QUASI-NORMAL MODES
OF TOROIDAL, CYLINDRICAL AND PLANAR
BLACK HOLES IN ANTI-DE SITTER SPACETIMES:
SCALAR, ELECTROMAGNETIC AND GRAVITATIONAL
PERTURBATIONS

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Abstract
We study the quasi-normal modes (QNM) of scalar, electromagnetic and gravitational perturbations of black holes in general relativity whose horizons have toroidal, cylindrical or planar topology in an asymptotically anti-de Sitter (AdS) spacetime. The associated QNM frequencies describe the decay in time of the corresponding test field in the vicinities of the black hole. In terms of the AdS/CFT conjecture, the inverse of the frequency is a measure of the dynamical timescale of approach to thermal equilibrium of the corresponding conformal field theory.
1 Introduction

Black holes in anti-de Sitter (AdS) spacetimes in several dimensions have been recently study. One of the reasons for this intense study is the AdS/CFT conjecture which states that there is a correspondence between string theory in AdS spacetime and a conformal field theory (CFT) on the boundary of that space. For instance, M-theory on AdS$_4 \times S^7$ is dual to a non-abelian superconformal field theory in three dimensions, and type IIB superstring theory on AdS$_5 \times S^5$ seems to be equivalent to a super Yang-Mills theory in four dimensions [1, 2] (for a review see [3]).

All dimensions up to eleven are of interest in superstring theory, but experiment singles out four dimensions as the most important. In four-dimensional (4D) general relativity, an effective gravity theory in an appropriate string theory limit, the Kerr-Newman family of four-dimensional black holes can be extended to include a negative cosmological constant [4]. The horizon in this family has the topology of a sphere. There are, however, other families of black holes in general relativity with a negative cosmological constant with horizons having topology different from spherical. Here we want to focus on the family of black holes whose horizon has toroidal, cylindrical or planar topology [5, 6, 7] (see [8] for a review). We are going to perturb these black holes with scalar, electromagnetic and gravitational fields.

Perturbations of known solutions are very important to perform in order to study their intrinsic properties, such as the natural frequencies of the perturbations, and to test for the stability of the solutions themselves. For gravitational objects, such as a black hole, the vibrational pattern set by the perturbation obliges the system to emit gravitational waves. Thus, for black holes, the study of perturbations is closely linked to the gravitational wave emission. Due to the dissipative character of the emission of the gravitational waves the vibrational modes do not form a normal set, indeed in the spectrum each frequency is complex whose imaginary part gives the damping timescale. These modes are called quasi-normal modes (QNMs). The QNMs of a black hole appear naturally when one deals with the evolution of some field in the black hole spacetime, and serve as a probe to the dynamics outside its event horizon.

Much work has been done for black holes in asymptotically flat spacetimes (see [9] for a review), the main interest being related to the gravita-
tional waves emitted when these astrophysical objects form. In turn, the recent AdS/CFT correspondence conjecture has attracted much attention to the investigation of QNMs in anti-de Sitter spacetimes. According to it, the black hole corresponds to a thermal state in the conformal field theory, and the decay of the test field in the black hole spacetime, correlates to the decay of the perturbed state in the CFT. The dynamical timescale for the return to thermal equilibrium can be computed using the black hole characteristics plus the AdS/CFT correspondence. Since one can always compactify extra unused dimensions, the study of black holes in any permitted dimension (from 2 to 11) is useful. In 3D QNMs were studied in [10, 11, 12, 13] (see also [14]) for scalar perturbations and in [13] for Maxwell and Weyl perturbations. In 4D QNMs were studied in [13, 10, 17] for Schwarzschild-AdS black holes with minimally coupled scalar field perturbations and box type boundary conditions and in [19] with conformally coupled scalar field and asymptotically flat boundary conditions, in [18, 13, 20] for Reissner-Nordström black holes with scalar perturbations, in [21] for Schwarzschild-AdS black holes with Maxwell and gravitational perturbations, and in [22] for topological black holes with scalar perturbations. For higher dimensions, such as 5 and 7D see [11, 15], and for a recent work on super-radiance in the Kerr-Newman-anti-de Sitter geometry see [23].

