AN IMPROVEMENT OF A THEOREM OF HEINRICH, MANKIEWICZ, SIMS, AND YOST

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Abstract. Heinrich, Mankiewicz, Sims, and Yost proved that every separable subspace of a Banach space $Y$ is contained in a separable ideal in $Y$. We improve this result by replacing the term “ideal” with the term “almost isometric ideal”. As a consequence of this we obtain, in terms of subspaces, characterizations of diameter 2 properties, the Daugavet property along with the properties of being an almost square space and an octahedral space.

1. Introduction

Let $Y$ be a Banach space and $X$ a subspace of $Y$. Recall that $X$ is an ideal in $Y$ if $X^\perp$, the annihilator of $X$, is the kernel of a contractive projection on the dual $Y^*$ of $Y$. A linear operator $\varphi$ from $X^*$ to $Y^*$ is called a Hahn-Banach extension operator if $\varphi(x^*)(x) = x^*(x)$ and $\|\varphi(x^*)\| = \|x^*\|$ for all $x \in X$ and $x^* \in X^*$. We denote by $\mathcal{HB}(X, Y)$ the set of all Hahn-Banach extension operators from $X^*$ to $Y^*$. We say that $X$ is locally 1-complemented in $Y$ if for every $\varepsilon > 0$ and every finite dimensional subspace $E$ of $Y$ there exists a linear operator $T : E \to X$ such that $Te = e$ for all $e \in E \cap X$ and $\|T\| \leq 1 + \varepsilon$. The fact that a Banach space is locally 1-complemented in its bidual is commonly referred to as the Principle of Local Reflexivity (PLR).

The following theorem is a collection of known results.

Theorem 1.1. Let $X$ be a subspace of a Banach space $Y$. The following statements are equivalent.

(a) $X$ is an ideal in $Y$.
(b) There exists $\varphi \in \mathcal{HB}(X, Y)$.
(c) $Y$ is locally 1-complemented in $X$.
(d) There exists $\varphi \in \mathcal{HB}(X, Y)$ such that for every $\varepsilon > 0$, every finite dimensional subspace $E$ of $Y$ and every finite dimensional subspace $F$ of $X^*$ there exists a linear operator $T : E \to X$ such that

\begin{enumerate}
  \item[(d1)] $Te = e$ for all $e \in E \cap X$,
  \item[(d2)] $\|Te\| \leq (1 + \varepsilon)\|e\|$ for every $e \in E$, and
\end{enumerate}

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(d3) \( \varphi f(e) = f(Te) \) for every \( e \in E \), \( f \in F \).

The equivalence of (a), (b), and (c) were independently discovered by Fakhoury [Fak72] and Kalton [Kal74]. Later Oja and Pöldvere [OP07] showed that these in turn are equivalent to statement (d).

The following result is essentially due to Heinrich and Mankiewicz [HM82]. Sims and Yost, however, gave in [SY89] another proof of this result, using a finite dimensional lemma and a compactness argument due to Lindenstrauss [Lin66]. As stated below the result appears for the first time in [HWW93, III. Lemma 4.3]).

**Theorem 1.2.** Let \( Y \) be a Banach space, \( X \) a separable subspace of \( Y \), and \( W \) a separable subspace of \( Y^* \). Then there exists a separable subspace \( Z \) of \( Y \) containing \( X \) and \( \varphi \in \mathcal{H}B(Z, Y) \) such that \( \varphi(Z^*) \supset W \).

In the language of ideals this result says that every separable subspace of a Banach space \( Y \) is contained in a separable ideal in \( Y \). Thus every non-separable Banach space contains an infinite number of ideals. Looked upon in this way ideals seems to occur quite frequently.

The following stronger form of an ideal was introduced and studied in [ALN14].

**Definition 1.3.** A subspace \( X \) of a Banach space \( Y \) is said to be an almost isometric ideal (ai-ideal) if for every \( \varepsilon > 0 \) and every finite dimensional subspace \( E \) of \( Y \), there exists a linear operator \( T : E \to X \) which satisfies (d1) in Theorem 1.1 as well as

(d2') \( (1 + \varepsilon)^{-1} \|e\| \leq \|Te\| \leq (1 + \varepsilon)\|e\| \) for \( e \in E \).

In [ALN14] the following was shown.

**Theorem 1.4.** Let \( X \) be a subspace of a Banach space \( Y \). The following statements are equivalent.

(a) \( X \) is an ai-ideal in \( Y \).

(b) There exists \( \varphi \in \mathcal{H}B(X, Y) \) such that for every \( \varepsilon > 0 \), every finite dimensional subspace \( E \) of \( Y \), and every finite dimensional subspace \( F \) of \( X^* \) there exists a linear operator \( T : E \to X \) which satisfies \( (d1) \) and \( (d3) \) in Theorem 1.1 and (d2') in Definition 1.3.

