Subregular characters of the group UT(n, ℝ)

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1 Introduction

Notion of character of representation plays an important role in the representation theory of groups. For finite dimensional representation a character \( \chi(g) \) is defined as a trace of matrix of operator \( T_g \). This definition cannot be directly applied for infinite dimensional representations. For some representations of Lie groups one can define a character as a generalized function on the group as follows.

Extend a representation \( T_g \) of a group \( G \) to the representation of the group algebra \( L^1(G) \) by the formula

\[
T_\varphi = \int \varphi(g) T_g dg,
\]

where \( dg \) is the left-invariant measure on the group. Suppose that for any finite function the operator \( T_\varphi \) has the trace. Then the formula

\[
(\chi, \varphi) = \text{Tr}(T_\varphi)
\]

defines the trace \( \chi(g) \) as a generalized function on the Lie group \( G \).

It was proved in the paper [1] that any irreducible representation of a connected nilpotent Lie group has a trace in mentioned sense. For any connected nilpotent Lie group and for a character of irreducible representation, which is associated with a coadjoint orbit \( \Omega \), the formula of A.A.Kirillov is valid

\[
(\chi, \varphi) = \int \hat{\varphi}(a) d_\Omega \mu(a), \quad (1)
\]

where

\[
\hat{\varphi}(a) = \int \varphi(\exp(x)) e^{2\pi i(a,x)} dx
\]

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is the Fourier transformation with respect to a given Lebesgue measure $dx$ on the Lie algebra $\mathfrak{g}$ of $G$, $d_\Omega \mu$ is the invariant measure on the orbit $\Omega$ (see [1] and [2, chapter 3]). But it may be very difficult to use this formula for calculation of the character, when the orbit is a complicated manifold, defined by a system of many equations. It will be better to use the other approach: if we represent the operator $T_\phi$ in the integral form

$$T_\phi F(x) = \int A(x, y)F(y)dy,$$  \hspace{1cm} (2)

then the trace is equal to $\int A(x, x)dx$ (see [1]). The $\delta$-function type multipliers rise in the character; it implies that the support of character doesn’t coincide with the group. Calculations of characters for different connected nilpotent Lie groups imply the following general conjecture.

**Conjecture 1.1.** Given a connected nilpotent Lie group, the support of a character of irreducible representation, associated with an coadjoint orbit $\Omega$, coincides with the closure of union of stabilizers $G^f$, where $f \in \Omega$.

The regular characters (i.e. characters of irreducible representations, associated with orbits of maximal dimension) of $\text{UT}(n, \mathbb{R})$ were calculated in the paper [1]. Recall that any coadjoint orbit has even dimension. An coadjoint orbit is called subregular if its dimension equals to $d - 2$, where $d$ is a maximal dimension of coadjoint orbits. A character of irreducible representation, associated with a subregular coadjoint orbit, is called a subregular character. Subregular coadjoint orbits of the group $\text{UT}(n, \mathbb{R})$ were classified in [4]. Basing on this classification we obtain formulas (theorems 3.3 and 3.6) for subregular characters. In this paper we also calculate regular characters (see theorems 2.1 and 2.2), since there is no proof of this formulas in [1] and there is some unexpected misprint (in the case of odd $n$). The obtained formulas confirm Conjecture 1.1 in the case of regular and subregular characters of the group $\text{UT}(n, \mathbb{R})$ (see Theorem 3.7).

Note that subregular characters for the case of finite field were obtained in [3]. We use the following notations:

1) if $C$ is a matrix with functional entries, then $\delta(C)$ is a product of $\delta$-functions at zero of its entries;
2) if $C$ is an unitriangular matrix with functional entries upper the diagonal, then we preserve the notation $\delta(C)$ for the product of $\delta$-functions at zero of its entries upper the diagonal;
3) if $C_1, \ldots, C_m$ is a system of matrix, then $\delta(C_1, \ldots, C_m) = \delta(C_1) \cdots \delta(C_m)$. 

2
2 Regular characters

The group \( G = \text{UT}(n, \mathbb{R}) \) is a group of upper triangular matrices of size \( n \) with units on the diagonal. Its Lie algebra \( \mathfrak{g} = \text{ut}(n, \mathbb{R}) \) consists of all upper triangular matrices with zeros on the diagonal. Applying the Killing form \((\cdot, \cdot)\) we identify the conjugate space \( \mathfrak{g}^* \) with the set of all lower triangular matrices with zeros on the diagonal. According to the orbit method of A.A.Kirillov, there exists one to one correspondence between irreducible representations of a connected nilpotent Lie group and its coadjoint orbits. Any element \( f \in \mathfrak{g}^* \) has a polarization \( \mathfrak{p} \) (i.e. a subalgebra that is a maximal isotropic subspace with respect to the skew symmetric bilinear form \( f([x, y]) \)). The irreducible representation, associated with the coadjoint orbit \( \Omega(f) \), is induced from one dimensional representation

\[
\xi(\exp(x)) = e^{2\pi i f(x)}, \text{ where } x \in \mathfrak{p},
\]

of the subgroup \( \exp(\mathfrak{p}) \).

