ON GRADED $K$-THEORY, ELLIPTIC OPERATORS
AND THE FUNCTIONAL CALCULUS

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ABSTRACT. Let $A$ be a graded $C^*$-algebra. We characterize Kasparov’s $K$-theory group $\hat{K}_0(A)$ in terms of graded $*$-homomorphisms by proving a general converse to the functional calculus theorem for self-adjoint regular operators on graded Hilbert modules. An application to the index theory of elliptic differential operators on smooth closed manifolds and asymptotic morphisms is discussed.

1. Introduction.

Let $A$ be a graded $\sigma$-unital $C^*$-algebra with grading automorphism $\alpha$. We characterize Kasparov’s $K$-group in the category of graded $C^*$-algebras, $\hat{K}_0(A) = KK(\mathbb{C}, A)$, as the group of graded homotopy classes of graded $*$-homomorphisms from $C_0(\mathbb{R})$, the $C^*$-algebra of continuous functions on the real line with the even-odd function grading, to the graded tensor product $A \hat{\otimes} K$, where $K \cong M_2(\mathcal{K})$ is the $C^*$-algebra of compact operators graded into diagonal and off-diagonal matrices. Addition is given by direct sum.

The isomorphism is established in Section 3 by proving a general converse to the functional calculus theorem [1] for self-adjoint regular operators on graded Hilbert modules in Section 2. We will indicate in Section 4 how this characterization is useful in simplifying calculations with asymptotic morphisms of $C^*$-algebras and elliptic differential operators $D$ with coefficients in a trivially graded $C^*$-algebra $A$ over a smooth closed manifold $M$. The functional calculus will give an explicit formulation as (nontrivial) compatible graded $*$-homomorphisms of the generalized Fredholm index $\text{Index}_A(D) \in K_0(A)$ and the symbol class $[\sigma(D)] \in K_0^0(T^*M)$ (the topological $K$-theory of vector $A$-bundles of the cotangent bundle $T^*M$) in a form which is suitable for composing directly with asymptotic morphisms, with no rescaling or suspensions as in the general theory. Since the product in $E$-theory is given by composition, this approach to index theory is simpler than using the Kasparov product in $KK$-theory [10], which can be very technical.

We should note that Kasparov’s graded $K$-theory is unrelated to van Daele’s version, except when $A$ is trivially graded [19]. This paper represents work that partially began in the author’s thesis [17], although the material in Section 2 is new. The author would like to thank his advisers Nigel Higson and Paul Baum for their invaluable help and encouragement and also Erik Guentner for helpful suggestions.

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2. Graded $C^*$-algebras and Hilbert modules.

In this section we collect some definitions and results on graded $C^*$-algebras and Hilbert modules and fix notation. For a complete discussion, see the books [3,9].

Let $A$ be a $C^*$-algebra. Recall that $A$ is graded if there is a $\ast$-automorphism $\alpha : A \to A$ such that $\alpha^2 = \text{id}_A$. Equivalently, there is a decomposition of $A$ as a direct sum $A = A_0 \oplus A_1$, where $A_0$ and $A_1$ are self-adjoint closed linear subspaces with the property that if $x \in A_m$ and $y \in A_n$ then $xy \in A_{m+n}$ (mod 2). In fact, $A_n = \{x \in A : \alpha(x) = (-1)^n x\}$. We write $\partial x = n$ if $x \in A_n$. If there is a self-adjoint unitary $\epsilon$ (called the grading operator) in the multiplier algebra $M(A)$ such that $\alpha(x) = \epsilon x \epsilon^*$, then $A$ is said to be evenly graded. A $\ast$-homomorphism $\phi : A \to B$ of graded $C^*$-algebras is graded if $\phi(A_n) \subset B_n$ for $n = 0, 1$.

Example 2.1. The following are the main examples we will be concerned with.

a.) Every $C^*$-algebra $A$ can be trivially graded by setting $A_0 = A$ and $A_1 = \{0\}$. This is an even grading with grading operator $\epsilon = 1$. The complex numbers $\mathbb{C}$ are always assumed to be trivially graded.

b.) The $C^*$-algebra $C_0(\mathbb{R})$ of continuous complex-valued functions on $\mathbb{R}$ vanishing at infinity is graded into the even and odd functions by defining $\alpha(f)(t) = f(-t)$ for all functions $f \in C_0(\mathbb{R})$ and $t \in \mathbb{R}$.

c.) Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space. By choosing an isomorphism $\mathcal{H} \cong \mathcal{H} \oplus \mathcal{H}$ we obtain the standard even grading on the $C^*$-algebra of compact operators $\mathcal{K} = \mathcal{K}(\mathcal{H}) \cong M_2(\mathcal{K})$, with grading operator

$$\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is determined uniquely up to conjugation by a unitary homotopic to the identity.

Let $A$ and $B$ be graded $C^*$-algebras. Define a graded product and graded involution on the vector space tensor product $A \otimes B$ by the formulas

$$(a \hat{\otimes} b)(a' \hat{\otimes} b') = (-1)^{\partial b \partial a'} (aa' \hat{\otimes} bb')$$

$$(a \hat{\otimes} b)^* = (-1)^{\partial a \partial b} (a^* \hat{\otimes} b^*).$$

The resulting $\ast$-algebra is denoted $A \hat{\otimes} B$. A grading on $A \hat{\otimes} B$ is defined by setting

$$\partial(a \hat{\otimes} b) = \partial a + \partial b \pmod{2}.$$ 

Now faithfully represent $A$ and $B$ by $\rho_1$ and $\rho_2$ on graded Hilbert spaces $H_1$ and $H_2$ with grading operators $\epsilon_1$ and $\epsilon_2$, respectively. Then $A \hat{\otimes} B$ is faithfully represented on $H_1 \otimes H_2$ (graded by $\epsilon_1 \otimes \epsilon_2$) via the formula

$$\rho(a \hat{\otimes} b) = \rho_1(a) \epsilon_1^{\partial a} \otimes \rho_2(b).$$

The $C^*$-algebra completion is denoted $A \hat{\otimes} B$ and is called the (minimal) graded tensor product. It does not depend on the choice of representations. (There is also a maximal graded tensor product [3] but it will not be needed for our purposes since one of the factors will always be nuclear.)
**Lemma 2.2.** (Proposition 15.5.1 [3]) If $B$ is evenly graded, then $A \hat{\otimes} B \cong A \otimes B$. If $A$ is also evenly graded, then under this isomorphism $A \otimes B$ is also evenly graded.

