COHOMOLOGICAL FUNCTORS ASSOCIATED WITH GENERAL HEARTS ON EXACT CATEGORIES

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Abstract. In the previous article "Hearts of twin cotorsion pairs on exact categories", we introduced the notion of the heart for any cotorsion pair on an exact category with enough projectives and injectives, and showed that it is an abelian category. This is an analog of Nakaoka’s result for triangulated categories [8]. In this paper, we construct a cohomological functor from the exact category to the heart. This is an analog of the construction of Abe and Nakaoka for triangulated categories. When the cotorsion pair comes from a cluster tilting subcategory, our cohomological functor coincides with the canonical quotient functor.

1. Introduction

Cotorsion pairs play an important role in representation theory (see [2] and see [6] for more recent examples). In [7], we define hearts of cotorsion pairs on exact categories and proved that they are abelian. It is natural to ask whether we can find any relationship between the hearts and the original exact categories. Abe and Nakaoka have already given an answer by constructing a cohomological functor in the case of triangulated categories [1]. In this paper we will construct an associated cohomological functor \( H \) from an exact category \( B \) to the heart of a cotorsion pair on it, which can be considered as a generalization of the construction of Abe and Nakaoka.

Throughout this paper, let \( B \) be a Krull-Schmidt exact category with enough projectives and injectives.

Let \( \mathcal{P} \) (resp. \( \mathcal{I} \)) be the full subcategory of projectives (resp. injectives) of \( B \).

We recall the definition of a cotorsion pair on \( B \) [7, Definition 2.3]:

**Definition.** Let \( \mathcal{U} \) and \( \mathcal{V} \) be full additive subcategories of \( B \) which are closed under direct summands. We call \( (\mathcal{U}, \mathcal{V}) \) a cotorsion pair if it satisfies the following conditions:

(a) \( \text{Ext}^1_B(\mathcal{U}, \mathcal{V}) = 0 \).

(b) For any object \( B \in B \), there exist two short exact sequences

\[
V_B \rightarrow U_B \rightarrow B, \quad B \rightarrow V^B \rightarrow U^B
\]

satisfying \( U_B, U^B \in \mathcal{U} \) and \( V_B, V^B \in \mathcal{V} \).

For any cotorsion pairs \( (\mathcal{U}, \mathcal{V}) \), let \( \mathcal{W} := \mathcal{U} \cap \mathcal{V} \). We denote the quotient of \( B \) by \( \mathcal{W} \) as \( \mathcal{B} := B/\mathcal{W} \). For any morphism \( f \in \text{Hom}_B(X, Y) \), we denote its image in \( \text{Hom}_\mathcal{B}(X, Y) \) by \( f \). For any subcategory \( \mathcal{C} \supseteq \mathcal{W} \) of \( B \), we denote by \( \mathcal{C} \) the subcategory of \( \mathcal{B} \) consisting of the same objects as \( \mathcal{C} \).

Let

\[
B^+ := \{ B \in B \mid U_B \in \mathcal{W} \}, \quad B^- := \{ B \in B \mid V^B \in \mathcal{W} \}.
\]

Let

\[
\mathcal{H} := B^+ \cap B^-.
\]

Since \( \mathcal{H} \supseteq \mathcal{W} \), we have an additive quotient subcategory \( \mathcal{H} \), which we call the heart of cotorsion pair \( (\mathcal{U}, \mathcal{V}) \).

Now we introduce the definition of cohomological functor on \( B \).

**Definition.** A covariant functor \( F \) from \( B \) to an abelian category \( A \) is called cohomological if \( F(\mathcal{P}) = 0 = F(\mathcal{I}) \) and for any short exact sequence

\[
A \rightarrow f \rightarrow B \rightarrow g \rightarrow C
\]

Key words and phrases. exact category, abelian category, cotorsion pair, heart, cohomological functor.
in \( \mathcal{B} \), there exists morphisms \( h : C \to \Omega^{-} A \) and \( h' : \Omega C \to A \) such that we can get a long exact sequence
\[
\cdots \xrightarrow{F(\Omega h')} F(\Omega A) \xrightarrow{F(\Omega f)} F(\Omega B) \xrightarrow{F(\Omega g)} F(\Omega C) \xrightarrow{F(\Omega h)} \cdots
\]
in \( \mathcal{A} \) where \( \Omega \) is the syzygy and \( \Omega^{-} \) is the cosyzygy.

Although cohomological functors are usually defined on triangulated categories, the functor we define has the similar property as normal cohomological functor. And when \( \mathcal{B} \) is Frobenius, the functor we construct induces a cohomological functor from the stable category of \( \mathcal{B} \), which is triangulated by [5], to the heart of a cotorsion pair.

We will prove the following theorem (see Theorem 4.5 for details).

**Theorem.** For any cotorsion pair \((\mathcal{U}, \mathcal{V})\) on \( \mathcal{B} \), there exists an associated cohomological functor
\[
\mathcal{H} : \mathcal{B} \to \mathcal{H}.
\]

2. Preliminaries

For briefly review of the important properties of exact categories, we refer to [7, §2]. For more details, we refer to [3].

We introduce the following properties used a lot in this paper, the proofs can be found in [3, §2]:

**Proposition 2.1.** Consider a commutative square
\[
\begin{array}{c}
A \rightarrowtail B \\
\downarrow f \\
A' \rightarrowtail B'
\end{array}
\]
in which \( i \) and \( i' \) are inflations. The following conditions are equivalent:

(a) The square is a push-out.

(b) The sequence \( A \xrightarrow{(i, -f')} B \oplus A' \xrightarrow{(f', i')} B' \) is short exact.

(c) The square is both a push-out and a pull-back.

(d) The square is a part of a commutative diagram
\[
\begin{array}{c}
A \rightarrowtail B \rightarrowtail C \\
\downarrow f \downarrow f' \\
A' \rightarrowtail B' \rightarrowtail C
\end{array}
\]
with short exact rows.

