Dots & Boxes is PSPACE-complete

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Abstract

Exactly 20 years ago at MFCS, Demaine posed the open problem whether the game of Dots & Boxes is PSPACE-complete. Dots & Boxes has been studied extensively, with for instance a chapter in Berlekamp et al. Winning Ways for Your Mathematical Plays, a whole book on the game The Dots and Boxes Game: Sophisticated Child’s Play by Berlekamp, and numerous articles in the Games of No Chance series. While known to be NP-hard, the question of its complexity remained open. We resolve this question, proving that the game is PSPACE-complete by a reduction from a game played on propositional formulas.

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1 Introduction

Dots & Boxes is a popular paper-and-pencil game that is played by two players on a grid of dots. The players take turns connecting two adjacent dots. If a player completes the fourth side of a unit box, the player is awarded a point and an additional turn. When no more moves can be made, the player with the highest score wins the game.

Originally described in 1883 [29], Dots & Boxes has since received a considerable amount of attention in the research community. In Winning Ways for Your Mathematical Plays, Berlekamp, Conway, and Guy [6] were among the first to discuss a number of interesting mathematical properties of the game. Later, Berlekamp [5] wrote an entire book The Dots-and-Boxes game: Sophisticated Child’s Play about the game, in particular discussing winning strategies in particular positions. Since then, the mathematics of Dots & Boxes and variants has been discussed in many papers and books [1, 2, 7, 12, 16, 21, 22, 25, 31, 33, 34]. There is also a rich body of work on solvers for Dots & Boxes [3, 4, 11, 27, 35].

Berlekamp et al. [6] argue that deciding the winner of a generalized version of Dots & Boxes, called Strings-and-Coin, is NP-hard. In this game, players take turns in removing edges of a given graph, scoring a point when they isolate a vertex. When restricted to the dual graph of a square grid, this corresponds to a dual formulation of Dots & Boxes. Eppstein [17] notes that the reduction given by Berlekamp et al. should extend to Dots & Boxes, and a formal proof of the NP-hardness is given in [8].

Exactly 20 years ago at MFCS, Demaine posed the open problem whether Dots & Boxes is PSPACE-complete [13]. Bounded two-player games, like Dots & Boxes, (that is, games in which the number of moves is bounded) naturally lie in PSPACE, since a Turing machine using space polynomial in the board size is able to search the entirety of the game.

1 For a visual explanation of the game see https://youtu.be/KboGyIilP6k last accessed 6.5.2021
space. Often, these games are also PSPACE-hard \cite{13}. PSPACE-hardness of many bounded two-player games is shown by a reduction from Generalized Geography, which is proven PSPACE-complete by Lichtenstein and Sipser \cite{28}. For example, the PSPACE-completeness of Reversi \cite{24}, uncooperative UNO \cite{14}, and Tic-Tac-Toe \cite{23} were shown by a reduction from Generalized Geography. However, unlike Dots & Boxes, the setting of Generalized Geography prescribes a stricter order on players’ moves, making a reduction to Dots & Boxes challenging to obtain.

In their seminal work, Hearn and Demaine \cite{20,21} introduce Constraint Logic, a framework for analyzing complexity of games and puzzles. Inspired by Flake and Baum’s proof of Rush Hour \cite{18}, it specifies a type of game played on a constraint graph. The framework includes bounded/unbounded state spaces and single/two-player variations. In the same work, Hearn and Demaine go on to provide a number of simpler reductions for various known PSPACE-complete games (including Rush Hour), as well as new proofs for several PSPACE-complete games. However, the Constraint Logic framework is intended for proving hardness of partisan games (games in which the moves available to the two players are different), whereas Dots & Boxes is not a partisan game.

Strings-and-Coins and the related game of Nimstring were very recently (while we were preparing this submission) proven to be PSPACE-complete by Demaine and Diomidov \cite{15} by a reduction from a game on a DNF formula $G_{pos}(\text{POS DNF})$ \cite{32}. But, as they point out, their results do not apply to Dots & Boxes, since the game positions they construct rely on multi-graphs (which additionally are neither planar nor have a maximum degree of 4). Specifically, they propagate signals through multi-edges consisting of a polynomial number of parallel edges, and the winner is the player who removes the last edge. As consequence, our reduction bears little commonalities with theirs.

In this paper, we prove that Dots & Boxes is PSPACE-complete by a reduction from $G_{pos}(\text{POS CNF})$. The starting point of our construction are strategies for Dots & Boxes endgames that were also used to prove NP-hardness. However, the NP-hardness is proven by having one player be in control, and there being only one way for the other player to respond. This de facto makes the game to be 1-player game. For PSPACE-hardness we need both players to have choices, making it a true 2-player game. This gives a lot of freedom to the players, and makes it much more difficult to construct gadgets to control the gameplay, in particular because moves and scoring opportunities for one player—if not played immediately—are also available to the other player.

In Section 1.1 we discuss the gameplay of Dots & Boxes in detail, and introduce terminology coined by Berlekamp et al. \cite{6}. In Section 2 we present the general structure of our reduction, and then describe our gadgets in Section 3. In Section 4 we first show that the players’ strategies, which we intend the players to use, are optimal for them and finally prove PSPACE-hardness.

1.1 Dots & Boxes

On the surface, Dots & Boxes is quite a simple game. The starting and a typical final position for a $10 \times 10$ grid are shown in Figure 1. We refer to the players playing the blue and the red colors as Trudy and Fred, respectively. The color of a line connecting two dots indicates which player drew it, and the color of a box—which player closed it.

Consider a dual graph $G$ of a board of Dots & Boxes, where a node in $G$ corresponds to a box or the unbounded face, and a pair of nodes in $G$ is connected with an edge if the corresponding move is still available, i.e., the line between the boxes has not been drawn. Let the degree of a box be the degree of the corresponding node in $G$. 

In Dots & Boxes, a typical game usually results in a board state that consists exclusively of moves that open the possibility for the opponent to claim a number of boxes in their next turn (see Figure 1b). That is, in this state there are no degree-1 boxes, but any move made by a player creates a degree-1 box that can be immediately claimed by the opponent. Consider such a board configuration $S$ and any available move $\ell$ in it. At least one box $b$ incident to $\ell$ has degree two in $S$ (before the move $\ell$ is made). Consider a maximal component $\sigma$ of degree-2 boxes in $S$ containing $b$. There are two cases, either $\sigma$ is a chain ending in boxes of degree higher than two (or the outer face), or $\sigma$ is a cycle. Then we say that a player making the move $\ell$ opens the chain (cycle) $\sigma$ for the opponent.

