Existence results of solution for fractional Sturm–Liouville inclusion involving composition with multi-maps

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1. Introduction

Most researchers in applied mathematics are using differential operators to describe a lot of kinds of modelling that have strong effects in various applied sciences. Many scholar teams are attracted to study partial differential operators having useful extents to present the physical equations. They use different methods to calculate their solutions, for instance, the obtained solutions in the shape of hyperbolic, trigonometric, elliptic functions including dark, bright, singular, combined, kink wave solitons, travelling wave, solitary wave and periodic wave. These kinds of solutions play vital role in mathematical physics, optical fiber, plasma physics and other various branches of applied sciences. There are some noteworthy results investigated by Seadawy et al. in [1–11]. They studied by different mathematical methods the general form of solutions for modified and the nonlinear damped modified Korteweg–de Vries, (2+1)-dimensional nonlinear Nizhnik–Novikov–Veselov, Kadomtsev–Petviashvili modified equal width, modified unstable nonlinear Schrodinger, Zakharov–Kuznetsov-modified equal width, and Kudryashov-Sinelshchikov dynamical equations, Camassa–Holm and the nonlinear longitudinal wave equations. They also [12–14] studied some systems of equations like dynamical system of nonlinear wave propagation, three coupled system of nonlinear partial differential equations, and the system of dynamical equations.

As a mathematical analysis, fractional calculus studies some different possibilities to define real number powers or complex number powers of the differential operator \(D = \frac{d}{dt}\) and then some strong generalization by fractional number powers, see [15]. For instance, Sturm–Liouville, Langevin, Evolution, Duffing, Navier–Stokes, and Hybrid operators are all of the most important and famous fractional differential operators [16–20].

The importance of fractional powers attracted mathematical scientific teams to study different considerable results of fractional differential equations and inclusions that have gained substantial noteworthiness derived from their applications in various sciences. Not long ago, Zhou [21] has investigated some results of the solvability for a Cauchy problem of Riemann–Liouville type fractional differential equations given by:

\[
^RD^\alpha w(i, w(i)), \quad i \in (0, \infty),
\]

\[
(I^{1-\alpha} w)(0) = w_0,
\]

where \(\alpha \in (0, 1)\), \(^RD^\alpha\) and \(I^{1-\alpha}\) are respectively Riemann–Liouville derivative and integral of orders \(\alpha\) and \(1 - \alpha\). After that, he in [22] has improved the previous problem up to more general by the next Riemann–Liouville fractional differential evolution equations:

\[
^RD^\alpha w(i) = Aw(i) + f(i, w(i)), \quad i \in (0, \infty),
\]

\[
(I^{1-\alpha} w)(0) = w_0,
\]
where $\alpha \in [0, 1)$, $\mathcal{D}_\alpha^p$ and $I^{1-\alpha}$ are respectively Riemann–Liouville derivative and integral of orders $\alpha$ and $1 - \alpha$.

In [23], he has initiated the attractively question of solutions for fractional evolution equations with almost sectorial operators. In fact, these equations have been studied first before more than 30 years ago.

There are cited results for fractional differential inclusions addressed by Ahmad et al. [24]. For example, Hadamard boundary value problems given by [24, 2.14–2.15] and (2.16–2.17)–p. 22] and presented, respectively, by:

$$hD^\alpha_x w(i) \in \mathcal{F}(i, w(i)), \quad i \in [1, b],$$

$$(1^{1-\alpha} w)_{i=1} = 0, \quad w(i) = \vartheta(i), \quad i \in [1 - r, 1],$$

and

$$hD^\alpha_x [w(i) - g(i, w(i))] \in \mathcal{F}(i, w(i)), \quad i \in [1, b],$$

$$(1^{1-\alpha} w)_{i=1} = 0, \quad w(i) = \vartheta(i), \quad i \in [1 - r, 1],$$

where $0 < \alpha < 1$, and $hD^\alpha_x$ denotes Hadamard derivative of order $\alpha$. Kamenskii has amazing works and adopted results on fractional inclusion problems introduced in [25–27]. See more in references therein.

The most interesting and the finest of our knowledge is recalling general forms of problems that are studying more cases at the same time (for example: equation and inclusion, ordinary and fractional operators). Here, we pick out one kind of nonlinear fractional problems that can be studied as equations and inclusions at one time.

It is considered with a jointly continuous composed functions $\Psi(t, \omega, K)$ with multi-valued maps $K$. This kind is not be presented before and new in inclusion field.

Consider the following problem:

$$cD^\alpha_x \left[p(i) cD^\beta_y \omega(i)\right] \in \Psi(t, \omega(i), K(t, \omega(i))), \quad t \in [0, \tau],$$

$$cD^\beta_y \omega(0) = \omega'(0) = 0, \quad \omega^{(i)}(0) = 0, \quad i = 0, 2, \ldots, n - 1, \quad n \in \mathbb{N},$$

where $0 < \alpha \leq 1, 0 < \beta \leq n, n > 1, cD^\alpha_x$ is the symbol of the Caputo fractional derivative with respect to the order $r$, $p(i)$ is a positive function such that $p(i) \in C[0, \tau]$,

$$K : [0, \tau] \times \mathbb{R} \rightarrow 2^\mathbb{R}$$

is a multi-valued map, and

$$\Psi : [0, \tau] \times \mathbb{R} \times 2^\mathbb{R} \rightarrow 2^\mathbb{R}.$$

We say that

$$\psi(i) \in \Psi(t, \omega(i), K(t, \omega(i))),$$

if and only if $\exists k(t, \omega(i)) \in K(t, \omega(i))$, in which that:

$$\psi(i) = \Psi(t, \omega(i), k(t, \omega(i))).$$

Up to now, fractional Sturm–Liouville problems are the main problems of applied science. They become more advantageous than the classical models and has drawn interest so much. Furthermore, Sturm–Liouville operators and its properties have attractive huge applications in physics, applied mathematics, engineering filed and science, applications of wide in quantum, classical mechanics and wave phenomena. Gerald in [2009] [28], has explored some mathematical methods in quantum mechanics under the vision of Sturm–Liouville operator. In the previous scientific studies and contributions we can see some related results in [29] given in (2019) for the following problems:

$$cD^\alpha_x \left[p(i) cD^\beta_y \omega(i)\right] + q(i) \omega(i) = h(i)f(\omega(i)), \quad t \in (0, \tau),$$

$$\sum_{k=0}^{m} \xi(k) \omega(a_k) = \nu \sum_{j=0}^{n} \eta(j) \omega(b_j), \quad \omega'(0) = 0, \quad \omega^{(i)}(0) = 0,$$

where $0 < \alpha \leq 1$, $cD^\alpha_x$ is Caputo fractional derivative, $0 \neq p(i) \in C^1[0, \tau]$, $h(i), q(i)$ are both absolute continuous functions in $[0, \tau]$, $0 < a_1 < a_2 < \cdots < a_m < a, \quad d \leq b_1 < b_2 < \cdots < b_n \leq \tau, \quad c \leq d, \quad \xi_k, \eta_j$ and $\nu \in \mathbb{R}$.

