Conformal infinitesimal bendings of submanifolds

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Abstract

This paper belongs to the realm of conformal geometry and deals with Euclidean submanifolds that are conformally bendable. Our first result is a Fundamental theorem for conformal infinitesimal bendings. It is established that the integrability condition for the differential equation of the bending is a system of three equations for a certain pair of tensors that are determined by the bending. Our second result is a rigidity theorem for conformal infinitesimal bendings of submanifolds that lay in low codimension.

This is a paper in conformal geometry that deals with smooth variations by immersions of an Euclidean submanifold of any dimension and in any codimension when the variation is an infinitesimal conformal bending. Until now the study of this class of bendings has received limited attention; see [17] for an exception. This is not the situation for the more restricted case of isometric infinitesimal bendings. In fact, the study of these bendings for hypersurfaces is a classical subject already considered by Sbrana [13] at the beginning of the 20th century after the earlier rigidity result contained in Cesàro’s book [3]. For recent results on the subject, we refer to [11] and [12] in the hypersurface case and for submanifolds in higher codimension to [6] and [7].

First Sbrana [14] and subsequently Cartan [1] classified isometrically bendable Euclidean hypersurfaces. Shortly after Cartan [2] gave a classification of the Euclidean hypersurfaces that admit nontrivial conformal bendings; see also [9] and [10]. A conformal bending of a given isometric immersion $f: M^n \to \mathbb{R}^m$ of a Riemannian manifold $(M^n, \langle , \rangle)$ into Euclidean space is

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a smooth variation \( F : I \times M^n \to \mathbb{R}^m \), where \( 0 \in I \subset \mathbb{R} \) is an open interval and \( f_t = F(t, \cdot) \) with \( f_0 = f \) is a conformal immersion for any \( t \in I \). Hence, there is a positive function \( \gamma \in C^\infty (I \times M^n) \) with \( \gamma(0, x) = 1 \) such that

\[
\gamma(t, x) \langle f_t \ast X, f_t \ast Y \rangle = \langle X, Y \rangle
\]

for any tangent vector fields \( X, Y \in \mathfrak{X}(M) \). Here and in the sequel, we use the same notation for the inner products in \( \mathbb{R}^m \) and \( M^n \), and denote by \( \tilde{\nabla} \) and \( \nabla \) the respective associated Levi-Civita connections. The derivative of (1) computed at \( t = 0 \) gives that the variational vector field \( T = F_\ast \partial / \partial t |_{t=0} \) of \( F \) has to satisfy the condition

\[
(\tilde{\nabla}_X T, f_\ast Y) + (f_\ast X, \tilde{\nabla}_Y T) = 2 \rho \langle X, Y \rangle
\]

for any \( X, Y \in \mathfrak{X}(M) \), where \( \rho \in C^\infty (M) \) is the conformal factor of \( T \) given by \( \rho(x) = -(1/2) \partial \gamma / \partial t(0, x) \).

Trivial conformal bendings are the ones induced by a composition of the immersion with a smooth family of conformal transformations of the Euclidean ambient space. In this case, the variational vector field is locally the restriction to the submanifold of a conformal Killing vector field of the ambient space. Recall that conformal transformations of Euclidean space are characterized by Liouville’s theorem; see [16] for a nice discussion of this classical result.

A variation \( F : I \times M^n \to \mathbb{R}^m \) of an isometric immersion \( f : M^n \to \mathbb{R}^m \) is called an infinitesimal conformal variation if there is \( \gamma \in C^\infty (I \times M^n) \) satisfying \( \gamma(0, x) = 1 \) and such that

\[
\frac{\partial}{\partial t} |_{t=0} (\gamma(t, x) \langle f_t \ast X, f_t \ast Y \rangle) = 0
\]

for any \( X, Y \in \mathfrak{X}(M) \). This concept is just the infinitesimal analogue to a conformal bending. It is well-known from classical differential geometry of submanifolds that the right approach to study infinitesimal variations is to deal with the variational vector field. In the conformal case, that this field satisfies condition (2) leads to the following definition:

A conformal infinitesimal bending with conformal factor \( \rho \in C^\infty (M) \) of an isometric immersion \( f : M^n \to \mathbb{R}^m \) of a Riemannian manifold \( M^n \) into Euclidean space is a smooth section \( \mathcal{T} \in \Gamma (f^* T \mathbb{R}^m) \) that satisfies

\[
(\tilde{\nabla}_X \mathcal{T}, f_\ast Y) + (f_\ast X, \tilde{\nabla}_Y \mathcal{T}) = 2 \rho \langle X, Y \rangle
\]
for any $X, Y \in \mathfrak{X}(M)$. Associated to a conformal infinitesimal bending of $f: M^n \to \mathbb{R}^m$ we have that the variation $F: \mathbb{R} \times M^n \to \mathbb{R}^m$ given by

$$F(t, x) = f(x) + t\mathcal{T}(x)$$

is an infinitesimal conformal variation with variational vector field $\mathcal{T}$ since (3) is satisfied for $\gamma(t, x) = e^{-2t\rho(x)}$.

We observe that carrying a conformal infinitesimal bending is indeed a concept in conformal geometry. In fact, let $\mathcal{T}$ be a conformal infinitesimal bending of $f: M^n \to \mathbb{R}^m$ with conformal factor $\rho$. Then let $\psi$ be a conformal transformation of $\mathbb{R}^m$ with conformal factor $\lambda$ and set $g = \psi \circ f$. We have that $\mathcal{T}' = \psi_* \mathcal{T}$ is a conformal infinitesimal bending of $g: M^n \to \mathbb{R}^m$. To see this, use the well-known formula that relates the Levi Civita connections to conclude that

$$\langle \tilde{\nabla}'_{X} \mathcal{T}', g_* Y \rangle + \langle \tilde{\nabla}'_{Y} \mathcal{T}', g_* X \rangle = 2(\rho + \langle \mathcal{T}, \tilde{\nabla} \log \lambda \rangle)\langle g_* X, g_* Y \rangle$$

for any $X, Y \in \mathfrak{X}(M)$.

We call trivial a conformal infinitesimal bending of $f: M^n \to \mathbb{R}^m$ if locally it is the restriction of a conformal Killing vector field of the Euclidean ambient space to the submanifold. If any conformal infinitesimal bending of $f$ is trivial we say that the submanifold is **conformally infinitesimally rigid**.

For a conformal infinitesimal bending $\mathcal{T} \in \Gamma(f^*T\mathbb{R}^m)$ with conformal factor $\rho \in C^\infty(M)$ of an isometric immersion $f: M^n \to \mathbb{R}^m$, we first show that $\mathcal{T}$ together with the second fundamental form $\alpha: TM \times TM \to N_fM$ of $f$ determine an *associate pair* of tensors $(\beta, \mathcal{E})$, where $\beta: TM \times TM \to N_fM$ is symmetric and $\mathcal{E}: TM \times N_fM \to N_fM$ satisfies the compatibility condition

$$\langle \mathcal{E}(X, \eta), \xi \rangle + \langle \mathcal{E}(X, \xi), \eta \rangle = 0$$

(5)

for any $X \in \mathfrak{X}(M)$ and $\eta, \xi \in \Gamma(N_fM)$. Subsequently, we prove that the pair $(\beta, \mathcal{E})$ satisfies the following fundamental system of equations, where by the term fundamental we mean that they are the integrability condition for the
existence of a conformal infinitesimal bending.

\[
\begin{aligned}
A_{\beta(Y,Z)}X + B_{\alpha(Y,Z)}X - A_{\beta(X,Z)}Y &- B_{\alpha(X,Z)}Y + \langle Y, Z \rangle \nabla_X \nabla \rho \\
+ \text{Hess} \rho(Y, Z)X - \langle X, Z \rangle \nabla_Y \nabla \rho - \text{Hess} \rho(X, Z)Y & = 0 \\
(\nabla^\perp_X \beta)(Y, Z) - (\nabla^\perp_Y \beta)(X, Z) & = \mathcal{E}(Y, \alpha(X, Z)) - \mathcal{E}(X, \alpha(Y, Z)) \\
+ \langle Y, Z \rangle \alpha(X, Y, \nabla \rho) - \langle X, Z \rangle \alpha(Y, Y, \nabla \rho) & = 0 \\
(\nabla^\perp_X \mathcal{E})(Y, \eta) - (\nabla^\perp_Y \mathcal{E})(X, \eta) & = \beta(X, A_\eta Y) - \beta(A_\eta X, Y) \\
+ \alpha(X, B_\eta Y) - \alpha(B_\eta X, Y) & = 0
\end{aligned}
\tag{5}
\end{equation}

where \( A_\xi, B_\xi \in \Gamma(\text{End}(TM)) \) are the pair of tensors \( \langle A_\xi X, Y \rangle = \langle \alpha(X, Y), \xi \rangle \) and \( \langle B_\xi X, Y \rangle = \langle \beta(X, Y), \xi \rangle \).

