The Linearisation Map in Algebraic $K$-Theory

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16.08.2004

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Let $X$ be a pointed connected simplicial set with loop group $G$. The linearisation map in $K$-theory as defined by Waldhausen uses $G$-equivariant spaces. This paper gives an alternative description using presheaves of sets and abelian groups on the simplex category of $X$. In other words, the linearisation map is defined in terms of $X$ only, avoiding the use of the less geometric loop group. The paper also includes a comparison of categorical finiteness with the more geometric notion of finite CW objects in cofibrantly generated model categories. The application to the linearisation map employs a model structure on the category of abelian group objects of retractive spaces over $X$.

AMS subject classification (2000): primary 19D10, secondary 55U35

Keywords: Algebraic $K$-Theory of spaces, linearisation, finiteness conditions

to appear in Forum Mathematicum

Introduction

Let $W$ be a simplicial set with a (right) action of the simplicial monoid $G$. The category of $G$-equivariant retractive spaces over $W$ is denoted $\mathcal{R}(W, G)$; objects are triples $(Y, r, s)$ where $Y$ is a simplicial set equipped with a $G$-action, $r: Y \to W$ is a $G$-equivariant map, and $s$ is a $G$-equivariant section of $r$. If $G$ is the trivial monoid, it is omitted from the notation.

The category $\mathcal{R}(W, G)$ has been introduced by Waldhausen to study the algebraic $K$-theory of spaces. For a pointed connected simplicial set $X$, the $K$-theory space $A(X)$ is defined as $A(X) := \Omega hS_* \mathcal{R}(X)$ where $\mathcal{R}(X)$ is a certain subcategory of $\mathcal{R}(X)$ consisting of “finite” objects. It is proved in §2.3 that one can replace $\mathcal{R}(X)$ by the category $\mathcal{R}(*, G)$ where $G$ is a loop group of $X$. The latter category is the domain of the linearisation functor which defines a natural map from $A(X)$ to the algebraic $K$-theory space $K(\mathbb{Z}[G])$ of the simplicial group ring $\mathbb{Z}[G]$.

Since the loop group $G$ is a less geometric object than the simplicial set $X$ itself, it is useful to have a description of the linearisation map purely in terms of spaces over $X$ as suggested by Waldhausen §2.3, p. 400]. The domain of the new linearisation map is the category $\mathcal{R}(X)$, the codomain is the category $\mathcal{R}^{ab}(X)$ of abelian group objects in $\mathcal{R}(X)$, and the map is induced by applying the (reduced) free abelian group functor to the fibres of the retractions $Y \to X$. 

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The goal of this paper is to provide a detailed account of the linearisation map. In particular, we discuss finiteness conditions for objects in $\mathcal{R}^{ab}(X)$ (not treated in [8, §2.3]), and give an intrinsic definition of weak equivalences in $\mathcal{R}^{ab}(X)$ (avoiding the use of simplicial $\mathbb{Z}[G]$-modules as in loc.cit.).

In §1 we introduce the relevant categories and functors. In particular, we prove that retractive objects are nothing but presheaves on the simplex category:

**Proposition 1.1.3 and 1.2.1** Suppose $W$ is a simplicial set equipped with an action of the simplicial group $G$. Then there are equivalences of categories $\mathcal{R}(W, G) \cong \text{Fun}(\text{Simp}_G(W)^{\text{op}}, \text{Set}_*)$ and $\mathcal{R}^{ab}(W, G) \cong \text{Fun}(\text{Simp}_G(W)^{\text{op}}, \text{Ab})$. Here $\text{Simp}_G(W)$ is the equivariant simplex category of $W$ (Definition 1.1.1), and the superscript “ab” denotes abelian group objects.

In §2 we set up the necessary machinery to discuss $K$-theory. A main ingredient is the model structure on the category of abelian group objects in $\mathcal{R}(X)$ established by DWYER and KAN [2], and generalised here to cover the equivariant version $\mathcal{R}(W, G)$ as well (Theorem 2.4.2); this allows us to characterise weak equivalences in the category $\mathcal{R}^{ab}(W, G)$ intrinsically instead of using the pullback procedure suggested in [8, §2.3].

Another advantage of the model category setup is that we can give a conceptual treatment of finiteness conditions (2.1.1). An object $Y$ of a model category $\mathcal{C}$ is called **categorically finite** if the representable functor $\mathcal{C}(Y, \cdot)$ commutes with filtered colimits. For $\mathcal{R}(X)$ this recovers the usual notion of a finite object as used in [8, §2.1].

**Theorem 2.1.1** Suppose $\mathcal{C}$ is a cofibrantly generated model category where all generating cofibrations have categorically finite domain and codomain. Then an object is cofibrant and categorically finite if and only if it is a retract of a finite CW object.

Finally, we prove that the new and old description of the linearisation map are equivalent:

**Theorem 2.5.4** Suppose $X$ is a pointed simplicial set with loop group $G$. There is a diagram of categories, commuting up to natural isomorphism,

$$
\begin{array}{ccc}
\mathcal{R}^{ab}(X)_{\text{cf}} & \longrightarrow & \mathcal{R}^{ab}(\ast, G)_{\text{cf}} \\
\uparrow & & \uparrow \\
\mathcal{R}(X) = \mathcal{R}(X)_{\text{cf}} & \longrightarrow & \mathcal{R}(\ast, G)_{\text{cf}}
\end{array}
$$

such that the horizontal arrows induce equivalences of $K$-theory spaces, and the right vertical functor induces the usual linearisation map. The $K$-theory space of $\mathcal{R}(X)_{\text{cf}}$ is $A(X)$. (The subscript “cf” refers to the subcategories of cofibrant, categorically finite objects.)

Throughout the paper we will freely use the language of model categories [5, 8] and of categories with cofibrations and weak equivalences [8].
1 Retractive Spaces, Presheaves and Linearisation

1.1 The equivariant simplex category and presheaves of sets

We start by generalising the usual simplex category of a simplicial set to the equivariant setting.