For the sake of finding new influential results and applications, we pick out the fractional inclusion given in (1) associated with composite functions with multi-valued maps.

In fact, a continuous maps composite with a multi-valued maps maybe take a single-valued for every points in their domains as follows:

$$K(t, \omega) = \pm \sqrt{\omega(t)},$$

$$\Psi(t, \omega(t), K(t, \omega(t))) = \left[\pm \sqrt{\omega(t)}\right]^2 = |\omega(t)|, \quad \omega(t) \geq 0.$$  

(4)

And absolutely on other times, they give multi-valued maps like:

$$K(t, \omega) = \left[\frac{4}{b^2} \omega\right]_{b=1}^\eta = Q_b(\omega), \quad b \in \mathbb{N},$$

$$\Psi(t, \omega(t), K(t, \omega(t))) = \left[\frac{K(t, \omega)}{1 + K(t, \omega)} + b^2\right]_{b=1}^\eta$$

$$= \left[\frac{4\omega b}{b^2 + 4\omega b^2} + b^2\right]_{b=1}^\eta.$$  

(5)

Basically, we are going to study the existence results under the inclusion arguments clarified as well as in [30–34]. The results as in equation case will be easy and clearly. It should be mentioned the existence results for the higher ordinary case in sense that:

$$\left[p(i) \omega^{(n)}(i)\right] \in \Psi(t, \omega(i), K(t, \omega(i))), \quad t \in [0, \tau],$$

$$n > 1, \quad n \in \mathbb{N}.$$  

(6)
under same conditions in (2) can be included in the provided results.

This paper is organized to start with some needed preliminaries in the next section. After that, we will illustrate the main results by section three. It should be add a section to gived some related examples for the adopted results. That will be in section four. Finally, in section five we try to conclude all studied points and mention to new open problem.

2. Prefatory Facts

As necessary for the contemporary paper, we will introduce some definitions, basic facts, and some useful rules.

First, let $(\Sigma, ||||)$ be denoted as a normed space. It is worth to note some needed symbols coming in the next table: [24]

| Set               | Symbol $\Sigma$ |
|-------------------|-----------------|
| Power set         | $P(\Sigma)$    |
| Closed subsets    | $P_{cl}(\Sigma)$|
| Bounded subsets   | $P_{b}(\Sigma)$|
| Compact subsets   | $P_{cp}(\Sigma)$|
| Convex subsets    | $P_{cv}(\Sigma)$|

Given $C, E \in P_{cl}(\Sigma)$, then

$$h(C, E) = H_d(C, E) = d_{H}(C, E)$$

where $h$ absorbs the Pompeiu-Housdorff distance of $C, E$ [35].

A multi-valued map $K : [0, \gamma] \rightarrow P_{cl}(\Sigma)$ is selected to be measurable if for every $v \in \Sigma$, the function $t \rightarrow d(v, K(t)) = \inf(d(v, y) : y \in K(t))$ is $\mathcal{L}$-measurable function [33].

A multi-valued $K : \Sigma \rightarrow P_{cl}(\Sigma)$ is known as convex (closed) if for every $v \in \Sigma$, $K(v)$ is convex (closed) [24, 36]. It is completely continuous if $K(\Omega)$ is relatively compact for every $\Omega \in P_{b}(\Sigma)$.

The map $K$ assimilates to be upper semi-continuous if $\forall W \in P_{cl}(\Sigma)$, $K^{-1}(W) \in P_{cl}(\Sigma)$. By the other word, $K$ is said to be upper semi-continuous if the set $\{v \in \Sigma : K(v) \subseteq O\}$ is open for all open subsets $O \subset \Sigma$. The condition: for each open subset $Z \subset \Sigma$, $K^{-1}(Z)$ is open subset of $\Sigma$, is comprehending $K$ as a lower semi-continuous map. Equivalently, $K$ is a lower semi-continuous if the set $\{v \in \Sigma : K(v) \cap O \neq \emptyset\}$ is open for all open subsets $O \subset \Sigma$.

If we adopt $K$ as a completely continuous function with non-empty compact values, then it is upper semi-continuous if and only if its graph is closed [37] (i.e.), if $v_0 \rightarrow v_n, y_n \rightarrow y$, then $y_n \in K(v_0)$ implies that $y \in K(v_n)$.

For any $E \subset \mathbb{R}$, the characteristic function of $E$ is defined as follows [38]:

$$\chi_E(x) = \begin{cases} 
0, & x \in E \\
1, & x \in \mathbb{R}.
\end{cases}$$

The function $\chi_E$ is measurable if and only if $E$ is measurable. A set $A \subset [0, b] \times \mathbb{R}$ is conformable as a decomposable set [24] if $\forall a_1, a_2 \in A$ and $I^* \subseteq [0, b]$ is $\mathcal{L}$-measurable, then $a_1 \chi_{I^*} + a_2 \chi_{I-I^*} \in A$, $I = [0, b]$.

**Definition 2.1 (Jointly Continuity [39]):** Suppose $A_1, A_2, \ldots, A_n, C$ are all topological spaces and the map

$$f : A_1 \times A_2 \times \cdots \times A_n \rightarrow C.$$

Then, the sentences come after are all equivalent:

- $f$ is jointly continuous,
- $f$ is continuous mapping in $A_1 \times A_2 \times \cdots \times A_n$ equipped with the product topology.
- there exists $v_i : \Omega \rightarrow A_i$ continuous maps represented with the function $f(\omega) = f(v_1(\omega), v_2(\omega), \ldots, v_n(\omega))$ is continuous, where $\Omega$ is topological space.

**Definition 2.2 ($L^1$-Caratheodory Function [33, 40]):** A map $\Psi : [0, \gamma] \times \mathbb{R} \times \mathbb{R}^2 \rightarrow P(\mathbb{R})$ is coming as a Caratheodory multi-valued map if:

1. $\forall z, w \in \mathbb{R}; t \rightarrow \Psi(i, z, w)$ satisfies the measurability condition.
2. for a.e $t \in [0, \gamma]; (z, w) \rightarrow \Psi(i, z, w)$ is upper semi-continuous.

And ergo it is $L^1$-Caratheodory if in addition of (1) and (2) it satisfies for each $R > 0$, $\exists \phi_R(t) \in L^1([0, \gamma], \mathbb{R}^+)$ whereas that:

$$\|\Psi(i, z, w)\| = \sup \{||\psi| : \psi \in \Psi(i, z, w)\} \leq \phi_R(i), \forall \|z\|, \|w\| \leq R, t \in [0, \gamma].$$

**Definition 2.3 (Lipschitz Condition [24]):** Take $(\Sigma, |||)$ as a normed space, and $d$ be the metric map conformed from the norm. Then, a multi-valued map $\Psi : \Sigma \rightarrow P_{cl}(\Sigma)$ is adopted as:

1. $\gamma$-Lipschitz if there exists $\gamma > 0$ such that:
   $$h(\Psi(z), \Psi(w)) \leq \gamma d(z, w), \ \forall z, w \in \Sigma.$$
2. a contraction if the first statement is hold with $\gamma < 1$.

