Envy-Free Cake Cutting with Graph Constraints

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Abstract

We study the classic problem of fairly dividing a heterogeneous and divisible resource—represented by a cake, [0, 1]—among n agents. This work considers an interesting variant of the problem where agents are embedded on a graph. The graphical constraint entails that each agent evaluates her allocated share only against her neighbor’s share. Given a graph, the goal is to efficiently find a locally envy-free allocation where every agent values her share to be at least as much as any of her neighbor’s share.

The best known algorithm (by Aziz and Mackenzie [AM16a]) for finding envy-free cake divisions has a hyper-exponential query complexity. One of the key technical contributions of this work is to identify a non-trivial graph structure—tree graphs with depth at-most two (DEPTH2TREE)—on n agents that admits a query efficient cake-cutting protocol (under the Robertson-Webb query model). In particular, we develop a discrete protocol that finds a locally envy-free allocation among n agents on DEPTH2TREE with at-most $O(n^3 \log(n))$ cuts on the cake. For the special case of DEPTH2TREE where every non-root agent is connected to at-most two agents (2-STARR), we show that $O(n^2)$ queries suffice. We complement our algorithmic results with establishing a lower bound of $\Omega(n^2)$ (evaluation) queries for finding a locally envy-free allocation among n agents on a 1-STARR graph (under the assumption that the root agent partitions the cake into n connected pieces).

1 Introduction

The cake-cutting problem provides an elegant mathematical abstraction to many real world situations where a divisible resource—modeled as a cake, [0, 1]—is to be allocated among agents with heterogeneous preferences. These situations include divorce settlements, division of land, allocation of radio and television spectrum, allocation of advertisement space on search platforms and so on (see [Adj] for implementations of cake-cutting methods). A central notion of fairness in resource-allocation settings is of envy-freeness that deems a partition of the cake to be fair if every agent prefers her share over that of any other agent [Fol67].

Cake-cutting problem has been extensively studied over the past seven decades across various disciplines like economics, mathematics and computer science; see [BT96, PM16, RW98] for excellent expositions. While it is known that an envy-free cake division is guaranteed to exist [Str80, ES99], corresponding efficient algorithmic results remain elusive. Even though the known lower bound on the query complexity of finding envy-free cake divisions is $\Omega(n^2)$ [Pro09], the best known algorithm has a hyper-exponential query complexity [AM16a]. This leaves a huge gap in our understanding of the query complexity of the underlying problem.

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In this work, we aim to (partially) address this gap by exploring the graphical framework of envy-freeness in cake cutting, where envy comparisons are restricted by an underlying (social) network on the agents. In contrast with the standard notion of envy-freeness, the goal here is to find a locally envy-free allocation of the cake such that no agent envies her neighbor(s) in a given network graph $G$. Note that when $G$ is a complete graph, we recover the classical setting of envy-free cake division.

The above-described graphical framework is a well-studied formulation of cake-cutting (see [AKP17, BQZ17, BSW+20, BKN22, Tuc21]) and it opens various interesting directions for understanding the problem of fairness in cake division. The notion of local envy-freeness is relevant in many natural scenarios where agents’ envy towards other agents can be restricted by external constraints such as social connections, relative rank hierarchy and overlap in their expertise/skill levels. For instance, when a network graph represents social connections between a group of people, it is reasonable to assume that agents only envy the agents whom they know (i.e., friends or friends of friends). Similarly, when a network represents a rank hierarchy in an organization, it is reasonable to assume that agents only envy their immediate neighbours (i.e. colleagues). The study of local envy-freeness is also interesting from a purely theoretical standpoint. Given that the state of the art protocols for finding envy-free divisions are highly complicated and require hyper-exponential queries, a natural line of research—and the focus of this work—is to find interesting network graph structures for which the problem instance admits query-efficient cake cutting protocols.

We remark here that our goal in this paper is to improve the theoretical understanding of locally envy-free cake cutting problems. In particular, we focus on investigating the graph structures that admit efficient protocol to find envy-free allocations. To the best of our knowledge, this is the first work to develop a discrete and efficient protocol for finding locally envy-free allocations among $n$ agents wherein the envy-constraints are specified via a non-trivial class of graphs. We believe that understanding the query complexity of interesting graph structures will help us improve our understanding of the general envy-free cake division problem by providing the necessary building blocks and identifying the bottleneck where the problem becomes computationally challenging.

Our Results and Techniques: The design of efficient locally envy-free protocols for interesting graph structures is listed as an open problem in [BSW+20] and [AKP17]. In this work, we address the aforementioned open problem and develop a novel algorithm (ALG2) that finds a locally envy-free allocation with $O(n^3 \log(n))$ cut queries among $n$ agents lying on a tree graph with depth at-most 2 (DEPTH2TREE) (see Theorem 3). The main technical contribution of this work is to identify an interesting non-trivial graph structure over $n$ agents that admits a discrete and efficient (locally) envy-free protocol. We further show that in a special case of DEPTH2TREE where each non-root agent is connected to at-most two other agents (2-STAR), one can find a locally envy-free allocation in $O(n^2)$ cut queries; see Theorem 4. Finally, for a star graph, we establish a lower bound of $\Omega(n^2)$ evaluation queries (when the root agent makes the cuts) for finding locally envy-free allocation. (see Theorem 5).

Interestingly, at the core of our results for DEPTH2TREE and 2-STAR graphs, lies a simple protocol (ALG1) for finding a locally envy-free allocation among four agents on a LINE graph (Theorem 1). Our protocols run in multiple rounds and certain agents are assigned the role of either a cutter or a trimmer. Typically, the most connected agent (root agent in DEPTH2TREE and 2-STAR and a center agent in LINE graph) is designated as a cutter and other non-leaf agents are trimmers. It is relevant to note that our protocols assign a single agent as the cutter. The protocol computes a partial locally envy-free allocation of the cake in each round until the remaining unallocated piece (a.k.a. residue) becomes insignificant enough so as to not invoke local envy if given to a specific subset of agents on a graph. Our algorithms operate under the standard Robertson-Webb query model [RW98] wherein an algorithm has access to agents’ valuations only through cut and evaluation queries; see Section 2.1 for details.
1.1 Related Literature:

Fairness in resource-allocation settings is extensively studied in economics, mathematics and computer science literature (see [CKM+19, BT96, Mou04]). While strong existential guarantees are known for envy-free cake divisions [Str80], the corresponding computational problem remain challenging [DQS12, Str08]. For three agents, the celebrated Selfridge-Conway protocol finds an envy-free allocation with 5 cut queries. However, despite significant efforts, developing efficient envy-free cake cutting protocols for $n$ agents remains largely open: the current known upper bound has hyper-exponential dependency on $n$ [AM16a], whereas the best known lower bound is $\Omega(n^3)$ [Pro09]. This leaves a huge gap in our understanding of the problem. Attempts have been made to address this gap for four agents. Aziz and Mackenzie [AM16b] proposed a cake cutting protocol that finds an envy-free allocation in (close to) 600 queries. This bound was recently improved by [ACF+18] to 171 queries, which is the best known query complexity bound for four agents. Barman et al. [ABKR19] developed an efficient algorithm that finds a cake division (with connected pieces) wherein the envy is multiplicatively within a factor of $3 + o(1)$. Fair cake cutting protocols for special classes of valuations have been developed in [KL13, BR21].

