Analytical treatment of spin-1/2 particle subject to a combination of potentials

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Abstract. By using the fundamental principles of supersymmetric quantum mechanics methodology and parametric Nikiforov-Uvarov method respectively, we obtained both the positive and negative (spin and pseudospin symmetries) energy equations of the Dirac equation with scalar and vector potentials. The nonrelativistic limit of the spin symmetry was obtained, and this was used to calculate Fisher information for both position space and momentum space respectively. The behaviour of eigenvalue energy with both the angular frequency and equilibrium bond length respectively were studied. Five special cases of the potential under consideration were also studied. Finally, we examined the feature of Fisher information with the angular frequency which was found to obey Heisenberg uncertainty principle.

Keywords: Dirac equation; Schrödinger equation; Uncertainty; Fisher information.

1. Introduction
A vast examination on different wave equations of relativistic quantum mechanics together with various potential interactions have triggered lots of attention in the relativistic regime over the years. The interest is due to the half-integral spin fermionic particles in the Dirac equation which discussed the most frequent building blocks of atoms and molecules. A study of the concept of the Dirac equation has been done in the nuclear theory under spin and pseudospin symmetries [1-3] which was used to construe the attributes of deformed nuclei, superdeformation, and the establishment of effective shell model coupling scheme [4-6]. The concept also explained identical bands and magnetic moment [7, 8]. The spin symmetry was used to explain mesons [9] which occurs only when \( \Delta(r) \) is a constant or when the scalar potential \( \Sigma(r) \) is almost the same as the vector potential \( V(r) \), \( \Sigma(r) \approx V(r) \). The antinucleon spectra symmetric characteristic of a nucleus was examined by studying how the Dirac spin doublets wave functions relate to the actual nuclei in the relativistic average field. In the case of the pseudospin symmetry, the sum potential is a constant, that is, \( \Sigma(r) = \text{constant} \) [10, 11]. These symmetries were studied with various potential models under different traditional methodologies such as Nikiforov-Uvarov method [12-16], asymptotic iteration method [17-19], supersymmetry quantum mechanics method [20-23] and others. Among the potentials reported include Yukawa potential [24], Hellmann potential [25], Manning-Rosen potential [26], Second Pöschl-Teller potential [27], Frost-Musulin potential [28]. Motivated by the interest in spin-1/2 particles, we attempt to study and obtain the exact solutions of the Dirac equation under spin and pseudospin symmetries with a combination of isotropic harmonic oscillator, Pseudoharmonic potential and inverse potential which has not been reported yet. It is the intention of the authors to use supersymmetric quantum mechanics approach to obtain the exact analytical solutions of the Dirac equation which as much as is known by us, have not yet been reported by authors. We equally want to study the behaviour of Fisher information both in position space and momentum space under the proposed potential. The combination of the isotropic harmonic oscillator, Pseudoharmonic and inverse potentials is given as:
\[ V(r) = \frac{m\omega^2 r^2}{2} + \frac{g}{r^2} + D_e \left( \frac{r}{r_e} - \frac{r_e}{r} \right)^2, \quad (1) \]

where \( r_e \) is equilibrium bond length, \( D_e \) is dissociation energy, \( r \) is the internuclear separation, \( m \) is a reduced mass and \( \omega \) is the angular frequency which is equal to \( 2\pi f \).

2. Dirac equation

The Dirac equation for a half-integral spin particle having a mass \( M \) and travelling in the field of a repulsive vector potential \( V(r) \) and attractive scalar potential \( S(r) \) in relativistic units \(( c = \hbar = 1)\) is given as

\[ \left[ \vec{\alpha} \cdot \vec{p} + \beta \left( M + S(r) \right) + V(r) - E \right] \psi(r) = 0, \quad (2) \]

where \( \vec{p} = -i\vec{\nabla} \) is the momentum operator, \( E \) denotes the relativistic energy of the system, \( \alpha \) and \( \beta \) are the \( 4 \times 4 \) regular Dirac matrices \([29]\). In view of equation (2), the spinor components of the Dirac equation can be put as written in Equation 3 below:

\[ \psi_{nk}(\vec{r}) = \begin{pmatrix} f_{nk}(\vec{r}) \\ g_{nk}(\vec{r}) \end{pmatrix} = \frac{1}{r} \begin{pmatrix} F_{nk}(r)Y_{jm}^\ell(\theta, \varphi) \\ iG_{nk}(r)Y_{jm}^7(\theta, \varphi) \end{pmatrix}, \quad (3) \]

where \( f_{nk}(\vec{r}) \) represents the upper component and \( g_{nk}(\vec{r}) \) represents the lower component of the Dirac spinors. \( Y_{jm}^\ell(\theta, \varphi) \) and \( Y_{jm}^7(\theta, \varphi) \) are spin and pseudospin spherical harmonics, respectively, and \( m \) is the angular momentum projection on the \( z \) – axis. Following the paper of Ikhdair and Sever \([30]\), the below equation is obtained:

