NEW ZETA FUNCTIONS OF REIDEMEISTER TYPE AND TWISTED BURNSIDE-FROBENIUS THEORY

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Abstract. We introduce new zeta functions related to an endomorphism $\phi$ of a discrete group $\Gamma$. They are of two types: counting numbers of fixed ($\rho \sim \rho^0 \circ \phi^n$) irreducible representations for iterations of $\phi$ from an appropriate dual space of $\Gamma$ and counting Reidemeister numbers $R(\varphi^n)$ of different compactifications. Many properties of these functions and their coefficients are obtained. In many cases it is proved that these zeta functions coincide. The Gauss congruences are proved. Useful asymptotic formulas for the zeta functions are found. Rationality is proved for some examples, which give also the first counterexamples simultaneously for TBFT ($R(\phi)$=the number of fixed irreducible unitary representations) and TBFT$_f$ ($R(\phi)$=the number of fixed irreducible unitary finite-dimensional representations) for an automorphism $\phi$ with $R(\phi) < \infty$.

INTRODUCTION

Let $\phi$ be an endomorphism of a group $\Gamma$. The Reidemeister number $R(\phi)$ of $\phi$ is the number of its Reidemeister or twisted conjugacy classes

$$\{g\}_{\phi} := \{xg\phi(x^{-1}), \quad x \in \Gamma\}.$$ 

In this paper we will be mostly interested in discrete groups (using the notation $\Gamma, \phi$) and in compact (Hausdorff) groups (using the notation $G, \varphi$).

The first group of problems related Reidemeister numbers includes a study of validity of the TBFT (twisted Burnside-Frobenius theorem (or theory)) for different classes of groups, a proof of the Gauss congruences for the Reidemeister numbers of iterations (mostly using the TBFT), and a study of rationality of the corresponding Reidemeister zeta function.

The first formulation of TBFT, due to A. Fel’shtyn and R. Hill, says that the $R(\phi)$=the number of fixed irreducible unitary representations, if one of them is finite. It was proved for automorphisms of abelian, compact, and abelian-by-finite groups in [6, 7, 11]. In [14] a counterexample to the TBFT was detected and it led to a new form of the problem TBFT$_f$, where it is conjectured that if $R(\phi) < \infty$, then it coincides with the number of $\rho \sim \rho \circ \phi$, where $\rho$ is a finite-dimensional irreducible representation. The TBFT$_f$ was proved for polycyclic groups (for automorphisms in [10] and for endomorphisms in [13]). In [11] it was observed that TBFT$_f$ is not true for some infinite groups with finite number of ordinary conjugacy classes. In [26] a counterexample to TBFT$_f$ was detected among

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infinitely generated residually finite groups. (We will give a common counterexample for TBFT and TBFT in the present paper.) For related results we refer to [25, 26]. Concerning rationality of the Reidemeister zeta function we refer to the presentation in [5] and a recent paper [3].

The second group of problems, related to the first one, is to determine groups with the $R_{\infty}$-property (each automorphism has infinite Reidemeister number). The approaches differ for various classes under consideration (e.g., branch groups, lattices, linear groups, solvable groups, etc.), so in fact we have a splitted system of problems, rather one entire problem. For endomorphisms the definition is not appropriate for a direct consideration, because each group has an endomorphism $\psi$ with $R(\psi) = 1$, namely $\psi : g \mapsto e$, for any $g$ (see the beginning of [13] for more information). Many papers were devoted to this problem recently. This is not a main subject of the present paper and we only refer to the following selection of papers on the problem and bibliography therein [18, 8, 15, 16, 19, 24, 2, 9, 17, 12].

We introduce here several interrelated zeta functions determined by an endomorphism $\phi$ of a discrete group $\Gamma$. First we introduce dynamic representation theory zeta functions with $n$th coefficients equal to the number of those irreducible unitary representations (respectively, finite-dimensional, respectively, finite irreducible unitary representations) $\rho$ such that $\rho \sim \rho \circ \phi^n$ (supposing these numbers to be finite) (Definition 2.2). We prove the Gauss congruences for these coefficients, using the existence of an appropriate dynamical system on a part of the unitary dual with periodic points of period $n$ being exactly the above mentioned representation classes (Theorem 2.7).

Then we introduce a natural notion of an admissible compactification of $\Gamma$ (Definition 3.3) with the profinite completion and the universal compactification as main examples. We define the corresponding compactification Reidemeister zeta function. Then we prove three statements about Reidemeister classes of an endomorphism $\varphi$ of a compact Hausdorff group $G$: 1) calculation of a finite Reidemeister number as the number of irreducible representations $\rho$ such that $\rho \sim \rho \circ \phi$ (Theorem 3.8); 2) description of those matrix coefficients, which are constant on Reidemeister classes (Lemma 3.9); and 3) proof of the TBFT in this situation (Corollary 3.10).

We prove that if $R(\varphi) < \infty$ the above mentioned representations must be finite (not only finite-dimensional) (Theorem 4.1).

Using these facts, we prove under finiteness conditions the coincidence of representation theory zeta-functions (related to finite-dimensional and finite representations) and compactification zeta-functions (for the profinite and the universal compactifications) (Theorem 5.5).

We obtain an asymptotic formula for these zeta-functions in Theorem 6.1 and Corollary 6.2.

Then we develop an example from [26] to give a proof of rationality in this situation of the zeta functions under consideration (Proposition 7.4). As a corollary of these calculations, we obtain the first example of a group and its automorphism, giving a counterexample to the TBFT and the TBFT simultaneously (Theorem 7.3).

