Preservation of no-signalling principle in parity-time symmetric quantum systems

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We look into the possibility of entanglement generation in a parity($P$)-time($T$)-symmetric framework and demonstrate the non-violation of non-signalling principle for the case of bipartite systems when at least one is guided by $PT$-symmetric quantum mechanics. Our analysis is based on the use of the $CPT$-inner product to construct the reduced density operators both before and after the action of time evolution operator.

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I. INTRODUCTION

The notion of quantum entanglement [1–3] speaks of a shared existence of particles having their properties interlinked with each other. An interesting manifestation of entanglement is that the correlation survives even when the particles get separated by a large distance after once having come into contact. Different aspects of quantum entanglement have been studied and there is a large literature in this subject [4–8]. In this article we want to address quantum entanglement in the framework of a complex extension of quantum mechanics. In particular we will concentrate on the aspect of quantum entanglement has been responsible to maintain the non-physicality of the issue of entanglement; the no-signalling principle is shared among observers. It is important to emphasise that our result runs counter to the recent claims in the literature that in bipartite systems [6, 7] the feature of no-signalling is violated whenever one of the subsystems is governed by $PT$-symmetric quantum mechanics ($PTQM$). Interestingly, an experimental search has pointed to the contrary evidence [9].

In standard quantum mechanics ($SQM$) the concept of Hermiticity holds preserving the reality of the associated energy spectrum. Almost a decade and a half ago, a typical $PT$-symmetric Hamiltonian was also shown by Bender and Boettcher [10] to possess a real bound-state spectrum. In fact, they observed that a system admitting an exact $PT$-symmetry generally preserved the reality of their bound-state eigenvalues while if opposite was the case then $PT$ was broken with the eigenvectors of the Hamiltonian ceasing to be the same for the $PT$-operator. In such a situation, complex eigenvalues spontaneously developed in conjugate pairs (see, for example, [11–13]) and the system underwent a $PT$-broken phase. In fact, the $PT$-transition causes a system to switch over from an equilibrium to a non-equilibrium state. The idea of $PT$-symmetry has also found experimental support (see, for example, [14] and earlier references therein): in particular research in optical systems has been a major source [15–17] wherein balancing gain and loss has uncovered the relevance of $PT$-structure in them.

From the theoretical point of view, $PTQM$ systems could be plagued with negative norms [18]. The reason is that the difference in the definition of the inner product in $SQM$ as introduced in the Dirac sense i.e.

$$
(f, g) = \int \mathbb{R} dx [Tf(x)]g(x), \quad f, g \in L_2(\mathbb{R})
$$

(1)

where $Tf(x) = f^*(x)$, from the of $PTQM$ i.e.

$$
(f, g)_{PT} = \int \mathbb{R} dx [PTf(x)]g(x), \quad f, g \in L_2(\mathbb{R})
$$

(2)

where $PTf(x) = [f(-x)]^*$, implies an indefinite norm and hence $PT$-systems lack a probabilistic interpretation. It was shown in [19] that an introduction of a linear operator $C$ to construct a $CPT$ inner product in the following sense

$$
(f, g)_{CPT} = \int \mathbb{R} dx [CPTf(x)]g(x)
$$

(3)

with the positive-definiteness of the associated norm, enabled one to get rid of this handicap. Note that $C$ commutes with both the Hamiltonian and the operator $PT$. Further it is idempotent and has eigenvalues $\pm 1$. A $PT$-symmetric system evolves in a manner wherein the accompanying time evolution of the state vector is unitary with respect to the $CPT$ inner product. For a
II. PRELIMINARIES

We focus on multi-partite systems which compose of a macroscopic number of subsystems [3]. Given a set of Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_N$, a state of composite system $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_N$ is entangled if we cannot carry out a decomposition like the product states $|\psi\rangle = |\chi_1\rangle \otimes |\chi_2\rangle \otimes ... \otimes |\chi_N\rangle$ with $|\chi_i\rangle \in \mathcal{H}_i$. Such states are then not separable.

