A Correlation-Breaking Interleaving of Polar Codes

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Abstract—It’s known that the bit errors of polar codes with successive cancellation (SC) decoding are coupled. However, existing concatenation schemes of polar codes with other error correction codes rarely take this coupling effect into consideration. To achieve a better BER performance, a concatenation scheme, which divides all \(N_l\) bits in a LDPC block into \(N_l\) polar blocks to completely de-corrrelate the possible coupled errors, is first proposed. This interleaving is called the blind interleaving (BI) and can keep the simple SC polar decoding while achieving a better BER performance than the state-of-the-art (SOA) concatenation of polar codes with LDPC codes. For better balance of performance and complexity, a novel interleaving scheme, named the correlation-breaking interleaving (CBI), is proposed by breaking the correlation of the errors among the correlated bits of polar codes. This CBI scheme 1) achieves a comparable BER performance as the BI scheme with a smaller memory size and a shorter turnaround time; 2) and enjoys a performance robustness with reduced block lengths. Numerical results have shown that CBI with small block lengths achieves almost the same performance at BER=10^{-4} compared with CBI with block lengths 8 times larger.

Index Terms—Polar codes, SC decoding, BP decoding, interleaving, code concatenation

I. INTRODUCTION

Polar codes, which are proposed in [1], provably achieve the capacity of symmetric binary-input discrete memoryless channels (B-DMCs) with a low encoding and decoding complexity. The encoding and decoding process (with successive cancellation, SC) can be implemented with a complexity of \(O(N \log N)\). The idea of polar codes is to transmit information bits on those noiseless channels while fixing the information bits on those completely noisy channels. The fixed bits are made known to both the transmitter and receiver. Later, the systematic version of polar codes was proposed in [2]. The construction of polar codes is studied in [3–6] and the hardware implementation is presented in [7–9].

The asymptotic behavior of polar codes are analyzed in [10], where the polarization rate is characterized in both the block length and the code rate. The finite scaling of polar codes is presented in [11] in which the relationship between the error probability, the code rate, and the block length is analyzed. However, it has been shown that the performance of polar codes in the finite domain is not satisfactory [12–15]. To improve the polar code performance in the finite domain, various decoding processes [12, 13, 16, 17] and concatenation schemes [14, 18, 19] were proposed. The decoding processes in these works have higher complexity than the original SC decoding of [1]. On the other hand, systematic polar codes [2] can achieve a better BER performance than non-systematic polar codes while still maintain almost the same decoding complexity. In non-systematic encoding, the codeword \(x\) is obtained by \(x = uG\), where \(u\) is the source vector and \(G\) is the generator matrix. The basic idea of systematic polar codes is to use some part of the codeword \(x\) to transmit information bits instead of directly using the source bits \(u\) to transmit them. In [2], it’s shown that systematic polar codes achieve better BER performance than non-systematic polar codes. But, theoretically, this better BER performance is not expected from the indirect decoding process: first decoding \(\hat{u}\) (\(\hat{u}\) is the estimation of \(u\) from the normal SC decoding) then re-encoding \(\hat{x}\) as \(\hat{u}G\). One would expect that any errors in \(\hat{u}\) would be amplified in this re-encoding process.

The bit errors occurring in the SC decoding affect the sequel bit errors, causing the problem of the error propagation. The better BER performance of systematic polar codes can be intuitively thought of coming from the error-decoupling. From the two-step decoding of systematic polar codes, this decoupling must be accomplished through the re-encoding \(\hat{x} = uG\). From \(\hat{x} = uG\) and that the number of errors in \(\hat{x}\) is smaller than that of \(u\), we can conclude that the coupling of the errors in \(\hat{u}\) are controlled by the columns of \(G\). We provide a proposition of this coupling pattern in this paper.

In this paper, two schemes are proposed to de-corrrelate the coupled errors. A concatenation scheme, which divides all \(N_l\) bits in a LDPC block into \(N_l\) polar blocks to completely de-corrrelate the possible coupled errors, is first introduced. The blind interleaving (BI) can keep the simple SC polar decoding while achieving a better BER performance than the state-of-the-art (SOA) concatenation of polar codes with LDPC codes. To better balance the performance and complexity, a novel interleaving scheme, named the correlation-breaking interleaving (CBI), is introduced to improve the performance of polar codes with finite block lengths while still maintaining the low complexity of the SC decoding. This CBI scheme is based on the correlation pattern proven in this paper. LDPC and BCH codes are used as the outer codes and polar codes are the inner codes. Note that the concatenation of polar
codes with LDPC codes is studied in [14] and [15] where no interleaving is used and BP (belief-propagation) decoding is applied for polar codes. For the ease of description, let’s denote polar codes applying SC decoding as POLAR-SC and polar codes applying BP decoding as POLAR-BP. Also let’s denote the direct concatenation system with a LDPC code as the outer code and a polar code as the inner code as LDPC+POLAR. If a CBI interleaving is used between the outer and the inner code, then we denote such a system as LDPC+CBI+POLAR. Similarly, a blind interleaving system is denoted as LDPC+BI+POLAR. The simulation results verified that the LDPC+CBI+POLAR-SC scheme achieves a better BER performance than the direct concatenation scheme of LDPC+POLAR-BP. With coupled errors being decoupled, this CBI scheme applied to shorter LDPC and polar codes achieves a great BER performance which shows the promising potentials of this scheme in power-limited areas. To further explore the advantage of the CBI scheme, a short BCH code is used in place of the short LDPC code, which again shows a surprising good performance.

Note that portions of this work are investigated in [20] where the theorems of the error patterns are not proven. In [20], details of the CBI algorithm and some of the key parameters involved are omitted either because of the space limit. In this paper, we provide the proofs of the theorems, implementation details, and examples of the CBI scheme. The contribution of this paper can be summarized as: 1) Theoretically, we prove that the errors from the SC decoding are coupled. The coupling pattern is found; 2) A BI scheme and a universal CBI scheme (based on the coupling pattern) are proposed; 3) A complete implementation of the CBI is provided, with details and examples to illustrate the key parameters.

