Hilbert-Schmidt Operators vs. Integrable Systems of Elliptic Calogero-Moser Type. II. The $A_{N-1}$ Case: First Steps

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Abstract: This is the second part of a series of papers concerning Hilbert-Schmidt integral operators acting on the Hilbert spaces associated with elliptic Calogero-Moser type Hamiltonians. We present an explicit diagonalization of special Hilbert-Schmidt operators arising in the free $A_{N-1}$ case, and a spectral structure analysis of the commuting family of Hilbert-Schmidt operators associated with the general $A_{N-1}$ case.

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1. Introduction

In a previous paper [1] we have laid the algebraic groundwork for the Hilbert space analysis to which further papers in this series are devoted. It also contains an introduction to the general setting at issue, and collects some information on the elliptic gamma function and allied functions. In this paper we will use this material without further ado, referring back to sections and equations in [1] by using a prefix I.

As explained in Section I.1, for the $A_{N-1}$ case the main idea to be substantiated is that the orthonormal base of joint eigenvectors of the commuting normal HS
(Hilbert-Schmidt) operators consists of joint eigenfunctions of the commuting Hamiltonians of elliptic Calogero-Moser type with real eigenvalues. The simplest case in which this scenario can be tested is the free case. One of the principal results of this paper is that the conjectured scenario is indeed borne out in this case.

More precisely, in Sect. 2 we first show that there exists a special orthonormal base for $\mathcal{H}$ on which the free $A_{N-1}$ Hamiltonians take real eigenvalues, hence yielding an interpretation as self-adjoint operators. (In physical terms, this interpretation amounts to free fermions with $N$-dependent boundary conditions.) As we prove next, this base is also the base on which the commuting HS operators are diagonal. Once this has been established, it follows that the definition of the Hamiltonians as self-adjoint operators via this special base is the unique one ensuring commutativity with the HS operators. Indeed, by contrast to the unbounded Hamiltonians, the latter bounded normal operators have a joint eigenvector base that is unique up to phase choices.

For a quite special subfamily of the commuting HS operators we are able to find also the eigenvalues in closed form, cf. Subsect. 2.3. This state of affairs is intimately connected to the elliptic Cauchy identity (Frobenius identity): It can be exploited to arrive at the eigenvalues. A slightly more involved reasoning, however, also yields the eigenvalues directly, hence yielding one more proof of the Frobenius identity (earlier ones occurring in [2–6]).

Section 3 concerns structural features of the general $A_{N-1}$ case. By contrast to the $BC_N$ case, the choice of Hilbert space for the $A_{N-1}$ Hamiltonians is not unique. Physically speaking, one may view the particles as moving on the line or on a ring; in the latter case one still has a choice between distinguishable and indistinguishable particles. Thus one arrives at three Hilbert spaces. As detailed in Subsect. 3.1, they differ in their center-of-mass description, which gives rise to $N$ distinct internal/reduced Hilbert spaces.

Already for $N = 2$ it is crucial to understand how the HS operators behave with respect to the external/internal decomposition. In Subsect. 3.2 we study this for arbitrary $N$ in a quite general setting. Indeed, to arrive at a rather detailed picture of salient spectral features, only a few general properties of the HS operators are essential. The latter serve as the assumptions for the main result of this subsection (Theorem 3.1), which clarifies the structure of the eigenvectors and yields information on vanishing eigenvalues.

Section 3 is concluded with Subsect. 3.3, in which we briefly sketch a more Lie-algebraic perspective for the various decompositions.

2. The Free $A_{N-1}$ Case: An Explicit Diagonalization

2.1. Preliminaries. From (1.1) it is immediate that for $g = 1$ we are dealing with a free Hamiltonian. More generally, in this case the nonrelativistic commuting Hamiltonians may be taken to be

$$H_{l,0}(x) = (-i)^l \sum_{1 \leq j_1 < \ldots < j_l \leq N} \frac{\partial}{\partial x_{j_1}} \cdots \frac{\partial}{\partial x_{j_l}}, \quad l = 1, \ldots, N. \tag{2.1}$$

The choice $g = 1$ in (2.8) also leads to free (constant coefficient) Hamiltonians. This is not immediate from (2.1), but can be seen as follows. First, from (1.13), (2.3), (1.16)
and the $G$-A$\Delta$Es I(2.10) one readily calculates

\[ H_{l,\delta}(\mu; x) = \sum_{|I|=l} \prod_{m \notin I} \left( \frac{R_{\delta}(x_m - x_n - \mu + ia_{\delta}/2) R_{\delta}(x_m - x_n - ia_{\delta} - \mu - ia_{\delta}/2)}{R_{\delta}(x_m - x_n + ia_{\delta}/2) R_{\delta}(x_m - x_n - ia_{\delta} - i a_{\delta}/2)} \right)^{1/2} \times \prod_{m \in I} \exp(-ia_{-\delta} \partial_{x_m}). \tag{2.2} \]

Using I(2.22), this implies

\[ H_{l,\delta}(ia_{-}; x) = \exp(-l(N - l)ra_{-\delta}) \sum_{|I|=l} \prod_{m \in I} \exp(-ia_{-\delta} \partial_{x_m}). \tag{2.3} \]

Thus the $\mu$-value

\[ \mu = ia_{-} \tag{2.4} \]

leads to free Hamiltonians, as announced. Obviously, the choice $\mu = ia_{+}$ yields the same Hamiltonians.

In the remainder of this section, the choice (2.4) is understood, so that we suppress $\mu$-dependence from now on. Clearly, I(2.3) and I(2.6) reduce to

\[ c(x) = \prod_{j < k} \frac{1}{R_{+}(x_j - x_k + ia_{+}/2)}, \tag{2.5} \]

\[ S_{\xi}(x, y) = \prod_{j, k} \frac{1}{R_{+}(x_j - y_k + \xi)}. \tag{2.6} \]

Since these functions do not depend on $a_{-}$, they are equal to their nonrelativistic limits, cf. I(2.8)–I(2.10). For $x, y$ in the fundamental domain $F$ I(1.7), the function $\Psi_{\xi}$ corresponding to $S_{\xi}$ (cf. I(1.14) and I(1.16)) can be written

\[ \Psi_{\xi}(x, y) = p_{+}^{N(N-1)} \prod_{j < k} \frac{s_{+}(x_j - x_k)s_{+}(y_j - y_k)}{\prod_{j, k} R_{+}(x_j - y_k + \xi)}, \tag{2.7} \]

where we used I(1.21) to trade $R$ for $s$ in the numerator. We recall that its relation to the Hamiltonians is given by

\[ (H_{l,\delta}(x) - H_{l,\delta}(-y))\Psi_{\xi}(x, y) = 0, \quad l = 1, \ldots, N, \quad \delta = +, -, 0, \tag{2.8} \]

cf. Sect. I.2 and [7].

From now on we restrict $\xi$ by requiring

\[ |\Im \xi| < a_{+}/2. \tag{2.9} \]

This ensures that $\Psi_{\xi}(x, y)$ is smooth for $x, y \in \mathbb{R}^{N}$. With $F$ given by I(1.7), we therefore have

\[ \int_{F \times F} |\Psi_{\xi}(x, y)|^{2} dx dy < \infty. \tag{2.10} \]
Hence the integral operator

\[(I_{\xi} f)(x) \equiv \int_{F} \Psi_{\xi}(x, y) f(y) dy, \quad f \in \mathcal{H},\]  

yields a HS operator on the Hilbert space \(\mathcal{H} = L^2(F, dx)\).

In this section we study the family of HS operators \(I_{\xi}\) in more detail. To this end it is convenient to identify \(\mathcal{H}\) with the antisymmetric subspace of the ‘big’ Hilbert space

\[\mathcal{H}_b \equiv L^2(\mathbb{T}^N; dx), \quad \mathbb{T} \equiv (-\pi/2r, \pi/2r].\]  

(2.12)

Specifically, denoting by \(P_{\sigma}\) the permutation operator

\[(P_{\sigma} f)(x) \equiv f(\sigma^{-1}(x)), \quad \sigma \in S_N, \quad f \in \mathcal{H}_b,\]  

(2.13)

the antisymmetric subspace \(\mathcal{H}_{b,a}\) equals the range of the operator \(\sum_{\sigma} (-)^{\sigma} P_{\sigma}\), and the identification operator

\[I : \mathcal{H}_{b,a} \rightarrow \mathcal{H}, \quad f(x) \mapsto (N!)^{1/2} f(x), \quad x \in F,\]  

(2.14)

yields an isometric isomorphism with inverse

\[(I^{-1} g)(x) = (-)^{\sigma} (N!)^{-1/2} g(\sigma^{-1}(x)), \quad x \in \sigma(F), \quad g \in \mathcal{H}.\]  

(2.15)

Since the kernel \(\Psi_{\xi}(x, y)\) of \(I_{\xi}\) is antisymmetric in \(x\) and \(y\), we have

\[(I^{-1} I_{\xi} f)(x) = \frac{1}{N!} \int_{\mathbb{T}^N} \Psi_{\xi}(x, y) f(y) dy, \quad f \in \mathcal{H}_{b,a}.\]  

(2.16)

(Clearly, the rhs yields a well-defined operator on \(\mathcal{H}_b\) as well, but this operator annihilates the orthocomplement \(\mathcal{H}_{b,a}^\perp\) and maps \(\mathcal{H}_{b,a}\) into itself.) For the remainder of this section we work with the transformed operator

\[J_{\xi} \equiv I^{-1} I_{\xi} I\]  

(2.17)

on \(\mathcal{H}_{b,a}\).

