ADE string vacua with discrete torsion

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ABSTRACT

We complete the classification of (2,2) string vacua that can be constructed by diagonal twists of tensor products of minimal models with ADE invariants. Using the Landau–Ginzburg framework, we compute all spectra from inequivalent models of this type. The completeness of our results is only possible by systematically avoiding the huge redundancies coming from permutation symmetries of tensor products. We recover the results for (2,2) vacua of an extensive computation of simple current invariants by Schellekens and Yankielowicz, and find 4 additional mirror pairs of spectra that were missed by their stochastic method. For the model (1)$^9$ we observe a relation between redundant spectra and groups that are related in a particular way.

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1 Introduction

In addition to the geometrical compactification on Calabi–Yau manifolds, it soon became clear that abstract conformal field theories can be used to construct perfectly sensible 4-dimensional string vacua. When starting from the standard heterotic construction with gauge group $SO(10) \times E_8$, there are basically three approaches for getting a modular invariant string theory with space–time supersymmetry. For tensor products of $N = 2$ minimal models, Gepner used an orbifold-type construction [1]. Aiming at twisted Landau–Ginzburg models, Vafa [2] could prove modular invariance and charge integrality for a very general class of orbifolds. Alternatively, Schellekens and Yankielowitz [3] used simple currents in their construction of consistent string vacua. Of course, there is a large overlap between these methods. In particular, Gepner’s models are all contained in the other two classes. Since the symmetries of $N = 2$ minimal models are generated by simple currents, the last approach is the most general one using only diagonal symmetries of ADE models [4]. Permutation symmetries of identical factors [5], on the other hand, can be moded in a simple and general way using the Landau–Ginzburg framework [6], but are outside the class of the simple current invariants. For diagonal symmetries of ADE models, our results indicate that, at least for $(2,2)$ vacua, these two approaches are actually equivalent.

In the present work we complete the construction of string vacua with a global $(2,2)$ worldsheet supersymmetry that can be obtained by diagonal twists of tensor products of $N = 2$ minimal models with ADE modular invariants, including the most general discrete torsions. Our motivation for focusing on this class of models is twofold: First, there have been extensive, but incomplete, computations in the literature [3, 5, 7, 8, 9, 10], which allow a comparison of results. Here our main contribution is to systematically avoid redundancies in the crucial cases for which a naive count of the number of possibilities seems to render completeness elusive. Furthermore, concerning this huge degeneracy due to permutation symmetries, the ADE models obviously also contain the crucial cases of the much larger class of Landau–Ginzburg string vacua. Thus our work is also a necessary first step for a complete enumeration of the corresponding large set of spectra [11].

As we use the Landau–Ginzburg description of $N = 2$ minimal models for the calculation of spectra, we first recall some results on Landau–Ginzburg orbifolds and describe the most general structure of spectra that we have to expect. In this framework simple formulas exist for the charge degeneracies of the chiral primary fields (or the Ramond ground states) [1]. It turns out that it is sufficient to consider only the well-known 168 potentials that describe pure diagonal invariants [4], plus the 12 potentials that in addition contain one or more $E_7$ invariants. In fact, we can argue that we can also omit the more symmetric of the two tensor products with 8 factors. The remaining models then only require modest computation time, except for the unique product of 9 factors.

It is thus only in the case of the model $(1)^9$, i.e. the tensor product of 9 minimal models at level 1, that we need to eliminate the redundancy coming from permutation symmetries. This model is treated in some detail in section 3. We first describe a simple method that allows us to eliminate all redundancies in the twist groups as well as in the corresponding allowed

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1 We use the term ‘spectrum’ for the non-singlet part of the massless string modes. The numbers of $E_6$ singlets could, if necessary, easily be obtained by just replacing (or extending) the subroutine for the computation of spectra in our computer program, which is only a small part of the code.
values of the discrete torsions. We end up with about $10^5$ orbifolds that are all inequivalent from the conformal field theory point of view. The massless spectra, however, are still highly degenerate. In section 4 we present and discuss our results. As expected for our complete calculation, we find perfect mirror symmetry for this class of $(2,2)$ models. Studying the explicit group structures, we observe that a group only produces new spectra if its projection in the (untwisted) chiral ring is irreducible in a certain sense. If true in general, this could explain most of this remaining degeneracy and provide an efficient computational tool.

