Ordering $Q$-indices of graphs: given size and girth

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Abstract: The signless Laplacian matrix in graph spectra theory is a remarkable matrix of graphs, and it is extensively studied by researchers. In 1981, Cvetković pointed 12 directions in further investigations of graph spectra, one of which is “classifying and ordering graphs”. Along with this classic direction, we pay our attention on the order of the largest eigenvalue of the signless Laplacian matrix of graphs, which is usually called the $Q$-index of a graph. Let $\mathcal{G}(m, g)$ (resp. $\mathcal{G}(m, \geq g)$) be the family of connected graphs on $m$ edges with girth $g$ (resp. no less than $g$), where $g \geq 3$. In this paper, we firstly order the first $(\lfloor \frac{g}{2} \rfloor + 2)$ largest $Q$-indices of graphs in $\mathcal{G}(m, g)$, where $m \geq 3g \geq 12$. Secondly, we order the first $(\lfloor \frac{g}{2} \rfloor + 3)$ largest $Q$-indices of graphs in $\mathcal{G}(m, \geq g)$, where $m \geq 3g \geq 12$. As a complement, we give the first five largest $Q$-indices of graphs in $\mathcal{G}(m, 3)$ with $m \geq 9$. Finally, we give the order of the first eleven largest $Q$-indices of all connected graphs with size $m$.

Keywords: Ordering, Size, $Q$-index, Girth

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1 Introduction

All graphs considered here are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$, where $n(G) = |V(G)|$ denote the order and $|E(G)| = m(G)$ the size of $G$. The set of the neighbors of a vertex $v \in V(G)$ is denoted by $N_G(v)$, and the degree of $v$ is denoted by $d_G(v)$ or $d(v)$. Let $\Delta = \Delta(G)$ be the maximum degree of $G$. The girth of a graph $G$, denoted by $g$, is the length of the shortest cycle in $G$. Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal degree matrix of $G$, respectively. The signless Laplacian matrix of $G$ is defined as $Q(G) = D(G) + A(G)$. The largest eigenvalue of $Q(G)$ is called the $Q$-index of $G$, denoted by $q(G)$. Note that $Q(G)$ is a non-negative matrix. From Perron-Frobenius theorem, there exists a non-negative unit
eigenvector $x$ corresponding to $q(G)$. Such eigenvector $x$ is called the Perron vector of $Q(G)$ and the entry of $x$ corresponding to vertex $u$ is denoted by $x_u$. Moreover, if $G$ is connected then the Perron vector $x$ is a positive vector. As usual, let $K_{i,n-1}$, $K_n$, $C_n$ and $P_n$ be respectively the star, complete graph, cycle and the path of order $n$.

In 1981, Cvetković [3] indicated 12 directions in further investigations of graph spectra, one of which is “classifying and ordering graphs”. Hence ordering graphs with various properties by their spectra becomes an attractive topic (see 2[19,21,27,29,30]). The signless Laplacian matrix is a remarkable matrix of graphs, and it is extensively studied by researchers. Cvetković and Simić presented a series of surveys on the signless Laplacian spectral theory [5–7]. Up to now, a simple and generality method on ordering graphs according to their spectra (or $Q$-spectra) has not yet been obtained.

In particular, there exists a significant amount of research on determining the graph with the first largest $Q$-index in a giving graph set. The giving graph set usually contains two aspects: $H$-free graph set, for example, $K_{i,n-1}$-free, $C_4$-free, $C_6$-free, $C_8$-free for $k$ is odd, and minors-free [8,10,11,23,25,32,33]. On the other side, the graph set with giving order and some graph invariant, such as, diameter [14,26], clique number [16], chromatic number [28], graphic degree sequence [35] and so on. One of other graph invariants is the girth, which has a rich research. Given girth and order, the maximum $Q$-index of unicycle, bicycle graph are determined in [22] and [36], one of tricycle, $k$-cyclic graph are determined in [24] and [18], respectively. Also, the graph set with giving size and graph invariant, one can see, diameter [20], clique number (resp. chromatic number) [31], girth (resp. circumference) [9], matching number [34].

All these results as mentioned above determined the graph with the first largest $Q$-index. However, there has little progress in the study of the spectral radius ordering problem, and there are few results related to the second largest, third largest, etc. In [19,21,29,30], the authors compared the $Q$-indices of two graphs by comparing their maximum degrees. In [19], Liu, Liu and Cheng stated that the ordering of trees according to their $Q$-indices can be transferred to the ordering of trees with large maximum degree. However, there is no way to compare the $Q$-indices of two graphs with the same maximum degree. Moreover, there are rarely results and methods for ordering in a giving graph set. In 2006, Guo [13] determined the first $(\lfloor \frac{d}{2} \rfloor + 2)$-th largest $Q$-indices (resp. Laplace spectral radius) of trees of giving order and diameter $d$. Very recently, Jia, Li and Wang [15] ordered the second to the $(\lfloor \frac{d}{2} \rfloor + 1)$-th $Q$-indices of graphs of giving size and diameter.

Inspired in above researches, in this paper we intend to order of $Q$-indices in a giving graph set. Girth is a graph variant that are widely concerned by researchers in graph spectral theory. Chen, Wang and Zhai in 2022 gave the first largest $Q$-index of graph of giving size and girth. Let $\mathbb{G}_m$ be the family of connected graphs with $m$ edges, $\mathbb{G}(m,g)$ and $\mathbb{G}(m,\geq g)$ are respectively the subset of $\mathbb{G}_m$ with girth equals to $g$ and girth no less than $g$, where $g \geq 3$. In this paper, we respectively consider the order of the three families $\mathbb{G}(m,g)$, $\mathbb{G}(m,\geq g)$ and $\mathbb{G}_m$ via their $Q$-indices.

For $g \geq 3$, we always denote by $C_g = 012 \cdots (g-2)(g-1)0$ the cycle of length $g$. For $0 \leq i \leq \lfloor \frac{g}{2} \rfloor$, let $G_i \in \mathbb{G}(m,g)$ be a graph obtained from $C_g$ by attaching $m-g-1$ pendant edges to the vertex 0 and simultaneously adding a pendant edge, say $iw$, at the vertex $i$ (see Fig.1). Let $G_r$ be a graph obtained from $C_g$ by attaching $m-g-2$ pendant edges and a $P_3$, respectively, at the vertex 0, where $P_3 = 0vvv_1$ (see Fig.1). The main results of this paper are presented as follows.
Theorem 1.1. Among all graphs in $\mathbb{G}(m, g)$ with $m \geq 3g \geq 12$, the order of the first $\left(\lfloor \frac{3g}{2} \rfloor + 2\right)$ largest $Q$-indices of graphs is given by:

$$q(G_0) > q(G_1) > q(G_v) > q(G_2) > q(G_3) > \cdots > q(G_{\lfloor \frac{3g}{2}\rfloor}).$$

We use $G_{i,g}$ and $G_{v,g}$ instead of $G_i \left(0 \leq i \leq \lfloor \frac{3g}{2}\rfloor\right)$ and $G_v$, respectively, if we emphasize that its girth equals to $g$. For the order of graphs in $\mathbb{G}(m, g)$, we get the following result.

