The universal algebra generated by a power partial isometry

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Dedicated to Albrecht Böttcher on the occasion of his 60th birthday.

Abstract

A power partial isometry (PPI) is an element \(v\) of a \(C^*\)-algebra with the property that every power \(v^n\) is a partial isometry. The goal of this paper is to identify the universal \(C^*\)-algebra generated by a PPI with (a slight modification of) the algebra of the finite sections method for Toeplitz operators with continuous generating function, as first described by Albrecht Böttcher and Bernd Silbermann in [1].

Keywords: power partial isometry, universal algebra, finite sections algebra

2010 AMS-MSC: 46L05, 47B35, 65R20

1 Introduction

Let \(A\) be a \(C^*\)-algebra. An element \(V\) of \(A\) is called a partial isometry if \(vv^*v = v\). Simple examples show that a power of a partial isometry needs not to be a partial isometry again. One therefore calls \(v\) a power partial isometry (PPI) if every power of \(v\) is a partial isometry again.

Examples. (a) In a \(C^*\)-algebra with identity element \(e\), every unitary element \(u\) (i.e. \(u^*u = uu^* = e\)) is a PPI. In particular, the function \(u : t \mapsto t\) is a unitary element of the algebra \(C(\mathbb{T})\) of the continuous functions on the complex unit circle \(\mathbb{T}\), and the operator \(U\) of multiplication by the function \(u\) is a unitary operator on the Hilbert space \(L^2(\mathbb{T})\) of the squared integrable functions on \(\mathbb{T}\).

(b) In a \(C^*\)-algebra with identity element \(e\), every isometry \(v\) (i.e. \(v^*v = e\)) and every co-isometry \(v\) (i.e. \(vv^* = e\)) is a PPI. In particular, the operators \(V : (x_0, x_1, \ldots) \mapsto (0, x_0, x_1, \ldots)\) and \(V^* : (x_0, x_1, x_2, \ldots) \mapsto (x_1, x_2, \ldots)\) of forward and backward shift, respectively, are PPIs on the Hilbert space \(l^2(\mathbb{Z}^+)\) of the squared summable sequences on the non-negative integers.

(c) The matrix \(V_n := (a_{ij})\) with \(a_{i+1,i} = 1\) and \(a_{ij} = 0\) if \(i \neq j + 1\), considered as
an element of the algebra \( \mathbb{C}^{n \times n} \) of the complex \( n \times n \) matrices, is a PPI.

(d) If \( v_i \) is a PPI in a \( C^* \)-algebra \( \mathcal{A}_i \) for every \( i \) in an index set \( I \), then \((v_i)_{i \in I}\) is a PPI in the direct product \( \prod_{i \in I} \mathcal{A}_i \). In particular, the operator \((V, V^*)\), considered as an element of \( L(l^2(\mathbb{Z}^+)) \times L(l^2(\mathbb{Z}^+)) \), is a PPI.

Note that the PPI \( V_n \) in (c) and \((V, V^*)\) in (d) are neither isometric nor co-isometric.

The goal of the present paper is to describe the universal \( C^* \)-algebra generated by a PPI. Recall that a \( C^* \)-algebra \( \mathcal{A} \) generated by a PPI \( v \) is universal if, for every other \( C^* \)-algebra \( \mathcal{B} \) generated by a PPI \( w \), there is a \( * \)-homomorphism from \( \mathcal{A} \) to \( \mathcal{B} \) which sends \( v \) to \( w \). The universal algebra generated by a unitary resp. isometric element is defined in an analogous way. The existence of a universal algebra generated by a PPI is basically a consequence of Example (d).

It follows from the Gelfand-Naimark theorem that the universal algebra generated by a unitary element is \( * \)-isomorphic to the algebra \( C(\mathbb{T}) \), generated by the unitary function \( u \). Coburn [5] identified the universal algebra generated by an isometry as the Toeplitz algebra \( T(C) \) which is the smallest \( C^* \)-subalgebra of \( L(l^2(\mathbb{Z}^+)) \) which contains the isometry \( V \), the shift operator. This algebra bears its name since it can be described as the smallest \( C^* \)-subalgebra of \( L(l^2(\mathbb{Z}^+)) \) which contains all Toeplitz operators \( T(a) \) with generating function \( a \in C(\mathbb{T}) \). Recall that the Toeplitz operator with generating function \( a \in L^1(\mathbb{T}) \) is given the matrix \((a_{i-j})_{i,j=0}^\infty \) where \( a_k \) stands for the \( k \)th Fourier coefficient of \( a \). This operator is bounded on \( l^2(\mathbb{Z}^+) \) if and only if \( a \in L^\infty(\mathbb{T}) \) (see [2, 3]).

We will see that the universal algebra of a PPI is also related with Toeplitz operators, via the finite sections discretization with respect to the sequence of the projections \( P_n : (x_0, x_1, \ldots) \mapsto (x_0, \ldots, x_{n-1}, 0, 0, \ldots) \) on \( l^2(\mathbb{Z}^+) \). Write \( \mathcal{F} \) for the set of all bounded sequences \((A_n)_{n \geq 1}\) of operators \( A_n \in \text{L}(\text{im} P_n) \) and \( \mathcal{G} \) for the set of all sequences \((A_n) \in \mathcal{F} \) with \( \|A_n\| \to 0 \). Provided with entry-wise defined operations and the supremum norm, \( \mathcal{F} \) becomes a \( C^* \)-algebra and \( \mathcal{G} \) a closed ideal of \( \mathcal{F} \). Since \( \text{L}(\text{im} P_n) \) is isomorphic to \( \mathbb{C}^{n \times n} \), we can identify \( \mathcal{F} \) with the direct product and \( \mathcal{G} \) with the direct sum of the algebras \( \mathbb{C}^{n \times n} \) for \( n \geq 1 \). Now consider the smallest \( C^* \)-subalgebra \( \mathcal{S}(T(C)) \) of \( \mathcal{F} \) which contains all sequences \((P_n T(a) P_n)\) with \( a \in C(\mathbb{T}) \) and its \( C^* \)-subalgebra \( \mathcal{S}_{\geq 2}(T(C)) \) which is generated by the sequence \((P_n V P_n)\) (note that \( V \) is the Toeplitz operator with generating function \( t \mapsto t \)). With these notations, the main result of the present paper can be formulated as follows.

