Weakly irreducible subgroups of $\text{Sp}(1, n + 1)$

Natalia I. Bezvitnaya

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Abstract

Connected weakly irreducible not irreducible subgroups of $\text{Sp}(1, n + 1) \subset \text{SO}(4, 4n + 4)$ that satisfy a certain additional condition are classified. This will be used to classify connected holonomy groups of pseudo-hyper-Kählerian manifolds of index 4.

Keywords: pseudo-hyper-Kählerian manifold of index 4, weakly irreducible holonomy group

Mathematical subject codes: 53C29, 53C50

1 Introduction

The classification of connected holonomy groups of Riemannian manifolds is well known [4, 5, 6, 10]. A classification of holonomy groups of pseudo-Riemannian manifolds is an actual problem of differential geometry. Very recently were obtained classifications of connected holonomy groups of Lorentzian manifolds [3, 11, 8] and of pseudo-Kählerian manifolds of index 2 [9]. These groups are contained in $\text{SO}(1, n + 1)$ and $\text{U}(1, n + 1) \subset \text{SO}(2, 2n + 2)$, respectively. As the next step, we study connected holonomy groups contained in $\text{Sp}(1, n + 1) \subset \text{SO}(4, 4n + 4)$, i.e. holonomy groups of pseudo-hyper-Kählerian manifolds of index 4.

By the Wu theorem [12] and the results of Berger for connected irreducible holonomy groups of pseudo-Riemannian manifolds [4], it is enough to consider only weakly irreducible not irreducible holonomy groups (each such group does not preserve any proper non-degenerate vector subspace of the tangent space, but preserves a degenerate subspace).

In the present paper we classify connected weakly irreducible not irreducible subgroups of $\text{Sp}(1, n + 1) \subset \text{SO}(4, 4n + 4)$ ($n \geq 1$) that satisfy a natural condition. The case $n = 0$ will be considered separately. We generalize the method of [7, 9]. Let $G \subset \text{Sp}(1, n + 1)$ be a weakly irreducible not irreducible subgroup and $\mathfrak{g} \subset \text{sp}(1, n + 1)$ the corresponding subalgebra. The results of [9] allow us to expect that if $\mathfrak{g}$ is the holonomy algebra, then $\mathfrak{g}$ contains a certain 3-dimensional ideal $\mathcal{B}$. We will prove this in another paper. Consider the action of $G$ on the space $\mathbb{H}^{1,n+1}$, then $G$ acts on the boundary of the quaternionic hyperbolic space, which is diffeomorphic to the $4n + 3$-dimensional sphere $S^{4n+3}$ and $G$ preserves a point of this space.

We define a map $s_1 : S^{4n+3} \setminus \{\text{point}\} \rightarrow \mathbb{H}^n$ similar to the usual stereographic projection. Then any $f \in G$ defines the map $F(f) = s_1 \circ f \circ s_2 : \mathbb{H}^n \rightarrow \mathbb{H}^n$, where $s_2 : \mathbb{H}^n \rightarrow S^{4n+3} \setminus \{\text{point}\}$ is the inverse of the usual stereographic projection restricted to $\mathbb{H}^n \subset \mathbb{H}^n \oplus \mathbb{R}^3 = \mathbb{R}^{4n+3}$. We get
that $F(G)$ is contained in the group $\text{Sim} \mathbb{H}^n$ of similarity transformations of $\mathbb{H}^n$. We show that $F(G)$ preserves an affine subspace $L \subset \mathbb{R}^{4n} = \mathbb{H}^n$ such that the minimal affine subspace of $\mathbb{H}^n$ containing $L$ is $\mathbb{H}^n$. Moreover, $F(G)$ does not preserve any proper affine subspace of $L$. Then $F(G)$ acts transitively on $L$. We describe subspaces $L$ with such property and using results of [9] we find all connected Lie subgroups $K \subset \text{Sim} \mathbb{H}^n$ preserving $L$ and acting transitively on $L$. Note that the kernel of the Lie algebra homomorphism $dF : \mathfrak{g} \rightarrow \mathcal{L}A(\text{Sim} \mathbb{H}^n)$ coincides with the ideal $\mathcal{B}$. Consequently, $\mathfrak{g} = (dF)^{-1}(\mathfrak{h})$, where $\mathfrak{h} \subset \mathcal{L}A(\text{Sim} \mathbb{H}^n)$ is the Lie algebra of one of the obtained Lie subgroups $K \subset \text{Sim} \mathbb{H}^n$.

Note that we classify weakly irreducible not irreducible subgroups of $\text{Sp}(1, n + 1)$ up to conjugacy in $\text{SO}(4, 4n + 4)$. It is also possible to classify these subgroups up to conjugacy in $\text{Sp}(1, n + 1)$, see Remark [1].

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2 Preliminaries

First we summarize some facts about quaternionic vector spaces. Let $\mathbb{H}^m$ be an $m$-dimensional quaternionic vector space and $e_1, ..., e_m$ a basis of $\mathbb{H}^m$. We identify an element $X \in \mathbb{H}^m$ with the column $(X_i)$ of the left coordinates of $X$ with respect to this basis, $X = \sum_{i=1}^{m} X_i e_i$.

Let $f : \mathbb{H}^m \rightarrow \mathbb{H}^m$ be an $\mathbb{H}$-linear map. Define the matrix $\text{Mat}_f$ of $f$ by the relation $f e_i = \sum_{t=1}^{m} (\text{Mat}_f)_{it} e_t$. Now if $X \in \mathbb{H}^m$, then $fX = (X^t \text{Mat}_f^t)^t$ and because of the non-commutativity of the quaternions this is not the same as $\text{Mat}_f X$. Conversely, to an $m \times m$ matrix $A$ of the quaternions we put in correspondence the linear map $O^p A : \mathbb{H}^m \rightarrow \mathbb{H}^m$ such that $O^p A \cdot X = (X^t A^t)^t$. If $f, g : \mathbb{H}^m \rightarrow \mathbb{H}^m$ are two $\mathbb{H}$-linear maps, then $\text{Mat}_{fg} = (\text{Mat}_f)^t \text{Mat}_g^t$. Note that the multiplications by the imaginary quaternions are not $\mathbb{H}$-linear maps. Also, for $a, b \in \mathbb{H}$ holds $ab = b\overline{a}$. Consequently, for two square quaternionic matrices we have $(\overline{A}B)^t = B^t A^t$.

