Quasi-Exact Solvability in Local Field Theory

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Abstract

The quantum mechanical concept of quasi-exact solvability is based on the idea of
partial algebraizability of spectral problem. This concept is not directly extendable
to the systems with infinite number of degrees of freedom. For such systems a new
concept based on the partial Bethe Ansatz solvability is proposed. In present paper
we demonstrate the constructivity of this concept and formulate a simple method for
building quasi-exactly solvable field theoretical models on a one-dimensional lattice.
The method automatically leads to local models described by hermitian hamiltonians.

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1 Introduction

Quasi-exactly solvable quantum models are distinguished by the fact that they can be solved exactly only for some limited parts of the spectrum but not for the whole spectrum. The initial motivation for looking for such models was associated with a natural desire of physicists and mathematicians to extend the set of exactly solvable Schrödinger type equations by relaxing the usual requirements to exact solvability. It was quite obvious that the problem of finding models with only a few number of exactly constructable states should be simplier than the problem of finding new models admitting exact solution for all the spectrum. This led to a natural conclusion that the set of quasi-exactly solvable models should be wider than that of exactly solvable ones. The first non-trivial examples of quasi-exactly solvable models in non-relativistic quantum mechanics constructed in the middle of eighteenes (see e.g. reviews [1, 2, 3] and book [4]) clearly demonstrated that this conclusion is true. Remember, for example, that in the class of one-dimensional models with polynomial potentials there is only one exactly solvable model — the simple harmonic oscillator. At the same time, the number of quasi-exactly solvable models belonging to this class is infinitely large [4].

Of course, it would be extremely important to try to realize the idea of quasi-exact solvability in field theory or in other systems with an infinite number of degrees of freedom. It is however clear that the quantum mechanical concept of quasi-exact solvability based on the idea of a partial algebraizability of spectral problem (see e.g. [1]) is not directly extendable to the field theoretical case\(^3\). The main difficulty lies in the fact that the physically interesting examples of quasi-exactly solvable field theoretical models should necessarily contain an infinite number of explicitly constructable states, forming, for example, some branches of excitations. However, the construction of such branches cannot, generally, be qualified as an algebraic problem. This means that the first thing which we need is to elaborate a constructive concept of quasi-exact solvability for systems with infinitely many degrees of freedom. The following reasonings show how to do this.

First note that the equations of mathematical physics can conventionally be divided into three large classes distinguished from each other by the level of their complexity. These are:

1) the equations for a finite set of numbers (level 1)
2) the equations for an infinite set of numbers or finite set of functions (level 2),
3) the equations for an infinite set of functions or finite set of functionals (level 3).

In principle, this list can be continued further. Sometimes it is possible to reduce a problem of the \(n\)th level of complexity to that (or those) of the \((n - 1)\)th level. The problems admitting such a reduction are usually called exactly solvable. Consider two particular cases of this general “definition”.

1. We usually call a quantum mechanical model exactly solvable if each its energy level and corresponding wavefunction admits a purely algebraic construction. This does not contradict the general ”definition” since the Schrödinger equation in non-relativistic

\(^3\)Remember that a quantum problem is called partially algebraizable if its hamiltonian admits an explicitly constructable finite-dimensional invariant subspace of Hilbert space. Then a certain finite part of its spectrum can be constructed algebraically.
quantum mechanics is an equation for a function (wavefunction) and thus the standard level of its complexity is two. At the same time, the level of complexity of all algebraic problems is one.

2. We usually call a field theoretical model exactly solvable if the problem of construction of all its energy levels and corresponding wave functionals is reduced to the problem of solving the so-called Bethe ansatz equations (as a most recent review see e.g. [6]). This is again in full accordance with the general "definition" since the Schrödinger equation in field theory is an equation for a functional (wave functional) and therefore the level of its complexity is three. At the same time, the Bethe ansatz equations have usually the form of integral equations and therefore belong the second level of complexity.

So, we see that in quantum mechanics and field theory the term "exact solvability" has different meaning. In quantum mechanics it means the algebraic solvability, while in quantum field theory it means the Bethe ansatz solvability.

