CONSERVATION LAWS AND LINE SOLITON SOLUTIONS OF
A FAMILY OF MODIFIED KP EQUATIONS

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Abstract. A family of modified Kadomtsev-Petviashvili equations which includes the in-
tegrable case is studied. The explicit line soliton solution and all conservation laws of low
order are derived and compared to their counterparts in the integrable case.

1. Introduction

An integrable generalization of the modified Korteweg-de Vries (mKdV) equation in 2+1
dimensions is the modified Kadomtsev-Petviashvili (mKP) equation \cite{10}

\( (u_t - \alpha u^2 u_x \pm \sqrt{2\alpha \gamma} u_x \partial_x^{-1} u_y + \beta u_{xxx})_x + \gamma u_{yy} = 0 \) \hspace{1cm} (1)

where \( \alpha, \beta, \gamma \) are non-zero constants. This equation arises in several physical applications
\cite{6, 7, 15} pertaining to dispersive nonlinear wave phenomena. Unlike the better known
Kadomtsev-Petviashvili (KP) equation \cite{9}, the mKP equation contains a nonlocal term and
has no obvious Lagrangian structure. Its line soliton solutions and some conservation laws
can be found in Refs. \cite{8, 11, 12, 17}.

In the present paper, we consider a family of mKP equations

\( (u_t - \alpha u^2 u_x + \kappa u_x \partial_x^{-1} u_y + \beta u_{xxx})_x + \gamma u_{yy} = 0 \) \hspace{1cm} (2)

with arbitrary non-zero constant coefficients \( \alpha, \beta, \gamma, \kappa \). The integrable mKP equation is given
by the case

\( \kappa^2 = 2\alpha \gamma > 0 \) \hspace{1cm} (3)

The main goals will be to determine the conservation laws and line soliton solutions of the
mKP family \cite{2} and compare the results to the integrable mKP case. First, in section \textsuperscript{2},
the mKP family \cite{2} is formulated as a local PDE by use of the potential \( w \), with \( u = w_x \).

Next, in section \textsuperscript{3}, all low-order conservation laws of the mKP family in potential form
are derived. The admitted conservation laws are found to consist of a topological charge and
a generalized momentum for arbitrary \( \kappa \), plus an energy in the integrable case of the mKP
equation. Computational aspects are summarized in an appendix.
In section 4, all line solitons \( u = U(x + \mu y - \nu t) \) are derived, where the parameters \( \mu \) and \( \nu \) determine the direction and the speed of the line soliton. The basic kinematical properties of these solutions are discussed and compared to the mKP line solitons.

Finally, a few concluding remarks are made in section 5.

2. Potential form

The mKP family (2) is equivalent to a local PDE system

\[
\begin{align*}
  u_t - \alpha u^2 u_x + \kappa u_x v + \beta u_{xxx} + \gamma v_y &= 0, \quad v_x = u_y \\
  \end{align*}
\]

This system can be expressed as a single PDE by the introduction of a potential \( w \) given by

\[
\begin{align*}
  u &= w_x, \quad v = w_y, \\
  \end{align*}
\]

which yields

\[
0 = w_{tx} - \alpha w_x^2 w_{xx} + \kappa w_{xx} w_y + \beta w_{xxx} + \gamma w_{yy}
\]

By applying a general scaling transformation \( t \rightarrow \lambda_1 t, \quad x \rightarrow \lambda_2 x, \quad y \rightarrow \lambda_3 y, \quad w \rightarrow \lambda_4 w \), where \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \neq 0 \), we can fix three of the four coefficients \( \alpha, \beta, \gamma, \kappa \): \( |\alpha| = \beta = |\gamma| = 1 \) and also we can fix the sign \( \kappa > 0 \), without loss of generality.

Hence, we will consider the mKP family in the scaled potential form

\[
0 = w_{tx} + (\sigma_1 w_x^2 + \kappa w_y) w_{xx} + w_{xxx} + \sigma_2 w_{yy}, \quad \sigma_1, \sigma_2 = \pm 1, \quad \kappa > 0
\]

which is a one-parameter family where \( \kappa \) (rescaled) is an arbitrary positive constant. We will refer to \( \sigma_1 = 1 \) as the focussing case, and \( \sigma_1 = -1 \) as the defocussing case; this distinction will be significant when line soliton solutions are considered.