A pseudo-quaternionic-Hermitian metric $g$ on $\mathbb{H}^m$ is a non-degenerate $\mathbb{R}$-bilinear map $g : \mathbb{H}^m \times \mathbb{H}^m \rightarrow \mathbb{H}$ such that $g(aX, Y) = ag(X, Y)$ and $g(Y, X) = g(X, Y)$, where $a \in \mathbb{H}$, $X, Y \in \mathbb{H}^m$. Hence, $g(X, aY) = g(X, Y)\overline{a}$. There exists a basis $e_1, ..., e_m$ of $\mathbb{H}^m$ and integers $(r, s)$ with $r + s = m$ such that $g(e_i, e_j) = 1$ if $t \neq i$, $g(e_i, e_t) = -1$ if $1 \leq t \leq p$ and $g(e_i, e_t) = 1$ if $p + 1 \leq t \leq m$. The pair $(r, s)$ is called the signature of $g$. In this situation we denote $\mathbb{H}^m$ by $\mathbb{H}^{r,s}$. The realification of $\mathbb{H}^m$ gives us the vector space $\mathbb{R}^m$ with the quaternionic structure $(i, j, k)$. Conversely, a quaternionic structure on $\mathbb{R}^m$, i.e. a triple $(I, J, K)$ of endomorphisms of $\mathbb{R}^m$ such that $I^2 = J^2 = K^2 = -\text{id}$ and $K = IJ = -JI$, allows us to consider $\mathbb{R}^m$ as $\mathbb{H}^m$.

A pseudo-quaternionic-Hermitian metric $g$ on $\mathbb{H}^m$ of signature $(r, s)$ defines on $\mathbb{R}^m$ the $i, j, k$-invariant pseudo-Euclidean metric $\eta$ of signature $(4r, 4s)$, $\eta(X, Y) = \text{Re} g(X, Y)$, $X, Y \in \mathbb{R}^m$. Conversely, a $I, J, K$-invariant pseudo-Euclidean metric on $\mathbb{R}^m$ defines a pseudo-quaternionic-Hermitian metric $g$ on $\mathbb{H}^m$,

$$g(X, Y) = \eta(X, Y) + i\eta(X, JY) + j\eta(X, JY) + k\eta(X, KY).$$
The Lie group $\text{Sp}(r, s)$ and its Lie algebra $\mathfrak{sp}(r, s)$ are defined as follows
\[
\text{Sp}(r, s) = \{ f \in \text{Aut}(\mathbb{H}^{r,s}) \mid g(fX, fY) = g(X, Y) \text{ for all } X, Y \in \mathbb{H}^{r,s} \},
\]
\[
\mathfrak{sp}(r, s) = \{ f \in \text{End}(\mathbb{H}^{r,s}) \mid g(fX, Y) + g(X, fY) = 0 \text{ for all } X, Y \in \mathbb{H}^{r,s} \}.
\]

3 The Main Theorem

**Definition 1.** A subgroup $G \subset \text{SO}(r, s)$ (or a subalgebra $\mathfrak{g} \subset \mathfrak{so}(r, s)$) is called weakly irreducible if it does not preserve any non-degenerate proper vector subspace of $\mathbb{R}^{r,s}$.

Let $\mathbb{R}^{4,4n+4}$ be a $(4n+8)$-dimensional real vector space endowed with a quaternionic structure $I, J, K \in \text{End}(\mathbb{R}^{4,4n+4})$ and an $I, J, K$-invariant metric $\eta$ of signature $(4, 4n+4)$. We identify this space with the $(n+2)$-dimensional quaternionic space $\mathbb{H}^{1,n+1}$ endowed with the pseudo-quaternionic-Hermitian metric $g$ of signature $(1, n+1)$ as above.

Obviously, if a Lie subgroup $G \subset \text{Sp}(1, n+1)$ acts weakly irreducibly not irreducibly on $\mathbb{R}^{4,4n+4}$, then $G$ acts weakly irreducibly not irreducibly on $\mathbb{H}^{1,n+1}$. The converse is not true, see Example 2 below. If $G$ acts weakly irreducibly not irreducibly on $\mathbb{H}^{1,n+1}$, then $G$ preserves a proper degenerate subspace $W \subset \mathbb{H}^{1,n+1}$. Consequently, $G$ preserves the intersection $W \cap W^\perp \subset \mathbb{H}^{1,n+1}$, which is an isotropic quaternionic line.

Fix a Wit basis $p, e_1, \ldots, e_n, q$ of $\mathbb{H}^{1,n+1}$, i.e. the Gram matrix of the metric $g$ with respect to this basis has the form
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & E_n & 0 \\
1 & 0 & 0
\end{pmatrix},
\]
where $E_n$ is the $n$-dimensional identity matrix.

Denote by $\text{Sp}(1, n+1)_{\mathbb{H}p}$ the Lie subgroup of $\text{Sp}(1, n+1)$ acting on $\mathbb{H}^{1,n+1}$ and preserving the quaternionic isotropic line $\mathbb{H}p$. Note that any weakly irreducible and not irreducible subgroup of $\text{Sp}(1, n+1)$ is conjugated to a weakly irreducible subgroup of $\text{Sp}(1, n+1)_{\mathbb{H}p}$.

The Lie subalgebra $\mathfrak{sp}(1, n+1)_{\mathbb{H}p} \subset \mathfrak{sp}(1, n+1)$ corresponding to the Lie subgroup $\text{Sp}(1, n+1)_{\mathbb{H}p} \subset \text{Sp}(1, n+1)$ has the following form
\[
\mathfrak{sp}(1, n+1)_{\mathbb{H}p} = \left\{ \begin{pmatrix}
\tilde{a} & -X^t & b \\
0 & \text{Mat}_h & X \\
0 & 0 & -a
\end{pmatrix} \right\} \text{ for } a \in \mathbb{H}, X \in \mathbb{H}^n, h \in \mathfrak{sp}(n), b \in \text{Im } \mathbb{H}.
\]

Let $(a, A, X, b)$ denote the above element of $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$. Define the following vector subspaces of $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$:
\[
\mathcal{A}_1 = \{(a, 0, 0, 0) | a \in \mathbb{R} \}, \quad \mathcal{A}_2 = \{(a, 0, 0, 0) | a \in \text{Im } \mathbb{H} \},
\]
\[
\mathcal{N} = \{(0, 0, X, 0) | X \in \mathbb{H}^n \}, \quad \mathcal{B} = \{(0, 0, 0, b) | b \in \text{Im } \mathbb{H} \}.
\]

