Exact universal excitation waveform for optimal enhancement of directed ratchet transport

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(Dated: May 11, 2020)

The aim of the present paper is to show the existence and properties of an exact universal excitation waveform for optimal enhancement of directed ratchet transport (in the sense of the average velocity). This is deduced from the criticality scenario giving rise to ratchet universality, and confirmed by numerical experiments in the context of a driven overdamped Brownian particle subjected to a vibrating periodic potential. While the universality scenario holds regardless of the waveform of the periodic vibratory excitations involved, it is shown that the enhancement of directed ratchet transport is optimal when the impulse transmitted by those excitations (time integral over a half period) is maximum. Additionally, the existence of a frequency-dependent optimal value of the relative amplitude of the two excitations involved is illustrated in the simple case of harmonic excitations.

PACS numbers:
Criticality scenario by providing two alternative derivations, and explore its implications in the case of a driven Brownian particle moving in a back-and-forth travelling periodic potential [2] described by the overdamped model

\[ \dot{x} + \sin [x - \gamma f(t)] = \sqrt{\sigma} \xi(t) + \gamma g(t), \]

where \( f(t), g(t) \) are temporal excitations with zero mean, \( f(t) \) is \( T \)-periodic, \( \gamma \) is an amplitude factor, \( \xi(t) \) is a Gaussian white noise with zero mean and \( \langle \xi(t) \xi(t+s) \rangle = \delta(s) \), and \( \sigma = 2k_b T' \) with \( k_b \) and \( T' \) being the Boltzmann constant and temperature, respectively. Note that Eq. (2) is equivalent to

\[ \dot{z} + \sin z = \sqrt{\sigma} \xi(t) + \gamma F(t), \]

\[ F(t) \equiv g(t) - f(t), \]

where \( z(t) \equiv x(t) - \gamma f(t) \), and \( z \) and \( x \) are the particle phases relative to the vibrating potential frame and the laboratory frame, respectively. Since the mean velocity on averaging over different realizations of noise is the same in both frames, \( \langle \dot{x} \rangle = \langle \dot{z} \rangle \), we shall consider Eq. (3) for convenience in our analysis. For the sake of clarity, we shall confine ourselves to the regime where the DSB mechanism dominates over the thermal inter-well activation mechanism [15]. Also, we shall show how RU allows the dependence of DRT velocity on the system’s parameters to be explained quantitatively, and works effectively in two significant cases: (1) when \( F(t) \) is a truncated Fourier series of the exact universal periodic excitation after \( N \geq 2 \) terms, and (2) when \( f(t) \) and \( g(t) \) are harmonic excitations. For deterministic ratchets, the effectiveness of the theory of RU has been demonstrated in diverse physical contexts in which the driving excitations are chosen to be biharmonic. Examples are cold atoms in optical lattices [17], topological solitons [9], Bose-Einstein condensates exposed to a sawtooth-like optical lattice potential [18], matter-wave solitons [11], and one-dimensional granular chains [19].

**Exact universal excitation waveform.**—Let us assume in this section that the excitation’s amplitude and period are fixed. The criticality scenario giving rise to the existence of a universal excitation waveform which optimally enhances DRT is a consequence of two competing reshaping-induced effects: the increase in DSB and the decrease in the (normalized) maximal transmitted impulse over a half-period [16]. This means that the greater the impulse transmitted by a periodic excitation having its shift symmetry broken, the lower the DSB needed to yield the same strength of DRT, and vice versa. Since the strength of any transport (induced by symmetry breaking or not, i.e., by non-zero-mean forces), in the sense of the mean kinetic energy per unit of mass on averaging over different realizations of noise \( \langle \dot{x}^2 \rangle / 2 \), depends upon the impulse transmitted by the driving excitation (see the Appendix for a detailed deduction), and the waveform yielding maximal transmitted impulse is that of a square-wave, the exact universal waveform should present a constant positive value, \( A \), over a certain range \( t \in [0, \tau] \), \( 0 < \tau < T \), and a constant negative value, \(-B\), over the remaining range \( t \in [\tau, T] \), i.e., it should belong to the parameterized family of functions

\[ F(t) \equiv \frac{2(A + B)}{\pi} \sum_{n=1}^{\infty} \frac{\sin (\pi n t/T)}{n} \cos \left[ \frac{2n\pi}{T} (t - \tau/2) \right] \]

