UNIFORMLY S-NOETHERIAN RINGS

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Abstract. Let $R$ be a ring and $S$ be a multiplicative subset of $R$. Then $R$ is called a uniformly $S$-Noetherian ring if there exists $s \in S$ such that, for any ideal $I$ of $R$, $sI \subseteq K$ for some finitely generated subideal $K$ of $I$. We give the Eakin-Nagata-Formanek theorem for uniformly $S$-Noetherian rings. In addition, the uniformly $S$-Noetherian properties on several ring constructions are given. The notion of $u$-$S$-injective modules is also introduced and studied. Finally, we obtain the Bass-Papp theorem for uniformly $S$-Noetherian rings.

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1. Introduction. Throughout this article, $R$ is always a commutative ring with identity. For a subset $U$ of an $R$-module $M$, we denote by $\langle U \rangle$ the submodule of $M$ generated by $U$. A subset $S$ of $R$ is called a multiplicative subset of $R$ if $1 \in S$.

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and \( s_1 s_2 \in S \) for any \( s_1, s_2 \in S \). Recall from [1] that a ring \( R \) is called an \( S \)-Noetherian ring if, for every ideal \( I \) of \( R \), there exists a finitely generated subideal \( K \) of \( I \) such that \( sI \subseteq K \) for some \( s \in S \). Cohen’s theorem, the Eakin-Nagata theorem, and the Hilbert basis theorem for \( S \)-Noetherian rings are given in [1]. Many algebraists have paid considerable attention to the notion of \( S \)-Noetherian rings, especially in the \( S \)-Noetherian properties of ring constructions. In 2007, Liu [13] characterized when the generalized power series ring is an \( S \)-Noetherian ring under some additional conditions. In 2014, Lim and Oh [11] obtained some \( S \)-Noetherian properties on amalgamated algebras along an ideal. They [12] also studied \( S \)-Noetherian properties on the composite semigroup rings and the composite generalized series rings next year. In 2016, Hamed and Hizem [8] gave an \( S \)-version of the Eakin-Nagata-Formanek theorem for \( S \)-Noetherian rings in the case where \( S \) is finite or \( S \) is a countable set and ideals of \( R \) are comparable. Recently, Kim, Mahdou, and Zahir [10] gave a necessary and sufficient condition for a bi-amalgamation to inherit the \( S \)-Noetherian property. Some generalizations of the \( S \)-Noetherian ring can be found in [3, 9].

However, in the definition of \( S \)-Noetherian rings, the choice of \( s \in S \) such that \( sI \subseteq K \subseteq I \) with \( K \) finitely generated depends on the ideal \( I \). This dependence poses many obstacles to the further study of \( S \)-Noetherian rings. The main motivation of this paper is to introduce and study a “uniform” version of \( S \)-Noetherian rings. In fact, we say that a ring \( R \) is uniformly \( S \)-Noetherian if there exists \( s \in S \) such that for any ideal \( I \) of \( R \), \( sI \subseteq K \) for some finitely generated subideal \( K \) of \( I \). Trivially, we have the following implications:

\[
\text{Noetherian} \Rightarrow \text{uniformly } S\text{-Noetherian} \Rightarrow \text{ } S\text{-Noetherian}
\]

However, none of these implications is reversible. Some counterexamples are given in Example 2.2 and Example 2.6. We also consider the notion of uniformly \( S \)-Noetherian modules (see Definition 2.7), and then obtain the Eakin-Nagata-Formanek theorem for uniformly \( S \)-Noetherian modules (see Theorem 2.8), which generalizes some part of the result in [8, Corollary 2.1]. The \( S \)-extension property of \( S \)-Noetherian modules is given in Proposition 2.14. In Section 3, we mainly consider the uniformly \( S \)-Noetherian properties on some ring constructions, including trivial extensions, pullbacks, and amalgamated algebras along an ideal (see Proposition 3.1, Proposition 3.2, and Proposition 3.4). In Section 4, we first introduce the notion of \( u \)-\( S \)-injective modules \( E \) for which \( \text{Hom}_R(\_, E) \) preserves \( u \)-\( S \)-exact sequences (see Definition 4.2), and then characterize them by uniformly \( S \)-torsion properties of the “\( \text{Ext} \)” functor in Theorem 4.3. The Baer’s criterion for \( u \)-\( S \)-injective modules is given in Proposition 4.9. Finally, we obtain the Bass-Papp theorem for uniformly \( S \)-Noetherian rings as follows (see Theorem 4.10):

**Theorem.** Let \( R \) be a ring and \( S \) be a multiplicative subset of \( R \) consisting of non-zero-divisors. Then the following conditions are equivalent:

1. \( R \) is uniformly \( S \)-Noetherian.
2. Every direct sum of injective modules is \( u \)-\( S \)-injective.
3. Every direct union of injective modules is \( u \)-\( S \)-injective.
2. Uniformly $S$-Noetherian rings and uniformly $S$-Noetherian modules. Let $R$ be a ring and $S$ be a multiplicative subset of $R$. Recall from [1] that $R$ is called an $S$-Noetherian ring (resp., $S$-PIR) if, for any ideal $I$ of $R$, there exist $s \in S$ and a finitely (resp., principally) generated subideal $K$ of $I$ such that $sI \subseteq K$. Note that the choice of $s$ is determined by the ideal $I$. Now we introduce some “uniform” versions of $S$-Noetherian rings and $S$-PIRs.

**Definition 2.1.** Let $R$ be a ring and $S$ be a multiplicative subset of $R$.

1. $R$ is called a uniformly $S$-Noetherian ring ($u$-$S$-Noetherian ring for short) if for every ideal $I$ of $R$, there exist $s \in S$ such that $sI \subseteq K$ for some finitely generated subideal $K$ of $I$.

2. $R$ is called a uniformly $S$-principal ideal ring ($u$-$S$-PIR for short) if there exists $s \in S$ such that for any ideal $I$ of $R$, $sI \subseteq (a)$ for some $a \in I$.

If the element $s$ can be chosen to be the identity $1$ in the definition of uniformly $S$-Noetherian rings, then uniformly $S$-Noetherian rings are exactly Noetherian rings. Thus every Noetherian ring is uniformly $S$-Noetherian. However, the converse is not generally true, as the next example shows.

**Example 2.2.** Let $R = \prod_{i=1}^{\infty} \mathbb{Z}_2$ be the countably infinite direct product of the field $\mathbb{Z}_2$. Then $R$ is not Noetherian. Denoted by $e_i$ the $i$th unit vector: Its $i$th component is 1, and others are 0. Set $S := \{ 1, e_i \mid i \geq 1 \}$. Then $R$ is uniformly $S$-Noetherian. Indeed, let $I$ be an ideal of $R$. If every element in $I$ has the first component 0, then we have $e_1I = 0$. Otherwise $e_1I = e_1R$. So $e_1I$ is principally generated. Consequently $R$ is a uniformly $S$-PIR, and thus is uniformly $S$-Noetherian.

Let $R$ be a ring, $M$ be an $R$-module, and $S$ be a multiplicative subset of $R$. For any $s \in S$, consider a multiplicative subset $\langle s \rangle := \{ 1, s, s^2, \ldots \}$ of $S$. We denote by $M_s$ the localization of $M$ at $\langle s \rangle$. Certainly $M_s \cong M \otimes_R R_s$.

**Lemma 2.3.** Let $R$ be a ring and $S$ be a multiplicative subset of $R$. If $R$ is a uniformly $S$-Noetherian ring (resp., uniformly $S$-PIR), then there exists $s \in S$ such that $R_s$ is a Noetherian ring (resp., PIR). Consequently $R_S$ is a Noetherian ring (resp., PIR).

**Proof.** Since $R$ is uniformly $S$-Noetherian, there exists $s \in S$ such that for any ideal $I$ of $R$, there exists a finitely (resp., principally) generated subideal $K$ of $I$ such that $sI \subseteq K$. Let $J$ be an ideal of $R_s$. Then there exists an ideal $I'$ of $R$ such that $J = I'_s$, and hence $sI' \subseteq K'$ for some finitely (resp., principally) generated subideal $K'$ of $I'$. So $J = I'_s = K'_s$ is a finitely (resp., principally) generated ideal of $R_s$. Consequently $R_s$ is a Noetherian ring (resp., PIR). Note that $R_S$ is a localization of $R_s$. So $R_S$ is also a Noetherian ring (resp., PIR).

**Remark 2.4.** The converse of Lemma 2.3 is not true in general. Indeed, let $p$ be a prime number, $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at $\mathbb{Z} \setminus p\mathbb{Z}$, and $x$ be an indeterminate
over the rational field \( \mathbb{Q} \). Set \( R := \mathbb{Z}_{(p)} + x\mathbb{Q}[[x]] \). Then \( pR \) is the maximal ideal of \( R \). Set \( S := \{1, p, p^2, \ldots\} \). Denote by \( R_p \) the localization of \( R \) at \( S \). Then \( R_p \cong \mathbb{Q}[[x]] \), which is a Dedekind domain. We claim that \( R \) is not \( u\)-\( S \)-Noetherian. Indeed, let \( I = \langle x, \frac{2}{7}, \frac{x}{p^2}, \ldots \rangle \). One can easily verify that \( I \) is not \( S \)-finite, and thus \( R \) is not \( S \)-Noetherian. Therefore, \( R \) is not \( u\)-\( S \)-Noetherian.

