A geometrical construction for the polynomial invariants of some reflection groups

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Abstract

In these notes we investigate the ring of real polynomials in four variables, which are invariant under the action of the reflection groups $[3, 4, 3]$ and $[3, 3, 5]$. It is well known that they are rationally generated in degree 2, 6, 8, 12 and 2, 12, 20, 30. We give a different proof of this fact by giving explicit equations for the generating polynomials.

0 Introduction

There are four groups generated by reflections which operate on the four-dimensional Euclidian space. These are the symmetry groups of some regular four dimensional polytopes and are described in [Co2] p. 145 and table I p. 292-295. With the notation there the groups and their orders are

| Group             | $[3, 3, 3]$ | $[3, 3, 4]$ | $[3, 4, 3]$ | $[3, 3, 5]$ |
|-------------------|------------|------------|------------|------------|
| Order             | 120        | 384        | 1152       | 14400      |

They operate in a natural way on the ring of polynomials $R = \mathbb{R}[x_0, x_1, x_2, x_3]$ and it is well known that the ring of invariants $R^G$ ($G$ one of the groups above) is algebraically generated by a set of four independent polynomials (cf. [Bi] p. 357). Coxeter shows in [Co1] that the rings $R^G, G = [3, 3, 3]$ or $[3, 3, 4]$ are generated in degree 2, 3, 4, 5 resp. 2, 4, 6, 8 and since the product of the degrees is equal to the order of the group, any other invariant polynomial is a combination with real coefficients of products of these invariants (i.e., in the terminology of [Co1], the ring $R^G$ is rationally generated by the polynomials). Coxeter also gives equations for the generators. In the case of the groups $[3, 4, 3]$ and $[3, 3, 5]$ he recalls a result of Racah, who shows with
the help of the theory of Lie groups that the rings $R^G$ are rationally generated in degree 2, 6, 8, 12 resp. 2, 12, 20, 30 (cf. [Ra]). Neither Coxeter nor Racah give equations for the polynomials. In these notes we construct the generators and give a different proof of the result of Racah. The invariant of degree two is well known (cf. [Co1]) and can be given as

$$q = x_0^2 + x_1^2 + x_2^2 + x_3^2.$$ 

We construct the other invariants in a completely geometrical way. For proving that our polynomials together with the quadric generate the ring $R^G$, we show some relations between them and the invariant forms of the binary tetrahedral group and of the binary icosahedral group.

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1 Notations and preliminaries

Denote by $R$ the ring of polynomials in four variables with real coefficients $\mathbb{R}[x_0, x_1, x_2, x_3]$, by $G$ a finite group of homogeneous linear substitutions, and by $R^G$ the ring of invariant polynomials.

1. A set of polynomials $F_1, \ldots, F_n$ in $R$ is called algebraically dependent if there is a non trivial relation

$$\sum \alpha_I (F_1^{i_1} \cdot \ldots \cdot F_n^{i_n}) = 0,$$

where $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, $\alpha_I \in \mathbb{R}$.

2. The polynomials are called algebraically independent if they are not dependent. For the ring $R^G$, there always exists a set of four algebraically independent polynomials (cf. [Bu], thm. I, p. 357).

3. We say that $R^G$ is algebraically generated by a set of polynomials $F_1, \ldots, F_4$, if for any other polynomial $P \in R^G$ we have an algebraic relation

$$\sum \alpha_I (P^{i_0} \cdot F_1^{i_1} \cdot \ldots \cdot F_4^{i_4}) = 0.$$

4. We say that the ring $R^G$ is rationally generated by a set of polynomials $F_1, \ldots, F_4$, if for any other polynomial $P \in R^G$ we have a relation

$$\sum \alpha_I (F_1^{i_1} \cdot \ldots \cdot F_4^{i_4}) = P, \ \alpha_I \in \mathbb{R}$$