**Proposition 2.2.** (a) If \( X \xrightarrow{i} Y \xrightarrow{d} Z \) and \( N \xrightarrow{g} M \xrightarrow{f} Y \) are two short exact sequences, then there is a commutative diagram of short exact sequences
\[
\begin{array}{c}
N \rightarrowtail N \\
\downarrow i \downarrow f \\
Q \rightarrowtail M \rightarrowtail Z
\end{array}
\]
where the lower-left square is both a push-out and a pull-back.
(b) If \( X \xrightarrow{i} Y \xrightarrow{d} Z \) and \( Y \xrightarrow{g} K \xrightarrow{f} L \) are two short exact sequences, then there is a commutative diagram of short exact sequences

\[
\begin{array}{c}
X \xrightarrow{i} Y \xrightarrow{d} Z \\
\downarrow \quad \downarrow \\
X \xrightarrow{g} K \xrightarrow{f} L \\
\downarrow \quad \downarrow \\
\end{array}
\]

where the upper-right square is both a push-out and a pull-back.

We recall some definitions and results of \[7\].

**Definition 2.3.** For any \( B \in \mathcal{B} \), define \( B^+ \) and \( b^+ : B \to B^+ \) as follows:

Take two short exact sequences:

\[
\begin{array}{c}
V_B \xrightarrow{u_B} U_B \xrightarrow{w_B} B \\
\downarrow \quad \downarrow \\
\end{array}
\]

\[
\begin{array}{c}
V_B \xrightarrow{u_B} U_B \xrightarrow{w_B} \mathcal{W}_0 \\
\downarrow \quad \downarrow \\
\end{array}
\]

where \( U_B, \mathcal{W}_0 \in \mathcal{U}, V_B \in \mathcal{V}, \mathcal{W}_0 \in \mathcal{W} \). By Proposition 2.2, we get the following commutative diagram

\[
\begin{array}{c}
V_B \xrightarrow{u_B} U_B \xrightarrow{w_B} B \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\end{array}
\]

where the upper-right square is both a push-out and a pull-back.

By \[7\], Lemma 3.2, \( B^+ \in \mathcal{B}^+ \), and if \( B \in \mathcal{B}^- \), then \( B^+ \in \mathcal{H} \).

**Proposition 2.4.** \[7\] Proposition 3.3 For any \( B \in \mathcal{B} \) and \( Y \in \mathcal{B}^+ \), \( \text{Hom}_B(b^+, Y) : \text{Hom}_B(B^+, Y) \to \text{Hom}_B(B, Y) \) is surjective and \( \text{Hom}_B(b^+, Y) : \text{Hom}_B(B^+, Y) \to \text{Hom}_B(B, Y) \) is bijective.

By Proposition 2.4, we can define a functor \( \sigma^+ \) from \( \mathcal{B} \) to \( \mathcal{B}^+ \) as follows:

For any object \( B \in \mathcal{B} \), since all the \( B^+ \)'s are isomorphic to each other in \( \mathcal{B} \) by Proposition 2.4, we fix a \( B^+ \) for \( B \). Let

\[
\sigma^+ : B \to B^+ \\
B \mapsto B^+.
\]

and for any morphism \( f : B \to C \), we define \( \sigma^+(f) \) as the unique morphism given by Proposition 2.4.

**Proposition 2.5.** The functor \( \sigma^+ \) has the following properties

(a) For any objects \( A \) and \( B \) in \( \mathcal{B} \), \( \sigma^+(A \oplus B) \simeq \sigma^+(A) \oplus \sigma^+(B) \) in \( \mathcal{B} \).

(b) \( \sigma^+|_{\mathcal{B}^+} = \text{id}_{\mathcal{B}^+} \)
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(c) $\sigma^+(B) = 0$ if and only if $B \in \mathcal{U}$.

Proof. (a), (b), can be concluded easily by definition, (c) is followed by [7, Lemma 3.4]. □

Definition 2.6. For any object $B \in \mathcal{B}$, we define $b^- : B^- \to B$ as follows:

Take the following two short exact sequences

$$
\begin{align*}
B &\to V^B &\to U^B, \\
V_0 &\to W_0 &\to V^B
\end{align*}
$$

where $W_0 \in \mathcal{W}, V^B, V_0 \in \mathcal{V},$ and $U^B \in \mathcal{U}$. By Proposition 2.7 we get the following commutative diagram:

By duality, we get:

Proposition 2.7. [7, Proposition 3.6] For any $B \in \mathcal{B}$, we obtain $B^- \in \mathcal{B}^-$. Moreover, $B \in \mathcal{B}^+$ implies $B^- \in \mathcal{H}$. For any $X \in \mathcal{B}^-$, $	ext{Hom}_\mathcal{B}(X, b^-) : \text{Hom}_\mathcal{B}(X, B^-) \to \text{Hom}_\mathcal{B}(X, B)$ is surjective and $	ext{Hom}_\mathcal{B}(X, b^-) : \text{Hom}_\mathcal{B}(X, B^-) \to \text{Hom}_\mathcal{B}(X, B)$ is bijective.

we define a functor $\sigma^-$ from $\mathcal{B}$ to $\mathcal{B}^-$ as the dual of $\sigma^+$:

$$
\sigma^- : \mathcal{B} \to \mathcal{B}^-
$$

For any morphism $f : B \to C$, we define $\sigma^-(f)$ as the unique morphism given by Proposition 2.7

$$
\begin{array}{c}
B^- \xrightarrow{\sigma^-(f)} C^- \\
\downarrow b^- & \downarrow \xi^- \\
B \xrightarrow{f} C
\end{array}
$$

Proposition 2.8. The functor $\sigma^-$ has the following properties

(a) For any objects $A$ and $B$ in $\mathcal{B}$, $\sigma^-(A \oplus B) \simeq \sigma^-(A) \oplus \sigma^-(B)$ in $\mathcal{B}$.