In this paper we shall study scalar, electromagnetic and gravitational perturbations of the toroidal, cylindrical or planar black holes in an AdS spacetime found in [6]. The motivation to perturb with a scalar field can be seen as follows. If one has, e.g., 11-dimensional M-theory, compactified into a (toroidal BH)\(_4\) x (compact space) the scalar field used to perturb the black hole, can be seen as a type IIB dilaton which couples to a CFT field operator \(\mathcal{O}\). Now, the black hole in the bulk corresponds to a thermal state in the boundary CFT, and thus the bulk scalar perturbation corresponds to a thermal perturbation with nonzero \(\langle \mathcal{O} \rangle\) in the CFT. Similar arguments hold for the electromagnetic perturbations since they can be seen as perturbations for some generic gauge field in the low energy limit of 11-dimensional M-theory. On the other hand, gravitational perturbations are always of importance since they belong to the essence of the spacetime itself.

We will find that the QNM frequencies for scalar perturbations scale with the horizon radius, at least for large black holes. In the case of electromagnetic perturbations of large black holes, the characteristic QNM frequencies have only an imaginary part, and scale with the horizon radius. As for grav-
itational perturbations, there are two important features. First, contrary to
the asymptotically flat spacetime case, odd and even perturbations no longer
have the same spectra, although in certain limits one can still prove that the
frequencies are almost the same. The second result is that, for odd perturba-
tions, there is a mode with a totally different behavior from that found in the
scalar and electromagnetic case: in this mode the frequency scales with \( \frac{1}{r_+} \),
just as in asymptotically flat Schwarzschild spacetime. These features were
also found in our study of Schwarzschild-AdS black holes [21]. One could
have predicted that these features would also appear here, at least for large
black holes, because in the large horizon limit the spherical-AdS black holes
have the geometry of the cylindrical (planar or toroidal) ones.

2 Scalar, electromagnetic and gravitational
perturbations in a toroidal, cylindrical or
planar black hole in an AdS background

Throughout this paper, we shall deal with the evolution of some perturbation
in a spacetime geometry in general relativity with a background metric given
by [4]:

\[
 ds^2 = f(r) \, dt^2 - f(r)^{-1} \, dr^2 - r^2 \, dz^2 - r^2 \, d\phi^2
\]  

(1)

where

\[
 f(r) = \frac{r^2}{R^2} - \frac{4MR}{r},
\]  

(2)

\( M \) is the ADM mass of the black hole, and \( R \) is the AdS lengthscale \( R^2 = -\frac{3}{\Lambda} \), \( \Lambda \) being the cosmological constant. There is a horizon at \( r_+ = (4M)^{1/3} R \). The range of the coordinates \( z \) and \( \phi \) dictates the topology of the black hole
spacetime. For a black hole with toroidal topology, a toroidal black hole,
the coordinate \( z \) is compactified such that \( z/R \) ranges from 0 to \( 2\pi \), and \( \phi \)
ranges from 0 to \( 2\pi \) as well. For the cylindrical black hole, or black string,
the coordinate \( z \) has range \( -\infty < z < \infty \), and \( 0 \leq \phi < 2\pi \). For the planar
black hole, or black membrane, the coordinate \( \phi \) is further decompactified
\( -\infty < R \phi < \infty \) [4]. We will work with the cylindrical topology but the
results are not altered for the other two topologies.
According to the AdS/CFT correspondence solution (1) ($\times S^7$) is dual to a superconformal field theory in three dimensions with $\mathcal{N} = 8$. These toroidal black holes with Ricci flat horizon we are considering in (1) can be seen as the large horizon radius limit, $r_+/R >> 1$, of the spherical AdS black holes. The large black holes are the ones that matter most to the AdS/CFT correspondence [2]. Perturbations of spherical black holes were studied in [21] (see also references in the Introduction). Therefore the results we obtain here should be similar to the results we have obtained in [21] for large black holes. This will be confirmed below. For small black holes we will show that the spherical and toroidal yield different results.

