Prospective mathematics teachers’ concept image on the limit of a function

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Abstract. The concept of limit is one of the prerequisite subjects in studying calculus. Previous studies indicate that the use of the limit concept in learning that is not in accordance with the limit concept in mathematics. This study aims to describe the concept image of prospective mathematics teachers on the concept of limit functions and the causes of the concept image. This qualitative study used the phenomenology approach, involving seven mathematics students in the fifth semester at a university in Bandung, Indonesia, who were selected based on the results of a written test. The data were obtained from the results of written tests on the concept of limit function and interviews and analysed descriptively. The results showed that students understood the concept of limit function in the same way that limit is an unreachable function. This causes the definition of limit formally using ε-δ is not properly understood.

1. Introduction
Defining limits personally, which is different from formal definitions affects the ability to fully understand formal definitions when found in the Real Analysis course [1]. Students’ understanding of the concept of limits is often incomplete and unstructured [2]. In addition, the concept of limits is often developed without being linked to the formal definition of limits in introductory courses of calculus and real analysis. In this case, many advanced mathematics students may not have a deep conceptual definition of the limits for learning and utilize them in advanced calculus subjects [3]. This also happens to students who have completed introductory calculus courses [4][5][6].

This causes the concept of limits to be considered difficult material [7][8][9]. Even though the concept of limit has an important role in studying calculus, both in explaining functions and as a condition for studying other calculus topics [10]. Many students do not consider the formal definition of limits in solving limit problems, and most of the personal limit concept definitions cannot be used to solve limit problems because they are not consistent with the formal definition of limits [3]. The definition of the formal concept of limit refers to the original verbal description of Cauchy [11], which is now translated into symbols as the definition of δ-ε. Meanwhile, the definition of the personal concept refers to a collection of words and symbols used by students in explaining limits [3].
In fact, students who study calculus and understand the formal definition of limit concepts conceptually will be more successful in solving problems related to limit [12]. In addition, students who have a formal understanding and can operate the concept of limits will be able to recognize, communicate, investigate, evaluate, apply, and make reasoning in advanced mathematics [3]. Conversely, students who do not understand the formal definition of limits will have obstacles so they must continue to struggle to understand the concept [5]. In mathematics, students should be encouraged to develop a formal, conceptual understanding of a concept rather than personal procedural and conceptual knowledge [1].

Personal understanding of the concept can be related to one's perception of an object in which the way a person perceives, understands, and interprets something. According to, perception is not unique because one person and another can interpret the same object in different ways. The process of perceiving an object can go through three phases, namely selection (focusing on certain characteristics), organization (organizing certain aspects based on experience), and interpretation (views of the selection process and organization). Related to this, [14] states that the development of mathematical knowledge is processed by the biological brain in the long term starting with perception as input and action as output. This shows that perception is the first step for students to accept a mathematical concept where this concept can vary between students depending on the interpretation. The interpretation of a concept is often called a conception.

[15] interprets concepts and conceptions similar to [16] in interpreting concept images and concept definitions. Conception is an internal representation that appears in a person's mind against a concept built by previous experience. These interrelated conceptions form the concept image. In describing the concept image, [12] uses the words association, intuition, efficiency, degeneration, and key elements. [16] describe knowledge about the meaning of something in a person as a concept image or concept image. Concept image consists of all cognitive structures in the individual's mind that are related to a given concept.

Many of the concepts used are not formally defined, so we must learn to recognize them by experience and use them in appropriate contexts. During the mental process of remembering and manipulating a concept, many associated processes are played out, consciously and unconsciously influencing its meaning and usage. In other words, the concept image is a mental picture, characteristics and characteristics, as well as the processes associated with the concept [16]. A person's concept image will be wider if the conception they have is getting richer [17].

Cognitive conflict can develop because of the difference between concept images (ideas formed in students' minds) and concept definitions (more precise mathematical ideas) [16]. In this case it is found that students who hold a concept image that is contrary to the concept definition cannot resolve this discrepancy in understanding a concept. To have a formal definition in each concept image, each student must do more than just pronounce the definition such as using a formal definition in constructing a more precise and valid mathematical argument [3].

