OPTIMAL TRANSPORT ON THE PROBABILITY SIMPLEX WITH LOGARITHMIC COST

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Abstract. Motivated by the financial problem of building financial portfolios which outperform the market, Pal and Wong [2] considered optimal transport on the probability simplex $\Delta^n$ where the cost function is induced by the free energy. We study the regularity of this problem and find that the associated MTW tensor is non-negative definite and in fact constant on $\Delta^n \times \Delta^n$. We further find that relative $c$-convexity corresponds to the standard notion of convexity in the probability simplex. Hence, we are able to use standard results in optimal transport to establish regularity for the optimal transport maps considered by Pal and Wong. We also provide several new examples of costs satisfying the MTW(0) condition.

1. Introduction

Recently, a series of papers by Pal and Wong ([1]-[5]) have used information-geometric ideas to try to find portfolio maps (investment strategies) which outperform the market portfolio under mild and realistic assumptions. They have shown that solving such a problem is equivalent to solving optimal transport problems where the cost function is given by certain divergences from information geometry. One of the main results of [1] is to show that a portfolio map $\pi$ outperforms the market iff it is $c$-cyclic monotone for a specific cost function $c$. They further show that these portfolio maps are $c$-cyclic monotone iff they are generated by an exponentially-convex function $\psi$. This observation relates the problem of finding investment portfolios to that of solving optimal transport problems on the probability simplex for the cost $c$.

In their work, Pal and Wong discuss the information geometric properties divergences induced by exponentially convex functions. For this paper, we are primarily interested in the cost function they use to dualize their divergences. Unless otherwise stated, in this paper we will consider the cost function $c : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}$:

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If we consider $x_i$ and $y_i$ to be the natural parameters of the multinomial distribution, then this cost has statistical meaning. Given natural parameters $\{x_i\}_{i=1}^{n-1}$, we can compute the associated probabilities using the following formulas.

\[
p_i = \frac{e^{x_i}}{1 + \sum_{j=1}^{n-1} e^{x_j}} \quad \text{for } 1 < i < n
\]

\[
p_n = \frac{1}{1 + \sum_{j=1}^{n-1} e^{x_j}}
\]

We can similarly write the probabilities $q_i$ associated to the $y_i$-parameters. Fixing $\pi = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \in \Delta^n$ and rewriting our cost in terms of the $p_i$s and $q_i$s, we have the following:

\[
T(p \mid q) = \log \left( \sum_{i=1}^{n} \pi_i \frac{p_i}{q_i} \right) - \sum_{i=1}^{n} \pi_i \log \left( \frac{p_i}{q_i} \right)
\]

This quantity is known as the free energy in statistical physics \cite{1} and by various different names in finance (such as the “diversification return”, the “excess growth rate,” the “rebalancing premium” and the “volatility return”). Since Pal and Wong consider this in the context of logarithmic divergences, we refer to this cost as the logarithmic cost. In view of Lemma 2 of \cite{2}, it may have been more appropriate to refer to this cost as an “entropic” cost. However, that terminology was already used by Gentil, Léonard, and Ripani \cite{22} to refer to something slightly different, so we chose a different name. This cost is given by any distance function as it is not symmetry. However, it is a divergence via Jensen’s inequality.

Pal and Wong’s work considers more general divergences induced by log-convex functions, of which $T(\cdot \mid \cdot)$ is only a single example. For any log-convex function, one can define a divergence $D[\cdot \mid \cdot]$, which has a self-dual representation in terms of the logarithmic cost $c$ (after a possible change of coordinates). In order to study optimal transport, we do not specify the log-convex function a priori. In fact, such a function induces the solution to the optimal transport problem. As such, we are primarily interested in the cost function itself and do not utilize much of the structure of \cite{5}, which studies the
curvature properties of the divergence once such an exponentially convex function has been specified.

Our motivation for this work is two-fold. Firstly, Pal and Wong briefly discuss the regularity theory of such transport in [2], leaving it open for further investigation. Our main goal here is to address that question. Recall that for optimal transport, we consider two probability spaces \((\mathcal{X}, \mu)\) and \((\mathcal{Y}, \nu)\) and a transport map \(T : \mathcal{X} \to \mathcal{Y}\) such that \(T_\# \mu = \nu\) and \(T\) minimizes some cost function. We wish to study the regularity of the transport, and determine conditions on the probability spaces that force \(T\) to be smooth. For the logarithmic cost, we show the following.

