Balanced Spanning Caterpillars

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Abstract

A $p$-caterpillar is a caterpillar such that every non-leaf vertex is adjacent to exactly $p$ leaves. We give a tight minimum degree condition for a graph to have a spanning $p$-caterpillar.

1 Introduction

Every connected graph contains a spanning tree, yet quite often it is desirable to find a spanning tree which satisfies certain additional conditions. There are many results giving sufficient minimum degree conditions for graphs to contain very special spanning trees. For example, Dirac's theorem from [5] states that any graph on $n \geq 3$ vertices with minimum degree at least $(n - 1)/2$ has a spanning path. In [9], S. Win generalized this fact and proved the following theorem.

Theorem 1.1. Let $k \geq 2$ and let $G$ be a graph on $n$ vertices such that $\sum_{x \in I} d(x) \geq n - 1$ for every independent set $I$ of size $k$. Then $G$ contains a spanning tree of maximum degree at most $k$.

In particular, if the minimum degree of $G$ is at least $(n - 1)/k$, then $G$ contains a spanning tree of maximum degree at most $k$. In fact, as showed in [9], the degree condition from Theorem 1.1 implies that either $G$ has a spanning caterpillar of maximum degree at most $k$ or $G$ belongs to a special exceptional class. We refer the reader to [7] for a comprehensive survey of spanning trees.

Another way of thinking about caterpillars is by looking at domination problems. A set $S \subseteq V$ is a dominating set in a graph $G = (V, E)$ if every vertex in $V \setminus S$ has a neighbor in $S$. A dominating set $S$ is called a connected dominating set if, in addition, $G[S]$ is connected. In the special case when $G[S]$ contains a path, we say that $G$ has a dominating path. In [1], Broersma proved a result on cycles passing within a specified distance of a vertex and stated an analogous result for paths from which, as one of the corollaries, we get the following fact.

Theorem 1.2. If $G$ is a $k$-connected graph on $n$ vertices such that $\delta(G) > \frac{n - k}{k + 2} - 1$, then $G$ contains a dominating path.

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In particular, if \( G \) is connected then \( \delta(G) > \frac{n-1}{2} - 1 \) implies that \( G \) has a spanning caterpillar. In this paper we will be concerned with a minimum degree condition that implies existence of spanning balanced caterpillar.

A \( p \)-caterpillar is a tree such that the graph induced by its internal vertices is a path and every internal vertex has exactly \( p \) leaves. The spine of a caterpillar is the graph induced by its internal vertices. The length of a caterpillar is the length of its spine. Recently, Faudree et. al. proved the following fact in [6].

**Theorem 1.4.** For \( p \in Z^+ \) there exists \( n_0 \) such that for every \( n \in (p+1)Z \) such that \( n \geq n_0 \) the following holds. If \( G \) is a graph on \( n \) vertices such that \( \delta(G) \geq \left(1 - \frac{p}{(p+1)^2}\right)n \), then \( G \) contains a spanning \( p \)-caterpillar.

The authors of [6] ask for the tight minimum degree condition which implies that \( G \) has a spanning 1-caterpillar. In addition, they ask for a tight minimum degree condition which gives a nearly balanced \( p \)-caterpillar (every vertex on the spine has \( p \) or \( p+1 \) leaf neighbors). We will settle the first problem and answer the second question in the case when \( n \) is divisible by \( p+1 \). In this paper we will substantially improve the minimum degree bound from Theorem 1.3 and give a tight minimum degree condition which guarantees existence of a spanning \( p \)-caterpillar. Our main result is the following fact.

**Theorem 1.4.** For \( p \in Z^+ \), there exists \( n_0 \) such that for every \( n \in (p+1)Z \) with \( n \geq n_0 \) the following holds. If \( G \) is a graph on \( n \) vertices such that

\[
\delta(G) \geq \begin{cases} \frac{n}{2} & \text{if } n/(p+1) \text{ is even} \\ \frac{n+1}{2} & \text{if } n/(p+1) \text{ is odd and } p > 2 \\ \frac{n-1}{2} & \text{if } n/(p+1) \text{ is odd and } p \leq 2 \\ \end{cases}
\]

then \( G \) contains a spanning \( p \)-caterpillar.

It’s not difficult to see that the minimum degree condition in Theorem 1.4 is best possible.

**Example 1.5.** First note that \( K_{n/2} \cup K_{n/2} \) in the case \( n \) is even and \( K_{(n-1)/2} \cup K_{(n+1)/2} \) in the case \( n \) is odd have no spanning caterpillars. Thus the degree condition in the case \( p \leq 2 \) is tight. Now suppose \( p \geq 3 \). Let \( n/(p+1) \) be even. Then \( n/2 \) is an integer. Consider \( K_{n/2-1,n/2+1} \). Clearly \( n/(2(p+1)) \) of spine vertices must be in one of the partite sets, because the spine is a path and its maximum independent set is of size \( n/(2(p+1)) \), but then the two partite sets must have the same size. Another example is \( K_{n/2} \cup K_{n/2} \). Now, suppose \( n/(p+1) \) equals \( 2k+1 \) for some \( k \in Z^+ \). If \( n \) is even, then consider \( K_{n/2,n/2} \). Clearly one of the partite sets must have \( k+1 \) spine vertices and so the other set must contain \((k+1)(p+1)-1 = \frac{n+p-1}{2} > \frac{n}{2}\) as \( p > 1 \). If \( n \) is odd then consider \( K_{(n-1)/2,(n+1)/2} \). Now, \( k+1 \) of the spine vertices must be in the partite set of size \( (n-1)/2 \). Consequently, the other set must have at least \( \frac{n+2-1}{2} > (n+1)/2 \) as \( p > 2 \).

We will prove Theorem 1.4 using the absorbing method from [8]. In this method, we first analyze the non-extremal case and then address two extremal cases, when \( G \) is "close to" \( 2K_{[n/2]} \) or \( K_{[n/2],[n/2]} \).

We will use \(|G|, ||G||\) to denote the order and the size of a graph \( G \). For two, not necessarily disjoint, sets \( U,W \subseteq V(G) \), we will use \(||U,W||\) to denote the number of edges in \( G \) with one endpoint in \( U \), another in \( W \). We say that a graph \( G \) is \( \beta \)-extremal if either \( V(G) \) contains a set
Let $W$ such that $|W| \geq (1/2 - \beta)n$ and $||G[W]|| \leq \beta n^2$ or if $V(G)$ can be partitioned into sets $V_1, V_2$ so that $|V_i| \geq (1/2 - \beta)n$ for $i = 1, 2$ and $||V_1, V_2|| \leq \beta n^2$. In addition, the following notation and terminology will be used. A $u, v$-caterpillar is a $p$-caterpillar where the first vertex in the spine is $u$ and the last is $v$.

