Abstract We define highest weight categorical actions of $\mathfrak{sl}_2$ on highest weight categories and show that basically all known examples of categorical $\mathfrak{sl}_2$-actions on highest weight categories (including rational and polynomial representations of general linear groups, parabolic categories $\mathcal{O}$ of type $A$, categories $\mathcal{O}$ for cyclotomic Rational Cherednik algebras) are highest weight in our sense. Our main result is an explicit combinatorial description of (the labels of) the crystal on the set of simple objects. A new application of this is to determining the supports of simple modules over the cyclotomic Rational Cherednik algebras starting from their labels.

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1 Introduction

Categorical actions of Kac-Moody algebras were introduced by Chuang and Rouquier, [8], in the case of $\mathfrak{sl}_2$ and by Rouquier, [22], in general. These actions proved to be very useful in Representation Theory. For example, in [8] they were used to prove the Broué abelian defect conjecture. It is worth mentioning that related techniques were used in several papers before [8] although they were not formalized.

Roughly speaking, a categorical $\mathfrak{sl}_2$-action on an abelian category $C$ is a pair of biadjoint functors $E$, $F$ satisfying the defining $\mathfrak{sl}_2$-relations together with certain natural transformations. A categorical action of an arbitrary Kac-Moody algebra includes functors $E_i$, $F_i$ corresponding to Chevalley generators and can be regarded as a collection of categorical $\mathfrak{sl}_2$-actions subject to some compatibility conditions that reflect, in particular, the usual...
compatibility of the defining $\mathfrak{sl}_2$-triples in a Kac-Moody algebra. Examples of the categories $\mathcal{C}$ that can be equipped with categorical actions of a Kac-Moody Lie algebra $\mathfrak{g}$ (below such categories are called $\mathfrak{g}$-categorifications) include many categories of interest for Representation theory, see, e.g., [8, Section 7]. For instance, one can consider the sum $\bigoplus_{n \geq 0} \mathbb{K}S_n - \text{mod}$ of the categories of all finite dimensional $\mathbb{K}S_n$-modules, where $\mathbb{K}$ is an algebraically closed field. This category comes equipped with a categorical action of $\mathfrak{g}$, where $\mathfrak{g} = \widehat{\mathfrak{sl}}_p$ if $\mathbb{K}$ is a field of characteristic $p > 0$, and $\mathfrak{g} = \mathfrak{gl}_\infty$ if the characteristic is 0. We emphasize that here $\mathfrak{g}$ is defined over $\mathbb{Q}$ not over $\mathbb{K}$. This categorification comes from the induction and restriction functors. There is a similar in spirit “higher level” construction for cyclotomic Hecke algebras.

Another class of examples considered in [8] comes from the representation theory of algebraic groups or Lie algebras. For example, we can consider the category $\text{GL}_n(\mathbb{K}) - \text{mod}$ of rational representations of $\text{GL}_n(\mathbb{K})$ (over the field $\mathbb{K}$). It comes with a categorical $\mathfrak{g}$-action (with $\mathfrak{g}$ as above) that is induced from tensoring with $\mathbb{K}^n$ and $(\mathbb{K}^n)^*$. This category has a polynomial analog $\bigoplus_{d \geq 0} \text{Rep}^d(\text{GL})$, where $\text{Rep}^d(\text{GL})$ stands for the “stable” category of polynomial representations of $\text{GL}$ of degree $d$ (“stable” means that we consider the representations of $\text{GL}_n$ with $n \geq d$). Also there are higher level analogs of these categories: parabolic categories $\mathcal{O}$ over $\mathfrak{gl}_n$, where one has categorical actions of $\mathfrak{gl}_\infty$, see [5,6]. Yet another, more recent, example comes from the representation theory of cyclotomic Rational Cherednik algebras, [13,23].

The categories described in the previous paragraph all have an additional structure, a highest weight structure (another name: a quasi-hereditary structure). That is, they have a distinguished collection of objects, standard objects, that have properties of Verma modules in the BGG category $\mathcal{O}$. Two natural questions then arise. First, what are reasonable compatibility relations between highest weight and categorification structures? Second, assuming that the structures are compatible, what implications for the representation theory does this have? In this paper we give some version of an answer to the first question (in the case of $\mathfrak{sl}_2$) and also describe an application: a combinatorial description of the crystal associated to a categorical action.

The crystal under consideration is on the set of the simples in $\mathcal{C}$. The crystal operators will be recalled below. In all of highest weight categories recalled above the simples are parameterized by some combinatorial objects. For instance, the simples in $\text{Rep}(\text{GL}_n(\mathbb{K}))$ are parameterized by (dominant) weights, while the simples in $\bigoplus_{d \geq 0} \text{Rep}^d(\text{GL})$ are parameterized by partitions. In the case of the latter category, the crystal has an explicit representation theoretic meaning: it describes the $d - 1$ degree part of the socle (i.e., the sum of all simple subobjects) in the restriction of an irreducible object in $\text{Rep}^d(\text{GL}_n(\mathbb{K}))$ to $\text{GL}_{n-1}(\mathbb{K})$, see, for example, [2, Theorem C].

The combinatorial description of the crystal was known previously for all the categories above, see [16] (for $\text{Rep}(\text{GL}_n)$), [4] (for the parabolic categories $\mathcal{O}$), with an exception of the categories $\mathcal{O}$ of cyclotomic Rational Cherednik algebras, there the description was only known under restrictions on parameters participating in the definition of the algebra, see, for example, [12]. For example, in the case of polynomial representations of GL the description is given in terms of addable and removable boxes in Young diagrams and we will see that this is a more or less general pattern. We remark that the descriptions of [16,4] require some non-trivial and technical computations. In the Cherednik case, the crystal has a very transparent representation theoretic meaning: it carries some information about the supports of irreducible modules.
There are also other examples of categorical actions on highest weight categories, see, for example, [25, 26]. We remark that results of [25] yield a description of the crystal of the categorification considered there (that may be regarded as a special case of the Cherednik category $\mathcal{O}$).

The main goal of this paper is to produce a combinatorial description of the crystal of a highest weight $\mathfrak{sl}_2$-categorification in a uniform way. The crucial part of the argument will be to verify that certain $\text{Ext}$'s between standard objects and irreducible objects vanish.

We would like to remark that our goal is not to describe the crystal of a categorification up to an isomorphism: in all cases of interest, this easily follows from some general arguments, compare with [23, Section 6]. Instead, we present some explicit combinatorial recipes to compute the crystal operators. For some categorifications, see, for example, [18], the crystals are connected, and then determining them up to an isomorphism gives a description as explicit as possible. However, for the categorifications considered in this paper, crystals are never connected (even if instead of $\mathfrak{sl}_2$ we consider a larger algebra acting, such an action is present for basically all our examples).

Let us now describe the structure of this paper. Sect. 2, 3 do not contain any new material. In Sect. 2 we will recall some standard facts about $\mathfrak{sl}_2$-categorifications including the crystal structure on the set of simple objects. In Sect. 3 we will recall the definition of a highest weight category and examples of highest weight categories mentioned above together with categorical actions. Then in Sect. 4 we will define highest weight categorical $\mathfrak{sl}_2$-actions and explain why the actions recalled in Sect. 3 are highest weight. Finally, in Sect. 5 we will state and prove our main result, Theorem 5.1, on the combinatorial description of the crystal. Then we will recall a relationship between the crystal and the supports of irreducible modules for the cyclotomic Rational Cherednik algebras obtained in [23,24].

2 $\mathfrak{sl}_2$-categorifications

2.1 $\mathfrak{sl}_2$-Categorifications: definitions

Our exposition here follows [8], where $\mathfrak{sl}_2$-categorifications (=categorical $\mathfrak{sl}_2$-actions) were introduced. Below when we consider the Lie algebra $\mathfrak{sl}_2$ we always mean an algebra over $\mathbb{Q}$.

Let $K$ be a field. Let $\mathcal{C}$ be an artinian $K$-linear abelian category. Following [8, 5.1], by a weak $\mathfrak{sl}_2$-categorification on $\mathcal{C}$ one means a pair of exact endofunctors $E, F$ of $\mathcal{C}$, where $E$ is left adjoint to $F$ with fixed unit and counit morphisms $\epsilon : \text{Id} \to FE, \eta : EF \to \text{Id}$. These data are supposed to satisfy

- The action of the operators $e := [E], f := [F]$ on $[\mathcal{C}] := \mathbb{Q} \otimes_{\mathbb{Z}} K(\mathcal{C})$ induced by the functors $E, F$, respectively, produces a locally finite action of $\mathfrak{sl}_2$.
- The classes of simples in $[\mathcal{C}]$ are weight vectors.
- $F$ is isomorphic to the left adjoint of $E$.

