The New Generalized Difference Sequence Space $\chi^2$ over $p$–Metric Spaces Defined by Musielak Orlicz Function Associated with a Sequence of Multipliers

Deepmala N1*, Subramanian N2 and Mishra VN3,4

1SQC and OR Unit, Indian Statistical Institute, Kolkata, India
2Department of Mathematics, SASTRA University, Thanjavur, India
3Department of Mathematics, National Institute of Technology, Assam, India
4Industrial Training Institute (ITI), Faizabad, Uttar Pradesh, India

Abstract

In the present paper, we introduce new sequence spaces by using Musielak-Orlicz function and a generalized $B^*_p$–difference operator on $p$–metric space. Some topological properties and inclusion relations are also examined.

Keywords: Analytic sequence; Double sequences; $\chi^2$ space; Difference sequence space; Musielak-Orlicz function; $p$–metric space; Lacunary sequence; Ideal

MSC 2010 No: 40A05; 40C05; 40D05

Introduction

Throughout $w$, $x$ and $L^2$ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write $w'$ for the set of all complex sequences $(x_{mn})$ where $m,n \in \mathbb{N}$, the set of positive integers. Then $w'$ is a linear space under the coordinate wise addition and scalar multiplication. For some approximations results in Musielak-Orlicz-Sobolev spaces and some applications to nonlinear partial differential equations see equation 22. The growing interest in this field is strongly stimulated by the treatment of recent problems in elasticity, fluid dynamics, calculus of variations, and differential equations.

Some initial works on double sequence spaces is found in Bromwich [1]. Later on it was investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankani [5], Tripathy et al. [6-10], Turkmenoglu [11], Raj [11-14], and many others [15].

Let $(x_{mn})$ be a double sequence of real or complex numbers. Then the series $\sum_{m,n} x_{mn}$ is called a double series. The double series $\sum_{m,n} x_{mn}$ gives one space is said to be convergent if and only if the double sequence $(S_{mn})$ is convergent, where:

$$ S_{mn} = \sum_{m,n} x_{mn} (m,n = 1,2,3,...) $$

A double sequence $x = (x_{mn})$ is said to be double analytic if,

$$ x_{mn} \rightarrow 0 $$

The vector space of all double analytic sequences are usually denoted by $L^2$. A sequence $x = (x_{mn})$ is called double entire sequence if $x_{mn} \rightarrow 0$ as $m,n \rightarrow \infty$.

The vector space of all double entire sequences are usually denoted by $G^2$. Let the set of sequences with this property be denoted by $L^2$ and $G^2$ is a metric space with the metric

$$ d(x,y) = \sup_{m,n} \left\{ \left| x_{mn} - y_{mn} \right| : m,n \geq 1,2,3,... \right\}, $$

for all $x = (x_{mn})$ and $y = (y_{mn})$ in $G^2$. Let $\varphi = \left\{ \text{finite sequences} \right\}$.

Consider a double sequence $x = (x_{mn})$. The $(m,n)^{th}$ section $x^{(m,n)}$ of the sequence is defined by $x^{(m,n)} = \sum_{i,j=1}^{m,n} x_{ij} \delta_{ij}$ for all $m,n \in \mathbb{N}$, with 1 in the $(m,n)^{th}$ position and zero otherwise.

A double sequence $x = (x_{mn})$ is called double gai sequence if

$$ \left( (m+n)! x_{mn} \right)^{1/m+n} \rightarrow 0 \text{ as } m,n \rightarrow \infty. $$

The double gai sequences will be denoted by $\chi^2$.