2.1 Scalar field perturbations

For scalar perturbations, we are interested in solutions to the minimally coupled scalar wave equation

$$\Phi_{\mu\nu}^{\mu} = 0,$$

(3)

where, a comma stands for ordinary derivative and a semi-colon stands for covariant derivative. We make the following ansatz for the field $\Phi$

$$\Phi = \frac{1}{r}P(r)e^{-i\omega t}e^{ikz}e^{il\phi},$$

(4)

where $\omega$, $k$, and $l$, are the frequency, the wavenumber and the angular quantum numbers of the perturbation. If one is dealing with the toroidal topology then $k$ should be changed into an angular quantum number $\bar{l}$, $e^{ikz} \rightarrow e^{i\bar{l}z}R$. For the planar topology $e^{il\phi} \rightarrow e^{ikR\phi}$, where $\bar{k}$ is now a continuous wave number.

It is useful to use the tortoise coordinate $r_*$ defined by the equation $dr_* = dr / (r^2 / R^2 - 4MR/r)$. With the ansatz (4) and the tortoise coordinate $r_*$, equation (3) is given by,

$$\frac{d^2P(r)}{dr_*^2} + \left[\omega^2 - V_{\text{scalar}}(r)\right]P(r) = 0,$$

(5)

where,

$$V_{\text{scalar}}(r) = f \left(\frac{l^2}{r^2} + \frac{R^2k^2}{r^2} + \frac{f'}{r}\right),$$

(6)
with \( r = r(r_*) \) given implicitly and \( f' \equiv df/dr \). The rescaling to the radial coordinate \( \hat{r} = \frac{r}{R} \) and to the frequency \( \hat{\omega} = \omega R \) is equivalent to take \( R = 1 \) in (5) and (6), i.e., through this rescaling one measures the frequency and other quantities in terms of the AdS lengthscale \( R \).

### 2.2 Maxwell field perturbations

We consider the evolution of a Maxwell field in a cylindrical-AdS black hole spacetime with a metric given by (1). The evolution is governed by Maxwell’s equations:

\[
F^{\mu\nu} = 0, \quad F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}.
\]

One can again separate variables by the ansatz:

\[
A_{\mu}(t, r, \phi, z) = \begin{bmatrix}
g^{kl}(r) \\
h^{kl}(r) \\
k^{kl}(r) \\
j^{kl}(r)
\end{bmatrix} e^{-i\omega t} e^{ikz} e^{il\phi},
\]

When we put this expansion into Maxwell’s equations (7) we get the 2nd order differential equation for the perturbation:

\[
\frac{\partial^2 \Psi(r)}{\partial r^2} + \left[ \omega^2 - V_{\text{maxwell}}(r) \right] \Psi(r) = 0,
\]

where the wavefunction \( \Psi \) is a combination of the functions \( g^{lm}, h^{lm}, k^{lm} \) and \( j^{lm} \) as appearing in (8), and \( \Psi = -i\omega h^{lm} - \frac{dg^{lm}}{dr} \) (see [24] for further details). The potential \( V_{\text{maxwell}}(r) \) appearing in equation (9) is given by

\[
V_{\text{maxwell}}(r) = f(r) \left( \frac{l^2}{r^2} + \frac{R^2 k^2}{r^2} \right).
\]

Again we can take \( R = 1 \) and measure everything in terms of \( R \).

### 2.3 Gravitational perturbations

In our analysis of gravitational perturbations, we shall adopt a procedure analogous to that of Chandrasekhar [25], generalizing the calculation by the
introduction of the cosmological constant $\Lambda = -\frac{3}{R^2}$. The perturbed metric will be taken to be

$$ds^2 = e^{2\nu}dt^2 - e^{2\psi}(d\phi - wdt - q_2dr - q_3dz)^2 - e^{2\mu_2}dr^2 - e^{2\mu_3}dz^2.$$  \hspace{1cm} (11)

