CATEGORIAL MIRROR SYMMETRY
FOR K3 SURFACES

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Abstract. We study the structure of a modified Fukaya category
\( \mathcal{F}(X) \) associated with a K3 surface \( X \), and prove that whenever \( X \) is an
elliptic K3 surface with a section, the derived category of \( \mathcal{F}(X) \) is equivalent
to a subcategory of the derived category \( D(\hat{X}) \) of coherent sheaves on the
mirror K3 surface \( \hat{X} \).

1. Introduction

In 1994 M. Kontsevich conjectured that a proper mathematical formulation of
the mirror conjecture is provided by an equivalence between Fukaya’s category
of a Calabi-Yau manifold \( X \) and the derived category of coherent sheaves of the
mirror Calabi-Yau manifold \( \hat{X} \). Thus in some sense mirror symmetry relates
the symplectic structure of a Calabi-Yau manifold with the holomorphic structure
of its mirror. It is expected that special Lagrangian tori on \( X \) are mapped by
mirror symmetry to skyscraper sheaves on the mirror \( \hat{X} \).

This conjecture found some physical evidence with the discovery of D-branes
and the description of their role in mirror symmetry \([14, 17]\). Moreover, in a recent
Kontsevich’s conjecture has been proved in the case of the simplest Calabi-Yau manifolds, the elliptic curves.

Our approach to mirror symmetry follows the geometric interpretation due to Strominger, Yau and Zaslow. According to their construction, given a Calabi-Yau manifold admitting a foliation in special Lagrangian tori, its mirror manifold should be obtained by relative T-duality. In the case of K3 surfaces this formulation has been given a rigorous treatment in [2, 4], proving that Strominger, Yau and Zaslow’s approach is consistent with previous descriptions of mirror symmetry (this is also related to work by Aspinwall and Donagi).

We show here how the constructions described in [2, 4] can be given a categorial interpretation which provides a proof of Kontsevich’s conjecture in the case of K3 surfaces. More precisely, we show that, under some assumptions which will be spelled out in the following Sections, the derived category of a Fukaya-type category built out of special Lagrangian submanifolds of an elliptic K3 surface $X$ is equivalent to a subcategory of the derived category of coherent sheaves on the mirror surface $\hat{X}$. This subcategory is formed by the complexes of sheaves whose zeroth Chern character vanishes.

2. Special Lagrangian submanifolds and Fukaya’s category

**Definition 2.1.** Let $X$ be a Calabi-Yau $n$-fold, with Kähler form $\omega$ and holomorphic $n$-form $\Omega$. A (real) $n$-dimensional submanifold $\iota: M \hookrightarrow X$ of $X$ is said to be special Lagrangian if the following two conditions are met:

— $X$ is Lagrangian in the symplectic structure given by $\omega$, i.e. $\iota^*\omega = 0$;

— there exists a multiple $\Omega'$ of $\Omega$ such that $\iota^*\Im \Omega' = 0$.

It can be shown that the second condition is equivalent to requiring that the real part of $\Omega'$ restricts to the volume form of $M$ induced by the Riemannian metric of $X$. This exhibits special Lagrangian submanifolds as a special type of calibrated submanifolds.

There are not many explicit examples of special Lagrangian submanifolds. The simplest ones are the 1-dimensional submanifolds of an elliptic curve: the first condition is trivial, and the multiple $\Omega'$ of the global holomorphic one-form $\Omega$ is readily obtained by a holomorphic change of coordinates in the universal covering of the elliptic curve. Additional examples are provided by Calabi-Yau manifolds equipped with an antiholomorphic involution. Since the involution changes the sign of both the Kähler form and the imaginary part of the holomorphic $n$-form,
the fixed point sets of the involution are special Lagrangian submanifolds. A third example, and the most relevant in our case, arises when considering Calabi-Yau manifolds endowed with a hyper-Kähler structure. This is always the case in dimension 2, i.e. for K3 surfaces. In this case special Lagrangian submanifolds are just holomorphic submanifolds with respect to a different complex structure compatible with the same hyper-Kähler metric. This example will be discussed at length in the next section.