5. The four polynomials of 3 are called a basic set if they have the smallest possible degree (cf. [Co1]).

6. There are two classical $2:1$ coverings

$$\rho : SU(2) \to SO(3) \quad \text{and} \quad \sigma : SU(2) \times SU(2) \to SO(4),$$

where
we denote by $T, O, I$ the tetrahedral group, the octahedral group and the icosahedral group in $SO(3)$ and by $\tilde{T}, \tilde{O}, \tilde{I}$ the corresponding binary groups in $SU(2)$ via the map $\rho$. The $\sigma$-images of $\tilde{T} \times \tilde{T}$, $\tilde{O} \times \tilde{O}$ and $\tilde{I} \times \tilde{I}$ in $SO(4)$ are denoted by $G_6$, $G_8$ and $G_{12}$. By abuse of notation we write $(p, q)$ for the image in $SO(4)$ of an element $(p, q) \in SU(2) \times SU(2)$. As showed in [Sa] (3.1) p. 436, the groups $G_6$ and $G_{12}$ are subgroups of index four respectively two in the reflections groups $[3, 4, 3]$ and $[3, 3, 5]$.

2 Geometrical construction

Denote by $\tilde{G}$ one of the groups $\tilde{T}$, $\tilde{O}$ or $\tilde{I}$. Clearly, the subgroups $\tilde{G} \times 1$ and $1 \times \tilde{G}$ of $SO(4)$ are isomorphic to $\tilde{G}$. Moreover, each of them operates on one of the two rulings of the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ and leaves invariant the other ruling (as shown in [Sa]). We recall the lengths of the orbits of points under the action of the groups $T$, $O$ and $I$

| group | $T$ | $O$ | $I$ |
|-------|-----|-----|-----|
| lengths of the orbits | 12, 6, 4 | 24, 12, 8, 6 | 60, 30, 20, 12 |

These lines are fixed by elements $(p, 1) \in \tilde{G} \times 1$ on one ruling, resp. $(1, p') \in 1 \times \tilde{G}$ on the other ruling of the quadric. Recall that these elements have two lines of fixed points with eigenvalues $\alpha$, $\bar{\alpha}$ which are in fact the eigenvalues of $p$ and $p'$. We call two lines $L$, $L'$ of $\mathbb{P}_1 \times \mathbb{P}_1$ a couple if $L$ is fixed by $(p, 1)$ with eigenvalue $\alpha$ and $L'$ is fixed by $(1, p)$ with the same eigenvalue.

2.1 The invariant polynomials of $G_6$ and of $G_{12}$

Consider the six couples of lines $L_1, L'_1, \ldots, L_6, L'_6$ in $\mathbb{P}_1 \times \mathbb{P}_1$ which form one orbit under the action of $\tilde{T} \times 1$, resp. $1 \times \tilde{T}$, and denote by $f_{11}^{(6)}, \ldots, f_{66}^{(6)}$ the six planes generated by such a couple of lines (and by abuse of notation their equation, too). Now set

$$F_6 = \sum_{g \in T \times 1} g(f_{11}^{(6)} \cdot f_{22}^{(6)} \cdot \ldots \cdot f_{66}^{(6)}) = \sum_{g \in T \times 1} g(f_{11}^{(6)}) \cdot g(f_{22}^{(6)}) \cdot \ldots \cdot g(f_{66}^{(6)}).$$

Observe that an element $g \in \tilde{T} \times 1$ leaves each line of one ruling invariant and operates on the six lines of the other ruling. A similar action is given by an element of $1 \times \tilde{T}$. Since we sum over all the elements of $\tilde{T} \times 1$, the action of $1 \times \tilde{T}$ does not give anything new, hence $F_6$ is $G_6$-invariant. Furthermore observe that $F_6$ has real coefficients. In fact, in the above product, for each plane generated by the lines $L_i$, $L'_i$ we also take the plane generated by the lines which consist of the conjugate points. The latter has equation $\bar{f}_{ii}^{(6)}$, i.e., we have an index $j \neq i$ with $f_{jj}^{(6)} = \bar{f}_{ii}^{(6)}$ and the products $f_{ii}^{(6)} \cdot \bar{f}_{ii}^{(6)}$
have real coefficients.
Consider now the orbits of lengths eight and twelve under the action of $\tilde{O} \times 1$ and $1 \times \tilde{O}$ and the planes $f_{ii}^{(8)}$, $f_{jj}^{(12)}$ generated by the eight, respectively by the twelve couples of lines. As before the polynomials