**Corollary.** Let \(\mu\) and \(\nu\) be smooth probability measures supported on compact strictly convex subsets \(X\) and \(Y\) of the probability simplex. Suppose further that \(d\mu\) and \(d\nu\) are bounded away from zero and infinity on their support. Let \(\tilde{c}(p, q)\) be the cost function given by

\[
\tilde{c}(p, q) = \log \left( \frac{1}{n} \sum_{i=1}^{n} q_i \right) - \frac{1}{n} \sum_{i=1}^{n} \log \frac{q_i}{p_i}.
\]

Then the optimal transport map \(T\) taking \(\mu\) to \(\nu\) is smooth.

Secondly, for this cost function, the MTW tensor ended up being extremely simple and so we believe that it might be of independent interest in optimal transport theory. This illustrates connection between optimal transport, statistical mechanics, and information geometry. We do not have a good explanation for it, but it seems too good to be pure chance, so we felt that it was worth exploring.

### 2. Background on Optimal Transport

We briefly discuss some foundational work in the regularity theory of optimal transport. Optimal transport is an enormous and very active field, so this section is only intended to be the bare minimum needed for our problem, not an attempt to do the topic justice. For a more complete picture, we highly recommend the book by Villani [6], especially Chapters 10 and 12.

The following two theorems are directly from the survey paper by De Phillipis and Figalli [11]. The first theorem explains how the solutions to optimal transport problems

\footnote{For general pairs of probability spaces, such a transport map need not exist. However, in our setting Theorem 1 establishes the existence of such maps}
are given by solutions to Monge-Ampere equations and the second provides a sufficient condition for a $C^2$ estimate on the solution.

**Theorem 1.** Let $X$ and $Y$ be two open subsets of $\mathbb{R}^n$ and consider a cost function $c : X \times Y \to \mathbb{R}$. Suppose that $d\mu$ is a smooth probability density supported on $X$ and that $d\nu$ is a smooth probability density supported on $Y$. Suppose that the following conditions hold:

1. The cost function $c$ is of class $C^4$ with $\|c\|_{C^4(X \times Y)} < \infty$.
2. For any $x \in X$, the map $Y \ni y \mapsto D_x c(x,y) \in \mathbb{R}^n$ is injective.
3. For any $y \in Y$, the map $X \ni x \mapsto D_y c(x,y) \in \mathbb{R}^n$ is injective.
4. $\det(D_{x,y} c)(x,y) \neq 0$ for all $(x,y) \in X \times Y$.

Then there exists a $c$-convex function $u : X \to \mathbb{R}$ such that the map $T_u : X \to Y$ defined by $T_u(x) := c - \exp_x(\nabla u(x))$ is the unique optimal transport map sending $\mu$ onto $\nu$. Furthermore, $T_u$ is injective $d\mu$-a.e.,

$$|\det(\nabla T_u(x))| = \frac{d\mu(x)}{d\nu(T_u(x))} \quad d\mu - a.e.,$$

and its inverse is given by the optimal transport map sending $\nu$ onto $\mu$.

Theorem 1 uses the $c$-exponential map, denoted $c - \exp_x$, which is defined as follows.

**Definition (c-exponential map).** For any $x \in X, y \in Y, p \in \mathbb{R}^n$, the $c$-exponential map satisfies the following identity.

$$c - \exp_x(p) = y \iff p = -D_x c(x,y).$$

For the regularity theory for optimal transport, the breakthrough work was done by Ma, Trudinger and Wang [9]. Their paper gave three conditions that ensure $C^2$ regularity to the solutions of Monge-Ampere equations. For this work, we use a slightly different version of these conditions which we can use for our cost function. In this theorem, $c_{I,J} = \partial_{x^I} \partial_{y^J} c$ for multi-indices $I$ and $J$. Furthermore, we denote $c^{i,j}$ is the matrix inverse of the mixed derivative $c_{i,j}$.

**Theorem 2.** Suppose that $c : X \times Y \to \mathbb{R}$ satisfies the hypothesis of the previous theorem, and that the densities $d\mu$ and $d\nu$ are bounded away from zero and infinity on their respective supports $X$ and $Y$. Suppose further that the following holds.

1. $X$ and $Y$ are smooth.
(2) $D_x c(x, Y)$ is uniformly convex for all $x \in X$.
(3) $D_y c(X, y)$ is uniformly convex for all $y \in Y$.
(4) The following condition (more commonly known as MTW(0)) holds:
For all vectors $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$, the following inequality holds.
$$\sum_{i,j,k,l,p,q,r,s} (c_{ij,p}c_{p,q}c_{q,rs} - c_{ij,rs})c^{r,k}c^{s,l}\xi_i\xi_j\eta_k\eta_l \geq 0$$

Then $u \in C^\infty(X)$ and $T : X \to Y$ is a smooth diffeomorphism, where $T(x) = c - \exp_x(\nabla u(x))$.