2.1 Case of even \( n \).

Let \( n = 2k \). Let us represent the general element \( g \in \text{UT}(n, \mathbb{R}) \) in the form of block matrix

\[
g = \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix}, \tag{3}
\]

where \( C_{ij} \) are blocks of size \( k \times k \). Any regular coadjoint orbit of the group \( \text{UT}(n, \mathbb{R}) \), where \( n = 2k \), contains a unique element of the form

\[
f = \begin{pmatrix} 0 & 0 \\ \Lambda & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & \ldots & \lambda_k \\ \vdots & \ddots & \vdots \\ \lambda_1 & \ldots & 0 \end{pmatrix}, \tag{4}
\]

where \( \lambda_1, \ldots, \lambda_k \in \mathbb{R} \) and \( \lambda_1, \ldots, \lambda_{k-1} \neq 0 \) (see \cite{4 §3}). We use the following notations. For every \( 1 \leq s \leq k \) we denote

1) \( \Delta_s = \begin{vmatrix} c_{s,1} & \cdots & c_{s,k-s+1} \\ \vdots & \ddots & \vdots \\ c_{k,1} & \cdots & c_{k,k-s+1} \end{vmatrix} \) the left lower minor of the block \( C_{12} = (c_{ij})_{i,j=1}^k \),

2) \( P_s = \frac{(-1)^{k-s} \Delta_s}{\Delta_{s+1}} \).
Theorem 2.1. Let $n = 2k$. The character of the irreducible representation, associated with the orbit of element $f \in \mathfrak{u}(n, \mathbb{R})^*$ of (4), has the form
\[
\chi(g) = \delta(C_{11}, C_{22}) \chi_\Lambda^*(C_{12}),
\]
where
\[
\chi_\Lambda^*(C_{12}) = \frac{e^{2\pi i (\lambda_1 p_1 + \ldots + \lambda_k p_k)}}{|\Delta_2 \Delta_3 \ldots \Delta_k|}, \quad \mu_0 = |\lambda_1^{k-1} \lambda_2^{k-2} \cdots \lambda_k^0|.
\]

Proof. The subalgebra \( \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \) is a polarization of \( f \). The irreducible representation, associated with the orbit \( \Omega(f) \), is induced from the one-dimensional representation \( e^{2\pi i (\Lambda, B)} \) of subgroup \( \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \). This representation is realized in the space \( L^2(\mathcal{X}) \), where \( \mathcal{X} \) consists of matrices \( X = \text{diag}(X_{11}, X_{22}) \) with blocks of \( \text{UT}(k, \mathbb{R}) \), by the formula
\[
T_g F(X) = e^{2\pi i (\Lambda, X_{11}C_{12}(X_{22}C_{22})^{-1})} F(X_{11}C_{11}, X_{22}C_{22}).
\]
Operator \( T_g \) is extended to operator
\[
T_\phi F(X) = \int \phi(C_{11}, C_{12}, C_{22}) e^{2\pi i (\Lambda, X_{11}C_{12}(X_{22}C_{22})^{-1})} F(X_{11}C_{11}, X_{22}C_{22}) dC_{11} dC_{12} dC_{22}.
\]
After substitution \( Y_{11} = X_{11}C_{11}, \ Y_{22} = X_{22}C_{22} \) we obtain \( T_\phi \) in the form (2):
\[
T_\phi F(X) = \int \phi(X_{11}^{-1} Y_{11}, C_{12}, X_{22}^{-1} Y_{22}) e^{2\pi i (\Lambda, X_{11}C_{12}Y_{22}^{-1})} F(Y_{11}, Y_{22}) dY_{11} dY_{22} dC_{12}.
\]
Then
\[
(\chi, \phi) = \int \phi(E, C_{12}, E) e^{2\pi i (\Lambda, X_{11}C_{12}X_{22}^{-1})} dX_{11} dX_{22} dC_{12}.
\]
After substitution \( X_{22}^{-1} \) by \( X_{22} \) we have
\[
\chi(g) = \delta(C_{11}, C_{22}) \int e^{2\pi i (\Lambda, X_{11}C_{12}X_{22})} dX_{11} dX_{22}
\]
(6)
Let \( X_{11}, X_{22} \) be unitriangular matrices of size \( k \) with entries \( x_{ij}, \ y_{ij}, \ 1 \leq i < j \leq k \) respectively.
For every \( 1 \leq s \leq k \) and \( 1 \leq j \leq k - s + 1 \) we apply the notations
\[
\tilde{c}_{s,j} = c_{s,j} + x_{s,s+1}c_{s+1,j} + \ldots + x_{s,k}c_{k,j},
\]
\[
\chi_s(g) = \int e^{2\pi i \lambda_s(\tilde{c}_{s,1} y_{i,k-s+1} + \ldots + \tilde{c}_{s,k-s} y_{k-k-s+1} + \tilde{c}_{s,k-s+1})} d x_s d y_s,
\]
(7)
where $dx_s = dx_{s,s+1} \ldots dx_{s,k}$ and $dy_s = dy_{1,k-s+1} \ldots dy_{k,k-s+1}$. Taking into account (7), the formula (6) is rewritten in the form

