Current-component independent transition form factors for semileptonic and rare $D \to \pi(K)$ decays in the light-front quark model

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ABSTRACT

We investigate the exclusive semileptonic and rare $D \to \pi(K)$ decays within the standard model together with the light-front quark model (LFQM) constrained by the variational principle for the QCD-motivated effective Hamiltonian. The form factors are obtained in the $q^+ = 0$ frame and then analytically continue to the physical timelike region. Together with our recent analysis of the current-component independent form factors $f_+(q^2)$ for the semileptonic decays, we present the current-component independent tensor form factor $f_T(q^2)$ for the rare decays to make the complete set of hadronic matrix elements regulating the semileptonic and rare $D \to \pi(K)$ decays in our LFQM. The tensor form factor $f_T(q^2)$ are obtained from two independent sets ($J_T^{+\pm}, J_T^{-\mp}$) of the tensor current $J_T^{\mu\nu}$. As in our recent analysis of $f_+(q^2)$, we show that $f_T(q^2)$ obtained from the two different sets of the current components gives the identical result in the valence region of the $q^+ = 0$ frame without involving the explicit zero modes and the instantaneous contributions. The implications of the zero modes and the instantaneous contributions are also discussed in comparison between the manifestly covariant model and the standard LFQM. In our numerical calculations, we obtain the $q^2$-dependent form factors ($f_\perp, f_T$) for $D \to \pi(K)$ and branching ratios for the semileptonic $D \to \pi(K)\ell\nu_\ell$ ($\ell = e, \mu$) decays. Our results show in good agreement with the available experimental data as well as other theoretical model predictions.

I. INTRODUCTION

The three flavors of charged leptons, $(e, \mu, \tau)$, are the same in many respects. In the standard model (SM), the couplings of leptons to gauge bosons are supposed to be independent of lepton flavors, which is known as lepton flavor universality (LFU) [1]. The experimental tests of LFU in various semileptonic $B$ decays have been reported [2–6] by measuring the ratios of branching fractions $\mathcal{B}_{D^{(*)}(\ell)} = Br(B \to D^{(*)}\ell\nu_\ell)/Br(B \to D^{(*)}\ell\nu_\ell)$ ($\ell = e, \mu$). Currently, the SM prediction is roughly three standard deviations away from the global average of results from the BABAR, Belle, and LHCb experiments. Many theoretical efforts have been made in resolving the issue of $\mathcal{B}_{D^{(*)}(\ell)}$ anomaly and searching for new physics beyond the SM [7–10]. In view of this, tests of LFU in $D$ decays are also intriguing complementary endeavors.

Exclusive semileptonic and rare $D$ decays provide rigorous tests of the SM in the charm sector including not only the LFU but also the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements [11, 12], which describe the mixings among the quark flavors in the weak decays and hold the key to the CP violation issues in the quark sector. Compared to the semileptonic $D \to \pi(K)\ell\nu_\ell (\ell = e, \mu)$ decays induced by flavor-changing charged current, the rare $D \to \pi(K)\ell\ell$ decays are induced by the flavor-changing neutral current (FCNC). Since the rare decays are loop-suppressed in the SM as they proceed through FCNC, they are also pertinent to test the SM and search for physics beyond the SM. Recent BES III measurements [13–20] for many exclusive semileptonic charm decays also allow one to test the SM in the charm sector more precisely.

While the experimental measurements of exclusive decays are much easier than those of inclusive ones, the theoretical knowledge of exclusive decays is sophisticated essentially due to the hadronic form factors entered in the long distance non-perturbative contributions. Along with new particle effects beyond the SM, which may amend the Wilson coefficients of the effective weak Hamiltonian that describes physics below the electroweak scale, the reliable and precise calculations of the hadronic form factors are very important to constrain the SM and search for new physics effects beyond the SM.

The calculations of hadronic form factors for semileptonic and rare $D$ decays have been made by various theoretical approaches, such as lattice QCD (LQCD) [21–24], QCD sum rules [25, 26], QCD light-cone sum rules [27–29], symmetry-preserving continuum approach to the SM strong-interaction bound-state problem [30], quark potential model [31–33], relativistic quark model (RQM) based on the quasipotential approach [34], covariant confining quark model (CCQM) [35], chiral quark model [36], and constituent quark model [37] etc.

Perhaps, one of the most apt formulations for the analysis of exclusive processes involving hadrons may be provided in the framework of light-front (LF) quantization [38]. The semileptonic and rare $D$ decays have also been analyzed by the light-front quark model (LFQM) [39–47] based on the LF quantization.

In the standard LFQM that we use in this work, the constituent quark and antiquark in a bound state are required to be on-mass shells and the spin-orbit wave function (WF) is obtained by the interaction-independent Melosh transformation [48] from the ordinary equal-time static spin-orbit WF assigned by the quantum number $J^{PC}$. For the radial part, we use the phenomenologically accessible Gaussian WF $\phi(x, k_\perp)$. Since the standard LFQM itself is not amenable to pin down the zero modes, the exactly solvable manifestly covariant Bethe-Salpeter (BS) model with the simple multipole type $q\bar{q}$ vertex was utilized [45, 46, 49, 50] to help identify the zero modes in a systematic way. On the other hand, this

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BS model is less realistic than the standard LFQM. Thus, as an attempt to apply the zero modes found in the BS model to the standard LFQM, the effective replacement [45, 46, 49] of the LF vertex function \( \chi(x, k_\perp) \) obtained in the BS model with the more realistic Gaussian WF \( \phi(x, k_\perp) \) in the standard LFQM has been made.

However, we found [51–53] that the correspondence relation between \( \chi \) and \( \phi \) proposed in [45, 46, 49] encounters the self-consistency problem, e.g. the vector meson decay constant obtained in the standard LFQM was found to be different for different sets of the LF current components and polarization states of the vector meson [51]. We also resolved [51–53] this self-consistency problem by imposing the on-shell shell condition of the constituent quark and antiquark in addition to the original correspondence relation between \( \chi \) and \( \phi \). Specifically, our new finding for the constraint of the on-mass shell condition corresponds to the replacement of physical meson mass \( M \) with the invariant mass \( M_0 \) in the calculation of the matrix element. The remarkable feature of our new additional correspondence relation \( (M \to M_0) \) between the two models in the calculations of the two-point functions such as the weak decay constants and the distribution amplitudes of mesons [51–53] was that the LF treacherous points such as the zero modes and the off-mass shell instanton contributions appeared in the BS model are absent in the standard LFQM. This prescription \( (M \to M_0) \) can be regarded as an effective inclusion of the zero modes in the valence region of the LF calculations.

As an extension our analysis of the two-point functions [51–53] to the three-point ones, in our very recent LFQM analysis [54] of the semileptonic \( B \to D\ell\nu_\ell \) decays, we presented the self-consistent descriptions of the weak transition form factors (TFFs) \( f_+(q^2) \) and \( f_-(q^2) \). Especially, \( f_-(q^2) \) should be obtained by using at least two components of the weak vector current \( J^\mu \) while \( f_+(q^2) \) can be obtained from the single and “good” component \( (J^\mu_+) \) of the current. Because of this, \( f_-(q^2) \) has been known to receive the zero mode mainly due to the unavoidable usage of the so called “bad” components of the current, i.e. \( J_{\mu V} = (J_\mu, J_\nu) \) and \( J_{\mu}^+ \), many efforts have been made to obtain the Lorentz covariant form factors [45, 46, 49] within the standard LFQM by properly handling the zero-mode as well as the instantaneous contributions.

