WELL-POSEDNESS FOR THE TWO DIMENSIONAL GENERALIZED ZAKHAROV-KUZNETSOV EQUATION IN WEIGHTED SOBOLEV SPACES

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ABSTRACT. We consider the well-posedness of the initial value problem associated to the $k$-generalized Zakharov-Kuznetsov equation in fractional weighted Sobolev spaces $H^s(\mathbb{R}^2) \cap L^2(|(x,y)|^{2r} \; dx \; dy)$, $s, r \in \mathbb{R}$. Our method of proof is based on the contraction mapping principle and it mainly relies on the well-posedness results recently obtained for this equation in the Sobolev spaces $H^s(\mathbb{R}^2)$ and a new pointwise commutator type formula involving the group induced by the linear part of the equation and the fractional weights to be considered.

1. Introduction

Our aim is to study persistence properties of solutions of the two dimensional $k$-generalized Zakharov-Kuznetsov equation (gZK) in fractional weighted spaces. More precisely we consider the initial value problem (IVP):

$$\begin{cases}
\partial_t u + \partial_x \Delta u + u^k \partial_x u = 0, & t \in \mathbb{R}, \; (x, y) \in \mathbb{R}^2, \; k \in \mathbb{Z}^+, \\
u(x, y, 0) = u_0(x, y).
\end{cases}$$

Let us introduce the weighted Sobolev spaces of our interest

$$Z_{s,r} = H^s(\mathbb{R}^2) \cap L^2(|(x,y)|^{2r} \; dx \; dy), \; s, r \in \mathbb{R},$$

We want to show that for initial data in this function space the associated IVP is either locally or globally well-posed. In general an IVP is said to be locally well-posed (LWP) in a function space $X$ if for each $u_0 \in X$ there exist $T > 0$ and a unique solution $u \in C([-T,T] : X) \cap \cdots = Y_T$ of the equation, such that the map data $\rightarrow$ solution is locally continuous from $X$ to $Y_T$.

This notion of LWP includes the “persistence” property, i.e. the solution describes a continuous curve on $X$. In particular, this implies that the solution flow of the considered equation defines a dynamical system in $X$. Whenever $T$ can be taken arbitrarily large we say that the IVP is globally well-posed (GWP).

It is important to mention that this family of dispersive equations includes the Zakharov-Kuznetsov (ZK) equation ($k=1$) and the modified Zakharov-Kuznetsov (mZK) equation ($k=2$) which are considered two dimensional versions of the famous Korteweg-de Vries (KdV) and modified Kortewg-de Vries (mKdV) equations respectively. The ZK equation was introduced by Zakharov and Kuznetsov in [22] in the context of plasma physics in order to model the propagation of ion-acoustic waves in magnetized plasma. On the other hand mKdV equation is used to describe the propagation of Alfvén waves at a critical angle to an undisturbed magnetic field (see [12]) and MZK is related to the same type of phenomena in two dimensions (see [20]).
In order to motivate our results we note that for the gKdV IVP:

\begin{equation}
\begin{aligned}
\partial_t u + \partial_x^3 u + u^k \partial_x u &= 0, \quad t, x \in \mathbb{R}, \quad k \in \mathbb{Z}^+,
\end{aligned}
\end{equation}

Kato showed in [13] the persistence of solutions in the weighted Sobolev spaces

\[ Z_{s,m} = H^s(\mathbb{R}) \cap L^2(|x|^{2m} \, dx), \quad s \geq 2m, \quad m = 1, 2, \ldots \]

The proof of this result is based on the commutative property of the operators

\begin{equation}
\Gamma = x - 3t \partial_x^2, \quad \mathcal{L} = \partial_t + \partial_x^3, \quad \text{so} \quad [\Gamma; \mathcal{L}] = 0.
\end{equation}

Let us consider the linear IVP

\begin{equation}
\partial_t v + \partial_x^3 v = 0, \quad t, x \in \mathbb{R}, \quad v(x, 0) = v_0(x),
\end{equation}

and let us denote by \( \{U(t) : t \in \mathbb{R}\} \) the unitary group of operators describing its solution, that is

\begin{equation}
U(t)v_0(x) = (e^{it\xi^3} v_0)(x)
\end{equation}

Then (1.4) is equivalent to

\begin{equation}
x U(t)v_0(x) = U(t)(xv_0)(x) + 3t U(t)(\partial_x^2 v_0)(x),
\end{equation}

This equality clearly suggests that regularity and decay are strongly related and furthermore in order to obtain persistent properties for the flow in (1.5) in those weighted spaces \( Z_{s,r} \), at least twice of the decay rate \( r \) is expected to be required in regularity, that is, \( s \geq 2r \).

Notice that Kato’s result strongly indicates us that this condition should hold even for the non-linear associated IVP. In fact, this was recently proved by Fonseca, Linares and Ponce in [8] where they extended (1.4) to fractional powers of \( |x| \) with the help of a point-wise version of the homogeneous derivative of order \( s \) introduced by Stein [21]. In that way, they improved Kato’s results for the gKdV in those fractional Sobolev weighted spaces. Their argument, via contraction principle, also required some detail on previous results on LWP and GWP results for the gKdV IVP on the classical Sobolev spaces \( H^s(\mathbb{R}) \) obtained by Kenig, Ponce and Vega, see [14], [15].

In view of the ideas just presented in the case of the gKdV equation, the first piece in our analysis is related to the existent theory on LWP and GWP for the gZK equation on classical Sobolev spaces \( H^s(\mathbb{R}^2) \). Let us define the regularity index \( s_k \) by:

\begin{equation}
s_k = \begin{cases} 
3/4 & \text{if } 1 \leq k \leq 7, \\
1 - 2/k & \text{if } k \geq 8.
\end{cases}
\end{equation}

We now state the results by Linares and Pastor [17], [18] and Farah, Linares, and Pastor [4] and notice that some detail of their proof will be included in Section 2.

**Theorem 1.** ([17], [18], [4]). For any \( u_0 \in H^s(\mathbb{R}^2), s > s_k \), there exist \( T = T(\|u_0\|_{H^s}) > 0 \), an space \( X_T \subset C([0, T] : H^s(\mathbb{R}^2)) \) and a unique solution \( u \in X_T \) of the IVP (1.1) defined in \([0, T]\). Moreover for any \( T' \in (0, T) \) there exists a neighborhood \( V \) of \( u_0 \) in \( H^s(\mathbb{R}^2) \) such that the map \( \bar{u}_0 \to \bar{u}(t) \) from \( V \) into \( X_{T'} \) is smooth.
Remarks: (a) The estimate for the length of the time interval of existence with respect of the size of the initial data in $H^s(\mathbb{R}^2)$ can be explicitly obtained in the proof of Theorem 1.

