On the application of copula in modeling maintenance contract

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Abstract. This paper deals with the application of copula in maintenance contracts for a non-repairable item. Failures of the item are modeled using a two dimensional approach where age and usage of the item and this requires a bi-variate distribution to modelling failures. When the item fails then corrective maintenance (CM) is minimally repaired. CM can be outsourced to an external agent or done in house. The decision problem for the owner is to find the maximum total profit whilst for the agent is to determine the optimal price of the contract. We obtain the mathematical models of the decision problems for the owner as well as the agent using a Nash game theory formulation.

Key words: Bivariate copula, maintenance contract, non-cooperative game theory, two-dimensional maintenance region

1. Introduction
In most mining company, for a complex and expensive equipments such as dump trucks, an economical way to carry out maintenance is to outsource the maintenance works to an external agent. The agent can do a partial or full coverage of the maintenance actions (Preventive Maintenance (PM) or/and Corrective maintenance (CM)). The benefits of an out-sourcing maintenance service are two folds: to assure the maximum availability and to reduce the maintenance cost as described in [2], [3], and [4]. To meet the required maintenance actions, the owner wants to do the maintenance action which ensures that the equipment availability target is achieved with reasonable maintenance cost or maximum profit. On the other hand, the agent's problem is to determining the price of each option that maximises its profit.

There are some literatures regarding maintenance service contract. The authors in [4], [5], [7], [8], [9] and [10] developed a Stackelberg game theory model to obtain an optimal cost strategy with the agent as a leader and consumer as the follower. Further, [3] and [4] developed a similar model with the inclusion of a preventive and corrective maintenance policy. However, all works previously only deals with a single parameter—i.e. age or usage. In this paper, we model a two-dimensional service contract, which consider age and usage for a non-repairable item—such as an electric function of a dump truck, copier.

The paper is organized as follows. Sections 2 and 3 deal with the model formulation and the model analysis. In Section 4 we present the mathematical model for the optimal solution. Finally, a brief conclusion and a discussion for future work are presented in Section 5.
2. Model Formulation
The following notation will be used in model formulation.

\( K, L \) : time and usage limits
\( z_i \) : downtime caused by the \( i \)-th failure and waiting time
\( \zeta \) : total repair time allowed
\( F(z) \) : distribution function of downtime
\( R \) : revenue
\( \alpha \) : usage rate
\( C_m \) : repair cost done by OEM
\( \omega(O) \) : profit owner
\( \pi(O) \) : revenue OEM
\( C_b \) : the product cost over the contract period
\( F(t,x) \) : marginal failure distribution
\( r(t,x), R(t,x) \) : hazard, and cumulative hazard functions associated with \( F(t,x) \)
\( C_p \) : penalty cost per unit of time

2.1. Failure Modeling
There are three approaches to modelling failure (i). One-dimensional (1-D) approach, (ii) Best scale method, and (iii) Two-dimensional (2-D) approach [1]. In the 2-D approach, failures are characterized by a bivariate distribution function. Let \((T, X)\) denote the age and usage of the truck at first failure. Then \((T, X)\) is modelled by \( F(t,x) \). \((T, X)\) can be dependent or independent. For the case where one has only marginal distributions of \( T \) and \( X \), then a bivariate distribution can be constructed using copula function [11]. We further call it as a bivariate copula distribution. In this paper, we use 2-D approach and model failures using bivariate copula distribution, \( F(t,x) \) as only marginal distributions are available. We describe \( F(t,x) \) as follows.

\[
F(t,x) = \tilde{C}(F_1(t), F_2(x), \Phi) = \sum_{i=0}^{2} p_i (C(F_1(t), F_2(x), \theta_i) - C(1-F_1(t), 1-F_2(x), \theta_i)) + p_2 (C(F_1(t), 1, \theta_2) - C(F_1(t), 1-F_2(x), \theta_2)) \tag{1}
\]

where \( \tilde{C}(F_1(t), F_2(x), \Phi) \) is the asymmetric copula function and \( \sum_{i=0}^{2} p_i = 1 \). \( C(F_1(t), F_2(x), \theta_2) \) is symmetric copula function and composed by symmetric copula. We use simple symmetric copula function such as Gumbel copula is given by:

\[
C'(F_1(t), F_2(x), \theta) = \exp\{-[(-\ln(F_1(t)))^{\theta} + (-\ln(F_2(x)))^{\theta}]^{1/\theta}\} \tag{2}
\]

