On torus homeomorphisms whose rotation set is an interval

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Abstract

We prove that for a torus homeomorphism with a lift whose rotation set is an interval, then either every rational point in the rotation set is realized by a periodic orbit, or the dynamics is annular. In the latter case we give a qualitative description of the dynamics.

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1 Introduction.

In [Poi52], Poincaré defined the rotation number for circle homeomorphisms, and he showed it to be a topological invariant carrying dynamical information. For a circle homeomorphism \( \tilde{f} \) with a lift \( f : \mathbb{R} \to \mathbb{R} \), the rotation number of \( f \), denoted \( \rho(f) \), is rational if and only if \( f \) has periodic points, and is irrational if and only if there exists a model for the dynamics of \( \tilde{f} \), in the sense that \( \tilde{f} \) is semi-conjugate to the irrational rotation \( x \mapsto x + \rho(f) \mod 1 \).

In [MZ89] Misiurewicz and Ziemian generalize the concept of the rotation number for homeomorphisms of \( T^n \), for any \( n \in \mathbb{N} \). For a torus homeomorphism \( \tilde{f} : T^n \to T^n \), the rotation set of some lift \( f : \mathbb{R}^n \to \mathbb{R}^n \), denoted \( \rho(f) \), is defined as the set of accumulation points of sequences of the form
\[
\left\{ \frac{f^{m_i}(x_i) - x_i}{m_i} \right\}_{i \in \mathbb{N}}
\]
where \( m_i \to \infty \) and \( x_i \in \mathbb{R}^2 \). The set \( \rho(f) \) is a compact subset of \( \mathbb{R}^n \), and in the case \( n = 2 \) it is also convex. In [MZ89], and in many other subsequent articles it is studied the relation between the rotation set and the dynamics of \( \tilde{f} \) (see for example [MZ91], [LM91], [Fra88], [Fra89], [KK08], [109], etc.). A lot more is known in the case that \( n = 2 \) thanks to the theory of surface homeomorphisms, like Brouwer theory, Thurston’s classification theory, etc. For this reason, we will restrict ourselves to the case \( n = 2 \).

A basic question, making an analogy with the theory of the circle, is whether there are periodic points associated to points with rational coordinates in \( \rho(f) \).

For a point \( v \in \rho(f) \cap \mathbb{Q}^2 \), expressed in the form \( v = (p_1/q, p_2/q) \) with \( \gcd(p_1, p_2, q) = 1 \), we say that \( v \) is realized by a periodic orbit of \( \tilde{f} \) if there exists \( x \in \mathbb{R}^2 \) such that
\[
f^q(x) = x + (p_1, p_2).
\]
This problem has been extensively studied. For example, in [Fra88], Franks proved that rational extremal points in the rotation set are realized by periodic orbits, and in [Fra89] he proved that rational points in the interior\(^1\) of the rotation set are also realized by periodic orbits. In the case that \( \rho(f) \) is an interval, it is not true that rational points are always realized by periodic orbits. However, in [Fra95] and [KK08], are given sufficient conditions under which, if \( \rho(f) \) is an interval, every rational point in \( \rho(f) \) is realized by a periodic orbit.

A second basic question is whether there are dynamical models associated to certain rotation sets. Unfortunately, this is not always the case. An example of this are the pseudo-rotations, that is, homeomorphisms whose rotation set is reduced to a single point, called rotation vector. There are examples of pseudo-rotations with the same rotation vector, but with very different behavior. The simplest example of a pseudo-rotation is a translation \( T_v : x \mapsto x + v \mod \mathbb{Z}^2 \).

Unlike the case of the circle, one may have that the deviations
\[
D(x, n) = |f^n(x) - x - nv|
\]
\(^1\)with respect to the topology of \( \mathbb{R}^2 \).
are unbounded, and this allows to create examples of pseudorotations with exotic properties, like Lebesgue weak-mixing [Fay02], topological expansive-type properties [KK09], etc. Therefore, in the case that the rotation set is reduced to a point, there seems not to be models for the dynamics associated to the rotation set.

In this work, we study the case that the rotation set is an interval. Suppose that \( \tilde{f} \) is a torus homeomorphism with a lift \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) whose rotation set is an interval of the form \( \{0\} \times I \), with \( 0 \in \text{int} \ I \) (the simplest example of such a homeomorphism is the twist \( (x, y) \mapsto (x, y + \sin(2\pi x)) \mod \mathbb{Z}^2 \)). As above, we could wonder if the horizontal deviations

\[
D_1(x, n) = |f^n(x)_1 - x_1|
\]

can be unbounded, and in this way construct examples with qualitatively different dynamics. We will show that if \( \rho(f) = \{0\} \times I \), then, either every rational point in \( \rho(f) \) is realized by a periodic orbit, or deviations \( D_1(x, n) \) are uniformly bounded in \( x \) and \( n \). With this, we will obtain the following, for the case that \( \rho(f) \) is any interval:

\textit{If \( \rho(f) \) is an interval, then either every rational point in \( \rho(f) \) is realized by a periodic orbit, or there exists a ‘model’ for the dynamics.}

A precise meaning of a ‘model’ for the dynamics is given in Theorems A and B. The simplest example of a homeomorphism with rational points in the rotation set not realized by a periodic orbit is a skew product \( \tilde{f} \) of a Morse-Smale circle homeomorphism (with fixed points), and a twist torus homeomorphism, as illustrated in Fig. 1a. This example has a lift \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) with \( \rho(f) = [-\pi, \pi] \), and any rational point contained in \( \rho(f) \) is not realized by a periodic orbit, since \( \tilde{f} \) has no periodic points.

![Figure 1](image1.png)

\textbf{Figure 1:} (a) \( \tilde{f}(x, y) = (\varphi(x), y + \pi \sin(2\pi x)) \), with \( \varphi : S^1 \to S^1 \) Morse-Smale. (b) a tentative to obtain unbounded horizontal displacements \( D_1(x, n) \).

One could try to modify this example, replacing the two invariant circles \( C_1 \) and \( C_2 \) by two non-connected invariant sets \( K_1 \) and \( K_2 \), so that orbits can pass through in hopes to obtain unbounded horizontal displacements \( D_1(x, n) \), but still having \( \rho(f) = \{0\} \times I \) (see Fig. 1b). However, we will see that this is
not possible; indeed, our main theorem implies that such a modification always leads to \( \max |\text{pr}_1(\rho(f))| > 0 \).

In Theorem A we deal with the particular case that \( \rho(f) \) is a vertical interval containing the origin in its interior, and the origin is not realized by a periodic orbit, and in Theorem B we deal with the case that \( \rho(f) \) is a general interval. In Theorem A we prove that the horizontal displacements \( D_1(x,n) \) are uniformly bounded, and to prove this we will show there exists an invariant vertical ‘wall’, that is, an invariant annular, essential and vertical set for \( \tilde{f} \). By an annular set we mean a nested intersection of compact annuli \( A_i \subset T^2 \), and by essential and vertical we mean that the annuli \( A_i \) are homotopic to the annulus \( \{ x \in T^2 : 0 \leq x_1 \leq 1/2 \} \). This ‘wall’ will also have the property of being a semi-attractor, which we now define:

**Definition 1.1.** An invariant, essential, annular set \( A \subset T^2 \) is a **semi-attractor** if there exist two simple, closed, essential curves \( \gamma_1, \gamma_2 \subset T^2 \) such that:

1. \( \omega(x, \tilde{f}) \subset A \) for all \( x \in \gamma_1 \), and
2. either \( \omega(y, \tilde{f}) \subset A \) for all \( y \in \gamma_2 \) or \( \alpha(y, \tilde{f}) \subset A \) for all \( y \in \gamma_2 \)

We say that a curve \( \gamma \subset T^2 \) is free forever for \( \tilde{f} \) if \( \tilde{f}^n(\gamma) \cap \gamma = \emptyset \) for all \( n \in \mathbb{Z} \). A closed curve \( \gamma \subset T^2 \) is vertical if it is homotopic to a straight vertical circle. Also, if \( \gamma_1, \gamma_2 \subset T^2 \) are vertical and disjoint curves, \( [\gamma_1, \gamma_2] \subset T^2 \) denotes the closed annulus whose ‘left’ border component is \( \gamma_1 \) and whose ‘right’ border component is \( \gamma_2 \) (for precise definitions see Section 2).

We now state our main theorem.

**Theorem A.** Let \( \tilde{f} \) be a homeomorphism of \( T^2 \) homotopic to the identity with a lift \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) such that:

1. \( \rho(f) = \{0\} \times I \), where \( I \) is a non-degenerate interval containing 0 in its interior, and
2. \( (0,0) \in \rho(f) \) is not realized by a periodic orbit.

There exists a non-empty finite family \( \{\tilde{l}_i\}_{i=0}^{r-1} \) of curves in \( T^2 \) which are simple, closed, vertical, pairwise disjoint and essential, and with the following properties. If

\[
\Theta_i := \bigcap_{n \in \mathbb{Z}} \tilde{f}^n([\tilde{l}_i, \tilde{l}_{i+1}]) \quad \text{for } i \in \mathbb{Z}/r\mathbb{Z},
\]

then,

1. at least one of the sets \( \Theta_i \) is an annular, essential, \( \tilde{f} \)-invariant set which is a semi-attractor,
2. the curves \( \tilde{l}_0, \tilde{l}_1, \ldots, \tilde{l}_{r-1} \) are free forever for \( \tilde{f} \),
3. there is \( \epsilon > 0 \) such that \( \rho(\Theta_i, f) \) is contained either in \( \{0\} \times (\epsilon, \infty) \), or in \( \{0\} \times (-\infty, -\epsilon) \), and
4. \( \Omega(\tilde{f}) \subset \cup \Theta_i \), (see Fig. 2).

For a definition of the rotation sets \( \rho(\Theta_i, f) \) see Section 3.1. In Theorem A, the curves \( \tilde{l}_i \) decompose the dynamics in a way similar to a filtration. If a point, under iteration by \( \tilde{f} \), leaves an annulus \([\tilde{l}_i, \tilde{l}_{i+1}]\), it never enters that annulus again, and if a point enters an annulus \([\tilde{l}_{i_0}, \tilde{l}_{i_0+1}]\) containing an essential set \( \Theta_{i_0} \), then it remains in that annulus forever.

As a corollary of Theorem A we will obtain the following theorem, dealing with the case that the rotation set is a general interval.

**Theorem B.** Let \( \tilde{f} : T^2 \to T^2 \) be a homeomorphism homotopic to the identity with a lift \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) whose rotation set is an interval.

Then, either every rational point in the rotation set is realized by a periodic orbit, or there is \( k \in \mathbb{N} \) such that \( \tilde{f}^k \) is topologically conjugate to a homeomorphism within the hypotheses of theorem A. In the latter case:

- there exists a simple, closed, essential curve in \( T^2 \) that is free forever for \( \tilde{f} \), and
- there exists an annular, essential set in \( T^2 \) that is periodic for \( \tilde{f} \).

**Remark 1.2.** In Theorem A from [KK08] it is proved that, if \( \rho(f) \) is an interval and there is a rational point in \( \rho(f) \) that is not realized by a periodic orbit, then for all \( n \in \mathbb{N} \) there is a simple, closed, essential curve disjoint from its first \( n \) iterates. Theorem A in this article generalizes that result, showing that there is actually a (simple, closed, essential) curve that is free forever.

We do not know if the property of the displacements \( D_1(x, n) \) being uniformly bounded is also present in the case that all the rational points in the rotation set are realized by periodic orbits:

**Question 1.3.** If \( \tilde{f} : T^2 \to T^2 \) is a homeomorphism with a lift \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( \rho(f) \) is an interval of the form \( \{0\} \times I \), then, are the deviations \( D_1(x, n) \) uniformly bounded?
This work is organized as follows. In section 3 we introduce the preliminary theory used in the proof of Theorem A, which is mainly the following: the rotation set for homeomorphisms of $\mathbb{T}^2$ and some results related to it, the Brouwer theory for planar homeomorphisms developed by Patrice Le Calvez, and Atkinson’s Lemma from ergodic theory. In section 4 we prove Theorem B assuming Theorem A. The proof of Theorem A is divided in sections 5 and 6.

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2 Notations.

By $\text{pr}_1, \text{pr}_2 : \mathbb{R}^2 \to \mathbb{R}$, we will denote the projections to the first and second coordinate, respectively. Also, if $x \in \mathbb{R}^2$, $x_1$ and $x_2$ will denote $\text{pr}_1(x)$ and $\text{pr}_2(x)$, respectively.

For a set $A \subset \mathbb{R}$, the diameter of $A$ is $\text{diam}(A) = \sup_{x,y \in A} |x - y|$. For $A \subset \mathbb{R}^2$, the horizontal diameter of $A$ is $\text{diam}_1(A) = \text{diam}(\text{pr}_1(A))$, and the vertical diameter of $A$ is $\text{diam}_2(A) = \text{diam}(\text{pr}_2(A))$.

For a set $A \subset \mathbb{R}^2$ and $x \in \mathbb{R}^2$, denote $d(x, A) = \inf_{y \in A} |y - x|$. For $x \in \mathbb{R}^2$ and $r > 0$, denote $B_r(x) = \{y \in \mathbb{R}^2 : |y - x| < r\}$, and for $A \subset \mathbb{R}^2$, denote $B_r(A) = \{x \in \mathbb{R}^2 : d(x, A) < r\}$.

For the circle $S^1 = \mathbb{R}/\mathbb{Z}$, and the two-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, denote by $\pi, \pi'$ and $\pi''$ the canonical projections

$$\mathbb{R}^2 \xrightarrow{\pi} \mathbb{R} \times S^1 \xrightarrow{\pi''} T^2,$$

and $\pi' = \pi'' \circ \pi$.

We will denote also by $d(\cdot, \cdot)$ the metric in $\mathbb{T}^2$ or in $\mathbb{R} \times S^1$ induced by the euclidean metric in $\mathbb{R}^2$.

Define $T_1, T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ to be the translations $T_1 : (x_1, x_2) \mapsto (x_1 + 1, x_2)$, $T_2 : (x_1, x_2) \mapsto (x_1, x_2 + 1)$. Also, $T_1$ and $T_2$ will denote the translations in $\mathbb{R} \times S^1$, $T_1 : (x_1, x_2) \mapsto (x_1 + 1, x_2)$, and $T_2 : (x_1, x_2) \mapsto (x_1, x_2 + 1 \mod 1)$.

By a curve $\gamma : I \to \mathbb{R}^2$, depending on the context, we mean either $\gamma$ or $\text{Im}(\gamma) \subset \mathbb{R}^2$. By an arc, we mean a compact curve, and if $\alpha$ is an arc, $\hat{\alpha}$ denotes the curve $\alpha$ without its endpoints.

A line $\ell$ is a proper embedding of $\ell : \mathbb{R} \to \mathbb{R}^2$. By Schoenflies’ Theorem ([Cal51]), given a line $\ell$ there exists an orientation preserving homeomorphism $h$ of $\mathbb{R}^2$ such that $h \circ \ell(t) = (0, t)$, for all $t \in \mathbb{R}$. Then, the open half-plane $h^{-1}((0, \infty) \times \mathbb{R})$ is independent of $h$, and we call it the right of $\ell$, and denote it by $R(\ell)$. Analogously, we define $L(\ell) = h^{-1}((-\infty, 0) \times \mathbb{R})$ the open half-plane to the left of $\ell$. The sets $\overline{R}(\ell)$ and $\overline{L}(\ell)$ denote the closures of $R(\ell)$ and $L(\ell)$, resp.
By \( \ell \prec \ell' \) we will mean \( \ell \subset L(\ell') \).

A closed curve \( \gamma \) in \( \mathbb{T}^2 \) or in \( \mathbb{R} \times S^1 \) is \textbf{essential} if it is not homotopic to a point, and we say that \( \gamma \) is \textbf{vertical} if \( \gamma \) is freely homotopic to a curve of the form \( c\beta \), where \( c \in \{1, -1\} \) and \( \beta(t) = (0, t) \).

A curve \( \gamma \) in \( \mathbb{T}^2 \) (or in \( \mathbb{R} \times S^1 \)) is \textbf{free} for \( f : \mathbb{T}^2 \to \mathbb{T}^2 \) (or \( f : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1 \)) if it is simple and closed, and \( f(\gamma) \cap \gamma = \emptyset \), and we say it is \textbf{free forever} for \( f \) if \( \gamma \) is disjoint from all its iterates by \( f \).

If \( \ell, \ell' \) are two lines in \( \mathbb{R}^2 \), we define \( (\ell, \ell') = R(\ell) \cap L(\ell') \) and \( [\ell, \ell'] = \overline{R(\ell) \cap L(\ell)} \). Similarly we define \( ([\ell, \ell'] = R(\ell) \cap L(\ell') \) and \( (\ell, \ell') = \overline{R(\ell) \cap L(\ell')} \). If \( \gamma \) and \( \gamma' \) are two disjoint, simple, closed and vertical curves in \( \mathbb{T}^2 \), we define the topological annuli \( (\gamma, \gamma') \subset \mathbb{T}^2 \) and \( [\gamma, \gamma'] \subset \mathbb{T}^2 \) in the following way. Let \( \tilde{\gamma} \subset \mathbb{R}^2 \) be any lift of \( \gamma \), and let \( \tilde{\gamma}' \) be the first lift of \( \gamma' \) to the right of \( \tilde{\gamma} \), that is, \( \tilde{\gamma}' \) is the lift of \( \gamma' \) with \( \tilde{\gamma} \prec \tilde{\gamma}' \prec T_{i}(\tilde{\gamma}) \). Orient \( \tilde{\gamma} \) and \( \tilde{\gamma}' \) as going upwards.

Define then \( (\gamma, \gamma') = \pi'(T_{\tilde{\gamma}}(\tilde{\gamma})) \) and \( [\gamma, \gamma'] = \pi'(T_{\tilde{\gamma}}(\tilde{\gamma}')) \). In a similar way, if \( \gamma \) and \( \gamma' \) are disjoint, simple, closed and vertical curves in \( \mathbb{R} \times S^1 \), define \( (\gamma, \gamma') \subset \mathbb{R} \times S^1 \) and \( [\gamma, \gamma'] \subset \mathbb{R} \times S^1 \).

Let \( M \) be either \( \mathbb{T}^2 \) or in \( \mathbb{R} \times S^1 \). A topological (open or closed) annulus \( B \) contained in \( M \) is \textbf{essential} if the inclusion \( B \hookrightarrow M \) induces a non trivial map \( \pi_1(B) \to \pi_1(M) \). A set \( A \subset M \) is \textbf{annular} if it is a nested intersection of topological compact annuli \( A_i \). An annular set is said to be \textbf{essential} if the \( A_i \) are essential annuli, and \( A \) is called \textbf{vertical} if the \( A_i \) are homotopic to the vertical annulus \( \{0 \leq x \leq 1/2\} \times S^1 \subset \mathbb{T}^2 \).

For a map \( f : X \to X \), where \( X \) is any metric space, we define an \( \epsilon \)-chain for \( f \) as a set \( \{x_i\}_{i=0}^{i_0} \subset X \) such that \( d(x_{i+1}, f(x_i)) < \epsilon \) for all \( i_0 \leq i < i_1 \). An \( \epsilon \)-chain \( \{x_i\}_{i=0}^{i_1} \) is \textbf{periodic} if \( x_{i_0} = x_{i_1} \). A point \( x \in X \) is \textbf{chain recurrent} for \( f \) if for all \( \epsilon > 0 \) there exists a periodic \( \epsilon \)-chain \( \{x_i\}_{i=0}^{i_1} \) for \( f \) with \( x_0 = x_n = x \).

The \textbf{chain recurrent set}, denoted by \( CR(f) \), is the set of chain recurrent points for \( f \).

