Part I. The Cosmological Vacuum from a Topological Perspective

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Abstract

This article examines how the physical presence of field energy and particulate matter can be interpreted in terms of the topological properties of space-time. The theory is developed in terms of vector and matrix equations of exterior differential systems, which are not constrained by tensor diffeomorphic equivalences. The first postulate defines the field properties (a vector space continuum) of the Cosmological Vacuum in terms of matrices of basis functions that map exact differentials into neighborhoods of exterior differential 1-forms (potentials). The second postulate requires that the field equations must satisfy the First Law of Thermodynamics dynamically created in terms of the Lie differential with respect to a process direction field acting on the exterior differential forms that encode the thermodynamic system. The vector space of infinitesimals need not be global and its compliment is used to define particle properties as topological defects embedded in the field vector space. The potentials, as exterior differential 1-forms, are not (necessarily) uniquely integrable: the fibers can be twisted, leading to possible Chiral matrix arrays of certain 3-forms defined as Topological Torsion and Topological Spin. A significant result demonstrates how the coefficients of Affine Torsion are related to the concept of Field excitations (mass and charge); another demonstrates how thermodynamic evolution can describe the emergence of topological defects in the physical vacuum.

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1 Preface

In 1993-1998, Gennady Shipov [1] presented his pioneering concept of a modified $A_n$ space of "Absolute Parallelism". Shipov implied that such geometric structures could exhibit "Affine Torsion", $[\mathbb{C}(x^a)] \wedge |dx^k\rangle$, where $[\mathbb{C}(x^a)]$ is the Cartan matrix of Connection 1-forms. Such non-Riemannian structures are different from those induced by Gauss curvature effects associated with Riemannian gravitational fields, for the $A_n$ space defines a domain of zero Gauss curvature. Although others had considered spaces with torsion, Shipov’s work was important to me for it stimulated the realization that the physical vacuum is not a void of nothingness, but indeed is something that can have structure and field properties that are physically measurable [2]. An objective of this article is to demonstrate how ”Affine Torsion”, defined in terms of Cartan’s matrix of connection 1-forms, $[\mathbb{C}(x^a)] \wedge |dx^k\rangle$, leads to interesting thermodynamic and topological conclusions. From a topological perspective, the similarities between the vector field continuum properties of hydrodynamics and electrodynamics are remarkable. The continuum fields with pair, or impair, Affine Torsion 2-forms can delineate topologically between the positive definite property of mass and the indefinite property of charge. In addition, pair and impair 3-forms of Topological Torsion and Topological Spin are to be found in non-equilibrium hydrodynamic systems as well as in non-equilibrium electromagnetic systems. The combination of the two continuum fields leads to the Cosmological Vacuum.

Recall that Arnold Sommerfeld in his formulation [3] of electromagnetism described quantities of field excitations (think D and H related to sources), now known to be representable as coefficients of certain exterior 2-form (impair) densities, $G$. In general, $dG$ is not zero and produces a 3-form (impair) density of ”charge-current”, $J = dG$. On closed domains where $dG = J = 0$, closed integrals of $G$ over cycles (not a boundary) define the number, $n$, of unit charges, $e$, contained within the closed cycle: $\int \int_{2\text{cycle}} G = ne$. Integrals over a boundary in the closed domains are zero, which implies that the charges confined to different cycles can be of opposite sign, and add to zero. Charge, although topologically quantized, has both plus and minus values. For every positive quantity of charge there is an equal and opposite negative quantity of charge, in agreement with the physical conclusion that the
electrodynamic universe is charge neutral in an isolated-equilibrium configuration. This result is due to the fact that impair densities are sensitive to the sign of the determinant of a 4-volume transformation; that is, impair densities are sensitive to the choice of orientation.

Sommerfeld also described fields of intensity, now known to be coefficients of (pair) 2-forms, \( F \) (think \( E \) and \( B \) related to forces). The 2-forms, \( F \), are exact, and therefore are closed over the domain of \( C2 \) definition. In thermodynamic terms, the additive objects of quantity, \( G \), are homogeneous of degree one, and the objects of intensity, \( F \), are homogeneous of degree zero. In deRham’s terminology, electromagnetic excitations, \( G \), are “impair” (exterior differential form densities whose closed integrals are sensitive to orientation) and the intensities, \( F \), are “pair” (exterior differential forms whose closed integrals do not depend upon orientation). It is the “impair” feature that leads to charges of different sign. These developments in topological electrodynamics (see Vol. 4 [4]) guide the development for topological hydrodynamics. However, there are some important differences.

Historically, charge, like mass, had been presumed to be a scalar, and therefore should not be orientation dependent. Closed impair densities, \( G \), yield a pseudoscalar integral \( \int \int_{\text{cycle}} \mathbb{G} \) when integrated over a closed, not bounded, domain. The values of the closed integrals have a sign dependent upon the \( \pm \) choice of orientation [5]. The historical assumptions of charge as a scalar are not compatible with the topological format that \( G \) is impair. E. Post, through his studies of magnetic effects in crystals, demonstrated long ago that if charge was a scalar, magnetic permeability would vanish in crystals that had a center of symmetry, counter to experimental observation of such effects [6]. Post, over the years has repeatedly championed this fact that charge is a pseudoscalar, but only recently has the physical community started to take note of this important, experimentally confirmed, result [7]. Most of the physics community still hangs on to the dogma that charge is a scalar.

As mass is considered to be a positive definite quantity, it should not be presented as having period integrals of 2-forms that are impair. Instead the mass-current 3-forms must be defined as pair 3-form densities, which are not sensitive to orientation, a fact that distinguishes them from the charge-current impair 3-form densities. What is the reason for such differences? Mirror images of mass are of the same sign.

One of the most significant results of the current work herein, which goes beyond Shipov’s focus on possible physical properties of the ”vacuum”, is the fact that the vector of Affine Torsion 2-forms, \( [\mathbb{C}(x^a)] \wedge dx^k \), can consist of either pair 2-form densities, \( |\Sigma\rangle \), or impair 2-form densities, \( |\mathbb{G}\rangle \).

Many investigations that appear in the literature force a self-duality (or even anti-self-duality) on the distinct thermodynamic concepts of quantities and intensities; these constraints can limit the thermodynamic generality and applicability of the constrained theories. Indeed, in classic hydrodynamics, the analog of the Affine Torsion 2-forms are a missing link,
for the theoretical assumptions (based on classical elasticity theory) which invoke symmetric metric hypotheses (the definition of stress) do not permit the generation of Affine Torsion in classical fluids. Once it is recognized that the continuum fields of the physical vacuum need not be a vector space generated by "symmetric" collineations, then fluids, as well as plasmas, can support Affine Torsion or its equivalent: to repeat, the topological theories of both hydrodynamics and electrodynamics, formally, are almost the same! In much of this article, both the hydrodynamic notation and the electrodynamic notation will be used side by side. For most readers I expect the electrodynamic notation (in its engineering format of $E$ and $B$ fields) will be the most readily comprehended. Over the years the concepts of fluids with Affine Torsion structure have stimulated interest in General Relativity, in superconductivity, and even in string theory: the current buzz-word is: "spin fluids" [8], [9].

In electromagnetism the idea of "string-like" particles was utilized, if not invented, by Bateman (1913): ". According to this idea a corpuscle has a kind of tube or a thread attached to it..." (see p131, [23]). What Bateman recognized was that there were two classes of solutions (representing corpuscles of two types!) of the fundamental equations of electromagnetism. The first solution class involves the classic vector ideas, and is related to the standard constitutive ideas relating the two thermodynamic species: $D = \varepsilon E$ and $B = \mu H$. The second solution class was constructed in terms of complex solutions with "zero" length. It is remarkable that this second model of a corpuscle could be related to a "self dual" constitutive relationship of the form, $D = -\sqrt{-1}\gamma B$ and $H = \sqrt{-1}\gamma E$. At the time, Bateman did not recognized that the second solutions were what Cartan (a bit later) defined as Isotropic Spinors. Bateman also failed to realize that such Spinor solutions were generators of conjugate pairs of minimal surfaces (tangential discontinuities). In addition, it is known that minimal surfaces in a domain with a Minkowski signature can exhibit structures that appear to be branes connected by strings (see Vol 2. [4]).
Similar string like ideas were developed by Post who assumed that there were paths of connected points that could represent "collectively" the electron, and that these "topologically collective paths" could be described as "cyclic time" orbits, when evaluated in terms of closed integrals of closed 2-forms. The classic solution to the Gauss period integral corresponds to charges within volumes contained by 2D–cycles. There are however other solutions that correspond to linking of pairs, or knotting of single cyclic paths, and even to "points". These ideas can be extended to spaces of Pfaff Topological Dimension 3, with both impair and pair 3-dimensional period integrals, leading to both the linking of charge triplets of cyclic time paths and the linking of mass triplets of cyclic time paths.

String theorists in elementary particle physics speculate about brane walls being connected by "strings", and introduce concepts such as "dilatons" and "axions", without any experimental evidence of their existence. All of this is done without regard to macroscopic "engineering" examples of fluids that can support Affine Torsion, but are fluids (or plasmas) in non-equilibrium configurations. Fluids that admit Affine Torsion can produce macroscopic structures that match the verbiage of elementary particle "string theorists". Perhaps the most vivid (and most easily produced) macroscopic example of strings and branes is given by the creation of Falaco Solitons (see Vol. 2 [4]). Falaco Solitons are minimal surface dimples (branes) connected by strings. Similar structures were observed by Hopfinger [10] in a rotating tank of a turbulent fluid, which produced Hasimoto vortex kinks. Models of the photon can be constructed with similar topological configurations [11]. Indeed, the

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1For dilatons, think expansion about a fixed point. For axions, think rotations about a fixed point.
chemical make up of soap films appears to be that of a double layer (branes) connected by
molecular strings. These examples are real world exhibitions of a string theory, but they do
not require many dimensions, or many worlds, or quantum mechanics, but they do suggest
that ”string theorists” should apply their kraft to the real world. In the examples that
follow, the common connection between these string and particle concepts relates to the
ability to construct homogeneous exterior differential forms which are closed.

In summary, this article examines how the physical presence of field energy and particulate
matter could influence the topological properties (not only the geometrical properties) of
space-time to form a ”Cosmological Vacuum”. The field part of the Cosmological Vacuum
is defined as a non-global 4D vector space, a continuum, in which are embedded topological
defect structures that play the role of particles. It becomes apparent that the topological
method is not intrinsically dependent on size or shape, and therefore can be used as a universal
tool for studying non-equilibrium continuum systems at all scales.

1.1 The Point of departure

The point of departure in this article consists of three parts:

I. Shipov’s constraint of Absolute Parallelism, is extended to include a larger set of
admissible systems. The larger set is based on the sole requirement that infinitesimal
neighborhoods of a ”Cosmological Vacuum” are elements of a 4D vector space, defined by the
mapping of a vector arrays of exact differentials into a vector arrays of non-exact differential
1-forms. Some authors have called this a ”local mapping”, but in order to preserve the
topological implications, I prefer the words ”infinitesimal mappings”. The method permits
the inclusion of non-trivial bundles when the 1-forms are not integrable.

String theorists in elementary particle physics speculate about brane walls being con-
nected by ”strings”, and introduce concepts such as ”dilatons” and ”axions”\footnote{For dilatons, think expansion about a fixed point. For axions, think rotations about a fixed point.}, without any
experimental evidence of their existence. All of this is done without regard to macroscopic
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molecular strings. These examples are real world exhibitions of a string theory, but they do not require many dimensions, or many worlds, or quantum mechanics, but they do suggest that "string theorists" should apply their kraft to the real world. In the examples that follow, the common connection between these string and particle concepts relates to the ability to construct homogeneous exterior differential forms which are closed.

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The arbitrary matrix of C2 functions with non-zero determinant (that establishes the vector space mapping of differentials) is defined as a Basis Frame, $\mathbb{B}(x^a)$. The class of matrices with non-zero determinant falls into two disjoint sets: those for which the determinant is greater than zero, and those for which the determinant is less than zero. The compliment of the vector space is then defined as those subspace domains where the determinant of $\mathbb{B}(x^a)$ is equal to zero. In other words, the vector space representing the continuum field properties of the Cosmological Vacuum need not be global. When a global object (like a Lie group) is replaced by its infinitesimal version, (the Lie "infinitesimal group"), the system is called the Lie algebra.

The basis vectors that make up the collineation Basis Frame for infinitesimal neighborhoods exhibit topological, differential closure. That is, the differential of any column vector of the Basis Frame is a linear combination of all of the column vectors that make up the Basis Frame. The set of admissible Basis Frames for the vector space of infinitesimal neighborhoods is richer than the set of basis frames for global neighborhoods. The infinitesimal maps need not be integrable, and therefore could represent non-trivial bundle concepts. In this article, the matrix format of exterior differential forms is chosen as the mathematical vehicle of choice, thus removing the "debauch des indices" associated with tensor analysis, and admitting evolutionary processes that are not diffeomorphisms. Moreover, the "twisting" of the "fibers" producing chiral effects (often missed by classical tensor methods) becomes evident, as for any $\mathbb{B}(x^a)$ there are usually two connections that satisfy the conditions of differential closure.

The Basis Frames of interest are not necessarily Symmetric or Orthonormal, which are properties associated with specific gauge constraints imposed on the general system of infinitesimal mappings. This topological point of view emphasizes the concept of a connection,
and minimizes investigation of the concepts that depend upon a metric. However it is recognized that the concept of a signature is of importance to thermodynamic evolution, due to the production of conjugate spinors. The orthogonal group, O(2n), associated with a signature (+ + + +), preserves the euclidean structure. The General Linear group, associated with a signature (+++ -), of complex elements preserves the complex structure [11]. The Symplectic group can be associated with the signature (- - - +). It is to be recognized that the signature (+++ -) leads to Majorana Spinors, and the signature (- - - +) leads to Dirac Spinors.

II. In certain domains of base variables \{x\} the Basis Frame matrix, \([B(x)]\), can be singular, and then one or more of its four (possibly complex) eigenvalues is zero. These singular domains (or objects) may be viewed as topological defects of 3 (topological) dimensions - or less - embedded in the field domain of a 4 dimensional "Cosmological Vacuum". These topological defects can be thought of as condensates, or particles, or field discontinuities, or Spinor Null spaces. The major theme of this article examines the continuum field properties of the "Cosmological Vacuum", which is the domain free of singularities of the type \(\det[B] = 0\). The Basis Frame Matrix \([B]\) will be assumed to consist of C2 functions, but only C1 differentiability is required for deriving a linear connection that defines infinitesimal differential closure. If the functions are not C2, singularities can occur in second order terms, such as curvatures (and accelerations).

Although more complicated, the singular sets admit analysis, for example, in terms of propagating discontinuities and topologically quantized period integrals [13]. These topics will be considered in more detail in a subsequent article.
III. It is recognized that topological coherent structures (fields, and particles, along with fluctuations) in a "Cosmological Vacuum" can be put into correspondence with the concepts of topological thermodynamics based on Continuous Topological Evolution (see Vol. 1, [4], [14]. Perhaps surprising to many, topology can change continuously in terms of processes that are not diffeomorphic. For example, a blob of putty can be deformed continuously into a cylindrical rope, and then the ends can be "pasted" together to create a non-simply connected object from a simply connected object. Topological continuity requires only that the limit points of the initial state topology be included in the closure of the topology of the final state. Such continuous maps are not necessarily invertible; it is important to remember that topology need not be conserved by such continuous processes. Diffeomorphic processes require continuity of the map and its inverse and therefore are specializations of homeomorphisms which preserve topology. This observation demonstrates why tensor constraints cannot be applied to problems of irreversible evolution and topological change [15].

In this article, the use of a Lagrange density and a variational principle to define field equations is not of primary importance. The field equations are generated with the sole constraint that they must represent thermodynamic, continuous, topological evolution and satisfy the dynamical format of the First Law of Thermodynamics. The dynamical format of the First Law of Thermodynamics is generated by use of the Lie differential with respect to a process direction field, $V$, acting on a system of differential forms, $\Xi$, that encode a specific thermodynamic system. The method of thermodynamic evolution can be applied to equilibrium or non-equilibrium thermodynamic systems, to describe reversible or thermodynamically irreversible processes.

2 Topological Structure of a Cosmological Vacuum

2.1 The Fundamental Postulates

2.1.1 Preliminaries

Given a variety of base variables, $\{x^a\}$, projective geometry teaches us that there are two kinds of maps: collineations, $\phi$, and correlations, $\psi$. These concepts are readily delineated by considering the differentials of maps that define contravariant objects and differentials of maps that define covariant objects.
\[ \phi : x^a \Rightarrow V^k(x^a) \]  
\[ d\phi : |dx^a| \Rightarrow [\partial V^k/\partial x^a] \circ |dx^a| = [B^k_a] \circ |dx^a|, \]  
\[ \psi : x^a \Rightarrow A_k(x^a) \]  
\[ d\psi : |dx^a| \Rightarrow [\partial A_k/\partial x^a] \circ |dx^a| = [J^k_{ka}] \circ |dx^a| \]

The matrix \([B^k_a]\) represents a Collineation; The matrix \([J^k_{ka}]\) represents a Correlation. In summary:

\[ [B^k_a] \text{ is a projective Collineation}, \]  
\[ [J^k_{ka}] \text{ is a projective Correlation}, \]  
\[ [g_{ab}] = [B^k_a]^T \circ [\eta] \circ [B] \text{ is a symmetric Metric correlation}, \]  
\[ [g_{ab}] = [g_{ka}] \text{ a metrical congruence}, \]  
\[ [J^k_{ka}] \text{ is not necessarily symmetric}. \]

In that which follows a more general idea is examined: the infinitesimal collineation, \([B^k_a] \circ |dx^a| = |\sigma^k|\), may not be uniquely integrable. On a 4 dimensional variety, this means that for each 1-form \(\sigma^k\) there are 4 possibilities:

\[ [B^k_a] \circ |dx^a| = |\sigma^k| \quad \text{but} \quad \text{Pfaff Sequence} \quad \text{PTD} \]
\[ \sigma = d\chi \quad \text{Case 1} \quad d\sigma = 0 \quad 1 \]
\[ \sigma = \phi d\chi \quad \text{Case 2} \quad d\sigma \neq 0, \quad \sigma^* d\sigma = 0 \quad 2 \]
\[ \sigma = \phi d\chi + d\beta \quad \text{Case 3} \quad \sigma^* d\sigma \neq 0, \quad d\sigma^* d\sigma = 0 \quad 3 \]
\[ \sigma = \phi d\chi + \alpha d\beta \quad \text{Case 4} \quad d\sigma^* d\sigma \neq 0 \quad 4 \]

The Pfaff Topological Dimension (PTD) is a topological property that depends on the functions that define the Collineation \([B^k_a]\); The PTD can vary from point to point in the domain \(\{x^a\}\). In cases 3 and 4 the Frobenius unique integrability criteria fails, and the 1-forms are said to be anholonomic. The classic examples of the failure of unique integrability are given by porisms of envelope solutions, which are not unique, and of characteristics, which represent discontinuities, such as an edge of regression.