$$
\chi(g) = \delta(C_{11}, C_{22}) \prod_{s=1}^{k} \chi_s(g) \tag{8}
$$

Apply the well known equality

$$
\int e^{2\pi i \lambda(a,x)} dx = \delta(a) \frac{1}{|\lambda|}
$$
in (7), we obtain

$$
\chi_s(g) = \frac{1}{|\lambda_s|^{k-s}} \int \delta(\tilde{c}_{s,1}, \ldots, \tilde{c}_{s,k-s}) e^{2\pi i \lambda_s \tilde{c}_{s,k-s+1}} dx_s. \tag{9}
$$

Let $(x_{0,s+1}^0, \ldots, x_{s,k}^0)$ be a solution of the system of linear equations

$$
\begin{aligned}
\tilde{c}_{s,1} &= c_{s,1} + x_{s,s+1}c_{s+1,1} + \ldots + x_{s,k}c_{k,1} = 0, \\
&\vdots \\
\tilde{c}_{s,k-s} &= c_{s,k-s} + x_{s,s+1}c_{s+1,k-s} + \ldots + x_{s,k}c_{k,k-s} = 0.
\end{aligned} \tag{10}
$$

Denote

$$
P'_s = c_{s,k-s+1} + x_{s,s+1}^0c_{s+1,k-s+1} + \ldots + x_{s,k}^0c_{k,k-s+1}.
$$

Let us introduce a new variable $x_{s,s}$. The vector $(1, x_{s,s+1}^0, \ldots, x_{s,k}^0)$ is a solution of the system of linear equations

$$
\begin{aligned}
x_{s,s}c_{s,1} + x_{s,s+1}c_{s+1,1} + \ldots + x_{s,k}c_{k,1} &= 0, \\
&\vdots \\
x_{s,s}c_{s,k-s} + x_{s,s+1}c_{s+1,k-s} + \ldots + x_{s,k}c_{k,k-s} &= 0, \\
x_{s,s}c_{s,k-s+1} + x_{s,s+1}c_{s+1,k-s+1} + \ldots + x_{s,k}c_{k,k-s+1} &= P'_s.
\end{aligned} \tag{11}
$$

Using the Cramer formulas in (11), we have

$$
1 = \frac{(-1)^{k-s} \Delta_{s+1} P'_s}{\Delta_s}.
$$

It implies

$$
P'_s = \frac{(-1)^{k-s} \Delta_s}{\Delta_{s+1}} = P_s, \quad \chi_s(g) = \frac{1}{|\lambda_s|^{k-s}|\Delta_{s+1}|} e^{2\pi i \lambda_s P_s} P'_s.
$$

Substituting $\chi_s(g)$ into (8), we obtain the formula for the regular character for even $n$. □
2.2 Case of odd \( n \).

Let \( n = 2k + 1 \). Let us represent the general element \( g \in \text{UT}(n, \mathbb{R}) \) in the form of block matrix
\[
g = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ 0 & 1 & C_{23} \\ 0 & 0 & C_{33} \end{pmatrix},
\]
where the partition of matrix into blocks corresponds to the partition \((k, 1, k)\) of its rows and columns. Any regular coadjoint orbit of the group \( \text{UT}(n, \mathbb{R}) \), where \( n = 2k + 1 \), contains a unique element of the form
\[
f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Lambda & 0 & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & \ldots & \lambda_k \\ \vdots & \ddots & \vdots \\ \lambda_1 & \ldots & 0 \end{pmatrix},
\]
where \( \lambda_1, \ldots, \lambda_k \in \mathbb{R}^* \) (see [4, §3]).