The problem of cake cutting with graphical (envy) constraints was first introduced by Abebe et al. [AKP17]. They study a special case of directed acyclic graphs and show an upper bound of $O(n^2)$ on the query complexity under a special setting where a single agent make cuts on the cake. In contrast, our work studies a class of undirected graphs that are significantly harder to analyze, and surprisingly develops comparable upper bounds. In another closely related paper, Bei et al. [BQZ17] develops a moving-knife protocol that outputs an envy-free allocation on tree graphs. In a more recent work, Bei et al. [BSW+20] develop a discrete and bounded locally proportional protocol for any given graph. In contrast, our work addresses stronger guarantee of local envy-freeness. We address the open question raised in [BSW+20] by developing an efficient envy-free protocol that finds a complete cake division that is locally envy-free among $n$ agents on tree graphs with depth at-most 2.

2 The Setting

We consider the problem of fairly dividing a heterogeneous and divisible resource—modeled as a line segment $[0, 1]$, and called as a *cake*—among $n$ agents, denoted by the set $\mathcal{N} = \{a_1, a_2, \ldots, a_n\}$. For an agent $a_i \in \mathcal{N}$, we write $v_i$, defined over Borel measurable subsets of $[0, 1]$, to denote her (cardinal) valuations of agent over the cake. For an interval $I \subseteq [0, 1]$, $v_i(I)$ represents the valuation of agent $a_i$ for $I$. Following the standard convention, we assume that $v_i$s are non-negative, additive and nonatomic\(^3\). Additionally, without loss of generality, we assume that the valuations are normalized i.e., we have $v_i(0, 1) = 1$ for all $i \in [n]$. For brevity, we will write $v_i(x, y)$ instead of $v_i([x, y])$ to denote agent $a_i$’s value for an interval $[x, y] \subseteq [0, 1]$.

We write $G := (V, E)$ to denote an undirected graph where vertex $i \in V$ represents agent $a_i \in \mathcal{N}$ and an edge $(i, j) \in E$ represents a connection between agents $a_i$ and $a_j$.

2.1 Preliminaries

**Problem Instance:** A cake-division instance $I$ with graph constraints is denoted by a tuple $(\mathcal{N}, G, \{v_i\}_{i \in [n]})$. Here, $\mathcal{N}$ denotes the set of $n$ agents, $G$ denotes the underlying graph over the agents and $v_i$s specify

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\(^1\)For two agents the a simple cut-and-choose protocol returns the envy free allocation with a single cut on the cake.

\(^2\)For a given graph, an allocation is said to be locally proportional if every agent values her share that is at least as high as her average value of her neighbours’ shares.

\(^3\)For any two disjoint intervals $I_1, I_2 \subseteq [0, 1]$, we have $v_i(I_1 \cup I_2) = v_i(I_1) + v_i(I_2)$.

\(^4\)For an interval $[x, y] \subseteq [0, 1]$ and $\lambda \in [0, 1]$ there exists a $y'$ such that $v_i(x, y') = \lambda \cdot v_i(x, y)$.
the valuations of agents over the cake $[0, 1]$.

**Allocations:** For cake-division instances, we define an allocation $A := \{A_1, A_2, \ldots, A_n\}$ to be a collection of $n$ pair-wise disjoint pieces such that $\cup_{i \in [n]} A_i = [0, 1]$. Here, piece or bundle $A_i$ (a finite union of intervals of the cake $[0, 1]$) is assigned to agent $a_i \in \mathcal{N}$. We say $A$ is a partial allocation if the union of $A_i$s form a strict subset of $[0, 1]$.

In this work, we study protocols for finding locally envy-free allocations, a natural extension of the well studied notion of envy-freeness defined below.

**Definition 1** (Envy-freeness). For a cake-division instance, an allocation $A$ is said to be envy-free if we have $v_i(A_i) \geq v_i(A_j)$ for all agents $i, j \in [n]$.

Note that in a cake division instance with graph constraints, the protocol also has access to underlying network graph over agents.

**Definition 2** (Local Envy-freeness). Given a cake-division instance with a network graph $G$, an allocation $A := \{A_1, A_2, \ldots, A_n\}$ is said to be locally envy-free (on $G$) if for all agents $i \in [n]$, we have $v_i(A_i) \geq v_i(A_j)$ for all $j$ such that $(i, j) \in E$.

Local envy-freeness ensures that every agent prefers her own piece over that of any of her neighbors in $G$. When $G$ is the complete graph over $n$ agents, we recover the classical fairness guarantee of envy-freeness. Our algorithms operate under the Robertson-Webb model [RW98] defined below.

**Definition 3** (Robertson-Webb query model). Our protocols access the agents’ valuations via the following two types of queries:

1. **Cut query:** Given a point $x \in [0, 1]$ and a target value $\tau \in [0, 1]$, $\text{cut}_i(x, \tau)$ asks agent $a_i$ to report the rightmost subset $[x, y]$ such that $v_i(x, y) = \tau$.

2. **Evaluation query:** Given $0 \leq x < y \leq 1$, $\text{eval}_i(x, y)$ asks agent $a_i$ to report her value $v_i(x, y)$ for the interval $[x, y]$ of the cake.

**Terminology:** For cake-division instances with agents embedded on a tree graph with depth at most two (DEPTH2TREE), our work develops a novel algorithm (ALG2) for computing local envy-free allocation. Here, we state the terminologies used to describe our protocols (ALG1 and ALG2). An agent is called a cutter if she makes initial cuts on the unallocated piece (i.e. current residue) of the cake in a given round. Interestingly, the proposed protocols require that the same agent (typically an agent with largest degree) acts as a cutter in every round. Furthermore, the cutter agent makes cuts to divide the cake into equal parts i.e., performs $\text{EQ-DIV}(\cdot)$ procedure described in Sec. 2.2. An agent is called a trimmer agent if she performs either a TRIM($\cdot$) or EQUAL($\cdot$) operation. All the non-root agents with degree $\geq 2$ act as trimmers in our protocols. The leaf agents perform SELECT($\cdot$) operation and are neither cutters nor trimmers. We will now define four important subroutines that are used throughout in our algorithms.

### 2.2 Important Subroutines

**Equal Division:** Given an agent $a_i$, a piece of cake $U$ and an integer $n$, the $\text{EQ-DIV}(\cdot)$ procedure divides $U$ in $n$ equal pieces according to the valuation function of $a_i$ i.e., $v_i(X_j) = v_i(U)/n$ for all $j \in [n]$. This subroutine requires $n - 1$ cut queries and 1 eval query (to compute $v_i(U)$). When valuation of agent $a_i$ for $U$ is clear from context we do not need the eval query. We remark here that it is only the
cutter agent who performs EQ-DIV in our algorithms.

**Select:** Given a collection of pieces $\mathcal{X}$ and an agent $a_i$, SELECT procedure returns $m$ largest valued pieces (for integer $m \leq |\mathcal{X}|$) according to the valuation of $a_i$. It is easy to see that SELECT requires zero cut queries and maximum of $|\mathcal{X}|$ eval queries.