2 Landau–Ginzburg orbifolds and their spectra

The Landau–Ginzburg description of superconformal field theories relies on the assumption that an $N = 2$ supersymmetric theory with a quasi-homogeneous superpotential $W(\lambda^m X_i) = \lambda^d W(X_i)$ will flow to an $N = 2$ superconformal model at the renormalization group fixed point [12]. For transverse polynomials $W$, the central charge $c = 3 \sum_i (1 - 2q_i)$ can be computed from the charges $q_i = n_i/d$ of the superfields $X_i$ [13]. The potentials with $c < 3$, which are easily enumerated, are in one-to-one correspondence with the $N = 2$ supersymmetric minimal models, including all non-diagonal modular invariants:

$$
A_n : W = X^{n+1}, \quad D_n : W = X^{n-1} + XY^2, \\
E_6 : W = X^3 + Y^4, \quad E_7 : W = X^3 + XY^3, \quad E_8 : W = X^3 + Y^5.
$$

The chiral ring structure and in particular the charge degeneracies, as derived from the Landau–Ginzburg description, coincide with those from conformal field theory. Therefore the uniqueness of the ADE classification implies that these models must coincide if appropriate fixed points exist. In terms of the above superfields the discrete symmetries of the minimal models are realized by simple phase symmetries. The minimal model $(k)$ at level $k$ with the diagonal invariant, which has a discrete $\mathbb{Z}_{k+2}$ symmetry, thus corresponds to $A_{k+1}$. The non-diagonal models $(k)_D$ exist at even levels $k \in 2\mathbb{Z}$ and are described by the potentials $D_{4k+2}$, and finally there are the exceptional invariants $E_6$, $E_7$ and $E_8$ at levels $k = 10, 16, 28$.

By inspection of the superpotentials (2), one sees that $E_6$ is a direct product of $A_2$ and $A_3$, and that $E_8$ is a direct product of $A_2$ and $A_4$. This has also been shown directly for the corresponding conformal field theories, provided that Ramond and Neveu–Schwarz states are paired only among themselves, i.e. for all models with at least $(1,1)$ superconformal invariance on the world sheet [14]. It is also well-known that $A$ and $D$ invariants are $\mathbb{Z}_2$ orbifolds of one another [2]. As we want to twist these models by all phase symmetries using the full set of consistent discrete torsions, the $D$ invariants should not yield anything new. Indeed, we checked in a large number of cases that the set of spectra we obtained for models that are related by exchange of $A$ and $D$ invariants coincide exactly (this is true also in the presence of $E_7$). Therefore in principle we only need to consider the 168 different combinations of minimal models with central charges adding up to 9 and with diagonal invariants, which were listed in refs. [7,8], and the additional 12 cases where one or more of the models at level 16 come with the $E_7$ invariant.

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2 From the $D$ invariant we can get back to the $A$ invariant by modding the “quantum” $\mathbb{Z}_2$ [3]; in case of extended chiral algebras this is less obvious in the simple current formalism than for the Landau–Ginzburg description.
For space–time supersymmetric vacua we require that all (left) charges are integral. To this end we orbifoldize these models using all diagonal symmetries and discrete torsions that are consistent with (2,2) superconformal invariance and integral charges [6]. For the construction of the phase symmetries we use the procedure that is already discussed in [10].

