Incomputability of Simply Connected Planar Continua

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Abstract

Le Roux and Ziegler asked whether every simply connected compact nonempty planar \( \Pi_0^1 \) set always contains a computable point. In this paper, we solve the problem of le Roux and Ziegler by showing that there exists a planar \( \Pi_0^1 \) dendroid without computable points. We also provide several pathological examples of tree-like \( \Pi_0^1 \) continua fulfilling certain global incomputability properties: there is a computable dendrite which does not \( \ast \)-include a \( \Pi_0^1 \) tree; there is a \( \Pi_0^1 \) dendrite which does not \( \ast \)-include a computable dendrite; there is a computable dendroid which does not \( \ast \)-include a \( \Pi_0^1 \) dendrite. Here, a continuum \( A \ast \)-includes a member of a class \( P \) of continua if, for every positive real \( \varepsilon \), \( A \) includes a continuum \( B \in P \) such that the Hausdorff distance between \( A \) and \( B \) is smaller than \( \varepsilon \).

1 Background

Every nonempty open set in a computable metric space (such as Euclidean \( n \)-space \( \mathbb{R}^n \)) contains a computable point. In contrast, the Non-Basis Theorem asserts that a nonempty co-c.e. closed set (also called a \( \Pi_0^1 \) set) in Cantor space (hence, even in Euclidean 1-space) can avoid any computable points. Non-Basis Theorems can shed new light on connections between local and global properties by incorporating the notions of measure and category. For instance, Kreisel-Lacombe [4] and Tanaka [17] showed that there is a \( \Pi_0^1 \) set with positive measure that contains no computable point. Recent exciting progress in Computable Analysis [18] naturally raises the question whether Non-Basis Theorems exist for connected \( \Pi_0^1 \) sets. However, we observe that, if a nonempty \( \Pi_0^1 \) subset of \( \mathbb{R}^1 \) contains no computable points, then it must be totally disconnected. Then, in higher dimensional Euclidean space, can there exist a connected \( \Pi_0^1 \) set containing no computable points? It is easy to construct a nonempty connected \( \Pi_0^1 \) subset of \([0, 1]^2\) without computable points, and a nonempty simply connected \( \Pi_0^1 \) subset of \([0, 1]^3\) without computable points. An open problem, formulated by Le Roux and Ziegler [14] was whether every nonempty simply connected compact planar \( \Pi_0^1 \) set contains a computable point. As mentioned in Penrose’s book “Emperor’s New Mind” [12], the Mandelbrot set is an example of a simply connected compact planar \( \Pi_0^1 \) set which contains a computable point, and he conjectured that the Mandelbrot set is not computable as a closed set. Hertling [5] observed that the Penrose conjecture has an implication for a

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famous open problem on local connectivity of the Mandelbrot set. Our interest is which topological assumption (especially, connectivity assumption) on a \( I^0 \) set can force it to possess a given computability property. Miller \cite{[10]} showed that every \( I^0 \) sphere in \( \mathbb{R}^n \) is computable, and so it contains a dense c.e. subset of computable points. He also showed that every \( I^0 \) ball in \( \mathbb{R}^n \) contains a dense subset of computable points. Iljazović \cite{[7]} showed that chainable continua (e.g., arcs) in certain metric spaces are almost computable, and hence there always is a dense subset of computable points. In this paper, we show that not every \( I^0 \) dendrite is almost computable, by using a tree-immune \( I^0 \) class in Cantor space. This notion of immunity was introduced by Cenzer, Weber Wu, and the author \cite{[4]}. We also provide pathological examples of tree-like \( I^0 \) continua fulfilling certain global incomputability properties: there is a computable dendrite which does not \( * \)-include a \( I^0 \) tree; there is a computable dendroid which does not \( * \)-include a \( I^0 \) dendrite. Finally, we solve the problem of Le Roux and Ziegler \cite{[13]} by showing that there exist a planar \( I^0 \) dendroid without computable points. Indeed, our planar dendroid is contractible. Hence, our dendroid is also the first example of a contractible Euclidean \( I^0 \) set without computable points.

2 Preliminaries

**Basic Notation:** \( 2^{<\mathbb{N}} \) denotes the set of all finite binary strings. Let \( X \) be a topological space. For a subset \( Y \subseteq X \), \( cl(Y) \) (\( int(Y) \), resp.) denotes the closure (the interior, resp.) of \( Y \). Let \( (X,d) \) be a metric space. For any \( x \in X \) and \( r \in \mathbb{R} \), \( B(x;r) \) denotes the open ball \( B(x;r) = \{ y \in X : d(x,y) < r \} \). Then \( x \) is called the *center* of \( B(x;r) \), and \( r \) is called the *radius* of \( B(x;r) \). For a given open ball \( B = B(x;r) \), \( \tilde{B} \) denotes the corresponding closed ball \( \tilde{B} = \{ y \in X : d(x,y) \leq r \} \). For \( a,b \in \mathbb{R} \), \( [a,b] \) denotes the closed interval \( [a,b] = \{ x \in \mathbb{R} : a \leq x \leq b \} \), \( (a,b) \) denotes the open interval \( (a,b) = \{ x \in \mathbb{R} : a < x < b \} \), and \( (a,b) \) denotes a point of Euclidean plane \( \mathbb{R}^2 \). For \( X \subseteq \mathbb{R}^n \), \( \text{diam}(X) \) denotes \( \max \{ d(x,y) : x,y \in X \} \).

**Continuum Theory:** A *continuum* is a compact connected metric space. For basic terminology concerning *Continuum Theory*, see Nadler \cite{[11]} and Illanes-Nadler \cite{[8]}.

Let \( X \) be a topological space. The set \( X \) is a *Peano continuum* if it is a locally connected continuum. The set \( X \) is a *dendrite* if it is a Peano continuum which contains no Jordan curve. The set \( X \) is *unicoherent* if \( A \cap B \) is connected for every connected closed subsets \( A,B \subseteq X \) with \( A \cup B = X \). The set \( X \) is *hereditarily unicoherent* if every subcontinuum of \( X \) is unicoherent. The set \( X \) is *a dendroid* if it is an arcwise connected hereditary unicoherent continuum. For a point \( x \) of a dendroid \( X \), \( r_X(x) \) denotes the cardinality of the set of arc-components of \( X \setminus \{x\} \). If \( r_X(x) \geq 3 \) then \( x \) is said to be a *ramification point of \( X \). The set \( X \) is a *tree* if it is dendrite with finitely many ramification points. Note that a topological space \( X \) is a dendrite if and only if it is a locally connected dendroid. Hahn-Mazurkiewicz’s Theorem states that a Hausdorff space \( X \) is a Peano continuum if and only if \( X \) is an image of a continuous curve.

**Example 1** (Planar Dendroids).
1. Put $B_t = \{2^{-t}\} \times [0, 2^{-t}]$. Then the following set $B \subseteq \mathbb{R}^2$ is dendrite.

$$B = \bigcup_{t \in \mathbb{N}} B_t \cup \{-1, 1\} \times \{0\}. \tag{1}$$

We call $B$ the basic dendrite. The set $B_t$ is called the $t$-th rising of $B$. See Fig. 1.

2. The set $\mathcal{H} = \text{cl}((\{1/n : n \in \mathbb{N}\} \times [0, 1]) \cup ([0, 1] \times \{0\}))$ is called a harmonic comb. Then $\mathcal{H}$ is a dendroid, but not a dendrite. The set $\{1/n\} \times [0, 1]$ is called the $n$-th rising of the comb $\mathcal{H}$, and the set $[0, 1] \times \{0\}$ is called the grip of $\mathcal{H}$. See Fig. 2.

3. Let $C \subseteq \mathbb{R}^1$ be the middle third Cantor set. Then the one-point compactification of $C \times (0, 1]$ is called the Cantor fan. (Equivalently, it is the quotient space $\text{Cone}(C) = (C \times [0, 1])/(C \times \{0\})$.) The Cantor fan is a dendroid, but not a dendrite. See Fig. 3.

Let $X$ be a topological space. $X$ is $n$-connected if it is path-connected and $\pi_i(X) \equiv 0$ for any $1 \leq i \leq n$, where $\pi_i(X)$ is the $i$-th homotopy group of $X$. $X$ is simply connected if $X$ is 1-connected. $X$ is contractible if the identity map on $X$ is null-homotopic. Note that, if $X$ is contractible, then $X$ is $n$-connected for each $n \geq 1$. It is easy to see that the dendroids in Example 1 are contractible.

**Computability Theory:** We assume that the reader is familiar with Computability Theory on the natural numbers $\mathbb{N}$, Cantor space $2^\mathbb{N}$, and Baire space $\mathbb{N}^\mathbb{N}$ (see also Soare [16]). For basic terminology concerning Computable Analysis, see Weihrauch [13], Brattka-Weihrauch [3], and Brattka-Presser [2].

Hereafter, we fix a countable base for the Euclidean $n$-space $\mathbb{R}^n$ by $\rho = \{B(x; r) : x \in \mathbb{Q}^n \& r \in \mathbb{Q}^+\}$, where $\mathbb{Q}^+$ denotes the set of all positive rationals. Let $\{\rho_n\}_{n \in \mathbb{N}}$ be an effective enumeration of $\rho$. We say that a point $x \in \mathbb{R}^n$ is computable if the code of its principal filter $\mathcal{F}(x) = \{i \in \mathbb{N} : x \in \rho_i\}$ is computably enumerable (hereafter c.e.) A closed subset $F \subseteq \mathbb{R}^n$ is $\Pi^0_1$ if there is a c.e. set $W \subseteq \mathbb{N}$ such that $F = X \setminus \bigcup_{r \in W} \rho_r$. A closed subset $F \subseteq \mathbb{R}^n$ is computably enumerable (hereafter c.e.) if $\{e \in \mathbb{N} : F \cap \rho_e \neq \emptyset\}$ is c.e. A closed subset $F \subseteq \mathbb{R}^n$ is computable if it is $\Pi^0_1$ and c.e. on $\mathbb{R}^n$.

**Almost Computability:** Let $A_0, A_1$ be nonempty closed subsets of a metric space $(X, d)$. Then the Hausdorff distance between $A_0$ and $A_1$ is defined by

$$d_H(A_0, A_1) = \max_{i \leq 2} \sup_{x \in A_i, y \in A_{1-i}} d(x, y).$$

Let $\mathcal{P}$ be a class of continua. We say that a continuum $A$ *includes a member of $\mathcal{P}$* if $\inf\{d_H(A, B) : A \supseteq B \in \mathcal{P}\} = 0$. 

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Proposition 2. Every Euclidean dendroid *-includes a tree.

