PATTERN RECOGNITION ON ORIENTED MATROIDS: SUBTOPES AND DECOMPOSITIONS OF (SUB)TOPES

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Abstract. For a symmetric 2t-cycle in the tope graph of a simple oriented matroid \( \mathcal{M} \) on the ground set \( \{1, \ldots, t\} \), where \( t \) is even, we describe decompositions of topes and subtopes of \( \mathcal{M} \) with respect to the subtopes corresponding to the edges of the symmetric cycle.

1. Introduction

Let \( \mathcal{M} := (E_t, \mathcal{T}) \) be a simple oriented matroid, with set of topes \( \mathcal{T} \subseteq \{1, -1\}^t \), on the ground set \( E_t := [t] := [1, t] := \{1, \ldots, t\} \), where \( t \geq 4 \). See, e.g., [1, 2, 3, 4, 5, 6, 10, 11] on oriented matroids. Using a nonstandard terminology, by “simple” we mean that the oriented matroid \( \mathcal{M} \) has no loops, parallel or antiparallel elements. We regard the set of covectors \( \mathcal{L} \subseteq \{1, -1, 0\}^t \) of \( \mathcal{M} \) as a set of row vectors of the real Euclidean space \( \mathbb{R}^t \).

Let \( \mathcal{T}(+) := (1, \ldots, 1) \) denote the \( t \)-dimensional row vector of all 1's; if the oriented matroid \( \mathcal{M} \) is acyclic, then \( \mathcal{T}(+) \) is called the positive tope. For a subset \( A \subseteq E_t \), we denote by \( -A \mathcal{T}(+) \) the tope \( T \) whose negative part \( T^- := \{e \in E_t : T(e) = -1\} \) is the set \( A \). If \( s \in E_t \), then we write \( -s \mathcal{T}(+) \) instead of \( -(s) \mathcal{T}(+) \).

If \( T' \) and \( T'' \) are adjacent topes in the tope graph of \( \mathcal{M} \) (i.e., the Hamming distance between the words \( T' \) and \( T'' \) is 1 or, equivalently, the standard scalar product \( \langle T', T'' \rangle \) of the vectors \( T' \) and \( T'' \) of the space \( \mathbb{R}^t \) is \( t - 2 \)), then their common subtope \( S \in \{1, -1, 0\}^t \) defined as the meet

\[
S := T' \wedge T''
\]

of the elements \( T' \) and \( T'' \) in the big face lattice of the oriented matroid \( \mathcal{M} \), can be interpreted as the midpoint

\[
S = \frac{1}{2}(T' + T'')
\]

of a straight line segment in \( \mathbb{R}^t \), with the endpoints \( T' \) and \( T'' \). From this viewpoint, if we let \( S^{t-1}(r) \) denote the \((t - 1)\)-dimensional sphere of radius \( r \) in \( \mathbb{R}^t \), centered at the origin, then \( S \) is the point of tangency of the tangent line to the sphere \( S^{t-1}(\sqrt{t - 1}) \) passing through the points \( T' \) and \( T'' \) that lie on the sphere \( S^{t-1}(\sqrt{t}) \).

The corresponding edge \( \{\tilde{T}', \tilde{T}''\} \) of the hypercube graph \( \tilde{\mathcal{H}}(t, 2) \) on the vertex set \( \{0, 1\}^t \), where \( \tilde{T}' := \frac{1}{t}(T(+) - T') \) and \( \tilde{T}'' := \frac{1}{2}(T(+) - T'') \), could be labeled by the row vector \( \tilde{S} := \frac{1}{2}(\tilde{T}' + \tilde{T}'') = \frac{1}{2}(T(+) - S) \in \{0, 1, \frac{1}{2}\}^t \).
A symmetric cycle $D$ in the tope graph of $M$ is defined to be its $2t$-cycle with vertex sequence

$$V(D) := (D^0, D^1, \ldots, D^{2t-1})$$

such that

$$D^{k+t} = -D^k, \quad 0 \leq k \leq t - 1.$$  \hfill (1.3)

The sequence $V(D)$ is a maximal positive basis of the space $\mathbb{R}^t$, see [7, §11.1]. We will see that if the cardinality $t$ of the ground set $E_t$ is even, then the set of subtopes, of the form $\mathbb{L}[\mathbb{L}]$, associated with the edges of the cycle $D$ is also a maximal positive basis of $\mathbb{R}^t$. We describe related (de)composition constructions for topes and subtopes of the oriented matroid $M$, and we give a detailed example to illustrate them. In addition, we consider vertex decompositions in hypercube graphs with respect to the edges of their distinguished symmetric cycles.