**Trim:** Given a collection of pieces $\mathcal{X}$ and an agent $a_i$, TRIM procedure returns a collection of pieces where each piece is equally valued by agent $a_i$. The procedure first finds the lowest valued piece according to $a_i$ and makes the remaining pieces of value equal to it by trimming. Trimmings are returned separately. The TRIM procedure requires $|\mathcal{X}| - 1$ cut queries and $|\mathcal{X}|$ eval queries.

**Equal:** Given a collection of pieces $\mathcal{X}$ and an agent $a_i$, EQUAL procedure redistributes among these pieces such that each piece is equally valued by $a_i$. It also identifies one bundle in the original collection that has a value larger than the average value of the bundles. Note the distinction between EQUAL and EQ-DIV procedures. While both EQUAL and EQ-DIV procedures return an allocation where all the pieces are equally valued by $a_i$, the EQ-DIV may generate a residue whereas EQUAL procedure redistributes all the cake into the same number of pieces without leaving any part unallocated. The EQUAL procedure requires $|\mathcal{X}| - 1$ cut queries and $|\mathcal{X}|$ eval queries.

| PROC: EQ-DIV($a_i, U, n$) |
|--------------------------------|
| 1 Set $\tau := \frac{v_i(U)}{n}$ |
| 2 for $j = 1 \rightarrow n$ do |
| 3 Let $X_j := \text{cut}_i(U, \tau)$ |
| 4 Update $U \leftarrow U \setminus X_j$ |
| 5 return $(X_1, X_2, \ldots, X_n)$ |

| PROC: SELECT($a_i, \mathcal{X}, m$) |
|-------------------------------------|
| 1 Initialize $\mathcal{X}_i \leftarrow \emptyset$ |
| 2 for $j = 1 \rightarrow m$ do |
| 3 Update $\mathcal{X}^{(i)} \leftarrow \mathcal{X}_i \cup \arg \max_{X \in \mathcal{X}} v_i(X)$, and $\mathcal{X} \leftarrow \mathcal{X} \setminus \mathcal{X}^{(i)}$ |
| 4 return $(\mathcal{X}^{(i)}, \mathcal{X})$ |

| PROC: EQUAL($a_i, \mathcal{X}$) |
|--------------------------------|
| 1 Let $\mathcal{X}^s := \{ X \in \mathcal{X} : v_i(X) < v_i(\mathcal{X})/|\mathcal{X}| \}$, $\mathcal{X}^\ell := \{ X \in \mathcal{X} : v_i(X) \geq v_i(\mathcal{X})/|\mathcal{X}| \}$ |
| 2 Initialize $\mathcal{T} \leftarrow \emptyset$ and let $\tau = v_i(\mathcal{X})/|\mathcal{X}|$ |
| 3 Let $X^*$ be arbitrary element of $\mathcal{X}^\ell$ |
| 4 for $X_j \in \mathcal{X}^s$ do |
| 5 $a_i$ divides $X_j$ into $X'_j$ and $T_j$ such that $v_i(X'_j) = \tau$ using a single cut |
| 6 $\mathcal{T} \leftarrow \mathcal{T} \cup T_j$ and $X_j \leftarrow X'_j$ |
| 7 for $X_j \in \mathcal{X}^s$ do |
| 8 while $v_i(X_j(j)) < \tau$ do |
| 9 $X_j \leftarrow X_j \cup T$ for some $T \in \mathcal{T}$ |
| 10 $\mathcal{T} \leftarrow \mathcal{T} \setminus T$ |
| 11 Let $T' \in \mathcal{T}$ be last piece added to $X_j$ |
| 12 Cut $T'$ into two pieces $T'_1$ and $T'_2$ such that $v_i(X'_1) = \tau$ where $X'_j = X_j \setminus T'_2$ |
| 13 $T \leftarrow T \cup T'_2$ and $X_j \leftarrow X'_j$ |
| 14 return $(\mathcal{X}, X^*)$; |

| PROC: TRIM($a_i, \mathcal{X}$) |
|--------------------------------|
| Let $\mathcal{X} = \{X_0, X_1, \ldots, X_{\ell} \}$, such that $X_0 = \arg \min_{X \in \mathcal{X}} v_i(X)$ |
| 1 Set $\tau = v_i(X_0)$ |
| 2 Initialize $U \leftarrow \emptyset$ |
| 3 for $j = 1 \rightarrow \ell$ do |
| 4 Let cut$_i(X_j, \tau) = X'_j$ with $v_i(X'_j) = v_i(X_0)$, and let $T_j = X_j \setminus X'_j$ |
| 5 Update $X_j \leftarrow X'_j$ and $U \leftarrow U \cup T_j$ |
| 6 return $(\mathcal{X}, U)$ |
3 Local Envy-Freeness on LINE graphs

We begin by developing a simple and efficient protocol, ALG1, that finds locally envy-free allocation among four agents on a LINE graph (see Section 3.1). Towards the end of this section, we extend the ideas of ALG1 to produce a similar result for five agents on a LINE graph.

3.1 Four agents on a LINE graph

We denote the agents as \( a_1, a_2, a_3 \) and \( a_4 \) lying on a LINE graph from left to right i.e., \( a_1 - a_2 - a_3 - a_4 \) is the underlying graph structure. We designate \( a_2 \) as the cutter and \( a_3 \) as the trimmer agent. Our protocol, ALG1 runs in two rounds and finds a locally envy-free allocation among fours agents on a LINE using 8 cut queries and 16 eval queries (see Theorem 1).

In the first round, \( a_2 \) cuts the cake into four equi-valued pieces using EQ-DIV(\( \cdot \)) procedure, \( a_1 \) selects her favourite piece and then \( a_3 \) makes her favorite piece (among remaining three pieces) equal to the second largest using the TRIM(\( \cdot \)) procedure. The trimming obtained from this procedure is called as residue and is considered in the second round. Here, similar to the first round, it is first cut by \( a_2 \) into four equal pieces using EQ-DIV(\( \cdot \)) procedure and \( a_1 \) selects her favourite piece. Then, unlike in the first round where the trimmer agent \( a_3 \) performs TRIM(\( \cdot \)), she makes her top two pieces of equal value using EQUAL(\( \cdot \)) procedure. Finally, the pieces from these two rounds are grouped together to form a complete partition of the cake \([0, 1] \), see Step 9 of ALG1.

The idea is to create a partition such that the cutter agent \((a_2)\) is indifferent between her and \(a_1\)'s bundle and deems the bundle allocated to \(a_3\) at-most as much as her own bundle. The trimmer agent \((a_3)\) similarly is indifferent between her and \(a_4\)'s bundle whereas deems the bundle allocated to \(a_2\) at-most as valuable as her own bundle. ALG1 ensures the above properties and hence finds a locally envy-free allocation (see Theorem 1).

**Theorem 1.** For cake-division instances with four agents on a LINE graph, ALG1 finds a locally envy-free allocation using 8 cut queries and 16 eval queries.

**Proof** We begin by showing that the allocation \( \{A_1, A_2, A_3, A_4\} \) returned by ALG1 is a complete partition of the cake, i.e., \( \bigcup_{i=1}^{4} A_i = [0, 1] \). To see this, note that \( T_1 \cup T_2 \cup T_3 \cup T_4' = T \) and \( P_3' \cup T = P_3 \). That is, we have \( \bigcup_{i=1}^{4} A_i = \bigcup_{i=1}^{5} P_i = [0, 1] \).

