To the memory of Yuri Grigorievich Borisovich

Hilbert $C^*$-Modules Related to Discrete Metric Spaces

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Abstract—It is shown that the metric on the union of the sets $X$ and $Y$ defines a Hilbert $C^*$-module over the uniform Roe algebra of the space $X$ with a fixed metric $d_X$. A number of examples of such Hilbert $C^*$-modules are described.

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INTRODUCTION

Hilbert $C^*$-modules ([1]) are a natural generalization of Hilbert spaces, in which the “scalar” product takes values in some $C^*$-algebra instead of the field of complex numbers. Although many properties of Hilbert $C^*$-modules are similar to those of Hilbert spaces, there are several important differences, among which are the following: not every closed submodule is orthogonally complemented, and not every functional is defined as a “scalar” product by some element (Riesz theorem). If $M$ is a (right) Hilbert $C^*$-module over the $C^*$-algebra $A$, then it is natural to call a bounded $A$-linear map of $M$ into $A$ an $A$-linear functional. The set of all such mappings constitutes the dual module $M'$, on which there is the structure of a right $A$-module, but, in general, there is no $A$-valued “scalar” product. Moreover, the bidual module $M''$, dual to the module $M'$ is a Hilbert $C^*$-module, and there is an isometric embedding $M \subset M'' \subset M'$ ([2], [3]).

The standard Hilbert $C^*$-module $l_2(A)$ is a (right) $A$-module of sequences $(a_n)_{n \in \mathbb{N}}$, where $a_n \in A$, $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} a_n^* a_n$ converges in $A$ (in the norm). In this case, the dual module $l_2(A)'$ consists of sequences $(a_n)_{n \in \mathbb{N}}$, for which the partial sums $\sum_{n=1}^{m} a_n^* a_n$, $m \in \mathbb{N}$, are uniformly bounded, but for the bidual module $l_2(A)''$ there is generally no good description (but the description of $l_2(A)''$ is known in the case when $A$ is commutative [4]).

Another important difference between Hilbert $C^*$-modules and Hilbert spaces is that a finitely generated Hilbert $C^*$-module does not have to be free, and a countably generated Hilbert $C^*$-module does not have to be standard, for example, $C_0(0, 1)$ is not isomorphic to $l_2(C[0, 1])$ as modules over $C[0, 1]$. The purpose of this paper is to demonstrate the variety of Hilbert $C^*$-modules using the example of modules over uniform Roe algebras. Information about Hilbert $C^*$-modules can be found in [5], and about Roe algebras and underlying metric spaces in [6], [7].

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1. HILBERT C*-MODULES OVER UNIFORM ROE ALGEBRAS

We denote the algebra of bounded (respectively, compact) operators of the Hilbert space $H$ by $\mathbb{B}(H)$ (respectively, $\mathbb{K}(H)$).

Let $X = (X, d_X)$ be a discrete countable metric space $X$ with the metric $d_X$, $H_X = l^2(X)$ be the Hilbert space of square summable complex-valued functions on $X$ with a standard orthonormal basis consisting of delta functions of points, $\delta_x$, $x \in X$. A bounded operator $T$ on $H_X$ with the matrix $(T_{x,y})_{x,y \in X}$ with respect to the standard basis, i.e. $T_{x,y} = (\delta_x, T\delta_y)$, has a propagation not exceeding $L$ if $d_X(x, y) \geq L$ implies that $T_{x,y} = 0$. The $*$-algebra of all bounded operators of finite propagation is denoted by $C_u[X]$, and its norm completion in $\mathbb{B}(H_X)$ is called the uniform Roe algebra $C_u^*(X)$.

For the set $Y$, let $d$ be the metric on $X \sqcup Y$ coinciding on $X$ with $d_X$, i.e. $d|_X = d_X$.

We denote by $M_{Y,d}$ the set of all bounded operators of finite propagation $T : H_X \rightarrow H_Y$, and by $M_{Y,d}$ its norm closure in the set $\mathbb{B}(H_X, H_Y)$ of all bounded operators from $H_X$ to $H_Y$.

If the operators $T \in \mathbb{B}(H_X, H_Y)$ and $R \in \mathbb{B}(H_X)$ have a finite propagation with respect to the metrics $d$ and $d_X$, respectively, then their composition $TR$ obviously also has a finite propagation with respect to the metric $d$. It follows from the continuity of the composition that the action of the $C^*$-algebra $C_u^*(X)$ on $M_{Y,d}$ is well defined and provides the structure of a right $C_u^*(X)$-module.