**Lemma 2.1 ([38, 41]):** We introduce the following

1. Let $(Z, \mu)$ be measurable space. Then, if $k : Z \rightarrow \mathbb{R}$ is measurable and $g : \mathbb{R} \rightarrow \mathbb{R}$ is Borel measurable, then $g(k(z)) : Z \rightarrow \mathbb{R}$ is measurable.
(II) Every continuous function is Borel measurable.

(III) Suppose \( k : [a, b] \rightarrow \mathbb{R} \) is Lebesgue integrable and \( g : \mathbb{R} \rightarrow \mathbb{R} \) is continuous. Then, \( g(z, k(z)) \) is Lebesgue integrable if \( |k(z)| < a + b |z|, \forall a, b \) ve constant.

(IV) Suppose that \( k : \Sigma \rightarrow \mathbb{R} \) is continuous and \( g : \mathbb{R} \rightarrow \mathbb{R} \) is continuous, then \( g(x, k(x)) : Z \rightarrow \mathbb{R} \) is so is.

**Definition 2.4 (Riemann–Liouville Fractional Integral [24]):** The fractional integral of order \( \alpha \) of Riemann–Liouville vision is given by the relation:

\[
F^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g(s) \, ds,
\]

in case that the integral exists.

**Definition 2.5 (Caputo Derivative [24]):** Caputo formula for fractional derivative of order \( \alpha \) for \( n \)-times absolutely continuous map \( g \) is defined as:

\[
^{\alpha}D^\alpha g(t) = -\frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-1}} \, ds,
\]

where \( g^{(n)} \) is continuous, then \( g \) is integrable if \( \alpha > n \) and \( g \) is continuous. Then, \( g(\cdot) \) is continuous.

**Lemma 2.2 ([15, 24]):** Let \( \omega(i) \in L^1[0, \gamma] \) and \( \alpha, \beta > 0 \) reals. Then,

1. \( F^\beta F^\alpha \omega(i) = F^{\alpha + \beta} \omega(i) \),
2. \( D^\beta D^\alpha \omega(i) = D^{\alpha + \beta} \omega(i) \),
3. For \( \omega(i) \in AC^0[0, \gamma] \),

\[
F^\alpha D^\beta \omega(i) = \omega(i) - \sum_{i=0}^{n-1} c_i (i),
\]

where \( c_i = \omega(i)(0)/\beta \) and \( AC^0[a, b] \) be class of all absolutely continuous functions up to \( n-1 \) order derivative.

See more details in the books [15, 37, 42–44].

**Definition 2.6:** Let \( i \in [0, \gamma] \), \( \omega(i) \in C[0, \gamma] \), and consider that \( K : [0, \gamma] \times C[0, \gamma] \rightarrow \mathbb{R} \) be a multi-valued map. Let \( \Psi : [0, \gamma] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is jointly continuous function such that:

\[
\psi(i) \in \Psi(i, \omega(i), K(i, \omega(i)))
\]

\[\iff\]

\[
\exists k(i, \omega(i)) \in K(i, \omega(i))
\]

\[\Rightarrow\]

\[
\psi(i) = \Psi(i, \omega(i), k(i, \omega(i)))).
\]

Then, \( \Psi \) has non-empty values if

1. \( S_K \neq \emptyset \),
2. \( S_K \subset dom(\Psi) \),

where \( S_K = \{k | k(i) \in K(i, \omega(i)) \} \).

Now, Lemma 2.2 allows us to present the next Lemma.

**Lemma 2.3:** Let \( \alpha \) and \( \beta \) be positive reals and \( p(i) \in C[0, \gamma] \) such that \( p(i) > 0 \). Then, the differential equation

\[
^{\alpha}D^\beta \{p(i) \, ^{\alpha}D^\beta \omega(i)\} = 0
\]

has the solution

\[
\omega(i) = \beta \left( \frac{1}{p(i)} \sum_{i=0}^{n-1} a_i i! \right) + \sum_{j=0}^{n-1} b_j i!,
\]

where \( m = [\alpha] + 1, n = [\beta] + 1 \).

**Proof:**

Let \( \omega(i) \in AC^n[0, \gamma], n = [\alpha] + 1 \) and \( p(i) \) \( ^{\alpha}D^\beta \)

\( \omega(i) \in AC^n[0, \gamma], m = [\beta] + 1 \). Operating \( F^\beta \) on both sides with using Lemma 2.2(3) implies that

\[
p(i) \, ^{\alpha}D^\beta \omega(i) = \sum_{i=0}^{n-1} a_i i!,
\]

and then

\[
^{\alpha}D^\beta \omega(i) = \frac{1}{p(i)} \sum_{i=0}^{n-1} a_i i!.
\]

Again by using the third item of Lemma 2.2, we get the desired result. \( \blacksquare \)

**Lemma 2.4:** Let \( i \in [0, \gamma] \) and \( \omega(i) \in C^n[0, \gamma] \). Then, for each \( \psi(i) \in C([0, \gamma], \mathbb{R}) \cap L^1([0, \gamma], \mathbb{R}) \), and \( (1/p(i)) \) \( F^\beta \psi(i) \) \( \in AC[0, \gamma] \), the unique solution based on separation boundary value problem

\[
^{\alpha}D^\beta \{p(i) \, ^{\alpha}D^\beta \psi(i)\} = \psi(i), \ i \in [0, \gamma],
\]

\[
^{\alpha}D^\beta \psi(0) = \psi(\gamma) = 0, \ \psi^{(i)}(0) = 0,
\]

where \( i = 0, 2, \ldots, n - 1, 0 < \alpha < 1, \ n - 1 < \beta \leq n, \ n > 1, n \in N, \) is taken in the form:

\[
\omega(i) = \int_0^i \frac{(s - i)^{\alpha-1}}{\Gamma(\beta)} \left[ \frac{1}{p(s)} \int_0^s \frac{\psi(m) \, dm}{\Gamma(\alpha)} \right] ds - \int_i^\gamma \frac{(\gamma - s)^{\beta-2}}{\Gamma(\alpha)} \left[ \frac{1}{p(s)} \int_0^s \frac{\psi(m) \, dm}{\Gamma(\alpha)} \right] ds.
\]

**Proof:** The general solution for \( 0 < \alpha < 1 \) associated with the problem (7)–(8) can be formed as:

\[
p(i) \, ^{\alpha}D^\beta \omega(i) = a + \int_0^i \frac{(i - m)^{\alpha-1}}{\Gamma(\alpha)} \psi(m) \, dm,
\]

where \( a = 0 \) according to the condition \( ^{\alpha}D^\beta \psi(0) = 0 \). This shows that the solution based on \( n - 1 < \beta <\)
n, n > 1 and based on the conditions ω^(i)(0) = 0, i = 0, 2, . . . , n − 1 is taking the formula:

\[ \omega(i) = b_i + \int_0^s (t - s)^{\beta - 1} \frac{1}{\Gamma(\alpha)} (s - m)^{a - 1} \psi(m) \, dm \, ds. \]

It is clear from the condition \( \omega'(\Upsilon) = 0 \) that \( b \) can be computed by:

\[ b = -\int_0^\Upsilon (\Upsilon - s)^{\beta - 2} \frac{1}{\Gamma(\beta - 1)} \times \left[ \frac{1}{p(s)} \int_0^s (s - m)^{a - 1} \psi(m) \, dm \right] \, ds. \]

Now we are going to prove that \( \omega(i) \) has \( n \)-derivatives. For \( n > 1 \), we have:

\[ D^i\omega(i) = \beta - i \left( \frac{1}{p(s)} \int_0^s (s - m)^{a - 1} \psi(m) \, dm \right) (s) \]

and then

1. If \( i = 1 \) we can see:

\[ D\omega(i) = \beta - 1 \left( \frac{1}{p(s)} \int_0^s (s - m)^{a - 1} \psi(m) \, dm \right) (s) \]

2. If \( 1 < i \leq n - 1 \) implies that:

\[ D^i\omega(i) = \beta - i \left( \frac{1}{p(s)} \int_0^s (s - m)^{a - 1} \psi(m) \, dm \right) (s). \]

Finally, it remains to prove that \( \omega(i) \) satisfies all conditions given in (8) which completes the proof.