The rest of the paper is structured as follows. In Section 2 we prove the absorbing lemma which is the key to handle the non-extremal case. In Section 3 we prove the non-extremal case and in Section 4 we address the extremal cases.

### 2 Absorbing Lemma

In this section we will prove an absorbing lemma and a few additional facts which are used in the next section to complete the proof in the case a graph is not extremal. We will start with the following observation.

**Lemma 2.1.** For $1/8 > \beta > 0$ there is $n_0 > 0$ such that the following holds. If $G$ is a graph on $n \geq n_0$ vertices such that $\delta(G) \geq (1/2 - \beta^2)n$ which is not $\beta$-extremal, then for any (not necessarily distinct) vertices $u, v \in G$, $||N(u), N(v)|| \geq \beta^2 n^2/32$.

**Proof.** We have $||G[N(u)]|| > \beta n^2$ from the definition of a $\beta$-extremal graph. Now suppose $u, v$ are two distinct vertices. If $\beta n/2 \leq ||N(u) \cap N(v)|| \leq (1/2 - \beta/2)n$, then $||N(u) \cup N(v)|| \geq 2(1/2 - \beta^2)n - (1/2 - \beta/2)n \geq (1/2 + \beta/4)n$. Thus every vertex $x \in N(u) \cap N(v)$ has at least $\beta n/8$ neighbors in $N(u) \cup N(v)$. Consequently, $||N(u), N(v)|| \geq \beta^2 n^2/32$. If $||N(u) \cap N(v)|| < \beta n/2$, then $||N(v) \setminus N(u)|| \geq (1/2 - 2\beta/3)n$. Thus, since $G$ is not $\beta$-extremal, $||N(u), N(v)|| \geq \beta^2 n^2/32$. If $||N(u) \cap N(v)|| \geq (1/2 - \beta/2)n$, then $||G[N(u) \cap N(v)]|| \geq \beta n^2$. □

Our next objective is to establish the following connecting lemma.

**Lemma 2.2** (Connecting Lemma). For $1/8 > \beta > 0$ there is $n_0 > 0$ such that the following holds. If $G$ is a graph on $n \geq n_0$ vertices such that $\delta(G) \geq (1/2 - \beta^2)n$ which is not $\beta$-extremal, then for any two vertices $u, v \in G$ there are at least $\alpha n^{4p+2}$, $u, v$-caterpillars of length three in $G$.

**Proof.** Let $u, v$ be two distinct vertices. By Lemma 2.1 $||N(u), N(v)|| \geq \beta^2 n^2/32$. Let $(x, y) \in E(N(u), N(v))$. Since each vertex in $(x, y, u, v)$ has degree at least $(1/2 - \beta^2)n$, the number of different $p$-caterpillars with spine $u, x, y, v$ is at least $\gamma n^{4p}$ for some $\gamma > 0$. Thus the total number of $u, v$-caterpillars of length three in $G$ is at least $\alpha n^{4p+2}$ for some $\alpha > 0$ which depends on $\beta$ only. □

We will be connecting through a small subset of $V(G)$ called a reservoir set.

**Lemma 2.3** (Reservoir Set). For $1/64 > \beta > 0$ and $\beta^4 > \gamma > 0$ there is $n_0$ such that if $G$ is a graph on $n \geq n_0$ vertices satisfying $\delta(G) \geq (1/2 - \beta^2)n$ which is not $\beta$-extremal then there is a set $Z \subset V(G)$ such that the following holds:

(i) $|Z| = (\gamma \pm \gamma^2)n$;

(ii) For every $v \in V$, $|N(v) \cap Z| \geq (1/2 - 2\beta^2)\gamma n$;

(iii) For every $u, v \in V$, $||N(u) \cap Z, N(v) \cap Z|| \geq \beta^6 \gamma^2 n^2/4$.

**Proof.** Let $Z$ be a set obtained by selecting every vertex from $V$ independently with probability $p := \gamma$. By the Chernoff bound [2], with probability $1 - o(1)$, the following facts hold:
(a) \((\gamma - \gamma^2)n \leq |Z| \leq (\gamma + \gamma^2)n\);

(b) For every vertex \(v\), \(|N(v) \cap Z| \geq (1/2 - 2\beta^2)\gamma n\).

To prove the third part let \(u, v \in V\) and let \(X_{u,v} := \{ w \in N(u) \mid |N(w) \cap N(v)| \geq \beta^3 n\}\). Since \(G\) is not \(\beta\)-extremal by Lemma 2.5, \(||N(u), N(v)|| \geq \beta^3 n^2/32\). Thus \(|X_{u,v}| \geq \beta^3 n\). Indeed, if \(|X_{u,v}| < \beta^3 n\), then \(|N(u), N(v)| < 2\beta^3 n^2 < \beta^2 n^2/32\). Consequently, by Chernoff’s inequality, with probability at least \(1 - o(1/\gamma^2)\), \(|X_{u,v} \cap Z| \geq \beta^3 \gamma n/2\). Thus with probability at least \(1 - o(1)\) for every \(u, v\), \(|X_{u,v}| \geq \beta^3 \gamma n/2\). Let \(u \in V\) be arbitrary and let \(w \in V\) be such that \(|N(w) \cap N(u)| \geq \beta^3 n\). Then with probability at least \(1 - o(1/\gamma^2)\), \(|N(w) \cap N(u) \cap Z| \geq \beta^3 \gamma n/2\). Thus with probability at least \(1 - o(1)\), we have

\[|N(u) \cap Z, N(v) \cap Z| \geq \beta^6 \gamma^2 n^2/4\]

every \(u, v\). Therefore there is a set \(Z\) such that (i)-(iii) hold. □

We will continue with our proof of the absorbing lemma. We shall assume that \(0 < \beta < 1/64\), \(G = (V, E)\) is a graph on \(n\) vertices where \(n\) is sufficiently large which is not \(\beta\)-extremal and which satisfies \(\delta(G) \geq (n-1)/2\). In addition, we will use an auxiliary constant \(\tau\) such that \(0 < \tau < \beta/16\).

**Lemma 2.4.** Let \(u, v\) be two vertices in \(G\) such that \(|N(u) \cap N(v)| \geq 2\tau n\). Then, at least one of the following conditions holds.

1. At least \(\tau n\) vertices \(x \in N(u) \cap N(v)\) are such that \(|N(x) \cap N(u)| \geq \tau^2 n\).
2. All but at most \(3\tau n\) vertices \(x \in N(v)\) satisfy \(|N(x) \cap N(v)| \geq \tau^3 n\).