(b) The critical index for the gZK equation (1.4) turns out to be $s_k = 1 - \frac{2}{k}$ which can easily be computed by an scaling argument, therefore it coincides with the regularity index $s_k$ in [18] within the range $k \geq 8$. Hence we have that for $k \geq 8$ these results are optimal. Actually, in [4] it was proven that the gZK IVP is ill-posed for $s = s_k$ in the sense that the map data to solution is not uniformly continuous so other approach different from contraction arguments is required in order to lower the LWP regularity. For the ZK equation, $k = 1$, Grünrock and Herr [10] were able to show LWP in $H^s(\mathbb{R}^2)$ for $s > 1/2$ by performing and appropriate linear change of variables and some bilinear refinements of Strichartz type in the context of Bourgain spaces $X^{s,b}$. However, notice that for $2 \leq k \leq 7$ there is still a gap to be filled in the expected LWP theory.

(c) Regarding GWP results, it is important to mention that for the ZK equation local solutions can be globally defined in $H^1$ with the help of the conserved energy. In the case of the gZK equation, global solutions in $H^1$, and even in a larger space in the case of the mZK, $k = 2$, are obtained if in addition it is assumed that the initial data is small enough, see [17], [18] and [4].

Next, let us explicitly introduce the group associated to the linear ZK equation:

\begin{equation}
W(t)v_0(x, y) = (e^{it(x^2 + y^2)}c_0)^\vee(x, y).
\end{equation}

Following the strategy used to deal with the gKdV equation in weighted spaces [8], our second task is directed to get an extension of formula (1.7) but with the linear group in (1.9) instead. More precisely we have our first result:

**Theorem 2.** Let $r \in (0, 1)$ and \{W(t) : t \in \mathbb{R}\} be the unitary group of operators defined in (1.9). If

\begin{equation}
|\{\Phi_{1,t,r}(\tilde{u}_0)(\xi)\}\vee||_2 \leq c(1 + |t|)(||u_0||_2 + ||D_x^{2r}u_0||_2 + ||D_y^{2r}u_0||_2).
\end{equation}

and

\begin{equation}
|\{\Phi_{2,t,r}(\tilde{u}_0)(\xi)\}\vee||_2 \leq c(1 + |t|)(||u_0||_2 + ||D_x^{2r}u_0||_2 + ||D_y^{2r}u_0||_2).
\end{equation}

Moreover, if in addition to (1.10) we suppose that for $\beta \in (0, r)$

\begin{equation}
D^\beta(|x|^ru_0), D^\beta(|y|^ru_0) \in L^2(\mathbb{R}^2) \text{ and } u_0 \in H^{\beta+2r}(\mathbb{R}),
\end{equation}

then for all $t \in \mathbb{R}$ and for almost every $(x, y) \in \mathbb{R}^2$
then for all $t \in \mathbb{R}$ and for almost every $(x, y) \in \mathbb{R}^2$

$$D^\beta(|x|^r u)(x, y)$$

(1.16)

$$= W(t)(D^\beta|x|^r u_0)(x, y) + W(t)(D^\beta(\Phi_{1,t,r}(\hat{u}_0)))(x, y)$$

and

$$D^\beta(|y|^s u)(x, y)$$

(1.17)

$$= W(t)(D^\beta|y|^s u_0)(x, y) + U(t)(D^\beta(\Phi_{2,t,s}(\hat{u}_0)))(x, y)$$

with

(1.18) \[ \|D^\beta(\Phi_{j,t,r}(\hat{u}_0))\|_2 \leq c(1 + |t|)(\|u_0\|_2 + \|D^\beta u_0\|_2 + \|D^\beta u_0\|_2), \]

for $j = 1, 2$.

Remarks: (a) As we mentioned above, this type of formula was recently established by Fonseca, Linares and Ponce in the context of the Airy group and more generally it also holds for the group associated to the dispersion generalized Benjamin-Ono equation:

(1.19)

\[
\begin{cases}
\partial_t u - D^\beta_x u = 0, & t, x \in \mathbb{R}, \quad 0 \leq \alpha < 1, \\
u(x, 0) = u_0(x),
\end{cases}
\]

where $D^\alpha$ denotes the homogeneous derivative of order $\alpha \in \mathbb{R}$,

$$D^\alpha = (-\partial_x^2)^{\alpha/2} \quad \text{so} \quad D^\alpha f = c_\alpha(|\xi|^\alpha \hat{f}), \quad \text{with} \quad D^\alpha = (\mathcal{H} \partial_x)^\alpha,$$

and $\mathcal{H}$ denotes the Hilbert transform,

$$\mathcal{H}f(x) = \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \int_{|y| > \epsilon} \frac{f(x-y)}{y} dy = (-i \text{sgn}(\xi) \hat{f})(\xi).$$

For the problem we are dealing with we adapt those 1 dimensional situations and carefully handle estimates in the two Fourier space variables.

(b) The proof of Theorem 2 is based on a characterization of the generalized Sobolev space

(1.20) \[ L^{\alpha,p}(\mathbb{R}^n) = (1 - \Delta)^{-\alpha/2}L^p(\mathbb{R}^n), \quad \alpha \in (0, 2), \quad p \in (1, \infty), \]

due to E. M. Stein (see Theorem 4 below).

Now we state our second result concerning local well-posedness of the gZK equation in weighted spaces:

**Theorem 3.** Let $u \in C([0, T]; H^s(\mathbb{R}^2)), s > s_k$ denote the local solution of the IVP (1.1) provided by Theorem 1. Let us assume that $u_0, \,(x, y)^r u_0 \in L^2(\mathbb{R}^2)$ with $r$ satisfying $0 < r \leq s/2$, then

(1.21)

$$u \in C([0, T]; Z_{s,r}).$$

For any $T' \in (0, T)$ there exists a neighborhood $V$ of $u_0$ in $H^s(\mathbb{R}) \cap L^2((x, y)|^2 r \, dx)$ such that the map $\tilde{u}_0 \rightarrow \tilde{u}(t)$ from $V$ into the class defined by $X_T$ in Theorem 1 and (1.21) with $T'$ instead of $T$ is smooth.
Remarks: (a) We observe that Theorem 3 guarantees that the persistent property in the weighted space $Z_{s,r}$ holds in the same time interval $[0, T]$ given by Theorem 1, where $T$ depends only on $\|u_0\|_{H^s}$.

