Mesoscopic Spin Hall Effect

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We investigate the spin Hall effect in ballistic chaotic quantum dots with spin-orbit coupling. We show that a longitudinal charge current can generate a pure transverse spin current. While this transverse spin current is generically nonzero for a fixed sample, we show that when the spin-orbit coupling time is large compared to the mean dwell time inside the dot, it fluctuates universally from sample to sample or upon variation of the chemical potential with a vanishing average. For a fixed sample configuration, the transverse spin current has a finite typical value $\approx e^2 V/h$, proportional to the longitudinal bias $V$ on the sample, and corresponding to about one excess open channel for one of the two spin species. Our analytical results are in agreement with numerical results in a diffusive system [W. Ren et al., Phys. Rev. Lett. 97, 066603 (2006)] and are further confirmed by numerical simulation in a chaotic cavity.

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**Introduction.** The novel and rapidly expanding field of spintronics is interested in the creation, manipulation, and detection of polarized or pure spin currents \[.\] The conventional methods of doing spintronics are to use magnetic fields and/or ferromagnets as parts of the creation-manipulation-detection cycle, and to use the Zeeman coupling and the ferromagnetic-exchange interactions to induce the spin dependency of transport. More recently, ways to generate spin accumulations and spin currents based on the coupling of spin and orbital degrees of freedom have been explored. Among these proposals, much attention has been focused on the spin Hall effect (SHE), where pure spin currents are generated by applied electric currents on spin-orbit (SO) coupled systems. Originally proposed by Dyakonov and Perel [2], the idea was resurrected by Hirsch [3] and extended to crystalline SO field (the intrinsic SHE) by Sinova et al. [4] and Murakami et al. [5]. The current agreement is that the SHE vanishes for bulk, $k$-linear SO coupling for diffusive two-dimensional electrons [4, 5, 8]. This result is however specific to these systems [8], and the SHE does not vanish for impurity-generated SO coupling, two-dimensional hole systems with either Rashba or Dresselhaus SO coupling, and for finite-sized electronic systems [6, 8]. These predictions have been, to some extent, confirmed by experimental observations of edge spin accumulations in electron [10] and hole [11] systems, and electrical detection of spin currents via ferromagnetic leads [12].

Most investigations of the SHE to date focused on disordered conductors with spin-orbit interaction, where the disorder-averaged spin Hall conductivity was calculated using either the Kubo formalism or a diffusion equation approach [3, 4, 6, 7, 8, 12, 13, 14]. Few numerical works alternatively used the scattering approach to transport [15] to calculate the average spin Hall conductance of explicitly finite-sized samples connected to external electrodes. These investigations were however restricted to tight-binding Hamiltonians with no or weak disorder in simple geometries [16, 16, 16]. The data of Ref. 16 in particular suggest that diffusive samples with large enough SO coupling exhibit universal fluctuations of the spin Hall conductance $G_{SH}$ with $\text{rms}[G_{SH}] \approx 0.18e^2/4\pi$. These numerical investigations call for an analytical theory of the SHE in mesoscopic systems. It is the purpose of this article to provide such a theory.

We analytically investigate the DC spin Hall effect in mesoscopic cavities with SO coupling. We calculate both the ensemble-average and the fluctuations of the transverse spin current generated by a longitudinal charge current. Our approach is based on random matrix theory (RMT) [20], and is valid for ballistic chaotic and mesoscopic diffusive systems at low temperature, in the limit when the spin-orbit coupling time is much shorter than the mean dwell time of the electrons in the cavity, $\tau_{\text{so}} \ll \tau_{\text{dwell}}$ [21]. We show that while the transverse spin current is generically nonzero for a typical sample, its sign and amplitude fluctuate universally, from sample to sample or upon variation of the chemical potential with a vanishing average. We find that for a typical ballistic chaotic quantum dot, the transverse spin current corresponds to slightly less than one excess open channel for one of the two spin species. These analytical results are confirmed by numerical simulations for a stroboscopic model of a ballistic chaotic cavity.

**Scattering approach.** We consider a ballistic chaotic quantum dot coupled to four external electrodes via ideal point contacts, each with $N_i$ open channels ($i = 1, \ldots, 4$). The geometry is sketched in Fig. 1. Spin-orbit coupling exists only inside the dot, and the electrochemical potentials in the electrodes are spin-independent. A bias voltage $V$ is applied between the longitudinal electrodes labeled 1 and 2. The voltages $V_3$ and $V_4$ are set such that
We replace the scattering matrix \( S \),

We introduced generalized transmission probabilities

no net charge current flows through the transverse electrodes 3 and 4. Spin-orbit coupling is active only in the gray region.

