Basics of BRST quantization on inner product spaces

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Abstract

There is an elaborated abstract form of BRST quantization on inner product spaces within the operator formalism which leads to BRST invariant states of the form

$$|ph\rangle = e^{[Q,\psi]}|\phi\rangle$$

where $\psi$ is a gauge fixing fermion, and where $|\phi\rangle$ is a BRST invariant state determined by simple hermitian conditions. These state representations are closely related to the path integral formulation. Here we analyse the basics of this approach in detail. The freedom in the choice of $\psi$ and $|\phi\rangle$ as well as their properties under gauge transformations are explicitly determined for simple abelian models. In all considered cases $SL(2,\mathbb{R})$ is shown both to be a natural extended gauge symmetry and to be useful to determine $|ph\rangle$. The results are also applied to nonabelian models.

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1 Introduction and results.

The operator version and the path integral version of BRST quantization look quite different. The reason is that the standard operator version does not take into account all consequences of the necessity to work on inner product spaces. Therefore, one of us has been developing a BRST quantization within the operator formalism in which general conditions are extracted for BRST invariant states to be inner product states [1]-[8]. Only when these conditions are satisfied do we have a precise connection with the path integral representation. In fact, this more elaborated operator version provides for a precise interpretation of BRST quantization within the path integral approach which is not exactly the naive correspondence [3]. In the present paper we give a coherent picture of this operator version of BRST quantization on inner product spaces. All basic properties are explained in detail and explicitly demonstrated to be true for simple abelian gauge models. Previous analyses are thereby considerably extended. We investigate more general gauge fixings in simple abelian models from a global point of view, as well as properties under global gauge transformations in these models. We determine the exact conditions under which a gauge fixing is satisfactory as well as its global uniqueness. These are basic properties underlying any gauge theory since there always exists a local abelianization [3]. The only additional feature in a more general gauge theory is its topological properties which are specific for each model. All operator formulas obtained are abstract and formal. To find a precise result and a precise corresponding path integral representation we have to choose a consistent state space representation and this we do for the considered models here. The general formulas only allow for specific classes of representations which are referred to as quantization rules of the basic variables. These rules automatically lead to precise, satisfactory path integral representations, which are expected to include all known rigorous results of BRST quantization within its the path integral version. This has been shown to be the case for all treated models including the ones in this paper. However, in some cases new representations have also been found. In fact, a given model can have inequivalent representations (see e.g. [4]) which means that it may be some physics also in the choice of state space representations.

The basic ingredients in the operator version of BRST quantization of general gauge theories are the odd, hermitian, and nilpotent BRST charge operator $Q$ and the even, antihermitian ghost number operator $N$ [10]. They satisfy

$$Q^2 = 0, \quad [N, Q] = Q. \quad (1.1)$$

Ghost numbers of operators $A$ and states $|\phi\rangle$, denoted $n_A$ and $n$, are defined by $[A, N] = n_A A$ and $N|\phi\rangle = n|\phi\rangle$. Hence, $Q$ has ghost number plus one according to (1.1). Usually one also has an even, hermitian Hamiltonian operator $H$ with ghost number zero which commutes with $Q$. However, any gauge theory with a nontrivial Hamiltonian may be embedded in a corresponding reparametrization invariant formulation with a vanishing Hamiltonian. We discard therefore Hamiltonians in the following. Some details how they may be introduced into the formalism are given in [8].

The basic condition in BRST quantization is that the physical degrees of freedom are described by BRST invariant states, i.e. by states satisfying

$$Q|\phi\rangle = 0 \quad (1.2)$$
where $|ph\rangle$ is required to be decomposable into eigenstates of the ghost number operator $N$. Formally, there are solutions of (1.2) of the form $Q|\chi\rangle$ for any state $|\chi\rangle$. Since $Q|\chi\rangle$ formally are zero norm states, the true physical state space is $\text{Ker}Q/\text{Im}Q$. In order for the last statements to be true it is absolutely necessary that all states belong to an inner product space in which all inner products are well defined. This is of course well known. However, what is not so well known is that the last conditions allow us to derive more precise formulas which have general solutions which may be specified precisely. This is described below.

The first approach to BRST quantization on inner product spaces was based on the condition that the hermitian BRST charge $Q$ must be possible to decompose as follows

$$Q = \delta + \delta^\dagger, \quad \delta^2 = 0, \quad [\delta, \delta^\dagger]_+ = 0,$$

(1.3)

and that the physical inner product states are solutions to the conditions

$$\delta|ph\rangle = \delta^\dagger|ph\rangle = 0.$$

(1.4)

(See [1], [2], [1].) These conditions severely restrict the way the model is represented in $Q$ as well as the solutions of (1.2). In fact, within the BFV prescription for $Q$ given in [13, 14] the condition (1.3) necessarily requires an even number of first class constraints. This is also what one has in the standard BFV-formulation with dynamical Lagrange multipliers and antighosts (the so called non-minimal sector). A typical example is Yang-Mills theories. Within standard BFV-formulation it has so far been shown that there indeed exists a $\delta$-operator satisfying the properties (1.3) for Lie group theories [1, 3]. This $\delta$-operator was constructed explicitly in [1, 3] which also made it possible to solve the conditions (1.4). The solutions turned out to have the form

$$|ph\rangle = e^{[Q, \psi]}|\phi\rangle,$$

(1.5)

where $\psi$ is a hermitian fermionic gauge fixing operator with ghost number minus one, and where $|\phi\rangle$ is a simple BRST invariant state determined by a set of hermitian operators and is decomposable into eigenstates of the ghost number operator. (The conditions determining $|\phi\rangle$ appear as boundary conditions in the corresponding path integral representation [2].) $|\phi\rangle$ is not a well defined inner product state by itself. The inner product is only well defined with the hermitian regulator or gauge fixing factor $e^{[Q, \psi]}$ present. ($e^{[Q, \psi]}$ has ghost number zero.) Locally $\langle ph|ph\rangle$ is independent of $\psi$ and this should also be true globally for topologically trivial models. However, even for topologically trivial models there are two distinct sectors characterized by $\psi$ and $-\psi$. Sometimes they have opposite norms as will be shown in section 5. The formula (1.3) has been applied to several models [1, 3], and it has been shown to provide for a natural connection between operator quantization and the path integral formulation [2]. The decomposition (1.3) is not unique. If (1.3) is true then we also have

$$Q = \delta' + \delta'^\dagger, \quad \delta'^2 = 0, \quad [\delta', \delta'^\dagger]_+ = 0,$$

$$\delta' \equiv U\delta U^\dagger, \quad [Q, U] = 0,$$

(1.6)

where $U$ is a unitary operator. In fact, from (1.4) we find that

$$\delta'|ph'\rangle = \delta'^\dagger|ph'\rangle = 0 \quad \Rightarrow \quad |ph'\rangle = U|ph\rangle,$$

(1.7)
which expresses the fact that the BRST invariant states are determined up to unitary
BRST invariant transformations. From (1.5) we find then

$$|p\phi'\rangle = e^{iQ,\psi'}|\phi\rangle, \quad \psi' \equiv U\psi U^\dagger, \quad |\phi'\rangle \equiv U|\phi\rangle. \quad (1.8)$$

If $U$ is of the form $U \equiv e^{iQ,\bar{\psi}}$, where $\bar{\psi}$ is a hermitian, fermionic operator with ghost
number minus one exactly like $\psi$, then $U$ represents a unitary gauge transformation. This
gauge group is much larger than the original one but has the original one as a subgroup.
All possible $|\phi\rangle$-states and all possible gauge fixing fermions $\psi$ which can be reached by
means of such $U$-operators defines a gauge equivalent sector. Here we investigate this
gauge equivalence in detail and show that only the original gauge group acts effectively
on $|\phi\rangle$.

The second approach to BRST quantization on inner product states is based on the
results of [6]. All solutions of the BRST condition (1.2) are here assumed to be of the
form (1.5) where the $|\phi\rangle$-states are determined by conditions of the form

$$B_i|\phi\rangle = 0, \quad B_i \equiv [Q, C_i], \quad (1.9)$$

where $\{B_i\}$ is the maximal set of independent simple hermitian operators of this form.
(The appropriate number of $B_i$ operators is one-fourth of the unphysical degrees of freedom
in phase space.) The set $\{B_i\}$ must be maximal in order to imply $Q|\phi\rangle = 0$ which then
implies (1.2). The number of $B_i$-operators must be even. In fact, half of the $B_i$-operators
should be bosonic and half fermionic ones. This condition corresponds to (1.3) (see (1.11)
below). This condition is also natural from the properties of a Dirac quantization since
a bosonic condition of the form (1.9) leads to an infinite norm (if the spectrum of $B_i$
is continuous) while a fermionic condition of the form (1.9) leads to a zero norm. Hence, the
norm of the $|\phi\rangle$-state in (1.3) is a product of equally many zeros as infinities which may
be reduced to a definite finite value by means of the regulator or gauge fixing factor $e^{iQ,\psi}$.
Notice that (1.9) requires

$$[B_i, B_j] = C_{ij}^k B_k \quad (1.10)$$

for consistency. ($C_{ij}^k$ are BRST invariant and commute with $B_i$.) This in turn requires
the $B_i$-operators and the corresponding $C_i$-operators in (1.9) to be in involution. In
section 4 we show that for simple abelian gauge theories in arbitrary linear gauges the two
approaches are equivalent. The connection between (1.9) and (1.4) follows according to
(1.2) in section 4 from relations like

$$\delta = A\left(e^{iQ,\psi}B_a e^{-iQ,\psi}\right)^\dagger \left(e^{iQ,\psi}B'_a e^{-iQ,\psi}\right), \quad (1.11)$$

where $B_a$ and $B'_a$ are the bosonic and fermionic parts of the $B_i$-operators ($B_i = \{B_a, B'_a\}$).
$A$ is a bosonic factor which commutes with the other factor in $\delta$. The conditions (1.4)
yield then always (1.3) where $|\phi\rangle$ satisfies (1.4). However, it is doubtful that the expression
(1.11) satisfies the properties (1.3) for all possible $B$-operators and all possible gauge fixings
$\psi$. The general relation between (1.3), (1.4) and (1.5), (1.9) remains to be investigated.

The BRST invariant states in (1.5), (1.7), and (1.8) contain zero norm states of the form
$Q|\chi\rangle$. A simple way to determine or project out the BRST singlets was given in [6]. The
BRST singlets $|s\rangle$ are BRST invariant inner product states representing the true physical
degrees of freedom of the BRST cohomology ($|s\rangle \in \text{Ker} Q / \text{Im} Q, |ph\rangle \in \text{Ker} Q$). The basic
condition used in [3] was that the BRST singlets, $|s\rangle$, are inner product states provided they are determined by nonhermitian BRST doublet operators which are in involution in such a way that they together with the corresponding hermitian conjugate BRST doublets form BRST quartets in a general sense (cf [10]). In [3] it was shown that the BRST singlets have the simple representation

$$|s\rangle = e^{[Q,\psi]}|\phi\rangle_s$$

for general gauge theories with finite number of degrees of freedom. (The generalization to infinite degrees of freedom should be straight-forward.) $\psi$ is the same gauge fixing fermion as in (1.5). However, $|\phi\rangle_s$ is determined by the conditions

$$D_r|\phi\rangle_s = 0,$$

where $D_r$ is a maximal set of hermitian BRST doublet operators in involution. Since $D_r \equiv \{B_i \equiv [Q, C_i], C_i\}$ and since there are equally many bosonic as fermionic $B_i$-operators, there are equally many bosonic as fermionic operators in the BRST doublets $D_r$ and the total number of BRST doublets is a multiple of four. The conditions

$$C_i|\phi\rangle_s = 0$$

in (1.13) are gauge fixing conditions to (1.3) (cf [15]). Like for the $B_i$-operators there are equally many bosonic as fermionic $C_i$-operators. Eq. (1.14) not only implies gauge fixing in the usual sense but also ghost fixing. In fact, $|\phi\rangle_s$ has ghost number zero. The precise basic criterion for $|s\rangle$-states to be BRST singlets is that $D'_r$ defined by

$$D'_r \equiv e^{[Q,\psi]}D_re^{-[Q,\psi]}$$

is such that $D'_r$ and $(D'_r)^\dagger$ form BRST quartets, i.e.

$$[D'_r, (D'_r)^\dagger]$$

is an invertible matrix operator.

Note that the regulator factor $e^{[Q,\psi]}$ is hermitian. Since (1.12) has been shown to exist for all gauge theories which have the general BFV-form of $Q$ [3], also (1.5) where $|\phi\rangle$ satisfies (1.9) should generalize to all kinds of gauge theories.

In the above formulations of BRST quantization on inner product spaces the physical states are determined purely algebraically. They are therefore formal since it remains to find an explicit realization of the basic operators and states consistent with these results. The approach is therefore different from the way mathematicians attack cohomology problems. They prescribe the basic operators and states from the very beginning. The above approaches leave some freedom for physical intuitions to act in the final explicit realization. Here we make a general investigation of state representations of the unphysical variables in which the basic variables even may have complex eigenvalues.

Some allowed forms of the gauge fixing fermion $\psi$ in (1.5) and (1.12) were given in [3] and further discussed in [7]. However, in the present paper we consider a much more general set of gauge fixing fermions $\psi$. We investigate in detail the freedom in the choices of $\psi$ and $|\phi\rangle$ in the formula (1.12) , as well as their properties under global extended gauge transformations for simple abelian models. We discuss gauge equivalences and the nilpotence of $\psi$. (If $\psi$ is nilpotent it may be interpreted as a coBRST charge (see [7])). The allowed forms of $\psi$ turns out to be in agreement with the allowed forms obtained in
within the path integral formulation, which again demonstrates the close connection between the two formulations. Some limiting cases are discussed in detail.