Consider now the kernel of the HS operator \(N!^2 J_{\xi_1} J_{\xi_2}\). It is given by

\[\int_{\mathbb{T}^N} \Psi_{\xi_1}(x, z) \Psi_{\xi_2}(z, y) dz = \int_{\mathbb{T}^N} \Psi_0(x, z - \xi_1 \zeta) \Psi_0(z + \xi_2 \zeta, y) dz, \quad \zeta \equiv (1, \ldots, 1),\]  

(2.18)

cf. (2.7). The integrand is \(\pi/r\)-periodic in \(z_1, \ldots, z_N\), and so we can take \(z_j \rightarrow z_j - \xi_2 + \xi_1\) and shift contours, yielding

\[\int_{\mathbb{T}^N} \Psi_0(x, z - \xi_2 \zeta) \Psi_0(z + \xi_1 \zeta, y) dz = \int_{\mathbb{T}^N} \Psi_{\xi_2}(x, z) \Psi_{\xi_1}(z, y) dz.\]  

(2.19)

This is the kernel of \(N!^2 J_{\xi_2} J_{\xi_1}\), so we have proved that \(J_{\xi_1}\) and \(J_{\xi_2}\) commute:

\[\{J_{\xi_1}, J_{\xi_2}\} = 0.\]  

(2.20)

(Of course, this implies that \(I_{\xi_1}\) and \(I_{\xi_2}\) commute as well, but this cannot be seen as easily.) It is also clear that the adjoint of \(J_{\xi}\) equals \(J_{-\xi}\):

\[J_{-\xi}^* = J_{\xi}.\]  

(2.21)

Hence we are dealing with a family of commuting HS operators, which are self-adjoint for \(\xi \in ia_+(-1, 1)/2\) and normal otherwise.
2.2. The orthonormal base of joint eigenvectors. It follows from the spectral theorem for normal HS operators that \( \mathcal{H}_{b,a} \) has an ONB (orthonormal base) of joint eigenvectors for the family \( \{ \mathcal{J}_\xi \} \), but the spectral theorem yields no explicit information on the eigenvectors and eigenvalues. The purpose of this subsection is to make the eigenvector ONB explicit. (More information on eigenvalues can be found in Subsects. 2.3 and 2.4.) In fact, we will show that the ONB is shared by a family that is larger, namely, the family obtained by allowing extra factors of the form \( I(2.7) \) in the kernel.

To begin with, take \( N = 3, 5, 7, \ldots \). Obviously, the functions

\[
b_k(x) \equiv \left( \frac{r}{\pi} \right)^{N/2} \exp(2irx \cdot k), \quad k \in \mathbb{Z}^N, \quad (N \text{ odd}),
\]

are an ONB for \( \mathcal{H}_b \) (2.12). Introducing the vector

\[
\rho \equiv (N - 1, N - 3, \ldots, -N + 1)/2,
\]

it follows that the functions

\[
en(x) \equiv (N!)^{-1/2} \sum_{\sigma \in S_N} (-)^\sigma b_{n+\rho}(\sigma(x)), \quad n \in \mathbb{Z}_{\geq}^N,
\]

where

\[
\mathbb{Z}_{\geq}^N \equiv \{ n \in \mathbb{Z}^N \mid n_1 \geq \cdots \geq n_N \},
\]

are an ONB for the antisymmetric subspace \( \mathcal{H}_{b,a} \).

Taking next \( N = 2, 4, 6, \ldots \), the functions

\[
b_k(x) \equiv \left( \frac{r}{\pi} \right)^{N/2} \exp(2irx \cdot k), \quad k \in \left( \mathbb{Z} + \frac{1}{2} \right)^N, \quad (N \text{ even}),
\]

are an ONB for \( \mathcal{H}_b \). Defining once again \( e_n(x) \) via (2.23)–(2.25), we obtain an ONB for \( \mathcal{H}_{b,a} \) for even \( N \).

It should be stressed that at this stage it is not at all clear that the \( \mathcal{H}_{b,a} \)-ONB we have just defined has a relation to the family \( \mathcal{J}_\xi \). But as we shall now show, this ONB is in fact an ONB of joint eigenvectors for \( \mathcal{J}_\xi \).

First of all, the dependence on \( \xi \) can be easily understood. Indeed, we have

\[
(\mathcal{J}_\xi e_n)(x) = \frac{1}{N!} \int_{\mathbb{T}^N} \Psi_\xi(x, y)e_n(y)dy,
\]

and the integrand is \( \pi/r \)-periodic in \( y_1, \ldots, y_N \). Thus we can change variables \( y_j \to y_j + \xi \), and shift the \( y_j \)-contours over \( \mathbb{R}^N \) to deduce

\[
\mathcal{J}_\xi e_n = \exp(2ir\xi \sum_{j=1}^N n_j)\mathcal{J}_0 e_n.
\]

Therefore, it suffices to prove that \( \{ e_n \} \) is an ONB of eigenvectors for \( \mathcal{J}_0 \).

The key to the proof is the relation of the \( \mathcal{J}_0 \)-kernel to the above free Hamiltonians, as encoded in (2.8). To explain this in more detail, we first note that the functions \( e_n(x) \) are
joint eigenfunctions for the Hamiltonians, with eigenvalues that are easily calculated. Specifically, we have

\[ H_{l, \delta} e_n(x) = E_{l, \delta, n} e_n(x), \quad j = 1, \ldots, N, \quad \delta = +, -, 0, \tag{2.29} \]

where

\[ E_{l, \pm, n} = \exp(-l(N - l)ra_\pm) \sum_{1 \leq i_1 < \cdots < i_l \leq N} \exp(2(n + \rho)i_r a_\pm) \times \cdots \exp(2(n + \rho)i_1 a_\pm), \tag{2.30} \]

\[ E_{l, 0, n} = (2r)^l \sum_{1 \leq i_1 < \cdots < i_l \leq N} (n + \rho)i_1 \cdots (n + \rho)i_l. \tag{2.31} \]

Viewing the functions \( e_n(x) \) as an ONB for \( \mathcal{H}_{b,a} \) (by restricting \( x \) to \( T^N \)), it follows that we can associate self-adjoint operators \( \hat{H}_{l, \delta} \) to \( H_{l, \delta} \) via (2.29), linear extension, and taking closures.

The point is now that the relations (2.8) can be exploited to show that \( J_0 \) commutes with the self-adjoint unbounded operators \( \hat{H}_{l, 0} \) just defined. Taking this for granted, we deduce

\[ \hat{H}_{l, 0} J_0 e_n = E_{l, 0, n} J_0 e_n, \quad l = 1, \ldots, N. \tag{2.32} \]

Therefore, \( J_0 e_n \) is an \( \hat{H}_{l, 0} \)-eigenvector with eigenvalue \( E_{l, 0, n} \). Since the eigenvalue vectors \( (E_{1, 0, n}, \ldots, E_{N, 0, n}) \) separate the points of \( \mathbb{Z}_+^N \), it follows that \( J_0 e_n \) is proportional to \( e_n \). Thus we have

\[ J_0 e_n = \lambda_n e_n, \quad n \in \mathbb{Z}_+^N, \quad \lambda_n \in \mathbb{R}. \tag{2.33} \]

We proceed to fill in the details of the reasoning just sketched.