**1.1.1 Definition.** Let $G$ denote a simplicial monoid acting from the right on a simplicial set $W$. The *equivariant simplex category* of $W$, denoted $\text{Simp}_G(W)$, is the following category: Objects are the simplices of $W$, formally pairs $(n, w) \in W_n$; a morphism $(m, v) \rightarrow (n, w)$ is a pair $(\alpha, g)$ where $\alpha: [m] \rightarrow [n]$ is a morphism in $\Delta$ and $g$ is an element of $G_m$ such that $v = W(\alpha)(w) \cdot g$. Composition of morphisms is defined as $(\beta, h) \circ (\alpha, g) := (\beta \circ \alpha, G(\alpha)(h) \cdot g)$.

It is clear from the definition that the category $\text{Simp}_G(W)$ depends covariantly on $W$ and $G$; a semi-equivariant map $(W, G) \rightarrow (V, H)$ induces a functor $\text{Simp}_G(W) \rightarrow \text{Simp}_H(V)$.

There is a more conceptual description of the simplex category. Let $H$ be a (discrete) monoid acting from the right on a set $S$. The *transport category* of $S$ is the category $\text{Tr}_H(S)$ with objects the elements of $S$, and a morphism $s \rightarrow t$ is an element $h \in H$ such that $s = t \cdot h$. Composition of morphisms is given by multiplication in $H$.

The transport category is functorial in $S$ and $H$. In particular, a $G$-equivariant simplicial set $W$ gives rise to a simplicial category

$$\text{Tr}: \Delta^{\text{op}} \rightarrow \text{Cat}, \quad [n] \mapsto \text{Tr}_{G_n}(W_n).$$

If $C$ is a category, we associate to a functor $F: C^{\text{op}} \rightarrow \text{Cat}$ into the category of (small) categories a new category, the *Grothendieck construction* $\text{Gr}(F)$ of $F$. Objects are the pairs $(c, x)$ with $c$ an object of $C$ and $x$ an object of $F(c)$. A morphism $(c, x) \rightarrow (d, y)$ consists of a morphism $f: c \rightarrow d$ in $C$ and a morphism $\alpha: x \rightarrow F(f)(y)$ in $F(c)$ with composition given by the formula $(f, \alpha) \circ (g, \beta) := (f \circ g, F(g)(\alpha) \circ \beta)$. The category $\text{Gr}(F)$ is called the lax colimit $F \int C^{\text{op}}$ in $[? \text{ §3}].$

**1.1.2 Lemma.** The category $\text{Simp}_G(W)$ is the Grothendieck construction of the functor $\text{Tr}: \Delta^{\text{op}} \rightarrow \text{Cat}, [n] \mapsto \text{Tr}_{G_n}(W_n).$ \hfill $\Box$

**1.1.3 Proposition.** The category $\mathcal{R}(W, G)$ is the category of set-valued presheaves on $\text{Simp}_G(W)$, i.e., $\mathcal{R}(W, G)$ is equivalent to $\text{Fun}(\text{Simp}_G(W)^{\text{op}}, \text{Set}_*)$. The functor corresponding to $(Y, r, s) \in \mathcal{R}(W, G)$ sends $(n, w) \in \text{Simp}_G(W)$ to the fibre $r^{-1}(w)$ over $w \in W_n$. In particular, the category $\mathcal{R}(W, G)$ is complete and cocomplete; limits and colimits are computed fibrewise.

**Proof.** An object $Y = (Y, r, s) \in \mathcal{R}(W, G)$ determines a functor

$$\text{Simp}_G(W)^{\text{op}} \rightarrow \text{Set}_*$$

by sending $(n, w)$ to the set $r^{-1}(w)$ with basepoint given by $s(w)$.
Conversely, starting from a functor $F$, we define the associated simplicial set $Z$, in degree $n$, as the disjoint union of sets $\coprod_{w \in W_n} F(n, w)$. From the definition of $\text{Simp}_G(W)$ one can check that $Z$ is actually a $G$-equivariant simplicial set. The $G$-action is specified by the morphisms of type $(\text{id}, g)$; explicitly, an element $y \in F(n, w)$ of the $w$-summand of the disjoint union is mapped to the element $F(1, g)(y)$ of the $wg$-summand. The simplicial structure is similarly encoded by morphisms of the type $(\alpha, 1)$. The constant functor sending everything into the one-point-set has $W$ as associated simplicial set. Thus the canonical maps to and from the one-point-set induce equivariant maps to and from $W$, thereby making $Z$ into an object of $\text{R}(W, G)$.

A calculation shows that these assignments are actually inverse to each other. We omit the details. 

In a similar way one can show that the category of $G$-equivariant simplicial sets over $W$ (with no specified section) is equivalent to the category $\text{Fun}(\text{Simp}_G(W)^{\text{op}}, \text{Set})$.

1.1.4 Definition. An equivariant $n$-simplex or an $n$-cell in $\text{R}(W, G)$ is an object of $\text{R}(W, G)$ isomorphic to

$$\Delta[n, w] = \chi_w + \text{id}: (\Delta^n \times G) \amalg W \longrightarrow W$$

where $\chi_w: \Delta^n \times G \longrightarrow W$ is obtained from the characteristic map of the simplex $w \in W_n$ by forcing equivariance. Structural section is the inclusion of $W$ as the second summand. The boundary is the restriction of $\Delta[n, w]$ to the subspace $(\partial \Delta^n \times G) \amalg W$, denoted $\partial \Delta[n, w]$. Similarly, we can define the horns $\Lambda_i[n, w]$.

1.1.5 Proposition. The equivalence of Proposition 1.1.3 identifies the $n$-cell $\Delta[n, w] \in \text{R}(W, G)$ with the representable functor $\text{Simp}_G(W) (\cdot, (n, w))_+$. 