\[ \left[ \frac{d^2}{dr^2} - \frac{\kappa(\kappa + 1)}{r^2} - \frac{d\Delta(r)}{dr} \frac{d}{dr} + \frac{\kappa}{r} - U(r) \right] F_{nk}(r) = \left[ (M + E_{nks} - \Delta(r))(M + E_{nkp} + \Sigma(r)) \right] F_{nk}(r), \quad (4) \]

for \( \kappa(\kappa + 1) = \ell(\ell + 1), r\in(0, \infty) \), and

\[ \left[ \frac{d^2}{dr^2} - \frac{\kappa(\kappa - 1)}{r^2} - \frac{d\Sigma(r)}{dr} \frac{d}{dr} + \frac{\kappa}{r} + U(r) \right] G_{nk}(r) = \left[ (M + E_{nks} - \Delta(r))(M + E_{nkp} + \Sigma(r)) \right] G_{nk}(r), \quad (5) \]

for \( (\kappa - 1) = \tilde{\ell}(\tilde{\ell} + 1), r\in(0, \infty) \). Here, the change in potential is mathematically written as \( \Delta(r) = V(r) - S(r) \), while the total potential is expressed as \( \Sigma(r) = V(r) + S(r) \), and for the spin and the pseudospin symmetric models, \( \kappa \) (quantum number of orbit-spin) is related to the orbital quantum numbers \( \ell \) and \( \tilde{\ell} \) respectively as:
In the same vein, \( \kappa \) in the quasi-degenerate doublet structure can be yet illustrated in terms of \( \tilde{s} = \frac{1}{2} \) and \( \tilde{\ell} \), the pseudospin and the pseudo-orbital angular momenta are given as

\[
\kappa = \begin{cases} 
- \left( \ell + 1 \right) = - \left( j + \frac{1}{2} \right) \left( s_1, p_3, \text{etc.} \right) j = l + \frac{1}{2}, \text{aligned spin} (\kappa < 0), \\
+ \ell = + \left( j + \frac{1}{2} \right) \left( p_1, d_1, \text{etc.} \right) j = l - \frac{1}{2}, \text{unaligned spin} (\kappa > 0)
\end{cases}
\]

where \( \kappa = \pm 1, \pm 2, \pm 3, \ldots, \ldots \).

3. Exact Solutions of the Dirac Equation

3.1. Bound State Solution of the Dirac Equation using Supersymmetric Quantum Mechanics

3.1.1. The Spin Symmetry Limit. For the limit of spin symmetry to occur, \( \frac{d\Delta(r)}{dr} = 0 \) and \( \Delta(r) = C_s \) [29]. The total potential \( \Sigma(r) \) is taken to be the potential under consideration, i.e. \( \Sigma(r) = V(r) \). To obtain the exact solution for the spin symmetry limit, we substitute equation (1) into equation (4), to have

\[
\frac{d^2 \Phi_{s}(r)}{dr^2} = \beta \left( g + D \sigma_0 r^2 \right) + \left( m \omega^2 + D \right) \beta + \beta (M - E_{\text{KKS}} - 2D),
\]

where \( \beta = M + E_{\text{KKS}} - C_s \). (7)

In this case, the basic concepts and formalism of the supersymmetric quantum mechanics are applied, the shape invariance technique is also implemented. For the ground state \( F_{0, \kappa}(r) \), the logarithmic derivative \( F_{0, \kappa}(r) \) is essentially equal to the superpotential [30–32] in solving equation (6). Hence the wave function of the ground state wave for the upper component \( F_{0, \kappa}(r) \) can be expressed as

\[
F_{0, \kappa}(r) = \exp(- \int Q(r) dr),
\]

where \( Q(r) \) is referred to as the superpotential function in supersymmetric quantum mechanics. To proceed to the next level, we propose the superpotential function as follows

\[
Q(r) = \rho_0 + \frac{\Delta}{r},
\]

where \( \rho_0 \) and \( \rho_1 \) are two superpotential constants which will be determined as we proceed. If equation (8) is substituted into equation (6), we arrive at a non-linear Riccati equation having the form
\[
\frac{d^2 F_{\text{ext}}(r)}{dr^2} = Q^2(r) - \frac{dQ(r)}{dr}, \quad (10)
\]

Substituting equation (9) into equation (10) and then, comparing equation (10) with equation (6), we easily determine the values of the two parametric constants in equation (9) as follows

\[
\rho_0^2 = \beta(M - E_{\text{exc}} - 2D_e), \quad (11)
\]

\[
\rho_1 = \frac{1 \pm \sqrt{(1 + 2\kappa)^2 + 4\beta g D_e r_e^2}}{2}, \quad (12)
\]

\[
\rho_0 = \sqrt{\frac{\beta(m_0^2 - D_e r_e^2)}{2}}, \quad (13)
\]