A multiplicative subset \( S \) of \( R \) is said to satisfy the maximal multiple condition if there exists an \( s \in S \) such that \( t|s \) for each \( t \in S \). Both finite multiplicative subsets and multiplicative subsets consisting of units satisfy the maximal multiple condition.

**Proposition 2.5.** Let \( R \) be a ring and \( S \) be a multiplicative subset of \( R \) satisfying the maximal multiple condition. Then the following statements are equivalent.

1. \( R \) is a uniformly \( S \)-Noetherian ring (resp., uniformly \( S \)-PIR).

2. \( R \) is an \( S \)-Noetherian ring (resp., \( S \)-PIR).

3. \( R_S \) is a Noetherian ring (resp., \( S \)-PIR).

**Proof.** (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) These are trivial.

(3) \( \Rightarrow \) (1) Suppose \( R_S \) is a Noetherian ring (resp., \( S \)-PIR). Let \( s \in S \) such that \( t|s \) for any \( t \in S \). Then for any ideal \( I \) of \( R_1 \), \( I_2 \) is a finitely generated (resp., principally generated) ideal of \( R_1 \). Let \( I_2 = \langle r_1, \ldots, r_n \rangle \) (resp., \( I_2 = \langle r \rangle \)) with every \( r_i \in I \) and \( t_i \in S \) (resp., \( r \in I \) and \( t \in S \)). It is easy to see that \( sI \subseteq \langle r_1, \ldots, r_n \rangle \subseteq I \) (resp., \( sI \subseteq \langle r \rangle \subseteq I \)). Consequently, \( R \) is a uniformly \( S \)-Noetherian ring (resp., uniformly \( S \)-PIR). \( \square \)

The following example shows that \( S \)-Noetherian rings are not uniformly \( S \)-Noetherian in general.

**Example 2.6.** Let \( R = k[x_1, x_2, \ldots] \) be the polynomial ring in countably infinite indeterminates over a field \( k \). Set \( S := R \setminus \{0\} \). Then \( R \) is an \( S \)-Noetherian ring. However, \( R \) is not uniformly \( S \)-Noetherian.

**Proof.** Certainly, \( R \) is an \( S \)-Noetherian ring. Indeed, let \( I \) be a non-zero ideal of \( R \). Take \( 0 \neq s \in I \). Then \( sI \subseteq sR \subseteq I \). Thus \( I \) is \( S \)-principally generated. So \( R \) is an \( S \)-PIR, and thus an \( S \)-Noetherian ring.

We claim that \( R \) is not uniformly \( S \)-Noetherian. On the contrary, suppose that \( R \) is uniformly \( S \)-Noetherian. Then \( R_s \) is a Noetherian ring for some \( s \in S \) by Lemma 2.3. If \( n \) is the smallest number such that \( x_m \) does not divide any monomial of \( s \) for any \( m \geq n \). Then \( R_s \cong T[x_n, x_2, \ldots, x_{n-1}] \). Obviously \( R_s \cong T[x_n, x_2, \ldots] \) is not Noetherian, since the ideal generated by \( \{x_n, x_{n+1}, \ldots\} \) is not a finitely generated ideal of \( T[x_n, x_{n+1}, \ldots] \). So \( R \) is not uniformly \( S \)-Noetherian. \( \square \)
Recall from [1] that an $R$-module $M$ is called an $S$-Noetherian module if every submodule of $M$ is $S$-finite, i.e., for every submodule $N$ of $M$ there exist $s \in S$ and a finitely generated $R$-module $F$ such that $sN \subseteq F \subseteq N$. Note that the choice of $s$ is determined by the submodule $N$. The rest of this section mainly studies a “uniform” version of $S$-Noetherian modules. Let $\{M_j\}_{j \in \Gamma}$ be a family of $R$-modules and $N_j$ be the submodule of $M_j$ generated by $\{m_{i,j}\}_{i \in \Lambda_j} \subseteq M_j$ for each $j \in \Gamma$. Recall from [17] that a family of $R$-modules $\{M_j\}_{j \in \Gamma}$ is uniformly $S$-generated (with respect to $s$) by $\{\{m_{i,j}\}_{i \in \Lambda_j}\}_{j \in \Gamma}$ if $sM_j \subseteq N_j$ for each $j \in \Gamma$, where $N_j = \langle \{m_{i,j}\}_{i \in \Lambda_j} \rangle$. We say that a family of $R$-modules $\{M_j\}_{j \in \Gamma}$ is uniformly $S$-finite (with respect to $s$) if the set $\{m_{i,j}\}_{i \in \Lambda_j}$ can be chosen as a finite set for each $j \in \Gamma$.

**Definition 2.7.** Let $R$ be a ring and $S$ be a multiplicative subset of $R$. An $R$-module $M$ is called a uniformly $S$-Noetherian $R$-module if the set of all submodules of $M$ is uniformly $S$-finite.

Let $R$ be a ring and $S$ be a multiplicative subset of $R$. Recall from [17] that an $R$-module $T$ is called a uniformly $S$-torsion module if there exists $s \in S$ such that $sT = 0$. Obviously, uniformly $S$-torsion modules are uniformly $S$-Noetherian. Thus a ring $R$ is uniformly $S$-Noetherian if and only if it is uniformly $S$-Noetherian as an $R$-module. It is well known that an $R$-module $M$ is Noetherian if and only if $M$ satisfies the ascending chain condition on submodules, if and only if $M$ satisfies the maximal condition (see [14]). In 2016, Hamed et al. [8] obtained an $S$-version of this result under the condition that $S$ is a finite set, and called it the $S$-version of the Eakin-Nagata-Formanek theorem. Next, we will give a uniform $S$-version of the Eakin-Nagata-Formanek theorem for any multiplicative subset $S$ of $R$.

First, we recall from [8, Definition 2.1] some modified notions of $S$-stationary ascending chains of $R$-modules and $S$-maximal elements of a family of $R$-modules. Let $R$ be a ring, $S$ be a multiplicative subset of $R$, and $M$ be an $R$-module. Denote by $M^\bullet$ an ascending chain $M_1 \subseteq M_2 \subseteq \cdots$ of submodules of $M$. An ascending chain $M^\bullet$ is said to be stationary with respect to $s$ if there exists $k \geq 1$ such that $sM_n \subseteq M_k$ for any $n \geq k$. Let $\{M_i\}_{i \in \Lambda}$ be a family of submodules of $M$. We say that an $R$-module $M_0 \in \{M_i\}_{i \in \Lambda}$ is maximal with respect to $s$ if whenever $M_0 \subseteq M_j$ for some $M_j \in \{M_i\}_{i \in \Lambda}$, we have $sM_j \subseteq M_0$.

**Theorem 2.8.** (Eakin-Nagata-Formanek theorem for uniformly $S$-Noetherian modules) Let $R$ be a ring, $S$ be a multiplicative subset of $R$, and $M$ be an $R$-module. Then the following conditions are equivalent:

1. $M$ is uniformly $S$-Noetherian.

2. There exists $s \in S$ such that any ascending chain of submodules of $M$ is stationary with respect to $s$.

3. There exists $s \in S$ such that any nonempty subset of submodules of $M$ has a maximal element with respect to $s$.
Proof.  (1) $\Rightarrow$ (2) Let $M_1 \subseteq M_2 \subseteq \cdots$ be an ascending chain of submodules of $M$. Set $M_0 := \bigcup_{i=1}^{\infty} M_i$. Then there exist an element $s \in S$ and a finitely generated submodule $N_i$ of $M_i$ such that $sM_i \subseteq N_i$ for every $i \geq 0$. Since $N_0$ is finitely generated, there exists $k \geq 1$ such that $N_0 \subseteq M_k$. Thus $sM_0 \subseteq M_k$. So $sM_n \subseteq M_k$ for any $n \geq k$.

(2) $\Rightarrow$ (3) Let $\Gamma$ be a nonempty subset of submodules of $M$. Assume, on the contrary, that (3) does not hold. Take any $M_1 \in \Gamma$. Then $M_1$ is not a maximal element with respect to $s$ for any $s \in S$. Thus there is $M_2 \in \Gamma$ such that $M_1 \subseteq M_2$, but $sM_2 \not\subseteq M_1$. Since $M_2$ is not a maximal element with respect to $s$, there exists $M_3 \in \Gamma$ such that $M_2 \subseteq M_3$, but $sM_3 \not\subseteq M_2$. Similarly we can get an ascending chain $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \cdots$ such that $sM_{n+1} \not\subseteq M_n$ for any $n \geq 1$. Obviously, this ascending chain is not stationary with respect to any $s \in S$.

