Simple Zeros Of The Zeta Function

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Abstract: This note studies the Laurent series of the inverse zeta function \(1/ζ(s)\) at any fixed nontrivial zero \(ρ\) of the zeta function \(ζ(s)\), and its connection to the simplicity of the nontrivial zeros.

1 Introduction

The set of zeros of the zeta function \(ζ(s) = \sum_{n≥1} n^{-s}, \Re(s) > 1\) is defined by \(Z = \#\{s \in \mathbb{C} : ζ(s) = 0\}\). This subset of complex numbers has a disjoint partition as \(Z = Z_T \cup Z_N\), the subset of trivial zeros and the subset of nontrivial zeros.

The subset of trivial zeros of the zeta function is completely determined; it is the subset of negative even integers \(Z_T = \{-2n : ζ(s) = 0 \text{ and } n ≥ 1\}\). The trivial zeros are extracted using the symmetric functional equation

\[
π^{-s/2}Γ(s/2)ζ(s) = π^{-(1-s)/2}Γ((1-s)/2)ζ(1-s),
\]

where \(Γ(t) = \int_0^∞ x^{t-1}e^{-x}dx\) is the gamma function, and \(s \in \mathbb{C}\) is a complex number. Various levels of explanations of the functional equation are given in [33, p. 222], [31, p. 328], [17, p. 13], [22, p. 55], [17, p. 8], [41], [27], and other sources. In contrast, the subset of nontrivial zeros

\[Z_N = \{s \in \mathbb{C} : ζ(s) = 0 \text{ and } 0 < \Re(s) < 1\}\]

and myriad of questions about its properties remain unknown. The nontrivial zeros are computed using the Riemann-Siegel formula, see [18, Eq. 25.10.3]. Some recent development in this area appears in [19] and [20].

Let \(N(T) = \#\{ρ = σ + it : ζ(ρ) = 0, \ 0 < σ < 1, \text{ and } |t| ≤ T\}\) be the counting function of the subset of nontrivial zeros, and let

\[N_0(T) = \#\{ρ = σ + it : ζ(ρ) = 0, \ \text{ and } |t| ≤ T\}\]

and

\[N_1(T) = \#\{ρ = σ + it : ζ(ρ) = 0, \ \zeta''(ρ) ≠ 0, \ \text{ and } |t| ≤ T\}\]

be the counting functions of the subsets of nontrivial distinct zeros, and simple zeros respectively. It is clear that \(N_1(T) ≤ N_0(T) ≤ N(T)\), and the RH predicts that \(N_1(T) = N_0(T) = N(T)\), a proof conditional on the RH appears in [34, p. 5], and [41]. The Riemann Hypothesis (RH) claims that the nontrivial zeros of the zeta function \(ζ(s)\) are of the

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form \( \rho, t \in \mathbb{R} \).

A very recent result in [5] shows, that assuming the RH, the subset of nontrivial simple zeros has positive density in the set of nontrivial zeros. Specifically,

\[
N_1(T) \geq (19/29)N(T) \quad \text{and} \quad N_0(T) \geq 0.84665N(T)
\]

for all large real number \( T \geq 1 \). The earliest results on the number of the nontrivial simple zeros are the Selberg result of positive density for nontrivial zeros of odd multiplicities, and the Montgomery result \( N_1(T) \geq (2/3 + o(1))N(T) \). The former is based on the Fourier analysis of the convolution of certain functions, refer to [28]. A simpler and expanded version of this analysis is given in [15]. Other methods and further progress on this problem appear in [9], and the references within. A survey on the theory of the zeros of the zeta function and recent developments appears in [38], and certain associated problems such as the moment of the derivatives are considered in [29]. Another related result, given in [36], claims that the short interval \([T, T + T^{1.552}]\) contains a positive proportion of simple zeros and on the critical line. This is a vast improvement over the naive estimate \( N_1(T + T^{1.552}) - N_1(T) \geq cT^{1.552}\log T \), with \( c > 0 \) constant.

The objective of this short note is to offer a simpler and indirect approach to the verification of the simplicity of the nontrivial zeros of the zeta function. The main result is the following.

