Asymptotics of the Number of Endpoints of a Random Walk on a Certain Class of Directed Metric Graphs

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Abstract. A certain class of directed metric graphs is considered. The asymptotics for a number of possible endpoints of a random walk at large times is found.

1. INTRODUCTION

Let us consider a directed metric graph. Denote the length of the edge $e_k$ by $t_k$ and suppose that all lengths $\{t_k\}_{k=1}^E$ are linearly independent over the field of rational numbers $\mathbb{Q}$.

One can consider a random walk (see [3]) on a directed metric graph (see, for example, [2] for references on metric graphs). The main unlikeness with the often considered case (see, e.g., [1]) is that the endpoint of a walk can be any point on an edge of a metric graph, and not only one of the vertices. Let a point start its motion along the graph from a vertex (a source) at the initial moment of time. The passage time for each individual edge is chosen. At every vertex, the point selects, with some nonzero probability, one of the outgoing edges for further movement. Backward turns on the edges are prohibited in this model. Our aim is to analyze the asymptotics of the number $N(T)$ of possible endpoints of such random walk as time $T$ increases. The only assumption about the probabilities of choosing an edge is that it is nonzero for all edges, i.e., a situation is in general position. Such a random walk can naturally arise in the study of dynamical systems on various networks.

The asymptotics for finite compact metric nondirected graphs was constructed in [5, 6]. Moreover, in [4], the problem for wave propagation on singular manifolds was reduced to considering graphs with a finite number of vertices, but with infinite valences (however, only finitely many edges are involved at any finite time). Thus, the present paper is the first in which the asymptotics of possible endpoints on digraphs is discussed.

We consider a finite directed graph $G = (V, E)$ of the following form: let there be an outgoing tree with vertices $v$ and a root $s$, oriented in the direction from the root. The graph $G$ is obtained from this tree by adding finitely many edges leading from some vertices $s$ to $s$. These graphs have an important property: there is only one route (that does not return to the root) from the root to any vertex.

Definition 1. We call a directed strongly connected metric graph a one-way Sperner graph (see [7] for details about Sperner graphs) if it consists of one-way tree starting from the source vertex $S$ and has backward edges leading to the source only.

Thus, in what follows, we consider only one-way Sperner graphs.

2. EXACT COMBINATORIAL FORMULA FOR $N(T)$

Let us introduce some notation. For any subgraph $G'$ of a graph $G$ and a vertex $v$, we denote by $\rho_{in}(G', v)$ and $\rho_{out}(G', v)$ the number of edges incoming to $v$ and the number of edges outgoing from $v$ for the subgraph $G'$.

For any route $\mu$, we denote by $t(\mu)$ the time of this route, i.e., the sum of the times of passage of the edges included in $\mu$.

In our graph $G$, for any vertex $v \in V$, there is a unique simple chain $l_v$ (i.e., a route in which all vertices are pairwise distinct) from $s$ to $v$ (in particular, $l_s = \emptyset$).
Let $c_1, \ldots, c_k$ be the elementary cycles of the graph $G$ (in our case, $k = \rho_{in}(G, s)$). Denote the set of all elementary cycles by $C = \{c_1, \ldots, c_k\}$.

In what follows, we assume that the passage times of all elementary cycles and all simple chains of the form $l_v$ in the aggregate are linearly independent over $\mathbb{Q}$. This is a situation of general position.

Consider the linear inequality of the form $n_1a_1 + \cdots + n_ja_j \leq T$, where $n_1, \ldots, n_j \in \mathbb{N}$. Denote by $\#\{n_1a_1 + \cdots + n_ja_j \leq T\}$ the number of positive integer solutions of this inequality.

**Theorem 1.**

$$N(T) = \sum_{v \in V} \sum_{I \subseteq \{1, \ldots, k\}} (\rho_{out}(G, v) - \rho_{in}(G', v)) \# \left\{ t(l_v) + \sum_{i \in I} n_it(c_i) \leq T \right\},$$

where the subgraph $G' = G'(v, I)$ of the graph $G$ is formed by the union of the edges of the simple chain $l_v$ and elementary cycles $\{c_i\}_{i \in I}$.

