Remarks on generalized Fedosov algebras

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Abstract

The variant of Fedosov construction based on fairly general fiberwise product in the Weyl bundle is studied. Here, we analyze generalized star products of functions, of sections of endomorphisms bundle, and those generating deformed bimodule structure as introduced previously by Waldmann. Isomorphisms of generalized Fedosov algebras are considered and their relevance for deriving Seiberg-Witten map is described. The existence of the trace functional is established. For star products and for the trace functional explicit expressions, up to second power of deformation parameter, are given. As an example, we discuss the case of symmetric part of non-commutativity tensor.

1 Introduction

Fedosov quantization [1], [2] is the beautiful construction providing powerful tools for studying various aspects of deformation quantization on symplectic manifolds. Quite recently, in author’s previous works [3], [4], it was advocated that Fedosov formalism is well suited for building global, geometric field theories on non-commutative spacetimes. The reason for such claim is that Fedosov theory admits generalization producing geometric star products in endomorphism bundles of vector bundles, and this is the place where (some) gauge fields can be put to live. Moreover, there exist convenient isomorphism theory, as well as trace functional construction for Fedosov quantization. These structures turn out to encode local and global versions of Seiberg-Witten map [5], [3], which can be viewed as a quite natural consequence of Fedosov formalism (without postulating it separately). On the other hand, generic Fedosov construction seems to be in some aspects still too rigid for field-theoretic applications. This is because it is “canonical” – taking minimal input of geometric data.
(symplectic form and symplectic connection) it produces simplest geometric deformation quantization. But such canonicality means that the construction “knows nothing” about other structures on underlying manifold, eg. about metric. The aim of the present paper is to introduce into Fedosov formalism fairly general additional degrees of freedom, which can be interpreted in various ways, possibly also in field-theoretic context. This is achieved by modifying core structure of Fedosov construction, namely the star product ◦ in the fibers of tangent bundle. In our setting it could be multiplication different from the Moyal one. There were various approaches investigating some specific non-Moyal fiberwise products, eg. [6], [7], [8], [9]. Here, we especially follow that of [6], but in more general context. The construction is carried out to further point then usually – the isomorphism theory and the trace functional are studied, as well as generalizations to products involving sections of vector and endomorphism bundles, as in [10]. Being primarily interested in further applications, we provide explicit formulas for generalized star-products and trace functional, up to second power of deformation parameter.

The paper is organized as follows. First (section 2), basic structure of generalized Fedosov construction is described. The existence of star-products for factors of various type (function, endomorphism, vector) is established. Then (section 3), the isomorphism theory is analyzed in some detail, as it is indispensable component for deriving Seiberg-Witten relations (discussed in the same section) and for constructing trace functional. The latter one is studied in section 4. As has been mentioned before, the special attention is paid to explicit formulas (including one for the trace functional), which are presented in section 5. It also contains example which illustrates single, very specific interpretation of the present generalization – introduction of symmetric part of non-commutativity tensor. Some concluding remarks are given in section 6.

2 Generalized Fedosov construction

In this section the variant of Fedosov construction is described. The reader interested in studying conceptual structure of Fedosov quantization (both geometric and algebraic) is referenced to [11], [12], [13], [14]. Here, taking original formulation of [2] together with generalization developed in [10], we extend them using methods of [6]. The main concept of this extension can be summarized as follows – replace fiberwise Moyal product ◦ in the Weyl bundle by some other product ∼ ◦, and allow it to vary across the fibers. As all fiberwise products must be equivalent to Moyal product, the required ∼ ◦ can be introduced by choosing an isomorphism g to the standard Moyal algebra. This technique was used in [6] for the description of deformation quantizations originating in different operator orderings. We make use of it to introduce fairly general fiberwise product ∼ ◦.
2.1 Weyl bundles and their basic properties

The initial data for the original Fedosov construction are given by the following structures. Let \((M, \omega, \partial)\) be the Fedosov manifold \([15, 16]\) of dimension \(2n\), for which components of symplectic curvature tensor will be denoted by \(\hat{R}^{i}_{jkl}\). Consider a finite dimensional complex vector bundle \(E\) over \(M\). Let \(\partial E\) be a linear connection in \(E\) and \(\partial \text{End}(E)\) connection induced by \(\partial E\) in endomorphisms bundle \(\text{End}(E)\). We are going to use \(R^{E}_{kl}\) to denote components of curvature of \(\partial E\). They are given by the local formula

\[
R^{E}_{kl} = \partial \Gamma^{E}_{l} \partial x^{k} - \partial \Gamma^{E}_{k} \partial x^{l} + [\Gamma^{E}_{k}, \Gamma^{E}_{l}] \quad \text{for} \quad \partial = d + \Gamma^{E} \quad \text{with} \quad \Gamma^{E}_{i} dx^{i} \text{being local connection 1-form.}
\]

As it will be seen, for the purpose of our generalization some extra fields will be required.

Introduce over \(M\) the Weyl bundle \(W_{C}\). Also, let us have \(W_{E} = W_{C} \otimes E\) and \(W_{\text{End}(E)} = W_{C} \otimes \text{End}(E)\). Sections of \(W_{C} \otimes \Lambda\), \(W_{E} \otimes \Lambda\) and \(W_{\text{End}(E)} \otimes \Lambda\) can be locally written as a formal sums

\[
a(x, y) = \sum_{k, p, q \geq 0} h^{k} a_{i_{1}...i_{p}, j_{1}...j_{q}}(x) y^{i_{1}} ... y^{i_{p}} dx^{j_{1}} \wedge \cdots \wedge dx^{j_{q}}
\]

where \(a_{i_{1}...i_{p}, j_{1}...j_{q}}(x)\) are components of some (respectively) \(C\), \(E\)- or \(\text{End}(E)\)-valued covariant tensor field at \(x \in M\), and \(y \in T_{x}M\). We will repeatedly encounter statements that hold true for all variants of these target spaces. To avoid redundant repetitions let us introduce notation that \(X\) stands for \(C, E\) and \(\text{End}(E)\) if not otherwise restricted.

The “degree counting” combined with “iteration method” are basic tools of Fedosov construction for controlling behavior of formal series \([11]\). Consider monomial in \([11]\) with \(p\)-fold \(y^{i}\) and \(k\)-th power of \(h\). One can prescribe degree to it by the rule

\[
\text{deg}(h^{k} a_{i_{1}...i_{p}, j_{1}...j_{q}}(x) y^{i_{1}} ... y^{i_{p}} dx^{j_{1}} \wedge \cdots \wedge dx^{j_{q}}) = 2k + p
\]

For a general inhomogeneous element of Weyl bundle, the degree is defined as the lowest degree of its nonzero monomials. The iteration method can be described in the following way. Let \(P_{m}\) denote the operator which extracts monomials of degree \(m\) from given \(a\)

\[
P_{m}(a)(x, y) = \sum_{2k + p = m} h^{k} a_{i_{1}...i_{p}, j_{1}...j_{q}}(x) y^{i_{1}} ... y^{i_{p}} dx^{j_{1}} \wedge \cdots \wedge dx^{j_{q}}.
\]

We frequently consider equations of the form

\[
a = b + K(a)
\]

and try to solve them iteratively with respect to \(a\), by putting \(a^{(0)} = b\) and \(a^{(n)} = b + K(a^{(n-1)})\). If \(K\) is linear and raises degree (i.e. \(\text{deg} a < \text{deg} K(a)\) or \(K(a) = 0\)) then one can quickly deduce that the unique solution of \((3)\) is given by the series of relations \(P_{m}(a) = P_{m}(a^{(m)})\). However, in the case of nonlinear \(K\) (as in \([13]\)) the more careful analysis must be performed.
The useful property of Weyl bundles is the existence of global “Poincaré decomposition”. It can be verified that operators $\delta$ and $\delta^{-1}$ defined by relations

\[
\delta a = dx^k \wedge \frac{\partial a}{\partial y^k} \quad \text{and} \quad \delta^{-1}a_{km} = \frac{1}{k+m} y^r \left( \frac{\partial}{\partial x^r} \right) a_{km}
\]

(4)

for $a_{km}$ with $k$-fold $y$ and $m$-fold $dx$ (and by linear extension for inhomogeneous $a$) provide for arbitrary $a \in W_X \otimes \Lambda$ decomposition

\[
a = \delta \delta^{-1}a + \delta^{-1}\delta a + a_{00}
\]

(5)

where $a_{00}$ denotes homogeneous part of $a$ containing no $y^i$ and $dx^j$. Both $\delta$ and $\delta^{-1}$ are nilpotent, i.e., $\delta\delta = \delta^{-1}\delta^{-1} = 0$.

Let $\circ$ denote usual fiberwise Moyal product

\[
a \circ b = \sum_{m=0}^{\infty} \left( -\frac{i\hbar}{2} \right)^m \frac{\partial^m a}{m! \partial y^{i_1} \ldots \partial y^{i_m}} \omega^{i_1j_1} \ldots \omega^{i_mj_m} \frac{\partial^m b}{\partial y^{j_1} \ldots \partial y^{j_m}}
\]

(6)

Here $\omega^{ij}$ are components of the Poisson tensor corresponding to symplectic form $\omega = \frac{1}{2} \omega_{ij} dx^i \wedge dx^j$. Notice that above formula is meaningful not only for $a, b \in W_C \otimes \Lambda$ or $a, b \in W_{End(E)} \otimes \Lambda$. Following [10] we admit case of $a \in W_{End(E)} \otimes \Lambda$, $b \in W_E \otimes \Lambda$ which would provide deformation of action of an endomorphism on a vector, and $a \in W_{End(E)} \otimes \Lambda$, $b \in W_C \otimes \Lambda$ corresponding to deformation of scaling a vector field by a function. For all these cases the Moyal product is associative $(a \circ b) \circ c = a \circ (b \circ c)$, as long as both sides of this relation are well defined. The generalized deformed fiberwise product can be introduced in the following way. Let

\[
g = \text{id} + \sum_{2s-k \geq 0, s, k \geq 0} h^s g_{(s)}^{i_1 \ldots i_k} \frac{\partial^k}{\partial y^{i_1} \ldots \partial y^{i_k}}
\]

(7)

be fiberwise star equivalence isomorphism with $g_{(s)}^{i_1 \ldots i_k}$ being components of some $k$-contravariant $\mathbb{C}$-valued tensors on $M$. Formula (7) implies that $g$ preserves degree of elements of Weyl bundle\(^1\), i.e. $\deg(ga) = \deg(a)$, and that $g$ is formally invertible. Indeed, writing

\[
g^{-1} = \text{id} + \sum_{2s-k \geq 0, s, k \geq 0} h^s g_{(s)}^{i_1 \ldots i_k} \frac{\partial^k}{\partial y^{i_1} \ldots \partial y^{i_k}}
\]

\(^1\)We will also allow case of $a \in W_C \otimes \Lambda$, $b \in W_{End(E)} \otimes \Lambda$ but this fiberwise product does not correspond to deformation of right multiplication of a vector field by a function on the manifold.

