Chapter 1

CLASSICAL AND RELATIVISTIC LAWS OF MOTION FOR SPHERICAL SUPERNOVAS

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Abstract

We derive some first order differential equations which model the classical and the relativistic thin layer approximations. The circumstellar medium is assumed to follow a density profile which can be exponential, Gaussian, Plummer-like, self-gravitating of Lane–Emden ($n = 5$) type, or power law. The first order differential equations are solved analytically, or numerically, or by a series expansion, or by recursion, or by Padé approximation. The initial conditions are chosen in order to model the temporal evolution of SN 1993J over ten years. The Padé approximated equations of motion are applied to four SNRs: Tycho, Cas A, Cygnus loop, and SN 1006.

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1. Introduction

The absorption features of supernovae (SN) allow the determination of their expansion velocity, \( v \). We select, among others, some results. The spectropolarimetry (CA II IR triplet) of SN 2001el gives a maximum velocity of \( \approx 26000 \, \text{kms}^{-1} \), see [1]. The same triplet when searched in seven SN of type Ia gives \( 10400 \, \text{kms}^{-1} \leq v \leq 17700 \, \text{kms}^{-1} \), see Table I in [2]. A time series of eight spectra in SN 2009ig allows asserting that the velocity at the CA II line, for example, decreases in 12 days from \( 32000 \, \text{kms}^{-1} \) to \( 21500 \, \text{kms}^{-1} \), see Figure 9 in [3].

A recent analysis of 58 type Ia SN in Si II gives \( 9660 \, \text{kms}^{-1} \leq v \leq 14820 \, \text{kms}^{-1} \), see Table II in [4]. Other examples for the maximum velocity of expansion are: \( \approx 3700 \, \text{kms}^{-1} \), see Fig. 20.21 in [5] and Fig. 6 in [6].

The previous analysis allow saying that the maximum velocity so far observed for SN is \( \frac{v}{c} \approx 0.123 \), where \( c \) is the speed of light; this observational fact points to a relativistic equation of motion.

The temporal observations of SN such as SN 1993J establish a clear relation between the instantaneous radius of expansion \( r \) and the time \( t \), of the type \( r \propto t^{0.82} \), see [7], and therefore allows exploring variants of the thin layer approximation. The previous observational facts exclude an SN propagation in a circumstellar medium (CSM) with constant density; two solutions of this type are the Sedov solution which, scales as \( r \propto t^{0.4} \), see [8, 9], and the momentum conservation in a thin layer approximation, which scales as \( r \propto t^{0.25} \), see [10, 11]. Previous efforts to model these observations in the framework of the thin layer approximation in a CSM governed by a power law, see [12], or in the framework in which the CSM has a constant density but swept mass regulated by a parameter called porosity, see [13], have been successfully explored.

An important feature of the various models is based on the type of CSM which surrounds the expansion. As an example in the framework of classical shocks, [14] and [15] analyzed self-similar solutions with a CSM of type \( r^{-s} \), which means an inverse power law dependence. In the framework of the Kompaneyets equation, see [16], for the motion of a shock wave in different plane-parallel stratified media, [17] considered four types of CSM. It is therefore interesting to take into account a self-gravitating CSM, which will give a physical basis to the considered model.

The relativistic treatment has been concentrated on the determination of the Lorentz factor, \( \gamma \), for the ejecta in GRB; we report some research in this regard: [18] found \( 30 < \gamma < 50 \) for a significant number of GRBs, [19] found \( \gamma = 305 \) for GRB 970828 and \( \gamma = 384 \) for GRB 990510, [20] found high values for the sample of the GRBs considered 30.5\( \gamma < 900 \), [21] in the framework of a high-energy spectral cutoff originating from the creation of electron–positron pairs found \( \gamma \approx 600 \) for GRB 080916C, [22] found \( \gamma \approx 6.7 \times 10^{2} \) for GRB 090510. The late stage of an SN is called supernova remnant (SNR) and a lifetime of \( \approx 1000 \, \text{yr} \) separates young SNRs from old SNRs.

In the present paper we review the standard two-phase model for the expansion of an SN, see Section 2., and five density profiles, see Section 3. In Section 4. we derive the differential equations which model the thin layer approximation for an SN in the presence of five types of medium. A relativistic treatment is carried out in Section 5. The application of the developed theory to SN 1993J is split into the classical case, see Section 6.
relativistic case, see Section 7. Section 8 closes the derived equations of motion to four SNRs.

2. The standard model

An SN expands at a constant velocity until the surrounding mass is of the order of the solar mass. The time this takes, \( t_M \), is

\[
t_M = 186.45 \frac{\sqrt{M_\odot}}{\sqrt{n_0 v_{10000}}} \text{ yr} ,
\]

where \( M_\odot \) is the number of solar masses in the volume occupied by the SN, \( n_0 \) is the number density expressed in particles \( \text{cm}^{-3} \), and \( v_{10000} \) is the initial velocity expressed in units of 10000 km/s, see [9]. A first law of motion for the SN is the Sedov solution

\[
R(t) = \left( \frac{25}{4} \frac{E t^2}{\pi \rho} \right)^{1/5} ,
\]

where \( E \) is the energy injected into the process and \( t \) is the time, see [8,9]. Our astrophysical units are: time, \( (t_1) \), which is expressed in years; \( E_{51} \), the energy in \( 10^{51} \text{ erg} \); \( n_0 \), the number density expressed in particles \( \text{cm}^{-3} \) (density \( \rho = n_0 m \), where \( m = 1.4 m_H \)). In these units, equation (2) becomes

\[
R(t) \approx 0.313 \left( \frac{E_{51} t_1^2}{n_0} \right)^{1/5} \text{ pc} .
\]

The Sedov solution scales as \( t^{0.4} \). We are now ready to couple the Sedov phase with the free expansion phase

\[
R(t) = \begin{cases} 
0.0157 t \text{ pc} & \text{if } t \leq 2.5 \text{ yr} \\
0.0273 \sqrt{t} \text{ pc} & \text{if } t > 2.5 \text{ yr}
\end{cases}
\]

This two-phase solution is obtained with the following parameters \( M_\odot = 1 \), \( n_0 = 1.127 \times 10^5 \), \( E_{51} = 0.567 \) and Figure 1 presents its temporal behavior as well as the data.

A similar model is reported in [23] with the difference that the first phase ends at \( t = 60 \) yr against our \( t = 2.5 \) yr. A careful analysis of Figure 1 reveals that the standard two-phase model does not fit the observed radius–time relation for SN 1993J.

3. Density profiles for the CSM

This section introduces five density profiles for the CSM: an exponential profile, a Gaussian profile, a Plummer-like profile, a self-gravitating profile of Lane–Emden type, and a power law profile.

3.1. The exponential profile

This density is assumed to have the following exponential dependence on \( r \) in spherical coordinates:

\[
\rho(r; r_0, b, \rho_0) = \rho_0 \exp \left( -\frac{r - r_0}{b} \right) ,
\]
where $b$ represents the scale. The piece-wise density is

$$
\rho(r; r_0, b, \rho_0) = \begin{cases} 
\rho_0 & \text{if } r \leq r_0 \\
\rho_0 \exp\left(-\frac{(r-r_0)}{b}\right) & \text{if } r > r_0
\end{cases}
$$

(6)

The total mass swept, $M(r; r_0, b, \rho_0)$, in the interval $[0, r]$ is

$$
M(r; r_0, b, \rho_0) =
\frac{4}{3} \rho_0 \pi r_0^3 - 4 b \left(2 b^2 + 2br + r^2\right) \rho_0 e^{-\frac{r_0 - r}{b}} \pi + 4 b \left(2 b^2 + 2br_0 + r_0^2\right) \rho_0 \pi .
$$

(7)

### 3.2. The Gaussian profile

This density has the Gaussian dependence

$$
\rho(r; r_0, b, \rho_0) = \rho_0 \exp\left(-\frac{1}{2} \frac{r^2}{b^2}\right)
$$

(8)

and the piece-wise density is

$$
\rho(r; r_0, b, \rho_0) = \begin{cases} 
\rho_0 & \text{if } r \leq r_0 \\
\rho_0 \exp\left(-\frac{1}{2} \frac{r^2}{b^2}\right) & \text{if } r > r_0
\end{cases}
$$

(9)

The total mass swept, $M(r; r_0, b, \rho_0)$, in the interval $[0, r]$ is

$$
M(r; r_0, b, \rho_0) =
\frac{4}{3} \rho_0 \pi r_0^3 + 4 \rho_0 \pi \left( -e^{-\frac{1}{2} \frac{r_0^2}{b^2}} r b^2 + \frac{1}{2} b^3 \sqrt{\pi} \sqrt{2} \text{erf}\left(\frac{1}{2} \frac{\sqrt{2} r}{b}\right) \right)
$$

$$
-4 \rho_0 \pi \left( -e^{-\frac{1}{2} \frac{r_0^2}{b^2}} r_0 b^2 + \frac{1}{2} b^3 \sqrt{\pi} \sqrt{2} \text{erf}\left(\frac{1}{2} \frac{\sqrt{2} r_0}{b}\right) \right)
$$

