The influence of cosmological transitions on the evolution of density perturbations

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Abstract

We study the influence of the reheating and equality transitions on superhorizon density perturbations and gravitational waves. Recent criticisms of the ‘standard result’ for large-scale perturbations in inflationary cosmology are rectified. The claim that the ‘conservation law’ for the amplitude of superhorizon modes was empty is shown to be wrong. For sharp transitions, i.e. the pressure jumps, we rederive the Deruelle-Mukhanov junction conditions. For a smooth transition we correct a result obtained by Grishchuk recently. We show that the junction conditions are not crucial, because the pressure is continuous during the reheating transition. The problem occurred, because the perturbed metric was not evolved correctly through the smooth reheating transition. Finally, we derive the ‘standard result’ within Grishchuk’s smooth (reheating) transition.

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I. INTRODUCTION

The discovery of anisotropies in the Cosmic Microwave Background Radiation (CMBR) \cite{1} was a major advance for Cosmology. Presently, a series of experiments is being conducted to measure these anisotropies on smaller angular scales. It is likely that the CMBR anisotropies originate from primordial fluctuations of the metric and matter fields which have been amplified later on. Therefore, the outcomes of the ongoing and planned observations may have important consequences on the theoretical models explaining the origin and the evolution of cosmological perturbations, and more generally, on our ideas about the early Universe.

Among these ideas is the theory of perturbations generated quantum-mechanically during an inflationary stage. In the present work our attention shall be restricted to inflationary models with one scalar field. In the subsequent evolution of the Universe matter shall be described by a perfect fluid. With the help of various models definite predictions for the amplitudes and spectra of the different types of perturbations (density perturbations, rotational perturbations and gravitational waves) can be made. In particular, allowing perfect fluids only, one can show that rotational perturbations have to decay. Moreover, single scalar field inflation cannot seed rotational perturbations.

The power spectrum of density perturbations in an inflationary Universe was first calculated by Mukhanov and Chibisov \cite{2}. This first computation was confirmed in Refs. \cite{3–6}. Essentially, the ‘standard result’ lies in the following: the closer the spectrum is to the flat (Harrison-Zeldovich) spectrum, the larger the amplitude of density perturbations is in comparison with the amplitude of gravitational waves \cite{7,8}. On the other hand, if the spectrum is tilted away from scale-invariance, the contribution due to gravitational waves can become important. (For some specific examples of this behavior see, e.g. \cite{9}.)

This result was recently challenged by Grishchuk in Ref. \cite{10} who found a similar amplitude for density perturbations and gravitational waves. However, his calculations were criticized by Deruelle and Mukhanov \cite{11}. They argued that Grishchuk did not properly consider the joining conditions at the transitions ‘inflation-radiation’ and ‘radiation-matter’.

Then, Grishchuk published a comment \cite{12} in which he stated that the equation expressing the ‘standard result’ is mathematically wrong. Short after, this claim was contested by Caldwell \cite{13} who expressed his agreement with Deruelle and Mukhanov. Finally, in the appendix of Ref. \cite{14}, Grishchuk re-stated his criticisms of the ‘standard result’ in more details.

The aim of this paper is to clear up this controversy and to study the influence of cosmological transitions undergone by the Universe during its history on superhorizon cosmological perturbations.

This article is organized as follows: the second section is devoted to density perturbations and gravitational waves in the gauge-invariant formalism. This section can be skipped by specialists. The third section deals with the synchronous-gauge formalism. It was claimed by Grishchuk that the synchronous gauge and gauge-invariant results differ by an arbitrary constant of integration. Thus, a systematic comparison of the two formalisms is made and we show that they are equivalent. In the fourth section, we study the so-called ‘conservation law’ for superhorizon modes. We show that the ‘conserved quantity’ $\zeta$ often used in the literature is actually not empty as it was claimed in Refs. \cite{12}. In the fifth section, we
turn to the question of the matching conditions. In particular, we re-derive the Durnelle-Mukhanov junction conditions with a different method. The sixth section is devoted to the study of the smooth transition of Ref. [10]. We will argue that the joining conditions are not the essential point, but that the evolution of the metric perturbations through the smooth reheating transition was not done correctly. We correct the result, which is now the same as for a sharp transition. In the last section, we briefly present our conclusions, which confirm the ‘standard result’. Finally, an appendix reviews how the initial conditions are fixed when the perturbations are quantum-mechanically generated. For the readers convenience we summarize the notation of the gauge-invariant formulation and the synchronous-gauge formulation in two tables at the end of the paper.

II. GAUGE-IN Variant SCALAR AND TENSOR PERTURBATIONS

We will use the notation of [13] except the signature of the metric and the harmonic decomposition of the gauge-invariant potentials. The line element for the Friedmann-Lemaitre-Robertson-Walker (FLRW) background plus scalar perturbations reads ($c = 1$)

$$ds^2 = a(\eta)^2\{-(1 + 2\phi)d\eta^2 + 2B_{ij}dx^i dx^j + [(1 - 2\psi)\gamma_{ij} + 2E_{ij}] dx^i dx^j\}.$$ (2.1)

The conformal time $\eta$ is related to the cosmic time $t$ by $dt = a(\eta)d\eta$. A dot denotes a derivative with respect to $t$, whereas a prime stands for a derivative with respect to $\eta$. The tensor $\gamma_{ij}$ is the metric of the three-dimensional space-like hypersurfaces and the derivative $|_{i}$ in (2.1) is covariant with respect to $\gamma_{ij}$. The curvature of these sections is given by $K$ and takes the values $0, \pm 1$.

One can generate fictitious perturbations by performing an infinitesimal change of co-ordinates which preserves the scalar form of equation (2.1) (let us note that this is not, as written in [13] on page 2439, a “general” infinitesimal transformation of coordinates):

$$\bar{\eta} = \eta + \xi^0(\eta, x^k), \quad \bar{x}^i = x^i + \gamma^{ij}\xi^j(\eta, x^k).$$ (2.2)

Two independent gauge-invariant scalar metric potentials may be constructed from the metric. Following Ref. [13] we take them to be:

$$\Phi_Q \equiv \phi + \frac{1}{a}[\dot{B} - E']', \quad \Psi_Q \equiv \psi - \frac{a'}{a}(B - E').$$ (2.3)

In these expressions, we have chosen to extract the scalar harmonic $Q(x^i)$ of the gauge-invariant variables from the very beginning. The function $Q(x^i)$ satisfies the Helmholtz equation

$$\triangle Q = -k^2Q,$$ (2.4)

where $\triangle Q \equiv \gamma^{ij}Q_{ij}$ and $k$ is the comoving wave number. For convenience the quantity $\mathcal{H} \equiv \dot{a}/a$ is defined, which is related to the Hubble parameter $H \equiv \dot{a}/a = \mathcal{H}/a$. The perturbed Einstein equations can be expressed in terms of $\Phi$ and $\Psi$ alone (see below). Working in the gauge-invariant formulation of Bardeen [16] is equivalent to the longitudinal
gauge \((B = E = 0)\), which fixes the constant time hypersurfaces to be the hypersurfaces with vanishing shear.

In most inflationary models matter is described by a scalar field \(\varphi = \varphi_0(\eta) + \delta \varphi(\eta, x^i)\). The background energy density and pressure are

\[
\rho_0 = \frac{(\varphi_0')^2}{2a^2} + V(\varphi_0), \quad p_0 = \frac{(\varphi_0')^2}{2a^2} - V(\varphi_0) ,
\]

where \(V(\varphi)\) is the potential of the scalar field. If \(\varphi_0' \neq 0\), the covariant conservation of the energy-momentum tensor provides the Klein-Gordon equation

\[
\varphi_{0''} + 2\mathcal{H}\varphi_0' + a^2V_{,\varphi}\varphi_0 = 0 .
\]

The gauge-invariant scalar field perturbation is \(\delta \varphi^{(\text{gi})}Q \equiv \delta \varphi + \varphi_0'(B - E')\). The linearized Einstein equations for a fixed mode \(k\) may be written in terms of gauge-invariant quantities only:

\[
-3\mathcal{H}(\mathcal{H}\Phi + \Psi') - k^2\Psi + 3K\Psi = \frac{\kappa}{2}[-(\varphi_0')^2\Phi + \varphi_0'(\delta \varphi^{(\text{gi})})' + a^2V_{,\varphi}\delta \varphi^{(\text{gi})}] ,
\]

\[
\mathcal{H}\Phi + \Psi' = \frac{\kappa}{2}\varphi_0'\delta \varphi^{(\text{gi})} ,
\]

\[
\varphi - \Psi = 0 ,
\]

\[
(2\mathcal{H}' + \mathcal{H}^2)\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi' - K\Psi - \frac{1}{3}k^2(\Phi - \Psi) = \frac{\kappa}{2}[-(\varphi_0')^2\Phi + \varphi_0'(\delta \varphi^{(\text{gi})})' - a^2V_{,\varphi}\delta \varphi^{(\text{gi})}] ,
\]

where \(\kappa \equiv 8\pi G\). Everything can be expressed in terms of a single gauge-invariant quantity since equation (2.9) tells us that \(\Phi = \Psi\).

If \(\varphi_0' = 0\) the background solution is the de Sitter spacetime \((\rho_0 = -p_0)\). In that case, the solution of the system (2.7) – (2.10) is \(\Phi = 0\); there are no scalar metric perturbations. This does not mean that the scalar field cannot fluctuate but that these fluctuations do not couple to gravity.

For \(\varphi_0' \neq 0\) the linearized Einstein equations (2.7) – (2.10) reduce with help of the Friedmann equation and the Klein-Gordon equation (2.6) to the equation \([15]\)

\[
\Phi'' + 2(\mathcal{H} - \frac{\varphi_0''}{\varphi_0'})\Phi' + [k^2 + 2(\mathcal{H}' - \mathcal{H}\frac{\varphi_0''}{\varphi_0'} - 2K)]\Phi = 0 .
\]

For \(K = 0\), the introduction of the new variables\([7]\)

\[
\sigma \equiv \frac{2a^2\theta}{3}\mathcal{H}\Phi , \quad \theta \equiv \frac{1}{a} \left( \frac{\rho_0}{\rho_0 + p_0} \right)^{\frac{1}{2}} = \left( \frac{3}{\kappa} \right)^{\frac{1}{2}} \frac{\mathcal{H}}{a\varphi_0} ,
\]

allows us to express equation (2.11) in the form:

\[\text{1}\]The perturbation \(\sigma\) is related to \(u\) of Ref. \([15]\) by \(\sigma = (\kappa/3)^{1/2}u\).
\[
\sigma'' + (k^2 - \frac{\theta''}{\theta})\sigma = 0.
\] (2.13)

For modes \(k^2 \ll \theta''/\theta\) the solution of Eq. (2.13) may be expanded in powers of \(k^2\). At the leading orders we obtain

\[
\sigma = \tilde{A}_1(k)\theta \int \frac{1}{\theta^2} \left(1 - k^2 \int \eta \theta^2 \int \tilde{\eta} d\tilde{\eta} + \mathcal{O}(k^4)\right) d\eta + 
\]
\[
\tilde{A}_2(k)\theta \left(1 - k^2 \int \frac{1}{\theta^2} \int \eta^2 d\eta + \mathcal{O}(k^4)\right).
\]

Since \(\theta \to \infty\) for \(a \to 0\) in general, \(\tilde{A}_1\) is the arbitrary constant in front of the regular (growing) mode and \(\tilde{A}_2\) a constant associated with the singular (decaying) mode.

