(De)quantization of black hole charges

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Abstract
We argue that magnetic and electric charges of the Reissner–Nordström black hole are quantized in CP conserving theories. A dequantization phenomenon occurs when CP is broken either explicitly or effectively, and, as a result, pure magnetic black holes are not possible. Two examples illustrating this phenomenon are discussed. One is the electric charge induced by the neutral pion field in a magnetically charged black hole, and another is the electric charge induced due to the massive spin-two field emerging from possible higher curvature terms.

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1. Introduction

It is well known that quantum-mechanical consistency of a system of an electron with charge $e$ and a monopole of charge $g$ requires the quantization condition,

$$eg = \frac{1}{2}n, \quad n \in \mathbb{Z},$$

(1)

to be fulfilled [1]. The quantization condition (1) has been generalized to a system of two dyons with charges $(q_e, q_m)$ and $(q'_e, q'_m)$ [2, 3]:

$$q_e q'_m - q'_e q_m = \frac{1}{2}n, \quad n \in \mathbb{Z}.$$  

(2)

For the charge-monopole system (e.g. $q_m = 0$ and $q'_e = 0$) (2) reduces to (1). Consider now a dyon $(q_e, q_m)$ and its CP conjugated dyon $(-q_e, q_m)$. If the CP is an exact symmetry, then the quantization condition (2) applied to the pair of such dyons implies the dyon charge quantization:

$$q_e = \frac{1}{2}en, \quad n \in \mathbb{Z},$$  

(3)

where, according to (1), we have taken an elementary magnetic charge to be $q_m = \frac{1}{2}$. Thus dyon electric charges are either integer or half-integer in units of the elementary charge $e$. However, in CP non-conserving theories electric charge of a dyon acquires a non-integral shift $\delta$, $q_e = \frac{1}{2}n + \delta$, which is proportional to the strength of the CP violation [4]. This means that magnetically charged particles necessarily carry non-zero electric charge $q_e = \delta (n=0)$ when the CP symmetry is broken.
Monopoles [5, 6] and dyons [7] emerge as particle-like solutions in spontaneously broken gauge theories. On the other hand black holes can also carry electric and/or magnetic charges. Does the charge quantization hold for such an object? The answer is yes. The necessity of black hole charge quantization can be understood from the following heuristic consideration. As soon as we are interested in a distantly separated pair of dyonic black holes (with asymptotically flat spacetime), they can be considered as usual point-like particles and the quantization condition (2) must be applicable. Also, the quantization condition (2) is supported by the fact that the electromagnetic angular momentum in black hole spacetime has exactly the same form as in flat spacetime [9], together with the traditional semi-classical derivation of the condition (2) in terms of the quantization of angular momentum.

In this paper we argue that Dirac’s quantization condition does indeed depend on the topology rather than geometry of a black-hole spacetime. Thus it does hold for charged black holes as well as for dyonic systems in black-hole background spacetimes, as long as the electric and magnetic charges can be meaningfully defined (e.g. in static asymptotically flat spacetimes).

It turns out that electric charge quantization holds only in CP conserving theories. Once CP symmetry is broken the phenomenon of charge dequantization occurs. Witten showed that this was the case in flat spacetime [4], and we generalize the result to the spacetime of black holes. We discuss two further examples demonstrating this effect. One involves a neutral pion field in the background of a magnetic black hole. Due to the axial anomaly, the pion field couples topologically to two photons. While this interaction term does not contribute to the energy momentum tensor, it does affect the dynamics of the pion field so that the pion carries classical hair outside the black hole horizon which in principle is detectable. At spatial infinity the pion field approaches its non-trivial vacuum expectation, and thus the anomalous pion–photon interaction term turns into an effective CP-violating $\theta$-term. The later induces an electric charge on the black hole which is, in general, a non-integer multiple of the elementary charge $e$.

Another example illustrates the effect of charge dequantization within the theory with higher curvature terms. The higher curvature theories contain a massive spin-2 field. According to the no-hair theorem [10], this spin-2 field vanishes outside the (Schwarzschild) black hole horizon; however its longitudinal part carries quantum hair, as discovered in [11]. Assuming that the field providing the quantum hair couples to the photon through the topological interaction term, we again generate a non-integer electric charge on the black hole. Obviously, the charge (de)quantization phenomenon can be applied to any other U(1) charge which is supported by a gauge field.