In view of the potential under consideration and the superpotential function, it becomes easier to construct the supersymmetric partner potentials of the supersymmetric quantum mechanics as follows

\[
U_+(r) = \rho_0^2 + 2\rho_0 \rho_1 \frac{r}{r^2} + \frac{\rho_1(\rho_1 - 1)}{r^2}, \quad (14)
\]

\[
U_-(r) = \rho_0^2 + 2\rho_0 \rho_1 \frac{r}{r^2} + \frac{\rho_1(\rho_1 + 1)}{r^2}. \quad (15)
\]

From equations (14) and (15) we find that the partner potentials are shape-invariant and thus, they satisfy the shape invariance condition [30–33], and so, we establish the following

\[
U_+(a_0, r) = U_-(a_1, r) + R(a_1), \quad (16)
\]

via mapping of the form \( \rho_1 \rightarrow \rho_1 + 1 \), where \( \rho_1 = a_0 \). It is deduced that \( a_1 = f(a_0) \Rightarrow a_0 + 1 \), where \( a_1 \) is a new set of parameters which has been determined uniquely from the old set of parameters \( a_0 \) and the residual term \( R(a_1) \) is independent of the variable \( r \). From the established relationship which satisfied the shape invariance condition, another form of relationship can be written: \( a_1 = a_0 + 1, \ a_2 = a_0 + 2, \ a_3 = a_0 + 3, \) subsequently, \( a_n = a_0 + n \). The energy spectra of the \( U_-(r) \) can be obtained by making use of the shape invariance method [32, 33] via

\[
E_{0}^{(-)}(n) = 0, \quad (17)
\]

\[
E_{n}^{(-)} = \sum_{k=1}^{n} R(a_k) = R(a_1) + R(a_2) + R(a_3) + \cdots + R(a_n) = a_0^2 - a_1^2 - a_2^2 + \cdots + a_{n-1}^2 - a_n^2 = a_0^2 - a_n^2 = \rho_1(\rho_1 + n), \quad (18)
\]

where \( n = 0, 1, 2, 3, \ldots \). Hence, substituting for the value of \( \rho_1 \), the positive energy equation (spin symmetry) of the Dirac equation is obtain as

\[
\beta(M - E_{\text{exc}} + 2D_e \beta - 2 \sqrt{\frac{m_0^2 \beta}{2} + \frac{D_e}{r_e^2}} (n + \frac{1}{2} + \frac{1}{4} \sqrt{(1 + 2\kappa)^2 + 4\beta g D_e r_e^2})) = 0, \quad (19)
\]

The upper spinor component is given as

\[
\psi_0^{(-)} = s^{\pm}(1 + A_\lambda) e^{-\frac{1}{2} \beta} L_n^{(-1+\lambda)}(B, s), \quad (20)
\]

and the lower component of the wave function as
\[ G_{k\kappa}(r) = \frac{1}{M+E_{\kappa\kappa}-C_\kappa} \left( \frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{k\kappa}(r), \]  \tag{21} \]

where
\[ A_S = \sqrt{(1+2\kappa)^2 + 4\beta(g+D_\kappa r_c^2)}, \]  \tag{22} \]
\[ B_S = \frac{m_\omega^2 \beta}{2} + \frac{D_\kappa \beta}{r_c^2}. \]  \tag{23} \]

3.1.2. The Pseudospin Symmetry Limit. For the pseudospin symmetry limit to occur, \( \frac{d\Sigma(r)}{dr} = 0 \) and \( \Sigma(r) = C_p \). The difference potential \( \Delta(r) \) is taken as the potential under consideration, i.e. \( \Delta(r) = V(r) \). To obtain the exact solution for the pseudospin symmetry limit, we substitute equation (1) into equation (5), to have
\[ \frac{d^2 G_{k\kappa}(r)}{dr^2} = \left[ \frac{(\kappa-1)}{r^2} - r^2 \left( \frac{m_\omega^2 \beta}{2} + \frac{D_\kappa \beta}{r_c^2} \right) \right] + y(M + E_{\kappa\kappa} + 2D_\kappa) G_{k\kappa}(r), \]  \tag{24} \]

where
\[ y = M - E_{n,k} + C_p, \]  \tag{25} \]

In order to prevent repetition of word/algebra, the same procedures as in the spin symmetry are used, therefore having the negative energy equation of the Dirac equation as
\[ y(M + E_{n,k}) + 2D_\kappa y - 2 \frac{m_\omega^2 \beta}{2} + \frac{D_\kappa \beta}{r_c^2} \left( n + \frac{1}{2} + \frac{1}{4} \sqrt{(1-2\kappa)^2 - 4y(g+D_\kappa r_c^2)} \right). \]  \tag{26} \]