**Proof.** First suppose that \(|N(v) \setminus N(u)| < 2\tau n + 2\). Since \(G\) is not \(\beta\)-extremal, \(||G[N(v) \cap N(u)]|| \geq \beta n^2\) and so the first condition holds. Thus we may assume that \(|N(v) \setminus N(u)| \geq 2\tau n + 2\). Since \(|N(u) \cup N(v)| > (1/2 + 2\tau)n + 1\), every vertex \(x \in N(v) \cap N(u)\) has at least \(2\tau n\) neighbors in \(N(u) \cup N(v)\). Thus all but at most \(\tau n\) vertices in \(N(v) \cap N(u)\) have at least \((2\tau - \tau^2) > \tau^3 n\) neighbors in \(N(v)\).

Now, suppose the first condition fails and we claim that all but at most \(2\tau n\) vertices \(x \in N(v) \setminus N(u)\) satisfy \(|N(x) \cap N(v)| \geq \tau^3 n\). Let \(A := \{ x \in N(u) \cap N(v) : |N(x) \cap N(u)| < \tau^2 n \}\) and note that \(|A| \geq \tau n\). Therefore,

\[||A, N(v) \setminus N(u)|| \geq |A|(|N(v) \setminus N(u)| - \tau^2 n) \geq (1 - \tau)|A||N(v) \setminus N(u)|.\]

Let \(B := \{ y \in N(v) \setminus N(u) : |N(y) \cap A| < \tau^2 |A| \}\) then \(\tau^2 |B||A| + (|N(v) \setminus N(u)| - |B|)|A| > |A, N(v) \setminus N(u)||.\) Hence \(\tau^2 |B||A| + (|N(v) \setminus N(u)| - |B|)|A| > (1 - \tau)|A||N(v) \setminus N(u)|\) and so \(|B| < \tau|N(v) \setminus N(u)|/(1 - \tau^2) < 2\tau n\). For every vertex \(x \in (N(v) \setminus N(u)) \setminus B\),

\[|N(x) \cap N(v)| \geq |N(x) \cap A| \geq \tau^2 |A| \geq \tau^3 n,\]

which completes the proof. □

**Lemma 2.5.** Let \(T\) be a set of \(p + 1\) vertices in \(G\). Then there exists a vertex \(x \in T\) such that for every \(y \in T\), \(||N(x), N(y)|| \geq \tau^4 n^2\).

**Proof.** Suppose there is a vertex \(v \in T\) such that condition (2) in Lemma 2.4 is satisfied. Let \(x := v\) and take \(y \in T\). If \(|N(y) \cap N(x)| \geq 5\tau n\), then \(||N(x), N(y)|| \geq \tau/4 \cdot (2\tau^4 n^2)\). If \(|N(y) \cap N(x)| < 5\tau n\), then since \(G\) is not \(\beta\)-extremal, \(||N(x), N(y)|| \geq \tau^4 n^2\). Therefore, we
may assume that there is no such $v$ in $T$. Let $x$ be an arbitrary vertex in $T$. Take $y \in T$. If $|N(x) \cap N(y)| \geq 2\tau n$, then by Lemma 2.4 (with $u := x$, $v := y$), $||N(x), N(y)|| \geq \tau^3 n^2 / 2$. If $|N(x) \cap N(y)| < 2\tau n$, then since $G$ is not $\beta$-extremal, $||N(x), N(y)|| \geq \gamma^4 n^2$. □

We say that an $x,y$-caterpillar $P$ absorbs a set $T$ of size $p + 1$, if $G[V(P) \cup T]$ contains an $x,y$-caterpillar on $|V(P)| + p + 1$ vertices. Let $M_q(T)$ denote the set of caterpillars of order $q$ which absorb $T$. A caterpillar $P$ is called $\gamma$-absorbing if $P$ absorbs every subset $W \subset V \setminus V(P)$ with $|W| \in (p + 1)Z$ and $|W| \leq \gamma n$. We will now prove our main lemma from which the absorbing lemma follows by using the deletion method.

**Lemma 2.6.** Let $p \in Z^+$. For every $\beta > 0$ there is $n_0$ and $\alpha > 0$ such that the following holds. If $G$ is a graph on $n \geq n_0$ vertices which is not $\beta$-extremal and such that $\delta(G) \geq (n - 1)/2$ and $T \subset V(G)$, $|T| = p + 1$, then

$$|M_q(T)| \geq \alpha n^q$$

where $q = (3p + 2)(p + 1)$.

**Proof.** Let $T = \{x, y_1, \ldots, y_p\}$ and in view of Lemma 2.5 suppose that for every $i$, $||N(x), N(y_i)|| \geq \tau^4 n^2$. We will construct a caterpillar $P$ which absorbs $T$. The counting fact follows easily from the way the construction works. To construct the caterpillar we will proceed in a few steps, selecting distinct vertices which have not been previously selected in each step. First take $v_1 \in N(y_i)$ so that $v_1, \ldots, v_p$ are distinct and $|N(v_i) \cap N(x)| \geq \tau^5 n$. Now let $u_1 \in N(v_i) \cap N(x)$ be such that $u_1, \ldots, u_p$ are distinct. Let $x_1x_2$ be an edge in $N(x)$. Use Lemma 2.2 to find $v_i, v_{i+1}$ caterpillars with spines $P_i$ for $2 \leq i \leq p - 1$, all vertices distinct, and let $P := v_2P_2v_3 \ldots v_{p-1}P_{p-1}v_p$. Use Lemma 2.2 to find a $v_2, x_2$-caterpillar and denote its spine by $Q_2$ and a $v_1, x_1$-caterpillar with spine $Q_1$. Let $Q := v_1Q_1x_1x_2Q_2v_2Pv_p$. Then $Q$ is a $v_1, v_p$-path. Disregard selected vertices not on $Q$. For every vertex $v_i$ select $p - 1$ distinct neighbors, so that together with $u_i$ they give $p$ leaves attached to $v_i$. For $x_1, x_2$ select $p$ distinct neighbors and let $S$ be the set containing all the vertices on $Q$, $u_1, \ldots, u_p$, and all the remaining neighbors. Then $G[S]$ contains a $v_1, v_p$-caterpillar of length $3p + 1$ which contains $(3p + 2)(p + 1)$ vertices. In addition, $G[S \cup T]$ contains a $v_1, v_p$-caterpillar of length $3p + 2$ obtained as follows. Insert $x$ between $x_1$ and $x_2$ in the spine $Q$, make $u_1, \ldots, u_p$ the neighbors of $x$, and let $y_i$ replace $u_i$ in the set of spikes of $v_i$. By Lemma 2.2 and in view of the construction, the number of such sets $S$ is at least $\alpha n^{(3p+2)(p+1)}$ for some $\alpha > 0$ which depends on $\beta$ and $p$ only. □

**Lemma 2.7.** (Absorbing Lemma) Let $p \in Z^+, q = (3p + 2)(p + 1), \beta > 0$ and $\alpha > 0$ be such that Lemma 2.6 holds. For any $\delta < \alpha/10q$, there is $n_0$ such that the following holds. If $G$ is a graph on $n \geq n_0$ vertices which is not $\beta$-extremal and such that $\delta(G) \geq (n - 1)/2$ then there is a caterpillar $P_{abs}$ in $G$ on at most $\delta n$ vertices which is $\delta^2$-absorbing.