We will discuss the assumptions of Theorem 2 in a bit more detail. The first condition is self-explanatory, while the second and third define the proper notions of convexity, which are necessary to establish regularity of optimal transport [16]. The importance of convex support was first recognized by Caffarelli in the context of Wasserstein distances. It took some time to recognize the correct notion of convexity for more general cost functions. To explain this in detail, we define the notion of $c$-convexity for sets [9].

**Definition (c-segment).** A $c$-segment in $X$ with respect to a point $y$ is a solution set $\{x\}$ to $D_y c(x, y) = z$ for $z$ on a line segment in $\mathbb{R}^n$. A $c^*$-segment in $Y$ with respect to a point $x$ is a solution set $\{y\}$ to $D_x c(x, y) = z$ for $z$ on a line segment in $\mathbb{R}^n$.

**Definition (c-convexity).** A set $E$ is $c$-convex relative to a set $E^*$ if for any two points $x_0, x_1 \in E$ and any $y \in E^*$, the $c$-segment relative to $y$ connecting $x_0$ and $x_1$ lies in $E$. Similarly we say $E^*$ is $c^*$-convex relative to $E$ if for any two points $y_0, y_1 \in E^*$ and any $x \in E$, the $c^*$-segment relative to $x$ connecting $y_0$ and $y_1$ lies in $E^*$.

Following the convention of [16], we denote $c$-convexity in the following way in Theorem 2. For all $x \in E$, the set $D_x c(x, E^*)$ is convex.

The fourth condition in Theorem 2 regards the sign of the so called MTW (short-hand for Ma-Trudinger-Wang) tensor. As suggested, this expression is in fact tensorial (coordinate-invariant) and transforms quadratically in $\eta$ and $\xi$. It is highly non-linear and non-local in the cost function. For optimal transport with the Riemannian distance squared, Loeper gave some insight into the behavior of the MTW tensor and showed that the tensor on the diagonal is proportional to the sectional curvature [16]. However, it is still not fully understood.
We should note that the $MTW(0)$ condition is a weaker version of the $MTW(\kappa)$ condition, which states that for any orthogonal vectors $\eta$ and $\xi$, $MTW(\eta, \xi) > \kappa |\eta|^2|\xi|^2$ for $\kappa > 0$. Ma, Trudinger and Wang’s original work used the $MTW(K)$, and this stronger assumption is needed to obtain regularity results for rough data [10]. The logarithmic cost is $MTW(0)$, which is why we focus on the weaker condition.

We mention two other important results before moving one. Firstly, Loeper proved that the $c$-convexity and non-negativity of the MTW tensor are essentially necessary conditions to prove regularity of optimal transport [16]. Secondly, De Phillipis and Figalli showed that while the optimal transport maps may be singular, the set of singularities is small in a precise sense [12].

3. The regularity of optimal transport with logarithmic cost

We now turn our attention back to the cost function $c(1)$ defined in the introduction. For this cost function, the latter terms do not affect optimal transport at all so only the first term is relevant to understanding optimal transport. To understand the potential regularity of optimal transport with cost $c$, we must first compute the associated MTW tensor. Somewhat surprisingly, we found the following simple formula for the MTW tensor.

\begin{equation}
MTW(\eta, \xi) = 2\langle \eta, \xi \rangle^2
\end{equation}

Here, $v$ and $w$ are two vectors in $\mathbb{R}^{n-1}$. The calculation of this tensor is computationally very intensive, so we recommend using Mathematica to verify it. We were only able to compute it explicitly in low dimensions ($n \leq 6$). For $n = 6$, it took our computer several minutes to perform the calculation. We have created a Mathematica notebook to compute the MTW tensor for different cost functions which is available online [13].

To rigorously prove this identity in all dimensions, it is an extremely tedious exercise of index juggling. To simplify the computation, one can instead compute the MTW tensor for the cost $c$ defined as

\[c(x, y) = \log(1 + \sum_{i=1}^{n} x_i \cdot y_i)\]

Since this cost can be obtained from our original cost by a change of coordinates, its MTW tensor is the same. To assist with the calculations, note that the inverse Hessian $c^{i,j}$ satisfies the following simple formula.
\[ c_{ij} = (1 + x \cdot y)(\delta_{ij} + x_j y_i) \]

The identity (2) shows that the MTW tensor is both non-negative definite and constant (it does not depend on either \( x \) or \( y \)). It vanishes whenever \( v \perp w \), which shows that this cost function satisfies the MTW(0) condition. This came as quite a surprise as costs with non-negative MTW tensor are of some interest in optimal transport (for instance, see the Conclusions and Open Problems chapter of [6]).