In section 2 the simple abelian models are presented and the results of [4] are generalized. In section 3 we consider a corresponding general class of gauge transformations and their properties. In section 4 we give the existence conditions for BRST invariant inner product spaces and compare the two approaches presented above. In section 5 we define wave function representations of the BRST singlets and demonstrate that they are consistent if half the fundamental hermitian operators are chosen to have imaginary eigenvalues. This is investigated in its most general form. In section 6 we consider the still more general gauge fixing fermions \( \psi \) which are allowed for bosonic and fermionic gauge theories and the properties of the corresponding BRST singlets. We derive the conditions under which these \( \psi \)'s may be derived from the \( \psi \)'s in section 2 by means of unitary gauge transformations. The main structure of section 2 is shown to be retained. In section 7 we show that our formulas also may be applied to nontrivial models. Finally the paper is concluded in section 8. In two appendices we display the basic formulas used in the text and indicate their derivations.

## 2 Physical states in simple abelian models

Consider as in [4] simple abelian models whose hermitian BRST charge operator is given by

\[
Q = \mathcal{C}^a p_a + \bar{\mathcal{P}}^a \pi_a,
\]

(2.1)

where \( p_a \) and \( \pi_a \) are hermitian conjugate momenta to the hermitian bosonic coordinates \( x^a \) and \( v^a \) respectively, and \( \mathcal{C}^a \) and \( \bar{\mathcal{P}}^a \) are hermitian fermionic operators conjugate to the hermitian operators \( \mathcal{P}_a \) and \( \bar{\mathcal{C}}_a \) respectively. The index \( a = 1, \ldots, n < \infty \) is assumed to be raised and lowered by a real, symmetric metric \( g_{ab} \). (Up to section 6 the explicit form of \( g_{ab} \) never enters our formulas since our treatment is manifestly covariant. In section 6 \( g_{ab} \) is restricted to be real, symmetric and constant.) The fundamental nonzero commutators are

\[
[x^a, p_a]_- = i\delta^a_b, \quad [v^a, \pi_b]_- = i\delta^a_b, \quad [\mathcal{C}^a, \mathcal{P}_b]_+ = \delta^a_b, \quad [\bar{\mathcal{C}}^a, \bar{\mathcal{P}}_b]_+ = \delta^a_b.
\]

(2.2)

One may think of (2.1) as the BRST charge operator of an abelian bosonic gauge theory where \( p_a \) are the gauge generators, \( v^a \) the Lagrange multipliers, and \( \mathcal{C}^a \) and \( \bar{\mathcal{C}}_a \) the ghosts and antighosts respectively. Alternatively one may view it as the BRST charge of a fermionic gauge theory with bosonic ghosts \( p_a \) and antighosts \( v^a \), or a mixture of these two interpretations.

Following the approach to BRST quantization on inner product spaces as proposed in [3], we first look for a maximal set of hermitian BRST doublets in involution. Here there are two natural sets, namely

\[
D_{(1)} = \{ x^a, \mathcal{C}_a, \bar{\mathcal{C}}_a, \pi_a \}, \quad D_{(2)} = \{ v^a, \bar{\mathcal{P}}_a, \mathcal{P}_a, p_a \}.
\]

(2.3)

They are dual in the sense that they together form BRST quartets since the matrix operator \( [D_{(1)}, D_{(2)}] \) is invertible. We are therefore led to two natural choices for \( |\phi\rangle \)-states. We have \( |\phi\rangle_1 \) and \( |\phi\rangle_2 \) determined by the conditions

\[
\mathcal{C}^a |\phi\rangle_1 = \pi_a |\phi\rangle_1 = 0,
\]

5
\[ \mathcal{P}^a |\phi\rangle_2 = p_a |\phi\rangle_2 = 0, \quad (2.4) \]

and

\[ x^a |\phi\rangle_1 = \bar{C}_a |\phi\rangle_1 = 0, \quad v^a |\phi\rangle_2 = \mathcal{P}_a |\phi\rangle_2 = 0, \quad (2.5) \]

The conditions (2.4) may equivalently be written as

\[ [Q, x^a] |\phi\rangle_1 = [Q, \bar{C}_a] |\phi\rangle_1 = 0, \quad [Q, v^a] |\phi\rangle_2 = [Q, \mathcal{P}_a] |\phi\rangle_2 = 0, \quad (2.6) \]

i.e. they are of the form (1.9), while (2.5) are gauge fixing conditions of the form (1.14). Notice also that (2.4) implies \( Q |\phi\rangle_1 = 0 \). Out of \( |\phi\rangle_1 \) and \( |\phi\rangle_2 \) we may construct BRST singlets according to the rule (1.12). We are then led to the following representations:

\[ |s\rangle_l = e^{[Q, \psi_l]} |\phi\rangle_l, \quad l = 1, 2, \quad (2.7) \]

where \( \psi_1 \) and \( \psi_2 \) are gauge fixing fermions. In [6, 7] it was shown that the choices

\[ \psi_1 = \alpha \mathcal{P}_a v^a, \quad \psi_2 = \beta \bar{C}_a x^a, \quad (2.8) \]

are satisfactory provided the real constants \( \alpha \) and \( \beta \) are different from zero, since only then do the states (2.7) satisfy the criterion (1.16). We notice that \( \psi_1 \) and \( \psi_2 \) are products of the gauge fixing variables to the conditions in (2.4). The inner products \( l \langle s | s \rangle_l \) are independent of \( \alpha \) and \( \beta \) for positive and negative values separately. However, they are undefined at \( \alpha = 0 \) and \( \beta = 0 \). In [7] it was shown that the two singlets in (2.7) are related as follows

\[ e^{\alpha K_1} |\phi\rangle_1 = e^{-\frac{1}{\alpha} K_2} |\phi\rangle_2, \quad (2.9) \]

where

\[ K_1 \equiv [Q, \mathcal{P}_a v^a] = v^a p_a + i \mathcal{P}_a \bar{\mathcal{P}}^a, \]
\[ K_2 \equiv [Q, \bar{C}_a x^a] = x^a \pi_a + i \bar{C}_a C^a. \quad (2.10) \]

One may easily check that the both sides in (2.9) satisfy the same conditions. Note that the BRST singlets are unique up to unitary transformations.

In [7] also the more general gauge fixing fermion

\[ \psi_1 = \psi_2 = \alpha \mathcal{P}_a v^a + \beta \bar{C}_a x^a \quad (2.11) \]

was investigated. It was found that

\[ |s\rangle_1 = e^{\alpha K_1 + \beta K_2} |\phi\rangle_1 = e^{\alpha' K_1} |\phi\rangle_1, \quad |s\rangle_2 = e^{\alpha K_1 + \beta K_2} |\phi\rangle_2 = e^{\beta' K_2} |\phi\rangle_2, \quad (2.12) \]

where

\[ \alpha' = \alpha \frac{\tan \sqrt{\alpha \beta}}{\sqrt{\alpha \beta}}, \quad \beta' = \beta \frac{\tan \sqrt{\alpha \beta}}{\sqrt{\alpha \beta}}. \quad (2.13) \]
for $\alpha\beta > 0$ and

$$\alpha' = \alpha \frac{\tanh \sqrt{-\alpha\beta}}{\sqrt{-\alpha\beta}}, \quad \beta' = \beta \frac{\tanh \sqrt{-\alpha\beta}}{\sqrt{-\alpha\beta}}$$  \hfill (2.14)

for $\alpha\beta < 0$. From (2.14) it follows that $|s\rangle_1$ and $|s\rangle_2$ are well defined inner product states provided $\alpha'$ and $\beta'$ are non-zero and finite. In [1] it was also shown that there are natural representations of the operators and states that make the inner products of $|s\rangle_{1,2}$ explicitly finite.

According to the BFV-formulation [13, 14] the general form of the gauge fixing fermion is

$$\psi = \mathcal{P}_a \Lambda^a + \bar{\chi}_a \chi^a$$  \hfill (2.15)

in the case when the simple abelian model (2.1) is considered to describe a bosonic gauge theory. $\Lambda^a$ and $\chi^a$ are then bosonic gauge fixing variables to the gauge generators $\pi_a$ and $p_a$. If on the other hand (2.1) is considered to describe a fermionic gauge theory, then the general form of $\psi$ is

$$\psi = x^a \bar{\chi}_a + v^a \tilde{\Lambda}_a,$$  \hfill (2.16)

where $\bar{\chi}_a$ and $\tilde{\Lambda}_a$ are fermionic gauge fixing variables to $\mathcal{C}^a$ and $\bar{\mathcal{P}}^a$. The forms (2.15) and (2.16) are different due to the different choices of ghost number operators together with the fact that the forms (2.15) and (2.16) follow from the requirement that $\psi$ must have ghost number minus one. The latter condition implies that the ghost number operator commutes with the regulator factor $e^{[Q,\psi]}$, a condition which always must be true. The ghost number operators in cases (2.15) and (2.16) are

$$N_b = \mathcal{C}^a \mathcal{P}_a - \bar{\mathcal{C}}_a \bar{\mathcal{P}}^a, \quad N_f = -ip_a x^a - iv^a \pi_a,$$  \hfill (2.17)

respectively. Note that $N_{b,f}$ is antihermitian, $N_{b,f} = -N_{b,f}$.

The most general form of $\psi$ consistent with both interpretations above, i.e. in which both ghost number operators in (2.17) are conserved (i.e. commuting with $[Q,\psi]$), and where $\Lambda^a$, $\chi^a$, $\bar{\chi}_a$, and $\tilde{\Lambda}_a$ are chosen to be linear and covariant is

$$\psi = \alpha \mathcal{P}_a v^a + \beta \bar{\mathcal{C}}_a x^a + \gamma (\bar{\mathcal{C}}_a v^a - \mathcal{P}_a x^a) + \delta / 2 (\bar{\mathcal{C}}_a v^a + \mathcal{P}_a x^a),$$  \hfill (2.18)

where $\alpha$, $\beta$, $\gamma$, and $\delta$ are real parameters. This $\psi$ is nilpotent, $\psi^2 = 0$. We notice that in order for the corresponding $\Lambda^a$, $\chi^a$, and $\bar{\chi}_a$, $\tilde{\Lambda}_a$, to be consistent gauge fixing variables classically, we must have $(\delta / 2)^2 \neq \alpha \beta + \gamma^2$ ($\Lambda^a$, $\chi^a \leftrightarrow v^a$, $x^a$ and $\bar{\chi}_a$, $\tilde{\Lambda}_a \leftrightarrow \bar{\mathcal{C}}_a$, $\mathcal{P}_a$ are then one-to-one). Although this condition will appear at several occasions in the following the inner products will always be well defined for $(\delta / 2)^2 = \alpha \beta + \gamma^2$ like in (2.8). The gauge fixing fermion (2.18) yields now

$$[Q,\psi] = \alpha K_1 + \beta K_2 + 2\gamma K_3 + \delta K_4,$$  \hfill (2.19)

where $K_1$ and $K_2$ are defined in (2.14) and where

$$K_3 = \frac{1}{2} [Q, \bar{\mathcal{C}}_a v^a - \mathcal{P}_a x^a] = \frac{1}{2} (v^a \pi_a - p_a x^a - i\mathcal{P}_a \mathcal{C}^a - i\bar{\mathcal{P}}^a \bar{\mathcal{C}}_a) =$$

$$= \frac{1}{2} (\pi_a v^a - x^a p_a + i\mathcal{C}^a \mathcal{P}_a + i\bar{\mathcal{C}}_a \bar{\mathcal{P}}^a),$$

$$K_4 = \frac{1}{2} [Q, \bar{\mathcal{C}}_a v^a + \mathcal{P}_a x^a] = \frac{1}{2} (v^a \pi_a + p_a x^a + i\mathcal{P}_a \mathcal{C}^a - i\bar{\mathcal{P}}^a \bar{\mathcal{C}}_a) =$$

$$= \frac{1}{2} (\pi_a v^a + x^a p_a - i\mathcal{C}^a \mathcal{P}_a + i\bar{\mathcal{C}}_a \bar{\mathcal{P}}^a).$$  \hfill (2.20)
The operators $K_i$, $i = 1, 2, 3, 4$, satisfy a closed algebra: The operators $K_1$, $K_2$, and $K_3$ satisfy the algebra

$$[K_1, K_2] = -2iK_3, \quad [K_1, K_3] = iK_1, \quad [K_2, K_3] = -iK_2,$$  \hspace{1cm} (2.21)

which is an SL(2,R) algebra. (By means of the identification $\phi_1 = 1/2(K_2 - K_1)$, $\phi_2 = 1/2(K_1 + K_2)$, $\phi_3 = K_3$, we arrive at the standard SL(2,R) algebra $[\phi_i, \phi_j] = i\epsilon_{ij}^k \phi_k$ with the metric $\text{Diag}(\eta_{ij}) = (-1, +1, +1, +1)$.) This was also shown in \[\text{[7]}\]. In addition we have

$$[K_4, K_i] = 0 \quad i = 1, 2, 3, 4.$$  \hspace{1cm} (2.22)

Hence, we may view $K_i$ as generators of SL(2,R)$\otimes$U(1). The $K_i$ operators satisfy also the properties

$$K_2|\phi_1\rangle = K_1|\phi_2\rangle = K_3|\phi_1\rangle = K_4|\phi_2\rangle = 0,$$

$$K_4|\phi_1\rangle = K_4|\phi_2\rangle = 0.$$  \hspace{1cm} (2.23)