\(^2\)Consider arbitrary $a \neq 0$. Let $\pi(a)$ be monomial in $a$ of lowest degree (i.e. $\deg \pi(a) = \deg a$), and with maximal number of $y$'s within this degree. Then it can be easily verified that $\pi(a) = \pi(ga)$ (because of the form of derivatives in (7)), hence $\deg(ga) = \deg(a)$.
and considering relation \( g^{-1}g = \text{id} \) one arrives at the formula

\[
g^{i_1 \ldots i_k}_{(a)} + \tilde{g}^{i_1 \ldots i_k}_{(a)} + \sum_{a-b=2a-k \geq 0 \atop a-s \geq 0 \atop b-k \geq 0} g^{(i_1 \ldots i_k b)}_{(a-s)} g^{(a-s)}_{(a)} = 0 \quad (8)
\]

which allows recursive computation of coefficients \( \tilde{g}^{i_1 \ldots i_k}_{(a)} \). The generalized fiberwise product is defined as

\[
a \tilde{\circ} b = g^{-1}(ga \circ gb) \quad (9)
\]

We are going to do denote graded commutator with respect to \( \tilde{\circ} \) by

\[
[a \tilde{\circ}, b] = a \tilde{\circ} b - (-1)^{r+s}b \tilde{\circ} a \quad (10)
\]

for \( r \)-form \( a \) and \( s \)-form \( b \). Similarly \([a \circ b]\) stands for commutator with respect to \( \circ \). Form of (7) implies that \( g \) commutes with \( \delta \) (ie. \( g\delta = \delta g \)) and consequently \( \delta \) is the +1-derivation with respect to \( \tilde{\circ} \)

\[
\delta(a \tilde{\circ} b) = \delta a \tilde{\circ} b + (-1)^{k}a \tilde{\circ} \delta b \quad (11)
\]

for \( k \)-form \( b \). The operator \( \delta \) can be represented as a commutator with respect to \( \circ \) by \( \delta a = \frac{i}{\hbar}[s \circ a] \) with \( s = -\omega_{ij}y^idy^j \). It follows that it is also commutator with respect to \( \tilde{\circ} \) because

\[
\delta a = g^{-1}\delta ga = \frac{i}{\hbar}g^{-1}[s \circ a] = \frac{i}{\hbar}[\tilde{s} \circ a]
\]

where \( \tilde{s} = g^{-1}s \).

Let us notice that in the case of \( \mathcal{W}_E \) one must deal with following subtlety related to initial non-commutativity of product of endomorphisms. For operators of the form \( K = \frac{1}{\hbar}[s \circ \cdot] \) one cannot use arbitrary \( s \), as this could yield negative powers of \( \hbar \). The only appropriate \( s \in \mathcal{C}^\infty(M, \mathcal{W}_E \otimes \Lambda) \) are those for which monomials of the form \( h_0 a_{i_1 \ldots i_p j_1 \ldots j_q} y^{i_1} \ldots y^{i_p} dy^{j_1} \wedge \ldots \wedge dy^{j_q} \) are defined by central endomorphisms \( a_{i_1 \ldots i_p j_1 \ldots j_q} \). (This statement can be easily verified for Moyal product \( \circ \), and then transported to generalized case by means of \( g \)). Let us call them \(\mathcal{C} \)-sections. Also, let us use the term \(\mathcal{C} \)-operator for mappings which transport \(\mathcal{C} \)-sections to \(\mathcal{C} \)-sections. Obviously \( \delta, \delta^{-1}, g \) and \( g^{-1} \) are \(\mathcal{C} \)-operators. The following lemma is an useful tool for controlling occurrence of negative powers of \( \hbar \).

**Lemma 1.** For arbitrary \(\mathcal{C} \)-section \( s \) the commutator \( \frac{1}{\hbar}[s \circ \cdot] \) is a \(\mathcal{C} \)-operator.

The proof is straightforward for \( \circ \), and using \(\mathcal{C} \)-operator \( g \) one can immediately extend it to generalized product \( \tilde{\circ} \).

We should also mention that the only central elements of \( \mathcal{W}_E \otimes \Lambda \) and \( \mathcal{W} \otimes \Lambda \) are these belonging to \( \Lambda \), ie. scalar forms on base manifold \( \mathcal{M} \). This fact is well known for \( \circ \) and can be trivially transfered to the case of \( \tilde{\circ} \).
Connections $\partial^S$ and $\partial^E$ give rise to the connections $\partial^W_x$ in all variants of Weyl bundle. Let $\Gamma^e_{jk}$ be local connection coefficients of $\partial^S$, and let $\Gamma^E$ be local connection 1-form of $\partial^E$, i.e. locally $\partial^E = d + \Gamma^E$ and $\partial^E = d + [\Gamma^E, \cdot]$. Then locally, in some Darboux coordinates, one can write

$$
\partial^W_x = d + \frac{i}{\hbar} [1/2\Gamma^E_{ijk} y^i y^j dx^k \circ \cdot]
$$

(12a)

$$
\partial^W_x = d + \frac{i}{\hbar} [1/2\Gamma^E_{ijk} y^i y^j dx^k \circ \cdot] + \Gamma^E
$$

(12b)

$$
\partial^{W_{\text{End}(\mathfrak{g})}} = d + \frac{i}{\hbar} [1/2\Gamma^E_{ijk} y^i y^j dx^k - i\hbar \Gamma^E \circ \cdot] \quad (12c)
$$

using local connection coefficients $\Gamma^e_{ijk} = \omega_i \Gamma^e_{jk}$ of $\partial^S$. Notice that if we consider $W_C$ as a subbundle of $W_{\text{End}(\mathfrak{g})}$, then (12a) gives same results as (12a). Connections $\partial^W_x$ are $+1$-derivations for Moyal product, and since we admitted various types of factors for $\circ$, this statement includes all corresponding “compatibilities” (eg. $\partial^W_x (a \circ b) = \partial^W_x (a \circ b + (-1)^k a \circ \partial^W_x b$ for $a \in W_{\text{End}(\mathfrak{g})} \otimes \Lambda^k$ and $b \in W_E \otimes \Lambda$). We need analogous $+1$-derivations for $\tilde{\circ}$, thus let us introduce the generalized connections

$$
\tilde{\partial}^W_x = g^{-1} \partial^W_x g
$$

(13)

for which the relation $\tilde{\partial}^W_x (a \tilde{\circ} b) = \tilde{\partial}^W_x a \tilde{\circ} b + (-1)^k a \tilde{\circ} \tilde{\partial}^W_x b$ holds. In general $\tilde{\partial}^W_x$ cannot be written in form analogous to (12) due to derivatives of fields $\gamma_{im}$. However, for $W_C$ and $W_{\text{End}(\mathfrak{g})}$ from $(\partial^W_x)^2 = \frac{i}{\hbar} [R^W_x \circ \cdot]$ with $R^C = \frac{i}{\hbar} \omega_{im} \tilde{R}^m_{jk} y^i y^j dx^k \wedge dx^l$ and $R^{\text{End}(\mathfrak{g})} = R^C - \frac{i\hbar}{2} R^E_{kl} dx^k \wedge dx^l$ we have

$$
(\tilde{\partial}^W_x)^2 = g^{-1} \partial^W_x \partial^W_x g = \frac{i}{\hbar} g^{-1} [R^W_x \circ \cdot, g(\cdot)] = \frac{i}{\hbar} [\tilde{R}^W_x \circ \cdot]
$$

(14)

for $\tilde{R}^W_x = g^{-1} R^W_x$. Notice that by lemma [4] both $\partial^{W_{\text{End}(\mathfrak{g})}}$ and $\tilde{\partial}^{W_{\text{End}(\mathfrak{g})}}$ are $C$-operators, while $R^{\text{End}(\mathfrak{g})}$ and $\tilde{R}^{\text{End}(\mathfrak{g})}$ are $C$-sections.

2.2 First Fedosov theorem – Abelian connections

One can analyze more general connections in $W_C$ and $W_{\text{End}(\mathfrak{g})}$ of the form

$$
\nabla^W_x = \tilde{\partial}^W_x + \frac{i}{\hbar} [\gamma_X \tilde{\circ} \cdot]
$$

(15)

with $\gamma_X \in C^\infty(\mathcal{M}, W_x \otimes \Lambda^1)$. It follows that $(\nabla^W_x)^2 = \frac{i}{\hbar} [\Omega^W_x \circ \cdot]$ for curvature 2-form defined as

$$
\Omega^W_x = \tilde{R}^W_x + \tilde{\partial}^W_x \gamma_X + \frac{i}{\hbar} \gamma_X \tilde{\circ} \gamma_X
$$

(16)

Abelian (flat) connections in Weyl bundles will be denoted by $D^W_x$. Flatness conditions $(D^W_x)^2 = 0$, $(D^{W_{\text{End}(\mathfrak{g})}})^2 = 0$ imply that for Abelian connections $D^W_C$ and $D^{W_{\text{End}(\mathfrak{g})}}$ their
curvatures must be scalar 2-forms. The first essential element of Fedosov formalism is explicit (although recursive) construction of Abelian connections. In our context this result can be summarized in the following theorem.

**Theorem 2.** Let $\tilde{\partial}^{Wx}$ be arbitrary connections of type $[13]$, let $\kappa \in \mathcal{C}^{\infty}(\mathcal{M}, \Lambda^2)[[h]]$ be formal power series of closed 2-forms (i.e. $d\kappa = 0$), and let

$$\mu^C \in \mathcal{C}^{\infty}(\mathcal{M}, W_C), \quad \mu^{\text{End}(\mathcal{E})} = \mu^C + ih\Delta \mu^{\text{End}(\mathcal{E})} \in \mathcal{C}^{\infty}(\mathcal{M}, W_{\text{End}(\mathcal{E})})$$

be arbitrary $C$-sections such that $\deg \mu^X \geq 3$ and $\mu^X|_{y=0} = 0$. There exist unique Abelian connections

$$D^{W_C} = -\delta + \tilde{\partial}^{W_C} + \frac{i}{h}[r^C \circ \cdot]$$

(17a)

$$D^{W_{\text{End}(\mathcal{E})}} = -\delta + \tilde{\partial}^{W_{\text{End}(\mathcal{E})}} + \frac{i}{h}[r^{\text{End}(\mathcal{E})} \circ \cdot]$$

(17b)

satisfying

- $\Omega^{W_C} = \Omega^{W_{\text{End}(\mathcal{E})}} = -\omega + h\kappa$
- $r^{\text{End}(\mathcal{E})}$ is $C$-section,
- $\delta^{-1}r^X = \mu^X$,
- $\deg r^X \geq 2$.

The 1-forms $r^C \in \mathcal{C}^{\infty}(\mathcal{M}, W_C \otimes \Lambda^1)$, $r^{\text{End}(\mathcal{E})} \in \mathcal{C}^{\infty}(\mathcal{M}, W_{\text{End}(\mathcal{E})} \otimes \Lambda^1)$ can be calculated as the unique solutions of equations

$$r^X = r^X_0 + \delta^{-1}(\tilde{\partial}^{Wx} r^X + \frac{i}{h}[r^C \circ \cdot])$$

(18)

with $r^X_0 = \delta^{-1}(R^X - h\kappa) + \delta\mu^X$. Moreover $D^{W_C}$ and $D^{W_{\text{End}(\mathcal{E})}}$ define Abelian connection

$$D^{W_{\mathcal{E}}} = -\delta + \tilde{\partial}^{W_{\mathcal{E}}} + \frac{i}{h}[r^C \circ \cdot] + \frac{i}{h}r^{\mathcal{E}}$$

(19)

with $\mathcal{C}^{\infty}(\mathcal{M}, W_{\text{End}(\mathcal{E})} \otimes \Lambda^1) \ni r^{\mathcal{E}} = r^{\text{End}(\mathcal{E})} - r^C$ and $\frac{i}{h}r^{\mathcal{E}}$ not containing negative powers of $h$, such that following compatibility conditions hold true

$$D^{W_{\mathcal{E}}}(A \circ X) = (D^{W_{\text{End}(\mathcal{E})}} A) \circ X + (-1)^k A \circ D^{W_{\mathcal{E}}} X$$

(20a)

$$D^{W_{\mathcal{E}}}(X \circ a) = (D^{W_{\mathcal{E}}} X) \circ a + (-1)^l X \circ D^{W_{\mathcal{E}}} a$$

(20b)

for $A \in \mathcal{C}^{\infty}(\mathcal{M}, W_{\text{End}(\mathcal{E})} \otimes \Lambda^k)$, $X \in \mathcal{C}^{\infty}(\mathcal{M}, W_{\mathcal{E}} \otimes \Lambda^l)$ and $a \in \mathcal{C}^{\infty}(\mathcal{M}, W_C \otimes \Lambda)$. 