The corresponding scaled mKP family has the form

\[
(u_t + \sigma_1 u^2 u_x + \kappa u_x \partial_x^{-1} u_y + u_{xxx})_x + \sigma_2 u_{yy} = 0, \quad \sigma_1, \sigma_2 = \pm 1, \quad \kappa > 0
\]

in which the scaled mKP equation is the case

\[
\kappa^2 = 2, \quad \sigma_1 \sigma_2 = -1
\]

namely,

\[
(u_t + \sigma_1 u^2 u_x + \sqrt{2} u_x \partial_x^{-1} u_y + u_{xxx})_x - \sigma_1 u_{yy} = 0, \quad \sigma_1 = \pm 1
\]

3. Conservation laws

Conservation laws are of basic importance for nonlinear evolution equations because they provide physical, conserved quantities as well as conserved norms. A general treatment of conservation laws is given in Refs. [13, 4, 5, 2].

For the mKP family in potential form (7), a local conservation law is a continuity equation

\[
D_t T + D_x X + D_y Y = 0
\]

holding for all solutions \( w(x, y, t) \) of equation (7), where \( T \) is the conserved density, and \( (X, Y) \) is the spatial flux, which are functions of \( t, \quad x, \quad y, \quad w \), and derivatives of \( w \). When solutions \( w(x, y, t) \) are considered in a given spatial domain \( \Omega \subseteq \mathbb{R}^2 \), every local conservation law yields a corresponding conserved integral

\[
C[w] = \int_{\Omega} T \, dx \, dy
\]
satisfying the global balance equation
\[
\frac{d}{dt} C[w] = - \int_{\partial \Omega} (X, Y) \cdot \hat{n} \, ds
\]  
(13)
where \( \hat{n} \) is the unit outward normal vector of the domain boundary curve \( \partial \Omega \), and where \( ds \) is the arclength on this curve with clockwise orientation. This global equation (13) has the physical meaning that the rate of change of the quantity (12) on the spatial domain is balanced by the net outward flux through the boundary of the domain.

A conservation law is locally trivial [13, 5, 2] if, for all solutions \( w(x, y, t) \) in \( \Omega \), the conserved density \( T \) reduces to a spatial divergence \( D_x \Psi^x + D_y \Psi^y \) and the spatial flux \( (X, Y) \) reduces to a time derivative \(-D_t(\Psi^x, \Psi^y)\), since then the global balance equation (13) becomes an identity. Likewise, two conservation laws are locally equivalent [13, 5, 2] if they differ by a locally trivial conservation law, for all solutions \( w(x, y, t) \) in \( \Omega \). We will be interested only in locally non-trivial conservation laws.

Any non-trivial conservation law (11) can be expressed in an equivalent characteristic form [13, 5, 2] which is given by a divergence identity holding off of the space of solutions \( w(x, y, t) \). For the mKP family in potential form (7), conservation laws have the characteristic form
\[
D_t \tilde{T} + D_x \tilde{X} + D_y \tilde{Y} = (w_{tx} + (\sigma_1 w_x^2 + \kappa w_y)w_{xx} + w_{xxxx} + \sigma_2 w_{yy})Q
\]  
(14)
where \( \tilde{T}, \tilde{X}, \tilde{Y} \) are functions of \( t, x, y, w \), and derivatives of \( w \), and where the conserved density \( \tilde{T} \) and the spatial flux \( (\tilde{X}, \tilde{Y}) \) reduce to \( T \) and \( (X, Y) \) when restricted to all solutions \( w(x, y, t) \) of equation (7). This divergence identity is called the characteristic equation for the conservation law, and the function \( Q \) is called the conservation law multiplier. Note that, when a conservation law is non-trivial, \( Q \) will be non-singular when it is evaluated on any solution \( w(x, y, t) \).