$$F_8 = \sum_{g \in \tilde{T} \times 1} g(f_{11}^{(8)} \cdots f_{88}^{(8)})$$

$$F_{12} = \sum_{g \in \tilde{T} \times 1} g(f_{11}^{(12)} \cdots f_{1212}^{(12)})$$

are $G_6$-invariant and have real coefficients.
Finally we consider the lines of $\mathbb{P}_1 \times \mathbb{P}_1$ which form orbits of length 12, 20 and 30 under the action of $\tilde{I} \times 1$ resp. $1 \times \tilde{I}$. The planes generated by the couples of lines produce the $G_{12}$-invariant real polynomials

$$\Gamma_{12} = \sum_{g \in \tilde{I} \times 1} g(h_{11}^{(12)} \cdots h_{1212}^{(12)})$$

$$\Gamma_{20} = \sum_{g \in \tilde{I} \times 1} g(h_{11}^{(20)} \cdots h_{2020}^{(20)})$$

$$\Gamma_{30} = \sum_{g \in \tilde{I} \times 1} g(h_{11}^{(30)} \cdots h_{3030}^{(30)})$$

### 2.2 The invariant polynomials of the reflection groups

We consider the matrices

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad C' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

as in [Sa] (3.1) p. 436, the groups generated by $G_6$, $C$, $C'$ and $G_{12}$, $C$ are the reflections groups [3, 4, 3] respectively [3, 3, 5].

**Proposition 2.1** 1. The polynomials $F_6$, $F_8$, $F_{12}$, $\Gamma_{12}$, $\Gamma_{20}$, $\Gamma_{30}$ are $C$ invariant.
2. The polynomials $F_6$, $F_8$, $F_{12}$ are $C'$ invariant.

**Proof.** 1. The matrix $C$ interchanges the two rulings of the quadric, hence the polynomials $F_i$ and $\Gamma_j$ are invariant by construction. We prove 2 by a direct computation in the last section. \(\Box\)

From this fact we obtain
Corollary 2.1 The polynomials \( q, F_6, F_8, F_{12} \) are \([3, 4, 3]\)-invariant and the polynomials \( q, \Gamma_{12}, \Gamma_{20}, \Gamma_{30} \) are \([3, 3, 5]\)-invariant.

Here we denote by \( q \) the quadric \( \mathbb{P}_1 \times \mathbb{P}_1 \).

3 The rings of invariant forms

Identify \( \mathbb{P}_3 \) with \( \mathbb{P}M(2 \times 2, \mathbb{C}) \) by the map

\[
(x_0 : x_1 : x_2 : x_3) \mapsto \left( \begin{array}{ccc} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{array} \right).
\]

Furthermore consider the map

\[
\mathbb{C}^2 \times \mathbb{C}^2 \rightarrow M(2 \times 2, \mathbb{C}) \\
((z_0, z_1), (z_2, z_3)) \mapsto \left( \begin{array}{cc} z_0 z_2 & z_0 z_3 \\ z_1 z_2 & z_1 z_3 \end{array} \right) = Z.
\]

Then \( Z \) is a matrix of determinant \( x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0 \) which is the equation of \( q \). Now denote by \( \mathcal{O}_{\mathbb{P}_3}(n) \) the sheaf of regular functions of degree \( n \) on \( \mathbb{P}_3 \) and by \( \mathcal{O}_q(n,n) \) the sheaf of regular function of be-degree \((n,n)\) on the quadric \( q \). We obtain a surjective map between the global sections

\[
\phi : H^0(\mathcal{O}_{\mathbb{P}_3}(n)) \rightarrow H^0(\mathcal{O}_q(n,n))
\]

by doing the substitution

\[
x_0 = \frac{z_0 z_2 + z_1 z_3}{2}, \quad x_1 = \frac{z_0 z_2 - z_1 z_3}{2}, \\
x_2 = \frac{z_0 z_3 - z_1 z_2}{2}, \quad x_3 = \frac{z_0 z_3 + z_1 z_2}{2}.
\]