**Theorem 2.1.** The ONB \( \{e_n\}, n \in \mathbb{Z}_+^N, \) of \( \mathcal{H}_{b,a} \), defined by (2.24), consists of joint eigenvectors for the HS operators \( J_\xi \) given by

\[ (J_\xi f)(x) \equiv \frac{1}{N!} \int_{T^N} \Psi_\xi(x, y) f(y) dy, \quad f \in \mathcal{H}_{b,a}, \tag{2.34} \]

where \( \Psi_\xi \) is defined by (2.7). Specifically, we have

\[ J_\xi e_n = \lambda_n(\xi) e_n, \quad n \in \mathbb{Z}_+^N, \tag{2.35} \]

where the eigenvalues satisfy

\[ \lambda_n(\xi) = \exp(2i \xi \cdot (\xi, n)) \lambda_n, \quad \lambda_n \equiv \lambda_n(0) \in \mathbb{R}. \tag{2.36} \]

**Proof.** Since we have already handled the \( \xi \)-dependence in relation to \( e_n \) (see (2.28)), it remains to prove (2.33). Consider the functions \( (J_0 e_n)(x) \). They extend to functions that are analytic in the polystrip \( S_{\alpha /2} \), where

\[ S_\alpha \equiv \{ x \in \mathbb{C}^N \mid |3x_j| < \alpha, \quad j = 1, \ldots, N \}, \quad \alpha > 0. \tag{2.37} \]

These functions are also antisymmetric and \( \pi / r \)-periodic/\( \pi / r \)-antiperiodic in \( x_1, \ldots, x_N \) for \( N \) odd/even. (Recall \( s_+(x) \) is \( \pi / r \)-antiperiodic, while \( R_+(x) \) is \( \pi / r \)-periodic.)
We now define the following space of functions:
\[ D_\alpha \equiv \{ f(x) \text{ analytic in } S_\alpha \mid f(x) \text{ antisymmetric and } \pi/r-\text{periodic/}\pi/r-\text{antiperiodic in } x_1, \ldots, x_N \text{ for } N \text{ odd/even} \}. \tag{2.38} \]

The subspace of \( H_{b,a} \) obtained by restricting functions in \( D_\alpha \) to \( T_N \subset \mathbb{R}^N \) will be denoted again by \( D_\alpha \). This subspace is dense in \( H_{b,a} \), since we have \( e_n \in D_\alpha \) for all \( n \in \mathbb{Z}^N \). Moreover, \( D_\alpha \) belongs to the domains of the self-adjoint operators \( \hat{H}_{l,0} \), indeed, the PDOs \( H_{l,0} \) have a symmetric action on \( D_\alpha \), as follows from integration by parts; hence their action coincides with that of \( \hat{H}_{l,0} \).

Now as we have already seen, we have
\[ J_0 e_n \in D_{a+}/2, \quad \forall n \in \mathbb{Z}^N, \tag{2.39} \]
and, more generally,
\[ J_0 D_{a+}/2 \subset D_{a+}/2. \tag{2.40} \]

Using (2.8) and integration by parts, we now obtain
\[ (f, \hat{H}_{l,0} J_0 e_n) = (J_0 f, \hat{H}_{l,0} e_n) = E_{l,0,n}(J_0 f, e_n) = E_{l,0,n}(f, J_0 e_n), \quad \forall f \in D_{a+}/2. \tag{2.41} \]

As a consequence, we have
\[ \hat{H}_{l,0} J_0 e_n = E_{l,0,n} J_0 e_n, \tag{2.42} \]
so \( J_0 e_n \) is an eigenvector of \( \hat{H}_{l,0} \) with eigenvalue \( E_{l,0,n} \).

Consider now the characteristic polynomial
\[ P_n(\lambda) \equiv |\lambda I_N + 2r \text{diag}(n_1 + \rho_1, \ldots, n_N + \rho_N)|. \tag{2.43} \]

By (2.31), it satisfies
\[ P_n(\lambda) = \sum_{k=0}^{N-1} \lambda^k E_{N-k,0,n} + \lambda^N. \tag{2.44} \]

Assuming we have \( E_{l,0,n} = E_{l,0,m} \) for \( l = 1, \ldots, N \), we deduce \( P_n(\lambda) = P_m(\lambda) \). This implies that the roots of \( P_n \) and \( P_m \) coincide, so that \( m = n \) by the ordering restriction. Therefore, \( J_0 e_n \) must be proportional to \( e_n \), and so the proof is complete. \( \square \)

We continue with some observations connected to the above theorem and its proof. First, instead of working with the spaces \( D_\alpha \), we could just as well have chosen to work with
\[ D_0 \equiv \{ f(x) \text{ smooth on } \mathbb{R}^N \mid f(x) \text{ antisymmetric and } \pi/r-\text{periodic/}\pi/r-\text{antiperiodic in } x_1, \ldots, x_N \text{ for } N \text{ odd/even} \}. \tag{2.45} \]

Once more, the point is that the ONB vectors \( e_n \) belong to \( D_0 \) and the PDOs \( H_{l,0} \) have a symmetric action on \( D_0 \), so that \( D_0 \) belongs to the domains \( D(\hat{H}_{l,0}) \) of \( \hat{H}_{l,0} \), \( l = 1, \ldots, N \).
Second, we have introduced the spaces $\mathcal{D}_{\alpha}$ with an eye on the operators $\hat{H}_{l,\pm}$. Indeed, we have
\[ \alpha > a_\delta \Rightarrow \mathcal{D}_{\alpha} \subset D(\hat{H}_{l,\delta}), \quad l = 1, \ldots, N, \quad (2.46) \]
again by symmetry of the $H_{l,\delta}$-action on $\mathcal{D}_{\alpha}$ (as now follows by shifting the $x_j$-contours and Cauchy’s theorem). If we now choose $a_-$ small enough, we can just as well use the operators $\hat{H}_{l,-}$ to prove the theorem. To explain this, we first point out that $J_0$ and $e_n$ do not depend on $a_-$, so that we are free to choose $a_-$ smaller than $a_+/2$. Then (2.48) and contour shifts yield
\[ (f, \hat{H}_{l,-} J_0 e_n) = (J_0 f, \hat{H}_{l,-} e_n) = E_{l,-,n}(J_0 f, e_n) = E_{l,-,n}(f, J_0 e_n), \quad \forall f \in \mathcal{D}_{a+/2}, \quad (2.47) \]
as the analog of (2.41). Thus we can complete the proof as before, noting that the eigenvalue vector $(E_{1,-,n}, \ldots, E_{N,-,n})$ again separates points on $\mathbb{Z}_N$. (This follows from (2.30) by the reasoning in the last paragraph of the proof.)

Third, now that we have proved that $\{e_n\}$ is an ONB of $J_\xi$-eigenvectors, we can pin down a key functional-analytic consequence of the algebraic relations (2.8) for the special $\mu$-value (2.4). Specifically, since the family $\{J_\xi\}$ consists of normal HS operators, its eigenvector ONB is essentially unique. (Only if all of the operators have common eigenspaces of dimension greater than one, there is an ambiguity in the choice of base vectors.) By contrast, there are various ways to associate unbounded self-adjoint commuting Hilbert space operators to the free PDOs and $A/D_\alpha$s. But only when we include the ONB $\{e_n\}$ in the definition domain (yielding uniquely determined self-adjoint operators) we obtain a Hilbert space reinterpretation as self-adjoint operators commuting with $J_\xi$. This, then, is what we expect to happen more generally for $\mu \in i(0, a_+/a_-)$.

Fourth, let us consider the more general operators obtained by allowing extra factors of the form $I(2.7)$. We assume that the meromorphic function $\phi(z)$ is $\pi/r$-periodic and has no real poles. Thus we obtain once again a HS operator given by
\[ (J_\xi^{(\phi)} f)(x) \equiv \frac{1}{N!} \int_{\mathbb{T}^N} \Psi_\xi(x, y) \phi((\xi, x-y)) f(y) dy, \quad f \in H_{b,a}, \quad (2.48) \]
which satisfies
\[ (J_\xi^{(\phi)} e_n)(x) \in \mathcal{D}_0. \quad (2.49) \]
Now the identities (2.8) still hold for the modified kernel. Using again integration by parts, we therefore get
\[ (f, \hat{H}_{l,0} J_\xi^{(\phi)} e_n) = E_{l,0,n}(f, J_\xi^{(\phi)} e_n), \quad \forall f \in \mathcal{D}_0, \quad (2.50) \]
as the analog of (2.41). From this we deduce as before that $e_n$ is an eigenvector:
\[ J_\xi^{(\phi)} e_n = \lambda_n(\xi, \phi) e_n, \quad \forall n \in \mathbb{Z}_N^+. \quad (2.51) \]
This implies in particular that all of these HS operators commute. This can no longer be derived in the same way as the commutativity of the family $J_\xi$, cf. (2.19)–(2.20). On the other hand, we show in Subsect. 3.2 how these commutativity features can be understood in a far more general setting.

