Single-Query Verifiable Proof-of-Sequential-Work

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Abstract. We propose a proof-of-sequential-work (PoSW) that can be verified with only a single query to the random oracle for each random challenge. Proofs-of-sequential-work are protocols that facilitate a verifier to efficiently verify if a prover has executed a specified number of computations sequentially. Denoting this number of sequential computations with $N$, the prover with poly($N$) parallelism must take $\Omega(N)$-sequential time while the verifier verifies the computation in $O(\log N)$-sequential time using upto $O(\log N)$ parallelism. We propose a PoSW that allows any verifier, even the one with no parallelism, to verify using just a single sequential computation on a single challenge.

All the existing PoSWs mandate a prover to compute a sequence of responses from a random oracle against $N$-rounds of queries. Then the prover commits this sequence using a commitment scheme (e.g., Merkle root (like) commitment) predefined in the PoSWs. Now the verifier asks the prover to provide a set of proofs against $t$ randomly chosen checkpoints, called challenges, in the computed sequence. The verifier finds out the commitment from each of these proofs spending $O(\log N)$ rounds of queries to the oracle. It can be reduced to a single round of queries only if the verifier owns $O(\log N)$ parallelism.

The verifier in our PoSW demands no parallelism but uses a single query to the random oracle in order to verify each of the $t$ challenges. The key observation is that the commitment schemes themselves in the prior works demand $O(\log N)$ oracle queries to verify. So our PoSW asks the prover to undergo an additional efficient binary operation $\otimes$ on the responses from the random oracle against $N$-rounds of queries. The cumulative result of $\otimes$, represented as a map $f$, on all such responses serves the purpose of the commitment. The verifier verifies this cumulative result with a single query to the oracle exploiting some special properties of $f$. Thus the prover still needs $\Omega(N)$-rounds of queries to compute the commitment but any (non-parallel) verifier needs only a single query to the random oracle to verify. We instantiate $\otimes$ along with $f$ practically.

We stress that the sequentiality of this proposed PoSW does not depend on the choice of the operation $\otimes$ or the map $f$ but proven under the random oracle model. However, its soundness demand some specific properties, which are specified in the end, for $\otimes$ and $f$. 
Keywords: Proofs of Sequential Work · Soundness · Sequentiality · Modulo exponentiation · Random Oracle

1 Introduction

The notion of proofs of sequential work (PoSW) was introduced by Mahmoody et al. in [10]. A PoSW is a protocol that enables a verifier $V$ to efficiently check if a prover $P$ has gone through $N$ sequential steps after receiving some statement $x$. Upon receiving $x$, $P$ computes some sequence spending at least $N$ sequential steps (i.e., time) and commits the sequence to some commitment $\phi$ using a specified commitment scheme. Observing the $\phi$, $V$ challenges $P$ to supply $t$ number of proofs $\pi_i$ of $V$’s choice. $V$ verifies the integrity of $\phi$ through each of the proofs $\pi_i$. If all of them are correct then $V$ accepts that $P$ has spent $N$ sequential steps on the input $x$; rejects otherwise. In order to verify efficiently, $V$ keeps $t$ as small as possible but sufficient to catch an adversary $A$ intending to skip a fraction (say $\alpha$) of $N$ with non-negligible probability.

This fraction $\alpha$ and the sufficiency of the commitment largeness of $t$ ties an important property with every PoSW. It is known as the soundness of a PoSW. Soundness demands a PoSW to guarantee that no adversary $A$ making only $(1-\alpha)$ fraction of $N$ sequential steps would be accepted with the probability $>(1-\alpha)^t$. The soundness of all the PoSWs [10, 5, 2, 6] are proven on the assumption that given a string $x \in \{0, 1\}^*$ and a random oracle $H$, no adversary $A^H$ making only $(1-\alpha)N$ queries to $H$, can compute a sequence of responses $H^i(x) = H(H^{i-1}(x))$ for $i = 1, 2, \ldots, N$ with the probability $>(1-\alpha)^t$.

The verification, in all these PoSWs, need at least $\log N$ queries to the random oracle for the time parameter $N$. Döttling et al. reduces it to a single round of query only if the verifier has $O(\log N)$ parallelism [6]. Our PoSW reduces it to a single query to the random oracle even if the verifier has no parallelism.

1.1 Organization of the Paper

Section 2 discusses all the existing PoSWs. In Section 3 we describe a succinct review of the technicalities PoSW and random oracle. Section 4 demonstrates the design of our single-query verifiable PoSW. In Section 5 we analyze the security of the PoSW. We compare the efficiency of the proposed PoSW and the existing ones in Section 6. We generalize the map $f$ in Section 7 in order to explore the other possibilities to have such a PoSW. Finally, Section 8 concludes the paper.

2 Related Work

With its introduction, the first PoSW by Mahmoody et al. asks a prover to compute the labels against all the vertices of a directed depth robust graph of $N$ nodes using a random oracle [10]. The label against a vertex requires to be computed recursively from the labels of all of its parents. Now the prover sends a
Merkle root commitment of all the labels to the verifier. The verifier challenges the prover on some of these labels. Given such a challenge node, the verifier needs to provide the labels of the challenge node, its parents, and the siblings of the nodes that lie over the path from the challenge node to the Merkle root, as a proof $\pi$. The verifier finds the label of the node using its parents’ labels. Then (s)he reconstructs the Merkle root using the labels of the siblings in $O(\log N)$-time.

The soundness of this PoSW is based on the property that an $(\alpha, (1 - \alpha))$-depth-robust graph always has a path of length of $(1 - \alpha)N$ even after removing $\alpha N$ of vertices where $\alpha < 1$. So a prover has to evaluate the labels along a path of length $(1 - \alpha)N$ spending $(1 - \alpha)N$ sequential time.

Cohen and Pietrzak propose another PoSW using a directed binary tree with some additional edges [5]. Each of these additional edges ends at each of the leaves of the starting from the left sibling of the nodes on the paths from the leaves to the root. Essentially the idea is similar to the one in [10], however, they use the Merkle tree not only for verification but also to guarantee the $N$ sequential computations. Moreover, the labels in the graph can be computed in topological order with the help of only $O(\log N)$ labels. Thus a prover spends $N$-sequential time to label the tree. The effort for verification is $O(\log N)$-time via Merkle root verification as mentioned above. We discuss this work in more detail in Sect. 3.4 as it is at the heart of our PoSW.

In the recent past, Abusalah et al. designed a reversible PoSW in [2] with the help of skip list graph where for each edge $(i, j)$ there exists a $k > 0$ such that $j - i = 2^k$ and $2^k | i$. The protocol asks the prover to label a skip list of $N$ nodes using a random permutation oracle. The verifier selects some challenge nodes uniformly at random and checks if the paths containing the challenge nodes are consistent. It has been shown that random oracle and random permutation oracle are indistinguishable [8]. So the prover needs $N$-sequential time to label the skip list. The verifier exploits the typical property of a skip list to verify in $O(\log N)$-time.

At the same time, incremental proof-of-sequential-work was introduced by Döttling et al. [6]. This additional feature allows a prover to continue the computation from some earlier checkpoints. Their construction is also based on the PoSW in [5]. They make it incremental by choosing the challenge leaves dynamically while labeling the graph. So the rest of the graph can be pruned gradually in the run-time. In order to determine the challenge leaves under a node it randomly chooses the set of leaves from both of its subtrees using another random oracle on the label of that node. So only the challenge paths are required to be stored for verification. The effort verification is $O(\log N)$ as the challenge paths are of length $O(\log N)$ at most.

The verification in all of the above PoSWs can be parallelized upto the availability of $O(\log N)$-parallelism.
2.1 Overview of Our PoSW

Like [8], our PoSW is based on the PoSW by Cohen and Pietrzak [4]. We call the PoSW in [8] as the CP construction and modify it as follows.

First we need an efficient binary operation $\otimes$ and a binary map $f$ defined over the set $\{0,1\}^\lambda$ such that,

1. for all $g,a$ and $b$, $f(f(g,a),b) = f(g,(a \otimes b))$.
2. given the result $f(g,a)$ and $a$, it is computationally hard (w.r.t. $\lambda$) to find $g$.
3. the quantity $f(s_0,((s_1 \otimes \ldots \otimes s_N) \otimes s_i^{-1}))$ is defined if and only if $s_i \in \{s_1, s_2, \ldots, s_N\}$.