For gravitational waves the line element reads:

\[
ds^2 = a(\eta)^2 \left\{-d\eta^2 + \left[\gamma_{ij} + h_{ij}\right]dx^i dx^j\right\},
\]
(2.15)

the tensor \(h_{ij}\) being symmetric, traceless and transverse. The tensor sector is gauge-invariant. We write \(h_{ij} = h_{gw}Q_{ij}\), where \(Q_{ij}\) is a symmetric, transverse, and traceless spherical harmonic, and \(h_{gw}\) is the amplitude of a gravitational wave. The decay of the amplitude due to the expansion of the Universe is taken into account by defining \(\mu_{gw} \equiv a(\eta)h_{gw}(\eta)\). The equation of motion for \(\mu_{gw}\) reads:

\[
\mu_{gw}'' + \left(k^2 - \frac{a''}{a}\right)\mu_{gw} = 0.
\] (2.16)

A physical interpretation of Eqs. (2.13) and (2.16) is parametric amplification of the perturbations while the Universe is expanding [17]. The scale factor plays the role of a ‘pump field’ and the ‘interaction’ between the background and the perturbations is described by the ‘potentials’ \(\theta''/\theta\) and \(a''/a\). The potential of the scalar perturbations involves not only the scale factor but also the derivatives of \(a(\eta)\) up to \(a^{(iv)}(\eta)\).

For simple models where the scale factor is given by \(a(\eta) = l_0|\eta|^{1+\beta}\), the exact solution to Eq. (2.16) can be found. It reads:

\[
\mu_{gw} = (k\eta)^{1/2}[A_{1}^{gw}(k)J_{\beta+1/2}(k\eta) + A_{2}^{gw}(k)J_{-(\beta+1/2)}(k\eta)],
\]
(2.17)

where \(J_{\pm(\beta+1/2)}\) are Bessel functions.

For an arbitrary scale factor, as for the scalar metric perturbations, the solution for modes \(k^2 \ll a''/a\) is given for gravitational waves by replacing \(\theta\) with \(a\) in Eq. (2.14):

\[
\mu_{gw} = \tilde{A}_1^{gw}(k)a + \tilde{A}_2^{gw}(k)a \int \eta d\tilde{\eta} + \mathcal{O}(k^2).
\]
(2.18)

Now \(\tilde{A}_1^{gw}\) corresponds to the regular and \(\tilde{A}_2^{gw}\) to the singular mode. It should be noticed that the \(A^{gw}\)'s differ from the \(\tilde{A}_i^{gw}\)'s, although they are connected. At superhorizon scales the dominant mode is constant in time, independent of the matter content of the Universe.

\[\text{An example where both modes are singular as } a \to 0 \text{ is provided by the model with scale factor behavior } a \propto |\eta|^{1+\beta}, \text{ for } -2 < \beta < -1 \text{ and } \eta < 0. \text{ However, this inflationary model violates the weak energy condition } \rho_0 + p_0 \geq 0 \text{ and cannot be realized with a single real scalar field.}\]
III. SCALAR PERTURBATIONS IN SYNCHRONOUS GAUGE

Let us describe the scalar perturbations using the class of the synchronous gauges. In order to make contact with previous works, we will use the notation of Ref. \[10\]. There the quantity $\mathcal{H}$ is denoted by $\alpha$, $\alpha \equiv \mathcal{H} \equiv a'/a$ and the function $\gamma(\eta)$ is defined by:

$$\gamma \equiv 1 - \frac{\alpha'}{\alpha^2}. \quad (3.1)$$

This function reduces to a constant for scale factors which are proportional to a power of the conformal time. The function $\gamma$ is zero if $\varphi_0' = 0$ (de Sitter). In Ref. \[10\], the line element is written as:

$$ds^2 = a^2(\eta)\{-d\eta^2 + [(1 + hQ)\gamma_{ij} + \frac{h_t}{k^2 - K}Q_{ij}]dx^idx^j\}. \quad (3.2)$$

Choosing the synchronous gauge (SG) means setting the perturbed lapse and shift functions to zero. The notation of \[10\] is related to the notation of \[15\] by

$$\phi = 0, \quad B = 0, \quad \psi = -\frac{h}{2}Q, \quad E = \frac{h_t}{2(k^2 - K)}Q. \quad (3.3)$$

Thus, the gauge-invariant variables expressed in the synchronous gauge are given by:

$$\Psi^{(SG)} = -\frac{1}{2}(h - \frac{\alpha}{k^2 - K}h'_t), \quad (3.4)$$

$$\Phi^{(SG)} = -\frac{1}{2(k^2 - K)}(h''_t + \alpha h'_t), \quad (3.5)$$

$$\delta\varphi^{(g)}(SG) = (\varphi_1 - \frac{1}{2(k^2 - K)}\varphi_0' h'_t), \quad (3.6)$$

where the scalar field is written as: $\varphi \equiv \varphi_0(\eta) + \varphi_1(\eta)Q$. Inserting these formulas into the system (2.7) – (2.10) provides the correct perturbed Einstein equations in the synchronous gauge (see Ref. \[10\]), namely:

$$h''_t + 2\alpha h'_t - (k^2 - K)h = 0, \quad (3.7)$$

$$-h' + \frac{K}{k^2 - K}h_t = \kappa\varphi_0'\varphi_1, \quad (3.8)$$

$$3\alpha h' - \alpha h_1' + (k^2 - 4K)h = \kappa(\varphi_0'\varphi_1' + a^2\varphi_1 V,_{\varphi}), \quad (3.9)$$

$$-h'' - 2\alpha h' + Kh = \kappa(\varphi_0'\varphi_1' - a^2\varphi_1 V,_{\varphi}). \quad (3.10)$$

This shows that, as expected, the two formalisms are completely equivalent. It is worth noticing that equation (3.7) expresses the fact that $\Psi = \Phi$. This is due to the vanishing longitudinal (anisotropic) pressure (see Ref. \[10\]).

Let us now consider the question of the residual gauge. It has been known for a long time that the condition $h_{0\mu} = 0$ does not fix the gauge completely. This condition is preserved under the change of coordinates

$$\bar{\eta} = \eta - \frac{C}{2a}Q, \quad \bar{x}^i = x^i - \frac{C}{2}Q^i \int \frac{d\eta}{a} - \frac{D}{2}Q^i, \quad (3.11)$$
where \( C \) and \( D \) are arbitrary constants for a fixed mode \( k \). The corresponding changes for \( h \) and \( h_l \) are given by:

\[
\bar{h} = h + \frac{\alpha}{a}, \quad \bar{h}_l = h_l + k^2 C \int \frac{d\eta}{a} + k^2 D .
\]  \hspace{1cm} (3.12)

According to Ref. [10], one can thus introduce two ‘residual-gauge-invariant’ quantities \( u \) and \( v \) defined by:

\[
u = h' + \alpha \gamma h, \quad v = h_l' - \frac{k^2}{\alpha} h .
\]  \hspace{1cm} (3.13)

The relations between the gauge-invariant quantities and the residual-gauge-invariant quantities can be expressed as:

\[
\Psi^{(SG)} = \frac{\alpha}{2k^2} v, \quad \Phi^{(SG)} = -\frac{u}{2\alpha} - \frac{1}{2k^2}(v' + \alpha v) .
\]  \hspace{1cm} (3.14, 3.15)

In Ref. [10] it has been claimed that \( u \) and \( v \) are genuine gauge-invariant quantities [i.e., invariant under the transformation (2.2)]. This claim is not correct. Indeed, a direct check shows that:

\[
\bar{u} = u + 2\alpha \zeta^0, \quad \bar{v} = v + 2k^2(\zeta' - \zeta^0). 
\]  \hspace{1cm} (3.16, 3.17)

The equations (3.14) – (3.15) just give the value \( \Psi \) and \( \Phi \) calculated in the synchronous gauge, \( \Psi^{(SG)} \), \( \Phi^{(SG)} \). It does not come as a surprise that what remains in a fixed gauge from the gauge invariance is simply the residual gauge invariance.

In Refs. [12] and [14], it has been argued that equation (2.11) is incorrect since it could be expressed as a combination of the derivatives of the correct equations and hence would contain a non-physical constant. This claim is incorrect as well. Let us demonstrate why. If we insert the expression of \( \Phi \) in terms of \( u \) and \( v \) [equation (3.15)] in equation (2.11), we find that this one transforms identically into:

\[
\frac{1}{2k^2\alpha}[k^2u + \alpha(v' + 2\alpha v)]'' + \frac{1}{k^2}(2\frac{\alpha'}{\alpha^2} + \frac{\gamma'}{2\alpha} - \frac{1}{2})[k^2u + \alpha(v' + 2\alpha v)]'
\]

\[
+ (\frac{\alpha'}{2k^2\alpha} - \frac{2\alpha''}{k^2\alpha^2} - \frac{\alpha'\gamma'}{2k^2\alpha^2} + \frac{\gamma''}{2k^2\alpha^2} - \frac{1}{2\alpha})[k^2u + \alpha(v' + 2\alpha v)]
\]

\[-\frac{1}{2}[u' - \alpha v - (\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma})u] = 0 .
\]  \hspace{1cm} (3.18)

Of course we must also take into account the equation \( \Psi = \Phi \) which is, in terms of \( u \) and \( v \), equal to: \( -k^2u = \alpha(v' + 2\alpha v) \). Therefore, we see that from the gauge-invariant formalism, we can reduce the whole problem to a set of two coupled first order differential equations (of course, this could have been done directly in the synchronous gauge) for the variables \( u \) and \( v \):
\[ -k^2u = \alpha(v' + 2\alpha v) , \]  
\[ v = \gamma \left( \frac{u}{\alpha \gamma} \right)' . \]  
\[ (3.19) \]
\[ (3.20) \]

From this system, we can generate two decoupled second order differential equations:

\[ u'' + u'(2\alpha \gamma - \frac{\gamma'}{\gamma}) + u[k^2 - 2\alpha' - \alpha \frac{\gamma'}{\gamma} - (\frac{\gamma'}{\gamma})'] = 0 , \]  
\[ (3.21) \]
\[ v'' + v'(2\alpha - \frac{\gamma'}{\gamma}) + k^2 v + v(2\alpha' - 2\alpha \frac{\gamma'}{\gamma}) = 0 . \]  
\[ (3.22) \]

The last equation could have been guessed from the very beginning by inserting the expression \( \Phi(SG) = \Psi(SG) = (\alpha v)/(2k^2) \) in equation (2.11). The variable change (see Ref. [10])

\[ u \equiv \frac{\alpha \sqrt{\gamma}}{a} \mu \]  
\[ (3.23) \]

in formula (3.21) allows us to obtain the correct equation for \( \mu \):

\[ \mu'' + [k^2 - \frac{(a \sqrt{\gamma})''}{a \sqrt{\gamma}}] \mu = 0 . \]  
\[ (3.24) \]

Therefore, we have proved that the gauge-invariant framework leads to the same (correct) equation of motion for the variable \( \mu \) as the calculation in synchronous gauge. Let us note that the residual-gauge-invariant variable \( \mu \) is nothing but the value of the gauge-invariant variable

\[ v_M \equiv a(\delta \varphi^{(gi)} + \frac{\varphi'_0}{\mathcal{H}}\Phi) , \]  
\[ (3.25) \]

which was defined by Mukhanov in Refs. [18, 15], expressed in the synchronous gauge. Inserting the synchronous gauge values of \( \delta \varphi^{(gi)} \) and \( \Phi \) into (3.25) and using (3.19) yields:

\[ v_M^{(SG)} = -\frac{\mu}{\sqrt{2\kappa}} . \]  
\[ (3.26) \]

In order to be as complete as possible we examine what was the mistake of Refs. [12,14]. From the equation (3.14) and the definition of \( \mu \), we can re-write equation (3.20) as:

\[ \Phi^{(SG)} = \frac{\alpha \gamma}{2k^2} \left( \frac{\mu}{a \sqrt{\gamma}} \right)' . \]  
\[ (3.27) \]

If one inserts this expression in formula (2.11), we obtain a third order differential equation for \( \mu \):

\[ \{ \mu'' + [k^2 - \frac{(a \sqrt{\gamma})''}{a \sqrt{\gamma}}] \mu \}' - \frac{(a \sqrt{\gamma})'}{a \sqrt{\gamma}} \{ \mu'' + [k^2 - \frac{(a \sqrt{\gamma})''}{a \sqrt{\gamma}}] \mu \} = 0 , \]  
\[ (3.28) \]

which can be integrated and leads to:
\[ \mu'' + [k^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}}] \mu = X a\sqrt{\gamma}, \] 

(3.29)

where \( X \) is a constant of integration. Comparing equations (3.24) and (3.29) shows that \( X = 0 \).