**Theorem 2.2** ([1]). Let \( n = 2k + 1 \). The character of the irreducible representation, associated with the orbit of element \( f \in \text{ut}(n, \mathbb{R})^* \) of (13), has the form
\[
\chi(g) = \delta(C_{11}, C_{33}, C_{12}, C_{23}) \frac{\chi^*_\Lambda(C_{13})}{|\det \Lambda|},
\]
where \( g \) as in (12), \( \chi^*_\Lambda(C_{13}) \) as in (5).

**Proof.** Subalgebra \( \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \) is a polarization for \( f \). The irreducible representation, associated with the orbit \( \Omega(f) \), is induced from the one-dimensional representation \( e^{2\pi i (\Lambda, B_{13})} \) of the subgroup \( \left\{ \begin{pmatrix} 0 & 0 & B_{13} \\ 0 & 0 & B_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\} \). This representation is realized in the space \( L^2(\mathcal{X}) \), where \( \mathcal{X} \) consists of matrices
\[
X = \begin{pmatrix} X_{11} & X_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & X_{33} \end{pmatrix},
\]
by the formula
\[
T_g F(X) = e^{2\pi i (\Lambda, (X_{11}C_{13} + X_{12}C_{23})X_{33}C_{33})^{-1})} F(X_{11}, X_{11}C_{13} + X_{12}, X_{33}C_{33}).
\]
Arguing as in subsection 2.1, we prove that
\[ \chi(g) = \delta(C_{11}, C_{33}, C_{12}) \int e^{2\pi i \langle \Lambda, (X_{11}C_{13} + X_{12}C_{23})X_{33} \rangle} dX_{11} dX_{12} dX_{33}. \]

Rewrite this formula in the form
\[ \chi(g) = \delta(C_{11}, C_{33}, C_{12}) \cdot J \int e^{2\pi i \langle \Lambda, X_{11}C_{13}X_{33} \rangle} dX_{11} dX_{33}, \tag{15} \]
where
\[ J = \int e^{2\pi i \langle \Lambda, X_{12}C_{23}X_{33} \rangle} dX_{12}. \]

Let us show that
\[ J = \frac{1}{|\lambda_1 \cdots \lambda_k|} \delta(C_{23}). \tag{16} \]

Let
\[
X_{12} = \begin{pmatrix} x_{1,k+1} \\ \vdots \\ x_{k,k+1} \end{pmatrix}, \quad C_{23} = (c_1, \ldots, c_k), \quad X_{23} = \begin{pmatrix} 1 & y_{12} & \cdots & y_{1k} \\ 0 & 1 & \cdots & y_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.
\]

Then
\[
(\Lambda, X_{12}C_{23}X_{33}) = \sum_{s=1}^{k} \lambda_s M_s,
\]
where
\[ M_s = x_{s,k+1} (c_1y_{1,k-s+1} + \cdots + c_{k-s}y_{k-s,k-s+1} + c_{k-s+1}). \]
In particular, \( M_k = x_{k,k+1} c_1 \), \( M_{k-1} = x_{k-1,k+1} (c_1y_{12} + c_2) \). Denote
\[
(\Lambda, X_{12}C_{23}X_{33})_t = \sum_{s=t}^{k} \lambda_s M_s, \quad J_t = \int e^{2\pi i \langle \Lambda, X_{12}C_{23}X_{33} \rangle_t} dx_{t,k+1} \cdots dx_{k,k+1}
\]

Note that
\[
(\Lambda, X_{12}C_{23}X_{33})_1 = (\Lambda, X_{12}C_{23}X_{33})_1, \quad J_1 = J,
\]
\[
(\Lambda, X_{12}C_{23}X_{33})_t = \lambda_t M_t + (\Lambda, X_{12}C_{23}X_{33})_{t+1}.
\]