These properties of $K_4$ may also be understood from the equality

$$K_4 = \frac{1}{2}i(N_f - N_b),$$  \hspace{1cm} (2.24)

since $K_i$ and the $|\phi\rangle$-states have ghost number zero with respect to both ghost number operators in \((2.17)\). The above properties of $K_i$ and $K_4$ imply now (see appendix A)

$$|s\rangle_1 = e^{\alpha K_1 + \beta K_2 + 2\gamma K_3 + \delta K_4}|\phi_1\rangle = e^{\alpha K_1 + \beta K_2 + 2\gamma K_3}|\phi_1\rangle = e^\alpha K_1|\phi_1\rangle,$$

$$|s\rangle_2 = e^{\alpha K_1 + \beta K_2 + 2\gamma K_3 + \delta K_4}|\phi_2\rangle = e^{\alpha K_1 + \beta K_2 + 2\gamma K_3}|\phi_2\rangle = e^\beta K_2|\phi_2\rangle,$$  \hspace{1cm} (2.25)

where the second equalities trivially follow from \((2.22)\) and the last two equalities in \((2.23)\). The last equalities in \((2.25)\) follow from the formulas \((A.10)-(A.17)\) in appendix A. The parameters $\alpha'$ and $\beta'$ are here in general complex and given by

$$\alpha' = \alpha \frac{\tan \sqrt{\alpha \beta + \gamma^2}}{\sqrt{\alpha \beta + \gamma^2 + i\gamma \tan \sqrt{\alpha \beta + \gamma^2}}},$$

$$\beta' = \beta \frac{\tan \sqrt{\alpha \beta + \gamma^2}}{\sqrt{\alpha \beta + \gamma^2 - i\gamma \tan \sqrt{\alpha \beta + \gamma^2}}},$$  \hspace{1cm} (2.26)

for $\alpha \beta + \gamma^2 > 0$ and

$$\alpha' = \alpha \frac{\tanh \sqrt{-\alpha \beta - \gamma^2}}{\sqrt{-\alpha \beta - \gamma^2 + i\gamma \tanh \sqrt{-\alpha \beta - \gamma^2}}},$$

$$\beta' = \beta \frac{\tanh \sqrt{-\alpha \beta - \gamma^2}}{\sqrt{-\alpha \beta - \gamma^2 - i\gamma \tanh \sqrt{-\alpha \beta - \gamma^2}}},$$  \hspace{1cm} (2.27)

for $\alpha \beta + \gamma^2 < 0$. Finally for $\alpha \beta + \gamma^2 = 0$ these expressions reduce to

$$\alpha' = \frac{\alpha}{1 + i\gamma}, \quad \beta' = \frac{\beta}{1 - i\gamma}.$$  \hspace{1cm} (2.28)

From \((2.23)\) it follows that provided the real parts of $\alpha'$ and $\beta'$ are non-zero and finite, $|s\rangle_1$ and $|s\rangle_2$ are well defined inner product states. (Only the real parts contribute to the inner products.) These formulas reduce to \((2.13)\) and \((2.14)\) for $\gamma = 0$ as they should.
3 Unitary gauge transformations

BRST invariant unitary operators, $U$, transform our singlet states as follow

$$|s\rangle' = U|s\rangle = e^{[Q,\tilde{\psi}]}U|\phi\rangle, \quad \psi' = U\psi U^\dagger,$$

which shows that changes in $|\phi\rangle$-states and gauge fixings are related. Gauge transformations are performed by BRST invariant operators of the form

$$U = e^{[Q,\tilde{\psi}]}$$

where $\tilde{\psi}$ is an odd, hermitian operator with ghost number minus one like the gauge fixing fermion $\psi$. If we only consider gauge transformations which commute with the two ghost number operators in (2.17) then the general covariant form of $\tilde{\psi}$ is given in (2.18). We are then led to consider gauge transformations obtained from the unitary operator

$$U(a, b, c, d) = \exp (iaK_1 + ibK_2 + 2icK_3 + idK_4).$$

This operator performs $SL(2, R) \otimes U(1)$ transformations. This might be confusing since we are considering abelian gauge models. If $a^\alpha$ and $v^a$ (or $P_a$ and $\bar{C}_a$) were parameters in $\psi$ we would obtain expected abelian gauge transformations. However, as will be seen below only an abelian subgroup of (3.3) acts effectively on the BRST invariant states.

From [A.13] in appendix A we find that

$$e^{\alpha'K_1 + \beta'K_2 + 2\gamma'K_3} = U(a, b, c, d)e^{\alpha K_1 + \beta K_2 + 2\gamma K_3}U^\dagger(a, b, c, d),$$

where

$$\alpha' = \alpha(A + cB)^2 - \beta a^2 B^2 - 2\gamma a(A + cB)B,$$

$$\beta' = -\alpha b^2 B^2 + \beta(A - cB)^2 + 2\gamma b(A - cB)B,$$

$$\gamma' = -\alpha b(A + cB)B + \beta a(A - cB)B + \gamma(1 + 2abB^2),$$

where in turn $A$ and $B$ are given in [A.13]-[A.14]. Thus, a general gauge fixing factor with $\delta = 0$ may be obtained from specific ones by means of these unitary gauge transformations. Note that $\alpha'\beta' + (\gamma')^2 = \alpha\beta + \gamma^2$ always.

We determine now what (3.3) does on the $|\phi\rangle$-states. On $|\phi\rangle_1$ and $|\phi\rangle_2$ we find

$$|\phi\rangle_1' = U(a, b, c, d)|\phi\rangle_1 = U(a, b, c, 0)|\phi\rangle_1 = e^{iaK_1}|\phi\rangle_1,$$

$$|\phi\rangle_2' = U(a, b, c, d)|\phi\rangle_2 = U(a, b, c, 0)|\phi\rangle_2 = e^{ibK_2}|\phi\rangle_2,$$

where

$$a' = a \frac{\tanh \sqrt{ab + c^2}}{\sqrt{ab + c^2} - c \tanh \sqrt{ab + c^2}},$$

$$b' = b \frac{\tanh \sqrt{ab + c^2}}{\sqrt{ab + c^2} + c \tanh \sqrt{ab + c^2}},$$

for $ab + c^2 > 0$ and

$$a' = a \frac{\tan \sqrt{-ab - c^2}}{\sqrt{-ab - c^2} - c \tan \sqrt{-ab - c^2}},$$

$$b' = b \frac{\tan \sqrt{-ab - c^2}}{\sqrt{-ab - c^2} + c \tan \sqrt{-ab - c^2}}$$

for $ab + c^2 < 0$.
for \(ab + c^2 < 0\). Finally for \(ab + c^2 = 0\) these expressions reduce to

\[
a' = \frac{a}{1 - c}, \quad b' = \frac{b}{1 + c}. \tag{3.9}
\]

Although the reduced parameters \(a'\) and \(b'\) are infinite for some values of \(a, b\) and \(c\) in all these cases, the transformations (3.6) are well defined even for these values. Due to the equality

\[
e^{iaK_1}|\phi\rangle_1 = e^{\frac{i\theta}{a}K_2}|\phi\rangle_2, \tag{3.10}
\]

valid for any parameter \(a\) (both sides satisfy the same equations), an infinite value of \(a'\) or \(b'\) in (3.6) implies \(|\phi\rangle_1' = |\phi\rangle_2\) and \(|\phi\rangle_2' = |\phi\rangle_1\) respectively. Thus, (3.3) is well defined and reduces to abelian gauge transformations on the states \(|\phi\rangle_1\) and \(|\phi\rangle_2\). Below we show that the same result may be obtained by an abelian subgroup of (3.3).

Consider the particular abelian unitary gauge operator

\[
U_R(\theta) \equiv e^{i\theta(K_1 - K_2)}. \tag{3.11}
\]

It rotates the basic variables an angle \(\theta\). From the above formulas we find

\[
U_R(\theta)|\phi\rangle_1 = e^{i\theta K_1}|\phi\rangle_1, \quad U_R(\theta)|\phi\rangle_2 = e^{-i\theta K_2}|\phi\rangle_2, \tag{3.12}
\]

where \(\theta' = \tan \theta\). Now since \(-\infty < \tan \theta < \infty\) it follows that the general gauge transformations (3.6) on \(|\phi\rangle_1\) and \(|\phi\rangle_2\) may be represented by states of the form

\[
|\phi\rangle_{\theta} \equiv U_R(\theta)|\phi\rangle_1 \tag{3.13}
\]

for some value of the parameter \(\theta\) \((-\frac{1}{2}\pi < \theta \leq \frac{1}{2}\pi\)). The state (3.13) is annihilated by the hermitian BRST doublets

\[
D_\theta \equiv \left\{ \begin{array}{c} U_R(\theta) x^a U_R(\theta)^\dagger, \\
U_R(\theta) C^a U_R(\theta)^\dagger, \\
U_R(\theta) C^a U_R(\theta)^\dagger, \\
U_R(\theta) a U_R(\theta)^\dagger \end{array} \right\} = \\
\begin{array}{c} x^a \cos \theta + v^a \sin \theta, \\
C^a \cos \theta + \bar{P}^a \sin \theta, \\
\bar{C}^a \cos \theta - P^a \sin \theta, \\
\pi_a \cos \theta - p_a \sin \theta \end{array} \tag{3.14}
\]

in accordance with (2.4). The states (3.13) represent therefore the most general class of \(|\phi\rangle\)-states determined by linear, covariant equations. Note that although \(U_R(\theta)\) on operators naturally is defined for \(-\pi < \theta \leq \pi\), the \(|\phi\rangle\)-states require only \(-\frac{1}{2}\pi < \theta \leq \frac{1}{2}\pi\) since \(|\phi\rangle_{\theta}\) is independent of the sign of \(D_\theta\).

From (3.13) and (3.14) we have

\[
|\phi\rangle_2 = U_R(\pi/2)|\phi\rangle_1, \tag{3.15}
\]

i.e. \(|\phi\rangle_1\) and \(|\phi\rangle_2\) are related by a rotation 90 degrees. This shows that the cases 1 and 2 above are unitarily equivalent. This is also true for the BRST singlets themselves. Since

\[
U_R(\theta) K_1 U_R^\dagger(\theta) = K_1 \cos^2 \theta - K_2 \sin^2 \theta + K_3 \sin 2\theta, \\
U_R(\theta) K_2 U_R^\dagger(\theta) = K_2 \cos^2 \theta - K_1 \sin^2 \theta + K_3 \sin 2\theta, \\
U_R(\theta) K_3 U_R^\dagger(\theta) = K_3, \tag{3.16}
\]

10
we find
\[ U_R(\pi/2)|s\rangle_1 = U_R(\pi/2)\alpha_{K_1}|\phi\rangle_1 = e^{-\alpha K_2}|\phi\rangle_2 = |s\rangle_2. \] (3.17)

Although the other SL(2,R) transformations do not generate other states than \( (3.13) \) from \(|\phi\rangle_1\), they do change \( K_i \) in the regulator factor. We have in particular
\[ U_S(\rho) = e^{i\rho K_3}, \] (3.18)
which scales the basic operators since
\[ U_S(\rho)K_1U_S^{-1}(\rho) = e^\rho K_1, \quad U_S(\rho)K_2U_S^{-1}(\rho) = e^{-\rho} K_2. \] (3.19)
This combined with \( U_S(\rho)|\phi\rangle_1 = |\phi\rangle_1 \) implies that singlets of the form \( (2.7) \) with different parameter values are unitarily equivalent. Note also that
\[ U(a, b, c, d)\alpha_{K_1}U(a, b, c, d)\dagger|\phi\rangle_1 = e^{\alpha'_{K_1}}|\phi\rangle_1 \] (3.20)
where
\[ \alpha' = \alpha \frac{(A + cB)^2}{1 + \beta B(A + cB)} \] (3.21)
from \( (A.15) \) and \( (2.25) \). Thus, the state \( (3.20) \) is up to a simple gauge transformation of the form \( U = e^{\alpha''_{K_1}} \) equal to a state \( e^{\alpha'_{K_1}}|\phi\rangle_1 \) where the real parameter \( \alpha'' \) is equal to \( (A + cB)^2/(1 + \beta B^2(A + cB)^2) \). Note also the peculiar identity transformation
\[ U_S(\ln \alpha)U_R(\pi/2)\alpha_{K_1}|\phi\rangle_1 = e^{\frac{i}{4}\lambda K_2}|\phi\rangle_2 = e^{\alpha K_1}|\phi\rangle_1 \] (3.22)
due to the equality \( (2.9) \).

A third basic abelian unitary gauge operator is
\[ U_H(\lambda) \equiv U(\lambda, \lambda, 0) = e^{i\lambda(K_1 + K_2)}, \] (3.23)
which performs hyperbolic transformations, i.e.
\[ D_\lambda = \left\{ U_H(\lambda)C^aU_H(\lambda)^\dagger, \quad U_H(\lambda)c_\lambda U_H(\lambda)^\dagger, \quad U_H(\lambda)\pi_a U_H(\lambda)^\dagger \right\} = \left\{ x^a \cosh \lambda + v^a \sinh \lambda, \quad C^a \cosh \lambda + \bar{\theta}^a \sinh \lambda, \quad \bar{\theta}^a \cosh \lambda - \bar{\theta}^a \sinh \lambda \right\}. \] (3.24)
From \( (A.15) \) in Appendix A we have also
\[ U_H(2\lambda)K_1U_H^{-1}(2\lambda) = \cosh^2 \lambda K_1 + \sinh^2 \lambda K_2 - \sinh 2\lambda K_3, \]
\[ U_H(2\lambda)K_2U_H^{-1}(2\lambda) = \cosh^2 \lambda K_2 + \sinh^2 \lambda K_1 + \sinh 2\lambda K_3, \]
\[ U_H(2\lambda)K_3U_H^{-1}(2\lambda) = (1 - \sinh^2 \lambda)K_3 + \frac{1}{2} \sinh 2\lambda(K_1 + K_2). \] (3.25)
Note that
\[ U_H(\lambda)|\phi\rangle_1 = U_R(\theta)|\phi\rangle_1, \quad D_\lambda U_H(\lambda)|\phi\rangle_1 = D_\theta U_R(\theta)|\phi\rangle_1 = 0 \] (3.26)
for \( \theta = \arctan(\tanh \lambda) \). However, there is no unitary operator of the form \( (3.2) \) which changes the sign of \( \alpha \) in \(|s\rangle_1 = e^{\alpha K_1}|\phi\rangle_1 \) since the existence of such a unitary operator would mean that \(|\phi\rangle_1 \) is a well defined inner product state which it is not. Thus, singlet states with opposite signs of \( \alpha \) are not gauge equivalent. There are two options here: either we may identify states with opposite signs of \( \alpha \) by hand, or one of them may be excluded (see next section).
4 Existence conditions of the inner product solutions

