Crack Fault Diagnosis and Location Method for a Dual-Disk Hollow Shaft Rotor System Based on the Radial Basis Function Network and Pattern Recognition Neural Network

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Abstract

The crack fault is one of the most common faults in the rotor system, and researchers have paid close attention to its fault diagnosis. However, most studies focus on discussing the dynamic response characteristics caused by the crack rather than estimating the crack depth and position based on the obtained vibration signals. In this paper, a novel crack fault diagnosis and location method for a dual-disk hollow shaft rotor system based on the Radial basis function (RBF) network and Pattern recognition neural network (PRNN) is presented. Firstly, a rotor system model with a breathing crack suitable for a short-thick hollow shaft rotor is established based on the finite element method, where the crack's periodic opening and closing pattern and different degrees of crack depth are considered. Then, the dynamic response is obtained by the harmonic balance method. By adjusting the crack parameters, the dynamic characteristics related to the crack depth and position are analyzed through the amplitude-frequency responses and waterfall plots. The analysis results show that the first critical speed, first subcritical speed, first critical speed amplitude, and super-harmonic resonance peak at the first subcritical speed can be utilized for the crack fault diagnosis. Based on this, the RBF network and PRNN are adopted to determine the depth and approximate location of the crack respectively by taking the above dynamic characteristics as input. Test results show that the proposed method has high fault diagnosis accuracy. This research proposes a crack detection method adequate for the hollow shaft rotor system, where the crack depth and position are both unknown.

Keywords Hollow shaft rotor, Breathing crack, Radial basis function network, Pattern recognition neural network, Machine learning

1 Introduction

Rotating machinery is an integral part of modern industry, which has many applications in gas turbines, aero-engine, wind turbines, and other critical machinery equipment. Because the working environment of rotating machinery is harsh, it is easy to generate mechanical failure. The crack fault is one of the most common failure forms in the rotor system [1], which can seriously threaten the reliability of rotating machinery operating. Therefore, timely and accurate fault diagnosis of the crack fault is of great significance in improving the operation reliability of rotating machinery. Since the 1970s, many researchers have studied the crack fault diagnosis of the rotor system in many aspects. Mayes et al. [2] first used a more accurate cosine breathing function to describe the breath behavior of the crack and studied
the dynamic response of the rotor system containing a breathing crack. Then Al-Shudeifat et al. [3] proposed a new type of breathing function using the Fourier series and compared it with Mayes’ results. The comparative results show that this respiration function could more accurately represent the breathing process of crack. Besides, it is found that the super-harmonic resonance can be regarded as the vibration characteristics caused by the crack, and the phenomenon of the natural frequency change with the increase of the crack depth is reported. Darpe et al. [4] analyzed the influence of the interaction of the two cracks on the breathing behavior and the dynamic response of the Jeffcott rotor based on the concepts of fracture mechanics. Significant transformations in the rotor’s dynamic response are observed when the angle between the two cracks’ directions changes. Xiang et al. [5] considered an asymmetric rotor-bearing system with crack and rub-impact coupling faults under oil-film forces. The effects of crack depth on the onset of instability and nonlinear responses of the rotor-bearing system are studied. Hou et al. [6] found the super-harmonic resonance phenomenon of the cracked rotor due to maneuver load. Lu et al. [7] studied the dynamic response of a hollow shaft dual-rotor system with breathing crack and unbalanced excitation. In the spectrum, it is found that the peak value of the super-harmonic resonance is related to the dimensionless depth of the crack. Fu et al. [8] reported similar results in the dynamic response of a cracked rotor system with uncertain crack parameters. Wang et al. [9] considered the anisotropy in a cracked rotor system and analyzed its parameter instability phenomena. Unlike traditional dynamic analysis methods, Liu et al. [10] developed a novel Nonlinear Output Frequency Response Functions (NOFRFs) based criterion and discussed its application to the cracked rotor system. Through simulations and experiments, they found that some specific index is sensitive to the degree of crack propagation. Most of these studies focus on the dynamic response characteristics of the cracked rotor system or only qualitative analysis of the influence of the crack parameters (such as the depth and location of the crack) [11–13]. However, quantitatively identifying the fault properties based on the dynamic response is necessary for many practical situations. This type of problem is known as fault diagnosis [14].

In recent years, various machine learning methods, such as the artificial neural network (ANN) [15], support vector machine (SVM) [16], and decision-making tree (DT), have been widely applied in various fields. Many researchers also adopt these methods to study the fault diagnosis of the rotor system. Munoz-Abella et al. [17] used the ANN and a large number of simulation data to make the crack fault diagnosis for a simple Jeffcott rotor and achieved good results. Guo et al. [18] proposed a fault diagnosis method for a Jeffcott rotor with a breathing crack at the early stage of crack propagation based on the empirical mode decomposition (EMD) technology combined with ANN and conducted experimental verification. Vashisht et al. [19] investigated the effect of cracks on a complex rotor system with ball bearing and advanced a crack detection method using the switching control strategy and Short Time Fourier Transform. Yan et al. [20] extracted multi-domain features from the vibration signals by combining multiple signal processing technologies (such as statistical analysis, Variational Mode Decomposition, and Fast Fourier Transform). Then, a novel optimized SVM is adopted to study the fault diagnosis problem of the rotor system. Fault types include a crack in the outer race, an inner race with the spall, and pitting in balls. Comparative test results show that the proposed method is better than the traditional SVM. Bin et al. [21] proposed a new approach for rotating machinery fault diagnosis based on wavelet packet decomposition (WPD)-EMD fault feature extraction and the neural network. Ma et al. [22] presented a diagnosis method for rotor and bearing faults of rotating machinery based on ensemble learning. In this study, the method of weighting and integrating the Convolution Residual Network (CRN), Deep Belief Network (DBN), and Deep AutoEncoder (DAE) obtains a significant effect on the problem of multi-fault classification. Wang et al. [23] determined the crack parameters, including crack location, depth, and angle for a solid shaft by using Kriging Surrogate Model and improved Nondominated Sorting Genetic Algorithm-III (NSGA-III), which has high parameter identification accuracy. Wang et al. [24] used the K-means clustering method to classify crack faults for a planetary gearbox. Li et al. [25] studied multiple crack identification based on the three steps meshing, and experimental verification was also carried out. Most researchers’ objects are simple Jeffcott rotors [26–28] or solid shaft rotors [29–31]. Nevertheless, to improve the rotor operating efficiency in practical problems, most large complex rotor systems adopt the hollow shaft. There is relatively little research on crack fault diagnosis of hollow shaft rotor systems.

