Vortices and bags in 2 + 1 dimensions

C. D. Fosco\textsuperscript{a} \textsuperscript{*} and A. Kovner\textsuperscript{b} \textsuperscript{†}

\textsuperscript{a}Centro Atómico Bariloche - Instituto Balseiro, Comisión Nacional de Energía Atómica 8400 Bariloche, Argentina

\textsuperscript{b}Department of Physics, Theoretical Physics University of Oxford 1 Keble Rd, Oxford OX1 3NP, UK

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Abstract

We consider the effect of the (heavy) fundamental quarks on the low energy effective Lagrangian description of nonabelian gauge theories in 2+1 dimensions. We show that in the presence of the fundamental charges, the magnetic $Z_N$ symmetry becomes local. We construct the effective Lagrangian representing this local symmetry in terms of magnetic vortex fields, and discuss its physical consequences. We show that the finite energy states described by this Lagrangian have distinct bag-like structure. The point-like quarks are confined to the region of space where the value of the vortex field is much smaller than in the surrounding vacuum.

\textsuperscript{*}Electronic address: fosco@cab.cnea.gov.ar
\textsuperscript{†}Electronic address: a.kovner1@physics.ox.ac.uk
1 Introduction.

The understanding of confinement and, more generally, of the principles that govern the low energy dynamics in QCD is, perhaps, the most interesting unanswered question in the theory of strong interactions. The strongly coupled dynamics of the four dimensional non Abelian gauge theories has so far proven to be forbiddingly complicated. A fruitful approach to cope with such situation can be to turn to simpler, more tractable models, that nevertheless capture at least some of the important features of the real theory.

From this point of view, three dimensional non Abelian gauge theories are of particular interest since they are, indeed, confining, and have a strong coupling dynamics. On the other hand they can be continuously deformed into the weak coupling region without encountering any phase transition. The prototypical case is the SU(2) gauge theory with an adjoint Higgs field - the Georgi-Glashow model. These weakly coupled theories are confining, just like their strongly coupled counterparts. Here, however, confinement can be studied using semiclassical methods, as first done by Polyakov for the SU(2) case [1], and later on generalized to SU(N) theories [2].

A related feature of these models is that their spectra has a large gap between the light (light ‘photons’) and heavy (heavy massive gauge bosons) states. This makes the identification of the relevant low energy degrees of freedom relatively easy, and also allows one to construct the low energy effective Lagrangian explicitly. It turns out that this effective Lagrangian exhibits the confinement phenomenon in a very simple and straightforward way on the classical level. In a nutshell, the natural degrees of freedom turn out to be magnetic vortex fields first introduced by ’tHooft [3], and the confinement mechanism is realized as confinement of topological solitons for those vortex fields. The structure of the effective Lagrangian is robustly determined by the spontaneously broken magnetic $Z_N$ symmetry [3]. This approach has been motivated by earlier work [4, 5]. It was advanced and formulated in [6], and further clarified in [7]. It has also been argued in [6], that the same type of effective Lagrangian remains valid in the strongly coupled limit of pure gluodynamics. In this situation, the components of the vortex field are naturally identified with the scalar and pseudoscalar glueballs, which according to lattice results are indeed the two lowest states in the spectrum of 2+1 dimensional gluodynamics [8]. The confinement mechanism for the strongly coupled regime remains, in this picture, essentially the same as for the weakly coupled phase.
All previous work dealt exclusively with theories having no matter fields in fundamental representation. The purpose of this note is to address the question of what effect dynamical fundamental ‘quarks’ have on this low energy structure. That effect exists and must be nontrivial can be seen from one basic observation: in the presence of fundamental charges, the vortex field (which is the effective low energy degree of freedom) is no longer a local field. The effective Lagrangian cannot, therefore, be a simple local scalar theory as in [4, 5]. We will show below that, in the presence of fundamental matter, the magnetic $Z_N$ in the effective Lagrangian is gauged, the value of the gauge coupling being inversely proportional to the mass of the (lightest) dynamical quark. We will construct such an effective Lagrangian, which moreover in the limit of the infinitely heavy quarks reduces to the effective theory of [5]. The presence of fundamental quarks implies the existence of a conserved baryon number charge. We will see how this charge is represented in the low energy theory, and will study the structure of finite energy baryon states. Quite surprisingly, we find that these states naturally have a bag-like structure. The value of the vortex field in the region of space which contains the quarks, is much lower than its value in the vacuum. The potential energy associated with this difference in the expectation values can also be naturally identified with the bag constant.

The plan of this paper is the following. In section 2, we give a lightning review of the results of [5]. We discuss the magnetic $Z_N$ symmetry, its order parameter (the vortex operator), and the structure of the effective Lagrangian in pure gluodynamics. In section 3 we explain the basic physics of the changes in this structure in the presence of the fundamental charges. In section 4 we show how to define the relevant low energy degrees of freedom, and how to construct the effective Lagrangian in this case. In section 5, we study the structure of the baryon states. Finally we conclude in section 6 with a short discussion.

2 Pure glue.

2.1 The magnetic $Z_N$.

Consider the pure SU($N$) gauge theory in $2 + 1$ dimensions

$$L = -\frac{1}{4} \text{Tr} F^2$$

(1)
As was argued by ’tHooft in [3], this theory possesses a global $Z_N$ symmetry which is spontaneously broken in the vacuum. The explicit realization of this symmetry was found in [7]. The generator of the $Z_N$ group is (up to a multiplicative constant factor) the fundamental Wilson loop along the spatial boundary of the system

$$G = \lim_{C \to \infty} \text{Tr} P \exp \left\{ i \oint_C dx_i A^i(x) \right\}. \tag{2}$$

In [7], the conservation of $G$ was proved taking the pure gluodynamics limit of the Georgi-Glashow model. This, however, can also be shown directly by calculating the commutator of $G$ with the Yang-Mills Hamiltonian $H = 1/2(E^2 + B^2)$:

$$[G, H] = \lim_{C \to \infty} \oint_C dx_i \text{Tr} \left[ PE_i(x) \exp \left\{ i \oint_{C(x,x)} dy_i A^i(y) \right\} \right] \rightarrow C \to \infty 0. \tag{3}$$

Here, the integral in the exponential on the right hand side starts and ends at the point of insertion of the electric field. The vanishing of the commutator follows from the fact that it only involves electric fields at spatial infinity, and in a theory with a finite mass gap those fields should vanish at infinity.