2.1.2 Postulate 1: The Cosmological Vacuum Field space

The first postulate of the Cosmological vacuum, assumes the existence of a matrix array of 0-forms (C2 functions), \([B] = [B^\text{conf}_a(x)] = [B^k_a(x)]\), on a 4D variety of points \(\{x^a\}\). The
arbitrary matrix array of functions divides the 4D variety into two topological regions: the Continuum, or field domain, where the determinants of the matrix of functions are not zero; and the compliment of the Continuum, or the domain of topological defects in the continuum, where the determinant of the matrix of functions is zero. This complement of the Continuum can be used to represent particles, condensates, wakes, solitons, and propagating or stationary discontinuities in 4D.

The Field domain is a vector space defined by the invertible matrix basis frames, \([B]\), which map a vector of exact differentials, \(\langle dx^k \rangle\), into a vector of exterior differential 1-forms, \(\langle \sigma^k \rangle\). The matrix, \([B]\), defines a Basis Frame for a vector space constrained to those (not necessarily global) neighborhoods where the inverse Frame, \([B]^{-1}\), exists:

Infinitesimal mappings: \([B] \circ \langle dx^a \rangle = \langle \sigma^k \rangle\).  \(\text{(11)}\)

If the four 1-forms \(\sigma^k\) are integrable, then the Basis Frame describes “holonomic” Frames and a linear connection,

Integrable mappings: \([B] \circ \langle X^a \rangle = \langle Y^k \rangle\),  \(\text{(12)}\)

relating vectors of functions \(\langle X^a \rangle\) to vectors of functions \(\langle Y^k \rangle\). If the four 1-forms \(\sigma^k\) are NOT integrable, then the Basis Frame describes maps to anholonomic 1-forms “which appear naturally even in (pseudo) Riemannian geometry if off–diagonal metrics are considered” \([16]\).

The fundamental assumption, Eq. (11) is interpreted as a map of the differential tangent vector, \(\langle dx^k \rangle\), into differential fibers, \(\langle \sigma^k \rangle\), where, from thermodynamics, the 1-forms, \(\langle \sigma^k \rangle\), will have physical dimensions of Action per unit source, \(\hbar/m_0\).

\([B] \circ \langle dx^a \rangle = \langle \sigma^k \rangle\)  \(\text{(13)}\)

Physical dimension of \(\sigma^k\) = \((\hbar/m_0)\).  \(\text{(14)}\)

The quantity \((\hbar/m_0)\) (the physical dimensions of the 1-forms \(\langle \sigma^k \rangle\)) has the units similar to those of a kinematic viscosity in hydrodynamics. Hence division of \(\sigma^k\) by \(\hbar/m_0\) yields a dimensionless 1-form analogous to the classic Reynolds number approach in classical hydrodynamic theory. The Reynolds number idea is equivalent to the philosophy of self-similarity and homogeneity incorporated into the Buckingham Pi theorem. As in the Renormalization Group, the critical points can be determined in terms of dimensionless variables.

For an angular momentum measured in terms of characteristic mass of galaxies, \(M_0\), velocity, \(V\), and length, \(L\), the characteristic Reynolds number can be quite large, indicating that the “fluid” is more than likely turbulent:

\(^{3}\text{Perhaps first utilized by Onsager in the description of superfluids}\)
Rey = \frac{VL}{\mu_B} = \frac{VL}{(h/M_0)}.
\tag{15}

For particles with characteristic mass, \(m_0\), velocity \(v\), and characteristic size, \(\lambda/2\pi = \hbar/m_0c\), the effective Reynolds number is the ratio of velocities, \(\beta = v/c < 1\). The 1-forms, \(\sigma^k\), can be compared directly to the electromagnetic potentials. The unit "source" or "mole number" then becomes "unit charge".

**Claim 1** The domain for which the determinant of those Basis frames that define infinitesimal maps defines a non-global Vector space. It is assumed that such regions will serve as the domain of the Cosmological Vacuum field.

The Basis Frame, \([B]\), maps a vector array of exact differentials into a vector array of exterior differential forms. Each component of \(\sigma^k\) may or may not be exact. If the components of the 1-forms are integrable in the sense of Frobenius, \(\langle \sigma^* d\sigma^k \rangle = 0\), then the differential mappings are said to define a "trivial" bundle of "tangent vectors". When the components, \(\sigma^k\), are not uniquely integrable, the mappings are said to define a "non-trivial" bundle of tangent vectors. The non-integrability, or anholonomic, feature of the 1-forms, \(\sigma^k\), have led to "N" (for non-linear) connections in Finsler Spaces which can be designed to produce off-diagonal terms in metric congruences of the Basis Frames.

The class of Basis Frames to be studied may have a determinant which is positive definite, or which consists of either a negative domain or a positive domain. In the latter case, it may be true that there exist non-unique (more than one) solutions which can describe the "twisting of the tangent vectors". It is this feature that leads to possible Chiral properties of the Cosmological Vacuum, and enantiomorphisms of orientation.

### 2.1.3 The Cosmological Vacuum Particle space

The compliment of the Cosmological Vacuum field domain, is a singular domain, defined as that set of points where the determinant of the Basis Frame for infinitesimals goes to zero. The singular domain can have many sub-structures. It is assumed that this singular domain is the realm of topologically-coherent defect structures (such as particle condensations, field discontinuities, open strings connecting branes, one dimensional cyclic paths, and even Null vectors, such as Cartan Spinors). The concept of topological coherence implies that these structures have recognizable properties under deformations (where distance is not preserved), and long, if not infinite, evolutionary lifetimes associated with solitons. Topological evolution occurs as these singular objects emerge form the Cosmological Vacuum field, or as one singular structure evolves into another. Thermodynamically, irreversible processes can cause the emergence of the topological defects from the field domain of Pfaff Topological
dimension 4. Examples of such continuous processes can produce the defect structures in finite time, emulating phase changes.

**Claim 2** The Cosmological Vacuum Particle domain is included in the compliment of the Vector Space that defines the Cosmological Vacuum Field domain. It is part of a "singular" domain where the determinant of the Basis Frame that defines the continuum field is zero.

### 2.1.4 Postulate 2: Cosmological Evolution $\approx$ Thermodynamic Evolution

The Cosmological Vacuum is presumed to be a thermodynamic system of exterior differential forms, and its dynamics relative to a process, $V$, must be described in terms of a topological realization of the First Law of Thermodynamics. No other constraints of symmetry, parallel transport, isometry, or diffeomorphic equivalences are placed on the dynamics, except to specialize a particular problem. It is usual to determine dynamical constraints by integral variational methods imposed on some Lagrange density. Such methods are not assumed herein, for typically such methods do not lead to representations of non-equilibrium systems and thermodynamically irreversible processes.

Given any p-form, $\omega$, Cartan’s magic formula expressing the exterior Lie differential of a p-form, defines the First law as a topological statement in terms of deRham cohomology:

$$L(V)\omega = i(V)d\omega + d(i(V)\omega) = W + dU = Q.$$  \hspace{1cm} (16)

When $\omega = A$, defines 1-form of "Action per unit mole (source)", Cartan’s magic formula can be compared, explicitly, to conventional concepts associated with (reversible or irreversible) processes, $V$, acting on (equilibrium or non-equilibrium) thermodynamic systems (see Vol. 1 [4]).

$$W = \text{the virtual Work 1-form},$$  \hspace{1cm} (17)

$$U = \text{the internal energy 0-form},$$  \hspace{1cm} (18)

$$Q = \text{the Heat 1-form}.$$  \hspace{1cm} (19)

This method, based on Continuous Topological Evolution as generated by the Lie differential, transcends the diffeomorphic constraints of tensor analysis. For the Vector space of infinitesimal mappings, any field equations are required to obey the rules of thermodynamic evolution, whereby the physical system is described by a system of 1-forms, $|\sigma\rangle$ and the dynamics is defined with respect to a process direction field, $V$, in in terms if the Lie differential:

$$L(V)|\sigma\rangle = i(V)d|\sigma\rangle + d(i(V)|\sigma\rangle) = W + dU = Q.$$  \hspace{1cm} (20)
2.2 Beyond Diffeomorphic Equivalence

2.2.1 Matrix Methods

The idea is to use matrix representations of differential forms and the exterior matrix product to replicate and expand the ideas generated by the ubiquitous Tensor analysis. Diffeomorphic equivalence - used to define tensors - eliminates the possibility of topological change. Matrix methods, with elements composed of exterior differential forms, do not impose the constraint of diffeomorphic equivalence.

The fundamental assumption is that the Basis Frames, which lead to the Cartan connection, are representations of infinitesimal mappings.

The formalism includes matrices of 0-forms (to defined Basis Frames and vector spaces), matrices of 1-forms (used to define a Connection) matrices of 2-forms (used in the definition of Curvature) as well as matrices of 3-forms (used to define Bianchi identities).

1. Matrices of 0-forms: Basis Frames $[B]$ and metric $[g]$

2. Matrices of 1-forms: Connections, $[C]_{\text{Cartan based on } [B]}$, $[\Gamma]_{\text{Christoffel on } [g]}$, $[T]_{\text{Residue } = [C] - [\Gamma]}$.

3. Matrices of 2-forms: Curvatures, $[\Theta]_{\text{based on } [C]}$, $[\Phi]_{\text{based on } [\Gamma]}$, $[\Sigma]_{\text{based on } [T]}$.

In addition, the formalism utilizes a number of vector arrays of p-forms generated from the fundamental assumption, Eq. (11):

1. Vectors of pair 1-forms are used to define tangent spaces and Action per unit source,

$$ |\sigma^k\rangle \approx e/m_0 |A^k\rangle; $$

the potentials.

2. Vectors of exact pair 2-forms are used to define Field Intensity 2-forms,

$$ F^k = |d\sigma^k\rangle \approx e/m_0 |dA^k\rangle. $$

3. Vectors of pair 3-forms are used to define the Topological Torsion generated by

$$ |\sigma^*d\sigma\rangle \approx (e/m)^2 |A^*dA\rangle = (e/m)^2 |A^*F\rangle = i(T_4)\Omega_4. $$

4. Vectors of pair 4-forms are used to define Topological Parity and bulk dissipation,

$$ |d\sigma^*d\sigma\rangle \approx (e/m)^2 |dA^*dA\rangle = (e/m)^2 |F^*F\rangle = -2(E \circ B)\Omega_4 = PoincareII. $$
5. Vectors of impair 2-form densities are used to define the Affine Torsion fields, \([C] \wedge |dx^n\), of hydrodynamics, as \(|G^k\).

6. Vectors of impair 3-form densities are used to define the closed 3-form currents of electromagnetism, \(|J^k\) = \(|dG^k\)

7. Vectors of impair 3-form pseudoscalar densities are used to define Topological Spin in electromagnetism, \(|A^\ast G\).

8. Vectors of impair 4-form densities are used to define the Poincare Lagrange pseudoscalar density of electromagnetism,

\[
PoincareI = d(A^\ast G) = (F^\ast G - A^\ast J).
\] (25)

9. Vectors of pair 2-form densities are used to define the Affine Torsion fields, \([C] \wedge |dx^n\), of hydrodynamics, as \(|\xi^k\).

10. Vectors of pair 3-form densities are used to define the Torsion Spin 3-forms, \(|\sigma^\ast \xi\).

Each of the formulas are vector indexed \(k\) to represent the elements of the Vectors so described by the assumptions of the Cosmological Vacuum.

Using tensor methods (that impose a Connection arbitrarily) it is easy to overlook the fact that for every Basis Frame, \([B]\), of an infinitesimal vector space, there exists both a Right Cartan Connection, \([C]\), and a Left Cartan Connection, \([\Delta]\). Each Connection is a matrix of 1-forms.

\[
d[B] = [B][C] = [\Delta][B],
\] (26)

\[
[C] = [B]^{-1}[\Delta][B].
\] (27)

The two connections are related by a similarity transformation, which preserves their symmetric properties, but does not exhibit the antisymmetry and chiral properties associated with torsion. The matrix methods, utilized below, correct these deficiencies, and will demonstrate Chirality effects associated with both matrices of 3-forms and with matrices of 1-forms.

### 2.3 Constructive Results for any Basis Frame

By applying algebraic and exterior differential processes, developed by E. Cartan, to each element, \([B]\), of given equivalence class of Basis Frames of C2 functions, the following concepts are derived, not postulated. It is possible to encode these results in terms of
existence theorems, but they are best demonstrated by constructive proofs. Starting from
the fundamental assumption, eq. (11), the following constructions are possible:

A "flat" Right Cartan Connection as a matrix of 1-forms, [C], can be derived, leading
to structural equations of curvature 2-forms and Cartan Torsion 2-forms, both of which are
zero relative to the Cartan Connection. However, the Cartan Connection can support the
concept of non-zero "Affine Torsion". The Right Cartan Connection, [C], is algebraically
compatible with the Basis Frame, and the Vector space that it defines. From the identity
[B] ⋅ [B]−1 = [I], use exterior differentiation to derive the (right) Cartan Connection [C] as
a matrix of 1-forms:

$$\text{Right Cartan Connection : } [C]$$

$$d\ [B] ⋅ [B]^{-1} + [B] ⋅ d\ [B]^{-1} = d\ [I] = 0$$  \hspace{1cm} (28)

hence $$d\ [B] = [B] ⋅ [C]$$,  \hspace{1cm} (29)

where $$[C] = -d\ [B]^{-1} ⋅ [B]$$  \hspace{1cm} (30)

$$= + [B]^{-1} ⋅ d\ [B]$$  \hspace{1cm} (31)

$$= [C_{\text{row}}^{b} \ m(y)dy^{m}] = [C_{\text{column}}^{b} \ m(y)dy^{m}]$$.  \hspace{1cm} (32)

The Connection leads to the idea of differential closure, in the sense that the differential of
any column vector of the Basis Frame is (at a point) a linear combination of the column
vectors that make up the Basis Frame.

It is also possible to construct a Left Cartan Connection matrix of 2-forms, [Δ], relative
to the Frame Matrix, [B] such that:

$$\text{Left : Cartan Connection } [\Delta]$$

$$d\ [B] = [\Delta] ⋅ [B]$$,  \hspace{1cm} (33)

$$[\Delta] = - [B] ⋅ d\ [B]^{-1}$$  \hspace{1cm} (34)

$$= + d\ [B] ⋅ [B]^{-1} = -[C]$$$.  \hspace{1cm} (35)

The coefficients that make up the matrix of 1-forms, [C], can be associated with what have
been called the Weitzenboch Connection coefficients.

The Right and Left Cartan connections are not (usually) identical. They are equivalent
in terms of the similarity transformation:

$$[C] = [B]^{-1} ⋅ [\Delta] ⋅ [B]$$$.  \hspace{1cm} (37)

The left Cartan Connection, in general, is not the same as the transpose of the right Cartan
Connection.

Also note that inverse matrix also enjoys differential closure properties.

$$d\ [B]^{-1} = [B]^{-1} ⋅ [-\Delta] = [-C] ⋅ [B]^{-1}$$$.  \hspace{1cm} (38)
2.3.1 Orthonormal Frames - a special case

If the Basis Frame is orthogonal such that the transpose $[\mathbb{B}]^T$ is equal to the inverse $[\mathbb{B}]^{-1}$, then the Cartan matrix of Connection 1-forms is anti-symmetric.

Orthonormal Tetrads: $[\mathbb{B}]^T = [\mathbb{B}]^{-1}$, $\det[\mathbb{B}] = 1$  \hspace{1cm} (39)

$[\mathbb{C}] = -[\mathbb{C}]^T$.  \hspace{1cm} (40)

The determinant of the orthogonal group can be $\pm 1$, but the orthonormal group implies that the determinant is positive definite. At this level, there is no need to presume that the Basis Frame is an element of the orthonormal group, but this choice is often made as a particular "gauge" constraint in metrical theories. The idea of gauge can be associated with the constraint that the Basis Frame $[\mathbb{B}]$ be an element of a particular subgroup of the general linear group of transformations.

It is somewhat surprising to me, but quite often it is claimed that this anti-symmetric Connection is a "Spin connection". A different point of view is taken in this article.

2.3.2 Symmetric Congruences - the metric

A symmetric quadratic congruence of functions, relative to a signature matrix, $[\eta]$, can be deduced by algebraic methods from Basis Frame. This symmetric congruence can play the role of a metric of 0-forms, $[g]$, compatible with the Basis Frame:

$[g] = [\mathbb{B}]^T \circ [\eta] \circ [\mathbb{B}]$.  \hspace{1cm} (41)

The quadratic form, $[g]$, can be used to generate a Christoffel Connection, $[\Gamma(g)]$, not equal to the Cartan Connection. The Christoffel computation is metric dependent and depends upon the partial derivatives of the symmetric congruence. The Christoffel Connection is not (necessarily) flat, and can produce Cartan curvature equations of structure which are not zero. The Christoffel Connection will not produce Affine Torsion. For a given Basis Frame, $[\mathbb{B}]$, the Cartan Connection, $[\mathbb{C}]$ can be decomposed into the sum of the Christoffel Connection, $[\Gamma(g)]$, and a residue $[T]$.

Connection Decomposition formula $[\mathbb{C}] = [\Gamma(g)] + [T]$  \hspace{1cm} (42)

If the metric is diagonal, then it is possible to construct a special compatible Basis Frame which will generate the diagonal metric as symmetric congruence (see example 1, below).
2.3.3 Matrices of Curvature 2-forms

It is common to define the Cartan matrix of curvature 2-forms, $[\Phi]$, for any arbitrary Connection $[\Gamma]$, in terms of the formula:

$$\text{Matrix of Curvature 2-forms } [\Phi] = [\Gamma] \wedge [\Gamma] + d[\Gamma].$$

(43)

This equation is said to define Cartan’s second equation of structure. Note that exterior differentiation of the Cartan structure matrix of curvature 2-forms is equivalent to a Bianchi identity:

$$[d\Phi] + [d\Gamma] \wedge [\Gamma] - [\Gamma] \wedge [d\Gamma] = 0.$$  \hspace{1cm} (44)

$$[d\Phi] + [\Phi] \wedge [\Gamma] - [\Gamma] \wedge [\Phi] \Rightarrow 0.$$  \hspace{1cm} (45)

This concept of a Bianchi identity is valid for all forms of the Cartan structure equations. The Bianchi statements are essentially definitions of cohomology, in that the difference between two non-exact p-forms is equal to a perfect differential (an exterior differential system). In this case the Bianchi identity describes the cohomology established by two matrices of 3-forms, $[J2] - [J1]$.

$$[d\Phi] = [J2] - [J1],$$  \hspace{1cm} (46)

where $[J1] = [d\Gamma] \wedge [\Gamma]$  \hspace{1cm} (47)

and $[J2] = [\Gamma] \wedge [d\Gamma].$  \hspace{1cm} (48)

These formulas can be interpreted in terms of Chiral properties of the Structure when the closed integrals of each 3-form is not zero, but the closed integrals have equal and opposite signs. Note that it is possible that each 3-form is not exact, but their difference is always exact. Such is the stuff of deRham cohomology theory.