The basic abstract criterion for the BRST singlets to be inner product states is given by (1.16). The BRST doublets \( D_r \) in (1.15) are here e.g. given by

\[
\begin{align*}
\mathcal{C}'_a &= e^{[Q,\psi]}\mathcal{C}_a e^{-[Q,\psi]}, & \pi'_a &= e^{[Q,\psi]}\pi_a e^{-[Q,\psi]}, \\
\mathcal{C}'^a &= e^{[Q,\psi]}\mathcal{C}^a e^{-[Q,\psi]}, & x'_a &= e^{[Q,\psi]}x_a e^{-[Q,\psi]},
\end{align*}
\]

(4.1)

where

\[
e^{[Q,\psi]} = e^{\alpha K_2 + \beta K_2 + 2\gamma K_3 + \delta K_4},
\]

(4.2)

\( \psi \) is the gauge fixing fermion (2.18). The operators in (4.1) are explicitly given in (A.9) appendix A. We find now that the only non-zero commutators between \( D'_r \) and \((D'_r)^\dagger\) are

\[
\begin{align*}
[\pi'_a, (x'^b)^\dagger] &= \alpha F e^{i\delta} \delta^b_a, & [\mathcal{C}'^a, (\mathcal{C}'^b)^\dagger] &= i\alpha F e^{i\delta} \delta^a_b,
\end{align*}
\]

(4.3)

where \( \alpha \) and \( \delta \) are the parameters in (4.2) and where

\[
F \equiv \begin{cases}
\frac{\sin 2\sqrt{\alpha \beta + \gamma^2}}{\sqrt{\alpha \beta + \gamma^2}}, & \alpha \beta + \gamma^2 > 0, \\
\frac{\sinh 2\sqrt{-\alpha \beta - \gamma^2}}{\sqrt{-\alpha \beta - \gamma^2}}, & \alpha \beta + \gamma^2 < 0, \\
2, & \alpha \beta + \gamma^2 = 0.
\end{cases}
\]

(4.4)

also in terms of the parameters in (4.2). The criterion (1.16) requires the commutators (4.3) to be non-vanishing. Thus, in order for the BRST singlets \(|s\rangle_1\) in (2.25) to be inner product states we must have

\[
\alpha F \neq 0 \quad \Leftrightarrow \quad \alpha \neq 0 \text{ and } \sin 2\sqrt{\alpha \beta + \gamma^2} \neq 0.
\]

(4.5)

If we define the operators \( \phi_a, \rho^a, \xi^a, k_a \) by

\[
\phi_a \equiv \pi'_a, \quad \rho^a \equiv \mathcal{C}'^a, \quad \xi^a \equiv \frac{e^{i\delta}}{\alpha F} x'^a, \quad k_a \equiv \frac{e^{i\delta}}{\alpha F} \bar{\mathcal{C}}'^a,
\]

(4.6)

then \( \phi_a, \rho^a, \xi^a, k_a \) and their hermitian conjugates commute among themselves except for the following two commutators

\[
[\phi_a, (\xi^b)^\dagger] = \delta_a^b, \quad [\rho^a, k_b^\dagger] = \delta_a^b.
\]

(4.7)

The BRST singlets \(|s\rangle_1\) in (2.25) satisfy then

\[
\phi_a |s\rangle_1 = \rho^a |s\rangle_1 = \xi^a |s\rangle_1 = k_a |s\rangle_1 = 0,
\]

(4.8)

which means that the states \(|s\rangle_1\) represent a Fock vacuum with respect to these operators and have well defined inner products. The complete Fock space consists then of half positive and half indefinite metric states (see e.g. section 5 in [7]). (The same reasoning is true for \(|s\rangle_2\) in (2.27) and the doublet set \(D(2)\) in (2.3).) Another Fock vacuum is

\[
(\phi_a)^\dagger |s\rangle_3 = (\rho^a)^\dagger |s\rangle_3 = (\xi^a)^\dagger |s\rangle_3 = (k_a)^\dagger |s\rangle_3 = 0.
\]

(4.9)
The solutions are here of the form
\[ |s⟩_3 = e^{-[Q,ψ]|φ⟩_1}. \] (4.10)

So far we have only considered the approach to BRST quantization on inner product spaces as presented in [6] which is for BRST singlets. As was mentioned in the introduction there is also a gauge invariant approach which yield similar solutions that are not gauge fixed. A crucial issue in this approach is to find the decomposition (1.3), \( Q = δ + δ† \).

As was also mentioned in the introduction if one such solution is found we have a whole set of solutions of the form
\[ δ' = UδU†, \quad [Q, U] = 0 \] (4.11)
where \( U \) is a unitary operator. For the simple abelian model (2.1) there is indeed a decomposition (1.3) which is such that the solutions of (1.4) will contain the singlets \( |s⟩_1 \) in (2.25). It is
\[ δ = \frac{i e^{iδ}}{αF} (π′_a)† C′_a, \] (4.12)
where \( C′_a \) and \( π′_a \) are given by (4.1), and where \( F \) is given by (4.4). α and δ on the right-hand side are parameters defined in (2.18), (2.19). The fact that \( C′_a, π′_a \) and their hermitian conjugates commute makes the δ in (4.12) satisfy the properties \( δ^2 = 0 \) and \( [δ, δ†] = 0 \). One may easily check that \( Q = δ + δ† \). The existence of the δ-operator (4.12) requires (4.5) which is identical to the condition required by the criterion (1.16) for the BRST singlets which here demands the nonvanishing of (4.3). Also the existence of the oscillators (4.6) requires the same condition. (For nonlinear gauges there is no such simple relation as (4.12).)

Now there are two ways to solve δ\( |ph⟩ = δ†|ph⟩ = 0 \). From (4.12) we have either
\[ C′_a |ph⟩_1 = π′_a |ph⟩_1 = 0 \quad \Rightarrow \quad |ph⟩_1 = e^{[Q,ψ]} |φ⟩_1, \] (4.13)
or
\[ (C′_a)† |ph⟩_1 = (π′_a)† |ph⟩_1 = 0 \quad \Rightarrow \quad |ph⟩_2 = e^{-[Q,ψ]} |φ⟩_1, \] (4.14)
where \( e^{[Q,ψ]} \) is given by (1.2), and where \( |φ⟩_1 \) satisfies the conditions (2.3). However, \( |φ⟩_2 \) does not satisfy (2.3). As was explained in [5] conditions like \( B|ph⟩ = 0, B ≡ [Q, C] \) allow for gauge fixing conditions \( C|ph⟩ = 0 \) provided \( B \) and \( C \) satisfy a closed algebra. By means of gauge transformations it is always possible to shift the gauge fixing conditions. We conclude that the solutions (4.13) do contain the singlets \( |s⟩_1 \) in (2.25), and that (4.14) contain the singlets (4.10).

For the abelian model we are considering one implication of the unitary ambiguity (1.11) is as follows: δ in (4.12) may be replaced by δ′ = \( UδU† \) where \( U = e^{i[Q,ψ]} \) where \( ψ \) is odd and hermitian. Thus, δ′ is a gauge transformed δ. The corresponding solutions of \( δ′|ph⟩′ = (δ′)†|ph⟩′ = 0 \) is then given by \( |ph⟩_{1,2} = U|ph⟩_{1,2} \) where \( |ph⟩_{1,2} \) are the original solutions in (1.13) and (1.14). For \( U = UR(π/2) \) in (1.11) we have e.g.
\[ |ph⟩′_1 = e^{-αK_2 − βK_1 + 2γK_3 + δK_4} |φ⟩_2 \] (4.15)
where
\[ p_a |\bar{\phi}\rangle_2 = \bar{P}^a |\bar{\phi}\rangle_2 = 0, \] (4.16)
which are identical to the conditions on |\phi\rangle_2 in (2.4). The states (4.15) do therefore contain the singlets |s\rangle_2 in (2.27).

Although \( \delta \) and the solutions of (1.4) are defined up to unitary transformations there are two distinct solutions of (1.4) which are not connected by any unitary operator. In the abelian case they are given by (4.13) and (4.14). Either there is an additional condition which excludes one of these conditions or one has to require them to be equivalent. The latter option was used in e.g. [4]. In the next section we show that the two solutions have opposite norms in some cases.

5 Wave function representations

We consider now wave function representations of the considered BRST singlets. These wave functions will be expressed in terms of the coordinates which are eigenvalues of \( x^a, v^a, C^a, \) and \( \bar{P}^a. \) Since the state space as a Fock space contains half positive and half indefinite metric states the proposed general rule is that we can only work with eigenstates which are such that half of the bosonic and fermionic coordinates have real eigenvalues and half imaginary eigenvalues [4]. The question is which halves to choose. For the simple abelian model which we are considering the results will be that the gauge fixing fermion \( \psi \) to a large extent determines the freedom in this choice.

Some inverses of the unitary gauge transformations in (A.12) appendix A are given by
\[
\begin{pmatrix}
   x^a \\
   v^a
\end{pmatrix} = \begin{pmatrix}
   \tilde{\alpha} & \tilde{\beta} \\
   \tilde{\gamma} & \tilde{\delta}
\end{pmatrix} \begin{pmatrix}
   z^a \\
   w^a
\end{pmatrix} \quad (5.1)
\]
\[
\begin{pmatrix}
   C^a \\
   \bar{P}^a
\end{pmatrix} = \begin{pmatrix}
   \tilde{\alpha} & \tilde{\beta} \\
   \tilde{\gamma} & \tilde{\delta}
\end{pmatrix} \begin{pmatrix}
   \eta^a \\
   \theta^a
\end{pmatrix} \quad (5.2)
\]
where \( z^a \equiv x^a, w^a \equiv v^a, \eta^a \equiv C^a, \) and \( \theta^a \equiv \bar{P}^a. \) The real constants \( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta} \) may be obtained from (A.12) in appendix A. They satisfy the property
\[
\det \begin{pmatrix}
   \tilde{\alpha} & \tilde{\beta} \\
   \tilde{\gamma} & \tilde{\delta}
\end{pmatrix} = \tilde{\alpha}\tilde{\delta} - \tilde{\gamma}\tilde{\beta} = e^{-2d} > 0 \quad (5.3)
\]
\( z^a, w^a, \eta^a \) and \( \theta^a \) are hermitian operators like \( x^a, v^a, C^a, \bar{P}^a. \) We choose \( z^a \) and \( \eta^a \) to have real eigenvalues, and \( w^a \) and \( \theta^a \) to have imaginary eigenvalues. The eigenstates \( |z, u\rangle \) and \( |\eta, \theta\rangle \) satisfy then
\[
\begin{align*}
   z^a |z', u\rangle &= z'^a |z', u\rangle, \\
   w^a |z', u\rangle &= i u^a |z', u\rangle, \\
   \eta^a |\eta', \theta'\rangle &= \eta'^a |\eta', \theta'\rangle, \\
   \theta^a |\eta', \theta'\rangle &= i \theta'^a |\eta', \theta'\rangle. \quad (5.4)
\end{align*}
\]
Note that \( |z, u\rangle \) are eigenstates to \( x^a \) and \( v^a \) with complex eigenvalues. We have
\[
\begin{align*}
   x^a |z, u\rangle &= (\tilde{\alpha} z^a + i \tilde{\beta} u^a) |z, u\rangle, \\
   v^a |z, u\rangle &= (\tilde{\gamma} z^a + i \tilde{\delta} u^a) |z, u\rangle. \quad (5.5)
\end{align*}
\]
Similarly $|\eta, \theta\rangle$ are eigenstates to $C^a$ and $\bar{P}^a$ with complex eigenvalues.

$$
C^a|\eta, \theta\rangle = (\bar{\alpha}\eta^a + i\bar{\beta}\theta^a)|\eta, \theta\rangle, \quad \bar{P}^a|\eta, \theta\rangle = (\bar{\gamma}\eta^a + i\bar{\delta}\theta^a)|\eta, \theta\rangle
$$

(5.6)

Note that the state $|\eta, \theta\rangle$ may be written as

$$
|\eta', \theta\rangle = e^{-\eta' a P_{\eta a} - i\theta' a P_{\theta a}} |0\rangle_{\eta\theta}
$$

(5.7)

where the vacuum state $|0\rangle_{\eta\theta}$ is Grassmann even satisfying the conditions

$$
\eta^a|0\rangle_{\eta\theta} = 0, \quad \theta^a|0\rangle_{\eta\theta} = 0.
$$

(5.8)

The states $|z, u\rangle$ and $|\eta, \theta\rangle$ satisfy also

$$
(|z, u\rangle)^\dagger = \langle z, -u|, \quad (|\eta, \theta\rangle)^\dagger = \langle \eta, -\theta|
$$

as well as the completeness relations

$$
\int d^n u d^n z |z, u\rangle\langle z, u| = 1, \quad \int d^n \theta d^n \eta |\eta, \theta\rangle\langle \eta, \theta| = 1.
$$

(5.9)

If we define the combined eigenstates by

$$
|z, u, \eta, \theta\rangle \equiv |z, u\rangle \otimes |\eta, \theta\rangle,
$$

(5.11)

then we may associate a wave function to any state $|\psi\rangle$ through the relation

$$
\psi(z, u, \eta, \theta) \equiv \langle z, u, \eta, \theta|\psi\rangle.
$$

(5.12)

The BRST singlet $|s\rangle_1$ satisfy the conditions

$$
(x^a + M v^a)|s\rangle_1 = 0, \quad (C^a + M \bar{P}^a)|s\rangle_1 = 0,
$$

$$
(\pi_a - M p_a)|s\rangle_1 = 0, \quad (\bar{C}_a - M \bar{P}_a)|s\rangle_1 = 0,
$$

(5.13)

where

$$
M \equiv -i \frac{\alpha B}{A + i\gamma B}.
$$

(5.14)

where in turn $A$ and $B$ are given by (A.6) and (A.7) in appendix A. The singlet $|s\rangle_2$ satisfies on the other hand

$$
(u^a + N x^a)|s\rangle_2 = 0, \quad (\bar{P}^a + N C^a)|s\rangle_2 = 0,
$$

$$
(p_a - N \pi_a)|s\rangle_2 = 0, \quad (\bar{P}_a - N \bar{C}_a)|s\rangle_2 = 0,
$$

(5.15)

where

$$
N \equiv -i \frac{\beta B}{A - i\gamma B}.
$$

(5.16)