Let us demonstrate why the introduction of \( \mu \) via (3.27) and equation (2.11) seems to give \( X \neq 0 \) at first sight. Consider a set of two first order differential equations [they play the role of equations (3.19) and (3.20), the latter is equivalent to (3.27)]: \( x' = -y \), \( y' = x \).

From them we can generate two second order differential equations [they play the role of equations (3.21) and (3.22), which is nothing but equation (2.11)]: \( y'' + y = 0 \) and \( x'' + x = 0 \). But we can also insert \( y' = x \) into the second of the two last equations [as we inserted (3.27) into (2.11)]. We will obtain a third order differential equation, \( y''' + y' = 0 \), which may be integrated to yield \( y'' + y = X \). There is no harm to do that as long as we do not forget to use equation \( y'' + y = 0 \) and therefore \( X \) must be equal to zero.

However, in Refs. [12,14] the derivation of equation (3.27) was not given, instead it was assumed that (3.27) may be considered to be the definition of \( \mu \). In that case, this 'new' \( \mu \) has nothing to do with the \( \mu \) defined previously and has not to satisfy Eq. (3.24). The equation of motion for this 'new' variable \( \mu \) is Eq. (3.28). This definition of \( \mu \) is unique up to a shift \( \mu \rightarrow \mu + Ya\sqrt{\gamma} \) only, where \( Y \) is an arbitrary constant for a fixed mode \( k \).

Demanding that \( \mu \) should fulfill (3.24) fixes \( Y = X/k^2 \). The choice \( Y = 0, X \neq 0 \) that implicitly was made in [12,14] is inconsistent.

In conclusion, let us emphasize again the main result of this section. The gauge-invariant formalism and the synchronous gauge formalism are completely equivalent for all values of \( k \) including the zero-mode. Using one or the other (or any further gauge) is only a question of taste or of prejudices.

IV. THE CONSTANCY OF \( \zeta \) FOR SUPERHORIZON MODES

In this section we turn to the study of the so-called 'conservation law'. Let us start by introducing matter that can be described by a hydrodynamical equation of state, in order to follow the evolution of the perturbations from reheating till today.

A. Perfect fluids and the 'standard result'

Assume that anisotropic stresses can be neglected, thus matter is described by a perfect fluid. The pressure of the perfect fluid is related to its energy density by the equation of state \( p = p(\rho, S) \). \( S \) is the entropy per baryon\(^3\), \( n \) is the density of baryons. For a reversible expansion of the background (there are no unbalanced creation/annihilation processes) the entropy per baryon is constant in time. Due to the isotropy of the background, the first law of thermodynamics reads \( d(\rho/n) = -pd(1/n) \), thus \( TdS = 0 \) from the second law for reversible

\(^3\)Of course, this makes sense when baryon number is conserved only.
processes. Therefore, the background equation of state has to be isentropic ($\nabla_i S = 0$), thus we may write $p = p(\rho) \equiv w(\rho)\rho$ for an adiabatically expanding background.

The pressure perturbation reads $\delta p = c_s^2 \delta \rho + \tau \delta S$, where $c_s^2 \equiv (\partial p/\partial \rho)_s$ is the isentropic sound speed and $\tau \equiv (\partial p/\partial S)_\rho$. For an adiabatically expanding background $p' = c_s^2 \rho'$. Below we use the relation $c_s^2 = p'/\rho'$, thus the following results hold true for negligible entropy production only.

The equations of motion for the gauge-invariant metric potentials for a perfect fluid are (see Ref. [15]) $\Phi = \Psi$ and

$$\Phi'' + 3(1 + c_s^2) \mathcal{H} \Phi' + [2 \mathcal{H}' + (1 + 3c_s^2)(\mathcal{H}^2 - K)] \Phi + c_s^2 k^2 \Phi = \frac{\kappa}{2} a^2 \tau \delta S , \quad (4.1)$$

except for the de Sitter universe, where $\Phi = 0$. Written in terms of the variable $\sigma$, which is defined by (2.12), the latter equation reads

$$\sigma'' + \left( c_s^2 k^2 - \frac{\theta''}{\theta} \right) \sigma = \frac{\kappa}{3 \mathcal{H}} a^4 \tau \delta S . \quad (4.2)$$

For isentropic perturbations the leading order solution is easily obtained to be $\sigma = \tilde{C}_1 \int d\eta/\theta^2 + \tilde{C}_2 \theta$, whereas the next to leading terms differ from the solution (2.14), because $c_s^2$ may be time dependent. [There should be a factor $c_s^2$ in the $\bar{\eta}$ integration in (2.14).]

If $w$ and the sound speed are constant an approximate solution of $\sigma$ is not of much use, because then the exact solution to all orders in $k$ can be given in terms of Bessel functions. Two examples of a perfect fluid are the radiation fluid ($w = c_s^2 = 1/3$) and dust ($w = c_s^2 = 0$). These examples have vanishing entropy perturbations ($\delta S = 0$). In general, $\delta S$ does not vanish for more than one fluid. The scalar field $\varphi$ is another form of matter that can be described by a perfect fluid, see (2.5). Formally, the sound speed is defined as above. With help of the Klein-Gordon equation (2.6) it reads

$$c_s^2(\varphi_0) = -\frac{1}{3} \left( 1 + \frac{2\varphi''_0}{\mathcal{H}\varphi'_0} \right) . \quad (4.3)$$

Now we define the ‘entropy perturbation’ through $\tau \delta S = \delta p - c_s^2 \delta \rho$. With the expressions for $\delta \rho$ and $\delta p$ and using Eqs. (2.7) and (2.8) we arrive at

$$\frac{\kappa}{2} a^2 \tau \delta S = \left( 1 - c_s^2(\varphi_0) \right) \left( 3K - k^2 \right) \Phi . \quad (4.4)$$

We obtain the scalar-field equation of motion for the metric potential, Eq. (2.11), by inserting (4.4) into (1.1) and replacing $c_s^2$ by (4.3). Thus, we may study scalar metric perturbations by means of Eq. (4.4) from their generation during the inflation epoch to its observation in the CMBR today. However, during reheating [19] the scalar field may oscillate, thus $\varphi'_0$ has zeros and Eq. (2.11) is singular at these points. This situation has been discussed in [20].

Let us assume in this work that this does not happen.

\[4\] With the definition $\theta \equiv 1/a[\rho_0/(\rho_0 + p_0)]^{1/2}(1 - 3K/\kappa \rho_0 a^2)^{1/2}$ Eq. (4.2) holds true for all values of $K$. 

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**Footnotes**

4With the definition $\theta \equiv 1/a[\rho_0/(\rho_0 + p_0)]^{1/2}(1 - 3K/\kappa \rho_0 a^2)^{1/2}$ Eq. (4.2) holds true for all values of $K$. 

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**Equations**

1. $\Phi'' + 3(1 + c_s^2) \mathcal{H} \Phi' + [2 \mathcal{H}' + (1 + 3c_s^2)(\mathcal{H}^2 - K)] \Phi + c_s^2 k^2 \Phi = \frac{\kappa}{2} a^2 \tau \delta S . \quad (4.1)$
2. $\Phi'' + \left( c_s^2 k^2 - \frac{\theta''}{\theta} \right) \sigma = \frac{\kappa}{3 \mathcal{H}} a^4 \tau \delta S . \quad (4.2)$
3. $c_s^2(\varphi_0) = -\frac{1}{3} \left( 1 + \frac{2\varphi''_0}{\mathcal{H}\varphi'_0} \right) . \quad (4.3)$
4. $\frac{\kappa}{2} a^2 \tau \delta S = \left( 1 - c_s^2(\varphi_0) \right) \left( 3K - k^2 \right) \Phi . \quad (4.4)$
We are now able to derive the ‘standard result’, i.e. the amplification of $\Phi$ during the reheating transition. From comparison of (2.14) with the solution of (4.2) it is clear that the leading superhorizon term of the solution does not depend on the sound speed $c_s$. After some time the decaying mode is unimportant and the leading superhorizon term, using (2.12), reads:

$$\Phi \approx \frac{3}{2} \tilde{A}_1(k) \frac{\mathcal{H}}{a^2} \int^a_0 a^2 (1 + w) d\tilde{\eta} = \tilde{A}_1(k) \frac{\mathcal{H}}{a^2} \int^a_0 a^2 \left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right) d\tilde{\eta}. \quad (4.5)$$

For a power law behavior of the scale factor, i.e. $a \propto |\eta|^{1+\beta}$, the equation of state is given by $w = (1 - \beta)/[3(1 + \beta)]$. Then, the ‘growing’ mode is constant in time:

$$\Phi \approx \frac{3}{2} \tilde{A}_1(k) \frac{1 + \beta}{2\beta + 3}(1 + w). \quad (4.6)$$

Let us assume that the evolution of the scale factor may be described by such a power law far away from the transitions inflation-radiation and radiation-matter. In between it changes smoothly. The ratio of the values of $\Phi$ during inflation and matter domination (again far away from the transitions) is then given by

$$\frac{\Phi_m}{\Phi_i} \approx \frac{2}{5} \left(\frac{2\beta_i + 3}{\beta_i + 1 + w_i}\right) \approx \frac{2}{5} \frac{1}{1 + w_i}, \quad (4.7)$$

where $\beta_m = 1, \beta_i \approx -2,$ and $w_i \approx -1$. Therefore, scalar perturbations are magnified by a big factor during the reheating transition. For de Sitter spacetime the amplification coefficient goes to infinity. This simply expresses the trivial fact that $\Phi$ goes from zero to a constant without prejudice to the numerical value of this constant. Note that we can not conclude from this argument that $\Phi_m$ is large because $\Phi_i \to 0$ as $w_i \to -1$! The real (absolute) value of $\Phi$ after the transition can be only known after having determined the initial conditions from the quantization of density fluctuations, that is to say after having fixed $\tilde{A}_1$.