The wave function’s upper component is
\[ F_{\kappa\exp}(r) = \frac{1}{M-E_{\kappa\exp}+C_p} \left( \frac{d}{dr} + \frac{\kappa}{r} + U(r) \right) G_{\kappa\exp}(r). \]  \tag{27} \]

and the spinor’s lower component is given as
\[ G_{n\kappa}(r) = s \frac{i^{A_p}}{2} e \frac{1}{2} B_p F_n \left( -1 + A_p \right) (B_p s), \]  \tag{28} \]
\[ A_p = \sqrt{(1-2\kappa)^2 - 4y(g+D_\kappa r_c^2)}, \]  \tag{28a} \]
\[ B_p = \frac{m_\omega^2 \gamma}{2} + \frac{D_\kappa \gamma}{r_c^2}. \]  \tag{28b} \]

3.2. Bound State Solution of the Dirac Equation using Parametric Nikiforov-Uvarov Method

Following the parametric Nikiforov-Uvarov method by Tezcan and Sever [34], the following general differential equation is considered for any solvable potential:
\[ \left[ \frac{d^2}{dx^2} + \frac{c_1-c_2 x d}{x(1-c_3 x)^2} + \frac{-A x^2 + B x - \epsilon}{x^2(1-c_3 x)^2} \right] \psi(x) = 0. \]  \tag{29} \]

The condition for energy equation and wave function according to Tezcan and Sever [34] are
\[ nc_2 - (2n+1)c_5 + n(n-1)c_3 + c_7 + 2c_3 c_8 + (2n+1)(\sqrt{c_9} + c_3 \sqrt{c_8}) + 2\sqrt{c_8 c_9} = 0, \]  \tag{30} \]
and
\[ \psi(x) = N_n e^{z_1} (1 - x)^{-c_{12} - c_{13} / 3} P_n \left( \epsilon^{10 - 1, c_{11} / 3 - c_{10} - 1} \right) (1 - 2c_3), \] (31)

respectively. The parameters in the energy equation and wave function are obtained as follows
\[
\begin{align*}
\begin{bmatrix}
  c_4 \\
  c_5 \\
  c_6 \\
  c_7 \\
  c_8 \\
  c_9 \\
  c_{10} \\
  c_{11} \\
  c_{12} \\
  c_{13}
\end{bmatrix} &= \begin{bmatrix}
  1 - c_1 \\
  c_2 - 2c_3 \\
  c_2^2 + c_4 + 2c_5 \\
  2c_4 c_5 - B, \\
  c_2^2 + C, c_0 = c_6 + c_7 + c_8 c_9, c_{10} = c_1 + 2c_4 + 2\sqrt{c_0}, \\
  c_1 = c_2 - 2c_5 + 2\sqrt{c_0}, c_8 = c_4 + c_8, \\
  c_9 = c_5 - \left( \sqrt{c_0} + c_3 \sqrt{c_8} \right), \\
  c_{12} = \frac{1}{4} + \frac{1}{4} \sqrt{(1 + 2\kappa)^2 + 4\beta (g + D_\epsilon \tau_3^2)}, c_{13} = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{m^2 \beta^2 + D_\epsilon^2}{\tau_3^2}}.
\end{bmatrix},
\end{align*}
\] (32)

3.2.1. The spin symmetry limit. To derive the exact solution for the spin symmetry limit, we substitute equation (1) into equation (4) and when a variable which is of the form \( x = r^2 \) is defined, a differential equation of the form in equation (33) is obtained
\[
\frac{d^2 F_{n,k}(s)}{ds^2} + \frac{1}{2s} \frac{dF_{n,k}(s)}{ds} + \left( \frac{\langle (x-1) + D_\epsilon \tau_3^2 \beta + g \beta \rangle}{4} + \frac{1}{4} \right) \beta \frac{s^2 - \left( \frac{m^2 \beta^2 + 2D_\epsilon}{4 \tau_3^2} \right) s^2}{s^2} F_{n,k}(s) = 0, (33)
\]

If equation (33) is compared with equation (29), the parametric constants’ values in equation (32) are obtained as follows
\[
\begin{align*}
c_1 &= 0.5, c_2 = c_3 = 0, c_4 = 0.25, c_5 = 0, c_6 = \left( \frac{m^2 \beta^2 + 2D_\epsilon}{8 \tau_3^2} \right) \beta, \\
c_7 &= \left( \frac{2D_\epsilon - M + E_{n,k}}{4} \right) \beta, c_8 = \frac{1}{16} + \left( \frac{\langle (x-1) + D_\epsilon \tau_3^2 \beta + g \beta \rangle}{4} \right) \beta, c_9 = c_6, \\
c_{10} &= \frac{1}{2} + \frac{1}{2} \sqrt{(1 + 2\kappa)^2 + 4\beta (g + D_\epsilon \tau_3^2)}, c_{11} = \left( \frac{m^2 \beta^2 + D_\epsilon \beta}{2} + \frac{D_\epsilon \beta}{\tau_3^2} \right), c_{12} = \left( \frac{m^2 \beta^2 + D_\epsilon \beta}{2} + \frac{D_\epsilon \beta}{\tau_3^2} \right), \\
c_{13} &= \frac{1}{2} - \frac{1}{2} \sqrt{\frac{m^2 \beta^2 + D_\epsilon \beta}{\tau_3^2}}.
\end{align*}
\] (34)