### 3 Preliminaries.

#### 3.1 The rotation set.

Denote by \( \text{Homeo}(\mathbb{T}^2) \) the set of homeomorphisms of \( \mathbb{T}^2 \), and by \( \text{Homeo}_*(\mathbb{T}^2) \) the elements of \( \text{Homeo}(\mathbb{T}^2) \) which are homotopic to the identity. Let \( \tilde{f} \in \text{Homeo}_*(\mathbb{T}^2) \) and let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a lift of \( \tilde{f} \).

**Definition 3.1 ([MZ89]).** The \textbf{rotation set of} \( f \) is defined as

\[
\rho(f) = \bigcap_{m=1}^{\infty} \text{cl} \left( \bigcup_{n=m}^{\infty} \left\{ \frac{f^n(x) - x}{n} : x \in \mathbb{R}^2 \right\} \right) \subset \mathbb{R}^2.
\]

The \textbf{rotation set of a point} \( x \in \mathbb{R}^2 \) is defined by

\[
\rho(x, f) = \bigcap_{m=1}^{\infty} \text{cl} \left\{ \frac{f^n(x) - x}{n} : n > m \right\}.
\]

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If the above set consists of a single point \( v \in \mathbb{R}^2 \), we call \( v \) the rotation vector of \( x \). If \( \Lambda \subset \mathbb{T}^2 \) is a compact \( \tilde{f} \)-invariant set, we define the rotation set of \( \Lambda \) as

\[
\rho(\Lambda, f) = \bigcap_{m=1}^{\infty} \text{cl} \left( \bigcup_{n=m}^{\infty} \left\{ \frac{f^n(x) - x}{n} : x \in \pi'^{-1}(\Lambda) \right\} \right) \subset \mathbb{R}^2.
\]

**Remark 3.2.** It is easy to see that for integers \( n,m \),

\[
\rho(T^{m_1} \tilde{f} T^{m_2} f^n) = n\rho(f) + (m_1, m_2).
\]

Then, the rotation set of any other lift of \( \tilde{f} \) is an integer translate of \( \rho(f) \), and we can talk of the ‘rotation set of \( \tilde{f} \)’ if we keep in mind that it is defined modulo \( \mathbb{Z}^2 \).

**Theorem 3.3.** (MZ89) Let \( \tilde{f} : \mathbb{T}^2 \to \mathbb{T}^2 \) be a homeomorphism, let \( \Lambda \subset \mathbb{T}^2 \) be a compact \( \tilde{f} \)-invariant set, and let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a lift of \( \tilde{f} \). Then:

- The set \( \rho(\Lambda, f) \) is compact.
- The set \( \rho(f) \) is compact and convex, and every extremal point of \( \rho(f) \) is the rotation vector of some point.

Given \( A \in \text{GL}(2, \mathbb{Z}) \), we denote by \( \hat{A} \) the homeomorphism of \( \mathbb{T}^2 \) lifted by \( A \). If \( \hat{h} \in \text{Homeo}(\mathbb{T}^2) \), there is a unique \( A \in \text{GL}(2, \mathbb{Z}) \) such that for every lift \( h \) of \( \hat{h} \), the map \( h - A \) is bounded (in fact, \( \mathbb{Z}^2 \)-periodic). Then \( \hat{h} \) is isotopic to \( A \).

**Lemma 3.4.** Let \( \tilde{f} \in \text{Homeo}^*(\mathbb{T}^2) \), \( A \in \text{GL}(2, \mathbb{Z}) \) and \( \hat{h} \in \text{Homeo}(\mathbb{T}^2) \) isotopic to \( A \). Let \( f \) and \( h \) be lifts of \( \tilde{f} \) and \( \hat{h} \) to \( \mathbb{R}^2 \). Then

\[
\rho(hfh^{-1}) = A\rho(f).
\]

In particular, \( \rho(AfA^{-1}) = A\rho(f) \).

For a proof of this lemma, see for example [KK08].

**Remark 3.5.** If \( \rho(f) \) is segment of rational slope, there exists \( A \in \text{GL}(2, \mathbb{Z}) \) such that \( A\rho(f) \) is a vertical segment. Indeed, if \( \rho(f) \) is a segment of slope \( p/q \) (with \( p \) and \( q \) coprime integers), we can find \( x, y \in \mathbb{Z} \) such that \( px + qy = 1 \), and letting

\[
A = \begin{pmatrix} p & -q \\ y & x \end{pmatrix}
\]

we have that \( \det(A) = 1 \), and since \( A(q, p) = (0, 1) \), \( A\rho(f) \) is vertical.

### 3.1.1 The rotation set and periodic orbits.

Recall that we say that a rational point \( (p_1/q, p_2/q) \in \rho(f) \) (with \( \gcd(p_1, p_2, q) = 1 \)) is realized by a periodic orbit if there exists \( x \in \mathbb{R}^2 \) such that

\[
f^q(x) = x + (p_1, p_2).
\]

We mention the following realization results.
Theorem 3.6 ([Fra88]). If a rational point of $\rho(f)$ is extremal, then it is realized by a periodic orbit.

Theorem 3.7 ([Fra89]). Any rational point in the interior of $\rho(f)$ is realized by a periodic orbit.

The following theorem is stated for diffeomorphisms in [Cal91], p. 106, but its proof remains valid for homeomorphisms using the results in [Cal05] (see p. 9 of that article).

Theorem 3.8. If a rational point belongs to a line of irrational slope which bounds a closed half-plane that contains $\rho(f)$, then this point is realized by a periodic orbit.

### 3.1.2 The rotation set and invariant measures.

For a compact $\hat{f}$-invariant set $\Lambda \subset T^2$, we denote by $\mathcal{M}_{\hat{f}}(\Lambda)$ the family of $\hat{f}$-invariant probability measures with support in $\Lambda$, and $\mathcal{M}_{\hat{f}} = \mathcal{M}_{\hat{f}}(T^2)$. Define the displacement function $\phi : T^2 \rightarrow \mathbb{R}^2$ by

$$\phi(\hat{x}) = f(x) - x, \quad \text{for } x \in \pi'^{-1}(\hat{x}).$$

This is well defined, as any two preimages of $\hat{x}$ by the projection $\pi' : \mathbb{R}^2 \rightarrow T^2$ differ by an element of $\mathbb{Z}^2$, and $f$ is $\mathbb{Z}^2$-periodic. Now, for $\mu \in \mathcal{M}_{\hat{f}}$, we define the rotation vector of $\mu$ as

$$\rho(\mu, f) = \int \phi \, d\mu.$$

Then, we define the sets

$$\rho_{\text{mes}}(\Lambda, f) = \left\{ \rho(\mu, f) : \mu \in \mathcal{M}_{\hat{f}}(\Lambda) \right\},$$

and

$$\rho_{\text{erg}}(\Lambda, f) = \left\{ \rho(\mu) : \mu \text{ is ergodic for } \hat{f} \text{ and } \text{supp}(\mu) \subset \Lambda \right\}.$$

When $\Lambda = T^2$ we simply write $\rho_{\text{mes}}(f)$ and $\rho_{\text{erg}}(f)$.

Proposition 3.9 ([MZ89]). It holds the following:

$$\rho(f) = \rho_{\text{mes}}(f) = \text{conv}(\rho_{\text{erg}}(f)).$$

When $\Lambda$ is a proper (compact, invariant) subset of $T^2$, the set $\rho(\Lambda, f)$ is not necessarily convex. However, we have the following.

Proposition 3.10. It holds

$$\text{conv}(\rho(\Lambda, f)) = \rho_{\text{mes}}(\Lambda, f),$$

and therefore, if $v \in \mathbb{R}^2$ is an extremal point of $\text{conv}(\rho(\Lambda, f))$, there exists an ergodic measure $\mu$ for $\hat{f}$ with $\rho(\mu, f) = v$. 

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Proof. We first observe that \( \rho_{mes}(\Lambda, f) \) is convex. To see this, let \( r_1, r_2 \in \rho_{mes}(\Lambda, f) \), and let \( \mu_1, \mu_2 \in \mathcal{M}_f(\Lambda) \) be such that \( \rho(\mu_1, f) = r_1 \) and \( \rho(\mu_2, f) = r_2 \). The set \( \mathcal{M}_f(\Lambda) \) is convex, and then for all \( t \in [0, 1] \), \( t \mu_1 + (1 - t) \mu_2 \) belongs to \( \mathcal{M}_f(\Lambda) \). Also, for \( t \in [0, 1] \),

\[
  t \cdot r_1 + (1 - t) r_2 = t \int \phi \, d\mu_1 + (1 - t) \int \phi \, d\mu_2 = \]

\[
  = \int \phi \, d(t \mu_1 + (1 - t) \mu_2) = \rho(t \mu_1 + (1 - t) \mu_2, f),
\]

and then \( tr_1 + (1 - t) r_2 \in \rho_{mes}(\Lambda, f) \). Therefore \( \rho_{mes}(\Lambda, f) \) is convex.

Thus, to prove the inclusion \( \text{conv}(\rho(\Lambda, f)) \subset \rho_{mes}(\Lambda, f) \) it suffices to prove that \( \rho(\Lambda, f) \subset \rho_{mes}(\Lambda, f) \). Let \( v \in \rho(\Lambda, f) \). There exists then a sequence \( \{x_n\}_n \) in \( \pi^{-1}(\Lambda) \) and a sequence of natural numbers \( \{m_n\}_n \) such that

\[
  \lim_n \frac{f^{m_n}(x_n) - x_n}{m_n} = v.
\]

Define a sequence of probability measures \( \{\delta_n\}_n \) by

\[
  \delta_n = \frac{\delta_{x_n} + \delta_{f(x_n)} + \cdots + \delta_{f^{m_n-1}(x_n)}}{m_n},
\]

and let \( \mu \) be an accumulation point of \( \{\delta_n\}_n \) in the space \( \mathcal{M}_f(\Lambda) \) of Borel probability measures in \( \Lambda \), equipped with the weak-* topology. Then, \( \mu \) is \( \tilde{f} \)-invariant. Choosing a subsequence, we can assume that \( \delta_n \to \mu \). Then

\[
  \rho(\mu, f) = \lim_n \int \phi \, d\delta_n = \lim_n \frac{f^{m_n}(x_n) - x_n}{m_n} = v,
\]

and therefore \( \rho(\Lambda, f) \subset \rho_{mes}(\Lambda, f) \).

Now we prove the inclusion \( \rho_{mes}(\Lambda, f) \subset \text{conv}(\rho(\Lambda, f)) \). As \( \rho_{mes}(\Lambda, f) \) is convex, it suffices to show that the extremal points of \( \rho_{mes}(\Lambda, f) \) are contained in \( \text{conv}(\rho(\Lambda, f)) \). Actually, we will show that the extremal points of \( \rho_{mes}(\Lambda, f) \) are contained in \( \rho(\Lambda, f) \).

Consider the vector space \( C(T^2) \) of continuous maps from \( T^2 \) to \( \mathbb{R} \), and consider the dual vector space \( C'(T^2) \) of \( C(T^2) \), that is, the space of linear functionals from \( C(T^2) \) to \( \mathbb{R} \). We know that \( C'(T^2) \) is isomorphic to the vector space \( \mathcal{M}_s(T^2) \) of signed measures in \( T^2 \) (see for example [Fol84]). Consider the linear map \( L_f : \mathcal{M}_s(T^2) \to \mathbb{R}^2 \) given by

\[
  L_f(\mu) = \int \phi \, d\mu.
\]

The map \( L_f \) is linear, and

\[
  L_f(\mathcal{M}_f(\Lambda)) = \rho_{mes}(\Lambda, f).
\]
Let \( w \) be an extremal point of \( \rho_{\text{mes}}(\Lambda, f) \). We show now that there is \( x \in \Lambda \) such that \( \rho(x, f) = w \). Recall the following fact from convex analysis: if \( T : E_1 \to E_2 \) is a linear map between vector spaces, and \( C \subset E_1 \) is convex, then \( T(C) \) is a convex subset of \( E_2 \) and for any extremal point \( v \in T(C) \), the set \( T^{-1}(v) \subset C \) contains an extremal point of \( C \) (see for ex. [Roc97]). As \( L_{\tilde{f}} \) is linear and the sets \( M_{\tilde{f}}(\Lambda) \) and \( \rho_{\text{mes}}(\Lambda, f) \) are convex, we have that the preimage by \( L_{\tilde{f}} \) of the extremal point \( w \) contains an extremal point of \( M_{\tilde{f}}(\Lambda) \), that is, an ergodic measure \( \mu \) for \( \tilde{f} \) with support in \( \Lambda \). By Birkhoff’s Theorem, there exists \( x \in \text{supp}(\mu) \subset \Lambda \) such that

\[
\int \phi \, d\mu = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(\tilde{f}^n(x)) = \lim_{n \to \infty} \frac{f^n(x) - x}{n} = \rho(x, f).
\]

As \( w = L_{\tilde{f}}(\mu) = \int \phi \, d\mu \), we have that \( \rho(x, f) = w \), as desired. With this we have that the set of extremal points of \( \rho_{\text{mes}}(\Lambda, f) \) is contained in \( \rho(\Lambda, f) \), and as mentioned above, this gives us the inclusion

\[
\rho_{\text{mes}}(\Lambda, f) \subset \text{conv}(\rho(\Lambda, f)).
\]

Then \( \rho_{\text{mes}}(\Lambda, f) = \text{conv}(\rho(\Lambda, f)) \), and this finishes the proof of the first claim of the proposition.

For the second claim, let \( w \) be an extremal point of \( \text{conv} \rho(\Lambda, f) = \rho_{\text{mes}}(\Lambda, f) \). Then, just notice that we proved that the set \( L_{\tilde{f}}^{-1}(w) \) contains an ergodic measure \( \mu \), such that \( \rho(\mu, f) = L_{\tilde{f}}(\mu) = w \), as desired. This finishes the proof of the lemma.

### 3.2 Brouwer Theory

In [Bro12], Brouwer proved the following theorem for homeomorphisms of the plane, known as the Brouwer Translation Theorem:

**Theorem 3.11.** Let \( h \) be a homeomorphism of \( \mathbb{R}^2 \) without fixed points. Then:

1. For all point \( x \in \mathbb{R}^2 \) there exists a line \( \ell \) passing through \( x \) such that
   \[
   \ell \prec h(\ell) \quad \text{and} \quad h^{-1}(\ell) \prec \ell.
   \]
   A line satisfying item (1) is called a Brouwer line for \( h \). By item (2) we have that \( h \) has no periodic points, and moreover, every point is wandering for \( h \). The proofs of this theorem use the Brouwer Translation Lemma, which states that if a homeomorphism of the plane has no fixed points, then it has no periodic points. In [Fra88] Franks proved the following stronger property of non-recurrence:

\[
\rho_{\text{mes}}(\Lambda, f) = \text{conv}(\rho(\Lambda, f)).
\]
Theorem 3.12 (Franks’ Lemma). Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism. If there exist a sequence $(U_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ of pairwise disjoint open disks and a sequence of integers $(m_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ such that

$$f^{m_i}(U_i) \cap U_{i+1} \neq \emptyset \text{ for all } i \in \mathbb{Z}/n\mathbb{Z},$$

then $h$ has a fixed point.

As a corollary one obtains the following.

Theorem 3.13 ([Fra89]). Let $h$ be an orientation preserving homeomorphism of the plane, without fixed points and which is the lift of a homeomorphism of $\mathbb{T}^2$. Then, there exists $\epsilon > 0$ such that there are no periodic $\epsilon$-chains for $h$.

In [Cal04], Le Calvez showed the following remarkable and much stronger version of the Brouwer Translation Theorem.

Theorem 3.14. Let $h$ be a plane homeomorphism without fixed points. There exists a topological oriented foliation $\mathcal{F}$ of the plane such that each leaf of $\mathcal{F}$ is a Brouwer line for $h$.

Then, in [Cal05] it is proved the following improvement of Theorem 3.14.

Theorem 3.15. Let $M$ be a surface and $(H_t)_{t \in [0,1]}$ an isotopy in $M$ joining the identity to a homeomorphism $f$. For all $z \in M$ we define the arc $\gamma_z : t \mapsto H_t(z)$. We suppose that $f$ does not have any contractible fixed point $z$, that is, a fixed point $z$ such that $\gamma_z$ is a closed curve homotopic to a point. Then there exists an oriented topological foliation $\mathcal{F}$ in $M$ and for all $z \in M$ an arc positively transverse to $\mathcal{F}$ joining $z$ to $f(z)$ that is homotopic with fixed extremes to the arc $\gamma_z$.

In [Cal05], as an application of Theorem 3.15, it is proved the following Theorem. The statement in [Cal05] (Theorem 9.1) is for orbits, instead of $\epsilon$-chains as it is stated here. However, the result is easily adaptable for the case of $\epsilon$-chains (see Proposition 8.2 in that article). We include a sketch of the proof.

Theorem 3.16. Let $\tilde{f} : \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism isotopic to the identity without contractible fixed points. Fix an isotopy $(\tilde{H}_t)_{t \in [0,1]}$ in $\mathbb{T}^2$ between $\tilde{f}$ and the identity, and let $\mathcal{F}$ be the foliation of $\mathbb{T}^2$ transverse to $(\tilde{H}_t)_{t \in [0,1]}$ given by Theorem 3.15. Let $(H_t)_{t \in [0,1]}$ be the isotopy in $\mathbb{R}^2$ which is the lift of $(\tilde{H}_t)$ and satisfies $H_0 = \text{Id}$, and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the lift of $\tilde{f}$ given by $f = H_1$. Let $\hat{\mathcal{F}}$ be the lift of $\mathcal{F}$ to $\mathbb{R} \times S^1$.

There exists $\epsilon > 0$ such that, if $\hat{x}, \hat{y} \in \mathbb{R} \times S^1$, are points with lifts $x, y \in \mathbb{R}^2$ and:

- there is an $\epsilon$-chain for $f$ from $x$ to $x + (0,m)$ for some $m \in \mathbb{N}$, and
- there is an $\epsilon$-chain for $f$ from $y$ to $y + (0,-n)$ for some $n \in \mathbb{N}$,
then there exists a compact leaf \( l \in \tilde{F} \) which is an essential curve that separates \( \hat{x} \) from \( \hat{y} \) (that is, \( \hat{x} \) and \( \hat{y} \) belong to different connected components of \( \mathbb{R} \times S^1 \setminus l \)). In particular \( \hat{x} \neq \hat{y} \).

**Sketch of the Proof.** Let \( F : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1 \) be the lift of \( \tilde{f} \) such that \( F \circ \pi = \pi \circ f \). Let \((\tilde{H}_t)_{t \in [0,1]}\) be the isotopy in \( \mathbb{R} \times S^1 \) between \( F \) and the identity which is the lift of the isotopy \((H_t)_{t \in [0,1]}\). By Theorem 3.15, for every \( \hat{x} \in \mathbb{R} \times S^1 \) there exists an arc which is positively transverse to \( \tilde{F} \), joins \( \hat{x} \) to \( F(\hat{x}) \) and is homotopic with fixed extremes to the arc \( \gamma_{\hat{x}} : \to \tilde{H}_t(\hat{x}) \). By this, one can easily see that, for any \( \hat{x} \in \mathbb{R} \times S^1 \) there exists \( \epsilon > 0 \) such that any point \( \hat{z} \) in \( B_\epsilon(\hat{x}) \) can be joined to any point \( \hat{z}' \) in \( B_\epsilon(F(\hat{x})) \) by an arc which is positively transverse to \( \tilde{F} \) and homotopic to an arc of the form \( \gamma_{\hat{x}} \gamma_{\hat{x}}' \gamma_{\tilde{F}(\hat{x})} ' \), where \( \gamma_{\hat{x}} \) joins \( \hat{x} \) to \( \hat{x} \) in \( B_\epsilon(\hat{x}) \) and \( \gamma_{\tilde{F}(\hat{x})} ' \) joins \( F(\hat{x}) \) to \( \hat{z}' \) in \( B_\epsilon(F(\hat{x})) \), and where the product of two arcs stands for their concatenation.

