Bose-condensed Bright Solitons under Transverse Confinement

L. Salasnich, A. Parola and L. Reatto

1Istituto Nazionale per la Fisica della Materia, Unità di Milano
Dipartimento di Fisica, Università di Milano,
Via Celoria 16, 20133 Milano, Italy

2Istituto Nazionale per la Fisica della Materia, Unità di Como
Dipartimento di Scienze Fisiche, Università dell’Insubria,
Via Valleggio 11, 23100 Como, Italy

We investigate the dynamics of Bose-condensed bright solitons made of alkali-metal atoms with negative scattering length and under harmonic confinement in the transverse direction. Contrary to the 1D case, the 3D bright soliton exists only below a critical attractive interaction which depends on the extent of confinement. Such a behavior is also found in multi-soliton condensates with box boundary conditions. We obtain numerical and analytical estimates of the critical strength beyond which the solitons do not exist. By using an effective 1D nonpolynomial nonlinear Schrödinger equation (NPSE), which accurately takes into account the transverse dynamics of cigar-like condensates, we numerically simulate the dynamics of the "soliton train" reported in a recent experiment (Nature 417 150 (2002)). Then, analyzing the macroscopic quantum tunneling of the bright soliton on a Gaussian barrier we find that its interference in the tunneling region is strongly suppressed with respect to non-solitonic case; moreover, the tunneling through a barrier breaks the solitonic nature of the matter wave. Finally, we show that the collapse of the soliton is induced by the scattering on the barrier or by the collision with another matter wave when the density reaches a critical value, for which we derive an accurate analytical formula.

03.75.Fz; 32.80.Pj; 42.50.Vk

I. INTRODUCTION

The experimental achievement of Bose-Einstein condensation with alkali-metal atoms at ultra-low temperatures [1,2] has opened the exciting possibility of studying topological configurations of the Bose-Einstein condensate (BEC), like solitary waves (solitons) with positive or negative scattering length \( a_s \) [3]. Dark solitons \((a_s > 0)\) of Bose condensed atoms have been experimentally observed few years ago [4], while bright solitons \((a_s < 0)\) have been detected only very recently with \(^7\)Li using an optical red-detuned laser beam along the axial direction of the sample to impose a transverse (radial) confinement [5].

Bright solitons have been studied in one-dimensional (1D) models [6] and no investigation has been performed on the effects a finite transverse width which is always present in experiment. In this paper the dynamics of a Bose condensate is investigated by using the full 3D Gross-Pitaevskii equation [7] and an effective 1D non-polynomial nonlinear Schrödinger equation (NPSE) [8,9], which accurately takes into account the transverse dynamics for cigar-like condensates. We analyze the existence, stability and collective oscillations of these 3D Bose-condensed bright solitons starting from a BEC with transverse confinement. We also study the existence and stability of multi-soliton condensates in a box and simulate the dynamics of the "soliton train" experimentally observed in Ref. [5]. Then we investigate the macroscopic quantum tunneling of a BEC on a Gaussian barrier showing that the bright-soliton strongly reduces interference fringes in the tunneling region. We show that the collapse of BEC can be induced by its scattering on a barrier if its density reaches a critical value, which can be analytically predicted. Finally we study the scattering of bright solitons and the conditions for their collapse at the collision.

II. BRIGHT SOLITONS UNDER TRANSVERSE CONFINEMENT

The dynamics of a BEC at zero temperature is well described by the 3D Gross-Pitaevskii equation (3D GPE) [7]. In many cases the numerical solution of 3D GPE is a hard task and approximate procedures are needed. Starting from the 3D GPE we have recently derived and studied [8,9] an effective time-dependent 1D nonpolynomial nonlinear Schrödinger equation (NPSE). NPSE describes very accurately Bose condensates confined by a harmonic potential with frequency \( \omega_\perp \) and harmonic length \( a_\perp = (\hbar/\omega_\perp)^{1/2} \) in the transverse direction and by a generic potential \( V(z) \) in the axial one. The total wave function of the condensate is \( \psi(x, y, z, t) = f(z, t)\phi(x, y, t) \), where the transverse wavefunction \( \phi(x, y, t) \) is a Gaussian with a width \( \eta \) given by \( \eta^2 = a_\perp^2 \sqrt{1 + 2a_sN|f|^2} \) and \( f(z, t) \) satisfies the NPSE equation