Let $M$ and $\Phi$ be mutually complementary Orlicz functions. Then, we have:

(i) For all $u,v \geq 0$,

$$ uv \leq M(u) + \Phi(v), \text{ (Young's inequality) [16]} \tag{2} $$

(ii) For all $u \geq 0$,

$$ \Phi(u) = M(u) + \Phi(u), \tag{3} $$

(iii) For all $u \geq 0$ and $0 < \lambda < 1$,

$$ M(\lambda u) \leq \lambda M(u) \tag{4} $$

Lindenstrauss and Tzafriri [16] used the idea of Orlicz function to construct Orlicz sequence space

*Corresponding author: Deepmala N, SQC and OR Unit, Indian Statistical Institute, 203 BT Road, Kolkata, West Bengal, India, Tel: +91 9913397604; E-mail: dmrai23@gmail.com; deepmaladm23@gmail.com

Received June 08, 2016; Accepted August 18, 2016; Published October 08, 2016

Citation: Deepmala N, Subramanian N, Mishra VN (2016) The New Generalized Difference Sequence Space $\chi^2$ over $p$–Metric Spaces Defined by Musielak Orlicz Function Associated with a Sequence of Multipliers. J Appl Comput Math 5: 331. doi: 10.4172/2168-9679.1000331

Copyright: © 2016 Deepmala N, et al. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.
Citation: Deepmala N, Subramanian N, Mishra VN (2016) The New Generalized Difference Sequence Space \( \chi^2 \) over \( p \)–Metric Spaces Defined by Musielak Orlicz Function Associated with a Sequence of Multipliers. J Appl Computat Math 5: 331. doi: 10.4172/2168-9679.1000331

\[
l_{\infty} = \left\{ x \in \mathbb{w} : \sum_{i=1}^{\infty} M \left( \frac{|x_i|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\},
\]

The space \( l_{\infty} \) with the norm

\[
\|x\| = \inf \left\{ \rho > 0 : \sum_{i=1}^{\infty} M \left( \frac{|x_i|}{\rho} \right) \leq 1 \right\},
\]

becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = \rho(1+e^{-t}) \), the spaces \( l_{\infty} \) coincide with the classical sequence space \( l_p \).

A sequence \( f=(f_{mn}) \) of Orlicz function is called a Musielak-Orlicz function. A sequence \( g=(g_{mn}) \) defined by \( g_{mn} = \sup \{|u| - \min (a; u \geq 0)\} \), \( m,n=1,2, \ldots \) is called the complementary function of a Musielak-Orlicz function \( f \). For a given Musielak Orlicz function \( f \), the Musielak-Orlicz sequence space \( c(\Delta) \) is defined as:

\[
t_f = \left\{ x \in \mathbb{w} : I_f \left( \left\{ x_{mn} \right\} \right)^{\infty} \rightarrow 0 \text{ as } m,n \rightarrow \infty \right\},
\]

where \( I_f \) is a convex modular defined by

\[
I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \left( \frac{1}{\rho} \right)^{\rho} \left\{ x_{mn} \right\} = (x_{mn}) \in t_f.
\]

We consider \( t_f \) equipped with the Luxemburg metric

\[
d(x,y) = \sup_{m,n} \left\{ \inf \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \left( \frac{1}{\rho} \right)^{\rho} \left\{ x_{mn} \right\} \right) \right\}. \leq 1 \right\}.
\]

The notion of difference sequences (for single sequences) was introduced by Kizmaz [17] as follows

\[
Z(\Delta) = \{ x(\Delta) \in \mathbb{w} : \Delta x(\Delta) \in Z \},
\]

where \( \Delta x = x - x_{-1} \) for all \( k \in \mathbb{N} \).

Here \( c_0 \) and \( c_0(\Delta) \) denote the classes of convergent, null and bounded scalar valued single sequences respectively. The spaces \( c_0(\Delta) \) and \( b_{v}\) are Banach spaces normed by:

\[
\|x\|_p = \left\{ \sum_{i=1}^{\infty} |x_i|^p \right\}^{1/p}. \leq 1 \}
\]

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by:

\[
Z(\Delta) = \{ x(\Delta) \in \mathbb{w} : \Delta x(\Delta) \in Z \},
\]

where \( Z = \{1, \chi, \Delta \chi, \Delta^2 \chi, \ldots \} \) and \( \Delta x = x - x_{-1} \) for all \( m,n \in \mathbb{N} \). The generalized difference double notion has the following representation:

\[
\Delta^k x = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{\rho} \right)^{\rho} \left\{ x_{ij} \right\}, \leq 1 \right\}.
\]