The solutions of (5.13) and (5.15) are in the wave function representation (5.12)

$$
\psi_1(z, u, \eta, \theta) = A \delta^{(n)}(z + \frac{(\bar{\beta} + M \bar{\delta})}{(\bar{\alpha} + M \bar{\gamma})} i u) \delta^{(n)}(\eta + \frac{(\bar{\beta} + M \bar{\delta})}{(\bar{\alpha} + M \bar{\gamma})} i \theta),
$$

$$
\psi_2(z, u, \eta, \theta) = A' \delta^{(n)}(z + \frac{(\bar{\delta} + N \bar{\beta})}{(\gamma + N \alpha)} i u) \delta^{(n)}(\eta + \frac{(\bar{\delta} + N \bar{\beta})}{(\gamma + N \alpha)} i \theta).
$$

(5.17)
In order for these solutions to make sense the bosonic delta function must have a real argument. This restricts the gauge transformations above, or in other words the choice of gauge fixing fermion $\psi$ governs the quantization rules, i.e. it determines which variables may be chosen to have imaginary eigenvalues. These conditions are explicitly

\[ Re\{(\tilde{\alpha} + M\tilde{\gamma})(\tilde{\beta} + M^*\tilde{\delta})\} = 0, \quad Re\{(\tilde{\delta} + N\tilde{\beta})(\tilde{\gamma} + N^*\tilde{\alpha})\} = 0. \] (5.18)

Note that

\[ \text{sign}(\text{Im}\{\tilde{\alpha} + M\tilde{\gamma}\} (\tilde{\beta} + M^*\tilde{\delta})\}) = -\text{sign}(\text{Im}M) = \text{sign}(\alpha \sin 2\sqrt{\alpha\beta + \gamma^2}), \]

\[ \text{sign}(\text{Im}\{\tilde{\delta} + N\tilde{\beta}\} (\tilde{\gamma} + N^*\tilde{\alpha})\}) = -\text{sign}(\text{Im}N) = \text{sign}(\beta \sin 2\sqrt{\alpha\beta + \gamma^2}), \] (5.19)

When calculating the norms of $|s\rangle_1,2$ we find

\[ \int d^nud^nzd^n\theta d^n\eta \psi_1^*(z,-u,\eta,-\theta) \psi_1(z,u,\eta,\theta) \propto (\text{sign}(\alpha \sin 2\sqrt{\alpha\beta + \gamma^2}))^n, \]

\[ \int d^nud^nzd^n\theta d^n\eta \psi_2^*(z,-u,\eta,-\theta) \psi_2(z,u,\eta,\theta) \propto (-\text{sign}(\beta \sin 2\sqrt{\alpha\beta + \gamma^2}))^n, \] (5.20)

which shows that the norms are undefined for exactly those values of $\alpha, \beta, \gamma$ and $\delta$ for which we could not do the decomposition (1.3) or equivalently for those values which do not satisfy the criterion (1.16). (There is also an infinite factor present when the sign factors are zero.) The results (5.20) also suggest that the physical vacuum norm changes sign when we cross the singularity points in odd dimension $n$. (Such changes for the fermionic vacua was given in eq.(4.18) in [16].) However, the relations between the bosonic vacua are not obvious since they are normally not related.

The above results are considerable generalizations of previous results which were for $\beta = \gamma = 0$ [4]. One may notice that when $\gamma = 0$, in which case $ReM = ReN = 0$, the unitary transformation in (5.1)-(5.2) may be chosen to be the identity transformation. This is not the case when $\gamma \neq 0$. Note that the conditions (5.18) leave a three parameter freedom in the unitary gauge transformations. Inserting (A.12) in (5.18) one finds a relation between the parameters $a, b$ and $c$ ($d$ may be chosen arbitrary).

We conclude that the basic quantization rule in [4] that half of the bosonic and half of the fermionic unphysical hermitian operators should have imaginary eigenvalues leads to perfectly consistent solutions even in the general situation which we are considering here. However, we had to use this principle in its most general form in which the original operators were chosen to have complex eigenvalues.

6 Still more general forms of $\psi$

We may consider a still more general gauge fixing fermion if we view (2.4) as the BRST charge of a bosonic gauge theory. In this case only the first ghost number operator $N_b$ in (2.17) has to be conserved. The most general invariant quadratic gauge fixing fermion allowed by the general expression (2.15) is then

\[ \psi' = \psi + \zeta_1 \mathcal{P}_a \pi^a + \zeta_2 \mathcal{C}_a \pi^a + \zeta_3 \mathcal{P}_a \pi^a + \zeta_4 \mathcal{C}_a \pi^a, \] (6.1)
where $\psi$ is given by (2.18). This expression yields
\[ [Q, \psi'] = \alpha K_1 + \beta K_2 + 2\gamma K_3 + \delta K_4 + \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3, \]
(6.2)
where in turn
\[ V_1 \equiv [Q, P_a p^a] = p^a p_a, \quad V_2 \equiv -[Q, \bar{C}_a \pi^a] = -\pi_a \pi^a, \]
\[ V_3 \equiv [Q, \bar{P}_a \pi^a] = [Q, \bar{C}_a p^a] = p_a \pi^a. \]
(6.3)

Note that $P_a \pi^a = \bar{C}_a p^a + [Q, \bar{C}_a \pi^a]$. In distinction to the case in the previous sections the metric $g_{ab}$ enters here explicitly. In fact, in order to have quadratic expressions in (6.1) and (6.2) the metric $g_{ab}$ must be constant, and this was assumed in (6.3). Comparison between (6.1) and (6.2) implies the following relations between the parameters $\zeta$, and $\lambda$:
\[ \lambda_1 = \zeta_1, \quad \lambda_2 = -\zeta_2 \quad \text{and} \quad \lambda_3 = \zeta_3 + \zeta_4. \]
If $\delta \neq 0$ we may choose $\zeta_3 - \zeta_4$ such that $\psi^2 = 0$ without affecting the gauge fixing factor $e^{[Q, \psi]}$.

Obviously $V_k$ commute among themselves. Note that they cannot provide for a good gauge fixing by themselves since they do not contain ghost terms. The combined algebra of $K_i$ and $V_k$ is given by (2.21), (2.22) and
\[
[K_1, V_1] = 0, \quad [K_1, V_2] = -2iV_3, \quad [K_1, V_3] = iV_1, \quad [K_2, V_1] = 2iV_3, \quad [K_2, V_2] = 0, \quad [K_2, V_3] = -iV_2, \quad [K_3, V_1] = -iV_1, \quad [K_3, V_2] = iV_2, \quad [K_3, V_3] = 0, \quad (6.4)
\]
\[ [K_4, V_k] = iV_k \quad k = 1, 2, 3. \]
(6.5)

Eq.(6.4) implies that $V_k$ transforms as a vector under the SL(2,R) generated by $K_1$, $K_2$, and $K_3$, and (6.3) that $K_4$ generates scale transformations of $V_k$. The $V_k$-operators satisfy furthermore the relations
\[ V_1|\phi\rangle_2 = 0, \quad V_2|\phi\rangle_1 = 0, \quad V_3|\phi\rangle_1 = V_3|\phi\rangle_2 = 0. \]
(6.6)

The general gauge fixing (6.1) involving all the terms presented so far leads to the following BRST singlets (see appendix B):
\[
|s, \mu\rangle_1 = e^{[Q, \psi']}|\phi\rangle_1 = e^{\alpha' K_1 + \mu p^a p_a}|\phi\rangle_1, \quad |s, \nu\rangle_2 = e^{[Q, \psi']}|\phi\rangle_2 = e^{\beta' K_2 + \nu \pi^a \pi_a}|\phi\rangle_2, \quad (6.7)
\]
$\alpha'$ and $\beta'$ are still given by (2.26)-(2.28), but $\mu$ and $\nu$ are expressions which also involve the parameters $\lambda_i$ in (6.2) (see (3.17) appendix B). Nonzero $\mu$ or $\nu$ leads to a regular effective BRST invariant Hamiltonian or Lagrangian in a bosonic gauge theory. We note that the duality property (2.3) here generalizes to
\[
e^{\alpha K_1} e^{\mu p^a p_a}|\phi\rangle_1 = e^{-\frac{1}{\alpha} K_2 - \frac{\mu}{\alpha^2} \pi^a \pi_a}|\phi\rangle_2. \]
(6.8)

This implies (cf.(3.23))
\[
U_S(\ln \alpha^2) U_R(\pi/2) e^{\alpha K_1 + \mu p^a p_a}|\phi\rangle_1 = e^{-\frac{1}{\alpha} K_2 - \frac{\mu}{\alpha^2} \pi^a \pi_a}|\phi\rangle_2 = e^{\alpha K_1 - \mu p^a p_a}|\phi\rangle_1. \]
(6.9)

Thus, states with opposite signs of $\mu$ ($\nu$) are unitarily equivalent. In fact, here we have a much larger set of unitary gauge transformations than those in section 3 since we only have
to require $U = e^{iQ,\tilde{\psi}}$ to commute with the first ghost number operator $N_b$ in (2.17). $\tilde{\psi}'$ may therefore have the same form as $\psi'$ in (6.1). We have now e.g. gauge transformations of the form

$$U(p_i) = e^{ip_1V_1+ip_2V_2+ip_3V_3},$$

(6.10)

where $p_i$ are real constants, corresponding to the last terms in (6.1). In fact, in appendix B it is shown that arbitrary values of $\lambda_i$ in (6.2) may be reached from $\lambda_i = 0$ by unitary transformations with (6.10) except if $\delta = 0$ or $(\delta^2/2) = \alpha\beta + \gamma^2$. ($p_i$ are infinite for these values.) We note that $\delta \neq 0$ allows for $\psi'$ in (6.1) to be chosen nilpotent, and that $(\delta^2/2) \neq \alpha\beta + \gamma^2$ is required from the classical requirement that $\Lambda^a = 0$ and $\chi^a = 0$ together with the original first class constraints must constitute a set of second class constraints. We have the relations

$$U(p_i)e^{\alpha K_1+\beta K_2+2\gamma K_3+\delta K_4}\phi_{1,2} = e^{\alpha \tilde{K}_1+\beta \tilde{K}_2+2\gamma \tilde{K}_3+\delta \tilde{K}_4}\phi_{1,2},$$

$$|\tilde{\phi}_{1,2} = U(p_i)|\phi_{1,2},$$

(6.11)

where $\tilde{K}_i \equiv U(p_i)K_iU^d(p_i)$ satisfy the same algebra as $K_i$. For $p_1 = p_2 = 0$, $p_3 = \lambda$ we have in particular

$$|\tilde{\phi}_{1,2} = |\phi_{1,2}, \quad \tilde{K}_1 = K_1 + \lambda p^a p_a,$$

$$\tilde{K}_2 = K_2 + \lambda \pi^a \pi_a, \quad \tilde{K}_3 = K_3, \quad \tilde{K}_4 = K_4 + \lambda \pi^a \pi_a.$$  

(6.12)

In this case the reduction formulas in section 2 may be used since $\tilde{K}_i$ annihilate the same $|\phi\rangle$-states as $K_i$. This corresponds to the parameter values

$$\lambda_1 = \alpha \lambda, \quad \lambda_2 = -\beta \lambda, \quad \lambda_3 = \delta \lambda$$

(6.13)

in (6.2). We find here

$$U(\lambda)|s,0\rangle_1 = |s,\alpha'\lambda\rangle_1, \quad U(\lambda)|s,0\rangle_2 = |s,\beta'\lambda\rangle_2, \quad U(\lambda) \equiv e^{i\Lambda V_3},$$

(6.14)

in terms of the states (6.7). Hence, the parameters $\mu$ and $\nu$ in (6.7) are pure gauge parameters.

A still more general gauge fixing fermion of the general form (2.15) is

$$\psi = P_a \Lambda^a + C_a \chi^a,$$

(6.15)

where

$$\Lambda^a = \alpha \nu^a + (\delta^2/2 - \gamma) x^a - \zeta_1 p^a + \zeta_3 \pi^a - \Lambda^a_0,$$

$$\chi^a = \beta x^a + (\delta^2/2 + \gamma) \nu^a + \zeta_2 \pi^a + \zeta_5 p^a - \chi^a_0.$$  

(6.16)

where in turn $\Lambda^a_0$ and $\chi^a_0$ are constants. (For $\Lambda^a_0 = 0$ and $\chi^a_0 = 0$ (6.15) agrees with (6.1).)