Figure 1-8 show failure data collected at a mining site, marginal distributions for age, and usage, Gumbel copula which is best fit compared to other copula. Asymmetric copula function can handle tail dependence between time and usage in a given direction, which can be applied in modeling reliability data. Based on the collected data, we could easily estimate the parameters of \( F_1(t), F_2(x) \) which are marginal distributions.
Figure 1. Claim data from Tuhup site

Figure 2. The five primary types of copulas

Figure 3. Plot data with Gumbel

Figure 4. Transformation kernel to Gumbel scale

Figure 5. Simulation with Gumbel with n = 1000

Figure 6. Transformation to the origin data

Figure 7. Probability plot of age

Figure 8. Probability plot of usage
Note:
The data follow Gumbel copula which is given by
\[ C(F_1(t), F_2(x), \theta) = \exp\left\{-[\ln(F_1(t))]^\theta + [\ln(F_2(x))]^\theta\right\} \] with the marginal cumulative failure given by Weibull distribution are
\[ F_1(t; \alpha_1) = 1 - \exp(-t/0.1982)^{1.053}, \quad F_2(x; \alpha_2) = 1 - \exp(-x/9567)^{0.763} \] and \( \theta = 3.9053 \).

2.2. A New Maintenance Contract

We consider that an OEM (original equipment manufacturer) offers maintenance for a non-repairable item such as a copier or an electric (i.e. a subsystem of a dump truck). For a fixed price \( P_0 \), OEM provides maintenance actions for \( K \) (age) or \( L \) (usage), whichever comes first. The maintenance actions include routine and corrective maintenances. It is assumed that a routine maintenance only keeps the basic condition of the item (does not improve reliability) and a corrective maintenance (CM) is done if the item fails. We consider that the contract given by the agent (OEM) covers all CM and hence, during the maintenance contract CM actions are done by the OEM without any charge to the owner. This two-dimensional contract region forms a rectangle region \( \Omega = [0, K] \times [0, L] \) where \( K \) and \( L \) are the age and the usage limits (e.g. \( K = 1 \) year and \( L = 50,000 \) km, \( K=1 \) year and \( L \), number of pages copied for the case of a copier) (see Figure 9). Let \( y \) denote the usage rate of a dump truck. Define \( \gamma = L/K \). The contract terminates due to the age limit, at \((K, K/y)\) if \( y \leq \gamma \), and due to the usage limit, at \((L/y, L)\) if \( y > \gamma \) (See Figure 10). The usage rate at the contract ends is given by

\[
y = \begin{cases} 
\frac{L}{t} > \gamma , & 0 < t < K \\
\frac{L}{K} = \gamma , & t = K, x = L \\
\frac{x}{K} < \gamma , & 0 < x < L 
\end{cases}
\] (3)

We assume that \( y \) is random with distribution function \( G(y) \). If any failure occurring under the contract is minimally repaired, then the expected total number of failures (and hence minimal repair) is given by

\[ N(K, L) = \int_0^K \int_0^L r(t, x) \, dt \, dx \] (4)

where \( r(t, x) \) is a bivariate failure rate function associated with \( F(t, x) \) given by

\[ f(t, x) = \frac{\partial F(t, x)}{\partial t \partial x} \quad ; \quad r(t, x) = \frac{f(t, x)}{1 - F(t, x)} \]
Figure 9. The two dimensional maintenance contract

Figure 10. Variation of usages in two dimensional maintenance region

Case (i): $y < \gamma$
Here, the contract ends at $(K, yK), 0 < y \leq \gamma$. As all failures are minimally repaired then the expected number of failures is given by

$$N(K, yK | y < \gamma) = \int_0^K \int_0^{yK} r(t', x') dt'dx'$$

(5)