Special Lagrangian submanifolds have received remarkable attention in physics since the appearance of D-branes in string theory, and especially since their role turned out to be of a primary importance for the mirror conjecture \[3, 17\]. D-branes are special Lagrangian submanifolds of the Calabi-Yau manifold which serves as compactification space, and are equipped with a flat $U(1)$ line bundle. In the physicists’ language, special Lagrangian submanifolds of the compactification space are associated with physical states which retain part of the supersymmetry of the vacuum. For this (and other related) reasons, special Lagrangian submanifolds are often called supersymmetric cycles, or also BPS states.

Fukaya’s category, whose objects are Lagrangian submanifolds of a symplectic manifold, was introduced in connection with Floer’s homology \[6\]. Here, following closely the exposition of \[15\], we offer a description of a modified Fukaya category, built out of the special Lagrangian submanifolds of a Calabi-Yau manifold $X$. We shall call this the special Lagrangian Fukaya category (SLF category for short) of $X$, and will denote it by $\mathcal{F}(X)$. The objects in $\mathcal{F}(X)$ are pairs $(\mathcal{L}, \mathcal{E})$, where $\mathcal{L}$ is a special Lagrangian submanifold of $X$, and $\mathcal{E}$ is a flat vector bundle on $\mathcal{L}$. The morphisms in this category are a little bit more complicate to define. Since special Lagrangian submanifolds are $n$-cycles in a compact complex $n$-dimensional manifold, two special Lagrangian cycles generically intersect at a finite number of points. The basic concept is that a morphism between two objects in the SLF category is a way to pass from the vector bundle defined on one cycle to the bundle on the other.

**Definition 2.2.** Let $U_1 = (\mathcal{L}_1, \mathcal{E}_1)$, $U_2 = (\mathcal{L}_2, \mathcal{E}_2)$ be two objects in the SLF category. Then the space of morphisms $\text{Hom}(U_1, U_2)$ is defined to be

$$\text{Hom}(U_1, U_2) = \bigoplus_{x \in \mathcal{L}_1 \cap \mathcal{L}_2} \text{Hom}(\mathcal{E}_1|_x, \mathcal{E}_2|_x).$$

Thus the space of morphisms between two objects in the SLF category turns out to be the direct sum of vector spaces, each one being the space of homomorphisms
between the fibers of the two vector bundles at the intersection points of the two special Lagrangian cycles.

**Maslov index.** The space of morphisms between two objects is naturally graded over \(\mathbb{Z}\) by the Maslov index of the tangent spaces to the special Lagrangian submanifolds at the intersection points \([13]\). Let us recall some basic facts about the Maslov index. Let \(V\) be a \(2n\)-dimensional real symplectic vector space, and denote by \(\mathcal{G}(V)\) the Grassmannian of Lagrangian \(n\)-planes in \(V\). One has an isomorphism \(\mathcal{G}(V) \cong U(n)/O(n)\), so that \(\pi_1(\mathcal{G}(V)) \cong \mathbb{Z}\). The Maslov index is the unique integer-valued function on the space of loops in \(\mathcal{G}(V)\) satisfying some naturality conditions \([13]\) which include its homotopic invariance; thus the Maslov index provides an explicit isomorphism \(\pi_1(\mathcal{G}(V)) \rightarrow \mathbb{Z}\). In order to define a Maslov index for the intersection of Lagrangian cycles one has to slightly modify its definition so as to consider open paths. One first notices that the Lagrangian Grassmannian is naturally stratified by the dimension of the intersection of the Lagrangian \(n\)-planes with a fixed Lagrangian \(n\)-plane. Then one can define a Maslov index for the intersection of two Lagrangian planes as a \(\mathbb{Z}\)-valued function on the space of paths in \(\mathcal{G}(V)\) which is homotopy invariant under deformations of the paths that do not move the extrema out of their strata.