2. Symmetric cycles in tope graphs: vertices, edges and (sub)topes

Given a symmetric cycle $D := (D^0, D^1, \ldots, D^{2t-1}, D^0)$ in the tope graph of a simple oriented matroid $M := (E_t, T)$, with the sequence $V(D)$ of its vertices $\mathbb{L}[\mathbb{L}]$, we denote by $E(D)$ the edge sequence of the cycle $D$:

$$E(D) := \{ (D^0, D^1), (D^1, D^2), \ldots, (D^{2t-2}, D^{2t-1}), (D^{2t-1}, D^0) \}.$$  \hfill (2.1)

Let $S \subset \{1, -1, 0\}^t$ denote the set of subtopes of the oriented matroid $M$. By means of the map

$$E(D) \to S, \quad \{D', D''\} \mapsto D' \wedge D'' = \frac{1}{2}(D' + D''),$$

we associate with the edge sequence of the cycle $D$ the corresponding sequence of subtopes $S(D)$:

$$S(D) := (S^0 := D^0 \wedge D^1, S^1 := D^1 \wedge D^2, \ldots, S^{2t-2} := D^{2t-2} \wedge D^{2t-1}, S^{2t-1} := D^{2t-1} \wedge D^0).$$  \hfill (2.2)

where we have

$$S^{k+t} = -S^k, \quad 0 \leq k \leq t - 1.$$  \hfill (2.3)

Recall that the sequence of topes $(D^0, D^1, \ldots, D^{t-1})$ is a basis of the space $\mathbb{R}^t$. Looking at the matrix expression

$$\begin{pmatrix}
S^0 \\
S^1 \\
S^2 \\
S^{t-3} \\
S^{t-2} \\
S^{t-1}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 \\
-1 & 0 & 0 & \cdots & 0 & 0 & 1
\end{pmatrix},$$

$$W := W(D) \begin{pmatrix}
D^0 \\
D^1 \\
D^2 \\
D^{t-3} \\
D^{t-2} \\
D^{t-1}
\end{pmatrix},$$

we see that this relation

$$W = \frac{1}{2} N(t) \cdot M$$
implies that
\[ \text{rank } W = \text{rank } N(t) = \begin{cases} t - 1, & \text{if } t \text{ is odd} \\ t, & \text{if } t \text{ is even} \end{cases} \]

Thus, if \( t \) is even, then the sequence of subtopes \((S^0, S^1, \ldots, S^{t-1})\) is a basis of the space \( \mathbb{R}^t \), the sequence \( S(D) \) defined by (2.2) is a maximal positive basis of \( \mathbb{R}^t \), and we have
\[ \text{M} = P(t) \cdot W, \quad (2.4) \]
where the \((i, j)\)th entries of the Toeplitz matrix (see, e.g., [9])
\[ P(t) := 2N(t)^{-1} \in \mathbb{R}^{t \times t} \]
are
\[ \begin{cases} (-1)^{i+j}, & \text{if } i \leq j \\ (-1)^{i+j+1}, & \text{if } i > j \end{cases}, \quad (2.5) \]
that is, we have
\[ \begin{pmatrix} D^0 \\ D^1 \\ D^2 \\ \vdots \\ D^{t-3} \\ D^{t-2} \\ p^{t-1} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & \cdots & -1 & 1 & -1 \\ -1 & -1 & 1 & \cdots & 1 & -1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & -1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & -1 & 1 & \cdots & 1 & -1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & -1 & 1 \\ -1 & -1 & 1 & \cdots & 1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} S^0 \\ S^1 \\ S^2 \\ \vdots \\ S^{t-3} \\ S^{t-2} \\ S^{t-1} \end{pmatrix} \]

For example, the matrices \( P(4) \) and \( P(6) \) are
\[ P(4) = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad P(6) = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \]

Given a vector \( z := (z_1, \ldots, z_t) \in \mathbb{R}^t \), we denote its support \( \{ e \in E_t : z_e \neq 0 \} \) by \( \text{supp}(z) \).

