Weinberg propagator
of a free massive particle with an arbitrary spin
from the BFV–BRST path integral

V.G.Zima\textsuperscript{1} and S.O.Fedoruk\textsuperscript{2}

\textsuperscript{1} Kharkov State University,
310077 Kharkov, 4 Svoboda Sq., Ukraine
e-mail: zima@postmaster.co.uk

\textsuperscript{2} Ukrainian Engineering–Pedagogical Academy,
310003 Kharkov, 16 Universitetskaya Str., Ukraine
e-mail: fed@postmaster.co.uk

Abstract
The transition amplitude is obtained for a free massive particle of arbitrary spin by calculating the path integral in the index–spinor formulation within the BFV–BRST approach. None renormalizations of the path integral measure were applied. The calculation has given the Weinberg propagator written in the index–free form with the use of index spinor. The choice of boundary conditions on the index spinor determines holomorphic or antiholomorphic representation for the canonical description of particle/antiparticle spin.

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1 Introduction

In the development of extended object theories, the problem of covariant description of spinning particles, in particular, the problem of covariant quantization of these particles, plays a double role. On the one hand, it is an educational model which allows one to illustrate the progress achieved and to train oneself in application of the developing methods. On the other hand, it is a starting point and, in a certain sense, the desired result of these theories intended to realize consistently the fundamental quantum and relativistic principles, so that one could speak about the construction of the interaction theory for particles remaining the only actually observable manifestation of the fundamental structure of matter.

An important part of the quantization problem of the particle is the calculation of its propagator. The most powerful modern method for solving this problem, just as for the problem of quantization in general, is the BFV-BRST approach [1]. However, up to now calculations of transition amplitudes for massive spinning particles in this approach have been carried out only in rather limited number of papers [2] in the framework of pseudoclassical mechanics [3, 4] and were restricted to the spin 1/2. For earlier paper with the old method of the calculation see, for example [5]. For field strength of massless particle there are calculations of the propagator for arbitrary spins [6] and also in pseudoclassical formulation.

In this paper we apply the BFV-BRST quantization procedure to the free arbitrary-spin massive particle moving in the space-time of the dimension $D = 4$. As in above-cited papers the question is in an obtaining equivalent of known expressions by novel methods and finding for an adequate representation of the result. In our opinion, the approach accepted here to the description of spin in terms of the index spinor [7] is very helpful in solution of the quantization problem. In view of an universal character of such description and relative novelty of the index spinor conception we give some details of our construction.

It is well known that at the classical level the spin is putting in the theory by introducing a few
of additional coordinates, a part of which can be auxiliary. These spin variables can be commuting or anticommuting Lorentz scalars of the target space-time, spinors, vectors etc. Among them one can extract subsets describing the internal geometry of the particle world line and spinors of the internal symmetry group, if such is present. If the postulated group of space-time symmetry is wider than the ordinary one, the several Lorentz representations can be collected into more complicated formations such as (super)twistors of (super)conformal group. A choice of using spin variables is in essence the matter of convenience. It may be useful to have various formulations of free spinning particle mechanics and then to study how interaction are “switched on” in each case.

Nevertheless, the spinors as basic spin variables are especially attractive. Actually, in the space of states the notion of spin is associated with the space of irreducible representation of the small group. In the case of a massive particle it is natural to take the rest frame momentum as a standard one. Then the small group is the group of the space rotations SO(3) or its quantum mechanical counterpart, i.e. the spinor group Spin(3) ≈ SU(2). Topologically irreducible representations of this group, as for each other compact group, are finite dimensional and since the group is linear they have the canonical realization in the space of multilinear forms (or tensorial degrees of the fundamental representation). An irreducible representation is determined by its highest weight or, equally, a Young tableau or by an eigenvalue of the Casimir operator, which is namely the spin for the case. The Young tableau visually determines the degree and the type of symmetry for the multilinear form, in particularly, the number of its different spinor arguments, which is equal to the rank of the small group \( r = 1 \). This circumstance points to the preferred and fundamental role of the set of variables which consists of \( r \) spinors of small group in the “mechanical” spin theory.

In the relativistic theory such a set of variables retains its role because the quantum mechanical group of rotations SU(2) has, as the proper complex envelope, the quantum mechanical Lorentz group SL(2, C), which is thus the relevant hipercompact complex group. So the space of a unitary representation of the rotational group is simultaneously the space of a nonunitary finite-dimensional repre-
sentation of the Lorentz group. Thus an application of the Weyl “unitary trick” connects irreducible representations of the compact group SU(2) with analytic and antianalytic irreducible representations of the corresponding complex group SL(2, C).

Until recently, the bosonic spinors have been used mostly as twistor–like variables that resolve the mass constraint in the massless case [8]. However, the potential of these variables is far from being exhausted. For example, in the theory with bosonic spinors there is, at least at the classical level, a very simple solution of the problem of infinite reducibility of fermionic κ–symmetry due to a possibility of construction of the projectors with such spinors. Its connection with the solution in the framework of doubly supersymmetric models is still unclear. This circumstance justifies a further analysis of bosonic–spinor particle models since it implies a possible existence of more subtle geometric and group–theoretical aspects.

In construction of the mechanical theory of particle spin the well known Borel-Weyl-Bott theorem is in a certain sense the Ariadnian thread. The theorem states that for any given irreducible representation of any compact connected Lie group there exists a classical dynamical system which quantization yields this representation as the quantum Hilbert space. In spin theory that group is the spinor group Spin(3). The factual construction of the mechanical system mentioned in the theorem for the massive particle with spin can be realized in the rest frame in two ways by using of either commuting (bosonic) coordinates or anticommuting (fermionic or Grassmann) ones. We have as bosonic construction the first order Lagrangian of the form $\alpha i(\dot{\zeta} \bar{\zeta} - \dot{\bar{\zeta}} \zeta) - \lambda(\zeta \bar{\zeta} - 1)$, where a dot denotes derivative with respect to the development parameter $\tau$, $\lambda$ is a Lagrange multiplier, $\alpha$ is a real constant and standard index free notations are used to contraction of the dimensionless spinor $\zeta$ and its complex conjugate $\bar{\zeta}$. In the kinetic term the minus sign provides its irreducibility to a total derivative. The potential term, i. e. the spin constraint $\zeta \bar{\zeta} - 1 \approx 0$ entering the action, restricts the configuration space to the group manifold because then the $2 \times 2$ matrix with the lines $\zeta$ and $\epsilon \bar{\zeta}$ is a unimodular unitary one. Here $\epsilon$ is the unit antisymmetric spinor.
The pair of the complex conjugated primary constraints \( p_\zeta \approx -i\alpha \bar{\zeta} \) and \( p_{\bar{\zeta}} \approx i\alpha \zeta \) belongs to the second class. Upon quantization with the use of Dirac brackets, \( \zeta \) and \( \bar{\zeta} \) are realized up to a multiplier as bosonic creation-annihilation operators \( \{\bar{\zeta}, \zeta\}_{DB} = -i/2\alpha \). The spin constraint \( \zeta \bar{\zeta} - 1 \approx 0 \) fixes the value of the “particle number operator” up to an ordering constant. Upon quantization the eigenvalue \( J \) of the modulus of the angular momentum \( M_i = \alpha \zeta \sigma_i \bar{\zeta} \) must be half-integer. Thus, under quantization bosonic theory leads to the states of arbitrary spin because the ordering constant is indetermined.

The fermionic Lagrangian can be taken quite similar \( \alpha i(\dot{\theta} \bar{\theta} - \theta \bar{\dot{\theta}}) \) without any potential term. Here \( \theta \) denotes an odd spinor. This results in spins up to \( 1/2 \) over a spinless ground state. To extract a definite spin it should be introduced the spin constraint \( \theta \bar{\theta} = 0 \) also but now there is only a finite set of eigenvalues of the corresponding quantum operator.

Using of bosonic and fermionic spinor variables simultaneously one can take the spinless vacuum, then the spin of bosonic subsystem is regarded as the spin of the Clifford vacuum for the fermionic subsystem.