An isometric infinitesimal bending is a conformal infinitesimal bending with conformal factor \( \rho = 0 \). It is said to be trivial if it is locally the restriction to the submanifold of a Killing vector field of the Euclidean ambient space. Let \( \mathcal{T}_1 \) be a conformal infinitesimal bending of \( f \) with conformal factor \( \rho \) and let \( \mathcal{T}_0 \) be an isometric infinitesimal bending of \( f \). Then \( \mathcal{T}_2 = \mathcal{T}_1 + \mathcal{T}_0 \) satisfies (2), and therefore it is also a conformal infinitesimal bending of \( f \) with conformal factor \( \rho \). From now on, we identify two conformal infinitesimal bendings of \( f \) with the same conformal factor as well as the corresponding pairs of associated tensors if the bendings differ by a trivial isometric infinitesimal bending.

**Theorem 1.** Let \( f: M^n \rightarrow \mathbb{R}^m, n \geq 3, \) be an isometric immersion of a simply connected Riemannian manifold. If there is a symmetric tensor \( \beta: TM \times TM \rightarrow N_f M, \) a tensor \( \mathcal{E}: TM \times N_f M \rightarrow N_f M \) that verifies (3) and \( \rho \in C^\infty(M) \) such that \( (\beta, \mathcal{E}, \rho) \neq 0 \) satisfy system (5) then there is a unique conformal infinitesimal bending \( \mathcal{T} \) of \( f \) with conformal factor \( \rho \) whose associated pair is \( (\beta, \mathcal{E}) \).

The above result takes a rather simpler form in the hypersurface case. In fact, let \( f: M^n \rightarrow \mathbb{R}^{n+1} \) be a hypersurface with shape operator \( A \) corresponding to the Gauss map \( N \in \Gamma(N_f M) \). Associated to a conformal infinitesimal bending we are now reduced to consider the tensor \( \mathcal{B} \in \Gamma(\text{End}(TM)) \) given by \( \beta(X, Y) = \langle \mathcal{B} X, Y \rangle N \). Then the fundamental system of equations takes the form

\[
\mathcal{B} X \wedge A Y - \mathcal{B} Y \wedge A X + X \wedge \nabla_Y \nabla \rho - Y \wedge \nabla_X \nabla \rho = 0
\tag{9}
\]
and

$$(\nabla_X \mathcal{B})Y - (\nabla_Y \mathcal{B})X + (X \wedge Y)A\nabla \rho = 0$$

for any $X, Y \in \mathfrak{X}(M)$.

Corollary 2. Let $f: M^n \to \mathbb{R}^{n+1}, n \geq 3$, be an isometric immersion of a simply connected Riemannian manifold. If there exists a symmetric tensor $0 \neq \mathcal{B} \in \Gamma(\text{End}(TM))$ and $\rho \in C^\infty(M)$ such that (9) and (10) are satisfied, then there exists a unique conformal infinitesimal bending $\mathcal{T}$ of $f$ with associated tensor $\mathcal{B}$ and conformal factor $\rho$.

The second main result in this paper is a rigidity theorem for conformal infinitesimal bendings of submanifolds of low codimension. The limitation on the codimension is due to the use of a result in the theory of flat bilinear forms that is known to be false for higher codimensions.

The conformal $s$-nullity $\nu^c_s(x)$ at $x \in M^n$, $1 \leq s \leq m - n$, of an isometric immersion $f: M^n \to \mathbb{R}^m$ is defined as

$$\nu^c_s(x) = \max\{\dim N(\alpha_{U^s} - \langle \cdot, \cdot \rangle \zeta)(x): U^s \subset N_f M(x) \text{ and } \zeta \in U^s\}$$

where $\alpha_{U^s} = \pi_{U^s} \circ \alpha$, $\pi_{U^s}: N_f M \to U^s$ is the orthogonal projection and $N(\alpha_{U^s} - \langle \cdot, \cdot \rangle \zeta)(x) = \{Y \in T_x M: \alpha_{U^s}(Y, X) - \langle Y, X \rangle \zeta = 0 \text{ for all } X \in T_x M\}$.

The conformal $s$-nullity is a concept in conformal geometry since it is easily seen to be invariant under a conformal change of the metric of the ambient space.

With respect to the next result, we observe that it has been shown in [11] that the set of Euclidean hypersurfaces admitting a nontrivial isometric infinitesimal bending is much larger than the ones allowing an isometric variation. We believe that the situation for hypersurfaces in the conformal case is similar.

**Theorem 3.** Let $f: M^n \to \mathbb{R}^{n+p}, n \geq 2p + 3$, be an isometric immersion with codimension $1 \leq p \leq 4$. Assume that the conformal $s$-nullities of $f$ at any point of $M^n$ satisfy $\nu^c_s \leq n - 2s - 1$ for all $1 \leq s \leq p$. Then $f$ is conformally infinitesimally rigid.

By the above result, in the case of hypersurfaces $f: M^n \to \mathbb{R}^{n+1}, n \geq 5$, the existence of a nontrivial conformal infinitesimal bending requires the presence at any point of a principal curvature of multiplicity at least $n - 2$. 

Theorem 3 is the version for conformal infinitesimal bendings of the rigidity result for conformal immersions due to do Carmo and Dajczer [4], where the concept of conformal s-nullity was introduced. Moreover, the corresponding result for isometric infinitesimal bendings was given by Dajczer and Rodríguez [8]. Both results, as well as additional information, can be found in [10]. With respect to the latter result, in sharp contrast with the situation in this paper a very short proof was possible in [10] by the use of a classical trick that fails completely in the conformal case.

1 The fundamental equations

In this section, we define a pair of tensors $(\beta, \mathcal{E})$ associated to a conformal infinitesimal bending $T$ of $f: M^n \to \mathbb{R}^m$ and show that they satisfy the system $(S)$ of equations.

Let $L \in \Gamma(\text{Hom}(TM, f^*T\mathbb{R}^m))$ be the tensor defined by

$$LX = \tilde{\nabla}_X T - \rho f_* X = \mathcal{J}_*X - \rho f_* X$$

for any $X \in \mathfrak{X}(M)$. Then (2) in terms of $L$ has the form

$$\langle LX, f_* Y \rangle + \langle f_* X, LY \rangle = 0$$

for all $X, Y \in \mathfrak{X}(M)$. Let $B: TM \times TM \to f^*T\mathbb{R}^m$ be the tensor given by

$$B(X, Y) = (\tilde{\nabla}_X L)Y = \tilde{\nabla}_X LY - L \nabla_X Y$$

for any $X, Y \in \mathfrak{X}(M)$. Then the tensor $\beta: TM \times TM \to N_f M$ is defined by

$$\beta(X, Y) = (B(X, Y))_{N_f M}$$

for any $X, Y \in \mathfrak{X}(M)$. Flatness of the ambient space and that

$$\beta(X, Y) = (\tilde{\nabla}_X \tilde{\nabla}_Y T - \tilde{\nabla}_{\nabla_X Y} T)_{N_f M} - \rho \alpha(X, Y)$$

give that $\beta$ is symmetric.

Let $\gamma \in \Gamma(\text{Hom}(N_f M, f_* TM))$ be defined by

$$\langle \gamma \eta, f_* X \rangle + \langle \eta, LX \rangle = 0.$$

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Then, let $E : TM \times N_f M \to N_f M$ be the tensor given by

$$E(X, \eta) = \alpha(X, \eta) + (L\alpha_X)_{N_f M}.$$

We have

$$\langle E(X, \eta), \xi \rangle = \langle \alpha(X, \eta) + L\alpha_X, \eta \rangle = \langle A\xi X, \eta \rangle - \langle Y, \eta \rangle = -\langle E(X, \xi), \eta \rangle,$$

and hence condition (5) is satisfied.

**Lemma 4.** We have that

$$(B(X, Y))_{TM} = \eta \alpha(X, Y) - \langle X, Y \rangle f_* \nabla \rho + \langle Y, \nabla \rho \rangle f_* X,$$

where $\nabla \rho$ denotes the gradient of $\rho$.