The crack model of the hollow shaft is more complex than the solid shaft, which leads to a more complex dynamic response, making crack fault diagnosis more difficult. Besides, many researchers focus on the pattern recognition of the crack fault in the various faults of the rotor system or calculating the crack depth with the fixed crack position. However, in practice, the crack failure may occur at any shaft position. The depth and position of the crack can have a coupling effect on the system’s dynamic response, which may confuse the
crack diagnosis results. Therefore, identifying the hollow shaft crack’s parameters based on the system's dynamic response when the crack depth and position are both unknown remains challenging.

The motivation of this study is to develop a novel crack fault diagnosis method for a two-disk hollow shaft rotor system. In which both the crack depth and location are uncharted. Considering the crack's periodic opening and closing pattern and different degrees of crack depth, we establish the hollow shaft crack model with the breathing function. The dynamic response of the cracked rotor system is obtained by adopting the Harmonic Balance Method, and some dynamic characteristics related to the crack properties are summarized. Based on this, the Radial basis function (RBF) neural network and pattern recognition network are utilized to solve the crack fault diagnosis problem when the crack's depth and location are both unknown. The effectiveness of the proposed method is verified by simulation.

The paper is organized as follows, in Section 2, the motion equations of the dual-disk hollow shaft rotor system with a breathing crack are constructed by the finite element method. Secondly, in Section 3, the harmonic balance method (HBM) is used to solve the dynamic response of the rotor system, and the effect of crack depth and position is analyzed, respectively. The Runge-Kutta method is used to verify the results of HBM. The crack depth estimation problem is discussed based on the RBF neural network, and the problem of crack location is solved using the pattern recognition network in Section 4. Finally, Section 5 summarizes the primary results of this paper.

2 Dynamic Model of a Dual-disk Hollow Shaft Rotor with a Breathing Crack

2.1 Finite Element Model of the Rotor System

The finite element model of the cracked dual-disk hollow shaft rotor system is shown in Figure 1, divided into 20 elements and 21 nodes. Each end of the rotor (node 1 and node 21) has a bearing and support. There are two disks in this rotor system, which locate at node 5 and node 17, respectively. The shaft of this model is hollow, and its inner radius is denoted as $r$ while its outer radius is $R$. To fit the actual failure status better, the crack may locate in any element of the shaft. The physical parameters of the model are shown in Table 1.

| Parameters        | Value                      |
|-------------------|----------------------------|
| Shaft length $l$ (m) | 0.8562                     |
| External radius $R$ (m) | 0.0441                     |
| Inside radius $r$ (m) | 0.0110                     |
| Material density $\rho$ (kg/m$^3$) | 7800                       |
| Young’s modulus $E$ (GPa) | 2.09                        |
| Shear modulus $G$ (GPa) | 80.38                      |
| Bearing stiffness $K$ (N/m) | 5 \times 10^7             |
| Disk 1 diameter $d_1$ (m) | 0.1215                     |
| Disk 1 mass $m_1$ (kg) | 6.5157                     |
| Disk 2 diameter $d_2$ (m) | 0.1493                     |
| Disk 2 mass $m_2$ (kg) | 3.9543                     |
| Amount of unbalance $m_d$ (kg-m) | 5 \times 10^{-7}          |
| Unbalance deflection $\theta$ | 0                          |
| Bearing damping $c_{xx}$ (Ns/m) | 1000                     |

![Figure 1](image1.png)

**Figure 1** Finite element model of a cracked two-disk hollow shaft rotor system

![Figure 2](image2.png)

**Figure 2** Cross-section diagram of hollow shaft crack: (a) Non-penetrating crack, (b) Passthrough crack. The dashed area represents the uncracked segment

2.2 Crack Model

The crack in the hollow shaft is more complex than that in the solid shaft. Figure 2 shows different crack conditions under disparate depths, and the dashed area represents the uncracked segment. When the crack is relatively shallow, it is a non-penetrating crack (Figure 2a), and when the crack depth is large, it is a passthrough crack (Figure 2b). $A_1$ and $A_C$ represent the cross-sectional area of uncracked and cracked segments, respectively. $O-\Omega$ is the fixed coordinate system, and $C-x\bar{y}$ is the centroid coordinate system. $C$ is the centroid of section $A_1$, $\Omega$, $h$ and $e$ represent rotor speed, crack depth, and eccentricity. $\alpha$ and $\alpha_1$ is the angle between the crack edge and $O$. From Figure 2, the moment of inertia of the non-penetrating crack across the $x$ and $y$-axis can be obtained as follows.
\[ A_1 = R^2(\pi - \cos^{-1}(1 - \lambda) + (1 - \lambda)\gamma) - \pi r^2, \quad (1) \]

\[ I_x^C = \frac{\pi R^2}{8} - \frac{R^4}{4}((1 - \lambda)(2\lambda^2 - 4\lambda + 1)\gamma + \sin^{-1}(1 - \lambda)), \quad (2) \]

\[ I_y^C = \frac{R^4}{12}((1 - \lambda)(2\lambda^2 - 4\lambda + 3)\gamma + 3\sin^{-1}(1 - \lambda)), \quad (3) \]

\[ e = \frac{2R^3}{3A_1}y^3, \quad (4) \]

where \( \lambda = h/R \) represent the dimensionless crack depth and \( \gamma = \sqrt{\lambda(2 - \lambda)} \). \( I_x^C \) and \( I_y^C \) is the moment of inertia of the cracked segment \( A_C \) across the \( x \) and \( y \)-axis.

When the crack is deep, the above equations can be rewritten as

\[ A_1 = R^2(\pi - \cos^{-1}(1 - \lambda) + (1 - \lambda)\gamma) -(\pi r^2 - \alpha_1 r^2/2 + (R - h)\sqrt{r^2 - (R - h)^2}), \quad (5) \]

\[ I_x^C = R^2(\pi - \cos^{-1}(1 - \lambda)(2\lambda^2 - 4\lambda + 1)\gamma + \sin^{-1}(1 - \lambda)) - \int_{-r}^{h-R} 2x^2 \sqrt{r^2 - x^2} \, dx, \quad (6) \]

\[ I_y^C = \frac{R^4}{12}((1 - \lambda)(2\lambda^2 - 4\lambda - 3)\gamma + 3\sin^{-1}\gamma) - \int_{-\sqrt{r^2 - (R - h)^2}}^{\sqrt{r^2 - (R - h)^2}} 2x^2 (\sqrt{r^2 - x^2} + h - R) \, dx, \quad (7) \]

\[ e = \frac{2R^3}{3A_1}y^3 - \frac{1}{A_1} \int_h^r 2x^2 \sqrt{r^2 - x^2} \, dx, \quad (8) \]

Supposing \( R = 1 \) and \( r = 0.5 \), Figure 3 shows the curve of \( I_x^C \) and \( I_y^C \) when \( \lambda \) changes from 0 to 1. It can be seen that both \( I_x^C \) and \( I_y^C \) increase with the increase of \( \lambda \), but the change trends are different. Furthermore, note that \( I_x^C \) and \( I_y^C \) are the same value at the dimensionless depth of the crack \( \lambda = 1 \). This result can be explained by the fact that the cross section of the crack element is a semicircle in this case which has an equal moment of inertia across the \( x \) and \( y \)-axis. Similar results are also reported in Refs. [3] and [7].