A weak $\mathfrak{sl}_2$-categorification on $\mathcal{C}$ is called an $\mathfrak{sl}_2$-categorification, [8, 5.2], if it comes equipped, in addition, with functor morphisms $X \in \text{End}(E), T \in \text{End}(E^2)$ and numbers $q \in K^\times$ and $a \in K$ with $a \neq 0$ provided $q \neq 1$ subject to the following conditions:

- $(1_E T) \circ (T 1_E) \circ (1_E T) = (T 1_E) \circ (1_E T) \circ (T 1_E)$ in $\text{End}(E^3)$. Here and below $1_E T$ denotes the endomorphism of $E^3$ that is obtained by applying $T$ to the second and third copies of $E$. The notation $T 1_E$ has a similar meaning.
- $(T + 1_E T) \circ (T - q 1_E T) = 0$. 

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• $T \circ (1_E X) \circ T = \begin{cases} qX1_E, & q \neq 1, \\ X1_E - T, & q = 1. \end{cases}$

• $X - a$ is nilpotent.

The notion of a morphism of (weak) categorifications is introduced in a natural way, see [8, 5.1.5.2].

Below we sometimes, following [8], write $E_+$ for $E$ and $E_-$ for $F$.

To finish the section let us provide a prototypical example of a categorification, see [8, 7.2]. Consider the affine Hecke algebra $H_q^n(n)$, where $q \neq 1$, generated by elements $T_1, \ldots, T_{n-1}, X_1, \ldots, X_n$ subject to the usual relations: the elements $T_1, \ldots, T_{n-1}$ satisfy the braid relations, the elements $X_1, \ldots, X_n$ pairwise commute, and, further, $T_i X_j = X_j T_i$ if $j - i \neq 0, 1$, while $T_i X_j T_i = q X_i X_j$. Fix a collection $Q$ of nonzero complex numbers $Q_0, \ldots, Q_{\ell-1}$ and consider the cyclotomic quotient $H_q^\ell(n)$ of $H_q^n(n)$ by the relation $\prod_{i=0}^{\ell-1}(X_1 - Q_i) = 0$. Consider the category $C := \bigoplus_{n \geq 0} H_q^0(n)$-$\text{mod}$. For every $n$ we have a natural inclusion $H_q^0(n - 1) \hookrightarrow H_q^0(n)$ making $H_q^0(n)$ into a free (left or right) $H_q^0(n - 1)$-module. So the category $C$ comes equipped with exact restriction (from $n$ to $n - 1$) and induction (from $n - 1$ to $n$) endofunctors. Clearly, $X_n \in H_q^0(n)$ commutes with $H_q^0(n - 1)$.

Pick $a \in \mathbb{K}^\times$ and consider the functors $E = \bigoplus_n E_n$, $F = \bigoplus_n F_n$. Here $E_n$ is the generalized eigen-functor for the action of $X_n$ on $\text{Res}_n : H_q^0(n)$-$\text{mod} \to H_q^0(n - 1)$-$\text{mod}$ with eigenvalue $a$. The functor $F_n$ is the direct summand of $\text{Ind}_n : H_q^0(n - 1)$-$\text{mod} \to H_q^0(n)$-$\text{mod}$ that is biadjoint to $E_n$.

2.2 $sl_2$-Categorifications: properties

Here we will list some properties of $sl_2$-categorifications.

**Proposition 2.1** ([8], Proposition 5.5) Let $C$ be a weak $sl_2$-categorification. Let $C_a$ denote the full subcategory of $C$ consisting of all objects whose class in $[C]$ lies in the $a$-weight space. Then $C = \bigoplus_{a \in 2C} C_a$ and $EC_a \subset C_{a+2}$, $FC_a \subset C_{a-2}$.

Following [8], we introduce some notation. For a simple object $S$ in a weak $sl_2$-categorification $C$ set $h_+(S) := \max\{|i| E_i^+ S \neq 0\}$ with $? = +, -$ and $d(S) = h_-(S) + h_+(S) + 1$. It is clear, in particular, that for $S \in C_a$ we have $a = h_-(S) - h_+(S)$.

**Lemma 2.2** ([8], Lemma 5.11) Let $C$ be a weak $sl_2$-categorification. Let $M$ be an object in $C$ such that for any its simple quotient $S$ one has $d(S) \geq d$. Then for any simple quotient $T$ of $E_\pm M$ we have $d(T) \geq d$. The same is true for subobjects instead of quotients.

The following theorem summarizes some results obtained in [8, Lemma 5.13, Proposition 5.20] (a part of (3) is actually contained in the proof of Proposition 5.20).

**Proposition 2.3** Let $C$ be an $sl_2$-categorification, $i \leq n$ be non-negative integers, and $S$ a simple in $C$ with $h_+(S) = n$. Then the following holds.

1. The functor $E^i$ decomposes into the sum of $i!$ copies of a functor $E^{(i)}$.
2. The socle and the head of $E^{(i)} S$ are isomorphic to the same simple object, say $T$ (depending on $i$).
3. Furthermore, for any other simple subquotient $T'$ of $E^{(i)} S$ we have $E^{n-i} T' = 0$. The same holds if we replace $E$ with $F$ and $h_+(\bullet)$ with $h_-(\bullet)$.
Another useful and interesting property of an $\mathfrak{sl}_2$-categorification $C$ that will not be used in this paper is that on $C_0$ one has $EF \oplus \text{Id}_{\mathfrak{g}^{\text{max}(0,-a)}} \cong FE \oplus \text{Id}_{\mathfrak{g}^{\text{max}(a,0)}}$. This is a categorical version of the relation $[e, f] = h$ in $\mathfrak{sl}_2$.

2.3 More general categorifications

To a simply laced quiver $Q$ with finitely many vertices one can assign the Kac-Moody algebra $\mathfrak{g}(Q)$. Let $I$ be the set of vertices of $Q$ and let $e_i, f_i$ be the Chevalley generators of $\mathfrak{g}(Q)$. Then one can introduce the notion of a weak $\mathfrak{g}(Q)$-categorification on $C$ similarly to the above ("locally finite" becomes "integrable"). This structure includes exact functors $E_i, F_i, i \in I$, together with fixed adjointness morphisms. The notion of a genuine $\mathfrak{g}(Q)$-categorification is more complicated than in the $\mathfrak{sl}_2$-case (with an exception of finite and affine type $A$), see [22]. We will not need the definition of a $\mathfrak{g}(Q)$-categorification. The only thing that we will use is that if $E_i, F_i, i \in I$, define a $\mathfrak{g}(Q)$-categorification, then for each $i$ the pair $E_i, F_i$ defines an $\mathfrak{sl}_2$-categorification.

2.4 Crystals

In this paper we consider $\mathfrak{sl}_2$-crystals. An $\mathfrak{sl}_2$-crystal is a set $C$ equipped with maps $\text{wt} : C \to \mathbb{Z}, \tilde{e}, \tilde{f} : C \to C \cup \{0\}, h_+, h_- : C \to \mathbb{Z}_{\geq 0}$. These maps should satisfy the following conditions

(i) $\text{wt} = h_- - h_+$.
(ii) $\tilde{e}c = 0$ if and only if $h_+(c) = 0$. Similarly, $\tilde{f}c = 0$ if and only if $h_-(c) = 0$.
(iii) For $c, c' \in C$ the equality $\tilde{e}c = c'$ is equivalent to $\tilde{f}c' = c$.
(iv) If $c' = \tilde{e}c$, then $h_+(c') = h_+(c) - 1, h_-(c') = h_-(c) + 1$.

Let $C, C'$ be $\mathfrak{sl}_2$-crystals. A map $\varphi : C \to C'$ is said to be a morphism of crystals if it intertwines $h_-, h_+, \tilde{e}, \tilde{f}$, i.e., $h_-(\varphi(c)) = h_-(c), h_+(\varphi(c)) = h_+(c), \text{wt}(\varphi(c)) = \text{wt}(c), \tilde{e}\varphi(c) = \varphi(\tilde{e}c), \tilde{f}\varphi(c) = \varphi(\tilde{f}c)$ (of course, we set $\varphi(0) = 0$). In particular, if $C \subset C'$ and the inclusion is a morphism of crystals one says that $C$ is a subcrystal of $C'$.

Example 2.4 The following example will be of great importance. Consider the set $C = \{+, -, 0\}^n$. So an element of $C$ is an ordered $n$-tuple of $+$’s and $-$’s. Let us define the reduced form of an element $t \in C$ as follows. This will be an $n$-tuple whose elements are $+, -, 0$. We transform $t = (t_1, \ldots, t_n)$ step by step as follows: for any $a < b$ with $t_a = -, t_b = +, t_i = 0$ for $a < i < b$ we replace $t_a, t_b$ with 0’s. We continue these transformations while possible and so we stop when in no + appears to the right of a -. This is the reduced form of interest to be denoted by $t^{red}$. It is easy to check that $t^{red}$ is well-defined. Also we remark that $t^{red}_i = t_i$ or 0. Define $h_+(t), h_-(t)$ as the number of $+$’s and $-$’s in the reduced form of $t$. Further, let $\tilde{e}t$ be the sequence obtained from $t$ by changing $t_i$ from + to -, where $i$ is the largest index such that $t^{red}_i = +$. We set $\tilde{e}t = 0$ if no such index $i$ exists. Similarly, let $\tilde{f}t$ be the sequence obtained from $t$ by changing $t_j$ from $-$ to $+$, where $j$ is the smallest index such that $t^{red}_j = -$. We set $\tilde{f}t = 0$ if no such index $j$ exists. It is straightforward to check that $C$ together with these structures is a crystal.