\[ = \frac{A + B}{\pi} \sum_{n=1}^{\infty} \left[ a_n(\tau) \cos (n \omega t) + b_n(\tau) \sin (n \omega t) \right], \]

where \( \omega \equiv 2\pi/T \). Clearly, the constraints \( A \neq B \) and \( \tau \neq T/2 \) are necessary conditions to satisfy two requirements: the breaking of the shift symmetry and the zero-mean property of the exact universal excitation \( f_u(t) \). This further requirement implies the relationship

\[ \tau = T/(1 + A/B), \]

i.e., one only has to obtain the suitable value of either the asymmetry parameter \( \tau \) or \( A/B \) that makes the DSB maximally effective, thus providing the exact universal excitation waveform.

The suitable value of \( \tau \) can be calculated from the observation that the exact universal excitation waveform cannot be independent of the biharmonic universal excitation waveform due to the unique character of both waveforms. This is due to the latter should inevitably be contained in the Fourier series of the former in the form of an infinity of harmonic pairs whose frequencies are one double the other while having the same waveform than that of the biharmonic universal excitation. Indeed, the biharmonic universal excitation is equivalently described by the expressions [16]

\[ f_{\sin, \sin \pm}(t) \equiv \varepsilon \left[ \sin(\omega t) \pm \frac{1}{2} \sin(2\omega t) \right], \]

\[ f_{\cos, \sin \pm}(t) \equiv \varepsilon \left[ \cos(\omega t) \pm \frac{1}{2} \sin(2\omega t) \right], \]

which satisfy the symmetries

\[ f_{\sin, \sin \pm}(t + T/2) = -f_{\sin, \sin \pm}(t), \]

\[ f_{\cos, \sin \pm}(t + T/2) = -f_{\cos, \sin \pm}(t), \]

\[ f_{\sin, \sin \pm}(t + T/4) = f_{\cos, \sin \mp}(t) \]

(see Fig. 1, top panel). From the Fourier series of \( F(t) \) [Eq. (4)], one has four harmonic pairs having frequencies \( \omega \) and \( 2\omega \) in each pair:

\[ b_1(\tau) \sin(\omega t) + b_2(\tau) \sin(2\omega t), \]

\[ a_1(\tau) \cos(\omega t) + b_2(\tau) \sin(2\omega t), \]

\[ b_1(\tau) \sin(\omega t) + a_2(\tau) \cos(2\omega t), \]

\[ a_1(\tau) \cos(\omega t) + a_2(\tau) \cos(2\omega t). \]

We see that the waveforms of the biharmonic expressions (8c) and (8d) do not correspond to that of the biharmonic
universal excitation (cf. Eq. (6)), the biharmonic expression (8b) with \( a_1 (\tau) = \pm 2b_2 (\tau) \) does but presents a phase difference of \( T/4 \) with respect to \( F(t) \), while the biharmonic expression (8a) with \( b_1 (\tau) = \pm 2b_2 (\tau) \) does and is in phase with \( F(t) \). Therefore, the compatibility between the exact universal excitation waveform and the biharmonic universal excitation requires that \( |b_1 (\tau) / b_2 (\tau)| = 2 \), i.e.,

\[
1 - \cos (\omega \tau) = \pm \frac{1 - \cos (2\omega \tau)}{2}.
\] (9)

After defining \( z \equiv \cos (\omega \tau) \), Eq. (9) can be put into the form \( 2z^2 - z - 1 = 0 \), \( 2z^2 - z - 3 = 0 \), for the signs \(+\) and \(-\), respectively. The solutions \( z = -3/2 \) and \( z = 1 \) of the latter algebraic equation lack mathematical acceptability between the exact universal excitation waveform and the biharmonic universal excitation regarding the periodicity of the coefficients, while property (11c) suggests that the complete Fourier series of the exact universal excitation \( f_A(t) \) can be understood as the sum of two complementary series: a series consisting only of sine terms containing all the ratcheting effect, and another series consisting only of cosine terms yielding the maximization of the transmitted impulse. Indeed, for the case \( A/B = 1/2 \) for instance, one has