Substituting the parametric constants in equation (34) into equations (30), we have positive energy equation of the Dirac equation (spin symmetry) as
\[
\beta (M - E_{n,k}) + 2D_\epsilon \beta - 2 \sqrt{\frac{m^2 \beta^2 + D_\epsilon \beta}{\tau_3^2} n + \frac{1}{2} + \frac{1}{4} \sqrt{(1 + 2\kappa)^2 + 4\beta (g + D_\epsilon \tau_3^2)}} = 0. (35)
\]

3.2.2. The Pseudospin Symmetry Limit. To derive the exact solution for the limit of pseudospin symmetry, we substitute equation (1) into equation (5) and by defining a variable which is of the form \( s = r^2 \), a differential equation of the form in equation (36) is obtained
\[
\frac{d^2 G_{n,k}(s)}{ds^2} + \frac{1}{2s} \frac{dG_{n,k}(s)}{ds} + \left( \frac{\langle (x-1) - D_\epsilon \tau_3^2 \beta - g \beta \rangle}{4} + \frac{1}{4} \right) \beta \frac{s^2 + \left( \frac{m^2 \beta^2 + 2D_\epsilon}{4 \tau_3^2} \right) s^2}{s^2} G_{n,k}(s) = 0, (36)
\]

Using the same procedures as in the spin symmetry, we have the negative energy equation of the Dirac equation as
\[ \gamma(M + E_{n,k}) + 2D_e \gamma = -2\sqrt{\frac{m_0 \alpha^2}{2} + \frac{D_x}{r_x^2}} \left( n + \frac{1}{2} + \frac{1}{4} \sqrt{(1 - 2\kappa)^2 - 4\gamma(g + D_x r_x^2)} \right). \]  

(37)

3.3. Non-relativistic limit

The Dirac equation is known for half integral spin particles while Schrödinger equation (the non-relativistic equation) is known for whole integral particles. Thus, certain relationship exists between the two fundamental equations. Therefore, using the following transformation for the positive energy equation of the Dirac equation (spin symmetry):

\[ M + E_{n,k} = \frac{2m}{h^2}, E_{n,k} \rightarrow E_{n,\ell} \] and

\[ M - E_{n,k} = -E_{n,\ell} \]
as \( \kappa \rightarrow \ell \) and \( C_0 \rightarrow 0 \), the nonrelativistic energy eigenvalue equation becomes

\[ E_{n,\ell} = \sqrt{m \hbar^2 \omega^2 + \frac{2m D_x h^2}{\ell^2}} \left( 2n + 1 + \frac{1}{2} \sqrt{(1 + 2\ell)^2 + \frac{8m(g + D_x r_x^2)}{h^2}} \right) - 2D_e. \]  

(38)

The corresponding radial wave function is given as

\[ R_{n,\ell}(r) = s^{\ell+1} A e^{-\frac{\ell}{2}} L_n^\ell(Bs), \]  

(39)

where

\[ A = \sqrt{(1 + 2\ell)^2 + \frac{8m(g + D_x r_x^2)}{h^2}}, \]  

(40)

\[ B = \sqrt{m \hbar^2 \omega^2 + \frac{2m D_x h^2}{\ell^2}}. \]  

(41)

Special Cases:

Case I: When \( g = 0 \), our potential (1) reduces to

\[ V(r) = \frac{m_0 \alpha^2}{2} + D_e \left( 1 - \frac{r}{r_g} \right)^2, \]  

(42)

and the energy equation (38) turns to

\[ E_{n,\ell} = \sqrt{m \hbar^2 \omega^2 + \frac{2m D_x h^2}{\ell^2}} \left( 2n + 1 + \frac{1}{2} \sqrt{(1 + 2\ell)^2 + \frac{8m D_x r_x^2}{h^2}} \right) - 2D_e. \]  

(43)

Case II: When \( D_x = 0 \), our potential (1) becomes

\[ V(r) = \frac{m_0 \alpha^2}{2} + \frac{g}{r^2}, \]  

(44)

The energy equation then becomes

\[ E_{n,\ell} = \sqrt{m \hbar^2 \omega^2} \left( 2n + 1 + \frac{1}{2} \sqrt{(1 + 2\ell)^2 + \frac{8mg}{h^2}} \right) - 2D_e. \]  

(45)