To apply Theorem 2 to the transport problem with this cost, we must also be able to determine when the support of a probability distribution is cost convex. Solving for the \( c \)-exponential map and setting \( p_n = 1 - \sum_{j=1}^{n-1} p_j \), we find the following

\[ y_i = x_i + \log \left( \frac{(1 - \sum_j p_j)}{p_i} \right) = x_i - \log \left( \frac{p_i}{p_n} \right) \]

Here, \( p \in \Delta^n \) is an element of the probability simplex and the subscripts denote components.

Recall that a set \( Y \) is relatively \( c \)-convex when for any \( x \in X \) and \( y_1, y_2 \in Y \), the \( c \)-segment connecting \( y_1 \) and \( y_2 \) is contained entirely in \( Y \). For this cost function, the relative \( c \)-convexity of \( Y \) is independent of the set \( X \), which merely acts to translate points. Therefore, we want to find conditions on a set \( Y \) so that if \( y_1 \) and \( y_2 \) are points in \( Y \), the connecting \( c \)-segment (which is the linearly interpolation of the \( p_i \)'s) lies in \( Y \). It turns out that this notion of convexity has a natural interpretation and is well understood in the context of information geometry.

If we consider \( Y \) to be a smooth subset of natural parameters of the multinomial distribution, then the \( p_i \)'s exactly correspond to the associated probabilities. As such, \( Y \) is \( c \)-convex if and only if it is geodesically convex with respect to the so-called mixture connection. In more elementary terms, \( Y \) is \( c \)-convex when it corresponds to a convex subset of the probability simplex (recall that \( Y \) is originally a subset of the natural parameters of the multinomial distribution). The analysis for \( X \) is similar, and shows the same result. It is straightforward to verify the other hypotheses of Theorems 1 and 2, which implies the following.

**Theorem 3.** Suppose \( X \) and \( Y \) smooth bounded subsets of the natural parameters of the multinomial distribution. Suppose further that both \( X \) and \( Y \) are strictly geodesically convex with respect to the mixture connection (i.e. \( X \) and \( Y \) are strictly convex when
viewed as subsets of the probability simplex). Let \( d\mu \) and \( d\nu \) be smooth probability densities supported on \( X \) and \( Y \) and bounded away from zero and infinity on their support. Let \( T_u \) be the \( c \)-optimal transport map carrying \( \mu \) to \( \nu \) as in Theorem 1.

Then there is a constant \( C \) so that \( \|u\|_{C^2} < C \).

Once we have the \( C^2 \) estimate, we can linearize the Monge-Ampere equation at \( u \) and use standard elliptic bootstrapping. This yields estimates of all orders, so implies that \( T_u \) is smooth. Translating this into the setting of Pal and Wong’s paper [2] on the probability simplex, we have the following corollary.

**Corollary 4.** Let \( f \) and \( g \) be smooth probability densities supported on compact strictly convex subsets \( X \) and \( Y \) of the probability simplex. Suppose further that \( \mu \) and \( \nu \) are bounded away from zero and infinity on their support. Let \( \hat{c}(p,q) \) be the cost function given by

\[
\hat{c}(p,q) = \log \left( \frac{1}{n} \sum_{i=1}^{n} \frac{q_i}{p_i} \right) - \frac{1}{n} \sum_{i=1}^{n} \log \frac{q_i}{p_i}.
\]

Then the optimal transport map \( T_u \) taking \( \mu \) to \( \nu \) is smooth.

This answers the question of regularity raised after Theorem 17 [2] for \( t = 1 \). We should note that this can also use this to address the smoothness of the displacement interpolation. Since the displacement interpolation as defined in their work linearly interpolates the log-convex function, the displacement interpolation is also smooth.

This does not show that \( \text{supp } \mu_t \) is \( c \)-convex whenever \( \text{supp } \mu \) and \( \text{supp } \nu \) are. We leave this as an open question, but note that counterexamples to this have been found for Wasserstein optimal transport [14].