We use induction on \( t \), moving in decreasing order from \( k \) to 1, to prove
\[ J_t = \delta(c_1, \ldots, c_{k-t+1}) \frac{1}{|\lambda_t \cdots \lambda_k|}. \tag{17} \]
For \( t = k \) we have
\[
J_k = \int e^{2\pi i \lambda_k} dx_{k,k+1} = \int e^{2\pi i \lambda_{k,k+1} c_1} dx_{k,k+1} = \frac{1}{|\lambda_k|} \delta(c_1),
\]
this proves (17). Assume that (17) is proved for \( t + 1 \); let us prove it for \( t \):
\[
J_t = \int e^{2\pi i (\Lambda, X_{23} X_{33})} dx_{t,k+1} \cdots dx_{k,k+1} = \int e^{2\pi i \lambda_t} \delta(c_{t+1}, \ldots, c_{k-t+1}) \frac{1}{|\lambda_t \cdots \lambda_k|} dx_{t,k+1} = \delta(c_1, \ldots, c_{k-t+1}) \frac{1}{|\lambda_t \cdots \lambda_k|}
\]
This proves (17). Substitute (17) into (15). Calculation of the integral (15) concludes similarly subsection 2.1. \( \Box \)

3 Subregular characters

3.1 Case of even \( n \).

Let \( n = 2(k+m+2) \). Partition rows and columns into blocks \((m, 1, 1, k, k, 1, 1, m)\).

The general element \( g \in UT(n, \mathbb{R}) \) can be written as a block matrix
\[
g = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} & C_{17} & C_{18} \\
0 & 1 & C_{23} & C_{24} & C_{25} & C_{26} & C_{27} & C_{28} \\
0 & 0 & 1 & C_{34} & C_{35} & C_{36} & C_{37} & C_{38} \\
0 & 0 & 0 & C_{44} & C_{45} & C_{46} & C_{47} & C_{48} \\
0 & 0 & 0 & 0 & C_{55} & C_{56} & C_{57} & C_{58} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & C_{67} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{78} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\] (18)

It follows from \( \mathbb{H} \ §3 \) that any subregular coadjoint orbit of the group \( UT(n, \mathbb{R}) \), where \( n = 2(k + m + 2) \), contains a unique element of the form
\[
f = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\] (19)

where \( \Lambda_1 \) (resp. \( \Lambda_2 \)) is a quadratic matrix of the form (14) of size \( m \) (resp. \( k \)).

All entries of the second diagonal of the matrix \( \Lambda_1 \) and \( \gamma_1, \gamma_2 \) do not equal to
zero. All entries of the second diagonal of the matrix $\Lambda_2$, besides the last entry (see (4)), do not equal to zero.

Denote by $E_{45}$ a system of $k$ matrix units that correspond to the places of second diagonal of the block $C_{45}$ of the matrix $g$ (see (18)). Similarly, $E_{18}$ is a system of $k$ matrix units that correspond to the places of second diagonal of the clock $C_{18}$. The following lemma is proved by direct calculation.

**Lemma 3.1.** 1) Stabilizer $g^f$ in the Lie algebra $\mathfrak{g} = \mathfrak{ut}(n, \mathbb{R})$ is a subspace spanned by

$$E_{45}, E_{18}, E_{26}, E_{27}, E_{37}, \gamma_2 E_{23} + \gamma_1 E_{67};$$

(20)

2) Stabilizer $G^f$ in the Lie group $G = \text{UT}(n, \mathbb{R})$ equals to $E + g^f$.

Construct the following table of size $8 \times 8$. By symbol ”X” we mark all places $(a, b)$ such that $E_{a,b}$ rises in (20).

|   |   |   |   |   | X |
|---|---|---|---|---|---|
|   |   |   | X |   |   |
| X | X | X |   |   |   |
|   |   | X |   |   |   |
| X |   |   |   |   |   |
|   |   |   |   |   |   |

Table 1.

For any entry $c$ of the matrix (18) that is lying in one of the blocks, marked in the Table 1 by the symbol ”•” or that is lying in the block $C_{18}$, we define a rational function $\tilde{c}$ as follows. Denote $\Phi = \{(2, 3), (4, 5), (6, 7)\}$. Let $c$ be an entry of the block $C_{ij}$ and $\Phi(c)$ be a set of all pairs $(a, b) \in \Phi$ such that $a > i$ and $b < j$. Let $\Delta(c)$ be a minor of the matrix (18) with rows and columns of $\Phi(c)$. Let $\tilde{\Delta}(c)$ be a minor of the matrix (18), constructed by adding the row (resp. column) of the entry $c$ to the system of rows (resp. columns) of the minor $\Delta(c)$. Denote

$$\tilde{c} = \pm \Delta(c)^{-1}\tilde{\Delta}(c),$$

(21)

where the sign $\pm$ coincides with the sign of entry $c$ in the minor $\tilde{\Delta}(c)$. For example, if $m = k = 1$ (i.e. every block $C_{ij}$ is an entry $c_{ij}$), then