Since the entries of matrices \( P(t) \) belong to the set \( \{1, -1\} \), the rows of these matrices can be viewed as vertices of the hypercube graph \( H(t, 2) \) on the vertex set \( \{1, -1\}^t \).

**Remark 2.1.** If \( \mathcal{H} := (E_t, \{1, -1\}^t) \) is the oriented matroid realizable as the arrangement of coordinate hyperplanes in \( \mathbb{R}^t \) (see, e.g., [2] Example 4.1.4), where \( t \) is even, then the rows \( P(t)^i =: P^{i-1}, 1 \leq i \leq t, \) of the nonsingular matrix
\[ P(t) := \begin{pmatrix} p^0 \\ p^1 \\ \vdots \\ p^{t-3} \\ p^{t-2} \end{pmatrix} \in \{1, -1\}^{t \times t} \quad (2.6) \]
with entries (2.5) constitute a distinguished sequence of certain \( t \) subtopes of the oriented matroid \( \mathcal{H} \), for which, on the one hand, we have
\[ P(t) = M(D) \cdot W(D)^{-1}, \quad (2.7) \]
for any symmetric cycle $D$ in the hypercube graph $H(t, 2)$ of tope of the oriented matroid $H$.

On the other hand, for each row $P(t)^i =: P^{i-1}$, $1 \leq i \leq t$, of the matrix (2.6) with entries (2.5), regarded as a vertex of the hypercube graph $H(t, 2)$ on the vertex set $\{1, -1\}^t$, there exists a unique row vector $x^{i-1} := x^{i-1}(P^{i-1}) := x^{i-1}(P^{i-1}, D) \in \{-1, 0, 1\}^t$, with $|\text{supp}(x^{i-1})|$ odd, such that

$$P^{i-1} = x^{i-1} M(D),$$

see [7, §11.1]. Thus, we have

$$P(t) = \begin{pmatrix} x^0 \\ x^1 \\ \vdots \\ x^{t-3} \\ x^{t-2} \\ x^{t-1} \\ X(P(t), D) \end{pmatrix} \cdot M(D). \quad (2.8)$$

Relations (2.7) and (2.8) yield

$$M(D) \cdot W(D)^{-1} = X(P(t), D) \cdot M(D),$$

for any symmetric cycle $D$ of the hypercube graph $H(t, 2)$.

3. Vertices decompositions and edge decompositions in tope graphs with respect to the edges of their symmetric cycles

In this section we discuss (de)composition constructions related to the edge sequences of symmetric cycles in the tope graphs of simple oriented matroids. The results and their proofs are accompanied by a detailed example.

**Proposition 3.1.** Let $D := (D^0, D^1, \ldots, D^{2t-1}, D^0)$ be a symmetric cycle in the tope graph of a simple oriented matroid $M := (E_t, T)$, where $t$ is even. Let $S(D)$ be the corresponding sequence of subtopes defined by (2.2).

(i) If $T \in T$ is a tope of $M$, then there exist a unique subset of subtopes $T$ of $S(D) \subset S(D)$ and a unique set of integers $\{\lambda_{Q'}: Q' \in T, S(D)\}$ such that

$$\sum_{Q' \in T, S(D)} \lambda_{Q'} \cdot Q' = T, \quad (3.1)$$

where

$$|T, S(D)| = t,$$

and

$$1 \leq \lambda_{Q'} \leq t - 1, \quad \text{and} \quad \lambda_{Q'} \text{ are all odd}.$$
(ii) If $S \in \mathcal{S}$ is a subtop of $\mathcal{M}$, then there exist a unique inclusion-minimal subset of subtopes $\overline{Q}(S,S(D)) \subset S(D)$ and a unique set of integers $\{\lambda_{Q'}: Q' \in \overline{Q}(S,S(D))\}$ such that

$$
\sum_{Q'\in \overline{Q}(S,S(D))} \lambda_{Q'} \cdot Q' = S,
$$

where

$$
1 \leq \lambda_{Q'} \leq t - 1.
$$

Proof. (i) See Example 3.2(i).