The moment of inertia of the uncracked intact shaft section relative to the \( x \) and \( y \) axes is denoted as \( I \), then according to the principle of superposition of the moment of inertia, the inertia moment of uncracked segment \( A_1 \) can be given as

\[ I_1 = I - I_x^C, \quad (9) \]

\[ I_2 = I - I_y^C, \quad (10) \]

where \( I = \pi(R^4 - r^4)/4 \), subsequently, the area moment of inertia of cross-section \( A_1 \) relative to the fixed coordinate system \( O-xy \) need to convert to the centroid coordinate system \( C-xy \), which can be denoted as \( T_1 \) and \( T_2 \) respectively and obtained by the parallel axis shift theorem as follows

\[ T_1 = I_1 - A_1 \varepsilon_1^2, \quad (11) \]

\[ T_2 = I_2. \quad (12) \]

During the rotation of the rotor, the crack will periodically open and close due to the influence of the rotor’s weight. It is necessary to introduce a breathing function to approximate this process. In this paper, the cosine breathing function proposed by Mayes [1] is adopted, which is given as

\[ f(t) = \frac{1}{2}(1 + \cos(\Omega t)), \quad (13) \]

where \( \Omega \) is the rotational speed. Then the time-varying moment of inertia of the crack element can be calculated as
\[ I_\tau(t) = I - (I - \hat{T}_1)f(t) = I - (I^C + A_1 e^2)f(t), \quad (14) \]

\[ I_\tau(t) = I - (I - \hat{T}_2)f(t) = I - I^C f(t). \quad (15) \]

When a crack appears at element \( j \), the time-varying stiffness matrix of the crack element is calculated as follows

\[
K_{ce}^j = \frac{E}{l^3} \begin{bmatrix}
12l\tau_1 \frac{1}{1+\psi_1} & 0 & 0 & 6l\tau_1 \frac{1}{1+\psi_1} \\
0 & 12l\tau_1 \frac{1}{1+\psi_1} & 6l\tau_1 \frac{1}{1+\psi_1} & 0 \\
6l\tau_1 \frac{1}{1+\psi_1} & 0 & 0 & l^2 I_5 \frac{1}{1+\psi_1} \\
-12l\tau_1 \frac{1}{1+\psi_1} & 0 & -6l\tau_1 \frac{1}{1+\psi_1} & 0 \\
6l\tau_1 \frac{1}{1+\psi_1} & 0 & 0 & l^2 I_5 \frac{1}{1+\psi_1} \\
-12l\tau_1 \frac{1}{1+\psi_1} & 0 & -6l\tau_1 \frac{1}{1+\psi_1} & 0 \\
6l\tau_1 \frac{1}{1+\psi_1} & 0 & 0 & l^2 I_5 \frac{1}{1+\psi_1} \\
0 & 0 & 0 & l^2 I_5 \frac{1}{1+\psi_1}
\end{bmatrix}, \quad (16)
\]

where \( \psi_1 = \frac{12l\tau_1}{\mu A_e G_l^2} \) and \( \psi_2 = \frac{12l\tau_1}{\mu A_e G_l^2} \). \( \mu \) is the Poisson ratio of the material, and \( l \) is the element length.

For simplicity, Eq. (16) can also be rewritten in the following form

\[
K_{ce}^j = K^I - f(t)K_{c}^j = (K^I - \frac{1}{2} K^I_e) - \frac{1}{2} \cos(\Omega t)K_{c}^j, \quad (17)
\]

where \( K^I \) is the stiffness matrix of the intact shaft element, and \( K_{c}^j \) is the reduced stiffness matrix representing the effect of crack.

### 2.3 The Motion Equation of the Cracked Rotor System

According to the rotor dynamics finite element method, Timoshenko beam theory, and Lagrange equation, the motion equation of a dual-disk hollow shaft rotor with a breathing crack can be given as follows

\[
M\ddot{q}(t) + \hat{C}\dot{q}(t) + (K - \frac{1}{2} K_c \cos \Omega t)q(t) = F_1 \cos \Omega t + F_2 \sin \Omega t + F_g, \quad (18)
\]

and others are 0. \( q(t) \) is \( 4(N + 1) \) the displacement vector of nodes. Each node has 2 lateral displacements and 2 bending angles. \( F_1 \) and \( F_2 \) are the components of the unbalanced excitation in the \( x \) and \( y \) direction. \( F_g \) represents the equivalent gravity force on the node. The specific forms of each matrix and vector in Eq. (18) are shown in the Appendix.

### 3 Dynamic Response Analysis

#### 3.1 Equation Solving

This paper uses the Harmonic Balance Method (HBM) to solve Eq. (18). HBM supposes that the stable solution of the equation has the finite Fourier series solution

\[
q(t) = A_0 + \sum_{j=1}^{p} (A_j \cos(j \Omega t) + B_j \sin(j \Omega t)), \quad (19)
\]

where \( p \) is the number of harmonics used in HBM, it can be determined according to the accuracy requirement. This paper takes \( p = 4 \) and inserts Eq. (19) in Eq. (18) yields

\[
\begin{bmatrix}
\hat{K} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & P_{11} & P_{21} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & P_{12} & P_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & P_{23} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_{32} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & P_{33} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & P_{31} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{34}
\end{bmatrix}
\begin{bmatrix}
A_1 \\
B_1 \\
A_2 \\
B_2 \\
A_3 \\
B_3 \\
A_4 \\
B_4
\end{bmatrix}
= \begin{bmatrix}
A_1 \\
B_1 \\
A_2 \\
B_2 \\
A_3 \\
B_3 \\
A_4 \\
B_4
\end{bmatrix}, \quad (20)
\]
where \( \hat{K} = \tilde{K} - \Omega^2 M - 1/8K_c\tilde{K}^{-1}K_c, \)
\[ P_{1i} = \Omega C, \quad P_{2i} = \frac{1}{4}K_c, \quad P_{3i} = \tilde{K} - (i \Omega)^2 M, \]
\[ \tilde{F}_i = F_i - 1/2K_c\tilde{K}^{-1}F_g, \quad i = 1, 2, 3, 4. \] After solving the linear equations of Eq. (20) to obtain the coefficients of each harmonic component, the direct component \( A_\theta \) can be calculated as follow

\[ A_\theta = \tilde{K}^{-1}(F_g - \frac{1}{4}K_cA_1). \] (21)

The dynamic response of the cracked rotor can be obtained by Eqs. (19)–(21). When there is no crack fault in the dual-disk hollow shaft rotor system, the first 4 order critical speeds are \( \omega_1 = 825.1 \) rad/s, \( \omega_2 = 1359.2 \) rad/s, \( \omega_3 = 1415.6 \) rad/s, and \( \omega_4 = 2183.0 \) rad/s according to the physical parameters in Table 1.

