EXCEPTIONAL COLLECTIONS ON $\Sigma_2$

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Abstract. Structure theorems for exceptional objects and exceptional collections of the bounded derived category of coherent sheaves on del Pezzo surfaces are established by Kuleshov and Orlov in [KO94]. In this paper we propose conjectures which generalize these results to weak del Pezzo surfaces. Unlike del Pezzo surfaces, an exceptional object on a weak del Pezzo surface is not necessarily a shift of a sheaf and is not determined by its class in the Grothendieck group. Our conjectures explain how these complications are taken care of by spherical twists, the categorification of $(-2)$-reflections acting on the derived category.

This paper is devoted to solving the conjectures for the prototypical weak del Pezzo surface $\Sigma_2$, the Hirzebruch surface of degree 2. Specifically, we prove the following results: Any exceptional object is sent to the shift of the uniquely determined exceptional vector bundle by a product of spherical twists which acts trivially on the Grothendieck group of the derived category. Any exceptional collection on $\Sigma_2$ is part of a full exceptional collection. We moreover prove that the braid group on 4 strands acts transitively on the set of exceptional collections of length 4 (up to shifts).

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1. Introduction

Semiorthogonal decomposition is among the most fundamental notions of triangulated categories. The finest ones, i.e., semiorthogonal decompositions whose components are equivalent to the bounded derived category of a point, are identified with (full) exceptional collections of the triangulated category. If there is an exceptional collection in a triangulated category, then virtually always there are infinitely many of them because of the group action explained below. Hence the classification of exceptional collections is not obvious at all.

Let $X$ be a smooth projective variety and $D(X)$ be the bounded derived category of coherent sheaves on $X$. Concerning the classification of the exceptional collections of $D(X)$, the prototypical results are given in [GR87, Section 5] for $X = \mathbb{P}^2$. The similar results for $X = \mathbb{P}^1 \times \mathbb{P}^1$ are given in [Rud88]. Based on these works, Kuleshov and Orlov established in [KO94] the similar results for arbitrary del Pezzo surfaces (see also [GK04, Corollary 4.3.2, Theorem 4.3.3, Theorem 4.6.1]). They are summarized as follows.

**Theorem 1.1** ([KO94]). Let $X$ be a del Pezzo surface over an algebraically closed field $k$.

1. Any exceptional object $E \in D(X)$ is either a vector bundle or a line bundle on a $(-1)$-curve, up to shifts.
2. The isomorphism class of an exceptional object $E \in D(X)$ is determined by its class $[E] \in K_0(X)$, up to shifts by $2\mathbb{Z}$.
3. Any exceptional collection of $D(X)$ can be extended to a full exceptional collection.
4. Conjecture 1.2 below is true for $X$; i.e., the action $G_r \ltimes EC_r(D(X))$ is transitive.

The symbol $K_0(X)$ denotes the Grothendieck group of the triangulated category $D(X)$. We briefly explain (4). Let $EC_r(D(X))$ denote the set of isomorphism classes of exceptional collections of length $r$ of $D(X)$, where $r = \text{rank } K_0(X)$. An important fact relevant to the classification is that there is a standard action of the group $G_r := \mathbb{Z}^r \rtimes \text{Br}_r$ on $EC_r(D(X))$ by shifts and mutations, where $\text{Br}_r$ is the braid group on $r$ strands (see Section 2.3 for details).

Inspired by the earlier works mentioned above, Bondal and Polishchuk gave the following conjecture.

**Conjecture 1.2** ([BP93, Conjecture 2.2]). Suppose that $X$ is a smooth projective variety such that $D(X)$ admits a full exceptional collection of length $r$. Then the action $G_r \ltimes EC_r(D(X))$ is transitive.

In dimensions greater than 2, the classification of exceptional collections is widely open. Even for $\mathbb{P}^3$, only partial results seem to be known ([Pos95]) and Conjecture 1.2 is still open.

The aim of this paper is to investigate the generalization of Theorem 1.1 to weak del Pezzo surfaces; i.e., smooth projective surfaces $X$ whose anti-canonical bundle $\omega_X^{-1}$ is nef and big. As it turns out, the generalization is not straightforward at all.

A weak del Pezzo surface $X$ which is not a del Pezzo surface admits at least one (and only finitely many) $(-2)$-curve(s); i.e., a smooth curve $C \subset X$ with $C^2 = -2$ and $C \simeq \mathbb{P}^1$. Line bundles on $C$, as objects of $D(X)$, are 2-spherical objects and hence yield non-trivial autoequivalences of $D(X)$ called spherical twists due to Seidel and Thomas [ST01]. For a 2-spherical object $\alpha \in D(X)$, the corresponding spherical twist $T_\alpha$ acts as a reflection on $K_0(X)$ whereas it always satisfies $T_\alpha^2 \not\simeq \text{id}_{D(X)}$. For example, the exceptional object $T_{\mathcal{O}(\mathcal{O}_X)}$ has the same class as $\mathcal{O}_X$ in $K_0(X)$ but is not isomorphic to $\mathcal{O}_X$. In fact, by
direct computation one can confirm the following.

$$\mathcal{H}^i \left( T^2_{\mathcal{O}_C}(\mathcal{O}_X) \right) \simeq \begin{cases} 
\mathcal{O}_X & i = 0 \\
\mathcal{O}_C & i = 1, 2 \\
0 & \text{otherwise}
\end{cases}$$

Moreover, by applying $T^2_{\mathcal{O}_C}$ repeatedly, we obtain a collection of infinitely many exceptional objects of unbounded cohomological amplitudes which share the same class in $K_0(X)$. Hence Theorem 1.1 (1) (2) are not true for $X$ at all.

Conjecture 1.3 which is the main conjecture of this paper, generalizes Theorem 1.1 to weak del Pezzo surfaces while taking into account all the complications mentioned in the previous paragraph. In a word, it asserts that the failure of Theorem 1.1 (1) (2) on weak del Pezzo surfaces is remedied by spherical twists and that Theorem 1.1 (3) (4) should hold for weak del Pezzo surfaces too. The main theorem of this paper is that Conjecture 1.3 is true for $\Sigma_2 = \mathbb{P}^1(\mathcal{O} \oplus \mathcal{O}(2))$; namely, the Hirzebruch surface of degree 2, a paradigm of weak del Pezzo surfaces. In fact we formulated Conjecture 1.3 by generalizing the results we obtained for $\Sigma_2$, Theorem 1.1, and [CJ18, Theorem 1.2] simultaneously.

**Conjecture 1.3.** Let $X$ be a weak del Pezzo surface over an algebraically closed field $k$.

1. For any exceptional object $\mathcal{E} \in D(X)$, there exists a sheaf $\mathcal{F}$ on $X$ which is either an exceptional vector bundle or a line bundle on a $(-1)$-curve, a sequence of line bundles on $(-2)$-curves $L_1, \ldots, L_n$, and an integer $m \in \mathbb{Z}$ such that

$$\mathcal{E} \simeq (T_{L_n} \circ \cdots \circ T_{L_1})(\mathcal{F})[m].$$

2. For any pair of exceptional objects $\mathcal{E}, \mathcal{E}' \in D(X)$ such that $[\mathcal{E}] = [\mathcal{E}'] \in K_0(X)$, there are a product $b$ of spherical twists and inverse spherical twists which acts trivially on $K_0(X)$ and $m \in 2\mathbb{Z}$ such that $\mathcal{E}' \simeq b(\mathcal{E})[m]$. Moreover, for each exceptional object $\mathcal{E} \in D(X)$, there is a unique exceptional vector bundle $\mathcal{F}$ such that either $[\mathcal{E}] = [\mathcal{F}]$ or $[\mathcal{E}] = -[\mathcal{F}]$ holds.

3. Any exceptional collection on $X$ can be extended to a full exceptional collection.

4. Conjecture 1.3 is true for $X$.

As it is more or less visible, (1), (2), (3), and (4) of Conjecture 1.3 generalizes (1), (2), (3), and (4) of Theorem 1.1 respectively.

To avoid repetition, let us simply state the main result of this paper as follows.

**Theorem 1.4 (MAIN THEOREM).** Conjecture 1.3 is true for $X = \Sigma_2$.

More specifically, for $X = \Sigma_2$

- Conjecture 1.3 (1) is solved affirmatively in Section 3 as Theorem 3.1.
- Conjecture 1.3 (2) is solved affirmatively in Section 4 as Corollary 4.1.
- Conjecture 1.3 (3) is solved affirmatively in Section 5 as Corollary 5.1.
- Conjecture 1.3 (4) is solved affirmatively in Section 6 as Theorem 6.1.

**Remark 1.5.** We give some comments on the preceding works which are related to Conjecture 1.3 and Theorem 1.4, and one important consequence of Conjecture 1.3 (4) on the fullness of exceptional collections of maximal length.

- This paper is a continuation of [OU15] by the 2nd and the 3rd authors and almost completely supersedes it. Conjecture 1.3 (1) for $\Sigma_2$ is stated as [OU15] Conjecture 1.3, and is solved for exceptional sheaves in [OU15] Theorem 1.4.
Among others we show that there is the index $i$ for torsion exceptional sheaves is partially solved in [CJ18 Theorem 1.2]. A weaker version of Conjecture 1.3 is stated as [CJ18 Conjecture 1.1].

Exceptional objects and exceptional collections of vector bundles on weak del Pezzo surfaces is systematically studied in [Kul97]. In fact we use some results of this work in this paper.

It follows from the definition of mutations that the triangulated subcategory generated by an exceptional collection is invariant under mutation. In particular, the fullness of an exceptional collection is preserved by mutations. On the other hand, any $X$ as in Conjecture 1.3 admits a full exceptional collection. Hence Conjecture 1.3 would imply that any exceptional collection of length equal to rank $K_0(X) = \text{rank Pic}(X) + 2$ of $D(X)$ is full, though it would also follow from Conjecture 1.3 (3).

1.1. Summary of each section and structure of the paper. Section 2 is a preliminary section. We recall the rudiments of mutations and the group $B$ of autoequivalences generated by spherical twists from [IU05]. Among others, we prove in Corollary 2.11 that $B^{K_0-triv}$, the subgroup of $B$ acting trivially on $K_0(\Sigma_2)$, is generated by spheres of spherical twists by line bundles on $C$.

From Section 3 till the end of the paper, we restrict ourselves to the proof of Theorem 3.1 and in particular discuss the case of $\Sigma_2$ only.

Section 3 is the main component of the paper and devoted to the proof of Theorem 3.1. Namely, in this section, we prove that for each exceptional object $E \in D(\Sigma_2)$ there is a sequence of integers $a_1, \ldots, a_n$ such that $(T_{a_n} \circ \cdots \circ T_{a_1})(E)$ is isomorphic to a shift of a vector bundle. In accordance with the steps of the proof, Section 3 is divided into six subsections.

In Section 3.1 we prove some properties of the cohomology sheaf $H^*(E) = \bigoplus_{i \in \mathbb{Z}} H^i(E)$. Among others we show that there is the index $i_0 \in \mathbb{Z}$ such that $\text{Supp } H^{v0}(E) = \Sigma_2$ and for any $i \neq i_0$ the cohomology sheaf $H^i(E)$, if not 0, is a pure sheaf whose reduced support is $C$. Then in Section 3.2, we prove that actually the schematic support of $H^i(E)$ for $i \neq i_0$ is $C$. This is the most technical part of the paper, and actually this had been the main obstacle for the whole work. Fortunately one can use the result of this subsection as a black box to read the rest of the paper.

In Section 3.3 we prove that there is a decomposition $H^{v0}(E) \simeq T \oplus E$, where $E = E(v) \oplus E(v)^\vee \oplus E(v)^{\vee \vee}$ is an exceptional sheaf and $T$ is a torsion sheaf. We moreover show that if tors $E(v) \neq 0$, then $T$ is a direct sum of copies of $O_C(a)$ for some $a \in \mathbb{Z}$ and that tors $H^0(E) = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{a \in \mathbb{Z}} O_C(a)$ is a direct sum of copies of $O_C(a)$ and $O_C(a+1)$. This integer $a = a(E)$ plays a central role throughout the paper.

In Section 3.4 we investigate the relationship between the cohomology sheaves of $E$ and those of $E(v)$, the derived dual of $E$. We in particular show in Corollary 3.27 that if $E$ is not isomorphic to a shift of a vector bundle and tors $E(v) = 0$, then tors $E(v)^{\vee \vee} \neq 0$. In this paper we mainly discuss the case tors $E(v) \neq 0$, and by this result, we can settle the case where tors $E(v) = 0$ by passing to $E(v)^{\vee \vee}$.

In Section 3.5 we introduce the notion of the length of the “torsion part” of an object at the generic point $\gamma$ of $C$. Formally speaking, for $E$ it is defined as $\ell(E) = \sum_{i \in \mathbb{Z}} \text{length}_{\text{tors} H^i(E)}$. It follows that $E$ is isomorphic to a shift of a vector bundle if and only if $\ell(E) = 0$, and hence it suffices to show that $\ell(T_c(E)) < \ell(E)$ for some $c = c(E) \in \mathbb{Z}$ if $\ell(E) > 0$. This is exactly what we achieve in Section 3.6. We show in Theorem 3.32 that $c = a(E)$ works if tors $E(v) \neq 0$ and otherwise $c = -a(E(v)) - 3$ does.
Spherical objects on the minimal resolution of type $A$ singularity is classified in [IU05]. More specifically, the proof of Theorem 3.1 is an adaptation of the proof of [IU05, Proposition 5.1]. It is, however, much more involved than that of [IU05, Proposition 5.1]. This is due to the fact that the support of an exceptional object on $\Sigma_2$ is never concentrated in the $(-2)$-curve $C$. On the contrary the reduced support of a spherical object on $\Sigma_2$ is concentrated in $C$, and it immediately implies that the schematic supports of the cohomology sheaves of the spherical object coincide with $C$.

Section 4 is devoted to the proof of Theorem 4.3. It is almost immediately obtained by combining Theorem 3.1 with a small trick on squares of spherical twists (Proposition 2.8) and the fact that an exceptional vector bundle is uniquely determined by its class in $\text{K}_0(\Sigma_2)$ (Lemma 2.33).

Section 5 is devoted to the proof of Corollary 5.6. Take an exceptional collection $E$. We first show in Theorem 5.4 that there is a product of spherical twists $b$ such that $b(E)$ consists of vector bundles up to shifts. This is achieved in a one-by-one manner. The key is that if $(B, E)$ is an exceptional pair such that $B$ is a vector bundle and $\ell(E) > 0$, then, surprisingly enough, $T_c(E)(B)$ remains to be a vector bundle (though it may not be isomorphic to $B$). Recall that $c(E) \in \mathbb{Z}$ depends only on $E$ and that $T_c(E)$ strictly decreases the length of $E$.

Corollary 5.6 is known for exceptional collections of vector bundles by a slight generalization Theorem 5.5 of a result by Kuleshov in [Kul97]. Applying it to $b(E)$, we immediately obtain the proof of Corollary 5.6 for $E$.

Section 6 is devoted to the proof of Theorem 6.1. We use the deformation of $\Sigma_2$ to $\mathbb{P}^1 \times \mathbb{P}^1$. The difficulty is that there are infinitely many exceptional objects on $\Sigma_2$ which deform to the same exceptional object on $\mathbb{P}^1 \times \mathbb{P}^1$ (non-uniqueness of the specialization, which is translated into the non-separatedness of the moduli space of semiorthogonal decompositions introduced in [BOR20]). Actually, two exceptional objects have the same deformation to $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if they have the same class in $\text{K}_0(\Sigma_2)$ (up to shifts by $2\mathbb{Z}$).

In Step 1 of the proof, we use the fact that the corresponding result is already known by Theorem 6.1 (4) for the del Pezzo surface $\mathbb{P}^1 \times \mathbb{P}^1$. Since deformation of exceptional collections commutes with mutations, the result for $\mathbb{P}^1 \times \mathbb{P}^1$ immediately implies that for any exceptional collection of length 4 on $\Sigma_2$ there is a sequence of mutations which brings it to a collection which is numerically equivalent to the standard collection $E_{\text{std}}$ (i.e., having the same classes as $E_{\text{std}}$ in $\text{K}_0(\Sigma_2)$). See (2.29) for the definition of $E_{\text{std}}$.

It remains to show that an exceptional collection $E$ on $\Sigma_2$ which is numerically equivalent to $E_{\text{std}}$ can be sent to $E_{\text{std}}$ by mutations. In Step 2 as an intermediate step, we find $b \in B$ such that $E = b(E_{\text{std}})$. We construct such $b$ again in one-by-one manner. In Step 3 we prove that $b$ can be replaced by a sequence of mutations. Thanks to the fact that mutations commute with autoequivalences, it is enough to show the assertion only for $b \in \{ T_0, T_{-1} \}$. Recall that $T_0, T_{-1}$ generate $B$. At this point the problem is concrete enough to be settled by hand.

1.2. Some words on future directions. Though Conjecture 1.3 is stated for arbitrary weak del Pezzo surfaces, in this paper we restrict ourselves to the study of the case of $\Sigma_2$. This is partly because the proof is rather involved already in this case. We nevertheless think that the case of $\Sigma_2$ should serve as a paradigm for the further investigations of Conjecture 1.3.

The proof of Theorem 1.4 is based on the classification of rigid sheaves on the $(-2)$-curve; i.e., the fundamental cycle of the minimal resolution of the $A_1$-singularity. So far
such classification is achieved only for the minimal resolution of type $A$ singularities by [IU05]. If one wants to push the strategy of this paper, it seems inevitable to establish the similar classification for the minimal resolution of type $D$ and type $E$ singularities. See [Kaw19] for results in this direction.

A weak del Pezzo surface over an algebraically closed field $k$ is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$, $\Sigma_2$, or a blowup of $\mathbb{P}^2$ in at most eight points in almost general positions (see, say, [Dol12 Theorem 8.1.15, Corollary 8.1.24]). Hence our strategy based on the deformation to del Pezzo surfaces, in principle, is applicable to all weak del Pezzo surfaces.

Structure theorems for exceptional collections on $\mathbb{P}^2$ play an important role in showing the contractibility of (the main component of) the space of Bridgeland stability conditions $\text{Stab}(\mathbb{P}^2)$ in [Li17]. Our results should be similarly useful for studying $\text{Stab}(\Sigma_2)$. More specifically, they should be very closely related to the relationship between $\text{Stab}(\Sigma_2)$ and $\text{Stab}(\mathbb{P}^1 \times \mathbb{P}^1)$; see Remark 5.11 for details.

1.3. Notation and convention. We work over an algebraically closed field $k$, unless otherwise stated. To ease notation, we will write $\text{Ext}^i = \dim_k \text{Ext}^i$, $h^i = \dim_k H^i$, $E^{p,q}_2 = \dim_k E^{p,q}_2$, and so on. Below is a list of frequently used symbols.

- $\Sigma_d$ the Hirzebruch surface of degree $d$ (2.1)
- $C$ the $(-2)$-curve of $\Sigma_2$ (2.1)
- $f$ the divisor class of a fiber of the morphism $\Sigma_2 \to \mathbb{P}^1$ (2.3)
- $*^\vee$ the derived dual of $* \in D(\Sigma_2)$ (2.4)
- $T_a$ (resp. $T'_a$) $\in \text{Auteq}(\Sigma_2)$ the (inverse) spherical twist by $O_C(a)$ (2.6)
- $B < \text{Auteq}(\Sigma_2)$ the group of autoequivalences generated by spherical twists (Definition 2.10)
- $\text{EC}_N, \text{ECVB}_N, \text{FEC}, \text{FECVB}$ various sets of exceptional collections (Definition 2.14)
- $\text{Br}_N$ (resp. $G_N$) the braid group on $N$ strands (2.18) (resp. the extension of $\text{Br}_N$ by $\mathbb{Z}^N$ (2.20))
- $\text{gen}$ the generalization map for exceptional collections from the central fiber to the generic fiber (2.24)
- $\text{numEC}_N, \text{numFEC}$ various sets of numerical exceptional collections (Definition 2.34)
- $E^{\text{std}} \in \text{FECVB}(\Sigma_2)$ the standard full exceptional collection of $D(\Sigma_2)$ (Definition 2.36)
- $E^{\text{std}}_\xi \in \text{FECVB}(\mathcal{X}_{\text{gen}})$ the standard full exceptional collection of $D(\mathcal{X}_{\text{gen}})$ (2.30)
- $\mathcal{H}^i(\mathcal{E})$ (resp. $\mathcal{H}^i(\mathcal{E})$) the total (resp. $i$-th) cohomology of $\mathcal{E} \in D(\mathcal{X})$ with respect to the standard t-structure (2.24)
- $i_0 = i_0(\mathcal{E})$ the unique index such that $\text{Supp} \mathcal{H}^{i_0}(\mathcal{E}) = \Sigma_2$ (Definition 3.7)
- $(R, m)$ the complete local ring of the $A_1$-singularity (Notation 3.10)
- $(0 : I)_M \subseteq M$ the maximal submodule of $M$ annihilated by the ideal $I \subseteq R$ (3.19)
- $\mathcal{I}_C \subseteq \mathcal{O}_{\Sigma_2}$ the ideal sheaf of $C \subset \Sigma_2$ (3.21)
- $D(*)$ the dual $k$-vector space of $*$ (3.24)
- $\mathcal{H}^{i_0}(\mathcal{E}) \simeq E(\mathcal{E}) \oplus T(\mathcal{E})$ the canonical decomposition into an exceptional sheaf and a torsion sheaf (Lemma 3.21)
- $\mathcal{T}(\mathcal{E}), \mathcal{F}(\mathcal{E})$ the torsion (resp. the torsion free) part of $\mathcal{H}^{i_0}(\mathcal{E})$ (Definition 3.22)
a, s, t ∈ ℤ  integers specified by the irreducible decomposition of T
\[T \text{ (3.30)}\] (see also Lemma 3.21)

ℓ(*) “length of the torsion part of *” at the generic point γ of C (Definition 3.28)

b, r, s ∈ ℤ  integers specified by the irreducible decomposition of an
exceptional vector bundle restricted to C (Lemma 3.20)

Acknowledgements. During the preparation of this paper, A.I. was partially supported
by JSPS Grants-in-Aid for Scientific Research (19K03444). S.O. was partially sup-
ported by JSPS Grants-in-Aid for Scientific Research (16H05994, 16H02141, 16H06337,
18H01120, 20H01797, 20H01794). H.U. was partially supported by JSPS Grants-in-Aid
for Scientific Research (18K03249).