Let us now remember that the quasi-exact solvability is nothing else than a restricted version of the exact solvability. Restricted — only with respect to the number of exactly computable states. As to the level of simplification of the initial problem, it (intuitively) is expected to be the same for both exactly and quasi-exactly solvable systems. At least this is so in the case of non-relativistic quantum mechanics: the non-relativistic quasi-exactly solvable models are defined as models admitting an incomplete but still algebraic solution of spectral problem.

Taking these reasonings into account, we can propose the following extension of the notion of quasi-exact solvability to the field theoretical case. We shall call a field theoretical model quasi-exactly solvable if it admits the Bethe ansatz solution only for a certain limited part of the spectrum but not for the whole spectrum.

In the present paper we demonstrate the constructivity of this definition and present the simplest examples of local quasi-exactly solvable models of field theory on a one-dimensional lattice\footnote{Note that the Bethe ansatz approach to the problem of quasi-exact solvability first appeared many years ago in papers \cite{8}, where it was successfully used for building and solving the quasi-exactly solvable problems of one- and multi-dimensional quantum mechanics. For further development of this approach see refs. \cite{2, 4, 9}.}. Remember that the locality is a fundamental property of physically reasonable field theories. Couriously enough, namely this property turns out to be essential for constructing quasi-exactly solvable field theoretical models.

2 The basic idea

Consider a class of quantum spin models defined on a one-dimensional lattice and described by the hamiltonians of the following general form:

\[ H(s_1, \ldots, s_N) = \sum_{n,m=1}^{N} C_{nm} \vec{S}_n(s_n) \vec{S}_m(s_m). \]  

(2.1)

Here \( \vec{S}_n(s_n) = \{S_{n1}(s_n), S_{n2}(s_n), S_{n3}(s_n)\} \) denote the generators of \( su(2) \) algebra associated with \( n \)th site of the lattice and realizing the \((2s_n+1)\)-dimensional irreducible representation with spin \( s_n \). The corresponding representation space we denote by \( W_n(s_n) \).
The matrix $C_{nm}$ is assumed to be real and symmetric, $C_{nm} = C_{mn}$. The hamiltonian $H(s_1, \ldots, s_N)$ is thus hermitian and acts in a $\prod_{n=1}^{N}(2s_n + 1)$-dimensional space $W(s_1, \ldots, s_N) = \bigotimes_{n=1}^{N} W_n(s_n)$. The number $N$ (the length of the lattice) is assumed to be a free parameter which can be made arbitrarily large.

The set of parameters $C_{nm}$, $n, m = 1, \ldots, N$ and $s_n$, $n = 1, \ldots, N$ completely determines the model. For any values of these parameters the problem of solving the Schrödinger equation for $H(s_1, \ldots, s_N)$ in $W(s_1, \ldots, s_N)$ is equivalent to the problem of the diagonalization of a $\prod_{n=1}^{N}(2s_n + 1)$-dimensional matrix. Generally, this can be done only by the help of computer (of course if $N$ is not astronomically large). There are, however, two special cases when such a diagonalization can be performed analytically in the limit $N \to \infty$. Below we consider these cases separately.

**The Gaudin magnet.** Let the coefficients $C_{nm}$ be given by the formula

$$C_{nm} = \int \frac{\rho(\lambda)}{(\lambda - a_n)(\lambda - a_m)} d\lambda$$

(2.2)

in which $\rho(\lambda)$ is an arbitrary function and $a_n$, $n = 1, \ldots, N$ are arbitrary real numbers. Such models (which are known under the generic name of Gaudin models [5]) are obviously non-local, because each spin interacts with each other. It is known that for arbitrary collection of spins $s_n$, $n = 1, \ldots, N$ the model is completely integrable and can be solved exactly by means of the Bethe ansatz [5].

**The Heisenberg magnet.** Let now the coefficients $C_{nm}$ have the form

$$C_{nm} = 2J \delta_{n,m+1}, \quad n = 1, \ldots, N - 1, \quad C_{Nm} = 2J \delta_{1,m}.$$  

(2.3)

In this case the model (which is called the Heisenberg magnet) becomes local, because each spin interacts only with the nearest neighbours. It turns out that the condition of locality of the model drastically changes its integrability properties. The Heisenberg magnet is known to be integrable and solvable within Bethe ansatz only if $s_n = 1/2$, $n = 1, \ldots, N$ [6].