Our calculation of the spectra is based on the results of refs. [2, 3]: The chiral ring of a Landau–Ginzburg orbifold can be computed by summing over all twisted sectors, taking into account that the chiral states of each sector correspond to the reduced Landau–Ginzburg model that is obtained by setting to 0 all fields that are twisted by a non-trivial phase. Thus we only need the charges and the transformation properties of the twisted vacua $|h\rangle$ in the $(c, c)$ ring (Ramond ground states and anti-chiral states can be obtained from this by spectral flow). Let the field $X_i$ transform with a phase $\exp 2\pi i\theta^h_i$ under the group element $h$. Then the semi-classical analysis of refs. [2, 3] yields the left/right charges

$$Q_\pm = \sum_{\theta_i \in \mathbb{Z}} (\frac{1}{2} - q_i) \pm (\theta^h_i - \frac{1}{2})$$

and the transformations

$$g|h\rangle = (-1)^K gK \varepsilon(g, h) \frac{\det g_{hk}}{\det g} |h\rangle$$

for commuting group elements $g$ and $h$. Here $K : g \to K_g \in \{0, 1\}$ is a homomorphism of the group into $\mathbb{Z}_2$, which determines the signs of the actions of the symmetry generators on the Ramond sector. The discrete torsions $\varepsilon(g, h)$ have to be multiplicative in both entries and must fulfill $\varepsilon(g, h)\varepsilon(h, g) = \varepsilon(1, g) = 1$. The projector to integral left charges $j = e^{2\pi i\theta^0_j}$ always has to be part of the twist group. In order to get a (2,2) theory with integral left and right charges, we must require $\varepsilon(j, g) = (-1)^{K_gK_j} \det g$ and $(-1)^{K_g} = \det g$ (thereby restricting the possible group elements to having determinants $\pm 1$).

We denote by $p_{ij}$ the number of states with $U(1)$ charges $(q_L, q_R) = (i, j)$, and we now want the find the most general form of these numbers for (2,2) vacua that can come from Landau–Ginzburg orbifolds. First, charge conjugation and spectral flow imply the Poincaré duality $p_{i,j} = p_{D-i,D-j}$; i.e. for $D = 3$ we have $p_{11} = p_{22}$ and $p_{12} = p_{21}$. These numbers determine the numbers of generations and anti-generations. One can show that there are further restrictions for states with vanishing left or right charge, namely $p_{0,i} = p_{0,D-i}$ and $p_{i,0} = p_{D-i,0}$ [11]. We also know that $p_{33} = p_{30} = p_{03} = p_{00} = 1$ (of course this is exactly what we wanted to achieve with the restrictions on $\varepsilon$ and $K$). The remaining freedom is in $p_{01} = p_{02}$ and $p_{10} = p_{20}$. These numbers can only assume the values 0, 1 or 3. From conformal field theory we know that in the context of the heterotic string a positive $p_{01}$ or $p_{10}$ implies an extension of the gauge group to (at least) $E_7$ or extended space–time supersymmetry. As this excludes the possibility of having chiral fermion generations, $p_1 = p_{01} + p_{10}$ can be non-zero only if the Euler number $\chi = 2(p_{12} - p_{11})$ vanishes. In fact, if $p_{01} = 3$ or $p_{10} = 3$ we always find that $p_{ij}$ factorizes into left and right contributions, i.e. $p_{ij} = p_{00}p_{0j}$; otherwise this product only gives a lower bound for $p_{ij}$.

In [11] the possibility of twisting by group actions with negative determinants in case of odd $d$ was excluded because of the constraints on the discrete torsions with $j$. We think, however, that this is too restrictive for the following reasons: We can always make $d$ even by adding a trivial field $X_0$ with a contribution $X_0^2$ to the superpotential. Even without resorting to trivial fields, it is easy to see that the fields in the Ramond sector are quantized in units of
1/2d rather that in units of 1/d. (Accordingly, the simple current that corresponds to \( j \) is of order \( 2(k + 2) \) for minimal models at odd levels \( k \).) Both arguments show that, for odd \( d \), it is sensible to assume that \( j \) is of order \( 2d \) rather than of order \( d \), implying that the torsion constraints no longer exclude a negative determinant. We do, however, agree with ref. [6] in the conclusion that such twists can be ignored, because they cannot yield anything new (see also ref. [14]).