All multipliers \( Q \) are determined by
\[
E_w((w_{tx} + (\sigma_1 w_x^2 + \kappa w_y)w_{xx} + w_{xxxx} + \sigma_2 w_{yy})Q) = 0
\]  
(15)
holding off of solutions of equation (7), where \( E_w \) is the Euler operator [13, 5, 2] with respect to \( w \). For \( Q \) having any specified form, this determining equation (15) splits with respect to all variables that do not appear in \( Q \), yielding an overdetermined system to be solved for \( Q \). A variety of methods [16, 3, 5, 2] can be used to derive the conserved density \( \tilde{T} \) and spatial flux \( (\tilde{X}, \tilde{Y}) \) arising from any given multiplier \( Q \).

Here we will explicitly find all low-order conservation laws of the mKP family in potential form (7) by determining all multipliers having the general form
\[
Q(t, x, y, w, \partial w, \partial^2 w, \partial^3 w)
\]  
(16)
where \( \partial^k w \) denotes the set of all partial derivatives of order \( k \geq 0 \) of \( w \). Some remarks on the computations are provided in the appendix.

**Proposition 3.1.** All low-order multipliers (16) admitted by the potential form of the mKP family (7) with \( \kappa \neq 0 \), \( \sigma_1^2 = 1 \), \( \sigma_2^2 = 1 \) are given by
(i) \( \kappa \) arbitrary:
\[
Q_1 = f_1(t),
\]  
(17)
\[
Q_2 = \kappa w_x f_2(t) + y f'_2(t);
\]  
(18)
(ii) \( \kappa^2 = 2, \sigma_2 = -\sigma_1 \):

\[
\begin{align*}
Q_3 &= (4yw_x - 2\kappa\sigma_2 x)f_3(t) + \kappa y^2 f'_3(t), \\
Q_4 &= - \left((\kappa\sigma_1 w_{xxx} + \frac{1}{3}\kappa w_x^3 - \sigma_1 w_x w_y - \frac{1}{4}\kappa\sigma_1 w_t) + \right.
onumber \\
&\hspace{1cm} \left. \frac{1}{4}\kappa\sigma_1 x w_x \right)f_3(t) + \left(\frac{1}{8}\kappa y^2 w_x + \frac{1}{4}\sigma_1 xy \right)f''_4(t) + \frac{1}{24} y^3 f'''_4(t);
\end{align*}
\]

where \( f_1(t), f_2(t), f_3(t), f_4(t) \) are arbitrary functions.

These multipliers yield all non-trivial conservation laws of low order, summarized as follows.

**Theorem 3.1.** (i) The low-order conservation laws admitted by the mKP family in potential form (11) for arbitrary \( \kappa \) are given by (up to equivalence)

\[
\begin{align*}
T_1 &= 0, \\
X_1 &= (w_{xxx} + \kappa w_x w_y + \frac{1}{3}\sigma_1 w_x^3 + w_t)f_1(t), \\
Y_1 &= (\sigma_2 w_y - \frac{1}{2}\kappa w_x^2)f_1(t); \\
\end{align*}
\]

\[
\begin{align*}
T_2 &= \frac{1}{2}\kappa f_2(t)w_x, \\
X_2 &= (\kappa w_x w_{xxx} + \frac{1}{2}\kappa w_x^2 + \frac{1}{4}\kappa\sigma_1 w_x^4 + \frac{1}{2}\kappa^2 w_x^2 w_y - \frac{1}{2}\kappa\sigma_2 w_y^2)f_2(t) \\
&\hspace{1cm} + (w_{xxx} + \frac{1}{3}\sigma_1 w_x^3 + \kappa w_x w_y + w_t)y f'_2(t), \\
Y_2 &= (\kappa\sigma_2 w_x w_y - \frac{1}{6}\kappa^2 w_x^3)f_2(t) + ((\sigma_2 w_y - \frac{1}{2}\kappa w_x^2)y - \sigma_2 w)f'_2(t); \\
\end{align*}
\]

where \( f_1(t), f_2(t) \) are arbitrary functions.