There is another relationship between an object $Y \in \text{R}(W, G)$ and its associated functor $F: \text{Simp}_G(W)^{\text{op}} \longrightarrow \text{Set}$ which we describe next. Let $H$ be a discrete group acting on a set $S$. We define the homotopy orbit space of $S$, denoted $S_{hH}$, as the nerve of the transport category $\text{Tr}_H S$ of $S$. This definition is functorial in $S$ and $H$. If $G$ is a simplicial group acting on a simplicial set $Y$, we define the homotopy orbit space of $Y$, denoted $Y_{hG}$, as the diagonal of the bisimplicial set $[n] \mapsto (Y_n)_{hG_n}$. If the action of $G$ on $Y$ is free, there is a weak homotopy equivalence $Y/G \simeq Y_{hG}$.

1.1.6 Theorem. Let $F$ denote the functor corresponding to $Y \in \text{R}(W, G)$, considered as a functor into the category of sets (not pointed sets). Then $\text{hocolim } F \simeq Y_{hG}$.

Proof. First note that a set $S$ determines a discrete category; objects are the elements of the set, the only morphisms are identities. Then $NS = S$ as simplicial sets (here and in what follows, $N$ denotes the nerve of a category). Moreover, we can consider any set-valued functor as a functor into the category of (small) categories and thus speak of its Grothendieck construction.
By [7, Theorem 3.19 and preceding remarks] there are weak equivalences
\[ \text{hocolim}(F) = \text{hocolim}N F \simeq N \text{Gr}(F). \]
Direct calculation shows \( \text{Gr}(F) \cong \text{Simp}_G(Y) \). By Lemma [1.1.2] the latter category is the Grothendieck construction of the functor
\[ \text{Tr}: \Delta^{op} \longrightarrow \text{Cat}, \quad [n] \mapsto \text{Tr}_{G_n} Y_n \]
which is the simplicial category of transport categories determined by \( Y \). Thus, using [7, Theorem 3.19] again,
\[ N \text{Gr}(F) \cong N \text{Gr}(\text{Tr}) \simeq \text{hocolim}_{\Delta^{op}}(N \text{Tr}). \]
But the functor \( N \text{Tr}: \Delta^{op} \longrightarrow \text{sSet} \) determines a bisimplicial set, and its homotopy colimit is weakly equivalent to its diagonal [11, XII.4.3] which, by definition, is the homotopy orbit space \( Y_{hG} \).

### 1.2 Presheaves of abelian groups and linearisation

The category \( \mathcal{R}(W, G) \) is a category with products, given by pull back over \( W \) using structural retractions, and \((W, \text{id}, \text{id})\) is a terminal object. Hence it makes sense to speak of the category \( \mathcal{R}^{ab}(W, G) \) of abelian group objects in \( \mathcal{R}(W, G) \). For an object \((Y, r, s) \in \mathcal{R}^{ab}(W, G)\) the fibres of \( r \) are abelian groups, and \( \mathcal{R}^{ab}(W, G) \) is a category of presheaves on \( W \):

#### 1.2.1 Proposition. The category \( \mathcal{R}^{ab}(W, G) \) is equivalent to the functor category \( \text{Fun}(\text{Simp}_G(W)^{op}, \text{Ab}) \).

**Proof.** Using Proposition [1.1.3] we have a chain of equivalences of categories \( \mathcal{R}^{ab}(W, G) \cong \text{Fun}(\text{Simp}_G(W)^{op}, \text{Set}_*)^{ab} \cong \text{Fun}(\text{Simp}_G(W)^{op}, \text{Set}_*) \cong \text{Fun}(\text{Simp}_G(W)^{op}, \text{Ab}). \)

#### 1.2.2 Definition. The linearisation functor \( \tilde{Z}_W: \mathcal{R}(W, G) \longrightarrow \mathcal{R}^{ab}(W, G) \) is the left adjoint of the forgetful functor \( \mathcal{R}^{ab}(W, G) \longrightarrow \mathcal{R}(W, G) \).

For \( W = * \) this recovers Waldhausen’s definition of the linearisation map [8, p. 398]. In general, the functor \( \tilde{Z}_W \) is given by “applying \( \tilde{Z}_*[\cdot] \) to the fibres of \( r \)”. Here \( \tilde{Z}_*[\cdot] \) denotes the reduced free abelian group functor given by \( \tilde{Z}_*[S] := \mathbb{Z}[S]/\mathbb{Z}[*] \) for a pointed set \( S \). In the presheaf language, linearisation is particularly easy to describe: The functor \( \tilde{Z}_W \) sends \( F: \text{Simp}_G(W)^{op} \longrightarrow \text{Set}_* \) to the functor
\[ \tilde{Z}_W(F) := \tilde{Z}_*[\cdot] \circ F: \text{Simp}_G(W)^{op} \longrightarrow \text{Ab}, \quad (n, w) \mapsto \tilde{Z}_*[F(n, w)]. \]

#### 1.2.3 Definition. An \( n \)- Simplex or an \( n \)-Cell in the category \( \mathcal{R}^{ab}(W, G) \) is an object isomorphic to \( \Delta^{ab}[n, w] := \tilde{Z}_W(\Delta[n, w]) \) where \( \Delta[n, w] \) is an \( n \)-simplex in \( \mathcal{R}(W, G) \). The boundary of \( \Delta^{ab}[n, w] \) is \( \partial \Delta^{ab}[n, w] := \tilde{Z}_W(\partial \Delta[n, w]) \). Similarly, we can define the horns \( \Lambda_i^{ab}[n, w] := \tilde{Z}_W(\Lambda_i[n, w]) \).

#### 1.2.4 Proposition. The equivalence of Proposition [1.1.3] identifies the \( n \)-cell \( \Delta^{ab}[n, w] \) with the the representable functor \( \tilde{Z}_*[\text{Simp}_G(W)(\cdot, (n, w))_] \).

\[ \square \]
1.3 Functors

From now on, we suppose that $X$ is a connected pointed simplicial set, $G$ is the loop group $\Pi$ of $X$, and $\xi: P \to X$ is a universal $G$-bundle, i.e., $P$ is a weakly contractible free $G$-equivariant simplicial set, the map $\xi$ is constant on $G$-orbits, and $\xi$ induces an isomorphism $X \cong P/G$. An explicit construction is given in \cite[Lemma 9.3]{Huettemann}.