Following the notations in [1], in this paper, we use \( v_i^N \) to represent a row vector with elements \( (v_1, v_2, ..., v_N) \). We also use \( v \) to represent the same vector for notational convenience. For a vector \( v_i^N \), the vector \( v_i^j \) is a subvector \( (v_i, ..., v_j) \) with \( 1 \leq i, j \leq N \). If there is a set \( A \subseteq \{1, 2, ..., N\} \), then \( v_A \) denotes a subvector with elements in \( \{v_i, i \in A\} \).

The rest of the paper is organized as follows. Section II introduces the fundamentals of non-systematic and systematic polar codes. Also introduced in this section is the coupling pattern of polar codes with the SC decoding. Section III proposes the new interleaving scheme with detailed algorithms. Section IV presents the simulation results. The conclusion remarks are provided at the end.

II. NON-SYSTEMATIC AND SYSTEMATIC POLAR CODES

In the first part of this section, the relevant theories on non-systematic polar codes and systematic polar codes are presented. In the second part of this section, the correlation among the SC decoding errors is introduced, which is the basis of the proposed interleaving scheme in Section III.

A. Preliminaries of Non-Systematic Polar Codes

The generator matrix for polar codes is \( G_N = BF^{\otimes n} \) where \( B \) is a bit-reversal matrix, \( F = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \), \( n = \log_2 N \), and \( F^{\otimes n} \) is the \( n \)-th Kronecker power of the matrix \( F \) over the binary field \( \mathbb{F}_2 \). In this paper, we consider an encoding matrix \( G = F^{\otimes n} \) without the permutation matrix \( B \). The generator matrix is the basis for devising the new interleaving scheme in this paper.

Mathematically, the encoding is a process to obtain the codeword \( x \) through \( x = uG \) for a given source vector \( u \). The source vector \( u \) consists of the information bits and the frozen bits, denoted by \( u_A \) and \( u_{\bar{A}} \), respectively. Here the set \( A \) includes the indices for the information bits and \( \bar{A} \) is the complementary set. Each bit goes through a bit channel. In [1], there are detailed definitions of the bit channels and the corresponding Bhattacharyya parameter for each bit channel. The set \( A \) can be constructed by selecting indices of the bit channels with the smallest Bhattacharyya parameters in BEC channels. For all the other channels, construction methods can be found in [3–6]. Both sets \( A \) and \( \bar{A} \) are in \( \{1, 2, ..., N\} \) for polar codes with a block length \( N = 2^n \).

An encoding diagram is shown in Fig. 1. If the nodes in Fig. 1 are viewed as memory elements, the encoding process is to calculate the corresponding binary values to fill all the memory elements from the left to the right. This view is helpful when it comes to systematic polar codes in the following section.

B. Construction of Systematic Polar Codes

For systematic polar codes, we also focus on a generator matrix without the permutation matrix \( B \), namely \( G = F^{\otimes n} \).

The source bits \( u \) can be split as \( u = (u_A, u_{\bar{A}}) \). The first part \( u_A \) consists of user data that are free to change in each round of transmission, while the second part \( u_{\bar{A}} \) consists of data that are frozen at the beginning of each session and made known to the decoder. The codeword can then be expressed as

\[
x = u_AG_A + u_{\bar{A}}G_{\bar{A}}
\]  

where \( G_A \) is the sub-matrix of \( G \) with rows specified by the set \( A \). The systematic polar code is constructed by specifying a set of indices of the codeword \( x \) as the indices to convey the information bits. Denote this set as \( B \) and the complementary set as \( \bar{B} \). The codeword \( x \) is thus split as \( (x_B, x_{\bar{B}}) \). With some manipulations, we have
The matrix $G_{AB}$ is a sub-matrix of the generator matrix with elements $(G_{i,j})_{i \in A, j \in B}$. Given a non-systematic encoder $(A, u_A)$, there is a systematic encoder $(B, u_B)$ which performs the mapping $x_B \mapsto x = (x_B, x_T)$. To realize this systematic mapping, $x_B$ needs to be computed for any given information bits $x_B$. To this end, we see from (2) that $x_B$ can be computed if $u_A$ is known. The vector $u_A$ can be obtained as the following

$$u_A = (x_B - u_A G_{AB}) (G_{AB})^{-1}$$  \hfill (3)

From (3), it’s seen that $x_B \mapsto u_A$ is one-to-one if $x_B$ has the same elements as $u_A$ and if $G_{AB}$ is invertible. In [2], it’s shown that $B = A$ satisfies all these conditions in order to establish the one-to-one mapping $x_B \mapsto u_A$. In the rest of the paper, the systematic encoding of polar codes adopts this selection of $B = B = A$. Therefore we can rewrite (3) as

$$\begin{aligned}
x_A &= u_A G_{AA} + u_A G_{AB} \\
x_A &= u_A G_{AA} + u_A G_{AB} \\
\end{aligned}$$  \hfill (4)

Let’s go back to the diagram in Fig. 1. For systematic polar codes, the information bits are now conveyed in the right-hand side in $x_A$. To calculate $x_A$, $u_A$ in the left-hand side needs to be calculated first. Once $u_A$ is obtained, systematic encoding can be performed in the same way as the non-systematic encoding: performing binary additions from the left to the right. Therefore, compared with non-systematic encoding, systematic encoding has an additional round of binary additions from the right to the left. The detailed analysis of systematic encoding can be found in [21, 22].