Similarly, if $T, S \in \mathbb{B}(H_X, H_Y)$ are operators of finite propagation with respect to $d$, then $S^*T$ has a finite propagation on $l^2(X)$ with respect to $d_X$, so one can define $(S, T) = S^*T \in C_u^*(X)$ and extend it by continuity to a $C_u^*(X)$-valued inner product on the module $M_{Y,d}$.

**Lemma 1.** The module $M_{Y,d}$ is a Hilbert $C^*$-module over $C_u^*(X)$.

**Proof.** Evidently, $\|T\|^2 = \|(T, T)\|$. The remaining properties of Hilbert $C^*$-modules follow from associativity of operator multiplication. \qed

**Example 1.** Let $Y$ be a one-point space, $Y = \{y_0\}$. Any operator $T : H_X \rightarrow H_Y = \mathbb{C}$ is a functional on $H_X$ and can be approximated by a functional with a finite number of nonzero coordinates; therefore, it is the limit of operators of finite propagation, i.e., $M_{y_0}$ can be identified with functionals on $H_X$ and, by the Riesz theorem, with $H_X$.

In [8] it was shown that the structure of a Hilbert $C^*$-module $M$ extends to the dual module $M'$ (making the latter a Hilbert $C^*$-module) if and only if the $C^*$-algebra $A$ is monotone complete. Recall that monotone completeness of $A$ means that any bounded increasing set $\{a_\alpha : \alpha \in I\}$ of self-adjoint elements of the $C^*$-algebra $A$ has the least upper bound $a = \sup \{a_\alpha : \alpha \in I\}$ in $A$.

**Theorem 1.** A metric space $X$ is bounded if and only if the $C^*$-algebra $C_u^*(X)$ is monotone complete.

**Proof.** If the space $X$ is bounded, then any bounded operator $T : H_X \rightarrow H_X$ has a finite propagation, then $C_u^*(X) = \mathbb{B}(H_X)$ is a von Neumann algebra, hence is monotone complete.

Conversely, suppose that the space $X$ is unbounded and that the algebra $C_u^*(X)$ is monotone complete. Find a sequence of pairs of different points $\{(x_n, y_n)\}_{n \in \mathbb{N}}$ in $X$ satisfying the condition $d_X(x_n, y_n) > n$. Suppose that the pairs of points $(x_1, y_1), \ldots, (x_n, y_n)$ with the condition $d_X(x_i, y_i) > i$, $i = 1, \ldots, n$, have already been found. If the estimate $d_X(x, y) \leq n + 1$ were fulfilled for any $x, y \neq x_1, \ldots, x_n, y_1, \ldots, y_n$, then the diameter of $X$ would be finite. Hence, it is possible to find points $x_{n+1}, y_{n+1} \in X$ that do not coincide with any of the previous ones and satisfy the condition $d_X(x_{n+1}, y_{n+1}) \geq n + 1$. We find such pairs of points inductively for each $n \in \mathbb{N}$.

Set

$$T_{x,y}^{(n)}(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \{(x_i, y_i), (y_i, x_i) : i = 1, \ldots, n\}; \\ 0 & \text{otherwise}. \end{cases}$$

Then the matrix $(T_{x,y}^{(n)})$ defines a bounded self-adjoint operator $T^{(n)}$ of finite rank, hence, a finite propagation, for each $n \in \mathbb{N}$. Let $T \in C_u^*(X)$ be the least upper bound for the set $\{T^{(n)}\}_{n \in \mathbb{N}}$. Then $T_{x,y} \geq 1$ for any $n \in \mathbb{N}$, which contradicts the fact that $T \in C_u^*(X)$.

\[\square\]
Corollary. A metric space $X$ is bounded if and only if the structure of a Hilbert $C^*$-module extends from any Hilbert $C^*$-module $M$ over $C^*_u(X)$ to its dual module $M'$.