We instantiate $\otimes$ with the multiplication over integers, $f$ as modulo exponentiation and $s_i$ as the $i$-th prime in our PoSW in Sect. 4. However, we prefer to discuss the fundamental idea using $\otimes$ and $f$ showing that the elegance of our PoSW is independent of this instantiation.

The evaluation phase in our PoSW works exactly as CP except that it asks the prover to evaluate the function $f$ on the labels (labeled with random oracle) of each node of CP-graph (discussed in Sect. 5.3). For the time parameter $N$ we need a CP-graph $G_n = (N,E)$ with $N = 2^{n+1} - 1$ nodes w.l.o.g. for some $n \in \mathbb{Z}$. The security parameter $\lambda$ determines the random oracle $H : \{0,1\}^* \rightarrow \{0,1\}^\lambda$.

Given the input $x \in \mathcal{X}$ the prover samples a random oracle $H_x(\cdot) \overset{\text{def}}{=} H(x||\cdot)$. Now (s)he computes a $\lambda$-bit label $s_0 = H_x(0^n)$. Then (s)he labels the entire graph $G_n$ as $s_i = H_x(i||s_{k_1} \ldots ||s_{k_j})$ where the $\{k_j\}$s are the parents of the node $i$.

Along with with the labeling, our PoSW also asks the prover to compute the product $\rho = (s_1 \otimes s_2 \otimes \ldots \otimes s_N)$ and the commitment $\phi = f(s_0, \rho) = f(\ldots f(f(s_0, s_1), s_2) \ldots s_N)$. The prover is allowed to store any fraction (even fully) of all the labels $\{s_1, s_2, \ldots, s_N\}$.

During verification, the verifier chooses $t$ challenge leaves $\{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ of the graph $G_n$, uniformly at random. For each leaf $\gamma_i$, the verifier asks the prover for the proof $\pi_i = (\sigma_i, \tau_i)$ where $\sigma_i = \{s_{k_1}, s_{k_2}, \ldots, s_{k_j}\}$ such that $s_{\gamma_i} = H_x(\gamma_i||s_{k_1} \ldots ||s_{k_j})$ and $\tau_i = f(s_0, (\rho \otimes s_{\gamma_i}^{-1}))$. The verifier checks if $s_{\gamma_i} \overset{?}{=} H_x(\gamma_i||s_{k_1} \ldots ||s_{k_j})$ and if $\phi \overset{?}{=} f(\tau_i, s_{\gamma_i}^{-1}) = f(s_0, (\rho \otimes s_{\gamma_i}^{-1}))$. The quantity $f(s_0, (\rho \otimes s_{\gamma_i}^{-1}))$ is defined if and only if $s_{\gamma_i}$ is one of the labels of the CP graph. The integrity of the label $s_{\gamma_i}$ is confirmed by the single query $s_{\gamma_i} \overset{?}{=} H_x(\gamma_i||s_{k_1} \ldots ||s_{k_j})$. If both the checks are true for all the challenge leaves $\{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ then verifier accepts it, rejects otherwise.

Sect. 3.5 presents a detailed comparison among our design with the existing ones.

3 Preliminaries

We fix the notations first.
3.1 Notations
We take $P$ and $V$ as the prover and the verifier respectively. We denote the security parameter with $\lambda \in \mathbb{Z}^+$ and the sequential time parameter $N \in \mathbb{Z}^+$. Here $\text{poly}(\lambda)$ is some function $\lambda^{\mathcal{O}(1)}$, and $\text{negl}(\lambda)$ represents some function $\lambda^{-\omega(1)}$.

For some $x, z \in \{0, 1\}^*$, $x \parallel z$ implies the concatenation of elements $x$ and $z$. When $x \in \{0, 1\}^*$ is a string then $|x|$ denotes its bitlength. The alphabets $T, S, R$ and $U$ represents sets defined in the context. We denote $|S|$ as the cardinality of $S$.

If any algorithm $A$ outputs $y$ on an input $x$, we write $y \leftarrow A(x)$. By $x \leftarrow -X$, we mean that $x$ is sampled uniformly at random from $X$. We consider $A$ as efficient if it runs in probabilistic polynomial time (PPT) in $\lambda$. We assume (or believe) a problem to be hard if it is yet to have an efficient algorithm for that problem. We denote $H : \{0, 1\}^* \rightarrow \{0, 1\}^w$ as a random oracle. If an algorithm $A$ queries the random oracle $H$ it is denoted as $A^H$.

3.2 Random Oracle

**Definition 1. (Random Oracle $H$).** A random oracle $H : \{0, 1\}^* \rightarrow \{0, 1\}^w$ is a map that always maps any element from its domain to a fixed element chosen uniform at random from its range.

**Definition 2. ($H$-sequence).** An $H$-sequence of length $\mu$ is a sequence $x_0, x_1, \ldots, x_\mu \in \{0, 1\}^*$ where for each $i, 1 \leq i < \mu$, $H(x_i)$ is contained in $x_{i+1}$ as continuous substring, i.e., $x_{i+1} = a \parallel H(x_i) \parallel b$ for some $a, b \in \{0, 1\}^*$.

We mention the following theorem from [5] for the sake of completeness.

**Lemma 1. ($H$ is Sequential).** With at most $(N - 1)$ rounds of queries to $H$, where in each round one can make arbitrary many parallel queries. If $H$ is queried with at most $\mu$ queries of total length $Q$ bits, then the probability that $H$ outputs an $H$-sequence $x_0, \ldots, x_N \in \{0, 1\}^*$ is at most

$$\frac{Q + \sum_{i=0}^{N} |x_i|}{\mu \cdot 2^w}.$$

**Proof.** There are two ways to figure out an $H$-sequence $x_0, \ldots, x_\mu \in \{0, 1\}^*$ with only $(N - 1)$ sequential queries.

Random Guess: As $H$ is uniform, for some $a, b \in \{0, 1\}^*$ and some $i$, if the query $H(x_i)$ has not been made then,

$$\Pr[x_{i+1} = a \parallel H(x_i) \parallel b] \leq \frac{\sum_{i=0}^{N} |x_i|}{\mu \cdot 2^w}.$$

Collision: If $x_i$’s were not computed sequentially then, for some $1 \leq i \leq j \leq N - 1$, a query $a_i$ is made in round $i$ and query $a_j$ in round $j$ where $H(a_j)$ is a sub-element of $a_i$. As $H$ is uniform, the probability of this event $\leq \frac{\mu}{2^w}$. 

3.3 Proof of Sequential Work

Mahmoody et al. are the first to formalize the idea of PoSWs in [10]. All the existing PoSWs are defined in the random oracle model as they inherit their sequentiality from that of the random oracle. However, it is not necessary as there exist other sources of sequentiality e.g., time-lock or RSW puzzle [12]. Therefore, we define PoSW in general, irrespective of the random oracle model. Later, we translate the same definition in the random oracle model in Sect. 3.3.

Definition 3. (Proof of Sequential Work PoSW). Assuming $\mathcal{X}, \mathcal{Y} \subseteq \{0, 1\}^*$, a PoSW is a quadruple of algorithms $\text{Gen, Solve, Open, Verify}$ that implements a mapping $\mathcal{X} \rightarrow \mathcal{Y}$ as follows,

$\text{Gen}(1^\lambda, N) \rightarrow \text{pp}$ is an algorithm that takes as input a security parameter $\lambda$ and time parameter $N$ and produces the public parameters $\text{pp}$. These parameters $\text{pp}$ are implicit in each of the remaining algorithms.

$\text{Solve}(\text{pp}, x) \rightarrow (\phi, \phi_P)$ takes an input $x \in \mathcal{X}$ usually called statement, and produces a commitment $\phi \in \mathcal{Y}$. The triple $(x, N, \phi)$, often called commitment, is publicly announced by $P$. Additionally $P$ may produce some extra information $\phi_P \in \{0, 1\}^*$ and stores it locally in order to use in the $\text{Open}$ algorithm. The running time of $\text{Solve}$ must be at least $N$.