(b) It is expected that the condition $s \geq 2r$ in Theorem 3 is optimal as it was shown in [11] in the case of the gKdV equation. More precisely, (1.21) would hold only if and only if $s \geq 2r$.

(d) Notice that for $k = 1$ the LWP result by Grünrock and Herr holds in a much larger space involving Bourgain spaces $X^{s,b}$, $s > 1/2$, but so far it is not clear for us how to handle our weights in those spaces.

(e) It is interesting to mention an important difference in the way we obtain persistent properties in these weighted spaces for dispersive type equations. Theorem 3 is established via the contraction principle as it was made for gKdV, the regularized Benjamin-Ono equations and the fifth order KdV equations, see [8], [9] and [2] respectively. However, for the weaker dispersion equations DGBO in (1.19) with the same quadratic non-linearity structure and for the Benjamin-Ono equation, even that we have at hand Theorem 1, we couldn’t apply the contraction principle and optimal persistency results in weighted Sobolev spaces were indeed attained via energy estimates see [7] and [6]. See also [3] regarding a 2D ZK-BO equation.

The paper is organized as follows. In section 2 we introduce Stein’s derivatives and some detail on known results on LWP for the gZK equation. The proof of Theorem 2 will be given in Section 3. In Section 4 we will present the proof of Theorem 3.

Notations $\| \cdot \|_p$ denotes the norm in the Lebesgue space $L^p(\mathbb{R}^n)$.

Let $\alpha$ be a complex number, the homogeneous derivatives $D^\alpha_x$, $D^\beta_y$ for functions in $\mathbb{R}^2$ are defined via Fourier transform by $\hat{D^\alpha_x} f(\xi, \eta) = |\xi|^\alpha \hat{f}(\xi, \eta)$ and $\hat{D^\beta_y} f(\xi, \eta) = |\eta|^\beta \hat{f}(\xi, \eta)$ respectively.

We consider the Lebesgue space-time $L^p_T L^q_x L^r_y$ spaces with $1 \leq p, q, r < \infty$ endowed with the norm

$$\|f\|_{L^p_T L^q_x L^r_y} = \left( \int_0^T \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left| f(x, y, t) \right|^q \frac{dy}{y} \right)^{\frac{r}{q}} \frac{dx}{x} \right)^{\frac{p}{r}} dt \right)^{\frac{1}{p}}$$

with the usual modifications when $p = \infty$ or $q = \infty$ or $r = \infty$.

In general $c$ denotes a universal constant which may change, increase, from line to line.

2. Preliminary results

Let us start with a characterization of the Sobolev space

(2.1) $L^{\alpha,p}(\mathbb{R}^n) = (1 - \Delta)^{-\alpha/2} L^p(\mathbb{R}^n), \quad \alpha \in (0, 2), \quad p \in (1, \infty),$

due to E. M. Stein [21]. For $\alpha \in (0, 2)$ define

(2.2) $D^\alpha f(x) = \lim_{\epsilon \to 0} \frac{1}{c_\alpha} \int_{|y| \geq \epsilon} \frac{f(x + y) - f(x)}{|y|^{n+\alpha}} dy,$

where $c_\alpha = \pi^{n/2} 2^{-\alpha} \Gamma(-\alpha/2)/\Gamma((n + 2)/2)$. 
As it was remarked in [21] for appropriate \( f \), for example \( f \in \mathcal{S}(\mathbb{R}^n) \), one has
\[
(2.3) \quad \hat{D}_n f(\xi) = \hat{D}^n f(\xi) \equiv |\xi|^n \hat{f}(\xi).
\]

The following result concerning the \( L^{\alpha,p}(\mathbb{R}^n) = (1 - \Delta)^{\alpha/2} L^p(\mathbb{R}^n) \) spaces was established in [21].

**Theorem 4.** (21) *Let \( \alpha \in (0, 2) \) and \( p \in (1, \infty) \). Then \( f \in L^{\alpha,p}(\mathbb{R}^n) \) if and only if*
\[
(2.4) \quad \begin{cases}
(a) & f \in L^p(\mathbb{R}^n), \\
(b) & D_\alpha f \in L^p(\mathbb{R}^n),
\end{cases}
\]
*with*
\[
(2.5) \quad \|f\|_{\alpha,p} = \|(1 - \Delta)^{\alpha/2} f\|_p \lesssim \|f\|_p + \|D_\alpha f\|_p \lesssim \|f\|_p + \|D^\alpha f\|_p.
\]

Notice that if \( f, fg : \mathbb{R}^n \to \mathbb{R} \in L^{\alpha,p}(\mathbb{R}^n) \) and \( g \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n) \) and consider Stein’s derivatives en each \( j^- \)th direction in \( \mathbb{R}^n \), Stein’s partial derivatives, we have that
\[
D_{j,\alpha}(fg)(x) = \lim_{\epsilon \to 0} \frac{1}{d_\alpha} \int_{|y| \geq \epsilon} \frac{f(x + y \overrightarrow{e_j}) g(x + y \overrightarrow{e_j}) - f(x) g(x)}{|y|^{1+\alpha}} dy
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{d_\alpha} \int_{|y| \geq \epsilon} g(x) \frac{f(x + y \overrightarrow{e_j}) - f(x)}{|y|^{1+\alpha}} dy
\]
\[
+ \lim_{\epsilon \to 0} \frac{1}{d_\alpha} \int_{|y| \geq \epsilon} \frac{(g(x + y \overrightarrow{e_j}) - g(x)) f(x + y \overrightarrow{e_j})}{|y|^{1+\alpha}} dy
\]
\[
= g(x) D_{j,\alpha} f(x) + \Lambda_{j,\alpha} ((g(\cdot + y \overrightarrow{e_j}) - g(\cdot)) f(\cdot + y \overrightarrow{e_j})) (x).
\]