To dispel any doubt about the vanishing of this commutator we note that the situation is completely analogous to the commutator of any ‘conserved charge’ which is defined as an integral of a local charge density

$$Q = \lim_{C \to \infty} \int_{|x| \leq C} d^2x \rho(x). \tag{4}$$

The commutator of such a charge with a Hamiltonian contains also a surface term, since the charge density $\rho$ never commutes with the Hamiltonian density but rather gives a total derivative in the commutator. For a conserved charge, due to the continuity equation, this surface term is equal to the circulation of the spatial component of the current

$$[Q, H] = i \oint_C dx^i j_i. \tag{5}$$

The vanishing of this term again is the consequence of the vanishing of the physical fields at infinity in a theory with a mass gap. When the charge is not conserved, the commutator contains also a bulk term in addition to the surface contribution. It is the absence of the bulk terms that is the
distinctive property of a conserved charge. The same conclusion is reached if, rather than considering the generator of the algebra, one considers the commutator of the group element for either continuous or discreet symmetry groups. We see, therefore, that the commutator in (3) indeed tells us that $G$ is a conserved operator.

Next, a simple argument establishes that the magnetic $Z_N$ symmetry is spontaneously broken in the vacuum if the Wilson loop $W_C$ has an area law behaviour [7]. The argument goes like this. The VEV of $W_C$ is the overlap of the vacuum state $|0>$ with the state $|S>$ which is obtained from the vacuum by acting on it with $W_C$. When acting on the vacuum state, $W_C$ performs the $Z_N$ transformation at all points within the area $S$ bounded by the loop. If the vacuum wavefunction depends on the configuration of the $Z_N$ non invariant degrees of freedom (the state in question is not $Z_N$ invariant) the action of $W_C$ affects the state everywhere inside the loop. In the local theory with finite correlation length the overlap between the two states approximately factorizes into the product of the overlaps taken over the region of space of linear dimension of order of the correlation length $l$

$$<0|S> = \Pi_x <0_x|S_x>$$

where the label $x$ is the coordinate of the point in the center of a given small region of space. For $x$ outside the area $S$ the two states $|0_x>$ and $|S_x>$ are identical and therefore the overlap is unity. However for $x$ inside $S$ the states are different and the overlap is therefore some number $e^{-\gamma}$ smaller than unity. The number of such regions inside the area is obviously of order $S/l^2$ and thus

$$<W_C> = \exp\{-\gamma \frac{S}{l^2}\}$$

The VEV of $W_C$ then falls off as an area. Conversely, if the vacuum is $Z_N$ invariant, the wavefunction does not depend on the configuration of the non invariant degrees of freedom. The action of $W_C$ then alters the state only along the perimeter and $W_C$ has the perimeter law behaviour in the unbroken phase.

Thus the area law behaviour of the fundamental Wilson loop is tantamount to the breaking of the magnetic $Z_N$ symmetry in the vacuum of pure gluodynamics.
2.2 The vortex field.

The only requirement that this argument presupposes is the existence of local degrees of freedom which are non invariant under $Z_N$, or in other words the existence of a local order parameter. Such an operator can indeed be constructed explicitly. This is the magnetic vortex creation operator $V$ \([4, 5, 7]\). The defining property of $V$ is that it satisfies the following commutation relation with the spatial fundamental Wilson loop $W$

$$V^\dagger(x)W(C)V(x) = e^{i\frac{2\pi}{N}n(x, C)}W(C) \quad (8)$$

where $n(x, C)$ is the linking number of the curve $C$ and the point $x$, or in other words, $n = 1$ if $x \in S$, and $n = 0$ if $x \notin S$, where $S$ is the region of the plane bounded by $C$. An explicit representation for $V(x)$ is given by writing it as the operator that performs a ‘singular gauge transformation’

$$V(x) = \exp\left\{\frac{2i}{gN} \int dy \epsilon_{ij} \frac{x_i - y_j}{(x - y)^2} \text{Tr}(YE_j(y)) + \Theta(x - y)J^Y_0(y)\right\}. \quad (9)$$

Here, the hypercharge generator $Y$ is defined as

$$Y = \text{diag} (1, 1, \ldots, -(N - 1)), \quad (10)$$

the electric field is written in the matrix notation $E_i = \lambda^a E^a_i$, with $\lambda^a$ denoting the $SU(N)$ generator matrices in the fundamental representation, and $J^Y_0(x)$ is the hypercharge density due to gluons $J^Y_0 = ig\text{Tr}[A_i, E_i]$. The function $\Theta(x)$ is the planar angle of the vector $x$. Using Gauss' law

$$\partial_i E^i = gA_i \times E_i, \quad (11)$$

this operator (on the states that satisfy Gauss' law) may be written in an alternative representation

$$V(x) = \exp\left\{\frac{4\pi i}{gN} \int_C dy \epsilon_{ij} \text{Tr}(YE_i(y))\right\}. \quad (12)$$

The integration here is along the branch cut in the definition of the planar angle $\theta(x)$, which is an infinite line that starts at the point $x$ and goes to infinity.

There are two crucial properties of the definition eq.\([12]\) that insure that the operator $V$ is a local physical field:
1. Gauge invariance. The definition is not explicitly gauge invariant. Nevertheless, one can show [3],[7] that $V$ transforms a physical state into another physical state. That is to say that the matrix elements of $V$ between a physical and an unphysical state vanish. By physical state we mean a state which satisfies the non Abelian Gauss’ law. By virtue of this property the definition eq.(12) can be used as long as the matrix elements of $V$ (or any power of $V$ with insertions of an arbitrary number of gauge invariant operators) are calculated between physical states.