In the following we will be interested in the bipartite case $N = 2$ only. A measure of entanglement of states of a bipartite system ($N = 2$) is provided by the following definition of entropy

$$E(\psi) = -\text{Tr}_1(\rho_1 \log \rho_1) - \text{Tr}_2(\rho_2 \log \rho_2)$$

(4)

where the reduced density matrices $\rho_1$ and $\rho_2$ are given in terms of the quantity $\rho = |\psi\rangle \langle \psi|$ with $\rho_1 = \text{Tr}_2(\rho)$ and $\rho_2 = \text{Tr}_1(\rho)$. One should note that the scheme of calculating $\rho$ here refers $|\psi\rangle$ not from the usual transpose-conjugate operation, but as the biorthogonal counterpart, of the state $|\psi\rangle$. A legible way of writing $E(\psi)$ is

$$E(\psi) = -\sum_i \lambda_i \log_N \lambda_i$$

(5)

where $\lambda_i, i = 1, 2$, are the respective eigenvalues of the reduced density matrices, $\rho_i, i = 1, 2$.

Consider $\{|u_n\rangle\}$ and $\{|v_n\rangle\}$ as basis sets of the respective Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$. This implies the existence of a basis set of the composite Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ namely $\{|u_n\rangle \otimes |v_n\rangle\}$. A general pure bipartite state which is assumed to be entangled is given by:

$$|\psi\rangle = \sum_{n,m=1}^{N,M} C_{nm} |u_n\rangle \otimes |v_n\rangle, \quad \sum_{n,m} |C_{nm}|^2 = 1$$

(6)

where $N, M$ are the dimensions of the respective Hilbert Spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and $C_{nm}$ are constants. In the following we take $N = M = 2$.

For calculational simplicity we adopt the following form\footnote{It is not difficult to establish the equivalence of the model considered in \cite{19} with the above matrix representation of the Hamiltonian modulo an identity factor.} of a two-level PT-symmetric Hamiltonian \cite{20}:

$$\hat{H} = \begin{pmatrix} i\gamma & -\zeta \\ -\zeta & -i\gamma \end{pmatrix}$$

(7)

where both $\gamma > 0$ and $\zeta > 0$ are taken to be positive constants. With the matrix form of the parity operator $\hat{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the time-reversal operation operating like $T : i \rightarrow -i$, the PT-symmetric character of $\hat{H}$ is evident.

Because of the underlying $PT$-symmetry, the right and left eigenvectors of $\hat{H}$ are not the same. The right eigenvectors read

$$|\psi_{\pm}\rangle = \frac{1}{\sqrt{2\cos \phi}} \begin{pmatrix} 1 \\ \mp i e^{\mp i\phi} \end{pmatrix}$$

(8)

where $\sin \phi = \frac{\zeta}{\gamma}$. The eigenvalues of $\hat{H}$ are

$$\lambda_{\pm} = \pm \sqrt{\gamma^2 - \zeta^2}$$

(9)

which are purely real if the inequality $\gamma \leq \zeta$ is obeyed. The degeneracy of the eigenvalues occurs when $\gamma = \zeta$. However for $\gamma \geq \zeta$ the eigenvalues become purely imaginary complex conjugates.

For an operator $\hat{X}$ which has simultaneous eigenstates of $\hat{H}$, we define the X-inner product (or X-norm) $\langle \cdot | \cdot \rangle_X$ and the bra vector in X-norm as given below:

$$\langle \phi | \psi \rangle_X = \hat{X} | \phi \rangle \cdot | \psi \rangle, \quad \langle \phi |_X = (\hat{X} \langle \phi |)^T$$

(10)

Next, following \cite{19}, we adopt, up to a sign, the following form of the $\hat{C}$ operator in tune with the $P$-operator noted earlier

$$\hat{C} = \begin{pmatrix} -i \tan \phi & \sec \phi \\ \sec \phi & i \tan \phi \end{pmatrix}$$

(11)