**Theorem 1.1.** Every zero \( \rho = \sigma + it \) of the zeta function \( \zeta(s) \) is a simple zero.

The proof of this result is an immediate consequence of Theorem 2.1 in Section 2, which constructs the Laurent series of the inverse zeta function \( 1/\zeta(s) \). A second independent proof also appears in Theorem 3.1 in Section 3. A corollary of these results is that the kth derivative \( \zeta^{(k)}(s) \neq 0 \) for any nontrivial zero \( \rho = \sigma + it \), and \( k \geq 1 \). A couple of related theorems in [20] and [42] claim that the RH implies that the derivatives \( \zeta'(s) \neq 0 \) and \( \zeta''(s) \neq 0 \) for any \( s = \sigma + it, 0 < \sigma < 1/2 \). In general, the zeros of the kth derivatives \( \zeta^{(k)}(s) \) of the zeta function have complex patterns, confer [7] for recent works on this topic.

\[00\] 2 The Laurent Series Expansion

The development of the Laurent series

\[
\frac{1}{\zeta(s)} = \frac{c_m}{(s - \rho)m} + \frac{c_{m+1}}{(s - \rho)^{m+1}} + \cdots + \frac{c_k}{s - \rho} + \gamma_0 + c_1(s - \rho) + c_2(s - \rho)^2 + \cdots
\]

for the inverse zeta function \( 1/\zeta(s) \) at a fixed nontrivial zero \( \rho = \sigma + it \) of the zeta function of multiplicity \( m(\rho) = m \geq 1 \) is based on an extension of a technique used in [10] p. 206 to derive the Laurent series

\[
\zeta(s) = \frac{1}{s - 1} + \sum_{n \geq 0} \frac{(-1)^n \gamma_n}{n!} (s - 1)^n
\]

of the zeta function at \( s = 1 \), and the standard practices of the principle of analytic continuation. The first coefficient \( \gamma = \gamma_0 \) coincides with Euler constant. A related analysis yields
the Laurent series of the zeta function of a quadratic field $K = \mathbb{Q}(\sqrt{d})$, which has the form:

$$
\zeta_K(s) = \sum_{(m,n) \neq (0,0)} \frac{1}{Q(m,n)s} = \frac{2\pi}{s-1} + \sum_{n \geq 0} \frac{(-1)^n \beta_n}{n!} (s-1)^n,
$$

(8)

where the quadratic form is $Q(x,y) = ax^2 + bxy + cy^2$ with $a, c > 0$, and $b^2 - 4ac = -1$. The first coefficient $\beta_0 = 4\pi(\gamma + \log(c)/2 + \log|\eta(w)|^2)$ is a constant, where $\eta(z) = e^{i\pi z/12} \prod_{n \geq 1} (1 - e^{i\pi z n})$ is the Dedekind eta function, and $w = (b+i)/(2a)$, see [33, p. 160], and [16, p. 15]. Other information on the Stieltjes-Hermite method appear in [11, 2, 6, 10, 23], and similar sources. As in the case of the series (7), this technique for computing the power series of holomorphic and meromorphic functions, seems to show that any nontrivial zero must have multiplicity $m = 1$ unconditionally. Equivalently, it shows that the nontrivial zeros are simple zeros, unconditionally.

**Theorem 2.1.** For each fixed nontrivial zero $\rho = \sigma + it_0$ of the zeta function $\zeta(s)$, the Laurent series of the inverse zeta function $1/\zeta(s)$ at $\rho = \sigma + it_0$ has the form

$$
\frac{1}{\zeta(s)} = \frac{c_{-1}}{s-\rho} + \sum_{n \geq 0} \frac{(-1)^n \phi_n}{n!} (s-\rho)^n,
$$

(9)

where $c_{-1}$ is the residue at $\rho = \sigma + it_0$, and the $n$th coefficient $\phi_n$ is given by

$$
\phi_n = \sum_{n \geq 1} \left( \frac{\mu(k)}{k^n} \log(k)^n \right) \frac{1}{n+1} - \frac{\log(k+1)^{n+1} - \log(k)^{n+1}}{n+1}
$$

(10)

for $n \geq 0$. This is an analytic function on the domain $D(\rho) = \{ s : 0 < |s-\rho| < r \}$ of some radius $r > 0$.

**Proof.** Let $\rho = \sigma_0 + it_0$ be a fixed nontrivial zero, $1/2 \leq \sigma_0 < 1$, let $k \geq 1$ be an integer, and consider the sequence of complex valued functions

$$
v_k(s) = \frac{\mu(k)}{k^s} - c_{-1} \int_k^{k+1} \frac{1}{x^{s+\rho+1}} dx,
$$

(11)

for $s = \sigma + it \in \mathbb{C}$, and the corresponding inequality

$$
|v_k(s)| = \left| \frac{\mu(k)}{k^s} - c_{-1} \int_k^{k+1} \frac{1}{x^{s+\rho+1}} dx \right| \leq \frac{c}{k^\sigma},
$$

(12)

where $c > 0$ is a constant, and $\Re(s) = \sigma$.