Expected excess downtime:
Let $Z_t$ denote the downtime (consisting repair time and waiting time) caused by each failure occurring during the contract. It is assumed that $Z_t$ is i.i.d with distribution function $F(z)$. The OEM incurs penalty cost when the down time caused by a failure exceeds the predetermined target. If $\delta$
denotes the down time allowed, then the expected excess time (when \( Z > \gamma \)) is given by
\[
\psi(\gamma) = \int_{\gamma}^{\infty} (z - \gamma)dF_Z(z)dz = \int_{\gamma}^{\infty} [1 - F_Z(z)]dz.
\]
As a result, the expected excess downtime is given by
\[
EP(K, \Gamma_0) = \psi(\gamma)N(K, yK\mid y < \gamma)
\]  
(6)
where \( N(K, yK\mid y < \gamma) \) is the expected number of failure under the contract given by (5).

Case (ii): \( y > \gamma \)

In this, the contract ends at \((L/ y, L), y > \gamma\). Then the expected number of minimal repairs is given by
\[
N(L/ y, L\mid y \geq \gamma) = \int_{0}^{L} \int_{0}^{L} r(t', x')dt'dx'
\]  
(7)

Expected excess downtime:

Using a similar approach as in Case (i), the expected excess downtime is given by
\[
EP(L, U_0) = \psi(\gamma)N(L/ y, L\mid y \geq \gamma)
\]
(8)
where \( N(L/ y, L\mid y \geq \gamma) \) is given by (7).

3. Model analysis

We assume that OEM and consumer have the same attitudes to risk, and agree to achieve a win-win solution [3]. Then, we can use a Nash game theory formulation to obtain optimal solution.

3.1. OEM’s Decision Problem

The expected revenue of the OEM, \( \pi(O) \), is given by
\[
E[\pi(O)] = PC - E[\text{Penalty cost}] - E[\text{CM cost}]
\]
(9)

We first consider Case (i): \( y < \gamma \) and then Case (ii): \( y > \gamma \)

Case (i), the contract ends at \((K, yK), 0 < y \leq \gamma\).

Expected cost of rectification is expressed as
\[
EC_m = C_mN(K, yK\mid y < \gamma)
\]
(10)
if \( C_p \) is the penalty cost per unit time, then the expected cost of penalty is given by
\[
EP(K) = C_p\psi(\gamma)N(K, yK\mid y < \gamma)
\]
(11)
Then, total expected cost of the OEM becomes
\[
E[\pi(O)] = C_p\psi(\gamma)N(K, yK\mid y < \gamma) + C_mN(K, yK\mid y < \gamma)
\]
(12)
Case (ii), the contract ends at \((L/ y, L), y > \gamma\).

Total expected cost of the OEM is given by
Then the total expected revenue contract is given by

\[
E[\pi(K,L)] = P_G - \left[ \int_0^\infty E_2[\pi(O)]g(y)dy + \int_0^\infty E_1[\pi(O)]g(y)dy \right]
\]  

(14)

3.2. Owner’s Decision Problem

Here we need also to consider two cases based i.e. Case (i): \( y < \gamma \) and Case (ii): \( y > \gamma \)

Case (i), the contract ends at \((K,yK), 0 < y \leq \gamma\).

The expected profit of the Owner becomes

\[
E_1[\omega(O)] = R \left[ K - N(K,yK)y < \gamma\left(\int_0^\infty zg(z)dz\right) - C_b - P_G + C_P \psi(\mathcal{F})N(K,yK)y < \gamma \right)
\]  

(15)

Case (ii), the contract ends at \((L/y,L), y > \gamma\).