2. Locality. Again, the definition is not explicitly local, since it contains the integral of the electric field along an infinite line. However in a theory that does not contain fundamental charges, $V$ in fact does not depend on the curve $C$, but only on its endpoint $x$. Physically, this is due to the fact that the Bohm-Aharonov phase between $V$ and any charged state in the theory vanishes. More formally, consider changing the position of the curve $C$ to $C'$. This adds to the phase in the definition eq.(12) a contribution $\frac{4\pi}{g_N} \int_S d^2x \text{Tr} \partial_i Y E^i$, where $S$ is the area bounded by $C - C'$. In our normalization, the hypercharge of gluons is $0$ or $\pm gN/2$ and no particles with smaller value of the hypercharge are present in the theory. Therefore the hypercharge within any closed area is an integer multiple of the gauge coupling $\int_S d^2x \partial_i E^i = \frac{gN}{2} n$, and the extra phase factor is always unity. One can also show directly [5], using canonical commutation relations, that $V(x)$ commutes with all the local gauge invariant operators $O(y)$, unless $y = x$, which formally establishes the locality of $V$.

2.3 The low energy effective theory.

The universal properties of the low energy dynamics of the pure Yang Mills theory are determined by the spontaneous breaking of the magnetic $Z_N$ symmetry. The effective low energy Lagrangian is constructed in terms of the vortex field $V$\footnote{For $N > 4$ in principle we have to include in this Lagrangian higher order potential terms which stabilize the potential energy at large values of $V$. Although we will not indicate them explicitly, it is assumed in the following that they are indeed present whenever necessary.}

\[
\mathcal{L} = \partial_\mu V^* \partial^\mu V - \lambda (V^* V - \mu^2)^2 - \zeta (V^N + V^* N) + \ldots
\] (13)
This Lagrangian can be derived in a controllable way in the weakly coupled phase\cite{5,6}. The coupling constants in eq. (13) are determined in the weakly coupled region from perturbation theory and dilute monopole gas approximation. It was argued in \cite{6} that the same low energy Lagrangian is also valid in the strongly coupled regime (pure gluodynamics).

Several features of this Lagrangian are worth noting:

• First, \(<V>\neq 0\), and therefore the \(Z_N\) symmetry is spontaneously broken.

• Second, the field components of \(V\) can be reinterpreted in terms of the physical glueball fields. Using a polar decomposition for \(V\)

\[
V = \rho e^{ix},
\]

we find that the fluctuation field \(\rho\) is the charge conjugation even scalar, while the phase \(\chi\) is the conjugation odd pseudoscalar. Indeed, according to lattice results \cite{8}, the two lightest glueballs in the spectrum of pure gluodynamics are the scalar \(0^{++}\) and pseudoscalar \(0^{--}\) states. The Lagrangian (13), is thus the natural effective Lagrangian that one would write to describe the dynamics of the lightest particles. In this respect the identification of the glueball fields with the modulus and the phase of \(V\) constrains the possible terms in the effective Lagrangian by imposing the \(Z_N\) symmetry on possible interaction terms.

• Finally, this effective Lagrangian exhibits in a simple way confining properties of the theory. The colour charged ‘gluon’ states (or massive vector bosons in the weakly coupled regime) are represented in the low energy theory by topological solitons carrying unit winding of the field \(V\). If not for the \(U(1)\) symmetry breaking term in the potential in (13), the lowest lying state in the topologically nontrivial sector would have a rotationally invariant hedgehog-like structure. Its energy would be logarithmically divergent in the infrared, which corresponds to the logarithmic Coulomb potential in 2+1 dimensions. However since the symmetry of the potential is \(Z_N\) rather than \(U(1)\), the energy of the hedgehog configuration diverges quadratically with the volume. This is so since the number of vacuum states is discreet and in the hedgehog configuration the phase of the field \(V\) is far from its vacuum value everywhere in space. The configuration that actually minimizes
the energy in the one soliton sector breaks rotational invariance by
confining the winding of the phase of $V$ to some quasi one-dimensional
strip, like in figure 1. It is clear that, dynamically, the width of this
strip is of the order of the inverse glueball mass and the strip itself is
nothing but the confining electric flux string. The energy of such an
isolated soliton now diverges \textit{linearly} in the infrared, due to the finite
energy density per unit length of the string.

The soliton we are discussing is a dynamical particle in the effective theory
- it thus represents a particle with an adjoint color charge, and the string is
the adjoint string. This string is of course not absolutely stable since it can
break into a soliton-antisoliton pair when it is too long. The fundamental string also appears naturally in the low energy description. It is the stable domain wall which separates the regions of space with different expectation values of the vortex field $V^{\bar{3}, \bar{6}}$, see figure 2.

3 Magnetic $Z_N$ and fundamental quarks.

In this section we wish to discuss the modifications introduced by the presence of fundamental quarks in the structure just described. For simplicity we will consider scalar quarks. The Lagrangian of the theory we are interested in is

$$L = -\frac{1}{4} \text{Tr} F^2 + |D_\mu \Phi|^2 - M^2 \Phi^* \Phi$$

(15)

where the scalar field $\Phi^\alpha$ transforms according to the fundamental representation of the $SU(N)$ color group.

3.1 No local order parameter.

First thing to note is that the fundamental Wilson loop still commutes with the Hamiltonian. This is obvious, since the extra terms in the Hamiltonian in the presence of the quarks do not involve electric field operator, but only
vector potential. The Wilson loop commutes with the vector potential, and thus with the additional terms in the Hamiltonian. We thus conclude that the theory still has the magnetic $Z_N$ symmetry.

