The zero energy quantum information processing

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Abstract: In contradistinction with some plausible statements of the information theory, we point out the possibility of the zero energy quantum information processing. Particularly, we investigate the rate of the entanglement formation in the operation of the quantum ”oracles” employing ”quantum parallelism”, and we obtain that the relative maximum of the rate of the operation distinguishes the zero average energy of interaction in the composite system ”input register + output register”. This result is reducible to neither of the previously obtained bounds, and therefore represents a new bound for the nonorthogonal state transformations in the quantum information processing.

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1. Introduction

In the realm of computation, one of the central questions is ”what limits the laws of physics place on the power of computers?” [1]. Physically, this question refers to the minimum time needed for execution of the logical operations, i.e. to the maximum rate of transformation of state of a physical system implementing the operation. From the fundamental point of view, this question tackles the yet-to-be-understood relation between the energy (of a system implementing the computation) on the one, and the concept of information, on the other side. Eventually, answering this question might shed new light on (e.g., might sharpen) the standard ”paradoxes” of the quantum world [2].

Of special interest are the rates of the reversible operations (i.e. of the reversible quantum state transformations). To this end, the two bounds for the so-called ”orthogonal transformations (OT)” are known; by OT we mean a transformation of a (initial) state $|\Psi_i\rangle$ to a (final) state $|\Psi_f\rangle$, while $\langle\Psi_i|\Psi_f\rangle = 0$. First, the minimum time needed for OT can be characterized in terms of the spread in energy, $\Delta\hat{H}$, of the system implementing the transformation [3-7]. However, recently, Margolus and Levitin [8, 9] have extended this result to show that a quantum system with average energy $\langle\hat{H}\rangle$ takes time at least $\tau = h/4\langle\hat{H}\rangle$ to evolve to an orthogonal state. In a sense, the second bound is more restrictive: a system with zero average energy cannot perform a computation ever. This however stems nothing about the nonorthogonal evolution which is still of interest in quantum computation.

Actually, most of the efficient quantum algorithms [10-12] employ the so-called quantum ”oracles” (quantum ”black boxes”) employing the ”quantum parallelism” [13, 14]. These do not require orthogonality of the initial and the final states of the composite quantum system ”input register + output register $(I+O)$”. Rather, orthogonality of the final states of the subsystems’ (e.g. $O$’s) states is required, thus emphasizing a need for a new bound for the operation considered.
In this paper we show that, in general, the relative maximum of the rate of the operation of the quantum "oracles" may point out the zero average energy of interaction in the composite system \( I + O \). More precisely: it appears that the rate of the operation cannot be characterized in terms of the average energy of the composite system as a whole. Rather, it can be characterized in terms of the average energy of interaction Hamiltonian, still pointing out the zero average energy of interaction. Physically, in a sense, it means that as lower the average energy, the higher the rate of the operation. This result is in obvious contradistinction with the result of Margolus and Levitin [8, 9]. On the other side, our result is neither reducible to the previously obtained bound characterized in terms of the spread in energy [3-7], thus providing us with a new bound in the quantum information theory. Finally, the possibility of the zero energy quantum information processing is somewhat counterintuitive result, which, we believe might sharpen the distinction between the classical and the "quantum information".

2. The quantum "oracle" operation

It is worth emphasizing: we are concerned with the bounds characterizing the rate of (or, equivalently, the minimum time needed for) the reversible transformations of a quantum system’s states. Therefore, the bounds known for the irreversible transformations are of no use here. Still, it is a plausible statement that the information processing should be faster for a system with higher (average) energy, even if—as it is the case in the reversible information processing—the system does not dissipate energy. This intuition of the classical information theory is justified by the bound obtained by Margolus and Levitin [8, 9]. However, this bound refers to OT, and does not necessarily applies to the nonorthogonal evolutions.

The typical nonorthogonal transformations in the quantum computing theory are the operations of the quantum "oracles" employing "quantum parallelism" [10, 14]. Actually, the operation considered is defined by the following state transformation:

\[
|\Psi_i\rangle_{IO} = \sum_x C_x |x\rangle_I \otimes |0\rangle_O \rightarrow |\Psi_f\rangle_{IO} = \sum_x C_x |x\rangle_I \otimes |f(x)\rangle_O,
\]

where \(\{ |x\rangle_I \} \) represents the "computational basis" of the input register, while \( |0\rangle_O \) represents an initial state of the output register; by " \( f \) " we denote the oracle transformation.