(b) $\sigma^-|_{\mathcal{B}^-} = \text{id}_{\mathcal{B}^-}$

(c) $\sigma^-(B) = 0$ if and only if $B \in \mathcal{V}$.

3. Reflection sequences and coreflection sequences

In the following two sections we fix a cotorsion pair $(\mathcal{U}, \mathcal{V})$.

Let $\mathcal{C}$ be a subcategory of $\mathcal{B}$, denote by $\Omega \mathcal{C}$ (resp. $\Omega^- \mathcal{C}$) the subcategory of $\mathcal{B}$ consisting of objects $\Omega \mathcal{C}$ (resp. $\Omega^- \mathcal{C}$) such that there exists a short exact sequence

$$
\Omega \mathcal{C} \hookrightarrow P \xrightarrow{\epsilon} C \quad (P \in \mathcal{P}, C \in \mathcal{C})
$$

(resp. $C \xrightarrow{\iota} I \xrightarrow{\tau} \Omega^- \mathcal{C}$ ($I \in \mathcal{I}, C \in \mathcal{C}$)).

Lemma 3.1. We have $\Omega \mathcal{U} \subseteq \mathcal{B}^-$ and $\Omega^- \mathcal{V} \subseteq \mathcal{B}^+$. 
Definition 3.2. Let \( \Omega U \in \Omega \mathcal{U} \) admits two short exact sequences

\[
\begin{array}{c}
\Omega U \rightarrow^q P_U \rightarrow^p U,
\end{array}
\]

where \( U, U' \in \mathcal{U}, V' \in \mathcal{V} \) and \( P_U \in \mathcal{P} \). Since \( q \) is a \( \mathcal{V} \)-approximation of \( \Omega U \), there exists a morphism \( p : P_U \rightarrow V' \) such that \( pq = v' \).

\[
\begin{array}{c}
\Omega U \rightarrow^q P_U \rightarrow^p U, \\
\Omega U \rightarrow^q P'_U \rightarrow^p U' \\
\end{array}
\]

It is enough to show that \( V' \in \mathcal{U} \), so we apply \( \text{Hom}_B(-, \mathcal{V}) \) to the above diagram. We get a commutative diagram

\[
\begin{array}{c}
0 = \text{Ext}^1_B(U', \mathcal{V}) \rightarrow \text{Ext}^1_B(V', \mathcal{V}) \rightarrow \text{Ext}^1_B(\Omega U, \mathcal{V}) \\
0 = \text{Ext}^1_B(U, \mathcal{V}) \rightarrow \text{Ext}^1_B(P_U, \mathcal{V}) \rightarrow \text{Ext}^1_B(\Omega U, \mathcal{V}).
\end{array}
\]

Since \( \text{Ext}^1_B(P_U, \mathcal{V}) = 0 \), we get \( \text{Ext}^1_B(v', \mathcal{V}) = 0 \), which implies that \( \text{Ext}^1_B(V', \mathcal{V}) = 0 \). Hence \( V' \in \mathcal{V} \), which implies that \( \Omega U \in \mathcal{B}^- \).

\[\square\]

**Lemma 3.3.** Let \( B \) be any object in \( \mathcal{B} \).

(a) A **reflection sequence** for \( B \) is a short exact sequence

\[
B \rightarrow^z Z \rightarrow^u U
\]

where \( U \in \mathcal{U}, Z \in \mathcal{B}^+ \) and there exists a commutative diagram

\[
\begin{array}{c}
\Omega U \rightarrow^q P_U \rightarrow^p U \\
\end{array}
\]

with rows short exact and \( P_U \in \mathcal{P} \) and \( x \) factoring through \( \mathcal{U} \).

(b) A **coreflection sequence** for \( B \) is a short exact sequence

\[
V \rightarrow^K K \rightarrow^k B
\]

where \( V \in \mathcal{V}, K \in \mathcal{B}^- \) and there exists commutative diagram

\[
\begin{array}{c}
V \rightarrow^K K \rightarrow^k B \\
\end{array}
\]

with rows short exact and \( I^V \in \mathcal{I} \) and \( y \) factoring through \( \mathcal{V} \).

Let \( B \) be an object in \( \mathcal{B} \). Then

(a) \( B \rightarrow^b^+ B^+ \rightarrow^0 U^0 \) in \ref{2.3} is a reflection sequence for \( B \).

(b) \( V_0 \rightarrow^b^- B^- \rightarrow B \) in \ref{2.6} is a coreflection sequence for \( B \).

(c) For any reflection sequence \( B \rightarrow^z Z \rightarrow U \) for \( B \), we have \( Z \simeq B^+ \) in \( \mathcal{B} \).

(d) For any coreflection sequence \( V \rightarrow^K K \rightarrow B \) for \( B \), we have \( K \simeq B^- \) in \( \mathcal{B} \).
Proof. We only prove (a) and (c), the other two are dual.
(a) Since $U^0$ admits the following short exact sequence
\[ \Omega U^0 \rightarrow P_U \rightarrow U^0, \]
we get the following commutative diagram
\[ \Omega U^0 \rightarrow P_U \rightarrow U^0 \]
\[ x_0 \downarrow \quad p_0 \downarrow \quad U_B \rightarrow b^+ \rightarrow B^+ \rightarrow U^0. \]

Since $P_U$ is projective, there exists a morphism $p'_0 : P_U \rightarrow W^0$ such that $wp'_0 = p_0$, we get $b^+x_0 = p_0q_0 = wp'_0q_0$. By the definition of pull-back, $x_0$ factors through $U_B \in \mathcal{U}$.