Obviously, $\mathfrak{sp}(n)$ is a subalgebra of $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ with the inclusion
\[
h \in \mathfrak{sp}(n) \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & \text{Mat}_h & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sp}(1, n+1)_{\mathbb{H}p}.
\]

We obtain that $\mathcal{A}_1$ is a one-dimensional commutative subalgebra that commutes with $\mathcal{A}_2$ and $\mathfrak{sp}(n)$, $\mathcal{A}_2$ is a subalgebra isomorphic to $\mathfrak{sp}(1)$ and commuting with $\mathfrak{sp}(n)$, $\mathcal{B}$ is a commutative
ideal, which commutes with $\mathfrak{sp}(n)$ and $\mathcal{N}$. Also,

\[ [(a, 0, 0, 0), (0, X, b)] = (0, aX, 2\text{Im } ab), \quad [(0, 0, X, 0), (0, 0, Y, 0)] = (0, 0, 2\text{Im } g(X, Y)), \quad [(0, A, 0, 0), (0, 0, X, 0)] = (0, 0, (X^i A^j)_{ij}, 0), \]

where $a \in \mathbb{H}, X, Y \in \mathbb{H}^n, A = \text{Mat}_h, h \in \mathfrak{sp}(n), b \in \text{Im } \mathbb{H}$. Thus we have the decomposition

$$\mathfrak{sp}(1, n+1)_{\mathbb{H}^p} = (\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathfrak{sp}(n)) \times (\mathcal{N} + \mathcal{B}) \simeq (\mathbb{R} \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)) \times (\mathbb{H}^n + \mathbb{R}^3).$$

Now consider two examples.

**Example 1.** The subalgebra $\mathfrak{g} = \{(0, 0, X, b) \mid X \in \mathbb{R}^n, b \in \text{Im } \mathbb{H} \} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}^p}$ acts weakly irreducibly on $\mathbb{R}^{4n+4}$.

**Proof.** Assume the converse. Let $\mathfrak{g}$ preserve a non-degenerate proper vector subspace $L \subset \mathbb{R}^{4n+4}$. Suppose the projection of $L$ to $\mathbb{H}^q \subset \mathbb{H}^{1,n+1} = \mathbb{R}^{4n+4}$ is non-zero, then there is a vector $v \in L$ such that $v = v_0 p + v_1 q$, where $v_0, v_2 \in \mathbb{H}, v_2 \neq 0$ and $v_1 \in \mathbb{H}^n$. Consider elements $\xi_1 = (0, 0, X, 0) \in \mathfrak{g}$ with $g(X, X) = 1$ and $\xi_2 = (0, 0, 0, b) \in \mathfrak{g}$. Then, $\xi_1(\xi_1 v) = -v_2 p \in L$ and $\xi_2 v = v_2 p b \in L$. Since $v_2 \neq 0$, we have $\mathbb{H}^p \subset L$. It follows that $L^{\perp_n} \subset \mathbb{H}^p \oplus \mathbb{H}^n$ and $L^{\perp_n}$ is a $\mathfrak{g}$-invariant non-degenerate proper subspace. Now we can assume that $\mathfrak{g}$ preserves a non-trivial non-degenerate vector subspace $L \subset \mathbb{H}^p \oplus \mathbb{H}^n$. Let $v = v_0 p + v_1 \in L, v \neq 0$. If $v_1 = 0$, then $L$ is degenerate. If $v_1 \neq 0$, then there is $X \in \mathbb{R}^n$ with $g(v_1, X) \neq 0$. We get $(0, 0, X, 0)v = -g(v_1, X)p \in L$. Hence $L$ is degenerate. Thus we have a contradiction. □

**Example 2.** The subalgebra $\mathfrak{g} = \{(0, 0, X, 0) \mid X \in \mathbb{R}^n \} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}^p}$ acts weakly irreducibly on $\mathbb{H}^{1,n+1}$ and not weakly irreducibly on $\mathbb{R}^{4n+4}$.

**Proof.** The proof of the first statement is similar to the proof of Example 1. Clearly, the subalgebra $\mathfrak{g}$ preserves the non-degenerate vector subspace $\text{span}_\mathbb{R}\{p, e_1, ..., e_n, q\} \subset \mathbb{R}^{4n+4}$.

The classification of the holonomy algebras contained in $\mathfrak{u}(1, n+1)$ [9] gives us the following hypothesis: If $n \geq 1$ and $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}^p}$ is a holonomy algebra, then $\mathfrak{g}$ contains the ideal $\mathcal{B}$. We will prove this hypothesis in an other paper.

In the following theorem we denote the real vector subspace $L \subset \mathbb{R}^{4n} = \mathbb{H}^n$ of the form

$$L = \text{span}_\mathbb{R}\{e_1, ..., e_m\} \oplus \text{span}_\mathbb{R^{\mathbb{H}^n}}\{e_{m+1}, ..., e_{m+k}\} \oplus \text{span}_\mathbb{R}\{e_{m+k+1}, ..., e_n\}$$

by $\mathbb{R}^n \oplus \mathbb{C}^k \oplus \mathbb{R}^{n-m-k}$. Let $\mathfrak{u}(k)$ be the subalgebra of $\mathfrak{sp}(\text{span}_\mathbb{R}\{e_{m+1}, ..., e_{m+k}\})$ that consists of the elements $\text{Op}\left(\begin{array}{cc} A & 0 \\ 0 & A \end{array}\right)$, where $A \in \mathfrak{u}(\text{span}_\mathbb{R^{\mathbb{H}^n}}\{e_{m+1}, ..., e_{m+k}\})$ and we use the decomposition $\text{span}_\mathbb{R}\{e_{m+1}, ..., e_{m+k}\} = \text{span}_\mathbb{R^{\mathbb{H}^n}}\{e_{m+1}, ..., e_{m+k}\} + j\text{span}_\mathbb{R^{\mathbb{H}^n}}\{e_{m+1}, ..., e_{m+k}\}$. Similarly, let $\mathfrak{so}(n-m-k)$ be the subalgebra of $\mathfrak{sp}(\text{span}_\mathbb{R}\{e_{m+k+1}, ..., e_n\})$ that consists of the elements $\text{Op}\left(\begin{array}{ccc} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{array}\right)$, where $A \in \mathfrak{so}(\text{span}_\mathbb{R}\{e_{m+k+1}, ..., e_n\})$ and we use the decomposition $\mathbb{R}^{n-m-k} = \mathbb{R}^{n-m-k} \oplus i\mathbb{R}^{n-m-k} \oplus j\mathbb{R}^{n-m-k} \oplus k\mathbb{R}^{n-m-k}$. For a Lie algebra $\mathfrak{h}$ we denote by $\mathfrak{h}'$ the commutant $[\mathfrak{h}, \mathfrak{h}]$ of $\mathfrak{h}$. 