By the second item of Lemma 2.2, we find that

\[ \psi(i), \]

which is the Equation (7). For \( 0 \leq i \leq n - 1 \), we have

\[ D\omega(i) = \beta - i \left( \frac{1}{p(s)} \int_0^s (s - m)^{a - 1} \psi(m) \, dm \right) (s) \]

and thus \( \omega(i) \) is satisfy-

(10) we can calculate \( D^\beta f(i) \) as follows:

\[ D^\beta f(i) = \frac{\Gamma(\beta)}{\Gamma(\beta - n + 1)} f(0) \]

where \( 0 < n - \beta \leq 1 \). So, due to (10)–(13) and Definition 2.5, \( \forall i = 0, \ldots, n; \omega^{(i)}(i) \) are all well-defined if \( f(i) \in AC[0, \Upsilon] \) and then \( \omega(i) \) has \( n \)th derivatives.

Remark 2.1: Since \( p(i) > 0 \) and \( p(i) \in C[0, \Upsilon] \), then

\[ \inf_{i \in J} p(i) \leq \sup_{i \in J} p(i), \quad J = [0, \Upsilon]. \]

Thus,

\[ \frac{1}{\inf_{i \in J} p(i)} \geq \frac{1}{\sup_{i \in J} p(i)}, \quad J = [0, \Upsilon]. \]
Let
\[ \Sigma = C^{n}([0, \Gamma], \mathbb{R}) = \{ \omega(i) | \omega(i) \in C([0, \Gamma], i = 0, \ldots, n) \} \]
be the n-Continuity functions space defined on \([0, \Gamma]\) equipped with the norm
\[ \| \omega \|_{\Sigma} = \max_{i \in J} | \omega(i) |. \]
This space is a Banach space.

Finally, we provide needed Lemma and two fixed point Theorems using in existence results.

**Lemma 2.5** ([45, p. 781–786]): Let \( \Sigma^* \) be a Banach space,
\[ \Psi : [0, \Gamma] \times \Sigma^* \rightarrow P_{cp,cv}(\Sigma^*) \]
be a \( L^1 \)-Caratheodory multi-valued map and \( P \) be a continuous and linear map from \( L^1([0, \Gamma], \Sigma^*) \) to \( C([0, \Gamma], \Sigma^*) \).

Then, the operator:
\[ P \circ S_{\psi} : C([0, \Gamma], \Sigma^*) \rightarrow P_{cp,cv}(C([0, \Gamma], \Sigma^*)), \]
such that:
\[ y \mapsto (P \circ S_{\psi})(y) = P(S_{\psi,y}) \]
is an operator with closed graph in \( C([0, \Gamma], \Sigma^*) \) \( \times \) \( C([0, \Gamma], \Sigma^*) \).

Here,
\[ S_{\psi,y} = \{ \psi \in L^1([0, \Gamma], \mathbb{R}) : (\psi(i) \in \Psi(i,y(i))) \}. \]

**Theorem 2.1** (Leray–Schauder Nonlinear Alternative Type [46,p. 169,47,p. 188]): Assuming that \( \Sigma \) be Banach space, \( E \) be a convex closed subset of \( \Sigma \), and \( \Omega \) be an open subset of \( E \) with \( 0 \in \Omega \). If \( \Psi : \Omega \rightarrow P_{cp,cv}(E) \) is upper semi-continuous multi-compact map, then either

(i) there exists \( \omega \in \partial \Omega, \rho \in (0,1) \) such that \( \omega \in \rho \Psi(\omega) \),
or
(ii) there exists a fixed point \( \omega \in \Omega \).

**Theorem 2.2** (Covitz and Nadler [48, p. 9]): Define \((\Sigma, d)\) to be complete metric space. Then, \( K \) has a fixed point if \( K : \Sigma \rightarrow P_{cl}(\Sigma) \) is a contraction.

3. Discussion and Results

It is so interesting for some scientific teams to describe so many models by using the fractional (ordinary) differential operators. See for examples related details given in [49, 50]. In this field, the general form of exact solutions created under the vision of derivatives and corresponding integrals with their rules and arguments. For example, Riemann–Liouville, Caputo, Caputo-Fabrizio, Hadmard, and the ordinary derivatives.

In the previous literature, the researchers used different fixed point theorems to study separately the attractiveness of types of solutions for differential equations and inclusions and systems of equations and inclusions in different spaces. See [16, 19, 20, 23, 26, 27, 29] and the references there in.

By the actual work, we suggest to use composite functions with multi-valued maps to explore the solvability of some equation and inclusion problems at one time. We will start by the results associated with Sturm–Liouville operator and hope other researchers to work with different operators. Our results essentially reveal the characteristics of solutions with Caputo derivative. So, we are going to create some portability results of solving the problem (1)–(2).

Backing to Definition 2.6, we define
\[ \Phi(\omega) = S_{K,\omega} = \{ k(i) | k(i) \in L^1([0, \Gamma], \mathbb{R}) \cap K(i, \omega) \}, \]
and
\[ S_{\psi,\omega} = \{ \psi(i) | \psi(i) \in L^1([0, \Gamma], \mathbb{R}) \} \exists \psi(i) = \Psi(i, \omega, k(i, \omega)), k(i, \omega) \in \Phi(\omega) \} \).

Then, in view of Lemma 2.4 consider \( \Pi : \Sigma \rightarrow P(\Sigma) \) as
\[ \Pi(\omega) = \left\{ v \in \Sigma : v(i) = (S(\psi)(i), \psi(i) \in S_{\psi,\omega}) \right\}, \]
with:
\[ (S(\psi))_{i} = \int_{0}^{1} (\frac{i - s){\beta}^{i-1}}{\Gamma(\beta)} \times \left[ \int_{0}^{1} (\frac{i - m)^{\alpha-1}}{\Gamma(\alpha)} \psi(m) \,dm \right] ds \]
\[ - \int_{0}^{\Gamma} \frac{(\Gamma - s)}{\Gamma(\beta - 1)} \times \left[ \frac{1}{\Gamma(\alpha)} \psi(m) \, dm \right] ds. \]

At this time, we are ready to survey the main results.