The point strongly to be emphasized is that the transformation (1) does not [10, 12] require the orthogonality \( IO \langle \Psi_i | \Psi_f \rangle_{IO} = 0 \). Rather, orthogonality for the subsystem’s states is required [10, 12]:

\[
O \langle f(x) | f(x') \rangle_O = 0, x \neq x'
\]

for at least some pairs \( (x, x') \), which, in turn, is neither necessary nor a sufficient condition for the orthogonality \( IO \langle \Psi_i | \Psi_f \rangle_{IO} \) to be fulfilled.

Physical implementation of the quantum oracles is an open question of the quantum computation theory. However (and in analogy with the quantum measurement and the decoherence process [15-17]), it is well understood that the implementation should rely on
interaction in the system $I + O$ as presented by the following equality:

$$|\Psi_f\rangle_{IO} = \hat{U}(t)|\Psi_i\rangle_{IO} \equiv \hat{U}(t) \sum_x C_x |x\rangle_I |0\rangle_O = \sum_x C_x |x\rangle_I f(x,t)\rangle_O,$$

(3)

where $\hat{U}(t)$ represents the unitary operator of evolution in time (Schrodinger equation) for the combined system $I + O$; the index $t$ represents an instant of time, and we omit unnecessary symbol of the tensor product. Therefore, the operation (1) requires the orthogonality:

$$O\langle f(x,t)|f(x',t)\rangle_O = 0,$$

(4)

which substitutes the equality (2).

Therefore, our task in this paper reads: by the use of Eq. (4), we investigate the minimum time needed for establishing of the entanglement present on the r.h.s. of both Eq. (1) and of Eq. (3).

### 3. The optimal bound for the quantum oracle operation

In this Section we derive the bound for the minimum time needed for the execution of the transformation (1), i.e. (3), as distinguished by the expression (4).

Actually, we consider the composite system "input register + output register ($I + O$)" defined by the Hamiltonian:

$$\hat{H} = \hat{H}_I + \hat{H}_O + \hat{H}_{int}$$

(5)

where the last term on the r.h.s. of (5) represents the interaction Hamiltonian. For simplicity, we introduce the following assumptions: (i) $\partial \hat{H} / \partial t = 0$, (ii) $[\hat{H}_I, \hat{H}_{int}] = 0$, $[\hat{H}_O, \hat{H}_{int}] = 0$, and (iii) $\hat{H}_{int} = C\hat{A}_I \otimes \hat{B}_O$, where $\hat{A}_I$ and $\hat{B}_O$ represent unspecified observables of the input and of the output register, respectively, while the constant $C$ represents the coupling constant.

#### 3.1 Entanglement establishing

Given the above simplifications (i)-(iii), the unitary operator $\hat{U}(t)$ (cf. Eq. (3)) spectral form reads:

$$\hat{U}(t) = \sum_{x,i} \exp\{-it(\epsilon_x + E_i + C\gamma_{xi})/\hbar\} \hat{P}_{1x} \otimes \hat{\Pi}_{Oi}.$$

(6)

The quantities in Eq. (6) are defined by the following spectral forms: $\hat{H}_I = \sum_x \epsilon_x \hat{P}_{1x}$, $\hat{H}_O = \sum_i E_i \hat{\Pi}_{Oi}$, and $\hat{H}_{int} = C \sum_{x,i} \gamma_{xi} \hat{P}_{1x} \otimes \hat{\Pi}_{O}$; bearing in mind that $\hat{A}_I = \sum_x a_x \hat{P}_{1x}$ and $\hat{B}_O = \sum_i b_i \hat{\Pi}_{Oi}$, the eigenvalues $\gamma_{xi} = a_x b_i$.

From now on, we take the system’s zero of energy at the ground state by the exchange $E_{xi} \rightarrow E_{xi} - E_o$; $E_{xi} \equiv \epsilon_x + E_i + C\gamma_{xi}$, $E_o$ is the minimum energy of the composite system–which Margolus and Levitin [8, 9] have used, and Lloyd [1] as well. Then one easily obtains for the output-register’s states:

$$|f(x,t)\rangle_O = \sum_i \exp\{-it(\epsilon_x + E_i + C\gamma_{xi} - E_o)/\hbar\} \hat{\Pi}_{Oi}|0\rangle_O.$$

(7)
Substitution of Eq. (7) into Eq. (4) directly gives:

\[ D_{xx'}(t) \equiv O \langle f(x, t)|f(x', t)\rangle_O = \exp\left\{-it(\epsilon_x - \epsilon_{x'})/\hbar\right\} \times \sum_i p_i \exp\{-iCt(a_x - a_{x'})b_i/\hbar\} = 0, \quad \sum_i p_i = 1, \]

(8)

where \( p_i \equiv O \langle 0|\Pi_{Oi}|0\rangle_O \). The expression (8) is the condition of the ”orthogonal evolution” for the subsystem's (\( O \)'s) states bearing explicit time dependence, while the ground energy \( E_0 \) does not appear in (8).