\[
\begin{align*}
\frac{i}{\hbar} \frac{\partial f}{\partial t} &= \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V(z) + \frac{2\hbar^2 N a_s}{ma_\perp} \frac{|f|^2}{\sqrt{1 + 2a_sN|f|^2}} ight] f, \quad (1) \\
&\quad + \frac{\hbar \omega_\perp}{2} \left( \frac{1}{\sqrt{1 + 2a_sN|f|^2}} + \sqrt{1 + 2a_sN|f|^2} \right) f ,
\end{align*}
\]

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where $a_s$ is the s-wave scattering length. $N$ is the number of condensed bosons and the function $f(z,t)$ is normalized to one. In the weakly-interacting limit $a_s N |f|^2 \ll 1$, NPSE reduces to a 1D GPE. Instead, in the strongly-interacting limit, NPSE becomes a nonlinear Schrödinger equation with the nonlinear term proportional to $|f|^4$ [8,9].

Under transverse confinement and negative scattering length ($a_s < 0$) a bright soliton sets up when the negative inter-atomic energy of the BEC compensates the positive kinetic energy such that the BEC is self-trapped in the axial direction. The shape of this 3D Bose-condensed bright soliton can be deduced from NPSE [8]. Setting $V(z) = 0$, scaling $z$ in units of $a_\perp$ and $t$ in units of $\omega_\perp^{-1}$, with the position

$$f(z, t) = \Phi(z-vt)e^{i\nu(z-vt)}e^{i(v^2/2-\mu)t},$$

one finds the bright-soliton solution written in implicit form

$$\zeta = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1-\mu}} \arctg \left[ \frac{\sqrt{1-2\Phi^2-\mu}}{1-\mu} \right]$$

$$- \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1+\mu}} \arctanh \left[ \frac{\sqrt{1-2\Phi^2-\mu}}{1+\mu} \right],$$

where $\zeta = z-vt$ and $\gamma = |a_s|N/a_\perp$. This equation is well defined only for $\gamma \Phi^2 < 1/2$; at $\gamma \Phi^2 = 1/2$ the transverse size is zero. Moreover, by imposing the normalization condition one has

$$(1-\mu)^{3/2} - \frac{3}{2} (1-\mu)^{1/2} + \frac{3}{2\sqrt{2}} \gamma = 0.$$  

The normalization relates the chemical potential $\mu$ to the coupling constant $\gamma$, while the velocity $v$ of the bright soliton remains arbitrary. In the weak-coupling limit ($\gamma \Phi^2 \ll 1$), the normalization condition gives $\mu = 1 - \gamma^2/2$ and the bright-soliton solution reads

$$\Phi(\zeta) = \sqrt{\frac{\gamma}{2}} \text{sech} \left[ \gamma \zeta \right].$$

The above solution is the text-book 1D bright soliton of the 1D nonlinear (cubic) Schrödinger equation (1D GPE).

In Figure 1 the “exact” solitonic solution obtained by numerically solving the 3D GPE with a finite-difference predictor-corrector method [10] is compared with the analytical solution (Eq. 3) of the NPSE and the analytical solution (Eq. 5) of the 1D bright soliton. The density profile is plotted for increasing values of $\gamma$. The agreement between the “exact” solution and the analytical one obtained from the NPSE is always remarkably good. Instead, as expected, the profile of the 1D bright soliton deviates from the 3D one for large $\gamma$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Axial density profile $\rho(z)$ of the Bose-condensed bright soliton: 3D GPE (full line), Eq. 3 (dotted line), Eq. 5 (dashed line). Length in units $a_\perp = (\hbar/\bar{\omega}_\perp)^{1/2}$ and density in units $1/a_\perp$.}
\end{figure}