2. Preliminaries

2.1. The Hirzebruch surfaces. The Hirzebruch surface of degree \(d \in \mathbb{Z}_{\geq 0}\) is the ruled
surface
\[p: \Sigma_d := \mathbb{P}_{\mathbb{P}^1} (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)) \to \mathbb{P}^1.\]  \hspace{1cm} (2.1)

The history of Hirzebruch surfaces goes back, at least, to the first paper by Hirzebruch
[Hir51]. An explicit isotrivial degeneration of \(\Sigma_d\) to \(\Sigma_{d'}\) for \(d > d'\) and \(d - d' \in 2\mathbb{Z}\) is
constructed in [Kod63, p.86 Example]. As a special case, there exists a smooth projective
morphism (defined over Spec \(\mathbb{Z}\))
\[\mathcal{X} \to \mathbb{A}_t^1\]  \hspace{1cm} (2.2)
such that the fiber over \(t = 0\) is isomorphic to \(\Sigma_2\) and the restriction of the family over
the open subscheme \(G_m \hookrightarrow \mathbb{A}^1\) is isomorphic to the trivial family \((\mathbb{P}^1 \times \mathbb{P}^1) \times G_m \xrightarrow{pr_2} G_m\).

In this paper we investigate the bounded derived category of coherent sheaves on \(\Sigma_2\),
which is the most basic example of weak del Pezzo surfaces. We let \(C \subset \Sigma_2\) denote the
unique negative curve, and \(f\) the (linear equivalence class of) the fiber of \(p\). Recall that
\[\text{Pic } \Sigma_2 = \mathbb{Z}C \oplus \mathbb{Z}f,\]
where
\[C^2 = -2, \quad f^2 = 0, \quad C.f = 1.\]  \hspace{1cm} (2.3)
The anti-canonical bundle is given by
\[-K_{\Sigma_2} = 2C + 4f.\]

2.2. Derived category, spherical twist, and the autoequivalence group of \(\Sigma_2\).

Definition 2.1. For a quasi-compact scheme \(Y\), we let Perf \(Y\) denote the perfect derived
category of \(Y\) with the standard structure of a triangulated category. When \(Y\) is equipped
with a morphism to Spec \(k\), we think of Perf \(Y\) as a triangulated \(k\)-linear category. It
comes with the natural symmetric monoidal structure given by the tensor product over
\(\mathcal{O}_Y\), but we do not take it into account unless otherwise stated.

When \(Y\) is a smooth and projective variety over a field \(k\), we identify Perf \(Y\) with the
bounded derived category \(D(Y)\) of coherent sheaves on \(Y\).

The following equivalence of tensor triangulated categories
\[\forall : (\text{Perf } Y)^{\text{op}} \xrightarrow{\simeq} \text{Perf } Y; \quad \mathcal{E} \mapsto \mathcal{E}^{\vee} := \mathbb{R} \mathcal{H}om_Y (\mathcal{E}, \mathcal{O}_Y)\]  \hspace{1cm} (2.4)
will be called the derived dual.
One can easily verify that there exists a canonical natural isomorphism
\[ \text{id} \cong \id. \]

**Definition 2.2.** For smooth projective varieties \( X, Y \) over \( \text{Spec} k \) and an object \( K \in D(X \times_k Y) \), the integral transform by the kernel \( K \) will be denoted and defined as follows.

\[ \Phi_K := \Phi^X \to Y : D(X) \to D(Y); \quad E \mapsto R \pi_Y * (p_X^* E \otimes^L \Delta_{X \times_k Y} K) \]

Let \( X \) be a smooth projective variety over a field \( k \). Recall that an object \( \alpha \in D(X) \) is spherical if \( \alpha \otimes \omega_X \simeq \alpha \) and \( R \text{Hom}_X(\alpha, \alpha) \simeq k \oplus k[-\dim X] \). The spherical twist by \( \alpha \) is the endofunctor
\[ T_\alpha := \Phi_{K_\alpha} \]
defined by the kernel \( K_\alpha := \text{cone} \left( \alpha^\vee \otimes \alpha \xrightarrow{\cong} \mathcal{O}_{\Delta_X} \right) \).

Consider the exchange automorphism
\[ \text{swap}: X \times X \to X \times X; \quad (x, y) \mapsto (y, x). \]

Recall that
\[ (\text{swap}^* K_\alpha)^\vee \otimes p_1^* \omega_X[\dim X] \]
is called the right adjoint kernel of \( K_\alpha \) and it enjoys the following adjoint property (see, say, [Huy06, Definition 5.7]).

\[ T_\alpha = \Phi_{K_\alpha} \dashv \Phi_{(\text{swap}^* K_\alpha)^\vee \otimes p_1^* \omega_X[\dim X]} =: T'_\alpha \]

It follows that \( T_\alpha \) is an autoequivalence, so that \( T'_\alpha \simeq T_\alpha^{-1} \). Note that there exists the obvious isomorphism
\[ \text{swap}^* K_\alpha \simeq K_\alpha^\vee, \]
so that
\[ T'_\alpha \simeq \Phi_{K_\alpha^\vee \otimes p_1^* \omega_X[\dim X]} \]
\[ (2.5) \]

A typical example of a spherical object on \( \Sigma_2 \) is \( \mathcal{O}_{\Sigma_2}(a) \) for \( a \in \mathbb{Z} \). The corresponding (inverse) spherical twist will be denoted as follows, for short.

\[ T_a := T_{\mathcal{O}_{\Sigma_2}(a)}; \quad T'_a := T^{-1}_{\mathcal{O}_{\Sigma_2}(a)} \quad (2.6) \]

By definition, for each spherical object \( \alpha \) and any object \( E \in D(X) \), there are standard triangles as follows. To ease notation, we simply let \( \otimes \mathcal{O}_X \) denote the derived tensor product. The morphisms \( \varepsilon \) and \( \eta \) are the evaluation and the coevaluation maps respectively, both of which are obtained from the standard adjoint pair of functors \(- \otimes \mathcal{O}_X \dashv R \text{Hom}_X(-, -)\).

\[ R \text{Hom}_X(\alpha, E) \otimes k \alpha \xrightarrow{\varepsilon} E \to T_\alpha(E) \xrightarrow{+1} \]
\[ T'_\alpha(E) \to E \xrightarrow{\eta} R \text{Hom}_X(E, \alpha)^\vee \otimes k \alpha \xrightarrow{+1} \]
\[ (2.7) \]
\[ (2.8) \]

**Lemma 2.3** ([IU05, Lemma 4.14]). For each \( \Phi \in \text{Auteq}(X) \), there is an isomorphism of autoequivalences as follows.

\[ \Phi \circ T_\alpha \circ \Phi^{-1} \simeq T_{\Phi(\alpha)} \]

In particular, for any \( a, m \in \mathbb{Z} \) it holds that
\[ \mathcal{O}_{\Sigma_2}(mC) \otimes \mathcal{O}_{\Sigma_2} \simeq T_{a-2m} \circ (\mathcal{O}_{\Sigma_2}(mC) \otimes \mathcal{O}_{\Sigma_2} -). \]
\[ (2.9) \]
Lemma 2.7. \[ \text{ical twists are categorification of root reflections.} \]

For any \( D \) \( k \), any autoequivalence \( \Psi \) of \( D(X) \) is a Fourier-Mukai transform; i.e., it is isomorphic to the integral transform by an appropriate kernel by \([\text{Orl97, Theorem 2.2}]\). It then follows that \( \Psi \) is isomorphic to \( \text{id} \) if and only if \( \Psi(E) \simeq E \) holds for any \( E \in D(X) \). With this in mind, one can confirm the assertion by using \((2.7)\).

The operation of taking duals is related to (inverse) spherical twists nicely.

\[ \text{Lemma 2.4.} \]
\[ (1) \text{For any spherical object } \alpha \in D(X), \text{ there exists a natural isomorphism of functors} \]
\[ (T_{a-})^\vee \simeq T_{a\alpha} (\alpha^{-\vee}) \] \tag{2.10} \]
\[ (2) \text{When } X = \Sigma_2, \text{ for any } a \in \mathbb{Z}, \text{ there exists a natural isomorphism of functors} \]
\[ (T_{a-})^\vee \simeq T'_{-2-a} (\alpha^{-\vee}) \] \tag{2.11} \]

Proof. There is a natural isomorphism
\[ (T_{a-})^\vee = (\Phi_{K_a}(\cdot))^\vee = \mathcal{R} \text{Hom}_X (\mathcal{R} p_{2*} (p_1^* - \otimes \omega_{\alpha} \otimes \dim X), \mathcal{O}_X) \]
\[ \simeq \mathcal{R} \text{Hom}_X (\mathcal{R} p_{2*} (\cdot \otimes \omega_{\alpha} \otimes \dim X), \mathcal{O}_X) \]
\[ \simeq \mathcal{R} p_{2*} \mathcal{R} \text{Hom}_X (\cdot \otimes \omega_{\alpha} \otimes \dim X, \mathcal{O}_X) \]
\[ \simeq \mathcal{R} p_{2*} (p_1^* - \otimes \omega_{\alpha} \otimes \dim X) \]
\[ \simeq \Phi_{K_a}(\cdot)^\vee \]

where the isomorphism \( \simeq \) follows from \([\text{Har66, Chapter VII, Corollary 4.3 b}]\).

Comparing this with \((2.5)\), we obtain \((2.10)\).

The second item follows from \((2.10)\) and the isomorphism \( \mathcal{O}_C(a)[1] \simeq \mathcal{O}_C(-2-a) \). \( \square \)

\[ \text{Lemma 2.5 (\cite{IU05} Lemma 4.15 (i) (2))}. \text{ There is an isomorphism of autoequivalences} \]
\[ T_a \circ T_{a+1} \simeq \mathcal{O}_{\Sigma_2}(C) \otimes \mathcal{O}_{\Sigma_2} - , \]
which implies
\[ T'_a \simeq T_{a+1} \circ T_{a+3} \circ \mathcal{O}_{\Sigma_2}(-C) \otimes \mathcal{O}_{\Sigma_2} - , \]
\[ T_a^2 \simeq T_{a+1} \circ T_{a+3} \circ \mathcal{O}_{\Sigma_2}(-2C) \otimes \mathcal{O}_{\Sigma_2} - . \] \tag{2.13} \]

Definition 2.6. For a triangulated category \( D \), we define the group \( \text{Auteq}^{K_0-\text{triv}}(D) \) of \( K_0 \)-trivial autoequivalences by the following exact sequence.

\[ 1 \to \text{Auteq}^{K_0-\text{triv}}(D) \to \text{Auteq}(D) \to \text{Aut}(K_0(D)) \]

The following well-known lemma, which follows from \((2.7)\) and \((2.8)\), asserts that spherical twists are categorification of root reflections.

\[ \text{Lemma 2.7.} \text{ For any spherical object } \alpha \in D(\Sigma_2), \text{ it holds that} \]
\[ T_a^2, T_a^2 \in \text{Auteq}^{K_0-\text{triv}}(D(\Sigma_2)). \]

\[ \text{Proposition 2.8.} \text{ For any } a, b \in \mathbb{Z}, \text{ there exists a sequence } a_1, \ldots, a_\ell \in \mathbb{Z} \text{ and } m \in \mathbb{Z} \text{ such that} \]
\[ T_a T_b = \mathcal{O}_{\Sigma_2}(mC) \otimes \mathcal{O}_{\Sigma_2} - \circ T_{a_2} \cdots T_{a_1}. \]

Proof. By \((2.9)\), we may and will reduce the proof to that of the following claim.
Claim 2.9. \( T_a T_b \) is contained in the subgroup of \( \text{Auteq}(\mathcal{D}(\Sigma_2)) \) generated by \( \mathcal{O}_{\Sigma_2}(C) \otimes \mathcal{O}_{\Sigma_2} - \) and \( T_a^2 \) for all \( a \in \mathbb{Z} \).

In the rest, we prove this claim. Consider first the case \( a \leq b \). We induct on \( b - a \geq 0 \). If it is 0, we have nothing to show. In general, we have

\[
T_a T_b \simeq (T_a T_{a+1}) \left( T_{a+1}^2 \right) (T_{a+1} T_b).
\]

By (2.12), the first term of the right hand side is isomorphic to \( \mathcal{O}_{\Sigma_2}(C) \otimes \mathcal{O}_{\Sigma_2} - \). By the induction hypothesis, the 3rd term is also contained in the subgroup mentioned in the claim. Hence we are done with this case.

Now suppose that \( a > b \). Then

\[
T_a T_b \simeq T_a^2 (T_b T_a)' T_b^2,
\]

and we know that the middle term is contained in the subgroup as shown in the previous paragraph.

□

Definition 2.10. The subgroup of \( \text{Auteq}(\mathcal{D}(\Sigma_2)) \) generated by spherical twists will be denoted by \( B \).

Corollary 2.11. Fix any \( a_0 \in \mathbb{Z} \). Then elements of \( B \) are exhausted by those of the following form, where \( a_1, \ldots, a_n, m \in \mathbb{Z} \).

\[
\left( \mathcal{O}_{\Sigma_2}(mC) \otimes \mathcal{O}_{\Sigma_2} - \right) \circ (T_{a_1}^{+2} \circ \cdots \circ T_{a_n}^{+2}) \quad (2.14)
\]

\[
\left( \mathcal{O}_{\Sigma_2}(mC) \otimes \mathcal{O}_{\Sigma_2} - \right) \circ T_{a_0} \circ (T_{a_1}^{+2} \circ \cdots \circ T_{a_n}^{+2}) \quad (2.15)
\]

Moreover, the normal subgroup

\[
B^{K_0-\text{triv}} := B \cap \text{Auteq}^{K_0-\text{triv}}(\mathcal{D}(X)) \triangleleft B
\]

is generated by \( \{ T_a^2 \mid a \in \mathbb{Z} \} \subset B^{K_0-\text{triv}} \).

Proof. The first assertion immediately follows from (2.13), (2.14), and Proposition 2.8.

For the second assertion, take \( b \in B^{K_0-\text{triv}} \). If \( b \) is as in (2.14), then obviously \( m = 0 \). Also one can verify that \( b \) is never like (2.15), say, by using that it must preserve the classes \( [\mathcal{O}_{\Sigma_2}((a_0+1)f)] \) and \( [\mathcal{O}_{\Sigma_2}((a_0+2)f)] \). □

Lemma 2.3 implies that \( B \) is a normal subgroup of \( \text{Auteq}(\mathcal{D}(\Sigma_2)) \). As we explain next, sets of generators of \( B \) are well understood.

Theorem 2.12. For any \( a \in \mathbb{Z} \), the group \( B \) is generated by the two spherical twists \( T_a, T_{a+1} \).

Proof. The case \( a = -2 \) is explicitly mentioned in [IU05, Lemma 4.6]. The general case inductively follows from this by the isomorphism of functors

\[
T_{a-1} T_a \simeq \mathcal{O}_{\Sigma_2}(C) \otimes \mathcal{O}_{\Sigma_2} - \simeq T_a T_{a+1},
\]

which is (2.12). □

2.3. Deformation and mutation of exceptional collections. Let

\[
f : \mathcal{X} \to B \quad (2.16)
\]

be a smooth projective morphism of Noetherian schemes with a closed point \( 0 \in B \), and let

\[
X_0 = \mathcal{X} \times_{f,B} \{ 0 \}
\]
be the central fiber. Note that the properness and smoothness of \( f \) implies it is a perfect morphism, in the sense that the derived pushforward \( f_* \) respects perfect complexes ([LN07, Proposition 2.1]).

**Definition 2.13.** (1) An object \( E \in \text{Perf}(X) \) is \( f \)-exceptional if \( Rf_* \mathcal{H}om(E, E) \) is a line bundle on \( B \).

(2) A collection of \( f \)-exceptional objects \( E_1, \ldots, E_N \in \text{Perf}(X) \) is an \( f \)-exceptional collection if \( Rf_* \mathcal{H}om(E_i, E_j) = 0 \) for any \( i, j \) with \( 1 \leq i < j \leq N \).

(3) An \( f \)-exceptional collection as above is said to be strong if moreover \( Rf_* \mathcal{H}om(E_i, E_i) \) is isomorphic to a locally free sheaf (regarded as a complex concentrated in degree 0) for any \( i, j \).

(4) An \( f \)-exceptional collection as above is said to be full if the minimal \( B \)-linear (i.e., closed under \( - \otimes f^* \mathcal{F} \) by any \( \mathcal{F} \in \text{Perf}(B) \)) triangulated subcategory which contains all of the objects in the collection is equivalent to \( \text{Perf}(X) \).

**Definition 2.14.** We say that two \( f \)-exceptional collections \( \mathcal{E}_1, \ldots, \mathcal{E}_N \) and \( \mathcal{E}'_1, \ldots, \mathcal{E}'_N \) are isomorphic if \( \mathcal{E}_i \simeq \mathcal{E}'_i \) for each \( i = 1, \ldots, N \). We will let \( \mathcal{E}C_{N}(f) \) (or \( \mathcal{E}C_N(X) \), if \( f \) is obvious from the context) denote the set of isomorphism classes of \( f \)-exceptional collections of length \( N \). The set of isomorphism classes of \( f \)-exceptional collections of length \( N \) consisting entirely of locally free sheaves will be denoted by \( \mathcal{E}C_{N}(f) \) (or \( \mathcal{E}C_N(X) \)), which comes with the obvious injection \( \mathcal{E}C_{N}(f) \hookrightarrow \mathcal{E}C_{N}(f) \). Similarly, the set of equivalence classes of full \( f \)-exceptional collections will be denoted by either \( \mathcal{F}C(f) \) or \( \mathcal{F}C(X) \), and the set of isomorphism classes of full \( f \)-exceptional collections consisting entirely of locally free sheaves will be denoted by \( \mathcal{F}C_{N}(f) \) (or \( \mathcal{F}C_N(X) \)), which comes with the obvious injection \( \mathcal{F}C_{N}(f) \hookrightarrow \mathcal{F}C(f) \).

**Lemma 2.15.** (1) Let \( E \in \text{Perf}(X) \) be an \( f \)-exceptional object. Then the functor

\[
\Phi_E: \text{Perf}(B) \to \text{Perf}(X); \quad F \mapsto f^* F \otimes E
\]

is fully faithful and admits a right adjoint as follows.

\[
\Phi_E \dashv \phi_E^R := f_* (- \otimes \mathcal{E}^\vee)
\]

(2) Let \( (\mathcal{E}_1, \ldots, \mathcal{E}_N) \in \mathcal{E}C_N(X) \) be an \( f \)-exceptional collection. Then the smallest \( B \)-linear triangulated subcategory of \( \text{Perf}(X) \) which contains \( \mathcal{E}_1, \ldots, \mathcal{E}_N \) admits a \( B \)-linear semiorthogonal decomposition \((\Phi_{E_1}(\text{Perf}(B)), \ldots, \Phi_{E_N}(\text{Perf}(B)))\).

We say that a semiorthogonal decomposition \( \text{Perf}(X) = (A_1, \ldots, A_N) \) is \( B \)-linear if \( A_i \otimes f^* b \subseteq A_i \) holds for any \( i = 1, \ldots, N \) and \( b \in \text{Perf}(B) \) (see [Kuz11, Section 2.3]).

We will freely use the following very useful base change theorem from [Bon06, Corollary 2.1.4]. See also [Kuz06, Section 2.4] and [Sta16, Tag 081B] for treatise from different points of view.

**Lemma 2.16.** Consider the following Cartesian square of finite dimensional noetherian schemes, where \( f, g \) are perfect.

\[
\begin{array}{ccc}
Y & \xrightarrow{\psi} & X \\
\downarrow g & & \downarrow f \\
C & \xrightarrow{\varphi} & B
\end{array}
\]

Then the standard natural transformation of functors

\[
\varphi^* \circ f_* \Rightarrow g_* \circ \psi^*: \text{Perf}(X) \to \text{Perf}(C) \tag{2.17}
\]

is an isomorphism if either \( f \) or \( \varphi \) is flat.
Corollary 2.17. Consider a morphism of schemes as in (2.16) and a (strong) \( f \)-exceptional collection \((\mathcal{E}_1, \ldots, \mathcal{E}_N) \in \text{EC}_N(\mathcal{X})\). Take any morphism \( \varphi : C \to B \), where \( C \) is a finite dimensional noetherian scheme, and consider the Cartesian diagram as in Lemma 2.16. Then
\[
(\psi^*\mathcal{E}_1, \ldots, \psi^*\mathcal{E}_N)
\]
is a (strong) \( g \)-exceptional collection on \( Y \).

Proof. As \( f \) is flat, the assertion immediately follows from the following computation. All functors are derived.
\[
g_* \text{Hom}_Y(\psi^*\mathcal{E}_i, \psi^*\mathcal{E}_j) \cong g_*\psi^*\text{Hom}_X(\mathcal{E}_i, \mathcal{E}_j) \cong \varphi^*\text{Hom}_X(\mathcal{E}_i, \mathcal{E}_j)
\]
\[\square\]

Lemma 2.18. Suppose that \( B = \text{Spec } R \) for a complete local Noetherian ring \((R, \mathfrak{m}, k)\). Then the natural restriction maps
\[
\text{EC}_N(f) \to \text{EC}_N(X_0),
\]
\[
\text{ECVB}_N(f) \to \text{ECVB}_N(X_0)
\]
obtained in Corollary 2.17 are bijections for any \( N \).

Proof. Let us first show that any exceptional object \( \mathcal{E} \in \text{Perf } X_0 \) deforms to an object \( \mathcal{E}_R \in \text{Perf } \mathcal{X} \). By definition of exceptional object, we know that \( \text{Ext}^1_{X_0}(\mathcal{E}, \mathcal{E}) = \text{Ext}^2_{X_0}(\mathcal{E}, \mathcal{E}) = 0 \). By the deformation theory of objects (see, say, [H10, Corollary 3.4]), for each \( n \geq 1 \) one finds the unique lift in \( \text{Perf}(\mathcal{X} \otimes R/\mathfrak{m}^{n+1}) \) of \( \mathcal{E} \). Then it algebrizes uniquely to an actual object \( \mathcal{E}_R \in \text{Perf}(\mathcal{X}) \) by [Lie06, Proposition 3.6.1].

We next show that \( \mathcal{E}_R \) is an \( f \)-exceptional object. This is equivalent to the assertion \( C = 0 \), where \( C \in \text{Perf } B \) is defined as the cone of the following standard morphism.
\[
\mathcal{O}_B \to R\mathcal{f}_*\mathcal{R}\text{Hom}_X(\mathcal{E}_R, \mathcal{E}_R)
\]
Consider the following Cartesian diagram.
\[
\begin{array}{ccc}
X_0 & \xrightarrow{\iota} & \mathcal{X} \\
\downarrow f_0 & & \downarrow f \\
\text{Spec } k & \xrightarrow{0} & \text{Spec } R = B \\
\end{array}
\]
By Lemma 2.16 it follows that
\[
\mathbb{L}^0\mathbb{f}_*\mathbb{R}\text{Hom}_X(\mathcal{E}_R, \mathcal{E}_R) \cong \mathbb{R}\text{Hom}_{X_0}(\mathcal{E}, \mathcal{E}) = k \text{id}_{E[0]},
\]
so that \( \mathbb{L}^0C = 0 \). By Nakayama’s lemma, this implies that \( C = 0 \). The semiorthogonality of the collection \( \mathcal{E}_{1,R}, \ldots, \mathcal{E}_{N,R} \) is shown by similar arguments. \( \square \)

Remark 2.19. One can similarly show that the deformation of a strong collection is also strong. This follows from the fact that if \( P \in \text{Perf } B \) satisfies \( 0^*P \simeq k^{n^r}[0] \) for some \( r \geq 0 \), then \( P \simeq R^{n^r}[0] \). Also, it follows from Lemma 2.15 (2) that the deformation of a full exceptional collection is again full.