Recall that two metrics, $d_1,d_2$ on a space $Z$ are coarsely equivalent [6] if there exists a monotonically increasing function $\varphi$ on $[0,\infty)$ such that $\lim_{t\to\infty} \varphi(t) = \infty$ and one has $d_1(z_1,z_2) \leq \varphi(d_2(z_1,z_2))$ and $d_2(z_1,z_2) \leq \varphi(d_1(z_1,z_2))$ for any $z_1,z_2 \in Z$.

Proposition 1. Let $d_1,d_2$ are metrics on $X \sqcup Y$ with the same restriction on $Y$. They are coarsely equivalent if and only if $M_{Y,d_1} = M_{Y,d_2}$.

Proof. If the metrics are roughly equivalent, then having a finite propagation with respect to one of them is equivalent to having a finite propagation with respect to the other.

Conversely, suppose the metrics are coarsely nonequivalent. Then there is a sequence of pairs of points $(x_n,y_n)$, $x_n \in X$, $y_n \in Y$, $n \in N$, such that for one metric the values $d_1(x_n,y_n)$ are uniformly bounded by some constant $C > 0$, while the other metric satisfies the estimate $d_2(x_n,y_n) \geq n$. We claim that each point $x_k$ can occur in the sequence $(x_n)_{n \in N}$ only a finite number of times. Indeed, if $x_k = x_{n_1} = x_{n_2} = \cdots$ then

$$d_1(y_{n_i},y_{n_1}) \leq d_1(x_{k},y_{n_i}) + d_1(x_{k},y_{n_1}) \leq 2C$$

for any $i \in N$, while

$$d_2(y_{n_i},y_{n_1}) \geq d_2(x_{k},y_{n_i}) - d_2(x_{k},y_{n_1}) \geq n_i - n_1,$$

i.e., is not bounded, but the metrics $d_1$, $d_2$ are equal on $Y$, and this contradiction shows that the point $x_k$ can be repeated in the sequence $(x_n)_{n \in N}$ only a finite number of times. Passing to a subsequence, we may assume that the sequence $(x_n)_{n \in N}$ does not contain repeating points at all. The same may be assumed for the sequence $(y_n)_{n \in N}$.

Set

$$T_{x,y} = \begin{cases} 1, & \text{if } (x,y) \in \{(x_n,y_n) : n \in N\}; \\ 0 & \text{otherwise.} \end{cases}$$

Then the matrix $(T_{x,y})$ defines a bounded operator $T$ from $H_X$ to $H_Y$. It has a finite propagation with respect to the metric $d_1$, i.e. $T \in M_{Y,d_1}$. If $M_{Y,d_1} = M_{Y,d_2}$ then the operator $T$ should be the limit of finite propagation operators with respect to the metric $d_2$, but this is not the case.

The following statements are obvious.

Proposition 2. If $d_1(x,y) \leq d_2(x,y)$ for any $x,y \in X \sqcup Y$ then $M_{Y,d_2} \subset M_{Y,d_1}$.

Proposition 3. If $Y = Y_1 \sqcup Y_2$ then $M_Y = M_{Y_1} \oplus M_{Y_2}$.

2. CASE $Y = X$

Let $Y = X$. To avoid ambiguity, we will denote the first copy of $X$ by $X_0$ and the second copy by $X_1$. Accordingly, the point $x \in X$ will be denoted by $x_0 \in X_0$ if it lies in the first copy of $X$, and $x_1 \in X_1$ if it lies in the second copy. We will also identify $B(H_{X_0},H_{X_1})$ with the algebra $B(H_X)$.

The set $S(X)$ of coarse equivalence classes of metrics on $X_0 \sqcup X_1$ has the natural structure of an inverse semigroup [9], where the composition of metrics is given by the formula

$$d_1d_2(x_0,z_1) = \inf_{y \in X} [d_2(x_0,y_1) + d_1(y_0,z_1)],$$

the adjoint (pseudoinverse) metric is given by the formula $d^*(x_0,y_1) = d(y_0,x_1)$, the unit element is given by the metric $d(x_0,y_1) = d_X(x,y) + 1$ and the zero element is given by the metric $d(x_0,y_1) = d_X(x,u) + d_X(y,u) + 1$ with a fixed point $u \in X$ (recall that for a metric $X_0 \sqcup X_1$ it suffices to define distances between points lying in different copies of the space $X$).