In certain cases the two Chiral species are equivalent, for which the exterior differential of the Curvature vanishes. In other cases, the two Chiral species are NOT equal, and the exterior differential of the Curvature is not zero. It is remarkable that in all cases the exterior differential of each term, though possibly different, cancel each other. These 3-forms represent Vectors of 3-forms, which may or may not have Zero divergence, but in all non-zero cases the divergence of $[RH_3]$ is equal and opposite in sign to the other, $[LH_3]$.

\footnote{The arbitrary connection $[\Gamma]$ is not necessarily a Cartan Connection, $[C]$, defined by a Basis Frame $[B]$.}
2.4 Constructions for Basis Frames of infinitesimals

2.4.1 Infinitesimal Neighborhoods, Affine Torsion

Now (IMO) it is most remarkable that the exterior differential of Eq. (11) leads to the equations,

\[ [B^k_m] \circ [C] \cdot |dx^m \rangle = |F^k \rangle, \]  
\[ d[B^k_m] = [B^k_m] \circ [C], \]

where \([C]\) is the right Cartan matrix of connection 1-forms, relative to the matrix of Basis Functions. The fascinating result is that the formula,

\[ [C] \cdot |dx^m \rangle = |\text{Affine Torsion 2-forms} \rangle, \]

precisely defines the classic vector of Affine Torsion 2-forms!! The Cartan Curvature 2-forms, \(\Theta = \{d[C] + [C] \cdot [C]\}\) are zero, so the space is "flat", but Affine Torsion persists.

2.4.2 Pair differential 2-form densities (mass)

Suppose that the determinant of Basis Frame functions is positive definite. Then concepts of orientation are not important and there exists a "center of symmetry". In such cases, the vector of Affine Torsion 2-forms consists of "pair" exterior differential form densities with integrations that do not depend upon orientation. Such cases will utilize the notation,

\[ \text{Affine Torsion (pair)} \ [C] \cdot |dx^m \rangle = |\Sigma^m \rangle. \]

The period integrals of such forms are always positive, emulating the physical properties of mass.

\[ M = \int \int_{2\text{cycle}} \Sigma > 0 \]

2.4.3 Impair differential 2-form densities (charge)

If the determinant of the Basis Frame is negative, then the vector of Affine Torsion 2-forms will be given the notation:

\[ \text{Affine Torsion (impair)} \ [C] \cdot |dx^m \rangle = |G^m \rangle. \]
The 2-forms $|G^m\rangle$ are sensitive to the orientation and are impair differential form densities. It is this sensitivity to orientation of the volume element that permits the period integrals of $|G^m\rangle$ to have both positive and negative values.

$$Q = \int \int_{2\text{cycle}} G \geq 0.$$ (55)

The two equations relating Affine Torsion 2-forms to field intensities $|F^k\rangle$, appear to be the equivalent of a constitutive equation mapping, mapping the Affine Torsion 2-forms $[C] \wedge |dx^m\rangle$ into field intensities $|F^k\rangle$.

$$[B^k_m] \circ |\Sigma^m\rangle = |F^k\rangle,$$ (56)

$$|\Sigma^m\rangle = \text{pair density 2-forms, or} \quad (57)$$

$$[B^k_m] \circ |G^m\rangle = |F^k\rangle,$$ (58)

$$|G^m\rangle = \text{impair density 2-forms,} \quad (59)$$

In the first case, the unit source will only have positive values (of mass), and in the second case the unit source can have both positive and negative values (of charge).

**Conjecture 3** I contend if $\det([B^k_m]) > 0$ then $[C] \wedge |dx^m\rangle \Rightarrow |\Sigma^m\rangle$ is pair and has periods that represent mass. If $\det([B^k_m]) < 0$ then $[C] \wedge |dx^m\rangle \Rightarrow |G^m\rangle$ is impair and has periods that represent charge.

### 2.4.4 Curvature 2-forms

Note, however, that a second exterior differentiation of the infinitesimal constraint equation, eq. (55), yields:

$$d\{[B] \circ [C] \wedge |dx^k\rangle\} = [B] \circ \{[C] \wedge [C] + d[C]\} \wedge |dx^k\rangle$$ (60)

$$= [B] \circ [\Theta] \wedge |dx^k\rangle = |dd\sigma^k\rangle \approx e/m_0 \ |dA^k\rangle = 0.$$ (61)

Hence the matrix of Curvature 2-forms for a Right Cartan Connection of infinitesimals is zero. The Cartan equations of structure, based on the Right Cartan Connection, $[C]$, indicate that the Basis Frame of infinitesimals defines a vector space of zero curvature, but non-zero Affine torsion. This vector space is defined as the vector space of the fields of the Cosmological Vacuum.

On the other hand, the Christoffel connection, $[\Gamma(g)]$, based on a symmetric congruence always produces a symmetric connection, for which the structural equations permit non-zero curvature, but zero Affine torsion.
If the vector space defined by the infinitesimal mappings, \([B] \circ |dx^k\rangle = |\sigma^m\rangle\) is constrained by a metric field, \([g]\), of congruent mappings, then the Christoffel Connection, \([\Gamma(g)]\), can be generated from the metric coefficients. The Cartan Connection relative to a vector space of infinitesimals can be decomposed into two parts,

\[ [C] = [\Gamma] + [T]. \tag{62} \]

This decomposition will permit the study of how mass and gravity (due to \([\Gamma]\)) can be influenced by Affine Torsion (generated by \([T]\)).

### 2.4.5 Vectors of 2-forms and 3-forms

**Intensities** In both the hydrodynamic case and the electrodynamic case, the system of intensity 2-forms, \(|dA^b\rangle = m_0/e \ |d\sigma^k\rangle\), are formally equivalent to the Maxwell-Faraday equations of field intensities:

\[ |dA^b\rangle = |F^b\rangle \text{ a vector of 2-forms,} \tag{63} \]

\[ |dF^b\rangle \Rightarrow 0, \text{ a vector of 3-forms,} \tag{64} \]

or: the Maxwell Faraday PDE,s. \tag{65}

Each of the exact 2-forms \(|F^b\rangle\) in 4D will generate a Symplectic manifold, if they are of maximal rank, \(F^*F \neq 0\). The notations and equations are identical to within a constant factor. It is the 1-forms and the intensities that lead to the different topological structures, and so the EM notation will be used for both the electromagnetic and the hydrodynamic problems.

**Currents** In the electromagnetic case, the exterior differential of the impair 2-form, \(G\), leads to the concept of a conserved charge-current density. The resulting equations have the format of the Maxwell-Ampere PDE,s.

\[ |J^k_{em}\rangle = d |G^k\rangle, \tag{66} \]

\[ d |J^k_{em}\rangle = 0. \tag{67} \]

In hydrodynamic theory it is most remarkable that the exterior differential of the vector density of Affine Torsion 2-forms, \(|\mathfrak{T}\rangle\), leads to a vector of closed ”fluid mass”-current 3-forms:

\[ |J^k_{fluid}\rangle = d |\mathfrak{T}\rangle = [d[C]^\ast |dx\rangle = -[C]^\ast [C]^\ast |dx\rangle, \tag{68} \]

\[ d |J^k_{fluid}\rangle = 0. \tag{69} \]
This result is similar, but not identical to the "charge"-current 3-form density of electromagnetism.

The mass-current is pair density 3-form, but the charge-current density 3-form is impair. In electromagnetic theory, the excitation 2-forms, $|\mathcal{G}^b\rangle$, are not exact, and the equations that lead to the closed 3-form densities are defined as the Maxwell Ampere equations. In domains where $d|J^k_{\text{em}}\rangle = 0$, closed integrals of $|\mathcal{G}^b\rangle$ can be of opposite sign. For the closed "mass currents", $d|J^k_{\text{fluid}}\rangle = 0$, the 2-forms of Affine Torsion, $|\mathcal{T}\rangle$, need not be exact, but closed integrals in domains where the mass-current vanishes, $|J^k_{\text{fluid}}\rangle = 0$, are now of the same sign: mass is positive definite, charge is not.

The 4-component structure can lead to serious algebraic difficulties, best overcome with a symbolic math processor, such as Maple. In the second part of this article, a series of Maple programs are presented offering the details of many examples. If all but one of the four 1-form components of $|A^k\rangle$ are closed, then the formalism encodes the topological theory of electromagnetism. Several examples based on Particle Affine, Wave-Affine, and the Hopf map are presented in Part II all have this simplistic property, and deserve closed study. Moreover, if the totality of the four 1-form components of $|\sigma^k\rangle$ are not closed, the same starting point encodes the fields that are utilized by Yang Mills theory. Each of these specializations is a topological refinement.

**Spin 3-forms** The next thing to investigate is the concept of Spin 3-forms. Construct the two spin 3-form densities, and the 3-form of Topological Torsion,

$$
\text{Pair Spin 3-form densities} \quad |A^m \cdot \mathcal{T}^m\rangle = |A^m \cdot [B^k_m]^{-1}F^k\rangle, \\
\text{Impair Spin 3-form densities} \quad |A^m \cdot \mathcal{G}^m\rangle = |A^m \cdot [B^k_m]^{-1}F^k\rangle \\
\text{Topological Torsion 3-form} \quad i(T_4)\Omega_n = |A^m \cdot F^m\rangle
$$

**Topological Spin and the First Poincare 4-form** In electromagnetic format, it is possible to construct a vector of impair density 3-forms, $|S^m\rangle$. Each component, $S^m$, will define a value for the Topological Spin generated by each $A^m$ of the 1-forms, $A^m$. The 4-divergence of each component yields the first Poincare function of (twice) the Field Lagrangian density

---

5Realize that each 1-form, $\sigma^m$, can generate a distinct topological structure. Hence there are 4 possibly different topologies imposed upon 4D Space-Time simultaneously.
minus the Field Interaction energy [19].

\[ S^m = |A^m \cdot G^m\rangle_{em}, \quad \text{(no sum)} \]  
\[ (73) \]

\[ \text{Physical dimensions of } S^m = (\hbar) \quad \text{(angular momentum)} \]  
\[ (74) \]

\[ \text{Poincare I 4-form } |dS^m\rangle = \{|F^m \cdot G^m\} - |A^m \cdot dG^m\rangle\} = |L^m\rangle, \]  
\[ (75) \]

\[ \text{The 4-form coefficient of } dS^m = \text{Lagrange Action, } L^m, \]  
\[ (76) \]

\[ \text{In EM format } L^m = \{(B \circ H - D \circ E) - (A \circ J - \rho \phi)\}^m \]  
\[ (77) \]

For the hydrodynamic case, it is possible to construct a vector of pair density 3-forms, \( |S^m\rangle \). In fluid format, each component, \( S^m \), will define a value for the Topological Spin generated by each\(^6\) of the 1-forms, \( \sigma^m \). The 4-divergence of each component yields the first Poincare function of (twice) the Field Lagrangian density minus the Field Interaction energy [19].

\[ \text{Topological Spin 3-forms } |S^m\rangle = |\sigma^m \cdot \varpi^m\rangle_{\text{fluid}} \quad \text{(no sum)} \]  
\[ (78) \]

\[ \text{Physical dimensions of } S^m = (\hbar) \quad \text{(angular momentum)} \]  
\[ (79) \]

\[ \text{Poincare I 4-form } |dS^m\rangle = \{|d\sigma^m \cdot \varpi^m\} - |\sigma^m \cdot d\varpi^m\rangle\} = |L^m\rangle, \]  
\[ (80) \]

\[ \text{The 4-form coefficient of } dS^m = \text{Lagrange Action, } L^m \]  
\[ (81) \]

Note that the existence of Topological Spin depends upon a non-zero value of the vector of Affine Torsion 2-forms. In this sense, Affine Torsion is a necessary, but not a sufficient condition for the existence of Spin. When the Poincare density, \( L \), vanishes, then the Spin 3-form is closed. The 3-dimensional integrals of the closed Spin 3-forms lead to deRham period integrals and topological Quantization of the Spin.

**Topological Torsion and the Second Poincare 4-form** It is possible to construct the vector of 3-forms that represent Topological Torsion, \( |H^m\rangle \), and its divergence, defined as Topological Parity, \( |K^m\rangle \). \( |K^m\rangle \) is a vector array of 4-forms, with coefficients that define the second Poincare function. The physical dimensions of \( K \) are proportional to the square of the kinematic viscosity defined by

\[ \text{Topological Torsion 3-forms } |H^m\rangle = |\sigma^m \cdot d\sigma^m\rangle \approx (e/m_0)^2 \quad |A^m \cdot F^m\rangle, \]  
\[ (82) \]

\[ |K^m\rangle = d |H^m\rangle = |d\sigma^m \cdot d\sigma^m\rangle \approx (e/m_0)^2 \quad |F^m \cdot F^m\rangle, \]  
\[ (83) \]

\[ \text{The Physical dimensions of each } K^m = (\hbar/m_0)^2 = (\hbar/\hbar)^2 = (\hbar/\hbar)^2 = (\hbar/\hbar/\hbar)^2 \]  
\[ (84) \]

\(^6\)Realize that each 1-form, \( \sigma^m \), can generate a distinct topological structure. Hence there are 4 possibly different topologies imposed upon 4D Space-Time simultaneously.
Note that the physical dimensions of each Topological Torsion 3-form have the physical dimensions of a kinematic viscosity squared:

\[(\hbar/m_0)^2 = \mu_B^2.\]  \hspace{1cm} (85)

This kinematic velocity is not the same as a ”shear” viscosity, but is more sensibly described as a ”bulk viscosity” \([20]\). Further note the natural inclusion of the Hall impedance, \((\hbar/e_0^2)\), and its relationship to the ”bulk viscosity”, \((\hbar/m_0)^2\).

Poincare II 4-form \(\langle K^m \rangle = (e/m_0)^2 \langle F^m \cdot \hat{F}^m \rangle\), \hspace{1cm} (86)

In EM format, for each \(m\), \(K^m = (e/m_0)^2\{-2(\mathbf{B} \cdot \mathbf{E})^m\}\Omega\) \hspace{1cm} (87)

**Chiral 1-forms** Perhaps the most intriguing thing about Einstein’s formula\(^7\) (which is used to define the concept of gravity in terms of the deformation of space-time) is that the Einstein tensor \(G^c_m\) is deduced from contractions of the Riemann curvature tensor that produce the ”Ricci tensor”. The formulation of the Einstein tensor, \(G^c_m\), is said to have ”Zero divergence”. The Riemann curvature tensor depends only upon the connection, but the formulation of the Einstein tensor also implicates the metric structure (to raise and lower the indices), as well. Why Einstein chose the particular tensor combination to define gravity is not obvious, for there are many structures that have a zero exterior derivative.

As demonstrated below, it is possible to generate two matrices of 1-forms, with a sum equal to zero, but with a difference which is not zero, but is always exact. Some of the properties of the Einstein tensor can be replicated by a vector of 1-forms that has zero exterior derivative, and is constructed directly from the Connection without recourse to the metric. The formulas demonstrate by construction the concept of a vector of 1-forms, that always has a ZERO exterior differential, and yet can be composed of components, that may exhibit Chiral properties.

By exterior differentiation of the fundamental assumption, eq (11),

\[d\{[\mathbf{B}] \circ |dx^k\}\} = d\{[\mathbf{B}] \circ |dx^k\} = |d\sigma^k\},\] \hspace{1cm} (88)

\{[\mathbf{B}] \circ \mathbb{C}^\top |dx^k\} = [\mathbf{B}] \circ |\mathbb{X}^k\} = |d\sigma^k\}\] \hspace{1cm} (89)

Recall that the vector of 2-forms, \(\langle \mathbb{X}^k \rangle = \mathbb{C} \cdot |dx^k\}, defines the vector of Affine Torsion 2-forms, \(\mathbb{X}^k\), relative to the Right Cartan connection, \([\mathbb{C}]\).

It is to be noted that the inverse Basis Frame, \([\mathbb{B}]^{-1}\), maps differential forms into exact differentials,

\(^7\)The Einstein formula uses the contracted Ricci tensor and the Ricci scalar to construct

\[G^c_m = g^{cb}R^a_{bma} - \delta^c_m R/2\]
\[ dx^k = [B]^{-1} \circ |\sigma^k| \].

(90)

Exterior differentiation leads to the expression,

\[ -\Delta \circ |\sigma^k| = [-\Delta] \circ [B] \circ |dx^k| = -|d\sigma^k|, \]

(91)
to compare with,

\[ [B] \circ [C] \circ |dx^k| = |d\sigma^k|. \]

(92)

It follows that,

\[ \{ [B] \circ [C] + [\Delta] \circ [B] \} \circ |dx^k| = 0. \]

(93)

\[ \{ [B] \circ [C] - [\Delta] \circ [B] \} \circ |dx^k| = 2 |d\sigma^k| \]

(94)
\[ [B] \circ [C] = [RH_1]. \]

(95)
\[ [\Delta] \circ [B] = [LH_1]. \]

(96)

The remarkable result is that the vector of 1-forms, \{ [B] \circ [C] - [\Delta] \circ [B] \} \circ |dx^k|, is not necessarily ZERO, but it is always closed. The 1-form can be decomposed into two component vectors of 1-forms: \([RH_1]\) and \([LH_1]\). It is then possible that:

\[ \{ [B] \circ [C] - [\Delta] \circ [B] \} \circ |dx^k| = 2 |d\sigma^k| \neq 0, \]

(97)
\[ \{ [RH_1] - [LH_1] \} \circ |dx^k| = 0, \]

(98)
\[ d[RH_1] \circ |dx^k| \neq 0, \]

(99)
\[ d[LH_1] \circ |dx^k| \neq 0, \]

(100)
\[ \text{but } d\{ [RH_1] - [LH_1] \} \circ |dx^k| = 2 |dd\sigma^k| = 0. \]

(101)

The exterior differential form, \{ [B] \circ [C] - [\Delta] \circ [B] \} \circ |dx^k| is a vector of 1-forms that is differentially closed (has a zero exterior derivative). Closed forms integrated over cycles (not a boundary) lead to values which have integer ratios. They are a form of deRham period integrals, which are related to the Bohm-Aharanov concepts of quantized circulation.