(ii) Additional low-order conservation laws are admitted only when \( \kappa^2 = 2, \sigma_2\sigma_1 = -1 \). These conservation laws consist of (up to equivalence):

\[
\begin{align*}
T_3 &= (2yw_x^2 - 2\kappa\sigma_1 w)f_3(t), \\
X_3 &= (2\kappa\sigma_1 w_{xxx} + 4\sigma_1 w_x w_y + \frac{2}{3}\kappa w_x^3 + 2\kappa\sigma_1 w_t) + (\sigma_1 w_x^4 + 2\kappa w_x^2 w_y + 2\sigma_1 w_y^2 - 2w_{xx}^2 \\
&\hspace{1cm} + 4w_x w_{xxx})y - 2\kappa\sigma_1 w_x f_3(t) + (\kappa w_{xxx} + \frac{1}{3}\kappa\sigma_1 w_x^3 + 2w_x w_y + \kappa w_t)y^2 f'_3(t), \\
Y_3 &= - ((\kappa\sigma_1 w_t + w_x^2)y^2 - 2\kappa\sigma_1 yw)f'_3(t); \\
\end{align*}
\]

\[
\begin{align*}
T_4 &= \frac{3}{8}\kappa\sigma_1 w_{xx}^2 - \frac{1}{3}\sigma_1 w_x w_y - \frac{1}{16}\kappa w_x^4 - \frac{1}{8}\kappa w_x^2 f_4(t) + \frac{1}{8}\kappa\sigma_1 x w_x^2 f'_4(t) \\
&\hspace{1cm} + \left(\frac{1}{16}\kappa y^2 w_x^2 - \frac{1}{4}\sigma_1 yw\right)f''_4(t),
\end{align*}
\]

4
\[ X_4 = -\left(\frac{1}{2} \kappa \sigma_1 w_{xx}^2 + (\sigma_1 w_x w_y + \frac{1}{3} \kappa \sigma_1 w_t + \frac{1}{6} \kappa w_y^2) w_{xxx} + \frac{1}{2} \sigma_1 w_x^2 w_y - \frac{1}{2} \kappa w_{xy}^2 + \frac{1}{6} \kappa \sigma_1 w_t^2 \right) \\
- (\sigma_1 w_x w_{xy} - \frac{3}{4} \kappa \sigma_1 w_x + \kappa w_y) w_{xx} + (\frac{1}{12} \kappa w_x^3 + \frac{1}{2} \sigma_1 w_x w_y) w_t + \frac{5}{16} w_x^4 w_y \\
+ \frac{1}{18} \kappa \sigma_1 w_x^6 + \frac{1}{6} w_y^3 + \frac{1}{2} \kappa \sigma_1 w_x^2 w_y^2) f_1(t) + \left(\frac{1}{4} \kappa \sigma_1 w_x w_{xxx} - \frac{1}{8} \kappa \sigma_1 w_{xx}^2 \right) \\
+ \frac{1}{10} \kappa w_x^4 + \frac{1}{8} \kappa w_y^2 + \frac{1}{2} \sigma_1 w_x^2 w_y) x - \frac{1}{4} \kappa \sigma_1 w_x x w_x) f_1(t) \\
+ ((\frac{1}{2} \kappa w_x w_{xxx} + \frac{1}{16} \kappa \sigma_1 w_y^2 + \frac{1}{8} w_x^3 w_y - \frac{1}{16} \kappa w_{xx}^2 + \frac{1}{12} \kappa \sigma_1 w_x^3) y^2 \\
+ (\frac{1}{4} \kappa \sigma_1 w_x w_y + \frac{1}{3} \sigma_1 w_{xxx} + \frac{1}{8} \sigma_1 w_t + \frac{1}{12} w_x^3 x y - \frac{1}{4} \sigma_1 y w_{xx}) f_4(t) \\
+ (\frac{1}{24} w_{xxx} + \frac{1}{24} w_t + \frac{1}{12} \sigma_1 w_x^3 + \frac{1}{12} \kappa w_x w_y) y^3 f_4''(t), \right) \]

\[(24b)\]

\[ Y_4 = (-\frac{1}{2} \sigma_1 w_x w_{xx}^2 - \kappa w_x w_{xy} + (\frac{1}{4} \kappa w_y + \frac{1}{4} \sigma_1 w_x^2) w_t + \frac{1}{3} \kappa \sigma_1 w_x^3 w_y + \frac{1}{2} w_x w_y^2 + \frac{1}{12} w_x^5) f_1(t) \\
- (\frac{1}{4} \kappa w_x w_y + \frac{1}{12} \sigma_1 w_x^3) x f_1(t) - \left((\frac{1}{4} y w_y + \frac{1}{8} \kappa \sigma_1 y w_x^2 - \frac{1}{4} y^2) x \right) \\
+ (\frac{1}{8} \kappa \sigma_1 w_x w_y + \frac{1}{24} w_x^3 y^2) f_1''(t) - \left(\frac{1}{24} \sigma_1 w_y + \frac{1}{48} \kappa w_x^3 y^3 - \frac{1}{8} \sigma_1 y^2 w) f_4'''(t); \right) \]

\[(24c)\]

where \( f_3(t), f_4(t) \) are arbitrary functions.