Applying the same correspondence relations found in [51–53] to the \( B \to D\ell\nu_\ell \) decays, we found that the zero modes and instantaneous contributions to \( f_-(q^2) \) are made to be absent in the standard LFQM while they exist in the BS model. In other words, we obtained the current-component independent tensor form factor \( f_T(q^2) \) in the standard LFQM, i.e. \( f_-(q^2) \) obtained from \( (J^+ J^\perp) \) numerically and both are expressed as the convolution of the initial and final state LFWFs in the valence region of the \( q^+ = 0 \) frame. This verifies that our new correspondence relations found in the two-point functions are also applicable to the three-point functions.

The purpose of this paper is to extend our previous analysis [54] of the form factors \( f_{\pm}(q^2) \) for the semileptonic decays between the two pseudoscalar mesons to obtain the current-component independent tensor form factor \( f_T(q^2) \) for the rare decays, which complete the set of hadronic matrix elements regulating the exclusive semileptonic and rare decays between the two pseudoscalar mesons. We then apply our Lorentz covariant form factors \( f_{\pm}, f_T \) for the analysis of the semileptonic and rare \( D \to \pi(K) \) decays within the standard model and the light-front quark model (LFQM) constrained by the variational principle for the QCD-motivated effective Hamiltonian [49, 55–57].

The paper is organized as follows: In Sec. II, we introduce three form factors \( f_T(q^2) \) for the semileptonic and rare decays between two pseudoscalar mesons. In the \( q^+ = 0 \) frame, we define the form factors extracted from the various combinations of vector and tensor currents. In Sec. III, we set up the current matrix elements for the form factors in an exactly solvable model based on the covariant BS model of (3 + 1) dimensional fermion field theory. We then present our LF calculations of tensor form factor \( f_T \) in the BS model using the two different sets \( (J^+_T, J^-_T)\) of the tensor current \( J^\mu_T \). For completeness, we also present the results of the current-component independent form factors \( f_{\pm}(q^2) \) found [54]. We note that while \( f_T(q^2) \) obtained from \( J^+_T \) is immune to the zero mode and the instantaneous contribution, \( f_T(q^2) \) obtained from \( J^-_T \) cannot avoid those contributions in this BS model. Linking the covariant BS model to the standard LFQM with our new correspondence relations between the two models [51–53], however, we find that \( f_T(q^2) \) obtained from \( J^+_T \) in the standard LFQM is made to be free of the zero mode as well as the instantaneous contribution. In other words, we obtained the current-component independent tensor form factor \( f_T(q^2) \) in the standard LFQM regardless of using \( J^+_T \) or \( J^-_T \) as in the case of \( f_{\pm}(q^2) \) calculation [54]. Finally, we present the current-component independent TFFs \( f_{\pm}, f_T \) in the \( q^+ = 0 \) frame of the standard LFQM. In Sec. IV, we present our numerical results of the form factors for the semileptonic and rare \( D \to \pi(K)\ell\nu_\ell \) (\( \ell = e, \mu \)) transitions. Summary and discussion follow in Sec. V.

II. THEORETICAL FRAMEWORK

The matrix elements of the vector \( J_V^\mu = \bar{q}T^\mu c \) and the tensor \( J_T^{\mu\nu} = \bar{q}\sigma^{\mu\nu}c \) currents for the weak \( c \to q = u, d, s \) transitions between the initial \( D \) meson and the final \( \pi \) or \( K \) meson can be parametrized by the following set of invariant form factors, \( (f_+, f_{\pm}, f_T) \) [33]:

\[
\mathcal{M}_V^\mu \equiv \langle P_2 | J_V^\mu | P_1 \rangle = f_+(q^2) P^\mu + f_-(q^2) q^\mu, \quad (1)
\]

and

\[
\mathcal{M}_T^{\mu\nu} \equiv \langle P_2 | J_T^{\mu\nu} | P_1 \rangle = i s(q^2) [P^\mu q^\nu - q^\mu P^\nu], \quad (2)
\]

where \( P_1 + P_2 = q \) and \( q = P_1 - P_2 \) is the four-momentum transfer to the lepton pair(\( \ell\nu_\ell \)) with \( m_\ell^2 \leq q^2 \leq (M_1 - M_2)^2 \) for the semileptonic decays or to the pair (\( \ell^+\ell^- \)) with \( 4m_\ell^2 \leq q^2 \leq (M_1 - M_2)^2 \) for the rare decays, respectively. The anti-symmetric tensor in Eq. (2) is given by \( \sigma^{\mu\nu} = (i/2) [\gamma^\mu, \gamma^\nu] \).
On many occasions, it is useful to express Eq. (1) in terms of the form factors $f_+(q^2)$ and $f_0(q^2)$, which are related to the transition amplitude with the exchange of a vector (1-) and a scalar (0+) boson in the $t$-channel, respectively, and satisfy

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{M_1^2 - M_2^2} f_-(q^2).$$

(3)

Likewise, the tensor form factor $s(q^2)$ in Eq. (2) can also be redefined by

$$s(q^2) = \frac{f_T(q^2)}{M_1 + M_2},$$

(4)

to make $f_T(q^2)$ dimensionless.

Including the nonzero lepton mass ($m_\ell$), the differential decay rate for the semileptonic $P_1 \rightarrow P_2 \ell \nu_\ell$ process is given by [58, 59]

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2}{96\pi^2} |\vec{P}|^2 |V_{Q_1 \ell}|^2 \frac{q^2}{M_1^2} \left(1 - m_\ell^2/q^2\right)^2 \times \left[\left(1 + \frac{m_\ell^2}{2q^2}\right)|H_+|^2 + \frac{3m_\ell^2}{2q^2}|H_0|^2\right],$$

(5)

where $G_F = 1.166 \times 10^{-5}$ GeV$^{-2}$ is the Fermi constant, $V_{Q_1 \ell}$ is the relevant CKM mixing matrix element, and

$$|\vec{P}| = 1 \frac{2M_1}{2M_1} \sqrt{(M_1^2 + M_2^2 - q^2)^2 - 4M_1^2M_2^2}$$

(6)

is the modulus of the three-momentum of the daughter meson in the parent meson rest frame and the helicity amplitudes $H_+$ and $H_0$ are given by

$$H_+ = \frac{2M_1|\vec{P}|}{\sqrt{q^2}} f_+(q^2), \quad H_0 = \frac{M_1^2 - M_2^2}{\sqrt{q^2}} f_0(q^2).$$

(7)

We note that $q^2 = q^2_{\text{max}}$ corresponds to zero-recoil of the final meson in the initial meson rest frame and the $q^2 = 0$ corresponds to the maximum recoil of the final meson recoiling with the maximum three momentum $|P_2| = (M_1^2 - M_2^2)$.  

In the LF calculation of the form factors, we use the metric convention $a \cdot b = \frac{1}{2}(a^+ b^- + a^- b^+) - a_\perpendicular \cdot b_\perpendicular$. Performing the LF calculation in the $q^2 = 0$ frame (i.e. $q^2 = -q_{\perpendicular}^2 = -Q^2 < 0$) with $P_1 = (P_1^+, P_1^-)$ and $P_2 = (P_2^+, P_2^-)$, we utilize all three components ($\mu, \nu = +, -, \perpendicular$) of the current $J_{\mu}^V$ and $J_{\mu}^T$ in Eqs. (1) and (2) to obtain $f_+(q^2)$, $f_-(q^2)$ and $f_T(q^2)$. The form factors obtained in the spacelike region ($q^2 < 0$) are then analytically continued to the timelike region by changing $q_{\perpendicular}^2$ to $-q^2$ in the form factors as we show in our numerical calculations.

