Integrability test for spin chains

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Abstract

We examine a simple heuristic test of integrability for quantum chains. This test is applied to a variety of systems, including a generic isotropic spin-1 model with nearest-neighbor interactions and a multiparameter family of spin-1/2 models generalizing the XYZ chain, with next-to-nearest neighbor interactions and bond alternation. Within the latter family we determine all the integrable models with an $o(2)$ symmetry.

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1. Introduction

This work examines integrability criteria for quantum chains. In particular, we propose a simple heuristic test of integrability for quantum chains with short-range interactions, consisting essentially in proving or disproving the existence of a nontrivial conserved local charge involving three-point interactions. We indicate the conditions in which this might be a necessary condition of integrability, as well as circumstances in which it is expected to be a sufficient condition. The test is then applied to a number of models. Our approach is motivated by our recent extensive analysis of the structure of the conservation laws for integrable spin chains of the XYZ type and the Hubbard model [1].

For hamiltonian systems, the common definition of quantum integrability mimics the Liouville-Arnold definition of classical integrable hamiltonian systems: a quantum system with \( N \) degrees of freedom is called integrable, if it possesses \( N \) nontrivial, functionally independent and mutually commuting conservation laws.

For classical continuous systems, the proof of integrability usually amounts to displaying a Lax or zero-curvature formulation. Although this is not always easy, there are various other manifestations of integrability that can be probed, such as a bi-hamiltonian formulation, nontrivial symmetries, prolongation structures or higher order conserved charges (see e.g. [2]). Furthermore, one can apply a systematical integrability test, based on the Painlevé property [3]. For classical discrete systems, there are related integrability indicators [4]. But except for higher order conservation laws, these integrability signals are no longer available for quantum chains.

On the other hand, the integrability of quantum chains is usually demonstrated rather indirectly, by showing that the model can be solved by the coordinate Bethe ansatz or that the hamiltonian can be derived from a commuting family of transfer matrices related to the Yang-Baxter equation. But these are only sufficient conditions for integrability. Moreover, testing these sufficient conditions is often not easy.

It is thus clearly desirable to design a more general, simple and efficient integrability test for quantum chains. Here we propose a simple test based on the existence of a local nontrivial three-point charge \( H_3 \), the higher order conservation law just above \( H_2 \), the defining hamiltonian of the model. Locality means that the interaction involving a certain set of sites disappears when the distances between them become sufficiently large. We will

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1 Note however, that the Yang-Baxter equation implies a relatively simple equation, known as the Reshetikhin condition. See section 2 for a discussion of this point.
indicate below why for quantum chains, one can expect that the existence of $H_3$ should be a necessary and in some cases a sufficient condition for integrability. From the computational point of view, the advantage of this simple-minded approach is that the number of possible candidates for the three-point charge is usually not exceedingly large, and they are often restricted by the symmetries of the system.

A related integrability test has been studied in [5]. However, these authors confined their test to the search of one nontrivial conservation law (not necessarily $H_3$) in spin 1/2 models, with the simultaneous existence of a ladder operator providing a recursive scheme for the calculation of the other conservation laws. This is certainly less general and less constructive than the test proposed here.

Another integrability test has been formulated for a very special class of self-dual quantum chains. For such systems, there exists a simple sufficient condition for integrability, due to Dolan and Grady [6]. This condition actually applies to any type of self-dual systems, discrete or continuous, and defined in any number of space-time dimensions. These systems are described by an hamiltonian of the form:

$$H_2 = \alpha H + \beta \tilde{H};$$  \hspace{1cm} (1.1)

where $\tilde{H}$ is the dual of $H$, with duality being defined as any nontrivial linear operator with the property $\tilde{\tilde{H}} = H$. If such hamiltonians satisfy the relations:

$$[H, [H, [H, \tilde{H}]]] = 16 [H, \tilde{H}],$$ \hspace{1cm} (1.2)

there exists a (potentially infinite, for infinite systems) family of conservation laws and these can be constructed systematically. We stress that this sufficient condition for integrability is simply the requirement of the existence of a charge $H_3$ of a particular form. We have thus a neat situation here in which the the existence of a third-order charge $H_3$ guarantees the existence of an infinite family of conserved charges. Unfortunately, the applicability of this condition seems rather limited. For example, the $H_{XYZ}$ hamiltonian is self-dual, as it may be put in the form:

$$H_{XYZ} = \alpha H_{YZ} + \beta \tilde{H}_{YZ};$$  \hspace{1cm} (1.3)

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2 The condition (1.2) appears to apply only to the $Z_n$ generalization of the Ising model defined in [7]. Systems satisfying (1.2) are sometimes called superintegrable and the underlying algebraic structure is the so-called Onsager algebra [8].
where

\[ H_{YZ} = \sum_{j \in \Lambda} [\lambda_y \sigma^y_j \sigma^y_{j+1} + \lambda_z \sigma^z_j \sigma^z_{j+1}], \]

(1.4)

where \( \sigma^a_i \) are Pauli spin matrices acting nontrivially only on the site \( i \) of the lattice \( \Lambda \). The duality is defined by:

\[ \tilde{\sigma}^x_j = \sigma^y_j, \quad \tilde{\sigma}^y_j = \sigma^x_j, \quad \tilde{\sigma}^z_j = -\sigma^z_j. \]

(1.5)

However, the condition (1.2) is not satisfied unless \( \lambda_y = 0 \) (in which case (1.1) reduces to the Ising model hamiltonian), or \( \lambda_z = 0 \) (the XY model).

Our proposal is described in detail in the next section. It is formulated for the class of translationally invariant models with nearest-neighbor interactions. These models are defined on a lattice \( \Lambda \) which is either finite with periodic boundary conditions (i.e. \( \Lambda = \{1, \ldots, N\}, \) with \( N + 1 \equiv 1 \)) or infinite (i.e. \( \Lambda = \mathbb{Z} \)). Notice that the restriction to nearest-neighbor interactions is not as severe as it might appear at first sight, since this class of models actually contains, when appropriately reformulated, any model involving binary interactions with finite range. Indeed, any model with a finite interaction range \( k_0 \) (where \( k_0 = 2 \) corresponds to nearest-neighbor interactions) can be equivalently described in terms of nearest-neighbor interactions just by grouping together \( k_0 \) consecutive sites into a single one on which a vectorial spin-like variable would live. Similarly, any model with a more general invariance under a shift \( j \rightarrow j + j_0 \) can be made invariant with respect to a translation by a single unit, by grouping together \( j_0 \) consecutive sites.

Let \( \{S^a_i\} \) denote a set of quantum operators (distinguished by the index \( a \)), acting nontrivially only in a Hilbert space \( V_i \). (We will assume in this work that all the \( V_i \)'s are finite-dimensional; in general they don’t have to be identical). The full space of states of such chain is then the tensor product \( \bigotimes_{i \in \Lambda} V_j \). The class of hamiltonians under consideration has then the following form:

\[ H_2 = \sum_{j \in \Lambda} [g_j + h_{j,j+1}], \]

(1.6)

where \( g_j = g(S_j) \) describes interactions at site \( i \) and \( h_{j,j+1} \) is some site-independent function of \( S_j, S_{j+1} \) which describes nearest-neighbor interactions.
2. Conjectured integrability tests for quantum chains

2.1. A conjectured necessary condition for integrability of a quantum chain

From the early days of soliton theory, it has been clear that the existence of a non-trivial conservation law (beyond those associated with the standard conservation of mass or charge, momentum and energy) is a very strong indication of the existence of an infinite number of additional nontrivial conservation laws. But for continuous systems, either classical or quantum, one cannot focus the attention on the existence or non-existence of the conservation law with a degree just above the one which usually plays the rôle of the hamiltonian. The reason is simply that symmetry considerations can prevent the existence of $H_n$ for a set of values of $n$. Take for instance the KdV equation,

$$u_t + u_{xxx} + 6uu_x = 0,$$  \hspace{1cm} (2.1)

for which $u$ has degree 2 in the normalization where $\partial_x$ has degree 1. Due to a hidden $Z_2$ invariance, there are no conservation laws with densities of odd degree. More generally, for Toda systems related to Lie algebras, the values of $n$ for which $H_n$ is non-zero are related to the exponents of the corresponding Lie algebra [9]. But notice that when the conservation laws are calculated by a recursive method, the conserved densities associated to charges that vanish by symmetry are not found to be zero; rather they are total derivatives.