To obtain the relativistic extension of these models one should construct a Lagrangian whose “spinor part” in the rest frame is reduced to the expressions discussed above. This is achieved by an obvious transformation of the kinetic and potential parts

\[
\alpha i(\dot{\zeta} \bar{\zeta} - \dot{\bar{\zeta}} \zeta) \rightarrow i(\zeta \hat{p} \bar{\zeta} - \bar{\zeta} \hat{p} \zeta), \quad (\zeta \bar{\zeta} - 1) \rightarrow (\zeta \hat{p} \bar{\zeta} - j).
\]

Here \( p \) is the energy-momentum vector, \( \hat{p} \) is its contraction with the Pauli matrices \( \sigma \) so that in the rest frame \( \hat{p} = m\sigma_0 \), where \( m \) is the particle mass. In these conversions some natural redefinitions have been made and the spinor acquires the dimension and becomes a Weyl spinor. We call this spinor the index spinor because of the role which it plays after quantization. The particle Lagrangian arises after adding the kinetic \( p \dot{x} \) term and the potential \( -\frac{e}{2}(p^2 + m^2) \) one of the free spinless particle for coordinates of the phase space. Here \( e \) is an einbein. In this way one obtains the action of the paper [7], where the sign of the particle energy coincides with the sign of the “classical spin” \( j \) due to
the spin constraint $\zeta \hat{p} \bar{\zeta} - j = 0$. The spectrum of the model consists of the induced representations of the Poincaré group.

The kinetic term of this model can be written in terms of the bosonic superform $\omega = dx - i(d\zeta \sigma \bar{\zeta} - \zeta \sigma d\bar{\zeta})$ and obviously possesses bosonic supersymmetry $\zeta \rightarrow \zeta + \epsilon, x \rightarrow x + i(\zeta \sigma \bar{\epsilon} - \epsilon \sigma \bar{\zeta})$, where $\epsilon$ is a constant commuting spinor. This symmetry is destroyed by the spinorial potential term in the action.

With fermionic coordinates Casalbuoni-Brink-Schwarz supersymmetric action \[4, 10\] appears, if the constraint for extracting a particular value of spin has not been inserted. In known way \[11, 12\] this model can be generalized to the model with extended supersymmetry by introducing of isospinor coordinates and attaching indices of internal group to the odd spinor coordinates. The terms correspond to the central charges also can be added to the action.

In both, bosonic and fermionic, cases the massless particle appears when on takes the consistent limit $m \rightarrow 0$. Certainly, all these models are theories with the first and second class constraints. But in the presence of bosonic spinor coordinates (which are not nilpotent) there is no problem with the covariant and irreducible separation of constraints into first and second classes, at least at the classical level, because we can construct the projectors by using spinors $\zeta$ and $\hat{p} \bar{\zeta}$ for which $\zeta \hat{p} \bar{\zeta} = j$.

There are other approaches \[13\]-\[23\] which use commuting spinors as variables for the description of spin. First of all it is the use of entries of the fundamental representation matrix. Often these variables are called harmonics by some misuse of language. Literally the harmonics \[12\] are pure auxiliary gauge variables which parametrize an arbitrary frame with respect to the canonical one and can be also regarded as a bridge connecting the representations of the group with ones of its subgroup. They acquire dynamical status only if a gauge is fixed and some basic variables transmit to them a part of their functions. So the use of these variables as dynamical ones should be understood as such a choice of gauge in which some initial dynamical variables have been gauged away and their role passed to the harmonics. In principle together with the first class constraints, which provide the harmonics with a gauge nature, their matrix should be subject to constraints which place it in the corresponding
group (in higher space-time dimensions it is impossible to formulate these constraints as conditions of the conservation of quadratic forms). Thus the theory with harmonics is strongly restricted. Careful account of these constraints in the quantization procedure is often rather nontrivial \[22\]. Therefore, sometimes the consideration is carried out in the frame of a quasiharmonic approach with dynamical “harmonics” \[19\]. One takes some number of independent harmonic spinors assuming that implicit gauge conditions and second class constraints have been resolved with respect to other ones. Undoubtedly, any set of independent spinors for the construction of arbitrary irreducible representation can be found among lines or rows of the matrix of spinor representation because any representation can be realized in the space of functions on the group. From the index spinor point of view, in such a consideration there is an implicit use of all or a part of index spinors. The index variables form a system of independent quantities in terms of which one can construct the harmonic matrix taking into account all present restrictions. Nonclassical nature of the spinor group in higher dimensions \[17, 21\] is a powerful evidence on behalf of explicit exploitation of the index variable conception not the complete and consecutive harmonic one.

In the massless case adapting the harmonic frame of reference to the positive energy-momentum vector, i. e. directing one of its basic vector along the isotropic vector of energy-momentum, it is possible to resolve the mass constraint \( p^2 \approx 0 \) in terms of the harmonic spinors \( v \) and \( \bar{v} \) as \( p \sim v\sigma\bar{v} \). Then, after suitable gauge fixing a dynamical role of space-time variables is given to harmonics \( v \) and their canonical conjugate momenta. In lower critical dimensions of space-time one has succeeded to deal with spinors subjected to explicitly formulated constraints. Twistor formulations \[8, 13, 16, 18, 20\] arise just in this way (here we are interested in the connection of twistors with harmonics and not in their group theoretical aspects). Of course, introducing twistors, when it is possible in the stated sense, one should not follow the described scheme necessary, i. e. one can take twistors irrespectively to harmonics. In particular, it is possible to introduce the twistors in parallel to the index spinors which then can be gauged away. Only in this case one can obtain the classical twistor theory in a
unique way. In the theory without index spinor, where the sign of energy is indeterminated, a choice of sign is necessary in such a transition.

It is important that even the use of pure gauge harmonics essentially changes the situation, since it yields a topologically nontrivial configuration space in the case of pure gauge torus degrees of freedom. Precisely this makes it possible to obtain different spins in the massless case without introducing nongauge variables \[22\].

Mechanical systems, which describe a massless particle with arbitrary spin, have the same number of dynamical degrees of freedom as the system for spinless particle. Therefore at the classical level all the models with commuting spinors as basic spin variables can be sufficiently easy reduced to each other by fixing some gauge symmetries. But it would be incorrect to think that all these models are identical. Only in the approach with the index spinor the massive and massless particles of arbitrary spin have uniform description with natural generalization to higher space-time dimensions.

For example, if in the massless index spinor model \[7\] one identified the variables \(v = |j|^{-1/2} p_\zeta\) and \(\omega = |j|^{+1/2} \zeta\) as spinorial components of the twistors \[16\] then the first class spin constraint \(S \equiv \frac{i}{2}(\zeta p_\zeta - \bar{p}_\zeta \bar{\zeta}) - j \approx 0\) has been rewritten as the twistorial Hamiltonian \(H \equiv \frac{i}{2}(\omega v - \bar{v} \bar{\omega}) - j \approx 0\) with a “classical helicity” \(j\). The fundamental twistor constraints \(T_{a\dot{a}} \equiv p_{a\dot{a}} - v_a \bar{v}_{\dot{a}} = 0\), which solve the massless condition \(p^2 = 0\), can be projected onto the twistor spinors. The projections \(vT\bar{v}, vT\bar{\omega}, \omega T\bar{v}\) and \(\omega T\bar{\omega}\) are equivalent to the set of constraints of the theory with the index spinor \[7\] which consists the massless constraint \(p^2 \approx 0\) and a part of projections of the spinorial constraints \(d_\zeta \equiv ip_\zeta - \bar{p}_\zeta \approx 0\) and \(d_{\bar{\zeta}} \equiv -ip_\zeta - \zeta \bar{p} \approx 0\) onto the index spinors \(\zeta\) and \(\bar{\zeta}\), i. e. to the constraints \(p^2, \zeta \bar{p} d_\zeta, d_\zeta \bar{p} \bar{\zeta}\) and \(\frac{1}{2}(\zeta d_\zeta + d_{\bar{\zeta}} \bar{\zeta}) \approx -(\zeta \bar{p} \bar{\zeta} - j)\). The projection \(\frac{1}{2}(\zeta d_\zeta - d_{\bar{\zeta}} \bar{\zeta}) = \frac{i}{2}(\zeta p_\zeta + \bar{p}_\zeta \bar{\zeta})\) which falls out from the last listing is nothing but the conformal constraint \(\omega v + \bar{v} \bar{\omega} = 0\) for which the constraint \(\omega T\bar{\omega} = 0\) plays a role of gauge condition and vice versa.

It is intersecting that in terms of twistorial variables \(v\) and \(\omega\) the spinor constraints \(d_\zeta\) and \(d_{\bar{\zeta}}\) take the form \(v \sim \bar{p} \bar{\omega}\) and c. c. which is in a sense dual to the twistor conditions \(\omega = \hat{x} \bar{v}\) and c. c.
We can say that the index spinors $\zeta$, $p_\zeta$ and the twistor ones $\omega$, $v$ replace each other under Fourier transformation in massless particle description.

