WARPED-LIKE PRODUCT MANIFOLDS WITH EXCEPTIONAL HOLONYM GROUPS

SELMAN UGUZ

ABSTRACT. In this paper we review $G_2$ and $\text{Spin}(7)$ geometries in relation with a special type of metric structure which we call warped-like product metric. We present a general ansatz of warped-like product metric as a definition of warped-like product. Considering fiber-base decomposition, the definition of warped-like product is regarded as a generalization of multiply-warped product manifolds, by allowing the fiber metric to be non block diagonal. For some special cases, we present explicit example of $(3+3+2)$ warped-like product manifolds with $\text{Spin}(7)$ holonomy of the form $M = F \times B$, where the base $B$ is a two dimensional Riemannian manifold, and the fibre $F$ is of the form $F_i = F_1 \times F_2$ where $F_i$'s ($i = 1, 2$) are Riemannian 3-manifolds. Additionally an explicit example of $(3+3+1)$ warped-like product manifold with $G_2$ holonomy is studied. From the literature, some other special warped-like product metrics with $G_2$ holonomy are also presented in the present study. We believe that our approach of the warped-like product metrics will be an important notion for the geometries which use warped and multiply-warped product structures, and especially manifolds with exceptional holonomy.

Mathematics Subject Classification: 53C25, 53C29.

Key words: Holonomy, $\text{Spin}(7)$ and $G_2$ manifold, warped product, multiply-warped product, $(3+3+2)$ warped-like product.

1. INTRODUCTION

A warped product metric is an important notion in Riemannian geometry as well as in physics and related areas. Using its properties, the different works of geometry and physics use warped product geometries. Furthermore many basic solutions of the Einstein field equations are used of warped geometries, e.g. the Schwarzschild solution and the Robertson-Walker models [One]. In string theory, Yau in [Yau] discussed that “...the easiest way to partition the ten-dimensional space is to cut it cleanly, splitting it into four-dimensional space-time and six-dimensional hidden subspace... and in the non-kahler case, the ten-dimensional spacetime is not a Cartesian product but rather a warped product.” However, the fundamental rigidity theorems for manifolds of non-negative or positive Ricci curvature are the volume cone (metric cone) theorem, the maximal diameter theorem and the splitting theorem [Che]. Each theorem asserts that if a certain geometric value such as volume or diameter is possible large enough relative to the pertinent lower bound on Ricci curvature, then the metric of the manifold is a warped product metric of a special type (see details in [Che]).

On the other hand, the notion of holonomy group of a Riemannian manifold defined by Élie Cartan has exhibited to be an efficient tool in the study of Riemannian manifolds (see for the details [KN, Agr, SW]). The list of all possible holonomy groups of irreducible, simply-connected, non-symmetric spaces was given by M. Berger in 1955 [Ber]. Berger’s list (refined by Alekseevskii [Ale] and Gray-Brown [Gra]) includes the groups $\text{SO}(n)$ in $n$-dimensions, $U(n)$, $SU(n)$ in $2n$-dimensions, $Sp(n)$, $Sp(1)Sp(1)$ in $4n$-dimensions and two special cases, $G_2$ holonomy in 7-dimensions and $\text{Spin}(7)$ holonomy in 8-dimensions. Manifolds with holonomy $\text{SO}(n)$ constitute the generic case, all others are denoted as manifolds with “special holonomy” and the last two cases are described as manifolds with “exceptional holonomy”.

The existence of manifolds with exceptional holonomy was first demonstrated by R. Bryant [Bry], complete examples were given by R. Bryant and S. Salamon [RS] and the first compact examples were found by D. Joyce in 1996 [Joy]. The study of manifolds with exceptional holonomy and the construction of explicit examples with special types (e.g. Fernandez classifications [Fe1]) are still an active research area in mathematics and physics (see also references in [SW, Ug1, Ug3, Fe1, Sem, Cab]).

In physics, there exist a special interest in the construction of $G_2$ and $\text{Spin}(7)$ holonomy metrics due to their application in supergravity compactification (see more details [Agr, Bry, Gib, Job, Sem]). Since manifolds with
special holonomy provide some geometrical structures for reducing the number of super-symmetries, they are natural candidates for the extra dimensions in string and M-theory [Agr, Bry, Job, RS, GPI, Gib, Sem]. Following the constructions of Bryant and Salamon [Bry, RS] and Joyce [Job], the study of gauge fields on reduced holonomy manifolds was formulated in the physics and mathematics literature. Existence theorems have been given on both compact and non-compact spaces (see refs. in [Cla]).

In [Baz], complete Riemannian metrics with holonomy group $G_2$ on manifolds obtained by deformation of cones over $S^3 \times S^3$ are constructed in details. Their idea of the paper was also used for the purpose of constructing complete Riemannian metrics with holonomy group $Spin(7)$ (see details in [Baz]). In [Cla], it is given a construction of $G_2$ and $Spin(7)$ instantons on exceptional holonomy manifolds constructed by Bryant and Salamon [RS], by using an ansatz of spherical symmetry coming from the manifolds being the total spaces of rank-4 vector bundles. For the seven dimensional $G_2$ case, it is shown that the connections are asymptotic to Hermitian Yang-Mills connections on the nearly Kahler $S^3 \times S^3$ in the asymptotically conical model [Cla].

The motivation for this work was the results [BUT, Ug1, Ug2] and the explicit $Spin(7)$ metric special solution on $M = F_1 \times F_2 \times B = S^3 \times S^3 \times R^2$ in [YO], with the uniqueness of this solution proved in [BUT]. It is noticed that their metric ansatz (structure) was a generalization of warped products and we presented “$(3+3+2)$ warped-like product metrics” as a general framework for the special metrical ansatz as given in [BUT]. It is looked whether one could obtain other solutions by relaxing some of their assumptions, in particular without requiring the three dimensional submanifolds to be $S^3$. It is also proved that the connection of the fibers is determined by the Bonan form $\Omega$ given [BUT]. Considering suitable global assumptions, it is concluded that the fibers are 3-spheres with constant curvature $k > 0$. Similar computation is done for 7-dimensional manifold with $G_2$ holonomy as a $(3 + 3 + 1)$ special warped-like product manifold [Ug1].

In the present paper, we study more general metric structure called “warped-like product metric”. It is presented a general ansatz of warped-like product metric as a definition of warped-like product (see Definition 4.2). Using fiber-base decomposition, the definition of warped-like product is considered as a generalization of multiply-warped product manifolds, by allowing the fiber metric to be non block diagonal. For some special cases, we present explicit examples warped-like product manifolds with $G_2$ and $Spin(7)$ holonomies (i.e. exceptional holonomy).

The outline of paper is given as follows: we set up the basic classifications of $G_2$ and $Spin(7)$ geometries in Section 2. Warped product and a generalization of warped products which is called multiply-warped product [FS] are presented in Section 3. In Section 4, we study a generalization of multiply-warped product manifolds as a “warped-like product”, by allowing the fiber metric to be non block diagonal. Some of the explicit examples of warped-like product manifolds with $G_2$ and $Spin(7)$ holonomies are studied in Section 5. The conclusions and planning of the next studies are presented in Section 6.

2. Exceptional geometries in special dimensions $d = 7$ and $d = 8$

In this section we present the basics of $G_2$ and $Spin(7)$ geometries with their related properties [Agr, Bry, Job, Sal, Fe1, Iva2, Iva3]. Let us start with seven dimensional case. Suppose that $N$ indicates a 7-dimensional manifold with an invariant $G_2$ structure. Thus, $N$ is endowed with a non-degenerate 3-form $\varphi$ that induce a Riemannian metric. The fundamental material for the $G_2$ (also $Spin(7)$) geometry can be found in standard holonomy references books [Job, Sal]. Let us only recall that the Riemannian geometry of $N$ is completely determined by the special form (called Fundamental 3-form in 7-dimension)

$$\varphi = e^{125} - e^{345} + e^{567} + e^{136} + e^{246} - e^{237} + e^{147}.$$  

(2.1)

It has become customary to suppress wedge signs when writing differential forms, so $e^{ij...}$ indicates $e^i \wedge e^j \wedge \ldots$ from now on. The results of Fernandez and Gray [Fe2] give one to describe $G_2$ geometry exclusively in algebraic terms, by looking at the various components of $d\varphi$, $d \wedge \varphi$ in the irreducible summands. Many authors have studied special classes of $G_2$ structures, see for instance [Fe1, Fe2, Sem, Iva2, Cab]. Note that 16 classes of $G_2$ manifolds in the Fernandez classification can be described in terms of the Lee form summarized some of them as given in Table 1.
Moving up one dimension, we consider a product $M$ of $N$ with $\mathbb{R}$, endowed with metric $g$. Indicating by $e^8$ the unit 1-form on the real line one obtains a basis for the cotangent spaces $T^*_p M$. The manifold $M$ inherits a non-degenerate four-form

$$\Omega = \varphi \wedge e^8 + *\varphi$$

which defines a reduction to the Lie group \cite{iva2}. In equation (2.1), a special note that the Hodge dual map of $\varphi$ (i.e. $*\varphi$) is considered on 7-dimensional manifold $N$, as in the equation (2.1) \cite{job}.