3.2 Effects of Dimensionless Crack Depth

The position of the crack \( c_p \) is fixed at element 10, and Figure 4 shows the 3D waterfall diagram of rotor speed-dimensionless crack depth-amplitude at node 10 under the low speed (near the first critical speed \( \omega_1 \)). Node amplitude is defined as

\[ a_i = \sqrt{u_i^2 + v_i^2}, \] (22)

where \( a_i \) is the amplitude of node \( i \), \( u_i \) is the horizontal displacement of node \( i \), and \( v_i \) is the vertical displacement of node \( i \). Analysis from Figure 4 shows that the rotor system has obvious resonance peaks near the first critical speed. Besides, compared to the rotor system without the crack, the cracked rotor system appears super-harmonic resonances (or called subcritical resonances) near 1/2, 1/3, or even 1/4 first-order critical speed due to the crack’s occurrence. When the crack depth is relatively shallow, the rotor system first appears to have apparent super-harmonic resonance peaks at 1/2 \( \omega_1 \) and 1/3 \( \omega_1 \). As the crack depth gradually increases, a super-harmonic resonance peak appears at 1/4 \( \omega_1 \). But compared to the peaks at 1/2 \( \omega_1 \) and 1/3 \( \omega_1 \), the resonance peak at 1/4 \( \omega_1 \) is weak. Furthermore, another pattern we can observe in Figure 4 is that with the dimensionless crack depth improves, the super-harmonic resonance peaks near 1/n \( \omega_1 \) (\( n = 2, 3, 4 \)) tend to be obvious, indicating that the peak value of the super-harmonic resonances is closely related to \( \lambda \). Moreover, Figure 5 shows the amplitude-frequency response curves at node 11 with \( \lambda = 0, 0.3, 0.6, 0.9 \), and \( c_p = 10 \). It can be observed that the crack slightly reduces the first-order critical speed of the rotor system, and with the increase of crack depth, the super-harmonic resonance peaks become more evident.

Overall, the crack can lead to super-harmonic resonance phenomena near 1/n \( \omega_1 \) (\( n = 2, 3, 4 \)). The first critical speed and the peak value of super-harmonic resonance (or called subcritical speed amplitude) are closely related to the dimensionless crack depth. These dynamic characteristics can be used to estimate crack depth.
3.3 Influence of Crack Position

The dimensionless crack depth $\lambda$ is fixed at 0.5, and Figure 6 shows the 3D waterfall diagram of rotor speed-crack position-amplitude at node 10 under the low speed (near the first critical speed $\omega_1$) at the different crack positions. From Figure 6, it can be seen that when the crack is located at both ends of the rotor system, the effect on the dynamic response of the rotor system is weak, and the peak value of super-harmonic resonance in amplitude-frequency response is unobvious. Conversely, as the crack gradually approaches the middle of the rotor system, the subcritical speed amplitude increases. When the crack is located in the middle of the rotor system, the super-harmonic resonance is most apparent. The crack has the most significant impact on the system's dynamic response. These analysis results indicate that the subcritical speed amplitude near $1/n$ ($n = 2, 3, 4$) first critical speed of the rotor system is related to the crack position. The crack position also has a specific influence on the critical speed of the rotor system. Figure 7 shows the amplitude-frequency response curves at node 10 near the main

![Figure 7](image)

**Figure 7** Amplitude-frequency response curves for different crack positions at node 10 ($\lambda = 0.5$)

In short, the crack position affects the dynamic response of the rotor system. As the crack approaches the middle of the rotor system, the shift in the critical and subcritical speeds increases. Besides, the more obvious main resonance peak and subcritical speed amplitude in the amplitude-frequency response can also be observed. Similar conclusions have been confirmed in Ref. [3]. These dynamic characteristics can be tracked for the rotor system to identify the crack position.

3.4 Numerical Verification

In order to verify the accuracy of the HBM adopted in this paper, Section 3.4 uses the Runge-Kutta method (R-K) to solve Eq. (18) and compares its results with the HBM’s solutions. Figure 8 shows the time history, whirl orbits, and frequency spectrum solved by HBM and R-K for $\lambda = 0.3$, $c_p = 10$, and $\Omega = 500$ rad/s. It can be seen that the calculation results match well. The rotor's orbit is an ellipse, and there are no multiple frequency components in the frequency spectrum, which indicates that there is no super-harmonic resonance phenomenon in the rotor system.

![Figure 8](image)

**Figure 8** Comparison of time history, whirling orbits, and frequency spectrum for $\lambda = 0.3$, $c_p = 10$, and $\Omega = 500$ rad/s at node 10: (a) Time history, (b) Whirl orbits, (c) Frequency spectrum

Moreover, Figure 9 compares the time history, whirl orbits, and frequency spectrum solved by HBM and R-K for $\lambda = 0.3$, $c_p = 10$, and $\Omega = 410$ rad/s $\approx 1/2\omega_1$. The super-harmonic resonance occurs at this operating speed. It can be seen from Figure 9 that in the case of super-harmonic resonance, the calculation results of the HBM and R-K are still close, further demonstrating resonance area when the crack is located at elements 1, 3, 5, and 10, where the corresponding first critical speed is 824.3 rad/s, 822.1 rad/s, 819 rad/s, and 814.6 rad/s separately. As the position of the crack moves to the middle of the rotor system, the rotor system's critical speed decreases. When the crack is located in the middle of the rotor system, the critical speed of the rotor system is the lowest.
the accuracy of the HBM. The whirling orbits of the rotor system present shape with two closed loops. At the same time, the \(2\times\) frequency component can be observed in the frequency spectrum.

4 The Crack Fault Diagnosis Method for the Hollow Shaft Rotor System based on the Neural Network

4.1 Radial Basis Function Neural Network

The Radial Basis Function (RBF) neural network is a feedforward, fully connected neural network. Compared with the traditional Backpropagation (BP) neural network, the RBF network generally contains only one hidden layer. The radial basis function is utilized as the activation function in the hidden layer. Note that the RBF neural network adopts a direct connection between the input layer and hidden layer instead of a weight connection, as shown in Figure 10.

