Isoperimetric Properties of the Mean Curvature Flow

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Abstract

In this paper we show that the space-time track of a mean curvature flow in arbitrary co-dimension, starting from an integral $k$-current $T_0$ in $\mathbb{R}^n$ (in the precise sense of Ilmanen's undercurrent [Ilm94]) satisfies a Michael-Simon type parabolic isoperimetric inequality. More precisely we show that the $k+2$ parabolic measure of the space-time track is bounded by $C(k)M[T_0]^{k+2}$ for some universal constant $C(k)$. We further show that there exists some universal constant $c = c(k)$ such that the spatial track $S$ of a mean curvature flow in a simply connected, non-positively curved Riemannian manifold $(M, g)$ starting from an integral $k$-current $T_0$ provides a good filling of $T_0$ in the Gromov-Michael-Simon sense: $M[S] \leq cM[T_0]^{k+1}$. Comparing the filling $S$ to the optimal one, we also observe a criterion for (quantitative) non-disappearance of Brakke flows starting from cycles.

1 Introduction

A family of embeddings of a $k$ manifold $N$ in $\mathbb{R}^n$: $\phi_t : N \to \mathbb{R}^n$ is said to evolve by mean curvature if at every point and time it satisfies the equation $\frac{d\phi}{dt}(x, t) = \vec{H}$ where $\vec{H}$ is the mean curvature vector. Starting with a smooth, compact embedding, this flow exists (smoothly) for some finite time, at which the flow encounters a singularity. As the embeddings need not disappear altogether when a singularity occurs, notions of weak solutions were desirable. Two such notions are the varifold flow of Brakke (see [Bra78]) and the level-set flow of Evans-Spruck (see [ES91], [ES92a], [ES92b], [ES95]) and Chen-Giga-Goto (see [CGG91]). If the initial surface has no boundary, via elliptic regularization, Ilmanen developed a notion of enhanced motion, which unites information from both approaches.

Let $\mathbb{R}^{1,n} = \mathbb{R} \times \mathbb{R}^n$, $\mathbb{R}^{1,n}_+ = \mathbb{R}_+ \times \mathbb{R}^n$ and $M_+ = \mathbb{R}_+ \times M$ where $(M, g)$ is a Riemannian manifold. Given an integral $k$-cycle of finite mass in $M$, $T_0 \in I_k^{\text{loc}}(\{0\} \times M)$ Ilmanen’s idea (motivated by the level-set approach) (see [Ilm94]) was to approximate the mean curvature flow (MCF) starting from $T_0$ by a family of $(k+1)$-dimensional translating solutions in $M_+$ which become more and more “cylindrical” and tall (in the first co-ordinate). At the limit, their horizontal sections will become a $k$-dimensional Brakke flow in $M$ starting from $T_0$ while the limit of the “down-scalings” of these solutions will yield a current $T \in I_{k+1}^{\text{loc}}(M_+)$ with $\partial T = T_0$ that provides a measure-theoretic subsolution to the Brakke flow.

Theorem 1.1 ([Ilm94, 8.1]). Let $T_0 \in I_k^{\text{loc}}(\{0\} \times M)$ be a cycle of finite mass. There exists a tuple $(T, \{\nu_t\}_{t \geq 0})$ called the enhanced motion where $T \in I_{k+1}^{\text{loc}}(M_+)$ with $\partial T = T_0$ and $\{\nu_t\}_{t \geq 0}$ is a Brakke flow with $\nu_0 = \nu_{T_0}$ such that:

\begin{align*}
\nu_t & \geq \nu_{T_t}, \\
M[\pi_x \# (\nu_t)] & \leq |B|^{1/2}M[T_0]
\end{align*}
and:
\[ M_\pi(B) \leq (|B| + |B|^{1/2})M_\pi(T_0) \tag{1.4} \]

where \( \pi_x : M_+ \to M \) is the projection to the \( M \) component and \( T_0 = T|B \times M \) for \( B \subseteq \mathbb{R} \).

**Remark 1.5.** \( T \) is called the **undercurrent** and \( T_t \), the \( t \) time slice of \( T \), is called the **underflow**.

**Remark 1.6.** From a heuristic point of view, \( T \) should be seen as the space-time track of the mean curvature flow (see [Ilm94, 2.2] and the discussion in 2.3). Contrary to the intuition suggested by these heuristics, there may be some discrepancy between the underflow and the Brakke flow. When the undercurrent is unique however, the underflow is indeed a Brakke flow (see [Ilm94, 9.2]). This happens, in particular, in the important case where \( T_0 \) is the cycle corresponding to a boundary of an open set of finite perimeter, in the case that the level-set flow is non-fattening (see [Ilm94, 11]).

We recall the notions of parabolic metric and parabolic Hausdorff measure. If \( A \in \mathbb{R}^{1,n} \) is a space-time track of a mean curvature flow, then given \( \lambda > 0 \), \( \lambda A \) is not such a track, but \( \{ (\lambda^2 t, \lambda x) \mid (t, x) \in A \} \) is. The same is true (measure theoretically and in the sense of the underlying rectifiable sets) for the space-time track of a Brakke flow. To study scale-invariant properties we would therefore need a metric that respects those scalings.

**Definition 1.7 ([Whi97]).** The **parabolic metric** on \( \mathbb{R}^{1,n} \) is defined to be:
\[
d_{\text{par}}((t, x), (s, y)) = \max\{ \sqrt{|t - s|}, |x - y| \}. \tag{1.8}\]

The Hausdorff measure corresponding to \( d_{\text{par}} \) will be called the **parabolic Hausdorff measure** and will be denoted by \( \mathcal{H}^*_{\text{par}} \).

**Remark 1.9.** Note that a \( k+1 \) plane has parabolic Hausdorff dimension \( k+1 \) if it is perpendicular to \( \partial_t \) and parabolic Hausdorff dimension \( k+2 \) if it has some \( \partial_t \) component. As mean curvature flow is a flow in time, it is therefore reasonable to measure the \( (k+2) \)-Hausdorff measure of the space-time track \( \mathcal{H}^{k+2}_{\text{par}} \).

**Remark 1.10.** The parabolic Hausdorff measure was first introduced to the study of mean curvature flow by White for his dimension reduction principle (see [Whi97]). In there, the question concerned the parabolic Hausdorff dimension of certain sets (the singular stratum) and the only property of the measure that was used was the above mentioned scaling. The relationship between the total measure of a set and its time slices was used by Federer’s general co-area inequality (see [Fed69, 2.10.25]). It will be one of the main technical objectives of this current paper to relate the horizontal measures of the slices of a Euclidean rectifiable set in space-time to the parabolic measure of the entire set in a more precise way (i.e. to obtain a co-area formula type result).

We are now ready to state the first main theorem of this paper.

**Theorem A (Parabolic Measure Estimate for the Space-Time Track of a MCF).** There exists some universal constant \( C = C(k) \) with the following property: Let \( (T, \{ \nu_t \}_{t \geq 0}) \) be an enhanced motion starting from an integral \( k \)-cycle \( T_0 \) of finite mass in \( \mathbb{R}^n \). Letting \( T = \tau(X, \theta, \xi) \) (i.e. \( T \) is the integral current corresponding to the rectifiable set \( X \), the multiplicity \( \theta \) and the orientation \( \xi \)) and letting \( \mu \) be the rectifiable parabolic radon measure corresponding to \( (X, \theta) \), i.e.
\[
\mu = \theta \mathcal{H}^{k+2}_{\text{par}}[X]. \tag{1.11}\]

Then \( \mu \) is well defined (see remark below) and:
\[
\mu(\mathbb{R}^{1,n}_+) \leq C(k)M_\pi(T_0) \tag{1.12}\]
Several remarks are in order.

Remark 1.13. Rectifiable sets are defined as a union of Lipschitz images and a set of $H^{k+1}$ measure zero, which is inconsequential from the point of view of a current supported on the set. Thus, in order for the above theorem to make sense, one needs to show that for a set $B \subseteq \mathbb{R}^{1,n}$ we have $H^{k+1}(B) = 0$ implies $H^{k+2}(B) = 0$. This is part of the content of Lemma 5.2.

Remark 1.14. In order for a scale invariant parabolic isoperimetric inequality concerning the undercurrent to be meaningful, one must check that the undercurrent construction itself is scale invariant. This is done in 2.11.

Remark 1.15. Note that the constant $C$ depends only on the dimension of the current, and not on the dimension of the ambient space. Thus, the above estimate is reminiscent of the Michael-Simon isoperimetric inequality (see [MS73] and discussion below).

Moving forward, our second main theorem discusses the size of the spatial track.

**Theorem B** (MCF Spatial Isoperimetric Inequality). There exists a constant $c = c(k)$ with the following property: For every simply connected, non-positively curved Riemannian manifold $(M, g)$ and every enhanced motion $(T, \{\nu_t\}_{t \geq 0})$ starting from an integral $k$-cycle $T_0$ of finite mass in $M$, taking $S = (\pi_x)_#(T)$ we have $\partial S = T_0$ and

$$
M[S] \leq cM[T_0]^{k+1/k}. \quad (1.16)
$$

Furthermore, when $M = \mathbb{R}^n$, the above inequality holds with $c = \frac{1}{\sqrt{4\pi}}$.