2.4.6 Chirality and Topological Quantization

Perhaps one of the most interesting concepts is that to topological quantization. The idea is that integrals of closed p-forms over closed domains which are not boundaries have values whose ratios are rational. It is also remarkable that any map to a m dimensional vector valued function can be used to construct a current a volume element of dimension m.
\[ \Psi : x^k \Rightarrow V^m(x^k), \]
\[ \Omega_m = dV^1 \wedge ... \wedge dV^m. \]

Then it is possible to construct a m-1-form "current", \( C \), form the formula:

\[ C = i(\rho V^m)\Omega_m \]

(104)

If the density, \( \rho \) is defined as the inverse of a Holder norm, \( \lambda_H \),

\[ \rho = 1/\lambda_H, \]
\[ \lambda_H = \{a(V^1)^p + b(V^2)^p + ... + \varepsilon(V^m)^p\}^{M/p} \]

(105)
(106)

then the Current has zero divergence for any \( p \) and any anisotropic signature constants, \( a, b, ...\varepsilon, \)

\[ J = i(V^m/\lambda_H)\Omega_m, \]
\[ dJ = (div_m J)\Omega_m = 0. \]

(107)
(108)

Consider the 2D vector \( V = [\varphi, \chi] : \)

\[ V = [\varphi(x, y, z, T), \chi(x, y, z, T)] \]
\[ \Omega_2 = d\varphi^\wedge d\chi, \]
\[ \lambda_H = \{a(\varphi)^p + \varepsilon(\chi)^p\}^{2/p} \]
\[ J = \{\varphi d\chi - \chi d\varphi\}/\lambda_H \]
\[ dJ = 0. \]

(109)
(110)
(111)
(112)
(113)

Note that the conserved current, \( J \), consists of two terms,

\[ J1 = \varphi d\chi/\lambda_H, \quad J2 = \chi d\varphi/\lambda_H, \]

(114)

neither of which need be closed or exact. Yet the difference of the two terms is always closed. This result is another expression of deRham cohomology theory. The divergence condition is valid except in domains where the Holder norm vanishes. The two terms, \( J1 \) and \( J2 \), are related to chirality concepts. The method is easily extended to 4D.

A intriguing idea, as yet unexplored, is what do the Holder Norms with different signatures and exponents, \( p \), imply physically. There is a hint that \( \varepsilon = -1, p > 2 \) may be related to diffraction issues.
2.4.7 The Line Element

Note that the line element, $\delta s^2$, can be constructed from the formula,

$$\delta s^2 = \langle dx^j \circ [g] \circ dx^k \rangle = \langle dx^k \circ [\mathcal{B}] T \circ [\eta] \circ [\mathcal{B}] \circ dx^k \rangle,$$

(115)

$$\delta s^2 = \langle \sigma^m \circ [\eta_{mn}] \circ [\sigma^n] \rangle.$$

(116)

The line element, $\delta s^2$, may or may not be integrable. When the infinitesimal mapping generates non-integrable 1-forms, $|\sigma^n\rangle$, the metric $[g]$ is no longer diagonal. The anholonomic terms generated by the infinitesimal mappings lead to off-diagonal metric coefficients [16].

2.4.8 The Four Forces and the Off-diagonal metric structure.

A number of years ago, it was noticed that the four forces of physics could be put into correspondence with the topological structure of the off-diagonal structure defined in terms of the “timelike” components of the metric, [17]:

$$g_{4m} dx^m = g_{4x} dx + g_{4y} dx + g_{4z} dz - g_{4t} dt.$$

(117)

The analysis depends upon the topological structure generated by the 1-form, $g_{4m} dx^m$, and its Pfaff Topological Dimension of the off-diagonal 1-form, which can be 1, 2, 3, 4. In the original article, the thermodynamic importance of the differential topological structures were not appreciated. In fact the argument for the PTD 1 and 2 cases was made on the basis that they admitted ”infinite range forces”, where the PTD 3 and 4 cases implied ”short range forces”. Several years later, the concepts of topological thermodynamics made it apparent that the PTD 1 and 2 cases meant that the topology was a connected topology, such that infinite range was better stated as: all points were ”reachable”. In the PTD 3 and 4 cases, the thermodynamic topologies form a disconnected topology. Short range was better stated as: points in a disconnected component of the topology were not ”reachable” from points in another component of the disconnected topology. In other words, objects in a disconnected component interact with other objects in that disconnected component, but not with objects in other disconnected components. Parity is preserved for all but the PTD = 4 case.

The results are displayed below:

| PTD | Einstein Solution Examples | Thermodynamic system |
|-----|---------------------------|----------------------|
| 1   | Schwarzschild             | equilibrium          |
| 2   | Reissner-Nordstrum        | isolated             |
| 3   | Godel                     | closed               |
| 4   | Kerr-Taub-Nut             | open                 |

(118)
3 **Continuous Topological Evolution**

The usual technique for generating "equations of motion" is to construct a Lagrange density, $\mathcal{L}$, and integrate it over the volume element, $\Omega$, then determine "trajectories" that will minimize the integral. In addition, certain constraints can be placed upon the Lagrange density, producing trajectories that minimize the integral, subject to the constraints. Such constraints lead to the concept of Lagrange multipliers.

In this article, another approach is utilized. The approach is based upon the continuous evolution of exterior differential forms (see Vol. 1 [4]), and the fact that exterior differential systems contain topological information. The field equations, or equations of motion, must describe the topological dynamics inherent in the First Law of Thermodynamics. In the next section, the correspondence between the continuous topological evolution and the First Law is established in terms of the Lie differential with respect to a process direction field acting on a thermodynamic system encoded in terms of systems of exterior differential forms.

### 3.1 Axioms of Topological Thermodynamics

The theory of non-equilibrium thermodynamics from the perspective of continuous topological evolution, as utilized in this monograph, is based on four axioms.

**Axiom 1.** Thermodynamic physical systems can be encoded in terms of a 1-form of covariant Action Potentials, $A_k(x,y,z,t...)$, on a ≥ four-dimensional abstract variety of ordered independent variables, \{x,y,z,t\...}. The variety supports a differential volume element $\Omega_4 = dx^\wedge dy^\wedge dz^\wedge dt$...

**Axiom 2.** Thermodynamic processes are assumed to be encoded, to within a factor, $\rho(x,y,z,t...)$, in terms of contravariant vector and/or complex isotropic Spinor direction fields, $V_4(x,y,z,t...)$.

**Axiom 3.** Continuous topological evolution of the thermodynamic system can be encoded in terms of Cartan’s magic formula (see p. 122 in [28]). The Lie differential, when applied to an exterior differential 1-form of Action, $A = A_k dx^k$, is equivalent abstractly to the first law of thermodynamics.
Cartan’s Magic Formula \( L_{(\rho V_4)} A = i(\rho V_4) dA + d(i(\rho V_4)A) \).  

First Law : \( W + dU = Q \). 

Inexact Heat 1-form \( Q = W + dU = L_{(\rho V_4)} A \). 

Inexact Work 1-form \( W = i(\rho V_4) dA \). 

Internal Energy \( U = i(\rho V_4) A \).

Axiom 4. Equivalence classes of systems and continuous processes can be defined in terms of the Pfaff Topological dimension of the 1-forms of Action, \( A \), Work, \( W \), and Heat, \( Q \).

In effect, Cartan’s methods can be used to formulate precise mathematical definitions for many thermodynamic concepts in terms of topological properties - without the use of statistics or geometric constraints such as metric or connections. Moreover, the method applies to non-equilibrium thermodynamical systems and irreversible processes, again without the use of statistics or metric constraints. The fundamental tool is that of continuous topological evolution, which is distinct from the usual perspective of continuous geometric evolution.

It is important to remember that thermodynamic processes, encoded by the symbol, \( V_4 \), describing a ”vector” directional field, need not be characterized by a single parameter (as in Newtonian dynamics). In other words, continuum processes may exhibit topological fluctuations about the kinematic perfection of ”point” particles:

\[ \Delta x = dx - V_4 ds \neq 0. \]  

The concept of a thermodynamic process transcends the concept of diffeomorphic trajectories, which represent ”particle - like” Newtonian curves.

Note that the formula for the First Law is in effect a statement in deRham cohomology, where the difference between two inexact differential forms is an exact differential.

Claim 4 The Lie differential acting on a thermodynamic system encoded by the system of forms, \( \Xi \), is the dynamical equivalent of the First Law of thermodynamics.

The processes, \( V_4 \), can be reversible or irreversible in a thermodynamic sense. The definition of an irreversible process was motivated by Caratheodory and Morse \([21], [22]\), and is given by the statement that the heat 1-form, \( Q \), does not satisfy the Frobenius integrability...
theorem. Combining the Caratheodory definition and the Cartan magic formula yields the expressions:

The necessary condition for thermodynamic irreversibility of a process \( \mathbf{V}_4 \) acting on a thermodynamic system, \( \Xi \), is given by the expression,

\[
\text{Irreversible Processes: } Q^*dQ \neq 0
\]

\[
L(\mathbf{V}_4)\Xi^*L(\mathbf{V}_4)d\Xi = Q^*dQ \neq 0.
\]

These requirements can be used to demonstrate that many "dissipative" systems are not thermodynamically irreversible, but are merely time reversal invariant. Note that thermodynamic processes need not generate a diffeomorphism. In other words, thermodynamic processes can change the system topology, and that change can be occur continuously.

### 3.2 The Thermodynamic Evolution of the Cosmological Vacuum.

Consider the Cosmological Vacuum to be a thermodynamic system encoded by the infinitesimal mapping equation:

\[
\text{The Thermodynamic System: } [\mathbb{B}] \circ |dx^k\rangle = |\sigma^k\rangle = e/m_0 |A^k\rangle,
\]

Physical dimension of \( \sigma^k = (\hbar/m_0) \).

The thermodynamic equations of topological evolution are defined in terms of the action of the Lie differential with respect to of process \( \mathbf{V} \) acting on the thermodynamic system:

\[
L(\mathbf{V})\{[\mathbb{B}] \circ |dx^k\rangle\} = L(\mathbf{V})\{|\sigma^k\rangle\} = e/m_0 L(\mathbf{V})\{|A^k\rangle\}
\]

First, evaluate the bracket factor on the RHS which is assumed to represent the evolutionary properties of the field intensities:

\[
L(\mathbf{V})\{|A^k\rangle\} = i(\mathbf{V})|dA^k\rangle + d(i(\mathbf{V})|A^k\rangle,
\]

\[
= i(\mathbf{V})|F^k\rangle + d|U^k\rangle = W + dU = Q.
\]
Next, evaluate the LHS which is assume to represent the evolutionary properties of the field excitations:

\[ L(V)\{[B] \circ |dx^k\} = i(V^k)d([B] \circ |dx^k\}) + d(i(V^k)[B] \circ |dx^k\}) \]  
(133)

\[ = i(V^k)[B] \circ [C] \circ |dx^k\} + d\{[B] \circ |V^k\} \]  
(134)

\[ = [B] \{i(V^k)|T^b\} + ([C] \circ |V^k\} + |dV^k\}) \} \]  
(135)

\[ = [B] \{i(V^k)|C^b| \circ |dx^k\} - [C] \circ |V^k\} \]  
(136)

\[ + [C] \circ |V^k\} + |dV^k\}) \} \]  
(137)

So the fundamental thermodynamic equation describing the thermodynamic evolution of the Cosmological Vacuum becomes (a vector equation of 1-forms) balances the evolution of the field excitations with the evolution with the field intensities:

The Fundamental Thermodynamic Field Equations: 

\[ LHS, \quad \text{Excitations} \quad [B] \{i(V^k)|T^b\} + |dV^k\} = e/m_0 Q, \]  
(139)

\[ RHS, \quad \text{Intensities, where} \quad \{i(V^k)|d\sigma^k\} + d\{i(V^k)|\sigma^k\} = e/m_0 Q. \]  
(140)

The three terms in the first bracket of eq. (135) can be identified in terms of

\[ \text{Affine Torsion } "accelerations" : i(V)|T^b\}, \]  
(141)

\[ 1/2 \ Coriolis \ "accelerations" : [C] \circ |V^k\}, \]  
(142)

\[ "Acceleration 1-forms" : |Acc^k\} = |dV^k\} \]  
(143)

These identifications lead to the fundamental thermodynamic evolution formula of infinitesimal neighborhoods. The "accelerations" of the space-time properties related to source excitations are balanced by the field intensities generated by those 1-forms of Action per unit source, that encode the thermodynamic systems.

It should be remembered that there can be 4 Cartan topologies defined by the structural equations associated with each of the 4 1-forms that make up the components of |d\sigma^k\}. In examples to be explored below, certain simplifications can lead to Basis Frames for which only one of the 4 topological structures possible is not exact. The Hopf map furnishes such an example. In addition, there are notable differences between the Basis Frames that represent the 13 parameter groups of transitive Particle-Affine maps and intransitive Wave-Affine maps.

The Fundamental Formula is of the form:
\[ [B] | \text{Torsion} + \text{Coriolis}/2 + \text{Acc} \rangle = \frac{\varepsilon}{m_0} \{ i(V) | F \rangle + | dU \rangle \} = \frac{\varepsilon}{m_0} Q. \]

The formula is algebraically correct and should apply to all vector spaces on space time defined by the Fundamental Axiom. It is remarkable that this thermodynamic equation of evolution (a vector of 1-forms) formally links the Torsion, Coriolis, and Acceleration 1-forms to the 1-form of heat, \( Q \), generated by the RHS and the 1-form of Action per unit mass (mole), \( | \sigma^k \rangle \).

Note that when the Affine Torsion components do not exist, the RHS is zero, and the remaining terms lead to the expression where rotational motion is balancing an acceleration, in agreement with Newton’s laws of circular motion at constant angular velocity. In addition, note that the formula implies that the concept of Heat requires non-zero valued of the Affine Torsion 2-forms.

### 3.3 Processes that Conserve the 4D-Volume Element

Consider evolutionary processes, \( J \), that conserve the volume element:

\[ L(J)\Omega = di(J)\Omega = 0. \tag{144} \]

Note that the 3-form defined by the equation,

\[ J = i(J)\Omega, \tag{145} \]

must be closed if the volume element is to be preserved:

\[ dJ = di(J)\Omega = 0. \tag{146} \]

Also note that the 3-forms, \( | J^b \rangle \), defined by the exterior derivative of the Affine Torsion 2-forms, \( | J^b \rangle = d | G^b \rangle \) generates closed 3-forms.

**Claim 5** Hence the charge current densities generated by the field excitations (the Affine Torsion 2-forms) preserve the 4D volume element.

#### 3.3.1 Invariant Volume in Electrodynamical notation

It is important to recognize that the vector field, \( J_4 \), has components directly related to the component functions of the associated 2-form. In Electromagnetic notation:

\[ i(J_4)\Omega = | J^b \rangle = d | G^b \rangle, \tag{147} \]

\[ | \text{div}J_4\Omega \rangle = d | J^b \rangle = dd | G^b \rangle = 0. \tag{148} \]
Similar results are applicable for the vectors of Topological Torsion, $T_4$, and Topological Spin, $S_4$:

$$|i(T_4)\Omega| = |A^* dA| = |A^* F|,$$  \hspace{1cm} (149)

$$|\text{div}\, T_4\Omega| = |F^* F|,$$  \hspace{1cm} (150)

$$|i(S_4)\Omega| = |A^* G|,$$  \hspace{1cm} (151)

$$|\text{div}\, S_4\Omega| = |F^* G| - |A^* J|.$$  \hspace{1cm} (152)

However, these 4-vectors, $T_4$ and $S_4$, do not necessarily preserve the 4D-volume element.

### 3.3.2 Invariant volume in Hydrodynamic notation

Similarly, in hydrodynamic notation,

$$i(J_4)\Omega = |J^b| = d|\Sigma^b|,$$  \hspace{1cm} (153)

$$|\text{div}\, J_4\Omega| = d|J^b| = dd|\Sigma^b| = 0.$$  \hspace{1cm} (154)

Similar results are applicable for the vectors of Topological Torsion, $T_4$, and Topological Spin, $S_4$:

$$|i(T_4)\Omega| = |\sigma^* d\sigma|,$$  \hspace{1cm} (155)

$$|\text{div}\, T_4\Omega| = |d\sigma^* d\sigma|,$$  \hspace{1cm} (156)

$$|i(S_4)\Omega| = |\sigma^* \Sigma|,$$  \hspace{1cm} (157)

$$|\text{div}\, S_4\Omega| = |d\sigma^* \Sigma| - |\sigma^* J|.$$  \hspace{1cm} (158)

However, these 4-vectors, $\Sigma_4$ and $S_4$, do not necessarily preserve the 4D-volume element. Recall that for the general problem there are $k$ of these 4-vectors, one for each of 4 Cartan topologies defined by the four 1-forms, $|\sigma^k|$. This makes symbolic math program like Maple a necessity to overcome the very tedious algebra. Such examples will be presented in part II.

### 3.4 Processes that Expand/Contract the 4D Volume element

#### 3.4.1 Topological Torsion and conformal invariance of $\Omega$ in electrodynamical notation

The concept of Topological Torsion is defined by a Vector (process), $T_4$, and, for the electromagnetic notation, is composed entirely from the 1-form of Action per unit source, $A$. For
each component of the vector of intensity 1-forms, \(|A^k\rangle\)

\[
\begin{align*}
|i(T^k_4)\Omega\rangle &= |A^k \cdot F^k\rangle, \\
|\text{div}T^k_4 \Omega\rangle &= |F^{k \cdot F^k}\rangle,
\end{align*}
\]  

(159)

(160)

Each of the \(k\) 4-vectors, \(T^k_4\), do not necessarily preserve the 4D-volume element. In fact, for each \(A^k\),

\[
L_{(T_4)}\Omega = \kappa\Omega,
\]

where,

\[
L_{(T_4)}\Omega = \{2(B \circ E)\}_{\text{electromagnetism}}\Omega.
\]

(161)

(162)

3.4.2 Topological Torsion and conformal invariance of \(\Omega\) in hydrodynamic notation

The concept of Topological Torsion in hydrodynamic notation is composed entirely from the 1-form of Action per unit source, \(\sigma\). For each component of the vector of intensity 1-forms, \(|\sigma^k\rangle\)

\[
\begin{align*}
|i(T^k_4)\Omega\rangle &= |\sigma^k \cdot d\sigma^k\rangle, \\
|\text{div}T^k_4 \Omega\rangle &= |d\sigma^{k \cdot d\sigma^k}\rangle,
\end{align*}
\]

(163)

(164)

Each of the \(k\) 4-vectors, \(T^k_4\), do not necessarily preserve the 4D-volume element. In fact, for each \(\sigma^k\),

\[
L_{(T_4)}\Omega = \kappa\Omega,
\]

where,

\[
L_{(T_4)}\Omega = \{2(\text{vorticity} \circ \text{acceleration})\}_{\text{hydrodynamics}}\Omega.
\]

(165)

(166)

When the coefficient, \(\kappa\), is not zero, the processes acting on the volume element is said to produce conformal invariance. When \(\kappa\) is a constant, it can be interpreted as a homogeneity index of degree \(\kappa\). Note that \(\kappa\) need not be an integer, which can then be interpreted as a fractal self similarity condition. Further note that \(\kappa\) can be both spatially and time dependent, ultimately leading to zero value, and stability of the volume element.

