Isothermic surfaces in sphere geometries as Moutard nets

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Abstract. We give an elaborated treatment of discrete isothermic surfaces and their analogs in different geometries (projective, Möbius, Laguerre, Lie). We find the core of the theory to be a novel projective characterization of discrete isothermic nets as Moutard nets. The latter belong to projective geometry and are nets with planar faces defined through a five-point property: a vertex and its four diagonal neighbors span a three-dimensional space. Analytically this property is equivalent to the existence of representatives in the space of homogeneous coordinates satisfying the discrete Moutard equation. Restricting the projective theory to quadrics, we obtain Moutard nets in sphere geometries.

In particular, Moutard nets in Möbius geometry are shown to coincide with discrete isothermic nets. The five-point property in this particular case says that a vertex and its four diagonal neighbors lie on a common sphere, which is a novel characterization of discrete isothermic surfaces. Discrete Laguerre isothermic surfaces are defined through the corresponding five-plane property which requires that a plane and its four diagonal neighbors share a common touching sphere. Equivalently, Laguerre isothermic surfaces are characterized by having an isothermic Gauss map. We conclude with Moutard nets in Lie geometry.

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1 Introduction

This paper is a sequel to our paper “On organizing principles of discrete differential geometry. Geometry of spheres” [BS2], where the following discretization principles have been formulated:

- **Transformation group principle**: smooth geometric objects and their discretizations belong to the same geometry, i.e. are invariant with respect to the same transformation group.

- **Consistency principle**: discretizations of smooth parametrized geometries can be extended to multidimensional consistent nets.

Being applied to discretization of curvature line parametrizations of general surfaces, these principles led to the definition of principal contact element nets. These are nets of contact elements with the property that neighboring contact elements share a common sphere. In particular, it was shown that the points and the planes of principal contact element nets build circular and conical nets, respectively.

In the present paper we turn to isothermic surfaces which is a special class of surfaces admitting a conformal curvature line parametrization. This important class of surfaces has been studied by classics [Da]. In the 1990-s a relation to the theory of integrable systems has been discovered [CGS, BHPP, BP1]. An overview of the modern theory of isothermic surfaces can be found in [HJ]. The theory has been extended for isothermic surfaces in spaces of arbitrary dimension in [Bu, Sch]. In [BP1] the theory has been discretized: discrete isothermic surfaces were defined as special circular nets with factorized cross-ratios of elementary quadrilaterals. This property is manifestly Möbius-invariant. Moreover, it can be consistently imposed on three-dimensional nets [HHP, BP2]. Thus, discrete isothermic surfaces is an instance of geometry satisfying both discretization principles.

In this paper, we give an elaborated treatment of discrete isothermic surfaces and their analogs in different geometries (projective, Möbius, Laguerre, Lie), applying the discretization principles systematically. We find the core of the theory to be a novel projective characterization of discrete isothermic nets as Moutard nets. The latter belong to projective geometry and are nets with planar faces \( f : \mathbb{Z}^2 \to \mathbb{R}^N \) defined through a five-point property: a vertex and its four diagonal neighbors span a three dimensional space (thus, in comparison with a generic net with planar faces, the dimension drops by one). Analytically this property is equivalent to the existence of representatives in the space of homogeneous coordinates \( y : \mathbb{Z}^2 \to \mathbb{R}^{N+1} \)
satisfying the \emph{Moutard equation}

\[ \tau_1 \tau_2 y - y \parallel \tau_2 y - \tau_1 y, \]

where \( \tau_i \) is the shift in the \( i \)-th coordinate direction. The consistency of the five-point property follows from the consistency of the Moutard equation.

Restricting the projective theory to quadrics, we obtain Moutard nets in sphere geometries. In particular, Moutard nets in Möbius geometry are shown to coincide with discrete isothermic nets. The five-point property in this particular case says that a vertex and its four diagonal neighbors lie on a common sphere, which is a novel characterization of discrete isothermic surfaces.

In Laguerre geometry discrete surfaces are maps \( \mathbb{Z}^2 \to \{ \text{planes in } \mathbb{R}^3 \} \). The planarity of faces in the Laguerre quadric is equivalent to the conical property, while the five-plane property requires that a plane and its four diagonal neighbors share a common touching sphere. This is a definition of discrete Laguerre isothermic surfaces. Equivalently, discrete Laguerre isothermic surfaces are characterized by having an isothermic Gauss map. The latter class was independently introduced in \[ WP \]. Smooth Laguerre isothermic surfaces have been studied in \[ MN1 \] \begin{itemize} \item MN2 \end{itemize}.

We conclude with Moutard nets in Lie geometry. The latter are special Ribaucour sphere congruences \( \mathbb{Z}^2 \to \{ \text{spheres in } \mathbb{R}^3 \} \) with the corresponding five-sphere property. A particular case is S-isothermic surfaces \[ BP2 \] \begin{itemize} \item BHS \item ho \end{itemize}.

\section{Discrete Moutard nets}

In considerations of various nets \( f : \mathbb{Z}^2 \to X \), we use the following notational conventions: for some fixed \( u \in \mathbb{Z}^2 \), we write \( f \) for \( f(u) \), and further \( f_i \) for \( \tau_i f(u) = f(u + e_i) \), and \( f_{-i} \) for \( \tau_i^{-1} f(u) = f(u - e_i) \). Also, we freely use notations, definitions and results from \[ BS2 \].

\begin{definition}[Discrete Moutard net] A two-dimensional \( Q \)-net \( f : \mathbb{Z}^2 \to \mathbb{R}P^N \) \((N \geq 4)\) is called a discrete Moutard net, if for any \( u \in \mathbb{Z}^2 \) the five points \( f \) and \( f_{\pm 1, \pm 2} \) lie in a three-dimensional subspace \( V \subset \mathbb{R}P^N \), not containing some (and then any) of the four points \( f_{\pm 1}, f_{\pm 2} \).
\end{definition}

Thus, the defining condition of a discrete Moutard net deals with four elementary planar quadrilaterals adjacent to one vertex. As a consequence of this definition, all nine vertices of such four quadrilaterals of a discrete Moutard net lie in a four-dimensional subspace of \( \mathbb{R}P^N \).
Theorem 2 (Discrete Moutard equation) A discrete Moutard net \( f : \mathbb{Z}^2 \to \mathbb{R}P^N \) possesses a lift to the homogeneous coordinates space \( y : \mathbb{Z}^2 \to \mathbb{R}^{N+1} \) satisfying the discrete Moutard equation:

\[
y_{12} - y = a_{12}(y_2 - y_1),
\]

(1)

with some \( a_{12} : \mathbb{Z}^2 \to \mathbb{R} \) (it is natural to assign the real numbers \( a_{12} \) to the elementary squares of \( \mathbb{Z}^2 \)).

Proof. We start with the observation that for any \( Q \)-net \( f \) in \( \mathbb{R}P^N \) it is always possible (and almost trivial) to find homogeneous coordinates for the four vertices of one elementary quadrilateral satisfying the discrete Moutard equation on that quadrilateral. Moreover, one can do this for an arbitrary choice of homogeneous coordinates for any two neighboring vertices of the quadrilateral. Indeed, consider any homogeneous coordinates \( \tilde{f}, \tilde{f}_1, \tilde{f}_2, \tilde{f}_{12} \in \mathbb{R}^{N+1} \) for the vertices of a planar quadrilateral, connected by a linear relation

\[
\tilde{f}_{12} = c_{21}\tilde{f}_1 + c_{12}\tilde{f}_2 + \rho_{12}\tilde{f}.
\]

Keep the representatives \( y = \tilde{f}, y_1 = \tilde{f}_1 \), say, and set \( \tilde{f}_{12} = \rho_{12}y_{12} \) and \( \tilde{f}_2 = ay_2 \) with \( a = -c_{21}/c_{12} \), then \( y \) satisfies the discrete Moutard equation within one elementary quadrilateral.