where the unperturbed quantities are $e^{2\nu} = \frac{r^2}{R^2} - \frac{4MR}{r}$, $e^{2\psi} = r^2$, $e^{2\mu_2} = \left(\frac{r^2}{R^2} - \frac{4MR}{r}\right)^{-1}$, $e^{2\mu_3} = \frac{r^2}{R^2}$ and all the other unperturbed quantities are zero. By observing the effect of performing $\phi \to -\phi$, and maintaining the nomenclature of the spherical symmetric case [25], it can be seen that the perturbations fall into two distinct classes: the odd perturbations (also called axial in the spherical case) which are the quantities $w$, $q_2$, and $q_3$, and the even perturbations (also called polar in the spherical case) which are small increments $\delta \nu$, $\delta \mu_2$, $\delta \mu_3$ and $\delta \psi$ of the functions $\nu$, $\mu_2$, $\mu_3$ and $\psi$, respectively. Note that since $z$ is also an ignorable coordinate, one could, in principle, interchange $\phi$ with $z$ in the above argument.

In what follows, we shall limit ourselves to axially symmetric perturbations, i.e., to the case in which the quantities listed above do not depend on $\phi$. In this case odd and even perturbations decouple, and it is possible to simplify them considerably.

2.3.1 Odd perturbations

We will deal with odd perturbations first. As we have stated, these are characterized by the non vanishing of $w$, $q_2$ and $q_3$. The equations governing these quantities are the following Einstein’s equations with a cosmological constant

$$G_{12} + \frac{3}{R^2}g_{12} = 0, \hspace{1cm} (12)$$

$$G_{23} + \frac{3}{R^2}g_{23} = 0. \hspace{1cm} (13)$$

The reduction of these two equations to a one dimensional second order differential equation is well known (see [25]), and we only state the results. By defining

$$Z^-(r) = r \left(\frac{r^2}{R^2} - \frac{4MR}{r}\right) \left(\frac{dq_2}{dz} - \frac{dq_3}{dr}\right), \hspace{1cm} (14)$$

One can easily check that $Z^-(r)$ satisfies

$$\frac{\partial^2 Z^-(r)}{\partial r^2} + \left[\omega^2 - V_{\text{odd}}(r)\right] Z^-(r) = 0, \hspace{1cm} (15)$$

7
where $V_{\text{odd}}(r)$ appearing in equation (13) is given by

$$V_{\text{odd}}(r) = f(r) \left( \frac{R^2 k^2}{r^2} - \frac{12MR}{r^3} \right).$$

(16)

### 2.3.2 Even perturbations

Even perturbations are characterized by non-vanishing increments in the metric functions $\nu$, $\mu_2$, $\mu_3$ and $\psi$. The equations which they obey are obtained by linearizing $G_{01} + \frac{3}{R^2} g_{01}$, $G_{03} + \frac{3}{R^2} g_{03}$, $G_{13} + \frac{3}{R^2} g_{13}$, $G_{11} + \frac{3}{R^2} g_{11}$ and $G_{22} + \frac{3}{R^2} g_{22}$ about their unperturbed values. Making the ansatz

$$
\delta \nu = N(r)e^{ikz}, \\
\delta \mu_2 = L(r)e^{ikz}, \\
\delta \mu_3 = T(r)e^{ikz}, \\
\delta \psi = V(r)e^{ikz},
$$

(17) (18) (19) (20)

we have from $\delta (G_{03} + \frac{3}{R^2} g_{03}) = 0$ that

$$V(r) = -L(r).$$

(21)

Inserting this relation and using (20) in $\delta (G_{01} + \frac{3}{R^2} g_{01}) = 0$, we have

$$
\left( \frac{3}{r} - \frac{f'}{2f} \right) L(r) + \left( \frac{f'}{2f} - \frac{1}{r} \right) T(r) + L(r)_r - T(r)_r = 0.
$$

(22)

From $\delta (G_{13} + \frac{3}{R^2} g_{13}) = 0$ we have

$$
\left( \frac{1}{r} + \frac{f'}{2f} \right) L(r) + \left( \frac{1}{r} - \frac{f'}{2f} \right) N(r) - N(r)_r + L(r)_r = 0,
$$