In particular, if \( g(x) = e^{it \varphi(x)} \), then
\[
\Lambda_{j,\alpha}((g(\cdot + y \overrightarrow{e_j}) - g(\cdot)) f(\cdot + y \overrightarrow{e_j}))(x)
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{d_\alpha} \int_{|y| \geq \epsilon} \frac{(g(x + y \overrightarrow{e_j}) - g(x)) f(x + y \overrightarrow{e_j})}{|y|^{1+\alpha}} dy
\]
\[
= e^{it \varphi(x)} \lim_{\epsilon \to 0} \frac{1}{d_\alpha} \int_{|y| \geq \epsilon} \frac{e^{it(\varphi(x + y \overrightarrow{e_j}) - \varphi(x))} - 1}{|y|^{1+\alpha}} f(x + y \overrightarrow{e_j}) dy
\]
\[
= e^{it \varphi(x)} \Phi_{j,\varphi,\alpha}(f)(x).
\]

Thus, we obtain the identity
\[
(2.8) \quad D_{j,\alpha}(e^{it \varphi(\cdot)} f)(x) = e^{it \varphi(x)} D_{j,\alpha} f(x) + e^{it \varphi(x)} \Phi_{j,\varphi,\alpha}(f)(x).
\]

Now we restrict to \( \mathbb{R}^2 \) and choose as the phase function the one from the group associated to the linear ZK equation in \([1.9]\)
\[
(2.9) \quad \varphi(x_1, x_2) = x_1^3 + x_1 x_2^2.
\]
For $j = 1, 2$, we shall obtain a bound for

$$
\|\Phi_{j, \alpha}(f)\|_p = \left\| \lim_{\epsilon \to 0} \int_{|y| \geq \epsilon} e^{it(\varphi(x + y, e_j) - \varphi(x))} \frac{1}{|y|^{1+\alpha}} f(x + y, e_j) \, dy \right\|_p.
$$

Indeed this is achieved in our first result

**Lemma 1.** Let $\alpha \in (0, 1)$, and $p \in (1, \infty)$. If $f \in L^{\alpha,p}(\mathbb{R})$ and $f \in L^p((1 + x_1^2 + x_2^2)\,dx_1\,dx_2)$, then for all $t \in \mathbb{R}$ and for almost every $(x_1, x_2) \in \mathbb{R}^2$

$$
D_{j,\alpha}(e^{it(x_1^2 + x_1 x_2)} f)(x_1, x_2) = e^{it(x_1^2 + x_1 x_2)} D_{j,\alpha} f(x_1, x_2)
$$

with

$$
\Phi_{1,t,\alpha}(f)(x_1, x_2) = \lim_{\epsilon \to 0} \frac{1}{d_\alpha} \int_{|y| \geq \epsilon} e^{it(\varphi(x_1, x_2) - \varphi(x_1, x_2))} \frac{1}{|y|^{1+\alpha}} f(x_1 + y, x_2) \, dy
$$

and

$$
\Phi_{2,t,\alpha}(f)(x_1, x_2) = \lim_{\epsilon \to 0} \frac{1}{d_\alpha} \int_{|y| \geq \epsilon} e^{it(\varphi(x_1, x_2 + y) - \varphi(x_1, x_2))} \frac{1}{|y|^{1+\alpha}} f(x_1, x_2 + y) \, dy
$$

satisfying

$$
\|\Phi_{j,t,\alpha}(f)\|_p \leq c_\alpha (1 + \|t\|)(\|f\|_p + \| (1 + x_1^2 + x_2^2)^\alpha f\|_p),
$$

for $j=1,2$.

**Proof.** Since our interest resides in the parameter $\alpha \in (0, 1)$, then we are allowed to pass the absolute value inside the integral sign in (2.10).

At different parts of our work we will make use of either of the elementary estimates

$$
\left\{ \begin{array}{ll}
(a) & \forall \theta \in \mathbb{R} \quad |e^{i\theta} - 1| \leq 2, \\
(b) & \forall \theta \in \mathbb{R} \quad |e^{i\theta} - 1| \leq |\theta|.
\end{array} \right.
$$

Let us consider first Stein’s derivate with respect to $x_1$, i.e. $j = 1$. From (2.15) (a) and Minkowski’s integral inequality it follows that

$$
\left\| \int_{|y| \geq \frac{1}{\alpha \alpha}} e^{it(\varphi(x_1, x_2) - \varphi(x_1, x_2))} \frac{1}{|y|^{1+\alpha}} f(x_1 + y, x_2) \, dy \right\|_p 
$$

$$
\leq \int_{|y| \geq \frac{1}{\alpha \alpha}} \left\| \frac{2|f(x_1 + y, x_2)|}{|y|^{1+\alpha}} \right\|_p \, dy
$$

$$
\leq c \int_{|y| \geq \frac{1}{\alpha \alpha}} \frac{\|f(x_1 + y, x_2)\|_p}{|y|^{1+\alpha}} \, dy
$$

$$
\leq c \|f\|_p \int_{|y| \geq \frac{1}{\alpha \alpha}} \frac{1}{|y|^{1+\alpha}} \, dy
$$

$$
\leq c_\alpha \|f\|_p.
$$
Now let us consider the estimate

\[
\left\| \int_{|y| \leq \frac{1}{100}} e^{it(\varphi(x_1+y,x_2)-\varphi(x_1,x_2))} - \frac{1}{|y|^{1+\alpha}} f(x_1 + y, x_2) \, dy \right\|_p.
\]

(2.17) (b) yields

\[
\left| e^{it(\varphi(x_1+y,x_2)-\varphi(x_1,x_2))} - 1 \right| \leq |t(\varphi(x_1+y,x_2)-\varphi(x_1,x_2))| = |t||y| \left| \int_0^1 \partial_x \varphi(x_1 + sy, x_2) \, ds \right|
\]

For \( x_1, x_2 \) in the ball \( B_{100}(0) = \{(x_1, x_2)/ x_1^2 + x_2^2 < 100 \} \) we obtain

\[
|\partial_x \varphi(x_1 + sy, x_2)| = 3(x_1 + sy)^2 + x_2^2.
\]

(2.19)

\[
\leq 3x_1^2 + 6s|x_1||y| + 4x_2^2 \leq c,
\]

and therefore

\[
\left| e^{it(\varphi(x_1+y,x_2)-\varphi(x_1,x_2))} - 1 \right| \leq c |t| |y|
\]

Hence our estimate is summarized as:

\[
\int_{|y| \leq \frac{1}{100}} \left\| \frac{e^{it(\varphi(x_1+y,x_2)-\varphi(x_1,x_2))} - 1}{|y|^{1+\alpha}} f(x_1 + y, x_2) \, dy \right\|_{L^p(B_{100}(0))}
\]

\[
\leq c \int_{|y| \leq \frac{1}{100}} \left\| \frac{|t||y||f(x_1 + y, x_2)|}{|y|^{1+\alpha}} \right\|_{L^p(B_{100}(0))} \, dy
\]