Proof. Fix a Euclidean dendroid $D \subseteq \mathbb{R}^n$, and a positive rational $\varepsilon \in \mathbb{Q}$. Then $D$ is covered by finitely many open rational balls $\{B_i\}_{i<n}$ of radius $\varepsilon/2$. Choose $d_i \in D \cap B_i$ for each $i < n$ if $B_i$ intersects with $D$. Note that $\{B(d_i;\varepsilon)\}_{i<n}$ covers $D$. Since $D$ is dendroid, there is a unique arc $\gamma_{i,j} \subseteq D$ connecting $d_i$ and $d_j$ for each $i, j < n$. Then, $E = \bigcup_{(i,j) \subseteq n} \gamma_{i,j}$ is connected and locally connected, since $E$ is a union of finitely many arcs (i.e., it is a graph, in the sense of Continuum Theory; see also Nadler [1]). It is easy to see that $E$ has no Jordan curve, since $E$ is a subset of the dendroid $D$. Consequently, $E$ is a tree. Moreover, clearly $d_i \in (E,D) < \varepsilon$, since $d_i \in E$ for each $i < n$. 

The class $\mathcal{P}$ has the almost computability property if every $A \in \mathcal{P}$ *-includes a computable member of $\mathcal{P}$ as a closed set. In this case, we simply say that every $A \in \mathcal{P}$ is almost computable. Iljazović [7] showed that every $\Pi^0_1$ chainable continuum is almost computable, hence every $\Pi^0_1$ arc is almost computable.

3 Incomputability of Dendrites

By Proposition 2, topologically, every planar dendrite *-includes a tree. However, if we try to effectivize this fact, we will find a counterexample.

Theorem 3. Not every computable planar dendrite *-includes a $\Pi^0_1$ tree.

Proof. Let $A \subseteq \mathbb{N}$ be an uncomputable c.e. set. Thus, there is a total computable function $f_A : \mathbb{N} \to \mathbb{N}$ such that range$(f_A) = A$. We may assume $f_A(s) \leq s$ for every $s \in \mathbb{N}$. Let $A_s$ denote the finite set $\{f_A(u) : u \leq s\}$. Then $st^A : \mathbb{N} \to \mathbb{N}$ is defined as $st^A(n) = \min\{s \in \mathbb{N} : n \in A_s\}$. Note that $st^A(n) \geq n$ by our assumption $f_A(s) \leq s$.

Construction. Recall the definition of the basic dendrite from Example 1. We construct a computable dendrite by modifying the basic dendrite $B$. For every $t \in \mathbb{N}$, we introduce the width of the $t$-rising $w(t)$ as follows:

$$w(t) = \begin{cases} 2^{-(2+st^A(t))}, & \text{if } t \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Let $I_t$ be the closed interval $[2^{-t} - w(t), 2^{-t} + w(t)]$. Since $st^A(n) \geq n$, we have $I_t \cap I_s = \emptyset$ whenever $t \neq s$. We observe that $\{w(t)\}_{t \in \mathbb{N}}$ is a uniformly computable sequence of real numbers. Now we define a computable dendrite $D \subseteq \mathbb{R}^2$ by:

$$D_0^t = (\{2^{-t} - w(t)\} \cup \{2^{-t} + w(t)\}) \times [0,2^{-t}]$$
$$D_1^t = [2^{-t} - w(t), 2^{-t} + w(t)] \times \{2^{-t}\}$$
$$D_2^t = (2^{-t} - w(t), 2^{-t} + w(t)) \times (-1,2^{-t})$$
$$D = \bigcup_{t \in \mathbb{N}} (D_0^t \cup D_1^t) \cup \{[-1,1] \times \{0\}\} \setminus \bigcup_{t \in \mathbb{N}} D_{t,m}$$

We call $D_t = D_0^t \cup D_1^t$ the $t$-th rising of $D$. See Fig. 4.
The set \( T \) of tree, \( N \) This implies that incomputable.

\[ T \langle \{ \langle t \cap D \rangle \} \text{ moves in} \]
Note that a Hausdorff space (hence every metric space) is (locally) arcwise connected if and only if it is (locally) pathwise connected. However, Miller \cite{10} pointed out that the effective versions of arcwise connectivity and pathwise connectivity do not coincide. Theorem \ref{10} could give a result on effective connectivity properties. Note that effectively pathwise connectivity is defined by Brattka \cite{1}. Clearly, the dendrite $D$ is effectively pathwise connected. We now introduce a new effective version of arcwise connectivity property by thinking a tree as closed sets. Let $A_−(X)$ denote the hyperspace of closed subsets of $X$ with negative information (see also Brattka \cite{1}).

**Definition 4.** A computable metric space $(X,d,α)$ is semi-effectively arcwise connected if there exists a total computable multi-valued function $P : X^2 \Rightarrow A_−(X)$ such that $P(x,y)$ is the set of all arcs $A$ whose two end points are $x$ and $y$, for any $x,y \in X$.

Obviously $D$ is not semi-effectively arcwise connected. Indeed, for every $ε > 0$ there exists $x_0,x_1 \in [0,1]$ with $d(x_0,x_1) < ε$ such that $⟨x_0,0⟩, ⟨x_1,0⟩ \in D$ cannot be connected by any $Π^0_1$ arc. Thus, we have the following corollary.

**Corollary 1.** There exists an effectively pathwise connected Euclidean continuum $D$ such that $D$ is not semi-effectively arcwise connected.

**Theorem 5.** Not every $Π^0_1$ planar dendrite is almost computable.

To prove Theorem \ref{5} we need to prepare some tools. For a string $σ \in 2^{< N}$, let $lh(σ)$ denote the length of $σ$. Then

$$ψ(σ) = 2^{−1} \cdot 3^{−i} + 2 \sum_{i<lh(σ) & σ(i)=1} 3^{−(i+1)} \cdot 2^{−lh(σ)} \in \mathbb{R}^2.$$ 

For two points $x,y \in \mathbb{R}^2$, the closed line segment $L(x,y)$ from $x$ to $y$ is defined by $L(x,y) = \{(1-t)x + ty : t \in [0,1]\}$. For a (possibly infinite) tree $T \subseteq 2^{< N}$, we plot an embedded tree $Ψ(T) \subseteq \mathbb{R}^2$ by

$$Ψ(T) = cl \left( \bigcup \{L(ψ(σ), ψ(τ)) : σ,τ \in T & lh(σ) = lh(τ) + 1\} \right).$$

Then $Ψ(T)$ is a dendrite (but not necessarily a tree, in the sense of Continuum Theory), for any (possibly infinite) tree $T \subseteq 2^{< N}$. See Fig. \ref{5}.

We can easily prove the following lemmata.

**Lemma 6.** Let $T$ be a subtree of $2^{< N}$, and $D$ be a planar subset such that $ψ(⟨⟩) \in D \subseteq Ψ(T)$ for the root $⟨⟩ \in 2^{< N}$. Then $D$ is a dendrite if and only if $D$ is homeomorphic to $Ψ(S)$ for a subtree $S \subseteq T$. 

![Figure 5: The plotted tree $Ψ(2^{< N})$.](image)
Proof. The “if” part is obvious. Let \( D \) be a dendrite. For a binary string \( \sigma \) which is not a root, let \( \sigma^- \) be an immediate predecessor of \( \sigma \). We consider the set 
\[ S = \{ \langle \rangle \} \cup \{ \sigma \in 2^{<\omega} : \sigma \neq \langle \rangle \} \& D \cap (L(\psi(\sigma^-), \psi(\sigma)) \setminus \{\psi(\sigma^-)\}) \neq \emptyset. \] 
Since \( D \) is connected, \( S \) is a subtree of \( T \). It is easy to see that \( D \) is homeomorphic to \( \Psi(S) \).

**Lemma 7.** Let \( T \) be a subtree of \( 2^{<\omega} \). Then \( T \) is \( \Pi^0_1 \) (c.e., computable, resp.) if and only if \( \Psi(T) \) is a \( \Pi^0_1 \) (c.e., computable, resp.) dendrite in \( \mathbb{R}^2 \).

**Proof.** With our definition of \( \Psi \), the dendrite \( \Psi(2^{<\omega}) \) is clearly a computable closed subset of \( \mathbb{R}^2 \). So, if \( T \) is \( \Pi^0_1 \), then it is easy to prove that \( \Psi(T) \) is also \( \Pi^0_1 \). Assume that \( T \) is a c.e. tree. At stage \( s \), we compute whether \( L(\psi(\sigma^-), \psi(\sigma)) \) intersects with the \( e \)-th open rational ball \( \rho_e \), for any \( e < s \) and any string \( \sigma \) which is already enumerated into \( T \) by stage \( s \). If so, we enumerate \( e \) into \( W_T \) at stage \( s \). Then \( \{ e \in \mathbb{N} : \Psi(T) \cap \rho_e \neq \emptyset \} = W_T \) is c.e.

Assume that \( \Psi(T) \) is \( \Pi^0_1 \). We consider an open rational ball \( B_\sigma = B(\psi(\sigma); 2^{-\ell(\sigma)+2}) \) for each \( \sigma \in 2^{<\omega} \). Note that \( \overline{B_\sigma} \cap \overline{B_\tau} = \emptyset \) for \( \sigma \neq \tau \). Hence, for each \( \sigma \in 2^{<\omega} \), \( \Psi(T) \) is \( \Pi^0_1 \). The function \( \Psi(T) \) is c.e., and \( \Pi^0_1 \) on the root of \( \psi(\langle \rangle) \), otherwise it is c.e. For a binary string \( \sigma \) which is a root, let \( \sigma^- \) be an immediate predecessor of \( \sigma \). Pick an open rational ball \( B_\sigma \) such that \( \Psi(2^\omega) \cap B_\sigma \subseteq L(\psi(\sigma^-), \psi(\sigma)) \) for each \( \sigma \). Since \( \Psi(T) \) is c.e., \( T_\sigma = \{ \sigma \in 2^\omega : \Psi(T) \cap B_\sigma = \emptyset \} \) is c.e. If \( \sigma \) is not a root and \( \sigma \in T \) then \( L(\psi(\sigma^-), \psi(\sigma)) \subseteq \Psi(T) \), or \( \Psi(T) \cap B_\sigma = \emptyset \). We observe that if \( \sigma \not\in T \) then \( L(\psi(\sigma^-), \psi(\sigma)) \cap \Psi(T) = \emptyset \). Thus, we have \( T = T_\sigma \). In the case that \( \Psi(T) \) is computable, \( \Psi(T) \) is c.e. and \( \Pi^0_1 \), hence \( T \) is c.e. and \( \Pi^0_1 \), i.e., \( T \) is computable.

**Lemma 8.** Let \( D \) be a computable subdendrite of \( \Psi(2^{<\omega}) \). Then there exists a computable subtree \( T^* \subseteq 2^{<\omega} \) such that \( \Psi \) contains \( D \subseteq \Psi(T^*) \) and \( ([0, 1] \times \{0\}) \cap D = ([0, 1] \times \{0\}) \cap \Psi(T^*) \).