**Theorem 1** The universal algebra generated by a PPI is \( * \)-isomorphic to the \( C^* \)-algebra \( \mathcal{S}_{\geq 2}(T(C)) \) generated by the PPI \((P_n V P_n)\).

For a general account on \( C^* \)-algebras generated by partial isometries, with special emphasis on their relation to graph theory, see [4].

Before going into the details of the proof of Theorem 1 we provide some
basic (and well known) facts on the algebras $S(T(C))$ and $S_{\geq 2}(T(C))$. Since the first entry of the sequence $(P_n V P_n)$ is zero, the first entry of every sequence in $S_{\geq 2}(T(C))$ is zero. So we can omit the first entry and consider the elements of $S_{\geq 2}(T(C))$ as sequences labeled by $n \geq 2$ (whence the notation). In fact this is the only difference between the algebras $S(T(C))$ and $S_{\geq 2}(T(C))$.

**Proposition 2** $S_{\geq 2}(T(C))$ consists of all sequences $(A_n)_{n \geq 2}$ where $(A_n)_{n \geq 1}$ is a sequence in $S(T(C))$.

The sequences in $S(T(C))$ are completely described in the following theorem, where we let $R_n$ denote the operator $(x_0, x_1, \ldots) \mapsto (x_{n-1}, \ldots, x_0, 0, 0, \ldots)$ on $l^2(\mathbb{Z}^+)$. Further we set $\tilde{a}(t) := a(t^{-1})$ for every function $a$ on $\mathbb{T}$. This description was found by A. Böttcher and B. Silbermann and first published in their 1983 paper [1] on the convergence of the finite sections method for quarter plane Toeplitz operators (see also [6], Section 1.4.2).

**Proposition 3** The algebra $S(T(C))$ consists of all sequences $(A_n)_{n \geq 1}$ of the form

$$(A_n) = (P_n T(a) P_n + P_n KP_n + R_n LR_n + G_n)$$

(1)

where $a \in C(\mathbb{T})$, $K$ and $L$ are compact operators, and $(G_n) \in \mathcal{G}$. The representation of a sequence $(A_n) \in S(T(C))$ in this form is unique.

**Corollary 4** $\mathcal{G}$ is a closed ideal of $S(T(C))$, and the quotient algebra $S(T(C))/\mathcal{G}$ is *-isomorphic to the $C^*$-algebra of all pairs

$$(T(a) + K, T(\tilde{a}) + L) \in L(l^2(\mathbb{Z}^+)) \times L(l^2(\mathbb{Z}^+))$$

(2)

with $a \in C(\mathbb{T})$ and $K, L$ compact. In particular, the mapping which sends the sequence (1) to the pair (2) is a *-homomorphism from $S(T(C))$ onto $S(T(C))/\mathcal{G}$ with kernel $\mathcal{G}$.

It is not hard to see that the algebra of all pairs (2) is just the smallest $C^*$-subalgebra of $L(l^2(\mathbb{Z}^+)) \times L(l^2(\mathbb{Z}^+))$ that contains the PPI $(V, V^*)$.

**Corollary 5** The set $\mathcal{J}$ of all pairs $(K, L)$ of compact operators $K$, $L$ is a closed ideal of $S(T(C))/\mathcal{G}$. The quotient algebra $(S(T(C))/\mathcal{G})/\mathcal{J}$ is *-isomorphic to $C(\mathbb{T})$. In particular, the mapping which sends the pair (2) to the function $a$ is a *-homomorphism from $S(T(C))/\mathcal{G}$ onto $C(\mathbb{T})$ with kernel $\mathcal{J}$.

Observe that all of the above examples (a) - (d) appear somewhere in the algebra $S(T(C))$ and its quotients.
2 Elementary properties of PPI

Our first goal is a condition ensuring that the product of two partial isometries is a partial isometry again.

**Proposition 6** Let $u$, $v$ be partial isometries. Then $uv$ is a partial isometry if and only if
\[ u^*uv^* = vv^*u^*u, \] i.e. if the initial projection $u^*u$ of $u$ and the range projection $vv^*$ of $v$ commute.

**Proof.** Condition (3) implies that $(uv)(uv)^*(uv) = uv^*u^*uv = (uu^*u)(vv^*v) = uv$;

hence, $uv$ is a partial isometry. Conversely, if $uv$ is a partial isometry, then a simple calculation gives
\[ v^*(vv^*u^*u - u^*uvv^*)(u^*uvv^* - vv^*u^*u)v = 0. \]

With the $C^*$-axiom we conclude that $v^*(vv^*u^*u - u^*uvv^*) = 0$, hence $vv^*(vv^*u^*u - u^*uvv^*) = 0$, which finally gives
\[ vv^*u^*u = vv^*u^*uvv^*. \]

The right-hand side of this equality is selfadjoint; so must be the left-hand side. Thus, $vv^*u^*u = (vv^*u^*u)^* = u^*uvv^*$, which is condition (3).

In particular, if $v$ is a partial isometry, then $v^2$ is a partial isometry if and only if
\[ v^*vv^* = vv^*v^*v. \] (4)

**Proposition 7** Let $v$ be a partial isometry with property (1) (e.g. a PPI). Then
\[ e := v^*v + vv^* - v^*vv^* = v^*v + vv^* - vv^*v^*v \]
is the identity element of the $C^*$-algebra generated by $v$. Moreover,
\[ p := vv^* - vv^*v^*v = e - v^*v \] and \[ \tilde{p} := v^*v - v^*vvv^* = e - vv^* \]
are mutually orthogonal projections (meaning that $p\tilde{p} = \tilde{p}p = 0$).