(10)

where erf is the error function, see [24].
3.3. The Plummer profile

The Plummer-like density profile, after [25], is

$$\rho(r; R_{\text{flat}}) = \rho_c \left( \frac{R_{\text{flat}}}{R_{\text{flat}}^2 + r^2} \right)^{1/2} \eta$$

(11)

where $r$ is the distance from the center, $\rho$ is the density, $\rho_c$ is the density at the center, $R_{\text{flat}}$ is the distance up to which the density is nearly constant, and $\eta$ is the power law exponent at large values of $r$, see [26] for more details. The following transformation, $R_{\text{flat}} = \sqrt{3}b$, gives the Plummer-like profile, which can be compared with the Lane–Emden profile

$$\rho(r; b) = \rho_c \left( \frac{1 + \frac{\eta r^2}{b^2}}{b^2} \right)^{\eta/2}$$

(12)

At low values of $r$, the Taylor expansion of the Plummer-like profile can be taken:

$$\rho(r; b) \approx \rho_c \left( 1 - \frac{1}{6} \frac{\eta r^2}{b^2} \right)$$

(13)

and at high values of $r$, the behavior of the Plummer-like profile is

$$\rho(r; b) \sim \rho_c (\sqrt{3}b)^{\eta} \left( \frac{1}{r} \right)^{\eta}$$

(14)

The total mass $M(r; b)$ contained between 0 and $r$ is

$$M(r; b) = \int_0^r 4\pi r^2 \rho(r; b) dr = \frac{PN}{PD}$$

(15)

where

$$PN = -\sqrt{3}b \rho_c \pi (4 F_1(\eta/2, -3/2 + \eta/2; -1/2 + \eta/2; -3 \eta \frac{b^2}{r^2}) \times$$

$$\times \Gamma(-\eta/2 + 5/2) \Gamma(\eta/2) \cos(1/2 \pi \eta) r^{3-\eta} 3^{1/2+\eta/2} \eta^{\eta-1}$$

$$-9 \pi^{3/2} b^2 \eta + 27 \pi^{3/2} b^2)$$

and

$$PD = 3 \cos(1/2 \pi \eta) \Gamma(-\eta/2 + 5/2)(\eta - 3) \Gamma(\eta/2)$$

(16)

where $\, F_1(a, b; c; z)$ is the regularized hypergeometric function [27, 24]. The above expression simplifies when $\eta = 6$, $M(r; b)_6$.

$$M(r; b)_6 = \frac{27 \rho_c \pi b^7 \sqrt{3}}{2 (3 b^2 + r^2)^2} \arctan(1/3 \frac{r \sqrt{3}}{b}) + 9 \frac{\rho_c \pi b^5 \sqrt{3} r^2}{(3 b^2 + r^2)^2} \arctan(1/3 \frac{r \sqrt{3}}{b}) +$$

$$3/2 \frac{\rho_c \pi b^7 \sqrt{3} r^4}{(3 b^2 + r^2)^2} \arctan(1/3 \frac{r \sqrt{3}}{b}) - \frac{27 \rho_c \pi b^6 r}{2 (3 b^2 + r^2)^2} + 9/2 \frac{\rho_c \pi b^4 r^3}{(3 b^2 + r^2)^2}$$

The astrophysical version of the total mass is

$$M(r_{\text{pc}}; b_{\text{pc}}) = \frac{PNA}{PD} M_\odot$$

(17)
with

\[
PNA = -2.47 \times 10^{-10} b_{pc}^3 n_0 [1.02 \times 10^{10} \arctan(1.73 \frac{b_{pc}}{r_{pc}}) b_{pc}^4 \\
+ 6.8 \times 10^9 \arctan(1.73 \frac{b_{pc}}{r_{pc}}) b_{pc}^2 r_{pc}^2 + 1.13 \times 10^9 \arctan(1.73 \frac{b_{pc}}{r_{pc}}) r_{pc}^4 - 1.6 \times 10^{10} b_{pc}^4 \\
+ 5.89 \times 10^9 b_{pc}^3 r_{pc} - 1.06 \times 10^{10} b_{pc}^2 r_{pc}^2 - 1.96 \times 10^9 b_{pc}^4 r_{pc}^3 - 1.78 \times 10^9 r_{pc}^4]
\]

and

\[
PDA = \left(3.0 b_{pc}^2 + r_{pc}^2\right)^2,
\]

(18)

where \(b_{pc}\) is \(b\) expressed in pc, \(r_{pc}\) is \(r\) expressed in pc, and \(n_0\) is the same as in equation (3).

More details can be found in [28].

### 3.4. The Lane–Emden profile

The self gravitating sphere of polytropic gas is governed by the Lane–Emden differential equation of the second order

\[
\frac{d^2}{dx^2} Y(x) + 2 \frac{d}{dx} \frac{Y(x)}{x} + \left(\frac{Y(x)}{x}\right)^n = 0,
\]

(19)

where \(n\) is an integer, see [29, 30, 31, 32, 33].

The solution \(Y(x)_n\) has the density profile

\[
\rho = \rho_c Y(x)_n^n,
\]

(20)

where \(\rho_c\) is the density at \(x = 0\). The pressure \(P\) and temperature \(T\) scale as

\[
P = K \rho^{1+\frac{1}{n}},
\]

(21)

\[
T = K' Y(x),
\]

(22)

where \(K\) and \(K'\) are two constants, for more details, see [34].

Analytical solutions exist for \(n = 0, 1\) and 5; that for \(n = 0\) is

\[
Y(x) = \frac{\sin(x)}{x},
\]

(23)

and has therefore an oscillatory behavior. The analytical solution for \(n = 5\) is

\[
Y(x) = \frac{1}{\left(1 + \frac{x^2}{3}\right)^{1/2}},
\]

(24)

and the density for \(n = 5\) is

\[
\rho(x) = \rho_c \frac{1}{\left(1 + \frac{x^2}{3}\right)^{5/2}}.
\]

(25)

The variable \(x\) is non-dimensional and we now introduce the new variable \(x = r/b\)

\[
\rho(r; b) = \rho_c \frac{1}{\left(1 + \frac{r^2}{3b^2}\right)^{5/2}}.
\]

(26)
This profile is a particular case, \( \eta = 5 \), of the Plummer-like profile as given by equation (12). At low values of \( r \), the Taylor expansion of this profile is

\[
\rho(r; b) \approx \rho_c \left( 1 - \frac{5}{6} \frac{r^2}{b^2} \right),
\]

and at high values of \( r \), its behavior is

\[
\rho(r; b) \sim 9 \frac{\rho_c \sqrt{3} b^5}{r^3}.
\]

The total mass \( M(r; b) \) contained between 0 and \( r \) is

\[
M(r; b) = \int_0^r 4\pi r^2 \rho(r; b) dr = \frac{4b^3 \rho_c \pi \sqrt{3}}{(3 b^2 + r^2)^{3/2}},
\]

or in solar units

\[
M(r_{pc}; b_{pc}) = \frac{2.2 \times 10^{55} b_{pc}^3 r_{pc}^3 n_0}{(2.85 \times 10^{37} b_{pc}^2 + 9.52 \times 10^{30} r_{pc}^2)^{3/2}} M_\odot.
\]

The total mass of the profile can be found calculating the limit as \( r \to \infty \) of equation (29)

\[
M(\infty; b) = \lim_{r \to \infty} M(r; b) = 4b^3 \rho_c \pi \sqrt{3}.
\]

### 3.5. The power law profile

We now assume that the CSM around the SN scales with the following piecewise dependence (which avoids a pole at \( r = 0 \))

\[
\rho(r; r_0, d) = \begin{cases} 
\rho_c & \text{if } r \leq r_0 \\
\rho_c \left( \frac{r_0}{r} \right)^d & \text{if } r > r_0.
\end{cases}
\]

The mass swept, \( M_0 \), in the interval \([0, r_0]\) is

\[
M_0 = \frac{4}{3} \rho_0 \pi r_0^3.
\]

The total mass swept, \( M(r; r_0, d) \), in the interval \([0, r]\) is

\[
M(r; r_0, d) = -4 r^3 \rho_c \pi \left( \frac{r_0}{r} \right)^d (d - 3)^{-1} \\
+ 4 \frac{\rho_c \pi r_0^3}{d - 3} + \frac{4}{3} \rho_c \pi r_0^3.
\]

or in solar units

\[
M(r_{pc}; r_{0,pc}, d) = \frac{3.14 n_0 \left( 0.137 r_{pc}^3 \left( \frac{r_{0,pc}}{r_{pc}} \right)^d - 0.0459 r_{0,pc}^3 d \right)}{3 - d} M_\odot,
\]

where \( r_{0,pc} \) is \( r_0 \) expressed in pc.
4. Classical conservation of momentum

This section reviews the standard equation of motion in the case of the thin layer approximation in the presence of a CSM with constant density and derives the equation of motion under the conditions of each of the five density profiles for the density of the CSM.