The index spinor can be added to pseudoclassical mechanics \[3, 4\]. Then on the mass shell this theory becomes classically equivalent to the theory describing spin by both commuting and anticommuting spinors simultaneously. By now such a theory have not been developed enough. So we would like only to point out some of its interesting peculiarities and a way of establishing the equivalence. For simplicity we restrict ourself to the case of pseudoclassical mechanics with the single anticommuting vector $\psi_\mu$ usually refered to as describing spin $1/2$ particle. It is useful to represent the anticommuting variables of the pseudoclassical mechanics in the form

$$\psi_{\alpha\bar{\alpha}} = \psi_\mu \sigma^\mu_{\alpha\bar{\alpha}} = 2 j^{-1/2} \left[ (\hat{p}\bar{\zeta})_\alpha \bar{\theta}'_\bar{\alpha} + (\zeta \hat{p})_\alpha \theta''_\alpha \right] + 2 \rho \zeta_\alpha \bar{\zeta}_\bar{\alpha},$$

$$\psi_5 = - j^{-1/2} \frac{p^2}{m} \left[ \zeta \theta' + \bar{\theta} \zeta' \right] + \frac{1}{m} \rho (\zeta \bar{p} \bar{\zeta} + \bar{\psi}_5),$$

where primed thetas $\theta'$, $\bar{\theta}'$ are anticommuting, as well as $\rho$ and $\bar{\psi}_5$. These representations for $\psi_{\alpha\bar{\alpha}}$, $\psi_5$ are general on the surface of the constraints $\zeta \hat{p} \bar{\zeta} \approx j \neq 0$ and $p^2 \approx -m^2 \neq 0$. The quantities $\rho$ and $\bar{\psi}_5$ are unabiguously defined by $\psi_\mu$ and $\psi_5$, respectively. The spinor $\theta'_\alpha$ is defined up to a term of the form $i(\hat{p}\bar{\zeta})_\alpha \psi$ with the anticommuting $\psi$. Since now we have $p^\mu \psi_\mu = m \bar{\psi}_5$, the main constraint $p^\mu \psi_\mu + m \psi_5 \approx 0$ of the pseudoclassical mechanics takes an easy solved form $\bar{\psi}_5 \approx 0$. Furser we suppose $\bar{\psi}_5 = 0$.

Under the substitution of these expressions into the kinetic term $\frac{i}{2} (\psi^\mu \dot{\psi}_\mu + \psi_5 \dot{\bar{\psi}}_5)$ for the anticommuting (pseudo)vector $\psi_\mu$ and (pseudo)scalar $\psi_5$ we use also for the index spinor, $2 i \zeta \hat{p} + \lambda \zeta \hat{p} = 0$ and its c. c., the equations of motion for the energy-momentum vector, $\hat{p} = 0$, and the trivial identities $\zeta^2 = \bar{\zeta}^2 = 0$. One should redefine $\theta'$ and $\zeta$ by the mutually conjugated phase multipliers $k$ and $\bar{k} = k^{-1}$ in order to obtain the new index spinor $\zeta' = k \zeta$, which satisfies to the equations $\zeta' \hat{p} = 0$, and the new anticommuting spinor $\theta = k^{-1} \theta'$, which satisfies to the equations $\dot{\theta} = k^{-1} (\dot{\theta}' - \frac{\lambda}{2} \theta')$.

The equation for $k$ is $\dot{k} = - \frac{\lambda}{2} k$, and it can be easily solved as $k = C \exp(-\frac{\lambda}{2i} \int_{\tau_0}^\tau \lambda d\tau)$, where $C$ is a
constant of integration, \( \tau_0 \) is an initial moment of the “time” \( \tau \), and \( \lambda \) is a Lagrange multiplier at the spin constraint \( \zeta \hat{p} \tilde{\zeta} - j \).

After these redefinitions one obtains for the kinetic term an expression in which is contained the new anticommuting spinors \( \theta, \bar{\theta} \) and the anticommuting scalar \( \rho \) with their derivatives. The constant spinors \( \zeta', \tilde{\zeta'} \) enter in this expression as well. It is instructive to note that

\[
\dot{\psi}_{\alpha\dot{\alpha}} = 2j^{-1/2}[(\hat{p}\zeta')_\alpha \hat{\theta}_{\dot{\alpha}} + (\zeta'\hat{p})_{\dot{\alpha}} \hat{\theta}_\alpha] + 2\dot{\rho}\zeta'\tilde{\zeta'}, \quad \dot{\psi}_5 = j^{-1/2}m[\zeta'\hat{\theta} + \tilde{\theta}\tilde{\zeta'}] + \frac{1}{m}\dot{\rho}j.
\]

Regarding the expression as a Lagrangian one can find the equation of motion for \( \rho \) which is \( j^{3/2} \dot{\rho} = m^2(\zeta'\hat{\theta} + \tilde{\theta}\tilde{\zeta'}) \). This equation can be easily integrated but it is not required to the substitution of its solution into the action. The direct use of the equation of motion for \( \rho \) in the Lagrangian yields the expression

\[
i(\theta \hat{p}\tilde{\theta} - \hat{\theta}\tilde{p}\theta) + \frac{im^2}{j}(\hat{\theta}\zeta' \zeta'\hat{\theta} + \tilde{\theta}\tilde{\zeta'} \tilde{\zeta'}\tilde{\theta}).
\]

Here the first term originates from the vector part of the initial kinetic term of the pseudoclassical mechanics only and is nothing but the spinor kinetic term of the CBS superparticle [4, 10]. The second term originates from the both items in the kinetic term of the pseudoclassics and represents a term which corresponds to the second-rank-spinor central charge of superparticle. Tensor central charges in particle models have been considered in [24]. In our case we have a complex self-dual antisymmetric isotropic (singular) tensor of second rank.

For the massless particle the pseudoclassical description contains only anticommuting vector \( \psi_\mu \). Here we have \( p^\mu \psi_\mu = -j\rho \). So the imposition of the constraint \( p^\mu \psi_\mu \approx 0 \) yields \( \rho \approx 0 \) and the fermionic \( \kappa \)-symmetry of the model is achieved without involving any central charge. The further calculations in the massless case a quite similar to those have been made for the massive particle. Because of the gauge equivalence of the massless particle model, with the index spinor and with the twistor, which was mentioned above, our calculation can be regarded as analogous to those in the paper [25] but without a direct appeal to the twistors.
In this paper we obtain the propagator of the free arbitrary-spin massive $D = 4$ particle as the BFV-BRST path integral. The present scheme for the description of spin in terms of the index spinor [7], used in this paper, is obviously applicable to both the massless case and the case of higher space–time dimensions; so the problem we deal with is only a test to estimate the efficiency of the approach.

The consideration of this type within the framework of modern quantization methods is performed for the first time. Along with the extension to higher spins, the advantages of the Hamiltonian formulation have been first used for such a problem to full extent and the path integral has been calculated without resorting to arbitrary uncontrolled renormalizations of the integration measure. The derived propagator coincides with that found previously within the traditional field theory in the framework of $(2J + 1)$–component formalism [26].

We make no recourse to the conversion of second–class constraints, because it would be natural to carry out this consideration when studying the massless case, where the bosonic $\kappa$–symmetry of the model leads to a nontrivial algebra of first–class constraints.

The choice of the domain of integration over the gauge degrees of freedom, being the key point in a similar consideration, is made by finding and choosing the fundamental region of the modular group in the Teichmüller space [27]. This choice is not associated with the ambiguity of the procedure, it is rather the selection of a solution of the problem out of the set of possible ones for a fixed system. As a result, the causal propagator arises naturally.

A careful analysis of boundary conditions requires the modification of the expression for the transition amplitude in the path integral form by adding the boundary terms to the classical action [28]. The presence of second–class constraints gives rise to the canonical conjugation between the index spinor and its complex–conjugate one. Therefore, the boundary conditions are different for them, i. e., one is fixed at the initial moment of time, and the other is fixed at the final moment. It is shown that the resulting alternative corresponds to the choice of the particle spin description: either
holomorphic with undotted spinors or antiholomorphic with dotted ones. The transition from one choice to an other is equivalent to the exchange of the roles between particles and antiparticles.