As \( F \) is the lift of the homeomorphism \( f : T^2 \to T^2 \), and as \( T^2 \) is compact, there exists \( \eta > 0 \) such that for any point \( \hat{x} \in \mathbb{R} \times S^1 \), any point in \( B_\eta(\hat{x}) \) can be joined to any point in \( B_\eta(F(\hat{x})) \) by an arc positively transverse to \( \tilde{F} \) as above. Also, by the continuity of \( F \), there is \( 0 < \epsilon < \eta \) such that for any \( \hat{x} \in \mathbb{R} \times S^1 \), if \( \{\hat{x}_i\}_{i=0}^n \) is a periodic \( \epsilon \)-chain for \( F \) with \( \hat{x}_0 = \hat{x}_n = \hat{x} \), then \( \hat{x}_{n-1} \in B_\eta(F^{-1}(\hat{x})) \).

Suppose then that there are \( \hat{x}, \hat{y} \in \mathbb{R} \times S^1 \) with lifts \( x, y \in \mathbb{R}^2 \) such that there is an \( \epsilon \)-chain \( \{x_i\}_{i=0}^{n_1} \) for \( f \) with \( x_0 = x \) and \( x_{n_1} = x + (0, m) \) for some \( m \in \mathbb{N} \), and an \( \epsilon \)-chain \( \{y_i\}_{i=0}^{n_2} \) for \( f \) with \( y_0 = y \) and \( y_{n_2} = y + (0, -n) \) for some \( n \in \mathbb{N} \). Then, we can construct a sequence of arcs \( \{\gamma_n\}_{n=1}^{n_1} \) positively transverse to \( \tilde{F} \), and such that:

- \( \gamma_1 \) joins \( x_0 \) to \( f(x_0) \),
- \( \gamma_i \) joins \( f(x_{i-2}) \) to \( f(x_{i-1}) \) for \( 2 \leq i \leq n_1 - 2 \),
- \( \gamma_{n_1-1} \) joins \( f(x_{n_1-3}) \) to \( f^{-1}(x + (0, m)) \), and
- \( \gamma_{n_1} \) joins \( f^{-1}(x + (0, m)) \) to \( x + (0, m) \).

Then, letting \( \gamma = \prod_{i=1}^{n_1} \gamma_i \), we have that \( \gamma \) is an arc positively transverse to \( \tilde{F} \) joining \( x \) to \( x + (0, m) \).

Analogously, we construct an arc \( \beta \) positively transverse to \( \tilde{F} \) and joining \( y \) to \( y + (0, -n) \). In [Cal05] it is proved that \( \gamma \) and \( \beta \) project to disjoint (not necessarily simple) loops \( \hat{\gamma} \) and \( \hat{\beta} \) in \( \mathbb{R} \times S^1 \), and there is a connected component \( U \) of \( \mathbb{R} \times S^1 \setminus (\hat{\gamma} \cup \hat{\beta}) \) which is a topological essential annulus. As \( \hat{\gamma} \) and \( \hat{\beta} \) are positively transverse to \( \tilde{F} \), then \( \tilde{F} \) is transverse to the border of \( U \), either inwards or outwards. By the Poincaré Bendixon theorem, there exists a closed essential leaf \( l \) contained in \( U \). As the points \( \hat{x} \) and \( \hat{y} \) belong to the border of \( U \), \( l \) separates \( x \) from \( y \). ■

### 3.3 Atkinson’s Lemma.

Let \( T : X \to X \) be a measurable map of the metric space \( X \), and suppose that \( T \) is ergodic with respect to a probability measure \( \mu \). Let \( \varphi \in L^1(\mu) \). The
following Theorem, known as Atkinson’s Lemma, tells us that there is a total measure subset of \( X \) such that for any point \( x \) in this set we can find infinitely many iterates \( T^n(x) \) with both recurrence and small Birkhoff sums.

**Theorem 3.17 ([Atk76]).** Let \( (X, \mu) \) be a probability space, and suppose that \( \mu \) is ergodic with respect to a measurable transformation \( T : X \to X \). Let \( \phi : X \to \mathbb{R} \) be a measurable function with \( \int \phi d\mu = 0 \). Then, there exists a full measure set \( \tilde{X} \subset X \) such that for any \( x \in \tilde{X} \), any \( \epsilon > 0 \), and any set of positive measure \( A \subset X \) containing \( x \), it holds that

\[
T^n(x) \in A \quad \text{and} \quad \left| \sum_{i=0}^{n-1} \phi(T^i(x)) \right| < \epsilon
\]

for infinitely many values of \( n \in \mathbb{N} \).

4 Proof of Theorem B from Theorem A.

If \( \rho(f) \) has irrational slope and contains a rational point, then by Theorem 3.8 this point is realized by a periodic orbit. Then we are left then with the case that \( \rho(f) \) has rational slope and contains rational points.

We will prove now that if there is a rational point \( v \in \rho(f) \) that is not realized by a periodic orbit, then there is a power of \( \tilde{g} \) that is topologically conjugate to a homeomorphism \( \tilde{g} : T^2 \to T^2 \) satisfying the hypotheses of Theorem A; that is, \( \tilde{g} \) has a lift \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( \rho(g) \) is a vertical interval containing the origin in its interior, and such that \( (0,0) \in \rho(g) \) is not realized by a periodic orbit.

By Remark 3.5 there is \( A \in \text{GL}(2, \mathbb{Z}) \) such that \( \rho(AfA^{-1}) = Ap(f) \) is a vertical segment, containing the rational point \( v' = (p'_1/q, p'_2/q) \) given by \( v' = Av \). By Remark 3.2 if \( g_0 = (AfA^{-1})v' \), then \( \rho(g_0) = q' \rho(AfA^{-1}) \), and then \( \rho(g_0) \) is a vertical interval containing the point \( w = q'v' = (p'_1, p'_2) \in \mathbb{Z}^2 \). We know that \( \rho(T_1^{-p'_1}T_2^{-p'_2}g_0) = T_1^{-p'_1}T_2^{-p'_2}\rho(g_0) \), and therefore, if \( g = T_1^{-p'_1}T_2^{-p'_2}g_0 \), \( \rho(g) \) is a vertical interval containing the point \( T_1^{-p'_1}T_2^{-p'_2}(w) = (0,0) \). Let \( \tilde{A} \), \( \tilde{g} \) and \( \tilde{g}_0 \) be the homeomorphisms of \( T^2 \) lifted by \( A \), \( g \) and \( g_0 \), respectively. Then \( \tilde{g} = \tilde{g}_0 \), and as \( g_0 = (AfA^{-1})v' = Af'A^{-1} \) we have that \( \tilde{g} \) and \( f' \) are conjugate by \( A \).

It remains to see that \( (0,0) \in \rho(g) \) is not realized by a periodic orbit for \( \tilde{g} \). As \( v \in \rho(f) \) is not realized by a periodic point of \( f \), then \( v' = Av \in \rho(AfA^{-1}) \) is not realized by a periodic point for \( AfA^{-1} \), and then \( w = q'v' \in \rho(g_0) \) is not realized by a periodic point of \( \tilde{g}_0 \). Therefore \( (0,0) = T_1^{p'_1}T_2^{p'_2}(w) \in \rho(g) \) is not realized by a periodic point of \( \tilde{g} \), as we wanted.

**There is an annular, essential, set \( C \subset T^2 \) which is periodic for \( \tilde{f} \).**

As \( f' \) is conjugate to a homeomorphism satisfying the hypotheses of Theorem A, there is an annular, essential set \( C \) which is invariant for \( f' \), and therefore periodic for \( \tilde{f} \).
There is a simple, closed, essential curve in $T^2$ that is free forever for $\hat{f}$.

We saw above that $\hat{f}$ is topologically conjugate to a homeomorphism $\tilde{f}_0$ with a lift $f_0$ such that $\rho(f_0)$ is a vertical interval. Also, we saw that there is $k \in \mathbb{N}$ such that $\hat{g} = \hat{f}_0^k$ has an annular, essential, vertical invariant set $C$, which is also a semi-attractor. By the fact that $\hat{f}$ is isotopic to the identity, it follows that $\hat{f}_0^i(C)$ is also vertical, for every $i$. By Theorem A, and as $C$ is a semi-attractor for $\hat{g}$, there is an essential, vertical curve $L \subset T^2$ disjoint from $C$, free forever for $\hat{g}$ and such that $\omega(x, \hat{g}) \subset C$ for all $x \in L$.

**Lemma 4.1.** Let $m \in \mathbb{N}$ be the minimal period of $\hat{f}_0$. Then, either $m = 1$, or $\hat{f}_0^i(C) \cap C = \emptyset$ for every $0 < i < m$.

By the fact that $C$ is annular and essential, this lemma gives us that the complement of the orbit of $C$ for $\hat{f}_0$ is a union of open annuli. That is, we have

$$\left( \bigcup_{i=0}^{m-1} \hat{f}_0^i(C) \right)^c = \bigcup_{i=0}^{m} U_i,$$

where the $U_i$ are open, essential, vertical and pairwise disjoint annuli. The minimal period of each $U_i$ is $m$, and $L \subset U_{i_0}$, for some $i_0$. Using this, we will prove the following.

**Lemma 4.2.** There is a simple, closed, essential curve $\gamma$ contained in $U_{i_0}$, such that $\hat{f}_0^m(\gamma) \cap \gamma = \emptyset$.

Lemma 4.2 implies that there exists an essential curve that is free forever for $\hat{f}$. To see this, let $\gamma$ be as in Lemma 4.2. By the fact that $\hat{f}_0^m(U_{i_0}) = U_{i_0}$, we get that

$$\hat{f}_0^m(\gamma) \cap \gamma = \emptyset \quad \text{for all } i \in \mathbb{Z},$$

and as the minimal period of $U_{i_0}$ is $m$, we then get that $\hat{f}_0^{im}(\gamma) \cap \hat{f}_0^{im}(\gamma) = \emptyset$ for every $i \in \mathbb{Z}$ and $j \in \{0, \ldots, m-1\}$. That is, $\gamma$ is free forever for $\hat{f}_0$. As $\hat{f}_0$ is conjugate to $\hat{f}$, there is also an essential curve that is free forever for $\hat{f}$, as we wanted.

We are left with the proofs of lemmas 4.1 and 4.2.

**Proof of Lemma 4.1.** Recall that $k \in \mathbb{N}$ is such that $\hat{g} = \hat{f}_0^k$ satisfies the hypotheses of Theorem A, the set $C$ is annular, vertical and invariant for $\hat{g}$, and $m$ denotes the minimal period of $C$ for $\hat{f}_0$. Therefore $m$ divides $k$.

We claim that, for all $i \in \mathbb{Z}$, $\hat{f}_0^i(C) \cap L = \emptyset$. To see this, suppose on the contrary there is $j$ such that

$$\hat{f}_0^j(C) \cap L \neq \emptyset. \quad (1)$$

As $C$ is periodic with period $m$, we have that $\hat{f}_0^i(C) \cap \hat{f}_0^im(L) \neq \emptyset$ for every $i$, and therefore

$$\hat{f}_0^i(C) \cap \hat{f}_0^{ik}(L) \neq \emptyset \quad \text{for every } i.$$

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This means that there is a sequence of \( x_n \in L \) such that the sequence \( (f^{nk}(x_n))_n \) has an accumulation point in \( \tilde{f}_0^\ell(C) \). However, the curve \( L \) was chosen such that \( \omega(x, \tilde{y}) \subset C \) for all \( x \in L \), and we must have then that \( \tilde{f}_0^\ell(C) = C \), and thus, by \( \prod_1 C \cap L \neq \emptyset \). This is a contradiction, as by Theorem A we know that \( L \) is disjoint from \( C \). This contradiction proves our claim.

Now, suppose the lemma does not hold. Then, \( m > 1 \) and there is \( 0 < j < m \) such that \( \tilde{f}_0^j(C) \cap C \neq \emptyset \). Let \( C_0 \subset \mathbb{R}^2 \) be a connected component of \( \pi^{j-1}(C) \), and let \( f_0 \) be a lift of \( f_0 \) such that:

\[ f_0^j(C_0) \cap C_0 \neq \emptyset, \quad \text{and} \]
\[ \text{pr}_1(\rho(f_0)) \text{ lies either strictly to the left or strictly to the right of zero} \]
(such a lift exists; if \( \text{pr}_1(\rho(f)) = \{0\} \) then \( C \) must be \( \tilde{f}_0 \)-invariant, and \( m = 1 \)).
Without loss of generality, suppose that \( \text{pr}_1(\rho(f_0)) \subset \mathbb{R}^+ \). By this, and as consequence of the claim we just proved, we get that if \( \ell \) is a lift of \( L \), there is \( n \in \mathbb{Z} \) such that \( f_0^n(C_0) \subset L(\ell) \) and \( f_0^{n+1}(C_0) \subset R(\ell) \), and also \( \ell \prec f_0(\ell) \).
Therefore, \( f_0^n(C_0) \cap f_0^{n+j}(C_0) = \emptyset \), and then \( C_0 \cap f_0^j(C_0) = \emptyset \), which contradicts the properties of \( f_0 \). This contradiction finishes the proof of the lemma.

In [K90] it is proved the following lemma (Lemma 3.2 in that article). A vertical line in \( \mathbb{R}^2 \) is the lift of a vertical circle in \( \mathbb{T}^2 \).

**Lemma 4.3.** Let \( h : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be an orientation preserving homeomorphism. Let \( n \in \mathbb{N} \) and suppose that \( \ell \subset \mathbb{R}^2 \) is a vertical Brouwer curve for \( f^n \). Then, there is a vertical Brouwer curve \( \ell' \) for \( f \). Also, if \( S \subset \mathbb{R}^2 \) is an open set containing the curves \( \ell, h(\ell), \ldots, h^n(\ell) \), then the curve \( \ell' \) can be chosen to be contained in \( S \).

We will use Lemma 4.3 in the proof of Lemma 4.2.

**Proof of Lemma 4.3** We recall that \( k \in \mathbb{N} \) is such that \( \tilde{y} = \tilde{f}_0^k \) satisfies the hypotheses of Theorem A, and \( m \) is the minimal period for \( f_0 \) of the annular set \( C \).

If \( \tilde{f}_0^m(L) \cap L = \emptyset \), then setting \( \gamma := L \) the lemma follows. Otherwise, suppose that \( \tilde{f}_0^m(L) \cap L \neq \emptyset \). Let \( V \) be a connected component of \( \pi^{j-1}(U_{10}) \), and let \( F \) be a lift of \( \tilde{f}_0^m \) such that \( F(V) = V \). Let \( \ell \subset \mathbb{R}^2 \) be the lift of \( L \) contained in \( V \).
As \( m \) divides \( k \), \( \tilde{f}_0^k(U_{10}) = U_{10} \); and the curve \( L \) was chosen such that it is free forever for \( \tilde{f}_0^k \). In particular, \( \tilde{f}_0^k(L) \cap L = \emptyset \). Therefore, \( F^{k/m}(\ell) \cap \ell = \emptyset \), and \( F(\ell) \cap L \neq \emptyset \). By Lemma 4.3 there is a vertical Brouwer curve \( \Gamma \) for \( F \) contained in \( V \). Let \( \gamma = \pi'(\Gamma) \subset U_{10} \). Then \( \gamma \) is a vertical circle, and \( \tilde{f}_0^m(\gamma) \cap \gamma = \emptyset \), which concludes the proof of the Lemma.

**5 Proof of Theorem A, part I: construction of the curves \( \tilde{l}_i \), and items (2), (3) and (4).**

We begin by mentioning a related result for homeomorphisms of the compact annulus by Le Calvez. For a homeomorphism \( F : S^1 \times [0,1] \rightarrow S^1 \times [0,1] \) isotopic...
to the identity, the rotation set of some lift \( f : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1] \) is defined as the set of all accumulation points of sequences of the form

\[
\left\{ \frac{f_{m_i}(x_i)_1 - (x_i)_1}{m_i} \right\}_{i \in \mathbb{N}}
\]

where \( m_i \to \infty \) and \( x_i \in \mathbb{R} \times [0,1] \). In this case the rotation set \( \rho(f) \) is a compact interval \( I \subset \mathbb{R} \) (possibly degenerate). Also, if \( \Lambda \subset S^1 \times [0,1] \) is a compact invariant set, we can define the rotation set of \( \Lambda \), denoted \( \rho(\Lambda, f) \), as the set of all accumulation points of sequences of the form

\[
\left\{ \frac{f_{m_i}(x_i)_1 - (x_i)_1}{m_i} \right\}_{i \in \mathbb{N}}
\]

with \( m_i \to \infty \) and \( x \in \Pi^{-1}(\Lambda) \), where \( \Pi : \mathbb{R} \times [0,1] \to S^1 \times [0,1] \) is the canonical projection.

The following theorem was proven for \( C^1 \) diffeomorphisms in [Cal91], but by the results of [Cal05] it is also valid for homeomorphisms (see Theorem 9.1 in that article).

**Theorem 5.1.** Let \( F : S^1 \times [0,1] \to S^1 \times [0,1] \) be a homeomorphism isotopic to the identity with a lift \( f : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1] \) that has no fixed points and whose rotation set is an interval containing 0 in its interior.

Then, there exists a finite non-empty family \( \{ \gamma_i \} \) of essential, pairwise disjoint, free curves for \( F \) such that, the maximal invariant set contained between two consecutive curves has rotation set contained either strictly to the right or strictly to the left of 0 (see Fig. 3).

![Figure 3: Illustration for Theorem 5.1. The free curves \( \gamma_1 \) and \( \gamma_2 \) are free for \( F \).](image)

In Section 5.1 we will adapt this result for the torus case, and in this way we will construct a family of curves satisfying item (3) from Theorem A.
5.1 Construction of the curves $\tilde{l}_i$ satisfying item (3).

We will prove the following.

**Proposition 5.2.** Let $\tilde{f}$ and $f$ be as in Theorem A. There exists a finite family $\{\tilde{l}_i\}_{i=0}^r$ of pairwise disjoint, simple, closed, essential curves $\tilde{l}_i \subset T^2$ such that if $\Theta_i$ is the maximal invariant set of $[\tilde{l}_i, \tilde{l}_{i+1}]$, then $\Theta_i$ is non-empty, and $\rho(\Theta_i, f)$ is contained either in $\{0\} \times R^+$ or in $\{0\} \times R^-$.

Also, the family $\{\tilde{l}_i\}$ satisfies that, for any $i$, if $\rho(\Theta_i, f) \subset \{0\} \times R^+$ then $\rho(\Theta_{i+1 \mod r}, f) \subset \{0\} \times R^-$, and if $\rho(\Theta_i, f) \subset \{0\} \times R^-$ then $\rho(\Theta_{i+1 \mod r}, f) \subset \{0\} \times R^+$.

**Remark 5.3.** As the sets $\Theta_i$ are compact, the sets $\rho(\Theta_i, f)$ are also compact (see Theorem 3.3). Therefore, the fact that $\rho(\Theta_i, f)$ is contained either in $\{0\} \times (0, \infty)$ or in $\{0\} \times (-\infty, 0)$ means actually that $\rho(\Theta_i, f)$ is contained either in $\{0\} \times (\epsilon, \infty)$ or in $\{0\} \times (-\infty, -\epsilon)$, for some $\epsilon > 0$ and for any $i$. Therefore the family $\{\tilde{l}_i\}$ given by Proposition 5.2 satisfies item (3) from Theorem A.

**Remark 5.4.** By the Brouwer Translation Lemma (see Section 3.2), the hypothesis in Theorem A that $(0, 0) \in \rho(f)$ is not realized by a periodic orbit is equivalent to the fact that $f$ has no fixed points. Therefore all the results from Section 3.2 apply to $f$.