4.1. Motion with constant density

In the case of a constant density of the CSM, \( \rho_c \), the differential equation which models momentum conservation is

\[
\frac{4}{3} \pi (r(t))^3 \rho_c \frac{d}{dt} r(t) - \frac{4}{3} \pi r_0^3 \rho_c v_0 = 0 ,
\]

where the initial conditions are \( r = r_0 \) and \( v = v_0 \) when \( t = t_0 \). The variables can be separated and the radius as a function of the time is

\[
r(t) = \sqrt[\frac{3}{4}]{4 r_0^3 v_0 (t - t_0)} + r_0^4 ,
\]

and its behavior as \( t \to \infty \) is

\[
r(t) = \sqrt{2r_0^3 / 4 \sqrt{v_0 \sqrt{t - t_0} + \frac{1}{16} \frac{2r_0^7}{v_0^3 (t - t_0)^{3/4}}}} .
\]

The velocity as a function of time is

\[
v(t) = \frac{r_0^3 v_0}{(4 r_0^3 v_0 (t - t_0) + r_0^4)^{3/4}} .
\]

4.2. Motion with exponential profile

Assuming an exponential profile as given by equation (6), the velocity is

\[
\frac{dr}{dt} = \frac{NE}{DE} ,
\]

where

\[
NE = -r_0^3 v_0 ,
\]

and

\[
DE = 6 e^{r_0 - r} b^3 + 6 e^{r_0 - r} b^2 r + 3 e^{r_0 - r} b^2 - 3 r_0^3 b - 6 r_0 b^2 - 6 b^3 .
\]

In the above differential equation of the first order in \( r \), the variables can be separated and integration gives the following nonlinear equation:

\[
\frac{1}{r_0^3 v_0} \left( 18 e^{r_0 - r} b^4 + 12 e^{r_0 - r} b^3 r + 3 e^{r_0 - r} b^2 r^2 - r_0^4 - 3 r_0^3 b 
+ r_0^3 r - 9 r_0^2 b^2 + 3 r_0^2 b r - 18 b^3 r_0 + 6 r_0 b^2 r - 18 b^4 + 6 b^3 r \right)
= (t - t_0) \quad (39)
\]
In this case it is not possible to find an analytical solution for the radius, \( r \), as a function of time. We therefore apply the Padé rational polynomial approximation of degree 2 in the numerator and degree 1 in the denominator about the point \( r = r_0 \) to the left-hand side of equation (39):

\[
- \frac{(r_0 - r) (-5 br - br_0 - 2 r r_0 + 2 r_0^2)}{2 v_0 (2 br - 5 br_0 - r r_0 + r_0^2)} = t - t_0 .
\] (40)

The resulting Padé approximant for the radius \( r_{2,1} \) is

\[
r_{2,1} = \frac{1}{2 r_0 + 5 b} \left( r_0 t v_0 - r_0 t_0 v_0 - 2 b t v_0 + 2 b t_0 v_0 + 2 r_0^2 + 2 r_0 b \right. \\
+ \left( 4 b^2 r_0^2 v_0^2 - 8 b^2 t t_0 v_0^2 + 4 b^2 t_0^2 v_0^2 - 4 b t^2 r_0 v_0^2 \\
+ 8 b t t_0 r_0 v_0^2 - 4 b t_0^2 r_0 v_0^2 + r^2 t_0^2 v_0^2 - 2 t t_0 r_0^2 v_0^2 + t_0^2 r_0^2 v_0^2 \\
+ 42 b^2 t r_0 v_0^2 - 42 b^2 t_0 r_0 v_0^2 - 6 b t r_0^2 v_0^2 - 6 b t_0 r_0^2 v_0^2 + 9 r_0^2 b^2 \left. \right) \right)^{\frac{1}{2}},
\] (41)

and the velocity is

\[
v_{2,1} = \frac{dr_{2,1}}{dt} = \frac{NVE}{DVE},
\] (42)

\[
NVE = 4 v_0 \left\{ (-b/2 + 1/4 r_0) \times \right. \\
\sqrt{4 \left( b - \frac{1}{2} r_0 \right)^2 (t - t_0)^2 v_0^2 + 42 \left( b + 1/7 r_0 \right) (t - t_0) b r_0 v_0 + 9 r_0^2 b^2 + \\
(3/4 b + (t/4 - 1/4 t_0) v_0) r_0^2 + \frac{21 r_0 b}{4} \left( v_0 \left( -\frac{4 t}{21} + \frac{4 t_0}{21} \right) + b \right) \\
\left. + b^2 v_0 (t - t_0) \right\},
\] (43)

and

\[
DVE = \sqrt{4 \left( b - \frac{1}{2} r_0 \right)^2 (t - t_0)^2 v_0^2 + 42 \left( b + 1/7 r_0 \right) (t - t_0) b r_0 v_0 + 9 r_0^2 b^2 \times \\
(2 r_0 + 5 b) }.
\] (44)

### 4.3. Motion with Gaussian profile

Assuming a Gaussian profile as given by equation (8) the velocity is

\[
\frac{dr}{dt} = \frac{NG}{DG},
\] (45)

where

\[
NG = -2 r_0^3 v_0
\] (46)
and
\[
DG = -3b^3\sqrt{\frac{\pi}{2}}\text{erf}\left(\frac{1}{2}\frac{\sqrt{2}r}{b}\right) + 6b^3\sqrt{\frac{\pi}{2}}e^{-\frac{1}{2}\frac{r^2}{b^2}}
\]
\[
-6e^{-\frac{1}{2}\frac{r_0^2}{b^2}}r_0b^2 - 2r_0^3 .
\] (47)

The appropriate nonlinear equation is
\[
\frac{1}{2} v_0 \left(2b^2r - 5r_0b^2 - r_0^2 + r_0^3\right) \left(-9e^{-\frac{1}{2}\frac{r_0^2}{b^2}}b^2r
\right.
\]
\[
-9e^{-\frac{1}{2}\frac{r_0^2}{b^2}}r_0b^2 - 4b^2r + 10r_0b^2 + 2r_0^2 - 2r_0^3\right) = t - t_0 .
\] (48)

The Padé rational polynomial approximation of degree 2 in the numerator and degree 1 in
the denominator about \( r = r_0 \) for the left-hand side of the above equation gives
\[
\frac{1}{2} v_0 \left(2b^2r - 5r_0b^2 - r_0^2 + r_0^3\right) \left(-9e^{-\frac{1}{2}\frac{r_0^2}{b^2}}b^2r
\right.
\]
\[
-9e^{-\frac{1}{2}\frac{r_0^2}{b^2}}r_0b^2 - 4b^2r + 10r_0b^2 + 2r_0^2 - 2r_0^3\right) = t - t_0 .
\] (49)

The resulting Padé approximant for the radius \( r_{2,1} \) is
\[
r_{2,1} = \frac{1}{9e^{-\frac{1}{2}\frac{r_0^2}{b^2}}b^2 + 2r_0^2 - 4b^2}
\]
\[
\left\{9e^{-\frac{1}{2}\frac{r_0^2}{b^2}}r_0b^2 - 2b^2tv_0
\right.
\]
\[
+2b^2t_0v_0 + r_0^2t_0v_0 - r_0^2t_0v_0 - 7r_0b^2 + 2r_0^3
\]
\[
+\left\{54b^4r_0v_0(t - t_0)e^{-\frac{1}{2}\frac{r_0^2}{b^2}}
\right.
\]
\[
+4 \left( (t - t_0) \left(b^2 - \frac{1}{2}r_0^2\right)v_0 - \frac{3}{2}r_0b^2 \right)^2 \right\}^{\frac{1}{2}} \right\} ,
\] (50)

and the velocity is
\[
v_{2,1} = \frac{dr_{2,1}}{dt} = \frac{NVG}{DVG} ,
\] (51)

\[
NVG = - \left(-27e^{-\frac{1}{2}\frac{r_0^2}{b^2}}r_0b^4 + (2b^2 - r_0^2)(v_0(t - t_0)r_0^2
\right.
\]
\[
+3r_0b^2 - 2v_0b^2(t - t_0) + \left\{54b^4r_0v_0(t - t_0)e^{-\frac{1}{2}\frac{r_0^2}{b^2}}
\right.
\]
\[
+4 ((t - t_0)(b^2 - \frac{1}{2}r_0^2)v_0 - 3/2r_0b^2)^2) \right\}^{\frac{1}{2}} \right\} v_0 ,
\] (52)
and

\[ DVG = \left\{ 54 b^4 r_0 v_0 (t - t_0) e^{-1/2 \frac{r^2}{r_0^2}} + 4 ((t - t_0) (b^2 - \frac{1}{2} r_0^2)) v_0 \right. \]

\[ - \frac{3}{2} r_0^2 b^2 \right\}^{1/2} (9 e^{-1/2 \frac{r^2}{r_0^2}} b^2 + 2 r_0^2 - 4 b^2) \right. \ . \quad (53) \]

