THE $R$-MATRIX FOR (TWISTED) AFFINE QUANTUM ALGEBRAS

Ilaria Damiani

To the victims of the NATO aggression against Yugoslavia.

March 24$^{th}$, 1999: NATO starts bombing the Federal Republic of Yugoslavia, aiming at its civil and productive structures and killing thousands of people.

May 8$^{th}$, 1999: NATO missiles hit the Chinese Embassy in Belgrade, destroying it and killing 3 people.

I want to dedicate this paper to the Yugoslav students, whose right to life and culture is threatened, and to the Chinese students, who raised their voice against this criminal war.

Index

0. INTRODUCTION

1. GENERAL SETTING: DEFINITIONS AND PRELIMINARIES
   §1.1. Affine Kac-Moody algebras
   §1.2. Some notations, structures and general properties
      1.2.A. Root lattice
      1.2.B. Cartan matrix and symmetric bilinear form
      1.2.C. Weight lattices
      1.2.D. Weyl and braid group
      1.2.E. Root system
   §1.3. Dynkin diagrams and classification
   §1.4. The quantum algebra

2. COPIES OF $A^{(1)}_1$ IN $X_h^{(k)}$
   §2.1. Definition of the root vectors
   §2.2. The homomorphisms $\varphi_i$

3. THE CASE OF $A^{(2)}_2$
   §3.1. Real root vectors
   §3.2. The imaginary root vector $\tilde{E}_\delta$
   §3.3. The imaginary root vectors $\tilde{E}_{m\delta}$
   §3.4. Commutation between positive and negative real root vectors
   §3.5. The imaginary root vectors $E_{m\delta}$

4. $A^{(2)}_{2n}$ AND A COPY OF $A^{(2)}_2$
   §4.1. Definition of $\varphi_1$
   §4.2. Root system and Weyl group: some properties
   §4.3. $\varphi_1$ is well defined
   §4.4. Relations between $\varphi_1$ and the braid group action

5. RELATIONS BETWEEN $U^{(i)}_q$ AND $U^{(j)}_q$ ($i \neq j$)
   §5.1. First level of commutation
   §5.2. Some particular computations: low twisted cases
   §5.3. General commutation formulas

6. A PBW-BASIS
   §6.1. A convex ordering
   §6.2. PBW-basis and L-S formula
   §6.3. Some subspaces of $U^+_q$

1
7. THE $R$-MATRIX

§7.1. General strategy for the construction of the $R$-matrix

§7.2. The Killing form on the imaginary root vectors

§7.3. A “canonical” form for the $R$-matrix

§7.4. Linear triangular transformation: the $E_\alpha$'s

§7.5. The Killing form on the new imaginary root vectors

§7.6. The expression of $R$ in terms of the $E_\alpha$'s

8. REFERENCES

0. INTRODUCTION.

The aim of this paper is to give an exponential multiplicative formula for the $R$-matrix of the general affine quantum algebra $U_q$. The problem of describing $R$ has been attacked from different viewpoints, and different kinds of formulas for $R$ have already been given explicitly in many (general) cases.

For the finite type algebras, Rosso gave in [30] the first exponential multiplicative formula (for $\mathfrak{sl}_n$): this result was generalized in [22], where the strategy of partial $R$-matrices (that is the study of the connections between the coproduct and the braid group action) was introduced and developed, and the general (finite) case solved.

This exponential approach, mainly based on the description of $R$ via the Killing form (see [31]), has been extended to $A^{(1)}_1$ in [24], and to the general non twisted affine case in [6].

Other descriptions of $R$ in the affine case (both non twisted and twisted) are also known, by means of different techniques. Of particular relevance is the application of the theory of vertex algebras, which turns out to be a powerful instrument for studying the representations of the affine quantum algebras; in particular it is important to mention the works of Khoroshkin-Tolstoy (see [20] and [19]) which (together with [24]) are the first to face the study of $R$ for the affine algebras, and those of Delius-Gould-Zhang ([8]) and Gandenberger-MacKay-Watts ([12]), where the $R$-matrix was constructed in the twisted case.

The purpose of the present paper is to complete the “exponential picture” by including the twisted affine case in this description.

To this aim a quite precise analysis of the twisted affine quantum algebras is needed: since the PBW-bases (bases of type Poincaré-Birkhoff-Witt), and the possibility to make computations on them, are a fundamental tool for working in $U_q$ (and in particular for the construction of $R$), the most important step in this direction is producing a PBW-basis and stating its main properties by generalizing the well-known results in the finite and non twisted affine case.

While approaching this question, one easily finds out that in many key points the structure of the behaviour of the twisted affine quantum algebras is not so different (up to some minor adjustments involving no conceptual difficulty and requiring just some care in the notations; notations which are fixed in section 1) from what is already known in the non twisted case: the definition of the root vectors does not present any difference from the non twisted case (it only happens that the root system with multiplicities is slightly more complicated to describe), and the proof of most of the commutation formulas lies on a property of the root system ($\ast$: if $\alpha$ and $\delta + r\alpha$ are roots, then $r = \pm 1$) which is almost always true; when this is the case the contribution of this paper is just that of remarking the analogies and giving
references for the proofs. Thus, section 2, where copies of $\mathcal{U}_q(A_{1}^{(1)})$ are embedded in the affine quantum algebra, has been written following [1] (to which it refers heavily). The same can be said about paragraphs §5.1 and §5.3 (which are mainly based on [1]) and about section 6 (where the results previously found are gathered together to exhibit a PBW-basis). Similarly, the first two paragraphs of section 7 (§7.1 and §7.2, with the exception of lemma 7.2.3), show and use the validity for the twisted case of the results of [6] (hence those of [22] for what concerns the real root vectors) where the coproduct and the Killing form on the root vectors are computed.

But there is a case ($A_{2n}^{(2)}$) when $\delta - 2\alpha_1$ is a root, so that the property above denoted by $\ast$ is no more true. This situation must be dealt with more carefully, what is done in section 3 (where the case of $A_{2n}^{(2)}$ is discussed in details) and in section 4, where $\mathcal{U}_q(A_{2n}^{(2)})$ is embedded in $\mathcal{U}_q(A_{2n}^{(2)})$: $A_{2}^{(2)}$ plays here the same role that $A_{1}$ plays in the general frame (to each vertex there is attached a copy of $\mathcal{U}_q(A_{1}) = \mathcal{U}_q(sl_2)$) and $A_{1}^{(1)}$ in the generic affine picture (to “almost each” vertex of the associated finite diagram there is attached a copy of $\mathcal{U}_q(A_{1}^{(1)})$). The existence of a copy of $\mathcal{U}_q(A_{2}^{(2)})$ in $\mathcal{U}_q(A_{2n}^{(2)})$ was already proved in [4] by means of the language of vertex operators and through the Drinfeld realization of $\mathcal{U}_q$ (see [17] and [16]); here we give a different, direct proof, which has the advantage to provide a precise description of the commutation relations that we need.

Finally, paragraphs §5.2 (this in particular requires a deeper understanding of the twisted root systems and Weyl groups), §7.4 and §7.5 are devoted to the computations which, after establishing the general frame, are needed in order to make explicit the description of $R$, whose final formula is given in paragraphs §7.3 and §7.6.

I would like to thank Professors Chen Yu, Hu Naihong, Lin Zongzhu, Wang Jianpan and the other organizers of the International Conference on Representation Theory, held in Shanghai (China) in June/July 1998, for giving me the opportunity to participate in this meeting which has been for me of big interest, and for offering the possibility to publish a contribution on the Conference Proceedings.

Unfortunately last spring, during the work of drawing up of the paper, the country where I live (Italy) participated in the war against the Federal Republic of Yugoslavia, which has brutally involved also the People’s Republic of China: the decision of the NATO countries to isolate the scientific community of Yugoslavia is just one - yet hateful - of the “collateral effects” of this aggression. Being deeply concerned by the evident and arrogant injustice of this politics, I want to express, as a mathematician, my opposition to this choice, together with my solidarity to all those who are fighting every day to make the science survive, even in these hard times, for the present and future generations.

1. GENERAL SETTING: DEFINITIONS AND PRELIMINARIES.

§1.1. Affine Kac-Moody algebras.

In this paragraph we recall some basic generalities; for the proofs and for a deeper and more detailed investigation we refer to [18], where these notions have been introduced and studied.

Let $\mathfrak{g}$ be a Kac-Moody algebra of finite type. Then an automorphism of the Dynkin diagram of $\mathfrak{g}$ of order $k$ induces an automorphism of $\mathfrak{g}$ of the same order (it
is well known and immediate to see that \( k = 1, 2 \) or 3). This automorphism, whose eigenvalues are of course of the form \( e^{\frac{2\pi ir}{k}} \) with \( 0 \leq r < k \), is diagonalizable, that is \( \mathfrak{g} = \bigoplus_{r \in \mathbb{Z}/k\mathbb{Z}} \mathfrak{g}(r) \), \( \mathfrak{g}(r) \) being the eigenspace relative to \( e^{\frac{2\pi ir}{k}} \). It is straightforward to prove that this decomposition of \( \mathfrak{g} \) as a direct sum is a \( \mathbb{Z}/k\mathbb{Z} \)-grading, that is \([\mathfrak{g}(r), \mathfrak{g}(s)] \subseteq \mathfrak{g}(r+s) \) \( \forall r, s \in \mathbb{Z}/k\mathbb{Z} \); in particular \( \mathfrak{g}(0) \) is a Lie subalgebra of \( \mathfrak{g} \) (indeed a Kac-Moody algebra of finite type) and \( \mathfrak{g}(r) \) is a \( \mathfrak{g}(0) \)-module.

This remark allows to give a natural structure of Lie algebra to

\[
\bigoplus_{r \in \mathbb{Z}} \mathfrak{g}^{(r)} \otimes \mathbb{C}r;
\]

it is easy to construct a non trivial central extension

\[
\left( \bigoplus_{r \in \mathbb{Z}} \mathfrak{g}^{(r)} \otimes \mathbb{C}r \right) \oplus \mathbb{C}c
\]

of this algebra via the (non degenerate) Killing form on \( \mathfrak{g} \) and a natural structure of Lie algebra on

\[
\hat{\mathfrak{g}}^{(k)} \equiv \left( \bigoplus_{r \in \mathbb{Z}} \mathfrak{g}^{(r)} \otimes \mathbb{C}r \right) \oplus \mathbb{C}c \oplus \mathbb{C}\partial,
\]

where \( \text{ad}\partial = t \frac{d}{dt} \).

It was proved by Kac (see [18]) that \( \hat{\mathfrak{g}}^{(k)} \) is a Kac-Moody algebra of affine type depending just on \( \mathfrak{g} \) and \( k \) and not on the chosen Dynkin diagram automorphism (remark that \( \hat{\mathfrak{g}}^{(1)} \) is the non twisted affine Kac-Moody algebra associated to \( \mathfrak{g} \); in case that \( k > 1 \) the algebra \( \hat{\mathfrak{g}}^{(k)} \) is said to be twisted, that is relative to a non trivial automorphism of the Dynkin diagram); moreover all the affine Kac-Moody algebras are isomorphic to \( \hat{\mathfrak{g}}^{(k)} \) for some finite Kac-Moody algebra \( \mathfrak{g} \) and some \( k \in \{1, 2, 3\} \).

We also have that the Dynkin diagram \( \Gamma \) of \( \hat{\mathfrak{g}}^{(k)} \) is an extension of the Dynkin diagram \( \Gamma_0 \) of \( \hat{\mathfrak{g}}^{(0)} \) (which is called the finite diagram - or algebra - associated to the affine one) by adding an extra vertex, which will be denoted by 0; remark that if \( k = 1 \) then \( \hat{\mathfrak{g}}^{(0)} = \mathfrak{g} \).

\[\text{§1.2. Some notations, structures and general properties.}\]

Let us fix an affine Kac-Moody algebra \( \hat{\mathfrak{g}}^{(k)} \). We shall now introduce some notations.

First of all let us denote by \( \tilde{n} \) the number of vertices of the Dynkin diagram of \( \mathfrak{g} \); this means that \( \mathfrak{g} \) is of type \( X_{\tilde{n}} \) for some \( X \in \{A, B, C, D, E, F, G\} \), and in this case we say that \( \hat{\mathfrak{g}}^{(k)} \) is of type \( X^{(k)}_{\tilde{n}} \). We denote by \( n \) the cardinality of \( I_0 \), where \( I_0 \) is the set of vertices of \( \Gamma_0 \) (it means that the cardinality of \( I \), the set of vertices of \( \Gamma \), is \( n + 1 \)) and note that \( n = \tilde{n} \Leftrightarrow k = 1 \). In particular it is possible to identify \( I \) with the set \( \{0, 1, ..., n\} \) and \( I_0 \) with \( \{1, ..., n\} \) (so that \( I = I_0 \cup \{0\} \)). One such identification will be fixed in the next paragraph.

Furthermore, \( A = (a_{ij})_{i,j \in I} \) will denote the Cartan matrix of \( \hat{\mathfrak{g}}^{(k)} \) and \( A_0 \) will be \( (a_{ij})_{i,j \in I_0} \) (the Cartan matrix of \( \hat{\mathfrak{g}}^{(0)} \)). Remember that \( (I, A) \) determines \( \hat{\mathfrak{g}}^{(k)} \) completely; remark also that \( (I, A) \) can be extended to \( (I_\infty, A_\infty) \) where \( I_\infty \equiv I \cup \{\infty\} \) and \( A_\infty = (a_{ij})_{i,j \in I_\infty} \) with \( a_{i\infty} = a_{\infty i} = \delta_{i0} \forall i \in I_\infty \).

We have the following structures associated to \( (I, A) \):
1.2.A Root lattice.

The root lattice \( Q \) and the extended root lattice \( Q_\infty \) of \( \hat{\mathfrak{g}}^{(k)} \), and the finite root lattice \( Q_0 \) of \( \mathfrak{g}^{(0)} \) are defined respectively by:

\[
Q \doteqdot \bigoplus_{i \in I} \mathbb{Z} \omega_i, \quad Q_\infty \doteqdot \bigoplus_{i \in I_\infty} \mathbb{Z} \alpha_i, \quad Q_0 \doteqdot \bigoplus_{i \in I_0} \mathbb{Z} \alpha_i.
\]

Of course \( Q_0 \subseteq Q = Q_0 \oplus \mathbb{Z} \alpha_0 \subseteq Q_\infty = Q \oplus \mathbb{Z} \alpha_\infty \). We also define \( Q^+ \) and \( Q_0^+ \) as follows:

\[
Q^+ \doteqdot \sum_{i \in I} \mathbb{N} \alpha_i = \left\{ \sum_{i \in I} n_i \alpha_i \in Q | r_i \geq 0 \ \forall i \in I \right\}, \quad Q_0^+ \doteqdot Q_0 \cap Q^+.
\]

It can be useful to see \( Q_\infty, Q \) and \( Q_0 \) as lattices in a vector space over \( \mathbb{R} \) naturally defined as \( Q_\infty, Q \subseteq \mathbb{R} \otimes \mathbb{Z} Q_\infty \). Set also \( Q_\mathbb{R} \) and \( Q_0, \mathbb{R} \) to be the \( \mathbb{R} \)-subspaces of \( Q_\infty, \mathbb{R} \) generated respectively by \( Q \) and \( Q_0 \).

1.2.B Cartan matrix and symmetric bilinear form.

The Cartan matrix \( A \) is symmetrizable, that is there exists a diagonal matrix \( D = \text{diag}(d_i | i \in I) \) such that \( DA \) is symmetric; the diagonal entries \( d_i \)'s can be chosen to be coprime positive integers, and this condition determines them uniquely. Remark that if we put \( D_\infty = \text{diag}(d_i | i \in I) \) with \( d_\infty = d_0 \) then \( D_\infty A_\infty \) is symmetric.

Hence \( D_\infty A_\infty \) induces a symmetric bilinear form \( (\cdot | \cdot) \) on \( Q_\infty, \mathbb{R} \) (\( \mathbb{Z} \)-valued on \( Q_\infty \)) defined by \( (\alpha_i | \alpha_j) = d_i a_{ij} \). We have that \( (\cdot | \cdot) \) is non degenerate but it is not positive definite, while \( (\cdot | \cdot) \mid_{Q_0 \times Q_0} \) is positive definite (that is \( (\cdot | \cdot) \) induces a structure of Euclidean space on \( Q_0, \mathbb{R} \)) and \( (\cdot | \cdot) \mid_{Q_\infty, \mathbb{R}} \) is degenerate positive semidefinite: there exists a (unique) non zero element \( \delta \in Q^+ \) such that \( \delta - \alpha_0 \in Q_0^+ \) and \( (\delta | \delta) = 0 \); more precisely \( (\delta | \alpha) = 0 \ \forall \alpha \in Q \). Remark that \( Q \) can be decomposed as a direct sum also as \( Q = Q_0 \oplus \mathbb{Z} \delta \), and that of course \( (\alpha + r \delta | \beta + s \delta) = (\alpha | \beta) \ \forall \alpha, \beta \in Q \) and \( \forall r, s \in \mathbb{Z} \).

1.2.C Weight lattices.

\( (\cdot | \cdot) \) induces an identification of \( \text{Hom}(Q_0, \mathbb{Z}) \) with a subgroup \( P \) of \( Q_{0, \mathbb{R}} \). \( P \) is, by the very definition, the lattice \( P = \bigoplus_{i \in I_0} \mathbb{Z} \omega_i \) where the \( \omega_i \)'s, called fundamental weights, are defined as the elements of \( Q_{0, \mathbb{R}} \) characterized by the property that \( (\omega_i | \alpha_j) = d_{ij} \ \forall j \in I_0 \). \( Q_0 \) is obviously a sublattice of \( P \), and it is worth introducing another important sublattice \( \hat{P} \) of \( P \) as \( \hat{P} = \bigoplus_{i \in I_0} \mathbb{Z} \hat{\omega}_i \), where \( \forall i \in I_0 \omega_i = d_i \omega_i \); it is to be noticed that \( Q_0 \subseteq \hat{P} \subseteq P \). The elements of \( P \) are called weights and the elements of

\[
P_+ \doteqdot \left\{ \sum_{i \in I_0} r_i \omega_i \in P | r_i \geq 0 \ \forall i \in I_0 \right\}
\]

are called dominant weights; we also denote by \( P_+ \) the set \( P_+ \subseteq P_+ \cap P \). Remark that \( [P : P] = \prod_{i \in I_0} d_i \) and \( [P : Q_0] = \text{det}(A_0) \), what can be easily seen noticing that \( \forall i \in I_0 \) we have \( a_i = \sum_{j \in I_0} a_{ij} \omega_j \).

It is also important to mention the realization of \( P \) as a group of transformations of \( Q \) as follows: define \( t : P \rightarrow \text{Hom}(Q, Q) \) by setting \( t_x(\alpha) = \alpha - (x | \alpha) \delta \). Remark that \( t \) is an injective homomorphism of groups and that \( \forall \alpha \in P \ t_x \) preserves \( (\cdot | \cdot) \mid_{Q \times Q} \) and fixes \( \delta \); equivalently we can say that \( t \) maps \( P \) into \( \text{Aut}(Q_0, Q) \), where \( \text{Aut}(Q_0, Q) \) is the group of automorphisms of \( Q \) preserving \( (\cdot | \cdot) \mid_{Q \times Q} \) and fixing \( \delta \).
By abuse of notation, we shall usually denote by $x$ also its image $t_x$.

1.2.D Weyl and braid group.

The Weyl group $W$ is the subgroup of $\text{Aut}(\mathbb{Q}(\{\alpha_i\}_{i \in I}))$ generated by $\{s_i|i \in I\}$ where $s_i(\alpha_j) = \alpha_j - \alpha_{ij}\alpha_i$. $W_0$, the Weyl group of $\mathfrak{g}^{(0)}$, is the subgroup of $W$ generated by $\{s_i|i \in I_0\}$.

$\text{Aut}(\Gamma)$ is the group of automorphisms of the Dynkin diagram of $\mathfrak{g}^{(k)}$ (which can be naturally identified to a subgroup of $\text{Aut}(\mathbb{Q}(\{\alpha_i\}_{i \in I}))$).

The subgroups of $\text{Aut}(\mathbb{Q}(\{\alpha_i\}_{i \in I}))$ introduced above $(\mathcal{P}, W_0, \text{Aut}(\Gamma))$ have the following properties:

i) $W \cap \text{Aut}(\Gamma) = \{\text{id}\}$, and $\tau s_i \tau^{-1} = s_{\tau(i)}$ $\forall i \in I$ $\forall \tau \in \text{Aut}(\Gamma)$; hence $W \leq \langle W, \text{Aut}(\Gamma) \rangle$ and $\langle W, \text{Aut}(\Gamma) \rangle = W \rtimes \text{Aut}(\Gamma)$;

of course $\langle W, G \rangle = W \rtimes G \forall G \leq \text{Aut}(\Gamma)$.

ii) $W_0 \cap P = \{\text{id}\}$ and $w w^{-1} = w(x)$ $\forall x \in P \forall w \in W$; hence $P \leq \langle W_0, P \rangle$, so that $\langle W_0, P \rangle = P \rtimes W_0$; as above $\langle W_0, L \rangle = L \rtimes W_0 \forall L \leq P$ $W_0$-stable.

iii) $W \leq P \rtimes W_0$; more precisely

$$W = \begin{cases} Q_0 \rtimes W_0 & \text{in the non twisted case} \\ P \rtimes W_0 & \text{in case } A_{2n}^{(2)} \\ Q_0 \rtimes W_0 & \text{otherwise.} \end{cases}$$

$(Q_0 \subseteq Q = \oplus_{i \in I_0} \mathbb{Z}\alpha_i \subseteq P$ where $\alpha_i = \frac{\alpha_{ij}}{\alpha_j} )$.

iv) Let $\tau T \tilde{=} c \text{Aut}(\Gamma) \cap (P \rtimes W_0)$; then

$$\tilde{W} \tilde{= \tilde{\cup}} W \rtimes T = \begin{cases} P \rtimes W_0 & \text{in the non twisted case and in case } A_{2n}^{(2)} \\ P \vert W_0 & \text{otherwise.} \end{cases}$$

For a precise description of $T$ see [14].

The length function $l : \tilde{W} \rightarrow \mathbb{N}$ is defined by

$$l(\tilde{w}) = \min\{r \in \mathbb{N} \mid \exists i_1, \ldots, i_r \in I, \tau \in T \text{ s.t. } \tilde{w} = s_{i_1} \cdots s_{i_r} \tau\}.$$ 

Also, the braid group $\tilde{B}$ is the group generated by $\{T_{\tilde{w}}|\tilde{w} \in \tilde{W}\}$ with relations $T_{\tilde{w}}T_{\tilde{w}'} = T_{\tilde{w}w'}$ whenever $l(\tilde{w}w') = l(\tilde{w}) + l(\tilde{w}')$ (see [29]). Set $T_i \tilde{=} T_{s_i}.$

1.2.E Root system.

The root system $\Phi \subseteq Q$ divides into two components: $\Phi = \Phi^\text{re} \cup \Phi^\text{im}$ where

$$\Phi^\text{re} = W.\{\alpha_i|\alpha_i \in I\} = \tilde{W}.\{\alpha_i|\alpha_i \in I\} = \{\alpha \in \Phi|\alpha|\alpha| > 0\} = \{\text{real roots}\}$$

and

$$\Phi^\text{im} = \{m\delta|m \in \mathbb{Z} \setminus \{0\}\} = \{\alpha \in \Phi|\alpha|\alpha| = 0\} = \{\text{imaginary roots}\}.$$ 

$\Phi^\text{re}$ can be described in terms of $\Phi^\text{re} = W.\{\alpha_i|\alpha_i \in I_0\}$ (the root system of $\mathfrak{g}^{(0)}$) as follows:

$$\Phi^\text{re} = \begin{cases} \{m\delta + \alpha|\alpha \in \Phi_0, m \in \mathbb{Z}\} & \text{in the non twisted case} \\ \{m\delta + \alpha|\alpha \in \Phi_0, m \in \mathbb{Z}\} \cup \\ \{(2m + 1)\delta + 2\alpha|\alpha \in \Phi_0, (\alpha|\alpha| = 2, m \in \mathbb{Z}\} & \text{in case } A_{2n}^{(2)} \\ \{md, \delta + \alpha|\alpha \in \Phi_0, m \in \mathbb{Z}\} & \text{otherwise,} \end{cases}$$
where \( d_\alpha \overset{\alpha}{=} \frac{\langle \alpha | \alpha \rangle}{2} \) (or equivalently \( d_{\tilde{\alpha}(\alpha)} = d_i \forall i \in I, \tilde{w} \in \tilde{W} \)); it is worth remarking that all the subgroups of \( \text{Aut}(Q_{\perp}, \delta) \) considered in 1.2.C and 1.2.D leave \( \Phi \) stable.

The multiplicity of each real root is 1; on the other hand let us define, \( \forall i \in I_0, \)
\[
\tilde{d}_i \overset{\alpha}{=} \begin{cases} 1 & \text{in the non twisted case or in case } A_2^{(2)} \\ d_i & \text{otherwise} \end{cases}
\]
and set \( \tilde{I}_m \overset{\alpha}{=} \{ i \in I_0 | \tilde{d}_i | m \} \); then the multiplicity of \( m\delta \) is
\[
\# \tilde{I}_m = \begin{cases} n & \text{if } k | m \\ \frac{n-n}{k-1} & \text{if } k \not| m, \end{cases}
\]
which can be expressed also by saying that the root system with multiplicities is
\[
\tilde{\Phi} \overset{\alpha}{=} \Phi^{\alpha} \cup \tilde{\Phi}^{\alpha} \text{ where } \tilde{\Phi}^{\alpha} \overset{\alpha}{=} \{ (m\delta, i) | m \in \mathbb{Z} \setminus \{ 0 \}, \tilde{d}_i | m \} \]
(De note by \( p : \tilde{\Phi} \rightarrow \Phi \) the natural map).