While the form factor $f_+(q^2)$ can be obtained from the plus component ($J_\mu^+$) of the vector current, one cannot but use two different combinations of the current to obtain $f_-(q^2)$ such as $(J_1^+ J_\perpendicular)_V$ or $(J_1^+ J_{-})_V$. That is, using those sets of the current components in the $q^2 = 0$ frame, one obtains the relations between the weak form factors $f_{\pm}(q^2)$ and the current matrix elements in Eq. (1) as follows [54]:

$$f_+(q^2) = \frac{\mathcal{M}^+}{2P_1^+},$$

(8)

$$f_-(q^2) = \frac{\mathcal{M}^-}{2P_1^-} + \frac{\mathcal{M}_{\ell} \cdot q_\ell}{q_\ell^2},$$

(9)

$$f_T(q^2) = \frac{\mathcal{M}^T}{2P_1^T} \left(\frac{\Delta M_2^2 + q_\perpendicular^2}{\Delta M_2^2 - q_\perpendicular^2}\right) + \frac{P_1^+ \mathcal{M}_{\ell}}{\Delta M_2^2 - q_\perpendicular^2},$$

(10)

where $\Delta M_2^2 = M_2^2 \pm M_2^2$ and we denote $f_-(q^2)$ obtained from $(J_1^+ J_\perpendicular)_V$ and $(J_1^+ J_{-})_V$ as $f_-^{(+)}(q^2)$ and $f_-^{(-)}(q^2)$, respectively. It is prerequisite to show that $f_-^{(+)}(q^2) = f_-^{(-)}(q^2)$ to assert the Lorentz invariance of the form factor and the self-consistency of the model.

Likewise, the tensor form factor $s(q^2)$ can be obtained from using either $J_{\perpendicular}^+$ or $J_{\perpendicular}^-$. In this case, the relations between $s(q^2)$ and the current matrix element in Eq. (2) are given by

$$s^{(+)}(q^2) = -\frac{i \mathcal{M}_{\ell}^+ \cdot q_\perpendicular}{2P_1^+},$$

(11)

$$s^{(-)}(q^2) = -\frac{i \mathcal{M}_{\ell}^-}{2(\Delta M_2^2 - q_\perpendicular^2)},$$

(12)

where $s^{(+)}(q^2)$ and $s^{(-)}(q^2)$ represent the form factor $s(q^2)$ obtained from $J_{\perpendicular}^+$ and $J_{\perpendicular}^-$, respectively. Of course, $s^{(+)}(q^2) = s^{(-)}(q^2)$ should be satisfied in the self-consistent model calculation.

Our aim in this work is to show $s^{(+)}(q^2) = s^{(-)}(q^2)$ in addition to our previous verification of $f_-^{(+)}(q^2) = f_-^{(-)}(q^2)$ [54] in our LFQM, which completes the analysis of the exclusive semileptonic and rare decays between two pseudoscalar mesons. For this purpose, we start from the exactly solvable manifestly covariant BS model and then connect it to our phenomenologically accessible LFQM. Although we analyzed $f_{\pm}(q^2)$ in [54], we shall include them again in the next section for the completeness of the analysis.

III. MODEL DESCRIPTION

A. Manifestly covariant model

In the solvable model, based on the covariant BS model of $(3 + 1)$-dimensional fermion field theory [50, 54], the matrix elements $\mathcal{M} = (\mathcal{M}^\mu, \mathcal{M}^{\mu \nu})$ are given by

$$\mathcal{M} = iN_\ell \int \frac{d^3k}{(2\pi)^3} H_{P_1} \mathcal{F} H_{P_2},$$

(13)

where the corresponding trace terms $\mathcal{F} = (S^\mu, T^{\mu \nu})$ are given by

$$S^\mu = \text{Tr} [\gamma_5 (\not{P}_1 + m_1) \not{P}^\mu (\not{P}_2 + m_2) \gamma_5 (-\not{k} + m_q)],$$

(14)
for the vector current and

\[ T^\mu\nu = \text{Tr}[\bar{\psi} (p_1 + m_1) \sigma^{\mu\nu} (p_2 + m_2) \psi (-\sl{k} + m_q)], \quad (15) \]

for the tensor current, respectively. \( N_c \) is the number of colors and \( p_j = P_j - k (j = 1, 2) \) and \( k \) are the internal momenta carried by the quark and antiquark propagators of masses \( m_j \) and \( m_q \), respectively. The corresponding denominators are given by \( N_{p_j} = p_j^2 - m_j^2 + i\epsilon \) and \( N_k = k^2 - m_q^2 + i\epsilon \). We take the \( q\bar{q} \) bound-state vertex functions \( H_{p_j}(p_j^2, k^2) = g_j/(p_j^2 - \Lambda_j^2 + i\epsilon) \) of the initial \((j=1)\) and final \((j=2)\) state pseudoscalar mesons, where \( g_j \) and \( \Lambda_j \) are constant parameters in this manifestly covariant model.

Performing the LF calculation in the \( q^+=0 \) frame, one obtains the following identity \( g = q_{\text{on}} + \frac{1}{2} y \tau_\perp \Delta_q \), where \( \Delta_q^- = q^- - q_{\text{on}} \) and the subscript (on) denotes the on-mass shell quark propagating part and the “off”-mass shell instantaneous one, i.e. \( S^\mu = S_{\text{on}}^\mu + S_{\text{off}}^\mu \) for the vector current and \( T^\mu\nu = T_{\text{on}}^\mu\nu + T_{\text{off}}^\mu\nu \) for the tensor current.

The explicit LF calculation in parallel with the manifestly covariant calculation of Eq. (13) to compute \( f_{\pm}(q^2) \) can be found in [49] where \( f_{\pm}(q^2) \) was obtained from \( f^{(+\pm)}(q^2) \). The identical result for \( f^{(+\pm)}(q^2) \) was also obtained in [45, 46] using the so-called “covariant LFQM” analysis. As shown in Ref. [45, 46, 49], while \( f_{\pm}(q^2) \) obtained from the plus current was immune to the zero mode, the form factor \( f_{\pm}^-(q^2) \) received both instantaneous and zero-mode contributions. The same situation happens for \( f_{-}(q^2) \) although the zero-mode and the instantaneous contributions may differ quantitatively from \( f^{(+\pm)}(q^2) \). However, as we have shown in [54], \( f^{(+\pm)}(q^2) \) and \( f^{(+\pm)}(q^2) \) obtained in the standard LFQM by using our new correspondence relations between the BS model and the standard LFQM show identical result in the valence region of the \( q^+ = 0 \) frame without involving explicit zero modes and instantaneous contributions. In this work, we shall show that the tensor form factor \( s_{\pm}(q^2) \) in the standard LFQM is independent of the components of the current, i.e. \( s^{(+\pm)}(q^2) = s^{(\pm\pm)}(q^2) \). We should note that all of those equalities, i.e. \( f_{\pm}^{(+\pm)}(q^2) = f_{\pm}^{(+\pm)}(q^2) \) and \( s^{(+\pm)}(q^2) = s^{(\pm\pm)}(q^2) \), are derived from the constraint of the on-mass shellness of the quark and antiquark propagators together with the zero-binding energy limit (i.e. \( M = M_0 \)) used in the standard LFQM.