### 4.4. Motion with Plummer profile

The case of a Plummer-like profile for the CSM as given by (12) when \( \eta = 6 \) produces the differential equation

\[ \frac{d}{dt} r(t) = \frac{NDEP}{DNEP} \right. \ , \quad (54) \]

where

\[ NDEP = (9 \sqrt{3} \arctan(1/3 \frac{r_0 \sqrt{b}}{b}) b^4 + 6 \sqrt{3} \arctan(1/3 \frac{r_0 \sqrt{b}}{b}) b^2 r_0^2 + \]

\[ + \sqrt{3} \arctan(1/3 \frac{r_0 \sqrt{b}}{b}) r_0^4 - 9 b^3 r_0 + 3 b r_0^3) v_0 (3 b^2 + (r(t))^2) \right. \]

and

\[ NDEP = (3 b^2 + r_0^2)^2 (9 \sqrt{3} \arctan(1/3 \frac{r(t) \sqrt{b}}{b}) b^4 + 6 \sqrt{3} \arctan(1/3 \frac{r(t) \sqrt{b}}{b}) b^2 (r(t))^2 \right. \]

\[ + \sqrt{3} \arctan(1/3 \frac{r(t) \sqrt{b}}{b}) (r(t))^4 - 9 b^3 r(t) + 3 b (r(t))^3) \right. \]

There is no analytical solution to this differential equation, but the solution can be found numerically.

### 4.5. Motion with Lane–Emden profile

In the case of variable density for the CSM as given by the profile (25), the differential equation which models momentum conservation is

\[ 4 b^3 (r(t))^3 \rho c \pi \sqrt{3} \frac{d}{dt} r(t) = 4 b^3 r_0^3 \rho c \pi \sqrt{3} v_0 \right. \]

\[ - 4 \frac{b^3 r_0^3 \rho c \pi \sqrt{3} v_0}{(3 b^2 + (r(t))^2)^{3/2}} = 0 \ . \quad (55) \]

The variables can be separated and the solution is

\[ r(t; r_0, v_0, t_0, b) = \frac{N}{D} \right. \ , \quad (56) \]

where

\[ N = \sqrt{2} r_0^{3/4} (r_0^{13/2} + 2 r_0^{11/2} (t - t_0) v_0 \right. \]

\[ + r_0^{9/2} (t - t_0)^2 v_0^2 + 6 b_0^{10/2} \]

\[ + 18 b_0 r_0^{7/2} (t - t_0) v_0 + \sqrt{A} r_0^4 + \sqrt{A} r_0^3 (t - t_0) v_0 \right. \]

\[ + 9 b_0 r_0^{5/2} + 36 b_0 r_0^{3/2} v_0 (t - t_0) \]

\[ + 9 b_0^2 r_0^{1/2} + 18 \sqrt{A} b_0^{1/2} \]
and

\[ D = 2 \left( 3 b^2 + r_0^2 \right)^{3/2} , \]

with

\[ A(t - t_0) = r_0^3 (t - t_0)^2 v_0^2 + 36 b^4 (t - t_0) v_0 + 18 b^2 r_0^2 (t - t_0) v_0 + 2 r_0^4 (t - t_0) v_0 + 9 b^4 r_0 + 6 b^2 r_0^3 + r_0^5 . \]

This is the first solution and has an analytical form. The analytical solution for the velocity can be found from the first derivative of the analytical solution as represented by equation (56),

\[ v(t; r_0, v_0, t_0, b) = \frac{d}{dt} r(t; r_0, v_0, t_0, b) . \quad (57) \]

The previous differential equation (55) can be organized as

\[ \frac{d}{dt} r(t) = f(r; r_0, v_0, t_0, b) , \quad (58) \]

and we seek a power series solution of the form

\[ r(t) = a_0 + a_1 (t - t_0) + a_2 (t - t_0)^2 + a_3 (t - t_0)^3 + \ldots , \quad (59) \]

see [35, 36]. The Taylor expansion of equation (58) gives

\[ f(r; r_0, v_0, t_0, b) = b_0 + b_1 (t - t_0) + b_2 (t - t_0)^2 + b_3 (t - t_0)^3 + \ldots , \]

where the values of \( b_n \) are

\[
\begin{align*}
    b_0 &= f(r_0; r_0, v_0, t_0, b) \\
    b_1 &= \frac{\partial}{\partial t} f(r_0; r_0, v_0, t_0, b) \\
    b_2 &= \frac{1}{2!} \frac{\partial^2}{\partial t^2} f(r_0; r_0, v_0, t_0, b) \\
    b_3 &= \frac{1}{3!} \frac{\partial^3}{\partial t^3} f(r_0; r_0, v_0, t_0, b) \\
    \vdots & \quad \ldots .
\end{align*}
\]

The relation between the coefficients \( a_n \) and \( b_n \) is

\[
\begin{align*}
    a_1 &= b_0 \\
    a_2 &= \frac{b_2}{2} \\
    a_3 &= \frac{b_3}{3} \\
    \vdots & \quad \ldots .
\end{align*}
\]
The higher-order derivatives plus the initial conditions give

\[
\begin{align*}
a_0 &= r_0 \\
a_1 &= v_0 \\
a_2 &= -\frac{9}{2(3b^2+r_0^2)}r_0 \\
a_3 &= \frac{9}{2}v_0^3b^2(7b^2+r_0^2) \\
\ldots
\end{align*}
\]

These are the coefficient of the second solution, which is a power series.

A third solution can be represented by a difference equation which has the following type of recurrence relation

\[
\begin{align*}
r_{n+1} &= r_n + v_n \Delta t \\
v_{n+1} &= \frac{r_n^3v_0(3b^2+r_{n+1}^2)^{3/2}}{(3b^2+r_{n}^2)^{3/2}r_{n+1}^{3/2}},
\end{align*}
\]

where \( r_n, v_n, \) and \( \Delta t \) are the temporary radius, the velocity, and the interval of time.

The physical units have not yet been specified: pc for length and yr for time are the units most commonly used by astronomers. With these units, the initial velocity \( v_0 \) is expressed in \( \text{pc yr}^{-1} \), 1 yr = 365.25 days, and should be converted into \( \text{km s}^{-1} \); this means that \( v_0 = 1.02 \times 10^{-6}v_1 \) where \( v_1 \) is the initial velocity expressed in \( \text{km s}^{-1} \). In these units, the speed of light is \( c = 0.306 \text{ pc yr}^{-1} \).

### 4.6. Motion with the Lane–Emden profile, Padé approximation

Assuming a Lane–Emden profile, \( n = 5 \), as given by equation (26), the velocity is

\[
\frac{dr}{dt} = \frac{NL}{DL},
\]

where

\[
NL = r_0^3v_0\left(3b^2+r_0^2\right)^{3/2}\left(3b^2+r_0^2\right)^{3/2}
\]

and

\[
DL = -3\left(3b^2+r_0^2\right)^{3/2}\sqrt{3}r_0^3b^3 + 3\left(3b^2+r_0^2\right)^{3/2}\sqrt{3}b^3r_0^3 + (3b^2+r_0^2)^{3/2}\left(3b^2+r_0^2\right)^{3/2}r_0^3.
\]

The connected nonlinear equation is

\[
\frac{1}{r_0^3v_0(3b^2+r_0^2)^{3/2}\sqrt{3}b^2+r_0^2} \times \\
\left(54(b^2 + \frac{1}{3}r_0^2)(\frac{1}{18}r_0^3(r-r_0)\sqrt{3}b^2+r_0^2 + b^3\sqrt{3}(b^2 + \frac{1}{6}r_0^2))\sqrt{3}b^2+r_0^2 - 54\sqrt{3}b^2+r_0^2\sqrt{3}b^3(b^4 + \frac{1}{2}b^2r_0^2 + \frac{1}{18}rr_0^3)\right) = t - t_0.
\]
The Padé rational polynomial approximation of degree 2 in the numerator and degree 1 in the denominator for the left-hand side of the above equation gives

\[
\frac{NP}{2 \left(3b^2 + r_0^2\right)^2 v_0 \left(2rb^2 - 5b^2r_0 - rr_0^2\right)} = t - t_0 , \tag{66}
\]

where

\[
PN = -27 \left(r - r_0\right) \left(\frac{-4}{9} \left(rb^2 - \frac{5}{2} b^2 r_0 - \frac{1}{2} rr_0^2\right) \times \right.
\]
\[
\left(b^2 + \frac{1}{3} r_0^2\right) \sqrt{3b^2 + r_0^2 + b^5 \sqrt{3} \left(r - r_0\right)} \right) . \tag{67}
\]