To prove Proposition 5.2 it will be convenient to work on the lift $R \times S^1$ of $T^2$. Recall our notation for the canonical projections:

$$R^2 \xrightarrow{\pi} R \times S^1 \xrightarrow{\pi''} T^2, \quad \text{and} \quad \pi' = \pi'' \circ \pi.$$

We will first prove the following.

**Lemma 5.5.** For $\tilde{f}$ and $f$ as in Theorem A, let $F : R \times S^1 \to R \times S^1$ be the lift of $f$ such that $F \circ \pi = \pi \circ f$. Then:

1. $CR(F) \neq \emptyset$, and $CR(F) = \Lambda^+ \cup \Lambda^-$, where $\Lambda^+$ and $\Lambda^-$ are closed disjoint $F$-invariant sets such that, denoting $\Lambda^\pm = \pi''(\Lambda^\pm) \subset T^2$, we have $\rho(\Lambda^+, f) \subset \{0\} \times (\epsilon, \infty)$ and $\rho(\Lambda^-, f) \subset \{0\} \times (-\infty, -\epsilon)$, for some $\epsilon > 0$.

2. There exist simple, closed, essential curves $l_0 < l_1 < \cdots < l_r = T_1(l_0)$ on $R \times S^1$ which are free for $F$, and such that they ‘separate’ $\Lambda^+$ from $\Lambda^-$, that is:

   (a) $CR(F) \cap \bigcup_{i=0}^r l_i = \emptyset$,

   (b) for $0 \leq i < r$, the set $\Lambda_i := CR(F) \cap (l_i, l_{i+1})$ is compact, non-empty and $F$-invariant,

   (c) for $0 \leq i < r$, either $\Lambda_i \subset \Lambda^+$ or $\Lambda_i \subset \Lambda^-$, and

   (d) if $\Lambda_i \subset \Lambda^+$, then $\Lambda_{i+1} \subset \Lambda^-$, and if $\Lambda_i \subset \Lambda^-$, then $\Lambda_{i+1} \subset \Lambda^+$, for any $0 \leq i < r - 1$ (see Fig. 4).
Proof. First we observe the following elementary fact. There exists an isotopy $(\tilde{H}_t)_{t \in [0,1]}$ between the identity and $\tilde{f}$ with the property that if $(H_t)_{t \in [0,1]}$ is any isotopy between the identity and $f$, and if $(\tilde{H}_t)_{t \in [0,1]}$ is the lift of $(H_t)_{t \in [0,1]}$ with $H_0 = \text{Id}$, then $H_1 = f$. To see this just observe that if $(\tilde{H}_t)_{t \in [0,1]}$ is any isotopy between the identity and $\tilde{f}$, and if $(H_t)_{t \in [0,1]}$ is the lift of $(\tilde{H}_t)_{t \in [0,1]}$ with $H_0 = \text{Id}$, then $H_1 = f + (a, b)$, for some $a, b \in \mathbb{Z}$. Defining $H_t = H_t' + t(-a, -b)$, for $t \in [0,1]$, we have that $(H_t)_{t \in [0,1]}$ is an isotopy between the identity and $f$ which projects to an isotopy $(\tilde{H}_t)_{t \in [0,1]}$ on $T^2$ between the identity and $\tilde{f}$ with the desired properties.

Now, let $\mathcal{F}$ be the Brouwer foliation of $T^2$ transversal to $\tilde{H}$ given by Theorem 3.15. Let $\tilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to $\mathbb{R} \times S^1$.

Part 1. 

$CR(\mathcal{F})$ is non-empty. Let $\phi : T^2 \to \mathbb{R}$ be given by 

$$\phi(x) = f(x)_1 - x_1,$$

where $x \in \mathbb{R}^2$ is any lift of $\tilde{x}$, and let $\phi_1 = \text{pr}_1 \circ \phi$. Then 

$$\sum_{i=0}^{n-1} \phi_1(\tilde{f}^i(\tilde{x})) = f(x)_1 - x_1.$$ 

Let $\mu$ be any ergodic measure for $f$. By hypothesis, $\int \phi_1 \, d\mu = \rho(\mu, f) = 0$, and by Atkinson’s Theorem 3.17 there exists a full $\mu$-measure set $X \subset T^2$ such that for any $\tilde{x} \in X, x \in \pi^{-1}(\tilde{x})$ and $\epsilon > 0$ we have that 

$$d(\tilde{f}^n(\tilde{x}), \tilde{x}) < \epsilon, \quad \text{and} \quad \left| \sum_{i=0}^{n-1} \phi_1(\tilde{f}^i(\tilde{x})) \right| = |f^n(x)_1 - x_1| < \epsilon$$

for infinitely many values of $n \in \mathbb{N}$. This means that $\pi(x)$ is recurrent for $F$, and in particular $CR(F) \neq \emptyset$.

Definition of $\Lambda^+$ and $\Lambda^-$. As $f$ has no fixed points by hypothesis (see Remark 5.4), by Franks’ Lemma 3.13 there is $\epsilon_1 > 0$ such that $f$ has no periodic $\epsilon_1$-chains, and by Theorem 3.16 there is $\epsilon_2 > 0$ such that, if there are $\hat{x}, \hat{y} \in R \times S^1$ with lifts $x, y \in R^2$ such that:

$$\Lambda_i \quad \Lambda_{i+1}$$

$$\begin{array}{ccc}
\tilde{l}_i & l_{i+1} & l_{i+2} \\
\end{array}$$

Figure 4: The sets $\Lambda_i$ and the curves $l_i$. 

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there is an \( \epsilon_2 \)-chain for \( f \) from \( x \) to \( x + (0, m) \) for some \( m \in \mathbb{N} \), and

there is an \( \epsilon_2 \)-chain for \( f \) from \( y \) to \( y + (0, -n) \) for some \( n \in \mathbb{N} \),

then there exists a compact leaf of \( \hat{F} \) that separates \( \hat{x} \) from \( \hat{y} \). Let \( \epsilon_0 = \min\{\epsilon_1, \epsilon_2\} \).

We define \( \Lambda^+ \subset \mathbb{R} \times S^1 \) as the set of points \( \hat{x} \in CR(F) \) such that, if \( x \in \pi^{-1}(\hat{x}) \), there exists an \( \epsilon_0 \)-chain \( \{x_i\}_{i=0}^n \) for \( f \), with \( x_0 = x \) and \( x_n = x + (0, m) \) for some \( m \in \mathbb{N} \). Analogously, we define \( \Lambda^- \subset \mathbb{R} \times S^1 \) as the set of points \( \hat{x} \in CR(F) \) such that, if \( x \in \pi^{-1}(\hat{x}) \), there exists an \( \epsilon_0 \)-chain \( \{x_i\}_{i=0}^n \) for \( f \), with \( x_0 = x \) and \( x_n = x + (0, -m) \), for some \( m \in \mathbb{N} \).

\( \Lambda^+ \) and \( \Lambda^- \) are non-empty. We prove that \( \Lambda^+ \) is non-empty; the case of \( \Lambda^- \) is similar. As \( \rho(f) \) is a vertical interval containing the origin in its interior, by Proposition 3.10, there exists an ergodic measure \( \mu \) with respect to \( f \) with \( \rho(\mu)_2 > 0 \). By Birkhoff’s Theorem, there exists a set \( X \subset \mathbb{T}^2 \) of full \( \mu \)-measure such that for \( \hat{x} \in X \) and \( x \in \pi^{-1}(\hat{x}) \), we have

\[
\rho(\hat{x}, f) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(\tilde{f}^i(\hat{x})) = \int \phi d\mu = \rho(\mu, f).
\]

By Atkinson’s Lemma 3.17, there exists a full measure set \( X' \subset \mathbb{T}^2 \) such that if \( \hat{x} \in X' \) and \( x \in \pi^{-1}(\hat{x}) \), then for all \( \epsilon > 0 \) there are infinitely many values of \( n > 0 \) such that

\[
d(\tilde{f}^n(\hat{x}), \hat{x}) < \epsilon \quad \text{and} \quad \left| \sum_{i=0}^{n-1} \phi_1(\tilde{f}^i(\hat{x})) \right| = |f^n(x) - x_1| < \epsilon
\]

for and for infinitely many values of \( n \in \mathbb{N} \).

Let \( \hat{y} \in X \cap X' \) and \( y \in \pi^{-1}(\hat{y}) \). Then, given \( \epsilon > 0 \) there is an increasing sequence of naturals \( \{r_n\}_n \) and a sequence of integers \( \{s_n\}_n \) such that

\[
|f^{r_n}(y) - y - (0, s_n)| < \epsilon,
\]

and also \( \rho(\hat{y}, f)_2 = \rho(\mu, f)_2 > 0 \). Therefore \( \lim_n s_n = \infty \), and in particular \( s_n > 0 \) for \( n \) sufficiently large. As \( d(\hat{y}, F^r(\hat{y})) < \epsilon \), and as the choice of \( \epsilon > 0 \) was arbitrary, we have that \( \hat{y} \) is recurrent for \( F \), and in particular \( \hat{y} \in CR(F) \). Therefore \( \hat{y} \in \Lambda^+ \), and \( \Lambda^+ \) is non-empty.

**It holds** \( CR(F) = \Lambda^+ \cup \Lambda^- \). Observe that by the definition of the sets \( \Lambda^+ \) and \( \Lambda^- \), we have that \( \Lambda^+ \cup \Lambda^- \subset CR(F) \), and then we only need to prove that \( CR(F) \subset \Lambda^+ \cup \Lambda^- \). Suppose by contradiction that there is \( \hat{x} \in CR(F) \setminus (\Lambda^+ \cup \Lambda^-) \). Then, by definition of \( \Lambda^+ \) and \( \Lambda^- \), there exists an \( \epsilon_0 \)-chain for \( f \) starting and ending in \( x \), that is, a periodic \( \epsilon_0 \)-chain for \( f \). By definition of \( \epsilon_0 \), we have \( \epsilon_0 \leq \epsilon_1 \), where \( \epsilon_1 \) is the constant given by Franks’ Lemma 3.13 and therefore that lemma implies that there is a fixed point for \( f \), a contradiction. Therefore we must have \( CR(F) \subset \Lambda^+ \cup \Lambda^- \) as we wanted.

**The sets \( \Lambda^+ \) and \( \Lambda^- \) are disjoint and closed.** We will prove that \( \Lambda^+ \cap \Lambda^- = \emptyset \). As \( CR(F) = \Lambda^+ \cup \Lambda^- \) and \( CR(F) \) is closed, this will imply
that $\Lambda^+$ and $\Lambda^-$ are closed and disjoint. Suppose by contradiction that there is $\hat{x} \in \overline{\Lambda^+} \cap \overline{\Lambda^-}$. Let $\hat{y} \in \Lambda^+$ and $\hat{z} \in \Lambda^-$, be such that $d(\hat{y}, \hat{x}) < \epsilon_0/3$ and $d(\hat{z}, \hat{x}) < \epsilon_0/3$, and let $\{y_i\}_{i=0}^{n_1}$ and $\{z_i\}_{i=0}^{n_2}$ be $\epsilon_0/3$-chains for $f$ such that $y_0 \in \pi^{-1}(\hat{y})$, $y_{n_1} = y_0 + (0, m_1)$, $z_0 \in \pi^{-1}(\hat{z})$, $|z_0 - y_0| < 2\epsilon_0/3$ and $z_{n_2} = y_0 + (0, -m_2)$, for some $m_1, m_2 \in \mathbb{N}$. We now show that we can concatenate integer translates of these chains $\{y_i\}$ and $\{z_i\}$ to get a periodic chain for $f$. For each $0 \leq i < m_2$ define the $\epsilon_0/3$-chain $\{y_i\}_{i=0}^{n_1}$ for $f$ as the translate of $\{y_i\}_{i=0}^{n_1}$ by $T_2^{im_1}$, that is, 

$y_i' = T_2^{im_1}y_i$, for $0 \leq l < n_1$, 

and for each $0 \leq j < m_1$, define the $\epsilon_0/3$-chain $\{z_i\}_{i=0}^{n_2}$ for $f$ as the translate of $\{z_i\}_{i=0}^{n_2}$ by $T_2^{jm_2}$, that is, 

$z_i' = T_2^{jm_2}z_i$, for $0 \leq k < n_2$. 

Define then the $\epsilon_0$-chain $\{w_i\}_{i=0}^{n_1+n_2m_1}$ for $f$ as the concatenation of the chains defined above, given by 

$w_{in_1+l} = y_i'$, for $0 \leq i < m_2$ and $0 \leq l < n_1$, 

$w_{m_2n_1+jn_2+k} = z_i'$, for $0 \leq j < m_1$ and $0 \leq k < n_2$, and 

$w_{n_1m_2+n_2m_1} = w$. 

Then, $\{w_i\}_{i=0}^{n_1+n_2m_1}$ is a periodic $\epsilon_0$-chain for $f$. By Franks’ Lemma 3.13 this is a contradiction, and therefore there cannot be $\hat{x} \in \overline{\Lambda^+} \cap \overline{\Lambda^-}$. As we mentioned, this implies that $\Lambda^+$ and $\Lambda^-$ are closed and disjoint.

Before proving that the sets $\Lambda^+$ and $\Lambda^-$ are $F$-invariant and the last claim of Part 1, we will prove Part 2.

**Part 2. Construction of the family $\{l_i\}_{i=0}^{r}$.** By Theorem 3.16 and by the definition of the sets $\Lambda^+$ and $\Lambda^-$, for each $x \in \Lambda^+$, $y \in \Lambda^-$, there exists a compact leaf $l \in \mathcal{F}$ that separates $x$ from $y$. So, the set $\mathcal{F}_c$ of compact leaves of $\mathcal{F}$ is not empty. The union of the compact leaves of a foliation of $T^2$ is compact (see for ex. [Hae62]), and as $\mathcal{F}$ is a lift of a foliation of $T^2$, the set $\cup\mathcal{F}_c$ is closed as a subset of $\mathbb{R} \times S^1$ ($\cup\mathcal{F}_c$ denotes the union of the elements of $\mathcal{F}_c$). Observe that, as the leaves of $\mathcal{F}$ are Brouwer lines for $f$, the elements of $\mathcal{F}_c$ are free curves for $F$.

**Claim:** $CR(F) \cap \mathcal{F}_c = \emptyset$.

Let $l \in \mathcal{F}_c$, and without loss of generality, assume that $l \prec F(l)$. Let $\delta := d(l, F(l)) > 0$, and let $x \in l$. Observe that $F(R(l)) = R(F(l))$, and then if $\{x_i\}_{i=0}^r$ is any $\delta/2$-chain with $x_0 = x$, it holds that $d(x_i, l) > \delta/2$ for all $0 < i \leq r$, and therefore $x$ is not chain recurrent for $F$. As the choice of $l \in \mathcal{F}_c$ and $x \in l$ was arbitrary, we have that $CR(F) \cap \mathcal{F}_c = \emptyset$, which proves our claim.

This claim gives us that $CR(F)$ has an open cover $\mathcal{U}'$ whose elements are the connected components of $\mathbb{R} \times S^1 \setminus \cup\mathcal{F}_c$, which are sets of the form $(l, l')$, $\cdots$.
with \( l, l' \in \mathcal{F}_c \). By definition of the sets \( \Lambda^+ \) and \( \Lambda^- \), and by Theorem 3.16 we have that for any element \( (l, l') \) of \( \mathcal{U}' \),

\[
\text{either } CR(F) \cap (l, l') \subset \Lambda^+, \text{ or } CR(F) \cap (l, l') \subset \Lambda^-.
\]

Now, fix \( l_* \in \mathcal{F}_c \). The compact set \( CR(F) \cap [l_*, T_1(l_*)] \) has a finite subcover \( \mathcal{U}'' \subset \mathcal{U}' \), of the form \( \mathcal{U}'' = \{ (l_{i, 0}^{l_{i, 1}+1} ) \}_{i=0}^{r'} \). We reindex the curves \( l'_i \) in a way that \( l'_i < l'_{i+1} \) for \( 0 \leq i < 2^r - 1 \), and we extract from the family of compact leaves \( \{ l'_i \}_{i=0}^{r'} \) a subfamily \( \{ l_i \}_{i=0}^{r-1} \) which is minimal with respect to the following property: if \( l_* = T_1(l_0) \), then for each \( 0 \leq i < r \)

\[
either \emptyset \neq CR(F) \cap (l_i, l_{i+1}) \subset \Lambda^+, \text{ or } \emptyset \neq CR(F) \cap (l_i, l_{i+1}) \subset \Lambda^-.
\]

As a consequence we have that, if for \( 0 \leq i < r \) we define

\[
\Lambda_i = CR(F) \cap (l_i, l_{i+1}),
\]

then,

1. \( \Lambda_i \neq \emptyset \) for all \( 0 \leq i < r \), and
2. if \( \Lambda_i \subset \Lambda^+ \) then \( \Lambda_{i+1} \subset \Lambda^- \), and if \( \Lambda_i \subset \Lambda^- \) then \( \Lambda_{i+1} \subset \Lambda^+ \), for any \( i \in \mathbb{Z} / r \mathbb{Z} \).

This concludes the construction of the family \( \{ l_i \}_{i=0}^{r-1} \) satisfying items (a), (c) and (d) from Part 2 of the lemma. We also have that \( \Lambda_i \neq \emptyset \) for all \( 0 \leq i < r \), so to prove that \( \{ l_i \}_{i=0}^{r-1} \) satisfies item (b) it remains to prove that \( \Lambda_i \) is \( F \)-invariant, for each \( 0 \leq i < r \).

**For any** \( 0 \leq i < r \), \( \Lambda_i \) is **\( F \)-invariant**. Fix \( i \in \{ 0, \ldots, r - 1 \} \). First we prove that \( F(\Lambda_i) \subset \Lambda_i \). As \( CR(F) \) is \( F \)-invariant and \( \Lambda_i = CR(F) \cap (l_i, l_{i+1}) \), to show that \( F(\Lambda_i) \subset \Lambda_i \) it suffices to show that if \( x \in \Lambda_i \) then \( F(x) \in (l_i, l_{i+1}) \).

Suppose this is not true. Then there exists \( x_0 \in \Lambda_i \) such that \( F(x_0) \notin (l_i, l_{i+1}) \). Without loss of generality suppose that \( F(x_0) \in \overline{R(l_i)} \). Then, as \( l_{i+1} \) is free for \( F \) we must have that \( l_{i+1} < F(l_{i+1}) \). Let \( \delta_1 := d(l_i, F(l_i)) > 0 \). By the continuity of \( F \) there is \( \delta_2 > 0 \) such that if \( d(x, l_i) < \delta_2 \), then \( F(x) \in R(l_i) \) and \( d(F(x), l_i) > \delta_1 / 2 \). Let \( \delta = \min\{ \delta_2, \delta_1 / 2 \} \), and let \( \{ y_i \}_{i=0}^s \) be any \( \delta \)-chain for \( F \) with \( y_0 = x_0 \). Then \( d(y_1, l_i) < \delta_2 \) and then \( F(y_1) \in R(l_i) \) and \( d(F(y_1), l_i) > \delta_1 / 2 \). Therefore, \( y_2 \in R(l_i) \), and \( F(y_2) \in R(F(l_i)) \). Then \( y_3 \in R(l_i) \). By induction, we get that \( y_n \in R(l_i) \) for all \( n \geq 2 \). As \( \{ y_i \}_{i=0}^s \) was an arbitrary \( \delta \)-chain with \( y_0 = x_0 \), we then have that \( x_0 \) is not \( \delta \)-chain recurrent, which contradicts that \( x_0 \in \Lambda_i \subset CR(F) \). This contradiction gives us that \( F(x_0) \) must be contained in \( (l_i, l_{i+1}) \), and therefore \( F(\Lambda_i) \subset \Lambda_i \).