(3) $\Rightarrow$ (1) Let $N$ be a submodule of $M$ and $s \in S$ be the element in (3). Set $\Gamma = \{A \subseteq N \mid$ there exists a finitely generated submodule $F_A$ of $A$ which satisfies $sA \subseteq F_A\}$. Since $0 \in \Gamma$, $\Gamma$ is not empty. Thus $\Gamma$ has a maximal element $A$. If $A \neq N$, then there is $x \in N \setminus A$. Since $F_1 := F_A + Rx$ is a finitely generated submodule of $A_1 := A + Rx$ such that $sA_1 \subseteq F_1$, we have $F_1 \in \Gamma$, which contradicts the choice of maximality of $A$.

\begin{corollary}
Let $R$ be a ring and $S$ be a multiplicative subset of $R$. Then the following conditions are equivalent:

(1) $R$ is uniformly $S$-Noetherian.

(2) There exists $s \in S$ such that any ascending chain of ideals of $R$ is stationary with respect to $s$.

(3) There exists $s \in S$ such that any nonempty subset of ideals of $R$ has a maximal element with respect to $s$.

We can recover the following result by means of Proposition 2.5.

\begin{corollary} ([8, Corollary 2.1])
Let $R$ be a ring and $S$ be a multiplicative subset of $R$ satisfying maximal condition. Then the following conditions are equivalent:

(1) $R$ is an $S$-Noetherian ring.

(2) Every ascending chain of ideals of $R$ is $S$-stationary.

(3) Every nonempty set of ideals of $R$ has an $S$-maximal element.

Recall from [17] that an $R$-sequence $M \xrightarrow{f} N \xrightarrow{g} L$ is said to be $u$-$S$-exact if there is $s \in S$ such that $s\ker(g) \subseteq \text{Im}(f)$ and $s\text{Im}(f) \subseteq \ker(g)$. An $R$-homomorphism $f : M \to N$ is called a $u$-$S$-monomorphism (resp., $u$-$S$-epimorphism, $u$-$S$-isomorphism) if $0 \to M \xrightarrow{f} N$ (resp., $M \xrightarrow{f} N \to 0$, $0 \to M \xrightarrow{f} N \to 0$) is $u$-$S$-exact. It is easy to verify that an $R$-homomorphism $f : M \to N$ is a $u$-$S$-monomorphism (resp., $u$-$S$-epimorphism) if and only if $\ker(f)$ (resp., $\text{Coker}(f)$) is a uniformly $S$-torsion module.
LEMMA 2.11. ([17, Proposition 2.8]) Let $R$ be a ring, $S$ be a multiplicative subset of $R$, and $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a $u$-$S$-exact sequence of $R$-modules. Then $B$ is uniformly $S$-torsion if and only if $A$ and $C$ are uniformly $S$-torsion.

LEMMA 2.12. Let $R$ be a ring and $S$ be a multiplicative subset of $R$. Let

\[
\begin{array}{cccccc}
0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\
\downarrow^{i_A} & & \downarrow^{i_B} & & \downarrow^{i_C} & & \\
0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0
\end{array}
\]

be a commutative diagram of $R$-modules with exact rows, where $i_A, i_B,$ and $i_C$ are embedding maps. Suppose $s_A A_2 \subseteq A_1$ and $s_C C_2 \subseteq C_1$ for some $s_A, s_C \in S$. Then $s_A s_C B_2 \subseteq B_1$.

Proof. Let $x \in B_2$. Then $\pi_2(x) \in C_2$. Thus $s_C \pi_2(x) = \pi_2(s_C x) \in C_1$. So we have $\pi_1(y) = \pi_2(y) = \pi_2(s_C x)$ for some $y \in B_1$. Thus $s_C x - y = a_2$ for some $a_2 \in A_2$. It follows that $s_A s_C x = s_A y + s_A a_2 \in B_1$. Consequently $s_A s_C B_2 \subseteq B_1$.

LEMMA 2.13. Let $R$ be a ring, $S$ be a multiplicative subset of $R$, and $0 \to A \to B \to C \to 0$ be an exact sequence of $R$-modules. Then $B$ is uniformly $S$-Noetherian if and only if $A$ and $C$ are uniformly $S$-Noetherian.

Proof. It is easy to verify that if $B$ is uniformly $S$-Noetherian, then so are $A$ and $C$. Suppose that $A$ and $C$ are uniformly $S$-Noetherian. Let $\{B_i\}_{i \in \Lambda}$ be the set of all submodules of $B$. Then there exists $s_1 \in S$ such that $s_1 (A \cap B_i) \subseteq K_i \subseteq A \cap B_i$ for some finitely generated $R$-module $K_i$ and any $i \in \Lambda$, since $A$ is uniformly $S$-Noetherian. There also exists $s_2 \in S$ such that $s_2 (B_i + A)/A \subseteq L_i \subseteq (B_i + A)/A$ for some finitely generated $R$-module $L_i$ and any $i \in \Lambda$, since $C$ is uniformly $S$-Noetherian. Let $N_i$ be the finitely generated submodule of $B_i$ generated by the finite generators of $K_i$ and finite preimages of generators of $L_i$. Consider the following natural commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & K_i & \longrightarrow & N_i & \longrightarrow & L_i & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A \cap B_i & \longrightarrow & B_i & \longrightarrow & (B_i + A)/A & \longrightarrow & 0
\end{array}
\]

Set $s := s_1 s_2 \in S$. We have $s B_i \subseteq N_i \subseteq B_i$ by Lemma 2.12. So $B$ is uniformly $S$-Noetherian.

PROPOSITION 2.14. Let $R$ be a ring, $S$ be a multiplicative subset of $R$, and $0 \to A \to B \to C \to 0$ be a $u$-$S$-exact sequence of $R$-modules. Then $B$ is uniformly $S$-Noetherian if and only if $A$ and $C$ are uniformly $S$-Noetherian.
Proof. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a $u$-$S$-exact sequence. Then there exists $s \in S$ such that $s\ker(g) \subseteq \image(f)$ and $s\image(f) \subseteq \ker(g)$. Note that $\image(f)/s\ker(g)$ and $\ker(g)/s\image(f)$ are uniformly $S$-torsion. If $\image(f)$ is uniformly $S$-Noetherian, then the submodule $s\image(f)$ of $\image(f)$ is uniformly $S$-Noetherian. Thus $\ker(g)$ is uniformly $S$-Noetherian by Lemma 2.13. Similarly, if $\ker(g)$ is uniformly $S$-Noetherian, then $\image(f)$ is uniformly $S$-Noetherian. Consider the following three exact sequences:

$$0 \to \ker(g) \to B \to \image(g) \to 0,$$

$$0 \to \image(g) \to C \to \coker(g) \to 0,$$

$$0 \to \ker(f) \to A \to \image(f) \to 0$$

with $\ker(f)$ and $\coker(g)$ uniformly $S$-torsion. It is easy to verify that $B$ is uniformly $S$-Noetherian if and only if $A$ and $C$ are uniformly $S$-Noetherian by Lemma 2.13. \hfill $\square$

**Corollary 2.15.** Let $R$ be a ring, $S$ be a multiplicative subset of $R$, and $M \xrightarrow{f} N$ be a $u$-$S$-isomorphism of $R$-modules. If one of $M$ and $N$ is uniformly $S$-Noetherian, then the other is too.

**Proof.** This follows from Proposition 2.14, since $0 \to M \xrightarrow{f} N \to 0 \to 0$ is a $u$-$S$-exact sequence. \hfill $\square$

Let $p$ be a prime ideal of $R$. We say that an $R$-module $M$ is uniformly $p$-Noetherian if $M$ is uniformly $(R \setminus p)$-Noetherian. The next result gives a local characterization of Noetherian modules.

**Proposition 2.16.** Let $R$ be a ring and $M$ be an $R$-module. Then the following conditions are equivalent:

1. $M$ is Noetherian.
2. $M$ is uniformly $p$-Noetherian for any $p \in \spec(R)$.
3. $M$ is uniformly $m$-Noetherian for any $m \in \max(R)$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) These are trivial.

(3) $\Rightarrow$ (1) Let $N$ be a submodule of $M$. Then for each $m \in \max(R)$, there exist $s^m \in R \setminus m$ and a finitely generated submodule $F^m$ of $N$ such that $s^m N \subseteq F^m$. Since $\{s^m | m \in \max(R)\}$ generates $R$, there exist finite elements $\{s^{m_1}, \ldots, s^{m_n}\}$ such that $N = \langle s^{m_1}, \ldots, s^{m_n} \rangle N \subseteq F^{m_1} + \cdots + F^{m_n} \subseteq N$. So $N = F^{m_1} + \cdots + F^{m_n}$. It follows that $N$ is finitely generated, and thus $M$ is Noetherian. \hfill $\square$

**Corollary 2.17.** Let $R$ be a ring. Then the following conditions are equivalent:

1. $R$ is a Noetherian ring.
2. $R$ is a uniformly $p$-Noetherian ring for any $p \in \spec(R)$.
3. $R$ is a uniformly $m$-Noetherian ring for any $m \in \max(R)$.
3. Uniformly $S$-Noetherian properties on some ring constructions. In this section, we mainly consider the uniformly $S$-Noetherian properties on trivial extensions, pullbacks, and amalgamated algebras along an ideal. More on these ring constructions can be found in [2, 11].