The output of the RBF network can be defined as

\[
y_i = \sum_{j=1}^{q} W_{ij} \rho(x, \mathbf{c}_j) + b_i,
\]

(23)

where \(x\) is the input vector of the RBF network, and \(\rho(x, \mathbf{c}_j)\) is the activation function of the \(j\)th hidden layer neuron. \(\mathbf{c}_j\) is the center of radial basis function in the \(j\)th hidden layer neuron. \(W_{ij}\) represents the weight between the \(j\)th hidden layer neuron and the \(i\)th output layer neuron, and \(b_i\) is the bias parameter of the \(i\)th output layer neuron. \(q\) is the number of hidden layer neurons, and \(y_i\) is the output value of the \(i\)th output layer neuron. There are various forms of the radial basis functions in the RBF network.
network, and the most commonly used is the Gaussian function, which is given as
\[
\rho(x, c) = \exp\left(-\frac{||x - c||_2^2}{2\sigma^2}\right),
\]
where \(||x - c||_2\) represents the Euclidean norm, which is used to characterize the distance between the input vector and the center of the radial basis function. \(\sigma\) is the spread constant of the RBF, affecting the size of the radial basis function's activation region. When the input dimension is 2, take the spread constant as 1, and the center is \((0,0)\), then the Gaussian function is shown in Figure 11. It can be seen from Figure 11 that the Gaussian function has a relatively sensitive response near the center, while the function value in most areas far away from the center is almost 0 (which means inactive). The property of local activation allows the RBF network to set up more neurons without overfitting.

4.2 Calculate the Dimensionless Crack Depth with the RBF Neural Network

The dimensionless depth and position of the crack are assigned by random numbers. Then the dynamic response of the cracked rotor system can be obtained by adopting the methodology in Section 3. The dynamic characteristics related to the crack parameters (including the first critical speed, first subcritical speed, first critical speed amplitude, and the super-harmonic resonance peak at the first subcritical speed) are taken as the input of the RBF network. The output of the RBF network is the predicted dimensionless crack depth. Take 70% data as the training set, 15% as the validation set, and 15% as the test set. Some provided data are shown in Tables 2 and 3. Note that the rotor's amplitude is generally minimal (on the order of \(10^{-6}\)), which is detrimental to the training and computing of the network. Therefore, this paper takes the logarithm (denoted as log in Tables 2 and 3) of the first critical speed amplitude and the first subcritical speed amplitude.

| Table 2 | Part of the data used to train the RBF network |
|---------|-----------------------------------------------|
| \(\lambda\) | \(c_p\) | First critical speed (rad/s) | First critical speed amplitude (log) | First subcritical speed (rad/s) | First subcritical speed amplitude (log) |
| 0.4050 | 4 | 822 | -5.4783 | 411 | -8.0111 |
| 0.3086 | 10 | 820 | -5.4808 | 410 | -8.0141 |
| 0.3149 | 19 | 824 | -5.5058 | 412 | -8.1428 |
| 0.3384 | 20 | 825 | -5.5032 | 412 | -8.0971 |
| 0.3507 | 13 | 820 | -5.5021 | 410 | -7.9737 |
| 0.3919 | 11 | 817 | -5.4271 | 409 | -7.9061 |
| 0.4295 | 1 | 824 | -5.4968 | 412 | -7.9544 |
| 0.4328 | 3 | 823 | -5.4879 | 411 | -7.9721 |
| 0.4749 | 6 | 818 | -5.3933 | 409 | -7.8852 |
| 0.4947 | 17 | 821 | -5.4587 | 411 | -7.9143 |

| Table 3 | Part of the data used to validate and test the RBF network |
|---------|-----------------------------------------------|
| \(\Lambda\) | \(c_p\) | First critical speed (rad/s) | First critical speed amplitude (log) | First subcritical speed (rad/s) | First subcritical speed amplitude (log) |
| 0.5740 | 4 | 820 | -5.4097 | 410 | -7.8359 |
| 0.3204 | 6 | 821 | -5.4837 | 411 | -8.0734 |
| 0.3889 | 15 | 820 | -5.4652 | 410 | -7.9463 |
| 0.5191 | 10 | 813 | -5.0707 | 406 | -7.7704 |
| 0.32279 | 5 | 822 | -5.4847 | 411 | -8.0861 |
| 0.3319 | 8 | 820 | -5.4774 | 410 | -8.0090 |
| 0.3450 | 14 | 820 | -5.4742 | 410 | -7.9882 |
| 0.3488 | 3 | 823 | -5.4874 | 412 | -8.0571 |
| 0.3512 | 5 | 822 | -5.4925 | 411 | -8.0536 |
| 0.3536 | 9 | 819 | -5.4659 | 409 | -7.9957 |
The mean square error (MSE) function is used as the loss function to train the RBF neural network, which is described as

$$\text{MSE} = \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{o} (\hat{y}_{kj} - y_{kj})^2,$$

where $o$ is the number of output layer neurons, $m$ is the sample size, $\hat{y}_{kj}$ is the output of the $j$th neuron for the $k$th sample input, and $y_{kj}$ represents the actual $j$th output of the $k$th sample input. The training parameters are summarized in Table 4. The spread constant of the RBF network is 0.8. The steps of validation failure are 6, which means that the training process is stopped early when the loss function on the validation set is no longer descending for 6 consecutive epochs. The selection algorithm for the radial basis function's center adopts the k-means clustering algorithm. The training algorithm is the Ordinary Least Square (OLS) with a closed-form solution, which is very efficient. The RBF network gradually reduces the loss function by adding the number of hidden layer

### Table 4 Training parameters of the RBF network

| Loss function | Max epochs | Spread constant | Steps of validation failure | Clustering algorithm | Training algorithm |
|---------------|------------|-----------------|----------------------------|----------------------|--------------------|
| MSE           | 100        | 0.8             | 6                          | k-means              | OLS                |

![Figure 12](image1.png) Training and test results of the RBF network when the crack is located in the middle of the rotor system: (a) Loss function curve, (b) Regression output

![Figure 13](image2.png) Training and test results of the RBF network when the crack is located at the end of the rotor system: (a) Loss function curve, (b) Regression output
neurons one by one from 0, so there is no need to set the initial number of hidden layer neurons in advance.

Figures 12, 13, 14 illustrate the training and test results of the RBF network under three conditions: (1) the crack is in the middle of the rotor system \((c_p = 10)\); (2) the crack is at the end of the rotor system \((c_p = 1)\); (3) the crack position is uncertain.