The \( \varphi \) in Theorem 1.4 is called an almost isometric Hahn-Banach extension operator associated with the ai-ideal \( X \) in \( Y \). We denote by \( \mathcal{H}B_{ai}(X, Y) \) the set of such operators.

The main result of this paper is an improvement of Theorem 1.2 in which the Hahn-Banach extension operator is replaced by an almost isometric one.

**Theorem 1.5** (Main Theorem). Let \( Y \) be a Banach space, \( X \) a separable subspace of \( Y \), and \( W \) a separable subspace of \( Y^* \). Then there exists a separable subspace \( Z \) of \( Y \) containing \( X \) and \( \varphi \in \mathcal{H}B_{ai}(Z, Y) \) such that \( \varphi(Z^*) \supset W \).
So every separable subspace of a Banach space is contained in a separable ai-ideal. Thus ai-ideals seems to occur just as frequently as ideals. Nevertheless, being an ai-ideal is strictly stronger than being an ideal. This can e.g. be seen from Theorem 1.6 and the two paragraphs that follows. Theorem 1.6 is a collection of \cite[Proposition 3.4]{fak72} and \cite[Theorem 4.3]{aln14}.

**Theorem 1.6.** For a Banach space $X$ the following statements are equivalent:

- (i) $X$ is a Lindenstrauss (resp. Gurarii) space.
- (ii) $X$ is an ideal (resp. ai-ideal) in every superspace.

Recall that a **Lindenstrauss space** is a Banach space with a dual isometric to $L_1(\mu)$ for some positive measure $\mu$. A Banach space $X$ is called a **Gurarii space** if it has the property that whenever $\varepsilon > 0$, $E$ is a finite-dimensional Banach space, $T_E : E \to X$ is isometric and $F$ is a finite-dimensional Banach space with $E \subset F$, then there exists a linear operator $T_F : F \to X$ such that

\begin{align*}
& (i) \quad T_F(f) = T_E(f) \text{ for all } f \in E, \text{ and} \\
& (ii) \quad (1 + \varepsilon)^{-1}\|f\| \leq \|T_F f\| \leq (1 + \varepsilon)\|f\| \text{ for all } f \in F.
\end{align*}

It follows from Theorem 1.6 that Gurarii spaces are Lindenstrauss. Moreover, the class of Gurarii spaces is non-empty as was shown by Gurarii in \cite{gur66}. Here a separable Gurarii space was constructed. Later Lusky \cite{lus76} proved that all separable Gurarii spaces are in fact linearly isometric. Also non-separable Gurarii spaces exist as every Banach space embeds isometrically into a Gurarii space with the same density character \cite[Theorem 3.6]{gk11}. Nevertheless, no Gurarii space is a dual space (actually the unit ball of such a space contains no extreme points \cite[Proposition 3.3]{aln}). However, the bidual of a Lindenstrauss space is again a Lindenstrauss space \cite{lin64}. Thus it follows that the classes of separable and non-separable Gurarii spaces are non-empty proper subclasses of respectively the classes of separable and non-separable Lindenstrauss spaces.

Let us now relate the notion of an ai-ideal to the well established notion of a strict ideal (see e.g. \cite{gs93}, \cite{ll09}, \cite{rao01}, and \cite{abr14}). We say that $X$ is a **strict ideal** in $Y$ if $X$ is an ideal in $Y$ with an associated $\varphi \in \text{HB}(X, Y)$ whose range is 1-norming for $Y$, i.e. for every $y \in Y$ we have $\|y\| = \sup\{y^*(y) : y^* \in \varphi(X^*) \cap S_Y\}$. Let $\text{HB}_s(X, Y) = \{\varphi \in \text{HB}(X, Y) : \varphi \text{ is strict}\}$. Using the PLR it is straightforward to show that every strict ideal is an ai-ideal. However, the converse it not true (see e.g. \cite[Example 1]{aln14} and \cite[Remark 3.2]{alln}). We can sum up the last paragraphs by

$$
\text{HB}(X, Y) \supset \text{HB}_{ai}(X, Y) \supset \text{HB}_s(X, Y),
$$

where the containment may be proper.
In Section 2 we give a proof of Theorem 1.5 and in Section 3 we use the theorem to obtain characterizations of diameter 2 properties, the Daugavet property as well as the properties of being an almost square space and an octahedral space.

2. The main theorem

The proof of Theorem 1.5 depends on Lemma 2.1 below. The roots of this lemma go back to [Lin66] and [HWW93, III.Lemma 4.2]. Lemma 2.1 differs from [HWW93, III.Lemma 4.2] simply by the fact that the partial conclusion

ii) \( \|Tx\| \leq (1 + \varepsilon)\|x\| \) for every \( x \in E \),

in [HWW93, III.Lemma 4.2] is replaced by the stronger partial conclusion ii') in Lemma 2.1. The proof of Lemma 2.1 is interestingly enough already contained in the proof of Lemma [HWW93, III.Lemma 4.2]. This is perhaps not so easy to spot at first glance. So to make this clearer, we present a complete proof here.