Let $R$ be a commutative ring and $M$ be an $R$-module. Then the trivial extension of $R$ by $M$, denoted by $R(+)M$, is equal to $R \oplus M$ as $R$-modules with coordinate-wise addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. It is easy to verify that $R(+)M$ is a commutative ring with identity $(1,0)$. Let $S$ be a multiplicative subset of $R$. Then it is easy to verify that $S(+)M = \{(s,m) \mid s \in S, m \in M\}$ is a multiplicative subset of $R(+)M$. Now we give a uniformly $S$-Noetherian property on the trivial extension.

**Proposition 3.1.** Let $R$ be a commutative ring, $S$ be a multiplicative subset of $R$, and $M$ be an $R$-module. Then $R(+)M$ is a uniformly $S(+)M$-Noetherian ring if and only if $R$ is a uniformly $S$-Noetherian ring and $M$ is a uniformly $S$-Noetherian $R$-module.

**Proof.** Note that we have an exact sequence of $R(+)M$-modules:

$$0 \to 0(+)M \xrightarrow{i} R(+)M \xrightarrow{\pi} R \to 0.$$  

Suppose that $R(+)M$ is a uniformly $S(+)M$-Noetherian ring. Let $\{I_i\}_{i \in \Lambda}$ be the set of all ideals of $R$. Then $\{I_i(+)M\}_{i \in \Lambda}$ is a set of ideals of $R(+)M$. So there exist $(s,m) \in S(+)M$ and a finitely generated subideal $O_i$ of $I_i(+)M$ for each $i$ such that $(s,m)I_i(+)M \subseteq O_i$. Thus $sI_i \subseteq \pi(O_i) \subseteq I_i$. Say that $O_i$ is generated by $\{(r_{1,i},m_{1,i}),\ldots,(r_{n_i,i},m_{n_i,i})\}$. Then it is easy to verify that $\pi(O_i)$ is generated by $\{r_{1,i},\ldots,r_{n_i,i}\}$. So $R$ is a uniformly $S$-Noetherian ring. Let $\{M_i\}_{i \in \Gamma}$ be the set of all submodules of $M$. Then $\{0(+)M_i\}_{i \in \Gamma}$ is a set of ideals of $R(+)M$. Thus there exist $(s',m') \in S(+)M$ and a finitely generated subideal $O_i'$ of $0(+)M_i$ for each $i$ such that $(s',m')0(+)M_i \subseteq O_i'$. So $s'M_i \subseteq N_i \subseteq M_i$, where $O_i' = 0(+)N_i$. Say that $O_i'$ is generated by $\{(0,m'_{1,i}),\ldots,(0,m'_{n_i,i})\}$. Then it is easy to verify that $N_i$ is generated by $\{m'_{1,i},\ldots,m'_{n_i,i}\}$. Thus $M$ is a uniformly $S$-Noetherian $R$-module.

Suppose that $R$ is a uniformly $S$-Noetherian ring and $M$ is a uniformly $S$-Noetherian $R$-module. Let $O^* : O_1 \subseteq O_2 \subseteq \cdots$ be an ascending chain of ideals of $R(+)M$. Then there is an ascending chain of ideals of $R$: $\pi(O^*) : \pi(O_1) \subseteq \pi(O_2) \subseteq \cdots$. Thus there exists $s \in S$ which is independent of $O^*$ such that there exists $k \in \mathbb{Z}^+$ satisfying $s\pi(O_n) \subseteq \pi(O_k)$ for any $n \geq k$. Similarly $O^* \cap 0(+)M : O_1 \cap 0(+)M \subseteq O_2 \cap 0(+)M \subseteq \cdots$ is an ascending chain of subideals of $0(+)M$ which are isomorphic to some submodules of $M$. So there exists $s' \in S$ such that there exists $k' \in \mathbb{Z}^+$ satisfying $s'O_n \cap 0(+)M \subseteq O_{k'} \cap 0(+)M$ for any $n \geq k'$. Set $l := \max(k,k')$ and $n \geq l$. Consider the following natural commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \to & O_1 \cap 0(+)M & \to & O_1 & \to & \pi(O_1) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & O_n \cap 0(+)M & \to & O_n & \to & \pi(O_n) & \to & 0.
\end{array}
$$
Set $t := ss'$. Then we have $tO_n \subseteq O_l$ for any $n \geq l$ by Lemma 2.12. So $R(+)M$ is a uniformly $S(+)M$-Noetherian ring by Theorem 2.8. \hfill \square

Let $\alpha : A \to C$ and $\beta : B \to C$ be ring homomorphisms. Then the subring

$$D := \alpha \times_C \beta := \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$$

of $A \times B$ is called the pullback of $\alpha$ and $\beta$. Let $D$ be a pullback of $\alpha$ and $\beta$. Then there is a pullback diagram in the category of commutative rings:

$$
\begin{array}{ccc}
D & \overset{p_A}{\longrightarrow} & A \\
\downarrow{p_B} & & \downarrow{\alpha} \\
B & \overset{\beta}{\longrightarrow} & C.
\end{array}
$$

If $S$ is a multiplicative subset of $D$, then it is easy to verify that $p_A(S) := \{p_A(s) \in A \mid s \in S\}$ is a multiplicative subset of $A$. Note that if $\beta$ is surjective, then so is $p_A$. See [15, Section 8.1.1] for this particular type of pullback. Now we give a uniformly $S$-Noetherian property on this type of pullback diagram.

**Proposition 3.2.** Let $\alpha : A \to C$ be a ring homomorphism and $\beta : B \to C$ be a surjective ring homomorphism. Let $D$ be the pullback of $\alpha$ and $\beta$ and $S$ be a multiplicative subset of $D$. Then the following conditions are equivalent:

1. $D$ is a uniformly $S$-Noetherian ring.
2. $A$ is a uniformly $p_A(S)$-Noetherian ring and $\text{Ker}(\beta)$ is a uniformly $S$-Noetherian $D$-module.

**Proof.** Since $\beta$ is a surjective ring homomorphism, so is $p_A$. Then there is a short exact sequence of $D$-modules:

$$0 \to \text{Ker}(\beta) \to D \to A \to 0.$$ 

By Proposition 2.14, $D$ is a uniformly $S$-Noetherian $D$-module if and only if $\text{Ker}(\beta)$ and $A$ are uniformly $S$-Noetherian $D$-modules. Since $p_A$ is surjective, the $D$-submodules of $A$ are exactly the ideals of the ring $A$. Thus $A$ is a uniformly $S$-Noetherian $D$-module if and only if $A$ is a uniformly $p_A(S)$-Noetherian ring. \hfill \square

Let $f : A \to B$ be a ring homomorphism and $J$ be an ideal of $B$. By [4], the *amalgamation* of $A$ with $B$ along $J$ with respect to $f$, denoted by $A \rhd^f J$, is defined as

$$A \rhd^f J = \{(a, f(a) + j) \mid a \in A, j \in J\},$$

which is a subring of of $A \times B$. By [4, Proposition 4.2], $A \rhd^f J$ is the pullback $\hat{f} \times_{B/J} \pi$, where $\pi : B \to B/J$ is the natural epimorphism and $\hat{f} = \pi \circ f$:

$$
\begin{array}{ccc}
A & \overset{\hat{f}}{\longrightarrow} & A \\
\downarrow{p_A} & & \downarrow{\pi} \\
B & \overset{\pi}{\longrightarrow} & B/J.
\end{array}
$$
For a multiplicative subset $S$ of $A$, set $S' := \{(s, f(s)) \mid s \in S\}$ and $f(S) := \{f(s) \in B \mid s \in S\}$. Then it is easy to verify that $S'$ and $f(S)$ are multiplicative subsets of $A \bowtie^f J$ and $B$, respectively.

**Lemma 3.3.** Let $\alpha : R \to R'$ be a surjective ring homomorphism and $S$ be a multiplicative subset of $R$. If $R$ is a uniformly $S$-Noetherian ring, then $R'$ is a uniformly $\alpha(S)$-Noetherian ring.