Let us now associate with hamiltonians (2.1) a new model, which is formally described by the same formula,

$$H = \sum_{n,m=1}^{N} C_{nm} \vec{S}_n \vec{S}_m$$

(2.4)

but in which the spin operators $\vec{S}_n$ have different meaning. Now $\vec{S}_n = \{S_{n1}, S_{n2}, S_{n3}\}$ will denote spin operators realizing a certain completely reducible representation of algebra $su(2)$. Let $\sigma$ denote the set of spins characterizing the irreducible components of this representation. This set may be finite or infinite. The corresponding representation space associated with the $n$th site of the lattice we denote by $W_n$. For each $n$ we can write $W_n = \bigoplus_{s_n \in \sigma} W_n(s_n)$ where $W_n(s_n)$ denote the irreducible components of $W_n$. Thus the Hilbert space $W = \bigotimes_{n=1}^{N} W_n$ in which the operator $H$ acts can be represented as the direct sum

$$W = \bigoplus_{s_1 \in \sigma, \ldots, s_N \in \sigma} W(s_1, \ldots, s_N).$$

(2.5)
Now note that each of the spaces \( W(s_1, \ldots, s_N) \) is a common invariant subspace for the operators of the total spin \( \vec{S}_n^2, \ n = 1, \ldots, N \), which, obviously, commute with each other. All these operators commute also with the hamiltonian \( H \). But this means that the spaces \( W(s_1, \ldots, s_N) \) are simultaneously the invariant subspaces for \( H \). Therefore, the spectral problem for \( H \) in \( W \) decomposes into an infinite number of independent spectral problems of the type (2.1). After this remark let us return to cases 1 and 2 and consider them again from the point of view of model (2.4).

### The non-local case.

Let the coefficients \( C_{nm} \) in model (2.4) be given by the formula (2.2). Then the hamiltonian \( H \) takes a block-diagonal form with respect to the decomposition (2.5). The spectral problem for each given block \( H(s_1, \ldots, s_N) \), \( s_1, \ldots, s_N \in \sigma \) is the Gaudin problem. We know that it can be solved exactly (by means of Bethe ansatz) for any values of spins \( s_1, \ldots, s_N \). But this means that the problem of constructing the entire spectrum of the model (2.4) is exactly solvable.

### The local case.

Let now the coefficients \( C_{nm} \) in model (2.4) be given by the formula (2.3). As before, the hamiltonian \( H \) takes a block diagonal form with respect to the decomposition (2.3). But now not any block \( H(s_1, \ldots, s_N) \), \( s_1, \ldots, s_N \in \sigma \) can be explicitly diagonalized by means of Bethe ansatz. If \( 0 \not\in \sigma \), then only the Heisenberg blocks \( H(1/2, \ldots, 1/2) \) admit such a diagonalization. Now it becomes clear that the solvability properties of model (2.4) are completely determined by the structure of the set \( \sigma \).

Assuming that \( 0 \not\in \sigma \), consider the following three cases.

a) The spin \( s = 1/2 \) does not belong to the set \( \sigma \). In this case the hamiltonian \( H \) does not contain the exactly solvable blocks and the model (2.4) is exactly non-solvable.

b) The set \( \sigma \) consists only of the spins \( s = 1/2 \). In this case the hamiltonian \( H \) contains only the exactly solvable Heisenberg blocks and the model (2.4) is exactly solvable.

c) The set \( \sigma \) contains the spin \( s = 1/2 \) and at least one differing spin \( s \neq 1/2 \). In this case only a part of hamiltonian blocks admit explicit diagonalization so that we deal with a typical case of quasi-exactly solvable model!

Summarizing, one can claim that the model with hamiltonian

\[
H = 2J \sum_{n=1}^{N} \vec{S}_n \vec{S}_{n+1} \quad (2.6)
\]

is quasi-exactly solvable, provided that the representation in which the spin operators \( \vec{S}_n \) act is completely reducible and contains at least one representation of spin \( s = 1/2 \) and at least one representation of other spin \( s \neq 1/2 \). Below we shall refer this condition to as condition of 1/2-reducibility.