\[
\frac{2\pi f_A(t)}{3\sqrt{3}} = - \cos (\omega t) + \frac{1}{2} \cos (2\omega t) - \frac{1}{4} \cos (4\omega t) + \frac{1}{5} \cos (5\omega t) - \frac{1}{7} \cos (7\omega t) + \frac{1}{8} \cos (8\omega t) - \ldots + \frac{\sqrt{3}}{2} \sin (2\omega t) + \frac{\sqrt{3}}{4} \sin (4\omega t) + \frac{\sqrt{3}}{5} \sin (5\omega t) + \frac{\sqrt{3}}{7} \sin (7\omega t) + \frac{\sqrt{3}}{8} \sin (8\omega t) + \ldots \equiv C_u(t) + S_u(t),
\] (12)

where \( C_u(t), S_u(t) \) represent the aforementioned complementary series, while \( C_{u,N}(t), S_{u,N}(t) \) denote the corresponding truncated series after \( N \) terms, respectively (see Fig. 1, middle and bottom panels).

Alternatively, the suitable value of \( A/B \) can be calculated from the quantifier of the DSB associated with the shift symmetry of \( f_A(t) \), \( D_s(f_A) \) [cf. Eq. (1)]. To this end, we properly require that the (positive and negative) amplitudes of \( F(t) \) and a suitable (symmetry-breaking-inducing) biharmonic excitation, for example \( f_{bh}(t) = \gamma [\eta \sin (\omega t) + (1 - \eta) \sin (2\omega t + \varphi)] \) with \( \gamma > 0, \eta \in [0, 1], \varphi = \varphi_{opt} \equiv \pi/2 \) [16], should be the same, i.e., \( A = \max_s f_{bh}(t; \varphi_{opt} = \pi/2), B = - \min_s f_{bh}(t; \varphi_{opt} = \pi/2) \). One thus obtains straightforwardly

\[
D_s(f_A) \equiv \frac{1}{T} \int_0^T \frac{f_A(t + T/2) - f_A(t)}{f_A(t)} dt = \frac{T - \tau}{A/B} = \left\{ \begin{array}{ll} 1 - \eta + \frac{\pi^2}{\ln^2 \eta}, & \eta \leq \frac{4}{5} \\ \frac{\pi^2}{4 \eta - 1}, & \eta > \frac{4}{5} \end{array} \right\},
\] (13)

with \( A < B \) (and hence \( T/2 < \tau < T \)), and where an increase in the deviation of \( D_s(f_A) \) from 1 (unbroken symmetry) indicates an increase in the DSB. One finds that \( D_s(f_A) \) has the value \( D_s(f_A) \mid_{\eta=0.1} = 1 \), and presents, as a function of \( \eta \), a single extremum at \( \eta = 2/3 \) (see Fig. 2, top panel), and hence the DSB is maximum when \( A/B = 1/2, \tau = 2T/3 \) [cf. Eqs. (5) and (13)]. As expected from a symmetry analysis, we obtained the same behaviour when using any other alternative form for \( f_{bh}(t) \) together with the corresponding suitable values of \( \varphi_{opt} \) in each case [16]. In particular, for the other optimal value, \( \varphi_{opt} \equiv 3\pi/2 \) corresponding to \( f_{bh}(t) = \gamma [\eta \sin (\omega t) + (1 - \eta) \sin (2\omega t + \varphi)] \), one straightforwardly obtains \( \text{max}_s f_{bh}(t; \varphi_{opt} = 3\pi/2) = - \text{min}_s f_{bh}(t; \varphi_{opt} = \pi/2), \text{min}_s f_{bh}(t; \varphi_{opt} = 3\pi/2) = - \text{max}_s f_{bh}(t; \varphi_{opt} = \pi/2), \text{and } D_s(f_A) = B/A = \tau / (T - \tau) = A > B \) (and hence \( 0 < \tau < T/2 \)). This value of \( D_s(f_A) \) presents the same dependence on \( \eta \) than that corresponding to \( \varphi_{opt} \equiv \pi/2 \) [Eq. (13)], and hence the DSB is maximum when \( A/B = 2, \tau = T/3 \) and the DRT has the same strength but opposite di-
after decreasing thickness), Saddle: Truncations of the series that relate the different representations [cf. Eq. (7)]. Middle: Truncations of the series $S_u(t)$ and $C_u(t)$ [cf. Eq. (12)] after $N = 2, 6, 14$ terms vs $t$ (solid curves of respectively decreasing thickness), $S_{u,N}(t)$ (upper panel) and $C_{u,N}(t)$, respectively. Bottom: Functions $S_{u,N=10}(t)$, $C_{u,N=10}(t) + S_{u,N=10}(t)$ (upper panel, solid and dashed lines, respectively) and $C_{u,N=10}(t)$, $S_{u,N=10}(t) + S_{u,N=10}(t)$ (solid and dashed lines, respectively) vs $t$.

