On a conjecture on shifted primes with large prime factors, II

Yuchen Ding

Abstract. Let $\mathcal{P}$ be the set of primes and $\pi(x)$ the number of primes not exceeding $x$. Let also $P^+(n)$ be the largest prime factor of $n$ with convention $P^+(1) = 1$ and $T_c(x) = \# \{ p \leq x : p \in \mathcal{P}, P^+(p-1) \geq p^c \}$.

Motivated by a 2017 conjecture of Chen and Chen, the author [6] proved that there exists some absolute constant $c < 1$ such that
$$\limsup_{x \to \infty} T_c(x)/\pi(x) < 1/2.$$ In this note, the prior result is considerably improved to
$$\limsup_{x \to \infty} T_c(x)/\pi(x) \to 0, \text{ as } c \to 1.$$

1. Introduction

The investigations on shifted primes with large prime factors was opened up in a brilliant article of Goldfeld [8]. Historically, this topic had aroused great concern among the community due to its unexpected connection with the first case of Fermat’s last theorem, thanks to the theorems of Fouvry [7], Adleman and Heath-Brown [1].

For any positive integer $n$, let $P^+(n)$ be the largest prime factor of $n$ with convention $P^+(1) = 1$. Let $\mathcal{P}$ be the set of primes and $\pi(x)$ the number of primes not exceeding $x$. For $0 < c < 1$ let $T_c(x) = \# \{ p \leq x : p \in \mathcal{P}, P^+(p-1) \geq p^c \}$. As early as 1969, Goldfeld [8] proved
$$\liminf_{x \to \infty} T_1^2(x)/\pi(x) \geq 1/2.$$ Goldfeld further remarked that his argument also leads to another fact
$$\liminf_{x \to \infty} T_c(x)/\pi(x) > 0, \quad (1)$$ provided that $c < \frac{7}{12}$. It turns out that exploring large $c$ to satisfy Eq. (1) is rather difficult and important. For improvements on the values of $c$, see Motohashi [16], Hooley [11, 12], Deshouillers–Iwaniec [5] and Fouvry [7]. Up to now, the best numerical value of $c$ satisfying Eq. (1) is 0.677, obtained by Baker and Harman [2].

In a former note [6] of this sequel, I showed that the following inequality
$$\limsup_{x \to \infty} T_c(x)/\pi(x) < 1/2 \quad (2)$$ holds for some absolute constant $c < 1$. As a corollary, I disproved a 2017 conjecture of Chen and Chen [4] who conjectured that
$$\liminf_{x \to \infty} T_c(x)/\pi(x) \geq 1/2.$$
for any $1/2 \leq c < 1$. The proof in my former note is based on the following deep result which is a corollary of the elaborate Brun–Titchmarsh inequality.

**Proposition 1.** [19, Lemma 2.2] There exist two functions $K_2(\theta) > K_1(\theta) > 0$, defined on the interval $(0, 17/32)$ such that for each fixed real $A > 0$, and all sufficiently large $Q = x^\theta$, the inequalities

$$K_1(\theta) \frac{\pi(x)}{\varphi(m)} \leq \pi(x; m, 1) \leq K_2(\theta) \frac{\pi(x)}{\varphi(m)}$$

hold for all integers $m \in (Q, 2Q]$ with at most $O \left( Q(\log Q)^{-A} \right)$ exceptions, where the implied constant depends only on $A$ and $\theta$. Moreover, for any fixed $\varepsilon > 0$, these functions can be chosen to satisfy the following properties:

- $K_1(\theta)$ is monotonic decreasing, and $K_2(\theta)$ is monotonic increasing.
- $K_1(1/2) = 1 - \varepsilon$ and $K_2(1/2) = 1 + \varepsilon$.