4
4 Relation with the group of similarity transformations of $\mathbb{H}^n$

Let $\mathbb{H}^n$ be the $n$-dimensional quaternionic vector space endowed with a quaternionic-Hermitian metric $g$. For elements $a_1 \in \mathbb{R}_+$, $a_2 \in \text{Sp}(1)$, $f \in \text{Sp}(n)$ and $X \in \mathbb{H}^n$ consider the following transformations of $\mathbb{H}^n$: $d(a_1) : Y \mapsto a_1 Y$ (real dilation), $a_2 : Y \mapsto a_2 Y$ (quaternionic dilation), $f : Y \mapsto fY$ (rotation), $t(Y) : Y \mapsto Y + X$ (translation), where $Y \in \mathbb{H}^n$. Note that the elements $a_2 \in \text{Sp}(1)$ act on $\mathbb{H}^n$ as $\mathbb{R}$-linear (but not $\mathbb{H}$-linear) isomorphism. These transformations generate the Lie group $\text{Sim} \mathbb{H}^n$ of similarity transformations of $\mathbb{H}^n$. We get the decomposition

$$\text{Sim} \mathbb{H}^n = (\mathbb{R}_+ \times \text{Sp}(1) \cdot \text{Sp}(n)) \ltimes \mathbb{H}^n.$$
The Lie group Sim\(\mathbb{H}^n\) is a Lie subgroup of the connected Lie group Sim\(^0\mathbb{R}^{4n}\) of similarity transformations of \(\mathbb{R}^{4n}\), Sim\(^0\mathbb{R}^{4n} = (\mathbb{R}_+ \times \text{SO}(4n)) \ltimes \mathbb{R}^{4n}\).

The corresponding Lie algebra \(\mathcal{L}A(\text{Sim} \mathbb{H}^n)\) to the Lie group Sim\(\mathbb{H}^n\) has the following decomposition

\[
\mathcal{L}A(\text{Sim} \mathbb{H}^n) = (\mathbb{R} \oplus \text{sp}(1) \oplus \text{sp}(n)) \ltimes \mathbb{H}^n.
\]

Let \(p, e_1, ..., e_n, q\) be the basis of \(\mathbb{H}^{1,n+1}\) as above. Consider also the basis \(e_0, e_1, ..., e_n, e_{n+1}\), where \(e_0 = \sqrt{2}(p - q)\) and \(e_{n+1} = \sqrt{2}(p + q)\). With respect to this basis the Gram matrix of \(g\) has the form

\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & E_{n+1}
\end{pmatrix}.
\]

The subset of the \((n + 1)\)-dimensional quaternionic projective space \(\mathbb{H}P^{1,n+1}\) that consists of all quaternionic isotropic lines is called the *boundary* of the quaternionic hyperbolic space and is denoted by \(\partial \mathbb{H}^{1,n+1}\).

Let \(h_0, ..., h_{n+1}\), where \(h_s = x_s + iy_s + jz_s + kw_s \in \mathbb{H}\) \((0 \leq s \leq n + 1)\) be the coordinates on \(\mathbb{H}^{1,n+1}\) with respect to the basis \(e_0, ..., e_{n+1}\). Denote by \(\mathbb{H}^n\) and \(\mathbb{H}^{n+1}\) the subspaces of \(\mathbb{H}^{1,n+1}\) spanned by the vectors \(e_1, ..., e_n\) and \(e_1, ..., e_{n+1}\), respectively. Note that the intersection \((e_0 + \mathbb{H}^{n+1}) \cap \{X \in \mathbb{H}^{1,n+1} | g(X, X) = 0\}\) is given by the system of equations:

\[
x_0 = 1, \quad y_0 = 0, \quad z_0 = 0, \quad w_0 = 0, \quad x_1^2 + y_1^2 + z_1^2 + w_1^2 + \ldots + x_{n+1}^2 + y_{n+1}^2 + z_{n+1}^2 + w_{n+1}^2 = 1,
\]

i.e. this set is the \((4n + 3)\)-dimensional unite sphere \(S^{4n+3}\). Moreover, each isotropic line intersects this set at a unique point, e.g. \(\mathbb{H}p\) intersects it at the point \(\sqrt{2}p\). Thus we identify the space \(\partial \mathbb{H}^{n+1}\) with the sphere \(S^{4n+3}\). Any \(f \in \text{Sp}(1,n+1) \mathbb{H}p\) takes quaternionic isotropic lines to quaternionic isotropic lines and preserves the quaternionic isotropic line \(\mathbb{H}p\). Hence it acts on \(\partial \mathbb{H}^{n+1} \setminus \{\mathbb{H}p\} = S^{4n+3} \setminus \{\sqrt{2}p\}\).