**Proof.** Let $n_0$ be such that Lemma 2.6 holds with $\alpha$. Let $G$ be a graph on $n \geq n_0$ vertices which is not $\beta$-extremal and such that $\delta(G) \geq (n - 1)/2$.

Let $\mathcal{F}$ be a family obtained by selecting every set from $\binom{V}{q}$ independently with probability $\mu := \delta n / 3q \binom{n}{q}$. By the Chernoff bound [2], with probability $1 - o(1),$

$$|\mathcal{F}| \leq 2\mu \binom{n}{q} = 2\delta n / 3q$$

Now, let $T$ be a set of size $p + 1$. Again by Chernoff bound, with probability $1 - o(1/n^{p+1}),$

$$|M_q(T) \cap \mathcal{F}| \geq \frac{1}{2} \mu n^q > 3\delta^2 n.$$
The expected number of pairs \( \{S_1, S_2\} \) such that \( S_1, S_2 \in F \) and \( S_1 \cap S_2 \neq \emptyset \) is at most \( p \binom{n}{q} \cdot q \binom{n-1}{q-1} p \leq \delta^2 n \) and so by Markov’s inequality, with probability at least \( 1/2 \), the number of such pairs is at most \( 2\delta^2 n \). Therefore, with positive probability, there exists a family \( F \) such that \( |F| \leq 2\delta n/3q \), for every set \( T \) of size \( p + 1 \), \( |M_q(T) \cap F| > \delta^2 n \), and the number of \( \{S_1, S_2\} \) such that \( S_1, S_2 \in F \) and \( S_1 \cap S_2 \neq \emptyset \) is at most \( 2\delta^2 n \). Let \( F' \) be obtained from \( F \) by deleting all intersecting sets and sets that do not absorb any \( T \). Then \( |F'| \leq 2\delta n/3q \), and for every set \( T \) of size \( p + 1 \), \( |M_q(T) \cap F'| > \delta^2 n \). For each \( S \in F' \), \( G[S] \) contains a caterpillar on \( q \) vertices, so by using the minimum degree condition and Lemma \([2,2]\) we can connect the endpoints of these caterpillars to obtain a new caterpillar \( P_{abs} \). We also have that

\[
|P_{abs}| \leq |F'| \cdot q + 2|F'| \cdot p < |F'| \cdot (3q/2) \leq \delta n.
\]

To show that \( P_{abs} \) is \( \delta^2 \)-absorbing, consider \( W \subset V \setminus V(P_{abs}) \) such that \( (p + 1)|W| \) and \( |W| \leq \delta^2 n \). \( W = \{W_1, ..., W_m\} \) be an arbitrary partition of \( W \) into sets of size \( p + 1 \). We have that \( |M_q(W_i) \cap F'| > \delta^2 n \) for every \( i \in [m] \). Therefore, there exists a matching between \( W \) and \( F' \) so that every \( W_i \in W \) is paired with some \( S_i \in M_q(W_i) \). This implies that \( P_{abs} \) absorbs \( W \) and the proof is complete. \( \square \)

### 3 Non-extremal case

In this section we will finish proving the non-extremal case. The argument uses a similar approach as the proof of a corresponding fact in \([1]\).

Let \( p \in \mathbb{Z}^+, q = (3p + 2)(p + 1) \) and let \( \xi, \beta \) be such that \( 0 < \xi < 1/(4p + 5), 0 < \beta < \min\{\left(\frac{\beta}{30p}\right)^2, \left(\frac{\beta}{90}\right)^2\} \). Now, let \( \alpha > 0, n_0 \in \mathbb{N} \) be such that Lemma \([2,7]\) holds. Let \( \delta, \gamma > 0 \) be such that \( \delta < \min\{\left(\frac{\beta}{300}\right)^2, \left(\frac{\alpha}{100}\right)\}, \gamma < \frac{\delta^2}{4} \) and \( C \) be such that \( C > \frac{80(p + 1)}{\delta \gamma \beta^3} \). Let \( n > \max\{n_0, 4C^{-\frac{2(4+4C)}{\delta^4}}\} \) and \( G \) be a graph on \( n \) vertices which is not \( \beta \)-extremal and of minimum degree at least \( (n - 1)/2 \). Let \( P_{abs} \) be the absorbing caterpillar obtained in the previous section and let \( Z \) be the reservoir set from Lemma \([2,3]\) applied with \( \gamma \) which is less than \( \beta \) because \( \gamma < \frac{\delta^2}{4} < \beta \).

#### Claim 3.1

Let \( P_1, P_2 \) be disjoint caterpillars in \( G \) such that \( |Z \cap V(P_1)|, |Z \cap V(P_2)| < \frac{\beta^3}{100} \) and the endpoints of \( P_1 \) and \( P_2 \) are not in \( Z \). Then there is a caterpillar \( P \) containing \( V(P_1) \cup V(P_2) \) which has at most \( 2(p + 1) \) additional vertices in \( Z \) and such that its endpoints are not in \( Z \).

**Proof.** Let \( u_1, u_2 \) be the endpoints of \( P_1, P_2 \), respectively. Since \( \frac{\beta^3}{100} < 1/2 - 2\beta^2 \) and \( |(N(u_1) \cap Z) \cap (V(P_1) \cup V(P_2))| \cdot |(N(u_2) \cap Z) \cap (V(P_1) \cup V(P_2))| < \beta^6 \gamma^2 n^2/16 \), by Lemma \([2,3]\) there exists \( x_1 \in (N(u_1) \cap Z) \setminus (V(P_1) \cup V(P_2)), x_2 \in (N(u_2) \cap Z) \setminus (V(P_1) \cup V(P_2)) \) such that \( \{x_1, x_2\} \in E(G) \). Then we can construct a new caterpillar \( P \) using \( \{u_1, x_1\}, \{x_1, x_2\}, \{u_2, x_2\} \in E(G) \) and adding \( p \) vertices from \( N(x_1) \cap Z \setminus (V(P_1) \cup V(P_2)) \) and another \( p \) vertices from \( N(x_2) \cap Z \setminus (V(P_1) \cup V(P_2)) \) as leaf vertices of \( x_1, x_2 \). \( \square \)