The constant $c$ in Eq. (2) is not specified therein due to the indeterminate amounts $K_1(\theta)$ in Proposition 1. In fact, $K_1(\theta)$ (and hence $c$) can be explicitly given if one checks carefully the articles of Baker–Harman [3] for $1/2 \leq \theta \leq 13/25$ and Mikawa [14] for $13/25 \leq \theta \leq 17/32$. Actually, $K_1(\theta) \geq 0.16$ for $1/2 \leq \theta \leq 13/25$ [3, Theorem 1] and $K_1(\theta) \geq 1/100$ for Mikawa’s range [14, Eq. (4)]. However, it seems that the constant $c$ in Eq. (2) obtained upon this way will be very close to 1 (see the proofs in [6]).

In [6], I also pointed out that Chen and Chen’s conjecture is already in contradiction with the Elliott–Halberstam conjecture according to the works of Pomerance [17], Granville [9], Wang [18] and Wu [19]. In fact, one has

$$\limsup_{x \to \infty} \frac{T_c(x)}{\pi(x)} = \limsup_{x \to \infty} \frac{T_c(x)}{\pi(x)} = 1 - \rho \left( \frac{1}{c} \right) \to 0, \quad \text{as } c \to 1$$

under the assumption of the Elliott–Halberstam conjecture, where $\rho(u)$ is the Dickman function, defined as the unique continuous solution of the equation differential–difference

$$\begin{cases} 
\rho(u) = 1, & 0 \leq u \leq 1, \\
up'(u) = -\rho(u - 1), & u > 1.
\end{cases}$$

In the present note, we shall prove unconditionally Eq. (3) in part (see Corollary 2).

**Theorem 1.** Let

$$\mathcal{G} = \prod_{p > 2} \left( 1 - \frac{1}{(p - 1)^2} \right) = 0.66016 \cdots \quad \text{and} \quad \mathcal{D} = \prod_{p > 2} \left( 1 + \frac{1}{p^2} \right) = 1.21585 \cdots .$$

Then for $c_0 \leq c < 1$, we have

$$\limsup_{x \to \infty} \frac{T_c(x)}{\pi(x)} \leq 16(1 - c)\mathcal{G}\mathcal{D}/c,$$

where

$$c_0 = \left( 1 + \frac{1}{16\mathcal{G}\mathcal{D}} \right)^{-1} = 0.92775 \cdots .$$

We note that the restriction of the range on $c$ in our theorem is natural since the upper bound would exceed 1 beyond this range which is certainly meaningless. Theorem 1 can
also be compared with the prior results of Goldfeld [8], Luca et al. [13] and Chen–Chen [4] which state that
\[
\liminf_{x \to \infty} \frac{T_c(x)}{\pi(x)} \geq 1 - c
\]
for any \(0 < c \leq 1/2\). From Theorem 1, we clearly have the following two corollaries.

**Corollary 1.** Let \(c\) be a constant strictly greater than \((1 + \frac{1}{2 \zeta(1)})^{-1} = 0.96252\ldots\). Then we have
\[
\limsup_{x \to \infty} \frac{T_c(x)}{\pi(x)} < 1/2.
\]

**Corollary 2.** We have
\[
\limsup_{x \to \infty} \frac{T_c(x)}{\pi(x)} \to 0, \quad \text{as } c \to 1.
\]

Corollary 1 revisits the main result Eq. (2) of my former note in a quantitative form. It should be mentioned that the method used in my former note is far from the deduction of Corollary 2.

2. Proofs

From now on, \(p\) will always be a prime. The proof of Theorem 1 is based on the following lemma deduced from the sieve method (see e.g. [10, page 172, Theorem 5.7]).

**Lemma 1.** Let \(g\) be a natural number, and let \(a_i, b_i (i = 1, 2, \ldots, g)\) be integers satisfying
\[
E := \prod_{i=1}^{g} a_i \prod_{1 \leq r < s \leq g} (a_r b_s - a_s b_r) \neq 0.
\]
Let \(\rho(p)\) denote the number of solutions of
\[
\prod_{i=1}^{g} (a_i n + b_i) \equiv 0 \pmod{p},
\]
and suppose that\(\rho(p) < p\) for all \(p\).