This article is organized as follows. In sect. 2 we discuss the classical formulation of a spinning particle with the index spinor, proposed for the first time in paper [7], and carry out the Hamiltonian analysis in such a framework, which is necessary for the quantum path–integral consideration. In sect. 3 we construct the BFV–BRST path–integral expression for the transition amplitude in the “relativistic” gauge and evaluate it in sect. 4 for the holomorphic and antiholomorphic boundary conditions on the index–spinor variables. This calculation includes the determination of the integration domain and properly the integration over Teichmüller parameters. In sect. 5 we establish the links between the obtained transition amplitude and the propagator of a massive arbitrary–spin particle in the \((2J+1)\)–component formalism of the conventional field theory.

Here we use the spinor conventions of ref. [29].

2 Classical consideration of a spinning particle with the index spinor

In the usual space–time \((D = 4)\), a spinning particle can be described with the commuting coordinates \((z^A) = (x^\mu, \zeta^\alpha, \bar{\zeta}^{\dot{\alpha}})\), where \(x\) is a four–vector and \(\zeta\) is a Weil index spinor. We write the Lagrangian of the particle in the form [7]

\[
L = p\dot{\omega} - \frac{e}{2}(p^2 + m^2) - \lambda(\zeta\bar{\rho}\hat{\zeta} - j),
\]

where the bosonic “superform” is

\[
\omega \equiv \dot{\omega} \, d\tau = dx - id\zeta\sigma\bar{\zeta} + i\zeta\sigma d\bar{\zeta}.
\]

The kinetic term \(p\dot{\omega}\) represents the sum of the standard kinetic term for the spinless particle \(p\dot{x}\), where \(p_\mu\) is an auxiliary energy–momentum vector, and the spinning addend, which takes the standard oscillator form \(im(\zeta\sigma_0\hat{\zeta} - \hat{\zeta}\sigma_0\zeta)\) in the rest frame. As a result, the form \(\omega\) coincides with the N=1 SUSY superform, if one replaces the Grassmannian spinor by the index one there. It should be
stressed that this coincidence is not the result of some direct or naive generalization of the well–known expressions of the supersymmetric theory. Actually, this circumstance reflects an essential common feature of spin descriptions in terms of commuting and anticommuting variables. Namely, both these descriptions arise quite directly as neat “relativizations” of well known representations of the small group in terms of $c$–numbers and $a$–numbers, respectively, i. e., by construction of the corresponding induced representations of the Poincaré group. In natural way this inducing leads to the bosonic and fermionic supersymmetry of the respective kinetic terms in the language of theoretical mechanics. “Unfortunately” the bosonic supersymmetry is destroyed by the necessary restriction of the bosonic configuration space imposed by the spin constraint \([7]\); the “relativistic” form $\hat{\zeta} \hat{p} \zeta - j \approx 0$ of this constraint is explicitly involved in the Lagrangian \([11]\) with the Lagrange multiplier.

The einbein $e$ and $\lambda$ are the Lagrange multipliers in the Lagrangian \([11]\). The dimensionless constant $j$ is the classical spin the sign of which determines the sign of energy. Our action

$$A = \int_{\tau_i}^{\tau_f} L \, d\tau$$

universally describes both massless and massive cases, but in this work we restrict ourselves to consideration of the massive particle only, so that $m^2 > 0$. In the absence of the last term in the Lagrangian \([11]\), the massless particle action coincides with the Casalbuoni–Brink–Schwarz action \([4, 10]\) if one will interpret $\zeta$ as the Grassmannian spinor.

Apart from the constraints inserted into the action explicitly, i. e., the mass constraint

$$T \equiv \frac{1}{2}(p^2 + m^2) \approx 0 \quad (2)$$

and the spin one

$$\zeta \hat{p} \zeta - j \approx 0, \quad (3)$$

the Hamiltonization \([30]\) of the theory reveals the spinor Bose–constraints as well

$$d_\zeta \equiv ip \zeta - \hat{p} \zeta \approx 0, \quad \bar{d}_\zeta \equiv -i\bar{p} \zeta - \zeta \hat{p} \approx 0. \quad (4)$$
We omit obvious first class constraints on the momenta, which are canonically conjugate to the Lagrange multipliers, and the second–class constraints on the momenta conjugated to auxiliary variables $p$. Accounting of the last constraints in the strong sense by introducing the Dirac brackets is trivial and does not modify the brackets for fundamental variables. On the constraints surface the spin constraint (3) is equivalent to the following

$$S \equiv S_\zeta - j \equiv \frac{i}{2}(\zeta p_\zeta - \bar{p}_\zeta \bar{\zeta}) - j \approx 0,$$

(5)
since $S \equiv \frac{1}{2}(\zeta d_\zeta - \bar{d}_\zeta \bar{\zeta}) + (\zeta \hat{p}_\zeta - j)$.

The fundamental brackets are $\{z^A, p_B\} = \delta^A_B; \bar{p}_\zeta \equiv p_\zeta$.

The constraint algebra is found immediately, its nontrivial brackets are

$$\{d_\zeta, \bar{d}_\zeta\} = 2i\hat{p}, \quad \{S, d_\zeta\} = \frac{i}{2}d_\zeta, \quad \{S, \bar{d}_\zeta\} = -\frac{i}{2}\bar{d}_\zeta.$$

(6)

Thus, the constraints $(F_a) = (F_1, F_2) \equiv (T, S)$ belong to the first class, whereas the spinor constraints $(G_i) = (d_\zeta \alpha, \bar{d}_\zeta \dot{\alpha})$ are the second–class ones. The latter implies the consideration of the nonzero mass particle, i. e., $\hat{p}\bar{p} = m^2 > 0$. Certainly in the procedure of Hamiltonization, the spinor constraints (3) are primary, whereas the mass constraint (2) and the spin one (3) are the constraints of the second step of the procedure. The total Hamiltonian is a linear combination of the first–class constraints. This is due to the reparametrization invariance of the action.

The mass constraint (2) generates usual reparametrizations of space–time coordinates in the phase space

$$\delta x^\mu = p^\mu \epsilon, \quad \delta p_\mu = 0, \quad \delta e = \dot{\epsilon},$$

where the last equality follows from the invariance condition of the Hamiltonian action up to surface terms.

The spin constraint (3) generates phase transformations of phase space coordinates (in a sense of multiplying by the phase multiplier)

$$\delta \zeta^\alpha = \frac{i}{2}\zeta^\alpha \varphi, \quad \delta p_\zeta^\alpha = -\frac{i}{2}p_\zeta^\alpha \varphi; \quad \delta \lambda = \dot{\varphi}.$$
The corresponding variation of the action
\[ \delta A = \frac{1}{2}(p^2 - m^2)\epsilon \Big|_{\tau_i}^f \varphi \Big|_{\tau_i}^f + j \varphi \]
vanishes solely if \( \epsilon(\tau_i) = \epsilon(\tau_f) = 0 \) and \( \varphi(\tau_i) = \varphi(\tau_f) \). This circumstance makes directly admissible only “relativistic gauges” \([1]\), i.e., the gauges with derivatives which impose restrictions on \( \dot{\epsilon} \), expressing it in terms of other phase space variables. Then the condition of gauge conservation leads to the second–order equation on the parameter \( \epsilon \), which has the unique solution for any appropriate boundary conditions \([31]\). To use a canonical gauge without derivatives, one should consider it as a singular limit of a succession of admissible gauges \([32]\) or introduce appropriate boundary terms in the Hamiltonian action \([28]\).

3 BFV–BRST path integral for the transition amplitude

The most profound method for calculation of transition amplitude for constrained systems is the BFV–BRST formalism \([1]\). In this approach, for each first–class constraint \( F_a \) the set of coordinates of the initial phase space is supplemented by “dynamical” Lagrange multipliers

\( (\lambda^a) \equiv (\lambda_T, \lambda_S) \) with the same Grassmannian parity, their canonically conjugate momenta \( \pi_a \), \( \{\lambda^a, \pi_b\} = \delta^a_b \), and the ghost variables of the opposite parity. The ghost sector contains Grassmannian odd ghosts \( C^a \), antighosts \( \bar{C}_a \) and their canonically conjugate quantities \( \tilde{P}_a \) and \( P^a \), \( \{C^a, \tilde{P}_b\} = \{P^a, \bar{C}_b\} = \delta^a_b \). The variables \( \lambda, \pi, C, P \) are real, whereas \( \tilde{P}, \bar{C} \) are pure imaginary.