**Proof.** Condition (4) implies that $e$ is selfadjoint. Further,
\[ ve = vv^*v + vvv^* - vv^*vvv^* = v + vv^* - vv^* = v \]
and, similarly, $v^*e = v^*$. Taking adjoints it follows that $ev^* = v^*$ and $ev = v$, and $e$ is the identity element. The remaining assertions are also easy to check.

We will often use the notation $v^n$ instead of $(v^*)^n$.
Proposition 8  (a) If $v$ is a PPI, then
\[ v^k v^n v^* = v^n v^* v^k v^k \] for all $k, n \geq 1$.  \hspace{1cm} (5)

(b) If $v$ is a partial isometry and if (5) holds for $k = 1$ and for every $n \geq 1$, then $v$ is a PPI.

Proof. Assertion (a) is a consequence of Proposition 6 (the partial isometry $v^{n+k}$ is the product of the partial isometries $v^k$ and $v^n$). Assertion (b) follows easily by induction. For $k = 1$, condition (5) reduces to
\[ (v^* v)(v^n v^*) = (v^n v^*) (v^* v) \]
Thus if $v$ and $v^n$ are partial isometries, then $v^{n+1}$ is a partial isometry by Proposition 6.

Lemma 9  If $v$ is a PPI, then $(v^n v^*)_{n \geq 0}$ and $(v^* v^n)_{n \geq 0}$ are decreasing sequences of pairwise commuting projections.

Proof. The PPI property implies that the $v^n v^*$ are projections and that
\[ v^n v^* v^{n+k} (v^*)^{n+k} = (v^n v^* v^n) v^k (v^*)^{n+k} = v^n v^k (v^*)^{n+k} = n^{n+k} (v^*)^{n+k} \]
for $k, n \geq 0$. The assertions for the second sequence follow similarly.

3  A distinguished ideal

Let $\mathcal{A}$ be a $C^*$-algebra generated by a PPI $v$. By $\text{alg}(v, v^*)$ we denote the smallest (symmetric, not necessarily closed) subalgebra of $\mathcal{A}$ which contains $v$ and $v^*$. Further we write $\mathbb{N}_v$ for the set of all non-negative integers such that $pv^n \tilde{p} \neq 0$. From Proposition 7 we know that $0 \notin \mathbb{N}_v$. Finally, we set
\[ \pi_n := p v^n \tilde{p} v^n \tilde{p} \quad \text{and} \quad \tilde{\pi}_n := \tilde{p} v^n \tilde{p} v^n \tilde{p}. \]

Proposition 10  (a) The element $pv^n \tilde{p}$ is a partial isometry with initial projection $\tilde{\pi}_n$ and range projection $\pi_n$. Thus, the projections $\pi_n$ and $\tilde{\pi}_n$ are Murray-von Neumann equivalent in $\mathcal{A}$, and they generate the same ideal of $\mathcal{A}$. 

(b) $\pi_m \pi_n = 0$ and $\tilde{\pi}_m \tilde{\pi}_n = 0$ whenever $m \neq n$.

Proof. (a) By definition,
\[ \pi_n = p v^n \tilde{p} v^n \tilde{p} = p v^n (e - vv^*) v^n \tilde{p} = p(v^n v^* v^n - v^{n+1} (v^*)^{n+1}) \tilde{p}. \]
Since $p = e - vv^*$ and $v^n v^*$ commute by Proposition 8,
\[ \pi_n = p(v^n v^* v^n - v^{n+1} (v^*)^{n+1}) = (v^n v^* v^n - v^{n+1} (v^*)^{n+1}) \tilde{p}. \]
Being a product of commuting projections (Lemma 9), \( \pi_n \) is itself a projection. Analogously, \( \bar{\pi}_n \) is a projection. Thus, \( p v^n \bar{p} \) is a partial isometry, and \( \pi_n \) and \( \bar{\pi}_n \) are Murray-von Neumann equivalent. Finally, the equality
\[
\pi_n = \bar{\pi}_n = (p v^n \bar{p} v^n \bar{p})^2 = p v^n \bar{p} v^n p
\]
shows that \( \pi_n \) belongs to the ideal generated by \( \bar{\pi}_n \). The reverse inclusion follows analogously. Assertion (b) is again a simple consequence of Lemma 9.

Let \( C_n \) denote the smallest closed ideal of \( A \) which contains the projection \( \pi_n \) (like-wise, the projection \( \bar{\pi}_n \)). We want to show that \( C_n \) is isomorphic to \( C((n+1) \times (n+1)) \) whenever \( n \in \mathbb{N}_0 \) (Proposition 8 below). For we need to establish a couple of facts on (finite) words in \( \text{alg}(v, v^*) \).

**Lemma 11** Let \( a, b, c \) be non-negative integers. Then
\[
v^{a}v^{b}v^{c} = \begin{cases} 
(v^*)^{a-b+c} & \text{if } \min\{a, c\} \geq b, \\
v^{b-a}v^{c} & \text{if } a \leq b \leq c, \\
v^{a}v^{b-c} & \text{if } a \geq b \geq c 
\end{cases}
\]
and
\[
v^{a}v^{b}v^{c} = \begin{cases} 
v^{a-b+c} & \text{if } \min\{a, c\} \geq b, \\
v^{a}v^{b} & \text{if } a \geq b \geq c, \\
(v^*)^{b-a}v^{c} & \text{if } a \leq b \leq c. 
\end{cases}
\]

**Proof.** Let \( \min\{a, c\} \geq b \). Then
\[
v^{a}v^{b}v^{c} = (v^*)^{a-b}v^{a-b}v^{c} = (v^*)^{a-b}v^{c} = (v^*)^{a-b+c},
\]
where we used that \( v^{a} \) is a partial isometry. If \( a \leq b \leq c \), then
\[
v^{a}v^{b}v^{c} = v^{a}v^{a-b-a}(v^*)^{b-a}v^{c-b-a} = v^{b-a}(v^*)^{b-a}v^{a}(v^*)^{c-b-a}
\]
by Proposition 8 (a). Thus,
\[
v^{a}v^{b}v^{c} = v^{b-a}(v^*)^{b-a}v^{a}(v^*)^{c-b-a} = v^{b-a}(v^*)^{b-a}v^{a}(v^*)^{c-b-a} = v^{b-a}v^{c}.
\]
Similarly, \( v^{a}v^{b}v^{c} = v^{a}v^{b-c} \) if \( a \geq b \geq c \). The second assertion of the lemma follows by taking adjoints.