For instance, consider the 7-tuple $t = (+ - - - + - +)$. Here the reduced signature is $(-0000)$ and so $h_+(t) = 1, h_-(t) = 2$. We have $\tilde{e}t = (- - - + + + +)$ and $\tilde{f}t = (+ + + - + + +)$.

In fact, to any $\mathfrak{sl}_2$-categorification $C$ one can assign a crystal in a standard way. Namely, $C$ is the set of simples, the functions $h_+, h_-$ are as defined in 2.2 and $\text{wt}(S) = a$ if $S \in C_a$. 

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Further, $\delta S$ is 0 if $ES = 0$, and $\delta S$ is a unique simple object in the socle (equivalently, in the head) of $ES$ if $ES \neq 0$, see Proposition 2.3. The map $\delta f$ is defined similarly using $F$ instead of $E$. The condition (iii) follows from $\text{Hom}_C(ES, T) = \text{Hom}_C(S, FT)$.

Similarly, one can introduce a crystal for $g(Q)$ and produce such a crystal from a $g(Q)$-categorification. In this paper, however, we are going only to deal with crystals for $sl_2$, in particular, we will not need axioms for $g(Q)$-crystals.

3 Highest weight categories

3.1 General definition

As before, $\mathbb{K}$ stands for a field. Recall that by a highest weight category one means a pair $(C, \Lambda)$ of an artinian $\mathbb{K}$-linear abelian category $C$ and a poset $\Lambda$ equipped with a collection of objects $\Delta(\lambda) \in C$, one for each $\lambda \in \Lambda$. These data are supposed to satisfy the following conditions.

(HW1) $\text{End}_C(\Delta(\lambda)) = \mathbb{K}$.
(HW2) There is a unique simple quotient $L(\lambda)$ of $\Delta(\lambda)$, and each simple in $C$ is isomorphic to precisely one $L(\lambda)$.
(HW3) For each $\lambda \in \Lambda$ there is an indecomposable projective object $P(\lambda)$ equipped with a filtration $P(\lambda) = F_0 \supset F_1 \supset F_2 \ldots$ such that $F_i / F_{i+1} = \Delta(\lambda_i)$ with $\lambda_i > \lambda$ for all $i > 0$.
(HW4) The BGG reciprocity holds: for all $\lambda, \mu \in \Lambda$ the multiplicity of $L(\mu)$ inside $\Delta(\lambda)$ equals to the multiplicity of $\Delta(\lambda)$ inside $P(\mu)$.

For a highest weight category $C$ let $C^\Delta$ denote the full exact subcategory of $\Delta$-filtered objects, i.e., those that have a filtration whose successive quotients are of the form $\Delta(\lambda)$, one for each $\lambda \in \Lambda$. These data are supposed to satisfy the following conditions.

Now let us recall the definition of costandard objects. Following [21, Proposition 4.19], this is a unique set of objects $\nabla(\lambda)$ indexed by $\Lambda$ such that $(C^{\text{opp}}, \nabla(\lambda))$ is a highest weight category, and $\text{Ext}^i(\Delta(\lambda), \nabla(\mu)) = \mathbb{K}$ if $i = 0$, $\lambda = \mu$ and 0 else. Recall, [21, Lemma 4.21] that an object $N \in C$ lies in $C^\Delta$ if and only if $\text{Ext}^1(N, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$.

3.2 Representations of $GL$.

Assume from now on that the field $\mathbb{K}$ is algebraically closed. Consider the category $C := \text{Rep}(GL_n(\mathbb{K}))$ of all rational finite dimensional representations of the algebraic group $GL_n(\mathbb{K})$. This is a highest weight category: the standard objects are the Weyl modules $\Delta(\lambda)$, where $\lambda$ is a strictly dominant weight, i.e., $\lambda = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ are integers, see [15] for details (the highest weight of $\Delta(\lambda)$ is $\lambda - \rho$, where $\rho := (0, -1, -2, \ldots, 1 - n)$). For the ordering on the set of weights we can take the usual ordering on the dominant weights: $\lambda \leq \mu$ if $\mu - \lambda$ is a linear combination of positive roots with non-negative integral coefficients.

The categorification structure is introduced as follows, see [8, 7.5]. Consider the tensor Casimir $\Omega = \sum_{i,j} e_{ij} \otimes e_{ji} \in \mathfrak{gl}_n \times \mathfrak{gl}_n$, where $e_{ij}$ is the unit matrix $(\delta_{ii} \delta_{jj})_{i,j=1}$. For $M \in C$ and $i \in \mathbb{Z}$ let $F_i M$ (resp., $E_i M$) be the $i$-th (resp., $-n - i$-th) generalized eigenspace of $\Omega$ on $\mathbb{K}^n \otimes M$ (resp., on $(\mathbb{K}^n)^* \otimes M$). Of course, $E_i = E_j$, $F_i = F_j$ if $i = j$ in $\mathbb{K}$. So we get a $\mathfrak{gl}_\mathbb{K}$-categorification if $\text{char}\mathbb{K} = 0$ or an $\mathfrak{sl}_p$-categorification if $\text{char}\mathbb{K} = p$. 

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Let us now explain the categorification structure on the polynomial representations of GL, see [14] for details. Let $\text{Rep}^d(\text{GL}_n(\mathbb{K}))$ denote the subcategory in $\text{Rep}(\text{GL}_n(\mathbb{K}))$ consisting of all polynomial representations of degree $d$. The simple $L(\lambda)$ (equivalently, the standard $\Delta(\lambda)$) is polynomial of degree $d$ if and only if $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = d$. If $\lambda < \mu$ and $L(\mu)$ is a polynomial representation, then so is $L(\lambda)$. It follows that $\text{Rep}^d(\text{GL}_n(\mathbb{K}))$ is a highest weight category, whose standard objects are still the Weyl modules. The categories $\text{Rep}^d(\text{GL}_n(\mathbb{K}))$ are mutually equivalent (as highest weight categories) as long as $n \geq d$. Any of these categories is denoted by $\text{Rep}^d(\text{GL}(\mathbb{K}))$. In [14] it was shown how to modify the construction above for $\text{Rep}(\text{GL}_n(\mathbb{K}))$ to produce a categorical action on $\bigoplus_{d=0}^{\infty} \text{Rep}^d(\text{GL}(\mathbb{K}))$ (a modification is necessary because $E_i M$ may not be polynomial if $M \in \text{Rep}^d(\text{GL}_n(\mathbb{K}))$).

We remark that the simple objects in $(\text{GL}(\mathbb{K}))$ are naturally parameterized by Young diagrams.

3.3 Parabolic categories $\mathcal{O}$

Assume $\mathbb{K}$ has characteristic 0. Pick a positive integer $n$ and consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$. Next, pick a collection $n := (n_1, \ldots, n_k)$ of positive integers summing to $n$. Let $e_1, \ldots, e_n$ be the tautological basis in $\mathbb{K}^n$. Let $\mathfrak{p}$ be the parabolic subalgebra that stabilizes the subspaces

$$\text{Span}(e_1, \ldots, e_{n_1+n_2+\ldots+n_i}), \quad i = 1, \ldots, k$$

and let $\mathfrak{l}$ be its Levi subalgebra preserving the subspaces

$$\text{Span}(e_{n_1+\ldots+n_{i-1}+1}, \ldots, e_{n_1+\ldots+n_i}), \quad i = 1, \ldots, k.$$

Consider the category $\mathcal{O}^{\mathbb{Z}}$ consisting of all $\mathfrak{g}$-modules with integral central characters, where $\mathfrak{p}$ acts locally finitely and $\mathfrak{l}$ acts semisimply. This is a highest weight category, where standard objects are parabolic Verma modules $\Delta(\lambda)$ with $\lambda$ being a parabolically strictly dominant weight in the sense that $\lambda_1 > \ldots > \lambda_{n_1}, \lambda_{n_1+1} > \ldots > \lambda_{n_2}, \ldots, \lambda_{n_1+\ldots+n_k-1+1} > \ldots > \lambda_n$. An ordering on $\mathcal{O}^{\mathbb{Z}}$ is chosen as in Sect. 3.2.

The category $\mathcal{O}^{\mathbb{Z}}$ comes equipped with a $\mathfrak{gl}_\infty$-categorification, see [8, 7.4] and [6, Section 4], analogously to Sect. 3.2. A similar construction works for parabolic categories $\mathcal{O}^{\mathbb{Z}}_p$ for the Lusztig form $U_\epsilon(\mathfrak{gl}_n)$ of the quantized enveloping algebra at a root of unity $\epsilon$. There we get a categorical $\mathfrak{sl}_m$-action on the category $\mathcal{O}^{\mathbb{Z}}_p$, where $m$ is the order of $\epsilon$.

3.4 Cherednik categories $\mathcal{O}$: general case

We are going to start by recalling the definition of the Rational Cherednik algebras due to Etingof and Ginzburg, [9]. In this and a subsequent section we are going to assume that $\mathbb{K}$ is the field of complex numbers.

