STABLE ESTIMATION OF RIGID BODY MOTION

USING GEOMETRIC MECHANICS

BY

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DEDICATION

I dedicate this work to my mother Mahnaz, my father Col. E. Izadi, and my brothers Mohsen and Moein.
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ABSTRACT

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In this work, asymptotically stable state estimation schemes are proposed for rigid body motion, using the framework of geometric mechanics. Rigorous stability analyses of the estimation schemes presented here guarantee the nonlinear stability of these schemes. The stability of these schemes does not depend on the characteristics of the sensor measurement noise or external disturbances. In addition, they are robust to initial errors in the state estimates and do not need to be re-tuned when sensor noise properties change. In the first part of this dissertation, estimation of rigid body states is considered, given the dynamics
model of the rigid body. In the second part, an estimation scheme that does not require knowledge of the dynamics of the rigid body is derived, based on onboard sensor measurements obtained at an appropriate frequency. The frequency of such measurements must be suitably high to resolve the motion of the rigid body. These attitude and pose estimation schemes are obtained by applying the Lagrange-d’Alembert principle from variational mechanics, to a Lagrangian constructed from state estimation errors and a dissipative term linear in the velocity estimation errors.
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Angular and translational velocity estimation error for CASE 2.
1 INTRODUCTION

Estimation of rigid body motion is a long-standing problem of interest for a wide variety of mechanical systems. Specifically, these systems include aerial and under-water vehicles, spacecraft, or any other moving objects in three dimensions. Motion estimation for rigid bodies is challenging primarily because this motion is described by nonlinear dynamics and the state space is nonlinear. This nonlinearity arises from the intrinsic nature of rigid body attitude, which is represented by the special orthogonal group, SO(3). Throughout this dissertation, rigid body attitude is represented globally over the configuration space of rigid body attitude motion without using local coordinates or quaternions. Attitude estimators using unit quaternions for attitude representation may be unstable in the sense of Lyapunov, unless they identify antipodal quaternions with a single attitude. This is also the case for attitude control schemes based on continuous feedback of unit quaternions, as shown in [8, 20, 71]. One adverse consequence of these unstable estimation and control schemes is that they end up taking longer to converge compared with stable schemes under similar initial conditions and initial transient behavior. On the contrary, all the estimation schemes proposed here are stable in the sense of Lyapunov.

In the first phase of this work, which includes Chapters 2 and 3, three
instances of such estimation schemes are proposed for rigid body motion using knowledge of dynamics. This requires the knowledge of the physical properties of the rigid body, as well as all external forces and moments applied on it. In these chapters, exponential coordinates are used to represent rigid body configuration. An observer design for arbitrary rigid-body motion in the proximity of a spherical asteroid of unknown mass is considered in Chapter 2. This observer exhibits almost global convergence of state estimates in the state space of rigid body rotations and translations. Continuous observers cannot be globally asymptotically stable in this state space, which is the tangent bundle of the Lie group SE(3), due to topological obstructions arising from the fact that this state space is not contractible [10]. Most unmanned and manned vehicles can be accurately modeled as rigid bodies, and therefore this observer can be applied to such vehicles operating on air, underwater, and in space. In particular, such vehicles when operated in uncertain or poorly known environments, can be subject to unknown forces and moments. Therefore, estimation of parameters associated with such unknown forces and moments is also of value. Dynamical coupling between the rotational and translational dynamics, which occurs both due to the natural dynamics as well as control forces and torques, is treated directly in the geometric mechanics framework used for our observer design.

Relevant prior research on observer designs for rigid body dynamics in SE(3) is briefly covered here. A nonlinear observer for integration of GNSS and IMU
measurements in the presence of gyro bias was investigated in [34] by using inertial reading of acceleration and velocities, magnetometer measurements and satellite-based measurements. Using landmark measurements and noisy velocity data, a nonlinear observer for pose estimation in SE(3) is presented in [87]. Ideal inertial velocity readings decouples the position and attitude motions, whereas they are coupled in the presence of gyro rate bias. The work in [67] proposes an observer in the special Euclidean group SE(3) and considers the conditions under which the estimated states converge to the real states exponentially fast. It is also shown that in the case there exist some measurement noise, the estimate converges to a neighborhood of the real state. A global exponential stable attitude observer is presented in [7]. Although this observer does not evolve on SO(3), it yields estimates that converges asymptotically to SO(3) and as a result, it does not have any topological limitations. A nonlinear observer using active vision and inertial measurements that estimates the attitude of a rigid body is verified experimentally in [16, 17]. An almost globally convergent orientation estimator is presented in [84] when just a single body-fixed vector on the rotating rigid body is available. In [96], with the knowledge of a camera dynamics and recalling a system of partial differential equations describing the invariant dynamics of brightness and depth smooth fields, an SO(3)-invariant variational method to directly estimate the depth field is investigated. There are some novel methods to derive the nonlinear state observers designed directly on the Lie group structure of the
Special Euclidean group SE(3) called gradient-based observer design. A type of nonlinear state observers designed directly on the Special Euclidean group SE(3) (a Lie group) are gradient-based observers on Lie groups. Using these methods and considering right invariant kinematics along with left invariant cost functions, [38, 49] utilize position measurements to update the state estimates. A limitation of this approach is provided in [49] as well as a practical design methodology in the case where a non-invariant cost-function is considered. Dynamic attitude and angular velocity estimation for uncontrolled rigid bodies in gravity, using global representation of the equations of motion based on geometric mechanics, is reported in [68, 75]. This estimation scheme is used in [76] for feedback attitude tracking control.

In addition to estimating the states, in Chapter 2 the main gravitational parameter of an asteroid is also estimated using full state measurements, including pose and velocities of a spacecraft in an orbit around the asteroid. This parameter is a very important physical property of an asteroid, and is a critical piece of information in order to estimate the mass of the asteroid and predict the forces and moments applied to a mass particle in its gravity field. Estimation schemes for parameter estimation of asteroid based on measurements from exploring spacecraft have been developed in prior literature on this topic. Physical properties of the asteroid 433 Eros, describing its shape, spin rate and gravity field, were estimated in [62] using the data provided by the NEAR spacecraft in an orbit around Eros.
Using LIDAR ranging instruments, mass and density of asteroid 25143 Itokawa have been estimated [85]. The gravitational acceleration of Itokawa turns out to be 18 times stronger than the acceleration as a result of solar radiation pressure at a distance of 1 km from the asteroid’s surface [85]. The strength of the gravity field of some small-bodies during a series of slow hyperbolic flybys around them were estimated in [1]. The work in [1] also analyzed how rapidly and precisely the gravitational parameter had been estimated for Itokawa, Eros and Didymos, and a new operational procedure called ΔV ranging was proposed.

The framework of geometric mechanics has not been used in the past for design of observers for the particular application of spacecraft exploring unknown or little known solar system bodies. This framework is beneficial for this application because the asteroid-spacecraft pair can execute large relative rotational motions. The use of homogeneous coordinates, which are not generalized coordinates and allow global representation of the configuration space SE(3), make it possible to represent the motion of bodies that are executing large, non-periodic motions [13, 58]. For the observer design here, the exponential coordinates in SE(3) are also used. Since the exponential coordinates are not defined for rigid body orientations that correspond to π radian rotations about a body-fixed axis, the convergence of the observer obtained is almost global over the state space. Prior work [75] has obtained attitude determination and filtering schemes from direction measurements with bounded attitude and angular velocity measurement
errors, given a dynamics model, in the framework of geometric mechanics. Nevertheless, in Chapter 2, no measurement model has been used and instead, the full state measurement has been exploited.

Finally, at the end of this chapter, another nonlinear observer that can accurately estimate the configuration and velocity states of a rigid body is presented. It is assumed that the rigid body has an onboard sensor suite providing measurements of configuration and velocities as well as forces and torques. Exponential convergence of the estimation errors is shown and boundedness of the estimation error under bounded unmodeled torques and forces is established. Since exponential coordinates can describe uniquely almost the entire group of rigid body motions, the resulting observer design is almost globally exponentially convergent.

In Chapter 3, a finite-time convergent observer design for arbitrary rigid-body motion is derived and presented, using rigid body’s pose and velocities measurements. This observer has an almost global domain of attraction of state estimates to actual states in the state space of rigid body rotations and translations. Since finite-time convergence is known to be more robust to noise in the dynamics model and noise in the measured states, this observer design has inherent stability properties to such noise. Besides, finite-time observers guarantee the time it takes for the system to converge to the actual states [11, 23, 36].

Unlike discontinuous sliding-mode observers that also provide finite-time convergence [21, 22, 91], the observer given here uses continuous feedback. Although
this observer is not Lipchitz continuous, it is Hölder continuous like the continuous attitude feedback stabilization scheme on TSO(3) presented in [69]. The observer presented is derived explicitly using the exponential coordinate representation of SE(3), which is almost global in its description of the motion, whereas [69] uses the coordinate-free representation of the attitude on the group of rigid body rotations in three-dimensional Euclidean space, SO(3). A few related works that exploit exponential coordinates to design observers or controllers are [15, 50]. The continuous observer proposed here is shown to provide finite-time convergence of state estimates, through a Lyapunov analysis using exponential coordinates. The proposed observer laws are shown to drive the estimation errors to the origin in a finite amount of time. Although the observer design is based on a given (known) dynamics model, robustness to noise in the dynamics and measurement process are shown through numerical simulations. These simulation results for the observer with noisy measurements and additive noise in the dynamics, show that the estimate errors remain bounded in the presence of noise.

In Chapter 3, we give the rigid body dynamics model in TSE(3), the tangent bundle of SE(3), along with the kinematics expressed in exponential coordinates on SE(3). We also present the nonlinear observer design, analyze its convergence properties, and show its finite-time convergence to actual states of the rigid body system. Numerical simulation results are presented for the noise-free case, when there is no measurement noise and no noise in the dynamics model. This chapter
also presents numerical simulation results when there is additive noise in angular and translational velocities measurements and disturbance inputs in the dynamics model.

Since the dynamics model of a mechanical system may not always be accurately known due to external disturbances, or as a result of motions of internal mechanisms, estimation schemes that do not require any knowledge of the dynamics model are of great importance. Such schemes, instead of a known dynamics model, rely on rich measurements provided by sensors (nowadays at low costs) onboard the rigid body. The second phase of this treatise (including Chapters 4-8) focuses on such dynamics model-free estimation schemes.

The earliest solution to the attitude determination problem from two vector measurements is the so-called “TRIAD algorithm”, which dates from the early 1960s [12]. This was followed by developments in the problem of attitude determination from a set of three or more vector measurements, which was set up as an optimization problem called Wahba’s problem [90]. This problem has been solved by different methods in prior literature, a sample of which can be obtained in [24, 55, 68].

Continuous-time attitude observers and filtering schemes on SO(3) and SE(3) have been reported in, e.g., [5, 6, 14, 46, 47, 53, 54, 57, 66, 75, 87, 88]. These estimators do not suffer from kinematic singularities [4, 83] like estimators using coordinate descriptions of attitude, and they do not suffer from unwinding as
they do not use unit quaternions. The maximum-likelihood (minimum energy) filtering method of Mortensen [65] was recently applied to attitude estimation, resulting in a nonlinear attitude estimation scheme that seeks to minimize the stored “energy” in measurement errors [2, 38, 93–95]. This scheme is obtained by applying Hamilton-Jacobi-Bellman (HJB) theory [48] to the state space of attitude motion [92]. Since the HJB equation can only be approximately solved with increasingly unwieldy expressions for higher order approximations, the resulting filter is only “near optimal” up to second order. Unlike filtering schemes that are based on approximate or “near optimal” solutions of the HJB equation and do not have provable stability, the estimation scheme obtained here can be solved exactly, and is shown to be almost globally asymptotically stable. Moreover, unlike filters based on Kalman filtering, the estimator proposed here does not presume any knowledge of the statistics of the initial state estimate or the sensor noise. Indeed, for vector measurements using optical sensors with limited field-of-view, the probability distribution of measurement noise needs to have compact support, unlike standard Gaussian noise processes that are commonly used to describe such noisy measurements.

All the estimation schemes proposed in Chapter 4 and onwards are model-free, which means that they do not depend on any knowledge of the dynamics of rigid body. In Chapter 4, the attitude determination problem from vector measurements is formulated on SO(3). Wahba’s cost function is generalized in two
ways: by choosing a symmetric matrix of weights instead of scalar weight factor for individual vector measurements, and by making the resulting cost function an argument of a continuously differentiable increasing scalar-valued function. It is shown that this generalization of Wahba’s function is a Morse function on SO(3) under certain easily satisfiable conditions on the weight matrix, which can be chosen appropriately to satisfy these desirable conditions. This chapter formulates the attitude estimation problem for continuous-time measurements of direction vectors and angular velocity on the state space of rigid body attitude motion, using the formulation of variational mechanics. A Lagrangian is constructed from the measurement residuals (between measured and estimated states) for the angular velocity measurements and attitude estimates obtained from the vector measurements. The Lagrange-d’Alembert principle applied to this Lagrangian, with a dissipative term linearly dependent on the angular velocity estimate error, leads to the state estimation scheme. This estimation scheme, when applied in the absence of measurement errors, is shown to provide almost global asymptotic stability of the actual attitude and angular velocity states, with a domain of attraction that is almost global over the state space. In fact, this domain of attraction is shown to be equivalent to that of the almost global asymptotic stabilization scheme for attitude dynamics in [20]. In the development of the attitude and angular velocity estimation schemes presented here, it is assumed that measurements of direction vectors and angular velocity are available in continuous time, or at a suitably high
sampling frequency. In such a measurement-rich estimation process, one need not use a dynamics model for propagation of state estimates between measurements.

In order to obtain attitude state estimation schemes from discrete-time vector and angular velocity measurements, we apply the discrete-time Lagrange-d’Alembert principle to an action functional of a Lagrangian constructed from the state estimate errors, with a dissipation term linear in the angular velocity estimate error. It is assumed that these measurements are obtained in discrete-time at a sufficiently high but constant sample rate. In this chapter, we consider the state estimation problem for attitude and angular velocity of a rigid body, assuming that known inertial directions and angular velocity of the body are measured with body-fixed sensors. The number of direction vectors measured by the body may vary over time. For most of the theoretical developments in this chapter, it is assumed that at least two directions are measured at any given instant; this assumption ensures that the attitude can be uniquely determined from the measured directions at each instant. The state estimation schemes presented here have the following important properties: (1) the attitude is represented globally over the configuration space of rigid body attitude motion without using local coordinates or quaternions; (2) the schemes developed do not assume any statistics (Gaussian or otherwise) on the measurement noise; (3) no knowledge of the attitude dynamics model is assumed; and (4) the continuous and discrete-time filtering schemes presented here are obtained by applying the Lagrange-d’Alembert
principle or its discretization [59] to a Lagrangian function that depends on the state estimate errors obtained from vector measurements for attitude and angular velocity measurements.

Three discrete-time versions of the filter introduced in [41] are obtained and compared in Chapter 4. The three discrete-time filters are as follows: (1) a first-order implicit Lie group variational integrator that was presented in [41]; (2) a first-order explicit integrator that is the adjoint of the implicit integrator; and (3) a second-order time-symmetric integrator obtained by composing the flows of the first order integrators. A variational integrator works by discretizing the (continuous-time) variational mechanics principle that leads to the equations of motion, rather than discretizing the equations of motion directly. A good background on variational integrators is given in the excellent treatise [59]. As described in the book [37], symplectic integrators (for conservative systems) are a subset within the class of variational integrators. Lie group variational integrators are variational integrators for mechanical systems whose configuration spaces are Lie groups, like rigid body systems. In addition to maintaining properties arising from the variational principles of mechanics, like energy and momenta, *Lie group variational integrator* (LGVI) schemes also maintain the geometry of the Lie group that is the configuration space of the system [51].

A comparison of the variational estimator is made with some of the state-of-the-art attitude filters, namely the Geometric Approximate Minimum-Energy
(GAME), the Multiplicative Extended Kalman Filter (MEKF) and the Constant Gain Observer (CGO), in the absence of bias in sensors readings in Chapter 5. A new measurement model of the problem which can be used for all the filters is explained first. These three state-of-the-art filters on SO(3) are presented in detail which are used to evaluate the performance of the LGVI, by comparing their principal angles of attitude estimate errors together. Such comparisons are carried out, and cases in which the variational estimator has advantages over other state-of-the-art filters are presented using numerical simulations. Numerical simulations show that the presented observer is robust and unlike the extended Kalman filter based schemes [25, 97], its convergence does not depend on the gains values. Besides, the variational estimator is shown to be the most computationally efficient attitude observer.

Since the Variational Estimator requires gyro measurements and these data are usually corrupted by bias in angular velocities, another generalized version of this estimation scheme is presented in Chapter 6, considering a constant bias in gyro measurements in addition to measurement noise. The measurement model for measurements of inertially-known vectors and biased angular velocity measurements using body-fixed sensors is detailed first. The problem of variational attitude estimation from these measurements in the presence of rate gyro bias is formulated and solved on SO(3). A Lyapunov stability proof of this estimator is given next, along with a proof of the almost global domain of convergence of the
estimates in the case of perfect measurements. It is also shown that the bias estimate converges to the true bias in this case. This continuous estimation scheme is discretized in the form of an LGVI using the discrete Lagrange-d’Alembert principle. The LGVI gives a first-order approximation of the continuous-time estimator. Numerical simulations are carried out using this LGVI as the discrete-time variational attitude estimator with a fixed set of gains.

Chapter 7 describes the details of experimental verification of the attitude estimator presented in Chapter 4. This chapter utilizes the smartphone’s inbuilt accelerometer, magnetometer and gyroscope as an Inertial Measurement Unit (IMU) for attitude determination. The primary motivation for using an open source smartphone is to create a cost-effective, generic platform for spacecraft attitude determination and control (ADCS), while not sacrificing on performance and fidelity. The PhoneSat mission of NASA’s Ames Research Center demonstrated the application of Commercial Off-The-Shelf (COTS) smartphones as the satellite’s onboard computer with its sensors being used for attitude determination and its camera for Earth observation [60]. University of Surrey’s Space Centre (SSC) and Surrey Satellite Technology Ltd (SSTL) developed STRaND-1, a 3U CubeSat containing a smartphone payload [18, 44]. Some advantages of using smartphones, on-board are:

1. compact form factor with powerful CPU, GPU etc.,
2. integrated sensors and data communication options,

3. long lasting batteries: reduces total mass budget,

4. cheap price and open source software development kit.

The attitude and angular velocity estimation scheme is based on inertial directions and angular velocity of the spacecraft measured by sensors in the body-fixed frame of the smartphone. The standalone mechatronics architecture performs the task of state sensing through embedded MEMS sensors, filtering, state estimation, to determine the cellphone’s attitude, while maintaining active uplink/downlink with a remote ground control station.

An important generalization of the Variational Estimation scheme is to derive an estimator for the most general motion of rigid body in 3 dimensional space \[ SE(3) \], which is the special Euclidean group, SE(3). Autonomous state estimation of a rigid body based on inertial vector measurement and visual feedback from stationary landmarks, in the absence of a dynamics model for the rigid body, is analyzed in Chapter 8. The estimation scheme proposed here can also be applied to relative state estimation with respect to moving objects \[ 64 \]. This estimation scheme can enhance the autonomy and reliability of unmanned vehicles in uncertain GPS-denied environments. Salient features of this estimation scheme are: (1) use of onboard optical and inertial sensors, with or without rate gyros, for autonomous navigation; (2) robustness to uncertainties and lack of knowledge of
dynamics; (3) low computational complexity for easy implementation with onboard processors; (4) proven stability with large domain of attraction for state estimation errors; and (5) versatile enough to estimate motion with respect to stationary as well as moving objects. Robust state estimation of rigid bodies in the absence of complete knowledge of their dynamics, is required for their safe, reliable, and autonomous operations in poorly known conditions. In practice, the dynamics of a vehicle may not be perfectly known, especially when the vehicle is under the action of poorly known forces and moments. The scheme proposed here has a single, stable algorithm for the coupled translational and rotational motion of rigid bodies using onboard optical (which may include infra-red) and inertial sensors. This avoids the need for measurements from external sources, like GPS, which may not be available in indoor, underwater or cluttered environments [3, 52, 61].

Chapter 8 applies the variational estimation framework to coupled rotational (attitude) and translational motion, as exhibited by maneuvering vehicles like UAVs. In such applications, designing separate state estimators for the translational and rotational motions may not be effective and may lead to poor navigation. For navigation and tracking the motion of such vehicles, the approach proposed here for robust and stable estimation of the coupled translational and rotational motion will be more effective than de-coupled estimation of translational and rotational motion states. Moreover, like other vision-inertial navigation
schemes [81, 82], the estimation scheme proposed here does not rely on GPS. However, unlike many other vision-inertial estimation schemes, the estimation scheme proposed here can be implemented without any direct velocity measurements. Since rate gyros are usually corrupted by high noise content and bias [9, 27–32], such a velocity measurement-free scheme can result in fault tolerance in the case of faults with rate gyros. Additionally, this estimation scheme can be extended to relative pose estimation between vehicles from optical measurements, without direct communications or measurements of relative velocities.

In this chapter, the problem of motion estimation of a rigid body using on-board optical and inertial sensors is introduced first. The measurement model is introduced and rigid body states are related to these measurements. Artificial energy terms are introduced next, representing the measurement residuals corresponding to the rigid body state estimates. The Lagrange-d’Alembert principle is applied to the Lagrangian constructed from these energy terms with a Rayleigh dissipation term linear in the velocity measurement residual, to give the continuous time state estimator. Particular versions of this estimation scheme are provided for the cases when direct velocity measurements are not available and when only angular velocity is directly measured. The stability of the resulting variational estimator is proved next. It is shown that, in the absence of measurement noise, state estimates converge to actual states asymptotically and the domain of attraction is an open dense subset of the state space. The variational
pose estimator is discretized as a Lie group variational integrator, by applying
the discrete Lagrange-d’Alembert principle to discretizations of the Lagrangian
and the dissipation term. This estimator is simulated numerically, for two cases:
the case where at least three beacons are measured at each time instant; and
the under-determined case, where occasionally less than three beacons are ob-
served. For these simulations, true states of an aerial vehicle are generated using
a given dynamics model. Optical/inertial measurements are generated, assuming
bounded noise in sensor readings. Using these measurements, state estimates are
shown to converge to a neighborhood of actual states, for both cases simulated.
Finally, the contributions and possible future extensions of this chapter are listed.
2 MODEL-BASED OBSERVER DESIGN WITH ASYMPTOTIC CONVERGENCE

This chapter is adapted from papers published in Proceedings of the 2013 ASME Dynamic Systems and Control (DSC) Conference [39] and the 52nd IEEE Conference on Decision and Control [15]. The author gratefully acknowledges Dr. Amit Sanyal, Dr. Daero Lee, Dr. Eric Butcher, Dr. Daniel Scheeres, Jan Bohn, Sérgio Brás, Dr. Paulo Oliveira and Dr. Carlos Silvestre for their participation.

Abstract  We consider an observer design for a spacecraft modeled as a rigid body in the proximity of an asteroid. The nonlinear observer is constructed on the nonlinear state space of motion of a rigid body, which is the tangent bundle of the Lie group of rigid body positions and orientations in three-dimensional Euclidean space. The framework of geometric mechanics is used for the observer design. States of motion of the spacecraft are estimated based on state measurements. In addition, the observer designed can also estimate the gravity of the asteroid, assuming the asteroid to have a spherically symmetric mass distribution. Almost global convergence of state estimates and gravity parameter estimate to their corresponding true values is demonstrated analytically, and verified numerically.
2.1 Rigid Body Dynamics

Consider a body fixed reference frame in the center of mass of a rigid spacecraft denoted as \( \{B\} \) and an inertial fixed frame denoted as \( \{I\} \). Let the rotation matrix from \( \{B\} \) to the inertial fixed frame \( \{I\} \) be given by \( R \) and the coordinates of the origin of \( \{B\} \) with respect to \( \{I\} \) be denoted as \( b \). The set of rotation matrices which contains \( R \) is denoted by \( \text{SO}(3) = \{ R \in \mathbb{R}^{3 \times 3} : R^T R = I, \det(R) = 1 \} \).

The rigid body kinematics are given by

\[
\dot{R} = R \Omega^\times, \quad (1)
\]
\[
\dot{b} = Rv, \quad (2)
\]

the linear and angular velocities expressed in the body fixed frame \( \{B\} \) are denoted by \( v \) and \( \Omega \), respectively, and the skew-symmetric operator \( (.)^\times : \mathbb{R}^3 \rightarrow \mathfrak{so}(3) \) satisfies

\[
\Omega^\times = \begin{bmatrix}
0 & -\Omega_z & \Omega_y \\
\Omega_z & 0 & -\Omega_x \\
-\Omega_y & \Omega_x & 0
\end{bmatrix}. \quad (3)
\]

Let \( g \) be the spacecraft configuration such that

\[
g = \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} \in \text{SE}(3), \quad (4)
\]

where the *Special Euclidean Group* \( \text{SE}(3) \) is the Lie group of rotations and translations whose matrix representation is given by the so-called homogeneous coord-
dinates

\[
\mathbf{SE}(3) = \left\{ g \in \mathbb{R}^{4 \times 4}, g = \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} : R \in \mathbf{SO}(3), b \in \mathbb{R}^3 \right\}. \tag{5}
\]

The dynamics equations of the spacecraft in the compact form are

\[
\dot{g} = g \xi^\vee
\]

\[
\mathbb{I} \dot{\xi} = \text{ad}^* \xi \mathbb{I} \xi + \varphi_G(g) = \text{ad}^* \xi \mathbb{I} \xi + \mu \psi_G(g), \tag{7}
\]

where \( \xi = [\Omega^T \ v^T]^T, \mathbb{I} = \begin{bmatrix} J & 0 \\ 0 & mI \end{bmatrix} \), \( m \) and \( J \) denote mass and inertia matrix of the spacecraft respectively, \( I \) is the \( 3 \times 3 \) identity matrix, \( \text{ad}^* \xi = (\text{ad} \xi)^T \) and \( \text{ad} \xi \) stands for the linear adjoint operator of the Lie algebra \( \mathfrak{se}(3) \) associated with the Lie group \( \mathbf{SE}(3) \) such that

\[
\text{ad} \xi = \begin{bmatrix} \Omega^\times & 0 \\ v^\times & \Omega^\times \end{bmatrix}. \tag{8}
\]

Besides, \( \varphi_G(g) = \begin{bmatrix} M_G \\ F_G \end{bmatrix} \) is the vector of external moments and forces, \( \mu \) is the unknown scalar gravity parameter and \( \psi_G(g) \) is a \( 6 \times 1 \) vector which satisfies the equation \( \varphi_G(g) = \mu \psi_G(g) \), where \( M_G, F_G \in \mathbb{R}^3 \) denote the gravity gradient moment and gravitational force applied on the spacecraft respectively, which are given by [78]:

\[
M_G = \mu \left\{ \frac{3}{\|b\|^5} (p \times Jp) \right\} \tag{9}
\]

\[
F_G = \mu \left\{ -\frac{m}{\|b\|^3} p - \frac{3}{\|b\|^5} \mathcal{J} p + \frac{15}{2} p^T J p \right\}, \tag{10}
\]

where \( p = R^T b \) and \( \mathcal{J} = \frac{1}{2} \text{trace}(J) I_{3 \times 3} + J \).
2.2 Observer Design for a Spherical Asteroid

Consider \((\hat{g}, \hat{\xi})\) to be the estimated values of the states \((g, \xi)\) of a rigid body’s motion on \(SE(3) \times \mathbb{R}^6\). Define

\[
h = \hat{g}^{-1} g \quad \text{and} \quad \eta^\vee = \logm_{SE(3)}(h),\]
(11)

where \(\logm_{SE(3)}(.) : SE(3) \to \mathfrak{se}(3)\) denotes the logarithmic map on \(SE(3)\) and \(\expm_{SE(3)}\) is its inverse. Therefore, we obtain:

\[
\dot{h} = h \tilde{\xi}^\vee \quad \text{where} \quad \tilde{\xi} = \xi - \Ad_{h^{-1}} \hat{\xi},
\]
(12)

and

\[
\Ad_g \xi^\vee = \left( \begin{bmatrix} R & 0 \\ b \times R & R \end{bmatrix} \xi \right)^\vee, \quad \zeta \in \mathbb{R}^6, \quad g \in SE(3).
\]
(13)

If we define \(\tilde{\xi} = \Ad_{h^{-1}} \hat{\xi}\), then \(\tilde{\xi} = \xi - \tilde{\xi}\). We express the exponential coordinate vector \(\eta\) for the pose estimate error as

\[
\eta = \begin{bmatrix} \Theta \\ \beta \end{bmatrix} \in \mathbb{R}^6 \simeq \mathfrak{se}(3),
\]
(14)

where \(\Theta \in \mathbb{R}^3\) is the exponential coordinate vector (principal rotation vector) for the attitude estimation error and \(\beta \in \mathbb{R}^3\) is the exponential coordinate vector for the position estimate error. The time derivative of the exponential coordinates of the configuration error is given by [19]

\[
\dot{\eta} = G(\eta) \tilde{\xi},
\]
(15)
The time derivative of the exponential coordinate $\Theta$ for the rotational motion is obtained from Rodrigues’ formula

$$R(\Theta) = I + \frac{\sin \theta}{\theta} \Theta^\times + \frac{1 - \cos \theta}{\theta^2} (\Theta^\times)^2 \text{ with } \theta = \|\Theta\|,$$ (20)

which is a well-known formula for the rotation matrix in terms of the exponential coordinates on $\text{SO}(3)$, the Lie group of special orthogonal matrices. In the context of equations (15)-(19), the matrix $R(\Theta) = \tilde{R}$, i.e., the attitude estimate error on $\text{SO}(3)$. We consider next a result that is important in obtaining the observer described later in this section.

**Lemma 2.1.** The matrix $G(\eta)$, which occurs in the kinematics equations (15)-(19) for the exponential coordinates on $\text{SE}(3)$, satisfies the relation

$$G(\eta)\eta = \eta.$$ (21)
Proof. Beginning with the expression for \( G(\eta) \) given by (16), we evaluate

\[
G(\eta)\eta = \begin{bmatrix}
A(\Theta)\Theta \\
T(\Theta, \beta)\Theta + A(\Theta)\beta
\end{bmatrix}.
\]

From the expression for \( A(\Theta) \), it is clear that

\[
A(\Theta)\Theta = \Theta.
\]

On evaluation of the other component, after some algebra, we obtain

\[
T(\Theta, \beta)\Theta = \beta - A(\Theta)\beta.
\]

Therefore, we obtain

\[
T(\Theta, \beta)\Theta + A(\Theta)\beta = \beta,
\]

which gives the desired result.