Now, for \( Q \)-nets with a special property formulated in Definition I this construction can be extended to the whole net. Start with arbitrary representatives \( y, y_1 \), and proceed clockwise around the vertex \( y \). We find consecutively: the representatives \( y_{-2}, y_{1,-2} \) which assure the Moutard equation on the quadrilateral \( (y, y_1, y_{1,-2}, y_{-2}) \), then the representatives \( y_{-1}, y_{1,-2} \) which assure the Moutard equation on the quadrilateral \( (y, y_{-1}, y_{1,-2}, y_{-2}) \), and then the representatives \( y_2, y_{-1,2} \) which assure the Moutard equation on the quadrilateral \( (y, y_{-1}, y_{-1,2}, y_2) \), see Fig. I. In the remaining quadrilateral \( (y, y_1, y_{12}, y_2) \), the representatives \( y, y_1, y_2 \) are already fixed on the previous steps of the construction, so that we can dispose of the representative \( y_{12} \) of \( f_{12} \) only. Observe that the point with the representative \( y_1 - y_2 \) belongs to the the plane \( \Pi \subset \mathbb{R}P^N \) of the quadrilateral \( (f, f_1, f_{12}, f_2) \) (obviously) and to the three-dimensional space \( V \subset \mathbb{R}P^N \) through the points \( f, f_{1,-2}, f_{-1,-2}, f_{-1,2} \), because of the equation

\[
y_1 - y_2 = (y_1 - y_2) + (y_{-2} - y_{-1}) + (y_{-1} - y_2)
\]

\[
= \alpha(y_{1,-2} - y) + \beta(y_{-1,-2} - y) + \gamma(y_{-1,2} - y).
\]

By the hypothesis of the theorem, the point \( f_{1,2} \) lies in the latter space \( V \). Therefore, the whole line through \( f \) and \( f_{1,2} \) lies in the intersection \( \Pi \cap V \).
Since \( N \geq 4 \), we conclude that in general position \( \Pi \cap V \) is the line through \( f \) and \( f_{12} \). Thus, the point with the representative \( y_1 - y_2 \) belongs to this line, therefore \( y_1 - y_2 \) is a linear combination of \( y \) and \( y_{12} \). By a suitable choice of representative \( y_{12} \) of \( f_{12} \), we can make \( y_1 - y_2 \) proportional to \( y_{12} - y \). Thus, the construction of representatives satisfying the Moutard equation closes up around any vertex. This allows to extend the construction to the whole lattice \( \mathbb{Z}^2 \). □

Definition \( \Pi \) is non-applicable in the case when some, and then all of the points \( f_{\pm 1}, f_{\pm 2} \) lie in the three-dimensional space \( V \) through \( f, f_{\pm 1, \pm 2} \); in particular, it cannot be used to define discrete Moutard nets in \( \mathbb{RP}^3 \). We will show that Theorem \( \Box \) remains valid if one defines discrete Moutard nets in \( \mathbb{RP}^3 \) as follows.

**Definition 3 (Discrete Moutard net in \( \mathbb{RP}^3 \))** A two-dimensional Q-net \( f : \mathbb{Z}^2 \to \mathbb{RP}^3 \) is called a discrete Moutard net, if for any \( u \in \mathbb{Z}^2 \) the following condition is satisfied: the three planes

\[
\Pi^{(\text{up})} = (f, f_{12}, f_{-1,2}), \quad \Pi^{(\text{down})} = (f, f_{1,-2}, f_{-1,-2}), \quad \Pi^{(1)} = (f, f_1, f_{-1})
\]

have a common line \( \ell^{(1)} \).

**Remark 1.** It is not difficult to see that in the context of Definition \( \Box \) with \( N \geq 4 \) the requirement of Definition \( \Box \) is automatically satisfied. Indeed, in this case all nine points \( f, f_{\pm 1}, f_{\pm 2} \) and \( f_{\pm 1, \pm 2} \) lie in a four-dimensional subspace of \( \mathbb{RP}^N \). In this subspace one can consider, along
with the three-dimensional subspace $V$, the three-dimensional subspaces $V^{(\text{up})}$ containing the two quadrilaterals $(f, f_1, f_{12}, f_2)$, $(f, f_{-1}, f_{-1,2}, f_2)$, and $V^{(\text{down})}$ containing the quadrilaterals $(f, f_1, f_{1,-2}, f_{-2})$, $(f, f_{-1}, f_{-1,-2}, f_{-2})$. Obviously, one has:

$$\Pi^{(\text{up})} = V^{(\text{up})} \cap V, \quad \Pi^{(\text{down})} = V^{(\text{down})} \cap V, \quad \Pi^{(1)} = V^{(\text{up})} \cap V^{(\text{down})}.$$ 

Generically, three three-dimensional subspaces $V$, $V^{(\text{up})}$ and $V^{(\text{down})}$ of a four-dimensional space intersect along a line $\ell^{(1)}$.

**Remark 2.** There is an asymmetry between the coordinate directions 1 and 2 in Definition 3. However, this asymmetry is apparent: the condition in Definition 3 is equivalent to the requirement that the three planes

$$\Pi^{(\text{left})} = (f, f_{-1,2}, f_{-1,-2}), \quad \Pi^{(\text{right})} = (f, f_{1,2}, f_{1,-2}), \quad \Pi^{(2)} = (f, f_2, f_{-2})$$

have a common line $\ell^{(2)}$. One way to see this is to consider a central projection of the whole picture from the point $f$ to some plane not containing $f$. In this projection, the planarity of elementary quadrilaterals $(f, f_i, f_{ij}, f_j)$ turns into collinearity of the triples of points $f_i$, $f_j$ and $f_{ij}$. The traces of the planes $\Pi^{(\text{up})}$, $\Pi^{(\text{down})}$ and $\Pi^{(1)}$ on the projection plane are the lines $(f_{12}, f_{-1,2})$, $(f_{1,-2}, f_{1,-2})$, and $(f_1, f_{-1})$, respectively, and the requirement of Definition 3 turns into the requirement for these three lines to meet in a point. Similarly, the traces of the planes $\Pi^{(\text{left})}$, $\Pi^{(\text{right})}$ and $\Pi^{(2)}$ on the projection plane are the lines $(f_{-1,2}, f_{-1,-2})$, $(f_{1,2}, f_{1,-2})$, and $(f_2, f_{-2})$, respectively. The requirement for the latter three lines to meet in a point is equivalent to the previous one – this is the statement of the famous Desargues theorem, see Fig. 2.

Another way to demonstrate the actual symmetry between the coordinate directions 1 and 2 in Definition 3 is to show that Theorem 2 still holds in $\mathbb{R}P^3$. Indeed, the discrete Moutard equation (1) is manifestly symmetric with respect to the flip $1 \leftrightarrow 2$.

**Proof of Theorem 2 for $N = 3$.** We start the proof exactly as in the general case $N \geq 4$. The only thing to be changed is the demonstration of the fact that the point with the representative $y_2 - y_1$ lies on the line through $f$ and $f_{12}$. To do this in the present situation, we first observe that, due to

$$y_1 - y_{-1} = (y_1 - y_{-2}) + (y_{-2} - y_{-1}) = \alpha(y_{1,-2} - y) + \beta(y_{-1,-2} - y),$$

the point with the representative $y_1 - y_{-1}$ lies in the plane $\Pi^{(\text{down})}$. Obviously, it lies also in $\Pi^{(1)}$. Therefore, it lies on the line $\ell^{(1)}$. As a consequence
of the property of Definition 3, it belongs also to the plane $\Pi^{(up)}$. Now from

$$y_2 - y_1 = (y_2 - y_{-1}) - (y_1 - y_{-1}) = \gamma(y_{-1,2} - y) - (y_1 - y_{-1})$$

we find that the point with the representative $y_2 - y_1$ belongs to $\Pi^{(up)}$, as well. Since the point with the representative $y_2 - y_1$ also belongs (obviously) to the plane of the quadrilateral $(f, f_1, f_{12}, f_2)$, we conclude that it lies in the intersection of the latter plane with $\Pi^{(up)} = (f, f_{12}, f_{-1,2})$, which is, in the generic case, the line through $f$ and $f_{12}$. □

Definitions 1, 3 are essentially dealing with two-dimensional Q-nets. However, the characterization of discrete Moutard nets given in Theorem 2 opens a way to define multi-dimensional Moutard nets, and, in particular, to define transformations of Moutard nets with remarkable permutability properties. Namely, it turns out that eq. 1 can be posed on multi-dimensional lattices.