We proceed to define the following diagram of categories:

\[
\begin{array}{cccccc}
\mathcal{R}^{ab}(X) & \xrightarrow{\Xi} & \mathcal{R}^{ab}(P, G) & \xrightarrow{C} & \mathcal{R}^{ab}(*, G) \cong \mathcal{M}(\mathbb{Z}[G]) \\
\mathcal{Z}_X & \xrightarrow{\sim} & \mathcal{Z}_P & \xrightarrow{\sim} & \mathcal{Z}_* \\
\mathcal{R}(X) & \xrightarrow{\Xi} & \mathcal{R}(P, G) & \xrightarrow{\sim} & \mathcal{R}(*, G) \cong G\text{-sSet}_* \\
\end{array}
\]

This diagram commutes up to natural isomorphism. Note that $\mathcal{R}(*, G)$ is the category of pointed $G$-equivariant simplicial sets, and that $\mathcal{R}^{ab}(*, G)$ is equivalent to the category $\mathcal{M}(\mathbb{Z}[G])$ of simplicial right $\mathbb{Z}[G]$-modules.—The vertical arrows denote the linearisation functors as defined in \S1.2.

The bundle projection $\xi: P \to X$ determines a functor

$$\Xi: \text{Simp}_G(P)^{\text{op}} \to \text{Simp}(X)^{\text{op}}.$$ 

Since $G$ acts freely on $P$, this functor is actually an equivalence of categories. By pre-composition $\Xi$ determines an equivalence of categories

$$\Xi^*: \text{Fun}(\text{Simp}(X)^{\text{op}}, \text{Set}_*) \to \text{Fun}(\text{Simp}_G(P)^{\text{op}}, \text{Set}_*) .$$

In the language of retractive spaces, $\Xi^*$ is given by

$$\mathcal{R}(X) \xrightarrow{\Xi} \mathcal{R}(P, G), \ Y \mapsto Y \times_X P .$$

The same description defines an equivalence of $\mathcal{R}^{ab}(X)$ and $\mathcal{R}^{ab}(P, G)$, denoted $\Xi^*$. By construction there is a natural isomorphism $\Xi^* \circ \mathcal{Z}_X \cong \mathcal{Z}_P \circ \Xi^*$.

The map $P \to *$ induces a functor $\kappa: \text{Simp}_G(P)^{\text{op}} \to \text{Simp}_G(*)^{\text{op}}$. Both $C$ and $\mathcal{C}$ are defined, using the presheaf language, by left KAN extension along $\kappa$. On the level of retractive spaces, $\mathcal{C}: \mathcal{R}(P, G) \to \mathcal{R}(*, G)$ is given by collapsing $P$, sending $Y$ to $Y/P$. For ABELian group objects the collapsing functor $C$ involves summation over the fibres of the structural retraction. Explicitly, $C(Y)_n = \bigoplus_{p \in P_n} r^{-1}(p)$ for an object $(Y, r, s) \in \mathcal{R}^{ab}(P, G)$. (An explicit description for $\mathcal{C}$ is given by the same expression if “$\bigoplus$” is interpreted as a one-point union of pointed sets.)

2 Model Structures and Algebraic $K$-Theory

2.1 Categorically finite objects and finite $CW$ objects

An object $Y$ of a cocomplete category $C$ is called categorically finite if the representable functor $C(Y, \cdot)$ commutes with filtered colimits. We want to
compare this categorical finiteness notion with the more geometric notion of a finite CW object which is defined whenever we have a notion of “cells” in the category $C$.

2.1.1 Theorem. Let $C$ be a model category. Suppose there is a set $I$ of generating cofibrations which have categorically finite domains. Let $Y \in C$ be an object. If $Y$ is categorically finite and cofibrant, then $Y$ is a retract of an object $Z$ which is a finite CW object in the following sense: There is a finite filtration

\[ *= Z_0 \longrightarrow Z_1 \longrightarrow \ldots \longrightarrow Z_n = Z \]

where each map $Z_k \longrightarrow Z_{k+1}$ is a pushout of a single map in $I$, and “*” denotes the initial object of $C$.

Conversely, if $Y$ is a retract of a finite CW object $Z$, and if the maps in $I$ have categorically finite domain and codomain, then $Y$ is categorically finite and cofibrant.

Proof. If $Y$ is a retract of a finite CW object then $Y$ is certainly cofibrant. If all the maps in $I$ have categorically finite domain and codomain, then since $Z$ is a finite colimit of categorically finite objects $Z$ is categorically finite, hence so is its retract $Y$.

To prove the first statement of the theorem, assume that $Y$ is categorically finite and cofibrant. Since $Y$ is cofibrant we know by QUILLEN’s small object argument \[5, \S II.3, 3, \S 10.5.16\] that $Y$ is a retract of an object $C$ which comes equipped with a filtration

\[ *= C_0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \ldots \]

with $\text{colim} C_i = C$ such that each map $C_k \longrightarrow C_{k+1}$ is a pushout of a (possibly infinite) coproduct of maps in $I$. In more details, the small object argument yields a factorisation of the map $* \longrightarrow Y$ as a cofibration $* \longrightarrow C$ and an acyclic fibration $C \longrightarrow Y$. Since $Y$ is cofibrant, we can find the dotted lift in the solid square diagram below exhibiting $Y$ as a retract of $C$.

\[
\begin{array}{c}
\ast \\
\Downarrow \\
Y
\end{array} 
\hspace{1cm}
\begin{array}{c}
C \\
\Downarrow \\
Y
\end{array} 
\]

By construction, the following square is a pushout diagram for all $k \geq 0$:

\[
\begin{array}{c}
\prod_{\lambda \in \Lambda_k} A^k_{\lambda} \xrightarrow{\Pi f^k_{\lambda}} \prod_{\lambda \in \Lambda_k} B^k_{\lambda} \\
\downarrow \sum g^k_{\lambda} \\
C_k \longrightarrow C_{k+1}
\end{array}
\]
Here \( \Lambda_k \) is an index set, \( f^k_{\lambda}: A^k_{\lambda} \rightarrow B^k_{\lambda} \) is a map in \( I \), and \( g^k_{\lambda}: A^k_{\lambda} \rightarrow C_k \) is the attaching map. For a subset \( M \subseteq \Lambda_k \) define \( C_k(M) \) by a similar pushout with indices \( \lambda \) ranging through \( M \) only.