To put the above result in context, we will briefly discuss the history of the isoperimetric inequality. In the fundamental paper [FF60] Federer and Fleming proved an isoperimetric inequality (now bearing their names) stating that there exists a constant $c = c(k, n)$, depending both on the dimension and the co-dimension, such that for every integral cycle $N \in I_k(\mathbb{R}^n)$ with finite mass, there exists some $F \in I_{k+1}(\mathbb{R}^n)$ such that:

$$
M[F] \leq cM[N]^{k+1/k}. \quad (1.17)
$$

with $\partial F = N$ (see also [Sim83, 29,30]). Thirteen years later, Michael and Simon showed that the constant $c$ can be taken to be independent of the co-dimension (see [MS73]). This was soon generalized to the case of non-positively curved Riemannian manifolds (see [HS74]). Another argument that applies to a much more general setting than Euclidean spaces was given in [Gro83] where one also gets the estimate $c = O(k^k)$. Finally, in [Alm86] Almgren proved an “optimal” isoperimetric inequality in the Euclidean case: the constant $c$ in (1.17) corresponds to the case of a standard sphere enclosing a disk (i.e $c_k = \omega_{k-1}/k \approx 1/\sqrt{k}$) and equality is achieved if and only if $N$ is a constant multiple of the standard sphere.

In light of the above, Theorem B shows that the spatial track of a singular MCF provides a filling that satisfies such a Gromov-Michael-Simon isoperimetric inequality. In the Euclidean case, the inequality is worse than the optimal one by a factor of only $\sqrt{k}$. Theorem B is particularly interesting when the underflow is in fact a Brakke flow (see remark 1.16 again) as then the filling is local.

Remark 1.18. The proof of Theorem B in the Euclidean case relies only on the Federer-Fleming isoperimetric inequality [FF60] and so it can be considered as a genuine proof of an isoperimetric inequality with a (relatively) good constant.

Prior uses of the mean curvature flow in studying the isoperimetric problem include [Sch08], where the level-set power MCF on mean convex hypersurfaces was used to derive the optimal
isoperimetric inequality in co-dimension one and a Euclidean isoperimetric inequality for surfaces in simply connected 3-manifolds with non-positive sectional curvature (which was proven originally in [Kle92] and [Top98] where the curve shortening flow was used to obtain an optimal isoperimetric inequality on surfaces. In both cases the argument is based on a monotonicity of certain surface-area volume functions, and is quite different from the one here.

Thinking about the spatial track of MCF as an isoperimetric filling gives the following funny lower bound on the extinction time of the Brakke flow in an enhanced motion:

**Theorem C** (Lower Bound on Extinction). If $T_0, (T, \{ \nu_t \}_{t \geq 0})$ are as in Theorem B and $F$ is the optimal filling of $T_0$ then the extinction time of $\{ \nu_t \}_{t \geq 0}$ satisfies:

$$\tau \geq \left( \frac{M[F]}{M[T_0]} \right)^2.$$  \hfill (1.19)

**Remark 1.20.** This estimate is, of course, the most interesting when there is an a priori information about the optimal filling of $T_0$, which is the case, for instance, when $T_0 - S^k = \partial W$ and $M[W]$ is small.

**Remark 1.21.** A related result, providing a lower bound on the extinction in the co-dimension one case by an inscribed sphere, was proven in [GYu93].

The proofs of Theorems B and C are more or less at the level of an observation on Ilmanen’s construction, while the proof of Theorem A is more involved. The proofs of A and the Euclidean case of B rely on a relatively standard use of Huisken’s monotonicity formula [Hui90] to estimate the extinction time of a Brakke flow. For the full Theorem B, a generalized version of Brakke’s clearing-out lemma is proved (see [ES92b]), from which point an argument of Evans and Spruck is used to derive an extinction bound. From that point, Theorem B follows by using the basic properties of the enhanced motion (1.1). For the parabolic Hausdorff measure estimate for the space-time track of the MCF more work is needed. This mostly consists of studying the geometric measure theory of Euclidean rectifiable currents in parabolic space and the relationship between the Euclidean Hausdorff measure of time slices of such sets and the total parabolic Hausdorff measure. As it turns out, the co-area formula in such a situation takes the form of Fubini’s theorem without any co-area factor. More precisely, we will have the following theorem, which is perhaps of some interest in its own right:

**Theorem D** (Parabolic Co-Area). Let $\mathcal{M} \subseteq \mathbb{R}^{1,n}$ be a Euclidean $(k+1)$-rectifiable set of finite $(k+1)$ dimensional Hausdorff measure and let $g : \mathcal{M} \to \mathbb{R}$. Then:

$$\int_{\mathcal{M}} g dH_{par}^{k+2} = c_1(k) \int_{\mathbb{R}^1} \left( \int_{\mathcal{M}_t} g dH^k \right) dH_{par}^2(t)$$

where $c_1(k)$ is some universal constant.

**Remark 1.23.** This should be compared with the Euclidean situation where the co-area formula takes the form:

$$\int_{\mathcal{M}} g| \nabla \mathcal{M}| dH^{k+1} = \int_{\mathbb{R}} \left( \int_{\mathcal{M}_t} g dH^k \right) dH^1(t).$$

**Remark 1.25.** The absence of a co-area factor is not too surprising: considering, say, a smooth $k+1$ submanifold in $\mathbb{R}^{1,n}$ we see that if the submanifold had a time-like direction, the parabolic blow-ups at a point will contain the vector $\partial_t$, and so we are always in the “split” Fubini situation infinitesimally. If it were perpendicular to time, then it should not contribute to the $H_{par}^{k+2}$ measure anyway.
The organization of the paper is as follows: In Section 2 we collect some preliminary results, in Section 3 we derive the extinction estimates and in Section 4 we give the proof of Theorems B and C and a proof of Theorem A assuming Theorem D. In Section 5, which is the most technically involved, we will study the geometric measure theory (GMT) of Euclidean currents in parabolic space and in particular prove Theorem D.

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2 Preliminaries:

We recall the notion of a Brakke flow (see [Bra78, 3.2]). We will follow the slightly different definition appearing in [Ilm94]. Letting

$$D_t f = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

for \( f : \mathbb{R} \to \mathbb{R} \),

**Definition 2.2** ([Ilm94, 6.2-3]). A family of Radon measures \( \{ \nu_t \}_{t \geq 0} \) on \((M, g)\) is a Brakke flow if for all \( t \geq 0 \) and all \( \phi \in C^1_c(M, \mathbb{R}^+) \) we have

$$B(\nu, \phi) = \int -\phi H^2 + \nabla \phi \cdot S^\perp \cdot \vec{H} \, d\nu$$

whenever \( \nu|\{\phi > 0\}\) is radon \( k \) rectifiable, \( |\delta V||\{\phi > 0\}\) is a radon measure absolutely continuous w.r.t \( \nu|\{\phi > 0\}\) for \( V = V_\nu|\{\phi > 0\}\) and when \( \phi H^2 \) is integrable. Here \( S \) is the approximate tangent space and we confuse a subspace with the projection operator to it. If either of the above conditions is not satisfied, we let \( B(\nu, \phi) = -\infty \). A Brakke flow is called integral if for a.e \( t \geq 0 \) \( \nu_t \in \text{IM}_k(M) \), the space of integer rectifiable radon measures.

We will further need the following generalization of Huisken’s monotonicity formula (see [Hui90]) to the context of Brakke flows (see [Ilm]):

**Theorem 2.4.** Let \( \{ \nu_t \}_{t \geq 0} \) be a \( k \)-dimensional integral Brakke flow in \( \mathbb{R}^n \) with \( \nu_0(\mathbb{R}^n) < \infty \) and let \( t_1 < t_2 < \tau \) and \( p \in \mathbb{R}^n \), then we have:

$$\int \frac{1}{(4\pi(\tau - t_1))^{k/2}} e^{-\frac{|x-p|^2}{4(\tau - t_1)}} \, d\nu_{t_1}(x) \leq \int \frac{1}{(4\pi(\tau - t_2))^{k/2}} e^{-\frac{|x-p|^2}{4(\tau - t_2)}} \, d\nu_{t_2}(x).$$

\( \Box \)

In what follows, we indicate why Ilmanen’s construction of the enhanced motion ([Ilm94]) respects the natural parabolic scalings on \( \mathbb{R}^{1,n} \). We also include, for the reader’s convenience, Ilmanen’s heuristics for why \( T \) should be seen as the space-time track of the mean curvature flow (see [Ilm94, 2.2]). For both purposes, we need to describe the construction in some more detail.

Let \( T_0 \in I_k(\{0\} \times \mathbb{R}^n) \) be an integral cycle of finite mass. For \( Q \in I_{k+1}(\mathbb{R}^{1,n}) \) and \( \epsilon > 0 \), Ilmanen ([Ilm94, 2.1]) defined the functional:

$$I^\epsilon[Q] = \frac{1}{\epsilon} \int e^{-\frac{|x|^2}{\epsilon}} \, d\nu_Q$$

(2.6)
where \((z, x) \in \mathbb{R}^{1,n}\). By the direct method of the calculus of variations, he produced currents \(P^\epsilon \in P^\epsilon_{\text{loc}}(\mathbb{R}^{1,n})\) minimizing \(I^\epsilon\) subject to the constraint of having boundary \(T_0\), which additionally turn out to be supported on \(\mathbb{R}^{1,n}_+\). The Euler-Lagrange equation of this functional is:

\[
\vec{H} + S^\perp \frac{\omega}{\epsilon} = 0
\]  

(2.7)

for \(\vec{H}\) the generalized mean curvature vector, \(\omega = (1, 0, \ldots, 0)\) and \(S\) the approximate tangent space (with the usual abuse of notation identifying a subspace with the projection to it). \(P^\epsilon\) is thus a translating solution for the MCF with velocity \(v = -\vec{\omega}\). Letting \(\kappa(z, x) = (\epsilon z, x)\) and \(t = \epsilon^2\), Ilmanen defines \(T^\epsilon = (\kappa_\epsilon \#(P^\epsilon))\) and Ilmanen’s undercurrent \(T\) is, by definition, a sub-limit of those \(T^\epsilon\) as \(\epsilon \to 0\).