**Corollary 2.3.** Let $K$ have the standard even grading. Then $A \hat{\otimes} K \cong M_2(A \otimes K)$. If $A$ is evenly graded by $\epsilon$, $A \hat{\otimes} K \cong M_2(A \otimes K)$ with standard even grading given by $\eta = \text{diag}(\epsilon \otimes 1, -\epsilon \otimes 1)$.

Let $B$ be another graded $C^*$-algebra with grading $\beta$. Then $B[0,1] = C([0,1], B)$ canonically inherits a grading by the formula $\hat{\beta}(f)(t) = \beta(f(t))$. Two graded $*$-homomorphisms $\phi_0, \phi_1 : A \to B$ are graded homotopic if there is a graded $*$-homomorphism $\Phi : A \to B[0,1]$ such that composition with the evaluation maps $ev_t : B[0,1] \to B$ for $t = 0, 1$ are equal to $\phi_0$ and $\phi_1$, respectively. We shall denote by $[A,B]$ the set of graded homotopy classes of graded $*$-homomorphisms from $A$ to $B$. If $\phi : A \to B$ is a graded $*$-homomorphism, then we denote by $[\phi]$ its equivalence class in $[A,B]$.

A Hilbert $A$-module $H$ is graded if there is a Banach space decomposition $H = H_0 \oplus H_1$ such that $H_n \cdot A_m \subseteq H_{n+m}$ and $\langle H_n, H_m \rangle \subseteq A_{n+m}$ (mod 2). We let $L(H)$ denote the $C^*$-algebra of all bounded $A$-linear maps $T : H \to H$ with an adjoint $T^*$ and let $K(H)$ denote the closed two-sided ideal of compact operators. The grading on $H$ induces gradings on $L(H)$ and $K(H)$ via the identities $\partial T = m$ if $T(H_n) \subseteq H_{n+m}$. We let $H^{\text{op}}$ denote $H$ with the opposite grading $H_n^{\text{op}} = H_{1-n}$. Note that if $A$ is trivially graded, $H$ is the direct sum of two orthogonal $A$-modules. If $\phi : B \to L(H)$ is a $*$-homomorphism, a closed submodule $E$ of $H$ is $\phi$-invariant if $\phi(b) : E \to E$ for all $b \in B$.

3. The Converse Functional Calculus.

Let $H$ be a (graded) Hilbert $A$-module. A regular operator on $H$ is a densely defined closed $A$-linear map $D : \text{Domain}(D) \to H$ such that the adjoint $D^*$ is densely defined and $1 + D^* D$ has dense range.\footnote{If $1 + D^* D$ is not invertible, then $D$ is sometimes called an unbounded multiplier [2, 3].} $D$ has degree one if $\partial(Dx) = \partial x + 1$ for all $x \in \text{Domain}(D)$.

**Proposition 3.1.** For any graded $*$-homomorphism $\phi : C_0(\mathbb{R}) \to A$, there is a maximal $\phi$-invariant closed graded Hilbert $A$-submodule $A_\phi$ of $A$ and a self-adjoint regular operator $D$ on $A_\phi$ of degree one such that for all $f \in C_0(\mathbb{R})$ we have $\phi(f)|_{A_\phi} = f(D)$.

**Proof.** Given a graded $*$-homomorphism $\phi : C_0(\mathbb{R}) \to A$, define

$$A_\phi = C_0(\mathbb{R}) \hat{\otimes} \phi A = \phi(C_0(\mathbb{R}))A$$

to be the closed right ideal generated by the image of $\phi$. This is a closed graded Hilbert submodule of $A$ (see Blackadar [3].) Let $C_c(\mathbb{R})$ denote the dense graded ideal of continuous functions on $\mathbb{R}$ with compact support. Define

$$\text{Domain}(D) = \phi(C_c(\mathbb{R}))A$$

which is a dense graded submodule of $A_\phi$. Let $d$ denote the function $d(t) = t$ on $\mathbb{R}$. Define $D : \text{Domain}(D) \to A_\phi$ by the formula $D\phi(f)x = \phi(df)x$ where $f \in C_c(\mathbb{R})$ (so $df \in C_c(\mathbb{R})$) and extend linearly. Suppose that $\phi(f)x = \phi(g)y$ for some other
Choose a function $d' \in C_c(\mathbb{R})$ such that $d = d'$ on the compact set $\text{supp}(f) \cup \text{supp}(g)$. Then we have

$$D\phi(f)x = \phi(d'f)x = \phi(d')\phi(f)x = \phi(d')\phi(g)y = \phi(d'g)y = D\phi(g)y.$$  

It follows that $D$ is well-defined and is clearly $A$-linear. Also, $D$ is degree one since $d$ is an odd function on $\mathbb{R}$. The computation

$$\langle D\phi(f)x, \phi(g)y \rangle = x^*\phi(df)^*\phi(g)y = x^*\phi(df)g = x^*\phi(f)^*\phi(g)y = \langle \phi(f)x, D\phi(g)y \rangle$$

shows that $D$ is symmetric on $\text{Domain}(D)$. This implies that $D$ is closeable, so we replace $D$ by its closure $\overline{D}$. Consequently, $(\overline{D} \pm i)$ are injective and have closed range by Lemma 9.7 [11]. Let $f \in C_c(\mathbb{R})$. For any $x \in A$ we have

$$(1 + D^2)\phi((1 + d^2)^{-1})\phi(f)x = \phi((1 + d^2)(1 + d^2)^{-1}f)x = \phi(f)x.$$ 

It follows that $\text{Range}(1 + D^2) \supset \text{Domain}(D)$ is dense and so $D$ is regular. We will show $D$ is self-adjoint by using a Cayley transform argument.

Extend $\phi$ to $\phi^+ : C_0(\mathbb{R})^+ \rightarrow A^+$ by adjoining a unit. Let $z \in C_0(\mathbb{R})^+$ denote the unitary

$$z(t) = \frac{t - i}{t + i} = 1 - 2i\tilde{r}_-(t), \text{ for } t \in \mathbb{R}$$

where $r_-(t) = (t - i)^{-1}$ denotes the resolvent. Let $U_D = \phi^+(z) = 1 - 2i\phi(r_-) \in A^+$.