Therefore, from now on, we discuss only for the on-mass shell contributions in the valence region of the \( q^+ = 0 \) frame between the manifestly covariant BS model and the standard LFQM. The on-shell contributions to \( S^\mu \) and \( T^\mu\nu \) are given by

\[ S_{\text{on}}^\mu = 4 \left[ p_{1\text{on}}^\mu (P_{2\text{on}} \cdot k_{\text{on}} + m_2 q_{\text{on}}) + p_{2\text{on}}^\mu (P_{1\text{on}} \cdot k_{\text{on}} + m_1 q_{\text{on}}) \right] + k_{\text{on}}^\mu (m_1 m_2 - P_{1\text{on}} \cdot P_{2\text{on}}), \quad (16) \]

and

\[ T_{\text{on}}^\mu\nu = 4 i \left[ p_{1\text{on}}^\mu (m_2 k_{\text{on}}^\nu + m_q p_{2\text{on}}^\nu) - p_{2\text{on}}^\mu (m_1 k_{\text{on}}^\nu + m_q p_{1\text{on}}^\nu) \right] + m_{\text{on}}^\nu (m_1 p_{2\text{on}}^\mu - m_2 p_{1\text{on}}^\mu), \quad (17) \]

respectively. The LF four-momenta of the on-mass shell quark and antiquark propagators in the \( q^+ = 0 \) (i.e. \( P_1^+ = P_2^+ \)) frame are given by

\[ p_{1\text{on}} = \left[ x p_1^0, \frac{m_1^2 + k_1^2}{x p_1^0}, -k_\perp \right], \quad (18) \]

\[ p_{2\text{on}} = \left[ x p_1^0, \frac{m_2^2 + (k_\perp + q_\perp)^2}{x p_1^0}, -k_\perp - q_\perp \right], \quad (19) \]

\[ k_{\text{on}} = \left[ (1-x) p_1^0, \frac{m_2^2 + k_\perp^2}{(1-x) p_1^0}, k_\perp \right], \quad (20) \]

where \( x = \frac{p_1^0}{p_2^0} \) and \( x = k_\perp^2 \) are the LF longitudinal momentum fractions of the quark and antiquark, which satisfy \( x + x = 1 \).

By the integration over \( k^- \) in Eq. (13) and closing the contour in the lower half of the complex \( k^- \) plane, one picks up the residue at \( k^- = k^-_{\text{on}} \) in the valence region of \( 0 < k^+ < P^+ \) (or \( 0 < x < 1 \)). We denote the on-mass shell contribution to \( \mathcal{M}^\mu \) and \( \mathcal{M}^\mu_T \) in the valence region as \( [\mathcal{M}^\mu_{\text{on}}]^\text{BS} \) and \( [\mathcal{M}^\mu_T]_{\text{on}} \), respectively. The explicit forms of \( [\mathcal{M}^\mu]_{\text{on}} \) and \( [\mathcal{M}^\mu_T]_{\text{on}} \) are obtained as [54]

\[ \mathcal{M}^\mu_{\text{on}} = N_c \int_0^1 dx \int \frac{d^2 k_\perp}{16\pi^2} \chi_1(x, k_\perp) \chi_2(x, k'_\perp) \mathcal{F}_{\text{on}}, \quad (21) \]

where \( \mathcal{M} = (\mathcal{M}^\mu, \mathcal{M}^\mu_T) \) pairs with \( \mathcal{F} = (S^\mu, T^\mu\nu) \). The LF quark-meson vertex function \( \chi_1(2) \) of the initial (final) state is given by

\[ \chi_1(2)(x, k_\perp) = \frac{s_{1(2)}}{x^2 (M_{1(2)}^2 - M_{1(2)}^0)^2 (M_{1(2)}^2 - M_{1(2)}^0)_{\Lambda_1(\Lambda_2)}}, \quad (22) \]

where \( k_\perp = k_\perp + (1-x)q_\perp \) and

\[ M_0^2 = \frac{k_\perp^2 + m_1^2}{x} + \frac{k_\perp^2 + m_q^2}{1-x}, \quad (23) \]

\[ M_0^2 = \frac{k_\perp^2 + m_1^2}{x} + \frac{k_\perp^2 + m_2^2}{1-x}, \quad (24) \]

are the invariant masses of the initial and final states, respectively. Likewise, \( M_{\Lambda_1(2)} \) are obtained as \( M_{\Lambda_1} = M_0 (m_1 \to \Lambda_1) \) and \( M_{\Lambda_2}' = M_0 (m_2 \to \Lambda_2) \).

For the trace \( \mathcal{F} = (S^\mu, T^\mu\nu) \) calculations relevant to the form factors, the on-mass shell contributions \( S_{\text{on}} \) obtained from all three components \( \mu = (+, \perp, -) \) of the vector current \( J^\mu \) are given by [54]

\[ S_{\text{on}} = \frac{4P^+_1}{x} (k_\perp \cdot k'_\perp + \alpha_1, \alpha_2), \quad (25) \]
other three form factors $f^{(+\perp)}$, $f^{(-\perp)}$, and $s^{(+\perp)}$ are contaminated by the zero modes as well as the instantaneous contributions, the on-mass shell contributions $[f^{(+\perp)}]_{\text{on}}$, $[f^{(-\perp)}]_{\text{on}}$, and $[s^{(+\perp)}]_{\text{on}}$ themselves can never be the same as the exact solutions unless the zero modes and the instantaneous contributions are taken into account. Furthermore, one can easily check that $[f^{(+\perp)}]_{\text{on}} \neq [f^{(-\perp)}]_{\text{on}}$ and $[s^{(+\perp)}]_{\text{on}} \neq [s^{(+\perp)}]_{\text{on}}$.

However, in the following subsection, we shall show in the standard LFQM (denoted by SLF) that $f^{\text{SLF}} = [f^{(+\perp)}]_{\text{SLF}}$ and $s^{\text{SLF}} = [s^{(+\perp)}]_{\text{SLF}}$ without involving explicit zero-mode and instantaneous contributions, which comes after using our new correspondence relations between the BS model and the standard LFQM.

B. The standard LFQM

In the standard LFQM [39, 41, 42, 55–57, 60, 61], the LF wave function (LFWF) of a ground state pseudoscalar meson as a $q\bar{q}$ bound state is given by

$$\Psi_{\lambda\bar{\lambda}}(x, k_\perp) = \phi(x, k_\perp)R_{\lambda\bar{\lambda}}(x, k_\perp),$$

(36)

where $R_{\lambda\bar{\lambda}}(x, k_\perp)$ is the spin-orbit WF that is obtained by the interaction-independent Melosh transformation from the ordinary spin-orbit WF assigned by the quantum number $J^{PC}$. The covariant form of $R_{\lambda\bar{\lambda}}$ with the definite spin $(S, C) = (0, 0)$ constructed out of the LF helicity $\lambda(\bar{\lambda})$ of a quark (antiquark) is given by

$$R_{\lambda\bar{\lambda}} = \frac{\tilde{u}_\lambda(p_\bar{q})\gamma_5 v_{\bar{\lambda}}(p_\bar{q})}{\sqrt{2[M_0^2 - (m_1 - m_\bar{q})^2]^{1/2}}},$$

(37)

which satisfies the unitarity condition, $\Sigma_{\lambda\bar{\lambda}} R_{\lambda\bar{\lambda}}^\dagger R_{\lambda\bar{\lambda}} = 1$. Its explicit matrix form is given by

$$R_{\lambda\bar{\lambda}} = \frac{1}{\sqrt{2\sqrt{k_\perp^2 + \alpha^2}}}
\begin{pmatrix}
k^\perp & -\alpha^2 & k^R \\
\alpha^2 & k^\perp & -k^R \\
k^R & -k^R & 0
\end{pmatrix},$$

(38)

where $k^R = k_x + ik_y$ and $k^\perp = k_x - ik_y$.