3.1. Convex Case

The result here is followed by assuming that both maps \( \Psi \) and \( K \) are convex and overviewed by applying (Leray–Schauder Theorem 2.1).

**Theorem 3.1:** Assuming the below statements:

\( (\mathcal{H}_1) \) \( \Psi(i,\omega,w) : [0, \Gamma] \times \mathbb{R} \times 2^\mathbb{R} \rightarrow P_{cp,cv}(\mathbb{R}) \) is \( L^1 \)-Caratheodory with \( \theta_{i}(i) \in L^\infty([0, \Gamma], \mathbb{R}^+) \), \( i = 1, 2 \) and non-decreasing function \( h_{1}(i) : [0, \infty) \rightarrow (0, \infty) \) with:
\[ |(\Psi(i,\omega,w))| = \sup_{i \in J} |\Psi(i,\omega,w)| \leq h_{1}(i)(\theta_{1}(\|\omega\|) + \theta_{2}(\|w\|)) \]
for \( \omega, w \in \mathbb{R} \) and a.e \( i \in J = [0, \Gamma] \).
According to convexity of \( S(\lambda) \) (which leads to \( F(\lambda, \beta) \)),

\[
\|K(t,\omega)\| = \sup\{k(t) : k(t) \in K(t,\omega)\} \leq h_2(t)\theta_2(\|\omega\|),
\]

for \( \omega \in \mathbb{R} \) and \( a \in J \in [0,\tau] \).

**Proof:** The fit technical to prove this argument is going through five steps. These steps together will show that \( \Pi \) defined by (16) and (17), satisfies all arguments of (Leray-Schauder Theorem 2.1).

**Step 1:** By this step, \( \Pi(\omega) \) will be convex valued. According to convexity of \( \Psi \), the set \( S_{\Psi,\omega} \) so does. Let \( \lambda \in [0,1], v_1, v_2 \in \Pi(\omega) \), which means that

\[
\exists \psi_i(i) \in S_{\Psi,\omega}, i=1,2 \ \exists \psi(i) = S(\psi(i)),
\]

where

\[
(S\psi)(i) = \int_0^i \left( (t-s)^{\beta-1} \right) \frac{1}{\Gamma(\beta)} \left( s - m \right)^{\alpha-1} \psi(m)\, dm \, ds
\]

For \( \mu(\beta - 1) (\alpha + \beta) M \),

\[
\frac{\mu(\beta - 1) (\alpha + \beta) M}{(\beta \gamma \tau)^{\beta+\beta}} \left( \|h_1\| (\theta_1(M) + \theta_2(M)) \right) > 1,
\]

\[
\psi = \inf_{i \in I} p(i).
\]

Then, the problem (1) tends to be solvable on \( [0,\tau] \).

**Step 2:** Through this step, we prove that \( \Pi(t,\omega) \) is bounded on a bounded set. In order to see that, consider the open ball \( B_R = \{\omega : \|\omega\| < R\} \) and \( v \in \Pi(t,\omega) \), then we get:

\[
\psi(i) = (S\psi)(i), \quad \psi(i) \in S_{\Psi,\omega}, i \in [0,\tau]. \quad (18)
\]

Through few simple calculations and by the inequality (15), we can see that:

\[
|\psi(i)| \leq \int_0^\tau \left( \frac{\Gamma(\beta)}{\Gamma(\beta - 1)} \right) \left( \frac{1}{\Gamma(\beta)} \right) \left( s - m \right)^{\alpha-1} \psi(m)\, dm \, ds.
\]

Hence, by taking \( \psi = \inf_{i \in I} p(i) \) we have:

\[
|\psi(i)| \leq \left( \frac{\Gamma(\beta + \beta)}{\Gamma(\beta - 1)} \right) \left( \frac{1}{\Gamma(\beta)} \right) \left( s - m \right)^{\alpha-1} \psi(m)\, dm \, ds,
\]

which implies that:

\[
\|\psi\| \leq \left( \frac{\Gamma(\beta + \beta)}{\Gamma(\beta - 1)} \right) \left( \frac{1}{\Gamma(\beta)} \right) \left( s - m \right)^{\alpha-1} \psi(m)\, dm \, ds.
\]

In fact, (19) will be true if and only if \( (H_3) \) be hold for the value \( \beta \).

**Step 3:** By this step, \( \Pi(\beta) \) will be equicontinuous. Take \( 0 < t_1 < t_2 < \tau \), \( t_2 - t_1 \rightarrow 0 \), \( \omega \in \beta \). For \( v \in \Pi(\omega) \) \( \psi(i) \in S_{\Psi,\omega} \) satisfies (17) and (18), we find that:

\[
|\psi(t_2) - \psi(t_1)|
\]

Thus,

\[
(\lambda \psi_1 + (1 - \lambda) \psi_2)(i)
\]

It is easy to see by using convexity of \( S_{\Psi,\omega} \), that:

\[
(\lambda \psi_1 + (1 - \lambda) \psi_2)(t) \in \Pi(\omega),
\]

which leads to \( \Pi(\omega) \) is convex.
By (15) we get

\[ |v(t_2) - v(t_1)| \leq \int_0^{t_1} \left( \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1 - s)^{\beta-1}}{\Gamma(\beta)} \right) \left[ \int_0^s (s - m)^{\alpha-1} |\psi(m)| \, dm \right] \, ds 
\]

\[ + \int_{t_1}^{t_2} \left( \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1 - s)^{\beta-1}}{\Gamma(\beta)} \right) \left[ \int_0^s (s - m)^{\alpha-1} |\psi(m)| \, dm \right] \, ds 
\]

\[ + \left( t_2 - t_1 \right) \int_0^{T} \frac{\Gamma(\beta - 1)}{\Gamma(\beta - 1)} \left[ \int_0^s (s - m)^{\alpha-1} |\psi(m)| \, dm \right] \, ds. \]

Independently of being \( \omega \in B_R, v \in \Pi(\omega) \) we get:

\[ |v(t_2) - v(t_1)| \to 0, \quad \text{as} \ t_2 - t_1 \to 0. \]

**Step 4:** By this step, the graph of \( \Pi \) will be closed and then \( \Pi \) is upper semi-continuous. Let \( \omega_n \to \omega \), \( v_n \in \Pi(\omega_n) \) with \( v_n \to v \), \( v \in \Pi(\omega) \) means:

\[ \exists \psi_n(t) \in S\Psi_{\omega_n}, \forall v_n(t) = S\psi_n(t). \]

Existence of \( \psi_n(t) \) is depending on the existence of a suitable sequence of the functions \( k_n(t) \in S_{K_{\omega_n}} \) where:

\[ \psi_n(t) = \Psi(i, \omega_n, k_n), \quad \text{a.e.} \ i \in J. \]