To devise a good strategy for Dots & Boxes, it is important to note that a player is not obliged to claim a box whenever they have the ability to do so. While seemingly counter-intuitive, it is sometimes beneficial for a player to sacrifice a small number of boxes for long-term gain. Consider the position in Figure 2 and let it be Fred’s (red) turn. Here, it may seem intuitive for Fred to claim the bottom three boxes (Figure 2 (top)). However, after doing so Fred has to make an extra move, allowing Trudy (blue) to claim the remaining four boxes and win the game. On the other hand, by sacrificing two boxes (Figure 2 (bottom)), Fred can force Trudy to make another move and open the middle chain for him to claim. That way, Fred loses two boxes in the bottom chain, but gains all four boxes in the middle chain, securing the win.

In Winning Ways, Berlekamp et al. [6] refer to the moves sacrificing a small number of boxes but passing the turn onto the opponent as double-dealing moves. Double-dealing moves can be made in chains of boxes, sacrificing two boxes, and in cycles, sacrificing four boxes.

![Figure 1](image1.png) Typical starting, intermediate, and final position of a Dots & Boxes game.

![Figure 2](image2.png) Two possible plays that Fred (red) can do. Fred can choose to claim all the available boxes (top) and lose the game, or to perform a double-dealing move sacrificing two boxes (bottom, second state, edge 2), and win the game. The order of the edges that are played by Trudy or Fred in one turn is indicated by edge labels. This example is borrowed from Winning Ways, chapter 16 [6].
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Figure 3 Double-dealing move by Fred (red). If Trudy (blue) opens a chain (or a cycle), Fred can claim a sequence of boxes. To pass the turn back to Trudy, Fred can leave two (or four) boxes unclaimed.

Each double-dealing move is usually immediately followed by the opponent making at least one double-cross move, i.e., a move that closes two boxes at once. These double-dealing and double-cross moves are essential for players that want to consistently win games of Dots & Boxes, and will be used in the reduction later.

Note that double-dealing moves are only possible in long chains of at least three boxes, and in cycles. (Chains of length one do not have enough boxes for a double-dealing move, and a chain of length two can be opened by selecting the middle edge, thus preventing the opponent from playing a double-dealing move.) Thus, opening a long chain or a cycle, if there are other moves available, is often a bad idea. Berlekamp et al. [6] refer to such moves as loony moves.

Making loony moves is not always a choice. If, at some point in the game, all unclaimed boxes are part of long chains and cycles, the only possible moves are loony moves (Figure 1b). Such positions are referred to as a loony endgames. Note that in chains of length \( \geq 4 \) and cycles of length \( \geq 8 \), the player making the double-dealing moves scores at least as many boxes as their opponent. Thus, in loony endgames with chains of length \( \geq 4 \) and cycles of length \( \geq 8 \), under optimal play, the game consists of one player making loony moves (opening chains and cycles), and the other player claiming all but two or four boxes, and making double-dealing moves to pass the turn back to the opponent [6]. Here, the player making the double-dealing moves is always better off, since each chain or cycle yields at least as many boxes to this player as it yields to their opponent. This player is thus referred to as being in control of the game. The benefit of being in control can be seen in Figure 1c, which is the end result of Trudy being in control of the loony endgame shown in Figure 1b.

In Winning Ways, Berlekamp et al. [6] state that finding a winning strategy in the loony endgame for the player who is not in control is NP-hard. They argue that maximizing the number of disjoint cycles will maximize the score of the player not in control, since double-dealing moves in cycles yield twice as many boxes as double-dealing moves in chains. Since this property is important for our reduction, we restate it here and, for completeness, present the argument in the appendix.

\[ \text{Lemma 1. Let the configuration of a loony endgame contain } k \text{ boxes with degree higher than } 2, \text{ let } T \text{ be the sum of the degrees of these boxes, and let } c \text{ be the maximum number of disjoint cycles in the configuration. Then, the player who is not in control can claim at most } 4c + T - 2k - 4 \text{ boxes.} \]

2 Structure of the construction

To show that Dots & Boxes is PSPACE-hard we reduce from the game \( G_{\text{pos}}(\text{POS CNF}) \), introduced and proven PSPACE-complete by Schaefer [32]. The game is played by two players, Trudy and Fred, on a positive CNF formula \( F \). The players take turns picking a
A choice of a cycle can encode the value of a signal. Variables picked by Trudy are set to TRUE, variables picked by Fred are set to FALSE. When all variables have been chosen, the game ends. Trudy wins if formula $F$ evaluates to TRUE, and Fred wins if formula $F$ evaluates to FALSE.

Given a positive CNF formula $F$ with $n$ variables and $m$ clauses, we construct an instance of Dots & Boxes in which Trudy has a winning strategy if and only if she also has a winning strategy in the corresponding instance of $G_{pos}(POS\ CNF)$. For simplicity we assume that $n$ is even, so that Trudy and Fred get to assign values to the same number of variables. If the number of variables in $F$ is odd, we can introduce dummy variables without changing the outcome of a game such that the total number of the variables becomes even. For each variable and clause of $F$ we construct a variable and a clause gadget, respectively. We place the variable gadgets in a row at the top of the board of Dots & Boxes, and the clause gadgets in a row at the bottom. We connect the variable gadgets to their corresponding clause gadgets using the wire gadgets, which transfer the values of the variables to the clauses. If a clause consists of more than one variable, the wires from these variables must pass through an or gadget. Since the signals propagating from the variables may need to cross each other, we construct a crossover gadget that preserves the values in the two crossing wires. In our instance of Dots & Boxes, only the gadgets contain available moves. The remaining boxes on the board have all the incident edges present.