Every word in \( \text{alg}(v, v^*) \) is a product of powers \( v^n \) and \( v^m \). Every product \( v^{a}v^{b}v^{c} \) and \( v^{a}v^{b}v^{c} \) of three powers can be written as a product of at most two powers if one of the conditions
\[
\min\{a, c\} \geq b \quad \text{or} \quad a \leq b \leq c \quad \text{or} \quad a \geq b \geq c
\]
(6)
in Lemma 11 is satisfied. Since (6) is equivalent to \( \max\{a, c\} \geq b \), such a product can not be written as a product of less than three powers by means of Lemma 11 if
max\{a, c\} < b. Since it is not possible in a product \(v^av^{*b}c^dv^d\) or \(v^av^{*b}c^dv^d\) of four powers that \(\max\{a, c\} < b\) and \(\max\{b, d\} < c\), one can shorten every product of powers \(v^n\) and \(v^m\) to a product of at most three powers. Summarizing we get the following lemma.

**Lemma 12** Every finite word in \(\text{alg}(v, v^*)\) is of the form \(v^av^{*b}\) or \(v^{*b}v^a\) with \(a, b \geq 0\) or of the form \(v^av^{*b}v^c\) or \(v^sv^{*b}v^c\) with \(0 < \min\{a, c\} \leq \max\{a, c\} < b\).

**Corollary 13** Let \(w\) be a word in \(\text{alg}(v, v^*)\).
1. If \(pwvp \neq 0\), then \(w = v^av^{*a}\) for some \(a \geq 0\).
2. If \(\tilde{p}w\tilde{w} \neq 0\), then \(w = v^av^{*a}\) with some \(a \geq 0\).

**Proof.** We only check assertion (1). By the preceding lemma, \(w\) is a product of at most three powers \(v^av^{*b}v^c\) or \(v^sv^{*b}v^c\). First let \(w = v^av^{*b}v^c\). Since \(vp = pv^* = 0\), we conclude that \(c = 0\) if \(pwwp \neq 0\). Writing

\[
pwwp = \begin{cases} 
    pv^av^{*a}(v^*)^{b-a}p = v^av^{*a}p(v^*)^{b-a}p & \text{if } a \leq b, \\
    pv^{a-b}v^bv^{*b}p = pv^{a-b}pv^b v^{*b} & \text{if } a \geq b,
\end{cases}
\]

we obtain by the same argument that \(a = b\) if \(pwwp \neq 0\). Thus, \(w = v^av^{*a}\). The case when \(w = v^sv^{*b}v^c\) can be treated analogously.

An element \(k\) of a \(C^*-\)algebra \(A\) is called an **element of algebraic rank one** if, for every \(a \in A\), there is a complex number \(\alpha\) such that \(kak = \alpha k\).

**Proposition 14** Let \(m, n \in \mathbb{N}_\nu\). Then
1. \(\pi_n\) is a projection of algebraic rank one in \(A\).
2. \(\pi_m\) and \(\pi_n\) are Murray-von Neumann equivalent if and only if \(m = n\).

Analogous assertions hold for \(\tilde{\pi}_n\) in place of \(\pi_n\).

**Proof.** (1) Every element of \(A\) is a limit of linear combinations of words in \(v\) and \(v^*\). It is thus sufficient to show that, for every word \(w\), there is an \(\alpha \in \mathbb{C}\) such that \(\pi_n w \pi_n = \alpha \pi_n\). If \(\pi_n w \pi_n = 0\), this holds with \(\alpha = 0\). If \(\pi_n w \pi_n = \pi_n pwwp \pi_n \neq 0\), then \(w = v^av^{*a}\) for some \(a \geq 0\) by Corollary 13. In this case,

\[
\pi_n w \pi_n = \pi_n v^av^{*a} \pi_n = p(v^n v^{*n} - v^{n+1}(v^*)^{n+1}) v^av^{*a}(v^n v^{*n} - v^{n+1}(v^*)^{n+1}) p.
\]

From Lemma 7 we infer that

\[
(v^n v^{*n} - v^{n+1}(v^*)^{n+1}) v^av^{*a} = \begin{cases} 
    v^n v^{*n} - v^{n+1}(v^*)^{n+1} & \text{if } a \leq n, \\
    v^av^{*a} - v^av^{*a} = 0 & \text{if } a \geq n + 1.
\end{cases}
\]

Thus,

\[
\pi_n w \pi_n = \begin{cases} 
    p(v^n v^{*n} - v^{n+1}(v^*)^{n+1})^2 p = \pi_n & \text{if } a \leq n, \\
    0 & \text{if } a \geq n + 1,
\end{cases}
\]
In particular, \( C \) for some (Proposition 17) describe this algebra exactly. For every \( \pi \in \mathcal{A} \) since \( \pi \) elements commute by Lemma 9. Since (Corollary 16) for some \( a \geq 0 \) by Corollary \[13\] the terms in parentheses in

\[
\pi_m w \pi_n = \pi_m v^a v^{*a} \pi_n = p(v^m v^{*m} - v^{m+1}(v^*)^{m+1})(v^a v^{*a})(v^n v^{*n} - v^{n+1}(v^*)^{n+1})p
\]

commute by Lemma \[9\]. Since

\[
(v^m v^{*m} - v^{m+1}(v^*)^{m+1})(v^n v^{*n} - v^{n+1}(v^*)^{n+1}) = 0
\]

for \( m \neq n \) we conclude that \( \pi_m w \pi_n = 0 \), a contradiction. 