**Proof.** We can assume \( \psi(\langle \rangle) = \psi(\langle \rangle) \in D \), otherwise we connect \( \psi(\langle \rangle) \) and the root of \( D \) by a subarc of \( \Psi(2^{<\omega}) \). Again we consider an open rational ball \( B_\sigma = B(\psi(\sigma); 2^{-\ell(\sigma)+2}) \), and an open rational ball \( B_\sigma \) such that \( \Psi(2^{<\omega}) \cap B_\sigma \subseteq L(\psi(\sigma^-), \psi(\sigma)) \) for each \( \sigma \). Since \( D \) is \( \Pi^0_1 \), \( U^* = \{ \sigma \in 2^{<\omega} : \exists D \cap B_\sigma = \emptyset \} \) is c.e. Since \( D \) is c.e., \( T^* = \{ \sigma \in 2^{<\omega} : D \cap B_\sigma = \emptyset \} \) is c.e., and it is a tree by Lemma 7. For every \( \sigma \in 2^{<\omega} \), either \( D \cap B_\sigma = \emptyset \) or \( D \cap B_\sigma = \emptyset \) holds. Therefore, we have \( T^* \cup U^* = 2^{<\omega} \). Moreover, for the set of leaves of \( T^* \), \( L_{T^*} = \{ \rho \in T^* : (\forall i < 2) \rho^- \not\in T^* \} \), we observe that \( T^* \cup U^* \subseteq L_{T^*} \). Recall that the pointclass \( \Sigma^1_1 \) has the reduction property, that is, for two c.e. sets \( T^* \) and \( U^* \), there exist c.e. subsets \( T \subseteq T^* \) and \( U \subseteq U^* \) such that \( T \cup U = T^* \cup U^* \) and \( T \cap U = \emptyset \). This is because, for \( \sigma \in T^* \cap U^* \), \( \sigma \) is enumerated into \( T \) when \( \sigma \) is enumerated into \( T^* \) before it is enumerated into \( U^* \); \( \sigma \) is enumerated into \( U \) otherwise. Since \( T^* \cup U^* \subseteq L_{T^*} \), \( T \) must be tree. Furthermore, \( T \) is c.e., and \( U = 2^{<\omega} \setminus T \) is also c.e. Thus, \( T \) is a computable tree. Therefore, \( T^* = \{ \sigma^- \in T^* \& i < 2 \} \) is also a computable tree. Then, \( D \subseteq \Psi(T^*) \), and we have \( ([0, 1] \times \{0\}) \cap D = ([0, 1] \times \{0\}) \cap \Psi(T^*) \) since the set of all infinite paths of \( T \) and that of \( T^* \) coincide.
A nonempty closed set has the following remarkable property.

For a nonempty $\Pi^0_1$ subset $P \subseteq 2^N$, the corresponding tree $T_P$ is $\Pi^0_1$, and it has no dead ends. The set of all complete consistent extensions of Peano Arithmetic is an example of a tree-immune $\Pi^0_1$ subset of $2^N$. Tree-immune $\Pi^0_1$ sets have the following remarkable property.

**Lemma 10.** Let $P$ be a tree-immune $\Pi^0_1$ subset of $2^N$ and let $D \subseteq \Psi(T_P)$ be any computable subdendrite. Then $([0,1] \times \{0\}) \cap D = \emptyset$ holds.

**Proof.** By Lemma 5 there exists a computable subtree $T \subseteq 2^{<\omega}$ such that $D \subseteq \Psi(T)$ and $\Psi(T)$ agrees with $D$ on $[0,1] \times \{0\}$. Since $D \subseteq \Psi(T_P)$, and since $T_P$ has no dead ends, $T \subseteq T_P$ holds. Since $P$ is tree-immune, $T$ must be finite. By using weak König’s lemma (or, equivalently, compactness of Cantor space), $T \subseteq 2^l$ holds for some $l \in \mathbb{N}$. Thus, $D \subseteq \Psi(T) \subseteq [0,1] \times [2^{-l},1]$ as desired. □

We fix a nonempty tree-immune $\Pi^0_1$ set $P \subseteq 2^N$. For $\sigma \subseteq 2^{<\omega}$, put $E(\sigma) = \{ \tau \subseteq 2^{<\omega} : \tau \supseteq \sigma \}$. For a $\Pi^0_1$ tree $T_P \subseteq 2^{<\omega}$, there exists a computable function $f_P : \mathbb{N} \to 2^{<\omega}$ such that $T_P = 2^{<\omega} \setminus \bigcup_n E(f_P(n))$ and such that for each $\sigma \subseteq 2^{<\omega}$ and $s \in \mathbb{N}$ we have $\sigma \in \bigcup_{t<s} E(f_P(t))$ whenever $\sigma^{-0}, \sigma^{-1} \in \bigcup_{t<s} E(f_P(t))$. For such a computable function $f_P : \mathbb{N} \to 2^{<\omega}$, we let $T_{P,s}$ denote $2^{<\omega} \setminus \bigcup_{t<s} E(f_P(t))$. Then $T_{P,s}$ is a tree without dead ends, and $\{T_{P,s} : s \in \mathbb{N}\}$ is computable uniformly in $s$.

**Construction.** Let $c_1$ denote $(1,0) \in \mathbb{R}^2$. For a tree $T \subseteq 2^{<\omega}$ and $w \in \mathbb{Q}$, we define $\Psi(T;w)$, the edge of the fat approximation of the tree $T$ of width $w$, by

$$\Psi(T;w) = \text{cl} \left( \bigcup \left\{ L \left( \psi(\sigma) \pm (3^{-|\sigma|} \cdot w)c_1, \psi(\tau) \pm (3^{-|\tau|} \cdot w)c_1 \right) : \pm \in \{-, +\} \& \sigma, \tau \in T \& \text{lh}(\sigma) = \text{lh}(\tau) + 1 \right\} \right).$$

If $\lim_s w_s = 0$ then we have $\lim_s \Psi(T;w_s) = \Psi(T)$. Moreover, if $\{w_s : s \in \mathbb{N}\}$ is a uniformly computable sequence of rational numbers, then $\{\Psi(T;w_s) : s \in \mathbb{N}\}$ is also a uniformly computable sequence of computable closed sets. Additionally, the set $\Psi(T;w,c,t,q)$, for a tree $T \subseteq 2^{<\omega}$, for $w,c,q \in \mathbb{Q}$, and for $t \in \mathbb{N}$, is defined by

$$\Psi(T;w,c,t,q) = \left\{ c + q \cdot \left( x - \frac{1}{2}, \frac{2 - y}{2^{t+1}} \right) \in \mathbb{R}^2 : (x,y) \in \Psi(T;w) \right\}.$$
There exists a nonempty $\Pi^0_1$ subset of $[0,1]^2$ which is contractible, locally contractible, and $*$-includes no connected computable closed subset.
4 Incomputability of Dendroids

Theorem 11. Not every computable planar dendroid *-includes a $\Pi^0_1$ dendrite.

Lemma 12. There exists a limit computable function $f$ such that, for every uniformly c.e. sequence $\{U_n : n \in \mathbb{N}\}$ of cofinite c.e. sets, we have $f(n) \in U_n$ for almost all $n \in \mathbb{N}$.

Proof. Let $\{V_e : e \in \mathbb{N}\}$ be an effective enumeration of all uniformly c.e. non-increasing sequences $\{U_n : n \in \mathbb{N}\}$ of c.e. sets such that $\min U_n \geq n$, where $(V_e)_n = U_n = \{x \in \mathbb{N} : (n,x) \in V_e\}$. The e-state of $y$ is defined by $\sigma(e,y) = \{i \leq e : y \in (V_i)_e\}$, and the maximal e-state is defined by $\sigma(e) = \max \sigma(e,z)$. The construction of $f : \mathbb{N} \to \mathbb{N}$ is to maximize the e-state. For each $e \in \mathbb{N}$, $f(e)$ chooses the least $y \in \mathbb{N}$ having the maximal e-state $\sigma(e,y) = \sigma(e)$. Since $\{\sigma(e,y) : e, y \in \mathbb{N}\}$ is uniformly c.e., and $\sigma(e,y) \in 2^e$, the function $e \mapsto \sigma(e) = \max \sigma(e,z)$ is total limit computable. Thus, $f$ is limit computable. It is easy to see that $\lim_n \sigma(e)(n)$ exists for each $n \in \mathbb{N}$. Let $U = \{U_n : n \in \mathbb{N}\}$ be a uniformly c.e. sequence of cofinite c.e. sets. Then $V = \{\bigcap_{m \leq n} U_m : n \in \mathbb{N}\}$ is a uniformly c.e. non-increasing sequence of cofinite c.e. sets. Thus, $V_i = V$ for some index $i$. Then $i \in \sigma(e,y)$ for almost all $e, y \in \mathbb{N}$. This ensures that $i \in \sigma(e)$ for almost all $e \in \mathbb{N}$ by our assumption $\min U_n \geq n$. Hence we have $f(n) \in U_n$ for almost all $n \in \mathbb{N}$.

Remark. The proof of Lemma 12 is similar to the standard construction of a maximal c.e. set (see Soare [16]). Recall that the principal function of the complement of a maximal c.e. set is dominant, i.e., it dominates all total computable functions. The limit computable function $f$ in Lemma 12 is also dominant. Indeed, for any total computable function $g$, if we set $U^n_g = \{y \in \mathbb{N} : y \geq g(n)\}$ then $\{U^n_g : n \in \mathbb{N}\}$ is a uniformly c.e. sequence of cofinite c.e. sets, and if $f(n) \in U^n_g$ holds then we have $f(n) \geq g(n)$.

Proof of Theorem 11. Pick a limit computable function $f = \lim_s f_s$ in Lemma 12. For every $t, u \in \mathbb{N}$, put $v(t, u) = 2^{-s}$ for the least $s$ such that $f_s(t) = u$ if such $s$ exists; $v(t, u) = 0$ otherwise. Since $\{f_s : s \in \mathbb{N}\}$ is uniformly computable, $v : \mathbb{N}^2 \to \mathbb{R}$ is computable.