From Eq. 4 it easy to show that for $\gamma > 2/3$ there are no solitary-wave solutions. Thus the NPSE gives the condition

$$-\frac{2}{3} < \frac{N a_s}{a_\perp} < 0,$$

for the existence of the 3D Bose-condensed bright soliton under transverse confinement. Note however that for $\gamma_c = 2/3$ the transverse size of the Bose-condensed soliton is not zero, in fact $\gamma_c \Phi^2 < 1/2$. We have numerically verified by solving the 3D GPSE that the critical value $\gamma_c = 2/3$ is very accurate. This is a remarkable result because, contrary to the 3D bright soliton (Eq. 3), the widely studied 1D bright soliton (Eq. 5) exists (and it is stable) at any $\gamma$: for large values of $\gamma$ the wavefunction simply becomes narrower.

In order to analyze the stability of our 3D soliton solution and its collective modes, it is useful to provide a simple analytical representation of its shape. The most natural choice is, as usual, a Gaussian with two variational parameters: the transverse width $\sigma$ and the axial width $\eta$ [11]. In this case, the energy per particle $E$ of the condensed atomic-cloud obtained from the 3D GPE energy functional is given by

$$E = \frac{1}{2} \left( \frac{1}{\sigma^2} + \frac{1}{2\eta^2} + \frac{\sigma^2}{\sigma^2\eta} \right),$$

where $g = \sqrt{2/\pi \gamma}$, lengths are in units $a_\perp$ and the energy is in units $\hbar \omega_\perp$. 

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The energy function $E = E(\sigma, \eta)$ has a local minimum, that is the 3D bright soliton condition, if the inter-atomic strength is smaller than a critical value: for larger values the soliton collapses. In Figure 2 we plot the widths $\sigma$ and $\eta$ of the approximated 3D bright soliton as a function of the inter-atomic strength. Note that the transverse width $\sigma$ does not change very much while the longitudinal width $\eta$ is divergent for $\gamma = 0$ and it is comparable with $\sigma$ for $\gamma$ close to $\gamma_c$. Moreover the critical value $\gamma_c$ of the collapse is slightly overestimated with respect to the analytical prediction $\gamma_c = 2/3$ of NPSE.

The Gaussian approximation of the BEC wavefunction can be used to study the dynamical stability of the 3D bright soliton by means of its collective oscillations. The diagonalization of the Hessian matrix of the energy function $E(\sigma, \eta)$ gives two frequencies $\omega_1$ and $\omega_2$ of collective excitations around the bright soliton solution. In Figure 2 we plot such frequencies and the widths $\sigma$ and $\eta$ of the solitonic configuration as a function of the strength $\gamma$. The frequencies are real and it means that the bright soliton is dynamically stable until it collapses at $\gamma_c$. Note that only for $\gamma = 0$ the frequencies $\omega_1$ and $\omega_2$ can be interpreted as transverse and axial collective oscillations of the bright soliton; nevertheless the mixing angle remains quite small also for finite values of $\gamma$ so they can be associated the transverse and axial motion respectively. The transverse frequency $\omega_1$ is practically constant but close to $\gamma_c$ it suddenly grows. The axial frequency $\omega_2$ is zero for $\gamma = 0$ it increases with $\gamma$ but close to $\gamma_c$ it goes to zero.

### III. Multi-soliton Bose Condensates in a Box

The solitary bright-soliton solution (Eq. 3) has been found by using Eq. 1 and Eq. 2. From these equations one obtains the Newtonian second-order differential equation

$$\frac{d^2}{ds^2} - 2\gamma \frac{\Phi^2}{1 - 2\gamma \Phi^2} + \frac{1}{2} \left( \frac{1}{1 - 2\gamma \Phi^2} + \frac{1}{\sqrt{1 - 2\gamma \Phi^2}} \right) \Phi = \mu \Phi, \quad (8)$$

where $\zeta = z - vt$ and $\gamma = |a_s| N / a_\perp$. The constant of motion of this equation is given by

$$E = \frac{1}{2} \left( \frac{d\Phi}{d\zeta} \right)^2 + \mu \Phi^2 - \Phi^2 \sqrt{1 - 2\gamma \Phi^2}. \quad (9)$$

By imposing the boundary condition $\Phi \to 0$ for $\zeta \to \infty$, which implies that $E = 0$, one has the solitary bright-soliton solution, that has only one peak.