**B. Definitions and use of $\zeta$**

As was first recognized by Bardeen, Steinhardt and Turner [6] and further elaborated in [21–24], the equation of motion (4.1) has a first integral for isentropic modes ($\delta S = 0$) that are much larger than the Hubble scale, i.e. $k^\text{phys} \equiv k/a \ll H$. Following Ref. [15] we define

$$\zeta \equiv \frac{2}{3} \frac{\mathcal{H}^{-1} \Phi' + \Phi}{1 + w} + \Phi, \quad (4.8)$$

which was introduced by Lyth [23] originally (a quantity differing by terms $O(k^2/\mathcal{H}^2)$ only was used by Brandenberger and Kahn [22]). $\zeta$ essentially is the perturbation of the intrinsic curvature in the comoving gauge [23]. Its first derivative reads

$$\frac{1}{\mathcal{H}} \zeta' = \frac{2}{3} \frac{1}{1 + w} \left[ \frac{K}{\mathcal{H}^2} \left(\frac{1 + 3w}{2\mathcal{H}} \Phi' + 3(c^2_s - w)\Phi\right) + \frac{\kappa a^2 \tau \delta S}{2 \mathcal{H}^2} - c^2_s \left(\frac{k}{\mathcal{H}}\right)^2 \Phi \right], \quad (4.9)$$

where we used the equation of motion (4.1) and the equations
\[ w' = 3H(1 + w)(w - c_s^2), \quad H' = -\frac{1 + 3w}{2}(H^2 + K). \quad (4.10) \]

Let us note that (4.9) is an equation for a fixed mode \( k \), i.e. the large scale limit \( k/H \to 0 \) in this equations means to make \( H \) large. Thus, \( \zeta \) is constant in time for superhorizon modes \( k/H \ll 1 \) if and only (i) \( K = 0 \), (ii) entropy perturbations are negligible, and (iii) the perturbation is given by the regular mode only. The conditions (i) and (ii) are obvious from (4.9). The last condition means that the last term on the r.h.s of (4.9) vanishes in the limit \( k/H \to 0 \). The decaying mode is \( \Phi \propto H^\sigma/(a^2\theta) \propto H/a^2 \) at leading order. Thus, the r.h.s is proportional to \( (k/H)^2H/a^2 \), which blows up in the limit \( a \to 0 \), except for very special cases\(^5\). Therefore, in general, one has to exclude the decaying mode in order to make use of the ‘conservation law’ \( \zeta' = 0 \). In fact \( \zeta \) is nothing more than a first integral of the equation of motion (4.1) when the conditions (i) – (iii) are fulfilled.

The above definition of \( \zeta \) differs from the original definition \[ \zeta_{BST} = \frac{1}{3} \frac{\delta \rho}{\rho_0 + p_0} - \psi. \] (4.11)

\( \zeta_{BST} \) is a hypersurface-independent quantity \cite{24}. Written in terms of the gauge-invariant metric potential it reads:

\[ \zeta_{BST} = -\frac{2}{3} \frac{H^2}{(1 + w)(H^2 + K)} \left[ H^{-1}\Phi' + (1 - \frac{K}{H^2} + \frac{1}{3} \left( \frac{k}{H} \right)^2 \Phi \right] - \Phi. \quad (4.12) \]

From its time derivative

\[ \frac{1}{H} \zeta'_{BST} = -\frac{2}{3} \frac{H^2}{(1 + w)(H^2 + K)} \left[ \frac{1}{3} \left( \frac{k}{H} \right)^2 (H^{-1}\Phi' + \Phi) + \frac{\kappa a^2 \tau \delta S}{H^2} \right] \quad (4.13) \]

the constancy of \( \zeta_{BST} \) follows if and only (i) there are no entropy perturbations, and (ii) there is the regular mode only.

Besides the advantage of \( \zeta_{BST} \) over \( \zeta \) to be conserved even if \( K \neq 0 \), \( \zeta_{BST} \) is a hypersurface-independent measure of the metric perturbations including regular and singular modes (singular in the limit \( k/H \to 0 \)), whereas \( \zeta \) does not measure singular modes, if one takes the leading order contribution into account only. This can be easily seen by rewriting \( \zeta \) in terms of \( \sigma \): Whatever the coefficient \( \tilde{C}_2 \) in front of the decaying mode is, it does not enter into \( \zeta \) in the leading order in \( k/H \), because

\[ \zeta = \theta^2 \left( \frac{\sigma}{\theta} \right) \left( \frac{\sigma}{\theta} \right)' \quad (4.14) \]

for \( K = 0 \). On the other hand

\(^5\) In the model given by the scale factor \( a \propto \eta^{1+\beta} \) with \(-1 < \beta < -1/2 \) and \( \eta > 0 \) the decaying mode does no harm, because \( (k/H)^2H/a^2 \propto \eta^{-2\beta-1} \). However, these models lead to equations of state where \( 3p_0 > \rho_0 \), which does not give rise to inflationary expansion.
\[ \zeta_{BST} = -\theta \left[ \sigma' - \left( \frac{\theta'}{\theta} - \frac{1}{3} \frac{k^2}{\mathcal{H}} \right) \sigma \right]. \]  

(4.15)

(for any value of \( K \)) does depend on both \( \tilde{C}_1 \) and \( \tilde{C}_2 \). Let us in the following discuss the properties of \( \zeta(k) \).

In Ref. [12] it has been claimed that “the conservation law \( \zeta(t_i) = \zeta(t_f) \) degenerates to an empty statement \( 0 = 0 \)”. In order to explain the line of reasoning of [12] and to show where the argument fails, let us restrict the discussion to matter in form of a scalar field. For an isentropic perfect fluid the line of reasoning is analogous. Since \( \zeta \) is a gauge-invariant variable its value is the same in all gauges. Let us compute it in the synchronous gauge. Inserting Eq. (3.27) in Eq. (4.8) we find that:

\[ \zeta = \frac{1}{2k^2a^2\gamma} [a^2\gamma (\frac{\mu}{a\sqrt{\gamma}})']'. \]  

(4.16)

In Ref. [12] Eq. (3.29), re-written as

\[ \frac{1}{a^2\gamma} [a^2\gamma (\frac{\mu}{a\sqrt{\gamma}})']' + k^2 \frac{\mu}{a\sqrt{\gamma}} = X, \]  

(4.17)

was inserted into (4.16) to obtain \( \zeta \sim X/(2k^2) \) in the limit \( k \to 0 \). According to Ref. [12], there is no mean to know that \( X = 0 \) in the gauge-invariant formalism. This would explain \( \zeta = \text{const.} \neq 0 \) in the considered limit. On the other hand, always according to Ref. [12], the synchronous gauge formalism could tell us that \( X = 0 \). Thus, we would be ‘betrayed’ by the gauge-invariant formalism in which \( X \) would appear. Only computations in the synchronous gauge formalism could reveal that \( \zeta = X/(2k^2) = 0 \). In Refs. [12, 14], the fact that \( \zeta = 0 \) at the ‘leading order’ \( k^{-2} \) was misused as a proof that it is not possible to use the quantity \( \zeta \) to learn something about the behavior of density perturbations.

We have shown in Sec. III that the synchronous gauge formalism and the gauge-invariant formalism are completely equivalent and that \( X = 0 \) in both approaches. Thus, there is no risk of confusion at all computing \( \zeta \) in the gauge-invariant formalism.

According to Ref. [13] (p. 6), the conclusion of Ref. [12] occurs because “the \( k \to 0 \) limit has not been taken consistently”. Let us show that this is not the case. Following Ref. [12] we assume \( X \neq 0 \) because we now define \( \mu \) by Eq. (3.27), but remember that this definition of \( \mu \) is not unique. In that case the equation of motion is given by Eq. (3.29). Let us expand the solution \( \mu(\eta) \) in powers of \( k^2 \):

\[ \frac{\mu}{a\sqrt{\gamma}} = \bar{A}_1(k) \left( 1 - k^2 \int \frac{1}{a^2\gamma} \int^\eta a^2\gamma d\tilde{\eta} d\eta + \mathcal{O}(k^4) \right) + \]

\[ + \bar{A}_2(k) \int \frac{1}{a^2\gamma} \left( 1 - k^2 \int^\eta a^2\gamma \int^\tilde{\eta} \frac{1}{a^2\gamma} d\tilde{\eta} d\eta + \mathcal{O}(k^4) \right) d\tilde{\eta} + \]  

(4.18)

\[ + X \int \frac{1}{a^2\gamma} \int^\eta a^2\gamma \left( 1 - k^2 \int^\eta \frac{1}{a^2\gamma} \int^\tilde{\eta} a^2\gamma d\tilde{\eta} d\eta + \mathcal{O}(k^4) \right) d\tilde{\eta} d\eta, \]

where \( \bar{A}_1(k), \bar{A}_2(k) \) are arbitrary integration constants fixed by the initial conditions. If we compare Eq. (4.16) with Eq. (4.17) we deduce that:
\[ \zeta = \frac{X}{2k^2} - \frac{\mu}{2a\sqrt{\gamma}} , \]  

(4.19)

and we find the beginning of the series giving \( \zeta \):

\[ \zeta = \frac{X}{2k^2} - \frac{1}{2} \left( \bar{A}_1(k) + \bar{A}_2(k) \int \frac{d\eta}{a^2\gamma} + X \int \frac{1}{a^2\gamma} \int_{\eta} d\eta d\eta + \mathcal{O}(k^2) \right) . \]  

(4.20)

This confirms that the limit had been taken properly in Ref. [12] and that the first term in the series contains \( X/k^2 \). This result does not imply \( \zeta \to \infty \) as \( k \to 0 \), because \( X = X(k) \) and we know nothing about the \( k \) dependence of \( X \). However, there is no argument telling us that \( \bar{A}_1(k) \) or \( \bar{A}_2(k) \) are subleading compared to \( X(k)/k^2 \).

Taking into account the correct value \( X = 0 \), Eq. (4.20) reads:

\[ \zeta = \frac{-\bar{A}_1}{2} - \frac{\bar{A}_2}{2} k^2 \int d\eta \theta^2 + \mathcal{O}(k^2) . \]  

(4.21)

The same expression could have been obtained directly within the gauge-invariant formalism. It is necessary to push the expansion in \( k^2 \) one step further because the leading order term of the decaying mode does not contribute in (4.14). Inserting Eq. (2.14) in Eq. (4.14), we obtain:

\[ \zeta = \bar{A}_1 - \bar{A}_2 k^2 \int d\eta \theta^2 + \mathcal{O}(k^2) . \]  

(4.22)

Comparison with (4.21) yields \( \bar{A}_1(k) = -2\bar{A}_1(k) \) and \( \bar{A}_2(k) = 2k^2\bar{A}_2(k) \). We would have obtained the same relations between the initial conditions by inserting (4.18) into (3.27) and comparing the result with (2.14). From (4.22) it is seen that \( \zeta \) does not vanish and is not constant in general. The second term of (4.22) is the decaying mode. We emphasize that it is crucial to neglect the decaying mode if one makes use of

\[ \zeta \simeq \bar{A}_1 . \]  

(4.23)

Let us show how to make use of the constancy of \( \zeta \). Assume conditions (i) – (iii) are fulfilled. An example is the ‘standard’ scenario: The initial perturbations are provided by quantum-fluctuations during inflation. At the first horizon crossing we denote them \( \Phi_k(t_i) \). Let \( k \) be a mode which re-enters the Hubble horizon after equality between matter and radiation. The theory of quantum fluctuations shows that \( \Phi_k(t_i) \) is due to isentropic (adiabatic) fluctuations. The decaying mode is negligible short after the first horizon crossing. Entropy perturbations due to the reheating transition and due to the transition from radiation to matter affect much smaller scales. We know that the growing mode in \( \Phi \) is a constant on superhorizon scales if the scale factor obeys a simple power law behavior. In the inflationary stage \( a \propto t^p \) where \( p \gg 1 \), in the matter stage \( a \propto t^{3/2} \). Thus, with \( w(t_m) \approx 0 \)

\[ \zeta \simeq \frac{\Phi_k(t_i)}{1 + w(t_i)} \Phi_k(t_i) \simeq \frac{5}{3} \Phi_k(t_m) . \]  

(4.24)

During inflation \( w \sim -1 \) and therefore the large amplification
follows, which is the same as (4.7).