As indicated above, these considerations apply for both classical and quantum systems. But in the quantum case, there is another possible source for the absence of a conserved charge of given order, which is that the density can be exactly proportional to a null vector. Again this is observed in the quantum KdV case, where for special values of the central charge, the free parameter in the defining commutation relation, some conserved densities become proportional to null vectors of the Virasoro algebra [10].

Now, our point here is that both these possibilities are absent in the case of quantum chains. In particular, it is clear that on the lattice there is no room for total derivatives. Furthermore, null vectors appear not to be relevant if the spaces $V_i$ are finite-dimensional.\footnote{Constraints imposed on the factor spaces $V_i$ (e.g. realized via a projection onto some subspace of $V_i$) may “project out” some charges of an integrable model - in other words some of the conserved charges evaluated in a restricted space of states may conceivably vanish. Such constraints would thus have an effect similar as null-vectors. However, situations like this can be avoided by considering the conservation laws in the full (unrestricted) Hilbert space.}

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On the other hand, for systems whose integrability can be traced back to commuting transfer matrices, the conserved charges are obtained from the expansion of the logarithm of the transfer matrix [11] and $H_3$ is never absent. This can be easily seen for fundamental systems [12], characterized by a Lax operator $L_n(\lambda)$ proportional to the model’s $R$ matrix $R_{n0}(\lambda)$, normalized so that $R_{n0}(0) = P_{n0}$:

$$L_n(\lambda) = R_{n0}(\lambda).$$

(2.2)

Here we use the standard notation in which $R_{ij}$ acts in the tensor product space $V_i \otimes V_j$, where $V_i$ is the vector space associated to the site $i$. The index zero refers to the quantum space $V_0$ on which the matrix entries of $L_i$ act. $P_{n0}$ denotes the permutation operator. (For simplicity all the vector spaces $V_i$ are assumed to be equivalent here). The transfer matrix is defined as:

$$T(\lambda) = \text{Tr}_{V_0} L_N(\lambda)...L_1(\lambda).$$

(2.3)

For the fundamental models, a simple calculation shows that the second logarithmic derivative of the transfer matrix at $\lambda = 0$ is proportional to:

$$Q_3 \sim \sum_{j \in \Lambda} [h_{j,j+1}, h_{j+1,j+2}]$$

(2.4)

where $h_{i,i+1}$ is defined in (1.6). (Here we follow the notation used in [1]: $Q_n$ stands for the $(n-1)$-th derivative of the logarithm of the transfer matrix and $H_n$ is obtained from $Q_n$ by stripping off contributions of the lower order charges.) If one excludes a pathological case in which all adjacent links commute, the three-point charge defined above is nontrivial. The existence of a nontrivial third order charge can also be proven for a more general class of integrable models, characterized by the condition

$$L_n(0) = P_{n0}.$$  

(2.5)

Note that such $L$ matrix is not necessarily itself a solution of the Yang-Baxter equation. An example of a transfer matrix which satisfies (2.5), but which does not define a fundamental

\footnote{For example, such pathological situation arises for the “chopped XXZ” model introduced in section 3.}
model, is provided by the transfer matrix of the Hubbard model found by Shastry [13]. For models satisfying (2.5), the second logarithmic derivative of the transfer matrix becomes:

$$Q_3 \sim \sum_{j \in \Lambda} [h_{j,j+1}, h_{j+1,j+2}] - \sum_{j \in \Lambda} h_{j,j+1} h_{j,j+1}$$

$$+ \sum_{j \in \Lambda} T^{-1}(0) \text{Tr}_0 (P_{N0} \ldots \frac{d^2L_j}{d\lambda^2}(0) \ldots P_{10}).$$ (2.6)

The last two terms involve only nearest-neighbor interactions, while the first is a sum over triples of consecutive spins. This term has the same form as (2.4), and it is clearly nontrivial except again for pathological systems with all adjacent links commuting.

The above considerations motivate the following conjecture for the class of systems described by (1.6):

**Conjecture 1:** A translationally invariant periodic quantum spin chain, with a hamiltonian $H_2$ involving at most nearest-neighbor interactions, is integrable only if there exists a nonvanishing local independent charge $H_3$, being a sum of terms coupling spins at most three sites, which commutes with $H_2$ for all chain sizes $N \geq 3$.

This conjecture implies then a simple test consisting in establishing the existence of such $H_3$. In particular, one may conclude that a spin chain is nonintegrable, by demonstrating the non-existence of a nontrivial $H_3$. On the other hand, if such a charge exists, the system is likely to be integrable; but of course its integrability has to be proven independently. Actually, it appears also that for a large class of systems (and in particular for all the models considered in sections 3 and 4), the mere existence of $H_3$ is enough to guarantee the existence of an infinite family of conserved charges in involution.

### 2.2. Clarifying comments related to the formulation of conjecture 1

Some elements entering in the formulation of the conjecture deserve clarification.

i) **Independence of the charges**

The charge $H_3$ in the above conjecture should be independent of the hamiltonian and of possible charges of lower order (such as, e.g. the components of the total spin). For general quantum integrable models, the issue of functional independence can be quite complicated; it is indeed a major difficulty in formulating a general definition of quantum integrability.
in a completely rigorous way. However, functional independence of the charges can be usually easily verified for spin chains with short range interactions: the leading term of $H_n$ contains $n$ adjacent interacting spins; the leading terms of $H_n$ and $H_m$ for $m \neq n$ being clearly distinct, the corresponding charges are linearly independent. Furthermore, such a cluster of $n$ adjacent spins cannot be obtained from a product of lower order charges (since that would also generate terms with $n$ non-adjacent spins).

ii) Stability of the charge under a variation of the chain length

The last item in the above conjecture ensures that the existence of $H_3$ should not be affected by a change $N \to N + k$, where $k$ is an arbitrary positive integer. This stability requirement can be simply illustrated for the XXX model defined by the hamiltonian

$$H = \sum_{i=1}^{N} \sigma_i \cdot \sigma_{i+1}$$  \hspace{1cm} (2.7)

(where as usual periodic boundary conditions are assumed). This hamiltonian commutes with any component of the total spin

$$S^a = \sum_{i=1}^{N} \sigma_i^a.$$  \hspace{1cm} (2.8)

A simple calculation shows that for $N = 4$ the quantity

$$H' \equiv \sum_{i=1}^{N} \sigma_i \cdot \sigma_{i+2}$$  \hspace{1cm} (2.9)

is conserved. $H'$ is also conserved for $N = 5$ but not for any higher value of $N$. The reason for this behavior is that for $N = 4$ and 5, $H'$ can be regarded as a nonlocal charge. Such nonlocal charges typically can be obtained from powers of the local ones. More exactly, for $N = 4$ we have the following identity (modulo an additive constant):

$$S^a S^a = 2H + H'.$$  \hspace{1cm} (2.10)

\footnote{Dropping the requirement of functional independence does not lead to a meaningful definition of integrability [14]. In the context of spin chains this can be easily seen. Consider an isotropic chain, for which the hamiltonian commutes with all the components of the total spin. Arbitrary powers of any of the spin components yield then a set of mutually commuting conserved charges. Removing the requirement of functional independence in the definition of integrability would therefore render any isotropic spin chain automatically integrable.}
Similarly, for \( N = 5 \)

\[
S^a S^a = 2H + 2H'.
\] (2.11)

However, for \( N > 5 \) \( H' \) is not related to \( S^a S^a \) or other nonlocal charges and it is no longer conserved. For \( N = 6 \) for instance, we have instead

\[
S^a S^a = 2H + 2H'',
\] (2.12)

where

\[
H'' \equiv \sum_{i=1}^{N} [\sigma_i \cdot \sigma_{i+2} + \frac{1}{2} \sigma_i \cdot \sigma_{i+3}]
\] (2.13)

contains contributions with one and two holes. Thus the conservation of \( H' \) is not preserved under the change \( N \to N + 2 \); the same is true for \( H'' \). This is actually typical for all such “accidental” charges, whose conservation for some particular values of \( N \) is due to an accidental identity (relating them to nonlocal charges), which is true only for these particular values of \( N \).

iii) Locality vs nonlocality

The above example illustrate another issue: it is not always easy to distinguish the nonlocal charges (e.g. \( H' \)) from the local ones (e.g. \( H \)) for finite chains. For an infinite chain, local and nonlocal expressions are quite distinct (the latter contain interactions between arbitrarily distant spins). On the other hand, for the XXX model with \( N = 4, 5 \), one hole in the expression of a 2-spin conserved law reflects “nonlocality”! However, as exemplified above, the form of those nonlocal charges that can be written as powers of local ones is strongly \( N \)-dependent; this property makes them easily detected. Notice also that the local charges for a finite chain may be defined non-ambiguously from the densities of the first \( N \) charges of the infinite chain.

