The $q$-Diode

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Abstract

The present work introduces the new function $R_{q, Q}(z)$, solution of the equation $R_{q, Q}(z) \times Q \exp_{q}(R_{q, Q}(z)) = z$. It is shown this new function can be used to construct a new disentropy as well it is used to model the $q$-diode, a hypothetical electronic device whose electrical current depends $q$-exponentially on the voltage between its terminals.

Key words – Lambert-Tsallis $W_q$ function; $q$-exponential; disentropy; diode

1. Introduction

The Lambert $W$ function is an important elementary mathematical function that finds applications in different areas of mathematics, computer Science and physics [1-6]. The Lambert $W$ function is defined as the solution of the equation

$$W(z)e^{W(z)} = z. \quad (1)$$

In the interval $-1/e \leq x \leq 0$ there exist two real values of $W(z)$. The branch for which $W(x) \geq -1$ is the principal branch named $W_0(z)$ while the branch satisfying $W(z) \leq -1$ is named $W_{-1}(z)$. For $x \geq 0$ only $W_0(z)$ is real and for $x < -1/e$ there are not real solutions. The point $(z_b = -1/e, W(z_b) = -1)$ is the branch point where the solutions $W_0$ and $W_{-1}$ have the same value.

On the other hand, the $q$-exponential function proposed by Tsallis [7] is given by

$$e_{q}^{z} = \begin{cases} e^{z} & q = 1 \\ \left[1+(1-q)z\right]^{1/(1-q)} & q \neq 1 \ & 1+(1-q)z \geq 0 \\ 0 & q \neq 1 \ & 1+(1-q)z < 0 \end{cases} \quad (2)$$
Using Tsallis $q$-exponential (2) in the Lambert equation (1), one has the Lambert-Tsallis equation

$$W_q(z)e_q^{W_q(z)} = z$$

(3)

whose solutions are the Lambert-Tsallis $W_q$ functions [8]. Using the definition of $exp_q$ given in eq. (2) in eq. (3), the $W_q$ function can be found solving the equation [8]

$$x(r+x)_r = r'z,$$

(4)

where $x = W_{(r-1)}(z)$, $r = 1/(1-q)$ and $(x)_r = \text{max}\{x,0\}$. When $q = 1$, one has $e_1(z) = e^z$

and, consequently, $W_1(z) = W(z)$. For example, for $q = \{2, 3, 3/2, 1/2\}$ one has the following Lambert-Tsallis $W_q$ upper branches

$$W_2(z) = \frac{z}{z+1}, \quad z > -1,$$

(5)

$$W_3(z) = z\sqrt{z^2 + 1 - z^2} \quad (z \geq 0).$$

(6)

$$W_{3/2}^+(z) = \frac{2(z+1) + 2\sqrt{2z+1}}{z}, \quad z > -1/2,$$

(7)

$$W_{3/2}^-(z) = \left[ \frac{3\sqrt{2z + \sqrt{\left(2z + \frac{8}{27}\right)^2 - \frac{64}{729} + \frac{8}{27} - 2}}}{9\sqrt{2z + \sqrt{\left(2z + \frac{8}{27}\right)^2 - \frac{64}{729} + \frac{8}{27}}} - \frac{64}{729} + \frac{8}{27} - 2} \right]^{1/2}, \quad z \geq -0.29629,$$

(8)

Figure 1 shows the plot of $W_{q=3/2}$ versus $z$. 
Fig. 1. $W_{q=3/2}$ versus $z$.

More details about the Lambert-Tsallis function and its applications can be found in [8-15].

In order to handle with the $exp_q$ function, one has to use the $q$-operations. The important ones used in this work are:

\[
\begin{align*}
\alpha \times_q b &= \max \left\{ \left[ a^{(1-q)} + b^{(1-q)} - 1 \right]^{1/(1-q)} , 0 \right\} = \left[ a^{(1-q)} + b^{(1-q)} - 1 \right]^{1/(1-q)} \\
(e_q^x)^\alpha &= e^{\alpha x_{1-(1-q)/q}} - (10)
\end{align*}
\]

2. The $R_{q,Q}$ function

In this section a new function is introduced. It is named $R_{q,Q}$ function and it is the solution of the following equation

\[
R_{q,Q}(z) \times_Q e_q^{R_{q,Q}(z)} = z. \tag{11}
\]

Equation (11) is the Lambert-Tsallis equation using the $q$-product operation. Obviously, $R_{q,Q=1}(z) = W_q(z)$. Using (2) and (9) in (11) one gets
\[ R^{i,q}_q(z) + \left[ 1 + (1-q)R_{q,q}(z) \right] \frac{z^{i-q}}{i-q} - (z^{i-q} + 1) = 0. \] (12)

The general solutions of (12) will be published elsewhere. Here, the important case for introduction of a new disentropy and the \( q \)-diode modelling is \( Q = q \). In this case eq. (12) is reduced to

\[ R^{i,q}_q(z) + (1-q)R_{q,q}(z) - z^{i-q} = 0. \] (13)

For example, for \( q = 2 \) and \( q = 1/2 \) one has

\[ R_{2,2}(z) = -\frac{1}{2z} \pm \frac{1}{2} \sqrt{\frac{1}{z^2} + 4}. \] (14)

\[ R_{q/2,1/2}(z) = 2 \left( z^{1/2} + 1 \right) - 2\sqrt{2z^{1/2} + 1}. \] (15)

Figure 2 shows the plot of the parts of the functions \( R_{2,2} \) and \( R_{1/2,1/2} \) that obey eq. (11).