To finish this section let us consider the concrete model studied in Ref. [14] (see the appendix of that paper). It consists in a transition from one power law scale factor, \( a(t) = a_1 t^{p_1} \) for \( t < t_1 \), to another power law scale factor, \( a = a_2 (t - t_*)^{p_2} \) for \( t > t_1 \). At the transition \( a \) and \( H \) are continuous, whereas \( \dot{H} \) jumps. Assuming the constancy of \( \zeta = \zeta_0 \) we may integrate the definition of \( \zeta \) (4.8) in terms of cosmic time to obtain the evolution of the Bardeen potential

\[
\Phi(t) = -\zeta_0 \frac{H}{a} \int_{t_i}^{t} \frac{a \dot{H}}{H^2} dt + \frac{H}{a} C.
\]

(4.26)

For \( t < t_1 \) we find:

\[
\Phi(t < t_1) = \frac{\zeta_0}{1 + p_1} [1 - \left( \frac{t_i}{t} \right)^{1+p_1}] + \frac{p_1}{a_1 t^{1+p_1}} C.
\]

(4.27)

Consistent use of the constancy of \( \zeta = \zeta_0 \) requires the vanishing of the decaying mode in Eq. (4.27) and therefore \( C = \zeta_0 a_1 t_1^{1+p_1} / (p_1 + p_2^2) \). We obtain:

\[
\Phi(t < t_1) = \frac{\zeta_0}{1 + p_1}.
\]

(4.28)

For the de Sitter case ('\( p_1 = \infty \)') we recover that \( \Phi = 0 \). Let us compare this result with the one of Ref. [14]. We do not agree that "The initial value of the potential is \( \Phi(t_i) = \zeta_0 + H(t_i)C/a(t_i), \ldots \)" and that "The constant \( C \) could be set to zero from the very beginning," as it is stated on page 31. This would lead to \( \Phi(t_i) = \zeta_0 \) and, would imply a non-vanishing Bardeen's potential for the de Sitter spacetime. Instead the initial value is \( \Phi(t_i) = \zeta_0 / (1 + p_1) \) which is clear from Eq. (4.28).

For \( t > t_1 \), we obtain the solution:

\[
\Phi(t > t_1) = \frac{\zeta_0}{1 + p_1} \left( \frac{p_2}{p_1} \right)^{1+p_2} \left( \frac{t_1}{t - t_*} \right)^{1+p_2} + \frac{\zeta_0}{1 + p_2} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{1+p_2} \left( \frac{t_1}{t - t_*} \right)^{1+p_2} \right].
\]

(4.29)

The Bardeen variable is continuous at \( t = t_1 \) (since we have integrated a Heaviside function). Long after the transition, the previous relation reduces to:

\[
\Phi(t \gg t_1) \simeq \frac{\zeta_0}{1 + p_2}.
\]

(4.30)

Therefore, we reach the conclusion that the amplification coefficient is given by:

\[
\frac{\Phi(t \gg t_1)}{\Phi(t < t_1)} \simeq \frac{1 + p_1}{1 + p_2}.
\]

(4.31)

Formula (4.31) should be compared with the third equation after Eq. (87) of Ref. [14]. In this expression the missing term \( 1 + p_1 \) in the numerator is due to the incorrect assumption \( \Phi(t_i) = \zeta_0 \).
V. A SHARP TRANSITION — JOINING CONDITIONS

There are two physical situations where a sharp transition in the equation of state is a good approximation. The first one is the study of the behavior of superhorizon modes \((k_{\text{phys}} \ll H)\). The duration \(T\) of processes like the change from the radiation dominated to the matter dominated universe, reheating at the end of inflation, or recombination typically take several expansion times, i.e. \(T \sim H^{-1}\). Superhorizon modes change on much larger time scales, i.e. \(1/k_{\text{phys}}\), thus they see a sharp transition. These sharp transitions may violate the second law of thermodynamics, e.g. at equality radiation entropy is destroyed instantaneously when we assume a sharp drop in the pressure. The second situation where sharp transitions are of interest in cosmology are phase transitions like the QCD transition or a GUT transition. In these transitions the pressure is continuous, but its derivatives may be discontinuous. An example is provided by a first order QCD transition where the sound velocity may jump \([25]\).

Recently, Deruelle and Mukhanov \([11]\) derived the joining conditions for scalar metric perturbations in a spatially flat FLRW model. For perfect fluids (no scalar fields) gauge-invariant joining conditions for cosmological perturbations have been derived before by Hwang and Vishniac \([26]\). The difficulty to state the correct joining conditions arises because the physical hypersurface of the transition is not necessarily that of constant coordinate time. In the equality and reheating transitions the physical hypersurface is the one of constant density contrast. The gauge-invariant variables describe zero shear hypersurfaces as constant time hypersurfaces. Below we derive the joining conditions for general, spatially non-flat metric perturbations. Although the method of Deruelle and Mukhanov might be simpler than ours, we think that it is worth to view the problem from a different perspective below.

Let us start with exposing the method in general. Assume the spatial transition hypersurface \(\Sigma\) is defined by its normal \(n^{\mu}\). In order to join two space-time manifolds along \(\Sigma\) without a surface layer two conditions have to be met \([27]\): The induced spatial metric \(h_{ij} \equiv g_{ij} + n_i n_j\) and the extrinsic curvature \(K_{ij}\) should be continuous on \(\Sigma\). The extrinsic curvature is defined as:

\[
K_{ij} = -\frac{1}{2} \mathcal{L}_n h_{ij},
\]

where \(\mathcal{L}_n\) denotes the Lie derivative with respect to the normal \(n^\mu\). In order to compute \(K_{ij}\) the system of coordinates (i.e. the gauge) and the vector \(n^\mu\) (i.e. the surface of transition) have to be specified. Different choices for \(n^\mu\) lead to inequivalent junction conditions. Our derivation of the joining conditions differs from the derivation in \([11]\), where the joining conditions are calculated in a coordinate system adapted to the surface of the transition. For a more general coordinate system the joining conditions have been obtained by a gauge transformation.

As a simple example we can apply the previous rules to the background model. The surface \(\Sigma\) is defined by \(d_0(\eta_0) = 0\) and the components \(n_\mu\) are given by: \(n_0 = -a, n_i = 0\). It is then straightforward to show that \(K^i_j(\eta) = -H(\eta) \delta^i_j\). The continuity of the induced spatial metric leads to \(\lim_{\epsilon \to 0} [a(\eta_\sigma + \epsilon) - a(\eta_\sigma - \epsilon)] \equiv [a]_{\pm} = 0\) whereas the continuity of the extrinsic curvature amounts to \([a']_{\pm} = 0\). From the Friedmann equations we see that the
energy density can not have a jump, whereas the pressure may jump. Let us turn now to the case of scalar perturbations.

A. Scalar perturbations

The perturbed transition hypersurface is defined by $q_0(\eta) + \delta q(\eta, x^i) = 0$. From the last expression we immediately get that the transition now occurs at time $\eta = \eta_0 + \delta \eta = \eta_0 - \delta q/q_0'$. In addition, the normal of $\Sigma$ now reads:

$$n_0 = -a(1 + \phi), \quad n_i = -a \frac{\partial_i \delta q}{q_0'},$$

(5.2)

where the scalar perturbations are parameterized by the line element (2.1). First we must write that the perturbed induced metric $h_{ij}$ is continuous on $\Sigma$. Expressing this condition for diagonal and off-diagonal terms leads to:

$$[\psi(\eta) + H\delta q/q_0']_\pm(\eta) = 0, \quad [E(\eta)]_\pm = 0.$$

(5.3)

(5.4)

Second, we must compute the perturbed extrinsic curvature for the vector whose components are given in Eq. (5.2). We obtain the following result:

$$\delta K^i_j(\eta) = \frac{1}{a}[(\psi' + H\phi)\delta^i_j + (B - E' + \delta q/q_0')i|j],$$

(5.5)

and therefore, on the surface of transition $\Sigma$, $\delta K^i_j$ takes the value:

$$\delta K^i_j(\eta) = \frac{1}{a}[(\psi' + H\phi + (H' - H^2)\frac{\delta q}{q_0'})\delta^i_j + (B - E' + \delta q/q_0')i|j].$$

(5.6)

Let us notice that the extrinsic curvature on the hypersurface $\Sigma$ is a gauge-invariant quantity. The two other (gauge-invariant) junction conditions are easily deduced from the previous expression and read:

$$[\psi' + H\phi + (H' - H^2)\frac{\delta q}{q_0'}]_\pm(\eta) = 0,$$

(5.7)

$$[B - E' + \delta q/q_0']_\pm(\eta) = 0.$$

(5.8)

These are the same conditions as obtained in Ref. [11]. We have shown that these are also valid for all ($K = 0, \pm 1$) FLRW models.

We can also establish what are the junction conditions if one chooses to match the perturbations on a surface of constant time as it was done in Ref. [10]. Since $\delta q$ no longer depends on spatial coordinates, $\partial_i \delta q = 0$, the normal to the surface of transition is now given by:

$$n_0 = -a(1 + \phi), \quad n_i = 0,$$

(5.9)
and the extrinsic curvature takes on the form:

\[ \delta K_i^j(\eta) = \frac{1}{a} \left[ (\psi' + \mathcal{H}\phi)\delta_i^j + (B - E')|_i|_j \right]. \tag{5.10} \]

Requiring the continuity of \(h_{ij}\) and \(K^i_j\) given by \(5.10\) at \(\eta_\Sigma = \eta_0\), a surface of constant time, and using Eq. \(3.3\) leads to the joining conditions in the synchronous gauge:

\[ [h]_\pm = [h']_\pm = [h^\prime]_\pm = 0. \tag{5.11} \]

These are exactly the conditions that have been used in Ref. [10]. The joining conditions \(5.3\), \(5.4\), \(5.7\), and \(5.8\) are not equivalent to the conditions \(5.11\). This confirms that for a sharp transition the choice of the surface of matching is crucial.

However, all joining hypersurfaces are equivalent if \([p]_\pm = 0\). This can be easily seen by considering a hypersurface given by \(5.2\) in the longitudinal gauge and then make gauge transformations to any other gauge. Thus, \(B^{(LG)} = E^{(LG)} = 0\) and \([\delta q^{(LG)}/q'_0]_\pm = 0\) from \(5.8\). From \([p]_\pm = 0\) and the Friedman equations \([\mathcal{H}']_\pm = 0\) follows. The joining conditions \(5.3\) and \(5.7\) imply

\[ [\psi^{(LG)}]_\pm = [\psi^{(LG)'} + \mathcal{H}\phi^{(LG)}]_\pm = 0. \tag{5.12} \]

For a perfect fluid the equation of motion \(\phi^{(LG)} = \psi^{(LG)}\) reduces the joining conditions to the continuity of the metric perturbations and its derivatives. Consider now all hypersurfaces that are related to the zero shear (longitudinal gauge) hypersurface by the gauge transformations \(2.2\) where \(\xi^0, \xi \in C^2\). Then from the gauge transformations \(2.2\) and the junction conditions it follows that all metric perturbations and its derivatives have to be continuous in any gauge that is smoothly connected to the longitudinal gauge. Therefore, all these hypersurfaces are equivalent to the constant time hypersurface \(5.3\) if the pressure does not jump. We conclude that Grishchuk’s joining conditions are correct, provided \([p]_\pm = 0\). We argue in the next section that this is the case in his reheating transition.