Claim 6 This result indicates that a topological torsion process, \(T^k_4\), can cause the cosmological vacuum to expand (or contract) irreversibly, depending on the sign of the bulk viscosity coefficient, \(2(B \circ E) \approx 2(\omega \circ a)\)
The contraction or expansion of the Universe (the 4D volume element) has been expressed in terms of a "dilaton" field, but the concept has its engineering roots in terms of "bulk viscosity" \[20\], without reference to General Relativity. The engineering idea forces attention on the fact that dilaton methods in General Relativity are thermodynamically irreversible.

Recall that for the general problem there are \(k\) of these 4-vectors, one for each of 4 Cartan topologies defined by the four 1-forms, \(\sigma^k\). This makes symbolic math program like Maple a necessity to overcome the very tedious algebra. A number of Maple examples will be attached as Part II.

3.4.3 **Topological Spin and conformal invariance of \(\Omega\) in electrodynamic notation**

The concept of Topological Spin is defined by a Vector (process), \(S_4\), that is composed from both the components of the 1-forms, \(A^k\) and the vector of two forms representing Affine Torsion, \(G^k\). Results similar to those enumerated above are applicable for the vectors of Topological Torsion, \(S_4\). For each component of the vector of intensity 1-forms, \(A^k\)

\[
\begin{align*}
|i(S_4^k)\Omega\rangle &= |A^k \cdot G^k\rangle, \\
|\text{div}S_4^k\Omega\rangle &= |F^k \cdot G^k\rangle - |A^k \cdot J^k\rangle, \\
|J^k\rangle &= d|G^k\rangle.
\end{align*}
\]

In electromagnetic notation, the 4-divergence of the Topological Spin process is to be recognized as the Field Lagrange density\(^8\) minus the Field (charge current and potential) interactions.

\[
|\text{div}S_4^k\Omega\rangle = (\{B \circ H - D \circ E\} - \{A \circ J - \rho \phi\})\Omega \quad (170)
\]

**Claim 7** This result indicates that a topological Spin process, \(T_4\), can cause the cosmological vacuum to expand (or contract), depending on the sign of the Lagrange coefficient, \(\{B \circ H - D \circ E\} - \{A \circ J - \rho \phi\}\).

It is obvious that a combination of the two processes should be studied.

\[
E_4 = T_4 + (\hbar/e^2)S_4. 
\]

\[
L_{(E_4)}\Omega = \{2(B \circ E) + (\hbar/e^2)(B \circ H - D \circ E - A \circ J + \rho \phi)\}\Omega. \quad (172)
\]

\(^8\)Note that expression is not \(\{B \circ H - D \circ E\}/2\) (see \[19\]).
Note that coefficient \( (\hbar/e^2) \) is the Hall coefficient, and is used for dimensional equivalence in the composite formula. Note that the identification of Affine Torsion with the excitation field, \( |G^k\rangle \), permits the formalism presented to be interpreted in terms of either topological electromagnetism, or topological hydrodynamics.

**Claim 8** In classical hydrodynamics, the concept of the excitation fields has "slipped through the net". Herein it is recognized that the concept of Affine Torsion defines the excitation fields for both hydrodynamics and electromagnetism. Hence the concept of Topological Spin can be developed for classical hydrodynamic systems, when Affine Torsion is taken into account. However, Affine Torsion 2-forms are necessary, but not sufficient, for the creation of 3-forms of Topological Spin.

It is remarkable that a combination of the Topological Spin and the Topological Torsion processes could influence the expansion rate of the universe. In particular it is conceivable that the two process paths, each of which is not divergence free, could yield a composite that is divergence free. This concept is another exhibition that there can exist two non-exact forms with a difference that is exact. Simply said, the idea is that a "rotation" can balance a "contraction". Resaid, the divergence of Topological Spin - a rotation due to a non-zero Lagrangian - could balance divergence of Topological Torsion - a contraction due to Bulk viscosity.

### 4 Diffusion and Conformal Evolution

In this section attention will be paid to the RHS of the fundamental equation, where certain constraints will produce a form of a diffusion equation. This idea goes back to Bateman [23], where he demonstrated that solutions to a diffusion equation in 2+1 space-time could be transformed into solutions to the wave equation in 3+1 space-time. It is easier (for me) to use the topological formulation of electromagnetism (which is formally equivalent to the topological theory of hydrodynamics). In thermodynamics the fundamental unit source is the mole. In electromagnetism the fundamental unit source is the electric charge; division of the 1-form of Action, \( |A^k\rangle \), by the factor \( (\hbar/e) \) leads to equations that are free from "physical dimensions". In hydrodynamics, the fundamental unit source is mass, and the division of the 1-form of Action, \( |\sigma^k\rangle \), by the "kinematic viscosity", \( (\hbar/m) \), leads to a Reynolds number type of formulation, where the Action 1-form is dimensionless in terms of the "physical dimensions".

To reduce the algebra, only one component of the vector of field intensities, \( |A^t\rangle \), will be treated as a non-exact 1-form of Pfaff Topological Dimension 4. The Basis Frame will
be a member of the W-Affine Group. The thermodynamic evolution of the field potentials generated by $A^t$ are given by the first law as:

$$L(V_4)A^t = i(V_4)dA^t + d(i(V_4)A^t) = Q.$$  \hspace{1cm} (173)

The thermodynamic evolution of the limit points $dA^t$ are given by the expression,

$$L(V_4)dA^t = d(i(V_4)dA^t) = dQ.$$  \hspace{1cm} (174)

The next step is to reconsider those processes which preserve the 4D volume element, and those which permit an expansion (or contraction) of the 4D volume element in a homogenous manner. There are three distinct classes of processes, each of which can be associated with a minimal hypersurface (mean curvature = 0). The wave equation, diffusion equation, and the equation of a minimal surface are all related to a form of a null divergence condition on a vector field, and are thereby related to some form of a conservation law, or minimization process. For example consider the variety \{x, y, z; \xi\} with a volume N-form, $\Omega = dx \hat{\times} dy \hat{\times} dz \hat{\times} d\xi$. Also consider a contravariant vector field $J$ with components

$$J = \rho V = \rho(x, y, z, t)[V^x, V^y, V^z, 1]$$  \hspace{1cm} (175)

Then the volume $\Omega$ is an invariant with respect to an evolutionary path generated by $J = \rho V$ if $\text{div}_4 J = 0$. That is,

$$L(\rho V)\Omega = d(i(J)\Omega) = \{\text{div}_4 J\}\Omega = \Omega \{\partial(\rho V^x)/\partial x, \partial(\rho V^y)/\partial y, \partial(\rho V^3)/\partial z, \partial\rho/\partial \xi\} \Omega$$

\Rightarrow 0, \text{ when } \{\text{div}_4 J\} = 0.  \hspace{1cm} (176-178)

Therefor, the invariant volume element, $\Omega$, is associated with a process that has zero 4-divergence.

However, suppose the process does not have zero divergence, but is equal to some function, $\Psi$. Then, the differential volume element is not invariant with respect to the process, but it is conformal:

$$L(\rho V)\Omega = d(i(J)\Omega) = \{\text{div}_4 J\}\Omega = \Psi \Omega.$$  \hspace{1cm} (179)

### 4.1 The Wave Equation

Examine the case where:

$$V_4 = \rho[\partial \phi/\partial x^k; \varepsilon S] \quad \rho = \text{constant}, \quad S = \partial \phi/\partial \xi.$$  \hspace{1cm} (180)
Then the null divergence condition becomes:

\[
div J = \rho \, \text{div} V = \rho \{ \partial^2 \phi/\partial x^2 + \partial^2 \phi/\partial y^2 + \partial^2 \phi/\partial z^2 + \varepsilon \partial^2 \phi/\partial \xi^2 \} \Rightarrow 0. \quad (181)
\]

When \( \varepsilon = -1 \) and \( \xi = ct \), the null divergence constraint is exactly the wave equation.

Suppose the density function was not constant. Then the zero divergence condition would be given by the expression:

\[
div J = \text{div}(\rho V) = \{ \partial^2 \phi/\partial x^2 + \partial^2 \phi/\partial y^2 + \partial^2 \phi/\partial z^2 + \varepsilon \partial^2 \phi/\partial \xi^2 \} + (\text{grad}_4 \ln \rho) \circ V = 0. \quad (182)
\]

This equation describes a modified wave equation, but if the density function is a first integral (a process invariant) then the term \((\text{grad}_4 \ln \rho) \circ V\) vanishes, and the standard form of the wave equation is recovered.

4.2 The Diffusion Equation

Similarly, examine the case where:

\[
\mathbf{V}_4 = \rho [\partial \psi/\partial x^k; \varepsilon \psi], \quad \rho = \text{constant}, \quad S = \phi. \quad (183)
\]

then the null divergence condition becomes:

\[
div J = \rho \, \text{div} V = \rho \{ \partial^2 \psi/\partial x^2 + \partial^2 \psi/\partial y^2 + \partial^2 \psi/\partial z^2 + \varepsilon \partial \psi/\partial \xi \} \Rightarrow 0. \quad (184)
\]

When \( \varepsilon = -1 \) and \( \xi = t/D \), the null divergence constraint is exactly the diffusion equation,

\[
D \partial \psi/\partial t = \partial^2 \psi/\partial x^2 + \partial^2 \psi/\partial y^2 + \partial^2 \psi/\partial z^2 \quad (185)
\]

Again if the density is not a constant, then a modified diffusion equation is the result,

\[
\{ \partial^2 \psi/\partial x^2 + \partial^2 \psi/\partial y^2 + \partial^2 \psi/\partial z^2 + \varepsilon D \partial \psi/\partial t \} + (\text{grad}_4 \ln \rho) \circ V \Rightarrow 0. \quad (186)
\]

4.3 The Minimal Surface equation

Examine the case where \((\lambda_H \text{ is a Holder norm})\) relative to any vector field, \(\mathbf{V}_4\):

\[
\mathbf{V}_4 = [V^k; V^s], \quad \rho \mathbf{V}_4 = \mathbf{V}_4/\lambda_H, \quad (187)
\]

\[
\lambda_H = [a(V^x)^p + b(V^y)^p + c(V^z)^p + \varepsilon(V^s)^p]^{N/p}. \quad (188)
\]

\( k = 1,2,3 \)
Consider the 3-form constructed from the equation,
\[ C = i(\mathbf{V}/\lambda_H)dV^x \wedge dV^y \wedge dV^z \wedge dV^s = i(\mathbf{V}/\lambda_H)\Omega_{(\mathbf{V})}. \] (189)

For any anisotropic signature \((a,b,c,\\epsilon)\) and any exponent \(p\), then if \(N = 4\), the 3-form is homogeneous of degree zero. It follows that the trace of the Jacobian matrix \([\text{Jac}(\mathbf{V}/\lambda_H)]\) is zero. The surface defined by the characteristic polynomial of \([\text{Jac}(\mathbf{V}/\lambda_H)]\) is a 4th order polynomial = 0, representing a hypersurface in 4D, which has zero mean curvature. A hypersurface with zero mean curvature is a Minimal Surface. Hence the renormalized vector, \((\mathbf{V}/\lambda_H)\), has zero divergence on \(\Omega_{(\mathbf{V})}\) for all values of the Holder norm, if \(N = 4\).

It has been assumed that the determinant of \([\text{Jac}(\mathbf{V}/\lambda_H)]\) is maximal rank and non-zero. Therefore, the zero divergence condition is also valid on differential volume element, \(\Omega = dx \wedge dy \wedge dz \wedge ds\). The wave equation and the diffusion equation are special cases of the minimal surface equation.

### 4.4 The Characteristic Polynomial

In 4D, the characteristic polynomial of \([\text{Jac}(\mathbf{V}/\lambda_H)]\) is of 4th degree, where for (possibly complex) eigen values, \(\gamma\), the polynomial (by the Cayley Hamilton theorem) generates the hypersurface,

\[ Ch[Jac(\mathbf{V}/\lambda_H)] = \gamma^4 - M\gamma^3 + G\gamma^2 - A\gamma + K = 0, \] (190)
\[ = \gamma^4 - (4 - N)\gamma^3/\lambda_H + (6 - 3N)\gamma^2/\lambda^2_H - (4 - 3N)\gamma/\lambda^3_H + (1 - N)/\lambda^4_H. \] (191)
\[ = (\gamma\lambda_H - 1)^3 (\gamma\lambda_H - 1 + N) = 0. \] (192)

For different values of \(N\), the Holder norm creates different homogeneity criteria. It was demonstrated above that for a minimal surface the trace of the matrix \([\text{Jac}(\mathbf{V}/\lambda_H)]\) vanishes. It is easily demonstrated (with Maple) that

\[
\begin{align*}
\text{Mean Curvature: } M &= (4 - N)/\lambda_H = \text{Trace } [\text{Jac}(\mathbf{V}/\lambda_H)], \tag{193} \\
\text{Gauss Curvature: } G &= (6 - 3N)/\lambda^2_H, \tag{194} \\
\text{Cubic Curvature: } A &= (4 - 3N)/\lambda^3_H, \tag{195} \\
\text{Quartic Curvature: } K &= (1 - N)/\lambda^4_H. \tag{196}
\end{align*}
\]

So, as mentioned above, for \(N = 4\), the trace of \(Ch[Jac(\mathbf{V}/\lambda_H)]\) vanishes and the Mean Curvature is zero of the hypersurface is zero; the 4-divergence of the process \(\mathbf{V}/\lambda_H\) is zero, and the volume element is invariant for such all such homogeneous processes (\(N=4\)).
If the Mean curvature is not zero, then divergence of the homogeneous process depends upon the Holder norm, $\lambda_H$. Hence the conformal factor, $\Psi$, in the equation,

\[ L(\rho V)\Omega = d(i(J))\Omega = \{\text{div}_4 J\}\Omega = \Psi \Omega, \]  

\[ \{\text{div}_4 J\} = \Psi = (4 - N)/\lambda_H \neq 0. \]

is well defined for any $N$, and for all forms of the Holder norm. When $N > 4$ the volume is contracting; when $N < 4$, the volume is expanding, due to the homogeneous process $V_4/\lambda_H$.

Now return to the “diffusion” format,

\[ V_4 = [\partial \psi/\partial x, \partial \psi/\partial y, \partial \psi/\partial z, \varepsilon \psi], \quad \rho V_4 = V_4/\lambda_H, \]  

\[ \lambda_H = [a(\partial \psi/\partial x)^p + b(\partial \psi/\partial y)^p + c(\partial \psi/\partial z)^p + \varepsilon (\psi)^{p^N/p}]. \]

Then the divergence of $V_4/\lambda_H$ then yields a modified diffusion equation:

\[ \text{div}_4 (V_4/\lambda_H) = (\text{div}_4 V_4)/\lambda_H - V_4 \circ \text{grad}(\lambda_H)/(\lambda_H)^2 = (4 - N)/\lambda_H, \]  

\[ = (\text{div}_4 V_4) - V_4 \circ \text{grad}_4(\lambda_H)/(\lambda_H) = (4 - N)/\lambda_H. \]  

\[ -(\varepsilon/D)\partial \psi/\partial t = \partial^2 \psi/\partial x^2 + \partial^2 \psi/\partial y^2 + \partial^2 \psi/\partial z^2 - (4 - N) - V_4 \circ \text{grad}_4(\ln \rho). \]

If the Characteristic Polynomial defines a minimal surface $M = 0$ for $N = 4$, then the Gauss curvature is negative for any signature, indicating the surface is unstable. The volume element is invariant, but the Minimal Surface is unstable. Suppose that $N \leq 2$; then the Gauss curvature is positive, indicating that the Hypersurface generated by the Characteristic polynomial is stable, but it is not a Minimal surface. However, the Volume element is expanding.

**Conjecture 9** Is the Expansion of the universe required to stabilize the Hypersurface generated by the Characteristic polynomial?

There is one singular case where the homogeneity index $N = 0$. Then the 4D differential volume element becomes singular. However, the analysis can be extended to the 3D case:

\[ V_3 = [U, V, W], \]  

\[ \lambda_H = [a(U)^p + b(V)^p + c(W)^p]^n/p \]  

\[ Ch [\text{Jac}(V_3/\lambda_H)] = \gamma^3 - M\gamma^2 + G\gamma^1 - A = 0. \]

Mean Curvature: $M = (3 - n)/\lambda_H$, 

Gauss Curvature: $G = (3 - 2n)/\lambda_H^2$, 

Cubic Curvature: $A = (1 - n)/\lambda_H^3$, 

\[ \lambda_H = [a(U)^p + b(V)^p + c(W)^p]^n/p \]  

\[ Ch [\text{Jac}(V_3/\lambda_H)] = \gamma^3 - M\gamma^2 + G\gamma^1 - A = 0. \]  

\[ Mean\ Curvature: \quad M = (3 - n)/\lambda_H, \]  

\[ Gauss\ Curvature: \quad G = (3 - 2n)/\lambda_H^2, \]  

\[ Cubic\ Curvature: \quad A = (1 - n)/\lambda_H^3, \]
where the curvature similarity invariants are independent from the constant anisotropy coefficients, \( \{a, b, c\} \) and the exponent \( p \).

5 The Continuum Field for a Plasma (or a Fluid)

5.1 Affine Torsion and Excitation 2-forms

As mentioned above, the vector of Affine Torsion 2-forms \( \mathbf{T}^b = [C^b_a] \cdot dx^a \) has coefficients that can be put into 1-1 correspondence (to within a factor) with the concept of ”excitation fields” \( G^b(D, H) \) in classical electromagnetism. It is important to realize that the use of the words ”Affine torsion” to describe the antisymmetric coefficients of a Cartan connection is unfortunate, and has nothing to do with whether or not the Basis Frame matrix is a member of the Affine group, or one of its subgroups. Classically, the affine group is a \textit{transitive} group of 13 parameters in 4D, (see p.162 in Turnbull \[24\]). The anti-symmetry concept related to Affine Torsion is described by the same formula that defines the vector of excitation 2-forms, \( \mathbf{T}^b = [C^b_a] \cdot dx^a \), for any Basis Frame \([B(x)]\). For example, the torsion formula holds equally well for Basis Frames which are elements of the 15 parameter projective group, which is not affine.

\[ \mathbf{T}^b = [B]^{-1} \circ d\sigma^k, \]

The vector of 2-forms \( \mathbf{T}^b \) is formally equivalent (for each index \( b \)) to the (impair, or odd) 2-form (density) of the field excitations (D and H) in electromagnetic theory (see Vol. 4 \[4\]). In the notation of electromagnetism, the source of field excitations (and, consequently, topological charge and and topological spin) is due to the Affine Torsion components of the Cartan Connection.