Further, define an auxiliary variable

\[
\ell = \tilde{\xi} + k_1 \eta.
\] (22)

Let the candidate Lyapunov function be

\[
V = \frac{1}{2}k_2 \eta^T \eta + \frac{1}{2} \ell^T \ell + \frac{1}{2} k_3 \tilde{\mu}^2,
\] (23)

where \( \tilde{\mu} = \mu - \hat{\mu} \) is the scalar estimation errors of the gravity parameter. Using this Lyapunov function we can show that the following observer design is stable.
Theorem 2.1. The states observer given in the form

\[ \dot{g} = \dot{g} \hat{\xi}^\vee \quad (24) \]

\[ \mathbb{I} \dot{\hat{\xi}} = \text{ad}^\ast \xi \mathbb{I} \xi + \dot{\mu} \psi_G(g) + s_\xi, \quad (25) \]

where

\[ s_\xi = -\text{ad}^\ast \eta \mathbb{I} \xi + k_2 G^T(\eta) \eta + k_1 G(\eta) \tilde{\xi} + k_4 \ell, \quad (26) \]

along with the following equations for estimating the unknown gravity parameter \( \mu \) ensures that the estimate errors converge to the origin:

\[ \dot{\hat{\mu}} = \hat{\mu} - \hat{\mu} = \frac{1}{k_3} \ell T \psi_G(g). \quad (27) \]

Proof. Using the equations (15) and (21) in [19] and calculating the time derivative of estimation error in velocities and the gravity parameter as follows

\[ \mathbb{I} \dot{\hat{\xi}} = \text{ad}^\ast \xi \mathbb{I} \xi + \dot{\mu} \psi_G(g) - k_2 G^T(\eta) \eta - k_1 G(\eta) \tilde{\xi} - k_4 \ell \quad (28) \]

\[ \dot{\hat{\mu}} = -\frac{1}{k_3} \ell T \psi_G(g), \quad (29) \]

and also the time derivative of the auxiliary parameter

\[ \mathbb{I} \dot{\ell} = \mathbb{I} \dot{\xi} + k_1 \mathbb{I} \dot{\eta} \]

\[ = \text{ad}^\ast \xi \mathbb{I} \xi + \dot{\mu} \psi_G(g) - k_2 G^T(\eta) \eta - k_1 G(\eta) \tilde{\xi} - k_4 \ell + k_1 G(\eta) \tilde{\xi} \]

\[ = \text{ad}^\ast \xi \mathbb{I} \xi + \dot{\mu} \psi_G(g) - k_2 G^T(\eta) \eta - k_4 \ell, \quad (30) \]
the first and second order time derivative of the proposed Lyapunov function can
be written as

\[
\dot{V} = k_2 \eta^T \dot{\eta} + \ell^T \dot{\ell} + k_3 \dot{\mu} \dot{\tilde{\mu}}
\]

\[
= k_2 \eta^T G(\eta) \dot{\xi} + \ell^T \text{ad}^*_\xi \xi + \mu^T \psi_G(g) - k_2 \ell^T G(\eta) \dot{\eta} - k_4 \ell^T \dot{\ell} - \mu^T \psi_G(g)
\]

\[
= k_2 \eta^T G(\eta) (\dot{\xi} - \ell) - k_4 \ell^T \dot{\ell} = -k_1 k_2 \eta^T G(\eta) \dot{\eta} - k_4 \ell^T \dot{\ell}
\]

\[
= -k_1 k_2 \|\eta\|^2 - k_4 \|\ell\|^2.
\]

(31)

Thus, \( \dot{V} \) is negative semi-definite. Besides,

\[
\ddot{V} = -k_1 k_2 \eta^T \ddot{\eta} - k_4 \ell^T \ddot{\ell}
\]

(32)

\[
= -k_1 k_2 \eta^T G(\eta) \dddot{\xi} - k_4 \ell^T \text{ad}^*_\xi \dot{\xi} + k_4 \mu \ell^T \text{ad}^*_\xi \xi + k_4 \mu \ell^T \frac{1}{2} \psi_G(g) + k_2 k_4 \ell^T \frac{1}{2} \psi_G(g)
\]

\[
+ k_4 \ell^T \frac{1}{2} \psi_G(g)
\]

which means that \( \ddot{V} \) is finite for any bounded pose and velocity vector. Using

Barbalat’s Lemma one can conclude that \( \dot{V} \to 0 \) which gives \( \|\eta\| \to 0 \) and \( \|\ell\| \to 0 \),

therefore \( \|\tilde{\xi}\| \to 0 \) in turn. Moreover, for initially bounded state estimate errors,

\( \|\tilde{\xi}\| \to 0 \) leads to \( \frac{d}{dt} \|\tilde{\xi}\| \to 0 \) which implies that \( |\tilde{\mu}| \to 0 \).

Note that this observer, which uses the exponential coordinate representation
of the pose in SE(3), is not defined when the exponential coordinate vector itself
is not defined. This happens whenever the attitude corresponds to a principal
rotation angle given by an odd multiple of \( \pi \) radians.
2.3 Numerical Simulations

In order to verify the performance of the observer, we used a set of realistic data for all the states of the system. One can consider a spherical asteroid with the gravitational constant $\mu = 1.729 \times 10^{10}$ m$^3$/s$^2$ and integrate the dynamics to mimic the true states of the spacecraft in an orbit around this spherical asteroid.

Mass and inertia matrix of the spacecraft is considered as $m = 21$ kg and $J = \text{diag}(2.56, 3.01, 2.98)^T$ kg m$^2$. The spacecraft is rotating in an elliptical orbit with semi-major axis $a = 330$ km and the radius at periapsis equal to $r_p = 310$ km. The initial configuration is given by

$$R_0 = \exp_{\text{SO}(3)}([0.4 \ 0.2 \ 0.1]^T)^\times, \ b_0 = r_p \times \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}^T$$

(33)

and the initial angular and linear velocities were set to

$$\Omega_0 = 10^{-3} \times [7 \ -4 \ 1]^T \text{ rad/s},$$

(34)

$$\nu_0 = R_0^T \left(v_p \times \frac{b_0}{r_p} \times ([0 \ \frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}}]^T)^\times \right) = [241.4 \ 16.7 \ -24.5]^T \text{ m/s},$$

(35)

where

$$v_p = \sqrt{-\frac{\mu}{a} + \frac{2\mu}{r_p}}.$$  

(36)

Using these numerical values, the simulated orbit is shown in the Fig. 1. As can be easily seen, the spacecraft’s path around the asteroid is an elliptical periodic orbit with the radius at periapsis $r_p = 310$ km and the semi-major axis $a = 330$ km.
Figure 1: The Generated Path for the Actual Orbit of the Spacecraft Rotating Around the Spherical Asteroid.

Note that the orbital period of the spacecraft is

\[ T = 2\pi \sqrt{\frac{a^3}{\mu}} = 9058.4 \text{ s} \]  

(37)

Integrating the logarithmic map of equation 6 along with 7, the exponential coordinates of the spacecraft in the vicinity of the asteroid could be plotted numerically as in Fig. 2. Note that the logarithmic map could be used to get the exponential coordinates of both absolute configuration and relative configuration and we have used the same notation for them and their components. In other words, \( \eta \) denotes the logarithmic map of both pose \( (g) \) and relative pose
\( h = \hat{g}^{-1}g \) and Θ and β are its angular and linear components.

Figure 2: Exponential Coordinates of the Spacecraft.

The angular and linear components of the spacecraft’s velocity are also depicted in Fig. 3. Both Fig. 2 and 3 show somehow periodic motion in their components which was expected from the motion in the elliptical orbit of Fig. 1. After mimicking the actual dynamics, another code was used to numerically integrate the observer ODEs which are equations (24)-(27).

Since the numerical values for the translational quantities (displacements and velocities) depend on the unit by which they are described and specially in the case a relatively small unit like meters has been used the quantities will be of a much higher order compared with the angular quantities, we should normalize
translational quantities to resolve numerical issues while dealing with the compact forms. The semi-major axis $a$ was used to make all linear quantities dimensionless. Note that the angular velocities are in radians and therefore dimensionless. In view of the fact that the dimension of the gravitational parameter is $L^3T^{-2}$, it was divided by $a^3$.

In order to better agreement between angular and linear components, and as a result get better convergence behavior, we also could use a block diagonal form for some gain factors. This helps different components in the compact form converge at almost the same rate. The tuning parameters are set to be $k_1 = \begin{bmatrix} 1.12 \times I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix}$, $k_2 = 1$, $k_3 = 0.2$ and $k_4 = \begin{bmatrix} 1.2 \times I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix}$.
In this step, the value of the gravity parameter is completely unknown like all other states. The initial values for the estimated quantities are set as follows. The initial values of estimated attitude and position vector of the spacecraft are 
\[ \hat{R}_0 = I_3 \times 3 \] and 
\[ \hat{b}_0 = [103 \ - \ 206 \ 206]^T \text{km}. \]
\[ \hat{\Omega}_0 = 10^{-3} \times [5 \ - \ 7 \ 3]^T \text{rad/s} \]
and 
\[ \hat{v}_0 = [59 \ - \ 19 \ - \ 27]^T \text{m/s} \] were set as the initial estimates for the angular and linear velocities of the spacecraft. The initial estimated value of gravitational parameter is set to 
\[ \hat{\mu}_0 = 1.6172 \times 10^{14} \text{m}^3/\text{s}^2 \]
which is almost 10,000 times bigger than the actual value and large enough to test the convergence behavior of the observer.

The estimation errors in the exponential coordinates obtained in this numerical simulation, are shown in Fig. 4. These estimation errors are seen to decrease at a satisfactory rate. As can be seen, these errors become negligible after about 50(s).

The errors in estimated angular and linear velocities are depicted in Fig. 5. Convergence can be seen in all components, even though the first few seconds show increasing errors for some components. Here, the observer is seen to have desirable behavior for velocity estimation of the spacecraft.

The estimate error in the gravitational parameter is plotted in Fig. 6, beginning with a large initial estimate error. Asymptotic convergence to the true value can be observed in this figure. Note that in Fig. 2-6, the length unit has been normalized to 1 unit= 310 km, which is the value of the semi-major axis of the orbit of the spacecraft around the asteroid.
2.4 Nonlinear Observer Design when Force Measurements are available

An observer design for pose and velocity estimation for three-dimensional rigid body motion, in the framework of geometric mechanics is presented here. Based on a Lyapunov analysis, a nonlinear observer on the Special Euclidean Group SE(3) is derived. This observer is based on the exponential coordinates which are used to represent the group of rigid body motions.

Assumption 2.1. The sensor suite available provides measurements about the configuration, velocity, forces and torques applied to the vehicle.
Figure 5: Estimation Errors of the Spacecraft’s Velocities.

Note that, even with full state measurements, the existence of an observer is valuable for any navigation and control system as, like the EKF, it can mitigate the effects of sensor uncertainties such as noise and bias. The configuration observer takes the form

\[
\dot{\hat{g}} = \hat{g} \hat{\xi}',
\]

\[
\dot{\hat{\xi}} = \text{ad}_{-K(k_1 \hat{\eta} - \hat{\xi})} \hat{\xi} + \varphi + k_1 G(\hat{\eta}) \hat{\xi} + G^T(K \hat{\eta}) \hat{\eta} + k_3 u,
\]

where \( K = \begin{bmatrix} I & 0 \\ 0 & k_d I \end{bmatrix} \), \( k_1, k_2, k_3 > 0 \), \( u = k_1 \hat{\eta} + \hat{\xi} \) and

\[
\hat{\zeta} = \text{Ad}_{\hat{g}}^{-1} \hat{\xi}, \quad \tilde{\xi} = \xi - \hat{\xi}.
\]
Figure 6: Estimation Errors of the Gravitational Parameter.

Proof. Consider the following Lyapunov function candidate

\[ V = \frac{1}{2} \tilde{\eta}^T K \tilde{\eta} + \frac{1}{2} (k_1 \tilde{\eta} + \tilde{\xi})^T K I (k_1 \tilde{\eta} + \tilde{\xi}), \]  

(41)

which motivates the development of the velocity observer. Letting \( u = k_1 \tilde{\eta} + \tilde{\xi} \)
and taking the time derivative produces

\[ \dot{V} = -k_1 \eta^T K \tilde{\eta} + u^T K (G^T (K \tilde{\eta}) \tilde{\eta} + k_1 I G(\tilde{\eta}) \tilde{\xi} + \text{ad}^* \tilde{\xi} + \varphi - I \dot{\tilde{\xi}}), \]  

(42)

where it is exploited the equality \( K^{-1} G^T (\tilde{\eta}) K = G^T (K \tilde{\eta}) \). Let

\[ \dot{\tilde{\xi}} = \text{ad}^*_{K(k_1 \tilde{\eta} - \tilde{\xi})} \tilde{\xi} + \varphi + k_1 I G(\tilde{\eta}) \tilde{\xi} + G^T (K \tilde{\eta}) \tilde{\eta} + k_3 u. \]  

(43)

Then, resorting to some algebraic manipulations, the time derivative of (41)
takes the negative definite form

\[ \dot{V} = -k_1 \tilde{\eta}^T K \tilde{\eta} - k_3 (k_1 \tilde{\eta} + \tilde{\xi})^T K (k_1 \tilde{\eta} + \tilde{\xi}). \] 

(44)

Thus, the point \((\tilde{\eta}, \tilde{\xi}) = (0, 0)\) is asymptotically stable in sense of Lyapunov [45].

Topological limitations precludes global asymptotic stability of the origin [10]. In fact, if \(\theta = \pi\), the exponential coordinates of the configuration error \(\tilde{\eta}\) cannot be computed without ambiguity. Sufficient conditions ensuring that for all \(t > t_0\), \(\theta(t) < \pi\) is provided in [15].

2.5 Conclusion

A nonlinear observer for rigid body motion in the presence of an unknown central gravity field due to a spherical asteroid was presented. In addition to estimating the states of an exploring spacecraft, modeled as a rigid body, in the proximity of a spherical asteroid, this observer also estimates the gravity parameter of this asteroid. Estimates obtained from this observer are shown to converge to true states and the true gravity parameter almost globally over the state space of motion of the rigid spacecraft. These convergence properties are verified by numerical simulation for a realistic scenario of a satellite in the proximity of an asteroid with spherical mass distribution. The following chapter presents another nonlinear observer for rigid body motion that has finite-time convergence.
3 MODEL-BASED OBSERVER DESIGN WITH FINITE-TIME CONVERGENCE

This chapter is adapted from a paper published in Proceedings of the 2014 ASME Dynamic Systems and Control (DSC) Conference [72]. The author gratefully acknowledges Dr. Amit K. Sanyal and Jan Bohn for their participation.

Abstract An observer that obtains estimates of the translational and rotational motion states for a rigid body under the influence of known forces and moments is presented. This nonlinear observer exhibits almost global convergence of state estimates in finite time, based on state measurements of the rigid body’s pose and velocities. It assumes a known dynamics model with known resultant force and resultant torque acting on the body, which may include feedback control force and control torque. The observer design based on this model uses the exponential coordinates to describe rigid body pose estimation errors on SE(3), which provides an almost global description of the pose estimate error. Finite-time convergence of state estimates and the observer are shown using a Lyapunov analysis on the nonlinear state space of motion. Numerical simulation results confirm these analytically obtained convergence properties for the case that there is no measurement noise and no uncertainty (noise) in the dynamics. The robustness of this observer to measurement noise in body velocities and additive noise in the force and torque
components is also shown through numerical simulation results.

3.1 Rigid Body Dynamics Model

Consider a body fixed reference frame in the center of mass of a rigid body denoted as \( \{B\} \) and an inertial fixed frame denoted as \( \{I\} \). Let the rotation matrix from \( \{B\} \) to the inertial fixed frame \( \{I\} \) be given by \( R \) and the coordinates of the origin of \( \{B\} \) with respect to \( \{I\} \) be denoted as \( b \).

3.1.1 Rigid Body Dynamics

The rigid body dynamics is given by

\[
\dot{J} \Omega = J \Omega \times \Omega + ^B \tau,
\]

\[
m \dot{\nu} = m \nu \times \Omega + ^B \phi,
\]

(45)

where \( m \) and \( J \) denote the rigid body mass and inertia matrix, respectively, \( ^B \phi \) denotes the force applied to the rigid body and \( ^B \tau \) the external torque, both expressed in the body reference frame. The dynamics equations (45) can be expressed in compact form as

\[
\tilde{I} \dot{\xi} = \text{ad}^{\xi}_{\tilde{I}} \xi + \phi,
\]

(46)

where \( \phi = [^B \tau^T \ ^B \phi^T]^T \).
3.1.2 Kinematics in Exponential Coordinates

The exponential coordinate vector \( \eta \in \mathbb{R}^6 \) for a given configuration \( g \in \text{SE}(3) \) is given by

\[
\eta^\wedge = \log_m(g) = \begin{bmatrix} \Theta^x & \beta \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad \eta = \begin{bmatrix} \Theta \\ \beta \end{bmatrix},
\]

and \( \log_m \) denotes the matrix logarithm, which is also the inverse of the exponential map \( \exp_m : \mathfrak{se}(3) \rightarrow \text{SE}(3) \). We can obtain the exponential coordinate vector \( \eta \) from \( g \) as follows:

\[
\Theta^x = \frac{\theta}{\sin(\theta)} (R - R^T), \quad \text{and} \quad \beta = S^{-1}(\Theta)b,
\]

where \( S^{-1}(\Theta) = I - \frac{1}{2} \Theta^x + \left( \frac{1}{\theta^2} - \frac{1 + \cos \theta}{2 \theta \sin \theta} \right) (\Theta^x)^2 \),

and \( \theta = \|\Theta\| \) is the principal angle of rotation corresponding to the rotation matrix \( R \). Note that \( \Theta \) in (48) cannot be obtained when \( \theta \) is an odd multiple of \( \pi \) radians. Since all of \( \text{SO}(3) \) can be represented by principal angle values in the range \( \theta \in [0, \pi] \), we can therefore obtain an unique exponential coordinate vector for all \( g \in \text{SE}(3) \) whose \( \text{SO}(3) \) component has a principal angle less than \( \pi \) radians, i.e., \( \theta \in [0, \pi) \). Therefore, the exponential coordinates can represent almost all poses in \( \text{SE}(3) \) excluding those with rotations of exactly \( \pi \) radians about any axis.

The exponential coordinate vector \( \eta \in \mathbb{R}^6 \) (corresponding to \( \eta^\wedge = \log_m(g) \in \mathfrak{se}(3) \)), satisfies (15). Note that \( \theta = 0 \) is a removable singularity in equations (17)-(19), and corresponds to the identity orientation on \( \text{SO}(3) \). An equivalent
expression for $G(\eta)$ given in [19] is as follows:

$$G(\eta) = I + \frac{1}{2} \text{ad}_{\eta} + \alpha(\theta) \text{ad}_{\eta}^{2} + \beta(\theta) \text{ad}_{\eta}^{4},$$  \hfill (49)

where

$$\alpha(\theta) = \frac{2}{\theta^{2}} - \frac{3}{4\theta} \cot(\theta/2) - \frac{1}{8} \csc^{2}(\theta/2),$$

$$\beta(\theta) = \frac{1}{\theta^{4}} - \frac{1}{2\theta^{3}} \cot(\theta/2) - \frac{1}{8\theta^{2}} \csc^{2}(\theta/2).$$  \hfill (50)

From the expression (50), it is clear that $G(\eta) \eta = \eta$ [39], a fact used in the observer design. The exponential coordinates on $\text{SE}(3)$ were used for observer design recently in [15]. However, the observer design in [15] had asymptotic (exponential) convergence, unlike the observer designed here, which exhibits finite-time convergence.

3.2 Finite-Time Convergent Observer Design

We assume that a sensor suite onboard a rigid body vehicle provides information about the configuration and velocities of the vehicle. Our aim is to design a dynamic observer which exploits the sensors measurements (pose and velocities) to estimate the configuration (pose) and the velocities, such that the estimated states converge in finite-time to their true values in the absence of measurement errors. Robustness to bounded measurement errors and noisy inputs to the dynamics model is obtained consequently, and is shown through numerical simulation results.
Consider \((\hat{g}, \hat{\xi})\) to be estimates of the states \((g, \xi)\) of a rigid body’s motion on \(\text{SE}(3) \times \mathbb{R}^6\). Define
\[
h = \hat{g}^{-1}g \quad \text{and} \quad \tilde{\eta}^\vee = \logm(h).
\] (51)

Therefore, we obtain:
\[
\dot{h} = h\tilde{\xi}^\vee \quad \text{where} \quad \tilde{\xi} = \xi - \text{Ad}_{h^{-1}}\hat{\xi}.
\] (52)

If we define \(\hat{\xi} = \text{Ad}_{h^{-1}}\hat{\xi}\), then \(\tilde{\xi} = \xi - \hat{\xi}\). From (52) and the kinematics in exponential coordinates given in the previous section, we conclude that
\[
\dot{\tilde{\eta}} = G(\tilde{\eta})\tilde{\xi}.
\] (53)

Further, define
\[
u = \xi - \text{Ad}_{h^{-1}}\hat{\xi} + k\frac{\tilde{\eta}}{\left(\tilde{\eta}^T\tilde{\eta}\right)^{1-\frac{1}{p}}} = \tilde{\xi} + k\frac{\tilde{\eta}}{\left(\tilde{\eta}^T\tilde{\eta}\right)^{1-\frac{1}{p}}},
\] (54)

where \(k > 0\) and \(p \in (1, 2)\) is a rational number (preferably a ratio of odd integers, to avoid sign mismatches when taking powers using a computer code). Let
\[
V = \frac{1}{2}\gamma\tilde{\eta}^T\tilde{\eta} + \frac{1}{2}\nu^T\mathbb{I}\nu
\] (55)

be a candidate Lyapunov function, where \(\gamma > 0\), and \(\mathbb{I}\) is the complete inertia matrix as given in eq. This Lyapunov function is used to show the finite-time convergence of the observer design that follows.
Theorem 3.1. The observer dynamics given by:

\[ \dot{\hat{g}} = \hat{g} \hat{\xi} \vee, \quad \text{OR} \quad \dot{\hat{\eta}} = G(\hat{\eta})\hat{\xi} \quad \text{and} \quad \dot{\hat{g}} = g \expm(-\hat{\eta} \vee), \]

(56)

\[ \dot{\hat{\xi}} = \text{ad}^*_{(\hat{\xi} - k(\hat{\eta} T \hat{\eta})^{1-p} \hat{\eta})} \hat{\xi} + \varphi + kH(\hat{\eta})G(\hat{\eta})\hat{\xi} + \gamma G^T(\hat{\eta})\hat{\eta} + k \frac{I u}{(u T I u)^{1-p}}, \]

(57)

where \( H(\hat{\eta}) = \frac{1}{(\hat{\eta} T \hat{\eta})^{1-p}} \left\{ I - 2(1 - \frac{1}{p}) \frac{\hat{\eta} T \hat{\eta}}{\hat{\eta} T \hat{\eta}} \right\}, \) and \( \varphi \) is the resultant of forces and moments acting on the rigid body, ensures that the estimate errors converge to the origin in finite time. Thus, \((\hat{\xi}, \hat{\eta}) = (0, 0) \in \mathbb{R}^6 \times \mathbb{R}^6\) for all time \( t \geq t_f \), where \( t_f \) is finite.

Proof. The time derivative of the Lyapunov function given by (55) is:

\[ \dot{V} = \gamma \hat{\eta}^T \dot{\hat{\eta}} + u^T \dot{I} \]

\[ = \gamma \hat{\eta}^T G(\hat{\eta})\dot{\hat{\xi}} + u^T \dot{I} \]

(58)

From (57) and the dynamics

\[ \dot{\hat{\xi}} = \text{ad}_{\hat{\xi}}^* \hat{\xi} \quad \varphi \]

(59)

of the rigid body, we obtain:

\[ \dot{I} \dot{u} = I \left\{ \dot{\hat{\xi}} - \hat{\xi} + \frac{d}{dt} \left( k(\hat{\eta} T \hat{\eta})^{\frac{1}{p}} \hat{\eta} \right) \right\} \]

\[ = \dot{I} \dot{\hat{\xi}} - \dot{\hat{\xi}} + kI H(\hat{\eta})\dot{\hat{\eta}} \]

\[ = \text{ad}_{\hat{\xi}}^* \hat{\xi} + \varphi - \dot{\hat{\xi}} + kI H(\hat{\eta})G(\hat{\eta})\dot{\hat{\xi}}, \]

(60)

using the kinematics \( \dot{\hat{\eta}} = G(\hat{\eta})\dot{\hat{\xi}} \) in (56), which holds for the configuration error expressed in exponential coordinates. Substituting for \( \dot{\hat{\xi}} \) from (57) into expression
(60), we obtain:
\[
\mathbb{I} \dot{u} = \text{ad}_u^* \mathbb{I} \xi - \gamma G^T(\tilde{\eta}) \tilde{\eta} - k \frac{\mathbb{I} u}{(u^T \mathbb{I} u)^{1 - \frac{1}{p}}} \tag{61}
\]
for the feedback dynamics of the variable \( u \). Now substituting for \( \mathbb{I} \dot{u} \) from (61) into the expression for \( \dot{V} \) in (58), we get:
\[
\dot{V} = \gamma \tilde{\eta}^T G(\tilde{\eta}) \tilde{\xi} - \gamma u^T G^T(\tilde{\eta}) \tilde{\eta} - k(u^T \mathbb{I} u)^{\frac{1}{p}}
= \gamma \tilde{\eta}^T G(\tilde{\eta}) (\tilde{\xi} - u) - k(u^T \mathbb{I} u)^{\frac{1}{p}}
= -k \frac{\tilde{\eta}^T G(\tilde{\eta}) \tilde{\eta}}{(\tilde{\eta}^T \tilde{\eta})^{1 - \frac{1}{p}}} - k(u^T \mathbb{I} u)^{\frac{1}{p}}
= -k \left[ \gamma (\tilde{\eta}^T \tilde{\eta})^{\frac{1}{p}} - (u^T \mathbb{I} u)^{\frac{1}{p}} \right] \tag{62}
\]
using the fact that \( G(\tilde{\eta}) \tilde{\eta} = \tilde{\eta} \). From (62), we note that
\[
\dot{V} = -2^{\frac{1}{p}} k \left[ \gamma^{1 - \frac{1}{p}} (\tilde{\eta}^T \tilde{\eta})^{\frac{1}{p}} + \left( \frac{1}{2} u^T \mathbb{I} u \right)^{\frac{1}{p}} \right] \leq -2^{\frac{1}{p}} k V^{\frac{1}{p}}, \tag{63}
\]
for \( \gamma \geq 1 \), using the binomial expansion theorem. Thus, \( V \) converges to zero in finite time, and hence the result.

Note that since the exponential coordinates are defined almost globally on the configuration space \( \text{SE}(3) \), the above observer can be used for all initial estimate errors \( (h(0), \tilde{\xi}(0)) \in \text{SE}(3) \times \mathbb{R}^6 \) such that the principal angle corresponding to the \( \text{SO}(3) \) component of \( h(0) \) is not \( \pi \) radians (or \( 180^\circ \)). Therefore, the above observer is finite-time convergent, and its domain of convergence is almost global on the state space \( \text{SE}(3) \times \mathbb{R}^6 \).

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3.3 Numerical Simulations

In this section, this observer numerically simulated for a rigid body that represents a maneuverable aerial vehicle with known mass and inertia. The observer needs a set of measured states (pose and rigid body velocities) to estimate the exponential coordinates as well as the rigid body’s velocities. Measurement data is generated by integrating the “true” (known) dynamics of the rigid body offline with known models of external torques and forces, and then adding noise. The rigid body’s mass is assumed to be $m = 21$ kg and its inertia matrix is

$$J = \text{diag}(2.56, 3.01, 2.98) \text{ kg.m}^2.$$

The rigid body is subjected to an external force and an external torque, which are expressed in the body-fixed frame as

$$\phi^0_D = [0.4 \ 0.5 \ 0.768]^T \text{ N and } \tau^0_D = [0.07 \ 0.0687 \ 0.02]^T \text{ N.m},$$

as well as a uniform gravity force directed towards the negative z-axis of the inertial frame. The initial configuration (pose) is given by

$$R_0 = \expm(0.1[4 \ 2 \ 1]^\times), \ b_0 = [1 \ 2 \ 3]^T \text{ m},$$

and the initial angular and linear velocities are

$$\omega_0 = [0.5 \ -0.5 \ 0.1]^T \text{ rad/s}, \ v_0 = 10^{-3}[-5 \ 25 \ 30]^T \text{ m/s}.$$
A discrete-time numerical integration scheme with constant time stepsize is used to propagate the true states as well as the estimated ones. The discrete time period for this numerical integration scheme, $h$, is chosen to be 0.01 s. The true states are propagated for $T = 0$ to $10$ s.

Using the above initial states, integration scheme, time interval and step size, the dynamics of the rigid body is integrated and its trajectory in three dimensional space is depicted in Fig. 7. The body frame axes are also plotted on this path every 1 second to show the attitude motion.

![Figure 7: Rigid body actual trajectory with its attitude.](image)

The initial estimated pose and velocity have been taken to be $\hat{g}_0 = I_{4 \times 4}$ and $\hat{\xi}_0 = 0_{6 \times 1}$ respectively. In order to get fast convergence and smooth estimation
error with a reasonable overshoot, the best design parameters of the observer arrived at using trial and error are:

\[ k = 50, \quad p = \frac{23}{21}, \quad \gamma = 0.03. \quad (67) \]

These parameter values were used in simulating the observer’s performance with and without measurement noise and external disturbance.

3.3.1 In the absence of noise and disturbance

Here, we assume that there is no measurement noise or external disturbance and the sensors are ideal. The measurement sampling period is taken to be 0.01 s. This simulation runs for 1.5 s, which is long enough for the estimation errors in exponential coordinates and velocities to converge to zero. The principal angle of the attitude estimation error as well as the estimate errors in the Cartesian coordinates of the rigid body are depicted in Fig. 8. All these components have been derived from the exponential coordinates estimation errors proposed in Theorem 3.1. The estimation errors in the angular and translational velocities of the rigid body during the simulation are shown in Fig. 9.

These figures show that the estimation errors converge to zero in a very small and finite time, which is almost 0.3 s here. Just due to the numerical artifacts, the errors will not be exactly zero, but after the mentioned finite time all of their components are negligible.
Figure 8: Principal angle of attitude estimate error and rigid body’s position estimate error in the absence of noise or disturbance.

3.3.2 In the presence of noise and disturbance

In this section, we show that the observer is robust to measurement noises and external disturbances. Presence of measurement errors and external disturbances is always the case in reality, where the sensors data contain some certain levels of measurement noise. First of all, the dynamics of the system is mimicked to generate the pure states. Next, some noise signals with a realistic level of available rough sensors are added to each set of states. The noises in the position and attitude data are sinusoidal signals with the amplitudes of 10 cm and 2°, respectively. Their frequencies also are both 100 Hz. The same kind of signals are added
Figure 9: Rigid body’s angular and translational velocities estimate errors in the absence of noise or disturbance.

to the pure velocities, but with different amplitudes, which are $1^\circ/s$ and 1 cm/s, respectively. Note that the observer proposed in Theorem 3.1 does not use the pose in all time steps. Therefore, just the noise in the initial value of position and attitude affect the estimated states. On the other hand, the noisy angular and translational velocities are used as a feedback in each step of the estimation. The external disturbances are assumed to be added to the previous external torques, which make the total torques applied to the rigid body equal to

$$\tau_D = \tau_D^0 + 0.01 \sin(0.1t) \begin{bmatrix} 0.424 \\ 0.9 \\ 0.1 \end{bmatrix} \text{ N.m.} \quad (68)$$

The total external forces applied on the rigid body also are taken to have
disturbance as

\[ \phi_D = \phi_D^0 + 0.1 \sin(0.1t) \begin{bmatrix} 0.1 \\ 0.2 \\ 0.975 \end{bmatrix} \text{ N.} \quad (69) \]

In Fig. 10 the principal angle corresponding to the rigid body attitude estimate error is plotted. This figure also depicts the rigid body position estimation errors by components. The estimation errors of the rigid body velocities are shown in Fig. 11. These two figures show that the estimation errors in all the pose and velocities converge to zero in a finite time almost as fast as the noise-free case. Thus, the proposed observer can estimate the real states even in the presence of additive noise in the dynamics model. This was expected, since the finite-time convergent systems have been shown to be robust to bounded external disturbances and measurement errors.