The variables of original phase space are subjected to the second–class constraints \([4]\), but the algebra of the first–class constraints \( F_a \) remains Abelian even after introducing the Dirac brackets

\[ \{A, B\}_D = \{A, B\} - \frac{i}{2p^2} \{A, \dot{d}_\zeta\} \tilde{P}\{d_\zeta, B\} + (-1)^{AB} \frac{i}{2p^2} \{B, \dot{d}_\zeta\} \tilde{P}\{d_\zeta, A\}. \]

Thus, the BRST charge has a zero rank and is a linear combination of the first–class constraints, \( F_a \) and \( \pi_a \), of the extended phase space

\[ \Omega = F_a C^a + \pi_a P^a; \]

(7)
\[ \{\Omega, \Omega\} = \{\Omega, \Omega\}_D = 0, \quad \overline{\Omega} = \Omega. \]

The BRST charge is Grassmannian odd, \( \epsilon(\Omega) = 1 \), and has the ghost number one, \( \text{gh}(\Omega) = 1 \), as it is supposed that

\[ \text{gh}(C) = \text{gh}(\mathcal{P}) = -\text{gh}(\tilde{\mathcal{P}}) = -\text{gh}(\tilde{C}) = 1. \]

The path integral for the transition amplitude

\[ Z_\Psi = \int D[z, p_z; \lambda, \pi; C, \tilde{P}; \mathcal{P}, \tilde{\mathcal{P}}] \prod_i \delta(G_i) \prod_\tau (2\pi)^2 |\det\{G_i, G_j\}|^{1/2} \exp(iA_{eff}) , \]  

includes the usual Liouville measure. Let us describe it in more detail. This means that in the standard finite–dimensional approximations of the path integral, the product of differentials of each pair of the canonically conjugate real bosonic variables in the measure is divided by \( 2\pi \). The differential of each variable that remains without its pair, in accordance with the boundary conditions under consideration, is also divided by \( 2\pi \). Here, all what has been said relates to the variables \( p_\mu \) and \( \lambda^a \). Similar multipliers are absent for the Grassmannian quantities. In the Hamiltonian approach, the multipliers corresponding to the realification Jacobian of the using complex variables do not appear in the measure.

Fulfillment of the second–class constraints (4) in expression (8) is provided by the functional \( \delta \)–functions. The multipliers corresponding to the realification Jacobian do not arise in the product \( \prod_i \delta(G_i) \) of \( \delta \)–functions of the complex second–class constraints. The measure is normalized by the determinant of Poisson brackets matrix for the second–class constraints

\[ \det\{G_i, G_j\} = (\det\{d_\zeta, \bar{d}_\zeta\})^2 = \left(4\pi^2\right)^2. \]

In addition, for every time “moment” \( \tau \) the factor \( 2\pi \) should be introduced into the measure on each pair of real bosonic second–class constraints.

The effective Hamiltonian action is

\[ A_{eff} = \int_{\tau_i}^{\tau_f} \left( p\dot{x} + \dot{\zeta}p_\zeta + \tilde{\zeta}p_\tilde{\zeta} + \pi\dot{\lambda} + \tilde{\mathcal{P}}\dot{C} + \tilde{C}\dot{\mathcal{P}} - H_\Psi \right) d\tau + A_{b.t.}. \]
The choice of the BRST Hamiltonian $H_{\Psi}$ and the boundary term $A_{b.t.}$ is argued below.

For the theory with a reparametrization invariance, the BRST Hamiltonian $H_{\Psi}$ is the BRST “derivative” of the gauge fermion $\Psi$:

$$H_{\Psi} = \{\Omega, \Psi\}.$$  

In the amplitude (8), one may use on equal footing both Poisson and Dirac brackets because, in our case, the Poisson brackets of the first–class constraints (entering into $\Omega$) and the arbitrary function are different from the Dirac brackets by addends which are proportional to the second–class constraints only. Thus these terms vanish on the second–class constraint surface. The gauge fermion is Grassmannian odd, $\epsilon(\Psi) = 1$, pure imaginary, $\overline{\Psi} = -\Psi$, and has a negative ghost number, $\mathrm{gh}(\Psi) = -1$. As it is known [3], the transition amplitude does not depend on the choice of a gauge fermion if the path integral is taken over the paths which belong to the one class of equivalence with respect to the BRST transformation. Such class is extracted by choosing the appropriate gauge and boundary conditions. The relativistic gauge with derivatives for the Lagrange multipliers ($\dot{\lambda}^a = 0$) corresponds to

$$\Psi = \tilde{P}_a \lambda^a,$$  

then

$$H_{\Psi} = F_a \lambda^a + \tilde{P}_a P^a.$$  

It should be stressed that an attempt to simplify further the expression for $\Psi$ by excluding some addends is rather undesirable. In such a way one loses the restriction to the only equivalence class of the paths and, as a result, arrives at “averaging” over many classes. Then an infinite renormalization of the integration measure becomes necessary [33].

We carry out the calculation of transition amplitude in the coordinate representation for the variables $z^A$ and in the mixed representation for the ghosts, i. e., we choose the boundary conditions

$$x^\mu(\tau_i) = x_i^\mu, \quad x^\mu(\tau_f) = x_f^\mu;$$  

$$\zeta^\alpha(\tau_1) = \zeta_1^\alpha, \quad \bar{\zeta}^\dot{\alpha}(\tau_2) = \bar{\zeta}_2^\dot{\alpha},$$  

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where the marks (1, 2) of spinors must be understood as \((f, i)\) for the holomorphic choice and as \((i, f)\) for the antiholomorphic one;

\[
\pi_a(\tau_i) = \pi_a(\tau_f) = 0; \quad C^a(\tau_i) = C^a(\tau_f) = 0; \quad \tilde{C}_a(\tau_i) = \tilde{C}_a(\tau_f) = 0. \tag{14}
\]

The boundary values are not fixed for the rest of variables. The boundary conditions imposed are BRST–invariant and ensure vanishing of the BRST charge on the boundaries. This provides the form–invariance of amplitude (8). One can understand the vanishing of the boundary values of the BRST charge as a classical manifestation of the quantum condition \(\hat{\Omega} |\psi_{\text{phys}}\rangle = 0\) [34]. The choice of boundary conditions for the index spinor is covariant and consistent with canonical conjugacy of \(\zeta\) and \(\bar{\zeta}\) (appearing due to the second–class constraints). Such choice is not unique. Using combinations of the index spinor and its conjugate momentum with other variables of the phase space, one can propose a variety of covariant boundary conditions on index variables. All they are in essence equivalent and reflect a concrete choice of the quantum description of a spin (i. e., realization of the Hilbert space of quantum states). We restrict ourselves to the consideration of two basic variants (13). As the simplest ones, they are described in the literature now [7].

With boundary conditions (13), the correctness of the variational principle, i. e., independence of any variation of the action from the boundary values of the variation for variables which are not fixed at the boundary, needs introducing the boundary term

\[
A_{b.t.} = -\frac{\varepsilon_\zeta}{2} (\zeta_i \bar{p}_\zeta i + \zeta_f \bar{p}_\zeta f - \bar{p}_\zeta i \zeta_i - \bar{p}_\zeta f \zeta_f ). \tag{15}
\]

Here \(\varepsilon_\zeta = +1\) corresponds to the holomorphic choice of the boundary condition (13) and \(\varepsilon_\zeta = -1\) corresponds to the antiholomorphic one.

4 Calculation of the path integral

In the gauge (10), the path integral (8) is factorized

\[
Z_\Psi = Z \cdot Z_{gh}. \tag{16}
\]
The path integral over the odd ghost variables has a simple Gaussian form

\[ Z_{gh} = \int D[C, \tilde{P}; P, \tilde{C}] \cdot \exp \{ i \int_{\tau_i}^{\tau_f} (\tilde{P}_a \dot{C}^a - \dot{\tilde{C}}_a P^a - \dot{P}_a C^a) d\tau \}. \]  

(17)

In Eq. (17), integration by parts has been performed in the index of the exponent with the boundary conditions for \( \tilde{C} \) being taken into account. This integral can be calculated by partition of the variation interval for the evolution parameter \( \tau \) into \( N \) equal parts. Put \( T_{\tau} = \tau_f - \tau_i \) and \( \Delta \tau = T_{\tau}/N \). Now the integrations over \( P \) and \( \tilde{P} \) automatically determine the normalization multiplier \( (i\Delta \tau)^2 N \) for the measure in the intermediate integral

\[ Z_{gh} = \int \tilde{D}[C, \tilde{C}] \exp \{ -i \int_{\tau_i}^{\tau_f} \dot{\tilde{C}}_a \dot{C}^a d\tau \} \]  