**Outline of the proof.** The theorem combines results of Fedosov (theorems 5.2.2, 5.3.3 of [2]) and Waldmann (theorem 3 of [10]) with present generalized setting prototyped in [6]. There is nothing substantially new in the proof, thus let us restrict to its key ingredients and few points for which some care due to our generalization should be taken.
Notice first that connections \((17)\) can be written as
\[
D^W_x = \tilde{\partial}^W_x + \frac{i}{h}\tilde{s} + r^X \tilde{\circ} : \cdot
\]
and thus are of type \((15)\). Their curvatures \((16)\) are given by
\[
\Omega^W_x = \tilde{R}^X - \omega - \delta r^X + \tilde{\partial}^W_x r^X + \frac{i}{h} r^X \tilde{\circ} r^X
\]
and the requirement \(\Omega^W_x = -\omega + \hbar \kappa\) yields
\[
\delta r^X = \tilde{R}^X - \hbar \kappa + \tilde{\partial}^W_x r^X + \frac{i}{h} r^X \tilde{\circ} r^X
\]
This formula together with decomposition \((5)\) and condition \(\delta^{-1} r^X = \mu^X\) gives relation \((18)\).

In order to show that unique recursive solution of \((18)\) exists, one could use lemma 5.2.3 of \([2]\). Its proof relies on “degree counting” and stays valid in generalized case. This is because both \(g\) and \(g^{-1}\) preserve degree. In turn \(\tilde{\partial}\) does not lower degree and \(\deg(a \tilde{\circ} b) = \deg(a) + \deg(b)\) if \(a, b \neq 0\). Consequently, whole reasoning of the proof remains intact. To ensure that \(r^X\) is a \(C\)-section one may observe that it is calculated by recursive application of \(\text{id} + \delta^{-1} \left(\tilde{\partial} + \frac{i}{h}(\cdot)^2\right)\) which for 1-forms is a \(C\)-operator. The initial point for this iterative procedure is given by \(\text{End}(E)\)

The immediate consequence of above theorem and lemma \([1]\) is that \(D^W_{\text{End}(E)}\) is a \(C\)-operator.
2.3 Second Fedosov theorem – star products

Abelian connections allow to construct nontrivial liftings of functions, sections of \( \mathcal{E} \) and sections of \( \text{End}(\mathcal{E}) \) to sections of corresponding Weyl bundles. The key point is that these liftings form subalgebra of algebra of all sections. In this way desired star products are constructed – we take two objects we want to multiply, lift them to Weyl bundles, multiply liftings using \( \sim \circ \) and project the result back to the appropriate space of functions or sections of suitable bundle.

More precisely, let us call \( a \in C^\infty(\mathcal{M}, W_X) \) flat if \( DW_X a = 0 \). Flat sections form subalgebra of \( C^\infty(\mathcal{M}, W_X) \) (this is obvious consequence of Leibniz rule and \( (DW_X)^2 = 0 \)) denoted by \( W^D_X \). Let \( Q_X(a) \) be the solution of equation

\[
Q_X(a) = a + \delta^{-1}(DW_X + \delta)a.
\]

with respect to \( b \). The iteration method ensures that \( Q_X : C^\infty(\mathcal{M}, W_X) \to C^\infty(\mathcal{M}, W_X) \) is well-defined linear bijection. It follows that the inverse mapping is given by \( Q_X^{-1}a = a - \delta^{-1}(DW_X + \delta)a \). The following theorem holds.

**Theorem 3.** \( Q_X \) bijectively maps \( C^\infty(\mathcal{M}, \mathcal{X})[[h]] \) to \( W^D_X \).

This is just theorem 5.2.4 of [2], combined with extensions of [10] and phrased in our generalized context. We omit the proof, as it does not require any changes comparing to original formulation.

Now let us define all variants of generalized Fedosov product. The second Fedosov theorem, properties of \( \sim \) and \( DW_X \) (notice importance of compatibility conditions (20) for (25c) and (25d)) yield that

\[
f \ast g = Q_X^{-1}(Q_X f \sim Q_X g)
\]

for \( f, g \in C^\infty(\mathcal{M}, \mathcal{C})[[h]] \)

\[
A \ast B = Q^{-1}_{\text{End}(\mathcal{E})}(Q_{\text{End}(\mathcal{E})}A \sim Q_{\text{End}(\mathcal{E})}B)
\]

for \( A, B \in C^\infty(\mathcal{M}, \text{End}(\mathcal{E}))[h] \)

\[
A \ast X = Q^{-1}_{\mathcal{E}}(Q_{\text{End}(\mathcal{E})}A \sim Q_{\mathcal{E}}X)
\]

for \( A \in C^\infty(\mathcal{M}, \text{End}(\mathcal{E}))[h] \), \( X \in C^\infty(\mathcal{M}, \mathcal{E})[[h]] \)

\[
X \ast f = Q^{-1}_{\mathcal{E}}(Q_{\mathcal{E}}X \sim Q_{\mathcal{E}}f)
\]

for \( X \in C^\infty(\mathcal{M}, \mathcal{E})[[h]] \), \( f \in C^\infty(\mathcal{M}, \mathcal{C})[[h]] \)

are associative (in all meaningful ways) star products.

Writing decomposition (5) for \( (DW_X + \delta)a \) and using \( (DW_X)^2 = \delta^2 = 0 \) one can derive identity

\[
Q_X^{-1}DW_X + \delta Q_X^{-1} = 0
\]

It follows that corollary 5.2.6 of [2] holds true in our context.

**Theorem 4.** For given \( b \in C^\infty(\mathcal{M}, W_X \otimes \Lambda^p) \), \( p > 0 \) equation \( D^{W_X} a = b \) has a solution if and only if \( D^{W_X} b = 0 \). The solution may be chosen in the form \( a = -Q_X \delta^{-1}b \).
3 Isomorphisms of generalized Fedosov algebras

Now it is time to analyze isomorphisms of generalized Fedosov algebras. Again, the original presentation of [2] is followed, supplemented with some necessary additions and modifications. The material is presented in rather detailed way. This is because we want to provide solid basis for further statements on Seiberg-Witten map and the trace functional. Here, bundles $W_{\text{End} (\mathcal{E})}$ and $W_C$ are dealt with, thus we restrict $\mathcal{X}$ to denote $\mathcal{C}$ or $\text{End} (\mathcal{E})$ within this section.

Let $g_t$ be homotopy of isomorphisms of type (7) parametrized by $t \in [0, 1]$. Corresponding homotopy of deformed fiberwise products will be denoted by $\tilde{\gamma}_t$. We would like to study evolution with respect to parameter $t$, and for this purpose let us introduce

$$\Delta_t = g_t^{-1} \frac{d}{dt} g_t = \frac{d}{dt} + g_t^{-1} \frac{dg_t}{dt} \tag{26}$$

It can be immediately observed that $\Delta_t$ is $0$-derivation with respect to $\tilde{\gamma}_t$

$$\Delta_t (a \tilde{\gamma}_t b) = \Delta_t a \tilde{\gamma}_t b + a \tilde{\gamma}_t \Delta_t b \tag{27}$$

Let us also introduce homotopies of connections $\tilde{\partial}_t^{\mathcal{S}}$, $\partial_t^{\mathcal{E}}$ and corresponding homotopies of connections in Weyl bundles, i.e. $\tilde{\partial}_t^{W_X}$ and $\tilde{\partial}_t^{W_X} = g_t^{-1} \partial_t^{W_X} g_t$. Connections (12a) and (12b) can be rewritten in the uniform manner as $\partial_t^{W_X} = d + \frac{i}{\hbar} [\Gamma_t^{W_X} \tilde{\gamma}_t , \cdot ]$ with $\Gamma_t^{W_E} = 1/2 \Gamma_{ijk}(t) y^i y^j dx^k$ and $\Gamma_t^{W_{\text{End} (\mathcal{E})}} = 1/2 \Gamma_{ijk}(t) y^i y^j dx^k + i b \Gamma_{ij}^{\mathcal{E}} (t)$. Denoting $\tilde{\Gamma}_t^{W_X} = g_t^{-1} \Gamma_t^{W_X}$ one may check that

$$\Delta_t \tilde{\partial}_t^{W_X} a = \tilde{\partial}_t^{W_X} \Delta_t a + \frac{i}{\hbar} [\Delta_t \tilde{\Gamma}_t^{W_X} \tilde{\gamma}_t , a] \tag{28}$$

Notice that unlike connection coefficients, $\frac{d}{dt} \Gamma_t^{W_X}$ define global, coordinate and frame independent sections of Weyl bundles. In turn the same stays true for $\Delta_t \tilde{\Gamma}_t^{W_X}$. The following theorem (5.4.3 of [2]) builds isomorphism theory for Fedosov algebras.

**Theorem 5.** Let

$$D_t^{W_X} = \tilde{\partial}_t^{W_X} + \frac{i}{\hbar} [\gamma_{\mathcal{X}}(t) \tilde{\gamma}_t , \cdot ]$$

be a homotopy of Abelian connections parameterized by $t \in [0, 1]$, and let $H_{\mathcal{X}}(t)$ be a $t$-dependent $C$-section of $W_{\mathcal{X}}$ (called Hamiltonian) satisfying $\deg(H_{\mathcal{X}}(t)) \geq 3$ and such that

$$D_t^{W_X} H_{\mathcal{X}}(t) = \Delta_t (\tilde{\Gamma}_t^{W_X} + \gamma_{\mathcal{X}}(t)) = \lambda(t) \tag{29}$$

for some scalar $1$-form $\lambda(t)$. Then, the equation

$$\Delta_t a + \frac{i}{\hbar} [H_{\mathcal{X}}(t) \tilde{\gamma}_t , a] = 0 \tag{30}$$

has the unique solution $a(t)$ for any given $a(0) \in W_{\mathcal{X}} \otimes \Lambda$ and the mapping $a(0) \mapsto a(t)$ is isomorphism for any $t \in [0, 1]$. Moreover, $a(0) \in W_{\mathcal{X}}^{D_0}$ if and only if $a(t) \in W_{\mathcal{X}}^{D_t}$. 

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Proof. Using definition of $\Delta_t$ and integrating (30) one obtains
\[
a(t) = g_t^{-1} g_0 a(0) - \frac{i}{\hbar} g_t^{-1} \int_0^t g_\tau [H_X(\tau) a(\tau)] d\tau \quad (31)
\]
The operator $\frac{i}{\hbar} g_t^{-1} \int_0^t g_\tau [H_X(\tau) a(\tau)]$ raises degree if $\deg (H_X(t)) \geq 3$, and defines unique iterative solution $a(t)$. Thus, the mapping $a(0) \mapsto a(t)$ is indeed bijective. From Leibniz rule we infer that $a(0) \mapsto a(t)$ is indeed bijective. From Leibniz rule we infer that $a(0) \mapsto a(t)$ holds if and only if $a(t) = 0$.

The very first observation concerning theorem 5 is that there is some consistency condition for $\lambda(t)$. Indeed, calculating $\frac{d}{dt} H_X(t)$ one obtains from Abelian property of $D_t^{Wx}$ and (29)
\[
0 = \frac{d}{dt} H_X(t) = D_t^{Wx} \Delta_t (\gamma_X(t)) + d\lambda(t) \quad (32)
\]
Straightforward but a bit longish calculation shows that
\[
D_t^{Wx} \Delta_t (\gamma_X(t)) = \Delta_t \Omega_t^{Wx} = \frac{d}{dt} \Omega_t^{Wx} \quad (33)
\]
Hence
\[
d\lambda(t) = -\frac{d}{dt} \Omega_t^{Wx} \quad (34)
\]
This observation results in the following theorem.

**Theorem 6.** Let $D_t^{Wx}$ be homotopy of Abelian connections

\[
D_t^{Wx} = \tilde{\partial}_t^{Wx} + \frac{i}{\hbar} [\gamma_X(t) \tilde{\gamma}_t \cdot]
\]

A Hamiltonian generating isomorphism between corresponding algebras of flat sections exists iff there is some scalar 1-form $\lambda(t)$ such that $d\lambda(t) = -\frac{d}{dt} \Omega_t^{Wx}$.