For the radial WF $\phi(x, k_\perp)$ in Eq. (36), we use the Gaussian WF

$$\phi(x, k_\perp) = \frac{4\pi^{3/2}}{\beta^{3/2}} \sqrt{\frac{d_{k_\perp}}{dx}} \exp\left(-\frac{k^2}{2\beta^2}\right),$$

(39)

where $k^2 = k_\perp^2 + k_z^2$ and $\beta$ is the variational parameter. The longitudinal component $k_z$ is defined by $k_z = (x - \frac{1}{2})M_0 + \frac{m_1 - m_\bar{q}}{M_0}$, and the Jacobian of the variable transformation $\{x, k_\perp\} \rightarrow \tilde{k} = (k_\perp, k_z)$ is given by $\frac{dk}{dx} = \frac{M_0}{4\pi(1-x)}[1 - (\frac{m_1 - m_\bar{q}}{M_0})^2]$. The normalization of our Gaussian radial WF is...
then given by
\[ \int_0^1 dx \int \frac{d^2 k_{\perp}}{16\pi^3} |\phi(x, k_{\perp})|^2 = 1. \] (40)

In particular, the key idea in our LFQM [49, 55–57] for mesons is to treat \( \phi(x, k_{\perp}) \) as a trial function for the variational principle to the QCD-motivated effective Hamiltonian saturating the Fock state expansion by the constituent quark and antiquark. Using this Hamiltonian, we analyze the meson mass spectra and various wave-function-related observables, such as decay constants, electromagnetic form factors of mesons in a spacelike region, and the weak form factors, such as in Eq. (43) together with the same operators given by Eqs. (32), (33), and (35) to the other form factors \( \mathcal{F} = \{ f^{(+)}_{-}, f^{(-)}_{-}, f^{(+)}_{+}, f^{(-)}_{+} \} \) obtained from the only on-mass shell contributions, one encounters the same problems as the BS model, i.e. \( [f^{(+)}_{-}]_{\text{SLF}} \neq [f^{(-)}_{-}]_{\text{SLF}} \) and \( \{[s^{(+)}_{-}]_{\text{SLF}}, \} \neq \{[s^{(-)}_{-}]_{\text{SLF}} \} \) implying that the same physical quantities obtained from different components of the current yield different results.

In our previous analysis [51–53], however, we found that the correspondence relation including only LF vertex functions given by Eq. (43) brings about the self-consistency problem, i.e. the same physical quantity obtained from different components of the current and/or the polarization vectors yields different results in the standard LFQM. Furthermore, we also discovered the additional requirement for the correct correspondence relation between the two models to obtain the current-component independent physical observables in the standard LFQM.

Our new correspondence relation (denoted by “CJ-scheme” for convenience) to restore the self-consistency in the standard LFQM is given by [51–54]:
\[ \sqrt{2N_c} \frac{\lambda_{\perp}(x, k_{\perp})}{1-x} \rightarrow \frac{\phi_{1(2)}(x, k_{\perp})}{\sqrt{\lambda_{\perp}(2) + k_{\perp}^2}} M_{1(2)} \rightarrow M_0^0, \] (44)

that is, the physical mass \( M_{1(2)} \) included in the integrand of the BS amplitude, e.g., the operators \( \langle \sigma \rangle_{\text{on}}^{\text{BS}} \) in Eqs. (31)-(35) should be replaced by the invariant mass \( M_0^0 \) as all constituent quark and antiquarks are required to be on their respective mass shell in the standard LFQM. We should note that this “CJ-scheme” has been verified through our previous analyses for the decay constants and the twist-2 and-3 DAs of pseudoscalar and vector mesons [51–53] and the form factor \( f_{-}(q^2) \) for the semileptonic B decays [54].

We now show in this work that the “CJ-scheme” is also valid to obtain the current-component independent tensor form factor \( s(q^2) \) in addition to \( f_{-}(q^2) \) [54]. That is, applying Eq. (44) to the form factors \( \mathcal{F} = \{ f^{(+)}_{-}, s^{(+)}_{-} \} \) defined in Eq. (30) implies the replacements of the current operators \( \langle \sigma \rangle_{\text{on}}^{\text{BS}} = \{ \sigma^{(+)}_{-}, \sigma^{(-)}_{-} \} \) in the BS model with \( \langle \sigma \rangle_{\text{SLF}} = \{ \sigma^{(+)}_{-}, \sigma^{(-)}_{-} \} \) in the standard LFQM, i.e.,
\[ \sigma^{(+)}_{-} = \frac{S_{on}^+}{2P_{on}^+} \left( \frac{\Delta M_{5-0}^++q_{5-0}^2}{\Delta M_{5-0}^+-q_{5-0}^2} + \frac{P_{on+}^-S_{on}^-}{\Delta M_{5-0}^+-q_{5-0}^2} \right), \] (45)
\[ \sigma^{(-)}_{-} = -\frac{IT_{on}^-}{2(\Delta M_{5-0}^+-q_{5-0}^2)} \] (46)
where \( \Delta M_{5-0}^0 = M_0^0 \pm M_0^2 \). Then, we obtain from Eqs. (42), (45), and (46) the current-component independent form factors, i.e. \( [f^{(+)}_{-}]_{\text{SLF}} = [f^{(-)}_{-}]_{\text{SLF}} \) and \( [s^{(+)}_{-}]_{\text{SLF}} = [s^{(-)}_{-}]_{\text{SLF}} \) in the standard LFQM, where “±” represents the equality of both sides numerically. The additional requirement in the “CJ-scheme”, i.e. \( M_{1(2)} \rightarrow M_0^0 \), can therefore be regarded as the effective inclusion of the zero modes in the valence region of
the $q^+ = 0$ frame in the standard LFQM. This replacement $M_{1(2)} \to M_{0}^{(t)}$ is not possible in the BS model due to the form of the LF vertex function $\chi$ given by Eq. (22).

\begin{equation}
 f_+(q^2) = \int_0^1 dx \int \frac{d^2k_\perp}{16\pi^3} \frac{\Phi_1(x,k_\perp)}{\sqrt{k^2_\perp + k^2_\perp}} \Phi_2(x,k') \left( \sigma_1 \sigma_2 + k_\perp \cdot k'_\perp \right),
\end{equation}

\begin{equation}
 f^{(+1)}_{(-)}(q^2) = \int_0^1 dx \int \frac{d^2k_\perp}{16\pi^3} \frac{\Phi_1(x,k_\perp)}{\sqrt{k^2_\perp + k^2_\perp}} \Phi_2(x,k') \left( \sigma_1 \sigma_2 + k_\perp \cdot k'_\perp \right)
 \times \left\{ \tilde{M}^2_0 + (m_2 - m_q)\sigma_1 - m_q(m_1 - m_q) + \frac{k_\perp \cdot q_\perp}{q^2} [M^2_0 + M^2_0 - 2(m_1 - m_q)(m_2 - m_q)] \right\},
\end{equation}

\begin{equation}
 f^{(-)}_{(-)}(q^2) = \int_0^1 dx \int \frac{d^2k_\perp}{16\pi^3} \frac{\Phi_1(x,k_\perp)}{\sqrt{k^2_\perp + k^2_\perp}} \Phi_2(x,k') \left( \sigma_1 \sigma_2 + k_\perp \cdot k'_\perp \right)
 \times \left\{ a_0 \left[ x^2 \tilde{M}^2_0 + k_\perp \cdot q_\perp \tilde{M}^2_0 + k_\perp \cdot k'_\perp + k_\perp \cdot q_\perp - \tilde{M}^2_0 \right] + x^2 q_\perp \left[ (m_1 - m_2) \frac{k_\perp \cdot q_\perp}{q^2} + \sigma_1 \right] \right\},
\end{equation}