Consider the simplest example. Let for any \( n \) \( W_n \) be a direct product of two representation spaces of irreducible representations with spins \( s = 2 \) and \( s = 3/2 \): \( W_n = W_n(2) \otimes W_n(3/2) \). This means that the hamiltonian (2.6) can be represented in the form

\[
H = 2J \sum_{n=1}^{N} (\vec{S}_n(2) + \vec{S}_n(3/2)) (\vec{S}_{n+1}(2) + \vec{S}_{n+1}(3/2)) \quad (2.7)
\]

The case when \( 0 \in \sigma \) is a little bit richer. We consider it separately in section 4.
where $S_n(2)$ and $S_n(3/2)$ denote the corresponding spin operators. For any $n$ the representation $W_n$ is completely reducible and its decomposition in irreducible components reads: $W_n = W_n(1/2) \oplus W_n(3/2) \oplus W_n(5/2) \oplus W_n(7/2)$. We see that it satisfies the condition of $1/2$-reducibility and therefore the model (2.7) is quasi-exactly solvable.

Before completing this section let us reduce the model (2.6) to a little more convenient form. Taking into account the fact that the hamiltonian $H$ commutes with Casimir invariants $\vec{S}_n^2$ of algebra $su(2)$ (whose spectrum in each block is trivial), we can conclude that the modified hamiltonians $H = H + \sum_{n=1}^{M} b_n S_n^2$ with arbitrary coefficients $b_n$, will be quasi-exactly solvable, as well. In particular, taking $b_n = 2|J|$, $n = 1, \ldots, N$ we obtain a new hamiltonian

$$H = |J| \sum_{n=1}^{N} (\vec{S}_n^2 \pm \vec{S}_{n+1}^2)^2 \quad (2.8)$$

which is bounded from below and in which $\pm \equiv \text{Sign } J$. We see that sign $'-'$ corresponds to a ferromagnet case $J > 0$ and the sign $'+'$ describes the anti-ferromagnet case $J < 0$.

### 3 The QES families

It is known that the quasi-exactly solvable models of non-relativistic quantum mechanics usually appear in the form of infinite sequences of models which look more or less similarly but differ from each other by the number of exactly computable states. This number, which is usually called the order of a quasi-exactly solvable model, is governed by the set of discrete parameters in the potential. For example, the potentials of simplest sequence of quasi-exactly solvable sextic anharmonic oscillator models are parametrized by a semi-integer parameter $s$ and read $V(x) = x^6 - (8s + 3)x^2$. For any given $s = 0, 1/2, 1, 3/2, \ldots$ the model admits $2s + 1$ algebraically constructable solutions. It is naturally to call this phenomenon the phenomenon of quantization of potential.

Now it is naturally to ask, if there is any analogue of the quantization of potential in the field theoretical case? Or, more concretely, if the model we constructed in the previous section is a member of a certain infinite sequence of similar models?

The answer to this question is positive. In order to show this it is sufficient to remember the well known result in the theory of completely integrable quantum systems about the generalization of completely integrable Heisenberg chain to the case of arbitrary spin. This result is highly non-trivial because the naive substitution of $1/2$ spin operators $S_n(1/2)$ by the arbitrary spin operators $S_n(s_n)$ does lead to integrable model. The correct generalization includes the change of potential describing the interaction of neighbouring spins. The form of the generalized hamiltonian is

$$H_{P}(s_1, \ldots, s_N) = J \sum_{n,m=1}^{N} P_{2s}[\vec{S}_n(s) \vec{S}_{n+1}(s)] \quad (3.1)$$

where $\vec{S}_n(s)$ are the operators of spin $s$ and $P_{2s}[t]$ is a very specific polynomial of degree $2s$ defined by the formula

$$P_{2s}[t] = 2 \sum_{i=0}^{2s} \left( \sum_{j=i+1}^{2s} \frac{1}{j} \right) \prod_{k=0}^{2s} \frac{t - t_k}{t_i - t_k} \quad (3.2)$$

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with
\[ t_k = \frac{1}{2} k(k + 1) - l(l + 1). \]  

(3.3)

It turns out that if the form of polynomial \( P_2s[t] \) differs from that given by formula (3.2) then the spin system (3.1) becomes non-integrable and exactly non-solvable.

Repeating the reasonings of section 2, let us now consider the hamiltonian

\[ H_P = J \sum_{n=1}^{N} P_{2s} \langle \tilde{S}_n \tilde{S}_{n+1} \rangle \]  

(3.4)

in which, as before, \( \tilde{S}_n \) denote the spin operators acting in a certain completely reducible representation of algebra \( su(2) \). In full analogy with the previous case the model (3.4) is quasi-exactly solvable if this representation contains at least one irreducible representation of spin \( s \) and at least one representation of spin \( s' \neq s \). Such representations we shall call \( s \)-reducible.