$$\tilde{c}_{12} = c_{12}, \quad \tilde{c}_{34} = c_{34}, \quad \tilde{c}_{56} = c_{56}, \quad \tilde{c}_{78} = c_{78}, \quad \tilde{c}_{14} = -\frac{c_{23}}{c_{23}}, \quad \begin{vmatrix} c_{13} & c_{14} \\ c_{23} & c_{23} \end{vmatrix}.$$
Denote:
1) $S_0$ is the set of entries lying upper the diagonal in the blocks $C_{11}, C_{44}, C_{55}, C_{88}$;
2) $S_1$ is the set of all rational functions $\tilde{c}$, where $c$ runs through all entries in the blocks that is marked in the Table 1 by the symbol "\$";
3) $S_2 = \{C_{23}C_{35} + C_{24}C_{45}, C_{45}C_{57} + C_{46}C_{67}, \gamma_1C_{23} - \gamma_2C_{67}\}$;
4) $S = \{S_0, S_1, S_2\}$,
5) $d(g) = \det C_{23} \cdot \det C_{45} \cdot \det C_{67}$.

The algebra $\mathbb{R}[G]$ of regular functions on the group $G = UT(n, \mathbb{R})$ admits localization $\mathbb{R}'[G]$ by the denominator system generated by $d(g)$. Note that $S \subset \mathbb{R}'[G]$. Let $I'$ be an ideal in $\mathbb{R}'[G]$ generated by $S$. The annihilator of the ideal $I'$ is the subset in $G' = \{g \in G : d(g) \neq 0\}$.

**Proposition 3.2.** Let $f$ be of the form (19). Then the closure in $G$ of the annihilator of the ideal $I'$ coincides with the closure of the set $\text{Ad}_G(G')$.

**Proof.** The Lemma 3.1 implies that a function of $S$ annihilate $G'$. One can prove directly that $S$ annihilate $\text{Ad}_G(G')$. Then $\text{Ann}I' \supset \text{Ad}_G(G') \cap G'$. The ideal $I'$ is prime and its dimension coincides with the dimension of the set $\text{Ad}_G(G')$. □

**Theorem 3.3.** Let $n = 2(k + m + 2)$. The character of the irreducible representation, associated with the orbit of element $f \in \text{ut}(n, \mathbb{R})^*$ of (19), has the form

$$
\chi(g) = \delta(S) \cdot \chi_{\Lambda_1}(\widetilde{C}_{18}) \cdot \chi_{\Lambda_2}(C_{45}) \cdot \chi_0(g)
$$

where $\chi_{\Lambda_2}(C_{45})$ as in (5), the matrix $\widetilde{C}_{18}$ is filled by entries $\tilde{c}$, where $c$ is an corresponding entry of the block $C_{18}$ (see (21), $\chi_{\Lambda_1}(\widetilde{C}_{18})$ as in (5), and

$$
\chi_0(g) = \frac{1}{|\gamma_1\gamma_2|^k \cdot d(g)^m} e^{2\pi i \left(\frac{\gamma_1}{c_{67}}Q_0 + \gamma_3C_{67}\right)},
$$

where $Q_0 = c_{23}c_{37} + c_{24}c_{47} + c_{25}c_{57} + c_{26}c_{67}$.

Remark, since the element $\gamma_1C_{23} - \gamma_2C_{67}$ belongs to the set $S$, one can substitute $\frac{\gamma_1}{c_{67}}$ by $\frac{\gamma_2}{c_{23}}$ in the formula for $\chi_0(g)$.  

10
Proof scheme.

Item 1. Consider the special case $m = 0$. Partition of rows and columns has the form $(1, 1, k, k, 1, 1)$. The general element $g \in G = \text{UT}(n, \mathbb{R})$ and $f_0 \in g^*$ are written as block matrices

$$
g = \begin{pmatrix}
1 & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
0 & 1 & C_{23} & C_{24} & C_{25} & C_{26} \\
0 & 0 & C_{33} & C_{34} & C_{35} & C_{36} \\
0 & 0 & 0 & C_{44} & C_{45} & C_{46} \\
0 & 0 & 0 & 0 & 1 & C_{56} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad f_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\gamma_1 & 0 & 0 & 0 & 0 \\
0 & \gamma_2 & 0 & 0 & \gamma_3 & 0
\end{pmatrix}
$$