Remark that for \( m \neq 0 \) we have
\[
(m\delta, i) \in \tilde{\Phi}^{\alpha} \iff \tilde{d}_i | m \Rightarrow m\delta + \alpha_i \in \Phi \iff m\omega_i \in \tilde{W}.
\]

There is also another decomposition of \( \Phi \): \( \Phi = \Phi_+ \cup \Phi_- \), where \( \Phi_+ = \Phi \cap Q^+ = (\text{positive roots}) \) and \( \Phi_- = \Phi \cap Q^- = (\text{negative roots}) \). It induces an analogous decomposition of \( \tilde{\Phi} \) and of its real and imaginary parts, with obvious notations.

Recall also the relation between the length function and the root system:
\[
l(\tilde{w}) = \# \Phi_+(\tilde{w}) \text{ where } \Phi_+(\tilde{w}) = \{ \alpha \in \Phi_+ | \tilde{w}^{-1}(\alpha) < 0 \} ;
\]
recall that if \( s_{i_1} \cdots s_{i_r} \) is a reduced expression then
\[
\Phi_+(s_{i_1} \cdots s_{i_r}) = \{ s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) | 1 \leq k \leq r \}.
\]
Finally, \( l(\tilde{w}\tilde{w}') = l(\tilde{w}) + l(\tilde{w}') \Rightarrow \Phi_+(\tilde{w}) \subseteq \Phi_+(\tilde{w}\tilde{w}') \).

§1.3. Dynkin diagrams and classification.

In the following we list the affine Dynkin diagrams. The labels under the vertices fix an identification between \( I \) and \( \{ 0, 1, \ldots, n \} \) such that \( I_0 \) corresponds to \( \{ 1, \ldots, n \} \).
For each type we also recall the coefficients \( r_i \) (for \( i \in I_0 \): recall that \( r_0 \) is always 1) in the expression \( \delta = \sum_{i \in I} r_i \alpha_i \).

| \( X_{\tilde{n}=\tilde{n}(n)}^{(k)} \) | \( n \) | \( (\Gamma, I) \) | \( (r_1, \ldots, r_n) \) |
|---|---|---|---|
| \( A_1^{(1)} \) | 1 | \( \frac{q}{n} = \frac{q}{n} \) | (1) |
| \( A_n^{(1)} \) | > 1 | \( \frac{q}{n} \vdash \ldots \vdash \frac{q}{n} \vdash \frac{q}{n} \) | (1, ..., 1) |
\[ B_n^{(1)} > 2 \quad (2, ..., 2, 1) \]

\[ C_n^{(1)} > 1 \quad (1, 2, ..., 2) \]

\[ D_n^{(1)} > 3 \quad (1, 1, 2, ..., 2, 1) \]

\[ E_6^{(1)} \quad 6 \quad (2, 1, 2, 3, 2, 1) \]

\[ E_7^{(1)} \quad 7 \quad (2, 2, 3, 4, 2, 1) \]

\[ E_8^{(1)} \quad 8 \quad (3, 2, 4, 6, 5, 4, 3, 2) \]

\[ F_4^{(1)} \quad 4 \quad (2, 4, 3, 2) \]

\[ G_2^{(1)} \quad 2 \quad (3, 2) \]

\[ A_2^{(2)} \quad 1 \quad (2) \]

\[ A_{2n}^{(2)} > 1 \quad (2, ..., 2) \]

\[ A_{2n-1}^{(2)} > 2 \quad (1, 2, ..., 2) \]

\[ D_{n+1}^{(2)} \quad (1, ..., 1) \]

\[ F_6^{(2)} \quad 4 \quad (2, 3, 2, 1) \]

\[ D_4^{(3)} \quad 2 \quad (2, 1) \]
§1.4. The quantum algebra.

In this paragraph we recall the definition of the quantum group $\mathcal{U}_q$ associated to the affine Kac-Moody algebra $\hat{\mathfrak{g}}^{(k)}$ of type $X_n^{(k)}$, and the main structures that $\mathcal{U}_q$ can be endowed with.

**Definition 1.4.1.**
Let $\hat{\mathfrak{g}}^{(k)}$ be the affine Kac-Moody algebra of type $X_n^{(k)}$.

We denote by $\mathcal{U}_q = \mathcal{U}_q(\hat{\mathfrak{g}}^{(k)}) = \mathcal{U}_q(X_n^{(k)})$ the quantum algebra of $\hat{\mathfrak{g}}^{(k)}$, that is the associative $\mathbb{C}(q)$-algebra generated by

$$\{ E_i, F_i, K_i^{\pm 1}, D^\pm = K_i^\pm | i \in I \}$$

with relations:

$$K_\lambda K_\mu = K_{\lambda+\mu}, \forall \lambda, \mu \in Q_{\infty},$$
$$K_\lambda E_i = q^{\langle \lambda, \alpha_i \rangle} E_i K_\lambda, \forall \lambda \in Q_{\infty}, \forall i \in I,$$
$$K_\lambda F_i = q^{-\langle \lambda, \alpha_i \rangle} F_i K_\lambda, \forall \lambda \in Q_{\infty}, \forall i \in I,$$
$$[E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \forall i, j \in I.$$

where we set:

i) $K_\lambda = \prod_{i \in I} K_i^{m_i}$ if $\lambda = \sum_{i \in I} m_i \alpha_i \in Q_{\infty}$;

ii) $q_\alpha \doteq q^{\alpha}$, $q_\alpha = q^\alpha, \forall \alpha \in \Phi^\vee, \forall i \in I, q_{(m,i)} = q_i, \forall (m,i) \in \tilde{\Phi}_+^{im}$;

iii) $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$ and $[m]_q! = \prod_{r=1}^m [r]_q \forall m, r \in \mathbb{Z}$;

moreover $\left[ \begin{array}{c} \lambda \\ \alpha \end{array} \right]_q = \frac{[\lambda]_q [\lambda_\alpha]_q!}{[\alpha]_q [\lambda - \alpha]_q!} \forall \alpha \in \mathfrak{t}, \forall \lambda \in \mathfrak{t}$;

iv) for further use let us also define $\forall \alpha \in \tilde{\Phi}_+, \forall x \in \mathcal{U}_q$

$$\exp_\alpha(x) \doteq \sum_{m \geq 0} \frac{x^m}{(m)_\alpha!},$$

where $\forall \alpha \in \tilde{\Phi}_+, \forall m \in \mathbb{N}$ we have put

$$(m)_\alpha \doteq \left\{ \begin{array}{ll} \frac{q_\alpha 2^m - 1}{q_\alpha - 1} & \text{if } \alpha \in \Phi^\vee_+ \text{ and } (m)_\alpha! \doteq \prod_{s=1}^m (s)_\alpha. \\
\frac{2^m - 1}{m} & \text{if } \alpha \in \tilde{\Phi}_+^{im} \end{array} \right.$$
3) the \( \mathbb{C} \)-anti-linear anti-involution \( \Omega : \mathcal{U}_q \to \mathcal{U}_q \) defined by
\[
\Omega(E_i) = F_i, \quad \Omega(F_i) = E_i \quad \forall i \in I; \quad \Omega(K_i) = K_i^{-1} \quad \forall i \in I_\infty; \quad \Omega(q) = q^{-1};
\]
4) the \( \mathbb{C}(q) \)-linear anti-involution \( \Xi : \mathcal{U}_q \to \mathcal{U}_q \) defined by
\[
\Xi(E_i) = E_i, \quad \Xi(F_i) = F_i \quad \forall i \in I; \quad \Xi(K_i) = K_i^{-1} \quad \forall i \in I_\infty;
\]
5) a braid group action commuting with \( \Omega \), verifying \( \Xi T_i = T_i^{-1} \Xi \quad \forall i \in I \), and preserving the graduation (that is \( T_w(\mathcal{U}_{q;\eta}) = \mathcal{U}_{q;\eta(w)} \))—more precisely
\[
T_w(E_i) \in \mathcal{U}_{q,w(\alpha_i)}^+ \quad \text{if} \quad w \in W \quad \text{and} \quad i \in I \quad \text{are such that} \quad w(\alpha_i) \in Q^+ \quad (\text{i.e.} \quad l(ws_i) > l(w)),
\]
and
\[
T_i(E_i) = -F_i K_i \quad \forall i \in I, \quad T_i(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{-r} E_i (-a_{ij} - r) E_j E_i^r \quad \forall i \neq j \in I
\]
where \( \forall m \in \mathbb{N} \ E_i^{(m)} = \frac{E_i^m}{m!} \);
6) a Hopf-structure \( (\Delta, \epsilon, S) \), whose coproduct \( \Delta : \mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q \) is given by
\[
\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i; \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{-1}; \quad \Delta(K_i) = K_i \otimes K_i;
\]
Let us remark that \( \Delta \circ \Omega = \sigma \circ (\Omega \otimes \Omega) \circ \Delta \) where \( \sigma : \mathcal{U}_q \otimes \mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q \) is defined by \( \sigma(x \otimes y) = y \otimes x \);
7) the Killing form, that is the unique bilinear form \( (\cdot, \cdot) : \mathcal{U}_q^{<0} \otimes \mathcal{U}_q^{<0} \to \mathbb{C}(q) \) such that

i) \( (x, y_1 y_2) = (\Delta(x), y_1 \otimes y_2) \quad \forall x \in \mathcal{U}_q^{<0}, \forall y_1, y_2 \in \mathcal{U}_q^{<0}; \)

ii) \( (x_1 x_2, y) = (x_2 \otimes x_1, \Delta(y)) \quad \forall x_1, x_2 \in \mathcal{U}_q^{<0}, \forall y \in \mathcal{U}_q^{<0}; \)

iii) \( (\cdot, \cdot)|_{\mathcal{U}_q^{<0} \otimes \mathcal{U}_q^{<0}} = 0 \quad \text{if} \quad \eta \neq \tilde{\eta}; \)

iv) \( (K_\lambda, K_\mu) = q^{\lambda(\lambda, \mu)} \quad \forall \lambda, \mu \in Q_\infty; \)

v) \( (E_i, F_i) = \frac{1}{q_i - q_i^{-1}} \quad \forall i \in I; \)

(\cdot, \cdot) has the following important properties:

a) \( (\cdot, \cdot)|_{\mathcal{U}_q^{<0} \otimes \mathcal{U}_q^{<0}} \) is non degenerate \( \forall \eta \in Q_+; \)

b) \( (x K_\lambda, y) = (x, y K_\lambda) \quad \forall x \in \mathcal{U}_q^+, \forall y \in \mathcal{U}_q^-, \forall \lambda \in Q_\infty. \)

2. COPIES OF \( A_1^{(1)} \) IN \( X^{(k)}_h \).

§ 2.1. Definition of the root vectors.

In this section we shall shortly recall Beck’s results for non-twisted affine quantum algebras (see [1] and [2]) and prove that his method applies word by word to the twisted case, that is we can find a copy of \( \mathcal{U}_q(A_1^{(1)}) \) in \( \mathcal{U}_q(X^{(k)}_h) \) \( \forall i \in I_0 \), provided that \( (X^{(k)}_h, i) \neq (A_2^{(2)}, 1)\).
To this aim let us first define, \( \forall i \in I_0 \), an element \( \lambda_i \in P \) as follows:
\[
\lambda_i = \bar{d}_i \omega_i
\]
where we recall that \( \bar{d}_i = \min\{m > 0 | m \omega_i \in \bar{W} \} \) (see 1.2.E).

The next proposition and remark is what one needs in order to define the positive root vectors.

**Proposition 2.1.1**
\[ \forall m \in \mathbb{N} \text{ we have } l((\lambda_1 \cdots \lambda_n)^m) = m \sum_{i \in I_0} l(\lambda_i). \]

**Proof:** See [3]. \( \square \)

**Remark 2.1.2.**
\[
\Phi^+_+ = \{ m\delta - \alpha \in \Phi_+ | \alpha \in Q_0^+, m > 0 \} \cup \{ m\delta + \alpha \in \Phi_+ | \alpha \in Q_0^+, m \in \mathbb{N} \}.
\]
Moreover
\[
\{ m\delta - \alpha \in \Phi_+ | \alpha \in Q_0^+, m > 0 \} = \bigcup_{m > 0} \Phi_+((\lambda_1 \cdots \lambda_n)^m),
\]
\[
\{ m\delta + \alpha \in \Phi_+ | \alpha \in Q_0^+, m \in \mathbb{N} \} = \bigcup_{m > 0} \Phi_+((\lambda_1 \cdots \lambda_n)^{-m}).
\]

**Proof:** See [2]. \( \square \)

The preceding results suggest to define the following sequence, by means of which it will be easy to give a suitable definition of the positive real root vectors.

**Definition 2.1.3.**
Let \( N \subseteq l(\lambda_1 \cdots \lambda_n) \) and define \( \iota : \mathbb{Z} \to I_0 \) to be such that
\[
\begin{cases}
\lambda_1 \cdots \lambda_n = s_{i_1} \cdots s_{i_N} \tau \\
\iota_{r+N} = \tau(\iota_r) \quad \forall r \in \mathbb{Z},
\end{cases}
\]
where \( \tau \) is a suitable element of \( \mathcal{T} \) (uniquely determined by the above condition); for further determinations of \( \iota \) see remark 2.1.5 and notation 4.2.5.

Notice that \( \iota \) induces functions
\[
\mathbb{Z} \ni r \mapsto w_r \in W \quad \text{and} \quad \mathbb{Z} \ni r \mapsto \beta_r \in \Phi^+_+
\]
by
\[
w_r = \begin{cases}
s_{i_1} \cdots s_{i_{r-1}} \text{ if } r \geq 1 \\
s_{i_0} \cdots s_{i_{r-1}} \text{ if } r \leq 0,
\end{cases}
\]
and \( \beta_r = w_r(\alpha_{i_r}) \).

It is well known (see [2]) that \( \beta \) is a bijection.

Then we arrive at the definition of the root vectors:

**Definition 2.1.4.**
The positive real root vectors \( E_\alpha \) (with \( \alpha \in \Phi^+_+ \)) are defined by
\[
E_\alpha = \begin{cases}
T_{w_r}(E_{i_r}) & \text{if } r \geq 1 \\
T_{w_r}^{-1}(E_{i_r}) = \Xi T_{w_r}(E_{i_r}) & \text{if } r \leq 0.
\end{cases}
\]
Also, positive imaginary root vectors \( E_{(m,\delta,i)} \) (with \( m > 0, i \in I_0 \)) are defined by

\[
E_{(m,\delta,i)} \not\sim E_{m,\delta-\alpha}_i + q_i^{-2} E_{m,\delta-\alpha}_i.
\]

Similarly, the negative root vectors are defined by

\[
F_\alpha \not\sim \Omega(E_\alpha) \quad \text{if } \alpha \in \Phi^{\text{re}}, \quad \tilde{F}_\alpha \not\sim \Omega(\tilde{E}_\alpha) \quad \text{if } \alpha \in \Phi^{\text{im}}.
\]

**Remark 2.1.5.**

It is worth remarking, for later use, that \( \epsilon \) can be chosen (and will be chosen!) so that \( \exists N_0 = 0 < N_1 < \ldots < N_n = N \) and \( \exists \tau_1, \ldots, \tau_n \in T \) such that \( \forall i = 1, \ldots, n \), \( \lambda_i = s_{i_1} \tau_i \cdot \ldots \cdot s_{i_n} \tau_i \) (notice that \( \tau_1 \cdot \ldots \cdot \tau_n = \tau \)).

If this is the case then

\[
E_{m,\delta+\alpha_i} = T^{-m}_\lambda(E_i) \quad \forall m \in \mathbb{N},
\]

\[
E_{m,\delta-\alpha_i} = T^m_{\lambda_i^{-1}}(E_i) \quad \forall m > 0,
\]

which depends on the fact that

1) \( T_{\lambda_i} T_{\lambda_j} = T_{\lambda_j} T_{\lambda_i} \) \( \forall i, j \in I_0 \);
2) \( T_{\lambda_i}(E_i) = E_i \) and \( T_{\lambda_j} T_i = T_i T_{\lambda_j} \) \( \forall i, j \in I_0 \) such that \( j \neq i \).

**Proof:** See [25] and [6]. \( \square \)

**Definition 2.1.6.**

If \( i \in I_0 \) we define \( \mathcal{U}_q^{(i)} \) to be the \( \mathbb{C}(q^{d_i}) \)-subalgebra of \( \mathcal{U}_q = \mathcal{U}_q(X_n^{(k)}) \) generated by \( \{ E_i, E_{d,\delta-\alpha_i}, F_i, F_{d,\delta-\alpha_i}, K_{l,i}^{\pm 1}, K_{d,i}^{\pm 1} \} \).

Obviously \( \mathcal{U}_q^{(i)} \) is \( \Omega \)-stable and pointwise fixed by \( T_{\lambda_j} \) for \( j \in I_0 \setminus \{ i \} \). Moreover it will become clear that \( \mathcal{U}_q^{(i)} \) is the smallest \( \mathbb{C}(q^{d_i}) \)-subalgebra of \( \mathcal{U}_q \) containing \( E_i \) and \( K_i \) and stable under \( T_i^{\pm 1}, T_{\lambda_i}^{\pm 1} \); in particular we shall see that it contains \( E_{m,\delta \pm \alpha_i}, F_{m,\delta \pm \alpha_i}, \tilde{E}_{(m,\delta,i)} \) and \( \tilde{F}_{(m,\delta,i)} \) \( \forall m > 0 \).

**§2.2. The homomorphisms \( \varphi_i \).**

We are now ready to state the announced properties of the root vectors just defined and of the braid group action, which will allow us to construct homomorphisms \( \varphi_i : \mathcal{U}_q(A^{(1)}_n) \to \mathcal{U}_q^{(i)} \subseteq \mathcal{U}_q(X_n^{(k)}) \) for each \( i \in I_0 \), under the only condition that \( i \neq 1 \) if \( X_n^{(k)} = A^{(2)}_n \), condition that we shall assume in the remaining part of section 2. These homomorphisms \( \varphi_i \) behave “well” on the root vectors (that is, they transform root vectors in root vectors, see corollary 2.2.3), and this fact immediately implies some important consequences: in particular we shall translate the commutation relations from \( \mathcal{U}_q(A^{(1)}_n) \) to \( \mathcal{U}_q(X_n^{(k)}) \). We shall also underline the obstruction that we meet if we try to apply the same argument when \( X_n^{(k)} = A^{(2)}_n \) and \( i = 1 \): this last situation will be studied in sections 3 and 4.

**Lemma 2.2.1**

Assume, as required, that \( X_n^{(k)} \neq A^{(2)}_n \) or \( i \neq 1 \); then we have:

1) \( l(\lambda, s_i \lambda) = 2l(\lambda) - 1 \);
2) \( (T_{\lambda_i} T_i^{-1})^2 \in \langle T_{\lambda_i} \rangle \); \( j \neq i \);
3) \( [E_{d,\delta-\alpha_i}, F_i] = 0 \) and \( E_i E_{d,\delta+\alpha_i} = q_i^{-2} E_{d,\delta+\alpha_i} E_i \).
4) \(\forall j \in I_0 \setminus \{i\}\) we have \(\left(T_{\lambda_i} T_{E_i}^{-1}\right)^{2 T_{\lambda_i}} \in \langle T_{\lambda_i} | r \in I_0 \setminus \{i, j\}\rangle\); in particular

\[ T_{\lambda_i} T_{E_i}^{-1}(E_j) = T_{\lambda_i}(E_j). \]

**Proof:** For the proof see [25] and [1]. Here we want just to point out that all the statements are based on the fact that there is no root \(\alpha = m \delta + \varepsilon \alpha_i\), such that \(\bar{d}_i \delta + \alpha_i < \alpha \prec \alpha_i\).

Remark that this is no longer true in case \(A_n^{(2)}\) for \(i = 1\): indeed in this case we have \(\delta + \alpha_i < \delta + 2 \alpha_i < \alpha_i\). Hence the existence of roots of the form \(m \delta + 2 \alpha\) with \(\alpha \in \Phi_+ \cap Q_0\) (which never happens if we are not in case \(A_n^{(2)}\)) is what makes the difference between this case and the others.

Since Beck’s construction of the homomorphisms \(\varphi_i : U_q(A_1^{(1)}) \rightarrow U_q(X_n^{(k)})\) (and their properties) is based on the statements of lemma 2.2.1 the argument used in [1] applies exactly to the present situation, so that we have:

**Proposition 2.2.2**

If \(X_n^{(k)} \neq A_n^{(2)}\) or \(i \neq 1\), then there exists a \(\mathbb{C}\)-algebra homomorphism

\[ \varphi_i : U_q(A_1^{(1)}) \rightarrow U_q(X_n^{(k)}) \]

verifying the following properties:

1) \(\varphi_i(K_1) = K_1\), \(\varphi_i(K_1 K_0) = K_{\bar{d}_i \delta}\) and \(\varphi_i(q_i) = q_i = q_i^0\);

2) \(\varphi_i(E_i) = E_i\) and \(\varphi_i(E_0) = E_{\bar{d}_i \delta - \alpha_i}\);

3) \(\varphi_i(\Omega) = \Omega \varphi_i\);

4) \(\varphi_i(T_1 = T_i \varphi_i\) and \(\varphi_i T_{\tau} = T_{\lambda_i} T_{E_i}^{-1} \varphi_i\), where \(\tau\) is the non trivial Dynkin diagram automorphism of \(A_1^{(1)}\).

**Corollary 2.2.3**

Suppose again that \(X_n^{(k)} \neq A_n^{(2)}\) or \(i \neq 1\) and let \(\nu_i\) be the group homomorphism

\[ \nu_i : Q(A_1^{(1)}) \rightarrow Q(X_n^{(k)}) \]

defined in the obvious way by \(\nu_i(\alpha_i) \equiv \alpha_i\), \(\nu_i(\delta) \equiv \bar{d}_i \delta\); furthermore let the injection \(\tilde{\nu}_i : \Phi_+(A_1^{(1)}) \rightarrow \Phi_+(X_n^{(k)})\) be given by:

\[ \tilde{\nu}_i(\alpha) \equiv \begin{cases} \nu_i(\alpha) & \text{if } \alpha \text{ is real} \\ (\nu_i(\alpha), i) & \text{if } \alpha \text{ is imaginary} \end{cases}. \]

Then \(\varphi_i\) has the property that

\[ \varphi_i(K_\alpha) = K_{\nu_i(\alpha)} \forall \alpha \in Q(A_1^{(1)}), \]

\[ \varphi_i(E_\alpha) = E_{\tilde{\nu}_i(\alpha)} \forall \alpha \in \Phi_+(A_1^{(1)}) \]

and

\[ \varphi_i(E_\alpha) = E_{\tilde{\nu}_i(\alpha)} \forall \alpha \in \Phi_+^{\text{im}}(A_1^{(1)}), \]

where the vectors \(E_{(m \delta, \delta, i)}\) are defined by the relation

\[ 1 - (q_i - q_i^{-1}) \sum_{m > 0} \tilde{E}_{(m \delta, \delta, i)} u^m = \exp \left( (q_i - q_i^{-1}) \sum_{m > 0} E_{(m \delta, \delta, i)} u^m \right). \]
As an immediate consequence of corollary 2.2.3, we have the following proposition.

**Proposition 2.2.4**

Suppose as before that \((X^{(k)}_n, i) \neq (A^{(2)}_{2n}, 1)\); then we have the following relations in \(U_q(X^{(k)}_n)\):

1) \([\hat{E}_{(r \delta, \delta, i)}, \hat{E}_{(s \delta, \delta, i)}] = 0 = [E_{(r \delta, \delta, i)}, E_{(s \delta, \delta, i)}] \forall r, s \geq 0;\)

2) \(T_{A_1} (\hat{E}_{(r \delta, \delta, i)}) = \hat{E}_{(r \delta, \delta, i)}\) and \(T_{A_1} (E_{(r \delta, \delta, i)}) = E_{(r \delta, \delta, i)} \forall r > 0, j \in I_0;\)

3) \([E_{(r \delta, \delta, i)}, F_{(s \delta, \delta, i)}] = \delta_{r, s} \frac{[2r]_q}{q_{i} - q_{i}} \forall r, s \geq 0;\)

4) \([E_{(r \delta, \delta, i)}, E_{s \delta, \delta + \alpha_i}] = \frac{[2r]_q}{q_{i} - q_{i}} E_{(r + 1)\delta, \delta + \alpha_i}\) and \([E_{(r \delta, \delta, i)}, E_{(s + 1)\delta, \delta - \alpha_i}] = -\frac{[2r]_q}{q_{i} - q_{i}} E_{(r + 1)\delta, \delta - \alpha_i} \forall r > 0, s \geq 0.\)

**Proof:** The claim follows applying \(\varphi_1\) to the analogous relations holding in \(U_q(A^{(1)}_1)\) (see [5] and [1]).

3. **THE CASE OF** \(A^{(2)}_2\).

In this section we study the case of \(sl^{(2)}_3 = A^{(2)}_2\) (and \(U_q\) will indicate \(U_q(sl^{(2)}_3)\)). In particular we want to understand some important properties of the imaginary root vectors, their commutation rules, and their behaviour under the action of the braid group.