(23)

and from $\delta (G_{11} + \frac{3}{R^2} g_{11}) = 0$ we obtain

$$
-\frac{k^2 R^2}{f^2} N(r) + \left( \frac{k^2 R^2}{f^2} - \frac{\omega^2}{f^2} - \frac{\omega^2}{f^2} \right) L(r) + \frac{\omega^2}{f^2} T(r) + \\
\left( \frac{1}{r} + \frac{1}{r^2} \right) N(r)_r + \left( \frac{L'}{rf} - \frac{1}{r^2} \right) L(r)_r + \left( \frac{T'}{rf} + \frac{1}{r} \right) T(r)_r = 0.
$$

(24)

Multiplying equation (23) by $\frac{2}{r}$ and adding (24) we can obtain $N(r)$ and $N(r)_r$ in terms of $L(r)$, $L(r)_r$, $T(r)$ and $T(r)_r$. Using (22) and (23) we
can express $L(r)$, $N(r)$, and up to their second derivatives in terms of $T(r)$, $T(r)_r$ and $T(r)_{rr}$. Finally, we can look for a function

$$Z^+ = a(r)T(r) + b(r)L(r).$$

(25)

which satisfies the second order differential equation

$$\frac{\partial^2 Z^+(r)}{\partial r^2} + \left[\omega^2 - V_{\text{even}}(r)\right] Z^+(r) = 0,$$

(26)

Substituting (25) into (26) and expressing $L(r)$ and its derivatives in terms of $T(r)$ and its derivatives, we obtain an equation in $T(r)$, $T(r)_r$ and $T(r)_{rr}$ whose coefficients must vanish identically. If we now demand that $a(r)$ and $b(r)$ do not depend on the frequency $\omega$ we find

$$a(r) = \frac{r}{12Mr + k^2r^2},$$

(27)

$$b(r) = \frac{6M + k^2r}{72M^2 + 6k^2Mr},$$

(28)

and the potential $V_{\text{even}}(r)$ in (26) is

$$V_{\text{even}}(r) = f(r) \left[\frac{576M^3 + 12k^4Mr^2 + k^6r^3 + 144M^2r(k^2 + 2r^2)}{r^3(12M + k^2r)^2}\right].$$

(29)

As a final remark concerning the wave equations obeyed by odd and even gravitational perturbations, we note that it can easily be checked that the two potentials can be expressed in the form

$$V_{\text{odd}}_{\text{even}} = W^2 \pm \frac{dW}{dr^*} + \beta,$$

(30)

$$W = \frac{96M^2(k^2 + 3r^2)}{2k^2r^2(12M + k^2r)} + j,$$

(31)

where $j = -\frac{k^6 + 288M^2}{24k^4M^2}$ and $\beta = -\frac{k^8}{576M^2}$. It is worth of notice that the two potentials can be written in such a simple form (potentials related in this manner are sometimes called superpartner potentials [26]), a fact which seems to have been discovered by Chandrasekhar [25].
3 Quasi-normal modes and some of its properties

3.1 Boundary conditions

To solve (5), (9), (15) and (26) one must specify boundary conditions, a non-trivial task in AdS spacetimes. Consider first the case of a Schwarzschild black hole in an asymptotically flat spacetime (see [9]). Since the potential now vanishes at both infinity and the horizon, two independent solutions near these points are \( \Psi_1 \sim e^{-i\omega r_*} \) and \( \Psi_2 \sim e^{i\omega r_*} \), where the \( r_* \) coordinate now ranges from \(-\infty\) to \(\infty\). QNMs are defined by the condition that at the horizon there are only ingoing waves, \( \Psi_{\text{hor}} \sim e^{-i\omega r_*} \). Furthermore, one wishes to have nothing coming in from infinity (where the potential now vanishes), so one wants a purely outgoing wave at infinity, \( \Psi_{\text{infinity}} \sim e^{i\omega r_*} \). Clearly, only a discrete set of frequencies \( \omega \) meet these requirements.