$\text{Open}(\text{pp}, x, N, \phi, \phi_P, \gamma) \rightarrow \pi$ takes the challenge vector $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_t\} \in \mathbb{Z}_N^t$ uniformly at random.

$\text{Verify}(\text{pp}, \pi, x, N, \gamma, \phi) \rightarrow \{0, 1\}$ is an algorithm that takes a triple $(x, N, \phi)$, a challenge vector $\gamma$, a proof vector $\pi$ and either accepts (1) or rejects (0). We say the commitment triple is a valid one if and only if $\text{Verify}$ accepts it. The algorithm must be “exponentially” faster than $\text{Solve}$, in particular, must run in $\text{poly}(\lambda, \log N)$ time.

Before we proceed to the security of a PoSW we precisely model parallel adversaries that suit the context.

Definition 4. (Parallel Adversary) A parallel adversary $A = (A_0, A_1)$ is a pair of non-uniform randomized algorithms $A_0$ with total running time $\text{poly}(\lambda, N)$, and $A_1$ which runs in parallel time $\delta < N - o(N)$ on at most $\text{poly}(\lambda, N)$ number of processors.

Here, $A_0$ is a preprocessing algorithm that precomputes some state based only on the public parameters, and $A_1$ exploits this additional knowledge to solve in parallel running time $\delta$ on $\text{poly}(\lambda, N)$ processors.

The three necessary properties of a PoSW are now introduced.
Definition 7. (Sequentiality) A PoSW is δ-sequential if for all parallel algorithms $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$ (Def. 4) that finds a $\phi$ in parallel time $\delta(N) < N$, it holds that

$$\Pr \left[ \phi = \text{Solve}(\mathcal{pp}, x) \mid \mathcal{pp} \leftarrow \text{Gen}(1^\lambda, N), \text{state} \leftarrow \mathcal{A}_0(1^\lambda, N, \mathcal{pp}), x \overset{\$}{\leftarrow} \mathcal{X}, \phi \leftarrow \mathcal{A}_1(\text{state}, x) \right] \leq \text{negl}(\lambda).$$

Non-interactive PoSWs The Open phase is required only in the interactive version of a PoSW. It can be made non-interactive using another hash function $H'_x : \{0, 1\}^* \rightarrow \mathbb{Z}_N$ as per the Fiat-Shamir heuristic. In that case, $\mathcal{P}$ will compute

$$\gamma = \{H'_x(\phi\|1), H'_x(\phi\|2), \ldots, H'_x(\phi\|t)\}.$$ 

Subexponentiality of Time $N$ An adversary $\mathcal{A}$ running on $\text{poly}(\lambda, 2^{O(\lambda)})$ processors will always be able to efficiently find a valid commitment $(x, N, \phi)$ for any $N \in 2^{O(\lambda)}$. The trick is to brute-force the proof space using Verify which is efficient. Given a statement $x$ and target time $N$, $\mathcal{A}$ does not need to run Solve rather (s)he will choose a $\phi \in \mathcal{Y}$ uniformly at random. Now $\mathcal{V}$ will sample a challenge vector $\gamma$ for which $\mathcal{A}$ needs to find a proof vector $\pi$ such that, $\mathcal{Verify}(\mathcal{pp}, \pi, x, \gamma, \phi) = 1$. For each $\gamma_i$, $\mathcal{A}$ will run $2^{O(\lambda)}$ instances of Verify each with a different $\pi'$ on each of its processors and identify the correct $\pi'$ with $\mathcal{Verify}(\mathcal{pp}, \pi', x, \gamma, \phi) = 1$. So PoSWs restrict $N \in 2^{\omega(\lambda)}$ enforcing the complexity of this brute-force approach to be $2^\lambda / 2^{\omega(\lambda)} = 2^{O(\lambda)}$. 

Definition 6. (Soundness) A PoSW is sound if for all non-uniform parallel algorithms $\mathcal{A}$ (Def. 4) that run in $(1 - \alpha)N$ time, for some $0 < \alpha < 1$, we have

$$\Pr \left[ \phi \neq \text{Solve}(\mathcal{pp}, x) \mid \mathcal{pp} \leftarrow \text{Gen}(1^\lambda, N), \text{state} \leftarrow \mathcal{A}_0(1^\lambda, N, \mathcal{pp}), x \overset{\$}{\leftarrow} \mathcal{X}, (\phi, \pi) \leftarrow \mathcal{A}_1(\text{state}, x) \right] \leq (1 - \alpha)^t.$$ 

Using $t$ number of random challenges $\gamma$, the verifier $\mathcal{V}$ should catch all non-uniform parallel adversaries $\mathcal{A}$ with “non-negligible” probability.

Definition 5. (Correctness) A PoSW is correct, if for all $n, N, \mathcal{pp}$, and $x \in \mathcal{X}$, we have

$$\Pr \left[ \mathcal{Verify}(\mathcal{pp}, \pi, x, \gamma, \phi) = 1 \mid \mathcal{pp} \leftarrow \text{Gen}(1^\lambda, N), x \overset{\$}{\leftarrow} \mathcal{X}, (\phi, \pi) \leftarrow \text{Solve}(\mathcal{pp}, x), \pi \leftarrow \text{Open}(\mathcal{pp}, x, \phi, \gamma) \right] = 1.$$ 

$\mathcal{V}$ always accept a triple $(x, N, \phi)$ generated by $N$ sequential queries to $H$. 

Single-Query Verifiable PoSWs
PoSW in the Random Oracle Model PoSWs are not necessarily defined in the random oracle model, however traditionally all the existing PoSWs have been so [10,5,2,6]. Essentially in the random oracle model both the prover $P$, the verifier $V$ and also the parallel adversary $A$ are allowed to access a common random oracle $H$. So we write $\text{PoSW}^H = \{\text{Gen}^H, \text{Solve}^H, \text{Open}^H, \text{Verify}^H\}$ to emphasize that all these algorithms except $\text{Gen}$ may access the random oracle $H$. The input statement $x$ samples a random oracle $H_x$ where

$$H_x(\cdot) \overset{\text{def}}{=} H(x\|\cdot).$$

The notion of sequentiality in these PoSWs comes from the fact that $\text{Solve}^H$ requires $H_x$-sequence of length $N$ to compute the commitment $(x, N, \phi)$. By lemma 1, no parallel adversary $A$ running on $\text{poly}(\lambda, N)$ processors, can produce a $(x, N, \phi')$ in time $< N$ such that $\Pr[\phi = \phi'] > \text{negl}(\lambda)$. On the contrary, $\text{Verify}^H$ uses only $\text{poly}(\lambda, \log N)$ queries to $H$. So we call such a PoSW as a proof-of-sequential-work in the random oracle model.

3.4 The CP PoSW

It has two parts.

The CP Graph Suppose $N = 2^{n+1} - 1$ and $B_n = (V, E')$ is a complete binary tree of depth $n$ with edges pointing to the root from the leaves. So the set $V$ can be identified with $\{0, 1\}^n$ binary strings of length $\leq n$, identifying root with the null string $\epsilon$. As the edges point upward each of the internal nodes have 2 parents, left and right. The index of the left and right parents of a node $v$ are $v\|0$ and $v\|1$, respectively. Therefore essentially,

$$E' = \{ (v\|0, v) \cup (v\|1, v) \mid v \in \{0, 1\}^n \}.$$

The leaves are identified with $\{0, 1\}^n$. So, the node $w$ lies over the path from a node $u$ to the root if $u = w\|x$ for some $x \in \{0, 1\}^{n-|w|}$.

The CP graph $G_n = (V, E' \cup E'')$ is essentially the graph $B_n$ with the additional edges,

$$E'' = \{ (v, u) \mid u \in \{0, 1\}^n, u = w\|1\|w', v = w\|0 \}.$$

It means an edge $(v, u) \in E''$ if and only if the node $v$ is a left sibling of another node that lies over the path from the leaf $u$ to the root. We denote $\text{parent}(v) = \{u \mid (u, v) \in E' \cup E''\}$.

The CP Protocol Given the input $x \in X$ the prover samples a random oracle $H_x(\cdot) \overset{\text{def}}{=} H(x\|\cdot)$. Now (s)he computes a $\lambda$-bit label $s_0 = H_x(0^n)$. Then (s)he labels the entire graph $G_n$ as $s_i = H_x(i\|s_{k_1}\|\ldots\|s_{k_j})$ where the parent$(i) = \{k_1, \ldots, k_j\}$. The label of the root $\phi$ serves the purpose of the commitment.