It is easy to check that for all $x \in \text{Domain}(D)$ we have that the unitary $U_D$ satisfies

$$U_D(D + i)x = (D + i)U_Dx = (D - i)x.$$ 

By Lemma 9.8 and the discussion following Proposition 10.6 in Lance [11], the closed symmetric regular operator $D$ is self-adjoint and $U_D = (D + i)^{-1}(D - i)$.

To show $\phi(f)|_{A_\phi} = f(D)$, it suffices to show this for the resolvents $r_\pm(t) = (d \pm i)^{-1}(t)$. Let $\{f_n\}_{n=1}^\infty$ be an approximate unit for $C_0(\mathbb{R})$ consisting of compactly supported functions. Let $x \in A_\phi$ be given. Then $\phi(f_n)x \in \text{Domain}(D)$ for all $n$ and $\phi(f_n)x \rightarrow x$ as $n \rightarrow \infty$. We have that as $n \rightarrow \infty$,

$$(D \pm i)\phi((d \pm i)^{-1}f_n)x = \phi((d \pm i)(d \pm i)^{-1}f_n)x = \phi(f_n)x \rightarrow x.$$ 

Now since $\phi((d \pm i)^{-1}f_n)x = \phi((d \pm i)^{-1})\phi(f_n)x \rightarrow \phi((d \pm i)^{-1})x$ as $n \rightarrow \infty$ and $(D \pm i)$ is closed, we conclude that $\phi((d \pm i)^{-1})x = (D \pm i)^{-1}x$. Since $x \in A_\phi$ was arbitrary, we are done. \hfill \square

Let $B$ be a $C^*$-algebra. If $\mathcal{H}$ is a Hilbert $B$-module, a $*$-homomorphism $\phi : A \rightarrow \mathcal{L}(\mathcal{H})$ is called nondegenerate if $\phi(A)\mathcal{H}$ is dense in $\mathcal{H}$. It is called strict if $\{\phi(u_n)\}$ is Cauchy in the strict topology of $\mathcal{L}(\mathcal{H})$ for some approximate unit $\{u_n\}$ in $A$. Nondegeneracy implies strictness [11]. The following result may be considered the converse to the functional calculus for self-adjoint regular operators [2,4,11].

**Theorem 3.2.** (Converse Functional Calculus) Let $\phi : C_0(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ be graded. There is a closed graded $\phi$-invariant Hilbert submodule $\mathcal{H}_\phi$ of $\mathcal{H}$ and a self-adjoint regular operator $D$ on $\mathcal{H}_\phi$ of degree one such that for all $f \in C_0(\mathbb{R})$ we have $\phi(f)x = f(D)x$ for all $x \in \mathcal{H}_\phi$. Moreover, if $\phi$ is strict then $\mathcal{H}_\phi$ is complemented.
and $\phi(f) = f(D) \in \mathcal{L}(\mathcal{H}_\phi) \subseteq \mathcal{L}(\mathcal{H})$. If $\phi$ is nondegenerate then $\mathcal{H} = \mathcal{H}_\phi$. And if $\phi(C_0(\mathbb{R})) \subseteq \mathcal{K}(\mathcal{H})$ then $D$ has compact resolvents.

**Proof.** Let $A = \mathcal{L}(\mathcal{H})$. Let $D' : \text{Domain}(D') \to A_\phi$ be the self-adjoint regular operator on $A_\phi = C_0(\mathbb{R}) \hat{\otimes}_\phi A$ from the previous proposition such that $\phi(f) = f(D')$. Let $i : A \to \mathcal{L}(\mathcal{H})$ be the identity. Define $\mathcal{H}_\phi = \overline{\phi(C_0(\mathbb{R}))\mathcal{H}} = A_\phi \hat{\otimes}_i \mathcal{H}$ which is a closed Hilbert submodule of $\mathcal{H}$. Define $D = D' \hat{\otimes}_i 1$ on

$$\text{Domain}(D) = \text{Domain}(D') \hat{\otimes}_i \mathcal{H} \supseteq \phi(C_c(\mathbb{R}))\mathcal{H}.$$  

By Proposition 10.7 [11], $D$ extends to a self-adjoint regular operator on $\mathcal{H}_\phi$. ($D = i_*(D')$ in the notation of [11].) If $x \in \mathcal{H}_\phi$, we compute that

$$f(D)x = f(D' \hat{\otimes}_i 1)x = (f(D') \hat{\otimes}_i 1)x = f(D' \hat{\otimes}_i 1)x = \phi(f)x.$$  

If $\phi$ is strict then $\mathcal{H}_\phi$ is a complemented submodule of $\mathcal{H}$ by Proposition 5.8 [11] and so $\mathcal{L}(\mathcal{H}_\phi)$ includes as a graded subalgebra of $\mathcal{L}(\mathcal{H})$. The result now easily follows. □

Note that if $\phi$ is the zero homomorphism then $\mathcal{H}_\phi = \{0\}$ and $D = 0$, so $f(D) = 0 = \phi(f)$.

4. Graded $K$-theory.

**Standing Assumptions.** Throughout this section $A$ will denote a complex $\sigma$-unital graded $C^*$-algebra and $C_0(\mathbb{R})$ and $K$ will have the gradings as in Example 2.1.

Let $H_A$ denote the Hilbert $A$-module of all sequences $\{a_n\}_{1}^{\infty} \subseteq A$ such that $\{\sum_{1}^{n} a^*_n a_n\}_{1}^{\infty}$ converges in $A$. It has a natural grading into sequences of even and odd elements. Let $\hat{H}_A = H_A \oplus H^*_A$, where $H^*_A$ denotes $H_A$ with the opposite grading. This is the standard graded Hilbert module for $A$. We have the following very important result of Kasparov in the theory of graded Hilbert modules.

**Stabilization Theorem.** (Kasparov [10]) If $\mathcal{H}$ is a countably generated graded Hilbert $A$-module then $\mathcal{H} \oplus \hat{H}_A \cong \hat{H}_A$.