(18)

in its calculation by a discretization of the interval \([\tau_i, \tau_f]\). One can directly obtain (18) from (17) without discretization by sequential integrations over \( \tilde{P} \), which creates the \( \delta \)–function \( \delta(P - \dot{C}) \), and over \( P \), which annihilates this \( \delta \)–function, if no care is taken for normalization. When the vanishing boundary values of the ghost variables \( C \) and \( \tilde{C} \) are not assumed, the result of integration in (18) has the form

\[ Z_{gh} = -T_{\tau}^2 \exp \{ -i(\tilde{C}_f a - \tilde{C}_i a)(C^a_f - C^a_i)/T_{\tau} \}. \]  

(19)

Let us give some details of integration over the ghosts. As the integrand in (17) does not include any cross terms with the ghosts for different constraints, it is sufficient to restrict the consideration by the case of a unique first–class constraint. We have

\[ Z_{gh}^{(1)} = \lim_{N \to \infty} Z_{gh}^{(1)} [C_i, C_f, \tilde{C}_i, \tilde{C}_f; T_{\tau}, N] \]

where

\[ Z_{gh}^{(1)} [C_i, C_f, \tilde{C}_i, \tilde{C}_f; T_{\tau}, N] = \int \prod_{k=1}^{N-1} dC_k d\tilde{C}_k \cdot \prod_{k=1}^{N} d\tilde{P}_k dP_k \cdot \exp \left\{ i \sum_{k=1}^{N} \left[ \tilde{P}_k (C_k - C_{k-1}) - (\tilde{C}_k - \tilde{C}_{k-1}) P_k - \tilde{P}_k P_k \Delta \tau \right] \right\} ; \]

\( C_0 = C_i, \tilde{C}_0 = \tilde{C}_i, C_N = C_f, \tilde{C}_N = \tilde{C}_f. \) The superscript in \( Z_{gh}^{(1)} \) refers to the case of a unique constraint.
The shifts $\tilde{\mathcal{P}}_k \to \tilde{\mathcal{P}}_k - (\tilde{C}_k - \tilde{C}_{k-1})/\Delta \tau$, $\mathcal{P}_k \to \mathcal{P}_k + (C_k - C_{k-1})/\Delta \tau$ make possible integrations over $\mathcal{P}_k$ and $\tilde{\mathcal{P}}_k$:

$$Z_{gh}^{(1)}[C_i, C_f; \tilde{C}_i, \tilde{C}_f; T, N] = i\Delta \tau \int \prod_{k=1}^{N-1} dC_k d\tilde{C}_k (i\Delta \tau) \cdot \exp \left\{ -\frac{i}{\Delta \tau} \sum_{k=1}^{N} (\tilde{C}_k - \tilde{C}_{k-1})(C_k - C_{k-1}) \right\}.$$ 

As it is easily verified by the induction in $N$, this integral is independent of $N$:

$$Z_{gh}^{(1)}[C_i, C_f; \tilde{C}_i, \tilde{C}_f; T, N] = iT \exp \left\{ -i(\tilde{C}_f - \tilde{C}_i)(C_f - C_i)/T \right\} = Z_{gh}^{(1)}.$$ 

For zero boundary values of $C$ and $\tilde{C}$, and even for weaker conditions $C_f = C_i$ or $\tilde{C}_f = \tilde{C}_i$, we have

$$Z_{gh} = -T^2 \tau.$$ 

Thus the transition amplitude is

$$Z_{\Psi} = -T^2 \tau \int D[z, p; \lambda, \pi]\prod_{i, \tau} \delta(G_i) \cdot \prod_{\tau} 4|p^2|(2\pi)^2 \cdot \exp \left\{ i \int_{\tau_i}^{\tau_f} (p\dot{x} + \zeta p_\zeta + \bar{p}_\zeta \dot{\zeta} + \pi \dot{\lambda} - F_a \lambda^a) d\tau + iA_{b,t.} \right\},$$

where only the path integration over even variables remains to be done.

The integrals over the momenta $\pi_a$ of the Lagrange multipliers $\lambda^a$ give the $\delta$ – functions $\delta(\dot{\lambda}^a)$. So, after the path integration over $\lambda^a$ is performed, only usual integrals over zero modes of $\lambda^a$ remain in $Z_{\Psi}$. A precise determination of integration domain over zero modes of Lagrange multipliers, which plays a key role in our consideration, will be considered below.

It is convenient to carry out the integration by parts in the index of the exponent in [21]:

$$\int_{\tau_i}^{\tau_f} p\dot{x} d\tau = px \bigg|_{\tau_i}^{\tau_f} - \int_{\tau_i}^{\tau_f} \dot{p} x d\tau.$$ 

Then the path integrals over $x$ give the $\delta$ – functions $\delta(\dot{p})$, so that the path integrals over $p$ are reduced to usual integrals over zero modes of $p$. Hence, instead of the considered integrals in the index of the exponent, the expression $ip(x_f - x_i)$ appears.

The second–class constraints (4) have the form solved with respect to the spinor momenta $p_\zeta$ and $\bar{p}_\zeta$. So, we can easily integrate over these variables, using the functional $\delta$ – functions in the measure.
Now the transition amplitude (21) takes the form

\[ Z_\Psi = -T_\tau^2 \int \frac{d^4p}{(2\pi)^4} e^{ip(x_f - x_i)} d\lambda_T d\lambda_S \exp \left\{ -i \frac{T_\tau}{2} \lambda_T (p^2 + m^2) + iT_\tau \lambda_S J \right\} \cdot Z_\zeta \] (22)

with the path integrations over the index spinor

\[ Z_\zeta = \int \prod_\tau d^2\zeta d^2\bar{\zeta} |p^2| \cdot \exp \left\{ i \int_\tau i^\tau \left( -i \dot{\zeta}\hat{p}\bar{\zeta} + i \dot{\bar{\zeta}}\hat{p}\zeta - \lambda_S \hat{p}\bar{\zeta} \right) d\tau + i\tilde{A}_{b.t.} \right\} \] (23)

being factored. The boundary term (15) acquires the form

\[ \tilde{A}_{b.t.} = -i\varepsilon_\zeta (\zeta_i\hat{p}\bar{\zeta}_i + \zeta_f\hat{p}\bar{\zeta}_f) \] (24)

The quantum spin \( J \) is introduced in (22) (or rather from the very beginning in (8)) instead of \( j \) to stress the possibility of redefinition of the classical value of spin \( j \) by an ordering constant in the construction of a quantum theory corresponding to the classical one (1). As in the original functional Liouville measure (8), in Eq. (22) the differential of each coordinate of zero mode of the energy–momentum vector \( p \) is divided by \( 2\pi \). The same concerns the differential of zero mode of each Lagrange multiplier.

As usually, the exponential multiplier in the expression for Gaussian integral (23) can be easily found by the saddle–point method. When the boundary conditions are taken into account, the extremality of the exponent index with respect to \( \bar{\zeta} \) and \( \zeta \) is reached on the equations of motion for \( \zeta \) and \( \bar{\zeta} \):

\[ 2i\dot{\zeta}\hat{p} + \lambda_S \zeta\hat{p} = 0 \quad \text{and} \quad \text{c. c.} \] (25)

Only the boundary term contributes to the integrand exponent (23) after Eqs. (25) are taken into account. With boundary conditions (13) the solutions of equations (25) take the form

\[ \hat{\zeta}\hat{p} = e^{\frac{i}{2} \lambda_S (\tau - \tau_1)} \zeta_1\hat{p}, \quad \hat{\bar{\zeta}}\hat{p}\bar{\zeta} = e^{-\frac{i}{2} \lambda_S (\tau - \tau_2)} \bar{p}\zeta_2. \] (26)

Thus the integral (23) acquires the form

\[ Z_\zeta = \exp \{ 2\varepsilon_\zeta \zeta_1\bar{\zeta}_2 e^{-i\varepsilon_\zeta \lambda_S T_\tau/2} \}. \] (27)
The pre-exponential multiplier in (27) can be found from the prelimiting expression in the equation
\[ Z_\zeta = \lim_{N \to \infty} Z_\zeta(\zeta_1, \bar{\zeta}_2; T, N) \] for calculation of the considered Gaussian path integral (23) by discretization of the interval for the development parameter \( \tau \). For example, in the holomorphic case, we have
\[
Z_\zeta(\zeta_f, \bar{\zeta}_i; T, N) = \int \prod_{k=1}^{N} d^2\zeta_k \prod_{k=1}^{N} d^2\bar{\zeta}_k |p|^2 / \pi^2 \cdot \exp \left\{ -2 \left( 1 + i\lambda_S \frac{T}{2N} \right) \sum_{k=1}^{N} \zeta_k \hat{p} \bar{\zeta}_k + 2 \sum_{k=0}^{N} \zeta_k \hat{p} \bar{\zeta}_k \right\}
\]
with \( \zeta_0 = \bar{\zeta}_i, \zeta_N = \zeta_f \). Using mathematical induction it is not difficult to verify that
\[
Z_\zeta(\zeta_f, \bar{\zeta}_i; T, N) = \left[ 1 + \left( \frac{\lambda_S T}{2N} \right)^2 \right]^{-N} \exp \left\{ 2\zeta_f \hat{p} \bar{\zeta}_i \left( 1 + i\lambda_S \frac{T}{2N} \right)^{-N} \right\}
\]
from whence in a limit \( N \to \infty \) one obviously obtains (27).