Theorem 1.2. Among all graphs in $\mathbb{G}(m, g)$ with $m \geq 3g \geq 12$, the order of the first $\left(\lfloor \frac{3g}{2} \rfloor + 3\right)$ largest $Q$-indices of graphs is given by:

$$q(G_{0,g}) > q(G_{1,g}) > q(G_{v,g}) > q(G_{2,g}) > q(G_{3,g}) > \cdots > q(G_{\lfloor \frac{3g}{2}\rfloor,g}) > q(G_{0,g+1}).$$

Let $B_1$ be the bicycle graph obtained from two triangles with a common edge by join $m-5$ pendant edges to its one end-vertex (see Fig.5). Let $B_2$ be the bicycle graph obtained from two triangles with a common vertex by join $m-6$ pendant edges to it (see Fig.5). As supplement of Theorem 1.1 we give the order of the first five largest $Q$-indices for $g = 3$, which has some different with the result of Theorem 1.1.

Theorem 1.3. Among all graphs in $\mathbb{G}(m, 3)$ with $m \geq 9$, the order of the first five largest $Q$-indices is given by: $q(G_{0,3}) > q(B_1) > q(B_2) > q(G_{1,3}) > q(G_{v,3})$.

Zhai, Xue and Lou [31] showed that $K_{1,m}$ attains the maximum $Q$-index among all graphs in $\mathbb{G}_m$. At last, we extend the result by ordering the first eleven largest $Q$-indices among all graphs in $\mathbb{G}_m$.

Theorem 1.4. Let $T_1$, $T_2$, $T_3$ and $T_4$ be trees on $m$ edges with $\Delta(T_1) = m-1$ and $\Delta(T_2) = \Delta(T_3) = \Delta(T_4) = m-2$ (see Fig.6). Among all graphs in $\mathbb{G}_m$ and $m \geq 9$, the order of the first eleven largest $Q$-indices is given by: $q(K_{1,m}) > q(G_{0,3}) > q(T_1) > q(B_1) > q(B_2) > q(G_{1,3}) > q(G_{v,3}) > q(T_2) > q(G_{0,4}) > q(T_3) > q(T_4)$.

2 Some lemmas and a new upper bound of $q(G)$

In the section, we give some useful lemmas and then give a new bound of $q(G)$.
Lemma 2.1 (\cite{5}). If $H$ is the subgraph of a connected graph $G$, then $q(H) \leq q(G)$. Particularly, if $H$ is proper then $q(H) < q(G)$.

Lemma 2.2 (\cite{17}). Let $u$, $v$ be two distinct vertices of a connected graph $G$. Suppose $w_1, w_2, \ldots, w_t$ ($t \geq 1$) are some vertices of $N_G(v) \setminus N_G(u)$ and $x$ is the Perron vector of $Q(G)$. Let $G' = G - \{vw_i \mid i = 1, 2, \ldots, t\} + \{uw_i \mid i = 1, 2, \ldots, t\}$. If $x_u \geq x_v$ then $q(G) < q(G')$.

The following lemma gives an interesting transformation that could increase the $Q$-index. As usual, we call it as the ‘quadrangle’ principle, which is a useful tool in our proofs.

Lemma 2.3 (\cite{4}). Let $G'$ be a graph obtained from a connected graph $G$ by a local switching of edges $ab$ and $cd$ to the positions of non-edges $ad$ and $bc$. Let $x$ be the Perron vector of $Q(G)$. If $(x_a - x_c)(x_d - x_b) > 0$ then $q(G') > q(G)$.

The equitable partition is a significant tool for graph spectral theory. The largest eigenvalue of the quotient matrix corresponding to an equitable partition of matrix $M$ is the largest eigenvalue of $M$ (see \cite{1}, Lemma 2.3.1). Let $S_{n,3}$ be a graph on $n$ vertices obtained from $K_{1,n-1}$ by attaching three pendant paths of length 2 at the center vertex of $K_{1,n-1}$, and $H_0$ be an unicycle graph of order $n$ and girth 4 (see Fig. 2). It is routine to verify that $\phi_1(x, n)$ and $\phi_2(x, n)$ in Lemma 2.4 are respectively the characteristic polynomial of the quotient matrix of $S_{n,3}$ and $H_0$, naturally we have the following result.

Lemma 2.4. (i). The $Q$-index of $S_{n,3}$ is the largest root of $\phi_1(x, n) = x^3 - nx^2 + (3n-8)x - n$. (ii). The $Q$-index of $H_0$ is the largest root of

$$\phi_2(x, n) = x^5 - (n+5)x^4 + (7n+1)x^3 - (15n-17)x^2 + (10n-8)x - 2n.$$  

Let $x$ be the Perron vector of $Q(G)$ with respect to $q(G)$ and $x_u$ the coordinate of $x$ corresponds to vertex $u$ ($u \in V(G)$). By the eigenvalue equation $Q(G)x = q(G)x$, we have $q(G)x_u = d(u)x_u + \sum_{v \in N_G(u)} x_v$. Thus we can get a upper bound of $x_u$ for any $u \in V(G)$, which can be used to compare the coordinates of two vertices.

Lemma 2.5. Let $G$ be a connected graph. For any $u \in V(G)$, we have

$$x_u^2 \leq \frac{1}{1 + \left(\frac{q(G)-d(u)}{d(u)}\right)^2}.$$
Proof. From the Cauchy-Schwarz inequality, we have
\[ d(u) \sum_{v \in \mathcal{N}_G(u)} x_v^2 = \sum_{v \in \mathcal{N}_G(u)} 1^2 \sum_{v \in \mathcal{N}_G(u)} x_v^2 \geq \left( \sum_{v \in \mathcal{N}_G(u)} x_v \right)^2 = \left( (q(G) - d(u)) \cdot x_u \right)^2, \]
which implies
\[ \sum_{v \in \mathcal{N}_G(u)} x_v^2 \geq \frac{((q(G) - d(u))^2 \cdot x_u^2}{d(u)}. \]
Thus,
\[ 1 = \sum_{i \in V(G)} x_i^2 \geq x_u^2 + \sum_{v \in \mathcal{N}_G(u)} x_v^2 \geq x_u^2 + \frac{((q(G) - d(u))^2 \cdot x_u^2}{d(u)} = \left( 1 + \frac{(q(G) - d(u))^2}{d(u)} \right) x_u^2, \]
which follows the result.