![FIG. 3. Axial density profile $\rho(\zeta)$ of the bright multi-soliton Bose-condensate in a Box of length $L = 10$. From top to bottom: one, three and twelve solitons close to their critical strength. Units as in Fig. 1.](image)

It is important to observe that it is possible to obtain a multi-soliton static solution, i.e. a multi-peak solution, by imposing box boundary conditions: $\Phi(-L/2) = \Phi(L/2) = 0$, where $L$ is the length of the axial box. The density profile is periodic and the quantized number $N_s$ of peaks defines the number of solitons in the box for a given chemical potential $\mu$ and inter-atomic strength $\gamma$. In Figure 3 we plot the axial density profile obtained from the numerical integration of Eq. 9 for one, three
and twelve bright solitons in a box of length $L = 10a_\perp$. Also in the case of 3D bright multi-soliton Bose condensates, due to the finite transverse confinement, the solution exists only if the interaction strength $\gamma = N|a_s|/a_\perp$ is below a critical value. From the normalization condition it is easy to find that the condition of existence for a multi-soliton solution is given by

$$-\frac{L}{4} < Na_s < 0,$$

in the limit $N_s \to \infty$. Thus, in a large box, the strength $\gamma$ has the critical value $\gamma_c = 2/3$ for one soliton and $\gamma_c = L/(4a_\perp)$ for many-solitons.

| $N_s$ | $\mu$   | $\rho_0$ | $\gamma_c$ |
|-------|---------|----------|-------------|
| 1     | 0.4918  | 0.5694   | 0.6658      |
| 2     | 0.4449  | 0.3311   | 1.2481      |
| 3     | 0.4735  | 0.2745   | 1.6266      |
| 4     | 0.6505  | 0.2491   | 1.8588      |
| 5     | 0.9627  | 0.2353   | 2.0077      |
| 6     | 1.3827  | 0.2275   | 2.1083      |
| 7     | 1.8957  | 0.2231   | 2.1789      |
| 8     | 2.5987  | 0.2178   | 2.2319      |
| 9     | 3.3157  | 0.2159   | 2.2722      |
| 10    | 4.2019  | 0.2134   | 2.3032      |
| 11    | 5.2023  | 0.2113   | 2.3278      |
| 12    | 6.3053  | 0.2098   | 2.3473      |
| 13    | 7.4997  | 0.2085   | 2.3645      |
| 14    | 8.7045  | 0.2082   | 2.3793      |
| 15    | 10.1493 | 0.2069   | 2.3907      |
| 16    | 11.5817 | 0.2068   | 2.4006      |
| 17    | 13.2062 | 0.2061   | 2.4086      |
| 18    | 14.8693 | 0.2055   | 2.4173      |
| 19    | 16.6409 | 0.2051   | 2.4237      |
| 20    | 17.6887 | 0.2050   | 2.4330      |

Table 1. Bose-Einstein condensate with $N_s$ bright solitons in a box of length $L = 10a_\perp$. Chemical potential $\mu$ and maximum density $\rho_0$ of the multi-solitons at the critical strength $\gamma_c$. Units as in Fig. 1.

In Table 1 we show the chemical potential $\mu$, the maximum density $\rho_0$ and the critical strength $\gamma_c$ of a Bose condensate made by $N_s$ bright solitons in a box of length $L = 10a_\perp$. As expected the critical strength $\gamma_c$ grows with $N_s$ and for large $N_s$, it approaches $L/(4a_\perp) = 2.5$. In the case $N_s$ even, the multi-soliton solutions are also solutions with periodic boundary conditions (toroidal configuration with axial length $L$). In a torus, this train of solitons can travel without spreading with arbitrary velocity.