Let $\mathfrak{h}$ be a complex vector space, and $W \subset \text{GL}(\mathfrak{h})$ be a finite subgroup generated by complex reflections. Recall that $s \in \text{GL}(\mathfrak{h})$ is called a complex reflection if the dimension of the fixed point subspace $\mathfrak{h}^s$ equals $\dim \mathfrak{h} - 1$. Let $S_0, \ldots, S_r$ be all conjugacy classes of complex reflections in $W$. For each $S_i$ pick a complex number $c_i$. Also for a complex reflection $s$ let $\alpha_s \in \mathfrak{h}^*, \alpha_s^\vee \in \mathfrak{h}$ be elements vanishing on $\mathfrak{h}^s$, $(\mathfrak{h}^*)^s$, respectively, normalized by $\langle \alpha_s, \alpha_s^\vee \rangle = 2$.

Set $p := (c_0, \ldots, c_r)$. The rational Cherednik algebra $H_p(= H_p(\mathfrak{h}, W))$ is the quotient of the smash-product $T(\mathfrak{h} \oplus \mathfrak{h}^* )#W$ (= the semidirect tensor product of $\mathbb{K}W$ and $T(\mathfrak{h} \oplus \mathfrak{h}^*)$) by the relations

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\[ [x, x'] = [y, y'] = 0, \ [y, x] = \langle x, y \rangle - \sum_{i=0}^{r} c_i \sum_{s \in S_i} \langle x, \alpha_s \rangle \langle y, \alpha_s \rangle s. \] (3.1)

Following [11, 4.2] consider the map \( w \mapsto w^{-1} : W \to W \). It induces an involution on the set of conjugacy classes and hence an involution \( p \mapsto p^* \) on the set of parameters. The map \( x \mapsto x, y \mapsto -y, w \mapsto w^{-1}, x \in h^*, y \in h, w \in W \) gives rise to an isomorphism

\[ H_p(h, W) \xrightarrow{\sim} H_{p^*}(h^*, W)^{opp}. \] (3.2)

Let us proceed now to the category \( O \). According to [9], the natural homomorphism \( S(h) \otimes K W \otimes S(h^*) \to H_e \) is an isomorphism of vector spaces. So, following [11], we can consider the category \( O_p(W) := O_p(h, W) \) consisting of all finitely generated \( H_e \)-modules, where the action of \( h \) is locally nilpotent. We remark that any module in \( O_p(W) \) is finitely generated over \( K[h] = S(h^*) \). Let us give an important example of a module in \( O_p(W) \).

Take an irreducible \( W \)-module \( L \) and set \( \Delta(L) := (\Delta_p(L) :=) H_p \otimes S(h^*) \# W L \), where \( h^* \) is supposed to act by 0 on \( L \). This is a so called standard (or Verma) module in \( O_p(W) \).

It turns out that \( O_p(W) \) together with the collection of standard modules is a highest weight category. A partial order on \( \text{Irr}(W) \) can be defined as follows. Consider the \textit{deformed Euler element} \( \text{eu} \in H_e \) given by

\[ \text{eu} := \sum_{i=1}^{\dim h} x_i y_i + \frac{\dim h}{2} - \sum_{i=0}^{r} c_i \sum_{s \in S_i} \frac{2}{1 - \lambda_s} s. \] (3.3)

Here \( x_i, y_i \) are mutually dual bases of \( h^*, h \) and \( \lambda_s \) denotes the only non-unit eigenvalue of \( s \) in its action on \( h^* \). The element \( \text{eu} \) commutes with \( W \), while \( [\text{eu}, x] = x, [\text{eu}, y] = -y \) for \( x \in h^*, y \in h \). Given a parameter \( p \) define a \textit{c-function} \( c_p : \text{Irr}(W) \to K \) as follows: \( c_p(E) \) is the difference between the eigenvalues of \( \text{eu} \) on \( E \subset \Delta(E) \) and on \( \text{triv} \subset \Delta(\text{triv}) \). We set \( E >_p E' \) if \( c_p(E) - c_p(E') \in \mathbb{Z}_{\leq 0} \). Clearly, if \( L(E') \) appears in \( \Delta(E) \), then \( E \geq p \).

An important tool to study the category \( O_p(W) \) is the KZ functor from [11]. It is a surjective exact functor from \( O_p(W) \) to the category of modules over the Hecke algebra \( \mathcal{H}_p(W) \), whose parameters are recovered from \( p \). An important property of the KZ functor is that it is fully faithful on projectives.

Now let us recall the duality for the categories \( O \) from [11, 4.2]. Take a module \( M \in O_p(W) \). The space \( \text{Hom}_{K}(M, \mathbb{K}) \) has a natural structure of a right \( H_p(h, \mathbb{K}) \)-module and thanks to (3.2) of a left \( H_{p^*}(h^*, \mathbb{K}) \)-module. Let \( D(M) \) be the span of all generalized \( \text{eu} \)-eigenvectors in \( \text{Hom}_{K}(M, \mathbb{K}) \). This is a module in \( O_{p^*}(h^*, W) \). It is easy to show that \( D^2 \) is the identity functor. The costandard objects in \( O_p(W) \) are given by \( \nabla_p(E) = D(\Delta_{p^*}(E^*)) \).

### 3.5 Cherednik categories \( O \): cyclotomic case

The categories \( O \) for general Rational Cherednik algebras do not seem to give rise to categorifications. The latter appear only in the \textit{cyclotomic} case that we are going to describe in this section.

Suppose \( W = G(\ell, 1, n) \), where \( \ell > 1, n \geq 1 \), is the wreath product of the symmetric group \( S_n \) and the group \( \mu_\ell \) of \( \ell \)-th roots of 1. That is, \( W := S_n \times \mu_\ell^n \) acts on \( h := \mathbb{K}^n \) in a natural way. For \( n > 1 \) there are following \( \ell \) classes of complex reflections in \( W \):

- \( S_0 \) consisting of elements of the form \((ij)\gamma_i^{-1}\gamma_j\), where \((ij)\) is the transposition in \( S_n \) swapping \( i \) and \( j \), and \( \gamma_i, \gamma_j \) are elements in the \( i \)-th and \( j \)-th copies of \( \mu_\ell \) inside of \( \mu_\ell^n \),
- \( S_i, i = 1, \ldots, \ell - 1 \), consisting of the elements \( \gamma_j \) with \( \gamma = \exp(2\pi \sqrt{-1}/\ell), j = 1, \ldots, \ell \).
We remark that for \( n = 1 \) there are \( \ell - 1 \) conjugacy classes: \( S_0 \) is absent. In fact, it is convenient to use another set of parameters. Pick a complex number \( \kappa \) and set \( c_0 := -\kappa \). The case \( \kappa = 0 \) is non-interesting because the algebra \( H_p \) in this case decomposes as \( (H_p^1)^{\otimes n} # S_n \), where \( H_p^1 \) is the similar algebra for \( n = 1 \). Below we always assume that \( \kappa \neq 0 \). Also let \( s_0, \ldots, s_{\ell - 1} \) be complex numbers. Then we set
\[
c_i := -\frac{1}{2}(1 + \kappa \sum_{j=1}^{\ell-1} \exp(-i j \cdot 2\pi \sqrt{-1}/\ell) - 1) (s_j - s_{j-1}), \quad i = 1, \ldots, \ell - 1. \tag{3.4}
\]
We remark that two collections \( s = (s_0, \ldots, s_{\ell-1}) \), \( s' = (s_0, \ldots, s_{\ell-1}) \) give rise to the same parameters \( c_1, \ldots, c_{\ell-1} \) if and only if \( s_i' - s_i \) is independent of \( i \).

Let us proceed to the category \( O \). In this case we will write \( O_p(n) \) instead of \( O_p(W) \).

First, let us recall the classical combinatorial description of \( \text{Irr}(W) \): the \( W \)-irreducibles are parameterized by \( \ell \)-multipartitions \( \lambda := (\lambda^{(0)}, \ldots, \lambda^{(\ell-1)}) \) of \( n \). Namely, consider the subgroup \( G(\lambda, 1, \ell) = \prod_{i=0}^{\ell-1} G(\lambda^{(i)}, 1, \ell) \subset G(n, 1, \ell) \). Here and below for a partition \( \mu = (\mu_1, \mu_2, \ldots) \) we set \( |\mu| := \sum_i \mu_i \). View \( \lambda^{(i)} \) as a representation of \( S_{|\lambda^{(i)}|} \). Further, let \( \lambda^{(i)}(r) \) denote the representation of \( G(\lambda^{(i)}, 1, \ell) \), that coincides with \( \lambda^{(i)} \) as an \( S_{|\lambda^{(i)}|} \)-module, while for \( \gamma \in \mu_\ell \) the element \( \gamma_1 \) acts by the scalar \( \gamma^r \). The irreducible \( W \)-module corresponding to \( \lambda \) is induced from the \( G(\lambda, 1, \ell) \)-module \( \lambda^{(0)}(0) \boxtimes \lambda^{(1)}(1) \boxtimes \ldots \boxtimes \lambda^{(\ell-1)}(\ell - 1) \).