Lemma 2.6 ([12]). Let \( G \) be a connected graph. Then
\[ q(G) \leq \max_{u \in V(G)} \left\{ d(u) + \frac{\sum_{v \in \mathcal{N}_G(u)} d(v)}{d(u)} \right\}, \]
with equality if and only if \( G \) is either a semiregular bipartite graph or a regular graph.

We now give an upper and lower bound of \( Q \)-index by Lemma 2.6, from which we can deduce three useful corollaries for the later use.

Theorem 2.1. For any connected \( G \) with size \( m \geq 5 \), we have
(i) if \( \Delta(G) \leq s \) and \( s \geq \frac{2m}{3} \), then \( q(G) \leq s + 2 \).
(ii) if \( s \leq \Delta(G) \leq m - 1 \), then \( q(G) > s + 1 \).

Proof. Let \( z \in V(G) \) such that
\[ d(z) + \frac{\sum_{v \in \mathcal{N}_G(z)} d(v)}{d(z)} = \max_{u \in V(G)} \left\{ d(u) + \frac{\sum_{v \in \mathcal{N}_G(u)} d(v)}{d(u)} \right\}. \]
If \( d(z) = 1 \), by Lemma 2.6, we have \( q(G) \leq d(z) + \frac{\sum_{v \in \mathcal{N}_G(z)} d(v)}{d(z)} \leq 1 + \Delta(G) \leq s + 1 \). It follows the result. If \( d(z) = 2 \), by Lemma 2.6, we get \( q(G) \leq d(z) + \frac{\sum_{v \in \mathcal{N}_G(z)} d(v)}{d(z)} \leq 2 + \Delta(G) \leq s + 2 \). It follows the result. Next we consider the case of \( d(z) \geq 3 \). Note that
\[ q(G) \leq d(z) + \frac{\sum_{v \in \mathcal{N}_G(z)} d(v)}{d(z)} \leq d(z) + \frac{2m - d(z)}{d(z)} = d(z) + \frac{2m}{d(z)} - 1. \tag{1} \]
Clearly, \( 3 \leq d(z) \leq \Delta(G) \leq s \). Let \( f(x) = x + \frac{2m}{x} \). Then \( f(x) \) is decreased in the internal \( [3, \sqrt{2m}] \) and increased in the internal \( [\sqrt{2m}, +\infty) \). Since \( s \geq \frac{2m}{3} \), also noticed that \( f(3) = f(\frac{2m}{3}) \) and \( \frac{2m}{3} > \sqrt{2m} \) due to \( m \geq 5 \), we have
\[ d(z) + \frac{2m}{d(z)} - 1 \leq s + \frac{2m}{s} - 1 \leq s + 1 = s + 2. \tag{2} \]
Combining (1) and (2), we have the first result.
If \( s \leq \Delta(G) \leq m - 1 \), then \( G \) has a \( K_{1,s} \) as a proper subgraph, and so \( q(G) > q(K_{1,s}) = s + 1 \) from Lemma 2.1. It follows the results.
Recall that $\mathbb{G}(m, g)$ is the set of connected graphs on $m$ edges and girth no less than $g$, where $g \geq 3$, and $G_0 \in \mathbb{G}(m, g)$ is a graph obtained from $C_g$ by attaching $m - g$ pendant edges to 0. By simple observation, we see that $G_0$ is the unique graph among $\mathbb{G}(m, g)$ with maximum degree $\Delta(G_0) = m - g + 2$. Moreover, we have the following result.

**Corollary 2.1.** Let $G \in \mathbb{G}(m, g)$ with $m \geq 3g - 3$. Then $q(G) \leq q(G_0)$, with equality if and only if $G \cong G_0$.

**Proof.** By the definition of $\mathbb{G}(m, g)$, taking any $G \in \mathbb{G}(m, g) \setminus \{G_0\}$, we have $\Delta(G) \leq m - g + 1$. Note that $m - g + 1 \geq \frac{2m}{3}$ since $m \geq 3g - 3$. By Theorem 2.1(ii), we have $q(G) \leq m - g + 3$. On the other hand, we know that $m - g + 2 = \Delta(G_0) \leq m - 1$ since $g \geq 3$. By Theorem 2.1(ii), we have $q(G_0) > m - g + 3 \geq q(G)$.

Recently, by using different ways Chen, Wang and Zhai in [9] have obtained the result of Corollary 2.1. By Theorem 2.1, we will give a relation of $Q$-indices of graphs between two distinct girths.

**Corollary 2.2.** Let $G^*$ and $H^*$ respectively be graph with the maximum $Q$-index in $\mathbb{G}(m, g)$ and $\mathbb{G}(m, g')$. If $g < g'$ and $m \geq 3g' - 3$, then $q(G^*) > q(H^*)$.

**Proof.** Since $m \geq 3g' - 3$ and $g' > g$, we have $m \geq 3g - 3$. By Corollary 2.1, we get that $G^*$ (resp. $H^*$) is isomorphic to an unicycle graph $C_g$ (resp. $C_{g'}$) by attaching $m - g$ (resp. $m - g'$) pendant edges to the same vertex of the cycle. Clearly, $\Delta(G^*) = m - g + 2$ and $\Delta(H^*) = m - g' + 2$. Notice that $\Delta(G^*) = m - g' + 2 \geq \frac{2m}{3}$ since $m \geq 3g'^{\prime} - 6$. We have $q(G^*) \leq m - g' + 4$ from Theorem 2.1(i). Note that $\Delta(G') = m - g + 2 \leq m - 1$. By Theorem 2.1(ii), we have $q(G^*) > m - g + 3 \geq m - g' + 4 \geq q(H^*)$. It follows the result.

By Theorem 2.1, we can get a relation of $Q$-indices of graphs between two maximum degrees.

**Corollary 2.3.** Let $G$ and $H$ be graphs with size $m \geq 5$ and maximum degree $\Delta(G)$ and $\Delta(H)$, respectively. If $m - 1 \geq \Delta(G) > \Delta(H) \geq \frac{2m}{3}$, then $q(G) > q(H)$.

**Proof.** Since $\Delta(G) \leq m - 1$, we have $q(G) > \Delta(G) + 1 \geq \Delta(H) + 2$ by Theorem 2.1(ii). On the other hand, note that $\Delta(H) \geq \frac{2m}{3}$. By Theorem 2.1(i), we have $q(H) \leq \Delta(H) + 2 < q(G)$. It follows the result.

### 3 The order of $Q$-indices of graphs in $\mathbb{G}(m, g)$

In the section, we will give the order of graphs in $\mathbb{G}(m, g)$ via their $Q$-indices. For any $G \in \mathbb{G}(m, g)$, $C_g = 012 \cdots (g - 2)(g - 1)0$ is always denoted by one of a shortest cycle of $G$. If $m = g$ then $\mathbb{G}(m, g) = \{C_g\}$. If $m = g + 1$ then $\mathbb{G}(m, g) = \{C^+_g\}$, where $C^+_g$ is a graph obtained from $C_g$ by attaching a pendant edge at some vertex of $C_g$. In what follows, we consider $m \geq g + 2$ and the corresponding $|\mathbb{G}(m, g)| \geq 2$.