One can verify immediately that the respective actions of $PT$ and $\hat{C}$ operators on the eigenstates $|\psi_{\pm}\rangle$ are

$$\hat{P}\hat{T} |\psi_{\pm}\rangle = \frac{-1 \pm i\phi}{\sqrt{2 \cos \phi}} \begin{pmatrix} 1 \\ \mp i e^{\mp i\phi} \end{pmatrix}$$

$$\hat{C} |\psi_{\pm}\rangle = \frac{1}{\sqrt{2 \cos \phi}} \begin{pmatrix} \mp 1 \pm i\phi \end{pmatrix}$$

(12)

That these lead to the positive definiteness of the CPT-inner product, we first take an arbitrary state $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} r_a e^{i\theta_a} \\ r_b e^{i\theta_b} \end{pmatrix}$ and work out the following

$$\hat{C}\hat{P}\hat{T} |\psi\rangle = \frac{1}{\cos \phi} \begin{pmatrix} a^* - ib^* \sin \phi \\ b^* + ia^* \sin \phi \end{pmatrix}$$

(13)

Then the orthogonality of $|\psi\rangle$ under the CPT follows using (7) and (10). Indeed we find

$$\langle \psi | \psi \rangle_{\text{CPT}} = \frac{1}{\cos \phi} \left[ |a|^2 + |b|^2 - i(b^* a - a^* b) \sin \phi \right]$$

$$= \frac{1}{\cos \phi} \left[ r_a^2 + r_b^2 + 2r_a r_b \sin \phi \sin (\theta_a - \theta_b) \right] \geq 0$$

(14)
consistent with the result obtained in [22].

On an interesting note, as $\phi \rightarrow 0$, the transition of the concerned Hamiltonian from the framework of PTQM to SQM is observed, which is given below:

$$
\hat{H} \rightarrow -\zeta \sigma_x \\
\frac{1}{\sqrt{2 \cos \phi}} \left( \frac{1}{\mp e^{\mp i \phi}} \right) \rightarrow \frac{1}{\sqrt{2}} \left( \frac{1}{\mp i} \right)$$

$$
\pm \sqrt{\zeta^2 - \gamma^2} \rightarrow \pm \zeta
$$

$$
\hat{C} = \begin{bmatrix} -i \tan \phi & \sec \phi \\ i \tan \phi & \sec \phi \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \hat{P}
$$

$$
\langle -| - \rangle_T \rightarrow \langle -| - \rangle_T
$$

(15)

\langle -| - \rangle_T is identified as the usual Dirac norm which is used in SQM.

III. ENTANGLEMENT ISSUE

Against the background of the recent results in the literature [4, 6, 7] that have addressed the issue of the violation of the no-signalling principle concerning the PT-symmetric systems, we provide in this section a systematic demonstration to the contrary. In this regard, we consider two different possibilities one of which is related to the pair wherein each is guided by the PTQM while the other comprises the combination in which one is the PTQM while the other conforms to the SQM type.

For any general composite system, one may prove the no-signalling principle by showing that a time evolution produces no change in the system with respect to the measure of entropy. The underlying procedure goes as follows

1. First, we develop the entangled state $|\psi\rangle$ and determine the quantity $E(\psi_{t=0})$.

2. Second, we operate the time evolution operator over the composite state in the given Hilbert space $\mathcal{H}$.

3. Third, we calculate the reduced density matrix by performing partial trace of $|\psi\rangle \langle \psi|$ in $\mathcal{H}$. This will signify the measurement of the entanglement of $\psi_t$ in the other Hilbert space.

4. Finally, we estimate the time-dependent quantity $E(\psi_t)$. We are able to demonstrate the no-signalling theorem should the invariance $E(\psi_{t=0}) = E(\psi_{t=t})$ hold.

We now proceed to address the following subsystems as alluded to above.