This may seem somewhat surprising at first sight. We have seen in the previous section that spontaneous breaking of the magnetic $Z_N$ implies the area law for the Wilson loop, and conversely the perimeter law of $W$ implies unbroken $Z_N$. In the theory with fundamental charges the Wilson loop is known to have perimeter law due to breaking of the confining string at any finite value of the fundamental mass $M$. We might then conclude that the magnetic $Z_N$ is restored at any, arbitrarily large value of $M$ but is broken at $M \to \infty$. The common lore is that the $Z_N$ breaking phase transition for $N > 2$ is first order, and this then implies a discontinuous behaviour of the theory in the infinite mass limit. This of course is completely counterintuitive and in fact plain wrong. The caveat in this line of reasoning is the following. The relation between the behaviour of the Wilson loop and the mode of the realization of the magnetic symmetry hinges crucially on the existence of a local order parameter of the magnetic $Z_N$. In the absence of such an order parameter it is not true that the Wilson loop locally changes the quantum state inside the loop and this invalidates the whole argument. In particular in the absence of a local order parameter, the $Z_N$ symmetry can be spontaneously broken but the Wilson loop can have a perimeter law.

In fact it is easy to see that for any finite $M$ the magnetic $Z_N$ does not have a local order parameter. The only candidate for such an order parameter is the vortex operator $V(x)$ defined in eq.(12), since it has to be local also relative to the purely gluonic operators[^2]. However in the presence of fundamental quarks the operator $V(x)$ is not local anymore. To see this consider the dependence of $V_C(x)$ on the curve $C$ which enters its definition. As before the operators $V_C$ and $V_{C'}$ are related by

$$V_C(x) = V_{C'}(x) \exp\left\{\frac{4\pi i}{gN} \int_S d^2 x \text{Tr} \partial_i Y E^i\right\}$$

(16)

where $S$ is the area bounded by $C - C'$. As before, due to the Gauss’ law the integral in the exponential is equal to the total hypercharge in the area $S$. However the hypercharge of fundamental quarks has eigenvalues $\pm g/2$. The

[^2]: One can of course multiply $V(x)$ by any explicitly local gauge invariant operator. Such modifications are however irrelevant as far as the locality properties of $V$ are concerned, and we will not consider them in the following.
extra phase factor is therefore not unity anymore but can rather take values \( \exp\{2\pi i/N\} \) depending on the state and the choice of the contour \( C \).

The status of the magnetic \( Z_N \) is thus quite different in a theory with fundamental quarks - it does not have a local order parameter. Nevertheless it is clear that at least as long as the mass \( M \) is large \( M/g^2 > 1 \), the relevant degrees of freedom for the effective infrared dynamics should still be the vortices \( V \) and the main factor which determines their dynamics should still be the magnetic \( Z_N \). At large \( M \) the dependence of \( V \) on the curve \( C \) is weak, since the probability of the vacuum fluctuations which involve fundamental charges is small. The probability of appearance of a virtual \( q\bar{q} \) pair separated by a distance \( l \) is suppressed by the exponential factor \( \exp\{-Ml\} \). The typical distance scale for the ”glueball” physics is \( 1/g^2 \). Thus at these distances such fluctuations are unimportant and should not affect much the dynamics. Things are different however if one is interested also in the baryonic sector of the theory. The baryons are necessarily heavy and in order to be able to discuss their structure we must understand the main dynamical effects also at shorter distances.

Therefore our aim now is to understand what is the main effect of the non locality discussed above on the dynamics of magnetic vortices.

### 3.2 \( Z_N \) as a local symmetry.

The situation we have just described - a symmetry without a local order parameter - is not exceptional in quantum field theory. This is precisely the property of the global part of any Abelian gauge group. Consider for example quantum electrodynamics. The global electric charge is of course a physical gauge invariant charge with the corresponding gauge invariant local charge density.

\[
Q = \int d^2 x \rho
\]  \hspace{1cm} (17)

Nevertheless there is no local operator that carries this charge. This is a direct consequence of the Gauss’ law

\[
\partial_i E_i = g \rho \]  \hspace{1cm} (18)

Any physical, gauge invariant operator that carries \( Q \) must also carry the long range electric field, which can not fall off faster than a power of the distance. The gauge invariant QED Lagrangian is written in terms of ”local” charged fields \( \phi \). But appearances are deceptive: these fields are not gauge
invariant, and therefore not physical. A gauge invariant charged field can be constructed from $\phi$ by multiplying it by a phase factor

$$\phi_{\text{phys}}(x) = \phi \exp\{ig \int d^2y e_i(x - y) A_i(y)\}$$ (19)

with the $c$-number field $e_i$ satisfying

$$\partial_i e_i = \delta^2(x)$$ (20)

For any $e_i$ satisfying this condition the operator $\phi_{\text{phys}}$ is gauge invariant, and therefore physical. It is however necessarily nonlocal. Different choices of $e_i$ define different gauge invariant operators and correspond to different gauge fixings. Thus for $e_i(x) = x_i / x^2$ the field $\phi_{\text{phys}}$ is the field $\phi$ in the Coulomb gauge, while $e_i(x) = \delta_1 \delta(x_2) \theta(x_1)$ corresponds to the axial gauge $A_1 = 0$, and so on. Different definitions of $\phi_{\text{phys}}$ differ from each other by a phase factor, which is precisely the gauge ambiguity of the original field $\phi$.

The $U(1)$ gauge group is the most natural Abelian gauge symmetry to consider in continuum field theory. One can however also consider discrete groups like $Z_N$\textsuperscript{[9]}. In this case again no local operator that carries the global $Z_N$ charge exists. The various gauge invariant charged operators are nonlocal and differ from each other by a local $Z_N$ valued phase. These different operators again correspond to different gauge fixings of the local $Z_N$ group.

This is precisely the structure that emerged in our discussion in the earlier part of this section. We have

1. Global magnetic $Z_N$ symmetry generated by the fundamental Wilson loop.

2. The set of nonlocal vortex operators $V_C(x)$, which all carry the $Z_N$ charge and differ from each other by a $Z_N$ valued phase factors.

It is very suggestive therefore to think about $V_C(x)$ as of different gauge fixed versions of a field charged under local $Z_N$. This leads us to expect that the low energy theory we are after should be a $Z_N$ gauge theory of the magnetic vortex field $V$.