Proof. If a Hamiltonian exists, then consistency condition (34) holds as calculated before. Conversely if there is $\lambda(t)$ satisfying (34) then the Hamiltonian can be constructed as a solution of equation
\[
D_t^{Wx} H_X(t) = \Delta_t (\gamma_X(t)) + \lambda(t)
\]
Due to theorem 4 such solution exists because $D_t^{W_X} \left( \Delta_t (\tilde{\Gamma}^{W_X} + \gamma_X(t)) + \lambda(t) \right) = 0$ by relations (33) and (34). Thus, the Hamiltonian can be taken as

$$H_X(t) = -Q_X t \delta^{-1} \left( \Delta_t (\tilde{\Gamma}^{W_X} + \gamma_X(t)) + \lambda(t) \right)$$

where $Q_X$ denotes quantization map corresponding to $D_t^{W_X}$.

**Remark 1.** The immediate consequence of this theorem is that algebras generated by Abelian connections described in theorem 2 are isomorphic in the sense of theorem 5, if and only if their curvatures are in the same cohomology class. Indeed, if they are isomorphic then condition (34) holds, which, after integration, yields that the curvatures must be cohomological. On the other hand, given two sets of input data for theorem 2 (ie. fiberwise product, symplectic connection, connection in the bundle, normalizing section $\mu_X$ and curvature), we can always homotopically transform one to another and, in turn, obtain homotopy of Abelian connections. Thus, the only obstruction is the requirement (34), which can be satisfied for curvatures in the same cohomology class by homotopy $\Omega^{W_X} = \Omega^{W_X}_0 + t(\Omega^{W_X}_1 - \Omega^{W_X}_0)$.

Since locally one can always find $\lambda(t)$ such that $\frac{d}{dt} \Omega^{W_X} = -d\lambda(t)$, then locally all Abelian connections of theorem 2 give rise to isomorphic algebras. In particular, all such algebras are locally isomorphic to the trivial algebra, ie. Moyal algebra, which is generated in theorem 4 by fiberwise Moyal product, $\kappa = 0$, $\mu^X = 0$ and flat connections $\partial^S$, $\partial^E$. The useful consequence of this fact is the following observation (compare [2], corollary 5.5.2).

**Lemma 7.** If $[s \tilde{\circ}, a] = 0$ for all $a \in W^D_X$, then $s$ is some scalar form.

This fact can be easily verified for the trivial algebra. As a local statement it can be transported to arbitrary $W^D_X$, because scalar forms remain unmodified by considered isomorphisms.

The next step is to observe that the mapping introduced in theorem 5 can be written in a bit more explicit form as

$$T_t(a(0)) := a(t) = U^{-1}_X(t) \tilde{\circ}_t (g_t^{-1} g_0 a(0)) \tilde{\circ}_t U_X(t)$$

where $U_X(t)$ is the solution of the equation

$$\Delta_t U_X(t) = \frac{i}{\hbar} U_X(t) \tilde{\circ}_t H_X(t)$$

with $U_X(0) = 1$. Notice that (36) can be rewritten as

$$U_X(t) = 1 + \frac{i}{\hbar} g_t^{-1} \int_0^t g_t(U_X(\tau) \tilde{\circ}_\tau H_X(\tau)) d\tau$$

and by iteration method it follows that $U_X(t)$ is defined uniquely. However, the operator
\[ \frac{1}{\hbar} g_{-1} \int_0^1 g_r (\cdot) \tilde{\omega}_t \ H_X(\tau) d\tau \] may produce negative powers of \( \hbar \), but these terms do not introduce negative total degree, and appear in finite number for each total degree, thus we avoid problem of infinite series of monomials at fixed total degree. The inverse \( U_X^{-1}(t) \) is taken with respect to \( \tilde{\omega}_t \) and it is uniquely determined by the equation

\[ \Delta_t U_X^{-1}(t) = -\frac{i}{\hbar} H_X(t) \tilde{\omega}_t U_X^{-1}(t) \]  (38)

Notice that composing two mappings of type (35) one obtains a mapping of the same type, ie. for \( U_X \) and \( U_X' \) defining respectively mappings from \( \tilde{\omega}_0 \) to \( \tilde{\omega}_1 \) and from \( \tilde{\omega}_1 \) to \( \tilde{\omega}_2 \) their composition yields

\[ U_X'^{-1} \sim_2 \left( g_1^{-1} g_1 \left( U_X^{-1} \sim_1 \left( g_1^{-1} g_0 a \right) \sim_1 U_X \right) \right) \sim_2 U_X' = (U_X')^{-1} \sim_2 \left( g_2^{-1} g_0 a \right) \sim_2 U_X'' \]  (39)

with \( U_X'' = \left( g_2^{-1} g_1 U_X \right) \sim_2 U_X' \) describing mapping from \( \tilde{\omega}_0 \) to \( \tilde{\omega}_2 \) and \( (U_X'')^{-1} \) being inverse with respect to \( \tilde{\omega}_2 \). It follows that the inverse mapping \( T_t^{-1} \) can be written as

\[ T_t^{-1}(a) = \left( g_0^{-1} g_1 U_X(t) \right) \sim_0 \left( g_0^{-1} g_0 a \right) \sim_0 \left( g_0^{-1} g_1 U_X^{-1}(t) \right) \]  (40)

where \( U_X(t) \) and \( U_X^{-1}(t) \) are the same quantities that in (35).

**Remark 2.** For technical reasons which become clear later, we would like to emphasize the following fact. Let \( T_t \) be homotopy of isomorphisms generated by homotopy \( D_t^{W_X} \), Hamiltonian \( H_X(t) \) and scalar form \( \lambda(t) \). The inverse \( T_t^{-1} \) of an “endpoint” \( T_1 \) can be always represented as an “endpoint” \( T_1' = T_1^{-1} \) of homotopy \( T_t' \) generated by \( D_t^{W_X} = D_t^{W_X'} \), Hamiltonian \( H_X'(t) = -H_X(1 - t) \) and \( \lambda'(t) = -\lambda(1 - t) \). To verify that \( T_t' \) is well defined, we consequently put \( \Gamma_1 = \Gamma_1' \), \( \gamma_X'(t) = \gamma_X(1 - t) \), \( g_t' = g_1 - t \), \( \Delta_t' = g_t'^{-1} g_t \) and let \( \tilde{\omega}_t \) stand for \( \tilde{\omega}_1 - t \). Then \( D_t^{W_X} H_X'(t) = \Delta_t \Gamma_1 + \gamma_X'(t) \) holds as a result of analogous relation for \( T_1 \). Moreover, one can observe that \( U_X'(t) = \left( g_t'^{-1} g_1 U_X^{-1}(1) \right) \tilde{\omega}_1 U_X(1 - t) \) is the solution of \( \Delta_t U_X'(t) = \frac{i}{\hbar} U_X'(t) \tilde{\omega}_t H_X(t) \), satisfying \( U_X'(0) = 1 \), provided that \( U_X(t) \) is the solution of corresponding equation for \( T_1 \). Hence, by (40), the relation \( T_t'(a) = U_X'^{-1}(1) \tilde{\omega}_1 \left( g_0^{-1} g_0 a \right) \tilde{\omega}_1 U_X(1) = \left( g_0^{-1} g_1 U_X(1) \right) \tilde{\omega}_0 \left( g_0^{-1} g_0 a \right) \tilde{\omega}_0 \left( g_0^{-1} g_1 U_X^{-1}(1) \right) = T_t^{-1}(a) \) indeed holds.

Using isomorphism \( T_t' \) one can push-forward arbitrary connection \( \nabla^{W_X} \) (defined for fiber-wise product \( \tilde{\omega}_0 \))

\[ T_t' \nabla^{W_X} = T_t \nabla^{W_X} T_t^{-1} \]  (41)

3Strictly, this means that \( U_X(t) \) belongs to what is called extended Weyl bundle in [2]. As the notation of the present paper seems to be cumbersome enough without precise dealing with this issue, we are not going to introduce any separate symbol for those extended bundles.

4Here \( U_X^{-1}(1) \) is inverse with respect to \( \tilde{\omega}_1 \), ie. \( \tilde{\omega}_0 \), while \( U_X^{-1}(1) \) is inverse for \( \tilde{\omega}_1 \).
The immediate observation coming directly from the definition and properties of $T_t$ is that $T_t\nabla^{W_x}$ is $+1$-derivation with respect to $\tilde{\sigma}$, and also an Abelian connection if $\nabla^{W_x}$ is such. Inserting (35) and (10) into (11) one can compute that

$$T_t\nabla^{W_x} = g_t^{-1}g_0\nabla^{W_x}g_0^{-1}g_t + [U^{-1}_X(t)\tilde{\sigma}_t (g_t^{-1}g_0\nabla^{W_x}g_0^{-1}g_tU_X(t)) \tilde{\sigma}_t] \cdot$$

Let $\tilde{d}_t = g_t^{-1}dg_t$ and let

$$\nabla^{W_x} = \tilde{d}_0 + \frac{i}{h}[\nabla^{W_x} \tilde{\sigma}_0 \cdot]$$

be local representation of $\nabla^{W_x}$. Since

$$g_t^{-1}g_0\nabla^{W_x}g_0^{-1}g_t = \tilde{d}_t + \frac{i}{h}[g_t^{-1}g_0\nabla^{W_x} \tilde{\sigma}_t \cdot]$$

it is possible to write down the following local formula

$$T_t\nabla^{W_x} = \tilde{d}_t + \frac{i}{h}[g_t^{-1}g_0\nabla^{W_x} - ihU^{-1}_X(t)\tilde{\sigma}_t (g_t^{-1}g_0\nabla^{W_x}g_0^{-1}g_tU_X(t)) \tilde{\sigma}_t] \cdot$$

and we are justified to define local push-forward of $\nabla^{W_x}$

$$T_t^*\nabla^{W_x} = g_t^{-1}g_0\nabla^{W_x} - ihU^{-1}_X(t)\tilde{\sigma}_t (g_t^{-1}g_0\nabla^{W_x}g_0^{-1}g_tU_X(t))$$

Suppose that $T$ maps form $\tilde{\sigma}_0$ to $\tilde{\sigma}_1$ and $T'$ maps form $\tilde{\sigma}_1$ to $\tilde{\sigma}_2$. Then

$$T_t^*T_s\nabla^{W_x} = (T'T)_s\nabla^{W_x}$$

and also, after some calculations, one may check that

$$T_t^*T_s\nabla^{W_x} = (T'T)_s\nabla^{W_x}$$

Clearly, we are able to rewrite Abelian connections $D_0^{W_x}$ in the local form $D_t^{W_x} = \tilde{d}_t + \frac{i}{h}[\tilde{\nabla}_t^{W_x} + \gamma_X(t) \tilde{\sigma}_t \cdot]$. With above notations an useful lemma on push-forward of $D_0^{W_x}$ can be formulated (previously considered in [5]).

**Lemma 8.** Connections $T_t^*D_0^{W_x}$ and $D_t^{W_x}$ coincide. Moreover if $T_t$ is generated by Hamiltonian satisfying $D_t^{W_x}H_X(t) - \Delta_t(\tilde{\nabla}_t^{W_x} + \gamma_X(t)) = \lambda(t)$ with scalar 1-form $\lambda(t)$ then

$$T_t\left(\tilde{\nabla}_0^{W_x} + \gamma_X(0)\right) = \tilde{\nabla}_t^{W_x} + \gamma_X(t) + \int_0^t \lambda(\tau)d\tau$$

**Proof.** From theorem it follows that subalgebras of flat sections are the same for both considered connections, i.e. $W_\Delta^{D_1} = W_\Delta^{T_tD_0}$. Hence, for arbitrary $a \in W_\Delta^{D_1}$ it holds that $T_t^*D_0^{W_x}a = D_t^{W_x}a = 0$. Locally this relation yields

$$[T_t\left(\tilde{\nabla}_0^{W_x} + \gamma_X(0)\right) \tilde{\sigma}_0 a] = [\tilde{\nabla}_t^{W_x} + \gamma_X(t) \tilde{\sigma}_t a]$$
for all \( a \in W^D_X \). Thus, by lemma 4, \( T_{t*}(\hat{\Gamma}^W_X + \gamma_X(0)) \) and \( \hat{\Gamma}^W_X + \gamma_X(t) \) differ by some scalar form, and consequently \( T_{t*}D^W_0 a = D^W_t a \) for arbitrary, not necessarily flat, section \( a \).