Let us now provide a formula for the \( c \)-function, obtained in \([21]\). We will express the \( c \)-function \( c_p(\lambda) \) in terms of the presentation \( p = (\kappa, s_0, \ldots, s_{\ell-1}) \). We represent a partition \( \mu \) as a Young diagram with \( \mu_1 \) boxes in the first row, \( \mu_2 \) in the second row and so on. For a box \( x \) lying in the \( a \)th row and \( b \)th column of \( \lambda^{(i)} \) we define its \( s \)-shifted content by \( \text{cont}^s(x) := s_i + b - a \). Then set
\[
d^P(x) := \kappa \ell \text{cont}^s(x) - i - \kappa \sum_{i=0}^{\ell} s_i, \tag{3.5}
\]
\[
c_p(\lambda) := \sum_{x \in \lambda} d^P(x). \tag{3.6}
\]
Up to a scalar independent of \( p \) and \( n \) the function \( c_p(\lambda) \) coincides with the \( c \)-function introduced above, see \([12, (2.3.8)]\).

The Hecke algebra \( H_p(W) \) is just the cyclotomic Hecke algebra \( H_p^\mathcal{Q}(n) \) mentioned in Sect. 2.1. The parameters \( q, Q_0, \ldots, Q_{\ell-1} \) are determined by \( p = (\kappa, s_0, \ldots, s_{\ell-1}) \) in the following way: \( q := \exp(2\pi \sqrt{-1}\kappa) \), \( Q_i := \exp(2\pi \sqrt{-1}\kappa s_i) \).

Now we are in position to recall Shan’s categorification, \([23]\). Consider the category \( O_p := \bigoplus_{n \geq 0} O_p(n) \), where \( O_p(0) \) is just the category of finite dimensional vector spaces.

Etingof and Bezrukavnikov defined induction and restriction functors for rational Cherednik algebras in \([1]\). Namely, let \( W \) be an arbitrary complex reflection group acting on \( \mathfrak{h} \), and let \( W \) be its parabolic subgroup. Let \( \mathfrak{h} \) be a unique \( W \)-stable complement to the fixed point subspace \( \mathfrak{h}^W \) in \( \mathfrak{h} \). Abusing the notation we denote the restriction of \( p \) to \( S \cap W \) again by \( p \). According to \([1]\) there are exact functors \( \text{Res}^W_W : O_p(\mathfrak{h}, W) \rightarrow O_p(\mathfrak{h}, \mathfrak{h}_\mathfrak{g}, W) \) and \( \text{Ind}^W_W : O_p(\mathfrak{h}, W) \rightarrow O_p(\mathfrak{h}, W) \). Shan in \([23]\) (see also \([19]\)) checked that these functors are biadjoint.

In particular, we can consider the case \( W = G(n - 1, 1, \ell) \subset G(n, 1, \ell) \). The Bezrukavnikov-Etingof construction yields exact endofunctors \( \text{Res} \) of \( O_p \) with \( \text{Res} : O_p(n) \rightarrow O_p(n - 1) \). According to \([23, \text{Section 2}]\), the KZ functors intertwine the inductions/restrictions in \( O_p \) with the cyclotomic Hecke algebra.
inductions/restrictions. Since the KZ functors are fully faithful on projectives, this yields the
generalized eigen-functor decompositions \( \text{Res} = \bigoplus_{z \in \mathbb{K}} E_z \), \( \text{Ind} = \bigoplus_{z \in \mathbb{K}} F_z \). The functors \( E_z, F_z \) constitute an \( \mathfrak{s}_2 \)-categorification on \( \mathcal{O}_p \).

We are interested in the behavior of the functors \( E_z, F_z \) on the rational Grothendieck group \( \mathcal{O}_p \). We have the basis \( \{ \Delta(\lambda) \} \) of \( \mathcal{O}_p \) indexed by all multi-partitions \( \lambda \). We call a box \( x \) of a multipartition \( \lambda \) a \( z \)-box if \( \exp(2\pi \sqrt{-1}\text{cont}(x)) = z \). Recall that we represent \( \lambda \) as a collection of Young diagrams. A box lying in \( \lambda \) (i.e., in one of the diagrams \( \lambda^{(i)} \)) is called \textit{removable} if \( \lambda^{(i)} \setminus \{x\} \) is still a Young diagram (in other words, a box in \( j \)-th row and \( k \)-th column is removable if \( k = \lambda^{(j)}_j \) and \( \lambda^{(j)}_{j+1} < \lambda^{(j)}_j \)). Similarly, a box lying outside of \( \lambda^{(i)} \) is \textit{addable} if \( \lambda^{(i)} \cup \{x\} \) is again a Young diagram (equivalently, \( x \) lies in \( j \)-th row and \( \lambda^{(j)}_j + 1 \)-th column with either \( j = 1 \) or \( \lambda^{(j)}_j < \lambda^{(j+1)}_j \)).

We observe that [23, Proposition 4.4] has the following generalization whose proof basically repeats Shan’s.

**Proposition 3.1** We have \( \{ E_z \Delta(\lambda) \} = \bigoplus_x \{ \Delta(\lambda \setminus \{x\}) \} \), where the sum is taken over all removable \( z \)-boxes \( x \). Further, \( \{ F_z \Delta(\lambda) \} = \bigoplus_x \{ \Delta(\lambda \cup \{x\}) \} \), where the sum is taken over all addable \( z \)-boxes \( x \).

## 4 Highest weight \( \mathfrak{s}_2 \)-categorifications

### 4.1 Definition

Assume now that \( (\mathcal{C}, \Lambda) \) is a highest weight category and that \( \mathcal{C} \) is equipped with an \( \mathfrak{s}_2 \)-categorification, let \( E, F \) be the categorification functors. We say that \( \mathcal{C} \) is a highest weight categorification if there are

- a function \( c : \Lambda \to \mathbb{C} \),
- an index set \( \mathfrak{A} \), a collection of non-negative integers \( n_a, a \in \mathfrak{A} \), a partition \( \Lambda = \bigsqcup_{a \in \mathfrak{A}} \Lambda_a \),
- identifications \( \sigma_a : \{+,-\}^{n_a} \overset{\sim}{\to} \Lambda_a \), and functions \( d_a : \{1, 2, \ldots, n_a\} \to \mathbb{C} \)

such that the following conditions are satisfied.

**(HWC0)** The functors \( E, F \) preserve the subcategory \( \mathcal{C}^\Delta \) of \( \Delta \)-filtered objects.

**(HWC1)** The inequality \( \lambda < \mu \) implies \( c(\lambda) > c(\mu) \).

**(HWC2)** For \( a \in \mathfrak{A}, t \in \{+,-\}^{n_a} \), in the Grothendieck group of \( \mathcal{C} \) we have \( e[\Delta(\sigma_a(t))] = \sum_j [\Delta(\sigma_a(t^j))] \), where the sum is taken over all \( j \in \{1, 2, \ldots, n_a\} \) such that \( t_j = + \), where \( t^j \in \{+,-\}^{n_a} \) is given by \( t^j_k = t_k \) for \( k \neq j \) and \( t^j_j = - \). Similarly, \( f[\Delta(\sigma_a(t))] = \sum_l [\Delta(\sigma_a(t^l))] \), where the sum is taken over all \( l \in \{1, 2, \ldots, n_a\} \) such that \( t_l = - \), and \( t^l_k = t_k \) for \( k \neq j \) and \( t^l_j = + \).

**(HWC3)** For \( t^j, t^l \) as above we have \( c(t^j) = c(t) + d_a(j), c(t^l) = c(t) - d_a(l) \).

**(HWC4)** Finally, for any \( a \) we have \( d_a(1) < d_a(2) < \ldots < d_a(n_a) \).

Recall that for complex numbers \( \alpha, \beta \) we write \( \alpha < \beta \) if \( \beta - \alpha \) is a positive integer. The definition is, of course, obtained by generalizing examples.

We remark that for further results to be obtained in [20] we will use a finer ordering on \( \mathcal{C} \) and so will need to modify the definition of a highest weight categorification making it much more technical.
4.2 Examples

In this section we will show that all examples of \( \mathfrak{sl}_2 \)-categorifications we considered before are actually highest weight categorifications in the sense of Sect. 4.1. We are going to consider the Cherednik case in detail and only sketch the other (that are more standard).

Let \( \mathcal{O}_p \) be the sum \( \bigoplus_{n \geq 0} \mathcal{O}_p(n) \) of the categories \( \mathcal{O} \) for the cyclotomic Cherednik algebra \( H_p(n) \) as in Sect. 3.5. Assume that \( \kappa \) is not integral.

We set \( \mathcal{C} := \bigoplus_{n \geq 0} \mathcal{O}_p(n) \), \( E := F_z \), \( F := E_z \), where \( z \) is some complex number. Then \( E, F \) define an \( \mathfrak{sl}_2 \)-categorification on \( \mathcal{O}_p \).