The results of Sections 1 and 2 are obtained by A.F. and M.Z.
The results of Sections 3, 4, 5, 6, and 7 are obtained by E.T.

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1. Preliminaries

We will need the following basic observation.

**Lemma 1.1.** Suppose, $\phi$ is a (continuous) endomorphism of $\Gamma$ and $N$ is a normal $\phi$-invariant subgroup of $\Gamma$. Then the map $\Gamma \to \Gamma/N$ maps Reidemeister classes of $\phi$ onto Reidemeister classes of the induced endomorphism of $\Gamma/N$.

Also we need the following results [7, 5] about the Reidemeister classes of an endomorphism $\varphi$ of a finite group $F$. Consider the action of $\varphi$ on usual (non-twisted) class functions (i.e. functions, which are constant on usual conjugacy classes): $B: f \mapsto f \circ \varphi$. Using the orbit-stabilizer theorem and the TBFT for finite groups one can prove that $R(\varphi) = \text{Tr} B = \text{the number of } \varphi\text{-fixed usually conjugacy classes}$. This implies rationality of the Reidemeister zeta function (see [5, Theorem 17]) via the following calculation:

\[
R_\varphi(z) = \exp \left( \sum_{n=1}^{\infty} \frac{R(\varphi^n)}{n} z^n \right) = \exp \left( \sum_{n=1}^{\infty} \frac{\text{Tr} B^n}{n} z^n \right) = \det \exp \left( \sum_{n=1}^{\infty} \frac{B^n}{n} z^n \right) \\
= \det \exp(- \log(1 - Bz)) = \frac{1}{\det(1 - Bz)}
\]

(at least for $|z| < 1$).

Suppose, $X$ is a set and $\sigma : X \to X$ is a mapping. Denote by $\text{Fix}(\sigma^n)$ the set of fixed points of $\sigma^n$, i.e., $n$-periodic points of $\sigma$. Suppose, its cardinality $|\text{Fix}(\sigma^n)| < \infty$ for any $n$. Then the Artin–Mazur zeta function is defined by

\[
AM_\sigma(z) = \exp \left( \sum_{n=1}^{\infty} \frac{|\text{Fix}(\sigma^n)|}{n} z^n \right).
\]

It is well known (see, e.g. [1]) that using the Taylor series of $\log(1 - z^p)$ the function $AM_\sigma(z)$ can be written as a formal Euler product over all primitive periodic orbits $\tau = \{x_\tau, \sigma(x_\tau), \ldots, \sigma^{p-1}(x_\tau)\}$, with $p = p(\tau)$, such that $\sigma^p(x_\tau) = x_\tau$ and $\sigma^k(x_\tau) \neq x_\tau$ for $0 < k < p$. Namely,

\[
AM_\sigma(z) = \prod_{\tau \text{ primitive periodic orbit}} \frac{1}{1 - z^p(\tau)}.
\]

The elements of a primitive periodic orbit of length $p$ are called $p$-periodic elements. If $|X| < \infty$ this implies rationality of $AM_\sigma(z)$. In the case of a bijective $\sigma$ and finite $X$ the following functional equation is well-known in the field (see e.g. [28])

\[
AM_\sigma(1/z) = (-z)^{|X|} \det(\sigma_C) AM_\sigma(z),
\]
where $\sigma_C : C(X) \to C(X)$ is the induced linear mapping, $C(X) \cong C^{[X]}$. One can generalize (3) to the non-bijective case in the following way (see e.g. [5, Sect. 2.3.2])

\begin{equation}
AM_\sigma(1/z) = \prod_\tau \frac{1}{1 - z^{-p(\tau)}} = \prod_\tau \frac{-z^{p(\tau)}}{1 - z^{p(\tau)}} = \prod_\tau (-z^{p(\tau)}) \prod_\tau \frac{1}{1 - z^{p(\tau)}} = (-1)^a z^b AM_\sigma(z),
\end{equation}

where $a$ is the number of primitive orbits and $b$ is the number of periodic elements. If $\sigma$ is a bijection, each point is periodic and $b = |X|$, while

$$\det(\sigma_C) = \prod_\tau (-1)^{p(\tau) - 1} = \prod_\tau (-1)^{p(\tau)} \cdot (-1)^a = (-1)^{|X|} \cdot (-1)^a$$

and we arrive to (3).

## 2. Dynamic representation theory zeta functions

Suppose, $\phi$ is an endomorphism of a discrete group $\Gamma$. Generally the correspondence $\hat{\phi} : \rho \mapsto \rho \circ \phi$ does not define a dynamical system (an action of the semigroup of positive integers) on the unitary dual $\hat{\Gamma}$ or its finite-dimensional $\hat{\Gamma}_f$ part, or finite $\hat{\Gamma}_{ff}$ part, because in contrast with the automorphism case, the representation $\rho \circ \phi$ may be reducible. Here the unitary dual is the space of equivalence classes of unitary irreducible representations of $\Gamma$, equipped with the hull-kernel topology, $\hat{\Gamma}_f$ is its subspace formed by finite-dimensional representations, and $\hat{\Gamma}_{ff}$ is formed by finite representations.

Nevertheless we can consider representations $\rho$ such that $\rho \sim \rho \circ \phi$.

**Definition 2.1.** A representation theory Reidemeister number $RT(\phi)$ is the number of all $[\rho] \in \hat{\Gamma}$ such that $\rho \sim \rho \circ \phi$. Taking $[\rho] \in \hat{\Gamma}_f$ (respectively $[\rho] \in \hat{\Gamma}_{ff}$) we obtain $RT_f(\phi)$ (respectively $RT_{ff}(\phi)$). Evidently $RT(\phi) \geq RT_f(\phi) \geq RT_{ff}(\phi)$.