Hence by definition $B \rightarrow b^+ B^+ \rightarrow U^0$ is a reflection sequence for $B$.
(c) By Proposition 2.4, it is enough to show that there exists a unique morphism $f : Z \rightarrow B^+$ such that $b^+ = fz$. We only show the existence, the proof of uniqueness is similar as [7, Proposition 3.3]. The reflection sequence admits a commutative diagram
\[ \Omega U \rightarrow P_U \rightarrow U \]
\[ x \downarrow \quad p \downarrow \quad B \rightarrow z \rightarrow Z \rightarrow U \]
where the left square is a push-out by Proposition 2.1. Since $x$ factors through $\mathcal{U}$, and $u_B$ is a right $\mathcal{U}$-approximation of $B$, there exists a morphism $q : \Omega U \rightarrow U_B$ such that $x = u_By$. Since $q$ is a left $\mathcal{V}$-approximation of $\Omega U$, there exists a morphism $p' : P_U \rightarrow W^0$ such that $w'y = p'q$, thus $b^+x = b^+u_By = ww'y = wp'q$. Then by the definition of push-out, there exists a morphism $f : Z \rightarrow B^+$ such that $b^+ = fz$.

We need the following lemma, the proof is similar with [7, Proposition 3.3].
Lemma 3.4. Let $B$ be any object in $\mathcal{B}$. Then

(a) $\text{Hom}_\mathcal{B}(b^+, \Omega^-V) : \text{Hom}_\mathcal{B}(B^+, \Omega^-V) \to \text{Hom}_\mathcal{B}(B, \Omega^-V)$ is surjective.

(b) $\text{Hom}_\mathcal{B}(\Omega^U, b^-) : \text{Hom}_\mathcal{B}(\Omega^U, B^-) \to \text{Hom}_\mathcal{B}(\Omega^U, B)$ is surjective.

Proposition 3.5. There exists a natural isomorphism

$$\eta : \sigma^+ \circ \sigma^- \cong \sigma^- \circ \sigma^+.$$

Proof. By Lemma 3.3, we can take the following commutative diagram of short exact sequence

$$
\begin{array}{cccccc}
V_0 & \rightarrow & B^- & \rightarrow & b^- & \rightarrow \ B \\
\downarrow d \downarrow & & \downarrow y_0 & & \downarrow \\
V_0' & \rightarrow & j & \rightarrow & i & \rightarrow \Omega^-V_0
\end{array}
$$

where $y_0$ factors through $V^B$ since $v^B$ is a left $V$-approximation of $B$. By Lemma 3.4, there exists a morphism $t : B^+ \rightarrow \Omega^-V_0$ such that $y_0 = tb^+$. Since $b^+$ is a left $V$-approximation of $B^+$, there exists a morphism $v_0 : B^+ \rightarrow V^B$ such that $v^B = v_0b^+$. Thus $tb^+ = v'v^B = v'v_0b^+$, then we obtain that $t - v'v_0$ factors through $U^0$.

$$
\begin{array}{cccccc}
B & \rightarrow & B^+ & \rightarrow & U^0 \\
\downarrow v_B & & \downarrow v' & & \downarrow u \\
V^B & \rightarrow & \Omega^-V_0 & \rightarrow & \\
\downarrow v' & & \downarrow i & & \downarrow \\
V_0' & \rightarrow & j & \rightarrow & \Omega^-V_0
\end{array}
$$

Since $i$ is a right $U$-approximation of $\Omega^-V_0$, $u$ factors through $l^0 \in V$, we conclude that $t$ factors through $V$.

Take a pull-back of $t$ and $i$, we get the following commutative diagram

$$
\begin{array}{cccccc}
V_0 & \rightarrow & Q & \rightarrow & B^+ \\
\downarrow d' \downarrow & & \downarrow PB & & \downarrow t \\
V_0' & \rightarrow & l^0 & \rightarrow & \Omega^-V_0.
\end{array}
$$

By [7, Lemma 2.11], we obtain $Q \in B^+$. Now by Proposition 2.2, we get the following commutative diagram

$$
\begin{array}{cccccc}
V_0 & \rightarrow & V_0 & \rightarrow & \Omega^-V_0 \\
\downarrow & & \downarrow & & \downarrow \\
Q' & \rightarrow & Q & \rightarrow & U^0 \\
\downarrow & & \downarrow s & & \downarrow \\
B & \rightarrow & B^+ & \rightarrow & U^0.
\end{array}
$$
By the definition of pull-back, there exists a morphism $k : B \to Q$ such that $sk = b^+b^-$ and $d'k = d$. Hence we have the following diagram

where the upper-left square commutes. Hence $jv_0 = d'kv = dv = j$, we can conclude that $v_0 = id_{V_0}$ since $j$ is monomorphic. By the same method we can get the following commutative diagram

where $v'_0 = id_{V_0}$. Therefore $k'$ is isomorphic by [3, Corollary 3.2]. We can replace $V_0 \to B$ by $V_0 \to B$.

Now we have a short exact sequence $B \to k \to Q \to U^0$. We get $Q \in B$ by [7] Lemma 2.10, hence $Q \in H$. Since $t$ factors through $V$, $V_0 \to B^+$ is a coreflection sequence for $B^+$. By Lemma 3.3, we have the following commutative diagram

in $\mathcal{B}$ where $\alpha'$ is isomorphic. Thus $Q \simeq \sigma^-(B^+) = \sigma^-\sigma^+(B)$ in $\mathcal{H}$.

By duality we conclude that $B^- \to k \to Q \to U^0$ is a reflection sequence for $B^-$. By Lemma 3.3 we have the following commutative diagram

in $\mathcal{B}$ where $\beta'$ is isomorphic. Thus $Q \simeq \sigma^+(B^-) = \sigma^+\sigma^-(B)$ in $\mathcal{H}$. Hence there exists an isomorphism $\eta_B : \sigma^+\sigma^-(B) \to \sigma^-\sigma^+(B)$ in $\mathcal{H}$.