Now, let \( G' := G[V \setminus (Z \cup V(P_{abs}))] \) and let \( P \) be a longest caterpillar in \( G' \). Starting with \( P \) we will extend \( P \) iteratively, adding at least \( \delta C/2 \) vertices by using at most \( 10(p + 1) \) vertices from \( Z \) in each step, until the number of vertices left is at most \( \frac{\delta^2 n}{2} \). Since the number of iterations is at most \( 2n/(\delta C) \), and so the number of vertices used to construct \( P \) in \( Z \) is at most \( \frac{2n}{\delta C} \cdot 10(p + 1) < \frac{\beta^3 \gamma}{4} n \), by Claim \([3,1]\) the process can be completed. Moreover, \( P_{abs} \) can be connected with \( P \) using \( Z \) and the number of remaining vertices which are not on the caterpillar is at most \( |Z| + \frac{\delta^2 n}{2} \leq \delta^2 n \) and so
they can be absorbed by $P_{ab}$. For the general step, let $W := V(G') \setminus V(P)$ and suppose $|W| > \frac{2\delta}{3}$.
We partition $P$ into $l$ blocks $B_1, \ldots, B_l$ of consecutive caterpillars so that $C \leq |B_i| \leq (1 + \delta)C$.

**Claim 3.2.** If $|G[W]| \geq \gamma|W|^2$, then there is a caterpillar in $G[W]$ with at least $\gamma|W| - p$ vertices.

**Proof.** $G[W]$ contains a subgraph $H$ such that $\delta(H) > \gamma|W|$. Let $Q$ be a longest caterpillar in $H$. If $|Q| \leq \gamma|W| - p$, then each endpoint of $Q$ has a neighbor $x \in V(H) \setminus Q$, and every vertex not on $Q$ has at least $p$ neighbors outside $Q$. □

**Case 1.** $|G[W]| \geq \gamma|W|^2$.

By Claim 3.2 there is a caterpillar $Q$ in $G[W]$ on at least $\delta C/2$ vertices. Since $Q \cap Z, P \cap Z = \emptyset$, by Claim 3.1 we can construct a caterpillar containing both of them.

**Case 2.** There is a block $B_i$ such that $||B_i, W|| \geq (\frac{1}{2} + \delta)|B_i||W|$.

Let $W' := \{w \in W||N(w) \cap B_i| \geq \left(\frac{1}{2} + \frac{\delta}{2}\right)|B_i|\}$. Then $|W'| \geq \delta|W| > \frac{\delta n}{2}$. Consider $H := G[W', B_i]$. Since there are less than $2^{(1+\delta)C}$ subsets of $B_i$ of size $\left(\frac{1}{2} + \frac{\delta}{2}\right)|B_i|$, there is a set $X \subset B_i$ such that $|X| = \left(\frac{1}{2} + \frac{\delta}{2}\right)|B_i|$ and for at least $|W'|/2^{(1+\delta)C}$ vertices $w \in W'$, $X \subset N(w) \cap B_i$. Since $\frac{|W'|}{2^{(1+\delta)C}} \geq 2C \geq \left(\frac{1}{2} + \frac{\delta}{2}\right)|B_i|$, $H$ contains $K_{D,D}$ where $D = \left(\frac{1}{2} + \frac{\delta}{2}\right)|B_i|$ which in turn contains a caterpillar on $2D - p > \left(\frac{1}{2} + \frac{\delta}{2}\right)|B_i|$ vertices. By Claim 3.1 using at most $4(p + 1)$ vertices in $Z$ we can connect the endpoints of this caterpillar with the endpoints of $B_i - 1$ and $B_{i+1}$.

**Case 3.** For every block $B_i$, $||B_i, W|| < (\frac{1}{2} + \delta)|B_i||W|$. 

Since we are not in Case 1, $\sum_{v \in W} |N(v) \cap W| \leq 2\gamma|W|^2$ and so $\sum_{v \in W} |N(v) \cap P| > (1/2 - \delta - 2\gamma)n|W| - 2\gamma|W|^2$, so

$$||P, W|| \geq \left(\frac{1}{2} - 2\delta\right)n|W|.$$ 

A block $B$ is called **good** if $||B, W|| \geq \left(\frac{1}{2} - 2\sqrt{\delta}\right)|W||B|$. Let $q$ denote the number of good blocks.

We have $q \geq \left(1 - 3\sqrt{\delta}\right)\frac{n}{C}$ as otherwise

$$||P, W|| \leq q\left(\frac{1}{2} + \delta\right)(1 + \delta)C|W| + (l - q)\left(\frac{1}{2} - 2\sqrt{\delta}\right)(1 + \delta)C|W|$$

which is less than $\left(\frac{1}{2} - 2\delta\right)n|W|$. Using the same argument as in Case 2, for a good block $B_i$ we can find set $C_i \subset B_i$ and $D_i \subseteq W$ such that $G[C_i, D_i] = K_{|C_i|, |D_i|}$, $|C_i| = \left(\frac{1}{2} - 3\sqrt{\delta}\right)C$ and $|D_i| \geq C$.

Let $U := \cup(B_i \setminus C_i)$ where the union is taken over good blocks. We have

$$|U| \geq \left(1 - 3\sqrt{\delta}\right)\frac{n}{C} - C\left(\frac{1}{2} - 3\sqrt{\delta}\right)\frac{n}{C} = \frac{n}{2}.$$ 

Thus, since $G$ is not $\beta$-extremal, $|G[U]| \geq \beta n^2$, and so there exist two good blocks $B_s, B_t$ with $s < t$ such that $|G[(B_s \setminus C_s) \cup (B_t \setminus C_t)]| \geq \beta C^2/2$. Thus by Claim 3.2 $G[(B_s \setminus C_s) \cup (B_t \setminus C_t)]$ contains a caterpillar $S$ on $\beta C/4$ vertices. In addition $G[C_s \cup C_t, W]$ contains two disjoint caterpillars $S_1, S_2$, each on $(1 - 7\sqrt{\delta})C$ vertices. Thus $|S \cup S_s \cup S_t| - |B_s \cup B_t| \geq 2(1 - 7\sqrt{\delta})C + \frac{\beta C}{4} - 2(1 + \delta)C \geq \delta C$.