It is a standard result that $A \hat{\otimes} K$ is graded $*$-isomorphic to $K(\hat{H}_A)$, the $C^*$-algebra of compact operators on $\hat{H}_A$ (with the induced grading) (See 14.7.1 [3]). For the remainder of this section, we will identify $A \hat{\otimes} K = K(\hat{H}_A)$. From stabilization, conjugation by the graded isomorphism $\hat{H}_A \cong H_A \oplus \hat{H}_A$ determines a unitary in $\mathcal{L}(\hat{H}_A) = M(A \hat{\otimes} K)$ of degree zero.

**Lemma 4.1.** Let $u \in M(A \hat{\otimes} K)$ be a unitary of degree zero. There is a strictly continuous path of degree zero unitaries $\{U_t\}_{t \in [0,1]} \subset M(A \hat{\otimes} K)$ such that $U_1 = u$ and $U_0 = 1$.

**Proof.** Write $K = K(H \oplus H)$ where $H = L^2[0,1]$. Then $M(A \hat{\otimes} K)$ contains a copy of $\mathcal{L}(H \oplus H)$. Let $\{v_t\}$ be a strictly continuous path of isometries in $\mathcal{L}(H)$ with $p_t = v_t v^*_t \rightarrow 0$ strongly as $t \rightarrow 0$ as in Proposition 12.2.2 [3]. Set $V_t = v_t \oplus v_t \in \mathcal{L}(H \oplus H)$ and note that each $V_t$ has degree zero. Set $W_t = 1 \hat{\otimes} V_t$ which also has degree zero and let

$$U_t = W_t \circ W^{*}_t \circ (1 - W_t W^{*}_t).$$
for \( t > 0 \) and \( U_0 = 1 \). It is easy to check that this works. \( \square \)

**Definition 4.2.** Let \( A \) have grading automorphism \( \alpha \). Define

\[
K'(A) = K'(A, \alpha) = [\mathcal{C}_0(\mathbb{R}), A \hat{\otimes} \mathcal{K}].
\]

Define a binary operation on \( K'(A) \) by direct sum \([\phi] + [\psi] = [\phi + \psi]\), where the direct sum is with respect to the graded isomorphism \( \hat{H}_A \cong \hat{H}_A \oplus \hat{H}_A\).

**Theorem 4.3.** \( K'(A) \) is an abelian group under the direct sum operation and satisfies the relation

\[
-u[\phi] = [u\phi u^*]
\]

where \( u = u^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) on \( \hat{H}_A = H_A \oplus H_A\).

**Proof.** It follows from Lemma 4.1 and the proof of Theorem 3.1 in Rosenberg [15] carried over to the graded case that \( K'(A) \) is an abelian monoid with zero given by the zero (or any null-homotopic) \(*\)-homomorphism. We only need to show inverses.

Let \( \phi : \mathcal{C}_0(\mathbb{R}) \to \mathcal{K}(\hat{H}_A) \) be graded. Let \( D \) be the regular operator on \( \mathcal{H}_\phi \subset \hat{H}_A \) associated to \( \phi \) from the converse functional calculus. Via stabilization \( \mathcal{H}_\phi \oplus \hat{H}_A \cong \hat{H}_A \) and Lemma 4.1, we may assume (up to graded homotopy) that \( \phi \) is strict by Proposition 5.8 [11]. Thus \( \phi(f) = f(D) \) for all \( f \in \mathcal{C}_0(\mathbb{R}) \). Then \( D^{\text{op}} = uDu^* \) on the Hilbert module \( \mathcal{H}_\phi \) is the operator associated to \([u\phi u^*]\) since by the functional calculus

\[
f(D^{\text{op}}) = f(uDu^*) = uf(D)u^* = u\phi(f)u^*.
\]

Let \( \epsilon \) be the grading on \( \hat{H}_A \). For each \( t \geq 0 \), define

\[
\mathbb{D}_t = \begin{pmatrix} D & t\epsilon \\ t\epsilon & D^{\text{op}} \end{pmatrix}
\]

on \( \mathcal{H}_\phi \oplus \mathcal{H}_\phi^{\text{op}} \subset \hat{H}_A \) and let \( \mathbb{D}_t = 0 \) on the complement. Define \( \Phi_t : \mathcal{C}_0(\mathbb{R}) \to \mathcal{K}(\hat{H}_A) \) by

\[
\Phi_t(f) = f(\mathbb{D}_t).
\]

For \( t = 0 \) we have \( \Phi_0(f) = f(\mathbb{D}_0) = \phi \oplus u\phi u^* \). Note that

\[
\mathbb{D}_t^2 = \begin{pmatrix} D & t\epsilon \\ t\epsilon & D^{\text{op}} \end{pmatrix}^2 = \begin{pmatrix} D^2 + t^2 & 0 \\ 0 & D^{\text{op}}^2 + t^2 \end{pmatrix}
\]

and so the spectrum of \( \mathbb{D}_t \) is contained outside the interval \((-t, t)\). Therefore,

\[
||f(\mathbb{D}_t)|| \leq \sup\{|f(x)| : x \in \text{spec}(\mathbb{D}_t)\} \to 0 \text{ as } t \to \infty
\]

for all \( f \in \mathcal{C}_0(\mathbb{R}) \) and the result follows. \( \square \)

**Definition 4.4.** A \( K \)-cycle for a graded \( \mathcal{C}^*\)-algebra \( A \) is an ordered pair \((\mathcal{H}, T)\), such that \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) is a countably generated graded Hilbert \( A \)-module and \( T \in \mathcal{L}(\mathcal{H}) \), where \( \mathcal{L}(\mathcal{H}) \) is the graded \( \mathcal{C}^*\)-algebra of all bounded \( A \)-linear operators on \( \mathcal{H} \) with adjoint, which satisfies the following conditions:

i.) \( T \) is of degree one;

ii.) \( T - T^* \in \mathcal{K}(\mathcal{H}) \) is compact;

iii.) \( T^2 - 1 \in \mathcal{K}(\mathcal{H}) \) is compact.
The $K$-cycle is called degenerate if $T^2 = 1$.