Consider now our asymptotically AdS spacetime. The first boundary condition stands as it is, so we want that near the horizon \( \Psi_{\text{hor}} \sim e^{-i\omega r_*} \). However \( r_* \) has a finite range, so the second boundary condition needs to be changed. There have been a number of papers on which boundary conditions to impose at infinity in AdS spacetimes ([27]-[29]). We shall require energy conservation and adopt reflective boundary conditions at infinity [27] which means that the wavefunction is zero at infinity (see however [30]).

3.2 Numerical calculation of the QNM frequencies

To find the frequencies \( \omega \) that satisfy the previously stated boundary conditions we first change wavefunction to \( \phi = e^{i\omega r_*}Z \) (where, \( Z = P, \Psi, Z^+, Z^- \)). The wave equation then transforms into

\[
f(r) \frac{\partial^2 \phi}{\partial r^2} + (f' - 2i\omega) \frac{\partial \phi}{\partial r} - \frac{V}{f} \phi = 0. \tag{32}
\]

We now note that (32) has only regular singularities in the range of interest. It has therefore, by Fuchs theorem, a polynomial solution. To deal with the point at infinity, we first change the independent variable to \( x = \frac{1}{r_*} \). Now we can use Fröbenius method by looking for an indicial equation (for further
details see [15]), and force it to obey the boundary condition at the horizon \((x = \frac{1}{r_+} = h)\). We get

\[
Z(x) = \sum_{j=0}^{\infty} a_j(\omega)(x - h)^j,
\]

where \(a_j(\omega)\) is a function of the frequency. If we put (33) into (32) and use the boundary condition \(Z = 0\) at infinity \((x = 0)\) we get:

\[
\sum_{j=0}^{\infty} a_j(\omega)(-h)^j = 0
\]

Our problem is reduced to that of finding a numerical solution of the polynomial equation (34). The numerical roots for \(\omega\) of equation (34) can be evaluated, resorting to numerical computation. Obviously, one cannot determine the full sum in expression (34), so we have to determine a partial sum from 0 to \(N\), say and find the roots \(\omega\) of the resulting polynomial expression. We then move onto the next term \(N + 1\) and determine the roots. If the method is reliable, the roots should converge. We stop our search once we have a 3 decimal digit precision. One can label the roots \(\omega\) with the principal quantum number \(n\), \(\omega_n\). We have only looked for the roots with lowest imaginary part, e.g. \(n = 1, 2\). The \(n = 1\) solution is called the lowest QNM. By default we write the frequencies for the lowest \(n = 1\) and first excited \(n = 2\) QNMs as \(\omega\) simply, instead of \(\omega_1, \omega_2\).

As we will see there are frequencies with a vanishing real part, which makes it possible to use an approximation, due to Liu, to these highly damped modes [32, 33]. Although the method was originally developed for the asymptotically flat space, it is quite straightforward to apply it to our case. There is therefore a way to test our results. Unfortunately, this method relies heavily on having not only a pure imaginary frequency but also a frequency with a large imaginary part, so as we shall see it will only work for electromagnetic perturbations. We have computed the lowest frequencies for some values of the horizon radius \(r_+\), and \(l\). The frequency is written as \(\omega = \omega_r + i\omega_i\), where \(\omega_r\) is the real part of the frequency and \(\omega_i\) is its imaginary part. We present the results in tables 1-4.
3.2.1 Scalar:

| $r_+$ | $-\omega_i$   | $\omega_r$  |
|-------|---------------|------------|
| 0.1   | 0.266         | 0.185      |
| 1     | 2.664         | 1.849      |
| 5     | 13.319        | 9.247      |
| 10    | 26.638        | 18.494     |
| 50    | 133.192       | 92.471     |
| 100   | 266.373       | 184.942    |

Table 1. Lowest QNM of scalar perturbations for $l = 0$ and $k = 0$.