§3.1. **Real root vectors.**

Recall that the Dynkin diagram of \(A^{(2)}_2\) is

\[
\begin{array}{cc}
& \circ \circ \circ \\
1 & 0 \\
\end{array}
\]

and that we can describe its roots as follows: \(\delta = \alpha_0 + 2\alpha_1\) and

\[
\Phi^+ = \{m \delta + \alpha_1, (m + 1) \delta - \alpha_1, (2m + 1) \delta \pm 2\alpha_1 | m \in \mathbb{N}\},
\]

\[
\Phi^- = \{(m + 1) \delta | m \in \mathbb{N}\};
\]

moreover \(\omega_1 = \omega_1 = s_0 s_1\): indeed \(d_1 = 1\) and

\[
s_0 s_1 (\alpha_1) = -s_0 (\alpha_1) = -(\alpha_1 + \alpha_0) = -\delta + \alpha_1 = \omega_1 (\alpha_1).
\]

It follows that the real root vectors are given by:

\[
E_{m \delta + \alpha_1} = (T_0 T_1)^{-m} (E_1),
\]

\[
E_{(m + 1) \delta - \alpha_1} = (T_0 T_1)^m T_0 (E_1),
\]

\[
E_{(2m + 1) \delta + 2 \alpha_1} = (T_0 T_1)^{-m} T_1^{-1} (E_0),
\]

\[
E_{(2m + 1) \delta - 2 \alpha_1} = (T_0 T_1)^m (E_0).
\]
We shall now give the following proposition which will simplify many calculations.

**Proposition 3.1.1**

\[ \forall \alpha \in \Phi^+ \setminus \{ \alpha_0 \} \quad T_0 \Xi(E_\alpha) = E_{s_0(\alpha)}. \]

\[ \forall \alpha \in \Phi^+ \setminus \{ \alpha_1 \} \quad \Xi T_1(E_\alpha) = E_{s_1(\alpha)}. \]

**Proof:** \[ \Xi(E_\alpha) \in U_\alpha^+, \text{ hence } T_0 \Xi(E_\alpha) \in U_\alpha^{s_0(\alpha)}; \] it is then enough to prove that \( T_0 \Xi(E_\alpha) \) is a root vector, which is easily seen: \( T_0 \Xi(T_0 T_1)^{-m}(E_1) = (T_0 T_1)^{-m} T_0(E_1) \) and similarly for the other root vectors, so that the first assertion is proved. From this result and from the fact that \( (T_0 \Xi)(\Xi T_1) = T_0 T_1 \) the second assertion of the proposition follows immediately.

We shall now give some simple commutation relations between the real root vectors.

**Lemma 3.1.2**

\[ [E_\delta - \alpha_1, F_1] = -[4]_q K_1 E_0 \quad \text{and} \quad E_1 E_{\delta + \alpha_1} - q^{-2} E_{\delta + \alpha_1} E_1 = q^{-2}[4]_q E_{\delta + 2 \alpha_1}. \]

**Proof:** The first identity is a straightforward computation, using that \( E_\delta - \alpha_1 = T_0(E_1) = -E_0 E_1 + q^{-2} E_1 E_0; \) the second relation is \( \Xi T_1 \) applied to the first one.

As a first application of lemma 3.1.2 we state the following proposition:

**Proposition 3.1.3**

\( \{ E_1, E_{\delta - \alpha_1}, F_1, F_{\delta - \alpha_1}, K_1^{\pm 1}, K_0^{\pm 1} \} \) is a system of generators of \( U_q \) as a \( \mathbb{C}(q) \)-algebra.

**Proof:** Let \( \mathcal{A} \) be the \( \mathbb{C}(q) \)-subalgebra of \( U_q \) generated by \( \{ E_1, E_{\delta - \alpha_1}, F_1, F_{\delta - \alpha_1}, K_1^{\pm 1}, K_0^{\pm 1} \} \).

It is then enough to prove that \( E_0, F_0 \in \mathcal{A}. \) Since \( \mathcal{A} \) is \( \Omega \)-stable it is enough to prove that \( E_0 \in \mathcal{A}. \) But \( E_0 = -\frac{1}{[4]_q} K_1^{-1} [E_{\delta - \alpha_1}, F_1] \in \mathcal{A}. \)

**Corollary 3.1.4**

The least subalgebra of \( U_q \) containing \( U_q^0 \) and \( E_1 \) and stable under the action of \( T_0 T_1 \) and \( T_0 \Xi \) is \( U_q. \)

§3.2. The imaginary root vector \( \tilde{E}_\delta. \)

Here we want to study some properties of \( \tilde{E}_\delta, \) where we recall that

\[ \tilde{E}_\delta = -E_{\delta - \alpha_1} E_1 + q^{-2} E_1 E_{\delta - \alpha_1}. \]

**Proposition 3.2.1**

\[ T_0 \Xi(\tilde{E}_\delta) = \tilde{E}_\delta. \]

**Proof:** The claim follows immediately from the definitions and thanks to proposition 3.1.1.

**Proposition 3.2.2**
The following commutation rules hold:

\[ [\hat{E}_\delta, E_1] = -[3]_q E_{\delta+\alpha_1}, \quad [\hat{E}_\delta, F_1] = -[3]_q F_{\delta-\alpha_1}. \]

**Proof:** The proof consists in simple computations using the following ingredients: \( \hat{E}_\delta = E_0 E_1^2 - q^{-3} [2]_q E_1 E_0 E_1 + q^{-6} E_1^2 E_0 \) (by the definition and the fact that \( E_{\delta-\alpha_1} = -E_0 E_1 + q^{-4} E_1 E_0 \)),

\[ E_{\delta+\alpha_1} = T_1^{-1} T_0^{-1}(E_1) = (-E_1 E_0 + q^{-4} E_0 E_1) = -K_1^{-1}[T_1^{-1}(E_0), F_1] = \]

\[ = -K_1^{-1} \sum_{r=0}^{4} (-1)^r q^{-r} E_1^{(r)} E_0 E_1^{(4-r)}, F_1 \]

and

\[ [E_1^m, F_1] = [m]_q \frac{q^{1-m} K_1 - q^{m-1} K_1^{-1}}{q - q^{-1}} E_1^{m-1}. \]

\( \square \)

**Proposition 3.2.3**

\[ \Xi T_1(\hat{E}_\delta) = \hat{E}_\delta \] (hence \( T_0 T_1(\hat{E}_\delta) = \hat{E}_\delta \), thanks to proposition 3.2.1).

**Proof:** Since \( \Xi T_1(E_1) = -K_1^{-1} F_1 \) we have that

\[ \Xi T_1(\hat{E}_\delta) = K_1^{-1} F_1 E_{\delta+\alpha_1} - q^{-2} E_{\delta+\alpha_1} K_1^{-1} F_1 = -K_1^{-1}[E_{\delta+\alpha_1}, F_1] = \]

\[ = \frac{1}{[3]_q} K_1^{-1}[[\hat{E}_\delta, E_1], F_1] = -\frac{1}{[3]_q} K_1^{-1}[E_1, [[\hat{E}_\delta, F_1]]] = K_1^{-1}[E_1, K_1 E_{\delta-\alpha_1}] = \hat{E}_\delta. \]

\( \square \)

**Corollary 3.2.4**

Applying a power of \( T_0 T_1 \) to proposition 3.2.2 we get that \( \forall m \in \mathbb{N} \) the following commutation relations hold:

\[ [\hat{E}_\delta, E_{m\delta+\alpha_1}] = -[3]_q E_{(m+1)\delta+\alpha_1}, \quad [\hat{E}_\delta, E_{(m+1)\delta-\alpha_1}] = [3]_q E_{(m+2)\delta-\alpha_1}. \]

\( \square \)

We conclude this paragraph with the commutation rules between \( \hat{E}_\delta \) and the simple root vectors \( E_0 \) and \( F_0 \).

**Lemma 3.2.5**

\[ [\hat{E}_\delta, F_0] = -(q^2 - 1)[3]_q E_0^2 K_0^{-1} \] and \( [\hat{E}_\delta, E_0] = -(q^2 - 1)[3]_q E_0^2 \).

**Proof:** The first relation is found by straightforward calculations; the second relation is found by applying \( T_0 \Xi \) to the first one.

\( \square \)

**§3.3. The imaginary root vectors \( \hat{E}_{m\delta} \).**

Here we generalize to the vectors \( \hat{E}_{m\delta} = -E_{m\delta-\alpha_1} E_1 + q^{-2} E_1 E_{m\delta-\alpha_1} \), what we know for \( \hat{E}_\delta \).

We start by proving the equivalence between some commutation rules among the imaginary root vectors and their behaviour under the action of \( T_0 T_1 \).

This is completely similar to what happens in the case of \( A_1^{(1)} \).
Lemma 3.3.1
\( \forall m > 0 \) the following implications hold:
\[
[\tilde{E}_\delta, \tilde{E}_{m\delta}] = 0 \iff T_0 T_1(\tilde{E}_{(m+1)\delta}) = \tilde{E}_{(m+1)\delta}
\]
and
\[
T_0 T_1(\tilde{E}_{(m+1)\delta}) = \tilde{E}_{(m+1)\delta} \iff T_0 \Xi(\tilde{E}_{(m+1)\delta}) = \tilde{E}_{(m+1)\delta} = \Xi T_1(\tilde{E}_{(m+1)\delta}).
\]

Proof: See [5].

The following proposition is a fundamental tool in the study of the imaginary root vectors.

Proposition 3.3.2
Let \( P_m \) be the following statement:
- \( i) \) \( m \) \( T_0 T_1(\tilde{E}_{m\delta}) = \tilde{E}_{m\delta} = T_0 \Xi(\tilde{E}_{m\delta}) = \Xi T_1(\tilde{E}_{m\delta}); \)
- \( ii) \) \( m \) \( E_0 E_{(m-1)\delta+\alpha_1} - q^{-4} E_{(m-1)\delta+\alpha_1} E_0 = \)
  \[
  = \frac{1}{[4]_q} ((q^4 - q^2 - q^{-2}) E_{\delta-\alpha_1} \tilde{E}_{(m-1)\delta} + (q^2 + q^{-2} - q^{-4}) \tilde{E}_{(m-1)\delta} E_{\delta-\alpha_1} +
  q^2 E_{2\delta-\alpha_1} \tilde{E}_{(m-2)\delta} - q^{-2} \tilde{E}_{(m-2)\delta} E_{2\delta-\alpha_1});
  \]
- \( iii) \) \( m \) \( \tilde{E}_{m\delta}, E_1 = (q^4 - q^{-2}) \tilde{E}_{(m-1)\delta} E_{\delta+\alpha_1} + (q^2 - q^{-4}) E_{\delta+\alpha_1} \tilde{E}_{(m-1)\delta} +
  q^2 \tilde{E}_{(m-2)\delta} E_{2\delta+\alpha_1} - q^{-2} E_{2\delta+\alpha_1} \tilde{E}_{(m-2)\delta}; \)
- \( iv) \) \( m \) \( \tilde{E}_{m\delta}, E_{\delta-\alpha_1} = -(q^4 - q^{-2}) E_{2\delta-\alpha_1} \tilde{E}_{(m-1)\delta} - (q^2 - q^{-4}) \tilde{E}_{(m-1)\delta} E_{2\delta-\alpha_1} +
  q^2 E_{3\delta-\alpha_1} \tilde{E}_{(m-2)\delta} + q^{-2} \tilde{E}_{(m-2)\delta} E_{3\delta-\alpha_1}; \)
- \( v) \) \( m \) \( \tilde{E}_{r\delta}, \tilde{E}_{s\delta} = 0 \) \( \forall r, s \leq m. \)

Then, if we set \( \tilde{E}_{-\delta} = 0 \) and \( \tilde{E}_{0\delta} = -\frac{1}{q-q^{-1}} \), \( P_m \) is true for every \( m > 0 \).

Proof: The proof is an induction on \( m \).

If \( m = 1 \) remark that: \( i) \) \( 1 \) is proposition 3.2.1 and proposition 3.2.3; \( ii) \) \( 1 \) is the expression of \( E_{\delta-\alpha_1} \) in terms of \( E_0 \) and \( E_1 \); \( iii) \) \( 1 \) is proposition 3.2.2; \( iv) \) \( 1 \) is \( T_0 \Xi \) applied to \( iii) \) \( 1 \); \( v) \) \( 1 \) is obvious.

Let \( m \) be bigger than 1. Then:
- \( i) \) \( m \) follows immediately from \( v) \) \( m-1 \) (which in particular implies \( \tilde{E}_{\delta}, \tilde{E}_{(m-1)\delta} = 0 \)) and from lemma 3.3.1.
- \( ii) \) \( m \) is obtained by the following steps, where we shall indicate among the lines the results that we need to pass from one side to the other of the identities:
  \[
  E_0 E_{(m-1)\delta+\alpha_1} - q^{-4} E_{(m-1)\delta+\alpha_1} E_0 =
  \]
  \[
  \{ \text{corollary 3.2.4} \}
  \]
  \[
  = \frac{1}{[3]_q} (E_0 \tilde{E}_\delta E_{(m-2)\delta+\alpha_1} - E_0 E_{(m-2)\delta+\alpha_1} \tilde{E}_\delta +
  
\]
\[-q^{-4} \tilde{E}_\delta E_{(m-2)\delta + \alpha_1} E_0 + q^{-4} E_{(m-2)\delta + \alpha_1} \tilde{E}_\delta E_0 =
\{\text{lemma 3.2.5}\}\]
\[-\frac{1}{[3]_q} \left( [\tilde{E}_\delta, E_0 E_{(m-2)\delta + \alpha_1} - q^{-4} E_{(m-2)\delta + \alpha_1} E_0] +
+ (q^2 - 1)[3]_q (E_{\delta - \alpha_1} E_{(m-2)\delta + \alpha_1} - q^{-4} E_{(m-2)\delta + \alpha_1} E_{\delta - \alpha_1}) \right) =
\{i\}_{m-1}, \ ii\}_{m-1} \text{ and definition of } \tilde{E}_{(m-1)\delta}\]
\[-\frac{1}{[3]_q} \left( \frac{1}{[4]_q} [\tilde{E}_\delta, (q^4 - q^2 - q^{-2}) E_{\delta - \alpha_1} \tilde{E}_{(m-2)\delta} + (q^2 + q^{-2} - q^{-4}) \tilde{E}_{(m-2)\delta} E_{\delta - \alpha_1} +
+ q^2 E_{2\delta - \alpha_1} \tilde{E}_{(m-3)\delta} - q^{-2} \tilde{E}_{(m-3)\delta} E_{2\delta - \alpha_1} ] +
- (q^2 - 1)[3]_q E_{\delta - \alpha_1} \tilde{E}_{(m-1)\delta} - (1 - q^{-2})[3]_q \tilde{E}_{(m-1)\delta} E_{\delta - \alpha_1} \right) =
\{\text{corollary 3.2.4 and } v\}_{m-1}\]
\[-\frac{1}{[4]_q} ((q^4 - q^2 - q^{-2}) E_{\delta - \alpha_1} \tilde{E}_{(m-1)\delta} + (q^2 + q^{-2} - q^{-4}) \tilde{E}_{(m-1)\delta} E_{\delta - \alpha_1} +
+ q^2 E_{2\delta - \alpha_1} \tilde{E}_{(m-2)\delta} - q^{-2} \tilde{E}_{(m-2)\delta} E_{2\delta - \alpha_1} ) +
- \frac{1}{[4]_q} ([\tilde{E}_{(m-1)\delta}, E_{\delta - \alpha_1} ] + (q^4 - q^2) E_{2\delta - \alpha_1} \tilde{E}_{(m-2)\delta} + (q^2 - q^{-4}) \tilde{E}_{(m-2)\delta} E_{2\delta - \alpha_1} +
+ q^2 E_{3\delta - \alpha_1} \tilde{E}_{(m-3)\delta} - q^{-2} \tilde{E}_{(m-3)\delta} E_{3\delta - \alpha_1} ) =
\{iv\}_{m-1}\]
\[-\frac{1}{[4]_q} ((q^4 - q^2 - q^{-2}) E_{\delta - \alpha_1} \tilde{E}_{(m-1)\delta} + (q^2 + q^{-2} - q^{-4}) \tilde{E}_{(m-1)\delta} E_{\delta - \alpha_1} +
+ q^2 E_{2\delta - \alpha_1} \tilde{E}_{(m-2)\delta} - q^{-2} \tilde{E}_{(m-2)\delta} E_{2\delta - \alpha_1} ), \]
which is the claim.

iii) \_m is the result of the following computations, where the first identity is true thanks to i) \_m:

\[ [\tilde{E}_{m\delta}, E_1] = [-E_{(m-1)\delta - \alpha_1} E_{\delta + \alpha_1} + q^{-2} E_{\delta + \alpha_1} E_{(m-1)\delta - \alpha_1}, E_1] =
- E_{(m-1)\delta - \alpha_1} E_{\delta + \alpha_1} E_1 + q^{-2} E_{\delta + \alpha_1} E_{(m-1)\delta - \alpha_1} E_1+ \]
+ E_{1} E_{(m-1)\delta+ \alpha_{1}} E_{\delta+ \alpha_{1}} - q^{-2} E_{1} E_{\delta+ \alpha_{1}} E_{(m-1)\delta- \alpha_{1}} = \\
\{\text{lemma 3.1.2 and definition of } \hat{E}_{(m-1)\delta}\}

= -q^{2} E_{(m-1)\delta- \alpha_{1}} E_{1} E_{\delta+ \alpha_{1}} + [4]_{q} E_{(m-1)\delta- \alpha_{1}} E_{\delta+ 2\alpha_{1}} + q^{-4} E_{\delta+ \alpha_{1}} E_{1} E_{(m-1)\delta- \alpha_{1}} + \\
- q^{-2} E_{\delta+ \alpha_{1}} \hat{E}_{(m-1)\delta} + E_{1} E_{(m-1)\delta- \alpha_{1}} E_{\delta+ \alpha_{1}} - q^{-2} E_{1} E_{\delta+ \alpha_{1}} E_{(m-1)\delta- \alpha_{1}} = \\
\{\text{definition of } \hat{E}_{(m-1)\delta} \text{ and lemma 3.1.2}\}

= q^{2} \hat{E}_{(m-1)\delta} E_{\delta+ \alpha_{1}} - q^{-2} E_{\delta+ \alpha_{1}} \hat{E}_{(m-1)\delta} + \\
+[4]_{q} E_{(m-1)\delta- \alpha_{1}} E_{\delta+ 2\alpha_{1}} - q^{-4} E_{\delta+ 2\alpha_{1}} E_{(m-1)\delta- \alpha_{1}} = \\
\{\Xi T_{1}(ii)_{m} \text{ and } \iota)_{m-1, m-2}\}

= q^{2} \hat{E}_{(m-1)\delta} E_{\delta+ \alpha_{1}} - q^{-2} E_{\delta+ \alpha_{1}} \hat{E}_{(m-1)\delta} + (q^{4} - q^{2} - q^{-2}) \hat{E}_{(m-1)\delta} E_{\delta+ \alpha_{1}} + \\
+ (q^{2} + q^{-2} - q^{-4}) E_{\delta+ \alpha_{1}} \hat{E}_{(m-1)\delta} + q^{2} \hat{E}_{(m-2)\delta} E_{2\delta+ \alpha_{1}} - q^{-2} E_{2\delta+ \alpha_{1}} \hat{E}_{(m-2)\delta} = \\
= (q^{4} - q^{-2}) \hat{E}_{(m-1)\delta} E_{\delta+ \alpha_{1}} + (q^{2} - q^{-4}) E_{\delta+ \alpha_{1}} \hat{E}_{(m-1)\delta} + \\
+ q^{2} \hat{E}_{(m-2)\delta} E_{2\delta+ \alpha_{1}} - q^{-2} E_{2\delta+ \alpha_{1}} \hat{E}_{(m-2)\delta},

which is \((iii)_{m}\).

\(iv)_{m}\): we have to prove that \([\hat{E}_{m\delta}, \hat{E}_{r\delta}] = 0 \forall r < m\). Since \([\hat{E}_{m\delta}, \hat{E}_{r\delta}] \in U_{q}^{\ast} (m+r)\delta\)

it is enough to prove that it commutes with \(F_{0}\) and \(F_{1}\) (see [27]): in particular, proving that it belongs to the center of \(U_{q}\) will solve the problem; but the set

of the elements of \(U_{q}\) commuting with \([\hat{E}_{m\delta}, \hat{E}_{r\delta}]\) is a subalgebra of \(U_{q}\) obviously containing \(U_{q}^{0}\) and stable by the action of \(T_{0} T_{1}\) and \(T_{0} \Xi\) (because \([\hat{E}_{m\delta}, \hat{E}_{r\delta}]\) is fixed by \(T_{0} T_{1}\) and \(T_{0} \Xi\)). Hence the problem reduces (see corollary 3.1.4) to show that

\([\hat{E}_{m\delta}, \hat{E}_{r\delta}], E_{1} = 0\), that is \([\hat{E}_{m\delta}, E_{1}] = [\hat{E}_{r\delta}, [\hat{E}_{m\delta}, E_{1}]]\): this will be done by induction on \(r\):

\(r = 1\): proposition 3.2.2, \(i)_{m}\), \((T_{0} T_{1})^{\ast} (iii)_{m}\) and \(v)_{m-1}\) imply that

\([\hat{E}_{m\delta}, [\hat{E}_{\delta}, E_{1}]] = -[3]_{q}[\hat{E}_{m\delta}, E_{\delta+ \alpha_{1}}] = \\
= -[3]_{q}((q^{2} - 1)[3]_{q} \hat{E}_{(m-1)\delta} E_{2\delta+ \alpha_{1}} + (1 - q^{-2})[3]_{q} E_{2\delta+ \alpha_{1}} \hat{E}_{(m-1)\delta} + \\
+ q^{2} \hat{E}_{(m-2)\delta} E_{3\delta+ \alpha_{1}} - q^{-2} E_{3\delta+ \alpha_{1}} \hat{E}_{(m-2)\delta}) = \\
= [\hat{E}_{\delta}, (q^{2} - 1)[3]_{q} \hat{E}_{(m-1)\delta} E_{\delta+ \alpha_{1}} + (1 - q^{-2})[3]_{q} E_{\delta+ \alpha_{1}} \hat{E}_{(m-1)\delta} + \\
+ q^{2} \hat{E}_{(m-2)\delta} E_{2\delta+ \alpha_{1}} - q^{-2} E_{2\delta+ \alpha_{1}} \hat{E}_{(m-2)\delta}] = \\
= [\hat{E}_{\delta}, [\hat{E}_{m\delta}, E_{1}]],

so that \([\hat{E}_{m\delta}, \hat{E}_{\delta}] = 0\).
\[ r > 1: \text{ notice that if we put } x_1 = q^4 - q^{-2}, \ y_1 = q^2 - q^{-4}, \ x_2 = q^2, \ y_2 = -q^{-2}, \]
we have that \( \forall u \leq m, \forall v \in \mathbb{N} \) (applying \((T_0T_1)^{-v}\) to \(iii)\))

\[
[E_{u\delta}, E_{v\delta+\alpha_1}] = \sum_{s=1}^{2} (x_s \tilde{E}_{(u-s)\delta} E_{(v+s)\delta+\alpha_1} + y_s E_{(v+s)\delta+\alpha_1} \tilde{E}_{(u-s)\delta});
\]
hence, using the inductive hypothesis and \(v)_{m-1}\), we get that

\[
[E_{m\delta}, [E_{r\delta}, E_{1}]] = \left[\tilde{E}_{m\delta}, \sum_{s=1}^{2} (x_s \tilde{E}_{(r-s)\delta} E_{s\delta+\alpha_1} + y_s E_{s\delta+\alpha_1} \tilde{E}_{(r-s)\delta})\right] = \]

\[
= \sum_{s,t=1}^{2} (x_s x_t \tilde{E}_{(r-s)\delta} \tilde{E}_{(m-t)\delta} E_{(s+t)\delta+\alpha_1} + x_s y_t \tilde{E}_{(r-s)\delta} E_{(s+t)\delta+\alpha_1} \tilde{E}_{(m-t)\delta} +
\]

\[
y_s x_t \tilde{E}_{(m-t)\delta} E_{(s+t)\delta+\alpha_1} \tilde{E}_{(r-s)\delta} + y_s y_t E_{(s+t)\delta+\alpha_1} \tilde{E}_{(m-t)\delta} \tilde{E}_{(r-s)\delta}) = \]

\[
= \left[\tilde{E}_{r\delta}, \sum_{t=1}^{2} x_t \tilde{E}_{(m-t)\delta} E_{t\delta+\alpha_1} + y_t E_{t\delta+\alpha_1} \tilde{E}_{(m-t)\delta}\right] = [E_{r\delta}, [E_{m\delta}, E_{1}]],
\]

which is the claim. \( \square \)

\section{3.4. Commutation between positive and negative real root vectors.}

The preceding proposition is first of all useful in proving other commutation relations, now involving the real root vectors.

\textbf{Lemma 3.4.1}

\( \forall r, s \in \mathbb{N} \) we have that:

\( a)_{r,s} [E_{r\delta+\alpha_1}, F_{s\delta+\alpha_1}] = \begin{cases} 
K_{r\delta+\alpha_1} - K_{s\delta+\alpha_1} & \text{if } r = s \\
q - q^{-1} & \text{if } r > s \\
-K_{s\delta+\alpha_1} \tilde{E}_{(r-s)\delta} & \text{if } r < s;
\end{cases} \)

\( b)_{r,s} [E_{(r+1)\delta-\alpha_1}, F_{(s+1)\delta-\alpha_1}] = \begin{cases} 
K_{(r+1)\delta-\alpha_1} - K_{(s+1)\delta-\alpha_1} & \text{if } r = s \\
q - q^{-1} & \text{if } r > s \\
-K_{(s+1)\delta-\alpha_1} \tilde{E}_{(r-s)\delta} & \text{if } r < s;
\end{cases} \)

\textbf{Proof}: \( a)_{r,r} \) is proved by applying \((T_0T_1)^{-r}\) to the identity \([E_{1}, F_{1}] = \frac{K_{1} - K_{-1}}{q - q^{-1}}\) if \( r > s a)_{r,s} \) follows from the identity \(-E_{(r-s)\delta-\alpha_1} E_{1} + q^{-2}E_{1} E_{(r-s)\delta-\alpha_1} = \tilde{E}_{(r-s)\delta}\) again applying to it \((T_0T_1)^{-r}\); if \( r > s a)_{r,s} \) is nothing but \( \Omega \) applied to \( a)_{s,r} \).