**Remark 2.8.** Ilmanen sees those \(T^\epsilon\) as an approximation for the space-time track of the mean curvature flow starting form \(T_0\). The reason is the following (see [Ilm94, 2.2]): As it turns out, for \(\epsilon << 1\), \(P^\epsilon\) are of height \(\approx \frac{\epsilon}{2}\) and are almost cylindrical. Slicing \(P^\epsilon\) at some \(z\) we obtain:

\[
\vec{H}_{T^\epsilon} = \vec{H}_{P^\epsilon} \approx \vec{H}_{P^\epsilon}.
\]

(2.9)

Letting \(\text{MCF}^k(M, t)\) be flow by mean curvature of a \(k\) sub-manifold \(M\) for time duration \(t\) and \(\text{HMCF}^k(M, t)\) be flow by only the horizontal part of the mean curvature, we see that for \(s > t\):

\[
T^\epsilon_s = P^\epsilon_{s/t} = \text{MCF}^{k+1}(P^\epsilon, s - t)_{t/s} \approx \text{HMCF}^{k+1}(P^\epsilon, s - t)_{t/s} = \text{MCF}^k(P^\epsilon_{s/t}, s - t) = \text{MCF}^k(T^\epsilon_t, s - t)
\]

(2.10)

\(T\), being a sub-limit of the \(T^\epsilon\) is thus seen (intuitively) as the space-time track. See Remark 1.10 for cases when it is known to be valid.

Both classical mean curvature and Brakke flow are invariant under parabolic rescalings. The same is true for the undercurrent:

**Lemma 2.11.** Let \(\lambda > 0\) and let \(\eta_\lambda, S_\lambda : \mathbb{R}^{1,n} \to \mathbb{R}^{1,n}\) be parabolic and Euclidean rescaling by \(\lambda\), i.e. \(\eta_\lambda(t, x) = (\lambda^2 t, \lambda x)\) and \(S_\lambda(t, x) = (\lambda t, \lambda x)\). If \(T\) is an undercurrent corresponding to \(T_0\) then \((\eta_\lambda \#(T)) = (\eta_\lambda \#(T_0))\).

**Proof.** Letting \(P^\epsilon(T_0)\) be a minimizer of \(I^\epsilon\) with boundary \(T_0\) we see that \((S_\lambda \#(P^\epsilon))\) is a minimizer of \(I^\epsilon\) with boundary \((S_\lambda \#(T_0))\). Thus one can take \(P^\lambda((S_\lambda \#(T_0))) = (S_\lambda \#(P^\epsilon(T_0)))\) and so:

\[
T^\lambda((S_\lambda \#(T_0))) = (\kappa_\lambda \circ S_\lambda)(S_\lambda \#(P^\epsilon(T_0))) = (\kappa_\lambda \circ S_\lambda)(T^\epsilon(T_0)) = (\eta_\lambda \#(T^\epsilon(T_0)))
\]

(2.12)

so we get the desired scaling in the level of the subsequences and so at a (possible) limit. \(\square\)

**Remark 2.13.** Without the above lemma, a parabolic Hausdorff measure estimate regarding the undercurrent would have been rather meaningless. This is not the case.

We also need a slight generalization of Brakke’s perpendicularity theorem

**Theorem 2.14 ([Bra78 5.8]).** Let \((M, g)\) be a smooth Riemannian manifold and let \(V \in IV_\epsilon(M)\) with \(||\delta V||\) a radon measure, absolutely continuous w.r.t. \(||V||\). Then for \(||V||\)-a.e. \(x\) we have \(T_x N \perp \vec{H}\) where \(N\) is the rectifiable set underlying \(V\).
Remark 2.15. Brakke only states the the theorem for $M = \mathbb{R}^n$. However, in view of the Nash embedding theorem [Nas56], we can find an isometric embedding of $M$ into $\mathbb{R}^d$. Given a $C^1$ vector field in $\mathbb{R}^d$ we can write $X = Y + Z$ where $Y$ is tangent to $M$ and $Z$ is perpendicular to $M$. Then $\text{div}_N Z = -\langle \nabla_x e_i, Z \rangle = -\langle A(e_i, e_i), Z \rangle$ where $A$ is the second fundamental form of $M$ in $\mathbb{R}^d$ and $\{e_i\}$ is an orthonormal basis for $T_x N$. Thus $V$ has a generalized mean curvature $\vec{H}$ in $\mathbb{R}^d$ and $P_M(\vec{H}) = \vec{H}$. By the $M = \mathbb{R}^n$ case, we know that $\vec{H} \perp T_x N$ and so $\vec{H} \perp T_x N$.

Finally, we will need the following version of the Michael-Simon Sobolev inequality, which is due to Hoffman and Spruck [HS74]:

**Theorem 2.16.** There exists some constant $c = c(k)$ such that if $(M, g)$ is a simply connected non-positively curved manifold and $V$ is an integral $k$ varifold in $M$ with generalized mean curvature $\vec{H}$ and with underlying rectifiable set $N$, then for every $h \in C^1_0(M)$ with $h \geq 0$ we have:

$$\left( \frac{1}{k} \int h^{k/(k-1)} d|V| \right)^{(k-1)/k} \leq c \int \left( |\nabla^N h| + h|\vec{H}| \right) d|V|$$

(2.17)

Remark 2.18. Both [MS73] and [HS74] discuss the Sobolev inequality for the case of submanifolds, the latter generalizing for submanifolds in curved spaces. In [Sim83, Sec. 18] the Sobolev inequality in the Euclidean case is generalized to varifolds. Although it is well known, we were not able to find an account in the literature for the corresponding generalization in the simply connected, non-positively curved case. We remark here that the proof in [Sim83, Sec. 18] easily generalizes to this case: Indeed, the radial field used in the proof is $X_x = h(x) \gamma(r) r\nabla r$ for $x \in M$ and $r$ the distance from a point $p \in M$. Computing the $N$- divergence of $X$ at $x$, all the terms will be exactly the same as in the Euclidean case, except for $h(x) \gamma(r) \text{Hess} \, \delta(r)(e_i, e_i)$ (where $\{e_i\}$ is an orthonormal basis for $T_x N$). By Hessian comparison and since $\gamma, h \geq 0$ this term is “more positive” than what it would have been in the Euclidean case. This inequality goes in the “right direction” and the rest of the proof remains the same.

3 Extinction Time Estimates:

In this section we will derive two estimates for the extinction time of Ilmanen’s underflow. Both estimates come from corresponding extinction estimates for Brakke flows. The first one is simpler and sharper, but works only in the Euclidean case, while the second one gives a worse constant, but works for mean curvature flows in arbitrary simply connected non-positively curved manifolds.

In the Euclidean case, we can use Huisken’s monotonicity formula (see [Hui90]) to derive an extinction time estimate for a Brakke flow. In the case of smooth flows this is standard (see [Man10, 3.2.16] for instance), and in the general case of Brakke flow the generalization is straightforward. We will include it here for the sake of completeness.

**Lemma 3.1.** Let $\{\nu_t\}_{t \geq 0}$ be an integral Brakke flow with $\nu_0(\mathbb{R}^n) < \infty$ then the flow becomes extinct in finite time $\tau$ (that is $\nu_t = 0$ for $t > \tau$) and:

$$\tau \leq \frac{\nu_0(\mathbb{R}^n)^2}{4\pi}$$

(3.2)

**Proof.** Take $t > 0$ at which the flow is integral (which happens a.e.) and not extinct, let $p$ be a point at which the approximate tangent space of $\nu_t$ exists and has multiplicity $\theta_0 \geq 1$ and let $s > t$. Then by the monotonicity formula we get:

$$\nu_0(\mathbb{R}^n)^2$$
\[
\int \frac{1}{(4\pi(s-t))^{k/2}} e^{-\frac{|x-p|^2}{4(s-t)}} \, dv_t(x) \leq \int \frac{1}{(4\pi s)^{k/2}} e^{-\frac{|x-p|^2}{4s}} \, dv_0(x) \leq \frac{1}{(4\pi s)^{k/2}} \nu_0(\mathbb{R}^n) \tag{3.3}
\]
and taking the limit \(s \to t\) we obtain:
\[
1 \leq \theta_0 \leq \frac{1}{(4\pi t)^{k/2}} \nu_0(\mathbb{R}^n) \tag{3.4}
\]
so \(t \leq \frac{\nu_0(\mathbb{R}^n)}{4\pi}\) holds for a.e. time prior to the extinction time and we are done. \(\square\)

**Corollary 3.5** (first undercurrent extinction). Given an enhanced motion \((T, \{\nu_t\}_{t \geq 0})\) starting from \(T_0\) we have:
\[
\text{spt} T \subseteq \left[0, \frac{M[T_0]^2}{4\pi}\right] \times \mathbb{R}^n \tag{3.6}
\]

**Proof.** By (1.1) we know that \(M[T_0] = \nu_0(\mathbb{R}^n)\). Moreover, by (1.2) we know that \(\nu_T \leq \nu_t\) and since the latter disappears when \(t > \frac{M[T_0]^2}{4\pi}\) we get the desired result. \(\square\)

The second estimate is based on the observation in [ES92], that Brakke’s clearing-out lemma [Brak78, 6.3] implies an extinction estimate. The point made here is that the clearing-out lemma is still valid for Brakke flows in simply connected, non-positively curved manifolds. To this end, it would be easier to follow the statement (and proof) from [Eck91].

**Lemma 3.7** (Clearing-Out). Given a natural number \(k\), there exist \(\epsilon_0, \alpha, c\) depending on it with the following property: For every smooth, simply connected, Riemannian manifold with non-positive curvature \((M, g)\) and a \(k\)-integral Brakke flow \(\{\nu_t\}_{t \geq 0}\) in \(M\) such that \(\nu_0(B(p, R)) \leq cR^k\) for \(\epsilon < \epsilon_0\) we have \(\nu_0(B(p, R/4)) = 0\) for some \(t \leq c\epsilon^\alpha R^2\).