A. Subsystems governed by PTQM

We focus on two subsystems each coming under the purview of PTQM. Let them be controlled by the following set of PT-symmetric Hamiltonians

$$
\hat{H}_1 = \begin{bmatrix} i\gamma & -\zeta \\ -\zeta & -i\gamma \end{bmatrix}, \quad \hat{H}_2 = \begin{bmatrix} i\gamma' & -\zeta' \\ -\zeta' & -i\gamma' \end{bmatrix}
$$

(16)

one for each subsystem. The associated time evolution operator $U = e^{-iH_i t}$, $i = 1, 2$ maps $\hat{H}_1$ and $\hat{H}_2$ to their time-dependent counterparts. Here we first construct an entangled state and then apply the time evolution operator $I \otimes U(t)$, $U(t) = e^{-iH_2 t}$ on it.

The eigenstates of $\hat{H}_i$ which serve as a basis set of $\mathcal{H}_i$, $i = 1, 2$ are given by the following entries

$$
\{ |p_1\rangle, |p_2\rangle \} = \left\{ \frac{1}{\sqrt{2 \cos \phi}} \left[ \begin{array}{c} 1 \\ -e^{-i \phi} \end{array} \right], \frac{1}{\sqrt{2 \cos \phi}} \left[ \begin{array}{c} 1 \\ +e^{i \phi} \end{array} \right] \right\}
$$

$$
\{ |q_1\rangle, |q_2\rangle \} = \left\{ \frac{1}{\sqrt{2 \cos \phi}} \left[ \begin{array}{c} 1 \\ -e^{-i \phi'} \end{array} \right], \frac{1}{\sqrt{2 \cos \phi'}} \left[ \begin{array}{c} 1 \\ +e^{i \phi'} \end{array} \right] \right\}
$$

(17)

(18)

where $\sin \phi = \frac{\sqrt{2}}{\sqrt{1 + \zeta^2}}$ and $\sin \phi' = \frac{\sqrt{2}}{\sqrt{1 + \zeta'^2}}$. Using now the definition of $|\psi\rangle$ the entangled state emerges

$$
|\psi\rangle = \sum_{n,m=1}^{2,3} C_{nm} |p_n\rangle \otimes |q_m\rangle, \quad \sum_{n,m} |C_{nm}|^2 = 1
$$

(19)

where $p_i$'s and $q_i$'s, $i = 1, 2$, are as given in (17) and (18). One needs to perform necessary calculations in the CPT-norm with appropriate definition of the conjugate state (i.e. the bra vector). The relevant expressions are given in the Appendix. Note that with $|\psi\rangle = |\psi\rangle_{CPT}$ the results for the CPT conjugate of the state $|\psi\rangle$ and the full density matrix are provided below

$$
\langle \psi | = \left( \hat{C} \hat{P} \hat{T} \otimes \hat{C} \hat{P} \hat{T} |\psi\rangle \right)^T = \sum_{n,m=1}^{2,3} C_{nm}^* |p_n\rangle \otimes |q_m\rangle
$$

(20)

$$
|\psi\rangle \langle \psi | = \sum_{n,m,a,b=1}^{2} C_{ab} C_{nm}^* |p_a\rangle \langle p_n| \otimes |q_b\rangle \langle q_m|
$$

(21)

Applying the partial trace in $\mathcal{H}_2$ gives us the reduced density operator for $\mathcal{H}_1$,

$$
\rho_1 = \text{Tr}_2[|\psi\rangle \langle \psi |] = \sum_{a,b,n=1}^{2} C_{ab} C_{nb}^* |p_a\rangle \langle p_n|
$$

(22)
where \( \rho_1 \) stands for the matrix

\[
\frac{1}{2 \cos \phi} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}
\]  

(23)

with its elements given by

\[
\begin{align*}
N_{11} &= (\alpha + \gamma)e^{i\phi} + (\beta + \delta)e^{-i\phi} \\
N_{12} &= (\beta - \alpha - \gamma) \\
N_{21} &= (\delta - \alpha - \beta)e^{-2i\phi} + \gamma e^{2i\phi} \\
N_{22} &= (\delta - \gamma)e^{i\phi} + (\alpha - \beta)e^{-i\phi} \\
\alpha &= C_{11}C_{11}^* + C_{12}C_{12}^* \\
\beta &= C_{11}C_{21}^* + C_{12}C_{22}^* \\
\gamma &= C_{21}C_{11}^* + C_{22}C_{12}^* = \beta^* \\
\delta &= C_{21}C_{21}^* + C_{22}C_{22}^* \\
\alpha + \delta &= 1
\end{align*}
\]