Consider the connected Lie subgroups \(A_1, A_2, \text{Sp}(n)\) and \(P\) of \(\text{Sp}(1,n+1) \mathbb{H}p\) corresponding to the subalgebras \(A_1, A_2, \text{sp}(n)\) and \(N + B\) of the Lie algebra \(\text{sp}(1,n+1) \mathbb{H}p\). With respect to the basis \(p, e_1, ..., e_n, q\) these groups have the following matrix form:

\[
A_1 = \left\{ \text{Op} \begin{pmatrix}
a_1 & 0 & 0 \\
0 & E_n & 0 \\
0 & 0 & a_{n+1}^{-1}
\end{pmatrix} \mid a_1 \in \mathbb{R}_+ \right\}, \quad A_2 = \left\{ \text{Op} \begin{pmatrix}
e^{-a_2} & 0 & 0 \\
0 & E_n & 0 \\
0 & 0 & e^{-a_2}
\end{pmatrix} \mid a_2 \in \text{Im} \mathbb{H} \right\}.
\]

\[
\text{Sp}(n) = \left\{ \text{Op} \begin{pmatrix}
1 & 0 & 0 \\
0 & \text{Mat}_f & 0 \\
0 & 0 & 1
\end{pmatrix} \mid f \in \text{Sp}(n) \right\},
\]

\[
P = \left\{ \text{Op} \begin{pmatrix}
1 & -Y^t & b - \frac{1}{2}Y^tY \\
0 & E_n & Y \\
0 & 0 & 1
\end{pmatrix} \mid Y \in \mathbb{H}^n, \quad b \in \text{Im} \mathbb{H} \right\}.
\]

We have the decomposition

\[
\text{Sp}(1,n+1) \mathbb{H}p = (A_1 \times A_2 \times \text{Sp}(n)) \ltimes P \simeq (\mathbb{R}_+ \times \text{Sp}(1) \times \text{Sp}(n)) \ltimes (\mathbb{H}^n \cdot \mathbb{R}^3).
\]

Let \(s_1 : S^{4n+3} \setminus \{\sqrt{2}p\} \rightarrow e_0 + \mathbb{H}^n\) be the map defined as the usual stereographic projection, but using quaternionic lines. More precisely, for \(s \in S^{4n+3} \setminus \{\sqrt{2}p\}\) we define \(s_1(s)\) to be the point of the intersection of \(e_0 + \mathbb{H}^n\) with the quaternionic line passing through the points \(\sqrt{2}p\)
and s. It is easy to see that this intersection consists of a single point. Let \( s_2 : e_0 + \mathbb{H}^n \to S^{4n+3} \setminus \{ \sqrt{2}p \} \) be the restriction to \( e_0 + \mathbb{H}^n \) of the inverse to the usual stereographic projection from \( S^{4n+3} \setminus \{ \sqrt{2}p \} \) to \( e_0 + \mathbb{H}^n \oplus (\text{Im} \mathbb{H}) e_{n+1} \). Note that \( s_1 \circ s_2 = \text{id}_{e_0 + \mathbb{H}^n} \), but unlike in the usual case, \( s_1 \) is not surjective. We have \( s_2 \circ s_1|_{\text{Im} e_2} = \text{id}_{\text{Im} e_2} \). Also, let \( e_0 \) and \(-e_0\) denote the translations \( \mathbb{H}^n \to e_0 + \mathbb{H}^n \) and \( e_0 + \mathbb{H}^n \to \mathbb{H}^n \), respectively.

For \( f \in \text{Sp}(1,n+1)_{\text{H}} \) define the map

\[
F(f) = (-e_0) \circ s_1 \circ f \circ s_2 \circ e_0 : \mathbb{H}^n \to \mathbb{H}^n.
\]

Now we will show that \( F \) is a surjective homomorphism from the Lie group \( \text{Sp}(1,n+1)_{\text{H}} \) to the Lie group \( \text{Sim} \mathbb{H}^n \) and \( \ker F = \mathbb{Z}_2 \times B \), where \( \mathbb{Z}_2 = \{ \text{id}, - \text{id} \} \in \text{Sp}(1,n+1)_{\text{H}} \) and \( B \) is the connected Lie subgroup of \( \text{Sp}(1,n+1)_{\text{H}} \) corresponding to the ideal \( B \subset \text{Sp}(1,n+1)_{\text{H}} \).

First of all, the computations show that for \( a_1 \in \mathbb{R} \), \( a_2 \in \text{Im} \mathbb{H} \), \( f \in \text{Sp}(n) \) and \( Y \in \mathbb{H}^n \) it holds

\[
F \left( \text{Op} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & a_1^{-1} \end{pmatrix} \right) = d(a_1) \in \mathbb{R}_+ \subset \text{Sim} \mathbb{H}^n,
\]

\[
F \left( \text{Op} \begin{pmatrix} e^{-a_2} & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & a^{-a_2} \end{pmatrix} \right) = e^{a_2} \in \text{Sp}(1) \subset \text{Sim} \mathbb{H}^n,
\]

\[
F \left( \text{Op} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Mat}_f & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = f \in \text{Sp}(n) \subset \text{Sim} \mathbb{H}^n,
\]

\[
F \left( \text{Op} \begin{pmatrix} 1 & -\bar{Y}^t & b - \frac{1}{2} \bar{Y}^t \bar{Y} \\ 0 & E_n & \bar{Y} \\ 0 & 0 & 1 \end{pmatrix} \right) = t \left( -\frac{\sqrt{2}}{2} Y \right) \in \mathbb{H}^n \subset \text{Sim} \mathbb{H}^n.
\]

It follows that if \( f_1, f_2 \in P \), then \( F(f_1 f_2) = F(f_1) F(f_2) \), i.e. \( F \) is a homomorphism from \( P \) to \( \text{Sim} \mathbb{H}^n \). It can easily be checked that any \( f \in A_1 \times A_2 \times \text{Sp}(n) \) considered as a map from \( S^{4n+3} \setminus \{ \sqrt{2}p \} \) to itself preserves \( \text{Im} s_2 \subset S^{4n+3} \setminus \{ \sqrt{2}p \} \). Hence if \( f_1 \) is from \( P \) or \( A_1 \times A_2 \times \text{Sp}(n) \) and \( f_2 \in A_1 \times A_2 \times \text{Sp}(n) \), then

\[
F(f_1 f_2) = (-e_0) \circ s_1 \circ f_1 \circ f_2 \circ s_2 \circ e_0 = (-e_0) \circ s_1 \circ f_1 \circ s_2 \circ e_0 \circ (-e_0) \circ s_1 \circ f_2 \circ s_2 \circ e_0 = F(f_1) F(f_2),
\]

since \( s_2 \circ s_1|_{\text{Im} e_2} = \text{id}_{\text{Im} e_2} \). Therefore it is enough to prove that \( F(f_1 f_2) = F(f_1) F(f_2) \), for \( f_1 \in A_1 \times A_2 \times \text{Sp}(n) \) and \( f_2 \in P \). Let

\[
f_1 = \text{Op} \begin{pmatrix} a_1 e^{-a_2} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a_1 e^{-a_2} \end{pmatrix} \in A_1 \times A_2 \times \text{Sp}(n), \quad f_2 = \text{Op} \begin{pmatrix} 1 & -\bar{Y}^t & b - \frac{1}{2} \bar{Y}^t \bar{Y} \\ 0 & E_n & \bar{Y} \\ 0 & 0 & 1 \end{pmatrix} \in P.
\]