Assume now that the completely reducible representation in which the operators \( \tilde{S}_n \) act contains all irreducible representations of spins \( s_n = 0, 1/2, 1, 3/2, \ldots \). There are many ways for building such a representation. One of the simples ways is to write

\[
S_{n1} = \frac{1}{2} (a_n^+ b_n + b_n^+ a_n), \\
S_{n2} = \frac{1}{2i} (a_n^+ b_n - b_n^+ a_n), \\
S_{n3} = \frac{1}{2} (a_n^+ a_n - b_n^+ b_n),
\]  

(3.5)

where \( a_n, a_n^+ \), and \( b_n, b_n^+ \) denote two independent groups of hermite conjugated annihilation and creation bosonic operators obeying the Heisenberg commutation relations \([a_n, a_n^+] = 1\) and \([b_n, b_n^+] = 1\) (the commutators between the \( a \)- and \( b \)-operators and also between the operators associated with different sites are zero). Indeed, it is obvious that the operators defined by formulas (3.5) are hermitian and obey the standard commutation relations of \( su(2) \) algebra. Moreover, if we denote by \( |M_{a_n}, M_{b_n}\rangle \) the states with given numbers of \( a \)-bosonic and \( b \)-bosonic quants, then the set of all such states with \( M_{a_n} + M_{b_n} = 2s_n \) will form the basis of the \((2s_n + 1)\)-dimensional irreducible representation of algebra \( su(2) \) with spin \( S_n \). This means that the representation defined by formulas (3.5) is completely reducible and contains all finite-dimensional irreducible representations of algebra \( su(2) \) with multiplicity 1. This means that for any given \( s \) this representation is \( s \)-reducible, which, in turn, implies the quasi-exact solvability of model (3.4) for any given \( s \).

The quasi-exactly solvable models constructed above describe the interaction of two bosonic fields on a lattice. So we see that these models form an infinite sequence. The role of a semi-integer parameter \( s \) quantizing the potentials of quasi-exactly solvable sextic anharmonic oscillator is now played by the function \( P_{2s}[t] \). This is quite natural because for field theoretical problems (having the third level of complexity) the functions play the same role as numbers for quantum mechanical problems (whose level of complexity is two).

It is not difficult to construct for any \( s \) the analogs of modified hamiltonians (2.8). To do this, let us introduce the new polynomials \( Q_{2s}[t] \) related to the old ones, \( P_{2s}[t]\), by the
formula:

\[ Q_{2s}[2s(s + 1) ± 2t] = P_{2s}[t] \] (3.6)

Repeating the reasonings of section 2 we can easily conclude that the models with hamiltonians

\[ \mathcal{H}_Q = \sum_n^N Q_{2s}[(\vec{S}_n ± \vec{S}_{n+1})^2] \] (3.7)

will be quasi-exactly solvable if the model (3.4) is.

It is worth stressing that for integer spins the polynomial \( Q_{2s}[t] \) is even and, according to formulas (3.2), (3.3) and (3.6) has positive coefficient at the leading term. Therefore, for \( J > 0 \) the spectrum of such models is bounded from below irrespective of the sign ‘±’. If the spin is semi-integer, then the leading term of polynomial \( Q_{2s}[t] \) is positive for sign ‘+’ and negative for sign ‘−’. Therefore, the spectrum of the model will be bounded from below if \( \text{Sign } J = \pm \).

## 4 Possible generalizations

The method discussed above admits many natural generalizations. We divide them into three groups. The first group concerns the form of the initial hamiltonian. One can consider the following important subcases:

1. **The anisotropic case.** In this paper we considered only the isotropic XXX-magnets invariant under global \( su(2) \)-rotations. However, all the reasonings given above remain valid for anisotropic XXY magnets and their generalizations for higher spins constructed in ref. 7.

2. **The case of higher integrals of motion.** The XXX and XXY magnets admit an infinite set of integrals of motion. These integrals can be considered as hamiltonians of local spin chains with different number of interacting spins. All the reasonings of the present paper can be repeated for these hamiltonians.