By using at most $10(p + 1)$ vertices in $Z$, we can connect the endpoint of $B_{s-1}$ to the endpoint of $S \cap B_s$, connect the endpoint of $S \cap B_t$ to the endpoint of $B_{t-1}$, connect the endpoint of $B_{s+t}$ to the endpoint of $S_1$, and connect the endpoint of $S_1$ to the endpoint of $S_2$. Finally, by connecting the endpoint of $S_2$ to the endpoint of $B_{t+1}$, we form a longer caterpillar having more than at least $\delta C$ vertices than previous caterpillar. □
4 Extreme case

In this section we will address the extremal cases. First we will deal the the case when vertices of $G$ can be partitioned into two sets $V_1, V_2$ such that $|V_1, V_2| \leq \beta n^2$ and so, $G$ is close to a union of two complete graphs and then we address the case when $G$ has a large almost independent set.

We will start with the following lemma.

**Lemma 4.1.** Let $p \in Z^+$. For any $\xi < 1/(4p + 5)$ there is $n_0 \in N$ such that the following holds. Let $H$ be a graph on $n \geq n_0$ vertices such that $(p + 1)||H| and $\delta(H) \geq (1 - \xi)n$. Let $x, y \in V(H)$. Then there is a spanning $p$-caterpillar in $H$ connecting $x$ and $y$.

**Proof.** Let $P$ be a largest $p$-caterpillar in $G$ connecting $x$ and $y$. Let $S = (x = u_1 \ldots u_q (= y)$ denote the spine of $P$ and let $C[i]$ denote the set of spikes of $u_i$. For any two $u, v \in G$, $|N(u) \cap N(v)| \geq (1-2\xi)n$ and so $q \geq (1-2\xi)n/(p+1)$. Indeed, if $q < (1-2\xi)n/(p+1)$ then $|V(P)| < (1-2\xi)n$ and there exists $u'_1 \in (N(u_1) \cap N(u_2)) \setminus V(P), |N(u'_1) - V(P)| \geq (1-\xi)n - (1-2\xi)n > p$. If $V(P) = V(H)$ then we are done, so assume that there exists $\{v', y_1, ..., y_p\} \subset V(H) \setminus V(P)$.

Since $d(v') \geq (1-\xi)n$ there exists $i \in [q]$ such that $u_i, u_{i+1} \in N(v')$. Otherwise, since $\xi < 1/(4p + 5)$,

\[(1-\xi)n \leq d(v') \leq (n - q/2) \leq (1 - \frac{1 - 2\xi}{2(p+1)})n \leq (1 - \frac{1 - \xi}{4(p+1)})n < (1 - \xi)n,
\]

a contradiction. Moreover, there are $p$ distinct vertices $f_1, ..., f_p \in [q] \setminus \{i, i + 1\}$ such that for each $j \in [p], |N(v') \cap C[f_j]| > 0$ and $f_jy_j \in E(H)$, which gives us the caterpillar $P'$ such that $V(P') = V(P) \cup \{v', y_1, ..., y_p\}$ and $P'$ still connects $x$ and $y$. \(\square\)

A $p$-star is a star which has exactly $p$ leaves.

**Lemma 4.2.** Let $p \in Z^+$. There is $\beta > 0$ and $n_0$ such that if $G$ is a graph on $n \geq n_0$ vertices such that $(p + 1)n, \delta(G) \geq n^{1/2},$ and for some partition $V_1, V_2$ of $V(G)$ with $|V_i| \geq (1/2 - \beta)n$, $||G[V_1, V_2]|| \leq \beta n^2$, then $G$ contains a spanning $p$-caterpillar.

**Proof.** Let $\xi$ and $\beta$ be such that $0 < \xi < 1/(4p + 5)$ and $0 < \beta \leq (\frac{\xi}{3p})^2$. Let $W_i := \{v \in V_i : |N(v) \cap V_i| < (1/2 - 5\sqrt{\beta})n\}$. We have $\sum_{v \in V_i} |N(v) \cap V_i| \geq (1/4 - \beta/2)n(n-1) - \beta n^2 \geq (1/4 - 2\beta)n^2$ and

\[\sum_{v \in V_i} |N(v) \cap V_i| < |W_i|(1/2 - 5\sqrt{\beta})n + (|V_i| - |W_i|)|V_i|\]

and so $|W_i| \leq \sqrt{\beta}n - 1$. In addition, for every $v \in W_i$, $|N(v) \cap (V_{3-i} \setminus W_{3-i})| \geq 4\sqrt{\beta}n$. Let $U_i := V_i \setminus W_i$ and $X_i := W_{3-i}$. Then

- for every $v \in U_i$, $|N(v) \cap U_i| \geq (1/2 - 6\sqrt{\beta})n$,
- for every $v \in X_i$, $|N(v) \cap U_i| \geq 4\sqrt{\beta}n$.

Without loss of generality, suppose $|U_1 \setminus X_1| \leq |U_2 \setminus X_2|$. Then for every $v \in U_1 \cup X_1$, $|N(v) \cap (U_2 \setminus X_2)| \geq 1$. Let $r_i := |U_i \cup X_i| \mod (p+1)$. Since every vertex in $U_1 \cup X_1$ has at least one neighbor in $U_2 \cup X_2$, we pick $r_1$ vertices $u_1, ..., u_{r_1}$ in $U_1 \cup X_1$ and choose one neighbor in $U_2 \cup X_2$ for each. Note that clearly these neighbors do not need to be distinct. Let $w_1, ..., w_l$ denote distinct vertices in $U_2 \cup X_2$ chosen in this way. We have $l \leq p$ and each $w_i$ was chosen by at most $r_1 \leq p$ vertices. We will construct a spanning caterpillar by starting with $(U_1 \cup X_1) \setminus \{u_1, ..., u_{r_1}\}$. Since $|X_1| \leq \sqrt{\beta}n - 1$, there is a matching from $X_1 \setminus \{u_1, ..., u_{r_1}\}$ to $U_1 \setminus \{u_1, ..., u_{r_1}\}$. The matching
can be easily extended to a caterpillar $P$ in $G[(U_1 \cup X_1) \setminus \{u_1, \ldots, u_r\}]$ on at most $2(p + 1)\sqrt{\beta n}$ vertices. Let $b$ be the starting point of $P$. Let $G^\prime = G[(U_1 \cup X_1) \setminus \{u_1, \ldots, u_r\}] \cup V(P)$ and $b' \in N(b) \cap V(G^\prime)$. Since $\delta(G^\prime) \geq (1/2 - (8 + 2p)\sqrt{\beta})n \geq \left(\frac{1-\xi n}{2}\right) \geq (1 - \xi)|G^\prime|$, by Lemma 4.1 $G^\prime$ contains a spanning $p$–caterpillar $P^\prime$ starting at $b'$.