When combined, these results are very useful: We can omit the model \((1)^7(4)\) from our computation, which would be the worst case concerning redundancies, except for \((1)^9\). In a first step, we can replace \((1)^7(4)_A\) by \((1)^7(4)_D\). The phase symmetries of \( X_1^3 + X_1X_2^2 \), however, are contained in the symmetries of \( X_1^3 + X_2^3 \) except for a \( \mathbb{Z}_2 \) factor with negative determinant. That generator can be omitted without losing any spectra, and thus all possible spectra are already present in the model \((1)^9\). (The authors of ref. [14] arrive at a similar conclusion within a rather different framework.)

For simple types of superpotentials, in particular the present case of ADE models, it is quite straightforward to implement the explicit sum over the chiral ring and the projection to invariant states in a computer program. An alternative, which is independent of the singularity type of \( W \) and therefore does not require the construction of a basis of the chiral ring, is based on the index formula

\[
-\chi = \frac{1}{|G|} \sum_{gh=hg} (-1)^{N_h + K_h + K_g} \varepsilon(g, h) \prod_{\theta^g = \theta^h = 0} \frac{n_i - d}{n_i},
\]

for the Euler number \( \chi \). A similar formula can be derived for the dimension \( \chi = \sum_{ij} p_{ij} \) of the chiral ring

\[
\chi = \frac{1}{|G|} \sum_{gh=hg} (-1)^{N_h + K_h + K_g} \varepsilon(g, h) \prod_{\theta^g = \theta^h = 0} \frac{n_i - d}{n_i},
\]

where \( N_h \) denotes the number of fields that are invariant under \( h \). Together with \( p_1 \) these two quantities allow computation of \( p_{11} \) and \( p_{12} \). Contributions to \( p_{10} \) and \( p_{01} \), which must be zero for \( \chi \neq 0 \), can only come from twisted ground states [16], so we do not need any explicit knowledge of the ring structure for the calculation of the complete non-singlet spectrum. As a check we compared the results of the two programs in many examples. The algorithm that constructs the chiral ring turns out to be much faster.

## 3 Redundancies and the model \((1)^9\)

If all factors in the tensor product are different, the algorithm for computing subgroups that we gave in ref. [10] constructs all inequivalent subgroups of an abelian symmetry group exactly once. For each such group we then have to generate all antisymmetric matrices that satisfy the quantization conditions appropriate for discrete torsions [17]. In case of permutation symmetries, however, there can be a huge redundancy both in the groups and in the discrete torsions that are generated. In the present case of the model \((1)^9\), for example, we would already get \( 3^{21} \) different torsion matrices for the single maximal symmetry group with determinant 1, which is \((\mathbb{Z}_3)^8\), if the torsion constraint with \( j \) is taken into account. Unfortunately, finding an algorithm that directly constructs exactly the inequivalent cases is very difficult in general.
From a pragmatic point of view, however, we do not need a direct construction, because we can let the computer sort out the redundant cases. Indeed, there is a conceptually very simple algorithm to avoid any redundancies, and with refinements it also works sufficiently fast. The idea is to write the group action as an array and to impose an ordering (lexicographical, for example). Then we consider all the transformations of this array induced by permutations of identical factors in the tensor product and keep a particular group action only if it is minimal in the set of equivalent actions. By remembering the automorphisms that are found as a by-product in this process we can then also eliminate the redundant torsions in a similar way.

Generating all the groups and then all permutations for each of them would of course take much too long; but we can organize the generation of groups in such a way that we only get a modest redundancy in the first step and at the same time only have to check for a subset of the permutations that have a chance to give a “smaller” group. Fortunately, we only had to implement this procedure for the single model \((1)^9\), for which the whole computation took a few days on a workstation. The elimination of redundancies for that model is illustrated in table 1: Altogether, the naïve counting overestimates the number of models by more than 5 orders of magnitude. The only other critical case could be omitted, as we discussed in section 2. The remaining 178 models together also needed a few days. We now describe in more detail how the above concept can be applied to the model with the maximal permutation symmetry \(S_9\).