By Proposition 2.4 there exists a morphism $\theta : B^- \to \sigma^-\sigma^+(B)$ in $\mathcal{B}$ such that $\alpha\theta = b^+b^-$. Then by Proposition 2.4 there exists a morphism $\eta_B : \sigma^+\sigma^-(B) \to \sigma^-\sigma^+(B)$ such that $\eta_B\beta = \theta$. Hence we get
the following commutative diagram

\[
\begin{array}{ccc}
\sigma^+\sigma^-(B) & \xrightarrow{\beta} & B^-\\
\downarrow{\eta_B} & & \downarrow{\gamma} \\
\sigma^-\sigma^+(B) & \xrightarrow{\alpha} & B^+.
\end{array}
\]

Then \(\alpha\eta_B = b^+b^- = sk = \alpha\alpha'\beta'\beta\), then \(\eta_B = \alpha'\beta'\beta\) by Proposition 2.4 and 2.7 and thus is isomorphic. Let \(f : B \rightarrow C\) be a morphism in \(\mathcal{B}\), then we can get the following commutative diagram by Proposition 2.4 and 2.7.

\[
\begin{array}{ccc}
\sigma^+\sigma^-(B) & \xrightarrow{\beta} & B^- & \xrightarrow{\lambda} & C^- & \xrightarrow{\gamma} & \sigma^+\sigma^-(C) \\
\downarrow{\eta_B} & & \downarrow{\lambda} & & \downarrow{\eta_C} & & \\
\sigma^-\sigma^+(B) & \xrightarrow{\alpha} & B^+ & \xrightarrow{\lambda^+} & C^+ & \xrightarrow{\delta} & \sigma^-\sigma^+(C)
\end{array}
\]

Since

\[
\delta(\sigma^-\sigma^+(f))\eta_B = (\sigma^+(f))\lambda^+ = \lambda^+\lambda^- = \lambda^-\lambda^+ = (\sigma^-(f))\eta_B = \delta(\sigma^+\sigma^-(f))\beta
\]

we get \((\sigma^-\sigma^+(f))\eta_B = \eta_C(\sigma^+\sigma^-(f))\beta\) by Proposition 2.4 and 2.7. Thus \(\eta\) is a natural isomorphism. \(\square\)

4. Cohomological functor

By Proposition 3.5, we have a natural isomorphism of functors

\[
\sigma^+ \circ \sigma^- \circ \pi \simeq \sigma^- \circ \sigma^+ \circ \pi
\]

where \(\pi : \mathcal{B} \rightarrow \mathcal{B}\) denotes the quotient functor. We denote \(\sigma^- \circ \sigma^+ \circ \pi\) by

\[
H : \mathcal{B} \rightarrow \mathcal{H}
\]

In the rest of this section, we show that \(H\) is cohomological.

**Proposition 4.1.** The functor \(H\) has the following properties:

(a) For any objects \(A\) and \(B\) in \(\mathcal{B}\), \(H(A \oplus B) \simeq H(A) \oplus H(B)\) in \(\mathcal{H}\).

(b) \(H|_{\mathcal{H}} = \pi|_{\mathcal{H}}\).

(c) \(H(\mathcal{V}) = 0\) and \(H(\mathcal{U}) = 0\).

(d) For any reflection sequence \(B \xrightarrow{z} Z \xrightarrow{U} B\) for \(B\), \(H(z)\) is an isomorphism in \(\mathcal{H}\).

(e) For any coreflection sequence \(V \xleftarrow{K} B \xrightarrow{k} B\) for \(B\), \(H(k)\) is an isomorphism in \(\mathcal{H}\).

**Proof.** All the results are followed directly by Proposition 2.5 and 2.8. \(\square\)

**Lemma 4.2.** Let \(B\) be any object in \(\mathcal{B}\).

(a) If \(B \in \mathcal{B}^+\) and \(U \in \mathcal{U}\), then \(\text{Hom}_B(U, B) = 0\).

(b) If \(B \in \mathcal{B}^-\) and \(V \in \mathcal{V}\), then \(\text{Hom}_B(B, V) = 0\).
Proof. We only show (a), (b) is dual. Since \( B \in B^+ \), it admits a short exact sequence
\[
V_B \rightarrow W_B \rightarrow B
\]
where \( W_B \in W \). Let \( f \) be any morphism from \( U \in \mathcal{U} \) to \( B \), since \( w_B \) is a right \( \mathcal{U} \)-approximation, there exists a morphism \( g : U \rightarrow W_B \) such that \( f = w_B g \). Hence \( f = 0 \), which implies \( \text{Hom}_\mathcal{U}(U, B) = 0 \). □

Lemma 4.3. Let
\[
\begin{array}{c}
\Omega U \xrightarrow{g} P_U \rightarrow U \\
\downarrow f \downarrow \downarrow p \\
A \xrightarrow{g} B \xrightarrow{h} U
\end{array}
\]
be a commutative diagram with rows short exact satisfying \( U \in \mathcal{U} \) and \( P_U \in \mathcal{P} \). Then the sequence
\[
H(\Omega U) \xrightarrow{H(f)} H(A) \xrightarrow{H(g)} H(B) \rightarrow 0
\]
is exact in \( H \).