Definition 2.20. Let \( f \) be a morphism as in (2.16). An \((f-)\)-exceptional pair is an \((f-)\)-exceptional collection of length 2. For an \( f \)-exceptional pair \( \mathcal{E}, \mathcal{F} \), the left mutation \( L_{\mathcal{E}}\mathcal{F} \)
of $F$ through $E$ and the right mutation $R_F E$ of $E$ through $F$ are defined by the following distinguished triangles.

$$f^* \mathcal{R}_f \mathcal{R} \mathcal{H}om_X(E, F) \otimes \mathcal{O}_X \xrightarrow{\cong} F \rightarrow \mathcal{L}_E F,$$

$$R_F E \rightarrow E \xrightarrow{\eta} f^* \mathcal{R}_f \mathcal{R} \mathcal{H}om_X(E, F) \vee \otimes \mathcal{O}_X F$$

**Remark 2.21.** The definition of mutations given above differs by shifts from the one in [Bon89, Section 2], but is slightly simpler in that for an orthogonal exceptional pair, the mutations just exchange the two objects without any shift.

By the base change theorem Lemma 2.16, one can easily verify that mutations commute with base change.

**Lemma 2.22.** Under the notation and the assumptions of Lemma 2.16, suppose that $f$ is flat and hence the natural transformation (2.17) is an isomorphism. For any $f$-exceptional pair $(E, F)$, it follows that $(\psi^* E, \psi^* F)$ is an $g$-exceptional pair and the following isomorphisms hold.

$$L_{\psi^* E} (\psi^* F) \simeq \psi^*(L_E F)$$

$$R_{\psi^* F} (\psi^* E) \simeq \psi^*(R_F E)$$

Next, let us recall a group action on $E_{C_N}(f)$ from [BP93, Proposition 2.1]. Let $B_{r_N}$ be the braid group on $N$ strands, which admits the following famous presentation by generators and relations.

$$B_{r_N} = \langle \sigma_1, \ldots, \sigma_{N-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad i = 1, \ldots, N - 1, \sigma_i \sigma_j = \sigma_j \sigma_i \mid |i - j| \geq 2 \rangle$$

Consider the action

$$B_{r_N} \curvearrowright E_{C_N}(f)$$

given by

$$\sigma_i : E_i, E_{i+1} \mapsto E_{i+1}, R_{E_{i+1}} E_i,$$

so that

$$\sigma_i^{-1} : E_i, E_{i+1} \mapsto L_{E_i} E_{i+1}, E_i.$$  

On the other hand, through the standard surjective homomorphism $B_{r_N} \twoheadrightarrow S_N; \quad \sigma_i \mapsto (i, i+1)$ to the symmetric group of degree $N$, the group $B_{r_N}$ acts naturally on the abelian group $\mathbb{Z}^N = \text{Map}(\{1, \ldots, N\}, \mathbb{Z})$ from the left. Let

$$G_N := \mathbb{Z}^N \rtimes B_{r_N}$$

be the semi-direct product corresponding to the action.

One can verify that this, together with the action $\mathbb{Z}^N \curvearrowright E_{C_N}$, where $[a_1, \ldots, a_N]^T \in \mathbb{Z}^N$ sends a collection $(E_1, \ldots, E_N) \in E_{C_N}$ to $(E_1[a_1], \ldots, E_N[a_N])$, extends to an action $G_N \curvearrowright E_{C_N}(f)$. In particular, one has the following induced action.

$$B_{r_N} \curvearrowright E_{C_N}(f)/\mathbb{Z}^N$$

When Perf $X$ admits a full $f$-exceptional collection of length $r$, then one similarly obtains the action $G_r \curvearrowright FEC(X)$.

**Remark 2.23.** [BP93, Conjecture 2.2] asserts that this action should be transitive. Note that this conjecture is equivalent to the transitivity of the action (2.21). Theorem 6.1 below is nothing but the affirmative answer to [BP93, Conjecture 2.2] for $X = \Sigma_2$. 

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Let \( X \) be a smooth projective variety over the field \( k \). The autoequivalences of \( D(X) \) and the notion of exceptional collections are nicely compatible as we explain next.

**Lemma 2.24.** If \( \mathcal{E}_1, \ldots, \mathcal{E}_N \in D(X) \) is an exceptional collection and \( \Phi \in \text{Auteq}(D(X)) \), then so is \( \Phi(\mathcal{E}_1), \ldots, \Phi(\mathcal{E}_N) \in D(X) \). In particular, there is the natural action

\[
\text{Auteq}(D(X)) \lesssim \text{EC}_N(X).
\]

(2.22)

The following lemma is easy to verify and plays an important role in this paper.

**Lemma 2.25.** The actions (2.19) and (2.22) commute.

### 2.4. Results obtained via deformation of \( \Sigma_2 \) to \( \mathbb{P}^1 \times \mathbb{P}^1 \)

Consider the (isotrivial) degeneration (2.2) over an algebraically closed field \( k \). Consider the discrete valuation ring \( R = (k[[t]], (t), k) \) and take the base change by \( B = \text{Spec} R \to \mathbb{A}^1 \) of the family. We write

\[
f : \mathcal{X} = \mathcal{X} \times_{\mathbb{A}^1} B \to B
\]

(2.23)

by abuse of notation. The central fiber \( \mathcal{X}_0 \) of \( f \) is isomorphic to \( \Sigma_2 \). Also, let

\[
\xi : \overline{K} := k((t)) \to B
\]

be the geometric generic point of \( B \). The isotriviality of the family (2.2) outside the origin implies that the geometric generic fiber \( \mathcal{X}_0 \to \text{Spec} \overline{K} \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \) over \( \overline{K} \). Throughout this section, we freely use the symbols introduced in this paragraph.

Since the generic fiber \( \mathbb{P}^1 \times \mathbb{P}^1 \) is a del Pezzo surface, the properties of exceptional collections on it is very well known by [KO94]. We list the known properties.

**Theorem 2.26.** (1) Any exceptional object on \( \mathbb{P}^1 \times \mathbb{P}^1 \) is isomorphic to a shift of an exceptional vector bundle, so that the natural map \( \text{EC}_N(\mathbb{P}^1 \times \mathbb{P}^1) \to \text{ECV}_N(\mathbb{P}^1 \times \mathbb{P}^1) \) is a bijection.

(2) \( \text{EC}_4(\mathbb{P}^1 \times \mathbb{P}^1) = \text{FEC}(\mathbb{P}^1 \times \mathbb{P}^1) \), and the action \( G_4 \lesssim \text{FEC}(\mathbb{P}^1 \times \mathbb{P}^1) \) is transitive.

(3) Any exceptional collection on \( \mathbb{P}^1 \times \mathbb{P}^1 \) can be extended to a full exceptional collection.

The non-triviality of the group \( \text{Auteq}^{K_0 \text{–triv}}(D(\Sigma_2)) \) implies that an exceptional object on \( \Sigma_2 \) is not uniquely determined by its class in \( K_0 \), even modulo shifts by \( \mathbb{Z}[2] \). However, if one considers only exceptional vector bundles, then it is the case:

**Lemma 2.27** (=a weaker version of [OU15] Lemma 3.5). Let \( \mathcal{E}, \mathcal{E}' \) be exceptional vector bundles on \( \Sigma_2 \) such that \( [\mathcal{E}] = [\mathcal{E}'] \in K_0(\Sigma_2) \). Then \( \mathcal{E} \simeq \mathcal{E}' \).

The degeneration (2.23) allows one to compare various invariants of \( \Sigma_2 \) to those of \( \mathbb{P}^1 \times \mathbb{P}^1 \). Recall that \( \mathcal{X}_0 \simeq \Sigma_2 \).

**Definition 2.28.** For an exceptional object \( \mathcal{E} \in D(\mathcal{X}_0) \), let \( \mathcal{E}_R, \text{gen}(\mathcal{E}) \) denote the unique deformation of \( \mathcal{E} \) to \( \mathcal{X} \) and its restriction to the geometric generic fiber \( \mathcal{X}_0 \), respectively. For an exceptional collection \( \mathcal{E} \) on the central fiber, we will similarly write \( \mathcal{E}_R, \text{gen}(\mathcal{E}) \) to mean its (unique) deformation to \( \mathcal{X} \) and its restriction to \( \mathcal{X}_0 \), respectively.

Let

\[
\text{gen} : \text{EC}_4(\mathcal{X}_0) \to \text{FEC}(\mathcal{X}_0)
\]

(2.24)

be the map which sends (an isomorphism class of) an exceptional collection \( \mathcal{E} \) of length 4 on \( \Sigma_2 \) to \( \text{gen}(\mathcal{E}) \in \text{FEC}(\mathcal{X}_0) \), which is obtained by restricting the deformation of \( \mathcal{E} \) to the \( f \)-exceptional collection, whose existence and uniqueness is guaranteed by Lemma 2.18.
to the geometric generic fiber (see Corollary 2.17). See Corollary 2.37 below for the surjectivity of gen.

One similarly defines the map $\text{gen}|_{\text{ECVB}_4(X_0)}: \text{ECVB}_4(X_0) \to \text{FECVB}(X_\xi)$, to obtain the following diagram.

We next compare $K_0(\text{Perf})$ of the surfaces. Note that we have the following diagram of schemes (the labels of the arrows in the diagram will be freely used).

Applying the functor $K_0(\text{Perf}(-))$, we obtain the first two rows of Figure 2.25, which is a commutative diagram of commutative rings with units.

The derived dual $\vee$ defined in (2.4) induces an automorphism of commutative rings $K_0(\text{Perf}X) = K_0((\text{Perf}X)^{\text{op}}) \xrightarrow{\vee} K_0(\text{Perf}X)$; $[E] \mapsto [E^\vee]$.

The Euler pairing on $K_0(\text{Perf}X) = K_0(X)$ (note that $X$ is a regular scheme) is the following bilinear pairing.

We can similarly define

We define

Lemma 2.29.

Let $E$ be a full exceptional collection of $X_0$, and $E_R, E_\xi$, be the deformation of $E$ to $X$ and the restriction of $E_R$ to $X_\xi$ as defined in Definition 2.28. As pointed out in Remark 2.19, both $E_R, E_\xi$ are full exceptional collections.

Lemma 2.30. $K_0(E), K_0(E_R), K_0(E_\xi)$ are bases of $K_0(X_0), K_0(X), K_0(X_\xi)$, respectively. In particular, the horizontal maps $i^*, j^*$ in the first row of (2.25) are isomorphisms.
Proof. Immediately follows from the fact that the collections $\mathcal{E}, \mathcal{E}_R, \mathcal{E}_\xi$ are full exceptional collections of the triangulated categories over the base of length 4. □

On the other hand we obtain the bottom row of the diagram Figure 2.25, where the vertical maps to the bottom row are isomorphisms of rings. Let

$$K_0(\text{gen}) : K_0(\mathcal{X}_0) \to K_0(\mathcal{X}_\xi)$$

(2.26)

be the isomorphism of abelian groups obtained from the diagram Figure 2.25. Also, regard $\chi_{\mathcal{X}_0}, \chi_{\mathcal{X}_\xi}$ as $\mathbb{Z}$-valued bilinear pairings by the diagram Figure 2.25. With all the preparations above, we can show the desired properties of the map $K_0(\text{gen}).$

**Proposition 2.31.** The isomorphism $K_0(\text{gen})$ of (2.26) respects the pairings $\chi_{\mathcal{X}_0}, \chi_{\mathcal{X}_\xi}$ on the source and the target abelian groups. Moreover, it fits in the following commutative diagram.

$$\begin{array}{ccc}
\text{EC}_1(\mathcal{X}_0) & \xrightarrow{\text{gen}} & \text{EC}_1(\mathcal{X}_\xi) \\
\downarrow K_0(\cdot) & & \downarrow K_0(\cdot) \\
K_0(\mathcal{X}_0) & \cong & K_0(\mathcal{X}_\xi)
\end{array}$$

**Proposition 2.32.** Let $\mathcal{E}, \mathcal{E}' \in D(\mathcal{X}_0)$ be exceptional objects. The following conditions are equivalent.

1. $[\mathcal{E}] = [\mathcal{E}'] \in K_0(\mathcal{X}_0)$.
2. $\text{gen}(\mathcal{E}) \cong \text{gen}(\mathcal{E}')[2m]$ for some $m \in \mathbb{Z}$.

Proof. (2) ⇒ (1) is a consequence of the commutative diagram above and that $K_0(\text{gen})$ is an isomorphism.

Conversely, assume (1). It then follows that $[\text{gen}(\mathcal{E})] = [\text{gen}(\mathcal{E}')] \in K_0(\mathcal{X}_\xi)$, which in turn implies (2) by Theorem 1.1 (2). □

For exceptional vector bundles on $\Sigma_2 \cong \mathcal{X}_0$, we have the following reconstruction result. This is an immediate corollary of Lemma 2.27.

**Lemma 2.33.** For any $N = 1, \ldots, 4$, the map $\text{ECVB}_N(\Sigma_2) \xrightarrow{K_0(\cdot)} \text{numEC}_N(\Sigma_2)$ is injective.

See Definition 2.34 for the definition of the set $\text{numEC}_N(\Sigma_2)$.

**Definition 2.34.** Let $S$ be a smooth projective variety, and for simplicity let us assume that $K_0(S)$ is isomorphic to the numerical Grothendieck group; i.e., the Euler pairing $\chi$ is non-degenerate. This in particular implies that $K_0(S)$ is a free abelian group of finite rank.

An exceptional vector is an element $e \in K_0(S)$ such that $e^2 = \chi_S(e, e) = 1$.

A numerical exceptional collection on $S$ is a sequence of exceptional vectors $e_1, \ldots, e_N \in K_0(S)$ such that $\chi_S(e_i, e_j) = 0$ for any $1 \leq i < j \leq N$. A numerical exceptional collection is said to be full if it is a basis of $K_0(S)$; i.e., when $N = \text{rank} K_0(S)$.

The set of numerical exceptional collections of length $N$ (resp. full) on $S$ will be denoted by $\text{numEC}_N(S)$ and $\text{numFEC}(S)$, respectively.

For a numerical exceptional collection $(e, f)$ of length 2, which will also be called a numerical exceptional pair, its right and left mutations are the new numerical exceptional pairs defined and denoted as follows.

$$(f, R_f(e) := e - \chi_S(e, f)f) \quad (2.27)$$

$$(L_e(f) := f - \chi_S(e, f)e, e) \quad (2.28)$$
By similar arguments for the action $G_N \curvearrowright \text{EC}_N$, one can verify that for a surface $S$ with $K_0(S) \cong \mathbb{Z}^N$ there is an action $G_N \curvearrowright \text{numFEC}(S)$, where the subgroup $\text{Br}_N$ acts by the mutations \[(2.27)\quad (2.28)\] and $\mathbb{Z}^N$ by the change of signs (hence the action descends to the quotient $(\mathbb{Z}/2\mathbb{Z})^N \times \text{Br}_N$).

At last we claim that everything goes together.

**Proposition 2.35.** There exists the following commutative diagram of sets equipped with the action of the group $G_4 = \mathbb{Z}_4 \rtimes \text{Br}_4$.

\[
\begin{array}{ccc}
\text{EC}_4(\mathcal{X}_0) & \overset{\text{gen}}{\longrightarrow} & \text{FEC}(\mathcal{X}_\xi) \\
\downarrow K_0(\cdot) & & \downarrow K_0(\cdot) \\
\text{numFEC}(\mathcal{X}_0) & \overset{\simeq}{\longrightarrow} & \text{numFEC}(\mathcal{X}_\xi)
\end{array}
\]

Let us now introduce a particular ($f$-)exceptional collection of invertible sheaves, which will be called the standard collection and serve as the base point of the sets FEC and FECVB in this paper.

**Definition 2.36.** The standard full exceptional collection of invertible sheaves on $\Sigma_2 \cong \mathcal{X}_0$ is defined as follows.

\[
\mathcal{E}^\text{std} := (\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(f), \mathcal{O}_{\Sigma_2}(C+2f), \mathcal{O}_{\Sigma_2}(C+3f)) \in \text{FECVB}(\Sigma_2).
\]  

(2.29)

By Lemma 2.18 and Remark 2.19 it uniquely deforms to a full $f$-exceptional collection of invertible sheaves. We will write it $\mathcal{E}^\text{std}_\xi$, and its pullback to $\mathcal{X}_\xi$ will be denoted by $\mathcal{E}^\text{std}_\xi$.

Using the deformation invariance of the intersection numbers, one can easily confirm that

\[
\mathcal{E}^\text{std}_\xi = (\mathcal{O}_{\mathcal{X}_\xi}, \mathcal{O}_{\mathcal{X}_\xi}(1,0), \mathcal{O}_{\mathcal{X}_\xi}(1,1), \mathcal{O}_{\mathcal{X}_\xi}(2,1))
\]

(2.30)

under an isomorphism $\mathcal{X}_\xi \cong \mathbb{P}^1 \times \mathbb{P}^1$.

**Corollary 2.37.** The map $\text{gen}: \text{EC}_4(\mathcal{X}_0) \rightarrow \text{FEC}(\mathcal{X}_\xi)$ is surjective.

**Proof.** By Theorem 2.26 (2), we know that $\text{FEC}(\mathcal{X}_\xi) = G_4 \cdot \mathcal{E}^\text{std}_\xi$. Since the diagram of Proposition 2.35 is $G_4$-equivariant and $\text{gen}(\mathcal{E}^\text{std}) = \mathcal{E}^\text{std}_\xi$, we obtain the conclusion. □

**Corollary 2.38.** For any $\mathcal{E} \in \text{EC}_4(\mathcal{X}_0)$, there exists $\sigma \in G_4$ such that $K_0(\sigma(\mathcal{E})) = K_0(\mathcal{E}^\text{std}) \in \text{numFEC}(\mathcal{X}_0)$.

**Proof.** Again by Theorem 2.26 (2), one can find an element $\sigma \in G_4$ such that $\sigma \text{gen}(\mathcal{E}) = \mathcal{E}^\text{std}_\xi$. Again by the equivariance, it follows that $\text{gen}(\sigma(\mathcal{E})) = \mathcal{E}^\text{std}_\xi = \text{gen}(\mathcal{E}^\text{std})$, which means that $K_0(\sigma(\mathcal{E})) = K_0(\mathcal{E}^\text{std})$ by Proposition 2.32. □

3. **Twisting exceptional objects down to exceptional vector bundles**

The purpose of this section is to show the following theorem.

**Theorem 3.1.** For any exceptional object $\mathcal{E} \in \mathcal{D}(\Sigma_2)$, there exists an exceptional vector bundle $\mathcal{F}$ and a sequence of integers $a_1, \ldots, a_n, m$ such that

\[
\mathcal{E} \cong (T_{a_n} \circ \cdots \circ T_{a_1})(\mathcal{F})[m].
\]  

(3.1)

The similar result for spherical objects is given in [IU05, Proposition 1.6]. In fact, we prove Theorem 3.1 by suitably modifying the proof of [IU05, Proposition 1.6].
Notation 3.2. Let $X$ be an integral noetherian scheme. For $\mathcal{E} \in \text{coh } X$, we define

$$\text{Supp } \mathcal{E} := \text{Spec } \text{Im} \left( O_X \to \text{End}_X(\mathcal{E}) \right) \subset X$$

and call it the schematic support of $\mathcal{E}$. It is universal among the closed subschemes $i: Z \to X$ which admits a coherent sheaf $\mathcal{E}' \in \text{coh } Z$ such that $i_* \mathcal{E}' \sim \mathcal{E}$. The underlying closed subset of $X$, or equivalently the reduced closed subscheme $(\text{Supp } \mathcal{E})_{\text{red}} \subset X$, is called the reduced support of $\mathcal{E}$. Also we let $\text{tors } \mathcal{E} \subset \mathcal{E}$ be the maximum torsion subsheaf of $\mathcal{E}$. For an object $\mathcal{E} \in D(X)$, we use the following notation.

$$\mathcal{H}^i(\mathcal{E}) := \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i(\mathcal{E}) \text{ Supp } \mathcal{E} := \bigcup_{i \in \mathbb{Z}} (\text{Supp } \mathcal{H}^i(\mathcal{E}))_{\text{red}} \quad (3.2)$$

3.1. First properties of $\mathcal{H}^*(\mathcal{E})$. As the first step toward the proof of Theorem 3.4, in this subsection we prove some basic properties of $\mathcal{H}^*(\mathcal{E})$. Part of them concern $(\text{Supp } \mathcal{H}^i(\mathcal{E}))_{\text{red}}$, which can be summarized as follows.

There exists the unique integer $i_0 \in \mathbb{Z}$ with the following properties.

- $\text{Supp } \mathcal{H}^{i_0}(\mathcal{E}) = \Sigma_2$
- $\left( \text{Supp } \text{tors } \mathcal{H}^{i_0}(\mathcal{E}) \right)_{\text{red}} = C$
- $\left( \text{Supp } \mathcal{H}^i(\mathcal{E}) \right)_{\text{red}} = C$ for $\forall i \neq i_0$

Based on this, the similar statements for the schematic supports will be proved in the next subsection; namely, we will remove $\text{red}$ from the second and the third items. In fact, this step has been the main obstacle for the project.

The similar results for spherical objects appear in [IU05, Lemma 4.8], where it is rather easily shown that the schematic support of the cohomology sheaves of a spherical object coincides with $C$. However, unfortunately, the proof of [IU05, Lemma 4.8] does not immediately apply to our situation. The fact $\text{Supp } \mathcal{E} = \Sigma_2$ prevents us from studying the problem locally around the curve $C$.

Lemma 3.3. The cohomology sheaves of an exceptional object $\mathcal{E} \in D(\Sigma_2)$ enjoy the following properties.

1. $\mathcal{H}^*(\mathcal{E})$ is rigid ($\iff E_2^{1,q} = 0 \forall q \in \mathbb{Z}$ in the spectral sequence (3.3)).
2. There exists a unique integer $i_0 \in \mathbb{Z}$ such that
   - $\text{Supp } \mathcal{H}^{i_0}(\mathcal{E}) = \Sigma_2$,
   - $\mathcal{H}^i(\mathcal{E})$ for $i \neq i_0$ and tors $\mathcal{H}^{i_0}(\mathcal{E})$ are pure sheaves with $(\text{Supp }(-))_{\text{red}} = C$, unless $i = 0$.
3. $\text{tors}(\mathcal{H}^{i_0}(\mathcal{E}))$ is rigid, and $\mathcal{H}^{i_0}(\mathcal{E})/\text{tors}(\mathcal{H}^{i_0}(\mathcal{E}))$ is an exceptional vector bundle.

See the following definition for the notion of rigidity.

Definition 3.4. An object $\mathcal{E} \in D(X)$ is said to be rigid if $\text{Ext}^1_X(\mathcal{E}, \mathcal{E}) = 0$.

We will freely use the following standard fact on rigid objects.