We first introduce a convenient notation for a general diagonal symmetry group. Such a group of rank \(r\) can be represented by a matrix

\[
G^{(r)} = (1_r, M) \tag{7}
\]

of integers modulo 3, where \(1_r\) is the unit matrix and \(M\) is an \(r \times (9 - r)\) matrix. Here we already used the freedom to redefine generators and to permute the identical factors in the tensor product, i.e. the columns of the \(r \times 9\) matrix \(G^{(r)}\). The determinant condition is taken into account by requiring that the sum of the numbers in each line is a multiple of 3; furthermore, as the generator \(j\) has to be contained, the last line can be computed in terms of the remaining ones by requiring that the sum of the numbers in each column is 1 modulo 3, where of course \(1 \leq r \leq 8\).

Now we define the normal form of a group as the smallest number whose \(r(9 - r)\) digits, which are 0, 1 or 2, are the lines of a matrix \(M\) in an equivalence class (we let this number begin with the last line). The remaining permutations that are consistent with the choice (7) induce three types of transformations of the matrix \(M\): Permutations of the last \(9 - r\) factors simply permute the columns of \(M\). Permutations of the first \(r\) factors, on the other hand, induce a permutation of the lines of \(M\). It is straightforward to make efficient use of this freedom already in the construction of candidate matrices \(M\) by choosing their entries in such a way that no permutation of a line can be ‘smaller’ than the last one and that within each line the permutations of columns that are not yet fixed by the previous lines are taken into account. When we finally check for possible smaller matrices, we only need to consider those permutations of lines that replace the last one by a line consisting of the same digits. Furthermore, if we keep for each line the information about the residual permutation symmetry, we can automatically use an optimal permutation of the columns and thereby considerably reduce the number of transformations that have to be checked. The whole procedure of generating all inequivalent groups then only takes a fraction of a second.
Table 1: Numbers of (inequivalent) groups and torsions for the model (1)

| rank | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | together |
|------|---|---|---|---|---|---|---|---|----------|
| groups | 1 | 1093 | 99463 | 925771 | 925771 | 99463 | 1093 | 1 | $2.1 \times 10^6$ |
| inequiv. | 1 | 5 | 21 | 49 | 49 | 21 | 5 | 1 | 152 |
| spectra | 1 | 5 | 14 | 22 | 22 | 14 | 5 | 1 | 23 |
| torsions | 1 | 1093 | 298389 | $2.5 \times 10^7$ | $6.7 \times 10^8$ | $5.9 \times 10^9$ | $1.6 \times 10^{10}$ | $1.1 \times 10^{10}$ | $3.3 \times 10^{10}$ |
| inequiv. | 1 | 5 | 43 | 370 | 3517 | 20604 | 48285 | 30867 | 103692 |
| spectra | 1 | 5 | 20 | 29 | 31 | 31 | 31 | 31 | 31 |

Finally, there are $9!/(r!(9-r)!)$ possibilities for choosing subsets of $r$ factors to become the first $r$ factors in the tensor product. Together with the permutations within each sector, they generate the full symmetric group $S_9$. Since we require that the matrix of generators is of the form $(1, M)$, such a choice is only consistent if the square matrix consisting of the corresponding $r$ columns can be transformed into the unit matrix, i.e. is invertible. For a given matrix $M$, this excludes, in general, some of the choices; for each of the remaining ones we have to perform the matrix inversion and then repeat the procedure described in the last paragraph. Using all these transformations to check for the normal form of $M$, we end up with 152 inequivalent groups (see table 1). The calculation of all orbifolds without torsion for the model (1)$^9$ now takes 9 seconds (instead of almost one week if we ignore permutation symmetries $[10]$).