**Definition 4 (T-net)** A map $y : \mathbb{Z}^m \to \mathbb{R}^N$ is called an $m$-dimensional T-net (trapezoidal net), if for any $u \in \mathbb{Z}^m$ and for any pair of indices $i \neq j$ there holds the discrete Moutard equation

$$y_{ij} - y = a_{ij}(y_j - y_i),$$

(2)

with some $a_{ij} : \mathbb{Z}^m \to \mathbb{R}$, in other words, if all elementary quadrilaterals $(y, y_i, y_{ij}, y_j)$ are planar and have parallel diagonals.
Of course, coefficients $a_{ij}$ have to be skew-symmetric, $a_{ij} = -a_{ji}$. We show that three-dimensional T-nets are described by a well-defined three-dimensional system.

**Theorem 5 (Elementary hexahedron of a T-net)** Given seven points $y, y_i, y_{ij}$ (1 ≤ $i \neq j ≤ 3$) in $\mathbb{R}^N$, such that eq. (2) is satisfied on the three quadrilaterals $(y, y_i, y_{ij}, y_j)$ adjacent to the vertex $y$, there exists a unique point $y_{123}$ such that eq. (2) is satisfied on the three quadrilaterals $(y_i, y_{ij}, y_{123}, y_{ik})$ adjacent to the vertex $y_{123}$.

**Proof.** Three equations (2) for the faces of an elementary cube of $\mathbb{Z}^3$ adjacent to $y_{123}$ give:

$$\tau_i y_{jk} = (1 + (\tau_i a_{jk})(a_{ij} + a_{ki}))(y_i) - (\tau_i a_{jk})a_{ij}y_j - (\tau_i a_{jk})a_{ki}y_k.$$  

They lead to consistent results for $y_{123}$ for arbitrary initial data, if and only if the following conditions are satisfied:

$$1 + (\tau_1 a_{23})(a_{12} + a_{31}) = -(\tau_2 a_{31})a_{12} = -(\tau_3 a_{12})a_{31},$$
$$1 + (\tau_2 a_{31})(a_{23} + a_{12}) = -(\tau_3 a_{12})a_{23} = -(\tau_1 a_{23})a_{12},$$
$$1 + (\tau_3 a_{12})(a_{23} + a_{31}) = -(\tau_1 a_{23})a_{31} = -(\tau_2 a_{31})a_{23}.$$  

These conditions constitute a system of 6 (linear) equations for 3 unknown variables $\tau_i a_{jk}$ in terms of the known ones $a_{jk}$. A direct computation shows that this system is not overdetermined but admits a unique solution:

$$\frac{\tau_1 a_{23}}{a_{23}} = \frac{\tau_2 a_{31}}{a_{31}} = \frac{\tau_3 a_{12}}{a_{12}} = -\frac{1}{a_{12}a_{23} + a_{23}a_{31} + a_{31}a_{12}}.$$  

(3)

With $\tau_i a_{jk}$ so defined, eqs. (2) are fulfilled on all three quadrilaterals adjacent to $y_{123}$. □

Eqs. (3) represent a well-defined birational map $\{a_{jk}\} \mapsto \{\tau_i a_{jk}\}$, which can be considered as the fundamental 3D system related to T-nets. It is sometimes called the "star-triangle map".

Theorem 5 means that the defining condition of T-nets (parallel diagonals of elementary planar quadrilaterals) yields a discrete 3D system with fields on vertices taking values in an affine space $\mathbb{R}^N$. This system can be considered as an admissible reduction of the 3D system describing Q-nets in $\mathbb{R}^N$. Indeed, if one has an elementary hexahedron of an affine Q-net $y : \mathbb{Z}^3 \rightarrow \mathbb{R}^N$ such that its elementary quadrilaterals $(y, y_i, y_{ij}, y_j)$ have parallel diagonals, then the elementary quadrilaterals $(y_i, y_{ij}, y_{123}, y_{ik})$ have this property, as well. To see this, observe that the point $y_{123}$ from Theorem 5
satisfies the planarity condition, and therefore it has to coincide with the unique point defined by planarity of the quadrilaterals \((y_i, y_{ij}, y_{123}, y_{ik})\).

The 4D consistency of T-nets is a consequence of the analogous property of Q-nets, since T-constraint propagates in the construction of a Q-net from its coordinate surfaces. On the level of formulas we have for T-nets with \(m \geq 4\) the system \(2\), while the map \(\{a_{jk}\} \mapsto \{\tau_i a_{jk}\}\) is given by

\[
\frac{\tau_i a_{jk}}{a_{jk}} = -\frac{1}{a_{ij}a_{jk} + a_{jka_{ki}} + a_{ki}a_{ij}}.
\]

(4)

All indices \(i, j, k\) vary now between 1 and \(m\), and for any triple of pairwise different indices \((i, j, k)\), equations involving these indices solely, form a closed subset.

The multidimensional consistency of T-nets yields in a usual fashion Darboux transformations with permutability properties (which in the present context should be called discrete Moutard transformations). We refer to \([BS1]\) for the background on the relation of multidimensional consistency to Darboux transformations, and give here only the formulas for the discrete Moutard transformation of eq. \(2\) into

\[
y^+_{ij} - y^+_i = a^+_{ij}(y^+_j - y^+_i).
\]

(5)

These formulas read:

\[
y^+_i - y = b_i(y^+_i - y_i),
\]

(6)

where the quantities \(b_i\) and the transformed coefficients \(a^+_{ij}\) are defined by equations

\[
\frac{\tau_i b_j}{b_j} = \frac{a^+_{ij}}{a_{ij}} = \frac{1}{(b_i - b_j)a_{ij} + b_ib_j}.
\]

(7)

It is not difficult to recognize in eqs. \(5\), \(6\) the same Moutard equations \(2\) on the \((m + 1)\)-dimensional lattice, with the superscript "\(^+\)" used to denote the shift \(\tau_{m+1}\). Similarly, eqs. \(7\) are nothing but the star-triangle formulas \(4\) with \(b_i = a_{i, m+1}\).

\textbf{Remark.} We learned about the projective five-point characterization of two-dimensional Moutard nets from conversations with A. Doliwa. It should be noted that three- and higher-dimensional Moutard nets admit a different projective characterization (planarity of tetrahedra formed by odd or by even vertices of any elementary cube), see \([Do3]\). Also the paper \([Do2]\) by A. Doliwa deals with a closely related notion of discrete Koenigs nets.
2.1 Discrete Moutard nets in quadrics

We have seen that discrete Moutard nets (or, more precisely, their T-net representatives) constitute an admissible reduction of Q-nets. The restriction to a quadric constitutes another admissible reduction [Do1]. Imposing two admissible reductions simultaneously, one comes to T-nets in quadrics.

Let $\mathbb{R}^N$ be equipped with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ (which does not need to be positive-definite), and let

$$Q = \{ y \in \mathbb{R}^N : \langle y, y \rangle = \kappa_0 \}$$

be a quadric in $\mathbb{R}^N$. We study T-nets $y : \mathbb{Z}^m \to Q$. This leads to a *discrete 2D system*, since constructing elementary quadrilaterals of T-nets in $Q$ corresponding to elementary squares of the lattice $\mathbb{Z}^m$ admits a well-posed initial value problem: given three points $y, y_1, y_2 \in Q$, one finds a unique fourth point $y_{12} \in Q$, $y_{12} \neq y$, satisfying the discrete Moutard equation

$$y_{12} - y = a_{12}(y_2 - y_1).$$

Indeed, the condition

$$\langle y_{12}, y_{12} \rangle = \langle y + a_{12}(y_2 - y_1), y + a_{12}(y_2 - y_1) \rangle = \kappa_0$$

leads to a quadratic equation for $a_{12}$, which has one trivial solution $a_{12} = 0 \iff y_{12} = y$, and one non-trivial:

$$a_{12} = \frac{\langle y, y_1 - y_2 \rangle}{\kappa_0 - \langle y_1, y_2 \rangle}.$$

This elementary construction step, i.e., finding the fourth vertex of an elementary quadrilateral out of the known three vertices, is symbolically represented on Fig. 3.

![Figure 3: 2D system on an elementary quadrilateral](image-url)
Turning to an elementary cube of dimension $m \geq 3$, we see that one can prescribe all points $y$ and $y_i$ for all $1 \leq i \leq m$. Indeed, these data are independent, and one can construct all other vertices of an elementary cube from these data, provided one does not encounter contradictions. To see the possible source of contradictions, consider in detail the case of $m = 3$. From $y$ and $y_i$ ($1 \leq i \leq 3$) one determines all $y_{ij}$ by

$$y_{ij} - y = a_{ij}(y_j - y_i), \quad a_{ij} = \frac{\langle y, y_i - y_j \rangle}{\kappa_0 - \langle y_i, y_j \rangle}$$

(8)

After that one has, in principle, three different ways to determine $y_{123}$, from three squares adjacent to this point; see Fig. 4. These three values for $y_{123}$ have to coincide, independently of initial conditions.