By a standard argument we may assume that $T = T^*$ is self-adjoint. There is an obvious notion of unitary equivalence for two $K$-cycles [3,10]. Two $K$-cycles $(\mathcal{H}_0, T_0)$ and $(\mathcal{H}_1, T_1)$ are homotopic if there is a $K$-cycle $(\mathcal{H}, T)$ for $A[0,1]$ such that $(\mathcal{H} \otimes_{ev} A, T \otimes_{ev} 1)$ are unitarily equivalent to $(\mathcal{H}_i, T_i)$ where $\text{ev}_t : A[0,1] \rightarrow A$ are the evaluation maps. A collection $\{(\mathcal{H}, T_t)\}_{t \in [0,1]}$ of $K$-cycles for $A$ is called an operator homotopy if $t \mapsto T_t$ is norm continuous in $t$. An operator homotopy induces a homotopy $(\mathcal{H}', T)$ by defining $\mathcal{H}' = C([0,1], \mathcal{H})$ and $T(f)(t) = T_t(f(t))$ for $f : [0,1] \rightarrow \mathcal{H}$.

**Proposition 4.5.** (Theorem 4.1 [10]) The set $KK(C, A)$ of all equivalence classes of $K$-cycles for $A$ under the equivalence relation (generated by) homotopy is an abelian group under the relations

\[
(\mathcal{H}_1, T_1) + (\mathcal{H}_2, T_2) = (\mathcal{H}_1 \oplus \mathcal{H}_2, D_1 \oplus D_2) \\
-(\mathcal{H}, T) = (\mathcal{H}^\text{op}, -T).
\]

The class of any degenerate $K$-cycle is zero in $KK(C, A)$.

Let $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ be the degree one unitary with respect to the grading on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$

**Lemma 4.6.** $-(\mathcal{H}, T) = (\mathcal{H}, T^\text{op}) \in KK(C, A)$, where $T^\text{op} = uTu^*$.

**Proof.** In the complex world, $(\mathcal{H}, T) = (\mathcal{H}, -T)$ since they are operator homotopic (but not through self-adjoint $K$-cycles in general.) It follows that

\[
-(\mathcal{H}, T) = (\mathcal{H}^\text{op}, -T) = (\mathcal{H}^\text{op}, T) = (\mathcal{H}, uTu^*) = (\mathcal{H}, T^\text{op})
\]
since $u : \mathcal{H}^\text{op} \rightarrow \mathcal{H}$ implements a unitary equivalence. \qed

**Theorem 4.7.** $K'(A)$ is isomorphic to $KK(C, A)$.

**Proof.** Let $G(t) = t(t^2 + 1)^{-1/2}$ which defines a degree one, self-adjoint element in $C_0(\mathbb{R}) = M(C_0(\mathbb{R}))$, the continuous bounded functions on $\mathbb{R}$. Define a map $K'(A) \rightarrow KK(C, A)$ via

\[
[\phi] \mapsto (\mathcal{H}_\phi, G(D))
\]

where $D$ is the regular operator associated to $\phi : C_0(\mathbb{R}) \rightarrow K(\mathcal{H}_\phi) \subset K(\mathcal{H}_A)$ via the converse functional calculus. (As in Theorem 4.3, we may assume that $\phi$ is strict.) The operator $G(D)$ is a degree one, self-adjoint element of $M(K(\mathcal{H}_A)) = L(\mathcal{H}_A)$ and $G(D)^2 - 1$ is compact since

\[
G(D)^2 - 1 = (D^2 + 1)^{-1} = \phi(G) \in K(\mathcal{H}_\phi).
\]

This map is easily seen to be well-defined since applying the construction to a graded homotopy $\Phi : C_0(\mathbb{R}) \rightarrow K(\mathcal{H}_A)[0,1]$ yields a homotopy of $K$-cycles by using the graded isomorphism

\[
K(\mathcal{H}_A)[0,1] \sim (A \hat{\otimes} K)[0,1] \sim A[0,1] \hat{\otimes} K \sim K(\mathcal{H}_A).
\]
It is also distributes over direct sums and maps

\[-\phi = \phi^* \mapsto (\mathcal{H}_\phi, G(D)_{\text{op}}) = -(\mathcal{H}_\phi, G(D))\]

via properties of the functional calculus and Lemma 4.6.

The reverse map is defined using the techniques of Baaj and Julg [2]. Let \((\mathcal{H}, F)\) be a \(K\)-cycle for \(A\). We may assume that \(F = F^*\) and \(\mathcal{H} = \mathcal{H}_A\). Let \(T > 0\) be a strictly positive element of \(\mathcal{K}(\mathcal{H}_A)\) of degree zero which commutes with \(F\). Any two such operators are operator homotopic via the straight line homotopy. Let \(D = FT^{-1}\). Note that \(\text{Domain}(D) = \text{Range}(T)\) is a dense submodule of \(\mathcal{H}_A\). One has that \(D = D^*\) and \((D^2 + 1)^{-1} = (T^2(F^2 + T^2))^{-1}\) is compact. We have the identity \(G(D) = F(F^2 + T^2)^{-1/2}\) and so it also follows that \((\mathcal{H}_A, F)\) and \((\mathcal{H}_A, G(D))\) are operator homotopic. It follows from the identity

\[(D \pm i)^{-1} = D(D^2 + 1)^{-1} \mp i(D^2 + 1)^{-1}\]

that the resolvents are also compact. Define

\[KK(C, A) \to K'(A)\]

by sending \((\mathcal{H}_A, F)\) to the graded homotopy class of the graded \(*\)-homomorphism

\[\phi : f \mapsto f(D) \in \mathcal{K}(\mathcal{H}_A).\]

As above, \(\mathcal{K}(\mathcal{H}_A[0,1]) \cong \mathcal{K}(\mathcal{H}_A)[0,1]\), so a homotopy \((\mathcal{H}_A[0,1], F)\) is mapped to a homotopy \(\Phi : C_0(\mathbb{R}) \to \mathcal{K}(\mathcal{H}_A)[0,1]\). Thus the reverse map is well-defined. One checks easily that these two maps are inverses to each other. \(\square\)

If \(A\) is trivially graded and unital then \(A \hat{\otimes} K \cong M_2(A \otimes K)\) with even grading given by \(\epsilon = \text{diag}(1, -1)\). That is, \(M_2(A \otimes K)\) is graded into diagonal and off-diagonal matrices. It follows from the above that

\[K'(A) = [C_0(\mathbb{R}), A \otimes K] \cong K_0(A).\]

We will describe the isomorphism directly via the more familiar language of projections. It is a standard result that \(K_0(A)\) is the group of formal differences of homotopy classes of projections \(p = p^* = p^2 \in A \otimes K\) with addition given by direct sum \([p] + [q] = [p' + q']\) where \(p \sim_h p', q \sim_h q'\) and \(p' \perp q'\). Let \(u \in M_2(M(A \otimes K))\) be the degree one unitary

\[u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Recall that for any self-adjoint involution \(w\) (i.e., \(w^* = w, w^2 = 1\)) there is an associated projection \(p(w) = \frac{1}{2}(w + 1)\).