Now we prove that \( F^{-1}(\Lambda_i) \subset \Lambda_i \). Applying the arguments in last paragraph to \( F^{-1} \) we get that \( F^{-1}(CR(F^{-1}) \cap (l_i, l_{i+1})) \subset CR(F^{-1}) \cap (l_i, l_{i+1}) \), and as \( CR(F) = CR(F^{-1}) \) we get that \( F^{-1}(\Lambda_i) \subset \Lambda_i \).

As the choice of \( i \) was arbitrary, we conclude that for any \( i \), \( \Lambda_i \) is \( F \)-invariant, as we wanted. This finishes the proof of Part 2.
Now we proceed to the remaining part of the proof of Part 1.

**The sets \( \Lambda^+ \) and \( \Lambda^- \) are \( F \)-invariant.** We proved that for each \( i \) the set \( \Lambda_i \) is \( F \)-invariant, and contained either in \( \Lambda^+ \) or in \( \Lambda^- \). As both \( \Lambda^+ \) and \( \Lambda^- \) are contained in \( \cup \Lambda_i \), we conclude that \( \Lambda^+ \) and \( \Lambda^- \) are \( F \)-invariant.

There is \( \epsilon > 0 \) such that \( \rho(\Lambda^+, f) \subset \{0\} \times (\epsilon, \infty) \), and \( \rho(\Lambda^-, f) \subset \{0\} \times (-\infty, -\epsilon) \). We will deal only with the case of \( \rho(\Lambda^+, f) \); the case of \( \rho(\Lambda^-, f) \) is analogous. As \( \Lambda^+ \) is closed and \( F \)-invariant, \( \tilde{\Lambda}^+ \) is a compact \( f \)-invariant set. Let \( v^- \) the lower endpoint of \( \text{conv} \rho(\Lambda^+, f) \). It suffices to prove that \( (v^-)_2 > 0 \).

Suppose by contradiction that \( (v^-)_2 \leq 0 \). By Proposition 3.10, as \( v^- \) is an extremal point of \( \text{conv} \rho(\Lambda, f) \), there exists an ergodic measure \( \mu \) for \( f \) with \( \rho(\mu, f) = v^- \) and therefore \( \text{supp}(\mu) \subset \pi''(\Lambda^+) \). As \( (v^-)_2 \leq 0 \), then, as in the proof that \( \Lambda^+ \) and \( \Lambda^- \) are non-empty, with the aid of Birkhoff’s Theorem and Atkinson’s Lemma 3.17 we find a point \( \tilde{x} \in \text{supp}(\mu) \) such that

\[
f^n(x) - x - (0, -m) < \epsilon_0
\]

for \( x \in \pi''^{-1}(\tilde{x}) \) and for some \( n \in \mathbb{N} \) and \( m \in \mathbb{N}_0 \). If \( m > 0 \) this means that \( \pi(x) \in \Lambda^- \), and as \( x \in \pi''^{-1}(\text{supp}(\mu)) \) and \( \text{supp}(\mu) \subset \pi''(\Lambda^+) \), we have that \( \pi(x) \in \pi(\pi''^{-1}(\text{supp}(\mu))) = \pi''^{-1}(\pi''(\Lambda^+)) = \Lambda^+ \). Therefore \( \Lambda^+ \cap \Lambda^- \neq \emptyset \), which is a contradiction, and then we cannot have that \( m > 0 \). If \( m = 0 \) we have that \( x \) is \( \epsilon_0 \)-chain recurrent for \( f \), which by the definition of \( \epsilon_0 \) and by Franks’ Lemma 3.13 is a contradiction. Therefore we cannot have that \( (v^-)_2 \leq 0 \), as we wanted. This finishes the proof of Part 1, and of the lemma. \( \blacksquare \)

**Remark 5.6.** The fact that the sets \( \Lambda_i \subset (l_i, l_{i+1}) \) are non-empty and \( F \)-invariant implies the following. If \( \ell \subset \mathbb{R}^2 \) is a lift of \( l_i \) and if \( \ell' \subset \mathbb{R}^2 \) is the lift of \( l_{i+1} \) such that \( \ell \prec \ell' \prec T_1(\ell) \), then

\[
\bigcap_{n \in \mathbb{Z}} f^n((\ell, \ell')) = \pi^{-1}\left( \bigcap_{n \in \mathbb{Z}} F^n((l_i, l_{i+1})) \right) \supset \pi^{-1}(\Lambda_i) \neq \emptyset.
\]

**Proof of Proposition 5.2** By construction, the curves \( l_i \) are lifts of leaves from a foliation of \( \mathbb{T}^2 \); that is, the curves \( \tilde{l}_i = \pi''(l_i) \subset \mathbb{T}^2 \) are also compact leaves from a foliation of \( \mathbb{T}^2 \) (and therefore pairwise disjoint). As \( \pi'' : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{T}^2 \) is a covering map, the curves \( \tilde{l}_i \) are also essential, and by the definition of \( \pi'' \) it is easy to see that they are vertical. For any \( 0 \leq i < r \), if \( \Theta_i \subset \mathbb{T}^2 \) is as in Theorem A and \( \Lambda_i \subset \mathbb{R} \times \mathbb{S}^1 \) is as in Lemma 5.5, we observe that \( \Theta_i \) is non-empty: by item 2-(b) of Lemma 5.5, \( \emptyset \neq \pi''(\Lambda_i) \subset (l_i, l_{i+1}) \) is \( f \)-invariant, and then \( \pi''(\Lambda_i) \subset \Theta_i \).

We now prove that it holds item (3) from Theorem B; that is, \( \rho(\Theta, f) \) is contained either in \( \{0\} \times (0, \infty) \) or in \( \{0\} \times (-\infty, 0) \) for each \( i \).

Let \( i \in \{0, 1, \ldots, r-1\} \), and suppose first that \( \Lambda_i \subset \Lambda^+ \), where \( \Lambda^+ \) is as in Lemma 5.5. Then we will prove that \( \rho(\Theta_i, f) \subset \{0\} \times (0, \infty) \). Let \( v^- \) be the lower endpoint of the interval \( \text{conv}(\rho(\Theta_i, f)) \). To prove that \( \rho(\Theta_i, f) \subset \{0\} \times (0, \infty) \), it suffices then to prove that \( (v^-)_2 > 0 \). By contradiction, suppose that \( (v^-)_2 \leq 0 \).

By Proposition 3.10 we can find an ergodic measure \( \mu \) with support contained in \( \Theta_i \) and with \( \rho(\mu, f) = v^- \). As in the proof in Lemma 5.5 that the
sets $\Lambda^+$ and $\Lambda^-$ are non-empty, with the use of Atkinson’s Lemma we can find a point $\hat{x} \in \text{supp}(\mu)$ such that, for any $\epsilon > 0$ and $x \in \pi^{l-1}(\hat{x})$, it holds

$$|f^n(x) - x - (0, m)| < \epsilon$$

for some $m \leq 0$ and $n > 0$. 

Therefore if $\hat{x} \in \pi^{l-1}(\hat{x}) \subset R \times S^1$, $\hat{x}$ is recurrent for $F$, and in particular $\hat{x} \in CR(F)$. As $\hat{x} \in \text{supp}(\mu) \subset [\hat{l}_i, \hat{l}_{i+1}]$, $\hat{x}$ belongs to some integer translate of $\in [l_i, l_{i+1}]$. Therefore, there is an integer translate $\hat{x}'$ of $\hat{x}$ such that $\hat{x}' \in CR(F) \cap [l_i, l_{i+1}] = \Lambda_i$. As in (2) $\epsilon > 0$ is arbitrary and $m \leq 0$, we have that $\hat{x}' \in \Lambda^-$, and then $\Lambda_i \cap \Lambda^- \neq \emptyset$, which contradicts our assumption that $\Lambda_i \subset \Lambda^+$. Therefore we must have $(\nu^-)_2 > 0$, and this proves that $\rho(\Theta_i, f) \subset \{0\} \times (0, \infty)$, as we wanted.

Similarly, if $\Lambda_i \subset \Lambda^-$ we prove that $\rho(\Theta_i, f)$ is contained in $\{0\} \times (-\infty, 0)$. The choice of $i \in \{0, \ldots, r-1\}$ was arbitrary, and then by Remark 5.3 we have that, for each $i$, either $\rho(\Theta_i, f) \subset \{0\} \times (-\infty, -\epsilon)$ or $\rho(\Theta_i, f) \subset \{0\} \times (\epsilon, \infty)$ for some $\epsilon > 0$, and then for the family $\{\hat{l}_i\}_{i=0}^{r-1}$ it holds item (3) from Theorem A.

### 5.2 Proof of items (2) and (4), assuming item (1).

We recall that, in Theorem A, the sets $\Theta_i$ are the maximal invariant sets in $[\hat{l}_i, \hat{l}_{i+1}]$ for $\hat{f}$. Also, we recall items (1), (2) and (4) from that theorem.

1. One of the sets $\Theta_i$ is an essential, vertical, annular set.
2. The curves $\hat{l}_i$ are free forever for $\hat{f}$.
3. $\Omega(f) \subset \cup_i \Theta_i$.

We will prove items (2) and (4) assuming it holds item (1).

(1) $\Rightarrow$ (2). As we suppose it holds item (1), there is $i_0$ such that $\Theta_{i_0}$ is an essential, vertical, annular set. Let $C \subset R^2$ be a connected component of $\pi^{l-1}(\Theta_{i_0})$. Then $T^n_2(C) = C$ for all $n$, and $\pi^{l-1}(\Theta_{i_0}) = \cup_n T^n_1(C)$. By the fact that $\Theta_{i_0}$ is annular, essential and vertical, $C$ is a connected set such that $R^2 \setminus C$ has two unbounded connected components. By our hypothesis that $\rho(f) = \{0\} \times I$ we can easily deduce that for each $n$, $T^n_1(C)$ is $f$-invariant.

Let $S \subset R^2$ be the $f$-invariant open strip bounded by $C$ and $T_1(C)$. In the construction of the curves $\hat{l}_i$, we saw that $\hat{l}_i$ is disjoint from $\Theta_i$ for every $i$ and $j$. By this, by the fact that $\Theta_{i_0}$ is essential and vertical, and as the curves $\hat{l}_i$ are vertical, we have that for any $i$ there is only one lift $\hat{l}_i \subset R^2$ of the curve $\hat{l}_i$ which intersects $S$, and actually $\hat{l}_i \subset S$. By the invariance of $S$ we have that $f^n(\hat{l}_i) \subset S$ for all $n$ and $i$, and then $f^n(\hat{l}_i) \cap T^n_1(\hat{l}_i) = \emptyset$ for every $i$ and for all $n \in Z$ and $m \neq 0$. Also, by the construction of $\hat{l}_i$, the lifts $\hat{l}_i$ are Brouwer curves for $f$, and then $f^n(\hat{l}_i) \cap \hat{l}_i = \emptyset$ for every $i$ and all $n \in Z$. We conclude that $f^n(\hat{l}_i) \cap T^n_1(\hat{l}_i) = \emptyset$ for every $i$, and all $n, m \in Z$. This implies that the curves $\hat{l}_i$ are free forever for $\hat{f}$, and it holds item (2) from Theorem A.
(1) ⇒ (4). To prove it holds item (4) we let \( \tilde{x} \in T^2 \setminus \cup_i \Theta_i \), and we will prove that \( \tilde{x} \) is wandering. Let \( x \in \mathbb{R}^2 \) be a lift of \( \tilde{x} \) contained in the open strip \( S \) defined above. By the invariance of \( S \), we have that if \( B \subset S \) is a ball containing \( x \), then the iterates of \( B \) do not meet any integer translate of \( B \) that is outside \( S \), that is, \( f^n(B) \cap T_s^r(T_s^r(B)) = \emptyset \) for all \( r \neq 0 \) and \( n, s \in \mathbb{Z} \). Then, to get that \( \tilde{x} \) is wandering for \( \tilde{f} \) it suffices to show that there is an open ball \( B' \subset S \) containing \( x \) such that \( f^n(B') \cap T_s^r(B') = \emptyset \) for all \( n > 0 \) and \( r \in \mathbb{Z} \).

Let \( i_1 \) be an integer such that \( x \in [\ell_{i_1}, \ell_{i_1+1}] \). By hypothesis, \( \tilde{x} = \pi'(x) \notin \Theta_{i_1} \), and then there exists \( n_0 \in \mathbb{Z} \) such that \( \tilde{x} \) is contained in the open annulus bounded by \( \tilde{f}^{n_0}(\ell_{i_1}) \) and \( \tilde{f}^{n_0+2}(\ell_{i_1}) \). Then, \( x \) is contained in the open strip \( S' \subset \mathbb{R}^2 \) bounded by \( f^{n_0}(\ell_{i_1}) \) and \( f^{n_0+2}(\ell_{i_1}) \). As the curve \( \ell_{i_1} \) is a Brouwer curve for \( f \) (by construction of the curves \( \ell_i \)), then

\[
f^n(S') \cap S' = \emptyset \quad \text{for all } n \geq 2. \tag{3}
\]

Also, as the curve \( \ell_{i_1} \) is a lift of a vertical curve in \( T^2 \), we have that if \( B' \subset S' \) is an open ball containing \( x \) then \( T^2_s(B') \subset S' \) for all \( j \in \mathbb{Z} \). By this and by (3) we have that \( f^n(B') \cap T^2_s(B') = \emptyset \) for all \( n \geq 2 \), and then, there is a ball \( B'' \subset B' \) containing \( x \) such that \( f^n(B'') \cap T^2_s(B'') = \emptyset \) for all \( n > 0 \), which by last paragraph implies that \( \tilde{x} = \pi'(x) \) is wandering for \( \tilde{f} \). As \( \tilde{x} \in T^2 \setminus \cup_i \Theta_i \) was arbitrary, we get that \( \Omega(\tilde{f}) \subset \cup_i \Theta_i \), and it holds item (4) from Theorem A, as desired.

6 Proof of Theorem A, part II: proof of item (1).

6.1 Strategy and outline of the proof.

We recall that, for each \( i \), the set \( \Theta_i \) is the maximal invariant set of \([\tilde{l}_i, \tilde{l}_{i+1}]\) for \( \tilde{f} \). Also, we recall item (1) from Theorem A:

1. At least one from the sets \( \Theta_i \) is an essential, vertical, annular set.

Without loss of generality, from now on we make the following assumption:

**Assumption 6.1.** The family of curves \( \{\tilde{l}_i\} \) constructed in section 5 consists only of two curves \( \tilde{l}_0 \) and \( \tilde{l}_1 \), which are straight vertical circles.

To prove item (1) we will work in \( \mathbb{R}^2 \) with lifts of the curves \( \tilde{l}_i \), so we start by fixing a family of such lifts.

**Definition 6.2** (The curves \( \ell_i \)). For \( i \in \mathbb{N}_0 \) we define a lift \( \ell_i \subset \mathbb{R}^2 \) of the curve \( \tilde{l}_i \text{ mod } 2 \) in the following way. First define \( \ell_0 \subset \mathbb{R}^2 \) as any lift of \( \tilde{l}_0 \). Then, define \( \ell_1 \) as the lift of \( l_1 \) such that \( \ell_0 \prec \ell_1 \prec T_1(\ell_0) \). Then, for \( i = 0, 1 \), and \( j \in \mathbb{N} \) we define \( \ell_{2j+i} = T_1^j \ell_i \) (see Fig. 5).
Figure 5: The curves $\ell_i$. By Assumption 6.1 they are straight lines.

**Strategy.** The proof of item (1) will be by contradiction. We will suppose that none of the sets $\Theta_i$ is essential, and we will obtain that $\max |\text{pr}_1(\rho(f))| > 0$, which contradicts our hypothesis that $\rho(f)$ is an interval of the form $\{0\} \times I$.

We start by noting the following.

**Claim 6.3.** If none of the sets $\Theta_i$ is essential, then there is $n_0 \in \mathbb{N}$ such that
1. $f^{n_0}(\ell_i) \cap \ell_i+1 \neq \emptyset$ for all $i \geq 0$, or
2. $f^{n_0}(\ell_i) \cap \ell_{i-1} \neq \emptyset$ for all $i > 0$.

**Proof.** If there was $i_0$ such that $f^n(\ell_{i_0}) \cap \ell_{i_0+1} = \emptyset$ for all $n \in \mathbb{Z}$, then the maximal invariant set of $(\ell_{i_0}, \ell_{i_0+1})$ would be a connected set $A$ such that $\pi'(A) = \Theta_{i_0}$ is essential. Details are left to the reader. ■

Therefore, if we assume that none of the sets $\Theta_i$ is essential, we are in case 1 or 2 from Claim 6.3. From now on, without loss of generality we will assume that we are in case 1:

**Assumption 6.4.** There is $n_0 > 0$ such that $f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset$ for all $i \geq 0$. In particular $\ell_i \prec f(\ell_i)$ for all $i \geq 0$.

**Basic/ideal model.** We will now describe a basic model for the dynamics of $f$ under Assumption 6.4. We will give some hypotheses satisfied by this model, and we will see that these hypotheses imply that $\max \text{pr}_1(\rho(f)) > 0$. Finally, we will see that these hypotheses are satisfied by any homeomorphism within the hypotheses of Theorem A. This will conclude the proof of item (1) of Theorem A.

The hypotheses defining our basic model are the following:
H1 In each strip \([\ell_i, \ell_{i+1}]\) there exists a compact, connected set \(E_i\), with the following property: there is \(N_1 \in \mathbb{N}\) such that \(f^{N_1}(E_i) \subset R(\ell_{i+1})\) (see Fig. 6). Moreover, the integer \(N_1\) is independent of \(i\), and therefore, by the periodicity of \(f\),

\[
f^{N_1}(T_2^n(E_i)) \subset R(\ell_{i+1}) \quad \text{for all} \ i \geq 0 \text{ and } n \in \mathbb{Z}.
\]

The sets \(T_2^n(E_i)\) will be called rides.

H2 In each strip \([\ell_i, \ell_{i+1}]\) there exists a compact, connected set \(F_i\), with the property that for all \(n \geq 0\), \(f^n(F_i) \subset [\ell_i, \ell_{i+1}]\) (see Fig. 6). By the periodicity of \(f\) we have that

\[
f^n(T_2^m(F_i)) \subset [\ell_i, \ell_{i+1}] \quad \text{for all} \ n \geq 0 \text{ and } m \in \mathbb{Z}.
\]

The sets \(T_2^m(F_i)\) will be called anchors.

H3-1 The curve \(\ell_0\) contains an arc \(\gamma_0\) with one endpoint contained in a ride and the other in an anchor, that is, \(\gamma_0(0) \in T_2^n(E_0)\) and \(\gamma_0(1) \in T_2^n(F_0)\), for some \(r, s \in \mathbb{Z}\). Therefore, \(f^{N_1}(\gamma_0(0)) \in R(\ell_1)\) and \(f^{n}(\gamma_0(1)) \in [\ell_0, \ell_1]\) for all \(n \geq 0\).

Also, there is \(N_2 \in \mathbb{N}\) such that the image of \(f^{N_1}(\gamma_0)\) by \(f^{N_2}\) gets ‘stretched’, and it intersects the rides and anchors of the next strip \([\ell_1, \ell_2]\) in a ‘good’ way. By this we mean that \(f^{N_1+N_2}(\gamma_0)\) contains an arc \(\gamma_1\) such that \(\gamma_1 \subset R(\ell_1)\), \(\gamma_1(0) \in T_2^n(E_1)\), and \(\gamma_1(1) \in T_2^n(F_1)\) for some \(r, s \in \mathbb{Z}\) (see Fig. 7).