Since \( Y \) is categorically finite, the section \( i: Y \rightarrow C \) factors through a finite stage \( C_{k+1}, k \geq -1 \), of the filtration of \( C \). Now \( C_{k+1} \) is obtained from \( C_k \) by attaching a coproduct of cells, indexed by \( \Lambda_k \). Hence \( C_{k+1} \) is the filtered colimit of the objects \( C_k(M) \) where \( M \) varies over the (directed) poset of finite subsets of \( \Lambda_k \). Thus there exists a finite subset \( M_k \subseteq \Lambda_k \) and a factorisation of \( i \) as \( Y \rightarrow C_k(M_k) \rightarrow C \).

Since all maps in \( I \) have categorically finite domain by hypothesis, the object \( \Pi_{\lambda \in M_k} A^k_{\lambda} \) is categorically finite. By a similar argument as above, the attaching map \( h: \Pi_{\lambda \in M_k} A^k_{\lambda} \rightarrow C_k \) factors through \( C_{k-1}(M') \) for some finite subset \( M' \subseteq \Lambda_{k-1} \). Thus \( h \) factors compatibly through the objects \( C_{k-1}(M) \) where \( M \) is a finite subset of \( \Lambda_{k-1} \) containing \( M' \). Consequently, \( C_{k-1}(M)(M_k) \) is defined for such \( M \), and \( C_k(M_k) \) is the filtered colimit of the \( C_{k-1}(M)(M_k) \). Thus there exists a finite subset \( M_{k-1} \subseteq \Lambda_{k-1} \) containing \( M' \) and a factorisation of \( i \) as \( Y \rightarrow C_{k-1}(M_{k-1})(M_k) \rightarrow C \).

Iterating this argument shows that \( Y \) is a retract of an object that can be obtained from the initial object \( * \) by attaching cells indexed by the finite set \( \Pi_{i=0}^{n} M_i \), hence \( Y \) is a retract of a finite \( CW \) object as claimed.

\[ \square \]

2.2 Model structure and simplicial structure of \( \mathcal{R}(W, G) \)

For the rest of the paper, we will use the notational conventions of \([1,3]\). That is, \( X \) is a connected pointed simplicial set with loop group \( G \), and \( P \rightarrow X \) is a universal \( G \)-bundle.

The category \( \mathcal{R}(W, G) \) admits the structure of a simplicial model category where a map \( f: (Y, r, s) \rightarrow (Y', r', s') \) is a weak equivalence (resp., a fibration) if and only if the underlying map \( Y \rightarrow Y' \) of simplicial sets is a weak equivalence (resp., a Kan fibration). This model structure is cofibrantly generated; a set of generating cofibrations is given by the inclusion maps

\[ \partial \Delta[n, w] \rightarrow \Delta[n, w] \quad \text{for } n \geq 0, \ w \in W_n, \]

and a set of generating acyclic cofibrations is given by the inclusion maps

\[ \Lambda_i[n, w] \rightarrow \Delta[n, w] \quad \text{for } n \geq 0, \ w \in W_n, \ 0 \leq i \leq n. \]

Since these maps have categorically finite domain and codomain, Theorem 2.1.1 completely characterises categorically finite cofibrant objects in \( \mathcal{R}(W, G) \).

Given \( (Y, r, s) \in \mathcal{R}(W, G) \) and a simplicial set \( K \) the object \((Y, r, s) \otimes K\) is given by the pushout \((Y \times K) \cup_{(W \times K)} W\) (where the map \( W \times K \rightarrow W \) is the projection), equipped with the obvious structure maps to and from \( W \). In particular, if \( Y = \Delta[n, w] \) is an equivariant \( n \)-cell \([1,4]\), then \((Y, r, s) \otimes K\) is the object

\[ (\Delta^n \times K \times G) \amalg W \rightarrow W \]

with retraction induced by projection onto \( \Delta^n \) and the characteristic map of \( w \in W_n \), and section given by inclusion into the second summand. This shows:
2.2.1 Lemma. If \((Y, r, s) \in \mathcal{R}(W, G)\) is a finite colimit of cells (representable functors), so is \((Y, r, s) \otimes \Delta^k\) for all \(k \geq 0\). \qed

2.3 Simplicial structure of \(\mathcal{R}^{\text{ab}}(W, G)\)

For two objects \(Y, Z \in \mathcal{R}^{\text{ab}}(W, G)\) we define their tensor product \(Y \otimes Z\) by taking fibrewise tensor product of abelian groups. In the presheaf language this is the functor \(Y \otimes Z: \text{Simp}_G(W)^{\text{op}} \rightarrow \text{Ab}\), \((n, w) \mapsto Y(n, w) \otimes Z(n, w)\).