As we detail in Section 4, after the values of the variables are set, the game enters a loony endgame where Fred is in control. Then Trudy’s winning strategy reduces to selecting a maximum set of disjoint cycles $C_{\text{max}}$ in the remaining configuration (Lemma 1). To maximize her score, Trudy opens all the chains outside of $C_{\text{max}}$ first, gaining two boxes per chain, and opens the chosen cycles last, gaining four boxes per cycle. The optimal play for Fred is to ensure that he will be in control when the loony endgame starts. After entering the loony endgame, simply making double-dealing moves until his very last turn is optimal for Fred.

Signal representation Most of our gadgets consist of partially overlapping cycles of boxes. The choice of a set of disjoint cycles determines the value of a signal. For example, in Figure 4 the choice of the left vs. right cycle can encode the value TRUE vs. FALSE. Of course, Trudy could join the cycles together to select the outermost cycle, but this, as we show later, will not be more beneficial.

Variable assignment As both players must have a choice in picking which variable to set, the instance of Dots & Boxes cannot yet be in a loony endgame. Thus, the variable gadgets, which we describe in detail in Section 3.4, contain non-loony moves instrumental in setting the value of a variable. We ensure that the optimal behavior of both players results in the variables being set in alternating fashion, where Trudy sets them to TRUE, and Fred sets them to FALSE. Once all variables are set, the loony endgame is entered. At this point Fred is in control of the game, and it is up to Trudy to maximize her score by maximizing the number of disjoint cycles in $C_{\text{max}}$. The optimal play by Trudy results in a correct propagation
of the signals from the variables to the clauses.

**Remaining constraints and scoring** To ensure that optimal play by both players in the instance of Dots & Boxes corresponds to a valid $G_{\text{pos}}$ (POS CNF) game, our gadgets need to give a specific number of boxes to Trudy depending on the signal values. We will show that after the variable values have been set, under optimal play, Trudy can maximize her score only if the signals are propagated correctly. Every gadget, except for the clause, yields the same number of disjoint cycles independent of the values of the signals passing through the gadget. Only the clause gadget gives more cycles to Trudy if a true signal reaches it. Exactly half of the variables are set to true, and half to false. Thus we can tune the starting score count between Trudy and Fred such that the game is won by Trudy if and only if all the clauses are satisfied.

## 3 Gadgets

In this section we provide the details of the gadgets used in our reduction. When describing the gadgets below, for a simpler exposition, we assume that the moves that Trudy and Fred make follow the following sequence. First, in the first $n$ moves Trudy and Fred set all the variables to true and false respectively. Afterwards, when the loony endgame is entered, the order in which Trudy selects which cycles to add to the disjoint set of cycles $C_{\text{max}}$ is from the top to bottom, that is, from the variables, through the outgoing wires, through the crossover and or gadgets, and finally down to the clause gadgets. Later, in Lemma 7, we will show that, indeed, under optimal gameplay Trudy and Fred start by setting all the variables. Furthermore, we will argue that the outcome of the game depends only on the choice of the cycles in $C_{\text{max}}$, and not on the order in which Trudy selects them.

### 3.1 Basic wiring

Signals from the variable gadgets are propagated to the clause gadgets through wires. A wire consists of a chain of an even number of partially overlapping cycles (see Figure 5). The first cycle in the wire overlaps with the gadget from which the signal is propagated, and the last cycle overlaps with the gadget towards which the signal is propagated. Consider some wire $w$, let $C_i$ be its first cycle overlapping with gadget $G_i$, and let $C_o$ be its last cycle.

![Figure 5](image.png)

**Figure 5** A wire gadget consisting of four overlapping cycles and two ways of selecting disjoint cycles. Shown in green are the connections to the adjacent gadgets. Selecting odd cycles in $C_{\text{max}}$ corresponds to true (top), and selecting even cycles corresponds to false (bottom).
overlapping with gadget $G_o$. If $C_i$ is disjoint from the cycles of $G_i$ that Trudy adds to $C_{\text{max}}$, then we say that the input signal to the wire is TRUE; otherwise, if $C_i$ overlaps with one of the cycles of $G_i$ in $C_{\text{max}}$, the input value is FALSE. If Trudy does not add $C_o$ to $C_{\text{max}}$, then the output signal is TRUE, and the output signal is FALSE otherwise.

To ensure that Fred always follows the strategy of double-dealing moves, we require that each maximal chain of degree-2 boxes in a wire gadgets contain at least four boxes. That way, Fred receives at least as many boxes in each chain (and cycle) as Trudy, and thus for Fred being in control is always beneficial [6].

Note that, besides the lower bound on the length of a chain, the size and the embedding of the overlapping cycles in a wire can be chosen freely. Thus wires are very flexible in connecting components together, which facilitates the construction.

Lemma 2. Let a wire $w$ consist of $2k$ partially overlapping cycles. Then, under optimal play, if the signal in $w$ changes from FALSE to TRUE, then Trudy can select at most $k - 1$ disjoint cycles from $w$ to add to $C_{\text{max}}$. Otherwise, under optimal play, Trudy can select $k$ disjoint cycles from $w$ to add to $C_{\text{max}}$.

Proof. As we show in Lemma 7 after the first $n$ moves, which Trudy and Fred make in the variable gadgets, the game enters a loony endgame with Fred in control. If the output signal in the wire matches the input signal, then only one of $C_i$ or $C_o$, of $w$ are in $C_{\text{max}}$. Then Trudy can select all odd (if $C_i \in C_{\text{max}}$) or all even (if $C_o \in C_{\text{max}}$) cycles to add to $C_{\text{max}}$, which results in $k$ disjoint cycles. If the the input signal is TRUE, and the output signal is FALSE, then both $C_i$ and $C_o$ are in $C_{\text{max}}$. Then Trudy can, for example, select $k - 1$ odd cycles and $C_o$ to add to $C_{\text{max}}$, which again results in $k$ cycles in total.

If, however, the input signal is FALSE, and the output signal is TRUE, then neither $C_i$ nor $C_o$ can be in $C_{\text{max}}$. This leaves a chain of $2k - 2$ cycles, of which at most $k - 1$ disjoint cycles can be selected to be added to $C_{\text{max}}$.

In our construction we ensure that Trudy can win only if she gets $k$ disjoint cycles from a wire, and thus under optimal play she cannot flip a signal propagating from a variable from FALSE to TRUE. Flipping a signal from TRUE to FALSE is not beneficial for Trudy, as her goal is to satisfy all the clauses. Nevertheless, flipping a signal from TRUE to FALSE leads to the same number of boxes for her (at least locally within a wire), and is thus allowed.

