$ supported on collections of non-simple loops) can now be described as follows: If $\Gamma$ is a nested $\text{CLE}_{\kappa'}$ then the law of its boundary touching loops $\Xi$ is a BCLE$_{\kappa'}(0)$ and the conditional law of $\Xi$ given $\Xi$ is given by the union of $\Xi$ together with an independent nested $\text{CLE}_{\kappa'}$ within each loop of $\Xi$ and within each false loop of $\Xi^*$. This formulation can also be phrased as an iterative construction. Just as for BCLEs, CLEs are defined in any disjoint union of Jordan domains by conformal invariance and taking independent samples in the connected components.

Recall from the previous Section 2.4 that in the discrete setting, a percolation configuration can be encoded as a collection of nested loops. The following classical convergence conjecture states that nested CLE arise in the limit from these collections of discrete loops; this is so far only known for $q = 2$, i.e. $\kappa' = 16/3$ by [Smi10, KS19, KS16]. We stress that the encoding of percolation configurations in terms of loops is convenient precisely because this description is explicit using SLE tools in the continuum.

**Conjecture 2.8.** Consider $q \in (0,4]$ and let $\kappa' = 4\pi / \arccos(-\sqrt{q}/2) \in [4,8)$. Suppose that $D$ is a Jordan domain, $\varepsilon_n \to 0$ as $n \to \infty$ and that $D_n$ is a discrete domain in $\varepsilon_n \mathbb{Z}^2$ for each $n \geq 1$ such that $D_n$ converges to $D$ as $n \to \infty$. Let $\omega^n \sim \Phi^0_{D_n,q}$, let $\Gamma$ be a nested CLE$_{\kappa'}$ in $D$ and let $\Gamma^\partial$ be the loops in $\Gamma$ intersecting $\partial D$. Then $(\Gamma^\omega_n \setminus \Gamma^\partial_n, \Gamma^\omega_n)$ converges in distribution to $(\Gamma \setminus \Gamma^\partial, \Gamma^\partial)$ with respect to the metric $d_\mathcal{L}$.

**2.6. CLE Percolations**

We now define the divide and color model in the continuum setting. This is based on the seminal work [MSW17]. Throughout this text, we will refer to the construction performed in this section as the continuum fuzzy Potts model.

In the discrete setting, we assigned colors to the percolation clusters and then agglomerated them into divide and color clusters. In the continuum setting, we will construct the coupling of divide and color cluster boundaries and percolation cluster boundaries in one step. Fix $\kappa' \in (4,8)$ and $r \in (0,1)$. We will first define all the relevant parameters for the continuum construction. Let

\begin{align}
\kappa &= 16/\kappa' \in (2,4) , \\
q(\kappa') &= 4\cos^2(4\pi/\kappa') \in (0,4) , \\
\rho_B(\kappa', r) &= \frac{2}{\pi} \arctan \left( \frac{\sin(\pi \kappa/2)}{1 + \cos(\pi \kappa/2) - 1/r} \right) - 2 \in (-2, \kappa - 4) , \\
\rho_R(\kappa', r) &= \kappa - 6 - \rho_B(\kappa', r) \\
&= \frac{2}{\pi} \arctan \left( \frac{\sin(\pi \kappa/2)}{1 + \cos(\pi \kappa/2) - 1/(1 - r)} \right) - 2 \in (-2, \kappa - 4) , \\
\rho'_B(\kappa', r) &= -\frac{\kappa'}{4} \rho_B + 2 \in (\kappa'/2 - 4, 0) , \\
\rho'_R(\kappa', r) &= -\frac{\kappa'}{4} (\rho_R + 2) \in (\kappa'/2 - 4, 0) .
\end{align}

When $\kappa'$ and $r$ are clear from the context, we will sometimes drop them from the notation. In the coupling constructed below, $\Gamma^{OR}$ (resp. $\Gamma^{IR}$) correspond to the outer (resp. inner) boundaries of percolation clusters which have been colored red (similarly in the case when we consider blue clusters) and $\Sigma$ will be the collection of continuum fuzzy Potts interfaces.

We will now perform a rather involved construction. See Figure 3 for an illustration.

Let $\Xi_B \sim \text{BCLE}_{\kappa}(\rho_B)$ in the unit disk $\mathbb{D}$ and let $\Xi := \Xi_B \sim \text{BCLE}_{\kappa}(\rho_R)$ be the collection of its false loops. Within $\Xi_B$ we make the following definition.

- Let $\Xi'_B \sim \text{BCLE}_{\kappa'}(\rho'_B)$ in $\cup_{\eta \in \Xi_B} \eta^\rho$ and let $\Xi''_B$ be its false loops which then forms a $\text{BCLE}_{\kappa'}(\kappa' - 6 - \rho'_B)$ in $\cup_{\eta \in \Xi_B} \eta^\rho$.
- Moreover, let $\Xi'''_B$ be an non-nested $\text{CLE}_{\kappa'}$ in $\cup_{\eta \in \Xi''_B} \eta^\rho$. 

We make the analogous definition with \((\Xi_B, \rho_B)\) replaced by \((\Xi_R, \rho_R)\) to obtain \((\Xi'_R, \Xi''_R)\). Using conformal invariance and by taking independent samples in connected components, we can sample the tuple \(\Xi := (\Xi_B, \Xi'_B, \Xi''_B, \Xi_R, \Xi'_R, \Xi''_R)\) in any domain which is a disjoint union of Jordan domains.

Fix a Jordan domain \(D\) and write \(\eta_0\) for the loop tracing the boundary. To perform the iteration, we first let \(\Upsilon_B^0 = \emptyset\) and \(\Upsilon_R^0 = \{\eta_0\}\). Suppose that we have already constructed \((\Sigma_i, \Upsilon_i^B, \Upsilon_i^R, \Gamma_i^{OB}, \Gamma_i^{IR}, \Gamma_i^B, \Gamma_i^R): 1 \leq i \leq n - 1\) for \(n \geq 1\). Then we proceed as follows:

- Sample a copy of \(\Xi\) within \(\cup_{\eta \in \Upsilon^B_{n-1}} \eta^\rho\) and call it \(\Xi^B_{n\to B}\). We also sample an independent copy of \(\Xi\) in \(\cup_{\eta \in \Upsilon^R_{n-1}} \eta^\rho\) and call the resulting tuple \(\Xi^R_{n\to B}\). We write
  \[
  \Xi^B_{n\to B} = (\Xi^B_{RB}, \Xi^m_{BB}, \Xi^m_{BR}, \Xi^m_{RR}, \Xi^m_{RR}),
  \Xi^R_{n\to B} = (\Xi^R_{RB}, \Xi^m_{BB}, \Xi^m_{BR}, \Xi^m_{RR}, \Xi^m_{RR}),
  \]

- Define \(\Sigma_n, \Upsilon^B_n, \Upsilon^R_n, \Gamma^{OB}_n, \Gamma^{IR}_n, \Gamma^B_n, \Gamma^R_n\) by
  \[
  \Sigma_n = \Xi^m_{RB} \cup \Xi^m_{BR},
  \Upsilon^B_n = \Xi^m_{RB} \cup \Xi^m_{BB} \cup \Xi^m_{BR} \cup \Xi^m_{BR},
  \Upsilon^R_n = \Xi^m_{BB} \cup \Xi^m_{BR} \cup \Xi^m_{RR} \cup \Xi^m_{RR},
  \Gamma^{OB}_n = \Xi^m_{BR} \cup \Xi^m_{RB},
  \Gamma^{IR}_n = \Xi^m_{BR} \cup \Xi^m_{RR},
  \Gamma^B_n = \Xi^m_{BB} \cup \Xi^m_{BR},
  \Gamma^R_n = \Xi^m_{BB} \cup \Xi^m_{RR}.
  \]

The reader is encouraged to look at Figure 3 where the first step of this iteration is displayed and the inductive definition mentioned. Lastly, we define \(\Sigma, \Gamma^{OB}, \Gamma^{IR}, \Gamma^B, \Gamma^R\) as the unions of the collection \((\Sigma_n), (\Gamma^{OB}_n), (\Gamma^{IR}_n), (\Gamma^B_n), (\Gamma^R_n)\) respectively. We also set
  \[
  \Gamma^O = \Gamma^{OB} \cup \Gamma^{IR},
  \Gamma^I = \Gamma^{IR} \cup \Gamma^B,
  \Gamma = \Gamma^O \cup \Gamma^I.
  \]
collections of loops. Then a,b distinct

Remark 2.10. For a, b ∈ ∂D we can define the interface γa,b from a to b associated to Ξ1RB = Σ1. By definition, we marginally have γa,b ∼ SLEκ(ρB, κ − 6 − ρB).

The next result is the continuum version of the classical Edwards-Sokal coupling with wired boundary conditions. Indeed, the appearance of Γ0 in the statement corresponds on the discrete side to the inner boundaries of the boundary cluster (when considering wired boundary conditions). The result was proved in [MSW17, Theorem 7.8] with the inexplicit parameter there having been determined in [MSW21, Theorem 1.2].

Theorem 2.11 ([MSW17, MSW21]). Suppose that κ′ ∈ (4, 6) and let r = 1/q(κ′). Let Γ0 be a non-nested CLEκ, r in a Jordan domain D and for each connected component C of Σ′ where Σ′ ∈ Γ0 we take an independent copy of Σ and conformally map it into C. Let Σ′ be the union of all these collections of loops. Then Σ′ forms a nested CLEκ in D.

Remark 2.13. Let us make two additional observations about this proposition here: It is a consequence of the definition of CLEκ that a.s. no two distinct loops in Γ contain the same boundary point and that no loop in Γ hits a boundary point twice. Hence in the proposition above we have

ηi+1 = ηi+1 ∈ ∂D for all i < n .
Figure 4. Top left. This figure illustrates Proposition 2.12; the green curve is $\gamma^{a,b}$, the area shaded in green is $\gamma^{a,b}(\{s,t\}) + B_r(0)$ and the dashed blue curve is $\gamma$. Top right. This graphic explains Corollary 2.14. The green loop is $\eta$, the shaded area in green is $\eta(\partial D) + B_r(0)$ and the dashed blue loop is $\tilde{\eta}$. The bottom row illustrates the argument which can be used to derive Corollary 2.14 from Proposition 2.12. Bottom left. The green curve is $\gamma^{a,b}$ restricted to $[s,t]$ and the dashed blue curve is the approximating curve appearing in the proposition (see also the top left part of this figure). Crucially, the restriction of $\gamma^{a,b}$ restricted to $[s,t]$ yields a segment of the loop $\eta$. Bottom right. The green curve is $\gamma^{b,a}$ restricted to $[s',t']$ and the dashed blue curve is again the approximating curve as in the proposition. Again this curve segment forms part of $\eta$. The key is that the approximating curves appearing in the two bottom figures intersect which readily implies the corollary.

This comment will also equally apply to Corollary 2.14. Furthermore, there is no loop in $\Gamma$ that lies right of $\gamma^{a,b}$ and which intersects the set $\gamma^{a,b}([0,1]) \cap ([a,b])$; this is a consequence of the inductive construction of $\Gamma$ in this section in terms of iterated BCLEs. Hence, by the local finiteness of $\Gamma$, for any given $\delta > 0$ we can take $\epsilon > 0$ in the proposition sufficiently small such that $\text{diam}(\eta^{1}), \text{diam}(\eta^{n}) < \delta$. 
There is also a version of the result above in the case of the actual BCLE loops rather than the interfaces. Indeed, still in the setting $D = \mathbb{D}$, for any $\eta \in \Xi^i_{RB}$ we can take $a, b \in \mathbb{Q} := \{e^{i\theta} : \theta \in \mathbb{Q}\}$ such that $\eta(\partial \mathbb{D})$ intersects both $\langle (a, b) \rangle$ and $\langle (b, a) \rangle$.

Then $\eta(\partial \mathbb{D}) = \gamma_{a,b}([s, t]) \cup \gamma_{b,a}([s', t'])$ where

$$s = \inf \{u \geq 0 : \gamma_u \in \langle (a, b) \rangle \cap \eta(\partial \mathbb{D})\}, \quad t = \sup \{u \geq 0 : \gamma_u \in \langle (a, b) \rangle \cap \eta(\partial \mathbb{D})\},$$

$$s' = \inf \{u \geq 0 : \gamma_u \in \langle (b, a) \rangle \cap \eta(\partial \mathbb{D})\}, \quad t' = \sup \{u \geq 0 : \gamma_u \in \langle (b, a) \rangle \cap \eta(\partial \mathbb{D})\}.$$

By applying Proposition 2.12 to the segments $\gamma_{a,b}([s, t])$ and $\gamma_{b,a}([s', t'])$ we obtain curves $\tilde{\gamma}_{a,b}$ and $\tilde{\gamma}_{b,a}$ (for any prespecified $\epsilon > 0$). By noting that $\tilde{\gamma}_{a,b}$ and $\tilde{\gamma}_{b,a}$ intersect for $\epsilon > 0$ sufficiently small, we readily obtain the following corollary. See also Figure 4 where this is illustrated.

**Corollary 2.14.** Consider $D = \mathbb{D}$ and let $\Gamma^Q_f$ be defined as in Proposition 2.12. Then the following statement is almost surely true: Suppose that $\eta \in \Xi^i_{RB}$ surrounds a point $z \in \mathbb{D}$ and fix $\epsilon > 0$. Then there are $\eta^1, \ldots, \eta^n \in \Gamma^Q_f$ (all oriented counterclockwise say) and $s^1_1, \ldots, s^1_n \in \partial \mathbb{D}$ such that the concatenation of the curves $\gamma^1|_{\langle (s^1_1, s^1_n) \rangle}, \ldots, \gamma^n|_{\langle (s^n_1, s^n_n) \rangle}$ defines a simple loop $\tilde{\eta}$, viewed as a function on $\partial \mathbb{D}$, such that $\tilde{\eta}$ surrounds the curve $\tilde{\eta}$, $\tilde{\eta}$ surrounds $z$ and $\tilde{\eta}(\partial \mathbb{D}) \subset \eta(\partial \mathbb{D}) + B_\epsilon(0)$.

### 2.7. Imaginary Geometry Results

In this short section, we collect some results from the theory of imaginary geometry which will be used in Section 5. We encourage the reader to skip this section and refer back to it whenever necessary when reading Section 5.

The following two results appear as part of [MS16] (see also [MSW17, Section 8]) and in [DMS21, Theorem 5.6] respectively. They are illustrated in Figure 5.

**Lemma 2.15 (MS16).** Let $\kappa \in (0, 4), \rho, \rho_-, \rho_+ > -2$ and $\rho > 0$. Then one can couple two curves $\gamma_- \sim \text{SLE}_\kappa(\rho - 2, 2, \rho_-, \rho_+)$ and $\gamma_+ \sim \text{SLE}_\kappa(\rho_-, \rho_+)$ such that conditionally on $\gamma_-$, the restrictions of $\gamma_+$ to the components right of $\gamma_-$ are independent SLE$_\kappa(\rho_-, \rho_+)$ curves.

**Lemma 2.16 (DMS21).** Fix $\kappa \in (0, 4), \rho \in (-2, \kappa/2 - 2)$ and define $\rho' = \kappa - 4 - \rho$. Let $\gamma \sim \text{SLE}_\kappa(0, \rho)$ from $-i$ to $i$ in $\mathbb{D}$, write $\zeta$.

$$\zeta_1 = \inf \{t \geq 0 : \gamma_t \in \langle (1, i) \rangle\}, \quad \zeta_{-1} = \sup \{t < \zeta_1 : \gamma_t \in \langle (-i, 1) \rangle\}.$$ 

Then the conditional law of $\gamma([\zeta_{-1}, \zeta_1])$ given $\gamma([0, \zeta_{-1}] \cup [\zeta_1, 1])$ is that of the image of a SLE$_\kappa(0, \rho')$ from $\gamma_{\zeta_{-1}}$ to $\gamma_{\zeta_1}$ in the component of $\mathbb{D} \setminus \gamma([0, \zeta_{-1}] \cup [\zeta_1, 1])$ with 1 on its boundary.
Let us remark that the final lemma makes sense from the Loewner chain description as well since an SLE_{κ}(0, ρ) can be encoded in terms of a Bessel process of dimension δ = 1 + 2(ρ + 2)/κ and the excursion measure of such a Bessel process can be constructed from Bessel processes of dimension 4 − δ = 1 + 2(ρ + 2)/κ (i.e., the dimension is reflected around 2).

Another input on relations between different SLE curves that we will state here is the following change of coordinates property of SLE_{κ}(ρ, κ − 6 − ρ) curves (see [MSW17, Section 7] and [SW05]). This is related to the target invariance property of SLE_{κ}(ρ, κ − 6 − ρ) curves which has been instrumental in [MSW17].

**Lemma 2.17** ([MSW17]). Consider κ ∈ (2, 4) and ρ ∈ (−2, κ − 4). One can couple γ ∼ SLE_{κ}(ρ, κ − 6 − ρ) from −i to i in D with initial force points at −i (on the left) and x ∈ (−i, i] with γ′ ∼ SLE_{κ}(ρ, 0) from −i to x in D such that the following is true: If ζ_{x} = \inf\{t ≥ 0: γ_{t} \in (x, i)\} and ζ′_{x} = \inf\{t ≥ 0: γ′_{t} \in (x, i)\} then γ(0, ζ_{1}) = γ′(0, ζ′_{1}).

The final two lemmas we state here say that SLE curves stay within prespecified tubes with positive probability.

The proofs of both results are based on imaginary geometry (IG) techniques. The first result is a special case of [MW17, Lemma 2.5] and the second one follows from a straightforward adaptation of the proof of [MW17, Lemma 2.3].

**Lemma 2.18** ([MW17]). Consider κ ∈ (0, 4), ρ_{−} ∈ (−2, κ/2 − 2) and ρ_{+} > −2. Let γ ∼ SLE_{κ}(ρ_{−}, ρ_{+}) in D from −i to i with initial force points at x_{−} ∈ (−i, i) \cup \{−i\} and x_{+} ∈ (−i, i) \cup \{i\}. Let ν^{±}: [0, 1] → D be two simple curves with disjoint images such that only their endpoints are on \partial D. Also assume that ν_{0}^{−} ∈ ((−i, i), ν_{0}^{+} ∈ ((i, ν_{0}^{+})) \cap ((i, x_{−})), ν_{0}^{+} ∈ ((−i, i)) and ν_{1}^{−} ∈ ((i, ν_{1}^{−})). Then with positive probability, γ hits (ν_{1}^{−}, ν_{1}^{−}) before ν^{−}(0, 1) ∪ ν^{+}(0, 1).

**Lemma 2.19** ([MW17]). Consider κ ∈ (0, 4) and ρ_{±} > −2. Let γ ∼ SLE_{κ}(ρ_{−}, ρ_{+}) from −i to i in D with initial force points at x_{−} ∈ (−i, i) \cup \{−i\} and x_{+} ∈ (−i, i) \cup \{i\}. Let ν^{±}: [0, 1] → D be disjoint simple curves with only their endpoints on \partial D. Assume that ν_{0}^{−} ∈ ((i, −i)), ν_{1}^{−} ∈ ((i, ν_{0}^{−})) and ν_{1}^{+} ∈ ((i, ν_{1}^{+})), then with positive probability, γ does not hit ν^{−}(0, 1) ∪ ν^{+}(0, 1).

3. Fuzzy Potts model: Quasi-multimultiplicativity and other discrete ingredients

In this section, we study the fuzzy Potts model based on the precise understanding of critical FK percolation, summarized in Section 2.1, that relies to a large extent on [DCMT21]. Throughout the section, the cluster weight q ∈ (1, 4) and the coloring parameter r ∈ (0, 1) are fixed and we drop them from the notation. We will prove the ‘discrete’ ingredients needed to establish our main results. First, we will show separation, extension and localization properties of arms and these lead to quasi-multimultiplicativity for arm event probabilities. This part follows the strategy known for Bernoulli percolation (see [Kes87, Nol08]) and FK percolation (see [CDCH16, DM20, DCMT21]). Secondly, we will bound the alternating six-arm exponent for the fuzzy Potts model and explain how to reduce the study of arm exponents in the fuzzy Potts model to the case of alternating color sequences.

In comparison to the study of FK percolation, there are several difficulties arising due to the additional layer of randomness. To explain them and to motivate our approach, let us describe a natural first attempt: One could try to establish an analogue of the strong crossing estimates from Theorem 2.3 for the fuzzy Potts model, in order to then follow in much the same way the arguments used for FK percolation. In particular, one would show that conditional on any color configuration on the boundary of the rectangle, the probability of a red path crossing a

---

1 In the setting of [MW17], one considers a GFF h in D with IG boundary conditions −λ(1 + ρ_{−}) on (i, x_{−}), −λ on (x_{−}, i), λ on (i, x_{+}) and λ(1 + ρ_{+}) on (x_{+}, i), and the flow line γ from −i to i. One also considers a GFF h’ in the domain between ν_{−} and ν_{+} which has the same IG boundary conditions on ((ν_{−}, ν_{0}^{−})) ∪ ((ν_{1}^{−}, ν_{1}^{−})), −λ on ν^{−}(0, 1) and λ on ν^{+}(0, 1). The flow line γ’ from −i to i associated to h’ does not hit ν^{−}(0, 1) ∪ ν^{+}(0, 1) a.s. and the proof proceeds as in [MW17] by showing absolute continuity of h and h’ when both are restricted to a connected set which is a positive distance away from ν^{−}(0, 1) ∪ ν^{+}(0, 1).
rectangle from left to right is bounded away from 0 and 1, with the bounds only depending on the rectangle’s aspect ratio. This turns out to be wrong: It is expected (and known in the case of the Ising model) that if we condition on red boundary conditions, then the crossing probability will converge to 1 along a sequence of increasing rectangles with the same aspect ratio. Another aspect which makes the study of the fuzzy Potts model more challenging in general is the lack of a domain Markov property (except in special cases like Potts models).

Let us mention that softer Russo-Seymour-Welsh arguments (as proven in [KT20]) can be applied directly to the fuzzy Potts model using its positive association (see [Hi99] for the Bernoulli case $q = 1$ (due to Häggström and Schramm) and [KW07] for the general case $q \geq 1$) but they yield no control of boundary effects and so we will not make use of them here.

Our guiding principle in the study of the fuzzy Potts model will be the following: Whenever we want to condition on the color configuration on a subset $S \subset \mathbb{Z}^2$, we instead condition on the percolation configuration in $S$ and the color of clusters in the interior of $S$. Importantly, we never condition on the color of a cluster that intersects the boundary $\partial S$. In this way, we only obtain information about the induced FK boundary condition, i.e. the partition of $\partial S$, and we can apply the domain Markov property for FK percolation, the mixing property for FK percolation, as well as the crossing estimates of Theorem 2.3 to extend (resp. block) clusters using open (resp. dual open) paths.

In Section 2.2, we have introduced two closely related versions of arm events, $A^*_\tau(m,n)$ and $A_\tau(m,n)$, which denote the existence of $|\tau|$ arms in the annulus $\Lambda_{m,n}$ with colors prescribed by $\tau$. In the case of $A^*_\tau(m,n)$, all arms correspond to strong paths, whereas for $A_\tau(m,n)$, red arms correspond to strong paths and blue arms to weak paths. In this section, we choose to present most proofs for the latter version but it will be clear that the arguments also apply to the first version (which involves only strong paths).