In fact coming back to the discussion in the beginning of this section, we see that from this vantage point it is obvious why the Wilson loop has a perimeter law, even if the global $Z_N$ is broken spontaneously. The action of a Wilson loop of a finite size inside the contour is a gauge transformation. Thus in physical terms locally inside the contour the new state is the same as the old one and so the overlap between the two locally is unity. The only
nontrivial contributions to the overlap come from the region close to the contour, thus giving the perimeter law.

In the next section we shall construct the effective gauge theory and discuss its relation to the original QCD Lagrangian.

4 Gauging the Wilson loop.

Although it is possible to give a more ‘analytic’ derivation of the effective Lagrangian, we prefer to write it down directly guided by the previous discussion. We will then explain the physical meaning of the various fields that appear in it.

4.1 The Lagrangian.

The easiest way to construct a $Z_N$ gauge theory in the continuum is to consider a $U(1)$ gauge theory with the Higgs field of charge $N$ which has a large expectation value $\xi$. Consider therefore the following Lagrangian:

\[
L = -\frac{1}{4e^2} f_{\mu\nu}^2 + |(\partial_\mu - i \frac{1}{N} b_\mu)V|^2 + |(\partial_\mu - ib_\mu)U|^2 - \lambda (V^*V - \mu^2)^2 - \xi (V^NU^* + V^*NU) - \tilde{\lambda}(U^*U - u^2)^2 .
\]  

(21)

Here $f_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$. We take the parameters such that, $\tilde{\lambda} >> \lambda$ and $u^2 >> \mu^2$.

The ‘Higgs’ field $U$ has a large expectation value and breaks the $U(1)$ gauge symmetry down to its $Z_N$ subgroup $V(x) \to \exp\{i\frac{2\pi n(x)}{N}\}V(x)$. It should be noted that this gauge subgroup does not correspond to ‘singular’ gauge transformations, since partial derivatives commute on the gauge parameter, despite the fact that there are $\delta$-like singularities.

Below the scale determined by the expectation value $u$, the field $U$ is practically frozen and its fluctuations are unimportant. In this regime the model indeed describes the locally $Z_N$ invariant theory. The global part of the gauge group is our $Z_N$ magnetic symmetry generated by the Wilson loop. The larger $U(1)$ gauge structure at this point is just an auxiliary trick which enables us to write down a discrete gauge theory in continuous notations. We will however see later that it does in fact has a real physical meaning of its own and arises naturally in the effective theory.
Let us first discuss how this Lagrangian reduces to the effective Lagrangian of the pure Yang-Mills theory eq.(13) in the limit of the large quark mass. In this limit not only the field $U$ must decouple, but also the gauge interactions of the field $V$ must vanish. There is another consistency requirement. In the limit of zero gauge coupling $e^2 \to 0$ the gauge $U(1)$ symmetry becomes global $U(1)$ and is broken due to non vanishing expectation value of $U$. The spectrum therefore contains a massless Goldstone boson. This Goldstone boson is of course the longitudinal component of the gauge field $b_\mu$. There is however no such massless particle in the pure gluodynamics nor in the effective Lagrangian eq.(13). This means that the couplings in eq.(21) should depend on the quark mass in such a way that the vector particles remains heavy for any finite $M$ and its mass goes to zero very sharply only in the limit when it is completely decoupled. In terms of the Goldstone boson couplings it means that $f_\pi$ must be larger than any scale relevant to the dynamics of the vortex field $V$. All these conditions can be met by choosing for example

$$e^2 = \frac{y}{M}, \quad u^2 = xM, \quad \xi = \frac{\zeta}{u}$$

With this choice the mass of the vector boson $m^2 = e^2u^2$ stays finite as $M \to \infty$ and can be arbitrarily large. The Goldstone boson in the decoupling limit has an infinite $f_\pi$ and is completely decoupled just as the "invisible axion".

4.2 The symmetries and the fields.

We now want to understand how the basic fields present in the effective Lagrangian arise in the fundamental theory eq.(15). We start with considering the symmetries.

The Lagrangian eq.(21) has two global $U(1)$ symmetries. One is the global part of the local $U(1)$ with the conserved current $e_j^T_\mu = \partial_\nu f_{\nu\mu}$. Note that due to the direct coupling between the $V$ and $U$ fields the $U(1)$ rotations of $V$ and $U$ separately are not symmetries. The other current, conserved by virtue of the homogeneous Maxwell equation is $\tilde{f}_\mu = \epsilon_{\mu\nu\lambda}f_{\nu\lambda}$. The two charges have quite different nature. The second one, the dual magnetic flux

$$\Phi_D = \int d^2x \tilde{f}_0$$

has a local order parameter. It can be constructed in a way similar to the
vortex field in QED [4],[5]. The global $U(1)$ gauge charge, on the other hand does not have a local order parameter, as discussed earlier.

Both these conserved currents should also exist in the fundamental theory. It is fairly straightforward to identify them. The QCD Lagrangian has one obvious global $U(1)$ charge - the baryon number. This charge has local order parameters - gauge invariant baryon fields of QCD, and is therefore identified with the dual magnetic flux

$$\frac{1}{2\pi}f_\mu = J^B_\mu, \quad Q_B = \Phi_D$$ (24)

The second charge can be expressed in terms of the spatial current components of the first one

$$Q^T = \int d^2x j_0^T = \int d^2x \partial_i \left[ \frac{1}{e^2} \epsilon_{ij} \tilde{f}_j \right]$$ (25)

It is thus the vorticity associated with the baryon number current.

The vortex operator $V$ which appears in eq.(21) is not gauge invariant and is only physical after complete gauge fixing of the $U(1)$ gauge group. After such a gauge fixing (which amounts to multiplying $V$ by an operator valued phase) the Gauss law requires that on the physical states the physical operator $V$ carry the charge $Q^T$. Due to the identification eqs.(24,25) this leads to a surprising conclusion that any physical operator $V$ in the effective theory creates a vortex of the original baryon number current.