As for assertion \( b)\), we have that \( b)_{r,s} = \Xi T_1(a)_{r+1,s+1} \).

\section{3.5. The imaginary root vectors \( E_{m\delta}\).}

Recall that the subalgebra of \( \mathcal{U}_q \) generated by the \( \tilde{E}_{m\delta} \)'s is a commutative subalgebra pointwise fixed by \( T_0T_1 \) and \( T_0 \Xi \) (see proposition 3.3.2). Hence it makes sense to introduce new imaginary root vectors by the following equation:

\textbf{Definition 3.5.1}
Let us define some new elements in the subalgebra of $\mathcal{U}_q$ generated by the $\tilde{E}_{m\delta}$'s by the following identity:

$$1 - (q - q^{-1}) \sum_{m > 0} \tilde{E}_{m\delta} u^m = \exp \left( (q - q^{-1}) \sum_{m > 0} E_{m\delta} u^m \right).$$

Remark that of course $\forall m > 0$ $E_{m\delta} \in \mathcal{U}_q^+$ and $E_{m\delta} + \tilde{E}_{m\delta}$ belongs to the subalgebra of $\mathcal{U}_q$ generated by $\{ \tilde{E}_{r\delta} \mid r < m \}$.

**Proposition 3.5.2**

$\forall m > 0$ we have the following commutation relations:

$$[E_{m\delta}, E_1] = \frac{[2m]_q}{m} (q^{2m} + (-1)^{m-1} + q^{-2m}) E_{m\delta + \alpha_1}.$$

**Proof:** The idea of this proof is completely similar to the proof of the analogous result for $A^{(1)}_1$ (see [1]). Let $\tilde{S}_m^\pm, S_m^\pm : \mathcal{U}_q \to \mathcal{U}_q$ be the operators of left ($^+$) and right ($^-$) multiplication respectively by $E_{m\delta}$ and $\tilde{E}_{m\delta}$ (where we keep on indicating by $\tilde{E}_0\delta$ and $\tilde{E}_{-\delta}$ respectively $-\frac{1}{q-q}$ and 0) and let us set

$$\tilde{S}_m^\pm(u) \doteq \sum_{m \geq 0} \tilde{S}_m^\pm u^m, \quad S_m^\pm(u) \doteq \sum_{m > 0} S_m^\pm u^m;$$

moreover let us indicate by $T$ the automorphism $(T_0 T_1)^{-1}$: remark that $\forall m$

$$\tilde{S}_m^\pm T = T \tilde{S}_m^\pm \text{ and } S_m^\pm T = T S_m^\pm.$$

Now observe that point iii) of proposition 3.3.2 can be written by saying that $\forall m$

$$(\tilde{S}_m^+ - \tilde{S}_m^-)(E_1) =$$

$$= ((q^4 - q^{-2}) \tilde{S}_{m-1}^+ T + (q^2 - q^{-4}) \tilde{S}_{m-1}^- T + q^2 \tilde{S}_{m-2}^+ T^2 - q^{-2} \tilde{S}_{m-2}^- T^2)(E_1)$$

or, also, (multiplying by $u^m$ and summing over $m$)

$$\tilde{S}_m^+(u)(1 + q^{-2} T u)(1 - q^4 T u)(E_1) =$$

$$= \tilde{S}_m^-(u)(1 + q^2 T u)(1 - q^{-4} T u)(E_1);$$

multiplying both sides of this identity by $-(q - q^{-1})$ and considering that

$$\log(-(q - q^{-1})\tilde{S}_m^+(u)) = (q - q^{-1})S_m^+(u)$$

we get that

$$(q - q^{-1})(S_m^+(u) - S_m^-(u))(E_1) =$$

$$= (\log(1 + q^2 T u) + \log(1 - q^{-4} T u) - \log(1 + q^{-2} T u) - \log(1 - q^4 T u))(E_1),$$
that is
\[ [E_{m\delta}, E_1] = \frac{1}{m(q - q^{-1})}((-1)^{m-1}q^{2m} - q^{-4m} + (-1)^m q^{-2m} + q^{4m})E_{m\delta + \alpha_1} = \]
\[ = \frac{[2m]_q}{m}(q^{2m} + (-1)^{m-1} + q^{-2m})E_{m\delta + \alpha_1}. \]

The last proposition generalizes immediately to the following more general relations:

**Corollary 3.5.3**

Let \( m \) be bigger than 0; then \( \forall r \in \mathbb{N} \) we have:

\( a_r \) \( [E_{m\delta}, E_{r\delta + \alpha_1}] = \frac{[2m]_q}{m}(q^{2m} + (-1)^{m-1} + q^{-2m})E_{(m+r)\delta + \alpha_1}; \)

\( b_r \) \( [E_{m\delta}, E_{(r+1)\delta - \alpha_1}] = -\frac{[2m]_q}{m}(q^{2m} + (-1)^{m-1} + q^{-2m})E_{(m+r+1)\delta - \alpha_1}; \)

\( c_r \) \( [E_{m\delta}, F_{r\delta + \alpha_1}] = \]
\[ = \left\{ \begin{array}{ll}
\frac{[2m]_q}{m}(q^{2m} + (-1)^{m-1} + q^{-2m})K_{r\delta + \alpha_1}E_{(m-r)\delta - \alpha_1} & \text{if } r < m \\
-\frac{[2m]_q}{m}(q^{2m} + (-1)^{m-1} + q^{-2m})K_{m\delta}F_{(r-m)\delta + \alpha_1} & \text{if } r \geq m;
\end{array} \right. \]

\( d_r \) \( [E_{m\delta}, F_{(r+1)\delta - \alpha_1}] = \]
\[ = \left\{ \begin{array}{ll}
-\frac{[2m]_q}{m}(q^{2m} + (-1)^{m-1} + q^{-2m})E_{(m-r-1)\delta + \alpha_1}K_{(r+1)\delta - \alpha_1}^{-1} & \text{if } r < m \\
\frac{[2m]_q}{m}(q^{2m} + (-1)^{m-1} + q^{-2m})K_{m\delta}^{-1}F_{(r-m+1)\delta - \alpha_1} & \text{if } r \geq m.
\end{array} \right. \]

**Proof:** \( a_r \) is \((T_0T_1)^{-r}\) applied to proposition 3.5.2; \( b_r \) is \( \Xi T_1 \) applied to \( a\) \( r+1; \)

\( c_r \) is \((T_0T_1)^{-r}\Xi T_1 \) applied to \( a \) \( 0; \) finally \( d_r \) is \( \Xi T_1 \) applied to \( c\) \( r+1. \)

**Corollary 3.5.4**

The imaginary root vectors form a “Heisenberg” algebra, with commutation relations given by:

\[ [E_{r\delta}, F_{s\delta}] = \delta_{r,s} \frac{[2r]_q}{r}(q^{2r} + (-1)^{r-1} + q^{-2r})K_{r\delta} - K_{s\delta}^{-1}. \]

**Proof:** The claim is a straightforward consequence of proposition 3.3.2 and corollary 3.5.3; see also [4].

**4. \( A_{2n}^{(2)} \) AND A COPY OF \( A_2^{(2)} \).**

§4.1. Definition of \( \varphi_1 \).

The aim of this section is to prove that the particular situation of \( \mathfrak{sl}_2^{(2)} = A_2^{(2)} \)
studied in the preceding section plays, in some cases, the same role as \( \mathfrak{sl}_2^{(1)} \) does in general. More precisely we know that if \( i \) is a vertex of the finite Dynkin diagram associated to an affine algebra \( \hat{\mathfrak{g}}^{(k)} \), then, provided that \( \hat{\mathfrak{g}}^{(k)} \) is not of type \( A_{2n}^{(2)} \) or that \( i \) is not 1, there exists a homomorphism of C-algebras \( \varphi_i : \mathcal{U}_q(\mathfrak{sl}_2^{(1)}) \rightarrow \mathcal{U}_q(\hat{\mathfrak{g}}^{(k)}) \)
such that \( \varphi_i(E_1) = E_i, \varphi_i(E_0) = E_{\delta - \alpha_i}, \varphi_i(q) = q_i, \Omega \varphi_i = \varphi_i \Omega; \) moreover \( \varphi_i \) is such that \( T_i \varphi_i = \varphi_i T_i \) and \( T_{\lambda_i} \varphi_i = \varphi_i T_{\lambda_i} (= \varphi_i T_0 T_i) \) (that is \( T_i \) corresponds to \( T_i, T_i = T_0 T_i T_i^{-1} \) to \( T_{\lambda_i} T_i^{-1} \) and \( T_0 = T_0 T_2^{-1} \) to \( T_{\lambda_i} T_i^{-1} \)) (see proposition 2.2.2).

Here we want to study the remaining cases, namely the situation where \( \hat{g}^{(k)} \) is of type \( A_{2n}^{(2)} \) and \( i = 1 \): in this case we construct a homomorphism of \( \mathbb{C} \)-algebras \( \varphi_1 : \mathcal{U}_q(\mathfrak{sl}_3^{(2)}) \to \mathcal{U}_q(\hat{g}^{(k)}) \) such that \( \varphi_1(q) = q_1 (= q) \) (\( \varphi_1 \) is indeed a homomorphism of \( \mathbb{C}(q) \)-algebras), \( \varphi_1(E_1) = E_1 \) and \( \Omega \varphi_1 = \varphi_1 \Omega \); moreover \( \varphi_1 \) is such that \( T_i \varphi_1 = \varphi_i T_i \) and \( T_{\omega_1} \varphi_1 = \varphi_1 T_{\omega_1} (= \varphi_1 T_0 T_i) \) (that is \( T_1 \) corresponds to \( T_1 \) and \( T_0 \) to \( T_{\omega_1} T_i^{-1} \)). Of course now the image of \( E_0 \) can’t be \( E_{\delta - \alpha_1} \) (since \( \delta - \alpha_1 \) has the same length as \( \alpha_1 \) while \( \alpha_0 \) is longer): in this case we will have that

\[ \varphi_1(E_0) = E_{\delta - 2\alpha_1}. \]

**Definition 4.1.1.**

Let \( \varphi_1 : \mathcal{U}_q(\mathfrak{A}_{2n}^{(2)}) \to \mathcal{U}_q(\mathfrak{A}_{2n}^{(2)}) \) be the homomorphism of \( \mathbb{C}(q) \)-algebras defined on the generators as follows:

- \( \varphi_1(E_i) = E_i, \quad \varphi_1(F_i) = F_i, \quad \varphi_1(K_i^{\pm 1}) = K_i^{\pm 1}, \)

- \( \varphi_1(E_0) = E_{\delta - 2\alpha_1}, \quad \varphi_1(F_0) = F_{\delta - 2\alpha_1}, \quad \varphi_1(K_0^{\pm 1}) = K_{\delta - 2\alpha_1}^{\pm 1}. \)

Our program is to prove first of all that \( \varphi_1 \) is well defined (of course if this is the case then it commutes with \( \Omega \)), point to which we devote paragraph 4.3, after §4.2, whose aim is recalling and describing some (mainly combinatorial) properties of the root system and Weyl group of type \( A_{2n}^{(2)} \); then, in §4.4, we shall study the relations between the braid group action and \( \varphi_1 \). These are the main tools which will allow us to construct a PBW-basis.

**§4.2. Root system and Weyl group: some properties.**

In this paragraph we shall study the properties of the root system and the Weyl group of \( A_{2n}^{(2)} \) which will enable us to attack the central problem of this section, that is the research of a copy of \( \mathcal{U}_q(\mathfrak{A}_{2n}^{(2)}) \) in \( \mathcal{U}_q(\mathfrak{A}_{2n}^{(2)}) \). What we need in particular is a more explicit description of the root vectors (in particular of some root vectors) and their behaviour under the action of the braid group, goal which can be achieved via a closer analysis of the Weyl group. More precisely our attention will concentrate on the study of a reduced expression of \( \omega_1 \) (remark that \( \omega_1 = \omega_1 \), since \( d_1 = 1 \)) and of some of its properties, what will help us in manipulating the root vectors.

**Remark 4.2.1.**

\( l(\omega_1) = n(n + 1). \)

**Proof:** It is enough to compute the cardinality of \( \Phi_+(\omega_1) \):

\[ \Phi_+(\omega_1) = \{ \alpha \in \Phi_+ \mid \omega_1^{-1}(\alpha) < 0 \} = \left\{ \delta - \varepsilon \sum_{r=1}^{k} \alpha_r \mid 1 \leq k, \varepsilon = 1, 2 \right\} \cup \]

\[ \cup \left\{ \varepsilon \delta - 2 \sum_{r=1}^{k} \alpha_r - \sum_{r=k+1}^{l} \alpha_r \mid 1 \leq k < l \leq n, \varepsilon = 1, 2 \right\}, \]

so that

\[ \#\Phi_+(\omega_1) = 2 \left( n + \frac{n(n - 1)}{2} \right) = n(n + 1), \]
Furthermore \(\forall\), and \(w\) consequence of lemma 4.2.1.

**Definition 4.2.2.**

In the following \(w\) will denote the element of \(W\) given by:

\[
w \doteq s_0 s_n s_{n-1} \cdots s_1.
\]

**Lemma 4.2.3**

The action of \(w\) on \(Q\) can be described as follows:

\[
w(\alpha_i) = \begin{cases} 
\delta - 2 \sum_{r=1}^{n-1} \alpha_r & \text{if } i = 0 \\
-\delta + \alpha_1 + \cdots + \alpha_n & \text{if } i = 1 \\
\delta - (\alpha_2 + \cdots + \alpha_n) & \text{if } i = 2 \\
\alpha_{i-1} & \text{if } 3 \leq i \leq n.
\end{cases}
\]

Furthermore \(\forall k \in \mathbb{Z}\)

\[
w^k(\alpha_i) = \begin{cases} 
(1 + 2 \left[ \frac{k}{n} \right]) \delta - 2 \sum_{r=1}^{n-k} \alpha_r & \text{if } i = 0 \\
\left[ -\frac{k}{n} \right] \delta + \sum_{r=1}^{n+1-k} \alpha_r & \text{if } i = 1 \\
\alpha_{i-k} & \text{if } 2 \leq i \leq n \text{ and } i-k \neq 1 \\
\delta - (\alpha_2 + \cdots + \alpha_n) & \text{if } 2 \leq i \leq n \text{ and } i-k = 1,
\end{cases}
\]

where \(\mathbb{Z} \rightarrow \{1, \ldots, n\}\) is the application such that \(k \equiv k \pmod{n}\) \(\forall k \in \mathbb{Z}\).

**Proof:** It is clear that if \(3 \leq i \leq n\) then

\[
w(\alpha_i) = s_0 s_n \cdots s_1(\alpha_i) = s_0 s_n \cdots s_{i-1}(\alpha_i) = s_0 s_n \cdots s_{i+1}(\alpha_{i-1}) = \alpha_{i-1};
\]

also, it is straightforward to see that \(w(\alpha_1) = -\delta + (\alpha_1 + \cdots + \alpha_n), w(\alpha_2) = \delta - (\alpha_2 + \cdots + \alpha_n)\) and \(w(\alpha_0) = \alpha_0 + 2\alpha_n = \delta - 2\sum_{r=1}^{n-1} \alpha_r\), which proves the first assertion.

Remark that \(w^2(\alpha_2) = \alpha_n\), so that the second assertion immediately follows for \(2 \leq i \leq n\).

The fact that \(w(\alpha_1 + \alpha_2) = \alpha_1\) implies that \(w(\alpha_1 + \cdots + \alpha_r) = \alpha_1 + \cdots + \alpha_{r-1}\) \(\forall r > 1\), which easily leads to the remaining assertions for \(i = 0, 1\).

**Corollary 4.2.4**

\(\omega_1 = w^n\) and \((s_0 s_n \cdots s_1)^n\) is a reduced expression of \(\omega_1\); in particular we have that \(T_{\omega_1} = T^n_w = (T_0 T_n \cdots T_1)^n\).

**Proof:** That \(\omega_1 = w^n\) follows from the fact that \(w^n(\alpha_i) = \alpha_i\) when \(i \in I_0 \setminus \{1\}\) and \(w^n(\alpha_1) = -\delta + \alpha_1\) (see lemma 4.2.3). The remaining assertion is an immediate consequence of lemma 4.2.1.

**Notation 4.2.5.**

We fix \(\iota\) so that \(\iota r + r \equiv 1 \pmod{n + 1}\) whenever \(1 \leq r \leq n(n + 1)\) (this can be done thanks to corollary 4.2.4; see also remark 2.1.5).
Corollary 4.2.6
\[ w^{n-1}(a_0) = \delta - 2a_1; \text{ in particular } E_{\delta - 2a_1} = T_{w^{n-1}}(E_0). \]

**Proof:** That \( w^{n-1}(a_0) = \delta - 2a_1 \) is a direct consequence of lemma 4.2.3. Since \( T_{w^{n-1}}(E_0) \) is a root vector, it must then be \( E_{\delta - 2a_1}. \)

The lemma that we are now going to propose is devoted to the study of the reduced expressions of some suitable elements of the Weyl group. Here lies perhaps the main difference between this case and the general case that has been recalled in section 2: indeed in that case \( l(\omega_1 s_1 \omega_1) = 2l(\omega_1) - 1 \), and this implied that \( (T_{\omega_1} T_{\omega_1}^{-1})^2 \) acted as the identity on the subalgebra of \( U_q \) generated by \( \{ E_1, E_{\delta - \alpha_1} \} \) (see lemma 2.2.1 and remark 2.1.5); this is no more true here. However we can still say something.

**Lemma 4.2.7**
\[ l(s_0 w s_1 \omega_1) = l(s_0 w s_1) + l(\omega_1), \] so that \( T_{s_0 w s_1 w} = T_{s_0 w s_1} T_w^k \ \forall k \in \{0, \ldots, n\} \);
\[ l(s_0 w s_1 \omega_1) = l(w s_1) + l(s_1 w^{-1} s_0 w s_1 \omega_1), \] that is \( T_{s_0 w s_1 \omega_1} = T_{w s_1} T_{s_1 w^{-1} s_0 w s_1 \omega_1} \); moreover \( s_0 w s_1 w^{n-1}(a_0) = a_0 \) and \( s_1 w^{-1} s_0 w s_1 \omega_1 (\alpha_1) = \alpha_1 \).

**Proof:** The first statement means that \( \Phi^+ (s_0 w s_1) \subseteq \Phi^+(s_0 w s_1 \omega_1). \)
Now \( (s_0 w s_1)^{-1}(\Phi^+(s_0 w s_1)) \subseteq \sum_{i=2}^{n} \mathbb{Z} \alpha_i \) which is pointwise fixed by \( \omega_1 \), so that
\[ \alpha \in \Phi^+(s_0 w s_1 \omega_1) \implies (s_0 w s_1 \omega_1)^{-1}(\alpha) = (s_0 w s_1)^{-1}(\alpha) < 0 \implies \alpha \in \Phi^+(s_0 w s_1 \omega_1). \]

The second part of the lemma is equivalent to saying that \( \Phi^+(w s_1) \subseteq \Phi^+(s_0 w s_1 \omega_1) \).

Of course
\[ \Phi^+(w s_1) = \{ s_0 s_n \cdots s_{r+1}(\alpha_r) \mid 2 \leq r \leq n+1 \} = \{ \alpha_0 + \alpha_n + \ldots + \alpha_r \mid 2 \leq r \leq n+1 \} \]
so that, if \( 2 \leq r \leq n+1 \), \( (s_0 w s_1 \omega_1)^{-1}(\alpha_0 + \alpha_n + \ldots + \alpha_r) = \)
\[ = \begin{cases} \omega_1^{-1}(\alpha_0 + 2\alpha_n + \ldots + 2\alpha_{r-1} + \alpha_r + \ldots + \alpha_2) & \text{if } r \leq n \\ \omega_1^{-1}(\alpha_0 + 2\alpha_n + \ldots + 2\alpha_2) & \text{if } r = n+1 \end{cases} \]
\[ = \begin{cases} -\delta - (\alpha_r + \ldots + \alpha_2 + 2\alpha_1) & \text{if } r \leq n \\ -\delta - 2\alpha_1 & \text{if } r = n+1. \end{cases} \]

Now we have that
\[ s_0 w s_1 w^{n-1}(\alpha_0) = s_0 w s_1 (\delta - 2\alpha_1) = s_0 w (\delta + 2\alpha_1) = s_0 (2\delta - (w(\delta - 2\alpha_1))) = \]
\[ = s_0 (2\delta - \omega_1 (\alpha_0)) = s_0 \omega_1 (\delta + 2(\alpha_1 + \ldots + \alpha_n)) = s_0 (-\alpha_0) = \alpha_0. \]

Finally we get that
\[ s_1 w^{-1} s_0 w s_1 \omega_1 (\alpha_1) = s_1 w^{-1} s_0 w (\delta + \alpha_1) = \]
\[ = s_1 w^{-1} (\alpha_0 + \alpha_1 + \ldots + \alpha_n) = s_1 (-\alpha_1) = \alpha_1, \]
which concludes the proof.

**Corollary 4.2.8**
\( E_{\delta - 2a_1} = T_{s_0 w s_1}^{-1}(E_0) \) and \( E_{\delta - \alpha_1} = T_{s_0 w s_1}^{-1} T_{w s_1}(E_1). \)

**Proof:** Straightforward consequence of lemma 4.2.7.
\[4.3. \varphi_1 \text{ is well defined.}\]

Here we want to prove that \(\varphi_1\) is well defined, which means that the relations between the "canonical" generators of \(\mathcal{U}_q(A_2^{(2)})\) (that is \(E_i, F_i, K^\pm_i\) for \(i = 0, 1\)) are preserved by \(\varphi_1\) (so that \(\varphi_1 : \mathcal{U}_q(A_2^{(2)}) \rightarrow \mathcal{U}_q(A_{2n}^{(2)})\) is well defined), and that \(E_{\delta-2\alpha_1} \in \mathcal{U}_q^{(1)}\) (so that \(\varphi_1\) effectively maps \(\mathcal{U}_q(A_2^{(2)})\) in \(\mathcal{U}_q^{(1)}\)).

For some of these relations this result is immediate as indicated in proposition 4.3.2. The others will require some more work.

Let us recall which are the relations that we are speaking about:

**Definition 4.3.1.**
\(\mathcal{U}_q(A_2^{(2)})\) is the \(\mathbb{C}(q)\)-algebra generated by \(\{E_i, F_i, K^\pm_i | i = 0, 1\}\) with the following relations:

\[
\begin{align*}
 [KK] & \quad K_1K_0 = K_0K_1; \\
 [KE] & \quad K_1E_1 = q^2E_1K_1, \quad K_1E_0 = q^{-1}E_0K_1, \quad K_0E_1 = q^{-4}E_1K_0, \quad K_0E_0 = q^8E_0K_0; \\
 [KF] & \quad K_1F_1 = q^{-2}F_1K_1, \quad K_1F_0 = q^4F_0K_1, \quad K_0F_1 = q^4F_1K_0, \quad K_0F_0 = q^{-8}F_0K_0; \\
 [EF] & \quad [E_1, F_1] = \frac{K_1 - K_1^{-1}}{q - q^{-1}}, \quad [E_0, F_0] = \frac{K_0 - K_0^{-1}}{q^4 - q^{-4}}, \quad [E_1, F_0] = 0 = [E_0, F_1]; \\
 [EE] & \quad \sum_{r=0}^{2} (-1)^r E_0^{(r)}E_1E_0^{(2-r)} = 0 = \sum_{r=0}^{5} (-1)^r E_1^{(r)}E_0E_1^{(5-r)}; \\
 [FF] & \quad \sum_{r=0}^{2} (-1)^r F_0^{(r)}F_1F_0^{(2-r)} = 0 = \sum_{r=0}^{5} (-1)^r F_1^{(r)}F_0F_1^{(5-r)}. 
\end{align*}
\]

**Proposition 4.3.2**
The relations [KK], [KE], [KF] and [EF] are obviously preserved by \(\varphi_1\).

**Proof:** The subalgebra of \(\mathcal{U}_q(A_{2n}^{(2)})\) generated by \(\{E_i | i \neq 1\}\) (to which \(E_{\delta-2\alpha_1}\) belongs) commutes with \(F_1\), so that \([E_{\delta-2\alpha_1}, F_1] = 0\) (and of course \([E_1, F_{\delta-2\alpha_1}] = = \Omega([E_{\delta-2\alpha_1}, F_1]) = 0\)). The other relations are trivial. \(\Box\)

The point is now to prove the remaining relations.

**Remark 4.3.3.**
Once we have proved the compatibility of relations [EE] with the definition of \(\varphi_1\), it is enough to apply \(\Omega\) in order to find the same result for [FF]; indeed both the
left and the right hand sides of [FF] are found by applying $\Omega$ to the corresponding expressions of [EE].

**Lemma 4.3.4**
In $\mathcal{U}_q(A_{2n}^{(2)})$ we have that $E_{\delta-\alpha_1}E_{\delta-2\alpha_1} = q^{-4}E_{\delta-2\alpha_1}E_{\delta-\alpha_1}$.

**Proof:** Since $E_{\delta-2\alpha_1} = T_{w_{n-1}}(E_0)$ and $E_{\delta-\alpha_1} = T_{w_1}^{-1}(E_1) = T_{w_1}^{-1}(E_1)$ we can apply Levendorskii-Soibelman formula (see [23]): the claim follows from the fact that $(\delta - 2\alpha_1)(\delta - \alpha_1) = 4$, and that the only possibility for a sum of positive roots of the form $m\delta - \alpha$ (with $\alpha$ a positive multiple of a root of the finite system associated to $A_{2n}^{(2)}$) to be equal to $2\delta - 3\alpha_1$ is that these positive roots are $\delta - 2\alpha_1$ and $\delta - \alpha_1$.