We will focus on the magnitude of the transverse currents flows through the transverse leads 3 and 4.

Random matrix theory. We calculate the average and fluctuations of the transverse spin currents \( J_i^{(\mu)} \), \( \mu = x, y, z \) within the framework of RMT. Accordingly, we replace the scattering matrix \( S \) by a random unitary matrix, which, in our case of a system with time reversal symmetry (absence of magnetic field) and totally broken spin rotational symmetry (strong spin-orbit coupling), has to be taken from the circular symplectic ensemble (CSE) \([20, 22, 23, 24]\). We rewrite the generalized transmission probabilities \( T_{ij}^{(\mu)} \) as a trace over \( S \)

\[
\langle T_{ij}^{(\mu)} \rangle = \frac{2\delta_{\mu 0}}{N_T - 1/2} \left( N_iN_j - \frac{1}{2}N_i\delta_{ij} \right),
\]

while variances and covariances are given by
where \( \langle \delta T^{(\mu)}_{ij} \delta T^{(\nu)}_{kl} \rangle \) is the transverse potentials \( \tilde{V}_{3,4} \) are spin-independent, they are not correlated with \( T^{(\mu)}_{ij} \). Additionally taking Eq. (3) into account, one concludes that the average transverse spin current vanishes \((i = 3, 4)\),

\[
\langle J_{i}^{(\mu)} \rangle = \frac{1}{4} \left( \langle \delta T^{(\mu)}_{12} \rangle - \langle \delta T^{(\mu)}_{11} \rangle - \sum_{j=3,4} \langle \delta T^{(\mu)}_{ij} \rangle \langle \tilde{V}_{j} \rangle \right) = 0. \tag{9}
\]

However, for a given sample at a fixed chemical potential \( J_{i}^{(\mu)} \) will in general be finite. We thus calculate \( \text{var} [J_{i}^{(\mu)}] \). We first note that \( \langle \tilde{V}_{3,4} \rangle / V = (N_{1} - N_{2}) / (2(N_{1} + N_{2}) \), and that \( \langle \tilde{V}_{3,4} \rangle \) vanishes to leading order in the inverse number of channels. One thus has

\[
\text{var} [J_{i}^{(\mu)}] = \frac{1}{4} \sum_{j=1,2} \text{var}[\delta T^{(\mu)}_{ij}] - \frac{1}{2} \text{covar}[\delta T^{(\mu)}_{i1}, \delta T^{(\mu)}_{i2}] \tag{10}
\]

\[
+ \sum_{j=3,4} \left\{ \text{var}[\delta T^{(\mu)}_{ij}] \langle \tilde{V}_{j} \rangle^{2} + \text{covar}[\delta T^{(\mu)}_{i1}, \delta T^{(\mu)}_{i2}] \langle \tilde{V}_{j} \rangle \right\} + 2 \text{covar}[\delta T^{(\mu)}_{i3}, \delta T^{(\mu)}_{i4}] \langle \tilde{V}_{3} \rangle \langle \tilde{V}_{4} \rangle.
\]

From Eq. (3) it follows that

\[
\text{var} [J_{i}^{(\mu)}] = \frac{4N_{1}N_{2}(N_{T} - 1)}{N_{T}(2N_{T} - 1)(2N_{T} - 3)(N_{1} + N_{2})}. \tag{11}
\]

Eqs. (3) and (11) are our main results. They show that, while the average transverse spin current vanishes, it exhibits universal sample-to-sample fluctuations. The origin of this universality is the same as for charge transport \cite{20}, and relies on the fact expressed in Eq. (3) that to leading order, spin-dependent transmission correlators do not scale with the number of channels. The spin current carried by a single typical sample is given by \( \text{rms}[J_{i}^{(\mu)}] \times e^{2}V/h \), and is thus of order \( e^{2}V/h \) in the limit of large number of channels. In other words, for a given sample, one spin species has of order one more open transport channel than the other one. For a fully symmetric configuration, \( N_{i} \equiv N \), the spin current fluctuates universally for large \( N \), with \( \text{rms}[J_{i}^{(\mu)}] \approx (e^{2}V/h) / \sqrt{2} \). This translates into universal fluctuations of the transverse spin conductance with \( \text{rms}[G_{\mu\nu}] = (e / 4\pi \sqrt{2}) \approx 0.18(e / 4\pi T) \) in agreement with Ref. \cite{19}.