In terms of exponential functions, the $k$th function $v_k(s)$ has the form

$$
v_k(s) = \frac{\mu(k)}{k^s} + \frac{c_{-1}}{s-\rho} \left( \frac{k}{(k+1)^{s-\rho}} - \frac{1}{k^{s-\rho}} \right)
$$

$$
= \frac{\mu(k)}{k^s} e^{-(s-\rho) \log(k)} + \frac{c_{-1}}{s-\rho} \left( e^{-(s-\rho) \log(k+1)} - e^{-(s-\rho) \log(k)} \right)
$$

(13)

$$
= \frac{\mu(k)}{k^\rho} \sum_{n \geq 0} \frac{(-1)^n (s-\rho)^n}{n!} \log(k)^n + \frac{c_{-1}}{s-\rho} \sum_{n \geq 0} \frac{(-1)^n (s-\rho)^n}{n!} (\log(k+1)^n - \log(k)^n)
$$

$$
= \frac{\mu(k)}{k^\rho} \sum_{n \geq 0} \frac{(-1)^n (s-\rho)^n}{n!} \log(k)^n + c_{-1} \sum_{n \geq 0} \frac{(-1)^{n+1} (s-\rho)^n}{(n+1)!} (\log(k+1)^{n+1} - \log(k)^{n+1}).
$$
The summation index of the second power series was shifted because the first term
\[ \frac{(-1)^0(s - \rho)^0}{0!} (\log(k + 1)^0 - \log(k)^0) = 0 \] (14)
vanishes for all \( k \geq 1 \), this is the convention in the Stieltjes-Hermite method, see \[6\, p. 282\].

Summing the right side of the sequence of functions (11) over the index \( k \geq 1 \), and using the absolute convergence of both the power series and the integral yield
\[
\sum_{k \geq 1} v_k(s) = \sum_{k \geq 1} \left( \frac{\mu(k)}{k^s} - c_{-1} \int_{k}^{k+1} \frac{1}{x^{s-\rho+1}} dx \right)
= \sum_{k \geq 1} \frac{\mu(k)}{k^s} - c_{-1} \int_{1}^{\infty} \frac{1}{x^{s-\rho+1}} dx
= \frac{1}{\zeta(s)} \frac{c_{-1}}{s - \rho}
\] (15)
for any complex number \( s = \sigma + it \in \mathbb{C} \) such that \( \Re(s) = \sigma > 1 \).

Summing the right side of the sequence of functions (13) over the index \( k \geq 1 \), and using the absolute convergence of the power series for \( \Re(s) > 1 \), it follows that the double sum can be summed in any order:
\[
\sum_{k \geq 1} v_k(s) = \sum_{k \geq 1} \frac{\mu(k)}{k^s} \sum_{n \geq 0} \frac{(-1)^n(s - \rho)^n}{n!} \log(k)^n
+ c_{-1} \sum_{n \geq 0} \sum_{k \geq 1} \frac{(-1)^n+1(s - \rho)^n}{(n+1)!} (\log(k+1)^{n+1} - \log(k)^{n+1})
\] (16)
\[
= \sum_{n \geq 0} \frac{(-1)^n(s - \rho)^n}{n!} \sum_{k \geq 1} \left( \frac{\mu(k) \log(k)^n}{k^s} - c_{-1} \frac{\log(k+1)^{n+1} - \log(k)^{n+1}}{n+1} \right)
\]
\[
= \sum_{n \geq 0} \frac{(-1)^n \phi_n}{n!} (s - \rho)^n,
\]
where the \( n \)th coefficient \( \phi_n \) is written in the form
\[
\phi_n = \sum_{k \geq 1} \left( \frac{\mu(k) \log(k)^n}{k^s} - c_{-1} \frac{\log(k+1)^{n+1} - \log(k)^{n+1}}{n+1} \right),
\] (17)
for \( n \geq 0 \). The right side of equation (16) is an analytic, and absolutely convergent function for all complex numbers \( s \in \mathbb{C} \).