But this expression is already known from, e.g., the decoherence theory [15-17]. Actually, one may write:

\[ D_{xx'}(t) = \exp\left\{-it(\epsilon_x - \epsilon_{x'})/\hbar\right\} z_{xx'}(t), \]

(9)

where

\[ z_{xx'}(t) \equiv \sum_i p_i \exp\{-iCt(a_x - a_{x'})b_i/\hbar\} \]

(10)

represents the so-called ”correlation amplitude”, which appears in the off-diagonal elements of the (sub)system’s (\( O \)'s) density matrix [15]:

\[ \rho_{Oxx'}(t) = C_x C_x^* z_{xx'}(t). \]

So, we could make direct application of the general results of the decoherence theory. However, our aim is to estimate the minimum time for which \( D_{xx'}(t) \) may approach zero, rather than calling for the qualitative limit of the decoherence theory [15]:

\[ \lim_{t \to \infty} |z_{xx'}(t)| = 0, \]

(11)

or equivalently \( \lim_{t \to \infty} z_{xx'}(t) \to 0 \).

So, here, we will use the inequality \( \cos x \geq 1 - (2/\pi)(x + \sin x) \), valid only for \( x \geq 0 \) [8, 9]. However, the use cannot be straightforward.

Actually, the exponent in the ”correlation amplitude” is proportional to:

\[ (a_x - a_{x'})b_i, \]

(12)

which need not be strictly positive. That is, for a fixed term \( a_x - a_{x'} > 0 \), the expression Eq. (12) can be both positive or negative, depending on the eigenvalues \( b_i \). For this reason, we will refer to the general case of the eigenvalues of the observable \( \hat{B}_O \), \( \{b_i, -\beta_j\} \), where both \( b_i, \beta_j > 0 \).

In general, Eq. (10) reads:

\[ z_{xx'}(t) = z_{xx'}^{(1)}(t) + z_{xx'}^{(2)}(t), \]

(13a)

where

\[ z_{xx'}^{(1)} = \sum_i p_i \exp\{-iCt(a_x - a_{x'})b_i/\hbar\}, \]

(13b)
\[ z^{(2)}_{xx'} = \sum_j p'_j \exp\{iCt(a_x - a'_x)\beta_j/\hbar\}, \] (13c)

while \( \sum_i p_i + \sum_j p'_j = 1 \). Now, since both \((a_x - a'_x)b_i > 0, (a_x - a'_x)\beta_j > 0, \forall i, j\), one may apply the above distinguished inequality.

The relaxed equality (4) (i.e. relaxed equality (11)) is equivalent with \( Rez_{xx'} \equiv 0 \) and \( Imz_{xx'} \equiv 0 \). Now, from Eq. (13a-c) it directly follows:

\[ Rez_{xx'} = \sum_i p_i \cos[C(a_x - a'_x)b_i t/\hbar] + \sum_j p'_j \cos[C(a_x - a'_x)\beta_j t/\hbar], \] (14)

which after applying the above inequality gives:

\[ Rez_{xx'} > 1 - \frac{4}{\hbar} C(a_x - a'_x)(B_1 + B_2)t - \frac{2}{\pi} Imz_{xx'} - \frac{4}{\pi} \sum_i p_i \sin[C(a_x - a'_x)b_i t/\hbar], \] (15)

where \( B_1 \equiv \sum_i p_i b_i \), and \( B_2 \equiv \sum_j p'_j \beta_j \).

Now, since \(|\sum_i p_i \sin[C(a_x - a'_x)b_i t/\hbar]| \leq \sum_i p_i \equiv \alpha < 1, \forall t\), from Eq. (11) and Eq. (15) it follows:

\[ 0 \cong Rez_{xx'} + \frac{2}{\pi} Imz_{xx'} > 1 - \frac{4\alpha}{\pi} - \frac{4}{\hbar} C(a_x - a'_x)(B_1 + B_2)t. \] (16)

From (16) it is obvious that the condition Eq. (4) cannot be fulfilled in the time intervals shorter than \( \tau_{xx'} \):

\[ \tau_{xx'} > \frac{(1 - 4\alpha/\pi)h}{4C(a_x - a'_x)(B_1 + B_2)}, \] (17)

which is strictly positive for \( \alpha < \pi/4 \), and which directly defines the optimal bound \( \tau_{ent} \) as:

\[ \tau_{ent} = \sup\{\tau_{xx'}\}. \] (18)

The assumption \( \alpha < \pi/4 \) is not very restrictive. Actually, above, we have supposed that neither \( \sum_i p_i \equiv 1 \), nor \( \sum_j p'_j \equiv 1 \), while the former is automatically satisfied with the condition \( \alpha < \pi/4 \).