The corresponding inner product states $e^{iQ,\psi}|\phi\rangle$ may be obtained by means of unitary gauge transformations of the form ($x^a_0$ and $\nu^a_0$ are real constants)

$$U = e^{i[Q,\Phi_a x^a_0 + C_a \nu^a_0]} = e^{i2\nu^a_0 p_a + i \nu^a_0 \pi_a}$$

(6.17)

provided $(\delta^2/2) \neq \alpha\beta + \gamma^2$. This condition we recognize as the condition for $\Lambda^a$ and $\chi^a$ to be consistent gauge fixing variables classically. If $(\delta^2/2) = \alpha\beta + \gamma^2$ then the inner product
inner product states (6.7) are well defined at these values. States from (6.1) and (6.15) seems to be gauge inequivalent although the corresponding \( \delta \) fermionic integration yields zero. Although we have found difficulties with certain gauge the normalizations are undefined since the bosonic Jacobi determinantal is infinite while the

$$\langle \alpha | (\bar{a} + \bar{M} \bar{\gamma})^{2} \{ z^a + \sqrt{\alpha \beta + \gamma^2} \} u^a \rangle^2 \times \delta^{(n)}(\eta + \frac{\delta + M \tilde{\delta}}{\tilde{\alpha} + M \tilde{\gamma}}) \theta),$$

$$\psi_2(z, u, \eta, \theta) = A' e^{K'((\bar{N} + \bar{a})}{z^a + \sqrt{\alpha \beta + \gamma^2} \} u^a \rangle^2 \times \delta^{(n)}(\eta + \frac{\delta + N \tilde{\delta}}{\tilde{\gamma} + N \tilde{\alpha}}) \theta),$$

where

$$K = \frac{A}{2i(R + SM)}, \quad K' = \frac{A'}{2i(R' + S'N)}.$$  

If we still impose conditions (5.19) we find for the norms of \( |s\rangle_{1,2} \)

$$\int d^n u d^n z d^n \theta d^n \eta \psi_1^*(z, -u, \eta, -\theta) \psi_1(z, u, \eta, \theta) \propto \int d^n u d^n z d^n \theta d^n \eta \psi_2^*(z, -u, \eta, -\theta) \psi_2(z, u, \eta, \theta) \propto \int d^n u d^n z d^n \theta d^n \eta \psi_1^*(z, -u, \eta, -\theta) \psi_2(z, u, \eta, \theta) \propto \int d^n u d^n z d^n \theta d^n \eta \psi_2^*(z, -u, \eta, -\theta) \psi_2(z, u, \eta, \theta) \propto \int d^n u d^n z d^n \theta d^n \eta \psi_2^*(z, -u, \eta, -\theta) \psi_2(z, u, \eta, \theta) \propto \int d^n u d^n z d^n \theta d^n \eta \psi_2^*(z, -u, \eta, -\theta) \psi_2(z, u, \eta, \theta) \propto \int d^n u d^n z d^n \theta d^n \eta \psi_2^*(z, -u, \eta, -\theta) \psi_2(z, u, \eta, \theta) \propto$$

This is well defined for \( \alpha \sin 2\sqrt{\alpha \beta + \gamma^2} \neq 0 \) (\( \beta \sin 2\sqrt{\alpha \beta + \gamma^2} \neq 0 \)) provided \( K((\bar{a} + \bar{M} \bar{\gamma})^2 \) \( K'((\bar{N} + \bar{a})^2 \) has a negative real part. This seems always possible to achieve with an appropriate choice of \( \bar{a} \) and \( \bar{\gamma} \). However, for \( \alpha \sin 2\sqrt{\alpha \beta + \gamma^2} = 0 \) (\( \beta \sin 2\sqrt{\alpha \beta + \gamma^2} = 0 \)) the normalization are undefined since the bosonic Jacobi determinantal is infinite while the fermionic integration yields zero. Although we have found difficulties with certain gauge transformations at the values \( \delta = 0 \) and \( (\delta/2)^2 = \alpha \beta + \gamma^2 \) the inner products are perfectly well defined at these values.

Above we have investigated some applications of a general gauge fixing fermion

$$\psi = \mathcal{P}_a \Lambda^a + \bar{c}_a \chi^a,$$

where \( \Lambda^a \) and \( \chi^a \) are linear and covariant in the basic variables for a simple abelian bosonic gauge theory. We have then shown that the properties of the BRST singlets expressed in terms of \( e^{[Q, \psi]} \) in general are determined by the SL(2,R) properties of the commutators
A characteristic feature of these cases is also that the gauge fixing variables $\Lambda^a$ and
\[ \chi^a \]
may in general be chosen to commute without affecting $[Q, \psi]$ which in turn implies that $\psi$ in (6.22) may be chosen to be nilpotent. (Here this was true if $\delta \neq 0$.) These properties are expected to be valid also in more general gauge theories, which to some extent is verified in the next section.

If we instead interpret $Q$ as the BRST charge of a fermionic gauge theory then the general form (2.16) allows for the gauge fixing fermion
\[ \psi' = \psi + \rho_1 C_a x^a + \rho_2 \bar{P}_a x^a + \rho_3 C_a \psi^a + \rho_4 \bar{P}_a \psi^a, \]
(6.23)
where $\psi$ is given by (2.18). This expression yields in the case of a symmetric metric
\[ [Q, \psi'] = \alpha K_1 + \beta K_2 + 2\gamma K_3 + \delta K_4 + i\rho \bar{P}_a C^a, \]
(6.24)
where in turn $\rho = \rho_2 - \rho_3$. Note that
\[ [Q, C_a x^a] = [Q, \bar{P}_a x^a] = 0, \]
\[ [Q, \bar{P}_a x^a] = -[Q, C_a \psi^a] = i\bar{P}_a C^a. \]
(6.25)
Note also that $K_i$ commute with $\bar{P}_a C^a$, but that
\[ [K_4, \bar{P}_a C^a] = -i\bar{P}_a C^a, \]
(6.26)
which means that $\bar{P}_a C^a$ is an SL(2,R) scalar that scales under $K_4$. $\psi'$ in (6.24) may be chosen to be nilpotent without affecting $[Q, \psi']$ provided $\delta \neq 0$ and $(\delta/2)^2 \neq \alpha \beta + \gamma^2$. The last condition follows also if we require $\chi_a$ and $\Lambda_a$ in (2.16) to be consistent gauge fixing variables. Also here we have a much larger set of unitary gauge transformations than those in section 3 since we only have to require $U = e^{i\sum [Q, \psi']}$ to commute with the second ghost number operator $N_f$ in (2.17). $\psi'$ may therefore have the same form as $\psi'$ in (6.23). We have now gauge transformations of the form
\[ U(\lambda) = e^{\lambda \bar{P}_a C^a}, \]
(6.27)
where $\lambda$ is a real constant, corresponding to the last terms in (6.24). We find from (6.27)
\[ U(\lambda)e^{\alpha K_1 + \beta K_2 + 2\gamma K_3 + \delta K_4} |\phi\rangle_{1,2} = e^{\alpha \tilde{K}_1 + \beta \tilde{K}_2 + 2\gamma \tilde{K}_3 + \delta \tilde{K}_4} |\phi\rangle_{1,2} = e^{\alpha \lambda \bar{P}_a C^a} |\phi\rangle_{1,2}, \]
(6.28)
since $\tilde{K}_i \equiv U(\lambda) K_i U^+(\lambda) = K_i$ and $\tilde{K}_4 \equiv U(\lambda) K_4 U^+(\lambda) = K_4 + i\lambda \bar{P}_a C^a$. Hence, arbitrary values of $\rho$ in (6.25) may be reached from the gauge fixing $\psi$ provided $\delta \neq 0$ a condition which also the nilpotence of $\psi'$ in (6.23) required.

7 Applications to nontrivial models

In this section we give two examples of non-trivial bosonic gauge theories to which our formulas may easily be applied.
7.1 Example 1: Cohomological dynamics

Any regular dynamical system may be written in a reparametrization invariant form [17]. The BFV-BRST charge for such a theory is

\[ Q = \mathcal{C}(\pi + H) + \bar{\mathcal{P}}\pi_v, \]  

(7.1)

where \( H \) is the Hamiltonian of the original theory and \( \pi \) the conjugate momentum to time, \( t \), which here is a dynamical variable. \( \pi_v \) is a conjugate momentum to a Lagrange multiplier \( v \). All variables are hermitian. Since \( H \) commutes with \( t \) we may treat \( \pi + H \) as conjugate momentum to \( t \). It is then quite obvious that all formulas in their most general form given in section 6 and appendix B apply here. The BRST singlets are of the form

\[ \langle s | = e^{[Q, \tilde{\psi}]} |\phi\rangle, \]  

(7.2)

where \( \tilde{\psi} \) and \([Q, \tilde{\psi}]\) are given by (6.1)-(6.3). The \( |\phi\rangle \)-state may e.g. be chosen to be \( |\phi\rangle_1 \) or \( |\phi\rangle_2 \) determined by the conditions

\[ t|\phi\rangle_1 = \mathcal{C}|\phi\rangle_1 = \tilde{\mathcal{C}}|\phi\rangle_1 = \pi_v|\phi\rangle_1 = 0, \]  

(7.3)

\[ v|\phi\rangle_2 = \bar{\mathcal{P}}|\phi\rangle_2 = \mathcal{P}|\phi\rangle_2 = (\pi + H)|\phi\rangle_2 = 0. \]  

(7.4)

Note that the last condition is the Schrödinger equation. In this case we have no covariant indices to care about. We may therefore without problems consider transformations obtained from the BRST invariant operator \( V \) given by (cf (6.17))

\[ V \equiv e^{i[Q, t_0\mathcal{P} + v_0\tilde{\mathcal{C}}]}, \]  

(7.5)

where \( t_0 \) and \( v_0 \) are two constants. We find then

\[ V|s\rangle = e^{i[Q, \tilde{\psi}]}|\tilde{\phi}\rangle, \quad \tilde{\psi} = V\psi V^{-1}, \quad |\tilde{\phi}\rangle = V|\phi\rangle. \]  

(7.6)

\( \tilde{\psi} \) is equal to \( \psi' \) with \( t \) and \( v \) replaced by \((t - t_0)\) and \((v - v_0)\), and \( |\tilde{\phi}\rangle_{1,2} \) satisfy (7.3) and (7.4) with the first conditions replaced by

\[ (t - t_0)|\tilde{\phi}\rangle_1 = 0, \quad (v - v_0)|\tilde{\phi}\rangle_2 = 0. \]  

(7.7)

We have then arrived at the gauge fixings considered in [8]. However, the singlets \( V|s\rangle \) are a considerable generalization of those given in [8].

7.2 Example 2: Nonabelian gauge theory

The BFV-BRST charge for a general bosonic nonabelian gauge theory with a finite number of degrees of freedom is given by (a, b, c = 1, ..., n < \infty [13]

\[ Q = \theta_a \mathcal{C}^a - \frac{1}{2} iU_{bc}^a \mathcal{P}_a \mathcal{C}^b \mathcal{C}^c - \frac{1}{2} iU_{ab}^c \mathcal{C}^a + \bar{\mathcal{P}}\pi_a \]  

(7.8)

where \( \theta_a \) are the hermitian bosonic gauge generators (constraints) satisfying

\[ [\theta_a, \theta_b] = iU_{ab}^c \theta_c \]  

(7.9)
where $U_{ab}^c$ are real structure constants. (We consider Lie group theories.) Remarkably enough there exists a simple abelianization of the BRST charge (7.8) by means of which all our results for abelian models can be directly applied to (7.8) (see also [15]). Introduce canonical group coordinates, $x^a$, which are hermitian operators satisfying the properties

$$[x^a, x^b] = 0, \quad [x^a, \theta_b] = iM^a_b(x),$$

(7.10)

where the hermitian matrix operators $M^a_b(x)$ satisfy $M^a_b(x)x^b = x^a$, $M^a_b(0) = \delta^a_b$ and the Maurer-Cartan equations

$$(\partial_d M^c_a)M^d_b - (\partial_d M^c_b)M^d_a = U_{ab}^d M^c_d.$$  (7.11)

We may then define hermitian conjugate momenta, $p_a$, to $x^a$ by

$$p_a \equiv (M^{-1})^b_a(x)\theta_b + i\frac{1}{2}(M^{-1})^b_a(x)\partial_c M^c_b(x)$$

(7.12)

in terms of which we have

$$\theta_a = \frac{1}{2}\left[p_b M^b_a(x) + M^b_a(x)p_b\right].$$  (7.13)

Consider then also the following unitary transformation which only affects $C^a$, $P_a$, and $p_a$

$$C^a \to \tilde{C}^a = M^a_b(x)C^b, \quad P_a \to \tilde{P}_a = (M^{-1})^b_a(x)P_b$$

$$p_a \to \tilde{p}_a = p_a + i\frac{1}{2}\partial_a M^c_b(x)(M^{-1})^d_b (C^c P_d - P_d C^c).$$  (7.14)

If one now inserts (7.13) into (7.8) and replaces $C^a$, $P_a$, and $p_a$ by $\tilde{C}^a$, $\tilde{P}_a$, and $\tilde{p}_a$ using (7.14), one finds

$$Q = \tilde{C}^a\tilde{p}_a + \pi_a\bar{\tilde{P}}^a,$$

(7.15)

which is the BRST charge (2.1) for an abelian model. The complete set of canonical operators are \{ $x^a, \tilde{p}_a; v^a, \pi_a; C^a, \tilde{P}_a; \tilde{C}^a, \bar{P}^a$ \}. Since we here have a bosonic gauge theory we may consider the general gauge fixing fermion (6.1). The BRST singlets are given by expressions of the form

$$|s\rangle = e^{[Q, \psi']}|\phi\rangle,$$

(7.16)

where $[Q, \psi']$ is given by (6.2). The $K_i$-operators (2.10) and (2.20) may then be written in the following invariant forms

$$K_1 \equiv [Q, \tilde{P}a^v] = [Q, P_a (M^{-1})^a_b v^b] = \theta_a (M^{-1})^a_b v^b - i\frac{1}{2}U_{bc}^\epsilon (M^{-1})^b_a v^\epsilon -$$

$$- i\partial_a M^\epsilon_b (M^{-1})^d_b \partial_d C^\epsilon a + iP_a (M^{-1})^a_b \tilde{P}^b,$$

$$K_2 \equiv [Q, \tilde{C}^a x^\pi] = x^\pi \pi_a + i\tilde{C}^a M^a_b C^b,$$

$$K_3 \equiv \frac{1}{2}[Q, \tilde{C}^a v^\pi - \tilde{P}_a x^\pi] = \frac{1}{2}[Q, \tilde{C}^a v^\pi - P_a x^\pi] = \frac{1}{2}\left(v^\pi \pi_a - \theta_a x^\pi + \frac{1}{2}U_{ab}^x x^a -
$$

$$- ix^a U_{ab} dP_a^c - iP_a M^a_b \tilde{C}^b - i\bar{P}^a \tilde{C}^a\right) = \frac{1}{2}\left(\pi_a v^a - x^a \theta_a -
$$

$$- \frac{1}{2} iU_{ab}^x x^a + ix^a U_{ab}^d C^b P_a + iM^a_b \tilde{C}^b P_a + i\bar{C}^a \bar{P}^a\right),$$

22
\[ K_4 \equiv \frac{1}{2} [Q, \tilde{C}_a v^a + \tilde{P}_a x^a] = \frac{1}{2} [Q, \tilde{C}_a v^a + \mathcal{P}_a x^a] = \frac{1}{2} \left( v^a \pi_a + \theta_a x^a - \frac{1}{2} i U^{ab} b x^a + \right. \\
+ i x^a U^{ab} d \mathcal{P}_d c^b + i \mathcal{P}_a M^a c^b - i \mathcal{P}_a C_a \right) = \frac{1}{2} \left( \pi_a v^a + x^a \theta_a + \\
+ \frac{1}{2} i U^{ab} b x^a - i x^a U^{ab} d \mathcal{P}_d c^b - i M^a U^{ab} b \mathcal{P}_b + i \mathcal{C}_a \mathcal{P}_a \right) , \quad (7.17) \]

However, the \( V \)-operators (6.3) are noncovariant. The reason is that (6.1) - (6.3) require \( \tilde{p}^a \) to be obtained from \( \tilde{p}_a \) by means of a constant metric \( \theta^{ab} \), which is unnatural since \( \tilde{p}_a \) has a curved index.