The effect of charge (de)quantization discussed in this paper can be contrasted with recent observation on the quantization of global charges [12]. Dvali argued that a global charge, i.e. the charge which is not coupled to a long-range (massless) field, is necessarily quantized with maximal periodicity $n_{\text{max}} = \left[ \frac{M^2}{P/m^2} \right]$. The origin of this charge quantization (see [12] for details) is entirely different from the one applying to fields which couple to massless fields, which we discuss in this paper. Note also that gauge theories with both electric ($g$) and magnetic ($1/g$) charges do not have a sensible global limit, $g \to 0$, and thus our result on dequantization is not in contradiction with the quantization phenomenon discussed in [12].

2. Dyonic Reissner–Nordström black hole

Let us consider a Reissner–Nordström black hole with electric and magnetic charges. It is described by the metric tensor
\[ g_{\mu\nu} = \text{diag}[A, -A^{-1}, -r^2, -r^2 \sin^2 \theta], \quad (4) \]

where \( A = 1 - G_N \left( \frac{2M}{r} - \frac{q_e^2}{r^2} - \frac{q_m^2}{r^2} \right), \) and by the electromagnetic field strength 2-form,

\[ F = -\frac{q_e}{r^2} \, dr \wedge dr - q_m \sin \theta \, d\theta \wedge d\phi. \quad (5) \]

The dual 2-form, \( *F = \frac{\sqrt{-g}}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \, dx^\mu \wedge dx^\nu, \) reads:

\[ *F = -\frac{q_m}{r^2} \, dt \wedge dr + \frac{q_e}{r^2} \sin \theta \, d\theta \wedge d\phi. \quad (6) \]

Note that the above 2-forms satisfy the source-free equation of motion \( d \ast F = 0. \) Electric-magnetic duality is manifest in the above equations, since \( *F \) is obtained from \( F \) by the transformation \( q_e \rightarrow q_m, q_m \rightarrow -q_e. \)

The metric (4) has a physical singularity at the origin \( r = 0, \) and thus topologically spacetime is \( \mathbb{R}^2 \times S^2. \) On the other hand (since the space-time is asymptotically flat), we can define the global electric

\[ q_e = \frac{1}{4\pi} \int_{S^2} *F, \quad (7) \]

and magnetic

\[ q_m = \frac{1}{4\pi} \int_{S^2} F \]

charges as the total electric and magnetic fluxes through the 2-sphere at infinity.

In quantum theory the primary field is the vector potential 1-form \( A = A_\mu dx^\mu, \) and the 2-form is given as \( F = dA. \) Thus alongside the equation of motion \( d \ast F = 0 \) we have a Bianchi identity, \( dF = 0. \) The duality symmetry allows us to alternatively define a dual 1-form, \( \tilde{A} \) from \( *F = d\tilde{A}, \) as a primary field. With the later choice the equation of motion and the Bianchi identity are interchanged, but both equations still hold.

In the original derivation of the quantization condition (1) Dirac made use of the singular gauge potential. Equation (1) readily follows from the condition of non-observability of this singularity (Dirac’s string). Indeed the Dirac string is known to be a gauge artefact, and the appropriate gauge exists where the potential is a regular, but multiply defined object [14]. Thus the ‘true’ nature of Dirac’s quantization is encoded in the non-trivial topology, namely \( R^1 \times R^3/[0], \) which is diffeomorphic to \( R^2 \times S^2. \) But this is exactly the topology of black hole spacetimes. This simple topological argument again leads to the conclusion that Dirac’s quantization must hold in black-hole spacetimes (contrary to some claims in the literature, see e.g. [14]).