The Padé approximant for the radius is

\[
r_{2,1} = \frac{NR}{DR} \tag{68}
\]

where

\[
NR = -18 \left(b^2 + \frac{1}{3} r_0^2\right)^2 b^2 \left(-\frac{1}{2} r_0^3 - \frac{1}{2} v_0(t - t_0)r_0^2 \right.
\]
\[
+ \frac{7}{2} b^2 r_0 + b^2 v_0(t - t_0)) \sqrt{3b^2 + r_0^2} + (81b^9 r_0 + 27 b^7 r_0^3) \sqrt{3}
\]
\[
+ \sqrt{972} \left(\left(b^2 + \frac{1}{3} r_0^2\right)^4 b^4 \left(\frac{9}{2} \sqrt{3} r_0^5 v_0(t - t_0) \sqrt{3b^2 + r_0^2} \right.
\]
\[
+ \left(\frac{1}{2} r_0^3 - \frac{1}{2} v_0(t - t_0)r_0^2 - \frac{3}{2} b^2 r_0 \right.
\]
\[
+ b^2 v_0(t - t_0))^2 b^2 + \frac{1}{3} r_0^2\right)^{\frac{1}{2}} , \tag{69}
\]

and

\[
DR = b^2 \left(3b^2 + r_0^2\right) \left(27 b^5 \sqrt{3} - 12 b^4 \sqrt{3b^2 + r_0^2}
\]
\[
+ 2 b^2 r_0^2 \sqrt{3b^2 + r_0^2} + 2 r_0^4 \sqrt{3b^2 + r_0^2}\right) , \tag{70}
\]

and the velocity is

\[
v_{2,1} = \frac{dr_{2,1}}{dt} = \frac{NVL}{DVL} , \tag{71}
\]

where

\[
NVL = -18 \sqrt{3} \left(3b^2 + r_0^2\right) v_0 \left(\left(-\frac{1}{2} r_0^3 - \frac{1}{2} v_0(t - t_0)r_0^2 - \frac{3}{2} b^2 r_0 \right.
\]
\[
+ b^2 v_0(t - t_0))^2 b^2 + \frac{1}{3} r_0^2\right)^{\frac{1}{2}} \left(\left(b^2 + \frac{1}{3} r_0^2\right)^4 b^4 \left(\frac{9}{2} \sqrt{3} r_0^5 v_0(t - t_0) \sqrt{3b^2 + r_0^2} \right.
\]
\[
+ \left(\frac{1}{2} r_0^3 - \frac{1}{2} v_0(t - t_0)r_0^2 - \frac{3}{2} b^2 r_0 \right.
\]
\[
+ b^2 v_0(t - t_0))^2 b^2 + \frac{1}{3} r_0^2\right)^{\frac{1}{2}} , \tag{72}
\]

\[\text{and the velocity is} \quad v_{2,1} = \frac{dr_{2,1}}{dt} = \frac{NVL}{DVL} , \tag{71}\]

where

\[
NVL = -18 \sqrt{3} \left(3b^2 + r_0^2\right) v_0 \left(\left(-\frac{1}{2} r_0^3 - \frac{1}{2} v_0(t - t_0)r_0^2 - \frac{3}{2} b^2 r_0 \right.
\]
\[
+ b^2 v_0(t - t_0))^2 b^2 + \frac{1}{3} r_0^2\right)^{\frac{1}{2}} \left(\left(b^2 + \frac{1}{3} r_0^2\right)^4 b^4 \left(\frac{9}{2} \sqrt{3} r_0^5 v_0(t - t_0) \sqrt{3b^2 + r_0^2} \right.
\]
\[
+ \left(\frac{1}{2} r_0^3 - \frac{1}{2} v_0(t - t_0)r_0^2 - \frac{3}{2} b^2 r_0 \right.
\]
\[
+ b^2 v_0(t - t_0))^2 b^2 + \frac{1}{3} r_0^2\right)^{\frac{1}{2}} , \tag{72}\]

\[\text{and the velocity is} \quad v_{2,1} = \frac{dr_{2,1}}{dt} = \frac{NVL}{DVL} , \tag{71}\]

where

\[
NVL = -18 \sqrt{3} \left(3b^2 + r_0^2\right) v_0 \left(\left(-\frac{1}{2} r_0^3 - \frac{1}{2} v_0(t - t_0)r_0^2 - \frac{3}{2} b^2 r_0 \right.
\]
\[
+ b^2 v_0(t - t_0))^2 b^2 + \frac{1}{3} r_0^2\right)^{\frac{1}{2}} \left(\left(b^2 + \frac{1}{3} r_0^2\right)^4 b^4 \left(\frac{9}{2} \sqrt{3} r_0^5 v_0(t - t_0) \sqrt{3b^2 + r_0^2} \right.
\]
\[
+ \left(\frac{1}{2} r_0^3 - \frac{1}{2} v_0(t - t_0)r_0^2 - \frac{3}{2} b^2 r_0 \right.
\]
\[
+ b^2 v_0(t - t_0))^2 b^2 + \frac{1}{3} r_0^2\right)^{\frac{1}{2}} , \tag{72}\]
\[
-b^2 v_0 (t - t_0)^2 \right) \left\{ \frac{1}{2} (2b^2 - r_0^2) \right\} \sqrt{3b^2 + r_0^2} \r
-108 \left( b^2 + \frac{1}{3} r_0^2 \right) b^2 (-1/2 r_0^3 - 1/2 v_0 (t - t_0) r_0^2 - 3/2 b^2 r_0 \\
+ b^2 v_0 (t - t_0) (b^2 - 1/2 r_0^2), \tag{72}\n\]

and

\[
DVL = 18 \sqrt{972} \sqrt{3} \left\{ (b^2 + \frac{1}{3} r_0^2)^4 b^4 (9/2 \sqrt{3} r_0 b^5 v_0 \\
(t - t_0) \sqrt{3b^2 + r_0^2} + b^2 + \frac{1}{3} r_0^2) (-1/2 r_0^3 - 1/2 v_0 (t - t_0) r_0^2 \\
-3/2 b^2 r_0 + b^2 v_0 (t - t_0))^2 \right\}^{\frac{1}{2}} \left\{ (\frac{1}{4} b^4 + 2 b^2 r_0^2 + 2 r_0^4) \sqrt{3b^2 + r_0^2} \\
+ 27 b^5 \sqrt{3} \right\}. \tag{73}\n\]

### 4.7. Motion with a power law profile

The differential equation which models momentum conservation in the presence of a power law behavior of the density, as given by (32), is

\[
\frac{d}{dt} \left( \frac{\rho_c \pi}{d - 3} \left( \frac{r_0}{r(t)} \right)^d \right) + 4 \frac{\rho_c \pi r_0^3}{d - 3} \frac{d}{dt} r(t) \\
-4/3 \rho_c \pi r_0^3 v_0 = 0. \tag{74}\n\]

A first solution can be found numerically, see [12] for more details. A second solution is a truncated series about the ordinary point \( t = t_0 \) which to fourth order has coefficients

\[
a_0 = r_0 \\
a_1 = v_0 \\
a_2 = -\frac{3 v_0 r_0^2}{2 r_0} \\
a_3 = \frac{(d+7)v_0^3}{2 r_0^d}. \tag{75}\n\]

A third approximate solution can be found assuming that \( 3r_0^d r^{4-d} \gg -(4r_0^3 d - r_0^3 d^2) r \)

\[
r(t) = \left( r_0^{4-d} - \frac{1}{3} dr_0^4 (4 - d) \right) \\
+ \frac{1}{3} (4 - d) v_0 r_0^{3-d} (3 - d) (t - t_0) \frac{1}{(4 - d)}. \]

This is an important approximate result because, given the astronomical relation \( r(t) \propto t^\alpha \), we have \( d = 4 - \frac{1}{\alpha} \).
5. Conservation of the relativistic momentum

The thin layer approximation assumes that all the swept mass during the travel from the initial time, \( t_0 \), to the time \( t \), resides in a thin shell of radius \( r(t) \) with velocity \( v(t) \). The relativistic conservation of momentum, see [37, 38, 39], is formulated as

\[
M(r_0)\gamma_0\beta_0 = M(r)\gamma\beta, \tag{76}
\]

where

\[
\gamma_0 = \frac{1}{\sqrt{1 - \beta_0^2}}; \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \tag{77}
\]

and

\[
\beta_0 = \frac{v_0}{c}; \quad \beta = \frac{v}{c}. \tag{78}
\]