3. **The case of higher Lie algebras.** Instead of local \( su(2) \) magnets one can consider their generalizations for arbitrary simple Lie algebras.

The second group of generalizations concerns the realization of completely reducible unitary representations of Lie algebras. The two examples of such a realization considered in sections 2 and 3 do not exhaust all the existing possibilities whose variety is very large. Below we consider some of them restricting ourselves to the simplest \( su(2) \) case.

1. **The spin realization.** In this realization the extended spin operators have the form

\[ \vec{S}_n = \vec{S}_n(s^{(1)}) + \ldots + \vec{S}_n(s^{(K)}) \] (4.1)

where \( s^{(1)}, \ldots, s^{(K)} \) are arbitrary spins. This representation is completely reducible and contains all representations with spins lying (with spacing 1) between \( s_{\text{min}} = \min |s^{(1)} ± \ldots ± s^{(K)}| \) and \( s_{\text{max}} = \max |s^{(1)} ± \ldots ± s^{(K)}| \). The resulting quasi-exactly solvable models form a finite family with \( s_{\text{max}} - s_{\text{min}} + 1 \) members. In formula (4.1) the extension is
assumed to be homogeneous, i.e., the spins $s^{(1)}, \ldots, s^{(K)}$ are assumed to be independent on $n$. However, equally well we could consider the inhomogeneous case.

2. The bosonic realizations. There are several ways of expressing the generators of $su(2)$ algebra via the generators of Heisenberg algebra. One of such ways we considered in section 3 (see formula (3.5)). Another way can be based on the following realization of spin operators

\[
S_{n1} = p_{n2}q_{n3} - p_{n3}q_{n2}, \\
S_{n2} = p_{n3}q_{n1} - p_{n1}q_{n3}, \\
S_{n3} = p_{n1}q_{n2} - p_{n2}q_{n1}. \tag{4.2}
\]

Here $\vec{q}_n$ are the components of a certain (real) bosonic vector field and $\vec{p}_n$ denote the components of the corresponding (real) generalized momentum. These components satisfy the Heisenberg commutation relations $[p_{ni}, q_{njk}] = i\delta_{nm}\delta_{ik}$. The representation in which the operators (4.2) act is infinite-dimensional and completely reducible, but, in contrast with the representation defined by formula (3.5), it does not contain any irreducible components with half integer values of $s$. As before, the realization (4.2) leads to an infinite sequence of models of the type (3.7), but now these models are quasi-exactly solvable only for integer values of $s$. As to the models with half integer values of $s$, they all become exactly non-solvable.

3. The fermionic realizations. Along with the bosonic realizations considered above, we can consider the fermionic ones. The simplest fermionic realization looks like the bosonic one given by formula (3.5) and reads

\[
S_{n1} = \frac{1}{2}(f^+_n g_n + g^+_n f_n), \\
S_{n2} = \frac{1}{2i}(f^+_n g_n - g^+_n f_n), \\
S_{n3} = \frac{1}{2}(f^+_n f_n - g^+_n g_n). \tag{4.3}
\]

Here the operators $f^+_n$ and $g^+_n$ satisfy the Heisenberg anti-commutation relations $\{f_n, f^+_m\} = 1$, $\{g_n, g^+_m\} = 1$ (the anti-commutators between the $f$- and $g$-operators and also between the operators associated with different sites are zero). As before, the operators defined by formulas (4.3) are hermitian and obey the standard commutation relations of $su(2)$ algebra. The representation in which the operators (4.3) act is four-dimensional and decomposes into three irreducible representation with spins 1/2, 0 and 0. This means that it does not satisfy the condition of $s$-reducibility and thus cannot lead to any quasi-exactly solvable model. In order to improve the situation one can consider a tensor product of several representations of the type (4.3). Then the resulting (composite) representation will satisfy the $s$-reducibility condition for a certain finite set of spins. This will lead to a finite family of quasi-exactly solvable models describing the interaction of several fermionic fields on a lattice.

