ON THE S-MATRIX CONJECTURE

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Abstract. Motivated with a problem in spectroscopy, Sloane and Harwit conjectured in 1976 what is the minimal Frobenius norm of the inverse of a matrix having all entries from the interval \([0, 1]\). In 1987, Cheng proved their conjecture in the case of odd dimensions, while for even dimensions he obtained a slightly weaker lower bound for the norm. His proof is based on the Kiefer-Wolfowitz equivalence theorem from the approximate theory of optimal design. In this note we give a short and simple proof of his result.

Key words: matrices, Frobenius norm, inequalities

Math. Subj. Classification (2010): 15A45, 15A60

A Hadamard matrix is a square matrix with entries in \([-1, 1]\) whose rows and hence columns are mutually orthogonal. In other words, a Hadamard matrix of order \(n\) is a \([-1, 1]\)-matrix \(A\) satisfying \(AA^T = nI\), i.e., \(\frac{1}{\sqrt{n}}A\) is a unitary matrix.

An S-matrix of order \(n\) is a \([0, 1]\)-matrix formed by taking a Hadamard matrix of order \(n + 1\) in which the entries in the first row and column are 1, changing 1’s to 0’s and \(-1\)’s to 1’s, and deleting the first row and column.

The Frobenius norm of a real matrix \(A = [a_{i,j}]_{i,j=1}^n\) is defined as

\[
\|A\|_F = \left(\sum_{i=1}^n \sum_{j=1}^n a_{i,j}^2\right)^{1/2}.
\]

It is associated to the inner product defined by

\[
\langle A, B \rangle = \text{tr}(AB^T) = \sum_{i=1}^n \sum_{j=1}^n a_{i,j}b_{i,j}.
\]

Let \(D_n\) denote the set of all matrices \(A = [a_{i,j}]_{i,j=1}^n\) whose entries are in the interval \([0, 1]\).

In 1976, Sloane and Harwit [6] posed the following conjecture. See also [2, p.59] or [7, Conjecture 11].

Date: May 7, 2014.

The paper will appear in Linear Algebra and Its Applications.
Conjecture. If $A \in D_n$ is a nonsingular matrix, then
\[ \|A^{-1}\|_F \geq \frac{2n}{n + 1}, \]
where the equality holds if and only if $A$ is an S-matrix.

This conjecture arose from a problem in spectroscopy. A detailed discussion of its applications in spectroscopy can be found in [2]. The conjecture has been proved in recent papers [9], [8] and [3] for some special matrices. Apparently, the authors of these papers were not aware of the fact that for odd dimensions the conjecture has already been proved in [1], while for even dimensions a slightly weaker lower bound for the norm has been derived; see [1, Corollary 3.4]. The proof is based on the celebrated equivalence theorem due to Kiefer and Wolfowitz [5] that connects the problem with the approximate theory of optimal design. For an extensive treatment of this theory we refer to [4].

In this note we give a short and transparent proof of the conjecture when $n$ is odd, while for even $n$ our method gives the same (weaker) lower bound as in [1, Corollary 3.4].

Theorem. Let $A \in D_n$ be a nonsingular matrix.

If $n \geq 3$ is an odd integer, then
\[ (1) \quad \|A^{-1}\|_F \geq \frac{2n}{n + 1}, \]
where the equality holds if and only if $A$ is an S-matrix.

If $n \geq 4$ is an even integer, then
\[ (2) \quad \|A^{-1}\|_F > \frac{2\sqrt{n^2 - 2n + 2}}{n}, \]
If $n = 2$ then
\[ (3) \quad \|A^{-1}\|_F \geq \sqrt{2}, \]
where the equality holds if and only if $A$ is either the identity matrix or \[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \].

Proof. Let $e = (1, 1, 1, \ldots, 1)^T \in \mathbb{R}^n$ and $J = ee^T$. We divide the proof into three cases.

CASE 1: $n \geq 3$ is an odd integer, so that $n = 2k - 1$ for some $k \in \mathbb{N}$.

Define the matrices $M$ and $N$ of order $n + 1$ by
\[ M = \begin{bmatrix} 1 \\ e \\ -kA^{-1} \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 1 \\ e \\ -(2I - \frac{1}{k}J)A^T \end{bmatrix}. \]

Since
\[ MN^T = \begin{bmatrix} n + 1 & * \\ * & (n + 1)I \end{bmatrix}, \]
we have
\[ \langle M, N \rangle = \text{tr}(MN^T) = (n + 1)^2. \]
By the Cauchy-Schwarz inequality, we then obtain
\[(n + 1)^4 = (\langle M, N \rangle)^2 \leq \|M\|_F^2 \cdot \|N\|_F^2 = (1 + 2n + k^2\|A^{-1}\|_F^2)(1 + 2n + \|(2I - \frac{1}{k}J)A^T\|_F^2).\]

If we show that
\[\|(2I - \frac{1}{k}J)A^T\|_F \leq n,\]
then (4) gives the inequality
\[(n + 1)^2 \leq 1 + 2n + k^2\|A^{-1}\|_F^2,\]
and so
\[\|A^{-1}\|_F^2 \geq \frac{n^2}{k^2} = \left(\frac{2n}{n + 1}\right)^2\]
completing the proof of (1).