3.2 Crossover gadget

Since the graph representing $G_{\text{pos}}$ (POS CNF) is not necessarily planar, wires may need to cross each other in our construction. We describe a crossover gadget that allows two signals to cross while preserving the signal values. The gadget has two inputs and two outputs on the opposite sides of the gadget. Let $C_{1,1}$ and $C_{2,1}$ be the input cycles of the gadget, and $C_{1,0}$ and $C_{2,0}$ be the output cycles (see Figure 3). An input cycle $C_{*,1}$ is in $C_{\text{max}}$ if the corresponding input signal is TRUE, and otherwise it is FALSE. An output cycle $C_{*,0}$ is not in $C_{\text{max}}$ if the output signal is TRUE, and otherwise it is FALSE.

There are four pairwise overlapping cycles $C_a$, $C_b$, $C_c$, and $C_d$ in the middle of the gadget, forming a cross shape. Only one of these cycles can be added to $C_{\text{max}}$. A choice of which of these cycles is added to $C_{\text{max}}$ is in one-to-one correspondence to the input signal values (see Figure 7).

Lemma 3. Under optimal play, if a signal in a crossover gadget changes from FALSE to TRUE, then Trudy can select at most 4 disjoint cycles from the gadget to add to $C_{\text{max}}$. Otherwise, under optimal play, Trudy can select 5 disjoint cycles from the gadget.
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Proof. If the output signals in the crossover gadget match the input signals, then only one of each pair \( \{C_{1,i}, C_{1,o}\} \) and \( \{C_{2,i}, C_{2,o}\} \) are in \( C_{\text{max}} \). Since the four center cycles \( C_a, C_b, C_c, \) and \( C_d \) all share a single square, only one of these four cycles can be chosen. Then Trudy can select a corresponding cycle from the middle of the gadget, and two more cycles from each signal. For example, a selection of five disjoint cycles for the case when the first input signal is false and the second is true is shown in Figure 7. If an input signal is true, and the corresponding output signal is false, then both \( C_{*,i} \) and \( C_{*,o} \) are in \( C_{\text{max}} \). Then Trudy can, for example, make exactly the same choice as in the case where the output signal would have been true.

Assume now, w.l.o.g., that the signal corresponding to \( C_{1,i} \) and \( C_{1,o} \) changes from false to true in the gadget. That is, neither \( C_{1,i} \) nor \( C_{1,o} \) are in \( C_{\text{max}} \). Let \( C' \) and \( C'' \) be the cycles in the gadget adjacent to \( C_{1,i} \) and \( C_{1,o} \) respectively. Thus, among cycles \( C', C'', C_a, C_b, C_c, \) and \( C_d \) at most two cycles can be in \( C_{\text{max}} \), and therefore at most four cycles can be chosen to be in \( C_{\text{max}} \).

3.3 Or gadget

The or gadget consists of three pairwise overlapping cycles (see Figure 8 (left)). Two of the cycles partially overlap with an end cycle of an input wire, and one cycle partially overlaps with the output cycle. Let \( C_{1,w} \) and \( C_{2,w} \) be the last cycles of the two input wire gadgets, and let \( C_{1,i} \) and \( C_{2,i} \) be the cycles of an or gadget adjacent to these two wires respectively. Let \( C_o \) be the third cycle of the or gadget, which is adjacent to an output wire. Cycles \( C_{1,w} \)

![Figure 6](image1.png) The crossover gadget. Connections to the adjacent wires are shown in green.

![Figure 7](image2.png) A possible choice of a set of disjoint cycles. The selection has fourth degree rotational symmetry.

![Figure 8](image3.png) The or gadget (left) and the three possible combinations of the input values. From left to right: two true inputs, one true and one false input, and two false inputs. The boxes highlighted in grey belong to a cycle in an adjacent wire gadget.
and $C_{2,w}$ are not in $C_{\text{max}}$ if the input from their corresponding wire is \textsc{true}, and are in $C_{\text{max}}$ if their input is \textsc{false}. If $C_o$ is not in $C_{\text{max}}$ then the output of the or gadget is \textsc{true}, and if it is in $C_{\text{max}}$ then the output value is \textsc{false}. Only one of the three cycles in the or gadget can be selected to be added to $C_{\text{max}}$, and thus the output of the gadget can be \textsc{true} only if one of $C_{1,w}$ or $C_{2,w}$ is in $C_{\text{max}}$.

▶ Lemma 4. Under optimal play, if both input signals in an or gadget are \textsc{false} but the output signal is \textsc{true}, then Trudy cannot add a single cycle from the gadget to $C_{\text{max}}$. Otherwise, under optimal play, Trudy can select 1 cycle from the gadget to add to $C_{\text{max}}$.

Proof. First consider the case when one of the input signals in the or gadget is \textsc{true}. W.l.o.g., let the signal from the first wire be \textsc{true}, that is $C_{1,w}$ is not in $C_{\text{max}}$. Then Trudy can select $C_{1,i}$ to add to $C_{\text{max}}$ and thus the output from the or gadget would correspond to \textsc{true}. Trudy may as well choose $C_o$ to add to $C_{\text{max}}$ and make the output of the gadget to be \textsc{false}. In either case, one cycle from the gadget is in $C_{\text{max}}$.

If both input signals are \textsc{false}, then both cycles $C_{1,w}$ and $C_{2,w}$ are in $C_{\text{max}}$. Thus none of $C_{1,i}$ and $C_{2,i}$ can be in $C_{\text{max}}$. If at the same time the output of the or gadget is \textsc{true}, then $C_o$ is not in $C_{\text{max}}$, and thus Trudy cannot select a single cycle to add to $C_{\text{max}}$ from this or gadget. ◀

3.4 Variable gadget

The variable gadget is responsible for the assignment of \textsc{true} and \textsc{false} values to the variables of the $G_{\text{pos}}$ (POS CNF) instance. It consists of two components: the value-setting component (see Figure 9) designed to set the value of the variable, and the fan-out component designed to duplicate the variable signal. The whole construction is presented in Figure 10.