This somewhat unexpected conclusion is in fact quite natural for an eigenoperator of the magnetic $Z_N$. Consider the vortex operator $V$ defined by eq.(12). In QCD just like in pure gluodynamics, it has an alternative representation of an operator of the singular gauge transformation of the form eq.(9)

$$V_C(x) = \exp\left\{ \frac{2i}{gN} \int d^u \epsilon_{ij} \frac{x_i - y_j}{(x - y)^2} \text{Tr}(YE_j(y)) + \theta(x - y)J^Y_0(y) \right\}$$ (26)

The only difference is that now $J^Y_0$ is the hypercharge operator due to both, gluons and fundamental quarks

$$J^Y_0 = ig\left[ \text{Tr}Y[A_i, E_i] + Y_{\alpha\beta}(\Phi^*_\alpha\Pi_\beta - \Phi_\alpha\Pi^*_\beta) \right]$$ (27)

Consider the action of this operator on a quark field $\Phi$. The transformed quark field $\Phi' = V^\dagger_C \Phi V_C$ is a gauge transform of $\Phi$ everywhere except along
the branch cut of the function $\theta$. Across this cut the phase of $\Phi'$ is discontinuous - it jumps by $2\pi/N$ for all color components of $\Phi'$. Due to this discontinuity the baryon current - the global $U(1)$ current of $\Phi$ - does not vanish at points along the cut. Calculating explicitly the action of $V$ on the baryon number current we find

$$V_C^I J^B_i(x) V_C = i V_C^I (\Phi^* \partial_i \Phi - \partial_i \Phi^* \Phi) V_C = J^B_i(x) + \frac{2\pi}{N} n^C_i(x) \delta(x \in C) \Phi^* \Phi(x)$$  

(28)

where $n^C_i(x)$ is a unit vector normal to the branch cut $C$ at the point $x$.

It is natural to define the local vorticity associated with the baryon number as

$$\rho_T = i \epsilon_{ij} \partial_i \left[ \frac{\Phi^* \partial_j \Phi - \partial_j \Phi^* \Phi}{\Phi \Phi^*} \right]$$  

(29)

The vortex operator $V$ therefore creates a vortex of baryon number current with fractional vorticity $2\pi/N$

$$V_C^I(x) \rho_T(y) V_C(x) = \rho_T(y) + \frac{2\pi}{N} \delta^2(x - y)$$  

(30)

The operator $U$ due to the Gauss’ law also carries baryon vorticity. Since its gauge coupling is $N$ times the coupling of $V$, it creates one unit of vorticity.

In fact this simple exercise also helps us to identify the value of the gauge coupling constant $e^2$ in the effective theory. Comparing eq.(29) with eq.(25) we find

$$e^2 \propto \Phi^* \Phi$$  

(31)

The same relation is obtained by comparing the current algebra in the fundamental and the effective theories. The commutator of the baryon charge density with the baryon current density in the fundamental theory is

$$[J^B_0(x), J^B_i(y)] = i \Phi^* \Phi \partial_i \delta^2(x - y)$$  

(32)

In the effective theory using the canonical commutators that follow from eq.(21) and the identification eq.(24) we find

$$[J^B_0(x), J^B_i(y)] = i e^2 \frac{4}{4\pi^2} \partial_i \delta^2(x - y)$$  

(33)

For a one component field $\Phi$ the vorticity defined in this way is precisely the circulation of the phase of $\Phi$.  

\[3\]
Again we deduce eq.\((31)\). The operator on the right hand side of this equation in the effective theory is indeed a constant. Recall, that our effective theory should be valid at long distances. In this regime in the leading order in the derivative expansion the operator \(\Phi^* \Phi\) should be approximated by its expectation value. Taking into account fluctuations of \(\Phi^* \Phi\) is tantamount to including higher derivative terms in the effective Lagrangian eq.\((21)\). At this, higher order in derivative expansion the gauge coupling constant would become a dynamical field. While this is perfectly legitimate, it is certainly beyond our present framework.

When taking the expectation value of \(\Phi^* \Phi\) we should remember that it has to be calculated with the cutoff \(\Lambda\) appropriate for the effective theory. This cutoff must be above the characteristic scale of the pure gluodynamics (determined by the string tension) but below the heavy quark mass. With this in mind we get

\[
<\Phi^* \Phi>_{\Lambda} = \int_0^\Lambda \frac{d^2p}{8\pi^2} \frac{1}{(p^2 + M^2)^{1/2}} \propto \frac{\Lambda^2}{M} \tag{34}
\]

So that finally

\[
e^2 \propto \frac{\Lambda^2}{M} \tag{35}
\]

which is consistent with the expected scaling in the large mass limit eq.\((22)\).

5  The baryon and the bag.

5.1  The baryon.

Having understood the origin of the fields and the symmetries in the effective theory, we would like to see how it encodes the qualitative features of the low energy QCD physics. Since we are considering a heavy quark theory, below the fundamental mass scale the spectrum should be the same as in pure gluodynamics. We have already seen that, in the infinite mass limit, the effective theory reduces to that of \((13)\). Indeed, even at finite but large \(M\), this is the case at low energies. Formally, this can be seen as follows. Since the VEV of the field \(U\) is large, we can impose the unitary gauge condition on it. In this unitary gauge the phase of \(U\) disappears. The modulus of \(U\) is very heavy, and so is the vector field \(b_\mu\). Thus at low energies we recover the effective theory of pure glue sector. There is however a set of configurations,
on which the unitary gauge can not be imposed. Those are configurations in which $U$ vanishes at some points in space. Indeed it is these configurations that are important for the baryonic sector. Recall that $U$ is a vortex of baryon number current. Thus one expects that the baryon charge is associated with the vortex configuration of the field $U$. In the core of the vortex the field $U$ of course has to vanish, and so the unitary gauge is not admissible. Thus in the baryon sector we can not think of $U$ as frozen at its expectation value and instead have to treat it as a dynamical field.