3.4 Conclusion

An observer that exhibits finite-time convergence for arbitrary rigid body motion states in the tangent bundle of the special Euclidean group SE(3) is presented. The observer design is based on use of the exponential coordinates, which are defined almost globally over the configuration space SE(3). Therefore, the domain of convergence is almost global on this state space, and excludes only those initial pose errors whose rotation component has a principal angle of exactly \( \pi \) radians. Since finite-time convergence has been shown to be more robust to noise in the
dynamics or in measurements, this observer is expected to be robust to both measurement noise and process noise. Such robustness properties are indicated in simulation results obtained from a numerical implementation of this observer. The estimation errors in the absence of measurement noise and in the presence of measurement noise are seen to converge in a finite-time that depends on the observer design parameters.

Figure 10: Principal angle of attitude estimate error and rigid body’s position estimate error in the presence of noise and disturbance.
Figure 11: Rigid body’s angular and translational velocities estimate errors in the presence of noise and disturbance.
4 MODEL-FREE RIGID BODY ATTITUDE ESTIMATION BASED ON THE LAGRANGE-D’ALEMBERT PRINCIPLE

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Abstract Estimation of rigid body attitude and angular velocity without any knowledge of the attitude dynamics model, is treated using the Lagrange-d’Alembert principle from variational mechanics. It is shown that Wahba’s cost function for attitude determination from two or more non-collinear vector measurements can be generalized and represented as a Morse function of the attitude estimation error on the Lie group of rigid body rotations. With body-fixed sensor measurements of direction vectors and angular velocity, a Lagrangian is obtained as the difference between a kinetic energy-like term that is quadratic in the angular velocity estimation error and an artificial potential obtained from Wahba’s function. An additional dissipation term that depends on the angular velocity estimation error is introduced, and the Lagrange-d’Alembert principle is applied to the Lagrangian
with this dissipation. A Lyapunov analysis shows that the state estimation scheme so obtained provides stable asymptotic convergence of state estimates to actual states in the absence of measurement noise, with an almost global domain of attraction. These estimation schemes are discretized for computer implementation using discrete variational mechanics. A first order implicit Lie group variational integrator is obtained as a discrete-time implementation and its adjoint flow yields an explicit first order LGVI. Composing these two first order flows, a symmetric second-order version of this discrete-time filtering scheme is also presented. In the presence of bounded measurement noise, numerical simulations show that the estimated states converge to a bounded neighborhood of the actual states. A comparison between the performances of the second-order filter and the first-order filter is also carried out.

4.1 Attitude Determination from Vector Measurements

Rigid body attitude is determined from \( k \in \mathbb{N} \) known inertial vectors measured in a coordinate frame fixed to the rigid body. Let these vectors be denoted as \( u_i^m \) for \( i = 1, 2, \ldots, k \), in the body-fixed frame. The assumption that \( k \geq 2 \) is necessary for instantaneous three-dimensional attitude determination. When \( k = 2 \), the cross product of the two measured vectors is considered as a third measurement for applying the attitude estimation scheme. Denote the corresponding known inertial vectors as seen from the rigid body as \( e_i \), and let the true vectors in
the body frame be denoted \( u_i = R^T e_i \), where \( R \) is the rotation matrix from the body frame to the inertial frame. This rotation matrix provides a coordinate-free, global and unique description of the attitude of the rigid body. Define the matrix composed of all \( k \) measured vectors expressed in the body-fixed frame as column vectors,

\[
U^m = [u_1^m \ u_2^m \ u_1^m \times u_2^m] \text{ when } k = 2, \quad \text{and}
\]

\[
U^m = [u_1^m \ u_2^m \ ...u_k^m] \in \mathbb{R}^{3 \times k} \text{ when } k > 2, \tag{70}
\]

and the corresponding matrix of all these vectors expressed in the inertial frame as

\[
E = [e_1 \ e_2 \ e_1 \times e_2] \text{ when } k = 2, \quad \text{and}
\]

\[
E = [e_1 \ e_2 \ ...e_k] \in \mathbb{R}^{3 \times k} \text{ when } k > 2. \tag{71}
\]

Note that the matrix of the actual body vectors \( u_i \) corresponding to the inertial vectors \( e_i \), is given by

\[
U = R^T E = [u_1 \ u_2 \ u_1 \times u_2] \text{ when } k = 2, \quad \text{and}
\]

\[
U = R^T E = [u_1 \ u_2 \ ...u_k] \in \mathbb{R}^{3 \times k} \text{ when } k > 2.
\]

4.1.1 Generalization of Wahba’s Cost Function for Instantaneous Attitude Determination from Vector Measurements

The optimal attitude determination problem for a set of vector measurements at a given time instant, is to find an estimated rotation matrix \( \hat{R} \in \text{SO}(3) \) such that
a weighted sum of the squared norms of the vector errors

\[ s_i = e_i - \hat{R}u_i^m \]  

are minimized. This attitude determination problem is known as Wahba’s problem, and is the problem of minimizing the value of

\[ U^0(\hat{R}, U^m) = \frac{1}{2} \sum_{i=1}^{k} w_i(e_i - \hat{R}u_i^m)^\mathsf{T}(e_i - \hat{R}u_i^m) \]  

with respect to \( \hat{R} \in \mathbb{SO}(3) \), where the weights \( w_i > 0 \). Defining the trace inner product on \( \mathbb{R}^{m \times n} \) as

\[ \langle A_1, A_2 \rangle = \text{trace}(A_1^\mathsf{T}A_2), \]  

we can re-express equation (73) for Wahba’s cost function as

\[ U^0(\hat{R}, U^m) = \frac{1}{2} \langle E - \hat{R}U^m, (E - \hat{R}U^m)W \rangle, \]  

where \( U^m \) is given by equation (114), \( E \) is given by (115), and \( W = \text{diag}(w_i) \) is the positive diagonal matrix of the weight factors for the measured directions.

From the expression (75), note that \( W \) may be generalized to be any positive definite matrix, not necessarily diagonal. Another generalization of Wahba’s cost function is given by

\[ U(\hat{R}, U^m) = \Phi\left( \frac{1}{2} \langle E - \hat{R}U^m, (E - \hat{R}U^m)W \rangle \right), \]  

where \( \Phi : [0, \infty) \mapsto [0, \infty) \) is a \( C^2 \) function that satisfies \( \Phi(0) = 0 \) and \( \Phi'(x) > 0 \) for all \( x \in [0, \infty) \). Furthermore, \( \Phi'(\cdot) \leq \alpha(\cdot) \) where \( \alpha(\cdot) \) is a Class-\( \mathcal{K} \) function.
Note that these properties of $\Phi(\cdot)$ ensure that the indices $U^0(\hat{R}, U^m)$ and $U(\hat{R}, U^m)$ have the same minimizer $\hat{R} \in \text{SO}(3)$. In other words, minimizing the cost $U$, which is a generalization of the cost $U^0$, is equivalent to solving Wahba’s problem. Here, $W$ is positive definite (not necessarily diagonal), and $E$ and $U^m$ are assumed to be of rank 3, which is true under the assumption that $k \geq 2$ vectors are measured. The solution to Wahba’s problem is given in [68] and [55].

4.1.2 Properties of Wahba’s Cost Function in the Absence of Measurement Errors

In the absence of measurement errors, $U^m = U = R^T E$, and let $Q = R\hat{R}^T \in \text{SO}(3)$ denote the attitude estimation error. The following lemmas give the structure of Wahba’s cost function in this case.

**Lemma 4.1.** Let $\text{rank}(E) = 3$, where $E$ is as defined in (115). Let the singular value decomposition of $E$ be given by

$$E = U_E \Sigma_E V_E^T$$

where $U_E \in \text{O}(3)$, $V_E \in \text{O}(m)$,

$$\Sigma_E \in \text{Diag}^+(3, m),$$

(77)

and $\text{Diag}^+(n_1, n_2)$ is the vector space of $n_1 \times n_2$ matrices with positive entries along the main diagonal and all other components zero. Let $\sigma_1, \sigma_2, \sigma_3$ denote the main diagonal entries of $\Sigma_E$. Further, let the positive definite weight matrix $W$
in the generalization of Wahba’s cost function (76) be given by

\[ W = V_E W_0 V_E^T \]  \quad \text{where } W_0 \in \text{Diag}^+(m,m) \]  

(78)

and the first three diagonal entries of \( W_0 \) are given by

\[ w_1 = \frac{d_1}{\sigma_1^2}, \quad w_2 = \frac{d_2}{\sigma_2^2}, \quad w_3 = \frac{d_3}{\sigma_3^2} \text{ where } d_1, d_2, d_3 > 0. \]  

(79)

Then \( K = E W E^T \) is positive definite and

\[ K = U_E \Delta U_E^T \]  \quad \text{where } \Delta = \text{diag}(d_1, d_2, d_3), \]  

(80)

is its eigendecomposition. Moreover, if \( d_i \neq d_j \) for \( i \neq j \) and \( i, j \in \{1, 2, 3\} \), then \( \langle I - Q, K \rangle \) is a Morse function whose critical points are

\[ Q \in \{I, Q_1, Q_2, Q_3\} \]  \quad \text{where } Q_i = 2U_E a_i a_i^T U_E^T - I, \]  

(81)

and \( a_i \) is the \( i \)th column vector of the identity \( I \in \text{SO}(3) \).

**Proof:** It is straightforward to show that (80) holds given (77)-(243). It is shown here that \( \langle I - Q, K \rangle \) has the isolated non-degenerate critical points given by (81). Consider a first differential in \( Q \) given by

\[ \delta Q = Q \Sigma^\times, \]  

(82)

where \( \Sigma \in \mathbb{R}^3 \). The first variation of \( \langle I - Q, K \rangle \) with respect to \( Q \) is given by

\[ \partial_Q \langle I - Q, K \rangle = \langle K, -\delta Q \rangle = \text{trace} \left( \frac{1}{2}(Q^T K - KQ) \Sigma^\times \right) \]
\[ = \frac{1}{2} \langle KQ - Q^T K, \Sigma^\times \rangle = S_K^T(Q) \Sigma, \]  

(83)
where

\[ S_K(Q) = \vex(KQ - Q^T K) \quad (84) \]

and \( \vex(\cdot) : \mathfrak{so}(3) \to \mathbb{R}^3 \) is the inverse of the \((\cdot)^\times\) map. The critical points of \( \langle I - Q, K \rangle \) on \( \text{SO}(3) \) are therefore given by

\[ S_K(Q) = 0 \Rightarrow KQ = Q^T K. \quad (85) \]

Substituting the eigendecomposition of \( K \) given by (80) in equation (85), we obtain

\[ U_E \Delta U_E^T Q = Q^T U_E \Delta U_E^T \Rightarrow \Delta P = P^T \Delta, \quad (86) \]

where \( P = U_E^T Q U_E \in \text{SO}(3) \). Given that \( \Delta \) is a positive diagonal matrix with distinct diagonal entries, the solution set for \( P \) that satisfies the condition (86) is

\[ C_P = \{ I, \text{diag}(1, -1, -1), \text{diag}(-1, 1, -1), \text{diag}(-1, -1, 1) \} \]

\[ = \{ I, 2a_1 a_1^T - I, 2a_2 a_2^T - I, 2a_3 a_3^T - I \}. \quad (87) \]

Thus, the set of critical points of \( \langle I - Q, K \rangle \) is given by

\[ C_Q = U_E C_P U_E^T = \{ I, Q_1, Q_2, Q_3 \}, \quad (88) \]

where \( Q_1, Q_2 \) and \( Q_3 \) are as given by (81). These critical points are clearly isolated. To show that they are non-degenerate, we evaluate the second variation of \( \langle I - Q, K \rangle \) at \( Q \in C_Q \subset \text{SO}(3) \), as follows:

\[ \partial_Q^2 \langle I - Q, K \rangle = -\langle Q^T K, \delta \Sigma^\times \rangle + \langle \Sigma^\times Q^T K, \Sigma^\times \rangle. \]
Since $Q^T K$ is symmetric at the critical points according to (85), and since $\delta \Sigma^\times$ is clearly skew-symmetric, the first term on the right-hand side of the above expression vanishes, as symmetric and skew-symmetric matrices are orthogonal under the trace inner product. Therefore the second variation of $\langle I - Q, K \rangle$ evaluated at the critical points $Q \in C_Q$ is given by

$$
\partial^2_Q \langle I - Q, K \rangle = \langle \Sigma^\times Q^T K, \Sigma^\times \rangle = -\langle Q^T K, (\Sigma^\times)^2 \rangle.
$$

(89)

Since $(\Sigma^\times)^2$ is symmetric, the second variation vanishes for arbitrary non-zero $\Sigma^\times$ if and only if $Q^T K = 0$ for $Q \in C_Q$. However, that possibility would contradict the positive definiteness of $K$, which we have already established. Therefore, the critical points of $\langle I - Q, K \rangle$ are non-degenerate and isolated, which makes this a Morse function on $SO(3)$ [63].

Note that this lemma specifies the weight matrix $W$ according to the SVD of the matrix $E$ and selected eigenvalues $d_1, d_2, d_3 > 0$ for the matrix $K = EWE^T$. As the following lemma shows, these eigenvalues play a special role in determining the overall properties of Wahba’s cost function and its generalization.

Note that since $\langle I - Q, K \rangle$ is a Morse function on $SO(3)$ by Lemma 4.1, by the properties of the function $\Phi$, one can conclude that $\Phi(\langle I - Q, K \rangle) : SO(3) \to \mathbb{R}$ is also a Morse function with the same critical points as those of $\langle I - Q, K \rangle$. The following result gives the characteristics of the critical points of $\Phi(\langle I - Q, K \rangle)$.

**Lemma 4.2.** Let $K = EWE^T$ have the properties given by Lemma 4.1. Then
the critical points of $\Phi(\langle I - Q, K \rangle) : \text{SO}(3) \to \mathbb{R}$ given by (81) consist of a global minimum at the identity $I \in \text{SO}(3)$, a global maximum, and two hyperbolic saddle points whose indices depend on the distinct eigenvalues $d_1, d_2, \text{and} d_3$ of $K$.

Proof: The characteristics of these critical points are obtained from the second variation of $\Phi(\langle I - Q, K \rangle)$ with respect to $Q \in C_Q$, which was obtained in (89). We express (89) as follows:

$$\partial_Q^2 \langle I - Q, K \rangle = \langle \Sigma^\times Q^T K, \Sigma^\times \rangle = F_K^T(Q, \Sigma)\Sigma,$$

where $F_K(Q, \Sigma) = \text{vex}(KQ\Sigma^\times + \Sigma^\times Q^T K)$. To express $F_K(Q, \Sigma)$ as a vector in $\mathbb{R}^3$, the following identity is useful:

$$\text{vex}(A^T\Sigma^\times + \Sigma^\times A) = (\text{trace}(A)I - A)\Sigma,$$

for $A \in \mathbb{R}^{3\times3}$ and $\Sigma \in \mathbb{R}^3$. Using identity (91) in the expression (90), one obtains

$$F_K(Q, \Sigma) = H_K(Q)\Sigma,$$

where

$$H_K(Q) = \text{trace}(Q^T K)I - Q^T K.$$

Note that $H_K(Q)$ corresponds to the Hessian matrix of $\langle I - Q, K \rangle$ for $Q \in C_Q$. Moreover, at the critical points $Q_i (i = 1, 2, 3)$ defined by (81), $\Delta P_i = \Delta(2a_i a_i^T - I)$ is a diagonal matrix that is not positive definite. The Hessian at these critical points
points is therefore evaluated to be:

\[
H_K(Q_i) = U_E \Lambda_i U_E^T, \quad \Lambda_i = \text{trace}(\Delta P_i)I - \Delta P_i, \quad i = 1, 2, 3, \text{ such that }
\]

\[
\Lambda_1 = \text{diag}(-d_2 - d_3, d_1 - d_3, d_1 - d_2),
\]

\[
\Lambda_2 = \text{diag}(d_2 - d_3, -d_3 - d_1, d_2 - d_1),
\]

and \( \Lambda_3 = \text{diag}(d_3 - d_2, d_3 - d_1, -d_1 - d_2) \). \quad (93)

Clearly, the indices of these critical points depend on the distinct eigenvalues \( d_1, d_2 \) and \( d_3 \). For example, if \( d_1 > d_2 > d_3 \), then the index of \( Q_1 \) is one, the index of \( Q_2 \) is two, and the index of \( Q_3 \) is three, which makes \( Q_3 \) the global maximum of \( \langle I - Q, K \rangle : \text{SO}(3) \to \mathbb{R} \). Note that the identity \( I \in \text{SO}(3) \) is the global minimum of this function since the Hessian evaluated at the identity is

\[
H_K(I) = \text{trace}(K)I - K = U_E \Lambda_0 U_E^T,
\]

where \( \Lambda_0 = \text{diag}(d_2 + d_3, d_3 + d_1, d_1 + d_2) \), \quad (94)

and therefore the identity is a critical point with index zero. Finally, note that the second variation of \( \Phi(\langle I - Q, K \rangle) : \text{SO}(3) \to \mathbb{R} \) evaluated at its critical points is given by

\[
\hat{\partial}_Q^2 \Phi(\langle I - Q, K \rangle) = \Phi'(\langle I - Q, K \rangle) \hat{\partial}_Q^2 \langle I - Q, K \rangle
\]

\[
= \Phi'(\langle I - Q, K \rangle) \Sigma^T H_K(Q) \Sigma \quad \text{for } Q \in C_Q.
\]

Since \( \Phi \) is a Class-\( \mathcal{K} \) function, the critical points and their indices are identical for \( \Phi(\langle I - Q, K \rangle) \) and \( \langle I - Q, K \rangle \). \quad \square
4.2 Attitude State Estimation Based on the Lagrange-d’Alembert Principle

Let $\Omega \in \mathbb{R}^3$ be the angular velocity of the rigid body expressed in the body-fixed frame. The attitude kinematics is given by Poisson’s equation:

$$\dot{R} = R\Omega \times.$$  \hspace{1cm} (96)

In order to obtain attitude state estimation schemes from continuous-time vector and angular velocity measurements, we apply the Lagrange-d’Alembert principle to an action functional of a Lagrangian of the state estimate errors, with a dissipation term in the angular velocity estimate error. This section presents an estimation scheme obtained using this approach, as well as stability properties of this estimator.

4.2.1 Action Functional of the Lagrangian of State Estimate Errors

The “energy” contained in the errors between the estimated and the measured inertial vectors is given by $\mathcal{U}(\hat{R}, U^m)$, where $\mathcal{U} : \text{SO}(3) \times \mathbb{R}^{3 \times k} \rightarrow \mathbb{R}$ is defined by (76) and depends on the attitude estimate. The “energy” contained in the vector error between the estimated and the measured angular velocity is given by

$$\mathcal{T}(\hat{\Omega}, \Omega^m) = \frac{m}{2}(\Omega^m - \hat{\Omega})^T (\Omega^m - \hat{\Omega}).$$  \hspace{1cm} (97)
where \( m \) is a positive scalar. One can consider the Lagrangian composed of these “energy” quantities, as follows:

\[
L(\hat{R},U^m,\hat{\Omega},\Omega^m) = T(\hat{\Omega},\Omega^m) - U(\hat{R},U^m)
\]

\[
= \frac{m}{2} (\Omega^m - \hat{\Omega})^T (\Omega^m - \hat{\Omega}) - \Phi\left(\frac{1}{2} \langle E - \hat{R}U^m, (E - \hat{R}U^m)W \rangle\right).
\]

If the estimation process is started at time \( t_0 \), then the action functional of the Lagrangian (98) over the time duration \([t_0, T]\) is expressed as

\[
S(L(\hat{R},U^m,\hat{\Omega},\Omega^m)) = \int_{t_0}^{T} \left( T(\hat{\Omega},\Omega^m) - U(\hat{R},U^m) \right) ds
\]

\[
= \int_{t_0}^{T} \left\{ \frac{m}{2} (\Omega^m - \hat{\Omega})^T (\Omega^m - \hat{\Omega}) - \Phi\left(\frac{1}{2} \langle E - \hat{R}U^m, (E - \hat{R}U^m)W \rangle\right) \right\} ds.
\]

### 4.2.2 Variational Filtering Scheme

Consider attitude state estimation in continuous time in the presence of measurement noise and initial state estimate errors. Applying the Lagrange-d’Alembert principle to the action functional \( S(L(\hat{R},U^m,\hat{\Omega},\Omega^m)) \) given by (99), in the presence of a dissipation term on \( \omega := \Omega^m - \hat{\Omega} \), leads to the following attitude and angular velocity filtering scheme.

**Proposition 4.1.** The filter equations for a rigid body with the attitude kinematics (96) and with measurements of vectors and angular velocity in a body-fixed frame,
are of the form

\[
\begin{align*}
\dot{\hat{R}} &= \hat{R}\dot{\Omega} = \hat{R}(\Omega^m - \omega)^x, \\
m\dot{\omega} &= -m\hat{\Omega} \times \omega + \Phi'(U^0(\hat{R}, U^m))S_L(\hat{R}) - D\omega, \\
\dot{\Omega} &= \Omega^m - \omega,
\end{align*}
\]

(100)

where \(D\) is a positive definite filter gain matrix, \(\hat{R}(t_0) = \hat{R}_0, \omega(t_0) = \omega_0 = \Omega_0^m - \hat{\Omega}_0\), \(S_L(\hat{R}) = \text{vex}(L^T\hat{R} - \hat{R}^T L) \in \mathbb{R}^3\), \(L = EW(U^m)^T\) and \(W\) is chosen to satisfy the conditions in Lemma 4.1.

**Proof:** In order to find a filter equation which reduces the measurement noise in the estimated attitude, one may take the first variation of the action functional (99) with respect to \(\hat{R}\) and \(\hat{\Omega}\). Consider the potential term \(U^0(\hat{R}, U^m)\) as defined by (75). Taking the first variation of this function with respect to \(\hat{R}\) gives

\[
\delta U^0 = \langle -\delta \hat{R}U^m, (E - \hat{R}U^m)W \rangle = \frac{1}{2} \langle \Sigma^x, U^mWE^T \hat{R} - \hat{R}^T EW(U^m)^T \rangle = \frac{1}{2} \langle \Sigma^x, L^T \hat{R} - \hat{R}^T L \rangle = S_L^T (\hat{R}) \Sigma.
\]

(101)

Now consider \(U(\hat{R}, U^m) = \Phi(U^0(\hat{R}, U^m))\). Then,

\[
\delta U = \Phi'(U^0(\hat{R}, U^m))\delta U^0 = \Phi'(U^0(\hat{R}, U^m))S_L^T (\hat{R}) \Sigma.
\]

(102)
Taking the first variation of the kinematic energy term associated with the artificial system (97) with respect to \( \hat{\Omega} \) yields

\[
\delta T = -m(\Omega^m - \hat{\Omega})^T \delta \hat{\Omega} = -m(\Omega^m - \hat{\Omega})^T (\dot{\Sigma} + \hat{\Omega} \times \Sigma) = -m\omega^T (\dot{\Sigma} + \hat{\Omega} \times \Sigma), \tag{103}
\]

where \( \omega = \Omega^m - \hat{\Omega} \). Applying Lagrange-d’Alembert principle leads to

\[
\delta S + \int_{t_0}^{T} \tau_D^T \Sigma dt = 0 \tag{104}
\]

\[
\Rightarrow \int_{t_0}^{T} \left\{ -m\omega^T (\dot{\Sigma} + \hat{\Omega} \times \Sigma) - \Phi'(\mathcal{U}^0(\dot{\hat{R}}, \hat{U}^m)) S_E^T (\hat{R}) \Sigma + \tau_D^T \Sigma \right\} dt = 0 \Rightarrow
\]

\[
-m\omega^T \Sigma \bigg|_{t_0}^{T} + \int_{t_0}^{T} m\dot{\omega}^T \Sigma dt = \int_{t_0}^{T} \left\{ m\omega^T \hat{\Omega}^\times + \Phi'(\mathcal{U}^0(\dot{\hat{R}}, \hat{U}^m)) S_E^T (\hat{R}) - \tau_D^T \right\} \Sigma dt,
\]

where the first term in the left hand side vanishes, since \( \Sigma(t_0) = \Sigma(T) = 0 \), and after replacing the dissipation term \( \tau_D = D\omega \) gives the second equation in (100).

\[\square\]

### 4.2.3 Stability of Filter

Next consider the stability of the estimation scheme (filter) given by Proposition 4.1. The following result shows that this scheme is stable, with almost global convergence of the estimated states to the real states in the absence of measurement noise.

**Theorem 4.1.** The filter presented in Proposition 4.1, with distinct positive eigenvalues for \( K = EW^T E^T \), is asymptotically stable at the estimation error state \( (Q, \omega) = (I, 0) \) in the absence of measurement noise. Further, the domain of attraction of \( (Q, \omega) = (I, 0) \) is a dense open subset of \( \text{SO}(3) \times \mathbb{R}^3 \).
\textit{Proof}: In the absence of measurement noise, \( U^m = U = R^T E \) and therefore \( \mathcal{U}^0(\hat{R}, U^m) = \frac{1}{2} \langle E - \hat{R}U, (E - \hat{R}U)W \rangle = \langle I - Q, K \rangle = \mathcal{U}^0(Q) \) where \( K = EWE^T \) and \( Q = R\hat{R}^T \). Therefore, \( \Phi(\langle I - Q, K \rangle) \), is a Morse function on \( \text{SO}(3) \). The stability of this filter can be shown using the following candidate Morse-Lyapunov function, which can be interpreted as the total energy function (equal in value to the Hamiltonian) corresponding to the Lagrangian (98):

\[
V(\hat{R}, \omega, U) = \Phi\left(\frac{1}{2} \langle E - \hat{R}U, (E - \hat{R}U)W \rangle \right) + \frac{m}{2} \omega^T \omega = \Phi(\langle I - Q, K \rangle) + \frac{m}{2} \omega^T \omega = V(Q, \omega).
\tag{105}
\]

Note that \( V(Q, \omega) \geq 0 \) and \( V(Q, \omega) = 0 \) if and only if \( (Q, \omega) = (I, 0) \). Therefore, \( V(Q, \omega) \) is positive definite on \( \text{SO}(3) \times \mathbb{R}^3 \). Using (96) and (100)

\[
\frac{d}{dt} \Phi(\langle I - Q, K \rangle) = \frac{d}{dt} \Phi(\langle I - R\hat{R}^T, K \rangle)
= \Phi'(\langle I - Q, K \rangle) \langle K, -R\Omega^x \hat{R}^T + R\hat{\Omega}^x \hat{R}^T \rangle
= \Phi'(\langle I - Q, K \rangle) \left( \frac{1}{2} (\hat{R}^T K R - R^T K \hat{R}, \omega^x) \right)
= -\Phi'(\langle I - Q, K \rangle) S^T_L(\hat{R}) \omega.
\tag{106}
\]

Therefore, the time derivative of the candidate Morse-Lyapunov function is

\[
\dot{V}(Q, \omega) = \frac{d}{dt} \Phi(\langle I - Q, K \rangle) + m\omega^T \dot{\omega}
= \omega^T \left( -\Phi'(\mathcal{U}^0(Q)) S_L(\hat{R}) - m\hat{\Omega} \times \omega + \Phi'(\mathcal{U}^0(Q)) S_L(\hat{R}) - D\omega \right).
\tag{107}
\]

Noting that \( m\omega^T (\hat{\Omega} \times \omega) = 0 \), this yields

\[
\dot{V}(Q, \omega) = -\omega^T D\omega.
\tag{108}
\]
Hence, the derivative of the Morse-Lyapunov function is negative semi-definite.