In $Spin(7)$ geometry, we recall that the special 4-form $\Omega$ is self-dual $*\Omega = \Omega$, where $*$ is the Hodge operator and the 8-form $*\Omega \wedge \Omega$ coincides with the volume form. It is well known that the subgroup of $GL(8, \mathbb{R})$ which fixes $\Omega$ is isomorphic to the double covering $Spin(7)$ of $SO(7)$ \cite{sal}. Moreover, $Spin(7)$ is a compact simply-connected Lie group of dimension 21 \cite{agr}.

The 4-form $\Omega$ corresponds to a real spinor $\phi$ and therefore, $Spin(7)$ can be identified as the isotropy group of a non-trivial real spinor \cite{cab}. A 3-fold vector cross product $P$ on $R^8$ can be defined by

$$< P(x \wedge yu \wedge z), t > = \Omega(x, y, z, t), \text{ for } x, y, z, t \in R^8.$$\label{3.3}

Then $Spin(7)$ is also characterized by

$$Spin(7) = \{ au \in O(8) | P(axu \wedge ayu \wedge az) = P(xu \wedge yu \wedge z), x, y, zu \in R^8 \}.$$\label{3.1}

The inner product $<,>$ on $R^8$ can be reconstructed from $\Omega$ \cite{fe1, cab}, which corresponds with the fact that $Spin(7)$ is a subgroup of $SO(8)$. A $Spin(7)$ structure on an 8-dimensional manifold $M$ is by definition a reduction of the structure group of the tangent bundle to $Spin(7)$, we shall also say that $M$ is a $Spin(7)$ manifold. This can be described geometrically by saying that there is a 3-fold vector cross product $P$ \cite{fe1} defined on $M$, or equivalently there exists a nowhere vanishing differential 4-form $\Omega$ on $M$ which can be locally written as

$$\Omega = e^{1258} + e^{3458} + e^{1368} - e^{2468} + e^{1478} + e^{2378} - e^{5678} - e^{1267} - e^{3467} + e^{1357} - e^{2457} - e^{1456} - e^{2356} + e^{1234}.$$\label{3.2}

This special 4-form $\Omega$ is called “Bonan (Cayley or Fundamental) form” of the $Spin(7)$ manifold $M$ \cite{bry, bon, bui}. We also recall that a $Spin(7)$ manifold $(M, g, \Omega)$ is said to be parallel if the holonomy of the metric $Hol(g)$ is a subgroup of $Spin(7)$. This is equivalent to saying that the fundamental form $\Omega$ is parallel with respect to the Levi-Civita connection $\nabla^{LC}$ of the metric $g$. Moreover, $Hol(g)u \subset Spin(7)$ if and only if $d\Omega = 0$ \cite{bry} and any parallel $Spin(7)$ manifold is Ricci-flat \cite{bon}.

According to the classification given by Fernandez \cite{fe1}, there are four classes of $Spin(7)$ manifolds obtained as irreducible representations of $Spin(7)$ of the space $\nabla^{LC}\Omega$. By using the fact that given by Cabrera et. al. \cite{cab}, it is considered the 1-form of the 8-manifold defined by

$$7\Theta = -* (*d\Omega \wedge \Omega) = *(d\Omega \wedge \Omega).$$\label{3.4}

It is called the Lee form (this 1-form is denoted by $\Theta$) of a given $Spin(7)$ structure \cite{iva1}. The four classes of $Spin(7)$ manifolds in the Fernandez classification can be described in terms of the Lee form as below \cite{ug1, iva1}: $W_0 : d\Omega = 0; \ W_1 : \Theta = 0; \ W_2 : d\Omega = \Theta \wedge \Omega; \ W_4 = W_1 \oplus W_2$. We summarize these facts in the following Table 2.
The classes of Spin(7) manifolds

| Conditions  |
|-----------------|
| Parallel case: $W_0$ | $d\Omega = 0, \Theta = 0$ |
| Balanced case: $W_1$ | $\Theta = 0$ |
| Locally conformally parallel case: $W_2$ | $d\Omega = \Theta \wedge \Omega$ |
| Mixed type: $W_4 = W_1 + W_2$ | - |

Table 2: Fernandez classification table of Spin(7) manifolds.

3. Multiply-warped and warped product manifolds

We start with the definition of multiply-warped and warped product manifolds. In the next subsection, we present the definition of our special chosen warped product; warped-like product manifolds as a generalization of a multiply-warped product.

3.1. Warped product manifolds. Let $(F, g_F)$, $(B, g_B)$ be Riemannian manifolds and $f > 0$ be smooth function on $B$. A warped product manifold is a product manifold $M = F \times B$ equipped with the metric

$$ g = \pi_2^* g_B + (f \circ \pi_2)^2 \pi_1^* g_F, $$

where $\pi_1 : F \times B \to F$ and $\pi_2 : F \times B \to B$ are the natural projections.

That is, in local coordinates the first block that depends on the coordinates of the first group of coordinates is multiplied by a function of the second group of coordinates. If the definition holds an open subset of $M$, then $M$ is called locally warped product manifold. Basic properties of warped product manifolds can be found in [One].

3.2. Multiply-warped product manifolds. A generalization of the notion of warped product metrics is the “multiply-warped products”, defined as follows [FS]. Let $(F_i, g_i), i = 1, 2, \ldots, k$ and $(B, g_B)$ be Riemannian manifolds and $f_i > 0$ be smooth functions on $B$. A multiply-warped product manifold is the product manifold

$$ F_1 \times F_2 \times \ldots \times F_k \times B, $$

equipped with the metric

$$ g = \pi_B^* g_B + \sum_{i=1}^{k} (f_i \circ \pi_B)^2 \pi_i^* g_i, $$

where $\pi_B : F_1 \times F_2 \times \ldots \times F_k \times B \to B$ and $\pi_i : F_1 \times F_2 \times \ldots \times F_k \times B \to F_i$ are the natural projections on $B$ and $F_i$ respectively. In this scheme, the metric is block diagonal, with the metrics of the $F_i$’s are multiplied by a conformal factor depending on the coordinates of the base.

We shall first give a local description of the generalization of the warped product structure. We shall use the summation convention whenever appropriate. The base-fiber decomposition suggests that the local formulation of the line element is of the form

$$ ds^2 = (g_F)^{ab}_{ij} dy_i^a \otimes dy_j^b + (g_B)_{ij} dx^i \otimes dx^j, $$

where $g_B$ depends on the base coordinates and $g_F$ depends both on the base and the fiber coordinates. For a generalized warped product the local coordinate expression of $g_F$ is of the form

$$ g = \begin{pmatrix} g_F & 0 \\ 0 & g_B \end{pmatrix} = \begin{pmatrix} A_1 g_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 g_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 g_3 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & A_k g_k & 0 \\ 0 & 0 & 0 & 0 & 0 & g_B \end{pmatrix}. $$

(3.5)
where the $A_a$’s are functions of the base coordinates and each $g_a$ the metric on $F_a$ hence is a symmetric matrix whose entries are functions of the coordinates of the fiber $F_a$.

We will study a special extension of multiply warped product type warped metric construction case called warped-like product manifold in the following.