The described procedure to avoid redundancies becomes crucial when we include discrete torsions, as the number of consistent torsions per group and the computation time per orbifold both grow exponentially with the rank of the group. Here we have to generate all anti-symmetric $r \times r$ matrices with the constraint that the numbers in each line add up to 0 modulo 3 (with our conventions, this is necessary to have $\varepsilon(j, g) = 1$ for all group elements $g$). We again represent these matrices as single numbers $T$, in this case with $r(r-1)/2$ digits. Then we consider all automorphisms of the group and the transformations of the torsion matrix they induce. As soon as any of these transformations makes $T$ smaller we can omit that torsion. In practice we follow a strategy analogous to the one described above: We use permutations of the generators, i.e. permutations of the lines of $M$ that can be compensated by permutations of its columns, to restrict the torsions that are generated in the first place. In the crucial case of rank 8 we even have the full symmetry $S_8$. Fortunately, all automorphisms of the group have already been generated as a by-product in the check for redundant groups, and can now be used to eliminate all redundant torsions.$^3$

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$^3$For rank 8 there are 9! automorphisms, which would require a considerable amount of memory. Furthermore, a check for redundant groups is not necessary in this case; we therefore explicitly generate the automorphisms, or rather the obvious transformations they induce, in the torsion part of our program.
4 Results and discussion

We calculated all (2,2) spectra coming from diagonal twists for all potentials consisting of $A$, $D$- and $E$-type contributions, with the two exceptions $(1)\hat{7}(4)_A$ and $(1)\hat{7}(4)_D$. As the diagonal invariant and the $D$ invariant are $\mathbb{Z}_2$ orbifolds of one another, the respective sets of spectra that can arise when arbitrary twists and torsions are used, should be identical. We verified for many examples, as well as for the complete set of spectra, that exchanging $A$ and corresponding $D$-type contributions indeed does not change the result. This allowed us to omit the $A$-type potential corresponding to $(1)\hat{7}(4)$; for the corresponding $D$-type invariant we can show that its spectra are contained in those of the model $(1)^9$. For the latter model we constructed the complete set of orbifolds, which was possible by eliminating all redundancies as described in section 3. Thus we indeed enumerated all (2,2) spectra for ADE orbifolds.

All in all, we got only 943 different spectra (to be compared, for example, with the 3798 spectra found in [10] for general Landau–Ginzburg orbifolds without torsion). Our results are most easily represented graphically. Figure 1 shows all spectra that we found in a plot that displays $n_{27} + n_{\overline{27}}$ over the Euler number. It turns out that spectra coming from potentials that contain the $E_7$ term $X_1^3 + X_1 X_2^3$ are all in the region of a relatively low total number of non-singlets. Figure 2, which is an enlarged view of this region, contains all these spectra. Spectra that come only from potentials that do not contain the $E_7$ term $X_1^3 + X_1 X_2^3$ are represented by small dots, spectra coming only from potentials containing that term by large dots, and spectra occurring for both types of potentials by circles. Figure 3 displays in a similar way all spectra that cannot be obtained without the use of discrete torsion. Here large dots represent spectra that cannot come from any Landau–Ginzburg orbifold without torsion [10], whereas circles represent those of the remaining spectra that do not come from ADE models without torsion. All other ADE spectra, as far as they are in the range of the plot, are represented by small dots.

As we expected, figures 1 and 2 show complete mirror symmetry. This symmetry under the exchange of $n_{27}$ and $n_{\overline{27}}$ has been shown to be present for orbifolds of $A$-type potentials without torsion by Greene and Plesser [18]. More recently, their construction was extended to more general types of potentials, including $E_7$, by Berglund and Hübsch [19]. Using the properties of quantum symmetries, it should be straightforward to extend these constructions to cover the case of arbitrary discrete torsions. This is indeed confirmed by our results, and the complete mirror symmetry for $D$-type potentials is then a consequence of the $A-D$ equivalence.