3.1. Almost-arm events: Definitions and first results

The occurrence of $A_\tau(m,n)$ provides information about the colors at the inner boundary $\partial \Lambda_m$ and the outer boundary $\partial \Lambda_n$, and it is therefore convenient to introduce a variant of the arm event $A_\tau(m,n)$ that does not include such information.

Let us consider the subgraph induced by $S \subset \mathbb{R}^2$ and denote its edge set by $E(S)$. For a vertex $v \in S$ and a percolation configuration $\omega$, we denote by $C_{v,S}(\omega)$ the $\omega$-cluster of $v$ in $S$, i.e. the connected component of $v$ in the subgraph with edge set $\{e \in E(S) : \omega(e) = 1\}$. For a path $\gamma \subset S$, we define its cluster hull

$$H_{\gamma,S}(\omega) := \bigcup_{v \in \gamma} C_{v,S}(\omega)$$

as the union of all $\omega$-clusters intersecting $\gamma$. We will write it simply $H_\gamma(\omega)$ when no confusion can arise.

**Definition 3.1** (Almost-arm). Let $S \subset \mathbb{Z}^2$ and $A,B \subset \partial S$. A finite sequence of vertices $\gamma = (\gamma_i)_{i=0}^\ell \subset S$ with $\gamma_0 \in A$ and $\gamma_\ell \in B$ is called a red (resp. blue) almost-arm from $A$ to $B$ in $S$ with respect to $\mathbb{Z}^2$ if there exist $0 \leq k_A \leq k_B \leq \ell$ such that

(i) $(\gamma_0, \ldots, \gamma_{k_A})$ is an $\omega$-open path, $(\gamma_{k_A}, \ldots, \gamma_{k_B})$ is a strong (resp. weak) path, and $(\gamma_{k_B}, \ldots, \gamma_{\ell})$ is an $\omega$-open path;

(ii) the vertices in $\tilde{\gamma} := (\gamma_{k_A+1}, \ldots, \gamma_{k_B-1})$ are colored in red (resp. blue);

(iii) the $\omega$-clusters of the colored vertices in $\tilde{\gamma}$ do not intersect the boundary of $S$ with respect to $\mathbb{Z}^2$, i.e.

$$H_{\tilde{\gamma},S}(\omega) \cap \partial S = \emptyset.$$

For $S \subset \mathbb{Z} \times \mathbb{Z}_+$ and $A,B \subset \partial_+ S$, we analogously define a red (resp. blue) almost-arm from $A$ to $B$ in $S$ with respect to $\mathbb{Z} \times \mathbb{Z}_+$ by requiring that the $\omega$-clusters of the colored vertices in $\tilde{\gamma}$ do not intersect the boundary of $S$ with respect to $\mathbb{Z} \times \mathbb{Z}_+$, i.e. $H_{\tilde{\gamma},S}(\omega) \cap \partial_+ S = \emptyset$. 
Figure 6. Two examples of red almost-arms from $\partial \Lambda_m$ to $\partial \Lambda_n$ in $\Lambda_{m,n}$ (with respect to $\mathbb{Z}^2$) are highlighted in gray.

In other words, a red almost-arm is a concatenation of an (FK) arm of type 1 starting at $A$, a (fuzzy Potts) red arm using only interior clusters, and an (FK) arm of type 1 ending at $B$. We refer to Figure 6 for an illustration. Our definition is inspired by the notion of almost-crossing introduced in [Tas14, Chapter 3] and the idea of considering cluster hulls was already present in [BCM09].

Let us describe a simple way to check if a finite sequence of vertices $\gamma = (\gamma_i)_{i=0}^\ell$ is actually a red (resp. blue) almost-arm:

(i) Starting from $\gamma_0 \in A$, explore $\gamma$ until the first $\gamma_i \gamma_{i+1}$ that is not an $\omega$-open edge (either because it is closed or a ‘diagonal step’). If $\gamma_i \gamma_{i+1}$ does not exist, $\gamma$ is an $\omega$-open path from $A$ to $B$, hence an almost-arm. Otherwise, set $k_A := i$.

(ii) Starting from $\gamma_\ell \in B$, explore $\gamma$ backwards until the first $\gamma_j \gamma_{j+1}$ that is not an $\omega$-open edge, and set $k_B := j+1$. If $(\gamma_{k_A}, \ldots, \gamma_{k_B})$ is not a strong (resp. weak) path, $\gamma$ is not an almost-arm.

(iii) For every $v \in \tilde{\gamma} = (\gamma_{k_A+1}, \ldots, \gamma_{k_B-1})$, explore the $\omega$-cluster of $v$ in $S$. If $C_{v,S}(\omega) \cap \partial S \neq \emptyset$ for some vertex $v \in \tilde{\gamma}$, $\gamma$ is not an almost-arm. Otherwise, explore the colors of these clusters.

If they are all red (resp. blue), $\gamma$ is an almost-arm.

Consequently, the event ‘$\gamma$ is an almost-arm’ depends only on the status of the edges in $\gamma$, the edges incident to clusters visited by $\gamma$ (except the clusters of $\gamma_0$ and $\gamma_\ell$), and the colors of the interior clusters among them. The fact that it does not depend on the color of boundary-touching clusters is the main advantage of working with almost-arms compared to arms.

In Sections 3.2 and 3.3, we will naturally encounter almost-arms with respect to $\mathbb{Z}^2$ when studying arm events under the measures $\mu_{\mathbb{Z}^2}$ and $P_{\mathbb{Z}^2}$. In Section 3.4, we study arm events under the halfplane measures $\mu_{\mathbb{Z}_+ \times \mathbb{Z}}^0$ and $P_{\mathbb{Z}_+ \times \mathbb{Z}}^0$ with free boundary conditions and in this context, we will also encounter almost-arms with respect to $\mathbb{Z} \times \mathbb{Z}^+$. Therefore, we also make the following definitions.

**Definition 3.2** (Almost-arm event in the plane). Let $1 \leq m \leq n$ and let $\tau$ be a color sequence. The almost-arm event $B_\tau(m,n)$ denotes the existence of $|\tau|$ counterclockwise-ordered, disjoint almost-arms $\gamma^1, \ldots, \gamma^{|\tau|}$ from $\partial \Lambda_m$ to $\partial \Lambda_n$ in $\Lambda_{m,n}$ (with respect to $\mathbb{Z}^2$) such that

(i) the almost-arm $\gamma^i$ has color $\tau_i$, for every $1 \leq i \leq |\tau|$;

(ii) the cluster hulls $H_{\gamma^1, \Lambda_{m,n}}(\omega), \ldots, H_{\gamma^{|\tau|}, \Lambda_{m,n}}(\omega)$ are pairwise disjoint.
The almost-arm event ordered (starting with the rightmost), disjoint almost-arms with the bound is uniform in with respect to $3.5$

Remark where it is one part of showing that the two-arm exponent in the halfplane is universal and equal

Throughout the proof, we abbreviate $P$. The upper bound stated above will be sufficient for our purposes.

Proof. Throughout the proof, we abbreviate $P_{\mathbb{Z}^2}$ by $P$ and $B_{RB}^{++}(m, n)$ by $B(m, n)$.

**Definition 3.3** (Almost-arm events in the halfplane). Let $1 \leq m \leq n$ and let $\tau$ be a color sequence. The almost-arm event $B_{\tau}^{++}(m, n)$ (resp. $B_{\tau}^{++}(m, n)$) denotes the existence of $|\tau|$ counterclockwise-ordered (starting with the rightmost), disjoint almost-arms $\gamma^1, \ldots, \gamma^{|\tau|}$ from $\partial \Lambda^+_{m, n}$ to $\partial \Lambda^+_{m, n}$ in $\Lambda^+_{m, n}$ with respect to $\mathbb{Z} \times \mathbb{Z}_+$ (resp. $\mathbb{Z}^2$) such that

(i) the almost-arm $\gamma^i$ has color $\tau_i$, for every $1 \leq i \leq |\tau|$;

(ii) the cluster hulls $H_{\gamma^1, \Lambda^+_{m, n}}(\omega), \ldots, H_{\gamma^{|\tau|, \Lambda^+_{m, n}}}(\omega)$ are pairwise disjoint.

We emphasize the difference between the two almost-arm events in the halfplane: $B_{\tau}^{++}(m, n)$ involves almost-arms with respect to $\mathbb{Z}^2$ which implies that clusters intersecting the segments $[-m, -m] \times \{0\}$ and $[m, n] \times \{0\}$ are considered as boundary-touching clusters. This is not the case for $B_{\tau}^{+}(m, n)$. Furthermore, it is important to note that two almost-arms are required to have disjoint cluster hulls (even if they have the same color). As before, we remark that assuming $m \geq |\tau|$ is sufficient to guarantee that $B_{\tau}(m, n), B_{\tau}^{+}(m, n)$ and $B_{\tau}^{++}(m, n)$ are non-empty. We begin our study of almost-arms with an upper bound on the probability that there exist a red and a blue almost-arm with respect to $\mathbb{Z}^2$ in the upper halfplane.

**Proposition 3.4.** For all $1 \leq m \leq n$,

$$P_{\mathbb{Z}^2}(B_{RB}^{++}(m, n)) \lesssim \frac{m}{n},$$

where the bound is uniform in $r$ and $m, n$.

The principal idea of the proof is standard and well-known in the case of Bernoulli percolation, where it is one part of showing that the two-arm exponent in the halfplane is universal and equal to 1 (see, e.g. [Wer09, Nol08]). We remark that these arguments also imply certain universal arm exponents for the fuzzy Potts model.

**Remark 3.5.** Uniformly in $n \geq 1$, it holds that

$$\mu_{\mathbb{Z}^2}(A_{RB}^{++}(1, n)) \sim \mu_{\mathbb{Z} \times \mathbb{Z}_+}(A_{RB}^{++}(1, n)) \sim n^{-1}, \quad \mu_{\mathbb{Z}^2}(A_{RB}^{++}(1, n)) \sim n^{-2}.$$

Let us emphasize that the three-arm exponent $(RBR)$ in the halfplane is only universal under the measure $\mu_{\mathbb{Z}^2}$ and so there is no contradiction with Theorem 1.2. We include the proof of the proposition since working with almost-arms requires additional care compared with arms. One indication for it being the (maybe surprising) observation that the standard proof of the corresponding lower bound does not apply to the almost-arm event $B_{RB}^{++}(m, n)$. In any case, the upper bound stated above will be sufficient for our purposes.

**Proof.** Throughout the proof, we abbreviate $P_{\mathbb{Z}^2}$ by $P$ and $B_{RB}^{++}(m, n)$ by $B(m, n)$. 

![Figure 7](image_url) Illustrations of the proof of Proposition 3.4. Left. Six translated versions of the event $B'(m, n)$ occur ($|X| = 6$); inducing three disjoint dual open paths that cross the semi-annulus $\Lambda_{m, 2n'}^+$. Right. Construction of the event $B'(4m, n)$ from the event $B(m, n)$. The almost-arms $\Gamma_R$ and $\Gamma_B$ are highlighted.
Let us define a variant of the event $B(m, n)$ by considering $\Lambda^+_n$ instead of $\Lambda^+_{m,n}$ and by requiring the two almost-arms (with respect to $\mathbb{Z}^2$) to start from vertices on the line segment $[-m, m] \times \{0\}$, and denote it by $B'(m, n)$.

**Step 1:** $P(B'(m, n)) \leq m/n$.

We set $n' := \lfloor n/3 \rfloor$ and we may assume $n' \geq 4m$ without loss of generality. Let $N_{n'}$ be the number of disjoint dual open paths that cross the semi-annulus $\Lambda^+_{n', 2m'}$ from $\partial \Lambda_{n'}$ to $\partial \Lambda_{2m'}$.

First, let us argue that

$$\sum_{x \in 2m\mathbb{Z} : |x| \leq n' - m} 1_{(x, 0) + B'(m, n)} \leq 2 \cdot N_{n'}, \tag{3.1}$$

where we use the notation $(x, y) + F$ to denote the translate of the event $F$ by the vector $(x, y) \in \mathbb{Z}^2$. In words, it says that if there are many pairs of RB almost-arms in $\Lambda^+_{2m'}$ from $[-n', n'] \times \{0\}$ to $\partial \Lambda_{2m'}$, then this induces many dual open paths crossing the semi-annulus $\Lambda^+_{n', 2m'}$ from the inner to the outer boundary. For the reader familiar with the proof in the case of Bernoulli percolation, we mention that the factor $2$ on the right-hand side is necessary since we work with almost-arms (see the left of Figure 7).

Let $X = \{x \in 2m\mathbb{Z} : |x| \leq n' - m\}$ be the set of points for which the event $(x, 0) + B'(m, n)$ occurs. For $x \in X$, let $\gamma_{B,x}$ denote the leftmost blue almost-arm in $\Lambda^+_{2m'}$ from $[x - m, x + m] \times \{0\}$ to $\partial \Lambda_{2m'}$. Clearly, there exists a red almost-arm $\gamma_{R,x}$ to the right of $\gamma_{B,x}$ that also crosses $\Lambda^+_{2m'}$ and so that the cluster hulls are disjoint. Hence, the dual open path $\gamma_{d,x}$ which follows the right boundary of the cluster hull of $\gamma_{B,x}$ is sandwiched between $\gamma_{B,x}$ and $\gamma_{R,x}$. While arms of different colors cannot cross each other, this might happen for almost-arms. Indeed, the uncolored segments at the beginning (resp, the end) which correspond to $\omega$-open paths might intersect (this is exactly why we have insisted on disjoint cluster hulls in Definitions 3.2 and 3.3). Therefore, $x < y \in X$ does not imply that $\gamma_{R,x}$ stays to the left of $\gamma_{B,y}$ (if true, it would have allowed us to deduce directly that $\gamma_{d,x}$ is strictly to the left of $\gamma_{d,y}$). Nonetheless, it is true that for all $x < y < z \in X$, the dual open path $\gamma_{d,x}$ is strictly to the left of $\gamma_{d,z}$. This uses the existence of $\gamma_{B,y}$ and $\gamma_{R,y}$ starting from the intermediate boundary segment $[y - m, y + m] \times \{0\}$ and can be checked on a case-by-case basis (by looking at the boundary touching clusters at which the different paths start and end). It readily implies $N_{n'} \geq |X|/2$, and so equation (3.1) follows.

Taking expectation on both sides in equation (3.1), we get

$$\frac{n'}{2m} \cdot P(B'(m, n)) \leq \frac{n'}{m} \cdot P(B'(m, n)) \leq 2 \cdot \phi_{\mathbb{Z}^2}(N_{n'}). \tag{3.2}$$

To complete Step 1, it is now sufficient to observe $\phi_{\mathbb{Z}^2}(N_{n'}) \leq C < \infty$, which is a standard consequence of the domain Markov property and the crossing estimates in Theorem 2.3.

**Step 2:** $P(B(m, n)) \leq P(B'(4m, n))$.

In this step, we show that the probabilities of $B(m, n)$ and $B'(m, n)$ are comparable. One direction follows readily from the inclusion $B'(m, n) \subset B(m, n)$ and we will now establish the other direction. Let us work on the event $B(m, n)$. In this case, we can introduce the following notation (see also the right of Figure 7):

(i) Let $\Gamma_R$ be the rightmost red almost-arm in $\Lambda^+_{m,n}$ from $\partial \Lambda_m$ to $\partial \Lambda_n$, and let $E(\Gamma_R)$ denote the set of edges with respect to which $\Gamma_R$ is measurable, i.e. all edges in $\Lambda^+_{m,n}$ that are on or to the right of $\Gamma_R$ and all edges that are incident to interior clusters visited by $\Gamma_R$.

(ii) Let $\Gamma_d$ be the rightmost dual open simple path in $\Lambda^+_{2m,n}$ from $\partial \Lambda_{2m}$ to $\partial \Lambda_n$ that is to the left of $\Gamma_R$, and let $E(\Gamma_d)$ denote all primal edges in $\Lambda^+_{2m,n}$ that intersect or are to the right of $\Gamma_d$.

(iii) Let $\Gamma_B$ be the rightmost blue almost-arm in $\Lambda^+_{3m,n}$ from $\partial \Lambda_{3m}$ to $\partial \Lambda_n$ that is to the left of $\Gamma_d$, and let $E(\Gamma_B)$ denote the set of edges with respect to which $\Gamma_B$ is measurable, i.e. all edges in $\Lambda^+_{3m,n}$ that are on or to the right of $\Gamma_B$ and all edges that are incident to interior clusters visited by $\Gamma_B$. 

Let us condition on $\Gamma_R, \Gamma_d, \Gamma_B$, the status of the edges in $E(\Gamma_R) \cup E(\Gamma_d) \cup E(\Gamma_B)$ and the colors of the interior clusters in the subgraph induced by this edge set.

As shown in right of Figure 7, the occurrence of the event $B'(4m, n)$ can now be guaranteed by the existence of an open path in $\Lambda^+_{m,2m} \setminus E(\Gamma_R)$, a dual open path in $\Lambda^+_\eta,3m \setminus E(\Gamma_d)$ and an open path in $\Lambda^+_{3m,4m} \setminus E(\Gamma_B)$, and we denote by $F(m,4m)$ the event that such paths exist.

Furthermore, the domain Markov property implies that the configuration outside of $E(\Gamma_R) \cup E(\Gamma_d) \cup E(\Gamma_B)$ depends on the conditioning only through the induced FK boundary conditions. Therefore, using that the extremal distances of the considered discrete domains are bounded below by a constant $\epsilon > 0$ which does not depend on $m, n$, on $r$, or on the configuration we have conditioned on.\textsuperscript{2} By plugging this lower bound into the previous equation, we get $P(B'(4m, n)) \geq \epsilon \cdot P(B(m, n))$ which completes the second step and thereby the proof. \hfill $\square$

The following useful corollary is a standard consequence of the previous proposition. We refer to Figure 8 for an illustration of the main idea of the proof.

**Corollary 3.6.** Let $\tau \notin \{RRR,BBB\}$ be a color sequence of length 3. There exists a constant $\beta_2 = \beta_2(q) > 0$ such that for all $1 \leq m \leq n$,

$$P_{2\zeta}(B^{\tau+\tau}(m, n)) \lesssim \left(\frac{m}{n}\right)^{1+\beta_2},$$

where the bound is uniform in $r$ and $m, n$.

**Remark 3.7.** To obtain the same upper bound for the almost-arm event corresponding to $A^{\tau+\tau}(m, n)$, denoted by $B^{\tau+\tau}(m, n)$, it suffices to note that $B^{\tau+\tau}(m, n) \subset B^{\tau+\tau}(m, n)$.

### 3.2. Quasi-Multiplicativity

In this subsection, we prove quasi-multiplicativity for the fuzzy Potts model, which is the main ‘discrete’ ingredient needed to establish our main results.

\textsuperscript{2}More precisely, let $\mathcal{D} = (V,E)$ be the largest discrete domain in $\Lambda^+_{m,2m} \setminus E(\Gamma_R)$ (where we allow $\partial \mathcal{D}$ to use edges of $\gamma_\mathcal{D}$ that correspond to the open paths touching the boundary of $\Lambda^+_{m,n}$). We then consider the approximate discrete domain $\mathcal{D}'$ obtained from it by adding all edges in $\Lambda^+_{m,2m} \setminus E(\Gamma_R)$ with exactly one endpoint in $V$. One can then check that an open path from $[-2m,-m] \times \{0\}$ to the segment of $\partial \mathcal{D}'$ neighboring $E(\Gamma_R)$ guarantees the existence of a red almost-arm from $[-2m,-m] \times \{0\}$ to $\partial \Lambda_n$. The approximate discrete domains in $\Lambda_{2m,3m}$ and $\Lambda_{3m,4m}$ can be defined similarly.
Theorem 3.8 (Quasi-multiplicativity). Let $\tau$ be an alternating color sequence. For all $|\tau| \leq \ell \leq m \leq n$ we have
\[
\mu_{\mathbb{Z}^2}(A_\tau(\ell, n)) \asymp \mu_{\mathbb{Z}^2}(A_\tau(\ell, m)) \cdot \mu_{\mathbb{Z}^2}(A_\tau(m, n)),
\]
where the bounds are uniform in $\ell, m, n$. The same holds true for the arm event version $A^a_\tau$.

We present all proofs in this section for the arm event version $A_\tau$ but the same reasoning also applies to the arm event version $A^a_\tau$. We will always work with the infinite-volume measures $P_{\mathbb{Z}^2}$ and $\mu_{\mathbb{Z}^2}$, so we also drop $\mathbb{Z}^2$ from the notation and simply write $P$ and $\mu$.

The quasi-multiplicativity of arm events is usually obtained by first proving that arms can be separated, chosen to land at prescribed boundary segments and extended.

This strategy has been developed for Bernoulli percolation in [Kes87] and it has been extended to FK percolation in [CDCH16] and [DCMT21]. As we will see in the following subsection, arm separation in itself is a useful tool, e.g. to compare arm exponents in the plane. Our proof follows closely the well-known approach of Kesten [Kes87] (see also [Nol08, Man12]) and we have also been inspired by the presentation in [DMT21]. The main novelty is the use of almost-arm events, which allows to extend this approach to the fuzzy Potts model.

We begin with introducing some notation. Fix $\beta := \beta_1 \wedge \beta_2 \wedge (1/2)$ according to Corollaries 2.5 and 3.6. We say that a finite sequence $I^{(s)} \subset \mathbb{R}^2$ of counterclockwise-ordered points with $\|x\|_\infty = s$, i.e. on the boundary of the square $[-s, s]^2$, is $\delta$-separated for $\delta > 0$ if the points in $I^{(s)}$ are at distance at least $2\delta^{\beta/2}s$ from each other and at distance at least $\delta^{\beta/4}s$ from the four corners of the square.

Let $x \in \mathbb{R}^2$ with $\|x\|_\infty = s$. For $t > 0$, we denote by $R^m_t(x)$ the rectangle $x + [-t, 0] \times [0, 2t]$ (resp. $x + [-2t, 0] \times [-t, 0]$, $x + [0, 2t] \times [-2t, 0]$) if $x$ is on the right (resp. top, left, bottom) side of the boundary of the square $[-s, s]^2$. These rectangles are contained in $\Lambda_s$, have one corner at $x$ and ‘stretch’ from there in counterclockwise order. We denote by $R^{out}_t(x)$ the analogous rectangle that is positioned outside of $\Lambda_s$. It is obtained by interchanging $[0, t]$ and $[-t, 0]$ in the previous definition. Finally, we denote by $S^m_t(x)$ and $S^{out}_t(x)$ the slightly translated versions of $R^m_t(x)$ and $R^{out}_t(x)$ that are obtained if we replace $[0, 2t]$ by $[2t, 4t]$ and $[-2t, 0]$ by $[-4t, -2t]$ in the previous definitions.