FIG. 1: Top: Functions $f_{\sin,\sin,\pm}(t)$ and $f_{\cos,\sin,\pm}(t)$ [cf. Eq. (6)] representing the biharmonic universal excitation vs $t$. The horizontal and vertical arrows indicate the symmetries that relate the different representations [cf. Eq. (7)]. Middle: Truncations of the series $S_u(t)$ and $C_u(t)$ [cf. Eq. (12)] after $N = 2, 6, 14$ terms vs $t$ (solid curves of respectively decreasing thickness), $S_{u,N}(t)$ (upper panel) and $C_{u,N}(t)$, respectively. Bottom: Functions $S_{u,N=10}(t)$, $C_{u,N=10}(t) + S_{u,N=10}(t)$ (upper panel, solid and dashed lines, respectively) and $C_{u,N=10}(t)$, $S_{u,N=10}(t) + S_{u,N=10}(t)$ (solid and dashed lines, respectively) vs $t$.

Therefore, the values $A/B = 1/2, \tau = 2T/3$ (or equivalently $A/B = 2, \tau = T/3$) again fix the exact universal waveform of the excitation $f_u(t)$ as well as the properties of the associated ratchet potential $U_u(x) \equiv -\int f_u(x)dx$ (see Fig. 2, middle and bottom panels). In this regard, it is worth mentioning that the biparametric $(A, B)$ family of dichotomous driving waveforms predicted in Ref. [20] for optimal enhancement of DRT in overdamped, adiabatic rocking ratchets includes (without indicating that it is a special case) the exact universal waveform of $f_u(t)$ for the particular choice $A/B = 1/2$. Also, the exact universal waveform was used (without indicating the reason of its choice) in the experimental realization of a relativistic-flux-quantum-based diode [12]. After calculating the Fourier series of the universal excitation and potential,

$$f_u(t) = \frac{6A}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\pi/3)}{n} \cos[2n\pi (t/T - 1/3)] , \quad (14)$$

$$U_u(x) = -\frac{3A\lambda}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(2n\pi/3)}{n^2} \sin[2n\pi (x/\lambda - 1/3)] , \quad (15)$$

where $\lambda$ is the spatial period, one obtains the geometric properties of the universal ratchet potential per unit of amplitude and unit of spatial period [Eq. (15); see Fig. 2, bottom panel].

Next, we consider the case $g(t) = 0$ and $f(t) = -(1/A) \int f_{u,N}(t)dt$ [cf. Eq. (15)], i.e., $F(t) \equiv f_{u,N}(t)/A$ in Eq. (3), with $f_{u,N}(t)$ being the Fourier series of $f_u(t)$ truncated after $N$ terms [cf. Eq. (14)]. Our numerical results systematically indicate an overall increase of the maximum value of $\langle \frac{\Delta x}{\Delta t} \rangle$ with the number of terms $N$, while keeping the remaining parameters constant. Moreover, the typical instance shown in Fig. 3 (top panel) indicates that the average velocity (absolute value) quickly increases with $N$, and reaches its asymptotic value for $N \sim 13$. This behaviour is found to be correlated with that of the impulse per unit of amplitude transmitted by $f_{u,N}(t)$ over a half-period,

$$I_N \equiv \frac{1}{T} \int_0^{T/2} f_{u,N}(t)dt = \frac{3}{\pi^2} \sum_{i=1}^{N} \sin\left(\frac{i\pi}{3}\right) \left| \sin\left(\frac{i\pi}{6}\right) + \sin\left(\frac{2i\pi}{3}\right) \right| , \quad (16)$$

as expected from the theory of RU [16] (see Fig. 3, bottom panel).