**Definition 6 (3D consistency)** A 2D system is called 3D consistent, if it can be imposed on all two-dimensional faces of an elementary cube of $\mathbb{Z}^3$.

![Figure 4: 3D consistency of 2D systems](image)

There holds a quite general theorem, analogous to Theorem 5 of [BS2]:

**Theorem 7 (3D consistency yields consistency in all higher dimensions)** Any 3D consistent discrete 2D system is also $m$-dimensionally consistent for all $m > 3$.

**Proof** goes by induction in $m$ and is analogous to the proof of Theorem 5 from [BS2].

**Theorem 8 (T-nets in quadrics are 3D consistent)** The 2D system governing T-nets in $Q$ is 3D consistent.
Proof. This can be checked by a tiresome computation, which can be however avoided by the following conceptual argument. T-nets in $\mathcal{Q}$ are a result of imposing two admissible reductions on Q-nets in $\mathbb{R}^N$, namely the T-reduction and the restriction to a quadric $\mathcal{Q}$. This reduces the effective dimension of the system by 1 (allows to determine the fourth vertex of an elementary quadrilateral from the three known ones), and transfers the original 3D equation into the 3D consistency of the reduced 2D equation. Indeed, after finding $y_{12}$, $y_{23}$ and $y_{13}$, one can construct $y_{123}$ according to the planarity condition (as intersection of three planes). Then both the T-condition and the Q-condition are fulfilled for all three quadrilaterals adjacent to $y_{123}$, according to Theorem 5 and the result of [Do1]. Therefore, these quadrilaterals satisfy our 2D system. □

We mention also an important property of T-nets in quadrics used in the sequel: the functions

$$\alpha_i = \langle y, y_i \rangle,$$

defined on edges of $\mathbb{Z}^m$ parallel to the $i$-th coordinate axes, satisfy

$$\tau_i \alpha_j = \alpha_j, \quad i \neq j,$$

i.e, any two opposite edges of any elementary square carry the same value of the corresponding $\alpha_i$. Indeed, equations

$$\langle y_{ij}, y_j \rangle = \langle y_i, y \rangle, \quad \langle y_{ij}, y_i \rangle = \langle y_j, y \rangle,$$

follow from $\S$ by a direct computation.

3 Isothermic surfaces in Möbius geometry

Definition 9 (Discrete isothermic surface) A two-dimensional circular net $f : \mathbb{Z}^2 \to \mathbb{R}^N$ is called a discrete isothermic surface, if the corresponding $\hat{f} : \mathbb{Z}^2 \to \mathbb{L}^{N+1,1}$ is a discrete Moutard net.

From Definitions $\S$ there follows a geometric characterization of discrete isothermic nets:

Theorem 10 (Central spheres for discrete isothermic nets) (a) A circular net $f : \mathbb{Z}^2 \to \mathbb{R}^N$ not lying on a two-dimensional sphere is a discrete isothermic net, if and only if for any $u \in \mathbb{Z}^2$ the five points $f$ and $f_{\pm1, \pm2}$ lie on a two-dimensional sphere not containing some (and then any) of the four points $f_{\pm1}, f_{\pm2}$.  

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(b) A circular net \( f: \mathbb{Z}^2 \to \mathbb{S}^2 \subset \mathbb{R}^N \) is a discrete isothermic net, if and only if for any \( u \in \mathbb{Z}^2 \) the three circles through \( f \),

\[
C^{(\text{up})} = \text{circle}(f, f_{12}, f_{-1,2}), \quad C^{(\text{down})} = \text{circle}(f, f_{1,-2}, f_{-1,-2}), \quad C^{(1)} = \text{circle}(f, f_1, f_{-1}),
\]

have one additional point in common, which is also equivalent for the three circles through \( f \),

\[
C^{(\text{left})} = \text{circle}(f, f_{-1,2}, f_{-1,-2}), \quad C^{(\text{right})} = \text{circle}(f, f_{1,2}, f_{1,-2}), \quad C^{(2)} = \text{circle}(f, f_2, f_{-2}),
\]

to have one additional point in common.

Figure 5: Four circles of a generic discrete isothermic surface, with a central sphere.

The cases a), b) of Theorem\(^{[10]}\) are illustrated on Figs.\(^{[5,6]}\) respectively.

Another characterization of discrete isothermic surfaces can be given in terms of the cross-ratios. Recall that for any four concircular points \( f, f_1, f_2, f_{12} \in \mathbb{R}^N \) their (real-valued) cross-ratio can be defined as

\[
q(f, f_1, f_{12}, f_2) = (f_1 - f)(f_{12} - f_1)^{-1}(f_{12} - f_2)(f_2 - f)^{-1}. \quad (11)
\]

Here multiplication is interpreted as the Clifford multiplication in the Clifford algebra \( \text{Cl} \left( \mathbb{R}^N \right) \). Recall that for \( x, y \in \mathbb{R}^N \) the Clifford product satisfies
$xy + yx = -2(x, y)$, and that the inverse element of $x ∈ \mathbb{R}^N$ in the Clifford algebra is given by $x^{-1} = -x/|x|^2$. Alternatively, one can identify the plane of the quadrilateral $(f, f_1, f_{12}, f_2)$ with the complex plane $\mathbb{C}$, and then interpret multiplication in eq. (11) as the complex multiplication. An important property of the cross-ratio is its invariance under Möbius transformations.

**Theorem 11 (Four cross-ratios of a discrete isothermic net)** A circular net $f : \mathbb{Z}^2 → \mathbb{R}^N$ is a discrete isothermic net, if and only if the cross-ratios $q = q(f, f_1, f_{12}, f_2)$ of its elementary quadrilaterals satisfy the following condition:

$$q \cdot q_{-1,-2} = q_{-1} \cdot q_{-2}. \quad (12)$$

Here, like in Sect. 2, the negative indices $-i$ are used to denote the backward shifts $\tau_i^{-1}$, so that, e.g., $q_{-1} = q(f_{-1}, f, f_2, f_{-1,2})$, see Fig. 7.

**Proof.** Perform a Möbius transformation sending $f$ to $∞$. Under such a transformation, the four adjacent circles through $f$ turn into four straight lines $f_{±1}f_{±2}$, containing the corresponding points $f_{±1,±2}$. Formula (12) turns into the following relation for the quotients of (directed) lengths:

$$\frac{|f_{2}f_{12}|}{|f_{12}f_{1}|} \cdot \frac{|f_{1}f_{1,-2}|}{|f_{1,-2}f_{-2}|} \cdot \frac{|f_{-2}f_{-1,-2}|}{|f_{-1,-2}f_{-1}|} \cdot \frac{|f_{-1}f_{-1,2}|}{|f_{-1,2}f_{2}|} = 1. \quad (13)$$

If the affine space through the points $f_{±1}, f_{±2}$ is three-dimesnional, then eq. (13) is equivalent to the fact that the four points $f_{±1,±2}$ lie in a plane, which...
Figure 7: Four adjacent quadrilaterals of a discrete isothermic net: the cross-ratios satisfy $q \cdot q_{-1,-2} = q_{-1} \cdot q_{-2}$, the five points $f$ and $f_{\pm 1,\pm 2}$ lie on a sphere.

is a sphere through $f = \infty$. This is a three-dimensional generalization of the Menelaus theorem; since this generalization is not very well known, we give it, with a proof, in appendix to this section. If, on the contrary, the four points $f_{\pm 1}, f_{\pm 2}$ are co-planar, then we are in the situation of Fig. 2, described by the Desargues theorem. Here, we apply the Menelaus theorem twice, to the triangle $f_{-1}f_{2}f_{1}$ intersected by the line $f_{-1,2}f_{12}$, and to the triangle $f_{-1}f_{-2}f_{1}$ intersected by the line $f_{-1,-2}f_{1,-2}$: both lines meet the line $f_{-1}f_{1}$ at the same point $\ell^{(1)}$, if and only if

$$\frac{|f_{2}f_{12}|}{|f_{12}f_{1}|} \cdot \frac{|f_{-1}f_{-1,2}|}{|f_{-1,2}f_{2}|} = -\frac{|f_{-1}\ell^{(1)}|}{|\ell^{(1)}f_{1}|} = \frac{|f_{-2}f_{1,-2}|}{|f_{1,-2}f_{1}|} \cdot \frac{|f_{-1}f_{-1,-2}|}{|f_{-1,-2}f_{-2}|}.$$ 

This yields (13). □

The claim of Theorem 11 can be re-formulated as follows.