in a polynomial \( p(x_0, x_1, x_2, x_3) \in H^0(\mathcal{O}_{\mathbb{P}_3}(n)) \). Observe that \( \phi(q) = 0 \). Now let

\[
t = z_0 z_1 (z_0^4 - z_1^4), \\
W = z_0^8 + 14 z_0 z_1^4 + z_1^8, \\
\chi = z_0^{12} - 33 (z_0^8 z_1^4 + z_0^4 z_1^8) + z_1^8
\]

denote the \( \bar{T} \)-invariant polynomials of degree 6,8 and 12 and let

\[
f = z_0 z_1 (z_0^{10} + 11 z_0^5 + z_1^5), \\
H = -(z_0^{20} + z_1^{20}) + 228(z_0^{15} z_1^5 - z_0^5 z_1^{15}) - 494 z_0^{10} z_1^{10}, \\
T = (z_0^{30} + z_1^{30}) + 522(z_0^{25} z_1^5 - z_0^5 z_1^{25}) - 10005(z_0^{20} z_1^{10} + z_0^{10} z_1^{20})
\]

be the \( \bar{T} \)-invariant polynomials of degree 12,20,30 given by Klein in [K] p. 51-58. Put \( t_1 = t(z_0, z_1), \ t_2 = t(z_2, z_3), \ W_1 = W(z_0, z_1), \ W_2 = W(z_2, z_3) \) and analogously for the other invariants.
Proposition 3.1 If $p \in H^0(\mathcal{O}_\mathbb{P}^3(n))$ is $G_6$-invariant, then:

$$\phi(p) = \sum_i \alpha_i t_1^{\alpha_i} t_2^{\alpha_i} W_1^{\alpha_1} W_2^{\alpha_2} \chi_1 \chi_2$$

If $p$ is $G_{12}$-invariant, then:

$$\phi(p) = \sum_j \beta_j f^\beta_1 f_2^{\beta_2} H_1^{\beta_1} H_2^{\beta_2} \tau_1^{\beta_3} \tau_2^{\beta_4}$$

where

$$(\alpha_1, \alpha_1', \alpha_2, \alpha_2', \alpha_3, \alpha_3')|\begin{array}{c} (\alpha_i, \alpha_i') \\
\in \mathbb{N}, 6\alpha_2 + 12\alpha_3 = n, \ 6\alpha_1' + 8\alpha_2' + 12\alpha_3' = n
\end{array}$$

$$I = \{(\alpha_1, \alpha_1', \alpha_2, \alpha_2', \alpha_3, \alpha_3')|\begin{array}{c} (\alpha_i, \alpha_i') \\
\in \mathbb{N}, 6\alpha_1' + 8\alpha_2' + 12\alpha_3' = n
\end{array}\}$$

$$J = \{(\beta_1, \beta_1', \beta_2, \beta_2', \beta_3, \beta_3')|\begin{array}{c} (\beta_i, \beta_i') \\
\in \mathbb{N}, 12\beta_1 + 20\beta_2 + 30\beta_3 = n, \ 12\beta_1' + 20\beta_2' + 30\beta_3' = n
\end{array}\}$$

Proof. Put

$$\phi(p) = p'(z_0, z_1, z_2, z_3).$$

An element $g = (g_1, g_2)$ in $G_6$ or $G_{12}$ operates on $(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}_3$ by the matrix multiplication

$$g_1\left(\begin{array}{cc}
x_0 + ix_1 & x_2 + ix_3 \\
-x_2 + ix_3 & x_0 - ix_1
\end{array}\right) g_2^{-1}$$

and on the matrix $Z$ of $[2]$ by

$$g_1\left(\begin{array}{cc}
z_0 z_2 & z_0 z_3 \\
z_1 z_2 & z_1 z_3
\end{array}\right) g_2^{-1} = g_1\left(\begin{array}{c}
z_0 \\
z_1
\end{array}\right) \cdot \left(\begin{array}{c}
z_2 \\
z_3
\end{array}\right) g_2^{-1}.$$}