**Definition 2.2.** If these numbers are finite for all powers of $\phi$, we define the corresponding dynamic representation theory zeta functions

\begin{align*}
RT_\phi(z) &:= \exp \left( \sum_{n=1}^{\infty} \frac{RT(\phi^n)}{n} z^n \right), & RT_f(\phi)(z) &:= \exp \left( \sum_{n=1}^{\infty} \frac{RT_f(\phi^n)}{n} z^n \right), \\
RT_{ff}(\phi)(z) &:= \exp \left( \sum_{n=1}^{\infty} \frac{RT_{ff}(\phi^n)}{n} z^n \right).
\end{align*}

The importance of these numbers is justified by the following dynamical interpretation. In [13] the following “dynamical part” of the dual space, where $\hat{\phi}$ and all its iterations $\hat{\phi}^n$ define a dynamical system, was defined.

**Definition 2.3** (Def. 2.1 in [13]). A class $[\rho]$ is called a $\hat{\phi}$-f-point, if $\rho \sim \rho \circ \phi$ (so, these are the points under consideration in above definitions).

**Definition 2.4** (Def. 2.2 in [13]). An element $[\rho] \in \hat{\Gamma}$ (respectively, in $\hat{\Gamma}_f$ or $\hat{\Gamma}_{ff}$) is called $\phi$-irreducible if $\rho \circ \phi^n$ is irreducible for any $n = 0, 1, 2, \ldots$.

Denote the corresponding subspaces of $\hat{\Gamma}$ (resp., $\hat{\Gamma}_f$ or $\hat{\Gamma}_{ff}$) by $\hat{\Gamma}_\phi$ (resp., $\hat{\Gamma}_f^\phi$ or $\hat{\Gamma}_{ff}^\phi$).
Lemma 2.5 (Lemma 2.4 in [13]). Suppose, the representations $\rho$ and $\rho \circ \phi^n$ are equivalent for some $n \geq 1$. Then $[\rho] \in \hat{\Gamma}$.

Corollary 2.6 (Corollary 2.5 in [13]). Generally, there is no dynamical system defined by $\hat{\phi}$ on $\hat{\Gamma}$ (resp., $\hat{\Gamma}_f$, or $\hat{\Gamma}_f^f$). We have only the well-defined notion of a $\hat{\phi}^n$-f-point.

A well-defined dynamical system exists on $\hat{\Gamma}$ (resp, $\hat{\Gamma}_f$, or $\hat{\Gamma}_f^f$). Its $n$-periodic points are exactly $\hat{\phi}^n$-f-points.

Let us remark the number of $\hat{\phi}^n$-f-points was denoted in [13] by $\mathbf{F}(\hat{\phi}^n)$, but here we denote it more conceptually by $RT(\hat{\phi}^n)$. We refer to [13] for proofs and details.

Once we have identified the coefficients of representation theory zeta functions with the standard argument with the M"obius inversion formula (see e.g. [5, p. 104], [13]) gives the following statement.

Theorem 2.7. Suppose, $RT(\hat{\phi}^n) < \infty$ for any $n$. Then we have the following Gauss congruences for representation theory Reidemeister numbers:

$$\sum_{d|n} \mu(d) \cdot RT(\hat{\phi}^{n/d}) \equiv 0 \mod n$$

for any $n$.

A similar statement is true for $RT^f(\hat{\phi}^n)$ and $RT^{ff}(\hat{\phi}^n)$.

Here the above M"obius function is defined as

$$\mu(d) = \begin{cases} 
1 & \text{if } d = 1, \\
(-1)^k & \text{if } d \text{ is a product of } k \text{ distinct primes}, \\
0 & \text{if } d \text{ is not square-free.}
\end{cases}$$

The following statement evidently follows from the definitions.

Proposition 2.8. Suppose, $\phi : \Gamma \rightarrow \Gamma$ is an endomorphism and $R(\phi) < \infty$. If TBFT (resp., TBFT$_f$) is true for $\Gamma$ and $\phi$, then $R(\phi) = RT(\phi)$ (resp, $R(\phi) = RT^f(\phi) = RT^{ff}(\phi)$).

If the suppositions keep for $\phi^n$, for any $n$, then $R_\phi(z) = RT_\phi(z)$ (resp., $R_\phi(z) = RT^f_\phi(z) = RT^{ff}_\phi(z)$).

Proof. The only non-trivial fact is that $RT^f(\phi) = RT^{ff}(\phi)$ under our suppositions. We postpone this till a more general consideration in Theorem 5.4. $\square$

Theorem 2.9. Suppose, TBFT (resp., TBFT$_f$) is true for $\Gamma$ and $\phi^n$; and $R(\phi^n) < \infty$ for any $n$. If $R(\phi(z))$ is rational, then $RT_\phi(z)$ (resp., $RT^f_\phi(z) = RT^{ff}_\phi(z)$) is rational.

In particular, $RT^f_\phi(z) = RT^{ff}_\phi(z)$ is rational in the following cases:

1. $\Gamma$ is a finitely generated abelian group;
2. $\Gamma$ is a finitely generated torsion free nilpotent group;
3. $\Gamma$ is a crystallographic group with diagonal holonomy $\mathbb{Z}_2$ and $\phi$ is an automorphism.

Proof. The first part follows immediately from Proposition 2.8.