**Remark 3.8.** The statement in [Eck91] is for co-dimension one smooth sub-manifolds of \(\mathbb{R}^n\) moving by mean curvature. The proof, however, generalizes to the situation above, using Hessian comparison, the appropriate Sobolev inequality (2.10), the perpendicularly of the mean curvature (2.14), and working with Brakke motion instead of smooth co-dimension one mean curvature flow. We will include the proof for the sake of completeness.

**Proof of Lemma 3.7.** Since the condition of being non-positively curved is scaling invariant, it will suffice to assume \(R = 1\). We deal with the case \(k = 2\) first: Take \(\phi(t, x) = (1 - (r^2 + 2kt)^2)^{\frac{1}{2}}\) where \(r = d(x, p)\) and let \(h(s) = (s)^{\frac{1}{2}}\). Note that \(h', h'' \geq 0\). Whenever \(B(\nu_t, \phi) \neq -\infty\) we have, by Brakke’s perpendicularly (2.14):
\[
\nabla \nu_t \nu_t(\phi) \leq \int -\phi H^2 + D\phi \cdot \mathbf{H} + \frac{\partial \phi}{\partial t} \, dv_t = \int -\phi H^2 - \sum_{i=1}^{k} \text{Hess}(\phi)(e_i, e_i) + \frac{\partial \phi}{\partial t} \, dv_t \tag{3.9}
\]
where \(\{e_i\}\) is an orthonormal basis for the approximate tangent space. Now, \(-\text{Hess}(\phi)(e_i, e_i) = -(h'')h'(r^2) + h'\text{Hess}(r^2)(e_i, e_i) \leq 2h'\) by Hessian comparison. We also have \(\frac{\partial \phi}{\partial t} = -2kh'\). Thus we conclude that:
\[
\nabla \nu_t \nu_t(\phi) \leq \int -\phi H^2 \, dv_t. \tag{3.10}
\]
Now, let \(t_0 \leq 1/16\) and suppose that for every \(t < t_0\) we have \(\nu_t(B(p, 1/4)) > 0\). If for such \(t' < t_0\) we have \(\nabla \nu_t \nu_t(\phi) > -\delta\) we must have that \(B(t', \phi) > -\delta\) and so
\[
\frac{1}{8} \int_{B(p, 1/2)} H^2 \, dv_t \leq \int \phi H^2 \, dv_t < \delta. \tag{3.11}
\]
In light of the Sobolev inequality (2.16), when $\delta$ is very small this will imply that $\nu_t(B(p, 1/2))$ is very big, which will contradict the fact that $\nu_t(\phi)$ is decreasing. More precisely (letting $t=t'$), $f(r) = \nu_t(B(p, r))$ is increasing and therefore differentiable a.e. Moreover, by choosing $h$ to be a radial cut-off function in the Sobolev inequality (2.16) and by using Cauchy-Schwartz we get the inequality

$$f(r)^{1/2} \leq c(f'(r) + (8\delta)^{1/2} f(r)^{1/2})$$

(3.12)

for $r < 1/2$. Since $f(r) > 0$ for $r > 1/4$, we obtain:

$$c^{-1}(1 - c(8\delta)^{1/2}) \leq \frac{d}{dr} \sqrt{f}$$

(3.13)

so $(c^{-1}(1 - c(8\delta)^{1/2}))^2 / 16 \leq \nu_t(B(p, 1/2))$. Thus, if $\delta < 1/(32c^2)$ and choosing $c_0 = 1/(1024c^2)$ we get $16c_0 \leq \nu_t(B(p, 1/2)) \leq 8\nu_t(\phi)$ which is a contradiction. Thus, choosing such $c_0$, for as long as $\nu_t(B(p, 1/4)) \neq 0$ we have $D_t \nu_t(\phi) \leq -\delta$ so $\nu_t(B(p, 1/4))$ must become 0 at some $t < \delta = 32c^2$. In the case $k \geq 3$ take $\phi(t, x) = (1 - (r^2 + (2k + m - 1))t)^m$ for some $m > 1$ such that $m(m-1) \leq 1$ and let $\psi(t, x) = (1 - (r^2 + (2k + m - 1))t)$ and $h(s) = s^m$ so that $\phi = h(\psi)$. Note that $\phi \in C^1_0$ and that it is smooth on the set $\{\phi > 0\}$, where we can compute, as before:

$$\sum_{i=1}^k \text{Hess}(\phi)(e_i, e_i) = \sum_{i=1}^k \left[-(h''(e_i(r^2)))^2 + h'(r^2)(e_i, e_i)\right] \leq$$

$$- m(m-1)\psi^{m-2}\nabla^N\psi^2 + 2mk\psi^{m-1} = - m(m-1)\psi^{m-2}\nabla^N\psi^2 + 2mk\psi^{m-1}$$

(3.14)

and $\frac{\phi}{\psi} = -m\psi^{m-1}(2k + m - 1)$. Thus, whenever $B(\nu_t, \phi) \neq -\infty$, and as the value $B(\nu_t, \phi)$ depends only on the set $\{\phi > 0\}$, we have:

$$D_t \nu_t(\psi^m) \leq -m(m-1) \int \psi^m H^2 + \psi^{m-1} + \psi^{m-2}\nabla^N\psi^2 d\nu_t.$$ 

(3.15)

Taking $p = \frac{2k}{k+2m}$, which is always smaller than 2 and can be made bigger than 1 by choosing $m$ to be very close to 1 (since $k \geq 3$), and letting $\lambda = m\frac{L_p}{L_p}$, using the $L^p$ version of the Sobolev inequality we get (again, when $B(\nu_t, \phi) \neq -\infty)$:

$$\left[\int \psi^m d\nu_t \right]^{1-2/(k+2m)} = \left[\int \psi^\lambda d\nu_t \right]^{(k-p)/k} \leq$$

$$c(k, m) \int \psi^{(\lambda-1)p}\nabla^N\psi^p + |H|^p \psi^{\lambda p} d\nu_t =$$

$$c(k, m) \int |\nabla^N\psi|^p \psi^m + |H|^p \psi^{m+1} \psi^{\lambda p} d\nu_t =$$

$$c(k, m) \int |\nabla^N\psi|^p \psi^m + |H|^p \psi^{m+1} \psi^{\lambda p} d\nu_t \leq$$

(3.16)

Finally, denoting by $f(t)$ the decreasing (hence differentiable a.e) function $\nu_t(\psi^m)$ we obtain:

$$f'(t) \leq -c(k, m)f(t)^{1-2/(k+2m)}$$

(3.17)
and so \( f(t) \) will have become zero at time
\[
t = c^{-1} f(0)^{2/(k+2m)} \leq c^{-1} \epsilon^{2/(k+2m)}.
\]

\[\square\]

**Corollary 3.19.** Given a natural number \( k \), there exist \( \epsilon_0, c \) depending on it with the following property: For every smooth, simply connected, Riemannian manifold with non-positive curvature \((M,g)\) and a \( k \)-integral Brakke flow \( \{\nu_t\}_{t \geq 0} \) in \( M \), if \( \nu_0(B(p,R)) \leq \epsilon R^k \) then \( \nu_{cR^2}(B(p,R/8)) = 0 \).

**Proof.** This follows since the time at which \( \nu_t(B(p,R/4)) = 0 \) can be made arbitrarily small compared to \( R^2 \) by choosing \( \epsilon \) very small in (3.7). After that time, by choosing an appropriate test function, we see \( \nu_t(B(p,R/8)) \) will remain zero for time proportional to \( R^2 \). \[\square\]

From this, by an argument from [ES92b] we get an extinction estimate:

**Corollary 3.20.** There exists \( c = c(k) \) such that every \( k \)-integral Brakke flow with finite mass \( \{\nu_t\}_{t \geq 0} \) in a simply connected, non-positively curved Riemannian manifold \((M,g)\) becomes extinct in time \( t \leq c \nu_0(M)^{2/k} \).

**Proof.** We can take \( R \) large enough such that \( \nu_0(M) = \epsilon R^k \) and so for that \( R \) we obtain that for every \( p \): \( \nu_{cR^2}(B(p,R/8)) = 0 \) so the flow gets extinct at time \( t \leq c R^2 = c \left( \epsilon^{-1} \nu_0(M) \right)^{2/k} \). \[\square\]

As before, this gives us undercurrent extinction:

**Corollary 3.21** (second undercurrent extinction). Given \( k \), there exists \( c \) such that if \((T,\{\nu_t\}_{t \geq 0})\) is an enhanced motion starting from a \( k \)-cycle of finite mass \( T_0 \) in a simply connected, non-positively curved Riemannian manifold we have:
\[
spt T \subseteq \left[ 0, cM[T_0]^2 \right] \times M \quad (3.22)
\]

## 4 Proof of Theorems B, C and A assuming D:

We first give a proof of the parabolic measure estimate for the space-time track of the MCF A assuming the validity of the parabolic co-area theorem D. The next section will deal with proving the latter.