### C. Correlated Bits

In [21, 22], it’s shown that the re-encoding process of $\hat{x} = \hat{u} G$ after decoding $\hat{u}$ does not amplify the number of errors in $\hat{u}$. Instead, there are less errors in $\hat{x}$ than in $\hat{u}$. This clearly shows that the coupled errors in $\hat{u}$ are de-coupled (or cancelled) in the re-encoding process. In this section, we first restate a corollary from [22] and then provide a proposition to show the coupling pattern of the errors in $\hat{u}$. This coupling pattern is used in Section III to design the interleaving scheme.

#### Corollary 1. The matrix $G_{AA} = 0$.

The proof of this corollary can be found in [22]. The following proposition shows the pattern of the coupling of the errors in $\hat{u}$ from the SC decoding process.

#### Proposition 1. Let the indices of the non-zero entries of column $i \in A$ of $G$ be $A_i$. Then, the errors of $u_{A_i}$ are dependent (or coupled).

**Proof:** Assume the errors in $\hat{u}_{A_i}$ are independent. For non-systematic polar codes, we define a set $A_i \subset A$ containing the indices of the information bits in error in an error event. In the same way, we define a set $A_{sys, t} \subset A$ containing the corresponding indices of the information bits in error for systematic polar codes. Let $v$ be an error indicator vector: a

$$
\begin{aligned}
x_B &= u_A G_{AB} + u_A G_{AB} \\
x_B &= u_A G_{AB} + u_A G_{AB} \\
\end{aligned}
$$

$N$-element vector with 1s in the positions specified by the error event $A_i$ and 0s elsewhere. Let the error probability being: $Pr(v_m = 1) = p_m$ ($0 \leq p_m \leq 0.5$). Correspondingly, we set a vector $q$ with 1s in the positions specified by $A_{sys, t}$ and 0s elsewhere. From the systematic encoding process, we have $q = v G$. Correspondingly, $\bar{q}_A = v A G_{AA}$. In this way, we convert the errors of non-systematic polar codes and systematic polar codes to the weight of the vectors $v$ and $q$. Denote the Hamming weight of the vector $v$ as $\omega_H(v)$. Specifically, the element $q_i (i \in A)$ is one if $v_{A_i}$ has an odd number of ones. With the independent assumption of errors in $\hat{u}_{A_i}$, the probability that the $i$th information bit $\hat{x}_i$ is in error is

$$\hat{p}_i = 1 - \frac{1}{2} \prod_{m=1}^{K_i} (1 - 2p_m)$$  \hfill (5)

where $K_i = |A_i|$. In (5), we can order the probabilities $\{p_m\}_{m=1}^{K_i} \ (0 \leq p_m \leq 0.5)$ in the ascending order. Applying the Monotone Convergence Theorem to real numbers [24], we have:

$$\lim_{K_i \to \infty} \hat{p}_i = \lim_{K_i \to \infty} \frac{1}{2} \prod_{m=1}^{K_i} (1 - 2p_m) = \frac{1}{2}$$  \hfill (6)

Thus, the mean weight of $q$: $\omega_H(q) = \frac{K}{2} \geq \omega_H(v)$, meaning the average number of errors of the systematic polar codes is larger than the average number of errors of non-systematic polar codes. This contradicts with the existing results that systematic polar codes outperform non-systematic polar codes. Thus, we can conclude the errors of $u_{A_i}$ are dependent.

### From Proposition 1 an error pattern among the errors in $\hat{u}$ can be deduced. We call bits $\hat{u}_{A_i}$ the correlated estimated bits. This says that statistically, the errors of bits $\hat{u}_{A_i}$ are coupled. To show this coupling, we give an example of $N = 16$ and $R = 0.5$ in a BEC channel with an erasure probability 0.2. The indices selected in this case for information bits are $\{8, 10, 11, 12, 13, 14, 15, 16\}$ (indexed from 1 to 16). The coupling effect (similar to the correlation coefficient) of bits indicated by non-zero positions of column 10, 11, and 13 is recorded in simulations and is shown in Table I. From Table I, we can see that if there are errors in $\hat{u}_{A_{10}}$, then 76% of times these errors happen simultaneously, resulting in a coupling coefficient 0.76 for errors in $\hat{u}_{A_{10}}$. Both the coupling coefficient for $\hat{u}_{A_{11}}$ and $\hat{u}_{A_{13}}$ is 0.74 in Table II.

To the authors’ knowledge, there is no attempt yet to utilize this coupling pattern to improve the performance of polar codes. In the next section of this paper, we propose a novel interleaving scheme to break the coupling of errors to improve

#### Table I

| Columns of $G$ | Coupling coefficient |
|----------------|----------------------|
| 10             | 76%                  |
| 11             | 74%                  |
| 13             | 74%                  |
We give an example in Fig. 2 where blocks, which guarantees that the errors in each LDPC block is 1 length is $N$. These $N_l$ bits are divided into $N_l$ polar code blocks, which guarantees that the errors in each LDPC block are independent as they come from different polar code blocks. We give an example in Fig. 2 where $K_l = 64$ and $N_l = 155$. Polar code in this example has $N = 256$, $K = 64$ and a rate $R = 1/4$. In Fig. 2, bit $i$ of all the $K = 64$ LDPC code blocks forms the input vector to the $i$th polar code encoder. In the receiver side, to collect the decoded bits of the first LDPC block, $N_l$ polar blocks are needed. For the rest $N_l - 1$ LDPC blocks, the decoder does not need to wait since all polar bits of one circulation are obtained already. For this BI scheme, the decoder needs to store the real LR values of a block of [155, 64]. The turnaround time of this scheme is $N_l \times N \times T_s$ where $T_s$ is the symbol duration.

**B. The Correlation-Breaking Interleaving (CBI) Scheme**

The BI scheme in Section III-A occupies a big memory and has the longest possible turnaround time. From Section II-C we know that it is not necessary to scatter all bits in a LDPC block into different polar blocks since not all bits in a polar block are correlated. Only those bits in $\{A_i\}_{i=1}^K$ are correlated. The interleaving scheme in this section is to make the bits $\{A_i\}_{i=1}^K$ of one polar block composed of different LDPC blocks. Or in other words, the interleaving scheme is to scatter the information bits $\{A_i\}_{i=1}^K$ of each polar block into different LDPC blocks.