In the second case we have a polycyclic group and in the third case we have an almost polycyclic group. TBFT$_f$ for the cases under consideration is proved in [6] (for endomorphisms of abelian groups) (see also [11]), in [10] (for automorphisms of almost polycyclic groups) and in [13] (for endomorphisms of polycyclic groups).
Rationality of \( R_\phi(z) \) is proved in [7] for the first and second cases (see [5]) and for the third case in [3].

\[ \square \]

**Remark 2.10.** In the same way with the help of Proposition 2.8 one can extract from [5] more information in the first case above. Namely, in this case all irreducible representations are 1-dimensional and \( RT_\phi(z) = RT'_\phi(z) = RT''_\phi(z) \). Let \( T \) be the characteristic torsion subgroup with \( Z^n = T / T \). Denote by \( \phi_T : T \to T \) and by \( \phi' : Z^n \to Z^n \) the induced endomorphisms. Then

\[ RT_\phi(z) = L_\phi(\sigma z)^{-r}, \]

where \( L \) is the Lefschetz zeta function of \( \hat{\phi} : \hat{T} \to \hat{T} \), \( \sigma = (-1)^p \) where \( p \) is the number of real eigenvalues of the linear operator on the Lie algebra of \( \hat{G} \), corresponding to \( \hat{\phi}' \) such that \( \lambda < -1 \) and \( r \) is the number of its real eigenvalues \( \lambda \) such that \( |\lambda| > 1 \) (see [5, Theorems 28 and 29] for details).

Since, by the definition, \( RT_\phi(z) \) (resp, \( RT'_\phi(z) \), or \( RT''_\phi(z) \)) is the Artin-Mazur zeta function of \( \hat{\phi} \) on \( \hat{T} \) (resp, \( \hat{T}' \), or \( \hat{T}'' \)), we obtain immediately from (2) the following

**Proposition 2.11.**

\[ RT_\phi(z) = \prod_{\tau \text{ primitive periodic orbit}} \frac{1}{1 - z^{p(\tau)}}, \]

and similarly for \( RT'_\phi(z) \) and \( RT''_\phi(z) \).

Now we shall prove rationality of zeta functions and related facts under restriction on \( \hat{T} \) rather than on \( T \) itself, as above. More precisely, we will suppose that \( \hat{T} \) (resp, \( \hat{T}' \), or \( \hat{T}'' \)) is finite. In this case \( RT_\phi(z) \) (resp, \( RT'_\phi(z) \), or \( RT''_\phi(z) \)) is the Artin-Mazur function on a finite set and from Proposition 2.11 and (4) we obtain the following statement.

**Theorem 2.12.** Suppose that \( \hat{T} \) (resp, \( \hat{T}' \), or \( \hat{T}'' \)) is finite. Then \( RT_\phi(z) \) (resp, \( RT'_\phi(z) \), or \( RT''_\phi(z) \)) is rational and satisfies the following functional equation

\[ RT_\phi(1/z) = (-1)^a z^b RT_\phi(z), \]

where \( a \) is the number of primitive orbits and \( b \) is the number of periodic elements of \( \hat{\phi} \) on \( \hat{T} \). Similarly for \( RT'_\phi(z) \) and \( RT''_\phi(z) \).

**Example 2.13.** An evident example is a finite group.

**Example 2.14.** Less evident examples give some infinite groups with finitely many conjugacy classes: Osin group [22], Ivanov group and some HNN extensions described in [10]. Then \( R(\text{Id}) \) is finite. These groups have only one finite-dimensional representation (the trivial one) and thus enter conditions of Theorem 2.12 (see [10] for necessary proofs).

### 3. Compactifications and Reidemeister numbers

**Definition 3.1.** A \( \phi \)-class function is a function being constant on Reidemeister classes.

**Lemma 3.2.** If \( G \) is compact Hausdorff and \( R(\phi) < \infty \), then Reidemeister classes are clopen (closed and open). In particular, \( \phi \)-class functions are continuous.
Proof. Reidemeister classes are orbits of the continuous twisted action \( g \mapsto xg\varphi(x^{-1}) \) of \( G \) on itself. Thus, they are compact. Hence, closed, because \( G \) is Hausdorff. The complement to each of them is a finite union of closed sets.

Definition 3.3. A compactification \( C \) of a group \( G \) is a couple consisting of a compact Hausdorff group \( C(G) \) and a (continuous) homomorphism \( \alpha_C : G \to C(G) \) with dense image. We say that \( C \) is admissible for an endomorphism \( \phi : G \to G \) if

- \( \alpha_C \) has a \( \phi \)-invariant kernel (may be non-trivial);
- the induced homomorphism \( C(\phi) : C(G) \to C(G) \) is continuous.

If \( C \) is admissible for any endomorphism \( \phi : G \to G \), we say that it is admissible. The induced homomorphism \( C(\phi) \) is the compactification of \( \phi \). Denote by \( R_C(\phi) \) the corresponding Reidemeister numbers, i.e. \( R_C(\phi) = R(C(\phi)) \).

The corresponding compactification Reidemeister zeta function is defined as

\[
R^C_\phi(z) := \exp \left( \sum_{n=1}^{\infty} \frac{R_C(\phi^n)}{n} z^n \right)
\]

supposing that all \( R_C(\phi^n) \) are finite.

Example 3.4. The main examples of admissible compactifications come from the profinite completion \( \mathcal{P} \) and the universal compactification \( \mathcal{U} \) (see Lemma 3.5).