**Lemma 2.1.** Let \( Y \) be a Banach space, \( B \) a finite dimensional subspace of \( Y \), \( k \in \mathbb{N} \), \( \varepsilon > 0 \), and \( C \) a finite subset in \( Y^* \). Then there is a finite dimensional subspace \( Z \) containing \( B \) such that for every subspace \( E \) of \( Y \) containing \( B \) and satisfying \( \dim E/B = k \) one can find a linear operator \( T : E \to Z \) such that

i) \( Ty = y \) for every \( y \in B \),

ii') \( (1 - \varepsilon)\|y\| \leq \|Ty\| \leq (1 + \varepsilon)\|y\| \) for every \( y \in E \),

iii) \( |f(Ty) - f(y)| \leq \varepsilon\|y\| \) for every \( y \in E \) and \( f \in C \).

**Proof.** Choose \( \delta > 0 \) such that \( \delta < \varepsilon \) and \( (1 + \delta)^{-1} > 1 - \varepsilon \). Let \( C = \{f_1, \ldots, f_m\} \subset Y^* \) and \( P \) a projection on \( Y \) onto \( B \). Put \( U = \ker P \). Then we can write \( Y = B \oplus U \). Choose \( M \) so large that

\[ M > \frac{5k\|I - P\|}{\delta} \quad \text{and} \quad \frac{M + 1}{M - 1} < 1 + \delta. \]

Let \((b_\rho)_{\rho \leq r}\) and \((\lambda_\sigma)_{\sigma \leq s}\) be finite \( 1/M \)-nets for \( \{b \in B : \|b\| \leq M\} \) and \( S_{\ell_1(k)} \) respectively. Define \( \phi : (B_U)^k \to \mathbb{K}^{rs} \times \mathbb{K}^{mk} = \mathbb{K}^{rs+mk} \), by

\[ \phi(u_1, \ldots, u_k) = \left( (\|b_\rho + \sum_{k=1}^k \lambda_{k,\mu}u_\mu\|)_{\rho \leq r, \sigma \leq s}, (f_\mu(u_\mu))_{\mu \leq m, \kappa \leq k} \right). \]

Since \( \phi(B_U)^k \) is totally bounded, we can find \( (u_\nu)_{\nu \leq n} \subset (B_U)^k \) such that \( (\phi u_\nu)_{\nu \leq n} \) is a finite \( 1/M \)-net for \( \phi(B_U)^k \) where we may take any norm on \( \mathbb{K}^{rs+mk} \) for which the coefficient functionals have norm \( \leq 1 \).

Put

\[ Z = B \oplus \text{span}\{u_{\kappa,\nu} : \kappa \leq k, \nu \leq n\}. \]

Now, given a subspace \( E \supset B \) with \( \dim E/B = k \), there are \( u_1, \ldots, u_k \in U \) such that \( E = B \oplus \text{span}\{u_\kappa : \kappa \leq k\} \). By Auerbach’s lemma we can
choose \( u = (u_1, \ldots, u_k) \) such that

\[ \|u_\kappa\| = 1, 1 \leq \kappa \leq k \text{ and } \| \sum_{\kappa=1}^k u_\kappa \| \geq \frac{1}{k} \sum_{\kappa=1}^k |\lambda_\kappa| \] for all \( (\lambda_\kappa) \in \mathbb{K}^k \).

Indeed, find \( u_\kappa^* \in \text{span}\{u_\kappa : 1, \ldots, k\} \) with \( u_\kappa^*(u_j) = 0 \) if \( \kappa \neq j \) and \( \text{sign}(\lambda_\kappa) \) otherwise. Then the norm of \( u^* = \frac{1}{k} \sum_{\kappa=1}^k u_\kappa^* \) is \( \leq 1 \) and we have

\[ \| \sum_{\kappa=1}^k u_\kappa \| \geq u^*(\sum_{\kappa=1}^k u_\kappa) = \frac{1}{k} \sum_{\kappa=1}^k |\lambda_\kappa|. \]

This means that there is \( \nu \leq n \) such that

\[ \| \phi u - \phi u_\nu \| < \frac{1}{M}, \]

i.e. with \( \lambda u = \sum_{\kappa=1}^k \lambda_\kappa u_\kappa \) where \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{K}^k \) we have

\[ \| b_\rho + \lambda_\sigma u \| - \| b_\rho + \lambda_\sigma u_\nu \| < \frac{1}{M} \] for all \( \rho \leq r, \sigma \leq s, \)

\[ \| f_\mu(\kappa) - f_\mu(\kappa, \nu) \| < \frac{1}{M} \] for all \( \mu \leq m, \kappa \leq k. \)

