Moment Approach to Determining the Orbital Elements of an Astrometric Binary with a Low Signal-to-Noise Ratio

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Abstract

A moment approach to the orbit determination of an astrometric binary with a low signal-to-noise ratio from astrometric observations alone is proposed, especially aiming at a close binary system with a short orbital period, such as Cyg X-1, and also at a star wobbled by planets. As an exact solution to the nonlinearly coupled equation, the orbital elements are written in terms of the second and third moments of projected positions that are measured by astrometry. This may hold the possibility of the true orbit estimate.

Key words: astrometry — celestial mechanics — methods: analytical — stars: binaries: close

1. Introduction

Space-astrometry on such missions as Gaia and JASMINE are expected to be accurate to a precision of a few μas (Mignard 2005; Perryman 2005; Gouda et al. 2007). Moreover, high-accuracy VLBI is also available.

For visual binaries, the formulation of orbit determinations has been well developed since the nineteenth century (Thiele 1883; Binnendijk 1960; Aitken 1964; Danby 1988; Roy 1988). At present, numerical methods are successfully used (Eichhorn & Xu 1990; Catovic & Olevic 1992; Olevic & Cvetkovic 2004). Furthermore, an analytic solution for an astrometric binary, where one object can be observed and the other such as a black hole and a neutron star is unseen, has been found (Asada et al. 2004; Asada 2008). The solution requires that sufficiently accurate positions of a star (or a photocenter of the binary) are measured at more places than four during an orbital cycle of the binary system.

For the close binary system with a short orbital period, we have a relatively large uncertainty about the determination of the position. For instance, the orbital period of Cyg X-1 is nearly 6 d, which is extremely shorter than that of the normal binary star, e.g., a few months, or even several years. Because of such an extreme condition, it is interesting to seek the other method in addition to the standard one. Moreover, stars with planets are also interesting targets.

What can we do for orbit determinations from position measurements with a low signal-to-noise (SN) ratio? It is expected that the position of the object is measured at many spots during an orbital cycle. The dense region of the observed points corresponds to the vicinity of the apastron of the Keplerian orbit, because the motion of the source star is slower according to Kepler’s second law. On the other hand, a region of fewer points includes the periastron, where the source star moves faster. Therefore, a statistical analysis that includes the variance of the measured positions and their correlation will bring information about the orbital elements of the binary system. Thought experiments suggest that the second moments are useful for exploring the shape of the orbit, but they are not sufficient to determine a full orbit. At least the third moments seem to be needed. See figures 1 and 2.

Therefore, the main purpose of this paper is to propose a method for the orbit determination of an astrometric binary with a low SN ratio by using the second and third moments. We shall also provide an exact solution for the coupled equations. As a result, the orbital element of the binary is written in terms of the second and third moments.

This paper is organized as follows. We present a formulation and its solution in section 2. In section 3, numerical tests are also described, in order to see how reliable the analytic result is in practical application, when we take account of observational noises. Section 4 is devoted to conclusion. Throughout this paper, the spatial coordinates are the angular positions normalized by the distance to the celestial object.

2. Basic Formulation

2.1. Motion in the Orbital Plane

Before projecting on the celestial sphere, we consider a Kepler orbit, where the semimajor axis and semiminer one are denoted by $a_K$ and $b_K$. We choose the $(X, Y)$ coordinates on the orbital plane, such that the $X$ and $Y$ axes are drawn along the semimajor and semiminer axes of the elliptic orbit, respectively, and the center of the ellipse is chosen as the origin of the coordinates. The orbit is expressed by

\[ X = a_K \cos u, \]
\[ Y = b_K \sin u, \]

where $u$ denotes the eccentric anomaly.