Using (36), (38) and (42) we calculate

\[
\Delta t T_{t*} \left( \hat{\Gamma}^W_0 + \gamma_X(0) \right) = -i h \Delta t U^{-1}_X(t) \tilde{\gamma}_t (g_t^{-1} g_0 D^W_0 g_0^{-1} g_t \Delta t U_X(t)) - i h U^{-1}_X(t) \tilde{\gamma}_t (g_t^{-1} g_0 D^W_0 g_0^{-1} g_t \Delta t U_X(t))
\]

\[
= [U^{-1}_X(t) \tilde{\gamma}_t (g_t^{-1} g_0 D^W_0 g_0^{-1} g_t \Delta t U_X(t))] - \gamma_X(t) + g_t^{-1} g_0 \Delta t U^W_X + g_t \lambda(t)
\]

and the formula (49) can be obtained by application of \( g_t^{-1} \int_0^t g_\tau \cdot d\tau \) to above relation.

Now we are about formulating quite important fact concerning automorphisms of Fedosov algebras. (The lemma given below is essentially generalized variant of proposition 5.5.5 of [2]. Here, we reproduce it with different proof, which employs lemma 8 and explicitly relates function \( f \), used for rescaling, to 1-forms \( \lambda^{(i)} \) corresponding to involved isomorphisms).

**Lemma 9.** Let \( D^W_0, \ldots, D^W_{n-1} \) be Abelian connections defined with respect to fiberwise products \( \tilde{\gamma}_0, \ldots, \tilde{\gamma}_{n-1} \) for \( n \geq 1 \). Consider homotopies of isomorphisms \( T^{(i)}_t \), such that \( T^{(1)}_t : W^D_X \to W^D_X, T^{(2)}_t : W^D_X \to W^D_X, \ldots, T^{(n)}_t : W^D_X \to W^D_X \). Their composition \( T^{(n)}_t \ldots T^{(1)}_t \) can always locally represented as

\[
T^{(n)}_t \ldots T^{(1)}_t a = V^{-1}_X \tilde{\gamma}_0 a \tilde{\gamma}_0 V_X
\]

for arbitrary \( a \in W_X \), and with \( V_X \) being flat section belonging to \( W^D_X \).

**Proof.** For sake of more compact notation let us define \( D^W_X = D^W_0 \) and let \( \tilde{\gamma}_n \) stand for \( \tilde{\gamma}_0 \). To construct \( V_X \), we proceed as follows. Isomorphisms under consideration can be written as \( T^{(i)}_t a = (U^{(i)}_X)^{-1} \tilde{\gamma}_i a \tilde{\gamma}_i U^{(i)}_X \). For their composition formula (39) yields

\[
T^{(n)}_t \ldots T^{(1)}_t a = (U^{(n)}_X)^{-1} \tilde{\gamma}_0 a \tilde{\gamma}_0 (U^{(n)}_X)
\]

We are going to show that \( U_X \) can be locally rescaled by some scalar function and become a flat section with respect to \( D^W_0 X \). For this purpose lemma 8 can be used. First, one immediately gets that \( (T^{(i)}_t \ldots T^{(1)}_t)_* D^W_X = D^W_X \). A bit less straightforward relation holds for local coefficients of these connections. Let \( D^W_X = \hat{\Delta}_t + \frac{i}{h} \tilde{\gamma}_t \hat{\Gamma}^W_0 + \gamma_X(t) \tilde{\gamma}_t \). Using definition 46 we obtain

\[
\left( T^{(n)}_t \ldots T^{(1)}_t \right)_* \left( \hat{\Gamma}^W_0 + \gamma_X(0) \right) = \hat{\Gamma}^W_0 + \gamma_X(0) - i h U^{-1}_X(0) \tilde{\gamma}_0 (D^W_X U_X)
\]

On the other hand, since each isomorphism \( T^{(i)}_t \) is generated by some Hamiltonian \( H_X(t) \) satisfying condition 20, with some scalar 1-form \( \lambda^{(i)}(t) \), lemma 8 can be consecutively
applied
\[
T_{1}^{(n)} \ldots T_{1}^{(2)} T_{1}^{(1)} \left( \hat{T}_{0}^{W_{X}} + \gamma X_{0} \right) = T_{1}^{(n)} \ldots T_{1}^{(2)} \left( \hat{T}_{0}^{W_{X}} + \gamma X_{1} + \int_{0}^{1} \lambda^{(1)}(\tau)d\tau \right)
\]
\[
= \ldots = T_{1}^{(n)} \left( \hat{T}_{n-1}^{W_{X}} + \gamma X_{n-1} + \int_{0}^{1} \lambda^{(1)}(\tau)d\tau + \ldots + \int_{0}^{1} \lambda^{(n-1)}(\tau)d\tau \right)
\]
\[
= \hat{T}_{0}^{W_{X}} + \gamma X_{0} + \int_{0}^{1} \lambda^{(1)}(\tau)d\tau + \ldots + \int_{0}^{1} \lambda^{(n)}(\tau)d\tau \quad (53)
\]
In virtue of (48) we can combine (52) with (53) and obtain
\[- \frac{i}{\hbar} U_{X}^{-1} \sim_{0} (D_{0}^{W_{X}} U_{X}) = \int_{0}^{1} \lambda^{(1)}(\tau)d\tau + \ldots + \int_{0}^{1} \lambda^{(n)}(\tau)d\tau \quad (54)
\]
The right hand side of above formula is a closed 1-form. Indeed, integration of relation (44) for each \(\lambda^{(i)}\) gives
\[d \int_{0}^{1} \lambda^{(i)}(\tau)d\tau = -\Omega_{W_{X}}^{W_{X}} + \Omega_{W_{X}}^{W_{X}}\]
Summing all these terms produces
\[d \left( \int_{0}^{1} \lambda^{(1)}(\tau)d\tau + \ldots + \int_{0}^{1} \lambda^{(n)}(\tau)d\tau \right) = 0 \quad (55)
\]
because \(\Omega_{W_{X}}^{W_{X}} = 0\). Thus (54) can be always locally rewritten as
\[D_{0}^{W_{X}} U_{X} = \frac{i}{\hbar} U_{X} df \quad (56)
\]
for some scalar function \(f\), such that \(df = \int_{0}^{1} \lambda^{(1)}(\tau)d\tau + \ldots + \int_{0}^{1} \lambda^{(n)}(\tau)d\tau\). Finally, we define \(V_{X} = e^{-\frac{i}{\hbar} f} U_{X}\). From (56) it follows that \(D_{0}^{W_{X}} V_{X} = 0\), while
\[V_{X}^{-1} \sim_{a} \sim_{a} V_{X} = U_{X}^{-1} \sim_{a} \sim_{a} U_{X} = T_{1}^{(n)} \ldots T_{1}^{(1)} a \quad (57)
\]
Let us additionally observe, that – by remark [2] – in above lemma one (or more) isomorphism \(T_{1}^{(i)} : W_{X_{i+1}}^{D_{i}} \rightarrow W_{X_{i}}^{D_{i}}\) can be safely replaced by some \(T_{1}^{(i)}\), provided that
\(T_{1}^{(i)} : W_{X_{i}}^{D_{i}} \rightarrow W_{X_{i}}^{D_{i}}\).

\[5\]We should comment on the lack of negative powers of \(h\) in the term \(e^{-\frac{i}{\hbar} f}\). This fact comes quite easily from conditions \(\deg(H_{X}(t)) \geq 3\) and (29). Indeed, analyzing right hand side of relation \(\lambda(t) = D_{1}^{W_{X}} H_{X}(t) - \Delta_{i}(\hat{T}_{i}^{W_{X}} + \gamma X_{i}(t))\) we observe that the term \(\Delta_{i}(\hat{T}_{i}^{W_{X}} + \gamma X_{i}(t))\) does not introduce monomials of degree 0 if \(\deg(\gamma X_{i}(t)) \geq 1\). (We want \(\gamma X_{i}(t)\) to be a \(C\)-section. Hence, there could be only scalar terms of degree 0, but such terms does not affect \(D_{1}^{W_{X}}\) and can be safely omitted). The same stays true for the term \(D_{1}^{W_{X}} H_{X}(t)\) provided that \(\deg(H_{X}(t)) \geq 3\). Thus \(\lambda(t)\) must be a scalar form of degree 2 or greater and consequently \(f\) can be taken without zeroth power of \(h\).
3.1 Seiberg-Witten map

As a quite direct application of above considerations on isomorphisms of Fedosov algebras one can make the following observation concerning Seiberg-Witten map [17], [18], [5]. Suppose that there is some fiberwise adjoint Lie group action in the endomorphism bundle, realized by invertible elements of the fiber, i.e. one is dealing with $T_x : \text{End}(\mathcal{E})_x \to \text{End}(\mathcal{E})_x$ given smoothly for all $x \in \mathcal{M}$ and acting by

$$T_x(A) = gAg^{-1}$$

for both $A$ and $g$ belonging to the fiber $\text{End}(\mathcal{E})_x$. Action of $T_x$ can be extended naturally to the Weyl bundle $W_{\text{End}(\mathcal{E})}$. Let $D^W_{\text{End}(\mathcal{E})}$ be arbitrary Abelian connection with curvature $\Omega^W_{\text{End}(\mathcal{E})}$, and let $D^W_{\text{End}(\mathcal{E})}$ be some flat Abelian connection with the same curvature, but originating in flat connection $D^F_{\mathcal{E}}$ in the bundle $\mathcal{E}$. Due to Fedosov isomorphism theory, one is able to establish local isomorphism $T$ mapping $D^W_{\text{End}(\mathcal{E})}$ to $D^W_{\text{End}(\mathcal{E})}$, with $\lambda(t) = 0$ in theorem [5]. On the other hand, using gauge transformation $T_x$, the Abelian connection

$$D^W_{\text{End}(\mathcal{E})}' = T_x(T^W_{\text{End}(\mathcal{E})}) = T_x(D^W_{\text{End}(\mathcal{E})}T^{-1})$$

is obtained. It can be easily observed, that if $D^W_{\text{End}(\mathcal{E})}$ comes from theorem [2] for connection $\partial^W_{\text{End}(\mathcal{E})}$ and section $\mu^W_{\text{End}(\mathcal{E})}$, then $D^W_{\text{End}(\mathcal{E})}'$ also originates in this theorem with $\Omega^W_{\text{End}(\mathcal{E})}' = \Omega^W_{\text{End}(\mathcal{E})}$, but for the gauge transformed objects $\partial^W_{\text{End}(\mathcal{E})}' = T_x(\partial^W_{\text{End}(\mathcal{E})})$, $T_x(\mu^W_{\text{End}(\mathcal{E})})^{-1}$ and $\mu^W_{\text{End}(\mathcal{E})}' = T_x(\mu^W_{\text{End}(\mathcal{E})})$. For $D^W_{\text{End}(\mathcal{E})}'$ one can also set up local isomorphism $T'$ mapping $D^W_{\text{End}(\mathcal{E})}$ to $D^W_{\text{End}(\mathcal{E})}'$, again with $\lambda'(t) = 0$. Then the following question arises: how $T$ and $T'$ are related to each other?