The concept image of students about the concept of limits can be influenced by the use of words that are not appropriate in expressing the meaning of limit, such as "approaching" or "getting closer," [7][16][9]. Students have unclear representations of limits [18], and students are confused about the various properties of the limit concept [19]. Students' conception of the concept of limits is formed since high school, and this error continues until university [12]. In addition, certain elements of the image are often not conceptually connected to students, who are then unaware of the contradiction between them.

Research concept image about limits (see [18][7][16][9][12][20]) was conducted on students who were studying the concept of limits in basic mathematics or introductory calculus courses or students who had already completed the course. Image classification regarding the concept of limit as environment, graphical approach, approximate value, defined at x₀, limit f at x₀ equals f (x₀), and algorithms [12].

To develop previous research, this study was conducted on mathematics students as prospective teachers who will practice teaching in schools. They have completed a number of mathematics and
education courses. In addition, this research study focuses on the experience of students in studying limit functions. The purpose of this study is to describe the concept image of the prospective mathematics teacher on the concept of limit functions, and the causes of the conceptual image emergence.

2. Method
This research is qualitative research with a phenomenological approach based on the experiences of research subjects in studying the concept of limit functions in calculus courses. The research subjects were students as prospective teachers who will practice at school from the mathematics education department in the fifth semester at a university in Bandung, Indonesia. Subjects were selected based on completeness and variation of written test answers, and willingness to be involved in the research. The research data were obtained from the results of tests and in-depth interviews. This test question was adapted from [20] and [21], which consists of 3 questions about the concept of limit functions. Before the test questions were given to students, they were validated by mathematicians and limited testing was carried out on a few students. The test was conducted to select students who met the requirements to be involved in the research, which was attended by 54 students. Based on the test results, 7 students were obtained which could be continued at the interview stage to examine in-depth understanding of the concept of limit functions and the depth of test questions. Data analysis was carried out using the basic analysis stages of [22], namely managing data, reading-memoing, describing-classifying-interpreting, and representing-visualizing, then presented descriptively for the results of the test answers and interviews conducted.

3. Results and discussion
Separate interviews among students showed several answers in the same context, but there were also variations in some cases. At the initial stage, the researcher asked students' understanding of the limit concept. This is because the limit material has been studied in the second semester of Differential Calculus, and now they are still studying limits in the Real Analysis course. Students' understanding of limits can be seen in the following interview results between the researcher (P) and student (S).

| P      | What is understood about limits? |
|--------|----------------------------------|
| S1     | The limit is approximate         |
| S5     | Limit                            |
| S6     | The value of the function (pause) is approximate |
| S4     | The value of a function that approximates to a value of x, eh, not necessarily x, when it approaches a value |
| S7     | Usually there are values that are difficult to reach, what are their names, not integers, so we need approximate values. Hear the limit, ooh approach values or the values around. |
| S2     | As far as I know limit, it is close to a number, for example, the limit x approaches zero, it means that x is close to zero but does not touch zero |
| S3     | Limit is a limit in everyday life, in mathematics it is not like a limit but almost as close to the limit as possible. Values that are close to the x-value, for example. |

From the students' answers, it can be understood that they understand limit, unlike the formal definition of limit in epsilon-delta. This is because in the learning process, the lecturer does not discuss the formal definition of limits in full and in depth. In discussion with one of the lecturers, it was said that at the undergraduate level for mathematics education, students did not need the concept of limits in depth except at the advanced level or students majoring in pure mathematics. The results of interviews with students also found that the cause of the lack of understanding of the concept of limit was because they only learned in class and no longer deepened the material after being explained by the lecturer. Even though, the lecturers have explained the concept of limit in the formal definition,
and they easily understand the limit as constraints or approximations. In addition, students see limit as a function that moves when x is moved towards a certain value. It represents limit as a process, not as a number or result of a process. As a result, students are confused about whether the limit value can be achieved [10].

For the first question in the test stage, there are two answers that are mostly chosen by students, namely the choice of "all real numbers", and "all real numbers except 0". One of the answers of students who chose 'all real numbers' for almost the same reason as those who chose this answer, because the value of a limit will approach a point so that all real numbers if substituted in the function will have a value. Some of the answers are as follows.