**Proof of Thm. A assuming Thm. D.** By (3.5) we know that the undercurrent is supported on finite time. By (1.4) this implies that it has a finite Euclidean mass, and so we are in the position to use the parabolic co-area formula D. Letting \( \tau \) be the extinction time from Corollary (3.5) we see:
\[
\int_X \theta dH_{\text{par}}^{k+2} = c_1 \int_0^\tau \int_X \theta dH^k dH_{\text{par}}^2(t) = c_1 \int_0^\tau M[T_0] dH_{\text{par}}^2(t) \quad (4.1)
\]

and as \( M[T_t] \leq \nu_t(\mathbb{R}^n) \leq \nu_0(\mathbb{R}^n) \) we obtain:
\[
\int_X \theta dH_{\text{par}}^{k+2} \leq c_1 \int_0^\tau \nu_0(\mathbb{R}^n) dH_{\text{par}}^2(t) = c_1 \frac{\alpha(2)}{2\alpha(1)} \tau M[T_0] \quad (4.2)
\]

by (5.5). By (5.5) we have \( \tau \leq \frac{M[T_0]^2}{4\pi} \) and we are done. \[\square\]
Proof of Thm. B. By Corollary 3.21 we have $\tau \leq c M[T_0]^{\frac{k-1}{k}}$ where $\tau$ is the extinction time of the underflow and so $(\pi_x)_#(T_{[0,\tau)}) = (\pi_x)_#(T) = S$. By (1.3) we have $M[S] \leq M[T_0]^{1/2}$ so indeed:

$$M[S] \leq c M[T_0]^{\frac{k+1}{k}}$$  \hspace{1cm} (4.3)

The Euclidean case follows from using 3.5 instead of 3.21. $\Box$

Proof of Thm. C. If the extinction time was $\tau < \tau_0 = (M[F]/M[T_0])^2$, by (1.2) the undercurrent will be supported on $[0,\tau] \times M$, and so $(\pi_x)_#(T_{[0,\tau)}) = (\pi_x)_#(T) = S$. By (1.3) this implies that:

$$M[S] < M[T_0]^{1/2} = M[F]$$  \hspace{1cm} (4.4)

which would provide a filling of $T_0$ that is better than the optimal one. $\Box$

5 Parabolic GMT of Euclidean Rectifiable Sets:

This section is divided as follows: In Section 5.1 we will explore some basic properties of the parabolic Hausdorff measure, In Section 5.2 we will show that infinitesimally spatial Euclidean $(k+1)$-rectifiable sets (see Def. 5.13) with finite volume are negligible in the parabolic setting, in Section 5.3 we will deal with sets that have a time-like component a.e. and in Section 5.4 we will prove the parabolic co-area formula (Thm. D).

5.1 Basic properties and Examples:

Recall that we are considering the space $\mathbb{R}^{1,n} = \mathbb{R} \times \mathbb{R}^n$. Points $p \in \mathbb{R}^{1,n}$ will be denoted by $p = (t, x)$. The $\mathbb{R}$ factor is called the time direction and the $\mathbb{R}^n$ factor is called the space direction. On $\mathbb{R}^{1,n}$ we consider two metrics: the standard Euclidean one $d$ with corresponding Hausdorff measure $\mathcal{H}^*$ and the parabolic metric $d_{par}$ with corresponding Hausdorff measure $\mathcal{H}^*_{par}$. $diam$ will stand for the Euclidean diameter while $diam_{par}$ will stand for the parabolic one. Rectifiable will mean, unless otherwise stated, Euclidean-rectifiable.

Remark 5.1. By the Caratheodory criterion it is clear that $\mathcal{H}^*_{par}$ is Borel.

The first thing we will see is the following:

Lemma 5.2. Let $A$ be a $(k+1)$-rectifiable set in $\mathbb{R}^{1,n}$.

1. If $\mathcal{H}^{k+1}(A) = 0$ then $\mathcal{H}^{k+2}_{par}(A) = 0$.

2. If $\mathcal{H}^{k+1}(A) < \infty$ then $\mathcal{H}^{k+2}_{par}(A) < \infty$.

Proof. For (1), since $\mathcal{H}^{k+1}(A) = 0$, for every $\delta > 0$ and $\epsilon$ there is a $\delta$ cover of $A$ by cubes $\{C_i\}$, parallel to the axes with $\sum diam(C_i) < \epsilon$ (by enlarging an initial small covering and swallowing the constant multiplicative factor). Looking at $C_i$ in the parabolic metric, we see that $diam_{par}(C_i) \equiv \sqrt{diam(C_i)}$. Now, slice each $C_i$ to rectangular boxes $\{D_i\}$ of time-like sides of length $diam(C_i)^2$ and space-like sides of length $diam(C_i)$. This way, the parabolic diameter of the boxes will be smaller than the Euclidean diameter of the original cube and so they provide a parabolic $\delta$ cover of $A$. We will need $\frac{1}{diam(C_i)}$ such boxes to cover the cube $C_i$, and:

$$\sum_{j=1}^{\frac{1}{diam(C_i)}} diam_{par}(D_i)^{k+2} \leq \frac{1}{diam(C_i)}diam(C_i)^{k+2} = diam(C_i)^{k+2} \hspace{1cm} (5.3)$$
and so $H^k_{par,\delta}(A) < \epsilon$ and we are done. The proof of (2) is similar (see also (5.14)) □

Remark 5.4. The first part of the lemma allows us to measure the $(k+2)$-parabolic measure of a $(k+1)$-rectifiable set in a well defined manner.

The following example shows that in $\mathbb{R}^{1,0}$ there is no real difference between $H^2_{par}$ and the standard one dimensional Lebesgue measure $L^1$:

Example 5.5. On $\mathbb{R}^{1,0} \cong \mathbb{R}$ we have:

$$H^2_{par} = \frac{\alpha(2)}{2\alpha(1)} L^1. \quad (5.6)$$

Proof. It suffices to check it for intervals. Taking $\delta > 0$ and a parabolic $\delta$ cover of $[a,b]$ we have $diam_{Euc}(C_i) = diam_{par}(C_i)$ and so it is a Euclidean $\delta$ cover of $[a,b]$ and:

$$\alpha(2) \sum \left( \frac{diam_{par}(C_i)}{2} \right)^2 = \frac{\alpha(2)}{2\alpha(1)} \alpha(1) \sum diam_{Euc}(C_i) \quad (5.7)$$

□

More generally in the full dimensional case, the parabolic Hausdorff measure is identical, up to a constant, to the Lebesgue measure:

Lemma 5.8 (Top dimensional compatibility). There exist some constants $c_i = c_i(k) > 0$ ($i = 1, 2$) such that on $\mathbb{R}^{1,k}$ we have:

$$H^{k+2}_{par} = c_2 L^{k+1} = c_1 H^2_{par} \times H^k. \quad (5.9)$$

Proof. The second equality is clear from Example 5.5. For the first equality, note that by Lemma 5.2 $H^{k+2}_{par}$ is a Radon measure that is absolutely continuous w.r.t the Lebesgue measure. As both $H^{k+2}_{par}$ and $L^{k+1}$ are invariant under translations, by Radon-Nikodym we obtain:

$$H^{k+2}_{par} = c_2 L^{k+1} \quad (5.10)$$

for some $c_2 \geq 0$. In order to conclude, it will suffice to show that $H^{k+2}_{par}([0,1] \times [0,1]^k) > 0$. Otherwise, for every $\epsilon$, there would be a cover $\{C_i\}$ such that $\sum diam_{par}(C_i)^{k+2} < \epsilon$, but then by perhaps enlarging $C_i$ a little bit, we get $\tilde{C}_i$ of the form $\tilde{C}_i = [a_i, b_i] \times D_i$ and with

$$\sqrt{|b_i - a_i|} = diam_{par}(\tilde{C}_i) = diam_{par}(C_i) = diam(D_i) \quad (5.11)$$

but then:

$$\sum L^{k+1}(\tilde{C}_i) = \sum diam_{par}(\tilde{C}_i)^{k+2} < \epsilon \quad (5.12)$$

so the $\tilde{C}_i$ can not be a cover of the unit cube. □

The situation with lower dimensional parabolic Hausdorff measures is very different, as Example 5.33 (and indeed the entire Section 5.3) will indicate.

5.2 Infinitesimally spatial rectifiable sets:

This subsection deals with the validity of the parabolic co-area formula (Thm. D) for rectifiable sets which are infinitesimally spatial.

Definition 5.13. A $k+1$ rectifiable set $B$ in $\mathbb{R}^{1,n}$ is called infinitesimally spatial if for $H^{k+1}$ a.e $p \in B$ we have $\partial_t \perp T_p B$. 

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Lemma 5.14. If $B$ be is a $(k + 1)$-infinitesimally spatial set in $\mathbb{R}^{1,n}$ and $\mathcal{H}^{k+1}(B) < \infty$ then $\mathcal{H}_{par}^{k+2}(B) = 0$.

Proof. The argument is a refined version of the one in Lemma (5.2). Fix $\beta > 0$. For $l = 1, 2, \ldots$ let $B^l_\beta$ be the set of points at which the set it $\beta$ close to being spatial at scales $< 1/l$. More precisely, $B^l_\beta$ consists of those $(t, x)$ in $B$ such that for every $r < 1/l$ we have:

$$1 - \beta \leq \frac{\mathcal{H}^{k+1}(B \cap I^{n+1}(t,x), r)}{(2r)^{k+1}} \leq 1 + \beta$$

and also:

$$1 - 2\beta \leq \frac{\mathcal{H}^{k+1}(B \cap I^{n+1}(t,x) \cap ([t - \beta r, t + \beta r] \times \mathbb{R}^n))}{(2r)^{k+1}} \leq 1 + 2\beta$$

where $I^{n+1}(t,x)$ is the rectangle parallel to the axes with center $(t,x)$ and (Euclidean) sides $2r$. We claim that for $\beta$ sufficiently small, if $(t, x) \in B^2_\beta$ then for every $r < 1/(2l)$

$$B^l_\beta \cap I^{n+1}((t,x), r) \cap ([t - 3\beta r, t + 3\beta r] \times \mathbb{R}^n)^c = \emptyset$$

(5.17)

Otherwise, given $(s, y)$ in the left hand side of (5.17) we get that:

$$B \cap I^{n+1}((s,y), r) \cap ([s - \beta r, s + \beta r] \times \mathbb{R}^n) \subseteq B \cap I^{n+1}((t,x), 2r)$$

(5.18)

but:

$$(I^{n+1}((s,y), r) \cap ([s - \beta r, s + \beta r] \times \mathbb{R}^n)) \cap (I^{n+1}((t,x), 2r) \cap ([t - 2\beta r, t + 2\beta r] \times \mathbb{R}^n)) = \emptyset.$$  

(5.19)

For small $\beta$, this would contradict (5.15) for the point $(t, x)$ at scale $2r$, as by the above disjointness and by using (5.16) first for $(t, x)$ at scale $2r$ and then for $(s, y)$ at scale $r$ we get:

$$\mathcal{H}^{k+1}(B \cap I^{n+1}(t,x), 2r) \geq (1 - 2\beta) + (1 - 2\beta)/2^{k+1}$$

(5.20)

which is bigger than $(1 + \beta)$ when $\beta$ is very small.