Next, we will establish local envy-freeness of this allocation. Recall that agents have additive valuations. Therefore, since \( a_2 \) is the cutter, she values her piece, \( A_2 = P_2 \cup T_2 \) and \( a_1 \)'s piece, \( A_1 = P_1 \cup T_1 \) equally, that is, \( v_2(A_2) = v_2(A_1) \). Furthermore, Steps 2 and 6 ensures that \( a_1 \) values her piece at least as much as that of \( a_2 \), i.e., \( v_1(A_1) \geq v_1(A_2) \). Therefore, there is no envy between \( a_1 \) and \( a_2 \). Next, to see that there is no envy between \( a_2 \) and \( a_3 \), observe

\[
v_2(A_2) = v_2(P_2) + v_2(T_2) = v_2(P_1) + v_2(T_3) \geq v_2(P_1) + v_2(T_3') = v_2(A_4) \quad \text{(by Steps 1 and 5 of ALG1)}
\]

We also have

\[
v_2(A_4) = v_2(P_4) + v_2(T_3') \geq v_2(P_3) \geq v_2(P_3') + v_2(T_4') = v_2(A_3) \quad \text{(since } v_2(P_3) = v_2(P_3') \text{)}
\]

That is, we have \( v_2(A_2) \geq v_2(A_4) \geq v_2(A_3) \) and since \( a_3 \) is allocated either \( A_3 \) or \( A_4 \), \( a_2 \) does not envy \( a_3 \). Now, for the trimmer agent \( a_3 \), Steps 4 and 8 ensures that she values pieces \( A_3 \) and \( A_4 \) equally, i.e., we have \( v_3(A_3) = v_3(A_4) \). Furthermore,

\[
v_3(A_4) = v_3(P_4) + v_3(T_3') \geq v_3(P_2) + v_3(T_2) = v_3(A_2) \quad \text{(by Steps 3 and 7 of ALG1)}
\]
**Algorithm:** Local Envy-freeness for 4 agents on a LINE (ALG1)

**Input:** A cake-division instance \( I \) with 4 agents on a LINE

**Output:** A locally envy-free allocation

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1. Let \( \{ P_1, P_2, P_3, P_4 \} \leftarrow \text{EQ-Div}(a_2, [0, 1], 4) \)
2. \( a_1 \) chooses her favorite piece, say \( P_1 \)
3. Let \( \{ P_3, P_4 \} \leftarrow \text{SELECT}(a_3, \{ P_2, P_3, P_4 \}, 2) \)
4. Let \( \{ \{ P_3', P_4' \}, T \} \leftarrow \text{TRIM}(a_3, \{ P_3, P_4 \}) /\ast \) \( P_3' := P_3 \setminus T \), \( v_3(P_3') = v_3(P_4) \) */

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**Trimming Phase**

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5. Let \( \{ T_1, T_2, T_3, T_4 \} \leftarrow \text{EQ-Div}(a_2, T, 4) \)
6. \( a_1 \) chooses her favorite piece, say \( T_1 \)
7. Let \( \{ T_3, T_4 \} \leftarrow \text{SELECT}(a_3, \{ T_2, T_3, T_4 \}, 2) \)
8. \( \{ T_3', T_4' \} = \text{EQUAL}(a_3, \{ T_2, T_3 \}) /\ast \) Let \( T_3' = T_3 \setminus T' \) and \( T_4' = T_4 \cup T' \) such that \( v_3(T_3') = v_3(T_4') \) */

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**Equalizing Phase**

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9. Set \( A_1 := P_1 \cup T_1, A_2 := P_2 \cup T_2, A_3 := P_3' \cup T_4' \) and \( A_4 := P_4 \cup T_3' \)

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**Final Allocation**

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10. \( A_1 \) is allocated to \( a_1 \) and \( A_2 \) is allocated to \( a_2 \)
11. \( a_4 \) picks her favourite piece between \( A_3 \) and \( A_4 \). The remaining piece is allocated to \( a_3 \)

Hence, we have that \( v_3(A_3) = v_3(A_4) \geq v_3(A_2) \). This implies that \( a_3 \) does not envy any of \( a_2 \) and \( a_4 \). Finally, \( a_4 \) does not envy \( a_3 \) since she picks her favorite piece between \( A_3 \) and \( A_4 \) and \( a_3 \) gets the remaining piece.

**Counting queries:** It is easy to see that Steps 1 and 5 each require 3 cut queries (for the cutter agent \( a_2 \)) to divide the cake \([0, 1]\) and the residue \( T \), respectively, into four equi-valued pieces. For the trimmer agent \( a_2 \), we require 1 cut query to execute Step 4 and 1 more cut query to execute Step 8, making a total of 8 cut queries. Next, observe that ALG1 requires a total of 16 eval queries: a total of 7 (3 and 4 in Steps 2 and 6 respectively) queries for agent \( a_1 \) to select her favorite piece, 6 queries for agent \( a_3 \) (3 each in Steps 3 and 7) to select her two highest-valued pieces, 1 query for agent \( a_2 \) to evaluate her value for \( T \) in Step 5, and 2 queries for agent \( a_4 \) in Step 11 to select her favorite piece. \( \square \)

### 3.2 Five agents on a LINE graph

In this section, we present a simple protocol that finds a locally envy-free allocation among five agents on a LINE using 18 cut queries and 29 eval queries (see Theorem 2). We do so by extending our ideas from the previous section when we had four agents on a LINE. This section acts as a bridge and provide an intuition about our main algorithm (ALG2) that establishes the fact that the problem of finding a locally envy-free allocation among agents are on a DEPTH2TREE admits an efficient algorithm (see Theorem 3).

Consider a cake-division instance with five agents \( a_1 - a_2 - a_3 - a_4 - a_5 \) on a LINE. Here, we make \( a_3 \) as the cutter and \( a_2 \) and \( a_4 \) as trimmers. Observe the contrast from the 4-agents case where we had a single cutter and a single trimmer agent. In 5-agents case, the residue consists of trimmings from both \( a_2 \) and \( a_4 \). This residue is then redistributed in multiple rounds until the cutter dominates (formally defined later in the proof) both her neighbouring agents. Once the dominance is achieved—which happens in at-most two rounds of the Trimming phase—the current residue can be distributed without creating local envy in the Equaling phase. The distinction from the 4-agents case (which has a single trimmer agent) is that it requires an additional round of Trimming to create dominance of the
cutter agent over the two trimmer agents. We state our result now.

**Theorem 2.** For cake-division instances with five agents on a `LINE` graph, there exists a discrete protocol that finds an envy-free allocation using 18 cut queries and 29 eval queries.

**Proof** We begin by describing the protocol, followed by the analysis for its correctness and query complexity. As mentioned earlier, this protocol is an extension of `ALG1`, and hence we will a short description while detailing out the key differences.

**The Protocol:** A locally envy-free protocol for five agents on a `LINE` graph consists of the center agent (\(a_3\)) as the cutter and her neighbours (\(a_2\) and \(a_4\)) as trimmer agents. It begins with the cutter agent dividing the cake into 5 equal pieces using `EQ-DIV`, followed by \(a_2\) and \(a_4\) each picking their two highest-valued (available) pieces one after the other. The remaining last piece in added to the cutter agent’s bundle. Both the trimmer agents then perform `TRIM()` procedure in the first round (trimming phase). After this round, the cutter agent starts to *dominate* at least one trimmer agent, say \(a_2\) (see Claim 1), and therefore \(a_2\) performs the `EQUAL()` procedure for the remaining rounds. In the next round, \(a_4\) continues to perform the `TRIM()` procedure. Finally, in the third round, we show that the dominance is established over \(a_4\) as well, and both \(a_2\) and \(a_4\) perform `EQUAL()` procedure in the last round.