Construction. For each $t \in \mathbb{N}$, the center position of the $u$-th rising of the $t$-th comb is defined as $c_s(t, u) = 2^{-(2t+1)} + 2^{-(2t+u+1)}$, and the width of the
The set \( K \) occurs for finitely many \( K \). Then we show this claim, we first observe that \( p \) and \( q \) are connected by a subarc of \( K \). For each \( t, u \in \mathbb{N} \), we consider the following subcontinuum

\[
K^*_t = \left\{ 2^{-2(t+1)} \right\} \times [0, 2^{-t}]
\]

\[
K^i_{t,u} = \{ c_* (t, u) - v_* (t, u), c_* (t, u) + v_* (t, u) \} \times [0, 2^{-t}]
\]

\[
K^2_{t,u} = \{ c_* (t, u) - v_* (t, u), c_* (t, u) + v_* (t, u) \} \times (-1, 2^{-t})
\]

\[
K_t = \left( K^*_t \cup \bigcup_{i < 2} \bigcup_{u \in \mathbb{N}} K^i_{t,u} \right) \cup \left( \left\{ 2^{-2(t+1)}, 2^{-2t} \right\} \times \{ 0 \} \cup \bigcup_{u \in \mathbb{N}} K^2_{t,u} \right).
\]

Note that \( K_t \) is homeomorphic to the harmonic comb \( H \) for each \( t \in \mathbb{N} \). This is because, for fixed \( t \in \mathbb{N} \), since \( \lim s f_s (t) \) exists we have \( v(t, u) = 0 \) for almost all \( u \in \mathbb{N} \). Then the desired computable dendroid is defined as follows.

\[
K = \left( [-1, 0] \times \{ 0 \} \right) \cup \bigcup_{t \in \mathbb{N}} \left( \left\{ 2^{-2(t+2)}, 2^{-2(t+1)} \right\} \times \{ 0 \} \cup K_t \right).
\]

**Claim.** The set \( K \) is a computable dendroid.

Clearly \( K \) is a computable closed set. To show that \( K \) is pathwise connected, we consider the following subcontinuum \( K^*_t \), the grip of the comb \( K_{t,m} \), for each \( t \in \mathbb{N} \).

\[
K^*_t = \left( \bigcup_{i < 2} \bigcup_{v(t,u) > 0} K^i_{t,u} \right) \cup \left( \left\{ 2^{-2(t+1)}, 2^{-2t} \right\} \times \{ 0 \} \right) \cup \bigcup_{v(t,u) > 0} K^2_{t,u}.
\]

Then \( K^- = \left( [-1, 0] \times \{ 0 \} \right) \cup \bigcup_{t \in \mathbb{N}} \left( \left\{ 2^{-2(t+2)}, 2^{-2(t+1)} \right\} \times \{ 0 \} \cup K^- \right) \) has no ramification points. We claim that \( K^- \) is connected and \( K^- \) is even an arc. To show this claim, we first observe that \( K^- \) is an arc for any \( t \in \mathbb{N} \), since \( v(t,u) > 0 \) occurs for finitely many \( u \in \mathbb{N} \). Moreover \( K^- \subseteq S(t) \), and \( \lim \text{diam}(S(t)) = 0 \) holds. Therefore, we see that \( K^- \) is locally connected and, hence, an arc. For points \( p, q \in K \), if \( p, q \in K^- \) then \( p \) and \( q \) are connected by a subarc of \( K^- \). In the case \( p \in K \setminus K^- \), the point \( p \) lies on \( K^0_{t,u} \) for some \( t, u \) such that \( v(t,u) = 0 \). If \( q \in K^- \) then there is a subarc \( A \subseteq K^- \) (one of whose endpoints must be \( \langle c_* (t, u), 0 \rangle \)) such that \( A \cup K^0_{t,u} \) is an arc containing \( p \) and \( q \). In the case \( q \in K \setminus K^- \), similarly we can connect \( p \) and \( q \) by an arc in \( K \). Hence, \( K \) is
pathwise connected. K is hereditarily unicoherent, since the harmonic comb is hereditarily unicoherent. Thus, K is a dendroid.

Claim. The computable dendroid K does not \( \ast \)-include a \( \Pi^0_1 \) dendrite.

What remains to show is that every \( \Pi^0_1 \) subdendrite \( R \subseteq K \) satisfies \( d_H (R, K) \geq 1 \). Let \( R \subseteq K \) be a \( \Pi^0_1 \) dendrite. Put \( S(t) = \{ 2^{-(2t+1)}, 2^{-2t} \} \times [0,2^{-t}] \). Since \( R \) is locally connected, \( R \cap S(t) = R \cap K_t \) is also locally connected for each \( t \in \mathbb{N} \) and \( m < 2^t \). Thus, for fixed \( t \in \mathbb{N} \), let \( K_{t,u}^* = [c_\ast(t,u) - 2^{-(2t+u+3)}, c_\ast(t,u) + 2^{-(2t+u+3)}] \times \{ 2^{-t} \} \). For any continuum \( R^* \subseteq K_t \), if \( R^* \cap K_{t,u}^* \neq \emptyset \) for infinitely many \( u \in \mathbb{N} \), then \( R^* \) must be homeomorphic to the harmonic comb \( H \). Hence, \( R^* \) is not locally connected. Therefore, we have \( R \cap K_{t,u}^* = \emptyset \) for almost all \( u \in \mathbb{N} \). Since \( K_{t,u}^* \) and \( K_{t,v}^* \) is disjoint whenever \( (t,u) \neq (s,v) \), and since \( R \) is \( \Pi^0_1 \), we can effectively enumerate \( U_i = \{ u \in \mathbb{N} : R \cap K_{t,u}^* = \emptyset \} \), i.e., \( \{ U_i : t \in \mathbb{N} \} \) is uniformly c.e. Moreover, \( U_t \) is cofinite for every \( t \in \mathbb{N} \). Then, by our definition of \( \ast \) in Lemma 12 there exists \( t^* \in \mathbb{N} \) such that \( f(t) \in U_t \) for all \( t \geq t^* \). Note that \( v(t,f(t)) > 0 \) and thus the condition \( f(t) \in U_t \) (i.e., \( R \cap K_{t,f(t)}^* = \emptyset \)) implies that, for every \( t \geq t^* \), either \( R \subseteq [c_\ast(t,u) + v_\ast(t,u), 1] \times [0,1] \) or \( R \subseteq [c_\ast(t,u) - v_\ast(t,u), 1] \times [0,1] \) holds. Thus we obtain the desired condition \( d_H (R, K) \geq 1 \).

Remark. It is easy to see that the dendroid constructed in the proof of Theorem 11 is contractible.

Corollary 3. There exists a nonempty constructible planar computable closed subset of \( [0,1]^2 \) which \( \ast \)-includes no \( \Pi^0_1 \) subset which is connected and locally connected.

Theorem 13. Not every nonempty \( \Pi^0_1 \) planar dendroid contains a computable point.

Proof. One can easily construct a \( \Pi^0_1 \) Cantor fan \( F \) containing at most one computable point \( p \in F \), and such \( p \) is the unique ramification point of \( F \). Our basic idea is to construct a topological copy of the Cantor fan \( F \) along a pathological located arc \( A \) constructed by Miller [10, Example 4.1]. We can guarantee that moving the fan \( F \) along the arc \( A \) cannot introduce new computable points. Additionally, this moving will make a ramification point \( p^* \) in a copy of \( F \) incomputable.

Fat Approximation. To archive this construction, we consider a fat approximation of a subset \( P \) of the middle third Cantor set \( C \subseteq \mathbb{R}^1 \), by modifying the standard construction of \( C \). For a tree \( T \subseteq 2^{\leq \mathbb{N}} \), put \( \pi(\sigma) = 3^{-1} + 2 \sum_{i<lh(\sigma) \& \sigma(i)=1} 3^{-(i+2)} \) for \( \sigma \in T \), and \( J(\sigma) = [\pi(\sigma) - 3^{-lh(\sigma)+1}, \pi(\sigma) + 2 \cdot 3^{-lh(\sigma)+1}] \). If a binary string \( \sigma \) is incomparable with a binary string \( \tau \) then \( J(\sigma) \cap J(\tau) = \emptyset \). We extend \( \pi \) to a homeomorphism \( \pi_* \) between Cantor space \( 2^\mathbb{N} \) and \( C \cap \left[ 1/3, 2/3 \right] \) by defining \( \pi_*(\sigma) = 3^{-1} + 2 \sum_{f(i)=1} 3^{-(i+2)} \) for \( f \in 2^\mathbb{N} \). Let \( P^* \subseteq 2^\mathbb{N} \) be a nonempty \( \Pi^0_1 \) set without computable elements. Then there exists a computable tree \( T_P \) such that \( P^* \) is the set of all paths of \( T_P \), since \( P^* \) is \( \Pi^0_1 \). A fat approximation \( \{ P_s : s \in \mathbb{N} \} \) of \( P = \pi_*(P^*) \) is defined as \( P_s = \bigcup \{ J(\sigma) : lh(\sigma) = s \& \sigma \in T_P \} \). Then \( \{ P_s : s \in \mathbb{N} \} \) is a computable decreasing sequence of computable closed sets, and we have \( P = \bigcap_s P_s \).

Since \( P \) is a nonempty bounded closed subset of a real line \( \mathbb{R}^1 \), both \( \min P \)
and max \( P \) exist. By the same reason, both \( l_s^- = \min P_s \) and \( r_s^+ = \max P_s \) also exist, for each \( s \in \mathbb{N} \), and \( \lim s l_s^- = \min P \) and \( \lim s r_s^+ = \max P \), where \( \{ l_s : s \in \mathbb{N} \} \) is increasing, and \( \{ r_s : s \in \mathbb{N} \} \) is decreasing. Let \( l_s = l_s^- + 3^{-s(s+1)} \) and \( r_s = r_s^+ - 3^{-s(s+1)} \). We also set \( l_0 = l_0^- + 3^{-0(0+1)} \) and \( r_0 = r_0^+ - 3^{-0(0+2)} \). Note that \( l_s < r_s \), \( \lim s l_s = \min P \), and \( \lim s r_s = \max P \). Since \( \min P, \max P \in P \) and \( P \) contains no computable points, \( \min P \) and \( \max P \) are non-computable, and so \( l_s < \min P < \max P < r_s \) holds for any \( s \in \mathbb{N} \). The fat approximation of \( P \) has the following remarkable property:

\[
[l_s^-, l_s] \subseteq P_s, \quad [l_s^-, l_s] \cap P = \emptyset, \quad [r_s, r_s^+] \subseteq P_s, \quad \text{and} \quad [r_s, r_s^+] \cap P = \emptyset.
\]

To simplify the construction, we may also assume that \( P \) has the following property:

\[
P = \{ 1-x \in \mathbb{R} : x \in P \}
\]

Because, for any \( \Pi^0_1 \) subset \( A \subseteq C \), the \( \Pi^0_1 \) set \( A^* = \{ x/3 : x \in A \} \cup \{ 1-x/3 : x \in A \} \subseteq C \) has that property.