Note that the matrix \([B]^{-1}\) plays the role of the Constitutive map between \( E, B \) and \( D, H \).

\[ \mathbf{T}^b = [B]^{-1} \circ d\sigma^k, \]

If the global (integrability) assumption, \( [F(x)] \circ x^a \Rightarrow y^k \), is imposed, then it is possible by exterior differentiation to show that a constraint must be established between the excitation 2-forms and the Cartan Curvature 2-forms constructed from the globally integrable Basis Frames, \([F(x)]\):

\[ [B]^{-1} \approx \text{a Constitutive map} \]
\[
[F(x)] \circ \{[[C_F] \circ |x^a\rangle + |dx^a\rangle\} = |dy^b\rangle,
\]

such that \([C_F] \circ |dx^a\rangle = -\{d[C_F] + [C_F] \cdot [C_F]\} \circ |x^a\rangle.
\]

Hence as the vector of excitation 2-forms \(|G^b\rangle\) has been defined in terms of the Cartan Connection, two different results are obtained for the two different types of Basis Frames:

\[
|T^b\rangle_B = [C_B] \cdot |dx^a\rangle
\]
\[
|T^b\rangle_F = [C_F] \cdot |dx^a\rangle.
\]

The integrability condition places a constraint on the Cartan Curvature and the Affine torsion coefficients of the Cartan Connection, \([C_F]\), which is not equivalent to the constitutive map (eq 213) given above:

\[
|T^b\rangle_F = -\{d[C] + [C] \cdot [C]\} \circ |x^a\rangle
\neq [B]^{-1} \circ |d\sigma^k\rangle.
\]

The result demonstrates that the set of infinitesimal Basis Frames is much different from the global set of Basis Frames.

None of this development depends upon the explicit specification of a metric, reinforcing the fact that Maxwell’s theory of Electrodynamics is a topological, not a geometric theory. Again, remember that the electromagnetic notation is used as a learning crutch to emphasize the universal ideas of the Cosmological Vacuum. The formulas are valid topological descriptions of the field structures of all continuum “fluids”.

Herein, for simplicity, it is assumed that all functions of the Basis Frame are at least C2. However, note that the definitions of the matrix of connection 1-forms \([C]\) and the vector of 2-forms \(|d\sigma^k\rangle\) only require C1 functions.

### 5.2 The Lorentz Force and the Lie differential (EM notation)

The Lorentz force is a derived, universal, concept in terms of the thermodynamic cohomology. It is generated by application of Cartan’s Magic formula [28] to the 1-form \(A\) that that encodes all or part of a thermodynamic system. The system, \(A\), can be interpreted as an electromagnetic system, or a hydrodynamic system, or any other system that supports continuous topological evolution. For the purposes herein apply Cartan’s magic formula to the formula for infinitesimal mapping produced by the matrix multiplication of a vector of perfect (exact) differentials of the base variables given by eq(??).
Recall that the exterior derivative of any specific 1-form, $A^k$, if not zero, can be defined as a 2-form with coefficients of the type

$$F = dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j \wedge dx^k = F_{jk}dx^j \wedge dx^k \quad (223)$$

The specialized notation for the coefficients used above is that often used in studies of electromagnetism, but the topological 2-form concepts are universal, independent from the notation.

Given any process that can be expressed in terms of a vector direction field, $V = \rho[V, 1]$, and for a physical system, or component of a physical system, that can be encoded in terms of a 1-form of Action, $A$, the topological evolution of the 1-form relative to the direction field can be described in terms the Lie differential:

$$L(V)A = i(V)dA + d(i(V)A) \quad (224)$$
$$= i(V)F + d(i(V)A) \quad (225)$$
$$= \rho\{E + V \times B\}dx^k - \rho\{E \cdot V\}dT \quad (226)$$
$$+ d(\rho A \cdot V - \rho \phi) \quad (227)$$
$$= \text{Work due to Lorentz force - dissipative power} \quad (228)$$
$$+ \text{change of internal interaction energy}. \quad (229)$$

Note that if the notation is changed (such that the vector potential is designated as the velocity components of a fluid), then the "Lorentz force" represents the classic expression to be found in the formulation of the hydrodynamic Lagrange Euler equations of a fluid (see Vol. 3 [4]). A fluid, based upon a 1-form of Action of Pfaff Topological dimension 2 (or greater) obeys the topological equivalent of a Maxwell Faraday induction law!

**Remark 10** The universal concept of a Lorentz force is derived from the properties of a "Cosmological Vacuum", and does not require a separate postulate of existence.
5.3 Processes that leave the intensity 2-form, \( F \), invariant, or conformally invariant

If a process \( V_4 \) is to preserve the 2-form of field intensities, \( F \), then

\[
L(V_4)dA = L(V_4)F = d(i(V_4)dA) = dQ = 0. \tag{230}
\]

This constraint identifies the process as expressing the Helmholtz theorem (conservation of vorticity) in hydrodynamics.

Now consider the equations in EM format,

\[
V_4 = [V_3, 1] \tag{231}
\]

\[
i(V_4)F = -(\mathbf{E} + \mathbf{V}_3 \times \mathbf{B})_k dx^k - (\mathbf{E} \circ \mathbf{V}_3) dT, \tag{232}
\]

\[
di(V_4)F = (\partial(\mathbf{E} + \mathbf{V}_3 \times \mathbf{B})/\partial T + \nabla(\mathbf{E} \circ \mathbf{V}_3))_k dT^* dx^k + \{\text{curl}(\mathbf{E} + \mathbf{V}_3 \times \mathbf{B})\}^{x} dy^x dz \tag{233}
\]

\[
+ \{\text{curl}(\mathbf{E} + \mathbf{V}_3 \times \mathbf{B})\}^{y} dz^y dx \tag{234}
\]

\[
+ \{\text{curl}(\mathbf{E} + \mathbf{V}_3 \times \mathbf{B})\}^{z} dz^z dy, \text{ where} \tag{235}
\]

\[
(\text{curl}(\mathbf{E} + \mathbf{V}_3 \times \mathbf{B}) = -\partial B/\partial T + \mathbf{B} \circ [\nabla V_3] - V_3 \circ [\nabla B] + (\text{div} V_3) \mathbf{B} \tag{236}
\]

If \( dQ \) is equal to zero, then there are 6 equations of constraint:

\[
\partial(\mathbf{E} + \mathbf{V}_3 \times \mathbf{B})/\partial T + \nabla(\mathbf{E} \circ \mathbf{V}_3) = 0, \tag{238}
\]

\[
-\partial \mathbf{B}/\partial T + \mathbf{B} \circ [\nabla V_3] - V_3 \circ [\nabla B] + (\text{div} V_3) \mathbf{B} = 0. \tag{239}
\]

It is apparent that the last equation represents a diffusion equation for the components of the vector \( \mathbf{B} \). This method was utilized in a 2+1 space to demonstrate that the Schroedinger equation admitted and exact mapping that permitted the square of the wave function to be interpreted as the Enstrophy (square of the vorticity) in a hydrodynamic format [27]. The method is remindful of Ricci flows, but does not depend upon metric explicitly.

The next concept is to examine conformal invariance of the 2-form, \( F \). The equations are similar to those above:

\[
L(V_4)dA = L(V_4)F = d(i(V_4)dA) = \kappa F. \tag{240}
\]

The analysis leads to the constraints:

\[
\partial(\mathbf{E} + \mathbf{V}_3 \times \mathbf{B})/\partial T + \nabla(\mathbf{E} \circ \mathbf{V}_3) = \kappa \mathbf{E}, \tag{241}
\]

\[
-\partial \mathbf{B}/\partial T + \mathbf{B} \circ [\nabla V_3] - V_3 \circ [\nabla B] + (\text{div} V_3) \mathbf{B} = \kappa \mathbf{B}. \tag{242}
\]
Again, the conformality factor can represent a homogeneity condition, or a fractal self-similarity condition under the evolutionary process. Such processes are modifications of a diffusion equation.

5.4 Processes that leave the excitation 2-form, $G$, invariant, or conformally invariant

If a process $V_4$ is to preserve the 2-form of field excitations, $G$, then

$$L(V_4)G = i(V_4)dG + d(i(V_4)G) = 0. \quad (243)$$

This constraint identifies the process as expressing the Helmholtz theorem (conservation of vorticity) in hydrodynamics.

Now consider the equations in EM format,

$$V_4 = [V_3, 1] \quad (244)$$

$$i(V_4)G = -\{(H + V_3 \times D)_k dx^k - (H \circ V_3)dt\}, \quad (245)$$

$$di(V_4)G = (\partial(H + V_3 \times D) / \partial T + \nabla(H \circ V_3))_k dT^k dx^k + \{\text{curl}(H + V_3 \times D)\}^z dy^z dz \quad (246)$$

$$+ \{\text{curl}(H + V_3 \times D)\}^y dz^y dx \quad (247)$$

$$+ \{\text{curl}(H + V_3 \times D)\}^z dx^z dy, \text{ where} \quad (248)$$

$$(\text{curl}(H + V_3 \times D) = \partial D / \partial T + J_3 + D \circ [\nabla V_3] - V_3 \circ [\nabla D] + (\text{div} V_3)D - (\text{div} D)V_3, \quad (249)$$

$$i(V_4)dG = (V_3 \times J_3)_k dT^k dx^k \quad (250)$$

$$+ (J_3^z - \rho \nabla V_3^z) dy^z dz + (J_3^y - \rho \nabla V_3^y) dz^y dx + (J_3^z - \rho \nabla V_3^z) dx^z dy. \quad (251)$$

If $L(V_4)G$ is equal to zero, then there are 6 equations of constraint:

$$\partial(H + V_3 \times D) / \partial T + \nabla(H \circ V_3) + (V_3 \times J_3) = 0, \quad (253)$$

$$\partial D / \partial T + D \circ [\nabla V_3] - V_3 \circ [\nabla D] + (\text{div} V_3)D = 0. \quad (254)$$

It is apparent that the last equation represents a diffusion equation for the components of the vector $D$.

The next concept is to examine conformal invariance of the 2-form, $F$. The equations are similar to those above:

$$L(V_4)G = \varepsilon G. \quad (255)$$
The analysis leads to the constraints:

\[ \frac{\partial (H + V_3 \times D)}{\partial T} + \nabla (H \circ V_3) + (V_3 \times J_3) - \varpi H = 0, \quad (256) \]
\[ \frac{\partial D}{\partial T} + D \circ [\nabla V_3] - V_3 \circ [\nabla D] + (\text{div} V_3)D - \varpi D = 0. \quad (257) \]

Again, the conformality factor, \( \varpi \), can represent a homogeneity condition, or a fractal self-similarity condition under the evolutionary process. As before, such processes are modifications of a diffusion equation.

It is important to note that these equations represent the process constraints that preserve the 2-forms of Affine Torsion \( \approx |G| \) conformally.

### 5.5 Period Integrals and Topological Quantization.

Although the main interest of this article is associated with the field properties of the Cosmological Vacuum, a few words are appropriate about the topological defect structures (that will be treated in more detail in another article). Of specific interest are those topological structures represented by closed, but not exact, differential forms. Such exterior differential forms are homogeneous of degree zero expressions. Such closed structures can lead to deRham period integrals \[13\], whose values have rational ratios, when the integration chain, \( z_1 \), is also closed, and not equal to a boundary. For example, the Flux Quantum of EM theory is given by the closed integral of the (electrodynamic) 1-form of Action:

\[ \text{Flux quantum} = \oint A = \oint A = n \frac{\hbar}{e} \quad (258) \]

In domains where \( F = dA \Rightarrow 0 \quad (259) \)

This period integral is NOT dependent upon the electromagnetic field intensities, \( F \). Stokes theorem does not apply if the integration chain is not a boundary. All integrals of exact forms over boundaries would yield zero.

As another example, the Period integral of quantized charge is given by the expression,

\[ \text{Charge quantum} = \iint_{z_2} G = n \left( \frac{\hbar}{e} \right)^2 \quad (260) \]

In domains where \( J = dG \Rightarrow 0. \quad (261) \)

The integration is over a closed chain, \( z_2 \), which is not a boundary. Again, the quantized Period integral does not depend upon the charge current 3-form, \( J \), and the expression is
valid only in domains where $J \Rightarrow 0$. These same properties of topological quantization are
universal ideas independent from the notation, or some topological refinement to a specific
types of physical systems. Note that the concept of the Charge quantum depends upon
the existence of $G$ which in turn depends upon the existence of Affine Torsion of the Cartan
Connection $[C]$ based on $[B]$.

Similar results apply to the 3-forms of Topological Torsion and Topological Spin:

$$\text{Chirality (Helicity) quantum} = \iiint_{z^3} A \wedge F = n \left( \frac{\hbar}{e} \right)^2$$

In domains where $d(A \wedge F) = 0$. 

$$\text{Spin quantum} = \iiint_{z^3} A \wedge G = n \left( \frac{\hbar}{e} \right)$$

In domains where $d(A \wedge G) = 0$.

5.6 The Source of Charge and Spin

A number of years ago, it became apparent to me that the origin of charge was to be asso-
ciated with topological structures of space time $[25]$. The concepts exploited in that study
assumed the existence of an impair 2-form of field intensities $G$. Based on the developments
above, it can be stated:

Remark 11 *The theory of a Cosmological Vacuum asserts that the existence of charge is
dependent upon the Affine Torsion of the Cartan connection $[C]$.*

This result is based upon the formal correspondence between equations of electromagnetic
field excitations $|G\rangle$ and the ”Affine torsion” coefficients (not the Cartan torsion coefficients)
deduced for the Cartan Connection matrix of 1-forms $[C]$.

$$\text{Excitation 2-forms } |\Sigma\rangle = [C] \wedge |dy\rangle \text{ Affine Torsion 2-forms.}$$

The impair 2 forms that compose the elements of $|\Sigma\rangle$ formally define the field excitations
in terms of the coefficients of ”Affine torsion”. The closed period integrals of those (closed
but not exact) components of $|\Sigma\rangle$, which are homogeneous of degree 0, lead to the deRham
integrals with rational, quantized, ratios. Such excitation 2-forms $\Sigma$ do not exist if the
coefficients of Affine torsion are zero.
The topological features of the Cosmological Vacuum are determined by the structural properties of the Basis Frames, \([\mathcal{B}]\), and the derived Cartan Connection matrix of 1-forms, \([\mathcal{C}]\).

**The Topological Structure of the Cosmological Vacuum in terms of \([\mathcal{B}], [\mathcal{C}], \text{and } [\Gamma]\):**

- **Mass** \(\sim\) Non Zero Curvature \([\Phi]\) based on metric generated Christoffel Connection \([\Gamma]\).
  \[ [\Phi] = [\Gamma]^\dagger [\Gamma] + d[\Gamma]. \]
  Note that the Cartan Curvature based on \([\mathcal{C}]\) is zero.

- **Charge** \(\sim\) Non Zero Affine Torsion of \([\Xi]\) based on the Cartan right Connection \([\mathcal{C}].\)
  \[ [\Xi] = [\mathcal{C}] \wedge [dy]. \]
  The Cartan Torsion is Zero, but the Affine Torsion need not be Zero.

Where the presence of mass is recognized in terms of the Riemannian curvature of the quadratic congruences of a Cosmological Vacuum, the presence of charge is recognized in terms of the coefficients of Affine Torsion of a Cosmological Vacuum.

### 5.7 Quadratic Congruences and Metrics

Starting from the existence of a Linear Basis Frame, it is remarkable that symmetric properties of \([\mathcal{B}]\) can be deduced in terms of a quadratic congruence (see p. 36 in Turnbull and Aitken [26]). The quadratic congruence is related to the concept of strain in elasticity theory, and is quite different from the linear definition of matrix symmetries in terms of the sum of a matrix \([\mathcal{B}]\) and its transpose \([\mathcal{B}]^T\). The algebraic quadratic congruence will be used to define 0compatible symmetric (metric) qualities in terms of the structure of the Basis Frames, \([\mathcal{B}]\):

\[
[g] = [\mathcal{B}]^T \circ [\eta] \circ [\mathcal{B}].
\]  \hspace{1cm} (267)

The matrix \([\eta]\) is a (diagonal) Sylvester signature matrix whose elements are \(\pm 1\). Recall that in projective geometry the congruence transformation based upon \([\mathcal{B}]^T\) defines a correlation, where the similarity transformation based upon \([\mathcal{B}]^{-1}\) defines a collineation. Note that the ubiquitous choice of an orthonormal basis frame, where \([\mathcal{B}]^T = [\mathcal{B}]^{-1}\), would place limits the topological generality of the concept of a Cosmological Vacuum.

In a later section it will be demonstrated how the similarity invariants of the Basis Frame find use in representing thermodynamic phase functions appropriate to the Cosmological Vacuum.

**Remark 12** *Note that this quadratic (multiplicative) symmetry property is not the equivalent to the (additive) symmetry property defined by the linear sum of the matrix \([\mathcal{B}]\) and its transpose.*
However, from the Basis Frame, it is also possible to construct a topological exterior differential system that defines a quadratic form of the law of differential closure. It then follows that, for \( d [\eta] = 0 \),

\[
\begin{align*}
  d [g] &= d \left( [\mathbb{B}]^T \circ [\eta] \circ [\mathbb{B}] + [g] \right) = [\mathbb{B}]^T \circ [\eta] \circ d [\mathbb{B}], \\
  d [g] &= [\tilde{\mathbb{C}}_r] \circ [g] + [g] \circ [\mathcal{C}_r].
\end{align*}
\]

The fact that the differential \( d [g] \) is the sum of two 1-forms is a topological property, known as the metricity condition:

**Metricity condition:** \( d [g] - [\tilde{\mathbb{C}}_r] \circ [g] - [g] \circ [\mathcal{C}_r] \Rightarrow 0. \) (270)

The thermodynamic evolution of the metric can be expressed in terms of Cartan’s magic formula:

\[
L_{(\text{V}_4)} [g] = i(\text{V}_4) d [g] \text{ thermodynamic evolution of the metric}
\]

\[
= \{ i(\text{V}_4) [\tilde{\mathbb{C}}_r] \} \circ [g] + [g] \circ \{ i(\text{V}_4) [\mathcal{C}_r] \}. \] (272)

It is apparent that the thermodynamic evolution of the metric can depend upon the path, \( \text{V}_4 \), as well as the connection, \( [\mathcal{C}_r] \). It is apparent that if \( d[g] = 0 \), then the thermodynamic evolution of the metric is invariant.