Denote by $x$ the another starting point of $P^\prime$, i.e $P^\prime$ is $b, x$–caterpillar. Now, pick $y \in N(x) \cap (U_2 \cup X_2)$. If $y \in X_2$ then construct a star $Y_0$ centered at $y$ such that $Y_0 \subset (X_2 \cup Y_2) \setminus \{w_1, \ldots, w_t\}$ and choose $y' \in N(y) \cap (U_2 \setminus \{w_1, \ldots, w_t\} \cup Y_0)$, otherwise $y' = y$. We will now construct a caterpillar in $G[\cup X_2 \cup \{u_1, \ldots, u_r\}]$. If $a_i$ denotes the number of vertices which choose $w_i$, then select $p - a_i$ neighbors of $w_i$ in $U_2 \cup X_2 \setminus \{y, w_1, \ldots, w_t\}$, all vertices distinct for different values of $i$. Let $S_i$ denote the $p$–star with center at $w_i$. Note that $y$ can be among $w_1, \ldots, w_t$ but it cannot be among the remaining vertices of $S_1, \ldots, S_t$. Since $|X_2| \leq \sqrt{3n}$, there is a matching from $X_2 \setminus \{y, y'\} \cup Y_0 \cup \bigcup_{i \in [t]} S_i$ to $U_2 \setminus \{y, y'\} \cup Y_0 \cup \bigcup_{i \in [t]} S_i$. The matching and $S_1, \ldots, S_t$ also can be extended to a caterpillar $P^\prime$ in $G[\cup X_2 \cup \{u_1, \ldots, u_r\}]$. Denote by $y''$ the other endpoint of the spine of $P^\prime$ and let $y''' \in (N(y'') \setminus (V(P^\prime') \cup Y_0)) \cap U_2$. Let $G''' = G[U_2 \cup X_2 \setminus (V(P^\prime') \cup \{w_1, \ldots, w_t\})]$. Since $\delta(G''') \geq (1 - \xi)|G'''|$, again by Lemma 4.1 there exists a spanning $p$–caterpillar $P'''$ connecting $y'$ and $y'''$. Then we get a spanning $p$–caterpillar of $G$ by linking $P', P''$ and $P'''$. □

We will now proceed to prove the other extremal case. We have the following lemma.

**Lemma 4.3.** Let $p \in Z^+$. For any $\xi < 1/(4p + 5)$ there is $n_0 \in \mathbb{N}$ such that the following holds. Let $H = (A_1, A_2)$ be a bipartite graph on $n \geq n_0$ vertices with $(p + 1)n$ such that $|A_1| = |A_2| = \frac{n}{2}$ if $n/(p + 1)$ is even and $|A_2| = \frac{n + p - 1}{2}$ if $n/(p + 1)$ is odd. Suppose that for any $v \in A_1, d(v) \geq (1 - \xi)|A_3 - i|$. Then for any $x \in A_1$, there exists a spanning $p$-caterpillar starting at $x$ in $H$.

**Proof.** First suppose $n/(p + 1)$ is even. Let $B_1$ be an arbitrary set of $n/(2(p + 1))$ vertices in $A_1$ such that $x \in B_1$. For any vertex $v \in B_1$ and any set $C \subseteq X_3 - i$ of size $n/(2(p + 1))$, $|N(v) \cap C| \geq |C| - \xi n \geq |C|/2$. Consequently, by Hall’s theorem, there is a set of pairwise disjoint $p$-stars with centers in $B_1$ and leaves in $X_3 - i \setminus W_3 - i$. In addition, $G[B_1, B_2]$ has a Hamilton cycle and so a spanning path which starts at $x$. The path, in connection with stars, gives a $p$-caterpillar starting at $x$. Now suppose $|A_2| = \frac{n + p - 1}{2}$. Let $B_2$ be a subset of $A_2$ of size $(n - p - 1)/(2(p + 1))$ and let $B_1$ be a subset of $A_1$ of size $(n + p - 1)/(2(p + 1))$. Note that $|B_1|p = |A_3 - i| - |B_3 - i|$. As before, by Hall’s theorem there are pairwise disjoint $p$-stars with centers in $B_1$ and leaves in $A_3 - i \setminus W_3 - i$ and $G[B_1, B_2]$ has a spanning path. □

**Lemma 4.4.** Let $p \in Z^+$. There is $\beta > 0$ and $n_0$ such that if $G$ is a graph on $n \geq n_0$ vertices such that $(p + 1)n$, for some set $S$ of $V(G)$ with $|S| \geq (1/2 - \beta)n$, $|G[S]| \leq \beta n^2$, and

$$
\delta(G) \geq \begin{cases} 
\frac{n}{2} & \text{if } n/(p + 1) \text{ is even} \\
\frac{n + p - 1}{2} & \text{if } n/(p + 1) \text{ is odd and } p > 2 \\
\frac{n}{2} & \text{if } n/(p + 1) \text{ is odd and } p \leq 2
\end{cases}
$$

then $G$ contains a spanning $p$-caterpillar.

**Proof.** Let $\xi$ and $\beta$ be such that $0 < \xi < 1/(4p + 5)$ and $0 < \beta \leq \text{min}\left\{\left(\frac{\xi}{10+3p}\right)^2, \left(\frac{\xi}{10+3p}\right)^2\right\}$. We may assume that $|S| \leq n/2$. Let $U_1 := S$ and $U_2 := V \setminus S$. We have

$$
||G[U_1, U_2]| \geq (1/2 - \beta)n^2/2 - 2\beta n^2 \geq (1 - 10\beta)|U_1||U_2|.
$$

Let $W_i := \{u \in U_i \setminus N(u) \cup U_3 - i\} \leq (1 - 10\sqrt{\beta})|U_3 - i|$. Then

$$
||G[U_1, U_2]| \leq |W_1||U_2|(1 - 10\sqrt{\beta}) + (|U_1| - |W_1||U_2|
$$
and so $|W_1| \leq \sqrt{3}|U_1|$ and similarly $|W_2| \leq \sqrt{3}|U_2|$.

We define $s$ to be $n/2$ when $n/(p+1)$ is even and $(n-p+1)/2$ when $n/(p+1)$ is odd. Let $W := W_1 \cup W_2$. Distribute vertices from $W$ to $X_1, X_2$ so that the following holds.

(a) If $x \in X_i$, then $|N(x) \cap U_{3-i}| \geq 10\sqrt{3}n$.

(b) $|\min\{|X_1 \cup (U_1 \setminus W_1)|, |X_2 \cup (U_2 \setminus W_2)|\} - s|$ is the least possible.

If the quantity in the second condition is positive, we further move vertices from $U_i \setminus W_i$ to $X_{3-i}$ which satisfy (a) to make $|\min\{|X_1 \cup (U_1 \setminus W_1)|, |X_2 \cup (U_2 \setminus W_2)|\} - s|$ as small as possible. Let $Y_i := X_i \cup (U_i \setminus W_i)$ and suppose $|Y_1| \leq |Y_2|$. 