By introducing the eccentricity, $e_K$, the semiminor axis becomes

\[ b_K = a_K \sqrt{1 - e_K^2}. \]

For an object in Keplerian motion, the time $t$ is related to the position $u$ through Kepler equation,

\[ t = t_0 + \frac{P}{2\pi} (u - e_K \sin u), \]
where $t_0$ denotes the time of periastron passage. This equation represents a transcendental equation in the sense that it cannot be solved analytically without any approximation. In other words, $u$ cannot be expressed by using elementary functions of $t$. This makes the inverse problem in astrometry difficult.

2.2. Projection on the Plane Perpendicular to the Line of Sight

We assume a plane perpendicular to the line of sight, where we safely consider a small part of the celestial sphere because angular position shifts of extra-solar objects are sufficiently small.

A reference frame in astrometric observations is described by the $(x, y)$ coordinates. As usual, the following equations relate the $(X, Y)$ coordinates on the orbital plane and the $(x, y)$ coordinates on the reference plane,

\begin{align}
  x &= (X - a_K \epsilon_K) (\cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i) \\
  &= x_0 + a \cos u + \beta \sin u, \\
  y &= (X - a_K \epsilon_K) (\cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i) \\
  &= y_0 + y \cos u + \delta \sin u,
\end{align}

where $x_0$, $y_0$, $\alpha$, $\beta$, $\gamma$, and $\delta$ are defined as follows:

\begin{align}
  x_0 &\equiv -a_K \epsilon_K (\cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i), \\
  y_0 &\equiv -a_K \epsilon_K (\cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i), \\
  \alpha &\equiv a_K (\cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i), \\
  \beta &\equiv -b_K (\sin \omega \cos \Omega + \cos \omega \cos \Omega \cos i), \\
  \gamma &\equiv b_K (\cos \omega \sin \Omega - \sin \omega \cos \Omega \cos i), \\
  \delta &\equiv -b_K (\sin \omega \sin \Omega - \cos \omega \cos \Omega \cos i).
\end{align}

Here, $\Omega$, $\omega$, and $i$ denote the longitude of the ascending node, the argument of the periastron, and the inclination, respectively. See figure 3.

We cannot distinguish between the ascending and descending nodes from astrometric observations alone. Therefore, both $\pm i$ for $i$ are possible. Furthermore, because this paper focuses on the moments only, the clockwise motion cannot be distinguished from the anticlockwise. Two pairs of $(\Omega, \omega)$ are also possible. Nevertheless, the shape of the orbit is uniquely determined as shown below.

2.3. Moment Formalism

Let us assume frequent observations of the angular position. Namely, we consider a large number of observed points. In such a case, the statistical average, expressed as a summation, is taken as the temporal average in an integral form; that is,

\begin{align}
  < F > &\equiv \frac{1}{T_{\text{obs}}} \int_0^{T_{\text{obs}}} F dt,
\end{align}

where $< >$ denotes the mean and $T_{\text{obs}}$ the total time duration of observations.

In this paper, since we focus on the periodic motion, the above expression becomes the integration over one orbital period. Thus we obtain

\begin{align}
  < F > &= \frac{1}{P} \int_0^P F dt = \frac{1}{2\pi} \int_0^{2\pi} F (1 - \epsilon_K \cos u) du,
\end{align}

where we used

\begin{align}
  dt &= (1 - \epsilon_K \cos u) du &\text{derived from equation (4)}.
\end{align}

Let us consider statistical moments. Figure 1 suggests that the moments $M_{xx}$ and $M_{yy}$ are useful for distinguishing between two different orbits. Next, we consider the periastron for the orbit denoted by the closed solid curve. For the sake of simplicity, we assume that it is located at positive $x$ (in the right-hand side of the ellipse). In this case, the object moves faster around the periastron and slower around the apastron. Hence, the dots in the figure schematically show asymmetry in the number of observed points. In order to display, such an asymmetry, the third moment, such as $M_{xxy}$, seems to be useful because of the odd parity. Figure 2 suggests that $M_{xxy}$ is needed via a thought experiment. The second moments are defined as follows:

\begin{align}
  M_{xx} &\equiv < (x - < x >)^2 > = \frac{1}{2} (\alpha^2 + \beta^2) - \frac{1}{4} \epsilon_K^2 \alpha^2,
\end{align}

\begin{align}
  M_{yy} &\equiv < (y - < y >)^2 > = \frac{1}{2} (\alpha^2 + \beta^2) - \frac{1}{4} \epsilon_K^2 \beta^2,
\end{align}

\begin{align}
  M_{xxy} &\equiv < (x - < x >) (y - < y >) > = \frac{1}{2} (\alpha \beta) + \frac{1}{4} \epsilon_K^2 \alpha \beta,
\end{align}

\begin{align}
  M_{xxy} &\equiv < (x - < x >) (y - < y >) > = \frac{1}{2} (\alpha \beta) - \frac{1}{4} \epsilon_K^2 \alpha \beta.
\end{align}

Fig. 1. Comparison between the second moments, $M_{xx}$ and $M_{yy}$. The orbit denoted by the closed solid curve has a larger variance along the $x$ axis, where $M_{xx}$ is larger than $M_{yy}$. On the other hand, for the orbit denoted by the closed dashed curve, the $y$ components of the position have a larger scatter, where $M_{yy}$ is larger than $M_{xx}$. 

Fig. 2. Dashed curves denote two orbits: one is the semimajor axis along the $x$ axis; the other is along the $y$ axis. The orbit denoted by the solid curve is not distinguishable by using the second moments, $M_{xx}$ and $M_{yy}$. Thus the moment $M_{xxy}$ is needed.
The definition of $I_1$ by equation (22) is rewritten as

$$\alpha^2 = -\beta^2 + I_1,$$

which gives us $\alpha$ as a function of $\beta$,

$$\alpha = \pm \sqrt{-\beta^2 + I_1}. \tag{33}$$

It is obvious that the sign of the right-hand side of equation (33) must be the same as that of the left-hand side, namely $\alpha$. For $e_K \neq 0$, the sign of $\alpha$ is the same as that of $e_K\alpha$ that has been obtained above. This is expressed by

$$\text{sgn}(\alpha) = \text{sgn}(e_K\alpha). \tag{34}$$

where $\text{sgn}$ denotes the sign. Therefore, the sign of the right-hand side of equation (33) is obtained uniquely as $\text{sgn}(e_K\alpha)$. Equation (33) thus becomes

$$\alpha = \text{sgn}(e_K\alpha) \sqrt{-\beta^2 + I_1}. \tag{35}$$

What is the difference between equations (33) and (35)? Equation (33) means two different ones because of $\pm$ on the right-hand side, whereas equation (35) is a single one.

For convenience of calculation, we define $\Gamma$ as

$$\Gamma \equiv \frac{\gamma}{\alpha}, \tag{36}$$

which is obtained from known quantities, $e_K\alpha$ and $e_K\gamma$, as $\Gamma = (e_K\gamma)(e_K\alpha)^{-1}$. 

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$$I_1 = \frac{1}{2} e_K^2 \alpha^2 + 2 M_{xx}, \tag{25}$$

$$I_2 = \frac{1}{2} e_K^2 \gamma^2 + 2 M_{yy}, \tag{26}$$

$$I_3 = \frac{1}{2} e_K^2 \alpha \gamma + 2 M_{xy}. \tag{27}$$

Then, equation (25) is substituted into equation (18) to obtain a cubic equation for $e_K\alpha$ as

$$(e_K\alpha)^3 - 12 M_{xx} (e_K\alpha) + 16 M_{xxx} = 0. \tag{28}$$

This cubic equation gives three roots as $e_K\alpha$, which can be substituted into equations (26) and (29), whereas not all of them fulfill the equations of observed third moments. They have to fulfill the remaining set of third moments as follows:

$$M_{xxx} = \frac{1}{8} (e_K\gamma) I_1 + \frac{1}{4} (e_K\alpha) I_3 - \frac{1}{4} (e_K\alpha)^2 (e_K\gamma). \tag{30}$$

$$M_{xyy} = \frac{1}{8} (e_K\alpha) I_2 + \frac{1}{4} (e_K\alpha) I_3 - \frac{1}{4} (e_K\alpha)^2 (e_K\gamma)^2. \tag{31}$$

which can be used for picking up the correct $e_K\alpha$ and $e_K\gamma$ from multiple candidate values.