For $g \geq 3$ and $m \geq g + 2$, let $\mathbb{G}_\Delta(m, g)$ be the set of graphs in $\mathbb{G}(m, g)$ with maximum degree $\Delta = m - g + 1$. Recall that $G_i \in \mathbb{G}(m, g)$ is obtained from $C_g$ by attaching $m - g - 1$ pendant edges at vertex 0 and simultaneously adding a pendant edge at the vertex $i$, say $iw$, for $1 \leq i \leq \lfloor \frac{g}{2} \rfloor$ (see Fig. 1), and $G_{iv}$ is obtained from $C_g$ by attaching, respectively, $m - g - 2$ pendant edges and a $P_3$ at vertex 0, where $P_3 = 0vv_1$ (see Fig. 1).
Lemma 3.1. \( \mathbb{G}_\Delta(m, g) = \{G_1, G_2, \ldots, G_{\lfloor \frac{g}{2} \rfloor}, G_v\} \), where \( g \geq 4 \) and \( m \geq g + 2 \).

Proof. Let \( G \in \mathbb{G}_\Delta(m, g) \) with a cycle \( C_g = 012 \cdots (g-2)(g-1)0 \). Without loss of generality, we may assume that \( d(0) = \Delta(G) = m - g + 1 \geq 3 \). Let \( G' \) be a subgraph of \( G \) induced by \( V(C_g) \cup N_G(0) \). We have \( m(G') = |C_g| + d(0) - 2 = m - 1 \). Thus \( G \) can be obtained from \( G' \) by adding an edge \( e \). Denote by \( v \not\in C_g \) a vertex adjacent with 0. By the minimality of the length of \( C_g \) and \( g \geq 4 \), we get that \( e \) must be a pendant edge attaching one vertex of \( \{1, 2, \ldots, g-1, v\} \). By considering the symmetry of the vertices \( i \) and \( g - i \) in \( C_g \), we have \( \mathbb{G}_\Delta(m, g) = \{G_1, G_2, \ldots, G_{\lfloor \frac{g}{2} \rfloor}, G_v\} \). \( \square \)

Lemma 3.2. Let \( G_i \in \mathbb{G}_\Delta(m, g) \) shown in Fig.1 where \( 1 \leq i \leq \lfloor \frac{g}{2} \rfloor \). Then \( q(G_1) > q(G_2) > \cdots > q(G_{\lfloor \frac{g}{2} \rfloor}) \).

Proof. For any \( 2 \leq i \leq \lfloor \frac{g}{2} \rfloor \), let \( x = (x_u) \) be the Perron vector of \( Q(G_i) \), where \( u \in V(G_i) \). It suffices to show \( q(G_i) < q(G_{i-1}) \). To prove our result, first we give the following claim.

Claim 1. If there exists \( 1 \leq j \leq i-1 \) such that \( x_{i-j} < x_{i+j-1} \) and \( x_{i-j-1} \geq x_{i+j} \), then \( q(G_i) < q(G_{i-1}) \).

Proof. In fact, let \( G' = G_i - \{(i-j-1)(i-j), (i+j-1)(i+j)\} \cup \{(i-j-1)(i+j-1), (i-j)(i+j)\} \) (see Fig.3). One can observe that \( G' \cong G_{i-1} \). By Lemma 2.3 we have \( q(G_i) < q(G') = q(G_{i-1}) \). \( \square \)

We start to prove by firstly assuming \( x_{i-1} \geq x_i \). Now we construct \( G'' = G_i - w_{i-1} \) from \( G_i \). It is clear that \( G'' \cong G_{i-1} \). By Lemma 2.2 we have \( q(G_i) < q(G'') = q(G_{i-1}) \). Otherwise \( x_{i-1} < x_i \), if \( x_{i-2} \geq x_{i+1} \) then from Claim 1 we get \( q(G_i) < q(G_{i-1}) \) by taking \( j = 1 \). Otherwise \( x_{i-2} < x_{i+1} \), if \( x_{i-3} \geq x_{i+2} \) then from Claim 1 we get \( q(G_i) < q(G_{i-1}) \) by taking \( j = 2 \). Repeating \( i \) steps we come to the assumption \( x_0 < x_{2i-1} \) for \( j = i \). Let \( N_G(0) = \{1, g-1, w_1, \ldots, w_{m-g-1}\} \) and \( G'''' = G_i - \{0w_t \mid 1 \leq t \leq m-g-1\} \cup \{(2i-1)w_t \mid 1 \leq t \leq m-g-1\} \). Clearly, \( G'''' \cong G_{i-1} \). By Lemma 2.2 we have \( q(G_i) < q(G''') = q(G_{i-1}) \). It completes the proof. \( \square \)

![Fig. 3: \( G_i \) and \( G' \) used in the proof of Lemma 3.2 where the edge with “X” represents it is deleted.](image_url)

Next we will give a lower and upper bound for \( q(G_i) \).

Lemma 3.3. Let \( G_i \in \mathbb{G}_\Delta(m, g) \) with \( 1 \leq i \leq \lfloor \frac{g}{2} \rfloor \) shown in Fig.1. If \( m \geq g + 3 \), we have

\[
m - g + 2 < q(G_i) \leq m - g + 2 + \frac{2}{m - g + 1} < m - g + 3.
\]
Proof. Since \( \Delta(G_i) = m - g + 1 \leq m - 1 \), by Theorem 2.1 ii) we have \( q(G_i) > m - g + 2 \). It suffices to show \( q(G_i) \leq m - g + 2 + \frac{2}{m-g+1} \). We consider \( d(u) + \sum_{v \in N_{G_i}(u)} d(v) \) by distinguishing the following situations.

If \( u = 0 \), then \( d(u) = m - g + 2 + \frac{2}{m-g+1} \). If \( u = i \), we have

\[
d(u) + \sum_{v \in N_{G_i}(u)} d(v) \leq d(i) + \frac{d(w) + d(0) + d(i) + 1}{d(i)} \leq 3 + \frac{1}{3} (m - g + 1) + \frac{2}{m - g + 1} \text{ (because } m \geq g + 3)\]

If \( u \) is a pendent vertex of \( G_i \), then \( d(u) + \sum_{v \in N_{G_i}(u)} d(v) \leq 1 + d(0) = m - g + 2 < m - g + 2 + \frac{2}{m-g+1} \).

If \( u \in V(G_i) \setminus \{0, i\} \) is not a pendent vertex, we have \( d(u) + \sum_{v \in N_{G_i}(u)} d(v) \leq 2 + \frac{d(0) + d(i)}{2} \leq 2 + \frac{m-g+4}{m-g+1} \) for \( m \geq g + 3 \).

