Nonclassical Total Probability Formula and Quantum Interference of Probabilities

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Abstract
Interpretation of the nonclassical total probability formula arising in some quantum experiments is provided based on stochastic models described by means of a sequence of random vectors changing in the measurement procedures.

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1. Introduction

We are interested in giving meaning, within Kolmogorov’s axiomatics, to nonclassical versions of the total probability formula. This problem was considered in Khrennikov, 2001a, 2001b, on physical level of rigor, using the frequency approach, see von Mises, 1964.

Almost any treatise on quantum mechanics dwells on the fundamental two-slot experiment with e.g. electrons passing through one or two open slits (see, e.g., d’Espagnat, 1999, p.6). Let \( A_i \) denote an event of passing through the slit with label \( i \), here \( i = 1, 2 \). Interpretation of the results of this experiment and the related ones (see, e.g., Feynman and Hibbs, 1965, p.11) has led to the following formula for the probability of electron passing through the two open slots:

\[
P(A_1 \cup A_2) = P(A_1) + P(A_2) + 2\sqrt{P(A_1)P(A_2)} \cos \theta
\]

where \( P \) is a symbol of probability and \( \theta \) a certain parameter. Usually the origin of the term \( 2\sqrt{P(A_1)P(A_2)} \cos \theta \) is explained by referring to ”self-interference” inherent to the ”wave nature” of an electron. To be more precise, calculating relative frequencies of the considered events after repetitions of the experiment led to conclusion that the classical formula of addition for probabilities of disjoint events should be modified.
A remarkable achievement of physics was the employment of Hilbert space methods for description of quantum mechanical systems. In the standard formalism (see, e.g., d’Espagnat, 1995, p. 63-65) one derives formula (1) with the help of superposition principle. This elegant proof using complex-valued coefficients of a ”state-vector” decomposition with respect to the basis of eigenvectors of a self-adjoint operator corresponding to certain ”observable” rather loses links with traditional concepts of probability theory. Moreover, a number of specialists in quantum mechanics believe that with the Kolmogorov axiomatics it is impossible to justify formula (1). The discussion continues from the early days of the subject and still the joint viewpoint is not generated, see, e.g., Heisenberg, 1930, Dirac, 1930, Bohr, 1934, von Neumann, 1955, Feynman and Hibbs, 1965, Jauch, 1968, Beltrametti and Cassinelli, 1981, Holevo, 1980, 2001, Accardi, 1984, Ballentine, 1986, Parthasaraty, 1992, Meyer, 1993, Peres, 1994, d’Espagnat, 1995, 1999, Busch et al., 1995.

The careful analysis of probability spaces and families of random variables used for models of stochastic experiments gives the key to understanding the origin of the formulas like (1) (ch. Khrennikov, 1999, p.100-102, 2000, p. 5936, Thorisson, 2000, p.29, Accardi and Regoli, 2001, p.8-9). The main role here is played by the following contextualism principle: introduction of a random variable or a collection of random variables describing some properties of a physical system or an ensemble of systems should take into account the total collection (class) of the conditions, i.e. context, under which the values of these variables (or their distributions) were specified.

In fact the contextualism principle can also be found in the fundamental work by Kolmogorov, 1933, or in his paper, 1965 (p. 249): ”Thus, to say that an event A is ”random” or ”stochastic” and to assign it a definite probability $p = P(A | S)$ is possible only when we have already determined the class of allowable ways of setting up the series of experiments. The nature of this class will be assumed to be included in the conditions $S$“. So, the role of a specified class of conditions which can be reproduced is stressed (see details in Gnedenko, 1962, Shiryaev, 1998).

The aim of this paper is to show that a simple stochastic model of measurement of characteristics of some physical systems provides a quite natural explanation for the ”interferential” term arising in the formula (1). Let us remark once more that besides the probability space $(\Omega, \mathcal{F}, P)$ the essential role in construction of probabilistic models is played by families of random variables used for description of studied phenomena. Of course, having specified the model, one can consider the new (canonical) probability space with coordinate random variables.

It should be emphasized that neither we are going to analyze concrete quantum mechanical systems, nor impose on them a classical probability space $(\Omega, \mathcal{F}, P)$. Our goal is not to transform the physical phenomena into ”classical” probability events, i.e. elements of a $\sigma$-algebra $\mathcal{F}$ of subsets of a space $\Omega$, where $P$ is a measure defined on events and normalized so that $P(\Omega) = 1$. Deep study of the quantum probability problems can be found, e.g., in Accardi, 1984, 2001, Holevo, 1980, 2001, Meyer, 1993, Parthasaraty, 1992.