### Numerical simulation

In the diffusive setup of Ref. \cite{19} the universal regime is not very large and thus it is difficult to unambiguously identify it. We therefore present numerical simulations in chaotic cavities to further illustrate our analytical predictions \cite{9} and \cite{11}.

We model the electronic dynamics inside a chaotic ballistic cavity by the spin kicked rotator \cite{26,27}, a one-dimensional quantum map which we represent by a \( 2M \times 2M \) Floquet (time-evolution) matrix \cite{25}

\[
\mathcal{F}_{ll'} = (\Pi U X U^{\dagger} \Pi)_{ll'}, \quad l, l' = 0, 1, \ldots, M - 1, \quad \text{for} \quad \Pi_{ll'} = \delta_{ll'} e^{-i\pi l(l+1)^{2}/M}, \tag{12a}
\]

\[
U_{ll'} = M^{-1/2} e^{-i\pi ll'/M} \sigma_{0}, \quad \text{for} \quad \sigma_{0}, \tag{12b}
\]

\[
X_{ll'} = \delta_{ll'} e^{-i(\pi / 4) V l(l+1)/M} \sigma_{0}, \quad \text{for} \quad \Pi_{ll'} = \delta_{ll'} e^{-i(\pi / 4) V l(l+1)/M}. \tag{12c}
\]

The matrix size \( 2M = 2L / \lambda_{F} \gg 1 \) is given by twice the ratio of the linear system size to the Fermi wavelength. The matrix \( X \) with

\[
V(p) = K \cos p \sigma_{0} + K_{so} \sigma_{x} \sin 2p + \sigma_{z} \sin p \tag{13}
\]

corresponds to free SO coupled motion interrupted periodically by kicks described by the matrix \( \Pi \), corresponding to scattering off the boundaries of the quantum dot. In this form, the model is time-reversal symmetric, and the parameter \( l_{0} \) ensures that no additional symmetry exists in the system. The map is classically chaotic for kicking strength \( K \gtrsim 7.5 \), and \( K_{so} \) is related to the SO coupling time \( \tau_{so} \) (in units of the stroboscopic period) through \( \tau_{so} = 32\pi^{2} / K_{so}^{2}M^{2} \). From \cite{12}, we construct the quasienergy-dependent scattering matrix \cite{25}.

\[
S(\varepsilon) = P[e^{-i\varepsilon} - \mathcal{F}(1 - P^{T}P)^{-1} \mathcal{F}P^{T}], \tag{14}
\]

with \( P \) a \( 2N_{T} \times 2M \) projection matrix

\[
P_{\alpha k l' \beta} = \begin{cases} 
\delta_{\alpha \beta} & \text{if } k' = l^{(k)}, \\
0 & \text{otherwise}.
\end{cases} \tag{15}
\]

The \( l^{(k)} \) \( (k = 1, 2, \ldots, 2N_{T}) \) labels the modes) give the position in phase space of the attached leads. The mean dwell time \( \tau_{dwell} \) (in units of the stroboscopic period) is given by \( \tau_{dwell} = M / N_{T} \). At large enough SO coupling, this model has been shown to exhibit the universality of the CSE. We refer the reader to Ref. \cite{27} for further details on the model.

Averages were performed over 35 values of \( K \) in the range \( 41 < K < 48 \), 25 values of \( \varepsilon \) uniformly distributed
metric configuration with vs. the number of modes. Top panel: longitudinally symmetric configuration with $N_1 = N_2 = 2N_3 = 2N_4 = 2N$; bottom panel: longitudinally asymmetric configuration with $N_2 = N_1 = 2N_1 = 2N_3 = 2N$. In both cases the total number of modes $N_T = 6N$. The solid (dashed) lines give the analytical predictions [11] for the mean (variance) of the spin currents. Empty diamonds correspond to $\langle J^{(i)} \rangle$, circles to $\text{var} [J^{(i)}]$ and triangles to $\text{var} [J^{(i)}]$

Figure 2: Average and variance of the transverse spin current vs. the number of modes.

in $0 < \varepsilon < 2\pi$, and 10 different lead positions $l^{(k)}$. We set the strength of $K_{so}$ such that $\tau_{so} = \tau_{\text{dwell}}/1250$, and fixed values of $M = 640$ and $l_0 = 0.2$.