It is a priori not obvious that we are dealing with the same class of (2,2) models as were constructed with simple current techniques by Schellekens and Yankielowicz [3]. It turns out, however, that we recover exactly all their (2,2) spectra, and find four additional mirror pairs. These four spectra come from only two different potentials: In $(1)(16)^3$, with no $E_7$ invariant, we find the mirror pairs $\{p_{11}, p_{12}\} = \{47, 29\}$, $\{101, 11\}$ and $\{110, 2\}$ with Euler numbers $\pm 36$, $\pm 180$ and $\pm 216$ (this is the model #117 in the list of ref. [7]); the spectrum with $\chi = -180$ was already found in ref. [20]. In addition, for $(2)(4)(12)(82)$, i.e. the model #127, we find $\{p_{11}, p_{12}\} = \{39, 33\}$ with $\chi = \pm 12$. This indicates that the two approaches indeed cover

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4In ref. [19] the mirror spectra for $D$-type potentials are constructed in terms of the corresponding diagonal invariant; in our case of arbitrary torsion, however, this makes no difference and we find perfect symmetry for the spectra of each potential.
Table 2: The 31 spectra for the model (1)$^9$ and the respective minimal ranks $r$ ($r_0$) of twist groups with (without) torsion. The last two columns of spectra have $\chi = 0$, and the last one is left–right asymmetric.

| $n_{27}$ | $n_{27}^{-}$ | $\chi$ | $r$ | $r_0$ | $n_{27}$ | $n_{27}^{-}$ | $\chi$ | $r$ | $r_0$ | $n_{27}$ | $p_{1/2}$ | $r$ | $r_0$ | $n_{27}$ | $p_{01}$ | $p_{10}$ | $r$ |
|----------|--------------|--------|-----|-------|----------|--------------|--------|-----|-------|----------|---------|-----|-------|----------|---------|---------|-----|
| 84       | 0            | 168   | 1   | 1     | 0        | 84           | 168   | 4   | 4     | 9        | 3       | 3   | 3     | 3        | 3       | 1       | 4   |
| 40       | 4            | 72    | 2   | 2     | 4        | 40           | 72    | 3   | 3     | 21       | 2       | 1   | 2     | 3        | 1       | 3       | 4   |
| 36       | 0            | 72    | 2   | 2     | 0        | 36           | 72    | 4   | 4     | 9        | 1       | 3   | 3     | 0        | 3       | 0       | 5   |
| 27       | 3            | 48    | 3   | 4     | 3        | 27           | 48    | 3   | 4     | 3        | 1       | 3   | 3     | 0        | 0       | 3       | 5   |
| 24       | 12           | 24    | 2   | 2     | 12       | 24           | 24    | 3   | 3     | 0        | 12      | 0   | 3     | 12       | 0       | 1       | 3   |
| 18       | 6            | 24    | 3   | 3     | 6        | 18           | 24    | 3   | 3     | 13       | 0       | 3   | 3     | 12       | 0       | 1       | 3   |
| 16       | 4            | 24    | 3   | 3     | 4        | 16           | 24    | 4   | 4     | 9        | 0       | 3   | 4     | 6        | 1       | 0       | 4   |
| 12       | 0            | 24    | 3   | 4     | 0        | 12           | 24    | 4   | 5     | 7        | 0       | 4   | 4     | 6        | 0       | 1       | 4   |

The results of the computation for the model (1)$^9$ are listed in table 2. Although we find more than $10^5$ inequivalent models, we only end up with 31 different spectra (all of which already appear in other tensor products). In the first two columns we list the mirror pairs of the chiral spectra, whereas columns 3 and 4 contain the symmetric and the asymmetric spectra with vanishing Euler number, respectively. We also list the minimal rank for which a given spectrum occurs with or without torsion. Note that, in this model, only the asymmetric spectra actually require torsion. Considering the sets $S_r$ of all spectra that occur for groups of rank $r$ and arbitrary allowed torsions, we find that $S_r \subset S_{r'}$ if and only if $r \leq r'$. Without torsion, this is true only up to rank 4, where we have 22 of the 23 symmetric spectra. At that point we must, according to the well-known construction of the mirror model for diagonal invariants [13], switch to the mirror spectra, i.e. exchange the numbers $n_{27}$ and $n_{27}^{-}$. Indeed, when we go from rank 4 to rank 5, we trade the spectrum $(12,0;−24)$ for its mirror partner, whereas all other spectra occur in a mirror-symmetric way; beyond rank 5 the set of spectra decreases exactly the way it should if we use only trivial torsions.