To proceed, we choose $\phi$ such that we can determine the eigenvalues $\lambda_n(\xi, \phi)$ in closed form.
2.3. Explicit eigenvalues for a special subfamily. Using (1.20)–(1.21), we see that the
denominator of $\Psi_1^\xi$ (2.7) can be rewritten as

$$[-ip_\star \exp(ir\xi - ra_\star)]N^2 \exp(irN(\xi, x - y))\prod_{j,k} s_+(x_j - y_k + ia_\star/2 + \xi). \quad (2.52)$$

We now recall the Frobenius identity (elliptic Cauchy identity)

$$\det(C) = s_+(\beta)^{N-1} s_+(\beta + N\gamma + (\xi, x - y)) \frac{\prod_{j<k} s_+(x_j - x_k)s_+(y_k - y_j)}{\prod_{j,k} s_+(x_j - y_k + \gamma)}, \quad (2.53)$$

where

$$C_{jk} = \frac{s_+(x_j - y_k + \gamma + \beta)}{s_+(x_j - y_k + \gamma)}, \quad j, k = 1, \ldots, N. \quad (2.54)$$

To exploit this identity, we consider the kernel

$$C_{\beta,\gamma}(x, y) = \frac{1}{N!} \left(\frac{i}{2\pi}\right)^N \frac{s_+(\beta + N\gamma + (\xi, x - y))}{s_+(\beta)} \exp[ir(1 - N)(\xi, x - y)] \times \prod_{j<k} s_+(x_j - x_k)s_+(y_k - y_j) \prod_{j,k} s_+(x_j - y_k + \gamma), \quad \exists \beta, \exists \gamma \in (0, a_\star). \quad (2.55)$$

The corresponding HS operator

$$(C_{\beta,\gamma} f)(x) = \int_{\mathbb{H}_{b,a}} C_{\beta,\gamma}(x, y) f(y) dy, \quad f \in \mathcal{H}_{b,a}. \quad (2.56)$$

is of form (2.48), where

$$\xi = \gamma - ia_\star/2, \quad (2.57)$$

$$\phi(z) = c_N \frac{s_+(\beta + N\gamma + z)}{s_+(\beta)} \exp(irz), \quad (2.58)$$

with the constant given by (cf. (2.7) and (2.52))

$$c_N = \frac{1}{N!} \left(\frac{i}{2\pi}\right)^N (-)^{N(N-1)/2} [-ip_\star \exp(ir\xi - ra_\star)]N^2 p_\star^{-N(N-1)} \exp[N^2r(i\gamma - a_\star/2)]. \quad (2.59)$$

Therefore, the vectors $\{e_n\}$ furnish an ONB of eigenvectors for $C_{\beta,\gamma}$. In our next result, we supply not only an independent proof of this fact, but also obtain the associated eigenvalues explicitly.

**Theorem 2.2.** The family of HS operators $C_{\beta,\gamma}$ (given by (2.55)–(2.56)) satisfies

$$C_{\beta,\gamma} e_n = \kappa_n e_n, \quad \forall n \in \mathbb{Z}_N, \quad (2.60)$$

where

$$\kappa_n = \exp(ir\gamma[N(N - 1) + 2(\xi, n)]/ \prod_{k=1}^N [1 - \exp(2ir\beta - 2ra_\star[N - k + n_k])]. \quad (2.61)$$
Proof. In view of (2.53), we have

$$C_{\beta, \gamma}(x, y) = \frac{1}{N!} \sum_{\sigma} (-)^{\sigma} \prod_{j=1}^{N} M(x_{\sigma(j)} - y_j), \quad (2.62)$$

with

$$M(z) \equiv \frac{is_+ (z + \gamma + \beta) \exp(ir(1 - N)z)}{2\pi s_+(\beta)s_+(z + \gamma)}. \quad (2.63)$$

Recalling the definition (2.22)–(2.26) of $e_\phi(x)$, we deduce

$$(C_{\beta, \gamma}e_n)(x) = (N!)^{-3/2} \left( \frac{r}{\pi} \right)^{N/2} \sum_{\sigma, \tau} (-)^{\sigma\tau} \int_{T^N} dy \prod_{j=1}^{N} M(x_{\sigma(j)} - y_j)$$

$$\times \exp(2iry_j(n + \rho)_{\tau(j)})$$

$$= (N!)^{-1/2} \left( \frac{r}{\pi} \right)^{N/2} \sum_{\pi} (-)^{\pi} \exp[2ir(\pi(x), n + \rho)] \prod_{k=1}^{N} I(n_k + \rho_k), \quad (2.64)$$

where

$$I(l) \equiv \int_{-\pi/2r}^{\pi/2r} dz M(-z) \exp(2irlz). \quad (2.65)$$

We proceed to calculate $I(l)$ for the pertinent $l$-values. From (2.63) we have

$$I(l) = \frac{i}{2\pi} \int_{-\pi/2r}^{\pi/2r} dz s_+(z - \beta - \gamma) s_+(z + \gamma) \exp[2ir(l + (N - 1)/2)z]. \quad (2.66)$$

Recalling the definition (2.23) of $\rho$, we see that the integrand is $\pi/r$-periodic when $l$ equals $n_k + \rho_k$. Thus we can use $I(1.23)$ and Cauchy’s theorem to relate its integral over the rectangular contour connecting $-\pi/2r, \pi/2r, \pi/2r, \pi/2r + ia_+, -\pi/2r + ia_+$ and $-\pi/2r$ to $I(l)$. This yields

$$I(l) = \frac{\exp[2ir(l + (N - 1)/2)\gamma]}{1 - \exp[-2r(l + (N - 1)/2)a_+ + 2ir\beta]}. \quad (2.67)$$

Substituting this in (2.64), we obtain (2.60)–(2.61). \qed

To continue, we add some observations connected to the above theorem. First, from (2.61) we deduce that all of the elliptic HS operators $C_{\beta, \gamma}$ have a trivial kernel, and when we let $\beta$ vary over the strip $\Im \beta \in (0, a_+)$, their eigenvalues separate the points of $\mathbb{Z}^N$. (I. e., if $\kappa_n(\beta)$ equals $\kappa_m(\beta)$ for all $\beta$ in the strip, then $n$ equals $m$.) For other choices of $\phi$, however, infinite-dimensional kernels can arise. This can be exemplified by (2.55)–(2.58): When we multiply by $s_+(\beta)$ and take $\beta$ to 0, the resulting HS operator only has eigenvalues different from 0 when we have

$$n_l = l - N, \quad l \in \{1, \ldots, N\}. \quad (2.68)$$
By (2.61) the latter are given by
\[ \kappa_n = i \exp(i r \gamma [N(N - 1) + 2(\zeta, n)])/2r \prod_{k \neq l} [1 - \exp(-2ra_+[N - k + n_k])]. \] (2.69)

Second, using the limit
\[ \lim_{a_+ \to \infty} s_+ (z) = \frac{\sin r z}{r}, \] (2.70)
which holds uniformly on \( \mathbb{C} \)-compacts, we obtain from (2.55) the trigonometric kernel
\[ C_{\beta, \gamma}^{tr}(x, y) = \frac{1}{N!} \left( \frac{ir}{2\pi} \right)^N \frac{\sin r (\beta + N \gamma + (\zeta, x - y))}{\sin(r \beta)} \exp[i r (1 - N)(\zeta, x - y)] \times \prod_{j < k} \frac{\sin r(x_j - x_k) \sin r(y_k - y_j)}{\prod_{j, k} \sin r(x_j - y_k + \gamma)}, \] \( \beta, \gamma \in (0, \infty). \) (2.71)

It yields a HS operator
\[ (C_{\beta, \gamma}^{tr} f)(x) \equiv \int \mathbb{T}_N C_{\beta, \gamma}^{tr}(x, y) f(y) dy, \quad f \in \mathcal{H}_{b, a}, \] (2.72)
satisfying
\[ C_{\beta, \gamma}^{tr} e_n = \kappa_n^{tr} e_n, \quad \forall n \in \mathbb{Z}_N^+, \] (2.73)
with
\[ \kappa_n^{tr} = 0, \quad n_N < 0, \] (2.74)
\[ \kappa_n^{tr} = \exp(i r \gamma [N(N - 1) + 2(\zeta, n)])/(1 - e^{2ir \beta}), \quad n_N = 0, \] (2.75)
\[ \kappa_n^{tr} = \exp(i r \gamma [N(N - 1) + 2(\zeta, n)]), \quad n_N > 0. \] (2.76)

(This easily follows from (2.60)–(2.61).)