The universal compactification \( \mathcal{U}(\Gamma) \) is defined as the closure of the image of the diagonal homomorphism from \( \Gamma \) to the topological (Tikhonoff) product of unitary groups being the ranges of all (equivalence classes of) finite dimensional irreducible unitary representations of \( \Gamma \). Denote by \( \alpha_{\mathcal{U}} \) the natural map \( \alpha_{\mathcal{U}} : \Gamma \to \mathcal{U}(\Gamma) \). The pair \( (\mathcal{U}(\Gamma), \alpha_{\mathcal{U}}) \) enjoys the following universal property: for any homomorphism \( \alpha' : \Gamma \to G \), where \( G \) is some compact group there exists a unique continuous homomorphism \( \gamma \) such that the diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\alpha_{\mathcal{U}}} & \mathcal{U}(\Gamma) \\
\downarrow{\alpha'} & & \downarrow{\gamma} \\
G & & \end{array}
\]

commutes and is uniquely defined by it (see, e.g. [4, §16] for details).

The profinite completion \( \alpha_{\mathcal{P}} : \Gamma \to \mathcal{P}(\Gamma) \) of the image of the diagonal homomorphism to the Tikhonoff product of all finite quotients of \( \Gamma \). It enjoys a universality property, similar to (5), but with a profinite (i.e. Hausdorff, compact, and totally disconnected) group \( G \) instead of a general compact group \( G \) (see e.g. [23, 27] for details). The profinite completion also can be defined in the same way as the universal completion: namely we need to take all finite representations instead of all finite-dimensional representations. The equivalence follows now from the universal property of profinite completion and the decomposition of the regular representation of a finite group.

Lemma 3.5. The profinite completion \( \mathcal{P} \) and the universal compactification \( \mathcal{U} \) are admissible compactifications.
Proof. This follows immediately from the universal properties. Indeed, a continuous homomorphism \( \phi' \) making the diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\alpha_C} & \mathcal{U}(\Gamma) \\
\downarrow \phi & & \downarrow \phi' \\
\Gamma & \xrightarrow{\alpha_C} & \mathcal{U}(\Gamma)
\end{array}
\]

commutative can exist if and only if both properties in the definition of admissibility are fulfilled. Similarly, for the profinite completion. □

Lemma 3.6. For any admissible \( \mathcal{C} \), we have \( R_\mathcal{C}(\phi) \leq R(\phi) \).

Proof. Consider the image \( \alpha_C(\Gamma) \) of the compactification homomorphism \( \alpha : \Gamma \to \mathcal{C}(\Gamma) \). Then \( \alpha_C : \Gamma \to \alpha_C(\Gamma) \) is an epimorphism, and Lemma 1.1 implies \( R(\phi) \geq R(\mathcal{C}(\phi)|_{\alpha_C(\Gamma)}) \).

Evidently, any Reidemeister class in \( \alpha_C(\Gamma) \) is inside some class of \( \mathcal{C}(\phi) \), which is a compact set being an orbit of the twisted action of the compact group \( \mathcal{C}(\Gamma) \). Hence, the closure of a class in \( \alpha_C(\Gamma) \) still is inside a class of \( \mathcal{C}(\phi) \). On the other hand, the density of \( \alpha_C(\Gamma) \) implies that each class of \( \mathcal{C}(\Gamma) \) contain a class of \( \alpha_C(\Gamma) \). □

From the universal property (5) we obtain an epimorphism \( \mathcal{U}(\Gamma) \to \mathcal{P}(\Gamma) \) and Lemma 1.1 immediately implies the following statement.

Lemma 3.7. \( R_\mathcal{U}(\phi) \geq R_\mathcal{P}(\phi) \).

Now we need to generalize Theorem 4.3 and Lemma 5.1 from [13] from finite to compact groups in the following way.

Theorem 3.8 (cf. [6]). Let \( \varphi : G \to G \) be an endomorphism of a compact Hausdorff group \( G \). Suppose, the Reidemeister number \( R(\varphi) < \infty \). Then \( R(\varphi) \) coincides with the number of \( \hat{\varphi} \)-f-points on \( \hat{G} \).

Proof. Let us note that \( R(\varphi) \) is equal to the dimension of the space of \( \varphi \)-class functions (i.e. those functions that are constant on Reidemeister classes). By Lemma 3.2 these functions are continuous. Thus, they can be described as fixed elements of the action \( a \mapsto ga\varphi(g^{-1}) \) on the group algebra \( C^*(G) \). For the latter algebra we have the Peter-Weyl decomposition

\[ C^*(G) \cong \bigoplus_{[\rho] \in \hat{G}} \text{End} V_\rho, \quad \rho : G \to U(V_\rho), \]

which respects the left and right \( G \)-actions and the right-hand side is equipped with the sup-norm. Hence,

\[ R(\varphi) = \sum_{[\rho] \in \hat{G}} \dim T_\rho, \quad T_\rho := \{ a \in \text{End} V_\rho \mid a = \rho(g)a\varphi(g^{-1}) \text{ for all } g \in G \}, \]

where we allow at this stage infinitely many non-zero summands. Thus, if \( 0 \neq a \in T_\rho \), then \( a \) is an intertwining operator between the irreducible representation \( \rho \) and some representation \( \rho \circ \varphi \). This implies that \( \rho \) is equivalent to some (irreducible) subrepresentation \( \pi \) of \( \rho \circ \varphi \) (cf. [20, VI, p.57]). Hence, \( \dim \rho = \dim \pi \), while \( \dim \rho = \dim \rho \circ \varphi \). Thus, \( \pi = \rho \circ \varphi \), and is irreducible. In this situation \( \dim T_\rho = 1 \) by the Schur lemma. Evidently, vice versa, if
\[
\rho \sim \rho \circ \varphi \text{ then } \dim T_{\rho} = 1. \text{ Hence,}
\]

\[
R(\varphi) = \sum_{[\rho] \in \hat{G}} \begin{cases} 
1, & \text{if } \rho \sim \rho \circ \varphi \\
0, & \text{if } \rho \not\sim \rho \circ \varphi 
\end{cases} = \text{number of } \hat{\varphi}\text{-f-points.}
\]

**Lemma 3.9.** Let \( \rho \) be a (finite-dimensional) irreducible representation of a compact group \( G \). It is a \( \hat{\varphi}\text{-f-point of an endomorphism } \varphi : G \to G \), if and only if there exists a non-zero \( \varphi \) class function being a matrix coefficient of \( \rho \).