The difficulty in designing a CBI scheme is that the sets $\{A_i\}_{i=1}^K$ are different for different block lengths and data rates. They are also different for different underlying channels for which polar codes are designed. A CBI scheme is dependent on at least three parameters: the block length $N$, the data rate $R$, and the underlying channel $W$. Let’s denote a CBI scheme as CBI($N, R, W$) to show this dependence. A CBI($N, R, W$) optimized for one set of $(N, R, W)$ is not necessarily optimized for another set $(N', R', W')$. It may not even work for the set $(N', R', W')$ if $N' \neq N, R' \neq R$. In the following, we provide a CBI scheme which works for any sets of $(N, R, W)$, but not necessarily optimal for one specific set of $(N, R, W)$.

As $A_i$ are the indices of the non-zero entries of column $i \in A$, we first extract the $K = |A|$ columns of $G$ and denote it as the submatrix $G(:, A)$. Divide the submatrix $G(:, A) = [G_{AA} G_{A\bar{A}}]$. Since the submatrix $G_{AA} = 0$ from Corollary 1 we only need to analyze the submatrix $G_{A\bar{A}}$. If a CBI needs to look at each individual set $A_i$, then a general CBI is beyond reach. However, we can simplify this problem by dividing the information bits only into two groups: the correlated bits $A_c$ and the uncorrelated bits $A_{\bar{c}}$. The following proposition can be used to find the sets $A_c$ and $A_{\bar{c}}$.

**Proposition 2.** For the submatrix $G_{A\bar{A}}$, the row indices (relative to the submatrix $G_{A\bar{A}}$) with Hamming weight greater than one is denoted as the set $A_{cs}$. The corresponding set of $A_{cs}$ with respect to the matrix $G$ is the set $\bar{A}_{cs}$.

**Proof:** For $1 \leq i \leq K$, assume the weight of the $i$th row of $G_{A\bar{A}}$ is $w_i$. We need to prove:

$$\left\{ \begin{array}{ll} i \in A_{cs}, & \text{if } w_i = 1 \\ i \in \bar{A}_{cs}, & \text{if } w_i > 1 \end{array} \right.$$  \hspace{1cm} (7)

where $\bar{A}_{cs}$ is the complementary set of $A_{cs}$. Now, for $w_i = 1$, the $i$th information bit appears only in the $i$th column of $G_{A\bar{A}}$. This means that the $i$th information bit does not affect other information bits. Nor do other information bits affect the $i$th information bit. For $w_j = k (k > 1)$, we define the set $\Gamma_j$ containing the positions of ones in the $j$th row of $G_{A\bar{A}}$. Then, the $j$th information bit appears not only in the $j$th column but also in other columns of $G_{A\bar{A}}$. Divide $\Gamma_j$ into two parts: $\Gamma_{j1}$ and $\Gamma_{j2}$ where $\Gamma_{j1}$ containing indices less than $j$ and $\Gamma_{j2}$ containing indices greater than $j$. In the decoding process, the $j$th information bit is affected by the previously decoded bits in $\Gamma_{j1}$. In the mean time, the decoded $j$th bit affects the bits in $\Gamma_{j2}$ since all indices in $\Gamma_{j2}$ are greater than $j$. Therefore, the information bits in $\Gamma_j$ are in the set $A_{cs}$.

We give an example of how to use Proposition 2 to find the set $A_c$ and $A_{\bar{c}}$. Let the block length be $N = 16$, the code rate $R = 0.5$, and the underlying channel is the BEC channel with an erasure probability 0.2. The set $A$ is the same as the example in Section II-C With Proposition 2 we can easily find...
that $\mathcal{A}_{cc} = \{4, 6, 7, 8\}$ for the submatrix $G_{AA}$. Relative to the matrix $G_{16}$, this set is $\mathcal{A}_c = \{12, 14, 15, 16\}$. The uncorrelated set is thus $\mathcal{A}_c = \{8, 10, 11, 13\}$.

With the sets $\mathcal{A}_c$ and $\mathcal{A}_{uc}$ obtained for any $(N, R, \lambda)$, we can devise a CBI scheme. Let $K_c = |\mathcal{A}_c|$ and $K_{uc} = |\mathcal{A}_{uc}|$. Fig.3 is a general CBI scheme. To describe the algorithms, the following parameters are needed and defined: $n_d = \lceil N_l / K_c \rceil$, $n_u = \lfloor N_l / K_c \rfloor$, $k_l = N_l / K_c$ (modulo operation), $K_n = K_c + 1$, $d_m = k_l - K_{uc}$, $d_n = k_l - K_n$.

$$n_p = \begin{cases} n_u \times K_n, & \text{if } k_l \geq K_n \\ n_d \times K_n + k_l, & \text{o.w.} \end{cases} \quad (8)$$

and

$$\epsilon(d_n) = \begin{cases} 1, & \text{if } d_n \geq 0 \\ 0, & \text{o.w.} \end{cases} \quad (9)$$