It is clear that if the metrics $d_1$ and $d_2$ are coarsely equivalent then $M_{X,d_1} = M_{X,d_2}$. Thus, we have the Hilbert $C^*$-module $M_{X,d}$ for each coarse equivalence class $s = [d] \in S(X)$. The collection of these Hilbert $C^*$-modules forms a *Fell bundle* in the sense of Definition 2.1 from [10] (the mapping $M_{X,d_1} \otimes M_{X,d_2} \to M_{X,d_1d_2}$ is given by composition, see [11]).
Example 2. Let $A \subset X$. Define the metric $d^A$ on $X_0 \sqcup X_1$ by
\[
d^A(x_i, y_i) = d_X(x, y), \quad i = 0, 1;
\]
\[
d^A(x_0, y_1) = \inf_{z \in A} [d_X(x, z) + d_X(y, z) + 1]
\]
for any $x, y \in X$.

Denote the $k$-neighborhood of $A$ by $N_k(A)$, i.e.
\[
N_k(A) = \{x \in X : d_X(x, A) \leq k\}.
\]
For $B \subset X$, denote by $H_B = l_2(B) \subset l_2(X) = H_X$ the closed subspace in $l_2(X)$ generated by the functions $\delta_x, x \in B$. To simplify notation, we will identify an operator $T \in \mathcal{B}(H_B)$ with the operator in $\mathbb{B}(H_X)$ equal to $T$ on $H_B$ and equal to 0 on $H_B^\perp$.

Proposition 4. The module $M_{X,d^A}$ is canonically isomorphic to the norm closure of the set
\[
\bigcup_{k=1}^\infty C_u^a(N_k(A)).
\]

Proof. Let $T \in C_u^a(N_k(A))$ be an operator of propagation not exceeding $L$. If $T_{x_0,y_1} \neq 0$ then $d_X(x,y) \leq L$ and $x, y \in N_k(A)$, hence there exists a point $u \in A$ such that $d_X(x,u) \leq k + 1$. Then, taking $z = u$, we obtain
\[
d^A(x_0, y_1) = \inf_{z \in A} [d_X(x, z) + d_X(z, y) + 1] \leq d_X(x, u) + d_X(u, y) + 1
\]
\[
\leq k + 1 + d_X(u, x) + d_X(x, y) + 1 \leq L + 2k + 3,
\]
i.e., $T$ is of finite propagation, $T \in M_{X,d^A}$.

Let now $S \in M_{X_1,d^A}$ be an operator of propagation not exceeding $L$. If $S_{x_0,y_1} \neq 0$ then
\[
d^A(x_0, y_1) = \inf_{z \in A} [d_X(x, z) + d_X(z, y) + 1] \leq L.
\]
The triangle inequality implies that
\[
d_X(x,y) \leq d_X(x,z) + d_X(z,y)
\]
for any $z \in X$, hence, passing in (1) to the infimum with respect to $z \in X$, we get
\[
d_X(x,y) \leq d^A(x_0, y_1) - 1 \leq L - 1.
\]
As a metric is always non-negative, we have
\[
d_X(x,A) = \inf_{z \in A} d_X(x, z) \leq \inf_{z \in A} [d_X(x, z) + d_X(z, y)] = d^A(x_0,y_1) - 1 \leq L - 1.
\]
Similarly, we obtain that $d_X(y,A) \leq L$. Thus, $S_{x_0,y_1} \neq 0$ implies that $x, y \in N_L(A)$ and $d_X(x,y) \leq L - 1$, i.e. $S \in C_u^a(N_L(A))$. \hfill $\blacksquare$

Lemma 2. Let $A \subset X$. If $X \setminus N_k(A)$ is not empty for any $k \in \mathbb{N}$ then the submodule $M_{X,d^A}$ in the module $C_u^a(X)$ is not orthogonally complemented.

Proof. Let $B \subset X$, $P_B$ a projection onto $l_2(B)$ in $l_2(X)$. Evidently, $P_{N_k(A)} \in M_{X,d^A}$ for any $k \in \mathbb{N}$. If $S \in C_u^a(X)$ is orthogonal to $M_{X,d^A}$ then $P_{N_k(A)}S = 0$ for any $k \in \mathbb{N}$, but, as $\bigcup_{k=1}^\infty N_k(A) = X$, the sequence $P_{N_k(A)}$ of projections is convergent to $1 = P_X$ with respect to the strong topology, whence $S = 0$.