Let $x := x(T) := x(T,D) := (x_1, \ldots, x_t) \in \{-1,0,1\}^t$ be the unique row vector such that

$$
T = xM(D),
$$

where $|\text{supp}(x)|$ is odd; see [7, §11.1]. In other words,

$$
T = xP(t)W(D) := xW(D),
$$

where the vector $\overline{x} := \overline{x}(T) := \overline{x}(T,S(D)) := (\overline{x}_1, \ldots, \overline{x}_t) \in \mathbb{Z}^t$ is defined by

$$
\overline{x} := xP(t).
$$

Since the entries of the matrix $P(t)$ belong to the set $\{1, -1\}$, the components of $\overline{x}$ are all odd integers and, as a consequence, $|\text{supp}(\overline{x})| = t$. We have

$$
\overline{Q}(T,S(D)) = \{\text{sign}(\overline{x}_i) \cdot S^{i-1}: 1 \leq i \leq t\},
$$

$$
Q' \in \{-S^{j-1}, S^{j-1}\} \implies \lambda_{Q'} = |\overline{x}_j|, \ 1 \leq j \leq t.
$$

(ii) See Example 3.2(ii).

Let $T'$ and $T''$ be the two topes of $\mathcal{M}$ such that $S = \frac{1}{2}(T' + T'')$. If $x_{T'} := x(T') \in \{-1,0,1\}^t$ and $x_{T''} := x(T'') \in \{-1,0,1\}^t$ are the unique row vectors such that $T' = x_{T'}M(D)$ and $T'' = x_{T''}M(D)$, then we have

$$
S = \frac{1}{2}(x_{T'} + x_{T''})M(D)
$$

$$
= \frac{1}{2}(x_{T'} + x_{T''})P(t)W(D),
$$

or

$$
S = \overline{x}W(D),
$$

where the vector $\overline{x} := \overline{x}(S) := \overline{x}(S,S(D)) := (\overline{x}_1, \ldots, \overline{x}_t) \in \mathbb{Z}^t$ is the vector

$$
\overline{x} := \frac{1}{2}(\overline{x}_{T'} + \overline{x}_{T''}) = \frac{1}{2}(x_{T'} + x_{T''})P(t).
$$

Thus,

$$
\overline{Q}(S,S(D)) = \{\text{sign}(\overline{x}_{T',i} + \overline{x}_{T'',i}) \cdot S^{i-1}: \overline{x}_{T',i} \neq -\overline{x}_{T'',i}, 1 \leq i \leq t\},
$$

$$
Q' \in \{-S^{j-1}, S^{j-1}\} \implies \lambda_{Q'} = \frac{1}{2}|\overline{x}_{T',j} + \overline{x}_{T'',j}|, \ 1 \leq j \leq t.
$$

□
A symmetric cycle $D := (D^0, D^1, \ldots, D^{2t-1}, D^0)$ in the hypercube graph $H(t, 2)$ on the vertex set $\{1, -1\}^t$, where $t := 6$. The edges of the cycle $D$ are labeled by the corresponding subtopes of the oriented matroid $\mathcal{H} := (E_6, \{1, -1\}^6)$; for example, the edge $\{D^0, D^1\}$ is labeled by the subtope $S^0 := (-1, 0, 1, 1, -1, 1)$. Here the sign components 1, -1 and 0 of covectors of $\mathcal{H}$ are substituted by the familiar symbols ‘+’, ‘-’ and ‘0’, respectively.

For the vertex $T^+ := (1, 1, 1, 1, 1, 1)$ of the graph $H(6, 2)$, on the one hand, we have $T^+ = Q^0 + Q^1 + Q^2 + Q^3 + Q^4$ for a unique inclusion-minimal subset $Q(T^+, D) =: \{Q^0, \ldots, Q^4\}$ of five topes in the vertex sequence $V(D)$ of the cycle $D$, where $Q^0 := D^0$, $Q^1 := D^2$, $Q^2 := D^4$, $Q^3 := D^7$ and $Q^4 := D^9$; see [7, §11.1].