4. WARPED-LIKE PRODUCT MANIFOLDS

Let $M$ be the topologically product manifold
\begin{equation}
M = F_1 \times F_2 \times \cdots \times F_k \times B,
\end{equation}
where $\dim F_a = n_a$, ($a = 1, \ldots, k$) and $\dim B = n$. The manifold $B$ will be called the “base” and the manifolds $F_a$ will be called “fibers”. We shall denote the local coordinates on the base $B$ by
\begin{equation}
\{x^1, \ldots, x^n\}
\end{equation}
and the local coordinates each fiber $F_a$ by
\begin{equation}
\{y^1_a, \ldots, y^{n_a}_a\}.
\end{equation}

We shall work with Riemannian metrics and use the the summation convention whenever appropriate. A local “moving frame” on an $n$-dimensional Riemannian manifold consists of $n$ local orthonormal sections of its cotangent bundle of. If we denote these local sections by $e_i$, the metric is given by
\begin{equation}
\begin{aligned}
ds^2 &= e_1 \otimes e_1 + e_2 \otimes e_2 + \cdots + e_n \otimes e_n.
\end{aligned}
\end{equation}
Thus, the metric written with respect to the moving frame has constant coefficients. The moving frame has redundancies; we may transform the frame as
\begin{equation}
e_i' = P_{ij} e_j
\end{equation}
where $P$ belongs to $SO(n)$; such transformations leave the metric invariant and they are called the “gauge transformations”. 

The “warped-like” manifold structure that we are trying to define aims to express the metric with respect to a frame in such a way that the functions that appear in the metric depend only on the coordinates of the base.

To start with, let us assume that each manifold $F_a$ is equipped with a Riemannian metric. Hence we have local orthonormal frames on each of them. We denote these local orthonormal frames by $\{\theta^i_a\}$ for $a = 1, \ldots, k$ and $i = 1, \ldots, n_a$. Then
\begin{equation}
\{\theta^1_1, \ldots, \theta^{n_1}_1, \theta^2_2, \ldots, \theta^{n_2}_2, \ldots, \theta^1_k, \ldots, \theta^{n_k}_k, dx^1, \ldots, dx^n\}
\end{equation}
are linearly independent local sections of $T^*M$. Note the notation is chosen such that in $\theta^i_a$, the lower index indicates the frame while the upper index runs through the dimension of the fiber. We can define the metric of the manifold $M$ by defining linearly independent local sections of $T^*M$ and declaring these $lo$ sections to be orthonormal: We shall define a “locally warped-like product” metric on $M$ by declaring the following local sections to be orthonormal:
\begin{equation}
e_a^i = \sum_{b=1}^k \sum_{j=1}^{n_b} A_{ab}^{ij}(x^l) \theta^j_b, \quad a = 1, \ldots, k, \quad i = 1, \ldots, n_a.
\end{equation}
\begin{equation}
e_a = \sum_{\beta=1}^n A_{a}^{\beta} dx^\beta.
\end{equation}
Thus, the metric written with respect to (4.2) has coefficients depending on the base coordinates only.

Remark 4.1. The form of the $e_a^i$ given above is non invariant under local orthonormal transformations of the fibers. If
\begin{equation}
e_a^i \to \sum_{l=1}^{n_b} P_{al}(y^l_a) \theta^l_a,
\end{equation}
then the moving frame transforms as

\[ e_i^a = \sum_{k=1}^n \sum_{j=1}^{n_k} \sum_{l=1}^{n_b} A_{aj}^{lb}(x^l)P_{bl}(y_b^l)\theta_b^l, \quad a = 1, \ldots, k, \quad i = 1, \ldots, n_a. \]

Therefore we should define a manifold to be warped-like if it admits a moving frame of the form (4.3).

The functions that appear in (4.4) that are sums of products of two different groups of coordinates are called “separable functions”.

**Definition 4.2.** Let \( M \) be the topologically product manifold

\[ M = F_1 \times F_2 \times \ldots \times F_k \times B, \]

where \( \dim F_a = n_a, \quad (a = 1, \ldots, k) \) and \( \dim B = n \). Let the local coordinates on the base \( B \) be \( \{x^1, \ldots, x^n\} \).

A metric on \( M \) is called to be locally warped-like product, if there are local frames \( \{\theta_a^i\} \) on each \( F_a \) that are orthonormal with respect to the metric of the \( F_a \), such that

\[ e_i^a = \sum_{k=1}^n \sum_{j=1}^{n_b} A_{aj}^{lb}(x^l)\theta_b^l, \quad a = 1, \ldots, k, \quad i = 1, \ldots, n_a. \]

is an orthonormal frame for \( M \). Note that

\[ A_{aj}^{lb} = A_{aj}^{lb}(x_1, x_2, \ldots, x_n), A_{lj}^{jb} = A_{lj}^{jb}(x_1, x_2, \ldots, x_n). \]

We are looking for a generalization of the form where the local coordinate expression of the metric is no longer block diagonal, but still with a certain simple structure. Since we want to have a fiber structure, we shall allow dependencies on the coordinates of the base everywhere. We shall discuss possible generalizations below.

- One possibility is to have each entry of \( g_F \) to depend on all fiber coordinates, but be of the form

  \[ (g_F)_{ij} = \phi(x^1, \ldots, x^n)\psi(y_1^1, \ldots, y_b^{n_b}). \]

  This means that there is a fiber-base decomposition, but no internal decomposition of the fiber.

- Another possibility is to partition \( g_F \) according to the dimensions of the \( F_a \)'s. If we denote the block in this partition as \( A_{ab} \) where \( A_{ab} \) is a matrix with \( n_a \) rows and \( n_b \) columns, then we may require that

  \[ (A_{ab})_{ij} = \phi_{ij}(x^1, \ldots, x^n)\psi_{ij}(y_a^1, \ldots, y_a^{n_a}, y_b^1, \ldots, y_b^{n_b}). \]

- A stronger form of this is to require that

  \[ (A_{ab})_{ij} = \phi(x^1, \ldots, x^n)(\psi_{ab})_{ij}(y_a^1, \ldots, y_a^{n_a}, y_b^1, \ldots, y_b^{n_b}), \]

  that is to require that the same function of the base coordinates multiplies the whole submatrix.

In the case of generalized warped products, the local expression of the metric is as in case (3) with only diagonal blocks. Then each block is the metric of the fiber multiplied by a function of the base coordinates and we can write the metric in terms of pull-backs of the metrics of the components.

In all of the schemes above, above a fixed point \( \{x^i\} \) of the base we have the same situation. That is for each \( a \), \( A_{aa} \) is a metric on the fiber component \( F_a \) and similarly for each pair \((a, b)\) the submatrix

\[ g_{ab} = \begin{pmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{pmatrix} \]

is the metric on \( F_a \times F_b \) (above this point). But if we want to write the metric as a pull back, we should use the third scheme and in addition the off-diagonal block \( A_{ab} \) should be a “well defined” tensor on the product manifold.
$F_a \times F_b$. Under these conditions we can write the metric on $M$ in terms of various pull-backs. If $g_a$ be the metric on the $F_a$ and $h_{ab}$ the tensor corresponding to the off-diagonal block, then

\begin{equation}
(4.9) 
    g = \pi_B^* g_B + \sum_{a=1}^{k} \pi_a^* g_a + \sum_{a<b=1}^{k} \pi_{ab}^* h_{ab}.
\end{equation}

Thus the main problem is the characterization of $h_{ab}$ (see for special $3 + 3 + 2$ case in (4.12) and (5.7)). Our future research is on towards that direction.

We shall now study the construction of these metrics in terms of an orthonormal frame. If $\{e_a\}$, $a = 1, \ldots, N$ is an orthonormal frame then the metric is

\[
    ds^2 = e_1 \otimes e_1 + \cdots + e_N \otimes e_N.
\]

Denoting by $e$ the vector whose entries are the 1-forms $e_i$ and omitting the tensor product sign, we can write this as

\[
    ds^2 = e^t e.
\]

If the frame transforms as

\[
    e = P\tilde{e},
\]

then

\[
    ds^2 = (P\tilde{e})^t P\tilde{e}.
\]

Thus if we have the metric to be invariant we should have

\[
    P^t P = I,
\]

that is the allowable frame rotations should belong to the orthogonal group if an arbitrary Riemannian metric has to be preserved. This is the reduction of the structure group from $Gl(N, R)$ to $O(N, R)$.

4.1. **Special warped-like manifolds with 6-dimensional fiber space.** We will consider the case where the fiber $F$ is a 6-manifold of the form

\[
    F = F_1 \times F_2,
\]

where $F_i$'s ($i = 1, 2$) are 3-manifolds each equipped with Riemannian metrics. Since all 3-manifolds are parallelizable, then the fiber $F$ is also parallelizable 6-manifold.