(2.20)

\[
\leq c |t| \int_{|y| \leq \frac{1}{100}} \frac{1}{|y|^\alpha} \left\| f(x_1 + y, x_2) \right\|_{L^p(B_{100}(0))} \, dy
\]

\[
\leq c |t| \|f\|_p \int_{|y| \leq \frac{1}{100}} \frac{1}{|y|^\alpha} \, dy
\]

\[
\leq c_\alpha |t| \|f\|_p.
\]

From the above estimates we now have to consider in (2.10) the region:

\[
|y| \leq 1/100, \quad \text{and} \quad x_1^2 + x_2^2 \geq 100.
\]

We sub-divide it into two parts:

\[
(2.21) \quad (a) \quad |y| \leq \frac{1}{1 + x_1^2 + x_2^2}, \quad (b) \quad |y| \geq \frac{1}{1 + x_1^2 + x_2^2}.
\]

We first assume \(|y| \leq \frac{1}{1 + x_1^2 + x_2^2}\) and observe that \( |\partial_x \varphi(x_1 + sy, x_2)| \leq c(1 + x_1^2 + x_2^2) \) in this region. With the help of the change of variable \( \tilde{y} = (1 + x_1^2 + x_2^2)y \), inequality
(2.15) (b), the argument in (2.18) and Minkowski’s inequality we obtain

\[
\left\| \int_{|y| \leq \frac{1}{1 + x_1^2 + x_2^2}} \frac{e^{it(\varphi(x_1+y, x_2) - \varphi(x_1, x_2)) - 1}{|y|^{1+\alpha}} f(x_1 + y, x_2) \, dy \right\|_{L^p(B_{100}(0)^c)} \\
\leq c_\alpha \left\| \int_{|y| \leq \frac{1}{1 + x_1^2 + x_2^2}} \frac{|t| |y| (1 + x_1^2 + x_2^2) f(x_1 + y, x_2)|}{|y|^{1+\alpha}} \, dy \right\|_{L^p(B_{100}(0)^c)} \\
\leq c_\alpha \left\| \int_{|\tilde{y}| \leq 1} \frac{|t| (1 + x_1^2 + x_2^2) f(x_1 + \frac{\tilde{y}}{1 + x_1^2 + x_2^2}, x_2)|}{|\tilde{y}|^\alpha} \, d\tilde{y} \right\|_{L^p(B_{100}(0)^c)} \\
\leq c_\alpha \left\| \int_{|\tilde{y}| \leq 1} \frac{|t| \left( 1 + \frac{\tilde{y}}{1 + x_1^2 + x_2^2} \right)^{2\alpha} f(x_1 + \frac{\tilde{y}}{1 + x_1^2 + x_2^2}, x_2)|}{|\tilde{y}|^{2\alpha}} \, d\tilde{y} \right\|_{L^p(B_{100}(0)^c)} \\
+ c_\alpha \left\| \int_{|\tilde{y}| \leq 1} \frac{|t| \left( 1 + \frac{\tilde{y}}{1 + x_1^2 + x_2^2} \right)^{2\alpha} f(x_1 + \frac{\tilde{y}}{1 + x_1^2 + x_2^2}, x_2)|}{|\tilde{y}|^{2\alpha}} \, d\tilde{y} \right\|_{L^p(B_{100}(0)^c)} 
\] \]

Now we perform a second change of variable \( u = x_1 + \frac{\tilde{y}}{1 + x_1^2 + x_2^2}, v = x_2 \) and since

\[
(2.22) \quad \frac{\tilde{y}}{1 + x_1^2 + x_2^2} = y, \quad |y| \leq 1/100, \quad x_1^2 + x_2^2 \geq 100, \quad \text{so} \ du \sim dx_1, 
\]

we conclude that

\[
\left\| \int_{|y| \leq \frac{1}{1 + x_1^2 + x_2^2}} \frac{e^{it(\varphi(x_1+y, x_2) - \varphi(x_1, x_2)) - 1}{|y|^{1+\alpha}} f(x_1 + y, x_2) \, dy \right\|_{L^p(B_{100}(0)^c)} \\
\leq c_\alpha |t| \left\| \int_{|\tilde{y}| \leq 1} \frac{|(1 + x_1^2 + x_2^2)^{\alpha} f(x_1, x_2)|}{|\tilde{y}|^\alpha} \right\|_p d\tilde{y} \\
+ c_\alpha |t| \left\| \int_{|\tilde{y}| \leq 1} \frac{|f(x_1, x_2)|}{|\tilde{y}|^\alpha} \right\|_p d\tilde{y} \\
\leq c_\alpha |t| (\|f\|_p + |(1 + x_1^2 + x_2^2)^{\alpha} f\|_p). 
\]

Next suppose that \(|y| \geq \frac{1}{1 + x_1^2 + x_2^2} \). Changing variable, \( \tilde{y} = (1 + x_1^2 + x_2^2)y \), using (2.15) part (a), Minkowski’s inequality, and a second change of variable as in (2.22)
we get
\[
\left\| \int_{\frac{x_1}{1+x_1^2+x_2^2}}^1 e^{\frac{x_1}{1+x_1^2+x_2^2}} \frac{f(x_1 + y, x_2)}{|y|^{1+\alpha}} dy \right\|_{L^p(B_{100}(0)^c)} 
\leq c \left\| \int_{\frac{x_1}{1+x_1^2+x_2^2}}^1 \frac{|f(x_1 + y, x_2)|}{|y|^{1+\alpha}} dy \right\|_{L^p(B_{100}(0)^c)} 
\leq c \left\| \int_{|\tilde{y}| \leq \frac{1+x_1^2+x_2^2}{100}} (1 + x_1^2 + x_2^2)^\alpha |f(x_1 + \frac{\tilde{y}}{1+x_1^2+x_2^2}, x_2)| \frac{dy}{|\tilde{y}|^{1+\alpha}} \right\|_{L^p(B_{100}(0)^c)} 
\leq c_\alpha \left\| |f|_p + \|(1 + x_1^2 + x_2^2)^\alpha f\|_p \right\|_{L^p(B_{100}(0)^c)} 
\leq c_\alpha (\|f\|_p + \|(1 + x_1^2 + x_2^2)^\alpha f\|_p)
\]

On the other hand, regarding Stein’s derivative with respect to $x_2$, i.e. $j = 2$, we notice that the useful inequality, $2x_1x_2 \leq x_1^2 + x_2^2$, allows us to perform the same computations and obtain exactly the same bound $2.14$. □

Next we focus in the local $H^s$ theory in Theorem 1 for the gZK IVP in 2D from the works by Linares, Pastor and Farah, see [17, 18, 4] and references therein. Although for every $k = 1, 2, 3, \ldots$ the proof is accomplished by the contraction principle, the $X_T$ space is different according to the nonlinearity degree $k$ (different space-time norms are involved in the choice of $X_T$).