That the baryon does indeed carry vorticity of the field $U$, can be seen by the following simple argument. The baryon number is represented in the effective theory by the dual magnetic flux. The baryonic state must therefore be the dual magnetic vortex. Such a vortex in a nonsingular gauge has a vector potential of the form $b_i = \epsilon_{ij} \frac{x_j}{x^2}$. To have a finite energy it must be accompanied by the winding of the phases of both $V$ and $U$. In fact since $U$ carries $N$ times the charge of $V$, the only states that are allowed energetically are those that carry $N$ vortices of $U$.

This is natural in view of the ‘dual’ relation between the effective and the fundamental theories. The field $U$ is the ‘vortex’ dual to the fundamental quark. We thus expect that its elementary vortex would represent the fundamental quark itself, and so finite energy states must contain $N$ such elementary vortices. A single vortex must be confined. In the same sense, $V$ is ‘dual’ to the adjoint gluon field. The elementary vortex of $V$ is then, in a sense, the ‘constituent’ gluon. In fact, such single gluon should not exist as a finite energy state either, and we expect it also to be confined. The single vortex of $V$ should therefore bind either with the anti-vortex of the same type, or with $N$ vortices of $U$. The former type of state is a glueball, and exists in the pure glue theory [6], while the latter type is a baryon.

Interestingly enough, this line of reasoning leads us to expect that the baryon must have a bag-like structure. Namely, the quarks are bound to the $V$ field vortex. Inside this vortex, the value of $V$ is small - in fact it exactly vanishes in the middle, and than rises quite slowly (relative to the scale of $1/M$) towards the edges. Recall that $V = 0$ corresponds to a non confining state [6]. The quarks are therefore sitting in the ‘perturbative’ region of space - where there are no confining forces. Only when they separate far from each other - into the region with non vanishing $V$, the linear potential pulls them inside again.

Let us look at this more carefully. Consider for simplicity the $Z_2$ symmetric case $N = 2$. The baryon is the dual vortex with dual magnetic flux
$2\pi/e$. Far from the vortex core, the field configuration is pure gauge, with the phases of $V$ and $U$ following the vector potential:

$$b_i = \epsilon_{ij} \frac{x_j}{x^2}, \quad V(x) = ve^{i\alpha(x)}, \quad U(x) = ue^{2i\alpha(x)}.$$  

(36)

The parameters of the model are such that the field $V$ is much lighter than both $U$ and $b_i$. Thus the size of the vortex core of $V$ is large - of the order of the inverse glueball mass. The two $U$-vortices which have a very small size core, sit inside this core. Since the length associated with the dual magnetic field is much smaller than the core size of the $V$ - vortex, the dual flux is concentrated on the $U$ - vortices. From the low energy point of view, the picture is that two point-like magnetic vortices sit inside a soft core of a $V$ field vortex. The field configuration looks roughly as depicted on figure 3.

The magnetic flux is concentrated in the points $A$ and $B$. The phase of the field $U$ follows the variation of the vector potential very closely. The phase of $V$ is also trying to do that, but it can not quite follow it all the way, since on the line between the two vortices it would have to be discontinuous. The most important energy contribution (apart form the core energy of small vortices) comes therefore from the vicinity of this line. The phase of $V$ obviously has to interpolate across this line between the values $0$ and $\pi$. Since the modulus of $V$ is not extremely rigid, it will be smaller along this line than in the immediate neighborhood. Both, the variation of the modulus and the phase of $V$ along the line connecting the small vortices contribute to the energy which is clearly linear in the distance $|A - B|$.

5.2 The string and the bag.

To study the structure of the baryon more quantitatively let us fix the gauge such that

$$U(x) = u \exp[i \theta(x)]$$

$$\theta(x) = \arctan\left(\frac{y}{x-a}\right) + \arctan\left(\frac{y}{x+a}\right).$$

(37)

This is a valid gauge choice for the configuration with two vortices of unit winding at the positions $(a,0)$ and $(-a,0)$.

Due to the conditions on the parameters of our model, the vector field $b_\mu$ follows the phase of $U$ in the whole space. Thus

$$b_0 = 0, \quad b_j = -\left[\epsilon_{jk} \frac{(x - x^{(l)})_k}{|x - x^{(l)}|^2} + \epsilon_{jk} \frac{(x - x^{(r)})_k}{|x - x^{(r)}|^2}\right]$$

(38)
Figure 3: The ‘baryon’ configuration for $Z_2$. Arrows represent the direction (phase) of $V$, and the big dots at $A$ and $B$ correspond to the positions of the $U$ field vortices.
where: $x_1 \equiv x$, $x_2 \equiv y$, and $l$, $r$ are the positions of the left and right vortices, respectively: $x^{(l)} = (-a, 0)$, $x^{(r)} = (a, 0)$.

There are two interesting limiting situations. The first is when the distance $a$ is larger than the dynamical distance scales $(\lambda \mu^2)^{-1/2}$ and $(\xi u^2)^{-1/2}$ which determine the masses of the glueballs. The second interesting situation is the reverse, that is when the two vortices are sitting well inside the glueball correlation length. Let us look at them in turn.

When the distance between the vortices is large we expect the potential between them to be linear with the string tension calculated in the theory without the dynamical $U$ field. We will study the interquark potential in our low energy theory in the semiclassical approximation.

To find the minimal energy configuration, we have to solve the classical equations of motion for the field $V$ at a fixed configuration of $U$ and $b_i$ given by eqs. (37,38). Let us concentrate on the points which are close to the $x$-axis, with $|x| << a$. The main contribution to the energy comes from this region of space. In this region the vector potential $b_i$ vanishes, and the phase of the field $U$ is zero. Thus the equations of motion for the field $V$ are the same as in the pure gluodynamics. Also as long as we stick to this region, the configuration of $V$ is translationally invariant in the $x$ direction. What determines the energy then are the boundary conditions on the field $V$. In this configuration clearly the phase of the field $V$ is $\pi/2$ far above the $x$ axis and $-\pi/2$ far below the axis, fig.3. Thus both, the equations and the boundary conditions are precisely the same as for the domain wall separating the two degenerate vacua in the effective theory of pure gluodynamics ($e^2 = 0$, $\tilde{\lambda} \to \infty$). There is an extra contribution to the energy that comes from the region of space close to the points $A$ and $B$. But this energy does not depend on $a$ and is subleading for large $a$. The rest of the space does not contribute to the energy, since the field configuration there is pure gauge.