Let $\theta^i, \tilde{\theta}^i$ ($i = 1, 2, 3$) be (global) orthonormal sections of the cotangent bundles of $F_1$ and $F_2$ respectively. Then the set of 3-forms in the fiber $F$, i.e. $\Lambda^3(F)$, includes two closed 3-forms. These are the volume forms of the $F_i$'s ($i = 1, 2$) given by

\[
    vol_{F_1} = \theta^{123} \quad \text{and} \quad vol_{F_2} = \tilde{\theta}^{123}.
\]

We choose an **almost complex structure** $J$ in the automorphism group of $TF$ (i.e. $J \in \text{End}(TF)$ and $J^2 = -I_d$) as follows,

\[
    J : \quad TF \longrightarrow TF \\
    (\theta_1, \theta_i) \longmapsto (\theta_1, -\theta_i),
\]

where $\theta_1$ and $\theta_i$ are the dual of the $\theta^i$ and $\tilde{\theta}^i$ respectively.

An almost complex structure $J$ is called an **orthogonal almost complex structure** if $J$ admits a Hermitian metric,

\[
    g(JX, JY) = g(X, Y), \quad \forall X, Y \in \chi(F).
\]

Note that every almost complex structure admits a Hermitian metric on a paracompact manifold and the Hermitian metric $h$ is given by

\[
    h(X, Y) = g(X, Y) + g(JX, JY).
\]

Then the almost complex structure admits a Hermitian metric.

If we fix the Hermitian metric $g$ for the orthogonal almost complex structure $J$, we obtain a non-degenerate 2-form $\omega$ on $F$ and it is defined by

\[
    \omega(X, Y) = g(JX, Y), \quad \forall X, Y \in \chi(F).
\]
Note that the 2-form $\omega$ is also $J$ invariant, i.e. $\omega(JX, JY) = \omega(X, Y)$. Then we get $\omega$ as
\[
\omega = \sum_{i=1}^{3} \theta^i \wedge J\theta^i = \theta^1 \theta^\hat{1} + \theta^2 \theta^\hat{2} + \theta^3 \theta^\hat{3}.
\]

Using orthogonal almost complex structure $J$, it is defined a complex volume form $\Psi$ on $F$ as
\[
\Psi = \Psi^+ + i\Psi^- = (\theta^1 + iJ\theta^1)(\theta^2 + iJ\theta^2)(\theta^3 + iJ\theta^3).
\]

Then we obtain
\[
\Psi^+ = \theta^{123} - \theta^{\hat{1}\hat{2}\hat{3}} - \theta^{1\hat{2}\hat{3}} - \theta^{1\hat{1}2},
\]
\[
\Psi^- = \theta^{1\hat{2}\hat{3}} + \theta^{1\hat{1}2} + \theta^{1\hat{2}\hat{3}} - \theta^{1\hat{1}2}.
\]

Hence the fiber space $F$ admits a special almost Hermitian 6 dimensional manifold structure. Such a manifold is characterized by its complex volume form $\Psi = \Psi^+ + i\Psi^-$ and its Kähler form.

Let us consider a $(6 + n)$-dimensional warped-like product manifold $M = F_1 \times F_2 \times B$ where $F_1, F_2$ are 3-dimensional and $B$ is an $n$-dimensional manifold. As we shall use this structure, we define it separately for ease of reference in the next studies.

**Definition 4.3.** Let $M = F_1 \times F_2 \times B$ be a $(6 + n)$-dimensional topologically product manifold where $F_1, F_2$ are 3-manifolds and $B$ is an $n$-manifold, each equipped with Riemannian metrics. Let $\theta^i, \theta^\hat{i}$ be orthonormal sections of the cotangent bundles of $F_1$ and $F_2$ respectively and $x_1, x_2, ..., x_k$ be local coordinates on $B$. If the metric on $M$ is defined by the following orthonormal frame
\[
e^i = A(x_1, x_2, ..., x_n)\theta^i + B(x_1, x_2, ..., x_n)\theta^\hat{i}, \quad i = 1, 2, 3
\]
\[
e^\hat{i} = \hat{A}(x_1, x_2, ..., x_n)\theta^i + \hat{B}(x_1, x_2, ..., x_n)\theta^\hat{i}, \quad i = 1, 2, 3
\]
(4.10)
\[
e^{i+6} = a_{i1}(x_1, x_2, ..., x_n)dx^1 + ... + a_{in}(x_1, x_2, ..., x_k)dx^n, \quad i = 1, 2, ..., n
\]
where $A, B, \hat{A}, \hat{B}$ and $a_{ij} (i, j = 1, 2, ..., n)$ are functions on base manifold $B$. Then we call $(M, e^i)$ $(i = 1, 2, ..., 6 + n)$ a “$(3 + 3 + n)$ warped-like product” manifold.

In the next section, we will consider a special case for the base manifold $B$ is a two dimensional, that is, the fiber space $F$ is a six dimensional manifold with $n = 2$ dimensional base space.

4.2. 8-dimensional warped-like manifolds with 6-dimensional fiber space. The problem we are dealing with is modelled on an 8-dimensional warped-like product manifold $M = F_1 \times F_2 \times B$ where $F_1, F_2$ are 3-dimensional and base manifold $B$ is a 2-dimensional manifold. Since we will use this structure often, we define it for ease of reference.

**Definition 4.4.** [BUT] Let $M = F_1 \times F_2 \times B$ be an 8-dimensional topologically product manifold where $F_1, F_2$ are 3-manifolds and $B$ is a 2-manifold, each equipped with Riemannian metrics. Let $\theta^i, \theta^\hat{i}$ be orthonormal sections of the cotangent bundles of $F_1$ and $F_2$ respectively and $x, y$ be local coordinates on $B$. If the metric on $M$ is defined by the following orthonormal frame
\[
e^i = A(x, y)\theta^i + B(x, y)\theta^\hat{i}, \quad i = 1, 2, 3
\]
\[
e^\hat{i} = \hat{A}(x, y)\theta^i + \hat{B}(x, y)\theta^\hat{i}, \quad i = 1, 2, 3
\]
(4.11)
\[
e^{i+6} = a_{i1}(x, y)dx + a_{i2}(x, y)dy, \quad i = 1, 2
\]
where $A, B, \hat{A}, \hat{B}$ and $a_{ij} (i, j = 1, 2)$ are functions on base manifold $B$. Then we call $(M, e^i)$ $(i = 1, 2, ..., 8)$ a “$3+3+2$ warped-like product” manifold.

The metric given in the equation (4.11) is written as
\[
g = \sum_{i=1}^{3} e^i \otimes e^i + e^\hat{i} \otimes e^\hat{i} + \sum_{i=1}^{2} e^{i+6} \otimes e^{i+6}
\]
structure gives a decomposition of the exterior algebra as follows.

\[ \Lambda = (a_1^2 + a_2^2) dx \otimes dx + (a_1 a_2 + a_2 a_1) dx \otimes dy + (a_1 a_2 + a_2 a_1) dy \otimes dx \]

\[ + (A^2 + \hat{A}^2) \sum_{i=1}^{3} \theta^i \otimes \theta^i + (B^2 + \hat{B}^2) \sum_{i=1}^{3} \theta^i \otimes \theta^i + 2(AB + \hat{A}\hat{B}) \sum_{i=1}^{3} \theta^i \otimes \theta^i \]

\[ = \pi_B^* g_B + (f_1 \circ \pi_B) \pi^*_1 g_{F_1} + (f_2 \circ \pi_B) \pi^*_2 g_{F_2} + (h_{12} \circ \pi_B) \pi^*_1 \omega, \]

where

\[ f_1 = A^2 + \hat{A}^2, \quad f_2 = B^2 + \hat{B}^2, \quad h_{12} = 2(AB + \hat{A}\hat{B}), \]

\[ \pi_B : F_1 \times F_2 \times B \to B, \quad \pi_b : F_1 \times F_2 \times B \to F_b \quad \text{and} \quad \pi_{12} : F_1 \times F_2 \times B \to F_1 \times F_2 \]

are the natural projections on \( B, F_b \) and \( F_1 \times F_2 \) respectively, and \( \omega = \sum_{i=1}^{3} \theta^i \otimes \theta^i \) on \( F_1 \times F_2 \).