S4 : The limit value is the approximate value so that in the function one per x all x real number members have a value.
S6 : Because all real number values have their limit value. This is the context of limit so when a is 0, one per x will have a value because x is only approaching 0.
S7 : Because whatever the value of a still has a limit.

Different answers were given by several students with the answer "all real numbers except 0". The reason is almost the same, namely because if x is equal to zero it will produce one by zero which has no value. Furthermore, the results of interviews between the researcher (P) and the student (S) who chose this answer and their understanding of infinity are as follows.

P : For the first question, why choose the answer to all real numbers except zero?
S1 : Because for x members of real numbers without zero will have a value, while for x equal to 0, the limit value is one per zero.
S2 : Because one by zero is undefined.
S3 : Limit values approaching zero yields infinite values.
P : If so, is it not up to a number?
S1 : Numbers, ... but I don't know how much that goes to where. It is a number but it is not clear how much it is, only a very large number. Yes, numbers, because so far, we, for example, studying like that, we explain that it is close to infinity, approaching a very large number.
S6 : Real numbers, we don't know how much they are worth. It is the real member number but not part of the set. Yes, real numbers, if it is interrupted so.
S4 : So, it does not mean that it represents a very large number, which does not know where, how, has no end. If the definition of a number must be the number and the infinite definition represents a number that does not know its value. If the number line gets to the right of the end of the end, we don't know how long it will stop, because it is called infinite because we don't know the point it stops. The definition of a number that doesn't know how much.
S5 : It's not a number, it's a number, of course, you can say what number it is. But not to how much, so I don't know so it's not a number

The concept image of 'infinite' causes students to be less precise in defining or deciding whether infinity is a number or not. From some of these answers, S4's concept image of 'infinity' is correct and he maintains his principle that infinity cannot be said to be a number, only a representation of a large number.

The second question relates to the limit at a point. It turns out that students' understanding of the concept of limit and function is still wrong and contradicting each other. For the value of the limit function which is known in the figure, there are five out of seven students who answered correctly, namely 3 (one of them is in Figure 1).
Figure 1. One student's answer to the second question.

The wrong answer to the limit value in the third question is due to understanding the blank circle on the graph which is considered the limit value. Like S4's answer that "there is no limit value because f(x) is discontinuous at x equal to 2". On the other hand, S3 adds or takes the known function f(x) on the graph with another function, namely f(x) = x - 1, because it is considered in accordance with the graph sketch. This shows that they do not understand the concept of limit, and cannot distinguish between the concept of limit and the concept of a function.

As for the purpose of the known limit function in the third question, most students answered with the same intention that the value of the function f(x) when around x = 2. The other answers "a function that approaches the value 2 but does not touch, because limit equals to approaching".

For the third question about the limit function on infinity, the same answer is obtained in the first point where the limit f(x) approaches 0 (Figures 2). Some of the reasons are as follows.

Three questions about the concept of a limit function and the results of the interview show the image of the same students who think that limits can never be achieved in problem-solving or in theoretical situations. The use of formal definitions has not been properly understood because they are rarely used or introduced in-depth, only as symbols. Related to this, the research results of [23] state that the subject rejects pure rote learning and instead emphasizes a deep understanding of mathematical concepts so that it makes sense to understand the meaning of mathematics outside of the definition. This picture is also influenced when studying limits in high school to university. Similar to the results of research conducted by [18] that students experience problems in connecting theory to problem-solving, such as determining whether limits can be achieved for functions or not. Many students interpret inequality in formal definitions to say that limits cannot be reached, or perceive limits as a process of estimation [24]. Students' conceptual problems about limits are closely related to their perceptions of infinity. They use them but actually only refer to very large or very small numbers as indefinite [25].
a. What do you understand by \( \lim_{x \to \infty} \frac{1}{x} = 0 \)?

\[
\lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} \text{ Where } \frac{1}{\infty} \text{ will produce 0.}
\]

Since 1 is divided by a very large number, the value will approach 0.

b. Explain, can we write \( f(\infty) = 0 \)?