Let \(y\) and \(z\) be real numbers satisfying
\[
1 < y \leq z.
\]
Then
\[
\{n : z - y < n \leq z, a_i n + b_i \text{ prime for } i = 1, 2, \ldots, g\}
\]
\[
\leq 2^g g! \prod_p \left(1 - \frac{\rho(p) - 1}{p - 1}\right) \left(1 - \frac{1}{p}\right)^{-g+1} \left(\frac{y}{\log^g y} \left(1 + O\left(\frac{\log \log 3y + \log \log 3|E|}{\log y}\right)\right)\right),
\]
where the constant implied by the \(O\)-symbol depends at most on \(g\).

We also need the following important relation established by Wu [19, Theorem 2].

**Lemma 2.** For \(0 < c < 1\), let
\[
T'_c(x) = \#\{p \leq x : p \in \mathcal{P}, P^+(p - 1) \geq x^c\}.
\]

Then for sufficiently large \(x\) we have
\[
T_c(x) = T'_c(x) + O\left(\frac{x \log \log x}{(\log x)^2}\right),
\]
We now turn to the proof of Theorem 1.

**Proof of Theorem 1.** Let $x$ be a sufficiently large number throughout our proof. Instead of investigating $T_c(x)$ directly, we deal with $T'_c(x)$ firstly. For $1/2 \leq c < 1$ it is easy to see, on putting $p - 1 = qh$, that

$$T'_c(x) = \sum_{x \leq q < x/h} \sum_{p \leq x/h} 1 = \sum_{x \leq q < x/h} \sum_{q \in \mathbb{P}} \sum_{p \leq x/h} \sum_{qh+1 \in \mathbb{P}} 1 \leq \sum_{h < x^{1-c}/2} \sum_{q < x/h} 1.$$  \hfill (1)

For any $h$ with $2|h$ and $h < x^{1-c}$, let $\rho(p)$ denote the number of solutions of $n(hn + 1) \equiv 0 \pmod{p}$.

Then

$$\rho(p) = \begin{cases} 1, & \text{if } p|h, \\ 2, & \text{otherwise}. \end{cases}$$

Now, by Lemma 1 with $g = 2, a_1 = 1, b_1 = 0, a_2 = h, b_2 = 1$ and $z = y = x/h$ we have

$$3|E| = 3h \ll x, \quad 3y = 3x/h \ll x \quad \text{and} \quad y = x/h \geq \sqrt{x},$$

from which it follows that

$$\sum_{2 < q < x/h} 1 \leq 16\mathcal{G} \prod_{p \mid h} \left(1 + \frac{1}{p-2}\right) \frac{x/h}{\log^2(x/h)} \left(1 + O\left(\frac{\log \log x}{\log x}\right)\right), \hfill (2)$$

where the empty product for $\prod_{p \mid h}$ above denotes 1 as usual and

$$\mathcal{G} = \prod_{p > 2} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p} - \frac{1}{p^2}\right)^{-1} = \prod_{p > 2} \left(1 - \frac{2}{(p-1)^2}\right).$$

Inserting Eq. (2) into Eq. (1) we obtain

$$T'_c(x) \leq (1 + o(1))16\mathcal{G} \sum_{h < x^{1-c}/2} \prod_{p \mid h} \left(1 + \frac{1}{p-2}\right) \frac{x/h}{\log^2(x/h)}. \hfill (3)$$

It can be noted that

$$\prod_{p \mid h} \left(1 + \frac{1}{p-2}\right) \leq 2 \prod_{p \mid h} \left(1 + \frac{1}{p}\right) \hfill (4)$$

since the gaps between odd primes are at least 2. Thus, we deduce from Eq. (3) that

$$T'_c(x) \leq (1 + o(1))32\mathcal{G} \sum_{h < x^{1-c}/2} \prod_{p \mid h} \left(1 + \frac{1}{p}\right) \frac{x/h}{\log^2(x/h)}. \hfill (5)$$