In this situation this function is unique up to scaling and is defined by the formula

\[
(6) \quad T_{S,\rho} : g \mapsto \text{Tr}(S \circ \rho(g)),
\]

where \( S \) is an intertwining operator between \( \rho \) and \( \rho \circ \varphi \):

\[
\rho(\varphi(x))S = S\rho(x) \quad \text{for any } x \in G.
\]

In particular, TBFT is true for \( \varphi \) if and only if the above matrix coefficients form a base of the space of \( \varphi \)-class functions.

**Proof.** First, let us note that (6) defines a class function:

\[
T_{S,\rho}(xg\varphi(x^{-1})) = \text{Tr}(S \rho(xg\varphi(x^{-1}))) = \text{Tr}(\rho(\varphi(x))S \rho(g)\rho(\varphi(x^{-1}))) = \text{Tr}(S \rho(g))
\]

If \( S \neq 0 \), then \( \rho(a) = S^* \) for some \( a \in C^*(G) \), and \( \text{Tr}(SS^*) \neq 0 \). Thus, the \( \varphi \)-class function is non-zero. On the other hand, any non-trivial matrix coefficient of \( \rho \), i.e., a functional \( T : \text{End}(V_\rho) \to \mathbb{C} \), has the form \( g \mapsto \text{Tr}(D \rho(g)) \) for some fixed matrix \( D \neq 0 \). If it is a \( \varphi \)-class function, then for any \( g \in G \), or similarly, \( a \in C^*(G) \),

\[
\text{Tr}(D \rho(a)) = \text{Tr}(D \rho(xa\varphi(x^{-1}))) = \text{Tr}(\rho(\varphi(x^{-1}))D \rho(x)\rho(a)).
\]

Since \( \rho(a) \) runs over the entire matrix algebra, this implies

\[
D = \rho(\varphi(x^{-1}))D \rho(x), \quad \text{or} \quad \rho(\varphi(x))D = D \rho(x),
\]

i.e., \( D \) is the desired non-zero intertwining operator.

The uniqueness up to scaling follows now from the explicit formula and the Shur lemma.

The last statement follows from linear independence of matrix coefficients of non-equivalent representations. \( \square \)

**Corollary 3.10.** TBFT is true for endomorphisms of compact Hausdorff groups.

As it was explained above, this implies

**Corollary 3.11.** If \( R_C(\phi^n) < \infty \) for any \( n \), they satisfy the Gauss congruences.

4. **Finite representations versus finite-dimensional ones**

We will extend in this section a result from [12] to compact groups and endomorphisms.

**Theorem 4.1.** Suppose, \( \phi : G \to G \) is an endomorphism of a compact group \( G \) and \( R(\phi) < \infty \). If a matrix coefficient of some finite-dimensional irreducible representation \( \rho \) of \( G \) is a \( \phi \)-class function, then \( \rho \) is finite.
Proof. We give only a sketch of the proof, because it is more or less the same as in [12]. Suppose, \( f_\rho \) is a non-trivial matrix coefficient. Hence, its left translations, being once again matrix coefficients of \( \rho \), generate a translation invariant subspace \( W \) of the finite-dimensional space \( V_\rho \otimes V_\rho^* \). Hence, \( W \) is a space of a finite-dimensional representation, which is isomorphic to a direct sum of several copies of \( \rho \) (see, e.g. [20, Ch. IV]). The space \( W \) has a basis \( L_g f_\rho, \ldots, L_g f_\rho \). Thus, all functions from \( W \) take only finitely many values (with level sets of the form \( \cap g_i U_j \), where \( U_j \) are the level sets of \( f_\rho \)). Thus, there exists a finite partition \( G = V_1 \sqcup \cdots \sqcup V_m \) such that elements of \( W \) are constant on the elements of the partition and for each pair \( V_i \neq V_j \) there exists a function from \( W \) taking distinct values on them. Note that these sets \( V_i \) are closed and open, because matrix coefficients are continuous. Hence, left translations map \( V_i \) onto each other. This means that the representation \( G \) on \( W \) factorizes through (a subgroup of) the permutation group on \( m \) elements, i.e. a finite group. The same is true for its subrepresentation \( \rho \), thus it is finite by definition. \( \square \)

The following statement was known for automorphisms of locally connected compact groups (see [11]), while here our main subject is the class of totally disconnected groups.

Corollary 4.2. If \( \varphi : G \to G \) is an endomorphism of a compact group \( G \) with \( R(\varphi) < \infty \), then there exists a closed normal \( \varphi \)-invariant subgroup of finite index \( G_0 \) such that the epimorphism

\[
p : G \to G/G_0 =: F
\]

onto a (finite) group \( F \) gives a bijection of Reidemeister classes.

Proof. The intersection of group kernels of all finite representations \( \rho \) such that \( \rho \sim \rho \circ \varphi \) can be taken as \( G_0 \) by Theorem 3.8, Lemma 3.9, and Theorem 4.1. \( \square \)

5. COMPARING ZETA FUNCTIONS

Now we will use Theorem 4.1 to prove the following statement.