By assumption, $M_{X,d^A} \neq C_u^a(X)$. Indeed, if $1 \in C_u^a(X)$ would belong to $M_{X,d^A}$ then there would exist a sequence $T^{(k)} \in C_u^a(N_k(A))$ such that $T^{(k)}$ would converge to the unit with respect to the norm topology. But if $x \notin N_k(A)$ then $T^{(k)}\delta_x = 0$, so the convergence may take place only with respect to the strong topology, but not in norm. \hfill $\blacksquare$
Two extreme examples are the cases when \( A = X \) and when \( A \) consists of a single point. In the first case \( M_{X,dX} = C_u^*(X) \), and the second case is given by the following statement.

**Proposition 5.** Let \( x_0 \in X \). If \( X \) is proper, i.e., if each ball contains only a finite number of points, then \( M_{X,d(x_0)} = \mathbb{K}(H_X) \).

**Proof.** Properness of \( X \) means that the operator \( T \in M_{Y,d(x_0)} \) is of finite propagation if and only if it is of finite rank. Passing to the closure, we obtain the required statement. \( \square \)

3. CASE \( Y = X \times \mathbb{N} \)

Consider the case \( Y = X \times \mathbb{N} \). Let us introduce the notation \( \overline{\mathbb{N}} = \mathbb{N} \cup \{0\} \). For convenience, we write \( X_n \) instead of \( X \times \{n\} \subset Y \), and \( X_0 = X \). For the point \( x \in X \), we denote the point \( (x,n) \in X \times \mathbb{N} \), \( n \in \overline{\mathbb{N}} \), by \( x_n \in X_n \).

In this case \( H_{X \times \mathbb{N}} = \bigoplus_{n=1}^\infty H_{X_n} \). By \( Q_n : H_{X \times \mathbb{N}} \to H_{X_n} \) we denote the projection onto the \( n \)-th direct summand. For \( T : H_X \to H_{X \times \mathbb{N}} \), put \( T_n = Q_n T : H_X \to H_{X_n} \). In what follows, we identify \( T \) with the sequence \( (T_n)_{n \in \mathbb{N}} \).

Consider first the metric \( d_1 \) on \( X \sqcup X \times \mathbb{N} = X \times \overline{\mathbb{N}} \) given by the formula \( d_1(x_n, y_m) = d_X(x, y) + |nm| \) for any \( x, y \in X \), \( n, m \in \overline{\mathbb{N}} \), i.e., coinciding on the factors \( X \) and \( \overline{\mathbb{N}} \) with the metric \( d_X \) and with the standard metric on \( \overline{\mathbb{N}} \), respectively.

**Proposition 6.** The module \( M_{X \times \mathbb{N},d_1} \) coincides with the standard Hilbert \( C^* \)-module \( l_2(C_u^*(X)) \).

**Proof.** Let the operator \( T = (T_n)_{n \in \mathbb{N}} \) have a propagation not exceeding \( L \). Then \( T_n = 0 \) for any \( n > L \), therefore the sequence \( (T_n) \) consists of zeroes for \( n > L \). On the other hand, any sequence \( (T_1, T_2, \ldots, T_n, 0, 0, \ldots) \), where \( T_i \in C_u^*(X) \), \( i = 1, \ldots, n \), lies in \( M_{X \times \mathbb{N},d_1} \). Thus, \( M_{X \times \mathbb{N},d_1} \) is the completion of the set of finitely supported sequences with elements from \( C_u^*(X) \), and therefore coincides with \( l_2(C_u^*(X)) \). \( \square \)

As a second example, let us consider the metric \( d_0 \) on \( X \sqcup X \times \mathbb{N} \) determined by the formulas \( d_0(x_n, y_m) = d_X(x, y) + 1 \) for \( m \neq n \) and \( d_0(x_n, y_n) = d_X(x, y) \) for any \( x, y \in X \).

**Theorem 2.** If the space \( X \) is bounded then the module \( M_{X \times \mathbb{N},d_0} \) coincides with \( l_2(B(H))^\prime \). If the space \( X \) is unbounded then there exists an element \( T \in M_{X \times \mathbb{N},d_0} \) such that \( T \notin l_2(C_u^*(X))^\prime \).