**Lemma 15**  
(a) If \( a > n \) or \( b > a \), then \( v^b v^{*a} \pi_n = 0 \).
(b) If \( b \leq a \leq n \), then \( v^b v^{*a} \pi_n = (v^*)^{a-b} \pi_n \).

**Proof.**  
(a) One easily checks that \( (v^*)^{n+1}p = 0 \), which gives the first assertion. Let \( b > a \). Then, since \( p \) commutes with \( v^k v^k \) and \( vp = 0 \),

\[
v^b v^{*a} \pi_n = v^{b-a} v^{a} v^{*a} (v^n v^{*n} - v^{n+1}(v^*)^{n+1})p = v^{b-a} p v^a v^{*a} (v^n v^{*n} - v^{n+1}(v^*)^{n+1}) = 0.
\]

(b) Applying Lemma \[11\] to the terms in inner parentheses in

\[
v^b v^{*a} \pi_n = ((v^b v^{*a} v^n) v^{*n} - (v^b v^{*a} v^{n+1})(v^*)^{n+1})p,
\]

one can simplify this expression to

\[
((v^*)^{a-b} v^n v^{*n} - (v^*)^{a-b} v^{n+1} v^{*n+1})(v^*)^{n+1}p = (v^*)^{a-b} \pi_n.
\]

**Corollary 16**  
(a) If \( w \) is a word in \( v, v^* \), then \( w \pi_n \in \{0, \pi_n, v^* \pi_n, \ldots, v^{*n} \pi_n\} \).

(b) For every \( w \in \mathcal{A} \), \( w \pi_n \) is a linear combination of elements \( v^i \pi_n \) with \( i \in \{0, 1, \ldots, n\} \).

(c) Every element of the ideal \( \mathcal{C}_n \) generated by \( \pi_n \) is a linear combination of elements \( v^i \pi_n v^j \) with \( i, j \in \{0, 1, \ldots, n\} \).

In particular, \( \mathcal{C}_n \) is a finite-dimensional \( C^* \)-algebra. We are now in a position to describe this algebra exactly.

**Proposition 17**  
(a) For \( n \in \mathbb{N}_0 \), the algebra \( \mathcal{C}_n \) is \( * \)-isomorphic to \( \mathbb{C}^{(n+1) \times (n+1)} \).

(b) \( \mathcal{C}_m \mathcal{C}_n = \{0\} \) whenever \( m \neq n \).
Proof. (a) The elements \( e_{ij}^{(n)} := v^* \pi_n v^j \) with \( i, j \in \{0, 1, \ldots n\} \) span the algebra \( \mathcal{C}_n \) by Corollary 16 (c). Thus, the assertion will follow once we have shown that these elements form a system of \((n + 1) \times (n + 1)\) matrix units in the sense that
\[
(e_{ij}^{(n)})^* = e_{ji}^{(n)} \quad \text{and} \quad e_{ij}^{(n)} e_{kl}^{(n)} = \delta_{jk} e_{il}^{(n)}
\]
for all \( i, j, k, l \in \{0, 1, \ldots n\} \), (7) with \( \delta_{jk} \) the standard Kronecker delta. The symmetry property is clear. To check (7), first let \( j = k \). Then
\[
e_{ij}^{(n)} e_{jl}^{(n)} = v^* \pi_n (v^j v^* j \pi_n) v^l = v^* \pi_n^2 v^l = e_{il}^{(n)}
\]
by Lemma 15 (b). If \( j > k \), then
\[
e_{ij}^{(n)} e_{kl}^{(n)} = v^* \pi_n (v^j v^* k \pi_n) v^l = 0
\]
by Lemma 15 (a). Finally, if \( j < k \), then
\[
e_{ij}^{(n)} e_{kl}^{(n)} = v^* (\pi_n v^j \pi_n) v^l = v^* (v^k v^* j \pi_n)^* \pi_n v^l = 0,
\]
again by Lemma 15 (a). This proves (a). Assertion (b) follows from Proposition 14 (b).

Given a PPI \( v \), we let \( \mathcal{G}_v \) stand for the smallest closed ideal which contains all projections \( \pi_n \). If \( N_v \) is empty, then \( \mathcal{G}_v \) is the zero ideal. Let \( N_v \neq \emptyset \). The ideal generated by a projection \( \pi_n \) with \( n \in N_v \) is isomorphic to \( \mathbb{C}^{(n+1) \times (n+1)} \) by Proposition 17 and if \( u, w \) are elements of \( \mathcal{A} \) which belong to ideals generated by two different projections \( \pi_m \) and \( \pi_n \), then \( uw = 0 \) by Proposition 14 (b). Hence, \( \mathcal{G}_v \) is then isomorphic to the direct sum of all matrix algebras \( \mathbb{C}^{(n+1) \times (n+1)} \) with \( n \in N_v \).

If \( \mathcal{A} \) is the universal \( C^* \)-algebra generated by a PPI \( v \), then \( N_v \) is the set of all positive integers. Indeed, the algebra \( S_{\geq 2}(\mathbb{T}(C)) \) introduced in the introduction is generated by the PPI \( v := (P_n V P_n) \), and \( N_v = \mathbb{N} \) in this concrete setting.

**Corollary 18** If \( \mathcal{A} \) is the universal \( C^* \)-algebra generated by a PPI \( v \), then \( N_v = \mathbb{N} \), and \( \mathcal{G}_v \) is isomorphic to the ideal \( \mathcal{G}_{\geq 2} := S_{\geq 2}(\mathbb{T}(C)) \cap \mathcal{G} \).

4 **PPI with \( N_v = \emptyset \)**

Our next goal is to describe the \( C^* \)-algebra \( \mathcal{A} \) which is generated by a PPI \( v \) with \( N_v = \emptyset \). This condition is evidently satisfied if one of the projections \( p = e - v^* v \) and \( \bar{p} = e - vv^* \) is zero, in which cases the algebra generated by the PPI \( v \) is well known:
• If \(p = 0\) and \(\tilde{p} = 0\), then \(v\) is unitary, and \(A\) is \(*\)-isomorphic to \(C(X)\) where \(X \subseteq T\) is the spectrum of \(v\) by the Gelfand-Naimark theorem.