Definition 3.9. Let $1 \leq m \leq n$ and $\tau = \tau_1 \cdots \tau_k$ be a color sequence. Let $\delta > 0$ and let $I^{(m)} = (I_1, \ldots, I_k)$, $J^{(n)} = (J_1, \ldots, J_k)$ be $\delta$-separated. The well-separated almost-arm event with extensions $B_{\delta, \ell, 1}^{\delta, 1, J}(m, n)$ denotes the subset of $B_{\tau}(m, n)$ for which the almost-arms $\gamma^i$ for $1 \leq i \leq k$ can be chosen as follows:
(i) The almost-arm $\gamma^i$ starts at some $x_i \in \partial \Lambda_m$ with $d_\infty(x_i, I_i) \leq \delta m$ and ends at some $y_i \in \partial \Lambda_n$ with $d_\infty(y_i, J_i) \leq \delta n$.

(ii) There is an almost-arm $\hat{\gamma}_d^i$ in $\Lambda_{m, n} \cup R^\text{in}_{2\delta m}(I_i) \cup R^\text{out}_{2\delta n}(J_i)$ from $\partial \Lambda_{m-\delta m}$ to $\partial \Lambda_{n+\delta n}$ whose colored vertices belong to the interior clusters of $\hat{\gamma}_d^i$ (we call $\hat{\gamma}_d^i$ an extension of $\gamma^i$) and the rectangles $R^\text{in}_{2\delta m}(I_i)$ and $R^\text{out}_{2\delta n}(J_i)$ are crossed in the long direction.

(iii) There is a dual open simple path $\hat{\gamma}_d^i$ in $\Lambda_{m, n} \cup S^\text{in}_{\delta m}(I_i) \cup S^\text{out}_{\delta n}(J_i)$ from $\partial \Lambda_{m-\delta m}$ to $\partial \Lambda_{n+\delta n}$ that separates $\hat{\gamma}_d^i$ from $\hat{\gamma}_d^{i+1}$ and the rectangles $S^\text{in}_{\delta m}(I_i)$ and $S^\text{out}_{\delta n}(J_i)$ are dual crossed in the long direction.

We refer to the left of Figure 9 for an illustration. Importantly, extensions of almost-arms use only $\omega$-open edges in $\Lambda_{m-\delta m, m}$ and $\Lambda_{n, n+\delta n}$, and this allows for the relevant gluing constructions.

**Proposition 3.10 (Extendability for $\hat{B}_\tau$).** Let $\tau$ be a color sequence and let $\delta > 0$. For all $|\tau| \leq m \leq n$ and for all $I^{(m/2)}$, $I^{(m)}$, $J^{(n)}$, $J^{(2n)}$ $\delta$-separated, 

\[
P\left(\hat{B}_\tau^{\delta, I, J}(m, n)\right) \lesssim P\left(\hat{B}_\tau^{\delta, I', J'}(m, 2n)\right),
\]

\[
P\left(B_\tau^{\delta, I, J}(m, n)\right) \lesssim P\left(\hat{B}_\tau^{\delta, I', J'}(m, 2n)\right),
\]

uniformly in $I', I, J, J'$ and in $m, n$.

**Proof.** The two bounds can be proven in the same way and we only present the argument for the second bound here. The idea is to use the extensions of the almost-arms in $\Lambda_{n, n+\delta n}$ to further extend them to the boundary segments of $\partial \Lambda_{2n}$ prescribed by $J^{(2n)}$. To this end, we assume that the event $B_\tau^{\delta, I, J}(m, n)$ occurs and we condition on the percolation configuration $\omega$ in $\Lambda_{m-\delta m, n}$ as well as the colors of the interior clusters. For each $1 \leq i \leq |\tau|$, we now want to explore a suitable extension of $\gamma^i$. We refer to the right of Figure 9 for an illustration. For simplicity, assume that $J_i$ is a point on the right side of $[-n, n]^2$. In the rectangle $R_i := R^\text{out}_{\delta n}(J_i)$, we explore the percolation configuration $\omega$ to find the leftmost crossing $\lambda$ from bottom to top. In this way, the edges of each $\delta$-arm remain unexplored, and by the definition of $B_\tau^{\delta, I, J}(m, n)$, the extension $\hat{\gamma}_d^i$ must intersect $\lambda$. Analogously, we can explore the leftmost dual crossing $\lambda_d$ in $S_i := S^\text{out}_{\delta n}(J_i)$ which by definition intersects the dual open path $\gamma_d^i$. It becomes clear from the right of Figure 9 that $2|\tau|$ suitable tubes of thickness $\delta n$ can be placed in $\Lambda_{2n, 2n+2\delta n}$ such that for each $1 \leq i \leq |\tau|$, the almost-arm $\hat{\gamma}_d^i$ and the dual open path $\gamma_d^i$ can be extended inside the tubes to the boundary segment nearby $J'_i$ in order to guarantee the occurrence of the event $\hat{B}_\tau^{\delta, I, J'}(m, 2n)$. Under the previous conditioning, the probability of the open resp. dual open crossings inside the tubes can be bounded from below using Theorem 2.3. This shows for some $\epsilon' = \epsilon'(\delta, \tau) > 0$, 

\[
P\left(\hat{B}_\tau^{\delta, I, J'}(m, 2n) \mid \hat{B}_\tau^{\delta, I, J}(m, n)\right) \geq \epsilon',
\]

and thereby concludes the proof. \qed

The bounds corresponding to the other direction in Proposition 3.10 can also be proven. In fact, much more is true: $\delta$ can be chosen so that the probabilities of arm events and well-separated almost-arm events with extensions are comparable. This is the most technical result in this section and we postpone the proof to Section 3.3.

**Theorem 3.11 (Arm separation).** Let $\tau$ be an alternating color sequence. There exists $\delta_0 > 0$ such that for all $|\tau| \leq m \leq n$, for all $\delta \in (0, \delta_0)$ and for all $I^{(m)}$, $J^{(n)}$ $\delta$-separated, 

\[
P\left(\hat{B}_\tau^{\delta, I, J}(m, n)\right) \asymp \mu(A_\tau(m, n)) \asymp P\left(B_\tau(m, n)\right),
\]

uniformly in $I, J$ and in $m, n$. 

Remark 3.12. The theorem is stated under the assumption that \( \tau \) is alternating but this is only to ensure that on the event \( A_\tau(m,n) \), all arms have disjoint cluster hulls. More precisely
\[
P\left( \hat{B}_\tau^{\delta,I,J}(m,n) \right) \asymp P\left(B_\tau(m,n)\right)
\]
for any color sequence \( \tau \) as the cluster hulls of almost-arms are disjoint by definition.

Remark 3.13. As for every other result in this section, arm separation also holds true for the arm event version \( A_\tau^*(m,n) \) and the corresponding versions of the almost-arm events.

By combining the previous two results, we obtain the following direct corollary.

**Corollary 3.14 (Extendability for \( A_\tau \) and \( B_\tau \)).** Let \( \tau \) be an alternating color sequence. For all \(|\tau| \leq m \leq n\),
\[
\mu(A_\tau(m,n)) \asymp \mu(A_\tau(m/2,2n)) \quad \text{and} \quad P(B_\tau(m,n)) \asymp P(B_\tau(m/2,2n)),
\]
uniformly in \( m, n \).

We are now ready to prove quasi-multiplicativity.

**Proof of Theorem 3.8.** Fix \( \delta > 0 \) for which Theorem 3.11 applies. First, we prove the upper bound
\[
\mu(A_\tau(\ell,n)) \lesssim \mu(A_\tau(\ell,m)) \cdot \mu(A_\tau(m,n)).
\]
In the case of Bernoulli percolation, it follows trivially from independence. Here, we apply the mixing property for FK percolation to almost-arm events, which is possible since conditioning on an almost-arm event provides no information about the colors on the boundary but only about the FK boundary condition. More precisely,
\[
\mu(A_\tau(\ell,n)) \leq P(B_\tau(\ell,m/2) \cap B_\tau(m,n)) \asymp P(B_\tau(\ell,m/2)) \cdot P(B_\tau(m,n))
\asymp \mu(A_\tau(\ell,m/2)) \cdot \mu(A_\tau(m,n)) \asymp \mu(A_\tau(\ell,m)) \cdot \mu(A_\tau(m,n)),
\]
where we have used the mixing property (Corollary 2.1) in the second comparison as well as arm separation and extendability in the third and fourth comparison.

To prove the lower bound, we fix any \( I^{(\ell)}, J^{(m/2)}, I^{(m)}, J^{(n)} \) \( \delta \)-separated. Then,
\[
\mu(A_\tau(\ell,m)) \cdot \mu(A_\tau(m,n)) \asymp P\left( \hat{B}_\tau^{\delta,I,J}(\ell,m/2) \right) \cdot P\left( \hat{B}_\tau^{\delta,I',J'}(m,n) \right)
\asymp P\left( \hat{B}_\tau^{\delta,I,J}(\ell,m/2) \cap \hat{B}_\tau^{\delta,I',J'}(m,n) \right),
\]
where we have applied arm separation and extendability in the first comparison and the mixing property (Corollary 2.1) in the second comparison. Now, one can use the similar gluing constructions as in the proof of Proposition 3.10 to connect the almost-arms (resp. dual open paths) in the inner annulus \( \Lambda_{\ell,m/2} \) with the corresponding almost-arms (resp. dual open paths) in the outer annulus \( \Lambda_{m,n} \). By coloring in red resp. blue the \(|\tau| \) interior clusters intersecting the intermediate annulus \( \Lambda_{m/2,m} \) that have been created to connect the almost-arms, one can guarantee the occurrence of the event \( \hat{B}_\tau^{\delta,I,J}(\ell,n) \). We note that the coloring step has probability at least \((r(1-r))^{|\tau|}\). Hence, we have just argued that
\[
P\left( \hat{B}_\tau^{\delta,I,J}(\ell,m/2) \cap \hat{B}_\tau^{\delta,I',J'}(m,n) \right) \lesssim P\left( \hat{B}_\tau^{\delta,I,J}(\ell,n) \right).
\]
By plugging this bound into the previous comparison, we conclude that
\[
\mu(A_\tau(\ell,m)) \cdot \mu(A_\tau(m,n)) \lesssim P\left( \hat{B}_\tau^{\delta,I,J}(\ell,n) \right) \asymp \mu(A_\tau(\ell,n)),
\]
where we have once more used arm separation in the last comparison.
3.3. Arm separation

In this section, we present the proof of Theorem 3.11. The very rough idea of the proof is to apply the a priori bound on the three-arm event appearing in Corollary 3.6 at each scale to control the probability of two arms landing close to each other. By iterating over the scales, it is possible to deduce that almost-arm events with extensions are comparable up to constants with arm events.

**Proof of Theorem 3.11.** First, we note that the occurrence of the arm event $A_\tau(m,n)$ implies the occurrence of the almost-arm event $B_\tau(m,n)$ since the color sequence $\tau$ is alternating. Second, if the event $\hat{B}_\tau^J(m,n)$ occurs, then one can use a similar gluing construction as in the proof of Proposition 3.10 to connect the dual open paths $\gamma^i_d$, $1 \leq i \leq |\tau|$, to a dual open circuit in $\Lambda_{m-28m,m-6m}$ and to a dual open circuit in $\Lambda_{n+\delta n,n+26n}$. Conditional on the event $\hat{B}_\tau^J(m,n)$, this construction has probability bounded below by some $c'=c'(\delta,\tau)>0$ according to Theorem 2.3 and it ensures that the boundary-touching clusters (in $\Lambda_{m,n}$) of different almost-arms $\gamma^i_d$, $1 \leq i \leq |\tau|$, are disjoint. By coloring the boundary-touching clusters of $\gamma^i_d$ with color $\tau_i$ for each $1 \leq i \leq |\tau|$, one can guarantee the occurrence of the arm event $A_\tau(m,n)$. This coloring step has probability at least $(r(1-r))^{|\tau|}$. In summary, we have obtained

$$P\left(\hat{B}_\tau^J(m,n)\right) \lesssim \mu(A_\tau(m,n)) \leq P(B_\tau(m,n)) = P(B_\tau(m,n)).$$

It remains to prove that there exists $\delta_0 > 0$ such that for every $\delta \in (0,\delta_0)$ and every $I^{(m)}$, $J^{(n)}$ $\delta$-separated,

$$P(B_\tau(m,n)) \lesssim P\left(\hat{B}_\tau^J(m,n)\right).$$

Since the separation and extension of the almost-arms at the inner boundary $\partial \Lambda_m$ resp. the outer boundary $\partial \Lambda_n$ follows from the same arguments, we focus on the latter. More precisely, we want to show that for every $\delta$-separated $J^{(n)}$,

$$P(B_\tau(m,n)) \lesssim P\left(\hat{B}_\tau^J(m,n)\right),$$

where we write $\hat{B}_\tau^J(m,n)$ for the well-separated almost-arm event with extensions, which only requires separation and extensions at the outer boundary $\partial \Lambda_n$.

Throughout the rest of the proof, we assume that $n$ is of the form $n = 2^km$ for some $k \geq 1$. Once we have established the result for $n$ of this form, the general case follows from the gluing constructions as in the proof of Proposition 3.10. From now onward, it will be important to keep track which constants depend on $\delta$. Therefore, we write $c_i = c_i(\delta,\tau)$, $i \geq 1$ for constants depending on $\delta$ and $c_i = c_i(\tau)$, $i \geq 1$ for constants not depending on $\delta$.

We begin with the following generalization of Proposition 3.10: There exist $\epsilon_1 > 0$ and $c_1 > 0$ such that for every $1 \leq i \leq \log(n/m)$ and for any $\delta$-separated $J, J'$,

$$P\left(\hat{B}_\tau^J(m,n)\right) \geq \epsilon_1 \cdot c_1 \cdot P\left(\hat{B}_\tau^{J'}(m,n/2^i)\right).$$

This lower bound can be obtained by iterative extension of the almost-arms as in the proof of Proposition 3.10. Importantly, the dependence on $\delta$ is only via the constant $\epsilon_1$ because only the first and the last extension require tubes of size $\delta$ and for the intermediate extensions, it is possible to work with tubes of some fixed size $\delta'$ (only depending on $\tau$). We note that equation (3.5) implies equation (3.4) when the ratio $n/m$ is bounded by a constant.

Let us introduce two intermediate types of almost-arm events. First, $B_\tau^J(m,n)$ denotes the subset of $B_\tau(m,n)$ such that for every $1 \leq i \leq |\tau|$, the almost-arm $\gamma^i_d$ can be chosen to end at some $y_i \in \partial \Lambda_n$ with $d_\infty(y_i,J_i) \leq \delta n$ and there is a dual open simple path $\gamma^i_d$ in $\Lambda_{m,n}$ from $\partial \Lambda_m$ to $\partial \Lambda_n$ that ends next to $y_i$ (in counterclockwise order). We refer to the left part of Figure 10 for an illustration. Second, for an almost-arm $\gamma$ in $\Lambda_{m,n}$, let us denote by $\hat{H}_\gamma$ the cluster
hull of its interior clusters. The event $\tilde{B}^\delta_{\gamma^i}(m, n)$ is the subset of $B^\delta_{\gamma^i}(m, n)$ such that for any $1 \leq i \neq j \leq |\tau|$, 

$$(\gamma^i \cup \tilde{H}_{\tau^i}) \cap (J_j + \Lambda_{4\delta n}) = \emptyset,$$

where $\gamma^i$ is the almost-endpoint ending at $y_i$. In words, the almost-arms (and their interior clusters) stay away from the endpoints of the other almost-arms (see the center of Figure 10 for an illustration). While the event $\tilde{B}^\delta_{\gamma^i}(m, n)$ was chosen so that the arms can be extended, the event $B^\delta_{\gamma^i}(m, n)$ is chosen in such a way that an analogue of the three-arm event in the halfplane occurs whenever the event $B_r(m, n)$, but none of the events $\tilde{B}^\delta_{\gamma^i}(m, n), J^{(n)} \subset J^{(n)}$, holds. This will be explained in more detail in Steps 2.2 and 2.3. To begin with, let us show how to construct the well-separated almost-arm event with extensions $\tilde{B}^\delta_{\gamma^i}(m, n)$ from the event $B^\delta_{\gamma^i}(m, n)$.

**Step 1:** $P(B^\delta_{\gamma^i}(m, n)) \geq \varepsilon_2 \cdot P(\tilde{B}^\delta_{\gamma^i}(m, n))$ for some $\varepsilon_2 > 0$.

The condition on the event $\tilde{B}^\delta_{\gamma^i}(m, n)$ ensures that every almost-arm $\gamma^i$, $1 \leq i \leq |\tau|$, can be chosen so that it is measurable with respect to the percolation configuration $\omega$ restricted to $\Lambda_{m,n} \setminus \cup_{j \neq i} (J_j + \Lambda_{4\delta n})$ and the colors of the interior clusters of $\omega$. We proceed as follows.

Assume that the event $B^\delta_{\gamma^i}(m, n)$ occurs and condition on the percolation configuration $\omega$ and the colors of the interior clusters in $\Lambda_{m,n} \setminus \cup_{j \neq i} (J_j + \Lambda_{4\delta n})$. We now construct the extensions for each $i$ separately and refer to the right of Figure 10 for an illustration. Let $1 \leq i \leq |\tau|$ and assume for simplicity that $J_i$ belongs to the right side of the boundary of $[-n, n]^2$. We condition on the lowest almost-arm $\gamma^i$ of color $\sigma_i$ from $\partial \Lambda_m$ to the boundary segment of length $2\delta n$ centered at $J_i$, for which $\gamma^i \cup \tilde{H}_{\tau^i}$ is contained in $\Lambda_{m,n} \setminus \cup_{j \neq i} (J_j + \Lambda_{4\delta n})$. Let us emphasize that this additional conditioning is only on parts of the configuration in $\Lambda_{m,n} \cap (J_i + \Lambda_{4\delta n})$ and so referring to the lowest almost-arm with respect to the ordering in this semi-annulus makes sense. Finally, we explore the lowest dual open simple path from $\partial \Lambda_m$ to $\partial (J_i + \Lambda_{2\delta n})$ that is above $\gamma^i$. As shown on the right in Figure 10, it is now easy to construct an $\omega$-open path inside $J_i + \Lambda_{5\delta n, 2\delta n}$ together with an $\omega$-open path inside $R^{out}_{\delta n}(J_i)$ from top to bottom. In particular, this creates an extension of $\gamma^i$ and a crossing of $R^{out}_{\delta n}(J_i)$ in the long direction. In the same way, we create a dual open path inside $J_i + \Lambda_{5\delta n, 2\delta n}$ to $\partial \Lambda_{n+\delta n}$ and a crossing of $S^{out}_{\delta n}(J_i)$ in the long direction. Under the conditioning, the probability of these constructions are bounded below by $\varepsilon_2 > 0$ thanks to the crossing estimates of Theorem 2.3 and the FKG inequality. This completes Step 1.

In the following Steps 2.1, 2.2 and 2.3, we argue that when the event $B_r(m, n) \setminus \tilde{B}^\delta_{\gamma^i}(m, n)$ occurs, then an event of small probability (going to 0 as $\delta \to 0$) must occur in the outermost dyadic annulus $\Lambda_{n/2,n}$.

**Step 2.1:** Almost-arms stay away from corners.

We denote by $E^\delta_{\gamma^i}(n)$ the event that there exists an almost-arm in $\Lambda_{3\delta n, 4\delta n}$ from the inner to the outer boundary that ends at distance $< \delta^{3/4} n$ from a corner of $\Lambda_n$. 

**Figure 10.** Left. The event $B^\delta_{RB}(m, n)$. Center. The event $\tilde{B}^\delta_{RB}(m, n)$. Right. This shows the construction of the event $\tilde{B}^\delta_{\gamma^i}(m, n)$. 


Let us argue that the probability of \( E^\delta_2(n) \) goes to 0 as \( \delta \to 0 \). For a corner \( z \) of \( \Lambda_n \), we consider the quarter-annulus \( (z + \Lambda_{3\delta^4/n,4}) \cap \Lambda_n \) if there is an \( \omega \)-open path \( \lambda \) which crosses the quarter-annulus from one boundary segment of \( \partial \Lambda_n \) to the other and if the cluster hull \( H_{\lambda \Lambda_n} \) is entirely contained in the quarter-annulus (which is the case if \( \lambda \) lies between two dual-open paths), then there cannot exist an almost-arm from \( \partial \Lambda_n/4 \) to \( \partial \Lambda_n \) that ends at distance \( < \delta^{3/4}n \) from \( z \). Indeed, this would mean that the almost-arm intersects \( \lambda \) but does not connect to the boundary using \( H_{\lambda \Lambda_n} \), contradicting the definition of an almost-arm. By the crossing estimates of Theorem 2.3, the probability of an \( \omega \)-open (resp. dual-open) circuit in a dyadic annulus \( \Lambda_n/2^{i+1,n/2^i} \) is uniformly bounded from below. Hence, \( P(E^\delta_1(n)) \to 0 \) follows since the number of dyadic annuli in \( \Lambda_{3\delta^4/n,4} \) goes to \( \infty \) as \( \delta \to 0 \).

**Step 2.2: Almost-arms end at some \( \delta \)-separated \( J^{(n)} \).**

To partition \( \partial \Lambda_n \) into boundary segments of length \( 2\delta n \), we consider a sequence of counterclockwise-ordered points \( J^{(n)} \subset \mathbb{R}^2 \) on the boundary of \([-n,n]^2 \) that are at distance \( 2\delta n \) from each other. In addition, we denote by \( X \subset \mathbb{R}^2 \) a sequence of counterclockwise-ordered points on the boundary of \([-n,n]^2 \) that are at distance \( 2\delta^{3/2}n \) from each other and from which we have removed all points that are at distance \( < \delta^{3/4}n \) from a corner. We define

\[
E^\delta_2(n) := B_{\gamma}(m, n) \setminus \left( \bigcup_{J^{(n)} \subset J^{(n)} \delta \text{-separated}} B^\delta_{e,J}(m, n) \cup E^\delta_1(n) \right)
\]

We now argue that \( P(E^\delta_2(n)) \to 0 \) as \( \delta \to 0 \). The idea is as follows: Either the almost-arms can be chosen to go to well-separated boundary segments, there exists an almost-arm that ends close to a corner, or there occurs a variant of an alternating three-almost-arm event somewhere at the boundary \( \partial \Lambda_n \). We refer to left of Figure 11 for an illustration. We also point out that we always choose \( \gamma^\delta \) to end at the furthest point of its boundary-touching cluster (in counterclockwise-order), thereby guaranteeing the existence of a dual open path \( \gamma^\delta \) ending next to the endpoint of \( \gamma^\delta \).

Let us explain why the occurrence of the event \( E^\delta_2(n) \) implies that for some \( x \in X \), there occurs a variant of an alternating three-almost-arm event in the semi-annulus \( (x + \Lambda_{3\delta^2/n,4}) \cap \Lambda_n \):

Assume that the first \( i-1 \) almost-arms were chosen to end at well-separated boundary segments around the points \( J_1, \ldots, J_{i-1} \in J^{(n)} \). If the almost-arm \( \gamma^\delta \) must be chosen to end at the boundary segment of some \( J_i \) that is at distance less than \( 2\delta^{3/2}n \) from the previously chosen points, we denote the point in \( X \) that is closest to \( J_i \) by \( x_i \) and observe that for some \( n' \in 2^f \delta^{3/2}n \) with \( 3\delta^{3/2}n \leq n' < n/4 \), there occurs an alternating three-almost-arm event \( B^+_{\gamma^\delta}(2n', n/4) \) in the semi-annulus \( (J_i + \Lambda_{2n',n/4}) \setminus \Lambda_n \) (with \( \tau' \in \{RBR, BBR \} \) depending on \( \tau_i \)) and an alternating FK three-arm event \( A^+_{\gamma^\delta}(2\delta^{3/2}n, n') \) in the semi-annulus \( (J_i + \Lambda_{3\delta^{3/2}/n,n'}) \cap \Lambda_n \). Here, \( \ell = 0 \) corresponds to the case that the boundary-touching cluster of \( \gamma^\delta \) is contained in \( (J_i + \Lambda_{3\delta^{3/2}/n}) \cap \Lambda_n \).