Then \( f_1 f_2 f_1^{-1} = f_2' \in P \), where \( f_2' = \text{Op} \begin{pmatrix} 1 & -((A^{-1})^t \bar{Y} a_1 e^{-a_2})^t & a_1^2 e^{a_2} (b - \frac{1}{2} \bar{Y}^t \bar{Y} e^{-a_2}) \\ 0 & E_n & a_1 e^{a_2} (Y^t A)^t \\ 0 & 0 & 1 \end{pmatrix} \).
We have
\[ F(f_1 f_2) = F(f_2 f_1) = F(f_2) F(f_1) = t \left( -\frac{\sqrt{2}}{2} a_1 e^{a_2} (Y^t A^t) \right) a_1 e^{a_2} \text{Op} A \]
\[ = t \left( -\frac{\sqrt{2}}{2} a_1 e^{a_2} \text{Op} A \cdot Y \right) a_1 e^{a_2} \text{Op} A = a_1 e^{a_2} \text{Op} A \cdot t \left( -\frac{\sqrt{2}}{2} Y \right) = F(f_1) F(f_2), \]

since for any \( f \in \mathbb{R}_+ \times SO(4n) \) and \( X \in \mathbb{R}^{4n} \) it holds \( ft(X) f^{-1} = t(fX) \) or \( t(fX) f = ft(X) \). Thus \( F \) is the homomorphism from the Lie group \( \text{Sp}(1, n+1)_{\mathbb{H}^p} \) to the Lie group \( \text{Sim} \mathbb{H}^n \). Obviously, \( F \) is surjective. The claim is proved.

Let \( L \subset \mathbb{H}^n \) be a vector (affine) subspace. We call the subset \( L \subset \mathbb{H}^n \) a \textit{real vector (affine) subspace}.

**Theorem 2.** Let \( G \subset \text{Sp}(1, n+1)_{\mathbb{H}^p} \) act weakly irreducibly on \( \mathbb{H}^{1,n+1} \). Then if \( F(G) \subset \text{Sim} \mathbb{H}^n \) preserves a proper real affine subspace \( L \subset \mathbb{H}^n \), then the minimal affine subspace of \( \mathbb{H}^n \) containing \( L \) is \( \mathbb{H}^n \).

**Proof.** First we prove that the subgroup \( F(G) \subset \text{Sim} \mathbb{H}^n \) does not preserve any proper affine subspace of \( \mathbb{H}^n \). Assume that \( F(G) \) preserves a vector subspace \( L \subset \mathbb{H}^n \). Choosing the basis \( e_1, \ldots, e_n \) of \( \mathbb{H}^n \) in a proper way, we can suppose that \( L = \mathbb{H}^m = \text{span}_{\mathbb{H}} \{ e_1, \ldots, e_m \} \). Consequently, \( F(G) \subset (\mathbb{R}_+ \times (\text{Sp}(1) \cdot (\text{Sp}(m) \times \text{Sp}(n-m)))) \setminus \mathbb{H}^m \). Hence, \( G \subset (\mathbb{R}_+ \times \text{Sp}(1) \times \text{Sp}(m) \times \text{Sp}(n-m)) \setminus (\mathbb{H}^m \cdot \mathbb{R}^3) \) and \( G \) preserves the non-degenerate vector subspace \( \text{span}_\mathbb{H} \{ e_{m+1}, \ldots, e_n \} \subset \mathbb{H}^{1,n+1} \). Now suppose that \( F(G) \) preserves an affine subspace \( L \subset \mathbb{H}^n \). Let \( L = Y + L_0 \), where \( Y \in L \) and \( L_0 \subset \mathbb{H}^n \) is the vector subspace corresponding to \( L \). We may assume that \( L_0 = \mathbb{H}^m = \text{span}_{\mathbb{H}} \{ e_1, \ldots, e_m \} \). Consider \( f = \text{Op} \left( \begin{array}{cc} 1 & \sqrt{2} Y^t \vspace{1ex} \mathbf{0} \\ \mathbf{0} & E_n \end{array} \right) \in P \) and the subgroup \( \tilde{G} = f^{-1} G f \subset \text{Sp}(1, n+1)_{\mathbb{H}^p} \). For \( F(\tilde{G}) \) we get that \( F(\tilde{G}) = -t(Y) F(G) t(Y) \). By the above \( \tilde{G} \) preserves the non-degenerate vector subspace \( \text{span}_\mathbb{H} \{ e_{m+1}, \ldots, e_n \} \subset \mathbb{H}^{1,n+1} \). Hence \( G \) preserves the non-degenerate vector subspace \( f(\text{span}_\mathbb{H} \{ e_{m+1}, \ldots, e_n \}) \subset \mathbb{H}^{1,n+1} \). Since \( G \) is weakly irreducible, we get \( m = n \).

Let \( F(G) \) preserve a real affine subspace \( L \subset \mathbb{H}^n \) and let \( L_0 \subset \mathbb{H}^n \) be the corresponding real vector subspace. Consider the vector subspace \( \text{span}_{\mathbb{H}} L_0 \perp \subset \mathbb{H}^n \). As above, it can be proved that \( G \) preserves the non-degenerate vector subspace \( f(\text{span}_{\mathbb{H}} L_0 \perp) \subset \mathbb{H}^{1,n+1} \). Since \( G \) is weakly irreducible, we have \( (\text{span}_{\mathbb{H}} L_0 \perp) = 0 \) and \( \text{span}_{\mathbb{H}} L_0 = \mathbb{H}^n \). The theorem is proved. \( \square \)