**Lemma 4.1** The \( \mathfrak{sl}_2 \)-categorification \( \mathcal{C} \) satisfies (HWC0).

**Proof** From the definition of the functors \( E_z, F_z \) it follows that one only needs to prove that all Bezrukavnikov-Etingof functors \( \text{Res}^{W}_{W}, \text{Ind}^{W}_{W} \) preserve \( \mathcal{C}^{\Delta} \). By [23, Proposition 1.9], the restriction functors \( \text{Res}^{W}_{W} \) preserves \( \mathcal{C}^{\Delta} \). Recall that \( \text{Ind}^{W}_{W} \) is right adjoint to \( \text{Res}^{W}_{W} \) and both \( \text{Ind}^{W}_{W}, \text{Res}^{W}_{W} \) are exact. So for any \( M, N \in \mathcal{C} \) we \( \text{Ext}^i(\text{Res}^{W}_{W}(M), N) = \text{Ext}^i(M, \text{Ind}^{W}_{W}(N)) \).

Now [21, Lemma 4.21] implies that \( \text{Ind}^{W}_{W} \) preserves the subcategory \( \mathcal{C}^{V} \subset \mathcal{C} \) of all standardly filtered objects. Recall that Bezrukavnikov and Etingof, [1], introduced functors \( \text{res}^{W}_{W} := D \circ \text{Res}^{W}_{W} \circ D, \text{ind}^{W}_{W} := D \circ \text{Ind}^{W}_{W} \circ D, \) where \( D \) is the duality functor recalled in Sect. 3.4. Since the standard and costandard objects are related via \( D \), we see that \( \text{res}^{W}_{W} \) preserves \( \mathcal{C}^{V} \), while \( \text{ind}^{W}_{W} \) preserves \( \mathcal{C}^{\Delta} \). The main result of [19] says that \( \text{Res}^{W}_{W} \cong \text{res}^{W}_{W} \) and \( \text{Ind}^{W}_{W} \cong \text{ind}^{W}_{W} \). This completes the proof.

Let us explain the choice of \( c, \mathfrak{A}, n_a, \Lambda_a, \sigma, d_a \) making \( \mathcal{C} \) into a highest weight categorification. For \( c \) we just take the \( c \)-function recalled in Sect. 3.4. The condition (HWC1) follows. Two multipartitions \( \lambda, \mu \) belong to the same set \( \Lambda_a \) if the multipartitions obtained from \( \lambda \) and \( \mu \) by removing all removable \( z \)-boxes coincide. The following easy combinatorial lemma shows that the sets of addable and removable \( z \)-boxes in \( \lambda \) and \( \mu \) coincide.

**Lemma 4.2** For a (multi)partition \( \mu \) let \( B_z(\mu) \) denote the set of all addable and removable \( z \)-boxes in \( \mu \). Then for any addable \( z \)-box \( x \) we have \( B_z(\mu \sqcup x) = B_z(\mu) \).

**Proof** It is easy to see that adding a \( z \)-box affects only the sets \( B_{q^\pm 1}^z(\mu) \), where \( q = \exp(2\pi \sqrt{-1}\kappa) \).

For \( n_a \) we take the cardinality of \( B_z(\lambda), \lambda \in \Lambda_a \), from the previous lemma. For a \( z \)-box \( x \) of \( \lambda \) we set \( d_a(x) := d^p(x) \) (where the last number is equal to \( \kappa \ell \text{cont}^p(x) - r - \kappa \sum_{i=0}^{r-1} s_i \) if \( x \) is in \( \lambda^{(r)} \)). If \( x \in \lambda^{(r)} \), \( y \in \lambda^{(r')} \) are \( z \)-boxes, then \( \kappa \text{cont}^r(x) - \kappa \text{cont}^{r'}(y) \in \mathbb{Z} \). Also there is at most one addable/removable box with a given content in each diagram. So for different \( x, y \in B_z(\lambda) \) the numbers \( d_a(x), d_a(y) \) differ by a nonzero integer. Let us number the boxes \( x_1, \ldots, x_{n_a} \) so that the sequence \( d_a(x_j) \) increases. Now we can define the map \( \sigma_a : \{+,-\}^{n_a} \rightarrow \Lambda_a \). By definition, it sends an \( n_a \)-tuple \( t \) to the only multipartition \( \lambda(t) \in \Lambda_a \), where the box \( x_i \) is in \( \lambda(t) \) if and only if \( t_i = - \) (and so it is removable, the boxes \( x_i \) with \( t_i = + \) are addable). (HWC2) follows now from Proposition 3.1. Finally, set \( d_a(j) := d_a(x_j) \). (HWC4) is tautological, and (HWC3) is a consequence of (3.5, 3.6). So we have checked that \( \mathcal{C} \) is a highest weight categorification.

Let us briefly outline the other examples. Let \( \mathbb{K} \) be an algebraically closed field. Consider the category \( \text{Rep}(\text{GL}_n) \) and fix an integer \( i \). Set \( E := F_i, F := E_i \). Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be a highest weight. Let \( I_j(\lambda) \) be the subset of \( \{1, \ldots, n\} \) consisting of all indices \( j \) with
λ_j - i = 0 in \mathbb{K}. Then F_1 \Delta(λ) has a filtration whose successive quotients are the Weyl modules \Delta(λ + ε_j), j ∈ I_1(λ), appearing with multiplicity 1 if λ + ε_j is dominant (and with multiplicity 0 otherwise). Similarly, E_j \Delta(λ) has a filtration whose successive quotients are the Weyl modules \Delta(λ - ε_j), j ∈ I_{i+1}(λ), appearing with multiplicity 1 if λ - ε_j is still in \Lambda. See [3, Theorems A,A'] for details. This shows (HWC0). For the c-function we take c(λ) = \sum_{i=1}^{n} i \lambda_i which implies (HWC1). Two dominant weights λ and \mu lie in the same \Lambda_a if for each j = 1, \ldots, n exactly one of the following possibilities holds:

- λ_j = µ_j in \mathbb{Z}.
- λ_j = µ_j + 1 in \mathbb{Z} and µ_j = i in \mathbb{K}.
- µ_j = λ_j + 1 in \mathbb{Z} and λ_j = i in \mathbb{K}.

Let j_1 < j_2 < \ldots < j_{n_a} be all indexes j such that λ_j \neq µ_j for λ, µ ∈ \Lambda_a. The n_a-tuple t := σ_a^{-1}(λ) for λ ∈ \Lambda_a is given by t_j := + if λ_{j_j} = i in \mathbb{K} and t_j = - if λ_{j_j} = i + 1 in \mathbb{K}. We set d_a(l) := j_j. Now it is easy to verify the remaining axioms (HWC2)-(HWC4).

For the remaining two categories, \bigoplus_{d=0}^{\infty} \operatorname{Rep}^d(\text{GL}) and \mathcal{O}_n, one introduces the additional structures and checks that (HWC0)-(HWC4) hold in a similar way (using [14] and [6] instead of [3]).

5 Structure of the crystal

5.1 Main result

Suppose that (\mathcal{C}, \Lambda) is a highest weight sl_2-categorification in the sense of Sect. 4.1. Recall the index set \mathfrak{A}, the subsets \Lambda_a ⊂ \mathfrak{A}, the integers n_a, and the bijections σ_a : {+,-}^{n_a} \sim \Lambda_a introduced in Sect. 4.1. Recall the crystal structure on \{+,-\}^{n_a} introduced in Example 2.4.

**Theorem 5.1** For each a ∈ \mathfrak{A} the set \{L(λ), λ ∈ \Lambda_a\} is a subcrystal in the crystal \{L(λ), λ ∈ \Lambda\} of \mathcal{C}. Further, the map t ↦ L(σ_a(t)) : \{+,-\}^{n_a} \sim \{L(λ), λ ∈ \Lambda_a\} is an isomorphism of crystals.

We remark that for \mathcal{C} = \operatorname{Rep}(\text{GL}_n) Theorem 5.1 gives the same description of the crystal as [16], see also [3, Theorems B,B'], while for \mathcal{C} = \mathcal{O}_2 we recover [4, 4.3]. We also expect that it is possible to extend the techniques used in the proof to the setting of standardly stratified categories and functors more general than sl_2-categorification functors. This should allow to recover the results from [17] that give a combinatorial description of the crystal for representations of the supergroup Q(n). We remark that categorical actions on standardly stratified categories has been already studied, see [10] for the case of sl_2 and [26] for a more general case. The results of the former paper yield a description of the crystal.

5.2 Preliminary considerations

The claim that \Lambda_a is a subcrystal follows from the following proposition.

**Proposition 5.2** Fix t = (t_1, \ldots, t_{n_a}) ∈ \{+,-\}^{n_a}. Then either \hat{e}L(σ_a(t)) = 0 or there is j with t_j = + such that \hat{e}L(σ_a(t)) = L(σ_a(t^j)), where t^j ∈ \{+,-\}^{n_a} is given by t^j_l = t_l for l \neq j and t^j_j = -. Similarly, either \hat{f}L(σ_a(t)) = 0 or there is k with t_k = - such that \hat{f}L(σ_a(t)) = L(σ_a(t^k)), where t^k ∈ \{+,-\}^{n_a} is given by t^k_l = t_l for l \neq k and t^k_k = +.
In the proof we will need the following lemma.