Clearly if $p$ is $G_6$- or $G_{12}$-invariant then also the projection $\phi(p)$ with the previous operation is. In particular for $g = (g_1, 1)$ in $\bar{T} \times 1$, resp. in $\bar{I} \times 1$ the polynomial $p'$ is $\bar{T} \times 1$-, respectively $\bar{I} \times 1$-invariant as polynomial in the coordinates $(z_0 : z_1) \in \mathbb{P}_1$ and for any $(z_2 : z_3) \in \mathbb{P}_1$. On the other hand for $g = (1, g_2)$ in $1 \times \bar{T}$, resp. in $1 \times \bar{I}$ the polynomial $p'$ is $1 \times \bar{T}$-, respectively $1 \times \bar{I}$-invariant as polynomial in the coordinate $(z_2 : z_3) \in \mathbb{P}_1$ and for any $(z_0 : z_1) \in \mathbb{P}_1$. Hence $p'$ must be in the form of the statement.

By a direct computation in section 4 we prove the following

**Proposition 3.2** The quadric $q$ does not divide the polynomials $F_1$, $\Gamma_j$. Moreover, $F_6$ does not divide $F_{12}$.

**Corollary 3.1** We have $\phi(q) = 0$, $\phi(F_6) = t_1 \cdot t_2$, $\phi(F_8) = W_1 \cdot W_2$, $\phi(F_{12}) = \chi_1 \cdot \chi_2$, $\phi(\Gamma_{12}) = f_1 \cdot f_2$, $\phi(\Gamma_{20}) = H_1 \cdot H_2$, $\phi(\Gamma_{30}) = T_1 \cdot T_2$ (up to some scalar factor).
Proof. This follows from Proposition 3.1 and 3.2.

**Proposition 3.3** The polynomials $q, F_6, F_8, F_{12}$, resp. $q, \Gamma_{12}, \Gamma_{20}, \Gamma_{30}$ are algebraically independent.

Proof. Let $\sum_I \alpha_I q^{i_1} F_6^{i_2} F_8^{i_3} F_{12}^{i_4} = 0$ and $\sum_J \beta_J q^{j_1} \Gamma_{12}^{j_2} \Gamma_{20}^{j_3} \Gamma_{30}^{j_4} = 0$ be algebraic relations, $I = (i_1, i_2, i_3, i_4) \in \mathbb{N}^4$, $J = (j_1, j_2, j_3, j_4) \in \mathbb{N}^4$, $\alpha_I, \beta_J \in \mathbb{R}$, then

$$0 = \phi(\sum_I \alpha_I q^{i_1} F_6^{i_2} F_8^{i_3} F_{12}^{i_4}) = \sum_I' \alpha_I' t_1^{i_2} t_2^{i_3} W_1^{i_4} W_2^{i_4} \chi_1 \chi_2$$

similarly

$$0 = \phi(\sum_J \beta_J q^{j_1} \Gamma_{12}^{j_2} \Gamma_{20}^{j_3} \Gamma_{30}^{j_4}) = \sum_J' \beta_J' f_1^{j_2} f_2^{j_3} H_1^{j_4} H_2^{j_4} T_1^{j_4} T_2^{j_4}.$$ 

If the polynomials $t_1, W_1, \chi_1$ are fixed, we obtain a relation between $t_2, W_2$ and $\chi_2$, which is the same relation as for $t_1, W_1$ and $\chi_1$ if we fix $t_2, W_2$ and $\chi_2$. The same holds for the polynomials $f_1, H_1, T_1$ and $f_2, H_2, T_2$. From [K] p. 55 and p. 57 there are only the relations

$$108 t_1^4 - W_1^3 + \chi_1^2 = 0, \quad 108 t_2^4 - W_2^3 + \chi_2^2 = 0$$

and

$$T_1^2 + H_1^3 - 1728 f_1^5 = 0, \quad T_2^2 + H_2^3 - 1728 f_2^5 = 0$$

between these polynomials. By multiplying these relations, however, it is not possible to obtain expressions like (4) and (5). □

**Corollary 3.2** The polynomials $q, F_6, F_8, F_{12}$, resp. $q, \Gamma_{12}, \Gamma_{20}, \Gamma_{30}$ generate rationally the ring of invariant polynomials of $[3, 4, 3]$, resp. $[3, 3, 5]$.