Thus as expected the energy in this regime is $E = a\sigma$, where $\sigma$ is the domain wall tension (fundamental string tension) calculated in pure gluodynamics [6].

It is perhaps more interesting to consider the opposite situation, that is when the distance between the quarks is smaller than the glueball correlation length. This is the regime in which we do not expect to see any stringy structure. Instead we can ask whether the lowest energy configuration has any resemblance to a bag. To study this question we take the limit $a \to 0$. In this case clearly the phase of the field $V$ will follow the phase of $U$ in the
Given this condition, only the variation of the radial component of $V$ has to be determined. Since the problem has rotational symmetry, the equation of motion for the modulus $\rho$ becomes

$$ - \frac{d^2 \rho}{dr^2} - 2\eta \rho + 2\lambda \rho^3 = 0. $$

with $\eta = \lambda \mu^2 - \xi u$, which is assumed to be positive throughout this paper. The relevant boundary condition in this case is that at infinity $\rho$ approaches its vacuum value $\rho_{r \to \infty} \to v$ while in the vortex core it vanishes $\rho(0) = 0$. With these boundary conditions, eq. (40) has the familiar form of the $\varphi^4$ static kink equation in 1 + 1 dimensions with the solution

$$ \rho(r) = v \frac{1 - e^{-2\sqrt{\eta}r}}{1 + e^{-2\sqrt{\eta}r}}. $$

The energy of this solution is

$$ E[V]_{a=0} = \lambda \int_0^\infty 2\pi drr (v^4 - \rho^4) = \frac{\pi^3 \eta}{12 \lambda} = \frac{\pi^3}{12} \left( \mu^2 - \frac{\xi u}{\lambda} \right). $$

The picture is thus just as described in the previous subsection. Since the two quark state is accompanied by the vortex of the field $V$, the quarks effectively ‘dig a hole’ in the vacuum. In their immediate vicinity the modulus $\rho$ vanishes, and therefore there is a ‘bag’ of the nonconfining state. The radius of this bag is given by the mass of the scalar glueball $2\sqrt{\eta}$.

It is interesting to note that, although for large separation $a$ the energy of the string gets contributions from both, the scalar and the pseudoscalar glueballs (the modulus and the phase of $V$), the ‘bag constant’ is determined solely by the scalar glueball. For small $a$ the phase of $V$ is not excited and only $\rho$ deviates from the vacuum state inside the ‘bag’. This is consistent with the common lore that the inside of the bag is distinguished from the outside by the value of the $F^2$ condensate. In fact since $\rho$ has vacuum quantum numbers and interpolates in our effective theory the scalar glueball, it is naturally associated with the operator $F^2$ which has a large overlap with the scalar glueball in QCD.
6 Discussion

In this note we have considered the effect of heavy fundamental quarks on the effective low energy theory description of nonabelian gauge theories in 2+1 dimensions. We found that the status of the magnetic $Z_N$ symmetry changes. It becomes a local symmetry with the ‘gauge coupling’ inversely proportional to the quark mass. Thus the low energy theory becomes a discrete gauge theory.

We have also studied the structure of the baryon in the framework of the effective theory. The picture that emerges is very reminiscent to the ‘bag’. The quarks in the baryon sit in the middle of the $V$-vortex, where the energy density differs from the vacuum energy density. The ‘bag constant’ is equal to the difference of these two energy densities, and parametrically in the effective theory is given by $\eta v^2$, where $\eta$ is the mass of the scalar glueball and $v$ is the expectation value of the vortex radial field $\rho$.

An important thing to note is that the bag we are talking about here arises in a very different situation than in the usual bag model [10]. There the bag describes the structure of the baryon containing light quarks. The radius of the bag in this situation is determined by the balance of the vacuum pressure and the pressure due to the free motion of the light quarks inside, and in fact depends on the quark wave function. In our case the inside of the bag contains heavy quarks. Their kinetic energy is small, and we have treated them here as static. The radius of the bag thus is determined purely by the dynamics of the scalar glueball field and is not sensitive to the state of the heavy quarks. This is true for low lying excitations for which the radius of the quark state is smaller than the inverse glueball mass. When these two scales are comparable presumably the quark pressure will also be important and will play a role in the energy balance. Thus in this intermediate regime we expect the $V$-vortex to be similar to the bag in the usual bag model. For states of even larger size the potential between quarks is linear with the fundamental string tension. The bag picture should therefore go smoothly into the string picture. It would be interesting to study these questions further in the framework of the effective theory discussed in this note.

Lastly we note that although we have so far considered scalar quarks, extending this discussion to spin 1/2 quarks is fairly straightforward. It is well known [11] that in the Abelian Higgs model the spin of the vortex is controlled by the coefficient of the Chern-Simons term of the vector field.
Thus adding the Chern Simons term for the vector field $b_\mu$

$$\delta L = \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda,$$

with $\kappa = 1/(2\pi)$, will endow the vortex of the $U$-field with a half integer spin\(^4\). Our effective model with this extra term is relevant to the description of QCD with one flavour of heavy spinor quarks. Just like the fermion mass term in 2+1 dimensions, the Chern Simons term breaks the parity symmetry in the baryonic sector of the theory. The extra term does not affect the classical analysis we have performed here, but is clearly relevant for determination of the spin of the baryon as well as the spectrum of the excitations.

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\(^4\)The spin will reside on the $U$-vortex and not the $V$-vortex, since in the dynamics of our effective model the vector potential $b_\mu$ follows the phase of the field $U$. 

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