In the sequel it will be convenient to give another form to relation (1). Set $C = A_1 \cup A_2$.

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1In quantum theory this principle was introduced by N. Bohr who underlined that experimental conditions plays the crucial role in determining physical observables, see e.g. Bohr, 1934.

2The choice of the principal features (conditions) leads to various models.
where $A_1 \cap A_2 = \emptyset$ and rewrite (1) as a "nonclassical total probability formula":

$$P(C) = P(C|A_1)P(A_1) + P(C|A_2)P(A_2) + 2\sqrt{P(C|A_1)P(A_1)P(C|A_2)P(A_2)} \cos \theta,$$

(2)

where, as usual,

$$P(C|A_i) = P(CA_i)/P(A_i)$$

and $P(A_i) > 0$, $i = 1, 2$. The trivial case $P(A_i) = 0$ for some $i$ is henceforth excluded.

2. The model description

For the sake of simplicity consider some physical system $S$ possessing only properties $A$ and $C$ described by a pair of two-valued random variables ($A$ and $C$ "characteristics"). Suppose that there is a procedure permitting to reproduce the copies $S_1, S_2, \ldots$ of $S$ to get a statistical ensemble of $S$ in the following sense. Let $(a^{(1)}, c^{(1)}), (a^{(2)}, c^{(2)}), \ldots$ be a sequence of i.i.d. random vectors defined on some probability space $(\Omega, \mathcal{F}, P)$. Assume that for some $a_i, c_i \in \mathbb{R}$, $i = 1, 2$

$$P(a^{(1)} = a_1) = p_1^{(a)}, \quad P(a^{(1)} = a_2) = p_2^{(a)},$$

(3)

$$P(c^{(1)} = c_1) = p_1^{(c)}, \quad P(c^{(1)} = c_2) = p_2^{(c)},$$

(4)

where $p_1^{(a)} + p_2^{(a)} = 1$ and $p_1^{(c)} + p_2^{(c)} = 1$.

For $j \in \mathbb{N}$ and elementary event $\omega \in \Omega$ the vector $(a^{(j)}(\omega), c^{(j)}(\omega))$ characterizes the properties $A$ and $C$ of the system $S_j$. The independence assumption for the sequence $\{(a^{(j)}, c^{(j)})\}_{j \geq 1}$ reflects the usual noninteraction hypothesis for elements of stochastic ensemble. Further we can use more general assumptions as well. For each $j \in \mathbb{N}$ no conditions are imposed on the joint distribution of random variables $a^{(j)}$ and $c^{(j)}$. Moreover, note that for any $\omega \in \Omega$ the sequence $\{(a^{(j)}(\omega), c^{(j)}(\omega))\}_{j \geq 1}$ is not, in general, available. In fact, to get the data certain measurement procedures should be used.

Suppose that it is possible to apply to systems $S_j$ a measurement procedure $\mathcal{M}_A$ permitting for each $\omega \in \Omega$ to fix the value $a^{(j)}(\omega)$ of the characteristic describing property $A$. In other words, for every $\omega \in \Omega$ we have integers $1 \leq k_1(\omega) < k_2(\omega) < \ldots$ such that

$$a^{(k_1(\omega))}(\omega) = a_1, \quad a^{(k_2(\omega))}(\omega) = a_1, \ldots$$

and, respectively,

$$a^{(m_1(\omega))}(\omega) = a_2 \quad \text{for} \quad m_1(\omega) < m_2(\omega) < \ldots$$

where $m_j(\omega) \in \mathbb{N} \setminus \cup_i \{k_i(\omega)\}$.

It is worth to note that after the procedure $\mathcal{M}_A$ the property $C$ of initial systems $S_j$ in general can be changed. Thus, new systems $S'_j$ ($j \in \mathbb{N}$) arise. To describe the property $C$ of these systems it is natural to use random sequences $\{\tilde{\tau}^{(j)}\}_{j \geq 1}$ and $\{\tilde{\tau}^{(j)}\}_{j \geq 1}$ such that

$$P(\tilde{\tau}^{(j)} = \tau_r) = p_r^{(\tilde{\tau})}, \quad P(\tilde{\tau}^{(j)} = \tau_r) = p_r^{(\tilde{\tau})}, \quad j, r = 1, 2,$$

(5)

3 On the notion of a property see, e.g. Khrennikov, 1999, p. 59.

4 This is very natural from quantum viewpoint.
where \( p_i^1 + p_i^2 = 1 \), \( p_i^1 + p_i^3 = 1 \).

Assume that for each \( \omega \in \Omega \) and \( i = i(\omega) \in \mathbb{N} \) such that \( a(i)(\omega) = a_i \) the value \( \bar{c}^{(i)}(\omega) \) appears instead of \( c^{(i)}(\omega) \). Other values \( c^{(i)}(\omega) \) (for \( i = i(\omega) \in \mathbb{N} \setminus \cup_i \{ k_i(\omega) \} \)) are replaced by \( \hat{c}^{(i)}(\omega) \).