**Proof.** Since $R$ is uniformly $S$-Noetherian, there is $s \in S$ such that for any ideal $J$ of $R$, there exists a finitely generated subideal $F_J$ of $J$ satisfying $sJ \subseteq F_J$. Let $I$ be an ideal of $R'$. Since $\alpha : R \to R'$ is a surjective ring homomorphism, there exists an ideal $\alpha^{-1}(I)$ of $R$ such that $\alpha(\alpha^{-1}(I)) = I$. Thus there exists a finitely generated subideal $F_{\alpha^{-1}(I)}$ of $\alpha^{-1}(I)$ which satisfies $s\alpha^{-1}(I) \subseteq F_{\alpha^{-1}(I)}$. So $\alpha(F_{\alpha^{-1}(I)})$ is a finitely generated subideal of $I$ satisfying $\alpha(s)I \subseteq \alpha(F_{\alpha^{-1}(I)})$. \hfill $\square$

**Proposition 3.4.** Let $f : A \to B$ be a ring homomorphism, $J$ be an ideal of $B$, and $S$ be a multiplicative subset of $A$. Set $S' := \{(s, f(s)) \mid s \in S\}$ and $f(S) := \{f(s) \in B \mid s \in S\}$. Then the following conditions are equivalent:

1. $A \bowtie^f J$ is a uniformly $S'$-Noetherian ring.

2. $A$ is a uniformly $S$-Noetherian ring and $J$ is a uniformly $S'$-Noetherian $A \bowtie^f J$-module (with the $A \bowtie^f J$-module structure naturally induced by $p_B$, where $p_B : A \bowtie^f J \to B$ defined by $(a, f(a) + j) \mapsto f(a) + j$).

3. $A$ is a uniformly $S$-Noetherian ring and $f(A) + J$ is a uniformly $f(S)$-Noetherian ring.

**Proof.** (1) $\Leftrightarrow$ (2) This follows from Proposition 3.2.

(1) $\Rightarrow$ (3) By Proposition 3.2, $A$ is a uniformly $S$-Noetherian ring. By [4, Proposition 5.1], there is a short exact sequence $0 \to f^{-1}(J) \times \{0\} \to A \bowtie^f J \to f(A) + J \to 0$ of $A \bowtie^f J$-modules. Note that every $A \bowtie^f J$-submodule of $f(A) + J$ is exactly an ideal of $f(A) + J$. Since $p_B(S') = f(S)$, we conclude that $f(A) + J$ is a uniformly $f(S)$-Noetherian ring by Proposition 2.14.

(3) $\Rightarrow$ (2) Let $f(s)$ be an element in $f(S)$ such that for any ideal of $f(A) + J$ is uniformly $f(S)$-Noetherian with respect to $f(s)$. Then for any $A \bowtie^f J$-submodule $J_0$ of $J$, $J_0$ is an ideal of $f(A) + J$, since every $A \bowtie^f J$-submodule of $J$ is an ideal of $f(A) + J$. Because $f(A) + J$ is uniformly $f(S)$-Noetherian, there exist $j_1, \ldots, j_k \in J_0$ such that $f(s)J_0 \subseteq \langle j_1, \ldots, j_k \rangle(f(A) + J) \subseteq J_0$. Hence we obtain

$$(s, f(s))J_0 \subseteq A \bowtie^f Jj_1 + \cdots + A \bowtie^f Jj_k \subseteq J_0.$$

Thus $J$ is uniformly $S'$-Noetherian with respect to $(s, f(s))$. \hfill $\square$
4. Bass-Papp theorem for uniformly \(S\)-Noetherian rings. It is well known that an \(R\)-module \(E\) is \(inj\)ective if the induced sequence \(0 \rightarrow \text{Hom}_R(C, E) \rightarrow \text{Hom}_R(B, E) \rightarrow \text{Hom}_R(A, E) \rightarrow 0\) is exact for any exact sequence \(0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0\) of \(R\)-modules. The well-known Bass-Papp theorem states that a ring \(R\) is Noetherian if and only if any direct sum of injective modules is injective (see [5, Theorem 3.1.17]). To obtain the Bass-Papp theorem for uniformly \(S\)-Noetherian rings, we first introduce the \(S\)-analog of injective modules.

**Definition 4.1.** Let \(R\) be a ring and \(S\) be a multiplicative subset of \(R\). An \(R\)-module \(E\) is said to be \(u\)-\(S\)-injective if the induced sequence
\[
0 \rightarrow \text{Hom}_R(C, E) \rightarrow \text{Hom}_R(B, E) \rightarrow \text{Hom}_R(A, E) \rightarrow 0
\]
is \(u\)-\(S\)-exact for any \(u\)-\(S\)-exact sequence \(0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0\) of \(R\)-modules.

Let \(M\) be an \(R\)-module. We will always denote by \(\Omega^{-n}(M)\) the \(n\)-th cosyzygy of \(M\) (see [15, Theorem 3.2.18]).

**Lemma 4.2.** Let \(R\) be a ring and \(S\) be a multiplicative subset of \(R\). If \(T\) is a uniformly \(S\)-torsion module, then \(\text{Ext}^n_R(T, M)\) and \(\text{Ext}^n_R(M, T)\) are uniformly \(S\)-torsion for any \(R\)-module \(M\) and any \(n \geq 0\).

*Proof.* We will only prove that \(\text{Ext}^n_R(T, M)\) is uniformly \(S\)-torsion, since the case of \(\text{Ext}^n_R(M, T)\) is similar. Let \(T\) be a uniformly \(S\)-torsion module with \(sT = 0\). If \(n = 0\), then for any \(f \in \text{Hom}_R(T, M)\), we have \(sf(t) = f(st) = 0\) for any \(t \in T\). Thus \(sf = 0\), and so \(s\text{Hom}_R(T, M) = 0\). Let \(0 \rightarrow M \rightarrow E \rightarrow \Omega^{-1}(M) \rightarrow 0\) be a short exact sequence with \(E\) injective. Then \(\text{Ext}^1_R(T, M)\) is a quotient of \(\text{Hom}_R(T, \Omega^{-1}(M))\) which is uniformly \(S\)-torsion. Thus \(\text{Ext}^1_R(T, M)\) is uniformly \(S\)-torsion. For \(n \geq 2\), we have an isomorphism \(\text{Ext}^n_R(T, M) \cong \text{Ext}^1_R(T, \Omega^{-(n-1)}(M))\). Since \(\text{Ext}^1_R(T, \Omega^{-(n-1)}(M))\) is uniformly \(S\)-torsion by induction, \(\text{Ext}^n_R(T, M)\) is uniformly \(S\)-torsion. \(\square\)

**Theorem 4.3.** Let \(R\) be a ring, \(S\) be a multiplicative subset of \(R\), and \(E\) be an \(R\)-module. Then the following conditions are equivalent:

1. \(E\) is \(u\)-\(S\)-injective.

2. For any short exact sequence \(0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0\) of \(R\)-modules, the induced sequence \(0 \rightarrow \text{Hom}_R(C, E) \xrightarrow{g^*} \text{Hom}_R(B, E) \xrightarrow{f^*} \text{Hom}_R(A, E) \rightarrow 0\) is \(u\)-\(S\)-exact.

3. \(\text{Ext}^1_R(M, E)\) is uniformly \(S\)-torsion for any \(R\)-module \(M\).

4. \(\text{Ext}^n_R(M, E)\) is uniformly \(S\)-torsion for any \(R\)-module \(M\) and \(n \geq 1\).

*Proof.* (1) \(\Rightarrow\) (2) and (4) \(\Rightarrow\) (3) These are trivial.

(2) \(\Rightarrow\) (3) Let \(0 \rightarrow L \rightarrow P \rightarrow M \rightarrow 0\) be a short exact sequence with \(P\) projective. Then there exists a long exact sequence \(0 \rightarrow \text{Hom}_R(M, E) \rightarrow \text{Hom}_R(P, E) \rightarrow \text{Hom}_R(L, E) \rightarrow \text{Ext}^1_R(M, E) \rightarrow 0\). Thus \(\text{Ext}^1_R(M, E)\) is uniformly \(S\)-torsion by (2).
(3) ⇒ (2) Let \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) be a short exact sequence. Then we have a long exact sequence \( 0 \to \text{Hom}_{\mathcal{R}}(C, E) \xrightarrow{g^*} \text{Hom}_{\mathcal{R}}(B, E) \xrightarrow{f^*} \text{Hom}_{\mathcal{R}}(A, E) \xrightarrow{\delta} \text{Ext}^1_{\mathcal{R}}(C, E) \to 0 \). By (3), \( \text{Ext}^1_{\mathcal{R}}(C, E) \) is uniformly \( S \)-torsion, and so \( 0 \to \text{Hom}_{\mathcal{R}}(C, E) \xrightarrow{g^*} \text{Hom}_{\mathcal{R}}(B, E) \xrightarrow{f^*} \text{Hom}_{\mathcal{R}}(A, E) \to 0 \) is \( u \)-\( S \)-exact.

(3) ⇒ (4) Let \( M \) be an \( R \)-module. Denote by \( \Omega^{n-1}(M) \) the \((n-1)\)-th syzygy of \( M \). Then \( \text{Ext}^1_{\mathcal{R}}(M, E) \cong \text{Ext}^1_{\mathcal{R}}(\Omega^{n-1}(M), E) \) is uniformly \( S \)-torsion by (3).