Lemma 3.5. Let $\mathcal{E} \in D(X)$ be a rigid object. Then $g\mathcal{E} \simeq \mathcal{E}$ for any $g \in \text{Aut}^0_{X/k}$. In particular, $\text{Supp } g\mathcal{E} = \text{Supp } \mathcal{E} \subset X$ for any such $g$.

We need the following spectral sequence, which also plays the central role for the study of spherical objects in [IU05].

Lemma 3.6. For any object $\mathcal{E} \in D(X)$, there exists the following spectral sequence.

$$E_2^{p,q} = \bigoplus_i \text{Ext}_{\Sigma_2}^p(\mathcal{H}^i(\mathcal{E}), \mathcal{H}^{i+q}(\mathcal{E})) \Rightarrow E^{p+q} = \text{Hom}_{\Sigma_2}^{p+q}(\mathcal{E}, \mathcal{E}) \quad (3.3)$$
Moreover, using the classes $e^i = e^i(\mathcal{E}) \in \text{Ext}^2_\Sigma_2(\mathcal{H}^i(\mathcal{E}), \mathcal{H}^{i-1}(\mathcal{E}))$ canonically determined by $\mathcal{E}$, the $d_2$ maps of (3.3) are given by

$$d_2^{p,q} : (\phi_i)_i \mapsto ((-1)^{p+q+1} \phi_{i-1} \circ e^i - e^{i+q} \circ \phi_i)_i. \quad (3.4)$$

**Proof.** See [IU05, Section 4.1], in particular [IU05, Proposition 4.1]. \qed

**Proof of Lemma 3.3.** Throughout the proof, we consider the spectral sequence (3.3) for the exceptional object $\mathcal{E}$. We prove the four items one by one.

1. The exceptionality of $\mathcal{E}$ is translated into the following conditions.

$$E^n = \begin{cases} k \mathfrak{id}_{\mathcal{E}} & n = 0 \\ 0 & n \neq 0 \end{cases} \quad (3.5)$$

Since $\Sigma_2$ is a smooth projective surface, $E_2^{p,q} = 0$ unless $0 \leq p \leq 2$. Hence (3.3) is $E_3$-degenerate, and moreover is $E_2$-degenerate at $(1,q)$ for any $q \in \mathbb{Z}$. Combined with (3.5), this implies that $E_2^{1,q} = 0$ for any $q \neq -1$. In the next paragraph we also confirm $E_2^{-1,-1} = 0$, thereby concluding the rigidity of $\mathcal{H}^*(\mathcal{E})$.

It follows from the explicit description (3.3) of $d_2$ maps that $0 \neq \sum_{i} \mathfrak{id}_{\mathcal{H}^i(\mathcal{E})} \in E_2^{0,0}$ is in fact contained in $\ker d_2^{0,0} \simeq E_3^{0,0} \simeq E_\infty^{0,0}$, which implies that $E_\infty^{0,0} \neq 0$. Combined with the isomorphism $E_0 \simeq k$ from (3.5) and the epimorphism $E_0 \twoheadrightarrow E_0/F E_0 \simeq E_0^{0,0}$, this implies that $E_\infty^{0,0} \simeq E_0^{0,0}(\simeq k)$ and hence $E_2^{-1,-1} \simeq E_\infty^{-1,-1} \simeq F_1 E_0/F^2 E_0 = 0$.

2. We further obtain the following equalities from (3.5).

$$e_2^{0,q} = e_2^{2,q-1} \quad (\text{if } q \neq 0) \quad (3.6)$$

$$e_2^{0,0} = e_2^{2,-1} + 1 \quad (3.7)$$

To see (3.6) for $q = -1$, note that the arguments in the previous paragraph implies $0 = E_\infty^{-2,-2} \simeq E_3^{-2,-2}$. Moreover, (11) implies that either $\text{Supp}(\text{tors} \mathcal{H}^i(\mathcal{E}))_{\text{red}} = \Sigma_2$, or $\emptyset$. This follows from Lemma 3.5 and the orbit decomposition $\Sigma_2 = (\Sigma_2 \setminus C) \coprod C$ for the action $\text{Aut}_\Sigma_2/k \simeq \Sigma_2$.

Suppose for a contradiction that $\text{Supp} \mathcal{H}^i(\mathcal{E})_{\text{red}} \neq \Sigma_2$ for every $i$. Then $\text{Supp} \mathcal{E}$ coincides with $C$ and hence there is an isomorphism $\mathcal{E} \otimes \omega_{\Sigma_2} \simeq \mathcal{E}$, from which we deduce $\text{ext}^2_{\Sigma_2}(\mathcal{E}, \mathcal{E}) = \text{ext}^0_{\Sigma_2}(\mathcal{E}, \mathcal{E})$ by the Serre duality. This contradicts (3.5) and hence there must be at least one integer $i_0$ with $\text{Supp} \mathcal{H}^{i_0}(\mathcal{E})_{\text{red}} = \Sigma_2$.

Now consider the Serre duality

$$(E_2^{-2,q})^\vee \simeq \bigoplus_i \text{Hom}_{\Sigma_2}(\mathcal{H}^i(\mathcal{E}), \mathcal{H}^{i,q}(\mathcal{E}) \otimes \omega_{\Sigma_2}).$$

Fix a non-trivial morphism $s : \omega_{\Sigma_2} \rightarrow \mathcal{O}_{\Sigma_2}$ such that the support of its cokernel is disjoint from the set of associated points of $\mathcal{H}^i(\mathcal{E})$ for all $i \in \mathbb{Z}$. This is possible, as the linear system $| - K_{\Sigma_2} |$ is base point free and there are only finitely many (schematic) points to be avoided (see [LL10, p. 8]). The property which we required for $s$ implies that for each $i \in \mathbb{Z}$ the natural morphism

$$\mathcal{H}^{i,q}(\mathcal{E}) \otimes \omega_{\Sigma_2} \xrightarrow{id \otimes s} \mathcal{H}^{i,-q}(\mathcal{E})$$

is injective. Thus we obtain an injection of vector spaces

$$\phi_q : (E_2^{-2,q})^\vee \hookrightarrow E_2^{0,-q}$$

and hence an inequality

$$e_2^{2,q} \leq e_2^{0,-q} \quad (3.8)$$

for any $q$.\[19\]
Next we prove that \( \phi_0 \) is not surjective on the direct summands indexed by those \( i \) with \( \text{Supp} \mathcal{H}^i(\mathcal{E}) = \Sigma_2 \). To see this, (by slight abuse of notation) let \( i_0 \) be one of such indices and apply the functor \( \text{Hom}(\mathcal{H}^{i_0}(\mathcal{E}), -) \) to the following short exact sequence.

\[
0 \to \mathcal{H}^{i_0}(\mathcal{E}) \otimes \omega_{\Sigma_2} \xrightarrow{\text{id} \otimes s} \mathcal{H}^{i_0}(\mathcal{E}) \to \mathcal{H}^{i_0}(\mathcal{E})|_{Z(s)} \to 0.
\]

The rigidity of \( \mathcal{H}^{i_0}(\mathcal{E}) \), which we confirmed in (1), implies that we have the following short exact sequence.

\[
0 \to \text{Hom}(\mathcal{H}^{i_0}(\mathcal{E}), \mathcal{H}^{i_0}(\mathcal{E}) \otimes \omega_{\Sigma_2}) \to \text{Hom}(\mathcal{H}^{i_0}(\mathcal{E}), \mathcal{H}^{i_0}(\mathcal{E})) \to \text{Hom}(\mathcal{H}^{i_0}(\mathcal{E}), \mathcal{H}^{i_0}(\mathcal{E})|_{Z(s)}) \to 0
\]

The 3rd term is not 0 by the assumption \( \text{Supp} \mathcal{H}^{i_0}(\mathcal{E}) = \Sigma_2 \) and hence id \( \otimes s \), which is identified with the direct summand of \( \phi_0 \) of interest, is not surjective.

Since we showed above that there is at least one such index \( i_0 \), we have confirmed the non-surjectivity of \( \phi_0 \) and hence the following inequality.

\[
e_2^{2,0} < e_2^{0,0}
\]

Summarizing the results so far, we obtain the following sequence of (in)equalities.

\[
e^{q-1}_{2} - e^{q-1}_{2} \leq e_{2}^{0,1} \leq e_{2}^{1,0} \leq e_{2}^{0,0} - \text{rank} \phi_0 \leq e_{2}^{2,1} + 1 - \text{rank} \phi_0 \leq e_{2}^{2-1},
\]

which implies that

\[
\text{rank} \phi_0 = e_{2}^{0,0} - e_{2}^{2,0} = 1.
\]

This means that the number of the indices \( i_0 \) with \( \text{Supp} \mathcal{H}^{i_0}(\mathcal{E}) = \Sigma_2 \) is exactly one.

Finally, recall that rigid sheaf with one-dimensional support is pure by [Kul97, Corollary 2.2.3]. We already confirmed the rigidity of \( \mathcal{H}^i(\mathcal{E}) \) for \( i \neq i_0 \) above, and for tors \( \mathcal{H}^{i_0}(\mathcal{E}) \) it is proven below.

(3) Put \( \mathcal{H} := \mathcal{H}^{i_0}(\mathcal{E}), \mathcal{T} := \text{tors}(\mathcal{H}) \) and \( \mathcal{F} := \mathcal{H}/\mathcal{T} \). Consider the spectral sequence

\[
E_1^{p,q} = \bigoplus_j \text{Ext}^{p+q}_{\Sigma_2}(G_j, G_{j+p}) \Rightarrow E^{p+q} = \text{Ext}^{p+q}_{\Sigma_2}(\mathcal{H}, \mathcal{H})
\]

arising from the short exact sequence \( 0 \to G_2 \to \mathcal{H} \to G_1 \to 0 \), where \( G_1 = \mathcal{F} \) and \( G_2 = \mathcal{T} \). For obvious reasons we see \( E_1^{0,q} = 0 \) unless \( 0 \leq p + q \leq 2 \) and \( -1 \leq p \leq 1 \).

On the other hand, since \( \mathcal{H} \) is rigid, it is stable under the action of \( \text{Aut}_{\Sigma_2/k} = \text{Aut}_{\Sigma_2/k}^{0} \) by Lemma (3.5). Since the torsion part of a coherent sheaf is uniquely determined by the sheaf, it follows that \( \mathcal{T} \) is also stable under the same group action. Hence it follows that \( (\text{Supp} \mathcal{T})_{\text{red}} \subset C \), so that

\[
\mathcal{T} \otimes \omega_{\Sigma_2} \simeq \mathcal{T}.
\]

Combined with the Serre duality, this implies the equality

\[
e_1^{p,q} = e_1^{p,2-q}
\]

for \( p \neq 0 \).

It then follows that \( 0 = E_1^{-1,1} = E_1^{1,1} \), where the first equality is the consequence of the fact that \( G_2 \) is torsion and \( G_1 \) is torsion free, and the second equality is the case \( (p, q) = (1, 1) \) of (3.13).

Thus we have confirmed that the spectral sequence (3.11) is \( E_1 \)-degenerate at \((0, 1)\). Hence \( E_1^{0,1} \simeq E_1^{0,1} \simeq E_1^{1} = 0 \), where the last vanishing is nothing but the rigidity of \( \mathcal{H} \), which we confirmed in (1). Thus we have shown the rigidity of \( \mathcal{T} \) and \( \mathcal{F} \).

Again in the spectral sequence (3.11), the vanishing \( E_1 = 0 \) implies \( 0 = E_1^{1,0} = E_{2}^{1,0} \). Hence \( d_1^{0,0} \) is surjective. Also the vanishing \( E_1 = 0 \) implies \( E_{-1,2} = 0 \). Since \( 0 = E_1^{0,1} = \)}
$E_{2}^{1,1}$, the spectral sequence is $E_{2}$-degenerate at $(-1, 2)$ and hence $E_{\infty}^{-1,2} \simeq E_{2}^{-1,2}$. This implies that $d_{1}^{-1,2}$ is injective. Thus we obtain

$$\text{hom}(\mathcal{H}, \mathcal{H}) = e_{1}^{-1,1} + \ker d_{1}^{0,0} = e_{1}^{-1,1} + e_{0}^{0,0} - e_{1}^{1,0} \quad (3.14)$$

$$\text{ext}^{2}(\mathcal{H}, \mathcal{H}) = e_{1}^{-1,3} + \coker d_{1}^{-1,2} = e_{1}^{-1,3} + e_{1}^{0,2} - e_{1}^{1,2}. \quad (3.15)$$

Therefore substituting (3.14) and (3.15) into

$$\text{(3.10)} \iff \text{hom}(\mathcal{H}, \mathcal{H}) = \text{ext}^{2}(\mathcal{H}, \mathcal{H}) + 1$$

and using (3.13), we obtain

$$1 = \text{hom}(\mathcal{H}, \mathcal{H}) - \text{ext}^{2}(\mathcal{H}, \mathcal{H}) = e_{1}^{0,0} - e_{1}^{1,2} \quad \text{hom}(\mathcal{F}, \mathcal{F}) - \text{ext}^{2}(\mathcal{F}, \mathcal{F}).$$

Since $\mathcal{F}$ is rigid, the same proof as in [OU15, Lemma 2.2] shows that $\mathcal{F}$ is an exceptional vector bundle. \hfill \Box

**Definition 3.7.** For an exceptional object $\mathcal{E} \in D(\Sigma_{2})$, we will write $i_{0}(\mathcal{E}) = i_{0} \in \mathbb{Z}$ for the unique integer such that $\text{Supp} \mathcal{H}^{i_{0}}(\mathcal{E}) = \Sigma_{2}$.

Let $\text{rank}: K_{0}(\Sigma_{2}) \to \mathbb{Z}$ be the rank function. If we let $\iota: \text{Spec} k(\Sigma_{2}) \to \Sigma_{2}$ denote the embedding of the generic point, rank is concisely defined as the composition of the map $K_{0}(\iota^{*})$ and the isomorphism $K_{0}(\text{Spec} k(\Sigma_{2})) \simeq \mathbb{Z}$ which sends the class of $k(\Sigma_{2})$ to 1. As a corollary of Lemma 3.3 we obtain the following

**Corollary 3.8.** The equality

$$\text{rank} \mathcal{E} = (-1)^{i_{0}} \text{rank} \mathcal{H}^{i_{0}}(\mathcal{E})$$

holds for any exceptional object $\mathcal{E} \in D(\Sigma_{2})$. In particular, $\text{rank} \mathcal{E} \neq 0$.

### 3.2. Properties of the schematic support of tors $\mathcal{H}^{*}(\mathcal{E})$

The aim of this subsection is to prove Proposition 3.18 which asserts that the schematic support of tors $\mathcal{H}^{*}(\mathcal{E})$ coincides with $C$. This is the most technical part of this paper.

**Remark 3.9.** In this paper, Proposition 3.18 will be used only as a black box. Hence one can first assume Proposition 3.18 to understand the rest of the paper and then later come back to its proof.

To describe the schematic support of the cohomology sheaves of $\mathcal{E}$, we consider the anti-canonical morphism $f: \Sigma_{2} \to P(1,1,2)$ to the weighted projective plane of weight $(1, 1, 2)$, which contracts the $(-2)$-curve $C$ to the singularity $(0: 0: 1)$.

**Notation 3.10.** Let $(R, \mathfrak{m})$ denote the local ring of $P(1, 1, 2)$ at the singular point. It is isomorphic to $k[x, y, z]_{(x, y, z)}/(z^{2} - xy)$, the $A_{1}$ singularity.

Our first goal is to give an $R$-module structure on $E_{2}^{p,q}$ for $(p, q) \neq (0, 0)$ of the spectral sequence (3.3) with respect to which the differentials of the spectral sequence are $R$-linear. The similar fact for spherical objects is used in [UU05], in which case the existence of such an $R$-module structure is trivial. In fact, since the support of a spherical object is concentrated in $C$, one can use the pushforward along $f|_{\text{Spec} R}$.

**Remark 3.11.** For any $\mathcal{M} \in \text{coh} \Sigma_{2}$ with $(\text{Supp} \mathcal{M})_{\text{red}} \subseteq C$, there is a natural homomorphism of $k$-algebras $R \to \text{End}_{\Sigma_{2}}(\mathcal{M})$ which kills $\mathfrak{m}^{\ell}$ for some $\ell > 0$. Hence for $i \in \mathbb{Z}$ and $\mathcal{N} \in \text{coh} \Sigma_{2}$, there are natural $R$-module structures of finite length on $\text{Ext}_{\Sigma_{2}}^{i}(\mathcal{M}, \mathcal{N}) \simeq \text{Hom}_{D(\Sigma_{2})}(\mathcal{M}, \mathcal{N}[i])$ and $\text{Ext}_{\Sigma_{2}}^{i}(\mathcal{N}, \mathcal{M}) \simeq \text{Hom}_{D(\Sigma_{2})}(\mathcal{N}, \mathcal{M}[i])$ given by precomposition and postcomposition, respectively. If $\mathcal{N}$ is also supported in $C$, we may use $R \to \text{End}_{\Sigma_{2}}(\mathcal{N})$ instead. However, we end up with the same $R$-module structures defined via $R \to \text{End}_{\Sigma_{2}}(\mathcal{M})$. 


For $q \neq 0$, the reduced support of either $\mathcal{H}^i(\mathcal{E})$ or $\mathcal{H}^{i+q}(\mathcal{E})$ is contained in $C$ and therefore $E_{2}^{0,q}$ has a canonical $R$-module structure by Remark 3.11. For $q = 0$ and $p \neq 0, 2$, $E_{2}^{p,0} = 0$.

For $p = 2$, we have

$$E_{2}^{2,0} \simeq \bigoplus_{i \neq i_0} \text{Ext}_{\Sigma_2}^{2}(\mathcal{H}^i(\mathcal{E}), \mathcal{H}^i(\mathcal{E})) \oplus \text{Ext}_{\Sigma_2}^{2}(\mathcal{H}^{i_0}(\mathcal{E}), \mathcal{H}^{i_0}(\mathcal{E})).$$

Since the reduced support of $\mathcal{H}^i(\mathcal{E})$ for $i \neq i_0$ is contained in $C$, all the direct summands but the last one have canonical $R$-module structures again by Remark 3.11.

Recall the following short exact sequence from the proof of Lemma 3.3 (3).

$$0 \to \mathcal{T} := \text{tors} \mathcal{H}^{i_0}(\mathcal{E}) \to \mathcal{F} := \mathcal{H}^{i_0}(\mathcal{E})/\mathcal{T} \to 0$$

Take $\text{Hom}(\mathcal{T}, \mathcal{H}^{i_0}(\mathcal{E}))$ to obtain the following exact sequence

$$\text{Ext}_{\Sigma_2}^{2}(\mathcal{T}, \mathcal{H}^{i_0}(\mathcal{E})) \rightarrow \text{Ext}_{\Sigma_2}^{2}(\mathcal{H}^{i_0}(\mathcal{E}), \mathcal{H}^{i_0}(\mathcal{E})) \rightarrow \text{Ext}_{\Sigma_2}^{2}(\mathcal{T}, \mathcal{H}^{i_0}(\mathcal{E}))$$

where

$$\text{Ext}_{\Sigma_2}^{2}(\mathcal{T}, \mathcal{H}^{i_0}(\mathcal{E})) \rightarrow \text{Ext}_{\Sigma_2}^{2}(\mathcal{F}, \mathcal{H}^{i_0}(\mathcal{E}))$$

is the precomposition (think of $x$ as a morphism $\mathcal{H}^{0}(\mathcal{E}) \to \mathcal{H}^{0}(\mathcal{E})[2]$ in $\mathcal{D}(\Sigma_2)$). The following vanishing follows from the exceptionality of $\mathcal{F}$.

$$\text{Ext}_{\Sigma_2}^{2}(\mathcal{F}, \mathcal{H}^{i_0}(\mathcal{E})) \simeq \text{Ext}_{\Sigma_2}^{2}(\mathcal{F}, \mathcal{F}) = 0$$

Hence we conclude that the map $\iota^*$ is an isomorphism of $k$-vector spaces. Since the reduced support of $\mathcal{T}$ is contained in $C$, the right hand side of (3.16) has a canonical $R$-module structure.

**Definition 3.12.** We transfer the $R$-module structure on $\text{Ext}_{\Sigma_2}^{2}(\mathcal{T}, \mathcal{H}^{i_0}(\mathcal{E}))$ to $\text{Ext}_{\Sigma_2}^{2}(\mathcal{H}^{i_0}(\mathcal{E}), \mathcal{H}^{i_0}(\mathcal{E}))$ via the isomorphism $\iota^*$ of (3.16), thereby giving an $R$-module structure on $E_{2}^{2,0}$.

**Lemma 3.13.** For $q \neq 0$, the maps $d_{2}^{0,q} : E_{2}^{0,q} \to E_{2}^{2,q-1}$ in the spectral sequence (3.3) are $R$-linear.

**Proof.** For $q \neq 0, 1$, as already explained, the $R$-module structures on the source and the target of $d_{2}^{0,q}$ are naturally defined and hence the $R$-linearity of $d_{2}^{0,q}$ is rather obvious.

Let us show the $R$-linearity of $d_{2}^{0,1}$. Take an arbitrary element

$$(\phi_i) \in E_{2}^{0,1} = \bigoplus_{i} \text{Hom}(\mathcal{H}^i(\mathcal{E}), \mathcal{H}^{i+1}(\mathcal{E})).$$

It suffices to show that the maps as follows, which appear in the description of $d_{2}^{0,1}$ given in Lemma 3.6 are $R$-linear.

$$\phi_{i_0-1} \mapsto \phi_{i_0-1} \circ e^{i_0} \in \text{Ext}_{\Sigma_2}^{2}(\mathcal{H}^{i_0}(\mathcal{E}), \mathcal{H}^{i_0}(\mathcal{E}))$$

$$\phi_{i_0} \mapsto e^{i_0+1} \circ \phi_{i_0} \in \text{Ext}_{\Sigma_2}^{2}(\mathcal{H}^{i_0}(\mathcal{E}), \mathcal{H}^{i_0}(\mathcal{E}))$$

Under the isomorphism $\iota^*$ of (3.16), the map (3.17) is identified with

$$\phi_{i_0-1} \mapsto \phi_{i_0-1} \circ e^{i_0} \circ \iota \in \text{Ext}_{\Sigma_2}^{2}(\mathcal{T}, \mathcal{H}^{i_0}(\mathcal{E})).$$

and, similarly, the map (3.18) is identified with

$$\phi_{i_0} \mapsto e^{i_0+1} \circ \phi_{i_0} \circ \iota \in \text{Ext}_{\Sigma_2}^{2}(\mathcal{T}, \mathcal{H}^{i_0}(\mathcal{E})).$$

These maps are $R$-linear, for the reason that the $R$-module structures given in Remark 3.11 are by means of precompositions. As the $R$-module structure on $\text{Ext}_{\Sigma_2}^{2}(\mathcal{H}^{i_0}(\mathcal{E}), \mathcal{H}^{i_0}(\mathcal{E}))$
is given in Definition 3.12 by transferring the $R$-module structure on $\text{Ext}^2_{\Sigma_2}(\mathcal{F}, \mathcal{H}^0(\mathcal{E}))$ via $i^*$, this is exactly what we had to prove.