(Actually one should consider a Grassmannian of special Lagrangian (rather than just Lagrangian) planes, and restrict the Maslov index to it. This will be done in the next section in the case of K3 surfaces.)

**\(A^\infty\) structure.** Strictly speaking an SLF category, as it happens with ordinary Fukaya categories, is not a category at all, since in general the composition of morphisms fails to be associative. Associativity is replaced by a more complicated property, which makes Fukaya’s “category” into an \(A^\infty\) category.

**Definition 2.3.** An \(A^\infty\) category \(\mathfrak{F}\) consists of

- a class of objects \(\text{Ob}(\mathfrak{F})\);
- for any two objects \(\mathcal{X}, \mathcal{Y}\), a \(\mathbb{Z}\)-graded abelian group of morphisms \(\text{Hom}(\mathcal{X}, \mathcal{Y})\);
- composition maps

\[
m_k : \text{Hom}(\mathcal{X}_1, \mathcal{X}_2) \otimes \cdots \otimes \text{Hom}(\mathcal{X}_k, \mathcal{X}_{k+1}) \rightarrow \text{Hom}(\mathcal{X}_1, \mathcal{X}_{k+1}), \quad k \geq 1,
\]

of degree \(2 - k\), satisfying the condition

\[
\sum_{r=1}^n (-1)^r m_{n-r+1}(a_1 \otimes \cdots \otimes a_{s-1} \otimes m_r(a_s \otimes \cdots \otimes a_{s+r-1}) \otimes a_{s+r} \otimes \cdots \otimes a_n) = 0
\] (1)
for all \( n \geq 1 \), where

\[
\epsilon = (r + 1)s + r(n + \sum_{j=1}^{s-1} \deg(a_j)).
\]

Condition (I) implies that \( m_1 \) is a coboundary operator. The vanishing of the morphism \( m_1 \), together with condition (I) for the morphism \( m_3 \), implies that the composition law given by \( m_2 \) is associative.

Let us see how this \( A^\infty \) structure arises in Fukaya's category. Let us assume that the first object \( X_1 \) and the last object \( X_{k+1} \) have a nonvoid intersection, otherwise \( \text{Hom}(X_1, X_{k+1}) = 0 \) and the composition map is trivial. The composition maps are explicitly described as follows: Let \( u_j = (a_j, t_j) \in \text{Hom}(U_j, U_{j+1}) \), where \( a_j \in \mathcal{L}_j \cap \mathcal{L}_{j+1} \) and \( t_j \in \text{Hom}(E_j|_{a_j}, E_{j+1}|_{a_j}) \). One defines

\[
m_k(u_1 \otimes \cdots \otimes u_k) = \sum_{a_{k+1} \in \mathcal{L}_1 \cap \mathcal{L}_{k+1}} (C(u_1, \ldots, u_k), a_{k+1}).
\]

Here one has

\[
C(u_1, \ldots, u_k, a_{k+1}) = \sum_{\phi} \pm \exp[2\pi i(\int \phi^* \omega_c)]P \exp[\oint \phi^* \beta].
\]

This requires some explanation. The sum is performed over holomorphic and antiholomorphic maps \( \phi \) from the disc \( D^2 \) into the manifold \( X \), up to projective equivalence, with the following boundary condition: there are \( k + 1 \) points \( p_j = e^{2\pi a_j} \in S^1 = \partial D^2 \) such that \( \phi(p_j) = a_j \) and \( \phi(e^{2\pi \alpha}) \in \mathcal{L}_j \) for \( \alpha \in (\alpha_{j-1}, \alpha_j) \). The two-form \( \omega_c \) appearing in (2) is the complexified Kähler form, while \( \beta \) is the connection of the bundle restricted to the image of the boundary of the disc. \( P \) represents a path-ordered integration, defined by

\[
P \exp(\oint \phi^* \beta) = \exp(\int_{\alpha_k}^{\alpha_{k+1}} \beta_k d\alpha) t_k \exp(\int_{\alpha_{k-1}}^{\alpha_k} \beta_{k-1} d\alpha) t_{k-1} \ldots t_1 \exp(\int_{\alpha_1}^{\alpha_k} \beta_1 d\alpha).
\]