When the crack is located in the middle of the rotor system, it can be seen from Figure 12a that the loss function of the training set, validation set, and test set all show an oscillating downward trend with the increase in the number of training epochs. The loss function value of the validation set is minimized at the 48th step, and the training process terminates in advance at the 54th epoch. At the end of the training, the loss function value of the training set is \(1.21 \times 10^{-5}\), the validation set is \(3.16 \times 10^{-5}\), and the test set is \(2.24 \times 10^{-5}\). All three values are reduced to the order of \(10^{-5}\) and with little difference, indicating that the RBF network has great generalization ability. Besides, it can be seen from Figure 12b that the scatter points of the neural network output almost all fall on the ideal regression curve. The correlation coefficient (denoted as CC) between the output results of the neural network and the ground truth is 0.9985, and the maximum percentage error is 1.8%. These results indicate that the RBF neural network can accurately estimate the crack depth based on the provided dynamic characteristics. According to the analysis results in Section 3, when the crack (with the same dimensionless depth) is located in the middle of the rotor system, the impact on the system's dynamic response is the greatest. The shift in the critical and subcritical speeds is the most obvious, and the subcritical amplitude in the amplitude-frequency response is the most significant. More specifically, the dynamic characteristics related to the crack are the most obvious, which is a benefit for the fault diagnosis. Hence, when the crack is located in the middle of the rotor system, the RBF network can get a satisfactory result for calculating the crack depth.

When the crack is located at the end of the rotor system, related results are summarized in Figure 13. It can be seen from the data in Figure 13a that the loss function of the training set, test set, and validation set decreases with the increase of training epochs. The RBF neural network achieves the best performance at the 35th epoch, and the training process terminates prematurely at the 41st epoch. At the end of the training, the loss function value of the training set is \(5.42 \times 10^{-4}\), the verification set is \(1.08 \times 10^{-3}\), and the test set is \(8.40 \times 10^{-4}\). Compared with the results in Figure 12a, the loss function value is larger. In addition, as shown in Figure 13b, when the crack is relatively shallow, many output results of the RBF network deviate from the ideal regression curve compared with Figure 12b. The maximum percentage error is 9.3%, indicating that the network's performance declined slightly. These results are likely related to the bearing and support with great rigidity. As mentioned in Section 3, when the crack is located at the end of the rotor system, its effects on the system's dynamic response are relatively small. Therefore, the dynamic characteristics related to the crack parameters may be insignificant, which results in a decrease in the RBF network's fault diagnosis performance. However, on the other hand, cracks in this position are also less likely to cause serious consequences.

Note that Figures 12 and 13 are the results of estimating the dimensionless crack depth when the crack position is fixed. However, in actual engineering problems, the crack may appear in various positions in the rotor system. Figure 14 provides the training and test results of the RBF network when \(c_p\) is uncertain. From
Figure 14, it can be observed that during the training process, the loss function value of the training set oscillates and decays in a small range with the number of iteration epochs increasing. The loss function value of the validation set is minimized at the 48th epoch, and the training process terminates prematurely at the 56th epoch. When the training is terminated, the loss function value of the training set is $2.19 \times 10^{-4}$, and the test set is $2.23 \times 10^{-3}$. Both are relatively small and not significantly different, indicating no severe overfitting. The performance of the RBF network can be seen more intuitively from the regression output results in Figure 14(b). Most regression output results are around the ideal regression curve without significant deviation. The correlation coefficient $CC$ between the RBF network’s outputs and the sample results is 0.9916, indicating an excellent linear correlation. Besides, the maximum percentage error is 7.6%, which demonstrates an accurate estimation result. The above analysis results show that even in the case of uncertain crack locations, the RBF network can use dynamic characteristics to calculate the crack depth and achieve sufficient precision.

### 4.3 Pattern Recognition Neural Network

The pattern recognition neural network (PRNN) is a kind of neural network adopted to solve classification problems. Its architecture is set out in Figure 15.

The input first undergoes an affine transformation to the hidden layer. And then, to achieve the nonlinear mapping, an activation function should be introduced. The most common activation function used in the PRNN’s hidden layer is the Tan-sigmoid function, which is described as

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$ 

(26)

Then, another affine transformation is defined in the Softmax layer, which is utilized to adjust the dimension of the hidden layer’s output. Besides, to obtain the discrete probability density, there is a so-called Softmax function in the Softmax layer, which is given as

$$y_i = \text{softmax}(x_i) = \frac{\exp(x_i)}{\sum_{i=1}^{n} \exp(x_i)},$$

(27)

where $x = [x_1, x_2, ..., x_n]^T$ is the input vector. Eq. (27) has the following properties: (1) each element in the obtained output vector $y = [y_1, y_2, ..., y_n]^T$ is non-negative; (2) the sum of all elements in $y$ is 1. Therefore, $y$ can represent the discrete probability density, and the position of the maximum value in $y$ can be utilized as the classification result. For instance, if the PRNN’s output result is $[0.1, 0.6, 0.3]^T$, the corresponding input should be classified into the second category.

### 4.4 Identification of the Crack Position with PRNN

Similar to Section 4.2, the dynamic characteristics related to the crack position (the first critical speed, first critical speed amplitude, first subcritical speed, and first subcritical speed amplitude) are taken as the PRNN’s input. The output of the PRNN is the discrete probability density of the crack position (20 elements). The training, validation, and test sets are the same as in Section 4.2.

Calculating the dimensionless crack depth can be considered a regression problem, so MSE (shown in Eq. (25)) is adopted as the loss function to train the RBF network. Unlike this, identifying the crack position is a pattern recognition problem (known as a classification problem), so an additional loss function should be introduced to train the PRNN. Cross entropy is a popular candidate, which meaning is to characterize the similarity of the probability distribution of two events. Its formula can be calculated as

$$\text{Crossentropy}(P, Q) = -\sum_{i=1}^{n} P(x_i)\log(Q(x_i)),$$

(28)

where $P(x_i)$ is the actual probability distribution of event $x_i$, and $Q(x_i)$ is the discrete probability density of event $x_i$ obtained by the PRNN. In particular, in the classification problem, $P(x_i)$ is a sparse vector, where the position of the actual category is 1 and all the other elements are 0 (known as the one-hot encoding).

| Table 5 Training parameters of the PRNN |
|----------------------------------------|
| Loss function | Number of hidden neurons | Max epochs | Steps of validation failure | Initial learning rate | Training algorithm |
| Cross entropy | 15 | 100 | 6 | 0.001 | L-M |
Some other training parameters are summarized in Table 5. The number of hidden neurons is 15. The steps of validation failure are 6, which means that the training process is stopped early when the loss function of the validation set is no longer descending for 6 consecutive times. The initial learning rate is 0.001, and the training algorithm adopts Levenberg-Marquardt backpropagation (L-M), which takes up more storage space but is faster.