As opposed to the 4-agents case, an additional trimming round is attributed to the fact that trimming comes from two trimmer agents in this case. The final bundles \(\{A_1, A_2, A_3, A_4, A_5\}\) are created similar to Step 9 of `ALG1`. The cutter agent is allocated a bundle \(A_3\) that consists of the remaining pieces from all the rounds (i.e. both Trimming and Equaling phases) after \(a_2\) and \(a_4\) make their selections from `EQ-DIV`. Two—one from each side of the cutter agent—of the other four bundles, \(A_2\) and \(A_4\), contain untrimmed pieces from the Trimming phase and a trimmed piece from the Equaling phase. This ensures that \(a_3\) values his bundle \(A_3\) as high as \(A_2\) and \(A_4\). The remaining two bundles (again, one from each side of cutter agent) \(A_1\) and \(A_5\) contain trimmed pieces from the Trimming phase and appended piece from the Equaling phase. The *dominance* of the cutter agent \(a_3\) over \(a_2\) and \(a_4\) further ensures that \(a_3\) will not have any envy for these bundles as well.

As mentioned earlier, \(a_3\) is allocated bundle \(A_3\). Agent \(a_1\) picks her favorite bundle between \(A_1\) and \(A_2\), and the remaining bundle is allocated to agent \(a_2\), and similarly \(a_5\) pick her favourite bundle between \(A_4\) and \(A_5\), and the remaining bundle goes to agent \(a_4\) (similar to Step 11 of `ALG1`). Recall that, Trimming and Equaling phases always create equi-valued bundles for \(a_2\) and \(a_4\), and that is valued at least as high as the bundle \(A_3\), the protocol ensures local envy-freeness.

**Correctness:** We now show that the above algorithm returns a locally envy-free allocation of the cake. Recall that there are three rounds in total, in the first round both trimmer agents \(a_2\) and \(a_4\) perform `TRIM()` procedure, in the second round at least one of \(a_2\) or \(a_4\) performs `TRIM()` and the other performs `EQUAL()` procedure, while in the last round, both trimmer agents perform `EQUAL()` procedure. Let us assume the cutter agent \(a_3\) divides the current residue into five equal pieces \(\{P^1_1, P^2_1, \ldots, P^3_3\}\) in round \(j = \{1, 2, 3\}\). Also write \(\{A^1_1, A^2_1, \ldots, A^3_3\}\) to denote the partial partition of the cake obtained at the end of round \(j = \{1, 2, 3\}\). Next, we denote the trimmings in the first round as \(T_2\) and \(T_4\) from agent \(a_2\) and \(a_4\) respectively. We say that \(T = T_2 \cup T_4\) is the total residue of the first round. In the second round, since there is only one agent who performs `TRIM()`, we denote this residue by \(T'\).

Note that we have \(\bigcup_{i \in \{3\}} P^1_i = [0, 1]\) in the first round, \(\bigcup_{i \in \{3\}} P^2_i = T\) in the second round, and \(\bigcup_{i \in \{3\}} P^3_i = T'\) in the third round. Without loss of generality, let us assume \(a_2\) picks \(P^1_2\) and \(P^2_2\), and \(a_4\) picks \(P^2_2\) and \(P^3_2\) in round \(j\) in the `SELECT()` procedure. The following claim proves a crucial property of our algorithm, which would be formally defined as *dominance condition* (see Definition ??) when we discuss about `DEPTH2TREE` in Section 4.
Therefore, either one of the conditions holds true at the end of the first round, proving the stated

Claim 1. Consider the partial partition \( \{A_1, A_2, \ldots, A_5\} \) of the cake at the end of the first round of \( \text{TRIM(\).} \) along-with trimming \( T \). Then, at least one of the following two conditions must hold true.

\[
\begin{align*}
& a) \quad v_3(A_3^1) - v_3(A_2^1) \geq \frac{4}{5} \cdot v_3(T) \\
& b) \quad v_3(A_3^1) - v_3(A_1^1) \geq \frac{4}{5} \cdot v_3(T).
\end{align*}
\]

Proof Let us assume for contradiction that both of the stated conditions are not satisfied at the end of the first round. That is, we have \( v_3(A_3^1) - v_3(A_2^1) < \frac{4}{5} \cdot v_3(T) \) and \( v_3(A_3^1) - v_3(A_1^1) < \frac{4}{5} \cdot v_3(T) \). Summing these two inequalities, we obtain

\[2v_3(A_3^1) - v_3(A_2^1) - v_3(A_1^1) < \frac{4}{5} \cdot v_3(T)\]

Recall that \( A_2^1 = P_2^1 \setminus T_2 \) and \( A_1^1 = P_1^1 \setminus T_4 \) after the first \( \text{TRIMMING} \) round. Therefore, by additivity of valuations, we obtain

\[2v_3(A_3^1) - (v_3(P_2^1) - v_3(T_2)) - (v_3(P_1^1) - v_3(T_4)) < \frac{4}{5} \cdot v_3(T)\]

Since \( a_3 \) performs the \( \text{EQUAL(\.)} \) procedure, and \( A_3^1 = P_3^1 \) is of value \( 1/5 \) for \( a_3 \), we have

\[\frac{2}{5} - \left( \frac{1}{5} - v_3(T_2) \right) - \left( \frac{1}{5} - v_3(T_4) \right) < \frac{4}{5} \cdot v_3(T)\]

Finally, since \( T = T_2 \cup T_4 \), by additivity of valuations, we obtain \( v_3(T) < \frac{1}{5} \cdot v_3(T) \), a contradiction. Therefore, either one of the conditions holds true at the end of the first round, proving the stated claim.

If condition \((a)\) is true, we say that the cutter agent dominates agent \( a_3 \), otherwise we say that \( a_2 \) dominates agent \( a_4 \). If condition \((a)\) is satisfied, it essentially says that the cutter agent will not envy the bundle \( A_2^1 \), even if she adds \( 2/5 \)th of the residue \( T \) to it in the next round. Therefore, she proceeds to perform \( \text{EQUAL(\.)} \) procedure.

Following the similar arguments, we show that the cutter agent will start dominating the remaining trimmer agent after one additional round of \( \text{TRIMMING} \). Hence, by the above description of the protocol and the ideas used in proving Theorem 1, we establish the fact that \( \text{ALG1} \) can be extended to achieve local envy-freeness among five agents on a \( \text{LINE} \) graph.

**Counting Queries:** It is easy to count the number of queries required in this protocol. Due to similar arguments, we urge the readers to refer the 4-agents case for details, here we state the number of queries required in each round for completeness. The first round requires 6 cut queries and 7 eval queries, the second and third rounds each require 6 cut queries and 9 eval queries. Finally, it requires 4 eval queries to allocate the bundles at the end.