3. **The special Lagrangian Fukaya category for K3 surfaces**

The main purpose of this section is to give a description of the SLF category when the Calabi-Yau manifold is a K3 surface \( X \). In this case, due to the fact that K3 surfaces admit hyper-Kähler metrics, special Lagrangian submanifolds are very easily exhibited. Let us denote by \( \omega \) the Kähler form associated with given hyper-Kähler metric and complex structure. One also has a holomorphic 2-form \( \Omega = x + iy \). The three elements \( \omega, x, y \) can be regarded as vectors in
the cohomology space $H^2(X, \mathbb{R})$; if the latter is equipped with the scalar product of signature (3,19) induced by the intersection form on $H^2(X, \mathbb{Z})$, these three elements are spacelike, and generate a 2-sphere which can be identified with the set of complex structures compatible with the fixed hyper-Kähler metric.

It is very easy to check that what is special Lagrangian in the original complex structure is holomorphic in the complex structure in which the roles of $\omega$ and $x$ are exchanged (up to a sign) \[ \mathbb{R} \] (this corresponds to a rotation of 90 degrees around the $y$ axis). We shall call such a change of complex structure a hyper-Kähler rotation.

We want in particular to consider elliptic K3 surfaces $X$ which admit a section.\[ \mathbb{R} \] K3 surfaces arising as compactification spaces of string theories which admit mirror partners are always of this type \[ \mathbb{R} \]. So let us consider a K3 surface $X$ that in a complex structure $\mathcal{I}$ is elliptic and has a section. Let us denote by $X_{\mathcal{I}}$ this K3 surface. The Picard group of $X_{\mathcal{I}}$ is generated by the section, by the divisor of the generic fiber, and by the irreducible components of the singular fibers that do not intersect the section.\[ \mathbb{R} \] If we perform the hyper-Kähler rotation described above, and call $\mathcal{J}$ the new complex structure, the submanifolds which were holomorphic in the complex structure $\mathcal{I}$ are now special Lagrangian. Assuming that $X_{\mathcal{J}}$ is elliptic as well, it has been shown \[ \mathbb{R} \] that this hyper-Kähler rotation reproduces, at the level of the Picard lattice of an elliptic K3 surface, the effects of mirror symmetry previously described in an algebraic way \[ \mathbb{R} \]. So the varieties $X_{\mathcal{I}}$ and $X_{\mathcal{J}}$ can be regarded as a mirror pair of K3 surfaces.

In this way one has a very precise picture of the configuration of special Lagrangian submanifolds of $X_{\mathcal{J}}$. Moreover, the flat vector bundles one considers on special Lagrangian submanifolds of $X_{\mathcal{J}}$ are (flat) holomorphic bundles in the complex structure $\mathcal{I}$.

On a K3 surface the $A^\infty$ structure of the SLF category turns out to be trivial, that is, the SLF category is a true category. In fact due to the hyper-Kähler structure of a K3 surface $X$, the Grassmanian of special Lagrangian subspaces of the tangent space to $X$ at a point reduces to a copy of $\mathbb{P}^1$, hence is simply connected. Moreover, special Lagrangian 2-cycles always intersect transversally, so there is no stratification, and the Maslov index is trivial (cf. \[ \mathbb{R} \]). The Hom

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\[ \mathbb{R} \]This means that there exists an epimorphism $\pi: X \to \mathbb{P}^1$ whose generic fiber is a smooth elliptic curve and admitting a section $e: \mathbb{P}^1 \to X$.

\[ \mathbb{R} \]Actually one may have further generators of the Picard group provided by additional sections of the projection $\pi: X \to \mathbb{P}^1$. 