Proof. (cf. [Co1] p. 775) By Proposition 3.3 and Proposition 3.2 these are algebraically independent, moreover the products of their degrees are

$$2 \cdot 6 \cdot 8 \cdot 12 = 1152 \quad \text{and} \quad 2 \cdot 12 \cdot 20 \cdot 30 = 14400,$$

which are equal to the order of the groups $[3, 4, 3]$ and $[3, 3, 5]$. By [Co1] this implies the assertion. □
4 Explicit computations

We recall the following matrices of $SO(4)$ (cf. [Sa]).

\[
(q_2, 1) = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}, \quad (1, q_2) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
\]

\[
(p_3, 1) = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
-1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1
\end{pmatrix}, \quad (1, p_3) = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 & -1 & 1 \\
-1 & 1 & -1 & -1 \\
1 & 1 & 1 & -1 \\
-1 & 1 & 1 & 1
\end{pmatrix},
\]

\[
(p_4, 1) = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 1 & 1
\end{pmatrix}, \quad (1, p_4) = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 1 & 1
\end{pmatrix},
\]

\[
(p_5, 1) = \frac{1}{2} \begin{pmatrix}
\tau & 0 & 1 - \tau & -1 \\
0 & \tau & -1 & \tau - 1 \\
\tau - 1 & 1 & \tau & 0 \\
1 & 1 - \tau & 0 & \tau \\
\tau & 0 & \tau - 1 & 1 \\
0 & \tau & -1 & \tau - 1 \\
1 - \tau & 1 & \tau & 0 \\
-1 & 1 - \tau & 0 & \tau
\end{pmatrix}, \\
(1, p_5) = \frac{1}{2} \begin{pmatrix}
\tau & 0 & 1 - \tau & -1 \\
0 & \tau & -1 & \tau - 1 \\
\tau - 1 & 1 & \tau & 0 \\
1 & 1 - \tau & 0 & \tau \\
-1 & 1 - \tau & 0 & \tau
\end{pmatrix},
\]

where $\tau = \frac{1}{2}(1 + \sqrt{5})$. Then we have

| Group | Generators |
|-------|------------|
| $G_6$ | $(q_2, 1), (1, q_2), (p_3, 1), (1, p_3)$ |
| $G_8$ | $(q_2, 1), (1, q_2), (p_3, 1), (1, p_3), (p_4, 1), (1, p_4)$ |
| $G_{12}$ | $(q_2, 1), (1, q_2), (p_3, 1), (1, p_3), (p_5, 1), (1, p_5)$ |

Now we can write down the equations of the fix lines on $\mathbb{P}_1 \times \mathbb{P}_1$ and those of the planes which are generated by a couple of lines. The products of planes
of section 2.1 in the case of the group $G_6$ are

$$
\begin{align*}
\sum_{i=1}^{6} f_i^{(6)} & = (x_2 - i x_3)(x_1 + i x_3)(x_2 + i x_3)(x_1 - i x_3)(x_1 + i x_2), \\
\sum_{i=1}^{8} f_i^{(8)} & = (x_1 + a x_2 - b x_3)(x_1 + b x_2 - a x_3)(x_1 - a x_2 - b x_3)(x_1 - a x_3 - b x_2) \\
\sum_{i=1}^{12} f_i^{(12)} & = (x_3 - x_1 + c x_2)(x_3 - x_1 - c x_2)(x_2 + x_3 + c x_1)(x_2 + x_3 - c x_1) \\
& \quad (x_3 - x_2 + c x_1)(x_3 - x_2 - c x_1)(x_1 + x_2 + c x_3)(x_1 + x_2 - c x_3) \\
& \quad (x_1 + x_3 - c x_2)(x_1 + x_3 + c x_2)(x_1 - x_2 + c x_3)(x_1 - x_2 - c x_3),
\end{align*}
$$

with $a = (1/2)(1 + i \sqrt{3})$, $b = (1/2)(1 - i \sqrt{3})$, $c = i \sqrt{2}$.