The verification exploits two important properties of $G_n$. 


1. Given a leaf \( v \), the labels of parent\( (v) \) are necessary and sufficient to compute the label of the root \( \phi \).
2. For any \( 0 < \alpha < 1 \), there exists a path of length \( (1 - \alpha)N \) in the induced subgraph of \( G_n \) having \( (1 - \alpha)N \) nodes.

The verifier chooses \( t \) challenge leaves \( \{\gamma_1, \gamma_2, \ldots, \gamma_t\} \) of the graph \( G_n \), uniformly at random. For each leaf \( \gamma_i \), the verifier asks the prover for the proof \( \pi_i = \text{parent}(\gamma_i) \). The verifier finds the label of the root \( \phi' \) using the labels of the node parent\( (\gamma_i) \) using the first property of \( G_n \). If the committed and the computed labels of the root match i.e., \( \phi = \phi' \) then the verifier accepts, rejects otherwise. The soundness claim comes from the second property of \( G_n \). An adversary has to label a path of length \( (1 - \alpha)N \) if (s)he attempts to skip \( \alpha N \) nodes. The sequentiality of this PoSW stands on the sequentiality of \( H \). It takes at least \( (1 - \alpha)N \) sequential time to label a path of length \( (1 - \alpha)N \).

4 Single Query Verifiable PoSW

Here we present our PoSW that verifies each challenge with only a single query to the random oracle. We denote \( \lambda \in \mathbb{Z} \) as the security parameter and \( N \) as the targeted sequential steps. The four algorithms for this PoSW are,

4.1 The Gen\( (1^\lambda, N) \) Algorithm

The generated public parameters are \( pp = (H, G_n, \times, f, t) \) having the following meanings.

1. \( H: \{0, 1\}^* \rightarrow \mathbb{P}_\lambda \) is a random oracle that maps any arbitrary binary strings to the set of first \( 2^\lambda \) primes each denoted as \( p_i \).
2. \( G_n \) is a \( \mathbb{CP} \)-graph having \( N = 2^{n+1} - 1 \) nodes (w.l.o.g.).
3. \( \times: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) represents the multiplication over the real numbers. We observe that,
   (a) \( \langle \mathbb{R}, \times \rangle \) forms a group but \( \langle \mathbb{Z}, \times \rangle \) forms a monoid.
   (b) It allows efficient computation of,
      i. the product \( (a \times b) \) for all \( a, b \in \mathbb{R} \).
      ii. the inverse \( a^{-1} \) for all \( a \in \mathbb{R} \).
   (c) For any subset \( S_k = \{p_0, p_1, \ldots, p_k\} \subseteq \mathbb{P}_\lambda \) the product \( (p_0 \times \ldots \times p_k) \times p_i^{-1} \) \( \in \mathbb{Z} \) if and only if \( p_i \in S_k \).
4. We define \( f: \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow (\mathbb{Z}/\Delta \mathbb{Z})^\times \) as \( f(g, a) = g^a \mod \Delta \) where \( \Delta = pq \) is a product of two safe primes that needs \( \Omega(N) \)-time to be factored. The choices for \( \Delta \) has been reported in Table I. We stress that the integer factorization of \( \Delta \) is known to neither \( \mathcal{P} \) nor \( \mathcal{V} \). Secrecy of this factorization is important as \( \mathcal{P} \) may violate the soundness of this PoSW with this knowledge (See Lemma 2). However, this PoSW is a public coin because \( \mathcal{V} \) uses no secret information.
The map \( f \) requires these three properties.
   (a) For all \( a \) and \( b \), \( ((g^a)^b \mod \Delta = g^{a(xb)} \mod \Delta \).
(b) Given the result \( f(g, a) = g^a \mod \Delta \) and \( a \), finding \( g \) is known as the Root Finding Problem (Def. 8). This problem is believed to be hard in the multiplicative group \((\mathbb{Z}/\Delta \mathbb{Z})^\times\) and thus determines the size of \( \Delta \) as a function of \( N \) (See Table 1).

(c) By the third property of the operation \( \times \), the quantity \( f(p_0, ((p_1 \times p_2 \times \ldots \times p_k) \times p_i^{-1}) \) is defined if and only if \( p_i \in \{p_1, p_2, \ldots, p_k\} \), as \( f \) operates on the domain \( \mathbb{Z} \) only.

5. \( t \) is the total number of random challenges.

Here we mention two additional properties, specific to this choice for \( f(g, a) = g^a \mod \Delta \), that are not necessary for our PoSW but come for free.

**Discrete Logarithm Problem** Given the result \( f(g, a) = g^a \mod \Delta \) and \( g \) is another famous problem, namely the Discrete Logarithm Problem, which is believed to be hard under the assumption that the integer factorization of \( \Delta \) is unknown.

**Time-lock Puzzle** Modulo exponentiation \( g^a \mod \Delta \) is believed to be a sequential operation that takes \( \Omega(\log_2 a) \)-time when the integer factorization of \( \Delta \) is unknown. It is renowned as the time-lock or RSW assumption [12]. As mentioned already, our PoSW stands sequential even if \( f \) is an \( \mathcal{O}(1) \)-time operation. So we do not count the sequentiality of \( f \) in the required sequential effort. We elaborate on this issue in Sect. 5.3.

In what follows, right hand sides of all the \( \leftarrow \) represent the actual computations and that of all the \( = \) explain their mathematical equivalents.

### 4.2 The \( \text{Solve}^H(pp, x) \) Algorithm

\( \mathcal{P} \) does the following,

1. sample a random oracle \( H_x(\cdot) \stackrel{df}{=} H(x||\cdot) \) using \( x \).
2. compute \( p_0 \leftarrow H_x(0^n) \).
3. initialize \( \rho \leftarrow 1 \) and \( \phi \leftarrow f(p_0, 1) = p_0 \mod \Delta \).
4. repeat for \( 1 \leq i \leq N \):
   \hspace{1cm} (a) \( p_i \leftarrow H_x(i||p_{k_1}||\ldots||p_{k_j}) \) such that parent\((i) = \{k_1, \ldots, k_j\} \).
   \hspace{1cm} (b) compute \( \rho \leftarrow \rho \times p_i \).
   \hspace{1cm} (c) compute \( \phi \leftarrow f(\phi, p_i) = (\phi)^{p_i} \mod \Delta \).

Here the \( \rho = (p_1 \times p_2 \times \ldots \times p_N) \) and the commitment is \( \phi = (p_0)^{p_1 \times p_2 \times \ldots \times p_N} \mod \Delta = (p_0)^\rho \mod \Delta \).

### 4.3 The \( \text{Open}^H(pp, x, N, \phi, \phi_\mathcal{P}, \gamma) \) Algorithm

In this phase \( \mathcal{V} \) samples a set of challenge leaves \( \gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_t\} \) where each \( \gamma_i \in \mathbb{Z}/N \). For each \( \gamma_i, \mathcal{P} \) needs to construct the proof \( \pi = \{\sigma_i, \tau_i\}_{1 \leq i \leq t} \) by the following.
1. repeat for $1 \leq i \leq t$:
   (a) assign $\sigma_i \leftarrow \text{parent}(\gamma_i)$.
   (b) $p_{\gamma_i} \leftarrow H_x(i||p_{k_1}||\ldots||p_{k_j})$ s.t $\text{parent}(\gamma_i) = \sigma_i = \{k_1, \ldots, k_j\}$.
   (c) compute $\tau_i \leftarrow f(p_0, (\rho \times p_{\gamma_i}^{-1})) \mod \Delta$.
   (d) assign $\pi_i \leftarrow \{\sigma_i, \tau_i\}$.

$\pi$ is the proof. $P$ can not compute $\tau_i \leftarrow f(\phi, p_{\gamma_i}^{-1})$ because the inverse $p_{\gamma_i}^{-1} \notin \mathbb{Z}$. Also without the knowledge of $\rho$ it is hard to find a $\tau_i$ as it requires to invert $f$ on its first argument (root finding assumption). As mentioned in Sect. 3, in the non-interactive version of the protocol $P$ will compute $\gamma = \{H'_x(\phi ||1), H'_x(\phi ||2), \ldots, H'_x(\phi ||t)\}$.