Indeed, since the electric charges in (7) and (8) are defined on a topological \( S^2, \) we are primarily interested in induced gauge potentials and gauge transformations on a 2-sphere. Denote by \( \mathcal{U} \) the space of all such restricted gauge potentials, \( A^\phi = A + g^{-1} dg, \) and by \( \mathcal{G}, \) the space of all continuous gauge transformations \( g \) on \( S^2. \) Then the restricted gauge orbit space is the quotient \( \mathcal{U}/\mathcal{G}. \) Because \( \mathcal{U} \) is topologically trivial, i.e. \( \pi_n(\mathcal{U}) = 0 \) for any \( n, \) from the exact homotopy sequence one obtains

\[ \pi_3(\mathcal{U}/\mathcal{G}) = \pi_1(\mathcal{G}) = Z \quad (9) \]

for the \( G = U(1) \) gauge group. This simply means that the quantum-mechanical system of an elementary charge \( e \) moving in the field of magnetically charged black hole is consistent once the quantization condition

\[ q_m e = \frac{1}{2} n \quad (10) \]
is satisfied. Analogously, for the quantum-mechanical system of a monopole of charge \( g \) and an electrically charged black hole, consistency demands the quantization condition

\[
q_e g = \frac{1}{2} n. \tag{11}
\]

In the presence of both an elementary charge and an elementary monopole we can use Dirac’s quantization condition to obtain the elementary magnetic charge \( g = \frac{1}{2e} \). Then (11) becomes

\[
q_e = n e. \tag{12}
\]

This is the quantization condition for the electric charge of the black holes in terms of ‘elementary’ charge \( e \).

As we have mentioned above a charge dequantization phenomenon \([4]\) occurs in the case of CP violation. This can be most easily seen within the model where the CP violation is provided by a Chern–Simons term, i.e.

\[
L_{CS} = \frac{\theta e^2}{8\pi} F \wedge F. \tag{13}
\]

Although the above term is a boundary term and thus does not affect the dynamics, it does contribute to the electric charge (7) of the black hole, which now reads

\[
Q_e = \frac{1}{4\pi} \int_{S^2} \left( * F + \frac{\theta e^2}{2\pi} F \right). \tag{14}
\]

The arguments which lead to equation (12) are now applied to (14), and thus we obtain

\[
q_e = \left( n - \frac{m\theta}{4\pi} \right) e, \tag{15}
\]

where we have used \( q_m = \frac{m}{2\pi} \). The charge dequantization (15) implies the nonexistence of pure magnetic black holes in CP non-conserving theories.

### 3. Induced charge on the black hole

Interestingly, the phenomenon of dequantization (15) can effectively emerge in some theories even in the absence of a Chern–Simons term (13). We discuss below two examples:

**Electric charge induced by the neutral pion.** The no-hair theorem (conjecture actually) \([10]\) is one of the most important results in black hole physics. According to this theorem, the only dynamical field strengths present in the exterior of (stationary and asymptotically flat) black hole solutions are those required by a local gauge invariance. For example, the gauge invariance of electrodynamics coupled to gravity implies non-vanishing of electric and/or magnetic fields around the black hole. In the contrast, the fields which are not related to a local gauge invariance cannot be observed outside the black hole horizon. For example, the baryonic charge does not couple to a gauge field, and thus it cannot be measured by the exterior observer. This implies that the black hole made of baryons and the one made of anti-baryons are indistinguishable as long as they have the same electric charge. We will see below that in certain cases a global baryonic charge can induce an electric charge which can be detected by the corresponding electrostatic field.

Baryons and mesons are described reasonably well by effective chiral Lagrangians. We consider a gauged 2-flavour \( SU(2)_L \times SU(2)_R \) effective chiral theory. Upon the chiral symmetry breaking the low energy degrees of freedom (pions) are combined in the unitary matrix field

\[
U = \exp \left( i \pi(x)^a \tau^a \right) \]

which belongs to a quotient \( SU(2)_L \times SU(2)_R / SU(2)_V \) (here \( \tau^a \) are the Pauli matrices, such that \( \text{Tr}(\tau^a \tau^b) = 2\delta^{ab} \)).

\[
\frac{1}{\sqrt{-g}} L_{\text{chiral}} = L_{\text{pion}} + L_{\text{anom}} \tag{16}
\]
L_{\text{pion}} = \frac{f_\pi^2}{16} g^{\mu\nu} \text{Tr}(D_\mu U D_\nu U^\dagger) + \frac{(f_\pi m_\pi)^2}{16\hat{m}} \text{Tr}(M_q [U + U^\dagger - 2]),
(17)