Then the $G_6$-invariant polynomials $F_6$, $F_8$ and $F_{12}$ have the following expressions

$$
\begin{align*}
F_6 & = x^6 + x^6 + x^6 + 5x^2 x_1^2 (x_1^2 + x_1^2) + 5x^2 x_3^2 (x_1^2 + x_3^2) + 5x^2 x_2^2 (x_1^2 + x_2^2) \\
& + 6x_0 x_2^2 (x_2^2 + x_2^2) + 6x_0 x_3^2 (x_0^2 + x_3^2) + 6x_3 x_2^2 (x_2^2 + x_3^2) + 2x_0 x_2^2 x_3^2, \\
F_8 & = 3 \sum x^8 + 12 \sum x^6 x^2 + 30 \sum x^4 x^4 + 24 \sum x^4 x^4 x^2 + 144 x_0^2 x_1^2 x_2^2 x_3^2, \\
F_{12} & = \frac{123}{8} \sum x_1^{12} + \frac{231}{4} \sum x_i^{10} x^2 + \frac{21}{8} \sum x_i^8 x^4 - \frac{255}{2} \sum x_i^6 x^6 + \frac{949}{2} \sum x_i^8 x^2 x^2 \\
& + \frac{6111}{4} \sum x_i^8 x^4 x^4 + 1809 \sum x_i^6 x^2 x^2 x^2 + \frac{7281}{2} \sum x_i^8 x^2 x^2 x^2.
\end{align*}
$$

Here the sums run over all the indices $i, j, k, h = 0, 1, 2, 3$, always being different when appearing together. By applying the map $\phi$, a computer computation with MAPLE shows that

$$
\begin{align*}
\phi(F_6) & = -\frac{13}{16} t_1 \cdot t_2, \\
\phi(F_8) & = \frac{3}{64} W_1 \cdot W_2, \\
\phi(F_{12}) & = \frac{3}{256} \chi_1 \cdot \chi_2
\end{align*}
$$

as claimed in Corollary 3.3.

**Proof of Proposition 2.1.** The polynomials $F_6$, $F_8$, $F_{12}$ remain invariant by interchanging $x_2$ with $x_3$, which is what the matrix $C'$ does.

**Proof of Proposition 3.2.** We write the computations just in the case of the $[3, 4, 3]$-invariant polynomials. Consider the points $p_1 = (i \sqrt{2} : 1 : 1 : 0)$ and $p_2 = (1 : i : 0 : 0)$, then $q(p_1) = q(p_2) = 0$ and by a computer computation with MAPLE we get $F_6(p_1) = 26$, $F_8(p_2) = 12$ and $F_{12}(p_2) = 32$. This shows that $q$ does not divide the polynomials. Since $F_6(p_2) = 0$, $F_6$ does not divide $F_{12}$.
Remark 4.1 Observe that an equation for a \([3, 4, 3]\)-invariant polynomial of degree six and for a \([3, 3, 5]\)-invariant polynomial of degree twelve was given by the author in \([Sa]\) by a direct computer computation with MAPLE.

References

[Bu] Burnside, W.: *Theory of groups of finite order*, Dover Publications, Inc. (1955).

[Co1] Coxeter, H. S. M.: *The product of the generators of a finite group generated by reflections*, Duke Math. J. Vol. 18 (1951) 765-782.

[Co2] Coxeter, H. S. M.: *Regular polytopes (second edition)*, The Macmillan company, New York (1963).

[K] Klein, F.: *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, Nachdr. der Ausg. Leipzig, Teubner 1884, hrsg. mit einer Einführung und mit Kommentaren von Peter Slodowy, Birkhäuser-B. G. Teubner (1993).

[Ra] Racah, G.: *Sulla caratterizzazione delle rappresentazioni irriducibili dei gruppi semisemplici di Lie*, Rend. Acad. Naz. dei Lincei, Classe di Scienze fisiche, matematiche e naturali (8), vol. 8 (1950) 108-112.

[Sa] Sarti, A.: *Pencils of symmetric surfaces in \(\mathbb{P}^3(\mathbb{C})\)*, J. of Alg. 246, 429–452 (2001).