Let $C_1$, $C_2$, and $C_3$ be the three cycles in the value-setting component. The variable gadget is the only gadget that contains non-loony moves; there are two non-loony moves (shown in yellow in the figure) at the intersection of $C_1$ and $C_2$.

As we show later, optimal play by both Trudy and Fred is to set all the variables in the first $n$ moves, such that Fred always sets a variable to \textsc{false} and Trudy—to \textsc{true}. Figure 11 shows the two possible value assignments of the variable gadgets. To set a variable to \textsc{false}, Fred plays one of the non-loony moves in the corresponding variable gadget. Then Trudy responds by claiming the one box available (see Figure 11 (left)). This results in the cycles

![Figure 9](image-url) The value-setting component of the variable gadget. There are two non-loony moves (yellow) available, of which only one can be played as a non-loony move.

![Figure 10](image-url) The complete variable gadget consisting of the value-setting component and the fan-out component. Outgoing wires are shown in green.
$C_1$ and $C_2$ getting merged. To set a variable to $\text{true}$, Trudy opens a side chain of $C_2$ (see Figure 11 (right)). Then Fred responds by claiming every box in the opened chain, and proceeds to setting the next variable. Note that after Trudy’s move the non-loony moves in the gadget become loony moves (as they are now a part of a long chain).

At this point we make two observations which will be useful when proving correctness of the construction and the properties of the optimal play in Section 4. First, observe that the non-loony moves come in pairs, one in each variable, such that, for each pair, either both moves in the pair are still non-loony or neither is anymore. We refer to them as non-loony pairs. Second, note that in the process of assigning values to the variable gadgets, Trudy gets a box for each variable set to $\text{false}$ by Fred, and zero boxes for each variable set to $\text{true}$ by herself.

Once the value of a variable is set, it propagates to the outgoing wires through the fan-out component of the variable gadget. The fan-out component simply consists of one cycle $C_4$ overlapping with the cycle $C_3$ (see Figure 10), to which multiple wires can be attached. After the variable is set, Trudy can add at most two cycles from it to $C_{\text{max}}$. Then, if the variable is set to $\text{false}$, cycle $C_4$ has to be one of the two selected cycles, and thus the signal propagated into the wires is $\text{false}$. If the variable is set to $\text{true}$, Trudy can add $C_1$ and $C_3$ to $C_{\text{max}}$, and thus propagate the $\text{true}$ value into the wires.

▶ **Lemma 5.** Under optimal play, after a variable gadget is assigned a value, if it is set to $\text{false}$ but the output signal is $\text{true}$, then Trudy can add at most 1 cycle from the gadget to $C_{\text{max}}$. Otherwise, under optimal play, Trudy can add 2 cycles from the gadget to $C_{\text{max}}$.

**Proof.** As we show in Lemma 8, optimal play of both Trudy and Fred results in them setting all the variables according to the rules described above in the first $n$ moves. Afterwards the game enters a loony endgame with Fred in control.

If a variable gadget is set to $\text{true}$, then there are three cycles left in the gadget: two overlapping cycles $C_3$ and $C_4$, and the cycle $C_1$ connected to $C_3$ by a chain. Then Trudy can select $C_1$ and one of $C_3$ or $C_4$ to add to $C_{\text{max}}$.

If the variable gadget is set to $\text{false}$, then there are still three cycles left in the gadget, but now these cycles are forming a chain where each consecutive pair of cycles is overlapping. Now, if the output value is $\text{true}$ then $C_4$ cannot be in $C_{\text{max}}$, and from the remaining two cycles, only one can be selected to be added to $C_{\text{max}}$. ◀
Finally, we describe a clause gadget that yields more boxes to Trudy if the signal entering the clause corresponds to true. A clause gadget is simply an extra cycle extending the end of a wire gadget to an odd length. Figure 12 shows the gadget, and the two possible assignments of this gadget. Whenever the signal is true, it is possible for Trudy to create a disjoint cycle in the gadget which gives her four boxes. If the signal is false, Trudy can only make a chain in this gadget which yields only two boxes.

Lemma 6. Under optimal play, the clause gadget yields at most 4 boxes to Trudy if the input signal is true, and at most 2 boxes if the input signal is false.

Proof. If the input signal to the clause gadget is true, the adjacent cycle to the clause gadget is not in $C_{\text{max}}$. Therefore, a cycle of the gadget can be added to $C_{\text{max}}$. When in the loony endgame, this cycle yields four boxes to Trudy after Fred makes a double-dealing move.

Otherwise, if the input signal is false, the adjacent cycle is in $C_{\text{max}}$, and from the clause gadget only a chain is left. This chain yields only two boxes to Trudy after Fred makes a double-dealing move.

4 Players’ strategies and PSPACE-completeness

With the gadgets described above, we construct a Dots & Boxes instance for any $G_{\text{pos}}(\text{POS CNF})$ instance such that Trudy can win the Dots & Boxes instance if and only if she can win the corresponding $G_{\text{pos}}(\text{POS CNF})$ instance. We lay out the variable gadgets, attach a corresponding number of wire gadgets, pass the wires through or gadgets, using crossover gadgets to cross signals, and finally connect wires to the clause gadgets. An example of our construction is given in Figure 13 in the appendix.

The initial score we set to the Dots & Boxes instance depends on the number of gadgets of each type in the construction. By Lemma 4 the total score in the loony endgame depends on the number of disjoint cycles $c$, the number of boxes $k$ with degree higher than 2, and their total degree $T$. The configuration of the loony endgame, and thus the values $k$ and $T$, is changed only when the variable gadgets are being assigned their values. We will argue below, that under optimal play, exactly half of the variables are set to true and half are set to false. Thus the total values of $k$ and $T$ are the same, no matter which variables are assigned to which values. If Trudy can satisfy $\mathcal{F}$, by Lemma 1 she can claim $4c + T - 2k - 4$ boxes in the loony endgame, and $n/2$ boxes from the variables set to false. Let $N$ be the total number of unclaimed boxes in our Dots & Boxes instance. Then, Fred gets $N - n/2 - (4c + T - 2k - 4)$.
Dots & Boxes is PSPACE-complete

boxes. We set the initial scores of Trudy and Fred such that Trudy’s final score is one larger than Fred’s if she can satisfy $\mathcal{F}$. Otherwise, her score will be strictly less than Fred’s.