Case III: When \( \omega = 0 \), our potential (1) becomes
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\[ V(r) = \frac{g}{r^2} + D_e \left( \frac{r}{r_e} - \frac{v^*}{r} \right)^2, \quad (46) \]

with energy equation as

\[ E_{n,\ell} = \frac{2nD_e \hbar^2}{\ell^2} \left( 2n + 1 + \frac{1}{2} \sqrt{(1 + 2\ell)^2 + \frac{8n(g + D_e r_e^2)}{\hbar^2}} \right) - 2D_e. \quad (47) \]

**Case IV:** When \( g = D_e = 0 \), our potential (1) turns to

\[ V(r) = \frac{m \omega^2 r^2}{2}, \quad (48) \]

and the energy equation becomes

\[ E_{n,\ell} = \hbar \sqrt{m} \left( 2n + \ell + \frac{3}{2} \right) \omega. \quad (49) \]

**Case V:** When \( g = \omega = 0 \), the potential under consideration becomes

\[ V(r) = D_e \left( \frac{r}{r_e} - \frac{v^*}{r} \right)^2, \quad (50) \]

and energy equation (38) reduces to

\[ E_{n,\ell} = \frac{2nD_e \hbar^2}{\ell^2} \left( 2n + 1 + \frac{1}{2} \sqrt{(1 + 2\ell)^2 + \frac{8nD_e r_e^2}{\hbar^2}} \right) - 2D_e. \quad (51) \]

4. Fisher information

In this section, we calculate Fisher information based on uncertainty relation. Fisher information is a theoretic quantity that determines the accuracy for localizing a particle in a system. Generally, the computation of Fisher information is based on the probability density. In a situation of Schrödinger equation, the probability density is obtained from the radial wave function. The definition of the Fisher information in position space is given as \([35-37]\)

\[ I(\rho) = \int \left[ \frac{\partial \rho(r)}{\rho(r)} \right]^2 dr, \quad (52) \]

and the measured momentum space that corresponds is given by

\[ I(\gamma) = \int \left[ \frac{\partial \gamma(p)}{\gamma(p)} \right]^2 dp, \quad (53) \]

where \( \gamma(p) \) and \( \rho(r) \) respectively denotes the position and momentum densities.

Here, the calculation is based on the expectation values. To obtain the radial expectation values, we use the Hellmann Feynman theory \([38-40]\). If a certain quantum system’s Hamiltonian depends on the parameter \( V \), then, taking the eigenvalues as \( E_{n,\ell}(V) \) and eigenfunction as \( R_{n,\ell}(V) \) of the Hamiltonian, we can write from Hellmann Feynman theory that
\[
\frac{\partial E_{n,l}(V)}{\partial V} = \left\{ R_{n,l}(V) \left[ \frac{\partial H(V)}{\partial V} \right] R_{n,l}(V) \right\}, \quad (54)
\]

provided that \( R_{n,l} \) is continuous with respect to \( V \). In terms of our potential, the effective Hamiltonian is given as

\[
H = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \ell(\ell+1) \frac{1}{2m r^2} + \frac{g+D_\ell r^2}{r^2} + \frac{D_\ell r^2}{r^2} + \frac{m \omega^2 r^2}{2} - 2D_e. \quad (55)
\]

When \( V = D_e \) and \( V = \mu \), then the expectation values of \( r^2 \) and \( p^2 \) for multiple potential are obtain as follows

\[
\langle r^2 \rangle = \frac{m^3}{\hbar^2} + \frac{h^2 r_e^2}{2m \omega D_e} \left[ 2n + 1 + \frac{1}{2} \sqrt{(1 + 2\ell)^2 + \frac{8m(g+D_\ell r_e^2)}{\hbar^2}} \right] \omega, \quad (56)
\]

and

\[
\langle p^2 \rangle = \frac{2m D_e}{\omega} \left\{ \frac{h^2}{2m D_\ell r_e^2 + m^2} \left[ 2n + 1 + \frac{1}{2} \sqrt{(1 + 2\ell)^2 + \frac{8m(g+D_\ell r_e^2)}{\hbar^2}} \right] - \frac{4m D_e}{\sqrt{(1 + 2\ell)^2 + \frac{8m(g+D_\ell r_e^2)}{\hbar^2}}}
\right\} \quad (57)
\]

Using expectation value, Fisher information is given as [41]

\[
I(\gamma) = 4\langle p^2 \rangle \quad (58)
\]

for Fisher information in momentum space and

\[
I(\rho) = 4\langle r^2 \rangle \quad (59)
\]

for Fisher information in position space.
Fig 1: Energy of the spin symmetry for $1p_{3/2}$ (S1), $2d_{5/2}$ (S2) and $3p_{1/2}$ (S3) against the equilibrium bond length $r_e$.  