2.3. A conjectured sufficient condition for integrability based on the boost operator

The integrable points in a multi-parameter space of general spin chain hamiltonians can usually be simply characterized by the occurrence of a dynamical symmetry. Related to such symmetry is the existence of a ladder operator \( B \), acting on the conservation laws as

\[
[B, H_n] = H_{n+1},
\] (2.14)
where $H_n$ denotes a charge with at most $n$ adjacent interacting spins. This motivates the formulation of a simple conjectured sufficient condition for integrability, based on the existence of a ladder operator $B$, in conjunction with the presence of a nontrivial charge $H_3$:

**Conjecture 2:** A translationally invariant periodic quantum chain, with a hamiltonian $H_2$ involving at most nearest-neighbor interactions, is integrable if there exists an operator $B$ such that for all chains of length $N \geq 3$, $[B, H_2]$ is nontrivial and

$$[[B, H_2], H_2] = 0. \quad (2.15)$$

In contrast with the first conjecture, the approach here is more constructive: it indicates how $H_3$ can be built, i.e. as $[B, H_2]$. This constructive aspect presupposes that we can easily guess the form of $B$. In all the cases we have considered, such a $B$ turns out to be proportional to the first moment of an appropriately symmetrized form of the hamiltonian $H_2$, i.e. with

$$H_2 = \sum_{j \in \Lambda} h_{j,j+1}, \quad \text{for } h_{j,j+1} \text{ symmetric in } j, j + 1, \quad (2.16)$$

$B$ is found to be

$$B = \sum_{j \in \Lambda} j \, h_{j,j+1}. \quad (2.17)$$

The condition of commutativity of $H_2$ and $H_3$ assumes then a particularly simple form:

$$\sum_{j \in \Lambda} [h_{j,j+1} + h_{j+1,j+2}, [h_{j,j+1}, h_{j+1,j+2}]] = 0. \quad (2.18)$$

The derivation of this result is very simple. Starting from

$$H_3 = [B, H_2] = \sum_{j \in \Lambda} (j - 1)[h_{j-1,j}, h_{j,j+1}] + (j + 1)[h_{j+1,j+2}, h_{j,j+1}]$$

$$= - \sum_{j \in \Lambda} [h_{j,j+1}, h_{j+1,j+2}], \quad (2.19)$$

one then enforces

$$[H_3, H_2] = \sum_{j} [[h_{j-2,j-1}, h_{j-1,j}, h_{j,j+1}] + [[h_{j-1,j}, h_{j,j+1}], h_{j,j+1}]$$

$$+ [[h_{j,j+1}, h_{j+1,j+2}], h_{j,j+1}] + [[h_{j+1,j+2}, h_{j+2,j+3}], h_{j,j+1}] = 0. \quad (2.20)$$

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Note that we allow for the possibility of a linear combination of lower order charges $H_{m \leq n}$ on the rhs of (2.14).
Using the Jacobi identity, the first term on the rhs can be written in the form $-\left[h_{j-1,j}, h_{j,j+1}, h_{j+1,j}\right]$; with the shift $j \rightarrow j + 2$, it cancels exactly the fourth term. Then, by shifting $j$ by one unit in the second term, we recover (2.18).

From the Jacobi identity, this condition automatically ensures the existence of a second nontrivial conservation law $H_4$ which commutes with $H_2$:

\[
[H_4, H_2] = [[B, H_3], H_2] = -[[H_3, H_2], B] - [[H_2, B], H_3] = 0. \tag{2.21}
\]

However, showing that $[H_4, H_3] = 0$ and that higher charges commute with $H_2$ requires additional information. For example, if it is known beforehand that the commutant of $H_2$ is abelian, (2.18) actually implies the existence of an infinite tower of charges in involution.

2.4. Relation between conjecture 2 and the Reshetikhin condition

It is known (see [15]) that the existence of such a ladder operator is a direct consequence of the Yang-Baxter equation for nearest-neighbor interacting chains for which the transfer matrix is a product of $R$ matrices (the so-called fundamental spin chains). But the conjecture is a priori independent of the Yang-Baxter equation and in principle there could exists models satisfying (2.18) and not the Yang-Baxter equation. Furthermore, (2.18) is easier to test that the Yang-Baxter equation for the related $R$ matrix.

Actually, for fundamental spin systems, (2.18) can be viewed as a condition for the matrix

\[
R(\lambda) = P[I + \lambda H_2 + \mathcal{O}(\lambda^2)] \tag{2.22}
\]

to be a solution of the Yang-Baxter equation. This approach leads to the condition

\[
[h_{j,j+1} + h_{j+1,j+2}, [h_{j,j+1}, h_{j+1,j+2}]] = X_{j,j+1} - X_{j+1,j+2} \tag{2.23}
\]

for some quantity $X$. This relation first appeared in [16] (see eq. (3.20)) and it is attributed to Reshetikhin. An explicit derivation can be found in [17]. The Reshetikhin condition (2.23) is nothing but the local version of (2.18). It appears to be an anticipation of the boost construction of conservation laws. It is also pointed out in [16] that this equation is not satisfied by all integrable systems (which is by now understood from the fact that not all such systems are fundamental); in particular this is the case for the Hubbard model (in agreement with the conclusion in [1] concerning the non-existence of a boost operator).

In the rest of this work we examine the existence of a third-order charge for a number of models. Some of the calculations were performed using “Mathematica”.

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3. Example 1: spin-1/2 chains with next-to-nearest neighbors interactions and bond alternation

3.1. Definition of the model

We consider a 10-parameter family of spin-1/2 models, which contain, in addition to XYZ-type interactions, next-to-nearest neighbor interactions, bond alternation terms and a magnetic field coupling term:

$$H_2 = \sum_{j \in \Lambda} \left\{ \sum_{a=x,y,z} \left[ \lambda_a \sigma_j^a \sigma_{j+1}^a + \lambda'_a (-1)^j \sigma_j^a \sigma_{j+1}^a + \lambda''_a \sigma_j^a \sigma_{j+2}^a \right] + h \sigma_j^z \right\}. \quad (3.1)$$

This is the most general spin-1/2 model with interaction range shorter than two lattice units admitting bond alternations. To ensure translational invariance, we assume that the number of sites $N$ is even. These models can be equivalently represented as nearest-neighbor interactions on a lattice $\Lambda'$, whose bonds correspond to nonvanishing interactions (see Fig. 1).

![Diagram with lattice Λ′ corresponding to the hamiltonian (3.1), whose bonds correspond to nonvanishing interactions.](image)

A lattice with such a “railroad trestle” topology has been considered in [18] (in the case where all the couplings are equal). Notice that the lattice in Fig. 1 corresponds also to a generalization of (3.1) admitting bond alternation for next-to-nearest neighbors. A
particular case of such bond alternation yields then the “sawtooth” topology, which has been shown to possess an exact valence-bond ground state [19].