First, assume that $|Y_1| = s$. Since for each $v \in W_1 \cup W_2$, $d(v) \geq 10\sqrt{3}n > |W_1| + |W_2|$, there is a matching $M$ such that for any $e \in M$, $|e \cap W_1| + |e \cap W_2| = 1$. Then we extend this matching to $p$-caterpillar $P$ so that for any $e \in M$, $e \cap W_i$ is a vertex of spine. Let $G' = G[V \setminus V(P)] = (V', E')$ and note that $V' \cap W = \emptyset$. Let $x$ be a last vertex of $P$ and $x' \in N(x) \cap V'$. Let $Y'_i = Y_i \cap V'$. For any $v \in Y'_i$,

$$|N(v) \cap Y'_{3-i}| \geq (1 - 10\sqrt{3})|U_{3-i}| - 3p\sqrt{3}|U_{3-i}| \geq (1 - \xi)|Y'_{3-i}|.$$

By Lemma 4.3 there exists a spanning caterpillar $P'$ starting at $x'$ of $G'$, then we get a spanning caterpillar of $G$ by attaching $P$ to $P'$.

Now, we assume that $|Y_1| \neq s$. If $|Y_1| > s$ then $n/(p+1)$ is odd and since $|Y_1| \leq |Y_2|$, $p \geq 3$, i.e. $\delta(G) \geq \frac{n+1}{2}$. We have $\delta(G[Y_2]) \geq \delta(G) - |Y_1| \geq 1$. If $|Y_1| < s$ (and so $|Y_1| < n/2$), then $\delta(G[Y_2]) \geq \delta(G) - |Y_1| \geq 1$.

In the first case we proceed as follows. Let $l = \frac{n+p-1}{2} - |Y_1|$. Since $|Y_1| > \frac{n-p+1}{2}$, $l < p-1$. If there is a vertex $y \in Y_2$ such that $|N(y) \cap Y_2| \geq p-1$, then pick $p-l$ neighbors of $y$ from $Y_1$, $l$ from $Y_2$ to form a $p$ star $S$ centered at $y$ and let $x$ be one more neighbor of $y$ in $Y_2$. Deleting $x$ and all vertices in $S$ gives $Y'_1, Y'_2$ such that $|Y'_1| = \frac{n-p-1}{2}$, and so by Lemma 4.3 there is a spanning $p$-caterpillar in $G[Y'_1, Y'_2]$ starting at $x$. If no such vertex exists, then $\Delta(G[Y_2]) \leq p-2$. Since $\delta(G[Y_2]) \geq 1$, there is a matching in $G[Y_2]$ of size at least $n/2(p-1)(>p-1)$. Let $y \in Y_2$ be arbitrary and let $x$ be a neighbor of $y$ in $Y_2$. Let $M = \{a_1b_1, \ldots, a_kb_k\}$ be a matching in $G[Y_2]$ such that $x, y \notin V(M)$. We construct caterpillar $Q$ as follows. Start with $x$ and pick $p$ neighbors of $x$ in $Y_1$. We will use $y$ as a vertex on spine of $Q$. Pick a neighbor new vertex $y' \in N(y) \cap Y_1$ and a $a'_1 \in N(a_1) \cap Y_1$. Note that $\Delta(G[Y_1]) \leq 20\sqrt{3}n$ as we can’t move any vertices from $Y_1$ and so any two vertices in $Y_1$ have at least $n/4$ common neighbors in $Y_2 \setminus V(M)$. Select one such unused vertex $z$ which gives a $y, a_1$-path of length four which will be added to the spine of $Q$. Now select $p$ neighbors from the opposite set for each vertices except $a_1$. In the case of $a_1$, pick $p-1$ neighbors from $Y_1$ and make $b_1$ one of the spikes. Now continue to add additional vertices. Then $Q$ has $2l + 2$ spine vertices in $Y_2$, $2l$ spine vertices in $Y_1$ and $|V(Q) \cap Y_2| = (2 + 2l) + 2lp + l$, $|V(Q) \cap Y_1| = (2l + 2p + 2l - l)$. This concludes the construction of $Q$. Let $x'$ be one new neighbor of $a_1$ in $Y_1$. Note that $|Y_2 \setminus V(Q)| = \frac{n-p-1}{2} + l - (2 + 2l + 2lp + l) = \frac{n+p-1}{2}$ where $n' = n - (4l + 2)(p+1) = |V - V(Q)|$. Thus by Lemma 4.3 we can extend $Q$ to get a spanning caterpillar in $G$.

In the second case, we have $\delta(G[Y_2]) \geq s - |Y_1| \geq 1$ and because no vertex can be moved from $Y_2$ to $Y_1$, $\Delta(G[Y_2]) \leq 20\sqrt{3}n$. Let $M$ be a maximum matching in $G[Y_2]$ and suppose $|M| < s - |Y_1|$. Then the number of edges in $G[Y_2]$ incident to $V(M)$ is at most $40\sqrt{3}|M| < 40\sqrt{3}(s - |Y_1|)$, but $|G[Y_2]| \geq \frac{|Y_2|}{2}(s - |Y_1|)$, and $|Y_2| \geq 80\sqrt{3}n$. 

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The rest of the argument is similar to the previous case. For every \( y \in Y_2 \), we have \(|N(y) \cap Y_1| \geq (1/2 - 20\sqrt{\beta})n\). Let \( M = \{a_1 b_1, \ldots, a_q b_q\} \). We move \( b_1, \ldots, b_q \) from \( Y_2 \) to \( Y_1 \) so that after moving \(|Y_1| = s\). Note that \(|Y_1| = |Y_2|\) or \(|Y_1| = \frac{n-p+1}{2}, |Y_2| = \frac{n+p-1}{2}\). Then we extend this matching to a \( p \)-caterpillar \( P \) so that for any \( i \in [q] \), \( b_i \) is a spike in \( P \). Let \( G' = G[V \setminus V(P)] = (V', E') \). Let \( x \) be the last vertex of \( P \) in \( Y_2 \) and \( x' \in N(x) \cap V' \). Let \( Y'_i = Y_i \cap V' \). For any \( v \in Y'_i \), since \( q \leq 4\sqrt{\beta}n \),

\[ |N(v) \cap Y'_{3-i}| \geq (1/2 - 24\sqrt{\beta})n \geq (1 - \xi)|Y'_{3-i}|, \]

By Lemma 4.3 there exists a spanning caterpillar \( P' \) starting at \( x' \) of \( G' \), then we get a spanning caterpillar of \( G \) by attaching \( P \) to \( P' \). □

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