3.1. **Conserved quantities.** Conservation law \((22)\) yields

\[ \mathcal{P}[w, f] = \frac{1}{2} \kappa \int_{R^2} w_x^2 f(t) \, dx \, dy = \frac{1}{2} f(t) \int_{\Omega} \kappa u^2 \, dx \, dy \]

which is a generalized momentum quantity, in analogy with the same conserved integral known to hold for the potential form of the mKdV equation. Conservation law \((21)\) in contrast yields a spatial flux quantity which describes a conserved topological charge

\[ \mathcal{F}[w, f] = \int_{\partial \Omega} (w_{xxx} + \kappa w_x w_y + \frac{1}{3} \sigma_1 w_x^3 + w_t, \sigma_2 w_y - \frac{1}{2} \kappa w_x^2) \cdot \hat{n} \, ds = 0 \]

holding for all closed curves \( \partial \Omega \) in \( R^2 \), without any boundary conditions on \( w \).

The two additional conservation laws \((23)\), \((24)\) arise in the case of the mKP equation. Conservation law \((23)\) yields an additional momentum quantity

\[ \mathcal{Q}[w, f] = \int_{\Omega} (2 y w_x^2 - 2 \kappa \sigma_1 w) f(t) \, dx \, dy = 2 f(t) \int_{\Omega} (y u^2 - \kappa \sigma_1 w) \, dx \, dy \]

while conservation law \((24)\) yields an energy quantity

\[ \mathcal{E}[w, f] = \int_{\Omega} \left( \left(\frac{3}{8} \kappa \sigma_1 w_x^2 - \frac{1}{4} \sigma_1 w_x^2 w_y - \frac{1}{16} \kappa w_x^4 - \frac{1}{8} \kappa w_y^2 \right) f(t) \right) \\
+ \frac{1}{8} \kappa \sigma_1 x w_x^2 f(t) + \left(\frac{1}{16} \kappa y^2 w_x^2 - \frac{1}{4} \sigma_1 y w \right) f''(t) \right) \, dx \, dy \\
= f(t) \int_{\Omega} \left( \left(\frac{3}{8} \kappa \sigma_1 u_x^2 - \frac{1}{4} \sigma_1 u_x^2 w_y - \frac{1}{16} \kappa u^4 - \frac{1}{8} \kappa w_y^2 \right) \right) \, dx \, dy \\
+ \frac{1}{8} \kappa \sigma_1 f'(t) \int_{\Omega} x u^2 \, dx \, dy + f''(t) \int_{\Omega} \left(\frac{1}{16} \kappa y^2 u^2 - \frac{1}{4} \sigma_1 y w \right) \, dx \, dy \]

\[(28)\]
4. Line soliton solutions

A line soliton is a solitary wave in two dimensions,

\[ u = U(x + \mu y - \nu t) \]  

(29)

with

\[ U, U', U'', \text{etc.} \to 0 \text{ as } |x|, |y| \to \infty, \]  

(30)

where the parameters \( \mu \) and \( \nu \) determine the direction and the speed of the wave.

A more geometrical form for a line soliton is given by writing \( x + \mu y = (x, y) \cdot \mathbf{k} \) with \( \mathbf{k} = (1, \mu) \) being a constant vector in the \((x, y)\)-plane. The travelling wave variable can then be expressed as

\[ \xi = x + \mu y - \nu t = |\mathbf{k}|(\mathbf{k} \cdot (x, y) - ct) \]  

(31)

where the unit vector

\[ \mathbf{k} = (\cos \theta, \sin \theta), \quad \tan \theta = \mu \]  

(32)

gives the direction of propagation of the line soliton, and the constant

\[ c = \nu/|\mathbf{k}|, \quad |\mathbf{k}|^2 = 1 + \mu^2 \]  

(33)

gives the speed of the line soliton.