**Proof.** The first statement is obvious. Suppose that \( X \) is unbounded, then there is a sequence of pairs of distinct points \( (x^k, y^k)_{k \in \mathbb{N}} \) such that \( d_X(x^k, y^k) > k \). Put \( (T_n)_{x^k, y^k} = (T_n)_{y^k, x^k} = 1 \), and all other matrix elements of the matrix of \( T_n \) are equal to zero; \( (S_n)_{x^k, x^k} = (S_n)_{y^k, y^k} = 1 \), and all other matrix elements of the matrix of \( S_n \) are equal to zero. Let \( T = (T_1, T_2, \ldots) \), \( S = (S_1, S_2, \ldots) \). Note that \( T_n S_n = S_n \) for any \( n \in \mathbb{N} \), and that the series \( \sum_{n=1}^\infty T_n \) and \( \sum_{n=1}^\infty S_n \) are convergent with respect to the strong operator topology in \( B(H_X) \).

The sequence \( (T_n)_{n \in \mathbb{N}} \) defines an element of the dual module \( l_2(C_u^*(X))^\prime \). Indeed, each operator \( T_n \) has propagation \( d_X(x^n, y^n) \), therefore, lies in \( C_u^*(X) \), and the partial sums \( \sum_{i=1}^N T_i^* T_n \) are uniformly bounded in \( N \). However, this sequence does not lie in \( M_{X \times \mathbb{N},d_0} \) because the strong limit \( \sum_{i=1}^\infty T_i^* T_n \) does not lie in \( C_u^*(X) \).

The sequence \( (S_n)_{n \in \mathbb{N}} \) lies in \( M_{X \times \mathbb{N},d_0} \). Indeed, the propagation of each \( S_n \), \( n \in \mathbb{N} \), is equal to one, which means that the propagation of \( S \) is equal to one. Suppose that \( S \) lies in the second dual module \( l_2(C_u^*(X))^\prime \). There is a natural action of the first dual module on the second dual. Let \( T(S) \in C_u^*(X) \) be the result of this action of \( T \) on \( S \). This action extends the standard inner product \( \langle T, S \rangle = \sum_{i=1}^\infty T_i S_i \) when \( T, S \in l_2(C_u^*(X)) \), but in general the value of \( T(S) \) is not related to the series \( \sum_{i=1}^\infty T_i^* S_i \), even if this series converges in some topology. Let \( L_N \cong C_u^*(X)^N \subset l_2(C_u^*(X)) \) be a free submodule of finite sequences of length \( N \). Then \( l_2(C_u^*(X)) = L_N \oplus L_N^1 \), and similar
decompositions into direct sums hold for the first and second dual modules of the module $l_2(C^*_u(X))$. Let $T = T_N + T'_N$, $S = S_N + S'_N$ be the corresponding decompositions of $T$ and $S$, respectively. Let $K_M \subset H_X$ be the $2M$-dimensional linear subspace generated by functions $\delta_{x_n}$ and $\delta_{y_n}$, $n \leq M$. Let us fix $M \in \mathbb{N}$. Since the series $\sum_{n=1}^{\infty} S_n^* S_n$ is convergent with respect to the strong topology, for any $\varepsilon > 0$ one can find $N \in \mathbb{N}$ such that

$$\left| \left( \varepsilon, \sum_{n=N+1}^{\infty} S_n^* S_n \right) \right| < \varepsilon$$

for any unit vectors $\xi, \eta \in K_M$. Let $P_M$ be the projection in $H_X$ onto $K_M$. Evidently, $P_M \in C^*_u(X)$. Then

$$\|S_N^* P_M\|^2 = \sup_{\xi, \eta \in K_M, \|\xi\| = \|\eta\| = 1} \left| \left( \xi, \sum_{n=N+1}^{\infty} P_M S_n^* S_n P_M \eta \right) \right| \leq \varepsilon,$$

hence

$$\|S_N^* P_M\|^2 < \varepsilon$$

for any unit vectors $\xi, \eta \in K_M$. Let $P_M$ be the projection in $H_X$ onto $K_M$. Evidently, $P_M \in C^*_u(X)$. Then

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for any unit vectors $\xi, \eta \in K_M$. Let $P_M$ be the projection in $H_X$ onto $K_M$. Evidently, $P_M \in C^*_u(X)$. Then

$$\left| \left( \varepsilon, \sum_{n=N+1}^{\infty} P_M S_n^* S_n P_M \right) \right| \leq \varepsilon.$$
Denote by $l_2(\mathbb{D}(H_X))'_0 \subset l_2(\mathbb{D}(H_X))'$ the submodule consisting of sequences $(D_n)_{n \in \mathbb{N}}$,
\[ D_n = \text{diag}(d_n^1, d_n^2, \ldots) \in \mathbb{D}(H_X), \quad d_n^i \in \mathbb{C}, \quad i, n \in \mathbb{N}, \]
such that $d_n^i = 0$ for $i < n$.