6
groups in the SLF category have trivial grading, so \( m_k = 0 \) for \( k \neq 2 \), while condition (\( \Box \)) for \( m_3 \) yields the associativity of the composition of morphisms.

The triviality of this Fukaya category for K3 surfaces may be related, via Sadov’s claim [16] that the Floer homology of an almost Kähler manifold \( X \) with coefficients in the Novikov ring of \( X \) is equivalent to the quantum cohomology of \( X \), to the triviality of the quantum cohomology of K3 surfaces.

4. **The special Lagrangian Fukaya category and the derived category of coherent sheaves**

We want now to describe a construction which exhibits the relationship between the SLF category of a K3 surface and the derived category of coherent sheaves on the mirror K3 surface.

We start by briefly recalling the definition of derived category of an abelian category \( \mathcal{A} \) (cf. [18]). One starts from the category \( K(\mathcal{A}) \) whose objects are complexes of objects in \( \mathcal{A} \), while the morphisms are morphisms of complexes identified up to homotopies. Let \( Ac(\mathcal{A}) \) be the full subcategory of \( K(\mathcal{A}) \) formed by acyclic complexes (i.e. complexes such that all cohomology objects vanishes). The derived category \( D(\mathcal{A}) \) is by definition the quotient \( K(\mathcal{A}) / Ac(\mathcal{A}) \). A morphism between two objects \([\mathcal{X}], [\mathcal{Y}]\) in \( D(\mathcal{A}) \) is represented by a diagram of morphisms in \( K(\mathcal{A}) \)

\[
\mathcal{X} \xleftarrow{q} \mathcal{Z} \xrightarrow{m} \mathcal{Y}
\]

where \( q \) is a quasi-isomorphism, i.e., a morphism which induces an isomorphism between the cohomology objects of \( \mathcal{X} \) and \( \mathcal{Y} \). Two objects \( \mathcal{X}, \mathcal{Y} \) in \( K(\mathcal{A}) \) turn out to be equivalent in \( D(\mathcal{A}) \) whenever they are quasi-isomorphic, that is, whenever there is a diagram as above where \( m \) is also a quasi-isomorphism. If there exists a quasi-isomorphism between two complexes, these represent isomorphic objects in \( D(\mathcal{A}) \).

Now we consider a K3 surface \( X \) with a fixed hyper-Kähler metric, and a compatible complex structure \( J \). If we start from an object \((\mathcal{L}, \mathcal{E})\) in the SLF category \( \mathcal{F}(X_J) \), where \( \mathcal{L} \) is a special Lagrangian submanifold of real dimension 2, and \( \mathcal{E} \) a flat rank \( n \) vector bundle on \( \mathcal{L} \), in the complex structure \( I \) obtained by performing a hyper-Kähler rotation \( \mathcal{L} \) is a divisor, and \( \mathcal{E} \) may be regarded as a coherent sheaf on \( X_J \) concentrated on \( \mathcal{L} \), whose restriction to \( \mathcal{L} \) is a rank \( n \) locally free sheaf. This operation is clearly functorial: the sheaf of homomorphisms between two such objects is a torsion sheaf concentrated on the points where the
two divisors intersect. The stalks at such points are precisely the homomorphisms between the stalks of the two coherent sheaves. Thus the hyper-Kähler rotation induces a functor between the SLF category $\mathcal{F}(X_I)$ and the category $\mathcal{C}(X_I)$ of coherent sheaves supported on a divisor of $X_I$, whose restriction to the divisor is locally free. This functor is clearly faithful, free and representative and hence gives an equivalence of the two categories.

Remark 4.1. To take account of the singular divisors in $X$ we should consider torsion-free sheaves rather than just locally free ones. However, since any coherent sheaf on a singular curve over $\mathbb{C}$ has a projective resolution by locally free sheaves, what we miss by restricting to locally free sheaves will be recovered when we go to the derived categories.