Next, we describe the optimal strategies for Trudy and Fred, both before the loony endgame is entered and in the loony endgame.

**Optimal strategies for Trudy and Fred in the loony endgame** We start by summarizing both strategies in the loony endgame, assuming that all variables have already been assigned a value using the moves we have described in Section 3.4. As we argue below, Fred can always ensure that he is in control of the loony endgame. It is always beneficial for Fred to stay in control, as all the chains and cycles in the loony endgame configuration yield at least as many boxes to him than to Trudy.

In the loony endgame, Trudy can choose which chains and cycles to open. To maximize her score, Trudy is going to select a maximum number of disjoint cycles $C_{\text{max}}$ in the loony endgame (see Lemma 1). This can be done by first making a loony move in all chains, to which Fred responds by claiming all but two boxes, finishing with a double-dealing move in order to stay in control. Afterwards, Trudy makes loony moves in the remaining cycles, to which Fred responds again by claiming all but four boxes, finishing with a double-dealing moves each time, except for in the final cycle.

**Optimal strategy for Trudy before the loony endgame** Trudy’s strategy before the loony endgame is to set enough variable gadgets to true in order to satisfy all the clauses. By Lemmas 1 and 3, Trudy gains more boxes from each satisfied clause. Therefore, the optimal strategy for Trudy is to claim the boxes opened by Fred when setting variables to false, and to set variables to true, by using a loony move in a side chain of cycle $C_2$ of the variables.

As we show in Lemma 7, if Fred deviates from setting variables to false, and plays a loony move when there are non-loony moves available, Trudy can adopt Fred’s strategy and dominate the rest of the game by ensuring that she ends up in control when the loony endgame is entered.

**Optimal strategy for Fred before the loony endgame** Fred’s strategy is to ensure that he is in control when the loony endgame starts, and it can be described completely as responses to what Trudy does. By our assumption the number of variables in $\mathcal{F}$ is even, thus initially the number of non-loony move pairs is even. Fred’s strategy is then to keep the number of non-loony move pairs even at the start of every Trudy’s turn. Then, once the number of non-loony moves reaches zero (and the loony endgame is reached), it is Trudy’s turn, and Fred is in control. Specifically, Fred responds to Trudy’s moves in the following way:

- If Trudy follows optimal play and makes a loony move in a variable to set it to true, then Fred simply claims all boxes in the chain opened by Trudy (without making a double-dealing move), and makes a non-loony move in another variable to set it to false.

- If Trudy deviates from her strategy by making a non-loony move, setting a variable to false, there must be at least one other non-loony move pair available to Fred. Therefore, Fred claims the boxes opened by Trudy, and makes a non-loony move, thereby setting another variable to false. The number of non-loony pairs is again even at the start of Trudy’s next turn.

- If Trudy deviates from her strategy by opening a chain with a loony move that does not remove a non-loony pair, Fred responds with claiming all but two (or four in case of a cycle) boxes and ends with a double-dealing move. The number of non-loony pairs remains even before Trudy’s next turn.
Using this strategy, Fred can set a variable to `false` each time Trudy sets a variable to any value, as well as gain control in the loony endgame.

Note that the order of moves in these strategies is not enforced. Trudy can play loony moves she would play in the loony endgame even if there are still non-loony moves available, as long as these moves do not interfere with the values set (or to be set) in the corresponding variables. For Fred it is optimal to simply respond to these moves as if the game was already in the loony endgame, since otherwise he would be in danger of losing control. Indeed, if Fred does not make a double-dealing move, the number of non-loony moves will no longer be even at the start of Trudy’s turn, and Fred loses control of the loony endgame. Thus, it is not more beneficial for any player to make a move in any other gadget than the variable gadgets while there are still variables that have not been set.

Lemma 7. Deviating from the strategies described above is sub-optimal for Fred and cannot be more beneficial for Trudy.

Proof. Trivially, Trudy and Fred always claim open boxes before making their move, except when Fred makes double-dealing moves. Otherwise the opponent can claim these boxes in their next move.

First, consider the strategies in the loony endgame. If Trudy deviates from her strategy and does not select the maximum number of disjoint cycles, by Lemma 1 her score will be too low and she loses the game. Therefore, the loony endgame strategy for Trudy as described above is optimal.

If, at any point in the loony endgame, except for his last move, Fred does not make a double-dealing move, he loses control. Since being in control is always beneficial in our construction, this play is sub-optimal.

The strategies described for before the loony endgame are also optimal. Observe that, under the described strategies, the value-setting component of a variable yields the same number of boxes to Trudy independent whether it is set to `true` or to `false`. Indeed, if it is set to `true`, the component contains three boxes with degree 3, while setting the variable to `true` does not give any boxes to Trudy; if the variable is set to `false`, the component contains two boxes with degree 3, but setting the value gives Trudy one box. Thus, the value-setting component contributes the same number of points to Trudy’s final score independent of the value.

If Trudy deviates from her strategy by making a non-loony move and setting a variable to `false`, she loses one box to Fred. Furthermore, setting a variable to `false` can never help Trudy to satisfy formula $F$. Thus, such a move is sub-optimal.

If Trudy deviates from her strategy by making a loony move in any other gadget than the variable gadget, there are two options: either she makes a move that leads to the same score as the strategy described above, or she makes a move that contradicts the setting of the variables and reduces her total score. The former case does not have any bad repercussions for Trudy. Fred will respond with a double-dealing move, otherwise Trudy would take control of the endgame. Thus, we can reorder the sequence of Trudy’s moves and assume that she first sets all the variables. However, in the latter case, the move reduces the number of possible disjoint cycles, and thus leads to Trudy’s loss in the game. Therefore, deviating from the strategy above is never more beneficial for Trudy.

If Fred deviates from his strategy before the loony endgame, then Trudy can adopt his strategy and ensure that the number of non-loony move pairs is even at the start of each of Fred’s turn. Since, if Fred is not in control of the loony endgame, he loses the game, deviating from his strategy is not optimal.
Theorem 8. Dots & Boxes is PSPACE-complete.

Proof. A game of Dots & Boxes is finished after a polynomial number of turns. Thus, all possible sequences of moves can be explored using polynomial sized memory. This implies that Dots & Boxes is in PSPACE.