On the other hand, $T^+ = S^1 + S^2 + 5S^4 + 3S^6 + 3S^9 + 5S^{11}$ for the unique set $Q(T^+, S(D))$ of $t := 6$ subtopes $S^1$, $S^2$, $S^4$, $S^6$, $S^9$ and $S^{11}$ associated with edges of the cycle $D$, for which the corresponding integer coefficients are all positive and odd; see Proposition [3, II].
Example 3.2. Let \( D \) be a symmetric cycle, depicted in Figure 1, in the hypercube graph of topes of the oriented matroid \( \mathcal{H} := (E_6, \{1, -1\}^6) \). The corresponding matrices \( M \) and \( W \) that describe vertices and edges of the cycle \( D \) in the expression (2.3) are as follows:

\[
M := M(D) = \begin{pmatrix} D^0 \\ D^1 \\ D^2 \\ D^3 \\ D^4 \\ D^5 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix},
\]

\[
W := W(D) = \begin{pmatrix} S^0 \\ S^1 \\ S^2 \\ S^3 \\ S^4 \\ S^5 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 0 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 0 \end{pmatrix}.
\]

(i) Here we illustrate the proof of Proposition 3.1(i).

We would like to find the decomposition of the positive tope \( T^{(+)} := (1, 1, 1, 1, 1) \) with respect to the set \( S(D) \) of subtopes (2.2) associated with the edges of the cycle \( D \); see Figure 1.

We have

\[
x := x(T^{(+)}, D) = (1, -1, 1, -1, 1, 0)
\]

and

\[
\overline{x} := \overline{x}(T^{(+)}, S(D)) = xP(6) = (1, -1, 1, -1, 1, 0) \cdot \begin{pmatrix} 1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}
\]

\[
= (-3, 1, 1, -3, 5, -5).
\]

Thus, we have

\[
\overline{Q}(T^{(+)}, S(D)) = \{\text{sign}(\overline{x}_i) \cdot S^{i-1} : 1 \leq i \leq 6\} = \{-S^0_{S^0}, S^1_{S^0}, S^2_{S^0}, -S^3_{S^0}, S^4_{S^0}, -S^5_{S^0}\}
\]

\[
= \{S^1, S^2, S^4, S^6, S^9, S^{11}\},
\]

that is,

\[
T^{(+)} := (1, 1, 1, 1, 1, 1)
\]

\[
= [\overline{x}_2] \cdot S^1 + [\overline{x}_3] \cdot S^2 + [\overline{x}_5] \cdot S^4 + [\overline{x}_1] \cdot S^6 + [\overline{x}_4] \cdot S^9 + [\overline{x}_6] \cdot S^{11}
\]

\[
= S^1 + S^2 + 5S^4 + 3S^6 + 3S^9 + 5S^{11}
\]

\[
= (-1, -1, 1, 0, 1) + (-1, -1, 0, 1, 1, 1) + 5 \cdot (1, -1, -1, 0, 1, 1)
\]

\[
+ 3 \cdot (1, 0, -1, -1, 1, -1) + 3 \cdot (0, 1, 1, -1, -1, -1) + 5 \cdot (-1, 1, 1, 1, -1, 0),
\]

for the unique set \( \overline{Q}(T^{(+)}, S(D)) = \{S^1, S^2, S^4, S^6, S^9, S^{11}\} \) of 6 subtopes associated with edges of the cycle \( D \) for which the corresponding integer coefficients are all positive and odd.
(ii) Let us now illustrate the proof of Proposition 3.1 (ii).
Consider the subtope
$$S := (-1, -1, 0, 1, -1, -1)$$
of the oriented matroid $H := (E_6, \{1, -1\}^6)$ that corresponds to an edge $\{T', T''\}$ of its hypercube graph of topes $H(6, 2)$, where
$$T' := (-1, -1, 1, -1, -1) \quad \text{and} \quad T'' := (-1, -1, 1, -1, -1) \ .$$
We have
$$S = \frac{1}{2} \left( \left( x_{T'} + x_{T''} \right) W(D) \right) \mathfrak{X}(S) = \frac{1}{2} \left( x_{T'} + x_{T''} \right) P(6) W(D) \ .$$
Since $x_{T'} := x(T') = (-1, 1, 0, 0, 0, -1)$ and $x_{T''} := x(T'') = (-1, 1, -1, 1, 0, -1)$, we have
$$\mathfrak{X}(S) = \frac{1}{2} \left( (-1, 1, 0, 0, 0, -1) + (-1, 1, -1, 1, 0, -1) \right) \cdot \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{pmatrix}$$
$$= (-1, 1, -1, 1, 0, -1) \cdot \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{pmatrix}$$
$$= (0, 2, -3, 4, -4, 2) \ .$$
We see that
$$S := (-1, -1, 0, 1, -1, -1) = 2S^1 - 3 \cdot \begin{pmatrix} S^2 \\ -S^8 \end{pmatrix} + 4S^3 - 4 \cdot \begin{pmatrix} S^4 \\ -S^{10} \end{pmatrix} + 2S^5$$
or, in other words,
$$S := (-1, -1, 0, 1, -1, -1)$$
$$= 2S^1 + 4S^3 + 2S^5 + 3S^8 + 4S^{10}$$
$$= 2 \cdot (-1, -1, 1, 0, 1) + 4 \cdot (0, -1, -1, 1, 1, 1) + 2 \cdot (1, -1, -1, -1, 1, 0)$$
$$+ 3 \cdot (1, 1, 0, -1, -1, -1) + 4 \cdot (-1, 1, 1, 0, -1, -1) \ ,$$
for the unique inclusion-minimal set $\mathfrak{Q}(S, S(D)) = \{S^1, S^3, S^5, S^8, S^{10}\}$ of subtopes associated with edges of the cycle $D$ for which the corresponding integer coefficients are all positive.