**Proof.** Let us consider an arbitrary route $\mu$ from $s$ to $v$. This route can be represented in the form of passing along the edges of several elementary cycles and along the edges of a simple chain $l_v$. In this case, the passage time $t(\mu)$ of the route $\mu$ has the form $t(l_v) + \sum_{i \in I} n_it(c_i)$, where $n_i \in \mathbb{N}$. Note that any time this form is the time of passage of a route from $s$ to $v$. However, different routes can, certainly, have identical passage times. Thus, we have described the set $M$ of passage times for routes starting at the vertex $s$.

$$M = \bigcup_{v \in V} \bigcup_{I \subseteq \{1, \ldots, k\}} M_{v, I}, \text{ where } M_{v, I} = \left\{ t(l_v) + \sum_{i \in I} n_it(c_i) \mid n_i \in \mathbb{N} \forall i \in I \right\}$$

are the passage times for routes ending at the vertex $v$ and passing along all edges of the cycles $c_i (i \in I)$. The condition of linear independence over $\mathbb{Q}$ of the passage times of elementary chains and elementary cycles ensures that the union is disjoint.

Now note that the function $N(T)$ is piecewise constant, and jumps can occur only during the times $M$ of passing routes. The jump occurring at time $t \in M_{v, I}$ is equal to $\rho_{out}(G, v) - \rho_{in}(G', v)$ (where the subgraph $G'$ is formed by the edges of an elementary chain $l_v$ and the cycles $c_i (i \in I)$), since, for any $t \in M_{v, I}$ there is a route $\mu$ with $t(\mu) = t$ which ends along any given edge $G'$ entering the vertex $v$.

It remains to sum up the jumps over all passage times not exceeding $T$ and obtain the value of the function $N(T)$.

3. **ASYMPTOTIC FORMULA FOR** $N(T)$ **AS** $t \to \infty$

**Theorem 2.** Let $G$ be a finite one-way Sperner metric graph. Consider a random walk on it with initial vertex $s$. Then the following asymptotics holds for the number of possible endpoints at the time $T$:

$$N(T) = \frac{T^{\beta - 1}}{(\beta - 1)!} \cdot \frac{\sum_{e \in E} t(e)}{\prod_{j=1}^{\beta} t(c_j)} (1 + o(1)),$$

where $\beta$ is a number of elementary cycles in $\Gamma$ and $T$ tends to infinity.
We have (see [6] for references on the Barnes–Bernoilli polynomials) that the number of nonnegative solutions to the inequality \( n_1a_1 + \cdots + n_ma_m \leq T \) grows as a polynomial of degree \( n \). Accordingly, to find the leading coefficient, we need to take inequalities in the formula (1), in which either all cycles (\(|I| = \beta \)) or all cycles except for one (\(|I| = \beta - 1 \)) are involved.

Consider the term \( N_1(T) \) in formula (1) corresponding to \(|I| = \beta \) (then \( G' = G \)):

\[
N_1(T) = \sum_{v \in V} [\rho_{\text{out}}(G, v) - \rho_{\text{in}}(G, v)] \# \left\{ t(l_v) + \sum_{i=1}^{\beta} n_i t(c_i) \leq T \right\}.
\]

Let us write out the two leading terms in the expansion of \( \# \left\{ t(l_v) + \sum_{i=1}^{\beta} n_i t(c_i) \leq T \right\} \):

\[
\# \left\{ \sum_{i=1}^{\beta} n_i t(c_i) \leq T - t(l_v) \right\} = \frac{1}{\prod_{i=1}^{\beta} t(c_i)} \left( \frac{\chi^2}{\beta!} - \frac{1}{2} \sum_{i=1}^{\beta} t(c_i) \frac{\chi^{\beta-1}}{(\beta-1)!} + O(\chi^{\beta-2}) \right)
\]

\[= \frac{1}{\beta! \prod_{i=1}^{\beta} t(c_i)} T^\beta - \beta \left( t(l_v) + \frac{1}{2} \sum_{i=1}^{\beta} t(c_i) \right) T^{\beta-1} + O(T^{\beta-2}).\]