Now, define \( T : E \to Z \) by

\[ T(b + \lambda u) = b + \lambda u_\nu. \]

Clearly \( T \) is the identity on \( B \). To show \( (1-\varepsilon)\|y\| \leq \|Ty\| \leq (1+\varepsilon)\|y\| \) for all \( y \in E \), it suffices to prove

\[ (1-\varepsilon)\|b + \lambda u\| \leq \|b + \lambda u_\nu\| \leq (1+\varepsilon)\|b + \lambda u\| \] for \( \|\lambda\|_{\mathcal{E}_1(k)} = 1. \)

Assume first \( \|b\| \leq M \). Then \( \|b - b_\rho\| < \frac{1}{M} \) for some \( \rho \) and \( \|\lambda - \lambda_\sigma\| < \frac{1}{M} \) for some \( \sigma \). Thus

\[ \|b + \lambda u_\nu\| \leq \|b_\rho + \lambda_\sigma u_\nu\| + \|b - b_\rho\| + \|\lambda u_\nu - \lambda_\sigma u_\nu\| \]

\[ < \|b_\rho + \lambda_\sigma u_\nu\| + \frac{2}{M} \]

\[ < \|b_\rho + \lambda_\sigma u\| + \frac{3}{M} \]

\[ \leq \|b + \lambda u\| + \|b_\rho - b\| + \|\lambda_\sigma u - \lambda u\| + \frac{3}{M} \]

\[ < \|b + \lambda u\| + \frac{5}{M}. \]

Similarly we also get \( \|b + \lambda u\| < \|b + \lambda u_\nu\| + \frac{5}{M} \), so we have

\[ \|b + \lambda u\| - \frac{5}{M} < \|b + \lambda u_\nu\| < \|b + \lambda u\| + \frac{5}{M}. \]
Also
\[
\|b + \lambda u\| \geq \frac{1}{\|I - P\|} \|(I - P)(b + \lambda u)\|
\]
\[
= \frac{1}{\|I - P\|} \|\sum_{\kappa=1}^{k} \lambda_u \| \tag{1}
\]
\[
\geq \frac{1}{k\|I - P\|} \sum_{\kappa=1}^{k} |\lambda_u|
\]
\[
= \frac{1}{k\|I - P\|} \geq \frac{5}{\delta M},
\]
so \(\varepsilon\|b + \lambda u\| \geq \frac{5}{\delta M}\) and thus (4) holds for \(\|b\| \leq M\).

For \(\|b\| > M\) we have
\[
\|b\| - 1 \leq \|b + \lambda u\| \leq \|b\| + 1 \quad \text{and} \quad \|b\| - 1 \leq \|b + \lambda u\| \leq \|b\| + 1,
\]
so both
\[
\frac{\|b + \lambda u\|}{\|b + \lambda u\|} \quad \text{and} \quad \frac{\|b + \lambda u\|}{\|b + \lambda u\|} \leq \frac{\|b\| + 1}{\|b\| - 1} < \frac{M + 1}{M - 1} < 1 + \delta,
\]
as \(y \to \frac{y + 1}{y - 1}\) is a positive and decreasing function for \(y > 1\). Thus (4) holds also for \(\|b\| > M\).

Finally for any \(y = b \sum_{\kappa=1}^{k} \lambda_u \in E\)
\[
|f_\mu(y) - f_\mu(T y)| = |f_\mu \left( \sum_{\kappa=1}^{k} \lambda_u (u_\kappa - u_{\kappa, \nu}) \right) |
\]
\[
\leq \sum_{\kappa=1}^{k} |\lambda_u| |f_\mu(u_\kappa - u_{\kappa, \nu})| \tag{3}
\]
\[
\leq \frac{1}{M} \|\sum_{\kappa=1}^{k} |\lambda_u| \|
\]
\[
\leq \frac{k}{M} \|\sum_{\kappa=1}^{k} \lambda_u u_\kappa\|
\]
\[
= \frac{k\|I - P\|}{M} \|y\| < \frac{\varepsilon}{5\|y\|}.
\]

By the proof of [ALN14, Theorem 1.4] the following holds.

**Lemma 2.2.** Let \(X\) be an ideal in \(Y\) and let \(\varphi \in H_B(X, Y)\). Then the following statements are equivalent.

(a) \(\varphi \in H_B(I)(X, Y)\).