In summary, we obtain

\[
P(E^\delta_2(n)) \leq \sum_{x \in X} \sum_{\ell = 0}^{\log(\delta^{3/2}/12)} C \cdot P \left( A^+_{\gamma^\delta}(2\delta^{3/2}n, 2^f \delta^{3/2}n) \right) \cdot P \left( B^+_{\gamma^\delta}(2^f + 1, 2^f \delta^{3/2}n, n/4) \right) \leq c_3^{-1} \cdot |X| \cdot \log(\delta^{3/2}) \cdot \delta(1/2) \cdot \delta^{3/2} \to 0 \quad \text{as} \quad \delta \to 0,
\]

where we have used the mixing property of Corollary 2.2 in the first inequality, and then the upper bounds from Corollaries 2.5 and 3.6.

**Step 2.3: Almost-arms stay away from endpoints of the other almost-arms.**
Figure 11. Left. If the almost-arms cannot be chosen to end at some $\delta$-separated $J^{(n)} \subset J^{(n)}$, then for some $x \in X$, a variant of an alternating three-almost-arm event occurs in the semi-annulus $(x + \Lambda_{3\delta^3/2n,n/4}) \cap \Lambda_n$. Right. If the almost-arms cannot be chosen to stay away from each others endpoint, then for some $J_i \in J^{(n)}$, a variant of an alternating three-almost-arm event occurs in the semi-annulus $J_i \cap \Lambda_{4\delta n, \delta^3/2n} \cap \Lambda_n$.

For $J^{(n)} = (J_1, \ldots, J_{|\tau|})$ $\delta$-separated, we denote $E_3^{\delta,J}(n) := B^{\delta,J}_\tau(m,n) \setminus \tilde{B}^{\delta,J}_\tau(m,n)$ and $E_3^\delta(n) := \bigcup_{J^{(n)} \subset J^{(n)} \delta\text{-separated}} E_3^{\delta,J}(n)$.

Let us now argue that $P(E_3^\delta(n)) \to 0$ as $\delta \to 0$. If the event $E_3^{\delta,J}(n)$ occurs, then there exists $1 \leq i \leq |\tau|$ such that for any choice of the almost-arm $\gamma^i$, $(\gamma^i \cup \tilde{H}_{\gamma^i}) \cap \bigcup_{j \neq i}(J_j + \Lambda_{4\delta n}) \neq \emptyset$.

As before, this implies a variant of an alternating three-almost-arm event in the semi-annulus $(J_j \cap \Lambda_{4\delta n, \delta^3/2n}) \cap \Lambda_n$ for some $1 \leq j \leq |\tau|$ (see the right of Figure 11 for an illustration). If the almost-arm $\gamma^i$ must be chosen to intersect $J_j + \Lambda_{4\delta n}$, then we simply observe an alternating three-almost-arm event $B^{\delta,J}_\tau(\delta n, \delta^3/2n)$ with $\tau \in \{RBR, BRB\}$ depending on $\tau_i$ in the semi-annulus. But actually, the intersection of $\gamma^i \cup \tilde{H}_{\gamma^i}$ with $J_j + \Lambda_{4\delta n}$ might also be due to an interior cluster of $\gamma^i$. In any case, one can show that for some $n' = 2^4\delta n$ with $4\delta n \leq n' < \delta^3/2n$, there occurs an alternating three-almost-arm event $B^{\delta,J}_\tau(\delta n', \delta^3/2n)$ in the semi-annulus $(J_j + \Lambda_{2n', \delta^3/2n}) \cap \Lambda_n$ and an alternating FK three-arm event $A_{010}(\delta n', n')$ in the semi-annulus $(J_j + \Lambda_{4\delta n,n'}) \cap \Lambda_n$. In summary, we obtain

$$P(E_3^{\delta,J}(n)) \leq c_5^{-1} \log(\delta^{3/2-1}) \cdot \delta^{1-\beta/2}(1+\beta),$$

where we have used the mixing property of Corollary 2.2 in the first inequality.

Since $(1 - \beta/2)(1 + \beta) \geq 1 + \beta/4$, we get

$$P(E_3^\delta(n)) \leq \sum_{J^{(n)} \subset J^{(n)} \delta\text{-separated}} P(E_3^{\delta,J}(n)) \leq c_6^{-1} \log(\delta^{3/2-1}) \cdot \delta^{\beta/4},$$

which goes to 0 as $\delta \to 0$.

Step 3: Conclusion.
In Steps 2.1, 2.2 and 2.3, we have shown that for any \( n \geq 2m \geq 1 \),

\[
B_\tau(m, n) \subset \bigcup_{J^{(n)} \subset J^{(n)}} \hat{B}_\tau^{\delta,J}(m, n)
\]

\[
\quad \cup \left( B_\tau(m, n/2) \cap \left( E_1^\delta(n) \cup E_2^\delta(n) \cup E_3^\delta(n) \right) \right)
\]

with \( P(E_1^\delta(n) \cup E_2^\delta(n) \cup E_3^\delta(n)) \to 0 \) as \( \delta \to 0 \). Hence, the mixing property for FK percolation (which is applicable since we consider almost-arm events instead of arm events) and the bound of Step 1 imply

\[
(3.6) \quad P(B_\tau(m, n)) \leq \sum_{J^{(n)} \subset J^{(n)}} \epsilon_1^{-1} \cdot P\left( \hat{B}_\tau^{\delta,J}(m, n) \right) + P(B_\tau(m, n/2)) \cdot f(\delta),
\]

where the function \( f \) is chosen such that \( C \cdot P(E_1^\delta(n) \cup E_2^\delta(n) \cup E_3^\delta(n)) \leq f(\delta) \to 0 \) uniformly in \( n \) as \( \delta \to 0 \) (here \( C \) is the constant from Corollary 2.1).

We would now like to iteratively apply equation (3.6). Fix any \( \delta \)-separated \( J^{(n)} \subset J^{(n)} \). By applying equation (3.6) to \( (m, n/2) \), we obtain

\[
P(B_\tau(m, n)) \leq P(B_\tau(m, n/2))
\]

\[
\leq \sum_{J^{(n/2)} \subset J^{(n/2)}} \epsilon_1^{-1} \cdot P\left( \hat{B}_\tau^{\delta,J}(m, n/2) \right) + P(B_\tau(m, n/4)) \cdot f(\delta)
\]

\[
\leq (\epsilon_1 \epsilon_2 \epsilon_3) \cdot P\left( \hat{B}_\tau^{\delta,J}(m, n) \right) + P(B_\tau(m, n/4)) \cdot f(\delta),
\]

where we have applied equation (3.5) for each \( J^{(n/2)} \) in the second inequality and the constant \( \epsilon_3 \) accounts for the number of terms in the sum. By iteration, we get

\[
P(B_\tau(m, n)) \leq (\epsilon_1 \epsilon_2 \epsilon_3)^i P\left( \hat{B}_\tau^{\delta,J}(m, n) \right) \cdot \sum_{i=0}^{\log(n/m)-1} c_1^{-i} f(\delta)^i
\]

Finally, we fix \( \delta_0 > 0 \) sufficiently small such that \( f(\delta_0) \leq c_1/2 \). Then for \( \delta < \delta_0 \), this implies

\[
P(B_\tau(m, n)) \leq 2c_1^{-1} (\epsilon_1 \epsilon_2 \epsilon_3)^{-1} P\left( \hat{B}_\tau^{\delta,J}(m, n) \right),
\]

and thereby completes the proof of equation (3.4). In the same way, one can show that for any \( \delta \)-separated \( I^{(m)} \),

\[
(3.7) \quad P\left( \hat{B}_\tau^{\delta,J}(m, n) \right) \leq P\left( \hat{B}_\tau^{\delta,I,J}(m, n) \right),
\]

which implies equation (3.3) when combined with equation (3.4) and thereby concludes the proof of the theorem. \( \square \)

### 3.4. Quasi-Multiplicativity and Arm Separation for the Halfplane

In this section, we briefly explain how to establish the following theorem.

**Theorem 3.15** (Quasi-multiplicativity for the halfplane). Let \( \tau \) be an alternating color sequence for the halfplane. For all \( |\tau| \leq \ell \leq m \leq n \),

\[
\mu^0_{\mathbb{Z} \times \mathbb{Z}_+}(A^+_\tau(\ell, n)) \asymp \mu^0_{\mathbb{Z} \times \mathbb{Z}_+}(A^+_\tau(\ell, m)) \cdot \mu^0_{\mathbb{Z} \times \mathbb{Z}_+}(A^+_\tau(m, n)),
\]

where the bounds are uniform in \( \ell, m, n \). The same holds true for the arm event version \( A^+_{\tau} \).

In Sections 3.2 and 3.3, we always worked in the plane and we have therefore only considered almost-arms with respect to \( \mathbb{Z}^2 \). In the halfplane, it is natural to consider almost-arms with respect to \( \mathbb{Z} \times \mathbb{Z}_+ \), i.e. the almost-arm event \( B^+_\tau(m, n) \). The first step is then to define the well-separated almost-arm event with extensions \( \hat{B}^{\delta,I,J}_{\tau}(m, n) \) in the halfplane as a subset of \( B^+_\tau(m, n) \). The definition is identical to Definition 3.9, except that no dual open simple path is
needed to separate the last almost-arm $\gamma^{|m|}$ from the first almost-arm $\gamma^1$ since this separation will be guaranteed by the free boundary condition. Using the same reasoning as in the previous subsections, one then obtains:

**Theorem 3.16** (Arm separation for the halfplane). Let $\tau$ be an alternating color sequence for the halfplane. There exists $\delta_0 > 0$ such that for all $|\tau| \leq m \leq n$, for all $\delta \in (0, \delta_0)$ and for all $I^{(m)}$, $J^{(n)}$ $\delta$-separated,

\[
P_{Z \times Z^+}^0 \left( \hat{B}_{\tau}^{\delta,I,J} (m, n) \right) \approx P_{Z \times Z^+}^0 \left( A_{\tau}^+ (m, n) \right) \approx P_{Z \times Z^+}^0 \left( B_{\tau}^+ (m, n) \right),
\]

uniformly in $I, J$ and in $m, n$. The same holds true for the arm event version $A_{\tau}^{+\ast} (m, n)$ and the corresponding versions of the almost-arm events.

Actually, the proof can be simplified in this case since arms can be explored in counterclockwise order starting from the right end of the semi-annulus as explained in Step 2 of the proof of Proposition 3.4.

### 3.5. Bounding and Comparing Arm Exponents

In this section, we apply arm separation to prove several results about arm events in the fuzzy Potts model, which will be needed in Sections 4 and 6. First of all, arm separation also allows us to prove a simple upper bound on the alternating six-almost-arm event that will be useful in Section 4.

**Proposition 3.17.** There exists a constant $\beta_3 = \beta_3(r, q) > 0$ such that for all $n \geq m \geq 1$,

\[
P_{\Lambda_n}^\xi \left( B_{RBRBRBRB} (m, n) \right) \lesssim \left( \frac{m}{n} \right)^{2+\beta_3},
\]

where the bounds are uniform in $\xi$ and in $m, n$.

**Proof.** First, we show that uniformly for all $1 \leq m \leq n$,

\[
P_{Z^2} \left( B_{RBRBRBRB} (m, n) \right) \lesssim \left( \frac{m}{n} \right)^2.
\]

For some (FK) cluster $C_0 \in \mathcal{C}$ with $C_0 \subset \Lambda_n$, we denote by $E(C_0; 2n)$ the event that there exist five disjoint sequences of clusters $C^{(j)} = (C_i^{(j)})_{i=1}^{k_j}, j \in \{1, \ldots, 5\}$, in $\Lambda_{2n}$ such that

- $C_0$ and all clusters in $C^{(1)}$, $C^{(3)}$ and $C^{(5)}$ are red, all clusters in $C^{(2)}$ and $C^{(4)}$ are blue,
- the sequences of clusters start next to $C_0$ and intersect $\partial \Lambda_{2n}$, i.e. for $j = 1, 3, 5$,

\[
d_1(C_0, C^{(j)}_1) = 1 \quad \text{and} \quad d_1(C^{(j)}_i, C^{(j)}_{i+1}) = 1, \ \forall 1 \leq i < k_j,
\]

and for $j = 2, 4$,

\[
d_\infty(C_0, C^{(j)}_1) = 1 \quad \text{and} \quad d_\infty(C^{(j)}_i, C^{(j)}_{i+1}) = 1, \ \forall 1 \leq i < k_j,
\]

and for each $j$, only the last cluster $C^{(j)}_{k_j}$ touches the boundary $\partial \Lambda_{2n}$.

- $C^{(1)}_{k_1}$ and $C^{(5)}_{k_5}$ touch the top, $C^{(2)}_{k_2}$ touches the left, $C^{(3)}_{k_3}$ touches the bottom, and $C^{(4)}_{k_4}$ touches the left of the boundary $\partial \Lambda_{2n}$.

It is easy to verify that the event $E(C_0; 2n)$ occurs for at most one cluster $C_0 \in \mathcal{C}$, i.e.

\[
1 \geq \sum_{C_0 \subset \Lambda_n} 1_{E(C_0; 2n)}.
\]

The bound (3.8) for $m = 1$ follows by taking expectations and in the case of Bernoulli percolation, this is a well-known step in proving that the five-arm exponent is universal and equals 2. For general $m$, we cover $\Lambda_n$ with boxes of size $m$ to deduce

\[
1 \geq \sum_{x \in 2m \mathbb{Z}^2 : x + \Lambda_m \subset \Lambda_n} 1_{E(C_0; 2n)} \text{ holds for some } C_0 \subset x + \Lambda_m,
\]
The same holds true for the arm event version (see Remark 3.5).

Remark 3.19

Let $\hat{\tau}$ be any color sequence. We formally define a reduced version $\hat{\tau}$ for the plane and a reduced version $\hat{\tau}^+$ for the halfplane as follows. Whenever the letter $R$ (resp. $B$) occurs subsequently more than once, we replace this subsequence by a single letter. The last and the first letter are viewed as subsequent in the plane but not in the halfplane. Clearly, $\hat{\tau}$ is alternating and $\hat{\tau}^+$ is alternating for the halfplane.

Proposition 3.18. Assume that there exist $\beta_4 > 0$ such that uniformly for all $n \geq 1$,

$$\mu_{22}(A_{RBRBR}(1,n)) \lesssim n^{-(2+\beta_4)}.$$ 

Let $\hat{\tau}$ (resp. $\hat{\tau}^+$) be the reduction of $\tau$ in the plane (resp. halfplane). It then holds that uniformly for all $n \geq m \geq |\tau|,$

$$\mu_{22}(A_\tau(m,n)) \asymp \mu_{22}(A_\tau^+(m,n)).$$

The same holds true for the arm event version $A^\ast.$

Remark 3.19. The proof of Proposition 3.18 can analogously be applied in the halfplane case. It yields that uniformly for all $n \geq m \geq |\tau|,$

$$\mu_{2\times\mathbb{Z}_+}^0(A_\tau^+(m,n)) \asymp \mu_{2\times\mathbb{Z}_+}^0(A_{\tau^+}^+(m,n)),$$

and the same holds true for the arm event version $A^\ast_{RBR}.$ The only additional ingredient needed in the halfplane case is that there exist $\beta_5 > 0$ such that uniformly for all $n \geq 1$,

$$\mu_{2\times\mathbb{Z}_+}^0(A_{RBR}^+(1,n)) \lesssim n^{-(1+\beta_5)}.$$

This follows directly from the comparison with the alternating two-arm exponent in the halfplane (see Remark 3.5).
It is needed since the defects (see below for more details) might be close to \( \mathbb{Z} \times \{0\} \) and thereby induce an alternating three-arm event in the halfplane instead of an alternating four-arm event in the plane.

In the proof of this result, we will naturally encounter paths that look like red or blue arms except for a few defects. Therefore, we introduce \( A^{(d)}_{\tau}(m,n) \), the arm event with \( d \) defects, containing all colorings \( \sigma \) for which there is some \( \sigma' \in A_{\tau}(m,n) \) with \( \{|v:\sigma'_v \neq \sigma_v\}| \leq d \).

Since almost-arm events depend both on the percolation configuration and on the coloring, one might consider different notions of defects. In analogy with the previous definition, the almost-arm event with \( d \) defects, denoted by \( B^{(d)}_{\tau}(m,n) \), contains all configurations \((\omega,\sigma)\) for which changing the color of at most \( d \) vertices (formally by closing the four adjacent edges and coloring the isolated vertex) results in a configuration \((\omega',\sigma') \in B_{\tau}(m,n) \).

**Proposition 3.20.** Let \( \tau \) be an alternating color sequence and \( d \geq 1 \) be the number of defects. Uniformly for all \( n \geq m \geq |\tau| \), it holds that

\[
\mu_{Z^2} \left( A^{(d)}_{\tau}(m,n) \right) \leq \left( 1 + \log \left( \frac{n}{m} \right) \right)^d \mu_{Z^2} \left( A_{\tau}(m,n) \right),
\]

and

\[
P_{Z^2} \left( B^{(d)}_{\tau}(m,n) \right) \leq \left( 1 + \log \left( \frac{n}{m} \right) \right)^d P_{Z^2} \left( B_{\tau}(m,n) \right).
\]

The same holds true for the arm (resp. almost-arm) event versions \( A^\tau_+ \) and \( A^{\tau+}_{\tau} \).

The upper bound on (almost)-arm events with defects is a standard consequence of the arm separation techniques that were established in the previous subsections and we refer the reader to [Nol08, Proposition 18] for a proof in the case of Bernoulli percolation that can directly be adapted to our setting.

**Proof of Proposition 3.18.** Let us present the proof for the events \( A_{\tau}(m,n) \) and \( A^\tau_+ \). The other cases can be proven in exactly the same way. Hence, we work with the measures \( P_{Z^2} \) and \( \mu_{Z^2} \) throughout the proof and abbreviate them by \( P \) and \( \mu \).

The upper bound \( \mu(A_{\tau}(m,n)) \leq \mu(A^\tau_+(m,n)) \) is trivial by inclusion. For the lower bound, it will be sufficient to show that

\[
(3.10) \quad \mu(A_{\tau}(m,n)) \geq \mu(A^\tau_+(m,n))
\]

for \( m, n \) of the form \( 2^k \) and for \( \tau \) of the form \( \tau = \tilde{\tau}_1 \cdots \tilde{\tau}_1 \cdots \tilde{\tau}_\ell \), where \( \ell = |\tilde{\tau}| \) and each letter of \( \tilde{\tau} \) is repeated \( d \) times for some fixed \( d \geq 1 \). Then, the result follows for general \( m, n \) using the extendability for \( A_\tau \) and for general \( \tau \) by inclusion.

Depending on the length of \( \tilde{\tau} \), we define the event \( E_{\tilde{\tau}}(m,n) \) (resp. \( E'_{\tilde{\tau}}(m,n) \)) as follows:

- If \( |\tilde{\tau}| \leq 2 \), it denotes the existence of a closed weak (resp. strong) path surrounding \( \Lambda_{m,n} \) that is blue (resp. red) except for at most \( d - 1 \) defects.
- If \( |\tilde{\tau}| > 2 \), it denotes the existence of three counterclockwise ordered arms \( \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3 \) of colors \( BRB \) (resp. \( BBR \)) from \( \partial \Lambda_{m/4} \) to \( \partial \Lambda_{4n} \) in \( \Lambda_{m/4,4n} \) and a weak (resp. strong) path connecting \( \tilde{\gamma}^2 \) and \( \tilde{\gamma}^3 \) inside \( \Lambda_{m,n} \) in counterclockwise order that is blue (resp. red) except for at most \( d - 1 \) defects.

It is well-known that if the maximal number of red (resp. blue) arms crossing the annulus \( \Lambda_{m,n} \) from the inner to the outer boundary is \( j \), then there is a blue (resp. red) circuit with \( j \) defects in the annulus. Hence, we obtain that

\[
A_{\tau}(m/4,4n) \subset A_{\tau}(m,n) \cup \left( E_{\tilde{\tau}}(m,n) \cap A_{\tau}(m/4,4n) \right) \cup \left( E'_{\tilde{\tau}}(m,n) \cap A_{\tau}(m/4,4n) \right)
\]

In the following, we will show that there exists some \( m_0 \) such that for \( n \geq m \geq m_0 \),

\[
(3.11) \quad \mu \left( E_{\tilde{\tau}}(m,n) \cap A_{\tau}(m/4,4n) \right) + \mu \left( E'_{\tilde{\tau}}(m,n) \cap A_{\tau}(m/4,4n) \right) \leq \frac{1}{2} \mu \left( A_{\tau}(m/4,4n) \right).
\]
Once we have established this inequality, it will follow that for \( n \geq m \geq m_0 \),
\[
\mu(A_\star(m, n)) \geq \frac{1}{2} \mu(A_\star(m/4, 4n)) \sim \mu(A_\star(m, n)),
\]
and this directly implies the lower bound (3.10) for general \( n \geq m \geq |\tau| \) since \( \mu(A_\star(|\tau|, n)) \gtrsim \mu(A_\star(m_0, n)) \) can be deduced from the finite energy property of FK percolation.

To show (3.11), we note that on the event \( E_\tau(m, n) \cap A_\tau(m/4, 4n) \), a four-arm event with at most \( d - 2 \) defects occurs locally around each defect. More precisely,
\[
E_\tau(m, n) \cap A_\tau(m/4, 4n)
\]
\[
\subset \bigcup_{k = \log(m)}^{\log(n) - 1} A_\tau(m/4, 2^{k-2}) \cap \left( \bigcup_{x \in A_{2k,2^{k+1}}} \left\{ x + A_{RBBB_\tau}(1,2^{k-1}) \right\} \right) \cap A_\tau(2^{k+3}, 4n)
\]
We emphasize that the intermediate event depends only on the coloring in \( A_{2^{k-1},2^{k+2}} \).