In table 1 we list the numerical values of the lowest ($n = 1$) QNM frequencies for the $l = 0$ scalar field QNM and for selected values of $r_+$. As discussed in Horowitz and Hubeny (HH) [15] the frequency should be a function of the scales of the problem, $R$ and $r_+$. However, they showed and argued that due to additional symmetries in the scalar field case and for large Schwarzschild-AdS black holes the frequency scales as $\omega \propto T$, with the temperature of the large black hole given by $T \propto r_+/R^2$, i.e., $\propto r_+$ in our units. This behavior is a totally different behavior from that of asymptotically flat space, in which the frequency scales with $1/r_+$. Now, for cylindrical (planar or toroidal) black holes and scalar fields this symmetry is present for any horizon radius, so $\omega \propto r_+$ always. For these black holes the temperature is also proportional to $r_+$, no matter how small the black hole is [3, 31]. Thus, the scalar field QNM frequencies are proportional to $T$, as can be directly seen from table 1, $\omega \propto r_+ \propto T$. The imaginary part of the frequency determines how damped the mode is, and according to the AdS/CFT conjecture is a measure of the characteristic time $\tau = \frac{1}{\omega_i}$ of approach to thermal equilibrium in the dual CFT (moreover, the frequencies do not seem to depend on the angular quantum number $l$, we have performed calculations for higher values of $l$). In the dual CFT the approach to thermal equilibrium is therefore faster for higher temperatures, i.e., larger black holes. This scaling for all horizon radii with temperature only happens in the scalar field case. For the electromagnetic and some of the gravitational perturbations the frequency scales with the temperature only in the large black hole regime, as we will show.
In table 2 we list the numerical values of the lowest \((n = 1)\) QNM frequencies for \(l = 1\) and for selected values of \(r_+\). For frequencies with no real part, we list the values obtained in the "highly damped approximation" \cite{32, 33}. We can note from table 2 that \(\omega\) is proportional to \(r_+\) and thus to the temperature for large black holes, \(r_+ \gtrsim 5\), say.

\subsection*{3.2.2 Electromagnetic:}

\begin{footnotesize}
\begin{center}
| \(r_+\) | \(\omega_i\) | \(\omega_r\) | \(-\omega_i\) | \(\omega_r\) |
|---|---|---|---|---|
| 0.1 | 0.104 | 1.033 | - | - |
| 1 | 1.709 | 1.336 | - | - |
| 5 | 7.982 | \(\sim 0\) | 7.500 | \(\sim 0\) |
| 10 | 15.220 | \(\sim 0\) | 15.000 | \(\sim 0\) |
| 50 | 75.043 | \(\sim 0\) | 75.000 | \(\sim 0\) |
| 100 | 150.021 | \(\sim 0\) | 150.000 | \(\sim 0\) |
\end{center}
\end{footnotesize}

Table 2. Lowest QNM of electromagnetic perturbations for \(l = 1\) and \(k = 0\).

In the large black hole regime, both scalar and electromagnetic QNM frequencies for this geometry are very similar to those of the Schwarzschild-anti-de Sitter black hole \cite{13, 21}. This is a consequence of the fact that in this regime, the wave equation for the fields become identical in both geometries. Indeed we can compare table 1 for the scalar field with table 1 of HH \cite{13} for the same field. We see that for \(r_+ = 1\) one has for the toroidal black hole \(\omega = 1.849 - 2.664i\), whereas HH find \(\omega = 2.798 - 2.671i\). Thus they differ, \(r_+ = 1\) is not a large black hole. For \(r_+ \gg 1\) one expects to find very similar QNM frequencies. For instance, for \(r_+ = 100\) we obtain \(\omega = 184.942 - 266.373i\) for the toroidal black hole, while HH obtain \(\omega = 184.953 - 266.385i\) for the Schwarzschild-anti-de Sitter black hole. The same thing happens for electromagnetic perturbations. We can compare table 2 for the electromagnetic field with table I of Cardoso and Lemos \cite{21}. For large black holes one can see that the frequencies for toroidal black holes are very similar to the frequencies of the Schwarzschild-anti-de Sitter black hole. For instance, for \(r_+ = 100\) we find \(\omega = -150.021i\) for the toroidal black hole,
while in [21] we found $\omega = -150.048i$ for the Schwarzschild-anti-de Sitter black hole. We can also see that the QNM modes in the electromagnetic case are in excellent agreement with the analytical approximation for strongly damped modes.