4.4 The $\text{Verify}^H(pp, x, N, \gamma, \pi, \phi)$ Algorithm

$\mathcal{V}$ runs these steps.

1. sample a random oracle $H_x(\cdot) \overset{H}{\leftarrow} \mathbb{H}(x||\cdot)$ using $x$.
2. compute $p_0 \leftarrow H_x(0^n)$.
3. repeat for $1 \leq i \leq t$:
   (a) $p_{\gamma_i} \leftarrow H_x(i||p_{k_1}||\ldots||p_{k_j})$ s.t $\text{parent}(\gamma_i) = \sigma_i = \{k_1, \ldots, k_j\}$.
   (b) set a flag $r_i$ if and only if $\phi = f(\tau_i, p_{\gamma_i}) = (\tau_i)^{p_{\gamma_i}^{-1}} \mod \Delta$.
4. accept if $(r_1 \land r_2 \land \ldots \land r_t) \overset{?}{=} \top$, rejects otherwise.

5 Security of This PoSW

Three security properties of this single query verifiable PoSW are,

5.1 Correctness

According to Def. 5, any PoSW should always accept a valid proof. The following theorem establishes this correctness property of our PoSW scheme.

Theorem 1. The proposed PoSW is correct.

Proof. Since $H$ always outputs a fixed element, $p_0$ is uniquely determined by the input $x$. Moreover in $\text{Verify}$, for each $1 \leq i < t$,

$$f(\tau_i, p_{\gamma_i}) = (\tau_i)^{p_{\gamma_i}} \mod \Delta$$

$$= (\rho \times p_{\gamma_i}^{-1}) \times p_{\gamma_i} \mod \Delta$$

$$= p_0' \mod \Delta$$

$$= \phi.$$  

So, the correct labeling of $G_n$ and evaluation of $\phi$ assigns $r_i = \top$. Then for all $i$, $r_i = \top$ results into $r = \top$ in $\text{Verify}$. So $\mathcal{V}$ has to accept it. \qed
It therefore follows that

\[
\Pr \left[ \text{Verify}(pp, x, \phi, \gamma, \pi) = 1 \right] = 1.
\]

5.2 Soundness

The soundness warrants that an adversary \( A \) having even \( \text{poly}(\lambda, N) \) processors cannot produce an invalid commitment \( \phi' \) that convinces the verifier with non-negligible probability. We establish the soundness under the adaptive root assumption for \((\mathbb{Z}/\Delta\mathbb{Z})^\times\) [13].

**Definition 8. (\( \ell \)-th Root Finding Game \( G^{\ell}_1 \))** Let \( A = (A_0, A_1) \) be a player playing the game. The \( \ell \)-th root finding game \( G^{\ell}_1 \) goes as follows:

1. \( A_0 \) is given the output \((H, G_n, \times, f, t) \leftarrow \text{Gen}(1^\lambda, N)\).
2. \( A_0 \) chooses an element \( w \in (\mathbb{Z}/\Delta\mathbb{Z})^\times \) and computes some information state.
3. A prime \( \ell \in \mathbb{P}_\lambda \) is sampled uniform at random from the set of first \( 2^\lambda \) primes.
4. observing \( \ell \) and state, \( A_1 \) outputs an element \( v \in (\mathbb{Z}/\Delta\mathbb{Z})^\times \).

The player \( A \) wins the game \( G^{\ell}_1 \) if \( v^\ell = w \mod \Delta \).

This problem is shown to be hard in the generic group model [4]. In the game \( G^{\ell}_1 \), \( A \) obtains the \( \Delta = pq \) from the description \( f \) but learns nothing about the safe primes \( p \) and/or \( q \). So \( A \) does not know the order \((p - 1)(q - 1)\) of the group \((\mathbb{Z}/\Delta\mathbb{Z})^\times \). Thus this problem \( G^{\ell}_1 \) is also hard, but its relationship with any standard hard problems like integer factorization of \( \Delta \) or RSA problem is still unknown. However, a more generic problem of finding the \( k \)-th root over \((\mathbb{Z}/\Delta\mathbb{Z})^\times \) is believed to be hard. For \( k = 2 \), there exists a randomized reduction from the factoring of \( \Delta \) to this problem [11]. Observe that, this happens in the game \( G^{\ell}_1 \) with the probability

\[
\Pr[\ell = 2] = 1/|\mathbb{P}_\lambda| = \text{negl}(\lambda).
\]

For arbitrary \( k \), this effort can be reduced to \( L_{\Delta}(\frac{3}{4}, \frac{\sqrt{32}}{2}) \) than \( L_{\Delta}(\frac{1}{4}, \sqrt{\frac{64}{k}}) \) for factoring, however, only in the presence of subexponential access to an oracle that provides affine modular roots [9]. Thus the adaptive root assumption holds true for \((\mathbb{Z}/\Delta\mathbb{Z})^\times \) too.

**Definition 9. (Adaptive Root Assumption for \((\mathbb{Z}/\Delta\mathbb{Z})^\times \))** There exists no efficient algorithm \( A = (A_0, A_1) \) that wins the \( \ell \)-th Root Finding Game \( G^{\ell}_1 \) (Def. 8) with non-negligible probability in the security parameter \( \lambda \).

\[
\Pr \left[ v^\ell = w \neq 1 \right] \leq \text{negl}(\lambda).
\]

\(^1\) where \( L_{\Delta}(\beta, \delta) = \exp(\delta(1 + o(1))O(\log^\beta \Delta \log \log^{1-\beta} \Delta)) \).
However, we need another version of the game $G_{\mathcal{H}}^\downarrow$ in order to reduce the soundness-breaking game for our PoSW.

**Definition 10.** ($\ell$-th Root Finding Game with Random Oracle $G_{\mathcal{H}}^\downarrow$) Let $\mathcal{A} = (A_0, A_1)$ be a player playing the game. The $\ell$-th root finding game with random oracle $G_{\mathcal{H}}^\downarrow$ goes as follows:

1. $A_0$ is given the output $(H, G_n, x, f, t) \leftarrow \text{Gen}(1^\lambda, N)$.
2. $A_0$ chooses an element $w \in (\mathbb{Z}/\Delta \mathbb{Z})^\times$, $x \in \mathcal{X}$ and $u \in \{0, 1\}^*$. Also $A_0$ computes some information $\text{state}$.
3. a prime $\ell \leftarrow H_z(u)$ is sampled.
4. observing $\ell$ and $\text{state}$, $A_1$ outputs an element $v \in (\mathbb{Z}/\Delta \mathbb{Z})^\times$.

The player $\mathcal{A}$ wins the game $G_{\mathcal{H}}^\downarrow$ if $v^\ell = w \mod \Delta$.

The computational hardness of both the games $G^\downarrow$ and $G_{\mathcal{H}}^\downarrow$ are equivalent as the responses of $H$ are sampled uniformly at random from $\mathbb{P}_\lambda$. Thus the adaptive root assumption (Def. 9) holds for the game $G_{\mathcal{H}}^\downarrow$ also.

**Theorem 2.** Suppose $\mathcal{A}$ be an adversary who breaks the soundness of this proposed PoSW with probability $p_{\text{win}}$. Then there exists one of these two attackers,

1. $\sqrt{\mathcal{A}}$ winning the root finding game with random oracle $G_{\mathcal{H}}^\uparrow$,
2. $\mathcal{A}_{CP}$ breaking the soundness of the CP construction,

with the same probability $p_{\text{win}}$.