Hence all the path integrations have been made and we obtain for the transition amplitude
\[
Z_\Psi = -T^2 \int \frac{d^4p}{(2\pi)^4} e^{ip(x_f - x_i)} \frac{d\lambda_T d\lambda_S}{(2\pi)^2} \exp \left\{ -i\frac{T}{2} \lambda_T (p^2 + m^2) + i\lambda_S T^2 J \right\} \cdot \exp \left\{ 2\varepsilon \zeta_1 \hat{p} \bar{\zeta}_2 e^{-i\zeta \lambda_S T^2 / 2} \right\}.
\]
Now only integrations over zero modes remain to be performed.

To characterize the gauge group orbits in the extended phase space, we introduce Teichmüller parameters
\[
C_T = \frac{1}{2} \int_{\tau_i}^{\tau_f} \lambda_T(\tau) d\tau, \quad C_S = \frac{1}{2} \int_{\tau_i}^{\tau_f} \lambda_S(\tau) d\tau.
\]
The parameter \( C_T \) has a transparent physical sense. In a suitable gauge it is the proper time [31]. The parameter \( C_S \) appears due to the fact that internal quantum numbers, such as a spin, a charge, etc., are realized in classical terms as topological toroidal-path characteristics. Let the parameter \( C_S \) be called the proper spin phase angle. As a result of boundary conditions on the parameters of reparametrization symmetry \( \epsilon(\tau_i) = \epsilon(\tau_f) = 0 \) and the phase transformations of index spinors \( \varphi(\tau_i) = \varphi(\tau_f) = 0 \), the Teichmüller parameters cannot be altered by gauge transformations because \( \delta \lambda_T = \dot{\epsilon}, \delta \lambda_S = \dot{\varphi} \). Admissibility of using the gauge with derivatives \( \dot{\lambda}_T = \dot{\lambda}_S = 0 \) means that the
gauge group orbits are bijectively characterized by zero modes of the Lagrange multipliers, for which, obviously, one has

$$C_T = \lambda_T \cdot T_T/2, \quad C_S = \lambda_S \cdot T_T/2.$$  \hfill (30)

Since the evolution parameter must bijectively correspond to the points of the particle world line [31], only reparametrizations described by strictly monotonic functions are admissible. As a consequence, the reparametrization group falls into two connected components. One of them is the subgroup which preserves the world line orientation, the second one is the set of reparametrizations which change this orientation. The corresponding modular group (the quotient of the complete gauge group by the connected component of the unit) is $Z_2$. The BFV–BRST quantization includes only gauge transformations which are continuously connected with the identical one, so the integration is to be taken over the fundamental domain of the modular group in the Teichmüller space. Let us choose the domain for the parameter $C_T$ assuming $C_T > 0$, then positive-energy particles move forward in time and the transition amplitude (8) is the causal propagator.

If it is assumed that internal quantum numbers are independent of the state of particle motion, then the fundamental domain of the modular group for phase transformations of index spinors is obvious from the expression derived for the amplitude (28). Owing to the integrand periodicity in the parameter $C_S = \lambda_S T_T/2$ at half-integers $J$, any interval period in length, say $[0, 2\pi]$, can be taken as a fundamental domain. The modular group of phase transformations is the group $Z$. One can invert the consideration and regard the modular invariance of the transition amplitude as a condition on the quantum theory obtained from the classical formulation by means of path–integral calculation. Then the boundary conditions on the parameter $\varphi$ should be weakened as $\varphi(\tau_f) - \varphi(\tau_i) = 2\pi n, n \in Z$, and the requirement of single–valuedness for the transition amplitude leads immediately to quantization of the spin $J$ (see a consideration of similar type, e. g., in [35]).

In (28) integration over the Teichmüller parameter $C_T$ is performed by using the well–known
equality
\[ \int_0^\infty dC_T \exp\{-iC_T(p^2 + m^2)\} = -i/(p^2 + m^2 - i0). \] (31)

So, the choice of a fundamental domain is equivalent to the usual pole bypass rule in the integral representation of the causal propagator.

The integral over the parameter \(C_S\) is found by application of the Cauchy integral formula
\[ f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(z')}{(z' - z)^{n+1}} dz'. \]
for the \(n\)-th derivative of an analytic function \(f(z)\) of a complex variable \(z\) in the interior of the domain bounded by a contour \(C\). If \(f(z) = \exp(Az)\) and the contour \(C\) is the unit circle with the center at \(z\), so that \(z' = z + e^{i\alpha}\) can be used as its parametrization, then we easily arrive at
\[ \int_0^{2\pi} \exp(-i\alpha + Ae^{i\alpha}) d\alpha = 2\pi A^n/n!. \] (32)

Finally, integrating in (29) over Teichmüller parameters with the help of the found equalities (31) and (32), we obtain the transition amplitude
\[ Z_{\psi} = \frac{-i}{(2J\varepsilon_\zeta)!} \int d^4p \frac{e^{ip(x_f - x_i)}}{p^2 + m^2 - i0} \frac{(2\varepsilon_\zeta \zeta_1 \tilde{p} \zeta_2)^{2J\varepsilon_\zeta}}{2\varepsilon_\zeta \Gamma(2J\varepsilon_\zeta + 1)}, \] (33)
which is nothing but the index–free form of Weinberg propagator [26] received in the \((2J + 1)\)-component formalism of the field theory. In the holomorphic case, the correct values of \(J\) are positive [1], \(J \geq 0\), and particles are described by symmetric spinors of rank \(2J + 1\) with undotted indices. In the antiholomorphic case \(J \leq 0\), and particles are described by spinors with dotted indices. In line with common reasons [31], connection among the sign of \(J\) and the sign of energy shows that alternation of the choice of boundary conditions (13) is equivalent to alternation of the definition of particles and antiparticles.

It should be noted that the spin dependent multiplier in integrand (33) can be represented in the form
\[ \frac{(2\varepsilon_\zeta \zeta_1 \hat{p} \zeta_2)^{2J\varepsilon_\zeta}}{(2J\varepsilon_\zeta)!} \frac{(2\varepsilon_\zeta \zeta_1 \hat{p} \zeta_2)^{2J\varepsilon_\zeta}}{(2J\varepsilon_\zeta)!} = \frac{1}{\Gamma(2J\varepsilon_\zeta + 1)} \]
which is unified for the whole spin tower. It indicates a possibility of analytic continuation to “any” complex \(J\) [36, 37], this being important for the theory of moving Regge poles and the string theory.
The transition amplitude as an index–free form of the propagator

Comparison of the obtained result with the result of paper [26] can be realized as follows. For the sake of definiteness, we shall restrict ourselves to the holomorphic case. Characteristics of the Wigner wave function $u(p;\zeta;\sigma)$ are determined by the primary quantization procedure [7], thus it obeys the spin constraint $(\hat{S}_z - J) = 0$ and the spinor constraint $\hat{d}_\zeta u = 0$, where the index spinor operators are realized as multiplication operators, $\zeta = \zeta$, and operators of their canonically conjugate momenta are realized as differentiation operators, $\hat{p}_\zeta = -i\partial/\partial\zeta$. As a consequence [7], $u(p;\zeta;\sigma) = e^{-\zeta \hat{p}_\zeta [\zeta]}^{J,\sigma}$, where $[\zeta]^{J,\sigma}$ is the homogeneous polynomial in $\zeta$ with degree $2J$, $\sigma = -J, -J+1, \ldots, J-1, J$.