Unfortunately, our technique does not extend to more than 5 agents on \( \text{LINE} \) graph. Since, for more than 5 agents on a \( \text{LINE} \) as every agent has at-least 3 agents on at-least one of her sides, the \( \text{EQUAL(\.)} \) procedure may end up adding trimmings into multiple bundles. Hence, it may not be possible to guarantee the local envy-freeness while creating bundles (similar to Step 9 of \( \text{ALG1} \)) which requires that the pieces that gets appended in the \( \text{Equalizing} \) phase are put in bundles containing trimmed pieces from the \( \text{TRIMMING} \) phase. One of the interesting directions for future work is to develop an efficient protocol for local envy-freeness among \( n \) agents, in general, on a \( \text{LINE} \) graph. Our approach, even though is severely limited by the fact that it considers a single cutter agent, is easily generalizable to acyclic graphs with path length of at-most 5. We show this formally in the next section when we present our main result about local envy-freeness on \( \text{DEPTH2TREE} \).
4 Local Envy-freeness on Trees with Depth at-most 2

In this section, we present our main result which identifies a non-trivial graph structure of envy-constraints—trees with depth at-most 2 (DEPTH2TREE)—on \( n \) agents that admits a simple and query efficient protocol (ALG2) to compute a locally envy-free allocation (see Theorem 3).

Given a cake-division instance with \( n \) agents on a DEPTH2TREE, we write \( a_r \) to denote the root agent and the set \( D \) to denote her neighbours (See Figure 1). Each neighbour agent \( a_i \in D \) is connected to \( \ell_i + 1 \) agents including the root agent. That is, each non-root agent \( a_i \) is connected with \( \ell_i \geq 0 \) leaf agents. Furthermore, we write \( L(i) \) to denote the set of neighbours of agent \( a_i \in D \). For cake-division instances, we will specify a DEPTH2TREE by \( (n, a_r, D, \{ \ell_i \}_{a_i \in D}) \).

Our protocol ALG2 Designates \( a_r \) as the cutter and each \( a_i \in D \) as the trimmer agents (recall the definition from Sec. 2). For clarity, we call the subset of trimmer agents who perform TRIM(\( ) \) procedure as active trimmer agents, denoted by the set \( Tr \) and the agents who perform EQUAL(\( ) \) procedure as equalizer agents. Before giving an overview of ALG2, we first define a crucial condition of dominance used in our protocol.

![Figure 1: Tree graph with depth at-most two (DEPTH2TREE).](image-url)

**Definition 4 (Domination Condition).** Given a cake-division instance on a DEPTH2TREE, let us denote root agent’s bundle by \( A_r \) and the residue by \( R \). We say that the root agent dominates a bundle \( A^{(i)}_k \) if

\[
v_r(A_r) - v_r(A^{(i)}_k) \geq \min\left\{ \frac{\ell_i + 1}{|D| + 1}, 1 \right\} \cdot v_r(R). \tag{1}\]

That is, the domination condition implies that there is enough difference between the value of root agent’s bundle and the bundle \( A^{(i)}_k \) such that even after adding a certain fraction of residue, the root agent will not envy that bundle. We impose the domination condition on all the bundles that could possibly get more than a single piece in the equalizing phase. We say that the root agent dominates agent \( a_i \in Tr \) if the condition stated in Step 21 of ALG2 is satisfied.

4.1 An Overview of ALG2

The algorithm seeks to achieve the domination condition for the root agent on all her neighbor agents by iteratively dividing the current residue in multiple rounds. There are three types of agents: the root agent is the designated cutter, non-leaf neighbours of the root agent are the trimmers and the leaf-agents. In the beginning, the set of active trimmer agents \( Tr = D \) and the algorithm progressively removes agents from \( Tr \) by checking the dominance condition. As long as there exists some trimmer agent in \( Tr \), the while-loop in Steps 3 – 22 are executed.

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5In ALG2, we take this fraction to be the upper bound on the value of the pieces generated in equalizing phase.

6An agent \( a_i \in D \) is removed from the set \( Tr \) of active trimmer agents as soon as the root agent starts dominating her.
In the beginning of each round of the while-loop, the root agent divides the current residue into \( n \) equal pieces. Then, each neighbour-agent \( a_i \in D \) selects her top-valued \( \ell_i + 1 \) pieces. The remaining last piece in added to the root agent’s bundle \( A_r \). For each agent \( a_i \in D \), the algorithm maintains a collection \( A^{(i)} = \{ A_{0}^{(i)}, \ldots, A_{\ell_i}^{(i)} \} \) of \( \ell_i + 1 \) bundles so that each of them are of equal value in her view.

In every round of the while-loop, post-selection phase, there are possibly two phases for a trimmer agent: trimming and equaling. We first note that the algorithm maintains a special bundle \( A_{0}^{(i)} \), which gets a whole (untrimmed) piece in each step of the trimming phase. Towards the end of the trimming phase, the algorithm assigns each of the \( \ell_i \) trimmed pieces to \( \ell_i \) bundles in \( A^{(i)} \setminus A_{0}^{(i)} \). We remark here that, with each round, this operation ((see Steps 13 and 14)) increases the difference between root agent’s valuation for her own bundle and any of these bundles containing the trimmed pieces, see Lemma 2. We prove that, after a certain number of rounds, the root agent will start dominating a new trimmer agent from the set \( Tr \). This is when the algorithm moves this trimmer agent from the set \( Tr \) of active trimmer agents to the set of equalizers. We will prove in Lemma 4 that the set of active trimmer agents become empty (i.e., all the neighbour-agents become equalizers) in polynomially-many rounds.

In the equaling phase, agent \( a_i \) makes all the pieces (picked in the selection phase) of equal value and then allocates them to the bundles in \( A^{(i)} \) such that each bundle gets a single piece. Bundle \( A_{0}^{(i)} \) gets a trimmed piece so that the root agent will not envy its neighbour if she is allocated this bundle. By the domination condition, we ensure that the root agent will not envy its neighbouring agents irrespective of which bundle is assigned to her. At the end, each leaf agent connected to neighbour agent \( a_i \) chooses one bundle from \( A^{(i)} \) and the neighbouring agent gets the remaining bundle.

Our main result (Theorem 3) proves that the problem of finding a locally envy-free allocation among \( n \) agents on a Depth2Tree admits a polynomial-time discrete protocol.

**Theorem 3.** There exists a discrete protocol (Alg2) that finds a locally envy-free allocation among \( n \) agents on a Depth2Tree using at most \( O(n^3 \log(n)) \) cut and \( O(n^4 \log(n)) \) eval queries.

We begin by proving various useful properties of Alg2, and then the proof of Theorem 3 will appear in Section 4.3.

### 4.2 Properties of Alg2

We begin by establishing three crucial properties of Alg2 in Lemmas 1, 2 and 3. These lemmas would be useful in proving Theorems 3 and 4.

**Lemma 1.** In every round of the while-loop in Alg2, we make \( O(n) \) many cuts on the cake.

**Proof.** Consider any while loop in the execution of Alg2, there are three steps where cuts are made on the cake. To begin with, the root agent makes \( n - 1 \) cuts in Step 4 to equally divide \( R \) into \( n \) pieces. In the trimming part, each agent \( a_i \) makes \( \ell_i \) many cuts in Step 11. In total, it will be no more than \( n \) cuts.