**Proof:** We have to show that

$$C(X, Y, Z) = \langle (B - \eta \alpha)(X, Y), f_* Z \rangle + \langle X, Y \rangle \langle Z, \nabla \rho \rangle - \langle Y, \nabla \rho \rangle \langle X, Z \rangle$$

vanishes for any $X, Y, Z \in \mathfrak{X}(M)$. The derivative of (11) gives

$$0 = \langle \tilde{\nabla} Z L X, f_* Y \rangle + \langle L X, \tilde{\nabla} Z f_* Y \rangle + \langle \tilde{\nabla} Z L Y, f_* X \rangle + \langle L Y, \tilde{\nabla} Z f_* X \rangle$$

$$= \langle B(Z, X), f_* Y \rangle + \langle L \nabla Z X, f_* Y \rangle + \langle L X, f_* \nabla Z Y + \alpha(Z, Y) \rangle$$

$$+ \langle B(Z, Y), f_* X \rangle + \langle L \nabla Z Y, f_* X \rangle + \langle L Y, f_* \nabla Z X + \alpha(Z, X) \rangle$$

$$= \langle B(Z, X), f_* Y \rangle + \langle L X, \alpha(Z, Y) \rangle + \langle B(Z, Y), f_* X \rangle + \langle L Y, \alpha(Z, X) \rangle$$

$$= \langle (B - \eta \alpha)(Z, X), f_* Y \rangle + \langle (B - \eta \alpha)(Z, Y), f_* X \rangle.$$

On the other hand,

$$\langle B(X, Y), f_* Z \rangle = \langle \tilde{\nabla} X \tilde{\nabla} Y \mathcal{J} - \tilde{\nabla} \nabla X Y \mathcal{J}, f_* Z \rangle - \langle X, \nabla \rho \rangle \langle Y, Z \rangle.$$

It follows that

$$C(X, Y, Z) = C(Y, X, Z)$$

and

$$C(Z, X, Y) = -C(Z, Y, X)$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Then

$$C(X, Y, Z) = -C(X, Z, Y) = -C(Z, X, Y) = C(Z, Y, X) = C(Y, Z, X) = -C(Y, X, Z) = -C(X, Y, Z) = 0,$$

as we wished. □

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Proposition 5. The pair of tensors $(\beta, \mathcal{E})$ associated to a conformal infinitesimal bending satisfy the system \([S]\) where \((7)\) is equivalent to

\[
(\nabla_X B_\eta)Y - (\nabla_Y B_\eta)X - B\nabla_X^\eta Y + B\nabla_Y^\eta X = A_{\mathcal{E}(X, \eta)} Y - A_{\mathcal{E}(Y, \eta)} X + \langle A_\eta X, \nabla \rho \rangle Y - \langle A_\eta Y, \nabla \rho \rangle X
\]

for any $X, Y \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_f M)$.

Proof: We first show that

\[
(\tilde{\nabla}_X Y) \eta = -f_* B_\eta X - LA_\eta X + \mathcal{E}(X, \eta)
\]

where $(\tilde{\nabla}_X Y) \eta = \tilde{\nabla}_X Y\eta - \nabla^\perp_X \eta$. Then \((11)\) and the derivative of \((12)\) give

\[
0 = \langle (\tilde{\nabla}_X Y) \eta, f_* Y \rangle + \langle Y \eta, \tilde{\nabla}_X Y \rangle + \langle \tilde{\nabla}_X L Y, \eta \rangle + \langle L Y, \tilde{\nabla}_X \eta \rangle
\]

\[
= \langle (\tilde{\nabla}_X Y) \eta, f_* Y \rangle + \langle B_\eta X, Y \rangle + \langle L A_\eta X, f_* Y \rangle.
\]

In fact, since $\langle Y \eta, \xi \rangle = 0$ we have

\[
0 = \langle (\tilde{\nabla}_X Y) \eta, \xi \rangle + \langle Y \eta, \tilde{\nabla}_X \xi \rangle = \langle (\tilde{\nabla}_X Y) \eta, \xi \rangle - \langle \alpha(X, Y \eta), \xi \rangle
\]

\[
= \langle (\tilde{\nabla}_X Y) \eta, \xi \rangle + \langle L A_\eta X - \mathcal{E}(X, \eta), \xi \rangle
\]

for any $X \in \mathfrak{X}(M)$ and $\eta, \xi \in \Gamma(N_f M)$, and \((13)\) follows.

Since

\[
(\tilde{\nabla}_X B)(Y, Z) = \tilde{\nabla}_X (\tilde{\nabla}_Y L) Z - (\tilde{\nabla}_{\nabla_X Y} L) Z - (\tilde{\nabla}_Y L) \nabla_X Z
\]

it is easy to see that

\[
(\tilde{\nabla}_X B)(Y, Z) - (\tilde{\nabla}_Y B)(X, Z) = -LR(X, Y)Z
\]

for all $X, Y, Z \in \mathfrak{X}(M)$. It follows using Lemma \(4\) that

\[
\langle (\tilde{\nabla}_X B)(Y, Z), f_* W \rangle = \langle (\tilde{\nabla}_X Y) \alpha(Y, Z) + \langle \nabla^\perp_X \alpha \rangle(YZ) - f_* A_{\beta(Y, Z)} X, f_* W \rangle
\]

\[
+ \langle Y, W \rangle \text{Hess } \rho(Z, X) - \langle Y, Z \rangle \text{Hess } \rho(X, W)
\]

for any $X, Y, Z, W \in \mathfrak{X}(M)$. Then from \((16)\), the Gauss equation

\[
R(Y, X)Z = A_{\alpha(X, Z)} Y - A_{\alpha(Y, Z)} X
\]
and the Codazzi equation, we obtain
\[
\langle (\tilde{\nabla}_X Y)\alpha(Y, Z) - (\tilde{\nabla}_Y Y)\alpha(X, Z), f_*W \rangle \\
= \langle LA_{\alpha(X,Z)}Y - LA_{\alpha(Y,Z)}X + f_*A_{\beta(Y,Z)}X - f_*A_{\beta(X,Z)}Y, f_*W \rangle \\
+ \langle Y, Z \rangle \text{Hess } \rho(X, W) - \langle Y, W \rangle \text{Hess } \rho(Z, X) + \langle X, W \rangle \text{Hess } \rho(Y, Z) \\
- \langle X, Z \rangle \text{Hess } \rho(Y, W).
\]

On the other hand, it follows from (14) that
\[
\langle (\tilde{\nabla}_X Y)\alpha(Y, Z) - (\tilde{\nabla}_Y Y)\alpha(X, Z), f_*W \rangle \\
= \langle f_*B_{\alpha(X,Z)}Y + LA_{\alpha(X,Z)}Y - f_*B_{\alpha(Y,Z)}X - LA_{\alpha(Y,Z)}X, f_*W \rangle.
\]

The last two equations give
\[
\langle B_{\alpha(X,Z)}Y - B_{\alpha(Y,Z)}X, f_*W \rangle = \langle A_{\beta(Y,Z)}X - A_{\beta(X,Z)}Y, W \rangle \\
+ \langle Y, Z \rangle \text{Hess } \rho(X, W) - \langle Y, W \rangle \text{Hess } \rho(Z, X) \\
+ \langle X, W \rangle \text{Hess } \rho(Y, Z) - \langle X, Z \rangle \text{Hess } \rho(Y, W),
\]
and this is (6). Using (15) we obtain
\[
((\tilde{\nabla}_X Y)\alpha(Y, Z) - (\tilde{\nabla}_Y Y)\alpha(X, Z), f_*W) \\
= \langle f_*B_{\alpha(X,Z)}Y + LA_{\alpha(X,Z)}Y - f_*B_{\alpha(Y,Z)}X - LA_{\alpha(Y,Z)}X, f_*W \rangle.
\]