• If \(p = 0\) and \(\tilde{p} \neq 0\), then \(v\) is a non-unitary isometry, \(A\) is \(*\)-isomorphic to the Toeplitz algebra \(T(C)\) by Coburn’s theorem, and the isomorphism sends \(v\) to the forward shift \(V\).

• If \(p \neq 0\) and \(\tilde{p} = 0\), then \(v\) is a non-unitary co-isometry, again \(A\) is \(*\)-isomorphic to the Toeplitz algebra \(T(C)\) by Coburn’s theorem, and the isomorphism sends \(v\) to the backward shift \(V^*\).

Thus the only interesting case is when \(N_v = \emptyset\), but \(p \neq 0\) and \(\tilde{p} \neq 0\). Let \(C\) and \(\tilde{C}\) denote the smallest closed ideals of \(A\) which contain the projections \(p\) and \(\tilde{p}\), respectively. For \(i, j \geq 0\), set

\[ f_{ij} := v^* i p v^j \quad \text{and} \quad \tilde{f}_{ij} := v^i \tilde{p} v^* j. \]

**Lemma 19** If \(v\) is a PPI with \(N_v = \emptyset\), then \((f_{ij})_{i,j \geq 0}\) is a (countable) system of matrix units, i.e. \(f_{ij}^* = f_{ji}\) and

\[ f_{ij} f_{kl} = \delta_{jk} f_{il} \quad \text{for all } i, j, k, l \geq 0. \quad (8) \]

If one of the \(f_{ij}\) is non-zero (e.g. if \(f_{00} = p \neq 0\)), then all \(f_{ij}\) are non-zero.

An analogous assertion holds for the family of the \(\tilde{f}_{ij}\).

**Proof.** The symmetry condition is evident, and if \(f_{ij} = 0\) then \(f_{kl} = f_{ki} f_{ij} f_{jl} = 0\) for all \(k, l\) by (8). Property (8) on its hand will follow once we have shown that

\[ p v^j v^* k p = \delta_{jk} p \quad \text{for all } j, k \geq 0. \quad (9) \]

The assertion is evident if \(j = k = 0\). If \(j > 0\) and \(k = 0\), then

\[
\begin{align*}
p v^j p &= (e - v^* v) v^j (e - v^* v) \\
&= v^j - v^* v^{j+1} - v v^* v + v^* v^j (v v^* v) \\
&= v^j - v^* v^{j+1} - v^{j-1} v^* + v^* v^j v = 0,
\end{align*}
\]

and (9) holds. Analogously, (9) holds if \(j = 0\) and \(k > 0\). Finally, let \(j, k > 0\). The assumption \(N_v = \emptyset\) ensures that

\[
\begin{align*}
p v^{j-1} \tilde{p} &= (e - v^* v) v^{j-1} (e - v v^*) = v^{j-1} - v^* v^j - v^j v^* + v^* v^{j+1} v^* = 0
\end{align*}
\]

for all \(j \geq 1\). Employing this identity we find

\[
\begin{align*}
p v^j v^* k p &= (e - v^* v) v^j v^* k p = v^j v^* k p - (v^* v^{j+1} v^*) (v^*)^{k-1} p \\
&= v^j v^* k - (v^{j-1} - v^* v^j - v^j v^*) (v^*)^{k-1} p \\
&= (e - v^* v) v^{j-1} (v^*)^{k-1} p.
\end{align*}
\]
Thus, $pv^jv^*k^p = pv^j-1(v^*)^{k-1}p$ for $j, k \geq 1$. Repeated application of this identity finally leads to one of the cases considered before.

**Proposition 20** Let $\mathbb{N}_v = \emptyset$ and $p \neq 0$.

(a) The ideal $\mathcal{C}$ of $\mathcal{A}$ generated by $p$ coincides with the smallest closed subalgebra of $\mathcal{A}$ which contains all $f_{ij}$ with $i, j \geq 0$.

(b) $\mathcal{C}$ is $^*$-isomorphic to the ideal of the compact operators on a separable infinite-dimensional Hilbert space.

**Proof.** For a moment, write $\mathcal{C}'$ for the smallest closed subalgebra of $\mathcal{A}$ which contains all $f_{ij}$ with $i, j \geq 0$. The identities

$$f_{ij}v = v^*pv^jv = f_{i,j+1}, \quad vf_{0j} = vpv^j = v(e - v^*v)v^j = 0$$

and, for $i \geq 1$,

$$vf_{ij} = vv^*(e - v^*v)v^j = vv^*v^j - (v^*)^{i+1}v^j = vv^*v^j + ((v^*)^{i-1} - (v^*)^i v - vv^*)v^j = (v^*)^{i-1}v^j - (v^*)^i v^j = (v^*)^{i-1}(e - v^*v)v^j = f_{i-1,j}$$

(where we used the adjoint of (10)) and their adjoints show that $\mathcal{C}'$ is a closed ideal of $\mathcal{A}$. Since $p = f_{00}$ we conclude that $\mathcal{C} \subseteq \mathcal{C}'$. Conversely, we have $f_{ij} = v^*pv^j \in \mathcal{C}$ for all $i, j \geq 0$ whence the reverse inclusion $\mathcal{C}' \subseteq \mathcal{C}$. This settles assertion (a).

For assertion (b), note that every $C^*$-algebra generated by a (countable) system of matrix units (in particular, the algebra $\mathcal{C}'$) is naturally $^*$-isomorphic to the algebra of the compact operators on a separable infinite-dimensional Hilbert space (see, e.g., Corollary A.9 in Appendix A2 in [7]).

**Lemma 21** If $\mathbb{N}_v = \emptyset$, then $\mathcal{C} \cap \tilde{\mathcal{C}} = \{0\}$.

**Proof.** $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are closed ideals. Thus, $\mathcal{C} \cap \tilde{\mathcal{C}} = \mathcal{C}\tilde{\mathcal{C}}$, and we have to show that $f_{ij}\tilde{f}_{kl} = 0$ for all $i, j, k, l \geq 0$. Since

$$f_{ij}\tilde{f}_{kl} = (v^*pv^j)(pv^k\tilde{p}) = v^*(pv^{j+k}\tilde{p})v^l,$$

this is a consequence of $\mathbb{N}_v = \emptyset$.