Note that the error dynamics for the attitude estimate error is given by

\[ \dot{Q} = Q\psi \times \text{ where } \psi = \hat{R}\omega, \]  

(109)

while the error dynamics for the angular velocity estimate error \( \omega \) is given by the second of equations (100). Therefore, the error dynamics for \((Q, \omega)\) is non-autonomous, since they depend explicitly on \((\hat{R}, \hat{\Omega})\). Considering (270) and (272) and applying Theorem 8.4 in [45], one can conclude that \( \omega^T D\omega \to 0 \) as \( t \to \infty \), which consequently implies \( \omega \to 0 \). Thus, the positive limit set for this system is contained in

\[ E = \dot{V}^{-1}(0) = \{ (Q, \omega) \in SO(3) \times so(3) : \omega \equiv 0 \}. \]  

(110)

Substituting \( \omega \equiv 0 \) in the filter equations (100), we obtain the positive limit set where \( \dot{V} \equiv 0 \) (or \( \omega \equiv 0 \)) as the set

\[ \mathcal{I} = \{ (Q, \omega) \in SO(3) \times \mathbb{R}^3 : S_K(Q) \equiv 0, \omega \equiv 0 \} \]  

(111)

Therefore, in the absence of measurement errors, all the solutions of this filter converge asymptotically to the set \( \mathcal{I} \). Thus, the attitude estimate error converges to the set of critical points of \( \langle I - Q, K \rangle \) in this intersection. The unique global minimum of this function is at \( (Q, \omega) = (I, 0) \) (Lemma 4.2, see also [68, 70]), so this estimation error is asymptotically stable.
Now consider the set
\[ C = \mathcal{I} \setminus (I, 0), \] (112)
which consists of all stationary states that the estimation errors may converge to, besides the desired estimation error state \((I, 0)\). Note that all states in the stable manifold of a stationary state in \(C\) will converge to this stationary state. From the properties of the critical points \(Q_i \in C_Q \setminus (I)\) of \(\Phi(\langle K, I - Q \rangle)\) given in Lemma 4.2, we see that the stationary points in \(\mathcal{I} \setminus (I, 0) = \{(Q_i, 0) : Q_i \in C_Q \setminus (I)\}\) have stable manifolds whose dimensions depend on the index of \(Q_i\). Since the angular velocity estimate error \(\omega\) converges globally to the zero vector, the dimension of the stable manifold \(\mathcal{M}^S_i\) of \((Q_i, 0)\) is
\[ \dim(\mathcal{M}^S_i) = 3 + (3 - \text{index of } Q_i) = 6 - \text{index of } Q_i. \] (113)
Therefore, the stable manifolds of \((Q, \omega) = (Q_i, 0)\) are three-dimensional, four-dimensional, or five-dimensional, depending on the index of \(Q_i \in C_Q \setminus (I)\) according to (276). Moreover, the value of the Lyapunov function \(V(Q, \omega)\) is non-decreasing (increasing when \((Q, \omega) \notin \mathcal{I}\)) for trajectories on these manifolds when going backwards in time. This implies that the metric distance between error states \((Q, \omega)\) along these trajectories on the stable manifolds \(\mathcal{M}^S_i\) grows with the time separation between these states, and this property does not depend on the choice of the metric on \(\text{SO}(3) \times \mathbb{R}^3\). Therefore, these stable manifolds are embedded (closed) submanifolds of \(\text{SO}(3) \times \mathbb{R}^3\) and so is their union. Clearly, all
states starting in the complement of this union, converge to the stable equilibrium \((Q, \omega) = (I, 0)\); therefore the domain of attraction of this equilibrium is

\[
\text{DOA}\{(I, 0)\} = \text{SO}(3) \times \mathbb{R}^3 \setminus \left\{ \bigcup_{i=1}^{3} M_i^S \right\},
\]

which is a dense open subset of \(\text{SO}(3) \times \mathbb{R}^3\).

\[\square\]

### 4.3 Discrete-Time Variational Estimator

#### 4.3.1 Measurement Model

Consider an interval of time \([t_0, T] \in \mathbb{R}^+\) separated into \(N\) equal-length subintervals \([t_i, t_{i+1}]\) for \(i = 0, 1, \ldots, N\), with \(t_N = T\) and \(t_{i+1} - t_i = h\) is the time step size. Let \((\hat{R}_i, \hat{\Omega}_i) \in \text{SO}(3) \times \mathbb{R}^3\) denote the discrete state estimate at time \(t_i\), such that \((\hat{R}_i, \hat{\Omega}_i) \approx (\hat{R}(t_i), \hat{\Omega}(t_i))\) where \((\hat{R}(t), \hat{\Omega}(t))\) is the exact solution of the continuous-time filter at time \(t \in [t_0, T]\). Rigid body attitude is determined from \(k \in \mathbb{N}\) known inertial vectors measured in a coordinate frame fixed to the rigid body. Let these vectors at time \(t_i\) be denoted as \(u_{ji}^m\) for \(j = 1, 2, \ldots, k\), in the body-fixed frame. The assumption that \(k \geq 2\) is necessary for instantaneous three-dimensional attitude determination. When \(k = 2\), the cross product of the two measured vectors is considered as a third measurement for applying the attitude estimation scheme. Denote the corresponding known inertial vectors as seen from the rigid body at time \(t_i\) as \(e_{ji}\), and let the true vectors in the body frame at the same time instance be denoted \(u_{ji} = R_i^T e_{ji}\), where \(R_i\) is the rotation
matrix from the body frame to the inertial frame at time $t_i$. This rotation matrix provides a coordinate-free, global and unique description of the attitude of the rigid body. Define the matrix composed of all $k$ measured vectors expressed in the body-fixed frame at time $t_i$ as column vectors,

$$U^m_i = [u^m_{1i} \ u^m_{2i} \ u^m_{1i} \times u^m_{2i}] \text{ when } k = 2, \text{ and}$$

$$U^m_i = [u^m_{1i} \ u^m_{2i} \ u^m_{3i}] \in \mathbb{R}^{3 \times k} \text{ when } k > 2,$$

and the corresponding matrix of all these vectors at the same time instance expressed in the inertial frame as

$$E_i = [e_{1i} \ e_{2i} \ e_{1i} \times e_{2i}] \text{ when } k = 2, \text{ and}$$

$$E_i = [e_{1i} \ e_{2i} \ \ldots \ e_{ki}] \in \mathbb{R}^{3 \times k} \text{ when } k > 2.$$

Note that the matrix of the actual body vectors $u_{ji}$ corresponding to the inertial vectors $e_{ji}$, is given by

$$U_i = R_i^T E_i = [u_{1i} \ u_{2i} \ u_{1i} \times u_{2i}] \text{ when } k = 2, \text{ and}$$

$$U_i = R_i^T E_i = [u_{1i} \ u_{2i} \ \ldots \ u_{ki}] \in \mathbb{R}^{3 \times k} \text{ when } k > 2.$$

4.3.2 Discrete-Time Lagrangian

Let $\Omega_i \in \mathbb{R}^3$ be the angular velocity of the rigid body at time $t_i$ expressed in the body-fixed frame. The attitude kinematics is given by Poisson’s equation:

$$\dot{R}_i = R_i \Omega^x_i.$$

(116)
The term encapsulating the “energy” in the attitude estimate error is discretized as follows:

$$
\mathcal{U}(\hat{R}_i, U_{im}) = \Phi \left( \mathcal{U}^0(\hat{R}_i, U_{im}) \right) = \Phi \left( \frac{1}{2} (E_i - \hat{R}_i U_{im}, (E_i - \hat{R}_i U_{im}) W_i) \right), 
$$

(117)

where

$$
\langle A_1, A_2 \rangle = \text{trace}(A_1^T A_2)
$$

(118)
denotes the trace inner product on $\mathbb{R}^{m \times n}$, $W_i = \text{diag}(w_{ji})$ is the positive diagonal matrix of the weight factors for the measured directions at time $t_i$ which satisfies the eigendecomposition condition in [41], and $\Phi : [0, \infty) \mapsto [0, \infty)$ is a $C^2$ function that satisfies $\Phi(0) = 0$ and $\Phi'(x) > 0$ for all $x \in [0, \infty)$. Furthermore, $\Phi'(\cdot) \leq \alpha(\cdot)$ where $\alpha(\cdot)$ is a Class-$K$ function.

The term containing the “energy” in the angular velocity estimate error is discretized as

$$
\mathcal{T}(\hat{\Omega}_i, \Omega_i^m) = \frac{m}{2} (\Omega_i^m - \hat{\Omega}_i)^T (\Omega_i^m - \hat{\Omega}_i), 
$$

(119)

where $m$ is a positive scalar.

As with the continuous-time state estimation process in [41], one can express these “energy” terms in the state estimate errors for the case that perfect measurements (with no measurement noise) are available. In this case, these “energy” terms can be expressed in terms of the state estimate errors $Q_i = R_i \hat{R}_i^T$ and
\( \omega_i = \Omega_i - \dot{\Omega}_i \) as follows:

\[
\mathcal{U}(Q_i) = \Phi \left( \frac{1}{2} (E_i - Q_i^T E_i, (E_i - Q_i^T E_i) W_i) \right) = \Phi \left( (I - Q_i, K_i) \right)
\]

where \( K_i = E_i W_i E_i^T \), and \( T(\omega_i) = \frac{m}{2} \omega_i^T \omega_i \) where \( m > 0 \). \quad (120)

The weights in \( W_i \) can be chosen such that \( K_i \) is always positive definite with distinct (perhaps constant) eigenvalues, as in the continuous-time filter. Using these “energy” terms in the state estimate errors, the discrete-time Lagrangian can be expressed as:

\[
\mathcal{L}(Q_i, \omega_i) = T(\omega_i) - \mathcal{U}(Q_i) = \frac{m}{2} \omega_i^T \omega_i - \Phi \left( (I - Q_i, K_i) \right).
\]

(121)

In order to numerically implement the filtering scheme introduced in this paper, a discrete-time version is obtained to estimate the attitude states from vector measurements and angular velocity measurements. It is assumed that these measurements are obtained in discrete-time at a sufficiently high but constant sample rate. In this section, a discrete-time version of the filter introduced in Proposition 4.1 is obtained in the form of a Lie group variational integrator (LGVI). A variational integrator works by discretizing the (continuous-time) variational mechanics principle that leads to the equations of motion, rather than discretizing the equations of motion directly. A good background on variational integrators is given in the excellent treatise [59]. The correspondence between variational integrators and symplectic integrators (for conservative systems) is given in the
book [37]. Lie group variational integrators are variational integrators for mechanical systems whose configuration spaces are Lie groups, like rigid body systems. In addition to maintaining properties arising from the variational principles of mechanics, like energy and momenta, LGVI schemes also maintain the geometry of the Lie group that is the configuration space of the system [51].

4.3.3 Discrete-Time Lagrangian

As a first step to obtaining the LGVI that discretizes the filter in Proposition 4.1, a discrete-time counterpart of the (continuous-time) Lagrangian expressed in (98) is obtained. Consider an interval of time \([t_0, T] \in \mathbb{R}^+\) separated into \(N\) equal-length subintervals \([t_i, t_{i+1}]\) for \(i = 0, 1, \ldots, N\), with \(t_N = T\) and \(t_{i+1} - t_i = h\) is the time step size. Let \((\hat{R}_i, \hat{\Omega}_i) \in \text{SO}(3) \times \mathbb{R}^3\) denote the discrete state estimate at time \(t_i\), such that \((\hat{R}_i, \hat{\Omega}_i) \approx (R(t_i), \Omega(t_i))\) where \((\hat{R}(t), \hat{\Omega}(t))\) is the exact solution of the continuous-time filter at time \(t \in [t_0, T]\).

It is assumed that \(k \geq 2\) known inertial vectors are measured in the body frame, as in Proposition 4.1. The term encapsulating the “energy” in the attitude estimate error, given by (76), is discretized as follows:

\[
\mathcal{U}(\hat{R}_i, U_i^m) = \Phi \left( \frac{1}{2} (E_i - \hat{R}_i U_i^m, (E_i - \hat{R}_i U_i^m) W_i) \right),
\]

(122)

where \(E_i \in \mathbb{R}^{3 \times k}\) is the set of inertial vectors and \(U_i^m \in \mathbb{R}^{3 \times k}\) is the corresponding set of measured body vectors observed at time \(t_i\), and \(W_i\) is the corresponding
diagonal matrix of weight factors. The term containing the “energy” in the angular velocity estimate error is discretized as

\[ T(\hat{\Omega}_i, \Omega^m_i) = \frac{m}{2}(\Omega^m_i - \hat{\Omega}_i)^T(\Omega^m_i - \hat{\Omega}_i), \]  

(123)

which is the discrete-time version of equation (97).

As with the continuous-time state estimation process in Sections 2 and 3, one can express these “energy” terms in the state estimate errors for the case that perfect measurements (with no measurement noise) are available. In this case, these “energy” terms can be expressed in terms of the state estimate errors \( Q_i = R_i \hat{R}_i^T \) and \( \omega_i = \Omega_i - \hat{\Omega}_i \) as follows:

\[ U(Q_i) = \Phi\left(\frac{1}{2}\langle E_i - Q_i^T E_i, (E_i - Q_i^T E_i)W_i \rangle\right) = \Phi\left(\langle I - Q_i, K_i \rangle\right) \]

(124)

where \( K_i = E_i W_i E_i^T \), and \( T(\omega_i) = \frac{m}{2} \omega_i^T \omega_i \) where \( m > 0 \).

The weights in \( W_i \) can be chosen such that \( K_i \) is always positive definite with distinct (perhaps constant) eigenvalues, as in the continuous-time filter given by Proposition 4.1. Using these “energy” terms in the state estimate errors, the discrete-time Lagrangian can be expressed as:

\[ L(Q_i, \omega_i) = T(\omega_i) - U(Q_i) = \frac{m}{2} \omega_i^T \omega_i - \Phi\left(\langle I - Q_i, K_i \rangle\right). \]  

(125)

4.3.4 Discrete-Time Attitude State Estimation Based on the Discrete Lagrange-d’Alembert Principle

The following statement gives the discrete-time filter equations, in the form of a Lie group variational integrator, corresponding to the continuous-time filter given
by Proposition 4.1.

**Proposition 4.2.** Let two or more vector measurements be available, along with angular velocity measurements in discrete-time, at time intervals of length $h$. Further, let the weight matrix $W_i$ for the set of vector measurements $E_i$ be chosen such that $K_i = E_i W_i E_i^T$ satisfies the eigendecomposition condition (80) of Lemma 4.1. A discrete-time filter that approximates the continuous-time filter of Proposition 4.1 to first order in $h$ is

\[
\hat{R}_{i+1} = \hat{R}_i \exp(h\hat{\Omega}_i^x) = \hat{R}_i \exp \left( h(\Omega_i^m - \omega_i)^x \right),
\]

(126)

\[
m\omega_{i+1} = \exp(-h\hat{\Omega}_i^x) \left\{ (mI_{3 \times 3} - hD)\omega_i + h\Phi'(U^0(\hat{R}_{i+1}, U^m_{i+1}))S_{L_i+1}(\hat{R}_{i+1}) \right\},
\]

(127)

\[
\hat{\Omega}_i = \Omega_i^m - \omega_i,
\]

(128)

where $S_{L_i}(\hat{R}_i) = \text{vex}(L_i^T \hat{R}_i - \hat{R}_i^T L_i) \in \mathbb{R}^3$, $L_i = E_i W_i (U_i^m)^T \in \mathbb{R}^{3 \times 3}$ and $(\hat{R}_0, \hat{\Omega}_0) \in SO(3) \times \mathbb{R}^3$ are initial estimated states.

Proof: The action functional in expression (99) is replaced by the discrete-time action sum as follows:

\[
S_d(\mathcal{L}(\hat{R}_i, U_i^m, \hat{\Omega}_i, \Omega_i^m)) = h \sum_{i=0}^{N} \left\{ \frac{m}{2}(\Omega_i^m - \hat{\Omega}_i)^T(\Omega_i^m - \hat{\Omega}_i) - \Phi(U^0(\hat{R}_i, U_i^m)) \right\}.
\]

(129)

Discretize the kinematics of the attitude estimate as

\[
\hat{R}_{i+1} = \hat{R}_i \exp(h\hat{\Omega}_i^x),
\]

(130)
and consider a first variation in the discrete attitude estimate, \( R_i \), of the form
\[
\delta R_i = \hat{R}_i \Sigma_i^x, \tag{131}
\]
where \( \Sigma_i \in \mathbb{R}^3 \) gives a variation vector for the discrete attitude estimate. For fixed end-point variations, we have \( \Sigma_0 = \Sigma_N = 0 \). Further, a first order approximation is to assume that \( \hat{\Omega}_i^x \) and \( \delta \hat{\Omega}_i^x \) commute. With this assumption, taking the first variation of the discrete kinematics (130) and substituting from (131) gives:
\[
\delta \hat{R}_{i+1} = \delta \hat{R}_i \exp(h \hat{\Omega}_i^x) + \hat{R}_i \delta \left( \exp(h \hat{\Omega}_i^x) \right)
\]
\[
= \hat{R}_i \Sigma_i^x \exp(h \hat{\Omega}_i^x) + h \hat{R}_i \exp(h \hat{\Omega}_i^x) \delta \hat{\Omega}_i^x = \hat{R}_{i+1} \Sigma_{i+1}^x, \tag{132}
\]
Equation (132) can be re-arranged to obtain:
\[
h \delta \hat{\Omega}_i^x = \exp(-h \hat{\Omega}_i^x) \hat{R}_i^T \left[ \delta \hat{R}_{i+1} - \hat{R}_i \Sigma_i^x \exp(h \hat{\Omega}_i^x) \right] = \exp(-h \hat{\Omega}_i^x) \hat{R}_i^T \hat{R}_i \Sigma_{i+1}^x - \text{Ad}_{\exp(-h \hat{\Omega}_i^x)} \Sigma_i^x
\]
\[
= \Sigma_{i+1}^x - \text{Ad}_{\exp(-h \hat{\Omega}_i^x)} \Sigma_i^x. \tag{133}
\]
This in turn can be expressed as an equation on \( \mathbb{R}^3 \) as follows:
\[
h \delta \hat{\Omega}_i = \Sigma_{i+1} - \exp(-h \hat{\Omega}_i^x) \Sigma_i, \tag{134}
\]
since \( \text{Ad}_R \Omega^x = R \Omega^x R^T = (R \Omega)^x \).

Applying the discrete Lagrange-d’Alembert principle [59], one obtains
\[
\delta S_d + h \sum_{i=0}^{N-1} \tau_{D_i}^T \Sigma_i = 0
\]
\[
\Rightarrow h \sum_{i=0}^{N-1} m(\hat{\Omega}_i - \Omega_i^m)^T \delta \hat{\Omega}_i - \left\{ \Phi^0(\hat{R}_i, U_i^m) S_{L_i}^T(\hat{R}_i) - \tau_{D_i}^T \right\} \Sigma_i = 0. \tag{135}
\]
Substituting (131) and (134) into equation (135), one obtains

\[
\sum_{i=0}^{N-1} \left\{ m(\hat{\Omega}_i - \Omega^m_i)^T (\Sigma_{i+1} - \exp(-h\hat{\Omega}_i^x) \Sigma_i) \\
- h\Phi'(U^0(\hat{R}_i, \hat{U}^m_i)) S^T_{L_i}(\hat{R}_i) \Sigma_i + h\tau^T_{D_i} \Sigma_i \right\} = 0. 
\] (136)

For \(0 \leq i < N\), the expression (136) leads to the following one-step first-order LGVI for the discrete-time filter:

\[
m(\Omega^m_{i+1} - \hat{\Omega}_{i+1})^T \exp(-h\hat{\Omega}_{i+1}^x) + h\tau^T_{D_{i+1}} \\
- h\Phi'(U^0(\hat{R}_{i+1}, \hat{U}^m_{i+1})) S^T_{L_{i+1}}(\hat{R}_{i+1}) + m(\hat{\Omega}_i - \Omega^m_i)^T = 0
\]

\[\Rightarrow m \exp(h\hat{\Omega}_{i+1}^x)(\Omega^m_{i+1} - \hat{\Omega}_{i+1}) = m(\Omega^m_i - \hat{\Omega}_i) \\
+ h \left( \Phi'(U^0(\hat{R}_{i+1}, \hat{U}^m_{i+1})) S_{L_{i+1}}(\hat{R}_{i+1}) - \tau_{D_{i+1}} \right), \] (137)

which after substituting \(\omega_i = \Omega^m_i - \hat{\Omega}_i\) and \(\tau_{D_{i+1}} = D\omega_i\) gives the discrete-time filter presented in (126)-(128).

Note that the filter equations (126)-(128) given by the LGVI scheme are in the form of an implicit numerical integration scheme. The discrete kinematics equation, which is the equation (126), is solved first. Then the angular velocity estimate error is updated by solving the implicit discrete dynamics equation, which is the equation (127). The stability and convergence properties of this discrete-time filter are not shown here directly. Since this filter is a first-order discretization of the continuous-time filter in [41] which is almost globally asymptotically stable, its solution will be a first-order (in \(h\)) approximation to the continuous-time filter.
4.3.5 Discrete-time First Order Butterworth Filter for Angular Velocity

Since the proposed filter does not filter noise from angular velocity measurements, a symmetric linear filter in the form of a discrete first-order Butterworth filter is applied to these measurements. The filtered velocities are then used in place of the unfiltered $\Omega_i^m$ to enhance the nonlinear filter given by equations (126)-(128). This time-symmetric filter is of the form:

$$(2 + h)\bar{\Omega}_{i+1} = (2 - h)\bar{\Omega}_i + h(\Omega_i^m + \Omega_{i+1}^m), \quad (138)$$

where $\bar{\Omega}_i$ is the filtered angular velocity at time $t_i$ for $i = 0, 1, ..., N - 1$, and $\bar{\Omega}_0 = \Omega^m(t_0)$. $\bar{\Omega}_i$ is used in place of $\Omega_i^m$ in (126)-(128).

The stability and convergence properties of this discrete-time filter are not shown here directly. Since this filter is a first-order discretization of the continuous-time filter in Proposition 4.1, its solution will be a first-order (in $h$) approximation to the continuous-time filter.

4.3.6 Explicit First-Order Estimator

Note that the second equation in the first-order Lie group variational integrator, eq. (127), is an implicit equation with respect to $\omega_{i+1}$. One needs to solve this equation using an iterative method like Newton-Raphson at every time step. This can considerably increase the computational load and runtime, making the esti-
mator difficult to implement in applications requiring real-time estimation [89].

In order to find a solution for this issue, one can use the adjoint of this LGVI, which provides an explicit first order as given in the following statement.

**Proposition 4.3.** A discrete-time filter that gives an explicit first order numerical integrator for the filter presented in [41] is given by:

\[
\hat{R}_{i+1} = \hat{R}_i \exp \left( h(\Omega_{i+1}^m - \omega_{i+1})\right),
\]

\[
\omega_{i+1} = (mI_{3 \times 3} + hD)\left\{ \exp (-h\hat{\Omega}_i^x) m\omega_i + h\Phi'(U^0(\hat{R}_i, U_i^m)) S_{L_i}(\hat{R}_i) \right\},
\]

\[
\hat{\Omega}_i = \Omega_i^m - \omega_i,
\]

where \(S_{L_i}(\hat{R}_i)\) is defined in Proposition 8.3 and \((\hat{R}_0, \hat{\Omega}_0) \in SO(3) \times \mathbb{R}^3\) are initial estimated states.

**Proof.** Let \(\Xi_h\) denote the forward time map of the Lie group variational integrator given by equations (126)-(128) of the filter in Proposition 8.3. The adjoint of a numerical integration method whose one step forward time map is denoted \(\Xi_h\), is defined as \(\Xi_h^* = \Xi_{-h}^{-1} [37]\) for the time interval \([t_i, t_{i+1}]\). In other words, the adjoint scheme is obtained by interchanging indices \(i\) and \(i+1\) and replacing \(h\) with \(-h\) in the original scheme. This adjoint flow can be constructed from (126)-(128) as

\[
\hat{R}_{i+1} = \hat{R}_i \exp \left( h(\Omega_{i+1}^m - \omega_{i+1})\right),
\]

\[
(mI_{3 \times 3} + hD)\omega_{i+1} = \exp(-h\hat{\Omega}_i^x) m\omega_i + h\Phi'(U^0(\hat{R}_i, U_i^m)) S_{L_i}(\hat{R}_i).
\]

The filter equations (139)-(141) are easily concluded from (142)-(143). \(\square\)
One can easily verify that the trajectories of both the implicit and the explicit first order filters are very similar using numerical simulations. However, the explicit filter is computationally simpler and faster, which makes it more suited for real-time implementation. A second order discretization of this variational estimator is presented next.

4.3.7 Symmetric Numerical Integrator as Discrete-Time Filter

A symmetric numerical scheme is presented here as a higher-order discretization of the continuous-time filter. Symmetric numerical integrators have discrete flows that are time reversible, i.e., the composition of the one step forward time map with the one step backward time map is the identity map. Symmetric schemes have many useful properties (e.g., easier error analysis), as detailed in chapter 5 of [37]. The symmetric scheme presented here is obtained by composing the first-order LGVI scheme from Proposition 8.3, with its adjoint defined in Proposition 4.3. Clearly, a symmetric integration scheme is self-adjoint by definition. The following statement gives the discrete-time attitude and angular velocity filter that is obtained in the form of a symmetric integrator using the above-mentioned composition.

Proposition 4.4. A discrete-time filter that gives a second order numerical inte-
grator for the filter in Proposition 8.3 is given as follows:

\[ \hat{R}_{i+\frac{1}{2}} = \hat{R}_i \exp \left( h \left( \Omega^m_{i+\frac{1}{2}} - \omega_{i+\frac{1}{2}} \right) \right), \quad (144) \]

\[ m\omega_{i+1} = \exp \left( -\frac{h}{2} \hat{\Omega}^\times_{i+1} \right) \left\{ \left( mI_{3\times 3} - \frac{h}{2} D \right) \omega_{i+\frac{1}{2}} + \frac{h}{2} \Phi' \left( \mathcal{U}^0 \left( \hat{R}_{i+1}, U^m_{i+1} \right) \right) S_{L-i+1} \left( \hat{R}_{i+1} \right) \right\}, \quad (145) \]

\[ \hat{\Omega}_i = \Omega^m_i - \omega_i, \quad \Omega^m_{i+\frac{1}{2}} = \frac{1}{2} (\Omega^m_i + \Omega^m_{i+1}), \quad (146) \]

where

\[ \omega_{i+\frac{1}{2}} = \left( mI_{3\times 3} + \frac{h}{2} D \right)^{-1} \left\{ \exp \left( -\frac{h}{2} \hat{\Omega}^\times_i \right) m\omega_i + \frac{h}{2} \Phi' \left( \mathcal{U}^0 \left( \hat{R}_i, U^m_i \right) \right) S_L \left( \hat{R}_i \right) \right\}, \quad (147) \]

is a discrete-time approximation to the angular velocity at time \( t_{i+\frac{1}{2}} := t_i + \frac{h}{2} \).

**Proof.** As described in chapter 2 of [37], a second-order symmetric integrator is obtained by composing the flow of this first-order LGVI with its adjoint to obtain the forward time map:

\[ \Psi_h := \Xi_{h/2} \circ \Xi_h^{*}. \quad (148) \]

This composition method is referred to as the *Strang splitting* in [37]. The flow \( \Xi_h^{*} = \Xi_{-h/2}^{-1} \), for the time interval \([t_i, t_i + \frac{h}{2}]\), can be constructed from (126)-(128) as

\[ \hat{R}_{i+\frac{1}{2}} = \hat{R}_i \exp \left( \frac{h}{2} \left( \Omega^m_{i+\frac{1}{2}} - \omega_{i+\frac{1}{2}} \right) \right), \quad (149) \]

\[ (mI_{3\times 3} + \frac{h}{2} D)\omega_{i+\frac{1}{2}} = \exp \left( -\frac{h}{2} \hat{\Omega}^\times_i \right) m\omega_i + \frac{h}{2} \Phi' \left( \mathcal{U}^0 \left( \hat{R}_i, U^m_i \right) \right) S_L \left( \hat{R}_i \right). \quad (150) \]
Note that equations (149) and (150) give an explicit integrator, where (150) can be solved first to obtain $\omega_{i+\frac{1}{2}}$, following which (149) can be used to solve for $\hat{R}_{i+\frac{1}{2}}$. Besides, $\Omega^m_{i+\frac{1}{2}}$ could be approximated as the average of $\Omega^m_{i}$ and $\Omega^m_{i+1}$. The flow $\Xi_{h/2}$ is easily obtained from (126)-(128) as follows:

\[
\hat{R}_{i+1} = \hat{R}_{i+\frac{1}{2}} \exp \left( \frac{h}{2} (\Omega^m_{i+\frac{1}{2}} - \omega_{i+\frac{1}{2}})^x \right),
\]

\[
m\omega_{i+1} = \exp \left( -\frac{h}{2} \Omega^x_{i+1} \right) \left\{ (mI_{3x3} - \frac{h}{2} D)\omega_{i+\frac{1}{2}} + \frac{h}{2} \Phi'(U^0(\hat{R}_{i+1}, U^m_{i+1})) S L_{i+1}(\hat{R}_{i+1}) \right\}.
\]

Composing the discrete-time flows given by (149)-(150) and (151)-(152), in the order specified by (148), gives rise to the one-step forward time map given as in (144)-(146) and (147). The overall integration scheme given by (144)-(147) is implicit because (152) is implicit.

4.4 Numerical Simulations

4.4.1 First-order variational integrator

This section presents numerical simulation results of the discrete time estimator presented in Section 4.3, which is a first order Lie group variational integrator. The estimator is simulated over a time interval of $T = 300$ s, with a time stepsize of $h = 0.01$ s. The rigid body is assumed to have an initial attitude and angular velocity given by,

\[
R_0 = \exp_{SO(3)} \left( \left( \frac{\pi}{4} \times \left[ \frac{3}{7} \frac{6}{7} \frac{2}{7} \right]^T \right)^x \right),
\]
\[ \Omega_0 = \frac{\pi}{60} \times [-2.1 \ 1.2 \ -1.1]^T \text{ rad/s}. \]

The inertia scalar gain is \( m = 100 \) and the dissipation matrix is selected as the following positive definite matrix:

\[ D = \text{diag}([12 \ 13 \ 14]^T). \]

\( \Phi(\cdot) \) could be any \( C^2 \) function with the properties described in Section 2, but is selected to be \( \Phi(x) = x \) here. \( W \) is selected based on the measured set of vectors \( E \) at each instant, such that it satisfies the conditions in Lemma 4.1. The initial estimated states have the following initial estimation errors:

\[ Q_0 = \exp_{\text{SO}(3)} \left( \left( \frac{\pi}{2.5} \times \left[ \frac{3}{7} \frac{6}{7} \frac{2}{7} \right]^T \right)^\times \right), \]

and \( \omega_0 = [0.001 \ 0.002 \ -0.003]^T \text{ rad/s}. \) (153)

We assume that there are at most 9 inertially known directions which are being measured by the sensors fixed to the rigid body at a constant sample rate. The number of observed directions is taken to be variable over different time intervals. The dynamics equations produce the true states of the rigid body, assuming a sinusoidal force is applied to it. These true states are used to simulate the observed directions in the body-fixed frame, as well as the comparison between true and estimated states. Bounded zero mean noises are considered to be added to the true quantities to generate each measured component. A summation of three sinusoidal matrix functions is added to the matrix \( U = R^T E \), to generate a measured \( U^m \).
with measurement noise. The frequency of the noise signals are 1, 10 and 100 Hz, with different phases and amplitudes up to $2.4^\circ$, based on coarse attitude sensors like sun sensors and magnetometers. Similarly, two sinusoidal noise signals of 10 Hz and 200 Hz frequencies are added to $\Omega$ to form the measured $\Omega^m$. These signals also have different phases and their magnitude is up to $0.97^\circ/s$, which is close to the real noise levels for coarse rate gyros. In order to integrate the implicit set of equations in (126)-(128) numerically, the first equation is solved at each sampling step, then the result for $R_{i+1}$ is substituted in the second one. Using the Newton-Raphson method, the resulting equation is solved with respect to $\omega_{i+1}$ iteratively. The root of this nonlinear equation with a specific accuracy along with the $\hat{R}_{i+1}$ is used for the next sampling time instant. This process is repeated to the end of the simulation time. Using the aforementioned quantities and the integration method, the simulation is carried out. The principal angle $\phi$ corresponding to the rigid body’s attitude estimation error $Q$ is depicted in Fig. 28. Components of the estimation error $\omega$ in the rigid body’s angular velocity are shown in Fig. 29. All the estimation errors are seen to converge to a neighborhood of $(Q, \omega) = (I, 0)$, where the size of this neighborhood depends on the bounds of the measurement noise.
Figure 12: Principal Angle of the Attitude Estimation Error

Figure 13: Angular Velocity Estimation Error
4.4.2 Comparison between the First-Order and Second-Order Filters

A set of comparisons between performances of the first- and second-order filters are presented next. The same initial conditions and parameters as in Subsection 4.4.1 are used for both first order and second order filters. The noise type and levels are also identical to those introduced in Subsection 4.4.1. The simulations are carried out for a simulated duration of $T = 300$ s and for three different time stepsizes, namely $h = 0.005$ s, $h = 0.01$ s and $h = 0.05$ s. This shows the effect of the discretization time stepsize on each filter’s convergence behavior. In order to integrate the implicit set of equations in (144)-(146) numerically, $\omega_{i+\frac{1}{2}}$ is substituted from (147), the first equation is solved at each sampling step, then the result for $\hat{R}_{i+1}$ is substituted in the second one. Using the Newton-Raphson method, the resulting implicit equation is solved with respect to $\omega_{i+1}$ iteratively to a set tolerance. The root of this nonlinear equation along with $\hat{R}_{i+1}$ is used for the next sampling time instant. This process is repeated to the end of the simulated duration.