Then, we have from (16) and the Gauss equation that
\[
(\nabla_{X}^\perp \beta)(Y, Z) - (\nabla_{Y}^\perp \beta)(X, Z) \\
= (LR(Y, X)Z)_{N,M} - \alpha(X, Y\alpha(Y, Z)) + \alpha(Y, Y\alpha(X, Z)) \\
+ \langle Y, Z \rangle \alpha(X, \nabla \rho) - \langle X, Z \rangle \alpha(Y, \nabla \rho) \\
= (LA_{\alpha(X,Z)}Y - LA_{\alpha(Y,Z)}X)_{N,M} - \alpha(X, Y\alpha(Y, Z)) + \alpha(Y, Y\alpha(X, Z)) \\
+ \langle Y, Z \rangle \alpha(X, \nabla \rho) - \langle X, Z \rangle \alpha(Y, \nabla \rho),
\]
and this is (7). Since \(E\) satisfies the compatibility condition (5), then
\[
\langle E(X, \alpha(Y, Z)), \eta \rangle = -\langle A_{E(X,\eta)}Y, Z \rangle,
\]
and this gives (13).
We have
\[
(\nabla^\perp_X \mathcal{E})(Y, \eta) = \nabla^\perp_X \mathcal{E}(Y, \eta) - \mathcal{E}(\nabla_X Y, \eta) - \mathcal{E}(Y, \nabla^\perp_X \eta)
= (\nabla^\perp_X \alpha)(Y, \eta) + (L(\nabla_X A)(Y, \eta))_{N_f M} + \alpha(Y, \nabla_X Y \eta)
- \alpha(Y, Y \nabla^\perp_X \eta) - (L \nabla_X A \eta Y)_{N_f M} + \nabla_X (L A_\eta Y)_{N_f M}.
\]
Then (14) yields
\[
(\nabla^\perp_X \mathcal{E})(Y, \eta) = (\nabla^\perp_X \alpha)(Y, \eta) + (L(\nabla_X A)(Y, \eta))_{N_f M} - \alpha(Y, B_\eta X)
- \alpha(Y, (L A_\eta X)_{TM}) - (L \nabla_X (A_\eta Y)_{N_f M} + \nabla^\perp_X (L A_\eta Y)_{N_f M}.
\]
Using the Codazzi equation, we obtain
\[
(\nabla^\perp_X \mathcal{E})(Y, \eta) - (\nabla^\perp_X \mathcal{E})(X, \eta) = \alpha(X, B_\eta Y) - \alpha(Y, B_\eta X) + \alpha(X, (L A_\eta Y)_{TM})
- \alpha(Y, (L A_\eta X)_{TM}) - (L \nabla_X A_\eta Y)_{N_f M} + \nabla^\perp_X (L A_\eta Y)_{N_f M}
+ (L \nabla_Y A_\eta X)_{N_f M} - \nabla^\perp_Y (L A_\eta X)_{N_f M}.
\]
Since
\[
\beta(X, A_\eta Y) = \alpha(X, (L A_\eta Y)_{TM}) - (L \nabla_X (A_\eta Y)_{N_f M} + \nabla^\perp_X (L A_\eta Y)_{N_f M},
\]
then (8) follows.

2 Trivial infinitesimal bendings

In this section, we characterize the trivial conformal infinitesimal bendings in terms of the associate pair of tensors \((\beta, \mathcal{E})\) to the bending.

An isometric infinitesimal bending \(\mathcal{T}\) of \(f : M^n \to \mathbb{R}^m\) is trivial if we have \(\mathcal{T} = Df + w\) where \(D \in \text{End}(\mathbb{R}^m)\) is skew-symmetric and \(w \in \mathbb{R}^m\); see [8] or [10] for details. Then \(L = D|_{f, TM}\) and \(B(X, Y) = D\alpha(X, Y)\).

\[
\beta(X, Y) = D^N \alpha(X, Y) \quad \text{and} \quad \mathcal{E}(X, \eta) = -(\nabla^\perp_X D^N)\eta
\]
where \(D^N \in \Gamma(\text{End}(N_f M))\) given by \(D^N \eta = (D\eta)_{N_f M}\) is skew-symmetric.

**Proposition 6.** An isometric infinitesimal bending \(\mathcal{T}\) of \(f : M^n \to \mathbb{R}^m\) is trivial if and only if there is \(C \in \Gamma(\text{End}(N_f M))\) skew-symmetric such that
\[
\beta(X, Y) = C\alpha(X, Y) \quad \text{and} \quad \mathcal{E}(X, \eta) = -(\nabla^\perp_X C)\eta. \quad (17)
\]
Proof: See [7].

Notice that the associated pairs of tensors to two conformal infinitesimal bendings with the same conformal factor are identified when they differ by a pair of tensors as in the above result.

It is well-known that any conformal Killing field on an open connected subset of $\mathbb{R}^n$, $n \geq 3$, has the form

$$X(x) = (\langle x, v \rangle + \lambda)x - (1/2)\|x\|^2v + Cx + w$$

where $\lambda \in \mathbb{R}$, $v, w \in \mathbb{R}^n$, $C \in \text{End}(\mathbb{R}^n)$ is skew-symmetric and the conformal factor is $\rho = \langle x, v \rangle + \lambda$; cf. [15] for details.

Let $T$ be a trivial conformal infinitesimal bending of $f$, that is, locally

$$T(x) = (\langle f(x), v \rangle + \lambda)f(x) - 1/2\|f(x)\|^2v + Df(x) + w$$

where $\lambda \in \mathbb{R}$, $v, w \in \mathbb{R}^m$ and $D \in \text{End}(\mathbb{R}^m)$ is skew-symmetric. Then the conformal factor is $\rho(x) = \langle f(x), v \rangle + \lambda$ and

$$LX = \langle f_*X, v \rangle f(x) - \langle f_*X, f(x) \rangle v + Df_*X.$$

Hence

$$(\nabla_X L)Y = \langle f_*Y, v \rangle X - \langle X, Y \rangle v + \langle \alpha(X, Y), v \rangle f(x) - \langle \alpha(X, Y), f(x) \rangle v + D\alpha(X, Y).$$

If $\mathcal{D}' \in \Gamma(\text{End}(f^*T\mathbb{R}^n))$ is the skew-symmetric map given by

$$\mathcal{D}'(\sigma) = \langle \sigma, v \rangle f(x) - \langle \sigma, f(x) \rangle v + D\sigma,$$

then $LX = \mathcal{D}'X$. Moreover, we have $\mathcal{Y}\eta = (\mathcal{D}'\eta)_{TM}$ and

$$(\nabla_X L)Y = \langle f_*Y, v \rangle f_*X - \langle X, Y \rangle v + \mathcal{D}'\alpha(X, Y).$$

Let $\mathcal{D}^N \in \Gamma(\text{End}(N_fM))$ be given by $\mathcal{D}^N\xi = (\mathcal{D}'\xi)_{N,f}$. Then

$$\beta(X, Y) = \mathcal{D}^N\alpha(X, Y) - \langle X, Y \rangle v_N$$

where $v_N = (v)_{N,f}$.  

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We have
\[(\tilde{\nabla}_X D')\sigma = \tilde{\nabla}_X D'\sigma - D'\tilde{\nabla}_X \sigma = \langle \sigma, v \rangle f_* X - \langle \sigma, f_* X \rangle v\]
for any \(X \in \mathfrak{X}(M)\) and \(\sigma \in \Gamma(f^* T\mathbb{R}^m)\). Then
\[
\mathcal{E}(X, \xi) = \alpha(X, Y\xi) + (LA_{\xi}X)_{N_fM} = (\tilde{\nabla}_X D'\xi - \tilde{\nabla}_X D^N\xi)_{N_fM} + (LA_{\xi}X)_{N_fM} = ((\tilde{\nabla}_X D')\xi + D'\tilde{\nabla}_X \xi - \tilde{\nabla}_X D^N\xi + LA_{\xi}X)_{N_fM} = -(\nabla^\perp_X D^N)\xi
\]
for any \(X \in \mathfrak{X}(M)\) and \(\xi \in \Gamma(N_fM)\).

**Proposition 7.** A conformal infinitesimal bending \(\mathcal{T}\) of \(f: M^n \to \mathbb{R}^m\), \(n \geq 3\), is trivial if and only if there exist \(\delta \in \Gamma(N_fM)\) and \(C \in \Gamma(\text{End}(N_fM))\) skew-symmetric such that the associated pair has the form
\[
\beta(X,Y) = C\alpha(X,Y) - \langle X, Y \rangle \delta \quad \text{and} \quad \mathcal{E}(X,\xi) = -(\nabla^\perp_X C)\xi. \quad (18)
\]

**Proof:** If \((\beta, \mathcal{E})\) has the form \((18)\) and \(\rho\) is the conformal factor of \(\mathcal{T}\), we obtain from \((6)\) that
\[
\langle X, Z \rangle ((\alpha(Y, W), \delta) - \text{Hess} \rho(Y, W)) + \langle Y, W \rangle ((\alpha(X, Z), \delta) - \text{Hess} \rho(X, Z))
- \langle X, W \rangle ((\alpha(Y, Z), \delta) - \text{Hess} \rho(Y, Z)) - \langle Y, Z \rangle ((\alpha(X, W), \delta) - \text{Hess} \rho(X, W)) = 0.
\]
for any \(X, Y, Z, W \in \mathfrak{X}(M)\). For \(X, Y, W\) orthonormal and \(Z = X\) this gives
\[
\langle \alpha(Y, W), \delta \rangle - \text{Hess} \rho(Y, W) = 0
\]
whereas for \(X = Z\) and \(Y = W\) orthonormal this yields
\[
\langle \alpha(X, X), \delta \rangle - \text{Hess} \rho(X, X) = -\langle \alpha(Y, Y), \delta \rangle + \text{Hess} \rho(Y, Y) = 0.
\]
Therefore
\[
\langle \alpha(X, Y), \delta \rangle - \text{Hess} \rho(X, Y) = 0 \quad (19)
\]
for any \(X, Y \in \mathfrak{X}(M)\).