Thus by Lemma 2.6 we have

\[
q(G_i) \leq \max_{u \in V(G_i)} \left\{ d(u) + \sum_{v \in N_{G_i}(u)} d(v) \right\} = m - g + 2 + \frac{2}{m - g + 1}.
\]

Thus the result holds. \( \square \)

Lemma 3.4. For \( 1 \leq i \leq \lfloor \frac{q}{2} \rfloor \) and \( m \geq \max\{2g - 2, g + 7\} \), let \( \mathbf{x} \) be the Perron vector of \( Q(G_i) \). Then \( x_0 \) is the maximum entry of \( \mathbf{x} \).

Proof. Let \( q = q(G_i) \) and \( v, w \) be a pendant vertex attaching to the vertex 0 and \( i \) of \( G_i \), respectively (see Fig 11). By eigenvalue equation, we have \( x_v = \frac{1}{q-1}x_0 \) and \( x_w = \frac{1}{q-1}x_i \).

Note that \( d(i) = 3 \) and \( d(j) = 2 \) for \( 1 \leq j \leq g - 1 \) and \( j \neq i \). By Lemma 2.5 we have

\[
x_j^2 \leq \frac{1}{1 + \frac{(q-3)^2}{2}} \text{ and } x_j^2 \leq \frac{1}{1 + \frac{(q-2)^2}{2}}.
\]

Furthermore, we get that

\[
1 = \sum_{u \in V(G_i)} x_u^2 = x_0^2 + (m - g - 1)x_v^2 + \sum_{j=1, j \neq i}^{g-1} x_j^2 + x_i^2 + x_w^2
\]

\[
\leq x_0^2 + (m - g - 1) \cdot \left( \frac{1}{q-1}x_0 \right)^2 + (g - 2) \cdot \frac{1}{1 + \frac{(q-2)^2}{2}} + (1 + \frac{1}{(q-1)^2}) \cdot \frac{1}{1 + \frac{(q-3)^2}{2}}.
\]

It follows that

\[
x_0^2 \geq \frac{1 - (g - 2) \cdot \frac{1}{1 + \frac{(q-2)^2}{2}} - (1 + \frac{1}{(q-1)^2}) \cdot \frac{1}{1 + \frac{(q-3)^2}{2}}}{1 + \frac{m-g-1}{(q-1)^2}},
\]

which is equivalent to

\[
x_0^2 \geq \frac{h_1(q) + h_2(q)}{2h_3(q)} + \frac{1}{2}.
\]
Recall that $\Delta(G_i) = m - g + 1 \leq m - 1$, by Theorem 2.1(ii) we have $q > m - g + 2 \geq 9$ due to $m \geq g + 7$. Moreover, notice that the polynomials below are constant coefficients, using the computer we get for $q > 9$ that

$$h_1(q) > 0, q^4 - 10q^3 + 42q^2 - 84q + 72 > 0, q^2 - 6q + 12 > 0, q^2 - 4q + 6 > 0.$$ 

Since $m \geq 2g - 2$, we have $q > m - g + 2 \geq g$ and so $q - (m - g + 2), q - g > 0$. Thus $h_2(q) > 0$. Clearly, $m - g \geq 0$. Then we have $q^2 - 2q + m - g > 0$, and so $h_3(q) > 0$. From (3), we have $x_0^2 > \frac{1}{2}$ and thus $x_u^2 < \frac{1}{2}$ for any $u \in V(G_i) \setminus \{0\}$. It completes the proof. 

**Lemma 3.5.** If $g \geq 4$ and $m \geq \max\{2g - 2, g + 7\}$, then $q(G_1) > q(G_v) > q(G_2)$. 

**Proof.** We first prove $q(G_1) > q(G_v)$. Let $x$ be the Perron vector of $Q(G_v)$. The vertices $v$ and $v_1$ of $G_v$ are shown in Fig. 1. By the eigenvalue equation, we have

$$q(G_v)x_v = x_v + x, \quad q(G_v)x_v = 2x_v + x_v + x_0,$$

$$q(G_v)x_v = (m - g + 1)x_0 + \frac{m - g - 2}{q(G_v) - 1}x_0 + x_v + x_1 + x_{g_1 - 1}. \quad (4)$$

Note that $x_1 = x_{g_1 - 1}$ due to the symmetry of $G_v$. From (4), we have

$$\begin{cases}
  x_v = \frac{q(G_v) - 1}{(q(G_v) - 2)(q(G_v) - 1) - 1}x_0, \\
  x_1 = \frac{1}{2}(q(G_v) - (m - g + 1)) - \frac{m - g - 2}{q(G_v) - 1} - \frac{q(G_v) - 1}{(q(G_v) - 2)(q(G_v) - 1) - 1}x_0.
\end{cases}$$

Let

$$f(x) = \frac{1}{2}(x - (m - g + 1)) - \frac{m - g - 2}{x - 1} - \frac{x - 1}{(x - 2)(x - 1) - 1} - \frac{x - 1}{(x - 2)(x - 1) - 1}.$$ 

Then $x_1 - x_v = f(q(G_v))x_0$. Notice that

$$f(x) = \frac{x}{2(x^2 - 3x + 1)(x - 1)} \phi_1(x, m - g + 5), \quad (5)$$

where $\phi_1(x, m - g + 5)$ is defined by Lemma 2.4(i). If $g \geq 5$, then $S_{m-g+5,3}$ is a proper subgraph of $G_v$, and so $q(G_v) > q(S_{m-g+5,3})$. Recall that $\Delta(G_v) = m - g + 1 \leq m - 1$, by Theorem 2.1(ii) we have $q(G_v) > m - g + 2 \geq 9$ due to $m \geq g + 7$. It is easy to verify that $x^2 - 3x + 1 > 0$ for $x > 9$. Thus $f(q(G_v)) > 0$, which implies that $x_1 > x_v$. If $g = 4$, then (5) becomes

$$f(x) = \frac{x}{2(x^2 - 3x + 1)(x - 1)} \cdot \frac{\phi_2(x, m) + 2x - 2}{x^2 - 4x + 2},$$
where \( \phi_2(x, m) \) is defined by Lemma 2.4(ii). Clearly, \( \phi_2(q(G), m) = 0 \). On the other hand, we have \( 2x - 2 > 0 \) and \( x^2 - 4x + 2 > 0 \) for \( x > 9 \). Recall that \( q(G) > 9 \). Thus \( f(q(G)) > 0 \), and also \( x_1 > x_v \). Let \( G' = G - \{v_v\} + \{1v_1\} \). Clearly, \( G' \cong G_1 \). By Lemma 2.2 we have \( q(G_1) < q(G') = q(G) \).