The principal angles of attitude estimate errors are depicted in Fig. 23, Fig. 16 and Fig. 18. In these figures, $\phi_1$ and $\phi_2$ denote the principal angle of attitude estimate error for first and second order observers, respectively. It can be observed that the transient response of the second order filter has less oscillations when compared with the first order integrator. Besides, the higher order estimator has
a smoother behavior in the steady state.

In order to compare the convergence of angular velocity estimate errors of the two discrete filters, the norm of each vector is calculated as shown in Fig. 24, Fig. 17 and Fig. 19 corresponding to \( h = 0.005 \)s, \( h = 0.01 \)s and \( h = 0.05 \)s, respectively. In these figures, the norm of the first and second order filter’s angular velocity estimate errors are denoted by \( \| \omega_1 \| \) and \( \| \omega_2 \| \), respectively. The first order filter always has a higher overshoot and more oscillations in the transient and steady state phases. Moreover, the second order LGVI appears to converges faster to the steady state. This second order filter also shows a more robust behavior throughout the simulation.

Comparing these three pairs of figures (for different time stepsizes), one can notice that with increasing time stepsize \( h \), the difference between the behavior of first order and second order integrators increases, as is expected.

### 4.5 Conclusion

This work obtains an attitude and angular velocity estimation scheme on the Lie group of rigid body rotational motion, assuming that measurements of inertial vectors and angular velocity are available in continuous-time or at a high sample rate in discrete-time. It is shown that Wahba’s cost function for attitude determination from vector measurements can be generalized and cast as a Morse function on the Lie group of rigid body rotations. This Morse function can also be con-
Figure 14: Principal Angles of the Attitude Estimate Errors for $h = 0.005s$

Figure 15: Norms of Angular Velocity Estimate Errors for $h = 0.005s$
Figure 16: Principal Angles of the Attitude Estimate Errors for $h = 0.01s$

Figure 17: Norms of Angular Velocity Estimate Errors for $h = 0.01s$
Figure 18: Principal Angles of the Attitude Estimate Errors for $h = 0.05s$

Figure 19: Norms of Angular Velocity Estimate Errors for $h = 0.05s$
sidered as an artificial potential function. A kinetic energy-like term, quadratic in the angular velocity estimation errors, can be used along with this artificial potential to construct a Lagrangian dependent on state estimation errors. The estimator is obtained by applying the Lagrange-d’Alembert principle and its discretization to this Lagrangian and a dissipation term dependent on the angular velocity estimation error. This estimation scheme is shown to be almost globally asymptotically stable, with estimates converging to actual states in a domain of attraction that is open and dense in the state space. In the presence of bounded measurement noise, the numerical results show that state estimates converge to a bounded neighborhood of the actual states. An implicit first order discrete-time version of the continuous-time estimation algorithm is obtained by applying the discrete Lagrange-d’Alembert principle. An explicit filter is obtained as the adjoint method corresponding to this implicit filter. A symmetric second order version of this estimation algorithm is constructed by composing these two filters. Using a realistic set of data for a rigid body, numerical simulations show that the estimation errors in attitude and angular velocities converge to a bounded neighborhood of \((I,0)\) in the presence of a bounded measurement noise. Some numerical comparison results are presented to show the performances of these filters.
This chapter is adapted from a paper published in Proceedings of the 2015 IEEE International Conference on Robotics and Automation [40]. The author gratefully acknowledges Dr. Amit K. Sanyal, Dr. Vijay Kumar and Ehsan Samiei for their participation.

Abstract Discrete-time estimation of rigid body attitude and angular velocity without any knowledge of the attitude dynamics model, is treated using the discrete Lagrange-d’Alembert principle. Using body-fixed sensor measurements of direction vectors and angular velocity, a Lagrangian is obtained as the difference between a kinetic energy-like term that is quadratic in the angular velocity estimation error and an artificial potential obtained from Wahba’s function. An additional dissipation term that depends on the angular velocity estimation error is introduced, and the discrete Lagrange-d’Alembert principle is applied to the Lagrangian with this dissipation. An implicit and an explicit first-order version of this discrete-time filtering scheme is presented. A comparison of this estimator is made with certain state-of-the-art attitude filters in the absence of bias in sensors readings. Numerical simulations show that the presented observer is robust and
unlike the extended Kalman filter based schemes, its convergence does not depend on the gains values. Ultimately, the variational estimator is shown to be the most computationally efficient attitude observer.

5.1 Measurement Model

The vectors notation and discretization definitions are introduced in Sections 4.3.1. Besides, the measurement model for direction sensors is of the form

\[ u_{ji} = R^T_i e_{ji} + D_j \nu_{ji}, \]  

(154)

where the coefficient matrix \( D_j \in \mathbb{R}^{3 \times 3} \) allows for different weightings of the components of the output measurement error \( \nu_{ji} \). A common assumption is that the matrix \( D_j \) is full rank and \( D_j^{-1} = D_j D_j^T \) is positive definite. Let \( \Omega_i \in \mathbb{R}^3 \) be the angular velocity of the rigid body at time \( t_i \) expressed in the body-fixed frame. The attitude kinematics is given by Poisson’s equation:

\[ \dot{R}_i = R_i \Omega_i^x. \]  

(155)

The measurement model for angular velocity is also as follows

\[ \Omega_i^{m} = \Omega_i + B w_i, \]  

(156)

where \( w_i \in \mathbb{R}^3 \) is the measurement error in angular velocity and \( B \in \mathbb{R}^{3 \times 3} \) allows for different weightings for the components of the unknown input measurement \( w_i \).
5.2 Other State-of-the-Art Filters on SO(3)

Some other observers are available in the literature which can estimate the attitude of the rigid body using the same measurement as explained in Chapter 4. Three estimation schemes are used in comparisons with the variational filter: the geometric approximate minimum-energy (GAME), the multiplicative extended Kalman filter (MEKF) and a constant gain observer (CGO).

5.2.1 GAME Filter

Generalizing Mortensen’s maximum-likelihood filtering scheme, a near-optimal filter is proposed in [93]. This geometric approximate minimum-energy (GAME) filter in continuous form is given by

\[
\dot{\hat{R}} = \hat{R}(\Omega^m - P\ell)\times, \ell = \sum_{j=1}^{k} \left( \Omega_j (\hat{u}_j - u_j) \right) \times \hat{u}_j, \tag{157}
\]

\[
\dot{P} = Q + \mathcal{P}_s \left( P(2\Omega^m - P\ell)\times \right) + P \left( \text{trace} \left( \sum_{j=1}^{k} \mathcal{P}_s (\Omega_j (\hat{u}_j - u_j)\hat{u}_j^T) \right) I_{3\times3} \right.
\]

\[
- \sum_{j=1}^{k} \mathcal{P}_s (\Omega_j (\hat{u}_j - u_j)\hat{u}_j^T) + \sum_{j=1}^{k} \hat{u}_j^T \Omega_j \hat{u}_j^T \bigg) \bigg) \bigg) P, \tag{158}
\]

where $\hat{u}_j = \hat{R}^T e_j$, $Q = BB^T$ with $B$ defined in (156), $\mathcal{P}_s (X) = \frac{1}{2} (X + X^T)$ for $X \in \mathbb{R}^{3\times3}$, $\Omega_j = (\Omega_j \Omega_j^T)^{-1}$ with $\Omega_j$ defined in (154), $\hat{R}(0) = I_{3\times3}$, and $P(0) = \frac{1}{\varphi^2} I_{3\times3}$ where $\varphi^2$ is the variance of the principal angle corresponding to the initial attitude estimate.
5.2.2 The Multiplicative Extended Kalman Filter

The Multiplicative Extended Kalman Filter (MEKF) presented in [56, 79, 80] in continuous form is as follows

\[ \dot{\hat{R}} = \hat{R}(\Omega^m - P\ell)^\times, \quad \ell = \sum_{j=1}^{k} (\mathcal{D}_j(\hat{u}_j - u_j)) \times \hat{u}_j, \] (159)

\[ \dot{P} = Q + P_s(P(2\Omega^m)^\times) - P \left( \sum_{j=1}^{k} \hat{u}_j^\times \mathcal{D}_j \hat{u}_j^\times \right) P, \] (160)

where \( Q, P_s(X) \) and \( \mathcal{D}_j \) are as defined in Subsection 5.2.1, and \( \hat{R}(0) \) and \( P(0) \) are set to the same values as in the GAME filter.

5.2.3 The Constant Gain Observer

The Constant Gain Observer (CGO) presented in [53] in continuous form is also represented as

\[ \dot{\hat{R}} = \hat{R}(\Omega^m - K_P\ell)^\times, \quad \ell = \sum_{j=1}^{k} (u_j \times \hat{u}_j), \] (161)

where \( K_P \) is a constant gain and \( \hat{R}(0) = I_{3\times3} \).

Note that all the filters presented here are in continuous setting and a discretized version of them need to be implemented in numerical simulations, using the measurement model defined in Section 5.1. The discrete-time versions of these estimators presented in [92] use the unit quaternion representation.
5.3 Numerical Simulations and Discussion

The performance of the discrete-time variational estimator is compared against that of the estimation schemes presented in Section 5.2, under identical conditions. This means that all the estimation schemes work on the same rigid-body dynamics, have the same initial estimate errors, equal time steps, and identical measurement noise. The sampling period and the total simulation time are $h = 0.01s$ and $T = 20s$, respectively. A rigid body with prescribed dynamics and inputs is considered. Three inertially known directions are measured by the sensors and form the matrix $E = I_{3 \times 3}$ in inertial frame. These measurements contain known levels of noise, however, all sensors are assumed to be unbiased. The initial rotation matrix is selected randomly with zero mean and a standard deviation of $std_{R_0} = 60^\circ$. The rigid body also has the following angular velocity profile:

$$
\Omega = \begin{bmatrix}
\sin(\frac{2\pi}{15} t) \\
- \sin(\frac{2\pi}{18} t + \frac{\pi}{20}) \\
\cos(\frac{2\pi}{17} t)
\end{bmatrix}
$$

(162)

All the estimators start from the same initial attitude estimate, which is $\hat{R}_0 = I_{3 \times 3}$. The initial angular velocity estimates are also set to be identical, as follows. According to eqs. (157) and (159), the initial angular velocity estimate errors are given by $P(0) \times \ell(0)$ for GAME and MEKF. For the variational estimator, choosing $\omega_0 = P(0) \times \ell(0)$ and for the CGO, choosing $K_p = P(0)$ satisfies this condition. The corresponding initial value for covariance matrices in equations (158) and (160) are chosen as $P(0) = \frac{1}{std_{R_0}} I_{3 \times 3} = \frac{9}{\pi^2} I_{3 \times 3}$. 

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The inertia scalar gain for the variational observer is \( m = 0.5 \) kg and its dissipation matrix is selected as the following positive definite matrix:

\[
D = \text{diag}([1.8 \quad 1.95 \quad 2.1]) \text{ N.s.}
\]

According to [41], \( \Phi(\cdot) \) could be any \( C^2 \) function, and here it is set to \( \Phi(x) = x \).

The weight matrix for the three directions is also

\[
W = \text{diag}([1.67 \quad 1.11 \quad 0.56]) \text{ N.s.}
\]

As discussed in [92], GAME filter is designed based on rotation matrices, but the numerical implementation utilizes unit quaternions. The sensors readings are assumed to be bias-free in all cases. The constant gain for CGO is also set as \( k_P = P(0) \), to make its attitude and angular velocity estimates identical to the other filters. We compare the performance of these filters for two different cases.

### 5.3.1 CASE 1: High Noise Levels

Both directions measurement error \( \nu_j \) and angular velocity measurement error \( w \) are random zero mean signals whose probability distribution follow a bump function with unit maximum. The coefficient matrices \( D_j \) in (154) are chosen equal to 30°. The coefficient matrix \( B \) in (156) is also set as 25°/s. Since different estimation techniques have different (initial) filter gains, the equivalence in these parameters need to be defined reasonably. In this comparison, the filter gains are set such that all filters have the same initial rate of the attitude estimate, or
equivalently they have the same initial angular velocity estimate $\hat{\Omega}_0$. The principal angle profiles of the attitude estimate error are compared in Fig. 28.

Figure 20: Principal Angles of Attitude Estimation Error For Noise Levels In [92].

Based on the behavior of the filters shown in Fig. 28, the variational estimator converges fast enough, if the filter gains are chosen wisely. Although the transient behavior of the LGVI is not the best, the differences of the convergences are not remarkable. The settling time for this filter is as small as the other filters’ settling times. Besides, the steady state phase of the attitude estimate error is as smooth as other observers.
5.3.2 CASE 2: Low Noise Levels, with Filter Gains as Before

In this case, the noise signals are considered to be the same type as the previous case (normally-distributed random zero mean bump functions), but with much smaller levels, which are used in simulation implementations of [41]. These levels are close to common coarse attitude and angular velocity sensors for space applications. All the observer gains are kept the same as in case 1, to see the filters’ performance in the case that these gains are not designed for known noise statistics. The coefficient matrices $D_j$ and $B$ are chosen in such a way that the magnitude of each component of the signals $D_j \nu_j$ and $B \omega_i$ are $2.4^\circ$ and $0.97^\circ$/s, respectively. The principal angle of attitude estimate error for the mentioned filters are plotted in Fig. (29).

A magnified behavior of these filters are depicted in Fig. (22). As can be observed, in this case, the GAME filter and MEKF become singular. On the other hand, the CGO and the variational estimator are stable and filter noise out from the estimates. The settling times are also sufficiently small.

Comparing these two cases, one can conclude that although the MEKF and GAME filter perform nicely in the presence of measurement noise with known distribution and level; however, they may not be stable and their initial gains need to be reset, if the noise signal’s nature changes. Hence, one major downfall of these filters are their dependence on the value of the initial estimator gain.
Figure 21: Principal Angles of Attitude Estimation Error For Noise Levels In [41].

On the contrary, the variational estimator is a robust filter with proven stability regardless of the statistics of the noise. This is mainly because of the almost global asymptotically stable structure of this estimator, as has been shown in [41] using the total energy as a Lyapunov function.

5.3.3 CASE 3: Smaller Initial Estimate Error and Low Noise Levels, with Filter Gains as Before

5.3.4 Discussion

Besides, considering the run-times of these filters, one can notice that the explicit version of the variational observer (which has the same behavior as the implicit
version) is considerably faster than the others. Using the aforementioned initial conditions and filter gains, the run-times are depicted for a simulation duration of 20s for these four filters in Table 1.

| Filter  | GAME   | MEKF   | CGO     | Var. Est. |
|---------|--------|--------|---------|-----------|
| Run-Time| 0.6864 s | 0.6240 s | 0.4304 s | 0.1716 s |

It can be seen from this table that the explicit variational estimator is computationally faster than the other filters used in this comparison. This advantage make this LGVI the best choice in real-time experiments, where the computation...
time is a bottleneck [89].

5.4 Conclusion

This work presents both an implicit and an explicit discrete-time attitude and angular velocity estimation scheme on the Lie group of rigid body rotational motion, assuming that measurements of inertial vectors and angular velocity are available at a high sample rate in discrete-time. A discrete-time filter is obtained, in the form of an implicit first order Lie group variational integrator, by applying the discrete Lagrange-d’Alembert principle to the discrete Lagrangian and a dissipation term dependent on the angular velocity estimation error. An explicit
filter is also derived as the adjoint method corresponding to this implicit filter. The behavior of this estimation scheme is compared with three state-of-the-art observers for attitude estimation. Using a realistic set of data for a rigid body, numerical simulations show that the variational estimator performs as good as other filters, taking less computational budget. Furthermore, unlike the GAME filter and MEKF, it always is stable and its convergence is not dependent on the type and level of measurement noise.

Figure 24: Principal Angles of Attitude Estimation Error For Noise Levels In [41] For Initial Few Steps.
Abstract In this work, the variational attitude estimator is generalized to include angular velocity measurements that have a constant bias in addition to measurement noise. It is shown that the state estimates converge to true states almost globally over the state space if the measurements are perfect. Further, it is shown that the bias estimates converge to the true bias once the state estimates converge to the true states.

6.1 Measurement Model

For rigid body attitude estimation, assume that some inertially-fixed vectors are measured in a body-fixed frame, along with body angular velocity measurements having a constant bias. Let \( k \in \mathbb{N} \) known inertial vectors be measured in the coordinate frame fixed to the rigid body as introduced in 4.1. Moreover, the direction vector measurements are given by

\[
  u_j^m = R^T e_j + \nu_j \quad \text{or} \quad U^m = R^T E + N,
\]

(163)
where $\nu_j \in \mathbb{R}^3$ is an additive measurement noise vector and $N \in \mathbb{R}^{3 \times k}$ is the matrix with $\nu_j$ as its $j^{th}$ column vector.

The attitude kinematics for a rigid body is given by Poisson’s equation (96) and the measurement model for angular velocity is

$$\Omega^m = \Omega + w + \beta,$$

where $w \in \mathbb{R}^3$ is the measurement error in angular velocity and $\beta \in \mathbb{R}^3$ is a vector of bias in angular velocity component measurements, which we consider to be a constant vector.

### 6.2 Attitude State and Bias Estimation Based on the Lagrange-d’Alembert Principle

In order to obtain attitude state estimation schemes from continuous-time vector and angular velocity measurements, we apply the Lagrange-d’Alembert principle to an action functional of a Lagrangian of the state estimate errors, with a dissipation term linear in the angular velocity estimate error. This section presents an estimation scheme obtained using this approach, as well as stability and convergence properties of this estimator.

#### 6.2.1 Lagrangian Constructed from Measurement Residuals

The “energy” contained in the errors between the estimated and the measured inertial vectors is given by $U(\hat{R}, U^m)$, where $U : \text{SO}(3) \times \mathbb{R}^{3 \times k} \to \mathbb{R}$ is defined by
(76) and depends on the attitude estimate. Let $\hat{\Omega} \in \mathbb{R}^3$ and $\hat{\beta} \in \mathbb{R}^3$ denote the estimated angular velocity and bias vectors, respectively. The “energy” contained in the vector error between the estimated and the measured angular velocity is then given by

$$ T(\hat{\Omega}, \Omega^m, \hat{\beta}) = \frac{m}{2} (\Omega^m - \hat{\Omega} - \hat{\beta})^T (\Omega^m - \hat{\Omega} - \hat{\beta}). $$

(165)

where $m$ is a positive scalar. One can consider the Lagrangian composed of these “energy” quantities, as follows:

$$ L(\hat{R}, U^m, \hat{\Omega}, \Omega^m, \hat{\beta}) = T(\hat{\Omega}, \Omega^m, \hat{\beta}) - U(\hat{R}, U^m) $$

$$ = \frac{m}{2} (\Omega^m - \hat{\Omega} - \hat{\beta})^T (\Omega^m - \hat{\Omega} - \hat{\beta}) - \Phi \left( \frac{1}{2} \langle E - \hat{R}U^m, (E - \hat{R}U^m)W \rangle \right). $$

If the estimation process is started at time $t_0$, then the action functional of the Lagrangian (166) over the time duration $[t_0, T]$ is expressed as

$$ S(L(\hat{R}, U^m, \hat{\Omega}, \Omega^m)) = \int_{t_0}^{T} (T(\hat{\Omega}, \Omega^m, \hat{\beta}) - U(\hat{R}, U^m)) ds $$

$$ = \int_{t_0}^{T} \left\{ \frac{m}{2} (\Omega^m - \hat{\Omega} - \hat{\beta})^T (\Omega^m - \hat{\Omega} - \hat{\beta}) - \Phi \left( \frac{1}{2} \langle E - \hat{R}U^m, (E - \hat{R}U^m)W \rangle \right) \right\} ds. $$

6.2.2 Variational Filtering Scheme

Define the angular velocity measurement residual and the dissipation term:

$$ \omega := \Omega^m - \hat{\Omega} - \hat{\beta}, \quad \tau_D = D\omega, $$

(168)

where $D \in \mathbb{R}^{3 \times 3}$ is a positive definite filter gain matrix. Consider attitude state estimation in continuous time in the presence of measurement noise and initial
state estimate errors. Applying the Lagrange-d’Alembert principle to the action functional $S(\mathcal{L}(\hat{R}, U^m, \hat{\Omega}, \Omega^m))$ given by (167), in the presence of a dissipation term linear in $\omega$, leads to the following attitude and angular velocity filtering scheme.

**Theorem 6.1.** The filter equations for a rigid body with the attitude kinematics (96) and with measurements of vectors and angular velocity in a body-fixed frame, are of the form

$$
\begin{align*}
\dot{\hat{R}} &= \hat{R}\hat{\Omega}^\times = \hat{R}((\Omega^m - \omega - \hat{\beta})^\times), \\
\dot{\hat{m}}\omega &= -m\hat{\Omega} \times \omega + \Phi'(U^0(\hat{R}, U^m))S_L(\hat{R}) - D\omega, \\
\hat{\Omega} &= \Omega^m - \omega - \hat{\beta},
\end{align*}
$$

(169)

where $\hat{R}(t_0) = \hat{R}_0$, $\omega(t_0) = \omega_0 = \Omega^m_0 - \hat{\Omega}_0$, $S_L(\hat{R}) = \text{vex}(L^T\hat{R} - \hat{R}^TL) \in \mathbb{R}^3$, $L = EW(U^m)^T$ and $W$ is chosen to satisfy the conditions in Lemma 2.1 of [41].

**Proof:** In order to find an estimation scheme that filters the measurement noise in the estimated attitude, take the first variation of the action functional (167) with respect to $\hat{R}$ and $\hat{\Omega}$ and apply the Lagrange-d’Alembert principle with the dissipative term in (168). Consider the potential term $\mathcal{U}^0(\hat{R}, U^m)$ as defined by
(75). Taking the first variation of this function with respect to $\hat{R}$ gives

$$\delta \mathcal{U}^0 = \langle -\delta \hat{R} U^m, (E - \hat{R} U^m) W \rangle$$

$$= \frac{1}{2} \langle \Sigma, U^m W E^T \hat{R} - \hat{R}^T E W U^m \rangle,$$

$$= \frac{1}{2} \langle \Sigma, L^T \hat{R} - \hat{R}^T L \rangle = S^T_L (\hat{R}) \Sigma. \tag{170}$$

Now consider $\mathcal{U}(\hat{R}, U^m) = \Phi(\mathcal{U}^0(\hat{R}, U^m))$. Then,

$$\delta \mathcal{U} = \Phi'(\mathcal{U}^0(\hat{R}, U^m)) \delta \mathcal{U}^0 = \Phi'(\mathcal{U}^0(\hat{R}, U^m)) S^T_L (\hat{R}) \Sigma. \tag{171}$$

Taking the first variation of the kinematic energy term associated with the artificial system (165) with respect to $\hat{\Omega}$ yields

$$\delta \mathcal{T} = -m (\Omega^m - \hat{\Omega} - \hat{\beta})^T \delta \hat{\Omega} = -m (\Omega^m - \hat{\Omega} - \hat{\beta})^T (\hat{\Sigma} + \hat{\Omega} \times \Sigma)$$

$$= -m \omega^T (\hat{\Sigma} + \hat{\Omega} \times \Sigma), \tag{172}$$

where $\omega$ is as given by (168). Applying Lagrange-d’Alembert principle leads to

$$\delta \mathcal{S} + \int_{t_0}^T \tau_D^T \Sigma dt = 0 \tag{173}$$

$$\Rightarrow \int_{t_0}^T \left\{ -m \omega^T (\hat{\Sigma} + \hat{\Omega} \times \Sigma) - \Phi'(\mathcal{U}^0(\hat{R}, U^m)) S^T_L (\hat{R}) \Sigma + \tau_D^T \Sigma \right\} dt = 0 \Rightarrow$$

$$- m \omega^T \Sigma \bigg|_{t_0}^T + \int_{t_0}^T m \omega^T \Sigma dt = \int_{t_0}^T \left\{ m \omega^T \hat{\Omega}^x + \Phi'(\mathcal{U}^0(\hat{R}, U^m)) S^T_L (\hat{R}) - \tau_D^T \right\} \Sigma dt,$$

where the first term in the left hand side vanishes since $\Sigma(t_0) = \Sigma(T) = 0$. After substituting the dissipation term $\tau_D = D \omega$, one obtains the second equation in (169). \[\square\]
6.3 Stability and Convergence of Variational Attitude Estimator

The variational attitude estimator given by Theorem 6.1 can be used for constant or time-varying bias in the angular velocity measurements given by the measurement model (164). The following analysis gives the stability and convergence properties of this estimator for the case that \( \beta \) in equation 164 is constant.

6.3.1 Stability of Variational Attitude Estimator

Prior to analyzing the stability of this attitude estimator, it is useful and instructive to interpret the energy-like terms used to define the Lagrangian in equation (166) in terms of state estimation errors. The following result shows that the measurement residuals, and therefore these energy-like terms, can be expressed in terms of state estimation errors in the case of perfect measurements.

**Proposition 6.1.** Define the state estimation errors

\[
Q = R\hat{R}^T \quad \text{and} \quad \omega = \Omega - \hat{\Omega} - \hat{\beta},
\]

where \( \hat{\beta} = \beta - \hat{\beta} \).

In the absence of measurement noise, the energy-like terms (76) and (165) can be expressed in terms of these state estimation errors as follows:

\[
\mathcal{U}(Q) = \Phi \left( (I - Q, K) \right) \quad \text{where} \quad K = EWE^T,
\]

\[
\mathcal{T}(\omega) = \frac{m}{2} \omega^T \omega.
\]
Proof: The proof of the above statement is obtained by first substituting $N = 0$ and $w = 0$ in equations (163) and (164), respectively. The resulting expressions for $U^m$ and $\Omega^m$ are then substituted back into equations (76) and (165) respectively. Note that the same variable $\omega$ is used to represent the angular velocity estimation error for both cases: with and without measurement noise. Expression (176) is also derived in [41]. □

The stability of this estimator, for the case of constant rate gyro bias vector $\beta$, is given by the following result.

**Theorem 6.2.** Let $\beta$ in equation (164) be a constant vector. Then the variational attitude estimator given by equations (169), in addition to the following equation for update of the bias estimate:

$$\dot{\hat{\beta}} = \Phi'(U^0(\hat{R},U^m))P^{-1}S_L(\hat{R}),$$

(178)

is Lyapunov stable for $P \in \mathbb{R}^{3 \times 3}$ positive definite.

Proof: To show Lyapunov stability, the following Morse-Lyapunov function is considered:

$$V(U^m, \Omega^m, \hat{R}, \hat{\Omega}, \hat{\beta}) = \frac{m}{2}(\Omega^m - \hat{\Omega} - \hat{\beta})^T(\Omega^m - \hat{\Omega} - \hat{\beta})$$

$$+ \Phi(U^0(\hat{R},U^m)) + \frac{1}{2}(\beta - \hat{\beta})^TP(\beta - \hat{\beta}).$$

(179)

Now consider the case that there is no measurement noise, i.e., $N = 0$ and $w = 0$ in equations (163) and (164), respectively. In this case, the Lyapunov function
(179) can be re-expressed in terms of the errors $\omega$, $Q$ and $\tilde{\beta}$ defined by equations (174)-(175), as follows:

$$V(Q, \omega, \tilde{\beta}) = \frac{m}{2} \omega^T \omega + \Phi(\langle I - Q, K \rangle) + \frac{1}{2} \tilde{\beta}^T P \tilde{\beta}. \quad (180)$$

The time derivative of the attitude estimation error, $Q \in SO(3)$, is obtained as:

$$\dot{Q} = R(\Omega - \hat{\Omega})^x \hat{R}^T = Q(\hat{R}(\omega - \tilde{\beta}))^x, \quad (181)$$

after substituting for $\hat{\Omega}$ from the third of equations (169) in the case of zero angular velocity measurement noise (when $\Omega^m = \Omega + \beta$). The time derivative of the Morse-Lyapunov function expressed as in (180) can now be obtained as follows:

$$\dot{V}(Q, \omega, \tilde{\beta}) = m\omega^T \dot{\omega} - \Phi(\langle I - Q, K \rangle) S_L^T(\hat{R})(\omega - \tilde{\beta}) - \tilde{\beta}^T P \dot{\tilde{\beta}}. \quad (182)$$

After substituting equation (178) and the second of equations (169) in the above expression, one can simplify the time derivative of this Lyapunov function along the dynamics of the estimator as

$$\dot{V}(Q, \omega, \tilde{\beta}) = -\omega^T D\omega \leq 0. \quad (183)$$

The time derivative (183) is negative semi-definite in the states $(Q, \omega, \tilde{\beta}) \in TSO(3) \times \mathbb{R}^3$ of this estimator. This proves the result. \hfill \Box

Lemma 4.1 and 4.2 are required to show the convergence of estimation errors. Lemma 4.1 provides guidelines on how to choose the weight matrix $W$ in Wahba’s

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cost function such that it is a Morse function on SO(3). Choosing $W$ according to these guidelines, which depend on the set of inertial vector observed (denoted $E$ here), ensures that this is a smooth function on SO(3) that has the minimum possible number of critical points as dictated by the Morse lemma [63]. Note that the estimated attitude coincides with the true attitude when $Q = I$, which is the minimum of this Morse function according to Lemma 4.2.

6.3.2 Domain of Convergence of Variational Attitude Estimator

The domain of convergence of this estimator is given by the following result.

**Theorem 6.3.** In the absence of measurement noise, the variational attitude estimator with biased velocity measurements, given by eqs. (169) and (178), converges asymptotically to $(Q, \omega, \tilde{\beta}) = (I, 0, 0) \in TSO(3) \times \mathbb{R}^3$. Further, the domain of attraction is a dense open subset of $TSO(3) \times \mathbb{R}^3$.

**Proof:** Note that the error dynamics for the attitude estimate error is given by

$$\dot{Q} = Q\psi^\times \text{ where } \psi = \hat{R}(\omega - \tilde{\beta}),$$

while the error dynamics for the angular velocity estimate error $\omega$ is given by the second of equations (169) and the bias estimate error dynamics is obtained from (175) and (178) as

$$\dot{\tilde{\beta}} = -\Phi'(\hat{U}(\hat{R}, U^m))P^{-1}S_L(\hat{R}).$$
Therefore, the error dynamics for \((Q, \omega, \tilde{\beta})\) is non-autonomous, since they depend explicitly on \((\hat{R}, \hat{\Omega})\). In the absence of measurement noise, considering (180) and (183) and applying Theorem 8.4 in [45], one can conclude that \(\omega^T D\omega \to 0\) as \(t \to \infty\), which consequently implies \(\omega \to 0\). Thus, the positive limit set for this system is contained in

\[
\mathcal{E} = V^{-1}(0) = \{(Q, \omega, \tilde{\beta}) \in TSO(3) \times \mathbb{R}^3 : \omega \equiv 0\}. \tag{186}
\]

Substituting \(\omega \equiv 0\) in the filter equations (169) in the absence of measurement noise, we obtain the positive limit set where \(\dot{V} \equiv 0\) (or \(\omega \equiv 0\)) as the set

\[
\mathcal{I} = \{(Q, \omega, \tilde{\beta}) \in TSO(3) \times \mathbb{R}^3 : S_K(Q) \equiv 0, \omega \equiv 0, \psi = 0\}
\]

\[
= \{(Q, \omega, \tilde{\beta}) \in SO(3) \times \mathbb{R}^3 : Q \in C_Q, \omega \equiv 0, \tilde{\beta} = 0\}, \tag{187}
\]

where \(\psi \in \mathbb{R}^3\) is defined by (184) and \(C_Q\) is the set of critical points of \(\langle I - Q, K \rangle\), given by equation (12) or (20) in [41]. Therefore, in the absence of measurement errors, all the solutions of this filter converge asymptotically to the set \(\mathcal{I}\). Thus, the attitude estimate error converges to the set of critical points of \(\langle I - Q, K \rangle\) in this intersection. The unique global minimum of the Morse-Lyapunov function \(V\) is at \((Q, \omega, \tilde{\beta}) = (I, 0, 0)\) according to Lemma 4.2; therefore, this state estimation error is asymptotically stable.