(2) ⇒ (1) Let \( E \) be an \( R \)-module satisfying (2). Let \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) be a \( u \)-\( S \)-exact sequence of \( R \)-modules. Then there is an exact sequence \( B \xrightarrow{g} C \to T \to 0 \), where \( T := \text{Coker}(g) \) is uniformly \( S \)-torsion. Then we have an exact sequence

\[
0 \to \text{Hom}_{\mathcal{R}}(T, E) \to \text{Hom}_{\mathcal{R}}(C, E) \to \text{Hom}_{\mathcal{R}}(B, E).
\]

By Lemma 4.2, we have \( \text{Hom}_{\mathcal{R}}(T, E) \) is uniformly \( S \)-torsion. So \( 0 \to \text{Hom}_{\mathcal{R}}(C, E) \xrightarrow{g^*} \text{Hom}_{\mathcal{R}}(B, E) \xrightarrow{f^*} \text{Hom}_{\mathcal{R}}(A, E) \to 0 \) is \( u \)-\( S \)-exact at \( \text{Hom}_{\mathcal{R}}(C, E) \).

There are also two short exact sequences:

\[
0 \to \text{Ker}(f) \xrightarrow{i_A} A \xrightarrow{\pi_{\text{Im}(f)}} \text{Im}(f) \to 0 \quad \text{and} \quad 0 \to \text{Im}(f) \xrightarrow{i_B} B \to \text{Coker}(f) \to 0,
\]

where \( \text{Ker}(f) \) is uniformly \( S \)-torsion. Consider the induced exact sequences

\[
0 \to \text{Hom}_{\mathcal{R}}(\text{Im}(f), E) \xrightarrow{\pi_{\text{Im}(f)}} \text{Hom}_{\mathcal{R}}(A, E) \xrightarrow{i_A^*} \text{Hom}_{\mathcal{R}}(\text{Ker}(f), E)
\]

and

\[
0 \to \text{Hom}_{\mathcal{R}}(\text{Coker}(f), E) \to \text{Hom}_{\mathcal{R}}(B, E) \xrightarrow{i_B^*} \text{Hom}_{\mathcal{R}}(\text{Im}(f), E).
\]

Then \( \text{Im}(i_A^*) \) and \( \text{Coker}(i_B^*) \) are all uniformly \( S \)-torsion. We have the following pushout diagram:

\[
\begin{array}{ccccccccc}
0 & 0 \\
& \\
& \\
& \\
\downarrow & \downarrow \\
\text{Im}(i_B^*) & \text{Im}(i_B^*) \\
& \\
& \\
& \\
0 \longrightarrow & \text{Hom}_{\mathcal{R}}(\text{Im}(f), E) & \longrightarrow & \text{Hom}_{\mathcal{R}}(A, E) & \longrightarrow & \text{Im}(i_A^*) & \longrightarrow & 0 \\
& \\
& \\
& \\
& \\
& \downarrow \\
& \text{Coker}(i_B^*) & \longrightarrow & Y & \longrightarrow & \text{Im}(i_A^*) & \longrightarrow & 0 \\
& \\
& \\
& \\
& \\
0 & 0
\end{array}
\]

Since \( \text{Im}(i_A^*) \) and \( \text{Coker}(i_B^*) \) are all uniformly \( S \)-torsion, \( Y \) is also uniformly \( S \)-torsion by Lemma 2.11. Thus the natural composition \( f^* : \text{Hom}_{\mathcal{R}}(B, E) \to \text{Im}(i_B^*) \to \text{Hom}_{\mathcal{R}}(A, E) \) is a \( u \)-\( S \)-epimorphism. So \( 0 \to \text{Hom}_{\mathcal{R}}(C, E) \xrightarrow{g^*} \text{Hom}_{\mathcal{R}}(B, E) \xrightarrow{f^*} \text{Hom}_{\mathcal{R}}(A, E) \to 0 \) is \( u \)-\( S \)-exact at \( \text{Hom}_{\mathcal{R}}(A, E) \).
Since the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is $u$-$S$-exact at $B$ and $C$, there exists $s \in S$ such that $s\text{Ker}(g) \subseteq \text{Im}(f)$, $s\text{Im}(f) \subseteq \text{Ker}(g)$, and $s\text{Coker}(g) = 0$. We claim that $s^2\text{Im}(g^*) \subseteq \text{Ker}(f^*)$ and $s^2\text{Ker}(f^*) \subseteq \text{Im}(g^*)$. Indeed, consider the following diagram:

$$
\begin{array}{c}
0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \\
& & \uparrow{h} & & & & \downarrow{g} & & \\
& & \text{E} & & \text{B} & & \text{C}
\end{array}
$$

Let $h \in \text{Im}(g^*)$. Then there exists $u \in \text{Hom}_R(C, E)$ such that $h = u \circ g$. Thus for any $a \in A$, $sh \circ f(a) = su \circ g \circ f(a) = u \circ g \circ sf(a) = 0$ since $s\text{Im}(f) \subseteq \text{Ker}(g)$. So $sh \circ f = 0$ and then $s\text{Im}(g^*) \subseteq \text{Ker}(f^*)$. Thus $s^2\text{Im}(g^*) \subseteq \text{Ker}(f^*)$. Now let $h \in \text{Ker}(f^*)$. Then $h \circ f = 0$. Thus $\text{Ker}(h) \supseteq \text{Im}(f) \supseteq s\text{Ker}(g)$. So $sh \circ i_{\text{Ker}(g)} = 0$, where $i_{\text{Ker}(g)} : \text{Ker}(g) \to B$ is the natural embedding map. There is a well-defined $R$-homomorphism $v : \text{Im}(g) \to E$ such that $v \circ \pi_B = sh$, where $\pi_B$ is the natural epimorphism $B \to \text{Im}(g)$. Consider the exact sequence $\text{Hom}_R(\text{Coker}(g), E) \to \text{Hom}_R(C, E) \to \text{Hom}_R(\text{Im}(g), E) \to \text{Ext}^1_R(\text{Coker}(g), E)$ induced by $0 \to \text{Im}(g) \to C \to \text{Coker}(g) \to 0$. Since $s\text{Hom}_R(\text{Coker}(g), E) = s\text{Ext}^1_R(\text{Coker}(g), E) = 0$, it follows that $s\text{Hom}_R(\text{Im}(g), E) \subseteq i_{\text{Im}(g)}^*(\text{Hom}_R(C, E))$. Thus there is a homomorphism $v : C \to E$ such that $s^2h = v \circ g$. Then we have $s^2\text{Ker}(f^*) \subseteq \text{Im}(g^*)$.

Therefore $0 \to \text{Hom}_R(C, E) \xrightarrow{g^*} \text{Hom}_R(B, E) \xrightarrow{f^*} \text{Hom}_R(A, E) \to 0$ is $u$-$S$-exact at $\text{Hom}_R(B, E)$.

It follows from Theorem 4.3 that uniformly $S$-torsion modules and injective modules are $u$-$S$-injective.

**COROLLARY 4.4.** Let $R$ be a ring and $S$ be a multiplicative subset of $R$. Suppose that $E$ is a uniformly $S$-torsion $R$-module or an injective $R$-module. Then $E$ is $u$-$S$-injective.

The following example shows that the condition “$\text{Ext}^1_R(M, F)$ is uniformly $S$-torsion for any $R$-module $M$” in Theorem 4.3 cannot be replaced by “$\text{Ext}^1_R(R/I, F)$ is uniformly $S$-torsion for any ideal $I$ of $R$”.

**EXAMPLE 4.5.** Let $R = \mathbb{Z}$ be the ring of integers, $p$ be a prime number in $\mathbb{Z}$, and $S = \{p^n \mid n \geq 0\}$. Let $J_p$ be the additive group of all $p$-adic integers (see for example [6]). Then $\text{Ext}^1_R(R/I, J_p)$ is uniformly $S$-torsion for any ideal $I$ of $R$. However, $J_p$ is not $u$-$S$-injective.