### B. Vector perturbations

Let us define the line element for vector perturbations to read (here again we use the notations of Ref. [15] except for the signature of the metric):

\[ ds^2 = a(\eta)^2 \left\{ -d\eta^2 - 2S_i dx^i d\eta + [\gamma_{ij} + F_{ij} + F_{ji}] dx^i dx^j \right\}, \tag{5.13} \]

with \(S_i\) and \(F_i\) being transverse vectors, i.e. \(S_i|_i = F_i|_i = 0\). Fictitious perturbations can be generated by performing the infinitesimal change of coordinates:

\[ \bar{\eta} = \eta, \quad \bar{x}^i = x^i + \zeta^i, \tag{5.14} \]

where \(\zeta^i|_i = 0\). Under this transformation \(S_i\) and \(F_i\) change according to the equations:

\[ \bar{S}_i = S_i - \zeta^i, \quad \bar{F}_i = F_i + \zeta_i. \tag{5.15} \]

Therefore, we introduce the gauge-invariant dragging potential \([16]\) defined by
\[ \Xi_i \equiv S_i + F'_i. \] (5.16)

Since there is no possible vector contribution to the normal of the spatial hypersurface \( \Sigma \), the components of the normal are simply those of the background, i.e. \( n_0 = -a, n_i = 0 \). From the continuity of the induced metric we get that \( [F'_i]_{\pm}(\eta) = 0 \) and from the expression of the extrinsic curvature

\[ \delta K^i_{\cdot j} = \frac{1}{2a}(\Xi^j_{\cdot i} + \Xi^i_{\cdot j}), \] (5.17)

we deduce that the second junction condition is: \( [\Xi_i]_{\pm}(\eta) = 0 \). These junction conditions are gauge-invariant. In the synchronous gauge, they simply reduce to the continuity of the metric and its derivative. In the gauge-invariant formulation we have to demand the continuity of \( \Xi_i \) only. The fact that there is only a single condition reflects the very different behavior of rotational perturbations compared to density perturbations or gravitational waves. For perfect fluids the equation of motion for \( \Xi_i \) reads

\[ \Xi'_i + 2H\Xi_i = 0, \] (5.18)

giving rise to a decaying solution \( \propto 1/a^2 \) fixed by one initial condition only.

C. Tensor perturbations

Finally, we treat the case of the gravitational waves. The perturbed tensor line element is given by (2.15). As for the vector case, the normal to the surface of transition is simply the one of the background. The continuity of the induced metric leads to \( [h_{ij}]_{\pm}(\eta) = 0 \) and a straightforward calculation of the extrinsic curvature,

\[ \delta K^i_{\cdot j} = \frac{1}{2a}(h^i_{\cdot j})', \] (5.19)

shows that the derivative of the metric must be continuous as well, namely \( [h'_{ij}]_{\pm}(\eta) = 0 \). There are no gauge dependences in this sector anyhow.

VI. A SMOOTH TRANSITION

In this section, we analyze the approach taken by Grishchuk in Ref. [10]. He assumed that during inflation, the scale factor is given by \( a \propto |\eta|^{1+\beta} \) with \( 1 + \beta < 0 \). During this stage, the function \( \gamma(\eta) \) is constant and equal to \( (2 + \beta)/(1 + \beta) \). Then, instead of matching directly inflation to the radiation stage characterized by \( a(\eta) \propto (\eta - \eta_c) \), \( \gamma = 2 \), he introduced a smooth transition in between. Physically, this smooth transition represents the reheating of the Universe. It begins at \( \eta = \eta_1 - \epsilon \) (the end of inflation) and ends at \( \eta = \eta_1 + \epsilon \) (the beginning of radiation). Note that the parameter \( \epsilon \) introduced above is different from the one used in Ref. [10]. Here, \( \epsilon \) is small compared to \( \eta_1 \) because we assume that reheating is fast. In the limit \( \epsilon \) goes to zero, we recover the sharp transition considered before for which \( \gamma(\eta) \) becomes an Heaviside function jumping from \( (2 + \beta)/(1 + \beta) \) to 2. In the case of a smooth transition, without taking into account all the details of the reheating process, we
do not know how the scale factor evolves between \( \eta_1 - \epsilon \) and \( \eta_1 + \epsilon \). It is clear that the function \( a(\eta) \) [and therefore \( \gamma(\eta) \)] is probably complicated for this stage of the evolution. The idea of Ref. [10] was to assume that the function \( \gamma(\eta) \) is given by:

\[
\gamma(\eta) = \frac{4 + 3\beta}{2(1 + \beta)} + \frac{\beta}{2(1 + \beta)} \tanh \left( \frac{\eta - \eta_1}{s} \right),
\]

where \( s \) is a parameter controlling the sharpness of the transition. This equation holds for inflation and reheating, i.e. for \( \eta \) between \(-\infty\) and \( \eta_1 + \epsilon \). Such a postulated behavior for \( \gamma(\eta) \) can be justified with the help of the following two arguments. First, we have \( \gamma \simeq \frac{2 + \beta}{1 + \beta} \) when \( \eta < \eta_1 - \epsilon \) (the parameter \( s \) must be chosen such that the \( \tanh \) reaches quickly the value \(-1\); it is sufficient to take \(|\epsilon/s| \gg 1\) and we recover the fact that \( \gamma(\eta) \) is constant during inflation. During reheating \( \gamma(\eta) \) smoothly passes from \( \frac{2 + \beta}{1 + \beta} \) to its exiting value \( \gamma(\eta_1 + \epsilon) \approx 2 \). Second, more physically, \( \gamma(\eta) \) is related to \( w \) by the equation \( p/\rho = w = -1 + (2/3)\gamma(\eta) \). The introduction of the expression (6.1) in the last equation reproduces the expected behavior of \( w \) in a reasonable model. Therefore, Eq. (6.1) gives a reasonable approximation of the real (exact) complicated function \( \gamma(\eta) \) even if details of the reheating process cannot be taken into account in such a simple approach.

From the previous discussion, it is clear that \( \gamma(\eta) \) is always a continuous function, even at \( \eta = \eta_1 + \epsilon \) where the explicit joining was performed in Ref. [10]. This means that \([p]_\pm \) vanishes, see Eq. (3.4). In Ref. [11], Deruelle and Mukhanov criticized the calculations done in Ref. [10] by means of the smooth transition described before, arguing that the junction conditions were not taken into account properly. We have shown in the previous section that if the pressure is continuous, the two sets of matching conditions, Eqs. (5.3), (5.4), (5.7), (5.8) and Eqs. (5.11) are equivalent. Therefore, the claim of Deruelle and Mukhanov is not appropriate. For a smooth transition, the matching conditions used by Grishchuk are perfectly justified since they coincide with the ones derived in Ref. [11]. The argument of Deruelle and Mukhanov would be correct if the transition were sharp and \( \gamma(\eta) \) discontinuous at \( \eta = \eta_1 + \epsilon \). Moreover, the exiting values of the functions \( \gamma(\eta) \) and \( \gamma'(\eta) \) at the joining reheating-radiation are:

\[
\gamma(\eta_1 + \epsilon) = 2, \quad \gamma'(\eta_1 + \epsilon) = 0,
\]

in contradiction with the claims of Ref. [11] but in accordance with what is written by Grishchuk.

The next step would be to solve Eq. (3.24) for \( \gamma(\eta) \) given by Eq. (6.1). It was shown in Ref. [10] that the integration of Eq. (6.1) can be performed and provides us with the function \( \alpha(\eta) \). However, obtaining the corresponding \( a(\eta) \) is not possible. This is not a problem since the potential \( (a/\sqrt{\gamma})^\alpha/(a/\sqrt{\gamma}) \) depends only on \( \gamma, \alpha \) and their derivatives. However, even the simple form (6.1) is too complicated to allow a direct integration of Eq. (3.24). Nevertheless, we can follow the evolution of \( \mu \) through inflation and reheating. For \( \eta < \eta_1 - \epsilon \), \( \gamma(\eta) \) is a constant and the equation (3.24) can be solved. The solution reads:

\[
(3.24)
\]

\[\text{In this paper } \eta_1 \text{ was used instead of } \eta_1 + \epsilon \text{ to denote the end of reheating. It is very important to distinguish these two events.}\]
This solution is the same as for gravitational waves. The initial conditions \(A_{1,2}\) are fixed by the quantum-mechanical generation of the density and metric fluctuations: see the Appendix. The \(A_{1,2}\) differ from the \(\dot{A}_{1,2}\) and \(A_{1,2}\) introduced previously. From Eq. (6.3), we can determine the value of \(\mu(\eta)\) just before reheating:

\[
\mu(\eta_1 - \epsilon) \simeq \frac{A_1}{2^{\beta+\frac{3}{2}} \Gamma(\beta + \frac{3}{2})} [k(\eta_1 - \epsilon)]^{\beta + 1} \simeq \frac{A_1}{2^{\beta+\frac{3}{2}} \Gamma(\beta + \frac{3}{2})} (k\eta_1)^{\beta + 1},
\]

because \(k\eta_1 \ll 1\) and \(\epsilon \ll \eta_1\). Between \(\eta_1 - \epsilon\) and \(\eta_1 + \epsilon\) the function \(\gamma(\eta)\) is no longer a constant and the solution (6.3) can no longer be used. In order to evolve \(\mu\) through the reheating transition we use the superhorizon solution \(\mu \sim a\sqrt{\gamma}\) to obtain

\[
\mu(\eta_1 + \epsilon) \simeq \frac{\mu(\eta_1 - \epsilon)}{a(\eta_1 - \epsilon) \sqrt{\gamma(\eta_1 - \epsilon)}} a(\eta_1 + \epsilon) \sqrt{\gamma(\eta_1 + \epsilon)} \simeq \mu(\eta_1 - \epsilon) \sqrt{\frac{2}{\gamma_i}},
\]

because \(a(\eta_1 + \epsilon) \simeq a(\eta_1 - \epsilon)\). \(\gamma_i\) is the value of \(\gamma(\eta)\) during inflation. This relation should be compared to Eq. (81) and to the relation \(\mu|_{\eta_0 = 0} = \mu|_{\eta_0 + \epsilon}\) below Eq. (48) of Ref. [10]. From Eq. (6.3), it is clear that the ratio \(\mu(\eta_1 + \epsilon)/\mu(\eta_1 - \epsilon)\) is not 1 but proportional to \(1/\sqrt{\gamma_i}\). This factor is huge when \(\gamma_i\) is close to 0 (de Sitter case). Therefore, the mistake in Ref. [10] was not due to the use of wrong junction conditions but to the fact that the function \(\mu(\eta)\) was not evolved correctly through the reheating transition: actually \(\gamma(\eta_1 - \epsilon) \neq \gamma(\eta_1 + \epsilon)\) implies \(\mu(\eta_1 - \epsilon) \neq \mu(\eta_1 + \epsilon)\).

The value of \(\Phi\) for superhorizon modes during inflation is obtained by inserting (6.3) into (3.27), using the Taylor expansion of the Bessel functions and substituting (6.4) in the corresponding expression. The result reads:

\[
\Phi_i \simeq -\frac{\beta + 1}{2\beta + 3} \frac{\sqrt{\gamma_i} \mu(\eta_1 - \epsilon)}{a(\eta_1)}. \tag{6.6}
\]

This equation will be used below.