Proof. By Proposition 2.2, we get a commutative diagram by taking a pull-back of \( g \) and \( b^- \)
\[
\begin{array}{c}
V_0 \xrightarrow{g'} L \rightarrow U \\
\downarrow l \downarrow \downarrow b^- \\
A \xrightarrow{g} B \xrightarrow{h} U
\end{array}
\]
By [7, Lemma 2.10], \( L \in B^- \). By Lemma 3.3, we can obtain a commutative diagram of short exact sequences
\[
\begin{array}{c}
V_0 \xrightarrow{g} L \rightarrow A \\
\downarrow l \downarrow \downarrow b^- \\
V_0 \xrightarrow{b^-} B \rightarrow B \\
\downarrow j \downarrow \downarrow j
\end{array}
\]
where \( j \) factors through \( V \), hence
\[
\begin{array}{c}
V_0 \xrightarrow{g} L \rightarrow A
\end{array}
\]
is a coreflection sequence for \( A \). By Proposition 4.1, \( H(l) \) and \( H(b^-) \) are isomorphic in \( H \). Thus, replacing \( A \) by \( L \) and \( B \) by \( B^- \), we may assume that \( A, B \in B^- \). Under this assumption, we show \( H(g) \) is the
cokernel of $H(f)$. We have $\Omega U \in \mathcal{B}^-$ by Lemma 3.1. For any $Q \in \mathcal{H}$, we have a commutative diagram

$$
\begin{array}{ccccccc}
\text{Hom}_B(H(B), Q) & \xrightarrow{\text{Hom}_B(H(g), Q)} & \text{Hom}_B(H(A), Q) & \xrightarrow{\text{Hom}_B(H(f), Q)} & \text{Hom}_B(H(\Omega U), Q) \\
\cong & & \cong & & \cong \\
\text{Hom}_B(\sigma^+(B), Q) & \xrightarrow{\text{Hom}_B(\sigma^+(g), Q)} & \text{Hom}_B(\sigma^+(A), Q) & \xrightarrow{\text{Hom}_B(\sigma^+(f), Q)} & \text{Hom}_B(\sigma^+(\Omega U), Q) \\
\cong & & \cong & & \cong \\
\text{Hom}_B(B, Q) & \xrightarrow{\text{Hom}_B(g, Q)} & \text{Hom}_B(A, Q) & \xrightarrow{\text{Hom}_B(f, Q)} & \text{Hom}_B(\Omega U, Q).
\end{array}
$$

So it suffices to show the following sequence

$$
0 \to \text{Hom}_B(B, Q) \xrightarrow{\text{Hom}_B(g, Q)} \text{Hom}_B(A, Q) \xrightarrow{\text{Hom}_B(f, Q)} \text{Hom}_B(\Omega U, Q)
$$

is exact.

We first show that $\text{Hom}_B(g, Q)$ is injective. Let $r : B \to Q$ be any morphism such that $rg = 0$. Take a commutative diagram of short exact sequences

Since $rga$ factors through $W$, and $q_A$ is a left $\mathcal{V}$-approximation of $\Omega U^A$, it factors through $q_A$. Thus there exists $c : W^A \to Q$ such that $cw^A = rg$.

As $\text{Ext}^1_B(\mathcal{U}, \mathcal{V}) = 0$, there exists $d : B \to W^A$ such that $w^A = dg$. Hence $rg = cw^A = cdg$, then $r - cd$ factors through $U$.

Since $\text{Hom}_B(U, Q) = 0$ by Lemma 4.2, we get that $\mathfrak{z} = 0$.

Assume $r' : A \to Q$ satisfies $r'f = 0$, since $q$ is left $\mathcal{V}$-approximation of $\Omega U$, $r'f$ factors through $q$. As the left square of (3) is a push-out, we get the following commutative diagram.
Hence \( r' \) factors through \( g \). This shows the exactness of

\[
\text{Hom}_B(B, Q) \xrightarrow{\text{Hom}_B(g, Q)} \text{Hom}_B(A, Q) \xrightarrow{\text{Hom}_B(f, Q)} \text{Hom}_B(\Omega U, Q).
\]

\( \Box \)

Dually, we have the following:

**Lemma 4.4.** Let

\[
\begin{array}{cccc}
V & \xrightarrow{f} & A & \xrightarrow{g} & B \\
\downarrow & & \downarrow & & \downarrow \\
V & \xrightarrow{h} & I^Y & \xrightarrow{\Omega^{-}V} &
\end{array}
\]

be a commutative diagram with rows short exact satisfying \( V \in \mathcal{V} \) and \( I^Y \in \mathcal{I} \). Then the sequence

\[
0 \to H(A) \xrightarrow{H(g)} H(B) \xrightarrow{H(h)} H(\Omega^{-}V)
\]

is exact in \( \mathcal{H} \).

**Theorem 4.5.** For any cotorsion pair \((\mathcal{U}, \mathcal{V})\) in \( \mathcal{B} \), the functor

\[
H : \mathcal{B} \to \mathcal{H}
\]

is cohomological. We call \( H \) the associated cohomological functor to \((\mathcal{U}, \mathcal{V})\).

**Proof.** Let

\[
\begin{array}{cccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\end{array}
\]

be any short exact sequence in \( \mathcal{B} \). By Proposition 2.2, we can get the following commutative diagram:

\[
\begin{array}{cccc}
\Omega U^A & \xrightarrow{b} & P_UA & \xrightarrow{U^A} \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{c} & V^A & \xrightarrow{U^A} \\
\downarrow & & \downarrow & & \downarrow \\
B & \xrightarrow{e} & D & \xrightarrow{U^A} \\
\downarrow & & \downarrow & & \downarrow \\
C & = & C.
\end{array}
\]

By Proposition 4.1, Lemma 4.3 and 4.4, we have exact sequences

\[
H(\Omega U^A) \xrightarrow{H(a)} H(A) \to 0,
\]

\[
H(\Omega U^A) \xrightarrow{H(fa)} H(B) \xrightarrow{H(c)} H(D) \to 0,
\]

\[
0 \to H(D) \xrightarrow{H(d)} H(C).
\]

Now we can obtain an exact sequence \( H(A) \xrightarrow{H(f)} H(B) \xrightarrow{H(g)} H(C) \).