The definition of $I_1$ by equation (22) is rewritten as

$$\alpha^2 = -\beta^2 + I_1,$$ 

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For convenience of calculation, we define $\Gamma$ as

$$\Gamma \equiv \frac{\gamma}{\alpha}, \tag{36}$$

which is obtained from known quantities, $e_K\alpha$ and $e_K\gamma$, as $\Gamma = (e_K\gamma)(e_K\alpha)^{-1}$.

**Fig. 3.** Actual Keplerian orbit and apparent ellipse in three-dimensional space. We denote the inclination angle as $i$, the argument of periastron as $\omega$ and the longitude of ascending node as $\Omega$. These angles relate two coordinates, $(x', y')$ and $(\tilde{x}, \tilde{y})$, both of which choose the origin as the common center of mass. Here, the $x'$ axis is taken to lie along the semimajor axis of the apparent ellipse, while the $\tilde{x}$-axis is along the direction of the ascending node. This figure is the same as figure 3 in Asada (2008).
We substitute equation (36) into the definition of $I_3$, in order to eliminate $\gamma$ from equation (24). We obtain
\[
\alpha^2 = \frac{I_3 - \beta \delta}{\Gamma}.
\] (37)

This is substituted into the definition of $I_1$, equation (22). We obtain
\[
\delta = \frac{\beta^2 \Gamma + I_3 - I_1 \Gamma}{\beta}.
\] (38)

This is a function of only $\beta$.

Equations (36) and (38) are substituted into equation (23). After rather lengthy calculations, we obtain
\[
\beta^2 = \frac{(I_3 - I_1 \Gamma)^2}{I_1 \Gamma^2 + I_2 - 2I_3 \Gamma},
\] (39)

where we used equation (37) for $\alpha^2$. Interestingly, the right-hand side of this equation consists of only the known quantities $I_1$, $I_2$, $I_3$, and $\Gamma$. Therefore, we obtain the value of $\beta^2$, which determines $\alpha$ from equation (35).

Then, equation (39) of $\beta$ is solved as follows:
\[
\beta = \pm \frac{I_3 - I_1 \Gamma}{\sqrt{I_1 \Gamma^2 + I_2 - 2I_3 \Gamma}}.
\] (40)

Unfortunately, we do not know $e_K \beta$, contrary to $e_K \alpha$. Therefore, the sign of the right-hand side of equation (40) is not uniquely determined. The multiplicity of $\beta$ is two, and hence that of $\delta$ is also two according to equation (38).

Up to this point, we have known the values of $\alpha$ and $e_K \alpha$ from observed quantities. Thus we have found separately $e_K$ to be $e_K = (e_K \alpha) \alpha^{-1}$. It is crucial in the following procedure that the eccentricity, $e_K$, is determined at this stage.

It is very inconvenient that $\beta$ and $\delta$ are proportional to the semiminor axis, $b_K$, in their definition. Since we know $e_K$, $b_K$ can be expressed by $a_K$. Hence, we define renormalized quantities as
\[
\bar{\beta} \equiv \frac{\beta}{\sqrt{1 - e_K^2}} = -a_K (\sin \omega \cos \Omega + \cos \omega \sin \Omega \cos i),
\] (41)
\[
\bar{\delta} \equiv \frac{\delta}{\sqrt{1 - e_K^2}} = -a_K (\sin \omega \sin \Omega - \cos \omega \cos \Omega \cos i),
\] (42)

where we use equations (10) and (12). We know the values of $\beta$, $\delta$, and $e_K$. Therefore, one can estimate $\bar{\beta}$ and $\bar{\delta}$.