**Lemma 4.3.5**
$-E_0T_nT_{n-1} \cdots T_3(E_2) + q^{-4}T_nT_{n-1} \cdots T_3(E_2)E_0 = E_0T_nT_{n-1} \cdots T_3(E_2)$.

**Proof:** Since $T_nT_{n-1} \cdots T_3(E_2) \in U_{q_{\alpha_n + \ldots + \alpha_{2}}}^{\mathcal{U}}$ we can write it as $\sum xE_ny$ where $x, y \in (E_2, \ldots, E_{n-1})$; in particular $x$ and $y$ commute with $E_0$ and are fixed points for $T_0$. Then we have

$$-E_0T_nT_{n-1} \cdots T_3(E_2) + q^{-4}T_nT_{n-1} \cdots T_3(E_2)E_0 = \sum x(-E_0E_n + q^{-4}E_nE_0)y =$$

$$= \sum xT_0(E_n)y = T_0 \left( \sum xE_ny \right) = T_0T_nT_{n-1} \cdots T_3(E_2).$$

**Corollary 4.3.6**
In $\mathcal{U}_q(A_{2n}^{(2)})$ we have that $-E_{\delta-2\alpha_1}E_1 + q^{-4}E_1E_{\delta-2\alpha_1} = E_{\delta-\alpha_1}$.

**Proof:** The proof consists of a computation in which lemma 4.3.5 and corollary 4.2.8 play a fundamental role: if $n \geq 2$

$$-E_{\delta-2\alpha_1}E_1 + q^{-4}E_1E_{\delta-2\alpha_1} = -T_{s_{0\omega s_1}}^{-1}(E_0)E_1 + q^{-4}E_1T_{s_{0\omega s_1}}^{-1}(E_0) =$$

$$= T_{s_{0\omega s_1}}^{-1}(-E_0T_{s_{0\omega s_1}}E_1) + q^{-4}E_1T_{s_{0\omega s_1}}^{-1}(E_0) =$$

$$= T_{s_{0\omega s_1}}^{-1}(-E_0T_{s_{0\omega s_1}}E_1 + q^{-2}E_1E_2) + q^{-4}T_{s_{0\omega s_1}}^{-1}(-E_2E_1 + q^{-2}E_1E_2)E_0) =$$

$$= -T_{s_{0\omega s_1}}^{-1}(-E_0T_n \cdots T_3(E_2) + q^{-4}T_n \cdots T_3(E_2)E_0)E_1 +$$

$$-q^{-2}E_1(-E_0T_n \cdots T_3(E_2) + q^{-4}T_n \cdots T_3(E_2)E_0) =$$

$$= -T_{s_{0\omega s_1}}^{-1}(T_0T_nT_{n-1} \cdots T_3(E_2)E_1 - q^{-2}E_1T_0T_nT_{n-1} \cdots T_3(E_2)) =$$

$$= T_{s_{0\omega s_1}}^{-1}T_0T_nT_{n-1} \cdots T_3(E_2)E_1 = T_{s_{0\omega s_1}}^{-1}T_{s_{0\omega s_1}}E_1 = E_{\delta-\alpha_1},$$

where we have used that $[E_1, E_0] = 0$.

**Proposition 4.3.7**
The following relation holds in $\mathcal{U}_q(A_{2n}^{(2)})$:

$$\sum_{r=0}^{2} (-1)^r \binom{2}{r} q^r E_{\delta-2\alpha_1}E_1E_{\delta-2\alpha_1}^2 = 0.$$
Proof: The proof is a simple application of lemma 4.3.4 and corollary 4.3.6.

So we are now left to prove just the last relation [EE]. To this aim we start by recalling a very simple and basic combinatorial lemma.

**Lemma 4.3.8**

Let $a,b$ be elements of a $\mathbb{C}(q)$-algebra satisfying the relation

$$\sum_{r=0}^{k} (-1)^r q^{pr} \binom{k}{r} q a^r b^{k-r} = 0$$

for some natural number $k$ and some integer $p$. Then $a,b$ also satisfy the relation

$$\sum_{r=0}^{k+1} (-1)^r q^{(p+1)r} \binom{k+1}{r} q a^r b^{k+1-r} = 0.$$

**Proof:** The proof is a straightforward computation based on the identity

$$q^{(p+1)r} \binom{k+1}{r} q = q^{pr} \binom{k}{r} q + q^{k+p+1} q^{p(r-1)} \binom{k}{r-1} q.$$

Before going to the point, we study some other simple relations.

**Lemma 4.3.9**

The following is true in $\mathcal{U}_q(A^{(2)}_2)$:

$$[E_{\delta-2\alpha_1}, F_i] = 0 \quad \forall i \in \{3, 4, ..., n\};$$

$$T_0^{-1}(E_{\delta-2\alpha_1}) \in \mathcal{U}_q^+;$$

$$[T_{w_n^{-1}}(E_0), E_1] = 0 = [T_{w_n^{-1}}(E_0), F_2].$$

**Proof:** The proof consists in some easy remarks:

1) if $i \in \{3, 4, ..., n\}$ then $T_{w_n w_{i}}(F_i) = T_n T_{n-1} \cdot \cdot \cdot T_{i-1}(F_i) = F_{i-1}$, which commutes with $E_0$, so that

$$[E_{\delta-2\alpha_1}, F_i] = T_0^{-1}(E_{\delta-2\alpha_1}) \in \mathcal{U}_q^+;$$

2) $T_0^{-1}(E_{\delta-2\alpha_1}) = T_0^{-1}T_{w_n^{-1}}(E_0) = T_{w_n^{-1}}(E_0)$ which is obviously in $\mathcal{U}_q^+$;

3) $T_{w_n^{-1}}(E_0) \in (E_0, E_3, E_4, ..., E_n)$, so that it commutes with $E_1$ and $F_2$. □

The following proposition is what we still need for our goal; it concerns the last relation [EE], and the proof is adapted from the analogous situation in case $A^{(1)}_2$ (see [1]).

**Proposition 4.3.10**

In $\mathcal{U}_q(A^{(2)}_2)$ the following relation is verified:

$$\sum_{r=0}^{5} (-1)^r E_1^{(r)} E_{\delta-2\alpha_1} E_1^{(5-r)} = 0.$$
Proof: Let $E = \sum_{r=0}^{5}(-1)^r E_1^{(r)} E_{\delta-2\alpha} E_1^{(5-r)}$; we want to prove that $E = 0$. The strategy to achieve this goal is to prove that $T_i^{-1}(E) \in U_q^- \forall i = 0,...n$ (see [27]). Recall that $[E,F_i] = 0$ implies $T_i^{-1}(E) \in U_q^-$. Lemma 4.3.9 evidently implies that $[E,F_i] = 0$ when $i = 3,...,n$ and $T_0^{-1}(E) \in U_q^+$ (since $T_0^{-1}(E_1) \in U_q^+$).

Moreover, thanks to proposition 4.3.2, $[E,F_1] = 0$. To this aim we rewrite $E$ (when $n > 1$, which is the case we are interested in) noticing that

$$E_{\delta-2\alpha} = T_{\delta}\left(E_0\right) = T_{\delta-2\alpha} T_0\left(E_0\right) = T_{\delta-2\alpha} T_0^{-1}(E_0) = T_{\delta-2\alpha}(E_0) E_{\alpha}^{(2)} - q^{-2} E_2 T_{\delta-2\alpha}(E_0) E_2 + q^{-4} E_2^{(2)} T_{\delta-2\alpha}(E_0),$$

so that $E$ is given by the expression

$$\sum_{r=0}^{5}(-1)^r E_1^{(r)} (T_{\delta}(E_0) E_2^{(2)} - q^{-2} E_2 T_{\delta}(E_0)) E_1^{(5-r)}.$$

Then we are ready to conclude thanks to the last part of lemma 4.3.9:

$$[E,F_2] =$$

$$= \sum_{r=0}^{5}(-1)^r E_1^{(r)} \left(T_{\delta}(E_0) \frac{q^{-2} K_2 - q^2 K_2^{-1}}{q^2 - q^{-2}} E_2 - \frac{q^{-2} K_2 - K_2^{-1}}{q^2 - q^{-2}} T_{\delta}(E_0) E_2 + q^{-2} E_2 T_{\delta}(E_0) E_2 \right) +$$

$$- q^{-2} E_2 T_{\delta}(E_0) \frac{K_2 - K_2^{-1}}{q^2 - q^{-2}} E_2 + q^{-2} E_2 T_{\delta}(E_0) E_2 \right) E_1^{(5-r)} =$$

$$= T_{\delta}(E_0) \sum_{r=0}^{5}(-1)^r X E_1^{(r)} E_2 E_1^{(5-r)} - \sum_{r=0}^{5}(-1)^r E_1^{(r)} E_2 E_1^{(5-r)} Y T_{\delta}(E_0)$$

where

$$X = \frac{q^{2r-2} K_2 - q^{2r} K_2^{-1} - q^{2r-6} K_2 + q^{2r-2} K_2^{-1}}{q^2 - q^{-2}} = q^{2r-4} K_2.$$
and
\[ Y = \frac{q^{2r-8}K_2 - q^{4-2r}K_2^{-1} - q^{2r-12}K_2 + q^{4-2r}K_2^{-1}}{q^2 - q^{-2}} = q^{2r-10}K_2, \]
so that
\[ [E, F_2] = q^{-4}T_{w^a-z}(E_0)K_2 \left( \sum_{r=0}^{5} (-1)^r q^{2r}E_1^{(r)}E_2E_1^{(5-r)} \right) +
- q^{-10} \left( \sum_{r=0}^{5} (-1)^r q^{2r}E_1^{(r)}E_2E_1^{(5-r)} \right) K_2T_{w^a-z}(E_0) \]
which is zero thanks to lemma 4.3.8 and to the relation, valid in \( \mathcal{U}_q(A_{2n}^{(2)}) \) (if \( n > 1 \)),
\[ \sum_{r=0}^{3} (-1)^r E_1^{(r)}E_2E_1^{(3-r)} = 0. \]
This completely proves the claim, that is \( E = 0 \).

The paragraph can now be closed with the next proposition, which states the achievement of our goal:

**Proposition 4.3.11**
\( \varphi_1 \) is well defined.

**Proof:** The only thing to prove is that \( E_{\delta-2\alpha_1} \in \mathcal{U}_q^{(1)} \). This follows immediately from proposition 4.3.2 and corollary 4.3.6: indeed by straightforward computations
\[ [E_{\delta-\alpha_1}, F_1] = -[4]_q K_1E_{\delta-2\alpha_1}. \]

§4.4. Relations between \( \varphi_1 \) and the braid group action.

The results of paragraph 4.3 are the first step in our attempt to prove that the relations between the root vectors in \( \mathcal{U}_q(A_{2n}^{(2)}) \) found in section 3 are still valid in \( \mathcal{U}_q(A_{2n}^{(2)}) \). To conclude this piece of program we need to analyze how the braid group acts with respect to \( \varphi_1 \). As we already announced, we want to prove that \( T_1\varphi_1 = \varphi_1T_1 \) and \( T_{\omega_1}T_1^{-1}\varphi_1 = \varphi_1T_0 \). We start from this second statement.

**Proposition 4.4.1**

The action of \( T_0 \) on \( \mathcal{U}_q(A_{2n}^{(2)}) \) corresponds, under \( \varphi_1 \), to the action of \( T_{\omega_1}T_1^{-1} \), that is \( T_{\omega_1}T_1^{-1}\varphi_1 = \varphi_1T_0 \).

**Proof:** Of course it is enough to prove that \( T_{\omega_1}T_1^{-1}\varphi_1 = \varphi_1T_0 \) on the generators of \( \mathcal{U}_q(A_{2n}^{(2)}) \), that is on \( E_1, E_0, F_1, F_0, K_1, K_0 \). For the elements of \( \mathcal{U}_q^{(\ell)}(A_{2n}^{(2)}) \) the claim is trivial.

Remark that \( T_{\omega_1}T_1^{-1}, \varphi_1 \) and \( T_0 \) all commute with \( \Omega \), so that we are reduced to prove that \( T_{\omega_1}T_1^{-1}\varphi_1(E_1) = \varphi_1T_0(E_1) \) and \( T_{\omega_1}T_1^{-1}\varphi_1(E_0) = \varphi_1T_0(E_0) \); this means that we want to prove on one hand that
\[ \varphi_1(-E_0E_1 + q^{-3}E_1E_0) = E_{\delta-\alpha_1}. \]
(which is nothing but corollary 4.3.6), and on the other hand that
\[-F_{\delta-2\alpha_1}K_{\delta-2\alpha_1} = T_{\omega_1}T_{\omega_1}(E_{\delta-2\alpha_1}).\]
or equivalently (see corollaries 4.2.4, 4.2.6 and 4.2.8) that \(-F_0K_0 = T_{\omega_1}T_{\omega_1}(E_0);\) but of course \(T_{\omega_1}T_{\omega_1}(E_0) = T_0(E_0),\) which proves the claim. \(\Box\)

**Remark 4.4.2.**

We concentrate now on the question of proving that \(T_1\varphi_1 = \varphi_1T_1,\) or equivalently that \(T_1^{-1}\varphi_1 = \varphi_1T_1^{-1}.)\) The same considerations as before tell us that this problem reduces to the question whether \(T_1^{-1}\varphi_1(E_1) = \varphi_1T_1^{-1}(E_1)\) and \(T_1^{-1}\varphi_1(E_0) = \varphi_1T_1^{-1}(E_0);\) notice that the first relation is obviously true while the second one can be translated into

\[T_1^{-1}(E_{\delta-2\alpha_1}) \equiv \sum_{r=0}^{4}(-1)^{r}q^{-r}E_1^{(r)}E_{\delta-2\alpha_1}E_1^{(4-r)}.\]

To face this problem let us set the following definition:

**Definition 4.4.3.**

We introduce the elements \(X_0,...,X_n\) of \(U_q(A_{2n}^{(2)})\) as follows:

\[X_0 \overset{s}{=} E_0, \quad X_{s-1} \overset{s}{=} T_1^{-1}(X_s) \quad \forall s = 1,...,n.\]

Remarking that \(X_1 = E_{\delta-2\alpha_1}\) (corollary 4.2.8) and \(X_0 = T_1^{-1}(E_{\delta-2\alpha_1}).\)

Our strategy will divide in two steps:

1) proving that \(X_{s-1} = \sum_{r=0}^{2}(-1)^{r}q^{-2r}E_1^{(r)}X_rE_1^{(2-r)} \forall s = 2,...,n;\)
2) computing \(X_0\) thanks to the description of \(X_1\) given in 1): this will be the claim.

**Lemma 4.4.4**

Let \(s\) be \(\in\{2,...,n-1\}\) and let \(X \in U_q\) be an element of \(U_q(A_{2n}^{(2)})\) commuting with \(X_s\) and fixed by \(T_s\). Then

\[\sum_{r=0}^{2}(-1)^{r}q^{-2r}E_1^{(r)}E_{s+1}XE_{s+1}E_1^{(2-r)} = T_1^{-1}(E_{s+1}XE_{s+1}).\]

**Proof:** Using the Serre relation between \(E_s\) and \(E_{s+1}\) we have that

\[\sum_{r=0}^{2}(-1)^{r}q^{-2r}E_1^{(r)}E_{s+1}XE_{s+1}E_1^{(2-r)} =
\]

\[= E_{s+1}XE_{s+1}E_s - \frac{1}{q^2 + q^{-2}}E_{s+1}XE_{s+1}E_{s+1} + q^{-2}E_{s+1}XE_{s+1}E_{s+1}E_s +
\]

\[+ q^{-4}E_sE_{s+1}XE_{s+1}E_s - \frac{q^{-4}}{q^2 + q^{-2}}E_{s+1}XE_{s+1}E_{s+1} =
\]

\[= (-E_{s+1}E_s + q^{-2}E_{s+1}E_{s+1})X(-E_{s+1}E_s + q^{-2}E_{s+1}E_{s+1}) =
\]

\[= T_1^{-1}(E_{s+1}XE_{s+1}) = T_1^{-1}(E_{s+1}XE_{s+1}).\]
Corollary 4.4.5
Let $s$ and $X$ be as in lemma 4.4.4, and let $Y = \sum_{r=0}^{2} (-1)^r q^{-2r} E_{s+1}^{(r)} X E_{s+1}^{(2-r)}$; then
\[ \sum_{r=0}^{2} (-1)^r q^{-2r} E_{s}^{(r)} Y E_{s}^{(2-r)} = T_{s}^{-1}(Y). \]

Proof: The claim is an immediate application of lemma 4.4.4.

Corollary 4.4.6
Let $s$ be in $\{2, \ldots, n\}$; then $X_{s-1} = \sum_{r=0}^{2} (-1)^r q^{-2r} E_{s}^{(r)} X_{s} E_{s}^{(2-r)}$.
In particular $E_{s-2\alpha_1} = \sum_{r=0}^{2} (-1)^r q^{-2r} E_{2}^{(r)} X_{2} E_{2}^{(2-r)}$.

Proof: The statement is proved by induction on $s$: if $s = n$ the claim is obvious (it is the definition of $T_{n}^{-1}(E_{0})$).
If $2 \leq s < n$ define $X$ by $X = X_{s+1}$; then $X \in (E_{0}, E_{n}, \ldots, E_{s+2})$, hence it commutes with $E_{s}$ and is fixed by $T_{s}$. Moreover, by the inductive hypothesis, the element $Y$ defined in corollary 4.4.5 is nothing but $X_{s}$. Hence we have that
\[ X_{s-1} = T_{s}^{-1}(X_{s}) = T_{s}^{-1}(Y) = \sum_{r=0}^{2} (-1)^r q^{-2r} E_{s}^{(r)} Y E_{s}^{(2-r)} = \]
\[ = \sum_{r=0}^{2} (-1)^r q^{-2r} E_{s}^{(r)} X_{s} E_{s}^{(2-r)}. \]

We have thus solved the first step. For the second one let us develop some computations.

Lemma 4.4.7
Let $X$ be an element of $U_{q}$ commuting with $E_{1}$. Then we have the following identities:
\[ E_{2} X E_{2} E_{1}^{(4)} = \]
\[ = E_{2} E_{1}^{(2)} X E_{2} E_{1}^{(2)} = \frac{[2]_{q}}{[3]_{q}} E_{2} E_{1}^{(2)} X E_{1} E_{2} E_{1} + \frac{1}{q^2 + q^{-2}} E_{2} E_{1}^{(2)} X E_{1}^{(2)} E_{2}; \]
\[ E_{1} E_{2} X E_{2} E_{1}^{(3)} = \]
\[ = E_{1} E_{2} E_{1} X E_{2} E_{1}^{(2)} = \frac{1}{[2]_{q}} E_{1} E_{2} E_{1} X E_{1} E_{2} E_{1} + \frac{1}{[3]_{q}} E_{1} E_{2} E_{1}^{(2)} E_{2}; \]
\[ E_{1}^{(3)} E_{2} X E_{2} E_{1} = \]
\[ = E_{1}^{(2)} E_{2} X E_{1} E_{2} E_{1} = \frac{1}{[2]_{q}} E_{1} E_{2} E_{1} X E_{1} E_{2} E_{1} + \frac{1}{[3]_{q}} E_{1} E_{2}^{(2)} X E_{1} E_{2} E_{1}; \]
\[ E_{1}^{(4)} E_{2} X E_{2} = \]
\[ = E_{1}^{(2)} E_{2} X E_{1}^{(2)} E_{2} = \frac{2}{[3]_{q}} E_{1} E_{2} E_{1} X E_{1}^{(2)} E_{2} + \frac{1}{q^2 + q^{-2}} E_{2} E_{1}^{(2)} X E_{1}^{(2)} E_{2}. \]
Proof: It is a matter of manipulation of the left hand sides using the relation

\[ \sum_{r=0}^{3} (-1)^r E_1^{(r)}E_2 E_1^{(3-r)} = 0. \]

Corollary 4.4.8

Let \( X \) be an element of \( U_q \) commuting with \( E_1 \) and fixed by \( T_1 \). Then

\[ \sum_{r=0}^{4} (-1)^r q^{-r} E_1^{(r)}E_2 X E_2 E_1^{(4-r)} = T_1^{-1}(E_2 X E_2). \]

Proof: The claim is a consequence of lemma 4.4.7 once that one remarks that

\[ T_1^{-1}(E_2) = E_2 E_1^{(2)} - q^{-1} E_1 E_2 E_1 + q^{-2} E_2 E_1^{(2)} E_2; \]

indeed

\[ \sum_{r=0}^{4} (-1)^r q^{-r} E_1^{(r)}E_2 X E_2 E_1^{(4-r)} = \]

\[ = E_2 E_1^{(2)} X E_2 E_1^{(2)} - \frac{[2]_q}{[3]_q} E_2 E_1^{(2)} X E_1 E_2 E_1 + \frac{1}{q^2 + q} E_2 E_1^{(2)} X E_1^{(2)} E_2 + \]

\[ - q^{-1} E_1 E_2 E_1 X E_2 E_1^{(2)} + \frac{1}{[2]_q} E_1 E_2 E_1 X E_1 E_2 E_1 - \frac{1}{[3]_q} E_1 E_2 E_1 X E_1^{(2)} E_2 + \]

\[ + q^{-2} E_2 E_1^{(2)} E_2 X E_2 E_1^{(2)} + \]

\[ - q^{-3} E_1^{(2)} E_2 X E_1 E_2 E_1 + \frac{1}{[2]_q} E_1 E_2 E_1 X E_1 E_2 E_1 - \frac{1}{[3]_q} E_2 E_1^{(2)} X E_1 E_2 E_1 + \]

\[ + q^{-4} E_1^{(2)} E_2 X E_1^{(2)} E_2 - \frac{1}{[2]_q} E_1 E_2 E_1 X E_1^{(2)} E_2 + \frac{1}{q^2 + q} E_2 E_1^{(2)} X E_1^{(2)} E_2 = \]

\[ = E_2 E_1^{(2)} X E_2 E_1^{(2)} - q^{-1} E_2 E_1^{(2)} X E_1 E_2 E_1 + q^{-2} E_2 E_1^{(2)} X E_1^{(2)} E_2 + \]

\[ - q^{-1} E_1 E_2 E_1 X E_2 E_1^{(2)} + q^{-2} E_1 E_2 E_1 X E_1 E_2 E_1 - q^{-3} E_1 E_2 E_1 X E_1^{(2)} E_2 + \]

\[ + q^{-2} E_2 E_1^{(2)} E_2 X E_1 E_2 E_1 + q^{-4} E_2 E_1^{(2)} X E_1^{(2)} E_2 = \]

\[ = (E_2 E_1^{(2)} - q^{-1} E_1 E_2 E_1 + q^{-2} E_2 E_1^{(2)} X (E_2 E_1^{(2)} - q^{-1} E_1 E_2 E_1 + q^{-2} E_1^{(2)} E_2) = \]

\[ = T_1^{-1}(E_2 X E_2). \]

We can now conclude:

Proposition 4.4.9

\( \varphi_1 \) commutes with \( T_1 \).

Proof: As already remarked, we want to prove that

\[ T_1^{-1}(E_{\delta-2\alpha_1}) = \sum_{r=0}^{4} (-1)^r q^{-r} E_1^{(r)}E_{\delta-2\alpha_1} E_1^{(4-r)}; \]
following the notations of definition 4.4.3 and the result of corollary 4.4.6

\[ E_{\delta-2\alpha_1} = \sum_{r=0}^{2} (-1)^r q^{-2r} E_2^{(r)} X_2 E_2^{(2-r)} \quad \text{with} \quad [X_2, E_1] = 0 \quad \text{and} \quad T_1(X_2) = X_2; \]

then corollary 4.4.8 implies the claim.

To sum up, we can state the following proposition, which makes the importance of the present section explicit: indeed proposition 4.4.10 allows us to read in \(A_{2n}^{(2)}\) all the relations found in section 3 for \(A_2^{(2)}\); at the same time it allows to define \(E_{(m,1)} \forall m > 1\).

**Proposition 4.4.10**

Let \(\nu\) be the group homomorphism \(\nu : Q(A_2^{(2)}) \to Q(A_{2n}^{(2)})\) defined in the obvious way by \(\nu(\alpha_1) = \alpha_1, \nu(\delta) = \delta\) and let the injection \(\tilde{\nu} : \Phi_+(A_2^{(2)}) \to \Phi_+(A_{2n}^{(2)})\) be given by:

\[ \tilde{\nu}(\alpha) = \begin{cases} \nu(\alpha) & \text{if } \alpha \text{ is real} \\ (\nu(\alpha), 1) & \text{if } \alpha \text{ is imaginary}; \end{cases} \]

Then \(\varphi_1\) has the property that

\[ \varphi_1(E_\alpha) = E_{\tilde{\nu}(\alpha)} \quad \forall \alpha \in \Phi_+(A_{2n}^{(2)}). \]

Moreover \(\varphi_1(\tilde{E}_\alpha) = \tilde{E}_{\tilde{\nu}(\alpha)} \quad \forall \alpha \in \Phi_+^{im}(A_2^{(2)}).\]

**Proof:** The claim for \(\tilde{E}_\alpha\) when \(\alpha \in \Phi_+^{im}(A_2^{(2)})\) follows from the assertion on the real roots. Then \(E_{(m,1)} \in \mathcal{U}_q(A_{2n}^{(2)})\) can be defined \(\forall m > 0\) (see the analogous definition, corollary 2.2.3) and the claim for \(E_{m\delta}\) is also obvious.