In the Equaling part, each agent \( a_i \) requires at most \( \ell_i \) cuts to execute Step 17. For each piece larger than the average, it requires one cut to make it equal to the average. For each piece less than the average, we add some pieces and make at most one cut. In total, there are no more than \( n \) cuts. Therefore, in total there are \( O(n) \) cuts in each while loop of Alg2. \( \Box \)

Consider the \( j \)th round of the while-loop in Alg2. We write \( R^j \) to denote the residue at the beginning of round \( j \).
Lemma 2. Consider any round \( j \) of the while-loop in Alg2. We have the following bound on the valuation of the root agent for the residues from two consecutive iterations of the while-loop,

\[
v_r(R^{j+1}) \leq \left(1 - \frac{|D| + 1}{n}\right)v_r(R^j)
\]
In other words, the residue is decreasing exponentially with respect to the root agent with each iteration of the while loop.

Proof Consider $j$th round of the while-loop. It begins with the root agent dividing the residue $R_j$ into $n$ equal pieces, each of value $v_r(R_j)/n$.

Recall that $D$ is the set of the neighbours of $a_r$. We will prove that there are at least $|D| + 1$ whole pieces that do not generate any residue. The root agent chooses one whole piece. For each agent $a_i \in D$, that is still a trimmer agent, she would reserve a whole piece for $A_0(i)$. Otherwise, for the remaining agents in the set $D$, there is no residue in the Equaling phase. Therefore, each agent $i \in D$ and the root agent keep at least one whole piece without generating any residue. Thus, there are at least $|D| + 1/n$ proportion less residue towards the end of the $j$th round of the while-loop. Therefore, it follows that $v_r(R_j+1) \leq (1 - \frac{|D|+1}{n})v_r(R_j)$.

Lemma 3. Consider any round $j$ of the while-loop in ALG2. According to the root agent $a_r$, the total value of all the pieces selected by any agent $a_i \in D \setminus Tr$ in rounds $t \geq j$ is at most $\min\{\ell_i + 1, 1\} \cdot v_r(R_j)$.

Proof Consider the $j$th round of the while loop in ALG2. Fix any agent $a_i \in D \setminus Tr$. She executes the Equaling procedure and does not generate any residue. Therefore, it follows trivially that according to $a_r$, the total value of all the pieces selected by $a_i$ in rounds $t \geq j$ is at most $R_j$.

Next, at the beginning of round $j$, $a_r$ divides the residue $R_j$ into $n$ equal pieces according to her and $a_i$ selects $\ell_i + 1$ out of them. By Lemma 2, we know that $v_r(R_j+1) \leq (1 - \frac{|D|+1}{n})v_r(R_j)$. Therefore, the total value of all the pieces selected by $a_i$ in the subsequent rounds (including round $j$) is at-most

$$\frac{\ell_i + 1}{n} \sum_{m=j}^{\infty} \left(1 - \frac{|D|+1}{n}\right)^{m-j} v_r(R_j) \leq \frac{\ell_i + 1}{|D|+1} \cdot v_r(R_j).$$

This proves the stated claim.

4.3 Proof of Theorem 3

Equipped with the properties established in Section 4.2, we can now prove Theorem 3.

Theorem 3. There exists a discrete protocol (ALG2) that finds a locally envy-free allocation among $n$ agents on a Depth2Tree using at most $O(n^3 \log(n))$ cut and $O(n^4 \log(n))$ eval queries.

Proof (Correctness of ALG2) We will argue envy-freeness for the three agent-types: the root agent ($a_r$), neighbour agents of $a_r$ ($a_i, s \in D$), and the leaf agents $(\cup_{a_i \in D} L(i))$ separately.

(a) Root agent: In each round of the while-loop, root agent $a_r$ divides the current residue into $n$ equal pieces, denoted by the set $X$. For any agent $a_i \in Tr$, the piece added to any bundle $A_k(i) \in A(i)$ is a subset of a piece from $X$. Therefore, $a_r$ will not value any bundle $A_k(i)$ (for $k \geq 0$) larger than her own bundle $A_r$. When $a_r$ starts dominating an agent $a_i \in Tr$, then it is removed from $Tr$ and is therefore in the set $D \setminus Tr$. For $k \geq 1$, the bundles $A_k(i)$ (for $k \geq 1$) must therefore satisfy the domination condition (line 21) in the round in which agent $a_i$ is removed from the set $Tr$, i.e., $v_r(A_r) - v_r(A_k(i)) \geq \min\{\frac{\ell_i + 1}{|D|+1}, 1\} \cdot v_r(R)$. Note that if $v_r(A_r) - v_r(A_k(i)) \geq v_r(R)$, no matter how the residue is allocated, the root agent will not value $A_k(i)$ higher than $A_r$. On the other hand, if $v_r(A_r) - v_r(A_k(i)) \geq \frac{\ell_i + 1}{|D|+1} \cdot v_r(R)$, Lemma 3 ensures that the root agent again prefers $A_r$ to $A_k(i)$. 

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Finally, observe that the algorithm ensures that the bundle $A_0^{(i)}$ is a subset of a piece from $\mathcal{X}$ (see lines 11, 17 and 18). Therefore, the root agent will not envy any of her neighbours in the output allocation.

(b) **Neighbour agents:** For any agent $a_i \in D$, every bundle $A_k^{(i)} \in \mathcal{A}^{(i)}$ is of equal value in the view of agent $a_i$. So no matter how the leaf agents (that are her neighbours) choose, agent $a_i$ will have no envy towards them. In the selection phase, agent $a_i$ chooses $\ell_i + 1$ of her favorite pieces from $\mathcal{X}$, before the root agent. In each round of the while-loop, the increment for each bundle $A_k^{(i)}$ for $k \geq 0$ is as large as the increment in bundle $A_r$ in the view of agent $a_i$. So agent $a_i$ will not envy the root agent. Therefore, there is no envy for agent $a_i$ in the final allocation.

(c) **Leaf agents:** Every leaf agent chooses her favorite bundle before the neighbour agent she is connected to.

Therefore, ALG2 outputs a locally envy-free allocation among $n$ agents on a DEPTH2TREE.

Next, let us analyse the query complexity of ALG2. We begin by proving that in every $O(n \log n)$ rounds, there must be an agent in the active trimmer set $Tr$ that the root agent starts dominating. The following claim serves as a prerequisite for the above.

**Claim 2.** Let us denote $t^j := \max_{a_i \in Tr, k \leq \ell_j} \{v_r(R_{j}^k) - v_r(X_k^{(i)})\}$ to be the maximum trimmed value in round $j$ of the while-loop in ALG2. After $O(n \log n)$ rounds of the while-loop, we have $v_r(R_{j}^{i+O(n \log n)}) \leq v_r(t^j)$.

**Proof** As there are at most $n$ trimmed pieces and $t^j$ is the maximum value among them in round $j$ of the while-loop, we have $v_r(R_{j}^{i+1}) \leq n \cdot t^j$.