From four variables ($\alpha$, $\beta$, $\gamma$, and $\delta$), one can construct some quantities that are dependent on the inclination, $i$, but not on any other angles, $\omega$ nor $\Omega$. One example is
\[
C \equiv a^2 + \beta^2 + \gamma^2 + \delta^2 = a_K^2 (1 + \cos^2 i).
\] (43)

Another is
\[
D \equiv a \bar{\delta} - \bar{\beta} \gamma = a_K^2 \cos i.
\] (44)

These relations can be verified by direct calculations. Note that $D$ must be positive because $\cos i \geq 0$. This positivity chooses one pair of $(\bar{\beta}, \bar{\delta})$ and rejects the other pair. Only the pair of $(\beta, \delta)$ is thus obtained. Since we know $\alpha$, $\gamma$, $\beta$, and $\delta$, $C$ and $D$ can be estimated.

By eliminating $a_K^2$ from equations (43) and (44), we obtain
\[
\cos^2 i - \frac{C}{D} \cos i + 1 = 0.
\] (45)

This is a quadratic equation for $\cos i$. Apparently, two cases of $\cos i$ are possible. However, this is not the case. By using Newton’s identities for equation (45), we obtain the identity; that is,
\[
(\cos i_1) \times (\cos i_2) = 1,
\] (46)

where $\cos i_1$ and $\cos i_2$ denote the two roots of equation (45). Thus the inequality,
\[
|\cos i| \leq 1,
\] (47)

leads to a unique value of $\cos i$, because the other exceeds unity as its absolute value.

The single value of $\cos i$ provides a positive $i$ and a negative one. They correspond to the ascending node and the descending one, respectively. We cannot distinguish between the two nodes from astrometric observations alone. In order to distinguish the ascending node from the descending, the radial-velocity measurement, for instance, is needed.

After we obtain the value of the inclination, the semimajor axis is obtained from equation (44); that is,
\[
a_K = \sqrt{\frac{D}{\cos i}}.
\] (48)

This suggests that the value of $a_K$ is uniquely determined, but not doubly.

In order to determine $\omega$ and $\Omega$, let us consider the other combinations of $\alpha$, $\gamma$, $\beta$, and $\delta$. Direct calculations lead to
\[
\alpha^2 + \beta^2 = a_K^2 (\cos^2 \Omega + \sin^2 \Omega \cos^2 i),
\] (49)
\[
\gamma^2 + \delta^2 = a_K^2 (\sin^2 \Omega + \cos^2 \Omega \cos^2 i).
\] (50)
The ratio between them is denoted by
\[ r_1 \equiv \frac{\gamma^2 + \delta^2}{\alpha^2 + \beta^2} = \frac{\sin^2 \Omega + \cos^2 \Omega \cos^2 i}{\cos^2 \Omega + \sin^2 \Omega \cos^2 i}. \]  
(51)

This equation of \( \Omega \) is solved as follows:
\[ \tan^2 \Omega = \frac{r_1 - \cos^2 i}{1 - r_1 \cos^2 i}, \]  
(52)

which gives us the values of \( \Omega \), because we have already determined \( i \) and \( r_1 \). As already mentioned, both \( \pm \Omega \) are allowed.