(b) For every \(\delta, \varepsilon > 0\), for every finite dimensional subspace \(E\) of \(Y\), and every finite dimensional subspace \(F\) of \(X^*\) there exists a linear operator \(T : E \to X\) which satisfies
Clearly and put we have that is contained in some $E$. Finally we check that \( \delta \)-nets. Choose $n$ so large that $B_n$ contains a $\delta$-net for $S_{H \cap M}$ and then

\( (d1') \quad \|Te - e\| \leq \epsilon \|e\| \text{ for every } e \in E \cap X, \)

\( (d2') \text{ in Definition } 1.3 \) and

\( (d3') \quad |\varphi(e) - f(Te)| < \delta \|e\| \cdot \|f\| \text{ for every } e \in E, f \in F. \)

Along with Lemma 2.1 we will use Lemma 2.2 to prove our Main Theorem.

**Proof of Theorem 1.5.** The first part of the proof is identical to that of \cite{HWW93} III. Lemma 4.3 except at the crucial point where we use Lemma 2.1 in place of \cite{HWW93} III. Lemma 4.2. This is what in the end makes it possible to establish Lemma 2.2 (b).

Let \( (x_n) \) be a sequence dense in $X$ and \( (f_n) \) a sequence dense in $W$. Starting with $M_1 = \{0\}$ we inductively define subspaces $M_n$ as follows: Put $B_n = \text{span}(M_n, x_n)$, $C_n = \{f_1, \ldots, f_n\}$, and let $M_{n+1}$ be the subspace $Z$ given by Lemma 2.1 when $B = B_n, k = n, \epsilon = \frac{1}{n}$, and $C = C_n$. Without loss of generality assume $\dim M_{n+1}/B_n \geq n+1$. Clearly $M = \overline{\bigcup M_n}$ is separable and contains $X$. For $n \in \mathbb{N}$ define

\[ I_n = \{ E \subset Y : B_n \subset E, \dim E/B_n \leq n \} \]

and put

\[ I = \bigcup I_n. \]

Since

\[ E \in I_n, F \in I_m \Rightarrow E \oplus F \oplus B_{\dim E + \dim F} \in I_{\dim E + \dim F}, \]

we have that $I$ is a directed set. Moreover, it is clear that every $y \in Y$ is contained in some $E \in I$. Just take $E = \text{span}(x_1, y)$. Then $E \in I_1$.

Note that the condition $\dim M_{n+1}/B_n \geq n+1$ implies $\dim B_{n+1}/B_n \geq n+1$. This easily gives that for each $E \in I$ there is a unique $n \in \mathbb{N}$ such that $E \in I_n$. So by Lemma 2.1 there exists a linear operator $T_E : E \rightarrow M_{n+1}$ such that $T_E|B_n = I_Bn, (1 - \frac{1}{n})\|y\| \leq \|T_Ey\| \leq (1 + \frac{1}{n})\|y\|$, and $|f_i(T_Ey) - f_i(y)| < \frac{1}{n}\|y\| \cdot \|f_i\|$ for every $y \in E$ and $1 \leq i \leq n$. Extend $T_E$ (nonlinearly) to $Y$ by setting $S_E(y) = T_E(y)$ if $y \in E$ and $S_E(y) = 0$ otherwise. Since $\|S_E(y)\| \leq 2\|y\|$ and regarding $S_E(y) \in M$ as an element in $M^{**}$ we have

\[ (S_Ey)_{y \in Y} \in \Pi_{y \in Y} B_{M^{**}}(0, 2\|y\|). \]

By Tychonoff’s compactness theorem $(S_Ey)_{y \in Y} \in E$ has a convergent subnet, for simplicity, also denoted $(S_Ey)_{y \in Y} \in E$, i.e. for every $y \in Y$ there is $m_y \in M^{**}$ such that $S_Ey \rightarrow E m_y$ with respect to the weak* topology on $M^{**}$. Define $S : M^* \rightarrow Y^*$ by

\[ Sm^*(y) = m_y(m^*). \]

It is now straightforward to check that $S \in \text{IB}(M, Y)$ and $S(M^*) \supset F$.

Finally we check that $S$ satisfies condition (b) in Lemma 2.2. To this end let $H$ be a finite dimensional subspace of $Y$ and $G$ a finite dimensional subspace of $M^*$. Let $(h_i)_{i=1}^k \subset S_H$ and $(g_j)_{j=1}^l \subset S_G$ be $\delta$-nets. Choose $n$ so large that $B_n$ contains a $\delta$-net for $S_{H \cap M}$ and then
choose a finite dimensional subspace $H' \in I$ with $H' \supset \text{span}(H, B_u)$. By Lemma 2.1 there is a unique $N > n$ such that $H' \in I_N$ and a linear operator $T_H : H' \to M$ such that $T_H|_{B_N} = I_{B_N}$, $(1 - 1/N)\|h\| \leq \|T_H h\| \leq (1 + 1/N)\|h\|$ and $|\varphi g_j(h_i) - g_j(T_H h_i)| < \delta$ (weak* convergence of $(T_H)$). For $h \in S_{H \cap M}$ we can find $h_i \in S_H \subset B_N$ such that $\|h - h_i\| < \delta$. Thus we get

$$
\|T_H h - h\| \leq \|T_H h - T_H h_i\| + \|T_H h_i - h_i\| + \|h_i - h\| \\
\leq (2 + 1/N)\delta.
$$