Suppose $\mathcal{M}$ is a pure coherent sheaf on $\Sigma_2$ with $(\text{Supp} \mathcal{M})_{\text{red}} = C$. Then by Remark 3.11 $M := H^0(\Sigma_2, \mathcal{M}) \simeq \text{Hom}_{\Sigma_2}(\mathcal{O}_{\Sigma_2}, \mathcal{M})$ has a standard $R$-module structure of finite length and hence there is an integer $\ell$ such that $m^\ell M = 0$. In general, for an $R$-module $M$ and an ideal $I \subset R$, we let $(0 : I)_M$ denote the annihilator
\[ (0 : I)_M := \{ x \in M \mid IX = 0 \} \subseteq M. \quad (3.19) \]
Geometrically speaking, this is the maximum submodule of $M$ which is “supported on the closed subscheme $\text{Spec } R/I$”.

**Lemma 3.14.** For a pure coherent sheaf $\mathcal{M}$ on $\Sigma_2$ with $(\text{Supp} \mathcal{M})_{\text{red}} = C$, assume
\[ n := \min\{ \ell \mid m^\ell M = 0 \} > 1, \]
where $M = H^0(\Sigma_2, \mathcal{M}) \in \text{mod } R$ as above. Then the following strict inequality holds.
\[ \dim_k \frac{M}{(0 : m^{n-1})_M} < \dim_k m^{n-1}M \quad (3.20) \]

**Proof.** Without loss of generality, we may assume that $f^*f_* \mathcal{M} \not\simeq \mathcal{M}$ is surjective; i.e., $\mathcal{M}$ is $f$-globally generated. In fact, if $\eta$ is not surjective, we can replace $\mathcal{M}$ with the image $\text{Im} \eta$. By standard arguments on the adjoint pair $f^* \dashv f_*$, there is a canonical isomorphism $f_* \text{Im} \eta \xrightarrow{\sim} f_* \mathcal{M} = M$ and $\text{Im} \eta$ is automatically $f$-globally generated.

The following fact about the $A_1$-singularity $R$ is known well: for each $i > 0$, the $i$-th powers of the ideal sheaf
\[ T_C = \mathcal{O}_{\Sigma_2}(-C) \subseteq \mathcal{O}_{\Sigma_2} \quad (3.21) \]
of $C$ and $m$ are related to each other as follows.
\[ m^i \mathcal{O}_{\Sigma_2} = \mathcal{O}_{\Sigma_2}(-iC) = T_C^i \subseteq \mathcal{O}_{\Sigma_2} \quad (3.22) \]

The vanishing $m^nM = 0$ means that $M = f_* \mathcal{M}$ is a sheaf on $\text{Spec}(R/m^n)$, from which we deduce that $f^*f_* \mathcal{M}$ and hence its quotient $\mathcal{M}$ are supported on $f^{-1}(\text{Spec}(R/m^n)) \overset{(3.22)}{=} nC$. This means $T^n_C \mathcal{M} = 0$.

On the other hand, note that the image of the canonical morphism
\[ H^0(\Sigma_2, T^{-1}_C) \otimes_k H^0(\Sigma_2, \mathcal{M}) \to H^0(\Sigma_2, T^{-1}_C \mathcal{M}), \]
as a submodule of $H^0(\Sigma_2, \mathcal{M}) = M$, is $m^{n-1}M$ by (3.22). Thus we see $H^0(\Sigma_2, T^{-1}_C \mathcal{M}) \supset m^{n-1}M \neq 0$, hence $T^{-1}_C \mathcal{M} = 0$. Therefore we conclude
\[ n = \min\{ \ell \mid T^{-1}_C \mathcal{M} = 0 \}. \]

Since $T_C(T^{-1}_C \mathcal{M}) = 0$, we can naturally think of $T^{-1}_C \mathcal{M}$ as an object of coh $C$. Since we assumed that $\mathcal{M}$ is pure, so is its subsheaf $T^{-1}_C \mathcal{M}$. Hence $T^{-1}_C \mathcal{M}$ is a vector bundle on $C$ of positive rank.

Likewise $\frac{\mathcal{M}}{(0 : T^{-1}_C)_{\mathcal{M}}}$ can be thought of an object of coh $C$, where $(0 : T^{-1}_C)_{\mathcal{M}}$ is the sheaf version of the annihilator $(3.19)$; i.e., the maximum subsheaf of $\mathcal{M}$ whose schematic support is contained in $(n-1)C$. We claim that there is an isomorphism
\[ \varphi: \mathcal{O}_C(-(n-1)C) \otimes \mathcal{O}_C \frac{\mathcal{M}}{(0 : T^{-1}_C)_{\mathcal{M}}} \xrightarrow{\sim} T^{-1}_C \mathcal{M} \quad (3.23) \]
of vector bundles on $C$. Indeed $\varphi$ is induced from the product morphism $T_{C}^{-1} \otimes_{O_{S}} M \to T_{C}^{-1} M$, whose surjectivity implies that of $\varphi$. At the generic point $\xi$ of $C$, the stalk $(\mathcal{I}_{C})_{\xi}$ is the maximal ideal of the discrete valuation ring $O_{S,\xi}$ and $M_{\xi}$ is a finite length $O_{S,\xi}/(\mathcal{I}_{C})_{\xi}$-module. Hence one sees that $\varphi$ is an isomorphism at $\xi$ (use the structure theorem for finitely generated modules over a discrete valuation ring). Therefore it is enough to show that the left hand side of (3.23) is pure; i.e., there is no zero-dimensional subsheaf.

Assume for a contradiction that $\mathcal{M}/(0 : T_{C}^{-1})_{\mathcal{M}}$ is not pure. Then there is a subsheaf $(0 : T_{C}^{-1})_{\mathcal{M}} \subseteq S \subseteq \mathcal{M}$ such that Supp $(S/(0 : T_{C}^{-1})_{\mathcal{M}})$ is zero-dimensional. Then $T_{C}^{-1} S \subset \mathcal{M}$ is non-zero and zero-dimensional, which contradicts the purity of $\mathcal{M}$. To see dim $T_{C}^{-1} S = 0$, note that there is an epimorphism $T_{C}^{-1} \otimes_{O_{S}} S / (0 : T_{C}^{-1})_{\mathcal{M}} \to T_{C}^{-1} S$

and that dim $S/(0 : T_{C}^{-1})_{\mathcal{M}} = 0$.

Now consider the following $R$-module.

$$ V := \frac{M}{H^{0}(\Sigma_{2},(0 : T_{C}^{-1})_{\mathcal{M}})} = \frac{M}{(0 : m^{n-1})_{\mathcal{M}}} $$

To see the equality, note that the denominator of the left hand side, as a submodule of $M = H^{0}(\Sigma_{2}, \mathcal{M})$, coincides with the following subset.

$$ \{ s \in M \mid s_{p} \in (0 : T_{C,p}^{-1})_{\mathcal{M}}, \forall p \in C \}. $$

Since $T_{C,p}^{-1} = m^{n-1} O_{S_{p}}$, this coincides with the denominator of the right hand side.

If we put $L := O_{C}(-(n-1)C)$ and $E := \mathcal{M}/(0 : T_{C}^{-1})_{\mathcal{M}}$, then (3.23) is rewritten as $L \otimes_{O_{C}} E \simeq T_{C}^{-1} \mathcal{M}$.

Noting $H^{0}(C, L) \simeq m^{n-1}/m^{n}$, we see that $m^{n-1} M$ is the image of the product map $\psi : H^{0}(C, L) \otimes_{k} V \to H^{0}(C, T_{C}^{-1} \mathcal{M})$.

Then the assertion follows from Lemma 3.15 below.

**Lemma 3.15.** Let $E$ and $L$ be a vector bundle and a line bundle on a smooth projective curve $C$, respectively. Suppose dim $H^{0}(C, L) > 1$ and let $V \subset H^{0}(C, E)$ be a non-zero linear subspace. Then we have a strict inequality

$$ \dim V < \text{rank} \psi, $$

where $\psi : H^{0}(C, L) \otimes_{k} V \to H^{0}(C, L \otimes_{O_{C}} E)$ denotes the product map.

**Proof.** Without loss of generality, we may replace $E$ with the subsheaf generated by $V$. Take a pair of linearly independent global sections $s, t \in H^{0}(C, L)$. It follows that $O_{C} \cdot t \not\subseteq O_{C} \cdot s$ as subsheaves of $L$. Since $E$ is locally free, this implies that $(O_{C} \cdot t) \otimes_{O_{C}} E \not\subseteq (O_{C} \cdot s) \otimes_{O_{C}} E$ as subsheaves of $L \otimes_{O_{C}} E$. Since $(O_{C} \cdot t) \otimes_{O_{C}} E$ and $(O_{C} \cdot s) \otimes_{O_{C}} E$ are the subsheaves of $L \otimes_{O_{C}} E$ generated by $\psi(kt \otimes V)$ and $\psi(ks \otimes V)$ respectively, we conclude $\psi(kt \otimes V) \not\subseteq \psi(ks \otimes V)$. Thus we see

$$ V \simeq \psi(ks \otimes V) \not\subseteq \psi(ks \otimes V) + \psi(kt \otimes V) \subseteq \text{Im} \psi. $$

Taking $\dim_{k}$, we obtain the assertion. \qed

Let $\text{mod}_{0} R \subset \text{mod} R$ be the category of Artinian ( $\Longleftrightarrow \dim_{k} < \infty$) $R$-modules. There is an (anti-)involution of categories

$$ D : (\text{mod}_{0} R)^{\text{op}} \to \text{mod}_{0} R $$

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which is defined as follows.

\[ DM := \text{Hom}_k(M, k) \] (3.24)

**Corollary 3.16.** Under the same notation and assumption as in Lemma 3.14, put \( W := DM \). Then the following strict inequality holds.

\[ \dim_k W > \dim_k m_{n-1}W \]

**Proof.** To obtain the assertion, apply the anti-involution \( D \) to both sides of the equality (3.20) (before applying \( \dim_k \)) and then use the following Lemma 3.17. Note that \( D \) preserves dimensions. \( \square \)

**Lemma 3.17.** For \( M \in \mod_0 R \) and integer \( \ell \geq 0 \), there are isomorphisms as follows.

\[ D(m^\ell M) \cong DM/(0 : m^\ell)_DM \]

\[ D(M/(0 : m^\ell)_M) \cong m^\ell DM \]

**Proof.** To obtain the second isomorphism, replace \( M \) with \( DM \) in the first isomorphism and then use \( D^2 \cong \text{id} \). To see the first isomorphism, consider the following short exact sequence. The inclusion \( i \) is the canonical one.

\[ 0 \to (0 : m^\ell)_DM \xrightarrow{i} DM \to C := \text{coker } i \to 0 \]

By applying the anti-involution \( D \) to this, we obtain the following short exact sequence.

\[ 0 \to DC \to M \to D(0 : m^\ell)_DM \to 0 \]

Then, as a submodule of \( M \), \( DC \) is computed as follows.

\[ DC = \{ x \in M \mid y \in (0 : m^\ell)_DM \Rightarrow i(y)(x) = 0 \} = \{ x \in M \mid y \in DM, m^\ell y = 0 \Rightarrow y(x) = 0 \} \]

Note that \( y \in DM \) satisfies \( m^\ell y = 0 \) if and only if \( y(m^\ell M) = 0 \); i.e., \( y \in D(M/m^\ell M) \xrightarrow{q^*} DM \), where \( q : M \to M/m^\ell M \) is the quotient map. Hence

\[ DC = \{ x \in M \mid y \in D(M/m^\ell M) \Rightarrow y(x) = 0 \} = D(M/m^\ell M)^\perp = m^\ell M, \]

so that \( C \cong D(m^\ell M) \). \( \square \)

Now we are ready to prove the following

**Proposition 3.18.** For an exceptional object \( E \in D(\Sigma_2) \), it holds that

\[ \text{Supp tors } \mathcal{H}^{i_0}(E) = \text{Supp } \mathcal{H}^i(E) (i \neq i_0) = C \text{ or } \emptyset. \]

**Proof.** We first discuss \( \text{Supp } \mathcal{H}^i(E) \) for \( i \neq i_0 \). We may and will assume \( \bigoplus_{i \neq i_0} \mathcal{H}^i(E) \neq 0 \), since otherwise there is nothing to prove. Put

\[ n := \min \{ \ell \mid I_{C_0}^\ell \mathcal{H}^i(E) = 0 \} \geq 1. \]

In the spectral sequence (3.3) for the exceptional object \( E \), we have the isomorphism

\[ d^0_{2,1} : E_{2}^{0,1} = \bigoplus_i \text{Hom}_{\Sigma_2}(\mathcal{H}^i(E), \mathcal{H}^{i+1}(E)) \cong E_{2}^{2,0} = \bigoplus_i \text{Ext}_{\Sigma_2}^2(\mathcal{H}^i(E), \mathcal{H}^i(E)) \]

of \( R \)-modules by Lemma 3.13. Note that \( m^n E_{2}^{0,1} = 0 \), since for each \( i \) either \( I_{C_0}^\ell \mathcal{H}^i(E) = 0 \) or \( I_{C_0}^\ell \mathcal{H}^{i+1}(E) = 0 \) holds.
On the other hand, for each $i \neq i_0$ there is an isomorphism of $R$-modules given by the Serre duality:

$$\text{Ext}^i_{\mathcal{O}_C}(\mathcal{H}^i(\mathcal{E}), \mathcal{H}^i(\mathcal{E})) \simeq D \text{Hom}(\mathcal{H}^i(\mathcal{E}), \mathcal{H}^i(\mathcal{E}))$$

This implies that $m^{n-1}E_2^{2,0} \neq 0$. Combining this with the isomorphism of $R$-modules (3.25), we obtain

$$n = \min\{\ell \mid m_2^{\ell}E_2^{0,1} = 0\} = \min\{\ell \mid m_2^\ell E_2^{2,0} = 0\}.$$

Now assume for a contradiction that $n > 1$. Let

$$\mathcal{M} := \bigoplus_i \text{Hom}_{\Sigma_2}(\mathcal{H}^i(\mathcal{E}), \mathcal{H}^{i+1}(\mathcal{E})), $$

so that $E_2^{0,1} \simeq f_* \mathcal{M}$. We can apply Lemma 3.14 to $\mathcal{M}$, to obtain the strict inequality

$$\dim_k \frac{E_2^{0,1}}{(0 : m^{n-1})_{E_2^{0,1}}} < \dim_k m^{n-1}E_2^{0,1}. \quad (3.26)$$

On the other hand, let

$$\mathcal{M}' := \bigoplus_{i \neq i_0} \mathcal{E}nd_{\Sigma_2}(\mathcal{H}^i(\mathcal{E})) \oplus \text{Hom}_{\Sigma_2}(\mathcal{H}^{i_0}(\mathcal{E}), \mathcal{T}).$$

By Lemma 3.13 and the Serre duality, it follows that there is an isomorphism of $R$-modules $E_2^{2,0} \simeq Df_* \mathcal{M}'$. We can apply Corollary 3.16 to $\mathcal{M}'$, to obtain the strict inequality

$$\dim_k \frac{E_2^{2,0}}{(0 : m^{n-1})_{E_2^{2,0}}} > \dim_k m^{n-1}E_2^{2,0}. \quad (3.27)$$

The strict inequalities (3.26) and (3.27) contradict the isomorphism (3.25). Hence we obtain $n = 1$, which means that $\mathcal{H}^i(\mathcal{E})$ is an $\mathcal{O}_C$-module for any $i \neq i_0$. In fact, by rigidity, $\mathcal{H}^i(\mathcal{E})$ is a vector bundle on $C$ for any $i \neq i_0$.

Finally, to investigate tors $\mathcal{H}^{i_0}(\mathcal{E})$, we consider the derived dual $\mathcal{E}^\vee$. As we show in Lemma 3.25 below, there is an isomorphism as follows.

$$\mathcal{H}^{-i_0+1}(\mathcal{E}^\vee) \simeq (\text{tors} \mathcal{H}^{i_0}(\mathcal{E}))^\vee[1]$$

Since $-i_0 + 1 \neq -i_0$, by applying what we have just proved to the exceptional object $\mathcal{E}^\vee$, we see that the left hand side, hence the right hand side, is a vector bundle on $C$. Hence so is tors $\mathcal{H}^{i_0}(\mathcal{E})$. \hfill $\square$

**Remark 3.19.** Proposition 3.18 will not be used in the proof of Lemma 3.25 so that it is harmless to use Lemma 3.25 in the proof of Proposition 3.18 (as we did in the last paragraph). Actually, in the proof of Lemma 3.25 we only use Lemma 3.3 and some standard facts on homological algebra.

### 3.3. More on the structure of $\mathcal{H}^i(\mathcal{E})$.

In this subsection we give a structure theorem for $\mathcal{H}^{i_0}(\mathcal{E})$ in Lemma 3.21. It is then used to give a structure theorem for tors $\mathcal{H}^i(\mathcal{E})$ in Corollary 3.23.

Below is repeatedly used in this paper.

**Lemma 3.20** ([Kul97] Remark 2.3.4). Let $\mathcal{E} \in \mathcal{D}(\Sigma_2)$ be an exceptional vector bundle of rank $\mathcal{E} = r$. Then there is an isomorphism

$$\mathcal{E}|_C \simeq \mathcal{O}_C(b)\oplus s \oplus \mathcal{O}_C(b + 1)\oplus r-s \quad (3.28)$$

for some $b \in \mathbb{Z}$ and $s \in \mathbb{N}$ such that $1 \leq s \leq r$.

Note that the integers $b$ and $s$ in Lemma 3.20 are uniquely determined by $\mathcal{E}$. 

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*Note: The text above is a transcription of the content in the image provided, aiming to maintain the original meaning and structure as closely as possible.*
Lemma 3.21. Let $E \in \mathcal{D}(\Sigma_2)$ be an exceptional object. Then the unique non-torsion cohomology sheaf $\mathcal{H}^0(E)$ decomposes as

$$\mathcal{H}^0(E) \simeq E \oplus T,$$

where $E$ is an exceptional sheaf and $T$ is a vector bundle on $C$. Moreover, if $E$ is not locally free, then there is an integer $a \in \mathbb{Z}$ such that

- The torsion part $\text{tors} E$ is a direct sum of copies of $\mathcal{O}_C(a)$.
- $T$ is a direct sum of copies of $\mathcal{O}_C(a)$ and $\mathcal{O}_C(a+1)$.

Definition 3.22. For an exceptional object $E \in \mathcal{D}(\Sigma_2)$, let $E = E(\mathcal{E})$ denote the exceptional sheaf $E$ of Lemma 3.21. Also let $\mathcal{F} = \mathcal{F}(E)$ denote the torsion free part of the sheaf $\mathcal{H}^0(E)$, which is known to be an exceptional vector bundle by Lemma 3.3 (3).

Proof of Lemma 3.21. Consider the following standard short exact sequence.

$$0 \to T := \text{tors} \mathcal{H}^0(E) \to \mathcal{H}^0(E) \to \mathcal{F} := \mathcal{H}^0(E)/T \to 0 \quad (3.29)$$

Lemma 3.3 (3) asserts that $\mathcal{F}$ is an exceptional vector bundle. Hence we assume that $T \neq 0$, since otherwise there is nothing to prove.

$T$ is a rigid sheaf again by Lemma 3.3 (3). Combined with Proposition 3.18, this implies that there are $a \in \mathbb{Z}, s > 0, t \geq 0$ such that

$$T \simeq \mathcal{O}_C(a)^{\oplus s} \oplus \mathcal{O}_C(a+1)^{\oplus t}. \quad (3.30)$$

By Lemma 3.20, $\mathcal{F}|_C$ is also rigid and hence there are $b \in \mathbb{Z}, s' > 0, t' \geq 0$ such that

$$\mathcal{F}|_C \simeq \mathcal{O}_C(b)^{\oplus s'} \oplus \mathcal{O}_C(b+1)^{\oplus t'}.$$  

If $\text{Ext}^1_{\Sigma_2}(\mathcal{F}, T) = 0$, then the short exact sequence (3.29) splits and we are done. Therefore we assume $\text{Ext}^1_{\Sigma_2}(\mathcal{F}, T) \neq 0$, which implies

$$a \leq b - 1 \quad (3.31)$$

In order to make the argument conceptual, fix a $k$-vector space $V$ of dimension $s$ and replace $\mathcal{O}_C(a)^{\oplus s}$ with $V \otimes_k \mathcal{O}_C(a)$. Let

$$\mathcal{H}' := \mathcal{H}^0(E)/\mathcal{O}_C(a+1)^{\oplus t} \quad (3.32)$$

be the quotient by the subsheaf $\mathcal{O}_C(a+1)^{\oplus t} \subset T \subset \mathcal{H}^0(E)$, which fits in the following short exact sequence.

$$0 \to V \otimes_k \mathcal{O}_C(a) \to \mathcal{H}' \to \mathcal{F} \to 0 \quad (3.33)$$

From this we see

$$\text{Hom}_{\Sigma_2}(\mathcal{O}(a+1)^{\oplus t}, \mathcal{H}') = 0,$$

which implies that $\mathcal{H}'$ is rigid by Mukai’s lemma ([Knu97, Lemma 2.1.4. 2.(a)]), (which is obtained from the spectral sequence of the form (3.11)).

Let

$$[f : \text{Ext}^1_{\Sigma_2}(\mathcal{F}, \mathcal{O}_C(a))^\vee \to V] \in \text{Hom}_k(\text{Ext}^1_{\Sigma_2}(\mathcal{F}, \mathcal{O}_C(a))^\vee, V) \simeq \text{Ext}^1_{\Sigma_2}(\mathcal{F}, V \otimes_k \mathcal{O}_C(a))$$

correspond to the extension (3.33). We will show that $f$ is injective. In the long exact sequence obtained by applying $\text{Hom}_{\Sigma_2}(\mathcal{H}', -)$ to (3.33),

- the map $\text{Hom}_{\Sigma_2}(\mathcal{H}', \mathcal{H}') \to \text{Hom}_{\Sigma_2}(\mathcal{H}', \mathcal{F})$ is surjective since $\text{Hom}_{\Sigma_2}(\mathcal{H}', \mathcal{F}) \cong \text{Hom}_{\Sigma_2}(\mathcal{F}, \mathcal{F}) = k \cdot \text{id}_{\mathcal{F}}$ by the exceptionality of $\mathcal{F}$, and
- $\text{Ext}^1(\mathcal{H}', \mathcal{H}') = 0$ by the rigidity of $\mathcal{H}'$.  