**Lemma 5.3** We keep the notation of Proposition 5.2. Let \( j_1 > j_2 > \ldots > j_l \) be all indexes such that \( t_{j_i} = +. \) Then there is a filtration \( E \Delta(\lambda) = F_0 E \Delta(\lambda) \supset F_1 E \Delta(\lambda) \supset F_2 E \Delta(\lambda) \supset \ldots \supset \{0\} \) such that \( F_{i-1} E \Delta(\lambda)/F_i E \Delta(\lambda) = \Delta(\sigma(t^i)). \)

**Proof** By (HWC0), \( E \Delta(\lambda) \in C^\Delta. \) Since the classes of the standard objects are linearly independent in the Grothendieck group, (HWC2) implies that the successive quotients of a filtration by standards on \( E \Delta(\lambda) \) are exactly \( \Delta(\lambda^j) \) with \( \lambda^j = \sigma(t^j) \) each occurring with multiplicity 1. By (HWC3),(HWC4), we have \( c(\lambda^j) > c(\lambda^{j_2}) > \ldots > c(\lambda^1) \) in the ordering of the highest weight category. Now the claim of the lemma follows from (HWC1).

A filtration from Lemma 5.3 will be referred to as a **standard filtration**.

**Proof of Proposition 5.2** We will prove the first statement, the second one is completely analogous. Set \( \lambda := \sigma_a(t), \lambda^j := \sigma_a(t^j). \)

For an object \( M \in C \) let \( \text{head}(M) \) denote its head, that is, the maximal semisimple quotient. We remark that \( \text{head} \) can be viewed as an endofunctor of \( C \). This functor can be easily seen to be right exact. Indeed, we can write \( \text{head}(M) = \bigoplus_{\lambda \in \Lambda} \text{Hom}_C(M, L(\lambda))^\sigma \otimes_{K} L(\lambda). \)

Of course, \( \text{head}(\Delta(\mu)) = L(\mu) \) for all \( \mu \in \Lambda. \) From here, the right exactness of \( \text{head} \) and Lemma 5.3 we deduce that \( \text{head}(E \Delta(\lambda)) \subset \bigoplus_j L(\lambda^j). \)

Now we recall that \( E \) is an exact functor so \( E \Delta(\lambda) \rightarrow EL(\lambda). \) The right exactness of \( \text{head} \) implies that \( \text{head}(E \Delta(\lambda)) \rightarrow \text{head}(EL(\lambda)). \) But \( \text{head}(EL(\lambda)) = \bar{c}L(\lambda), \) by the definition of the right hand side. Thanks to the previous paragraph, we are done.

Until the end of the section we write \( n \) for \( n_a, \sigma \) for \( \sigma_a. \) Further, for \( t \in \{+, -, \}^n \) we write \( \Delta(t) := \Delta(\sigma(t)), L(t) := L(\sigma(t)). \)

**Lemma 5.4** We have \( \text{wt}(t) = \text{wt}(L(t)). \)

**Proof** We remark that \( L(t) \) lies in \( C_t \) if and only if \( \Delta(t) \in C_t. \) The inclusion \( \Delta(t) \in C_{\text{wt}(t)} \) can be easily deduced from (HWC2).

### 5.3 Ext vanishing

In this section we are going to prove an important technical result. For \( t \in \{+, -, \}^n \) and \( k \in \{1, 2, \ldots, n\} \) set \( h^k_-(t) := h_-(t_k, \ldots, t_n). \) Clearly, \( h_-(t) = h^1_-(t) \supseteq h^2_-(t) \supseteq \ldots \supseteq h^k_-(t). \)

We introduce a linear order on \( \{+, -, \}^n: \) we write \( t' > t \) if there is an index \( i \) such that \( t_j = t'_j \) for all \( j > i \) but \( t'_i = - \) and \( t_i = +. \) For example, \( (+-+-+) > (---++). \)

Pick \( \mu \in \Lambda \) with \( \text{wt}(\mu) = w \) and set \( L := L(\mu). \) Let \( m \) be a positive integer such that \( F^m L = 0. \) If \( \mu \notin \Lambda_a, \) set \( k := 1. \) Otherwise, let \( k \) be any integer such that \( h^k_-(s) \leq m - 1, \) where \( \sigma(s) = \mu. \)

**Proposition 5.5** Let \( L, k, m \) be as in the previous paragraph. If \( L = L(s), \) assume, in addition, that \( h_-(L(s')) = h_-(s') \) provided \( \text{wt}(s') < \text{wt}(s). \) Then \( \text{Ext}^i(\Delta(t), L) = 0 \) for \( i < k^+_-^-(t) - m. \)

In the proof we will need the following combinatorial lemma.

**Lemma 5.6** Let \( t \in \{+, -, \}^n \) and let \( l \) be an index with \( h^l_-(t) > h^{l+1}_-(t). \) Let \( \bar{t} \in \{+, -, \}^n \) be given by \( \bar{t}_i = t_i \) for \( i \neq l \) and \( \bar{t}_l = +. \) Further, let \( \Delta(t^1), \ldots, \Delta(t^N) \) be the successive subquotients of the standard filtration on \( E \Delta(\bar{t}) \) with \( t^1 > t^2 > \ldots > t^N. \) Finally, let \( j \) be such that \( t^j = t. \) Then the following holds
(1) \(h_{l+1}^l(t^j) + 1 = h_{l}^l(t^j)\).
(2) \(h_{l+1}^l(t^j) = h_{l+1}^l(t^j)\) for \(i < j\).
(3) \(h_{l}^l(t^j) \geq h_{l}^l(t^j) + 1\) for \(i > j\).

Proof (1) follows from \(h_{l}^l(t) > h_{l+1}^l(t)\). To prove (2) we just notice that \(t^k_i = t^j_i\) for all \(k > l\) provided \(i < j\).

Let us prove (3). First of all, let us remark that \(h_{l+1}^l(t^j) \geq h_{l+1}^l(t^j+1)\) for \(i > j\). So it is enough to consider the case \(i = j + 1\). Let us note that \(t^j_i = -\), \(t^j_i+1 = +\). Let \(p\) be the index with \(t^j_p = +\), \(t^j_p+1 = -\). Then \(t^j_k = t^j_k\) for all \(k\) different from \(l\), \(p\). Since \(t^j_i\) survives in the reduced form, \(p > l + 1\). Also \(p\) is the minimal index bigger than \(l\) with \(t^j_p = +\) and so \(t^j_{l+1} = \ldots = t^j_{p-1} = -\). Since \(t^j_i\) survives in the reduced form, we see that \(p > l + h_+(t^j_p, \ldots, t^j_n) = 1 + h_+(t^j_{p+1}, \ldots, t^j_n)\). Moreover, \(h_{l}^l(t^j) = p - l - 1 = h_+(t^j_{p+1}, \ldots, t^j_n)\). Similarly, \(h_{l}^l(t^j+1) = p - l = h_+(t^j_{p+1}, \ldots, t^j_n)\). So \(h_{l}^l(t^j+1) = h_{l}^l(t^j) + 1\) and we are done. \(\square\)

Proof of Proposition 5.5 We remark that the claim for \(i = 0\) just follows from \(\sigma(t) \neq \mu\). Indeed, even if \(\mu = \sigma(s)\) for some \(s\), we have \(h_{l}^l(t) \geq m > h_{l}^l(s)\).

To prove the statement of the proposition, we use decreasing induction on \(l = n, n - 1, \ldots, k\) to show that the following holds:

\(^(*)\) \(\text{Ext}^q(\Delta(t), L) = 0\) for \(i \leq h_{l}^l(t) - m\).

The base \(l = n\) follows from the previous paragraph.

In the proof we may assume that \(h_{l}^l(t) > h_{l+1}^l(t)\), otherwise we are done by induction. Also we only need to prove that \(\text{Ext}^q(\Delta(t), L) = 0\) for \(q = h_{l}^l(t) - m\), the vanishing of the remaining \(\text{Ext}\)'s follows from the inductive assumptions on \(l\) because \(h_{l+1}^l(t) = h_{l}^l(t) - 1\). We prove the claim in several steps.

Step 1. Let \(i, j\) be as in Lemma 5.6. Let \(\mathcal{F} \supseteq \mathcal{F}_0\) be the consecutive filtration subobjects of the standard filtration of \(E \Delta(i)\) such that \(\mathcal{F}/\mathcal{F}_0 = \Delta(t^j)\). Let us prove that \(\text{Ext}^q(\mathcal{F}, L) = 0\).

For this consider the exact sequence

\[\text{Ext}^q(E \Delta(i), L) \to \text{Ext}^q(\mathcal{F}, L) \to \text{Ext}^q(E \Delta(i)/\mathcal{F}, L).\]

In the next two steps we will prove that the left and the right terms in this sequence vanish, thus showing that \(\text{Ext}^q(\mathcal{F}, L) = 0\).