The models in (3.1) can be equivalently described by a Hamiltonian of the form (1.6). The structure of (1.6) is recovered if we express (3.1) in terms of the variables:

\[ S_i = \begin{pmatrix} \sigma_{2i-1} \\ \sigma_{2i} \end{pmatrix} \text{ or } S_i = \begin{pmatrix} \sigma_{2i} \\ \sigma_{2i+1} \end{pmatrix}. \] (3.2)

The family (3.1) contains many interesting systems, including some that are well known integrable models and some which are “exactly solvable” in some sense. In particular, among the class of isotropic (globally \( su(2) \)-invariant) models satisfying \( h = 0 \) and

\[ \lambda_a = \lambda, \quad \lambda'_a = \lambda', \quad \lambda''_a = \lambda'', \] (3.3)

for all \( a \), the following special cases are covered by (3.1):

(i) the Heisenberg (XXX) model \( (\lambda' = \lambda'' = 0) \):

\[ H_2 = \sum_{j \in \Lambda} \lambda \sigma_j^a \sigma_{j+1}^a, \] (3.4)

(ii) the staggered XXX model \( (\lambda'' = 0) \):

\[ H_2 = \sum_{j \in \Lambda} [\lambda + \lambda' (-1)^j] \sigma_j^a \sigma_{j+1}^a, \] (3.5)

(iii) the alternating XXX model \( (\lambda'' = \lambda = 0) \):

\[ H_2 = \sum_{j \in \Lambda} \lambda (-1)^j \sigma_j^a \sigma_{j+1}^a, \] (3.6)

(iv) the Majumdar-Ghosh model [20]: \( (\lambda' = 0, \lambda'' = \frac{1}{2} \lambda) \):

\[ H_2 = \sum_{j \in \Lambda} [\lambda \sigma_j^a \sigma_{j+1}^a + \frac{1}{2} \lambda \sigma_j^a \sigma_{j+2}^a]. \] (3.7)

As is well known this model possesses an exact valence-bond ground state.

For models invariant under global spin rotation around the \( z \)-axis:

\[ \begin{pmatrix} \sigma_i^x \\ \sigma_i^y \end{pmatrix} \rightarrow \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \sigma_i^x \\ \sigma_i^y \end{pmatrix}, \] (3.8)
(in which case the global $su(2)$ invariance is broken down to $o(2)$), the coupling constants satisfy:

$$\lambda_x = \lambda_y, \quad \lambda'_x = \lambda'_y, \quad \lambda''_x = \lambda''_y.$$ \hspace{1cm} (3.9)

Some interesting $o(2)$-symmetric models that can be obtained from a specialization of (3.1) are:

(v) the XXZ model ($\lambda'_a = \lambda''_a = 0$):

$$H_2 = \sum_{j\in\Lambda} [\lambda_x (\sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1}) + \lambda_z \sigma^z_j \sigma^z_{j+1}],$$ \hspace{1cm} (3.10)

(vi) the Lieb-Sultz-Matis model with alternating Heisenberg and Ising bonds [21]:

$$\lambda''_a = 0, \quad \lambda_x = \lambda_y = \lambda'_x = \lambda'_y = \frac{1}{2},$$

$$\lambda_z = (1 + U)/2, \quad \lambda'_z = (1 - U)/2,$$ \hspace{1cm} (3.11)

with the hamiltonian

$$H_2 = \sum_{j\in\Lambda} [\sigma^a_{2j} \sigma^a_{2j+1} + U \sigma^z_{2j+1} \sigma^z_{2j+2}],$$ \hspace{1cm} (3.12)

(vii) the Hubbard model:

$$\lambda_x = \lambda_y = \lambda'_x = \lambda'_y = 0, \quad \lambda''_x = \lambda''_y = 1,$$

$$\lambda'_z = \pm \lambda_z = U,$$ \hspace{1cm} (3.13)

with the hamiltonian

$$H_2 = \sum_{j\in\Lambda} [\sigma^+_{j} \sigma^-_{j+2} + \sigma^-_{j} \sigma^+_{j+2} + U [1 + (-1)^j] \sigma^z_j \sigma^z_{j+1}],$$ \hspace{1cm} (3.14)

The equivalence of (3.14) (for a lattice with an even number of sites) with the usual formulation of the Hubbard model

$$H = \sum_{j\in\Lambda} [s^x_j s^x_{j+1} + s^y_j s^y_{j+1} + t^x_j t^x_{j+1} + t^y_j t^y_{j+1} + U s^z_j t^z_j]$$ \hspace{1cm} (3.15)

(where $s_i$ and $t_i$ are two independent sets of Pauli matrices at site $i$), can be seen by redefining the spin variables in (3.14) as

$$s^a_j \rightarrow \sigma^a_{2j}, \quad t^a_j \rightarrow \sigma^a_{2j+1}.$$ \hspace{1cm} (3.16)
3.2. Strategy of the test

For the completely general anisotropic case, a natural candidate for the third order charge involves only nearest neighbor interactions on the the lattice in Fig. 1 and has the form:

\[
H_3 = \sum_{j \in \Lambda} \sum_{a_i=x,y,z} \left[ \left( \alpha_{a_1a_2a_3}^{(1)} + (-1)^j \alpha_{a_1a_2a_3}^{(1)} \right) \sigma_{j}^{a_1} \sigma_{j+1}^{a_2} \sigma_{j+2}^{a_3} + \left( \alpha_{a_1a_2a_3}^{(2)} + (-1)^j \alpha_{a_1a_2a_3}^{(2)} \right) \sigma_{j}^{a_1} \sigma_{j+2}^{a_2} \sigma_{j+3}^{a_3} + \left( \alpha_{a_1a_2a_3}^{(3)} + (-1)^j \alpha_{a_1a_2a_3}^{(3)} \right) \sigma_{j}^{a_1} \sigma_{j+1}^{a_2} \sigma_{j+3}^{a_3} + \left( \alpha_{a_1a_2a_3}^{(4)} + (-1)^j \alpha_{a_1a_2a_3}^{(4)} \right) \sigma_{j}^{a_1} \sigma_{j+2}^{a_2} \sigma_{j+4}^{a_3} \right]
\]

where \( \alpha_{a_1a_2a_3}^{(i)} \) and \( \alpha_{a_1a_2a_3}^{(i)} \) are arbitrary coefficients. We search for integrable systems by imposing the condition of commutativity of \( H_3 \) with the hamiltonian. This leads to an over-determined system of equations for the set of parameters in \( H_3 \). This system has nontrivial solutions only for special values of the parameters of the hamiltonian. However, the analysis of this system is very cumbersome, since in absence of additional symmetries, (3.17) contains 216 free parameters! Henceforth, we consider only three special situations, in which the analysis simplifies significantly: the isotropic case, the \( o(2) \)-symmetric case, and the anisotropic case without bond alternation nor next-to-nearest neighbor interaction - that is, the XYZh model.

One might consider other candidate charges, involving other triples of sites than those in (3.17). However, suppose that one cluster different from those appearing in (3.17) is introduced. Then (considering an infinite chain), to cancel the new terms arising in the commutator with the hamiltonian, an infinite sequence of other clusters would have to be added, with the distance between spins in such clusters growing arbitrarily. This would violate the requirement of locality. (The above reasoning is not so obvious however when some of the couplings vanish.)

3.3. The XYZh model

We first consider (3.1) in absence of bond alternation and next-to-nearest neighbor terms. The most general candidate charge \( H_3 \), coupling three nearest-neighbor spins, is:

\[
H_3 = \sum_{j \in \Lambda} \sum_{a_i=x,y,z} \alpha_{a_1a_2a_3} \sigma_{j}^{a_1} \sigma_{j+1}^{a_2} \sigma_{j+2}^{a_3},
\]

which contains 27 free parameters. By enforcing the commutativity of this candidate \( H_3 \) with the hamiltonian, we found that if \( h \neq 0 \) and no two couplings are equal, the only
solution is: \( \alpha_{a_1a_2a_3} = 0 \) for all triples \( a_1a_2a_3 \). Thus there is no nontrivial charge \( H_3 \) for the anisotropic XYZ chain in a nonzero magnetic field. This suggests thus that the XYZh model is nonintegrable, in agreement with the fact that the Bethe Ansatz solution for the XYZ model cannot be generalized to the case with a nontrivial magnetic field.