Since every short exact sequence

\[
\begin{array}{cccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\end{array}
\]

in \( \mathcal{B} \) admits two commutative diagrams
we get two short exact sequences by Proposition 2.1

\[ \Omega C \xrightarrow{(\alpha, \beta)} P_C \oplus A \xrightarrow{(p, f)} B, \quad B \xrightarrow{(\gamma, \delta)} I^A \oplus C \xrightarrow{(j, h)} \Omega^A. \]

Since \( H(\mathcal{P}) = H(\mathcal{I}) = 0 \) by Proposition 4.1, we get two exact sequences

\[ \Omega C \xrightarrow{H(\alpha)} H(A) \xrightarrow{H(f)} H(B), \quad H(B) \xrightarrow{H(g)} H(C) \xrightarrow{H(h)} H(\Omega^A). \]

Repeat such step, we get the following commutative diagrams:

\[
\begin{array}{ccccc}
\Omega C & \xrightarrow{\Omega^B} & P_B & \xrightarrow{j} & B \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} \\
\Omega C & \xrightarrow{\Omega^B} & P_B & \xrightarrow{j} & B \\
\end{array}
\quad
\begin{array}{cccc}
P_C \oplus A & \xrightarrow{(p, f)} & B & \xrightarrow{\gamma} \\
\downarrow{\alpha} & & \downarrow{\beta} & \\
P_C \oplus A & \xrightarrow{(p, f)} & B & \xrightarrow{\gamma} \\
\end{array}
\]

which induce the following two exact sequences

\[ H(\Omega B) \xrightarrow{H(x)} H(\Omega C) \xrightarrow{H(\alpha)} H(A), \quad H(C) \xrightarrow{H(h)} H(\Omega^A) \xrightarrow{H(\gamma)} H(\Omega^B). \]

It is easy to check that \( x = \Omega g \) in \( \mathcal{B} / \mathcal{P} \) (resp. \( y = \Omega f \) in \( \mathcal{B} / \mathcal{I} \)), hence we can get a long exact sequence

\[ \cdots \xrightarrow{H(\Omega^g)} H(\Omega A) \xrightarrow{H(\Omega f)} H(\Omega B) \xrightarrow{H(\Omega g)} H(\Omega C) \xrightarrow{H(h)} H(A) \xrightarrow{H(f)} H(B) \xrightarrow{H(g)} H(\Omega^A). \]

in \( \mathcal{H} \).

Corollary 4.6. The cohomological functor \( H \) induces a morphism from \( \text{Ext}^1_B(C, A) \) to \( \text{Hom}_H(H(C), H(\Omega^A)) \) (resp. \( \text{Hom}_H(H(\Omega C), H(A)) \)).

Proof. \( H \) induces a morphism

\[ \alpha : \text{Hom}_B(C, \Omega^A) \rightarrow \text{Hom}_H(H(C), H(\Omega^A)) \]

and \( H(\mathcal{I}) = 0 \), it induces a morphism from \( \text{Hom}_B(C, \Omega^A) / \text{Im} \text{Hom}_B(C, j) \) to \( \text{Hom}_H(H(C), H(\Omega^A)) \) since \( j \) factors \( \mathcal{I} \). Apply \( \text{Hom}_B(C, -) \) to the short exact sequence \( A \xrightarrow{j} \Omega A \), we get the following exact sequence

\[ \text{Hom}_B(C, I^A) \xrightarrow{\text{Hom}_B(C, j)} \text{Hom}_B(C, \Omega^A) \rightarrow \text{Ext}^1_B(C, A) \rightarrow 0. \]

Hence \( \text{Ext}^1_B(C, A) \simeq \text{Hom}_B(C, \Omega^A) / \text{Im} \text{Hom}_B(C, j) \). Thus \( H \) induces a morphism from \( \text{Ext}^1_B(C, A) \) to \( \text{Hom}_H(H(C), H(\Omega^A)) \).

The proof for \( \text{Hom}_H(H(\Omega C), H(A)) \) is by dual.

We show the following corollary which gives some information of the kernel of \( H \).

Corollary 4.7. For any cotorsion pair \( (\mathcal{U}, \mathcal{V}) \) in \( \mathcal{B} \) and any object \( B \in \mathcal{B} \), the following are equivalent.

(a) \( H(B) = 0 \) in \( \mathcal{H} \).
(b) For any \( \Omega U \in \Omega \mathcal{U} \), any morphism \( u \in \text{Hom}_B(\Omega U, B) \) factors through \( \mathcal{U} \).
For any \( \Omega^- V \in \Omega^- \mathcal{V} \), any morphism \( v \in \text{Hom}_B(B, \Omega^- V) \) factors through \( \mathcal{V} \).

**Proof.** We only show the equivalence of (a) and (b), the equivalence of (a) and (c) is dual. Suppose (a) holds, by dual of [7, Lemma 3.4], we obtain \( B^+ \in \mathcal{V} \).

Let \( \Omega U \) admit a short exact sequence

\[
\Omega U \rightarrow q \rightarrow U
\]

where \( P_U \in \mathcal{P} \) and \( U \in \mathcal{U} \), then for any morphism \( u \in \text{Hom}_B(\Omega U, B) \), since \( q \) is a left \( \mathcal{V} \)-approximation of \( \Omega U \), there exists a morphism \( p : P_U \rightarrow B^+ \) such that \( pq = b^+ u \).

Take the following commutative diagram

\[
\begin{array}{ccc}
\Omega U & \xrightarrow{q} & P_U \\
\downarrow{u} & & \downarrow{p} \\
B & \xrightarrow{b^+} & B^+ \\
\end{array}
\]

with \( u \) factoring through \( \mathcal{U} \), we obtain a short exact sequence

\[
\Omega U \xrightarrow{q_0} B \oplus P (b^+ p_0) \rightarrow B^+
\]

Since \( P_U \) is projective, \( q \) factors through \( (b^+ p_0) \).