No, because the value of \( \infty \) can be anything. The result is not exactly zero but is close to 0.

**Figure 2.** One student's answer to the third question, both points are correct.

Research conducted by [27] on the development of the concept of limits shows that some students have vague conceptions of limits and are even wrong in some cases. In addition, students who study limit functions perceive and treat limits differently. This is because students receive different levels of understanding because they have different priorities and abilities. In addition, understanding the concept of limits that is not good enough in forming a coherent concept image is obtained from the imprint of the concept image of students who are confused about various sides of the limit concept [19]. As stated by [27] that students make knowledge of mathematical concepts and ideas differently from one another or from what is considered by mathematicians.

Some students see limits as limits that cannot be passed by a function, limits are not reachable where limits approach a point but never reach a certain point, limit as a dynamic process and not as a static object, and limit as an estimate [20]. In general, this is the same as students' understanding in this study about the concept of limits. In this case, there are also some misconceptions regarding limits, namely a function must be defined at a point in order to have a limit at that point, a function must be continuous at a point in order to have a limit at that point, the limit value at a point is the same as the function value at that time, and students see boundary discovery and the substitution process as exactly the same thing. Another case is the student's representation of the concept of limits that is not clear but they do not recognize it themselves [18]. The concept of limit is not a single process but a more complicated scheme because it involves a coordinated process of function values that approach the limit [28].

In teaching the concept of limit using an intuitive approach, students are usually directed to the idea of limit (sequence) as a process of consecutive estimates leading to a value that may not be reached using a sequence of smaller and smaller numbers, or when the limit function at a point is described in the form of graphical representation of a function. In this case, there are many representations of the idea of limits which imply that there is room for potential conflict in developing different concept images [24].
The limit problem can be solved through an exploratory approach with a table of function values and graphs from the start and then as a symbolically represented entity. According to [28] limit function learning demands a leap between operational and static perceptions. Learning limits requires skills from many areas of mathematics [26]. In this case, students must be able to understand formal exposition, perform algebraic manipulation, understand the meaning of numerators and absolute values, and relate theory to everyday problem solving. This means that their concept image needs to be well developed and broad, that is, they require strong abstraction skills and relationships between various topic areas. They also need to find inspiration and reason through hard work to make meaningful knowledge in their concept image [26].

Regarding the development of the concept image, [29] has introduced three areas of mathematics to differentiate mathematical thinking which aims to obtain an overview of various cognitive developments in mathematics. The theory emphasizes the construction of mental representations of concepts from several theories about concept development. For example, [15] who worked on the encapsulation of processes to objects and Piaget's theory of abstraction. The three fields are hierarchical, which means there is a progression from simply understanding a concept through action to a formal understanding of the concept. First, an embodied field in which individuals use their physical perceptions of the real world to carry out mental experiments in constructing mental conceptions of mathematical concepts. Second, the procedural field in which the individual begins with procedural actions on the mental conception of the first field, such as counting using symbols, becomes encapsulated as a concept. Symbols represent processes and concepts called procept [13], for example counting and sum or addition. Symbols are used as processes and concepts depending on the context. Third, the formal field is expressed by the formal definition as a Juter [26] axiom.

4. Conclusion
The results showed that prospective mathematics teachers understood the concept of limit functions in the same way that limits were as unreachable functions. This resulted in the formal definition of limit using ε-δ not well understood. Although solving limit problems using function value tables and graphs have been taught from high school to college, this method is rarely used. However, this method is very helpful in understanding the limit concepts. They prefer to solve boundary problems by direct substitution. In addition, the concept of limit is not taught in detail and deeply so that they do not understand it well. This is what causes the appearance of an inaccurate concept image related to functional limitations. Therefore, prospective teachers need to relearn the concept of limit functions in depth and relate the concept of limits in problem-solving with the theory being studied. In addition, one must distinguish between the concept of limits in mathematics and the context of everyday life. In this case, a comprehensive study of the limit function is needed at the beginning of calculus learning and should be an important concern for calculus lecturers.

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