for the vector current, and

\begin{equation}
 s^{(+1)}(q^2) = -\int_0^1 (1-x) dx \int \frac{d^2k_\perp}{16\pi^3} \frac{\Phi_1(x,k_\perp)}{\sqrt{k^2_\perp + k^2_\perp}} \Phi_2(x,k') \left[ (m_1 - m_2) \frac{k_\perp \cdot q_\perp}{q^2} + \sigma_1 \right],
\end{equation}

\begin{equation}
 s^{(-)}(q^2) = \int_0^1 dx \int \frac{d^2k_\perp}{16\pi^3} \frac{\Phi_1(x,k_\perp)}{\sqrt{k^2_\perp + k^2_\perp}} \Phi_2(x,k') \left[ (m_1 - m_2) \frac{k_\perp \cdot q_\perp}{q^2} + \sigma_1 \right]
 \times \left\{ \frac{a_0}{2} \left[ (1-x)(m_1 - m_2)k^2_\perp + 2(1-x)\sigma_1 k_\perp \cdot q_\perp + (1-x)\sigma_1 q^2_\perp + (m_2 - m_1)\sigma_1 \sigma_2 \right] \right\}.
\end{equation}

for the tensor current, where $a_0 = \frac{2}{M^2_0 - M^2_0 - q^2_\perp}$ and $b_0 = \frac{M^2_0 + M^2_0 + q^2_\perp}{M^2_0 - M^2_0 - q^2_\perp}$. Indeed, our prescription $M_{1(2)} \to M_{0}^{(t)}$ is applied through the two terms ($a_0b_0$) in $f^{(+1)}_{(-)}$ and $s^{(-)}$. Finally, we confirm from the numerical calculations the current independencies of the form factors, i.e. $f_-(q^2) = f^{(+1)}_{(-)} = f^{(-)}$ and $s(q^2) = s^{(+1)} \cong s^{(-)}$, which supports the universality of the “CJ-scheme” given by Eq. (44) and the self-consistency of our standard LFQM.

For our numerical calculations in the following section, we use the tensor form factor $f_T(q^2) = s(q^2)(M_1 + M_2)$ as defined in Eq. (4). We should emphasize that the physical masses $M_{1(2)}$ used in defining $f_T$ is nothing to do with our correspondence relations. Only the physical masses $M_{1(2)}$ appeared as a result from the choice of minus component ($\mu, v = -$) of the vector and tensor currents given by Eqs. (1) and (2) are eligible for the transformation into the corresponding invariant masses $M_{0}^{(t)}$ as shown in Eq. (10).

The final results for $f_+$, $f_- = f^{(+1)}_{(-)} = f^{(-)}$, and $s = s^{(+1)} \cong s^{(-)}$ in the standard LFQM are given by

IV. NUMERICAL RESULTS

In our numerical calculations for the semileptonic and rare $D \to (\pi,K)$ decays, we use the model parameters $(m_{q\bar{q}}, \beta_{q\bar{q}})$ for the harmonic oscillator (HO) confining potential given in Table I obtained from the calculation of the ground state meson mass spectra [49, 57]. The decay constants of $(\pi,K,D)$ mesons obtained from the HO parameters are given by $(f_\pi, f_K, f_D) = (131, 155, 197)$ MeV compared to the experimental data [62], $(f_\pi^{exp}, f_K^{exp}, f_D^{exp}) = (130.2(1.2), 155.7(3), 212.6(7))$ MeV. While the decay constant of $D$ meson is not quite sensitive to the quark mass variation, e.g. $f_D = 199.2$ MeV for $m_c = 1.7^{+0.1}$ GeV, we find that the form factors are somewhat sensitive to $m_c$. Thus, as a sensitivity check of our LFQM, we use this charm quark mass variation for the calculations of the form factors and the branching ratios. For the physical $(D,K,\pi)$ meson masses, we use the central values quoted by the Particle Data Group (PDG) [62].

In principle, it is possible to use the $q^+ \neq 0$ frame satisfy-
ing $q^2 = q^+q^- - q_T^2 > 0$ for this timelike semileptonic and rare decays. However, in this $q^+ ≠ 0$ frame, it is inevitable to confront the particle-number-nonconserving Fock state (or nonvalence) contribution [63, 64]. The main source of difficulty in the LFQM phenomenology is the paucity of information on the non-wave-function vertex [50] in the nonvalence diagram arising from the quark-antiquark pair creation/annihilation. This should contrast with the usual LWFV used in the valence region. Contrary to the $q^+ ≠ 0$ frame, the $q^+ = 0$ frame does not suffer from the nonvalence contribution although one needs to be cautious about the zero-mode problem as we discussed already. Once the zero-mode issue is resolved as we proved in this work, it is straightforward to analytically continue the form factors given by Eqs. (47)-(51) obtained in the spacelike region to the timelike physical region.

Our results of the form factors $(f_\pm, f_0, f_T)$ obtained from Eqs. (47)-(51) can also be compared with several parametric forms. Among several forms, a more systematic and model-independent parametrization of semileptonic form factors, often referred to as “$z$-expansion” or “$z$-parametrization” [65, 66], has been developed based on general properties of analyticity, unitarity, and crossing symmetries. Especially, this $z$-parametrization provides better control of theoretical uncertainties in LQCD calculations [23, 24].

Our direct LFQM results for the form factors $f_i(q^2)$ ($i = \pm, 0, T$) are also well described by the “$z$-parametrization”, which takes the form [23, 24]

$$f_i(q^2) = \frac{f_i(0) + c_j(z - z_0)(1 + \frac{z - z_0}{2})}{1 - b_i q^2},$$

where

$$z = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}},$$

and $z_0 = z(q^2 = 0)$ with $t_\pm = (M_1 \pm M_2)^2$ and $t_0 = t_+ (1 - \sqrt{1 - t_+ / t_0})$.

The fitted parameters $(b_i, c_j)(j = +, 0, T)$ for the $D \to \pi$ and $D \to K$ TFFs $(f_\pm, f_0, f_T)$ are summarized in Tables II and III, respectively, where the errors occur due to the choice of $m_c = 1.7^{+0.1}_{-0.1}$ GeV. In Table IV, we also compare the form factors $f_\pi(0)$ and $|f_T(0)|$ for $D \to (\pi, K)$ transitions at $q^2 = 0$ with those obtained from various theoretical model predictions and experimental data [13, 67, 68].

In Fig. 1, we show the $q^2$ dependences of $f^D_{\pi}(q^2)$ (black lines), $f^D_0(q^2)$ (blue lines), and $f^D_T(q^2)$ (red lines) for $D \to \pi$ decay, where the solid and dashed lines represent the results obtained from $m_c = 1.8$ GeV and 1.6 GeV, respectively. That is, the bands correspond to the sensitivity coming from the charm quark mass variation, $m_c = 1.7^{+0.1}_{-0.1}$ GeV in our LFQM. We should note that the form factors are displayed not only for the whole timelike kinematic region $[0 \leq q^2 \leq (M_D - M_\pi)^2]$ (in unit of GeV$^2$) but also for the spacelike region $(-0.5 \leq q^2 \leq 0)$ (in unit of GeV$^2$) to demonstrate the validity of our analytic continuation from spacelike region to the timelike one by changing $q^2$ to $-q^2$ in the form factors. For comparison, the data (circles) of the form factors $(f_\pm, f_0, f_T)$ from the LQCD (for the ETM Collaboration) [23, 24] and the data of $f_\pi$ (squares) extracted from the BABAR [67] are shown. Our results for $f^D_{\pi}(0) = 0.613^{+21}_{-22}$ and $|f^D_0(0)| = 0.501^{+36}_{-39}$ are in good agreement with $f^D_{\pi}(0) = 0.610(25)$ from the BABAR [67] and $f^D_0(0) = 0.637(24)$ from the BES III [13], as well as $f^D_{\pi}(0) = 0.612(35)$ and $|f^D_0(0)| = 0.506(79)$ from the LQCD [23, 24]. As one can see from Fig. 1, the sensitivity to the charm quark mass is more pronounced at the zero-recoil ($q^2 = q^2_{\text{max}}$) of the final meson than the maximum recoil ($q^2 = 0$). Especially, the $q^2$-dependent behaviors of our results show better agreement with the data from the BABAR [67] and LQCD [23, 24] when we use $m_c \simeq 1.6$ GeV rather than 1.8 GeV.