Let us turn now to the second transition, i.e. the transition radiation-matter taking place at equality. The matter era is described by \(a(\eta) \propto (\eta - \eta_m)^2\) and \(\gamma = 3/2\). In principle, one should do the same interpolation at equality as was done at reheating. In Ref. [10], this was not done and the second transition was treated as a sharp transition. As was shown in [11], the joining conditions (5.11) are not correct in general (i.e., for any residual gauge fixing). However, if one specifies the synchronous gauge in the matter dominated epoch to be the comoving one, then the joining conditions (5.11) are fine at the equality transition [11]. That is because the density contrast vanishes at the leading order [i.e., it is proportional to \((k\eta)^2\)]. This was actually done in [10]. Neglecting the decaying mode leads to the solution

\[
h = C_1, \quad h_t = \frac{1}{10} C_1 k^2 (\eta - \eta_m)^2. \tag{6.7}
\]

The coefficient \(C_1\) is related to \(\mu(\eta_1 + \epsilon)\), the value of \(\mu\) at the end of reheating, by (see Eq. (84) of Ref. [10]):

\[
\mu = (k\eta)^{1/2} [A_1 J_{\beta + \frac{1}{2}}(k\eta) + A_2 J_{\beta - \frac{1}{2}}(k\eta)] . \tag{6.3}
\]
\[ C_1 \simeq \frac{1}{\sqrt{2} a(\eta_1)} \mu(\eta_1 + \epsilon). \]  

(6.8)

Therefore, everything is known at the matter stage. Let us calculate the Bardeen’s gauge-invariant potential. Inserting Eq. (6.7) in the formulae (3.13) – (3.15) gives:

\[ \Phi_m \simeq - \frac{3}{10} C_1 = - \frac{3}{10 \sqrt{2} a(\eta_1)} \mu(\eta_1 + \epsilon). \]  

(6.9)

We may now take into account Eqs. (6.6) and (6.9) to arrive at

\[ \frac{\Phi_m}{\Phi_i} \simeq \frac{2 \beta + 3}{1 + \beta} \frac{3 \mu(\eta_1 + \epsilon)}{5 \sqrt{2} \gamma_i \mu(\eta_1 - \epsilon)} \simeq \frac{2 \beta + 3}{5} \frac{1}{1 + w_i}. \]  

(6.10)

Thus, we obtained the ‘standard result’ (4.7) entirely within the synchronous gauge, without any reference to the constancy of \( \zeta \) or the joining conditions of Deruelle and Mukhanov. From the last expression, it is clear that it is crucial to evaluate correctly the ratio \( \mu(\eta_1 + \epsilon)/\mu(\eta_1 - \epsilon) \).

Finally, it is also interesting to calculate the ratio of \( \Phi \) and \( h_{gw} \) at superhorizon scales today. This quantity is of relevance for observations since it is related to the ratio of gravitational waves to density perturbations contributing to the CMBR quadrupole anisotropy. The leading term of the ‘growing’ mode of the gravitational waves is constant in time, thus its value today is equal to its value at time \( \eta = \eta_1 - \epsilon \). Therefore we obtain:

\[ h_{gw}(\text{today}) \simeq \frac{\mu_{gw}(\eta_1 - \epsilon)}{a(\eta_1 - \epsilon)} \approx A_{gw}^1 \frac{\mu(\eta_1 - \epsilon)}{a(\eta_1)}, \]  

(6.11)

where we used the fact that the equations of motion of \( \mu \), Eq. (3.24), and \( \mu_{gw} \), Eq. (2.16), are the same during inflation (as long as \( \gamma \) is almost constant) until the onset of reheating at \( \eta_1 - \epsilon \). From Eqs. (A24) and (A26) of the Appendix, it follows that the rms amplitude of gravitational waves, which takes into account both polarizations, reads:

\[ h_{rms} \simeq \frac{\sqrt{2}}{\pi} \left| A_{gw}^1 \right| \frac{\mu(\eta_1 - \epsilon)}{a(\eta_1)} \left| k^2 \right| \]  

(6.12)

On the other hand, Eqs. (6.5), (6.9) and (A13) imply that:

\[ \Phi_{rms} \simeq \frac{3}{20 \pi} \sqrt{\frac{2}{\gamma_i}} \left| a(\eta_1) / \mu(\eta_1 - \epsilon) \right| k^2. \]  

(6.13)

Therefore, the value of the ratio \( h_{rms}/\Phi_{rms} \) today is:

\[ \frac{h_{rms}}{\Phi_{rms}} \simeq \frac{20}{3} \sqrt{\frac{\gamma_i}{\gamma_i}} \left| A_{gw}^1 / A_1 \right| = \frac{20}{\sqrt{6}} \sqrt{1 + w_i}, \]  

(6.14)

where the equations (A25) of the Appendix have been used. This finally proofs that the closer the inflationary epoch is to the de Sitter space-time, the less important are large-scale gravitational waves in the CMBR today.
VII. CONCLUSION

In Ref. [10] Grishchuk claimed that the magnitude of superhorizon scalar metric perturbations is most likely smaller than the amplitude of superhorizon gravitational waves. We have shown in Sec. VI that this result is wrong, because the time evolution of the scalar metric perturbation through the (smooth) reheating transition was not calculated correctly. With the appropriate correction we recover the ‘standard result’ for the rms amplitudes at superhorizon scales

$$\frac{\Phi_{\text{rms}}}{\Phi_{\text{rms}}} \Big|_{\text{today}} = \frac{20}{\sqrt{6}} \sqrt{1 + w_i} \sim \frac{m_{\text{Pl}}V_{,\phi}}{V} \Big|_{\text{slow-roll}},$$

where the slow-roll approximation is valid if $\gamma_i \ll 1$. However, in the limit $\gamma_i \to 0$ linear perturbation theory breaks down since Eq. (6.13) blows up. Thus, for power-law inflation, the model which we considered in detail, one cannot make the slow-roll approximation arbitrarily precise by making $\gamma_i$ arbitrarily small.

Recently, Deruelle and Mukhanov [11] corrected the result of [10] within the framework of sharp transitions. We rederived their joining conditions in Sec. V and extended them to non-flat FLRW models ($K \neq 0$). Moreover, we derive the joining conditions for the vector and tensor perturbations. According to Deruelle and Mukhanov, Grishchuk made two mistakes: He took the wrong joining conditions and he used the wrong equation of state (expressed in terms of $\gamma$) at the reheating transition. However, Grishchuk introduced a tanh to interpolate the pressure between the inflation and radiation epochs. Therefore, both his joining conditions at the reheating transition and the equation of state after reheating have been used correctly.

A commonly used derivation of the ‘standard result’ has been criticized by Grishchuk in Refs. [12,14]. This derivation is based on the conservation of certain quantities for superhorizon modes. These ‘conservation laws’ are essential in Refs. [6,22–24]. We have investigated Grishchuk’s arguments in Sec. III and IV, where we have shown that his criticisms are not correct.

Note added: After our work was finished, a paper by M. Goetz (astro-ph/970427) appeared. In this paper he independently reaches one of the conclusions of our paper, namely that Grishchuk’s claim on the emptiness of the ‘conservation law’ is wrong.

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APPENDIX: QUANTIZATION

In this appendix, we briefly review how the perturbations are generated quantum-mechanically in the early Universe. This mechanism fixes the initial conditions, i.e. the
coefficients \( A_{1}^{gw} \), \( A_{2}^{gw} \) and \( A_{1}, A_{2} \) in Eqs. (2.17) and (6.3). It has been emphasized in the text how crucial the precise values of these coefficients are to obtain of the final (standard) result.

Let us first consider density perturbations. The normalization of the (perturbed) scalar field operator is fixed by the uncertainty principle of Quantum Mechanics. In a spatially flat FLRW model, this leads to the following expression:

\[
\delta \hat{\varphi}(\eta, \mathbf{x}) \equiv \frac{1}{(2\pi)^{3/2}} \int d\mathbf{k} \hat{\varphi}_{1}(\eta, \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} = \frac{\sqrt{\hbar}}{a(\eta)} \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}}{\sqrt{2k}} [c_{k}(\eta)e^{i\mathbf{k} \cdot \mathbf{x}} + c_{k}^{\dagger}(\eta)e^{-i\mathbf{k} \cdot \mathbf{x}}], \tag{A1}
\]

where \( c_{k} \) and \( c_{k}^{\dagger} \) are the annihilation and creation operators satisfying the usual commutation relation. This equation agrees with Eq. (95) of Ref. [10]. The initial conditions are determined by demanding that at some initial time \( \eta_{0} \) (e.g., at the beginning of inflation), the scalar field be placed in the vacuum state: \( c_{k}(\eta = \eta_{0})|0\rangle = 0 \). Then, solving the Heisenberg equation of motion shows that the operator \( \hat{\varphi}_{1}(\eta, \mathbf{k}) \) can be written as:

\[
\hat{\varphi}_{1}(\eta, \mathbf{k}) = \frac{\sqrt{\hbar}}{a(\eta)} [c_{k}(\eta_{0}) \frac{u_{k} + v_{k}^{*}}{\sqrt{2k}} + c_{-k}^{\dagger}(\eta_{0}) \frac{u_{k}^{*} + v_{k}}{\sqrt{2k}}], \tag{A2}
\]

where the function \((u_{k} + v_{k}^{*})(\eta)\) satisfies:

\[
(u_{k} + v_{k}^{*})'' + (k^{2} - \frac{a''}{a})(u_{k} + v_{k}^{*}) = 0. \tag{A3}
\]

The initial conditions translate into the statement that \( u_{k}(\eta_{0}) = 1 \) and \( v_{k}(\eta_{0}) = 0 \). In the high frequency regime, this implies the asymptotic behavior \( \lim_{k \to +\infty}(u_{k} + v_{k}^{*}) = e^{-ik(\eta - \eta_{0})} \).

The normalization of the perturbed scalar field fixes automatically the normalization of the scalar perturbations of the metric since they are linked through Einstein’s equations. In the high frequency limit, this link is expressed through the formula:

\[
\lim_{k \to +\infty} \mu(\eta, \mathbf{k}) = -\sqrt{2a} \hat{\varphi}_{1}(\eta, \mathbf{k}), \tag{A4}
\]

which follows most easily from Eqs. (3.26) and (3.27). This equation is the same as written at the bottom of p. 7168 of Ref. [10]. This allows to find immediately the asymptotic behavior of the operator \( \hat{\mu}(\eta, \mathbf{k}) \):

\[
\lim_{k \to +\infty} \hat{\mu}(\eta, \mathbf{k}) = -4\sqrt{\pi}l_{Pl}[c_{k}(\eta_{0}) \frac{e^{-ik(\eta - \eta_{0})}}{\sqrt{2k}} + c_{-k}^{\dagger}(\eta_{0}) \frac{e^{ik(\eta - \eta_{0})}}{\sqrt{2k}}], \tag{A5}
\]

where \( l_{Pl} = (G\hbar)^{1/2} \) is the Planck length. For simple models where the scale factor is \( a(\eta) = l_{0}|\eta|^{1+\beta} \), the exact solution for \( \hat{\mu}(\eta, \mathbf{k}) \) can be expressed as:

\[
\hat{\mu}(\eta, \mathbf{k}) = (k\eta)^{1/2} [\hat{A}_{1}(\mathbf{k})J_{\beta+1/2}(k\eta) + \hat{A}_{2}(\mathbf{k})J_{-(\beta+1/2)}(k\eta)], \tag{A6}
\]

where \( J_{\pm(\beta+1/2)} \) is a Bessel function of order \( \pm(\beta + 1/2) \). The two last equations imply that the (so far arbitrary) operators \( \hat{A}_{1}(\mathbf{k}) \) and \( \hat{A}_{2}(\mathbf{k}) \) are given by:
\begin{align}
\hat{A}_1(k) &= \frac{i\sqrt{8\pi}l_{Pl}}{\cos \beta \pi} \frac{1}{\sqrt{2k}} [e^{i(k\eta_0 + \frac{\beta \pi}{2})}c_k(\eta_0) - e^{-i(k\eta_0 + \frac{\beta \pi}{2})}c_{-k}(\eta_0)], \tag{A7}
\hat{A}_2(k) &= -\frac{\sqrt{8\pi}l_{Pl}}{\cos \beta \pi} \frac{1}{\sqrt{2k}} [e^{i(k\eta_0 - \frac{\beta \pi}{2})}c_k(\eta_0) + e^{-i(k\eta_0 - \frac{\beta \pi}{2})}c_{-k}(\eta_0)]. \tag{A8}
\end{align}

Essentially, these relations agree with Eqs. (102) of Ref. [10].