**Proof.** Let $\langle n \rangle$ be an ideal of $\mathbb{Z}$. Let $n = p^km$ with $(p, m) = 1$. Then $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/\langle n \rangle, J_p) \cong J_p/nJ_p \cong \mathbb{Z}/(p^k) \cong \mathbb{Z}/(p^{km})$ by [6, Exercise 1.3(10)]. So $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/\langle n \rangle, J_p)$ is uniformly $S$-torsion for any ideal $\langle n \rangle$ of $\mathbb{Z}$. However, $J_p$ is not $u$-$S$-injective. Indeed, let $\mathbb{Z}(p^\infty)$ be the quasi-cyclic group (see [6] for example). Then $\mathbb{Z}(p^\infty)$ is a divisible group and $J_p \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty))$. So

$$
\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty), J_p)
$$
\[ \cong \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}(p^\infty), \text{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty))) \]
\[ \cong \text{Hom}_{\mathbb{Z}}(\text{Tor}_1^\mathbb{Z}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)), \mathbb{Z}(p^\infty)) \]
\[ \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) \cong J_p, \]

where the third isomorphism above comes from the direct limit of [6, (F), p. 238]. Note that for any \( p^k \in S \), we have \( p^k J_p \neq 0 \). So \( J_p \) is not \( u \)-\( S \)-injective. \( \square \)

**Remark 4.6.** It is well known that every direct product of injective modules is injective. However, the direct product of \( u \)-\( S \)-injective modules need not be \( u \)-\( S \)-injective. Indeed, let \( R \) and \( S \) be as in Example 4.5. Let \( \mathbb{Z}/\langle p^k \rangle \) be a cyclic group of order \( p^k \) \((k \geq 1)\). Then every \( \mathbb{Z}/\langle p^k \rangle \) is uniformly \( S \)-torsion, and thus is \( u \)-\( S \)-injective. Let \( \mathbb{Q} \) be the rational number group. Then, by [6, Theorem 9.6.2], we have
\[ \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}/\mathbb{Z}, \prod_{k=1}^\infty \mathbb{Z}/\langle p^k \rangle) \cong \prod_{k=1}^\infty \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}/\langle p^k \rangle) \cong \prod_{k=1}^\infty \mathbb{Z}/\langle p^k \rangle \]since each \( \mathbb{Z}/\langle p^k \rangle \) is a reduced cotorsion group. It is easy to verify that \( \prod_{k=1}^\infty \mathbb{Z}/\langle p^k \rangle \) is not uniformly \( S \)-torsion. So \( \prod_{k=1}^\infty \mathbb{Z}/\langle p^k \rangle \) is not \( u \)-\( S \)-injective.

**Proposition 4.7.** Let \( R \) be a ring and \( S \) be a multiplicative subset of \( R \). Then the following statements hold.

1. Every finite direct sum of \( u \)-\( S \)-injective modules is \( u \)-\( S \)-injective.
2. Let \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) be a \( u \)-\( S \)-exact sequence of \( R \)-modules. If \( A \) and \( C \) are \( u \)-\( S \)-injective modules, then so is \( B \).
3. Let \( A \to B \) be a \( u \)-\( S \)-isomorphism of \( R \)-modules. If one of \( A \) and \( B \) is \( u \)-\( S \)-injective, then the other is too.
4. Let \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) be a \( u \)-\( S \)-exact sequence of \( R \)-modules. If \( A \) and \( B \) are \( u \)-\( S \)-injective, then \( C \) is \( u \)-\( S \)-injective.

**Proof.**

1. Let \( E_1, \ldots, E_n \) be \( u \)-\( S \)-injective modules. Let \( M \) be an \( R \)-module. Then there exists \( s_i \in S \) such that \( s_i \text{Ext}_R^1(M, E_i) = 0 \) for each \( i = 1, \ldots, n \). Set \( s := s_1 \cdots s_n \). Then \( s \text{Ext}_R^1(M, \bigoplus_{i=1}^n E_i) \cong \bigoplus_{i=1}^n s \text{Ext}_R^1(M, E_i) = 0 \). Thus \( \bigoplus_{i=1}^n E_i \) is \( u \)-\( S \)-injective.

2. Suppose that \( A \) and \( C \) are \( u \)-\( S \)-injective modules and let \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) of \( R \)-modules be a \( u \)-\( S \)-exact sequence. Then there are three short exact sequences:

\[ 0 \to \ker(f) \to A \to \text{im}(f) \to 0, \]
\[ 0 \to 
\[ \ker(g) \to B \to \text{im}(g) \to 0, \]
\[ 0 \to \text{im}(g) \to C \to \text{coker}(g) \to 0. \]
Then $\ker(f)$ and $\coker(g)$ are all uniformly $S$-torsion and $s\ker(g) \subseteq \text{Im}(f)$ and $s\text{Im}(f) \subseteq \ker(g)$ for some $s \in S$. Let $M$ be an $R$-module. Then

$$\text{Ext}^1_R(M, A) \to \text{Ext}^1_R(M, \text{Im}(f)) \to \text{Ext}^2_R(M, \ker(f))$$

is exact. Since $\ker(f)$ is uniformly $S$-torsion and $A$ is $u$-$S$-injective, $\text{Ext}^1_R(M, \text{Im}(f))$ is uniformly $S$-torsion. Note that

$$\text{Hom}_R(M, \coker(g)) \to \text{Ext}^1_R(M, \text{Im}(g)) \to \text{Ext}^1_R(M, C)$$

is exact. Since $\coker(g)$ is uniformly $S$-torsion, $\text{Hom}_R(M, \coker(g))$ is uniformly $S$-torsion by Lemma 4.2. Thus $\text{Ext}^1_R(M, \text{Im}(g))$ is uniformly $S$-torsion since $\text{Ext}^1_R(M, C)$ is uniformly $S$-torsion. We also note that

$$\text{Ext}^1_R(M, \ker(g)) \to \text{Ext}^1_R(M, B) \to \text{Ext}^1_R(M, \text{Im}(g))$$

is exact. Thus, to verify that $\text{Ext}^1_R(M, B)$ is uniformly $S$-torsion, we need only show that $\text{Ext}^1_R(M, \ker(g))$ is uniformly $S$-torsion. Denote $N := \ker(g) + \text{Im}(f)$. Consider the following two exact sequences

$$0 \to \ker(g) \to N \to N/\ker(g) \to 0 \text{ and } 0 \to \text{Im}(f) \to N \to N/\text{Im}(f) \to 0.$$  

Then it is easy to verify that $N/\ker(g)$ and $N/\text{Im}(f)$ are all uniformly $S$-torsion. Consider the following induced two exact sequences

$$\text{Hom}_R(M, N/\text{Im}(f)) \to \text{Ext}^1_R(M, \ker(g)) \to \text{Ext}^1_R(M, N) \to \text{Ext}^2_R(M, N/\text{Im}(f)),$$

$$\text{Hom}_R(M, N/\ker(g)) \to \text{Ext}^1_R(M, \text{Im}(f)) \to \text{Ext}^1_R(M, N) \to \text{Ext}^2_R(M, N/\ker(g)).$$

Thus $\text{Ext}^1_R(M, \ker(g))$ is uniformly $S$-torsion if and only if $\text{Ext}^1_R(M, \text{Im}(f))$ is uniformly $S$-torsion. Consequently $B$ is $S$-injective since $\text{Ext}^1_R(M, \text{Im}(f))$ is uniformly $S$-torsion.

(3) Considering the $u$-$S$-exact sequences $0 \to A \to B \to 0 \to 0$ and $0 \to 0 \to A \to B \to 0$, it follows by (2) that $A$ is $u$-$S$-injective if and only if $B$ is $u$-$S$-injective.

(4) Let $0 \to A \xrightarrow{\bar{f}} B \xrightarrow{g} C \to 0$ be a $u$-$S$-exact sequence. Then, as in the proof of (3), there are three short exact sequences:

$$0 \to \ker(f) \to A \to \text{Im}(f) \to 0,$$

$$0 \to \ker(g) \to B \to \text{Im}(g) \to 0,$$

$$0 \to \text{Im}(g) \to C \to \coker(g) \to 0.$$  

Then $\ker(f)$ and $\coker(g)$ are all uniformly $S$-torsion and $s\ker(g) \subseteq \text{Im}(f)$ and $s\text{Im}(f) \subseteq \ker(g)$ for some $s \in S$. Let $M$ be an $R$-module. Note that

$$\text{Hom}_R(M, \coker(g)) \to \text{Ext}^1_R(M, \text{Im}(g)) \to \text{Ext}^1_R(M, C) \to \text{Ext}^1_R(M, \coker(g))$$

is exact. Since $\coker(g)$ is uniformly $S$-torsion, it follows by Lemma 4.2 that $\text{Hom}_R(M, \coker(g))$ and $\text{Ext}^1_R(M, \coker(g))$ are uniformly $S$-torsion. We just need to verify that $\text{Ext}^1_R(M, \text{Im}(g))$ is uniformly $S$-torsion. Note that

$$\text{Ext}^1_R(M, B) \to \text{Ext}^1_R(M, \text{Im}(g)) \to \text{Ext}^2_R(M, \ker(g))$$
is exact. Since $\text{Ext}^1_R(M, B)$ is uniformly $S$-torsion, we only need to prove that $\text{Ext}^2_R(M, \text{Ker}(g))$ is uniformly $S$-torsion. By the proof of (2), we only need to show that $\text{Ext}^3_R(M, \text{Im}(f))$ is uniformly $S$-torsion. Note that

$$\text{Ext}^2_R(M, A) \to \text{Ext}^2_R(M, \text{Im}(f)) \to \text{Ext}^3_R(M, \text{Ker}(f))$$

is exact. Since $\text{Ext}^2_R(M, A)$ and $\text{Ext}^3_R(M, \text{Ker}(f))$ are uniformly $S$-torsion, it follows that $\text{Ext}^2_R(M, \text{Im}(f))$ is uniformly $S$-torsion. So $C$ is $u$-$S$-injective. \hfill $\Box$

Let $p$ be a prime ideal of $R$. We say that an $R$-module $E$ is $p$-injective shortly if $E$ is $(R \setminus p)$-injective. The next result gives a local characterization of injective modules.