The parameter $n_p$ is the total number of polar blocks to transmit $K_n$ LDPC blocks. We have two $n_u$ cycles in Algorithm II to collect the uncorrelated and correlated bits respectively for polar blocks. For both cycles, the first $n_p$ cycles run every $K_n$ polar blocks. In the meantime, we define two new sets $\mathcal{A}_{cc}$ and $\mathcal{A}_{cp}$, which control the collection of the correlated information bits for each polar block. The sets $\mathcal{A}_{cc}$ and $\mathcal{A}_{cp}$ contain the indices of the bits before and after the set $\mathcal{A}_c$, respectively. Two examples are given in Table III and Table II to explain these parameters. The polar code is a $(32,16)$ code and the LDPC is a $(21,8)$ code in the example shown in Table II. The correlated set $\mathcal{A}_c = \{16, 24, 26, 27, 28, 29, 30, 31, 32\}$. Therefore $K_c = |\mathcal{A}_c| = 9$ and $K_{nc} = K_c + 1 = 10$. In this case, $k_l = 5 < K_n$. In Table II, the first row contains indices of the LDPC blocks and the first column is the indices of the polar blocks. Therefore, in this example, to transmit $K_n = 10$ LDPC blocks, $n_p = 15$ polar blocks are needed. From Table II, we can see that for polar block 1, 7 bits are taken from LDPC block 1, and the other 9 bits are taken from LDPC blocks 2 to 10. The 7 bits from LDPC block 1 are placed at the uncorrelated positions $\mathcal{A}_c$ of polar block 1. The other information bits of polar block 1 are divided into two parts: the positions before and after $\mathcal{A}_c$, which are collected in the sets $\mathcal{A}_{cc}$ and $\mathcal{A}_{cp}$ respectively. For polar block 1, the set $\mathcal{A}_{cc} = \emptyset$ and the set $\mathcal{A}_{cp} = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$. The other polar blocks follow the same fashion in collecting the input bits. Table III shows another example when $k_l \geq K_n$. In this example, the polar code (32,8) has an $\mathcal{A}_c = \{28, 30, 31, 32\}$ with $K_c = 4$. The parameter $k_l = 5$ and $K_n = K_c + 1 = 5$. The total polar blocks $n_p = n_u \times K_n = 3 \times 5$ are used to transmit $K_n = 5$ LDPC blocks. For both examples, there are 0s at the left low corner, which means that there are polar positions which are not used. These positions are wasted which are the cost of the universal CBI design. However, some of these positions can be turned to frozen bits to increase the performance of the overall interleaving scheme. This operation is just taking these bits as the frozen bits in the encoding process. Therefore it does not change the complexity of the decoding.

Pseudocodes in Algorithms I–V show a detailed implementation of the proposed interleaving scheme. The top function is given in Algorithm I which contains two $n_u$ cycles. Specifically, lines from 1 to 7 are collecting the encoded LDPC bits as the uncorrelated information bits of polar groups. Lines from 18 until the end collect the encoded LDPC bits for the correlated information bits of polar groups. The principle in collecting the information bits for polar groups is that the correlated information bits come from different LDPC blocks while the uncorrelated information bits can be from the same LDPC block. Therefore the $K_{uc}$ uncorrelated information bits of one polar block can be directly taken from a continuous chunk of a LDPC block. However, taking the $K_c$ correlated information bits for each polar encoding block needs a fine design which are controlled by the sets $\mathcal{A}_{cc}$ (line 20) and $\mathcal{A}_{cp}$ (line 23).

| Table II | The CBI Scheme for LDPC (21,8) and Polar (32,16) |
|---|---|
| Polar | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 11 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 12 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 13 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 14 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 15 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 16 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| Table III | The CBI Scheme for LDPC (21,8) and Polar (32,8) |
|---|---|
| LDPC | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 |
| 6 | 1 | 1 | 1 | 1 | 1 |
| 7 | 1 | 1 | 1 | 1 | 1 |
| 8 | 1 | 1 | 1 | 1 | 1 |
| 9 | 1 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 |
| 11 | 1 | 1 | 1 | 1 | 1 |
| 12 | 1 | 1 | 1 | 1 | 1 |
| 13 | 1 | 1 | 1 | 1 | 1 |
| 14 | 1 | 1 | 1 | 1 | 1 |
| 15 | 1 | 1 | 1 | 1 | 1 |
| 16 | 1 | 1 | 1 | 1 | 1 |
same fashion as the first part.

C. Complexity Analysis

**Proposition 3.** For the CBI scheme, the memory size is \([n_p, K]\) LR values of the polar code and the decoder only needs to wait \(n_p\) polar blocks to decode the \(K_n\) LDPC blocks. The turnaround time is therefore \(n_p \times N \times T_s\). For the BI scheme, the memory size is \([N_l, K]\) and the turnaround time is \(N_l \times N \times T_s\).

**Proof:** For the memory size and the turnaround time of the CBI scheme, the calculation details are as the following.

- **Case 1:** When \(k_l < K_{uc}\), & \(k_l! = 0\). There are two parts. For the first part \(k_l > K_{uc}\), to collect the decoded bits of the first LDPC block, \((K_n - 1) \times n_d + n_u + d_m\) polar blocks are needed. From the second to \((d_m + 1)\)th LDPC block, the decoder does not need to wait and uses the decoded bits of \((K_n - 1) \times n_d + n_u + d_m\) polar blocks to decode directly. For LDPC block from \((d_m + 2)\) to \(K_n\), only another polar block is needed. For the second part \(k_l \leq K_{uc}\), to collect the decoded bits of the first LDPC block, \((K_n - 1) \times n_d + n_u\) polar blocks are needed. From the second to \(k_l\)th LDPC block, an additional polar block is needed. For LDPC blocks from \(k_l + 1\) to \(K_n\), the decoder does not need to wait since all polar bits of one circulation are obtained already. Therefore, the memory size is \([n_p, K]\) and the decoder only needs to wait \(n_p\) polar blocks to decode the \(K_n\) LDPC blocks. The turnaround time is therefore \(n_p \times N \times T_s\).

- **Case 2:** When \(k_l \geq K_{uc}\). Case 2 is also divided into two parts. For the first part \(k_l > K_{uc}\), to collect the decoded bits of the first LDPC block, \((K_n - 1) \times n_d + n_u + d_m\) polar blocks are needed. From the second to \((d_m + 1)\)th LDPC block, the decoder does not need to wait and uses the known decoded bits of polar blocks to decode directly. For LDPC block from \((d_m + 2)\) to \(K_n\), only another polar block is needed. For the second part \(k_l \leq K_{uc}\), the required polar blocks of the first LDPC block is the same as the required polar blocks of the first LDPC block in part two of case one. For the LDPC block from \(2\) to \(K_n\), another polar block is required. Therefore, the memory size is \([n_p, K]\) and the decoder only needs to wait \(n_p\) polar blocks to decode the \(K_n\) LDPC blocks. The turnaround time is the same as the previous case.