The expected profit for the Owner conditional on \( Y \) is given by

\[
E_2[\omega(O)|y \geq \gamma] = R \left[ L/y - N(L/y,L)y \geq \gamma\left(\int_0^\infty zg(z)dz\right) - C_b - P_G + C_P \psi(\mathcal{F})N(L/y,L)y \geq \gamma \right)
\]  

(16)

Finally the total expected profit for the Owner is given by

\[
E[\omega(K,L)] = \int_0^\gamma E_1[\omega(O)|y \leq \gamma]g(y)dy + \int_\gamma^\infty E_2[\omega(O)|y > \gamma]g(y)dy.
\]  

(17)

4. Optimization

We assume that the OEM and the consumer (owner) would like to negotiate in order to determine the value of the service contract, \( P_G \). In other words, there would be a negotiation between the two parties in the situation considered. As a result, the method of Nash equilibrium can used to obtain the optimal solution. In the presence of negotiation, for every option, the Owner and the OEM will receive the same profit or \( E[\omega] = E[\pi] \). Using \( E[\omega] = E[\pi] \), then we can find the optimal \( P_G \) given by

\[
P_G^* = \int_0^\gamma E_2[\pi(O)]g(y)dy + \int_\gamma^\infty E_1[\pi(O)]g(y)dy + E[\omega(K,L)]
\]  

(18)

Substituting (18) to (17) we have the optimal profit for the owner or the OEM is given by
\[ E[\omega(K, L)] = \\
R \left[ \left\{ \int_0^\infty zg(z)dz \right\} + \left\{ K - N(K, yK) \int_0^\infty zg(z)dz \right\} \right] \\
- \left\{ \int_0^\infty \left[ E_{\pi(O)} g(y)dy + \int_0^\gamma E_{\omega(O)|y \leq \gamma} g(y)dy \right] dy \right\} \\
+ \int_0^\infty E_{\omega(O)|y > \gamma} g(y)dy \right\} \\
+ \int_0^\infty \left[ N(L, L|y \geq \gamma) + N(K, yK|y < \gamma) \right] \\
\right) \\
\right) (,)
\] 

5. Conclusions
In this paper we study a two-dimensional maintenance contract for non-repairable items, where the maintenance contract coverage is characterized by two parameters – age and usage. Failures occurring during the contract is modelled using a bivariate copula distribution and the optimal price of the maintenance contract is obtained using a Nash equilibrium. One can extend this research to the case of repairable item and consider PM and CM provided by the OEM under a two dimensional maintenance contract. This is one topic for future research.

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References
[1] B.P. Iskandar, and N. Jack, 2011, Warranty Servicing with Imperfect Repair for Products Sold with a Two-dimensional Warranty, Replacement Models with Minimal Repair, Springer, 163-174
[2] B.P. Iskandar, U. S. Pasaribu and H. Husniah, 2014, Maintenance Service Contracts For Equipment Sold With Two Dimensional Warranties, Journal Quality Management and Quality Technology, vol. 11, Issue 3, Pages 321-3
[3] C. Jackson, and R. Pascual, 2008, Optimal maintenance service contract negotiation with aging equipment, European Journal of Operational Research, vol. 189, pp. 387–398, 2008
[4] D. N. P. Murthy, and V. Yeung, 1995, Modelling and analysis of maintenance service contracts,” Mathematical and Computer Modelling, vol. 22, pp. 219–225
[5] D. N. P. Murthy, and E. Ashgarizadeh, 1998, A stochastic model for service contract, International Journal of Reliability, Quality and Safety Engineering, vol. 5, pp. 29-45
[6] D. N. P. Murthy, and V. Yeung, 1995, Modelling and analysis of maintenance service contracts, Mathematical and Computer Modelling, vol. 22, pp. 219–225
[7] D. N. P. Murthy, and E. Ashgarizadeh, 1999, Optimal decision making in a maintenance service operation, European Journal of Operational Research, vol. 62, pp. 1–34
[8] E. Ashgarizadeh, and D. N. P. Murthy, 2000, Service contracts, Mathematical and Computer Modelling, vol. 31, pp. 11–20, 2000
[9] H. Husniah, U. S. Pasaribu, A. Cakravastia and B. P. Iskandar, 2014, Performance-based maintenance contract for equipment used in mining industry, Proc. of ICMIT, Singapore
[10] K. Rinsaka, and H. Sandoh, 2006, A stochastic model on an additional warranty service contract, Computers and Mathematics with Applications, vol. 51, pp. 179–188
[11] S. Wu, 2014, Construction of asymmetric copulas and its application in two-dimensional reliability modelling, European Journal of Operational Research