**Harmonic excitations.**—For the sake of completeness, we next explore the standard case [2] in which the two temporal excitations involved are harmonic: $f(t) \equiv \eta \cos(\omega t), g(t) \equiv (1-\eta) \cos(2\omega t + \varphi), \omega \equiv 2\pi/T, \eta \in [0, 1]$ in Eq. (2), i.e.,

$$F(t) \equiv \eta \omega \sin(\omega t) + (1-\eta) \cos(2\omega t + \varphi) \quad (17)$$
in Eq. (3). Leaving aside the effect of noise (an effective change of the potential barrier which is in turn controlled by the DSB mechanism [15]), RU predicts (for $\sigma = 0$) that the optimal value of the relative amplitude $\eta$ comes from the condition that the amplitude of $\sin (\omega t)$ must be twice as large as that of $\cos (2\omega t + \phi)$ in Eq. (3) with $F(t)$ given by Eq. (17), and the optimal values of the initial phase difference are $\phi = \phi_{opt} \equiv \{0, \pi\}$ [16]. Thus, RU predicts the existence of a frequency-dependent optimal value of $\eta$:

$$\eta_{opt} \equiv 2/(2 + \omega),$$

and, equivalently, an optimal frequency for each value of $\eta$: $\omega_{opt} = 2 (1 - \eta)/\eta$. Numerical simulations confirmed this prediction over a wide range of frequencies (see Fig. 4, top panel). As mentioned above, the numerical estimate of the $\eta$ value at which the average velocity presents an extremum, $\eta_{opt}^{\sigma > 0}$, is slightly lower than the corresponding value $\eta_{opt}$ [Eq. (18)], as expected [15] (see
velocity should scale as

tion coming from

Also, this finding is in sharp contrast with the prediction for the maximum average velocity \[ F(t) \equiv F(t) \] [cf. Eq. (18)] subjected to the requirement that both excitations have the same (positive and negative) amplitudes for each value of \( \eta \). Recall that varying the amplitudes of \( F(t) \) implies varying the asymmetry parameter \( \tau \), and vice versa [cf. Eq. (5)], whence both \( \tau \) and \( A/B \) will be \( \eta \)-dependent so as to allow a proper comparison of the ratchet effectiveness of these excitations. Indeed, the results shown in Fig. 5 indicate that the DRT strength of the dichotomous excitation is greater than that of the biharmonic excitation over (almost) the entire range of \( \eta \) values, i.e., enhancement of DRT occurs when the impulse transmitted is maximum regardless of the DSB of the two excitations. One clearly sees in Fig. 5 that the greater the impulse transmitted, the lower the DSB needed to yield the same strength of DRT, and vice versa, as predicted from the criticality scenario. Note that the noise-induced decrease of the optimal value of \( \eta \) with respect to the corresponding deterministic prediction, \( \eta_{\text{opt}}^{\text{det}} - \eta_{\text{opt}}^{\text{ord}} \) [cf. Eq. (18)], is slightly lower when the transmitted impulse is maximum. This provides additional evidence for the impulse being the main quantifier of the driving effectiveness of a periodic excitation. Additionally, robustness of the present universality scenario is also observed when the external periodic excitation is replaced by a chaotic excitation having the same underlying main frequency in its Fourier spectrum (see the Appendix).

**Conclusions.** In summary, from the criticality scenario giving rise to ratchet universality we have demonstrated the existence and properties of an exact universal excitation waveform for optimal enhancement of directed ratchet transport by providing two alternative derivations. Our numerical experiments confirmed those findings, as well as revealed other unanticipated properties for the standard case of harmonic excitations in the general context of a driven overdamped Brownian particle subjected to a vibrating periodic potential. The exact universal waveform is the simplest possible (a particular dichotomous waveform), and is far more efficient that its biharmonic approximation, and the waveform of the associated optimal ratchet potential is therefore a particular case of the simplest piecewise waveform as is used, for instance, in a flashing ratchet. Since most models of biological Brownian motors are compatible with a simplified description based on the flashing ratchet, we are tempted to conjecture that the universal optimal ratchet potential could underlie the complex biological machin-

![Graph](image-url)
we deduce an analytical expression for the excitation is substituted by a chaotic excitation.