For the (real) roots of the form \(m\delta - \alpha_1\) the claim follows from propositions 4.4.1 and 4.4.9, and from remark 2.1.5; on the other hand, since \(E_{\delta-2\alpha_1} \in \mathcal{U}_q^{(1)}\), \(T_{\omega_i}(E_{\delta-2\alpha_1}) = E_{\delta-2\alpha_1} \forall i \in I_0 \setminus \{1\};\) thus

\[ E_{(2n+1)\delta-2\alpha_1} = (T_{\omega_1} \cdot \ldots \cdot T_{\omega_n})^m T_{w_n}^{m-1}(E_0) = T_{\omega_1}^m(E_{\delta-2\alpha_1}) = \varphi_1(T_0 T_1)^m(E_0) \]

and analogously

\[ E_{(2n+1)\delta+2\alpha_1} = (T_{\omega_1} \cdot \ldots \cdot T_{\omega_n})^{-m-1} T_{w_n}^{m-1} T_0(E_0) = T_{\omega_1}^{-m-1}(-F_{\delta-2\alpha_1} K_{\delta-2\alpha_1}) = \varphi_1(T_0 T_1)^{-m-1} T_0(E_0), \]

which concludes the proof.

\[ \Box \]

**5. RELATIONS BETWEEN \(\mathcal{U}_q^{(i)}\) AND \(\mathcal{U}_q^{(j)}\) \((i \neq j)\).**

In this section we shall complete the description of the commutation relations among the root vectors. The basis for this section is, as for section 2, an application of Beck’s work (with something more to check in the twisted case: see §5.2); the argument works also in the case \(A_{2n}^{(2)}\).
In particular the aim of this part of the work is to understand the commutation rules between the vectors from $U_q^{(i)}$ and $U_q^{(j)}$ when $i \neq j$.

**Remark 5.1.**
$\forall i, j \in I_0$ such that $a_{ij} = 0$ we have $[U_q^{(i)}, U_q^{(j)}] = 0$.

**Proof:** $[E_i, \Omega^r(E_{\bar{d}, \delta - \alpha_j})] = T_{\lambda_j}([E_i, \Omega^T^{-1}(E_j)]) = 0$. 

Hence we are reduced to the case when $a_{ij} < 0$; recall also that, if this is the case, then either $a_{ij} = -1$ or $a_{ji} = -1$.

§5.1. First level of commutation.

In this paragraph we sketch Beck’s recursive approach to the problem.

**Lemma 5.1.1**
Let $i \neq j \in I_0$. Then $\forall r, s \in \mathbb{N}$ the following relations hold:

$$[E_{r\bar{d}, \delta + \alpha_i}, F_{s\bar{d}, \delta + \alpha_j}] = 0 \quad \text{and} \quad E_{r\bar{d}, \delta + \alpha_i} E_{(s+1)\bar{d}, \delta - \alpha_j} = q_i^{a_{ij}} E_{(s+1)\bar{d}, \delta - \alpha_j} E_{r\bar{d}, \delta + \alpha_i}.$$ 

**Proof:** It is enough to apply $T_{\lambda_i}^{-r} T_{\lambda_j}^{-s}$ and $T_{\lambda_i}^{-r} T_{\lambda_j}^{s+1}$ to the relation $[E_i, F_j] = 0$.

**Lemma 5.1.2**
Let $i, j \in I_0$ be such that $a_{ij} = -1$ (then $d_{ij} \in \mathbb{N}$). If $r \geq 1$, we have:

$$[\tilde{E}_{(r\bar{d}, \delta, i)}, E_j] = \begin{cases} -E_{d, \delta + \alpha_j} & \text{if } r = 1 \\ -\frac{d_{ij}}{q_i} T_{\lambda_j}^{-1} (q_i E_i \tilde{E}_{((r-1)\bar{d}, \delta, i)} - q_i^{-1} \tilde{E}_{((r-1)\bar{d}, \delta, j)} E_j) & \text{if } r > 1 \end{cases}$$

and

$$[\tilde{E}_{(r\bar{d}, \delta, i)}, F_j] = \begin{cases} -K_j E_{d, \delta - \alpha_j} & \text{if } r = 1 \\ \frac{d_{ij}}{q_i^{-1}} K_{\bar{d}, \delta} T_{\lambda_j}^{(s)} (q_i^{-1} F_j \tilde{E}_{((r-1)\bar{d}, \delta, i)} - q_i \tilde{E}_{((r-1)\bar{d}, \delta, j)} F_j) & \text{if } r > 1 \end{cases}$$

similarly, if $r \geq \frac{d_{ij}}{q_i}$, we get:

$$[\tilde{E}_{(r\bar{d}, \delta, j)}, E_i] = \begin{cases} [a_{ij}] q_i E_{\bar{d}, \delta + \alpha_i} & \text{if } r = \frac{d_{ij}}{q_i} \\ T_{\lambda_i}^{-1} (q_j^{-a_{ji}} E_i \tilde{E}_{((r\bar{d}, \delta - \alpha_i), \delta, j)} - q_j^{a_{ji}} \tilde{E}_{((r\bar{d}, \delta - \alpha_i), \delta, j)} E_i) & \text{if } r > \frac{d_{ij}}{q_i} \end{cases}$$

and

$$[\tilde{E}_{(r\bar{d}, \delta, j)}, F_i] = \begin{cases} [a_{ij}] q_j K_i E_{\bar{d}, \delta - \alpha_i} & \text{if } r = \frac{d_{ij}}{q_i} \\ K_{\bar{d}, \delta} T_{\lambda_i} (q_j^{a_{ji}} F_i \tilde{E}_{((r\bar{d}, \delta - \alpha_i), \delta, j)} - q_j^{-a_{ji}} \tilde{E}_{((r\bar{d}, \delta - \alpha_i), \delta, j)} F_i) & \text{if } r > \frac{d_{ij}}{q_i} \end{cases}.$$ 

**Proof:** The result on which this lemma is based on is that $T_{\lambda_i} T_{\lambda_j}^{-1}(E_j) = T_{\lambda_j} T_{\lambda_i}^{-1}(E_j)$ and $T_{\lambda_i} T_{\lambda_j}^{-1}(F_j) = T_{\lambda_j} T_{\lambda_i}^{-1}(F_j)$ (see [1] and point 4) of lemma 2.2.1): indeed notice that if $a_{ij} = -1$ then $(X_n^{(k)}, i) \neq (A_n^{(2)}, 1)$. The fact that $\frac{d_{ij}}{q_i} \in \mathbb{N}$ is
a consequence of the following simple remark: we have either $\tilde{d}_i = 1 \, \forall i \in I_0$ or $\tilde{d}_i = d_i \, \forall i \in I_0$, so that $\frac{d_i}{d_j}$ is either 1 or $\frac{d_i \omega_{ji}}{d_j} = -a_{ji}$. \(\square\)

§5.2. Some particular computations: low twisted cases.

What happens of $[\hat{E}_{(r\delta,j)}], E_i]$ and $[\hat{E}_{(r\delta,j)}, F_i]$ when $1 \leq r < \frac{d_i}{d_j}$ (and $i, j \in I_0$ are such that $a_{ij} = -1$) is the matter of the present paragraph; of course this situation occurs neither for the non twisted algebras, nor for $A_{2n}^{(2)}$, since in these cases we have $\tilde{k} = 1$. Hence we have now to figure out what happens in cases $A_{2n-1}^{(2)}, D_{n+1}^{(2)}, E_6^{(2)}$ and $D_{4}^{(3)}$ (when $1 \leq r < \frac{d_i}{d_j}$).

The aim of this paragraph is to prove that the brackets above are zero.

The problem will be first reduced to compute the commutation of $F_j$ with a suitable element lying in $\mathcal{U}_q^+$; this simple remark will immediately solve the case $D_{n+1}^{(2)}$ (see proposition 5.2.2).

In the other cases the question will be further modified in looking for an element of the Weyl group with “good” properties (see remark 5.2.3 and lemma 5.2.4); the last part of the paragraph will be devoted to a closer and more detailed investigation of $A_{2n-1}^{(2)}, D_{4}^{(3)}$ and $E_6^{(2)}$.

Remark 5.2.1.

Let $i, j \in I_0$ be such that $a_{ij} = -1 \neq a_{ji}$ and let $r$ be such that $1 \leq r < \frac{d_i}{d_j}$ (notice that $\tilde{d}_j = d_j = 1$, $d_i = k > 1$ and $\lambda_j = \omega_j = \tilde{\omega}_j$); we want to prove that $[\hat{E}_{(r\delta,j)}, E_i] = 0$ and $[\hat{E}_{(r\delta,j)}, F_i] = 0$.

Since, by a simple manipulation depending on $a_{ij} = -1$ (which implies that $-E_j E_i + q_{ij}^{-1} E_j E_i = T_i^{-1}(E_j)$), and on the definition of $\hat{E}_{(r\delta,i)}$, we see that

$$[\hat{E}_{(r\delta,j)}, E_i] = -T_i^{-1} T_j^{-1}([T_j T_i T_{\omega_j}]^{-1}(E_j), F_j) K_j$$

and

$$[\hat{E}_{(r\delta,j)}, F_i] = -q_i K\alpha_i - \alpha_j T_i^{-1}([T_j T_i T_{\omega_j}]^{-1}(E_j), F_j),$$

the problem reduces to prove that $[T_j T_i T_{\omega_j}]^{-1}(E_j), F_j] = 0$.

Remark that from $l(s_i s_j \omega_j) = l(\omega_j) + 2$ we get $T_j T_i T_{\omega_j}^{-1}(E_j) \in \mathcal{U}_q^{+,r\delta-s_i(\alpha_1+\alpha_2)}$.

Proposition 5.2.2

Let $X_n^{(k)} = D_{n+1}^{(2)}$ and let $i, j \in I_0$ be such that $a_{ij} = -1 \neq a_{ji}$ (this means that $i = 2, j = 1$).

Then we have that $[\hat{E}_{(\delta,j)}, E_i] = 0$ and $[\hat{E}_{(\delta,j)}, F_i] = 0$.

Proof: Thanks to remark 5.2.1 we need to show that $[T_j T_i T_{\omega_i}]^{-1}(E_j), F_i] = 0$, where $T_j T_i T_{\omega_i}^{-1}(E_i) \in \mathcal{U}_q^{+,r\delta-s_i(\alpha_1+\alpha_2)}$. But, since $\delta = \sum_{r=0}^{n} a_r$ and $a_{12} = -2$,

$$\delta - s_i(\alpha_1 + \alpha_2) = \delta - (\alpha_1 + \alpha_2) = a_0 + \sum_{r=3}^{n} a_r,$$

so that $T_j T_i T_{\omega_i}^{-1}(E_i)$ belongs to the subalgebra of $\mathcal{U}_q^{+}$ generated by $\{E_r | r \neq 1\}$, which immediately implies that it commutes with $F_1$ (since $E_r$ does for $r \neq 1$). \(\square\)

Remark 5.2.3.
Remark that for an element \( x \in U_q^+ \) the property \([x, F_j] = 0\) is equivalent to the condition \( T_j^{±1}(x) \in U_q^+ \) (see [27]); since for any \( i, j \in I_0 \) and \( r > 0 \) we obviously have that \( T_j^{-1}T_jT_j^rT_j^{-1}(E_j) = T_jT_j^rT_j^{-1}(E_j) \in U_q^+ \), remark 5.2.1 allows us to translate the problem of understanding the behaviour of \([\dot{E}_{(r,\delta,j)}, E_i] \) and \([\dot{E}_{(r,\delta,j)}, F_i] \) (when \( 1 \leq r < d_j \) and \( i, j \in I_0 \) are such that \( a_{ij} = -1 \neq a_{ji} \)) into the following question: is \( T_j^2T_j^rT_j^{-1}(E_j) \) an element of \( U_q^+ \)?

The strategy that we shall use to prove that the answer is “yes” is to find an element \( w \) of the Weyl group \( \dot{W} \) and an element \( t \in I \) with the property that \( l(s_jw^t_t) = l(w) + 2 \) and that \( T_j^2T_j^rT_j^{-1}(E_j) = T_js_jw^t_t(E_i) \).

The next lemma illustrates more precisely what we are looking for.

**Lemma 5.2.4**
Fix \( i \neq j \in I_0 \) and \( r > 0 \), and let \( w \in \dot{W} \) satisfy the following conditions:
1) \( l(w^{-1}s_js_i\omega_j^1s_js_j) = l(s_js_i\omega_j^1s_js_j) - l(w); \)
2) \( w^{-1}s_js_i\omega_j^1s_j(\alpha_j) \) is a simple root;
3) \( w^{-1}(\alpha_j) > 0. \)

Then \( T_j^2T_j^rT_j^{-1}(E_j) \in U_q^+ \).

**Proof:** The first hypothesis means that \( T_jT_j^rT_j^{-1} = T_wT_w^{-1}s_is_j\omega_j^1s_j \) and the third means that \( T_jT_w = T_wT_j \).

Then we have: \( T_j^2T_j^rT_j^{-1}(E_j) = T_js_jwT_w^{-1}s_is_j\omega_j^1s_j(E_j) = T_js_jw(E_j) \), where \( t \) is such that \( w^{-1}s_js_i\omega_j^1s_j(\alpha_j) = \alpha_i; \) hence the only thing to prove is that \( s_jw(\alpha_i) > 0, \)
which is obvious since \( s_jw(\alpha_i) = s_i\omega_j^1s_j(\alpha_j) = r\delta - (\alpha_i + \alpha_j) \).

The aim of the next lemmas is now to find an element \( w \) with the properties stated in lemma 5.2.4: in order to exhibit such an element we need to understand something about \( \omega_j \).

**Definition 5.2.5.**
Consider the case of \( A_{2n-1}^{(2)} \). Let \( \tau \) be the Dynkin diagram automorphism defined by
\[
\tau(t) \doteq \begin{cases} 
  t & \text{if } 1 \leq t \leq n - 1 \\
  0 & \text{if } t = n \\
  n & \text{if } t = 0 
\end{cases}
\]
and define elements \( y_r \in \dot{W} \) (for \( r = 1, \ldots, n \)) as follows: \( y_r \doteq \tau s_n s_{n-1} \cdots s_1 \).

**Lemma 5.2.6**
In \( A_{2n-1}^{(2)} \) the following assertions hold:
a) if \( 1 \leq s \leq r \leq n - 1 \) and \( r < i \leq n \) or \( i = 0 \), then
\[
y_{s+1} \cdots y_r(\alpha_i) = \begin{cases} 
  \alpha_{i-r+s} & \text{if } i \neq 0 \\
  \alpha_0 + \sum_{t=1}^{r-1}(\alpha_{n-t} + \alpha_{n-t+1}) & \text{if } i = 0; 
\end{cases}
\]
b) \( \forall r = 2, \ldots, n; \Phi_+(y_r) = \{\tau(\alpha_i + \cdots + \alpha_n) | r \leq i \leq n\} \);
c) if \( 1 \leq r < i \leq n \) we have
\[
y_2 \cdots y_r(\tau(\alpha_i + \cdots + \alpha_n)) = \delta - (\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{i-r}) + \alpha_{i-r+1} + \cdots + \alpha_{n-r+1});
\]
d) if \( 1 \leq r < i \leq n \) we have
\[
s_2\omega_2^{-1}s_1s_2y_2 \cdots y_r(\tau(\alpha_i + \cdots + \alpha_n)) =
\]
\[\begin{align*}
\alpha_2 & \quad \text{if } i - r = 1 = n - r \\
-\delta - (\alpha_1 + \ldots + \alpha_{n-r+1}) & \quad \text{if } i - r = 1 < n - r \\
-\omega - (\alpha_3 + \ldots + \alpha_{n-r+1}) & \quad \text{if } i - r = 2 \\
-\delta - (\alpha_1 + 2(\alpha_2 + \ldots + \alpha_{i-r}) + \alpha_{i-r+1} + \ldots + \alpha_{n-r+1}) & \quad \text{if } i - r > 2.
\end{align*}\]

**Proof:** Statement a) is easily proved by induction on \( r - s \).

For b) one immediately shows that \( y_i^{-1}(\tau(\alpha_1 + \ldots + \alpha_n)) = s_i \ldots s_n(\alpha_1 + \ldots + \alpha_n) < 0 \) if \( r \leq i \leq n \), which means that \( \{\tau(\alpha_1 + \ldots + \alpha_n)\}_{r \leq i \leq n} \subseteq \Phi_+(y_i) \); but \( \#\Phi_+(y_i) = l(y_i) \leq n - r + 1 \), which proves the claim.

Assertion c) follows from a) remarking that \( \tau(\alpha_1 + \ldots + \alpha_n) = \alpha_1 + \ldots + \alpha_{n-1} + \alpha_0 \)

and \( y_2 \ldots y_r(\alpha_0) = \delta - (\alpha_1 + 2(\alpha_1 + \ldots + \alpha_{n-r}) + \alpha_{n-r+1}) \).

d) is just \( s_2 s_1^{-1} s_1 s_2 \) applied to c).

**Proposition 5.2.7**

Let \( X_n^{(k)} = A_{2n-1}^{(2)} \) and let \( i, j \in I_0 \) be such that \( a_{ij} = -1 \neq a_{ji} \) (this means that \( i = 1, j = 2 \)).

Then we have:

1. \( l((y_2 \ldots y_{n-1})^{-1}s_2 s_1 \omega s_2 s_2) = l(s_2 s_1 \omega s_2 s_2) - l(y_2 \ldots y_{n-1}) \);
2. \( (y_2 \ldots y_{n-1})^{-1}s_2 s_1 \omega s_2 s_2(\alpha_2) = 0 \);
3. \( (y_2 \ldots y_{n-1})^{-1}(\alpha_2) > 0 \).

In particular \( T_2^2 T_1 T_2 T_2^{-1}(E_2) \in U_1^\omega \), so that \( [\hat{E}(\delta, 2), E_1] = 0 \) and \( [\hat{E}(\delta, 2), F_1] = 0 \).

**Proof:** To prove 1) it is enough to show that \( \Phi_+(y_r) \subseteq \Phi_+((y_2 \ldots y_{n-1})^{-1}s_2 s_1 \omega s_2) \) \( \forall r = 2, \ldots, n - 1 \), and this is a straightforward consequence of points b) and d) of proposition 5.2.6.

From point a) of lemma 5.2.6 we see also that \( y_2 \ldots y_{n-1}(\alpha_0) = \delta - (\alpha_1 + \alpha_2) = s_2 s_1 \omega s_2(\alpha_2) \), which is 2), and \( y_2 \ldots y_{n-1}(\alpha_0) = \alpha_2 \), which implies 3).

**Proposition 5.2.8**

Consider the case \( D_4^{(2)} \) and let \( i, j \in I_0 \) be such that \( a_{ij} = -1 \neq a_{ji} \); this means that \( i = 2 \) and \( j = 1 \). Then \( [\hat{E}(\delta, 2), E_2] = 0 \) and \( [\hat{E}(\delta, 2), F_2] = 0 \) for \( r = 1, 2 \).

**Proof:** First of all remark that \( \delta = \alpha_0 + 2 \alpha_1 + \alpha_2 \), hence \( s_1 s_2 \omega s_1(\alpha_1) = \alpha_0 \), so that proposition 5.2.4 applies with \( w = \text{id} \); it follows from remark 5.2.3 that \( [\hat{E}(\delta, 1), E_2] = 0 \) and \( [\hat{E}(\delta, 1), F_2] = 0 \).

Consider now the element \( s_0 s_1 s_2 \); then

1. \( l((s_0 s_1 s_2)^{-1}s_2 s_1 \omega^2 s_1) = l(s_1 s_2 \omega^2 s_1) - 3 \); indeed

\[s_1 \omega_1 s_2 s_1, s_1 \omega_1 s_2 s_1(\alpha_0 + \alpha_1) \text{ and } s_1 \omega_1 s_2 s_1(3 \alpha_0 + 3 \alpha_1 + \alpha_2)\]

are negative roots (they are respectively \(-\delta + \alpha_1, -3 \delta - (\alpha_1 + \alpha_2) \) and \(-3 \delta - \alpha_2)\);

2. \( (s_0 s_1 s_2)^{-1}s_1 s_2 \omega^2 s_1(\alpha_1) = \alpha_1 \) which is a simple root;

3. \( (s_0 s_1 s_2)^{-1}(\alpha_0) = \alpha_0 > 0 \).

As above, remark 5.2.3 and lemma 5.2.4 apply to the present situation and imply that \( [\hat{E}(\delta, 1), E_2] = 0 \) and \( [\hat{E}(\delta, 1), F_2] = 0 \).

**Proposition 5.2.9**

Consider the case \( E_6^{(2)} \) and let \( i, j \in I_0 \) be such that \( a_{ij} = -1 \neq a_{ji} \); this means that \( i = 3, j = 2 \). Then \( [\hat{E}(\delta, 2), E_3] = 0 \) and \( [\hat{E}(\delta, 2), F_3] = 0 \).
Proof: Consider the element \( w = s_0 s_1 s_2 s_3 s_2 s_1 s_0 s_2 s_3 s_2 s_1 s_2 s_3 s_2 \); then, through direct computations very similar (even though longer!) to those used for \( D_4^{(3)} \) (see the proof of proposition 5.2.8, 1), 2), 3)), one sees that:
1) \( l(w^{-1} s_2 s_3 \omega_2 s_2) = l(s_2 s_3 \omega_2 s_2) - l(w) \);
2) \( s_2 s_3 \omega_2 \alpha_2 = w(\alpha_2) \);
3) \( w^{-1}(\alpha_2)(= \alpha_0) > 0 \).
The claim follows. □

§5.3. General commutation formulas.
We are now ready to discuss the commutation relations, as announced at the beginning of this section. Before giving them explicitly, let us introduce some definitions and notations.

Definition 5.3.1.
In the twisted case different from \( A^{(2)}_{2n} \) let us define a matrix \( A^s = (a^s_{ij})_{ij \in I_0} \) (called the simply laced matrix associated to \((I, A)\)) as follows:
\[
a^s_{ii} = 2 \quad a^s_{ij} = 0 \quad \text{if} \quad a_{ij} = 0 \quad a^s_{ij} = -1 \quad \text{if} \quad a_{ij} < 0.
\]
This can be said equivalently by:
\[
\begin{cases}
a^s_{ii} = 2 \\
a^s_{ij} = 0 & \text{if } a_{ij} = 0 \\
a^s_{ij} = -1 & \text{if } a_{ij} < 0.
\end{cases}
\]
Remark that \((I_0, A^s)\) is a Kac-Moody datum of type \( A_n \).

Define also \( b_{ij} \) by:
\[
\begin{align*}
b_{ij} &= \begin{cases} 1 & \text{if } a_{ij} \geq -1 \\ d_j & \text{otherwise.} \end{cases}
\end{align*}
\]

Theorem 5.3.2
Let \( i, j \in I_0 \) and \( r, s > 0 \) be such that \( \tilde{d}_i | r \) and \( \tilde{d}_j | s \); then we have:
1) \( [E(r\delta, i), E(s-\tilde{d}_j)\delta+\alpha_j] = \begin{cases} x_{ij} E(s-\tilde{d}_j+r)\delta+\alpha_j & \text{if } \tilde{d}_j | r \\ 0 & \text{otherwise;} \end{cases} \)
2) \( [E(r\delta, i), E(s)\delta-\alpha_j] = \begin{cases} -x_{ij} E(s+r)\delta-\alpha_j & \text{if } \tilde{d}_j | r \\ 0 & \text{otherwise;} \end{cases} \)
3) \( [E(r\delta, i), F(s-\tilde{d}_j)\delta+\alpha_j] = \begin{cases} x_{ij} K(s-\tilde{d}_j)\delta+\alpha_j E(r-s+\tilde{d}_j)\delta-\alpha_j & \text{if } \tilde{d}_j | r \text{ and } r \geq s \\ -x_{ij} K_{r\delta} F(s-r-\tilde{d}_j)\delta+\alpha_j & \text{if } \tilde{d}_j | r \text{ and } r < s \\ 0 & \text{otherwise;} \end{cases} \)
4) \( [E(r\delta, i), F(s)\delta-\alpha_j] = \begin{cases} -x_{ij} E(r-s)\delta+\alpha_j K_{\alpha_j-s\delta} & \text{if } \tilde{d}_j | r \text{ and } r \geq s \\ x_{ij} F(s-r)\delta-\alpha_j K_{r\delta}^{-1} & \text{if } \tilde{d}_j | r \text{ and } r < s \\ 0 & \text{otherwise;} \end{cases} \)
(5) \[ [E_{(r\delta,i)}, E_{(s\delta,j)}] = 0; \]

(6) \[ [E_{(r\delta,i)}, F_{(s\delta,j)}] = \delta_{rs} x_{ijr} \frac{K_{r\delta} - K_{s\delta}^{-1}}{q_j - q_j^{-1}}; \]

where

\[ x_{ijr} = \begin{cases} 
(o(i)o(j))^{r\ell |ra_{ij}|_q} & \text{in the non twisted case and} \\
\frac{[2r]_q}{r} (q^{2r} + (-1)^{r-1} + q^{-2r}) & \text{in case } A_{2n}^{(2)} \text{ with } (i,j) \neq (1,1) \\
(o(i)o(j))^{\frac{r}{|ra_{ij}|_q}} d_{ij}^{\frac{|ra_{ij}|_q}{r_d}} & \text{otherwise}
\end{cases} \]

and \( o : I_0 \rightarrow \{ \pm 1 \} \) is such that \( a_{ij} < 0 \Rightarrow o(i)o(j) = -1. \)

**Proof:** For \( i = j \), the claim is nothing but proposition 2.2.4 and \( \varphi_1 \) applied to 3.5.3 and 3.5.4.

For \( a_{ij} = 0 \), the claim follows from remark 5.1.