In Fig. 2, we show the $q^2$ dependences of $(f^D_{\piK}, f^D_{0K}, f^D_{T})$ for $D \to K$ decay, compared with the results from the LQCD [23, 24]. The same line codes are used as in Fig. 1. Our predictions of $f^D_{\piK}(0) = 0.744^{+22}_{-23}$ and $|f^D_{0K}(0)| = 0.660^{+42}_{-45}$ agree with $f^D_{\piK}(0) = 0.737(4)$ from the BES III [13] and $f^D_{0K}(0) = 0.727(11)$ from the BABAR [68], as well as $f^D_{\piK}(0) = 0.764(31)$ and $|f^D_{0K}(0)| = 0.685(54)$ from the LQCD [23, 24] within the error bars. As in the case of $D \to \pi$ decay, the $q^2$-dependent behaviors of our results show better agreement with the data from the LQCD [23, 24] when
we use \( m_c \approx 1.6 \) GeV rather than 1.8 GeV.

Figs. 3 and 4 show our predictions for the differential decay rates of \( D \to \pi \ell \nu_\ell \) and \( D \to K \ell \nu_\ell \) decays, respectively, compared with the experimental data from the BABAR [67, 68] (black circles), CLEO [69] (blue squares), and BES III [13, 14] for neutral \( D^0 \) (red diamonds) and charged \( D^+ \) with the account of isospin factor (green triangles). In our numerical calculations of the branching ratios, we use the CKM matrix elements \( |V_{u}\ell| = 0.221 \pm 0.004 \) and \( |V_{c}\ell| = 0.987 \pm 0.011 \) quoted by the PDG [62]. Considering uncertainties coming from the CKM elements and the constituent charm quark mass \( m_c = 1.7^{+0.1}_{-0.3} \) GeV, we made band plots, i.e. the solid (dashed) lines represent the results obtained from \( m_c = 1.8 \) (1.6) GeV with lower (upper) limits of the CKM elements. Our results are shown to be consistent with the current available experimental data within those uncertainties.

In Tables V and VI, we summarize our results for the branching ratios for \( D \to \pi \ell \nu_\ell \) and \( D \to K \ell \nu_\ell \) (\( \ell = e, \mu \)), respectively, and compare with the experimental data from PDG [62]. Our results for \( Br(D \to \pi) \) are best fit to the data with \( m_c = 1.6 \) GeV but those for \( Br(D \to K) \) prefers \( m_c = 1.7 \) GeV.

\textbf{TABLE II. Fitted parameters} \((b_j, c_j)\) in Eq. \( (52) \) for the \( D \to \pi \) TFFs with \( m_c = 1.7^{+0.1}_{-0.3} \) GeV. \( b_j \) is in unit of \([\text{GeV}^{-2}]\).

| \( f(+) (0) \) | \( b_+ \) | \( c_+ \) | \( b_0 \) | \( c_0 \) | \( f_T (0) \) | \( b_T \) | \( c_T \) |
|----------------|--------|--------|--------|--------|-------------|--------|--------|
| \( 0.613^{+0.21}_{-0.22} \) | \( 0.189^{+0.206}_{-0.233} \) | \(-0.8200^{+0.375}_{-0.317} \) | \( 0.2986^{+0.0882}_{+11.8745} \) | \( 1.9051^{+0.5966}_{-1.07077} \) | \(-0.501^{+0.36}_{-0.39} \) | \( 0.1957^{+0.216}_{-0.242} \) | \( 0.6290^{+0.421}_{-0.481} \) |

\textbf{TABLE III. Fitted parameters} \((b_j, c_j)\) in Eq. \( (52) \) for the \( D \to K \) decay with \( m_c = 1.7^{+0.1}_{-0.3} \) GeV. \( b_j \) is in unit of \([\text{GeV}^{-2}]\).

| \( f(+) (0) \) | \( b_+ \) | \( c_+ \) | \( b_0 \) | \( c_0 \) | \( f_T (0) \) | \( b_T \) | \( c_T \) |
|----------------|--------|--------|--------|--------|-------------|--------|--------|
| \( 0.744^{+0.22}_{-0.23} \) | \( 0.178^{+0.157}_{-0.176} \) | \(-1.1711^{+0.571}_{-0.618} \) | \(-0.056^{+0.0348}_{+0.1426} \) | \(-2.3039^{+0.3509}_{+1.8289} \) | \(-0.660^{+0.42}_{-0.45} \) | \( 0.1826^{+0.163}_{-0.182} \) | \( 0.9893^{+0.794}_{-0.87} \) |

Finally, as a test for the LFU, the \( R \) ratios of the semileptonic \( D \to (\pi, K) \) decays is defined by

\[
R_P = \frac{Br(D \to P \mu \nu_\mu)}{Br(D \to P \ell \nu_\ell)},
\]

where \( P = (\pi, K) \). Our predictions for \( R_P \) obtained from using \( m_c = 1.7^{+0.1}_{-0.3} \) GeV are as follows: \( R_{\pi^0} = (0.980^{+0.165}_{-0.0003} 0.983^{+0.001}_{-0.003}), R_{K^-} = (0.974^{+0.002}_{-0.001} 0.973^{+0.001}_{-0.001}), R_{K^+} = 0.974 \pm 0.014 \) [18], as well as other theoretical predictions such as \( R_\pi = 0.985(2) \) and \( R_K = 0.975(1) \) from the LQCD [70], \( R_\pi = 0.985 \) and \( R_K = 0.980 \) from the RQM [34], and \( R_\pi = 0.98 \) and \( R_K = 0.97 \) from the CCQM [35].
TABLE IV. Form factors $f_+(0)$ and $|f_T(0)|$ for $D \rightarrow (\pi, K)$ transitions at $q^2 = 0$ compared with various model predictions and experimental data.

| Channel | This work | [23, 24] | [27] | [34] | [35] | [44] | [47] | BES III [13] | BABAR [67, 68] |
|---------|-----------|---------|-------|------|------|------|------|-------------|---------------|
| $f_+^{D\pi}(0)$ | 0.613$^{+0.21}_{-0.22}$ | 0.612 (35) | 0.63 (11) | 0.640 | 0.63 (9) | $-$ | $-$ | 0.66 (1) | 0.637 (24) | 0.610 (25) |
| $|f_T^{D\pi}(0)|$ | 0.501$^{+0.16}_{-0.39}$ | 0.506 (79) | $-$ | $-$ | $-$ | 0.84$^{+0.16}_{-0.15}$ | $-$ | $-$ | $-$ |
| $f_+^{DK}(0)$ | 0.744$^{+0.24}_{-0.23}$ | 0.765 (31) | 0.75 (12) | 0.716 | 0.77 (11) | $-$ | 0.79 (1) | 0.737 (4) | 0.727 (11) |
| $|f_T^{DK}(0)|$ | 0.660$^{+0.42}_{-0.45}$ | 0.687 (54) | $-$ | $-$ | $-$ | 0.96$^{+0.17}_{-0.15}$ | $-$ | $-$ | $-$ |