Based on these parameters, the loss function curves of the training and test sets are shown in Figure 16. It can be observed that the loss function of the training set oscillates and decreases with the training epochs increasing. Similar patterns are also presented in the validation and test sets. The loss function value of the validation set is minimized at the 66th epoch, and the training process terminates in advance at the 72nd epoch. When the training process is finished, the loss function values on the training, validation, and test sets are 0.0147, 0.0390, and 0.0415, respectively. The loss function values on the training and test sets are close, indicating that the trained PRNN has good generalization ability. The recognition accuracy in the test and validation sets is 78.6%, which achieves the basic requirements for identifying the crack position.

Furthermore, in pattern recognition problems, in addition to the obtained recognition accuracy, the causes of classification errors also need attention. Figure 17 shows

![Figure 16](image1.png)

**Figure 16** The loss function curve of the PRNN in the training process

![Figure 17](image2.png)

**Figure 17** The confusion matrix between the PRNN’s output results and the actual crack position
the confusion matrix between the PRNN’s output results and the actual crack positions. Note that only the part containing the misclassified results is provided for simplicity. As can be seen from Figure 17, most of the non-zero elements of the confusion matrix are concentrated on the diagonal (dashed area), which means that the PRNN accurately identifies the crack position in most cases. Moreover, careful analysis of the misclassified elements in the confusion matrix can be found that most of the incorrectly classified PRNN output values are only one unit away from the actual crack position. For example, as shown in row 14 in Figure 17, there are 4 cracks in element 15. However, they are misclassified in element 14 by the trained PRNN. A possible explanation for this might be that the dynamic characteristics of the PRNN’s input are related to both the crack depth and position. This coupling effect confuses the results of crack location. Nevertheless, when the shaft is divided into 20 elements, the element length is very small. Therefore, the deviation of 1 element length in determining the crack position may not cause a severe problem. Based on this consideration, the condition for evaluating the recognition accuracy can be extended to the recognition deviation within 1 element length. For instance, if the predicted crack position of the PRNN is 12 or 14, while the actual crack position is 13, it can also be considered that the PRNN’s output is approximately accurate. Accordingly, it can be seen from Figure 17 that almost all the elements in the confusion matrix fall in the approximately accurate area (red line region). Only 1 instance of the 56 samples has a recognition deviation of 2 element lengths (the actual crack location is 15, but the PRNN’s output is 13). The approximate recognition accuracy of the crack position is 98.2%, which is a brilliant result. The above analysis shows that PRNN can utilize the dynamic characteristics of the cracked rotor system to determine the crack position and achieve a high approximate identification accuracy.

4.5 Comparison with Other Crack Diagnosis Methods

To further illustrate the effectiveness and superiority of the proposed method, several alternative machine learning-based methods for crack diagnosis are implemented. As mentioned previously, calculating the dimensionless crack depth can be considered a regression problem. Correspondingly, the following methods are utilized to conduct a comparison test: (1) multiple linear regression (denoted as MLR); (2) support vector regression (denoted as SVR); (3) support vector regression with kernel function (denoted as KSVR). Similarly, identification of crack position is a pattern recognition problem, so the following methods are considered: (1) K-nearest neighbor algorithm (denoted as KNN); (2) support vector machine (denoted as SVM); (3) support vector machine with kernel function (denoted as KSVM).

The crack diagnosis results obtained from different methods on the test set are summarized in Tables 6 and 7. Note that for a fair comparison, all mentioned methods share the same training set, and their hyperparameters are optimized based on the validation set. As can be seen from Table 6, MLR gives the worst results for calculating the dimensionless crack depth under the same conditions, which indicates that the relationship between the dimensionless crack depth and the system’s dynamic response characteristics cannot be represented by a simple linear equation. More accurate diagnosis results can be obtained by adopting the support vector regression, in which introducing the kernel function can improve the nonlinear mapping capability of SVR, thus enhancing its diagnosis performance. However, the maximum percentage error of SVR and KSVR on the test set is still greater than 10%, which is higher than the proposed method (RBF).

In a similar manner, it can be seen from the data in Table 7 that the distance-based KNN method has a poor property for crack position identification. Only 44.6% accuracy is achieved. In contrast, SVM demonstrated better diagnosis performance on the test set, and the nonlinear kernel function is beneficial. However, the crack position identification accuracy of the KNN, SVM and KSVM is not higher than 50%, while our proposed method (PRNN) reaches 78.6%.

Overall, compared with other crack diagnosis methods, RBF neural network and PRNN model realize the best diagnosis results in dimensionless crack depth calculating

### Table 6 Dimensionless crack depth calculating results obtained by different crack diagnosis methods on the test set

| Method      | MSE on the test set | Maximum percentage error (%) |
|-------------|---------------------|------------------------------|
| MLR         | $9.75 \times 10^{-4}$ | 16.2                         |
| SVR         | $5.56 \times 10^{-4}$ | 13.2                         |
| KSVR        | $3.28 \times 10^{-4}$ | 10.6                         |
| RBF (proposed) | $2.23 \times 10^{-4}$ | 7.6                          |

### Table 7 Crack position identification results obtained by different crack diagnosis methods on the test set

| Method          | Accuracy (%) |
|-----------------|--------------|
| KNN             | 44.6         |
| SVM             | 48.2         |
| KSVM            | 50.0         |
| PRNN (proposed) | 78.6         |
and crack position identification, respectively. The above comparison further illustrates the effectiveness of our proposed method.

5 Conclusions
In this paper, a dual-disk hollow shaft rotor system model with a breathing crack is established, where two different crack forms (non-penetrating and passthrough crack) are considered. Then, the dynamic response of the cracked rotor system is obtained through the HBM, and the dynamic characteristics related to the crack parameters are summarized by analyzing the amplitude-frequency curve and waterfall plot. Based on this, a novel crack fault diagnosis and location method based on the RBF network and PRNN is proposed. The main conclusions are as follows.

(1) Due to introducing the time-varying stiffness, the crack can cause the super-harmonic resonance phenomenon in the rotor system near 1/n (n = 2, 3, 4) first critical speed. Besides, the crack can reduce the stiffness of the system, resulting in a decrease in the system's critical speed.

(2) The analysis results of the cracked rotor system's amplitude-frequency curves and waterfall plots with different crack parameters reveal the dynamic characteristics related to the crack depth and position. The first critical speed, first subcritical speed, first critical speed amplitude, and first subcritical speed amplitude can be utilized to detect the crack.