Let \( \eta(x) \) be a sequence of nonzero scalars. Then, for a sequence space \( E \), the multiplier sequence space \( E_\eta \) associated with the multiplier sequence \( \eta \) is defined as:

\[
E_\eta = \{ x(\Delta) \in \mathbb{w} : \eta(x) \in E \}.
\]

The notion of sequence spaces associated with multiplier sequences was introduced by. Later on this notion was studied from different aspects by Tripathy and Sen [18], Tripathy and Hazarika [19] and many others [20].

Let \( \eta(x) \) be a sequence of nonzero scalars. Then, for a sequence space \( E \), the multiplier sequence space \( E_\eta \) associated with the multiplier sequence \( \eta \) is defined as:

\[
E_\eta = \{ x(\Delta) \in \mathbb{w} : \eta(x) \in E \}.
\]

**Definition and Preliminaries**

Let \( n \in \mathbb{N} \) and \( X \) be a real vector space of dimension \( w \), where \( n \leq w \). A real valued function \( d_f \) is a sequence of multipliers, \( d_f(x_1, \ldots, x_n) \) is defined on \( X \) satisfying the following four conditions:

(i) \( \|d_f(x_1, \ldots, d_f(x_n))\|_\eta = 0 \) if and only if \( d_f(x_1, \ldots, d_f(x_n)) \) are linearly dependent,

(ii) \( \|d_f(x_1, \ldots, d_f(x_n))\|_\eta \) is invariant under permutation,

(iii) \( \|cd_f(x_1, \ldots, d_f(x_n))\|_\eta = \|d_f(x_1, \ldots, d_f(x_n))\|_\eta \) for \( c \in \mathbb{R} \),

(iv) \( d_f(x_1, \ldots, d_f(x_n)) = (d_f(y_1, \ldots, y_n)) \) for \( 1 \leq p < \infty \).

Let \( \phi(x, y) \) be a sequence of scalars. For \( x, y \in X \), \( x_1, \ldots, x_n \) is called the product metric of the Cartesian product of \( n \) metric spaces is the \( p \) norm of the \( n \) vector of the norms of the \( n \) subspaces.

A trivial example of \( p \) product metric of \( n \) metric space is the \( p \) norm space is \( X = \mathbb{R}^n \) equipped with the following Euclidean metric in the product space is the \( p \) norm:

\[
\|d_f(x_1, \ldots, d_f(x_n))\|_\eta = \sup \left( d_f(x_{i1}, \ldots, x_{in}) \right).
\]

Where \( x_i \in \mathbb{w} \) for each \( i = 1, \ldots, n \).

If every Cauchy sequence in \( X \) converges to some \( L \in X \), then \( X \) is said to be complete with respect to the \( p \)–metric. Any complete \( p \)–metric space is said to be \( p \)–Banach metric space.

Let \( X \) be a linear metric space. A function \( w : \mathbb{R}^n \) is called a Banach space, if:

1. \( w(x) = 0, \) for all \( x \in X \);

2. \( w(-x) = w(x), \) for all \( x \in X \);

3. \( w(x+y) \leq w(x) + w(y), \) for all \( x, y \in X \);

4. \( w(\sigma(x)) \leq \sigma w(x), \) for all \( x \in X \) and \( \sigma \) is a sequence of vectors with \( w(x_{n_i}) \rightarrow 0 \) then \( w(\eta(x_{n_i} - x)) \rightarrow 0 \) as \( n, m \rightarrow \infty \).

A paranorm \( w \) for which \( w(x) = 0 \) implies \( x = 0 \) is called total paranorm and the pair \((X, w)\) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm by (Willansky, 1984).

\( \eta(\phi_x) \) is a nondecreasing sequence of positive reals tending to infinity and \( \phi_x \leq 1 \) and \( \phi_x, \phi_y \leq \phi_{x+y} \).