TABLE V. Branching ratios (in $10^{-3}$) for $D \rightarrow \pi^0\nu\ell$ ($\ell = e, \mu$) obtained from using $m_c = 1.7^{+0.1}_{-0.1}$ GeV together with $|V_{cs}| = 0.221 \pm 0.004$ [62].

| Channel | Ours | PDG [62] |
|---------|------|----------|
| $D^+ \rightarrow \pi^0 e^+ \nu_e$ | 3.03$^{+0.57}_{-0.44}$ | 3.72$^{+0.17}_{-0.15}$ |
| $D^+ \rightarrow \pi^0 \mu^+ \nu_\mu$ | 2.97$^{+0.58}_{-0.44}$ | 3.50$^{+0.15}_{-0.14}$ |
| $D^0 \rightarrow \pi^- e^+ \nu_e$ | 2.37$^{+0.45}_{-0.34}$ | 2.91$^{+0.04}_{-0.03}$ |
| $D^0 \rightarrow \pi^- \mu^+ \nu_\mu$ | 2.33$^{+0.44}_{-0.34}$ | 2.67$^{+0.12}_{-0.11}$ |

TABLE VI. Branching ratios (in $10^{-3}$) for $D \rightarrow K^0\nu\ell$ ($\ell = e, \mu$) decays obtained from using $m_c = 1.7^{+0.1}_{-0.1}$ GeV together with $|V_{cs}| = 0.987 \pm 0.011$ [62].

| Channel | Ours | PDG [62] |
|---------|------|----------|
| $D^+ \rightarrow K^0 e^+ \nu_e$ | 8.88$^{+1.10}_{-0.80}$ | 8.73$^{+0.10}_{-0.10}$ |
| $D^+ \rightarrow K^0 \mu^+ \nu_\mu$ | 8.64$^{+1.07}_{-0.79}$ | 8.76$^{+0.19}_{-0.19}$ |
| $D^0 \rightarrow K^- e^+ \nu_e$ | 3.50$^{+0.43}_{-0.31}$ | 3.54$^{+0.034}_{-0.031}$ |
| $D^0 \rightarrow K^- \mu^+ \nu_\mu$ | 3.41$^{+0.41}_{-0.30}$ | 3.41$^{+0.004}_{-0.004}$ |

V. SUMMARY AND DISCUSSION

In this work, we discussed the self-consistency description on the weak form factors $f_+$, $f_-$ (or $f_0$), and $f_T$ for the exclusive semileptonic $D \rightarrow (\pi, K)\ell\nu_\ell (\ell = e, \mu, \tau)$ and rare $D \rightarrow (\pi, K)\ell\nu_\ell$ decays in the standard LFQM. It has been known in the LF formulation that while the plus component ($J^+$) of the LF current $J^\mu$ in the matrix element can be regarded as the “good” current, the perpendicular ($J^\perp$) and the minus ($J^-$) components of the current were known as the “bad” currents since ($J^+, J^-)$ are easily contaminated by the treacherous points such as the LF zero mode and the off-mass shell instantaneous contributions.

For a systematic analysis of such treacherous points in case one cannot avoid the use of $J^\perp$ or $J^-$, we utilized the exactly solvable manifestly covariant BS model to carry out the LF calculations for three form factors, ($f_+, f_-, f_T$). In particular, we obtained $f_-$ from two sets of the vector current, $(J^+, J^-)\nu$ and $(J^+, J^-)\nu$, and $f_T$ from two sets of tensor current, $J_T^\perp$ and $J_T^-$. In this BS model, we found that while $f_+$ obtained from $J^+$ and $f_T$ obtained from $J_T^\perp$ are free from the zero modes, $f_-$ obtained from both $(J^+, J^-)\nu$ and $(J^+, J^-)\nu$ sets and $f_T$ obtained from $J_T^\perp$ receive the zero-mode contributions as well as the instantaneous ones. We then linked the BS model to the standard LFQM using the “CJ-scheme” [51–54] given by Eq. (44) for the correspondences between the two models and replaced the LF vertex function in the BS model with the more phenomenologically accessible Gaussian wave function provided by the LFQM analysis of meson mass spectra [55, 56]. As in the case of previous analysis [51–54], it is astonishing to discover that the zero modes and the instantaneous contributions present in the BS model become absent in the LFQM. In other words, our LFQM results of $(f_-, f_T)$ are shown to be independent of the components of the current without involving any of those treacherous contributions. Since the absence of the zero mode found in the standard LFQM is mainly due to the replacement of the physical mass $M_{(1,2)}$ with the invariant mass $M_0^{(1,2)}$ in the course of linking the two models, this replacement could be regarded as an effective treatment of the zero mode in the standard LFQM.

In the standard LFQM, the constituent quark and antiquark in a bound state are required to be on-mass shell, which is different from the covariant formalism, in which the constituents are off-mass shell. The common feature of the standard LFQM is thus to use the sum of the LF energy of the
constituent quark and antiquark for the meson mass in the spin-orbit wave function, which is obtained by the interaction-independent Melosh transformation from the ordinary equal-time static spin-orbit wave function assigned by the quantum number $j^P_C$. Under these circumstances, it is natural to apply the replacement $M_{1(2)} \rightarrow M^I_0$ in the calculation of the physical observables in the standard LFQM. Indeed, we have shown explicitly that this correspondence relation for the calculations of the decay constants and weak transition form factors between two pseudoscalar mesons provide the current-component independent predictions in the standard LFQM.

We then apply our current-component independent form factors $(f_\pm, f_T)$ for the self-consistent analysis of semileptonic and rare $D \rightarrow (\pi, K)$ decays using our LFQM constrained by the variational principle for the QCD-motivated effective Hamiltonian with the HO plus Coulomb interaction [49, 57]. The form factors $(f_\pm, f_T)$ obtained in the $q^+ = 0$ frame ($q^2 = -q_T^2 < 0$) are then analytically continued to the timelike region by changing $q^2 \rightarrow -q^2_T$ in the form factors. In our numerical calculations, we also checked the sensitivity of the constituent charm quark mass $m_c = 1.7_{-0.7}^{+0.1}$ GeV through the analysis of the form factors $(f_\pm, f_T)$ for $D \rightarrow (\pi, K)$ decays. Our results for the form factors and branching ratios for $D \rightarrow (\pi, K)$ decays show in good agreement with the available experimental data as well as other theoretical predictions. Especially, the smaller charm quark mass $m_c \approx 1.6$ GeV seems preferable to larger $m_c = 1.8$ GeV for $D \rightarrow (\pi, K)$ decays while they are not much different for the analysis of the decay constant of the $D$ meson. Finally, we obtained the the $R$ ratios of the semileptonic $D \rightarrow (\pi, K)$ decays as a test for the LFU and our results are consistent with the recent measurements from the BES III [15, 18] as well as other theoretical results [34, 35, 70].

While the rare decay analyses including the tensor form factor can in principle be made, I just focused on the extraction of the current-component independent weak transition form factors as well as the comparison with the available experimental data in the present work. More complete phenomenological analyses regarding on the rare decays of heavy $D$ and $B$ mesons are also under consideration.

DATA AVAILABILITY

The data used to support the findings of this study are available from the author upon request.

CONFLICTS OF INTEREST

The author declares that there are no conflicts of interest regarding the publication of this paper.

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