Our persistence result in weighted spaces for gZK strongly depends on this theorem so for the sake of clearness we emphasize some of aspects of its proof for all included nonlinearities with $k = 1, 2, 3, \ldots$.

We start by noticing that the method of proof is performed via the Picard iteration applied to the integral version of the gZK IVP given by:

\begin{equation}
(2.23) \quad \Psi(u(t)) = W(t)u_0 - \int_0^t W(t - t')(u^k \partial_x u)(t') dt', \quad t \in [0, T],
\end{equation}

where the solution space $X_T \subset C([0, T] : H^s(\mathbb{R}^2))$ is determined by the norms

- For $k = 1, s > 3/4$.

\begin{equation}
(2.24) \quad \mu_{1,1}^T = \|u\|_{L_\infty^s H^s} + \|D_x^s \partial_x u\|_{L^p_\infty L^2_{\theta^r}} + \|D_y^s \partial_x u\|_{L^p_\infty L^2_{\theta^r}} + \|\partial_x u\|_{L^p_\infty L^2_{\theta^r}} + \|u\|_{L^p_\infty L^2_{\theta^r}}.
\end{equation}

- For $k = 2, s > 3/4$. 

then carried out in the closed ball Strichartz estimates, maximal function estimates, ... . The standard argument is stated group to the $gZK$ equation, i.e. linear estimates, like the smoothing effect, such that

\[ \gamma = \frac{12(k-1)}{7-12k} \text{ and } \gamma \in (0, 1/12). \]

For $k \geq 8$, $s > s_k = 1 - 2/k$.

The norms involved in these spaces reflect important properties of the associated group to the $gZK$ equation, i.e. linear estimates, like the smoothing effect, Strichartz estimates, maximal function estimates, ... . The standard argument is then carried out in the closed ball

\[ B_T^0 = \{ u \in X_T; \mu_{1,k}^T(u) \leq a = 2c\| u_0 \|_{H^s(\mathbb{R}^2)} \}, \]

of the metric space

\[ X_T = \{ u \in C([0,T]: H^s(\mathbb{R}^2)); \mu_{1,k}^T(u) < \infty \}. \]

By applying to the integral equation \((2.23)\) each norm in the definition of $\mu_{1,k}^T(u)$, linear estimates yield

\[ \mu_{1,k}^T(u) \leq c\| u_0 \|_{H^s} + cT^\gamma (\mu_{1,k}^T(u))^{k+1} \]

where $\gamma$ is a positive constant. From this point, the local existence time is chosen so that

\[ c\| u_0 \|_{H^s} \leq 1/2, \]

which implies that the time size $T \sim \frac{1}{2} \frac{1}{\| u_0 \|_{H^s}}$ and that the local solution satisfies $\mu_{1,k}^T(u) \leq 2c\| u_0 \|_{H^s}$.

3. Proof of Theorem 2

We consider the unitary group of operators $\{W(t) : t \in \mathbb{R}\}$ in $L^2(\mathbb{R}^2)$ defined as

\[ W(t)u_0(x,y) = (e^{it(\xi^3 + \xi^5)}u_0(\xi,\eta))^\vee(x,y). \]

Thus, for $\alpha \in (0,1)$, \((2.23)\) yields

\[ |x|^\alpha W(t)u_0(x,y) = |x|^\alpha (e^{it(\xi^3 + \xi^5)}u_0(\xi,\eta))^\vee(x,y) = (D_{1,\alpha}(e^{it(\xi^3 + \xi^5)}u_0(\xi,\eta)))^\vee(x,y). \]
and from Lemma 11 that
\[ D_{1,\alpha}(e^{it(\xi^2 + \xi_2^2)} \hat{u}_0)(\xi, \eta) = e^{it(\xi^2 + \xi_2^2)} D_{1,\alpha}\hat{u}_0(\xi, \eta) + e^{it(\xi^2 + \xi_2^2)} \Phi_{1, t, \alpha} \hat{u}_0(\xi, \eta), \]
with
\[ \|\Phi_{1, t, \alpha} \hat{u}_0\|_p \leq c_\alpha (1 + |t|)(\|\hat{u}_0\|_p + \| 1 + \xi^2 + \eta^2 \|^2 \|\hat{u}_0\|_p). \]
Hence, taking Fourier transform in (3.2) we obtain the identity
\[ |x|^\alpha W(t)u_0(x, y) = W(t)(|x|^\alpha u_0)(x, y) + W(t)(\{\Phi_{1, t, \alpha}(\hat{u}_0)(\xi, \eta)\}^\vee)(x, y). \]
with \( \Phi_{1, t, \alpha} \) as in (2.12) and
\[ \|\{\Phi_{1, t, \alpha}(\hat{u}_0)(\xi)\}^\vee\|_2 = \|\Phi_{1, t, \alpha}(\hat{u}_0)\|_2 \]
(3.4)
\[ \leq c_\alpha (1 + |t|)(\|\hat{u}_0\|_2 + \| 1 + \xi^2 + \eta^2 \|^2 \|\hat{u}_0\|_2) \]
(3.5)
\[ \leq c_\alpha (1 + |t|)(\|u_0\|_2 + \| D_x^{2\alpha} u_0\|_2 + \| D_y^{2\alpha} u_0\|_2). \]

On the other hand, if \( \beta \in (0, \alpha) \), then
\[ D_x^\beta(|x|^\alpha W(t)u_0)(x, y) = W(t)(D_x^\beta |x|^\alpha u_0)(x, y) \]
\[ + W(t)(D_x^\beta (\{\Phi_{1, t, \alpha}(\hat{u}_0)(\xi, \eta)\}^\vee)(x, y). \]