3.1. **Relation to the reflector antennae problem.** At this point, we have shown that if we consider the logarithmic cost function on the probability simplex, we can use standard theory to establish the regularity of optimal transport with this cost. After a change of variables, the cost \( c \) can be rewritten as

\[
c(x, y) = \log \left( 1 + \sum_{i=1}^{n-1} x_i y_i \right).
\]

This is similar to the well-known cost function \( \log \left( 1 - \sum_{i=1}^{n-1} x_i y_i \right) \), which was first introduced in order to study the reflector antennae problem [19] and has found applications in conformal geometry. For the reflector antennae problem, an extensive regularity
theory is known (for example, see [8] and [10]) and it was this precise problem that led to the discovery of the MTW tensor. We expect that many of those results can be directly adapted to the logarithmic cost.

There is an important difference between the logarithmic cost and the reflector antennae cost in terms of their domains. Traditionally, the reflector antennae problem considers subsets of the unit sphere. Some authors have also considered the reflector antennae cost as the restriction of the 0-power cost$\mathbb{S}^2$ to the sphere. Meanwhile, the logarithmic cost is defined on the probability simplex. It is worth noting that the probability simplex with the Fisher-Rao metric is the positive orthant of a hypersphere. However, if we rewrite our cost function in terms of the natural embedding to the sphere, then it is distinct from the reflector antennae cost. As such, it seems worthwhile to consider these two costs as being genuinely distinct, though their similarity may underlie a deeper connection.

For general cost functions, the MTW tensor is very complicated as it is both non-local and highly non-linear in the cost function. As such, many of the results establishing $MTW(0)$ are very deep and difficult (for instance, [17][18]). By contrast, showing that this particular cost function is $MTW(0)$ is much more straightforward, if tedious. We are interested in finding a geometric explanation why this formula holds without appealing to this calculation, and leave that for further exploration.

4. Optimal Transport and Information Geometry

Both optimal transport and the study of statistical divergences are thriving branches of research. In recent years, there has been quite a bit of work merging these two fields and we will mention some of the important contributions to the field. The pioneering work in this direction is due to Otto [20], who provided new insights into optimal transport theory and its relation to partial differential equations. Léonard discovered connections between the Schrödinger problem and optimal transport with Wasserstein distances [21]. In [22], Gentil, Léonard and Ripani considered optimal transport using entropic costs, which are similar to the costs that we consider in this paper. A running theme of their work is to study the behavior of displacement interpolation. For brevity, we have not made use of this approach in this paper but it is an important part of modern research into optimal transport.

$^2$This cost is defined as $-\log |\theta - \phi|$ and has positive MTW tensor [7].
In the past few years, statistical distances interpolating between entropic notions of distance and the Wasserstein distance were independently introduced by at least three different groups [23] - [26]. As all of these groups had different problems and applications in mind for considering these statistical metrics, this seems to be a very promising direction of future research.

The work of Khesin, Lenells, Misiolek, and Preston relates non-parametric information geometry and optimal transport in terms of diffeomorphism groups [15]. Apart from the analysis in their work, this paper also provides a natural infinite dimensional version of non-parametric information geometry on a compact space without needing to apply the machinery of Orlicz spaces. We found this paper very helpful to our understanding of non-parametric information geometry.

We should emphasize that our work is considering optimal transport on a statistical manifold, which is a different set up than most of the work relating optimal transport and information theory. Information geometry and optimal transport traditionally provide two alternative ways of comparing probability distributions (as in [15]). In this approach, we are studying optimal transport of probability densities on a parametrized space of probability measures. Implicitly, there are three different spaces in this construction.

\[
\begin{align*}
\mathcal{X} &\rightarrow \\
\mathcal{M} &\rightarrow \\
\mathcal{P}(\mathcal{M}), \mathcal{C} &\rightarrow \\
(P(M),C) &\rightarrow \\
M & \rightarrow \\
X & 
\end{align*}
\]

In this diagram, \(X\) is the sample space and \(M\) is a statistical manifold of probability measures on \(X\). For our purposes, \(X\) is a finite set and \(M\) is the probably simplex. We have drawn \(M\) as a torus to suggest that this can be applied to more general statistical manifolds. Finally \((P(M),C)\) is the space of probability measures on \(M\) with the statistical divergence \(C\) induced from optimal transport with cost function \(c\). This will always be infinite-dimensional in any case of interest.