The lines connecting the symbols are solely plotted to guide the eye. Fixed parameters: $\gamma = 8, T = 4\pi, \varphi = \varphi_{opt} = 0, \sigma = 4$.

### I. APPENDIX: SUPPLEMENTARY CALCULATION DETAILS AND RESULTS

This Appendix provides details on the energy analysis, the case where the roles of the harmonic excitations are interchanged, and the case where the external periodic excitation is substituted by a chaotic excitation.

#### A. Energy-based analysis

In this subsection we deduce an analytical expression for the mean kinetic energy per unit mass of a Brownian particle of mass $m$ which satisfies the general equation of motion

$$m \ddot{x} + \frac{dU}{dx} = -\mu \dot{x} + \gamma f(t) + \sqrt{\sigma} \xi(t), \quad (A1)$$

where $U(x)$ is a potential subject to a lower bound (i.e., $\exists \alpha \in \mathbb{R} / U(x) \geq \alpha \ \forall x$), $f(t)$ is a unit-amplitude $T$-periodic function with zero mean, $\xi(t)$ is a Gaussian white noise of zero mean and $\langle \xi(t) \xi(t+s) \rangle = \delta(s)$, and $\sigma = 2\mu k_b T'$ with $k_b$ and $T'$ being the Boltzmann constant and temperature, respectively. Also, we assume without loss of generality that $f(0 \leq t \leq T^*) \geq 0$ and redefine here the impulse transmitted by $f(t)$ (per unit of amplitude) as

$$I \equiv \int_{nT}^{nT+T^*} f(t) \, dt > 0, \ n = 0, 1, 2, \ldots . \quad (A2)$$

Equation (A1) has the associated energy equation

$$\frac{dE}{dt} = -\mu \dot{x}^2 + \gamma \dot{x} f(t) + \sqrt{\sigma} \dot{x} \xi(t), \quad (A3)$$

where $E(t) \equiv (m/2) \dot{x}^2(t) + U[x(t)]$ is the energy function. Integration of Eq. (A3) over the intervals $[nT, nT + T^*]$ and $[nT + T^*, (n+1)T], \ n = 0, 1, 2, \ldots , \ yields

$$E(nT + T^*) = E(nT) - \mu \int_{nT}^{nT+T^*} \dot{x}^2(t) \, dt$$

$$+ \sqrt{\sigma} \int_{nT}^{nT+T^*} \dot{x}(t) \xi(t) \, dt + \gamma \int_{nT}^{nT+T^*} \dot{x}(t) f(t) \, dt, \quad (A4)$$

$$E[(n+1)T] = E(nT + T^*) - \mu \int_{nT+T^*}^{(n+1)T} \dot{x}^2(t) \, dt$$

$$+ \sqrt{\sigma} \int_{nT+T^*}^{(n+1)T} \dot{x}(t) \xi(t) \, dt + \gamma \int_{nT+T^*}^{(n+1)T} \dot{x}(t) f(t) \, dt, \quad (A5)$$

respectively, where the second integrals in Eqs. (A4) and (A5) are considered in the Stratonovich sense. After applying the first mean value theorem for integrals [24] to the last integrals on the right-hand sides of Eqs. (A4) and (A5), using Eq. (A2), and recalling that $f(t)$ is a zero-mean function, one obtains

$$E(nT + T^*) = E(nT) - \mu \int_{nT}^{nT+T^*} \dot{x}^2(t) \, dt$$

$$+ \sqrt{\sigma} \int_{nT}^{nT+T^*} \dot{x}(t) \xi(t) \, dt + \gamma \dot{x}_n I, \quad (A6)$$

$$E[(n+1)T] = E(nT + T^*) - \mu \int_{nT+T^*}^{(n+1)T} \dot{x}^2(t) \, dt$$

$$+ \sqrt{\sigma} \int_{nT+T^*}^{(n+1)T} \dot{x}(t) \xi(t) \, dt - \gamma \dot{x}_n' I, \quad (A7)$$

respectively, where the discrete variables $\dot{x}_n \equiv \dot{x}(t_n), \dot{x}_n' \equiv \dot{x}(t_n')$, with $t_n$ and $t_n'$ being unknown instants which only have to satisfy the respective relationships $nT \leq t_n \leq nT + T^*$ and $nT + T^* \leq t_n' \leq (n+1)T$, according to the first mean value theorem for integrals.
After adding Eqs. (A6) and (A7) from $n = 0$ to $n = N - 1$ and dividing the result by $NT$, one obtains