3.4. The isotropic case

In the isotropic case, the most general third order charge involving only the nearest neighbors on the lattice in Fig. 1 has the following form:

\[
H_3 = \sum_{j \in \Lambda} \epsilon_{abc} (\alpha + (-1)^j \alpha') \sigma_j^a \sigma_{j+1}^b \sigma_{j+2}^c \\
+ (\beta + (-1)^j \beta') \sigma_j^a \sigma_{j+2}^b \sigma_{j+3}^c + (\tilde{\beta} + (-1)^j \tilde{\beta}') \sigma_j^a \sigma_{j+1}^b \sigma_{j+3}^c + \\
(\gamma + (-1)^j \gamma') \sigma_j^a \sigma_{j+2}^b \sigma_{j+4}^c,
\]

where \( \alpha, \beta, \tilde{\beta}, \gamma \) and their primed variants are arbitrary parameters. Solving the commutativity condition \([H_2, H_3] = 0\), we found that (for \( N \geq 8 \)) nontrivial solutions (for which not all parameters of \( H_3 \) are equal to zero) exist only in two cases: for the XXX model \((\lambda' = \lambda''\)) with the solution \( \alpha \neq 0, \beta = \beta' = \tilde{\beta} = \tilde{\beta}' = \gamma = \gamma' = 0 \), i.e.

\[
H_3 = \sum_{j \in \Lambda} \epsilon_{abc} \sigma_j^a \sigma_{j+1}^b \sigma_{j+2}^c,
\]

and for \( \lambda = \lambda' \), which corresponds to two decoupled XXX models on the even and odd sublattices (the solution is then \( \gamma, \gamma' \neq 0 \) and all other parameters of \( H_3 \) equal to zero).

These results suggest that the only integrable isotropic models within the family (3.1) are of the XXX-type. In particular, the Majumdar-Ghosh model, the alternating XXX model, and the staggered XXX model all fail the test of the existence of \( H_3 \) and thus seem to be nonintegrable.

Note that in the continuous limit the alternating term corresponds to \( \text{tr } g \) in the WZNW model [22] with a level one affine \( \text{su}(2) \) spectrum generating algebra. \( g \) stands for the basic field in the WZNW model, a \( 2 \times 2 \) matrix in the \( \text{su}(2) \) case. \( \text{tr } g \) turns out to be an integrable perturbation of this WZNW model [23]. However, the staggered XXX model in the continuous limit gives the WZNW model perturbed by both \( \text{tr } g \) and the marginal current-current term. Together, these two perturbations are incompatible with integrability. Note also that for \( \lambda = \pm \lambda' \) the staggered XXX model degenerates into a pathological “chopped XXX” chain consisting of disjoint bonds (see section 3.5 for a discussion of a similar case).
For \( N < 8 \) there exist nontrivial solutions for \( H_3 \) of the form (3.19). Let us illustrate this in the case \( \lambda_x' = 0 \), when the hamiltonian (3.1) contains only nearest and next-to-nearest neighbors,

\[
H_2 = H + \mu H',
\]  

where \( \mu \) is an arbitrary parameter, and \( H \) and \( H' \) are given by (2.7) and (2.9) respectively. For \( N = 5 \) we find that the XXX charge (3.20) commutes with (3.21). This is a consequence of the “accidental” identity (2.11) holding for \( N = 5 \). In other words, for \( N = 5 \) the next-to-nearest interaction is a “nonlocal” charge, (related to the square of the total spin), which commutes with all the XXX charges. For \( N > 5 \) this solution disappears (the three-spin XXX charge no longer commutes with the hamiltonian (3.21)). Similarly, for \( N = 6 \) there is a one parameter family of solutions

\[
H_3 = H_3^{XX} + \nu F_{3,1} - 1/3 (\mu + \mu \nu - \nu) F_{3,2}^s,
\]  

where \( \nu \) is an arbitrary parameter, and

\[
F_{3,1} = \sum_{j \in \Lambda} \epsilon^{abc} \sigma^a_j \sigma^b_{j+2} \sigma^c_{j+3} + \sigma^a_j \sigma^b_{j+1} \sigma^c_{j+3},
\]

\[
F_{3,2}^s = \sum_{j \in \Lambda} \epsilon^{abc} \sigma^a_j \sigma^b_{j+2} \sigma^c_{j+4}.
\]  

For \( N = 7 \) there is another solution:

\[
H_3 = H_3^{XX} + \mu F_{3,1} - \mu^2 F_{3,2}^s.
\]  

Similar accidental nonlocal three-spin charges, whose form changes with \( N \) exist in fact for all \( N \), but it is only for \( N < 8 \) that they “look local”, i.e. can be put in the form (3.19).

### 3.5. The o(2)-invariant case

We are searching again for a nonvanishing charge \( H_3 \) of the form (3.17). The requirement of the o(2) invariance imposes a number of restrictions on the parameters \( \alpha_{a_1 a_2 a_3}^{(i)} \) and \( \alpha_{a_1 a_2 a_3}^{(i)} \):

\[
\begin{align*}
\alpha_{yxy} &= \alpha_{zxx} = \alpha_{xyx} = \alpha_{xyz} = \alpha_{yyx} = \alpha_{yxz} = \\
\alpha_{zxx} &= \alpha_{zzx} = \alpha_{yzz} = \alpha_{xyy} = \alpha_{xxy} = \alpha_{xxx} = \alpha_{yyy} = 0,
\end{align*}
\]  

\footnote{\( F_{3,2}^s \) is the symmetric part of the quantity \( F_{3,2} \) introduced in [1].}
and
\[\alpha_{yxz} = -\alpha_{xyz}, \alpha_{zyx} = -\alpha_{zxy}, \alpha_{xzy} = -\alpha_{yzx},\]
\[\alpha_{yzy} = \alpha_{xzx}, \alpha_{zxx} = \alpha_{zyy}, \alpha_{xxz} = \alpha_{yyz}.\]
(3.27)

As a result, the number of free parameters in \(H_3\) is now decreased to 56. We then check whether there are nontrivial \(H_3\) of this form commuting with the Hamiltonian. For \(N = 8\) for example, this leads to a linear system of roughly 240 equations with 56 unknowns, which depends on the 6 free parameters of the Hamiltonian \((\lambda_x, \lambda_z, \lambda_x', \lambda_z', \lambda_x'', \lambda_z'')\). Again one is looking for the values of these parameters allowing for the existence of a nontrivial solution (persisting when \(N\) is increased). This system is best analyzed with a computer.

The results are given below.

If \(\lambda_x''\) is not zero, a nontrivial solution for \(H_3\) exists only in two cases:
(a) for two decoupled XXZ models on two disjoint (even and odd) sublattices: \((\lambda_x = \lambda_x' = \lambda_z = \lambda_z')\),
(b) the Hubbard model: \((\lambda_x = \lambda_x' = \lambda_z'' = 0, \lambda_z = \pm \lambda_z')\).

The solution for the charge \(H_3\) is then
\[
H_3 = \sum_{j \in \Lambda} \left[ 2\lambda_z' (\sigma_j^y \sigma_{j+1}^z + \sigma_j^z \sigma_{j+1}^y) - \sigma_j^y \sigma_{j+1}^y + \sigma_j^y \sigma_{j+1}^y - \sigma_j^x \sigma_{j+1}^x - \sigma_j^x \sigma_{j+1}^x \right] + \lambda_x'' (\sigma_j^y \sigma_{j+2}^y - \sigma_j^x \sigma_{j+2}^x)
+ \lambda_z' (1 - (-)^j) (\sigma_j^y \sigma_{j+3}^y - \sigma_j^x \sigma_{j+3}^x)
- \sigma_j^y \sigma_{j+1}^y + \sigma_j^x \sigma_{j+1}^x,\]
(3.28)
which can be translated, using (3.16), into the usual expression for the third-order charge in the Hubbard model [13, 1].