**IV. PROPAGATION OF A SOLITON TRAIN: COMPARISON WITH EXPERIMENTS**

In Ref. [5] it has been reported the formation of a multi-soliton Bose condensate of $^7$Li atoms created in a quasi-1D optical trap that can be modelled by an asymmetric harmonic potential with a large aspect ratio ($\lambda = \omega_\perp/\omega_z = 100$). In this experiment, a "soliton train", containing $N_s$ solitons ($N_s = 4, 5, 6$), has been created from a Bose-Einstein condensate by magnetically tuning the inter-atomic interaction from repulsive ($a_s = 10a_B$ with $a_B$ the Bohr radius) to attractive ($a_s = -3a_B$) using a Feshbach resonance. The soliton train has been set in motion and observed to propagate in the axial optical harmonic potential for many oscillatory cycles with a small spreading. Moreover, it has been found that the spacing between solitons is compressed at the turning points and spread out at the center of the oscillation, suggesting a repulsive interaction between neighboring solitons.

By using our time-dependent NPSE it is quite easy to simulate the dynamics of the soliton train described in [5]. In Figure 4 we plot three frames of the time evolution of a set of four $^7$Li bright solitons, each of them made of 5000 atoms, initially located far from the center of the axial harmonic potential with an alternating phase structure. Note that the train configuration ceases to be a true solitonic solution as soon as we switch off the axial confinement. In fact its shape and density change during the motion. The agreement between our numerical results of Figure 4 and the experimental data shown in Figure 4 of Ref. [5] is remarkable: the spacing between solitons increases near the center of oscillation and bunches at the end points. An alternating phase struc-
ture is essential to have a repulsion between neighboring solitons [12]. In fact, we have verified that a train of solitons with the same phase (attractive interaction between solitons) becomes a narrow blob of high density which can induce the collapse of the condensate. In the case of positive scattering length our calculations show that during the motion an initially confined Bose condensate made of many peaks becomes a blob which spreads along the axial direction is switched off and a Gaussian energy width. The BEC is moved towards the barrier by adding numerical findings.

V. TUNNELING WITH BRIGHT SOLITONS

Now we investigate the role of solitonic configurations and the effect of an attractive inter-atomic interaction on the behavior of a Bose-Einstein condensate during tunneling. The initial condition of the BEC is the ground-state of NPSE with a harmonic trapping potential also in the horizontal axial direction: \( V(z) = \frac{1}{2} m \omega_z^2 (z - z_0)^2 \). To have a cigar-shaped condensate we choose \( \lambda = \omega_z / \omega_x = 10 \). We set \( z_0 = 20 \), where \( z_0 \) is written in units of the harmonic length \( a_z = (\hbar / m \omega_z)^{1/2} \). For \( t > 0 \) the trap in the axial direction is switched off and a Gaussian energy barrier is inserted at \( z = 0 \). The potential barrier is given by

\[
V(z) = V_0 e^{-z^2 / \Sigma^2},
\]

where \( V_0 \) is the height of the potential barrier and \( \Sigma \) its width. The BEC is moved towards the barrier by adding an initial momentum \( p_0 \) in the axial direction:

\[
f(z, 0) \rightarrow f(z, 0) e^{i p_0 z / \hbar}.
\]

We have verified that the condensate with a scattering length close to the collapse value \( N a_s / a_z = -0.2 \) is a bright soliton: its shape does not depend on the axial harmonic trapping potential. In Figure 5 we plot the axial density profile of the Bose condensate at different instants for three values of the interaction strength: \( N a_s / a_z = 2 \cdot 10^{-1} \) (solid line), \( N a_s / a_z = 0 \) (dashed line), \( N a_s / a_z = 2 \cdot 10^{-1} \) (dotted line). Length in units \( a_z = (\hbar / m \omega_z)^{1/2} \), time in units \( \omega_z^{-1} \), and energy in units \( \hbar \omega_z \).