3.2 Analysis of the results

The desired bound \( \tau_{ent} \) is obviously determined by the minimum of the difference \( a_x - a'_x \). This difference however is virtually irrelevant. So, one may note that the bound Eq. (18) can be operationally decreased by increase of the coupling constant \( C \) and/or by the increase of the sum \( B_1 + B_2 \). As to the former, for certain quantum ”hardware” [18], the coupling constant \( C \) can, at least in principle, be manipulated by experimenter. On the other side, similarly–as it directly follows from the above definitions of \( B_1 \) and \( B_2 \)--by
the choice of the initial state of the output register, one could eventually increase the rate of the operation by the increase of the sum $B_1 + B_2$.

Bearing in mind the obvious equality:

$$\langle \hat{H}_{int} \rangle = \langle \hat{A}_I \rangle \langle \hat{B}_O \rangle = \langle \hat{A}_I \rangle (B_1 - B_2),$$

one directly concludes that adding energy to the composite system as a whole, does not necessarily increase the rate of the operation considered. Rather, the increase of the rate of the operation is related to the average energy of interaction, $\langle \hat{H}_{int} \rangle$. For instance, if $B_1 \neq 0$ while $B_2 = 0$, from Eq. (19) it follows that the increase of $B_1$ coincides with the increase of $|\langle \hat{H}_{int} \rangle|$, as well as with the decrease of the bound Eq. (18). This observation is in accordance with the bound obtained by Margolus and Levitin [8, 9]: the increase of the average energy (of interaction) gives rise to the increase of the rate of the operation.

However, for the general initial state of the output register, both $B_1 \neq 0$ and $B_2 \neq 0$. Then, e.g., for $B_1 > B_2$:

$$B_1 + B_2 = B_1(1 + \kappa) \leq 2B_1, \kappa \leq 1,$$

which obviously determines the relative maximum of the rate of the operation by the following equality:

$$B_1 = B_2, \kappa = 1,$$

which, in turn (for $\langle \hat{A}_I \rangle \neq 0$), is equivalent with:

$$\langle \hat{H}_{int} \rangle = 0.$$

But this result is in obvious contradistinction with the result of Margolus and Levitin [8, 9]. Actually, the expressions (21) stem that, apart from the particular values of $B_1$ and $B_2$, the relative maximum of the rate of the operation requires (mathematically: implies) the zero average energy of interaction, $\langle \hat{H}_{int} \rangle = 0$.

4. Discussion

Intuitively, the speed of change of a system’s state should be directly proportional to the average energy of the system. This intuition is directly justified for the quantum ”orthogonal transformations” by the bound obtained by Margolus and Levitin [8, 9]. Naively, one would expect this statement to be of relevance also for the nonorthogonal evolution. Actually, in the course of the orthogonal evolution, the system’s state ”passes” through a ”set” of nonorthogonal states, thus making the nonorthogonal evolution faster than the orthogonal evolution itself.

This intuition however is obviously incorrect for the cases studied. In a sense, the expressions (21) state the opposite: as lower difference $B_1 - B_2$ (i.e. as lower the average energy of interaction), the faster the operation considered. Therefore, our the main result, Eq. (21), is in obvious contradistinction with the conclusion drawn from the bound obtained by Margolus and Levitin [8, 9]: the zero average energy quantum information processing is possible and, in the sense of Eq. (21), even preferable. From the operational
point of view, the bound $\tau_{ent}$ can be decreased by manipulations of the interaction in the combined system $I + O$, as well as by the proper local operations (e.g., the proper state preparations increasing the sum $B_1 + B_2$) performed on the output register.

As it can be easily shown, the increase of the sum $B_1 + B_2$ coincides with the increase of the spread in $\hat{B}_O$, $\Delta \hat{B}_O$, i.e. with the increase in the spread $\Delta \hat{H}_{int}$. This observation, however, cannot be interpreted as to suggest reducibility of the bound Eq. (18) onto the bound characterized in terms of the spread in energy [3-7]—in the case studied, $\langle \hat{H}_{int} \rangle$. Actually, as it is rather apparent, the increase in the spread $\Delta \hat{H}_{int}$ does not pose any restrictions on the average value $\langle \hat{H}_{int} \rangle$. Therefore, albeit having a common element with the previously obtained bound [3-7], the bound Eq. (17), i.e. Eq. (18), represents a new bound$^2$ in the quantum information theory.