Now consider the set

\[
\mathcal{C} = \mathcal{I} \setminus (I, 0, 0), \tag{188}
\]

which consists of all stationary states that the estimation errors may converge to, besides the desired estimation error state \((I, 0, 0)\). Note that all states in the stable
manifold of a stationary state in $C$ will converge to this stationary state. From the properties of the critical points $Q_i \in C_Q \setminus (I)$ of $\Phi(\langle K, I - Q \rangle)$ given in Lemma 4.2, we see that the stationary points in $\mathcal{S} \setminus (I, 0, 0) = \{(Q_i, 0, 0) : Q_i \in C_Q \setminus (I)\}$ have stable manifolds whose dimensions depend on the index of $Q_i$ in $SO(3)$. Since the angular velocity estimate error $\omega$ and the bias estimate error $\tilde{\beta}$ converge globally to the zero vector according to (187), the dimension of the stable manifold $M^S_i$ of $(Q_i, 0, 0) \in SO(3) \times \mathbb{R}^3$ is

$$\dim(M^S_i) = 6 + (3 - \text{index of } Q_i) = 9 - \text{index of } Q_i. \quad (189)$$

Therefore, the stable manifolds of $(Q, \omega, \tilde{\beta}) = (Q_i, 0, 0)$ are six-dimensional, seven-dimensional, or eight-dimensional, depending on the index of $Q_i \in C_Q \setminus (I)$ according to (189). Moreover, the value of the Lyapunov function $V(Q, \omega, \tilde{\beta})$ is non-decreasing (increasing when $(Q, \omega, \tilde{\beta}) \notin \mathcal{S}$) for trajectories on these manifolds going backwards in time. This implies that the metric distance between error states $(Q, \omega, \tilde{\beta})$ along trajectories on the stable manifolds $M^S_i$ grows with the time separation between these states, and this property does not depend on the choice of the metric on $TSO(3) \times \mathbb{R}^3$. Therefore, these stable manifolds are embedded (closed) submanifolds of $TSO(3) \times \mathbb{R}^3$ and so is their union. Clearly, all states starting in the complement of this union, converge to the stable equilibrium $(Q, \omega, \tilde{\beta}) = (I, 0, 0)$; therefore the domain of attraction of this equilibrium is

$$\text{DOA}\{(I, 0, 0)\} = TSO(3) \times \mathbb{R}^3 \setminus \bigcup_{i=1}^{3} M^S_i,$$
which is a dense open subset of $\text{TSO}(3) \times \mathbb{R}^3$. \hfill \Box

6.4 Discrete-Time Estimator Based on the Lagrange-d’Alembert Principle

6.4.1 Discrete-Time Lagrangian

The “energy” in the measurement residual for attitude is discretized as:

$$U(\hat{R}_i, U_i) = \Phi\left(U\left(0, \hat{R}_i, U_i\right)\right) = \Phi\left(\frac{1}{2} \langle E_i - \hat{R}_i U_i, (E_i - \hat{R}_i U_i) W_i \rangle\right), \quad (190)$$

where $\Phi : [0, \infty) \mapsto [0, \infty)$ is as defined in Section 6.2. The “energy” in the angular velocity measurement residual is discretized as

$$\mathcal{T}(\hat{\Omega}_i, \Omega_i) = \frac{m}{2} (\Omega_i - \hat{\Omega}_i - \dot{\beta}_i)^T (\Omega_i - \hat{\Omega}_i - \dot{\beta}_i), \quad (191)$$

where $m$ is a positive scalar.

Similar to the continuous-time attitude estimator in [41], one can express these “energy” terms for the case that perfect measurements (with no measurement noise) are available. In this case, these “energy” terms can be expressed in terms of the state estimate errors $Q_i = R_i \hat{R}_i^T$ and $\omega_i = \Omega_i - \hat{\Omega}_i - \dot{\beta}_i$ as follows:

$$U(Q_i) = \Phi\left(\frac{1}{2} \langle E_i - Q_i^T E_i, (E_i - Q_i^T E_i) W_i \rangle\right) = \Phi\left(\langle I - Q_i, K_i \rangle\right)$$

where $K_i = E_i W_i E_i^T$, and $\mathcal{T}(\omega_i) = \frac{m}{2} \omega_i^T \omega_i$ where $m > 0$. \quad (192)

The weights in $W_i$ can be chosen such that $K_i$ is always positive definite with distinct (perhaps constant) eigenvalues, as in the continuous-time estimator of
Using these “energy” terms in the state estimate errors, the discrete-time Lagrangian is expressed as:

\[
\mathcal{L}(Q_i, \omega_i) = T(\omega_i) - U(Q_i) = \frac{m}{2} \omega_i^T \omega_i - \Phi\left(I - Q_i, K_i\right).
\]  \quad (193)

### 6.4.2 First-Order Discrete-Time Attitude State Estimation Based on the Discrete Lagrange-d’Alembert Principle

The following statement gives a first-order discretization, in the form of a Lie group variational integrator, for the continuous-time estimator of [41].

Proposition 6.2. Let discrete-time measurements for two or more inertial vectors along with angular velocity be available at a sampling period of \( h \). Further, let the weight matrix \( W_i \) for the set of vector measurements \( E_i \) be chosen such that \( K_i = E_i W_i E_i^T \) satisfies Lemma 2.1 in [41]. A discrete-time estimator obtained by applying the discrete Lagrange-d’Alembert principle to the Lagrangian (193) is:

\[
\begin{align*}
\hat{R}_{i+1} &= \hat{R}_i \exp\left(h(\Omega_i^m - \omega_i - \hat{\beta}_i) \times\right), \\
\hat{\beta}_{i+1} &= \hat{\beta}_i + h\Phi'(U^0(\hat{R}_i, U_i^m))P^{-1}S_{L_i}(\hat{R}_i), \\
\hat{\Omega}_i &= \Omega_i^m - \omega_i - \hat{\beta}_i, \\
m\omega_{i+1} &= \exp(-h\hat{\Omega}_i^\times) \left\{ (mI_{3\times3} - hD)\omega_i + h\Phi'(U^0(\hat{R}_{i+1}, U_{i+1}^m))S_{L_{i+1}}(\hat{R}_{i+1}) \right\},
\end{align*}
\]  \quad (194-197)

where \( S_{L_i}(\hat{R}_i) = \text{vex}(L_i^T \hat{R}_i - \hat{R}_i^T L_i) \in \mathbb{R}^3 \), \( \text{vex}(\cdot) : \mathfrak{so}(3) \rightarrow \mathbb{R}^3 \) is the inverse of the \((\cdot)^\times\) map, \( L_i = E_i W_i(U_i^m)^T \in \mathbb{R}^{3\times3} \), \( U^0(\hat{R}_i, U_i^m) \) is defined in (190) and
\((\hat{R}_0, \hat{\Omega}_0) \in \text{SO}(3) \times \mathbb{R}^3\) are initial estimated states.

**Proof:** Equation (168) is discretize as

\[
\omega_i := \Omega_i^m - \hat{\Omega}_i - \hat{\beta}_i, \quad \tau_{D_i} = D\omega_i, \tag{198}
\]

and (178) can be rewritten in discrete-time as

\[
\hat{\beta}_i = \frac{\hat{\beta}_{i+1} - \hat{\beta}_i}{h} = \Phi'(U_0(\hat{R}_i, U_i^m)) P^{-1} S_{L_i}(\hat{R}_i), \tag{199}
\]

The action functional in expression (167) is replaced by the discrete-time action sum as follows:

\[
S_d(L(\hat{R}_i, U_i^m, \hat{\Omega}_i, \Omega_i^m)) = h \sum_{i=0}^{N} \left\{ \frac{m}{2} (\Omega_i^m - \hat{\Omega}_i - \hat{\beta}_i)^T (\Omega_i^m - \hat{\Omega}_i - \hat{\beta}_i) - \Phi(U_0(\hat{R}_i, U_i^m)) \right\}. \tag{200}
\]

Discretize the kinematics of the attitude estimate as

\[
\hat{R}_{i+1} = \hat{R}_i \exp(h\hat{\Omega}_i^\times), \tag{201}
\]

and consider a first variation in the discrete attitude estimate, \(R_i\), of the form

\[
\delta \hat{R}_i = \hat{R}_i \Sigma_i^\times, \tag{202}
\]

where \(\Sigma_i \in \mathbb{R}^3\) gives a variation vector for the discrete attitude estimate. For fixed end-point variations, we have \(\Sigma_0 = \Sigma_N = 0\). Further, a first order approximation is to assume that \(\hat{\Omega}_i^\times\) and \(\delta \hat{\Omega}_i^\times\) commute. With this assumption, taking the first
variation of the discrete kinematics (201) and substituting from (202) gives:

\[
\delta \hat{R}_{i+1} = \delta \hat{R}_i \exp(h\hat{\Omega}^x_i) + \hat{R}_i \delta \left( \exp(h\hat{\Omega}^x_i) \right)
\]

\[
= \hat{R}_i \Sigma_i^x \exp(h\hat{\Omega}^x_i) + h \hat{R}_i \exp(h\hat{\Omega}^x_i) \delta \hat{\Omega}_i^x
\]

\[
= \hat{R}_{i+1} \Sigma^x_{i+1}.
\] (203)

Equation (203) can be re-arranged to obtain:

\[
h\delta \hat{\Omega}_i^x = \exp(-h\hat{\Omega}_i^x) \hat{R}_i^T \left[ \delta \hat{R}_{i+1} - \hat{R}_i \Sigma_i^x \exp(h\hat{\Omega}_i^x) \right]
\]

\[
= \exp(-h\hat{\Omega}_i^x) \hat{R}_i^T \hat{R}_{i+1} \Sigma^x_{i+1} - \text{Ad}_{\exp(-h\hat{\Omega}_i^x)} \Sigma_i^x
\]

\[
= \Sigma^x_{i+1} - \text{Ad}_{\exp(-h\hat{\Omega}_i^x)} \Sigma_i^x.
\] (204)

This in turn can be expressed as an equation on \( \mathbb{R}^3 \) as follows:

\[
h \delta \hat{\Omega}_i = \Sigma_{i+1} - \exp(-h\hat{\Omega}_i^x) \Sigma_i,
\] (205)

since \( \text{Ad}_R \Omega^x = R \Omega^x R^T = (R\Omega)^x \).

Applying the discrete Lagrange-d’Alembert principle [59], one obtains

\[
\delta S_d + h \sum_{i=0}^{N-1} \tau_{D_i}^T \Sigma_i = 0
\]

\[
\Rightarrow h \sum_{i=0}^{N-1} m(\dot{\Omega}_i - \Omega_i^m + \beta_i)^T \delta \hat{\Omega}_i - \left\{ \Phi'(U^0(\hat{R}_i, U_i^m)) S_{L_i}^T(\hat{R}_i) - \tau_{D_i}^T \right\} \Sigma_i = 0. \] (206)

Substituting (202) and (205) into equation (206), one obtains

\[
\sum_{i=0}^{N-1} \left\{ m(\dot{\Omega}_i - \Omega_i^m + \beta_i)^T (\Sigma_{i+1} - \exp(-h\hat{\Omega}_i^x) \Sigma_i)
\right.

\[
- h \Phi'(U^0(\hat{R}_i, U_i^m)) S_{L_i}^T(\hat{R}_i) \Sigma_i + h \tau_{D_i}^T \Sigma_i \right\} = 0.
\] (207)
For $0 \leq i < N$, the expression (207) leads to the following one-step first-order LGVI for the discrete-time filter:

\[
m(\Omega_{i+1}^m - \hat{\Omega}_{i+1} - \hat{\beta}_{i+1})^T \exp(-h\hat{\Omega}_{i+1}^x) + h\tau_{D_{i+1}}^T
- h\Phi'\left(U^0(\hat{R}_{i+1}, U_{i+1}^m)\right)S_{L_{i+1}}^T(\hat{R}_{i+1}) + m(\hat{\Omega}_i - \Omega_i^m + \hat{\beta}_i)^T = 0
\]

\[
\Rightarrow m \exp(h\hat{\Omega}_{i+1}^x)(\Omega_{i+1}^m - \hat{\Omega}_{i+1} - \hat{\beta}_{i+1}) = m(\Omega_i^m - \hat{\Omega}_i - \hat{\beta}_i)
+ h\left(\Phi'\left(U^0(\hat{R}_{i+1}, U_{i+1}^m)\right)S_{L_{i+1}}(\hat{R}_{i+1}) - \tau_{D_{i+1}}\right),
\]

which after substituting $\omega_i = \Omega_i^m - \hat{\Omega}_i - \hat{\beta}_i$ and $\tau_{D_{i+1}} = D\omega_i$ gives the discrete-time equation (197). This equation along with (198), (199) and (201) form the estimator equations (194)-(196).

Note that the estimator equations (194)-(196) given by the LGVI scheme are in the form of an implicit numerical integration scheme. The discrete kinematics (194) is solved first. Then the angular velocity estimate error is updated by solving the implicit discrete dynamics, equation (197). The stability and convergence properties of this discrete-time estimator are not shown here. This estimator is a first-order (in $h$) discretization of the continuous-time estimator given by eqs. (169) and (178), which is almost globally asymptotically stable.

### 6.5 Numerical Simulation

This section presents numerical simulation results of the discrete estimator presented in Section 6.4, in the presence of constant bias in angular velocity measure-
ments. In order to validate the performance of this estimator, a rigid body’s states are artificially generated using the kinematics and dynamics equations. The rigid body moment of inertia is selected to be $J_v = \text{diag}(\begin{bmatrix} 2.56 & 3.01 & 2.98 \end{bmatrix}) \text{ kg.m}^2$.

Moreover, a sinusoidal function is applied to it as the only external torque, which is expressed in body fixed frame as

$$\varphi(t) = \begin{bmatrix} 0 & 0.028 \sin(2.7t - \frac{\pi}{7}) & 0 \end{bmatrix}^T \text{ N.m.} \quad (209)$$

The rigid body is assumed to have an initial attitude and angular velocity given by,

$$R_0 = \expm_{SO(3)} \left( \left( \frac{\pi}{4} \times \begin{bmatrix} 3 & 6 & 2 \end{bmatrix}^T \right)^\times \right)$$

and $\Omega_0 = \frac{\pi}{60} \times [-2.1 \ 1.2 \ -1.1]^T \text{ rad/s.} \quad (210)$

A set of five inertial sensors and three gyros perpendicular to each other are assumed to be onboard the rigid body. The actual states generated from kinematics and dynamics of this rigid body are used to simulate the observed directions in the body fixed frame, as well as the comparison between true and estimated states.

We assume that there are five inertially known directions which are being measured by the five inertial sensors fixed to the rigid body at a constant sample rate.

These unit vectors for the constant inertially known matrix $E$ as follows:

$$E = \begin{bmatrix} -0.6543 & -0.6338 & -0.5978 & -0.5559 & -0.5138 \\ -0.5407 & -0.4559 & -0.4202 & -0.4253 & -0.3845 \\ 0.5287 & 0.6248 & 0.6827 & 0.7142 & 0.7669 \end{bmatrix}. \quad (211)$$

Bounded random zero mean signals whose probability distributions are normalized bump functions are added to the true direction vectors $U$ to generate each
measured direction $U^m$. The maximum error (width of bump function) in each component of a direction vector measurement is $2.4^\circ$ based on coarse attitude sensors like sun sensors and magnetometers. Similarly, random zero mean bump functions are added to each element of $\Omega$ to form the measured $\Omega^m$. The width of these bump functions is $0.97^\circ/s$, which corresponds to a coarse rate gyro. Besides, the gyro readings are assumed to contain a constant bias in three directions, as follows:

$$\beta = [-0.01 \quad -0.005 \quad 0.02]^T \text{ rad/s}.$$ (212)

The estimator is simulated over a time interval of $T = 20s$, with a time stepsize of $h = 0.01s$. The estimator’s inertia scalar gain is $m = 5$ and the dissipation matrix is selected as the following positive definite matrix:

$$D = \text{diag}([17.04 \quad 18.46 \quad 19.88]^T).$$ (213)

As in [41], $\Phi(x) = x$. The weight matrix $W$ is also calculated using the conditions in [41]. This matrix is given by:

$$W = \begin{bmatrix}
296.5458 & -296.8526 & -293.3936 & 150.4527 & 150.2987 \\
-296.8526 & 368.7300 & 341.0189 & -197.1644 & -221.0503 \\
-293.3936 & 341.0189 & 321.6729 & -179.3406 & -194.9746 \\
150.4527 & -197.1644 & -179.3406 & 107.4149 & 123.2687 \\
150.2987 & -221.0503 & -194.9746 & 123.2687 & 147.3057 \\
\end{bmatrix}.$$ (214)
The positive definite matrix for bias gain is selected as $P = 4 \times 10^3 I$. The initial estimated states are equal to:

$$
\hat{R}_0 = \exp_{\text{SO}(3)} \left( \left( \frac{\pi}{2.5} \times \begin{bmatrix} 3/7 & 6/7 & 2/7 \end{bmatrix}^T \right)^x \right),
$$

$$
\hat{\Omega}_0 = \begin{bmatrix} -0.26 & 0.1725 & -0.2446 \end{bmatrix}^T \text{rad/s},
$$

and $
\hat{\beta}_0 = \begin{bmatrix} 0 & -0.01 & 0.01 \end{bmatrix}^T \text{rad/s}.$

(215)

In order to integrate the implicit set of equations in (194)-(197) numerically, the first two equation are solved at each sampling step. Using (196), $\hat{\Omega}_{i+1}$ in (197) is written in terms of $\omega_{i+1}$ next. The resulting implicit equation is solved with respect to $\omega_{i+1}$ iteratively to a set tolerance applying the Newton-Raphson method. The root of this nonlinear equation along with $\hat{R}_{i+1}$ and $\hat{\beta}_{i+1}$ are used for the next sampling time instant. This process is repeated to the end of the simulated duration.

Results from this numerical simulation are shown here. The principal angle corresponding to the rigid body’s attitude estimation error is depicted in Fig. 25, and estimation errors in the rigid body’s angular velocity components are shown in Fig. 26. Finally, Fig. 27 portrays estimate errors in bias components. All the estimation errors are seen to converge to a neighborhood of $(Q, \omega, \hat{\beta}) = (I, 0, 0)$, where the size of this neighborhood depends on the bounds of the measurement noise.
Figure 25: Principal angle of the attitude estimate error

Figure 26: Angular velocity estimate error
6.6 Conclusion

The formulation of variational attitude estimation is generalized to include bias in angular velocity measurements and estimate a constant bias vector. The continuous-time state estimator is obtained by applying the Lagrange-d’Alembert principle of variational mechanics to a Lagrangian consisting of the energies in the measurement residuals, along with a dissipation term linear in angular velocity measurement residual. The update law for the bias estimate ensures that the total energy content in the state and bias estimation errors is dissipated as in a dissipative mechanical system. The resulting generalization of the variational attitude estimator is almost globally asymptotically stable, like the variational attitude
estimator for the bias-free case reported in [41]. A discretization of this estimator is obtained, in the form of an implicit first order Lie group variational integrator, by applying the discrete Lagrange-d’Alembert principle to the discrete Lagrangian with the dissipation term linear in the angular velocity estimation error. Using a realistic set of data for rigid body rotational motion, numerical simulations show that the estimated states and estimated bias converge to a bounded neighborhood of the true states and true bias when the measurement noise is bounded. A future extension of this work will be the formulation of an explicit discrete-time implementation of this variational attitude estimation in the presence of bias, and its real-time implementation with optical and inertial sensors.
7 EXPERIMENTAL VALIDATION OF THE VARIATIONAL ATTITUDE ESTIMATOR

This chapter is adapted from a paper published in Proceedings of the 2015 Indian Control Conference [89]. The author gratefully acknowledges Dr. Amit K. Sanyal and S.P. Viswanathan for their participation.

Abstract The attitude determination (estimation) scheme presented in Chapter 4 is experimentally verified here. Implementing this variational estimation scheme on an Android cellphone and using the data from its “onboard” sensors, the cellphone’s attitude is determined. This attitude is compared against the attitude derived from solving the Wahba’s problem at each time instant to show the performance of the estimator. These results, obtained in the Spacecraft Guidance, Navigation and Control Laboratory at NMSU, demonstrate the excellent performance of this estimation scheme with the noisy raw data from the smartphone sensors.

7.1 Definitions

The raw IMU measurements from the smartphone are fused/filtered through the variational attitude estimation. In Chapter 4, an estimation of rigid body attitude and angular velocity without any knowledge of the attitude dynamics model,
is presented using the Lagrange-d’Alembert principle from variational mechanics. This variational observer requires at least two body-fixed sensors to measure inertially known and constant direction vectors as well as sensors to read the angular velocity. First- and second-order Lie group variational integrators were introduced for computer implementation using discrete variational mechanics.

In order to determine three-dimensional rigid body attitude instantaneously, three known inertial vectors are needed. This could be satisfied with just two vector measurements. In this case, the cross product of the two measured vectors is considered as a third measurement for applying the attitude estimation scheme. Let these vectors be denoted as $u^m_1$ and $u^m_2$, in the body-fixed frame. Denote the corresponding known inertial vectors as seen from the rigid body as $e_1$ and $e_2$, and let the true vectors in the body frame be denoted $u_i = R^T e_i$ for $i = 1, 2$, where $R$ is the rotation matrix from the body frame to the inertial frame. This rotation matrix provides a coordinate-free, global and unique description of the attitude of the rigid body. Define the matrix composed of all three measured vectors expressed in the body-fixed frame as column vectors, $U^m = [u^m_1 \ u^m_2 \ u^m_1 \times u^m_2]$ and the corresponding matrix of all these vectors expressed in the inertial frame as $E = [e_1 \ e_2 \ e_1 \times e_2]$. Note that the matrix of the actual body vectors $u_i$ corresponding to the inertial vectors $e_i$, is given by $U = R^T E = [u_1 \ u_2 \ u_1 \times u_2]$. The symmetric second-order filter equations (144)-(147) has been used in order to verify its performance in practice.
7.2 Experiments

This estimation scheme is implemented off-board on a remote PC using the sensor measurements acquired and transmitted by the smartphone. The coordinates used for the inertial frame is ENU, which is a right-handed Cartesian frame formed by local east, north and up. The coordinates fixed to the COM of the cellphone with right direction of the screen as \( x \), upwards direction as \( y \) and the direction out of screen as \( z \) is considered to be the body fixed frame. As mentioned in the previous section, at least two inertially known and constant directions are required in order to estimate the rigid body attitude. Using the inertial sensors installed on the smartphone, the accelerometer is used to measure the gravity direction and the magnetometer is used to measure the geomagnetic field direction. The cross product of these two vectors is considered as the third measured vector. In order to find these directions, one could normalize the vector readings from accelerometer and magnetometer in the case that the cellphone is aligned with the true geographical directions and the body fixed frame coincides with the ENU frame. Note that the direction read by the accelerometer shows the local up direction, since an upward acceleration equal to \( g \) is applied to the phone in order to cancel the Earth’s gravity and keep the phone still. Therefore, the matrix of these three inertially constant directions as expressed in the ENU frame is found
to be

\[ E = [e_1 \, e_2 \, e_1 \times e_2] = \begin{bmatrix}
0 & 0.0772 & -0.9921 \\
0 & 0.6117 & 0.1251 \\
1 & -0.7873 & 0
\end{bmatrix}. \]

The three axis gyroscope also gives the angular velocity measurements. These three sensors produce measurement data at different frequencies. The filter’s time step is selected according to the fastest sensor, which is the accelerometer here. At those time instants where some of the sensors readings are not available because of the difference in sampling frequencies, the last read value from that sensor is used.

\( \Phi(\cdot) \) could be any \( C^2 \) function with the properties described in Section 2 of [41], but is selected to be \( \Phi(x) = x \) here. Further, \( W \) is selected based on the value of \( E \), such that it satisfies the conditions in [41] as below:

\[ W = \begin{bmatrix}
3.19 & 1.51 & 0 \\
1.51 & 3.19 & 0 \\
0 & 0 & 2
\end{bmatrix}. \]

The inertia scalar gain is set to \( m = 0.5 \) and the dissipation matrix is selected as the following positive definite matrix:

\[ D = \text{diag}(12 \, 13 \, 14^T). \]

Sensors outputs usually contain considerable levels of noise that may harm the behavior of the nonlinear filter. A Butterworth pre-filter is implemented in order to reduce these high-frequency noises. Note that the true quantities would not contain high-frequency signals, since they are related to a rigid body motion. A symmetric discrete-time filter for the first order Butterworth pre-filter is
implemented for filtering the measurement data as follows:

\[(2 + h)\bar{x}_{k+1} = (2 - h)\bar{x}_k + h(x^m_k + x^m_{k+1}),\]  

(216)

where \(h\) is the time stepsize, \(\bar{x}\) and \(x^m\) are the filtered and measured quantities, respectively, and the subscript \(k\) denotes the \(k\)th time stamp. The initial estimated states have the following initial estimation errors:

\[Q_0 = \expm_{SO(3)}\left(\left(2.2 \times [0.63 \ 0.62 \ -0.48]^T\right)^\times\right),\]

and \(\omega_0 = [0.001 \ 0.002 \ -0.003]^T\text{ rad/s}.

(217)

In order to integrate the implicit set of equations (144)-(147) numerically, the first equation is solved at each sampling step, then the result for \(\hat{R}_{i+1}\) is substituted in the second one. Using the Newton-Raphson method, the resulting equation is solved with respect to \(\omega_{i+1}\) iteratively. The root of this nonlinear equation with a specified tolerance along with the \(\hat{R}_{i+1}\) is used for the next sampling time instant. This process is repeated over the simulated time period. The results of this experiment are described next.

7.3 Results

Experimental results for the attitude estimation scheme, obtained from the experimental setup described in the previous subsection, are presented here. These experiments were carried out on the HIL simulator testbed in the Spacecraft Guidance, Navigation and Control laboratory at NMSU’s MAE department. The prin-
cipal angle corresponding to the rigid body’s attitude estimation error is depicted in Figure 28. Estimation errors in the rigid body’s angular velocity components are shown in Figure 29. All the estimation errors are seen to converge to a neighborhood of $(Q, \omega) = (I, 0)$, where the size of this neighborhood depends on the characteristics of the measurement noise.

### 7.4 Conclusion

This chapter presents a novel software architecture of a spacecraft attitude determination and control subsystem (ADCS), using a smartphone as the onboard computer. This architecture is being implemented using a HIL ground simulator for three-axis attitude motion simulation in the Spacecraft Guidance, Navigation and Control laboratory at NMSU. Theoretical and numerical results for the attitude control and attitude estimation schemes that are part of this architecture,
have appeared in recent publications. The attitude estimation scheme provides almost global asymptotic stability, and is robust to measurement noise and bounded disturbance inputs acting on the spacecraft. Experimental verification of the attitude estimation algorithm is presented here, and the experimental results show excellent agreement with the theoretical and numerical results on this algorithm that have appeared in recent publications.
8 MODEL-FREE RIGID BODY POSE ESTIMATION BASED ON THE LAGRANGE-D’ALEMBERT PRINCIPLE

This chapter is adapted from a paper to appear in Automatica [42]. The author gratefully acknowledges Dr. Amit K. Sanyal for his participation.

Abstract Stable estimation of rigid body pose and velocities from noisy measurements, without any knowledge of the dynamics model, is treated using the Lagrange-d’Alembert principle from variational mechanics. With body-fixed optical and inertial sensor measurements, a Lagrangian is obtained as the difference between a kinetic energy-like term that is quadratic in velocity estimation error and the sum of two artificial potential functions; one obtained from a generalization of Wahba’s function for attitude estimation and another which is quadratic in the position estimate error. An additional dissipation term that is linear in the velocity estimation error is introduced, and the Lagrange-d’Alembert principle is applied to the Lagrangian with this dissipation. A Lyapunov analysis shows that the state estimation scheme so obtained provides stable asymptotic convergence of state estimates to actual states in the absence of measurement noise, with an almost global domain of attraction. This estimation scheme is discretized for computer implementation using discrete variational mechanics, as a first order Lie group variational integrator. The continuous and discrete pose estimation schemes
require optical measurements of at least three inertially fixed landmarks or beacons in order to estimate instantaneous pose. The discrete estimation scheme can also estimate velocities from such optical measurements. Moreover, all states can be estimated during time periods when measurements of only two inertial vectors, the angular velocity vector, and one feature point position vector are available in body frame. In the presence of bounded measurement noise in the vector measurements, numerical simulations show that the estimated states converge to a bounded neighborhood of the actual states.

8.1 Navigation using Optical and Inertial Sensors

Consider a vehicle in spatial (rotational and translational) motion. Onboard estimation of the pose of the vehicle involves assigning a coordinate frame fixed to the vehicle body, and another coordinate frame fixed in the environment which takes the role of the inertial frame. Let \( O \) denote the observed environment and \( S \) denote the vehicle. Let \( S \) denote a coordinate frame fixed to \( S \) and \( O \) be a coordinate frame fixed to \( O \), as shown in Fig. 30. Let \( R \in \text{SO}(3) \) denote the rotation matrix from frame \( S \) to frame \( O \) and \( b \) denote the position of origin of \( S \) expressed in frame \( O \). The pose (transformation) from body fixed frame \( S \) to inertial frame \( O \) is then given by (4). Consider vectors known in inertial frame \( O \) measured by inertial sensors in the vehicle-fixed frame \( S \); let \( \beta \) be the number of such vectors. In addition, consider position vectors of a few stationary
Figure 30: Inertial landmarks on $O$ as observed from vehicle $S$ with optical measurements.

Points in the inertial frame $O$ measured by optical (vision or lidar) sensors in the vehicle-fixed frame $S$. Velocities of the vehicle may be directly measured or can be estimated by linear filtering of the optical position vector measurements [43]. Assume that these optical measurements are available for $j$ points at time $t$, whose positions are known in frame $O$ as $p_j, j \in \mathcal{I}(t)$, where $\mathcal{I}(t)$ denotes the index set of beacons observed at time $t$. Note that the observed stationary beacons or landmarks may vary over time due to the vehicle’s motion. These points generate \( \binom{j}{2} \) unique relative position vectors, which are the vectors connecting any two of these landmarks. When two or more position vectors are optically measured, the
number of vector measurements that can be used to estimate attitude is \( \binom{j}{2} + \beta \).

This number needs to be at least two (i.e., \( \binom{j}{2} + \beta \geq 2 \)) at an instant, for the attitude to be uniquely determined at that instant. In other words, if at least two inertial vectors are measured at all instants (i.e., \( \beta \geq 2 \)), then beacon position measurements are not required for estimating attitude. However, at least one beacon or feature point position measurement is still required to estimate the position of the vehicle. Note that the use of two vector measurements for attitude determination was first proposed by the TRIAD algorithm in the 1960s [12].