Since \(\mathcal{E}\) has the form \((18)\) we obtain from \((7)\) that
\[
\langle X, Z \rangle (\nabla^\perp_X \delta + \alpha(Y, \nabla \rho)) = \langle Y, Z \rangle (\nabla^\perp_X \delta + \alpha(X, \nabla \rho)).
\]

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Hence
\[ \nabla_X \delta + \alpha(X, \nabla \rho) = 0 \]  
(20)
for any \( X \in \mathfrak{X}(M) \).

Equations (19) and (20) are equivalent to \( f_* \nabla \rho + \delta = v \) being constant along \( f \). In particular \( \rho(x) = \langle f(x), v \rangle + \lambda \) for some \( \lambda \in \mathbb{R} \).

Let \( \mathcal{T}_1 \in \Gamma(f^* T\mathbb{R}^m) \) be the trivial conformal infinitesimal bending
\[ \mathcal{T}_1(x) = (\langle f(x), v \rangle + \lambda)f(x) - 1/2 \| f(x) \|^2 v. \]

Then \( \mathcal{T} - \mathcal{T}_1 \) is an isometric infinitesimal bending whose associated tensors have the form (17), and thus is trivial.  

We conclude this section with some nontrivial examples of conformal infinitesimal bendings of simple geometric nature.

**Examples 8.** (i) If \( f : M^n \to \mathbb{R}^m \) is an isometric immersion then a conformal Killing vector field of \( M^n \) is a conformal infinitesimal bending of \( f \).

(ii) Let \( g : M^n \to S^m \) be an isometric immersion. Then \( \mathcal{T} = \varphi f \) is a conformal infinitesimal bending of \( f = i \circ g : M^n \to \mathbb{R}^{m+1} \) where \( \varphi \in C^\infty(M) \) and \( i : S^m \to \mathbb{R}^{m+1} \) is the inclusion.

(iii) Let \( f, g : M^n \to \mathbb{R}^m \) be two conformal immersions such that the map \( h = f + g : M^n \to \mathbb{R}^m \) is an immersion. Then \( \mathcal{T} = f - g \) is a conformal infinitesimal bending of \( h \).

3 The Fundamental theorem

In this section we prove the Fundamental theorem for conformal infinitesimal bendings given in the Introduction.

**Proof of Theorem.** Given a pair \( (\beta, \mathcal{E}) \) as in the statement, we argue that there is \( \mathcal{D} \in \Gamma(\text{End}(f^* T\mathbb{R}^m)) \) satisfying
\[ (\tilde{\nabla}_X \mathcal{D})(Y + \eta) = f_* (\mathcal{E}(X, \nabla \rho) X - \langle X, Y \rangle \nabla \rho - B_\eta X) + \beta(X, Y) + \mathcal{E}(X, \eta) \]  
(21)
for any \( X, Y \in \mathfrak{X}(M) \) and \( \eta \in \Gamma(N_f M) \). To prove this, henceforth we check that its integrability condition
\[ (\tilde{\nabla}_X \tilde{\nabla}_Y \mathcal{D} - \tilde{\nabla}_Y \tilde{\nabla}_X \mathcal{D} - \tilde{\nabla}_{[X,Y]} \mathcal{D})(Z + \eta) = 0 \]
holds for any \(X, Y, Z \in \mathfrak{X}(M)\) and \(\eta \in \Gamma(N_fM)\). For simplicity, we now write \(X\) instead of \(f_*X\). We have

\[
(\tilde{\nabla}_X \tilde{\nabla}_Y \mathcal{D} - \tilde{\nabla}_Y \tilde{\nabla}_X \mathcal{D} - \tilde{\nabla}_{[X,Y]} \mathcal{D})(Z + \eta)
\]

\[
= \tilde{\nabla}_X((\tilde{\nabla}_Y \mathcal{D})(Z + \eta) - (\tilde{\nabla}_Y \mathcal{D})\tilde{\nabla}_X(Z + \eta) - \tilde{\nabla}_Y(\tilde{\nabla}_X \mathcal{D})(Z + \eta)) \
+ ((\tilde{\nabla}_X \mathcal{D})\tilde{\nabla}_Y(Z + \eta) - (\tilde{\nabla}_{[X,Y]} \mathcal{D})(Z + \eta)) \
= \tilde{\nabla}_X[(Z, \nabla \rho)Y - \langle Y, Z \rangle \nabla \rho - B_\eta Y + \beta(Y, Z) + \mathcal{E}(Y, \eta)] \
- \langle \nabla_X Z - A_\eta X, \nabla \rho \rangle Y + \langle Y, \nabla_X Z - A_\eta X \rangle \nabla \rho + B_{\alpha(X,Z) + \nabla X} Y \
- \beta(Y, \nabla_X Z - A_\eta X) - \mathcal{E}(Y, \alpha(X, Z) + \nabla X) \
- \tilde{\nabla}_Y[(Z, \nabla \rho)X - \langle X, Z \rangle \nabla \rho - B_\eta X + \beta(X, Z) + \mathcal{E}(X, \eta)] \
+ \langle \nabla_Y Z - A_\eta Y, \nabla \rho \rangle X - \langle X, \nabla_Y Z - A_\eta Y \rangle \nabla \rho - B_{\alpha(Y,Z) + \nabla Y} X \
+ \beta(X, \nabla_Y Z - A_\eta Y) + \mathcal{E}(X, \alpha(Y, Z) + \nabla Y) - \langle Z, \nabla \rho \rangle [X, Y] \
+ \langle [X, Y], Z \rangle \nabla \rho + B_\eta[X, Y] - \beta([X, Y], Z) - \mathcal{E}([X, Y], \eta).
\]

Making use of all equations in \((S)\) as well as \((13)\), we obtain

\[
(\tilde{\nabla}_X \tilde{\nabla}_Y \mathcal{D} - \tilde{\nabla}_Y \tilde{\nabla}_X \mathcal{D} - \tilde{\nabla}_{[X,Y]} \mathcal{D})(Z + \eta)
\]

\[
= -A_{\beta(Y,Z)} X + B_{\alpha(X,Z)Y} Y + A_{\beta(X,Z)} Y - B_{\alpha(Y,Z)} X \
+ \text{Hess } \rho(X, Z) Y - \text{Hess } \rho(Z, Y) X + \langle X, Z \rangle \nabla_Y \nabla \rho - \langle Y, Z \rangle \nabla_X \nabla \rho \
+ ((\nabla_X \beta)(Y, Z) - (\nabla_Y \beta)(X, Z) + \mathcal{E}(X, \alpha(Y, Z)) - \mathcal{E}(Y, \alpha(X, Z)) \
+ \langle X, Z \rangle \alpha(Y, \nabla \rho) - \langle Y, Z \rangle \alpha(X, \nabla \rho) + \langle A_\eta X, \nabla \rho \rangle Y - \langle A_\eta Y, \nabla \rho \rangle X \
- (\nabla_X B_\eta) Y + (\nabla_Y B_\eta) X + B_{\nabla X} Y - B_{\nabla Y} X + A_{\mathcal{E}(Y,Z)} Y \
+ (\nabla_X \mathcal{E})(Y, \eta) - (\nabla_Y \mathcal{E})(X, \eta) - \alpha(X, B_\eta Y) + \alpha(Y, B_\eta X) \
+ \beta(Y, A_\eta X) - \beta(X, A_\eta Y)
\]

\[= 0.\]

Fix a solution \(\mathcal{D}^* \in \Gamma(\text{End}(f^*T\mathbb{R}^m))\) of \((21)\) and a point \(x_0 \in M^n\). Set \(\mathcal{D}_0 = \mathcal{D}^*(x_0)\) and let \(\phi: f^*T\mathbb{R}^m \times f^*T\mathbb{R}^m \to \mathbb{R}\) be the tensor defined by

\[
\phi(\rho, \sigma) = (\langle \mathcal{D}^* - \mathcal{D}_0 \rangle \rho, \sigma) + (\langle \mathcal{D}^* - \mathcal{D}_0 \rangle \sigma, \rho).
\]

Using \((5)\) and \((21)\) we obtain \((\nabla_X \phi)(\rho, \sigma) = 0\). Hence \(\phi = 0\), and thus the maps \(\mathcal{D}(x) = \mathcal{D}^*(x) - \mathcal{D}_0\) are skew-symmetric endomorphisms of \(\mathbb{R}^m\).

Define \(L \in \Gamma(\text{Hom}(TM, f^*T\mathbb{R}^m))\) by \(L(x) = \mathcal{D}(x)|_{T_xM}\). Using \((21)\) we obtain

\[
(\nabla_X L) Y = \tilde{\nabla}_X \mathcal{D} Y - \mathcal{D} \tilde{\nabla}_X Y = \langle Y, \nabla \rho \rangle X - \langle X, Y \rangle \nabla \rho + \beta(X, Y) + \mathcal{D} \alpha(X, Y).
\]
Define \( L' \in \Gamma(\text{Hom}(TM, f^*T\mathbb{R}^m)) \) by
\[
L'X = LX + \rho X.
\]
Then, we have
\[
(\tilde{\nabla}_X L')Y = (\tilde{\nabla}_Y L')X.
\]
Hence, there is \( T \in \Gamma(f^*T\mathbb{R}^m) \) such that \( \tilde{\nabla}_X T = L'X \) for any \( X \in \mathfrak{X}(M) \).