The state \( |\phi\rangle \) in (7.16) may e.g. be chosen to be \( |\phi\rangle_1 \) or \( |\phi\rangle_2 \) determined by the conditions
\[
x^a |\phi\rangle_1 = \tilde{C}_a |\phi\rangle_1 = \bar{C}_a |\phi\rangle_1 = \pi_a |\phi\rangle_1 = 0, \\
v^a |\phi\rangle_2 = \tilde{\mathcal{P}}^a |\phi\rangle_2 = \bar{\mathcal{P}}_a |\phi\rangle_2 = \bar{p}_a |\phi\rangle_2 = 0. \quad (7.18) \]

In terms of the original variables these conditions are
\[
x^a |\phi\rangle_1 = C^a |\phi\rangle_1 = \bar{C}_a |\phi\rangle_1 = \pi_a |\phi\rangle_1 = 0, \\
v^a |\phi\rangle_2 = \bar{\mathcal{P}}^a |\phi\rangle_2 = \mathcal{P}_a |\phi\rangle_2 = \left( \theta_a + \frac{i}{2} U^{ab} b \right) |\phi\rangle_2 = 0. \quad (7.19) \]

To prove the equivalence of the last conditions in (7.18) and (7.19) we have used
\[
(M^{-1})^b_a \partial_b M^c - (M^{-1})^b_a \partial_a M^c_b = (M^{-1})^b_a U^{ab} U^{be} c , \quad (7.20) \]
which follows from (7.11). Concerning the last condition in (7.19) one may note that
\[
[Q, \mathcal{P}_a] |\phi\rangle_2 = \left( \theta_a + \frac{i}{2} U^{ab} b \right) |\phi\rangle_2 . \quad (7.21) \]

Thus, \( |\phi\rangle_1 \) and \( |\phi\rangle_2 \) are determined by manifestly covariant conditions. Note also the invariant properties (2.23). If furthermore the regulator factor \( e^{(Q, \psi)} \) is of the general form considered in sections 2-5 and appendix A, then also the singlets \( |s\rangle_{1,2} = e^{(Q, \psi)} |\phi\rangle_{1,2} \) satisfy manifestly covariant conditions which may be extracted from (A.9). (The special formulas (6.19)-(6.20) in [7] are not quite correct. The adjoint matrix representation \( A^a_b \) in (6.19) should have the argument \( +i \alpha^a \), and the right-hand side of the last equality in (6.20) should have a term \( i \beta^a M^a_b \pi_b \) added.) The singlets (7.16) are a considerable generalization of those given in [4].

8 Conclusions

We have made an extensive analysis of the abstract operator approaches to BRST quantization on inner product spaces as presented in the introduction. For simple abelian models which allow for explicit calculations we have verified the properties and interrelations described in the introduction. For these models we have considered the most general gauge fixing fermions expressed in terms of quadratic invariant terms, and then determined the exact conditions for the existence of BRST singlets as inner product states both abstractly and concretely within a specific state representation. The abstract conditions as well as
the explicit wave function integrations led to the same results. We have verified the general quantization rule that half of the fundamental hermitian operators are to be quantized with imaginary eigenvalues. The choices of gauge fixing fermions $\psi$ were shown to partly govern these quantization rules. Some choices of $\psi$ forced us even to consider complex eigenvalues of the basic variables. In this way we have to some extent explored the freedom this general quantization rule leaves us. We have also found that the importance of the $SL(2,R)$ symmetry in the gauge fixing factor to determine the BRST singlets, first noted in [7], was retained even in the much more general invariant gauge fixing fermions considered here. If we also introduce parameters in $\psi$ then this $SL(2,R)$ symmetry will be broken unless these parameters are introduced by means of unitary gauge transformations as was done in section 6 and at the end of subsection 7.1. We have also investigated unitary gauge transformations and shown that there are natural extended ones performing $SL(2,R)$ transformations although they effectively only act as abelian gauge transformations on the BRST invariant states. Conditions for gauge equivalences as well as whether or not $\psi$ may be chosen nilpotent without affecting the gauge fixing factor were determined. Although our treatment only is valid for simple abelian models the results should be valid in general since we always may perform an abelianization within a sector of a given model by an appropriate choice of $\psi$. In subsection 7.2 this was demonstrated for a general nonabelian model. Finally, it should be mentioned that the corresponding path integral results for the models considered here may be directly extracted from our treatment.
Appendix A

Transformation formulas used in sections 2 and 3

In section 2 and 3 we considered the general gauge fixing fermion \((2.18)\). It leads to the regulator factor

\[
e^{[Q,\psi]} = e^{\alpha K_1 + \beta K_2 + 2\gamma K_3 + \delta K_4},
\]

where \(\alpha, \beta, \gamma\) and \(\delta\) are real parameters. By means of the commutation relations \((2.21)-(2.22)\) of the \(K_i\)-operators we may factorize this regulator factor in various ways. We have e.g. the following alternative forms

\[
e^{[Q,\psi]} = e^{is_1 K_1 e_{s_2 K_2} e^{i\delta K_4}}, \quad \beta \neq 0 \quad (A.2)
\]

\[
e^{[Q,\psi]} = e^{it_1 K_2 e_{t_2 K_1} e^{i\delta K_4}}, \quad \alpha \neq 0 \quad (A.3)
\]

where

\[
s_1 = \frac{i(A - i\gamma B - 1)}{\beta B}, \quad s_2 = \beta B, \quad s_3 = \frac{i(A + i\gamma B - 1)}{\beta B}, \quad (A.4)
\]

and

\[
t_1 = \frac{i(A - i\gamma B - 1)}{\alpha B}, \quad t_2 = \alpha B, \quad t_3 = \frac{i(A + i\gamma B - 1)}{\alpha B}, \quad (A.5)
\]

where in turn

\[
A \equiv \cos \sqrt{\alpha \beta + \gamma^2}, \quad B \equiv \frac{\sin \sqrt{\alpha \beta + \gamma^2}}{\sqrt{\alpha \beta + \gamma^2}}. \quad (A.6)
\]

for \(\alpha \beta + \gamma^2 > 0\). For \(\alpha \beta + \gamma^2 < 0\) we have

\[
A \equiv \cosh \sqrt{-\alpha \beta - \gamma^2}, \quad B \equiv \frac{\sinh \sqrt{-\alpha \beta - \gamma^2}}{\sqrt{-\alpha \beta - \gamma^2}}. \quad (A.7)
\]

\((A = 1 \text{ and } B = 1 \text{ for } \alpha \beta + \gamma^2 = 0)\)

We have used the factorizations above to derive transformations of the form

\[
D' = e^{[Q,\psi]} D e^{-[Q,\psi]} = e^{\alpha K_1 + \beta K_2 + 2\gamma K_3 + \delta K_4} D e^{-\alpha K_1 - \beta K_2 - 2\gamma K_3 - \delta K_4}, \quad (A.8)
\]

where \(D\) is any operator. For the basic variables we find explicitly

\[
x^{a} = (x^a (A + i\gamma B) - i\alpha v^a B) e^{-\frac{i}{2}},
\]

\[
v^{a} = (v^a (A - i\gamma B) - i\beta x^a B) e^{-\frac{i}{2}},
\]

\[
C^{a} = (C^a (A + i\gamma B) - i\alpha \bar{P}^a B) e^{-\frac{i}{2}},
\]

\[
\bar{P}^{a} = (\bar{P}^a (A - i\gamma B) - i\beta C^a B) e^{-\frac{i}{2}},
\]

\[
\bar{C}^{a} = (\bar{C}_a (A + i\gamma B) + i\alpha \bar{p}^a B) e^{\frac{i}{2}},
\]

\[
\bar{p}^{a} = (\bar{p}_a (A - i\gamma B) + i\beta \bar{C}_a B) e^{\frac{i}{2}},
\]

\[
\pi^{a} = (\pi^a (A + i\gamma B) + i\alpha \pi_a B) e^{\frac{i}{2}},
\]

\[
p^{a} = (p_a (A - i\gamma B) + i\beta \pi_a B) e^{\frac{i}{2}}, \quad (A.9)
\]
which are well defined even at $\alpha = 0$ and/or $\beta = 0$.

These transformations imply
\begin{align*}
K'_1 &= (A - i\gamma B)^2 K_1 + \beta^2 B^2 K_2 + 2i\beta (A - i\gamma B)BK_3 \\
K'_2 &= (A + i\gamma B)^2 K_2 + \alpha^2 B^2 K_1 - 2i\alpha (A + i\gamma B)BK_3 \\
K'_3 &= (1 - 2\alpha\beta B^2)K_3 + i\alpha (A - i\gamma B)BK_1 - i\beta (A + i\gamma B)BK_2 \\
K'_4 &= K_4 \\
\end{align*}
(A.10)

The corresponding transformations for the unitary $\text{SL}(2,\mathbb{R}) \times \text{U}(1)$ transformations considered in section 3 are
\begin{align*}
D' &= e^{iaK_1+ibK_2+2icK_3+idK_4}De^{-iaK_1-ibK_2-2icK_3-idK_4}. \\
\end{align*}
(A.11)

The explicit expressions are obtained from (A.9) with the replacements $\alpha \rightarrow ia$, $\beta \rightarrow ib$, $\gamma \rightarrow ic$ and $\delta \rightarrow id$. We find
\begin{align*}
x'^a &= (x^a(A - cB) + av^a B) e^{\frac{d}{2}}, \\
v'^a &= (v^a(A + cB) + bx^a B) e^{\frac{d}{2}}, \\
C'^a &= (C^a(A - cB) + a\bar{P}^a B) e^{\frac{d}{2}}, \\
\bar{P}'^a &= (\bar{P}^a(A + cB) + bC^a B) e^{\frac{d}{2}}, \\
\bar{C}'^a &= (\bar{C}^a(A - cB) - a\bar{C}^a B) e^{-\frac{d}{2}}, \\
P'^a &= (P_a(A + cB) - b\bar{P}^a B) e^{-\frac{d}{2}}, \\
\pi'^a &= (\pi_a(A - cB) - ap_a B) e^{-\frac{d}{2}}, \\
p'^a &= (p_a(A + cB) - b\pi_a B) e^{-\frac{d}{2}}, \\
\end{align*}
(A.12)

where now
\begin{align*}
A &\equiv \cosh \sqrt{ab + c^2}, & B &\equiv \frac{\sinh \sqrt{ab + c^2}}{\sqrt{ab + c^2}} \\
\end{align*}
(A.13)

for $ab + c^2 > 0$. For $ab + c^2 < 0$ we have to use the replacement
\begin{align*}
\cosh \sqrt{ab + c^2} \rightarrow \cos \sqrt{-ab - c^2}, & & \sinh \sqrt{ab + c^2} \rightarrow \sin \sqrt{-ab - c^2}. \\
\end{align*}
(A.14)

The unitary transformations (A.12) imply
\begin{align*}
K'_1 &= (A + cB)^2 K_1 - b^2 B^2 K_2 - 2b(A + cB)BK_3 \\
K'_2 &= (A - cB)^2 K_2 - a^2 B^2 K_1 + 2a(A - cB)BK_3 \\
K'_3 &= (1 + 2abB^2)K_3 - a(A + cB)BK_1 + b(A - cB)BK_2 \\
K'_4 &= K_4 \\
\end{align*}
(A.15)
Derivation of the reduction formulas (2.25).