To apply the mixing property of Corollary 2.1, we consider almost-arm events instead of arm events. Thereby, we obtain
\[
\mu(E_\tau(m, n) \cap A_\tau(m/4, 4n))
\]
\[
\leq C^2 \sum_{k = \log(m)}^{\log(n) - 1} P \left( B_{\tau}(m/4, 2^{k-2}) \cdot |A_{2k,2^{k+1}}| \cdot P \left( B_{RBBB_\tau}(1,2^{k-1}) \right) \cdot P \left( B_{\tau}(2^{k+3}, 4n) \right) \right)
\]
\[
\lesssim \mu(A_\tau(m/4, 4n)) \cdot \sum_{k = \log(m)}^{\log(n) - 1} |A_{2k,2^{k+1}}| \cdot P \left( B_{RBBB_\tau}(1,2^{k-1}) \right),
\]
where we have used arm separation and quasi-multiplicativity in the second comparison. Using Proposition 3.20 and the assumption on the alternating four-arm event, it follows that
\[
\sum_{k = \log(m)}^{\log(n) - 1} |A_{2k,2^{k+1}}| \cdot P \left( B_{RBBB_\tau}(1,2^{k-1}) \right)
\]
\[
\lesssim \sum_{k = \log(m)}^{\log(n) - 1} 4^k \cdot k^{d-2} \cdot P \left( B_{RBBB_\tau}(1,2^{k-1}) \right) \lesssim \sum_{k = \log(m)}^{\infty} k^{d-2} \left( 2^{-\beta_4} \right)^k < \infty.
\]
In particular, there exists \( m_0 \) sufficiently large such that the sum is smaller than \( 1/4 \) for \( m \geq m_0 \).

The upper bound for \( E_\tau(m, n) \cap A_\tau(m/4, 4n) \) follows analogously. This establishes (3.11) and completes the proof. \( \square \)

4. Convergence results

As announced in the introduction, the aim of this section will be to prove the convergence of the (discrete) fuzzy Potts model to its continuum counterpart conditional on the conformal invariance conjecture for FK percolation (stated as Conjecture 2.8). There are two types of results that we present: Firstly, we will describe the scaling limit of a single fuzzy Potts interface and secondly, we will determine the scaling limit of the loop encoding of an entire discrete fuzzy Potts model. We will only make use of the former result to transport continuum critical exponents back to the discrete setting but emphasize that the latter result is needed to transport the one-arm exponent back but the corresponding continuum exponent is not known at the time of writing.

Throughout this section, we fix \( q \in [1,4) \) and assume Conjecture 2.8 for this value \( q \) and let \( \kappa' \in (4,6) \) be the parameter appearing in the statement of the conjecture. We will also fix \( r \in (0,1) \) and let \( \Gamma, \Gamma^0, \Gamma^I, \Gamma^{0R}, \Gamma^{0B}, \Gamma^{1R}, \Gamma^{1B}, \Sigma \) and \( (\gamma^{a,b}) \) be as in Section 2.6. Recall also that analogous objects have been defined in the discrete setting in Section 2.4.
We also fix a Jordan domain $D$, $\epsilon_n \to 0$ and discrete domains $D_n = (V_n, E_n)$ in $\epsilon_n \mathbb{Z}^2$ converging to $D$ (see Section 2.4 for the sense of convergence). Recall that we wrote $\partial D_n$ for the set of boundary vertices of $D_n$. By Skorokhod’s representation theorem and the scaling limit conjecture we can therefore couple $\omega^n \sim \phi^n_{D_n, q}$ with $\Gamma$ such that

$$(\Gamma^{\omega^n}_n \setminus \Gamma^n, \Gamma^n) \to (\Gamma \setminus \Gamma, \Gamma) \quad \text{a.s.}$$

with respect to $d_\mathcal{L}$. It is straightforward to see that nesting levels are preserved by limits with respect to $d_\mathcal{L}$ and in particular $\Gamma^{\omega^n}_n \setminus \Gamma^n \to \Gamma \setminus \Gamma$ with respect to $d_\mathcal{L}$. In this section we would like to consider fuzzy Potts configurations $(\omega^n, \sigma^n) \sim P^n_{D_n, q}$ which are coupled together such that

$$(\Gamma^{\omega^n, \sigma^n}_n \cap \Gamma, \Gamma^{\omega^n, \sigma^n}_n \cap \Gamma) \to (\Gamma \cap \Gamma, \Gamma \cap \Gamma),$$

and

$$(\Gamma^{\omega^n, \sigma^n}_n \setminus \Gamma, \Gamma^{\omega^n, \sigma^n}_n \setminus \Gamma) \to (\Gamma \setminus \Gamma, \Gamma \setminus \Gamma),$$

where

$$(\Gamma_{\omega^n, \sigma^n}^{R, \Gamma}, \Gamma_{\omega^n, \sigma^n}^{B, \Gamma}) \to (\Gamma_{\omega}^{R, \Gamma}, \Gamma_{\omega}^{B, \Gamma}) \quad \text{a.s.}$$

Let us quickly explain how to construct such a coupling. By the definition of $d_\mathcal{L}$ we may take $G_n \subset \Gamma^{\omega^n}_n$, $F_n \subset \Gamma$, $G'_n \subset \Gamma^{\omega^n}_n \setminus \Gamma$, $F'_n \subset \Gamma \setminus \Gamma$ and bijections $\pi_n: G_n \to F_n$, $\pi'_n: G'_n \to F'_n$ such that

$$(\sup_{\eta \in G_n} d_\mathcal{L}(\eta, \pi_n(\eta)) \vee \sup_{\eta \notin F_n} \sup_{\eta \notin G_n} \sup_{\eta \notin F_n} \sup_{\eta \notin G'_n} \diam(\eta) \leq 2 d_\mathcal{L}(\Gamma^{\omega^n}_n \setminus \Gamma, \Gamma \setminus \Gamma),$$

and such that $G_n, F_n, G'_n, F'_n, \pi_n, \pi'_n$ are measurable with respect to $\Gamma, \Gamma^{\omega^n}_n$. We leave the construction of these measurable mappings to the reader.

For fixed $n \geq 1$ we assign a color to the cluster $C \subset \mathcal{C}(\omega^n)$ as follows. Recall that $\eta^{C}_{\omega^n}$ denoted the outer boundary of $C$. If $\eta^{C}_{\omega^n} \notin G_n \cup G'_n$, we color $C$ (independently) all in red (R) with probability $r$ and all in blue (B) otherwise. If $\eta^{C}_{\omega^n} \in G_n$ and $\pi_n(\eta^{C}_{\omega^n}) \in \Gamma^{B, \Gamma}$ (resp. $\eta^{C}_{\omega^n} \in \Gamma^{R, \Gamma}$ we color $C$ in red (resp. blue). Similarly, if $\eta^{C}_{\omega^n} \in G'_n$ and $\pi'_n(\eta^{C}_{\omega^n}) \in \Gamma^{B, \Gamma}$ (resp. $\eta^{C}_{\omega^n} \in \Gamma^{R, \Gamma}$ we color $C$ in red (resp. blue). We call the obtained configuration $\sigma^n \in \{R, B\}^{V_n}$. By Theorem 2.9 we have $(\omega^n, \sigma^n) \sim P^n_{D_n, q}$ and it is direct from the definitions to check that (4.1) holds.

The results we prove on the convergence of fuzzy Potts ‘boundary to boundary’ interfaces and interface collections are as follows. We are formulating them here as convergence statements assuming that we are working with a coupling (4.1) but we can directly deduce convergence in distribution assuming Conjecture 2.8 by the discussion above.

**Theorem 4.1.**\ Let $a, b \in \partial D$ be distinct and let $a_n, b_n \in \partial D_n$ be such that $a_n \to a$ and $b_n \to b$. Then assuming (4.1), we have

$$(\gamma_{a_n, b_n}^+, \gamma_{a_n, b_n}^-) \to (\gamma^{a, b}_1, \gamma^{a, b}_2)$$

in probability with respect to the metric $d_\mathcal{L}$ in each coordinate.

**Theorem 4.2.**\ Assuming (4.1), we have $(\Sigma^+_a, \Sigma^-_a) \to (\Sigma, \Sigma)$ in probability with respect to the metric $d_\mathcal{L}$ in each coordinate.

By combining the two theorems above with (4.1), one directly obtains the joint (distributional) convergence of the FK percolation and fuzzy Potts loop encodings.

The idea of the proofs is to approximate the continuum fuzzy Potts interface with the use of Proposition 2.12 and Corollary 2.14. It will then suffice to argue that chains of touching continuum cluster boundaries are approximated by chains of touching discrete cluster boundaries. The key input (also in the convergence of all discrete fuzzy Potts interfaces in the next section) is that if clusters get very close in the discrete then they in fact, touch each other. The following lemma is folklore and is a standard consequence of the fact that the six-arm exponent of FK percolation is larger than 2 as stated in Corollary 2.5.
Lemma 4.3. For $0 < \delta' < \delta$, let $G_n(\delta', \delta)$ be the event that whenever for some $z \in \mathbb{C}$ with $B_{2\delta}(z) \subset D$, the annulus $B_{\delta}(z) \setminus B_{\delta'}(z)$ is crossed by two $\omega^n$-open clusters $C$ and $C'$ that are disjoint inside the annulus $B_{\delta}(z) \setminus B_{\delta'}(z)$, then $C$ and $C'$ contain vertices $v$ and $v'$ respectively that are nearest neighbors in $D_n$. For all $\delta > 0$ we have that $\inf \mathbb{P}(G_n(\delta', \delta)) \to 1$ as $\delta' \to 0$.

Proof. Let $A_r(z, \delta', \delta, n)$ be the event that we see an arm event of type $\tau = 100100$ with respect to the percolation configuration $\omega^n$ in the annulus $B_{\delta}(z) \setminus B_{\delta'}(z)$. Then

$$G_n(\delta', \delta)^c \subset \bigcup_{z \in \mathbb{C}: B_{2\delta}(z) \subset D} A_r(z, \delta', \delta, n) \subset \bigcup_{z' \in \delta'(\mathbb{Z} + i\mathbb{Z}) \cap D} A_r(z', 2\delta', 2\delta/2, n)$$

where for the second inclusion, we need to make the assumption $\delta' < \delta/4$. The first inclusion is immediate from the definition of $G_n(\delta', \delta)$ and the second one follows since if $B_{2\delta}(z) \subset D$ we can take $z' \in (\delta'(\mathbb{Z} + i\mathbb{Z}) \cap D)$ and in that case $A_r(z, \delta', \delta, n) \subset A_r(z', 2\delta', 2\delta/2, n)$ and $B_{\delta}(z') \subset D$. Therefore by Corollary 2.5 we get

$$\mathbb{P}(G_n(\delta', \delta)) \lesssim (\delta')^{-2}(\delta'/\delta)^{2+\beta_1} \to 0 \quad \text{as } \delta' \to 0$$

for any $\delta > 0$ and the claim follows. \qed

The appearance of the condition $B_{2\delta}(z) \subset D$ is only to ensure that the annuli are contained in the discrete domains for $n$ sufficiently large. The next lemma is the analogous result for the fuzzy Potts model and will play a role when we apply Lemma A.1 and A.2.

Lemma 4.4. For $0 < \delta' < \delta$, let $G'_n(\delta', \delta)$ be the event that for each $z \in D$ with $B_{2\delta}(z) \subset D$, there exist at most five disjoint loop segments of loops in $\Sigma_{n}^+$ crossing the annulus $B_{\delta}(z) \setminus B_{\delta'}(z)$. For all $\delta > 0$ we have that $\inf \mathbb{P}(G'_n(\delta', \delta)) \to 1$ as $\delta' \to 0$. The same statement holds with $\Sigma_{n}$ in place of $\Sigma_{n}^+$.

Proof. The proof is virtually identical to the one of Lemma 4.3 except that we make use of Proposition 3.17 instead of Corollary 2.5. We therefore omit the details here. \qed

We will also make use of a small deterministic lemma, the purpose of which is to remove the need to consider boundaries of fillings of loops in Proposition 2.12 and Corollary 2.14 so that we can work with the loops directly.

Lemma 4.5. Consider $\eta \in C^4(\partial \mathbb{D}, \mathbb{C})$ and suppose that the boundary of the unbounded connected component of $\mathbb{C} \setminus \eta(\partial \mathbb{D})$ is a simple curve $\eta'$ which surrounds a point $z \in \mathbb{C}$. Consider $\epsilon > 0$, then there exist distinct $s_0^+, s_1^+, \ldots, s_{n-1}^+, s_n^+ \in \partial \mathbb{D}$ ordered in a counterclockwise way such that

$$\eta_{s_i^+} = \eta_{s_{i+1}^+} \in \eta'(\partial \mathbb{D}) \quad \text{for all } i < n \ (\text{addition modulo } n),$$

$$\tilde{\eta}(\partial \mathbb{D}) \subset \eta'(\partial \mathbb{D}) + B_\epsilon(0)$$

and such that $\tilde{\eta}$ surrounds $z$ where $\tilde{\eta}$ is the concatenation of $\eta_{s_0^+, s_1^+, \ldots, s_{n-1}^+, s_n^+}$ parametrized to be a function on $\partial \mathbb{D}$.

Proof. By applying a conformal transformation to the domain enclosed by $\eta'$ (which extends continuously to the boundary of the domain) we may assume that $\eta'$ parametrizes $\partial \mathbb{D}$. Therefore $\eta(\partial \mathbb{D}) \subset \mathbb{D}$ and $\eta(\partial \mathbb{D}) \cap \partial \mathbb{D} = \partial \mathbb{D}$.

Suppose without loss of generality that $|\eta_1| \leq 1 - \epsilon$. It suffices to find $t_-, t_+ \in \partial \mathbb{D}$ such that $t_-, t_+ \in \partial \mathbb{D}$ are oriented counterclockwise and such that $\eta_{t_-} = \eta_{t_+}$ as well as $\eta((t_+ \cup \{t_-, t_+\}) \cap \partial \mathbb{D} = \partial \mathbb{D}$.

We can therefore cut out an excursion from $\eta$ without changing the outer boundary. Iterating this procedure a finite number of times yields the lemma.

By the definition of $C^4(\partial \mathbb{D}, \mathbb{C})$ as uniform limits of simple curves, it suffices to prove the following quantitative version for simple curves: Consider disjoint balls $B_i := B_{r_i}(x_i)$ for $i = 0, \ldots, N - 1$ centered on points $x_0, \ldots, x_{N-1} \in \partial \mathbb{D}$ (ordered counterclockwise), let $\eta$ be a simple counterclockwise oriented curve in $\mathbb{D}$ intersecting $B_{r_i}(x_i)$ for all $i$, and let $I_0, \ldots, I_{m-1} \in \{0, \ldots, N - 1\}$ be an order in which the balls are visited by $\eta$, i.e. there are $s_0, \ldots, s_{m-1} \in \partial \mathbb{D}$ (distinct and ordered counterclockwise) such that $\eta_{s_j} \in B_{r_j}$ for all $j$ and such that $\eta((s_j, s_{j+1}))$
intersects $B_i$ only if $i \in \{I_j, I_{j+1}\}$. Then there exists $1 \leq i < j \leq m - 1$ such that $I_i = I_j + 1$ (modulo $N$) and such that $\{I_1, \ldots, I_j\} = \{0, \ldots, N-1\}$. Constructing $i$ and $j$ is a straightforward combinatorial exercise and omitted here.

4.1. Convergence of divide and color interfaces

The proof of Theorem 4.1 is now rather straightforward by combining all the ingredients we have collected in the lemmas above. In the proofs, the reason why certain inclusions only hold when $n$ is sufficiently large (rather than for all values of $n$) is that the outer and inner boundaries of FK clusters are themselves not open paths but are a distance given by at most the mesh size of the lattice away from an open path in the percolation configuration. Another effect which results in some inclusions only holding for $n$ sufficiently large is the need to consider approximations where the boundary curve of the discrete domain is sufficiently close to the continuum one.

Let us point out that the assumption that boundary clusters converge to boundary clusters in Conjecture 2.5 is needed in the proof below to ensure that certain discrete paths end on, rather than only close to the discrete boundary. In the case of domains with piecewise linear boundaries, this property can also be deduced from the fact that the alternating boundary three-arm event with type sequence 010 for FK percolation has exponent $> 1$.

Proof of Theorem 4.1. Fix $\epsilon > 0$. By the SDE description of the marginal law of $\gamma^{a,b}$, the interior of $\gamma^{a,b}([0,1]) \cap \partial D$ is almost surely empty. Therefore, by Lemma A.2 and 4.4, it suffices to show that the probability of the event

$$A_n := \{\gamma^\pm_{\sigma_n, a_n, b_n}([0,1]) \subset \gamma^{a,b}([0,1]) + B_\epsilon(0), \gamma^-_{\sigma_n, a_n, b_n}([0,1]) \subset \gamma^{a,b}([0,1]) + B_\epsilon(0)\}$$

tends to 1 as $n \to \infty$. Let $I_B$ (resp. $I_R$) be the open counterclockwise (resp. clockwise) boundary segment from $a$ to $b$ along $\partial D$. By combining Proposition 2.12 (together with the observations in Remark 2.13) and Lemma 4.5 we get the following almost sure statement for any fixed $\epsilon' > 0$.

Suppose that $0 \leq s < t \leq 1$ is such that $\gamma^{a,b}_s, \gamma^{a,b}_t \in I_B$ and $\gamma^{a,b}(s, t) \cap I_B = \emptyset$; we call $\gamma^{a,b}|_{(s, t)}$ a right boundary excursion of $\gamma^{a,b}$.

Then there exists $\eta^1, \ldots, \eta^m \in \Gamma^{OB}$ (oriented clockwise by convention) and $s^1_1, \ldots, s^m_m \in \partial D$ such that the concatenation of $\eta^{11}((s^1_1, s^1_2), \ldots, \eta^{m1}((s^1_m, s^2_m))$ defines a continuous curve $\gamma$, viewed as a function on the interval $[0, 1]$, with the following properties.

- The curve $\gamma$ lies right of $\gamma^{a,b}$.
- We have $\gamma_0 \in B_{\epsilon'}(\gamma^{a,b}_s)$, $\gamma_1 \in B_{\epsilon'}(\gamma^{a,b}_t)$ and $\gamma((0, 1)] \subset \gamma^{a,b}([s, t]) + B_{\epsilon'}(0)$.
- It holds that diam ($\eta^1$) < $\epsilon'$, diam ($\eta^m$) < $\epsilon'$. $\gamma$
- For all $i < m$ we have $\gamma^i_{s^i_t} = \gamma^{i+1}_{s^{i+1}_t}$ $\notin \partial D$ and $\gamma^{1}_{s^1_t}, \gamma^{m}_{s^m_t} \in \partial D$.

Note that Proposition 2.12 was stated in the case $D = \mathbb{D}$ but the above statements follow since conformal transformations from the unit disk to Jordan domains extend to homeomorphisms on the closures of the domains. Let $0 < s_1 < t_1 < \cdots < s_k < t_k < 1$ be the times such that $\gamma^{a,b}|_{(s_i, t_i)}$ are all the right boundary excursions for which $\gamma^{a,b}(s_i, t_i) + B_{\epsilon'/2}(0)$ does not contain the connected component of $D \setminus \gamma^{a,b}((s_i, t_i))$ which lies right of $\gamma^{a,b}$. From Lemma 4.3 and (4.1), it follows that for each $\epsilon'' > 0$, the probability of the following event tends to 1 as $n \to \infty$:

For each $1 \leq i \leq k$ there exists a blue strong path $v^{n,i}_{B}$ in $D_n$ from $u^{n,i}_{B} \in \partial D_n \cap B_{\epsilon''/2}(\gamma^{a,b})$ to $v^{n,i}_{B} \in \partial D_n \cap B_{\epsilon''/2}(\gamma^{a,b})$ such that its image is contained in $\gamma^{a,b}((s_i, t_i)) + B_{\epsilon''/2}(0)$. In fact, by construction, the path $v^{n,i}_{B}$ is a concatenation of a finite number of blue percolation cluster outer boundary curves.

When $1 \leq i \leq k + 1$, let us write $\gamma^{n,i}_{B}$ for the simple curve parametrizing the counterclockwise boundary segment of $D_n$ from $v^{n,i-1}_{B}$ to $v^{n,i}_{B}$ where $v^{n,0}_{B} = a_n$ and $v^{n,k+1}_{B} = b_n$. 

Then $\tilde{\nu}_B^{n,i}$ is at distance $<\epsilon''/2$ with respect to $d_{C'}$ to the counterclockwise boundary segment from $v^{n,i}_B$ to $u_{i+1}^n$ in $\partial D$ for $n$ sufficiently large where

$$ v_B^0 = a, \quad u_B^i = \gamma_n^{a,b}, \quad v_B^i = \gamma_n^{a,b} \quad \text{for} \quad 1 \leq i \leq k, \quad v_B^{k+1} = b. $$

The analogous statement holds when $B$ and $R$, ‘clockwise’ and ‘counterclockwise’, ‘blue’ and ‘red’, as well as ‘left’ and ‘right’ are interchanged in the statements above. Let us define $\nu_B^n$ to be the concatenation of the paths

$$ \tilde{\nu}_B^{n,1}, \nu_B^{n,1}, \ldots, \tilde{\nu}_B^{n,k}, \nu_B^{n,k}, \ldots, \tilde{\nu}_B^{n,k+1} $$

and define $\nu_B^m$ analogously. The claim now follows from the fact that $\gamma_n^{\pm,a_n,b_n}$ is ‘sandwiched’ in between $\nu_B^n$ and $\nu_B^m$, i.e. $\gamma_n^{\pm,a_n,b_n}$ has to lie left of $\nu_B^n$ and right of $\nu_B^m$. \hfill $\Box$

4.2. LIMIT OF DIVIDE AND COLOR LOOPS

The proof of Theorem 4.2 will be rather similar to the one in the previous section. The significant difference is that we need to exclude the possibility of ‘small’ FK clusters (which disappear in the scaling limit) agglomerating into large fuzzy Potts clusters which need to be considered when describing the scaling limit of the fuzzy Potts model. This will be achieved by arguing that each discrete fuzzy Potts interface passes in between two macroscopic FK clusters of differing colors and hence has to approximate the continuum fuzzy Potts interface passing in between the scaling limit of the macroscopic clusters.

**Proof of Theorem 4.2.** We present the proof for the collection $(\Sigma_{\eta}^{\pm})$. The proof for $(\Sigma_{\eta}^{\pm})$ is analogous.

**Step 1:** Let $A_{\eta}^n$ be the event that for each $\eta \in \Sigma$ with diameter $> \epsilon$, there exists $\eta' \in \Sigma_{\eta}^{\pm}$ such that $d_C(\eta, \eta') < \epsilon$. Then for every $\epsilon > 0$, we have $P(A_{\eta}^n) \to 1$ as $n \to \infty$.

By the construction in Section 2.6, the following statement holds almost surely for any $\epsilon' > 0$.