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Hence we obtain
\[ \text{Ext}^1_{\Sigma_2}(\mathcal{H}', \mathcal{O}_C(a)) = 0. \]

Next we apply Hom\(_{\Sigma_2}(\mathcal{H}', \mathcal{O}_C(a))\) to (3.33) to obtain a surjective map
\[ V^\vee \cong \text{Hom}_{\Sigma_2}(V \otimes \mathcal{O}_C(a), \mathcal{O}_C(a)) \rightarrow \text{Ext}^1_{\Sigma_2}(\mathcal{F}, \mathcal{O}_C(a)). \] (3.34)

For any \( \varphi \in V^\vee \), the map \((3.33)\) sends \( \varphi \) to \((\varphi \otimes \text{id}_{\mathcal{O}_C(a)}) \circ e\), where \( e \in \text{Ext}^1_{\Sigma_2}(\mathcal{F}, V \otimes_k \mathcal{O}_C(a))\) denotes the extension class \(3.33\). Take a basis \( \{v_1, \ldots, v_s\}\) of \( V \) and decompose \( e \) as \( \sum_i e_i \otimes v_i \) under the isomorphism \( \text{Ext}^1_{\Sigma_2}(\mathcal{F}, V \otimes_k \mathcal{O}_C(a)) \cong \text{Ext}^1_{\Sigma_2}(\mathcal{F}, \mathcal{O}_C(a)) \otimes_k V \). Then one can confirm that \((3.34)\) sends \( \varphi \) to \( \sum_i \varphi(v_i)e_i \) and that \( f \) sends a linear form \( \xi \in \text{Ext}^1_{\Sigma_2}(\mathcal{F}, \mathcal{O}_C(a))^\vee \) to \( \sum_i \xi(e_i)v_i \). Hence the surjectivity of the map \((3.34)\) implies that \( e_1, \ldots, e_s \) generates \( \text{Ext}^1_{\Sigma_2}(\mathcal{F}, \mathcal{O}_C(a)) \), which in turn is equivalent to the injectivity of \( f \).

Now consider the universal extension of \( \mathcal{F} \) by \( \mathcal{O}_C(a) \).
\[ 0 \rightarrow \text{Ext}^1_{\Sigma_2}(\mathcal{F}, \mathcal{O}_C(a))^\vee \otimes_k \mathcal{O}_C(a) \rightarrow E \rightarrow \mathcal{F} \rightarrow 0 \] (3.35)

The inequality \((3.33)\) implies \( \text{Ext}^1_{\Sigma_2}(\mathcal{F}, \mathcal{O}_C(a)) \cong \text{Ext}^1_{\Sigma_2}(\mathcal{F}, \mathcal{O}_C(a))[-1] \), so that the distinguished triangle which \((3.35)\) yields is isomorphic to the defining distinguished triangle \((2.8)\) for the inverse spherical twist \( T_a(\mathcal{F}) \). In particular, there is an isomorphism \( E \cong T_a(\mathcal{F}) \) and hence \( E \) is an exceptional sheaf. Moreover, the injectivity of \( f \) and the basic properties of universal extensions imply that there is an isomorphism \( \mathcal{H}' \cong E \oplus (\text{coker} f \otimes_k \mathcal{O}_C(a)) \).

In what follows let \( d := \dim_k \text{Ext}^1_{\Sigma_2}(\mathcal{F}, \mathcal{O}_C(a)) \), so that
\[ \mathcal{H}' \cong E \oplus \mathcal{O}_C(a)^{\oplus s - d}. \]

If \( d = 0 \), then \( E \cong \mathcal{F} \) is a vector bundle and we are done. So, in the rest of the proof, we assume \( d > 0 \); i.e., we assume that \( \text{tors} E \neq 0 \).

Let us prove
\[ \text{Ext}^1_{\Sigma_2}(\mathcal{H}', \mathcal{O}_C(a + 1)) = 0, \] (3.36)

which together with \((3.32)\) implies
\[ \mathcal{H}^0(\mathcal{E}) \cong \mathcal{H}' \oplus \mathcal{O}_C(a + 1)^{\oplus s} \cong E \oplus \mathcal{O}_C(a)^{\oplus s - d} \oplus \mathcal{O}_C(a + 1)^{\oplus s}. \] (3.37)

Since \( \text{tors} E \cong \mathcal{O}_C(a)^{\oplus d} \), this is the desired conclusion.

\((3.36)\) follows from the local-to-global spectral sequence and the following vanishings.
\[ H^1(\Sigma_2, \text{Hom}_{\Sigma_2}(\mathcal{H}', \mathcal{O}_C(a + 1))) = 0 \] (3.38)
\[ H^0(\Sigma_2, \mathcal{E}xt^1_{\Sigma_2}(\mathcal{H}', \mathcal{O}_C(a + 1))) = 0 \] (3.39)

\((3.38)\), in turn, follows from \( \mathcal{H}' = E \oplus \mathcal{O}_C(a)^{\oplus s - d} \) and [OU15, Theorem 1.4(1)], which says that an exceptional sheaf \( E \) whose torsion part is a non-zero direct sum of copies of \( \mathcal{O}_C(a) \) satisfies
\[ E|_C \cong \mathcal{O}_C(a + 1)^{\oplus r} \] (3.40)

for some \( r \).

Finally, \((3.39)\) follows from the following isomorphisms.
\[ \mathcal{E}xt^1_{\Sigma_2}(\mathcal{H}', \mathcal{O}_C(a + 1)) \cong \mathcal{E}xt^1_{\Sigma_2}(V \otimes_k \mathcal{O}_C(a), \mathcal{O}_C(a + 1)) \cong V^\vee \otimes \mathcal{O}_C(-1) \]

\( \square \)

From Lemma 3.24 we immediately obtain
Corollary 3.23. Suppose that $E$ is an exceptional object with $\text{tors } E \neq 0$. Then, with the notation of Lemma 3.21, it holds that
\[ \text{tors } \mathcal{H}^s(E) = \bigoplus_{i \neq i_0} \mathcal{H}^i(E) \oplus T \oplus \text{tors } E \simeq \mathcal{O}_C(a)^{b_{i_0}} \oplus \mathcal{O}_C(a + 1)^{b_{i_0} - s} \]
for some $1 \leq s \leq \ell(E)$.

Remark 3.24. This is analogous to [1U05, Corollary 4.10] (see also [1U05 Section 5]).

Proof. If $i \neq i_0$, then $\text{Supp } \mathcal{H}^i(E) = C$ and hence $\mathcal{H}^i(E) \otimes \omega_{\Sigma_2} \simeq \mathcal{H}^i(E)$. Thus we see
\[ \text{Ext}^1_{\Sigma_2}(\mathcal{H}^{i_0}(E), \mathcal{H}^i(E)) \simeq \text{Ext}^1_{\Sigma_2}(\mathcal{H}^i(E), \mathcal{H}^{i_0}(E))^\vee \text{Lemma 3.3 (1)} \simeq 0. \]
Since $E$ is a direct summand of $\mathcal{H}^{i_0}(E)$, this implies $\text{Ext}^1_{\Sigma_2}(E, \mathcal{H}^i(E)) = 0$, $\text{Ext}^1_{\Sigma_2}(\mathcal{H}^i(E), E) = 0$. Moreover, since $\dim \text{Supp } \mathcal{H}^i(E) = 1$, the local to global spectral sequence for $\text{Ext}$ groups implies
\[
\begin{align*}
H^1(\Sigma_2, \text{Hom}_{\Sigma_2}(E, \mathcal{H}^i(E))) &= 0, \\
H^1(\Sigma_2, \text{Hom}_{\Sigma_2}(\mathcal{H}^i(E), E)) &= 0.
\end{align*}
\]
(3.41)

On the other hand, by Proposition 3.18 and the rigidity, there is a vector bundle $V$ on $C$ such that $\mathcal{H}^i(E) \simeq \iota_* V$. Hence there are isomorphisms as follows.
\[ \text{Hom}_{\Sigma_2}(E, \mathcal{H}^i(E)) \simeq \iota_* \text{Hom}_C(E|C, V) \simeq \iota_* \text{Hom}_C(\mathcal{O}_C(a + 1), V)^{\oplus r} \]
\[ \text{Hom}_{\Sigma_2}(\mathcal{H}^i(E), E) \simeq \iota_* \text{Hom}_C(\mathcal{H}^i(E), \mathcal{O}_C(a))^{\oplus d} \simeq \iota_* \text{Hom}_C(V, \mathcal{O}_C(a))^{\oplus d} \]
Combining these isomorphisms with (3.41), we obtain the following vanishings.
\[ H^1(C, \text{Hom}_C(\mathcal{O}_C(a + 1), V)) = 0 \]
\[ H^1(C, \text{Hom}_C(V, \mathcal{O}_C(a))) = 0 \]
From this we deduce that $V$ is of the form
\[ V \simeq \mathcal{O}_C(a)^{b_{i_0}} \oplus \mathcal{O}_C(a + 1)^{b_{i_0}} \]
for some $s_i$ and $t_i$, concluding the proof. \qed

3.4. Derived dual of exceptional objects. Let $E$ be an exceptional object which is not isomorphic to a shift of a vector bundle, and let $E = E(E)$ be the exceptional sheaf in Definition 3.22. In what follows we will mainly discuss the case where $\text{tors } E \neq 0$. If $\text{tors } E = 0$ ( $\iff$ $E$ is a vector bundle), we will replace $E$ with its derived dual $E^\vee$ and reduce the problem to the main case. What we mean by this will be made precise by Corollary 3.27.

Lemma 3.25. For an exceptional object $E$ on $\Sigma_2$, the cohomology sheaves of the derived dual $E^\vee$ are related to those of $E$ as follows:

- $i_0(E^\vee) = -i_0$.
- If $i \neq -i_0$, then $\mathcal{H}^i(E^\vee) \simeq \mathcal{E}xt^1_{\Sigma_2}(\mathcal{H}^{-i_0}(E), \mathcal{O}_{\Sigma_2})$.
- For $i = -i_0$, $\mathcal{H}^{-i_0}(E^\vee)$ fits into an exact sequence
\[ 0 \to \mathcal{E}xt^1_{\Sigma_2}(\mathcal{H}^{i_0+1}(E), \mathcal{O}_{\Sigma_2}) \to \mathcal{H}^{-i_0}(E^\vee) \to \text{Hom}_{\Sigma_2}(\mathcal{H}^{i_0}(E), \mathcal{O}_{\Sigma_2}) \to 0. \]
- For $i = -i_0 + 1$, the cohomology sheaf can also be written as
\[ \mathcal{H}^{-i_0+1}(E^\vee) \simeq \mathcal{E}xt^1_{\Sigma_2}(\text{tors } \mathcal{H}^{i_0}(E), \mathcal{O}_{\Sigma_2}) \simeq (\text{tors } \mathcal{H}^{i_0}(E))^\vee [1]. \]
Proof. Consider the following spectral sequence.

\[ E_2^{p,q} = \text{Ext}^p_{\mathcal{S}_2}(\mathcal{H}^{-q}(\mathcal{E}), \mathcal{O}_{\mathcal{S}_2}) \Rightarrow H^{p+q}(\mathcal{E}^\vee) \]

Since \( \Sigma_2 \) is a smooth projective surface, \( E_2^{p,q} = 0 \) if \( p < 0 \) or \( p > 2 \). Since \( \mathcal{H}^i(\mathcal{E}) \) is torsion for \( i \neq i_0 \) by Lemma 3.3 (2), \( E_2^{0,q} = 0 \) for \( q \neq -i_0 \). Moreover, for any \( i \neq i_0 \), \( \mathcal{H}^i(\mathcal{E}) \) is pure by Lemma 3.3 (2). Furthermore, tors \( \mathcal{H}^0(\mathcal{E}) \) is pure again by Lemma 3.3 (2) and \( \mathcal{H}^0(\mathcal{E})/\text{tors} \mathcal{H}^0(\mathcal{E}) \) is locally free by Lemma 3.3 (3). Hence by [HL10, Theorem 1.1.10, 1) \( E_2^{p,q} = 0 \) for any \( q \in \mathbb{Z} \).

Summing up, we see that \( E_2^{p,q} \neq 0 \) only if \( (p,q) = (0,-i_0) \) or \( p = 1 \). In particular, this spectral sequence is \( E_2 \)-degenerate. All assertions follow from these observations. \( \square \)

Lemma 3.26. Let \( \mathcal{E} \) be an exceptional object such that both \( E = E(\mathcal{E}) \) and \( E(\mathcal{E}^\vee) \) are vector bundles. Then \( \mathcal{E} \simeq E[-i_0] \).

Proof. We consider the spectral sequence

\[ E_2^{p,q} = \text{Ext}^p_{\mathcal{S}_2}(E, \mathcal{H}^q(\mathcal{E})) \Rightarrow \text{Hom}^{p+q}(E, \mathcal{E}). \]

As \( \Sigma_2 \) is a smooth surface, \( E_2^{p,q} \neq 0 \) only if \( 0 \leq p \leq 2 \). In particular, it is \( E_3 \)-degenerate. Since \( E_2^{0,-1} \simeq \text{Hom}_{\Sigma_2}(\mathcal{H}^{i_0-1}(\mathcal{E}), E)^\vee = 0 \) for \( \mathcal{H}^{i_0-1}(\mathcal{E}) \) being torsion and \( E \) being torsion free, it follows that \( d_2^{0,i_0} = 0 \) and hence

\[ E_2^{0,i_0} = \ker d_2^{0,i_0} \simeq E_3^{0,i_0} \simeq E_\infty^{0,i_0}. \] (3.42)

Take any

\[ [\varphi \colon E[-i_0] \to \mathcal{E}] \in \text{Ext}^{i_0}(E, \mathcal{E}) \]

whose image under the surjection \( \text{Ext}^{i_0}(E, \mathcal{E}) \to E_\infty^{0,i_0} \) of the spectral sequence, which in fact is \( \mathcal{H}^{i_0}(\varphi) \), corresponds to the natural inclusion \( E \hookrightarrow \mathcal{H}^{i_0}(\mathcal{E}) \) under the isomorphisms (3.42). Then the derived dual

\[ \varphi^\vee \colon \mathcal{E}^\vee \to E^\vee[i_0] \]

induces the surjection

\[ \mathcal{H}^{i_0}(\mathcal{E}^\vee) \to \text{Hom}_{\Sigma_2}(\mathcal{H}^{i_0}(\mathcal{E}), \mathcal{O}_{\Sigma_2}) \simeq E^\vee \]

in Lemma 3.25. As we assumed that \( E(\mathcal{E}^\vee) \simeq E^\vee \) is torsion free, by applying what we have just confirmed to \( \mathcal{E}^\vee \), we similarly obtain a morphism

\[ \psi : E^\vee[i_0] \to \mathcal{E}^\vee \]

such that \( \mathcal{H}^{i_0}(\psi) \) is the inclusion \( E^\vee \simeq E(\mathcal{E}^\vee) \to \mathcal{H}^{i_0}(\mathcal{E}^\vee) \) as a direct summand. Then it follows that the composite \( \varphi^\vee \circ \psi \) is an automorphism of \( E^\vee[i_0] \) and therefore \( \mathcal{E}^\vee \) splits as a direct sum of \( E^\vee[i_0] \) and \( \text{Cone} \psi \). Since the exceptional object \( \mathcal{E}^\vee \) is indecomposable, this implies \( \text{Cone} \psi = 0 \) and hence \( \mathcal{E}^\vee \simeq E^\vee[i_0] \), which is equivalent to \( \mathcal{E} \simeq E[-i_0] \). \( \square \)

Thus we immediately obtain the following

Corollary 3.27. Let \( \mathcal{E} \) be an exceptional object which is not isomorphic to a shift of a vector bundle. If \( E = E(\mathcal{E}) \) is torsion free, then \( \text{tors} E(\mathcal{E}^\vee) \neq 0 \).
3.5. Length of the torsion part. Now we introduce the notion of length, which measures for an exceptional object the distance from a shift of a vector bundle. The proof of Theorem 3.1 is reduced to the assertion Theorem 3.32 that one can always reduce the length by an appropriate spherical twist.

Definition 3.28. Let $\gamma$ be the generic point of $C$ and $O_{\Sigma_2, \gamma}$ the local ring of $\Sigma_2$ at $\gamma$. For a coherent sheaf $H$, put

$$\ell(H) = \text{length}_{O_{\Sigma_2, \gamma}} \text{tors} H_{\gamma}$$

where $H_{\gamma}$ is the stalk of $H$ at $\gamma$ and $\text{length}_{O_{\gamma}} \text{tors} H_{\gamma}$ is the length of its torsion part.

Let $\mathcal{E} \in D(\Sigma_2)$, we define

$$\ell(\mathcal{E}) := \sum_i \ell(\mathcal{H}^i(\mathcal{E})).$$

We give a bit more concrete description for $\ell(\mathcal{E})$. Let $\mathcal{E}_{\gamma} \in D(O_{\Sigma_2, \gamma})$ be the pull-back of $\mathcal{E}$ by the flat morphism $\text{Spec} O_{\Sigma_2, \gamma} \rightarrow \Sigma_2$. The length function $\ell$ is defined for objects in $D(O_{\Sigma_2, \gamma})$ as well, and it immediately follows from the exactness of the (underived) pull-back functor that $\ell(\mathcal{E}_{\gamma}) = \ell(\mathcal{E})$. On the other hand, since $O_{\Sigma_2, \gamma}$ is a DVR, for $M \in D(O_{\Sigma_2, \gamma})$ there is an isomorphism $M \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i(M)[-i]$ and hence the equality $\ell(M) = \sum_{i \in \mathbb{Z}} \ell(\mathcal{H}^i(M)).$ By the structure theorem for finitely generated modules over a DVR, $\mathcal{H}^i(M)$ is a direct sum of finite copies of $O_{\Sigma_2, \gamma}$ and $O_{\Sigma_2, \gamma}/(p)$ for various $p \geq 1$, where $t$ is a generator of the maximal ideal. Based on this, we obtain the following invariance.

Lemma 3.29. For any object $\mathcal{E} \in D(\Sigma_2)$, $\ell(\mathcal{E}^\vee) = \ell(\mathcal{E})$.

Proof. Note first that $\ell(\mathcal{E}^\vee) = \ell((\mathcal{E}^\vee)_\gamma) = \ell((\mathcal{E}_\gamma)^\vee)$. Hence it is enough to show $\ell(M^\vee) = \ell(M)$ for all $M \in D(O_{\Sigma_2, \gamma})$. By the explicit descriptions of $M$ we gave above, it is enough to show this for $M = O_{\Sigma_2, \gamma}/(p)$. In this case one can easily confirm $M^\vee \cong M[-1]$, so we are done. \[\square\]

If $\mathcal{E}$ is an exceptional object with $\text{Supp}(\mathcal{H}^{i_0}) = \Sigma_2$, then Proposition 3.18 implies

$$\ell(\mathcal{E}) = \sum_i \text{rank}_C \text{tors}(\mathcal{H}^i(\mathcal{E}))$$

$$= \sum_{i \neq i_0} \text{rank}_C \mathcal{H}^i(\mathcal{E}) + \text{rank}_C \text{tors} \mathcal{H}^{i_0}(\mathcal{E}),$$

where $\text{rank}_C$ denotes the rank of a coherent sheaf on $C$. Note that thus defined $\ell(\mathcal{E})$ is the same as $\ell(\mathcal{E})$ in Corollary 3.23. From this we immediately obtain the following characterization of exceptional vector bundles among exceptional objects.

Lemma 3.30. An exceptional object $\mathcal{E}$ is isomorphic to a shift of a vector bundle if and only if $\ell(\mathcal{E}) = 0$.

The following (sub)additivity of the length function with respect to short exact sequences will be useful later.

Lemma 3.31. For an exact sequence

$$0 \rightarrow \mathcal{H}_1 \rightarrow \mathcal{H}_2 \rightarrow \mathcal{H}_3 \rightarrow 0$$

in $\text{coh} \Sigma_2$, an inequality

$$\ell(\mathcal{H}_2) \leq \ell(\mathcal{H}_1) + \ell(\mathcal{H}_3)$$

holds. This is an equality if $\mathcal{H}_1$ is a torsion sheaf.
Thus we see that always possible to decrease the length of $E$ are as follows. By direct computations we easily see that $E$ can also write that $H^\cdot (E) \simeq E \oplus T$.

Theorem 3.32. Under the notation of the previous paragraphs, the following holds.

3.6. Proof of Theorem 3.1. Let us complete the proof of Theorem 3.1. Recall again the decomposition

$$H^i_0(E) \simeq E \oplus T.$$ from Lemma 3.21, where $E$ is an exceptional sheaf and $T$ is a torsion sheaf. There are two possibilities as follows.

1. $E$ is not torsion free. Then by Lemma 3.21 there are $a \in \mathbb{Z}, d \in \mathbb{Z}_{>0}$ such that $\text{tors } E \simeq \mathcal{O}_C(a) \oplus d$.

2. $E$ is torsion free. Then by Corollary 3.27 $E'$ fits into the case (1). Namely, the exceptional sheaf $E' := E(E')$ has a non-trivial torsion and there is a torsion sheaf $T', d' \in \mathbb{Z}$, and $d' > 0$ such that $H^{-i_0}(E) \simeq E' \oplus T'$ and $\text{tors } E \simeq \mathcal{O}_C(a') \oplus d'$.

Theorem 3.32 obviously follows from the following theorem, which asserts that it is always possible to decrease the length of $E$ by an appropriate spherical twist.

Theorem 3.32. Under the notation of the previous paragraphs, the following holds.

- In the case (1), i.e., if $E$ is not torsion free, then $\ell(T_aE) < \ell(E)$.
- In the case (2), i.e., if $E$ is torsion free, then $\ell(T_{-a}E) < \ell(E)$.

Proof. In this proof, to make life easy, we assume $i_0 = i_0(E) = 0$. The general case is easily reduced to this just by replacing $E$ with $E[i_0]$.

Consider the case (1). For each $i \neq 0$, by Corollary 3.27 there are $s_i, t_i \geq 0$ such that $H^i(E) \simeq \mathcal{O}_C(a)^{\oplus s_i} \oplus \mathcal{O}_C(a + 1)^{\oplus t_i}$. Putting $s_0 := s - d$ and $t_0 := t$ in (3.37), we can also write $H^0(E) \simeq E \oplus \mathcal{O}_C(a)^{\oplus s_0} \oplus \mathcal{O}_C(a + 1)^{\oplus t_0}$. Consider the following spectral sequence.

$$E_2^{p,q} = H^p(T_a(H^q(E))) \Rightarrow H^{p+q}(T_aE)$$

By direct computations we easily see that $E_2^{p,q}$ are as follows.

$$E_2^{p,q} \simeq \begin{cases} \mathcal{O}_C(a - 1)^{\oplus s_q} & p = -1 \\ \mathcal{O}_C(a)^{\oplus s_q} & p = 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus we see

$$\sum_{p,q} \ell(E_2^{p,q}) = \ell(E) - \ell(\text{tors } E) < \ell(E).$$ (3.43)

Noting that the differential map $d^{p,q}_2$ is non-zero only if $p = -1$, we easily see that $E_3^{p,q}$ are as follows.

$$E_3^{p,q} \simeq \begin{cases} \ker d_2^{1,q} & p = -1 \\ \text{coker } d_2^{1,q+1} & p = 1 \\ E_2^{p,q} & \text{otherwise} \end{cases}$$
For each $q \in \mathbb{Z}$, since $E_2^{-1,q}$ is torsion, it follows from Lemma 3.31 that $\ell(\text{coker } d_2^{-1,q+1}) \leq \ell(E_2^{-1,q-1})$. Also, the inclusion $\text{ker } d_2^{-1,q} \hookrightarrow E_2^{-1,q}$ implies $\ell(\text{ker } d_2^{-1,q}) \leq \ell(E_2^{-1,q})$. Thus we have confirmed the inequality
\[
\ell(E_2^{p,q}) \leq \ell(E_2^{p,q}) \quad \forall (p,q).
\] (3.44)

Finally, the spectral sequence degenerates at $E_3$ and hence Lemma 3.31 implies
\[
\ell(T_\alpha \mathcal{E}) \leq \sum_{p,q} \ell(E_3^{p,q}) = \sum_{p,q} \ell(E_3^{p,q}).
\] (3.45)

Combining (3.43), (3.44), and (3.45), we conclude $\ell(T_\alpha \mathcal{E}) \leq \ell(\mathcal{E})$.