Consider the subgroup $B$ of all matrices

$$
b = \begin{pmatrix}
1 & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\
0 & 1 & 0 & 0 & 0 & b_{26} \\
0 & 0 & b_{33} & b_{34} & 0 & b_{36} \\
0 & 0 & 0 & b_{44} & 0 & b_{46} \\
0 & 0 & 0 & 0 & 1 & b_{56} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Let $\pi_{\Lambda}(b_1)$ be the irreducible representation from subsection 2.1 of the unitriangular group $B_1$ of all matrices $b_1 = \begin{pmatrix}
b_{33} & b_{34} \\
0 & b_{44}
\end{pmatrix}$

The irreducible representation $T_{g}^{f_0}$, that corresponds to $f_0 \in g^*$, is induced from the representation

$$
e^{2\pi i(\gamma_1 b_{13} + \gamma_2 b_{26} + \gamma_3 b_{56})} \pi_{\Lambda}(b_1)
$$

of the subgroup $B$. Direct calculations in spirit of subsection 2.1 lead to proof of formula (22) for special case $m = 0$.

Item 2. Consider the special case $m = 1$. Partition of rows and columns has the form $(1, 1, 1, k, k, 1, 1, 1)$. The general element $g \in G = \text{UT}(n, \mathbb{R})$ and $f \in g^*$ are written as block matrices

$$
g = \begin{pmatrix}
1 & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} & C_{17} & C_{18} \\
0 & 1 & C_{23} & C_{24} & C_{25} & C_{26} & C_{27} & C_{28} \\
0 & 0 & 1 & C_{34} & C_{35} & C_{36} & C_{37} & C_{38} \\
0 & 0 & 0 & C_{44} & C_{45} & C_{46} & C_{47} & C_{48} \\
0 & 0 & 0 & 0 & C_{55} & C_{56} & C_{57} & C_{58} \\
0 & 0 & 0 & 0 & 0 & 1 & C_{67} & C_{68} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & C_{78} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad f = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

where $\lambda_1 \in \mathbb{R}^*$. 
Consider the subgroup $B$ of all matrices
\[
b = \begin{pmatrix}
1 & 0 & 0 & 0 & b_{15} & b_{16} & b_{17} & b_{18} \\
0 & 1 & b_{23} & b_{24} & b_{25} & b_{26} & b_{27} & b_{28} \\
0 & 0 & 1 & b_{34} & b_{35} & b_{36} & b_{37} & b_{38} \\
0 & 0 & 0 & b_{44} & b_{45} & b_{46} & b_{47} & b_{48} \\
0 & 0 & 0 & 0 & 0 & 1 & b_{57} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Let $b'$ be a matrix obtained from $b$ by deleting the first row and the last column, $B'$ be a group of all such matrices. Let $T_{f_0}^f(b')$ be an irreducible representation of the group $B'$ as in Item 1.

The irreducible representation $T^f_g$, that corresponds to $f \in g^*$, is induced from the representation
\[
e^{2\pi i(\lambda_1 b_{18})}T_{f_0}^f(b)
\]
of the subgroup $B$. Direct calculations in spirit of subsection 2.1 lead to proof of formula (22) for special case $m = 1$.

**Item 3.** Case of general $m$. Consider the normal subgroup $B_m$ of all $g \in G$, such that $C_{11} = C_{88} = E$. The character $\chi_m$ of irreducible representation of the subgroup $B_m$, that corresponds to the restriction of $f$ on Lie($B_m$), is calculated similarly to Item 2. The character $\chi$ is induced from the character $\chi_m$ of subgroup $B_m$. Calculation of character is similar to subsection 2.1. □

### 3.2 Case of odd $n$.

Let $n = 2(k+m+2)+1$. Partition rows and columns in blocks $(m, 1, 1, k, 1, k, 1, 1, m)$. The general element $g \in \text{UT}(n, \mathbb{R})$ can be written as a block matrix
\[
g = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} & C_{17} & C_{18} & C_{19} \\
0 & 1 & C_{23} & C_{24} & C_{25} & C_{26} & C_{27} & C_{28} & C_{29} \\
0 & 0 & 1 & C_{34} & C_{35} & C_{36} & C_{37} & C_{38} & C_{39} \\
0 & 0 & 0 & C_{44} & C_{45} & C_{46} & C_{47} & C_{48} & C_{49} \\
0 & 0 & 0 & 0 & 1 & C_{56} & C_{57} & C_{58} & C_{59} \\
0 & 0 & 0 & 0 & 0 & C_{66} & C_{67} & C_{68} & C_{69} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & C_{78} & C_{79} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & C_{89} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{99}
\end{pmatrix}
\] (23)
Any subregular coadjoint orbit of the group UT(n, \( \mathbb{R} \)), where \( n = 2(k + m) + 5 \), contains a unique element of the form

\[
f = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & 0 \\
\Lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]  

(24)

where \( \Lambda_1 \) (resp. \( \Lambda_2 \)) is a matrix of form (1) of size \( m \) (resp. \( k \)). All entries of second diagonals of the matrices \( \Lambda_1, \Lambda_2 \) and \( \gamma_1, \gamma_2 \) do not equal to zero [4, §3]).