**Corollary 12 (Factorized cross-ratios for a discrete isothermic net)**

A circular net $f : \mathbb{Z}^2 \rightarrow \mathbb{R}^N$ is a discrete isothermic net, if and only if the cross-ratios $q = q(f, f_1, f_{12}, f_2)$ of its elementary quadrilaterals satisfy the following condition:

$$q(f, f_1, f_{12}, f_2) = \frac{\alpha_1}{\alpha_2}, \quad (14)$$

with some edge functions $\alpha_i$ satisfying the labelling property (10).
Clearly, functions $\alpha_i$ are defined up to a common constant factor. Actually, it was this characterization of discrete isothermic nets that was used as a definition in the pioneering paper \[BP1\].

Actually, edge functions $\alpha_i$ in Corollary 12 admit a nice geometric expression. According to Theorem 2, a discrete isothermic net $f : \mathbb{Z}^2 \to \mathbb{R}^N$ can be characterized by the existence of representatives $\hat{\gamma} : \mathbb{Z}^2 \to \mathbb{L}^{N+1,1}$ in the light cone satisfying the discrete Moutard equation \[14\]. We will use for these T-net representatives the notation

$$\hat{\gamma} = s^{-1}\hat{f} = s^{-1}(f + e_0 + |f|^2 e_\infty).$$

(15)

Thus, $s^{-1}$ denotes the $e_0$-component of the T-net representative $\hat{\gamma}$ of an isothermic net $f$. The function $s : \mathbb{Z}^2 \to \mathbb{R}$ plays the role of a discrete metric of the isothermic net $f$. Now, eq. \[9\] defines functions on edges,

$$\alpha_i = -2\langle\hat{\gamma}_i, \hat{\gamma}_i\rangle = \frac{|f_i - f|}{s s_i},$$

(16)

possessing property \[10\]: any two opposite edges of any elementary square carry the same value of the corresponding $\alpha_i$.

**Theorem 13 (Cross-ratios through discrete metric)** Edge functions $\alpha_i$ participating in the factorization \[14\] of the cross-ratios of elementary quadrilaterals of a discrete isothermic net can be defined by eq. \[16\].

**Proof.** Comparing the $e_0$-components in the Moutard equation $\hat{\gamma}_{12} - \hat{\gamma} = a_{12}(\hat{\gamma}_2 - \hat{\gamma}_1)$, we find: $a_{12} = (s_{12}^{-1} - s^{-1})/(s_2^{-1} - s_1^{-1})$. Therefore, we can re-write the Moutard equation as

$$\left(\frac{1}{s_2} - \frac{1}{s_1}\right)\left(\hat{f}_{12} - \hat{f}\right) = \left(\frac{1}{s_{12}} - \frac{1}{s}\right)\left(\hat{f}_2 - \hat{f}_{1}\right),$$

which is equivalent to

$$\frac{\hat{f}_1 - \hat{f}}{s_{1}s_1} + \frac{\hat{f}_{12} - \hat{f}_1}{s_{12}s_1} = \frac{\hat{f}_2 - \hat{f}}{s_{2}s_2} + \frac{\hat{f}_{12} - \hat{f}_2}{s_{2}s_{12}}.$$

(17)

The $\mathbb{R}^N$-part of the latter equation, i.e.,

$$\frac{f_1 - f}{s s_1} + \frac{f_{12} - f_1}{s_1 s_{12}} = \frac{f_2 - f}{s s_2} + \frac{f_{12} - f_2}{s_2 s_{12}},$$

(18)

can be rewritten with the help of eq. \[16\] as

$$\alpha_1 \frac{f_1 - f}{|f_1 - f|^2} + \alpha_2 \frac{f_{12} - f_1}{|f_{12} - f_1|^2} = \alpha_2 \frac{f_2 - f}{|f_2 - f|^2} + \alpha_1 \frac{f_{12} - f_2}{|f_{12} - f_2|^2}.$$  

(19)
In terms of the inversion in the Clifford algebra \( \mathcal{C}(\mathbb{R}^N) \), this can be presented as
\[
\alpha_1(f_1 - f)^{-1} + \alpha_2(f_{12} - f_1)^{-1} = \alpha_2(f_2 - f)^{-1} + \alpha_1(f_{12} - f_2)^{-1}.
\] (20)

This latter equation is, in the generic case \( f_{12} + f \neq f_1 + f_2 \), equivalent to eq. (14). It is not quite straightforward to show this equivalence in case of non-commutative variables \( f \in \mathcal{C}(\mathbb{R}^N) \). But one can identify the plane of the quadrilateral \( (f, f_1, f_{12}, f_2) \) with \( \mathbb{C} \), and then eq. (19) is the (complex conjugate of) eq. (20), where now all variables are commutative (complex numbers), and in this case the equivalence to eq. (14) is immediate. \( \square \)

T-nets in the light cone \( \mathbb{L}^{N+1,1} \) are 3D-consistent. This yields also the 3D-consistency of the cross-ratio equation (14) with prescribed labelling \( \alpha_i \) of the edges, i.e., of the 2D equation
\[
q(f, f_i, f_{ij}, f_j) = \frac{\alpha_i}{\alpha_j}.
\] (21)

Both constructions provide us with a well-defined notion of multidimensional discrete isothermic nets, and therefore with Darboux transformations of discrete isothermic nets with the usual permutability properties.

We finish this section with the notion of duality for discrete isothermic nets.

**Theorem 14 (Dual discrete isothermic net)** Let \( f : \mathbb{Z}^m \to \mathbb{R}^N \) be a discrete isothermic net, with the T-net representatives in the light cone
\[
\hat{y} = s^{-1} f = s^{-1}(f + e_0 + |f|^2e_\infty) : \mathbb{Z}^m \to \mathbb{L}^{N+1,1}.
\]
Then the \( \mathbb{R}^N \)-valued discrete one-form \( \delta f^* \) defined by
\[
\delta_i f^* = \delta_i f \frac{s s_i}{\delta_i f^2}, \quad i = 1, \ldots, m,
\] (22)
is closed. Its integration defines (up to translation) a net \( f^* : \mathbb{Z}^m \to \mathbb{R}^N \), called dual to the net \( f \). The net \( f^* \) is a discrete isothermic net, with
\[
q(f^*, f_i^*, f_{ij}^*, f_j^*) = \frac{\alpha_i}{\alpha_j}.
\] (23)
Define also the function \( s^* : \mathbb{Z}^m \to \mathbb{R} \) as \( s^* = s^{-1} \). Then the net
\[
\hat{y}^* = (s^*)^{-1} \hat{f}^* = (s^*)^{-1}(f^* + e_0 + |f^*|^2e_\infty) : \mathbb{Z}^m \to \mathbb{L}^{N+1,1}
\]
is a T-net in the light cone.
Proof. Clearly, for any pair of indices \( i, j \) the function \( \hat{f} \) satisfies an equation analogous to eq. (17), which expresses the closeness of the \( \mathbb{R}^{N+1,1} \)-valued one-form defined by \( \delta_i \hat{f}/(ss_i) \). Unfortunately, the net obtained by integration of this one-form does not lie, in general, in the light cone \( \mathbb{L}^{N+1,1} \) and cannot be taken as the dual net \( \hat{f}^* \). We use the following trick for the construction of the dual net \( \hat{f}^* \) in the light cone. The \( \mathbb{R}^N \)-part of eq. (17), i.e., eq. (18), expresses the closeness of the \( \mathbb{R}^N \)-valued one-form \( \delta_i \hat{f}^* = \delta_i \hat{f}/(ss_i) \), whose integration gives a dual net \( f^* \) in \( \mathbb{R}^N \). Eq. (23) follows immediately from eq. (22) and implies that \( f^* \) is a discrete isothermic net. In particular, it is a circular net, so that \( f^* = f^* + e_0 + |f^*|^2 e_\infty \) is a conjugate net in the light cone. It remains to show that the so defined \( \hat{f}^* \) is a Moutard net, with the T-net representatives \( \hat{y}^* = (s^*)_1 \hat{f}^* \). This claim is equivalent to closeness of the discrete \( \mathbb{R}^{N+1,1} \)-valued one-form \( \delta_i \hat{f}^*/(s^*s_i^*) \). Since \( \hat{f}^* \) is a conjugate net in the light cone, it is enough to prove the closeness of the \( \mathbb{R}^N \)-valued one-form \( \delta_i f^*/(s^*s_i^*) \). But for \( s^* = s^{-1} \) we have:

\[
\frac{\delta_i f^*}{s^*s_i^*} = (ss_i) \frac{\delta_i f^*}{ss_i} = (ss_i) \frac{\delta_i f}{ss_i} = \delta_i f,
\]

which is automatically closed. □

3.1 Appendix: generalized Menelaus theorem

Theorem 15 Let \( P_0, \ldots, P_n \) be \( n+1 \) points in general position in \( \mathbb{R}^n \), so that the affine space through the points \( P_i \) is \( n \)-dimensional. Let \( P_{i,i+1} \) be some points on the lines \( P_i P_{i+1} \) (indices are read modulo \( n+1 \)). The \( n+1 \) points \( P_{i,i+1} \) lie in an \( (n-1) \)-dimensional affine subspace, if and only if the following relation for the quotients of the directed lengths holds:

\[
\prod_{i=0}^{n} \frac{|P_{i,i+1}|}{|P_{i,i+1}P_{i+1}|} = (-1)^{n+1}.
\]

Proof. The points \( P_{i,i+1} \) lie in an \( (n-1) \)-dimensional affine subspace, if there is a non-trivial linear dependence

\[
\sum_{i=0}^{n} \alpha_i P_{i,i+1} = 0 \quad \text{with} \quad \sum_{i=0}^{n} \alpha_i = 0.
\]

Substituting \( P_{i,i+1} = (1-\xi_i)P_i + \xi_i P_{i+1} \), and taking into account the general position condition, which can be read as linear independence of the vectors.
\[ P_i - P_0, \] we come to a homogeneous system of \( n + 1 \) linear equations for \( n + 1 \) coefficients \( \alpha_i \):

\[
\xi_i \alpha_i + (1 - \xi_{i+1}) \alpha_{i+1} = 0, \quad i = 0, \ldots, n
\]

(where indices are understood modulo \( n + 1 \)). Clearly it admits a non-trivial solution if and only if

\[
\prod_{i=0}^{n} \frac{\xi_i}{1 - \xi_i} = \prod_{i=0}^{n} \frac{|P_i P_{i+1}|}{|P_{i+1} P_{i+2}|} = (-1)^{n+1}.
\]

□

4 Isothermic surfaces in Laguerre geometry

Definitions and constructions of this section can be generalized for higher-dimensional nets

\[ P : \mathbb{Z}^m \to \{ \text{hyperplanes in } \mathbb{R}^N \}, \]

with the most natural geometric case \( m = N - 1 \) (discrete Laguerre geometry of hypersurfaces in \( \mathbb{R}^N \)); we restrict ourselves to the case of surfaces in \( \mathbb{R}^3 \), i.e., \( N = 3, m = 2 \).

**Definition 16 (Discrete L-isothermic surface)** A two-dimensional conical net \( P : \mathbb{Z}^2 \to \{ \text{planes in } \mathbb{R}^3 \} \) is called a discrete L-isothermic surface, if the corresponding \( \hat{p} : \mathbb{Z}^2 \to L^{4,2} \) is a discrete Moutard net.

Recall that for an (oriented) plane \( P = \{ x \in \mathbb{R}^3 : \langle v, x \rangle = d \} \) with the unit normal vector \( v \in S^2 \) and \( d \in \mathbb{R} \) its representative \( \hat{p} \) in the Lie quadric \( L^{4,2} \) is given by

\[
\hat{p} = v + 0 \cdot e_0 + 2d e_\infty + 1 \cdot e_6.
\]

Recall also that the vectors \( v : \mathbb{Z}^2 \to S^2 \) comprise the Gauss map for a given conical net \( P : \mathbb{Z}^2 \to \{ \text{planes in } \mathbb{R}^3 \} \).

From Definitions 1, 3 there follows a geometric characterization of discrete L-isothermic nets:

**Theorem 17 (Central spheres for discrete L-isothermic nets)**

(a) A conical net \( P : \mathbb{Z}^2 \to \{ \text{planes in } \mathbb{R}^3 \} \) not tangent to a two-dimensional sphere is a discrete L-isothermic net, if and only if for any \( u \in \mathbb{Z}^2 \) the five planes \( P \) and \( P_{\pm 1, \pm 2} \) are tangent to a two-dimensional sphere not touching some (and then any) of the four planes \( P_{\pm 1}, P_{\pm 2} \).
(b) A conical net \( P : \mathbb{Z}^2 \to \{\text{tangent planes of } S^2 \subset \mathbb{R}^3\} \) is a discrete L-isothermic net, if and only if for any \( u \in \mathbb{Z}^2 \) the three cones through \( P \),

\[
\begin{align*}
C^{(\text{up})} &= \text{cone}(P, P_{12}, P_{-1,2}), \\
C^{(\text{down})} &= \text{cone}(P, P_{1,-2}, P_{-1,-2}), \\
C^{(1)} &= \text{cone}(P, P_1, P_{-1}),
\end{align*}
\]

have one additional plane in common, which is also equivalent for the three cones through \( P \),

\[
\begin{align*}
C^{(\text{left})} &= \text{cone}(P, P_{-1,2}, P_{-1,-2}), \\
C^{(\text{right})} &= \text{cone}(P, P_{1,2}, P_{1,-2}), \\
C^{(2)} &= \text{cone}(P, P_2, P_{-2}),
\end{align*}
\]

to have one additional plane in common.

The (generic) case a) of Theorem 17 is illustrated on Fig. 8.

Theorem 18 (Gauss map of an L-isothermic net is an isothermic net in the sphere) Gauss map of an L-isothermic net is a discrete isothermic net in \( S^2 \). Conversely, if for any \( u \in \mathbb{Z}^2 \) the four planes \( P, P_1, P_2, P_{12} \) of a net \( P : \mathbb{Z}^2 \to \{\text{planes in } \mathbb{R}^3\} \) meet at a point, and the Gauss map of the net \( P \) is isothermic, then \( P \) is an L-isothermic conical net.
Proof. First, let \( P \) be an L-isothermic net. Then for some \( c : \mathbb{Z}^2 \to \mathbb{R} \) the net \( c\hat{p} \) is a T-net in the Lie quadric. As a consequence, \( c(v + 1 \cdot e_0 + 1 \cdot e_\infty) \) is a T-net in the light cone \( \mathbb{L}^{4,1} \) of the Minkowski space \( \mathbb{R}^{4,1} \) of the Möbius geometry for \( N = 3 \). Therefore, the net \( v : \mathbb{Z}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3 \) is isothermic.

Conversely, let the net \( v : \mathbb{Z}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3 \) be isothermic. This is equivalent to the existence of the function \( c : \mathbb{Z}^2 \to \mathbb{R} \) such that \( c(v,1) \) is a T-net. If now \( \langle v, x \rangle = d \) is the equation of the plane \( P \), then the existence of the common intersection point of the planes \( P, P_1, P_2, P_{12} \) yields that the function \( cd \) satisfies the same Moutard equation as the function \( cv \). Therefore, \( c(v, d, 1) \) is a T-net, so that \( \hat{p} \) is a discrete Moutard net. □

5 Lie geometry: S-isothermic nets

Two-dimensional nets in the Lie quadric \( \mathbb{L}^{4,2} \) are discrete congruences of spheres. An interesting class of such congruences is constituted by discrete Moutard nets in \( \mathbb{P}(\mathbb{L}^{4,2}) \). We leave a general study of this class for a future research, and describe here as an example a particularly interesting subclass, for which the T-net representatives in \( \mathbb{L}^{4,2} \) have a fixed \( e_6 \)-component:

\[
\hat{y} = \frac{\kappa}{r} \left( c + e_0 + (|c|^2 - r^2)e_\infty + re_6 \right).
\]

Omitting the constant and therefore non-interesting \( e_6 \)-component, we come to a T-net in a hyperboloid of the Lorentz space of the Möbius geometry,

\[
\mathbb{L}^{4,1}_\kappa = \{ \xi \in \mathbb{R}^{4,1} : \langle \xi, \xi \rangle = \kappa^2 \}.
\]

Definition 19 (S-isothermic net) A map

\[
S : \mathbb{Z}^2 \to \{ \text{oriented spheres in } \mathbb{R}^3 \}
\]

is called an S-isothermic net, if the corresponding map

\[
\hat{s} : \mathbb{Z}^2 \to \mathbb{L}^{4,1}_\kappa, \quad \hat{s} = \frac{\kappa}{r} \left( c + e_0 + (|c|^2 - r^2)e_\infty \right),
\]

is a T-net.