5.1. Lane–Emden case \((n = 5)\)

On assuming a Lane–Emden dependence \((n = 5)\), the total mass \( M(r; b) \) contained between 0 and \( r \) is given by equation \((29)\). The equation of the relativistic conservation of momentum is easily solved for \( \beta \) as a function of the radius:

\[
\beta = \sqrt{\frac{A}{4}} \left[ 3b^2 + r^2 \right] \left( 3b^2 + r_0^2 \right)^{3/2} \rho \pi r_0^3 \beta_0, \tag{79}
\]

with

\[
A = -27b^6r^6\beta_0^2 + 27b^6\beta_0^2r_0^6 - 27b^4r^6\beta_0^2r_0^2 + 27b^4r^2\beta_0^2r_0^6 - 9b^2r^6\beta_0^2r_0^4 + 9b^2r^4\beta_0^2r_0^6 + 27b^6r_0^6 + 27b^4r_0^2 + 9b^2r^6r_0^4 + r^6r_0^6.
\]

Inserting

\[
\beta = \frac{1}{c} \frac{d}{dt} r(t), \tag{80}
\]

the relativistic conservation of momentum can be written as the differential equation

\[
\frac{4b^3(r(t))^3\rho \pi \sqrt{3} \frac{d}{dt} r(t)}{(3b^2 + (r(t))^2)^{3/2}c} + \frac{4b^3r_0^3\rho \pi \sqrt{3}\beta_0}{(3b^2 + r_0^2)^{3/2}\sqrt{-\beta_0^2 + 1}}. \tag{81}
\]

This first order differential equation can be solved by separating the variables:

\[
\int_0^r \frac{A}{\sqrt{A(3b^2 + r^2)(3b^2 + r_0^2)r_0^3\beta_0}} dr = c(t - t_0). \tag{82}
\]

The previous integral does not have an analytical solution and we treat the previous result as a nonlinear equation to be solved numerically. The differential equation has a truncated series solution about the ordinary point \( t = t_0 \) which to fifth order is

\[
r_s(t) = \sum_{n=0}^{4} a_n(t - t_0)^n. \tag{83}
\]
The coefficients are

\[
\begin{align*}
  a_0 &= r_0 \\
  a_1 &= c\beta_0 \\
  a_2 &= \frac{9b^2\beta_0^2c^2(\beta_0^2-1)}{2r_0(3b^2+r_0^2)} \\
  a_3 &= \frac{9c^4(\beta_0-1)(\beta_0+1)(12b^2\beta_0^2-7b^2-r_0^2)\beta_0b^2}{2(3b^2+r_0^2)^2r_0^2} \\
  a_4 &= \frac{9b^2(\beta_0-1)(\beta_0+1)B\beta_0^4c^4}{8r_0^4(3b^2+r_0^2)^2}
\end{align*}
\]

**where** \( B = 756b^4\beta_0^4 - 927b^4\beta_0^2 - 117b^2\beta_0^2r_0^2 + 231b^4 + 69b^2r_0^2 + 4r_0^4 \)

The velocity approximated to the fifth order is

\[
v_s(t) = \sum_{n=1}^{4} \frac{a_n(t-t_0)^n}{(t-t_0)}
\]

The presence of an analytical expression for \( \beta \) as given by equation (79) allows the recursive solution

\[
\begin{align*}
  r_{n+1} &= r_n + c\beta_n \Delta t \\
  \beta_{n+1} &= \sqrt{A(3b^2+r_{n+1}^2)(3b^2+r_{n+1}^2)r_n^3\beta_n} \\
  A &= \frac{27b^6\beta_n^2r_n^6 - 27b^6\beta_n^2r_{n+1}^6 + 27b^4\beta_n^2r_{n+1}^2r_n^6}{27b^4\beta_n^2r_{n+1}^6r_n^2 + 9b^2\beta_n^2r_{n+1}^4r_n^6 - 9b^2\beta_n^2r_{n+1}^6r_n^4 + 27b^6r_{n+1}^6 + 9b^2r_{n+1}^6r_n^4 + r_{n+1}^6r_n^4}
\end{align*}
\]

where \( r_n, \beta_n \) and \( \Delta t \) are the temporary radius, the relativistic \( \beta \) factor, and the interval of time, respectively. Up to now we have taken the time interval \( t - t_0 \) to be that as seen by an observer on earth. For an observer which moves on the expanding shell, the proper time \( \tau^* \) is

\[
\tau^* = \int_{t_0}^{t} \frac{dt}{\gamma} = \int_{t_0}^{t} \sqrt{1-\beta^2}dt
\]

see [40, 41, 42, 43]. In the series solution framework, \( \beta = \frac{v_s}{c} \), and \( v_s \) is given by equation (85).

### 5.2. Plummer case (\( \eta = 6 \))

The relativistic conservation of momentum for a Plummer profile with \( \eta = 6 \) is

\[
\frac{AN}{AD} = \frac{BN}{BD}
\]

\[
AN = 3\pi b^3 \left( 9 \arctan \left( \frac{1}{3} \frac{r(t)\sqrt{3}}{b} \right) \sqrt{3}b^4 + 6 \arctan \left( \frac{1}{3} \frac{r(t)\sqrt{3}}{b} \right) \sqrt{3b^2(r(t))^2} \\
+ \arctan \left( \frac{1}{3} \frac{r(t)\sqrt{3}}{b} \right) \sqrt{3}(r(t))^4 - 9b^3r(t) + 3b(r(t))^3 \right) \frac{d}{dt}r(t)
\]

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\[ AD = 2 \left( 3b^2 + (r(t))^2 \right)^2 c \sqrt{1 - \left( \frac{d}{dt} r(t) \right)^2} , \quad (90) \]

\[ BN = 3 \pi b^3 \left( 9 \arctan \left( \frac{1}{3} \frac{R_0 \sqrt{3}}{b} \right) \sqrt{3} b^4 + 6 \arctan \left( \frac{1}{3} \frac{R_0 \sqrt{3}}{b} \right) \sqrt{3} b^2 R_0^2 \right. \]
\[ + \arctan \left( \frac{1}{3} \frac{R_0 \sqrt{3}}{b} \right) \sqrt{3} R_0^4 - 9 b^3 R_0 + 3 b R_0^3 \beta_0 \quad , \quad (91) \]

\[ BD = 2 \left( 3b^2 + R_0^2 \right)^2 \sqrt{1 - \beta_0^2} . \quad (92) \]

The above differential equation has a truncated series solution about the ordinary point \( t = t_0 \) which to fourth order is

\[ r_s(t) = \sum_{n=0}^{3} a_n (t - t_0)^n . \quad (93) \]

The coefficients are

\[ a_0 = r_0 \]
\[ a_1 = 36 c^2 \left( \beta_0^2 - 1 \right) b^3 r_0^2 \beta_0^2 \]
\[ a_2 = \frac{A2N}{A2D} \quad , \quad (94) \]

where

\[ A2N = 36 c^2 \left( \beta_0^2 - 1 \right) b^3 r_0^2 \beta_0^2 \quad , \quad (95) \]

and

\[ A2D = 27 \sqrt{3} b^6 \arctan \left( \frac{1}{3} \frac{r \theta \sqrt{3}}{b} \right) + 27 \sqrt{3} b^4 r \theta^2 \arctan \left( \frac{1}{3} \frac{r \theta \sqrt{3}}{b} \right) \]
\[ + 9 \sqrt{3} b^2 r \theta^4 \arctan \left( \frac{1}{3} \frac{r \theta \sqrt{3}}{b} \right) + \sqrt{3} r \theta^6 \arctan \left( \frac{1}{3} \frac{r \theta \sqrt{3}}{b} \right) \]
\[ - 27 b^5 r \theta + 3 b r \theta^5 \quad . \quad (96) \]

6. Classical applications to SN 1993J

This section introduces: the SN chosen for testing purposes, the astrophysical environment connected with the selected SN, two types of fit commonly used to model the radius–time relation in SN, and the application of the results obtained for the Lane–Emden, Plummer, and power law profiles.
6.1. The data

The data of SN 1993J, radius in pc and elapsed time in years, can be found in Table 1 of [7] and a multiple radio image is displayed in Figure 2.

The instantaneous velocity of expansion can be deduced from the formula

\[ v_i = \frac{r_{i+1} - r_i}{t_{i+1} - t_i}, \] (97)

where \( r_i \) is the radius and \( t_i \) is the time at the position \( i \). The uncertainty in the instantaneous velocity is found by implementing the error propagation equation, see [44]. A discussion of the thickness of the radio shell in SN 1993J in the framework of a reverse shock [14][15] can be found in [45]. The thickness of the radio shell can also be explained in the framework of the image theory, see Section 6.3 in [12].