Now, since $\mathcal{H}^{k+1}(B) < \infty$, by enlarging an efficient $\delta$-cover to become one with cubes (gaining a multiplicative factor) we see that there is a constant $A$ (independent of $\beta, l$) such that for every $\delta > 0$ there is a cover $\{C_i\}$ of $B^2_\beta$ by cubes such that $\sum diam(C_i)^{k+1} < A$ and $diam(C_i) < \delta$. Taking $\delta < 1/(4l)$ and such a good $\delta$-cover $\{C_i\}$, looking at $C_i$ in the parabolic metric, we see that $diam_{par}(C_i) \leq \sqrt{diam(C_i)}$. Now, slice each $C_i$ to rectangular boxes $\{D_i\}$ of time-like sides of length $diam(C_i)^2$ and the initial space-like sides. This way, the parabolic diameter of the boxes will be smaller than the Euclidean diameter of the original cube. We will need $\frac{1}{diam(C_i)}$ such rectangles and as in Lemma (5.2) we see that:

$$\sum_{j=1}^{\lfloor diam(C_i) \rfloor} diam_{par}(D_i^j)^{k+2} \leq \frac{1}{diam(C_i)} diam(C_i)^{k+2} = diam(C_i)^{k+1}$$

(5.21)

and so $\mathcal{H}_{par}^{k+2}(B^2_\beta) < A$. In fact, our situation is much better! Indeed, in light of (5.17), only $6\beta \lfloor \frac{1}{diam(C_i)} \rfloor$ out of the $\lfloor \frac{1}{diam(C_i)} \rfloor$ rectangles can contribute to covering $B^2_\beta$. For if $(t, x) \in C_i$ we have $C_i \subseteq I^{n+1}((t,x), diam(C_i))$ and so by (5.17):

$$C_i \cap B^l_\beta \subseteq [t - 3\beta diam(C_i), t + 3\beta diam(C_i)] \times \mathbb{R}^n$$

(5.22)
This gives
\[ H^{k+2}(B^t_\beta) < 6\beta A \] (5.23)
and so:
\[ H^{k+2}(\bigcup_{l=1}^{\infty} B^l_\beta) \leq 6\beta A \] (5.24)
and as \( H^{k+1}(B - \bigcup B^l_\beta) = 0 \) we get:
\[ H^{k+2}_{par}(B) \leq 6\beta A \] (5.25)
by the first part of Lemma (5.2). By the arbitrariness of \( \beta \) we are done. □

Computing the right hand side of the parabolic co-area formula in the infinitesimally spatial case is easier.

**Lemma 5.26.** For \( B \) infinitesimally spatial we have:
\[ \int_{\mathbb{R}^{1,n}} H^k(B_t) dH^2_{par}(t) = 0 \] (5.27)

**Proof.** This follows directly from the Euclidean co-area formula, as it implies that a.e. level set has \( H^k(B_t) = 0 \). □

### 5.3 The Time Advancing Part:

We first make several definitions:

**Definition 5.28.** A \( k+1 \) rectifiable set \( \mathcal{M} \) is called **time-advancing** if for \( H^{k+1} \) a.e. \( p \in \mathcal{M} \) we have that \( \partial_t \) is not perpendicular to \( T_p \mathcal{M} \).

For technical reasons, it will be easier to work with the definitions below:

**Definition 5.29.** A Lipschitz (w.r.t the standard Euclidean metric) map \( F : \mathbb{R}^{1,k} \supseteq A \rightarrow \mathbb{R}^{1,n} \) will be called **vertical** if \( \pi_t(F(t,x)) = t \) for every \( x \in \mathbb{R}^{k}, t \in \mathbb{R} \). Here \( \pi_t \) is the projection to the time factor.

**Definition 5.30.** A vertical map \( F : \mathbb{R}^{1,k} \supseteq A \rightarrow \mathbb{R}^{1,n} \) will be called \((M,m) \) Lipschitz if it is \( M \) Lipschitz in the Euclidean sense, and if its restriction to every time slice is \( m \) Lipschitz.

**Definition 5.31.** A set \( \mathcal{M} \subseteq \mathbb{R}^{1,n} \) is said to be \((1,k) \) **vertically rectifiable** if one can write \( \mathcal{M} = \mathcal{M}_0 \cup \bigcup_{i=1}^{\infty} \mathcal{M}_i \) where \( H^{k+1}(\mathcal{M}_0) = 0 \) and where \( \mathcal{M}_i = F_i(A_i), A_i \subseteq \mathbb{R}^{1,k} \) are measurable and \( F_i \) are \((M_i,m_i) \) Lipschitz.

The following lemma shows the equivalence between the geometric definition (5.28) and the technical definition (5.31):

**Lemma 5.32.** Let \( \mathcal{M} \) be a \( k+1 \) rectifiable in \( \mathbb{R}^{1,n} \). Then \( \mathcal{M} \) is time advancing iff it is vertically rectifiable.

**Proof.** Assume \( \mathcal{M} \) is time advancing. By rectifiability, write \( \mathcal{M} = \mathcal{M}_0 \cup \bigcup_{i=1}^{\infty} \mathcal{M}_i \) where \( H^{k+1}(\mathcal{M}_0) = 0 \) and \( \mathcal{M}_i \subseteq N_i \) where \( N_i \) is an embedded \( C^1 \) submanifold in \( \mathbb{R}^{1,n} \). We can therefore work on each \( \mathcal{M}_i \) separately. Given \( p \in \mathcal{M}_i \) with \( \partial_t \) not perpendicular to \( T_p \mathcal{M}_i \), this non-perpendicularity will also hold in an arbitrarily small ball around it in \( N_i \). In a yet smaller ball, we will be able to use the inverse function theorem with the first co-ordinate being \( t \). Restricting it a little further will give an \((M,m) \) Lipschitz map. By Vitali covering we can get
such a cover of the set and by the disjointedness of small balls, there are only countably many elements in that cover. The other implication is clear (and less important). □

At a stark contrast to the full dimensional case (see Lemma 5.38), the lower dimensional parabolic Hausdorff measures are far from the Euclidean ones, as the following example indicates:

**Example 5.33.** Let \( F : \mathbb{R}^{1,0} \supset [a, b] \to \mathbb{R}^{1,1} \) be vertical, Lipschitz and increasing. Then \( \mathcal{H}_{\text{par}}^2(F([a, b])) = \frac{a(2)}{a(1)} (b - a) \).

**Proof.** Let \( M \) be the Lipschitz constant of \( F \). For \( \delta_0 \) sufficiently small we have for every \( \delta < \delta_0 \) \( M \delta^2 < \delta \). Thus, for every parabolic \( \delta \)-cover of \([a, b] \subseteq \mathbb{R}^{1,0} \) by \( C_i \) and for every \( t, s \in C_i \) we have \( |t - s| < \delta^2 \) so \( |F(t) - F(s)| < \delta \). Thus, \( F(C_i) \) is a \( \delta \)-cover of \( F([a, b]) \). Similarly we see that \( \text{diam}_{\text{par}}(F(C_i)) = \text{diam}_{\text{par}}(C_i) \). Thus:

\[
\mathcal{H}_{\text{par}}^2(F([a, b])) \leq \mathcal{H}_{\text{par}}^2([a, b]) = \frac{a(2)}{a(1)} \mathcal{H}^1([a, b]). \tag{5.34}
\]

The other direction is trivial. □

The main difference between the parabolic and Euclidean Hausdorff measures is captured by the following volume dilation estimate.

**Lemma 5.35 (Basic volume estimate).** Let \( F : \mathbb{R}^{1,k} \supset A \to \mathbb{R}^{1,m} \) be an \((M, m)\) Lipschitz map. Then:

\[
\mathcal{H}_{\text{par}}^{k+2}(F(A)) \leq \max\{m, m^{k+2}\} \mathcal{H}_{\text{par}}^{k+2}(A). \tag{5.36}
\]

**Proof.** The proof is divided into four steps. In the first we consider what happens when \( \mathcal{H}_{\text{par}}^{k+2}(A) = 0 \), in the second and third we derive the weaker inequality:

\[
\mathcal{H}_{\text{par}}^{k+2}(F(A)) \leq \max\{1, m^{k+2}\} \mathcal{H}_{\text{par}}^{k+2}(A) \tag{5.37}
\]

and in the fourth we prove the strong inequality.

**Step 1:** If \( \mathcal{H}_{\text{par}}^{k+2}(A) = 0 \) then \( \mathcal{H}_{\text{par}}^{k+2}(F(A)) = 0 \): Let \( \{C_i\} \) be a parabolic \( \delta \)-cover of \( A \) then \( \text{diam}_{\text{par}}(F(C_i)) \leq M \cdot \text{diam}_{\text{par}}(C_i) \) from which it is clear.