H3-n For each \(n \in \mathbb{N}\), the curve \(f^{n(N_1+N_2)}(\ell_0)\) contains an arc \(\gamma_n\) that has ‘good’ intersection with the rides and anchors of the \(n\)-th strip \([\ell_n, \ell_{n+1}]\). That is, \(\gamma_n \subset R(\ell_n)\), \(\gamma_n(0) \in T_2^n(E_n)\), and \(\gamma_n(1) \in T_2^n(F_n)\) for some \(r, s \in \mathbb{Z}\). Therefore, \(f^{N_1}(\gamma_n(0)) \in R(\ell_{n+1})\) and \(f^n(\gamma_n(1)) \in [\ell_n, \ell_{n+1}]\) for all \(n \geq 0\).

Also, the image of \(f(\gamma_n)\) by \(f^{N_2}\) gets ‘stretched’, and it intersects the rides and anchors of the next strip \([\ell_{n+1}, \ell_{n+2}]\) in a ‘good’ way. That is, \(f^{N_1+N_2}(\gamma_n)\) contains an arc \(\gamma_{n+1}\) such that \(\gamma_{n+1} \subset R(\ell_n)\), \(\gamma_{n+1}(0) \in T_2^n(E_{n+1})\), and \(\gamma_{n+1}(1) \in T_2^n(F_{n+1})\) for some \(r, s \in \mathbb{Z}\). Observe that \(\gamma_{n+1} \subset f^{N_1+N_2}(\gamma_n) \subset f^{n+1(N_1+N_2)}(\ell_0)\).

From these hypotheses it holds in particular that \(f^{n(N_1+N_2)}(\ell_0) \cap [\ell_n, \ell_{n+1}] \neq \emptyset\), for each \(n \in \mathbb{N}\), and therefore \(\max |p_1(\rho(f))| > 0\).

Now we mention two main difficulties in showing that these hypotheses hold for a general homeomorphism within the hypotheses of Theorem A.

- The iterates of \(\ell_0\) could avoid the rides of the second strip \([\ell_1, \ell_2]\). Precisely, we could have in principle that \(f^n(\ell_0) \cap T_2^n(E_1) = \emptyset\) for all \(n \geq 0\) and \(r \in \mathbb{Z}\) (see Fig. 8).

- Even if the iterates of \(\ell_0\) intersect the rides of every strip \([\ell_n, \ell_{n+1}]\), the necessary amount of iterates \(k_n\) to intersect the rides of the \(n\)-th strip \([\ell_n, \ell_{n+1}]\) could be such that \(n/k_n \to 0\) as \(n \to \infty\), and this would not imply that \(\max |p_1(\rho(f))| > 0\).
Figure 6: The *rides* and *anchors*.

Therefore, our main objectives now are the following:

1. Define (and prove that there exist) the *rides* and *anchors*. This will be done in Section 6.2.

2. Define *good intersection* of an iterate of \( \ell_0 \) with the rides and anchors of each strip \([\ell_i, \ell_{i+1}]\). This will be done in Section 6.3.

3. Prove there is uniform advance; that is, prove that there is \( N \in \mathbb{N} \) such that for each \( n \in \mathbb{N} \), \( f^{nN}(\ell_0) \) has good intersection with the rides and anchors of the \( n \)-th strip \([\ell_n, \ell_{n+1}]\). This will be done in sections 6.3 and 6.4.

### 6.2 Definition of the Rides and Anchors.

We start with a definition of a property satisfied by compact sets in \( \mathbb{R}^2 \), with respect to the lift \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) of a torus homeomorphism.

#### 6.2.1 Properties PR and PL.

**Definition 6.5.** Let \( \tilde{h} : \mathbb{T}^2 \to \mathbb{T}^2 \) be a homeomorphism isotopic to the identity, and let \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) be a lift of \( \tilde{h} \). Let \( C \subset \mathbb{R}^2 \) compact and connected, \( k \in \mathbb{R}^+ \) and \( p \in \mathbb{R}^2 \). We say that \( C \) satisfies the property \( \text{PL}(k, p) \) if the following hold (see Fig. 9):

1. There exist horizontal (disjoint) straight lines \( r_1 \prec r_2 \), (oriented as going to the right) such that \( r_1 \cap C \neq \emptyset \), \( r_2 \cap C \neq \emptyset \), and such that the strip \((r_1, r_2) \subset \mathbb{R}^2 \) contains a ball of radius \( k \) centered in \( p \).
2. The point \( p \) belongs to the (unique) connected component of \((r_1, r_2) \setminus C\) which is unbounded to the left.

Analogously, we say that \( C \) satisfies the property \( \text{PR}(k, p) \) if it holds item (1) from property \( \text{PL}(k, p) \) and \( p \) belongs to the (unique) connected component of \((r_1, r_2) \setminus C\) which is unbounded to the right.

The following lemma will be an important tool in the proof of Theorem A.

**Lemma 6.6.** Let \( h : T^2 \to T^2 \) be a homeomorphism isotopic to the identity, let \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) be a lift of \( h \), and for \( x \in \mathbb{R}^2 \), denote by \( v(x) \subset \mathbb{R}^2 \) the vertical straight line that passes through \( x \). There exists \( k > 0 \) such that if a compact connected set \( C \subset \mathbb{R}^2 \) satisfies \( \text{PL}(k, p) \) (resp. \( \text{PR}(k, p) \)) for some \( p \in \mathbb{R}^2 \), then \( h(C) \cap R(v(h(p))) \neq \emptyset \) (resp. \( h(C) \cap L(v(h(p))) \neq \emptyset \), see Fig. 9).

**Proof.** First observe that as \( h \) is the lift of a homeomorphism of \( T^2 \), \( \|h - \text{Id}\|_0 < \infty \). Define \( k = 2 \|h - \text{Id}\|_0 + 1 \). Suppose that \( C \) satisfies the property \( \text{PL}(k, p) \) for some \( p \in \mathbb{R}^2 \) (the case of \( \text{PR}(k, p) \) is similar). Then there are two horizontal straight lines \( r_1 < r_2 \) intersecting \( C \) and such that \((r_1, r_2)\) contains a ball of radius \( k \) centered in \( p \), and \( p \) belongs to the connected component \( U_L \) of \((r_1, r_2) \setminus C\) which is unbounded to the left. Observe that by the definition of \( k, \min \text{pr}_2(h(r_1)) > h(p)_2 > \max \text{pr}_2(h(r_2)) \), and then if \( w \) is the horizontal
Figure 8: The curves $f^n(\ell_0)$ and $f^m(\ell_0)$, with $m \gg n$, do not intersect the rides of the strip $[\ell_1, \ell_2]$.

straight line passing through $h(p)$, we have

$$w \subset (h(r_1), h(r_2)).$$  \hfill (4)

Let $U_R$ be the connected component of $(r_1, r_2) \setminus C$ which is unbounded to the right. As $\|h - \text{Id}\|_0 < \infty$, $h(U_L)$ is unbounded to the left and bounded to the right, and also $h(U_R)$ is unbounded to the right and bounded to the left.

We claim that for this choice of $k$, we have $h(C) \cap R(v(h(p))) \neq \emptyset$. If this was not the case, then we would have that $C \cap w_+ = \emptyset$, where $w_+ = w \cap R(v(h(p))$. By (4), $w_+ \subset (h(r_1), h(r_2))$, and therefore $w_+$ is contained in $h(U_R)$. Then $h(p)$ belongs to $h(U_R)$, which is unbounded to the right, which contradicts the fact that $p$ belongs to the connected component $U_L$ of $(r_1, r_2) \setminus C$ which is bounded to the right. We must have then that $h(C) \cap R(v(h(p))) \neq \emptyset$, and this proves the lemma.

The following lemma relates the properties PL and PR and the curves $\ell_i$.

**Lemma 6.7.** Let $i, j \in \mathbb{N}$ and suppose that $n \in \mathbb{Z}$ is such that $f^n(\ell_i) \cap \ell_j \neq \emptyset$. Then, there exists a constant $K > 0$ such that, if $C \subset \mathbb{R}^2$ is a continuum contained in the open strip bounded by $\ell_i$ and $\ell_j$ and such that $\text{diam}_2(C) \geq K$, then:

- If $i < j$, then $f^n(C) \cap R(\ell_j) \neq \emptyset$. 

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• **If** \( j < i \), **then** \( f^n(C) \cap L(\ell_j) \neq \emptyset \).

**Proof.** Without loss of generality suppose that \( n > 0 \). By Lemma 6.6 applied to \( f^n \) there is a constant \( k > 0 \) such that, if \( C \) is a continuum that satisfies the property PL\((k,p)\) (or PR\((k,p)\)) for some \( p \in \mathbb{R}^2 \), then \( f^n(C) \cap R(v(f^n(p))) \neq \emptyset \) (resp., \( f^n(C) \cap L(f^n(p)) \neq \emptyset \)).

We treat the case \( i < j \), the case \( i > j \) being similar. By hypothesis \( f^n(\ell_i) \cap \ell_j \neq \emptyset \). Take \( x \in f^{-n}(\ell_j) \cap \ell_i \) and define \( K = k + 1 \). Suppose that \( C \) is a continuum contained in \( (\ell_i, \ell_j) \) and such that \( \text{diam}_2(C) \geq K \). Then there is \( s \in \mathbb{Z} \) such that \((T_s^2(x))_2 - k, (T_s^2(x))_2 + k) \subset \text{pr}_2(C)\).

As we are under Assumption 6.1, the line \( \ell_i \) is straight, and as \( C \subset R(\ell_i) \), it is easy to see that \( C \) satisfies property PL\((k,x)\). Therefore as \( f^n(T_s^2(x)) \in \ell_j \), Lemma 6.6 gives us that \( f^n(C) \cap R(\ell_j) = f^n(C) \cap R(v(f^n(T_s^2(x)))) \neq \emptyset \), as we wanted. \( \blacksquare \)

### 6.2.2 The sets \( L^i_\infty, R^i_\infty \), and \( X_i \).

For each \( i \in \mathbb{N} \), we define the sets \( L^i_\infty \) and \( R^i_\infty \), which in some sense are the ‘stable’ and ‘unstable’ sets (resp.) of the maximal invariant set in \([\ell_i, \ell_{i+1}]\) for \( f \). Let

\[
R^i_\infty = \bigcap_{n \in \mathbb{Z}} R(f^n(\ell_i)), \quad L^i_\infty = \bigcap_{n \in \mathbb{Z}} L(f^{-n}(\ell_{i+1})), \quad \text{and} \quad X_i = L^i_\infty \cup R^i_\infty
\]

(see Fig. 10)

By definition, the sets \( R^i_\infty, L^i_\infty \) and \( X_i \) are \( f \)-invariant. As we are under Assumption 6.4 \( \ell_i < f(\ell_i) \) for all \( i \), and therefore we have that

\[
R^i_\infty = \{ x \in \mathbb{R}^2 : f^{-n}(x) \in R(\ell_i) \ \forall n \geq 0 \},
\]

and

\[
L^i_\infty = \{ x \in \mathbb{R}^2 : f^n(x) \in L(\ell_{i+1}) \ \forall n \geq 0 \},
\]
Therefore, for each \( i \), the set \( R_i^\infty \cap L_i^\infty \) is the maximal invariant set of \([\ell_i, \ell_{i+1}]\) for \( f \), and then
\[
R_i^\infty \cap [\ell_i, \ell_{i+1}] = \{ x \in [\ell_i, \ell_{i+1}] : d(f^{-n}(x), L_i^\infty \cap R_i^\infty) \to 0 \text{ as } n \to \infty \},
\]
and
\[
L_i^\infty \cap [\ell_i, \ell_{i+1}] = \{ x \in [\ell_i, \ell_{i+1}] : d(f^n(x), L_i^\infty \cap R_i^\infty) \to 0 \text{ as } n \to \infty \}.
\]
That is, the set \( L_i^\infty \cap [\ell_i, \ell_{i+1}] \) can be thought as the ‘local stable set’ of \( R_i^\infty \cap L_i^\infty \), and \( R_i^\infty \cap [\ell_i, \ell_{i+1}] \) can be thought as the ‘local unstable set’ of \( R_i^\infty \cap L_i^\infty \). The following lemmas study some properties of these sets.

**Lemma 6.8.** For every \( i \geq 0 \):

1. if \( C \) is a connected component of \( R_i^\infty \), then \( \sup \text{pr}_1(C) = +\infty \),
2. if \( C' \) is a connected component of \( L_i^\infty \), then \( \inf \text{pr}_1(C') = -\infty \), and
3. the connected components of \( \mathbb{R}^2 \setminus X_i \) are simply connected.

Figure 10: Some examples of the sets \( L_i^\infty, R_i^\infty \) and \( X_i \).
Proof. Let $S = \mathbb{R} \times S^1 \cup \{\infty\} \cup \{-\infty\}$ be the two-point compactification of $\mathbb{R} \times S^1$, which is homeomorphic to $S^2$, and let $j : \mathbb{R} \times S^1 \rightarrow S$ be the inclusion. The curves $\pi(\ell_i) \subset \mathbb{R} \times S^1$ are vertical circles, and then the sets $D_n := j(\pi(R^n(\ell_i))))$ and $D'_n := j(\pi(L(f^{-n}(\ell_i))))$ are topological closed discs in $S$, for any $n$ and $i$. Observe that

$$\widehat{L}_i := j(\pi(L^1)) \cup \{-\infty\} = \bigcap_{n \in \mathbb{N}} D'_n,$$

and

$$\widehat{R}_i := j(\pi(R^1)) \cup \{\infty\} = \bigcap_{n \in \mathbb{N}} D_n.$$

As we are under Assumption 6.4, $\ell_i \prec f(\ell_i)$ for any $i$, and then $D_{n+1} \subset D_n$ for all $n$. Therefore the sets $\widehat{L}_i$ and $\widehat{R}_i$ are nested intersections of topological closed discs, and thus they are compact and connected.

Observe that, for every $i$, $L^1 \cap R^1 = \cap_{n \in \mathbb{N}} f^n(\ell_i, \ell_{i+1})$. By Remark 5.6, $L^1 \cap R^1 \neq \emptyset$, and then $\widehat{L}_i \cap \widehat{R}_i \neq \emptyset$. Therefore, $\widehat{X}_i := j(\widehat{X}_1) \cup \{\infty\} \cup \{-\infty\}$ is compact and connected, as it is the union of $\widehat{L}_i$ and $\widehat{R}_i$, which are connected sets with nonempty intersection.

(1). It suffices to show that, for $x \in \widehat{R}_i \setminus \{\infty\}$, if $C_x$ is the connected component of $\widehat{R}_i \setminus \{\infty\}$ containing $x$, then $\infty \in \overline{C_x}$. For each $n$, let $\alpha_n$ be an arc contained in $\widehat{R}_n$, such that $\alpha_n(0) = \infty$ and $\alpha_n(1) = x$. Let $B \subset S$ be a ball that contains $\infty$, and does not contain $x$. For each $n$, let $\beta_n \subset \alpha_n$ be an arc contained in $B^c$ with extremes $\beta_n(0) \in \partial B$ and $\beta_n(1) = x$. Then, take an accumulation point $C \subset S$ of the sequence $(\beta_n)$ in the Hausdorff topology. As $\beta_n \subset \alpha_n \subset \widehat{R}_n$, and as $\widehat{R}_n \subset \widehat{R}_{n-1}$ for all $n$, we have that $\beta_n \subset D_k$ for all $k \leq n$. Therefore $C$ is contained in $\widehat{R}_n$ for all $n$; that is, $C \subset \widehat{R}_i = \bigcap_{n \geq 0} D_n$. Also, $C$ is compact, connected, contains $x$, and intersects $\partial B$. Therefore, the connected component $C_x$ must contain $C$, and then $C_x$ intersects $\partial B$. As $B$ was an arbitrarily small ball containing $\infty$, this means that $\infty \in \overline{C_x}$, as we wanted.

(2). The proof is analogous to (1).

(3). First we will prove that the connected components of $S \setminus \widehat{X}_i$ are simply connected. To see this just note that if there was a connected component $U_0$ of $S \setminus \widehat{X}_i$ not simply connected, then there would exist a simple closed curve $\gamma \subset U_0$ separating two connected components of $\partial U_0$, but as $\partial U_0 \subset \widehat{X}_i$, we would have that $\widehat{X}_i$ is not connected, a contradiction. Then each connected component of $S \setminus \widehat{X}_i$ must be simply connected.

Now, let $V$ be any connected component of $X^i \setminus (\ell_i, \ell_{i+1})$. Then $j(V) \subset S$ is a connected component of $S \setminus \widehat{X}_i$, and therefore simply connected. As $j : \mathbb{R}^2 \rightarrow S \setminus \{\infty\}$ is a homeomorphism, $V$ must be also simply connected. \(\blacksquare\)

**Corollary 6.9.** For each $i \geq 0$, the connected components of $X^i \cap (\ell_i, \ell_{i+1})$ are simply connected.

Proof. If $A, B \subset \mathbb{R}^2$ are simply connected sets, it is easy to see that each connected component of $A \cap B$ is simply connected. Then, if $U$ is any connected
component of $X^c_i$, each connected component of $U \cap R(\ell_i)$ is simply connected, and then each connected component of $U \cap (\ell_i, \ell_{i+1}) = U \cap R(\ell_i) \cap L(\ell_{i+1})$ is simply connected. As any connected component of $X^c_i \cap (\ell_i, \ell_{i+1})$ is of the form $U_0 \cap (\ell_i, \ell_{i+1})$, for some connected component $U_0$ of $X^c_i$, we then have that any connected component of $X^c_i \cap (\ell_i, \ell_{i+1})$ is simply connected. ■

The following lemma is an application of Lemma 6.7.

**Lemma 6.10.** There exists a constant $M_0$ such that, for any $i \geq 0$, any connected component of $R^i_\infty \cap L(\ell_{i+1})$ has vertical diameter less than $M_0$, and also any connected component of $L^i_\infty \cap R(\ell_i)$ has vertical diameter less than $M_0$.

**Proof.** First we treat the case of $R^i_\infty \cap L(\ell_{i+1})$. As we are under Assumption 6.4, we have that $f^{n_0}(\ell_{i+1}) \cap \ell_i \neq \emptyset$ for all $i \geq 0$. By Lemma 6.7, there exists a constant $K_0 > 0$ such that if $C \subset \mathbb{R}^2$ is a continuum contained in $(\ell_i, \ell_{i+1})$ with $\text{diam}_2(C) > K_0$, then $f^{n_0}(C) \cap L(\ell_i) \neq \emptyset$. Therefore, for any $i \geq 0$, any connected component $C_0$ of $R^i_\infty \cap L(\ell_{i+1})$ must have vertical diameter less than $K_0$, because otherwise $f^{n_0}(C_0)$ would intersect $L(\ell_i)$, which contradicts the definition of $R^i_\infty$.

Analogously, by Assumption 6.4, we have that $f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset$ for all $i \geq 0$, and by Lemma 6.7, if $C \subset \mathbb{R}^2$ is a continuum contained in $(\ell_i, \ell_{i+1})$, with $\text{diam}_2(C) > K_0$, then $f^{n_0}(C) \cap R(\ell_i) \neq \emptyset$. Therefore, for any $i \geq 0$, any component $C_0$ of $L^i_\infty \cap R(\ell_i)$ must have vertical diameter less than $K_0$, because otherwise $f^{n_0}(C_0)$ would intersect $R(\ell_{i+1})$, which contradicts the definition of $L^i_\infty$.

Setting $M_0 = K_0$, the lemma follows. ■

**Lemma 6.11.** There exists $M_1 > 0$ such that for any $i \geq 0$, any connected component of $X^c_i \cap (\ell_i, \ell_{i+1})$ has vertical diameter less than $M_1$.