Remember that $p \neq 0$ and $\tilde{p} \neq 0$. From the preceding lemma we conclude that the mapping

$$\mathcal{A} \to \mathcal{A}/\mathcal{C} \times \mathcal{A}/\tilde{\mathcal{C}}, \quad w \mapsto (w + \mathcal{C}, w + \tilde{\mathcal{C}})$$

is an injective $^*$-homomorphism; thus $\mathcal{A}$ is $^*$-isomorphic to the $C^*$-subalgebra of $\mathcal{A}/\mathcal{C} \times \mathcal{A}/\tilde{\mathcal{C}}$ generated by $(v + \mathcal{C}, v + \tilde{\mathcal{C}})$. The element $v + \mathcal{C}$ is an isometry in $\mathcal{A}/\mathcal{C}$ (since $e - v^*v \in \mathcal{C}$), but it is not unitary (otherwise $e - vv^* \in \tilde{\mathcal{C}}$ would be
a non-zero element of $C$, in contradiction with Lemma 21. Analogously, $v + \tilde{C}$ is a non-unitary co-isometry. By Coburn’s Theorem, there are $^*$-isomorphisms $\mu: A/C \rightarrow T(C)$ and $\tilde{\mu}: A/\tilde{C} \rightarrow T(C)$ which map $v + C \mapsto V$ and $v + \tilde{C} \mapsto V^*$, respectively. But then

$$\mu \times \tilde{\mu}: A/C \times A/\tilde{C} \rightarrow T(C) \times T(C), \quad (a, \tilde{a}) \mapsto (\mu(a), \tilde{\mu}(\tilde{a}))$$

is a $^*$-isomorphism which maps the $C^*$-subalgebra of $A/C \times A/\tilde{C}$ generated by $(v + C, v + \tilde{C})$ to the $C^*$-subalgebra of $T(C) \times T(C)$ generated by the pair $(V, V^*)$. The latter algebra has been identified in Corollary 4. Summarizing we get:

**Proposition 22** Let the $C^*$-algebra $A$ be generated by a PPI $v$ with $N_v = \emptyset$ and $p \neq 0$ and $\tilde{p} \neq 0$. Then $A$ is $^*$-isomorphic to the algebra $S(T(C))/G$ (likewise, to $S_{\geq 2}(T(C))/G_{\geq 2}$), and the isomorphism sends $v$ to $(P_n V P_n)_{n\geq 1} + G$ (likewise, to $(P_n V P_n)_{n\geq 2} + G_{\geq 2}$).

5 The general case

We are now going to finish the proof of Theorem 1. For we think of $A$ as being faithfully represented as a $C^*$-algebra of bounded linear operators on a separable infinite-dimensional Hilbert space $H$ (note that $A$ is finitely generated, hence separable). As follows easily from (7), $z_n := \sum_{i=0}^{n} e_i$ is the identity element of $C_n$. So we can think of the $z_n$ as orthogonal projections on $H$. Moreover, these projections are pairwise orthogonal by Proposition 17 (b). Thus, the operators $P_n := \sum_{i=1}^{n} z_n$ form an increasing sequence of orthogonal projections on $H$. Let $P \in L(H)$ denote the least upper bound of that sequence (which then is the limit of the $P_n$ in the strong operator topology). Clearly, $P$ is an orthogonal projection again (but note that $P$ does not belong to $A$ in general).

**Lemma 23** (a) Every $z_n$ is a central projection of $A$.
(b) $P$ commutes with every element of $A$.

**Proof.** Assertion (b) is a consequence of (a). We show that

$$z_n = \sum_{i=0}^{n} v^i \pi_n v^i = \sum_{i=0}^{n} v^i p(v^n v^{*n} - v^{n+1}(v^*)^{n+1}) v^i$$

$$= \sum_{i=0}^{n} v^i(e - v^*) (v^n v^{*n} - v^{n+1}(v^*)^{n+1}) v^i$$

commutes with $v$. Indeed,

$$v z_n = v(e - v^*) (v^n v^{*n} - v^{n+1}(v^*)^{n+1})$$
\[ + \sum_{i=1}^{n} vv^i (e - v^* v) (v^n v^m - v^{n+1} (v^*)^{n+1}) v^i \]
\[ = \sum_{i=1}^{n} vv^i (e - v^* v) v^n (v^m - v (v^*)^{n+1}) v^i \]
\[ = \sum_{i=1}^{n} (v v^i v^n - v (v^*)^{i+1} v^{n+1}) (v^m - v (v^*)^{n+1}) v^i \]
\[ = \sum_{i=1}^{n} ((v^*)^{i-1} v^n - (v^*)^i v^{n+1}) (v^m - v (v^*)^{n+1}) v^i \] (by Lemma 11)
\[ = \sum_{i=1}^{n} (v^*)^{i-1} (e - v^* v) (v^n v^m - v^{n+1} (v^*)^{n+1}) v^i \]
\[ = \sum_{i=0}^{n-1} v^i (e - v^* v) (v^n v^m - v^{n+1} (v^*)^{n+1}) v^i v \]
\[ = \sum_{i=0}^{n} v^i (e - v^* v) (v^n v^m - v^{n+1} (v^*)^{n+1}) v^i v \]
\[ = (v^n v^m - v^{n+1} (v^*)^{n+1}) v^i v = z_n v \]

again by Lemma 11. Thus, \( vz_n = z_n v \). Since \( z_n = z_0^* \), this implies that \( z_n \) also commutes with \( v^* \) and, hence, with every element of \( A \).