### IV. Simulation Result

In this section, simulation results are provided to verify the performance of the CBI scheme shown in Algorithm 1. The

| Complexity Scheme | Memory size | Turnaround time |
|-------------------|-------------|-----------------|
| CBI scheme        | \([n_p, K]\) | \(n_p \times N \times T_s\) |
| BI scheme         | \([N_l, K]\)  | \(N_l \times N \times T_s\) |

| Algorithm 1 | The algorithm of a general correlation-breaking interleaving scheme. Transmitting \(K_n\) LDPC blocks needs \(n_p\) groups of polar codes |
|-------------|----------------------------------------------------------------------------------------------------------------------------------|
| **Input:**  | \(N_l, K, K_{uc}, A_{uc}, \tilde{A}_c, U_L;\)                                                                                     |
| **Output:** | the input bits \(U_A\) of polar codes;                                                                                           |
| 1:          | for \(j = 1\) to \(n_u\) do                                                                                                     |
| 2:          | \(/\) collect bits from LDPC blocks as input bits \(A_{uc}\) for the \(i\)th polar encoding.                                      |
| 3:          | if \(j == n_u\) & & \(k_l! = 0\) then \(k_l = 0\)                                                                               |
| 4:          | if \(k_l < K_n\) then \(U_A = F_{un}(U_L)\)                                                                                     |
| 5:          | \textbf{end if}                                                                                                                  |
| 6:          | if \(k_l >= K_n\) then \(U_A = F_{bn}(U_L)\)                                                                                     |
| 7:          | \textbf{end if}                                                                                                                  |
| 8:          | if \(j == n_u\) & & \(k_l! = 0\) then \(k_l = 0\)                                                                               |
| 9:          | for \(i = (j - 1) \times K_{uc} + 1\) to \((j - 1) \times K_{uc} + \tilde{A}_c\) do                                             |
| 10:         | \(ls = ((i - 1) \times K_{uc}) \times N_l\)                                                                                     |
| 11:         | \(U_A((i - 1) \times K_{uc}) = U_L(ls + i - j + K_{uc} \times (j - 1) + 1)\)                                                    |
| 12:         | \textbf{end for}                                                                                                                 |
| 13:         | if \(j == n_u\) & & \(k_l! = 0\) then \(k_l = 0\)                                                                               |
| 14:         | for \(i = 1, 2, ..., i - K_n \times (j - 1) - 1\)                                                                              |
| 15:         | \(l_m = (A_{cc} - 1) \times N_l\)                                                                                              |
| 16:         | \(U_A((i - 1) \times K + A_c(A_{cc} - 1)) = U_L(l_m + i - (j - 1) + K_{uc} \times (j - 1))\)                                 |
| 17:         | \textbf{end for}                                                                                                                 |
| 18:         | if \(j == n_u\) & & \(k_l! = 0\) then \(k_l = 0\)                                                                               |
| 19:         | for \(i = (j - 1) \times K_{uc} + 1\) to \((j - 1) \times K_{uc} + K_n\) do                                                      |
| 20:         | \(A_{cc} = 1, 2, ..., K_n \times (j - 1) - 1\)                                                                              |
| 21:         | \(l_c = (A_{cc} - 1) \times N_l\)                                                                                              |
| 22:         | \(U_A((i - 1) \times K + A_c(A_{cc} - 1)) = U_L(l_c + i - j + K_{uc} \times j)\)                                                |
| 23:         | \textbf{end for}                                                                                                                 |
| 24:         | if \(j == n_u\) & & \(k_l! = 0\) then \(k_l = 0\)                                                                               |
| 25:         | for \(i = 1, 2, ..., i - K_n \times (j - 1) + 1\)                                                                              |
| 26:         | \(l_m = (A_{cp} - 1) \times N_l\)                                                                                              |
| 27:         | \(U_A((i - 1) \times K + A_c(A_{cp} - 1)) = U_L(l_m + i - (j - 1) + K_{uc} \times (j - 1))\)                                 |
| 28:         | \textbf{end for}                                                                                                                 |
| 29:         | if \(j == n_u\) & & \(k_l! = 0\) then \(k_l = 0\)                                                                               |
| 30:         | for \(i = (j - 1) \times K_{uc} + 1\) to \((j - 1) \times K_{uc} + K_n\) do                                                      |
| 31:         | \(A_{cp} = 1, 2, ..., K_n \times (j - 1) + 1\)                                                                              |
| 32:         | \(l_m = (A_{cp} - 1) \times N_l\)                                                                                              |
| 33:         | \(U_A((i - 1) \times K + A_c(A_{cp} - 1)) = U_L(l_m + i - (j - 1) + K_{uc} \times (j - 1))\)                                 |
| 34:         | \textbf{end for}                                                                                                                 |
| 35:         | \textbf{end if}                                                                                                                  |
| 36:         | \textbf{end if}                                                                                                                  |
| 37:         | \textbf{end if}                                                                                                                  |
| 38:         | \textbf{end for}                                                                                                                 |
Algorithm 2 Function of $[U_A] = F_{u,sn}(U_L)$. Collect bits from LDPC blocks as input bits $A_c$ for the $i$th polar encoding while $k_i < K_n$.