Suppose that $\eta \in \Xi_{RB}^{n}$ surrounds a point $z \in D$ (the case of $\eta \in \Xi_{RB}^{n}$ is analogous). Then by Corollary 2.14 and Lemma 4.5 there exists $\eta^1, \ldots, \eta^m \in \Gamma^{OB}$ oriented counterclockwise and $s_1^\pm, \ldots, s_m^\pm \in \partial D$ such that the concatenation of $\eta^1|_{((s_1^-, s_1^+) \cup \ldots \cup (s_m^-, s_m^+))}$ defines a loop $\eta_B^{n}$ surrounding $z$ such that $\eta_B^{n}(\partial D) \subset \eta(\partial D) + B_{\epsilon'}(0)$, the loop $\eta$ surrounds $\eta_B^{n}$, and such that we have

$$ \eta_{s_i^+}^{n,i} = \eta_{s_{i+1}^-}^{n,i+1} \notin \partial D \quad \text{for all} \quad 1 \leq i \leq m $$

where addition in the indices is understood modulo $m$. To approximate $\eta$ from the outside, we need to consider two distinct cases. If $\eta$ is an outermost continuum fuzzy Potts loop, i.e. $\eta \in \Xi_{RB}^{n}$ (oriented counterclockwise) and $s, t \in \partial D$ are such that $\eta_s, \eta_t \in \partial D$ and $\eta((s, t)) \subset D$ then one can approximate $\eta|_{((s, t))}$ from the outside just like in the proof of Theorem 4.1 by making use of Proposition 2.12. Suppose now that $\eta \in \Xi_{RB}^{n}$ for $n > 1$ (again, the case $\eta \in \Xi_{RB}^{n}$ is the same). Let $\eta'$ denote the curve parametrizing the connected component of $\cup_{\eta \in \Gamma_{R}^{n}}\eta''$ containing $\eta$. There are two cases which are possible in the construction in Section 2.6.

- The curve $\eta'$ is the boundary of a bounded connected component of a loop $\eta'' \in \Xi_{RB}^{n(1)} \cup \Xi_{RB}^{n(n-1)} \subset \Gamma^{OB}$. We write $S = \{\eta''\}$ in this case.
- The curve $\eta'$ is the boundary of a bounded connected component of a loop in $\Xi_{RB}^{n(n-1)} \cup \Xi_{RB}^{n(1)}$. Note that these are the false loops of $S := \Xi_{RB}^{n(1)} \cup \Xi_{RB}^{n(n-1)} \subset \Gamma^{OB}$ and so a path which is a concatenation of segments of false loops also gives a path which is a concatenation of segments of true loops.

Lemma 4.5 was stated in the case of boundaries of unbounded connected components but (by applying an inversion in a pivot) it extends to the case of bounded connected complementary components. Combining the lemma with Proposition 2.12 yields that there exists $\eta^1, \ldots, \eta^m \in \Gamma^{OB} \cup S \subset \Gamma^{R}$ and $s_1^\pm, \ldots, s_m^\pm$ such that the concatenation of $\eta^1|_{((s_1^-, s_1^+) \cup \ldots \cup (s_m^-, s_m^+))}$ defines
a loop $\tilde{\eta}_R$ surrounding $\eta$ such that $\tilde{\eta}_R(\partial \mathbb{D}) \subset \eta(\partial \mathbb{D}) + B_\epsilon(0)$. Therefore the continuum interface is ‘sandwiched’ in between $\tilde{\eta}_R$ and $\tilde{\eta}_B$ and one concludes that $\mathbb{P}(A_n^\alpha) \to 1$ as $n \to \infty$ in the same way as in the proof of Theorem 4.1.

Step 2: Let $G_n^{\alpha,\alpha}$ be the event that each $\eta \in \Sigma_\alpha^+$ of diameter $>$ $\epsilon$ touches a red cluster with diameter $>$ $\alpha$ and a blue cluster with diameter $>$ $\alpha$ with respect to the underlying FK percolation configuration $\omega$, such that these clusters have graph distance 1. Then for every $\epsilon > 0$, we have $\lim_{n \to \infty} \mathbb{P}(G_n^{\alpha,\alpha}) \to 1$ as $\alpha \to 0$.

First, we show that fuzzy Potts interfaces do not stay near the boundary. For $\rho > 0$ we define $D_\rho = \{ z \in D : d(z, \partial D) > \rho \}$ and let $B_n^{\rho,\alpha}$ denote the event that each $\eta \in \Sigma_\alpha^+$ with diameter $>$ $\epsilon$ intersects $D_\rho$. We claim that $\lim_{n \to \infty} \mathbb{P}(B_n^{\rho,\alpha}) \to 1$ as $\rho \to 0$. One can prove this using the crossing estimates from Theorem 2.3 but the details are involved and we choose to derive it directly from the scaling limit assumption (4.1) here. To this end, fix a conformal transformation $\varphi : D \to \mathbb{D}$ which then extends continuously to the boundary. Then since $\Gamma$ is locally finite and the set of points in $\varphi(\partial D)$ is dense in $\partial D$, the following statement holds almost surely: For each $\rho' > 0$ there exist $0 < \rho' < \rho'' < 1$ such that for each boundary segment of $\partial \mathbb{D}$ of length $\epsilon'$ there exists $\eta \in \Gamma^{OR} \cap \Gamma^D$ such that $\text{diam}(\varphi(\eta)) < \rho''$ with the property that $\eta$ intersects $\varphi^{-1}((1 - \rho')\partial \mathbb{D})$. By (4.1), these loops in $\Gamma^{OR} \cap \Gamma^D$ are limits of loops in $\Gamma^{\omega,\sigma_n}$ in $\Gamma^{\omega,\sigma_n}$. The analogous statement holds for $\Gamma^{OB} \cap \Gamma^D$ and $\Gamma^{OB} \cap \Gamma^{\omega,\sigma_n}$ and $\Gamma^{\omega,\sigma_n}$. The claim now follows from the continuity of $\varphi$ and the fact that fuzzy Potts interfaces do not cross any loop in $\Gamma^{\omega,\sigma_n} \cap \Gamma^{\omega,\sigma_n} \cap \Gamma^{\omega,\sigma_n}$ or in $\Gamma^{\omega,\sigma_n} \cap \Gamma^{\omega,\sigma_n} \cap \Gamma^{\omega,\sigma_n}$.

Second, we show that paths cannot avoid macroscopic FK clusters. Let $\rho$ be as before and by decreasing it further, assume that $\rho < \epsilon$. We define the event $H_n^{\rho,\alpha}(z)$ as follows: There is an $\omega^\alpha$-clusters chain, each cluster having diameter $>$ $\alpha$, such that there is a loop following the annulus $B_{\rho/2}(z) \setminus B_{\rho/4}(z)$ which only uses vertices in the aforementioned percolation clusters. From Theorem 2.6, we deduce that the event $H_n^{\rho,\alpha}(z)$ has probability tending to 1 as $\alpha \to 0$ uniformly in $n$ and in $B_\rho(z) \subset D$. Let us now take $z_1, \ldots, z_m \subset B_\rho(z)$ such that $\bigcup_i B_{\rho/4}(z_i) \supset D_\rho$. Then the probability of the event $H_n^{\rho,\alpha} := \cap_i H_n^{\rho,\alpha}(z_i)$ also tends to 1 as $\alpha \to 0$ uniformly in $n$.

Finally, note that on the event $B_n^{\rho,\alpha} \cap B_{\rho/2}^\hat{\alpha}$, any $\eta \in \Sigma_\alpha^+$ of diameter $>$ $\epsilon$ intersects $D_\rho$, therefore intersects $B_{\rho/4}(z_i)$ for some $i \leq m$ and hence crosses the annulus $B_{\rho/2}(z_i) \setminus B_{\rho/4}(z_i)$ since we assumed that $\rho < \epsilon$. By combining the previous two arguments, we therefore deduce the claim that $\lim_{n \to \infty} \mathbb{P}(B_n^{\rho,\alpha}) \to 1$ as $\alpha \to 0$.

Step 3: Let $\eta_1, \ldots, \eta_m$ denote the loops in $\Gamma^{\omega^\alpha}$ of diameter $>$ $\alpha$. Define $F_n^{\alpha,\alpha'}$ to be the event that there exist distinct loops $\eta'_1, \ldots, \eta'_m \subset \Gamma$ of diameter $>$ $\alpha$ with the following properties:

- It holds that $d_C(\eta_i, \eta'_i) < \alpha'$ for every $i \leq m$.
- We have $\eta_i \in \Gamma^{\omega^\alpha,\sigma_n}$ if and only $\eta'_i \subset \Gamma^{\omega^\alpha}$, and similarly for $\Gamma^{OB}$, $\Gamma^{IB}$ and $\Gamma^{IB}$.
- If the loops $\eta_i$ and $\eta_j$ are at distance $\leq \epsilon_n$ from each other then $\eta'_i$ and $\eta'_j$ touch each other at a point.

Then for every $\alpha, \alpha' > 0$, $\mathbb{P}(F_n^{\alpha,\alpha'}) \to 1$ as $n \to \infty$.

This follows readily from (4.1).

Step 4: Conclusion

Fix $\epsilon > 0$. As a consequence of the previous three steps, the probability of the event $G_{n}^{\alpha,\alpha'} \cap F_{n}^{\alpha,\alpha'} \cap A_{n}^\alpha$ can be made arbitrarily close to 1 by choosing $\alpha, \alpha', n$ in this order. We conclude by arguing that on this event, $\Sigma_\alpha^+$ and $\Sigma$ are close with respect to the metric $d_C$.

Let $\eta$ be a loop in $\Sigma_\alpha^+$ of diameter $>$ $\epsilon$ and consider $\eta_i \in \Gamma^{\omega^\alpha,\sigma_n} \cup \Gamma^{IB}$ and $\eta_j \in \Gamma^{\omega^\alpha,\sigma_n} \cup \Gamma^{IB}$ of diameter $>$ $\alpha$ such that $\eta_i$ and $\eta_j$ are at distance $\leq \epsilon_n$ from each other and such that one is on the inside and the other one is on the outside of $\eta$. The corresponding loops $\eta'_i \subset \Gamma^{\omega^\alpha} \cup \Gamma^{IB}$ touch each other and there is a unique loop $\eta' \subset \Sigma$ passing between them. We note that $\eta'$ must have diameter $>$ $\alpha$ and we set $\pi(\eta) = \eta'$. This defines a map from the loops in $\Sigma_\alpha^+$ of diameter $>$ $\epsilon$ into $\Sigma$ and it is now sufficient to check that $d_C(\eta, \pi(\eta)) < \alpha'$ for every loop $\eta \in \Sigma_\alpha^+$, that $\pi$ is injective and that every loop in $\Sigma$ of diameter $>$ $\epsilon + \alpha'$ is contained in the
image of $\pi$. Indeed, this follows from the definition of the events $G_n^{x,\alpha}$, $F_n^{x,\alpha'}$ and $A_n^{\alpha'}$ together with the key observation that whenever one considers two FK percolation clusters of opposite colors that are graph distance 1 away from each other, there is a unique discrete fuzzy Potts loop passing between the two.

\[\Box\]

Remark 4.6. When $r = 1/q$, we can also prove a version of the theorem above in the case when we condition on each vertex in $\partial D_n$ being red. Indeed, let us sample $\sigma^n \sim \mu_{D_n,q,r}^n$ conditioned on $\sigma^n_v = R$ for all $v \in \partial D_n$. Then, assuming Conjecture 2.8 for the value $q$, we have $\left(\Sigma_{\sigma^n}, \Sigma_{\sigma^n}^-\right) \to \left(\Sigma', \Sigma'\right)$ in distribution where $\Sigma'$ is a nested CLE$_\kappa$ in $D$.

The key observation is that one can equivalently sample $\sigma^n$ by starting with $\omega^n \sim \mu_{D_n,q}^1$ (i.e. we consider a measure with wired boundary conditions) and by coloring each boundary cluster in red and all others independently in red or blue with respective probabilities $r$ and $1 - r$ to obtain the configuration $\sigma^n$. The proof of the result then goes through without changes and by making use of Theorem 2.11.

In the particular case of the FK-Ising model where $q = 2$ and hence $\kappa = 3$ this statement is unconditional by [Smi10, KS19, KS16] and it describes the scaling limit of the Ising model with ‘plus boundary conditions’. This result has already been established in [BH19] using rather different techniques (in particular without going via the Edwards-Sokal coupling) and building on the convergence results [CDCH+14, CS12] for Ising interfaces.

5. Continuum exponents via exploration paths

In this section, we start with results on SLE interior and boundary arm exponents for SLE$_\kappa(0,\rho)$ and SLE$_\kappa$ curves respectively as derived in [Wu18a] and obtain the interior and boundary arm exponents for SLE$_\kappa(\kappa - 6 - \rho, \rho)$ curves. The novelty of the argument in this section lies in the use of imaginary geometry (see Section 2.7) to show that the additional force points do not affect the value of the exponent.

We begin by recalling the relevant results on SLE exponents appearing in [Wu18a]. In this paper, several exponents are computed using stochastic analysis tools. Other papers where such SLE exponents are computed using suitable SLE martingales are [LSW01a, LSW01b, LSW02b, Bef08, MW17, Wu18b, WZ17] and [LSW02a, SSW09].

We state the results in the unit disk rather than the upper halfplane since it will be more convenient to transport the results in this setting.

For $r \in (0,2)$ we let $I_r \in \partial \mathbb{D}$ be such that $B_r(1) \cap \partial \mathbb{D} = \{(1/I_r, I_r)\}$. Consider a random curve $\gamma$ from $-i$ to $i$ in $\mathbb{D}$ and consider parameters satisfying

\[x \in ((-i, i)), R \in (|x + i|, 2), \ y \in ((i, -i)), \ 0 < r < |y - i| \land (|y + i| - R), \ c > 1, \ c_0 \in (0,1)\]

Whenever we work with such parameters, we assume that the above relations are satisfied. Write $\zeta_x = \inf\{t \geq 0: \gamma_t \in ((i, x))\}$ for the swallowing time of $x$. Set $\sigma_0 = 0$ and for $\epsilon > 0$ sufficiently small and $j \geq 1$ we make the following inductive definition:

- Let $t_j$ be the first time after $\sigma_{j-1}$ where $\gamma$ hits the connected component of $\partial B_\epsilon(x) \setminus \gamma([0, \sigma_{j-1}])$ containing $x I_\epsilon$.
- Let $J$ be the connected component of $(\partial B_\epsilon(y) \cap \partial \mathbb{D}) \setminus (\gamma([0, \tau_1]) \cap (\partial B_\epsilon(y)))$ containing $y(1 - r)$; the curve $J$ inherits the counterclockwise orientation of $\partial B_\epsilon(y)$. Let $\sigma_j$ be the first time after $\tau_j$ when $\gamma$ hits the most counterclockwise connected component of $J \setminus (\gamma([0, \tau_1]))$.

For $j \geq 1$ and $k \geq 1$ we define the events

\[B_{2j-1}(x, \epsilon, y, r) = \{t_j < \zeta_x\}, \ B_{2j}(x, \epsilon, r, y) = \{\sigma_j < \zeta_x\}, \ G_B(x, \epsilon, c, c_0, R) = \{\gamma([0, \tau_1]) \subseteq B_R(-i), \ gamma([0, \tau_1]) \cap ((1/I_\epsilon, I_\epsilon) + B_{c_0}(0)) = \emptyset\}, \ B_{2k}^\epsilon(x, \epsilon, y, r, c, c_0, R) = B_k(x, \epsilon, r, c, c_0, R) \cap G_B(x, \epsilon, c, c_0, R)\].
We can now state [Wu18a, Proposition 3.3] on the exponents in the boundary case; the result we state here is slightly weaker but will suffice for our purposes. We note that on the event $B_k^1(\gamma, x, \epsilon, y, r, c, c_0, R)$ we automatically have $J = \partial B_r(y) \cap \mathbb{D}$ in agreement with the statement of the proposition in [Wu18a].

**Theorem 5.1** ([Wu18a]). Let $\kappa \in (0, 4)$, $\rho \in (-2, \kappa/2 - 2)$ and for $j \geq 1$ define

$$\alpha_{2j-1}^\kappa = (2j + \rho)(2j + \rho - \kappa/2)/\kappa,$$

$$\alpha_{2j}^\kappa = 2j(2j + \kappa/2 - 2)/\kappa.$$ 

Consider $\gamma \sim \text{SLE}_\kappa(0, \rho)$ from $-i$ to $i$ in $\mathbb{D}$. For $k \geq 1$ there exist constants $y$, $r$, $c$, $c_0$ and $R$ such that

$$\epsilon^{\alpha_j^\kappa} \lesssim \mathbb{P}(B_k^\kappa(\gamma, 1, \epsilon, y, r, c, c_0, R))$$

for all $\epsilon > 0$ sufficiently small.

The story is similar in the case of interior exponents. Again, we consider a random curve $\gamma$ in $\mathbb{D}$ from $-i$ to $i$ and parameters $c \in (0, 1)$, $|z| < c$, $R \in (1 + c, 2)$, $y \in (i, -i)$, $0 < r < |y - i| \wedge (|y + i| - R)$.

Again, we assume that these relations are satisfied whenever we work with these parameters. Let $\sigma_0^\kappa = 0$. For $\epsilon > 0$ sufficiently small and $j \geq 1$ we make the following inductive definition.

- Let $C_j$ denote the connected component of $\mathbb{D} \setminus (\gamma([0, \tau_j^\kappa]) \cup \partial B_r(y))$ with $i$ on its boundary. If $z \notin C_j$ we let $\tau_j^\kappa = \infty$. Otherwise (i.e., if $z \in C_j$), we write $C_j^\prime$ for the connected component of $C_j \cap \partial B_r(z)$ which is connected to $i$ within $C_j \setminus B_r(z)$ and we let $\tau_j^\prime$ be the first time after $\tau_j^\kappa$ when $\gamma$ hits $C_j^\prime$.

- Let $J$ be the connected component of $(\partial B_r(y) \cap \mathbb{D}) \setminus (\gamma([0, \tau_j^\kappa])$ containing $y(1 - r)$; the curve $J$ inherits the counterclockwise orientation of $\partial B_r(y)$. Let $\tau_j^\prime$ be the first time after $\tau_j^\kappa$ when $\gamma$ hits the most counterclockwise connected component of $J \setminus (\gamma([0, \tau_j^\kappa])$.

For $j \geq 1$ we now define the events

$$I_{2j}(\gamma, z, \epsilon, y, r) = \{\tau_j^\prime < \zeta_z\}, \quad G_{1,2j}(\gamma, z, \epsilon, y, r, R) = \{\gamma([0, \tau_j^\kappa]) \subset B_R(-i)\},$$

$$I_{2j}(\gamma, z, \epsilon, y, r, R) = I_{2j}(\gamma, z, \epsilon, y, r) \cap G_{1,2j}(\gamma, z, \epsilon, y, r, R)$$

where $\zeta_z$ is the swallowing time of $z$. With all this notation set up, we can now state the result on interior exponents [Wu18a, Proposition 4.1]; this result is stated for a fixed interior point $z$, however, the proof also implies the stronger result as stated below.\(^3\)

**Theorem 5.2** ([Wu18a]). Let $\kappa \in (0, 4)$ and for $j \geq 1$ define

$$\alpha_{2j} = (16j^2 - (\kappa - 4)^2)/(8\kappa).$$

Let $\gamma \sim \text{SLE}_\kappa$ from $-i$ to $i$ in $\mathbb{D}$. For $j \geq 1$ there exist constants $y$, $r$, $c$ and $R$ such that

$$\mathbb{P}(I_{2j}(\gamma, z, \epsilon, y, r, R)) = \epsilon^{\alpha_{2j} + o(1)} \quad \text{as } \epsilon \to 0$$

uniformly in $|z| < c$.

The goal is now to transport these results to the setting of $\text{SLE}_\kappa(\rho, \kappa - 6 - \rho)$. Because of the proof strategy, the results are slightly weaker than in the theorems stated above, but this is not relevant for applications to the discrete fuzzy Potts models.

**Proposition 5.3.** Let $\kappa \in (2, 4)$, $\rho \in (-2, \kappa - 4)$ and suppose that $\gamma \sim \text{SLE}_\kappa(\kappa - 6 - \rho, \rho)$ from $-i$ to $i$ in $\mathbb{D}$. For $k \geq 1$ there are constants $y$, $r$, $r'$, $c'$, $c'_0$ and $R'$ such that

$$\epsilon^{\alpha_j^\kappa} \lesssim \mathbb{P}(B_k^\kappa(\gamma, 1, \epsilon, y, r)), \quad \mathbb{P}(B_k'(\gamma, 1, \epsilon, y, r', c', c'_0, R')) \lesssim \epsilon^{\alpha_j^\kappa}$$

for all $\epsilon > 0$ sufficiently small.

\(^3\)Indeed, the only place where the exact location of the interior point plays a role is in [Wu18a, Lemma 4.2] which makes use of [MW17, Lemma 4.1] and the latter is stated uniformly in the location of the interior point.
Figure 12. Left. With positive probability $\gamma_\sim \text{SLE}_\kappa(\bar{\rho}_-,\rho_+,\rho_+)$ hits the top of the red dashed box before it hits the sides and after that, it does not hit the the blue dashed lines. Right. This drawing shows $\gamma_-$ (in red) and $\gamma'_+$. The conditional law of $\gamma'_+$ given $\gamma_-$ is given by a SLE$_\kappa(\rho_-,\rho_+)$ from $w_0$ to $w_\infty$ in the connected component right of $\gamma_-$ which has 1 on its boundary. The sets $\partial B_r(y)$ and $\partial B_c(1)$ are drawn in green and $\partial B_{R}(\bar{\rho}_-)$ and $\partial((1/I_{\text{c}e}(I_{\text{c}e}(0))) + B_{\text{c}e}(0))$ are drawn dashed in green.

Proposition 5.4. Let $\kappa \in (2,4)$, $\rho \in (-2,\kappa-4)$ and suppose that $\gamma \sim \text{SLE}_\kappa(\kappa-6-\rho,\rho)$ from $-i$ to $i$ in $D$. For $j \geq 1$ there exists constants $y$, $r$, $r'$, $c$ and $R'$ such that

$$e^{o_2j+o(1)} \leq \mathbb{P}(I_2j(\gamma, z, \epsilon, y, r))$$

$$\mathbb{P}(I'_2j(\gamma, z, \epsilon, y, r', R')) \leq e^{o_2j+o(1)}$$

as $\epsilon \to 0$ uniformly in $|z| < c$.

The proof of these propositions will be split into three lemmas, each of which involves the use of imaginary results as listed in Section 2.7. The proofs of all the lemmas below follow roughly the same kind of strategy. We start with one SLE curve and construct another one from it with the following property: If the first one satisfies a certain arm event, so will the latter. The results on distortion estimates for conformal maps used below appear in Appendix B. Throughout the remainder of this section $k$ and $2j$ are fixed and all constants are allowed to depend on them (and we will not make this dependence explicit everywhere). Moreover, whenever $\rho_\pm > -2$ we let $\gamma_{\rho_-,\rho_+} \sim \text{SLE}_\kappa(\rho_-,\rho_+)$ from $-i$ to $i$ in $D$. Each event we consider will only depend on one of these curves so we do not specify a coupling.

In this first lemma, we will rely on Lemma 2.15 and 2.19 to perform the construction outlined in the previous paragraph.