If \(\lambda_x''\) is zero there are more possibilities. Nontrivial solutions for \(H_3\) exist in four cases. These are:
(c) the XXZ model: \(\lambda_x' = \lambda_z' = \lambda_z'' = 0\),
(d) the staggered XX model: \(\lambda_z = \lambda_z' = \lambda_z'' = 0\),
(e) the staggered XXZ model: \(\lambda_x = \lambda_z' = \lambda_z'' = 0\),
(f) the model with alternating XXZ and Ising bonds: \(\lambda_x = \pm \lambda_x'\).

The three-spin charge obtained for the XXZ model is:
\[
H_3 = \sum_{j \in \Lambda} \left[ \lambda_z (\sigma_j^x \sigma_{j+1}^y + \sigma_j^y \sigma_{j+1}^x - \sigma_j^y \sigma_{j+1}^y + \sigma_j^y \sigma_{j+1}^y - \sigma_j^x \sigma_{j+1}^x - \sigma_j^x \sigma_{j+1}^x) + \lambda_x (\sigma_j^x \sigma_{j+2}^x - \sigma_j^y \sigma_{j+2}^y) \right],\]
(3.29)
\[\text{Note that trivial } o(2)-\text{symmetric models with Hamiltonians involving only the } z\text{-components of the spin variables at each site are explicitly excluded from consideration.}\]
in agreement with [11,1].

For the staggered XX model

\[
H_2 = \sum_{j \in \Lambda} (\lambda + \lambda'(-1)^j)(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y),
\]

(3.30)

the three-spin charge obtained via our test is identical to the XX charge, that is

\[
H_3 = \sum_{j \in \Lambda} \alpha(\sigma_j^x \sigma_{j+1}^x \sigma_{j+2}^x + \sigma_j^y \sigma_{j+1}^y \sigma_{j+2}^y) + \beta(\sigma_j^x \sigma_{j+1}^x \sigma_{j+2}^y - \sigma_j^y \sigma_{j+1}^z \sigma_{j+2}^x),
\]

(3.31)

where \(\alpha\) and \(\beta\) are arbitrary coefficients. Another indication of integrability of this model is provided by its continuum limit, which corresponds to a theory of free massive fermions [24]. Since we are not aware of an explicit proof of the integrability of the lattice model (3.30) in the literature, we present a simple direct proof by exhibiting a family of mutually commuting conservation laws. These can all be expressed in terms of the densities:

\[
e_{\alpha\beta}^\alpha = \sigma_j^\alpha \sigma_{j+1}^\alpha \cdot \cdot \cdot \sigma_{j+n-2}^\beta \sigma_{j+n-1}^\beta,
\]

(3.32)

defined for \(n \geq 2\). In terms of these quantities, the scalar and pseudoscalar conserved charges [1] of the XX model are:

\[
h_n^{(+)} = \sum_{j \in \Lambda} e_{n,j}^{xx} + e_{n,j}^{yy} \quad n \text{ even},
\]

\[
= \sum_{j \in \Lambda} e_{n,j}^{xy} - e_{n,j}^{yx} \quad n \text{ odd},
\]

(3.33)

and

\[
h_n^{(-)} = \sum_{j \in \Lambda} e_{n,j}^{xy} - e_{n,j}^{yx} \quad n \text{ even},
\]

\[
= \sum_{j \in \Lambda} e_{n,j}^{xx} + e_{n,j}^{yy} \quad n \text{ odd}.
\]

(3.34)

We also define

\[
k_n^{(+)} = \sum_{j \in \Lambda} (-1)^j(e_{n,j}^{xx} + e_{n,j}^{yy}),
\]

\[
k_n^{(-)} = \sum_{j \in \Lambda} (-1)^j(e_{n,j}^{xy} - e_{n,j}^{yx}).
\]

(3.35)

The conserved charges of the staggered XX model contain two families \(H_n^{(\pm)}\). For \(n\) odd these charges coincide with the XX charges:

\[
H_{2m+1}^{(+)} = h_{2m+1}^{(+)},
\]

\[
H_{2m+1}^{(-)} = h_{2m+1}^{(-)}.
\]

(3.36)
For \( n \) even, the conserved charges for (3.30) are:

\[
\begin{align*}
H_{2m}^{(+)} &= \lambda h_{2m}^{(+)} + \lambda' k_{2m}^{(+)} + \lambda h_{2m-2}^{(+)} - \lambda' k_{2m-2}^{(+)}, \\
H_{2m}^{(-)} &= \lambda h_{2m}^{(-)} + \lambda' k_{2m}^{(-)} + \lambda h_{2m-2}^{(-)} - \lambda' k_{2m-2}^{(-)}.
\end{align*}
\] (3.37)

Mutual commutativity of the charges \( H_n^{(\pm)} \) as well as their commutation with the staggered XX Hamiltonian (for \(|\Lambda|\) even) can be verified directly as in [1]. Note also that the boost operator,

\[
B = \frac{1}{2i} \sum_{j \in \Lambda} j[\lambda(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \lambda'(-1)^j(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y)],
\] (3.38)

has the ladder property: acting on (3.30) it produces the scalar part of (3.31).

Under a spin rotation by \( \pi/2 \) around the \( z \)-axis restricted to odd sites, i.e.:

\[
\begin{align*}
\sigma_{2j+1}^x &\rightarrow \sigma_{2j+1}^y, \quad \sigma_{2j+1}^y &\rightarrow -\sigma_{2j+1}^x, \\
\sigma_{2j}^x &\rightarrow \sigma_{2j}^x, \quad \sigma_{2j}^y &\rightarrow \sigma_{2j}^y,
\end{align*}
\] (3.39)

the alternating part of the staggered XX Hamiltonian transforms into the two-spin pseudoscalar charge \( h_2^{(-)} \) of the XX model,

\[
\sum_{j \in \Lambda} (-1)^j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) \rightarrow h_2^{(-)},
\] (3.40)

where

\[
h_2^{(-)} = \sum_{j \in \Lambda} [\sigma_j^x \sigma_{j+1}^y - \sigma_j^y \sigma_{j+1}^x].
\] (3.41)

This is a special case of the Dzyaloshinski-Moriya interaction [25]. Notice that the transformation (3.39) can be interpreted as a duality in the sense of (1.1) if we define \( \tilde{\sigma}_{2j+1}^x = i\sigma_{2j+1}^y \), \( \tilde{\sigma}_{2j+1}^y = -i\sigma_{2j+1}^x \) (where the factor \( i \) has been introduced in order to have \( \tilde{\sigma} = \sigma \)). Then \( \tilde{h}_2^{(+)} = ik_2^{(-)} \) and \( \tilde{h}_2^{(-)} = ik_2^{(+)} \). One may then consider a general Hamiltonian:

\[
H = \lambda_1 h_2^{(+)} + \lambda_2 h_2^{(-)} + \lambda_3 k_2^{(-)} + \lambda_4 k_2^{(+)},
\] (3.42)

where \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) are arbitrary constants. This Hamiltonian is integrable, as can be seen from the existence of an infinite family of conservation laws, given again (for \( n \) odd) by (3.36). Notice that (3.42) is self-dual for \( \lambda_1 \lambda_4 = \lambda_2 \lambda_3 \); however, it does not satisfy the Dolan-Grady sufficient integrability condition (1.2).
For the staggered XXZ model, defined by the Hamiltonian:

$$H_2 = \sum_{j \in \Lambda} \left[ \lambda_z (-1)^j (\sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1}) + \lambda_z \sigma^z_j \sigma^z_{j+1} \right], \quad (3.43)$$

the three-spin charge obtained from the test has the following form:

$$H_3 = \sum_{j \in \Lambda} [\lambda_z (-1)^j (\sigma^x_j \sigma^y_{j+1} \sigma^z_{j+2} - \sigma^y_j \sigma^x_{j+1} \sigma^z_{j+2} + \sigma^z_j \sigma^x_{j+1} \sigma^y_{j+2} - \sigma^x_j \sigma^y_{j+1} \sigma^x_{j+2})$$

$$+ \lambda_x' (\sigma^x_j \sigma^z_{j+1} \sigma^y_{j+2} - \sigma^y_j \sigma^z_{j+1} \sigma^x_{j+2})]. \quad (3.44)$$