Consider the other case (2). By Corollary 3.27, $\mathcal{E}^\vee$ is then as in the case (1). By applying our conclusion for the case (1) to $\mathcal{E}^\vee$, we obtain the inequality
\[
\ell(T_{\alpha'}(\mathcal{E}^\vee)) < \ell(\mathcal{E}^\vee).
\] (3.46)

On the other hand, the left hand side is computed as follows.
\[
\ell(T_{\alpha'}(\mathcal{E}^\vee)) = \ell(\mathcal{O}_{\Sigma^2}(-C) \otimes T_{\alpha'}(\mathcal{E}^\vee)) \overset{\text{Lemma 3.29}}{=} \ell((T_{\alpha' + 1}(\mathcal{E}^\vee))^\vee) \overset{\text{2.11}}{=} \ell(T_{a'-3}(\mathcal{E}^\vee)) \overset{\text{3.44}}{<} \ell(\mathcal{E}^\vee) \overset{\text{Lemma 3.29}}{=} \ell(\mathcal{E}).
\] (3.47)

Thus we see that $\ell(T_{a'-3}(\mathcal{E})) \overset{\text{3.44}}{=} \ell(T_{\alpha'}(\mathcal{E}^\vee)) < \ell(\mathcal{E}^\vee) \overset{\text{Lemma 3.29}}{=} \ell(\mathcal{E})$. □

**Remark 3.33.** Suppose both tors $E(\mathcal{E}) \neq 0$ and tors $E(\mathcal{E}^\vee) \neq 0$. From the third item of Lemma 3.25 and (3.22) in the proof of Lemma 5.1 below, it actually follows that $a = -a' - 3$. This does not seem to be a mere coincidence, but the authors do not have a good account of this.

4. Exceptional objects sharing the same class in $K_0(\Sigma_2)$

The goal of this section is to prove Corollary 4.4 on the set of exceptional objects sharing the same class in $K_0(\Sigma_2)$.

Let $\mathcal{E} \in \mathcal{D}(\Sigma_2)$ be an exceptional vector bundle of rank $r$. Recall from (3.28) of Lemma 3.20 that there is an isomorphism
\[
\mathcal{E}|_C \simeq \mathcal{O}_C(b)^{\oplus s} \oplus \mathcal{O}_C(b + 1)^{\oplus r-s}
\] for some $b \in \mathbb{Z}$ and $s \in \mathbb{N}$ such that $1 \leq s \leq r$. We freely use this result, especially the symbols $r, s, b$, throughout this section.

**Lemma 4.1.** Let $\mathcal{E} \in \mathcal{D}(\Sigma_2)$ be an exceptional vector bundle. Then $T_{b-1}\mathcal{E}$ and $T_b\mathcal{E}$ are exceptional vector bundles.

**Proof.** From the defining distinguished triangle of spherical twists (2.7), it immediately follows that $T_{b-1}\mathcal{E}$ is an exceptional sheaf. Note that it is isomorphic to $(T_b\mathcal{E})(C)$. It then follows from the defining distinguished triangle for the inverse spherical twist (2.8) that $T_b\mathcal{E}$ is torsion free. Thus we see that $T_{b-1}\mathcal{E}$ and $T_b\mathcal{E}$ are both exceptional vector bundles. □

**Lemma 4.2.** Let $\mathcal{E} \in \mathcal{D}(\Sigma_2)$ be an exceptional vector bundle such that $s = r$ holds in (3.28). Then $T_{b-1}\mathcal{E} \simeq \mathcal{E}$ and $T_{b-2}\mathcal{E} \simeq \mathcal{E}(C)$.

**Proof.** The first assertion immediately follows from the vanishing $\mathbb{R}\text{Hom}_{\mathcal{O}_C}(\mathcal{O}_C(b-1), \mathcal{E}) = 0$. The second assertion follows from $\mathcal{E} \simeq T_{b-1}\mathcal{E}$, which is nothing but the first assertion, and the isomorphism $T_{b-2}\mathcal{E} \simeq \mathcal{O}_{\Sigma_2}(C) \otimes_{\mathcal{O}_{\Sigma_2}} T_{b-1}\mathcal{E}$ (which follows from (2.13) and (2.9)). □
Theorem 4.3. Let $E \in D(\Sigma_2)$ be an exceptional object with rank $E > 0$. Then there exists $b \in B^{K_0-\text{triv}}$ and $m' \in \mathbb{Z}$ such that $b(E)[2m']$ is an exceptional vector bundle.

Proof. Let us choose an isomorphism as in (3.1). If $n$ happens to be an odd number, noting that $F \simeq T_b(T_bF)$ and $T_bF$ is an exceptional vector bundle for suitable $b \in \mathbb{Z}$ by Lemma 4.1, we may assume without loss of generality that $n$ is even. Now the assertion is an immediate consequence of Proposition 2.8 since $T_a^{\pm 2} \in B^{K_0-\text{triv}}$ for any $a \in \mathbb{Z}$. □

Corollary 4.4. Let $E \in D(\Sigma_2)$ be an exceptional object. Then

1. There exists a unique exceptional vector bundle $F$ on $\Sigma_2$ such that
   
   \[ [E] = \begin{cases} 
   [F] \in K_0(\Sigma_2) & \text{if } \text{rank } E > 0 \\
   -[F] \in K_0(\Sigma_2) & \text{if } \text{rank } E < 0 
   \end{cases} \]

   (recall that $\text{rank } E \neq 0$ by Corollary 3.3).

2. The action of the group $B^{K_0-\text{triv}} \times 2\mathbb{Z}$ on the following set is transitive.
   
   \[ \{ E' \in D(\Sigma_2) \mid \text{exceptional object such that } [E'] = [E] \in K_0(\Sigma_2) \} \]

Proof. The existence of $F$ as in (1) is a direct consequence of Theorem 4.3. The uniqueness of such $F$ is Lemma 2.27. One can prove (2) again by Theorem 4.3 by showing that any $E'$ is in the same orbit of $F$ or $F[1]$, depending on $\text{rank } E > 0$ or $\text{rank } E < 0$. □

5. Constructibility of exceptional collections

The aim of this section is Corollary 5.6, which asserts that any exceptional collection on $\Sigma_2$ is extendable to a full exceptional collection.

We first show that any exceptional collection on $\Sigma_2$ is sent to an exceptional collection consisting of (shifts of) vector bundles by a sequence of spherical twists.

Lemma 5.1. Let $(\mathcal{B}, E) \in EC_2(\Sigma_2)$ be an exceptional pair such that $\mathcal{B}$ is a vector bundle and $E$ is not isomorphic to a shift of a sheaf. Suppose also that $E = E(\mathcal{E})$ defined in Definition 3.22 is not torsion free, so that tors $E$ is a direct sum of copies of $O_C(a)$ for some $a \in \mathbb{Z}$. Then $T_a\mathcal{B}$ is isomorphic to either $\mathcal{B}$ or $\mathcal{B}(C)$; in particular, it remains to be a vector bundle.

Proof. Recall the decomposition $\mathcal{H}^{\alpha}(\mathcal{E}) = E \oplus T$ from Lemma 3.21. Set tors $E \simeq O_C(a)^{\oplus d}$, where $d > 0$ by the assumption. By Corollary 3.23 there is an isomorphism

\[ \text{tors } \mathcal{H}^s(\mathcal{E}) \simeq \bigoplus_{i \neq i_0} \mathcal{H}^i(\mathcal{E}) \oplus T \oplus \text{tors } E \simeq O_C(a)^{\oplus s} \oplus O_C(a + 1)^{\oplus \ell(\mathcal{E}) - s} \]

for some $1 \leq s \leq \ell(\mathcal{E})$. Also, again by Lemma 3.20 there is an isomorphism

\[ \mathcal{B}|_C \simeq O_C(b)^{\oplus s'} \oplus O_C(b + 1)^{\oplus r' - s'} \]

for some $b \in \mathbb{Z}$ and $1 \leq s' \leq r'$.

Consider the following spectral sequence.

\[ E_2^{p,q} = \text{Ext}_{\Sigma_2}^p(\mathcal{H}^{-q}(\mathcal{E}), \mathcal{B}) \Rightarrow \text{Ext}_{\Sigma_2}^{p+q}(\mathcal{E}, \mathcal{B}) \]

(5.1)

The assumption $\mathbb{R}\text{Hom}_{\Sigma_2}(\mathcal{E}, \mathcal{B}) = 0$ implies the vanishing of the limit $\text{Ext}_{\Sigma_2}^\infty(\mathcal{E}, \mathcal{B}) = 0$ for all $n \in \mathbb{Z}$. Also, since $\Sigma_2$ is a smooth projective surface, $E_{2}^{p,q} \neq 0$ only if $0 \leq p \leq 2$ and hence (5.1) is $E_3$-degenerate everywhere and $E_2$-degenerate at $p = 1$. These imply that

- $E_0^{0,q} = 0$ for all $q \neq -i_0$, for $H^{-q}(\mathcal{E})$ being torsion and $\mathcal{B}$ being torsion free. Hence (5.1) is $E_2$-degenerate at $(2, q)$ for each $q \neq -i_0 - 1$, so that $E_{2}^{2,q} \simeq E_{\infty}^{2,q} = 0$.
- $E_2^{1,q} = 0$ for all $q$. 

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Thus we have confirmed that

- \( E^{p,q}_2 \neq 0 \) only if \((p,q) = (0,-i_0)\) or \((2,-i_0-1)\). Moreover,
- \( d_2^{i_0} : E^{0,-i_0}_2 \to E^{2,-i_0-1}_2 \) is an isomorphism by the \( E_3\)-degeneracy of \([5.1]\) and the vanishing of the limit.

Note that \( 0 = E^{2,0}_2 \cong \text{Ext}^2_{\Sigma_2} (E \oplus T, \mathcal{B}) \) and the Serre duality imply \( \text{Hom}_{\Sigma_2} (\mathcal{B}, \text{tors} \ E) = 0 \). Thus we see

\[
a \leq b - 1.
\]

Take any \( q \neq -i_0, -i_0 - 1 \). We know that \( \mathcal{H}^{-q}(\mathcal{E}) \) is a direct sum of invertible sheaves on \( C \cong \mathbb{P}^1 \), so the vanishings \( E^{1,q}_2 = 0 = E^{2,q}_2 \) imply \( \mathcal{H}^{-q}(\mathcal{E}) = 0 \) if \( s' < r' \), and that \( \mathcal{H}^{-q}(\mathcal{E}) \) is a direct sum of (possibly 0) copies of \( \mathcal{O}_C(b-1) \) if \( s' = r' \).

Let us first settle the case \( s' < r' \). As we mentioned in the previous paragraph, in this case \( \mathcal{H}^i(\mathcal{E}) \neq 0 \) only if \( i = i_0 \) or \( i_0 + 1 \). Note in fact that \( \mathcal{H}^{i_0+1}(\mathcal{E}) \neq 0 \), since it is assumed in the statement that \( \mathcal{E} \) is not isomorphic to a shift of a sheaf.

It follows from the vanishing \( 0 = E^{0,-1}_2 = \text{Hom}_{\Sigma_2}(\mathcal{H}^{i_0+1}(\mathcal{E}), \mathcal{H}^{i_0}(\mathcal{E})) \) in the spectral sequence \([3.3]\) that \( 0 = \text{Hom}_{\Sigma_2}(\mathcal{H}^{i_0+1}(\mathcal{E}), \text{tors} \ E) \) and thus

\[
\mathcal{H}^{i_0+1}(\mathcal{E}) \cong \mathcal{O}_C(a+1)^{\oplus t}
\]

for some \( t > 0 \). On the other hand, we know that

\[
0 = E^{1,-i_0-1}_2 = \text{Ext}^1_{\Sigma_2}(\mathcal{H}^{i_0+1}(\mathcal{E}), \mathcal{B}),
\]

so that \( a+1 \geq b \) (recall \( r' - s' > 0 \)). Thus we see that \( a = b - 1 \). Hence \( T_a(\mathcal{B}) \cong \mathcal{B} \) by Lemma 4.1.

Next consider the case \( s' = r' \). Note that at least one of the sheaves \( \mathcal{O}_C(a) \) or \( \mathcal{O}_C(a+1) \) appears as a direct summand of \( \bigoplus_{i \neq i_0} \mathcal{H}^i(\mathcal{E}) \), since \( \mathcal{E} \) is not a shift of a sheaf. Then the vanishing \( 0 = E^{1,q}_2 = \text{Ext}^1(\mathcal{H}^{-q}(\mathcal{E}), \mathcal{B}) \) for all \( q \neq -i_0 \) in \([5.1]\) implies that either \( a = b-1 \) or \( b-2 \). Then by Lemma 4.2 \( T_a(\mathcal{B}) \) is isomorphic to \( \mathcal{B} \) and \( \mathcal{B}(C) \), respectively. \( \square \)

**Lemma 5.2.** Let \( (\mathcal{B}, \mathcal{E}) \in EC_2(\Sigma_2) \) be an exceptional pair. Suppose that \( \mathcal{B} \) is a vector bundle and \( \mathcal{E} \) is a sheaf such that

\[
\mathcal{T} := \text{tors} \mathcal{E} = \text{tors} \ E \cong \mathcal{O}_C(a)^{\oplus d} \neq 0.
\]

Then \( T_a \mathcal{B} \) is isomorphic to either \( \mathcal{B} \) or \( \mathcal{B}(C) \) and it holds that \( (T_a \mathcal{B}, T_a \mathcal{E}) \in ECVB_2(\Sigma_2) \).

**Proof.** Consider the short exact sequence as follows.

\[
0 \to \mathcal{T} \to \mathcal{E} \to \mathcal{F} \to 0 \tag{5.3}
\]

By \([OU15]\) Theorem 1.4 (1)], we know that \( d = \text{hom}_{\Sigma_2}(\mathcal{O}_C(a), \mathcal{E}) \) and \( T_a \mathcal{E} \cong \mathcal{F} \) is an exceptional vector bundle. Hence all we have to show is that \( T_a \mathcal{B} \) is a vector bundle.

The vanishing \( \mathbb{R}\text{Hom}(\mathcal{E}, \mathcal{B}) = 0 \) implies \( \text{Ext}^1_{\Sigma_2}(\mathcal{T}, \mathcal{B}) \cong \text{Ext}^{1+1}_{\Sigma_2}(\mathcal{F}, \mathcal{B}) \). Especially one has

\[
\text{Ext}^1_{\Sigma_2}(\mathcal{T}, \mathcal{B}) \cong \text{Ext}^2_{\Sigma_2}(\mathcal{F}, \mathcal{B}), \tag{5.4}
\]

\[
\text{Ext}^2_{\Sigma_2}(\mathcal{T}, \mathcal{B}) = 0. \tag{5.5}
\]

If \( \text{Ext}^1_{\Sigma_2}(\mathcal{T}, \mathcal{B}) = 0 \), then it follows that \( \mathbb{R}\text{Hom}(\mathcal{O}_C(a), \mathcal{B}) = 0 \) and hence \( T_a \mathcal{B} = \mathcal{B} \).

Therefore we may assume \( \text{Ext}^1_{\Sigma_2}(\mathcal{T}, \mathcal{B}) \neq 0 \), or by taking the dual,

\[
\text{Ext}^1_{\Sigma_2}(\mathcal{B}, \mathcal{T} \otimes K_{\Sigma_2}) \neq 0. \tag{5.6}
\]
The Serre dual of (5.4) and its restriction to $C$ yield the commutative square as follows.

\[
\begin{array}{ccc}
\text{Hom}_{\Sigma_2}(\mathcal{B}, \mathcal{F} \otimes K_{\Sigma_2}) & \xrightarrow{\sim} & \text{Ext}^1_{\Sigma_2}(\mathcal{B}, \mathcal{T} \otimes K_{\Sigma_2}) \\
\downarrow & & \downarrow \sim \\
\text{Hom}_C(\mathcal{B}|_C, \mathcal{F}|_C) & \xrightarrow{\sim} & \text{Ext}^1_C(\mathcal{B}|_C, \mathcal{T})
\end{array}
\]  

(5.7)

In the second row of the diagram (5.7), we omit $\otimes K_{\Sigma_2}$ by fixing an isomorphism $K_{\Sigma_2} \otimes \mathcal{O}_C \simeq \mathcal{O}_C$ and regard $\mathcal{T}$ as a sheaf on $C$. By restricting the locally split short exact sequence (5.3) to $C$, we obtain the following short exact sequence.

\[
0 \rightarrow \mathcal{T} \rightarrow \mathcal{E}|_C \rightarrow \mathcal{F}|_C \rightarrow 0
\]  

(5.8)

From (5.8) and the surjectivity of the second row in (5.7), we obtain the following isomorphism.

\[
\text{Ext}^1_C(\mathcal{B}|_C, \mathcal{E}|_C) \simeq \text{Ext}^1_C(\mathcal{B}|_C, \mathcal{F}|_C)
\]  

(5.9)

As in the proof of Lemma 5.1, put

\[
\mathcal{B}|_C \simeq \mathcal{O}_C(b) \oplus \mathcal{O}_C(b+1) \oplus \mathcal{O}_C(b-1)
\]

for $b \in \mathbb{Z}$ and $1 \leq s' \leq r'$ where $r' = \text{rank } \mathcal{B}$. Then (5.5) implies

\[
a \leq b - 1.
\]

To determine the value of $a$, let us compute the dimensions of the both sides of (5.9).

In order to compute the dimension of the left hand side, recall that $\mathcal{E}|_C \simeq \mathcal{O}_C(a+1) \oplus \mathcal{O}_C(a) \oplus \mathcal{O}_C(a-1)$ by [OU15, Theorem 1.4(1)]. We know moreover that $e = d + r$ by (5.8), where $r = \text{rank } \mathcal{F}$. Thus we obtain the following descriptions.

\[
\text{ext}^1_C(\mathcal{B}|_C, \mathcal{E}|_C) = \begin{cases} 
0 & \text{if } a = b - 1 \\
e((r' - s')(b - a - 1) + s'(b - a - 2)) & \text{if } a \leq b - 2
\end{cases}
\]  

(5.10)

Let us compute the dimension of the right hand side of (5.9). Since $\mathcal{F}$ is an exceptional vector bundle, there are $f \in \mathbb{Z}$ and $1 \leq s \leq r$ such that

\[
\mathcal{F}|_C \simeq \mathcal{O}_C(f) \oplus \mathcal{O}_C(f+1) \oplus \mathcal{O}_C(f+2).
\]

Note that by (5.6) and (5.7) we have

\[
\text{Hom}_C(\mathcal{B}|_C, \mathcal{F}|_C) \neq 0,
\]

which implies $f \geq b - 1$.

Suppose for a contradiction that $\text{ext}^1_C(\mathcal{B}|_C, \mathcal{F}|_C) \neq 0$. Then we obtain $f = b - 1$ and

\[
0 \neq \text{ext}^1_C(\mathcal{B}|_C, \mathcal{F}|_C) = s(r' - s').
\]  

(5.11)

Using $s \leq r < r+d = e$, we immediately see that (5.11) is strictly smaller than the second line of the right hand side of (5.10). This contradicts the isomorphism (5.9). Thus we have confirmed

\[
\text{ext}^1_C(\mathcal{B}|_C, \mathcal{E}|_C) = \text{ext}^1_C(\mathcal{B}|_C, \mathcal{F}|_C) = 0.
\]

Then (5.10) implies either $a = b - 1$ or $(a, r' - s') = (b - 2, 0)$. Thus we see that $T_a \mathcal{B}$ is isomorphic to $\mathcal{B}$ or $\mathcal{B}(C)$, respectively, by Lemma 4.1 and Lemma 4.2. Thus we conclude the proof. \square

**Corollary 5.3.** Let $(\mathcal{B}, \mathcal{E}) \in \mathcal{E}_C(\Sigma_2)$ be an exceptional pair such that $\mathcal{B}$ is a vector bundle and $\ell(\mathcal{E}) > 0$.
(1) Suppose that tors \( E(\mathcal{E}) \) is non-zero and is a direct sum of copies of \( \mathcal{O}_C(a) \). Then \( \ell(T_a(\mathcal{E})) = \ell(\mathcal{E}) \) and \( T_a(\mathcal{B}) \) is isomorphic to either \( \mathcal{B} \) or \( \mathcal{B}(C) \); in particular, it is a vector bundle.

(2) Suppose that tors \( E(\mathcal{E}) = 0 \), so that tors \( E(\mathcal{E}') \) is non-zero by Lemma 3.24. Suppose that it is a direct sum of copies of \( \mathcal{O}_C(a') \). Then \( \ell(T_{a'-3}(\mathcal{E})) < \ell(\mathcal{E}) \) and \( T_{a'-3}(\mathcal{B}) \) is isomorphic to either \( \mathcal{B} \) or \( \mathcal{B}(C) \); in particular, it is a vector bundle.

In particular, there is \( c = c(\mathcal{E}) \in \mathbb{Z} \) which depends only on \( \mathcal{E} \), independent of \( \mathcal{B} \) in particular, such that \( \ell(T_c(\mathcal{E})) < \ell(\mathcal{E}) \) and \( T_c(\mathcal{B}) \) is isomorphic to either \( \mathcal{B} \) or \( \mathcal{B}(C) \).

**Proof.** The assertions on the lengths are already proven in Theorem 3.32.

Let us first assume tors \( E(\mathcal{E}) \neq 0 \). The case where \( \mathcal{E} \) is not isomorphic to a shift of a sheaf is settled in Lemma 5.1. The case \( \mathcal{E} \) is isomorphic to a shift of a sheaf is settled in Lemma 5.2 (we may assume \( i_0 = 0 \), without loss of generality).