A simplicial set \(K\) defines an object \((W \times K)\Pi W\) of \(\mathcal{R}(W, G)\) with structural retraction induced by projection onto \(W\). Given an object \(Y \in \mathcal{R}^{\text{ab}}(W, G)\) we define

\[
Y \otimes K := Y \otimes \tilde{Z}_W((W \times K) \amalg W),
\]

this determines the simplicial structure of \(\mathcal{R}^{\text{ab}}(W, G)\). There is a natural isomorphism

\[
\tilde{Z}_W((Y, r, s) \otimes K) \cong \tilde{Z}_W(Y, r, s) \otimes K
\]

for all \((Y, r, s) \in \mathcal{R}(W, G)\) and simplicial sets \(K\). In particular, \(\Delta^{\text{ab}}(n, w) \otimes \Delta^k\) is a finite colimit of cells (representable functors). More generally, we have:

2.3.1 Lemma. If \((Y, r, s) \in \mathcal{R}^{\text{ab}}(W, G)\) is a finite colimit of cells (representable functors), so is \((Y, r, s) \otimes \Delta^k\) for all \(k \geq 0\). \qed

2.4 Model structure of \(\mathcal{R}^{\text{ab}}(W, G)\)

Precomposition with the inclusion of categories \(\iota: \text{Simp}(W) \rightarrow \text{Simp}_G(W)\) induces functors \(\overline{V}: \mathcal{R}(W, G) \rightarrow \mathcal{R}(W)\) and \(V: \mathcal{R}^{\text{ab}}(W, G) \rightarrow \mathcal{R}^{\text{ab}}(W)\) which forget the \(G\)-action on retractive spaces. The left adjoints of these are given by left Kan extension along \(\iota\). Explicitly, the left adjoint of \(\overline{V}\) is described by

\[
\overline{G}_*: \mathcal{R}(W) \rightarrow \mathcal{R}(W, G), \quad Y \mapsto (Y \times G) \cup_{(W \times G)} W
\]

(pushout along the action \(W \times G \rightarrow W\) of \(G\) on \(W\)), and a similar formula describes \(G_*\), the left adjoint of \(V\). By general properties of adjoints, linearisation commutes (up to canonical isomorphism) with \(G_*\) and \(\overline{G}_*\). This implies:

2.4.1 Lemma. The functors \(G_*\) and \(\overline{G}_*\) preserve cells, boundaries of cells and horns. More explicitly, \(\overline{G}_*(\Delta[n, w]) \cong \Delta[n, w]\) and \(G_* (\Delta^{\text{ab}}[n, w]) \cong \Delta^{\text{ab}}[n, w]\), and similarly for boundaries and horns. \qed

Given an object \(Y \in \text{Fun}(\text{Simp}_G(W)^{\text{op}}, \text{Ab}) \cong \mathcal{R}^{\text{ab}}(W, G)\) we define a simplicial abelian group \(\bigoplus_W Y\) which in degree \(n\) is given by \((\bigoplus_W Y)_n := \bigoplus Y(n, w)\), the sum ranging over all \(n\)-simplices of \(W\) (i.e., objects of the form \((n, w)\) of \(\text{Simp}(W)\)). Given \(Y\) and an object \(K \in \mathcal{R}(W)\) we define

\[
\hat{H}_s(K; Y) := \pi_* \bigoplus_W \left( \tilde{Z}_W K \otimes Y \right).
\]

(Strictly speaking one should write \(V(Y)\) instead of \(Y\) in this formula.) The construction is functorial in \(K\) and \(Y\).
2.4.2 Theorem. The category $\mathcal{R}^{\text{ab}}(W,G)$ admits a simplicial model structure where a map $f$ is a weak equivalence if and only if $\tilde{H}_*(K;f)$ is an isomorphism for all fibrant objects $K \in \mathcal{R}(W)$, and $f$ is a cofibration if and only if it has the left lifting property with respect to all morphisms $g$ such that the underlying map of $g$ is an acyclic fibration of simplicial sets.

For $W = \ast$ this recovers the usual model structure for simplicial $\mathbb{Z}[G]$-modules [5, §II.6].—The Theorem equips all the categories in the upper row of (†) with model structures.

Proof. If $G$ is the trivial group this is the main result of [2, §5]. For the general case, we can reformulate the definitions of weak equivalences and cofibrations. Namely, a map $f$ is a weak equivalence if and only if $V(f)$ is a weak equivalence in $\mathcal{R}^{\text{ab}}(W)$, and $f$ is a cofibration if and only if it has the left lifting property with respect to all morphisms $g$ such that $V(g)$ is an acyclic fibration in $\mathcal{R}^{\text{ab}}(W)$. Using this, the proof follows the pattern of [2, 5.7] with virtually no changes. For the BOUSFIELD argument in [2, Proposition 5.8] the cardinal $c$ has to be at least as large as the cardinality of $G$ and the cardinality of the total space of the path fibration in Lemma 4.8 of [2].

2.4.3 Corollary. The set of inclusions

$$\partial \Delta^{\text{ab}}[n,w] \longrightarrow \Delta^{\text{ab}}[n,w] \in \mathcal{R}^{\text{ab}}(W,G)$$

is a set of generating cofibrations for the model structure of Theorem 2.4.2. □

Since the maps of the Corollary have categorically finite domain and codomain, Theorem 2.4.1 completely characterises categorically finite cofibrant objects in $\mathcal{R}^{\text{ab}}(W,G)$.

We return to the diagram (†). Recall the notational conventions from the beginning of §1.3.

2.4.4 Proposition. The functor $\Xi^*: \mathcal{R}^{\text{ab}}(X) \longrightarrow \mathcal{R}^{\text{ab}}(P,G)$ preserves and detects weak equivalences. In particular, it is the left adjoint of a QUILLEN equivalence.

Proof. Let $f: Y \longrightarrow Z$ be a morphism in $\mathcal{R}^{\text{ab}}(X)$. By [2, Proposition 4.8] the map $f$ is a weak equivalence if and only if the map of simplicial ABELIAN groups

$$\bigoplus_X \left( \mathcal{Z}_X(P \amalg X) \otimes Y \right) \longrightarrow \bigoplus_X \left( \mathcal{Z}_X(P \amalg X) \otimes Z \right)$$

is a weak homotopy equivalence (we consider $P \amalg X$ as an object of $\mathcal{R}(X)$ in the obvious way). But this map can also be described as the map

$$\bigoplus_P (P \times_X Y) \longrightarrow \bigoplus_P (P \times_X Z)$$

induced by $\Xi^*(f)$. The domain of this map can be rewritten as

$$\bigoplus_P \left( \mathcal{Z}_P(P \amalg P) \otimes (P \times_X Y) \right) = \bigoplus_P \left( \mathcal{Z}_P(P \amalg P) \otimes \Xi^*(Y) \right),$$
and similarly for the codomain. By [2, Proposition 4.8] again, applied to the path fibration \( P \rightarrow P \), this map is a weak homotopy equivalence if and only if \( \Xi^*(f) \) is a weak equivalence.