Let \(x = \{p\} - \{q\} \in K_0(A)\) where \(p\) and \(q\) are projections in \(A \otimes K\). Define a map

\[\phi_x : C_0(\mathbb{R}) \to M_2(A \otimes K)\]

by the formula

\[\phi_x(f) = \begin{pmatrix} f(0)p & 0 \\ 0 & f(0)q \end{pmatrix}; \quad f \in C_0(\mathbb{R}).\]
This defines a $\ast$-homomorphism since $p = p^2 = p^*$ (similarly for $q$) and is graded since $f(0) = 0$ for any odd function. Note that the homotopy class of $\phi_x$ depends only on the homotopy classes of $p$ and $q$. Now we define a map $\mu : K_0(A) \to K'(A)$ by mapping

$$x \mapsto [\phi_x].$$

It also follows that

$$\phi_{[p]}(f) \oplus \phi_{[q]}(f) = \left( \begin{array}{cc} f(0) \text{ diag}(p, q) & \text{ diag}(0, 0) \\ \text{ diag}(0, 0) & \text{ diag}(0, 0) \end{array} \right) \sim_h \left( \begin{array}{cc} f(0)(p' + q') & 0 \\ 0 & 0 \end{array} \right) = \phi_{[p' + q']}(f)$$

and so it is additive. For $x = [p] - [q]$ we have $-x = [q] - [p]$ maps to

$$\phi_{-x}(f) = \left( \begin{array}{cc} f(0)q & 0 \\ 0 & f(0)p \end{array} \right) = u\phi_x(f)u^*.$$

Thus, $\mu(-x) = [u\phi_x u^*] = -[\phi_x] = -\mu(x)$. One should note that with the grading present $\phi_x$ and $\phi_{-x}$ are not homotopic through graded $\ast$-homomorphisms since $u$ has degree one and the identity has degree zero.

Conversely, given $[\phi] \in K'(A)$, extend $\phi$ to a graded $\ast$-homomorphism

$$\phi^+ : C_0(\mathbb{R})^+ \to (A \otimes K)^+$$

by adjoining a unit. Let $z$ denote the unitary given by the “Cayley transform”

$$z(t) = \frac{t + i}{t - i} = 1 + 2ir_-(t)$$

where $r_-(t) = (t - i)^{-1}$ is the resolvent function. Let $u_\phi$ denote the unitary

$$u_\phi = \phi^+(z) = 1 + 2i\phi(r_-) \in (A \otimes K)^+$$

A simple computation shows that $(\epsilon u_\phi)^2 = 1$ and $(\epsilon u_\phi)^* = \epsilon u_\phi$. We also have that $\epsilon^* = \epsilon$ and $\epsilon^2 = 1$. Consider the associated projections

$$p(\epsilon), \ p(\epsilon u_\phi) \in (A \otimes K)^+.$$

By the definition of $u_\phi$ above, we see that $p(\epsilon) - p(\epsilon u_\phi) = 2i\phi(r_+) \in A \otimes K$. Also, a homotopy of $\phi$ induces a homotopy of the unitary $u_\phi$ and thus of $p(\epsilon u_\phi)$. We define $\nu : K'(A) \to K_0(A)$ by

$$\nu([\phi]) = [p(\epsilon)] - [p(\epsilon u_\phi)] \in K_0(A).$$

A simple computation shows that $\nu \circ \mu = 1$. We only need to show $\mu$ is onto. It then follows that $\nu = \mu^{-1}$ is a homomorphism.

Since $A$ is trivially graded $\hat{H}_A = H_A \oplus H_A$ with each factor determining the grading. Again identify $A \hat{\otimes} \mathcal{K} = \mathcal{K}(\hat{H}_A)$. Let $[\phi] \in K'(A)$. Up to graded homotopy we may assume that $\phi : C_0(\mathbb{R}) \to \mathcal{K}(\hat{H}_A)$ is strict (via stabilization). Let

$$D = \begin{pmatrix} 0 & D^*_+ \\ 0 & 0 \end{pmatrix}$$
on $\hat{H}_A$ be the self-adjoint regular operator of degree one with compact resolvents from the converse functional calculus such that $\phi(f) = f(D)$. Let $G(D) = D(D^2 + 1)^{-\frac{1}{2}}$, which is a self-adjoint bounded operator of degree one on $\hat{H}_A$ with $G(D)^2 - 1$ compact. By a graded homotopy, we may assume that $\phi(f) = (f \circ G)(D) = f(G(D))$. (Note that the diffeomorphism $G : \mathbb{R} \to (-1, 1)$ is the homotopy inverse to the inclusion $(-1, 1) \subset \mathbb{R}$.) Thus, we can write

$$G(D) = \begin{pmatrix} 0 & G^*_+ \\ G_+ & 0 \end{pmatrix}$$

on $H_A \oplus H_A$ where $G_+ : H_A \to H_A$ is a generalized Fredholm operator [18]. Up to a compact perturbation of $G_+$ (which would induce a graded homotopy), we may assume that $\text{Ker}(G(D)) = \text{Ker}(G_+) \oplus \text{Ker}(G^*_+)$ is a finite projective $A$-module in $\hat{H}_A$, and is thus complemented. Note that for $x \in \text{Ker}(G(D))$ we have $f(G(D))x = f(0)x$. Since $A$ is trivially graded, $\text{Ker}(G_+)$ and $\text{Ker}(G^*_+)$ are finite projective $A$-modules. Let $P_+^{(x)} \in \mathcal{K}(H_A)$ be the compact projections onto $\text{Ker}(G^{(x)}_+)$. Let $x = [P_+] - [P^*_+] = \text{Index}_A(G_+) \in K_0(A) [18]$. A graded homotopy connecting $\phi$ to the graded $*$-homomorphism

$$\phi_x(f) = \begin{pmatrix} f(0)P_+ & 0 \\ 0 & f(0)P^*_+ \end{pmatrix} \in \mathcal{K}(H_A \oplus H_A) = \mathcal{K}(\hat{H}_A)$$

is given by

$$\Phi_t(f) = \begin{cases} f(t^{-1}G(D)), & t > 0 \\ \phi_x(f), & t = 0. \end{cases}$$

Thus, $\mu(x) = [\phi]$ and so $\mu$ is onto as was desired.