Fig 2: Energy of the pseudospin symmetry for $1s_{1/2}$ (P1), $0g_{7/2}$ (P2) and $2s_{1/2}$ (P3) against the equilibrium bond length $r_e$. 
Fig 3: Energy of against the angular frequency for the ground state and the first two excited states with $g = D_c = r_e = m = \ell = \hbar = 1$. 
Fig 4: Energy of against the equilibrium bond length for the ground state and the first two excited states with $g = D_o = m = f = \ell = h = 1$. 
Fig 5: Energy of the pseudoharmonic potential against the angular frequency for the ground state and the first two excited states with $D_e = m = \ell = \hbar = 1$. 
Fig 6: Fisher information in momentum space $I(\gamma)$ against the angular frequency $2\pi f$ with $g = D_e = r_e = m = \ell = \hbar = 1$ for the first excited state.
Fig 7: Fisher information in position space $I(\rho)$ against the angular frequency $2\pi f$ with $g = D_e = r_e = m = \ell = \hbar = 1$ for the first excited state.
Table 1: The spin symmetric energy levels of a Dirac particle subject to a combination of the isotropic harmonic oscillator, Pseudoharmonic and inverse potentials for various values of $\kappa$ and $n$.

| $\ell \kappa n$ | ($\ell, f$) | $r_e = 0.5$  | $r_e = 1$  | $r_e = 1.5$  |
|-----------------|-------------|--------------|-------------|--------------|
| 0 -1 0          | 0s_{1/2}    | 13.335984    | 12.506661   | 12.410726    |
| 0 -1 1          | 1p_{1/2}    | 14.034929    | 13.013358   | 12.871811    |
| 0 -1 2          | 2s_{1/2}    | 14.710252    | 13.510580   | 13.321672    |
| 0 -1 3          | 3s_{1/2}    | 15.364713    | 13.993584   | 13.761275    |
| 1 -2 0          | 0p_{3/2}    | 13.911572    | 12.904387   | 12.750202    |
| 1 -2 1          | 1p_{3/2}    | 14.583255    | 13.398202   | 13.199515    |
| 1 -2 2          | 2p_{3/2}    | 15.234600    | 13.879977   | 13.638659    |
| 1 -2 3          | 3p_{3/2}    | 15.867750    | 14.350749   | 14.068449    |
| 2 -3 0          | 0d_{5/2}    | 14.798210    | 13.536107   | 13.301767    |
| 2 -3 1          | 1d_{5/2}    | 15.435992    | 14.010214   | 13.734871    |
| 2 -3 2          | 2d_{5/2}    | 16.056970    | 14.473967   | 14.159105    |
| 3 -4 0          | 0f_{7/2}    | 15.802458    | 14.269030   | 13.953273    |
| 3 -4 1          | 1f_{7/2}    | 16.409200    | 14.724141   | 14.370298    |
| 3 -4 2          | 2f_{7/2}    | 17.001942    | 15.170341   | 14.779605    |
| 3 -4 3          | 3f_{7/2}    | 17.581828    | 15.608259   | 15.181710    |
| 1 1 0           | 0p_{1/2}    | 13.911572    | 12.904387   | 12.750202    |
| 1 1 1           | 1p_{1/2}    | 14.583255    | 13.398202   | 13.199515    |
| 1 1 2           | 2p_{1/2}    | 15.234600    | 13.879977   | 13.638659    |
| 1 1 3           | 3p_{1/2}    | 15.867750    | 14.350749   | 14.068449    |
| 2 2 0           | 0d_{3/2}    | 14.798210    | 13.536107   | 13.301767    |
| 2 2 1           | 1d_{3/2}    | 15.435992    | 14.010214   | 13.734871    |
| 2 2 2           | 2d_{3/2}    | 16.056970    | 14.473967   | 14.159105    |
| 2 3 3           | 3d_{3/2}    | 16.662698    | 14.928174   | 14.575119    |
| 3 3 0           | 0f_{5/2}    | 15.802458    | 14.269030   | 13.953273    |
| 3 3 1           | 1f_{5/2}    | 16.409200    | 14.724141   | 14.370298    |
| 3 3 2           | 2f_{5/2}    | 17.001942    | 15.170341   | 14.779605    |
| 3 3 3           | 3f_{5/2}    | 17.581828    | 15.608259   | 15.181710    |
Table 2: The pseudospin symmetric energy levels of a Dirac particle subject to a combination of the isotropic harmonic oscillator, Pseudoharmonic and inverse potentials for various values of $\kappa$ and $n$.