In order to prove 1.18 we need to show that
\[ \|D_x^\beta f(x, \xi) - \hat{u}_0(\xi)\|_2 \]
(3.6)
\[ \leq c_{\alpha, \beta} (1 + |t|)(\|u_0\|_2 + \| D_x^{2\alpha} u_0\|_2 + \| D_y^{2\alpha} u_0\|_2). \]
Thus, we write
\[ \|D_x^\beta f(x, \xi) - \hat{u}_0(\xi)\|_2 \]
\[ = \| \int \frac{e^{it(\varphi(\xi, \eta) - \varphi(\xi, \eta))} - 1}{|\tau|^{1+\alpha}} \hat{u}_0(\xi + \tau, \eta) d\tau \|_2 \]
\[ \leq \| \int \frac{|\xi|^\beta |e^{it(\varphi(\xi, \eta) - \varphi(\xi, \eta))} - 1|}{|\tau|^{1+\alpha}} \hat{u}_0(\xi + \tau, \eta) |d\tau\|_2 \]
\[ \leq c_{\beta} \| \int \frac{|\xi + \tau|^\beta |e^{it(\varphi(\xi, \eta) - \varphi(\xi, \eta))} - 1|}{|\tau|^{1+\alpha}} \hat{u}_0(\xi + \tau, \eta) d\tau \|_2 \]
\[ = I_1 + I_2. \]

Following the argument used in the proof of Lemma 11 to get (2.14) it holds that
\[ I_1 \leq c_{\alpha, \beta} (1 + |t|)(\|\xi|^\beta \hat{u}_0\|_2 + \| 1 + \xi^2 + \eta^2 \|^2 \|\hat{u}_0\|_2) \]
(3.7)
\[ \leq c_{\alpha, \beta} (1 + |t|)(\|D_x^\beta u_0\|_2 + \| D_x^{2\alpha} u_0\|_2 + \| D_y^{2\alpha} u_0\|_2) \]
\[ \leq c_{\alpha, \beta} (1 + |t|)(\|u_0\|_2 + \| D_x^{2\alpha} u_0\|_2 + \| D_y^{2\alpha} u_0\|_2). \]
To bound $I_2$ we observe that this estimate is similar to that used in the proof of Lemma 1 with $\alpha - \beta$ instead of $\alpha$. Hence,

\begin{equation}
I_2 \leq c_{\alpha,\beta}(1 + |t|)(\|\hat{u}_0\|_2 + \| (1 + \xi^2 + \eta^2)^{(\alpha-\beta)}\hat{u}_0\|_2)
\end{equation}

\begin{equation}
\leq c_{\alpha,\beta}(1 + |t|)(\|u_0\|_2 + \| D_x^{2(\alpha-\beta)}u_0\|_2 + \| D_y^{2(\alpha-\beta)}u_0\|_2)
\end{equation}

\begin{equation}
\leq c_{\alpha,\beta}(1 + |t|)(\|u_0\|_2 + \| D_x^{2+2\alpha}u_0\|_2 + \| D_y^{2+2\alpha}u_0\|_2).
\end{equation}

For the weight in the $y$ direction the analysis follows similar arguments and hence we get that for $\alpha \in (0, 1)$

\begin{equation}
|y|^a W(t)u_0(x, y) = W(t)(|y|^a u_0)(x, y) + W(t)((\Phi_{2, t, \alpha}(\hat{u}_0) (\xi, \eta))^{\gamma})(x, y).
\end{equation}

with $\Phi_{j, t, \alpha}$ as in (2.13) and

\begin{equation}
\| \Phi_{2, t, \alpha}(\hat{u}_0) (\xi) \|_2 \leq c_\alpha (1 + |t|)(\|u_0\|_2 + \| D_x^{2\alpha}u_0\|_2 + \| D_y^{2\alpha}u_0\|_2),
\end{equation}

and similarly (3.18) for $j = 2$ is obtained.

This completes the proof of Theorem 2.

4. PROOF OF THEOREM 3

We consider the most interesting case $s = 2r$, with $s_k < s < 1$ as in the LWP theory in $H^s$.

Case 1: $k = 1$.

From the previous assumption $s = 2r, s > s_1 = 3/4$.

Let $u \in C([0, T] : H^{2r}(\mathbb{R}))$ be the unique solution of the ZK IVP satisfying the integral equation

\begin{equation}
u(t) = \Phi(u(t)) = W(t)u_0 - \int_0^t W(t - t')(u \partial_x u)(t')dt', \ t \in [0, T]
\end{equation}

with $T = T(\|u_0\|_{H^s}) < 1$ in (2.29) chosen to satisfy

\begin{equation}c_1 \mu_1(T) T^\gamma \leq 1/2,
\end{equation}

where $\gamma = 1/2$ for $k = 1, T \sim \|u_0\|_{H^s}^{-2}$ and $a = 2c\|u_0\|_{H^s}$ is the radius of the ball in the contraction argument in $H^s$ so that

\begin{equation}\mu_{1,1}^T(u) \leq a = 2c\|u_0\|_{H^s}.
\end{equation}

Now we suppose additionally that

$u_0 \in Z_{2r, r} = H^{2r}(\mathbb{R}^2) \cap L^2((x, y)^{2r} dx dy),$

and introduce the new norm

\begin{equation}\mu_2^T(u) = \mu_{1, 1}^T(u) + \| \sigma(x, y)^{\gamma} u(t) \|_{L^\infty_x L^2_y},
\end{equation}

Let us estimate $\mu_2^T(u)$ in the integral equation (4.1) noticing that for the second term in the definition of $\mu_2^T(u)$ it is enough to consider the norms

\begin{equation}i) \| \sigma u(t) \|_{L^\infty_x L^2_y} \quad \text{and} \quad ii) \| y^\gamma u(t) \|_{L^\infty_x L^2_y}.
\end{equation}

For $i)$ we have from Theorem 2 and (4.3) that
From Theorem 2 we can restrict our attention to the first norm above and obtain:

\[ ||x|^r u||_{L^2_y} \leq ||x|^r v_0||_2 + c(1 + T)||u_0||_{H^{2r}} + ||x|^r \int_0^T W(t - t')(u \partial_x u)(t') dt' ||_{L^2_y} \]

\[ \leq ||x|^r v_0||_2 + c(1 + T)||u_0||_{H^{2r}} + \int_0^T |||x|^r u \partial_x u||_{L^2_y} dt \]

\[ + c(1 + T) \int_0^T |||u \partial_x u||_{H^{2r}} dt \]

\[ \leq ||x|^r v_0||_2 + c(1 + T)||u_0||_{H^{2r}} + c(1 + T)T^2 (\mu_{1,1}^T(u))^2 + I_1, \]

where

\[ I_1 = \int_0^T |||x|^r u \partial_x u||_{L^2_y}. \]