### 8.1.1 Pose Measurement Model

Denote the position of an optical sensor and the unit vector from that sensor to an observed beacon in frame \( S \) as \( s^k \in \mathbb{R}^3 \) and \( u^k \in \mathbb{S}^2 \), \( k = 1, \ldots, \kappa \), respectively. Denote the relative position of the \( j^{th} \) stationary beacon observed by the \( k^{th} \) sensor expressed in frame \( S \) as \( q^k_j \). Thus, in the absence of measurement noise

\[
p_j = R(q^k_j + s^k) + b = Ra_j + b, \; j \in \mathcal{I}(t),
\]

(218)

where \( a_j = q^k_j + s^k \), are positions of these points expressed in \( S \). In practice, the \( a_j \) are obtained from range measurements that have additive noise; we denote as \( a^m_j \) the measured vectors. In the case of lidar range measurements, these are given by

\[
a^m_j = (q^k_j)^m + s^k = (g^k_j)^m u^k + s^k, \; j \in \mathcal{I}(t),
\]

(219)
where \((p_j^k)^m\) is the measured range to the point by the \(k^{th}\) sensor. The mean of the vectors \(p_j\) and \(a_j^m\) are denoted as \(\bar{p}\) and \(\bar{a}^m\) respectively, and satisfy

\[
\bar{a}^m = R^T(\bar{p} - b) + \varsigma, \tag{220}
\]

where \(\bar{p} = \frac{1}{j} \sum_{j=1}^{j} p_j\), \(\bar{a}^m = \frac{1}{j} \sum_{j=1}^{j} a_j^m\) and \(\varsigma\) is the additive measurement noise obtained by averaging the measurement noise vectors for each of the \(a_j\). Consider the \(\binom{j}{2}\) relative position vectors from optical measurements, denoted as \(d_j = p_\lambda - p_\ell\) in frame \(O\) and the corresponding vectors in frame \(S\) as \(l_j = a_\lambda - a_\ell\), for \(\lambda, \ell \in I(t)\), \(\lambda \neq \ell\). The \(\beta\) measured inertial vectors are included in the set of \(d_j\), and their corresponding measured values expressed in frame \(S\) are included in the set of \(l_j\). If the total number of measured vectors (both optical and inertial), \(\binom{j}{2} + \beta = 2\), then \(l_3 = l_1 \times l_2\) is considered a third measured direction in frame \(S\) with corresponding vector \(d_3 = d_1 \times d_2\) in frame \(O\). Therefore,

\[
d_j = Rl_j \Rightarrow D = RL, \tag{221}
\]

where \(D = [d_1 \cdots d_n]\), \(L = [l_1 \cdots l_n] \in \mathbb{R}^{3 \times n}\) with \(n = 3\) if \(\binom{j}{2} + \beta = 2\) and \(n = \binom{j}{2} + \beta\) if \(\binom{j}{2} + \beta > 2\). Note that the matrix \(D\) consists of vectors known in frame \(O\). Denote the measured value of matrix \(L\) in the presence of measurement noise as \(L^m\). Then,

\[
L^m = R^T D + \mathcal{L}, \tag{222}
\]

where \(\mathcal{L} \in \mathbb{R}^{3 \times n}\) consists of the additive noise in the vector measurements made in the body frame \(S\).
8.1.2 Velocities Measurement Model

Denote the angular and translational velocity of the rigid body expressed in body fixed frame $S$ by $\Omega$ and $\nu$, respectively. Therefore, one can write the kinematics of the rigid body as

$$
\dot{\Omega} = R\Omega \times, \quad \dot{b} = R\nu \Rightarrow \dot{g} = g\xi^\vee,
$$

(223)

where $\xi = \begin{bmatrix} \Omega \\ \nu \end{bmatrix} \in \mathbb{R}^6$ and $\xi^\vee = \begin{bmatrix} \Omega \times & \nu \\ 0 & 0 \end{bmatrix}$. For the general development of the motion estimation scheme, it is assumed that the velocities are directly measured. The estimator is then extended to cover the cases where: (i) only angular velocity is directly measured; and (ii) none of the velocities are directly measured.

8.2 Dynamic Estimation of Motion from Proximity Measurements

In order to obtain state estimation schemes from measurements as outlined in Section 8.1 in continuous time, the Lagrange-d’Alembert principle is applied to an action functional of a Lagrangian of the state estimate errors, with a dissipation term linear in the velocities estimate error. This section presents the estimation scheme obtained using this approach. Denote the estimated pose and its kinematics as

$$
\hat{g} = \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix}, \quad \hat{b} \in \text{SE}(3), \quad \hat{\dot{g}} = \hat{g}\xi^\vee,
$$

(224)
where \( \hat{\xi} \) is rigid body velocities estimate, with \( \hat{g}_0 \) as the initial pose estimate and the pose estimation error as

\[
\hat{\mathbf{h}} = \hat{g} \hat{g}^{-1} = \begin{bmatrix} Q & b - Q\hat{b} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} Q & x \\ 0 & 1 \end{bmatrix} \in \text{SE}(3),
\]

where \( Q = RR^T \) is the attitude estimation error and \( x = b - Q\hat{b} \). Then one obtains, in the case of perfect measurements,

\[
\begin{align*}
\dot{\hat{\mathbf{h}}} &= \mathbf{h}^\vee, \\
\text{where } \mathbf{h}(\hat{g}, \xi^m, \hat{\xi}) &= \begin{bmatrix} \omega \\ \upsilon \end{bmatrix} = \text{Ad}_{\hat{g}}(\xi^m - \hat{\xi}),
\end{align*}
\]

where \( \text{Ad}_g = \begin{bmatrix} R & 0 \\ \hat{g} \times R & R \end{bmatrix} \) for \( g = \begin{bmatrix} R & \hat{b} \\ 0 & 1 \end{bmatrix} \). The attitude and position estimation error dynamics are also in the form

\[
\begin{align*}
\dot{Q} &= Q\omega^\times, \\
\dot{x} &= Q\upsilon.
\end{align*}
\]

### 8.2.1 Lagrangian from Measurement Residuals

Consider the sum of rotational and translational measurement residuals between the measurements and estimated pose as a potential energy-like function. The rotational potential function (Wahba’s cost function [90]) is expressed as

\[
\mathcal{U}_r(\hat{g}, L^m, D) = \frac{1}{2} \langle D - \hat{R}L^m, (D - \hat{R}L^m)W \rangle,
\]

where \( W = \text{diag}(w_j) \in \mathbb{R}^{n \times n} \) is a positive diagonal matrix of weight factors for the measured \( l^m_j \). Consider the translational potential function

\[
\mathcal{U}_t(\hat{g}, \bar{a}^m, \bar{p}) = \frac{1}{2} \kappa y^T y = \frac{1}{2} \kappa \| \bar{p} - \bar{R}\bar{a}^m - \hat{b} \|^2,
\]

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where $\bar{p}$ is defined by (220), $y \equiv y(\hat{g}, \bar{a}^m, \bar{p}) = \bar{p} - \hat{R}\bar{a}^m - \hat{b}$ and $\kappa$ is a positive scalar. Therefore, the total potential function is defined as the sum of the generalization of (228) defined in [41, 73] for attitude determination on SO(3), and the translational energy (229) as

$$U(\hat{g}, \bar{L}^m, D, \bar{a}^m, \bar{p}) = U_r(\hat{g}, \bar{L}^m, D) + U_t(\hat{g}, \bar{a}^m, \bar{p}) = \Phi \left( \frac{1}{2} \langle D - \hat{R}L^m, (D - \hat{R}L^m)W \rangle \right) + \frac{1}{2} \kappa \| \bar{p} - \hat{R}\bar{a}^m - \hat{b} \|^2,$$

where $W$ is positive definite (not necessarily diagonal), and $\Phi : [0, \infty) \mapsto [0, \infty)$ is a $C^2$ function that satisfies $\Phi(0) = 0$ and $\Phi'(\chi) > 0$ for all $\chi \in [0, \infty)$. Furthermore, $\Phi'(\cdot) \leq \alpha(\cdot)$ where $\alpha(\cdot)$ is a Class-$K$ function [45] and $\Phi'(\cdot)$ denotes the derivative of $\Phi(\cdot)$ with respect to its argument. Because of these properties of the function $\Phi$, the critical points and their indices coincide for $U_r^0$ and $U_r$ [41]. Define the kinetic energy-like function:

$$T \left( \varphi(\hat{g}, \xi^m, \hat{\xi}) \right) = \frac{1}{2} \varphi(\hat{g}, \xi^m, \hat{\xi})^T J \varphi(\hat{g}, \xi^m, \hat{\xi}),$$

where $J \in \mathbb{R}^{6 \times 6} > 0$ is an artificial inertia-like kernel matrix. Note that in contrast to rigid body inertia matrix, $J$ is not subject to intrinsic physical constraints like the triangle inequality, which dictates that the sum of any two eigenvalues of the inertia matrix has to be larger than the third. Instead, $J$ is a gain matrix that can be used to tune the estimator. For notational convenience, $\varphi(\hat{g}, \xi^m, \hat{\xi})$ is denoted
as \( \varphi \) from now on; this quantity is the velocities estimation error in the absence of measurement noise. Now define the Lagrangian

\[
L(\dot{g}, L^m, D, \dot{a}^m, \bar{p}, \varphi) = T(\varphi) - U(\dot{g}, L^m, D, \dot{a}^m, \bar{p}), \tag{232}
\]

and the corresponding action functional over an arbitrary time interval \([t_0, T]\) for \( T > 0 \),

\[
S(L(\dot{g}, L^m, D, \dot{a}^m, \bar{p}, \varphi)) = \int_{t_0}^{T} L(\dot{g}, L^m, D, \dot{a}^m, \bar{p}, \varphi) dt, \tag{233}
\]

such that \( \dot{g} = \dot{\hat{g}}(\hat{\xi}) \). The following statement gives the form of the Lagrangian when perfect (noise-free) measurements are available, and derives the variational estimator for rigid body pose and velocities.

**Lemma 8.1.** In the absence of measurement noise, the Lagrangian is of the form

\[
L(h, D, \bar{p}, \varphi) = \frac{1}{2} \varphi^T J \varphi - \Phi(\langle I - Q, K \rangle) - \frac{1}{2} \kappa y^T y, \tag{234}
\]

where \( K = DWDT \) and \( y \equiv y(h, \bar{p}) = Q^T x + (I - Q^T)\bar{p} \).

**Proof:** Suppose that all the measured states are noise free. Therefore, one can replace \( L^m = L, \dot{a}^m = \dot{a} \) and \( \xi^m = \xi \). The rotational potential function (228) can be replaced by

\[
U_r^0(h, D) = \frac{1}{2} \langle D - \hat{R}L^m, (D - \hat{R}L^m)W \rangle = \frac{1}{2} \langle D - Q^T D, (D - Q^T D)W \rangle,
\]

\[
= \frac{1}{2} \langle I - Q^T, (I - Q^T)DWDT \rangle = \langle I - Q, K \rangle \tag{235}
\]
since $\hat{R}E = Q^T D$ for the noise-free case. In addition,

$$y(h, \bar{p}) = \bar{p} - \hat{R}\bar{a} - \hat{b} = \bar{p} - \hat{R}\bar{a} - \hat{b}$$  \hspace{1cm} (236)

$$= \bar{p} - Q^T R\bar{a} - Q^T (b - x) = Q^T x + (I - Q^T)\bar{p}.$$  

The translational potential function in the absence of measurement noise can be expressed as

$$U_t(h, \bar{p}) = \frac{1}{2} \kappa y^T y.$$  \hspace{1cm} (237)

Therefore, the total potential energy function is

$$U(h, D, \bar{p}) = U_r(h, D) + U_t(h, \bar{p}) = \Phi(U_r^0(h, D)) + U_t(h, \bar{p})$$

$$= \Phi(\langle I - Q, K \rangle) + \frac{1}{2} \kappa y^T y,$$  \hspace{1cm} (238)

and the kinetic energy function is

$$T(\varphi) = \frac{1}{2} \varphi^T J \varphi.$$  \hspace{1cm} (239)

Substituting (238) and (239) into:

$$L(h, D, \bar{p}, \varphi) = T(\varphi) - U(h, D, \bar{p}) = T(\varphi) - \Phi(U_r^0(h, D)) - U_t(h, \bar{p}),$$  \hspace{1cm} (240)

gives the Lagrangian (234) for the noise-free case. \qquad \Box

As in [41], the positive definite weight matrix $W$ can be selected according to the following lemma:
Lemma 8.2. Let \( \text{rank}(D) = 3 \). Let the singular value decomposition of \( D \) be given by

\[
D : = U_D \Sigma_D V_D^T \quad \text{where} \quad U_D \in O(3), \ V_D \in O(n),
\]

\[
\Sigma_D \in \text{Diag}^+(3, n), \tag{241}
\]

and \( \text{Diag}^+(n_1, n_2) \) is the vector space of \( n_1 \times n_2 \) matrices with positive entries along the main diagonal and all other components zero. Let \( \sigma_1, \sigma_2, \sigma_3 \) denote the main diagonal entries of \( \Sigma_D \). Further, let the positive definite weight matrix \( W \) be given by

\[
W = V_D W_0 V_D^T \quad \text{where} \quad W_0 \in \text{Diag}^+(n, n) \tag{242}
\]

and the first three diagonal entries of \( W_0 \) are given by

\[
w_1 = \frac{\varsigma_1}{\sigma_1}, \ w_2 = \frac{\varsigma_2}{\sigma_2}, \ w_3 = \frac{\varsigma_3}{\sigma_3} \quad \text{where} \quad \varsigma_1, \varsigma_2, \varsigma_3 > 0. \tag{243}
\]

Then, \( K = DWD^T \) is positive definite and

\[
K = U_D \Delta U_D^T \quad \text{where} \quad \Delta = \text{diag}(\varsigma_1, \varsigma_2, \varsigma_3), \tag{244}
\]

is its eigendecomposition. Moreover, if \( \varsigma_i \neq \varsigma_j \) for \( i \neq j \) and \( i, j \in \{1, 2, 3\} \), then \( \langle I - Q, K \rangle \) is a Morse function whose critical points are

\[
Q \in C_Q = \{I, Q_1, Q_2, Q_3\} \quad \text{where} \quad Q_i = 2U_D I_i U_D^T U_D^T - I, \tag{245}
\]

and \( I_i \) is the \( i^{th} \) column vector of the identity \( I \in SO(3) \).

The proof is presented in [41].
8.2.2 Variational Estimator for Pose and Velocities

The nonlinear variational estimator obtained by applying the Lagrange-d’Alembert principle to the Lagrangian (232) with a dissipation term linear in the velocities estimation error, is given by the following statement.

**Theorem 8.1.** The nonlinear variational estimator for pose and velocities is given by

\[
\begin{align*}
\dot{J} & = \text{ad}^*_\varphi \dot{\varphi} - Z(\hat{g}, L^m, D, \hat{a}^m, \hat{p}) - \mathbb{D} \varphi, \\
\dot{\xi} & = \xi^m - \text{Ad}_{\hat{g}}^{-1} \varphi, \\
\dot{\hat{g}} & = \hat{g}(\hat{\xi})^\vee,
\end{align*}
\]

where \( \text{ad}^*_\zeta = (\text{ad}_\zeta)^T \) with \( \text{ad}_\zeta \) defined by (250), and \( Z(\hat{g}, L^m, D, \hat{a}^m, \hat{p}) \) is defined by

\[
Z(\hat{g}, L^m, D, \hat{a}^m, \hat{p}) = \begin{bmatrix} \Phi^f\left(\mathcal{U}_r^0(\hat{g}, L^m, D)\right) S_\Gamma(\hat{R}) + \kappa \bar{p} y \end{bmatrix},
\]

where \( \mathcal{U}_r^0(\hat{g}, L^m, D) \) is defined as (228), \( y \equiv y(\hat{g}, \hat{a}^m, \hat{p}) = \bar{p} - \hat{R} \bar{a}^m - \hat{b} \) and

\[
S_\Gamma(\hat{R}) = \text{vex}(\Gamma \hat{R}^T - \hat{R} \Gamma^T) = \text{vex}(DW(L^m)^T \hat{R}^T - \hat{R}L^mWD^T),
\]

\( \Gamma = DW(L^m)^T \) and \( \text{vex}(\cdot) : so(3) \rightarrow \mathbb{R}^3 \) is the inverse of the \((\cdot)^\times \) map.

**Proof:** A Rayleigh dissipation term linear in the velocities of the form \( \mathbb{D} \varphi \) where \( \mathbb{D} \in \mathbb{R}^{6 \times 6} > 0 \) is used in addition to the Lagrangian (234), and the Lagrange-d’Alembert principle from variational mechanics is applied to obtain the estimator
Reduced variations with respect to $h$ and $\varphi$ \cite{13,58} are applied, given by

$$\delta h = h\eta^\vee, \quad \delta \varphi = \eta + \text{ad}_\varphi \eta,$$

where $\eta^\vee = \begin{bmatrix} \Sigma^\times & \rho \\ 0 & 0 \end{bmatrix}$ and $\text{ad}_\zeta = \begin{bmatrix} w^\times & 0 \\ v^\times & w^\times \end{bmatrix}$, \hspace{1cm} (249)

for $\eta = \begin{bmatrix} \Sigma \\ \rho \end{bmatrix} \in \mathbb{R}^6$ and $\zeta = \begin{bmatrix} w \\ v \end{bmatrix} \in \mathbb{R}^6$, with $\eta(t_0) = \eta(T) = 0$. This leads to the expression:

$$\delta_{h,\varphi} S(L(h, D, \bar{p}, \varphi)) = \int_{t_0}^T \eta^T \mathcal{D} \varphi dt.$$ \hspace{1cm} (251)

Note that the variations of the attitude and position estimation errors are of the form

$$\delta Q = Q\Sigma^\times, \quad \delta x = Q\rho,$$ \hspace{1cm} (252)

respectively. Applying reduced variations to the rotational potential energy term (235), one obtains

$$\delta_Q U^0_t(h, D) = \langle -Q\Sigma^\times, K \rangle = \frac{1}{2} \langle \Sigma^\times, KQ - Q^T K \rangle = S_K^T(Q)\Sigma,$$ \hspace{1cm} (253)

where

$$S_K(Q) = \text{vex}(KQ - Q^T K).$$ \hspace{1cm} (254)

Taking first variation of the translational potential energy term (237) with respect to $Q$ and $x$ yields:

$$\delta_h U_t(h, \bar{p}) = \kappa(\delta x + \delta Q \bar{p})^T \left\{ x + (Q - I)\bar{p} \right\} = \kappa(\rho^T y + \Sigma^T \bar{p}^\times y).$$ \hspace{1cm} (255)
Therefore, the first variation of the total potential energy (238) with respect to estimation errors is

\[ \delta_h U(h, D, \bar{p}) = Z^T(h, D, \bar{p})\eta, \]  

(256)

where \( Z(h, D, \bar{p}) \) is defined by

\[ Z(h, D, \bar{p}) = \left[ \Phi'\left(\langle I - Q, K\rangle\right)S_K(Q) + \kappa\bar{p}\times \{Q^T x + (I - Q^T)\bar{p}\} \right]. \]  

(257)

Taking the first variation of the kinetic energy term (239) with respect to \( \phi \) results in:

\[ \delta_\phi T(\phi) = \phi^T \mathcal{J} \delta \phi = \phi^T \mathcal{J}(\dot{\eta} + \text{ad}_\phi \eta), \]  

(258)

applying the reduced variation for \( \delta \phi \) as given in (249). Therefore, the first variation of the action functional (233) is obtained as

\[
\delta_{h,\phi} S(L(h, D, \bar{p}, \varphi)) = \int_{t_0}^{T} \left\{ \phi^T \mathcal{J}(\dot{\eta} + \text{ad}_\phi \eta) - \eta^T Z(h, D, \bar{p}) \right\} dt \\
= \int_{t_0}^{T} \eta^T \left( \text{ad}_\phi^* \mathcal{J} \varphi - Z(h, D, \bar{p}) - \mathcal{J} \dot{\varphi} \right) dt + \phi^T \mathcal{J} \eta|_{t_0}^{T} \\
= \int_{t_0}^{T} \eta^T \left( \text{ad}_\phi^* \mathcal{J} \varphi - Z(h, D, \bar{p}) - \mathcal{J} \dot{\varphi} \right) dt,  
\]  

(259)

applying fixed endpoint variations with \( \eta(t_0) = \eta(T) = 0 \). Substituting (259) in expression (251) one obtains

\[ \mathcal{J} \dot{\varphi} = \text{ad}_\phi^* \mathcal{J} \varphi - Z(h, D, \bar{p}) - \mathcal{D} \varphi, \]  

(260)

where \( Z(h, D, \bar{p}) \) is defined by (257). In order to implement this estimator using the aforementioned measurements, substitute \( Q^T D = \hat{R}L^m \). This changes the
rotational potential energy formed by the estimation errors in attitude (235) to (228). Equation (254) is also reformulated as

\[ S_K(Q) = \text{vex}(DWD^TQ - Q^TDWD^T) \]
\[ = \text{vex}(DW(L^m)^T\hat{R}^T - \hat{R}(L^m)WD^T) = S_\Gamma(\hat{R}). \]

Finally, the second row in the matrix \( Z(h, D, \bar{p}) \) is replaced by

\[ \kappa\{Q^Tx + (I - Q^T)\bar{p}\} = \kappa\{Q^Tb - \hat{b} + \bar{p} - Q^T\bar{p}\} \]
\[ = \kappa\{\hat{R}\hat{R}^T(b - \bar{p}) - \hat{b} + \bar{p}\} \]
\[ = \kappa\{-\hat{R}\bar{a}^m - \hat{b} + \bar{p}\}. \]

(262)

Taking these changes into account, one could obtain the first of equations (246) with \( Z(\tilde{g}, L^m, D, \bar{a}^m, \bar{p}) \) and \( S_\Gamma(\hat{R}) \) defined by (247) and (248), respectively. Thus, the complete nonlinear estimator equations are given by (246). \( \square \)

This is a fundamentally new idea of applying a principle from variational mechanics to obtain a state estimator, recently applied to rigid body attitude estimation in [41]. This approach differs from the “minimum-energy” approach to nonlinear estimation due to Mortensen [65] in some important ways. The minimum-energy approach applies Hamilton-Jacobi-Bellman (HJB) theory [48], which can only be “approximately solved.” This approach was recently applied to state estimation of rigid body attitude motion in [92]. This HJB formulation can only be approximately solved in practice, using a Riccati-like equation,
to obtain a near-optimal filter that has no guarantees on stability. In the pro-
posed approach, the time evolution of \((\hat{g}, \hat{\xi})\) has the form of the dynamics of a
rigid body with Rayleigh dissipation. This results in an estimator for the mo-
tion states \((g, \xi)\) that dissipates the “energy” content in the estimation errors 
\((h, \varphi) = (g\hat{g}^{-1}, Ad_g(\xi - \hat{\xi}))\) to provide guaranteed asymptotic stability in the case
of perfect measurements \([41]\). The differences between these two approaches were
detailed in \([40]\), for rigid body attitude estimation.

The proposed estimator combines certain desirable features of stochastic es-
timation and observer design approaches to state estimation for unmanned vehi-
cles, when simultaneous inertial vector measurements and optical measurements
of fixed beacons or landmarks are available. This nonlinear estimator is robust to
measurement noise and does not require a dynamics model for the vehicle; instead,
it estimates the dynamics of the vehicle given the measurement model in Section
8.1. The variational pose estimator can also be interpreted as a low-pass stable
filter (cf. \([86]\)). Indeed, one can connect the low-pass filter interpretation to the
simple example of the natural dynamics of a mass-spring-damper system. This
is a consequence of the fact that the mass-spring-damper system is a mechanical
system with passive dissipation, evolving on a configuration space that is the vec-
tor space of real numbers, \(\mathbb{R}\). In fact, the equation of motion of this system can be
obtained by application of the Lagrange-d’Alembert principle on the configuration
space \(\mathbb{R}\). If this analogy or interpretation is extended to a system evolving on a Lie
group as a configuration space, then the generalization of the mass-spring-damper system is a “forced Euler-Poincaré system” [13, 58] with passive dissipation, as is obtained here. Explicit expressions for the vector of velocities $\xi^m$ can be obtained for two common cases when these velocities are not directly measured. These two cases are dealt with in the next subsection.

8.2.3 Variational Estimator Implemented without Direct Velocity Measurements

The velocity measurements in (246) can be replaced by filtered velocity estimates obtained by linear filtering of optical and inertial measurements using, e.g., a second-order Butterworth filter. This is both useful and necessary when velocities are not directly measured. The filtered values $\xi^f$ are then used in place of $\xi^m$ to enhance the nonlinear estimator given by Theorem 8.1. Denote the measured vector quantity at time $t$ by $z^m$. A linear second-order filter of the form:

$$\ddot{z}^f + 2\mu\omega_n\dot{z}^f = \omega_n^2(z^m - z^f),$$  \hspace{1cm} (263)

is used, where $\omega_n$ is the natural (cutoff) frequency, $\mu$ is the damping ratio, and $z^f$ is the filtered value of $z^m$. Thereafter, $z^f$ is used in place of $z^m$ in equations (246).
8.2.3.1 Angular velocity is measured using rate gyros

For the case that rate gyro measurements of angular velocities are available besides the \( j \) feature point (or beacon) position measurements, the linear velocities of the rigid body can be calculated using each single position measurement by rewriting (266) as

\[
\nu^f = (a^f_j)^x \Omega^f - v^f_j. \tag{264}
\]

for the \( j^{th} \) point. Averaging the values of \( \nu \) derived from all feature points gives a more reliable result. Therefore, the rigid body’s filtered velocities are expressed in this case as

\[
\xi^f = \left[ \frac{1}{j} \sum_{j=1}^{j} (a^f_j)^x \Omega^f - v^f_j \right]. \tag{265}
\]

8.2.3.2 Translational and angular velocity measurements are not available

In the case that both angular and translational velocity measurements are not available or accurate, rigid body velocities can be calculated in terms of the inertial and optical measurements. In order to do so, one can differentiate (218) as follows

\[
\dot{p}_j = R\Omega^x a_j + R\dot{a}_j + \dot{b} = R(\Omega^x a_j + \dot{a}_j + \nu) = 0
\]

\[
\Rightarrow \dot{a}_j - a_j^x \Omega + \nu = 0
\]

\[
\Rightarrow v_j = \dot{a}_j = [a_j^x - I] \xi = G(a_j) \xi, \tag{266}
\]

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where $G(a_j) = [a_j^x - I]$ has full row rank. From vision-based or Doppler lidar sensors, one can also measure the velocities of the observed points in frame $S$, denoted $v_j^m$. Here, velocity measurements as would be obtained from vision-based sensors is considered. The measurement model for the velocity is of the form

$$v_j^m = G(a_j)\xi + \vartheta_j, \quad (267)$$

where $\vartheta_j \in \mathbb{R}^3$ is the additive error in velocity measurement $v_j^m$. Instantaneous angular and translational velocity determination from such measurements is treated in [73]. Note that $v_j = \dot{a}_j$, for $j \in \mathcal{I}(t)$. As this kinematics indicates, the relative velocities of at least three beacons are needed to determine the vehicle’s translational and angular velocities uniquely at each instant. However, when only one or two landmarks/beacons are measured, the estimator can propagate velocity estimates based on a least squares velocity determined from the available measurements. The rigid body velocities in both cases are obtained using the pseudo-inverse of $G(A^f)$:

$$G(A^f)\xi^f = V(V^f) \Rightarrow \xi^f = G^+(A^f)V(V^f), \quad (268)$$

where $G(A^f) = \begin{bmatrix} G(a_1^f) \\ \vdots \\ G(a_j^f) \end{bmatrix}$ and $V(V^f) = \begin{bmatrix} v_1^f \\ \vdots \\ v_j^f \end{bmatrix}$.

(269)

for $1, \ldots, j \in \mathcal{I}(t)$. When at least three beacons are measured, $G(A^f)$ is a full column rank matrix, and $G^+(A^f) = \left( G^T(A^f)G(A^f) \right)^{-1}G^T(A^f)$ gives its pseudo-inverse. For the case that only one or two beacons are observed, $G(A^f)$ is a full row
rank matrix, whose pseudo-inverse is given by $G^\dagger(A') = G^T(A')(G(A')G^T(A'))^{-1}$.

### 8.3 Stability and Robustness of Estimator

The stability of the estimator (filter) given by Theorem 8.1 is analyzed here. The following result shows that this scheme is stable, with almost global convergence of the estimated states to the real states in the absence of measurement noise.

**Theorem 8.2.** Let the observed position vectors from optical measurements be bounded. Then, the estimator presented in Theorem 8.1 is asymptotically stable at the estimation error state $(\mathbf{h}, \varphi) = (I, 0)$ in the absence of measurement noise. Further, the domain of attraction of $(\mathbf{h}, \varphi) = (I, 0)$ is a dense open subset of $\text{SE}(3) \times \mathbb{R}^6$.

