On the quality of random number generators with taps

Lev N. Shchur\textsuperscript{1,2}

\textit{Landau Institute for Theoretical Physics, 142432 Chernogolovka, Russia}

Abstract

Recent exact analytical results developed for the random number generators with taps are reported. These results are applicable to a wide class of algorithms, including random walks, cluster algorithms, Ising models. Practical considerations on the improvement of the quality of random numbers are discussed as well.

Key words: Random Number Generation, Shift Registers, Lagged Fibonacci, Cluster Algorithm, Random Walk, Ising Model

1 Introduction

The large-scale simulations in statistical physics use some deterministic procedures to generate a sequence of uniformly distributed (pseudo)random numbers. It is possible to generate $10^{8}$ numbers per second and $10^{15}$ numbers in 100 days on the best processors. The widely known linear congruential methods cannot be used in such simulations on 32-bit computers because of the short period $T < 2^{32} \approx 10^{9}$ of the generated sequence\textsuperscript{[1]}. The next most used class of generators are based on the linear recursion

$$X_n = (X_{n-q} \text{ OP } X_{n-p}) \mod m,$$

where $m$ is the word length and $p$ and $q$ are feedback taps $p > q$. Here \text{OP} stands for the exclusive-or operation for shift registers, + operation for the lagged Fibonacci (LF) and − operation for the substract-with-carry (SWC).

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\textsuperscript{2} lev@landau.ac.ru
recipes. In the latter case one has to add unity if the result of the previous subtraction was negative. The correlations in 2-tap generators were found to be crucial in large-scale simulations of Ising model, self-avoiding random walks (3), percolation (4) and cluster algorithms (2) (see, also, (5)). It was shown recently by Heringa, Blöte and author that these deviations can be rigorously estimated numerically and even analytically (6). Using this idea, Blöte and author discovered the scaling of systematic deviations for Ising model (7).

The method gives us some rigorous and quantitative way to estimate the possible level of deviations in the results of simulations. So, we propose some practical definition of goodness of a concrete recipe for generating random numbers. This definition is based on the concept of observability: the recipe should be considered as a good one if it passes all known tests using the sequence of the pseudo-random numbers generated in a couple of month on the nowadays computers. So, for today we need those recipes which do not produce observable deviations when $10^{15}$ random numbers are used.

2 Random walk test

The test is based on the one-dimensional directed random walk (6): a walker starts at some site of an one-dimensional lattice and, at discrete times $i$, either he takes a step in a fixed direction with a probability $\mu_i$ or he stops with a probability $\nu_i \equiv 1 - \mu_i$. In the case of unbiased probabilities $\mu_i = \mu$ and the probability of a walk with length $n$ is then $P(n) = \mu^{n-1} (1 - \mu)$. We could apply some particular recipe of random number generation to calculation of the probabilities $\mu_i$. Clearly, they would differ from $\mu$ because of the correlations in the recipe (1). Therefore, it was proposed in (6; 7) to calculate the relative deviations

$$\delta P(n) \equiv \frac{P^*(n)}{P(n)} - 1$$

(2)

of the probability of the walk $P^*(n) = \nu_i^* \prod_{i=1}^{n} (\mu_i^*)^{i-1}$ from the unbiased probability $P(n)$.

Some results for SR anf LF generators from (6) and for SWC one from (9) are presented in the Table 1. The first interesting fact is that the lagged Fibonacci and substract-with-carry leads to the same deviations up to the order of $2^{-m}$, which is usually of the order of $10^{-9}$. Next, the deviations for shift registers are of the same nature and of the same order as for LF and SWC. Both these statements contradict to the widespread belief that LF and SWC produce "better" randomness.
Deviations $\delta P(n)$ of the probability of walk length $n$ due to the correlations in the random number sequences. Corrections of the order of $2^{-m}$ are not included in results for LF and SWC.

| length $n$ | 2-tap SR | 2-tap LF = SWC |
|------------|-----------|-----------------|
| $p$        | $\frac{1-\mu}{\mu}$ | $\frac{1-2\mu}{2\mu}$ |
| $p + 1$    | $\frac{(2\mu-1)^2}{\mu^2} - 1$ | $\frac{(3\mu-1)^2}{4\mu^2} - 1$ |

3 Scaling of the deviations

We performed Wolff simulations of the 2D Ising model at criticality, using SR with feedback positions $(p, q) = (36, 11), (89, 38), (127, 64)$ and $(250, 103)$ for a variety of the lattice sizes $L$. We found that the data could be collapsed on a single curve with the appropriate rescaling of deviations, say, of the energy $\delta \tilde{E} \equiv p^{0.88} \delta E$ and the system size $\tilde{L} \equiv p^{-0.43(5)} L$. This enables us to bound deviations of energy $E$

$$\delta E \lesssim 0.3 \, L^{-0.84(5)} \, p^{-0.52},$$

of the specific heat $C$

$$- \delta C \lesssim 0.85 \, L^{-0.21(5)} \, p^{-0.42},$$

and of the universal ratio $Q = \langle m^2 \rangle^2 / \langle m^4 \rangle$, where $m$ is the magnetisation,

$$\delta Q \lesssim 0.244 \, L^{-0.45(5)} \, p^{-0.41(5)}.$$ 

The analysis has been performed also for 3D Ising model, and it yields

$$- \delta C \approx L^{-0.21} \, p^{-0.19}$$

and

$$\delta Q \approx L^{-1.15} \, p^{0.09}.$$ 

These estimations are very useful in the planning of computer experiments ([11]). Their knowledge is equivalent to the knowledge of the accuracy of the experimental equipment.
4 Conclusion

Using the method developed in (6) we found some support for the previously proposed modifications of the 2-tapped methods. Correlations would be sharply diminished by one of the following tricks:

i) concept of luxuries (8; 9), i.e., generate \( r > p \) random numbers but use only \( p \) of them;

ii) by decimation of the random number sequence, or, by using a combination of the two or more generators (10; 7; 11);

iii) by using four-tap generators (10; 4).

Our results are shown to be important for the simulations of Ising model, the percolation problem, for random walks and cluster algorithms.

So, instead of the qualifications "bad" or "good" we propose to apply to random number generators a more detailed description of their features (9), for example, that no deviations would be observed in the simulation of the above mentioned problems when up to \( 10^{15} \) random numbers are used.

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