It cannot be overemphasized: the zero (average) energy quantum information processing is possible, at least in principle. Moreover, the condition $\langle \hat{H}_{int} \rangle = 0$ determines the relative maximum of the operation considered. But this result challenges our classical intuition, because it is commonly believed that the efficient information processing presumes some "energy cost". In other words: one may wonder if "saving energy" might allow the efficient information processing ever. Without ambition to give a definite answer to this question, we want to stress: as long as the "energy cost" in the classical information processing (including the quantum-mechanical "orthogonal evolution") is surely necessary, this need not be the case with the quantum information processing, such as the entanglement establishing. Actually, the entanglement formation by no means represents acquiring the classical information about the (sub)system(s). So, without further ado, we stress that Eq. (21) exhibits the peculiar aspect of the "quantum information" (here: of the entanglement formation), so pointing to the necessity of its closer further investigation. To this end, the expression (21) might be interpreted as to point to the boundary between the "classical information" and the "quantum information".

The roles of the two registers ($I$ and $O$) are by definition asymmetric, as obvious from Eq. (1) and Eq. (3). This asymmetry is apparent also in the bound Eq. (17), which is the reason we do not discuss in detail the role of the average value $\langle \hat{A}_I \rangle$. Having in mind the told in Section 3, this discussion is really an easy task not significantly changing the above conclusions.

Finally, the simplifications (i)-(iii) of Section 2 do not prove restrictive for our considerations, as briefly discussed in Appendix I.

5. Conclusion

We show that the zero average energy quantum information processing is possible. Concretely, we show that the entanglement establishing in the course of operation of the quantum oracles employing "quantum parallelism", distinguishes the zero average energy of interaction in the composite system "input register + output register". More precisely: the zero average energy of interaction proves to be optimal for execution of the operation considered. This result challenges our classical intuition, which plausibly stems a need for the "energy cost" in the information processing. To this end, our result, which sets a new bound for the nonorthogonal evolution in the quantum information processing, might eventually be interpreted as to point to the boundary between the "classical information"
on the one, and of the "quantum information"–the concept yet to be properly understood–on the other side.

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**Appendix I**

Relaxing the simplifications (i)-(iii) of Section 2 does not lead to the significant changes of the results obtained. This can be seen by the use of the results of Dugić [17], but for completeness, we will briefly outline the main points in this regard.

First, for a time dependent Hamiltonian, which is still a “nondemolition observable”, 

\[ [\hat{H}(t), \hat{H}(t')] = 0, \]

the spectral form reads [17]:

\[
\hat{H} = \sum_{x,i} \gamma_{xi}(t) \hat{P}_{xi} \otimes \hat{\Pi}_{Oi}.
\]

\[(I.1)\]
This is a straightforward generalization of the cases studied.

Similarly, relaxing the exact compatibilities (cf. the point (ii) in Section 2) leads to approximate separability—i.e., in Eq. (I.1) appear the terms of the small norm—which does not change the results concerning the “correlation amplitude” $z_{xx'}(t)$ [15], and consequently concerning $D_{xx'}(t)$.

Finally, generalization of the form of the interaction Hamiltonian (cf. point (iii) of Section 2) does not produce any particular problem, as long as the Hamiltonian is of (at least approximately) separable kind, and also a nondemolition observable. E.g., from $\hat{H}_{\text{int}} = \sum_k C_k \hat{A}_{Ik} \otimes \hat{B}_{Ok}$, one obtains the term $\sum_k C_k (a_{kk} - a_{kk'})b_{ki}$, instead of the term Eq. (12).

The changes of the results may occur [17] if the Hamiltonian of the composite system is not of the separable kind and/or not a "nondemolition observable".

For completeness, let us emphasize: a composite-system observable is of the separable kind if it proves diagonalizable in a noncorrelated basis of the Hilbert state space of the composite system [17].
FOOTNOTES:

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2This bound is of interest also for the decoherence theory, but it does not provide us with the order of the "decoherence time", $\tau_D$. Actually, with inspection to Ref. [19], one may write—in terms of our notation—that $\tau_D \propto (a_x - a_{x'})^2$, while—cf. Eq. (17)—$\tau_{ent} \propto a_x - a_{x'}$, which therefore stems $\tau_D \gg \tau_{ent}$. This relation is in accordance with the general results of the decoherence theory: the entanglement formation should precede the decoherence effect.