The category $\mathcal{C}(X_I)$ that we obtained via a hyper-Kähler rotation is not abelian (kernels and cokernels of morphisms do not necessarily lie in the category). In order to introduce a related derived category, one should find a somehow natural abelian category $\tilde{\mathcal{C}}(X_I)$ containing $\mathcal{C}(X_I)$. The most obvious choice is the subcategory of the category $\mathcal{Coh}(X_I)$ of coherent sheaves on $X_I$ whose objects are sheaves of rank 0 (in particular we are adding all the skyscraper sheaves).

We assume that the K3 surface $X_I$ is elliptic and has a section. Since $X_I$ is elliptic any point $p \in X$ lies on a divisor $D$. The complex $0 \to k_p \to 0$ concentrated in degree zero, where $k_p$ is the length one skyscraper at $p$, is quasi-isomorphic to the complex of sheaves in $\mathcal{C}(X_I)$

$$0 \to \mathcal{O}_D(-p) \to \mathcal{O}_D \to 0$$

where $\mathcal{O}_D$ is the term of degree zero.

Since every coherent sheaf on a smooth curve is the direct sum of a locally free sheaf and a skyscraper sheaf, we obtain that all coherent sheaves whose support lies on a divisor are objects of $\tilde{\mathcal{C}}(X_I)$.

It is not always true the derived category of an abelian subcategory $\mathcal{C}'$ of an abelian category $\mathcal{C}$ is also a subcategory of the derived category of $\mathcal{C}$. However, this is indeed the case for the category $\tilde{\mathcal{C}}(X_I)$, as we shall next show. Let us recall the definition of thick subcategory (cf. e.g. [9]).

**Definition 4.2.** A subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is said to be thick if for any exact sequence $\mathcal{Y} \to \mathcal{Y}' \to \mathcal{W} \to \mathcal{Z} \to \mathcal{Z}'$ in $\mathcal{C}$ with $\mathcal{Y}, \mathcal{Y}', \mathcal{Z}, \mathcal{Z}'$ in $\mathcal{C}'$ then $\mathcal{W}$ belongs to $\mathcal{C}'$ as well.
Now, \( \tilde{\mathcal{C}}(X_I) \) is a thick subcategory of \( \mathcal{Coh}(X_I) \): in fact, the generic stalk of a sheaf in \( \tilde{\mathcal{C}}(X_I) \) is 0, and, since a sequence of sheaves is exact when it is so at the stalks, this implies that also the generic stalk of \( \mathcal{W} \) is 0, i.e. \( \mathcal{W} \) also is a rank 0 sheaf. Moreover, \( \tilde{\mathcal{C}}(X_I) \) is a full subcategory, so that we can apply the following theorem [3].

**Theorem 4.3.** Let \( \mathcal{C} \) be an abelian category, \( \mathcal{C}' \) a thick full abelian subcategory. Assume that for any monomorphism \( f: \mathcal{W}' \to \mathcal{W} \) with \( \mathcal{W}' \in \text{Ob}(\mathcal{C}') \), there exists a morphism \( g: \mathcal{W} \to \mathcal{Y} \), with \( \mathcal{Y} \in \text{Ob}(\mathcal{C}') \), such that \( g \circ f \) is a monomorphism. Then the derived category \( D(\mathcal{C}') \) is equivalent to the subcategory of \( D(\mathcal{C}) \) consisting of complexes whose cohomology objects belong to \( \mathcal{C}' \).

In our case the condition of this theorem is easily met, just take for \( g \) the evaluation morphism. Thus the derived category built up from \( \tilde{\mathcal{C}}(X_I) \) is a subcategory of the derived category of coherent sheaves.