| $\ell \, n$ | $(\ell, f)$ | $r_e = 0.5$ | $r_e = 1$ | $r_e = 1.5$ |
|---|---|---|---|---|
| 1 | -1 | 1 | -12.756543 | -12.241907 | -12.240687 |
| 2 | -2 | 1 | -10.898272 | -11.300774 | -11.538941 |
| 3 | -3 | 1 | -10.721270 | -10.471787 | -10.428414 |
| 4 | -4 | 1 | -10.954352 | -10.95184 | -10.525837 |
| 1 | -1 | 2 | -12.783584 | -12.255469 | -12.251597 |
| 2 | -2 | 2 | -11.044235 | -11.322134 | -11.553095 |
| 3 | -3 | 2 | -10.845284 | -10.540551 | -10.489042 |
| 4 | -4 | 2 | -11.089858 | -10.670447 | -10.589121 |
| 1 | 2 | 1 | -12.756543 | -12.241907 | -12.240687 |
| 2 | 3 | 1 | -10.898272 | -11.300774 | -11.538941 |
| 3 | 4 | 1 | -10.721270 | -10.471787 | -10.428414 |
| 4 | 5 | 1 | -10.954352 | -10.95184 | -10.525837 |
| 1 | 2 | 2 | -12.783584 | -12.255469 | -12.251597 |
| 2 | 3 | 2 | -11.044235 | -11.322134 | -11.553095 |
| 3 | 4 | 2 | -10.845284 | -10.540551 | -10.489042 |
| 4 | 5 | 2 | -11.089858 | -10.670447 | -10.589121 |

5. Discussion

In Table 1, we presented the positive energy eigenvalues of the (spin symmetry limit) Dirac equation. Since the study did not include tensor interaction, the usual degeneracies were obtained without considering tensor interaction: $0p_{3/2} = 0p_{1/2}, 1p_{3/2} = 1p_{1/2}, 2p_{3/2} = 2p_{1/2}, 3p_{3/2} = 0p_{1/2}, 0d_{5/2} = 0d_{3/2}, 1d_{5/2} = 1d_{3/2}, 2d_{5/2} = 2d_{3/2}, 3d_{5/2} = 3d_{3/2}, 0f_{7/2} = 1f_{5/2}$.

In Table 2, the negative energy eigenvalues for the (pseudospin symmetry limit) Dirac equation were indicated. The following energy degeneracies were produced: $1s_{1/2} = 0d_{3/2}, 1p_{3/2} = 0f_{5/2}, 1d_{5/2} = 0g_{7/2}, 1f_{7/2} = 0h_{9/2}, 2s_{1/2} = 1d_{3/2}, 2p_{3/2} = 1f_{5/2}, 2d_{5/2} = 1g_{7/2}, 2f_{7/2} = 1h_{9/2}$.

In Fig 1, the spin symmetry’s energy for $1p_{3/2}, 2d_{5/2}$ and $3p_{1/2}$ ($S1, S2$ and $S3$) was examined against different values of the equilibrium bond length $r_e$. In Fig 2, we also plotted energy of the pseudospin symmetry for $1s_{1/2}, 0g_{7/2}$ and $2s_{1/2}$ ($P1, P2$ and $P3$) against the equilibrium bond length respectively.

It is seen that the negative energies (energies of the pseudospin symmetry) for the three states are almost the same. In Fig 3, we examined the nonrelativistic energy of the combined potential with the angular frequency $2\pi f$ ($\omega$). It is noticed that the frequency is directly proportional to the energy of the system. A particle under this system vibrates more as the frequency increases because, the particle gains more energy as the frequency increases. This also indicates that the amplitude of the system decreases as the frequency increases which results to more vibrations. It is also noted that as the frequency increases, the spring becomes weaker. This can be seen as the energy tends to increase in a non-linear form. In Fig 4, we plotted energy of the combined potentials against the equilibrium bond length for the ground state and the first two excited states. It is noted that as the equilibrium bond length increases, the
system’s energy is decreased monotonically. However, as the equilibrium bond length increases from 0.4, the system’s energy tends to be constant. This indicates that a particle under this system of combined potentials shows less distinguishable behaviour for $r_e \geq 0.4$. The same effect is observed in the case of pseudoharmonic potential except that the energy of the combined potentials is higher than the energy of the pseudoharmonic potential as shown in Fig 5. In Fig 6, we plotted Fisher information in momentum space against the angular frequency for the ground state. It is noted that as the frequency of the system increases, Fisher information in momentum space decreases. This simply means that there is an increase in uncertainty of the system. This however, reduces how accurate the prediction of particle localization within this system can be. On the other hand, we examined Fisher information in position space in Fig 7, and it was observed that as the frequency increases, Fisher information in position space increases. This indicates less uncertainty in the system and it shows that the prediction of a particle’s localization within this system is more accurate. However, the results of Figs 6 and 7 obeyed Heisenberg uncertainty Principle which agreed with those of the literature.

6. Conclusion
The Dirac equation for a combination of the isotropic harmonic oscillator, Pseudoharmonic and inverse potentials with spin symmetry and pseudospin symmetry can be solved exactly for $\ell$ –wave bound states in terms of the basic concepts of the supersymmetric quantum mechanics approach. The energy and the corresponding two-components spinor wave functions for the $\ell$ –wave bound states have been obtained analytically. With a certain transformation, the energy equation of the Schrödinger equation was obtained from the energy equation of the spin symmetry. Five special cases of the potential were studied in detail. The results obtained for Dirac equation and Fisher information agreed with those of the literature.

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