So, to describe the properties \( A \) and \( C \) for the ensemble of systems \( S'_j \) \((j \in \mathbb{N})\) we obtain two-dimensional sequences

\[
\{(a_1, \bar{c}^{(k_j(\omega))}(\omega))\}_{j \geq 1} \quad \text{and} \quad \{(a_2, \hat{c}^{(m_j(\omega))}(\omega))\}_{j \geq 1}
\]

Note that since \( k_j = k_j(\omega) \) and \( m_j = m_j(\omega) \) are random variables defined on probability space \((\Omega, \mathcal{F}, P)\) the same is true of \( \bar{c}^{(k_j)}(\omega) = \bar{c}^{(k_j(\omega))}(\omega) \) and \( \hat{c}^{(m_j)} = \hat{c}^{(m_j(\omega))}(\omega) \).

It is always possible to construct a probability space \((\Omega, \mathcal{F}, P)\) where stochastic sequences \( \{(a^{(j)}, c^{(j)})\}_{j \geq 1}, \{\bar{c}^{(j)}\}_{j \geq 1}, \{\hat{c}^{(j)}\}_{j \geq 1} \) with the above-mentioned properties are defined. Moreover, using the standard product of probability spaces it is easy to guarantee the independence of terms of these sequences and independence of the collection of three sequences as well. However, we do not use the last opportunity.

Suppose also that a measurement procedure \( \mathcal{M}_C \) can be applied to systems \( S'_j \) \((j \in \mathbb{N})\) to fix the values of the characteristics of property \( C \). Thus we get systems \( S''_i \) \((i \in \mathbb{N})\) having \( \bar{c}^{(k_j(\omega))}(\omega) \) or \( \hat{c}^{(m_j(\omega))}(\omega) \) as \( C \)-characteristics for \( \omega \in \Omega \) and \( i = k_j(\omega) \) or \( i = m_j(\omega) \). Of course, the \( A \)-characteristics of systems \( S'_j \) \((j \in \mathbb{N})\) in general can not be conserved.

3. Nonclassical total probability formula

Now we consider from the view-point of limit theorems for random variables defined on \((\Omega, \mathcal{F}, P)\) the approach used in Khrennikov, 2001a, 2001b. Namely, we discuss statistical properties of \( C \)-characteristics for ensembles of systems \( \{S_j\}_{j \geq 1} \) and \( \{S''_j\}_{j \geq 1} \). In this regard it is useful to recall the following statement from d’Espagnet, 1999. (see p. 15): "Quantum mechanics is essentially a statistical theory. Except in special cases it makes no prediction that bears on individual systems. Rather, it predicts statistical frequencies. In other words, it predicts as a rule, the number \( n \) of times that a given event will be observed when a large number \( N \) of physical systems of the same type and satisfying specified conditions are subjected to a given measurement process”.

We are not going to analyze the measurement procedures, preparing (for measurement) procedures and the mechanism governing the modification of initial properties of systems under various influences (see, e.g., Holevo, 1980, 2001).

Our very simple model with changing random variables permits to explain easily the asymptotical behaviour for frequencies of events studied in Khrennikov, 2001a, 2001b.

Introduce some notation. Let \(| \cdot |\) be a number of elements of a finite set. For \( \omega \in \Omega \), \( N \in \mathbb{N} \) and \( j = 1, 2 \) let

\[
g_{jN}(\omega) = \frac{|\{i \in \{1, \ldots, N\} : c^{(i)}(\omega) = c_j\}|}{N}.
\]  (6)