Next, we take \( \beta \to i \infty \) in (2.71). The resulting HS operator has eigenvalues given by (2.74) and
\[ \kappa_n^{tr} = \exp(i r \gamma [N(N - 1) + 2(\zeta, n)]), \quad n_N \geq 0. \] (2.77)

Therefore, for \( \epsilon \to 0 \) the HS operator with kernel
\[ P_\epsilon(x, y) \equiv \frac{1}{N!} \left( \frac{r}{\pi} \right)^N \prod_{j < k} 4 \frac{\sin r(x_j - x_k) \sin r(y_j - y_k)}{\prod_{j, k} (1 - e^{-\epsilon + 2ir(x_j - y_k)})}, \quad \epsilon > 0, \] (2.78)
converges strongly to the projection on the subspace of \( \mathcal{H}_{b, a} \) that is spanned by the vectors \( e_n(x) \) with \( n_N \geq 0 \). Observe that we can also take \( \beta \to -i \infty \) in (2.71), obtaining similar conclusions. Likewise, we can multiply (2.71) by \( \sin(r \beta) \) and take \( \beta \) to 0, yielding a HS operator with explicit eigenvalues.

Third, we used the Frobenius identity to prove the theorem. But in fact we can also derive (2.60)–(2.61) directly, via a recursive integration procedure. We proceed to detail
this. We begin by noting that the kernel (2.55) is antisymmetric in \(x\) and \(y\). To calculate \(C_{\beta,\gamma}\epsilon_n\), therefore, it suffices to calculate the integral

\[
J_N = \int_{\mathbb{T}^N} dy_1 \cdots dy_N I_N(y_1, \ldots, y_N),
\]

where

\[
I_N(y) \equiv s_+ (\beta + N\gamma + \sum_{j=1}^N (x_j - y_j)) \prod_{1 \leq j < k \leq N} s_+(y_k - y_j) \prod_{j,k=1}^N s_+(x_k - y_j + \gamma)
\]

\[
\times \exp \left( i r (1 - N) \sum_{j=1}^N (x_j - y_j) + 2 ir \sum_{j=1}^N y_j (n_j + \rho_j) \right), \quad x \in F,
\]

and \(\rho\) is defined by (2.23). By Fubini’s theorem we are free to choose the order of the multiple integration. Consider the \(y_N\)-integral. The integrand \(I_N\) is \(\pi/r\)-periodic in \(y_N\), and when we shift \(y_N\) to \(y_N + ia_+\), it is invariant up to a multiplier

\[
\mu_N = \exp(2ir\beta - 2ra_n n_N),
\]

cf. I(1.23). Hence the \(y_N\)-integration can be done via Cauchy’s theorem, picking up residues at the \(N\) simple poles \(x_1 - \gamma, \ldots, x_N - \gamma\). The result can be written

\[
J_N = \frac{-2\pi i \exp(2ir\gamma n_N)}{1 - \mu_N} \sum_{l_N = 1}^N \exp(2irx_{l_N}(n_N + \rho_N)) \frac{\prod_{j \neq l_N} s_+(x_{j_N} - x_{l_N})}{\prod_{j=1}^{N-1} s_+(x_k - y_j + \gamma)} \prod_{j=1}^{N-1} s_+(y_k - y_j)
\]

\[
\times \exp \left( i r (1 - N) \left( \sum_{j \neq l_N} x_j - \sum_{j=1}^{N-1} y_j \right) + 2 ir \sum_{j=1}^{N-1} y_j (n_j + \rho_j) \right).
\]

The integrand \(I_{N-1}^{(l_N)}\) is \(\pi/r\)-periodic in \(y_{N-1}\), and when we shift \(y_{N-1}\) to \(y_{N-1} + ia_+\), it is invariant up to a multiplier

\[
\mu_{N-1} = \exp(2ir\beta - 2ra_+ (n_{N-1} + 1)),
\]

(again by I(1.23)). Hence the \(y_{N-1}\)-integration can again be done explicitly. Proceeding recursively, we wind up with

\[
J_N = (-2\pi i)^N s_+ (\beta) \prod_{j=1}^N \frac{\exp(i r \gamma (n_j + N - j))}{1 - \exp(2ir\beta - 2ra_+(n_j + N - j))} \prod_{j \neq l_N} s_+(x_{j_N} - x_{l_N}) \cdots \prod_{j_2 \neq l_2, \ldots, l_2} s_+(x_{j_2} - x_{l_2}).
\]
Here the prime signifies summation over
\[ l_N = 1, \ldots, N, \ l_{N-1} \neq l_N, \ldots, l_1 \neq l_N, \ldots, l_2. \tag{2.87} \]
(Thus \( j_2 \) equals \( l_1 \).) A moment’s thought suffices to see that the sum is equal to
\[ \prod_{1 \leq j < k \leq N} s_+ (x_j - x_k)^{-1} \sum_{\sigma \in S_N} (-)^\sigma \exp (2ir \sum_{j=1}^N x_{\sigma(j)} (n_j + \rho_j)). \tag{2.88} \]
When we now put the pieces together, we reobtain (2.60)–(2.61).

As a corollary we obtain a new proof of the elliptic Cauchy identity (2.53). Indeed, we have shown that the HS operators \( L \) and \( R \) with kernels given by the lhs and rhs, resp., have the same eigenvalues on the ONB \( \{e_n\} \). Since the kernels are antisymmetric in \( x \) and \( y \), and \( L - R \) vanishes on the ONB, it follows that the kernels are equal. To be sure, this proof of (2.53) is neither the shortest nor the most natural one.

3. The General \( A_{N-1} \) Case: Hilbert Spaces vs. HS Operators

3.1. Hilbert spaces galore. Already in the setting of classical mechanics, one can associate three state spaces to the (classical version of the) elliptic \( A_{N-1} \) Hamiltonian
\[
H_{nt}(x) \equiv -\frac{1}{2} \sum_{j=1}^N \partial_{x_j}^2 + g(g - 1) \sum_{1 \leq j < k \leq N} \wp(x_j - x_k; \pi/2r, i\alpha/2),
\]
and its relativistic generalization. (For the trigonometric specialization this is detailed in [8].) Physically speaking, one views the dynamics as describing particles moving on a line or on a ring, with the latter interpretation allowing two versions (distinguishable vs. indistinguishable particles).

On the quantum level this physical picture gives rise to three distinct Hilbert spaces. For a detailed analysis of the commuting Hamiltonians and the corresponding HS operators, it is however expedient to work with several more state spaces, in particular Hilbert spaces obtained (roughly speaking) by omitting the center-of-mass motion.

This subsection is concerned with the formalism associated with these choices. To a large extent this can be phrased in Lie-theoretic terms, but we have not tried to do so systematically. On the other hand, we do use some terminology and notation that is inspired by this perspective, and we add a few more ingredients in Subsect. 3.3.

First, let us view (3.1) as the dynamics of a system of \( N \) particles moving on the line. Then the natural state space is
\[
\tilde{\mathcal{H}} \equiv L^2(W, dx),
\]
where the choice of \( W \) (the ‘big’ Weyl alcove) encodes the invariance of the particle ordering over an interval of length \( \pi/r \). (Recall \( \wp(\pi/2r, i\alpha/2; x) \) is positive on its period interval \((0, \pi/r)\) and diverges at the endpoints.) Specifically, we work with
\[
W \equiv \{ x \in \mathbb{R}^N \mid x_j - x_{j+1} \in \mathbf{w}, \ j = 1, \ldots, n \},
\]
where \( \mathbf{w} \) denotes the ‘small’ Weyl alcove
\[
\mathbf{w} \equiv \{ \delta \in \mathbb{R}^n \mid \delta_1, \ldots, \delta_n > 0, \sum_{j=1}^n \delta_j < \pi/r \}. \tag{3.4} \]
Here and below, we use the convention
\[ n \equiv N - 1, \quad N > 1. \tag{3.5} \]

Next, we may view (3.1) as the dynamics of \( N \) particles on a ring, with the position of particle \( j \) encoded by the phase \( \exp(2i\pi x_j) \). Thus the coordinates \( x_1, \ldots, x_N \) are generalized angle coordinates varying over a torus
\[ \mathbb{T} \equiv (-\pi/2r, \pi/2r]. \tag{3.6} \]

Excluding coinciding particle positions (where the potential energy diverges), the coordinate vector varies over
\[ \mathbb{T}_*^N \equiv \{ x \in \mathbb{T}^N \mid x_j \neq x_k, j \neq k \}. \tag{3.7} \]

In the ring picture of the dynamics there are two natural choices of state space. Choosing a clockwise ordering of the ring positions \( \exp(2i\pi x_1), \ldots, \exp(2i\pi x_N) \), this ordering is fixed under the classical dynamics, so the state space should be restricted accordingly. Now when the particles are indistinguishable, we should still divide out by the cyclic permutation group \( \mathbb{Z}_N \). Thus we wind up with the configuration space
\[ F \equiv \{ x \in \mathbb{R}^N \mid -\pi/2r < x_N < \cdots < x_1 \leq \pi/2r \}, \tag{3.8} \]

cf. I(1.7).