L_{\text{anom}} = \frac{eN_c}{48\pi^2} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} A_\mu \text{Tr}[\hat{Q} (\partial_\nu U \partial_\rho U^\dagger \partial_\sigma U U^\dagger + U^\dagger \partial_\nu U \partial_\rho U^\dagger \partial_\sigma U)]

+ \frac{ie^2 N_c}{48\pi^2} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} A_\mu F_{\nu\rho} \text{Tr}[2 \hat{Q}^2 \{\partial_\sigma U, U^\dagger\} + \hat{Q} \{\partial_\sigma U U^\dagger, U^\dagger \hat{Q} U\dagger\}] + \frac{i e^2 N_c}{24\pi^2} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} \pi(x) F_{\mu\nu} F_{\rho\sigma}.
(18)

The various contributions in (16) are described as follows. The first term in (17) is the SU(2)_L × SU(2)_R invariant term for the pion fields. $D_\mu = \partial_\mu - ieA_\mu [\hat{Q}, \cdot]$ is the Hermitian covariant derivative with respect to electromagnetic gauge symmetry, with $A_\mu$ being the photon field and $\hat{Q}$ the charge matrix, $\hat{Q} \equiv \frac{1}{2} \tau_3 + \frac{1}{6} \hat{1} = \text{diag}[\frac{2}{3}, -\frac{1}{3}]$. The second term in (17) describes the mass of the pion fields which emerge as a result of explicit breaking of chiral symmetry by the current quark masses, $M_q = \text{diag}[m_u, m_d]$. $f_\pi$ and $m_\pi$ are the pion decay constant and the pion mass respectively, and $\hat{m} = (m_u + m_d)/2$ is the mean quark mass. The part of the total Lagrangian in (18) is the effective term coming from the $U(1)_A$ chiral anomaly \[15\], where $N_c = 3$ is the number of colours. It is easy to see that (18) correctly describes anomalous processes such as $\pi^0 \rightarrow \gamma\gamma$ or the anomalous $\gamma\pi^+\pi^-\pi^0$ vertex.

Here we are interested in excitations of a neutral pion field$^1$, $U(x) = \exp[i\pi(x) \tau^3]$, (19)

$\frac{1}{\sqrt{-g}} L_{\text{chiral}} = \frac{f_\pi^2}{8} g^{\mu\nu} \partial_\mu \pi(x) \partial_\nu \pi(x) + \frac{(f_\pi m_\pi)^2}{4} [\cos(\pi(x)) - 1] - \frac{e^2 N_c}{24\pi^2} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} \pi(x) F_{\mu\nu} F_{\rho\sigma}.
(20)

Since $\pi(x)$ is a pseudoscalar field, the above theory is evidently CP conserving. However, the boundary condition $\lim_{x \rightarrow \infty} |U(x)| = 1$ implies $\lim_{x \rightarrow \infty} \pi(x) = \pi n$, and thus the last term in (20) turns into the Chern–Simons term (13) with $\theta = -\frac{N_c}{2 \pi} n$. Indeed, it is easy to see that the spherically symmetric solution for the pion field goes as $\pi(r) = \pi n + O(1/r)$. Thus, according to (15), the neutral pion field does induce an electric charge, $q_e = +\frac{e}{2\pi} n$, on the black hole [note that we have set $N_c = 3$ here]. Interestingly, in the context of the gauged Skyrme model, the effective $\theta$ is ultimately related to the baryon number \[15\],

$n_B = \frac{eq_m}{2\pi} \left( \pi(\infty) - \pi(r_+) \right),
(21)

and the value of the pion field at the event horizon $r = r_+$, so that

$q_e = \frac{n}{2\pi q_m} \left( n_B + \frac{eq_m}{2\pi} \pi(r_+) \right).
(22)

Charge induced by higher spin fields. As a second example of induced charge on the black hole we consider the model with curvature terms of a higher degree. For simplicity we consider here only quadratic terms in the curvature. The most general gravitational Lagrangian can be written as