**Proof:** In the absence of measurement noise, $\hat{R}E = Q^T D$. Therefore, the function $\Phi(\mathcal{U}_0^0(\mathbf{g}, L^m, D)) = \Phi(\mathcal{U}_0^0(\mathbf{h}, D))$ is a Morse function on $\text{SO}(3)$. The stability of this estimator can be shown using the following candidate Morse-Lyapunov function, which can be interpreted as the total energy function (equal in value to the Hamiltonian) corresponding to the Lagrangian (232):

$$V(\mathbf{h}, D, \bar{p}, \varphi) = \mathcal{T}(\varphi) + \mathcal{U}(\mathbf{h}, D, \bar{p}) = \frac{1}{2} \varphi^T \mathbb{J} \varphi + \Phi(\langle I - Q, K \rangle) + \frac{1}{2} \kappa y^T y. \quad (270)$$

Note that $V(\mathbf{h}, D, \bar{p}, \varphi) \geq 0$ and $V(\mathbf{h}, D, \bar{p}, \varphi) = 0$ if and only if $(\mathbf{h}, \varphi) = (I, 0)$. Therefore, $V(\mathbf{h}, D, \bar{p}, \varphi)$ is positive definite on $\text{SE}(3) \times \mathbb{R}^6$. Using (227), one can
derive the time derivative of (238) as

\[
\frac{d}{dt} U(h, D, \bar{p}) = \Phi'(U^0_r(h, D)) \langle -Q\omega^\times, K \rangle + \kappa(\dot{x} + \dot{Q}\bar{p})^T(Qy)
\]

\[
= \Phi'(U^0_r(h, D)) \langle \omega^\times, -Q^T K \rangle + \kappa(Qv + Q\omega^\times\bar{p})^T(Qy)
\]

\[
= \frac{1}{2} \Phi'(U^0_r(h, D)) \langle \omega^\times, KQ - Q^T K \rangle + \kappa(v + \omega^\times\bar{p})^T y
\]

\[
= \Phi'(U^0_r(h, D)) S^T_K(Q)\omega + \kappa y^T v + \kappa(\bar{p}^\times y)^T \omega
\]

\[
= Z^T(h, D, \bar{p})\varphi, \quad (271)
\]

where \( S_K(Q) \) is defined as (254) and \( Z(h, D, \bar{p}) \) as (257). Therefore, the time
derivative of the candidate Morse-Lyapunov function is

\[
\dot{V}(h, D, \bar{p}, \varphi) = \varphi^T J\dot{\varphi} + \varphi^T Z(h, D, \bar{p})
\]

\[
= \varphi^T \left( \text{ad}_{\varphi}^* \varphi - Z(h, D, \bar{p}) - D\varphi + Z(h, D, \bar{p}) \right)
\]

\[
= -\varphi^T D\varphi. \quad (272)
\]

noting that \( \varphi^T \text{ad}_{\varphi}^* \varphi = 0 \). Hence, the derivative of the Morse-Lyapunov function
is negative semi-definite. Note that the error dynamics for the pose estimate
error \( h \) is given by (226), while the error dynamics for the velocities estimate
error \( \varphi \) is given by (260). Note that \( D(t) \), as a function of time, is piecewise
continuous and uniformly bounded. The first property (piecewise continuity) is
naturally satisfied by \( D(t) \), which is piecewise constant as the number and inertial
positions of beacons (or feature points) observed by body-fixed optical sensors is
piecewise continuous in time. The second property (uniform boundedness) is
satisfied by \( D(t) \) if the position vectors observed are bounded in \( \mathbb{R}^3 \), as assumed in the statement. Therefore, the error dynamics for \((h, \varphi)\) is non-autonomous. Considering (270) and (272), and applying Theorem 8.4 in [45], one can conclude that \( \varphi^T D \varphi \to 0 \) as \( t \to \infty \), which consequently implies \( \varphi \to 0 \). Thus, the positive limit set for this system is contained in

\[
\mathcal{E} = \dot{V}^{-1}(0) = \{(h, \varphi) \in \text{SE}(3) \times \text{se}(3) : \varphi \equiv 0 \}. \tag{273}
\]

Substituting \( \varphi \equiv 0 \) in the first equation of the estimator (246), we obtain the positive limit set where \( \dot{V} \equiv 0 \) as \( \varphi \equiv 0 \) as the set

\[
\mathcal{I} = \{(h, \varphi) \in \text{SE}(3) \times \mathbb{R}^6 : Z(h, D, \bar{p}) \equiv 0, \varphi \equiv 0 \}\tag{274}
\]

\[
= \{(h, \varphi) \in \text{SE}(3) \times \mathbb{R}^6 : Q \in C_Q, Q^T x = 0, \varphi \equiv 0 \},
\]

where \( C_Q \) is defined by (245). Therefore, in the absence of measurement errors, all the solutions of this estimator converge asymptotically to the set \( \mathcal{I} \). Define \( \mathcal{U}_r(Q) := \Phi(\langle I - Q, K \rangle) \), which is the attitude measurement residual in the case of perfect measurements. Thus, the attitude estimate error converges to the set of critical points of \( \mathcal{U}_r(Q) \) in this intersection, and the position estimate error \( x \) converges to zero. The unique global minimum of \( \mathcal{U}_r(Q) \) is at \( Q = I \) (Lemma 2.1 in [41]), so this estimation error is asymptotically stable.

Now consider the set

\[
\mathcal{C} = \mathcal{I} \setminus (I, 0), \tag{275}
\]
which consists of all stationary states that the estimation errors may converge to, besides the desired estimation error state \((I, 0)\). Note that all states in the stable manifold of a stationary state in \(C\) converge to this stationary state. From the properties of the critical points \(Q_\iota \in C_Q \setminus (I)\) of \(U_0^\iota(Q)\), \((\iota = 1, 2, 3)\) given in Lemma 2.1 of [41], we see that the stationary points in \(\mathcal{S} \setminus (I, 0) = \{([Q_\iota 0 0], 0) : Q_\iota \in C_Q \setminus (I)\}\) have stable manifolds whose dimensions depend on the index of \(Q_\iota\). Since the velocities estimate error \(\varphi\) converges globally to the zero vector, the dimension of the stable manifold \(\mathcal{M}_\iota^S\) of the critical points, i.e. \(([Q_\iota 0 0], 0) \in SE(3) \times \mathbb{R}^6\) is

\[
\dim(\mathcal{M}_\iota^S) = 9 + (3 - \text{index of } Q_\iota) = 12 - \text{index of } Q_\iota.
\] (276)

Therefore, the stable manifolds of \((h, \varphi) = ([Q_\iota 0 0], 0)\) are nine-dimensional, ten-dimensional, or eleven-dimensional, depending on the index of \(Q_\iota \in C_Q \setminus (I)\) according to (276). Moreover, the value of the Lyapunov function \(V(h, D, \varphi)\) is non-decreasing (increasing when \((h, \varphi) \not\in \mathcal{S}\)) for trajectories on these manifolds when going backwards in time. This implies that the metric distance between error states \((h, \varphi)\) along these trajectories on the stable manifolds \(\mathcal{M}_\iota^S\) grows with the time separation between these states, and this property does not depend on the choice of the metric on \(SE(3) \times \mathbb{R}^6\). Therefore, these stable manifolds are embedded (closed) submanifolds of \(SE(3) \times \mathbb{R}^6\) and so is their union. Clearly, all states starting in the complement of this union, converge to the stable equilibrium.
\[
\begin{bmatrix} Q_0 & 0 \\ 0 & 1 \end{bmatrix}, 0) = (I, 0); \text{ therefore the domain of attraction of this equilibrium is}
\]
\[
\text{DOA}\{(I, 0)\} = \text{SE}(3) \times \mathbb{R}^6 \setminus \{ \bigcup_{i=1}^3 \mathcal{M}_i \},
\]
which is a dense open subset of \( \text{SE}(3) \times \mathbb{R}^6 \).

Therefore, the domain of attraction for the variational estimation scheme at \((h, \varphi) = (I, 0)\) is almost global over the state space \( \text{TSE}(3) \simeq \text{SE}(3) \times \mathbb{R}^6 \), which is the best possible with continuous control and navigation schemes for systems evolving on a non-contractible state space \([20, 63]\). In the presence of measurement noise with bounded frequencies and amplitudes, one can show that the expected values of the state estimates converge to a bounded neighborhood of the true states. The size of this neighborhood, which can be considered as a measure of the robustness of this estimation scheme, depends on the values of the estimator gains \( J, W \) and \( D \). These estimator gains can be selected based on balancing the transient and steady-state behavior of the estimator.

Remark. In the special case that the weight matrix \( W \) in Wahba’s function is chosen as a piecewise time constant matrix according to Lemma 8.2, \( K = DWD^T \) is a constant matrix for all time. Therefore, the RHS of (260) is not explicitly dependent on time. This makes \((h, \varphi)\) an autonomous system and therefore the use of Theorem 8.4 of [45] is not required to prove asymptotic stability. One can apply LaSalle’s invariance principle (Theorem 4.4 in [45]) to prove the convergence of state estimates to the equilibrium \((I, 0)\) in this case.
8.4 Discretization for Computer Implementation

For onboard computer implementation, the variational estimation scheme outlined above has to be discretized. This discretization is carried out in the framework of discrete geometric mechanics, and the resulting discrete-time estimator is in the form of a Lie group variational integrator (LGVI), as in [75]. Since the estimation scheme proposed here is obtained from a variational principle of mechanics, it can be discretized by applying the discrete Lagrange-d’Alembert principle [59]. Consider an interval of time \([t_0, T]\) \(\in \mathbb{R}^+\) separated into \(N\) equal-length subintervals \([t_i, t_{i+1}]\) for \(i = 0, 1, \ldots, N\), with \(t_N = T\) and \(t_{i+1} - t_i = \Delta t\) is the time step size. Let \((\hat{g}_i, \hat{\xi}_i) \in \text{SE}(3) \times \mathbb{R}^6\) denote the discrete state estimate at time \(t_i\), such that \((\hat{g}_i, \hat{\xi}_i) \approx (\hat{g}(t_i), \hat{\xi}(t_i))\) where \((\hat{g}(t), \hat{\xi}(t))\) is the exact solution of the continuous-time estimator at time \(t \in [t_0, T]\). Let the values of the discrete-time measurements \(\xi^m, \bar{a}^m\) and \(L^m\) at time \(t_i\) be denoted as \(\xi_i^m, \bar{a}_i^m\) and \(L_i^m\), respectively. Further, denote the corresponding values for the latter two quantities in inertial frame at time \(t_i\) by \(\bar{p}_i\) and \(D_i\), respectively. The term representing the energy content of the pose estimation error, given by (230), is discretized as

\[
U(\hat{g}_i, L_i^m, D_i, \bar{a}_i^m, \bar{p}_i) = U_r(\hat{g}_i, L_i^m, D_i) + U_t(\hat{g}_i, \bar{a}_i^m, \bar{p}_i) = \Phi(U_r^0(\hat{g}_i, L_i^m, D_i)) + U_t(\hat{g}_i, \bar{a}_i^m, \bar{p}_i)
\]

\[
= \Phi \left( \frac{1}{2} \langle D_i - \hat{R}_i L_i^m, (D_i - \hat{R}_i L_i^m) W_i \rangle \right) + \frac{1}{2} \kappa \|\bar{p}_i - \hat{R}_i \bar{a}_i^m - \hat{b}_i\|^2,
\]

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where $W_i$ is the matrix of weight factors corresponding to $D_i$ at time $t_i$. The term encapsulating the energy in the velocities estimate error (231), is discretized as

$$T\left(\varphi(\hat{g}_i, \xi_i^m, \hat{\xi}_i)\right) = \frac{1}{2} \varphi(\hat{g}_i, \xi_i^m, \hat{\xi}_i)^T \mathbb{J} \varphi(\hat{g}_i, \xi_i^m, \hat{\xi}_i), \quad (278)$$

where $\mathbb{J} = \text{diag}(J, M)$ and $M, J$ are positive definite matrices.

**Lemma 8.3.** In the absence of measurement noise, the discrete-time Lagrangian is of the form

$$\mathcal{L}(h_i, D_i, \bar{p}_i, \varphi_i) = \frac{1}{2} \langle J \omega_i^\times, \omega_i^\times \rangle + \frac{1}{2} \langle M v_i, v_i \rangle - \Phi(\langle I - Q_i^T K_i \rangle) - \frac{1}{2} \kappa y_i^T y_i, \quad (279)$$

where $y_i \equiv y(h_i, \bar{p}_i) = Q_i^T x_i + (I - Q_i^T) \bar{p}_i$ and $J$ is defined in terms of the matrix $J$ by $J = \frac{1}{2} \text{trace}[J] I - J$.

A Lie group variational integrator (LGVI) introduced in [77] is applied to the discrete-time Lagrangian (279) to obtain the discrete-time filter.

**Theorem 8.3.** A first-order discretization of the estimator proposed in Theorem
8.1 is given by

\[ (J\omega_i)^\times = \frac{1}{\Delta t}(F_i \mathcal{J} - \mathcal{J} F_i^T), \]

\[ (M + \Delta t \mathcal{D}_r)\nu_{i+1} = F_i^T M \nu_i + \Delta t \kappa (\hat{b}_{i+1} + \hat{R}_{i+1} \hat{a}_{i+1} - \hat{p}_{i+1}), \]

\[ (J + \Delta t \mathcal{D}_r)\omega_{i+1} = F_i^T J \omega_i + \Delta t M \nu_{i+1} \times \nu_{i+1} \]

\[ + \Delta t \kappa \hat{p}_{i+1} (\hat{b}_{i+1} + \hat{R}_{i+1} \hat{a}_{i+1}^m) \]

\[ - \Delta t \Phi'(U_r^0(\hat{g}_{i+1}, L_{m_{i+1}}, D_{i+1})) S_{\Gamma_{i+1}}(\hat{R}_{i+1}), \]

\[ \hat{\xi}_i = \xi_i^m - \text{Ad}_{\hat{g}_{i+1}} \varphi_i, \]

\[ \hat{g}_{i+1} = \hat{g}_i \exp(\Delta t \hat{\xi}^\vee), \]

where \( F_i \in SO(3), (\hat{g}(t_0), \hat{\xi}(t_0)) = (\hat{g}_0, \hat{\xi}_0), \varphi_i = [\omega_i^T \nu_i^T] T, \) and \( S_{\Gamma_{i+1}}(\hat{R}_i) \) is the value of \( S_{\Gamma}(\hat{R}) \) at time \( t_i \), with \( S_{\Gamma}(\hat{R}) \) as defined by (248).

**Proof:** Consider first variations with fixed endpoints for the pose estimation errors in discrete time given by:

\[ \delta Q_i = Q_i \Sigma_i^\times, \quad \Sigma_0 = \Sigma_N = 0, \]

\[ \delta x_i = Q_i \rho_i, \quad \rho_0 = \rho_N = 0, \]

where \( \Sigma_i, \rho_i \in \mathbb{R}^3 \) are “discrete variation vectors”. It can be shown that for any \( \omega \in \mathbb{R}^3 \) we have

\[ (J\omega)^\times = \omega^\times \mathcal{J} + \mathcal{J} \omega^\times. \]
Discretizing (227) assuming that the angular velocity estimation error is constant in the time interval \([t_i, t_{i+1}]\) with a constant time step size \(\Delta t\), one gets

\[
Q_{i+1} = Q_i F_i, \quad i \in \{0, 1, 2, \ldots, N - 1\},
\] (288)

where \(F_i \in SO(3)\) is given by

\[
F_i = \exp(\Delta t \omega_i^\times) \approx I + \Delta t \omega_i^\times.
\] (289)

The variation of \(F_i\) can be derived from (288) and \(\delta Q_i = Q_i \Sigma_i^\times\). Thus

\[
\delta F_i = -\Sigma_i^\times F_i + F_i \Sigma_i^\times_{i+1}.
\] (290)

Using (287) and (289), one can enforce the skew-symmetry of \((J\omega_i)^\times\) by

\[
(J\omega_i)^\times = \omega_i^\times J + J \omega_i^\times \approx \frac{1}{\Delta t} (F_i - I) J - J (F_i^T - I) = \frac{1}{\Delta t} (F_i J - J F_i^T).
\] (291)

From (226), the continuous rate of change of the attitude estimation error is \(\dot{x} = Q\upsilon\), which can be approximated to first order in discrete-time as

\[
\frac{x_{i+1} - x_i}{\Delta t} \approx Q_i \upsilon_i \Rightarrow x_{i+1} = \Delta t Q_i \upsilon_i + x_i.
\] (292)

The first variation in \(\upsilon_i\) is then calculated using (292) as

\[
\delta \upsilon_i = \delta \left( \frac{1}{\Delta t} Q_i^T (x_{i+1} - x_i) \right)
\]
\[
= -\Sigma_i^\times \upsilon_i + \frac{1}{\Delta t} Q_i^T (\delta x_{i+1} - \delta x_i)
\]
\[
= -\Sigma_i^\times \upsilon_i + \frac{1}{\Delta t} F_i \rho_{i+1} - \frac{1}{\Delta t} \rho_i.
\] (293)
The discrete Lagrangian (279) can be rewritten as

\[
\mathcal{L}(h_i, D_i, \bar{p}_i, F_i, v_i) = \frac{1}{2\Delta t} \langle \mathcal{J}(F_i - I), (F_i - I) \rangle + \frac{\Delta t}{2} \langle M v_i, v_i \rangle - \Delta t \Phi^0(h_i, D_i) - \frac{\Delta t}{2} \kappa(Q_i y_i)^T(Q_i y_i).
\]

The action functional (233) is replaced by the action sum

\[
S_d(\mathcal{L}(h_i, D_i, \bar{p}_i, F_i, v_i)) = \Delta t \sum_{i=0}^{N-1} \mathcal{L}(h_i, D_i, \bar{p}_i, F_i, v_i).
\]

Applying the discrete Lagrange-d’Alembert principle with two Rayleigh dissipation terms for angular and translational motions gives

\[
\delta S_d(\mathcal{L}(h_i, D_i, \bar{p}_i, F_i, v_i)) + \Delta t \sum_{i=0}^{N-1} \left\{ \langle \Sigma_i, \tau_i \rangle + \langle \rho_i, f_i \rangle \right\} = 0
\]

\[
\Rightarrow \sum_{i=0}^{N-1} \left\{ \frac{1}{\Delta t} \langle \delta F_i, \mathcal{J}(F_i - I) \rangle + \Delta t \langle \delta v_i, M v_i \rangle - \frac{\Delta t}{2} \Phi^0(h_i, D_i) \langle \Sigma_i, S_i^x(Q_i) \rangle
\]

\[
- \Delta t \kappa \langle \rho_i, y_i \rangle - \Delta t \kappa \langle \Sigma_i^x, y_i \bar{p}_i^T \rangle + \frac{\Delta t}{2} \langle \Sigma_i^x, \tau_i^x \rangle + \Delta t \langle \rho_i, f_i \rangle \right\} = 0.
\]

As symmetric matrices are orthogonal to skew-symmetric matrices in the trace inner product, using (289) we can rewrite the first term in (294) as

\[
\langle \delta F_i, \mathcal{J}(F_i - I) \rangle = \langle \Sigma_i^x, \mathcal{J} F_i^T \rangle - \langle \Sigma_i^x, F_i^T \mathcal{J} \rangle
\]

\[
= \frac{1}{2} \langle \Sigma_i^x, \mathcal{J} F_i^T \rangle - \frac{1}{2} \langle \Sigma_i^x, F_i \mathcal{J} \rangle - \frac{1}{2} \langle \Sigma_{i+1}^x, F_i^T \mathcal{J} \rangle + \frac{1}{2} \langle \Sigma_{i+1}^x, \mathcal{J} F_i \rangle
\]

\[
= -\frac{\Delta t}{2} \langle \Sigma_i^x, (J \omega_i)^x \rangle + \frac{\Delta t}{2} \langle \Sigma_{i+1}^x, F_i^T (J \omega_i)^x F_i \rangle.
\]
Hence equation (296) can be re-expressed as
\[
\sum_{i=0}^{N-1} \left\{ -\frac{1}{2} \langle \Sigma_i^x, (J\omega_i)^x \rangle + \frac{1}{2} \langle \Sigma_{i+1}^x, F_i^T (J\omega_i)^x F_i \rangle - \frac{\Delta t}{2} \langle \Sigma_i^x, (v_i \times Mv_i)^x \rangle \\
+ \langle F_i \rho_{i+1}, Mv_i \rangle - \langle \rho_i, Mv_i \rangle - \frac{\Delta t}{2} \Phi'(U_0^r(h_i, D_i)) \langle \Sigma_i^x, S_{K_i}^x(Q_i) \rangle - \kappa \Delta t \langle \rho_i, y_i \rangle \\
- \frac{\kappa \Delta t}{2} \langle \Sigma_i^x, (\vec{p}_i^x y_i)^x \rangle + \frac{\Delta t}{2} \langle \Sigma_i^x, \tau_i^x \rangle + \Delta t \langle \rho_i, f_i \rangle \right\} = 0. \tag{298}
\]

Separating this equation into two (rotational and translational) parts leads to
\[
(M + \Delta t \mathbb{D}_i) v_{i+1} = F_i^T M v_i - \Delta t \kappa y_{i+1}, \tag{299}
\]
\[
(J + \Delta t \mathbb{D}_r) \omega_{i+1} = F_i^T J \omega_i + \Delta t M v_{i+1} \times v_{i+1} - \Delta t \kappa \vec{p}_{i+1}^x y_{i+1} \tag{300}
\]
\[- \Delta t \Phi'(U_0^r(h_{i+1}, D_{i+1})) S_{K_{i+1}}(Q_{i+1}),
\]
using the identity \( f^T w^x f = (f^T w)^x \) and by replacing \( \tau_i = -\mathbb{D}_r \omega_i \) and \( f_i = -\mathbb{D}_t v_i \), where \( \mathbb{D}_r \) and \( \mathbb{D}_t \) are positive definite matrices such that
\[
\mathbb{D} = \begin{bmatrix} \mathbb{D}_r & 0 \\ 0 & \mathbb{D}_t \end{bmatrix}.
\]

In the presence of measurement noise, \( Q_i^T D_i \) and \( y_i \) are replaced by \( \hat{R}_i L_i^m \) and \( \hat{p}_i - \hat{b}_i - \hat{R}_i \hat{a}_i^m \), respectively. These give the discrete-time state estimator in the form of the Lie group variational integrator (280)-(284).

Remark. In the absence of any direct velocity measurements or only angular velocity measurements, the expressions provided in Section 8.2.3 to calculate rigid body state

Model-based discrete-time rigid body state estimators using LGVI schemes for attitude estimation were reported in \([75, 76]\), but dynamics model-free state estimators using LGVIs have appeared only recently in \([41, 43]\).
velocities are still valid in discrete-time. One can use the discrete-time variables introduced in this section in place of their continuous-time counterparts. The second-order Butterworth filter \((263)\) is discretized using the Newmark-\(\beta\) Method as follows:

\[
\begin{align*}
\dot{z}_i^f &= \frac{1}{4} \Delta t \ddot{z}_i^f + \frac{\Delta t^2}{4} (\dddot{z}_i^f + \dddot{z}_{i+1}^f) \\
\ddot{z}_i^f &= \frac{1}{2} \Delta t \dot{z}_i^f + \frac{\Delta t^2}{2} (\ddot{z}_i^f + \ddot{z}_{i+1}^f)
\end{align*}
\]

Choosing \(\omega_n = 2\) and \(\mu = \frac{1}{2}\), this method gives the filtered positions and velocities as follows:

\[
\begin{bmatrix}
\dot{z}_{i+1}^f \\
\ddot{z}_{i+1}^f
\end{bmatrix} = \frac{1}{4 + 4 \mu \omega_n \Delta t + \omega_n^2 \Delta t^2} \begin{bmatrix}
4 + 4 \mu \omega_n \Delta t - \omega_n^2 \Delta t^2 & 4 \Delta t & \omega_n^2 \Delta t^2 \\
-4 \omega_n^2 \Delta t & 4 - 4 \mu \omega_n \Delta t - \omega_n^2 \Delta t^2 & 2 \omega_n^2 \Delta t
\end{bmatrix} \begin{bmatrix}
z_i^f \\
\dot{z}_i^f
\end{bmatrix} + \begin{bmatrix}
z_i^m \\
\dot{z}_i^m + z_{i+1}^m
\end{bmatrix}
\]

where \(z_i^m\) and \(z_i^f\) are the corresponding value of quantities \(z_i^m\) and \(z_i^f\) at time instant \(t_i\), respectively. As with the continuous time version, \(\xi_i^m\) can be replaced with \(\xi_i^f\) in the estimator equations.

### 8.5 Numerical Simulations

This section presents numerical simulation results for the discrete-time estimator obtained in Section 8.4. In order to numerically simulate this estimator, simulated true states of an aerial vehicle flying in a room are produced using the
kinematics and dynamics equations of a rigid body. The vehicle mass and moment of inertia are taken to be \( m_v = 420 \text{ g} \) and \( J_v = [51.2 \ 60.2 \ 59.6] \text{ g.m}^2 \), respectively. The resultant external forces and torques applied on the vehicle are \( \phi_v(t) = 10^{-3}[10 \cos(0.1t) \ 2 \sin(0.2t) \ -2 \sin(0.5t)]^T \text{ N} \) and \( \tau_v(t) = 10^{-6}\phi_v(t) \text{ N.m} \), respectively. The room is assumed to be a cubic space of size \( 10\text{m}\times10\text{m}\times10\text{m} \) with the inertial frame origin at the center of this cube. The initial attitude and position of the vehicle are:

\[
R_0 = \expm_{\text{SO}(3)} \left( \left( \frac{\pi}{4} \times \begin{bmatrix} \frac{3}{7} & 0 & 0 \\ 0 & \frac{6}{7} & 0 \\ 0 & 0 & \frac{2}{7} \end{bmatrix} \right) \times \right), \text{ and } b_0 = [2.5 \ 0.5 \ -3]^T \text{ m}. \tag{303}
\]

This vehicle’s initial angular and translational velocity respectively, are:

\[
\Omega_0 = [0.2 \ -0.05 \ 0.1]^T \text{ rad/s}, \text{ and } \nu_0 = [-0.05 \ 0.15 \ 0.03]^T \text{ m/s}. \tag{304}
\]

The vehicle dynamics is simulated over a time interval of \( T = 150 \text{ s} \), with a time stepsize of \( \Delta t = 0.02 \text{ s} \). The trajectory of the vehicle over this time interval is depicted in Fig. 31. The following two inertial directions, corresponding to nadir and Earth’s magnetic field direction, are measured by the inertial sensors on the vehicle:

\[
d_1 = [0 \ 0 \ -1]^T, \ d_2 = [0.1 \ 0.975 \ -0.2]^T. \tag{305}
\]

For optical measurements, eight beacons are located at the eight vertices of the cube, labeled 1 to 8. The positions of these beacons are known in the inertial frame and their index (label) and relative positions are measured by optical sensors
onboard the vehicle whenever the beacons come into the field of view of the sensors. 

Three identical cameras (optical sensors) and inertial sensors are assumed to be 
installed on the vehicle. The cameras are fixed to known positions on the vehicle, 
on a hypothetical horizontal plane passing through the vehicle, 120° apart from 
each other, as shown in Fig. 30. All the camera readings contain random zero 
mean signals whose probability distributions are normalized bump functions with 
width of 0.001m. The following are selected for the positive definite estimator 
gain matrices:

\[
J = \text{diag}([0.9 \ 0.6 \ 0.3]), \quad M = \text{diag}([0.0608 \ 0.0486 \ 0.0365]), \quad (306)
\]

\[
\mathbb{D}_r = \text{diag}([2.7 \ 2.2 \ 1.5]), \quad \mathbb{D}_i = \text{diag}([0.1 \ 0.12 \ 0.14]).
\]
\( \Phi(\cdot) \) could be any \( C^2 \) function with the properties described in Section 8.2, but is selected to be \( \Phi(x) = x \) here. The initial state estimates have the following values:

\[
\hat{g}_0 = I, \quad \hat{\Omega}_0 = [0.1 \ 0.45 \ 0.05]^T \text{ rad/s}, \quad \text{and} \quad \hat{\nu}_0 = [2.05 \ 0.64 \ 1.29]^T \text{ m/s}. \quad (307)
\]

The performance of the proposed estimator is presented for two different cases.

### 8.5.1 CASE 1: At least three beacons are observed at each time instant

Having three beacons measured at each time instant guarantees full determination of vehicle’s translational and angular velocities instantaneously. A conic field of view (FOV) of \( 2 \times 40^\circ \) for cameras can satisfy this condition. The vehicle’s velocity is calculated by (268) in this case. The discrete-time estimator (280)-(284) is simulated over a time interval of \( T = 20 \text{ s} \) with sampling interval \( \Delta t = 0.02 \text{ s} \).

At each time instant, (280) is solved using the Newton-Raphson iterative method to find an approximation for \( F_i \). Following this, the remaining equations (all explicit) are solved to generate the estimated states. The principal angle of the attitude estimation error and the position estimation error for CASE 1 are plotted in Fig. 32. Plots of the angular and translational velocity estimation errors are shown in Fig. 33.
Figure 32: Principal angle of the attitude and position estimation error for CASE 1.
8.5.2 CASE 2: Less than three beacons are measured at some time instants

To implement the variational estimator for the case that less than three optical measurements are available, the field of view of the cameras is decreased to limit the number of beacons observed. Assuming the cameras have conical fields of view of $2\times 25^\circ$, the minimum number of beacons observed instantaneously drops to 1 during the simulated time interval. The dynamics model for the aerial vehicle, simulated time duration, and sample rate are identical to CASE 1. Fig. 34 depicts the principal angle of the attitude estimation error and the position estimation error for CASE 2, and Fig. 35 shows the angular and translational velocity esti-
Figure 34: Principal angle of the attitude and position estimation error for CASE 2.

mation errors. All estimation errors are shown to converge to a neighborhood of $(h, \varphi) = (I, 0)$ in both cases, where the size of this neighborhood depends on the magnitude of measurement noise.

8.6 Conclusion

This chapter proposes an estimator for rigid body pose and velocities, using optical and inertial measurements by sensors onboard the rigid body. The sensors are assumed to provide measurements in continuous-time or at a sufficiently high frequency, with bounded measurement noise. An artificial kinetic energy quadratic in rigid body velocity estimate errors is defined, as well as two fictitious potential
energies: (1) a generalized Wahba’s cost function for attitude estimation error in the form of a Morse function, and (2) a quadratic function of the vehicle’s position estimate error. Applying the Lagrange-d’Alembert principle on a Lagrangian consisting of these energy-like terms and a dissipation term linear in velocities estimation error, an estimator is designed on the Lie group of rigid body motions. In the absence of measurement noise, this estimator is shown to be almost globally asymptotically stable, with estimates converging to actual states in a domain of attraction that is open and dense in the state space. The continuous estimator is discretized by applying the discrete Lagrange-d’Alembert principle on the discrete Lagrangian and dissipation terms linear in rotational and translational
velocity estimation errors. In the presence of measurement noise, numerical simulations show that state estimates converge to a bounded neighborhood of the true states. Future extensions of this work include higher-order discretizations of the continuous-time filter given here and obtaining a stochastic interpretation of the variational pose estimator.
9 IDEAS FOR FUTURE WORK

The following are ideas to extend the research presented in this PhD dissertation:

• **Combining model-free and model-based estimation**: Implementations of filtering schemes similar to the variational estimator in the presence of measurements of direction vectors and angular velocities at different rates, use of a dynamics model for propagation of state estimates when measurements are available at low sampling rates, and design of state-varying or time-varying filter gains for faster convergence of state estimates.

• **Showing robustness to bounded measurement noise for model-free estimators**: Proving robustness of the variational estimators presented in Chapters 4 and 8 in the presence of measurement noise. This proof could be obtained by finding a neighborhood of the origin to which the estimates converge, given known upper bounds on the measurement errors. Lyapunov analysis will be exploited to show that outside the derived neighborhood, all the states estimates tend to converge to a region inside that neighborhood.

• **Computationally efficient discretization of model-free filter in SE(3)**: The Lie group variational integrator presented in Chapter 8 is implicit and needs to be solved using iterative Newton-Raphson method. This involves a
complicated algebra to calculate the rate of change of the implicit equation, in order to approximate its root. Moreover, due to the iterative structure of the solution, the estimator will be computationally slow. A discretization scheme proposed in [77] will be utilized in order to obtain computationally more efficient LGVI. Numerical simulations for this variational estimator in SE(3) will be carried out in future research.

- **Experimental validation of model-free filter in SE(3):** The performance of the variational estimator presented in Chapter 4 has been verified experimentally in [89]. The comparison of the variational estimator with the state-of-the-art filters can also be verified using the same setup by implementing all the estimators on the Android device. A similar set of experiments could be conducted to show the performance of the SE(3) version of the filter using experimental equipments.

- **Experimental validation of model-free filter in SO(3) in the presence of bias in angular velocities:** Gyroscopes are used in practice to provide angular velocities. The output of such sensors usually contain constant or variable drift which harms the performance of the filter. The estimator presented in Chapter 6 is designed in such a way that it could be robust to bias in the sensor readings. The performance of this version of the Variational Attitude Estimator needs to be verified experimentally.
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