Since \( \mathcal{D} \) is skew-symmetric, then \( L' \) satisfies
\[
\langle L'X, Y \rangle + \langle L'Y, X \rangle = 2\rho \langle X, Y \rangle,
\]
proving that \( T \) is a conformal infinitesimal bending of \( f \). Moreover, its associate pair of tensors \( (\tilde{\beta}, \tilde{\varepsilon}) \) is
\[
\tilde{\beta}(X, Y) = \beta(X, Y) + \mathcal{D}^N \alpha(X, Y) \quad \text{and} \quad \tilde{\varepsilon}(X, \eta) = \varepsilon(X, \eta) - (\nabla^N_X \mathcal{D}^N)\eta.
\]
In fact, in this case \( \mathcal{Y} \eta = (\mathcal{D} \eta)_TM \). Using (21), we have
\[
\tilde{\varepsilon}(X, \eta) = \alpha(X, (\mathcal{D} \eta)_TM) + (LA_\eta X)_{N,M}
= (\tilde{\nabla}_X (\mathcal{D} \eta)_TM)_{N,M} + (LA_\eta X)_{N,M}
= (\tilde{\nabla}_X \mathcal{D} \eta)_{N,M} - \nabla^N_X \mathcal{D}^N \eta + (LA_\eta X)_{N,M}
= \varepsilon(X, \eta) + (\mathcal{D} \tilde{\nabla}_X \eta)_{N,M} - \nabla^N_X \mathcal{D}^N \eta + (LA_\eta X)_{N,M}
= \varepsilon(X, \eta) - (LA_\eta X)_{N,M} - \nabla^N_X \mathcal{D}^N \eta + (LA_\eta X)_{N,M}
= \varepsilon(X, \eta) - (\nabla^N_X \mathcal{D}^N)\eta.
\]
Another solution \( \mathcal{D}' \) of (21) gives rise to a conformal infinitesimal bending \( \mathcal{T}_1 \) of \( f \). By Proposition 6 we have that \( \mathcal{T} - \mathcal{T}_1 \) is a trivial isometric infinitesimal bending, and this concludes the proof.

**Proof of Corollary** For hypersurfaces \( \mathcal{E} = 0 \). Let \( \beta: TM \times TM \to N_{fM} \) be the symmetric tensor given by \( \beta(X, Y) = \langle \mathcal{B}X, Y \rangle N \). Then (S) trivially holds for \( (\beta, 0) \). Moreover, (9) and (10) give that \( (\beta, 0, \rho) \) satisfy system \( (S) \).

Hence, by Theorem 4 there is a conformal infinitesimal bending \( \mathcal{T} \) of \( f \) with conformal factor \( \rho \) having \( (\beta, 0) \) as associated pair.

### 4 Conformal infinitesimal rigidity

Let \( V^n \) be an \( n \)-dimensional real vector space and let \( W^{p,p} \) be a real vector space of dimension \( 2p \) endowed with an indefinite inner product of signature
A bilinear form $\gamma: V^n \times V^n \to W^{p,p}$ is said to be flat if
$$\langle \gamma(X, Z), \gamma(Y, W) \rangle - \langle \gamma(X, W), \gamma(Y, Z) \rangle = 0$$
for all $X, Y, Z, W \in V^n$. We say that the bilinear form $\gamma$ is null if
$$\langle \gamma(X, Z), \gamma(Y, W) \rangle = 0$$
for all $X, Y, Z, W \in V^n$. Thus a null bilinear form is flat.

Given $\gamma: V^n \times V^n \to W^{p,p}$ we denote
$$N(\gamma) = \{X \in V^n : \gamma(X, Y) = 0 \text{ for all } Y \in V^n\}$$
and
$$S(\gamma) = \text{span}\{\gamma(X, Y) : X, Y \in V^n\}.$$

We will need the following result from the theory of flat bilinear forms which is known to be false for $p = 6$; see [10].

**Lemma 9.** Let $\gamma: V^n \times V^n \to W^{p,p}$, $p \leq 5$, be a symmetric flat bilinear form. If $\dim N(\gamma) \leq n - 2p - 1$ then there is an orthogonal decomposition
$$W^{p,p} = W_{1,\ell} \oplus W_{2,-\ell}^{p-\ell}, 1 \leq \ell \leq p,$$
such that the $W_j$-components $\gamma_j$ of $\gamma$ satisfy:

(i) $\gamma_1$ is nonzero but null since $S(\gamma_1) = S(\gamma) \cap S(\gamma)^\perp$.

(ii) $\gamma_2$ is flat and $\dim N(\gamma_2) \geq n - 2p + 2\ell$.

**Proof:** See [5] or [10].

**Proposition 10.** Let $\mathcal{I}$ be a conformal infinitesimal bending of an isometric immersion $f: M^n \to \mathbb{R}^m$ with conformal factor $\rho$ and associated tensor $\beta$. Then the bilinear form $\theta: TM \times TM \to N_f M \oplus \mathbb{R} \oplus N_f M \oplus \mathbb{R}$ defined at any point of $M^n$ by
$$\theta = (\alpha + \beta, \langle , \rangle + \text{Hess } \rho, \alpha - \beta; \langle , \rangle - \text{Hess } \rho) \quad (22)$$
is flat with respect to the inner product $\langle \langle , \rangle \rangle$ of signature $(m-n+1, m-n+1)$ given by
$$\langle \langle (\xi_1, a_1, \eta_1, b_1), (\xi_2, a_2, \eta_2, b_2) \rangle \rangle = \langle \xi_1, \xi_2 \rangle_{N_f M} + a_1 a_2 - \langle \eta_1, \eta_2 \rangle_{N_f M} - b_1 b_2.$$
Proof: A straightforward computation yields

\[
\frac{1}{2} (\langle \theta(X, W), \theta(Y, Z) \rangle - \langle \theta(X, Z), \theta(Y, W) \rangle) = \langle \beta(X, W), \alpha(Y, Z) \rangle \\
+ \langle \alpha(X, W), \beta(Y, Z) \rangle - \langle \beta(X, Z), \alpha(Y, W) \rangle - \langle \alpha(X, Z), \beta(Y, W) \rangle \\
+ \langle X, W \rangle \text{Hess } \rho(Y, Z) + \langle Y, Z \rangle \text{Hess } \rho(X, W) - \langle X, Z \rangle \text{Hess } \rho(Y, W) \\
- \langle Y, W \rangle \text{Hess } \rho(X, Z)
\]

for any \( X, Y, Z, W \in \mathcal{X}(M) \), and the proof follows from (6).

Lemma 11. Let \( Z_1, Z_2 \in T_x M \) at \( x \in M^n \) be non-zero vectors such that either \( Z_1 = Z_2 \) or \( \langle Z_1, Z_2 \rangle = 0 \). If \( n \geq 4 \) and \( \nu^c_1(x) \leq n - 3 \), then

\[ N_f M(x) = \text{span}\{\alpha(X, Y) : X, Y \in T_x M; \langle X, Y \rangle = \langle X, Z_1 \rangle = \langle Y, Z_2 \rangle = 0\}. \]

Proof: First assume that \( \langle Z_1, Z_2 \rangle = 0 \). Let \( U^s \subset N_f M(x) \) be the subspace given by \( U^s \perp \alpha(X, Y) \) for any \( X, Y \in T_x M \) as in the statement. If in addition \( \langle X, Z_2 \rangle = \langle Y, Z_1 \rangle = 0 \) and \( \|X\| = \|Y\| \), then \( \alpha(X + Y, X - Y) \) gives \( \alpha_{U^s}(X, X) = \alpha_{U^s}(Y, Y) \).

Thus, there is \( \zeta \in U^s \) such that

\[ \alpha_{U^s}(X, Y) = \langle X, Y \rangle \zeta \]

for any \( X, Y \in \text{span}\{Z_1, Z_2\}^\perp \). Since \( \alpha_{U^s}(W, Z_1) = \alpha_{U^s}(W, Z_2) = 0 \) for any \( W \in \text{span}\{Z_1, Z_2\}^\perp \) by assumption, then

\[ \text{span}\{Z_1, Z_2\}^\perp \subset N(\alpha_{U^s} - \langle , \rangle \zeta), \]

and this contradicts our assumption on \( \nu^c_1 \) unless \( s = 0 \).