For \( a_{ij} < 0 \), (1) and (2) are found as an application of lemma 5.1.2 and of the results of paragraph 5.2 by a standard argument: see [1] and proposition 3.5.2; (3) and (4) are found applying a suitable power of \( T_{\lambda} \), respectively to (2) and (1); (5) follows immediately from (1) and (2) while (6) follows from (3) and (4) (using proposition 2.2.4,2)). \( \square \)

6. A PBW-BASE.

An important and useful consequence of the commutation relations given in the preceding section is the construction of a PBW-basis for the general quantum algebra of affine type. The aim of this section is thus to exhibit such a basis using a convex ordering of the root system, to recall its principal properties (such as the Levendorskii-Soibelman formula) and to introduce some notations.

§ 6.1. A convex ordering.

**Definition 6.1.1.**

We denote by \( \preceq \) the total ordering of \( \tilde{\Phi}_+ \) defined by:

\[ \forall r, s \in \mathbb{Z} : \ \beta_r \preceq \beta_s \iff \begin{cases} 
\begin{aligned}
s &\leq r &\leq 0 &\text{or} \\
1 &\leq s &\leq r,
\end{aligned}
\end{cases} \]

\[ \forall r \in \mathbb{Z}, \forall \alpha \in \tilde{\Phi}_+^{\text{im}} : \ \beta_r \preceq \alpha \iff r \leq 0, \]

\[ \forall (r\delta, i), (s\delta, j) \in \tilde{\Phi}_+^{\text{im}} : \ (r\delta, i) \preceq (s\delta, j) \iff \begin{cases} 
\begin{aligned}
r &> s &\text{or} \\
r &= s &\text{and } i \leq j.
\end{aligned}
\end{cases} \]

**Remark 6.1.2.**

\( \preceq \) is a convex ordering of \( \tilde{\Phi}_+ \) (see [2]).
Notation 6.1.3.
\( \forall \eta \in Q \) define \( \mathcal{P}(\eta) \) as follows:
\[
\mathcal{P}(\eta) \doteq \{ \gamma = (\gamma_1, \ldots, \gamma_r) \in \bigcup_{r \in \mathbb{N}} \Phi^+_r | \gamma_1 \leq \ldots \leq \gamma_r, \sum_{u=1}^{r} p(\gamma_u) = \eta \};
\]
define also \( \mathcal{P} \) to be \( \mathcal{P} \doteq \bigcup_{\eta \in Q} \mathcal{P}(\eta) \).

We denote by \( \bar{\mu} : \bigcup_{r \in \mathbb{N}} \Phi^+_r \rightarrow \mathcal{P} \) the natural (reordering) map; moreover \( \forall \alpha \in \Phi^+ \)
\( \mu^\alpha : \mathcal{P} \rightarrow \mathbb{N} (\gamma \mapsto \mu^\alpha(\gamma)) \) indicates the occurrence of \( \alpha \) in \( \gamma \); that is if \( \gamma = (\gamma_1, \ldots, \gamma_r) \)
then \( \mu^\alpha(\gamma) = \# \{ u = 1, \ldots, r | \gamma_u = \alpha \} \).

Definition 6.1.4.

By abuse of notation we denote by \( \preceq \) also the induced lexicographical ordering
of \( \mathcal{P} \): if \( \gamma = (\gamma_1, \ldots, \gamma_r) \) and \( \gamma' = (\gamma'_1, \ldots, \gamma'_r) \) we set
\[
\gamma \preceq \gamma' \Leftrightarrow \text{ if } l = \min \{ u = 1, \ldots, r | \gamma_u \neq \gamma'_u \} \text{ then } \gamma_l < \gamma'_l.
\]

§ 6.2. PBW-basis and L-S formula.

Notation 6.2.1.

Let \( x : \Phi^+ \rightarrow U_\eta \) (\( \alpha \mapsto x_\alpha \)) be any function. Then \( \gamma = (\gamma_1, \ldots, \gamma_r) \in \mathcal{P} \) we set
\[
x(\gamma) \doteq x_{\gamma_1} \cdot \ldots \cdot x_{\gamma_r}.
\]
We are now ready to state the main theorem of this section.

Theorem 6.2.2

The set \( \{ E(\gamma) | \gamma \in \mathcal{P} \} \) is a \( C(\eta) \)-basis of \( U^+_\eta \); more precisely \( \forall \eta \in Q \, \{ E(\gamma) | \gamma \in \mathcal{P}(\eta) \} \) is a \( C(\eta) \)-basis of \( U^+_\eta \), while more generally \( \{ E(\gamma)K_\lambda F(\gamma') | \lambda \in Q, \gamma, \gamma' \in \mathcal{P} \} \) is a \( C(\eta) \)-basis of \( U^+_\eta \).

Proof: The argument used in [26] applies here word for word.

The following theorem is the Levendorskii-Soibelman formula:

Theorem 6.2.3

1) \( \forall \alpha, \beta \in \Phi^+, \alpha \succ \beta \Rightarrow E_\alpha E_\beta - q^{p(\alpha)|p(\beta)} E_\beta E_\alpha = \sum_{\gamma \succ (\beta, \alpha)} a_\gamma(\gamma) \); 
2) \( \forall \gamma, \gamma' \in \mathcal{P} \), \( \gamma \succ \gamma' \Rightarrow E(\gamma)E(\gamma') - q^{p(\gamma')|p(\gamma)} E(\gamma')E(\gamma) = \sum_{\gamma \succ (\gamma', \gamma')} a_\gamma(\gamma) \),
where \( (\gamma, \gamma') \mapsto r(\gamma, \gamma') \) is the only function compatible with 1).

Proof: See [23] and [1].

§ 6.3. Some subspaces of \( U^+_\eta \).

In this paragraph we first list some notations and then give a characterization
of some subspaces of \( U^+_\eta \).

Notation 6.3.1.

\( \forall r \in \mathbb{Z} \cup \{ \pm \infty, \pm \infty, \text{im} \} \) we define a a subset \( \tilde{\Phi}^+_r(\cdot) \) of \( \Phi^+ \):
\[
\tilde{\Phi}^+_r(\cdot) = \begin{cases} 
\{ \beta_s | 1 \leq s < r \} & \text{if } r \geq 1 \\
\{ \beta_s | 0 \geq s > r \} & \text{if } r < 0 \\
\tilde{\Phi}^0_0 & \text{if } r = \text{im} \\
\cup_{s \geq 0} \tilde{\Phi}^s_+ & \text{if } r = \pm \infty \\
\tilde{\Phi}^0_+ \cup \tilde{\Phi}^\infty_+ & \text{if } r = \pm \infty;
\end{cases}
\]
moreover we define $\mathcal{U}^+_q(r)$ and $\mathcal{U}^{\geq 0}_q(r)$ by:

\[
\mathcal{U}^+_q(r) \doteq \text{span}_{(\gamma)} \{ E(\gamma) \mathbb{I}_2 = (\gamma_1, ..., \gamma_s) \in \mathcal{P}, \gamma_u \in \tilde{\Phi}_+ (r) \ \forall u = 1, ..., s \},
\]

\[
\mathcal{U}^{\geq 0}_q(r) \doteq \mathcal{U}^+_q(\mathcal{U}^+_q(r)).
\]

In order to describe $\mathcal{U}^+_q(r)$ we need the following lemma.

**Lemma 6.3.2**

$\forall \alpha \in \tilde{\Phi}_+^\im$ we have that

\[
T_{w^{-1}_r}(E_{\alpha}) \in \mathcal{U}^+_q \ \forall r \geq 1, \ T_{w^{-1}_r}(E_{\alpha}) \in \mathcal{U}^+_q \ \forall r \leq 0.
\]

**Proof:** Following [2], we have to prove that $\forall j \in I_0$ if $\lambda_j = \tau_j s_{j_1} \cdot ... \cdot s_{j_d}$ then $T_{j_1} \cdot ... \cdot T_{j_d}(E_{\alpha}) \in \mathcal{U}^+_q \ \forall r = 1, ..., d$; let $\alpha = (m \delta, i, \delta)$ with $i \in I_0$, $m > 0$.

If $i \neq j$ the claim is obvious; if $i = j$ and $(X^{(k)}_{\alpha}, i) = (A_{2n}^{(1)}, 1)$, see [2].

Finally we need to study the case when $(X^{(k)}_{\alpha}, i, j) = (A_{2n}^{(2)}, 1, 1)$: first of all remark that $s_d = 1$ and that $\forall m > 0$

\[
\varphi_1 \Xi(E_{md^{-\alpha}}) = \varphi_1 \Xi(T_0 T_1)^m T_1^{-1}(E_1) = \varphi_1 (T_0^{-1} T_1^{-1})^m T_1 T_0^{-1}(E_1) = \varphi_1 T_1 (T_0 T_1)^{-m} (E_1) = T_1 T_{\omega^{-1}_1}^{-m}(E_1)
\]

which belongs to $\mathcal{U}^+_q$; then

\[
T_1(E_{(md, 1)}) = \varphi T_1(E_{md}) = \varphi \Xi(E_{md}) = -E_{1} T_{1} T_{-\omega^{-1}_1}^{-m}(E_{1}) + q^{-2} T_{1} T_{-\omega^{-1}_1}^{-m}(E_{1}) E_{1} \in \mathcal{U}^+_q.
\]

The claim now follows remarking that $l(s_{j_1} \cdot ... \cdot s_{j_{d-1}, d}) = d$, that $m > 0$ and that

$T_{j_1} \cdot ... \cdot T_{j_{d-1}, d} T_{-\omega^{-1}_1} = (T_{j_1} \cdot ... \cdot T_{j_{d-1}, d})^{-1} \cdot$.

**Proposition 6.3.3**

$\forall r \in \mathbb{Z} \cup \{ \pm \infty, \pm \infty, \im \} \mathcal{U}^+_q (r)$ is a subalgebra of $\mathcal{U}^+_q$.

Moreover

\[
\mathcal{U}^+_q (\pm \infty) = \{ x \in \mathcal{U}_q^+ | \exists m > 0 \ s.t. \ T_{w^{-1}_r}(x) \in \mathcal{U}^{\leq 0}_q \} =
\]

\[
= \{ x \in \mathcal{U}_q^+ | T_{w^{-1}_r}(x) \in \mathcal{U}^{\leq 0}_q \ \forall m > 0 \},
\]

$\mathcal{U}^+_q (\pm \infty) = \{ x \in \mathcal{U}_q^+ | T_{w^{-1}_r}(x) \in \mathcal{U}^+_q \ \forall m > 0 \},$

$\mathcal{U}^+_q (\im) = \mathcal{U}_q^+(\im) \cap \mathcal{U}^+_q (-\im) = \{ x \in \mathcal{U}_q^+ | T_{\lambda_1}(x) = x \ \forall i \in I_0 \ \forall \alpha \in \tilde{\Phi}^\im_+ \}.$

**Proof:** The first assertion follows from L-S formula and from the convexity of $\preceq$; for the second it is enough to apply lemma 6.3.2 (see [2] and [6]), while the third follows immediately from the preceding ones, remarking that $T_{\lambda_1}(E_{\alpha}) = E_{\alpha} \ \forall i \in I_0 \ \forall \alpha \in \tilde{\Phi}^\im_+$. \hfill $\Box$

7. THE $R$-MATRIX.
Here it will be shown, applying to the present situation the strategy and the
results used for the non twisted quantum algebras (see [24] and [6]), how it is
possible to exhibit an explicit multiplicative formula for the $R$-matrix (denoted by
$R$) in the twisted case.

Hence most of this section will be devoted to follow the fundamental steps which
lead to the description of the $R$-matrix for the non twisted quantum algebras, to
recall the main properties used to establish them, and to remark how, thanks to
the results developed in the preceding parts of this paper, the same properties (and
consequently the same results) are still valid in this more general setting: we shall
refer to [22] and [6] for the proofs. Also, some minor differences occurring in the
 twisted case (mainly in case $A_{2n}^{(2)}$) will be discussed, and the computations needed
to complete the argument performed.

§7.1. General strategy for the construction of the $R$-matrix.

First of all we recall a very general result connecting the $R$-matrix to the Killing
form (see [31]).

**Theorem 7.1.1**

Let $C_\eta \in U^+_{q,\eta} \otimes U^-_{q,-\eta}$ (with $\eta \in Q^+$) be the canonical element of $(\cdot, \cdot)_{U^+_{q,\eta} \times U^-_{q,-\eta}}$. Then

$$R = \left( \sum_{\eta \in Q^+} C_\eta \right) q^{-t_{\infty}},$$

where $q^{-t_{\infty}}$ is the diagonal operator on $V \otimes \tilde{V}$ ($V$ and $\tilde{V}$ $U_q$-modules) which acts
as $q^{-(\eta|\eta)}$ on $V_{\eta} \otimes \tilde{V}_{\eta}$.

Then we recall the behaviour of the real root vectors with respect to the coprod-
uct and to the Killing form.

**Proposition 7.1.2**

If $\alpha = \beta \in \Phi^+_r$ then $\Delta(E_\alpha) - (E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha) \in \begin{cases} U^+_q \otimes U^+_q & \text{if } r \geq 1 \\ U^+_q \otimes U^+_q(r) & \text{if } r \leq 0. \end{cases}$

Moreover let $\gamma = (\gamma_1, ..., \gamma_r), \gamma' = (\gamma'_1, ..., \gamma'_s) \in \mathcal{P}$ be such that $\gamma_u, \gamma'_v \in \tilde{\Phi}_+(-\infty) \forall u, v$; then

$$(E(\gamma), F(\gamma')) = \delta_{\gamma, \gamma'} \prod_{\alpha \in \tilde{\Phi}_v} \frac{(\mu_\alpha^2)_{\alpha^1}}{(q_\alpha^2 - q_\alpha)^{\alpha^2}}.$$

**Proof:** The proof given in [22] and [6] depends only on the action of the braid
group on $U_q$ and on its connection with the coproduct $\Delta$ (that is on the construction
of "partial $R$-matrices" and on their properties), hence it applies word for word to
the present situation.

Before going to the next proposition we shall introduce a notation and give a simple lemma.

**Notation 7.1.3.**

We denote by $\det_r X^{(k)}$ the determinant of the matrix

$$\left( \begin{array}{ccc} r \left[d_{ij}^{(k)}\right] \\ d_{ij}^{(k)} \end{array} \right)_{ij \in I^r} = \frac{r}{\left[d_i^{(k)}\right]} \text{diag} \left( \begin{array}{c} d_i^{(k)} \end{array} \right)_{i \in I^r} \left(x_{ijr}^{(k)}\right)_{ij \in I^r}.$$
for the quantum algebra of type $X_n^{(k)}$.

**Lemma 7.1.4**

∀r > 0 we have $\det_r X_n^{(k)} \neq 0$ (equivalently $\det(x_{ij})_{i \in I, j \in r} \neq 0$).

More precisely

$$\det_r X_n^{(1)} = \begin{cases} [n + 1]q^r & \text{in case } A_n^{(1)} \\ [2]q^n [2]q^{r(2n + 1)} & \text{in case } B_n^{(1)} \\ [2]q^n [2]q^{r(n + 1)} & \text{in case } C_n^{(1)} \\ [2]q^n [2]q^{r(n + 1)} & \text{in case } D_n^{(1)} \\ [2]q^n [2]q^{r(n + 1)} & \text{in cases } E_0^{(1)} \text{ and } G_2^{(1)} \\ [2]q^n [2]q^{r(n + 1)} & \text{in case } E_7^{(1)} \\ [2]q^n [2]q^{r(n + 1)} & \text{in case } F_4^{(1)} \\ [2]q^n [2]q^{r(n + 1)} & \text{in case } A_{2n - 1}^{(2)} \text{ and } D_{n + 1}^{(2)} \text{ if } 2|r \\ [2]q^n [2]q^{r(n + 1)} & \text{in case } E_6^{(2)} \text{ if } 2|r \\ [2]q^n [2]q^{r(n + 1)} & \text{in case } D_4^{(3)} \text{ if } 3|r. \end{cases}$$

**Proof:** The proof consists in simple computations: in the non twisted case we have $\det_r X_n^{(k)} = \det([d_{ij}a_{ij}]_{q^r})_{i \in I}$: see [7] and [6]; the result when $k \not\in r$ follows from the remark that $\det_r X_n^{(k)} = \det([a_{ij}]_{q^r})_{i \in I}$ (and from the result for $A_n^{(1)}$). The remaining assertions follow considering that

$$\det_r A_{2n - 1}^{(2)} = (q^{2r} + (-1)^{r-1} + q^{-2r})[2]_{q^r}^{2n} \det_r A_{n - 1}^{(1)} - [2]_{q^r}^{2n} \det_r A_{n - 2}^{(1)},$$

$$1 < k|r, \ X_n^{(4)} = A_{2n - 1}^{(2)}, D_n^{(k)} \Rightarrow \det_r X_n^{(k)} = [2]_{q^r} \det_r A_{n - 1}^{(1)} - k \det_r A_{n - 2}^{(1)},$$

$$\det_r E_6^{(2)} = [2]_{q^r} \det_r D_3^{(2)} - \det_r A_2^{(1)}.$$

\[\square\]

**Proposition 7.1.5**

1) Let $\alpha \in \Phi^{\text{im}}_+$; then $\Delta(E_\alpha) - (E_\alpha \otimes 1 + K_{p(\alpha)} \otimes E_\alpha) \in U_q^\pm(\infty) \otimes U_q^\pm(-\infty)$;

2) Let $\alpha \in \Phi_+$ and $\gamma \in \mathcal{P}$ be such that $(E_\alpha, F(\gamma)) \neq 0$ or $(E(\gamma), F_\alpha) \neq 0$; then $\gamma = (\beta)$ with $p(\alpha) = p(\beta)$.

3) Let $\{E_\alpha|\alpha \in \Phi_+\}$ be elements of $U_q^+$ satisfying the following requirements:

i) $E_0 = E_\alpha \forall \alpha \in \Phi^{\text{im}}_+$;
\[\forall r > 0 \{ E_{(r, i)} | i \in I^r \} \text{ and } \{ E_{(r, i)} | i \in I^r \} \text{ are bases of the same } \mathbb{C}(q)\text{-vector subspace of } \mathcal{U}_q; \]

\[\text{iii) } (E_{(r, i)}, F_{(r, j)}) = 0 \text{ if } i \neq j \text{ (where } F_{\alpha} = \Omega(E_{\alpha}); \]

then \(\forall \gamma, \gamma' \in \mathcal{P}\) we have that

\[(E(\gamma), F(\gamma')) = \delta_{\gamma, \gamma'} \prod_{\alpha \in \Phi^+} \frac{\mu_{\alpha}!}{(q^{-1} - q_{\alpha})^{\mu_{\alpha}} \prod_{\alpha \in \Phi_+} \left(\frac{\mu_{\alpha}!}{q^{-1} - q_{\alpha}}\right)^{\mu_{\alpha}}}.\]

**Proof:** For the proof see [6]. Indeed the argument used in the non twisted case is based on the following properties, which are still valid in the twisted case:

\[a) \ T_{\lambda_1, \ldots, \lambda_n}(E_{\beta}) = E_{\beta+\lambda_1, \ldots, \lambda_n} \text{ if } r > 0 \text{ or } r \leq -N \text{ and } T_{\lambda_1, \ldots, \lambda_n}(E_{\alpha}) = E_{\alpha} \text{ if } \alpha \in \Phi^+_m; \]

more precisely see proposition 6.3.3;

\[b) \text{ the matrix } (x_{ij})_{ij} \text{ is non degenerate (see lemma 7.1.4);} \]

\[c) \prec \text{ is a convex ordering of } \Phi^+_+ \square. \]

**Corollary 7.1.6**

Let \(E_{\alpha}, F_{\alpha} (\alpha \in \Phi^+_+) \) be as in proposition 7.1.5; then

\[R = \left( \prod_{\alpha \in \Phi^+_+} \exp_{\alpha} \left( \frac{E_{\alpha} \otimes F_{\alpha}}{(E_{\alpha}, F_{\alpha})} \right) \right) q^{-\tau} = \left( \prod_{\alpha \in \Phi^+_+} \exp_{\alpha} \left( \frac{q_{\alpha} - q_{\alpha}^-}{c_{\alpha}} E_{\alpha} \otimes F_{\alpha} \right) \right) q^{-\tau} \]

where \(c_{\alpha} = \left( \frac{q_{\alpha} - q_{\alpha}^-}{\mu_{\alpha}} \right) \) if \(\alpha \in \Phi^+_+ \)

\[
\text{and } c_{\alpha} = \left( \frac{q_{\alpha} - q_{\alpha}^-}{\mu_{\alpha}} \right) \text{ if } \alpha \in \Phi^+_m. \]

**Proof:** See [22] and [6]. \(\square\)

**Remark 7.1.7.**

Remark that \(\forall r > 0\)

\[\prod_{i \in I^r} \exp_{(r, i)} \left( \frac{E_{(r, i)} \otimes F_{(r, i)}}{(E_{(r, i)}, F_{(r, i)})} \right) = \exp \left( \sum_{i \in I^r} \frac{E_{(r, i)} \otimes F_{(r, i)}}{(E_{(r, i)}, F_{(r, i)})} \right). \]

Hence proposition 7.1.6 can be rewritten as

\[R \prod_{m \leq 0} \exp_{\beta_m} \left( (q_{\beta_m} - q_{\beta_m}^-) E_{\beta_m} \otimes F_{\beta_m} \right) \prod_{r > 0} \exp_{C_r} \prod_{m \geq 1} \exp_{\beta_m} \left( (q_{\beta_m} - q_{\beta_m}^-) E_{\beta_m} \otimes F_{\beta_m} \right) q^{-\tau} \]

where \(C_r = \sum_{i \in I^r} \frac{E_{(r, i)} \otimes F_{(r, i)}}{(E_{(r, i)}, F_{(r, i)})} \) is the canonical element of the restriction of \(\langle , \rangle\) to

\[\text{span}_{\mathbb{C}(q)} \{ E_{(r, i)} | i \in I^r \} \times \text{span}_{\mathbb{C}(q)} \{ F_{(r, i)} | i \in I^r \}, \]

and both the first and the third products in the right hand side are performed in decreasing order. \(\square\)

**§7.2. The Killing form on the imaginary root vectors.**

We have seen how to reduce the explicit description of \(R\) to the construction of \(\mathcal{C}_m\) or equivalently to the construction of elements \(E_{\alpha}\) with the properties described in proposition 7.1.5, and to the computation of \((E_{\alpha}, F_{\alpha})\) for \(\alpha \in \Phi^+_m\). Of course for
both approaches the problem is that of understanding (\cdot, \cdot). Thus we start from the computation of \((E_{(r,\delta,i)}, F_{(r,\delta,j)})\), following the path indicated in [6] and remarking that in case \(A_{2n}^{(2)}\) it has to be slightly modified: the problem arises, here too (see lemma 2.2.1), from the fact that in this case \(\delta - 2\alpha_1\) is a root.

**Definition 7.2.1.**

Let \(i \in I_0\); we shall denote by \(\pi_i\) the natural projection \(\pi_i : U_q \otimes U_q^{\geq 0} \rightarrow U_q \otimes \mathbb{C}(q)E_i\) (with respect to the \(Q\)-gradation).

If \(r > 0\) and \(i, j \in I^r\) we define \(c_{ij}^r\) by the following property:

\[
\pi_j(\Delta(E_{(r,\delta,i)})) = c_{ij}^r E_{r\delta - \alpha_j} K_j \otimes E_j
\]

\((c_{ij}^r\) is well defined since \(\pi_j(\Delta(E_{(r,\delta,i)}))\) is indeed a multiple of \(E_{r\delta - \alpha_j} K_j \otimes E_j\); see [6], using proposition 7.1.5, 1) and the convexity of \(\prec\).

**Lemma 7.2.2**

Let \(r > 0\) and \(i, j \in I^r\); then \(E_{(r,\delta,i)}, F_{(r,\delta,j)}\) are \(-q_j^2(q_j - q_j^{-1})x_{ijr}\).

**Proof:** The proof is based on proposition 7.1.5, 2), on 1.4.2, 7.1), and on proposition 7.1.2 (see [6]).

**Lemma 7.2.3**

Let \(r > 0\) and \(i, j \in I^r\); then \(c_{ij}^r = q_j^{-2}(q_j - q_j^{-1})x_{ijr}\).

**Proof:** The proof consists in showing that \(c_{ij}^r = \frac{c_{ij}^r}{x_{jj1}} x_{jj1}\) and in computing \(c_{ij}^r\).

For \(c_{ij}^r = \frac{c_{ij}^r}{x_{jj1}} x_{jj1}\) see [6].

For \(c_{ij}^0 = q_j^{-2}(q_j - q_j^{-1})\) see [6], when we are not in the case \(A_{2n}^{(2)}\) with \(j = 1\).

So it remains to study \(c_{11}^1\) just in case \(A_{2n}^{(2)}\): remark that \(\pi_1(\Delta(E_{\delta - 2\alpha_1})) = 0\) (because \(\delta - 3\alpha_1 \notin \hat{\Phi}\)); then, using corollary 4.3.6, we have that

\[
\pi_1(\Delta(E_{\delta - \alpha_1})) = -E_{\delta - 2\alpha_1} K_1 \otimes E_1 + q^{-4} K_1 E_{\delta - 2\alpha_1} \otimes E_1 = -(1 - q^{-8})E_{\delta - 2\alpha_1} K_1 \otimes E_1;
\]

it follows that

\[
\pi_1(\Delta(E_{(\delta,1)})) = \pi_1(\Delta(E_{\delta - \alpha_1}))(E_1 \otimes 1) - q^{-2}(E_1 \otimes 1)\pi_1(\Delta(E_{\delta - \alpha_1})) + + E_{\delta - \alpha_1} K_1 \otimes E_1 - q^{-2} K_1 E_{\delta - \alpha_1} \otimes E_1 = (q^2 + 1 - q^{-4} - q^{-6})E_{\delta - \alpha_1} K_1 \otimes E_1,
\]

which means that \(c_{11}^1 = q^2 + 1 - q^{-4} - q^{-6} = q_1^{-2}(q_1 - q_1^{-1})x_{111}\).