Thus, S-isothermic nets are governed by equation

\[
\hat{s}_{12} - \hat{s} = a_{12}(\hat{s}_2 - \hat{s}_1), \quad a_{12} = \frac{\langle \hat{s}, \hat{s}_1 - \hat{s}_2 \rangle}{\kappa^2 - \langle \hat{s}_1, \hat{s}_2 \rangle} = \frac{\alpha_1 - \alpha_2}{\kappa^2 - \langle \hat{s}_1, \hat{s}_2 \rangle},
\]

(25)
with the quantities $\alpha_i = \langle \hat{s}, \hat{s}_i \rangle$ depending on $u_i$ only. If (oriented) radii of all hyperspheres become uniformly small, $r(u) \sim \kappa s(u)$, $\kappa \to 0$, then in the limit we recover discrete isothermic nets.

Consistency of T-nets in $\mathbb{L}^4_\kappa$ (which is a particular case of Theorem 8) yields, in particular, Darboux transformations for S-isothermic nets. A Darboux transform $\hat{s}^+ : \mathbb{Z}^m \to \mathbb{L}^4_\kappa$ of a given S-isothermic net $\hat{s} : \mathbb{Z}^m \to \mathbb{L}^4_\kappa$ is uniquely specified by a choice of one of its spheres $\hat{s}^+(0)$.

We turn now to geometric properties of S-isothermic nets. First of all, S-isothermic nets form a subclass of discrete R-congruence of spheres (see [BS2] for a geometric characterization of discrete R-congruences). Further, consider the quantities $\langle \hat{s}, \hat{s}_i \rangle$ which have the meaning of cosines of the intersection angles of the neighboring spheres (resp., of their so called inversive distances if they do not intersect). Then these quantities $\langle \hat{s}, \hat{s}_i \rangle$ have the labelling property, i.e., depend on $u_i$ only.

There holds the following generalization of Theorem 14.

**Theorem 20 (Dual S-isothermic net)** Let

$$S : \mathbb{Z}^m \to \{\text{oriented spheres in } \mathbb{R}^3\}$$

be an S-isothermic net. Denote the Euclidean centers and (oriented) radii of $S$ by $c : \mathbb{Z}^m \to \mathbb{R}^3$ and $r : \mathbb{Z}^m \to \mathbb{R}$, respectively. Then the $\mathbb{R}^3$-valued discrete one-form $\delta c^*$ defined by

$$\delta_i c^* = \frac{\delta_i c}{rr_i}, \quad 1 \leq i \leq m,$$

is closed, so that its integration defines (up to a translation) a function $c^* : \mathbb{Z}^m \to \mathbb{R}^3$. Define also $r^* : \mathbb{Z}^m \to \mathbb{R}$ by $r^* = r^{-1}$. Then the spheres $S^*$ with the centers $c^*$ and radii $r^*$ form an S-isothermic net, called dual to $S$.

**Proof.** Consider equation

$$\hat{s}_{ij} - \hat{s} = a_{ij}(\hat{s}_j - \hat{s}_i),$$

in terms of $\hat{s}$ from (24). Its $e_0$-part yields: $a_{ij} = (r_{ij}^{-1} - r_i^{-1})/(r_j^{-1} - r_i^{-1})$. This allows us to rewrite eq. (24) as

$$(r_j^{-1} - r_i^{-1})(\hat{s}_{ij} - \hat{s}) = (r_{ij}^{-1} - r^{-1})(\hat{s}_j - \hat{s}_i).$$

A direct computation shows that the $\mathbb{R}^3$-part of this equation can be rewritten as

$$\frac{c_i - c}{rr_i} + \frac{c_{ij} - c_i}{rr_{ij}} = \frac{c_j - c}{rr_j} + \frac{c_{ij} - c_j}{rr_{ij}},$$

(29)
which is equivalent to closeness of the form $\delta c^*$ defined by (26). In the same
way, the $e_\infty$-part of eq. (28) is equivalent to closeness of the discrete form
$\delta w$ defined by
$$
\delta_i w = \frac{\delta_i (|c|^2 - r^2)}{r r_i}, \quad 1 \leq i \leq m.
$$
For similar reasons, the second claim of the theorem is equivalent to closeness
of the form
$$
\delta_i w^* = \frac{\delta_i (|c^*|^2 - (r^*)^2)}{r^* r_i^*}, \quad 1 \leq i \leq m,
$$
where, recall, $r^* = 1/r$. With the help of $c_i^* - c^* = (c_i - c)/rr_i$, one easily
checks that the forms $\delta w$ and $\delta w^*$ can be written as
$$
\begin{align*}
\delta_i w &= \langle c_i^* - c^*, c_i + c \rangle - \frac{r_i}{r} + \frac{r}{r_i}, \\
\delta_i w^* &= \langle c_i - c, c_i^* + c^* \rangle - \frac{r_i}{r} + \frac{r}{r_i}.
\end{align*}
$$
The sum of these one-forms is closed:
$$
\delta_i (w + w^*) = 2\langle c_i^*, c_i \rangle - 2\langle c^*, c \rangle,
$$
therefore they are closed simultaneously. □

An interesting particular case of S-isothermic surfaces is characterized
by touching of any pair of neighboring spheres. In this case the limit of
small spheres is not relevant, therefore it is convenient to restrict the con-
siderations to a fixed value of $\kappa = 1$. Clearly, in this case both $\alpha_i = \langle \hat{s}, \tau_i \hat{s} \rangle$,  
$i = 1, 2$, can, in principle, take values $\pm 1$. However, it is easily seen from
(25) that in case $\alpha_1 = \alpha_2$ one gets only trivial nets. Thus, we assume that
$$
\langle \hat{s}, \hat{s}_1 \rangle = \langle \hat{s}_2, \hat{s}_1 \rangle = -1, \quad \langle \hat{s}, \hat{s}_2 \rangle = \langle \hat{s}_1, \hat{s}_{12} \rangle = 1.
$$
Interestingly, these touching conditions already yield that the linear depen-
dence of the spheres has the Moutard shape.

**Theorem 21** S-isothermic surfaces with touching spheres can be charac-
terized by any of the following equivalent descriptions:

- $Q$-congruence of spheres ($Q$-net in the Lorentz space of Möbius geom-
  etry) with touching spheres
- $R$-congruence ($Q$-net in Lie quadric) with touching spheres
- $T$-net in the Lie quadric with touching spheres

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Proof. Let \( \hat{s}, \hat{s}_1, \hat{s}_2, \hat{s}_{12} \) be four oriented spheres in \( \mathbb{R}^4_1 \), pairwise touching so that eq. (30) is fulfilled, and linearly dependent. (We remark that in the present situation the geometric meaning of linear dependence is the existence of a common orthogonal circle through the touching points.) We show that the linear dependence has to be of the Moutard form:

\[
\hat{s}_{12} - \hat{s} = a_{12}(\hat{s}_2 - \hat{s}_1), \quad a_{12} = -\frac{2}{1 - \langle \hat{s}_1, \hat{s}_2 \rangle}.
\]  

(31)

In this proof, we make a general position assumption that the spheres \( \hat{s} \) and \( \hat{s}_{12} \) do not touch and the spheres \( \hat{s}_1 \) and \( \hat{s}_2 \) do not touch. Write the linear dependence condition as

\[
\hat{s}_{12} = \lambda \hat{s} + \mu \hat{s}_1 + \nu \hat{s}_2.
\]  

(32)