**Remark 4.5.** Note that \( 3 + 3 + 2 \) warped-like metric is given as [BU1].

\[ g_{3+3+2} = \begin{pmatrix}
A\theta^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A\theta^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A\theta^3 & 0 & 0 & 0 & 0 & 0 \\
0 & \hat{A}\theta^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \hat{A}\theta^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \]

where \( \{\theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6, dx, dy\} \) are (global) frame on 8-dimensional manifold.

### 4.3. Fibre-base decomposition on a (3+3+2) warped-like product manifold.

We present a special decomposition as an example of 8-dimensional manifold (i.e. \( 8 = 3 + 3 + 2 \)). Corresponding to the decomposition of the manifold as “base” and “fiber”, the exterior algebra has the following direct sum decomposition,

\[ \Lambda^p(M) = \bigoplus_{a+k=p} \Lambda^{(a,k)}(M), \]

where \( a = 1, \ldots, 6 \) and \( k = 1, 2 \), i.e. in our case the fiber is 6-dimensional and the base is 2-dimensional, this fiber structure gives a decomposition of the exterior algebra as follows.

\[ \Lambda^1(M) = \Lambda^{1,0} \oplus \Lambda^{0,1} \]

\[ \Lambda^2(M) = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2} \]

\[ \Lambda^3(M) = \Lambda^{3,0} \oplus \Lambda^{2,1} \oplus \Lambda^{1,2} \]

\[ \Lambda^4(M) = \Lambda^{4,0} \oplus \Lambda^{3,1} \oplus \Lambda^{2,2} \]

\[ \Lambda^5(M) = \Lambda^{5,0} \oplus \Lambda^{4,1} \oplus \Lambda^{3,2} \]

\[ \Lambda^6(M) = \Lambda^{6,0} \oplus \Lambda^{5,1} \oplus \Lambda^{4,2} \]

\[ \Lambda^7(M) = \Lambda^{6,1} \oplus \Lambda^{5,2} \]

\[ \Lambda^8(M) = \Lambda^{6,2} \]

Under the exterior derivative these summands are mapped as

\[ d : \Lambda^{(a,k)}(M) \to \Lambda^{(a+1,k)} \oplus \Lambda^{(a,k+1)} \]

We can refine this decomposition by splitting the components for each fiber as

\[ \Lambda^p(M) = \bigoplus_{a+b+k=p} \Lambda^{(a,b,k)}(M), \]

where \( a \) and \( b \) range from 1 to 3 and \( k = 1, 2 \), i.e.,

\[ \Lambda^1(M) = \Lambda^{1,0,0} \oplus \Lambda^{0,1,0} \oplus \Lambda^{0,0,1} \]
\[ \Lambda^2(M) = \Lambda^{2,0,0} \oplus \Lambda^{1,1,0} \oplus \Lambda^{0,2,0} \oplus \Lambda^{1,0,1} \oplus \Lambda^{0,1,1} \oplus \Lambda^{0,0,2}, \]
\[ \Lambda^3(M) = \Lambda^{3,0,0} \oplus \Lambda^{2,1,0} \oplus \Lambda^{1,2,0} \oplus \Lambda^{0,3,0} \oplus \Lambda^{2,0,1} \oplus \Lambda^{1,1,1} \oplus \Lambda^{0,2,1} \oplus \Lambda^{1,0,2} \oplus \Lambda^{0,1,2}, \]
\[ \Lambda^4(M) = \Lambda^{3,1,0} \oplus \Lambda^{2,2,0} \oplus \Lambda^{1,3,0} \oplus \Lambda^{0,4,0} \oplus \Lambda^{3,0,1} \oplus \Lambda^{2,1,1} \oplus \Lambda^{1,2,1} \oplus \Lambda^{0,3,1} \oplus \Lambda^{2,0,2} \oplus \Lambda^{1,1,2} \oplus \Lambda^{0,2,2}, \]
\[ \Lambda^5(M) = \Lambda^{3,2,0} \oplus \Lambda^{2,3,0} \oplus \Lambda^{3,1,1} \oplus \Lambda^{2,2,1} \oplus \Lambda^{1,3,1} \oplus \Lambda^{0,4,1} \oplus \Lambda^{3,0,2} \oplus \Lambda^{2,1,2} \oplus \Lambda^{1,2,2} \oplus \Lambda^{0,3,2}, \]
\[ \Lambda^6(M) = \Lambda^{3,3,0} \oplus \Lambda^{3,2,1} \oplus \Lambda^{2,3,1} \oplus \Lambda^{1,3,2} \oplus \Lambda^{0,3,3} \oplus \Lambda^{2,2,2} \oplus \Lambda^{1,2,2} \oplus \Lambda^{0,3,2}, \]
\[ \Lambda^7(M) = \Lambda^{3,3,1} \oplus \Lambda^{3,2,2} \oplus \Lambda^{2,3,2}, \]
\[ \Lambda^8(M) = \Lambda^{3,3,2}. \]

The effect of the exterior derivative is given by

\[ (4.19) \quad d : \Lambda^{(a,b,k)}(M) \longrightarrow \Lambda^{(a+1,b,k)} \oplus \Lambda^{(a,b+1,k)} \oplus \Lambda^{(a,b,k+1)}. \]

The explicit example of \((3 + 3 + 2)\) warped-like product with Spin(7) manifold and \((3 + 3 + 1)\) warped-like product with \(G_2\) manifold will be presented in the next section (see details in [BU1, Ug2, Ug1]).

5. Examples of warped-like product manifolds with \(G_2\) and Spin(7) holonomy

In this section, we present some special examples of warped-like product manifolds with \(G_2\) and Spin(7) holonomy group.

5.1. \((3+3+2)\) warped-like product manifold with Spin(7) holonomy [BU1]. We recall that the Yasui-Ootsuka solution [YO] on

\[ M = S^3 \times S^3 \times \mathbb{R}^2 \]

is given by the following (global) orthonormal frame

\[ e^i = \frac{1}{2} b^4 \text{sech}(y) \theta^i, \quad i = 1, 2, 3 \]
\[ e^i = \frac{1}{2} ab^{-\frac{3}{4}} \left(1 - \tanh(y)\right) \theta^i + ab^{-\frac{1}{4}} \theta^i, \quad i = 1, 2, 3 \]
\[ e^7 = ab^2 dx, \]
\[ e^8 = b^3 \text{sech}(y) dy, \]

where the local sections of the cotangent bundle of each \(S^3\) respectively by \(\theta^i, \theta^i\) and the functions \(a(x), b(x)\) satisfy the differential equations

\[ \frac{da}{dx} = \frac{1}{2} \left( \frac{a^3}{b} - ab \right), \quad \frac{db}{dx} = -2a^2. \]

Thus the metric is

\[ g = \pi_B^{s_{3\times3}^3} + \frac{1}{4} \left[ \sqrt{a^3 b} dx^2 + \sqrt{b^3 \text{sech}^2(y)} dy^2 \right] \]
\[ + \frac{3}{b} \sum_{i=1}^3 \theta^i \sum_{i=1}^3 \theta^i \]
\[ + \sqrt{a^4 b} \left[1 - \tanh(y)\right] \sum_{i=1}^3 \theta^i \theta^i, \]

where \(\pi_B : S^3 \times S^3 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2\) and \(\pi_1 : S^3 \times S^3 \times \mathbb{R}^2 \longrightarrow S^3\) are the natural projections on \(\mathbb{R}^2\) and \(S^3\) respectively.
Defining the functions
\[
\begin{align*}
    f_1 &= \frac{1}{4} \left( \sqrt{b^3 \text{sech}^2(y)} + \sqrt{\frac{a^4}{b} (1 - \tanh(y))^2} \right), \\
    f_2 &= \sqrt{\frac{a^4}{b}}, \\
    h &= \sqrt{\frac{a^4}{b} [1 - \tanh(y)]}
\end{align*}
\]
(5.5)
and the 2-form \( \omega \)
\[
\omega = \sum_{i=1}^{3} \theta^i \delta^i,
\]
(5.6)
we can write \( g \) as
\[
g = \pi_B^* g_B + \sum_{i=1}^{2} (f_i \circ \pi_B) \pi_i^* g_{F_i} + h \omega,
\]
(5.7)
where \( \pi_B : S^3 \times S^3 \times R^2 \longrightarrow R^2 \) and \( \pi_i : S^3 \times S^3 \times R^2 \longrightarrow S^3 \) are the natural projections on \( R^2 \) and \( S^3 \) respectively. The matrix of \( g \) with respect to the (global) frame
\[
\{ \theta^1, \theta^2, \theta^3, \hat{\theta}^1, \hat{\theta}^2, \hat{\theta}^3, e^7, e^8 \}
\]
is
\[
g = \begin{pmatrix}
    f_1 I_3 & \frac{b}{2} I_3 & 0 \\
    \frac{b}{2} I_3 & f_2 I_3 & 0 \\
    0 & 0 & I_2
\end{pmatrix}
\]
(5.9)
where \( I_3 \) and \( I_2 \) are identity matrices of size 3 and 2, and zeroes denote zero matrices of appropriate sizes. Note that if \( h \) were zero, the metric given in the equation (5.7) would be a multiply warped product with a block diagonal matrix with respect to an appropriate frame \([FS]\).