**Corollary 4.8.** If $A$ is unital and trivially graded then the maps $\mu$ and $\nu$ are inverses.

### 5. Elliptic Operators over $C^*$-algebras.

In this section, we will show how the previous results and the functional calculus give explicit realizations as graded $*$-homomorphisms of the $K$-theory symbol class and Fredholm index of an elliptic differential operator with coefficients in a trivially graded $C^*$-algebra.

Let $A$ be a trivially graded *unital* $C^*$-algebra and $M$ be a smooth closed Riemannian manifold. Let $E \to M$ and $F \to M$ be smooth vector $A$-bundles, that is, smooth locally trivial fiber bundles on $M$ whose fibers $E_p$ and $F_p$ are finite projective $A$-modules for each $p \in M$. Let $C_\infty(E)$ denote the vector space of smooth sections of $E$, which is a module over $A$, similarly for $C_\infty(F)$. Let $D : C_\infty(E) \to C_\infty(F)$ be an elliptic differential $A$-operator of order $n$ on $M$ [13,17]. (If $A = \mathbb{C}$ then $D$ is an ordinary differential operator.) Let $\sigma = \sigma(D) : \pi^*(E) \to \pi^*(F)$ denote the principal symbol of $D$ which is a homomorphism of vector $A$-bundles, where $\pi : T^*M \to M$ is the cotangent bundle. The condition of ellipticity is the requirement that for each non-zero cotangent vector $\xi \neq 0 \in T_p^*M$ the principal symbol $\sigma_{\xi}(D) : E_p \to F_p$ is an isomorphism of $A$-modules.

Equipping the fibers $E_p$ (and $F_p$) with smoothly varying Hilbert $A$-module structures

$$\langle \cdot, \cdot \rangle : E \times E \to A$$

and $\mathcal{K}(E)$ as the $K$-theory of the $A$-module $E$. Note that $\mathcal{K}(E)$ is a complex algebra which acts on $E$ as $A$-module algebra. We will show that $\text{Ker}(\pi^*(D))$ is the trivial $A$-module.

Theorem 5.1. If $E \to M$ is a smooth vector $A$-bundle and $D : C_\infty(E) \to C_\infty(F)$ is an elliptic $A$-operator on $M$, then $\pi^*(D)$ is an isomorphism of $A$-modules.
defines a pre-Hilbert $A$-module structure on $C^\infty(E)$ via the formula

$$\langle s, s' \rangle = \int_M \langle s(p), s'(p) \rangle_p \, d\text{vol}_M \in A,$$

for $s, s' \in C^\infty(E)$, where $d\text{vol}_M$ is the Riemannian volume measure on $M$. (And any two such structures are homotopic via the straight line homotopy.) It follows that an adjoint differential operator $D : C^\infty(F) \to C^\infty(E)$ exists and is of the same order as $D$. The principal symbol of the adjoint is the adjoint of the principal symbol $\sigma_\xi(D^\dagger) = \sigma_\xi^*(D) \in \mathcal{L}(F_p, E_p)$ for $\xi \in T^*_pM$. Consider the formally self-adjoint differential $A$-operator of degree one

$$D = \begin{pmatrix} 0 & D^t \\ D & 0 \end{pmatrix} : C^\infty(E) \oplus C^\infty(F) \to C^\infty(E) \oplus C^\infty(F)$$

on the graded pre-Hilbert $A$-module $C^\infty(E) \oplus C^\infty(F)$. The principal symbol of $D$ is the self-adjoint bundle morphism of degree one

$$\sigma = \sigma(D) = \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix} : \pi^*(E) \oplus \pi^*(F) \to \pi^*(E) \oplus \pi^*(F)$$

on the graded pull-back vector $A$-bundle $\pi^*(E) \oplus \pi^*(F)$.

**Lemma 5.1.** The resolvents $(\sigma \pm i)^{-1} : \pi^*(E) \oplus \pi^*(F) \to \pi^*(E) \oplus \pi^*(F)$ are vector $A$-bundle morphisms which vanish at infinity on $T^*M$ in the operator norm induced by the Hilbert $A$-module structures on the fibers $E_p \oplus F_p$.

**Proof.** Follows from homogeneity $\sigma(p, t\xi) = t^n \sigma(p, \xi)$ and ellipticity. □

Form the Cayley transform [14]

$$u = (\sigma + i)(\sigma - i)^{-1} = 1 + 2i(\sigma - i)^{-1}.$$  

By complementing the vector $A$-bundles $E$ and $F$, e.g. $E \oplus G \cong M \times A^n$, we may embed $\pi^*(E \oplus F)$ in a trivial $A$-bundle

$$A = T^*M \times (A^n \oplus A^n).$$

Now extend the automorphism $u$ to the $A$-bundle $A$ by defining it to be equal to the identity on the complement of $\pi^*E \oplus \pi^*F$ in $A$. From the lemma above, it follows that $u$ extends continuously to the trivial $A$-bundle on the one-point compactification $(T^*M)^+$ by setting $u(\infty) = I$.

Let $\epsilon = \text{diag}(1, -1)$ be the grading of the trivial $A$-bundle $(T^*M)^+ \times (A^n \oplus A^n)$. Since $\epsilon \sigma = -\sigma \epsilon$ it follows, as in the previous section, $(u\epsilon)^2 = 1$ and $(u\epsilon)^* = u\epsilon$. (We also have obviously that $\epsilon^* = \epsilon$ and $\epsilon^2 = 1$.)