**Step 2:** \((5.37)\) holds if \( A \) is a box, i.e. a set of the form \( I \times B \) for \( I \subseteq \mathbb{R} \) and \( B \subseteq \mathbb{R}^k \): Take \( \delta > 0 \) and let \( \{C_i\} \) be a parabolic \( \delta \)-cover of \( A \). Then \( F(C_i) \) is a cover of \( FA \) and for every \((t, x), (s, y) \in A \) we have:

\[
d_{\text{par}}(F(t, x), F(s, y)) \leq \max\{\sqrt{|t - s|}, m|x - y| + M|t - s|\} \tag{5.38}
\]

Or

\[
\text{diam}_{\text{par}}(F(C_i)) \leq \max\{\text{diam}_{\text{par}}(C_i), m \cdot \text{diam}_{\text{par}}(C_i) + M \cdot \text{diam}_{\text{par}}(C_i)^2\} \leq \text{diam}_{\text{par}}(C_i) \cdot \max\{1, m + M\delta\} \tag{5.39}
\]

thus, assuming \( \delta < 1 \) we obtain:

\[
\mathcal{H}_{\text{par}, \max\{1, m + M\}}^{k+2}(F(A)) \leq \min\{1, m + M\delta\}^{k+2} \mathcal{H}_{\text{par}, \delta}^{k+2}(A) \tag{5.40}
\]

and the desired result is obtained by taking \( \delta \to 0 \).
Step 3: In the general case, write \( A = \bigcup A_i \cup B \) where:

\[
A_i = \{ x \in A \; s.t \; \Theta_{Euc}^{k+1}(x,A,r) \geq \frac{99}{100} \; \text{for every} \; 0 < r < \frac{1}{i} \}
\]

and

\[
\Theta_{Euc}^{k+1}(x,A,r) = \frac{\mathcal{H}_{par}^{k+1}(A \cap B(x,r))}{\omega_{k+1} r^{k+1}}.
\]

Then \( \mathcal{H}_{par}^{k+2}(B) = \mathcal{L}^{k+1}(B) = 0 \) and \( A_i \not\supset A - B \). Thus \( \mathcal{H}_{par}^{k+2}(F(B)) = 0 \) and \( F(A) = \bigcup F(A_i) \cup F(B) \) and \( F(A_i) \not\supset F(A) \) up to measure 0. Thus, it suffices to show the desired weak inequality \( \mathcal{H}_{par}^{k+2} \) for \( A_i \). Take \( 0 < \delta < \frac{1}{i} \) and let \( \{ C_j \} \) be a parabolic \( \delta \) cover of \( A_i \) and assume further that \( C_j \subseteq A_i \). Then \( \{ F(C_j) \} \) is a cover of \( F(A_i) \) and for every \( (t,x), (s,y) \in C_j \) (assume w.l.o.g. \( s \leq t \)) we have \( |t - s| \leq \text{diam}_{par}(C_j)^2 \). Since both \( (t,x) \) and \( (s,y) \) are points of density at scale \( s \leq t \) there will be some \( s \leq r \leq t \) and points \( x_r, y_r \in \mathbb{R}^k \) with \( |x_r - x| \leq \text{diam}_{par}(C_j)^2, |y - y_r| \leq \text{diam}_{par}(C_j)^2 \) and such that \( (r,x_r), (r,y_r) \in A \). Thus, by the triangle inequality we get:

\[
d(\pi_x(F(t,x)), \pi_x(F_2(s,y))) \leq M \sqrt{|x - x_r|^2 + |t - r|^2 + m|x_r - y_r| + M \sqrt{|y - y_r|^2 + |s - r|^2} \leq
4M \cdot \text{diam}_{par}(C_j)^2 + m(|x - y| + 2\text{diam}_{par}(C_j)^2)
\]

and the proof continues as in step 2.

Step 4: For the improved estimate \( 5.36 \) note first that we may assume \( m < 1 \) or else it is equivalent to \( 5.37 \). Turning \( C_j \) into a product set does not increase the parabolic diameter (because of the “max”). Note further that if \( \text{diam}_{par}(C_j) > \max_{(t,x), (s,y) \in C_j} |x - y| \), it will be worthwhile to split \( C_j \) into smaller product sets with the same time-like factor. Thus, in the product case (step 2) we can assume \( C_j = [s, s + a^2] \times B \) where \( \text{diam}(B) = b \) and \( a \leq b \). But then, we can split \( [s, s + a^2] \) into \( \frac{1}{m} \) intervals \( I_{j,k} \), each of which of length \( m^2a^2 \) and consider the cover \( F(I_{j,k} \times B) \) of \( F(C_j) \). Note that:

\[
\text{diam}_{par}(F(I_{j,k} \times B)) \leq \max\{ma, mb\} = mb = m \cdot \text{diam}_{par}(C_j)
\]

Keeping in mind that we obtained \( \frac{1}{m} \) such split boxes gives the desired result. In the general case, we argue as in step 3. \( \square \)

Remark 5.45. Note that in the \( (M,m) \) Lipschitz setting, there is no effective extension theorem, in contrast to the Euclidean Lipschitz case, in which Kirszbraun’s extension theorem (see [Fed69 Sec. 2.10.43]) allows one to assume that the map is defined on the entire space (with the same Lipschitz constant). Thus the general assertion did not follow trivially from the one on boxes, and the third step was indeed needed.

Motivated by the above, we make the following definition.

Definition 5.46. Suppose \( F : \mathbb{R}^{1,k} \supset A \rightarrow \mathbb{R}^{1,n} \) is \( (M,m) \) Lipschitz. The horizontal differential of \( F \) at \( (t_0,x_0) \in A \): \( D^h F|_{(t_0,x_0)} \) is the differential of the map \( F_2(t_0,-) : A \cap \{ t = t_0 \} \rightarrow \mathbb{R}^n \). The horizontal Jacobian \( J^h F|_{(t_0,x_0)} \) is the Jacobian of that map.

Remark 5.47. Note that the above is well defined a.e. Indeed, by Fubini \( A \cap \{ t = t_0 \} \) is measurable for almost every \( t_0 \) and we can Lipschitz extend in every such level set. The resulting differential is independent of the extension at points of density.
**Definition 5.48.** An \((n + 1) \times (k + 1)\) matrix \(B\) is called **vertically linear** if it is of the form
\[
B = \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix}
\] (5.49)
for \(v \in \mathbb{R}^n\) and \(A\) an \(n \times k\) matrix.

**Remark 5.50.** The differential of an \((M, m)\) Lipschitz map is vertically linear.

The following three auxiliary lemmas concerning the parabolic Hausdorff measure have their direct Euclidean analogues (see [EG92, Sec 3.3.1 Lemmas 1-3]) with almost identical proofs. We will shortly remark about the (essentially cosmetic) differences.

**Lemma 5.51.** Suppose \(F : \mathbb{R}^{1,k} \to \mathbb{R}^{1,n}\) is vertically linear, then:
\[
\mathcal{H}_{par}^{k+2}(F(A)) = (J^h F)\mathcal{H}_{par}^{k+2}(A).
\] (5.52)

**Sketch.** Writing \(A = O \circ S\) for \(S : \mathbb{R}^k \to \mathbb{R}^k\) symmetric and \(O : \mathbb{R}^k \to \mathbb{R}^n\) orthogonal we see that:
\[
F = \begin{pmatrix} 1 & 0 \\ v & Id \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & O \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} = \tilde{N}\tilde{O}\tilde{S}
\] (5.53)

As both \(\tilde{N}\) and its inverse are \((|v| + 1, 1)\) Lipschitz, \(\tilde{N}^{-1}\) will preserve \(\mathcal{H}_{par}^{k+2}\) (the proof of \(5.36\) will work the same for \(\tilde{O} \circ \tilde{S}(A)\) as it is rectifiable). Thus
\[
\mathcal{H}_{par}^{k+2}(F(A)) = \mathcal{H}_{par}^{k+2}(\tilde{O}^* \tilde{N}^{-1} F(A)) = \mathcal{H}_{par}^{k+2}(\tilde{S}(A))
\] (5.54)
so by \(5.36\) we are back in the Euclidean case \(\square\).

**Lemma 5.55.** Suppose \(k \geq 1\), \(F : \mathbb{R}^{1,k} \supset A \to \mathbb{R}^{1,n}\) is \((M, m)\) Lipschitz for \(A \mathcal{H}_{par}^{k+2}\) measurable. Then:
1. \(F(A)\) is \(\mathcal{H}_{par}^{k+2}\) measurable.
2. The mapping \(y \mapsto \mathcal{H}^0(A \cap F^{-1}\{y\})\) is \(\mathcal{H}_{par}^{k+2}\) measurable on \(\mathbb{R}^{1,n}\).
3. \(\int_{\mathbb{R}^{1,n}} \mathcal{H}^0(A \cap F^{-1}\{y\}) d\mathcal{H}_{par}^{k+2} \leq \max\{m^k, m^{k+2}\} \mathcal{H}_{par}^{k+2}(A)\).

**Sketch.** The only difference here is that the standard Euclidean estimate
\[
\mathcal{H}^n(F(A)) \leq \text{Lip}(F)^n \mathcal{H}^n(A)
\] (5.56)
is replaced by the corresponding parabolic estimate for \((M, m)\) Lipschitz functions \(5.36\). \(\square\)

**Lemma 5.57.** Let \(F : \mathbb{R}^{1,k} \supset A \to \mathbb{R}^{1,n}\) be an \((M, m)\) Lipschitz map , let \(\alpha > 1\) and let \(B = \{x \in A \text{ s.t. } D^h F \text{ exists and } J^h F > 0\}\). Then there is a countable collection of Borel subsets \(\{E_j\}\) of \(\mathbb{R}^{1,k}\) such that:
1. \(B = \bigcup_{j=1}^{\infty} E_j\).
2. \(F|_{E_j}\) is one to one.
3. for each \(j\) there is a symmetric automorphism \(T_j : \mathbb{R}^k \to \mathbb{R}^k\) such that (identifying it with the corresponding vertical map from \(\mathbb{R}^{1,k}\) to \(\mathbb{R}^{1,k}\)):
   (a) \(F|_{E_j} \circ T_j^{-1}\) is \((M_j, \alpha)\) Lipschitz.
(b) $T_j \circ (F|_{E_j})^{-1}$ is $(M_j, \alpha)$ Lipschitz.