**Proof.** We give the adversaries one by one.

Construction of $\sqrt{\mathcal{A}}$: Suppose $\mathcal{A}_{CP}$ does not exist. $\sqrt{\mathcal{A}}$ chooses an arbitrary $w \in (\mathbb{Z}/\Delta \mathbb{Z})^\times$. Now (s)he needs to choose a $u$ and to find a $\tau$ such that $\tau^\ell = w \in (\mathbb{Z}/\Delta \mathbb{Z})^\times$ where $\ell = H_z(u)$. So $\sqrt{\mathcal{A}}$ fixes $u = i\|p_{k_1}\|\ldots\|p_{k_j}$ where $i$ is a leaf in $G_n$ and $\text{parent}(i) = \{k_1, \ldots, k_j\}$. Then $\sqrt{\mathcal{A}}$ challenges $\mathcal{A}$ on the leaf $i$ against the commitment $(x, w, N)$. $\mathcal{A}$ breaks the soundness with the probability $p_{\text{win}}$, so $\mathcal{A}$ must find a $\tau$ such that $\tau^\ell = w \in (\mathbb{Z}/\Delta \mathbb{Z})^\times$ where $\ell = H_z(i\|p_{k_1}\|\ldots\|p_{k_j})$. When $\mathcal{A}$ returns $\tau$, $\sqrt{\mathcal{A}}$ outputs $\tau$ and wins the game $G_{\mathcal{H}}^\downarrow$ with probability $p_{\text{win}}$.

Construction of $\mathcal{A}_{CP}$: Suppose $\sqrt{\mathcal{A}}$ does not exist. On a random challenge $\gamma_i$, $\mathcal{A}_{CP}$ breaks the soundness of the CP-construction if $\mathcal{A}_{CP}$ succeeds to find the labels of the nodes that are required to construct the root label $\phi$ of $G_n$. By the second property of the CP-graph (See Sect. 3.2), $\text{parent}(\gamma_i)$ is necessary and sufficient to construct the root label. So, $\mathcal{A}_{CP}$ calls $\mathcal{A}$ on the commitment $(x, \phi, N)$ against the $\gamma_i$. $\mathcal{A}$ finds $\ell = H_z(\gamma_i\|p_{k_1}\|\ldots\|p_{k_j})$ such that $\text{parent}(\gamma_i) = \{k_1, \ldots, k_j\}$. These labels $p_{k_1}, \ldots, p_{k_j}$ are consistent with probability $p_{\text{win}}$. So, $\mathcal{A}_{CP}$ returns $p_{\gamma_i}$ and breaks the soundness of CP construction with the probability $p_{\text{win}}$.

$\square$
The adaptive root assumption implies that $\sqrt{A}$ has only negligible advantage. On the other hand, for $t$ number of challenges, the advantage of $A_{\text{PoSW}}$ is upper bounded by $(1 - \alpha)^t$ where $(1 - \alpha)$ is the fraction of honest queries. We ignore the negligible slack in soundness caused by the collision in $H$. Therefore, it holds that,

$$\Pr \left[ \phi \neq \text{Solve}(pp, x) \mid \text{Verify}(pp, \pi, x, \gamma, \phi) = 1 \right] = \left[ \begin{array}{c} pp \leftarrow \text{Gen}(1^\lambda, N) \\ \text{state} \leftarrow A_0(1^\lambda, N, pp) \\ x \leftarrow \mathcal{A} \\ (\phi, \pi) \leftarrow A_1(\text{state}, x) \end{array} \right] \leq (1 - \alpha)^t.$$

Additionally, the factorization of $\Delta$ should not be known.

**Lemma 2.** If $A$ knows the integer factorization of $\Delta = pq$ then $\text{Verify}(pp, x, \phi', \gamma, \pi) = 1$, for any arbitrary $\rho'$ and $\phi' \equiv g^{\rho'} \mod \Delta$.

**Proof.** Suppose $A$ chooses an arbitrary $\rho'$ and a commitment $\phi'$ without labelling $G_n$. Now, on a random challenge $\gamma_i$, $A$ finds $p_{\gamma_i} \leftarrow H_x(i \| p_{k_1} \| \ldots \| p_{k_j})$ s.t $\text{parent}(\gamma_i) = \sigma_i = \{k_1, \ldots, k_j\}$. Then $A$ computes

$$\tau_i' \leftarrow p_0^{(\rho' \times p_{\gamma_i}^{-1}) \mod \varphi(\Delta)} \mod \Delta$$

where $\varphi(\Delta) = (p - 1)(q - 1)$ is the Euler’s totient function and is the order of the group multiplicative group $(\mathbb{Z}/\Delta \mathbb{Z})^\times$. Therefore, $\phi' = (\tau_i')^{p_{\gamma_i}} \mod \Delta$. Therefore, we have $\text{Verify}(pp, x, \phi, \gamma, \pi) = 1$.

### 5.3 Sequentiality

By sequentiality we mean that no parallel adversary should be able to find a valid commitment and also proofs in sequential time $< N - o(N)$ with a non-negligible probability.

**Theorem 3.** Suppose $A$ be an adversary who breaks the sequentiality of this proposed PoSW with probability $p_{\text{win}}$. Then there exists an attacker $A_{\text{CP}}$ breaking the sequentiality of the CP construction, with the same probability $p_{\text{win}}$.

**Proof.** $A_{\text{CP}}$ wants to label the $G_n$ consistently in time $\delta < N - o(N)$. Here “consistently” means given any random node $v \in G_n$, $A_{\text{CP}}$ must be able to figure out the label $p_v$ that are consistent with the labeling of $G_n$. So, given the node $v$ on the input $x$, $A_{\text{CP}}'$ calls the $A$ on the input $x$ and a random challenge $u$ such that $v \in \text{parent}(u)$. $A$ finds the label $p_v$ such that $p_v = H_x(u \| \ldots \| p_u \| \ldots)$, as it breaks the sequentiality of the proposed PoSW. Thus $A_{\text{CP}}$ returns $p_v$ on any random $v \in G_n$ and breaks the sequentiality of the CP construction with the same probability $p_{\text{win}}$.

So we have,

$$\Pr \left[ \phi = \text{Solve}^H(pp, x) \mid pp \leftarrow \text{Gen}(1^\lambda, N) \\ \text{state} \leftarrow A_0(1^\lambda, N, pp) \\ x \leftarrow \mathcal{A} \\ \phi \leftarrow A_1(\text{state}, x) \right] = \text{negl}(\lambda).$$
The most important corollary of the theorem is,

**Source of Sequentiality:** Theorem 3 never borrow the sequentiality of \( f \). It is solely based on the sequentiality of CP-construction. So our PoSW stands sequential even if \( f \) is computable in \( \mathcal{O}(1) \)-time. In our design \( f \) happens to be a sequential function but not necessarily. We consider the random oracle \( H \) as the only source of sequentiality and measure the sequentiality with the rounds of queries to \( H \). The map \( f \) serves the purpose of fast verification using short proofs and the soundness. In fact, we want \( f \) to be as efficient as possible satisfying only a few properties mentioned in the Def. 1.4.

6 Efficiency Analysis

Here we discuss the efficiencies of both the prover \( P \) and the verifier \( V \) and the memory requirement for the commitment and the proof. We reiterate that we count the number of \( H_x \)-queries only in order to estimate the sequential effort made by \( P \) and \( V \). The efforts to execute \( f \) have been assumed to be efficient and need not be sequential.

First we need to determine the size of the group \( \mathbb{Z}/\Delta \mathbb{Z} \times \).

6.1 Size of \( \mathbb{Z}/\Delta \mathbb{Z} \times \)

The number of bits to encode an element of the group \( \mathbb{Z}/\Delta \mathbb{Z} \times \) is \( \log_2 \Delta \). By Lemma 2 it suffices to choose a \( \Delta \) that takes \( \Omega(N) \)-time to be factored. It is shown in Sect. 3.3 that \( N \in 2^{o(n)} \).

So we describe the size of \( \Delta = pq \) as a function of \( \log N \) reported in [3]. Here the sizes of the safe primes \( p \) and \( q \) are roughly \( \log_2 \Delta / 2 \).

| \( \log N \) | \( \log \Delta \) |
|---|---|
| 80 | 1024 |
| 112 | 2048 |
| 128 | 3072 |
| 192 | 7680 |
| 256 | 15360 |

Further, assuming the generalized number field sieve method to be the most efficient heuristic for integer factorization, analytically,

\[
\begin{align*}
\exp((64/9)^{1/3} + o(1))((\ln \Delta)^{1/3}(\ln \ln \Delta)^2/3 & \geq N \\
((64/9)^{1/3} + o(1))((\ln \Delta)^{1/3}(\ln \ln \Delta)^2/3 & \geq \ln N \\
\ln \Delta(\ln \ln \Delta)^2 & \geq \frac{\ln N}{(64/9)^{1/3} + o(1)}.
\end{align*}
\]
Roughly $\tilde{O}(\log \Delta) = O(\log^3 N)$ where $\tilde{O}(h(n)) = h(n) \cdot \log^k h(n)$ for some $k \in \mathbb{Z}$.