This charge can be also obtained using the boost operator as $[B, H_2]$. Note that the transformation (3.39) establishes the equivalence of the staggered XXZ chain with the model

$$H_2 = \sum_{j \in \Lambda} [\sigma^x_j \sigma^x_{j+1} - \sigma^y_j \sigma^y_{j+1} + \lambda_z \sigma^z_j \sigma^z_{j+1}], \quad (3.45)$$

which is a particular case of the XXZ model with Dzyaloshinski-Moriya interaction:

$$H_2 = \sum_{j \in \Lambda} J_x (\sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1}) + D (\sigma^y_j \sigma^x_{j+1} - \sigma^y_j \sigma^y_{j+1}) + J_z \sigma^z_j \sigma^z_{j+1}, \quad (3.46)$$

where $J_x, J_z$ and $D$ are arbitrary parameters. Integrability of (3.43) follows then from the integrability of (3.46), which has been proven in [26]. Note also that the model (3.46) is not an $O(2)$-invariant one, and by a spin rotation (3.8) with a suitably chosen angle $\alpha$ it may be transformed into the anisotropic Dzyaloshinski-Moriya system:

$$H_2 = \sum_{j \in \Lambda} [D_x \sigma^x_j \sigma^x_{j+1} + D_y \sigma^y_j \sigma^x_{j+1} + D_z \sigma^z_j \sigma^z_{j+1}]. \quad (3.47)$$

The model with alternating XXZ and Ising bonds,

$$H_2 = \sum_{j \in \Lambda} [(\lambda_x + \lambda_x') (\sigma^x_{2j} \sigma^x_{2j+1} + \sigma^x_{2j} \sigma^y_{2j+1}) + (\lambda_z + \lambda_z') \sigma^z_{2j} \sigma^z_{2j+1}$$

$$+ (\lambda_z - \lambda_z') \sigma^z_{2j+1} \sigma^z_{2j+2} + \lambda'' \sigma^z_j \sigma^z_{j+2}], \quad (3.48)$$

which is a slight generalization of the Lieb-Schultz-Mattis model, presents certain peculiarities. The model has been diagonalized in [21] (for $\lambda'' = 0, \lambda_x = \lambda_x' = 1/2, \lambda_z = (1 + U)/2, \lambda_z = (1 - U)/2$). As observed in [21], a convenient basis is provided by the eigenstates of $L_j = \sigma^x_{2j} + \sigma^x_{2j+1}$. Consider then subspaces of the space of states corresponding to a particular sequence $\{M_j\}$ of the eigenvalues of the third component of the
L_j’s. Since the Lieb-Schultz-Mattis type hamiltonian (3.48) commutes with each of the L_j’s it does not mix different subspaces; in other words it is block-diagonal in this basis, and it can be diagonalized separately in each subspace. The projection operators onto the subspaces corresponding to different sequences \ \{M_j\} provide then a mutually commuting set of operators, all commuting with the hamiltonian. Therefore, (3.48) satisfies the definition of integrability given in the introduction; admittedly the nature of these conserved charges appears a bit unusual and somewhat trivial.

The explicit form H_3 charge found via the the test is:

\[
H_3 = \sum_{j \in \Lambda} \left[ \alpha L_j^z \sigma_{2j+2}^z \sigma_{2j+3}^z \sigma_{2j+1}^z L_{j+1}^z \right],
\]

where \(\alpha\) and \(\beta\) are arbitrary coefficients. Interestingly, not only does the above sum commute with (3.48), but each term in it is separately conserved. Clearly, all these terms can be expressed as linear combinations of the projection operators discussed above. Let \(P^{(m)}_j\) denote a projection onto the states with \(M_j = m\). Then \(\sigma_{2j}^z \sigma_{2j+1}^z = \frac{1}{4}(P^{(1)}_j + P^{(-1)}_j - P^{(0)}_j)\) and \(L_j^z = P^{(1)}_j - P^{(-1)}_j\). In particular, in the \(2^{N/2}\)-dimensional subspace where all the \(M_i\)’s are zero (which is the sector containing the vacuum), (3.49) vanishes.

Clearly, the block-diagonal nature of the hamiltonian (and hence the existence of the set of commuting projections) is preserved by the addition to (3.48) of an arbitrary interaction involving only the z-components of the spin variables. Finally, we note that a particular case of (3.48) with \(\lambda_z - \lambda_z' = \lambda_z'' = 0\) provides a pathological “chopped XXZ” system, consisting of \(N/2\) disjoint XXZ bonds, with all neighboring links trivially commuting.

Summing up, it appears that the apart from the models (a)-(f) described above, all other o(2)-symmetric models within the family (3.1) are nonintegrable. Among the integrable models, there are three situations in which the first moment of the hamiltonian (the boost) acts as a ladder operator for conservation laws: the XXZ chain (cases (a) and (c)), the staggered XX model (d), and the staggered XXZ chain (e).

4. Example 2: isotropic spin-1 chains

Consider now a class of isotropic spin-1 chains with nearest neighbor interactions. The most general hamiltonian contains a bilinear and a biquadratic term:

\[
H_2(\beta) = \sum_{j \in \Lambda} [S_j^a S_{j+1}^a + \beta (S_j^a S_{j+1}^a)^2],
\]

(4.1)
where $S_j^a$’s are the $su(2)$ spin-1 matrices, acting nontrivially only on the $j$-th factor of the Hilbert space $\bigotimes_j \mathbb{C}^3$. For convenience we choose the representation in which $S^z$ is diagonal, i.e.

$$S^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^y = i \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (4.2)$$

Using the identity:

$$(S_j^a S_k^a)^2 = \frac{1}{4} D_{ab}^j D_{bc}^k - \frac{1}{2} S_j^a S_k^a, \quad (4.3)$$

where

$$D_{ab}^j \equiv S_j^a S_j^b + S_j^b S_j^a \quad (4.4)$$

the Hamiltonian can be expressed as:

$$H_2(\beta) = \sum_{j \in \Lambda} [(1 - \frac{\beta}{2}) S_j^a S_{j+1}^a + \frac{\beta}{4} D_{ab}^j D_{bc}^j]. \quad (4.5)$$

The boost operator yields the following candidate for the $H_3$ charge:

$$H_3(\beta) = \sum_{j \in \Lambda} \epsilon_{abc} [(-2 + 2\beta - \frac{\beta^2}{2}) S_j^a S_{j+1}^b S_{j+2}^c - \frac{\beta^2}{2} D_{ab}^j S_{j+1}^b D_{cd}^j]
+ (-\beta + \frac{\beta^2}{2})(D_{ab}^j D_{j+1}^b S_{j+2}^c + S_j^a D_{j+1}^{bd} D_{j+2}^{dc})]. \quad (4.6)$$

The commutator $[H_2(\beta), H_3(\beta)]$ vanishes only for $\beta = \pm 1$ or $\beta = \infty$; these cases have been already identified as integrable in the literature. For $\beta = -1$, (4.1) reduces to the isotropic version of the Fateev-Zamolodchikov chain, associated with the 19-vertex model [27], whose integrability follows from directly from the Yang-Baxter equation. For $\beta = 1$, (4.1) describes the Sutherland $su(3)$ symmetric chain [28], whose Hamiltonian can be rewritten in terms of the Gell-Mann matrices $t^a$:

$$H_2(1) \sim \sum_{j \in \Lambda} t_j^a t_{j+1}^a. \quad (4.7)$$

It can be solved by the nested Bethe Ansatz [28] and it is also related directly to the Yang-Baxter equation. Note that in this case, $H_3(1)$ can also be written in the form [1]

$$H_3 = \sum_{j \in \Lambda} f_{abc} t_j^a t_{j+1}^b t_{j+2}^c. \quad (4.8)$$
In the limit $\beta \to \infty$, (4.1) reduces to a system with purely biquadratic interactions, whose integrability has previously been established in [28, 29].