We obtain a similar estimate for ii) and therefore we conclude that

\[ \mu_2^T(u) \leq c(1 + T)(||u_0||_{H^{2r}} + ||(x, y)|^r u_0||_{L^2_y}) + c(1 + T)T^2 (\mu_{1,1}^T(u))^2 + cI_1. \]

Now we have for I_1:

\[ I_1 \leq |||x|^r u||_{L^2_y} \||\partial_x u||_{L^2_x L^\infty_y} \]

\[ \leq \mu_2^T(u) T^2 \||\partial_x u||_{L^2_x L^\infty_y} \]

\[ \leq T^2 \mu_{1,1}^T(u) \mu_2^T(u). \]

In summary we get

\[ \mu_2^T(u) \leq c(1 + T)(||u_0||_{H^{2r}} + ||(x, y)|^r u_0||_{L^2_y}) \]

\[ + c(1 + T)T^2 (\mu_{1,1}^T(u))^2 + cT^2 \mu_{1,1}^T(u) \mu_2^T(u). \]

From the time size in \[4.12\] we can pass the last term to the left side and obtain

\[ \mu_2^T(u) \leq 2c(1 + T)(||u_0||_{H^{2r}} + ||(x, y)|^r u_0||_{L^2_y}) + 2c(1 + T)||u_0||_{H^{2r}}. \]

This basically completes the proof of this theorem in the \( k = 1 \) case.

**Case 2: \( k \geq 2 \).**

For these nonlinearities the argument follows exactly the same ideas as in the former case and we provide some details at the points where the estimates depend on the norms involved in the associated local theory in \( H^s \).

We again define a new norm as in \[4.3\]

\[ \mu_2^T(u) = \mu_{1,k}^T(u) + ||(x, y)|^r u(t)||_{L^2_y L^\infty_x}, \]

with \( \mu_{1,k}^T \) the norm associated to the solution space \( X_T \) in Theorem \[3\] and given in section \[2\].

Now we consider the second term in \[1.11\] of the solution \( u \) represented in Duhamel’s formula and observe that it is enough consider the norms

\[ i) \ ||x|^r u(t)||_{L^2_y L^\infty_x} \]

\[ ii) \ ||y|^r u(t)||_{L^2_y L^\infty_x}. \]

From Theorem \[2\] we can restrict our attention to the first norm above and obtain:
Hence the following estimate for the norm in (4.11) holds

\[
\| |x|^r u\|_{L^2_{x,y}} \leq \| |x|^r u_0\|_2 + c(1 + T)\| u_0 \|_{H^{2r}} + \| |x|^r \int_0^T W(t - t')(u^k \partial_x u)(t') dt' \|_{L^2_{x,y}} \]
\[
\leq \| |x|^r u_0\|_2 + c(1 + T)\| u_0 \|_{H^{2r}} + \int_0^T \| |x|^r u^k \partial_x u\|_{L^2_{x,y}} dt 
\]
\[
+ c(1 + T) \int_0^T \| u^k \partial_x u\|_{H^{2r}} dt 
\]
\[
\leq \| |x|^r u_0\|_2 + c(1 + T)\| u_0 \|_{H^{2r}} + c(1 + T)T^\gamma (\mu^T_{1,k}(u))^{k+1} + I_k,
\]

where

\[
I_k = \int_0^T \| |x|^r u^k \partial_x u\|_{L^2_{x,y}}.
\]

Let us estimate \(I_k\) for the different values of \(k\):

- For \(k = 2\):
  \[
  I_k \leq \| |x|^r u\|_{L^\infty_T L^2_{x,y}} \| u \partial_x u\|_{L^1_T L^\infty_{x,y}}
  \leq \mu^T_2(u) \| u\|_{L^\infty_T L^\infty_{x,y}} \| \partial_x u\|_{L^1_T L^\infty_{x,y}}
  \leq T^\gamma \mu^T_2(u) \| u\|_{L^\infty_T L^\infty_{x,y}} \| \partial_x u\|_{L^1_T L^\infty_{x,y}}
  \leq T^\gamma (\mu^T_2(u))^2 \mu^T_2(u).
  \]

- For \(3 \leq k \leq 7\):
  \[
  I_k \leq \| |x|^r u\|_{L^\infty_T L^2_{x,y}} \| u^{k-1} \partial_x u\|_{L^1_T L^\infty_{x,y}}
  \leq \mu^T_2(u) \| u^{k-1}\|_{L^\infty_T L^\infty_{x,y}} \| \partial_x u\|_{L^1_T L^\infty_{x,y}}
  \leq T^\gamma \mu^T_2(u) \| u^{k-1}\|_{L^\infty_T L^\infty_{x,y}} \| \partial_x u\|_{L^1_T L^\infty_{x,y}}
  \leq T^\gamma (\mu^T_{1,k}(u))\mu^T_2(u).
  \]

- For \(k \geq 8\):
  \[
  I_k \leq \| |x|^r u\|_{L^\infty_T L^2_{x,y}} \| u^{k-1} \partial_x u\|_{L^1_T L^\infty_{x,y}}
  \leq \mu^T_2(u) \| u^{k-1}\|_{L^\infty_T L^\infty_{x,y}} \| \partial_x u\|_{L^1_T L^\infty_{x,y}}
  \leq \mu^T_2(u) \| u^{k-1}\|_{L^\infty_T L^\infty_{x,y}} \| \partial_x u\|_{L^1_T L^\infty_{x,y}}
  \leq T^\gamma \mu^T_2(u) \| u^{k-1}\|_{L^\infty_T L^\infty_{x,y}} \| \partial_x u\|_{L^1_T L^\infty_{x,y}}
  \leq T^\gamma (\mu^T_{1,k}(u))\mu^T_2(u).
  \]

Hence the following estimate for the norm in (4.11) holds
From the time size in in the contraction argument in Theorem 1, we can again pass the last term to the left side and obtain

\begin{equation}
\mu^T_2(u) \leq 2c(1+T)(\|u_0\|_{H^{2r}} + \|(x,y)\|_{L^2_{x,y}}^r) + 2c(1+T)\|u_0\|_{H^s}.
\end{equation}

Which completes the proof of the theorem.

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