Therefore, from (15) and (16), it follows that the Laurent series
\[
\frac{1}{\zeta(s)} = \frac{c_{-1}}{s - \rho} + \sum_{n \geq 0} \frac{(-1)^n \phi_n}{n!} (s - \rho)^n,
\] (18)
where \( c_{-1} = 1/\zeta'(\rho) \) is the residue at \( s = \rho \), represents an analytic continuation of the inverse zeta function \( 1/\zeta(s) \) to the domain \( D(\rho) = \{ s : 0 < |s - \rho| < r \} \).
On the contrary, suppose that the fixed nontrivial zero \( \rho = \sigma_0 + it_0 \) has multiplicity \( m > 1 \), and let
\[
v_k(s) = \frac{\mu(k)}{k^s} - c_{-1} \int_k^{k+1} \frac{1}{x^{s-\rho+1}} dx - \cdots - c_{-m} \int_k^{k+1} \frac{1}{x^{s-\rho+1}} dx. \tag{19}
\]
Then, using the same procedure as \((11)\) to \((18)\), the Laurent series is
\[
\frac{1}{\zeta(s)} - \frac{c_{-m}}{(s-\rho)^m} - \cdots - \frac{c_{-2}}{(s-\rho)^2} - \frac{c_{-1}}{s-\rho} = \sum_{k \geq 1} v_k
\]
\[
= \sum_{n \geq 0} \frac{(-1)^n R(n)}{n!} (s-\rho)^n,
\]
where the \( n \)th term
\[
R(n) = \sum_{k \geq 1} \left( \frac{\mu(k) \log(k)^n}{k^\rho} - \sum_{k \geq 1} \frac{c_{-1} + \cdots + c_{-m}}{(s-\rho)^m} \left( \log(k+1)^{n+1} - \log(k)^{n+1} \right) \right).
\]
The left hand side of \((20)\) is analytic (it has no poles) on a small disk \( D(\rho) \) of some radius \( r > 0 \). But the right side has a pole at \( s = \rho \) of multiplicity \( m - 1 \geq 1 \) as confirmed in \((21)\). These information imply that \( m = 1 \).

In synopsis, The first coefficient has the form
\[
\phi_0 = \sum_{k \geq 1} \left( \frac{\mu(k)}{k^\rho} - c_{-1} \log(1 + 1/k) \right).
\]
It is a constant attached to a fixed nontrivial zero \( \rho = \sigma_0 + it_0 \). This is analogous to the definition of the Euler constant: the first Stieltjes coefficient of the series \((7)\) defines the Euler constant
\[
\gamma = \gamma_0(1) = \sum_{k \geq 1} \left( \frac{1}{k} - \log(1 + 1/k) \right). \tag{23}
\]
However, the \( n \)th coefficient \( \phi_n = \phi_n(\rho) \) carries far more information than the Stieltjes \( n \)th coefficient \( \gamma_n = \gamma_n(1) \). For example, it includes important information on the Mertens sum \( \sum_{n \leq x} \mu(n) \).

The sequence of coefficients \( \{\phi_n(\rho) : n \geq 0\} \) is a complicated sequence of functions of \( n \geq 0 \) involving the partial sums of the \( k \)th derivatives of the inverse zeta function
\[
\frac{d^k}{ds^k} \zeta^{-1}(s) = (-1)^k \sum_{n \geq 1} \frac{\mu(n) \log^k n}{n^s}, \tag{24}
\]
which is a topic of current research, see \([7, 13]\), and the literature. This sequence is somewhat similar to the sequence of Stieltjes constants \( \{\gamma_n(1) : n \geq 1\} \), see \([18, Eq. 25.2.4]\). The properties of the sequence of Stieltjes constants \( \gamma_n \) are studied in \([1, 8, 39, 23, 24, 25]\).

In these studies it is shown that the Stieltjes constants \( \gamma_n = \gamma_n(1) \) are unbounded functions of \( n \geq 1 \), for example, each one satisfies the inequality \( |\gamma_n| \leq (e \cdot n!) / (2^n \sqrt{n}) \) is given in \([8]\).
A survey of these studies, and an improved bound and information on the sign changes are undertaken in \cite{1}. The generalized Stieltjes constants to $L$-functions appears in \cite{39}.

## 3 A Second Proof

The zeta function has a myriad of different representations. Among these is the Jensen integral representation of the zeta function

$$\zeta(s) = \frac{\pi}{2(s-1)} \int_{-\infty}^{\infty} \frac{(1/2 + it)^{1-s}}{\cosh(\pi t)^2} dt,$$  \hspace{1cm} (25)

where $s \in \mathbb{C}$, see \cite[Theorem 1]{21}, facilitates another way of showing the simplicity of the zeros.