For the spin-1 models (4.1) with finite $\beta \neq \pm 1$, the boost operator does not produce a conserved quantity. Furthermore, as we show below, there exists no nontrivial local conserved charge involving up to three nearest-neighbors. The most general expression for such a charge would have the form:

$$H_3 = \sum_{j\in\Lambda} [a_1 \epsilon^{abc} S^a_j S^b_{j+1} S^c_{j+2} + a_2 \epsilon^{abc} D^a_j D^b_{j+1} S^c_{j+2} + a_3 \epsilon^{abc} S^a_j D^{bd}_{j+1} D^{dc}_{j+2}$$

$$\phantom{H_3} + a_4 \epsilon^{abc} D^a_j S^b_{j+1} D^{cd}_{j+2} + a_5 D^a_j D^b_{j+1} D^{ac}_{j+2} + a_6 S^a_j S^b_{j+1} D^{ab}_{j+2}$$

$$\phantom{H_3} + a_7 D^a_j S^b_{j+1} S^c_{j+2} + a_8 S^a_j D^a_{j+1} S^b_{j+2} + a_9 S^a_j S^a_{j+1} + a_{10} (S^a_j S^a_{j+1})^2].$$

(4.9)

where the $a_i$’s are undetermined coefficients. Enforcing the commutativity of $H_3$ with the hamiltonian, we obtain as usual a number of constraints on these coefficients. In particular, the vanishing of the four-spin terms in this commutator requires:

$$a_2 - a_3 = a_5 = a_6 = a_7 = a_8 = 0,$$

(4.10)

and

$$a_2 = a_4 = 0 \quad \text{if} \quad \beta = 0,$$

$$a_1 = (1 + 4/\beta^2 - 4/\beta)a_4, \quad a_2 = (-1 + 2/\beta)a_4 \quad \text{if} \quad \beta \neq 0.$$  

(4.11)

The above conditions means that in order for $H_3$ to commute with $H_2$, its three-spin part must be proportional to the commutator of the boost and $H_2$ (both for $\beta = 0$ and $\beta \neq 0$). The vanishing of the terms with two and three spins in the commutator imposes two further restrictions. First, the two-spin part of $H_3$ must be proportional to the hamiltonian $H_2$. Second, unless $\beta^2 = 1$ or $\beta = \infty$ the three-spin part of $H_3$ must be trivial.

The nonexistence of $H_3$ for (4.1) with finite $\beta \neq \pm 1$, suggests that all these models are nonintegrable. In particular, the bilinear system ($\beta = 0$), as well as the one with $\beta = 1/3$, for which there exists an exact valence-bond ground state [30], all fail the above test.

It should be added that the necessary condition (2.23) for having a quantum chain related to a solution of the Yang-Baxter equation has been examined for isotropic spin-$s$ chains with $s < 14$ in [17, 31]. In particular, these authors found that the only spin-1 systems satisfying (2.23) are $\beta = \pm 1$ and $\beta = \infty$. As we have discussed above, this implies that for other values of $\beta$ the hamiltonian (4.1) cannot be a fundamental model. But a priori there might exists integrable but non-fundamental models for $\beta \neq \pm 1, \infty$.  

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Our results, showing that there is no nontrivial three-spin charge provide a much stronger evidence for the non-integrability of all the isotropic spin-1 models with $\beta \neq \pm 1, \infty$. It would be interesting to perform a similar analysis for $s > 1$; such analysis is however much more complicated for higher $s$.

We end this section with a short remark on the general spin-$s$ bilinear system. Is it possible that this system is, by a bizarre accident, an integrable fundamental model for some values of $s$? This has been answered negatively in [31] for all $s < 100$, using computer algebra. Here we present a simple calculation showing that for the bilinear systems, the boost operator can never produce a conserved quantity for $s \neq 1/2$, which thus excludes the possibility of such an accident. Acting on the Hamiltonian, the boost operator generates a candidate for the three-spin charge of the form:

$$H_3 = \sum_{j \in \Lambda} \epsilon^{abc} S_j^a S_{j+1}^b S_{j+2}^c.$$  \hfill (4.12)

The commutator of this quantity with the Hamiltonian is then

$$[H_3, H_2] = \sum_{j \in \Lambda} [D_{j}^{ab} S_{j+1}^a S_{j+2}^b - D_{j}^{aa} S_{j+1}^b S_{j+2}^b + S_{j}^a S_{j+1}^b D_{j+2}^{bb} - S_{j}^a S_{j+1}^b D_{j+2}^{ab}].$$ \hfill (4.13)

The vanishing of this sum requires that all the terms containing spins in an arbitrary cluster vanish separately, which is not the case in general. The sum can vanish only if the terms cancel two by two, which is possible only if $D_{j}^{ab} \sim \delta_{ab}$, a condition which is not true unless $s = 1/2$. Therefore, the bilinear Heisenberg chain is an integrable fundamental model only for $s = 1/2$.

5. Concluding remarks

The simple integrability test considered in this work appears to be applicable to a rather general class of quantum chains. However, the range of applicability of the conjectures 1 and 2 has not yet been determined rigorously. Regardless of that, this simple method seems to be a useful heuristic tool. It may be worthwhile to use it to try to identify integrable models within other physically relevant families of models. In particular, the completely anisotropic case of the 10-parameter family considered in section 3 remains to be studied in detail. However, we do not expect that it will reveal new types of integrable systems, beyond anisotropic generalizations of the systems found in the $o(2)$-symmetric
case. In particular, we expect that the XYZ chain, the staggered XY model, the staggered XYZ chain (with the $xy$ part alternating in sign), equivalent to the Dzyaloshinski-Moriya system (3.47), a generalized Lieb-Schultz-Mattis model, and a generalized Hubbard model (consisting of two copies of the XY chain interacting along their $z$ components) are, up to a relabeling of variables, the only nontrivial integrable anisotropic models within the family (3.1).

Conjecture 1 could be generalized in a natural way to include even models with long-range interactions [32], for which there seems to be no ladder operator. We would then require that if a hamiltonian $H_2$ is given by a sum of two-spin interactions, there should exist a conserved three-spin charge $H_3$. Observe that for models with long range interactions, the leading term in $H_n$ is also characteristic: although the $n$ interacting spins in the leading term are no longer adjacent, the prefactor specifying the interaction of these $n$ spins is distinctive. But for these models there can exist independent non-local charges also for finite chains (see e.g. [33]) and again it could be less obvious at first sight to assert that a three-spin quantity is not a product of lower order charges. The analysis of such models will be reported elsewhere.

Finally, we mention a completely different integrability test for spin chains based on the properties of the $n$-magnon excitations of the ferromagnetic vacuum, which has been conjectured by Haldane [34]. In spin-$s$ chains the bound-state $n$-magnon dispersion branches extend over $\min(N, 2s)$ Brillouin zones and integrability manifests itself in that all branches are real (i.e. lie outside the spin wave continuum) and are continuous through the zone-boundaries. This integrability criterion has been investigated for 2-magnon excitations in spin-$s$ chains in [35, 36]. In particular, in [35] the Haldane criterion was used to obtain a relation between the parameters of a class of isotropic spin-$s$ hamiltonians. This relation defines then a one parameter family of models, supposed to be integrable. But it contains only two out of the four known fundamental integrable systems for $s = 3/2$ [17]. Neither the $su(4)$ invariant chain [28] nor the hamiltonian $H_{II}$ of [31] belong to it. And apart from two points (describing the Bethe-Ansatz integrable system [37] and the chain related to the Temperley-Lieb algebra [38, 39, 40]), the other models within the above one-parameter $s = 3/2$ family are not fundamental integrable systems, and are very likely to be nonintegrable. Thus the condition in [35] turns out to be neither a necessary nor a sufficient condition for integrability. Presumably, further constraints should follow from the analysis of $n$-magnon excitations with $n > 2$. A better understanding of the Haldane criterion, as well as its relation to the test proposed in this work is clearly needed. Note
that the former approach implies a choice of a particular vacuum, while the $H_3$ test is insensitive to the ferromagnetic or antiferromagnetic character of the model.
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