The generalized de la Vallee-Pussin means is defined by:

\[
t_{\phi_{x+y}}(x) = \frac{1}{\phi_{x+y}} \sum_{m,n=1}^{\infty} \phi_{x+y} x_{mn},
\]

Where \( I_{\phi_x} = \{x \in X : \phi_x x = 1 \} \). For the set of sequences that are strongly summable to zero, strongly summable and strongly bounded by the de la Vallee-Poussin method.
The notion of $\lambda$– double gai and double analytic sequences as follows: Let $\lambda = (\lambda_n)_{n=0}^{\infty}$ be a strictly increasing sequences of positive real numbers tending to infinity, that is:

$$0 < \lambda_0 < \lambda_1 < \ldots < \lambda_n \rightarrow \infty$$

and said that a sequence $x = (x_n) \in w^2$ is $\lambda$–convergent to 0, called a the $\lambda$– limit of $x$, if $B_\lambda^n (x) \to 0$ as $m, n \to \infty$. Where:

$$B_\lambda^n (x) = \frac{1}{\lambda_n} \sum_{i=0}^{\lambda_n - 1} \sum_{j=0}^{\lambda_n - 1} (\Delta^i + \lambda_{i+1}^{-1} \Delta^j - \lambda_{i+1}^{-1} \Delta^j + \Delta_{i+1}^{-1} \lambda_{i+1}^{-1}) |x_{i,j}| \| \tau \|^m_n.$$

The sequence $x = (x_m) \in w^2$ is $\lambda$–double analytic if $\lim B_\lambda^n (x) < \infty$.

If $\lim_{m,n \to \infty} x_m = 0$ in the ordinary sense of convergence, then:

$$\lim_{m,n \to \infty} \frac{1}{\lambda_m} \sum_{i=0}^{\lambda_m - 1} \sum_{j=0}^{\lambda_m - 1} (\Delta^i - \lambda^{-1}_{i+1} \Delta^j - \lambda^{-1}_{i+1} \Delta^j + \Delta_{i+1}^{-1} \lambda_{i+1}^{-1}) |x_{i,j}| \| \tau \|^m_n = 0.$$

This implies that:

$$\lim_{m,n \to \infty} B_\lambda^n (x) = 0 \text{ if } \lim_{m,n \to \infty} \frac{1}{\lambda_m} \sum_{i=0}^{\lambda_m - 1} \sum_{j=0}^{\lambda_m - 1} (\Delta^i - \lambda^{-1}_{i+1} \Delta^j - \lambda^{-1}_{i+1} \lambda_{i+1}^{-1}) \| \tau \|^m_n = 0,$$

which yields that linmum lim,$\mu_m(x) = 0$ and hence $x = (x_m) \in w^2$ is $\lambda$–convergent to 0.

Let $f = (f_m)$ be a Musielak-Orlicz function and $|x| = (d(x_0), d(x_0), \ldots, d(x_0))$, be a $p$–metric space, $q = (q_m)$ be double analytic sequence of strictly positive real numbers. By $w^2(p – X)$ we denote the space of all sequences defined over:

$$X = \{d(x_0), d(x_0), \ldots, d(x_0)\}.$$

The following inequality will be used throughout the paper. If $0 \leq q_m \leq \sup_{m,n} H, K, m \geq 1, 2^m\geq 1$ then:

$$q_m v_n + P_m v_n \leq K \left( q_m v_n + P_m v_n \right)^{2^m}$$

for all $m,n$ and $amn$, $bmn \in \mathbb{C}$. Also $\| \tau \| \leq \max \{1, \| \tau \| \}$ for all $a \in \mathbb{C}$.