Our numerical results are presented in Fig. 2. Two cases were considered, the longitudinally symmetric ($N_1 = N_2$) and asymmetric ($N_1 \neq N_2$) configurations. In both cases, the numerical data fully confirm our predictions that the average spin current vanishes and that the variance of the transverse spin current is universal, i.e. it does not depend on $N$ for large enough value of $N$. In the asymmetric case $N_4 = 2N_3$, the variance of the spin current in lead 4 is twice as big as in lead 3, giving further confirmation to Eq. [11].

Conclusion. We have calculated the average and mesoscopic fluctuations of the transverse spin current generated by a charge current through a chaotic quantum dot with SO coupling. We find that, from sample to sample, the spin current fluctuates universally around zero average. In particular, for a fully symmetric configuration $N_i \equiv N$, this translates into universal fluctuations of the spin conductance with $\text{rms} [G_{Hi}] = \langle e/4\pi \sqrt{32} \rangle \approx 0.18(e/4\pi)$. This analytically establishes the universality observed numerically in Ref. [11].

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[1] I. Zutić et al., Rev. Mod. Phys. 76, 323 (2004).
[2] M.I. Dyakonov and V.I. Perel, Sov. Phys. JETP Lett. 13, 467 (1971); Phys. Lett. A 35, 459 (1971).
[3] J. E. Hirsch, Phys. Rev. Lett. 83, 1834 (1999).
[4] J. Sinova et al., Phys. Rev. Lett. 92, 126603 (2004).
[5] S. Murakami, Phys. Rev. B 69, 241202(R) (2004).
[6] J.-I. Inoue, G.E.W. Bauer, and L.W. Molenkamp, Phys. Rev. B 70, 041303(R) (2004).
[7] E.G. Mishchenko, A.V. Shytov, and B.I. Halperin, Phys. Rev. Lett. 93, 226602 (2004).
[8] A.A. Burkov, A.S. Núñez, and A.H. MacDonald, Phys. Rev. B 70, 155308 (2004).
[9] I. Adagideli and G.E.W. Bauer, Phys. Rev. Lett. 95, 256602 (2005).
[10] Y.K. Kato et al., Science 306, 1910 (2004); V. Sih et al., Nature Phys. 1, 31-35 (2005).
[11] J. Wunderlich et al., Phys. Rev. Lett. 94, 047204 (2005).
[12] E. Saitoh et al., Appl. Phys. Lett. 88, 182509 (2006); S.O. Valenzuela and M. Tinkham, Nature 442, 176 (2006); T. Kimura et al., cond-mat/0609304
[13] J. Schliemann and D. Loss, Phys. Rev. B 71, 085308 (2005).
[14] R. Raimondi and P. Schwab, Phys. Rev. B 71, 033311 (2005).
[15] M. Büttiker, Phys. Rev. Lett. 57, 1761 (1986).
[16] B.K. Nikolić, L.P. Zárobo, and S. Souma, Phys. Rev. B 72, 75361 (2005).
[17] E.M. Hankiewicz, L.W. Molenkamp, T. Jungwirth, and J. Sinova, Phys. Rev. B 70, 241301(R) (2004).
[18] L. Sheng, D.N. Sheng, and C.S. Ting, Phys. Rev. Lett. 94, 016602 (2005).
[19] W. Ren, Z. Qiao, J. Wang, Q. Sun, and H. Guo, Phys. Rev. Lett. 97, 066603 (2006).
[20] C. W. J. Beenakker, Rev. Mod. Phys. 69, 731 (1997).
[21] It is at present unclear to us whether the validity of RMT requires averaging over lead positions, in addition to disorder averaging.
[22] We assume that the SO coupling parameters are sufficiently nonuniform, so that SO cannot be removed from the Hamiltonian by a gauge transformation, see : [23, 24].
[23] I. L. Aleiner and V. I. Fal’ko, Phys. Rev. Lett. 87, 256801 (2001).
[24] P. W. Brouwer, J. N. H. J. Cremers, and B. I. Halperin, Phys. Rev. B 65, 081302(R) (2002).
[25] P. W. Brouwer and C. W. J. Beenakker, J. Math. Phys. 37, 4904 (1996).
[26] R. Scharf, J. Phys. A 22, 4223 (1989).
[27] J. H. Bardarson, J. Tworzydlo, and C. W. J. Beenakker, Phys. Rev. B 72, 235305 (2005).
[28] F. M. Izrailev, Phys. Rep. 196, 299 (1990).
[29] Y.V. Fyodorov and H.-J. Sommers, JETP Lett. 72, 422 (2000).