Now, let \( \Phi : L^1([0, T], \mathbb{R}) \to C([0, T], \mathbb{R}) \) and set

\[ \Phi \circ S\Psi : C([0, T], \mathbb{R}) \to P_{cp, cv}(C([0, T], \mathbb{R})) \]

with \( \Phi \circ S\Psi(\omega) = \Phi(S\Psi(\omega)) \). That is: \( \psi \mapsto \Phi(\psi)(t) \),

\[ \Phi(\psi)(t) = \int_0^{T} \left( \frac{(t - s)^{\beta-1}}{\Gamma(\beta)} \cdot \left[ \int_0^s (s - m)^{\alpha-1} \psi(m) \, dm \right] \right) \, ds 
\]

\[ - t \int_0^{T} \frac{\Gamma(\beta - 1)}{\Gamma(\beta - 1)} \left[ \int_0^s (s - m)^{\alpha-1} \psi(m) \, dm \right] \, ds. \]

Hence,

\[ \Phi(\psi_n)(t) = \int_0^{T} \left( \frac{(t - s)^{\beta-1}}{\Gamma(\beta)} \cdot \left[ \int_0^s (s - m)^{\alpha-1} \psi_n(m) \, dm \right] \right) \, ds 
\]

\[ - t \int_0^{T} \frac{\Gamma(\beta - 1)}{\Gamma(\beta - 1)} \left[ \int_0^s (s - m)^{\alpha-1} \psi_n(m) \, dm \right] \, ds. \]

Under the vision of Lemma 2.5, the operator \( \Phi \circ S\Psi \) has a closed graph which implies the existence of \( k_n \) and then \( \psi_n \) with \( \psi_n(i) = \Psi(i, \omega_n, k_n) \). That follows \( v_n(i) = \Phi(\psi_n)(i) \), \( \psi_n(i) \in S\Psi(\omega_n) \). Then, we conclude that \( v_n \in \Pi(\omega_n) \) which completes the proof. The previous steps can explain that there exists fixed point of the map \( \Pi \). It remains to make sure about the priori bounds on the solutions and that will be in the next step.

**Step 5:** Let \( \omega \in \partial B_R, v \in \lambda \Pi(\omega), \lambda \in (0, 1) \). Hence, there exists

\[ \psi(i) \in S\Psi(\omega), \psi(i) = \lambda(S\psi)(i), \quad i \in [0, T]. \]

In view of step 2, we have:

\[ R = \|v\| \leq \|h_1\| \left( h_1(R) + h_2(R) \right) \frac{\beta \gamma^{\alpha+\beta}}{\mu(\beta - 1) \Gamma(\alpha + \beta)} + 1 \]

which contradicts with (H3). Take

\[ R = \frac{(\beta \gamma^{\alpha+\beta})(\|h_1\| (h_1(M) + h_2(M)))}{\mu(\beta - 1) \Gamma(\alpha + \beta)} + 1 \]

and set

\[ B_R = \{ \omega \in \Sigma : \|\omega\| < R \}. \]

Since \( B_R \) is an open subset of \( \Sigma \) and due to first four steps (step 1–step 4) with Arzela Ascoli theorem, we conclude that:

\[ \Pi : \overline{B_R} \to P_{cp, cv}(\Sigma). \]

Backing to choice of \( B_R (20–21) \), we have no

\[ v \in \partial B_R \ni v \in \lambda \Pi(\omega) \]

for some \( \lambda \in (0, 1) \). Hence, it can be deduced by non-linear alternative Theorem 2.1 that there is a fixed point \( \omega \in \overline{B_R} \) of the operator \( \Pi \). Therefore, there is a solution at least for the problem (1)–(2).

**3.2. Lower Semi-Continuous Case**

While this case is similar to the convex case in some conditions and arguments, but we are here going through non-convex hypothesis.

**Theorem 3.2:** Consider in addition of (H1) and (H2) that the following condition is hold:

\begin{enumerate}
  \item \((1)\ (i, \omega, w) \mapsto \Psi(i, \omega, w) \in L_1 \times B_\Sigma \times B_R \) measurable.
  \item \((\omega, w) \mapsto \Psi(i, \omega, w) \) is lower semi-continuous.
\end{enumerate}

Then, the problem (1)–(2) is able to has solutions at least one.
**Proof:** Define the operator

\[ \Xi(\omega) : C([0, \gamma], \mathbb{R}) \rightarrow P(L^1([0, \gamma], \mathbb{R})) \]

such that

\[ \Xi(\omega) = \{ \psi(\omega) \in L^1([0, \gamma], \mathbb{R}) : \psi(\omega) \in \Psi(i, \omega, K(i, \omega)) \}, \quad \text{quada.e.i \in [0, \gamma]} \].

The operator \( \Xi(\omega) \) is called Nemyskii’s operator associated with \( \Psi \) (see Definitions 1.2 and 1.3 in [24]) where \( \Psi \) is lower semi-continuous if \( \Xi \) is lower semi-continuous and has closed and decomposable values, see Lemma 4.4 of [51]. Note that

\[ \Xi(\omega) = S_{\Psi,\omega}. \]

This drives us to see clearly that there exists a continuous selection [52]

\[ \psi : C([0, \gamma], \mathbb{R}) \rightarrow L^1([0, \gamma], \mathbb{R}), \]

for all

\[ \omega(i) \in \mathcal{C}_0[0, \gamma] \ni \psi(\omega)(i) = \Psi(k(\omega))(i), \quad \text{a.e. } i \in [0, \gamma]. \]

Now, consider the problem

\[ cD^p[p(i) cD^\beta\omega(i)] = \psi(\omega)(i) = \Psi(k(\omega))(i). \]

Define the operator \( \Pi : \mathcal{C}_0([0, \gamma], \mathbb{R}) \rightarrow C([0, \gamma], \mathbb{R}) \) as

\[ \Pi(\omega)(i) = (S\psi)(\omega)(i). \]

We need to show that \( \Pi \) is continuous and equicontinuous. The other steps to apply alternative theorem is will be similarly in convex case.

**First:** Claim that \( \Pi \) is continuous. Since \( \omega, k, \Psi \) are all continuous, then for \( \epsilon > 0 \) be given there is \( \delta, \sigma > 0 \) such that for \( \omega_1(i), \omega_2(i) \in \mathcal{C}_0[0, \gamma] \) with \( |\omega_1(i) - \omega_2(i)| \leq \delta \) implies that

\[ |k(\omega_1)(i) - k(\omega_2)(i)| \leq \sigma. \]

Thus,

\[ |\Psi(k(\omega_1))(i) - \Psi(k(\omega_2))(i)| \leq \epsilon. \]

Moreover, we can see that

\[ |\Pi(\omega_1)(i) - \Pi(\omega_2)(i)| = |S(\psi(\omega_1))(i) - S(\psi(\omega_2))(i)| \leq \frac{\epsilon \gamma^\alpha + \beta}{\mu(\beta - 1)\Gamma(\alpha + \beta)}. \]

**Second:** Our aim is showing that \( \Pi \) is equicontinuous. Let \( t_1, t_2 \in [0, \gamma] \ni t_1 - t_2 \rightarrow 0 \). Then, similarly to step 2 in convex case we will see that the limit is independently of \( \omega, \psi(\omega) \) a.e. \( i \in [0, \gamma] \) will turn to 0. (i.e.):

\[ |\Pi(\omega)(t_2) - \Pi(\omega)(t_1)| \rightarrow 0, \quad \text{as } t_2 - t_1 \rightarrow 0. \]

In fact, \( \Pi \) satisfies all hypotheses of Leray Schauder alternative for single-values which proves the existence of solutions.