In order to calculate the regulator factor (A.1) on \(|\phi\rangle\)-states as in (2.25) the following factorizations are convenient to use

\[
e^{[Q,\psi]} = e^{f_1 K_1 e^{(f_2 + i f_3) K_3} e^{f_4 K_2} e^{\delta K_4}},
\]

(A.16)

\[
e^{[Q,\psi]} = e^{g_1 K_2 e^{(g_2 + i g_3) K_3} e^{g_4 K_1} e^{\delta K_4}},
\]

(A.17)

where

\[
f_1 = \frac{\alpha B}{A + i \gamma B}, \quad f_2 = -\arctan\left(\frac{\gamma B}{A}\right),
\]

\[
f_3 = \ln |A + i \gamma B|, \quad f_4 = \frac{\beta B}{A + i \gamma B},
\]

(A.18)

\[
g_1 = \frac{\beta B}{A - i \gamma B}, \quad g_2 = -\arctan\left(\frac{\gamma B}{A}\right),
\]

\[
g_3 = -\ln |A + i \gamma B|, \quad g_4 = \frac{\alpha B}{A - i \gamma B}.
\]

(A.19)

Appendix B

Transformation formulas used in section 6

Here we explore the extra features of the theory that follows from the more general choice of gauge fixing fermion made in (6.1), which is valid for simple abelian bosonic gauge theories. As we will see all the relevant properties of the theory is still dictated by the \(SL(2, R)\)-sector of the gauge fixing considered in appendix A. The regulator factor is here given by

\[
\exp\{[Q,\psi]\} = \exp \{\alpha K_1 + \beta K_2 + 2\gamma K_3 + \delta K_4 + \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3\},
\]

(B.1)

where \(\lambda_1, \lambda_2\) and \(\lambda_3\) are new real parameters. It may e.g. be factorized in the following two forms

\[
e^{[Q,\psi]} = e^{ip_3 V_3} e^{ip_1 V_1} e^{ip_2 V_2} e^{i s_1 K_1} e^{s_2 K_2} e^{i s_3 K_1} e^{\delta K_4} e^{-ip_2 V_2} e^{-ip_1 V_1} e^{-ip_3 V_3},
\]

(B.2)

\[
e^{[Q,\psi]} = e^{ip_3 V_3} e^{ip_1 V_1} e^{ip_2 V_2} e^{it_1 K_1} e^{it_2 K_2} e^{it_3 K_1} e^{\delta K_4} e^{-ip_2 V_2} e^{-ip_1 V_1} e^{-ip_3 V_3},
\]

(B.3)

where the parameters \(s_i\) and \(t_i\) are given in (A.4) and (A.5), and where

\[
p_1 = \frac{\left(\frac{\delta}{\gamma}\right)^2 - \alpha \beta + \delta \gamma}{2 \delta \left(\frac{\delta}{\gamma}\right)^2 - \alpha \beta - \gamma^2} \lambda_1 - \alpha^2 \lambda_2 - \alpha \left(\frac{\delta}{\gamma} + \gamma\right) \lambda_3,
\]

(B.4)

\[
p_2 = \frac{-\beta^2 \lambda_1 + \left(\frac{\delta}{\gamma}\right)^2 - \alpha \beta - \delta \gamma}{2 \delta \left(\frac{\delta}{\gamma}\right)^2 - \alpha \beta - \gamma^2} \lambda_2 + \beta \left(\frac{\delta}{\gamma} - \gamma\right) \lambda_3.
\]

(B.5)
\[ p_3 = \frac{-\beta \left( \frac{\delta}{2} + \gamma \right) \lambda_1 + \alpha \left( \frac{\delta}{2} - \gamma \right) \lambda_2 + \left( \frac{\delta}{2} \right)^2 - \gamma^2 \right) \lambda_3}{\delta \left( \frac{\delta}{2} \right)^2 - \alpha \beta - \gamma^2}. \]  

(B.6)

These results imply that

\[ e^{[Q, \psi]} = U(p_i)e^{[Q, \psi]}U^\dagger(p_i), \]  

(B.7)

where

\[ U(p_i) \equiv e^{ip_1 V_1 + ip_2 V_2 + ip_3 V_3}. \]  

(B.8)

is a unitary operator for gauge transformations. Thus, the regulator factor (B.1) may be obtained from the one in appendix A by means of a unitary gauge transformation. However, note that the parameters \( p_i \) in (B.3)-(B.6) are infinite at \( \delta \rightarrow 0 \) and at \( \delta \rightarrow \pm 2 \sqrt{\alpha \beta + \gamma^2} \). (As was mentioned in section 6, \( \delta \neq \pm 2 \sqrt{\alpha \beta + \gamma^2} \) was required for \( \Lambda^a \) and \( \chi^a \) in (2.15) to be consistent gauge fixing conditions classically, and \( \delta \neq 0 \) allows us to choose \( \psi' \) in (B.1) to be nilpotent without affecting the gauge fixing factor \( e^{[Q, \psi']} \).

By means of (B.3) we find now the transformation formulas (A.9) in appendix A where \( x'^a \) and \( v'^a \) are generalized to

\[ x'^a = (x^a(A + i\gamma B) - i\alpha B v^a) e^{-i\frac{\delta}{2}} + R_1 p^a + R_2 \pi^a, \]

\[ v'^a = (v^a(A - i\gamma B) - i\beta B x^a) e^{-i\frac{\delta}{2}} + R_3 p^a + R_4 \pi^a, \]  

(B.9)

where

\[ R_1 \equiv 4i p_1 Im(e^{-i\frac{\delta}{2}}(A + i\gamma B)) - 2i \alpha B p_3 \cos \frac{\delta}{2}, \]

\[ R_2 \equiv -2i(A + i\gamma B)p_3 \sin \frac{\delta}{2} + 2i B(e^{-i\frac{\delta}{2}} p_2 \alpha - e^{i\frac{\delta}{2}} p_1 \beta), \]

\[ R_3 \equiv -2i(A - i\gamma B)p_3 \sin \frac{\delta}{2} + 2i B(e^{i\frac{\delta}{2}} p_2 \alpha - e^{-i\frac{\delta}{2}} p_1 \beta), \]

\[ R_4 \equiv 4i p_2 Im(e^{i\frac{\delta}{2}}(A + i\gamma B)) - 2i \beta B p_3 \sin \frac{\delta}{2}. \]  

(B.10)

where the parameters \( p_i \) are given by (B.3)-(B.6). The parameters \( R_i \), and therefore also \( x'^a \) and \( v'^a \), are perfectly well defined and finite at \( \delta \rightarrow 0, \delta \rightarrow \pm 2 \sqrt{\alpha \beta + \gamma^2} \) although the parameters \( p_i \) in (B.3)-(B.6) then are infinite. For \( \delta \rightarrow 0 \) we find the limiting values to be

\[ R_1 = -\frac{2i \gamma^2 \lambda_1 B}{\alpha \beta + \gamma^2} - \frac{i \alpha A(\beta \lambda_1 + \alpha \lambda_2 + \gamma \lambda_3)}{\alpha \beta + \gamma^2} + \frac{B}{2(\alpha \beta + \gamma^2)} \left\{ (2 \alpha \beta^2 + 2 \beta \gamma^2) \lambda_1 + (\alpha^2 + \alpha \beta + 2 \alpha \gamma^2) \lambda_2 + (2 \alpha \beta \gamma + 2 \gamma^3) \lambda_3 \right\}, \]

\[ R_2 = \frac{i \gamma(\beta \lambda_1 + \alpha \lambda_2)(B - A)}{\alpha \beta + \gamma^2} - \frac{i \gamma^2 A \lambda_3}{\alpha \beta + \gamma^2} + \frac{B}{2(\alpha \beta + \gamma^2)} \left\{ (2 \alpha \beta^2 + 2 \beta \gamma^2) \lambda_1 + (\alpha^2 + \alpha \beta + 2 \alpha \gamma^2) \lambda_2 + (2 \alpha \beta \gamma + 2 \gamma^3) \lambda_3 \right\}, \]

\[ R_3 = \frac{i \gamma(\beta \lambda_1 + \alpha \lambda_2)(B - A)}{\alpha \beta + \gamma^2} - \frac{i \gamma^2 A \lambda_3}{\alpha \beta + \gamma^2} - \frac{B}{2(\alpha \beta + \gamma^2)} \left\{ (2 \alpha \beta^2 + 2 \beta \gamma^2) \lambda_1 + (\alpha^2 + \alpha \beta + 2 \alpha \gamma^2) \lambda_2 + (2 \alpha \beta \gamma + 2 \gamma^3) \lambda_3 \right\}, \]

\[ R_4 = \frac{2i \gamma^2 \lambda_2 B}{\alpha \beta + \gamma^2} + \frac{i \beta A(\beta \lambda_1 + \alpha \lambda_2 + \gamma \lambda_3)}{\alpha \beta + \gamma^2}. \]  

(B.11)
where $A$ and $B$ is given in (A.6)-(A.7) appendix A. At $\delta \to \pm 2\sqrt{\alpha \beta + \gamma^2}$ we find

\[
R_1 = -\frac{\lambda_3 i \sin 2\sqrt{\alpha \beta + \gamma^2}}{\sqrt{\alpha \beta + \gamma^2}},
\]

\[
R_2 = (\beta \lambda_1 + \alpha \lambda_2 + \gamma \lambda_3) \frac{\sin^2 \sqrt{\alpha \beta + \gamma^2}}{\alpha \beta + \gamma^2} - i\lambda_3 \frac{\sin 2\sqrt{\alpha \beta + \gamma^2}}{\sqrt{\alpha \beta + \gamma^2}},
\]

\[
R_3 = -(\beta \lambda_1 + \alpha \lambda_2 + \gamma \lambda_3) \frac{\sin^2 \sqrt{\alpha \beta + \gamma^2}}{\alpha \beta + \gamma^2} - i\lambda_3 \frac{\sin 2\sqrt{\alpha \beta + \gamma^2}}{\sqrt{\alpha \beta + \gamma^2}},
\]

\[
R_4 = \frac{\lambda_3 i \sin 2\sqrt{\alpha \beta + \gamma^2}}{\sqrt{\alpha \beta + \gamma^2}}.
\]

That the remaining basic variables transform exactly as in (A.9) follows trivially from the algebra since the $V_k$ generators only couples to $x^a$ and $v^a$. The transformation of the basic generators $K_i$ and $V_k$ follows readily, (Below, $K'_i$, refers to the $SL(2, R) \times U(1)$-transformed part, given by (A.10).)

\[
K''_1 = K'_1 + (A - i\gamma B)R_3 e^{\frac{i\delta}{2}} V_1 - i\beta B R_1 e^{\frac{i\delta}{2}} V_2 +
+((A - i\gamma B) R_4 + i\beta B R_3)e^{\frac{i\delta}{2}} V_3,
\]

\[
K''_2 = K'_2 + i\alpha B R_1 e^{\frac{i\delta}{2}} V_1 - i(A + i\gamma B) R_2 e^{\frac{i\delta}{2}} V_2 +
+((A + i\gamma B) R_2 + i\alpha B R_2)e^{\frac{i\delta}{2}} V_3,
\]

\[
K''_3 = K'_3 + (i\alpha B R_3 - (A - i\gamma B) R_1)e^{\frac{i\delta}{2}} V_1 -
-(A + i\gamma B)(i R_4 - i\beta B R_2)e^{\frac{i\delta}{2}} V_2 +
+((A + i\gamma B) R_4 + i\alpha B R_4 - i\beta B R_1 - (A - i\gamma B) R_2)e^{\frac{i\delta}{2}} V_3,
\]

\[
K''_4 = K'_4 + (i\alpha B R_3 - (A - i\gamma B) R_1)e^{\frac{i\delta}{2}} V_1 -
-(A + i\gamma B) R_4 - i\beta B R_2)e^{\frac{i\delta}{2}} V_2 +
+((A + i\gamma B) R_4 + i\alpha B R_4 - i\beta B R_1 - (A - i\gamma B) R_2)e^{\frac{i\delta}{2}} V_3,
\]

\[
V''_1 = ((A - i\gamma B)^2 V_1 - \beta^2 B^2 V_2 + 2i\beta B(A - i\gamma B) V_3)e^{i\delta},
\]

\[
V''_2 = (A + i\gamma B)^2 V_2 - 2i\alpha B(A - i\gamma B) V_1 - i\beta B(A + i\gamma B) V_2)e^{i\delta},
\]

\[
V''_3 = ((A^2 + \gamma^2 B^2 - \alpha \beta B^2) V_3 + i\alpha B(A - i\gamma B) V_1 - i\beta B(A + i\gamma B) V_2)e^{i\delta}.
\]

**Derivation of the reduction formulas (B.7).**

In order to compute the action of the regulator factor (B.1) on the states $|\phi\rangle_1$ and $|\phi\rangle_2$ it is convenient to use other factorizations than (B.2)-(B.3). Two different factorizations suitable for actions on $|\phi\rangle_1$ and $|\phi\rangle_2$ follow from (B.7) and (A.16)-(A.17). They are given by

\[
e^{[Q, \psi]} = e^{ip_3 V_3} e^{ip_1 V_1} e^{ip_2 V_2} e^{f_1 K_1 e^{(f_2 + i f_3) K_2 e^{f_4 K_4 e^{-ip_2 V_2} e^{-ip_1 V_1} e^{-ip_3 V_3}}}, \quad (B.14)
\]

\[
e^{[Q, \psi]} = e^{ip_3 V_3} e^{ip_1 V_1} e^{ip_2 V_2} e^{g_1 K_2 e^{(g_2 + i g_3) K_3 e^{g_4 K_4 e^{-ip_2 V_2} e^{-ip_1 V_1} e^{-ip_3 V_3}}}, \quad (B.15)
\]

29
where the parameters $p_i$ are given by (3.4)-(3.6), and where $f_i$ and $g_i$ are given in (A.18) and (A.19) respectively. The factorizations (B.14) and (B.15) imply
\[ e^{[Q,\psi]}|\phi\rangle_1 = e^{f_1 K_1} e^{\nu V_1} |\phi\rangle_1, \quad e^{[Q,\psi]}|\phi\rangle_2 = e^{g_1 K_2} e^{\nu V_2} |\phi\rangle_2, \] (B.16)
where
\[\mu \equiv -ie^{(i\delta-2i(f_2+if_3))}p_1 + ip_1 + f_1 p_3 + if_1^2 p_2,\]
\[\nu \equiv -ie^{(i\delta-2i(g_2+ig_3))}p_2 - ip_2 + g_1 p_3 - ig_1^2 p_1.\] (B.17)
Note that $\mu$ and $\nu$ are finite at $\delta \to 0$ and $\delta \to \pm \sqrt{\alpha \beta + \gamma^2}$ although $p_1$, $p_2$ and $p_3$ are infinite in these limits. Since $\mu$ and $\nu$ may be introduced by unitary gauge transformations they do not affect the inner product properties (see below and section 6).

**Conditions for finite inner-products**

The non-hermitian operator doublets are here given by
\[\det(D'(1), D'(2)\dagger) = \left|\begin{array}{ccc}
-4\text{Re}(\mu) & 0 & 2\text{Re}(f_1) \\
0 & -2\text{Re}(f_1) & 0 \\
2\text{Re}(f_1) & 0 & 0
\end{array}\right| \otimes 1_{n \times n} = (16\text{Re}(f_1))^{4n},\] (B.21)
and similarly
\[\det(D'(2), D'(2)\dagger) = (16\text{Re}(g_1))^{4n}.\] (B.22)

The basic condition (4.16) for inner product spaces require then
\[\text{Re} f_1 \neq 0 \iff \alpha \neq 0, \quad F \neq 0,\]
\[\text{Re} g_1 \neq 0 \iff \beta \neq 0, \quad F \neq 0,\] (B.23)
where $F$ is defined in (4.3). Thus
\[F \neq 0 \iff \alpha \beta + \gamma^2 \neq \frac{n^2 \pi^2}{4} \text{ for any integer } n \neq 0.\] (B.24)
These conditions are identical to those in section 2.
References

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