When one uses the logarithmic as a cost in terms of the natural parameters of the multinomial, computing the \(c\)-exponential directly involves the dual coordinates, which are precisely the probabilities. As such, information geometry seems intrinsically linked to understanding the behavior of optimal transport with this cost. It is worth noting
that if we compute the MTW tensor for the Kullback-Liebler divergence (or relative entropy), it identically vanishes. The relative entropy is dual to the logarithmic, so this might provide a clue to this phenomena, but also shows that dualizing cost functions does not preserve the MTW tensor.

It is worth mentioning that Pal and Wong study the information geometry of a more general $\alpha$-family of logarithmic divergences. In [5], Wong studies the following costs in the context of this $\alpha$-family:

$$c^{(\alpha)}(x, y) = \frac{1}{\alpha} \log(1 + \alpha x \cdot y)$$

When $\alpha = 1$, this is equivalent to $c$. Furthermore, if $\alpha = -1$, then this is exactly the cost in the reflector antennae problem. For any $\alpha$, $c^{(\alpha)}$ has a simple MTW tensor.

$$MTW(\eta, \xi) = 2\alpha(\langle \eta, \xi \rangle)^2$$

Furthermore, a set is relatively $c^{(\alpha)}$-convex whenever it is convex with respect to curves of the form

$$\gamma_i(t) = \log \left( \frac{p_i(t)}{(1 - \alpha) + \alpha p_n(t)} \right)$$

where all of the $p_i(t)$s are linear in $t$. When $\alpha \neq \pm 1$, we are not aware of a nice interpretation of $c^{(\alpha)}$-convexity, and leave it open to further exploration.

5. Discussion

5.1. Examples of other costs with non-negative MTW tensor. Using the Mathematica code we used to verify that the logarithmic has non-negative MTW tensor [13], we tested other cost functions to try to find examples with non-negative MTW tensor. After some experimentation, we found the the following examples. We have not found applications for these costs, but may try to do so in future work.

$$c(x, y) = \sum_{i=1}^{n} e^{x_i^2} + e^{y_i^2} - 2e^{x_i y_i}$$

This cost is a divergence, whose MTW tensor satisfies the following

$$MTW(\eta, \xi) = \sum_{i=1}^{n} \frac{e^{-x_i y_i} (1 + (1 + x_i y_i)^2)}{2(1 + x_i y_i)^3} \eta_i^2 \xi_i^2$$

So long as all of the $x_i$ and $y_i$ are positive, this is non-negative definite, so this cost also satisfies $MTW(0)$.11
Following up from the last example, this cost also has non-negative MTW tensor, although it’s quite a bit more complicated than the previous example.

In 2-dimensions, the MTW tensor for this cost is non-negative. The calculation for this cost is somewhat involved so we have provided Mathematica code to verify the non-negativity [13]. It is worth noting that the Hessian of this cost function is non-singular when both \((x_1 - y_1)\) and \((x_2 - y_2)\) are non-zero.

Many of these examples can be written as the sum of MTW non-negative costs on each coordinate. In fact, this is a simple way to generate MTW(0) costs, due to the following.

**Observation 5.** Suppose that \(X_1, Y_1 \subset \mathbb{R}^m\) and \(X_2, Y_2 \subset \mathbb{R}^n\) and that \(c_1 : X_1 \times Y_1 \to \mathbb{R}\) and \(c_2 : X_2 \times Y_2 \to \mathbb{R}\) are two cost functions satisfying the MTW(0) condition. Then the cost function

\[
c_1 + c_2 : (X_1 \times X_2) \times (Y_1 \times Y_2) \to \mathbb{R}
\]

\[
(x_1, x_2) \times (y_1, y_2) \mapsto c_1(x_1, y_1) + c_2(x_2, y_2)
\]

also satisfies the MTW(0) condition.

This observation follows directly from the definition of the MTW tensor. It is worth noting that \(c_1 + c_2\) will not satisfy MTW(\(\kappa\)) for \(\kappa > 0\), so this cannot be used to generate examples with positive cost-sectional curvature.

5.2. **A question about curvature.** Going forward, we are interested whether we can use the MTW tensor to define a notion of “sectional curvature” for the logarithmic cost or other cost functions in information geometry. In the spirit of Loeper’s work for the Wasserstein distances, it may be possible to define a curvature tensor of a statistical divergence by computing its MTW tensor on the diagonal.

We do not currently have a proposed interpretation for such a tensor; it would be distinct from the curvature of the Fisher metric, but would hopefully still have statistical
meaning. It is worth noting that MTW tensor for the logarithmic identically vanishes whenever $\eta \perp \xi$, so if such an analogy holds, the curvature must be zero. The fact that the probability simplex is dually flat suggests that there may be an analogous result.

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