6.2 Proof Size

The size of the commitment $\phi$ is $\log_2 \Delta$. Each proof $\pi_i$ has two parts $s_i$ and $\tau_i$. The $\sigma_i = \text{parent}(\gamma_i)$ has at most $\log N$ number of $\lambda$-bit strings. Thus $|\sigma_i| = O(\lambda \log N)$. Additionally, $|\tau_i| = \log \Delta$. So $|\pi_i| = O(\lambda \log N) + \log \Delta \approx O(\lambda \log N + \log^3 N)$. Thus the dominating factor depends on the max{log $N$, $\sqrt{N}$}. So the total size of the proof $\pi = t \cdot \pi_i = O(t(\lambda \log N + \log^3 N))$.

6.3 Prover’s Efficiency

A prover $P$ may adopt the similar approach like the $\mathcal{CP}$-construction. On random challenge $P$ needs to find out consistent labels of its parent. $P$ may store $2^{m+1}$ labels at $m$ upmost levels into the local information $\phi_P$ for some $0 \leq m \leq n = \log N$.

- for $m = 0$, $P$ labels $G_n$ spending $N$ rounds of queries once again.
- for $0 < m < n$, $P$ computes $2^{n-m+1}$ labels once again sequentially.
- for $m = n$, $P$ stores the entire labeled CP-graph. No further queries are required.

In general, $P$ requires at most $\lambda N^{1-\beta}$ bits of memory for some $0 < \beta < 1$ if (s)he is willing to spend only $tN^\beta$ queries in OpenH.

6.4 Verifier’s Efficiency

This is where our PoSW outperforms all other existing PoSW demanding $O(\log N)$ oracle queries. Note that in algorithm [9] even a non-parallel $V$ needs only a single query to the random oracle $H_x$ in order to verify on a random challenge. Thus our PoSW needs only $t$ queries in total instead of $O(t \log N)$ queries as in the prior works. The memory requirement of $V$ is same as the proof size.

6.5 Our Contribution

We summarize the comparison among all the PoSWs in Table 2.

We show that it is possible to design a PoSW with non-parallel verifier that queries the random oracle $H_x$ only once per random challenge. In particular, the verifier in our PoSW demands no parallelism but verifies using a single query to $H_x$ in each of the $t$ trials. The state-of-the-art PoSW [5] verifies with a single round of queries to $H_x$ only if $O(\log N)$ parallelism is available to the verifier. Without the parallelism they make $O(\log N)$ rounds of queries to $H_x$. The prior works [9,7] require $O(\log N)$ rounds of queries $H_x$ even in the presence of parallelism.
Table 2. Comparison among all the PoSWs. Here $N$ is the rounds of queries to the random oracle $H_x$. Parallelism is upper bounded by $O(\log N)$ processors. The effort for verification is the effort to verify a single challenge among $t$ such challenges in total. All the quantities may be subjected to $O$-notation, if needed.

| Schemes (by authors) | Solve (Sequential) | Solve (Parallel) | Verify (Sequential) | Verify (Parallel) | Proof-size |
|----------------------|--------------------|------------------|--------------------|------------------|------------|
| Mahmoody et. al [10] | $N$                | $N$              | $\log^* N$        | $\log N$        | $\lambda \log N$ |
| Cohen and Pietrzak [5]| $N + \sqrt{N}$    | $N$              | $\log^* N$        | $\log N$        | $\lambda \log N$ |
| Abusalah et al. [2]  | $N$                | $N$              | $\log^* N$        | $\log N$        | $\lambda \log^* N$ |
| Döttling et. al [6]  | $N + \sqrt{N}$    | $N$              | $\log N$          | $1$              | $\lambda \log N$ |
| **Our work**         | $N + \sqrt{N}$    | $N$              | $1$                | $1$              | $\max\{\lambda \log N, \log^* N\}$ |

The fundamental observation is that the Merkle root commitment scheme, used in all the existing PoSWs, mandates $O(\log N)$ queries to verify. Thus they require $O(\log N)$ parallelism to reduce the verification time to a single round of queries. So we replace the Merkle root commitment with a binary operation $\otimes$ and a binary map $f$ that immediately reduces the non-parallel verification effort to a single oracle query from $O(\log N)$ queries.

Further, we mention an important class of PoSWs that mandates much more fine-grained notion of soundness. These are called verifiable delay functions (VDF) that are nothing but PoSWs but with unique proofs. That is, the soundness of such PoSWs has to be upper bounded by $\negl(\lambda)$ instead of $(1 - \alpha)^t$ only.

There exist VDFs based on modulo exponentiation in the groups of unknown orders [13,11] and isogenies over super-singular curves [7]. From the perspective of design, the key difference between these VDFs and the PoSWs based on random oracle is the source of sequentiality. Although these VDFs are more sound, their sequentiality are conditional. For example, the VDFs based on modulo exponentiation assume the condition that $x^{2^N \mod \Delta}$ takes $\Omega(N)$-sequential time. To the best of our knowledge, this condition has neither a proof nor a counter-example.

On the other hand, all the existing PoSWs including ours achieve unconditional sequentiality in the random oracle model.

7 Generalizing The Function $f$

An obvious question to the above-mentioned PoSW is that is it necessary to be $f = g^a \mod \Delta$ always? Or is there any other option to instantiate $f$? We give a concrete guideline to choose $f$ with respect to an operation $\otimes$ in general in a generic PoSW.

The four algorithms that specify our PoSW are now described.

7.1 The Gen$(1^\lambda, N)$ Algorithm

We need four sets $T, S \subset R \subset U \subset \{0, 1\}^*$ such that $\log_2 |S| = \text{poly}(\lambda)$. Although all of these sets are exponentially large, we neither enumerate nor include them
in the generated public parameters $\mathbf{pp}$. These sets are required in order to define the binary operation $\otimes$ and the binary map $f$. The algorithm $\text{Gen}(1^k)$ outputs the public parameters $\mathbf{pp} = (\mathcal{H}, G_n, \otimes, t, f)$ with the following meanings.

1. $\mathcal{H} : \{0,1\}^* \to \{0,1\}^\lambda$ is a random oracle (See Sect. 1).
2. $G_n$ is the CP graph (See Sect. 3.4) such that w.l.o.g., $N = 2^n + 1$.
3. $\otimes : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ is a commutative and associative binary operation such that
   (a) $(\mathcal{U}, \otimes)$ forms a group but $(\mathbb{R}, \otimes)$ forms a monoid.
   (b) It allows efficient computation of,
       i. the product $(s_i \otimes s_j)$ for all $s_i, s_j \in \mathcal{U}$.
       ii. the inverse $s_i^{-1}$ for all $s_i \in \mathcal{U}$.
   (c) For any subset $\mathcal{S}_k = \{s_0, s_1, \ldots, s_k\} \subseteq \{0,1\}^k$ the product $((s_0 \otimes \ldots \otimes s_k) \otimes s_i^{-1}) \in \mathbb{R}$ if and only if $s_i \in \mathcal{S}_k$.
4. $f : \mathbb{R} \times \mathbb{R} \to \mathbb{T}$ be an efficient binary map such that,
   (a) For all $g, a, b \in \mathbb{R}$, $f(f(g, a), b) = f(g, (a \otimes b))$.
   (b) Given the result $f(g, a)$ and $a$, it is hard to find $g$.
5. $t$ as the number of random challenges.

None of the public parameters needs to be computed.

7.2 The $\text{Solve}^H(\mathbf{pp}, x)$ Algorithm

The prover,

1. sample a random oracle $H_x(\cdot) \overset{\text{def}}{=} \mathcal{H}(x\cdot)$ using $x$.
2. compute $s_0 = H_x(0^n)$.
3. initialize $\phi = s_0$ and $\rho = 1_U$.
4. repeat for $1 \leq i < N$ to label the graph $G_n$,
   (a) $s_i = H_x(i\|s_{k_1}\| \ldots\|s_{k_j})$ where $\text{parent}(i) = \{k_1, \ldots, k_j\}$.
   (b) $\rho = \rho \otimes s_i$.
   (c) $\phi = f(\phi, s_i)$.