Set \( N_1(\omega) = \max\{i : k_i(\omega) \leq N\}, \ N_2(\omega) = \max\{i : m_i(\omega) \leq N\} \). For \( \omega \in \Omega \) and \( j = 1, 2 \) put

\[
n_{j1}(\omega) = |\{i \in \{1, \ldots, N_1(\omega)\} : c^{(h_1(\omega))}(\omega) = c_j\}|,
\]
\[ n_{j2}(\omega) = |\{i \in \{1, \ldots, N_2(\omega)\} : c^{(m_i(\omega))}(\omega) = c_j\}|, \]
\[ m_{j1}(\omega) = |\{i \in \{1, \ldots, N_1(\omega)\} : \bar{c}^{(k_i(\omega))}(\omega) = c_j\}|, \]
\[ m_{j2}(\omega) = |\{i \in \{1, \ldots, N_2(\omega)\} : \bar{c}^{(m_i(\omega))}(\omega) = c_j\}|. \]

To simplify the notation (for \( j, r = 1, 2 \) and \( \omega \in \Omega \)) we omit the dependence of \( n_{jr}(\omega) \) and \( m_{jr}(\omega) \) on \( N \).

Evidently, for all \( \omega \in \Omega, j = 1, 2 \) and \( N \in \mathbb{N} \) one has
\[ q_{jN}(\omega) = \frac{m_{j1}(\omega)}{N} + \frac{m_{j2}(\omega)}{N} + \gamma_{jN}(\omega) \quad (7) \]
where
\[ \gamma_{jN}(\omega) = \frac{n_{j1}(\omega) - m_{j1}(\omega)}{N} + \frac{n_{j2}(\omega) - m_{j2}(\omega)}{N}. \]

**Theorem 1.** Let \( \{(a^{(j)}, c^{(j)})\}_{j \geq 1}, \{(a^{(j)}, \bar{c}^{(j)})\}_{j \geq 1} \) and \( \{(a^{(j)}, \bar{c}(j))\}_{j \geq 1} \) be sequences of pair wise independent random vectors identically distributed within each sequence, i.e. for all \( j \in \mathbb{N} \)
\[ (a^{(j)}, c^{(j)}) \overset{D}{=} (a^{(1)}, c^{(1)}), (a^{(j)}, \bar{c}^{(j)}) \overset{D}{=} (a^{(1)}, \bar{c}^{(1)}), (a^{(j)}, \bar{c}(j)) \overset{D}{=} (a^{(1)}, \bar{c}(1)), \]
here \( D \) means the equality of distributions. Assume that (4) – (5) hold true and \( a^{(1)} \) is not degenerate. Then for all \( j, r = 1, 2 \) and \( P \)-a.e \( \omega \in \Omega \) there exist nonrandom limits
\[ p_{jr}^{c} := \lim_{N \to \infty} \frac{m_{jr}(\omega)}{N_{r}(\omega)}, \quad \gamma_{j} := \lim_{N \to \infty} \gamma_{jN}(\omega) \quad (8) \]
and besides
\[ p_{j}^{c} = p_{j1}^{c} a_{1} + p_{j2}^{c} a_{2} + \gamma_{j}. \quad (9) \]

**Proof.** Borelean functions of pairwise independent random vectors are pairwise independent. Therefore, by the Etemadi SLLN (see e.g. Borkar, 1995, p. 69) one has for \( P \)-a.e. \( \omega \in \Omega \) and \( j = 1, 2 \)
\[ q_{jN}(\omega) = \frac{1}{N} \sum_{i=1}^{n} I_{(c^{(i)}(\omega) = c_j)}(\omega) \to E I_{(c^{(1)} = c_j)} = p_{j}^{c} \quad \text{as} \quad N \to \infty, \quad (10) \]
where the indicator function of an event \( D \)
\[ I_{D}(\omega) = \begin{cases} 1, & \omega \in D; \\ 0, & \omega \notin D. \end{cases} \]

Analogously with probability one
\[ \frac{N_{1}(\omega)}{N} \to p_{1}^{a} \quad \text{and} \quad \frac{N_{2}(\omega)}{N} \to p_{2}^{a} \quad \text{as} \quad N \to \infty. \quad (11) \]
It is clear that
\[
\frac{m_{j1}(\omega)}{N} = \frac{1}{N} \sum_{i=1}^{N} I_{\{\tilde{c}^{(i)}(\omega) = c_j\}}(\omega)I_{\{a^{(i)}(\omega) = a_1\}}(\omega),
\]
\[
\frac{m_{j2}(\omega)}{N} = \frac{1}{N} \sum_{i=1}^{N} I_{\{\tilde{c}^{(i)}(\omega) = c_j\}}(\omega)I_{\{a^{(i)}(\omega) = a_2\}}(\omega).
\]