Next we prove \( q(G_i) > q(G_2) \). Let \( y \) be the Perron vector of \( Q(G_2) \). From Lemma 3.4, \( y_0 = \max_{x \in V(G_2)}(y_x) \). Taking any vertex \( u \) with degree 2 in \( G_2 \), by the eigenvalue equation, we have \( q(G_2)y_u = 2y_u + \sum_{v \in N_G(u)}y_v \leq 2y_u + 2y_0 \), which implies that

\[
y_u \leq \frac{2}{q(G_2) - 2}y_0. \tag{6}
\]

By the eigenvalue equation again, \( q(G_2)y_2 = y_w + y_2 \) and \( q(G_2)y_2 = 3y_2 + y_w + y_1 + y_3 \). Thus from (6) we have

\[
y_2 = \frac{q(G_2) - 1}{q^2(G_2) - 4q(G_2) + 2}(y_1 + y_3) \leq \frac{q(G_2) - 1}{q^2(G_2) - 4q(G_2) + 2} \cdot \frac{4}{q(G_2) - 2}y_0.
\]

Let \( v \) be a pendant vertex attaching 0 in \( G_2 \). Clearly, \( y_v = \frac{1}{q(G_2) - 1}y_0 \). On the other hand, one can easily verify that

\[
q(G_2) - \frac{1}{q^2(G_2) - 4q(G_2) + 2} \leq \frac{4}{q(G_2) - 2} < \frac{1}{q(G_2) - 1}
\]

for \( q(G_2) \geq 8 \). Since \( m \geq g + 7 \), by Lemma 3.3, we have \( q(G_2) > m - g + 2 \geq 8 \). Hence, \( y_2 < y_v \). Let \( G'' = G_2 - \{2w\} + \{vw\} \). Clearly, \( G'' \cong G_2 \). By Lemma 2.2 we have \( q(G_2) < q(G'') = q(G_v) \).

It completes the proof. \( \square \)

It is time to provide the proofs of our main results. First we prove Theorem 1.1 that orders the first \( \lfloor \frac{m}{2} \rfloor + 2 \) largest graphs according their \( Q \)-indices among \( \mathcal{G}(m, g) \).

**Proof of Theorem 1.1** Note that \( G_0 \) is a unique graph with maximum degree \( m - g + 2 \) among \( \mathcal{G}(m, g) \) and \( m \geq 3g \geq 12 \). By Lemma 3.1, \( \mathcal{G}_\Delta(m, g) = \{G_1, G_2, \ldots, G_{\lfloor \frac{m}{2} \rfloor}, G_{\lfloor \frac{m}{2} \rfloor} \} \) is the set of graphs with maximum degree \( m - g + 1 \). Since \( m - 1 \geq \Delta(G_0) = m - g + 2 > m - g + 1 = \Delta(G_1) \geq 2m \), by Corollary 2.3 we have \( q(G_0) > q(G_1) \). Moreover, notice that \( g \geq 4 \) and \( m \geq \max\{2g - 2, g + 7\} \) due to \( m \geq 3g \geq 12 \), by Lemmas 3.2 and 3.5 we have \( q(G_0) > q(G_1) > q(G_2) > q(G_3) > \cdots > q(G_{\lfloor \frac{m}{2} \rfloor}) \). Set \( \mathcal{G}_{m-g}(m, g) = \mathcal{G}(m, g) \setminus (\mathcal{G}_\Delta(m, g) \cup \{G_0\}) \). Then for any \( G' \in \mathcal{G}_{m-g}(m, g) \), we have \( \Delta(G') \leq m - g \). Since \( m \geq 3g \), we have \( m - 1 \geq \Delta(G_{\lfloor \frac{m}{2} \rfloor}) = m - g + 1 > m - g = \Delta(G') \geq 2m \). By Corollary 2.3 we get \( q(G_{\lfloor \frac{m}{2} \rfloor}) > q(G') \). Thus the first \( \lfloor \frac{m}{2} \rfloor + 2 \) largest \( Q \)-indices of all graphs among \( \mathcal{G}(m, g) \) belong to \( \mathcal{G}_\Delta(m, g) \cup \{G_0\} \), which is given by

\[
q(G_0) > q(G_1) > q(G_2) > (G_3) > \cdots > q(G_{\lfloor \frac{m}{2} \rfloor}).
\]

It completes the proof. \( \square \)

Next we prove Theorem 1.2 that orders the first \( \lfloor \frac{m}{2} \rfloor + 3 \) largest graphs according their \( Q \)-indices among \( \mathcal{G}(m, g) \). For \( 0 \leq i \leq \lfloor \frac{m}{2} \rfloor \), we use \( G_{i, g} \) and \( G_{i, g} \) instead of \( G_i \) and \( G_v \) to distinguish the girth of the graphs in the following proofs.
Theorem 2.1(i) we have $q > m - g + 2$. For any $G^\prime \in \mathcal{G}_1$, notice that $\Delta(G^\prime) \leq m - g$ and $m - g > \frac{2m}{3}$ (since $m \geq 3g$), from Theorem 2.1(ii) we have $q(G^\prime) \leq m - g + 2$. Thus each $Q$-index of the graph in $\mathcal{G}_1$ is more than that of the graph in $\mathcal{G}_m$. By Theorem 1.1, to complete the proof it remains to show $q(G_{1,3,5,7}) > q(G_{0,g+1})$.

Let $x$ be the Perron vector of $G_{0,g+1}$ corresponding to $q = q(G_{0,g+1})$. Notice that $m \geq 3g$ and $g \geq 3$. We have $q > m - g + 2 \geq 8$. If $g$ is odd, by symmetry of $G_{0,g+1}$, then $x_{\lfloor \frac{g}{2} \rfloor} = x_{\lfloor \frac{g}{2} \rfloor + 2}$ (see Fig. 4). Moreover, by the eigenvalue equation of $Q(G_{0,g+1})$, we have $(q - 2)x_{\lfloor \frac{g}{2} \rfloor} = x_{\lfloor \frac{g}{2} \rfloor} + x_{\lfloor \frac{g}{2} \rfloor + 2} = 2x_{\lfloor \frac{g}{2} \rfloor}$, and so $x_{\lfloor \frac{g}{2} \rfloor} \geq x_{\lfloor \frac{g}{2} \rfloor + 1}$ since $q > 8$. If $g$ is even, by the symmetry of $G_{0,g+1}$, then $x_{\lfloor \frac{g}{2} \rfloor} = x_{\lfloor \frac{g}{2} \rfloor + 1}$ (see Fig. 4). Let

$$G^\prime = G_{0,g+1} - \{(\lfloor \frac{g}{2} \rfloor + 1)(\lfloor \frac{g}{2} \rfloor + 2)\} + \{(\lfloor \frac{g}{2} \rfloor)(\lfloor \frac{g}{2} \rfloor + 2)\}.$$  

Clearly, $G^\prime \cong G_{1,3,5,7}$. By Lemma 2.2 we have $q(G_{1,3,5,7}) = q(G^\prime) > q(G_{0,g+1})$.