(24)

Now is the question of applying time evolution operation on \( \mathcal{H}_2 \). The reduced density matrix for \( \mathcal{H}_1 \) turns out to be

\[
|\psi_t\rangle = \sum_{n,m=1}^{2,2} e^{-i\lambda m t} C_{nm} |p_n\rangle \otimes |q_m\rangle
\]

\[
\lambda_m = \lambda_\pm = \pm \sqrt{\zeta^2 - \gamma^2}
\]

(25)

along with

\[
|\psi_t\rangle \langle \psi_t| = \sum_{n,m,a,b=1}^{2} e^{i(\lambda_m - \lambda_a) t} C_{ab} C_{nm}^* |p_a\rangle \langle p_n| \otimes |q_b\rangle \langle q_m|
\]  

(26)

\[
\rho_1(t) = \frac{1}{2 \cos \phi} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}
\]

(27)

(23) and (27) are the reduced density operators for \( \mathcal{H}_2 \), found before and after the action of time evolution operator. One notices that they are the same matrices, implying \( E(\psi) = E(\psi_t) \). In this way we find that the no-signalling principle is a valid principle in PTQM.

The answer to the query as to whether the eigenvalues of the density operators change if one goes towards the QM regime (by performing \( \phi, \phi' \to 0 \)) is evident if we look at the dependence of the eigenvalues on the parameters of the Hamiltonian \( [16] \) itself. We find

\[
\omega_\pm = \frac{1}{2} \left( (\alpha + \delta) \pm \sqrt{1 + 4(\beta \gamma - \alpha \delta)} \right)
\]

\[
= \frac{1}{2} \left( 1 \pm \sqrt{1 - 4|C_{11}C_{22} - C_{12}C_{21}|^2} \right)
\]

(28)

showing no dependence on the parameters. In short, the Hamiltonians \( [16] \) transform towards the spin system described by the eigenvectors of the Pauli matrix \( \sigma_x \).

B. Subsystems governed by PTQM and SQM

We now turn to the case when one subsystem is governed by SQM while other is by PTQM. For concreteness let the Hamiltonians \( \mathcal{H}_1 \) be defined for the PTQM and \( \mathcal{H}_2 \) for the SQM. We can then write the initial density matrix of the composite state \( [19] \) in the manner

\[
\rho_{1,2} = \frac{1}{2 \cos \phi} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \otimes \begin{bmatrix} 1 & \beta + \beta^* \\ \beta - \beta^* & 1 - (\beta + \beta^*) \end{bmatrix}
\]

(29)

It should be noted that the inner product structure in \( \mathcal{H}_2 \) is the same as in SQM. In mathematical terms, we have \( (K \text{ is usual conjugation and } T = K \text{ in our case}) \):

\[
\langle \psi | (\hat{C}\hat{P}\hat{T} \otimes K |\psi\rangle)^T = \sum_{n,m=1}^{2,2} C_{nm}^* (\hat{C}\hat{P}\hat{T} |p_n\rangle)^T \otimes (|u_m\rangle)^\dagger
\]

\[
\{|u_1\}, \{|u_2\} = \{(1), (0)\} = \left\{ \frac{1}{\sqrt{2}} \left[ 1 \quad 1 \right], \frac{1}{\sqrt{2}} \left[ -1 \quad 1 \right] \right\}
\]

(30)

It is straightforward to realise that finding the partial trace of \( \rho_{1,2} \) in either of the Hilbert Space would return the same set of eigenvalues as when acted upon by the time evolution operation. This of course means that the measure of entropy would remain the same even after the time evolution has taken place. Thus the no-signalling principle is preserved in this case too. As a final remark, let us take a special case of this system. It can be shown rather easily that a maximally entangled state of the Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), governed jointly by PTQM and SQM yields

\[
|\psi\rangle = \frac{1}{\sqrt{2}} \left( |p_1\rangle \otimes |1\rangle + |p_2\rangle \otimes |0\rangle \right)
\]