**Proposition 7.** $M_{X \times \mathbb{N}, d} = l_2(\mathbb{K}(H_X)) + l_2(\mathbb{D}(H_X))'_0$.

**Proof.** If $T = (T_n)_{n \in \mathbb{N}} \in l_2(\mathbb{K}(H_X))$ then for any $\varepsilon > 0$ there exist $K_1, \ldots, K_n \in \mathbb{K}(H_X)$ such that $\|T - K\| < \varepsilon$, where $K = (K_1, \ldots, K_n, 0, 0, \ldots)$. As the propagation of the operator $K$ is finite, we have $K \in M_{Y, d}$. If $T = (T_n)_{n \in \mathbb{N}} \in l_2(\mathbb{D}(H_X))'_0$ then the propagation of $T$ equals one, hence, $T \in M_{X \times \mathbb{N}, d}$.

Let the propagation of $T \in M_{Y, d}$ does not exceed $L$. It follows from $T_{x^0, x^l} \neq 0$ that $d(x^k, x^l) \leq L$, hence, if $n \geq L$ then $k = l \geq n$. Denote by $P_L$ the projection onto the linear span of the functions $\delta_{x^1}, \ldots, \delta_{x^L}$. Let $D_n$ be the diagonal part of $T_n$, i.e. the operator given by $D_n\delta_x = (T_n)_{x_0, x_n}\delta_x$, $x \in X$, and let $K_n = P_L(T_n - D_n)P_L$. Then the rank of $K_n$ does not exceed $L$ for $n \in \mathbb{N}$, and $K_n = 0$ for $n \geq L$. Set
\[ K = (K_1, K_2, \ldots), \quad D = (D_1, D_2, \ldots), \quad T = K + D. \]

Let now $\{T^{(L)}\}_{L \in \mathbb{N}}$ be norm convergent to $T$, and let the propagation of $T^{(L)}$ does not exceed $L$. Then the diagonal parts $D^{(L)}$ of the operators $T^{(L)}$ are norm convergent to the diagonal part $D$ of the operator $T$, and, therefore, the same holds for their compact parts: $K^{(L)} \to K$ as $L \to \infty$. Since $K_n^{(L)} = 0$ for $n > L$, the norm closure of the finite support sequence lies in $l_2(\mathbb{K}(H_X))$.

Let us show that the partial sums $\|\sum_{n=1}^N D_n^*D_n\|$ are uniformly bounded in $N$. If this would be false then for any $m > 0$ there would exist $N_m$ such that $\|\sum_{n=1}^m D_n^*D_n\| > m$. As $D^{(L)}$ is norm convergent to $D$, there exists $L_0 > 0$ such that $\|D^{(L)}\| \leq \|D\| + 1$ for any $L \geq L_0$. But
\[ \|\sum_{n=1}^m (D_n^{(L)})^*D_n^{(L)}\| \leq \|D^{(L)}\|^2 \leq (\|D\| + 1)^2. \]
Taking here $m > (\|D\| + 1)^2$, we obtain a contradiction. Thus, $D \in l_2(\mathbb{D}(H_X))'$. It is easy to see that $D$ lies in the submodule $l_2(\mathbb{D}(H_X))'_0$. \hfill \Box

**Example 4.** Let
\[ x_n^k = \begin{cases} (k^2, 0, \ldots, 0, 1, 0, 0, \ldots) & \text{for } k < n; \\ (k^2 - n, 0, \ldots, 0, n + 1, 0, 0, \ldots) & \text{for } k \geq n, \end{cases} \]
where the non-zero entries are at the 0-th and $n$-th place. Let $\rho$ be the metric on $\bigsqcup_{n=0}^\infty X_n$ induced by the metric on $l_1(\overline{\mathbb{N}})$. Then $\lim_{n \to \infty} d(x_n^k, x_{n}^{k'}) = \infty$, and $d(x_0^k, x_{n}^{k}) = 1$ for $k \geq n$.