**Basic Notation.** For each \( i, j < 2 \), for each \( a, b \in \mathbb{R}^2 \), and for each \( q, r \in \mathbb{R} \), the 2-cube \( \Delta_{ij}(a, b; q, r) \subseteq [a, a+q] \times [b, b+r] \) is defined as the smallest convex set containing the three points \( \{(a, b), (a+q, b), (a+b+r, q), (a+b+1-r, r-j)r\} \). Namely,

\[
\Delta_{ij}(a, b; q, r) = \{((-1)^i)x + a + ij, (-1)^jy + b + jr \in \mathbb{R}^2 : x, y \geq 0 \& rx + qy \leq qr \}.
\]

For a set \( R \subseteq \mathbb{R}^1 \) and real numbers \( r, y \in \mathbb{R} \), put \( \Theta(R; r, y) = \{ rx + y \in \mathbb{R} : x \in R \} \). Clearly \( \Theta(R; r, y) \) is computably homeomorphic to \( R \). Let four symbols \( +, -, \cdot, \) and \( \circ \) denote \( (10, 01), (01, 10), (00, 11), \) and \( (11, 00) \), respectively. For \( v \in \{ +, -, \cdot, \circ \} \) and for any \( R \subseteq [0, 1], a, b \in \mathbb{R}^2, \) and \( q, r \in \mathbb{R} \), we define \( [v](R; a, b; q, r) \subseteq [a, a+q] \times [b, b+r] \) as follows:

\[
[v](R; a, b; q, r) = (([a, a+q] \times \Theta(R; r, b)) \cap \Delta_{v(0)}(a, b; q, r)) \cup ((\Theta(R; q, a) \times [b, b+r]) \cap \Delta_{v(1)}(a, b; q, r)).
\]

**Sublemma 1.** \( [v](P; a, b; q, r) \) is computably homeomorphic to \( P \times [0, 1] \). In particular, \( [v](P; a, b; q, r) \) contains no computable points.

To simplify our argument, we use the normalization \( \bar{P}_t \) of \( P_t \) for \( t \geq s \), that is defined by \( \bar{P}_t = \{(x-l_s^-)/(r_s^+-l_s^-) : x \in P_t \} \), for each \( s \in \mathbb{N} \).
Note that $\tilde{P}_t^s \subseteq [0,1]$ for $t \geq s$, and $0,1 \in \tilde{P}_s^s$ holds for each $s \in \mathbb{N}$. Put $[v]^t_s(\{a,a+q\} \times [b,b+r]) = [v](\tilde{P}_s^t; a,b,q,r)$ for $t \geq s$. We also introduce the following two notions:

$$[-]^t_s([a,a+q] \times [b,b+r]) = [a,a+q] \times \Theta(\tilde{P}_s^t; r,b);$$

$$[\_]^t_s([a,a+q] \times [b,b+r]) = \Theta(\tilde{P}_s^t; q,a) \times [b,b+r].$$

Here we code two symbols $-\_-$ and $\_-$ as 0 and 1 respectively.

**Sublemma 2.** $[v]^t_s([a,a+q] \times [b,b+r]) \subseteq [a,a+q] \times [b,b+r]$ and $[v]^t_s([a,a+q] \times [b,b+r])$ intersects with the boundary of $[a,a+q] \times [b,b+r]$.

**Sublemma 3.** There is a computable homeomorphism between $[v]^t_s(a,b;q,r)$ and $P_t \times [0,1]$ for any $t \in \mathbb{N}$. Therefore, $\bigcap_t [v]^t_s(a,b;q,r)$ is computably homeomorphic to $P_t \times [0,1]$.

**Blocks.** A block is a set $Z \subseteq \mathbb{R}^2$ with a bounding box $\text{Box}(Z) = [a,a+q] \times [b,b+r]$. Each $\delta \in 2^2$ is called a direction, and directions $(00), (01), (10)$, and $(11)$ are also denoted by $[-\_], [-\_], [\_\_], \text{ and } [\_\_\_], \text{ respectively.}$ For $\delta \in 2^2$, $\delta^0 = (\delta(0), 1 - \delta(0))$ is called the reverse direction of $\delta$. Put $\text{Line}(Z;\delta) = \{a\} \times [b,b+r]; \text{Line}(Z;[-\_]) = \{a+q\} \times [b,b+r]; \text{Line}(Z;[\_\_]) = [a,a+q] \times \{b\}; \text{Line}(Z;[\_\_\_]) = [a,a+q] \times \{b+r\}$. Assume that a class $Z$ of blocks is given. We introduce a relation $\rightarrow$ on $Z$ for each direction $\delta$. Fix a block $Z_{\text{first}} \in Z$, and we call it the first block. Then we declare that $\rightarrow Z_{\text{first}}$ holds. We inductively define the relation $\rightarrow$ on $Z$. If $Z \rightarrow Z_0$ (resp. $Z_0 \rightarrow Z$) for some $Z$ and $\delta$, then we can also write it as $\rightarrow Z_0$ (resp. $Z_0 \rightarrow$). For any two blocks $Z_0$ and $Z_1$, the relation $Z_0 \rightarrow Z_1$ holds if the following three conditions are satisfied:

1. $Z_0 \cap Z_1 = \text{Line}(Z_0;\delta) \cap Z_0 = \text{Line}(Z_1;\delta^0) \cap Z_1 \neq \emptyset.$
2. $\rightarrow \_\_Z_0$ has been already satisfied for some direction $\_\_\_\_\_.$
3. $Z_1 \rightarrow \_\_Z_0$ does not satisfy for any direction $\_\_\_\_\_$

If $Z_0 \rightarrow \_\_Z_1$, then we say that $Z_1$ is a successor of $Z_0$ ($Z_0$ is a predecessor of $Z_1$), and we also write it as $\rightarrow Z_0 \rightarrow Z_1$.

We will construct a partial computable function $Z : \mathbb{N}^3 \rightarrow \mathcal{A}(\mathbb{R}^2)$ with a computable function $k : \mathbb{N} \rightarrow \mathbb{N}$ and $\text{dom}(Z) = \{(u,i,t) \in \mathbb{N}^3 : u \leq t \& i < k(u)\}$ such that $Z(u,i,t)$ is a block with a bounding box for any $(u,i,t) \in \text{dom}(Z)$, and the block $Z(u,i,t)$ is computably homeomorphic to $P_t \times [0,1]$ uniformly in $(u,i,t)$. Here $\mathcal{A}(\mathbb{R}^2)$ is the hyperspace of all closed subsets in $\mathbb{R}^2$ with positive and negative information. For each stage $t$, $Z_t(u) = \{Z(t,u,i) \mid i < k(u)\}$ for each $u \leq t$ is defined. Let $Z(u)$ denote the finite set $\{\lambda t. Z(t,u,i) \mid i < k(u)\}$ of functions, for each $u \in \mathbb{N}$. The relation $\rightarrow$ induces an ordering for each $u \in \mathbb{N}$ such as $Z_0 < Z_1$ if there is a finite path from $Z_0(t)$ to $Z_1(t)$ on the directed graph $(\bigcup_{u \in \mathbb{N}} Z_u, \rightarrow)$ at some stage $t \in \mathbb{N}$. We will assume that $\prec$ is a well-ordering of order type $\omega$, and $Z_0 < Z_1$ whenever $Z_0 \in Z(u)$, $Z_1 \in Z(v)$, and $u < v$. In particular, for every $Z \in \bigcup_{u \in \mathbb{N}} Z(u)$, the predecessor $Z_{\text{pre}}$ of $Z$ and the successor $Z_{\text{succ}}$ of $Z$ under $\prec$ are uniquely determined. If $Z_{\text{pre}}(t) \rightarrow Z(t) \rightarrow Z_{\text{succ}}(t)$, then we say that $Z$ moves from $\delta$ to $\varepsilon$, and that $(\delta, \varepsilon)$ is the direction of $Z$. 14
**Example 14.** Fig. 13 is an example satisfying $\rightarrow Z_{\text{first}} \rightarrow Z_0 \rightarrow Z_1 \rightarrow Z_2$.

**Destination Point.** Basically, our construction is similar as the construction by Miller [10]. Pick the standard homeomorphism $\rho$ between $2^\mathbb{N}$ and the middle third Cantor set, i.e., $\rho(M) = 2 \sum_{i \in M} (1/3)^{i+1}$ for $M \subseteq \mathbb{N}$, and pick a non-computable c.e. set $B \subseteq \mathbb{N}$ and put $\gamma = \rho(B)$. We will construct a Cantor fan so that the first coordinate of the unique ramification point is $\gamma$, hence the fan will have a non-computable ramification point. Let $\{B_s : s \in \mathbb{N}\}$ be a computable enumeration of $B$, and let $u_s$ denote the element enumerated into $B$ at stage $s$, where we may assume just one element is enumerated into $B$ at each stage. Put $\gamma_s^\min = \rho(B_s)$ and $\gamma_s^\max = \rho(B_s \cup \{i \in \mathbb{N} : i \geq n_s\})$. Note that $\gamma$ is not computable, and so $\gamma_s^\min \neq \gamma$ and $\gamma_s^\max \neq \gamma$ for any $s \in \mathbb{N}$. This means that for every $s \in \mathbb{N}$ there exists $t > s$ such that $\gamma_s^\min \neq \gamma_t^\min$ and $\gamma_s^\max \neq \gamma_t^\max$. By this observation, without loss of generality, we can assume that $\gamma_s^\min \neq \gamma_t^\min$ and $\gamma_s^\max \neq \gamma_t^\max$ whenever $s \neq t$. We can also assume $1/3 \leq \gamma_s^\min \leq \gamma_s^\max \leq 2/3$ for any $s \in \mathbb{N}$.

**Stage 0.** We now start to construct a $\Pi^0_0$ Cantor fan $Q = \bigcap_{t \in \mathbb{N}} Q_s$. At the first stage 0, and for each $t \geq 0$, we define the following sets:

$$Z_{0,t}^\text{st} = [-\frac{1}{3}, \frac{1}{3}] \times [l_0^-, r_0^+] ;$$
$$Z_{0,t}^\text{end} = [\frac{1}{3} - 1/3, \frac{1}{3}] \times [l_0^-, r_0^+] .$$

Moreover, we set $Q_0 = Z_{0,0}^\text{st} \cup Z_{0,0}^\text{end}$. By our choice of $P_0$, actually $Q_0 = [\frac{1}{3} - 1/3, \frac{1}{3}] \times [l_0^-, r_0^+]$. $Z_{0,0}^\text{st}$ is called the straight block from 2/3 to 1/3 at stage 0, and $Z_{0,0}^\text{end}$ is called the end box at stage 0. The bounding box of the block $Z_{0}^\gamma$ is defined by $[\frac{1}{3} - 1/3, \frac{1}{3}] \times [l_0^-, r_0^+]$. The collection of 0-blocks at stage $t$ is $Z_t(0) = \{Z_{0,t}^\gamma\}$. We declare that $Z_0^\text{st}$ is the first block, and that $\rightarrow Z_{0,t}^\gamma$.