To answer it one can use, mutatis mutandis, lemma [3]. Consider composition $T^*T_xT^{-1}$ which maps $W_{\text{End}(\mathcal{E})}$ to itself. We need to represent $T_x$ as an endpoint of some homotopy of isomorphisms. Clearly, this cannot be homotopy generated strictly by theorem [5]. Notice however that if $\mathfrak{g}$ belongs to connected component of identity, i.e. it can be written as $\mathfrak{g} = e^\alpha$ for some $\alpha \in C^\infty(\mathcal{M}, \text{End}(\mathcal{E}))$, then one can introduce homotopy $T_x(t)$ which acts on sections of $W_{\text{End}(\mathcal{E})}$ by $T_x(t)(\sigma) = e^{\alpha t}e^{-\alpha t}$. (Of course, the fiberwise product $\circ$ is constant with respect to $t$ here). This homotopy is generated by Hamiltonian $H = i\hbar\alpha$ of degree 2 and equation $\Delta \alpha + \frac{\hbar}{2}\{H, \alpha\} = 0$ in analytic (not formal) sense. The relation (29) holds true with $D^W_{\text{End}(\mathcal{E})} = D^W_{\text{End}(\mathcal{E})} + [e^{\alpha t}D^W_{\text{End}(\mathcal{E})}e^{-\alpha t}, \cdot]$ and $\lambda(t) = 0$. Also, the formula (49) remains valid, thus we are able to repeat reasoning of lemma [3] with $T_x(t)$ put in place of some $T_x(t)$. In particular, the local relation

$$T^s(T_x(t))T^{-1}a = V^{-1}_{\text{End}(\mathcal{E})} \circ a \circ V_{\text{End}(\mathcal{E})}$$

holds true for $V_{\text{End}(\mathcal{E})} \in W_{\text{End}(\mathcal{E})}^D$ and arbitrary $a \in W_x$. This however implies that

$$T^s(T_x(t))a = V^{-1}_{\text{End}(\mathcal{E})} \circ T a \circ V_{\text{End}(\mathcal{E})}$$

(59)
Projecting back to $\text{End}(E)$ and using relation $T(g)Q_{\text{End}}(E) = Q'_{\text{End}}(E)\bar{T}(g)$ one obtains for $A \in C^\infty(M, \text{End}(E))$ and $A' = \bar{T}(g)A = gAg^{-1}$

$$M'(A') = G \ast_T M(A) \ast_T G^{-1} \quad (60)$$

where $M = Q_T^{-1}TQ_{\text{End}}(E)$, $M' = Q_T^{-1}T'Q'_{\text{End}}(E)$ and $G = Q_T^{-1}V_{\text{End}}^{-1}$ with $Q_{\text{End}}(E), Q'_{\text{End}}(E), Q_T\text{End}(E)$ corresponding to $D^{W_{\text{End}}(E)}, D^{W_{\text{End}}(E)}', \text{and} D^T_{\text{End}}(E)$ respectively. But (60) says that the covariance relation for map $M$ is exactly the one which should occur for Seiberg-Witten map applied on endomorphisms. Indeed, if one chooses frame in $E$ for which connection coefficients of $\partial_ET$ vanish, then $\ast_T$ acts on matrices representing endomorphisms in this frame according to the usual “row-column” matrix multiplication rule, but with commutative product of entries replaced by non-commutative Fedosov star product of functions (which can be chosen to be Moyal product for flat $\partial_S$). Notice also, that the leading term of $G$ is given by $g$ as it comes from our application of modified lemma 9. Thus, one is able to rewrite (60) in more conventional notation as

$$\hat{A}' = \hat{g} \ast_T \hat{A} \ast_T \hat{g}^{-1} \quad (61)$$

with hat denoting appropriate Seiberg-Witten map. One can reproduce usual Seiberg-Witten equations for gauge objects (gauge potential and field strength) from formula (60). For this kind of considerations, as well as examples of explicit calculations of Seiberg-Witten map, the reader is referenced to [5]. Here, let us close this subsection with the statement that in our generalized context, Seiberg-Witten map again appeared as a local isomorphism, which is just some property of global Fedosov quantization of endomorphism bundle.

## 4 Trace functional

We are ready to define trace functional for generalized Fedosov algebras. Fortunately, we already know that the trace exists for the case of fiberwise Moyal product, ie. for the case of original formulation of [2]. This fact can be used to avoid difficulties related to gluing together local trivializations. However, the isomorphisms theory of previous section remains crucial. Thus, in what follows the term “isomorphism” will always refer to isomorphism described by theorem 5. Again, in this section $X$ stands for $C$ or $\text{End}(E)$.

For arbitrary Abelian connection $D^{W_X}$ generated by theorem 2 with fiberwise product $\tilde{o}$, symplectic connection $\partial^S$, connection in the bundle $\partial^F$, normalizing section $\mu^X$ and curvature $\Omega^{W_X}$, let $D^{W_X}_F$ denote Abelian connection obtained for fiberwise Moyal product $o$ and remaining data unchanged. Thus $D^{W_X}_F$ is Abelian connection in the sense of [2], generated by the first Fedosov theorem, and $Q_XF$ is corresponding quantization map. For a compactly supported section $a \in W^{D_F}_X$ its Fedosov trace will be denoted $\text{tr}_F a$. Recall
that for the trivial (i.e. Moyal) algebras the trace is just integral\(^6\) \[
\text{tr}_M a = \int_{\mathbb{R}^n} Q_{X_M}(a) \frac{d^n}{dn!}.
\]
For general case one introduces local isomorphisms \(T_i\) to trivial algebra and corresponding partition of unity \(\rho_i\). The trace is then defined as \(\text{tr}_F a = \sum_i \text{tr}_M T_i(Q_{X_F}(\rho_i) \circ a)\). One can show that such definition depends neither on the choice of \(T_i\) nor \(\rho_i\). The important properties of the trace \(\text{tr}_F\) are that
\[
\text{tr}_F(a \circ b) = \text{tr}_F(b \circ a)
\]
for all \(a, b \in W^{D_F}_X\), and that
\[
\text{tr}'_F T a = \text{tr}_F a
\]
provided that \(T\) maps one Fedosov algebra \(W^{D_F}_X\) with trace \(\text{tr}_F\), to another \(W^{D'_F}_X\) with trace \(\text{tr}'_F\). Notice however, that as to this point \(T\) must denote isomorphism which results from theorem \(5\) with constant homotopy of fiberwise Moyal products. This is because \(63\) has been proven in \([2]\) only in such context. We are interested in properties of the trace under present, broader class of isomorphisms. With following lemma all the required relations can be easily obtained.

**Lemma 10.** Let \(D^{W_F}_X, D^{W_1}_X, \ldots, D^{W_{n-1}}_X\) be Abelian connections defined with respect to fiberwise products \(\circ, \circ_1, \ldots, \circ_{n-1}\) for \(n \geq 1\). Consider homotopies of isomorphisms \(T^{(i)}_1\), such that
\[
T^{(1)}_1 : W^{D_F}_X \rightarrow W^{D_1}_X, T^{(2)}_1 : W^{D_1}_X \rightarrow W^{D_2}_X, \ldots, T^{(n)}_1 : W^{D_{n-1}}_X \rightarrow W^{D_F}_X.
\]
For their composition \(T^{(n)}_1 \ldots T^{(1)}_1\) and compactly supported \(a \in W^{D_F}_X\) the global relation
\[
\text{tr}_F T^{(n)}_1 \ldots T^{(1)}_1 a = \text{tr}_F a
\]
holds.

**Proof.** Clearly, \(T^{(n)}_1 \ldots T^{(1)}_1\) maps \(W^{D_F}_X\) to \(W^{D_F}_X\), hence lemma \([9]\) can be applied. Let \(\{O_i\}\) be covering of the support of section \(a\) (by compactness one can make this covering finite), such that in each \(O_i\) lemma \([7]\) holds with some \(V_{X_i} \in W^{D_F}_X\), and choose some compatible partition of unity \(\sum_i \rho_i = 1\). Consequently \(\sum_i Q_{X_F}(\rho_i) = 1\). Then
\[
\text{tr}_F T^{(n)}_1 \ldots T^{(1)}_1 a = \text{tr}_F T^{(n)}_1 \ldots T^{(1)}_1 \left( \sum_i Q_{X_F}(\rho_i) \circ a \right)
= \sum_i \text{tr}_F \left( V_{X_i}^{-1} \circ Q_{X_F}(\rho_i) \circ a \circ V_{X_i} \right) = \sum_i \text{tr}_F \left( Q_{X_F}(\rho_i) \circ a \right) = \text{tr}_F a
\]
where linearity of \(\text{tr}_F\) and property \((62)\) have been used. \(\square\)

Again, by remark \([5]\) above lemma covers also the case of appropriate inverses put in place of some isomorphisms \(T^{(i)}_1\)’s. Now, let \(T\) be “endpoint” of some homotopy of our generalized

\(^6\)Here we omit normalizing constant \((2\pi h)^{-n}\) in front of integral, as compared to \([2]\).
isomorphisms (i.e. $\sim_t$ does not have to be equal to $\circ$ for each $t$) mapping $W^D_{\mathcal{X}}$ with trace $\text{tr}_F$, to $W'^D_{\mathcal{X}}$ with trace $\text{tr}'_F$. Then, there must exist isomorphism $T': W^D_{\mathcal{X}} \rightarrow W'^D_{\mathcal{X}}$ being “endpoint” of homotopy for which $\sim_t$ is constantly equal to Moyal product. Applying (63) for $T'$ and lemma 10 for $T'^{-1}T$ one obtains

$$\text{tr}'_F T' a = \text{tr}'_F T'^{-1}T a = \text{tr}'_F T' T^{-1}T a = \text{tr}'_F a$$

Thus, relation (63) holds true for generalized isomorphisms.

**Definition 11.** For arbitrary $W^D_{\mathcal{X}}$, corresponding to Abelian connection $D^{W_{\mathcal{X}}}$ of theorem 3, the trace of compactly supported $a \in W^D_{\mathcal{X}}$ is given by

$$\text{tr} a = \text{tr}_F T(a)$$

where $T$ is arbitrary isomorphism mapping $W^D_{\mathcal{X}}$ to $W'^D_{\mathcal{X}}$.

First, algebras $W^D_{\mathcal{X}}$ and $W'^D_{\mathcal{X}}$ are indeed isomorphic, as guaranteed by remark 1. We should also check that above definition does not depend on particular choice of isomorphism $T$. Let $T'$ be some other isomorphism mapping $W^D_{\mathcal{X}}$ to $W'^D_{\mathcal{X}}$. Clearly, $T'T^{-1}$ maps $W^D_{\mathcal{X}}$ to $W'^D_{\mathcal{X}}$, hence lemma 11 can be applied (together with remark 2) yielding

$$\text{tr}'_F T'(a) = \text{tr}'_F T'^{-1}T(a) = \text{tr}'_F T(a)$$

Similar argument can be employed to show that relation (63) can be carried over to generalized trace. Indeed, consider arbitrary isomorphism $T_X$ mapping $W^D_{\mathcal{X}}$ with trace defined by $\text{tr} a = \text{tr}_F T(a)$ to $W'^D_{\mathcal{X}}$ with trace $\text{tr}' a = \text{tr}'_F T'(a)$. Since $W^D_{\mathcal{X}}$ and $W'^D_{\mathcal{X}}$ are isomorphic, then also $W^D_{\mathcal{X}}$ and $W'^D_{\mathcal{X}}$ are isomorphic, because, by definition, connections $D^{W_{\mathcal{X}}}$ and $D'^{W_{\mathcal{X}}}$ inherit curvatures from their generalized counterparts. Thus remark 1 can be applied, and let $T_F : W^D_{\mathcal{X}} \rightarrow W'^D_{\mathcal{X}}$ be corresponding isomorphism. Then

$$\text{tr}' T_X a = \text{tr}'_F T'_F T_X T a = \text{tr}'_F T'_F T'^{-1}T' T_X T^{-1}T a = \text{tr}'_F T'^{-1}T' T_X T^{-1}T a = \text{tr}_F T a = \text{tr} a$$

where we have first used (63) for $T_F$, and then lemma 11 for $T'^{-1}T' T_X T^{-1}$. Finally, the relation

$$\text{tr}(a \sim b) = \text{tr}(b \sim a)$$

for compactly supported sections $a, b \in W^D_{\mathcal{X}}$ is the direct consequence of (62).

---

7 This is because $D^{W_{\mathcal{X}}}_F$ and $D'^{W_{\mathcal{X}}}_F$ are generated by some input data for theorem 2 but both with fiberwise Moyal product. We can homotopically transform one set of such input data into another without modifying fiberwise product and generate homotopy of Abelian connections, each with respect to $\circ$. Then, by theorem 6 the compatible Hamiltonian must exists since we already know that condition (64) can be fulfilled due to existence of $T$.  