**Proof.** Let $i \geq 0$, and let $x \in L^i_\infty \cap R^i_\infty$. Let $C_1$ and $C_2$ be the connected components of $R^i_\infty \cap L(\ell_{i+1})$ and $L^i_\infty \cap R(\ell_i)$, respectively, that contain $x$. By Lemma 6.8, $C_1$ is unbounded to the right and $C_2$ is unbounded to the left, so $C_1$ intersects $\ell_{i+1}$ and $C_2$ intersects $\ell_i$. The set $C = C_1 \cup C_2$ is connected and as it intersects both $\ell_i$ and $\ell_{i+1}$, it separates $(\ell_i, \ell_{i+1})$, that is, $(\ell_i, \ell_{i+1}) \cap C$ is not connected. Also, by Lemma 6.10, there is a constant $M_0$ such that $\text{diam}_2(C_1) \leq M_0$ for $i = 1, 2$, and then $\text{diam}_2(C) \leq 2M_0$. Thus, $C \cap T^3_{M_0}(C) = C \cap T^3_{-M_0}(C) = \emptyset$.

Now, consider the set

$$A = \bigcup_{n \in \mathbb{Z}} T^3_{M_0n}(C).$$

The connected components of $(\ell_i, \ell_{i+1}) \setminus A$ have then vertical diameter less than $\text{diam}_2(C) + 3M_0 \leq 4M_0$. As $A \subset X_i$, any connected component of $X^c_i \cap (\ell_i, \ell_{i+1})$ is contained in a connected component of $(\ell_i, \ell_{i+1}) \setminus A$, and therefore has diameter less than $4M_0$. Therefore, making $M_1 := 4M_0$, the lemma follows. ■
6.2.3 Definition of the Rides and Anchors.

The definition of the rides and anchors will follow from the following lemma.

**Lemma 6.12** (The sets $U_i$ and the curves $\alpha_i$). For each $i \geq 0$ there exists a connected component $U_i$ of $X_i^c \cap (\ell_i, \ell_{i+1})$ and an arc $\alpha_i$ such that:

- $\alpha_i(0) \in \ell_i \setminus L^i_\infty$,
- $\alpha_i(1) \in R^i_\infty \setminus L^i_\infty$, and
- $\alpha_i(t) \in U_i$ for $0 < t < 1$ (see Fig. 11).

Also, there is $N_1 \in \mathbb{N}$ such that $f^{N_1}(\alpha_i) \subset R(\ell_{i+1})$ for any $i \geq 0$.

![Figure 11: The sets $U_i$ and the curves $\alpha_i$.](image)

In this example $U_i$ intersects $\ell_{i+1}$. In general we may have $\overline{U_i} \cap \ell_{i+1} = \emptyset$.

**Definition 6.13.** For $i \geq 0$, let $U_i$ and $\alpha_i$ be as in Lemma 6.12. For $n \in \mathbb{Z}$, the sets $T^i_n(\alpha_i)$ are called **rides**, and the sets $T^i_2(\partial U_i \setminus R^i_\infty)$ are called **anchors**.

Observe that, if $N_1$ is as in Lemma 6.12, a ride $T^i_n(\alpha_i)$ has the property that $f^{N_1}(T^i_n(\alpha_i)) \subset R(\ell_{i+1})$.

Also, as $\partial U_i \subset L^i_\infty \cup R^i_\infty$, we have that an anchor $T^i_2(\partial U_i \setminus R^i_\infty)$ is contained in $L^i_\infty$, and therefore by definition of the set $L^i_\infty$, we have

$$f^m(T^i_2(\partial U_i \setminus R^i_\infty)) \subset [\ell_i, \ell_{i+1}]$$

for all $m \geq 0$.

**Proof of Lemma 6.12** By Assumption 6.4, for each $i$ we have that $f^{n_0+1}(\ell_i) \cap R(\ell_{i+1}) \neq \emptyset$. Then there exists an arc $I \subset \ell_{i+1}$ such that $I \subset L(f^{n_0+1}(\ell_i))$, and then, by the definition of $R^i_\infty$, $I \cap R^i_\infty = \emptyset$. Also, as $f^{-1}(\ell_{i+1}) \subset L(\ell_{i+1})$, then $I \cap L^i_\infty = \emptyset$, and therefore $I \subset X_i^c$.

Let $\tilde{U}_i$ be the connected component of $X_i^c$ that contains $I$. Let $\tilde{J}$ be the maximal open arc contained in $\ell_{i+1} \cap \tilde{U}_i$ that contains $I$, and let $J$ be the
closure of \( \hat{J} \). By Lemma 6.11, \( \hat{J} \) is compact, and we can give \( J \) a parametrization \( J : [0, 1] \to \ell_{i+1} \), with orientation coinciding with the upwards orientation of \( \ell_{i+1} \). We observe that \( J(1) \notin L^i_\infty \), because \( L^i_\infty \cap \ell_{i+1} = \emptyset \), and then

\[
J(1) \in R^i_\infty \setminus L^i_\infty. \tag{5}
\]

Now, as \( I \subset L(f^{-n_0-1}(\ell_i)), f^{-n_0-1}(I) \subset L(\ell_i) \), and then as \( I \subset J \), \( f^{-n_0-1}(J) \cap L(\ell_i) \neq \emptyset \). Also, as \( J(1) \in R^i_\infty \),

\[
f^{-n_0-1}(J(1)) \in R(\ell_i),
\]

and we can define \( t^*_i = \max \{ t \in [0, 1] : f^{-n_0-1}(J(t)) \in \ell_i \} \). Let \( r \) denote the cardinality of the family of curves \( \{ \hat{\ell}_i \} \) constructed in Section 5.1. For \( 0 \leq i < r \), define \( \alpha_i : [0, 1] \to \mathbb{R}^2 \) as (any reparametrization of) \( f^{-n_0-1} \circ J|_{t^*_i, 1} \), and define \( U_i \) as the connected component of \( X^i_\ell \cap (\ell_i, \ell_{i+1}) \) whose closure contains \( \alpha_i \). By the invariance of the sets \( L^i_\infty \) and \( R^i_\infty \) and by (5) we have that

\[
\alpha_i(1) = f^{-n_0-1}(J(1)) \in \partial U_i \cap (R^i_\infty \setminus L^i_\infty),
\]

so the second item of the lemma holds for \( \alpha_i \). Also, as \( \hat{J} \subset X^i_\ell \) and by the invariance of \( X^i_\ell \), \( f^{-n_0-1}(\hat{J}) \subset X^i_\ell \), and in particular \( f^{-n_0-1}(J(t^*_i)) \in X^i_\ell \). Then

\[
\alpha_i(0) = f^{-n_0-1}(J(t^*_i)) \in \ell_i \setminus L^i_\infty,
\]

and by definition of \( U_i \) and \( \alpha_i \), \( \alpha(t) \in U_i \) for \( 0 < t < 1 \), so the first and third items from the lemma hold for \( U_i \) and \( \alpha_i \), and we have found, for \( 0 \leq i < r \), \( \alpha_i \) and \( U_i \) as required. Then, for \( 0 \leq i < r \) and \( j \in \mathbb{N} \), define \( U_i + j r = T^i_j U_i \) and \( \alpha_i = T^i_j \alpha_i \). By the periodicity of \( f \), items (1) to (3) hold for \( \alpha_i \) and \( U_i \), for any \( i \geq 0 \).

Finally, we define \( N_1 \). By the definition of the curves \( \alpha_i \), we have that, for any \( i \),

\[
\alpha_i \subset X^i_\ell \cup R^i_\infty.
\]

For any point \( x \in X^i_\ell \cup R^i_\infty \) there exists \( n \in \mathbb{N} \) such that \( f^n(x) \in R(\ell_{i+1}) \) (by the definition of the sets \( X_i \) and \( R^i_\infty \)). Then by the compactness of each curve \( \alpha_i \subset \mathbb{R}^2 \), for each \( 0 \leq i < r \) there exists \( n_i \in \mathbb{N} \) such that \( f^{n_i}(\alpha_i) \subset R(\ell_{i+1}) \). By definition of \( \alpha_i \) for \( i \geq r \) and by the periodicity of \( f \), \( f^{n_i}(\alpha_i) \subset R(\ell_{i+1}) \) for any \( n_i \geq 0 \) and \( 0 \leq i < r \). So taking \( N_1 = \max_{0 \leq i < r} \{ n_i \} \), we have that

\[
f^{N_1}(\alpha_i) \subset R(\ell_{i+1}) \text{ for any } i \geq 0,
\]

as we wanted. \( \blacksquare \)

### 6.3 Main Lemma.

Before stating our main lemma in the proof of item (1) from Theorem A, we define \textit{good intersection} of an arc with the rides and anchors. We now loosely explain this definition. For \( i \geq 0 \) let \( U_i \) be as in Lemma 6.12. We will say that a curve \( \hat{\gamma} \) has \textit{good intersection} with the rides and anchors of the \( i \)-th strip \([\ell_i, \ell_{i+1}]\) if \( \hat{\gamma} \) contains an arc \( \gamma \) such that \( \hat{\gamma} \) is contained in a vertical integer translate of \( U_i \), and such that one endpoint lies in a ride, and the other endpoint lies in an anchor.
Definition 6.14 (good intersection). Let \( \{U_i\}_{i \geq 0} \) and \( \{\alpha_i\}_{i \geq 0} \) be as in Lemma 6.12. Let \( j \in \mathbb{N}_0 \). We say that a curve \( \tilde{\gamma} \) has good intersection with the rides and anchors of the \( j \)-th strip \([\ell_j, \ell_{j+1}]\) if there is \( s \in \mathbb{Z} \) and an arc \( \gamma \subset \tilde{\gamma} \) such that:

- one endpoint of \( \gamma \) lies in \( T_2^s(\partial U_j) \cap L^j_{\infty} \),
- the other endpoint of \( \gamma \) lies in \( T_2^s(\alpha_j) \), and
- \( \dot{\gamma} \subset T_2^s(U_j) \setminus X_j \) (see Fig. 12).

Figure 12: Left: good intersection. Right: not good intersection.

Remark 6.15. Suppose that an arc \( \gamma_i \) has good intersection with the rides and anchors of the \( i \)-th strip \([\ell_i, \ell_{i+1}]\), and let \( N_1 \) be the constant given by Lemma 6.12. Then, as \( \gamma_i \) has one endpoint in an anchor, and the other endpoint in a ride, the arc \( f^{N_1}(\gamma_i) \) contains an arc \( \beta_i \) such that (see Fig. 13):

- one endpoint of \( \beta_i \) lies in an anchor,
- the other endpoint of \( \beta_i \) lies in \( \ell_{i+1} \), and
- \( \dot{\beta}_i \subset X^c_i \cap (\ell_i, \ell_{i+1}) \).

Our main lemma gives us a constant \( N_2 \in \mathbb{N} \) such that, for any \( i \) and any curve \( \beta_i \) as in Remark 6.15, we have that \( f^{N_2}(\beta_i) \) has good intersection with the rides and anchors of the next strip \([\ell_{i+1}, \ell_{i+2}]\).

Lemma 6.16 (Main Lemma). There exists \( N_2 > 0 \) such that for any \( i \geq 0 \), and any arc \( \beta_i \) such that:

- \( \beta_i(0) \in L^i_{\infty} \),
- \( \beta_i(1) \in \ell_{i+1} \), and
- \( \beta_i(t) \in X^c_i \cap (\ell_i, \ell_{i+1}) \) for \( 0 < t < 1 \),
then $f^{N_2}(\beta_i)$ has good intersection with the rides and anchors of the $(i+1)$-th
strip $[\ell_{i+1}, \ell_{i+2}]$ (see Fig. 14).

We emphasize that the constant $N_2$ is independent of $i$ and of the arc $\beta_i$. We
will prove Lemma 6.16 in Section 6.4. Now, we will see that the Main Lemma implies item (1) of Theorem A.

6.3.1 The Main Lemma implies item (1) of Theorem A.

Assuming the Main Lemma 6.16 we prove the following.
Lemma 6.17. There exists $N_3 > 0$ such that, for each $n \geq 0$, $f^{nN_3}(\ell_0)$ has good intersection with the rides and anchors of the $n$-th strip $[\ell_n, \ell_{n+1}]$.

Proof. Let $N_1$ and $N_2$ be the constants given by Lemmas 6.12 and 6.16 respectively, and set $N_3 := N_1 + N_2$. We proceed by induction.

Step $n = 0$. It follows by the definitions that $\ell_0$ has good intersection with the rides and anchors of the 0-th strip $[\ell_0, \ell_1]$.

Step $n$. We suppose that $f^{N_3(n-1)}(\ell_0)$ has good intersection with the rides and anchors of the $(n-1)$-th strip $[\ell_{n-1}, \ell_n]$, and we will prove that $f^{N_3n}(\ell_0)$ has good intersection with the rides and anchors of the $n$-th strip.

Let $\{U_i\}$ and $\{\alpha_i\}$ be as in Lemma 6.12. By the definition of good intersection, there exists an arc $\gamma_{n-1}$ contained in $f^{N_3(n-1)}(\ell_0)$ and $s_{n-1} \in \mathbb{Z}$ such that:

- $\gamma_{n-1}(0) \in \mathcal{L}_n^{-1}$ (and therefore $f^j(\gamma_{n-1}(0)) \in (\ell_{n-1}, \ell_n)$ for all $j \geq 0$),
- $\gamma_{n-1}(1) \in T_2^{s_{n-1}}(\alpha_{n-1})$, and
- $\gamma_{n-1} \subset T_2^{s_{n-1}}(\overline{U}_{n-1}) \setminus X_{i-1}$.

By Remark 6.15 the arc $f^{N_3}(\gamma_{n-1})$ contains an arc $\beta_{n-1}$ such that:

- $\beta_{n-1}(0)$ lies in an anchor (and in particular, $\beta_{n-1}(0) \in \mathcal{L}_n^{-1}$),
- $\beta_{n-1}(1) \in \ell_{n+1}$, and
- $\beta_{n-1}(t) \in X_t^i \cap (\ell_n, \ell_{n+1})$ for $0 < t < 1$.

Therefore $\beta_{n-1}$ is an arc satisfying the hypotheses of Lemma 6.16 and then by that lemma $f^{N_3}(\beta_{n-1})$ has good intersection with the rides and anchors of the $n$-th strip $[\ell_n, \ell_{n+1}]$ (see Fig. 15). As $\beta_{n-1} \subset f^{N_3(n-1)+N_1+N_2}(\ell_0)$, we then have that $f^{N_3(n-1)+N_1+N_2}(\ell_0) = f^{N_3n}(\ell_0)$ has good intersection with the rides and anchors of the $n$-th strip $[\ell_n, \ell_{n+1}]$, which finishes the $n$-th induction step, and therefore the proof of the lemma.

Using this lemma, we now prove it holds item (1) from Theorem A. By Assumption 6.1 the curves $\ell_i$ are straight and vertical. Also by that assumption, the family $\{\ell_1\}$ consists only of two curves, and then $\ell_{2n} = T_1^n(\ell_0)$ for all $n \in \mathbb{N}$.

Therefore, Lemma 6.17 implies that there exists a sequence of points $x_n \in \ell_0$ such that $f^{2nN_3}(x_n) \in [\ell_{2n}, \ell_{2n+1}] = [T_1^n(\ell_0), T_1^n(\ell_1)]$, and then $f^{2nN_3}(x_n) - (x_n) > n$, for all $n \geq 0$. This implies that $\max_{x_n}(\rho(f)) > 0$, and this finishes our proof by contradiction of item (1) from Theorem A.

6.4 Proof of Main Lemma.

We first prove some previous results (lemmas 6.18 and 6.20), and then we will proceed to the proof of Lemma 6.16.
Figure 15: Illustration for Lemma [6.17]. The curve $f^{N_1+N_2}(\gamma_{n-1})$ has good intersection with the rides and anchors of $[\ell_n, \ell_{n+1}]$.

Lemma [6.18] will be a key step in proving there is uniformity in the advance to the right of the iterates of $\ell_0$. It tells us that the points that remain under iteration by $f$ in a strip $(\ell_i, \ell_{i+1})$, must go either upwards or downwards uniformly. We recall that in Section 5.1 we proved that for the sets $\Theta_i$ from Theorem A, for each $i$ we have either $\rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^+$ or $\rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^-$. We also recall that from Assumption 6.1 the family $\{\Theta_i\}$ consists only of two sets $\Theta_0$ and $\Theta_1$.

**Notation.** If $i \in \mathbb{N}_0$, for the sake of simplicity we will write $\rho(\Theta_i, f)$, when we actually mean $\rho(\Theta_i \mod 2, f)$.

**Lemma 6.18** (Uniformity Lemma). Given $m > 0$ there exists $N \in \mathbb{N}$ such that, if:

- $i \geq 0$,
- $n \in \mathbb{Z}$, $|n| \geq N$,
- $x \in (\ell_i, \ell_{i+1})$ and $f^n(x) \in (\ell_i, \ell_{i+1})$,

then:

- If $\rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^+$, then $f^n(x) - x > m$ if $n > 0$ and $x - f^n(x) > m$ if $n < 0$.
- If $\rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^-$, then $x - f^n(x) > m$ if $n > 0$ and $f^n(x) - x > m$ if $n < 0$. 

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Corollary 6.19. There exists component \( V \) of \( X^i \cap (\ell_i, \ell_{i+1}) \), such that:

- \( s_n \to \infty \) as \( n \to \infty \),
- \( f^{s_n}(x_n) \in (\ell_{i_0}, \ell_{i_0+1}) \) for all \( n \in \mathbb{N} \),
- \( f^{s_n}(x_n) - (x_n) < m_0 \) for all \( n \in \mathbb{N} \), if \( \rho(\Theta_{i_0}, f) \subset \{0\} \times \mathbb{R}^+ \), and
- \( x_n - f^{s_n}(x_n) < m_0 \) for all \( n \in \mathbb{N} \), if \( \rho(\Theta_{i_0}, f) \subset \{0\} \times \mathbb{R}^- \).

Proof. First suppose that \( n > 0 \). By contradiction, if the lemma does not hold, there exist \( m_0 > 0 \), \( i_0 \geq 0 \), and sequences \( \{x_n\}_n \subset (\ell_{i_0}, \ell_{i_0+1}) \), \( \{s_n\}_n \subset \mathbb{Z} \), such that:

- \( s_n \to \infty \) as \( n \to \infty \),
- \( f^{s_n}(x_n) \in (\ell_{i_0}, \ell_{i_0+1}) \) for all \( n \in \mathbb{N} \),
- \( f^{s_n}(x_n) - (x_n) < m_0 \) for all \( n \in \mathbb{N} \), if \( \rho(\Theta_{i_0}, f) \subset \{0\} \times \mathbb{R}^+ \), and
- \( x_n - f^{s_n}(x_n) < m_0 \) for all \( n \in \mathbb{N} \), if \( \rho(\Theta_{i_0}, f) \subset \{0\} \times \mathbb{R}^- \).

By now, suppose that \( \rho(\Theta_{i_0}, f) \subset \{0\} \times \mathbb{R}^+ \). Then,

\[
\limsup_n (f^{s_n}(x_n) - (x_n))/s_n \leq 0,
\]

and there is a subsequence of \( \{x_n\}_n \), denoted also \( \{x_n\}_n \), such that

\[
\lim_n (f^{s_n}(x_n) - (x_n))/s_n = (0, a)
\]

with \( a \leq 0 \). Define the sequence of probability measures \( \{\delta_n\}_n \) in \( \mathbb{T}^2 \) by

\[
\delta_n = \frac{\delta_{\pi}(x_n) + \delta_{\pi}(f(x_n)) + \cdots + \delta_{\pi}(f^{s_n-1}(x_n))}{s_n},
\]

and let \( \delta \) be an accumulation point of \( \{\delta_n\}_n \) in \( \mathcal{M}_f(\mathbb{T}^2) \). Then \( \delta \) is \( \tilde{f} \)-invariant, and

\[
\rho(\delta, f) = \int \phi d\delta = \lim_n \int \phi d(\delta_n) = \lim_n \left( \frac{1}{s_n} (f^{s_n}(x_n) - x_n) = (0, a) \right)
\]

where \( \phi : \mathbb{T}^2 \to \mathbb{R}^2 \) is the displacement function defined in section 3.1.2. Also, as \( \text{supp}(\delta) \) is \( \tilde{f} \)-invariant and is contained in \( [\ell_{i_0}, \ell_{i_0+1}] \), \( \text{supp}(\delta) \) must be contained in \( \Theta_{i_0} \) (because \( \Theta_{i_0} \) is the maximal invariant set of \( [\ell_{i_0}, \ell_{i_0+1}] \)). This means that \( (0, a) \in \rho(\Theta_{i_0}, f) \), and this is a contradiction, as \( a \leq 0 \) and we are assuming that \( \rho(\Theta_{i_0}, f) \subset \{0\} \times \mathbb{R}^+ \).