1: for $i = (n_u - 1) \times K_n + 1$ to $(n_u - 1) \times K_n + k_i$ do
2: \hspace{1em} $ls = ((i-1)\%K_n) \times N_l$
3: \hspace{1em} $U_A(i-1)K + A_c) = U_L(ls+i-n_u + K_{uc} \times (n_u-1)+ 1 : ls+i-n_u + K_{uc} \times (n_u-1)+1 + (K_{uc} - i\%K_n-1))$
4: end for

Algorithm 3 Function of $[U_A] = F_{u,sn}(U_L)$. Collect bits from LDPC blocks as input bits $A_c$ for the $i$th polar encoding while $k_i \geq K_n$.

1: for $i = (n_u - 1) \times K_n + 1$ to $(n_u - 1) \times K_n + K_n$ do
2: \hspace{1em} if $i = (n_u - 1) \times K_n + 1$ to $(n_u - 1) \times K_n + 1 + d_m$ then
3: \hspace{2em} $ls = ((i-1)\%K_n) \times N_l$
4: \hspace{2em} $U_A(i-1)K + A_c) = U_L(ls+i-n_u + K_{uc} \times (n_u-1)+ 1 : ls+i-n_u + K_{uc} \times (n_u-1)+1 + (K_{uc} - i\%K_n-1))$
5: \hspace{1em} end if
6: \hspace{1em} if $i = (n_u - 1) \times K_n + 1+d_m+1$ to $(n_u - 1) \times K_n + K_n$ then
7: \hspace{2em} $ls = ((i-1)\%K_n) \times N_l$
8: \hspace{2em} $U_A((i-1)K + A_c) = U_L(ls+i-n_u + K_{uc} \times (n_u-1)+ 1 : ls+i-n_u + K_{uc} \times (n_u-1)+1 + (K_{uc} - (i-d_m-1)\%K_n-1))$
9: \hspace{1em} end if
10: end for

Algorithm 4 Function $[U_A] = F_{sn}(U_L)$. Collect bits from LDPC blocks as input bits $A_c$ for the $i$th polar encoding while $k_i < K_n$.

1: for $i = (n_u - 1) \times K_n + 1$ to $(n_u - 1) \times K_n + k_i-d_n \times \epsilon(d_n)$ do
2: \hspace{1em} $A_{cp}\{i - K_n \times (n_u - 1) + 1, \ldots, K_n\}$
3: \hspace{1em} $lm = (A_{cp} - 1) \times N_l$
4: \hspace{1em} $U_A((i-1)K + A_c(A_{cp}-1)) = U_L(lm+i-(n_u-1)+ K_{uc} \times (n_u-1))$
5: end for

Algorithm 5 Function $[U_A] = F_{sn}(U_L)$. Collect bits from LDPC blocks as input bits $A_c$ for the $i$th polar encoding while $k_i \geq K_n$.

1: for $i = (n_u - 1) \times K_n + 1$ to $(n_u - 1) \times K_n + K_n$ do
2: \hspace{1em} if $i = (n_u - 1) \times K_n + 1$ to $(n_u - 1) \times K_n + 1 + d_m$ then
3: \hspace{2em} $A_{cp}\{1, 2, \ldots, i - K_n \times (n_u - 1) - 1\}$
4: \hspace{2em} $lc = (A_{cp} - 1) \times N_l$
5: \hspace{2em} $U_A((i-1)K + A_c(A_{cp}-1)) = U_L(lc+i-n_u + K_{uc} \times (n_u-1))$
6: \hspace{2em} $A_{cp}\{i - K_n \times (n_u - 1) + 1, \ldots, K_n\}$
7: \hspace{2em} $lm = (A_{cp} - 1) \times N_l$
8: \hspace{2em} $U_A((i-1)K + A_c(A_{cp}-1)) = U_L(lm+i-(n_u-1)+ K_{uc} \times (n_u-1))$
9: \hspace{1em} end if
10: \hspace{1em} if $i = (n_u - 1) \times K_n + 1+d_m+1$ to $(n_u - 1) \times K_n + K_n$ then
11: \hspace{2em} $A_{cp}\{i - K_n \times (n_u - 1) + 1, \ldots, K_n\}$
12: \hspace{2em} $lm = (A_{cp} - 1) \times N_l$
13: \hspace{2em} $U_A((i-1)K + A_c(A_{cp}-1)) = U_L(lm+i-(n_u-1)+ K_{uc} \times (n_u-1))$
14: \hspace{1em} end if
15: end for

example we take is the same as the BI scheme in Fig. 2. All
the LDPC codes used in this section is the (155,64,20) Tanner
code [25]. Therefore $N_l = 155$ and $K_t = 64$. The polar code
has the block length $N = 256$ and code rate $R = 1/4$. The
underlying channel is the AWGN channel. The polar code
construction is based on [3] which produces the set $A$. Then
the submatrix $G_{AA}$ is formed from the generator matrix $G_{256}$.
Based on the submatrix $G_{AA}$ and Proposition 3 the correlated
set $A_c$ ($K_c = 36$) and the un-correlated set $A_c$ ($K_{uc} = 28$
) is obtained. Algorithm 1 is implemented with the following
details.