### 3.2.3 Gravitational:

The numerical calculation of the QNM frequencies for gravitational perturbations proceeds as outlined previously (the associated differential equation has only regular singularities, so it is possible to use an expansion such as (33)). In tables 3 and 4 we show the two lowest lying ($n = 1, 2$) QNM frequencies for $l = 2$ and $l = 3$ gravitational perturbations.

We first note that there is clearly a distinction between odd and even perturbations: they no longer have the same spectra, contrary to the asymptotically flat space case (see [34]). This problem was studied in some detail by Cardoso and Lemos [21] who showed that it is connected with the behavior of $W$ (see equation (31)) at infinity.

#### odd (axial) modes:

| $r_+$ | $\omega_i$ | $\omega_r$ | $-\omega_i$ | $\omega_r$ |
|-------|------------|------------|-------------|------------|
| 1     | 2.646      | $\sim 0$  | 2.047       | 2.216      |
| 5     | 0.2703     | $\sim 0$  | 13.288      | 9.355      |
| 10    | 0.13378    | $\sim 0$  | 26.623      | 18.549     |
| 50    | 0.02667    | $\sim 0$  | 133.189     | 92.482     |
| 100   | 0.0134     | $\sim 0$  | 266.384     | 184.948    |

Table 3. Lowest and second lowest QNMs of gravitational odd perturbations for $k = 2$.  

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even (polar) modes:

| $r_+$ | $-\omega_i$ | $\omega_r$ |
|-------|-------------|-------------|
| 1     | 1.552       | 2.305       |
| 5     | 12.633      | 9.624       |
| 10    | 26.296      | 18.696      |
| 50    | 133.124     | 92.512      |
| 100   | 266.351     | 184.963     |

Table 4. Lowest QNM of gravitational even perturbations for $k = 2$.

For odd gravitational QNMs the lowest one scales with $\frac{1}{r_i} \propto \frac{1}{T}$. This is odd, but one can see that it is a reflection of the different behavior of the potential $V_{\text{odd}}$ for odd perturbations. For the second lowest odd gravitational QNM table 3 shows that for large black holes it scales with $T$. For the lowest even gravitational QNM table 4 shows that it also scales with $T$ for large black holes. Note further that the scalar, odd second lowest and even gravitational QNMs are very similar in the large black hole regime. Indeed, tables 1, 3 and 4 show a remarkable resemblance even though the potentials are so different. Finally, let us compare tables 3 and 4 with tables III-V of [21]. We see that for large black holes the frequencies of toroidal black holes are again very similar to those of the Schwarzschild-anti-de Sitter black hole. For instance, for odd perturbations and $r_+ = 100$ we find from table 3 $\omega = -0.0134i$ for the toroidal black hole, while in [21] we found (table III) $\omega = -0.0132i$ for the Schwarzschild-anti-de Sitter black hole.

4 Conclusions

We have computed the scalar, electromagnetic and gravitational QNM frequencies of the toroidal, cylindrical or planar black hole in four dimensions. These modes dictate the late time behaviour of a minimally coupled scalar, electromagnetic field and of small gravitational perturbations, respectively. The main conclusion to be drawn from this work is that these black holes
are stable with respect to small perturbations. In fact, as one can see, the frequencies all have a negative imaginary part, which means that these perturbations will decay exponentially with time. For odd gravitational perturbations in the large black hole regime, the imaginary part of the frequency goes to zero scaling with $\frac{1}{r_+}$, just as in asymptotically flat space and in the odd gravitational perturbations of Schwarzschild-Ads black hole. In terms of the AdS/CFT correspondence, this implies that the greater the mass, the more time it takes to approach equilibrium, a somewhat puzzling result. Apart from this interesting result, the frequencies all scale with the horizon radius, at least in the large black hole regime, supporting the arguments given in [15]. The QNM for toroidal, cylindrical or planar black holes (in anti-de Sitter space) are quite similar to those of the Schwarzchild-anti-de Sitter black hole [15, 21].

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