Summing up, we observe that in both cases, with and without torsion, all respective spectra can be obtained with groups of rank 1 to 5. When torsions are involved, these groups only amount to a small fraction of the inequivalent models. This peculiar behaviour calls for an explanation. An obvious idea is that the groups with increasing rank could become redundant at some point. This cannot be true in the strict sense, of course, because the models correspond to different modular invariants (see ref. [4]). As the chiral rings, however, seem to be equivalent – at least as far as the charge degeneracies are concerned – this could be related to a redundancy of the group actions that only concerns the chiral rings.
Let us call a group $G$ reducible iff there is a strict subgroup $H$ such that each state in the (untwisted) chiral ring is invariant under $G$ if it is invariant under $H$, i.e. iff the untwisted chiral primary fields of the orbifolds defined by $G$ and $H$ coincide. For example, the ‘smallest’ group $G_1^{(3)}$ of rank 3,

$$G_1^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ \end{pmatrix},$$

which is the only reducible one of this rank, yields the same projection on the chiral ring as its subgroup of rank 2

$$G_2^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ \end{pmatrix}.$$

It is indeed easy to check by inspection that the 20 inequivalent groups with rank larger than 5 are all reducible in this sense. Furthermore, none of the groups in our list that contribute spectra that are not present at lower rank is reducible. This suggests the conjecture that, at least for (2,2) vacua, all reducible groups can be omitted if we are only interested in computing the set of (non-singlet) spectra. In the absence of a proof, such a rule appears plausible only if we find a mechanism to explain the spectra of a reducible group in terms of its subgroups. Having in mind the modding of ‘quantum symmetries’, which can undo part of the original orbifolding, it is conceivable that a similar mechanism could be at work in the present situation. This would mean that we should be able to understand the spectrum of a reducible group only in terms of the subgroup of rank $r - 1$ that still implements the full projection, and of the subgroups of rank $r - 2$ of that group. The naive guess that these subgroups (corresponding to a single reduction of the group) should yield all the spectra has counter-examples at rank 4. It turns out, however, that all counter-examples that we found allow more than one reduction; the union of the sets of spectra arising from different reductions indeed gave the complete set of spectra.

With the methods we developed in this paper it should be straightforward to completely enumerate more general, and phenomenologically more promising, classes of string vacua. In particular, these would be general (2,2) and (2,1) Landau–Ginzburg orbifolds, or maybe even all (2,0) ADE models. Unfortunately, for exceptional invariants, which exist for tensor products at certain levels [14, 20], a complete classification is still missing. Disregarding massless $E_6$ singlets, however, there are no known spectra that can come only from such exceptional invariants.

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5 This is, of course, independent of the torsions.

6 These symmetries act only on the twisted sectors, with phases that are consistent with the orbifold selection rules [14]; they can be modded using discrete torsions (for examples see ref. [22]).

7 On the other hand, there are examples where the full massless spectrum is different [14].
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Figures

Fig. 1: $n_{27} + n_{37}$ vs. Euler number $\chi$ for all 943 spectra.
Fig. 2: $n_{27} + n_{\overline{27}}$ vs. Euler number for the spectra that can come [only] from $E_7$ invariants (large dots [circles]); the remaining spectra with $n_{27} + n_{\overline{27}} \leq 120$ are indicated by small dots.

Fig. 3: $n_{27} + n_{\overline{27}}$ vs. Euler number for spectra not occurring for any Landau–Ginzburg/ADE orbifold without torsion (large dots/circles), and all others with $n_{27} + n_{\overline{27}} \leq 120$ (small dots).