![Figure 5](image-url)

FIG. 5. Axial density profile of the Bose condensate tunneling through the Gaussian barrier. Start-up kinetic energy per particle of the condensate: \( E_0 = p_0^2 / (2m) = 10 \). Gaussian barrier parameters: \( V_0 = 10 \) and \( \Sigma = 1 \). Three values of the interaction strength: \( N a_s / a_z = 2 \cdot 10^{-1} \) (solid line), \( N a_s / a_z = 0 \) (dashed line), \( N a_s / a_z = 2 \cdot 10^{-1} \) (dotted line). Length in units \( a_z = (\hbar / m \omega_z)^{1/2} \), time in units \( \omega_z^{-1} \), and energy in units \( \hbar \omega_z \).
that

\[ a_s N |f|^2 < -\frac{1}{2}, \]

(13)

that is the condition for which condensate shrinks to zero in the transverse direction. Our numerical computations show that this condition gives precisely the one-dimensional density at which there is the collapse of the condensate during tunneling: \( \rho_{1D} = 1/(2|a_s|) \).

VI. COLLISIONS WITH BRIGHT SOLITONS

Solitons show peculiar properties not only in the collision with a barrier but also in the collision with other waves. In Figure 6 we plot some frames of the collision of two Bose-condensed bright solitons obtained by numerically solving the time-dependent NPSE (Eq. 1).

![Collision of two bright solitons](image)

FIG. 6. Collision of two bright solitons with a different number of particles: \( N_1 = 2N_2 \) and \( N_1 |a_s|/a_s = 0.1 \). Start-up momentum: \( p_0 = 50 \). Units as in Fig. 5.

The two solitons have a different number of particles and start with opposite momenta of equal modulus \( p_0 \). As shown in Figure 6, at the impact many fringes of interference are produced. We have verified that the number of fringes grows with the momentum \( p_0 \) while the maximal density of the interference peak remains constant. Moreover, we have found that the only effect of a phase-difference between the two colliding solitons is a shift in the position of fringes. In particular, at a fixed time, by changing the phase difference by \( \pi \) the spatial locations of maxima of interference are shifted into the positions of minima. After the impact the two bright solitons continue their motion without any recollection of the impact: the solitons acquire again their initial shape and then continue the motion without shape deformations.

This phenomenon of transparency can be also seen in the collision between a Bose-condensed bright soliton and a non-solitonic Bose condensate while two colliding generic matter-waves are not transparent. The numerical solution of NPSE confirms that, apart the interference at the impact, the solitonic matter-wave and the non-solitonic matter-wave do not see each other. At large times the bright soliton moves with its initial density profile while the non-solitonic wave evolves with a spreading that is not influenced by the collision with the bright soliton.

In the collision process of two Bose-condensed bright solitons the maximal density of the interference peak depends on the interaction strength, i.e. on the initial density of the two colliding solitons. For two colliding bright solitons with the same number of particles the maximal density at impact is about four times the initial density of each soliton. We have verified that when the maximal density satisfies the condition of Eq. 13 then the collapse of the condensates at the impact sets in.

CONCLUSIONS

Our calculations suggest that, contrary to the 1D bright soliton, the 3D bright soliton under transverse confinement exists and it is dynamically stable only if the attractive inter-atomic interaction is smaller than a critical value. We have analytically determined this critical interaction strength and the density profile of the 3D bright soliton. We have also found the collective oscillations of the bright soliton as a function of the interaction strength by using a Gaussian approximation. Then we have investigated multi-soliton Bose condensates with box boundary conditions. Also these multi-soliton solutions exist if the interaction strength is below a critical threshold that grows with box length. The dynamical properties of a train of solitons have been investigated by numerically solving our effective 1D nonpolynomial nonlinear Schrödinger equation: the results are in qualitative agreement with the experimental data. In the case of soliton scattering, our theoretical results show that the collapse of 3D soliton can be induced by the scattering on a Gaussian barrier when the density at the impact reaches a critical value, which does not depend on the solitonic nature of the incident wave. A clear signature of solitonic behavior is the transparency during collisions. Analyzing the collision between bright solitons we have verified that, apart the interference at the impact, the Bose-condensed soliton is transparent, namely at large times it recovers its initial density profile and then travels without spreading. However, if the density of the interference peak exceeds a critical density the system collapses. Moreover we have shown that there is not transparency during tunneling: a bright soliton tunneling through a barrier splits into reflected and transmitted wave packets which are not solitonic.
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