Note that the coefficient at \( T^\beta \) does not depend on \( v \) and, therefore, \( [T^\beta] N_1(T) = 0 \), due to the fact that \( \sum_{v \in V} [\rho_{\text{out}}(G, v) - \rho_{\text{in}}(G, v)] = 0 \), i.e., by the hand-shaking lemma. Thus,

\[ [T^{\beta-1}] N_1(T) = -\frac{1}{(\beta - 1)! \prod_{i=1}^{\beta} t(c_i)} \sum_{v \in V} [\rho_{\text{out}}(G, v) - \rho_{\text{in}}(G, v)] t(l_v). \]

Consider now the term \( N_2(T) \) in the formula (1) corresponding to \( I = \{1, \ldots, \beta\} \setminus \{j\} \),

\[
N_2(T) = \sum_{v \in V} \sum_{j=1, \ldots, \beta} (\rho_{\text{out}}(G, v) - \rho_{\text{in}}(G', v)) \# \left\{ t(l_v) + \sum_{i \neq j} n_i t(c_i) \leq T \right\}.
\]

In this sum, \( \rho_{\text{in}}(G', v) = \rho_{\text{in}}(G, v) = 1 \) for \( v \neq s \) and \( \rho_{\text{in}}(G', s) = \rho_{\text{in}}(G, s) - 1 \).

We see that \( N_2(T) = N'_2(T) + N''_2(T) \), where

\[
N'_2(T) = \sum_{v \in V} (\rho_{\text{out}}(G, v) - \rho_{\text{in}}(G, v)) \left( \sum_{j=1}^{\beta} \# \left\{ t(l_v) + \sum_{i \neq j} n_i t(c_i) \leq T \right\} \right),
\]

\[
N''_2(T) = \sum_{j=1}^{\beta} \# \left\{ \sum_{i \neq j} n_i t(c_i) \leq T \right\}.
\]

We have \( [T^{\beta-1}] N'_2(T) = 0 \) and, therefore,

\[ [T^{\beta-1}] N_2(T) = \frac{\sum_{i=1}^{\beta} t(c_i)}{(\beta - 1)! \prod_{i=1}^{\beta} t(c_i)}. \]

It remains to prove that

\[
\sum_{e \in E} t(e) = \sum_{i=1}^{\beta} t(c_i) - \sum_{v \in V} [\rho_{\text{out}}(G, v) - \rho_{\text{in}}(G, v)] t(l_v).
\]

We claim that, when removing edges, the expressions on the left- and right-hand sides decrease by the same amount. First, remove the edge \( e = (v, s) \). The expression on the left-hand side decreases by \( t(e) \), the expression on the right-hand side decreases by \( t(l) + t(e) - t(l) \), where the simple chain \( l \) leads from \( s \) to \( v \). We continue until \( \beta \) becomes equal to zero, i.e., \( G \) becomes a directed tree.

We remove the hanging edge \( e \): the expression on the left-hand side decreases by \( t(e) \), and the expression on the right-hand side decreases by \( -t(l) + t(l) + t(e) \), where \( l \) is a simple chain from \( s \) to the beginning of \( e \).

Finally, we obtain a tree with one edge, and the equality holds for it.
3.1. Discussion and Examples

Earlier (see [4]), a formula was obtained for the leading asymptotic coefficient of $N(T)$ in the case of an ordinary (undirected) graph. Here is the formula:

$$N(T) = \frac{T^{E-1}}{2^{V-2}(E - 1)!} \frac{\sum_{i=1}^{E} q_j}{\prod_{i=1}^{E} q_j} (1 + o(1)),$$

where $E$ is the number of edges in the graph, $V$ is the number of vertices, and $q_j$ are the lengths of the edges of the undirected graph.