We will now derive the explicit line soliton solutions (29) for the scaled mKP family (8). It will be convenient to use the coordinate form of the travelling wave variable \( \xi = x + \mu y - \nu t \) for this derivation. Thus, we have \( u_x = U', \ u_y = \mu U', \ u_t = -\nu U' \), and so on, while \( \partial_x^{-1} u_y = \mu \partial_\xi^{-1} U' = \mu U \) by the solitary wave conditions (30). Substitution of the line soliton expression (29) into equation (8) yields a nonlinear fourth-order ODE

\[(\sigma_2 \mu^2 - \nu)U'' + \sigma_1 (U^2 U')' + \kappa \mu (UU')' + U'''' = 0 \]  

(34)

We can straightforwardly reduce this ODE to a separable form

\[ U'^2 = \left( (\nu - \sigma_2 \mu^2) - \frac{1}{3} \kappa \mu U - \frac{1}{6} \sigma_1 U^2 \right) U^2 \]  

(35)

after use of conditions (30).

**Proposition 4.1.** The general line soliton solution of the scaled mKP family (8) is given by

\[ u = \frac{6(\nu - \sigma_2 \mu^2)}{\sqrt{6\sigma_1 \nu + (\kappa^2 - 6\sigma_1 \sigma_2)\mu^2} \cosh(\sqrt{\nu - \sigma_2 \mu^2}(x + \mu y - \nu t)) + \kappa \mu} \]  

(36a)

where

\[ \nu - \sigma_2 \mu^2 > 0 \text{ if } \sigma_1 = 1; \quad 0 < \nu - \sigma_2 \mu^2 < \frac{1}{6} \kappa^2 \mu^2 \text{ and } \mu > 0 \text{ if } \sigma_1 = -1 \]  

(36b)

With respect to the \( x \) axis, the angle \( \theta \) of the direction of motion of the line soliton is given by \( \arctan(\mu) \), while the speed of the line soliton is given by \( \nu/\sqrt{1 + \mu^2} \). These two parameters obey the kinematic condition (36b) which depends crucially on the signs of \( \sigma_1 \) and \( \sigma_2 \).

In the case (9) representing the scaled mKP equation (10), the general line soliton solution (36) becomes

\[ u = \frac{3\sqrt{2}(\nu + \sigma_1 \mu^2)}{\sqrt{3\sigma_1 \nu + 4\mu^2} \cosh(\sqrt{\nu + \sigma_1 \mu^2}(x + \mu y - \nu t)) + \mu} \]  

(37a)
with the kinematic condition

\[ \nu > -\mu^2 \text{ if } \sigma_1 = -\sigma_2 = 1; \quad \mu^2 < \nu < \frac{4}{3} \mu^2 \text{ and } \mu > 0 \text{ if } \sigma_1 = -\sigma_2 = -1 \]  

(37b)

We will next discuss a few properties of the mKP family of line solitons (36) in comparison to the mKP line solitons (37).

4.1. Subfamily containing the mKP equation. To begin, we examine the case \( \sigma_1 \sigma_2 = -1 \), where the mKP family constitutes a one-parameter (\( \kappa \)) extension of the mKP equation. The line soliton (36) in this case is given by

\[ u = \frac{6(\nu + \sigma_1 \mu^2)}{\sqrt{6\sigma_1 \nu + (\kappa^2 + 6\mu^2) \cosh(\sqrt{\nu + \sigma_1 \mu^2(x + \mu y - \nu t)} + \kappa \mu)}, \quad \sigma_1 = \pm 1, \quad \kappa > 0 \]  

(38)

with the kinematic conditions

\[ \nu > -\mu^2 \text{ if } \sigma_1 = 1 \]  

(39)

\[ \mu^2 < \nu < \left(\frac{1}{6} \kappa^2 + 1\right) \mu^2 \text{ and } \mu > 0 \text{ if } \sigma_1 = -1 \]  

(40)

In the focussing case, \( \sigma_1 = 1 \), there is a minimum negative speed \( c > -\mu^2/\sqrt{1 + \mu^2} \) which is determined by the angle \( \theta = \arctan(\mu) \), while there is no maximum speed. In the defocussing case, \( \sigma_1 = -1 \), the speed has both a positive minimum and maximum, \( \mu^2/\sqrt{1 + \mu^2} < c < (1 + \frac{\kappa^2}{6}) \mu^2/\sqrt{1 + \mu^2} \). Note \( \kappa^2 = 2 \) recovers the mKP equation in both cases.