$$E(NT) - E(0) = -\frac{\mu}{NT} \int_0^{NT} \dot{x}^2(t) \, dt + \frac{\gamma I}{NT} \sum_{n=0}^{N-1} \left[ \frac{\dot{x}_n - \dot{x}'_n}{NT} \right] + \frac{\sqrt{\sigma}}{NT} \int_0^{NT} \dot{x}(t) \xi(t) \, dt. \quad (A8)$$

Upon taking the limit $N \to \infty$ in Eq. (A8), averaging over different realizations of noise, and recalling that the system (A1) is dissipative and that $\xi(t)$ is a stationary random process which cannot contain a shot noise component, one finally obtains

$$\left\langle \langle \dot{x}^2 \rangle \right\rangle = \frac{\gamma I}{\mu} \left[ \left\langle \langle \dot{x}_n \rangle \right\rangle - \left\langle \langle \dot{x}'_n \rangle \right\rangle \right] + \frac{\sqrt{\sigma}}{\mu} \left\langle \langle \dot{x} \xi \rangle \right\rangle. \quad (A9)$$

The following remarks are now in order. First, $\left\langle \langle \dot{x}_n \rangle \right\rangle$ provides the average of the particle’s velocity when $\dot{x}$ is measured exclusively at certain instants for which $f(t)$ has the same sign as the acceleration $\ddot{x}$ [cf. Eq. (A1)], i.e., when $f(t)$ tends to yield an increase in the particle’s velocity, while $\left\langle \langle \dot{x}'_n \rangle \right\rangle$ does the same when $f(t)$ has the opposite sign to $\ddot{x}$, i.e., when $f(t)$ tends to yield a decrease in the particle’s velocity. One sees from Eq. (A9) that the effect of the difference $\left\langle \langle \dot{x}_n \rangle \right\rangle - \left\langle \langle \dot{x}'_n \rangle \right\rangle$ on the average kinetic energy per unit of mass is modulated by the impulse per unit of amplitude, while keeping the remaining parameters constant. Second, increasing the noise strength from $\sigma = 0$ activates the term $\left\langle \langle \dot{x} \xi \rangle \right\rangle$, which can be positive or negative. Third, one has $\lim_{m \to 0} [E(NT) - E(0)] / (NT) = [U(NT) - U(0)] / (NT)$ and hence Eq. (A9) remains valid in the overdamped limiting case.

**B. Complementary case of harmonic excitations**

Let us consider the case of harmonic excitations in Eq. (2) when the roles of the excitations $\eta \cos (\omega t)$ and $(1 - \eta) \cos (2\omega t + \varphi)$ are interchanged, i.e., the Langevin equation now reads

$$\dot{x} + \sin [x - \gamma(1 - \eta) \cos (2\omega t + \varphi)] = \sqrt{\sigma \xi}(t) + \gamma \eta \cos (\omega t) \xi(t). \quad (A10)$$

In the reference frame associated with the vibrating potential, one then obtains

$$\dot{z} + \sin z = \gamma \eta \cos (\omega t) + 2\omega (1 - \eta) \sin (2\omega t + \varphi) + \sqrt{\sigma \xi}(t), \quad (A11)$$

where $z(t) \equiv x(t) - \gamma(1 - \eta) \cos (2\omega t + \varphi)$. Once again, ratchet universality predicts that the optimal value of the relative amplitude $\eta$ comes from the condition that the amplitude of $\cos (\omega t)$ must be twice as large as that of $\sin (2\omega t + \varphi)$ in Eq. (A11), while the optimal values of the initial phase difference are $\varphi = \varphi_{opt} = \{\pi/2, 3\pi/2\}$ [16]. It therefore predicts the existence of a different (with respect to the case considered above, cf. Eq. (18)) frequency-dependent optimal value of $\eta$:

$$\eta_{opt} \equiv \frac{4\omega}{1 + 4\omega}, \quad (A12)$$

and, equivalently, a different optimal frequency for each value of $\eta$:

$$\omega_{opt} \equiv \frac{\eta}{4(1 - \eta)}. \quad (A13)$$

Numerical simulations (as shown in Fig. 6) confirmed this new prediction over a wide range of frequencies.