If \( Z_1 = Z_2 \) we have again that there is \( \zeta \in U^s \) such that

\[ \alpha_{U^s}(X, Y) = \langle X, Y \rangle \zeta \]

for any \( X, Y \in \text{span}\{Z\}^\perp \). It follows that \( A_\zeta \) has an eigenspace of multiplicity at least \( n - 2 \) contradicting the assumption on \( \nu^c_1 \).

Proposition 12. Let \( f : M^n \to \mathbb{R}^m, n \geq 4 \), be an isometric immersion and let \( \mathcal{S} \) be a conformal infinitesimal bending of \( f \) with conformal factor \( \rho \) and
associated pair of tensors \((\beta, \mathcal{E})\). If \(\nu_1(x) \leq n - 3\) at any \(x \in M^n\), then \(\mathcal{E}\) is the unique tensor satisfying (5) and an equation of the form

\[
(\nabla^\perp_X \beta)(Y, Z) - (\nabla^\perp_Y \beta)(X, Z) = \mathcal{E}(Y, \alpha(X, Z)) - \mathcal{E}(X, \alpha(Y, Z)) + \langle Y, Z \rangle \psi(X) - \langle X, Z \rangle \psi(Y)
\]

where \(\psi \in \Gamma(\text{End}(TM, N_f M))\).

**Proof:** If also \(\mathcal{E}_0: TM \times N_f M \to N_f M\) satisfies (5) and (23), then (7) gives

\[
(\mathcal{E} - \mathcal{E}_0)(X, \alpha(Y, Z)) - (\mathcal{E} - \mathcal{E}_0)(Y, \alpha(X, Z)) + \langle Y, Z \rangle (\psi(X) - \alpha(X, \nabla \rho)) - \langle X, Z \rangle (\psi(Y) - \alpha(Y, \nabla \rho)) = 0.
\]

Then

\[
(\mathcal{E} - \mathcal{E}_0)(X, \alpha(Y, Z)) = (\mathcal{E} - \mathcal{E}_0)(Y, \alpha(X, Z))
\]

if \(Z\) is orthogonal to \(X\) and \(Y\). Writing

\[
\langle (\mathcal{E} - \mathcal{E}_0)(X_1, \alpha(X_2, X_3)), \alpha(X_4, X_5) \rangle = (X_1, X_2, X_3, X_4, X_5)
\]

and taking \(\langle X_1, X_3 \rangle = \langle X_2, X_3 \rangle = 0\) we have symmetry in the pairs \(\{X_1, X_2\}\), \(\{X_2, X_3\}\) and \(\{X_4, X_5\}\). Moreover, since \(\mathcal{E}\) and \(\mathcal{E}_0\) verify (5) we obtain

\[
(X_1, X_2, X_3, X_4, X_5) = -(X_1, X_4, X_5, X_2, X_3).
\]

Hence, if \(\{X_i\}_{1 \leq i \leq 5}\) satisfies

\[
\langle X_1, X_3 \rangle = \langle X_1, X_4 \rangle = \langle X_2, X_3 \rangle = \langle X_2, X_5 \rangle = \langle X_4, X_5 \rangle = 0,
\]

then

\[
(X_1, X_2, X_3, X_4, X_5) = -(X_1, X_4, X_5, X_2, X_3) = -(X_5, X_4, X_1, X_2, X_3) = (X_5, X_2, X_3, X_4, X_1) = -(X_3, X_4, X_1, X_2, X_5) = -(X_4, X_3, X_1, X_2, X_5) = (X_4, X_2, X_5, X_3, X_1) = -(X_2, X_3, X_1, X_4, X_5) = -(X_1, X_2, X_3, X_4, X_5) = 0.
\]

Thus

\[
\langle (\mathcal{E} - \mathcal{E}_0)(X_1, \alpha(X_2, X_3)), \alpha(X_4, X_5) \rangle = 0
\]
if (24) holds. We already have $\langle X_1, X_4 \rangle = \langle X_2, X_5 \rangle = \langle X_4, X_5 \rangle = 0$. Hence, if also $\langle X_1, X_2 \rangle = 0$ we obtain from Lemma 11 that

$$\left( \mathcal{E} - \mathcal{E}_0 \right)(X_1, \alpha(X_2, X_3)) = 0$$

for any $X_1, X_2, X_3 \in \mathfrak{X}(M)$ with $\langle X_1, X_2 \rangle = \langle X_1, X_3 \rangle = \langle X_2, X_3 \rangle = 0$. Then using Lemma 11 again, it follows that

$$\left( \mathcal{E} - \mathcal{E}_0 \right)(X, \eta) = 0$$

for any $X \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_f M)$.

**Lemma 13.** Let $S \subset \mathbb{R}^m$ be a vector subspace and let $T_0: S \to \mathbb{R}^m$ be a linear map that is an isometry between $S$ and $T_0(S)$. Assume there is no $0 \neq v \in S$ such that $T_0 v = -v$. Then there is an isometry $T \in \text{End}(\mathbb{R}^m)$ that extends $T_0$ and has 1 as the only possible real eigenvalue.

**Proof:** Extend $T_0$ to an isometry $T$ of $\mathbb{R}^m$. Suppose that the eigenspace of the eigenvalue $-1$ of $T$ satisfies $\dim E_{-1} = k > 0$. We have by assumption that $E_{-1} \cap S = E_{-1} \cap T_0(S) = \{0\}$. Let $\{e_1, \ldots, e_k\}$ be an orthonormal basis of $E_{-1}$ and set

$$P = T_0(S) \oplus \text{span}\{e_2, \ldots, e_k\}.$$  

Let $\xi \in P^\perp$ be a unit vector collinear with the $P^\perp$-component of $e_1$. Let $\eta \in \mathbb{R}^m$ be such that $T \eta = \xi$ and let $H$ be the hyperplane $\{\eta\}^\perp$. If $R$ is the reflection with respect to the hyperplane $\{\xi\}^\perp$, then the isometry $T_1 = RT$ satisfies $T_1 v = T v$ for any $v \in H$ since $Tv \in \{\xi\}^\perp$.

Since $\langle \eta, e_1 \rangle = -\langle \xi, e_1 \rangle \neq 0$, there is $v \in H$ such that $\eta + v$ is collinear with $e_1$. Hence

$$T(\eta + v) = \xi + T v = -\eta - v.$$  

We claim that no vector of the form $\eta + u$, $u \in H$, is an eigenvector of $T_1$ associated to $-1$. If otherwise

$$T_1(\eta + u) = -\xi + Tu = -\eta - u$$

for some $u \in H$. From the last two equations we obtain

$$T(u + v) = -2\eta - (u + v).$$

Then

$$\|T(u + v)\|^2 = 4 + \|u + v\|^2$$

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which contradicts that $T$ is an isometry and proves the claim.

We have proved that the eigenspace of $T_1$ associated to $-1$ is contained in $H$, in fact, it is $\text{span}\{e_2, \ldots, e_k\}$. Therefore, by composing $T$ with $k$ appropriate reflections we obtain an isometry as in the statement. ■

Proposition 14. Let $\mathcal{T}$ be a conformal infinitesimal bending of an isometric immersion $f: M^n \to \mathbb{R}^m$. If $\mathcal{T}$ is trivial then $\theta$ is null. Conversely, if $\theta$ is null, $n \geq 4$ and $\nu^z_c(x) \leq n - 3$ at any $x \in M^n$ then $\mathcal{T}$ is trivial.

Proof: If $\mathcal{T}$ is a trivial conformal infinitesimal bending of $f$, then

$$\mathcal{T}(x) = (\langle f(x), v \rangle + \lambda)f(x) - 1/2\|f(x)\|^2v + \mathcal{D}f(x) + w$$

for some $\lambda \in \mathbb{R}$, $v, w \in \mathbb{R}^m$ and $\mathcal{D} \in \text{End}(\mathbb{R}^m)$ skew-symmetric. Since $\rho(x) = \langle f(x), v \rangle + \lambda$, then $f_*\nabla \rho = \nu_{TM}$. Thus

$$\langle \nabla_X v, f_*Y \rangle = \text{Hess} \rho(X, Y) - \langle A_{v_{NfM}}X, Y \rangle = 0$$ (25)

for any $X, Y \in \mathcal{X}(M)$. Moreover, we have seen that

$$\beta(X, Y) = C\alpha(X, Y) - \langle X, Y \rangle \nu_{NfM}$$

where $C \in \Gamma(\text{End}(NfM))$ is skew-symmetric. Using (25) and that $C$ is skew-symmetric, we obtain that the bilinear form $\theta$ is null. In fact,

$$\frac{1}{2} \langle \theta(X, Y), \theta(Z, W) \rangle = \langle \alpha(X, Y), \beta(Z, W) \rangle + \langle \beta(X, Y), \alpha(Z, W) \rangle$$

$$+ \langle X, Y \rangle \text{Hess} \rho(Y, W) + \langle Z, W \rangle \text{Hess} \rho(X, Y)$$