The image of the category \( \tilde{\mathcal{C}}(X_I) \) in cohomology is \( H^{1,1}(\mathbb{Z}) \oplus H^4(\mathbb{Z}) \) and is an ideal in the algebraic cohomology ring. It is a trivial observation that adding the unit to an ideal yields the whole ring. Hence, since the Chern map is a ring morphism between K-theory and algebraic cohomology, it follows immediately that by adding the structure sheaf to \( \mathcal{C}(X_I) \) we recover the whole derived category of coherent sheaves.

Adding the structure sheaf of the surface has no motivation from a strictly geometric viewpoint, but has physical grounds in the necessity of having 0-branes in the spectrum of the theory. (The association between coherent sheaves and branes is usually done by taking the Poincaré dual of the support of the coherent sheaf.)

Let us check explicitly that every complex \( 0 \to \mathcal{F} \to 0 \), where \( \mathcal{F} \) is a coherent sheaf on \( X_I \), is quasi-isomorphic to a complex

\[
0 \to \oplus \mathcal{O}_{X_I} \to \mathcal{S} \to 0,
\]

where \( \mathcal{S} \) is a coherent sheaf supported on a divisor. Let us fix a very ample divisor \( H \) in \( X_I \). Every coherent sheaf \( \mathcal{F} \) admits a finite projective resolution by sheaves of the form \( \oplus_{j=1}^r \mathcal{O}_{X_I}(-m_j H) \) (cf. [7]). Moreover, due to the exactness of the sequence

\[
0 \to \mathcal{O}_{X_I}(-m_i H) \to \mathcal{O}_{X_I} \to \mathcal{O}_{m_i H} \to 0,
\]

the sheaf \( \oplus_{j=1}^r \mathcal{O}_{X_I}(-m_j H) \) is quasi-isomorphic to a complex

\[
0 \to \oplus \mathcal{O}_{X_I} \to \mathcal{S} \to 0
\]
where $\mathcal{S}$ is a coherent sheaf supported on a divisor (here $\bigoplus \mathcal{O}_{X_i}$ is concentrated in degree 0). This proves that the whole derived category of coherent sheaves is obtained by complexes whose elements are either direct sums of the structure sheaf or lie in the image of the SLF category.

Collecting all these results, we have eventually proved the following fact: the derived category of a “natural abelianization” of the SLF category $\mathfrak{F}(X_I)$ is equivalent to a subcategory of the derived category $\mathcal{D}(X_I)$ of coherent sheaves on $X_I$.

5. Conclusions

Mirror symmetry yields definite predictions about the transformations of branes \cite{14}, which can be given a precise mathematical interpretation in terms of transformations of the derived category of coherent sheaves. In \cite{2} it was indeed proved that the action of a Fourier-Mukai transform on the derived category of coherent sheaves mimics precisely the action of mirror symmetry on branes. In particular, this shows that on an elliptic K3 surface genus 1 special Lagrangian cycles are mapped to points, which is exactly the behaviour one expects from mirror symmetry \cite{12}.

Moreover, one can argue that the very essence of mirror symmetry is an equivalence between a suitable (derived) version of the Fukaya category of a Calabi-Yau manifold $X$ and the derived category of coherent sheaves of the mirror manifold $\hat{X}$. This is exactly what we have proved when $X$ is an elliptic K3 surface with a section, admitting also a fibration in special Lagrangian tori. After performing a hyper-Kähler rotation, we map the SLF category into a category whose “natural abelianization” is a thick full subcategory of the category of coherent sheaves. Now, if we consider an extension of this category adding the structure sheaf (which seems in some sense very natural) and derive this, we obtain the whole derived category of coherent sheaves. Applying a Fourier-Mukai transform (which at the level of derived categories is an equivalence) we obtain the desired transformation mapping 2-cycles of genus 1 to points. If, instead, we do not extend the SLF category by adding the structure sheaf, we obtain a subcategory of the derived category of coherent sheaves. This will be mapped by Fourier-Mukai transform to another subcategory, but again this will show the desired feature of mapping 2-cycles of genus 1 to points.

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