We now show that Dots & Boxes is PSPACE-hard. Given a $G_{pos}(\text{POS CNF})$ formula $F$, we construct a Dots & Boxes instance $\delta$ following the description above. We argue that Trudy can win $F$ if and only if Trudy can win $\delta$.

If Trudy can win $F$, then there must be a variable assignment following the $G_{pos}(\text{POS CNF})$ rules such that every clause is connected to at least one variable which has been set to True. Therefore, there can be at most $n/2$ variables that need to be set to true by Trudy. Hence, Trudy can set the corresponding variable gadgets in $\delta$ to true, and if needed set the remaining variables available to her to true in any order. Thus, by Lemmas 2-6, Trudy can propagate the true values down to all the clauses, that is, she can select the maximum number of disjoint cycles from all the gadgets, including all the clause gadgets, leading to the winning score in $\delta$.

In order for Trudy to win $\delta$, the set of disjoint cycles $C_{max}$ that she selects must contain a cycle from every clause gadget, and the maximum number of cycles from all the other gadgets. By Lemmas 2-6, this can be done only if the output signals from each gadget conform to their input signals, and thus there must be a set of variable gadgets set to true whose signal is propagated all the way down to all the clause gadgets. In $\delta$ Trudy and Fred have to alternate choosing which variable gadgets get set to true and false, respectively. This assignment can be used as a winning strategy for Trudy to win $G_{pos}(\text{POS CNF})$ game on $F$.

Thus, Dots & Boxes is PSPACE-complete.

5 Conclusion

In this paper we proved Dots & Boxes to be PSPACE-complete, resolving a long-standing open problem.

There exist a number of other intriguing open problems related to Dots & Boxes. Does restricting the game to a $k \times n$ grid for a small $k$ make the game easier? How large does $k$ need to be to make the problem PSPACE-hard or even just NP-hard? These are challenging questions, given that even for a $1 \times n$ grid Dots & Boxes is not yet fully understood [12, 19, 25].

Another direction of further research is the computational complexity of variants of Dots & Boxes, in particular misère Dots & Boxes [12], of Dots & Boxes on other grids or even of variants of Dots & Boxes with more than two players as it was originally described by Lucas [29]. One variant that our result resolves is Dots & Polygons, since the reduction from Dots & Boxes to Dots & Polygons that was used to prove NP-hardness [8] now directly also shows PSPACE-hardness.

Our result can be interpreted as proving that Strings and coins restricted to grid graphs is PSPACE-complete. What is the complexity of Strings and coins on other restricted graph classes, for instance outerplanar graphs (which generalize $1 \times n$ grids)?

This may also be a good moment to revisit other games, which are known to be PSPACE-complete on general graphs, but for which the complexity on grid graphs is open. This, for instance, includes NoGo, Fjords (on hexagonal grids), Cats-and-Dogs and GraphDistance, which are known to be PSPACE-complete for planar graphs [11].
References

1. Oswin Aichholzer, David Bremner, Erik D. Demaine, Ferran Hurtado, Evangelos Kranakis, Hannes Krasser, Suneeta Ramaswami, Saurabh Sethia, and Jorge Urrutia. Games on triangulations. *Theoretical Computer Science*, 343(1):42–71, 2005. Game Theory Meets Theoretical Computer Science. [doi:https://doi.org/10.1016/j.tcs.2005.05.007]

2. Michael H. Albert, Richard J. Nowakowski, and David Wolfe. *Lessons in play: an introduction to combinatorial game theory*. CRC Press, 2019.

3. Joseph Barker and Richard Korf. Solving 4x5 dots-and-boxes. In *Proc. 25th AAAI Conference on Artificial Intelligence*, pages 1756–1757, 2011.

4. Joseph Barker and Richard Korf. Solving dots-and-boxes. In *Proc. 26th AAAI Conference on Artificial Intelligence*, pages 414–419, 2012.

5. Elwyn R. Berlekamp. *The Dots and Boxes Game: Sophisticated Child’s Play*. AK Peters/CRC Press, 2000.

6. Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. Chapter 16: Dots-and-boxes. In *Winning Ways for your Mathematical Plays*, volume 3, pages 541–584. A K Peters/CRC Press, 2nd edition, 2003.

7. Elwyn R. Berlekamp and Katherine Scott. Forcing your opponent to stay in control of a loony Dots-and-Boxes endgame. In R. J. Nowakowski, editor, *More Games of No Chance*, Proc. MSRI Workshop on Combinatorial Games, volume 42, pages 317–330. Cambridge University Press, 2002.

8. Kevin Buchin, Mart Hagedoorn, Irina Kostitsyna, Max van Mulken, Jolan Rensen, and Leo van Schooten. Dots & polygons (media exposition). In *Proc. 36th Internat. Sympos. Computational Geometry (SoCG 2020)*, pages 79:1–79:4, 2020. [arXiv:2004.01235 doi:10.4230/LIPIcs.SoCG.2020.79]

9. Kyle Burke and Robert A. Hearn. PSPACE-complete two-color planar placement games. *Int. J. Game Theory*, 48(2):393–410, 2019. [doi:10.1007/s00182-018-0628-8]

10. Kevin Buzzard and Michael Clere. Playing simple loony dots-and-boxes endgames optimally. *INTEGERS*, 14:2, 2014.

11. Sébastien Collette, Erik D. Demaine, Martin L Demaine, and Stefan Langerman. Narrow misere Dots-and-Boxes. *Games of No Chance 4*, 63:57, 2015.

12. Erik D. Demaine. Playing games with algorithms: Algorithmic combinatorial game theory. In *International Symposium on Mathematical Foundations of Computer Science*, pages 18–33. Springer, 2001.

13. Erik D. Demaine, Martin L. Demaine, Ryuhei Uehara, Takeaki Uno, and Yushi Uno. UNO is hard, even for a single player. In Paolo Boldi and Luisa Gargano, editors, *Fun with Algorithms*, pages 133–144. Springer, 2010.