For distinguishable particles we can coordinatize all distinct system position vectors \( \exp(2i\pi x_1), \ldots, \exp(2i\pi x_N) \) by letting \( x \) vary over the configuration space
\[ F' \equiv \left\{ x \in \mathbb{R}^N \mid x_j - x_{j+1} \in \mathbb{W}, j = 1, \ldots, n, \frac{1}{N} \sum_{j=1}^{N} x_j \in (-\pi/2r, \pi/2r] \right\}. \tag{3.9} \]

Hence we may and will view \( N^{-1} \sum_{j=1}^{N} x_j \) as a coordinate on the torus \( \mathbb{T} \) (3.6). Besides \( \tilde{\mathcal{H}} \) (3.2), we can therefore associate two more Hilbert spaces to (3.1), namely,
\[ \mathcal{H} \equiv L^2(F, dx), \tag{3.10} \]
and
\[ \mathcal{H}' \equiv L^2(F', dx). \tag{3.11} \]

There are several other ways to arrive at the above configuration spaces in terms of group actions. To begin with, indistinguishability can be encoded by dividing out the obvious \( S_N \)-action on \( \mathbb{T}_*^N \) (3.7), yielding
\[ F \simeq \mathbb{T}_*^N / S_N. \tag{3.12} \]

Next, it is clear that the \( \mathbb{Z} \)-action on \( \mathbb{R}^N \) with generator
\[ G : x_1, \ldots, x_N \mapsto x_N + \pi/r, x_1, \ldots, x_n, \tag{3.13} \]
leaves \( \mathbb{W} \) invariant. It is easily seen that \( F \) can again be viewed as the corresponding quotient space:
\[ F \simeq \mathbb{W}/\mathbb{Z}. \tag{3.14} \]
It is also important to introduce the $\mathbb{Z}$-subgroup generated by

$$ G' \equiv G^N : x_1, \ldots, x_N \mapsto x_1 + \pi/r, \ldots, x_N + \pi/r. \quad (3.15) $$

Indeed, denoting it by $\mathbb{Z}'$, we deduce

$$ F' \simeq W/\mathbb{Z}'. \quad (3.16) $$

Moreover, dividing out the action of the cyclic permutation group

$$ \mathbb{Z}_N = \mathbb{Z}/\mathbb{Z}', \quad (3.17) $$

on $F'$, we obtain $F$:

$$ F \simeq F'/\mathbb{Z}_N. \quad (3.18) $$

We proceed to define the coordinate transformation

$$ s = \frac{1}{N} \sum_{j=1}^N x_j, \quad \delta_m = x_m - x_{m+1}, \quad m = 1, \ldots, n, \quad (3.19) $$

whose inverse reads

$$ x_k = s - \frac{1}{N} \sum_{j=1}^{k-1} j \delta_j + \frac{1}{N} \sum_{j=k}^n (N - j) \delta_j, \quad k = 1, \ldots, N. \quad (3.20) $$

Then we have (cf. (3.3), (3.9), (3.13) and (3.15))

$$ W \simeq \{ (s, \delta) \in \mathbb{R}^N \mid \delta \in \mathbb{W} \} = \mathbb{R} \times \mathbb{W}, \quad (3.21) $$

$$ F' \simeq \{ s \in (-\pi/2r, \pi/2r], \delta \in \mathbb{W} \} = \mathbb{T} \times \mathbb{W}, \quad (3.22) $$

$$ G : s, \delta_1, \ldots, \delta_n \mapsto s + \pi/Nr, \pi/r - \sum_{j=1}^n \delta_j, \delta_1, \ldots, \delta_{n-1}, \quad (3.23) $$

$$ G' : s, \delta \mapsto s + \pi/r, \delta. \quad (3.24) $$

Also, introducing the reduced Hilbert space

$$ h_r \equiv L^2(\mathbb{W}, d\delta), \quad (3.25) $$

we may view $\tilde{\mathcal{H}}$ (3.2) and $\mathcal{H}'$ (3.11) as tensor products

$$ \tilde{\mathcal{H}} \simeq L^2(\mathbb{R}) \otimes h_r, \quad (3.26) $$

$$ \mathcal{H}' \simeq L^2(\mathbb{T}) \otimes h_r. \quad (3.27) $$

Now the $\mathbb{Z}_N$-action generated by (3.23) gives rise to a unitary operator $U_G$ on $\mathcal{H}'$:

$$ (U_G \Psi)(s, \delta) \equiv \Psi(G^{-1}(s, \delta)), \quad \Psi \in \mathcal{H}'. \quad (3.28) $$

It leads to a decomposition

$$ \mathcal{H}' = \bigoplus_{m=0}^n \mathcal{H}_m, \quad (3.29) $$
in pairwise orthogonal subspaces

\[ \mathcal{H}_m = \{ \Psi \in \mathcal{H}' \mid U_G \Psi = \omega^m \Psi \}, \quad m = 0, \ldots, n, \]  

(3.30)

where

\[ \omega \equiv \exp(2\pi i / N). \]  

(3.31)

In particular, the invariant subspace \( \mathcal{H}_0 \) may and will be identified with \( \mathcal{H} \), which corresponds to the quotient relation (3.18).

We also need the decomposition

\[ h_r = \bigoplus_{m=0}^n h_m, \]  

(3.32)

\[ h_m = \{ \psi \in h_r \mid U_g \psi = \omega^m \psi \}, \quad m = 0, \ldots, n, \]  

(3.33)

where

\[ (U_g \psi)(\delta) \equiv \psi(g^{-1}(\delta)), \]  

(3.34)

with \( g \) the reduced \( \mathbb{Z}_N \)-action (cf. (3.23))

\[ g : \delta_1, \ldots, \delta_n \mapsto \pi / r - \sum_{j=1}^n \delta_j, \delta_1, \ldots, \delta_{n-1}. \]  

(3.35)

Defining the orthonormal base

\[ c_l(s) \equiv (r / \pi)^{1/2} \exp(2ilrs), \quad l \in \mathbb{Z}, \]  

(3.36)

for \( L^2(\mathbb{T}, ds) \), the \( \mathbb{Z}_N \)-invariant subspace \( \mathcal{H}_0 \cong \mathcal{H} \) of \( \mathcal{H}' \) (3.27) can be written

\[ \mathcal{H} = \bigoplus_{m=0}^n \bigoplus_{j \in \mathbb{Z}} c_{m+jN} \otimes h_m, \]  

(3.37)

as is readily verified.

3.2. Spectral analysis of the commuting HS operators. In this subsection we use the elliptic Hamiltonian (3.1) to motivate and illustrate the following developments, but the formalism applies with minor changes to the higher commuting Hamiltonians and their relativistic generalizations. Likewise, its application to the structure analysis of the above HS operators is to a considerable extent independent of their specific form. We therefore work with a quite general class of HS operators in this subsection.

Under the transformation (3.19) the PDO (3.1) becomes

\[ H \rightarrow -\frac{1}{2N} \frac{\partial^2}{\partial s^2} + H_r \equiv H_{cm} + H_r, \]  

(3.38)

where the reduced Hamiltonian \( H_r \) depends only on the internal variables \( \delta_1, \ldots, \delta_n \). Choosing \( g > 0 \) from now on, the domain choice for \( H \) and \( H_r \), viewed as operators on the Hilbert spaces \( \tilde{\mathcal{H}}, \mathcal{H}', \mathcal{H} \) and \( h_r \), resp., will be determined by the family of HS operators on \( \mathcal{H} \) with kernels \( \Psi_{nr,\xi}(x, y) \). As will be shown shortly, the eigenfunctions of the latter may be chosen to be of the factorized form

\[ F_{m, j, \lambda}(s, \delta) \equiv c_{m+jN}(s) f_{m, \lambda}(\delta), \quad m \in \{0, \ldots, n\}, \ j \in \mathbb{Z}, \ \lambda \in I_m, \]  

(3.39)
where \( f_{m,\lambda} \) yields an ONB for \( h_m \) when \( \lambda \) varies over a suitable index set \( I_m \). Accordingly, the domain and action of the center-of-mass kinetic energy operator \( H_{cm} \) decouple from that of \( H_r \). Specifically, on \( L^2(\mathbb{R}, ds) \) the operator \( H_{cm} \) is defined in the usual way, and on \( L^2(\mathbb{T}, ds) \) it is defined as a multiple of the periodic Laplacian, with eigenfunction ONB \( c_j, l \in \mathbb{Z} \).

The results of Sect. 2 imply that for the free case \( g = 1 \) the functions \( f_{m,\lambda} \) are eigenfunctions of the PDO \( H_r \) with real eigenvalues. Thus one can associate a self-adjoint operator \( \hat{H} \) on \( \mathcal{H} \) to the PDO \( H \) whose action on the ONB \( F_{m,j,\lambda} \) coincides with the \( H \)-action. We expect this to be true for all \( g > 0 \), but thus far we have only obtained a complete proof for the case \( N = 2 \).