(c) $\alpha^{-k}|\det T_j| \leq J^h F|_{E_j} \leq \alpha^k|\det T_j|$. 

**Sketch.** This is also similar to the corresponding lemma [EC92 3.3.1.3]. This time, fixing $\epsilon > 0$ we let $C$ be a countable dense subset of $B$, $S$ be a countable dense subset of the symmetric automorphisms of $\mathbb{R}^k$ and $W$ be a countable dense subset of the vectors in $\mathbb{R}^{1,n}$ with first coordinate 1. Then for $c \in C, T \in S, w \in W$ and $i = 1, 2, 3, \ldots$ we define $E(c, T, w, i)$ to be the set of all $b \in B \cap B(c, 1/i)$ satisfying

$$\alpha^{-1} + \epsilon |Tv| \leq |DG_w(b)v| \leq (\alpha - \epsilon) |Tv|$$

for all $v \in \mathbb{R}^{1,k}$ and

$$|G_w(a) - G_w(b) - DG_w(b)(a - b)| \leq \epsilon|T(a - b)|$$

for all $a \in B \cap B(b, 2/i)$. Here

$$G_w(y) = F(y) - \langle y, \partial_i \rangle w.$$

Then for $b \in E(c, T, w, i)$ we have

$$J^h F = J^h G_w$$

and just like in [EC92] we obtain:

$$(\alpha^{-1} + \epsilon)^k \det(T) \leq J^h G_w(b) \leq (\alpha - \epsilon)^k \det(T)$$

Now, choose any $b \in B$ and write $DF(b) = O + S(b, \partial_i) u$ (confusing $O, S$ with the corresponding vertical maps) and choose $T \in S$ with $\text{Lip}(T \circ S^{-1}) \leq (\alpha^{-1} + 3\epsilon/2)^{-1}$ and $\text{Lip}(S \circ T^{-1}) \leq \alpha - 3\epsilon/2$ and $w \in W$ with $|w - u| \leq \epsilon|T|/2$ and select $i \in 1, 2, \ldots$ and $c \in C$ such that $|b - c| < 1/i$ and:

$$|F(a) - F(b) - DF(b)(a - b)| \leq \frac{\epsilon}{\text{Lip}(T^{-1})} |a - b|$$

for all $a \in B \cap B(b, 2/i)$. Then $b \in E(c, T, w, i)$. Renaming the sets $E(c, T, w, i) - E_j$ will yield, just like in [EC92] a partition $\{E_j\}$ of $B$ with $\text{Lip}(G_{w_j}|_{E_j} \circ T_j^{-1}) < \alpha$ and $\text{Lip}(T_j \circ (G_{w_j}|_{E_j})^{-1}) < \alpha$ with the desired property. Translating $G_{w_j}$ back to $F$ will therefore give corresponding $(|w_j| + 1, \alpha)$ Lipschitz maps and by (5.61) we are done $\square$.

We now come to the actual parabolic area formula for $(M, m)$ Lipschitz maps. Its proof is (again) identical to the one of the usual area formula (see [EC92 Section 3.3.2]), with the above lemmas replacing the Euclidean ones and by using the parabolic $(M, m)$ Lipschitz estimate (5.56).

**Theorem 5.64 (Parabolic area formula).** Let $F : \mathbb{R}^{1,k} \supset A \rightarrow \mathbb{R}^{1,n}$ be $(M, m)$ Lipschitz then:

$$\int_A J^h F dH^{k+2}_{\text{par}} = \int_{\mathbb{R}^{1,n}} \mathcal{H}^0(A \cap F^{-1}(y)) dH^{k+2}_{\text{par}}(y).$$

Moreover, if $g : \mathbb{R}^{1,n} \rightarrow \mathbb{R}$ is measurable:

$$\int_A (J^h F)(g \circ F) dH^{k+2}_{\text{par}} = \int_{\mathbb{R}^{1,n}} \mathcal{H}^0(A \cap F^{-1}(y)) g(y) dH^{k+2}_{\text{par}}(y).$$

**Sketch.** We use (5.57) instead of the usual Euclidean partition lemma. Then, in the original proof, it is crucial to obtain that $\alpha$ Lipschitz maps do not increase volume by much. We have the corresponding result using (5.59) controlling the horizontal Lipschitz-constant. The full Lipschitz constant is of no interest, as it is absent from the estimate. $\square$
5.4 Parabolic co-area Formula:

Proof of Thm. D. By section 5.2 we know that the contribution of the infinitesimally spatial part of $\mathcal{M}$ to both sides is zero. We can thus suppose that $\mathcal{M}$ is time advancing or equivalently, vertically rectifiable and in fact, that $\mathcal{M} = F(A)$ for $A \subseteq \mathbb{R}^{1,k}$ and $F : A \to \mathbb{R}^{1,n}$ ($M, m$) Lipschitz and one to one. But in this case, by (5.64):

$$\int_{\mathcal{M}} gd\mathcal{H}^{k+2}_{par} = \int_{A} (J^h F)(g \circ F)d\mathcal{H}^{k+2}_{par} =$$

(5.67)

by Lemma 5.8:

$$c_1 \int_{\mathbb{R}^{1,o}} \left( \int_{\mathcal{M}} (J^h F)(g \circ F)d\mathcal{H}^{k}_{par} \right) d\mathcal{H}^{2}_{par}(t) =$$

(5.68)

$$c_1 \int_{\mathbb{R}^{1,o}} \left( \int_{\mathcal{M}} gd\mathcal{H}^{k}_{par} \right) d\mathcal{H}^{2}_{par}(t)$$

where the last equality is by the Euclidean area formula. □

References

[Alm86] F. Almgren. Optimal isoperimetric inequality. Indi. Univ. Math. Jour., 35(3):451–547, 1986.

[Bra78] K.A Brakke. The motion of a surface by its mean curvature. Princeton Univ. Press and Univ. of Tokyo Press, 1978.

[CGG91] Y.G. Chen, Y. Giga, and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. J. Diff. Geom., 33:749–784, 1991.

[Eck91] K. Ecker. Local techniques for mean curvature flow. Proceedings of the Centre for Mathematics and its Applications, 26:107–119, 1991.

[EG92] L.C Evans and R.F Gariepy. Measure theory and fine properties of functions. CRC press, 1992.

[ES91] L.C Evans and J. Spruck. Motion of level sets by mean curvature i. J. Diff. Geom., 33:635–681, 1991.

[ES92a] L.C Evans and J. Spruck. Motion of level sets by mean curvature ii. Trans. Amer. Math. Soc., 330:321–332, 1992.

[ES92b] L.C Evans and J. Spruck. Motion of level sets by mean curvature iii. J. Geom. Analysis, 2(2):121–150, 1992.

[ES95] L.C Evans and J. Spruck. Motion of level sets by mean curvature vi. J. Geom. Analysis, 5(1):77–114, 1995.

[Fed69] H. Federer. Geometric measure theory. Springer-Verlag, New York, 1969.

[FF60] H. Federer and W. H. Fleming. Normal and integral currents. Ann. of math, 72:458–520, 1960.

[Gro83] M. Gromov. Filling riemannian manifolds. J. Diff. Geom., 18(1):1–147, 1983.
[GYu93] Y. Giga and K. Yamauchi. On a lower bound for the extinction time of surfaces moved by mean curvature. *Calc. Var. and PDE*, 1(4), 1993.

[HS74] D. Hoffman and J. Spruck. Sobolev and isoperimetric inequalities for riemannian submanifolds. *Comm. Pure. Appl. Math.*, 27(6):715–727, 1974.

[Hui90] G. Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Diff. Geom.*, 31:285–299, 1990.

[Ilm] T. Ilmanen. Singularities for mean curvature flow for surfaces. preprint. http://www.math.ethz.ch/~ilmanen/papers/sing.ps.

[Ilm94] T. Ilmanen. *Elliptic Regularization and Partial Regularity for Motion by Mean Curvatures*, volume 108(520) of *Mem. Amer. Math. Soc.* AMS, 1994.

[Kle92] B. Kleiner. An isoperimetric comparison theorem. *Invent. Math.*, 108(1):37–47, 1992.

[Man10] C. Mantegazza. *Lecture notes on mean curvature flow*, volume 290 of *Progress in mathematics*. Birkhauser, 2010.

[MS73] J.H Michael and L. Simon. Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$. *Comm. Pure. Appl. Math.*, 26:361–379, 1973.

[Nas56] J. Nash. The imbedding problem for riemannian manifolds. *Ann. of Math*, 63(1):29–63, 1956.

[Sch08] F. Schulze. Nonlinear evolution by mean curvature and isoperimetric inequalities. *J. Diff. Geom.*, 79(2):197–241, 2008.

[Sim83] L. Simon. *Lectures on geometric measure theory*, volume 3 of *Proc. Centre Math. Analysis*. Austr. Nat. Univ., 1983.

[Top98] P. Topping. Mean curvature ow and geometric inequalities. *J. Reine Angew. Math.*, 503:47–61, 1998.

[Whi97] B. White. Stratification of minimal surfaces, mean curvature flows, and harmonic maps. *J. Reine Angew Math*, 488:1–36, 1997.

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