(3) Based on the RBF network and PRNN, the quantitative crack fault diagnosis method is proposed. In the case where both the crack depth and position are uncertain, adopting the analyzed dynamic characteristics as input, the maximum percentage error between the trained RBF network’s output results and the ground truth is 7.56%. Besides, the approximate recognition accuracy of the crack position obtained by the PRNN can reach 98.2%. The requirements of crack fault diagnosis are satisfied preliminarily.

(4) Several alternative machine learning-based crack fault diagnosis methods are considered in the comparison experiment. The results show that the approach developed in this paper achieves the optimal fault diagnosis performance, further demonstrating its effectiveness.

In future work, further research should focus on the following aspects. Firstly, the proposed method relies on the analyzed dynamic characteristics, such as the super-harmonic resonance peak, which maintain its theoretical basis and interpretability. However, at the same time, noise and nonlinearities can introduce disturbances to these dynamic characteristics and thus confuse the diagnosis results. The robustness of the proposed method regarding interference needs to be further improved. Combining the signal processing methods with the proposed approach may be a good solution. Secondly, limited by the finite element model, the proposed method can only give the approximate crack position. Modeling technology that can accurately describe the crack’s location should be further studied. Finally, experimental verification is necessary. In this paper, the validation of the proposed method is based on simulation data. In the future, we will establish the cracked rotor system experiment bench and validate our proposed method with experimental data.

Appendix
Timoshenko beam finite element matrices in Eq. (18)

\[
K_s^c = \frac{EI}{(1 + \varphi_s^2)l^3} \begin{bmatrix}
K_{11}^c & 0 & K_{12}^c & K_{13}^c \\
0 & -K_{12}^c & K_{13}^c & 0 \\
-K_{12}^c & 0 & 0 & K_{23}^c \\
-K_{11}^c & 0 & 0 & K_{22}^c \\
0 & -K_{23}^c & K_{24}^c & 0 & 0 & K_{34}^c \\
0 & K_{22}^c & K_{24}^c & 0 & 0 & K_{33}^c \\
-K_{22}^c & 0 & 0 & K_{34}^c & K_{24}^c & 0 & 0 & K_{33}^c \\
\end{bmatrix}
\]

\(K_{31}^c = 12\)

\(K_{i2}^c = 6l\)

\(K_{i3}^c = (4 + 2\varphi_s + \varphi_s^2)l^2\)

\(K_{i4}^c = (2 - 2\varphi_s - \varphi_s^2)l^2\)

\(\xi = \frac{z}{l}, \varphi_s = \frac{12EI}{\mu AGl^2}\)
\[ K_s^s = \frac{EI\varphi_s}{(1 + \varphi_s^2)I^3} \begin{bmatrix} K_{s1}^s & 0 & -K_{s2}^s & 0 & -K_{s3}^s \\ -K_{s2}^s & 0 & K_{s3}^s & 0 & K_{s4}^s \\ -K_{s1}^s & -K_{s2}^s & 0 & K_{s4}^s & 0 \\ -K_{s1}^s & 0 & K_{s2}^s & 0 & K_{s3}^s \\ -K_{s1}^s & 0 & K_{s2}^s & 0 & K_{s3}^s \end{bmatrix} \]

\[ K_{s1}^s = 12K_{s2}^s = 6l, K_{s3}^s = (2 - \varphi_s)l^2, K_{s4}^s = (4 + \varphi_s)l^2 \]

\[ J_{s1} = 36 \]

\[ M_c^c = \frac{\rho_s l}{(1 + \varphi_s^2)} \begin{bmatrix} M_{c1}^c & M_{c1}^c & 0 & M_{c1}^c & M_{c1}^c \\ 0 & -M_{c2}^c & 0 & 0 & M_{c2}^c \\ -M_{c3}^c & 0 & M_{c3}^c & 0 & 0 \\ 0 & -M_{c4}^c & M_{c4}^c & 0 & M_{c4}^c \\ 0 & M_{c5}^c & 0 & 0 & M_{c5}^c \end{bmatrix} \]

\[ M_{c1}^c = \frac{13}{35} + \frac{7}{10\varphi_s + \varphi_s^2} \]

\[ M_{c2}^c = (1/105 + 1/60\varphi_s + 1/120\varphi_s^2)l^2 \]

\[ M_{c3}^c = 9/70 + 3/10\varphi_s + 1/6\varphi_s^2 \]

\[ M_{c4}^c = (11/210 + 11/120\varphi_s + 1/24\varphi_s^2)l \]

\[ M_{c5}^c = (13/420 + 3/40\varphi_s + 1/24\varphi_s^2)l \]

\[ M_{c6}^c = (1/140 + 1/60\varphi_s + 1/120\varphi_s^2)l^2 \]

\[ J_{s2} = 3l - 15l\varphi_s \]

\[ J_{s3} = l^2 + 5l^2\varphi_s - 15l^2\varphi_s^2 \]

\[ J_{s4} = 4l^2 + 5l^2\varphi_s + 10l^2\varphi_s^2 \]

\[ M_s^s = \frac{j_d}{(1 + \varphi_s^2)I} \begin{bmatrix} M_{s1}^s & 0 & -M_{s2}^s & 0 & -M_{s3}^s \\ 0 & M_{s4}^s & 0 & 0 & M_{s4}^s \\ -M_{s1}^s & 0 & -M_{s2}^s & 0 & -M_{s2}^s \\ -M_{s3}^s & M_{s4}^s & 0 & 0 & M_{s4}^s \\ 0 & -M_{s3}^s & M_{s4}^s & 0 & -M_{s4}^s \end{bmatrix} \]

\[ M_{s1}^s = 6/5, M_{s2}^s = (2/15 + 1/6\varphi + 1/3\varphi^2)l^2, \]

\[ M_{s3}^s = (-1/30 - 1/6\varphi + 1/6\varphi^2)l^2, M_{s4}^s = (1/10 - 1/2\varphi)l \]

\[ J_s = \frac{j_p}{30(1 + \varphi_s^2)I} \begin{bmatrix} 0 & -J_{s1} & 0 & 0 & 0 \\ J_{s1} & 0 & -J_{s2} & 0 & 0 \\ 0 & J_{s2} & J_{s4} & 0 & 0 \\ -J_{s1} & 0 & J_{s2} & -J_{s4} & 0 \\ 0 & J_{s1} & -J_{s2} & J_{s4} & 0 \end{bmatrix} \]

\[ J_{s1} = 36 \]
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Author contributions
YJ was in charge of the whole analyses and wrote the manuscript; LH assisted with sample analyses and revised the manuscript; ZL assisted with simulation analyses; YC revised the final manuscript. All authors read and approved the final manuscript.

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