If the metricity condition is satisfied, then the metric is a thermodynamic evolutionary invariant. This constraint is not presumed in this article. Note that his equation is an exterior differential system, and therefore defines topological properties.

Compute the Christoffel connection, and its matrix of 1-forms, \( [\Gamma] \), from the quadratic "metric" matrix \( [g] \), using the Levi-Civita-Christoffel formulas.

**Coefficients : Christoffel Connection**

\[
\Gamma^b_{ac}(\xi^c) = g^{be} \{ \partial g_{ce}/\partial \xi^a + \partial g_{ea}/\partial \xi^c - \partial g_{ac}/\partial \xi^e \}, \] (274)

\[
[\Gamma] = \left[ \Gamma^b_{ac} dy^c \right] \text{ as a matrix of 1-forms} \] (275)

The Christoffel Connection also satisfies the metricity condition,

**Metricity condition:** \( d [g] - [\Gamma] \circ [g] - [g] \circ [\Gamma] \Rightarrow 0. \) (276)

### 5.7.1 Thermodynamic Killing Vectors

The concept of a homogeneous or conformal mapping is given by the expression,
Process direction fields that have this property are defined as Killing vectors. If $\kappa = 0$, then the metric is said to be a process invariant of the flow generated by the Killing vector. If $\kappa \neq 0$, then the mapping is said to be conformal. Note that if $\kappa = 1$, the process indicates that the thermodynamic evolution of the metric is self-similar.

5.8 The vector of zero forms (Internal Energy)

Once again consider the Lie differential with respect to a direction field $V$, operating on the formula for differential closure

\[ L(V) [g] = i(V) d [g] = \kappa [g]. \tag{277} \]

From Koszul’s theorem, $|W^a| = i(V) d |A^a|$ is a covariant differential based on some (abstract) connection (for each $a$). Hence, the difference between the Lie differential and the Covariant differential is the exact term, $d(i(V) |A^a|)$:

\[ L(V)(|A^a|) - i(V) d |A^a| = d(i(V) |A^a|) = d |h^a|. \tag{281} \]

This equation is another statement of Cohomology, another exterior differential system, where the difference of two non-exact objects is an exact differential.

From the topological formulation of thermodynamics in terms of Cartan’s magic formula,

\[ \text{Cartan’s Magic Formula } L(\rho V_4) A = i(\rho V_4) dA + d(i(\rho V_4) A) \tag{282} \]

\[ \text{First Law } : W + dU = Q, \tag{283} \]

\[ \text{Inexact Heat 1-form } Q = W + dU = L(\rho V_4) A \tag{284} \]

\[ \text{Inexact Work 1-form } W = i(\rho V_4) dA, \tag{285} \]

\[ \text{Internal Energy } U = i(\rho V_4) A, \tag{286} \]

Now consider particular process paths (defined by the directional field $\rho V_4$), and deduce
that in the direction of the process path

\[ i(\rho \mathbf{V}_4)W = 0, \quad (287) \]

Work : is transversal; \hspace{1cm} (288)

\[ i(\rho \mathbf{V}_4)Q = i(\rho \mathbf{V}_4)dU \neq 0 \quad (289) \]

Heat : is not transversal;

but if \[ i(\rho \mathbf{V}_4)Q = 0, \quad (291) \]

the process : is adiabatic. \hspace{1cm} (292)

It is the non-adiabatic components of a thermodynamic process that indicate that there is a change of internal energy and hence an inertial force in the direction of a process. This implies that the non-adiabatic processes are inertial effects, and could be related to changes in mass.

Now to paraphrase statements and ideas from Mason and Woodhouse, (see p. 49 [29]), and [30] :

**Remark 13** "Then there is a Higgs field \( \phi_V \) associated with each conformal Killing vector \( V \in \mathfrak{h}, \) (the Lie algebra of \( H \)) which measures the difference between the Covariant derivative along \( V \) and the Lie derivative along \( V \)."

The implication is that the concept of a Higgs field represents the difference between a process that is NOT dependent upon the constraint of a gauge group (the Lie differential), and a process that is restricted to a specific choice of a connection defined by some gauge group, (the Covariant differential).

It becomes apparent that:

\[ |W^a\rangle = \text{Vector of Work 1-forms. (transversal)} \quad (293) \]

\[ |h^a\rangle = \text{Higgs potential as vector of 0-forms (Internal Energy)} \quad (294) \]

\[ d|h^a\rangle = \text{Higgs vector of 1-forms.} \quad (295) \]

\[ i(V)d|h^a\rangle = \text{vector of longitudinal inertial accelerations (with mass)} \quad (296) \]

\[ = \text{non adiabatic components of a process} \quad (297) \]

The method of the "Cosmological Vacuum" and its sole assumption leads to inertial properties and the Higgs field, all from a topological perspective and without "quantum" fluctuations.
5.9 A Strong Equivalence Principle

At this point, there has been no indication that the problem being investigated has anything to do with the Gravitational Field. The gravity issue is to be encoded into how the quadratic congruent symmetries of $[B]$, and its topological group structures, are established. In general, different choices for the group structure of the Basis Frame will strongly influence the application to any particular physical system of fields and particles.

Without the Einstein Ansatz, it appears that the concept of a Cosmological Vacuum can lead to a Strong Equivalence principle. Substitute $[\Gamma] + [T]$ for $[C]$ in the definition of the matrix of curvature 2-forms, and recall that for the Cosmological Vacuum the Cartan matrix of curvature 2-forms, $[\Theta]$, is zero.

\[
[\Theta] = \{d[C] + [C] \wedge [C]\} \Rightarrow 0, \quad (298)
\]
\[
= \{d([\Gamma] + [T]) + ([\Gamma] + [T]) \wedge ([\Gamma] + [T])\} \quad (299)
\]
\[
= \{d[\Gamma] + [\Gamma] \wedge [\Gamma]\} + \{[T] \wedge [\Gamma] + [\Gamma] \wedge [T]\} + \{d[T] + [T] \wedge [T]\}. \quad (300)
\]

Separate the matrices of 2-forms into the metric based (Christoffel) curvature 2-forms, defined as

\[
[\Phi_{\Gamma}] = \{d[\Gamma] + [\Gamma] \wedge [\Gamma]\} = [\text{Field metric 2-forms}], \quad (301)
\]
and the remainder, defined as

\[
[-\Sigma_{\text{Inertial}}] = [\Theta] - [\Phi_{\Gamma}] \quad (302)
\]
\[
= \{[T] \wedge [\Gamma] + [\Gamma] \wedge [T]\} + \{d[T] + [T] \wedge [T]\} \quad (303)
\]
\[
= \{\text{interaction 2-forms}\} + \{\Sigma_{\text{Inertial}}\} \quad (304)
\]

The decomposition leads to the strong equivalence equation,

\[
\text{Principle of : Strong Equivalence} \quad (305)
\]
\[
[\text{Metric Field curvature 2-forms}] = [\text{Inertial curvature 2-forms}], \quad (306)
\]
\[
[\Phi_{\Gamma}] = [\Sigma_{\text{Inertial}}] \quad (307)
\]

The Cartan Connection is not necessarily equal to the metric connection, but the above formula generates a vector of 2-forms that are equal, for any decomposition. If the Basis Frame is chosen such that the Cartan Connection is equal to the Christoffel connection, then the residue term vanishes. This result implies that the vector of affine torsion 2-forms vanishes. In the general case, the decomposition of the right Cartan connection can have symmetric as well as anti-symmetric parts, but it is always the case that the antisymmetric part generates the 2-forms of Affine Torsion, and therefore the excitation 2-forms, $|G\rangle$. It is
suggested that in the choice of a symmetric basis frame such that the Cartan connection is equal to the metric, with \([T] = 0\), corresponds to matter free space, analogous to the Einstein metric based equations.

6 Remarks

This universal set ideas enumerated above startles me. There are only two assumptions:

1. The postulate that the domain of continuum fields is a vector space defined by infinitesimal mappings.

2. The postulate that the field equations are generated by processes that must satisfy the concepts of thermodynamic evolution (≈ the First Law of Thermodynamics as expressed in terms of continuous topological evolution and Cartan's magic formula).

The rest of the concepts are derived, following the rules of the Cartan exterior calculus. These results appear to be universal rules. Particulate concepts also appear, from the same fundamental postulates of a Cosmological Vacuum, in the form of topologically coherent defect structures in the fields. Quantization occurs in a topological manner from the deRham theorems as period integrals. The quantized topologically coherent structures form the basis of macroscopic quantum states.

Earlier, related, thoughts about the topological and differential geometric ideas associated with the four forces appeared in [17]. Not only do the long range concepts (fields) of gravity and electromagnetism have the a thermodynamic base, but so also do the short range concepts (fields) of the nuclear and weak force. The short range forces are artifacts of non-equilibrium thermodynamic systems with PTD > 2, where the long range forces are artifacts of thermodynamic systems with PTD < 3. The topological theory of a Cosmological Vacuum, as presented above, re-enforces this earlier work.

It is also remarkable that the methods described above also have applications in engineering format, which can lead to new applications governed by the non-equilibrium and perhaps chiral features of thermodynamic evolution. The field theory of the Cosmological Vacuum appears to be applicable at all scales. In electromagnetic language, the key design principle to produce stable plasmas is to minimize \( E \circ B \).

An important result is the recognition that the 2-forms of Affine torsion are in effect the source of the topologically distinct Excitation 2-forms, \(|\Sigma\rangle\) and \(|G\rangle\). It follows that the topological foundation of both hydrodynamics and electrodynamics can be put on the same footing. Yet \(|\Sigma\rangle\) the pair analogue of the impair \(|G\rangle\) does not appear in the classic literature.
of hydrodynamics. Why? The simplest reason is the dogmatic adherence to symmetric metric representations of stress and strain. Such methods eliminate the possibilities of Affine Torsion in the classical theories. As another example, the symmetric metric of GR theory utilizes Christoffel connections which are free of Affine Torsion, again eliminating the important thermodynamic concept of additive variables. However, Fluids and Electromagnetism, based on a Cartan connection generated by infinitesimal mappings, indeed can have the properties associated with Affine Torsion. In fluids, the Affine Torsion is the source of unit mass (mole), and in electromagnetism, Affine Torsion is the source of unit charge. However, remember that the Affine Torsion 2-forms are necessary (but not sufficient) for constructing the 3-forms of Topological Spin.

7 Examples

7.1 Example 1. The Schwarzschild Metric embedded in a Basis Frame, $[B]$, as a 10 parameter subgroup of an affine group.

7.1.1 The Metric - a Quadratic Congruent symmetry

The algebra of a quadratic congruence can be used to deduce the metric properties of a given Basis Frame. These deduced metric features may be used to construct a "Christoffel" or metric compatible connection, different from the Cartan Connection. The Christoffel connection constructed from the quadratic congruence of the Basis Frame may or may not generate a "Riemannian" curvature. By working backwards, this example will demonstrate how to construct the Basis Frame, given a diagonal metric. The method is algebraic and exceptionally simple for all diagonal metrics that represent a 3+1 division of space-time. The important result is that given a Basis Frame of a given matrix group structure, a metric and a compatible Christoffel Connection can be deduced. An important and unexpected result is that a basis frame compatible with a metric as a quadratic congruent symmetry, can generate a Cartan matrix that supports Affine Torsion.

Remark 14 All 3+1 metric structures are to be associated with the 10 parameter subgroup of the 13 parameter Affine groups.

In this example, it will be demonstrated how the isotropic form of the Schwarzschild metric can be incorporated into the Basis Frame for a Physical Vacuum, $[B]$. The technique is easily extendable for diagonal metrics. However, the symmetry properties of the Cartan Connection are not limited to metrics of the "gravitational" type. Once the Schwarzschild
metric is embedded in to the Basis Frame, then the universal methods described above will be applied to the representative Basis Frame, and each important result will be evaluated.

The isotropic Schwarzschild metric is a diagonal metric of the form,

\[
(\delta s)^2 = -(1 + m/2r)^4 \left\{ (dx)^2 + (dy)^2 + (dz)^2 \right\} + \frac{(2 - m/r)^2}{(2 + m/r)^2} (dt)^2
\]

\[
= -\alpha^2 \left\{ (dx)^2 + (dy)^2 + (dz)^2 \right\} + \beta^2 (dt)^2
\]

with \( r = \sqrt{(x)^2 + (y)^2 + (z)^2} \), \( \alpha \), and \( \beta \) give the expression.

As Eddington [18] points out, the isotropic form is palatable with the idea that the speed of light is equivalent in any direction. That is not true for the non-isotropic Schwarzschild metric, where transverse and longitudinal null geodesics do not have the same speed.

For the isotropic Schwarzschild example, the metric \([g_{jk}]\) can be constructed from the triple matrix product:

\[
[g_{jk}] = [\tilde{f}] \circ [\eta] \circ [f],
\]

where

\[
f = \begin{bmatrix}
\alpha & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 \\
0 & 0 & \alpha & 0 \\
0 & 0 & 0 & \beta
\end{bmatrix},
\]

and \( \alpha = (1 + m/2r)^2 = (\gamma/2r)^2 \), \( \beta = \frac{(2 - m/r)}{(2 + m/r)} = \frac{\delta}{\gamma} \), \( \eta \) gives the expression.

At first glance it would appear that the Schwarzschild metric forms a quadratic form constructed from the congruence of a 4 parameter (diagonal) matrix. It is not obvious that this may be a special case of a 10 parameter group with a fixed point. In order to admit the 10 parameter Poincare group (which is related to the Lorentz group), a map from spherical 3+1 space to Cartesian 3+1 space will be perturbed by the matrix \([f]\).
7.1.2 The Diffeomorphic Jacobian Basis Frame

At first, consider the diffeomorphic map \( \phi^k \) from spherical 3+1 space to Cartesian 3+1 coordinates:

\[
\{ y^a \} = \{ r, \theta, \varphi, \tau \} \Rightarrow \{ x^k \} = \{ x, y, z, t \}
\]

\( \phi^k : [r \sin(\theta) \cos(\varphi), r \sin(\theta) \sin(\varphi), r \cos(\theta), \tau] \Rightarrow [x, y, z, t] \)

\[
\{ dy^a \} = \{ dr, d\theta, d\varphi, d\tau \}.
\]

The Jacobian of the diffeomorphic map \( \phi^k \) can be utilized as an integrable Basis Frame matrix \( [B] \) which is an element of the 10 parameter F-Affine group (The Affine subgroup with a fixed point):

\[
[B] = \begin{bmatrix}
\sin(\theta) \cos(\varphi) & r \cos(\theta) \cos(\varphi) & -r \sin(\theta) \sin(\varphi) & 0 \\
\sin(\theta) \sin(\varphi) & r \cos(\theta) \sin(\varphi) & r \sin(\theta) \cos(\varphi) & 0 \\
\cos(\theta) & -r \sin(\theta) & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

The infinitesimal mapping formula based on \( [B] \) yields

\[
[B]^k_a(y) \circ \begin{pmatrix}
   dr \\
   d\theta \\
   d\varphi \\
   dt
\end{pmatrix} \Rightarrow |\sigma^k\rangle = \begin{pmatrix}
   dx \\
   dy \\
   dz \\
   dt
\end{pmatrix},
\]

such that all of the 1-forms \( |\sigma^k\rangle \) are exact differentials, and there are no non-zero field intensities, \( |d\sigma^k\rangle \).

7.1.3 The Perturbed Basis Frame with a Congruent Symmetry

**Theorem 15** The effects of a diagonal metric \([g_{jk}]\) can be absorbed into a re-definition of the Frame matrix:

\[
[\hat{B}] = [f] \circ [B].
\]

The integrable Jacobian Basis Frame matrix given above will be perturbed by multiplication on the left by the diagonal matrix, \([f]\). The perturbed Basis Frame becomes
\[
\hat{B} = [f] \circ [B] \text{ the Schwarzschild Cartan Basis Frame.} \tag{321}
\]

\[
\begin{pmatrix}
\sin(\theta) \cos(\varphi) \gamma^2/4 r^2 & \cos(\theta) \cos(\varphi) \gamma^2/4 r & -\sin(\theta) \sin(\varphi) \gamma^2/4 r & 0 \\
\sin(\theta) \sin(\varphi) \gamma^2/4 r^2 & \cos(\theta) \sin(\varphi) \gamma^2/4 r & \sin(\theta) \cos(\varphi) \gamma^2/4 r & 0 \\
\cos(\theta) \gamma^2/4 r^2 & -\sin(\theta) \gamma^2/4 r & 0 & 0 \\
0 & 0 & 0 & \delta/\gamma
\end{pmatrix} \tag{322}
\]

\[
\gamma = (2r + m), \quad \delta = (2r - m) \tag{323}
\]

Use of the congruent pullback formula based on the perturbed Basis Frame, \(\hat{B}\), yields,

\[
[g_{jk}] = [\hat{B}_{\text{transpose}}] \circ \eta \circ [\hat{B}], \tag{324}
\]

\[
[g_{jk}] = \begin{pmatrix}
-(\gamma^2/4 r^2)^2 & 0 & 0 & 0 \\
0 & -(\gamma^2/4 r)^2 & 0 & 0 \\
0 & 0 & -(\gamma^2/4 r)^2 \sin^2(\theta) & 0 \\
0 & 0 & 0 & + (\delta/\gamma)^2
\end{pmatrix}, \tag{325}
\]

\[
\gamma = (2r + m), \quad \delta = (2r - m) \tag{326}
\]

which agrees with formula given above for the isotropic Schwarzschild metric in spherical coordinates. It actually includes a more general idea, for the coefficients, \(\alpha\), and \(\beta\), can be dependent upon both \(r\) and \(\tau\).

The infinitesimal mapping formula based on \(\hat{B}\) yields

\[
\begin{pmatrix}
[\hat{B}^k_a(y)] \circ \\
dr \\
d\theta \\
d\varphi \\
dt
\end{pmatrix}
\Rightarrow
|\sigma^k\rangle = \begin{pmatrix}
(\gamma^2/4 r^2) dx \\
(\gamma^2/4 r^2) dy \\
(\gamma^2/4 r^2) dz \\
(\delta/\gamma) dt
\end{pmatrix}, \tag{327}
\]

such that all of the 1-forms \(|\sigma^k\rangle\) are NOT exact differentials. The 2-forms of field intensities, \(|d\sigma^k\rangle\) are not zero.