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5 Some explicit expressions

Our goal for this section is to give explicit expressions, up to second power of deformation parameter, for generalized star products and trace functional. They appear to be quite bulky, and calculations leading to them are cumbersome. In fact, it would be overwhelmingly tedious task to perform them manually. Instead, Mathematica system has been used together with useful tensor manipulation package – xAct [21]. The interested reader can find the Mathematica file relevant for these calculations on the author’s website [20]. This file also contains “intermediate” structures (Abelian connection, lifting to Weyl bundle, trivialization isomorphism), as well as some tests for validity of obtained results (including associativity of star product and isomorphicity of trivialization). The results are given for arbitrary fiberwise isomorphism, as well as some tests for validity of obtained results (including associativity of star product and isomorphicity of trivialization). The results are given for arbitrary fiberwise isomorphism [7], but are restricted to the case of vanishing normalizing section $\mu^X = 0$, and curvature corrections $\kappa = 0$. The main reason for this is that for more general cases of $\mu^X$, the calculations become too time-consuming, even for computer algebra system. (Probably, some optimization of code developed in [20] could be helpful in overcoming this obstacle).

Also, we have decided to use already known [19], [3] formula for trace, and this formula was derived for $\kappa = 0$. The interested reader can find the Mathematica xAct [21]. The interested reader can find the Mathematica file relevant for these calculations on the author’s website [20]. This file also contains “intermediate” structures (Abelian connection, lifting to Weyl bundle, trivialization isomorphism), as well as some tests for validity of obtained results (including associativity of star product and isomorphicity of trivialization). The results are given for arbitrary fiberwise isomorphism, as well as some tests for validity of obtained results (including associativity of star product and isomorphicity of trivialization). The results are given for arbitrary fiberwise isomorphism [7], but are restricted to the case of vanishing normalizing section $\mu^X = 0$, and curvature corrections $\kappa = 0$. The main reason for this is that for more general cases of $\mu^X$, the calculations become too time-consuming, even for computer algebra system. (Probably, some optimization of code developed in [20] could be helpful in overcoming this obstacle).

Thus, after all calculations, the following formula can be derived for the generalized Fedosov product of endomorphisms $A, B \in C^\infty(\mathcal{M}, \text{End}(\mathcal{E})[[\hbar]]$. (Here $\partial$ stands for covariant derivative which combines $\partial^S$ with $\partial^E$, i.e. acts by means of $\partial^S$ on (co)tangent space and by $\partial^\text{End}(\mathcal{E})$ on endomorphisms. The shortened notation $\omega_{im}\tilde{R}_{ijkl} = \tilde{R}_{ijkl}$ is also used).

\[
A \ast B = AB - \frac{1}{2}h \left( \omega^{ab} - 4ig^{ab}_{(1)} \right) \partial_a A \partial_b B + h^2 \left( 3 \left( g^{pr}_{(1)} g^{qr}_{(1)} - g^{pr}_{(2)} \right) \left( \partial_p A \partial_q \partial_r B + \partial_p \partial_q A \partial_r B \right) + \left( g^{pr}_{(1)} g^{qr}_{(1)} - 2g^{pr}_{(2)} \right) \left( \frac{1}{3} \tilde{R}_{rsab} + \frac{7}{6} R_{sab} \right) + \frac{1}{2} \tilde{R}_{sabr} \omega^{ab} \partial^c \partial^r \partial^c \partial^r \right) + R_{sabr} \partial^c \partial^r \partial^c \partial^r B + \frac{1}{8} \left( \omega^{pa} - 4ig^{pa}_{(1)} \right) \left( \omega^{qr} - 4ig^{qr}_{(1)} \right) \partial_p A \partial_q B R^c_{rs} + \frac{1}{4} \left( \omega^{pa} - 4ig^{pa}_{(1)} \right) \left( \omega^{qr} - 4ig^{qr}_{(1)} \right) \partial_p A R^c_{rs} B \\
+ \frac{1}{4} \left( \omega^{pa} - 4ig^{pa}_{(1)} \right) \left( \omega^{qr} - 4ig^{qr}_{(1)} \right) \partial_p A \partial_r B + \frac{1}{8} \left( \omega^{pa} - 4ig^{pa}_{(1)} \right) \left( \omega^{qr} - 4ig^{qr}_{(1)} \right) \partial_p A \partial_r B + \frac{1}{8} \left( \omega^{pa} - 4ig^{pa}_{(1)} \right) \left( \omega^{qr} - 4ig^{qr}_{(1)} \right) \partial_p A \partial_r B + O(h^3) \right. \]

It is straightforward to obtain formula for star product of functions from above expression – one has to put $R^c_{ij} \equiv 0$ and interpret $\partial$ as symplectic connection $\partial^S$. Hence, let us focus on $A \ast X$ and $X \ast f$, with $X \in C^\infty(\mathcal{M}, \mathcal{E})[[\hbar]]$, $f \in C^\infty(\mathcal{M}, \mathbb{C})[[\hbar]]$. Now, $\partial$ acts by means of
\( \partial E \) on \( X \). It turns out that using (25) one gets relations which are formally very similar to that for \( A \star B \). In fact, the formula for \( A \star X \) looks like if \( B \) was mechanically replaced by \( X \) in (71), and than all meaningless terms (involving expressions like “\( \partial_p A \partial_q X R^{E}_{rs} \)”) were dropped. The analogous statement holds also for \( X \star f \).

\[
A \star X = AX - \frac{i}{2} h \left( \omega^{ab} - 4i g_{ab} \right) \partial_a A \partial_b X + h^2 \left( 3 \left( g_{(1)}^{p} g_{(1)}^{q} - g_{(2)}^{p} \right) \left( \partial_p A \partial_q \partial_{
abla} X + \partial_p \partial_q A \partial_{
abla} X + \partial_p \partial_q A \partial_{
abla} X \right) \right.
\]

\[
+ \left( g_{(1)}^{p} g_{(1)}^{q} - 2 g_{(2)}^{p} \right) \left( g_{(1)}^{p} g_{(1)}^{a} \omega^{pb} + g_{(1)}^{p} g_{(1)}^{a} \omega^{pb} \right) \left( \frac{1}{3} \tilde{R}_{rsab} + \frac{7}{6} \tilde{R}_{rsab} \right) + \frac{i}{3} g_{(1)}^{p} \tilde{R}_{rsab} \omega^{pq} \omega^{rb} \right.
\]

\[
+ \tilde{R}_{sarb} \left( g_{(2)}^{p} \omega^{pr} + g_{(2)}^{p} \omega^{pr} \right) + \frac{i}{2} \left( \omega^{qr} + 4i q_{(1)}^{r} \right) \partial_q g_{(1)}^{p} - \frac{1}{2} \left( \omega^{qr} - 4i q_{(1)}^{r} \right) \partial_q g_{(1)}^{p} \partial_q A \partial_{
abla} X
\]

\[
+ \frac{1}{4} \left( \omega^{ps} - 4i q_{(1)}^{r} \right) \left( \omega^{qr} + 4i q_{(1)}^{r} \right) \partial_q A \partial_{
abla} X - \left( \omega^{ps} - 4i q_{(1)}^{r} \right) \partial_q A \partial_{
abla} X
\]

\[
+ \left( 3 g_{(1)}^{p} g_{(1)}^{q} - 6 g_{(2)}^{p} \right) \left( \omega^{qr} - 4i q_{(1)}^{r} \right) \partial_q A \partial_{
abla} X + \left( 2 g_{(1)}^{p} g_{(1)}^{q} - 4 g_{(2)}^{p} \right) \partial_q A \partial_{
abla} X + O(h^3) \quad (72)
\]

\[
X \star f = X f - \frac{i}{2} h \left( \omega^{ab} - 4i g_{ab} \right) \partial_a X \partial_b f + h^2 \left( 3 \left( g_{(1)}^{p} g_{(1)}^{q} - g_{(2)}^{p} \right) \left( \partial_p X \partial_q \partial_{
abla} f + \partial_p \partial_q X \partial_{
abla} f \right) \right.
\]

\[
+ \left( g_{(1)}^{p} g_{(1)}^{q} - 2 g_{(2)}^{p} \right) \left( g_{(1)}^{p} g_{(1)}^{a} \omega^{pb} + g_{(1)}^{p} g_{(1)}^{a} \omega^{pb} \right) \left( \frac{1}{3} \tilde{R}_{rsab} + \frac{7}{6} \tilde{R}_{rsab} \right) + \frac{i}{3} g_{(1)}^{p} \tilde{R}_{rsab} \omega^{pq} \omega^{rb} \right.
\]

\[
+ \tilde{R}_{sarb} \left( g_{(2)}^{p} \omega^{pr} + g_{(2)}^{p} \omega^{pr} \right) + \frac{i}{2} \left( \omega^{qr} + 4i q_{(1)}^{r} \right) \partial_q g_{(1)}^{p} - \frac{1}{2} \left( \omega^{qr} - 4i q_{(1)}^{r} \right) \partial_q g_{(1)}^{p} \partial_q X
\]

\[
+ \frac{1}{4} \left( \omega^{ps} - 4i q_{(1)}^{r} \right) \left( \omega^{qr} + 4i q_{(1)}^{r} \right) \partial_q X
\]

\[
\left. \partial_q f + \frac{i}{2} g_{(1)}^{p} \partial_q g_{(1)}^{p} \left( \omega^{qr} + 4i q_{(1)}^{r} \right) \partial_q X \partial_{
abla} f - \left( \omega^{ps} - 4i q_{(1)}^{r} \right) \partial_q X \partial_{
abla} f \right)
\]

\[
+ \left( 3 g_{(1)}^{p} g_{(1)}^{q} - 6 g_{(2)}^{p} \right) \left( \omega^{qr} - 4i q_{(1)}^{r} \right) \partial_q X \partial_{
abla} f + \left( 2 g_{(1)}^{p} g_{(1)}^{q} - 4 g_{(2)}^{p} \right) \partial_q X \partial_{
abla} f + O(h^3) \quad (73)
\]

In order to calculate the generalized trace functional, one could use definition (77). It requires computation of isomorphism \( T : W^{D}_A \rightarrow W^{D}_{E} \) and this is again laborious task, which should be gladly given to some computer algebra system. As in the case of star products we refer interested reader to the Mathematica file [20] for details. Here only the
final result is presented.