**C. Robustness against chaotic excitations**

In this subsection, we study the robustness of the universality scenario against the presence of a bounded chaotic excitation instead of an external periodic excitation. We shall consider the simple case $f(t) \equiv \eta \cos (\omega t + \varphi/2)$, $g(t) \equiv (1 - \eta) \alpha \dot{y}(t)$, $\omega \equiv 2\pi/T, \eta \in [0, 1]$ in Eq. (2), i.e.,

$$F(t) = F_{chaos}(t) \equiv \eta \omega \sin (\omega t + \varphi/2) + (1 - \eta) \alpha \dot{y}(t) \quad (A14)$$

in Eq. (3), where $\dot{y}(t)$ is a chaotic response of a master system exhibiting the same underlying main frequency, $2\omega$, in its Fourier spectrum [cf. Eq. (17)], but cannot itself yield DRT. The value of $\alpha$ is chosen in order for the excitations $\cos (2\omega t + \varphi)$ and $\alpha \dot{y}(t)$ to have similar ranges. We considered the following master system
with the parameter values $\omega_0 = 0.5, K = 2.25, \delta = 0.375, F = 2.48625$, for which the pendulum presents a chaotic attractor irrespective of the initial conditions. Figure 7(a) shows the time series corresponding to the velocity $\dot{y}(t)$, and Fig. 7(b) shows the corresponding power spectrum which presents its main peak at the frequency $2\omega_0$. Note the presence of additional peaks at the frequencies $6\omega_0, 10\omega_0, 14\omega_0, \ldots$, i.e., the underlying periodic solution, $f(t)$, only presents odd harmonics and hence satisfies the shift symmetry $f(t + T/2) = -f(t)$ with $T = \pi/\omega_0$. This means that the function $f(t)$ itself cannot yield directed ratchet transport.

We found numerically the same dependence of the average velocity on $\eta$ as in the biharmonic case [Eq. (17)], but with a drastic decrease of the DRT strength (see Fig. 8, top). Indeed, the presence of other noticeable harmonics in the Fourier spectrum of $\dot{y}(t)$ [cf. Fig. 7(b)] yields interferences with the excitation $\eta \omega \sin(\omega t)$ which leads $F_{\text{chaos}}(t)$ to deviate from the optimal biharmonic approximation [cf. Eq. (10)]. This phenomenon and the inherent noise background lead to $F_{\text{chaos}}(t)$ losing DRT effectiveness, but without deactivating the DSB mechanism, and also to an additional decrease in the optimal value of $\eta$ with respect to the corresponding deterministic prediction [cf. Eq. (18)]. This robustness is also manifest in the dependence of the average velocity on $\varphi$ (see Fig. 8, bottom).

![Figure 7](image_url)

**FIG. 7:** (a) Velocity time series of $\dot{y}(t)$, and (b) the corresponding power spectrum ($\log_{10}|S(\omega)|$ versus $\omega/\omega_0$) associated with the damped driven pendulum given by Eqs. (A14) and (A15). Fixed parameters: $\omega_0 = 0.5$, $K = 2.25$, $\delta = 0.375$, $F = 2.48625$.

\[ \ddot{y} + K \sin y = -\delta \dot{y} + F \cos(2\omega_0 t), \quad (A15) \]
FIG. 8: Top: Average velocity $\langle \langle z \rangle \rangle$ [cf. Eq. (3)] vs parameter $\eta$ for $f(t) = \eta \cos(\omega t + \varphi/2)$, $\varphi = \varphi_{opt} = 0$, and two excitations $g(t)$ having the same underlying main frequency, $2\omega$, in their Fourier spectrum: chaotic excitation [cf. Eqs. (A14) and (A15); dots] and biharmonic excitation [cf. Eq. (17); stars]. Bottom: $\langle \langle z \rangle \rangle$ vs $\varphi$ for the chaotic excitation and $\eta = \{0.7, 0.8\}$. The lines connecting the symbols are solely to guide the eye. Fixed parameters: $\gamma = 8, T = 4\pi, \sigma = 5, \alpha = 0.25$. 