The same reasoning shows that for \( P \)-a.e. \( \omega \in \Omega \) there exist
\[
\lim_{N \to \infty} \frac{m_{j1}(\omega)}{N} = P(\tilde{c}^{(1)} = a_j, a^{(1)} = a_1), \tag{12}
\]
\[
\lim_{N \to \infty} \frac{m_{j2}(\omega)}{N} = P(\tilde{c}^{(1)} = a_j, a^{(1)} = a_1). \tag{13}
\]

The event consisting of those \( \omega \) rendering \( a^{(i)}(\omega) = a_1 \) for all \( i \in \mathbb{N} \) or \( a^{(i)}(\omega) = a_2 \) for all \( i \in \mathbb{N} \) has \( P \)-measure zero. So, there exists an event \( U \) with \( P(U) = 1 \) such that both \( N_1(\omega) > 0 \) and \( N_2(\omega) > 0 \) for all \( \omega \in \Omega \) and \( N > M(\omega) \). Hence for \( \omega \in U \) and \( N > M(\omega) \) we have
\[
\frac{m_{jr}(\omega)}{N} = \frac{m_{jr}(\omega)N_r(\omega)}{N}, \quad j, r = 1, 2.
\]

Taking into account (11) – (13) we establish the existence of limits \( p_{jr}^{c|a} \) in (8) and due to (10) the validity of the second relation in (8) follows. Now (9) is obvious and the proof is complete.

**Remark 1.** Successive employment of procedures \( \mathcal{M}_A \) and \( \mathcal{M}_C \) for systems \( S_j \) and \( S_j' \) \((j \in \mathbb{N})\) permits to determine \( N_1(\omega), N_2(\omega) \) and \( m_{jr}(\omega) \) for \( j, r = 1, 2 \). Thus, according to (8) and (11) there are strongly consistent estimates for \( p_{jr}^{c|a} \) and \( p_r^{c} \) \((j, r = 1, 2)\). To determine \( q_{jN}(\omega) \) one should apply the procedure \( \mathcal{M}_C \) to systems \( S_1, \ldots, S_N \). However, it would change the (distribution of) \( A \)-characteristics of these systems. Therefore, to obtain the strongly consistent estimates for parameters \( p_j^{c} \) \((j = 1, 2)\) we can proceed in the following manner. Invoking the principle of reproducibility we can construct besides the copies \( S_j \) \((j \in \mathbb{N})\) of a system \( S \) the copies \( \tilde{S}_j \) \((j \in \mathbb{N})\) described by a sequence of random vectors \( \{\{\tilde{a}^{(i)}, \tilde{c}^{(i)}\}\}_{i \geq 1} \). The standard extension of a probability space \( (\Omega, \mathcal{F}, P) \) can be used to define this auxiliary random sequence. For an extension we keep the same notation \((\Omega, \mathcal{F}, P)\). Applying the procedure \( \mathcal{M}_C \) to systems \( \tilde{S}_j \) \((j \in \mathbb{N})\) and defining \( \tilde{q}_{jN}(\omega) \) analogously to (8) we see that for \( P \)-a.e. \( \omega \in \Omega \) and \( j = 1, 2 \)
\[
\tilde{q}_{jN}(\omega)/N \to p_j^{c} \quad \text{as} \quad N \to \infty.
\]

Thus under the conditions of Theorem 1 one can provide the strongly consistent estimates for the values \( \gamma_j \) \((j = 1, 2)\) appearing in formula (8).

**Remark 2.** The notation \( p_{jr}^{c|a} \) for limits in (8) seems natural as \( m_{jr}(\omega)/N_r(\omega) \) is the relative frequency of appearing value \( c_j \) for the characteristics of property \( C \) in a sequence of those systems which have \( a_r \) value for the characteristics of property \( A \). However, the sense of formula (8) is demonstrated much better by relations (12) and (13) showing that
\[ p_{j1}^{\text{cl}} = P(\bar{c}(1) = c_j|a(1) = a_1), \quad p_{j2}^{\text{cl}} = P(\bar{c}(1) = c_j|a(1) = a_2). \] (14)

Thus the additional (with respect to the classical total probability formula) term \( \gamma_j \) appears in (9) because the following equalities need not hold true
\[
\begin{align*}
P(\bar{c}(1) = c_j|a(1) = a_1) &= P(\bar{c}(1) = c_j|a(1) = a_1), \\
P(\bar{c}(1) = c_j|a(1) = a_2) &= P(\bar{c}(1) = c_j|a(1) = a_2),
\end{align*}
\]

In such a way it is more informative to rewrite (9) in the form
\[
p_c = p_c|a_j^1 + \hat{p}_c|a_j^2 + \gamma_j, \quad (15)
\]

setting now
\[
\begin{align*}
p_{j1}^{\text{cl}} &= P(\bar{c}(1) = c_j|a(1) = a_1), \\
p_{j2}^{\text{cl}} &= P(\bar{c}(1) = c_j|a(1) = a_2), \quad j = 1, 2. \quad (16)
\end{align*}
\]

Formula (15) clarifies as well that simultaneous measurement of characteristics \( A \) and \( C \) in general is not supposed.