**Stage $s+1$.** Inductively assume that we have already constructed the collection of $u$-blocks $Z_t(u)$ at stage $t \geq u$ is already defined for every $u \leq s$. For any $u$, we think of the collection $Z(u) = \{Z_t(u) : t \geq u\}$ as a finite set $\{Z_i^u\}_{i \leq \#Z_u(u)}$ of computable functions $Z_i^u : \{t \in \mathbb{N} : t \geq u\} \rightarrow \bigcup Z_t(u)$ such that $Z_t(u) = \{Z_i^u(t) : i < \#Z_u(u)\}$ for each $t \geq u$. We inductively assume that the collection $Z(u) = \{Z_t(u) : t \geq u\}$ satisfies the following conditions:

(IH1) For each $Z \in Z(u)$ and for each $t \geq u \geq v \geq Z(t) \subseteq Z(v)$.

(IH2) There is a computable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $f : \bigcup_{u \leq s} Z_t(u)$ is a homeomorphism between $\bigcup_{u \leq s} Z_t(u)$ and $P \times [0,1]$ for any $t \geq s$. 

Figure 13: Example [14]
(IH3) There are \( y, z, \zeta \in \mathbb{Q} \) such that the blocks \( Z_s^{st} \) and \( Z_s^{end} \) are constructed as follows:

\[
Z_s^{st} = \left[ -\frac{1}{s} \right] \left( \left[ \gamma_s^{min}, \gamma_s^{max} \right] \times [y + zl_s^-, y + zr_s^+] \right);
\]
\[
Z_s^{end} = \left[ \gamma_s^{min} - \zeta, \gamma_s^{min} \right] \times [y + zl_s^-, y + zr_s^+].
\]

Here, a computable closed set \( Q_s \), an approximation of our \( \Pi^0_1 \) Cantor fan \( Q \) at stage \( s \), is defined by \( Q_s = Z_s^{end} \cup \bigcup_{n \leq s} Z_s(n) \).

**Non-injured Case.** First we consider the case \([\gamma_{s+1}^{min}, \gamma_{s+1}^{max}] \subseteq [\gamma_s^{min}, \gamma_s^{max}]\), i.e., this is the case that our construction is not injured at stage \( s+1 \). In this case, we construct \((s+1)\)-blocks in the active block \( Z_s^{st} \cup Z_s^{end} \). We will define \( Z_t(s, i, j) \) and \( \text{Box}(s, i, j) = \text{Box}(Z_t(s, i, j)) \) for each \( j < 6 \). The first two corner blocks at stage \( t \geq s + 1 \) are defined by:

\[
\text{Box}(s, 0) = [\gamma_s^{min} - \zeta, \gamma_s^{min}] \times [y + zl_s^-, y + zr_s^+],
\]
\[
Z_t(s, 0) = \left[ \frac{1}{s} \right] \left( [\gamma_s^{min} - \zeta, \gamma_s^{min}] \times [y + zl_s^-, y + zr_s^+] \right) \cap \text{Box}(s, 0),
\]
\[
\text{Box}(s, 1) = [\gamma_s^{min} - \zeta, \gamma_s^{min}] \times [y + zr_s^+, y + zr_s^+],
\]
\[
Z_t(s, 1) = \left[ \frac{1}{s} \right] \left( \text{Box}(s, 1) \right).
\]

**Sublemma 4.** \( Z_t(s, 0) \cup Z_t(s, 1) \subseteq Z_s^{end} \) for any \( t \geq s + 1 \).

**Sublemma 5.** \( Z_s^{st} \xrightarrow{[\cdot]} Z_t(s, 0) \xrightarrow{[\cdot]} Z_t(s, 1) \), for any \( t \geq s + 1 \).

The next block is a straight block from \( \gamma_s^{min} \) to \( \gamma_{s+1}^{max} \) which is defined as follows:

\[
\text{Box}(s, 2) = [\gamma_s^{min}, \gamma_s^{max}] \times [y + zr_s^+, y + zr_s^+].
\]
\[
Z_t(s, 2) = \left[ -\left( \text{Box}(s, 2) \right) \right].
\]

For given \( a, b, \alpha, \beta \in \mathbb{Q} \), we can calculate \( N_{0,s}(a, b; \alpha, \beta) \) and \( N_{1,s}(a, b; \alpha, \beta) \) satisfying \( N_{0,s}(a, b; \alpha, \beta) + N_{1,s}(a, b; \alpha, \beta) \cdot l_s^+ = a + ba \), and \( N_{0,s}(a, b; \alpha, \beta) + N_{1,s}(a, b; \alpha, \beta) \cdot r_s^+ = a + b\beta \). Put \( y^* = N_{0,s}(y, z; r_s^+, r_s^+) \), and \( z^* = N_{1,s}(y, z; r_s^+, r_s^+) \).

**Sublemma 6.** \( \text{Box}(s, 2) = [\gamma_s^{min}, \gamma_s^{max}] \times [y^* + z^*l_s^-, y^* + z^*r_s^+] \).
Put $\zeta^* = (\gamma_{s+1}^\max - \gamma_{\min}^s) / 3^s$. Note that $\zeta^* > 0$ since $\gamma_{s+1}^\max > \gamma_{\min}^{s+1}$. We then again define corner blocks.

\[
\begin{align*}
\text{Box}(s, 3) &= [\gamma_{\min}^{s+1}, \gamma_{\max}^{s+1} + \zeta^*] \times [y^* + z_s^r, y^* + z_s^r], \\
Z_t(s, 3) &= \lfloor (\gamma_{\min}^{s+1}, \gamma_{\max}^{s+1} + \zeta^*) \times [y^* + z_s^r, y^* + z_s^r]) \cap \text{Box}(s, 3), \\
\text{Box}(s, 4) &= [\gamma_{\min}^{s+1}, \gamma_{\min}^s + \zeta^*] \times [y^* + z_s^r, y^* + z_s^r], \\
Z_t(s, 4) &= \lceil [\text{Box}(s, 4)].
\end{align*}
\]

Next, a straight block from $\gamma_{\min}^s$ to $\gamma_{\max}^{s+1}$ is defined as follows:

\[
\begin{align*}
\text{Box}(s, 5) &= [\gamma_{\min}^{s+1}, \gamma_{\max}^s] \times [y^* + z_s^r, y^* + z_s^r], \\
Z_t(s, 5) &= [-\zeta^*] [\text{Box}(s, 5)].
\end{align*}
\]

Put $y^{**} = N_0,s(y^*, z^r_s; r_s^r, r_s^r)$, and $z^{**} = N_1,s(y^*, z^r_s; r_s^r, r_s^r)$.

\textbf{Sublemma 7.} $\text{Box}(s, 5) = [\gamma_{\min}^{s+1}, \gamma_{\max}^s] \times [y^{**} + z^{**} l_s^-, y^{**} + z^{**} r_s^+]$.

Put $\zeta^{**} = (\gamma_{s+1}^\min - \gamma_s^\min) / 3^s$. Note that $\zeta^{**} > 0$ since $\gamma_{s+1}^\min > \gamma_s^\max$. The end box at stage $s+1$ is:

\[
Z(s, 6) = [\gamma_{\min}^{s+1} - \zeta^{**}, \gamma_{\min}^{s+1}] \times [y^{**} + z^{**} l_s^-, y^{**} + z^{**} r_s^+].
\]

Then put $Z(s, s+1, t) = Z_t(s, 5)$, $Z_{s+1} = Z_{s+1, s+1}$, and $Z_{s+1}^{\text{end}} = Z(s, 6)$. The active block at stage $s+1$ is the set $Z_{s+1, s+1} \cup Z_{s+1}^{\text{end}}$, and the collection of (s+1)-blocks at stage $t$ is defined by $Z_t(s+1) = \{Z_t(s, i) : i \leq 5\}$. Clearly, our definition satisfies the induction hypothesis (IH3) at stage $s+1$.

\textbf{Sublemma 8.} $Z_t(s, i) \subseteq Z_u(s, i)$ for each $t \geq v \geq s+1$ and $i \leq 5$.

\textbf{Sublemma 9.} For any $t \geq s+1$,

\[
Z_{s+1}^{\text{end}} \rightarrow Z_t(s, 5) \rightarrow Z_t(s, 1) \rightarrow Z_t(s, 2) \rightarrow Z_t(s, 3) \rightarrow Z_t(s, 4) \rightarrow Z_t(s, 5).
\]

\textit{Proof.} It follows straightforwardly from the definition of these blocks $Z_t(s, i)$, and Sublemmas 8 and 9.

\textbf{Sublemma 10.} $\bigcup_{2 \leq i \leq 6} Z_t(s, i) \subseteq Z_{s+1}^{\text{end}} \cap [\gamma_{\min}^s, \gamma_{\max}^s] \times (y + z r_s, y + z r_s^+)$. Hence, $\left(\bigcup_{2 \leq i \leq 6} Z_t(s, i)\right) \cap Z_{s+1}^{\text{end}} = \emptyset$.

\textbf{Figure 15:} The first two corner blocks $Z(s, 0)$ and $Z(s, 1)$. 

---

Note: The above text contains mathematical expressions and diagrams that are not rendered properly. The intended content is described in the text, but the visual elements (diagrams) are not included in the text representation.
Assume that we have a computable function $f$, and $s$, and $8$. By Sublemma 10, our construction at stage $\langle f, s \rangle$ will define blocks $Z$ that are two kinds of blocks: one is $\text{words, } Z_\text{def}$, and another is $\text{other blocks, } Z_\text{inside}$. In this case, indeed, we have $\gamma_i \in \text{min, max} \cap \text{min, max} = \emptyset$. Fix the greatest stage $p \leq s$ such that $\gamma_{\text{min, max}} \subseteq [\gamma_{p, \text{min, max}}]$. We again, inside the end box $Z_\text{end}$ at stage $s$, define corner blocks $Z_i(s, 0)$ and $Z_i(s, 1)$ as non-injuring stage, whereas the construction of $Z_i(s, i)$ for $i \geq 2$ differs from non-injuring stage. The end box of our construction at stage $s+1$ will turn back along all blocks belonging $Z_i(u)$ for $p < u \leq s$ in the reverse ordering of $\prec$. Let $\{Z_i : i < k_{s+1}\}$ be an enumeration of all blocks in $Z_i(u)$ for $p < u \leq s$, under the reverse ordering of $\prec$. In other words, $Z_i$ is the successor block of $Z_{i+1}$ under $\rightarrow$, for each $i < k_{s+1} - 1$. There are two kinds of blocks: one is a straight block, and another is a corner block. We will define blocks $Z_i(s, i, j)$ for $i < k_{s+1}$ and $j < 3$. Now we check the direction $\langle \delta_i, \varepsilon_i \rangle$ of $Z_i$. Here, we may consistently assume that the condition $Z_0^{\varepsilon_i}$ holds.