There is no direct analogue of the class of graphs considered by us now in the undirected case. However, we can consider a special case, namely, the class of graphs that are oriented disjoint “loops” (cycles with two vertices) connected at the source. Such a graph corresponds to an undirected star graph. In this case, we can assume that the sum of the lengths of the edges of the directed graph $\sum_{i=1}^{E} t_j$ is equal to $2 \sum_{i=1}^{E} q_j$, and $t(c_j) = 2q_j$. The number of vertices $V$ is equal to $E + 1$; using the theorem proved in this paper, we obtain

$$N(T) = \frac{T^{E-1}}{(E - 1)!} \frac{2 \sum_{i=1}^{E} q_j}{2^E \prod_{i=1}^{E} q_j} (1 + o(1)),$$

which, after shortening, gives the formula for the undirected case.

We could notice that the presented formula for the leading term of $N(t)$ is valid not only for graphs in the class of digraphs we considered above.

Let us consider a graph in the form of a circle with two points on it and two directed chords.

The starting vertex is the vertex $A = s$. Let us find the times of possible routes.

1. Times of routes from $A$ to $A$:

$$n_1(t_1 + f_1) + n_2(t_2 + f_2 + f_1) + n_3(f_3 + f_2 + f_1), \quad n_1, n_2, n_3 \geq 0.$$

At these times, $N(T)$ increases by 2.

2. Times of routes from $A$ to $B$:

$$n_1(t_1 + f_1) + n_2(t_2 + f_2 + f_1) + n_3(f_3 + f_2 + f_1) - f_1,$$

where not all $n_i$ are zero. The condition under which, at these times, the routes end at $t_1$ and $f_2$:

$$\left\{ \begin{array}{l}
  n_1 > 0, \\
  n_2 > 0, \\
  n_3 > 0.
  \end{array} \right.$$
Here $N(T)$ decreases by 1.

3. Times of routes from $A$ to $C$:

$$n_1(t_1 + f_1) + n_2(t_2 + f_2 + f_1) + n_3(f_3 + f_2 + f_1) - f_1 - f_2,$$

where $n_2$ or $n_3 \neq 0$.

The condition under which, at these times, the routes end at $t_2$ and $f_3$:

$$\begin{cases} n_2 > 0, \\ n_3 > 0. \end{cases}$$

Thus, we see that $N(T)$ is a quadratic function of $T$:

$$N(T) = 2 \# \left\{ n_1(t_1 + f_1) + n_2(t_2 + f_2 + f_1) + n_3(f_3 + f_2 + f_1) \leq T, \ n_1, n_2, n_3 \geq 0 \right\}$$

$$- \# \left\{ n_1(t_1 + f_1) + n_2(t_2 + f_2 + f_1) + n_3(f_3 + f_2 + f_1) - f_1 \leq T, \ n_1 > 0, \left[ \begin{array}{c} n_2 > 0 \\ n_3 > 0 \end{array} \right] \right\}$$

$$- \# \left\{ n_1(t_1 + f_1) + n_2(t_2 + f_2 + f_1) + n_3(f_3 + f_2 + f_1) - f_3 - f_2 \leq T, \ n_1 > 0, n_2 > 0, n_3 > 0 \right\}$$

$$= T^2 \frac{f_1 + f_2 + f_3 + t_1 + t_2}{2(f_1 + f_2 + f_3)(f_1 + t_1)(f_1 + f_2 + t_2)} + O(T).$$

CONCLUSIONS

We found the asymptotics for the number of possible endpoints of a random walk at large times in the case of a certain class of directed graphs. Examples show that the formula for the leading coefficient still holds for digraphs without the uniqueness of the path from source to every vertex. Thus, to find a class of metric graph for which the derived formula could be correct, or to modify the formula, could be the aim of further research.

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