**3.3. Lipschitz Case**

In the way to dispute the result of existence under Lipschitz condition, we will follow Covitz and Nadler Theorem 2.2.

**Theorem 3.3:** Consider the two adopted assumptions hereinafter

\[ (H_5) \quad \text{Let } \Psi(i, \omega, w) : [0, \gamma] \times \mathbb{R} \rightarrow \mathcal{P}\mathcal{C}(\mathbb{R}) \text{ under the ensuing cases} \]

1. The map \( i \mapsto \Psi(i, \omega, w) \) is \( \mathcal{L} \)-measurable \( \forall \omega, w \in \mathbb{R} \).
2. \( \exists m(i) \in L^1([0, \gamma] \times \mathbb{R}_+) \ni \text{a.e. } i \in [0, \gamma] \) and \( \forall \omega_1, \omega_2 \in \mathbb{R} \text{ we have:} \)

\[ H_d(\Psi(i, \omega_1, K(i, \omega_1)), \Psi(i, \omega_2, K(i, \omega_2))) \leq m(i)(|\omega_1 - \omega_2| + |K(i, \omega_1) - K(i, \omega_2)|). \]

3. \( K(i, \omega) : [0, \gamma] \times \mathbb{R} \rightarrow P_{\mathcal{C}}(\mathbb{R}) \ni \exists n(i) \in L^1([0, \gamma] \times \mathbb{R}_+) \text{ such that a.e. } i \in [0, \gamma] \text{ and for all } \omega_1, \omega_2 \in \mathbb{R} \text{ we have:} \)

\[ |K(i, \omega_1) - K(i, \omega_2)| \leq n(i)(|\omega_1 - \omega_2|). \]

**Proof:** Let \( \Pi \) be espoused as in convex case. Since \( t \mapsto \Psi(i, \omega, w) \) is measurable and has closed values, then there exists at least one measurable selection and then the set \( S_{\Psi,\omega} \neq \emptyset \).

Let for each \( \omega(i) \in \mathcal{C}_0([0, \gamma]) \) that \( \Psi(\omega) \in P_{\mathcal{C}}(\mathcal{C}_0([0, \gamma])) \) and \( v_0 \in \Pi(\omega) \) such that \( v_0 \rightarrow v \) in \( C([0, \gamma], \mathbb{R}) \). Then, if \( v \in \Pi(\omega) \) implies that there exists \( \psi_n(i) \in S_{\Psi,\omega} \) such that

\[ \psi_n(i) = S(\psi_n)(i) \quad \text{and} \quad \psi_n(i) = \Psi(i, \omega, k_n(i, \omega)), \quad k_n(i) \in S_{K,\omega}. \]

By the assumption \((H_5)\), \( v_n \) is integrable bounded. Because \( \Psi \) and \( K \) have compact values, \( v_n \rightarrow v \) in \( L^1([0, \gamma], \mathbb{R}) \), which implies that

\[ k_n \rightarrow k \in S_{K,\omega} \Rightarrow v(i) \in S_{\Psi,\omega}. \]

Thus, \( \forall i \in [0, \gamma] ; v_n \rightarrow v(i) = S(\psi)(i) \), where

\[ \psi(i) = \Psi(i, \omega, k(i, \omega)), \quad k(i) \in S_{K,\omega}, \]

which follows that \( v \in \Pi(\omega) \) and \( \Pi \) is closed.
Now, we are in a position to prove that there exists \( \lambda \in (0, 1) \) such that
\[
H_d(\Pi(\omega_1), \Pi(\omega_2)) \leq \lambda \|\omega_1 - \omega_2\|. \tag{22}
\]
For that, take \( \omega_1(i), \omega_2(i) \in C^0[0, \Upsilon], \nu_1 \in \Pi(\omega_2) \), which means that:
\[
\exists \psi_1(i) \in S_{\Psi,\omega} \Rightarrow \nu_1(i) = S(\psi_1(i), i \in [0, \Upsilon]).
\]
From (\( \mathcal{H}_5 \)), we get:
\[
H_d(\Psi(i, \omega_1, K(i, \omega_1)), \Psi(i, \omega_2, K(i, \omega_2))) \leq m(i)(1 + n(i))(|\omega_1(i) - \omega_2(i)|).
\]
Define
\[
W(i) = \{ \psi \in \mathbb{R} : |\psi(i) - \psi| \leq m(i)(1 + n(i))(|\omega_1(i) - \omega_2(i)|) \}.
\]
Since the functions \( \psi_1(i) \) and
\[
\rho(i) = m(i)(1 + n(i))(|\omega_1(i) - \omega_2(i)|)
\]
are measurable, then Theorem III.41 in [36] and the assumption (\( \mathcal{H}_5 \)) show that \( W(i) \) is measurable. Hence,
\[
i \mapsto W(i) \cap \Psi(i, \omega, K(i, \omega)
\]
is measurable with non-empty closed values. According to the previous result and Proposition 2.1.43 in [44], we can find that \( \psi_2(i) \in \Psi(i, \omega, K(i, \omega) \) and
\[
|\psi_1(i) - \psi_2(i)|
\leq m(i)(1 + n(i))(|\omega_1(i) - \omega_2(i)|), \quad a.e. \quad i \in [0, \Upsilon].
\]
Let \( \nu_2(i) \in (S\psi_2(i)) \) (i.e.) \( \nu_2 \in \Pi(\omega_2) \). Then,
\[
|\nu_1(i) - \nu_2(i)|
= |(S\psi_1(i)) - (S\psi_2(i))|
\leq \int_0^\Upsilon \left( (\Upsilon - s)^{\beta - 2} \times \frac{(\beta \Upsilon - s)}{\beta - 1} \right.
\times \left. \left( \frac{(s - m)^{\alpha - 1}}{\Gamma(\alpha)} \cdot |\psi_1(i) - \psi_2(i)| \cdot d\mu \right) ds \right.
\]
Using (\( \mathcal{H}_6 \)), we get:
\[
|\nu_1(i) - \nu_2(i)| \leq \frac{2\|m\|}{\mu(\beta - 1)\Gamma(\alpha + \beta)} (\Upsilon)^{\alpha + \beta}\|\omega_1 - \omega_2\|.
\]
Take
\[
\lambda = \frac{2\|m\|}{\mu(\beta - 1)\Gamma(\alpha + \beta)} (\Upsilon)^{\alpha + \beta}.
\]
Hence, (22) is hold. To see that, use an akin relation obtained by exchanging the roles of \( \omega_1, \omega_2 \). In view of (Covitz and Nadler Theorem 2.2), the operator \( \Pi \) has a fixed point. Therefore, the problem (1)–(2) has one or more solutions. \[ \blacksquare \]

**Remark 3.1:** Note that the sufficient conditions of Continuity in these results can be applicable to explore finite time stability conditions for the differential equations (inclusions). For instance, compare to the active stability conditions in the papers [53–55] and more stability results related to Continuity, Lipschitz continuity, non-lipschitz continuity and Holder continuity of the settling-time functions.