Therefore, we obtain two projection-valued sections

$$p(\epsilon), p(u\epsilon) : (T^*M)^+ \to \text{End}(A)$$

on $(T^*M)^+$ which are equal at infinity. We can view them as projection-valued functions $(T^*M)^+ \to M_2(M_n(A)) \cong M_{2n}(A)$. Both define elements in $K_0(C(T^*M)^+ \otimes A)$ and so their difference defines an element

$$\Sigma(\epsilon) = [p(\epsilon)], \ [p(u\epsilon)] \in K_0(C(T^*M)^+ \otimes A).$$
This is the symbol class of the elliptic $A$-operator $D$ as constructed in [7,14,17]. By Corollary 4.8 and stability, it follows that

$$K_0(C_0(T^*M) \otimes A) \cong [C_0(\mathbb{R}), C_0(T^*M) \otimes M_{2n}(A)]$$

and $\Sigma(D)$ is identified with the graded homotopy class of the graded $*$-homomorphism

$$\Phi_\sigma : C_0(\mathbb{R}) \to C_0(T^*M, M_{2n}(A)) \cong M_{2n}(C_0(T^*M) \otimes A)$$

given fiber-wise by the ordinary matrix functional calculus

$$\Phi_\sigma(f)(\xi) = f(\sigma_\xi(\mathbb{D})) \in M_{2n}(A), \text{ for } \xi \in T^*M.$$ 

The principal symbol $\sigma(D) : \pi^*(E) \to \pi^*(F)$ determines a class $[\sigma(D)] \in K^0_\mathcal{A}(T^*M)$ (the topological $K$-theory of $T^*M$ defined via vector $A$-bundles) since it is a bundle morphism that is an isomorphism off the compact zero-section $M \subset T^*M$. By the Mingo-Serre-Swan Theorem [12,16], we have $K^0_\mathcal{A}(T^*M) \cong K_0(C_0(T^*M) \otimes A)$, which is induced via the action of taking sections as for the case $A = \mathbb{C}$. It thus follows from this and the constructions in the previous section that all three of these symbol classes can be identified.

**Proposition 5.2.** $[\sigma(D)] = \Sigma(D) = [\Phi_\sigma] \in K^0_\mathcal{A}(T^*M) \cong K_0(C_0(T^*M) \otimes A)$.

Let $L^2(E)$ denote the completion of the pre-Hilbert $A$-module $C^\infty(E)$. The differential $A$-operator $\mathbb{D}$ defines an (essentially) self-adjoint regular operator of degree one on the graded Hilbert $A$-module $\mathcal{H}_\mathbb{D} = L^2(E) \oplus L^2(F)$. (We replace $\mathbb{D}$ by its closure $\mathbb{D}$ which is self-adjoint.) Since $\mathbb{D}$ is elliptic, the resolvents $(\mathbb{D} \pm i)^{-1}$ are compact. (This follows from the parallel Sobolev theory for differential $A$-operators [13].) The complementation of the bundles $E$ and $F$ above allows a coherent inclusion (with the previous constructions)

$$\mathcal{H}_\mathbb{D} \subset L^2(A) \cong L^2(M) \otimes A^{2n}$$

which induces a graded inclusion of $C^*$-algebras $\mathcal{K}(\mathcal{H}_\mathbb{D}) \hookrightarrow M_{2n}(\mathcal{K} \otimes A)$. By the functional calculus for self-adjoint regular operators [11] we obtain a graded $*$-homomorphism

$$\Phi_\mathbb{D} : C_0(\mathbb{R}) \to M_{2n}(\mathcal{K} \otimes A) : f \mapsto f(\mathbb{D})$$

Recall that the usual definition of the generalized Fredholm (analytic) index $\text{Index}_\mathcal{A}(D)$ in terms of kernel and cokernel modules requires compact perturbations for a general $C^*$-algebra $\mathcal{A}$ [13,18]. This is incorporated in the computations in the proof of Corollary 4.8, so we see that the functional calculus for $\mathbb{D}$ gives this index.

**Proposition 5.3.** $\text{Index}_\mathcal{A}(D) = [\Phi_\mathbb{D}] \in K_0(A)$.

Naturally associated to $M$ and $A$ is an asymptotic morphism of $C^*$-algebras

$$\{\Psi_t\}_{t \in [1, \infty]} : C_0(T^*M) \otimes A \to \mathcal{K}(L^2M) \otimes A,$$

which is defined via Fourier transforms and a partition of unity up to asymptotic equivalence. (For complete details on the construction see [5,7,17].) The induced map

$$\Psi_* : K^0_\mathcal{A}(T^*M) \cong K_0(C_0(T^*M) \otimes A) \to K_0(A)$$

on $K$-theory is useful for doing index-theoretic and $K$-theoretic calculations with elliptic operators. If $M = \mathbb{R}^n$, the induced map is Bott periodicity $K_0(C_0(\mathbb{R}^{2n}) \otimes A) \cong K_0(A)$ [17]. The following result implies the exact form of the Mishchenko-Fomenko index theorem [13], hence the Atiyah-Singer index theorem [1] when $A = \mathbb{C}$ as proved originally by Higson [7].
Theorem 5.4. (Lemma 4.6 [17]) If $D$ is an elliptic differential $A$-operator of order one on $M$ then
\[ \Psi_*([\sigma(D)]) = \text{Index}_A(D) \in K_0(A). \]

The proof reduces to composing the graded symbol homomorphism
\[ \Phi_\sigma : C_0(\mathbb{R}) \to M_{2n}(C_0(T^*M) \otimes A) : f \mapsto f(\sigma) \]
with the matrix inflation of this “fundamental” asymptotic morphism for $M$ and $A$
\[ \{\Psi_t\} : M_{2n}(C_0(T^*M) \otimes A) \to M_{2n}(K \otimes A). \]
and comparing this to the continuous family of graded operator $*$-homomorphisms
\[ \{\Phi_D^t\}_{t \in [1, \infty)} : C_0(\mathbb{R}) \to M_{2n}(A \otimes K) : f \mapsto f(t^{-1} \mathbb{D}). \]

One then proves [17] via Fourier analysis and a compactness argument that for any $f \in C_0(\mathbb{R})$,
\[ \lim_{t \to \infty} \|\Psi_t(f(\sigma)) - f(t^{-1} \mathbb{D})\| = 0 \]
and so the composition $\{\Psi_t \circ \Phi_\sigma\}$ is asymptotically equivalent to $\{\Phi_D^t\}$. Therefore, by stability and homotopy invariance of the induced map [5,6],
\[ \Psi_*[\Phi_\sigma] = [\Phi_D^t] = [\Phi_D] \in K_0(A). \]

The result now follows by Propositions 5.2 and 5.3.

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