**Subcase 1.** If $\delta_i(0) = \varepsilon_i(0)$ then $Z_i$ is a straight block. In this case, we only construct $Z_i(s, i, 0)$. Since $Z_i$ is straight, there are $y_i, z_i, \alpha, \beta \in Q$ and $u \leq s$ such that, for $B_1(0) = [\alpha, \beta]$ and $B_1(1) = [y_i + z_i, y_i + z_i, y_i, y_i, y_i]$, such that $\Box(Z_i) = B_i(\delta_i(0)) \times B_i(1 - \delta_i(0))$. If $\delta_i(1) = 0$, then set $y_i^* = N_{0, s}(y_i, z_i); t_i^*$ and $z_i = N_{1, s}(y_i, z_i; t_i^*, t_i^*)$. If $\delta_i(1) = 1$, then set $y_i^* = N_{0, s}(y_i, z_i; r_i^*, r_i^*)$ and $z_i = N_{1, s}(y_i, z_i; r_i^*, r_i^*)$.
$z_i^* = N_{1,s}(y_i, z_i; r_i^*, r_i^+)$. Then, we define $Z_t(s, i, 0)$ as the following straight block:

$$B_t^*(0) = B_t(0); \quad B_t^*(1) = [y_i^* + z_i^* l_i^-, y_i^* + z_i^* r_i^+];$$

$$Z_t(s, i, 0) = [\delta_t(0)]^* B_t^*(\delta_t(0)) \times B_t^*(1 - \delta_t(0)).$$

Here, $\text{Box}(Z_t(s, i, 0))$ is defined by $B_t^*(\delta_t(0)) \times B_t^*(1 - \delta_t(0)).$

**Sublemma 12.** $Z_t(s, i, 0) \subseteq Z_i$.

**Proof.** By our definition of $N_{0,s}$ and $N_{1,s}$, we have $B_t^*(1) = [y_i + z_i l_i^-, y_i + z_i l_i^+]$ or $B_t^*(1) = [y_i + z_i r_i^+, y_i + z_i r_i^+]$.

**Subcase 2.** If $\delta_t(0) \neq \delta_t(2)$ then $Z_i$ is a corner block. We will construct three blocks; one corner block $Z_t(s, i, 0)$, and two straight blocks $Z_t(s, i, 1)$ and $Z_t(s, i, 2)$. We may assume that $Z_i$ is of the following form:

$$Z_i = [e]^* [u^{x} l_{x}^{u} l_{x}^{r} r_{x}^{r} r_{x}^{u}];$$

or $Z_i = [e]^* [(x_i + \zeta l_{x}^{u}, x_i + \zeta r_{x}^{r}) \times [y_i + z_i l_{x}^{u}, y_i + z_i r_{x}^{r}]]$

Set $z = 0$ if the former case occurs; otherwise, set $z = 1$. Let $p_0 : n < 6$ be an enumeration of $\{u^{x} l_{x}^{u} l_{x}^{r} r_{x}^{r} r_{x}^{u}\}$ in increasing order, and let $p_0$ be $r_{y}^{u}$. First we compute the value $\text{rot} = 2|\epsilon_1(0) - \delta_t(1) - z_i(1)| + 1$. Note that $\text{rot} \in \{1, 3\}$, and, if $Z_i$ rotates clockwise then $\text{rot} = 1$; and if $Z_i$ rotates counterclockwise then $\text{rot} = 3$. If $\text{rot} = 1$, put $D(0) = 1$; otherwise put $D(0) = 3$. If $\text{rot} = 3$, then put $D(1) = 1$; otherwise put $D(1) = 3$. If $\text{rot} = 1$, then put $E(0) = 0$; otherwise put $E(0) = 5 - \text{rot}$. If $\text{rot} = 3$, then put $E(1) = 0$; otherwise put $E(1) = 5 - \text{rot}$. Then we now define $Z_t(s, i, j)$ for $j < 3$ as follows:

$$\text{Box}(s, i, 0) = [x_i + \zeta_{pD(0)} x_i + \zeta_{pD(0) + 2}] \times [y_i + z_i pD(1) + z_i pD(1) + 2];$$

$$\text{Box}(s, i, 1) = [x_i + \zeta_{pE(0)} x_i + \zeta_{pE(0) + \text{rot}}] \times [y_i + z_i pE(1) + z_i pD(1) + 2];$$

$$\text{Box}(s, i, 2) = [x_i + \zeta_{pD(0)} x_i + \zeta_{pD(0) + 2}] \times [y_i + z_i pE(1) + z_i pE(1) + \text{rot} + 2];$$

$$Z_t(s, i, 0) = [e]^* (\text{Box}(s, i, 0));$$

$$Z_t(s, i, 1) = [-]^* (\text{Box}(s, i, 1));$$

$$Z_t(s, i, 2) = [\text{Box}(s, i, 2)].$$
Intuitively, $D(0) = 1$ (resp. $D(0) = 3$) indicates that $Z_t(s, i, 0)$ passes the west (resp. the east) of $Z_i$; $D(1) = 1$ (resp. $D(1) = 3$) indicates that $Z_t(s, i, 0)$ passes the south (resp. the north) of $Z_i$; $E(0) = 0$ (resp. $E(0) = 5 - \text{rot}$) indicates that $Z_t(s, i, 1)$ passes the west (resp. the east) border of the bounding box of $Z_i$; and $E(1) = 0$ (resp. $E(1) = 5 - \text{rot}$) indicates that $Z_t(s, i, 2)$ passes the south (resp. the north) border of the bounding box of $Z_i$. Note that the corner block $Z_t(s, i, 0)$ leaves $Z_i$ on his right, and $Z_t(s, i, 0)$ has the reverse direction to $Z_i$.

**Sublemma 13.** $Z_t(s, i, 2 - \delta_t(0)) \xrightarrow{\leftarrow} Z_t(s, i, 0) \xrightarrow{\leftarrow} Z_t(s, i, 2)$

**Sublemma 14.** $Z_t(s, i, j) \subseteq Z_i$.

For each $i < k_{s+1}$, we have already constructed $Z_t(s + 1; i) = \{Z_t(s, i, j) : j < 3\}$. All of these blocks constructed at the current stage are included in $Z_t^{\text{end}} \cup \bigcup_{p < a \leq s} Z_t(u)$. Let $Z_t^0[i]$ (resp. $Z_t^1[i]$) be the $\prec$-least (resp. the $\prec$-greatest) element of $\{\lambda_t Z_t(s, i, j) : j < 3\}$. It is not hard to see that our construction ensures the following condition.

**Sublemma 15.** $Z_t^1[i] \rightarrow Z_t^0[i + 1]$.

Thus, $\bigcup_{i < k_{s+1}} Z_t(s + 1; i)$ is computably homeomorphic to $P_t \times [0, 1]$, uniformly in $t \geq s + 1$. Therefore, we can connect blocks $Z_t(s, i)$ for $i < k_{s+1}$, and we succeed to return back on the current approximation of the $\prec$-greatest $p$-block $Z_t(p) = Z_t^{\text{pt}} \in Z_t(p)$. Then we construct blocks $Z_t(s, k)$ for $2 \leq k \leq 6$ on the block $Z_t(s)$. The construction is essentially similar as the non-injuring case. By induction hypothesis (IH3), we note that $Z_t(s)$ must be of the following form for some $y_p, z_p \in \mathbb{Q}$:

$$Z_t(p) = [-|p|\gamma_{s+1}^{\min}, \gamma_{s+1}^{\max}] \times [y_p + z_p^{-1}, y_p + z_p^{-1}]$$

On $Z_t(p)$, we define a straight block from $\gamma_{s+1}^{\min}$ to $\gamma_{s+1}^{\max}$ as follows:

$$Z_t(s, 2) = [-|p|\gamma_{s+1}^{\min}, \gamma_{s+1}^{\max}] \times [y_p + z_p, y_p + z_p]$$

Here, by our assumption, $\gamma_{s+1}^{\max} < \gamma_{s+1}^{\max}$ holds since $\gamma_{s+1}^{\max} \leq \gamma_{s+1}^{\max}$. The blocks $Z_t(s, k)$ for $3 \leq k \leq 6$ are defined as in the same method as non-injuring case. The active block at stage $s + 1$ is $Z_{s+1}(s, 5)$, and the end box at stage $s + 1$
Overview of the upside of the frontier \( p \)-block.

The active block \( Z^*_{s+1} \).

Figure 22: Outline of our construction of the injured case.

is \( Z_{s+1}(s,6) \). \((s + 1)\)-blocks at stage \( t \) are \( Z_t(s,i) \) for \( i < 6 \), and \( Z_t(s,i,j) \) for \( i < k_{s+1} \) and \( j < 3 \) if it is constructed. \( Z_t(s + 1) \) denotes the collection of \((s + 1)\)-blocks at stage \( t \).

Sublemma 16. \( Z_{s+1}^{\text{end}} \cup \bigcup Z_{s+1}(s + 1) \subseteq Z_s^{\text{end}} \cup \bigcup_{u \leq s} Z_{s}(u) \).

Thus we again have the following:

\[
Q_{s+1} = Z_{s+1}^{\text{end}} \cup \bigcup_{u \leq s+1} \bigcup Z_{s+1}(u) \subseteq Z_s^{\text{end}} \cup Z_s^{\text{end}} \cup \bigcup_{u \leq s} Z_{s}(u) \subseteq Q_s.
\]

Sublemma 17. Assume that we have a computable function \( f_s : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( f_s \mid \bigcup Z_{s+1}(u) \) is a computable homeomorphism between \( \bigcup Z_{s+1}(u) \) and \( P_t \times [1/(s + 2), 1] \) for any \( t \geq s \). Then we can effectively find a computable function \( f_{s+1} : \mathbb{R}^2 \to \mathbb{R}^2 \) extending \( f_s \mid \bigcup Z_{s+1}(u) \) such that \( f_{s+1} \mid \bigcup Z_{s+1}(u) \) is a computable homeomorphism between \( \bigcup Z_{s+1}(u) \) and \( P_t \times [1/(s + 3), 1] \) for any \( t \geq s + 1 \).

Finally we put \( Q = \bigcap_{n \in \mathbb{N}} Q_s \) and \( Z^* = \bigcup_{u \in \mathbb{N}} Z(u) \). The construction is completed.

Verification. Now we start to verify our construction.

Lemma 15. \( Q \) is \( \Pi^0_4 \).

Sublemma 18. \( \bigcap_{n \in \mathbb{N}} \bigcup Z_{n, Z^*} \bigcap Z_{t} = \bigcup Z_{n, Z^*} \cap \bigcap_{n \in \mathbb{N}} Z_{t} \).