### 4. Related Examples

Over the current section, we will work to present examples related to the stated upshots.

**Example 4.1:** Let us have the problem below:
\[
\mathcal{C}^{\delta /2}\left[ p(i)D^{\delta /2} \omega(i) \right] \in \Psi(i, \omega(i), K(i, \omega)), i \in [0, \Upsilon]
\]
\[
\mathcal{C}^{\delta /2} \omega(0) = \omega'(\Upsilon) = 0, \omega^{(i)}(0) = 0, i = 0, 2,
\]
where
\[
K(i, \omega) = \left[ \frac{4|\omega|^b}{b^2|\omega|^b} \right]_{b=1} = g_b(\omega), \quad b \in \mathbb{N}
\]
and
\[
\Psi(i, \omega, K(i, \omega)) = \frac{K(i, \omega)}{1 + K(i, \omega)} + b^2
\]
\[
= \left[ \frac{4|\omega|^b}{b^2 + 4|\omega|^b} + b^2 \right]_{b=1}^{\gamma}.
\]
For any \( k(i) \in K(i, \omega) \), we have:
\[
|k(i)| \leq 4 \max_{1 \leq b \leq \eta} \left\{ \|\omega\|^b \right\},
\]
which implies from (\( \mathcal{H}_2 \)) that \( h_2(i) = 4 \) and
\[
\theta_3(\|\omega\|) = \max_{1 \leq b \leq \eta} \left\{ \|\omega\|^b \right\}.
\]
Back to (\( \mathcal{H}_1 \)), we have for any \( \psi(i) \in \Psi(i) \) that
\[
|\psi(i)| \leq 1 + b^2 \leq 1 + \eta^2.
\]
It follows that we can take:
\[
h_1(i) = 1, \theta_1(i) = 1, \theta_2(i) = \eta^2.
\]
Let
\[
\rho(i) = i^2 + \frac{3}{2} \Rightarrow \inf_{i \in J} \rho(i) = \mu = \frac{3}{2}.
\]
Thus,
\[
\|\omega\| \leq r \Rightarrow R = \max \left\{ r, 4 \max_{1 \leq b \leq \eta} b^2 \right\},
\]
and
\[
M > \frac{3\gamma^2(1 + \eta^2)}{2(2\gamma^2(1))} = 2(1 + \eta^2)^2
\]
due to (\( \mathcal{H}_3 \)). Under the convex result (Theorem 3.1), it is clearly that we can find one solution or more for the problem (23)–(24).
Example 4.2: Given the problem (23)–(24) with

\[ K(\omega)(i) = \left[ -\pi |\omega|^b \right]_{b=1}^\eta \]

and

\[ \Psi(K(\omega))(i) = \sin((K(\omega))(i)), \]

It is known that \( n \)-variable function \( G \) over convex set is a concave function if and only if \( -G \) is convex in the same set. That is why \( K(\omega) \) is concave. And because the \( \sin \nu \) is concave on the interval \([0, \pi]\) and \( 0 < |K| < \pi \), that makes \( \Psi(K(\omega)) \) is concave function.

Applying the lower semi-continuous case (Theorem 3.2), we have:

\( \star \) In view of \( (\mathcal{H}_4) \) since \( \omega, K, \Psi \) are all continuous, then the composite \( \Psi(K(\omega)) \) is also continuous. It means that the points in \( (\mathcal{H}_4) \) are all hold.

\( \star \star \) The property \( 0 < |K| < \pi \) helps us to see that:

\[ h_2(i) = \pi, \quad h_3(i) = 1, \]

which drives the ability of satisfying \( (\mathcal{H}_2) \).

\( \star \star \star \) Using the fact \( |\sin \nu| \leq |\nu| \) implies that:

\[ |\Psi(K(\omega))(i)| \leq \pi. \]

It leads to take

\[ h_1(i) = \pi, \quad \theta_1(i) = \theta_2(i) = \frac{1}{2}. \]

There for, \( (\mathcal{H}_1) \) is hold.

All of these points show the probability to solve the problem (23)–(24).

Example 4.3: Suggesting the problem (23)–(24) with:

\[ K(i, \omega) = \left[ \frac{4}{b^2} |\omega| \right]_{b=1}^\eta = Q_b(\omega), \quad b \in \mathbb{N}, \]

and

\[ \Psi(i, \omega, K(i, \omega)) = \left[ \frac{K(i, \omega)}{1 + K(i, \omega)} \right] \]

\[ \bigcup \left[ \frac{1}{(1+2)\pi} \sin \omega + b^2 \right], \]

\[ = \left[ \frac{4|\omega|}{b^2 + 4|\omega|} \right]_{b=1}^\eta \]

\[ \bigcup \left[ \frac{1}{(1+2)\pi} \sin \omega + b^2 \right]. \]

Then, due to \( (\mathcal{H}_3) \) we can see that

\[ |K(i, \omega) - K(i, \nu)| = |Q_b(\omega) - Q_b(\nu)| = \frac{4}{b^2} |\omega| - |\nu| \]

\[ \leq 4|\omega - \nu| \Rightarrow n(i) = 4, \]

and

\[ H(\Psi(i, \omega, K(i, \omega)), \Psi(i, \nu, K(i, \nu))) \]

\[ \leq \frac{1}{(1+2)\pi} |\omega - \nu| + |Q_b(\omega) - Q_b(\nu)|. \]

That is

\[ H(\Psi(i, \omega, K(i, \omega)), \Psi(i, \nu, K(i, \nu))) \]

\[ \leq 5 \left( \frac{1}{1+2} \right) \| \omega - \nu \|, \]

where

\[ m(i) = 1 + \frac{1}{(1+2)^2}. \]

Finally, \( (\mathcal{H}_6) \) shows us that

\[ \frac{5}{3} \frac{2\gamma^2}{\eta} < 1 \Rightarrow \frac{25}{3} \frac{2\gamma^2}{\eta} < 1 \]

\[ \Rightarrow \gamma^2 < \frac{3}{25} \Rightarrow \gamma \in \left( 0, \frac{\sqrt{3}}{5\sqrt{2}} \right). \]

By Lipschitz case (Theorem 3.3), the problem (23)–(24) takes at least one solution.

5. Conclusion

Based on composed functions with multi-valued maps, we discuss one of the strong suggestions to create new kind (1)–(2) of nonlinear fractional differential boundary value problems. We prove by some examples (4)–(5) that this form of problems is able to be a generalization of equations and inclusions problems related to this form. That is why we call this form by eq-inclusion problems. In inclusion field, we particularized per chances to solve this problem involving Sturm–Liouville operators on a bounded domain. The obtained solutions for this problem are subjected to Caputo derivative. The chosen argument surveyed the advanced results to convexty and non-convextity cases. And the suitable theorems used here are (Leray-Schauder nonlinear alternative type ) and (Covitz and Nadler). As necessary, we applied all provided results in some related examples. What is more, we mentioned how to connect these results with some applications in stability field. It is worth to invoke the new concept in the next work that will be about the positive solutions at resonance of nonlinear fractional differential inclusions in the half real line. Besides, we expect that our results would make improvements for the previous studies into new extents.

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