In a similar way, for the cases that \( n < 0 \) and \( \rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^- \), if we suppose the lemma does not hold we obtain a contradiction, and this concludes the proof of the lemma.

As a corollary, we get that there is a maximum amount of displacement downwards, if \( \rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^+ \), for points that remain in \( (\ell_i, \ell_{i+1}) \) under iteration by \( f \). An analogous statement is obtained for the case \( \rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^- \).

Corollary 6.19. There exists \( c > 0 \) such that for any \( i \geq 0 \), and any connected component \( V \) of \( X^i \cap (\ell_i, \ell_{i+1}) \), we have that:

- If \( \rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^+ \), \( f^n(V) \cap L(\ell_{i+1}) \cap A^- \) = \( \emptyset \) for all \( n \geq 0 \), where \( A^- \) is the half-plane \( A^- = \{ x \in \mathbb{R}^2 : y_2 - x_2 > c \) for all \( y \in V \)\).
Figure 16: Illustration of Corollary 6.19 for the case that \( \rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^+ \) and \( \rho(\Theta_{i+1}, f) \subset \{0\} \times \mathbb{R}^- \). Left: The sets \( V \) and \( A^-_c \). Right: \( f^n(V) \cap L(\ell_{i+1}) \cap A^-_c = \emptyset \) for all \( n \geq 0 \).

- If \( \rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^- \), \( f^n(V) \cap L(\ell_{i+1}) \cap A^+_c = \emptyset \) for all \( n \geq 0 \), where \( A^+_c \) is the half-plane \( A^+_c = \{x \in \mathbb{R}^2 : x_2 - y_2 > c \text{ for all } y \in V\} \) (see Fig. 16).

Proof. By Lemma 6.18 there exists \( N_0 > 0 \) such that if:

- \( i \geq 0 \),
- \( n > N_0 \), and
- \( x \in (\ell_i, \ell_{i+1}) \) and \( f^n(x) \in (\ell_i, \ell_{i+1}) \),

then \( f^n(x)_2 - x_2 > 0 \) if \( \rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^+ \), and \( x_2 - f^n(x)_2 > 0 \) if \( \rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^- \). Let \( i \geq 0 \) be such that \( \rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^+ \). Let \( V \) be any connected component of \( X^c_i \cap (\ell_i, \ell_{i+1}) \). Then we have that for any \( x \in V \), either \( f^n(x) \in R(\ell_{i+1}) \) or \( f^n(x)_2 > x_2 \), for any \( n > N_0 \). Similarly, if \( j \geq 0 \) is such that \( \rho(\Theta_j, f) \subset \{0\} \times \mathbb{R}^- \) and \( V' \) is any connected component of \( X^c_j \cap (\ell_j, \ell_{j+1}) \), we have that for any \( x' \in V' \) either \( f^n(x') \in R(\ell_{j+1}) \) or \( f^n(x')_2 < x_2 \), for any \( n > N_0 \). Making \( c = N_0\|f - \text{Id}\|_0 \), the lemma follows.

Now we give our second lemma before the proof of the Main Lemma 6.16.

It tells us that, for a curve \( \beta \) contained in \([\ell_i, \ell_{i+1}]\) satisfying the hypotheses of the Main Lemma 6.16, there is an iterate of \( \beta \) intersecting \( \ell_{i+2} \), and moreover, this iterate is independent of \( \beta \) and \( i \).

Lemma 6.20. There exists \( N_4 > 0 \) such that for any \( i \geq 0 \), and any arc \( \beta_i \) such that:

- \( \beta_i(0) \in L^i_\infty \),
- \( \beta_i(1) \in \ell_{i+1} \), and
- \( \beta_i(t) \in X^c_i \cap (\ell_i, \ell_{i+1}) \) for \( 0 < t < 1 \),
then \( f^{N_i}(\beta_i) \cap \ell_{i+2} \neq \emptyset \) (see Fig. 17).

**Proof.** Fix \( i \in \{0, 1\} \), and suppose first that \( \rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^+ \). By Assumption \( 6.4 \) there is \( n_0 > 0 \) such that \( f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset \) for all \( i \geq 0 \). Then, by Lemma \( 6.7 \) there is a constant \( K_0 > 0 \) such that if \( C \) is a continuum contained in \( (\ell_{i+1}, \ell_{i+2}) \), with \( \text{diam} \geq K_0 \), then \( f^{n_0}(C) \cap R(\ell_{i+2}) \neq \emptyset \).

By Lemma \( 6.11 \) there is \( M_1 > 0 \) such that, for any \( i \geq 0 \) and any connected component \( V \) of \( X^c_i \cap (\ell_i, \ell_{i+1}) \), we have \( \text{diam}(V) \leq M_1 \). By Corollary \( 6.19 \) there is \( c > 0 \) such that for any connected component \( V \) of \( X^c_i \cap (\ell_i, \ell_{i+1}) \), and for any \( n \geq 0 \) we have that

\[
 f^n(V) \cap A_c \subset R(\ell_{i+1}),
\]

where \( A_c = \{ x \in \mathbb{R}^2 : z_2 - x_2 > c, \forall z \in V \} \). By Lemma \( 6.18 \) there exists \( N_0 > 0 \) such that:

- if \( x \in \ell_{i+1}, \ell_{i+2} \) then for all \( n \geq N_0 \) such that \( f^n(x) \in [\ell_{i+1}, \ell_{i+2}] \) we have \( x_2 - f^n(x)_2 > M_1 + c + K_0 \).
- if \( y \in \ell_i, \ell_{i+1} \) then for all \( n \geq N_0 \) such that \( f^n(y) \in [\ell_i, \ell_{i+1}] \) we have \( f^n(y)_2 - y_2 > 0 \)

(recall that \( \rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^+ \) and \( \rho(\Theta_{i+1}, f) \subset \{0\} \times \mathbb{R}^- \).

As \( \beta_i(0) \in L^c_{x'} \), we have \( f^{N_0}(\beta_i(0)) \in (\ell_i, \ell_{i+1}) \), and by the choice of \( N_0 \),

\[
 f^{N_0}(\beta_i(0))_2 > \beta_i(0)_2.
\]

Now, either

\[
 f^{N_0}(\beta_i(1)) \in R(\ell_{i+2}),
\]

or \( f^{N_0}(\beta_i(1)) \in (\ell_{i+1}, \ell_{i+2}) \). In this case, by the choice of \( N_0 \), and as \( \beta_i(1) \in \ell_{i+1} \),

\[
 \beta_i(1)_2 - f^{N_0}(\beta_i(1))_2 > M_1 + c + K_0,
\]
Let $V_0 \subset \mathbb{R}^2$ be the connected component of $X^e_i \cap (\ell_i, \ell_{i+1})$ that contains $\beta_i$. Then $\text{diam}_2(\beta_i) \leq \text{diam}_2(V_0) \leq M_1$, and by (8) we have that
\[ z_2 - f^{N_0}(\beta_i(1))_2 > c + K_0 \quad \forall z \in V_0. \]
Then, by (7) and (8), we have
\[ f^{N_0}(\beta_i(0))_2 - f^{N_0}(\beta_i(1))_2 > c + K_0. \quad (9) \]

Let $A_0 = \{ x \in \mathbb{R}^2 : z_2 - x_2 \geq c \forall z \in V_0 \}$. Then, by (6) and (9) we have that there is a connected component $C_0$ of $f^{n_0 + N_0}(\beta_i) \cap A_0$ such that $C_0 \subset R(\ell_{i+1})$ and $\text{diam}_2(C_0) \geq K_0$. By the choice of the constant $K_0$ we conclude that $f^{N_0 + n_0}(\beta_i) \cap \ell_{i+2} \neq \emptyset$.

Observe that the constant $n_0$ from Assumption 6.4 is independent of $i$. Also, as the constants $K_0$, $c$ and $M_1$ are also independent of $i$, we have that the constant $N_0$ given by Lemma 6.18 can be chosen to be independent of $i$.

In a similar way we prove that if $\rho(\Theta_{i+1}) \subset \{ 0 \} \times \mathbb{R}^+$, there is a constant $N'_0 > 0$ independent of $i$, such that $f^{N_0 + n_0}(\beta_i) \cap \ell_{i+1} \neq \emptyset$. Then, letting $N_4 = \max\{ N_0, N'_0 \} + n_0$ we have that for any $i$ and any arc $\beta_i$ within the hypotheses of the lemma, $f^{N_4}(\beta_i) \cap \ell_{i+2} \neq \emptyset$, and the lemma follows. ■

Now we are ready to prove Lemma 6.16. We recall the statement of the lemma, and the definition of good intersection of an arc with a translate of some $U_i$.

**Definition** (good intersection). Let $\{ U_i \}_{i \geq 0}$ and $\{ \alpha_i \}_{i \geq 0}$ be as in Lemma 6.12.

Let $j \in \mathbb{N}_0$. We say that a curve $\tilde{\gamma}$ has good intersection with the rides and anchors of the $j$-th strip $[\ell_j, \ell_{j+1}]$ if there is $s \in \mathbb{Z}$ and an arc $\gamma \subset \tilde{\gamma}$ such that:
- one endpoint of $\gamma$ lies in $T^2_s(\partial U_j) \cap L^j_{\infty}$,
- the other endpoint of $\gamma$ lies in $T^2_s(\alpha_j)$, and
- $\gamma \subset T^2_s(U_j) \setminus X_j$ (see Fig. 12).

**Lemma** (Main Lemma 6.16). There exists $N_2 > 0$ such that, if $i \geq 0$, and if $\beta_i$ is an arc such that:
- $\beta_i(0) \in L^i_{\infty}$,
- $\beta_i(1) \in \ell_{i+1}$, and
- $\beta_i(t) \in X^e_i \cap (\ell_i, \ell_{i+1})$ for $0 < t < 1$,
then $f^{N_2}(\beta_i)$ has good intersection with $T^2_s(U_{i+1})$, for some $s \in \mathbb{Z}$.  

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Proof. Fix $i \in \{0, 1\}$. First we treat the case $\rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^+$. By Lemma \ref{lem:6.11} there exists a constant $M_1$ such that every connected component of $X_i^* \cap (\ell_i, \ell_{i+1})$ has diameter less or equal than $M_1$. Let $V$ be the connected component of $X_i^* \cap (\ell_i, \ell_{i+1})$ that contains $\beta_i$. By Lemma \ref{lem:6.19} there is a constant $c$ such that, if $S \subset \mathbb{R}^2$ be the half-plane given by

$$S = \{ x \in \mathbb{R}^2 : y_2 - x_2 > c \text{ for any } y \in V \},$$

then $f^n(\beta_i) \cap S \subset R(\ell_{i+1})$ for all $n \geq 0$.

Let $s \in \mathbb{Z}$ be such that

$$T^s_2(U_{i+1}) \subset S \quad \text{and} \quad T^{s+1}_2(U_{i+1}) \cap S^c \neq \emptyset.$$  

By Lemma \ref{lem:6.20} there is $N_2 > 0$ such that $f^{N_2}(\beta_i) \cap \ell_{i+2} \neq \emptyset$. Let

$$c_1 = 2M_1 + c + N_4\|f - \text{Id}\|_0 + 1.$$  

By Lemma \ref{lem:6.18} there exists $N_0 > 0$ such that if $x$ and $f^{-N_0}(x)$ are contained in $(\ell_{i+1}, \ell_{i+2})$ then $f^{-N_0}(x)_2 - x_2 > c_1$ (recall that $\rho(\Theta_{i+1}, f) \subset \{0\} \times \mathbb{R}^-$). In particular,

$$f^{-N_0}(z)_2 - z_2 > c_1 \quad \text{for any } z \in R^{i+1}_\infty \cap L(\ell_{i+2}).$$

(10)

As $\text{diam}_2(U_{i+1}) < M_1$ and by the definition of $s$, if $z \in \partial T^s_2(U_{i+1}),$

$$z + (0, M_1 + 1) \subset S^c.$$  

(11)

If $y \in V$ and $z \in f^{N_2}(\beta_i)$, we have

$$z_2 - y_2 \leq M_1 + N_4\|f - \text{Id}\|_0.$$  

(12)

Then, by the definition of $c_1$, by (10), (11) and (12), we have that for any point $z$ in $\partial T^s_2(U_{i+1}) \cap R^{i+1}_\infty,$

$$f^{-N_0}(z)_2 > y_2 \quad \text{for any } y \in f^{N_2}(\beta_i) \cap (\ell_{i+1}, \ell_{i+2})$$

(13)

(see Fig. \ref{fig:18}).

Now, let $\beta_i^1 : [0, \infty) \to \mathbb{R}^2$ be a proper embedding such that

- $\beta_i^1(0) = f^{N_4}(\beta_i(0)),$
- $\beta_i^1(t) \in L(f^{-N_0}(\ell_{i+1}))$ for all $t > 0$, and
- $-\infty < \inf \{ \beta_i^1(t) : t \geq 0 \}.$

Let $\beta_i^2$ be the arc contained in $f^{N_4}(\beta_i)$ with endpoints $f^{N_4}(\beta_i(0))$ and $f^{N_4}(\beta_i)(t_*)$, where $t_* = \min\{t : f^{N_4}(\beta_i(t)) \in \ell_{i+2} \}$. Let $\beta_i^3 : [0, \infty) \to \mathbb{R}^2$ be a curve contained in $\ell_{i+2}$, starting in $f^{N_4}(\beta(t_*))$ and going upwards to infinity. Consider then the open unbounded disc $D \subset \mathbb{R}^2$ whose boundary is $\beta_i^1 \cup \beta_i^2 \cup \beta_i^3$ (see Fig. \ref{fig:19}).
Figure 18: If \( z \in \partial T_2^s(U_{i+1}) \cap R_{i+1}^{i+1} \), then \( f^{-N_0}(z) \) is above \( f^{N_4}(\beta_i) \cap (\ell_{i+1}, \ell_{i+2}) \).

![Diagram](image1.png)

Figure 19: Illustration of the disk \( D \).

Observe that \( D \) is bounded from below (that is, \( \inf pr_2(D) > -\infty \)). By (13), \( f^{-N_0}(z) \in D \) for any \( z \in \partial T_2^s(U_{i+1}) \cap R_{i+1}^{i+1} \). In particular, if \( \alpha_{i+1} \) is as in Definition 19, then \( \alpha_{i+1}(1) \in R_{i+1}^{i+1} \) and \( f^{-N_0}(T^s_2(\alpha_{i+1}(1))) \in D \), or equivalently

\[
T^s_2(\alpha_{i+1}(1)) \in f^{N_0}(D). \tag{14}
\]

Note that by the definition of \( D \),

\[
f^{N_0}(\partial D) \cap (\ell_{i+1}, \ell_{i+2}) = f^{N_0}(\beta_i^2) \cap (\ell_{i+1}, \ell_{i+2}), \tag{15}
\]

and then, by the definition of \( S \subset \mathbb{R}^2 \) and by the choice of the constant \( c \),

\[
f^{N_0}(\partial D) \cap S = f^{N_0}(\beta_i^2) \cap S \subset R(\ell_{i+1}). \tag{16}
\]

So, we see that \( f^{N_0}(\beta_i^2) \) must intersect \( T^s_2(\alpha_{i+1}) \); otherwise, by (14), (15) and (16), \( f^{N_0}(D) \) would contain the connected set \( S \cap L(\ell_{i+1}) \cup T^s_2(\alpha_{i+1}) \), but this is not possible, because both \( D \) and \( f^{N_0}(D) \), are bounded from below.

Observe that, as \( \beta_i \subset \mathbb{L}(\ell_{i+1}) \), \( f^n(\beta_i) \cap R_{i+1}^{i+1} = \emptyset \), for all \( n \geq 0 \). Then,

\[
f^{N_0}(\beta_i^2) \cap R_{i+1}^{i+1} \subset f^{N_4+N_0}(\beta_i) \cap R_{i+1}^{i+1} = \emptyset, \tag{17}
\]

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and therefore \( \alpha_{i+1}(1) \notin f^{N_0}(\beta^2_i) \), which implies

\[
  f^{N_0}(\beta^2_i) \cap T^s_2(\alpha_{i+1}) \subset \text{int}(E),
\]

where \( E \subset \mathbb{R}^2 \) is the set \( E = (S \cap L(\ell_{i+1})) \cup T^s_2(U_{i+1}) \) (see Fig. 20).

![Figure 20: The set E (colored in gray).](image)

Observe that by the definition of the integer \( s \), \( E \subset S \). Also, as \( \beta^2_i(0) \in L^i_\infty \),

\[
  f^{N_0}(\beta^2_i(0)) \in L(\ell_{i+1}),
\]

and then by (16)

\[
  f^{N_0}(\beta^2_i(0)) \subset S^c \subset E^c.
\]

By this, by (16), and as \( f^{N_0}(\beta^2_i) \cap T^s_2(\alpha_{i+1}) \neq \emptyset \), there exist \( 0 \leq t_1 < t_2 \leq 1 \) such that:

- \( f^{N_0}(\beta^2_i(t_1)) \in T^s_2(\partial U_{i+1} \setminus R^{t+1}_\infty) \),
- \( f^{N_0}(\beta^2_i(t_2)) \in T^s_2(\alpha_{i+1}) \), and
- \( f^{N_0}(\beta^2_i(t)) \in T^s_2(U_{i+1}) \) for all \( t_1 < t < t_2 \).

This means that \( f^{N_0}(\beta^2_i) \subset f^{N_{i+1}N_0}(\beta_i) \) has good intersection with the rides and anchors of the \((i+1)\)-th strip \([\ell_{i+1}, \ell_{i+2}]\).

Now suppose that \( \rho(\Theta_i, f) \subset \{0\} \times \mathbb{R}^- \). In this case we have that

\[
  z_2 - f^{-N_0}(z)_2 > c_1 \quad \text{for any } z \in R^{t+1}_\infty.
\]

If we define

\[
  S' = \{ x \in \mathbb{R}^2 : x_2 - y_2 > c \text{ for any } y \in V \},
\]
and let \( s' \in \mathbb{Z} \) be such that
\[
T_2^{s'}(U_{i+1}) \subset S \quad \text{and} \quad T_2^{s'-1}(U_{i+1}) \cap S^c \neq \emptyset,
\]
then all the arguments above work to show that \( f^{N_4+N_0}(\beta_i) \) intersects \( T_2^{s'}(U_{i+1}) \) in a good way, and \( f^{N_4+N_0}(\beta_i) \) has good intersection with the rides and anchors of \([\ell_{i+1}, \ell_{i+2}]\).

The choice of this integer \( N_0 \) was made for a fixed \( i \). So, for any \( i \in \{0, 1\} \) we obtain in this way an integer \( N_0^i \), such that \( f^{N_4+N_0^i}(\beta_i) \) has good intersection with the rides and anchors of the \((i + 1)\)-th strip \([\ell_{i+1}, \ell_{i+2}]\). Setting \( N_2 = N_4 + \max\{N_0^0, N_0^2\} \), by the periodicity of \( f \) we have that, for all \( n \in \mathbb{N}_0 \), \( f^{N_2}(\beta_n) \) has good intersection with the rides and anchors of the \((n + 1)\)-th strip \([\ell_{n+1}, \ell_{n+2}]\). This concludes the proof of the lemma.

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