For $h \in S_H$, $g \in S_G$ find $h_i \in S_H$, $g_j \in S_G$ with $\|h - h_i\| < \delta$ and $\|g - g_j\| < \delta$. We get

$$
|\varphi g(h) - g(T_H h)| \leq |\varphi g(h) - \varphi g_j(h)| + |\varphi g_j(h) - \varphi g_j(h_i)| \\
+ |\varphi g_j(h_i) - g_j(T_H h_i)| + |g_j(T_H h_i) - g_j(T_H h)| \\
+ |g_j(T_H h) - g(T_H h)| \\
\leq \|g - g_j\| + \|h - h_i\| + \delta \\
+ \|T_H\| (\|h_i - h\| + \|g_j - g\|) \\
\leq \delta(5 + 2/N).
$$

Now the operator $T_H$, restricted to $H$ will do the work as $\delta$ can be chosen arbitrary small. \qed

Remark 2.3. Note that Theorem 1.5 can be by transfinite induction, just as \[\text{HWW93, IIILemma 4.4}\], be extended to a non-separable version similar to \[\text{HWW93, IIILemma 4.3}\] but with an almost isometric Hahn-Banach extension operator in place of a Hahn-Banach extension operator. The proof is like that of \[\text{HWW93, IIILemma 4.4}\], but uses Theorem 1.5 instead of \[\text{HWW93, IIILemma 4.3}\].

As pointed out in Section 1 examples of ideals which are not ai-ideals are plentiful. Similarly examples of ai-ideals which are not strict ideals are also plentiful. Indeed, take any non-separable space $X$ for which $X^*$ contains no proper 1-norming subspace (e.g. the case for spaces being M-ideals in their biduals \[\text{GSS88}\] or more generally for strict u-ideals in their biduals \[\text{GKS93}\]). Then for every separable subspace $Y$ of $X$ there exists a separable ai-ideal $Z$ in $X$ containing $Y$. This ai-ideal cannot be strict. Moreover, this reasoning actually shows that one cannot extend the main theorem replacing “ai-ideal” with “strict ideal”.

3. Characterizations in terms of subspaces

Let $X$ be a Banach space with unit ball $B_X$. By a slice of $B_X$ we mean a set $S(x^*, \varepsilon) = \{x \in B_X : x^*(x) > 1 - \varepsilon\}$ where $x^*$ is in the unit sphere $S_{X^*}$ of $X^*$ and $\varepsilon > 0$. A finite convex combination of slices of
B_X is a set of the form
\[ S = \sum_{i=1}^{n} \lambda_i S(x_i^*, \varepsilon_i), \quad \lambda_i \geq 0, \quad \sum_{i=1}^{n} \lambda_i = 1, \]
where \( x_i^* \in S_{X^*} \) and \( \varepsilon_i > 0 \) for \( i = 1, 2, \ldots, n \).

The relations between the following three successively stronger properties were investigated in [ALN13]:

**Definition 3.1.** A Banach space \( X \) has the

(i) *local diameter 2 property* (LD2P) if every slice of \( B_X \) has diameter 2.

(ii) *diameter 2 property* (D2P) if every non-empty relatively weakly open subset in \( B_X \) has diameter 2.

(iii) *strong diameter 2 property* (SD2P) if every finite convex combination of slices of \( B_X \) has diameter 2.

From [BGZ Theorem 2.4] it is known that LD2P \( \not\Rightarrow \) D2P and from [HJ14 Theorem 1] or [ABGLPZ Theorem 3.2] that D2P \( \not\Rightarrow \) SD2P.

**Proposition 3.2.** Let \( Y \) be a Banach space. Then \( Y \) has the SD2P (resp. the D2P, LD2P) if and only if every separable ai-ideal in \( Y \) does.

**Proof.** Sufficiency was established in [ALN14] Propositions 3.2, 3.3 and Corollary 3.4 where it was proved that every ai-ideal in \( Y \) has the SD2P (resp. the D2P, LD2P) whenever \( Y \) has.

First let us prove the result for the SD2P. To this end let \( \varepsilon_k > 0 \) for \( k = 1, \ldots, n \) and \( S = \sum_{k=1}^{n} \lambda_k S_k \) a finite convex combination of slices \( S_k = \{ y \in B_Y : y_k^*(y) > 1 - \varepsilon_k \} \) of the unit ball of \( Y \). By Theorem [1.5] find a separable ai-ideal \( X \) in \( Y \) such that \( \text{span}(y_k^*)_{k=1}^{n} \subset \varphi(X^*) \) where \( \varphi \in \mathcal{B}_m(X,Y) \). For \( k = 1, \ldots, n \) find \( x_k^* \in S_{X^*} \) such that \( y_k^* = \varphi(x_k^*) \).