6.2. Astrophysical Scenario

The progenitor of SN 1993J was a K-supergiant star, see [46] and probably formed a binary system with a B-supergiant companion star, see [47]. These massive stars have strong stellar winds, and blow huge bubbles (of \( \approx \) 20 to 40 pc in size) in their lives. From an analytic approximation, [48] obtained a formula for the radius of the bubble, their eq. (21), see too their Fig. 3. The inner region of the bubble has very low density, and the border of the bubble is the wall of a relatively dense shell which is in contact with the ISM. The circumstellar envelope of the Pre SN 1993J with which the SN shock front is interacting is a small structure within the big bubble created by the strong stellar winds of the SN progenitor.
(and probably of its binary companion) during its life. Therefore this envelope of the Pre SN 1993J would be the product of a recent event of stellar mass ejection suffered by the Pre SN 1993J. That is to say that the SN shock wave interacts with a CSM created by Pre supernova mass loss. In this respect, [49] gave evidence that significant mass loss had taken place before the explosion, see also [50]. In the scenario in which the Pre SN 1993J had formed an interacting binary system, this can be interpreted in terms of a process of mass transfer. It is possible that this type of supernova originates in interacting binary systems.

6.3. Two types of fit

The quality of the fits is measured by the merit function $\chi^2$

$$\chi^2 = \sum_j \frac{(r_{th} - r_{obs})^2}{\sigma_{obs}^2},$$

(98)

where $r_{th}$, $r_{obs}$ and $\sigma_{obs}$ are the theoretical radius, the observed radius, and the observed uncertainty, respectively. A first fit can be done by assuming a power law dependence of the type

$$r(t) = r_p t^{\alpha_p},$$

(99)

where the two parameters $r_p$ and $\alpha_p$ as well their uncertainties can be found using the recipes suggested in [12]. A second fit can be done by assuming a piecewise function as in Fig. 4 of [7].

$$r(t) = \begin{cases} r_{br} \left( \frac{t}{t_{br}} \right)^{\alpha_1} & \text{if } t \leq t_{br} \\ r_{br} \left( \frac{t}{t_{br}} \right)^{\alpha_2} & \text{if } t > t_{br}. \end{cases}$$

(100)

This type of fit requires the determination of four parameters: $t_{br}$ the break time, $r_{br}$ the radius of expansion at $t = t_{br}$, and the exponents $\alpha_1$ and $\alpha_2$ of the two phases. The parameters of these two fits as well the $\chi^2$ can be found in Table 1.

6.4. The Lane–Emden case

The radius of SN 1993J which represents the momentum conservation in a Lane–Emden profile of density is shown in Figure 3; $r_0$ and $t_0$ are fixed by the observations and the two free parameters are $b$ and $v_0$.

Figure 4 compares the theoretical solution and the series expansion about the ordinary point $t_0$.

The range of time in which the series solution approximates the analytical solution is limited. Figure 5 compares the theoretical solution and the recursive solution as represented by equation 62.

The recursive solution approximates the analytical solution over the entire range of time considered, and the error at $t = 10$ yr is $\approx 0.6\%$ when $\Delta t = 0.05$ yr and $\approx 0.1\%$ when $\Delta t = 0.0083$ yr. The time evolution of the velocity is shown in Figure 6.

6.5. Plummer and power law cases

The numerical solution of the differential equation connected with the Plummer-like profile, $\eta = 6$, is shown in Figure 7 when the data of Table 1 is adopted.
Table 1. Numerical values of the parameters of the fits and $\chi^2$; $N$ represents the number of free parameters.

| $N$ | values | $\chi^2$ |
|-----|--------|----------|
| 2   | power law as a fit | 6364 |
|     | $\alpha_p = 0.82 \pm 0.0048$ | |
|     | $r_p = (0.015 \pm 0.00011) \text{ pc}$ | |
| 4   | piecewise fit | 32 |
|     | $\alpha_1 = 0.83 \pm 0.01$ | |
|     | $\alpha_2 = 0.78 \pm 0.0077$ | |
|     | $r_{br} = 0.05 \text{ pc}; t_{br} = 4.10 \text{ yr}$ | |
| 2   | Plummer profile, $\eta = 6$ | 265 |
|     | $b = 0.0045 \text{ pc}; r_0 = 0.008 \text{ pc}; v_0 = 19500 \frac{\text{ km}}{\text{ s}}$ | |
| 2   | Lane – Emden profile | 471 |
|     | $b = 0.00367 \text{ pc}; r_0 = 0.008 \text{ pc}; v_0 = 19500 \frac{\text{ km}}{\text{ s}}$ | |
| 2   | Power law profile | 276 |
|     | $d = 2.93; r_0 = 0.0022 \text{ pc}; t_0 = 0.249 \text{ yr}; v_0 = 100000 \frac{\text{ km}}{\text{ s}}$ | |

Figure 3. Theoretical radius as given by equation (56) (full line), with data as in Table 1. The astronomical data of SN 1993J is represented with vertical error bars.
Figure 4. Theoretical radius as given by equation (56) (full line) and series solution as given by equation (59) (dashed line). Data as in Table 1.

Figure 5. Theoretical radius as given by equation (56) (full line) and recursive solution as given by equation (62) when $\Delta t = 0.05\text{yr}$ (dashed line). Data as in Table 1.

Figure 6. Instantaneous velocity of SN 1993J with uncertainty and theoretical velocity as given by equation (57) (full line). Data as in Table 1.
Figure 7. Theoretical radius for the Plummer-type profile as obtained by the solution of the nonlinear equation connected with (54) (full line). Data as in Table 1. The astronomical data of SN 1993J is represented with vertical error bars.

Table 2. Numerical values of the parameters used in three relativistic solutions for the Lane–Emden case ($n=5$)

| parameters |
|------------|
| $t_0 = 10^{-4}$ yr ; $r_0 = 0.0033$ pc ; $\beta_0 = 0.3333$ ; $b = 0.004$ pc |

A comparison with the power law behavior for the CSM is shown in Figure 8 which is built from the data in Table 1.

The series solution for the power law dependence of the CSM with coefficients as given by equation (75) is not reported because the range in time of its reliability is limited to $t - t_0 \approx 0.0003$ yr.

7. Relativistic applications to SN 1993J

We now apply the relativistic solutions derived so far to SN 1993J in the Lane–Emden and Plummer cases.

7.1. Relativistic Lane–Emden case ($n=5$)

The initial observed velocity, $v_0$, as deduced from radio observations, see [7], is $v_0 \approx 20000$ kms$^{-1}$ at $t_0 \approx 0.5$ yr. We now reduce the initial time $t_0$ and we increase the velocity up to the relativistic regime, $t_0 = 10^{-4}$ yr, and $v_0 = 100000$ kms$^{-1}$. This choice of parameters allows fitting the observed radius–time relation that should be reproduced. The data used in the simulation is shown in Table 2. The relativistic numerical solution of equation (82) is shown in Figure 9, the relativistic series solution as given by (83) is shown in Figure 10, and Figure 11 contains the recursive solution as given by equation (87).
Figure 8. Theoretical radius for the power law case as obtained by the solution of the nonlinear equation connected with \( (74) \) (full line). Data as in Table 1. The astronomical data of SN 1993J is represented with vertical error bars.

Figure 9. Theoretical relativistic radius as solution of the nonlinear equation \( (82) \) (full line), with data as in Table 2. Lane–Emden case \( (n = 5) \). The astronomical data of SN 1993J is represented with vertical error bars.
Figure 10. Theoretical relativistic radius as solution of the nonlinear equation (82) (full line), and series solution as given by equation (83) (dashed line): Lane–Emden case \((n = 5)\). Data as in Table 2. The time is expressed in days.

Figure 11. Theoretical relativistic radius as solution of the nonlinear equation (82) (full line), and recursive solution as given by equation (87) when \(\Delta t = 0.053\) yr (dashed line): Lane–Emden case \((n = 5)\). Data as in Table 2.
Table 3. Numerical values of the parameters used in the Plummer ($\eta = 6$) relativistic solution.

| parameters                      |
|---------------------------------|
| $t_0 = 1 \times 10^{-7}$ yr or $t_0 = 3.15$ seconds |
| $R_0 = 0.01$ pc                  |
| $\beta_0 = 0.666$               |
| $b = 0.028$ pc                  |
| $d = 1$                         |

Figure 12. Theoretical relativistic radius as solution of the nonlinear equation (88) (full line), with data as in Table 3 and astronomical data of SN 1993J with vertical error bars: Plummer case ($\eta = 6$)

7.2. Relativistic Plummer case ($\eta = 6$)

The relativistic numerical solution of equation (88) is shown in Figure 12 with parameters as in Table 3. Figure 13 shows the theoretical radius in units of $10^{-3}$ pc as a function of the time in seconds.