It is important that the Wigner wave function of arbitrary momentum can be obtained from the wave function of the standard momentum by some transformation of the index spinor only

$$u(p;\zeta;\sigma) = u(\hat{p},\zeta Bp;\sigma).$$

Here $B_p = B_p^\dagger$ is the Wigner operator, $\hat{p} = B_p B_p^\dagger$, and $\hat{p} = (m,0)$ is the standard momentum. This circumstance makes possible such easy to pass from the arbitrary momentum frame to the standard momentum frame and conversely that in the following we usually do not thoroughly specify in what namely frame the consideration is carried out.

In the rest frame, the polynomial $[\zeta]^{J,\sigma}$ satisfies just the same condition as the Wigner wave function does, i.e., $(\hat{M}_3 - \sigma)u(\hat{p},\zeta;\sigma) = 0$, where the spinor part of the third component of the angular momentum is $\hat{M}_3 = \frac{1}{2}(\zeta_1 \frac{\partial}{\partial \zeta_1} + \zeta_2 \frac{\partial}{\partial \zeta_2} + \text{c.c.})$. This equation determines the degree $(J \mp \sigma)$ of the coordinate $\zeta^{1,2}$ entering into $[\zeta]^{J,\sigma}$, hence $[\zeta]^{J,\sigma} = N_J(2J)^{1/2}(\zeta_1)^{J-\sigma}(\zeta_2)^{J+\sigma}$. Here $(2J_{{\sigma}^+})$ is a binomial coefficient which allows us to identify the contraction over spinor indices and over the spin projection $\sigma$. The normalization multiplier $N_J$ is found below.

For transition to an arbitrary frame of reference one has to use the relation

$$[\zeta B]^{J,\sigma} = [\zeta]^{J,\sigma'} D^J(B)_{\sigma'}^\sigma,$$

where $B$ is an arbitrary $2 \times 2$ matrix and $D^J$ is the Wigner D–function.
The standard sesquilinear form in the space of holomorphic functions of the index spinor induces the inner product for polynomials in $\zeta$:

$$ (\varphi, \psi) = N \int d^2 \zeta \, d^2 \bar{\zeta} \, e^{-2\zeta \bar{\zeta}} \varphi \psi. \quad (34) $$

For homogeneous functions of degree $2J$ this inner product can be written in terms of the differential operator

$$ (\varphi, \psi) = \frac{2^{-(2J+2)}}{(2J)!m^{2J}} \left( \frac{\partial}{\partial \zeta} \bar{\psi} \right)^{2J} \bar{\varphi} \psi \cdot N \cdot \frac{4\pi^2}{m^2} . \quad (35) $$

The right-hand side of (35) coincides (up to the multiplier) with the known expression, see, e.g. [37] where the common factor is not fixed. Now from the orthonormality condition $([\zeta]^{J',\sigma'}, [\zeta]^{J,\sigma}) = \delta_{J',J} \delta_{\sigma',\sigma}$ the normalization of basic symmetric spinors $[\zeta]^{J,\sigma}$ can easily be found. It is sufficient to restrict ourselves to the calculation for the values $\sigma = -J$:

$$ N_J^2 = \frac{2^{2J+2}}{(2J)!m^{2J}} \, \frac{m^2}{4\pi^2} \cdot \frac{1}{N}. \quad (36) $$

The normalization multiplier $N$ is found from the condition that the expression (36) has to be equal to unity for $J = 0$: $N = m^2/\pi^2$.

Then to obtain the Weinberg propagator it is necessary to integrate the integrand (33) multiplied by $[\zeta_i]^{J,\sigma} [\bar{\zeta}_f]^{J,\sigma}$ over initial $\zeta_i$ and final $\zeta_f$ index spinors with the measure defined by Eq. (34). In such a way we obtain the propagator [26]

$$ G^{J}_{\sigma' \sigma}(x) = -im^{-2J} \Pi^J_{\sigma' \sigma}(i\partial) \Delta^C(x) , \quad (37) $$

where

$$ \Delta^C(x) = (2\pi)^{-4} \int d^4p e^{ipx}/(p^2 + m^2 - i0) $$

is the causal Green’s function of a scalar field, and the $(2J+1) \times (2J+1)$ – component matrix $\Pi^{J}_{\sigma' \sigma}$ is determined by identities in the following chain

$$ \frac{1}{(2J)!} (2\zeta_f \bar{\zeta}_i)^{2J} = \frac{1}{(2J)!} (\zeta_f B_p \bar{\zeta}_i)^{2J} \equiv [\zeta_f B_p]^{J,\sigma'} \Pi^J_{\sigma' \sigma} (\bar{p}) [B_p \bar{\zeta}_i]^{J,\sigma} = \quad (38) $$
\[
\{f\}^{I,\sigma}_{\sigma'} \Pi^{J}_{\sigma' \sigma}(p) [\bar{\zeta}^I]^{J, \sigma'} = p_{\mu_1} \cdots p_{\mu_{2J}} [\{f\}^{I, \sigma'}_{\sigma'} \mu_{\mu_1} \cdots \mu_{\mu_{2J}}] [\bar{\zeta}^I]^{J, \sigma'} (-1)^{2J}.
\]

The properties of \(t^{\mu_1 \cdots \mu_{2J}}_{\sigma' \sigma}(p)\) and \(\Pi^{J}_{\sigma' \sigma}(p)\) have been described in detail in \[26\].

In particular, it is essential in the calculation \(\text{calculation}\) \((38)\) that the quantities \(t^{\mu_1 \cdots \mu_{2J}}_{\sigma' \sigma}(p)\) and \(\Pi^{J}_{\sigma' \sigma}(p)\) have the following properties.

i) \(t^{\mu_1 \cdots \mu_{2J}}_{\sigma' \sigma}(p)\) is symmetric with respect to the 4–vector indices, because it is defined by contraction with the tensor power of the energy–momentum vector.

ii) \(t^{\mu_1 \cdots \mu_{2J}}_{\sigma' \sigma}(p)\) is traceless with respect to the 4–vector indices, due to the identity
\[
\sigma_{\alpha \alpha'} \sigma_{\beta \beta'} = -2\epsilon_{\alpha \beta} \epsilon_{\alpha' \beta'}
\]
and automatic symmetrization of the tensor power of the commuting spinor in spinor indices.

iii) \(\Pi^{J}_{\sigma' \sigma}(p)\) is a tensor, i. e.
\[
D^{J}(A) \Pi^{J}_{\sigma}(p) D^{J}(A)^\dagger = \Pi^{J}_{\sigma}(p'),
\]
where \(A \in SL(2, C), \ p' = A \hat{p} A^\dagger\). The irreducibility of the representation of the small group \(SO(3)\), which follows naturally from the model considered, and the Schur lemma mean that \(\Pi^{0}_{\sigma}(p)\) is a multiple of the identity matrix and it is normalized as \(\Pi^{0}_{\sigma}(p) = m^{2J} \delta_{\sigma, \sigma'}\). It can be shown, \[26\] that \(\Pi(p)\) is a polynomial of degree \(2J\) in the helicity operator \(\vec{p} \cdot \vec{M}^{(J)}/|\vec{p}|\). On the mass shell we have
\[
\Pi^{J}(p) = m^{2J} D^{J}(B \vec{p})^2 = m^{2J} \exp \left(-2\theta \vec{p} \cdot \vec{M}^{(J)}/|\vec{p}|\right),
\]
where \(\theta\) is defined by \(\sinh \theta = |\vec{p}|/m\). An explicit expression for the matrix \(\Pi^{J}\) is given in \[26\]. In the derivation \((37)\) from \((33)\), one should include the additional multipliers \(1/\pi\) (given by comparison with the direct calculation for \(J = 0\)) and \(2i\) (found from comparison between expressions for \(J \neq 0\)), which display the differences in the insertion of the pole multiplier in the integrand and in the transition to the nonzero spin case in our approach and in ref. \[26\].

Now, the relation between expressions \((33)\) and \((37)\) is obvious.
6 Conclusion

Thus, as it should be expected, the above–obtained transition amplitude (33) coincides with the index–free form of the Weinberg propagator (17), (20) for the massive particle with any spin \( J \), found in the \((2J + 1)\)–component formalism of the field theory. This result is obtained with the use of the BFV–BRST path–integral approach for the first time. It should be noted that it has been obtained without arbitrary renormalizations of the path integral measure. A similar study of the massless spinning particle, the spinning particle in the formulation with Dirac index Bose–spinors (the \(2(2J+1)\)–component formalism of the field theory) and for the higher space–time dimensions, as well as the supersymmetric generalization will be the subject of further articles.

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