**Theorem 3.1.** Each nontrivial zero $\rho = \sigma + it$ of the zeta function $\zeta(s)$ is a simple zero. In particular, for each complex number $s \in \mathbb{C}$, the Taylor series at a fixed nontrivial zero $\rho_0 = \sigma_0 + it_0$ is the power series

$$\zeta(s) = \sum_{n \geq 1} \frac{\zeta^{(n)}(\rho_0)}{n!} (z - \sigma_0 - it_0)^n.$$  \hspace{1cm} (26)

**Proof.** Let $\rho = \sigma + it$ be a nontrivial zero. The change of variable $s = 1 + \sigma - it + z$ in (31) leads to

$$\zeta(1 - \sigma - it + z) = \frac{\pi}{2(z - \sigma - it)} \int_{-\infty}^{\infty} \frac{(1/2 + it)^{-\sigma+it-z}}{\cosh(\pi t)^2} dt,$$  \hspace{1cm} (27)

where $z \in \mathbb{C}$. It has a simple pole at $z = \sigma + it$. The integral

$$\left| \int_{\infty}^{\infty} \frac{(1/2 + it)^{-\sigma+it-z}}{\cosh(\pi t)^2} dt \right| \ll \int_{0}^{\infty} \frac{1}{\cosh(\pi t)^2} dt < \infty$$  \hspace{1cm} (28)

is absolutely bounded for any complex number $z \in \mathbb{C}$. Evaluation at a nontrivial zero $z = \sigma + it$, implies that the simple pole on the left side

$$\zeta(1) = \frac{\pi}{2(z - \sigma - it)} \int_{-\infty}^{\infty} \frac{(1/2 + it)^{-\sigma+it-z}}{\cosh(\pi t)^2} dt$$  \hspace{1cm} (29)

is matched with a simple pole on the right side. Accordingly, the Taylor series at a fixed nontrivial zero $\rho_0 = \sigma_0 + it_0$ is

$$\zeta(s) = \sum_{n \geq 1} \frac{\zeta^{(n)}(\rho_0)}{n!} (s - \sigma_0 - it_0)^n$$  \hspace{1cm} (30)

is well defined for all complex numbers. Here the $n$th derivative

$$\zeta^{(n)}(s) = \frac{d^n}{ds^n} \left( \frac{\pi}{2(s-1)} \int_{-\infty}^{\infty} \frac{(1/2 + it)^{1-s}}{\cosh(\pi t)^2} dt \right),$$  \hspace{1cm} (31)

is absolutely continuous for all complex numbers $s \neq 1$. The evaluation at $s = \sigma_0 + it_0$ yields the $n$th Taylor coefficient $\zeta^{(n)}(\rho_0)/n!$ for every $n \geq 0$. This proves the claim. ■
4 Generalization Of The Stieltjes-Hermite Method

The Stieltjes-Hermite method of the previous Section should be extendable to the Laurent series some other zeta functions and theirs inverses, and \( L \)-functions and theirs inverses. For example, a more general zeta function and its inverse are,

\[
\zeta_F(s) = \sum_{a \in \mathbb{Z}_F} \frac{1}{N(a)^s} \quad \text{and} \quad \frac{1}{\zeta_F(s)} = \sum_{a \in \mathbb{Z}_F} \frac{\mu_F(a)}{N(a)^s},
\]

(32)

where \( F \) is a global number fields, and \( \mathbb{Z}_F \) is its ring of integers, see [32, p. 315] for a generalization of the Mobius function. And an \( L \)-function and its inverse

\[
L(s, f) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} \quad \text{and} \quad \frac{1}{L(s, f)} = \sum_{n \geq 1} \frac{\mu_f(n)}{n^s},
\]

(33)

where \( \sum_{d|n} \mu_f(d) \lambda(n/d) = 0 \) for \( n > 1 \), respectively.

The generalization of the Stieltjes-Hermite method to the Laurent series of these functions is clearly and heavily dependent on many parameters such as the

(i) The coefficients local, global and functions fields \( F \) in characteristics \( \text{char} F = 0 \) and \( \text{char} F > 0 \).

(ii) The existence of Euler Products.

(iii) The existence of zeros or poles at \( s = 1/2, 1 \), etc.

(iv) The Galois groups \( \text{Gal}(E/F) \), and theirs irreducible representations.

(v) The regulator \( R_F \), the class number \( h_F(d) \geq 1 \) of \( F \), and and the discriminant \( d_F \).