**Corollary 7.2.4**

Let \(r > 0\) and let \(x\) (respectively \(y\)) belong to the linear span of \(\{E_{(r,\delta,i)}|i \in I^r\}\) (respectively \(\{F_{(r,\delta,i)}|i \in I^r\}\)).

Then \([x, y] = -(x, y)(K_{r\delta} - K_{r\delta}^{-1})\) and \((\Omega(y), \Omega(x)) = -\Omega((x, y))\).

**Proof:** The claim follows from lemmas 7.2.2 and 7.2.3, from theorem 5.3.2, from the bilinearity of \((\cdot, \cdot)\) and \([\cdot, \cdot]\) and from the fact that \(\Omega\) is an antiautomorphism.

\(\S 7.3\). A “canonical” form for the \(R\)-matrix.

Remark 7.1.7 reduces the problem of describing \(R\) to that of finding \(\tilde{C}_r\) \(\forall r > 0\).

The goal of the present paragraph, which can be achieved thanks to the results of \(\S 7.2\), is to describe \(\tilde{C}_r\) in terms of \(\{E_{(r,\delta,i)}, F_{(r,\delta,i)}|i \in I^r\}\).
The following is a simple lemma of linear algebra.

Lemma 7.3.1

Let \( r > 0 \) and let \( \tilde{C}_r = \sum_{i,j \in I^r} y^r_{ij} E_{(r \delta, i)} \otimes F_{(r \delta, j)} \). Then if \( y^r = \left( y^r_{ij} \right)_{ij \in I^r} \) and \( H^r = \left( (E_{(r \delta, i)}, F_{(r \delta, j)})_{ij \in I^r} \right) \) we have \( y^r = \left( H^r \right)^{-1} \).

Proposition 7.3.2

In the notation of lemma 7.3.1, if we put

\[
\tilde{d}_i^r = \frac{(o(i) o(j))^{r \delta, i} \tilde{d}_i}{r^{[d_1]} [d_j]_{q} (q^{-1} - q)} y^r_{ij} \text{ with } k'_r = \begin{cases} \frac{k}{r} & \text{if } k | r, X = D, E \\ 1 & \text{otherwise} \end{cases},
\]

we can describe \( z^r_{ij} \) as follows: of course \( z^r_{ij} \tilde{d}_j = z^r_{ji} \tilde{d}_i \) (because \( H^r \) is symmetric); if \( i \leq j \) then

\[
A_n^{(1)} \quad z^r_{ij} = \frac{[i]_{q^r} [n - j + 1]_{q^r}}{[n + 1]_{q^r}} \text{ if } i = 1
\]

\[
B_n^{(1)} \quad z^r_{ij} = \frac{[n-j+1]_{q^r}}{[2]_{q^r(2n-1)}} \text{ if } i = j = 1
\]

\[
C_n^{(1)} \quad z^r_{ij} = \frac{[n-j+1]_{q^r}}{[2]_{q^r(n+1)}} \text{ if } j > 1 = i
\]

\[
D_n^{(1)} \quad z^r_{ij} = \frac{[n]_{q^r}}{[2]_{q^r(n+1)}} \text{ if } i = j \leq 2
\]

\[
E_n^{(1)} \quad z^r_{ij} = \frac{[n-2]_{q^r}}{[2]_{q^r(n+1)}} \text{ if } i = 1, j = 2
\]

\[
F_n^{(1)} \quad z^r_{ij} = \frac{[n-j+1]_{q^r}}{[2]_{q^r(n+1)}} \text{ if } i \leq 2 < j
\]

\[
G_n^{(1)} \quad z^r_{ij} = \frac{[2]_{q^r(i-2)} [n-j+1]_{q^r}}{[2]_{q^r(n+1)}} \text{ otherwise}
\]
\[
E_{n}^{(1)} = \det E_{n}^{(1)} z_{ij}^{r} = \begin{cases} 
[n]_{q^{r}} & \text{if } i = j = 1 \\
[j-1]_{q^{r}} [n-3]_{q^{r}} & \text{if } i = 1 < j \leq 3 \\
[2]_{q^{r}} [2]_{q^{r}} [n-2] & \text{if } i = j = 2 \\
[i-1]_{q^{r}} [n-1]_{q^{r}} & \text{if } 2 \leq i \leq 3 = j \\
[3]_{q^{r}} [n-j+1]_{q^{r}} & \text{if } i = 1, j > 3 \\
[2]_{q^{r}} [i-1]_{q^{r}} [n-j+1]_{q^{r}} & \text{if } 2 \leq i \leq 4 \leq j \\
[3]_{q^{r}} [n-j+1]_{q^{r}} & \text{if } i = 5 \\
[2]_{q^{r}} [q^{r}]_{q^{r}} [n-j+1]_{q^{r}} & \text{if } i = 6 \\
[3]_{q^{r}} [2]_{q^{r}} [n-j+1]_{q^{r}} & \text{if } i = 7 \\
[2]_{q^{r}} [2]_{q^{r}} [n-j+1]_{q^{r}} & \text{if } i = 8 \\
\end{cases}
\]

\[
F_{4}^{(1)} z_{ij}^{r} = \begin{cases} 
[2]_{q^{r}} [2]_{q^{r}} [n]_{q^{r}} & \text{if } i = j = 1 \\
[2]_{q^{r}} [2]_{q^{r}} [5-j]_{q^{r}} & \text{if } i \leq 2 \leq j \\
[2]_{q^{r}} [3]_{q^{r}} [5-j]_{q^{r}} & \text{if } i = 3 \\
[2]_{q^{r}} [2]_{q^{r}} [q^{r}]_{q^{r}} & \text{if } i = j = 4 \\
\end{cases}
\]

\[
G_{2}^{(1)} (z_{ij}^{r})_{i,j \in \mathbb{I}_{0}} = \begin{cases} 
[2]_{q^{r}} [3]_{q^{r}} & \text{if } 2 \not| r \\
[2]_{q^{r}} [2]_{q^{r}} [3]_{q^{r}} & \text{if } 2| r \\
[2]_{q^{r}} [2]_{q^{r}} [2]_{q^{r}} [2]_{q^{r}} & \text{if } k \not| r \\
\end{cases}
\]

\[
A_{2n}^{(1)} z_{ij}^{r} = \begin{cases} 
\frac{[2]_{q^{r}} [2]_{q^{r}} [n]_{q^{r}}}{[2]_{q^{r}} [n-1]_{q^{r}} [n-1]_{q^{r}}} & \text{if } k| r, i = j = 1 \\
\frac{[2]_{q^{r}} [n-2]_{q^{r}} [n-j+1]_{q^{r}}}{[2]_{q^{r}} [2]_{q^{r}} [n-2]_{q^{r}} [n-j+1]_{q^{r}}} & \text{if } k| r, i = 1 < j \\
\frac{[2]_{q^{r}} [q^{r}]_{q^{r}} [n-2]_{q^{r}} [n-j+1]_{q^{r}}}{[2]_{q^{r}} [2]_{q^{r}} [n-2]_{q^{r}} [n-j+1]_{q^{r}}} & \text{if } \text{otherwise} \\
\end{cases}
\]

\[
D_{2n+1}^{(2)} z_{ij}^{r} = \begin{cases} 
\frac{[2]_{q^{r}} [n-j+1]_{q^{r}}}{[2]_{q^{r}} [n-j+1]_{q^{r}}} & \text{if } k| r, i = 1 \\
\frac{[2]_{q^{r}} [n-j+1]_{q^{r}}}{[2]_{q^{r}} [n-j+1]_{q^{r}}} & \text{if } k \not| r \\
\frac{[2]_{q^{r}} [n-j+1]_{q^{r}}}{[2]_{q^{r}} [n-j+1]_{q^{r}}} & \text{if } k \not| r \\
\end{cases}
\]

\[
E_{6}^{(2)} z_{ij}^{r} = \begin{cases} 
\frac{[2]_{q^{r}} [2]_{q^{r}} [5-j]_{q^{r}}}{[2]_{q^{r}} [5-j]_{q^{r}}} & \text{if } k| r, i = j \in \{1, 4\} \\
\frac{[2]_{q^{r}} [2]_{q^{r}} [5-j]_{q^{r}}}{[2]_{q^{r}} [5-j]_{q^{r}}} & \text{if } k| r, i = 1 < j \\
\frac{[2]_{q^{r}} [2]_{q^{r}} [5-j]_{q^{r}}}{[2]_{q^{r}} [5-j]_{q^{r}}} & \text{otherwise} \\
\end{cases}
\]

\[
D_{3}^{(2)} z_{ij}^{r} = \begin{cases} 
\frac{[2]_{q^{r}} [q^{r}]_{q^{r}} [3-j]_{q^{r}}}{[2]_{q^{r}} [q^{r}]_{q^{r}} [3-j]_{q^{r}}} & \text{if } k \not| r \\
\frac{[2]_{q^{r}} [q^{r}]_{q^{r}} [3-j]_{q^{r}}}{[2]_{q^{r}} [q^{r}]_{q^{r}} [3-j]_{q^{r}}} & \text{if } k| r \\
\end{cases}
\]
Proof: Thanks to lemma 7.3.1, the proof consists in the simple exercise of inverting the matrix \((x_{ijr})_{ijr}\). \(\square\)

**Theorem 7.3.3**

With the notations of proposition 7.3.2, we have that

\[
R_t \infty \in \prod_{r \leq 0} \exp(\beta r) \left( q^{-1} - q \right) E_{\beta r} \otimes F_{\beta r} \prod_{r > 0} \exp(\sum_{ijr} y_{ijr} E_{\beta r} \otimes F_{\beta r}) = \prod_{r > 0} \exp(\beta r) \left( q^{-1} - q \right) E_{\beta r} \otimes F_{\beta r}.
\]

Proof: The claim is an immediate consequence of remark 7.1.7, of lemma 7.3.1 and of proposition 7.3.2. \(\square\)

§7.4. Linear triangular transformation: the \(\bar{E}_\alpha\)’s.

It can be useful to determine explicitly the expression of \(R\) in the form given in corollary 7.1.6, that is to describe \(R\) in terms of “dual” bases. To this aim we shall look for elements \(\bar{E}_\alpha\) with \(\alpha \in \tilde{\Phi}_+\) satisfying the requirements of proposition 7.1.5.

**Lemma 7.4.1**

For \(r > 0\) let \(\bar{E}_{(r,i)}\) \((i \in I^r)\) be elements of \(U_{r\tilde{\Phi}}^+\) such that:

1) \(\forall i \in I^r\) \(\bar{E}_{(r,i)}\) is a linear combination of \(\{E_{(r,j)}|j \in I^r, j \geq i\}\);

2) \(\forall i < j \in I^r\) \((\bar{E}_{(r,i)}, F_{(r,j)}) = 0\).

Then, if \(F_{(r,i)} \equiv \Omega(\bar{E}_{(r,i)})\), we have that \((\bar{E}_{(r,i)}, F_{(r,j)}) = 0\) \(\forall i \neq j \in I^r\).

Proof: The assertion is clear for \(i < j\) once that one notices that \(\bar{E}_{(r,i)}\) lies in the linear span of \(\{F_{(r,l)}|l \in I^r, l \geq j\}\); for \(i > j\), it follows from corollary 7.2.4. \(\square\)

**Remark 7.4.2.**

Lemma 7.4.1 allows us to reduce the construction of \(\bar{E}_{(r,i)}\) to the solution of the following system of linear equations:

\[
(*)_{r,i} \sum_{l \geq i} A_{il}^{(r)} x_{ljr} = 0 \quad \text{for} \quad j \in I^r \quad \text{with} \quad i < j
\]

under the condition \(A_{ii}^{(r)} \neq 0\), where \(E_{(r,i)} = \sum_{l \geq 1} A_{il}^{(r)} E_{(r,l)}\).

Notice that in the non twisted case

\[
x_{ljr} = (o(l)o(j))^r [a_{ij}]_q \left[ r \right]_{d_l} \left[ q \right]_{r}
\]

so that \(*)_{r,i}\) can be written as

\[
(*)_{r,i}^{(1)} \sum_{l \geq i} \frac{A_{il}^{(r)} o(l) [d_l]}{[d_l]} [a_{ij}]_{q^r} = 0 \quad \text{for} \quad j \in I_0 \quad \text{with} \quad i < j.
\]

This system has been solved in [7], and those solutions will be used here in order to simplify the computations in the general case.

**Proposition 7.4.3**

Consider the non twisted case \((k = 1)\) and let \(r > 0, i \in I_0\).
A solution of $(*)_{r,i}^{(1)}$ is given by:

\[
A_{ij}^{(r)} o(j)^r \left[ \frac{d_2}{d_1} q \right]^{(1)} = \begin{cases} 
[n - j + 1] q^{2r} & \text{in case } B_n^{(1)} \text{ and } \text{if } (i,j) \neq (1,1) \text{ in case } E_4^{(1)} \\
n - 2 j q^r & \text{if } i = 1, j = 2 \text{ in case } D_n^{(1)} \\
[j - 1] q [n - 3] q^r & \text{if } 4 > j > i = 1 \text{ in case } E_n^{(1)} \\
n - j + 1] q^r [2] q^r & \text{if } j \geq 4, i = 1 \text{ in case } E_n^{(1)} \\
[2] q^{r(n+1)} & \text{if } i = j = 1 \text{ in cases } F_4^{(1)} \text{ and } G_2^{(1)} \\
n - j + 1] q^r & \text{otherwise,}
\end{cases}
\]

where $j \in I_0$ is such that $i \leq j$.

**Proof:** See [7].

We can now pass to the study of the twisted case. We shall consider three different situations: the case $A_2^{(2)}$, the twisted case different from $A_2^{(2)}$ with $k \not| r$ and the twisted case different from $A_2^{(2)}$ with $k| r$.

**Remark 7.4.4.**

Remark that $x_{ijr}(A_2^{(2)}) = x_{ijr}(B_n^{(1)})$ if $(i,j) \neq (1,1)$ and that in the equations $(*)_{r,i} x_{11r}$ never appears; then $(*)_{r,i}(A_2^{(2)}) = (A_2^{(2)})$; in particular the solutions of $(*)_{r,i}$ in case $A_2^{(2)}$ are the same as the solutions of $(*)_{r,i}$ in case $B_n^{(1)}$.

**Remark 7.4.5.**

Let $X_n^{(k)} \neq A_2^{(2)}$ and $r > 0$ be such that $k \not| r$. Then $\forall i \in I^r$ (that is $\forall i \in I$ such that $d_i = 1$) the system $(*)_{r,i}(X_n^{(k)}) = (A_2^{(2)})$ (with a shift of indices in case $A_2^{(2)}$) that is $A_{ij}^{(r)}(A_2^{(2) - 1}) = A_{i-1,j}^{(r)}(A_2^{(2) - 1}) \forall i, j \in I^r$.

**Remark 7.4.6.**

In the twisted case different from $A_2^{(2)}$, remarking that $d_i b_{ij} = d_i b_{ij}$ when $a_{ij} \neq 0$, the system $(*)_{r,i}$ becomes the following (under the condition that $k| r$):

\[
(*)_{r,i}^{(k)} \sum_{l \geq 1} A_{il}^{(r)} \left( \frac{o(l)}{d_1 q} \right) b_{il}^{(r)} [a_{ij}] q^{r} = 0 \quad \text{for } j \in I_0 \text{ with } i < j.
\]

Notice that if $i < j \in I_0$ are such that $b_{ij} = 1 \forall l \geq i$ such that $a_{ij} \neq 0$, then $d_i = k_r$ and each equation above can be written as

\[
(*)_{r,i}^{(k)} \sum_{l \geq 1} A_{il}^{(r)} \left( \frac{o(l)}{d_1 q} \right) [a_{ij}] q^{r} = 0 \quad \text{for } j \in I_0 \text{ with } i < j.
\]

This always happens when $X_n^{(k)} = D_n^{(k)} (k = 2,3)$; on the other hand when $X_n^{(k)} = A_2^{(2)}$ or $E_6^{(2)}$ we have that $b_{jl} \neq 1 \Rightarrow j = 2$, which implies that if $i \neq 1$ then
\[(*)_{r,i}^{w} = \{(*_{r,i}^{k})^{r} | j > i \}. \text{ Moreover we see also that} \]
\[
(*)_{r,1}^{w} = \{(*_{r,1}^{k,j})^{r} | j > 2 \} \cup \left\{ \sum_{l \in I_0} \frac{A_{ij}^{(r)}}{[d_{j}]} (o(2) o(l)) \frac{x_{ijr}}{q_{l} - q_{r}^{-1}} b_{21}[a_{21}^{*}]_{q_{r}} = 0 \right\}.
\]

Remark also that so that the only condition on \(A_{ij}^{(r)}\) is the one arising from the last equation, i.e. \[
\frac{A_{ij}^{(r)}}{[d_{j}]} (o(2) o(1)) \frac{x_{ijr}}{q_{l} - q_{r}^{-1}} b_{21} = \sum_{l > 1} \frac{A_{ij}^{(r)}}{[d_{j}]} (o(2) o(l)) \frac{x_{ijr}}{q_{l} - q_{r}^{-1}} b_{21}[a_{21}^{*}]_{q_{r}}, \]
where the \(A_{ij}^{(r)}\)'s with \((i, j) \neq (1, 1)\) are solutions of the system \{\((*)_{r,j}^{k} | i > j \) and \((i, j) \neq (1, 2)\}\}.

**Proposition 7.4.7**

Let \(r > 0\) and \(i \in I'\). A solution of \((*)_{r,i}^{w}\) is given by:

\[
A_{ij}^{(r)} o(j) \frac{x_{ijr}}{[d_{j}]} = \begin{cases} 
[n - j + 1]_{q_{r}} & \text{in cases } B_{n}^{(1)} \text{ and } A_{2n}^{(2)}, \text{ and} \\
[n - j + 1]_{q_{r}} [2]_{q_{r}} & \text{if } (i, j) \neq (1, 1) \text{ in case } F_{4}^{(1)} \\
[n - j + 1]_{q_{r}} & \text{if } j > i = 1 \text{ in case } C_{n}^{(1)} \text{ and} \\
[n - 1]_{q_{r}} & \text{if } j - 1 > i = 1 \text{ in case } D_{n}^{(1)} \\
[n - 2]_{q_{r}} & \text{if } i = 1, j = 2 \text{ in case } D_{n}^{(1)} \\
[j - 1]_{q_{r}} [n - 3]_{q_{r}} & \text{if } 4 > j > i = 1 \text{ in case } E_{n}^{(1)} \\
[n - j + 1]_{q_{r}} [3]_{q_{r}} & \text{if } j \geq 4, i = 1 \text{ in case } E_{n}^{(1)} \\
[2]_{q_{r}} & \text{if } i = j = 1 \text{ in case } F_{4}^{(1)} \\
[3 - j]_{q_{r}} & \text{if } i = 1 \text{ in case } G_{2}^{(1)} \\
[\frac{n - n}{k - 1} - j + 1]_{q_{r}} & \text{if } k \not \in \text{ cases } D_{n}^{(k)} \text{ and } E_{6}^{(k=2)} \\
(-\frac{1}{2} [n])_{q_{r}} & \text{if } 2| r \text{ and } i = j = 1 \text{ in case } A_{2n-1}^{(2)} \\
(-\frac{1}{2} [2])_{q_{r}} & \text{if } 2| r \text{ and } i = j = 1 \text{ in case } E_{6}^{(2)} \\
[n - j + 1]_{q_{r}} & \text{otherwise,}
\end{cases}
\]

where \(j \in I'\) is such that \(i \leq j\).

**Proof:** The only statements which do not follow immediately from proposition 7.4.3 and remarks 7.4.4, 7.4.5 and 7.4.6 are those concerning the case when \(X_{n}^{(2)} = A_{2n-1}^{(2)} \) or \(E_{6}^{(2)}\), \(2| r\) and \(i = j = 1\), and it is a matter of straightforward computations. \(\square\)

§7.5. The Killing form on the new imaginary root vectors.

The next (and last) step is computing \((\mathcal{E}_{\alpha}, \bar{F}_{\alpha})\) when \(\alpha\) is an imaginary root.

**Lemma 7.5.1**

Let \(r > 0\) and \(i \in I'\). Then

\[
(\mathcal{E}_{(r, i)}, \bar{F}_{(r, i)}) = -A_{ij}^{(r)} \sum_{j \in I' ; j \geq i} A_{ij}^{(r)} \frac{x_{ijr}}{q_{l} - q_{r}^{-1}} = -A_{ij}^{(r)} \frac{x_{ijr}}{q_{l} - q_{r}^{-1}} \sum_{j \in I' ; j \geq i} A_{ij}^{(r)} \frac{x_{ijr}}{[d_{j}]}.
\]
Proof: The definition of $E_i$ implies that $(E_{(r,s,i)}, F_{(r,s,i)}) = A_i^{(r)} (E_{(r,s,i)}, F_{(r,s,i)})$, because $(E_{(r,s,i)}, F_{(r,s,i)} - A_i^{(r)}(E_{(r,s,i)}, F_{(r,s,i)})) = 0$. But

$$(E_{(r,s,i)}, F_{(r,s,i)}) = \sum_{j \in I'} A_i^{(r)} (E_{(r,s,j)}, F_{(r,s,j)}) = - \sum_{j \in I'; j \geq i} A_i^{(r)} x_j q_i^{-1}.$$ \hfill \square$

Proposition 7.5.2

Let $r > 0$, $i \in I'$. Then we have

$$\frac{o(i)v_{r}(q^{-1} - q)r[d]_q}{d_i[r]q A_i^{(r)}} (E_{(r,s,i)}, F_{(r,s,i)}) =$$

$$\begin{cases}
[2]_{q^r(2n-1)} & \text{if } i = 1 \text{ in case } B_n^{(1)} \\
[n - i + 2][q^r]_2 & \text{if } i \neq 1 \text{ in cases } B_n^{(1)} \text{ and } A_{2n}^{(2)} \text{ and } \\
[2]_{q^r} & \text{if } i > 2 \text{ in case } F_{n=4}^{(1)} \\
[2]_{q^r(n+1)} & \text{if } i = 1 \text{ in case } C_n^{(1)} \\
[2]_{q^r(n-1)} & \text{if } i = 1 \text{ in case } D_0^{(1)} \\
(2-n)_{q^r} & \text{if } i = 1 \text{ in case } E_n^{(1)} \text{ with } n = 6, 7 \\
\frac{[2]_{q^r}}{[2]_{q^{2r}}} & \text{if } i = 1 \text{ in case } G_2^{(1)} \\
\frac{[2]_{q^r}}{[2]_{q^{2r}}} & \text{if } i = 1 \text{ in case } G_2^{(1)} \\
[2]_{q^r} & \text{if } i = 1 \text{ in case } A_{2n}^{(2)} \\
[2]_{q^r(2n+1)} & \text{if } i = 1 \text{ and } 2|r \text{ in case } A_{2n}^{(2)} \\
[2]_{q^r(2n+1)} & \text{if } i = 1 \text{ and } 2|r \text{ in case } A_{2n}^{(2)} \\
[2]_{q^r(2n+1)} & \text{if } i = 1 \text{ and } 2|r \text{ in case } A_{2n}^{(2)} \\
[2]_{q^r} & \text{if } i = 1 \text{ and } k = 2|r \text{ in case } D_n^{(k)} \\
[2]_{q^r} & \text{if } i = 1 \text{ and } k = 3|r \text{ in case } D_n^{(k)} \\
[2]_{q^r} & \text{if } i = 2 \text{ and } 2|r \text{ in case } E_0^{(2)} \\
(1-1)^{\frac{1}{2}} [2]_{q^r} & \text{if } i = 1 \text{ and } 2|r \text{ in case } A_{2n-1}^{(2)} \\
[2]_{q^r} & \text{if } i = 1 \text{ and } k = 2|r \text{ in case } D_n^{(k)} \\
[2]_{q^r} & \text{if } i = 1 \text{ and } k = 3|r \text{ in case } D_n^{(k)} \\
[2]_{q^r} & \text{if } i = 2 \text{ and } 2|r \text{ in case } E_0^{(2)} \\
[n - i + 2][q^r]_2 & \text{otherwise.}
\end{cases}$$

Proof: For the non twisted case see [7]. Then also the cases $A_{2n}^{(2)}$ with $i \neq 1$, $X_n^{(k)}$ with $k \not| r$, $A_{2n-1}^{(2)}$ and $D_n^{(k)}$ with $k \neq 1|r$ and $i \neq 1$, and $E_0^{(2)}$ with $2|r$ and $i \neq 1, 2$ are clear. \hfill \square
§7.6. The expression of \( R \) in terms of the \( \bar{E}_\alpha \)'s.

We can now collect the results obtained till now and get an explicit formula for \( R \).

**Theorem 7.6.1**

\[
R = \left( \prod_{\alpha \in \Phi_+} \exp_{\alpha} \left( \frac{q^{-1}_\alpha - q_\alpha}{c_{\alpha}} \bar{E}_\alpha \otimes \bar{F}_\alpha \right) \right) q^{-t_{\infty}}
\]

where \( c_{\alpha} = (q^{-1}_\alpha - q_\alpha)(\bar{E}_\alpha, \bar{F}_\alpha) = \begin{cases} 1 & \text{if } \alpha \in \Phi_{\text{re}}^+, \\ (q^{-1}_\alpha - q_\alpha)(\bar{E}_\alpha, \bar{F}_\alpha) & \text{if } \alpha \in \Phi_{\text{im}}^+, \end{cases} \) and \( c_{\alpha} \) is explicitly determined from proposition 7.5.2.

**Proof:** The claim is a straightforward consequence of the preceding results: corollary 7.1.6 and proposition 7.5.2. \( \square \)

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