Step 2. Let us prove that \(\text{Ext}^q(E \Delta(i), L) = 0\). By the biadjointness of \(E, F\), it is enough to show that \(\text{Ext}^q(E \Delta(i), FL) = 0\). This will follow if we show that \(\text{Ext}^q(E \Delta(i), L') = 0\) for any simple subquotient \(L'\) of \(FL\). But \(F^{m-1}(FL) = 0\) and hence \(F^{m-1}L' = 0\). Also we have \(h_{l}^l(t^j) = h_{l}^l(t) - 1\) and \(h_{l}^l(t^j) - (m - 1) = q\). To complete the proof it remains to show that, in the case when \(L' = L(s')\) for some \(s' \in \{+, -\}^n\), we have \(h_{l}^l(s') \leq m - 2\). But our assumption in the statement of the proposition says \(h_-(L') = h_-(s')\). Since \(F^{m-1}L' = 0\), we get \(m - 2 > h_-(L') = h_-(s') \geq h_{l}^l(s')\). We are done by induction.

Step 3. Let us prove that \(\text{Ext}^q(E \Delta(i)/\mathcal{F}, L) = 0\). The object \(E \Delta(i)/\mathcal{F}\) inherits a standard filtration from \(E \Delta(i)\). The successive quotients are (in the notation of Lemma 5.6) \(\Delta(t^j), i > j\). Now, thanks to assertion (3) of that lemma, for each \(i > j\) we have \(h_{l}^{i+1}(t^j) \geq h_{l}^l(t^j) + 1\). Therefore, by induction, \(\text{Ext}^q(\Delta(t^j), L) = 0\). It follows that \(\text{Ext}^q(E \Delta(i)/\mathcal{F}, L) = 0\).

Step 4. So now we know that \(\text{Ext}^q(\mathcal{F}, L) = 0\). To show that \(\text{Ext}^q(\Delta(t), L) = 0\) consider the short exact sequence
By Lemma 5.6, equivalent to

\[ \text{Ext}^{d-1}(F_0, L) \to \text{Ext}^d(\Delta(t^j), L) \to \text{Ext}^d(F, L). \]

It remains to prove that \( \text{Ext}^{d-1}(F_0, L) = 0 \) and we will do this in the next step.

**Step 5.** The object \( F_0 \) again inherits a filtration from \( E\Delta(t^i) \). The successive quotients are \( \Delta(t^i) \) with \( i < j \). According to (1) and (2) of Lemma 5.6, \( h_{i-1}^d(t^j) = h_{i-1}^d(t^i) = h^d(t^i) - 1 \). So \( h_{i-1}^d(t^i) - m = q - 1 \) and, by induction, we have \( \text{Ext}^{d-1}(\Delta(t^i), L) = 0 \). Hence \( \text{Ext}^{d-1}(F_0, L) = 0 \).

\[ \square \]

**Remark 5.7** In fact, when \( \mu \not\in \Lambda_a \) one can prove that \( \text{Ext}^i(\Delta(t), L) = 0 \) for \( i \leq h_-(t) - h_-(L) \) (while the proposition above only guarantees \( i \leq h_-(t) - h_-(L) - 1 \)). We will assume that Theorem 5.1 holds, this remark is not used to prove it. The proof closely follows that of the proposition, the only difference is in the proof of Step 2. Namely, let \( L' \) be as in that step. Then, according to (3) of Proposition 2.3, either \( L' = \tilde{f}L \) or \( F^{h_-(L)-1}L' = 0 \). To prove that \( \text{Ext}^d(\Delta\tilde{t}, L') = 0 \) in the first case we can use the inductive assumption since still \( L' = L(\mu') \) for \( \mu' \not\in \Lambda_a \). In the second case we can apply Proposition 5.5.

**5.4 Proof of the main theorem**

Let us prove that \( \tilde{f}L(t) = L(\tilde{f}t) \) by using the induction on \( w = \text{wt}(t) \). The case \( \text{wt}(t) = -n \) is obvious: both sides of the equality are zero. Now suppose that the claim is proved for all \( s \in \{+, -, 0\}^n \) with \( \text{wt}(s) < w \). This implies \( h_-(s) = h_-(L(s)) \) provided \( \text{wt}(s) < \text{wt}(t) = w \).

We are going to prove, first, that \( h_-(s) = h_-(L(s)) \) for all \( s \) with \( \text{wt}(s) = w \). Suppose that \( h_-(L(s)) < h_-(s) \). Let \( \Delta(t^1), \ldots, \Delta(t^N) \) be the successive subquotients of \( E\Delta(\tilde{f}s) \) with \( t^i > h^i \). But assertion (3) of Proposition 2.3 implies that the number of \( s \) with \( h_-(s) = h_-(L(s)) \) coincides with the number of \( s \) with \( h_-(s) = h \) for any \( h \). So we see that \( h_-(L(s)) = h_-(s) \) for all \( s \) with \( \text{wt}(s) = w \).

Now we are going to prove that \( \tilde{e}L(s) = L(\tilde{e}s) \) for all \( s \) with \( \text{wt}(s) = w - 2 \). This is equivalent to \( L(\tilde{f}t) = L(t) \) for all \( t \) with \( \text{wt}(t) = w \).

First of all, let us remark that \( \tilde{e}L(s) = 0 \) and \( \tilde{e}s = 0 \) are equivalent. Indeed, we know that \( h_-(L(s)) = h_-(s) \) and hence \( h_+(L(s)) = h_+(s) \). So we may assume that \( \tilde{e}s, \tilde{e}L(s) \neq 0 \). We may also assume that \( \tilde{e}L(s') = L(\tilde{e}s') \) is proved for all \( s' \) such that \( h_-(s') = h_-(s) \) and \( \tilde{e}s > \tilde{e}s' \).

Let \( \tilde{s} \) denote the \( n \)-tuple with \( \tilde{e}L(s) = L(\tilde{s}) \). By what we have seen above, \( h_-(\tilde{e}s) = h_-(\tilde{s}) = h_-(s) + 1 \). So \( \tilde{e}L(s) \) is one of the simple modules \( L \) appearing in \text{head}(E\Delta(s)) with \( h_-(L) = h_-(s) + 1 \).

Assume that \( \tilde{s} = \tilde{e}s \) or \( \tilde{e}s > \tilde{s} \). Indeed, otherwise \( \tilde{e}s > \tilde{s} \). Since \( h_-(\tilde{s}) - 1 = h_-(s) \geq 0 \), we see that \( s' := \tilde{f}s \neq 0 \) and so \( \tilde{s} > \tilde{e}s' \). Since \( \tilde{e}s > \tilde{e}s' \) and \( h_-(s') = h_-(\tilde{s}) - 1 = h_-(s) \), we can use the inductive assumption and get \( \tilde{e}L(s') = L(\tilde{e}s') = L(\tilde{s}) \). But \( \tilde{e}L(s) = L(s) \) hence \( s = s' \) or, equivalently, \( \tilde{s} = \tilde{e}s \).

So \( \tilde{s} > \tilde{e}s \). But then Lemma 5.6 implies \( h_-(\tilde{s}) > h_-(\tilde{e}s) \). So we get a contradiction which proves \( \tilde{e}L(s) = L(\tilde{e}s) \). The equality \( L(\tilde{f}t) = L(t) \) for all \( t \) with \( \text{wt}(t) = w \) follows and we have completed the induction step for our claim in the beginning of the section.
So we have \( \tilde{f}(L(t)) = L(\tilde{f}t) \) as well as \( h_-(t) = h_-(L(t)) \) for all \( t \). Together with standard properties of crystals, this implies Theorem 5.1.

5.5 Application to Cherednik algebras

Let \( \mathbb{K} \) be the field of complex numbers and \( C \) be the category \( O_p \) from Sect. 3.5. Pick a multipartition \( \mu \) of \( n \). Define the depth \( D(\mu) \) inductively by setting \( D(\mu) = 0 \) if \( \tilde{e}_z\mu = 0 \) for all \( z \in C \) and \( D(\mu) = 1 + \max_{z \in C}(D(\tilde{e}_z\mu)) \) else. We remark that \( D(\mu) \) does depend on \( p \) as the crystal structure on the set of multipartitions does.

Following [23] and [24] we will interpret \( D(\mu) \) representation theoretically. Namely, since any object in the category \( O_p \) is a finitely generated \( \mathbb{K}[h]\#W \)-module, we can view \( L(\mu) \) as a \( W \)-equivariant coherent sheaf on \( h = \mathbb{T}^n \). Its support is known to be of the form \( W^\mu \), where \( h_{i,j} \) is the space of all \( n \)-tuples \( (x_1, \ldots, x_n) \) with \( x_{i+1} = x_i + 2 = \ldots = x_i + e, x_i + e + 1 = x_i + e + 2 = \ldots = x_i + 2e, \ldots, x_i + (j-1)e + 1 = \ldots = x_i + je, x_i + je + 1 = \ldots = x_n = 0 \). Here \( e \) is the denominator of \( \kappa \) if \( \kappa \) is rational (recall that we assume that \( \kappa \) is non-integral) and \( e = \infty \) if \( \kappa \) is irrational (so we do not have \( e \)-tuples of equal coordinates). See [24, Remark 3.7] for a proof. Now [24, Proposition 3.16, Corollary 3.18] imply that \( i = D(\mu) \). In particular, if \( \kappa \) is irrational we can recover the support of \( L(\mu) \) completely. This generalizes [12, Corollary 6.9.3] and also strengthens [7, Corollary 5.8].

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