Let $f = (f_m)$ be a Musielak-Orlicz function and $\| \tau \| = (d(x_0), d(x_0), \ldots, d(x_0))$, be a $p$–metric space and let $s(w^2 – x)$ denote the space of $X$–valued sequences. Let $q = (q_m)$ be any bounded sequence of positive real numbers and $f = (f_m)$ be a Musielak-Orlicz function. We define the following sequence spaces in this paper:

$$\chi_{\lambda}^{p} = \left\{ \left( d(x_0), d(x_0), \ldots, d(x_0) \right) \right\} = \left\{ x = (x_m) \in X^w : \lim_{m \to \infty} \left[ f_m \left( B_\lambda^n (x), d(x_0), d(x_0), \ldots, d(x_0) \right) \right]^{(m)} = 0 \right\}.$$  

If we take $f_m = x_m$, we get:

$$\chi_{\lambda}^{p} = \left\{ d(x_0), d(x_0), \ldots, d(x_0) \right\} = \left\{ x = (x_m) \in X^w : \lim_{m \to \infty} \left[ f_m \left( B_\lambda^n (x), d(x_0), d(x_0), \ldots, d(x_0) \right) \right]^{(m)} = 0 \right\}.$$  

If we take $q = (q_m) = 1$, we get:

$$\chi_{\lambda}^{p} = \left\{ d(x_0), d(x_0), \ldots, d(x_0) \right\} = \left\{ x = (x_m) \in X^w : \lim_{m \to \infty} \left[ f_m \left( B_\lambda^n (x), d(x_0), d(x_0), \ldots, d(x_0) \right) \right]^{(m)} = 0 \right\}.$$  

In the present paper we plan to study some topological properties and inclusion relation between the above defined sequence spaces. $\chi_{\lambda}^{p}$ and $\chi_{\lambda}^{p}$ are linear spaces.

### Proof

It is routine verification. Therefore the proof is omitted.

### Theorem 2

Let $f = (f_m)$ be a Musielak-Orlicz function, $q = (q_m)$ be a double analytic sequence of strictly positive real numbers, the sequence spaces.

$$\chi_{\lambda}^{p} = \left\{ d(x_0), d(x_0), \ldots, d(x_0) \right\}$$

and, 

$$\chi_{\lambda}^{p} = \left\{ d(x_0), d(x_0), \ldots, d(x_0) \right\}$$

are linear spaces.

### Proof

Clearly $g(x) \geq 0$ for $x = (x_m) \in X^w$. Since $f_m(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then:

$$\inf \left[ f_m \left( B_\lambda^n (x), d(x_0), d(x_0), \ldots, d(x_0) \right) \right]^{(m)} = 0.$$  

Suppose that $B_\lambda^n (x) \neq 0$ for each $m,n \in \mathbb{N}$. Then 

$$B_\lambda^n (x) = (d(x_0), d(x_0), \ldots, d(x_0)) \rightarrow \infty.$$  

It follows that 

$$\inf \left[ f_m \left( B_\lambda^n (x), d(x_0), d(x_0), \ldots, d(x_0) \right) \right]^{(m)} \rightarrow \infty$$

which is a contradiction.

Therefore $B_\lambda^n (x) = 0$.

Let:

$$\left[ f_m \left( B_\lambda^n (x), d(x_0), d(x_0), \ldots, d(x_0) \right) \right]^{(m)} \leq 1$$

and

$$\left[ f_m \left( B_\lambda^n (x), d(x_0), d(x_0), \ldots, d(x_0) \right) \right]^{(m)} \leq 1.$$  

Then by using Minkowski’s inequality, we have:
by (Theorem 1).

Finally, to prove that the scalar multiplication is continuous. Let \( \lambda \) be any complex number. By definition,
\[
g(\lambda x) = \inf \left[ \left( f_m \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right)^{\tau} \right] \leq 1.
\]

Then,
\[
g(\lambda x) = \inf \left[ \left( f_m \left[ B^*_p(\lambda x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right)^{\tau} \right] \leq 1.
\]

Where \( t = \frac{1}{|\lambda|} \). Since \( |\lambda| \leq \max (1, |\lambda\|^{\tau}) \), we have:
\[
g(\lambda x) = \max \left(1, |\lambda|^{\tau}\right) \inf \left[ f_m \left[ B^*_p(\lambda x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau} \leq 1.
\]