\[ \text{Fig. 2. } R_{q,q}(z) \text{ versus } z \text{ for } q = 1/2 \text{ and } q = 2. \]
3. Disentropy

The disentropy based on the Lambert and Lambert-Tsallis functions and its applications in quantum and classical information theory, image processing and black hole, among others, have been discussed in [8-14]. Taking the $\log_q$ in both sides of eq. (11) with $q = Q$, one gets

$$\log_q(z) = R_{q,q}(z) + \log_q\left[R_{q,q}(z)\right].$$

(16)

Hence, Tsallis $q$-entropy can be written as

$$S_q = \sum_i p_i^q \log_q(p_i) = \sum_i p_i^q R_{q,q}(p_i) + \sum_i p_i^q \log_q\left[R_{q,q}(p_i)\right].$$

(17)

The term

$$D_q = \sum_i p_i^q R_{q,q}(p_i)$$

(18)

is a disentropy. It can be shown it is maximal for delta distribution and minimal for a uniform distribution. Its quantum version is

$$D_q(\rho) = \sum_i \lambda_i^q R_{q,q}(\lambda_i)$$

(19)

where $\lambda_i$ is the $i$-th eigenvalue of the density matrix $\rho$. The disentropy based on the $R_{q,q}$ function can be used in the same problems that the disentropy based on the Lambert-Tsallis function is used. For example, it can be used to measure the disentanglement of bipartite of qubit states [8]. Figure 3 shows the behaviour of $D_q$ for the distribution $\{p, 1-p\}$ using the values $q = 0.5$, $q = 1$ and $q = 2$. 
4. The $q$-Diode

For a semiconductor diode that obeys the Schottky’s model, the relation between current and voltage is given by

$$I = I_s e^{\frac{V_D}{\eta kT}},$$

(20)

where $I_s$ is the saturation current of the diode, $V_D$ is the voltage between the diode terminals, $V_T = kT/q_e$ ($q_e$ – electron charge, $k$ – Boltzman constant, $T$ - temperature in Kelvin) and, finally, $\eta$ is the diode ideality factor ($1 < \eta < 2$ for silicon diodes). Figure 4 shows the very basic electrical circuit composed by a power supply, a resistor and the diode.
The current that flows through the diode in the circuit shown in Fig. 3 is given by

$$I = I_s e^{\frac{V - RI}{\eta V_T}}. \tag{21}$$

Using the Lambert $W$ function in (21) one gets the following relation between electrical current ($I$) and power supply voltage ($V$)

$$I = \frac{\eta V_T}{R} W \left( \frac{I_s R}{\eta V_T} e^{\frac{V}{\eta V_T}} \right). \tag{22}$$

The $q$-diode, by its turn, is defined as the hypothetical device whose relation between current and voltage between its terminals ($V_D$) is given by

$$I = I_s e^{\frac{V_D}{q \eta V_T}}. \tag{23}$$

Using the $q$-diode in the circuit shown in Fig. 4, the value of the electric current flowing through the diode is given by
Using the $q$-operations in (20) one gets

\[
I = I_s e^{\frac{V-IR}{\eta T}}. \tag{24}
\]

Now, using the function $R_{q,q}$ in (25), after some algebra one gets the following solutions for the electrical current $I$, for $q = 2$ and $q = 0.5$,

\[
I_2(V) = \frac{\eta V_T}{R} \left[ -1 + \frac{1}{2} \sqrt{\frac{1}{2e_q^\frac{V}{\eta T}}} + \frac{4\eta V_T}{I_s R} \right] \tag{26}
\]

\[
I_{\frac{1}{2}}(z) = \frac{\eta V_T}{R} \left[ 2 \left( e_q^\frac{V}{\eta T} \right)^{\frac{1}{2}} + \frac{\eta V_T}{I_s R} \right] - 2 \sqrt{\frac{2\eta V_T}{I_s R} \left( e_q^\frac{V}{\eta T} \right)^{\frac{1}{2}} + \left( \frac{\eta V_T}{I_s R} \right)^2} \tag{27}
\]

One may note that (26) and (27) are, respectively, equal to (14) and (15) when $(\eta V_T/I_s R) = 1$. In Fig. 5 one can see the comparison between the cases $q = 1$, $q = 0.75$ and $q = \frac{1}{2}$. The smaller the value of $q$ the slower is the growth of the current. The $q$-diode with $q > 1$ operates at very low voltage since $\exp_q(x)$ goes too fast to zero. For example, for $q = 1.25$, one must have $V/(\eta V_T) < 4$ ($V < ~0.1 \text{mV}$).
Fig. 5 – $q$-Diode current versus voltage curve for $q \in [0.5, 0.75, 1]$.

5. Conclusions

Initially, the present work introduced the solutions of the equation $R_{q,q}(z) \times Q \exp_q(R_{q,q}(z)) = z$ and showed two applications of the function $R_{q,q}(z)$: 1) It was used to construct a new disentropy formula. This new disentropy can be applied in a large variety of problems in physics and engineering. A comparison between the disentropy based on the $R_{q,q}$ function and the disentropy based on the Lambert-Tsallis $W_q$ function is a question for future investigation. 2) It was used to model the $q$-diode. Basically, compared to the classical diode, the $q$-diode with $q > 1$ has to operate with lower voltage while the $q$-diode with $q < 1$ requires a larger voltage. Since, the $q$-diode shows the nonlinear behaviour (between $I$ and $V$) it can be used in an electronic circuit as modulator or mixer, for example. Which values of $q$ will result in a $q$-diode that can be realized physically is still a problem to be investigated.

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