We present a parallel \( Spin(7) \) manifold as a special warped-like product structure on
\[
M = S^3 \times S^3 \times R^2
\]
\([YO, BU1]\). If we choose the (global) orthonormal frame as in the equation (5.14), then the coefficient (in the sense of warping) functions are \([BU1]\),
\[
\begin{align*}
    A &= \frac{1}{2} b^2 \text{sech}(y), & B &= 0, & \hat{A} &= \frac{1}{2} ab^{-\frac{1}{4}} (1 - \tanh(y)), & \hat{B} &= ab^{-\frac{9}{4}}, \\
    a_{11} &= ab^2, & a_{12} &= 0, & a_{21} &= 0, & a_{22} &= b^2 \text{sech}(y).
\end{align*}
\]
(5.11)
Hence the metric on \( M \) is given by
\[
g = \left[ \sqrt{a^4 b^3} dx^2 + \sqrt{b^3 \text{sech}^2(y)} dy^2 \right] + \frac{1}{4} \left[ \sqrt{b^3 \text{sech}^2(y)} + \sqrt{\frac{a^4}{b} (1 - \tanh(y))^2} \right] \sum_{i=1}^{3} (\theta^i)^2 \\
+ \sqrt{\frac{a^4}{b}} \sum_{i=1}^{3} (\hat{\theta}^i)^2 + \sqrt{\frac{a^4}{b} [1 - \tanh(y)]} \sum_{i=1}^{3} \theta^i \delta^i,
\]
(5.12)
where the local sections of the cotangent bundle of each \( S^3 \) respectively by \( \theta^i, \hat{\theta}^i \) and the functions \( a(x), b(x) \) satisfy the differential equations
\[
\frac{da}{dx} = \frac{1}{2} \left( \frac{a^3}{b} - ab \right), & \frac{db}{dx} = -2a^2.
\]
(5.13)

The following theorem is an important result for Yasui-Ootsu solution and \((3+3+2)\) warped-like product manifold with \( Spin(7) \) holonomy. See details in \([BU1, BU2]\).
Theorem 5.1. [BU1] Let $M$ be diffeomorphic to $F \times B$, where the base $B$ is a two-dimensional Riemannian manifold diffeomorphic to $R^2$, the fiber $F$ is a 6-manifold of the form $F = F_1 \times F_2$, and the $F_i$ $(i = 1, 2)$ are complete, connected and simply connected 3-manifolds. Let the metric on $M$ be a $(3 + 3 + 2)$ warped-like product. Then the fibers $F_i$ are isometric to $S^3$ with constant curvature $k > 0$ for the Spin(7) structure determined by the Bonan form (5). Also there exists a unique metric in the class of $(3 + 3 + 2)$ warped-like product metrics admitting the same Spin(7) structure, and the metric is written as given by Yasui-Ootsuka (5.12) up to a gauge transformation.

5.2. $(3+3+1)$ warped-like product manifold with $G_2$ holonomy [Ug1]. Inspired by the previous work [BU1], moving down one dimension, we study $(3 + 3 + 1)$ warped-like product manifolds [Ug1]. In [BU1], we studied special type of warped-like product and its properties with Spin(7) holonomy in 8-dimensional manifolds. Here the problem we are dealing with is modeled on an 7-dimensional warped-like product manifold $M = F_1 \times F_2 \times B$ where $F_1, F_2$ are 3-dimensional and $B$ is a one dimensional manifold.

Using by the paper [BU1], we will define $(3+3+1)$ warped-like product manifolds in this section as an example of $G_2$ manifold [Ug1].

Definition 5.2. Let $M = F_1 \times F_2 \times B$ be an 7-dimensional topologically product manifold where $F_1, F_2$ are 3-manifolds and $B$ is a one dimensional manifold, each equipped with Riemannian metrics. Let $\theta^i, \hat{\theta}^i$ be orthonormal sections of the cotangent bundles of $F_1$ and $F_2$ respectively and $x$ be local coordinate on $B$. If the metric on $M$ is defined by the following orthonormal frame

\begin{align}
e^i &= A(x)\theta^i + B(x)\hat{\theta}^i, \\
e^\hat{i} &= \hat{A}(x)\theta^i + \hat{B}(x)\hat{\theta}^i, \\
e^7 &= a(x)dx,
\end{align}

(5.14)

then we call $(M, e^i) \ i = 1, 2, ..., 7$ a “$(3+3+1)$ warped-like product” manifold.

5.2.1. An example of $(3+3+1)$ warped-like manifold with $G_2$ holonomy. Recall that the Konishi-Naka solution [Nak] on

(5.15) $M = SU(2) \times SU(2) \times R$

is given by the following (global) orthonormal frame

\begin{align}e^i &= A(x)\theta^i, \quad i = 1, 2, 3 \\
e^\hat{i} &= \hat{A}\left(\theta^i - \frac{1}{2}\theta^\hat{3}\right), \quad i = 1, 2, 3 \\
e^7 &= dx,
\end{align}

(5.16)

where the local sections of the cotangent bundle of each $SU(2)$ respectively by $\theta^i, \hat{\theta}^i$ and the functions $A(x), \hat{A}$ satisfy the differential equations

\begin{align}\frac{dA}{dx} &= \frac{\hat{A}}{2A}, \quad \frac{d\hat{A}}{dx} = 1 - \frac{\hat{A}^2}{4A^2}.
\end{align}

(5.17)

Thus the metric is

\begin{align}g &= A(x)^2 \sum_{i=1}^{3}(\theta^i)^2 + \hat{A}(x)^2 \left(\sum_{i=1}^{3}\left[\theta^i - \frac{1}{2}\theta^\hat{3}\right]\right)^2 + dx^2.
\end{align}

(5.18)

We can take $e^7 = dx$, as in [Nak]. It is seen that this metric is an example of $(3+3+1)$ warped-like product manifold with $G_2$ holonomy. See details in [Ug1], [Nak].

The following result is important for Konishi-Naka solution and $(3+3+1)$ warped-like product manifold with $G_2$ holonomy.
Theorem 5.3. Let $M$ be diffeomorphic to $F \times B$, where the base $B$ is a one dimensional Riemannian manifold diffeomorphic to $R$, the fibre $F$ is a 6-manifold of the form $F = F_1 \times F_2$, and $F_i$ $(i = 1, 2)$ are complete, connected and simply connected 3-manifolds. Let the metric on $M$ be a $(3+3+1)$ warped-like product metric. If $M$ is the manifold with the $G_2$ holonomy or with the weak $G_2$ holonomy determined by the fundamental 3-form (1), then the fibers $F_i$'s are isometric to $S^3$ with constant curvature $k > 0$. Also there exists a unique metric in the class of special warped-like product metrics admitting the $G_2$ holonomy, and the metric is written as given by Konishi-Naka (5.18) up to gauge transformation.

In the next sections, we present some other explicit examples of special warped-like metric constructions.