Theorem 5.1. We have for an endomorphism \( \phi : \Gamma \to \Gamma \),

1) \( R_U(\phi) = RT^f(\phi) \);
2) \( R_U(\phi) = R_P(\phi) \) if \( R_U(\phi) \) is finite;
3) \( R_P(\phi) = RT^{ff}(\phi) \) if \( R_P(\phi) \) is finite.

Proof. The first statement follows immediately from Theorem 3.8.

The second statement follows from the same theorem in combination with Lemma 3.9 and Theorem 4.1.

For the third statement observe that, by Theorem 3.8, \( R_P(\phi) \) is equal to the number of (finite-dimensional) irreducible representations \( \rho \) of \( P(\Gamma) \), such that \( \rho \sim \rho \circ \phi \). Since \( R_P(\phi) < \infty \), by Theorem 4.1, these representations are finite. But finite representations of the profinite completion are just finite representation of \( \Gamma \) itself (cf. the argument before Lemma 3.5).

Remark 5.2. It seems possible to make the conditions weaker in the case of finitely generated groups using approaches from [21].

Remark 5.3. Combining Corollary 3.11 and Theorem 5.1 one obtains another proof of Theorem 2.7.

Theorem 5.4. If \( R_U(\phi) < \infty \), then \( R_U(\phi) = R_P(\phi) = RT^f(\phi) = RT^{ff}(\phi) \). In particular, these equalities are true, if \( R(\phi) < \infty \).
Proof. The main statement is a combination of items from Theorem 5.1. The other statement follows from Lemma 3.6. □

This statement implies immediately

**Theorem 5.5.** If $R_d(\phi^n) < \infty$, $n = 1, 2, \ldots$, then $R_d(\phi)(z) = R_d(z) = R_d^{ij}(z) = R_d^{ij}(z)$. In particular, these equalities are true, if $R(\phi^n) < \infty$ for all $n$.

**Remark 5.6.** One can immediately obtain several equalities for zeta functions under more weak conditions for powers, similar to those from Theorem 5.1.

### 6. Calculating zeta function

We will need the following evident observation. Consider two power series $\sum_{k=1}^{\infty} a_k z^k$ and $\sum_{k=1}^{\infty} b_k z^k$ with non-negative coefficients: $a_k \geq 0$, $b_k \geq 0$. Suppose, the first series is uniformly convergent on a closed disk $D$ of radius $d$ and $b_k \leq a_k$ for all $k$. Denote $z_0 = d$.

Then we have the following estimation, for $z \in D$:

\[ \left| \sum_{k=s}^{p} b_k z^k \right| \leq \sum_{k=s}^{p} b_k |z|^k \leq \sum_{k=s}^{p} a_k d^k. \]  

(7)

In particular, the second series is uniformly convergent on $D$.

Let $R(\phi^n) < \infty$ for any $n$, where $\phi : G \to G$ is an endomorphism of a compact group $G$. In accordance with the above results (Corollary 4.2), there is a countable collection of normal $\phi$-invariant subgroups $G_i$ of $G$ of finite index such that $p_i : G \to G/G_i =: F_i$ gives a bijection of Reidemeister numbers.

**Theorem 6.1.** Suppose, $R(\phi^n) < \infty$ for any $n$, where $\phi : G \to G$ is an endomorphism of a compact group $G$. Suppose, $\sum_{k=1}^{\infty} R(\phi^k) z^k$ is uniformly convergent on some closed disk $D$ of radius $d < 1$. Choose an above defined collection of subgroups, and let $B_i$ denote the operator $B$ from Section 1 on class functions on $F_i$. Then

\[ R(\phi)(z) = \lim_{i \to \infty} \frac{1}{\det(1 - B_i z)}, \quad z \in D. \]

Proof. Take $z \in D$ and arbitrary $\varepsilon > 0$. Denote $u = \sum_{k} z^k \frac{R(\phi^k)}{k}$ and choose $\delta > 0$ such that $|\exp(u) - \exp(u')| < \varepsilon$, if $|u - u'| < \delta$. Choose $k_0$ so large that

\[ \left| \sum_{k=k_0}^{\infty} \frac{R(\phi^k)}{k} z^k \right| = \sum_{k=k_0}^{\infty} \frac{R(\phi^k)}{k} < \frac{\delta}{2}. \]

(8)

Now find a sufficiently large $i_0$ such that

\[ R(\phi^k) = R((\phi_i)^k), \quad k = 1, \ldots k_0 - 1, \quad i = i_0, i_0 + 1, \ldots. \]

For $k = k_0, k_0 + 1, \ldots$, we have $R(\phi^k) \geq R((\phi_i)^k)$ and hence, by the observation in the beginning of this section,

\[ \left| \sum_{k=k_0}^{\infty} \frac{R((\phi_i)^k)}{k} z^k \right| \leq \sum_{k=k_0}^{\infty} \frac{R(\phi^k)}{k} < \frac{\delta}{2}. \]

(9)
Evidently,  
\[
\left| \sum_{k=k_0}^{\infty} \frac{R(\varphi^k)}{k} z^k \right| \leq \sum_{k=k_0}^{\infty} \frac{R(\varphi^k)}{k} d^k < \frac{\delta}{2}.
\]

From (9), (10), and (11) we obtain, for \( i \geq i_0 \),
\[
\left| u - \sum_{k=1}^{\infty} \frac{R((\varphi_i)^k)}{k} z^k \right| \leq \left| \sum_{k=k_0}^{\infty} \frac{R(\varphi^k)}{k} z^k \right| + \left| \sum_{k=k_0}^{\infty} \frac{R((\varphi_i)^k)}{k} z^k \right| < \delta + \frac{\delta}{2} = \delta.
\]