14. Erik D. Demaine and Yevhenii Diomidov. Strings-and-Coins and Nimstring are PSPACE-complete. *arXiv e-prints*, January 2021. [arXiv:2101.06361]

15. David Eppstein. Computational complexity of games and puzzles. Last accessed on 06/05/2021. URL: [https://www.ics.uci.edu/~eppstein/cgt/hard.html](https://www.ics.uci.edu/~eppstein/cgt/hard.html)

16. Gary William Flake and Eric B. Baum. Rush Hour is PSPACE-complete, or “Why you should generously tip parking lot attendants”. *Theoretical Computer Science*, 270(1):895–911, 2002. [doi:https://doi.org/10.1016/S0304-3975(01)00173-6]

17. K. Guy and Richard J. Nowakowski. Unsolved problems in combinatorial games. In R. J. Nowakowski, editor, *More Games of No Chance*, Proc. MSRI Workshop on Combinatorial Games, volume 42, pages 457–473. Cambridge University Press, 2002.
Robert A. Hearn. *Games, puzzles, and computation*. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, USA, 2006.

Robert A. Hearn and Erik D. Demaine. *Games, puzzles, and computation*. CRC Press, 2009.

Takashi Horiyama, Takashi Iizuka, Masashi Kiyomi, Yoshih Okamoto, Ryuhei Uehara, Takeaki Uno, Yushi Uno, and Yukiko Yamauchi. Sankaku-tori: An old western-japanese game played on a point set. *Journal of Information Processing*, 25:708–715, 2017.

Ming Yu Hsieh and Shi-Chun Tsai. On the fairness and complexity of generalized k-in-a-row games. *Theoretical Computer Science*, 385(1-3):88–100, 2007.

Takashi Horiyama, Takashi Iizuka, Masashi Kiyomi, Yoshih Okamoto, Ryuhei Uehara, Takeaki Uno, Yushi Uno, and Yukiko Yamauchi. Sankaku-tori: An old western-japanese game played on a point set. *Journal of Information Processing*, 25:708–715, 2017.

Shigeki Iwata and Takumi Kasai. The Othello game on an $n \times n$ board is PSPACE-complete. *Theoretical Computer Science*, 123(2):329–340, 1994. [doi:https://doi.org/10.1016/0304-3975(94)90131-7]

Adam S. Jobson, Levi Sledd, Susan Calcote White, and D. Jacob Wildstrom. Variations on narrow dots-and-boxes and dots-and-triangles. *Integers*, 17(G2), 2017.

Will Johnson. The combinatorial game theory of well-tempered scoring games. *International Journal of Game Theory*, 43(2):415–438, 2014.

Anthony Knittel, Terry Bossomaier, and Allan Snyder. Concept accessibility as basis for evolutionary reinforcement learning of dots and boxes. In *2007 IEEE Symposium on Computational Intelligence and Games*, pages 140–145. IEEE, 2007.

David Lichtenstein and Michael Sipser. Go is polynomial-space hard. *Journal of the ACM (JACM)*, 27(2):393–401, 1980.

Édouard Lucas. *Récréations mathématiques*, volume 2. Gauthier-Villars et fils, 1883.

Henry Meyniel and Jean-Pierre Roudneff. The vertex picking game and a variation of the game of dots and boxes. *Discrete Math.*, 70:311–313, 1988.

Richard J. Nowakowski. . . , Welter’s game, Sylvie coinage, dots-and-boxes, . . . In R. K. Guy, editor, *Combinatorial Games*, Proc. Symp. Appl. Math., volume 43, pages 155–182. Amer. Math. Soc., 1991.

Thomas J. Schaefer. On the complexity of some two-person perfect-information games. *Journal of Computer and System Sciences*, 16(2):185–225, 1978.

Aaron N. Siegel. *Combinatorial game theory*, volume 146. American Mathematical Soc., 2013.

Julian West. Championship-level play of dots-and-boxes. In R. J. Nowakowski, editor, *Games of No Chance*, Proc. MSRI Workshop on Combinatorial Games, volume 29, pages 79–84. Cambridge University Press, 1996.

Yimeng Zhuang, Shuqin Li, Tom Vincent Peters, and Chenguang Zhang. Improving monte-carlo tree search for dots-and-boxes with a novel board representation and artificial neural networks. In *2015 IEEE Conference on Computational Intelligence and Games (CIG)*, pages 314–321. IEEE, 2015.
A Example game

Figure 13 Example reduction from the $G_{pos}(\text{POS CNF})$ formula $(w \lor x) \land (w \lor y) \land (x \lor z)$. The construction can be divided into four sections: a variable, crossover, or, and clause section. Each section contains only the corresponding gadgets and wire gadgets that connect different gadgets together.
Lemma 1. Let the configuration of a loony endgame contain \( k \) boxes with degree higher than 2, let \( T \) be the sum of the degrees of these boxes, and let \( c \) be the maximum number of disjoint cycles in the configuration. Then, the player who is not in control can claim at most \( 4c + T - 2k - 4 \) boxes.

Proof. Let Fred be in control of the game. To simplify the argument, w.l.o.g., we assume that the last move made by Trudy is made in a cycle. Let \( c \) denote the number of loony moves made by Trudy in a disjoint cycle and let \( \ell \) be the number of loony moves made by Trudy in chains. All but the last loony move in a disjoint cycle or chain yield 4 or 2 boxes for Trudy, respectively. Thus, the score gained by Trudy in the loony endgame is

\[ 4c + 2\ell - 4. \]

Consider the dual graph \( G = (V, E) \) to the Dots & Boxes instance. In it, a node corresponds to a box, and an edge connects two nodes if the two corresponding adjacent boxes do not have a line drawn between them. Suppose \( G \) has \( k \) nodes with degree higher than 2. We define \( T \) to be the sum of the degrees of these nodes:

\[
T = \sum_{\{v \in V | \text{degree}(v) > 2\}} \text{degree}(v).
\]

A loony move on a disjoint cycle does not change \( T \), since all disjoint cycles only contain boxes of degree 2. A loony move on a chain, however, decreases the degree of the box at both ends of the chain by 1. Furthermore, whenever the degree of a box reduces from 3 to 2 the degree of this box is no longer counted in \( T \). Thus

\[ T = 2\ell + 2k, \]

which means the score for Trudy will be

\[ 4c + T - 2k - 4. \]

Since \( T \) and \( k \) are fixed, the score is maximized when the number of loony moves in disjoint cycles is maximized.

\[ \square \]