It completes the proof.\hfill \qed

Let $\mathcal{G}_\Delta(m, 3)$ be the set of graphs with maximum degree $m - 2$ in $\mathcal{G}(m, 3)$. $B_1, B_2, G_{1,3}$ and $G_{1,3,5}$ are shown in Fig 5. By direct observation, we have $\mathcal{G}_\Delta(m, 3) = \{B_1, B_2, G_{1,3,5}\}$. Thirdly we give the order of $Q$-indices among $\mathcal{G}(m, 3)$ as supplement of Theorem 1.1 for $g = 3$.

Proof of Theorem 1.3. By Corollary 2.1, $G_{0,3}$ (see Fig 1) attains the maximum $Q$-index among all graphs in $\mathcal{G}(m, 3)$ for $m \geq 6$. Note that $m - 1 = \Delta(G_{0,3}) > \Delta(B_1) = m - 2 \geq \frac{2m}{3}$. By Corollary 2.3, we have $q(G_{0,3}) > q(B_1)$.

We now prove $q(B_1) > q(B_2)$. Let $x$ be the Perron vector of $B_2$ and vertices $w_1, w_2$ and $1$ of $B_2$ are shown in Fig 5. By the symmetry of graph $Q(B_2)$, we have $x_1 = x_{w_1}$. Let $B^\prime = B_2 - \{w_1w_2\} + \{1w_2\}$. Clearly, $B^\prime \cong B_1$. By Lemma 2.2 we have $q(B_1) = q(B^\prime) > q(B_2)$.
By calculation, the characteristic polynomials of $\phi$. It is easy to verify that $q$.

Recall that $\Delta$. Thirdly, we prove $q(B_2) > q(G_{1,3})$. Thus $q(G_{1,3}) > q(G_{v,3})$. $G_{v,3}$ has the equitable partition $\Pi_3 : V(G_{v,3}) = \{0\} \cup \{v\} \cup \{v_1\} \cup \{V_4\} \cup \{V_3\}$ (see Fig$^5$). Thus the quotient matrix with respect to $\Pi_3$ is

$$M(G_{v,3}) = \begin{pmatrix}
m-2 & 1 & 0 & m-5 & 2 \\
1 & 2 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 3
\end{pmatrix}.$$  

By calculation, the characteristic polynomial of $M(G_{v,3})$ is $\varphi(x, G_{v,3}) = x^5 - (m + 5)x^4 + (6m + 4)x^3 - (10m - 2)x^2 + (3m + 12)x - 4$. Thus

$$\varphi(x, G_{v,3}) = \varphi(x, G_{1,3}) + x(x^2 - (m - 1)x + 4).$$  

Recall that $\Delta(G_{v,3}) = m - 2$. By Theorem 2.1(ii), we have $q(G_{v,3}) > m - 1$. Thus $q^2(G_{v,3}) - (m - 1)q(G_{v,3}) + 4 > 0$. Clearly, $\varphi(q(G_{v,3}), G_{v,3}) = 0$. From (8), we get $\varphi(q(G_{v,3}), G_{1,3}) < 0$. Note that $q(G_{1,3})$ is the largest root of $\varphi(x, G_{1,3})$. Thus $q(G_{v,3}) < q(G_{1,3})$. For any $G \in G(m, 3) \setminus \{G_\triangle(m, 3) \cup G_{0,3}\}$, we have $\Delta(G) \leq m - 3$. Notice that $m - 2 = \Delta(G_{v,3}) > m - 3 \geq \frac{2m}{3}$ due to $m \geq 9$, by Corollary 2.3 we have $q(G_{v,3}) > q(G)$. It completes the proof. $\square$

Fig. 5: The graphs $B_1$, $B_2$, $G_{1,3}$ and $G_{v,3}$

Secondly, we prove $q(B_2) > q(G_{1,3})$. It is clear that $B_2$ and $G_{1,3}$ have the equitable partitions $\Pi_1 : V(B_2) = V_1 \cup \{0\} \cup V_2$ and $\Pi_2 : V(G_{1,3}) = \{0\} \cup \{1\} \cup \{2\} \cup \{w\} \cup \{V_3\}$ (see Fig$^5$), respectively. Thus the corresponding quotient matrices are

$M(B_2) = \begin{pmatrix}
3 & 1 & 0 \\
4 & m - 2 & m - 6 \\
0 & 1 & 1
\end{pmatrix}$, $M(G_{1,3}) = \begin{pmatrix}
m - 2 & 1 & 1 & 0 & m - 4 \\
1 & 3 & 1 & 1 & 0 \\
1 & 1 & 2 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1
\end{pmatrix}$.

By calculation, the characteristic polynomials of $M(B_2)$ and $M(G_{1,3})$ are $\varphi(x, B_2) = x^3 - (m+2)x^2 + (3m-3)x - 8$ and $\varphi(x, G_{1,3}) = x^5 - (m+5)x^4 + (6m+3)x^3 - (9m-1)x^2 + (3m+8)x - 4$. It is easy to verify that

$$\varphi(x, G_{1,3}) = \varphi(x, B_2) \cdot (x^2 - 3x) + (3m - 16)x - 4.$$  

Clearly, $\varphi(q(G_{1,3}), G_{1,3}) = 0$. Note that $\Delta(G_{1,3}) = m - 2$. By Theorem 2.1(ii), we have $q(G_{1,3}) > m - 1 \geq 5$ due to $m \geq 6$. Thus $q^2(G_{1,3}) - 3q(G_{1,3}) > 0$ and $(3m-16)q(G_{1,3}) - 4 > 0$. From (7), we have $\varphi(q(G_{1,3}), B_2) < 0$. Note that $q(B_2)$ is the largest root of $\varphi(x, B_2)$. Thus $q(B_2) > q(G_{1,3})$.
Proof of Theorem 1.4. Denote by $\mathcal{G}_\Delta(m)$ and $\mathcal{G}_{\leq \Delta}(m)$ the set of graphs in $\mathcal{G}_m$ with maximum degree $\Delta$ and no more than $\Delta$, respectively. Clearly, $\mathcal{G}_m(m) = \{K_{1,m}\}$, $\mathcal{G}_{m-1}(m) = \{G_{0,3}, T_1\}$ and $\mathcal{G}_{m-2}(m) = \{B_1, B_2, G_{1,3}, G_{v,3}, T_2, G_{0,4}, T_3, T_4\}$. Moreover, we can partition

$$\mathcal{G}_m = \mathcal{G}_m(m) \cup \mathcal{G}_{m-1}(m) \cup \mathcal{G}_{m-2}(m) \cup \mathcal{G}_{\leq m-3}(m).$$

Note that $\Delta(G_{0,3}) = m - 1 \geq \frac{2m}{3}$ since $m \geq 9$. By Theorem 2.1(i), we have $q(G_{0,3}) \leq m + 1$. If $q(G_{0,3}) = m + 1$, then by the eigenvalue equation of $Q(G_{0,3})$, we deduce that $m = -2$ or $3$, which contradicts $m \geq 9$. Thus $q(G_{0,3}) < m + 1 = q(K_{1,m})$.