More generally, it is plausible that the \( \mathcal{H} \)-ONB (3.39) is a joint eigenvector ONB of the \( N \) commuting PDOs with real eigenvalues, hence yielding a solution to the problem to reinterpret the commuting PDOs as bona fide commuting self-adjoint operators on \( \mathcal{H} \). (It will be clear how to extend the definition to the larger spaces \( \mathcal{H}' \) and \( \mathcal{H} \); one need only exploit the \( H_r \)-eigenfunctions \( f_{m,\lambda} \) to do so.) Moreover, we believe that a similar scenario can be used for the relativistic case, where a direct definition of suitable dense domains appears intractable.

Especially in the relativistic setting, the key step is the similarity transformation involving the weight function \( w(x) \). Indeed, this removes the square roots in the coefficients, yielding \( A\Delta \Omega s \) with meromorphic coefficients, cf. (2.1). Thus one gets an unambiguous action on meromorphic functions, so that domain issues are simpler to handle when one switches from Lebesgue measure \( dx \) to the weighted measure \( w(x)dx \).

The HS kernels, however, involve the similarity factors in both settings (cf. (1.14)), and accordingly the HS operators are more easily studied on the Lebesgue measure Hilbert spaces \( \mathcal{H} \) and \( \mathcal{H}' \). More precisely, as they stand the pertinent kernels only yield well-defined HS operators on \( \mathcal{H} \), but after the change of variables (3.20) a natural extension to the larger space \( \mathcal{H}' \) (3.27) arises, which facilitates their study. In order to handle both the nonrelativistic and relativistic kernels at once, it is expedient to take \( \xi \) real from now on, and generalize the class of kernels under consideration by assuming solely some common features.

Specifically, we consider HS operators on \( \mathcal{H}' \) of the form

\[
(K_{\xi} f)(s, \delta) = \int_{-\pi/2r}^{\pi/2r} ds' \int_{\Omega} d\delta' \mathcal{K}(s - s' + \xi; \delta, \delta') f(s', \delta'),
\]

\[
f \in \mathcal{H}' = L^2(\mathbb{T}, ds) \otimes L^2(\omega, d\delta).
\]

Here, the assumptions on the kernel \( \mathcal{K}(u; \delta, \delta'), u \in \mathbb{R}, \delta, \delta' \in \omega \), are as follows:

(i) It is \( \pi/r \)-periodic in \( u \);
(ii) It is bounded and smooth on \( \mathbb{R} \times \omega^2 \);
(iii) The reparametrized kernel \( \mathcal{K}(s - s' + \xi; \delta, \delta') \), with \( \xi \in \mathbb{R} \), is invariant under the \( G \)-action (3.23) on \( (s, \delta) \) and under the same action on \( (s', \delta') \).

The only property that may not be immediately clear for the special kernels arising from the function \( S_\xi(x, y) \) is the third one. (More precisely, (i) and (ii) are obvious provided we require \( g > 0 \) and \( \mu \in i(0, a_+ + a_-) \) in the nonrelativistic and relativistic case, resp.) To explain it, recall that \( S_\xi(x, y) \) is \( \pi/r \)-periodic in \( x_1, \ldots, x_N, y_1, \ldots, y_N \), and invariant under permutations of \( x \) and \( y \). Hence it is invariant under the action (3.13) of \( G \) on \( x \) and also under the same action on \( y \). Furthermore, the weight function \( w(x(s, \delta)) \) depends only on \( \delta \) and is invariant under the \( g \)-action (3.35), as is readily verified.
We proceed to study HS operators on $\mathcal{H}'$ of the form (3.40), with a fixed kernel $K(u; \delta, \delta')$ satisfying (i)–(iii). (Note that (ii) already suffices for the HS property.) To begin, the $G$-invariance property (iii) entails that all of the operators $K_\xi$ map the invariant subspace $\mathcal{H}_0 \cong \mathcal{H}$ into itself and annihilate the remaining subspaces in the decomposition (3.29). To show this, we choose $f_m \in \mathcal{H}_m$. Then we obtain from (3.28)–(3.30)

$$K_\xi f_m = K_\xi U_G f_m = \omega^m K_\xi f_m,$$

so that $K_\xi f_m = 0$ for $m \neq 0$. Also, since $U_G K_\xi f_0$ equals $K_\xi f_0$, we have $K_\xi \mathcal{H}_0 \subset \mathcal{H}_0$.

Next, we observe that we have

$$[K_{\xi_1}, K_{\xi_2}] = 0, \quad \xi_1, \xi_2 \in \mathbb{R}. \quad (3.42)$$

Indeed, commutativity follows from the convolution structure in the $s$-variable and property (i) by an obvious change of variable.

Consider now the operators

$$\hat{K}_l \equiv \int_{\mathbb{T}} d\xi K_\xi e^{-2irl\xi}, \quad l \in \mathbb{Z}. \quad (3.43)$$

A moment’s thought suffices to see that they are pairwise commuting HS operators, with kernels

$$\int_{\mathbb{T}} d\xi K(s - s' + \xi; \delta, \delta') e^{-2irl\xi} = e^{2irl(s-s')\kappa_l(\delta, \delta')}, \quad (3.44)$$

$$\kappa_l(\delta, \delta') \equiv \int_{\mathbb{T}} du K(u; \delta, \delta') e^{-2irlu}, \quad (3.45)$$

that are invariant under the $G$-action on $(s, \delta)$ and $(s', \delta')$. Thus we can write

$$\hat{K}_l = (\pi/r) (c_l \otimes c_l) \otimes \hat{k}_l, \quad l \in \mathbb{Z}. \quad (3.46)$$

where the HS operator $\hat{k}_l$ with kernel $\kappa_l(\delta, \delta')$ acts on $\mathfrak{h}_r$ (3.25). It maps the subspace $\mathfrak{h}_m$ in the decomposition (3.32) with $m \equiv l \operatorname{(mod N)}$ into itself, and annihilates the remaining $n$ subspaces.

Now since the operators $\hat{K}_l$ commute and the rank-one projections in (3.46) clearly commute as well, it follows that the operators $\hat{k}_l$ on $\mathfrak{h}_c$ commute. We may therefore choose an ONB for $\mathfrak{h}_m$ consisting of functions $f_{m,\lambda}(\delta), \lambda \in I_m$, that are joint eigenfunctions of $\hat{k}_{m+jN}$ for all integers $j$. As a consequence, the orthonormal functions

$$c_l(s) f_{m,\lambda}(\delta), \quad m \equiv l \operatorname{(mod N)}, \quad \lambda \in I_m, \quad (3.47)$$

are eigenvectors for $\hat{k}_l$, and $\hat{K}_l$ annihilates the orthocomplement of the $\mathcal{H}$-subspace spanned by the vectors in (3.47).

We are now in the position to translate these results to the family $K_\xi$ of commuting HS operators on $\mathcal{H}$. Indeed, inverting (3.43) yields

$$K_\xi = \frac{\pi}{r} \sum_{l \in \mathbb{Z}} e^{2irl\xi} \hat{K}_l, \quad \xi \in \mathbb{R}, \quad (3.48)$$

with the series converging in HS norm. Hence we deduce that the functions $F_{m,j,\lambda}(s, \delta)$ given by (3.39) yield an ONB for $\mathcal{H}$ consisting of $K_\xi$-eigenvectors, as announced.
In fact, we can readily generalize the joint eigenvector feature of \( F_{m, j, \lambda} \). Retaining the same kernel \( \mathcal{K}(s - s' + \xi; \delta, \delta') \), let us multiply it by a factor \( \phi(No - Ns') \), where \( \phi \) belongs to the space of smooth periodic functions

\[
C_p \equiv \{ \phi \in C^\infty(\mathbb{R}) \mid \phi(u + \pi/r) = \phi(u) \}.
\]

(3.49)

Then we obtain a kernel

\[
\mathcal{K}^{(\phi)}(u; \delta, \delta') \equiv \phi(No - N\xi)\mathcal{K}(u; \delta, \delta').
\]

(3.50)

We now have the following theorem, which summarizes and extends the above analysis.