Define the slices \( S_k = \{ x \in B_X : \varphi x_k^*(x) > 1 - \varepsilon_k \} \). Now the convex combination of slices \( S = \sum_{k=1}^{n} \lambda_k S_k \) has diameter 2 by assumption. As \( S' \subset S \), we get that \( S \) has diameter 2 as well.

For the LD2P the result follows by taking \( k = 1 \) in the argument above.

For the D2P property let \( V \) be a relatively weakly open subset in \( B_Y \). Find \( y_0 \in V \) and \( y_i^* \in Y^* \) such that \( V = \{ y \in B_Y : |y_i^*(y - y_0)| < \varepsilon, \; i = 1, \ldots, n \} \subset V \). By Theorem [1.5] find a separable ai-ideal \( X \) in \( Y \) which contains \( y_0 \) and such that \( \text{span}(y_i^*)_{k=1}^{n} \subset \varphi(X^*) \) where \( \varphi \in \mathcal{B}_m(X,Y) \). Then a similar argument as above will finish the proof. \( \square \)

Recall that a Banach space \( X \) has the Daugavet property if for every rank one operator \( T : X \to X \) the equation \( \|I + T\| = 1 + \|T\| \) holds where \( I \) is the identity operator on \( X \). One can show that the Daugavet property is equivalent to the following statement (see [Shv00]): For
every slice $S = S(x_0^*, \varepsilon_0)$ of $B_X$, every $x_0 \in S_X$ and every $\varepsilon > 0$ there exists a point $x \in S$ such that $\|x + x_0\| \geq 2 - \varepsilon$.

Using Theorem [L5] we get a similar characterization of the Daugavet property as for the diameter 2 properties.

**Proposition 3.3.** A Banach space $X$ has the Daugavet property if and only if every separable ai-ideal in $X$ does.

**Proof.** That the Daugavet property is inherited by ai-ideals is proved in [ALN14, Proposition 3.8]. Let $\varepsilon > 0$ and choose positive $\delta < \varepsilon$ such that $(1 - \delta)^2 > 1 - \varepsilon$. Let $x_0^* \in S_{X^*}$ and $x_0 \in S_X$. We must show that the slice $S(x_0^*, \varepsilon) = \{x \in B_X : x_0^*(x) > 1 - \varepsilon\}$ contains $x$ such that $\|x + x_0\| > 2 - \varepsilon$. To this end, choose $x_1 \in S(x_0^*, \delta)$ and find a separable ai-ideal $Z$ which contains span$\{x_0, x_1\}$. By construction the slice $S(x_0^*|Z/\|x_0^*|Z\|, \delta)$ of $B_Z$ is non-empty and by assumption there exists $y \in S(x_0^*|Z/\|x_0^*|Z\|, \delta)$ with $\|x_0 + y\| > 2 - \delta$. Since $x_0^*(y) = \|x_0^*|Z/\|x_0^*|Z\|\|y\|/\|x_0^*|Z\| > \|x_0^*|Z\|(1 - \delta) > (1 - \delta)(1 - \delta) > 1 - \varepsilon$, we are done. \[\square\]

In [ALL] the notion of an almost square Banach space was introduced and studied.

**Definition 3.4.** A Banach space $Y$ is said to be almost square (asq) if for every $\varepsilon > 0$ and every finite set $(y_n)_{n=1}^N \subset S_Y$, there exists $y \in S_Y$ such that

$$\|y_n - y\| \leq 1 + \varepsilon.$$

The following characterization of (asq) spaces was obtained in [ALL, Theorem 3.6].

**Theorem 3.5.** Let $Y$ be a Banach space. If $Y$ is (asq) then for every finite dimensional subspace $E \subset Y$ and $\varepsilon > 0$ there exists $y \in S_Y$ such that

$$(1 - \varepsilon) \max(\|x\|, |\lambda|) \leq \|x + \lambda y\| \leq (1 + \varepsilon) \max(\|x\|, |\lambda|)$$

for all scalars $\lambda$ and all $x \in E$.

We will use this result and Theorem [L5] to obtain characterizations of the an (asq) space in terms of its subspaces.

**Theorem 3.6.** Let $Y$ be a Banach space. Then the following statements are equivalent.

(a) $Y$ is (asq).

(b) Every separable ai-ideal in $Y$ is (asq).

(c) Every subspace $X$ of $Y$ for which $Y/X$ does not contain a copy of $c_0$ is (asq).