If tors \( E(\mathcal{E}) = 0 \), we reduce the proof to the first case. In fact, it follows from Lemma 3.32 that tors \( E(\mathcal{E}') \neq 0 \). Note that

\[
(B'(K_{\Sigma_2}), E') \tag{5.12}
\]

is also an exceptional pair such that the first component is a vector bundle and \( \ell(\mathcal{E}') \) \( \ell(\mathcal{E}) > 0 \). Suppose that tors \( E(\mathcal{E}') \) is a direct sum of copies of \( \mathcal{O}_C(a') \). Then, by applying the conclusion of the previous paragraph to the pair \( \mathcal{E}', \mathcal{B} \), it follows that \( T_{a'}(B'(K_{\Sigma_2})) \) is isomorphic to either \( B'(K_{\Sigma_2}) \) or \( B'(K_{\Sigma_2} + C) \). Then from the following computation we see that \( T_{-3-a}(B) \) is isomorphic to either \( \mathcal{B} \) or \( \mathcal{B}(C) \).

\[
T_{a'}(B'(K_{\Sigma_2})) \overset{Lemma 3.32}{\simeq} (T_{a'}(B'))(K_{\Sigma_2}) \overset{Lemma 2.3}{\simeq} (T_{2-a'}(B))(\Sigma_2) \overset{Lemma 5.1}{\simeq} (T_{-2-a'}(B))(\Sigma_2) \overset{Lemma 2.3}{\simeq} (T_{3-a'}(B))(\Sigma_2 + C)
\]

\[\square\]

**Theorem 5.4.** For any \( N \in \mathbb{Z} \) with \( 1 \leq N \leq 4 \) and any exceptional collection \( \mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N) \in \mathcal{EC}_N(\Sigma_2) \), there exists a product of spherical twists of the form \( T_a \) for some \( a \in \mathbb{Z} \), denoted by \( b \) such that \( b(\mathcal{E}) \in \mathbb{Z}^N \cdot \mathcal{ECVB}_N(\Sigma_2) \).

**Proof.** Without loss of generality, we may and will assume \( i_0(\mathcal{E}_i) = 0 \) for all \( i = 1, \ldots, N \). We prove the assertion by an induction on \( N \).

The case when \( N = 1 \) is nothing but Theorem 3.1. Consider the case when \( N > 1 \). By applying the induction hypothesis to the subcollection \( (\mathcal{E}_1, \ldots, \mathcal{E}_{N-1}) \), we may and will assume that these are already vector bundles, say, \( (F_1, \ldots, F_{N-1}) \). Suppose \( \mathcal{E}_N \) is not a vector bundle; i.e., \( \ell(\mathcal{E}_N) > 0 \). Otherwise there is nothing to show. In this case, since \( (F_i, \mathcal{E}_N) \in \mathcal{EC}_2(\Sigma_2) \) for each \( 1 \leq i \leq N - 1 \), if we take \( c = c(\mathcal{E}_N) \) as in Corollary 5.3, then \( T_c F_i \) remains to be a vector bundle for all \( i = 1, \ldots, N - 1 \) and it holds that \( \ell(T_c \mathcal{E}_N) < \ell(\mathcal{E}_N) \). By repeating this process until \( \ell(\mathcal{E}_N) \) reaches 0, we achieve our goal.

The constructibility for exceptional collections consisting of vector bundles is shown in [Kul97, Theorem 3.1.8.2] for a class of weak del Pezzo surfaces. Though \( \Sigma_2 \) is not contained in the class, we can deduce the same assertion for \( \Sigma_2 \) from it:

**Theorem 5.5.** Any exceptional collection on \( \Sigma_2 \) consisting of vector bundles can be extended to a full exceptional collection.

**Proof.** Let

\[
\pi : Y \to \Sigma_2
\]

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be the blowup of $\Sigma_2$ in a point outside of the curve $C$. Let $E \subset Y$ be the exceptional curve.

Take $\mathcal{E} := (\mathcal{E}_1, \ldots, \mathcal{E}_N) \in ECVB_N(\Sigma_2)$, so that $(\mathcal{O}_E(-1), \pi^* \mathcal{E}) \in EC_{N+1}(Y)$. Write $F := \pi^* \mathcal{E}_1$, and consider the right mutation $R_F \mathcal{O}_E(-1)$. We claim that this is an exceptional vector bundle.

To see this, note first that the semiorthogonality $\mathbb{R}Hom_Y(F, \mathcal{O}_E(-1)) = 0$ implies $F|_E \cong \mathcal{O}_E^{\oplus r}$, where $r = \text{rank} F$. Thus we obtain the following short exact sequence.

$$0 \to F^{\oplus r} \to R_F \mathcal{O}_E(-1) \to \mathcal{O}_E(-1) \to 0.$$ 

Suppose for a contradiction that the torsion part of the exceptional sheaf $R_F \mathcal{O}_E(-1)$ is nontrivial. Then it should map injectively to $\mathcal{O}_E(-1)$. It then implies that the canonical morphism from the locally free sheaf $F^{\oplus r}$ to the torsion free part of $R_F \mathcal{O}_E(-1)$ is both injective and surjective in codimension 1, which hence is an isomorphism. This contradicts the indecomposability of $R_F \mathcal{O}_E(-1)$.

Thus we have obtained $(F, R_F \mathcal{O}_E(-1), \pi^* \mathcal{E}_2, \ldots, \pi^* \mathcal{E}_N) \in ECVB_{N+1}(Y)$. By [Kul97, Theorem 3.1.8.2], it extends to a full exceptional collection on $Y$ (note that $Y$ is a weak del Pezzo surface obtained by blowing up $\mathbb{P}^2$ first in a point and then in a point on the (-1)-curve, (hence) that $| - K_Y|$ is base point free and $K_Y^2 = 7 > 1$). By applying the left mutation again, we obtain a full exceptional collection $(\mathcal{O}_E(-1), \pi^* \mathcal{E}_1, \ldots, \pi^* \mathcal{E}_N, \mathcal{E}'_{N+1}, \ldots, \mathcal{E}'_1) \in FEC(Y)$. Since $\mathcal{E}'_{N+1}, \ldots, \mathcal{E}'_1 \in \perp(\mathcal{O}_E(-1))$, there are some objects $\mathcal{E}_{N+1}, \ldots, \mathcal{E}_4 \in \mathcal{D}(\Sigma_2)$ such that $\mathcal{E}'_{N+1} \cong \mathbb{L} \pi^* \mathcal{E}_{N+1}, \ldots, \mathcal{E}'_1 \cong \mathbb{L} \pi^* \mathcal{E}_1$. Since $\mathbb{L} \pi^*: \mathcal{D}(\Sigma_2) \to \perp(\mathcal{O}_E(-1))$ is an equivalence, we see that $\mathcal{E}_1, \ldots, \mathcal{E}_N, \mathcal{E}_{N+1}, \ldots, \mathcal{E}_4$ is a full exceptional collection of $\mathcal{D}(\Sigma_2)$. 

We finally obtain the following constructibility theorem for $\Sigma_2$.

**Corollary 5.6.** Any exceptional collection on $\Sigma_2$ can be extended to a full exceptional collection.

**Proof.** For any exceptional collection $\mathcal{E} \in EC_N(\Sigma_2)$, by Theorem 5.4 there exists $b \in B$ such that $b(\mathcal{E}) \in \mathbb{Z}^N \cdot ECVB_N(\Sigma_2)$. Then by Theorem 5.5 $b(\mathcal{E})$ can be extended to a full exceptional collection on $\Sigma_2$. Applying $b^{-1}$ to the extended collection, one obtains the desired full exceptional collection which extends $\mathcal{E}$. 

**Remark 5.7.** Contrary to Theorem 4.3, an exceptional collection (consisting of objects of rank $> 0$) of length at least 2 is not necessarily numerically equivalent to an exceptional collection of vector bundles. Namely, for each $N = 2, 3, 4$, there is an exceptional collection $(\mathcal{E}_1, \ldots, \mathcal{E}_N) \in EC_N(\Sigma_2)$ of length $N$ and rank $\mathcal{E}_i > 0$ for $i = 1, \ldots, N$ for which there is no exceptional collection of vector bundles $(F_1, \ldots, F_N) \in ECVB_N(\Sigma_2)$ such that $[\mathcal{E}_i] = [F_i] \in K_0(\Sigma_2)$ for $i = 1, \ldots, N$. Example 5.8 below is such an example for $N = 2$. Examples for $N = 3, 4$ are obtained by extending examples of length 2 by Corollary 5.6. For these exceptional collections, in particular, in Theorem 5.4 one can not take $b$ from $B^{K_0 - triv}$. This is in contrast to Theorem 6.2.

**Example 5.8.** Consider the following exceptional pair.

$$\mathcal{E} := (\mathcal{E}_1, \mathcal{E}_2) := T_{-1}(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(C + 4f)) \in EC_2(\Sigma_2).$$

Then there is no exceptional pair of vector bundles $(F_1, F_2) \in ECVB_2(\Sigma_2)$ such that $[\mathcal{E}_i] = [F_i] \in K_0(\Sigma_2)$ for $i = 1, 2$ for the following reason.

Note first that $\mathcal{E}_1 \cong \mathcal{O}_{\Sigma_2}$ and $\mathcal{E}_2$ is a sheaf with tors $\mathcal{E}_2 = \mathcal{O}_C(-2)$ by [Ou15, Theorem 1.4]. Suppose that there is a pair $(F_1, F_2)$ as above. Then $F_1 \cong \mathcal{O}_{\Sigma_2}$ by Lemma 2.27 and
since

\[ [F_2] = [E_2] = [O_{\Sigma_2}(C + 4f)] - \chi(O_{\Sigma_2}(C + 4f), O_C(-1)) [O_C(-1)] = [O_{\Sigma_2}(3C + 4f)], \]

it follows that \( F_2 \cong O_{\Sigma_2}(3C + 4f) \) again by Lemma 2.27. This, however, leads to the following contradiction.

\[ 0 = \text{Ext}^2_{\Sigma_2}(F_2, F_1) = \text{Ext}^2_{\Sigma_2}(O_{\Sigma_2}(3C + 4f), O_{\Sigma_2}) \neq 0 \]

Remark 5.9. Given an exceptional collection \( \mathcal{E} = (E_1, \ldots, E_N) \in \text{EC}_N(\Sigma_2) \) such that \( \text{rank} E_i > 0 \) for all \( i \), by Corollary 1.1 there is a unique sequence of exceptional vector bundles \( F_1, \ldots, F_N \) such that \( [E_i] = [F_i] \in K_0(\Sigma_2) \) for all \( i = 1, \ldots, N \). Example 5.8 implies that \( (F_1, \ldots, F_N) \) is \textit{not} necessarily an exceptional collection.

It also implies that the map \( \text{gen}|_{\text{ECVB}_N(\Sigma_2)} : \text{ECVB}_N(\Sigma_2) \to \text{EC}_N(\mathbb{P}^1 \times \mathbb{P}^1) \) (the case \( N = 4 \) appears in Figure 2.31) is \textit{not} surjective for \( N = 2, 3, 4 \), though it is for \( N = 1 \) by [14, Lemma 4.6] and Proposition 2.32. In fact, let \( \mathcal{E} = (E_1, \ldots, E_N) \in \text{EC}_N(\Sigma_2) \) be an exceptional collection (consisting of objects of rank \( > 0 \)) which is not numerically equivalent to an exceptional collection of vector bundles. Then \( \text{gen}(\mathcal{E}) \) is not in the image of \( \text{gen}|_{\text{ECVB}_N(\Sigma_2)} \). In fact, an exceptional collection of vector bundles \( \mathcal{F} = (F_1, \ldots, F_N) \in \text{ECVB}_N(\Sigma_2) \) such that \( \text{gen}(\mathcal{F}) = \text{gen}(\mathcal{E}) \in \text{EC}_N(\mathbb{P}^1 \times \mathbb{P}^1) \) must satisfy \( [E_i] = [F_i] \in K_0(\Sigma_2) \) for \( i = 1, \ldots, N \), which contradicts the choice of \( \mathcal{E} \).

Despite Example 5.8 by Corollary 2.11 one can always bring an arbitrary exceptional collection (consisting of objects of rank \( > 0 \)) to an exceptional collection of vector bundles by an element of \( B^{K_0 - \text{triv}} \) up to a twist by \( T_0 \).

Theorem 5.10. Let \( \mathcal{E} = (E_1, \ldots, E_N) \in \text{EC}_N(\Sigma_2) \) be an exceptional collection with \( 2 \leq N \leq 4 \) and rank \( E_i > 0 \) for all \( i = 1, \ldots, N \). Then there exists \( b \in B^{K_0 - \text{triv}} \) such that \( b(\mathcal{E}) \in (2\mathbb{Z})^N \cdot (\text{ECVB}_N(\Sigma_2) \cup T_0(\text{ECVB}_N(\Sigma_2))) \).

Proof. We may assume without loss of generality that \( i_0 = 0 \) for any member of the collection \( \mathcal{E} \). By Theorem 5.3 there is \( b \in B \) such that \( b(\mathcal{E}) \in \text{ECVB}_N(\Sigma_2) \). Now the assertion immediately follows from the general description of elements of \( B \) given in (2.14) and (2.15) (note that \( T_0 = T_0'(T_0^2) \)). \( \square \)

Remark 5.11. Here we give some speculations on the spaces of Bridgeland stability conditions and a resulting question.

To start with, it is conceivable that there is a local homeomorphism \( \varphi \) as in the following commutative diagram whose restriction to the subspaces of algebraic stability conditions is compatible with the generalization map

\[
\text{gen} : \text{FSEC}(\Sigma_2) \to \text{FSEC}(\mathbb{P}^1 \times \mathbb{P}^1),
\]

where \( \text{FSEC}(\bullet) \) denote the set of isomorphism classes of full strong exceptional collections on \( \bullet \) (recall that a full strong exceptional collection yields a chamber of algebraic stability conditions in \( \text{Stab}(\bullet) \)). The fact that \( \text{gen} \) restricts to the sets of \textit{strong} exceptional collections follows from Remark 2.19 and Corollary 2.17.

\[
\text{Stab}(\Sigma_2) \xrightarrow{\varphi} \text{Stab}(\mathbb{P}^1 \times \mathbb{P}^1)
\]

\[
\text{Hom}(K_0(\Sigma_2), \mathbb{C}) \xrightarrow{(\text{gen})^*} \text{Hom}(K_0(\mathbb{P}^1 \times \mathbb{P}^1), \mathbb{C})
\]

Conjecturally, the Galois group (= the group of fiber-preserving automorphisms of \( \text{Stab}(\Sigma_2) \)) of \( p \) coincides with \( B^{K_0 - \text{triv}} \times 2\mathbb{Z} \). As \( B^{K_0 - \text{triv}} \) do not deform to \( \mathbb{P}^1 \times \mathbb{P}^1 \), it
is conceivable that $B^{K_0-\text{triv}}$ coincides with the Galois group of $\varphi$. Therefore it seems reasonable to ask the following question, which is an analogue of Corollary 4.4 (2). Unfortunately, Theorem 5.10 is not strong enough to answer it in the affirmative.

**Question 5.12.** For each $\mathcal{E} \in \text{FSEC}(\Sigma_2)$, the action of the group $B^{K_0-\text{triv}} \times 2\mathbb{Z}$ on the following set is transitive.

$$\{ \mathcal{E}' \in \text{FSEC}(\Sigma_2) \mid [\mathcal{E}'] = [\mathcal{E}] \in K_0(\Sigma_2)^4 \}$$

6. **Braid group acts transitively on the set of full exceptional collections**

This section is devoted to the proof of the following theorem.

**Theorem 6.1.** The action $G_4 \curvearrowright \text{EC}_4(\Sigma_2)$ is transitive. Namely, Conjecture 1.2 holds true for $\Sigma_2$.

The proof is divided into 3 steps. Let $\mathcal{E} \in \text{EC}_4(\Sigma_2)$ be the given exceptional collection of length 4.

**Step 1.** By Corollary 2.38, there exists $\sigma \in G_4$ such that

$$[\sigma(\mathcal{E})] = [\mathcal{E}^{\text{std}}] \in \text{numFEC}(\Sigma_2),$$

where $\mathcal{E}^{\text{std}}$ is the standard full exceptional collection defined in (2.29). Recall that there might be a difference between $\sigma(\mathcal{E})$ and $\mathcal{E}^{\text{std}}$ which is invisible on the numerical level. The rest of the proof is devoted to killing this (possible) difference.

**Step 2.** Next, we show the following theorem.

**Theorem 6.2.** For any $\mathcal{E} \in \text{EC}_4(\Sigma_2)$ satisfying

$$[\mathcal{E}] = [\mathcal{E}^{\text{std}}] \in \text{numFEC}(\Sigma_2),$$

there exists $b \in B^{K_0-\text{triv}}$ such that $b(\mathcal{E}) \simeq \mathcal{E}^{\text{std}}$.

**Proof of Theorem 6.2.** Write $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4)$. We may and will assume that $i_0 = 0$ for all of the objects in the collection. By Corollary 4.4 (2), there exists $b \in B^{K_0-\text{triv}}$ such that $b(\mathcal{E}_1) \simeq \mathcal{O}_{\Sigma_2}$. Hence by replacing $\mathcal{E}$ with $b(\mathcal{E})$, we may and will assume that $\mathcal{E}_1 = \mathcal{O}_{\Sigma_2}$.

Next, by Corollary 5.3, there is $b \in B$ such that $b(\mathcal{O}_{\Sigma_2})$ and $b(\mathcal{E}_2)$ are both vector bundles. Recall that $b$ is like either (2.14) or (2.15). If $b$ is like (2.14), then it follows that both $b_0(\mathcal{O}_{\Sigma_2})$ and $b_0(\mathcal{E}_2)$ are line bundles for some $b_0 \in B^{K_0-\text{triv}}$. Since $[b_0(\mathcal{E}_2)] = [\mathcal{E}_2] = [\mathcal{O}_{\Sigma_2}(f)]$ and $[b_0(\mathcal{O}_{\Sigma_2})] = [\mathcal{O}_{\Sigma_2}]$, it follows from Lemma 2.27 that $b_0(\mathcal{E}_2) \simeq \mathcal{O}_{\Sigma_2}(f)$ and $b_0(\mathcal{O}_{\Sigma_2}) \simeq \mathcal{O}_{\Sigma_2}$.

If $b$ is like (2.15) for $a_0 = -1$, then both $T_{-1}b_0(\mathcal{O}_{\Sigma_2})$ and $T_{-1}b_0(\mathcal{E}_2)$ are line bundles for some $b \in B^{K_0-\text{triv}}$. Then it follows from the following computations and Lemma 2.27 that $T_{-1}b_0(\mathcal{O}_{\Sigma_2}, \mathcal{E}_2) = (\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(C + f))$.

$$[T_{-1}b_0(\mathcal{O}_{\Sigma_2})] = [T_{-1}(\mathcal{O}_{\Sigma_2})] = [\mathcal{O}_{\Sigma_2}],$$

$$[T_{-1}b_0(\mathcal{E}_2)] = [T_{-1}(\mathcal{E}_2)] = [T_{-1}(\mathcal{O}_{\Sigma_2}(f))] = [\mathcal{O}_{\Sigma_2}(C + f)].$$

This immediately implies $b_0(\mathcal{O}_{\Sigma_2}, \mathcal{E}_2) = (\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(f))$. Hence we may and will assume $\mathcal{E}_i = \mathcal{E}_i^{\text{std}}$ for $i = 1, 2$.

At this point, in fact, we are done. To see this, note that both $(\mathcal{E}_3, \mathcal{E}_4)$ and $(\mathcal{E}_3^{\text{std}}, \mathcal{E}_4^{\text{std}})$ are exceptional pairs of the triangulated subcategory $\perp \langle \mathcal{E}_1^{\text{std}}, \mathcal{E}_2^{\text{std}} \rangle \subset \text{D}(\Sigma_2)$, which is equivalent to $\text{D}(\mathbb{P}^1)$, satisfying $[\mathcal{E}_i] = [\mathcal{E}_i^{\text{std}}] \in K_0(\perp \langle \mathcal{E}_1^{\text{std}}, \mathcal{E}_2^{\text{std}} \rangle) \hookrightarrow K_0(\text{D}(\Sigma_2))$ for $i =$
3, 4. It is well known that any exceptional pair of \( D(P^1) \) is (up to shifts) of the form \((O_{P^1}(a), O_{P^1}(a + 1))\), hence is uniquely determined (up to shifts) by the class in the Grothendieck group. This immediately implies that \( E_i \simeq E^\text{std}_i \) for \( i = 3, 4 \), hence the conclusion.

**Step 3.** In the previous step, we killed the possible difference between \( \mathcal{E} \) and \( E^\text{std} \) by spherical twists; more precisely, we found \( b \in B_K - \text{triv} \) such that

\[
\mathcal{E} = b(E^\text{std})
\]

(here we put \( b \) on the right hand side intentionally). Recall that we wanted to kill the difference by a sequence of mutations and shifts, rather than spherical twists. In this last step, we confirm that \( b \in B \) in (6.1) can be replaced by a sequence of mutations. We begin with a lemma.

**Lemma 6.3.** The following isomorphisms hold.

\[
R_{O_{\Sigma^2}}(O_{\Sigma^2}(-C)) \simeq T_0O_{\Sigma^2} \simeq L_{O_{\Sigma^2}(f)}O_{\Sigma^2}(C + 2f).
\]

**Proof.** By a direct computation, one can check that \( T_0O_{\Sigma^2} \) is the cone of the (essentially) unique non-trivial morphism

\[
O_C[-2] \to O_{\Sigma^2}.
\]

The assertion immediately follows from this observation. \( \square \)

**Theorem 6.4.** For any \( b \in B \), there exists \( \sigma \in Br_4 \) such that \( b(E^\text{std}) = \sigma(E^\text{std}) \).

**Proof.** By Theorem 2.12, \( b \) is a product of copies of \( T_{-1}, T_0 \) and their quasi-inverses. On the other hand, since autoequivalences and the action of the braid group commutes by Lemma 2.25, it is enough to show the assertion only for the two cases \( b = T_{-1}, T_0 \). We treat each case separately.

For \( T_0 \), we have

\[
T_0(E^\text{std}) = (T_0O_{\Sigma^2}, O_{\Sigma^2}(f), O_{\Sigma^2}(C + 2f) \otimes T_0O_{\Sigma^2}, O_{\Sigma^2}(C + 3f)).
\]

By Lemma 6.3, one can easily verify that

\[
\left( \sigma_1^{-1} \circ \sigma_2 \circ \sigma_1 \right)(E^\text{std}) = T_0(E^\text{std}).
\]

For \( T_{-1} \), we have

\[
T_{-1}(E^\text{std}) = (O_{\Sigma^2}, O_{\Sigma^2}(C + f), O_{\Sigma^2}(C + 2f), O_{\Sigma^2}(2C + 3f)).
\]

By a direct computation, one can verify the following assertion.

\[
\left( \sigma_3 \circ \sigma_2 \circ \sigma_3^{-1} \right)(E^\text{std}) = \left( (\sigma_3 \circ \sigma_2 \circ \sigma_1) \circ \sigma_1^{-1} \circ \sigma_3^{-1} \right)(E^\text{std}) = T_{-1}(E^\text{std}).
\]

\( \square \)

Below is an important consequence of Theorem 6.1.

**Corollary 6.5.** \( EC_4(\Sigma_2) = FEC(\Sigma_2) \).

**Proof.** Since \( E^\text{std} \) is full and the fullness is preserved under the action of the group \( G_4 \), this immediately follows from Theorem 6.1. \( \square \)
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