$\frac{1}{\sqrt{-g}} L_{\text{high cur}} = a R^2 + b R_{\mu\nu} R^{\mu\nu}.
(23)

The third possible invariant $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ is removed using the Gauss–Bonnet theorem. We also assume the specific relation $4a + b = 0$ between the dimensionless couplings. With this

$^1$ Dvali has pointed out to us that the effect discussed here can be understood in terms of a topologically massive 2-form field dual to the pion field (see e.g. \[16\] for the dual description of the pseudoscalar axion field).
relation we remove an extra scalar degree of freedom which is not essential for our discussion here. Then Lagrangian (23) can be rewritten in an equivalent form expressed in terms of a massive spin 2 field\(^2\) \(\pi_{\mu\nu}(x)\),

\[
\frac{1}{\sqrt{-g}} L_{\text{high cur}} = -\frac{1}{4b} \pi_{\mu\nu} \pi^{\mu\nu} + \pi_{\mu\nu} \Pi^{\mu\nu},
\]

(24)

where \(\Pi_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R\). Using the equation of motion,

\[
\pi_{\mu\nu} = 2b \Pi_{\mu\nu},
\]

(25)

one can easily check that (23) and (24) are indeed equivalent. In the background of Schwarzschild black hole \(R_{\mu\nu} = 0\), and thus we have the trivial solution \(\pi_{\mu\nu}(x) = 0\). This solution reflects the well-known fact that quadratic in curvature terms does not affect the Schwarzschild black hole solution.

Recently, it has been demonstrated in [11] that massive higher spin fields, and in particular massive spin-2 fields, might carry quantum hair in the black hole background. This type of hair is detectable in quantum-mechanical Aharonov–Bohm-type experiments, providing a certain topological interaction of the longitudinal part of the spin-2 field and a string is present. We decompose \(\pi_{\mu\nu}(x)\) into transverse components \(\tilde{\pi}_{\mu\nu}(x)\) (contains two physical states) and the components carrying three massive longitudinal degrees \(B_{\mu}(x)\),

\[
\pi_{\mu\nu} = \tilde{\pi}_{\mu\nu} + \partial_{\mu} B_{\nu} + \partial_{\nu} B_{\mu},
\]

(26)

and allow the coupling [11],

\[
\alpha e^{\mu\nu\rho\sigma} B_{\mu\nu} F_{\rho\sigma},
\]

(27)

where \(\alpha\) is some constant, \(B_{\mu\nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu}\) and \(F_{\mu\nu}\) is the electromagnetic field strength. Note that the reason why we can write this interaction is that (27) is topological and does not affect the dynamics. It is perfectly consistent with the trivial solution \(\pi_{\mu\nu} = 0\) to have a nontrivial magnetic configuration for the 2-form field \(B = B_{\mu\nu} dx^\mu \wedge dx^\nu\),

\[
B = -q_m \sin \theta \, d\theta \wedge d\phi.
\]

(28)

It is clear now that because of (28) and the topological interaction (27) an electric charge \(q_e = 4\alpha q_m\) will be induced on a black hole. Note that \(q_m\) in this case is not of electromagnetic origin. Thus starting from the uncharged Schwarzschild black hole one arrives at the conclusion that it has a charge \(q_e\) which can be detected by measuring the local electrostatic field around the black hole.

4. Conclusion

In this paper we have argued that electric and magnetic charges of a dyonic black hole are quantized in CP conserving theories. If CP is broken the electric charge gets non-integer contribution proportional to the strength of the CP violation. In this case pure magnetic black holes do not exist. We also demonstrate that charge dequantization can be induced by the neutral pion field. Finally, the electric charge can be induced by higher spin fields on an uncharged black hole. In particular, this can happen in a theory with higher curvature terms, provided the topological interactions (27) are present. We find these effects rather amusing.

\(^2\) The massive spin-2 field \(\pi_{\mu\nu}(x)\) mixes with usual gravitons of Einstein theory. The physical massive spin-2 field has a wrong sign kinetic term, i.e. it is the ghost field. This issue is not important for our discussion in what follows.
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