8. Classical Applications to SNR

In the previous section, we derived three equations of motion in the form of nonlinear equations and three Padé approximated equations of motion. We now check the reliability of the numerical and approximate solutions on four SNRs: Tycho or 3C10, see [51] and the radio map in Figure 14, Cas A, see [52] and the radio map in Figure 15, Cygnus loop, see [53] and the X-map in 16 and SN 1006, see [54] and image in 17.

The three astronomical measurable parameters are the time since the explosion in years, $t$, the actual observed radius in pc, $r$, and the present velocity of expansion in km s$^{-1}$, see Table 4. The astrophysical units have not yet been specified: pc for length and yr for time.
Figure 13. Theoretical relativistic radius in units of $10^{-3}$ pc as solution of the nonlinear equation (88) (full line), with data as in Table 3 and time in seconds: Plummer case ($\eta = 6$).

Figure 14. Radio image of Tycho or 3C10.
Figure 15. Image of CasA at 3 different frequencies: 1.4 GHz (L band), 5.0 GHz (C band), and 8.4 GHz (X band).

Figure 16. X image of the Cygnusloop.
Table 4. Observed astronomical parameters of SNRs

| Name         | Age (yr) | Radius (pc) | Velocity (km s\(^{-1}\)) | References             |
|--------------|----------|-------------|---------------------------|------------------------|
| Tycho        | 442      | 3.7         | 5300                      | Williams et al. 2016   |
| Cas A        | 328      | 2.5         | 4700                      | Patnaude and Fesen 2009|
| Cygnus loop  | 17000    | 24.25       | 250                       | Chiad et al. 2015      |
| SN 1006      | 1000     | 10.19       | 3100                      | Uchida et al.2013      |

are the units most commonly used by astronomers. With these units, the initial velocity is \(v_0 (\text{km s}^{-1}) = 9.7968 \times 10^3 v_0 (\text{pc yr}^{-1})\). The determination of the four unknown parameters, which are \(t_0, r_0, v_0\) and \(b\), can be obtained by equating the observed astronomical velocities and radius with those obtained with the Padé rational polynomial, i.e.,

\[
\begin{align*}
  r_{2,1} & = \text{Radius (pc)}, \\
  v_{2,1} & = \text{Velocity (km s}^{-1}).
\end{align*}
\]

In order to reduce the unknown parameters from four to two, we fix \(v_0\) and \(t_0\). The two parameters \(b\) and \(r_0\) are found by solving the two nonlinear equations (101) and (102). The results for the three types of profiles here adopted are shown in Tables 5, 6 and 7.

The goodness of the approximation is evaluated through the percentage error, \(\delta\), which is

\[
\delta = \frac{|r_{2,1} - r_E|}{r_E} \times 100,
\]

where \(r_{2,1}\) is the Padé approximated radius and \(r_E\) is the exact solution which is obtained by solving numerically the nonlinear equation of motion, as an example equation (39) in the exponential case. The numerical values of \(\delta\) are shown in column 6 of Tables 5, 6 and 7.

Another useful astrophysical variable is the predicted decrease in velocity on the basis
Table 5. Theoretical parameters of SNRs for the Padé approximated equation of motion with an exponential profile.

| Name          | $t_0$(yr) | $r_0$(pc) | $v_0(km \cdot s^{-1})$ | b(pc) | $\delta$ (%) | $\Delta v(km \cdot s^{-1})$ |
|---------------|-----------|-----------|-------------------------|-------|--------------|-----------------------------|
| Tycho         | 0.1       | 1.203     | 8000                    | 0.113 | 5.893        | -1.35                       |
| Cas A         | 1         | 0.819     | 8000                    | 0.1   | 6.668        | -3.29                       |
| Cygnus loop   | 10        | 12.27     | 3000                    | 45.79 | 6.12         | -0.155                      |
| SN 1006       | 1         | 5.49      | 10000                   | 2.332 | 1.455        | -12.34                      |

Table 6. Theoretical parameters of SNRs for the Padé approximated equation of motion with a Gaussian profile.

| Name          | $t_0$(yr) | $r_0$(pc) | $v_0(km \cdot s^{-1})$ | b(pc) | $\delta$ (%) | $\Delta v(km \cdot s^{-1})$ |
|---------------|-----------|-----------|-------------------------|-------|--------------|-----------------------------|
| Tycho         | 0.1       | 1.022     | 8000                    | 0.561 | 8.517        | -10.469                     |
| Cas A         | 1         | 0.741     | 7000                    | 0.406 | 7.571        | -13.16                      |
| Cygnus loop   | 10        | 11.92     | 3000                    | 21.803 | 7.875       | -0.161                      |
| SN 1006       | 1         | 5.049     | 10000                   | 4.311 | 4.568        | -18.58                      |

Table 7. Theoretical parameters of SNRs for the Padé approximated equation of motion with a Lane–Emden profile.

| Name          | $t_0$(yr) | $r_0$(pc) | $v_0(km \cdot s^{-1})$ | b(pc) | $\delta$ (%) | $\Delta v(km \cdot s^{-1})$ |
|---------------|-----------|-----------|-------------------------|-------|--------------|-----------------------------|
| Tycho         | 0.1       | 0.971     | 8000                    | 0.502 | 3.27         | -14.83                      |
| Cas A         | 1         | 0.635     | 8000                    | 0.35  | 4.769        | -23.454                     |
| Cygnus loop   | 10        | 11.91     | 3000                    | 27.203 | 7.731      | -0.162                      |
| SN 1006       | 1         | 5         | 10000                   | 4.85  | 3.297        | -19.334                     |

of the Padé approximated velocity, $v_{2,1}$, over ten years, see column 7 of Tables 5, 6 and 7. More details can be found in [55].

9. Conclusions

**Classic Case: Lane–Emden type**

The thin layer approximation which models the expansion in a self-gravitating medium of the Lane–Emden type ($n = 5$) can be modeled by a differential equation of the first order for the radius as a function of time. This differential equation has an analytical solution represented by equation (56). A power law series, see equation (59), can model the solution of the Lane–Emden type for a limited range of time, see Figure 4 relative to SN 1993J. A recursive solution for the first order differential equation, as represented by equation (62), approximates quite well the analytical solution of the Lane–Emden type and at the time of
$t = 10$ yr a precision of four digits is reached when $\Delta t = 10^{-3}$ yr. The Padé rational polynomial approximation allows deriving an approximate solution for the advancing radius, see equation (68), and for the velocity, see equation (71).

**Classic Case: Plummer case**

The differential equation which represents the expansion in the Plummer case ($\eta = 6$) is equation [54] no analytical solution exists, but a numerical solution is shown in Figure 7 relative to SN 1993J.

**Relativistic case: Lane–Emden case**

The temporal evolution of an SN in a self-gravitating medium of the Lane–Emden type can be found by applying the conservation of relativistic momentum in the thin layer approximation. This relativistic invariant is determined by a differential equation of the first order, see equation (81). Three different relativistic solutions for the radius as a function of time have been derived: (i) a numerical solution, see equation (82) which covers the range $10^{-4}$ yr $< t < 10$ yr and fits the observed radius–time relation for SN 1993J; (ii) a series solution, see equation (82), which has a limited range of validity, $10^{-4}$ yr $< t < 1.2 \times 10^{-2}$ yr; (iii) a recursive solution in which the desired accuracy is reached by decreasing the time step $\Delta t$. The relativistic results here presented model SN 1993J and are obtained with an initial velocity of $v_0 = 100000$ km s$^{-1}$ or $\beta_0 = 0.333$ or $\gamma = 1.06$.

**Relativistic case: Plummer case**

The temporal evolution of an SN in a medium of the Plummer type, $\eta = 6$, can be found by applying the conservation of relativistic momentum in the thin layer approximation. This relativistic invariant is found via a differential equation of the first order, see equation (88). A series solution is shown in equation (93) relative to SN 1993J.

**Classic SNRs**

The application of the Padé approximant to the left-hand side of the complicated equation of motion allows finding three approximate laws of motion, see equations (41), (50) and (68), and three approximate velocities, see equations (42), (51) and (71). An astrophysical test was performed on four SNRs assumed to be spherical; the four sets of parameters were presented in Tables 5, 6 and 7. The percentage of error of the Padé approximated solutions for the radius is always less than 10% with respect to the numerical exact solution, see column 6 of the three last tables. In order to produce an astrophysical prediction, the theoretical decrease in velocity for the four SNRs here analysed has been evaluated, see column 7 of Tables 5, 6 and 7.

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