5.3. Special warped-like product manifold with $G_2$ holonomy [Bra]. The general metric ansatz compatible with $G_2$ holonomy is given [Bra] by

$$ds^2 = \sum_{a=1}^{7} e^a \otimes e^a$$

with the following

$$e^1 = A(r)(\sigma^1 - \Sigma^1), \quad e^2 = A(r)(\sigma^2 - \Sigma^2),$$
$$e^3 = D(r)(\sigma^3 - \Sigma^3), \quad e^4 = B(r)(\sigma^1 + \Sigma^1),$$
$$e^5 = B(r)(\sigma^2 + \Sigma^2), \quad e^6 = C(r)(\sigma^3 + \Sigma^3),$$
$$e^7 = \frac{dr}{C(r)}.$$ 

where $\sigma^i$ and $\Sigma^i (i = 1, 2, 3)$ are two sets of left invariant $SU(2)$ one-forms (see details in [Bra]) and their metric have made a particular choice for the radial coordinate. Their ansatz depends on four functions. Note that this special warped-like metric is given as [Bra] be

$$g_{\text{special}} = \begin{pmatrix}
A\sigma^1 & 0 & 0 & -A\Sigma^1 & 0 & 0 & 0 \\
0 & A\sigma^2 & 0 & 0 & -A\Sigma^2 & 0 & 0 \\
0 & 0 & D\sigma^3 & 0 & 0 & -D\Sigma^3 & 0 \\
B\sigma^1 & 0 & 0 & B\Sigma^1 & 0 & 0 & 0 \\
0 & B\sigma^2 & 0 & 0 & B\Sigma^2 & 0 & 0 \\
0 & 0 & C\sigma^3 & 0 & 0 & C\Sigma^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & dr
\end{pmatrix}$$

where $\{\theta^1, \theta^2, \theta^3, \bar{\theta}^\dagger, \bar{\theta}^\dagger, dr\}$ are (global) frame on 7-dimensional manifold. To prove that the above metric has $G_2$ holonomy, we have to impose that the associative three-form $\varphi$ is closed and co-closed (i.e. $d\varphi = d^* \varphi = 0$). These conditions imposed on the associative three-form $\varphi$ constructed from metric structure leads to the following system of first order differential equations

$$\frac{dA}{dr} = \frac{1}{4} \left( \frac{B^2 - A^2 + D^2}{BCD} + \frac{1}{A} \right),$$
$$\frac{dB}{dr} = \frac{1}{4} \left( \frac{A^2 - B^2 + D^2}{ACD} - \frac{1}{B} \right),$$
$$\frac{dC}{dr} = \frac{1}{4} \left( \frac{C}{B^2} - \frac{C}{A^2} \right),$$
$$\frac{dD}{dr} = \frac{1}{2} \left( \frac{A^2 + B^2 - D^2}{ABC} \right)$$

Special solutions is presented in their paper [Bra]. This metric structure present an example of special warped-like product manifold with $G_2$ holonomy.
5.4. Another special warped-like product manifold with $G_2$ holonomy [Cv1]. In this section we present $G_2$ metrics for the six-function $a_i, b_i$ ($i = 1, 2, 3$) on $S^3 \times S^3$ manifold. A rather general ansatz involving more functions was considered in Bra, Cv2, and first-order equations for $G_2$ holonomy were derived. The metric for the six-function $G_2$ space is given by

$$ds^2 = dt^2 + a_i^2(\sigma^i - \Sigma^i) + b_i^2(\sigma^i + \Sigma^i)$$

where $\sigma^i$ and $\Sigma^i$ are left-invariant 1-forms for two $SU(2)$ group manifolds. It is seen that this special warped-like metric is given as be

$$g_{\text{special}} = \begin{pmatrix}
    a_1 \sigma^1 & 0 & 0 & -a_1 \Sigma^1 & 0 & 0 & 0 \\
    0 & a_2 \sigma^2 & 0 & 0 & -a_2 \Sigma^2 & 0 & 0 \\
    0 & 0 & a_3 \sigma^3 & 0 & 0 & -a_3 \Sigma^3 & 0 \\
    b_1 \sigma^1 & 0 & 0 & -b_1 \Sigma^1 & 0 & 0 & 0 \\
    0 & b_2 \sigma^2 & 0 & 0 & -b_2 \Sigma^2 & 0 & 0 \\
    0 & 0 & b_3 \sigma^3 & 0 & 0 & -b_3 \Sigma^3 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & dt
\end{pmatrix}$$

where $\{\theta^1, \theta^2, \theta^3, \theta^1, \theta^2, \theta^3, dt\}$ are (global) basis on 7-dimensional manifold. It was found that for $G_2$ holonomy, $a_i$ and $b_i$ must satisfy the first-order equations

$$\begin{align*}
d a_1 &= \frac{a_1^2}{4a_3b_2} + \frac{a_1^2}{4a_2b_3} - \frac{a_2}{4b_3} - \frac{a_3}{4b_2} - \frac{b_2}{4a_3} - \frac{b_3}{4a_2} \\
da_2 &= \frac{a_1^2}{4a_3b_2} + \frac{a_1^2}{4a_2b_3} - \frac{a_2}{4b_3} - \frac{a_3}{4b_2} - \frac{b_2}{4a_3} - \frac{b_3}{4a_2} \\
da_3 &= \frac{a_1^2}{4a_3b_2} + \frac{a_1^2}{4a_2b_3} - \frac{a_2}{4b_3} - \frac{a_3}{4b_2} - \frac{b_2}{4a_3} - \frac{b_3}{4a_2} \\
d b_1 &= \frac{a_1^2}{4a_3b_2} + \frac{a_1^2}{4a_2b_3} - \frac{a_2}{4b_3} - \frac{a_3}{4b_2} - \frac{b_2}{4a_3} - \frac{b_3}{4a_2} \\
d b_2 &= \frac{a_1^2}{4a_3b_2} + \frac{a_1^2}{4a_2b_3} - \frac{a_2}{4b_3} - \frac{a_3}{4b_2} - \frac{b_2}{4a_3} - \frac{b_3}{4a_2} \\
d b_3 &= \frac{a_1^2}{4a_3b_2} + \frac{a_1^2}{4a_2b_3} - \frac{a_2}{4b_3} - \frac{a_3}{4b_2} - \frac{b_2}{4a_3} - \frac{b_3}{4a_2}.
\end{align*}$$

Under some special conditions and numerical analysis on this system given above, a special solution of these equations is presented in Cv1.

Remark 5.4. One can consider the relation cone metric structure with special warped-like product manifold with $G_2$ (or $Spin(7)$) holonomy. Consider the cone $M = F \times R$ with the metric

$$ds^2 = dt^2 + A_i^2(\sigma^i + \Sigma^i) + B_i^2(\sigma^i - \Sigma^i)$$

where $\sigma^i$ and $\Sigma^i$ are the standard coframe of 1-forms, $A_i(t)$ and $B_i(t)$ are positive functions defining the deformation of the standard cone metric on $M$ (see details in Baz).

Complete Riemannian metrics with holonomy group $G_2$ on manifolds obtained by deformation of cones over $S^3 \times S^3$ are constructed in Baz. Their idea of the paper is to consider the standard conic metric over a Riemannian manifold with special geometry. After then any deformation of this metric depends on a number of functional parameters using which the $G_2$ (or $Spin(7)$) structure can be explicitly specified. In Baz, they propose (regarding to Bra, Cvi) to consider the fiber space $F = S^3 \times S^3$. Then the cone metric can be written as in equation (5.25). A system of differential equations guaranteeing that the metric $ds^2$ has a holonomy group contained in $G_2$ was written out in Bra. A particular solution of this system corresponding to the metric with holonomy group $G_2$ on $S^3 \times R^4$ was also obtained in Bra. Note that more general metrics on cones over $S^3 \times S^3$ were also studied in Bra, Cvi (see also refs. in Baz). In the paper Baz, they continue the study of this class of metrics by setting the particular
case $A_2 = A_3$ and $B_2 = B_3$ and considering a boundary condition different from the one in [Bra], which leads to spaces with other topological structures.

**Remark 5.5.** It is also noted that the metrics of Bryant and Salamon [RS] are asymptotically conical. In other words, outside of sufficiently large compact sets, the metric is arbitrarily close to a conical metric $g_t = dt^2 + t^2 g$ defined on $M = S^3 \times S^3 \times \mathbb{R}^+$. The closeness is measured with respect to the conical metric $C_{tn}$. A consequence of $(M, g_t)$ having $G_2$ holonomy is that $(S^3 \times S^3, g)$ must be nearly Kahler (see details in [RS, Cla]). In each given case they were able to solve a system of ordinary differential equations so as to obtain a metric $g_t$ of exceptional holonomy.