Announce the triple $(x, N, \phi)$ and stores the $\rho = (s_1 \otimes \ldots \otimes s_N)$ locally. Here $\phi = f(\ldots f(f(s_0, s_1), s_2), \ldots, s_N) = f(s_0, \rho)$ serves as the commitment to the labels of $G_n$.

7.3 The $\text{Open}^H(\mathbf{pp}, x, N, \phi, \rho, \gamma)$ Algorithm

In this phase $\mathcal{V}$ samples a set of challenge leaves $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ where each $\gamma_i \overset{\$}{\in} \mathbb{Z}_N$. For each $\gamma_i$, $\mathcal{P}$ needs to construct the proof $\pi = \{\sigma_i, \tau_i\}_{0 \leq i \leq t-1}$ by the following,

1. initialize an empty set $\pi = \{\emptyset\}$.
2. repeat for $1 \leq i \leq t$:
   (a) assign $\sigma_i = \text{parent}(\gamma_i)$.
   (b) compute $\tau_i \leftarrow f(s_0, (\rho \otimes s_{\gamma_i}^{-1}))$.
   (c) assign $\pi_i = \{\sigma_i, \tau_i\}$.

$\mathcal{P}$ can not compute $\tau_i \leftarrow f(\phi, s_{\gamma_i}^{-1})$ because the inverse $s_{\gamma_i}^{-1} \notin \mathbb{R}$ as $(\mathbb{R}, \otimes)$ is a monoid.
7.4 The \textbf{Verify}(pp, x, N, \gamma, \pi, \phi) Algorithm

The \mathcal{V} runs these steps.

1. samples a random oracle \(H_x(\cdot) \overset{def}{=} H(x\|\cdot)\) using \(x\).
2. computes \(s_0 = H_x(0^n)\).
3. repeat for \(1 \leq i < t\):
   (a) compute \(s_{\gamma_i} = H_x(i\|s_{k_1}\| \ldots \|s_{k_j})\) where \(\text{parent}(\gamma_i) = \sigma_i = \{k_1, \ldots, k_j\}\).
   (b) assign a flag \(r_i\) if and only if \(\phi ? f(\tau_i, s_{\gamma_i})\).
4. Accept if \((r_0 \land r_1 \land \ldots \land r_{t-1}) ? T\), rejects otherwise.

The security and the efficiency analysis are exactly same as in Sect. 5 and Sect. 6. It just needs to replace \(\times\) with \(\otimes\) and taking \(f\) in general. Still we reiterate exactly the same proofs for the sake of completeness.

\textbf{Theorem 4.} The generalized PoSW is correct.

\textit{Proof.} In the algorithm Solve\(^H\), since \(H\) always outputs a fixed element, \(s_0\) is uniquely determined by the input \(x\). Moreover in Verify, for each \(1 \leq i < t\),

\[
\begin{align*}
   f(\tau_i, s_{\gamma_i}^{-1}) &= f(f(s_0, (\rho \otimes s_{\gamma_i}^{-1})), s_{\gamma_i}) \\
                         &= f(s_0, (\rho \otimes s_{\gamma_i}^{-1} \otimes s_{\gamma_i})) \\
                         &= f(s_0, \rho) \\
                         &= \phi.
\end{align*}
\]

So, the correct enumeration of the sequence \(\sigma\) and evaluation \(\phi\) assigns \(f_i = T\). Then for each \(i\), \(f_i\) must be \(T\) which results into \(f = T\) in \textbf{Verify}. So \(\mathcal{V}\) has to accept it. \(\square\)

\textbf{Theorem 5.} Suppose \(A\) be an adversary who breaks the soundness of this generalized PoSW with probability \(p_{\text{win}}\). Then there exists any one of these two attackers,

1. \(A^{-1}\) inverting \(f\) on its first argument \(g\),
2. \(A_{CP}\) breaking the soundness of the CP construction,

with the same probability \(p_{\text{win}}\).

\textit{Proof.} We give the adversaries one by one.

Construction of \(A^{-1}\): Suppose \(A_{CP}\) does not exist. \(A^{-1}\) chooses an arbitrary \(\phi \in T\) but \(u = i\|s_{k_1}\| \ldots \|s_{k_j}\) where \(i\) is a leaf in \(G_n\) and \(\text{parent}(i) = \{k_1, \ldots, k_j\}\). Then \(A^{-1}\) challenges \(A\) on the leaf \(i\) against the commitment \((x, w, N)\).

\(A\) breaks the soundness with the probability \(p_{\text{win}}\). So \(A\) computes \(s_i = H_x(i\|s_{k_1}\| \ldots \|s_{k_j}\) and finds a \(\tau\) such that \(f(\tau, s_i) = \phi\). \(A^{-1}\) returns \(\tau\) to invert \(f\) on its first argument with probability \(p_{\text{win}}\).
Construction of $\mathcal{A}_{\text{CP}}$: Suppose $\mathcal{A}^{-1}$ does not exist. On a random challenge $\gamma_i$, $\mathcal{A}_{\text{CP}}$ breaks the soundness of the CP-construction if $\mathcal{A}_{\text{CP}}$ succeeds to find the labels of the nodes that are required to construct the root label $\phi$ of $G_n$. By the second property of the CP-graph (See Sect. 3.4), $\text{parent}(\gamma_i)$ is necessary and sufficient to construct the root label. So, $\mathcal{A}_{\text{CP}}$ calls $\mathcal{A}$ on the commitment $(x, \phi, N)$ against the $\gamma_i$. $\mathcal{A}$ finds $s_{\gamma_i} = H_x(\gamma_i \parallel s_{k_1} \parallel \ldots \parallel s_{k_j})$ such that $\text{parent}(\gamma_i) = \{k_1, \ldots, k_j\}$. These labels $s_{k_1}, \ldots, s_{k_j}$ are consistent with probability $p_{\text{win}}$. So, $\mathcal{A}_{\text{CP}}$ returns $s_{\gamma_i}$ and breaks the soundness of CP construction with the probability $p_{\text{win}}$.

**Theorem 6.** Suppose $\mathcal{A}$ be an adversary who breaks the sequentiality of this generalized PoSW with probability $p_{\text{win}}$. Then there exists an attacker $\mathcal{A}'_{\text{CP}}$ breaking the sequentiality of the CP construction, with the same probability $p_{\text{win}}$.

**Proof.** $\mathcal{A}'_{\text{CP}}$ wants to label the $G_n$ consistently in time $\delta(N) < N$. Here “consistently” means given any random node $v \in G_n$, $\mathcal{A}'_{\text{CP}}$ must be able to figure out the label $s_v$ that are consistent with the labeling of $G_n$. So, given the node $v$ on the input $x$, $\mathcal{A}'_{\text{CP}}$ calls the $\mathcal{A}$ on the input $x$ and a random challenge $u$ such that $v \in \text{parent}(u)$. $\mathcal{A}$ finds the label $s_u$ such that $s_u = H_x(u \parallel \ldots \parallel s_v \parallel \ldots)$, as it breaks the sequentiality of the proposed PoSW. Thus $\mathcal{A}'_{\text{CP}}$ returns $s_v$ on any random $v \in G_n$ and breaks the sequentiality of the CP construction with the same probability $p_{\text{win}}$. $\square$

8 Conclusion and Open Problem

This paper presents a proof of sequential work that queries the random oracle only once while verifying. Our PoSW is based on the one in [5], however uses two additional primitives i.e., the operation $\otimes$ and the map $f$ in its design. We have been able to show that even a non-parallel verifier needs only a single oracle query to verify. So the effort for verification reduces to a single query from logarithmically proportional queries to $N$ (time) as in the existing PoSWs. The key idea is to replace the Merkle root based commitment with the operation $\otimes$ and the map $f$.

Our PoSW is proven to be correct, sound and sequential. Finally we give a proper guideline to choose $\otimes$ and $f$. However, it turns out to be a nice open question that how far we can minimize the computation time of the map $f$ maintaining the guidelines. In particular only the soundness depend on the choice of $f$. So we would always like to have $f$ that is as fast as possible. Is there any lower bound of this computation time beyond which the safeguard will be violated or we are free to have any $f$. If yes, then what are the other options?

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