Thus, by the choice of \( \delta \),
\[
\| R(\varphi) - R(\varphi_i) \| < \varepsilon, \quad i \geq i_0.
\]

Applying (1) to \( \varphi_i \) with \( i \geq i_0 \) we deduce from (12) the estimation
\[
\left| R(\varphi(z)) - \frac{1}{\det(1 - B_i z)} \right| < \varepsilon.
\]

This completes the proof. \( \square \)

By the definition of \( R_{\varphi}^C(z) \) and Theorem 5.5 we obtain the following

**Corollary 6.2.** The formula from Theorem 6.1 remains valid for \( R_{\varphi}^C(z) \), \( RT_{\varphi}^T(z) \), and \( RT_{\varphi}^{ff}(z) \), if \( R(\phi^n) < \infty \) for all \( n \).

### 7. Examples and Counterexamples

In the remaining part of this section we will develop an example from [26].

Let \( F \) be a finite non-trivial group and \( G = \bigoplus_{i \in \mathbb{Z}} F_i \), \( F_i \cong F \), i.e.
\[
\Gamma = \{ g = (\ldots, g_{-1}, g_0, g_1, g_2, \ldots) \mid g_i \in F_i, g_i \neq e \text{ only for a finite number of } i \}.
\]

Hence, \( G \) is an infinitely generated residually finite group.

Define \( \phi \) to be the right shift: \( \phi(g)_i = g_{i-1} \), \( i \in \mathbb{Z} \).

**Lemma 7.1** ([26, Lemma 3.1]). \( R(\phi) = |F| < \infty \).

*Sketch of the proof.* For \( a, g \in G \), the twisted conjugation has the form \((g a \phi(g^{-1}))_i = g_i a_i (g_{i-1})^{-1}\). For a twisted conjugation of two elements of the form
\[
\alpha : \quad \alpha_0 = x, \quad \alpha_i = e \text{ for } i \neq 0, \quad \beta : \quad \beta_0 = y, \quad \beta_i = e \text{ for } i \neq 0,
\]
the above formula gives
\[
g_0 x (g_{-1})^{-1} = y, \quad g_i (g_{i-1})^{-1} = e \text{ for } i \neq 0.
\]

So, \( g_0 = g_1 = \ldots, g_{-1} = g_{-2} = \ldots \) Since \( g_i = e \) for large \( i \), \( g = (\ldots, e, e, e, \ldots) \). Thus, \( \alpha \) and \( \beta \) are twisted conjugate if and only if they coincide.

Then we verify that any element \( a = (\ldots, a_i, \ldots) \), \( a_i = e \) for \( i < -m \) and \( i > n \), is twisted conjugate to some element of the same form as \( \alpha \) (with \( x = a_n \ldots a_{-m} \)). \( \square \)

**Lemma 7.2** ([26, Lemma 3.2]). Suppose, \( F \) has a trivial center. Then TBFT fails for \( G \).
We will obtain below a more strong statement. It is an easy well-known exercise to prove the following statement:

\[ \hat{\oplus}_i F_i = \prod_i \hat{F}_i, \]

where \( \prod \) means the topological (Tikhonoff) product. Evidently, \( \hat{\phi} \) is the left shift. Its fixed points evidently are elements of \( \prod \) with the same value \([\rho]\) for all \( i \). The corresponding representations \( \rho^\infty \) are finite-dimensional if and only if \( \rho \) is one-dimensional. If we denote by \( \hat{F}(1) \) the subset of \( \hat{F} \), formed by one-dimensional representations, we obtain:

(13) \[ RT(\phi) = |\hat{F}|, \quad RT_f(\phi) = |\hat{F}(1)|, \]

where we use \(|.|\) for cardinality of a set.

Immediately from (13) and Lemma 7.1 we obtain the following statement.

**Theorem 7.3.** If \( F \) is not abelian, the above group \( \oplus F \) is an infinitely generated residually finite group, for which neither TBFT, nor TBFT_H is true.

Consider \( \phi^n \) for this \( \phi \). Then is the left shift by \( n \) positions and \( \oplus F \) decomposes in a sum of \( n \phi^n \)-invariant summands and at each of them \( \phi^n \) acts in the same way as \( \phi \) on \( \oplus F \), i.e. by the left shift (by 1 position). Thus, we have the following explicit formulas:

(14) \[ R(\phi^n) = (R(\phi))^n = |F|^n, \quad RT(\phi^n) = (RT(\phi))^n = |\hat{F}|^n, \quad RT_f(\phi^n) = (RT_f(\phi))^n = |\hat{F}(1)|^n. \]

**Proposition 7.4.** For the above group and automorphism \( \phi \), zeta functions \( R_\phi(z) \), \( R_\phi^d(z) = R_\phi^d(z) = R_\phi^d(z) = R_\phi^d(z) \) and \( R_\phi(z) \) are rational. More precisely,

\[ R_\phi(z) = \frac{1}{1 - |F|z}, \quad R_\phi^d(z) = \frac{1}{1 - |\hat{F}(1)|z}, \quad RT_\phi(z) = \frac{1}{1 - |\hat{F}|z}. \]

**Proof.** Indeed, by (14),

\[ R_\phi(z) = \exp \left( \sum_{n=1}^{\infty} \left( \frac{|F|^n}{n} \right) \right) = \exp(-\log(1 - |F|z)) = \frac{1}{1 - |F|z}. \]

Similarly for the remaining cases, using (14). \( \square \)

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