The rest of the proposition follows easily since \( \Xi^* \) is an equivalence of categories. 

\[ \square \]

Let \( T \) denote the right adjoint of \( C \). It is given by

\[
T: \mathcal{R}^{ab}(\ast, G) \rightarrow \mathcal{R}^{ab}(P, G), \quad M \mapsto M \times P
\]

with structural section given by \( P = \{0\} \times P \rightarrow M \times P \), and retraction given by projection onto \( P \). In the presheaf language, \( T \) is given by precomposition with \( \kappa: \text{Simp}_G(P)^{\text{op}} \rightarrow \text{Simp}_G(\ast)^{\text{op}} \). The functor \( T \) preserves all colimits (because \( T \) itself has a right adjoint, sending \((Y, r, s)\) to the space of all non-equivariant sections of \( r \)).

\[ \star \]

\[ \begin{align*}
2.4.5 \text{ Proposition.} & \quad \text{The two functors} \\
C: \mathcal{R}^{ab}(P, G) & \rightarrow \mathcal{R}^{ab}(\ast, G) \quad \text{and} \quad T: \mathcal{R}^{ab}(\ast, G) \rightarrow \mathcal{R}^{ab}(P, G)
\end{align*} \]

\[ \text{preserve and detect weak equivalences. The unit and counit of the adjunction of} \ C \ \text{and} \ T \ \text{are natural weak equivalences. In particular,} \ C \ \text{is the left adjoint of a Quillen equivalence.} \]

\[ \text{Proof.} \quad \text{The functor} \ C \ \text{preserves and detects weak equivalences by [2, Proposition 4.8], applied to the path fibration} \ P \rightarrow P. \]

The counit of the adjunction of \( C \) and \( T \) is given by the natural map \( \epsilon_M: C \circ T(M) \cong Z[P] \otimes_Z M \rightarrow Z[\ast] \otimes_Z M \cong M \) for \( M \in \mathcal{R}^{ab}(\ast, G) \) induced by \( P \rightarrow \ast \). Since \( P \) is weakly contractible, the map \( f: Z[P] \rightarrow Z[\ast] \) is a weak equivalence. But both domain and codomain are free simplicial abelian groups, thus the map is in fact a homotopy equivalence in the strong sense: A choice of a basepoint in \( P \) determines a map \( g: Z[\ast] \rightarrow Z[P] \) with \( f \circ g = \text{id}_{Z[\ast]} \), and there exists a homotopy \( H: Z[\Delta^1] \otimes_Z Z[P] \rightarrow Z[P] \) from \( \text{id}_{Z[P]} \) to \( g \circ f \). This implies that \( \epsilon_M = f \otimes \text{id}_M \) is a homotopy equivalence of simplicial abelian groups with homotopy inverse \( g \otimes \text{id}_M \) and homotopy \( H \otimes \text{id}_M \) from \( \text{id}_{Z[P] \otimes_M} \) to \( (g \otimes \text{id}_M) \circ (f \otimes \text{id}_M) \). In particular, \( \epsilon_M \) induces isomorphisms on homotopy groups and hence is a weak equivalence in \( \mathcal{R}^{ab}(\ast, G) \). Moreover, since \( C \) preserves and detects weak equivalences this implies that \( T \) preserves and detects weak equivalences as well.

By adjointness, the composite \( C \xrightarrow{C(\eta_M)} C \circ T \circ C \xrightarrow{\epsilon_T} C \) is the identity, with \( \eta \) the unit of the adjunction. Since \( \epsilon_{T(M)} \) is a weak equivalence by the above, so is \( C(\eta_M) \), hence \( \eta_M \) is a weak equivalence. 

\[ \square \]

\[ \begin{align*}
2.5 \quad \text{Algebraic K-theory} \end{align*} \]

\[ \begin{align*}
2.5.1 \text{ Lemma.} & \quad \text{All the functors in the diagram (†) preserve cofibrant categorically finite objects.} \\
\text{Proof.} & \quad \text{This follows immediately from Theorem [2,1.1] together with the observation that all functors in (†) preserve representable functors and colimits.} \quad \square
\end{align*} \]
An object \( Y \) of a model category \( \mathcal{C} \) is called categorically finite up to homotopy if there is a chain of weak equivalences connecting \( Y \) and a cofibrant categorically finite object. The full subcategories of cofibrant objects which are categorically finite and categorically finite up to homotopy, respectively, are denoted by \( \mathcal{C}_{\text{cf}} \) and \( \mathcal{C}_{\text{hcf}} \).

2.5.2 Proposition. 1. The functors in the diagram \((\dagger)\) induce, by restriction, the following diagram of categories:

\[
\begin{array}{ccc}
\mathcal{R}^{ab}(X)_{\text{hcf}} & \xrightarrow{\Xi^*} & \mathcal{R}^{ab}(P, G)_{\text{hcf}} \\
\tilde{Z}_X & \downarrow & \tilde{Z}_P \\
\mathcal{R}(X)_{\text{hcf}} & \xrightarrow{\tilde{\Xi}} & \mathcal{R}(P, G)_{\text{hcf}} \\
& \downarrow & \mathcal{R}(*, G)_{\text{hcf}} \\
& \mathcal{R}^{ab}(*, G)_{\text{hcf}}
\end{array}
\]

There is also a similar diagram, denoted \((\ast_{\text{cf}})\), using categorically finite objects throughout. Both are diagrams of categories with cofibrations and weak equivalences, and exact functors.