6. **Conclusions**

In this paper we define warped-like product metrics as a generalization of multiply-warped products and present examples of special type of these metrics for $G_2$ and $Spin(7)$ holonomies (i.e. exceptional holonomy cases [Bra, Cv1, Job, Sal]). We investigate a generalization of the warped product as a special warped-like form where the local coordinate expression of the metric is no longer block diagonal, but still with a certain simple structure. Since we want to have a fiber structure, we shall allow dependencies on the coordinates of the base everywhere. To obtain more explicit examples, different types of fiber-base decompositions will be worked in the next studies. The study of special holonomy manifolds (i.e. Berger’s list [Job]) and some classes of $G_2$, $Spin(7)$ (see Tables 1-2) manifolds with warped-like metric structure are open problems for future research topics. Finally we believe that our approach of the warped-like product metrics will be an important notion on the manifolds with special and exceptional holonomies for future studies, and also other types of geometries which use warped and multiply-warped product.

**References**

[Agr] Agricola, I., The Srni lectures on non-integrable geometries with torsion, Arch. Math. 42 (2006), 5-84. math.DG/0606705

[Ale] Alekseevskii, D., Riemannian spaces with unusual holonomy groups, Funct. Anal. Appl., 2 (1968), 97-105.

[Baz] Bazaikin, Y.V., Bogoyavlenskaya, O.A., complete Riemannian metrics with holonomy group $G_2$ on deformations of cones over $S^3 \times S^3$, Mathematical Notes, 93 (2013), 643-653.

[Ber] Berger, M., Sur les groupes d’holonomie des variétés à connexion affine et des variétés Riemanniennes, Bull. Soc. Math. France, 83 (1955), 279-330.

[BU2] Bilge, A. H. and Uğuz, S., A Generalization of Warped Product Manifolds with Spin(7) Holonomy, Geometry And Physics, XVI International Fall Workshop. AIP Conference Proceedings, 1023, (2008), 165-171.

[Bra] Brandhuber, A., Gomis, J., Gubser, S. S., Gukov, S., Gauge theory at large $N$ and new $G_2$ holonomy metrics, Nuclear Physics B, 611 (2001), 179-204.

[Bry] Bryant, R.L., Metrics with exceptional holonomy, Ann. of Math., 126 (1987), 525-576.

[RS] Bryant, R.L. and Salamon, S.M., On the construction of some complete metrics with exceptional holonomy, Duke Math. J., 58, no. 3 (1989), 829 - 850.

[Bon] Bonan, E., Sur les variétés Riemanniennes à groupe d’holonomie $G_2$ ou $Spin(7)$, C.R. Acad. Sci. Paris, 262 (1966), 127-129.

[Cab] Cabrera, F., On Riemannian manifolds with Spin(7) structure, Publ. Math. Debrecen 46 (3-4) (1995), 271-283.

[Cab] Cabrera, F., Monar, M., Swann, A., Classification of $G_2$-structures, J. London Math. Soc. 53 (1996), 407-416.

[Che] Cheeger, J., Colding, T.H., Lower bounds on Ricci curvature and the almost rigidity of warped products, Ann. Math. 144 (1996), 189-237.

[Cla] Clarke, A., Instantons on the exceptional holonomy manifolds of Bryant and Salamon, Journal of Geometry and Physics, 82 (2014), 84-97.

[Cv1] Cvetic, M., Gibbons,G.W., Lu, H., Pope, C.N., Cohomogeneity one manifolds of $Spin(7)$ and $G_2$ holonomy, Phys. Rev. D, 65 (2002), 106004.

[Cv2] Cvetic, M., Gibbons,G.W., Lu, H., Pope, C.N., Supersymmetric M3-branes and G2 manifolds, Nuclear Physics B, 620 (2002), 3-28.

[Fernandez, M., A classification of Riemannian manifolds with structure group $Spin(7)$, Ann. Mat. Pura Appl. 143 (1986), 101-122.

[Fernandez, M., Riemannian manifolds with structure group $G_2$, Ann. Mat. Pura Appl. 132 (1982), 19-45.

[FS] Flores, J.L., and Sanchez, M., Geodesic connectedness of multiwarped spacetimes, J. Diff. Eqn., 186, (2002), 1-30.

[FI1] Friedrich T., Ivanov, S.: Parallel spinors and connections with skew-symmetric torsion in string theory, Asian J. Math., 6, no. 2, 303-335, (2002).

[FI2] Friedrich, T., Ivanov, S.: Killing spinor equations in dimension 7 and geometry of integrable $G_2$ manifolds, J. Geom. Phys. 48, 1-11, (2003).

[TF] Friedrich, T., Kath, I., Moroianu, A., Semmelmann, U., On nearly parallel $G_2$-structures, J. Geom. Phys. 23 (1997), 259-286.

[GPP] Gibbons, G. W., Page, D. N. and Pope, C. N., Einstein metrics on $S^3$, $R^3$ and $R^4$ bundles, Commun. Math. Phys. 127 (1990) 529-553.
[Gib] Gibbons, G.W., Lü, H., Pope, C.N., Stelle, K.S.: Supersymmetric domain walls from metrics of special holonomy, Nuclear Phys. B, 623, no. 1-2, 3-46, (2002).

[Gra] Gray, A. and Brown, R. B., Riemannian Manifolds with Holonomy Group Spin(9), Diff. Geometry in honor of K. Yano, Kinokuniya, Tokyo, (1972), 41-59.

[Hem] Hempel, J., 3-manifolds, Princeton Univ. Press, (1976).

[Iva1] Ivanov, S., Connections with torsion, parallel spinors and geometry of $Spin(7)$ manifolds, Mathematical Research Letters 11, (2004) 171–186.

[Iva2] Ivanov, S., M. Parton and P. Piccinni, Locally conformal parallel $G_2$ and $Spin(7)$ manifolds, Mathematical Research Letters 13, (2006) 167-177. math.DG/0509038.

[Iva3] Ivanov, S., F. Martin Cabrera, $SU(3)$-structures on submanifolds of a $Spin(7)$-manifold, Diff. Geom. Appl. 26 (2008), 113-132. math.DG/0510406.

[Joy] Joyce, D., Compact 8-manifolds with holonomy $Spin(7)$, Inventiones mathematicae, 123, (1996), 507-552.

[Job] Joyce, D., Compact manifolds with special holonomy, Oxford University Press, Oxford, 2000.

[KN] Kobayashi, S., Nomizu, K., Foundations of Differential Geometry Vol.I, Interscience, 1969.

[Nak] Konishi Y. and Naka M., Coset construction of $Spin(7)$, $G_2$ gravitational instantons, Class. Quantum Grav. 18 (2001) 5521-5544.

[One] O’Neill, B., Semi Riemannian Geometry, Academic Press Inc., London, 1983.

[Sal] Salamon, S.M., Riemannian geometry and holonomy groups, Pitman Research Notes Math., Longman-Oxford, 1989.

[Sem] Salur, S., and Santillan, O., New $Spin(7)$ holonomy metrics admitting $G_2$ holonomy reductions and M-theory/type-IIA dualities, Phys. Rev. D 79 (2009), 086009.

[SW] Schwachhöfer, L.J., Riemannian, symplectic and weak holonomy, Annals of Global Analysis and Geometry, 18 (2000), 291-308.

[BU1] Uğuz, S. and Bilge, A. H., $(3 + 3 + 2)$ warped-like product manifolds with $Spin(7)$ holonomy, J. Geom. Phys., 61, (2011), 1093-1103.

[Ug1] Uğuz, S., Special warped-like product manifolds with (weak) $G_2$ holonomy, Ukrainian Mathematical Journal, 65 (8), (2014), 1257-1272.

[Ug2] Uğuz, S., Lee form and special warped-like product manifolds with locally conformally parallel $Spin(7)$ structure, Annals of Global Analysis and Geometry, Volume 43, Issue 2, 123-141, (2013).

[Ug3] Uğuz, S. Conformally parallel $Spin(7)$ structures on solvmanifolds, Turkish Journal of Mathematics, 38 (1), (2014), 166-178.

[War] Warner, F.W, Foundations of Differentiable Manifolds and Lie Groups. Scott and Foresman, 1971.

[YO] Yasui, Y. and Ootsuka, T., $Spin(7)$ holonomy manifold and superconnection, Class. Quantum Grav., 18 (2001), 807-816.

[Yau] Yau, S.T. Nadis, S., The Shape of Inner Space: String Theory and the Geometry of the Universe’s Hidden Dimensions, Basic Books; First Trade Paper Edition, March 6, 2012.