Next, we consider different combinations,
\[ \alpha^2 + \gamma^2 = \alpha_K^2 (\cos^2 \omega + \sin^2 \omega \cos^2 i), \]  
(53)

and
\[ \beta^2 + \delta^2 = \alpha_K^2 (\sin^2 \omega + \cos^2 \omega \cos^2 i), \]  
(54)

the calculations of which can be verified directly. Their ratio is denoted as
\[ r_2 \equiv \frac{\beta^2 + \delta^2}{\alpha^2 + \gamma^2} = \frac{\sin^2 \omega + \cos^2 \omega \cos^2 i}{\cos^2 \omega + \sin^2 \omega \cos^2 i}. \]  
(55)

This equation of \( \omega \) is solved as follows:
\[ \tan^2 \omega = \frac{r_2 - \cos^2 i}{1 - r_2 \cos^2 i}, \]  
(56)

which gives us the values of \( \omega \), because we have already determined \( i \) and \( r_2 \). As mentioned above, both \( \pm \omega \) are possible.

At most four values of \( \Omega \) are possible. Similarly, the maximum multiplicity of \( \omega \) is four. In total, sixteen sets of \((\omega, \Omega)\) appear to exist. The multiplicity of \((\omega, \Omega)\) is reduced, because these sets with the uniquely determined \( a_K, e_K \), and \( \cos i \) must fulfill the definitions of \( \alpha, \beta, \gamma \), and \( \delta \). In particular, \( \alpha, \beta, \gamma, \) and \( \delta \) include different combinations of \( \sin \) and \( \cos \). Basically, the signs of \( \sin \) and \( \cos \) have four types: \((+, +), (++, -), (-, +), \) and \((-,-)\). Hence, the sixteen apparent sets are separated into four. In addition, it is obvious from equations (9)–(12) that the sign of \( \Omega \) depends on that of \( \omega \). The multiplicity is also reduced by half. As a consequence, we obtain only two pairs of \((\Omega, \omega)\). One pair corresponds to the clockwise motion, and the other to the anticlockwise motion.

3. Discussion

3.1. Numerical Test

The above formalism is discussed from an idealized perspective. Numerical tests are described below to show whether the analytic result works for practical cases. First, equations (13) and (14) assume that one can integrate observed quantities. In practice, however, since observations are discrete, the integration becomes a summation. The integration and the summation could agree on the condition that the number of observations, \( N \), approaches infinity. In the real world, \( N \) is a large number, but much smaller than infinity. Does the above formalism still give us a reliable answer? In order to pursue investigations into this point, we performed numerical simulations. According to the simulation of observations at intervals of certain times, the above formalism perfectly recovers the orbital parameters for \( N = 100 \).

Next, we consider observation noises. The above formalism assumes that the observed points are located on an apparent ellipse. However, position measurements are inevitably associated with observational errors. Therefore, we performed numerical simulations by adding Gaussian errors to position measurements; that is, \( x \rightarrow x + \Delta x \) and \( y \rightarrow y + \Delta y \), where \( \Delta x \) and \( \Delta y \) have the Gaussian distribution with the standard deviation, \( \sigma \). We considered two cases: \( \sigma = 0.1 \) (smaller case) and 0.5 (larger one) in units of \( \Delta_\kappa = 1 \). Table 1 gives a list of the orbital parameters that were recovered by using the above formalism. See figure 5 for simulated points in an \( N = 100 \) simulation for \( \sigma = 0.1 \) and 0.5. In the small observation-error case, the orbital parameters are well recovered. In the case of large observation errors comparable to half of the semimajor axis, however, the recovered angles \((i, \omega, \Omega)\) are far from the true ones. On the other hand, the eccentricity, \( e_K \), and the semimajor axis, \( a_K \), are recovered better than the three angles. The semimajor axis is overestimated, because the simulated second moments apparently become larger than the true ones, owing to such a large dispersion. We also numerically considered different parameter values. They led to similar results, numerical tables of which are omitted to economize on space.

Our numerical tests concerning the discreteness and noisiness of the observations suggest that the above formalism derived for an idealized system could work in practice if the observation noises are not very large.

The previous analytical method cannot guess the orbital parameter for such a large measurement error of \( \sigma = 0.5 \), mostly because \( \cos i \) apparently becomes larger than unity; namely, the data points fit better with open orbits (Asada et al. 2007).