Consequently, \( A = PAP \oplus (I - P)A(I - P) \) where \( I \) stands for the identity operator on \( H \). We consider the summands of this decomposition separately. The part \( (I - P)A(I - P) \) is generated by the PPI \( v' := (I - P)v(I - P) \). Since
\[
(I - P)pv^n pv^m p(I - P) = (I - P)\pi_n(I - P) = (I - P)z_n e^{(11)}_{00} \pi_n(I - P) = 0,
\]
we conclude that \( N_{v'} = \emptyset \). Thus, this part of \( A \) is described by Proposition 22.

The part \( PAP \) is generated by the PPI \( P v P \). It follows from the definition of \( P \) that \( N_{P v P} = N_v \) and that \( G_{P v P} = P G_v P = G_v \). We let \( \prod_{n \in N_v} C_n \) stand for the direct product of the algebras \( C_n \) and consider the mapping
\[
PAP \rightarrow \prod_{n \in N_v} C_n, \quad PAP \mapsto (z_n PAP z_n)_{n \in N_v} = (z_n Az_n)_{n \in N_v}. \tag{11}
\]
If \( z_n Az_n = 0 \) for every \( n \in N_v \), then
\[
PAP = \sum_{m, n \in N_v} z_m PAP z_n = \sum_{n \in N_v} z_n A z_n = 0.
\]
Thus, the mapping (11) is injective, and the algebra $PAP$ is $\ast$-isomorphic to the $C^\ast$-subalgebra of $\prod_{n \in \mathbb{N}_v} \mathcal{C}_n$ generated by the sequence $(z_n v z_n)_{n \in \mathbb{N}_v}$. Further we infer from Proposition 17 (a) that $\mathcal{C}_n$ is isomorphic to $\mathbb{C}^{(n+1) \times (n+1)}$ if $n \in \mathbb{N}_v$. We are going to make the latter isomorphism explicit. For we note that

$$e_{ii}^{(n)} v e_{jj}^{(n)} = v^* \pi_n v^{i+1} v^j \pi_n v^j$$

$$= \begin{cases} v^* \pi_n v^{i+1} (v^*)^{i+1} \pi_n v^{i+1} & \text{if } i + 1 = j, \\ 0 & \text{if } i + 1 \neq j \end{cases}$$

(by Corollary 13)

$$= \begin{cases} v^* \pi_n v^{i+1} & \text{if } i + 1 = j, \\ 0 & \text{if } i + 1 \neq j \end{cases}$$

(by Lemma 15)

$$= \begin{cases} e_{ii+1}^{(n)} & \text{if } i + 1 = j, \\ 0 & \text{if } i + 1 \neq j. \end{cases}$$

We choose a unit vector $e_i^{(n)}$ in the range of $e_{ii}^{(n)}$ (recall Proposition 13 (a)), and let $f_{ii}^{(n)}$ stand for the $n+1$-tuple $(0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 at the $i$th position. Then $(e_i^{(n)})_{n \in \mathbb{N}_v}$ forms an orthonormal basis of $\text{im } z_n$, $(f_{ii}^{(n)})_{n \in \mathbb{N}_v}$ forms an orthonormal basis of $\mathbb{C}^{n+1}$, the mapping $e_i^{(n)} \mapsto f_{n-i}$ extends to a linear bijection from $\text{im } z_n$ onto $\mathbb{C}^{n+1}$, which finally induces a $\ast$-isomorphism $\xi_n$ from $\mathcal{C}_n \cong L(\text{im } z_n)$ onto $\mathbb{C}^{(n+1) \times (n+1)} \cong L(\mathbb{C}^{n+1})$. Then

$$\xi : \prod_{n \in \mathbb{N}_v} \mathcal{C}_n \to \prod_{n \in \mathbb{N}_v} \mathbb{C}^{(n+1) \times (n+1)}, \quad (A_n) \mapsto (\xi_n(A_n))$$

is a $\ast$-isomorphism which maps the $C^\ast$-subalgebra of $\prod_{n \in \mathbb{N}_v} \mathcal{C}_n$ generated by the sequence $(z_n v z_n)_{n \in \mathbb{N}_v}$ to the $C^\ast$-subalgebra of $\prod_{n \in \mathbb{N}_v} \mathbb{C}^{(n+1) \times (n+1)}$ generated by the sequence $(V_{n+1})_{n \in \mathbb{N}_v}$, where $V_n$ is the matrix described in Example (c). Note that $V_n$ is just the $n \times n$th finite section $P_n V P_n$ of the forward shift operator.

If now $\mathcal{A}$ is the universal algebra generated by a PPI $v$, then $\mathbb{N}_v = \mathbb{N}$, as we observed in Corollary 18. Thus, in this case, the algebra $PAP$ is $\ast$-isomorphic to the smallest $C^\ast$-subalgebra of $\mathcal{F} = \prod_{n \geq 1} \mathbb{C}^{n \times n}$ generated by the sequence $(P_n V P_n)$, i.e. to the $C^\ast$-algebra $\mathcal{S}_{\geq 2}(\mathcal{T}(C))$.

It remains to explain what happens with the part $(I - P)\mathcal{A}(I - P)$ of $\mathcal{A}$. The point is that the quotient $PAP/P\mathcal{G}_v P$ is generated by a PPI $u$ for which $\mathbb{N}_u$ is empty. We have seen in Proposition 22 that both this quotient and the algebra $(I - P)\mathcal{A}(I - P)$ are canonically $\ast$-isomorphic to $\mathcal{S}_{\geq 2}(\mathcal{T}(C))/\mathcal{G}_{\geq 2}$. Thus, there is a $\ast$-homomorphism from $PAP$ onto $(I - P)\mathcal{A}(I - P)$ which maps the generating PPI $P v P$ of $PAP$ to the generating PPI $(I - P)v(I - P)$ of $(I - P)\mathcal{A}(I - P)$. Hence, if $\mathcal{A}$ is the universal $C^\ast$-algebra generated by a PPI, then already $PAP$ has the universal property, and $\mathcal{A} \cong PAP \cong \mathcal{S}_{\geq 2}(\mathcal{T}(C))$. \hfill \blacksquare
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