**Theorem 3**

(i) If the sequence \( f_m \) satisfies uniform \( \Delta_2 \)-condition, then:
\[
\left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau} = \left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau}.
\]

(ii) If the sequence \( g_m \) satisfies uniform \( \Delta_2 \)-condition, then:
\[
\left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau} = \left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau}.
\]

**Proof**

Let the sequence \( f_m \) satisfies uniform \( \Delta_2 \)-condition, we get:
\[
\left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau} = \left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau}.
\]

To prove the inclusion:
\[
\left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau} \subseteq \left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau}.
\]

Then for all \( x_m \) with \( (\varepsilon_n) \) in \( \left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau} \), we get:
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g_m a_m < \infty.
\]

Since the sequence \( f_m \) satisfies uniform \( \Delta_2 \)-condition, then:
\[
\left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau} < \infty.
\]

Hence:
\[
\left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau} < \infty.
\]

This gives that:
\[
\left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau} < \infty.
\]

From this, we get:
\[
\left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau} = \left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau}.
\]

(ii) Similarly, one can prove that:
\[
\left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau} < \infty.
\]

if the sequence \( g_m \) satisfies uniform \( \Delta_2 \)-condition.

**Proposition 1**

If \( 0 < q_m < p_m < \infty \) for each \( m \) and \( n \), then:
\[
\left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau} \subseteq \left[ \sum_{m=1}^{\infty} \left[ B^*_p(x, d(x_1, 0), d(x_2, 0), \ldots, d(x_5, 0) \right] \right]^{\tau}.
\]

**Proof**

The proof is standard, so we omit it.
Proposition 2
(i) If \(0 < \inf_{q_{mn}} \leq q_{mn} < 1\) then
\[
\left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right] \cap
\left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right] = \sup_{m} \left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right]^y
\]
(ii) If \(1 \leq q_{mn} \leq \sup_{q_{mn}} < \infty\), then
\[
\left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right] \supseteq \left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right]^y
\]
Proof
The proof is standard, so we omit it.

Proposition 3
Let \(f = (f_{mn})\) and \(g = (g_{mn})\) be sequences of Musielak Orlicz functions, we have
\[
\left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right] \cap
\left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right] = \sup_{m} \left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right]^y
\]

Proof
The proof is easy so we omit it.

Proposition 4
For any sequence of Musielak Orlicz functions \(f = (f_{mn})\) and \(q = (q_{mn})\) be double analytic sequence of strictly positive real numbers. Then
\[
\left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right] \cap
\left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right] = \sup_{m} \left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right]^y
\]

Proof
The proof is easy so we omit it.

Proposition 5
The sequence space \(\left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right]^y \) is solid.
Proof
Let \(x = (x_{mn}) \in \left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right]^y \). (i.e)
\[
\sup_{m} \left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right] < \infty
\]
Let \((a_{mn})\) be double sequence of scalars such that \(|a_{mn}| \leq 1\) for all \(m, n \in \mathbb{N} \times \mathbb{N}\). Then we get:
\[
\sup_{m} \left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right] \leq \sup_{m} \left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right]^y
\]

Proposition 6
The sequence space \(\left[ \Lambda^{q_{\eta}} \left[ \beta^{\infty}(x), (d(0, x), d(0, x), \ldots, d(x_{n-1}, 0)) \right] \right]^y \) is monotone.
Proof
The proof follows from Proposition 5.
Proposition 8

If \( f = (f_{nm}) \) be any Musielak Orlicz function. Then

\[
\left[ \sum_{m,n \in \mathbb{N}} \left| \int_{\mathbb{R}^+} \phi_{f_{nm}}(x) \left( d(x,0), d(x,0), \ldots, d(x,0) \right) dx \right|^q \right]^{1/q} = \left[ \sum_{m,n \in \mathbb{N}} \left( \int_{\mathbb{R}^+} \phi_{f_{nm}}(x) \left( d(x,0), d(x,0), \ldots, d(x,0) \right) dx \right)^q \right]^{1/q}
\]

if and only if

\[
\sup_{x \in \mathbb{R}^+} \frac{\phi_{f_{nm}}(x)}{\phi_a} < \infty, \quad \sup_{x \in \mathbb{R}^+} \frac{\phi_{f_{nm}}^q(x)}{\phi_a} < \infty.
\]

Proof

It is easy to prove so we omit.