2. The inclusion of the diagram \((\ast_{\text{cf}})\) into \((\ast_{\text{hcf}})\) induces a homotopy equivalence of \( K \)-theory spaces for each entry.

Proof. (1) It has been observed before that all the functors in the diagram \((\dagger)\) preserve categorical finiteness of cofibrant objects (Lemma 2.5.1), weak equivalences and cofibrations, hence they preserve cofibrancy and categorical homotopy finiteness. All the categories are categories with cofibrations and weak equivalences; the only non-trivial thing to verify is that for a diagram \( A \leftarrow B \xrightarrow{i} C \) of objects categorically finite up to homotopy with \( i \) a cofibration, the pushout \( A \cup_B C \) is categorically finite up to homotopy. This follows from [1 Proposition 3.2]. The hypothesis that \( \cdot \otimes \Delta^1 \) preserves finite objects is verified in Lemmas [2.2.1] and [2.3.1]. Note that the Proposition applies to \( R^{ab}(X) \) (and hence to the equivalent category \( R^{ab}(P, G) \)) although we do not have a set of generating acyclic cofibrations. There is a fibrant replacement functor defined by the process of “filling (linearised) horns” [2 Proposition 4.6]; this is enough for the proof of [1 Proposition 3.2].

(2) This is part of [1 Proposition 3.2].

2.5.3 Lemma. The functor \( \tilde{T} \) (defined in [2.4]) preserves cofibrations, weak equivalences between cofibrant objects, and categorically homotopy finite cofibrant objects.

Proof. The functor \( \tilde{T} \) preserves all weak equivalences by Proposition [2.4.5]. Taking product with \( P \) defines a functor \( \overline{T}: \mathcal{R}(*, G) \longrightarrow \mathcal{R}(P, G) \) with \( \tilde{Z}_P \circ \overline{T} = T \circ \tilde{Z}_* \) (on the level of presheaves, both \( T \) and \( \overline{T} \) are giving by precomposition with \( \kappa: \text{Simp}_G(P)^{\text{op}} \longrightarrow \text{Simp}_G(*)^{\text{op}} \)). Since \( P \) has a free \( G \)-action, the functor \( \overline{T} \) maps generating cofibrations to cofibrations. Since the generating cofibrations in \( \mathcal{R}^{ab}(*, G) \) are the images of the generating cofibrations in \( \mathcal{R}(*, G) \) under the functor \( \tilde{Z}_* \), and since \( \tilde{Z}_P \) preserves cofibrations, this implies that \( T \) preserves cofibrations.
To prove that $T$ preserves cofibrant objects which are categorically finite up to homotopy, it is enough to show that $T$ maps finite CW objects \(2.1.1\) to objects categorically finite up to homotopy. We first consider the effect of $T$ on a single cell $M := \tilde{Z}_*([\Delta^n \times G]) \in R^{ab}(*, G)$. Choose a simplex $p \in P_n$. Since $C$ preserves cells (representable functors) we have $C(\Delta^{ab}[n, p]) \cong M$, and the counit defines a weak equivalence (Proposition 2.4.5)

which proves that $T(M)$ is categorically finite up to homotopy.

Given a pushout diagram $A \leftarrow B \rightarrow C$ of cofibrant objects in $R^{ab}(*, G)$ with $i$ a cofibration, then if $T(A)$, $T(B)$ and $T(C)$ are categorically finite up to homotopy, so is $T(A \cup_B C) \cong T(A) \cup_{T(B)} T(C)$ since $R^{ab}(P, G)_{hcf}$ is closed under pushouts by Proposition 2.5.2 (1). Since the boundary of a cell is a finite colimit of cells of lower dimensions, the result now follows by a double induction on $n$ and the number of cells.

2.5.4 Theorem. All horizontal functors in the diagrams \(*_{cf}\) and \(*_{hcf}\) induce equivalences of $K$-theory spaces. In particular, the composite $C \circ \Xi_* \ (1.3)$ induces a homotopy equivalence $K(M(Z[G])_{cf}) \simeq \Omega hS hR^{ab}(X)_{cf}$, and the linearisation functor $\tilde{Z}_X$ induces a map

$$A(X) \longrightarrow \Omega hS hR^{ab}(X)_{cf}$$

which is identified by $C \circ \Xi_*$ and $\Xi \circ \Xi^*$ with the usual linearisation map in algebraic $K$-theory \(8\ p. 398\) defined using $\tilde{Z}_*$.

Proof. First observe that $R(X)_{cf} = R(X)$, the latter being the full subcategory of $R(X)$ whose objects are obtained from $X$ by attaching finitely many cells \(8 \ [2.1]\). In fact, for $Y \in R(X)_{cf}$ we know by Theorem 2.1.1 that $Y/X$ contains finitely many non-degenerate simplices, and we can reconstruct $Y$ by attaching these simplices to $X$ (no retracts necessary). Thus $\Omega hS hR(X)_{cf} = A(X)$.

For the remaining statements, it suffices by Proposition 2.5.2 (2) to consider the diagram \(*_{hcf}\). The functors $\Xi^*$ and $\Xi_*$ are exact equivalences of categories, and $C$ induces a homotopy equivalence on $K$-theory spaces by the argument of \(8\ Proposition 2.1.4\): Since $P \simeq *$, the functor $\overline{C}$ (cf. Proof of Lemma 2.5.3) provides a homotopy inverse.

We are left to consider the functor $C$. By the previous lemma the right adjoint $T$ of $C$ induces a map on $K$-theory spaces. Since unit and counit of the adjunction are both weak equivalences by Proposition 2.4.5, $T$ and $C$ induce mutually inverse homotopy equivalences on $K$-theory spaces. \(\square\)

Acknowledgements

The author has to thank S. Sagave and O. Renaudin for helpful discussions. The referee’s comments helped significantly to improve the exposition of the paper.
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