Up to this point, we have assumed that the orbital period is known. What happens in the case of unknown binary? We performed Fourier analyses of numerically simulated points with time, as shown in figure 5. In the time domain, the Fourier spectrum has two peaks; one peak corresponds to the orbital period, and the other is around the artificial time step in the numerical simulation. This suggests that the moment approach can also be applied to unknown binary systems, if a Fourier analysis is adequately used for knowing the orbital period. Namely, the present method could be used for searching new a binary system.

Let us consider Cyg X-1, \( a \sim 0.2 \) AU at 2 kpc from us. The expected angular accuracy in JASMINE is \( \sim 10 \mu \text{as} \), so that the semimajor axis of Cyg X-1 can become directly observable. Other known X-ray binaries seem too faint to be observed by JASMINE.

A Sun-like star at 20 pc with a Jupiter-like planet at 1 AU could produce a wobble of 0.001 AU, corresponding to 50 \( \mu \text{as} \), which must be an interesting target.

3.2. Proper Motion of the Binary System

In the main part of this paper, we ignore the proper motion of the binary system in the following period. This is mostly because, for a close binary, the proper motion of the binary in our Galaxy causes a larger cumulative displacement than the orbital motion of the component star, though the orbital velocity may be larger than the proper motion. In advance, therefore, we need to know the proper motion before determining the orbital elements. For
instance, the proper motion can be known by making a comparison between the Hipparcos data and future space astrometry. If one wishes to determine the proper motion (\( \dot{x}, \dot{y} \)) in the present formalism, however, the apparent position should be replaced by \( x \rightarrow x + v_x t \) and \( y \rightarrow y + v_y t \). By averaging out the observed position using equation (13), we obtain

\[
\begin{align*}
\langle x \rangle & = \frac{1}{2} T_{\text{obs}} v_x + \text{const.} + O \left( \frac{T_K}{T_{\text{obs}}} a_K \right), \\
\langle y \rangle & = \frac{1}{2} T_{\text{obs}} v_y + \text{const.} + O \left( \frac{T_K}{T_{\text{obs}}} a_K \right),
\end{align*}
\]

where the term of \( O(a_K T_K T_{\text{obs}}^{-1}) \) on the right-hand side comes from Keplerian motion. Hence, the term is none other than such a decaying term as \( T_{\text{obs}}^{-1} \cos \nu_{\text{obs}} \) for the eccentric anomaly, \( \nu_{\text{obs}} \), corresponding to \( t = T_{\text{obs}} \). Terms become negligible as \( T_{\text{obs}} T_K^{-1} \rightarrow \infty \). Furthermore, the cumulative translation by the proper motion exceeds the oscillatory displacement by Keplerian motion for a close binary. That is, \( a_K (T_{\text{obs}} v)^{-1} \ll 1 \), where \( v \equiv \sqrt{v_x^2 + v_y^2} \). Owing to this effect, the decaying part due to Keplerian motion vanishes in the case of a long observation period of \( T_{\text{obs}} > 1 \) yr.

The parts growing linearly with the observation time, \( T_{\text{obs}} \), give information about \((v_x, v_y)\), provided that \( T_{\text{obs}} \) is taken as a variable in the data analysis. If \( T_{\text{obs}} \gg T_K \), the above extraction of the linearly growing part will be possible, especially for a short-separation binary.