- Consider the $i$th polar encoding block for $1 \leq i \leq K_c + 1 = 37$. The information bits $A_c$ of the $i$th polar block is composed of bit $i$ to $(i+28+1)$ of the $((i-1)\%37+1)$th LDPC code block. The information bits $A_c$ for the $i$th polar block are collected through two sets $A_{cc}$ and $A_{cp}$ with $A_{cc} = \{i-1, i-2, \ldots, 1\}$ and $A_{cp} = \{37, 36, 35, \ldots, i+1\}$. These two sets $A_{cc}$ and $A_{cp}$ are the indices of LDPC blocks. The bits of $A_c$ of the $i$th polar block are from two parts: the $(i-1+28)$th bit of LDPC groups $A_{cc}$ and the $i$th bit of LDPC groups $A_{cp}$.
- Consider the $i$th polar group for $38 \leq i \leq 74$. The information bits $A_c$ for the $i$th polar code consists of bits $(i - 2 + 28 + 1)$ to $(i - 2 + 28 + 1) + 28 - 1$ of the $((i - 1)\%37 + 1)$th LDPC block. In this case $A_{cc} = \{(i - 37) - 1, (i - 37) - 2, \ldots, 1\}$ and $A_{cp} = \{37, 36, 35, \ldots, (i - 37) - 1\}$. Therefore the bits $A_c$ of the $i$th polar code are from bit $(i - 2 + 28 \times 2)$ of LDPC groups $A_{cc}$ and bit $(i-1+28)$ of LDPC groups $A_{cp}$.
- Now consider the $i$th polar group for $75 \leq i \leq 101$. The bits $A_c$ of the $i$th polar code is made up of bits from $(i-3+28 \times 2+1)$ to $(i-3+28 \times 2+1)+(28-i\%37)-1$.
of the \((i - 1)\% 37 + 1\)th LDPC block. In this case, there is only \(A_{cp} = \{37, 36, 35, ... (i - 37 \times 2) + 1\}\). The information bits \(A_i\) for the \(i\)th polar code are from bit \((i - 2 + 28 \times 2)\) of LDPC groups \(A_{cp}\).

In this example, the occupied memory size is \([101, 64]\), which is smaller than \([155, 64]\). In the mean time, the turnaround time is \(101 \times 256\) symbols which is smaller than \(155 \times 256\) symbols. The BER performance of the CBI scheme is shown in Fig. 4 where the dashed black line with circles is the performance of the CBI scheme. The legend for this scheme is: LDPC (155,64)+CBI+Polar (256,64)SC. Here SC means polar codes are decoded with the SC algorithm. The solid black line with circles is the the performance of polar code directly concatenated with the LDPC code (no interleaving being performed), with a legend of LDPC (155,64)+Polar (256,64)SC. Note that compared with the BI scheme (the solid red line with triangles and the legend LDPC (155,64)+BI+Polar (256,64)SC), the CBI scheme achieves a comparable BER performance while having a memory size \(N_i/p_p = 1.5\) times smaller. At BER = \(10^{-4}\), the proposed CBI scheme only requires about 1dB increase of SNR compared with the BI scheme. Both the CBI and the BI scheme have a better BER performance than the direct concatenation. The advantage of the BI scheme comes from the fact that there is still a possibility of the un-correlated bits to be in error simultaneously (the coupling coefficient is not 1).

To compare with a direct concatenation of polar codes (BP decoding) with LDPC codes, simulation for this scheme is carried out (shown in Fig. 4 by the solid black triangular line). The legend for this scheme is LDPC (155,64)+Polar (256,64)BP. The proposed CBI scheme of polar codes with SC decoding outperforms the LDPC (155,64)+Polar (256,64)BP scheme (direct concatenation of polar codes with BP decoding) with a lower computational complexity.

To show the advantage of the CBI scheme, simulations with weaker codes are also conducted. Here by “weaker” we mean shorter or higher rate codes. The results are shown in Fig. 5. The solid red line with diamonds is the performance of the CBI scheme of polar code (32,8) concatenated with the LDPC code (21,8), with a legend of LDPC (21,8)+CBI+Polar (32,8). Compared with the BI scheme (the solid blue line with triangles and the legend LDPC (21,8)+BI+Polar (32,8)), the CBI scheme achieves almost the same performance. In this example, the occupied memory size of the CBI scheme is \([15,8]\) which is smaller than \([21,8]\) of the BI scheme. The same BER performance and the low occupied memory clearly show the advantage of the CBI scheme compared with the BI scheme. As discussed in Section \[11B\], some of the wasted bits in the CBI scheme can be turned to frozen bits to improve the system performance. This is shown by the red starred line in Fig. 5 with a legend of Improved LDPC (21,8)+CBI+Polar (32,8). It can be seen that by turning one wasted bit into a frozen bit, the BER performance of the CBI scheme reaches that of the BI scheme while still maintaining the same complexity as the original CBI scheme with the legend of LDPC (21,8)+CBI+Polar (32,8). Clearly, the CBI scheme with a stronger LDPC code will have a better BER performance than the CBI scheme with a weaker code in the large SNR region (beyond 0.5 dB). However, for application scenarios operating with SNRs below 0.5 dB, the CBI scheme with weaker codes is a good choice because of the good performance and the low complexity. To further reduce the complexity of the outer codes, BCH codes are employed to replace the LDPC codes. The CBI scheme with BCH as the outer code is shown in Fig. 4 by the solid line with circles and the legend for this scheme is BCH(15,5)+CBI+Polar(32,8). At BER=\(10^{-4}\), this scheme requires a 1dB increase compared with short and long outer codes: LDPC (21,8) and LDPC (155,64), which is the cost of the lower decoding complexity.

V. CONCLUSION

In this paper, we first propose the blind interleaving scheme (B1) to completely de-correlate the possible bit errors in the SC decoding process of polar codes. This BI scheme achieves a better BER performance than the state-of-the-art (SOA) concatenation of polar codes with LDPC codes while still maintaining the low complexity of SC decoding. To get a better balance between the performance and complexity,
a novel interleaving scheme, the correlation-breaking interleaving (CBI), is also proposed in this paper. Both the BI and CBI scheme have a much better performance than the SOA concatenation schemes. The CBI scheme 1) achieves a comparable BER performance as the BI scheme with a smaller memory size and a shorter turnaround time; 2) and enjoys a performance robustness with reduced block lengths. Tradeoff can therefore be made between the complexity and the performance between the BI scheme and the CBI scheme. Simulation results are provided which verified that the concatenation of polar codes with SC decoding with the CBI scheme achieves a better BER performance than the direct concatenation scheme of polar codes with the BP decoding. With the CBI scheme, concatenation of polar codes can achieve a satisfactory performance at a very low block length, which provides an efficient implementation option for polar codes.

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