5 Proof of the Main Theorem

First of all, from Example 1 it follows that the algebras of Types I–VIII act weakly irreducibly on \( \mathbb{R}^{4,4n+4} \). For the algebras of Type IX it can be proved in the same way. Therefore we must only prove that any subalgebra \( g \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}^p} \) that acts weakly irreducibly on \( \mathbb{R}^{4,4n+4} \) and contains the ideal \( \mathcal{B} \) is conjugated (by an element from \( \text{SO}(4,4n+4) \)) to one of the algebras of Types I–IX. Suppose that \( g \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}^p} \) acts weakly irreducibly on \( \mathbb{R}^{4,4n+4} \) and contains the ideal \( \mathcal{B} \). Let \( G \subset \text{Sp}(1, n+1)_{\mathbb{H}^p} \) be the corresponding connected
Let $F(G)$ preserves a real affine subspace $L \subset \mathbb{H}^n$ such that the minimal affine subspace of $\mathbb{H}^n$ containing $L$ is $\mathbb{H}^n$. We already know that $G$ is conjugated to a subgroup $\tilde{G} \subset \text{Sp}(1, n + 1)_{\mathbb{H}}$, such that $F(\tilde{G})$ preserves a real vector subspace $L_0 \subset \mathbb{H}^n$ with $\text{span}_\mathbb{H} L_0 = \mathbb{H}^n$. Hence we can assume that $F(G)$ preserves a real vector subspace $L \subset \mathbb{H}^n$ and $\text{span}_\mathbb{H} L = \mathbb{H}^n$. Moreover, assume that $F(G)$ does not preserve any proper affine subspace of $L$. Then $F(G)$ acts transitively on $L$ \cite{1}. The connected transitively acting groups of similarity transformations of the Euclidean spaces are well known. In \cite{9} these groups were divided into three types. We describe real subspaces $L \subset \mathbb{H}^n$ with $\text{span}_\mathbb{H} L = \mathbb{H}^n$ and subalgebras $\mathfrak{f} \subset \mathcal{L}(\text{Sim} \mathbb{H}^n)$ such that the corresponding connected Lie subgroups $\mathcal{K} \subset \text{Sim} \mathbb{H}^n$ preserve $L$ and act transitively on $L$. Then the algebra $\mathfrak{g}$ must be of the form $(dF)^{-1}(\mathfrak{f})$ for a subalgebra $\mathfrak{f}$.

Now we describe real vector subspaces $L \subset \mathbb{H}^n$ with $\text{span}_\mathbb{H} L = \mathbb{H}^n$. Let $L$ be such subspace. Put $L_1 = L \cap iL \cap jL \cap kL$, i.e. $L_1$ is the maximal quaternionic vector subspace in $L$. Let $L_2$ be the orthogonal complement to $L_1$ in $L$, then $L = L_1 \oplus L_2$ and $L_2 \cap iL \cap jL \cap kL = 0$. Now let $L_3 = L_2 \cap iL_2$, i.e. $L_3$ is the maximal $i$-invariant real vector subspace in $L_2$. Let $L_4$ be its orthogonal complement in $L_2$, then $L_2 = L_3 \oplus L_4$. Similarly, define the spaces $L_5, L_6, L_7, L_8 \subset L$ such that $L_3 = L_4 \cap jL_4$, $L_4 = L_5 \oplus L_6$, $L_7 = L_6 \cap kL_6$ and $L_6 = L_7 \oplus L_8$. By construction, we get the orthogonal decomposition $L = L_1 \oplus L_3 \oplus L_5 \oplus L_7 \oplus L_8$ and there exists a $g$-orthogonal basis $e_1, ..., e_n$ of $\mathbb{H}^n$ such that this decomposition has the form

$$L = \text{span}_{\mathbb{H}} \{e_1, ..., e_m\} \oplus \text{span}_{\mathbb{R}^m_{\mathbb{H}}} \{e_{m+1}, ..., e_{m+1}\} \oplus \text{span}_{\mathbb{R}^2} \{e_{m+2}, ..., e_{m+2}\} \oplus \text{span}_{\mathbb{R}^3} \{e_{m+3}, ..., e_{m+3}\}. \tag{1}$$

Obviously, there is an $f \in \text{SO}(n)$ such that

$$fL = \text{span}_{\mathbb{H}} \{e_1, ..., e_m\} \oplus \text{span}_{\mathbb{R}^m_{\mathbb{H}}} \{e_{m+1}, ..., e_{m+k}\} \oplus \text{span}_{\mathbb{R}^3} \{e_{m+k+1}, ..., e_{m+k}\}, \tag{2}$$

where $m + k = m_3$. Since we consider the subgroups of $\text{Sp}(1, n + 1)_{\mathbb{H}}$ up to conjugacy in $\text{SO}(4, 4n + 4)$, we can assume that $L$ has the form (2). We will write for short

$$L = \mathbb{H}^n \oplus \mathbb{C}^k \oplus \mathbb{R}^{m-k}.$$  

Suppose that a subgroup $\mathcal{K} \subset \text{Sim} \mathbb{H}^n$ preserves $L$. Since $K \subset \text{Sim} \mathbb{H}^n \subset \text{Sim} \mathbb{R}^4 \mathbb{R}^n = (\mathbb{R}_+ \times \text{SO}(4n)) \times \mathbb{R}^n$, we have $K \subset (\mathbb{R}_+ \times \text{SO}(L) \times \text{SO}(L^\perp)) \times L$. But $\mathcal{K} \subset \text{Sim} \mathbb{H}^n$, hence $\text{pr}_{\text{SO}(4n)} K \subset \text{Sp}(1) \cdot \text{Sp}(n)$. Consequently, $\text{pr}_{\text{SO}(4n)} K = \text{pr}_{\text{Sp}(1) \cdot \text{Sp}(n)} K \subset \text{Sp}(1) \cdot \text{Sp}(n) \cap \text{SO}(L) \times \text{SO}(L^\perp)$. For the corresponding subalgebra $\mathfrak{f} \subset \mathcal{L}(\text{Sim} \mathbb{H}^n)$, we have $\text{pr}_{\text{Sp}(1) \cdot \text{Sp}(n)} \mathfrak{f} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^\perp)$. Considering the matrices of the elements of these algebras in the basis of $\mathbb{R}^{4n}$, we obtain

$$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^\perp) = \begin{cases} \mathfrak{sp}(1) \oplus \mathfrak{sp}(n), & \text{if } m = n; \\ \mathfrak{sp}(m) \oplus \mathfrak{u}(n-m) \oplus i\mathbb{R}, & \text{if } 0 \leq m < n, \\ \mathfrak{sp}(m) \oplus \mathfrak{u}(k) \oplus \mathfrak{so}(n-m-k), & \text{if } 0 \leq m < n, \\ \mathfrak{sp}(m) \oplus \mathfrak{u}(k) \oplus \mathfrak{so}(n-m-k), & \text{if } n - m - k = 0; \\ \mathfrak{sp}(m) \oplus \mathfrak{u}(k) \oplus \mathfrak{so}(n-m-k), & \text{if } n - m - k \geq 1. \end{cases}$$

The action of the Lie algebras $\mathfrak{u}(n-m)$ and $\mathfrak{so}(n-m-k)$ on $\mathbb{C}^{n-m}$ and $\mathbb{R}^{n-m-k}$, respectively, is described in Section 8.