**Remark 3.** Due to (15) and (16) one has
\[
|\gamma_j| \leq 1, \quad j = 1, 2.
\]

In the particular case when for each \( j \in \mathbb{N} \) random variables \( \bar{c}(j) \) and \( a(j) \) are independent and the same is true for \( \hat{c}(j) \) and \( a(j) \) Theorem 1 implies that
\[
p_{j1}^c = p_{j1}^\text{cl}p_1^a + p_{j2}^\text{cl}p_2^a + \gamma_j, \quad (17)
\]

So, the equality \( \gamma_j = 0 \) (if \( \gamma_1 = 0 \) then \( \gamma_2 = 0 \) and vice versa) is satisfied if and only if the point \( p_{j1}^c \) is appropriately located on the segment with end points \( p_{j1}^\text{cl} \) and \( p_{j2}^\text{cl} \). Evidently, for any given \( p_{j1}^c, p_{j2}^c \) it is always possible to indicate \( \bar{p}_{j1}^\text{cl} \) and \( \bar{p}_{j2}^\text{cl} \) \( (j = 1, 2) \) to guarantee the relation \( \gamma_j = 0 \). Namely, one takes \( \bar{p}_{j1} \) in such a way that
\[
\max\{0, (p_{j2}^c - p_{j1}^c)/p_1^a\} \leq \bar{p}_{j1} \leq \min\{1, p_{j2}^c/p_1^a\}
\]

and after that chooses \( \bar{p}_{j2} = (p_{j2}^c - \bar{p}_{j1}p_{j2}^a)/p_2^a, \quad j = 1, 2. \)

**Remark 4.** It is possible to write formula (15) analogously to (2) using the auxiliary parameters \( \lambda_j \) determined by relation
\[
\gamma_j = 2\sqrt{p_{j1}^\text{cl}p_1^a p_{j2}^\text{cl}p_2^a} \lambda_j, \quad j = 1, 2.
\]

Note that in general \( \lambda_j \) can take any real values. In the particular case described by formula (17) we establish that
\[
\lambda_j \leq 1 \iff t_{j1} + t_{j2} \geq \sqrt{p_j^c},
\]
\[ \lambda_j \geq -1 \iff |t_{j1} - t_{j2}| \leq \sqrt{p^c_j} \]

where \( t_{j1} = \sqrt{p_j^c p_1^a}, t_{j2} = \sqrt{p_j^c p_2^a} \) \((j = 1, 2)\). If, moreover, \( p_j^c = p_j^\circ \) then

\[ |\lambda_j| \leq 1 \iff \frac{p_j^c}{1 + 2\sqrt{p_1^a p_2^a}} \leq p_j^\circ \leq \frac{p_j^c}{1 - 2\sqrt{p_1^a p_2^a}}. \tag{18} \]

The last relation shows that the possibility to represent \( \lambda_j \) as the cosine of some angle in this special case means that the distribution of \( c^{(1)} \) is obtained from that of \( c^{(1)} \) by a "bounded perturbation".

**Remark 5.** It is not difficult to complicate the scheme of measuring characteristics of properties \( A \) and \( C \). For instance one can assume that procedures \( M_A \) and \( M_C \) do not permit to fix exactly in all experiments (i.e. for all systems) the values of characteristics for properties \( A \) and \( C \). Then for validity of Theorem 1 it is sufficient to suppose that the number of faults (or "nonsuccessful") measurements is \( P \)-a.e. \( o(N) \) (as \( N \to \infty \)) among the systems with labels \( 1, \ldots, N \). One can obtain the analogue of Theorem 1 assuming (see e.g. Khrennikov, 1999, p.66) that prior to measurement procedures \( M_A \) and \( M_C \) one has to implement some preparing procedures \( P_A \) and \( P_C \) (modifying in an appropriate manner the properties of systems under consideration).

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