(d) Every subspace of finite codimension in $Y$ is (asq).
Proof. (a) ⇒ (b). This is proved in [ALL] Lemma 4.5.
(b) ⇒ (a). Let \( \varepsilon > 0 \) and \( (x_i)_{i=1}^N \subset S_X \), and find by Theorem 3.3 a separable ai-ideal \( Z \) in \( Y \) containing \( (x_i)_{i=1}^N \). As \( Z \) is (asq) there exists \( z \in S_Z \) such that \( \|x_i - z\| \leq 1 + \varepsilon \) and we are done.
(c) ⇒ (d) is trivial and (d) ⇒ (a) is clear as every finite set of points is contained in a subspace of finite codimension in \( Y \).
(a) ⇒ (c). Let \( (y_n)_{n=1}^N \subset S_Y \), \( E = \text{span}(y_n)_{n=1}^N \), and \( \varepsilon > 0 \). Choose a sequence \( (\varepsilon_n)_{n=1}^\infty \) of positive reals such that \( (\varepsilon_n) \downarrow 0 \) and
\[
\Pi_{n=1}^\infty (1 - \varepsilon_n) > 1 - \varepsilon/2 \quad \text{and} \quad \Pi_{n=1}^\infty (1 + \varepsilon_n) < 1 + \varepsilon/2.
\]
Using Theorem 3.5 we can find a sequence \( (z_n) \subset S_Y \) such that for every \( y \in \text{span}(E \cup \{z_1, \ldots, z_n\}) \) and every \( \lambda \in \mathbb{R} \) we have
\[
(1 - \varepsilon_n) \max(||y||, |\lambda|) \leq \|y + \lambda z_{n+1}\| \leq (1 + \varepsilon_n) \max(||y||, |\lambda|).
\]
Now, if \( z = \sum_{n=1}^N \lambda_k z_k \) we get from (7) and (8) that
\[
(1 - \varepsilon/2) \max(||y||, |\lambda_n|) = 1 < \|y + z\|
\]
\[
< (1 + \varepsilon/2) \max(||y||, |\lambda_n|)
\]
for every \( y \in E \). It is clear that the space \( \overline{\text{span}}(z_n)_{n=1}^\infty \) is isomorphic to \( c_0 \). As \( Y/X \) does not contain a copy of \( c_0 \), the quotient map \( \pi : Y \to Y/X \) fails to be bounded below on \( \overline{\text{span}}(z_n)_{n=1}^\infty \). From this it follows that there exists a linear combination \( \sum_{k=1}^{N_1} \lambda_k z_k \) whose norm is 1 and with \( \|\pi(\sum_{k=1}^{N_1} \lambda_k z_k)\| \leq \varepsilon/8 \). Thus there is \( f \in X \) with \( \|f - \sum_{k=1}^{N_1} \lambda_k z_k\| \leq \varepsilon/4 \). Putting \( g = f/\|f\| \) we get \( \|g - \sum_{k=1}^{N_1} \lambda_k z_k\| \leq \varepsilon/2 \). Hence using (9) we get for \( n = 1, \ldots, N \)
\[
\|y_n - g\| \leq \|y_n + \sum_{k=1}^{N_1} \lambda_k z_k\| + \|g - \sum_{k=1}^{N_1} \lambda_k z_k\|
\]
\[
< (1 + \varepsilon/2) \max(||y_n||, |\lambda_k|) + \varepsilon/2
\]
\[
\leq 1 + \varepsilon,
\]
which is what we need. \( \square \)

Theorem 3.6 (b) ⇔ (a) should be compared with [ALL] Proposition 4.5.

Let us end the paper with a result similar to Theorem 3.6 and for octahedral spaces. Octahedral spaces were introduced by Godefroy in [Goe89].

Definition 3.7. A Banach space \( Y \) is said to be octahedral if for every \( \varepsilon > 0 \) and every finite set \( (y_n)_{n=1}^N \subset S_Y \), there exists \( y \in S_Y \) such that
\[
\|y_n - y\| \geq 2 - \varepsilon.
\]

In [Lüc11] Theorem 2.5] Lücking proved that separable almost Daugavet spaces satisfies statement (c) in Theorem 3.8 below. In [KSW11] it was proved that the almost Daugavet property in general implies
octahedrality, and that for separable spaces the converse is true. For non-separable spaces it is as far as the author knows unknown whether octahedrality implies almost Daugavet.

**Theorem 3.8.** Let $Y$ be a Banach space. Then the following statements are equivalent.

(a) $Y$ is octahedral.

(b) Every separable ai-ideal in $Y$ is octahedral.

(c) Every subspace $X$ of $Y$ for which $Y/X$ does not contain a copy of $\ell_1$ is octahedral.

(d) Every subspace of finite codimension in $Y$ is octahedral.

The proof of this result follows along the same lines as the proof of Theorem 3.6 using [HJP, Proposition 2.4] instead of Theorem 3.5 and otherwise adjusting to the $\ell_1$ setting. Therefore the proof will be omitted. Theorem 3.8 should be compared with [BGLPZ, Theorems 2.2 and 2.6].

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