(31)

\[
\langle \psi | = \frac{1}{\sqrt{2}} \left( \langle p_1|_{CPT} \otimes \langle 1| + \langle p_2|_{CPT} \otimes \langle 0| \right)
\]

(32)

\[
|\psi\rangle \langle \psi| = \frac{1}{2} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \otimes \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]
\]

(33)

As obvious as it seems, the entanglement measure \( E(\psi) \) is 1 in either of the Hilbert Spaces, even after acting the time evolution operation.

IV. CONCLUSION

The calculations showing the violation of the no-signalling principle in \( [15] \) considered the maximally entangled state of the bipartite systems having the Hilbert
space being subjected to the Dirac norm. However, the bell states in modified quantum mechanical scenarios, such as in the PTQM, may not remain maximally entangled when the underlying Hilbert space is controlled by a CPT norm. The major difference of our work with the previous ones is that the latter did not consider finding the entanglement measure before the operation of time evolution. In the present work we demonstrated the non-violation of non-signalling principle for the case of bipartite systems when at least one is governed by PTQM and employing CPT inner product along with an appropriate choice of the maximally entangled state.

Appendix A: Calculations under CPT norm

Here $\langle \psi |$ is the CPT conjugate ($\langle \psi |_{CPT}$) as calculated from (10) and (13)

$$\langle p_{i} | p_{j} \rangle = \langle q_{i} | q_{j} \rangle = \delta_{ij} \quad \text{(A1)}$$

$$| p_{1} \rangle \langle p_{1} | = \frac{1}{2 \cos \phi} \begin{bmatrix} e^{i\phi} & -1 \\ -1 & e^{-i\phi} \end{bmatrix} \quad \text{(A2)}$$

$$| p_{2} \rangle \langle p_{2} | = \frac{1}{2 \cos \phi} \begin{bmatrix} e^{-i\phi} & 1 \\ 1 & e^{i\phi} \end{bmatrix} \quad \text{(A3)}$$

$$| p_{1} \rangle \langle p_{2} | = \frac{1}{2 \cos \phi} \begin{bmatrix} e^{-i\phi} & 1 \\ -e^{-2i\phi} & -e^{-i\phi} \end{bmatrix} \quad \text{(A4)}$$

$$| p_{2} \rangle \langle p_{1} | = \frac{1}{2 \cos \phi} \begin{bmatrix} e^{i\phi} & -1 \\ e^{2i\phi} & -e^{i\phi} \end{bmatrix} \quad \text{(A5)}$$

$$| q_{1} \rangle \langle q_{1} | = \frac{1}{2 \cos \phi} \begin{bmatrix} e^{i\phi'} & -1 \\ -1 & e^{-i\phi'} \end{bmatrix} \quad \text{(A6)}$$

$$| q_{2} \rangle \langle q_{2} | = \frac{1}{2 \cos \phi} \begin{bmatrix} e^{-i\phi'} & 1 \\ 1 & e^{i\phi'} \end{bmatrix} \quad \text{(A7)}$$

$$| q_{1} \rangle \langle q_{2} | = \frac{1}{2 \cos \phi} \begin{bmatrix} e^{-i\phi'} & -1 \\ -e^{-2i\phi'} & -e^{-i\phi'} \end{bmatrix} \quad \text{(A8)}$$

$$| q_{2} \rangle \langle q_{1} | = \frac{1}{2 \cos \phi} \begin{bmatrix} e^{i\phi'} & -1 \\ e^{2i\phi'} & -e^{i\phi'} \end{bmatrix} \quad \text{(A9)}$$

$$\text{Tr}[| p_{i} \rangle \langle p_{j} |] = \text{Tr}[| q_{i} \rangle \langle q_{j} |] = \delta_{ij} \quad \text{(A10)}$$

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