These parameters can vary the multiplicities of the zeros and poles of the Laurent series for these functions. Hence, these parameters can turn the analysis into a very complicated subject, both algebraically and analytically. It appears that the most important parameters are the coefficients fields \( F \), the Euler products, the class number, and the Galois group \( \text{Gal}(E/F) \).

4.1 Abelian Zeta and \( L \)-Functions

The zeta functions associated with global fields with Abelian Galois groups \( \text{Gal}(E/F) \) have representations as products of linearly independent \( L \)-functions;

\[
\zeta_F(s) = \prod_{\chi} L(s, \chi),
\]

(34)

confer [32, p. 414]. All the numerical evidence, and theoretical information available in the vast literature demonstrate that abelian zeta functions over global fields of characteristic \( \text{char}(F) = 0 \) have simple zeros, see [27], [3], [30].

Example 3.1. The zeta function over the rational numbers is an ideal case: it has trivial coefficients, Galois group \( \text{Gal}(F/\mathbb{Q}) = 1 \), class number 1, and a perfect Euler product,
and so on. This is also demonstrated by the trivial residue \( r_{-1} = 1 \) of the Laurent series (7) at \( s = 1 \). In contrast, the Dedekind zeta

\[
\zeta_F(s) = \frac{\beta_{-1}}{s - 1} + \sum_{n \geq 0} \frac{(-1)^n \beta_n}{n!} (s - 1)^n, \tag{35}
\]

for a field extension \( F \) of \( \mathbb{Q} \) has a very complex residue \( \beta_{-1} = 2^{r_1}(2\pi)^{r_2} h_F R_F \sqrt{|d_F|} \), see [11, p. 37], and similar references. This can have significant effect on the coefficients of the inverse zeta function (32), and the Laurent series

\[
\frac{1}{\zeta_F(s)} = \frac{c_{-m}(F)}{(s - \alpha)^m} + \cdots + \frac{c_{-1}(F)}{s - \alpha} + \sum_{n \geq 0} \frac{(-1)^n c_n(F)}{n!} (s - \alpha)^n. \tag{37}
\]

### 4.2 Nonabelian Zeta and \( L \)-Functions

Let \( \rho : \text{Gal}(E/F) \to \text{GL}(n) \) be an irreducible representation of the Galois group \( \text{Gal}(E/F) \) of degree \( \text{deg}(\rho) = d \). The Dedekind zeta functions associated with fields with nonabelian Galois groups \( \text{Gal}(E/F) \) have representations as products of linearly independent Artin \( L \)-functions;

\[
\zeta_F(s) = \prod_{\rho} L(s, \rho)^d. \tag{38}
\]

The occurrence of one or more nonlinear representation of degree \( \text{deg}(\rho) = d > 1 \) forces the zeta function to have zeros of multiplicity \( d > 1 \).

**Example 3.2.** An example of a field extension \( F = \mathbb{Q}(\theta) \), where \( \theta \) is a root of \( f(x) = x^4 - 2x^2 + 2 \), which has the Galois group \( \text{Gal}(F/\mathbb{Q}) = D_4 \), and the irreducible representation \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}(n) \) of degree \( \text{deg}(\rho) = 2 \), is listed in [27]. The corresponding Dedekind zeta function has the representation

\[
\zeta_F(s) = L(s, \rho_0) \prod_{\rho} L(s, \rho)^2. \tag{39}
\]

In this case, the Laurent series of the inverse zeta function should have the form

\[
\frac{1}{\zeta_F(s)} = \frac{c_{-2}}{(s - \alpha)^2} + \frac{c_{-1}}{s - \alpha} + \sum_{n \geq 0} \frac{(-1)^n c_n(F)}{n!} (s - \alpha)^n. \tag{40}
\]

Here, the basic Stieltjes-Hermite method can fail for infinitely many nontrivial zeros \( \alpha \) of \( \zeta_F(s) \) of multiplicity 2. Thus, the basic Stieltjes-Hermite method has to be modified to handle these cases. However, it is not clear how the algebraic properties of the coefficients of the Laurent series, see [37], obstruct the analytic properties whenever the Galois group is nonabelian.

**Example 3.3.** An example of an \( L \)-function of degree 2 Artin \( L \)-function for a \( S_3 \) extension \( F \) of the rational numbers \( \mathbb{Q} \) with square local factors in its Euler product

\[
L(s, \rho) = \prod_{\rho \geq 2} L_p(s, \rho) \tag{41}
\]

was computed in [12, Lemma 4.8]. Introductions to the calculations of zeta and \( L \)-functions are given in [37, p. 167], and [4], and similar references.
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