The systems of matrix units \( E_{46}, E_{19} \) is defined by blocks \( C_{46}, C_{19} \) similarly as in subsection 3.1.

**Lemma 3.4.** 1) Stabilizer \( g^f \) in the Lie algebra \( \mathfrak{g} = \mathfrak{ut}(n, \mathbb{R}) \) is a subspace spanned by

\[
E_{46}, E_{19}, E_{27}, E_{28}, E_{38}, \gamma_2 E_{23} + \gamma_1 E_{78};
\]

(25)

2) Stabilizer \( G^f \) in the Lie group \( G = \text{UT}(n, \mathbb{R}) \) is equal to \( E + \mathfrak{g}^f \).

Construct the following table of size \( 9 \times 9 \). By symbol "X" we mark all places \((a, b)\) such that \( E_{a,b} \) rises in (25).

\[
\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
X & X & X & X & X & X & X & X & X \\
\end{array}
\]

Table 2.

Denote \( \Phi' = \{(2, 3), (4, 6), (7, 8)\} \). For any entry \( c \) of the matrix (18) that is lying in one of the blocks, marked in the Table 2 by the symbol "•", or that is lying in the block \( C_{18} \), we define a rational function \( \tilde{c} \) as in (21) changing \( \Phi \) by \( \Phi' \). Denote:

1) \( S_0 \) is the set of entries lying upper the diagonal in the blocks \( C_{11}, C_{44}, C_{66}, C_{99} \);
2) \( S_1 \) is the set of all rational functions \( \tilde{c} \), where \( c \) runs through all entries in the blocks that is marked in the Table 2 by the symbol "•";
3) \( S_2 = \{C_{23}C_{36} + C_{24}C_{46}, C_{46}C_{68} + C_{47}C_{78}, \gamma_1 C_{23} - \gamma_2 C_{78}\} \).
4) $S = \{S_0, S_1, S_2\}$,
5) $d_1(g) = \det C_{23} \cdot \det C_{46} \cdot \det C_{78}$.

The algebra $\mathbb{R}[G]$ of regular function on the group $G = UT(n, \mathbb{R})$ admits localization $\mathbb{R}'[G]$ by the denominator subset generated by $d_1(g)$. As above denote by $I'$ the ideal in $\mathbb{R}'[G]$ generated by $S$; note that $\text{Ann}(I') \subset \{g \in G : d(g) \neq 0\}$.

**Proposition 3.5.** Let $f$ has the form (24). Then the closure in $G$ of the annihilator of the ideal $I'$ coincides with the closure of the set $\text{Ad}_G(G')$.

**Proof.** Similarly to Proposition 3.2.

**Theorem 3.6.** Let $n = 2(k + m) + 5$. The character of the irreducible representation, associated with the orbit of element $f \in \mathfrak{ut}(n, \mathbb{R})^*$ of (24), has the form

$$\chi(g) = \delta(S) \cdot \chi_{1}^*(C_{19}) \cdot \frac{\chi_{1}^*(C_{46})}{|\det \Lambda|} \cdot \chi_{1}^*(g).$$  

(26)

Here $\chi_{1}^*(C_{46})$ as in (5), the matrix $\tilde{C}_{19}$ filled by the entries $\tilde{c}$, where $c$ is the corresponding entry of the block $C_{19}$ (see (21), $\chi_{1}^*(\tilde{C}_{19})$ as in (5), and

$$\chi_{1}^*(g) = \left\{\frac{i}{|\gamma_1 \gamma_2|^k \cdot |\gamma_1 c_{23}| \cdot \epsilon_1 d_1(g)^m} \right\} e^{2\pi i \left(\gamma_1 Q_1 + \gamma_3 c_{67}\right)}.$$

$Q_1 = c_{23}c_{38} + c_{24}c_{48} + c_{25}c_{58} + c_{26}c_{68} + c_{27}c_{78}$.

**Proof** is similar to Theorem 3.3.

**Theorem 3.7.** Conjecture 1.1 is valid for regular and subregular characters of the group $UT(n, \mathbb{R})$.

**Proof.** Propositions 3.2, 3.5 and Theorems 2.1, 2.2, 3.3, 3.6 imply the proof.

**References**

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