Proposition 9

The sequence space \( X_{\infty}^\phi \) is not solid.

Proof

The result follows from the following example.

Example 1

Consider

\[
x = (x_{nm}) = \begin{cases} 1 & n = m \\
0 & m 
eq n
\end{cases}
\]

Let

\[
\alpha_{nm} = \begin{pmatrix} 1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \text{for all } m,n \in \mathbb{N}.
\]

Then \( \alpha_{nm} x_{nm} \notin \left[ X_{\infty}^\phi \right]^q \). Hence

\[
\left[ X_{\infty}^\phi \right]^q \] is not solid.

Proposition 10

The sequence space \( X_{\infty}^\phi \) is not monotone.

Proof

The proof follows from Proposition 9.

A sequence \( x = (x_{nm}) \) is said to be \( \phi \)-statistically convergent or \( s_\phi \)-statistically convergent to 0 if for every \( \epsilon > 0 \),

\[
\lim_{n \to \infty} \left| \int_{\mathbb{R}^+} \phi_{f_{nm}}(x) \left( d(x,0), d(x,0), \ldots, d(x,0) \right) dx \right|^q \geq \epsilon
\]

where the vertical bars indicate the number of elements in the enclosed set. In this case we write \( s_\phi - \lim x = 0 \) or \( s_{\phi^*} = 0 \) and \( s_{\phi} = \{x: 30 \in \mathbb{R} : s_\phi - \lim x = 0\} \).

Proposition 11

For any sequence of Musielak Orlicz functions \( f = (f_{nm}) \) and \( q = (q_{nm}) \) be double analytic sequence of strictly positive real numbers. Then

\[
\left[ X_{\infty}^\phi \right]^q \cap \left[ X_{\infty}^{\phi'} \right]^q \subset \left[ X_{\infty}^{\phi \wedge \phi'} \right]^q
\]

Proof

Let \( x \in \left[ X_{\infty}^\phi \right]^q \) and \( \epsilon > 0 \). Then

\[
\left| f_{nm} \left( B_{\phi}(x), d(x,0), d(x,0), \ldots, d(x,0) \right) \right|^q \geq \epsilon
\]

from which it follows that

\[
\left| f_{nm} \left( B_{\phi}(x), d(x,0), d(x,0), \ldots, d(x,0) \right) \right|^q \geq \epsilon
\]

To show that \( \left[ X_{\infty}^{\phi \wedge \phi'} \right]^q \) strictly contain

\[
\left[ X_{\infty}^\phi \right]^q \cap \left[ X_{\infty}^{\phi'} \right]^q
\]

We define \( x = (x_{nm}) \) by

\[
(x_{nm}) = \begin{cases} 0 & m 
eq n \\
1 & m = n
\end{cases}
\]

if \( rs - \sqrt{\phi_{f_{nm}}} \geq mn \leq rs \) and \( (x_{nm}) = 0 \) otherwise. Then:

\[
\left| f_{nm} \left( B_{\phi}(x), d(x,0), d(x,0), \ldots, d(x,0) \right) \right|^q \geq \epsilon
\]

as \( r,s \to \infty \),

i.e. \( x \to 0 \) \( \left[ X_{\infty}^\phi \right]^q \cap \left[ X_{\infty}^{\phi'} \right]^q \)

where \( \lfloor \ldots \rfloor \) denotes the greatest integer function. On the other hand,

\[
\left| f_{nm} \left( B_{\phi}(x), d(x,0), d(x,0), \ldots, d(x,0) \right) \right|^q \to \infty
\]

as \( r,s \to \infty \),

i.e. \( s_{\phi} \to 0 \) \( \left[ X_{\infty}^\phi \right]^q \cap \left[ X_{\infty}^{\phi'} \right]^q \)

Conclusion

Approximations results in Musielak Orlicz spaces are applicable in nonlinear partial differential equations. We proposed a generalized triple sequence spaces and discuss general topological properties with respect to a sequence of Musielak-Orlicz function. Our result generalizes and unifies the results of several author's in the case of classical Orlicz spaces. One can extend our results for more general spaces.