In the focussing case, plots of the mKP line soliton for different speeds and angles are shown in Fig. 1. Comparison plots of the mKP family line soliton with \( \kappa^2 \neq 2 \) are shown in Fig. 2 and Fig. 3.

![Figure 1](image1.png)

**Figure 1.** mKP line soliton (37) in the focussing case \( \sigma_1 = 1 \) with fixed speed \( c \) and different angles \( \theta \)

A similar comparison in the defocussing case is shown in Fig. 4 and Fig. 5.
Figure 2. mKP family line soliton (38) in the focussing case $\sigma_1 = 1$ with $c = 1$ and different angles

Figure 3. mKP family line soliton (38) in the focussing case $\sigma_1 = 1$ with $c = -1$ and different angles

4.2. Subfamily excluding the mKP equation. Last, we examine the case $\sigma_1\sigma_2 = 1$, where the mKP family is a strict generalization of the mKP equation. The line soliton (36) in this case is given by

$$u = \frac{6(\nu - \sigma_1\mu^2)}{\sqrt{6\sigma_1\nu + (\kappa^2 - 6\mu^2)cosh(\sqrt{\nu + \sigma_1\mu^2}(x + \mu y - \nu t)) + \kappa\mu}}, \quad \sigma_1 = \pm 1, \quad \kappa > 0$$

(41)

with the kinematic conditions

$$\nu > \mu^2 \text{ if } \sigma_1 = 1$$

(42)
In the focussing case, \( \sigma_1 = 1 \), here the speed has a positive minimum \( c > \mu^2/\sqrt{1 + \mu^2} \) and no maximum. In the defocussing case, \( \sigma_1 = -1 \), there is a minimum negative speed \( c > -\mu^2/\sqrt{1 + \mu^2} \), while the maximum speed is either positive if \( \kappa^2 > 6 \) or negative if \( \kappa^2 < 6 \). These conditions are qualitatively different compared to the conditions (39) in the mKP-like case.

Plots of the mKP family line soliton (41) for different speeds and angles are shown in Fig. 6 for the focussing case, and in Fig. 7 and Fig. 8 for the defocussing case.
Figure 6. mKP family line soliton (41) in the focusing case $\sigma_1 = 1$ with $c = 2$ and different angles.

Figure 7. mKP family line soliton (41) in the defocusing case $\sigma_1 = -1$ with $c = 2$ and different angles.

5. Concluding remarks

We have obtained all line soliton solutions in an explicit form and all low-order conservation laws for the family (2) of mKP equations which includes the well-known integrable case of the mKP equation (11). Depending on the coefficient of the nonlocal term in this equation, the line solitons can have a qualitatively different kinematic behaviour compared to the mKP line solitons.
Our results can be used as a starting point to investigate the stability of the line soliton solutions and to determine how their stability may depend on the coefficient of the nonlocal term.

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Appendix

The determining equation (15) for multipliers (16) with differential order less than four splits with respect to the set of variables \{\partial^4 w, \partial^5 w, \partial^6 w\}. We have carried out the setting up and splitting of the determining equation by using Maple. This yields an overdetermined system consisting of 3356 equations to be solved for \(Q\) as well as for \(\kappa \neq 0\), with \(c_1^2 = c_2^2 = 1\). Solving the system is a nonlinear problem because \(Q\) appears linearly in products with \(\kappa\). We use the Maple package ‘rifsimp’ to find the complete case tree of solutions. For each solution case in the tree, we solve the system of equations by using Maple ‘pdsolve’ and ‘dsolve’, and we check that the solution has the correct number of free constants/functions and satisfies the original overdetermined system. Finally, we merge overlapping cases by following the method explained in Ref. [14].

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