Lemma 5.5. Let $\kappa \in (0,4)$, $\rho_\pm > -2$ and $\bar{\rho}_\pm \geq 0$. Then for all $\lambda > 1$ sufficiently close to 1 there exists $C_\lambda < \infty$ such that

$$\mathbb{P}(B_k'(\gamma_{\rho_-,\rho_+}, 1, \epsilon/\lambda, y, r/\lambda, \lambda c_0, R/\lambda)) \leq C_\lambda \mathbb{P}(B_k'(\gamma_{\rho_-,\rho_+}, 1, \epsilon, y, r, c, c_0, R))$$

for all $\epsilon > 0$ sufficiently small. Moreover, for all $\lambda > 1$ sufficiently close to 1 there exists $C'_\lambda < \infty$ such that

$$\inf_{|z| < \lambda c} \mathbb{P}(I'_2j(\gamma_{\rho_-,\rho_+}, z', \epsilon/\lambda, y, r/\lambda, R/\lambda)) \leq C'_\lambda \inf_{|z| < c} \mathbb{P}(I'_2j(\gamma_{\rho_-,\rho_+}, \rho_+, z, \epsilon, y, r, R))$$

for all $\epsilon > 0$ sufficiently small.
Proof. Let \( \gamma_\sim \sim \text{SLE}_6(\bar{\rho}_- - 2, 2 + \rho_- + \rho_+) \) be coupled with \( \gamma_+ \sim \text{SLE}_6(\rho_- + \bar{\rho}_-, \rho_+) \) as in Lemma 2.15. Let \( D \) be the connected component of \( \{0, 1\} \) with 1 on its boundary, define \( w_0 \) to be the most counterclockwise point on \((-1, 0) \cap \gamma_\sim([0, 1]) \) and \( w_\infty \) to be the most clockwise point on \((1, 0) \cap \gamma_\sim([0, 1]) \). Let \( \gamma_\sim_+ \) be the part of \( \gamma_+ \) from \( w_0 \) to \( w_\infty \). Then by the lemma, conditionally on \( \gamma_\sim \), the curve \( \gamma_\sim_+ \) is a \( \text{SLE}_6(\rho_- + \bar{\rho}_-, \rho_+) \) in \( D \) from \( w_0 \) to \( w_\infty \).

We write \( \phi : D \to \mathbb{D} \) for the unique conformal transformation with \( \phi(0) = 0 \) and \( \phi'(0) > 0 \). Then \( \phi \) extends continuously to \((w_0, w_\infty) \cup \{w_0, w_\infty\}\) and we write \( f \) for the unique Möbius transformation from \( \mathbb{D} \) to itself mapping \((\phi(w_0), \phi(1), \phi(w_\infty)) \to (-i, 1, i) \). We let \( \psi = f \circ \phi \).

From Lemma 2.19 (applied twice) and the strong Markov property for SLE curves, we deduce that for all \( \delta \in (0, 1/16) \) the following key observations:

1. The curve \( \gamma_\sim \) hits \( S_T := (1 - 2\delta) \cdot (-i/I_\delta, -iI_\delta) \) before \( S_{LR} := (1 - 2\delta, 1) \cdot \{-i/I_\delta, -iI_\delta\} \) and we write \( \tau \) for the hitting time.
2. The curve \( \gamma_\sim|_{[\tau, 1]} \) hits \( S'_T \) before \( S'_{LR} := S'_B \setminus S'_T \) where \( S'_T := ([i/I_\delta, iI_\delta], \{i/I_\delta, -iI_\delta\}) \)

\[
S'_T := (1 - 3\delta, 1 - \delta) \cdot (-i/I_\delta, -iI_\delta) \cup (1 - 3\delta, 1 - 2\delta) \cdot ([i/I_\delta, -iI_\delta])
\]

and where \( S'_B \) is the connected component of \( \partial S^\prime \setminus \gamma([0, \tau]) \) containing \( i \).

Let us call this event \( N_\delta \); it ensures that \( B_{1 - 3\delta}(0) \subset D \) and that \( \gamma_+ \) hits \( w_0 \) before exiting \((-i/I_\delta, -iI_\delta) \cdot (1 - 2\delta, 1) \). We will now argue that for some (small) \( \delta \) we have the inclusion

\[
N_\delta \cap B'_k(\psi \circ \gamma_+, 1, \epsilon/\lambda, y, r/\lambda, \lambda c_0, R/\lambda) \subset B'_k(\gamma_+, 1, \epsilon, y, r, c, c_0, R)
\]

for all sufficiently small \( \epsilon \). From this the result follows by taking probabilities and using the independence of \( \gamma_\sim \) and \( \psi \circ \gamma_+ \).

We will now work on the event \( N_\delta \). For all \( \delta' > 0 \) by Lemma B.1 there exists \( \delta \) such that we have \( |\phi(w) - w| \leq \delta' \) for all \( w \in D_{(4\delta, 8\delta)} \) (we are using the notation from Appendix B). We make the following key observations:

1. The points \( w_0, w_\infty, 1 \) lie in the closure of \( D_{(4\delta, 8\delta)} \), so since \( \phi \) extends continuously to these points, we also have \( |\phi(w) - w| \leq \delta' \) for \( w \in \{w_0, w_\infty, 1\} \).
2. Let \( S \) denote the connected component of \( \partial B_y(\gamma_\sim([0, 1])) \) containing \( y(1 - r) \) (this is defined for \( 3\delta < r \)) and let \( S' \) denote the connected component of \( \partial B_y(\gamma_\sim([0, 1])) \) containing \( i(R - 1) \). Then we have \( S, S' \subset D_{(4\delta, 8\delta)} \).

It follows that for \( \delta \) sufficiently small, \( \psi(S) \cap B_{\epsilon/\lambda}(\gamma_+) = \emptyset \), \( \psi(S') \cap B_{R/\lambda}(-i) = \emptyset \). Moreover, \( B_{1/2}(1) \cap \mathbb{D} \subset D \) and so by further decreasing \( \delta \) and using Lemma B.2 applied to the function \( \psi \) we get that for all \( x \in ([1/I_{1/4}, I_{1/4}] \) and all sufficiently small \( \epsilon \),

\[
B_{\epsilon/\lambda}(\psi(x)) \cap \mathbb{D} \subset \psi(B_\epsilon(x) \cap \mathbb{D}) \subset B_{\epsilon/\lambda}(\psi(x)) \cap \mathbb{D}.
\]

In particular \( B_{\epsilon/\lambda}(1) \cap \mathbb{D} \subset \psi(B_\epsilon(1) \cap \mathbb{D}) \) and

\[
((1/I_{\lambda c_0}, I_{\lambda c_0})) + B_{\epsilon/\lambda}(0) \cap \mathbb{D} \subset \psi(((1/I_{\lambda c_0}, I_{\lambda c_0}) + B_{\epsilon/\lambda}(0)) \cap \mathbb{D})
\]

for \( \epsilon \) sufficiently small. Combining these inclusions readily implies (5.1) since one can see that if \( \psi \circ \gamma_+ \) satisfies the arm event and \( N_\delta \) holds then \( \gamma_+ \) will also satisfy an arm event.

The result on interior exponents follows similarly in the case where \( \bar{\rho}_- = 0 \) or when \( \bar{\rho}_+ = 0 \) (now using Lemma B.3 instead of Lemma B.2). The general case is obtained by bounding the arm event probability of a \( \text{SLE}_6(\rho_- + \bar{\rho}_-, \rho_+) \) by that of a \( \text{SLE}_6(\rho_- + \bar{\rho}_-, \rho_+) \) and then further by that of a \( \text{SLE}_6(\rho_- + \bar{\rho}_-, \rho_+) \). \( \square \)

The next lemma makes use of Lemma 2.16 and implements a very similar proof strategy to the one in the previous Lemma 5.5 to relate arm event probabilities for the two types of SLE appearing in Lemma 2.16. Note that this result will only be needed in the interior case.
Lemma 5.6. Suppose that $\kappa \in (0, 4)$ and $\rho \in (-2, \kappa/2 - 2)$. For all $\lambda > 1$ sufficiently close to 1, there exists $C_\lambda < \infty$ such that
\[
\inf_{|z'| < \lambda c} \mathbb{P}(I_{2j}^2(\gamma_{0, \kappa - 4 - \rho}, z', \epsilon/\lambda, y, r/\lambda, R/\lambda)) \leq C_\lambda \inf_{|z| < c} \mathbb{P}(I_{2j}^2(\gamma_{0, \rho}, z, \epsilon, y, r, R))
\]
for all $\epsilon > 0$ sufficiently small.

Proof. Let $\gamma \sim \text{SLE}_\kappa(0, \rho)$, $\zeta_1$ and $\zeta_{1-}$ be as in Lemma 2.16. We write $D$ for the connected component of $\mathbb{D} \setminus (\gamma([0, \zeta_{1-}] \cup [\zeta_1, 1]))$ having 1 on its boundary. We write $\gamma'$ for $\gamma|_{[\zeta_{1-}, \zeta_1]}$ (reparametrized so that it is a function on $[0, 1]$). Thus by the lemma, conditionally on $\gamma([0, \zeta_{1-}] \cup \gamma([\zeta_1, 1]))$, the curve $\gamma'$ is a SLE$\rho(0, \kappa - 4 - \rho)$ in $D$ from $w_0 := \zeta_{1-}$ to $w_\infty := \zeta_1$. We define $\phi$, $f$ and $\psi$ as in the proof of Lemma 5.5.

Consider $\delta \in (0, 1/16)$. We first argue that with positive probability $\gamma([0, \zeta_{1-}] \subset [-\delta, \delta] + i[-1, -1 + \delta]$ and $\gamma([\zeta_1, 1]) \subset [-\delta, \delta] + i[1 - \delta, 1]$; we call this event $N_\delta$. Indeed, Lemma 2.19 (applied three times) and the strong Markov property of SLE (applied twice) shows that the following event has positive probability (see also Figure 13):

- Let $S_T$ and $S_{LR}$ be the top and the union of the left and right sides of $S = ((-\delta, \delta) + i(-1, -1 + 2\delta)) \cap \mathbb{D}$ respectively. The curve $\gamma$ hits $S_T$ before $S_{LR}$ and at the time $\tau$. The curve $\gamma$ at $S_{LR}$ at a time $\tau'$.
- Let $S_T' = ((-\delta, \delta) + (1 - \delta)i$ and $S_{LR}' = S_B' \setminus S_T'$ where $S_T' = ((-\delta, \delta) + i(-1 - \delta, 1 - \delta)$ and where $S_B'$ denotes the connected component of $\partial S' \setminus \gamma([0, \tau])$ containing $i(1 - \delta)$. The curve $\gamma$ at $S_{LR}$ at a time $\tau'$.
- Let $S_B''$ denote the top of $S'' = ((-\delta, \delta) + i(1 - 2\delta, 1)) \cap \mathbb{D}$ and let $S_{LR}'' = S_B'' \setminus S_T''$ where $S_B''$ denotes the connected component of $\partial S'' \setminus \gamma([0, \tau'])$ containing $i$. Then $\gamma$ at $S_{LR}$ at a time $\tau'$.

There exists $\delta > 0$ such that for $|z| < c$ one has $|\psi(z)| < \lambda c$ on the event $N_\delta$ as well as the inclusion
\[
N_\delta \cap I_{2j}^2(\psi \circ \gamma', \psi(z), \epsilon/\lambda, y, r/\lambda, R/\lambda) \subset I_{2j}^2(\gamma, z, \epsilon, y, r, R)
\]
for all sufficiently small $\epsilon > 0$. This follows by a reasoning similar to the one in the proof of Lemma 5.5. The result now follows by taking probabilities and using the independence of $\psi \circ \gamma'$ and $\gamma([0, \zeta_{1-}] \cup \gamma([\zeta_1, 1])$. \qed

The last lemma leverages Lemma 2.17 in combination with Lemma 2.18 to compare the arm event probabilities (both in the interior and the boundary case) of the types of SLEs appearing in Lemma 2.17. The proof will make it transparent why we included the set $J$ in the definition of the arm events.

Lemma 5.7. Suppose that $\kappa \in (2, 4)$ and $\rho \in (-2, \kappa - 4)$. Then for all $\lambda > 1$ sufficiently close to 1, there exists $C_\lambda < \infty$ such that
\[
\mathbb{P}(B_k^i(\gamma_{0, \kappa - 6 - \rho}, 1, \epsilon/\lambda, y, r/\lambda, \lambda c_0, R/\lambda)) \leq C_\lambda \mathbb{P}(B_k(\gamma_{\kappa - 6 - \rho, \rho}, 1, \epsilon, y, r, y))
\]
for all $\epsilon > 0$ sufficiently small. Moreover, for all $\lambda > 1$ sufficiently close to 1 there exists $C'_\lambda < \infty$ such that
\[
\inf_{|z'| < \lambda c} \mathbb{P}(I_{2j}^2(\gamma_{0, \kappa - 6 - \rho}, z', \epsilon/\lambda, y, r/\lambda, R/\lambda)) \leq C'_\lambda \inf_{|z| < c} \mathbb{P}(I_{2j}^2(\gamma_{\kappa - 6 - \rho, \rho}, z, \epsilon, y, r, R))
\]
for all $\epsilon > 0$ sufficiently small.

Proof. Let $\gamma^0 \sim \text{SLE}_\kappa(\kappa - 6 - \rho, \rho)$ from $-i$ to $i$ in $\mathbb{D}$. Consider $\delta \in (0, 1/16)$. We first argue that the following event occurs with positive probability (see Figure 13):

- Let $S_T = ([i\delta, i\delta])$, $S_{LR} = S_B \setminus S_T$, $S = (1 - 2\delta, 1) \setminus ([i\delta, i\delta]) \cup (1 - 2\delta, 1) \cdot ([i\delta, i\delta])$ and $S_B = \partial S \setminus ([i\delta, i\delta])$. Then the curve $\gamma^0$ hits $S_T$ before hitting $S_{LR}$. We write $\tau$ for the hitting time.
Also, there exists a point \( \nu \) such that for all sufficiently small, we have

\[
N_\delta \cap B_k(\psi \circ \gamma', 1, \epsilon/\lambda, y, r/\lambda, \lambda c, \lambda C_0, R/\lambda) \subseteq B_k(\gamma_0', 1, \epsilon, y, r).
\]

Also, there exists \( \delta \) such that for \( |z| < c \) one has \( |\psi(z)| < \lambda c \) on the event \( N_\delta \) and for \( \epsilon > 0 \) sufficiently small,

\[
N_\delta \cap I_{2j}(\psi \circ \gamma', |0, \nu|, \psi(z), \epsilon/\lambda, y, r/\lambda, R/\lambda) \subseteq I_{2j}(\gamma_0', z, \epsilon, y, r, R).
\]

It remains to show that we can drop the restriction to the interval \([0, \sigma]\) since the result then follows by taking probabilities and using the independence of \( \gamma_0'(0, \sigma'] \) and \( \psi \circ \gamma' \).

In the case of the boundary arm events this is clear since the swelling time of the point \( 1 \) by the curve \( \psi \circ \gamma' \) appearing in the definition of the arm event occurs (not necessarily strictly) before \( \nu \). For the case of the interior arm event, we suppose that \( \sigma < \tau_j \), and hence \( \tau_1 < \sigma < \tau_j \) and therefore the point \( \psi(z) \) lies left of \( \psi \circ \gamma'(0, \nu) \) which contradicts the definition of the interior point arm event.
Proposition 5.3 and 5.4 are now obtained by combining Lemma 5.5, 5.6 and 5.7 in a rather straightforward way.

Proof of Proposition 5.3. The upper bound follows from the upper bound of Theorem 5.1 together with the first display in Lemma 5.5 with \( \rho_+ = \rho, \rho_- = \kappa - 6 - \rho \) and \( \bar{\rho}_+ = -\rho_- \). The lower bound follows from the lower bound in Theorem 5.1 and the first inequality of Lemma 5.7. \( \square \)

Proof of Proposition 5.4. The upper bound follows from the upper bound in Theorem 5.2 together with the second display in Lemma 5.5 with \( \rho_+ = \rho, \rho_- = \kappa - 6 - \rho, \bar{\rho}_+ = -\rho_+ \) and \( \bar{\rho}_- = -\rho_- \). The lower bound is obtained as follows: We use the lower bound from Theorem 5.2 and then apply the lemmas above in the following settings:

- Use Lemma 5.5 to bound the arm event probability of a \( \text{SLE}_\kappa \) by that of a \( \text{SLE}_\kappa(0, \kappa - 4 - \rho) \) using that \( \kappa - 4 - \rho > 0 \).
- Use Lemma 5.6 to bound the arm event probability of a \( \text{SLE}_\kappa(0, \kappa - 4 - \rho) \) curve by that of a \( \text{SLE}_\kappa(0, \rho) \).
- Use Lemma 5.7 to bound the arm event probability of a \( \text{SLE}_\kappa(\kappa - 6 - \rho, \rho) \).

Chaining these inequalities yields the claim. \( \square \)

6. Combining the discrete with the continuum results

The proofs of our main theorems are now standard by combining the tools collected in the previous sections.

Proof of Theorem 1.1. Let us first consider the case where \( \tau \) is alternating, i.e. \( |\tau| = I(\tau) = 2j \).

In this case, we see using Proposition 5.4, Theorem 4.1 and Theorem 3.11 (and mixing) by the same reasoning as in [Wu18a] that

\[
\liminf_{N \to \infty} \mu_{\mathbb{Z}^2}(A_+^\epsilon(\epsilon N, N)) \asymp \limsup_{N \to \infty} \mu_{\mathbb{Z}^2}(A_+^\epsilon(\epsilon N, N)) \asymp \epsilon^{\alpha_2(\rho) + o(1)} \quad \text{as} \quad \epsilon \to 0.
\]

In fact, we note that if the limiting curve appearing in the statement of Theorem 4.1 satisfies the arm event appearing in Proposition 5.4, then the two approximating discrete fuzzy Potts interfaces from Theorem 4.1 will also satisfy the arm event simultaneously for sufficiently small lattice mesh size. In particular, we obtain

\[
\liminf_{N \to \infty} \mu_{\mathbb{Z}^2}(A_+^\epsilon(\epsilon N, N)) \asymp \limsup_{N \to \infty} \mu_{\mathbb{Z}^2}(A_+^\epsilon(\epsilon N, N)) \asymp \epsilon^{\alpha_2(\rho) + o(1)} \quad \text{as} \quad \epsilon \to 0.
\]

The theorem in the case where \( \tau \) is alternating is then a consequence of quasi-multiplicativity (see Theorem 3.8) and follows again as in [Wu18a]. In particular, this shows that the exponent in the case of four alternating arms is \( > 2 \) and so the general case of the theorem is a consequence of Proposition 3.18. \( \square \)

Proof of Theorem 1.2. Again, let us first consider the case where \( \tau \) is alternating and \( I(\tau) = k \).

In this case, using Proposition 5.3, Theorem 4.1 and Theorem 3.16 (and mixing), we get by the same arguments as in [Wu18a] and as before that

\[
\liminf_{N \to \infty} \mu_{\mathbb{Z}^2 \times \mathbb{Z}^+}(A_+^\epsilon(\epsilon N, N)) \asymp \limsup_{N \to \infty} \mu_{\mathbb{Z}^2 \times \mathbb{Z}^+}(A_+^\epsilon(\epsilon N, N)) \\
\asymp \liminf_{N \to \infty} \mu_{\mathbb{Z}^2 \times \mathbb{Z}^+}(A_+^{j\rho}(\epsilon N, N)) \asymp \limsup_{N \to \infty} \mu_{\mathbb{Z}^2 \times \mathbb{Z}^+}(A_+^{j\rho}(\epsilon N, N)) \asymp \epsilon^{\alpha_k(\tau)}
\]

Again, the theorem follows in the case where \( \tau \) is alternating by quasi-multiplicativity (see Theorem 3.15) as in [Wu18a] and the general case can be deduced from Remark 3.19 by making use of Theorem 1.1 to see that the alternating four-arm exponent is \( > 2 \). \( \square \)
Also assume that for each more) by $z$ whenever
interior and that $C$ the proof relies on finding suitable reparametrizations of the curves
Proof.
\[(A.1) \sup_{s \in \partial D} \inf_{t \in \partial D} |\eta_t - \eta^n_t| \to 0 \quad \text{as} \quad n \to \infty.
\]
Also assume that for each $\delta > 0$ there is $\delta' \in (0, \delta)$ with the following property: For all $n \geq 1$, whenever $z \in D$ is such that $B_{2\delta}(z) \subset D$, the annulus $B_\delta(z) \setminus B_{\delta'}(z)$ is not crossed six times (or more) by $\eta^n$. Then $d_C(\eta^n, \eta) \to 0$ as $n \to \infty$.

**Proof.** The proof relies on finding suitable reparametrizations of the curves $\eta^n$. Without loss of generality, all loops are oriented counterclockwise around $z_0$. Consider $\epsilon > 0$ and let $M \geq 1$ be maximal such that there are points $t_0, \ldots, t_{M-1} \in \partial \mathbb{D}$ (ordered counterclockwise) with $|\eta_{t_i} - \eta_{t_{i+1}}| \geq \epsilon$ for all $i < M$ (addition modulo $M$).

Let $D_-$ and $D_+$ be the bounded and unbounded component of $\mathbb{C} \setminus \eta(\partial \mathbb{D})$ respectively. Let $\psi_- : \mathbb{D} \to D_-$ and $\psi_\infty : \mathbb{C} \setminus \mathbb{D} \to D_+$ be conformal transformations with $\psi_- (0) = z_0$. By Carathéodory’s theorem (see e.g. [Con95, Theorem 14.5.6]), $\psi_\pm$ extend continuously to the boundary since $\eta$ is a continuous simple loop. Since $\psi_\pm|_{\partial \mathbb{D}}$ are reparametrizations of $\eta$, for all $i < M$ there exist $t_i^\pm \in \partial \mathbb{D}$ with $\psi_+(t_i^+) = \psi_-(t_i^-) = \eta_{t_i}$. For $r \in (0, 1)$ and $i < M$ we let
\[U_r = \psi_- (\{z \in \mathbb{C} : 1 - r < |z| \leq 1\}) \cup \psi_+ (\{z \in \mathbb{C} : 1 \leq |z| < 1 + r\}),\]
\[I_r^i = \psi_- (t_i^- \cdot (1 - r, 1)] \cup \psi_+ (t_i^+ \cdot [1, 1 + r]),\]
\[S_r^i = I_r^i \cup \psi_- ((t_i, t_{i+1})) \cdot (1 - r, 1]) \cup \psi_+ ((t_i, t_{i+1}) \cdot [1, 1 + r]) \cup I_r^{i+1}.
\]

**Figure 14.** Left. Illustration of the proof of Lemma A.1. Right. Illustration of the proof of Lemma A.2. In the left (resp. right) figure, $\eta$ (resp. $\gamma$) is drawn in red, $U_r$ is shaded in red, $\eta^n$ (resp. $\gamma^n$) is drawn in green and the segments $I^0_1, \ldots, I^{M-1}_1$ are shown in blue. Both figures display the scenario where the green curve has at least three crossings between two blue segments; this results in six crossings of the annulus $B_\delta(\eta^n_t) \setminus B_{\delta'}(\eta^n_t)$ which by assumption does not occur for $\delta' \in (0, \delta)$ sufficiently small.

**Appendix A. A Technical Lemma on the Loop Topology**

In this section we will prove a simple lemma which gives a practical criterion for checking convergence with respect to $d_C$. Roughly speaking, if a curve is simple and is approximated in Hausdorff distance by a sequence of curves which do not oscillate too much, then one can deduce convergence with respect to the metric $d_C$. The scenario that needs to be excluded is that the approximating curves trace macroscopic segments of the limiting loop several times. The proofs are illustrated in Figure 14.