Proof. The intersection \( Z_h(p) \cap Z^i_t \) for \( i < 2 \) is included in some line segment \( L_i = \{0,1] \times \{b\}, \{b\} \times [0,1] : b \in \mathbb{R} \} \), and \( Z_h(p) \cap L_i = Z_h(p) \cap Z^i_t \) holds. \( \square \)

Sublemma 19. \( \bigcup Z_{n, Z^*(u)} \cap \bigcap_{n \in \mathbb{N}} Z_{t} \) is computably homeomorphic to \([0,1] \times P\), for each \( u \in \mathbb{N} \).

Proof. By the induction hypothesis (IH2). \( \square \)

Sublemma 20. \( \bigcup Z_{n, Z^*} \cap \bigcap_{n \in \mathbb{N}} Z_{t} \) is homeomorphic to \((0,1] \times P\).

Proof. By Sublemmas 11 and 17. \( \square \)
Lemma 16. $Q$ is homeomorphic to a Cantor fan.

Proof. By Sublemma [18] there exists a real $y_0 \in \mathbb{R}$ such that the following holds:

$$Q = \left( \bigcup_{Z \in \mathbb{Z}^*} \bigcap_{t \in \mathbb{N}} Z_t \right) \cup \{(\gamma, y_0)\}.$$  

Therefore, by Sublemma [20] $Q$ is homeomorphic to the one-point compactification of $[0, 1] \times P$.  

Lemma 17. $Q$ contains no computable point.

Proof. By Sublemma [19] $\bigcup_{Z \in \mathbb{Z}^*} \bigcap_{t \in \mathbb{N}} Z_t$ contains no computable point.  

By Lemmata [15, 16], and [17] $Q$ is the desired dendroid.

Remark. Since dendroids are compact and simply connected, Theorem [13] is the solution to the question of Le Roux and Ziegler [13]. Indeed, the dendroid constructed in the proof of Theorem [13] is contractible.

Corollary 4. Not every nonempty contractible $\Pi^0_1$ subset of $[0, 1]^2$ contains a computable point.

Question 18. Does every locally connected planar $\Pi^0_1$ set contain a computable point?

5 Immediate Consequences

5.1 Effective Hausdorff Dimension

For basic definition and properties of the effective Hausdorff dimension of a point of Euclidean plane, see Lutz-Weihrauch [9]. For any $I \subseteq [0, 2]$, let $\text{DIM}^I$ denote the set of all points in $\mathbb{R}^2$ whose effective Hausdorff dimensions lie in $I$. Lutz-Weihrauch [9] showed that $\text{DIM}^{[1, 2]}$ is path-connected, but $\text{DIM}^{(1, 2)}$ is totally disconnected. In particular, $\text{DIM}^{[1, 2]}$ has no nondegenerate connected subset. It is easy to see that $\text{DIM}^{(0, 2)}$ has no nonempty $\Pi^0_1$ simple curve, since every $\Pi^0_1$ simple curve contains a computable point, and the effective Hausdorff dimension of each computable point is zero.

Theorem 19. $\text{DIM}^{[1, 2]}$ has a nondegenerate contractible $\Pi^0_1$ subset.

Proof. For any strictly increasing computable function $f : \mathbb{N} \to \mathbb{N}$ with $f(0) = 0$ and any infinite binary sequence $\alpha \in 2^{\mathbb{N}}$, put $\iota_f(\alpha) = \prod_{i \in \mathbb{N}} (\alpha(i), \alpha(f(i)), \alpha(f(i) + 1), \ldots, \alpha(f(i + 1) - 1))$, where $\sigma \times \tau$ denotes the concatenation of binary strings $\sigma$ and $\tau$. Then, $r : 2^{\mathbb{N}} \to \mathbb{R}$ is defined as $r(\alpha) = \sum_{i \in \mathbb{N}} (\alpha(i) \cdot 2^{-i(i+1)})$. Note that $\alpha \neq \beta$ and $r(\alpha) = r(\beta)$ hold if and only if there is a common initial segment $\sigma \in 2^{\mathbb{N}}$ of $\alpha$ and $\beta$ such that $\sigma 0$ and $\sigma 1$ are initial segments of $\alpha$ and $\beta$ respectively, and that $\alpha(m) = 1$ and $\beta(m) = 0$ for any $m > \text{lh}(\sigma)$, where $\text{lh}(\sigma)$ denotes the length of $\sigma$. In this case, we say that $\alpha$ sticks to $\beta$ on $\sigma$. If $r(\alpha) \neq r(\beta)$, then clearly $r \circ \iota_f(\alpha) \neq r \circ \iota_f(\beta)$. Assume that $\alpha$ sticks to $\beta$ on $\sigma$. Then there are $m_0 < m_1$ such that $\iota_f(\alpha)(m_0) = \iota_f(\alpha)(m_1) = \alpha(\text{lh}(\sigma)) = 0$ and $\iota_f(\beta)(m_0) = \iota_f(\beta)(m_1) = \beta(\text{lh}(\sigma)) = 1$ by our definition of $\iota_f$. Therefore,
\[i_f(\alpha)\text{ does not stick to }i_f(\beta).\] Hence, \(r \circ i_f(\alpha) \neq r \circ i_f(\beta)\) whenever \(\alpha \neq \beta\). Actually, \(r \circ i_f : 2^\mathbb{N} \to \mathbb{R}\) is a computable embedding. For each \(n \in \mathbb{N}\), put \(k_f(n) = \#\{s : f(s) < n\}\). Then, there is a constant \(c \in \mathbb{N}\) such that, for any \(\alpha \in 2^\mathbb{N}\) and \(n \in \mathbb{N}\), we have \(K(i_f(\alpha) \upharpoonright n + k_f(n) + 1) \geq K(\alpha \upharpoonright n) - c\), where \(K\) denotes the prefix-free Kolmogorov complexity. Therefore, for any sufficiently fast-growing function \(f : \mathbb{N} \to \mathbb{N}\) and any Martin-Löf random sequence \(\alpha \in 2^\mathbb{N}\), the effective Hausdorff dimension of \(r \circ i_f(\alpha)\) must be 1. Thus, for any nonempty \(\Pi^0_1\) set \(R \subseteq 2^\mathbb{N}\) consisting of Martin-Löf random sequences, \(\{0\} \times (r \circ i_f(R))\) is a \(\Pi^0_1\) subset of \(\text{DIM}^{[1]}\). Let \(Q\) be the dendroid constructed from \(P = r \circ i_f(R)\) as in the proof of Theorem 13 where we choose \(\gamma = \rho(B)\) as Chaitin’s halting probability \(\Omega\). For every point \((x_0, x_1) \in Q\), the effective Hausdorff dimension of \(x_i\) for some \(i < 2\) is equivalent to that of an element of \(P\) or that of \(\Omega\). Consequently, \(Q \subseteq \text{DIM}^{[1, 2]}\).

5.2 Reverse Mathematics

Theorem 20. For every \(\Pi^0_1\) set \(P \subseteq 2^\mathbb{N}\), there is a contractible planar \(\Pi^0_1\) set \(Q\) such that \(Q\) is Turing-degree-equivalent to \(P\), i.e., \(\{\deg_T(x) : x \in P\} = \{\deg_T(x) : x \in D\}\).

Proof. We choose \(B\) as a c.e. set of the same degree with the leftmost path of \(P\). Then, the dendroid \(Q\) constructed from \(P\) and \(B\) as in the proof of Theorem 13 is the desired one.

A compact \(\Pi^0_1\) subset \(P\) of a computable topological space is Muchnik complete if every element of \(P\) computes the set of all theorems of \(T\) for some consistent complete theory \(T\) containing Peano arithmetic. By Scott Basis Theorem (see Simpson 15), \(P\) is Muchnik complete if and only if \(P\) is nonempty and every element of \(P\) computes an element of any nonempty \(\Pi^0_1\) set \(Q \subseteq 2^\mathbb{N}\).

Corollary 5. There is a Muchnik complete contractible planar \(\Pi^0_1\) set.

A compact \(\Pi^0_1\) subset \(P\) of a computable topological space is Medvedev complete (see also Simpson 15) if there is a uniform computable procedure \(\Phi\) such that, for any name \(x \in \mathbb{N}^\mathbb{N}\) of an element of \(P\), \(\Phi(x)\) is the set of all theorems of \(T\) for some consistent complete theory \(T\) containing Peano arithmetic.

Question 21. Does there exist a Medvedev complete simply connected planar \(\Pi^0_1\) set? Does there exist a Medvedev complete contractible Euclidean \(\Pi^0_1\) set?

Our Theorem 13 also provides a reverse mathematical consequence. For basic notation for Reverse Mathematics, see Simpson 14. Let \(\text{RCA}_0\) denote the subsystem of second order arithmetic consisting of \(\Sigma^0_1\) (Robinson arithmetic with induction for \(\Sigma^0_1\) formulas) and \(\Delta^0_1\)-CA (comprehension for \(\Delta^0_1\) formulas). Over \(\text{RCA}_0\), we say that a sequence \((B_i)_{i \in \mathbb{N}}\) of open rational balls is flat if there is a homeomorphism between \(\bigcup_{i < n} B_i\) and the open square \((0, 1)^2\) for any \(n \in \mathbb{N}\). It is easy to see that \(\text{RCA}_0\) proves that every flat cover of \([0, 1]\) has a finite subcover.

Theorem 22. The following are equivalent over \(\text{RCA}_0\).

1. Weak König’s Lemma: every infinite binary tree has an infinite path.
2. Every open cover of \([0, 1]\) has a finite subcover.

3. Every flat open cover of \([0, 1]^2\) has a finite subcover.

\textbf{Proof.} The equivalence of the item (1) and (2) is well-known. It is not hard to see that \(\text{RCA}_0\) proves the existence of the sequence \(\{Q_s\}_{s \in \mathbb{N}}\) as in our construction of the dendroid \(Q\) in Theorem \([13]\) by formalizing our proof in Theorem \([13]\) in \(\text{RCA}_0\). Here we may assume that \(\{Q_s\}_{s \in \mathbb{N}}\) is constructed from the set of all infinite paths of a given infinite binary tree \(T \subseteq 2^{<\mathbb{N}}\), and a c.e. complete set \(B \subseteq \mathbb{N}\). Note that \(\bigcup_{s<t}([0,1]^2 \setminus Q_s)\) does not cover \([0,1]^2\) for every \(t \in \mathbb{N}\). Over \(\text{RCA}_0\), there is a flat sequence \(\{[0,1]^2 \setminus Q_s^\ast\}_{s \in \mathbb{N}}\) of open rational balls such that \(\bigcap_{s<t} Q_s^\ast \supseteq \bigcap_{s<t} Q_s\) for any \(t \in \mathbb{N}\), and that an open rational ball \(U\) is removed from some \(Q_s^\ast\) if and only if an open rational ball \(U\) is removed from some \(Q_u\). However, if \(T\) has no infinite path, then \(Q\) has no element. In other words, \(\{[0,1]^2 \setminus Q_s^\ast\}_{s \in \mathbb{N}}\) covers \([0,1]^2\).

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