**Proposition 4.8.** Let $R$ be a ring and $E$ be an $R$-module. Then the following conditions are equivalent:

1. $E$ is injective.
2. $E$ is $p$-injective for any $p \in \text{Spec}(R)$.
3. $E$ is $m$-injective for any $m \in \text{Max}(R)$.

**Proof.** (1) $\Rightarrow$ (2) This follows from Theorem 4.7. 
(2) $\Rightarrow$ (3) This is trivial. 
(3) $\Rightarrow$ (1) Let $M$ be an $R$-module. Then $\text{Ext}^1_R(M, E)$ is uniformly $(R \setminus m)$-torsion. Thus, for every $m \in \text{Max}(R)$, there exists $s_m \in S$ such that $s_m \text{Ext}^1_R(M, E) = 0$. Since the ideal generated by all $s_m$ is $R$, $\text{Ext}^1_R(M, E) = 0$. So $E$ is injective. \hfill $\Box$

We say that an $R$-module $M$ is $S$-divisible if $M = sM$ for any $s \in S$. The well-known Baer criterion says that an $R$-module $E$ is injective if and only if $\text{Ext}^1_R(R/I, E) = 0$ for any ideal $I$ of $R$. The next result gives a uniform $S$-version of Baer’s criterion.

**Proposition 4.9.** (Baer’s criterion for $u$-$S$-injective modules) Let $R$ be a ring, $S$ be a multiplicative subset of $R$, and $E$ be an $R$-module. If $E$ is a $u$-$S$-injective module, then there exists $s \in S$ such that $s \text{Ext}^1_R(R/I, E) = 0$ for any ideal $I$ of $R$. Moreover, if $E$ is $S$-divisible, then the converse holds as well.

**Proof.** If $E$ is a $u$-$S$-injective module, then $\text{Ext}^1_R(\bigoplus_{I \subseteq R} R/I, E)$ is uniformly $S$-torsion by Theorem 4.3. Thus there is $s \in S$ such that $s \text{Ext}^1_R(\bigoplus_{I \subseteq R} R/I, E) = s \prod_{I \subseteq R} \text{Ext}^1_R(R/I, E) = 0$. So $s \text{Ext}^1_R(R/I, E) = 0$ for any ideal $I$ of $R$.

Suppose that $E$ is $S$-divisible. Let $B$ be an $R$-module, $A$ be a submodule of $B$, and $s$ be an element in $S$ that satisfies the necessity. Let $f : A \to E$ be an $R$-homomorphism. Set
\[ \Gamma := \{(C, d) \mid C \text{ is a submodule of } B \text{ containing } A \text{ and } d|_A = sf\}. \]

Since \((A, sf) \in \Gamma\), \(\Gamma\) is not empty. Define \((C_1, d_1) \leq (C_2, d_2)\) if \(C_1 \subseteq C_2\) and \(d_2|_{C_1} = d_1\). Then \((\Gamma, \leq)\) is a partially ordered set. For any chain \((C_j, d_j)\), set \(C_0 := \bigcup C_j\) and define \(d_0(c) = d_j(c)\) if \(c \in C_j\). Then \((C_0, d_0)\) is the upper bound of the chain \((C_j, d_j)\). By Zorn’s lemma, there is a maximal element \((C, d)\) in \(\Gamma\).

We claim that \(C = B\). Assume on the contrary that \(C \neq B\) and let \(x \in B \setminus C\). Set \(I := \{r \in R \mid rx \in C\}\). Then \(I\) is an ideal of \(R\). Since \(E = sE\), there exists a homomorphism \(h : I \to E\) satisfying that \(sh(r) = d(rx)\). Then there exists an \(R\)-homomorphism \(g : R \to E\) such that \(g(r) = sh(r) = d(rx)\) for any \(r \in I\). Set \(C_1 := C + Rx\) and define \(d_1(c + rx) = d(c) + g(r)\), where \(c \in C\) and \(r \in R\). If \(c + rx = 0\), then \(r \in I\), and thus \(d(c) + g(r) = d(c) + sh(r) = d(c) + d(rx) = d(c + rx) = 0\). Hence \(d_1\) is a well-defined homomorphism such that \(d_1|_A = sf\). So \((C_1, d_1) \in \Gamma\). But \((C_1, d_1) > (C, d)\), which contradicts the maximality of \((C, d)\).

We say that a multiplicative subset \(S\) of \(R\) is regular if it consists of non-zero-divisors. Now we give the main result of this section.

**Theorem 4.10.** (Bass-Papp theorem for uniformly \(S\)-Noetherian rings) Let \(R\) be a ring and \(S\) be a regular multiplicative subset of \(R\). Then the following conditions are equivalent:

1. \(R\) is uniformly \(S\)-Noetherian.
2. Every direct sum of injective modules is \(u\)-\(S\)-injective.
3. Every direct union of injective modules is \(u\)-\(S\)-injective.

**Proof.** (1) \(\Rightarrow\) (3) Let \(\{E_i, f_{i,j}\}_{i<j \in \Lambda}\) be a direct system of injective modules, where each \(f_{i,j}\) is the embedding map. Let \(\lim E_i\) be its direct limit. Let \(s\) be an element in \(S\) such that for every ideal \(I\) of \(R\), there exists a finitely generated subideal \(K\) of \(I\) such that \(sI \subseteq K\). Considering the short exact sequence \(0 \to I/K \to R/K \to R/I \to 0\), we have the following long exact sequence:

\[ \text{Hom}_R(I/K, \lim E_i) \to \Ext^1_R(R/I, \lim E_i) \to \Ext^1_R(R/K, \lim E_i) \to \Ext^1_R(I/K, \lim E_i). \]

Since \(R/K\) is finitely presented, we have \(\Ext^1_R(R/K, \lim E_i) \cong \lim \Ext^1_R(R/K, E_i) = 0\) by the Five Lemma and [16, Theorem 24.10]. By the proof of Lemma 4.2, it can be shown that \(s\Hom_R(I/K, \lim E_i) = 0\). Thus \(s\Ext^1_R(R/I, \lim E_i) = 0\) for any ideal \(I\) of \(R\). Since \(S\) is composed of non-zero-divisors, every \(E_i\) is \(S\)-divisible by the proof [15, Theorem 2.4.5]. Thus \(\lim E_i\) is also \(S\)-divisible. So \(\lim E_i\) is \(u\)-\(S\)-injective by Proposition 4.9.

(3) \(\Rightarrow\) (2) This is trivial.
(2) $\Rightarrow$ (1) Suppose $R$ is not a uniformly $S$-Noetherian ring. By Theorem 2.8, for any $s \in S$, there exists a strictly ascending chain $I_1 \subset I_2 \subset \cdots$ of ideals of $R$ such that for any $k \geq 1$ there exists $n \geq k$ satisfying $sI_n \not\subseteq I_k$. Set $I := \bigcup_{i=1}^{\infty} I_i$. Then $I$ is an ideal of $R$ and $I/I_i \neq 0$ for any $i \geq 1$. Denote by $E(I/I_i)$ the injective envelope of $I/I_i$. Let $f_i$ be the natural composition $I \rightarrow I/I_i \hookrightarrow E(I/I_i)$. Since $sI_n \not\subseteq I_i$ for any $i \geq 1$ and some $n \geq i$, we have $sf_i \neq 0$ for any $i \geq 1$. Define $f : I \rightarrow \bigoplus_{i=1}^{\infty} E(I/I_i)$ by $f(a) = (f_i(a))$. Not that for each $a \in I$, we have $a \in I_i$ for some $i \geq 1$. So $f$ is a well-defined $R$-homomorphism. Let $\pi_i : \bigoplus_{i=1}^{\infty} E(I/I_i) \rightarrow E(I/I_i)$ be the $i$th projection. The embedding map $i : I \rightarrow R$ induces an exact sequence

$$
\text{Hom}_R(R, \bigoplus_{i=1}^{\infty} E(I/I_i)) \xrightarrow{i^*} \text{Hom}_R(I, \bigoplus_{i=1}^{\infty} E(I/I_i)) \xrightarrow{\delta} \text{Ext}_R^1(R/I, \bigoplus_{i=1}^{\infty} E(I/I_i)) \rightarrow 0.
$$

Since $\bigoplus_{i=1}^{\infty} E(I/I_i)$ is $u$-$S$-injective, there is $s \in S$ such that

$$s\text{Ext}_R^1(R/I, \bigoplus_{i=1}^{\infty} E(I/I_i)) = 0.
$$

Thus there exists a homomorphism $g : R \rightarrow \bigoplus_{i=1}^{\infty} E(I/I_i)$ such that $sf = i^*(g)$. Thus for sufficiently large $i$, we have $s\pi_i f(a) = \pi_i i^*(g)(a) = a\pi_i i^*(g)(1) = 0$ for any $a \in I$. So for such $i$, $sf_i = s\pi_i f : I \rightarrow E(I/I_i)$ is a zero homomorphism, which is a contradiction. Hence $R$ is uniformly $S$-Noetherian.

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