4. **Vertex decompositions in hypercube graphs with respect to the edges of their distinguished symmetric cycles**

Let $R$ be a distinguished symmetric cycle in the hypercube graph $H(t, 2)$ of topes of the oriented matroid $H := (E_t, \{1, -1\}^t)$, where $t$ is even, defined
as follows:

\[ R^0 := T^{(+)} , \]
\[ R^s := -[s] R^0 , \quad 1 \leq s \leq t - 1 , \]

and

\[ R^{k+t} := -R^k , \quad 0 \leq k \leq t - 1 . \]  \hspace{1cm} (4.2)

No matter how large the dimension \( t \) of the discrete hypercube \( \{1, -1\}^t \) is, four assertions of [5, Prop. 2.4] allow us to find the linear algebraic decompositions

\[ T = \sum_{Q \in Q(T, R)} Q \]

of vertices \( T \) of the graph \( H(t, 2) \) on the vertex set \( \{1, -1\}^t \), by means of inclusion-minimal subsets \( Q(T, R) \subset V(R) \) of odd cardinality, in an explicit and computation-free way.

We now present a (subtope) companion to [5, Prop. 2.4]. As earlier, \( P(t)^s , \quad 1 \leq s \leq t \), denotes the \( s \)th row \( P^s \)−1 of the matrix \( P(t) \) given in (2.6), with entries (2.5).

**Proposition 4.1.** Let \( R \) be the distinguished symmetric cycle, defined by (4.1), (4.2), in the hypercube graph \( H(t, 2) \) on the vertex set \( \{1, -1\}^t \), where \( t \) is even.

Let \( A \) be a nonempty subset of the ground set \( E_t \), viewed as a disjoint union

\[ A = [i_1, j_1] \cup [i_2, j_2] \cup \cdots \cup [i_\varrho, j_\varrho] \]

of intervals of \( E_t \), such that

\[ j_1 + 2 \leq i_2 , \quad j_2 + 2 \leq i_3 , \quad \ldots , \quad j_{\varrho-1} + 2 \leq i_\varrho , \]

for some \( \varrho := \varrho(A) \).

Consider the vector \( \overline{\pi}(-A T^{(+)} , S(R)) \in \mathbb{Z}^t \) defined by

\[ -A T^{(+)} = : \overline{\pi}(-A T^{(+)} , S(R)) \cdot W(R) , \]

cf. (3.3) and (3.4).

(i) If \( \{1, t\} \cap A = \{1\} \), then

\[ \overline{\pi}(-A T^{(+)} , S(R)) = \sum_{1 \leq k \leq \varrho} P(t)^{j_k + 1} - \sum_{2 \leq \ell \leq \varrho} P(t)^{i_\ell} , \]

that is, for a component \( \overline{\pi}_e \) of this vector, where \( e \in E_t \), we have

\[ \overline{\pi}_e(-A T^{(+)} , S(R)) = \sum_{1 \leq k \leq \varrho} \begin{cases} (-1)^{e+j_k+1} , & \text{if } j_k < e , \\ (-1)^{e+j_k} , & \text{if } j_k \geq e \end{cases} \]