\[
\text{tr} Q_{\text{End} (\xi)} (A) = \int_M \text{Tr} \left( A + h \left( \frac{1}{2} R^\xi_{ab} \omega^{ab} - \partial_a g_{(1)}^{\alpha} + \partial_b \partial_a g_{(1)}^{\alpha} \right) + h^2 \left( -\frac{3}{8} R^\xi_{ab} R^\xi_{cd} \omega^{[ab} \omega^{cd]} \right) \right)
\]

\[
= \frac{1}{2} \omega^{bc} \partial_a \left( \tilde{g}^{ab} (1) R^\xi_{bc} \right) - i \omega^{bc} \partial_c \left( \tilde{g}^{ab} (1) R^\xi_{bc} \right) + \frac{1}{2} \omega^{cd} \partial_b \partial_a \left( \tilde{g}^{ab} (1) R^\xi_{cd} \right) + \frac{i}{4} \omega^{cd} \partial_b \partial_a \left( \tilde{g}^{ab} (1) R^\xi_{cd} \right)
\]

\[
- \frac{1}{2} \omega^{bc} \partial_a \left( g^{ab} (1) R^\xi_{bc} \right) + \frac{3i}{4} \omega^{cd} \partial_b \partial_a \left( g^{ab} (1) R^\xi_{cd} \right) + \frac{1}{48} \omega^{ab} \omega^{bc} \omega^{de} \partial_a \partial_b \partial_c \partial_d \tilde{R}_{acep} + \frac{1}{64} \tilde{R}^a_{\text{acq}} \tilde{R}^a_{\text{dpe}} \omega^{[eq}_{,pr]} + \partial_b \partial_a \left( \tilde{g}^{cd} \tilde{R}_{\text{cdac}} \omega^{ab} + \tilde{g}^{ab} \tilde{g}^{cd} \left( \frac{3}{2} \tilde{R}_{\text{acde}} + \frac{1}{2} \tilde{R}_{\text{eadc}} \right) \omega^{ep} \right)
\]

\[
- \omega^{bc} \partial_c \left( \tilde{g}^{ab} (1) \tilde{R}_{abcd} + \tilde{R}_{bced} \right) + \tilde{g}^{bc} \tilde{R}_{abcd} - \omega^{ep} \partial_a \left( \tilde{g}^{cd} \partial_b \tilde{R}_{bcea} + \tilde{g}^{ab} \tilde{g}^{cd} \partial_b \tilde{R}_{nabc} \right)
\]

\[
+ \omega^{cd} \partial_a \left( \tilde{g}^{ab} (1) \left( \frac{2}{3} \tilde{R}_{\text{acdp}} \partial_d g_{(1)}^{ep} + \frac{1}{2} \tilde{R}_{\text{acdp}} \partial_a g_{(1)}^{ep} \right) \right) - \partial_a g_{(2)}^{ab} - \partial_a \left( \tilde{g}^{ab} (1) \partial_a \tilde{g}_{(1)}^{bc} \right)
\]

\[
= \omega^{bc} \partial_c \left( \tilde{g}^{ab} (1) \partial_a \tilde{g}_{(1)}^{bc} \right) + \partial_a \partial_c \left( \tilde{g}^{ab} (1) \partial_a \tilde{g}_{(1)}^{bc} + 2 g^{bc} \partial_a \tilde{g}_{(1)}^{ab} \right) - \frac{1}{3} \partial_b \partial_a \partial_c \left( 2 \tilde{g}^{ab} \partial_a g_{(1)}^{cd} + 4 \tilde{g}^{ab} \partial_a \tilde{g}_{(1)}^{cd} \right)
\]

\[
+ \partial_a \partial_c \left( \tilde{g}^{ab} (1) \partial_a \tilde{g}_{(1)}^{cd} \right) + \partial_a \partial_b \partial_a g_{(2)}^{bc} - \partial_a \partial_b \partial_a g_{(2)}^{bc} + \partial_a \partial_b \partial_a g_{(2)}^{bc} \right) A + O(h^3) \frac{\omega^n}{n!} \quad (74)
\]

As in the case of star product, the trace of function can be obtained from right hand side of above formula, by dropping terms involving curvature \( R^E_{ij} \), and replacing \( A \) by function.

### 5.1 Example – symmetric part of non-commutativity tensor

From the general structure of definition \([7]\) of \( g \) it follows that our setting covers the case of “Moyal product with nonsymmetric \( \omega^{ij} \)”. More precisely, if one demands that \( \tilde{g} \) is given by \([6]\) with \( \omega^{ij} \) replaced by \( m^{ij} = \omega^{ij} + g^{ij} \) for antisymmetric (and with symplectic inverse) \( \omega^{ij} \) and symmetric \( g^{ij} \), then \( g \) can be taken as

\[
g = \exp \left( \frac{i h}{4} g^{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} \right) \quad (75)
\]

This means that, up to \( h^2 \), the only nonzero coefficients \( g^{ij}_{(1)} \) are

\[
g^{ij}_{(1)} = \frac{i}{4} g^{ij} \quad \text{and} \quad g^{ij}_{(2)} = -\frac{1}{32} g^{(ij)} g^{kl} \quad (76)
\]

Two cases can be considered.
5.1.1 Case of arbitrary symplectic connection

If there are no further assumptions about $\partial^S$, then after substitution (79), and due to symmetries of $\tilde{R}_{ijkl}$, the formula (71) can be brought to the following form.

\[
A * B = AB - \frac{1}{2} h (\omega^{ab} + g^{ab}) \partial_a \partial_b B + h^2 \left( - \frac{1}{8} g^{rs} \tilde{R}_{rs}^{ab} (\omega^{(p} g^{q)b} + \omega^{bq} \omega^{rh}) \partial_p \partial_q B + \right.
\]
\[
+ \frac{1}{8} (\omega^{ps} + g^{ps}) (\omega^{qr} + g^{qr}) \partial_p \partial_q B \tilde{R}_{rs}^e + \frac{1}{4} (\omega^{ps} + g^{ps}) (\omega^{qr} - g^{qr}) \partial_p AB \tilde{R}_{rs}^e \partial_q B
\]
\[
+ \frac{1}{8} (\omega^{ps} - g^{ps}) (\omega^{qr} - g^{qr}) \tilde{R}_s^e \partial_p \partial_q B - \frac{1}{8} \partial_a g^{qr} \left( (\omega^{ps} - g^{ps}) \partial_r \partial_S A \partial_p B - (\omega^{ps} + g^{ps}) \partial_p \partial_q B \right)
\]
\[
- \frac{1}{8} (\omega^{ps} + g^{ps}) (\omega^{qr} + g^{qr}) \partial_p \partial_q A \partial_r \partial_s B \right) + O(h^3) \quad (77)
\]

The trace functional reads now

\[
\text{tr} Q_{\text{Eud}(\xi)}(A) = \int_M \text{Tr} \left( A + \frac{ih}{2} \left( R_{ab}^{\xi} \omega^{ab} + \frac{1}{2} \partial_b \partial_a g^{ab} \right) A + h^2 \left( - \frac{3}{8} R_{ab}^{\xi} R_{cd}^{\xi} \omega^{[ab,\omega,cd]} \right.ight.
\]
\[
- \frac{1}{8} \omega^{cd} \partial_b \partial_a (g^{ab} R_{cd}^{\xi}) - \frac{1}{16} \omega^{cd} \partial_b \partial_a (g^{ab} R_{ac}^{\xi}) + \frac{1}{4} \omega^{cd} \partial_d (g^{ab} \partial_b R_{ac}^{\xi})
\]
\[
- \frac{3}{16} \omega^{cd} \partial_d \partial_a (g^{ab} R_{ac}^{\xi}) + \frac{1}{48} \omega^{cd} \omega^{ep} \omega^{ab} \partial_d \partial_b R_{acep} + \frac{1}{64} \tilde{R}_{ac}^{\xi} \tilde{R}_{cp}^{\xi} \omega^{[c,\omega,pr]} \n
\]
\[
- \frac{1}{32} \partial_b \partial_p \left( g^{(cd} g^{ep)} \tilde{R}_{cdace} \omega^{ab} + g^{ab} g^{cd} \left( 3 \tilde{R}_{acde} + \tilde{R}_{cdae} \right) \omega^{cp} \right)
\]
\[
+ \frac{1}{32} \omega^{cd} \partial_d \left( g^{(cb} g^{ad)} \partial_b \tilde{R}_{ace} + 2 g^{ab} g^{cd} \partial_d \tilde{R}_{ace} \right)
\]
\[
- \frac{1}{16} \omega^{cd} \partial_d \left( g^{ab} \left( \frac{2}{3} \tilde{R}_{ace} \partial_b g^{cp} + \frac{1}{2} \tilde{R}_{acep} \partial_c g^{dp} \right) \right) + \frac{1}{48} \partial_b \partial_a \partial_c (2 g^{ab} g^{cd} + 4 g^{ac} g^{bd})
\]
\[
- \frac{1}{16} \partial_a \partial_c \partial_b \partial_d \left( g^{ab} \partial_b \partial_a g^{cd} \right) - \frac{1}{32} \partial_a \partial_b \partial_c \partial_d \left( g^{(ab} g^{cd)} \right) A + O(h^3) \right) \left. \omega^n \right/ n! \quad (78)
\]

5.1.2 Case of symplectic connection preserving $g^{ij}$

It can be furthermore demanded that symplectic connection preserves $g^{ij}$, ie. $\partial^S g^{ij} = 0$. Notice, that in this specific way we reproduce one of generalizations considered in [23] and also cover Wick-type setting of [8]. It is assumed there, that fiberwise product is given by Moyal formula with nonsymmetric non-commutativity tensor $m^{ij} = \omega^{ij} + g^{ij}$, where $\omega^{ij}$ is nondegenerate, and that torsionfree connection in the tangent bundle preserves $m^{ij}$. This however implies, that the connection preserves separately $\omega^{ij}$ and $g^{ij}$, and that inverse of $\omega^{ij}$ is symplectic.

With such additional assumption we are able to simplify a bit the formula for the star
product of endomorphisms

\[ A \ast B = AB - \frac{i}{2} h (\omega^{ab} + g^{ab}) \partial_a A \partial_b B + h^2 \left( -\frac{1}{8} g^{rs} \tilde{R}_{rs} \left( \omega^{ab} g^{ph} + \omega^{ph} \omega^{ab} \right) \partial_p A \partial_q B \right. \\
+ \frac{1}{8} (\omega^{ps} + g^{ps}) (\omega^{qr} + g^{qr}) \partial_p A \partial_q B R^E_{rs} + \frac{1}{4} (\omega^{ps} + g^{ps}) (\omega^{qr} - g^{qr}) \partial_p A R^E_{rs} \partial_q B \\
+ \frac{1}{8} (\omega^{ps} - g^{ps}) (\omega^{qr} - g^{qr}) R^E_{rs} \partial_p A \partial_q B - \frac{1}{8} (\omega^{ps} + g^{ps}) (\omega^{qr} + g^{qr}) \partial_p A \partial_q B \partial_n \partial_\mu \frac{1}{16} \partial_n \partial_\mu S_{abc} \right) = O(h^3) \]

(79)

and to bring the trace to somehow more friendly form

\[ \text{tr} Q_{\text{End}(\mathcal{E})}(A) = \int_M \text{Tr} \left( A + \frac{i h}{2} R^E_{ab} \omega^{ab} A + h^2 \left( -\frac{3}{8} R^E_{ab} R^E_{cd} \omega^{ab} \omega^{cd} \right. \\
- g^{ab} \omega^{cd} \left( \frac{1}{8} \partial_a \partial_b R^E_{cd} + \frac{1}{16} \partial_a \partial_b R^E_{ac} - \frac{1}{16} \partial_a \partial_b R^E_{ac} \right) \right. \\
+ \frac{1}{48} g^{ab} \omega^{cd} \omega^{ep} \partial_a \partial_b \tilde{R}_{acpe} + \frac{1}{64} \hat{R}_{aceq} \tilde{R}_{dpe} \omega^{eq} A \right) \right) + O(h^3) \]

(80)

6 Final remarks

The general scheme for our approach (as well as for \cite{6}) was to establish isomorphism to standard fiberwise Moyal product, and then to pull-back almost all structures required for Fedosov construction. One can immediately observe, that if we pull-back exactly everything, then the result is strictly trivial – the star products are precisely those of generic Fedosov quantization. The key point is that we do not modify mappings \( \delta \) and \( \delta^{-1} \) leaving “Poincare decomposition” untouched. However, the \( \delta \) operator should remain +1-derivation for the new fiberwise product \( \tilde{\delta} \). This can be achieved by imposing condition \( g\delta = \delta g \), and definition (7) is consistent with it. (The same condition causes the relation \( \delta \tilde{\omega}^{Wx} + \tilde{\delta}^{Wx} \delta = 0 \) to hold true, which appears to be important in constructing Abelian connection).

The deformation quantization variant elaborated in this paper is isomorphic to the generic Fedosov quantization of \cite{2}, as can be easily seen both by the very construction, and due to general results on deformation quantization on symplectic manifolds \cite{12}. This however does not mean that presented results are trivial, especially from the point of view of possible applications in physics. Indeed, one can observe that introduced degrees of freedom enter rather non-trivially into formulas (71), (72), (73) and (74). Hence, when employed into any model-building, they should result in some physical input (compare also \cite{22}). The most straightforward (but not the only one) field-theoretic application can be conjectured from our example, with \( g^{ij} \) given the meaning of a metric on the spacetime. On the other hand
it could be quite interesting to study further consequences of obtained results in operator ordering problems, as in [6].

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