To show (5), we determine the maximum of the function \(f\) defined on \(D_n\) by
\[f(A) = \|(2I - \frac{1}{k}J)A^T\|_F^2 = \text{tr}(A(2I - \frac{1}{k}J)^2A^T) = \text{tr} \left(4AA^T - \frac{2k + 1}{k^2}(Ac)(Ac)^T \right) = 4 \text{tr}(AA^T) - \frac{2k + 1}{k^2}(Ac)(Ac)^T.\]

where \(A = [a_{i,j}]_{i,j=1}^n \in D_n\). Since \(f\) is a continuous function on a compact set, it attains its maximum at some matrix \(B = [b_{i,j}]_{i,j=1}^n \in D_n\). Assume that 0 < \(b_{i,j} < 1\) for some \(i, j \in \{1, 2, \ldots, n\}\). Then we have
\[\frac{\partial f}{\partial a_{i,j}}(B) = 0\quad \text{and}\quad \frac{\partial^2 f}{\partial a_{i,j}^2}(B) \leq 0.\]

However,
\[\frac{\partial f}{\partial a_{i,j}}(A) = 8a_{i,j} - \frac{2k + 1}{k^2} 2 \left(\sum_{i=1}^{n} a_{i,l}\right),\]
and so
\[\frac{\partial^2 f}{\partial a_{i,j}^2}(A) = 8 - \frac{2(2k + 1)}{k^2} = \frac{2(4k^2 - 2k - 1)}{k^2} > 0\]
for all \(A = [a_{i,j}]_{i,j=1}^n \in D_n\). Therefore, we conclude that \(B\) is necessarily a \(\{0, 1\}\)-matrix.

Let \(p_i\) be the number of ones in the \(i\)-th row of \(B\). Then
\[f(B) = 4 \sum_{i=1}^{n} p_i - \frac{2k + 1}{k^2} \sum_{i=1}^{n} p_i^2 = -\frac{2k + 1}{k^2} \sum_{i=1}^{n} \left(p_i - \frac{2k^2}{2k + 1}\right)^2 + \frac{4k^2}{2k + 1}(2k - 1).\]

Since \(k = \frac{2k^2}{2k + 1} \in (0, \frac{1}{2})\), we have \(|m - \frac{2k^2}{2k + 1}| > |k - \frac{2k^2}{2k + 1}|\) for all \(m \in \{1, 2, \ldots, n\} \setminus \{k\}\), implying that \(p_i = k\) for all \(i\). It follows that
\[f(B) = 4nk - \frac{2k + 1}{k^2} nk^2 = n(4k - 2k - 1) = n^2.\]
This completes the proof of the inequality (5).

Assume that the equality holds in (1). Then there are equalities in (4) and (5), that is, \( M = N \) and \( A \) is an invertible \( \{0,1\} \)-matrix with \( Ae = ke \). It follows that \( kA^{-1} = (2I - \frac{1}{k}J)A^T = 2A^T - J \), and so \( e = kA^{-1}e = 2A^Te - (2k - 1)e \) implying that \( A^Te = ke \). Therefore, we have

\[
NN^T = MN^T = \begin{bmatrix} 1 & e^T \\ e & -kA^{-1} \end{bmatrix} \begin{bmatrix} 1 & e^T \\ e & J - 2A \end{bmatrix} = \begin{bmatrix} n + 1 & 0 \\ 0 & (n+1)I \end{bmatrix}.
\]

This means that

\[
N^T = \begin{bmatrix} 1 & e^T \\ e & J - 2A \end{bmatrix}
\]
is a Hadamard matrix, and so \( A \) is an \( S \)-matrix. As the equality holds in (1) when \( A \) is an \( S \)-matrix, the proof is complete for odd dimensions.

**CASE 2:** \( n \geq 4 \) is an even integer, so that \( n = 2k \) for some integer \( k \geq 2 \).

Define the matrices \( M \) and \( N \) of order \( n + 1 \) by

\[
M = \begin{bmatrix} 0 & e^T \\ e & k\sqrt{k}k^{-1}A^{-1} \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & e^T \\ e & \sqrt{k-1} (k^{-1}I - J)A^T \end{bmatrix}.
\]

Then

\[
MN^T = \begin{bmatrix} 2k & * \\ * & \frac{k(2k-1)}{k-1}I \end{bmatrix},
\]

and so

\[
\langle M, N \rangle = \text{tr} (MN^T) = 2k + \frac{2k^2(2k-1)}{k-1} = 2k(2k^2 - 1).
\]