Scalar product of this with \( \hat{s}_1, \hat{s}_2 \) leads to:

\[
1 + \lambda = \mu + \nu \langle \hat{s}_1, \hat{s}_2 \rangle = -\mu \langle \hat{s}_1, \hat{s}_2 \rangle - \nu \quad \Rightarrow \quad \mu = -\nu = \frac{\lambda + 1}{1 - \langle \hat{s}_1, \hat{s}_2 \rangle}.
\]

Similarly, a scalar product of eq. (32) with \( \hat{s}, \hat{s}_{12} \) leads to:

\[
\mu - \nu = \lambda - \langle \hat{s}, \hat{s}_{12} \rangle = 1 - \lambda \langle \hat{s}, \hat{s}_{12} \rangle \quad \Rightarrow \quad \lambda = 1.
\]

This yields eq. (31). □

6 Remarks on the continuous limit

To perform a continuous limit in constructions of Discrete Differential Geometry, one should think of the underlying lattice \( \mathbb{Z}^2 \) of discrete surfaces as of \((\epsilon \mathbb{Z})^2 \) and then send \( \epsilon \to 0 \). In such a limit it is common to assume that the discrete functions \( y(n_1, n_2) \) on \( \mathbb{Z}^2 \) approximate sufficiently smooth functions \( y(u_1, u_2) \) on \( \mathbb{R}^2 \), if \( n_i \epsilon = u_i \).

Moutard nets. It turns out, however, that the discrete Moutard equation in the form (1) with minus signs does not admit a nice continuous limit, while its close relative – discrete Moutard equation with plus signs – does:

\[
y_{12} + y = a_{12}(y_1 + y_2).
\]  

(33)

Indeed, if \( a_{12} = 1 + \frac{\epsilon^2}{4} q_{12} \), then the above equation is re-written as

\[
\frac{1}{\epsilon^2}(y_{12} - y_1 - y_2 + y) = \frac{1}{4} q_{12}(y_{12} + y_1 + y_2 + y),
\]

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which in the limit $\epsilon \to 0$ tends to the Moutard equation \( M \):

\[
\partial_1 \partial_2 y = qy. \tag{34}
\]

Equations (11) and (33) are actually related by a very simple transformation of dependent variables: $y(n_1, n_2) \mapsto (-1)^{n_2} y(n_1, n_2)$. As long as $y$ play the role of homogeneous coordinates, such a transformation does not change the corresponding points in the projective space, therefore in the formulation of Theorem 2 we could use eq. (33) in place of eq. (11), as well. However, eq. (33) cannot be posed on all faces of a three-dimensional lattice, and it is not multidimensionally consistent. As a consequence, the right way to construct discrete Moutard transformations for nets governed by eq. (33) is to consider multidimensional T-nets and then perform a change of variables $y(n) \mapsto (-1)^{n_2} y(n)$ which breaks the symmetry among lattice directions. This leads to changing the signs in some of the formulas (6) for discrete Moutard transformations, which turn into

\[
y_1^+ - y = b_1(y^+ - y_1), \quad y_2^+ + y = b_2(y^+ + y_2). \tag{35}
\]

An advantage of this break of symmetry is again the possibility to perform a continuous limit in eqs. (35), recovering the formulas of the Moutard transformation for eq. (34), see \( M \):

\[
\partial_1 y^+ + \partial_1 y = p_1(y^+ - y), \quad \partial_2 y^+ - \partial_2 y = p_2(y^+ + y). \tag{36}
\]

**Isothermic nets.** A similar situation takes place for discrete isothermic nets. In order to enable the continuous limit to smooth isothermic surfaces, one should start with a discrete two-dimensional isothermic net with embedded elementary quadrilaterals. It is convenient to represent their negative cross-ratios as

\[
q(f, f_1, f_{12}, f_2) = -\frac{\alpha_1}{\alpha_2}, \tag{37}
\]

with positive labels $\alpha_1$ and $\alpha_2$. Formally, this means nothing more than changing the notation $\alpha_2 \mapsto -\alpha_2$. But this operation puts some formulas into the form which allows for a continuous limit. If one keeps the both formulas

\[
|f_1 - f|^2 = \alpha_1 s s_1, \quad |f_2 - f|^2 = \alpha_2 s s_2,
\]

then this implies a slight modification in the definition of the function $s$, namely $s(u) \mapsto (-1)^{u_2} s(u)$, which assures the positivity of $s$ in case of positive $\alpha_1$, $\alpha_2$. Only upon this modification does the positive function $s$ have a continuous limit. In this limit we recover a characterization of
isothermic surfaces as those curvature line parametrized surfaces for which the first fundamental form is conformal, possibly upon a re-parametrization of curvature lines:

\[ \langle \partial_1 f, \partial_2 f \rangle = 0, \quad |\partial_1 f|^2 = \alpha_1 s^2, \quad |\partial_2 f|^2 = \alpha_2 s^2, \quad (38) \]

with \( \alpha_i = \alpha_i(u_i) \). Eq. (22) for the dual surface turns into

\[ \delta_1 f^* = \alpha_1 \frac{\delta_1 f}{s}, \quad \delta_2 f^* = -\alpha_2 \frac{\delta_2 f}{s}, \quad (39) \]

which is a direct discrete analogue of equations

\[ \partial_1 f^* = \alpha_1 \frac{\partial_1 f}{s}, \quad \partial_2 f^* = -\alpha_2 \frac{\partial_2 f}{s}, \quad (40) \]

defining the dual isothermic surfaces in the smooth case.

**Remark.** In the smooth case the functions \( \alpha_1, \alpha_2 \) in eq. (38) can be absorbed into a re-parametrization of the independent variables \( u_i \mapsto \varphi_i(u_i) \) \((i = 1, 2)\), by which one can always achieve that \( \alpha_1 = \alpha_2 = 1 \), so that the first fundamental form of the surface \( f \) is conformal. Of course, such a re-parametrization is not possible in the discrete context. Nevertheless, one can consider a narrower class of discrete isothermic surfaces, characterized by eq. (37) with \( \alpha_1 = \alpha_2 = 1 \):

\[ q(f, f_1, f_{12}, f_2) = -1. \quad (41) \]

This condition (all elementary quadrilaterals of \( f \) are conformal squares) may be regarded as a discretization of the conformality of the first fundamental form. Eq. (11), being a particular case of eq. (21) with a special labelling, enjoys all the properties of the general case. However, it is important to observe that it is not 3D consistent with itself, i.e., it cannot be imposed on all faces of a 3D cube. Indeed, if \( \alpha_1/\alpha_2 = -1 \), then it is impossible to have additionally \( \alpha_2/\alpha_3 = -1 \) and \( \alpha_1/\alpha_3 = -1 \).

Note that the above change of signs yields also the similar modification in the lift \( \hat{s} \), which therefore satisfies the discrete Moutard equation with plus signs (33), admitting the continuous limit. This characterization of isothermic surfaces is due to Darboux [Da]:

**Theorem 22 (Isothermic surfaces as Moutard nets in the light cone)** A surface \( f : \mathbb{R}^2 \to \mathbb{R}^N \) is isothermic, if and only if there exists a function \( s : \mathbb{R}^2 \to \mathbb{R} \) such that the lift of \( f \) into the light cone

\[ \hat{s} = s^{-1}(f + e_0 + |f|^2e_\infty) : \mathbb{R}^2 \to \mathbb{L}^{N+1,1} \]

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satisfies a Moutard equation \([34]\). Darboux transformations of an isothermic surface \(f\) are obtained by projection to \(\mathbb{R}^N\) from Moutard transformations of \(\hat{s}\) within the light cone \(\mathbb{L}_{N+1,1}^+\).

**L-isothermic surfaces.** Analogously, we arrive at a (apparently new) characterization of smooth L-isothermic surfaces.

**Theorem 23 (L-isothermic surfaces as Moutard nets in the Laguerre quadric)** A surface enveloping a two-parameter family of planes \(P : \mathbb{R}^2 \to \{\text{planes in } \mathbb{R}^3\}\), with \(P = \{x \in \mathbb{R}^3 : \langle v, x \rangle = d\}\), is L-isothermic, if and only if there exists a function \(\rho : \mathbb{R}^2 \to \mathbb{R}\) such that

\[
\hat{p} = \rho^{-1}(v + 2de_\infty + |v|e_0) : \mathbb{R}^2 \to \mathbb{L}_{4,2}^+
\]

satisfies a Moutard equation \([34]\).

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