Secondly, we will show $q(G_{0,3}) > q(T_1)$. Let $x$ be the Perron vector of $T_1$. By the eigenvalue equation of $Q(T_1)$, we have $x_{u_2} = \frac{q^2(T_1) - 3q(T_1) + 1}{q(T_1) - 1} x_{u_1}$. Notice that $\Delta(T_1) = m - 1$. By Lemma 2.1(ii), we have $q(T_1) > q(K_{1,m-1}) = m \geq 9$. Thus $\frac{q^2(T_1) - 3q(T_1) + 1}{q(T_1) - 1} > 1$ and so $x_{u_2} > x_{u_1}$. Let $G' = T_1 - u_0u_1 + u_0u_2$. Clearly, $G' \backslash \{u_1\} \cong G_{0,3}$. By Lemma 2.2, we have $q(G_{0,3}) = q(G') > q(T_1)$.

Again notice that $m - 1 = \Delta(T_1) > \Delta(B_1) = m - 2 \geq \frac{2m}{3}$ since $m \geq 9$, we have $q(T_1) > q(B_1)$ by Corollary 2.3. Thus, from Theorem 1.3 we obtain

$$q(K_{1,m}) > q(G_{0,3}) > q(T_1) > q(B_1) > q(B_2) > q(G_{1,3}) > q(G_{v,3}).$$

Thirdly, we will show $q(G_{v,3}) > q(T_2) > q(G_{0,4})$ by comparing the largest root of their quotient matrices of two graphs. As shown in Fig. 6, $T_2$ and $G_{0,4}$ respectively have the equitable partition $\Pi_4 : V(T_2) = \{u_3\} \cup \{u_4\} \cup V_6 \cup V_7$ and $\Pi_5 : V(G_{0,4}) = \{0\} \cup V_8 \cup \{2\} \cup V_9$. The quotient matrices with respect to $\Pi_4$ and $\Pi_5$ are respectively given by

$$M(T_2) = \begin{pmatrix} m-2 & 1 & 0 & m-3 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M(G_{0,4}) = \begin{pmatrix} m-2 & 2 & 0 & m-4 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. $$

By calculation, the characteristic polynomials of $M(T_2)$ and $M(G_{0,4})$ are respectively

$$\varphi(x, T_2) = x^4 - (m + 3)x^3 + (4m - 3)x^2 - (m + 1)x \quad \text{and} \quad \varphi(x, G_{0,4}) = x^4 - (m + 3)x^3 + (4m - 2)x^2 - 2mx.$$

Combining (8), we obtain that

$$\varphi(x, G_{v,3}) = (x - 2) \cdot \varphi(x, T_2) + \frac{\varphi(x, T_2)}{x} + h(x), \quad (9)$$

Notice that $\mathcal{G}_m$ is the union of $\cup_{g \geq 3} \mathcal{G}(m, g)$ and some trees. The ordering of $Q$-indices of graphs in $\mathcal{G}(m, g)$ and $\mathcal{G}(m, g)$ would induce related result among graphs in $\mathcal{G}_m$. At last we prove Theorem 1.4 that orders of the first eleven $Q$-indices among $\mathcal{G}_m$.

Fig. 6: Some graphs used in the Proof of Theorem 1.4.
where \( h(x) = (13 - 3m)x + m - 3 \). Since \( m \geq 9 \), we have \( h(x) \) is decrease. Note that 
\[ \Delta(T_2) = m - 2 \leq m - 1, \]
by Lemma 2.7, we have \( q(T_2) > q(K_{1,m-2}) = m - 1 \) and then 
\[ h(q(T_2)) < h(m - 1) = -3m^2 + 17m - 16 < 0 \quad \text{due to} \quad m \geq 9. \]
From (9), we get \( \varphi(q(T_2), G_{v,3}) < 0 \). Note that \( q(G_{v,3}) \) is the largest root of \( \varphi(x, G_{v,3}) \). Thus \( q(G_{v,3}) > q(T_2) \). On the other hand, one also can verify that 
\[ \varphi(x, G_{0,4}) = \varphi(x, T_2) + x(x + (1 - m)). \tag{10} \]
Recall that \( q(T_2) > m - 1 \), we have \( \varphi(q(T_2), G_{0,4}) > 0 \). Since \( q(G_{0,4}) \) is the largest root of \( \varphi(x, G_{0,4}) \), we have \( q(T_2) > q(G_{0,4}) \) and thus \( q(G_{v,3}) > q(T_2) > q(G_{0,4}) \).

Fourthly, we show \( q(G_{0,4}) > q(T_3) \). Let \( y \) be the Perron vector of \( Q(T_3) \). One can see \( y_{u_6} = y_{u_1} \) from the symmetry of \( T_3 \) (see Fig 6). Let \( G'' = T_3 - u_5u_6 + u_5u_7 \). Clearly, \( G'' \backslash \{ u_6 \} \cong G_{0,4} \). By Lemma 2.2, we have \( q(G_{0,4}) = q(G'') > q(T_3) \).

Now we will show \( q(T_3) > q(T_4) \). Let \( z \) be the Perron vector of \( Q(T_4) \). One can verify \( z_{u_5} = \frac{q^3(T_4) - 5q^2(T_4) + 6q(T_4) - 1}{(q(T_4) - 1)^2} z_{u_0} \) from the eigenvalue equation of \( Q(T_4) \). Notice that 
\[ q(T_4) > q(K_{1,m-2}) = m - 1 \geq 8 \quad \text{due to} \quad m \geq 9. \]
We have \( \frac{q^3(T_4) - 5q^2(T_4) + 6q(T_4) - 1}{(q(T_4) - 1)^2} > 1 \) and so 
\[ z_{u_5} > z_{u_0}. \]
Let \( G''' = T_4 - u_{10}u_9 + u_{10}u_8 \). Clearly, \( G''' \cong T_3 \). By Lemma 2.2, we have 
\[ q(T_3) = q(G''') > q(T_4). \]

For any \( G \in \mathbb{G}_{<m-3}(m) \), we have \( \Delta(T_4) = m - 2 > m - 3 \geq \Delta(G) \geq \frac{2m}{3} \) since \( m \geq 9 \). By Corollary 2.3 we get \( q(T_4) > q(G) \). Therefore, by the above discussion, the first eleven largest \( Q \)-indices of graphs in \( \mathbb{G}_m \) are given by 
\[ q(K_{1,m}) > q(G_{0,3}) > q(T_1) > q(B_1) > q(B_2) > q(G_{1,3}) > q(G_{v,3}) > q(T_2) > q(G_{0,4}) \]
\[ > q(T_3) > q(T_4). \]

It completes the proof. \( \square \)

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