$$= - \langle Z, W \rangle \langle \alpha(X, Y), v_{NfM} \rangle - \langle X, Y \rangle \langle \alpha(Z, W), v_{NfM} \rangle$$

$$+ \langle X, Y \rangle \text{Hess} \rho(Y, W) + \langle Z, W \rangle \text{Hess} \rho(X, Y)$$

$$= 0.$$

For the converse, that $\theta$ is null means that

$$\langle \alpha(X, Y), \beta(Z, W) \rangle + \langle \beta(X, Y), \alpha(Z, W) \rangle + \langle X, Y \rangle \text{Hess} \rho(Z, W)$$

$$+ \langle Z, W \rangle \text{Hess} \rho(X, Y) = 0$$ (26)

for any $X, Y, Z, W \in \mathcal{X}(M)$. Let $S \subset NfM(x) \oplus \mathbb{R}$ be the subspace given by $S = \text{span}\{(\alpha(X, Y) + \beta(X, Y), \langle X, Y \rangle + \text{Hess} \rho(X, Y)) : X, Y \in T_xM\}$. 20
Then, the map $T_0$ defined by
\[ T_0(\alpha(X,Y) + \beta(X,Y), \langle X,Y \rangle + \text{Hess } \rho(X,Y)) \]
\[ = (\alpha(X,Y) - \beta(X,Y), \langle X,Y \rangle - \text{Hess } \rho(X,Y)) \]
is an isometry between $S$ and $T(S)$. We claim that $T_0$ does not possess the eigenvalue $-1$. If otherwise, then $T_0v = -v$ where
\[ 0 \neq v = \sum_i (\alpha(X_i,Y_i) + \beta(X_i,Y_i), \langle X_i,Y_i \rangle + \text{Hess } \rho(X_i,Y_i)) \in S \]
Hence $\sum_i \alpha(X_i,Y_i) = 0$ and $\sum_i \langle X_i,Y_i \rangle = 0$. Now (26) gives
\[ \sum_i (\beta(X_i,Y_i), \alpha(Z,W)) + \langle Z,W \rangle \sum_i \text{Hess } \rho(X_i,Y_i) = 0 \]
for any $Z,W \in \mathcal{X}(M)$. That is, we have $A_\eta = -hI$ where $\eta = \sum_i \beta(X_i,Y_i)$ and $h = \sum_i \text{Hess } \rho(X_i,Y_i)$. From our assumption on $\nu^c_1$ we obtain $\eta = h = 0$, hence $v = 0$ proving the claim.

Let $T$ be the isometry of $N_fM(x) \oplus \mathbb{R}$ extending $T_0$ given by Lemma 13. Then
\[ (I + T^t)(I - T) = (T^t - T) = -(I - T^t)(I + T) \]
where $T^t$ is the transpose of $T$. Thus
\[ (I - T)(I + T)^{-1} = -(I + T^t)^{-1}(I - T^t) = -((I - T)(I + T)^{-1})^t, \]
that is, $(I - T)(I + T)^{-1}$ is a skew-symmetric endomorphism of $N_fM(x) \oplus \mathbb{R}$.

It is easy to see that
\[ (I - T)(I + T)^{-1}(\alpha(X,Y),0) = (\beta(X,Y), \text{Hess } \rho(X,Y)) \]
for any $X,Y \in T_xM$ such that $\langle X,Y \rangle = 0$. Thus, there is $C \in \text{End}(N_fM(x))$ skew-symmetric such that $\beta(X,Y) = C\alpha(X,Y)$ for any $X,Y \in T_xM$ with $\langle X,Y \rangle = 0$. It follows from $\beta(X+Y,X-Y) = C\alpha(X+Y,X-Y)$ for any $X,Y \in T_xM$ orthonormal that
\[ \beta(X,X) - C\alpha(X,X) = \beta(Y,Y) - C\alpha(Y,Y). \]

Therefore, there is $\delta \in N_fM(x)$ such that
\[ \beta(X,Y) = C\alpha(X,Y) - \langle X,Y \rangle \delta \]
(27)
for any \( X, Y \in T_xM \).

By Lemma 11 there are smooth local vector fields \( X_i, Y_i, 1 \leq i \leq m - n \), satisfying \( \langle X_i, Y_i \rangle = 0 \) such that \( \alpha(X_i, Y_i), 1 \leq i \leq m - n \), span the normal bundle. Thus \( C \) and \( \delta \) are smooth.

Define \( \mathcal{E}_0: TM \times N_fM \to N_fM \) by \( \mathcal{E}_0(X, \eta) = - (\nabla_X C)\eta \). It follows from (7) that

\[
\mathcal{E}_0(Y, \alpha(X, Z)) - \mathcal{E}_0(X, \alpha(Y, Z)) - \langle Y, Z \rangle \nabla_X^\perp \delta + \langle X, Z \rangle \nabla_Y^\perp \delta
= \mathcal{E}(Y, \alpha(X, Z)) - \mathcal{E}(X, \alpha(Y, Z)) + \langle Y, Z \rangle \alpha(X, \nabla \rho) - \langle X, Z \rangle \alpha(Y, \nabla \rho).
\]

By Proposition 12 we have \( \mathcal{E} = \mathcal{E}_0 \), and \( \mathcal{J} \) is trivial by Proposition 7.

**Proof of Theorem 5.** Let \( \mathcal{J} \) be a conformal infinitesimal bending of \( f \) such that the flat bilinear \( \theta \) given by (22) is not null at \( x \in M^n \). Since \( N(\theta) = \{0\} \), there is an orthogonal decomposition

\[
W^{2p+2}_0 = N_fM(x) \oplus \mathbb{R} \oplus N_fM(x) \oplus \mathbb{R} = W^{\ell, \ell}_1 \oplus W^{p-\ell+1, p-\ell+1}_2, 1 \leq \ell \leq p,
\]

such that \( \theta \) splits as \( \theta = \theta_1 + \theta_2 \) as in Lemma 9. Denoting \( \Delta = N(\theta_2) \), we have \( \dim \Delta \geq n - 2(p - \ell + 1) \). Thus \( \theta(Z, X) = \theta_1(Z, X) \) for any \( Z \in \Delta \) and \( X \in T_xM \).

Let \( S \subset N_fM(x) \oplus \mathbb{R} \) be the vector subspace given by

\[
S = \text{span}\{ (\alpha(Z, X) + \beta(Z, X), \langle Z, X \rangle + \text{Hess} \rho(Z, X)) : Z \in \Delta \text{ and } X \in T_xM \}.
\]

If \( \Pi_1 \) denotes the orthogonal projection from \( W^{2p+2}_0 \) onto the first copy of \( N_fM(x) \oplus \mathbb{R} \), then \( S \subset \Pi_1(S(\theta) \cap S(\theta)^\perp) \) and, in particular, \( \dim S \leq \ell \).

That \( \theta_1 \) is null means that the map \( T: S \to N_fM(x) \oplus \mathbb{R} \) defined by

\[
T(\alpha(Z, X) + \beta(Z, X), \langle Z, X \rangle + \text{Hess} \rho(Z, X))
= (\alpha(Z, X) - \beta(Z, X), \langle Z, X \rangle - \text{Hess} \rho(Z, X)).
\]

is an isometry between \( S \) and \( T(S) \). We have that

\[
\frac{1}{2}(I + T)(\alpha(Z, X) + \beta(Z, X), \langle Z, X \rangle + \text{Hess} \rho(Z, X)) = (\alpha(Z, X), \langle Z, X \rangle).
\]

If \( S_1 = ((I + T)(S))^\perp \subset N_fM \times \mathbb{R} \), then \( \dim S_1 \geq p - \ell + 1 \). For \( (\eta, a) \in S_1 \)

\[
\langle \alpha(Z, X), \eta \rangle + a \langle X, Z \rangle = 0
\]

(28)
for any $Z \in \Delta$ and $X \in T_xM$. Let $U \subset N_fM$ be the orthogonal projection of $S_1$ in $N_fM$. Since $S_1$ does not posses elements of the form $(0, a)$ with $0 \neq a \in \mathbb{R}$, then
\[
\dim U \geq p - \ell + 1.
\]
It follows from (28) that there exists $\zeta \in U$ such that
\[
\alpha_U(Z, X) = \langle Z, X \rangle \zeta
\]
for any $Z \in \Delta$ and $X \in T_xM$. Hence $\alpha_U - \langle \cdot, \cdot \rangle \zeta$ has a kernel of dimension at least $\dim \Delta \geq n - 2(p - \ell - 1)$. But this contradicts the assumption on the conformal $s$-nullities, and hence $\theta$ is necessarily null at any point. We conclude from Proposition [4] that $\mathcal{I}$ is trivial.

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