Let $E$ be a Euclidean space. In \cite{9} subalgebras $\mathfrak{f} \subset \mathcal{L}(\text{Sim} E)$ corresponding to connected transitively acting subgroups of $\text{Sim} E$ were divided into the following three types:
Suppose that \( m = n \), i.e. \( L = \mathbb{H}^n \). If \( \mathfrak{t} \) is of Type \( \mathbb{R} \), then \( \mathfrak{t} = (\mathbb{R} \oplus \mathfrak{h}) \times L \), where \( \mathfrak{h} \subset \mathfrak{so}(E) \) is a subalgebra.

**Type \( \varphi \).** \( \mathfrak{t} = \{ \varphi(A) + A|A \in \mathfrak{h} \} \times E \), where \( \mathfrak{h} \subset \mathfrak{so}(E) \) is a subalgebra, \( \varphi \in \text{Hom}(\mathfrak{h}, \mathbb{R}) \), 
\[ \varphi|_{\mathfrak{h}'} = 0. \]

**Type \( \psi \).** \( \mathfrak{t} = \{ A + \psi(A)|A \in \mathfrak{h} \} \times U \), where we have an orthogonal decomposition \( E = W \oplus U \), 
\( \mathfrak{h} \subset \mathfrak{so}(W) \) is a subalgebra, \( \psi : \mathfrak{h} \rightarrow W \) is surjective linear map, \( \psi|_{\mathfrak{h}'} = 0. \)

Remark 1. It is also possible to classify weakly irreducible subalgebras of \( \mathfrak{sp}(1, n + 1) \mathbb{H}_p \) containing the ideal \( \mathcal{B} \) up to conjugacy by elements of \( \text{Sp}(1, n + 1) \). For this we should consider in addition the real vector subspace \( L \subset \mathbb{H}^n \) of the form \( \mathfrak{h} \) such that at least two of the inequalities \( m < m_1 < m_2 < m_3 \) hold. Note that
\[
\mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^+) = \mathfrak{sp}(\mathfrak{span}_\mathbb{H} \{ e_1, \ldots, e_m \}) \oplus \mathfrak{u}(\mathfrak{span}_{\mathbb{R} \oplus \mathbb{H}} \{ e_{m+1}, \ldots, e_m \})
\]
\[
\oplus \mathfrak{u}(\mathfrak{span}_{\mathbb{R} \oplus \mathbb{H}} \{ e_{m+1}, \ldots, e_{m_2} \}) \oplus \mathfrak{u}(\mathfrak{span}_{\mathbb{R} \oplus \mathbb{H}} \{ e_{m_2+1}, \ldots, e_{m_1} \}) \oplus \mathfrak{so}(\mathfrak{span}_{\mathbb{R}} \{ e_{m_3+1}, \ldots, e_n \}).
\]

We should generalize Type IX assuming that \( L \) has the form \( \mathfrak{h} \) and we should in addition add two types of Lie algebras:

**Type X.** \( \mathfrak{g} = \{ (a_1, A, X, b) | a_1 \in \mathbb{R}, A \in \mathfrak{h}, X \in L, b \in \text{Im} \mathbb{H} \} \), where \( \mathfrak{h} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^+) \) is a subalgebra.

**Type XI.** \( \mathfrak{g} = \{ (\varphi(A), A, X, b) | A \in \mathfrak{h}, X \in L, b \in \text{Im} \mathbb{H} \} \), where \( \mathfrak{h} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^+) \) is a subalgebra, \( \varphi \in \text{Hom}(\mathfrak{h}, \mathbb{R}) \), \( \varphi|_{\mathfrak{h}'} = 0. \)
References

[1] D. V. Alekseevsky, *Homogeneous Riemannian manifolds of negative curvature*. Mat. Sb. (N.S.) 96(138) (1975), 93–117.

[2] W. Ambrose, I. M. Singer *A theorem on holonomy*. Trans. Amer. Math. Soc. 75 (1953), 428–443.

[3] L. Berard Bergery, A. Ikemakhen, *On the Holonomy of Lorentzian Manifolds*. Proceeding of symposia in pure math., volume 54 (1993), 27–40.

[4] M. Berger, *Sur les groupers d’holonomie des variétés à connexion affine et des variétés riemanniennes*. Bull. Soc. Math. France 83 (1955), 279–330.

[5] A. L. Besse, *Einstein manifolds*. Springer-Verlag, Berlin-Heidelberg-New York, 1987.

[6] R. Bryant, *Metrics with exceptional holonomy*. Ann. of Math. (2) 126 (1987), 525–576.

[7] A. S. Galaev, *Isometry groups of Lobachevskian spaces, similarity transformation groups of Euclidian spaces and Lorentzian holonomy groups*. Rend. Circ. Mat. Palermo (2) Suppl. No. 79 (2006), 87–97.

[8] A. S. Galaev, *Metrics that realize all Lorentzian holonomy algebras*. International Journal of Geometric Methods in Modern Physics, Vol. 3, Nos. 5&6 (2006), 1025–1045.

[9] A. S. Galaev, *Classification of connected holonomy groups for pseudo-Kählerian manifolds of index 2*. [arXiv:math.DG/0405098](http://arxiv.org/abs/math.DG/0405098).

[10] D. Joyce, *Compact manifolds with special holonomy*. Oxford University Press, 2000.

[11] T. Leistner, *On the classification of Lorentzian holonomy groups*. J. Differ. Geom. 76, No. 3 (2007), 423–484.

[12] H. Wu, *Holonomy groups of indefinite metrics*. Pacific journal of math., 20 (1967), 351–382.

Department of Algebra and Geometry, Masaryk University in Brno, Janáčkovo nám. 2a, 66295 Brno, Czech Republic

E-mail address: bezvitnaya@math.muni.cz