By the Cauchy-Schwarz inequality, we then obtain

\[
\left( \frac{2k(2k^2 - 1)}{k-1} \right)^2 = \langle \langle M, N \rangle \rangle^2 \leq \| M \|^2_F \cdot \| N \|^2_F = (4k + \frac{k^3}{k-1}\| A^{-1} \|^2_F)(4k + g(A)),
\]

where \( g(A) \) is defined by

\[
g(A) = \left\| \sqrt{k-1} \left( \frac{k(2k-1)}{k-1}I - J \right)A^T \right\|_F^2.
\]

If \( A = [a_{i,j}]_{i,j=1}^n \in D_n \) then

\[
g(A) = \text{tr} \left( A \left( \frac{(2k-1)^2}{k(k-1)}I - \frac{2}{k}J \right)A^T \right) = \frac{(2k-1)^2}{k(k-1)} \text{tr} (AA^T) - \frac{2}{k}(Ae)^T(Ae) = \frac{(2k-1)^2}{k(k-1)} \sum_{i=1}^{2k} \sum_{j=1}^{2k} a_{i,j}^2 - \frac{2}{k} \sum_{i=1}^{2k} \left( \sum_{j=1}^{2k} a_{i,j} \right)^2.
\]

Since

\[
\frac{\partial^2 g}{\partial a_{i,j}^2}(A) = \frac{2(2k-1)^2}{k(k-1)} - \frac{4}{k} = \frac{2(4k^2 - 6k + 3)}{k(k-1)} > 0,
\]

...
we conclude (similarly as in Case 1) that the maximum of the function \( g \) on \( D_n \) is attained at some matrix \( B = [b_{i,j}]_{i,j=1}^n \in D_n \) with \( b_{i,j} \in \{0, 1\} \) for all \( i \) and \( j \). Let \( p_i \) be the number of ones in the \( i \)-th row of \( B \). Then

\[
g(B) = \left( \frac{(2k - 1)^2}{k(2k - 1)} \right)^2 \sum_{i=1}^{2k} p_i - \frac{2k}{k} \sum_{i=1}^{2k} p_i^2 = -\frac{2k}{k} \sum_{i=1}^{2k} \left( p_i - \frac{(2k - 1)^2}{k(2k - 1)} \right)^2 + \frac{(2k - 1)^4}{4(k - 1)^2}.
\]

Since \( k - \frac{(2k - 1)^2}{4(k - 1)} = -\frac{k}{2} \in (-\frac{1}{2}, 0) \), we obtain that \( p_i = k \) for all \( i \). It follows that

\[
g(B) = \frac{(2k - 1)^2}{k - 1} - 2k - 4k^2 = \frac{2(k^2 - 2k + 1)}{k - 1}.
\]

Now, since \( 4k + g(B) = \frac{2k(2k - 1)}{k - 1} \), the inequality (6) gives

\[
4k + \frac{k^3}{k - 1} \| A^{-1} \|_F^2 \geq \frac{2k(2k^2 - 1)}{k - 1},
\]

and so

\[
\| A^{-1} \|_F^2 \geq \frac{2(2k^2 - 2k + 1)}{k} = \frac{4(n^2 - 2n + 2)}{n^2}.
\]

To complete the proof of the inequality (2), we must exclude the possibility of the equality in (7). So, assume that for some matrix \( A \in D_n \) the equality holds in (7). Then \( A \) is a \( \{0, 1\} \)-matrix and \( M = N \). Therefore, we have

\[
\frac{k^3}{k - 1} A^{-1} = \left( \frac{k(2k - 1)}{k - 1} I - J \right) A^T
\]

or

\[
\frac{k^2}{2k - 1} I = \left( I - \frac{k - 1}{k(2k - 1)} J \right) A^T A,
\]

implying that

\[
A^T A = \frac{k^2}{2k - 1} \left( I - \frac{k - 1}{k(2k - 1)} J \right)^{-1} = \frac{k^2}{2k - 1} \left( I + \frac{k - 1}{k} J \right).
\]

It follows that the off-diagonal entries of the matrix \( A^T A \) are equal to the number \( \frac{k(k - 1)}{2k - 1} \) that is not an integer. This is a contradiction with the fact that \( A \) is a \( \{0, 1\} \)-matrix.

**CASE 3:** \( n = 2 \). If

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

is an invertible matrix in \( D_2 \), then

\[
A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},
\]

and so

\[
\| A^{-1} \|_F^2 = \frac{a^2 + b^2 + c^2 + d^2}{(ad - bc)^2}.
\]

Now, we have

\[
(ad - bc)^2 \left( \| A^{-1} \|_F^2 - 2 \right) = (a - d)^2 + (b - c)^2 + 2ad(1 - ad) + 2bc(1 - bc) + 4abcd \geq 0.
\]
We conclude that
\[ \| A^{-1} \|^2_F \geq 2 \]
and the equality holds if and only if \( a = d, b = c, ad \in \{0, 1\}, bc \in \{0, 1\} \) and \( abcd = 0 \). This implies the desired conclusions. \qed

Acknowledgment.

The author was supported in part by the Slovenian Research Agency.

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