**Lemma A.1.** Let $D$ be a Jordan domain and consider a simple loop $\eta : \partial \mathbb{D} \to \overline{\mathbb{D}}$ and $\eta^n \in C^\ast(\partial \mathbb{D}, \mathbb{C})$ all surrounding a fixed point $z_0 \in D$. Suppose that $Z = \{t \in \partial \mathbb{D} : \eta_t \in \partial D\}$ has empty interior and that
\[(A.1) \sup_{s \in \partial D} \inf_{t \in \partial D} |\eta_t - \eta^n_t| \to 0 \quad \text{as} \quad n \to \infty.
\]

**Proof.**
Note that $U_r$ has the topology of an annulus (which $\eta$ follows in a counterclockwise way) and that $I^i, \ldots, I^{M-1}$ are 'radial segments' within $U_r$ and splitting $U_r$ into $S^0, \ldots, S^{M-1}$. For each $n \geq 1$ with $\eta^n(\partial D) \subseteq U_r$, the curve $\eta^n$ follows the annulus $U_r$ in counterclockwise way and we may take $s^0, \ldots, s^{M-1} \in \partial D$ which are distinct and counterclockwise ordered such that

$$\eta^n_i \in I^i \quad \text{for all } i < M.$$ 

In this case, we pick an orientation-preserving homeomorphism $\phi_n : \partial D \to \partial D$ such that $\phi_n(t_i) = s^i$ for all $i < M$ and our goal will be to bound $\|\eta - \eta^n \circ \phi_n\|_{\infty}$.

Since $Z$ has empty interior, we see (arguing by contradiction) that there is $\epsilon' > 0$ such that for all $u, v \in \partial D$ with $|\eta_u - \eta_v| \geq \epsilon$ there exist $t \in ([u, v])$ such that $|\eta_u - \eta_t|, |\eta_v - \eta_t| \geq \epsilon/4$ and $\text{dist}(\eta_t, \partial D) > \epsilon'$. For any $i < M$, we apply this with $u = t_i$ and $v = t_{i+1}$ to obtain a point $t'_i \in ([t_i, t_{i+1}])$ with the stated property.

Fix any $\delta < (\epsilon'/2) \wedge (\epsilon/8)$ and let $\delta' \in (0, \delta)$ be chosen as in the statement of the lemma. Moreover, take $r \in (0, 1)$ sufficiently small such that

(A.2) \sup_{t \in \partial D} \sup_{c \in (1-r, 1]} |\psi_- (ct) - \psi_- (t)|, \quad \sup_{t \in \partial D} \sup_{c \in [1, 1+r]} |\psi_+ (ct) - \psi_+ (t)| < \delta'.

By (A.1), there exists $N \geq 1$ such that $\eta^n(\partial D) \subseteq U_r$ (and $\eta^n$ follows the annulus $U_r$ in a counterclockwise way) for all $n \geq N$. Fix any such $n \geq N$. Since $\delta' < \delta < \epsilon/8$ we get $\delta < \epsilon/4 - \delta'$ and we also have $2\delta < \epsilon'$. Hence

$$I^i \cap B_{\delta}(\eta_t) = I^{i+1} \cap B_{\delta}(\eta_t) = \emptyset \quad \text{and} \quad B_{2\delta}(\eta_t) \subseteq D$$

by (A.2) and the definition of $t'_i$. Therefore the maximal disjoint subintervals of $\partial D$ where $\eta^n$ (in its counterclockwise parametrization) crosses from $I^i$ to $I^{i+1}$ is 1 since otherwise the annulus $B_{\delta}(\eta_t) \setminus B_{\delta}(\eta_t)$ would be crossed at least six times by $\eta^n$ (see Figure 14). This implies that

$$\eta^n_i \in S^{i-1}_r \cup S^i_r \cup S^{i+1}_r \quad \text{for } s \in ((s^0, s^{i+1})].$$

Indeed, if this did not hold we would cross between two radial segments in counterclockwise direction more than once (again, see Figure 14).

If $t \in ([t_i, t_{i+1}])$ then by maximality of $t_0, \ldots, t_{M-1}$ we deduce that $|\eta_t - \eta_{t_i}| \leq 2\epsilon$. By maximality also $\text{diam}(\eta([t_i, t_{i+1}])) \leq 2\epsilon$ for all $i$ and hence for $s \in ((s^i, s^{i+1})$, $|\eta^n_i - \eta^n_s| \leq \text{diam}(S^0 \cup S^1 \cup S^{i+1}) \leq 3 \cdot 2\epsilon + 2\delta \leq 7\epsilon$.

Clearly $|\eta_{t_i} - \eta^n_{t'_i}| \leq \delta' \leq \epsilon/8 \leq \epsilon$. Combining everything yields $\|\eta - \eta^n \circ \phi_n\|_{\infty} \leq 10\epsilon$. \hfill $\square$

There is an analogous statement in the case of curves which we state here as well. Since the proof is very similar to the one for Lemma A.1 we will only sketch the proof.

**Lemma A.2.** Let $D$ be a Jordan domain. Consider a simple curve $\gamma : [0, 1] \to D$ and $\gamma^n \in C^\infty([0, 1], \mathbb{C})$. We suppose that $Z = \{t \in [0, 1] : \gamma_t \in \partial D\}$ has empty interior. We also assume that

$$\sup_{s \in [0, 1]} \inf_{t \in [0, 1]} |\gamma_t - \gamma^n_s| \to 0 \quad \text{and} \quad (\gamma^n_0, \gamma^n_1) \to (\gamma_0, \gamma_1) \quad \text{as } n \to \infty.$$ 

Again we assume that for each $\delta > 0$ there is $\delta' \in (0, \delta)$ with the following property: For all $n \geq 1$, whenever $z \in D$ is such that $B_{\delta}(z) \subseteq D$, the annulus $B_{\delta}(z) \setminus B_{\delta'}(z)$ is not crossed six times (or more) by $\gamma^n$. Then $d_C(\gamma^n, \gamma) \to 0$ as $n \to \infty$.

**Proof.** Fix $\epsilon > 0$ and let $M \geq 1$ be maximal such that there are points $t_0, \ldots, t_{M-1} \in (0, 1)$ (in increasing order) with $|\gamma_{t_i} - \gamma_{t_{i+1}}| \geq \epsilon$ for all $0 \leq i < M$. Let $\psi : \mathbb{C} \setminus \partial D \to \mathbb{C} \setminus \gamma([0, 1])$ be a conformal transformation. By a version of Carathéodory's theorem [Con95, Theorem 14.5.5] $\psi$ extends continuously to $\partial D$ and we may take $x, y \in \partial D$ such that $\psi(x) = \gamma_0$ and $\psi(y) = \gamma_1$. Let $I^+ = ((x, y))$ and $I^- = ((y, x))$. 

Since $\psi|_{I^\pm}: I^\pm \to \gamma((0,1))$ are homeomorphisms, there exist $t_0^\pm, \ldots, t_{M-1}^\pm \in I^\pm$ such that $\psi(t_i^\pm) = \psi(t_i^+) = \eta_i$ for all $i < M$. For $r \in (0,1)$ and $i < M$ we now let

$$U_r = \psi(\{z \in \mathbb{C} : |z| \in [1,1+r])\),
$$

$$I_r^\pm = \psi(t_i^\pm [1,1+r]) \cup \psi(t_i^\pm [1,1+r]).$$

If $\gamma^n((0,1)) \subset U_r$ then we can take $s_0^n, \ldots, s_{M-1}^n \in (0,1)$ (in increasing order) such that $\gamma^n_{s_i^\pm} \in I_r^\pm$ for all $i < M$ and we let $\phi_n: [0,1] \to [0,1]$ be an increasing homeomorphisms with $\phi_n(t_i) = s_i^n$ for all $i$. The remainder of the argument is virtually identical to the one of Lemma A.1 and is thus omitted.

**Appendix B. Distortion estimates up to the boundary**

In Section 5 we need to carefully consider the distortion of points, balls and semiballs by conformal transformations. The complex analysis input is given by the following lemma. We write $\phi_D: D \to \mathbb{D}$ for the unique conformal transformation satisfying $\phi_D(0) = 0$ and $\phi'_D(0) > 0$ whenever $D \subset \mathbb{D}$ is a simply connected domain containing $0$. By Schwarz reflection, $\phi_D$ continuously extends to the closure of $(\partial D \cap \partial \mathbb{D})^\circ$. We write $D_{(\delta,\delta)}$ for the union of $B_{1-\delta}(0)$ and the collection of all points in $z \in D \setminus B_{1-\delta}(0)$ which can be connected to $\partial B_{1-\delta}(0)$ by a curve which remains in $D \setminus B_{1-\delta}(0)$ and has diameter $\leq \delta_0$.

**Lemma B.1.** For all $\epsilon > 0$ there exists $\delta_0 \in (0,1)$ with the following property: For all $\delta \in (0,\delta_0)$ there exists $\delta' \in (0,\delta)$ such that $|\phi_D(z) - z| \leq \epsilon$ whenever $z \in D_{(\delta,\delta)}$ and $B_{1-\delta'}(0) \subset D \subset \mathbb{D}$.

**Proof.** We first define $\delta_0$. Let $B$ be a planar Brownian motion started from $0$ and write $\tau_K = \inf\{t \geq 0 : B_t \in K\}$ for the hitting time of $K$ by the Brownian motion. Consider $\delta \in (0,\delta_0)$, $z \in D_{(\delta,\delta_0)} \setminus B_{1-\delta}(0)$ and let $\gamma: [0,1] \to D \setminus B_{1-\delta}(0)$ be a curve from a point $z_0 \in \partial B_{1-\delta}(0)$ to $z$ with diameter $\leq \delta_0$. Then by conformal invariance of $B$ (for the first equality) and inclusion estimates, we get

$$P(\tau_{\phi_D(\gamma([0,1]))} < \tau_{\partial D}) = \mathbb{P}(\tau_{\phi(\gamma([0,1]))} < \tau_{\partial D}) \leq \mathbb{P}(\tau_{B_{\delta_0}(0) + [1-\delta_0,1]} < \tau_{\partial D}) =: p(\delta_0) \to 0 \quad \text{as} \quad \delta_0 \to 0.$$  

Note that $\phi_D \circ \gamma$ is a curve from $\phi_D(z_0)$ to $\phi_D(z)$. Let

$$q(\epsilon) = \inf \gamma^\prime P(\tau_{\gamma([0,1])} < \tau_{\partial D})$$

where the infimum is taken over all curves $\gamma': [0,1] \to \mathbb{D}$ such that $|\gamma' - \gamma'| \geq \epsilon/3$. One can check that $q(\epsilon) > 0$. Take $\delta_0$ sufficiently small such that $p(\delta_0) < q(\epsilon)$. Then by the definition of $q(\epsilon)$ we deduce that $|\phi_D(z_0) - \phi_D(z)| = |\phi_D(\gamma_0) - \phi_D(\gamma_1)| < \epsilon/3$. By further decreasing $\delta_0$ we can ensure that $\delta_0 < \epsilon/3$ and $\delta_0 < 1$.

By the Carathéodory kernel convergence theorem, given $\delta \in (0,\delta_0)$ there now exists $\delta' \in (0,\delta)$ such that $|\phi_D(z) - z| \leq \epsilon/3$ for all $z \in B_{1-\delta'}(0)$ whenever $B_{1-\delta'}(0) \subset D \subset \mathbb{D}$. The Carathéodory kernel convergence theorem is usually stated in sequential form [Con95, Theorem 15.4.10] but the version in the previous sentence follows by arguing via contradiction.

To conclude, let $z \in D_{(\delta,\delta_0)}$. If $z \in B_{1-\delta}(0)$ then the claim is immediate. Otherwise, take $z_0 \in \partial B_{1-\delta}(0)$ as in the first part of the proof and observe that

$$|\phi_D(z) - z| \leq |\phi_D(z_0) - z_0| + |z_0 - z| + |\phi_D(z) - \phi_D(z_0)| \leq \epsilon/3 + \delta_0 + \epsilon/3 \leq \epsilon$$

as required. \qed

We will now make use of the following distortion estimate (see [Con95, Theorem 14.7.9]). Let $\phi: B_{r_0}(z_0) \to \mathbb{C}$ be holomorphic and injective, then

$$\frac{|\phi'(z_0)| \cdot |z - z_0|}{(1 + |z - z_0|/r_0)^2} \leq |\phi(z) - \phi(z_0)| \leq \frac{|\phi'(z_0)| \cdot |z - z_0|}{(1 - |z - z_0|/r_0)^2}$$

It follows that for \( r \in (0, r_0) \) we have
\[
B_{|\psi'(z_0)|r/(1+r/r_0)^2}(\phi(z_0)) \subset \phi(B_r(z_0)) \subset B_{|\psi'(z_0)|r/(1-r/r_0)^2}(\phi(z_0)).
\]

The following two lemmas apply this result and explain how small balls around boundary and interior points are distorted by conformal transformations. We begin with the statement and proof in the boundary case.

**Lemma B.2.** Assume that \( \psi_D : D \rightarrow \mathbb{D} \) is a conformal transformation and that \( B_{r_0}(x) \cap \mathbb{D} \subset D \) for \( x \in \partial \mathbb{D} \). Fix \( \delta \in (0,1) \) and suppose that \( |\psi_D(z)−z| \leq \delta^2r_0/2 \) for all \( z \in B_{r_0}(x) \cap \mathbb{D} \). Then for all \( r' \leq r_0 \delta \),
\[
B_{r'(1-\delta)^3}(\psi_D(x)) \cap \mathbb{D} \subset \psi_D(B_{r'}(x) \cap \mathbb{D}) \subset B_{r'(1-\delta)^3}(\psi_D(x)) \cap \mathbb{D}.
\]

**Proof.** Let \( r = r_0 \delta < r_0 \) and \( \epsilon = r \delta/2 = \delta^2 r_0 / 2 \). We consider the Schwarz reflection of \( \psi_D \) across the unit circle. Formally, we extend \( \psi_D \) to a conformal transformation on a larger domain by
\[
\psi_D(z) = 1/\overline{\psi_D(1/z)} \quad \text{for} \quad z \in D := \mathbb{D} \cup I \cup \{1/\overline{z} : z \in \mathbb{D}\}
\]
where \( I \) is the connected component of \( x \) in \((\partial \mathbb{D} \cap \partial \mathbb{D})^0\). Our assumptions now imply that \( B_{r-2\epsilon}(\psi_D(x) \cap \mathbb{D}) \subset \psi_D(B_r(x) \cap \mathbb{D}) \subset B_{r+2\epsilon}(\psi_D(x)) \cap \mathbb{D} \). This statement extends via Schwarz reflection to \( B_{r-2\epsilon}(\psi_D(x)) \subset \psi_D(B_r(x)) \subset B_{r+2\epsilon}(\psi_D(x)) \). Note that \( x \in D' \) is an interior point of the extended domain. By the distortion estimate therefore
\[
\frac{|\psi_D'(x)|r}{(1+r/r_0)^2} \leq r + 2\epsilon \quad \text{and} \quad r - 2\epsilon \leq \frac{|\psi_D'(x)|r}{(1-r/r_0)^2}.
\]
Thus \((1-\delta)^3 \leq |\psi_D'(x)| \leq (1+\delta)^3 \). By the distortion theorem, we obtain for \( r' \leq r_0 \) that
\[
B_{r(1-\delta)^3}(\psi_D(x)) \subset \psi_D(B_{r'}(x)) \subset B_{r(1+\delta)^3}(\psi_D(x)).
\]
The result follows. \( \square \)

Finally, we also state the result in the case of an interior point. We omit the proof since it is identical to the one in the boundary case except that one does not need to define the Schwarz reflection in this case.

**Lemma B.3.** Suppose that \( D \) is a simply connected domain with \( B_{r_0}(z) \subset D \). Let \( \psi_D : D \rightarrow \mathbb{D} \) be a conformal transformation, fix \( \delta \in (0,1) \) and suppose that \( |\psi_D(w)−w| \leq \delta^2r_0/2 \) for \( w \in B_{r_0}(z) \). Then for all \( r' \leq r_0 \delta \), we have \( B_{r'(1-\delta)^3}(\psi_D(z)) \subset \psi_D(B_{r'}(z)) \subset B_{r'(1+\delta)^3}(\psi_D(z)) \).

**References**

[BBT13a] A. Bálint, V. Beffara, and V. Tassion. Confidence intervals for the critical value in the divide and color model. *ALEA Lat. Am. J. Probab. Math. Stat.*, 10(2):667–679, 2013, 1307.2575. MR3104913

[BBT13b] A. Bálint, V. Beffara, and V. Tassion. On the critical value function in the divide and color model. *ALEA Lat. Am. J. Probab. Math. Stat.*, 10(2):653–666, 2013, 1109.3403. MR3104912

[BCM09] A. Bálint, F. Camia, and R. Meester. Sharp phase transition and critical behaviour in 2D divide and colour models. *Stochastic Process. Appl.*, 119(3):937–965, 2009, 0708.3349. MR2499865

[BDC12] V. Beffara and H. Duminil-Copin. The self-dual point of the two-dimensional random-cluster model is critical for \( q \geq 1 \). *Probab. Theory Related Fields*, 153(3-4):511–542, 2012, 1006.5073. MR2948685

[Bef08] V. Beffara. The dimension of the SLE curves. *Ann. Probab.*, 36(4):1421–1452, 2008, math/0211322. MR2435854

[BH19] S. Benoist and C. Hongler. The scaling limit of critical Ising interfaces is CLE$_4$. *Ann. Probab.*, 47(4):2049–2086, 2019, 1604.06975. MR3980915

[CDCH'*14] D. Chelkak, H. Duminil-Copin, C. Hongler, A. Kemppainen, and S. Smirnov. Convergence of Ising interfaces to Schramm’s SLE curves. *C. R. Math. Acad. Sci. Paris*, 352(2):157–161, 2014, 1312.0533. MR3151886

[CDCH16] D. Chelkak, H. Duminil-Copin, and C. Hongler. Crossing probabilities in topological rectangles for the critical planar FK-Ising model. *Electron. J. Probab.*, 21:Paper No. 5, 28, 2016, 1312.7785. MR3485347

[CLM07] L. Chayes, J. L. Lebowitz, and V. Marinov. Percolation phenomena in low and high density systems. *J. Stat. Phys.*, 129(3):567–585, 2007. MR2351418
G. F. Lawler, O. Schramm, and W. Werner. Values of Brownian intersection exponents. I. Half-plane

J. Kahn and N. Weininger. Positive association in the fractional fuzzy Potts model.

L. Köhler-Schindler and V. Tassion. Crossing probabilities for planar percolation.

A. Kemppainen and S. Smirnov. Conformal invariance of boundary touching loops of FK Ising model.

A. Kemppainen and S. Smirnov. Conformal invariance in random cluster models. II. Full scaling limit

O. Häggström. Coloring percolation clusters at random.

G. Grimmett.

C. M. Fortuin and P. W. Kasteleyn. On the random-cluster model. I. Introduction and relation to

B. Duplantier, J. Miller, and S. Sheffield. Liouville quantum gravity as a mating of trees.

H. Duminil-Copin and I. Manolescu. Planar random-cluster model: fractal properties of random-cluster and Potts models with $q \geq 1$ are sharp. Probab. Theory Related Fields, 164(3-4):865–892, 2016, 1409.3748. MR3477782

H. Duminil-Copin, I. Manolescu, and V. Tassion. Planar random-cluster model: fractal properties of the critical phase. Probab. Theory Related Fields, 181(1-3):401–449, 2021, 2007.14707. MR4341078

H. Duminil-Copin, A. Raoufi, and V. Tassion. A new computation of the critical point for the planar random-cluster model with $q \geq 1$. Ann. Inst. Henri Poincaré Probab. Stat., 54(1):422–436, 2018, 1604.03702. MR3765895

H. Duminil-Copin, A. Raoufi, and V. Tassion. Sharp phase transition for the random-cluster and Potts models via decision trees. Ann. of Math. (2), 189(1):75–99, 2019, 1705.03104. MR3898174

H. Duminil-Copin, V. Sidoravicius, and V. Tassion. Continuity of the phase transition for planar random-cluster and Potts models with $1 \leq q \leq 4$. Comm. Math. Phys., 349(1):47–107, 2017, 1505.04159. MR3592746

H. Duminil-Copin and V. Tassion. Renormalization of crossing probabilities in the planar random-cluster model. Mosc. Math. J., 20(4):711–740, 2020, 1901.08294. MR4203056

H. Duminil-Copin and I. Manolescu. Planar random-cluster model: scaling relations. arXiv eprints, November 2020, 2011.15090.

B. Duplantier, J. Miller, and S. Sheffield. Liouville quantum gravity as a mating of trees. Astérisque, 427:vi+257, 2021, 1409.7055. MR4340069

H. Duminil-Copin, I. Manolescu, and V. Tassion. Near critical scaling relations for planar Bernoulli percolation without differential inequalities. arXiv eprints, November 2021, 2111.14414.

C. M. Fortuin and P. W. Kasteleyn. On the random-cluster model. I. Introduction and relation to other models. Physica, 57:536–564, 1972. MR359055

G. Grimmett. The random-cluster model, volume 333 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, 2006. MR2243476

C. Garban and H. Wu. On the convergence of FK-Ising percolation to SLE(16/3, (16/3) − 6). J. Theoret. Probab., 33(2):828–865, 2020, 1802.03939. MR4001584

O. Häggström. Positive correlations in the fuzzy Potts model. Ann. Appl. Probab., 9(4):1149–1159, 1999. MR1728557

O. Häggström. Coloring percolation clusters at random. Stochastic Process. Appl., 96(2):213–242, 2001. MR1865356

H. Kesten. A scaling relation at criticality for 2D-percolation. In Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), volume 8 of IMA Vol. Math. Appl., pages 203–212. Springer, New York, 1987. MR8994549

A. Kemppainen and S. Smirnov. Conformal invariance in random cluster models. II. Full scaling limit as a branching SLE. arXiv eprints, September 2016, 1609.08527.

A. Kemppainen and S. Smirnov. Conformal invariance of boundary touching loops of FK Ising model. Comm. Math. Phys., 369(1):49–98, 2019, 1509.08858. MR3959554

L. Köhler-Schindler and V. Tassion. Crossing probabilities for planar percolation. arXiv eprints, November 2020, 2011.04618.

J. Kahn and N. Weininger. Positive association in the fractional fuzzy Potts model. Ann. Probab., 35(6):2038–2043, 2007, 0711.3136. MR2363381

G. F. Lawler. Conformally invariant processes in the plane, volume 114 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005. MR2129588

M. Lehmkuehler. The trunks of CLE(4) explorations. arXiv eprints, July 2021, 2107.05310.

G. F. Lawler, O. Schramm, and W. Werner. Values of Brownian intersection exponents. I. Half-plane exponents. Acta Math., 187(2):237–273, 2001, math/0003156. MR1879850

G. F. Lawler, O. Schramm, and W. Werner. Values of Brownian intersection exponents. II. Plane exponents. Acta Math., 187(2):275–308, 2001, math/0011084. MR1879851

G. F. Lawler, O. Schramm, and W. Werner. One-arm exponent for critical 2D percolation. Electron. J. Probab., 7:no. 2, 13, 2002, math/0108211. MR1887622