### 7.1.4 The Schwarzschild-Cartan connection.

The Schwarzschild-Cartan (right) Connection \([\hat{C}]\), as a matrix of 1-forms relative to the perturbed Basis Frame \(\hat{B}\), becomes
$$\hat{C} = [\hat{B}^{-1}] \circ d[\hat{B}],$$  \hspace{1cm} (328)  

$$\hat{C} = \begin{bmatrix} -2mdr/r\gamma & -rd\theta & \sin^2(\theta)rd\phi & 0 \\ d\theta/r & \delta dr/\gamma & -\cos(\theta)\sin(\theta)d\phi & 0 \\ d\phi/r & \cot(\theta)d\phi & \cot(\theta)d\theta + \delta dr/\gamma & 0 \\ 0 & 0 & 0 & 4mdr/(\gamma\delta) \end{bmatrix}. \hspace{1cm} (329)$$  

$$\gamma = (2r + m), \hspace{0.5cm} \delta = (2r - m) \hspace{1cm} (330)$$  

**Perturbed Cartan Connection from Maple**

It is apparent that the Cartan Connection matrix of 1-forms is again a member of the 10 parameter matrix group, a subgroup of the 13 parameter affine matrix group.

### 7.1.5 Vectors of Closure 2-forms

Surprisingly, for the perturbed Basis Frame $[\hat{B}]$ which contains a the square root of a congruent metric field of a massive object, the vector of "excitation" torsion 2-forms, based on the 10 parameter affine subgroup with a fixed point, is not zero, and can be evaluated as:

$$|\hat{\Sigma}| = |\hat{C}|^{-1}|dy^a|$$  \hspace{1cm} (331)  

Torsion 2-forms : of the Affine subgroup with a fixed point  \hspace{1cm} (332)  

$$|\hat{\Sigma}| = \begin{vmatrix} 0 \\ (2m/r\gamma)(d\theta^* dr) \\ (2m/r\gamma)(d\phi^* dr) \\ (4m/\gamma\delta)(dr^* d\tau) \end{vmatrix} "Schwarzschild Excitations"$$  \hspace{1cm} (333)  

$$\gamma = (2r + m), \hspace{0.5cm} \delta = (2r - m) \hspace{1cm} (334)$$

The unexpected result is that the isotropic Schwarzschild metric admits coefficients of "Affine Torsion" relative to the Cartan Connection matrix, $|\hat{C}|$.

Similarly the vector of 2-form of field intensities $|\hat{F}|$ can be evaluated in terms of the
perturbed Basis Frame as:

\[ \hat{d}\sigma = d(\hat{B} \circ dy^a) \quad \text{"Schwarzschild Intensities"} \]

: Intensity 2-forms of the Affine subgroup with a fixed point

\[ \hat{d}\sigma = \begin{pmatrix}
+ \frac{m\gamma}{2r^2} \{(\sin(\phi) \cos(\theta) d\theta^* \, dr) - (\sin(\theta) \cos(\phi) d\phi^* \, dr) \\
+ \frac{m\gamma}{2r^2} \{(\sin(\phi) \cos(\theta) d\theta^* \, dr) + (\sin(\theta) \cos(\phi) d\phi^* \, dr)
\end{pmatrix}.
\]

\[ \gamma = (2r + m), \quad \delta = (2r - m) \]

The constitutive map relating the field intensities and the field excitations,

\[ \hat{\Sigma} = \hat{B}^{-1} \circ \hat{d}\sigma, \]

is determined by the inverse of the perturbed Basis Frame, \( \hat{B}^{-1} \):

Schwarzschild Constitutive map from Maple

\[ \hat{B}^{-1} = 4r/\gamma^2 \begin{bmatrix}
\cos(\theta) \cos(\phi) & r \sin(\theta) \sin(\phi) & \cos(\theta) & 0 \\
\sin(\phi) \sin(\theta) \cos(\theta) & r \sin(\theta) \cos(\phi) & \cos(\theta) & 0 \\
0 & \sin(\phi) \cos(\theta) & -\sin(\theta) & 0 \\
0 & 0 & 0 & \frac{\gamma^3}{4r\delta}
\end{bmatrix} \]

\[ \gamma = (2r + m), \quad \delta = (2r - m) \]

7.1.6 Vectors of 3-forms

The exterior derivative of the vector of excitations (Affine Torsion 2-forms) is zero, hence there are no current 3-forms of excitations, \(|J\rangle\), for the perturbed Basis Frame that encodes the Schwarzschild metric as a Congruent symmetry:

\[ \text{Charge Current 3-form} \quad |J\rangle = d|\Sigma\rangle = 0. \]

The Topological Torsion 3-form for this isotropic Schwarzschild example vanishes: \( H = |A^* F\rangle \Rightarrow 0 \). The implication is that the system is of Pfaff dimension 2 (and therefore is an equilibrium thermodynamic system).

Both Poincare 4-forms vanish, but the Topological Spin 3-form is NOT zero. The individual components are:
\[
\langle \sigma | \hat{\Sigma} \rangle = \begin{pmatrix}
0 \\
- m (2r + m) \cos(\phi) \sin(\theta) (d\theta^* d\phi^* dr)/(2r^2) \\
- m (2r + m) \sin(\theta) (d\theta^* d\phi^* dr)/(2r^2) \\
0
\end{pmatrix}
\]

"Schwarzschild Spin components"

(344)

The inner product of the components of potential 1-forms, \( \langle A \rangle \) and the Excitation 2-forms yields:

Total Topological Spin 3-form
\[
\mathcal{G} = \langle \sigma | \hat{\Sigma} \rangle \
dA = 0.
\]

(345)

\[
= (-m\gamma\sin(\theta)/(2r^2)\{\cos(\phi) + 1\}(d\theta^* d\phi^* dr)).
\]

(347)

The "Topological Spin" 3-form depends upon the "mass" coefficient, \( m \), and the 2-forms of Affine torsion.

7.1.7 The three Connection matrices

The three matrices of Connection 1-forms are presented below for each (perturbed) connection, \( \mathcal{\Gamma} \), \( \mathcal{C} \), \( \mathcal{T} \)

\[
[\hat{\mathcal{\Gamma}}] = \begin{pmatrix}
-2mdr/r\gamma & -\delta r d\theta/\gamma & \delta \sin^2(\theta) r d\phi/\gamma & 64\delta md\tau/\gamma^7 \\
\delta d\theta/r\gamma & \delta dr/r\gamma & -\cos(\theta) \sin(\theta) d\phi & 0 \\
\delta d\phi/r\gamma & \cot(\theta) d\phi & \cot(\theta) d\theta + \delta dr/\gamma & 0 \\
4md\tau/(\gamma\delta) & 0 & 0 & 4mdr/(\gamma\delta)
\end{pmatrix}
\]

(348)

\[
[\hat{\mathcal{C}}] = \begin{pmatrix}
-2mdr/r\gamma & -dr & \sin^2(\theta) r d\phi & 0 \\
\theta/r & \delta dr/\gamma & -\cos(\theta) \sin(\theta) d\phi & 0 \\
\delta d\phi/r & \cot(\theta) d\phi & \cot(\theta) d\theta + \delta dr/\gamma & 0 \\
0 & 0 & 0 & 4mdr/(\gamma\delta)
\end{pmatrix}
\]

(349)

\[
[\hat{\mathcal{T}}] = \begin{pmatrix}
0 & -2mr d\theta/\gamma & 2m \sin^2(\theta) r d\phi/\gamma & -64mr^4 \delta d\tau/\gamma^7 \\
2md\theta/r\gamma & 0 & 0 & 0 \\
4md\phi/r\gamma & 0 & 0 & 0 \\
-4md\tau/\delta\gamma & 0 & 0 & 0
\end{pmatrix}
\]

(350)

\[
\gamma = (2r + m), \quad \delta = (2r - m)
\]

Schwarzschild Perturbed Connections
The matrix of (metric) curvature 2-forms, $[\Phi_T]$, based on the formula

$$[\Phi_T] = [\Gamma] + [\Gamma] \cdot [\Gamma],$$  

is computed to be:

**Curvature 2-forms for the Schwarzschild Christoffel Connection**

$$[\Phi_T] = \frac{4m}{\gamma^2} \begin{bmatrix} 0 & -rdr^\ast d\theta & r \sin^2(\theta) dr^\ast d\phi & -32r^4 dr^\ast d\tau / \gamma^6 \\ 2mdr^\ast d\theta / r\gamma & 0 & -2 \sin^2(\theta) d\theta^\ast d\phi & 16\delta^2 r^3 d\theta^\ast d\tau / \gamma^6 \\ 4mdr^\ast d\phi / r\gamma & -2r d\theta^\ast d\phi & 0 & 16\delta^2 r^3 d\phi^\ast d\tau / \gamma^6 \\ -4mdr^\ast d\tau / \delta \gamma & 2r d\theta^\ast d\tau & -2 \sin^2(\theta) d\phi^\ast d\tau & 0 \end{bmatrix}$$

By the Strong Equivalence Principle,

$$\{d[\Gamma] + [\Gamma] \cdot [\Gamma]\} + \{[\Gamma] \cdot [\Gamma] + [\Gamma] \cdot [\Gamma]\} + \{d[\Gamma] + [\Gamma] \cdot [\Gamma]\} + \{[\Phi_T]\} + \{[ Interaction \ 2 - forms]\} + \{[\Phi_T]\} \Rightarrow 0,$$

which can be checked using Maple.

### 7.1.8 Summary Remarks

The idea that has been exploited is that the arbitrary Basis Frame (a linear form), without metric, can be perturbed algebraically to produce a new Basis Frame that absorbs the properties of a quadratic congruent metric system. This result establishes a constructive existence proof that compatible metric features of a Physical Vacuum can be derived from the structural format of the Basis Frame. The Basis Frame is the starting point and the congruent metric properties are deduced.

For the Schwarzschild example, another remarkable feature is that the 1-forms $|\sigma^k\rangle$ constructed according to the formula

$$|\hat{\mathcal{B}}\rangle \circ |dy^a\rangle \Rightarrow |\sigma^k\rangle \approx |A^k\rangle,$$

are all integrable (as the Topological Torsion term is Zero), but the coefficients of affine torsion are not zero. The symbol $|dy^a\rangle$ stands for the set $[dr, d\theta, d\phi, d\tau]$ (transposed into a column vector), and $|\hat{\mathcal{B}}\rangle$ is the "perturbed" Basis Frame which contains the Schwarzschild metric as a congruent symmetry. The integrability condition means that there exist integrating factors $\lambda^{(k)}$ for each $\sigma^k$ such that a new Basis Frame can be constructed from $|\hat{\mathcal{B}}\rangle$ algebraically. Relative to this new Basis Frame, the vector of torsion 2-forms is zero,
\[ |d\sigma^k| = |dA^k| = |F^k| = 0! \] The "Coriolis" acceleration which is related to the 2-form of torsion 2-forms \(|d\sigma^k|\) can be eliminated algebraically. Although this result is possible algebraically, it is not possible diffeomorphically.

Of course, this algebraic reduction is impossible if any of the 1-forms, \(\sigma^k\), is of Pfaff dimension 3 or more. The Basis Frame then admits Topological Torsion, which is irreducible.

### 7.2 Example 2: \([B]\) as a 13 parameter matrix group

#### 7.2.1 The Intransitive “Wave-Affine” Basis frame

The next set of examples considers the structure of those 4 x 4 Basis Frames that admit a 13 parameter group in 4 geometrical dimensions of space-time. There are 3 interesting types of 13 parameter group structures. This first example utilizes the canonical form of the 13 parameter “Wave Affine” Basis Frame. These Basis Frames will have zeros for the first 3 elements of the right-most column. Wave Affine Basis Frames exhibit closure relative to matrix multiplication. All products of Wave Affine Basis Frames have 3 zeros on the right column. From a projective point of view, these matrices are not elements of a transitive group. They are intransitive and have fixed points. The true affine group in 4 dimensions is a transitive group (without fixed points) and is discussed in [2].

For simplicity in display, the 9 parameter space-space portions of the Basis Frame will be assumed to be the 3 x 3 Identity matrix, essentially ignoring spatial deformations and spatially extended rigid body motions. In the language of projective geometry, this intransitive system has a fixed point. The first 3 elements in the bottom row can be identified (formally to within a constant factor) with the components of a vector potential in electromagnetic theory. The 4th (space-time) column will have three zeros, and the \(B_4\) component will be described in terms of a function \(\phi(x, y, z, t)\). For convenience this ”electrodynamic notation” for the field intensities will be utilized.

\[
[B_{wave\_affine}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ A_x & A_y & A_z & -\phi \end{bmatrix}.
\] (358)

#### 7.2.2 The Field Potential 1-forms

The projected 1-forms of potentials become

\[
[B_{wave\_affine}] \circ |dy^a| = |A^k| \] (359)
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
A_x & A_y & A_z & -\phi
\end{bmatrix}
\begin{bmatrix}
dx \\
dy \\
dz \\
dt
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
dx \\
dy \\
dz \\
d(\text{Action})
\end{bmatrix}
= |A^k\rangle,
\]

\[\text{Action} = A_x dx + A_y dy + A_z dz - \phi dt\]  \hspace{1cm} (361)

### 7.2.3 The Field Intensity 2-forms

The exterior derivative of the vector of 1-forms, \(|A^k\rangle\), produces the vector of 2-forms representing the field intensities \(|F^k\rangle\) of electromagnetic theory,

\[\textbf{Intensity 2-forms:} \quad d|A^k\rangle = |F^k\rangle = \begin{bmatrix}
dx \\
dy \\
dz \\
d(\text{Action})
\end{bmatrix}.\]  \hspace{1cm} (362)

Note that the Action 1-form produced by the wave affine Basis Frame is precisely the format of the 1-form of Action used to construct the Electromagnetic field intensities in classical EM theory.

### 7.2.4 The Cartan Right Connection of 1-forms

The Cartan right Connection matrix of 1-forms based upon the Basis Frame, \([\mathbb{B}_{\text{wave affine}}]\), given above is given by the expression

\[\left[C_{\text{wave affine}, \text{right}}\right] = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-d(A_x)/\phi & -d(A_y)/\phi & -d(A_z)/\phi & d(\ln \phi)
\end{bmatrix}.\]  \hspace{1cm} (363)

### 7.2.5 The Field Excitation 2-forms (Affine Torsion)

The general theory permits the vector of excitation 2-forms, \(|G\rangle\), to be evaluated as:

\[\left[C_{\text{wave affine}, \text{right}}\right] \wedge |dym\rangle \simeq |G\rangle \quad \text{Affine Torsion}\]  \hspace{1cm} (364)

\[\text{Excitation 2-forms} \quad |G\rangle = \begin{bmatrix}
0 \\
0 \\
0 \\
-(F)/\phi
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
-d(\text{Action})/\phi
\end{bmatrix} \quad \neq |\Sigma_{\text{Cartan, Torsion}}\rangle\]  \hspace{1cm} (365)
The coefficients of the vector of excitation 2-forms are precisely those ascribed to the coefficients of "Affine Torsion", even though the Basis Frame used as the example is not a member of the transitive affine group.

7.2.6 The Excitation Current 3-form

The exterior derivative of the vector of excitation two forms $|G\rangle$ produces the vector of 3-form currents $|J\rangle$. The result is

$$|J\rangle = \begin{bmatrix} 0 \\ 0 \\ d(-F)/\phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ +d\phi^2/F/\phi^2 \end{bmatrix}$$

(367)

Note that the charge-current 3-form is non-zero, but closed, producing the expected Maxwell-Ampere charge conservation law:

$$d|J\rangle = |0\rangle$$

(368)

The wave affine group of Basis Frames supports a vector of "Affine Torsion" 2-forms that are abstractly related to excitation 2-forms $|G\rangle$ representing the fields (the D and H fields) generated by the sources in classical EM theory. The charge-current 3-form need not vanish.

7.2.7 Topological Torsion and Topological Spin 3-forms

The 3-forms of Topological Spin and Topological Torsion are proportional to one another for the simple example, and are not zero if the Pfaff Topological Dimension of the 1-form of Action is 3 or more. Such systems are not in thermodynamic equilibrium.

$$\text{Topological Torsion: } A^\wedge F$$
$$\text{Topological Spin: } A^\wedge G = -A^\wedge F/\phi$$

If the Poincare invariants are to vanish it is necessary that the 4-form of Topological Parity is zero, $F^\wedge F \Rightarrow 0$.

7.2.8 A time independent simplification

For the simple time independent case, where, $Ax = 0, Ay = 0, Az(x, y, z) \neq 0, \phi(x, y, z) \neq 0$, the vector of affine torsion 2-forms (the excitations) becomes:
Affine Torsion 2-forms  $|G\rangle = \begin{vmatrix} 0 \\ 0 \\ 0 \\ (-dA_{z}^{\ast}dz + d\phi^{\ast}dt)/\phi \end{vmatrix}$, \hspace{1cm} (369)

and the 3-form of current induced becomes

Conserved Current 3-forms  $|J\rangle = |dG\rangle = \begin{vmatrix} 0 \\ 0 \\ 0 \\ (dA_{z}^{\ast}dz^{\ast}d\phi)/\phi^{2} \end{vmatrix}$, \hspace{1cm} (370)

The vector of field intensity 2-forms becomes:

Field Intensity 2-forms  $|F\rangle = \begin{vmatrix} 0 \\ 0 \\ 0 \\ (-dA_{z}^{\ast}dz + d\phi^{\ast}dt)/\phi \end{vmatrix}$, \hspace{1cm} (371)

The vector Topological Torsion 3-forms, becomes:

$|A^{\ast}F\rangle = \begin{vmatrix} 0 \\ 0 \\ 0 \\ (dA_{z} + A_{z}d\phi/\phi^{\ast}dz^{\ast}dt) \end{vmatrix}$, \hspace{1cm} (372)

The vector Topological Torsion 4-forms, becomes:

Topological Torsion 4-forms  $|F^{\ast}F\rangle = \begin{vmatrix} 0 \\ 0 \\ 0 \\ (dA_{z}^{\ast}d\phi/\phi)^{\ast}dz^{\ast}dt \end{vmatrix}$, \hspace{1cm} (373)

The second Poincare 4-form (the bulk viscosity ”expansion” coefficient) becomes proportional to the first Poincare 4-form,

\[ d(A^{\ast}F) = -2(\mathbf{E} \circ \mathbf{B}){\Omega}_{4} = \phi \cdot d(A^{\ast}G). \] \hspace{1cm} (374)

Hence neither the Topological Spin, $A^{\ast}G$, or the Topological Torsion, $A^{\ast}F$, define process direction fields that are leave the Topological Torsion or the the Topological Spin as evolutionary invariants. As the Topological Torsion 4-form is not zero, the system is of Pfaff Topological Dimension 4, and is a non-equilibrium thermodynamic system.
Recall, that all of the analysis above is equally applicable to fluids which admit Affine Torsion.

**Part II will consist of a collection of examples in terms of Maple programs formatted as pdf files.**

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