The power spectrum for the Bardeen potential can now be computed. Its definition is given in terms of the two-point correlation function for \( \Phi(\eta, x) \):

\[ \langle 0 | \Phi(\eta, x) \Phi(\eta, x + r) | 0 \rangle = \int_0^\infty \frac{dk}{k} \frac{dr}{kr} k^3 P_\Phi(k). \tag{A9} \]

Using Eq. (3.27) which expresses the link between \( \mu \) and \( \Phi \), we find the following expression valid for the simple models evoked previously and for long wavelengths:

\[ P_\Phi(k) = \frac{P_{\Phi}^2}{l_0^2} \frac{\gamma(1 + \beta)^2}{2^{2\beta+4} \cos^2(\beta \pi) \Gamma^2(\beta + 5/2)} k^{2\beta+1}. \tag{A10} \]

So far, this is the result during inflation on superhorizon scales. Although no explicit time dependence is visible in \((A10)\), the value of \( P_\Phi(k) \) changes during reheating and during the equality transition. In particular, the spectrum today is equal to: \( P_\Phi(\text{today}, k) = T(k)P_\Phi(\text{initial}, k) \). For super horizon modes, the transfer function is given by: \( \lim_{k \to 0} T(k) = [3(2\beta + 3)]^2/[5(1 + \beta)\gamma]^2 \). If space-time during inflation was close to a de Sitter phase, then the spectrum today is the well-known Harrison-Zeldovich (scale-invariant) spectrum (i.e. \( \beta < -2 \)).

This result permits us to carry the quantum-mechanical initial conditions to the classical level. Let us define the ‘classical spectrum’ for the classical quantity \( \Phi(\eta, x) \) by the following expression:

\[ \langle \Phi(\eta, x) \Phi(\eta, x + r) \rangle \equiv \frac{\int \text{d}x \Phi(\eta, x) \Phi(\eta, x + r)}{\int \text{d}x} = \int_0^\infty \frac{dk}{k} \frac{dr}{kr} k^3 P_{\Phi}^{cl}(k), \tag{A11} \]

where \( P_{\Phi}^{cl}(k) \) is given by:

\[ P_{\Phi}^{cl}(k) = \frac{1}{2\pi^2} |\Phi(\eta, k)|^2. \tag{A13} \]

In this expression, \( \Phi(\eta, k) \) can be calculated using Eqs. (3.27) and (6.3) where the unknown coefficients \( A_1, A_2 \) appear. Requiring that \( P_{\Phi}(k) = P_{\Phi}^{cl}(k) \) fixes \( |A_1| \). This last equation relies on the ergodic assumption, namely that ensemble averages are equal to spatial averages. In addition, if we have the following behavior for the classical \( \mu(\eta, k) \): \( \lim_{k \to \pm \infty} \mu = -4\sqrt{\pi}l_{Pl} e^{-ik(\eta - \eta_0)}/\sqrt{2k} \) as it is suggested by Eq. \((A3)\), then the classical initial conditions \( A_1, A_2 \) are completely determined. They read:

\[ A_1 = \frac{i\sqrt{8\pi}l_{Pl} e^{i(k\eta_0 + \frac{\beta \pi}{2})}}{\cos \beta \pi \sqrt{2k}}, \quad A_2 = iA_1 e^{-i\pi \beta}. \tag{A14} \]
Finally, we define the root mean square value $\Phi_{\text{rms}}$ by:

$$
\Phi_{\text{rms}} \equiv \sqrt{k^3 P_\phi(k)}.
$$

(A15)

This quantity is used at the end of Section VI.

Let us now consider gravitational waves. The problem is very similar to the previous one since the quantization of gravitational waves is equivalent to the quantization of two scalar fields (representing the two independent degrees of freedom of the wave). Assuming again that the initial state is the vacuum leads to the following equation for the gravitational wave operator:

$$
\hat{h}_{ij}(\eta, x) = \frac{1}{a(\eta)} \frac{1}{(2\pi)^{3/2}} \sum_s \int \frac{dk}{\sqrt{2k}} p^s_{ij}(k) \hat{\mu}^s_{gw}(\eta, k) e^{ikx} \tag{A16}
$$

$$
= 4\sqrt{\pi} \pi || \frac{1}{a(\eta)} \frac{1}{(2\pi)^{3/2}} \sum_s \int \frac{dk}{\sqrt{2k}} p^s_{ij}(k) \left[ (u_k^s + v_k^s) \epsilon_s^a(\eta_0) e^{ikx} + (u_k^s + v_k^s) c_k^s(\eta_0) e^{-ikx} \right]. \tag{A17}
$$

In these formulas, $p^s_{ij}(k)$ is the (transverse-traceless) polarization tensor and the summation over $s$ represents the summation over the two states of polarization of the wave. The polarization tensor is normalized as $p^s_{ij}(k) p^{s',ij}(k) = 2\delta^{ss'}$. The function $u_k^s + v_k^s$ satisfies Eq. (2.16) (and in fact does not depend on the state of polarization $s$). The initial conditions are: $u_k^s(\eta_0) = 1$ and $v_k^s(\eta_0) = 0$. This implies the following asymptotic behavior for the operator $\hat{\mu}^s_{gw}(\eta, k)$:

$$
\lim_{k \to \infty} \hat{\mu}^s_{gw}(\eta, k) = 4\sqrt{\pi} \pi || \frac{1}{a(\eta)} \frac{1}{(2\pi)^{3/2}} \sum_s \int \frac{dk}{\sqrt{2k}} \left[ c_k^s(\eta_0) e^{-ik(\eta-\eta_0)} + c^*_k(\eta_0) e^{ik(\eta-\eta_0)} \right]. \tag{A18}
$$

This equation is similar to Eq. (A5). Since the exact solution for $\hat{\mu}^s_{gw}(\eta, k)$ can be expressed as:

$$
\hat{\mu}^s_{gw}(\eta, k) = (k\eta)^{1/2} \left[ \hat{A}^s_1(k, s) J_{\beta+1/2}(k\eta) + \hat{A}^s_2(k, s) J_{-(\beta+1/2)}(k\eta) \right], \tag{A19}
$$

this fixes the operators $\hat{A}^s_1(k, s)$ and $\hat{A}^s_2(k, s)$:

$$
\hat{A}^s_1(k, s) = -\frac{i\sqrt{\pi} \pi ||}{\cos \beta \pi} \frac{1}{2k} \left[ e^{ik(k\eta_0 + \frac{\eta}{2})} c_k^s(\eta_0) - e^{-ik(k\eta_0 + \frac{\eta}{2})} c^*_k(\eta_0) \right], \tag{A20}
$$

$$
\hat{A}^s_2(k, s) = \frac{\sqrt{\pi} \pi ||}{\cos \beta \pi} \frac{1}{2k} \left[ e^{i(k\eta_0 - \frac{\eta}{2})} c_k^s(\eta_0) + e^{-i(k\eta_0 - \frac{\eta}{2})} c^*_k(\eta_0) \right]. \tag{A21}
$$

The result for gravitational waves is very similar to the result for density perturbations.

We can also compute the power spectrum for gravitational waves (it involves the calculation of $\langle 0 | \hat{h}_{ij}(\eta, x) \hat{h}_{ij}^*(\eta, x + r) | 0 \rangle$). For long wavelengths, we obtain:

$$
P_h(k) = \frac{l_P^2}{l_0^2} \frac{1}{2^{2\beta-2} \cos^2(\beta \pi) \Gamma^2(\beta + 3/2)} k^{2\beta+1}. \tag{A22}
$$

Interestingly enough, the $k$-dependence of the density perturbations and gravitational waves power spectrum is the same. This behavior is a special feature of power-law inflation. As
for the density perturbations case, the calculation of the classical spectrum leads to the
determination of the classical initial conditions. The classical spectrum is given by:

\[ \langle h_{ij}(\eta, x)h_{ij}(\eta, x + r) \rangle = \int_0^{\infty} \frac{dk \sin kr}{k} kr^3 P_{hl}^{cl}(k), \tag{A23} \]

where \( P_{hl}^{cl}(k) \) can be expressed as:

\[ P_{hl}^{cl}(k) = \frac{2}{\pi^2} \left| \frac{\mu_{gw}^{s}(\eta, k)}{a} \right|^2. \tag{A24} \]

Requiring \( P_h(k) = P_{hl}^{cl}(k) \) and demanding the same high frequency behavior for the quantum
and the classical solution leads to the determination of \( A_1^{gw} \), \( A_2^{gw} \) which appear as free
numbers in the expression of \( \mu_{gw}(\eta, k) \), Eq. (2.17). We obtain the following result:

\[ |A_1^{gw}| = |A_1|, \quad |A_2^{gw}| = |A_2| \tag{A25} \]

Thus we have derived the result used in Sec. VI: \( |A_1^{gw}/A_1| = 1 \).

Finally, the classical quantity \( h_{rms} \) is defined by:

\[ h_{rms} = \sqrt{k^3 P_h(k)}. \tag{A26} \]
TABLES

| Symbol | Definition | Reference |
|--------|------------|-----------|
| $\mathcal{H} \equiv a'/a$ | $\mathcal{H} = aH$, where $H$ is the expansion rate. | (2.1) |
| $\phi, B, \psi, E$ | Perturbed scalar metric coefficients, see (2.1). | |
| $\Phi, \Psi$ | Gauge-invariant scalar metric-perturbation, see (2.3). | |
| $\sigma$ | Proportional to $\Phi$, see (2.12). Denoted $u$ in [13]. | |
| $v_M$ | Gauge-invariant velocity potential, see (3.24). Denoted $v$ in [13]. | |
| $S_i, F_i$ | Perturbed vector metric coefficients, see (5.13). | |
| $\Xi_i$ | Gauge-invariant vector metric-perturbation, see (5.16). | |
| $h_{ij}$ | Perturbed tensor metric coefficient, see (2.15). | |

TABLE I. Notation of gauge-invariant formalism.

$\alpha \equiv a'/a$ \quad $\alpha = aH$, where $H$ is the expansion rate. $\alpha = aH$, where $H$ is the expansion rate.

$\gamma \equiv 1 - a'/a^2$ \quad $\gamma$ is related to $w = p_0/\rho_0$ by $\gamma = [3(1 + w)/2](1 - K/\alpha^2)$. $\gamma$ is related to $w = p_0/\rho_0$ by $\gamma = [3(1 + w)/2](1 - K/\alpha^2)$.

$h, h_l$ \quad Perturbed scalar metric coefficients, see (3.2). Perturbed scalar metric coefficients, see (3.2).

$u, v$ \quad Residual-gauge-invariant scalar metric-perturbation, see (3.13). Residual-gauge-invariant scalar metric-perturbation, see (3.13).

$\mu$ \quad Proportional to $u$, see (3.23). Proportional to $u$, see (3.23).

TABLE II. Notation of synchronous gauge.
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