**Theorem 3.1.** Let \( \mathcal{K}(u; \delta, \delta') \) be a kernel with the properties (i)–(iii) below (3.40). Then all HS operators on \( \mathcal{H} \) (3.37), given by kernels (3.50) with \( \phi \) any function in \( C_p \) (3.49), commute and admit an ONB of joint eigenvectors of the form (3.39). Moreover, if for some \( m \in \{0, \ldots, n\} \), \( j \in \mathbb{Z} \), and \( \lambda \in \mathbb{I}_m \), we have a zero eigenvalue relation

\[
K_\xi F_{m, j, \lambda} = 0, \quad \forall \xi \in \mathbb{R},
\]

(3.51)

then we have more generally

\[
K_\xi^{(\phi)} F_{m, j, \lambda} = 0, \quad \forall \xi \in \mathbb{R}, \quad \forall \phi \in C_p.
\]

(3.52)

**Proof.** Fixing \( \phi \in C_p \), it is clear that the kernel (3.50) also satisfies the assumptions (i)–(iii). Thus we can repeat the above analysis for the associated family of HS operators \( K_\xi^{(\phi)} \) on \( \mathcal{H}' \). We claim that the resulting joint eigenfunctions may be chosen equal to the previous ones (which correspond to choosing \( \phi(u) = 1 \)). To prove this claim, we consider the action of \( K_\xi^{(\phi)} \) on (3.47). Recalling (3.36), an obvious change of variables yields

\[
(K_\xi^{(\phi)} cl_{m, \lambda})(s, \delta) = e^{2irl\xi} c_l(s) \int_\mathcal{W} d\delta' k^{(\phi)}_l(\delta, \delta') f_{m, \lambda}(\delta'),
\]

(3.53)

where

\[
k^{(\phi)}_l(\delta, \delta') \equiv \int_\mathbb{T} du \phi(No - N\xi)\mathcal{K}(u; \delta, \delta')e^{-2irlu}.
\]

(3.54)

Now from (3.45) we deduce that the HS operator on \( \mathfrak{h}_r = L^2(\mathcal{W}, \delta) \) given by the (HS-norm convergent) series

\[
k_u = \frac{\pi}{r} \sum_{l \in \mathbb{Z}} e^{2irlu} \hat{k}_l, \quad u \in \mathbb{R},
\]

(3.55)

has kernel \( \mathcal{K}(u; \delta, \delta') \). Since \( f_{m, \lambda} \) is a joint eigenvector for all \( \hat{k}_l, l \in \mathbb{Z} \), it is also a joint eigenvector for all \( k_u, u \in \mathbb{R} \). Hence (3.54) amounts to an integral of commuting HS operators

\[
\hat{k}^{(\phi)}_l = \int_\mathbb{T} du \phi(No - N\xi)k_u e^{-2irlu},
\]

(3.56)

with joint eigenvectors \( f_{m, \lambda} \). Therefore, \( f_{m, \lambda} \) is a joint eigenvector for the HS operators \( \hat{k}^{(\phi)}_l \), too. In view of (3.53), this proves our claim. The proof also shows that all of the HS operators commute, cf. (3.56).
It remains to prove the last assertion. By (3.43), the assumption (3.51) implies
\[ \hat{K}_l F_{m,j,\lambda} = 0, \quad \forall l \in \mathbb{Z}. \] (3.57)
Recalling (3.44)–(3.46), this yields
\[ \hat{K}_l f_{m,\lambda} = 0, \quad \forall l \in \mathbb{Z}. \] (3.58)
Now (3.55) yields
\[ k_u f_{m,\lambda} = 0, \quad \forall u \in \mathbb{R}, \] (3.59)
so by (3.56) we obtain
\[ \hat{K}_{l}^{(\phi)} f_{m,\lambda} = 0, \quad \forall l \in \mathbb{Z}, \quad \forall \phi \in C_p. \] (3.60)
Hence the assertion follows from (3.53). \( \square \)

It is illuminating to see how the above spectral features apply to the special cases considered in Sect. 2. In particular, we point out that none of the eigenvalues \( \lambda_n(\xi) \) in (2.35) vanishes for all real \( \xi \), since \( \kappa_n(2.61) \) is nonzero for all \( n \). On the other hand, for the special choice of \( \phi \) associated with (2.68)–(2.69), we get an infinite-dimensional kernel. This shows by example that when we replace \( K_{\xi} \) in the assumption (3.51) by \( K_{\xi}^{(\phi_0)} \) with \( \phi_0 \) non-constant, then it is not generally true that (3.52) follows.

3.3. Some Lie-algebraic aspects. To conclude this section, we sketch a more Lie-theoretic viewpoint for the above. First, retaining the torus \( T \) (3.6) and position coordinates \( x_1, \ldots, x_N \), the obvious maximal abelian \( U(N) \) subgroup is given by
\[ \{ \text{diag}(e^{2irx_1}, \ldots, e^{2irx_N}) \mid x \in \mathbb{T}^N \} \simeq \mathbb{T}^N. \] (3.61)
Defining a \( \mathbb{Z}^N \)-action on \( \mathbb{R}^N \) by
\[ x \mapsto x + k\pi/r, \quad k \in \mathbb{Z}^N, \] (3.62)
we may view \( \mathbb{T}^N \) as \( \mathbb{R}^N / \mathbb{Z}^N \); alternatively, we can view \( \mathbb{Z}^N \) as the \( U(N) \) weight lattice, a weight \( k \) being defined by its values on \( \mathbb{T}^N \), namely by the character \( \exp(2irk \cdot x) \). With the natural action of \( S_N \) on \( \mathbb{Z}^N \), the semi-direct product \( \mathbb{Z}^N \rtimes S_N \) may be viewed as the affine Weyl group associated with \( U(N) \); it is generated by the map \( G \) (3.13) and the \( S_N \)-action. In addition to (3.12), (3.14) and (3.18), the fundamental domain \( F \) can be regarded as
\[ F \simeq \mathbb{R}^N_\times / \mathbb{Z}^N \rtimes S_N. \] (3.63)
(Here and below, the \( \times \) denotes exclusion of coinciding positions, cf. (3.7).)

Next we introduce the \( \mathbb{Z}^N \)-subgroups
\[ Q \equiv \{ k \in \mathbb{Z}^N \mid \sum_{j=1}^N k_j = 0 \}, \] (3.64)
\[ \mathbb{Z}' \equiv \{ k \in \mathbb{Z}^N \mid k_1 = \cdots = k_N \}. \] (3.65)
(We use the same symbol for the abstract group, its action as a transformation group on \( \mathbb{R}^N \), and its representation as a subgroup of \( \mathbb{R}^N \).) Then we can view \( Q \) as the root lattice and \( Q \cong S_N \) as the affine Weyl group associated with (the root system \( A_n \) of) \( SU(N) \). The group \( \mathbb{Z}_N \) (3.17) can also be obtained as

\[
\mathbb{Z}_N = \mathbb{Z}^N / (Q \oplus \mathbb{Z}').
\]

(Indeed, cosets can be labeled by \( \sum_{j=1}^N k_j = m \) (mod \( N \)).

The maximal \( U(N) \) torus (3.61) can be related to the maximal \( SU(N) \) torus via the coordinate transformation

\[
s = \frac{1}{N} \sum_{j=1}^N x_j, \quad s_1 = x_1 - s, \ldots, s_n = x_1 + \cdots + x_n - ns,
\]

with inverse

\[
x_1 = s + s_1, \quad x_k = s + s_k - s_{k-1}, \quad k = 2, \ldots, n, \quad x_N = s - s_n.
\]

This yields

\[
diag(e^{2irx_1}, \ldots, e^{2irx_N}) = e^{2irs} diag(e^{2irs_1}, e^{2ir(s_2-s_1)}, \ldots, e^{-2irs_n}),
\]

which shows that the \( SU(N) \) torus is obtained by letting \( (s_1, \ldots, s_n) \) vary over \( \mathbb{T}^n \), cf. (3.6).

Given \( x \in \mathbb{R}^N \), one can view the variables \( s_1, \ldots, s_n \) in (3.67) as coordinates w.r.t. the fundamental \( A_n \) weight vectors

\[
\omega_j = e_1 + \cdots + e_j - \frac{j}{N} \sum_{k=1}^N e_k, \quad j = 1, \ldots, n,
\]

whereas the variables \( \delta_1, \ldots, \delta_n \) in (3.19) are coordinates w.r.t. the dual basis of simple root vectors

\[
\alpha_j = e_j - e_{j+1}, \quad j = 1, \ldots, n.
\]

The \( A_n \) Weyl alcove \( \mathbb{W} \) is most easily coordinatized by the variables \( \delta_1, \ldots, \delta_n \), cf. (3.4).

It may also be viewed as

\[
\mathbb{W} \cong \mathbb{T}^n / S_N,
\]

where \( \mathbb{T}^n \) is the maximal \( SU(N) \) torus, cf. (3.69). In physical parlance, the \( S_N \) action on \( \mathbb{W} \) yields the Wigner-Seitz cell/first Brillouin zone of the lattice \( Q \).

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