4. The mixed cases. The mixed cases appear when the type of the realization (spin, bosonic, fermionic) changes from site to site. The mixed realizations may lead to quasi-exactly solvable models describing the interaction of different fields, say, fermionic and bosonic fields.
The third group of generalizations concerns the additional solutions which, up to now, we did not discuss in this paper. In order to explain the appearance of these solutions in the simplest $su(2)$ case, let us consider again the spectral problem for hamiltonian (3.4) discussed in section 3. Remember that this problem decomposes into a set of independent spectral problems for the spin hamiltonians

$$H_P(s_1, \ldots, s_N) = J \sum_{n,m=1}^{N} P_{2s}[\tilde{S}_n(s_n)\tilde{S}_{n+1}(s_{n+1})]. \quad (4.4)$$

Remember also that the hamiltonian with $s_n = s$ is explicitly diagonalizable within Bethe ansatz which gives us a part of explicit solutions of the initial problem (3.4). The additional solutions appear when the spin operators $\tilde{S}_n$ in formula (3.4) realize a completely reducible representation of algebra $su(2)$ containing a one-dimensional irreducible component with spin zero. Note that the generators of this one-dimensional component vanish: $\tilde{S}_n(0) = 0$. For this reason, those of models (4.4) which contain zero spins become equivalent to a system of several disconnected inhomogeneous spin chains with open ends. It is reasonable to distinguish between the following opposite cases.

1. The open sub-chains are homogeneous ($s_n = s$) and very long. In this case the boundary effects become negligibly small and all open sub-chains can be approximated by periodic ones. The latter are however exactly solvable and this extends the set of explicit solutions of model (3.4).

2. The open sub-chains are inhomogeneous ($s_n \neq s$) and short. Then the diagonalization can be performed algebraically. Note that algebraic diagonalizability of hamiltonians of short chains does not require the condition of Bethe ansatz solvability. This means that the condition of homogeneity of short chains becomes unnecessary. They equally well may consist of spins differing of $s$.

So we see that the presence of zero spin components in a completely reducible representation of a Lie algebra considerably extends the set of states admitting explicit solutions.

5 Concluding remarks

From the above consideration it follows that there are remarkable parallels between the methods of constructing quasi-exactly solvable problems in quantum mechanics (QM) and in field theory (FT) on a lattice. Both methods start with a certain spin system. This is a quantum top in QM case and infinite spin chain in FT case. Both systems are exactly solvable. This is the algebraic solvability in QM case and Bethe ansatz solvability in FT case. In both systems the spins realize a certain finite-dimensional and irreducible matrix representation $T_{fin}$ of algebra $su(2)$. In both cases one replaces the spin hamiltonian by a new extended hamiltonian which formally has the same form as the initial one but in which the spin operators have different meaning. Now they realize a certain infinite-dimensional and reducible representation $T_{inf}$ of algebra $su(2)$. In both cases however this representation contains $T_{fin}$ as an irreducible component. This finally leads to the quasi-exact solvability of the extended hamiltonian. As we noted above, both the QM and FT quasi-exactly solvable models appear in the form of infinite sequences.

It is worth stressing, however, that along with many common features there is a drastic difference between the quantum mechanical and field theoretical quasi-exactly solvable
models. This difference concerns the principle of building the infinite-dimensional representation spaces \( T_{\text{inf}} \). Indeed, in QM case there is no necessity of changing the form of the initial spin Hamiltonian if one changes the representation in which the spin acts. For any finite-dimensional representation \( T_{\text{fin}} \) the quantum top is exactly (algebraically) solvable. This means that the only way of reducing this Hamiltonian to a quasi-exactly solvable form is to use such an infinite dimensional and reducible representation \( T_{\text{inf}} \) which contains only a finite number of finite-dimensional irreducible components \( T_{\text{fin}} \). Such representations do actually exist but they are non-unitary and the corresponding extended spin operators become non-hermitian. This produces considerable difficulties in constructing hermitian quasi-exactly solvable models in quantum mechanics.

In FT case the situation is different. Now the Bethe ansatz solvability of the initial spin Hamiltonian strongly depends on the representation in which the spins act. For this reason the representation \( T_{\text{inf}} \) may now contain an infinite number of finite-dimensional irreducible components. At any rate the extended Hamiltonian will have only one invariant subspace in which it will be exactly solvable. This means that we obtain a great freedom in choosing the representation \( T_{\text{inf}} \). There are many unitary representations of such a sort which automatically lead to hermitian quasi-exactly solvable models of field theory. In this sense the proposed procedure of building quasi-exactly solvable problems in field theory is conceptually simpler than that in non-relativistic quantum mechanics.

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