\[ - \sum_{2 \leq \ell \leq \varrho} \begin{cases} (-1)^{e+i_\ell} , & \text{if } i_\ell \leq e , \\ (-1)^{e+i_\ell+1} , & \text{if } i_\ell > e . \end{cases} \]
(ii) If \( \{1, t\} \cap A = \{1, t\} \), then
\[
\varphi(-A^{T^{(+)}}, S(R)) = -P(t)^1 + \sum_{1 \leq k \leq \ell - 1} P(t)^{jk + 1} - \sum_{2 \leq \ell \leq \ell} P(t)^{ie},
\]
that is, for a component \( \varphi_e \) of this vector we have
\[
\varphi_e(-A^{T^{(+)}}, S(R)) = (-1)^e + \sum_{1 \leq k \leq \ell - 1} \begin{cases} (-1)^{e+1}, & \text{if } j_k < e, \\ (-1)^{e+j_k}, & \text{if } j_k \geq e \end{cases}
- \sum_{2 \leq \ell \leq \ell} \begin{cases} (-1)^{e+i_\ell}, & \text{if } i_\ell \leq e, \\ (-1)^{e+i_\ell + 1}, & \text{if } i_\ell > e. \end{cases}
\]

(iii) If \( |\{1, t\} \cap A| = 0 \), then
\[
\varphi(-A^{T^{(+)}}, S(R)) = P(t)^1 + \sum_{1 \leq k \leq \ell} P(t)^{jk + 1} - \sum_{1 \leq \ell \leq \ell} P(t)^{ie},
\]
that is, for a component \( \varphi_e \) of this vector we have
\[
\varphi_e(-A^{T^{(+)}}, S(R)) = (-1)^{e+1} + \sum_{1 \leq k \leq \ell} \begin{cases} (-1)^{e+1}, & \text{if } j_k < e, \\ (-1)^{e+j_k}, & \text{if } j_k \geq e \end{cases}
- \sum_{1 \leq \ell \leq \ell} \begin{cases} (-1)^{e+i_\ell}, & \text{if } i_\ell \leq e, \\ (-1)^{e+i_\ell + 1}, & \text{if } i_\ell > e. \end{cases}
\]

(iv) If \( \{1, t\} \cap A = \{t\} \), then
\[
\varphi(-A^{T^{(+)}}, S(R)) = \sum_{1 \leq k \leq \ell - 1} P(t)^{jk + 1} - \sum_{1 \leq \ell \leq \ell} P(t)^{ie},
\]
that is, for a component \( \varphi_e \) of this vector we have
\[
\varphi_e(-A^{T^{(+)}}, S(R)) = \sum_{1 \leq k \leq \ell - 1} \begin{cases} (-1)^{e+j_k + 1}, & \text{if } j_k < e, \\ (-1)^{e+j_k}, & \text{if } j_k \geq e \end{cases}
- \sum_{1 \leq \ell \leq \ell} \begin{cases} (-1)^{e+i_\ell}, & \text{if } i_\ell \leq e, \\ (-1)^{e+i_\ell + 1}, & \text{if } i_\ell > e. \end{cases}
\]

In particular, we have
\[
1 \leq j < t \implies \varphi(-[j]^{T^{(+)}}, S(R)) = P(t)^{j + 1};
\]
\[
\varphi(T^{(-)}, S(R)) = -\varphi(T^{(+)}, S(R)) = -P(t)^1;
\]
\[
1 < i < j < t \implies \varphi(-[i,j]^{T^{(+)}}, S(R)) = P(t)^1 - P(t)^i + P(t)^{j + 1};
\]
\[
1 < i \leq t \implies \varphi(-[i,j]^{T^{(+)}}, S(R)) = -P(t)^i.
\]

If \( s \in E_\ell \), then for a row vector \( \overline{y}(s) := \overline{y}(s; t) \) defined by
\[
\overline{y}(s) := \varphi(-s^{T^{(+)}}, S(R)) ,
\]
we have

\[
\mathcal{G}(s) = \begin{cases} 
P(t)^2, & \text{if } s = 1, \\
\frac{P(t)^1 - P(t)^s + P(t)^{s+1}}{2}, & \text{if } 1 < s < t, \\
-P(t)^t, & \text{if } s = t.
\end{cases}
\]

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