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

Acknowledgement

The authors are extremely grateful to the anonymous learned referee(s) for their keen reading, valuable suggestion and constructive comments for the improvement of the manuscript. The research of the first author Deepmala N is supported by the Science and Engineering Research Board (SERB), Department of Science and Technology (DST), Government of India under SERB National Post-Doctoral fellowship scheme File Number: PDF/2015/000799.
References

1. Bromwich TJ (1965) An introduction to the theory of infinite series. Macmillan and Co Ltd, New York.

2. Hardy GH (1917) On the convergence of certain multiple series. Proceedings of the Cambridge Philosophical Society 19: 86-95.

3. Moricz F (1991) Extentions of the spaces c and c₀ from single to double sequences. Acta Mathematica Hungarica 57: 129-136.

4. Moricz F, Rhoades BE (1988) Almost convergence of double sequences and strong regularity of summability matrices. Mathematical Proceedings of the Cambridge Philosophical Society 104: 283-294.

5. Basarir M, Solancan O (1999) On some double sequence spaces. Journal of Indian Academy Mathematics 21: 193-200.

6. Tripathy BC, Sarma B (2009) Vector valued double sequence spaces defined by Orlicz function. Mathematica Slovaca 59: 787-776.

7. Tripathy BC, Dutta AJ (2010) Bounded variation double sequence space of fuzzy real numbers. Computers and Mathematics with Applications 59: 1031-1037.

8. Tripathy BC, Sarma B (2011) Double sequence spaces of fuzzy numbers defined by Orlicz function. Acta Mathematica Scientia 31: 134-140.

9. Tripathy BC, Chandra P (2011) On some generalized difference paranormed sequence spaces associated with multiplier sequences defined by modulus function. Analysis in Theory and Applications 27: 21-27.

10. Tripathy BC, Dutta AJ (2013) Lacunary bounded variation sequence of fuzzy real numbers. Journal in Intelligent and Fuzzy Systems 24: 185-189.

11. Turkmenoglu A (1999) Matrix transformation between some classes of double sequences. Journal of Institute of Mathematics and Computer Science Mathematics Series 12: 23-31.

12. Raj K, Sharma SK (2011) Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function. Acta Universitatis Sapientiae Mathematica 3: 97-109.

13. Raj K, Sharma AK, Sharma SK (2012) Sequence spaces defined by Musielak-Orlicz function in 2-normed spaces. Journal of Computational Analysis and Applications pp: 14.

14. Raj K (2013) Lacunary sequence spaces defined by Musielak-Orlicz function. Le Matematiche 68: 33-51.

15. Kamthan PK, Gupta M (1981) Sequence spaces and series. Pure and Applied Mathematics, Marcel Dekker Inc, New York.

16. Lindenstrauss J, Tzafriri L (1971) On Orlicz sequence spaces. Israel Journal of Mathematics 10: 379-390.

17. Kizmaz H (1981) On certain sequence spaces. Canadian Mathematical Bulletin 24: 169-176.

18. Tripathy BC, Sen M (2014) Paralinear–convergent double sequence spaces associated with multiplier sequences. Kyungpook Mathematical Journal 54: 321-332.

19. Tripathy BC, Hazarika B (2008) l–convergent sequence spaces associated with multiplier sequence spaces. Mathematical Inequalities and Applications 11: 543-548.

20. Wilansky A (1984) Summability through Functional Analysis. North-Holland Mathematical Studies, North-Holland Publishing, Amsterdam.