Let $l_2(\mathbb{D}(H_X))'_1 \subset l_2(\mathbb{D}(H_X))'$ be a submodule consisting of sequences $(D_n)_{n \in \mathbb{N}}$,
\[ D_n = \text{diag}(d_n^1, d_n^2, \ldots) \in \mathbb{D}(H_X), \quad d_n^i \in \mathbb{C}, \quad i, n \in \mathbb{N}, \]
such that $d_n^i = 0$ for $i > n$.

**Proposition 8.** $M_{X \times \mathbb{N}, \rho} = l_2(\mathbb{K}(H_X)) + l_2(\mathbb{D}(H_X))'_1$.

**Proof** is similar to that of Proposition 7. \hfill \Box

Let a mapping $\varphi : \mathbb{N} \to \mathbb{N}$ satisfy the following conditions:

1) $\varphi$ takes each value infinitely many times,

2) $\varphi(k) \leq k$ for any $k \in \mathbb{N}$. 

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Example 5. Set
\[ x^k_n = (k^2 - \varphi(k), 0, \ldots, 0, \varphi(k), 0, 0, \ldots), \]
where \( \varphi(k) \) is the \((n + 1)\)-th coordinate. The metric on \( X_0 = \{x^k_n : k \in \mathbb{N}\} \) coincides with the standard metric on \( X = \mathbb{N}^2 \). Let \( b \) be the metric on \( \bigcup_{n=0}^{\infty} X_n \) induced by the metric on \( l_1(\mathbb{N}) \).

Let \( \{k_i\}_{i \in \mathbb{N}} \) be a sequence such that \( \varphi(k_i) = 1 \) for any \( i \in \mathbb{N} \). Then \( b(x^k_0, x^l_n) = 2 \) for any \( i \in \mathbb{N} \).

Put
\[
(T_n)_{x^k_0, x^l_n} = \begin{cases} 
1, & \text{if } k = l = k_n; \\
0 & \text{otherwise}.
\end{cases}
\]

Then \( T = (T_n)_{n \in \mathbb{N}} \in M_{X \times \mathbb{N}, b} \).

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REFERENCES
1. Paschke, W.L. “Inner product modules over \( B^*\)-algebras”, Trans. Amer. Math. Soc. 182, 443–468 (1972).
2. Frank, M. “Self-duality and \( C^*\)-reflexivity of Hilbert \( C^*\)-modules”, Zeitschr Anal. Anw. 9, 165–176 (1990).
3. Paschke, W.L. “The double \( B\)-dual of an inner product module over a \( C^*\)-algebra \( B\)”, Canad. J. Math. 26, 1272–1280 (1974).
4. Frank, M., Manuilov, V.M., Troitsky, E.V. “A reflexivity criterion for Hilbert \( C^*\)-modules over commutative \( C^*\)-algebras”, New York J. Math. 16, 399–408 (2010).
5. Manuilov, V.,Troitsky, E. “Hilbert \( C^*\)-and \( W^*\)-modules and their morphisms”, J. Math. Sci. 98 (2), 137–201 (2000).
6. Nowak, P.W., Guoliang, Yu. Large scale geometry (EMS Textbooks in Math. European Math. Soc., Zürich, 2012).
7. Roe, J. Lectures on coarse geometry (in: Univ. Lect. Ser. 31, Amer. Math. Soc., Providence, 2003).
8. Frank, M. “Hilbert \( C^*\)-modules over monotone complete \( C^*\)-algebras”, Math. Nachr. 175, 61–83 (1995).
9. Manuilov, V. “Metrics on doubles as an inverse semigroup”, J. Geom. Anal. 31 (2021).
10. Exel, R. “Noncommutative Cartan subalgebras of \( C^*\)-algebras”, New York, J. Math. 17, 331–382 (2011).
11. Manuilov, V. “Roe bimodules as morphisms of discrete metric spaces”, Russian J. Math. Phys. 26, 470–478 (2019).
12. Gromov, M. “Asymptotic Invariants of Infinite Groups” (in: Geometric Group Theory, Vol. 2. London Math. Soc. Lect. Note Ser. 182, CUP, Cambridge (1993)).