For $2 \leq z < x^{1-c}$, let

$$S(z) := \sum_{h < z} \prod_{p \mid h} \left(1 + \frac{1}{p}\right).$$
And for $1 \leq z < 2$, we appoint $S(z) = 0$. On writing $h = 2\ell$ and then exchanging the order of sums, we have

$$S(z) = \frac{1}{2} \sum_{\ell < z/2} \sum_{d|\ell, \ 2|d} \frac{\mu^2(d)}{d} = \frac{1}{2} \sum_{d=1}^{\infty} \sum_{\ell < z/2} \frac{\mu^2(d)}{d} \sum_{\ell|d} \frac{1}{\ell} = \frac{1}{2} D \log z + O(1),$$

where the constant implied by big-$O$ is absolute, $\mu(d)$ is the Möbius function and

$$D = \sum_{d=1, 2|d}^{\infty} \frac{\mu^2(d)}{d^2} = \prod_{p>2} \left(1 + \frac{1}{p^2}\right).$$

Here, we used the well-known facts (see e.g. [15, Corollary 1.15])

$$\sum_{\ell<m} \frac{1}{\ell} = \log m + O(1)$$

in the last equality of Eq. (6). Employing estimates (6) we get

$$\sum_{h<x-\epsilon, \ p|h \ p>2} \prod_{p>2} \left(1 + \frac{1}{p}\right) \frac{1/h}{\log^2(x/h)} = S(\frac{x^{1-\epsilon}}{\log^2(x/c)})^2 - \int_1^{x^{1-\epsilon}} S(z) \left(\log \frac{x}{z}\right)^{-2} dz$$

$$= \frac{(1 - c)D}{2c^2 \log x} - \int_1^{x^{1-\epsilon}} \frac{D \log z}{z} \left(\log \frac{x}{z}\right)^{-3} dz + O\left(\frac{1}{\log^2 x}\right)$$

$$= \frac{(1 - c)D}{2c \log x} + O\left(\frac{1}{\log^2 x}\right).$$

via partial summations with routine computations. Taking Eq. (7) into Eq. (5) we immediately obtain that

$$T'_c(x) \leq (1 + o(1)) \frac{16(1 - c)\mathcal{D}}{c} \frac{x}{\log x}.$$  

Therefore, by Lemma 2 we have

$$T_c(x) = T'_c(x) + O\left(\frac{x \log \log x}{(\log x)^2}\right) \leq (1 + o(1)) \frac{16(1 - c)\mathcal{D}}{c} \frac{x}{\log x}.$$  

Our theorem now follows from the prime number theorem. □

It is possible that the value of $c_0$ in Theorem 1 could be slightly improved if one pursues the full strength of (4). We do not proceed this here as it is not the main task of this short note.

3. Remarks

Under the assumption of the Elliott–Halberstam conjecture, it is reasonable to predict that the exact value of $c$ in Corollary 1 should be $e^{-1/2} = 0.60653 \cdots$ from Eq. (3) and the following recursion formula (see e.g. [15, Eq. (7.6)]) on Dickman’s function

$$\rho(v) = u - \int_u^v \frac{\rho(t-1)}{t} dt \quad (1 \leq u \leq v).$$

It therefore seems to be of independent interest to improve, as far as possible, the numerical value of $c$ in Corollary 1. We leave this as a challenge to dear readers who are
interested in this. Though we provided nontrivial upper bounds on $T_c(x)$ for $c_0 \leq c < 1$ in Theorem 1, the expansion of these bounds to $1/2 \leq c < 1$ is an unsolved problem.

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(YUCHEN DING) School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, People’s Republic of China

Email address: ycding@yzu.edu.cn