Thus we get \( b^+ u = pq = (b^+ c + p_0 d) q = b^+ c q + p_0 d q \), then we obtain the following commutative diagram

\[
\begin{array}{ccc}
\Omega U & \xrightarrow{q_0} & P_U \\
\downarrow{u} & & \downarrow{p} \\
B \xrightarrow{b^+} & B^+ \\
\end{array}
\]

where \( u = c q + u' e \). This implies \( u \) factors through \( \mathcal{U} \).

Now assume (c) holds. Since \( B \) admits a short exact sequence

\[
B \rightarrow V^B \rightarrow U^B
\]

we get the following commutative diagram

\[
\begin{array}{ccc}
\Omega U^B & \xrightarrow{q^a} & P_{U^B} \\
\downarrow{u} & & \downarrow{p} \\
B & \xrightarrow{V^B} & U^B \\
\end{array}
\]

where \( u \) factors through \( \mathcal{U} \). By Lemma 4.3 we get the following exact sequence

\[
H(\Omega U^B) \rightarrow H(B) \rightarrow H(V^B)
\]

in \( \mathcal{H} \). Since \( H(u) = 0 \) and \( H(V^B) = 0 \) by Proposition 4.1, we get \( H(B) = 0 \). \( \square \)
The kernel of $H$ becomes simple in the following cases.

**Proposition 4.8.** Let $(U, V)$ be a cotorsion pair on $B$, then

(a) If $U \subseteq V$, then $H(B) = 0$ if and only if $B \in V$.
(b) If $V \subseteq U$, then $H(B) = 0$ if and only if $B \in U$.

**Proof.** We only show (a), (b) is dual.

If $H(B) = 0$, then by dual of [7, Lemma 3.4], we obtain $B^+ \in V$. But since $U \subseteq V$, we have $B^+ = B$.

Hence $B = B^+ \in V$.

By Proposition 4.1, if $B \in V$, then $H(B) = 0$.

$\square$

5. Examples

The aim of this section is to describe the associated functor $H$ in a simple way for some special cases.

If $M$ is a cluster tilting subcategory of $B$, then $(M, M)$ is a cotorsion pair on $B$ (see [7, Proposition 10.5]). In this case we have $H = B^+ = B$, $\sigma^+ = id$ and $H = \pi$. By Theorem 4.5, we get:

**Proposition 5.1.** Let $M$ be a cluster tilting subcategory of $B$. Then the quotient functor

$$\pi : B \to B/M$$

is cohomological. More precisely, every short exact sequence

$$A \to B \to C$$

in $B$ induces a long exact sequence

$$\cdots \Omega^- \sigma^- A \to \Omega^- B \to \Omega^- C \to \Omega^- \sigma^- A \to \Omega^- B \to \Omega^- C \to \cdots$$

in an abelian category $B/M$.

The following is a corollary of Proposition 5.1.

**Corollary 5.2.** Let $M$ be a cluster tilting subcategory of $B$. Then there exists a cohomological functor

$$F \circ H : B \to \text{mod} M/P$$

where $F$ is the equivalence between $B/M$ and $\text{mod} M/P$ given in [3, Theorem 3.2]. $F \circ H(X) = 0$ if and only if $X \in M$.

A more general case is given as follows. If $M$ is a rigid subcategory of $B$ which is contravariantly finite and contains $P$, then by [7, Proposition 2.12], $(\mathcal{M}, \mathcal{M}^{\perp 1})$ is a cotorsion pair where $\mathcal{M}^{\perp 1} = \{X \in B \mid \text{Ext}^1_B(M, X) = 0\}$. Since $M$ is rigid, we have $M \subseteq \mathcal{M}^{\perp 1}$. In this case we have $B^+ = B$, $B^- = \mathcal{H}$, $\sigma^+ = id$ and $H = \sigma^+ \circ \pi$. By [3, Theorem 3.2], we get that there exists an equivalence between $\mathcal{H}$ and $\text{mod} \mathcal{M}/P$. Hence by Theorem 4.5, we get the following proposition:

**Proposition 5.3.** Let $M$ be a rigid subcategory of $B$ which is contravariantly finite and contain $P$. Then there exists a cohomological functor

$$F \circ H : B \to \text{mod} \mathcal{M}/P$$

$$X \mapsto \text{Ext}^1_B(\mathcal{H}, X)$$

and $F \circ H(X) = 0$ if and only if $X \in \mathcal{M}^{\perp 1}$.

Let $B$ be a Frobenius category with a cotorsion pair $(\mathcal{U}, \mathcal{V})$ on it.

**Corollary 5.4.** For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on a Frobenius category $B$, there exists a cohomological functor $H'$ from the stable category of $B$ to $\mathcal{H}$ which is induced by $H$. 
Proof. Let $\pi': B \to B/P$ be the quotient functor, since $H(P) = 0$ by Proposition 4.1, there exists a functor $H': B/P \to H$ such that $H = H'\pi'$. We shall prove that $H'$ is cohomological. Since every triangle in $B/P$ is isomorphic to some distinguished triangle

$$X \xrightarrow{\pi(f)} Y \xrightarrow{\pi(g)} Z \to \Omega^-X$$

which is induced by the following commutative diagram

$$\begin{array}{c}
X \xrightarrow{p} f^X \xrightarrow{\Omega^-} \Omega^-X \\
\downarrow f \downarrow \downarrow \downarrow P O \\
Y \xrightarrow{g} Z \xrightarrow{\Omega^-} \Omega^-X
\end{array}$$

it suffices to consider such distinguished triangle. But such commutative diagram induce a short exact sequence

$$X \xrightarrow{(p)} f^X \oplus Y \xrightarrow{(g g)} Z.$$ 

By Theorem 4.5 we get a exact sequence

$$H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z)$$

in $H$, which also has the form

$$H'(X) \xrightarrow{H'\pi'(f)} H'(Y) \xrightarrow{H'\pi'(g)} H'(Z).$$

□

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