### 3.3 Mildly Relativistic Compact Binary

Finally, we mention a mildly relativistic compact binary, in which a relativistic advance of the periastron occurs. Strictly speaking, we have to take account of the general relativistic equation. In practice, however, the above method may be applied. For instance, let us imagine 2 yr observations for the

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**Table 1.** Reconstructing the parameters for numerical simulations of three different eccentricity cases, \( e_K = 0.1, 0.3, \) and 0.5.

| \( \sigma \) | \( e_K \) | \( a_K \) | \( i [^\circ] \) | \( \omega [^\circ] \) | \( \Omega [^\circ] \) |
|---|---|---|---|---|---|
| 0 | 0.1 | 1.0 | 30.0 | 30.0 | 30.0 |
| 0.1 | 0.09454 ± 0.02518 | 1.011 ± 0.01423 | 30.02 ± 2.039 | 22.37 ± 17.45 | 29.81 ± 4.205 |
| 0.5 | 0.1445 ± 0.09343 | 1.257 ± 0.07624 | 28.87 ± 7.731 | 43.87 ± 25.79 | 40.75 ± 22.62 |
| 0 | 0.3 | 1.0 | 30.0 | 30.0 | 30.0 |
| 0.1 | 0.2936 ± 0.02269 | 1.011 ± 0.01526 | 30.04 ± 2.228 | 28.73 ± 8.557 | 30.20 ± 4.687 |
| 0.5 | 0.2327 ± 0.1211 | 1.266 ± 0.08215 | 29.63 ± 7.189 | 41.26 ± 25.13 | 35.45 ± 18.14 |
| 0 | 0.5 | 1.0 | 30.0 | 30.0 | 30.0 |
| 0.1 | 0.4793 ± 0.03103 | 1.005 ± 0.02253 | 30.34 ± 2.477 | 27.53 ± 8.234 | 31.72 ± 5.709 |
| 0.5 | 0.3065 ± 0.1223 | 1.233 ± 0.07314 | 31.33 ± 5.519 | 39.78 ± 26.46 | 40.78 ± 16.08 |

*In the table, the \( \sigma = 0 \) row indicates true orbital parameters, whereas the \( \sigma = 0.1 \) and 0.5 rows provide the recovered values for adding Gaussian errors (0.1 and 0.5 in units of the true semimajor axis, respectively). For each parameter set, 100 runs are done, and the mean value and the standard deviation are also evaluated.*

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**Fig. 5.** One hundred observed points of the same source star in the \( x-y \) plane for a simulation at an interval of constant times. The parameters are \( \omega = 30^\circ, \Omega = 30^\circ, i = 30^\circ, e_K = 0.1, a_K = 1.0, \) and \( N = 100 \). The black curve and the gray (red in color) one denote orbits for the true parameter and the mean value of the recovered parameters, respectively. Top, Gaussian errors \( \sigma = 0.1 \); bottom, \( \sigma = 0.5 \).
first-year data, the longitude of the periastron, $\varpi_1$, is derived. The second-year data give $\varpi_2$. The difference between the first- and second-year longitudes of the periastron suggests the periastron advance to be $(\varpi_2 - \varpi_1)/NUL_1$ yr$^{-1}$.

For instance, the Hulse–Taylor binary pulsar shows a large periastron shift rate of $\varpi = 4^\circ$ per year, though its angular radius of the orbit is a few $\mu$as, which is beyond the current measurement capability.

4. Conclusion

This paper proposes a moment approach to the orbit determination of a close binary system with a short orbital period from astrometric observations alone. As an exact solution to the coupled equation, the orbital elements are written in terms of the second and third moments of the projected position, which is measured by astrometry.

The moment formalism does not supersede the standard method using Kepler equation. It is safer to say that the present formalism is a supplementary tool for giving a rough parameter estimate, which can be used as a trial value for full numerical data fittings. It is interesting to make some numerical tests of the present method. It is left as future work.

In the moment approach, the temporal information relative to the orbital period cannot be acquired by this method. Another method, such as a Fourier analysis of position data with time (in figure 5 for example), would give a characteristic frequency that is the inverse of the orbital period. Fourier analyses recover the orbital period from numerically simulated data in figure 5. This suggests that the moment approach can be also applied to unknown binary systems, if a Fourier analysis is adequately used for knowing the orbital period. Namely, the method could be used for searching new binary systems.

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