As $|D| + 1 \geq 2$, by Lemma 2, the residue satisfies the inequality $v_r(R_{j}^{d+1}) \leq (1 - 2/n)v_r(R_{j}^{d})$ for any two consecutive rounds $d$ and $d + 1$. Therefore, after $O(n \log n)$ rounds, we obtain the desired bound on $a_i$’s value for the residue $v_r(R_{j}^{i+O(n \log n)}) \leq n \cdot t^j \cdot (1 - 2/n)^{O(n \log n)} \leq t^j$. □

**Lemma 4.** If $Tr \neq \emptyset$, then after $O(n \log n)$ rounds, there is an agent $a_i \in Tr$ that gets removed from the set $Tr$, i.e., the root agents starts dominating $a_i$.

**Proof** Recall that, ALG2 removes an agent $a_i$ from the set $Tr$ when the root agent starts dominating her; see the domination condition (4). Consider a round $j$ of the while-loop in ALG2 with residue $R_j$ at its beginning. Let us write $t^j := \max_{i \in Tr, k \leq \ell_j} \{v_r(R_{j}^k) - v_r(X_k^{(i)})\}$ to denote the maximum trimmed value in this round. Let $(i', k')$ be the pair such that $v_r(R_{j}^{k'}) - v_r(X_{k'}^{(i')}) = t^j$. Since $\sum_{a_i \in D} \ell_i \leq n$, therefore, by the pigeon hole principle, between the rounds $j$ to $j + n$, there must be an agent $a_{i'} \in Tr$ that appears at least $\ell_{i'}$ many times in the pairs $\{(i^h, k^h)\}_{j \leq h \leq j+n}$.

We will prove that agent $a_{i'}$ will be removed in round $j + O(n \log n)$. Let us define $K = \{k \mid (i', k) = (i^h, k^h) \text{ for some round } j \leq h \leq j+n\}$ to be the set of indices of the bundles of agent $a_{i'}$ that corresponds to the maximum trimmed value in rounds $j$ to $j + n$. By Lemma 2, it follows that for any $k \in K$, the root agent will start dominating the piece $A_k^{(i')}$ after $O(n \log n)$ rounds (from round $j$). If $\{1, \ldots, \ell_{i'}\} \subseteq K$, then all the bundles $\{A_k^{(i')}\}_{1 \leq k \leq \ell_{i'}}$ are dominated by the root agent, and therefore we obtain that the root agent dominates agent $a_{i'}$ after $O(n \log n)$ rounds.

Otherwise, there exists some index between 1 to $\ell_{i'}$ that is missing in $K$. This implies that there exists some $k' \in K$ that appears at least twice in pairs $\{(i^h, k^h)\}_{j \leq h \leq j+n}$. Let $h_1 < h_2$ be two indices such that $(i^{h_1}, k^{h_1}) = (i^h, k^h) = (i', k')$. After round $h_1$, we have $v_r(A_{h_1}^{(i')}) - v_r(A_{k'}^{(i')}) \geq t^{h_1}$. In round $h_2$, since the index $k'$ gets picked again, we have $v_r(A_{k'}^{(i')}) \leq v_r(A_{k'}^{(i')})$ for $1 \leq k \leq \ell_{i'}$ (by Steps 12 and 13). Therefore, we obtain that the difference $v_r(A_{h_1}^{(i')}) - v_r(A_{k'}^{(i')}) \geq t^{h_1}$ for all $k$ in round $h_2$. Hence, by Lemma 2, all of the pieces will be dominated after $O(n \log n)$ rounds, proving the stated claim. □
The above lemma therefore proves that, in every $O(n \log n)$ rounds, the size $|Tr|$ of the active trimmer agents will decrease at least by one. Hence, the while-loop will end after $O(n^2 \log n)$ many rounds. By Lemma 1, we know that each round of the while-loop requires $O(n)$ cuts. Hence, we conclude that ALG2 requires we have a total of $O(n^3 \log n)$ cut queries. For the eval queries, the worst case is that each agent evaluates all the contiguous pieces. Because there are $O(n^3 \log(n))$ cuts, there are $O(n^3 \log(n))$ contiguous pieces. Hence, the algorithm makes $O(n^4 \log(n))$ eval queries. Therefore, ALG2 finds a locally envy-free allocation among $n$ agents on a DEPTH2TREE using the stated number of queries, therefore completing the proof.

5 Local Envy-freeness on STAR Graphs

In this section, we analyse our protocol (ALG2) developed in the previous section for a special case of DEPTH2TREE and we prove that it only requires $O(n^2)$ cut queries and $O(n^3)$ eval queries to find a locally envy-free allocation among $n$ agents on a 2-STAR graph (see Theorem 4). In Section 5.1, we complement our algorithmic result by establishing a query lower bound of $\Omega(n^2)$ queries for finding locally envy-free allocation for a STAR graph (under the assumption where the root agent partitions the cake into $n$ connected pieces).

![Figure 2: A 2-STAR graph](image)

We begin by defining a $k$-STAR graph.

**Definition 5** ($k$-STAR graph). A graph $G = (V, E)$ is called a $k$-STAR graph if the following three conditions hold: (1) $G$ contains no cycles, (2) there exists a (root) vertex $a_r$ such that every other node is at-most $k$ hops away from $a_r$, and (3) the degree of each vertex except $a_r$ is at-most 2.

First, observe that a simple cut and choose protocol produces a locally envy-free allocation for 1-STAR graph. The root agent cuts the cake into $n$ equal pieces and each of the remaining agents pick their favourite piece one after other and the (last) remaining piece is allocated to the root agent. It is easy to see that this protocol produces a locally envy-free allocation with $n-1$ cut and $O(n^2)$ eval queries.

Now, let us focus on 2-STAR graphs that are a special case of DEPTH2TREE (Figure 2). Using the exact same protocol (ALG2) as used in Section 4, we will show a significantly lower query complexity for 2-STAR graphs. This lower query complexity for 2-STAR graphs is attributed to following two reasons: (a) 2-STAR graph has a large number of trimmer agents ($|D| \geq \frac{n-1}{2}$), which makes the residue decrease faster and (b) the trimmer agents are connected to at-most one leaf agent ($\ell_i \leq 1$). In particular, we have the following result.
Theorem 4. Alg2 finds a locally envy-free allocation for \( n \) agents on a 2-STAR graph using at most \( O(n^2) \) cut and \( O(n^3) \) eval queries.

Proof We direct the reader to refer to the proof of Theorem 3 for local envy-freeness guarantee of Alg2 on 2-STAR graph as it is a special case of Depth2Tree. We will analyze the number of cut and eval queries required by Alg2 for finding a locally envy-free allocation in a 2-STAR graph. We begin by proving that there are \( O(n) \) many cuts made on the cake in any execution of the while-loop and then, we will show that the while-loop ends in \( n \) rounds.

Lemma 5. When the set of trimmer agents is non-empty (\( Tr \neq \emptyset \)), at least one agent gets removed from the set \( Tr \) in every two rounds of the while-loop.

Proof Consider a 2-STAR graph with the root agent having \( |D| \) many neighbours and \( \ell \) leaf agents. That is, there are a total of \( n = 1 + |D| + \ell \) agents. Each neighbour agent is connected to at most one leaf agent. Therefore, the total number of leaf agents, \( \ell \leq |D| \) and hence \( |D| + 1 \geq n/2 \). Then, by Lemma 2, the residue will decrease at least by half in each round.

Consider some round \( j \) of the while-loop, with \( R^j \) residue at its beginning. We will identify a special trimmer agent related to \( R^{j+1} \) who can be removed from the set \( Tr \) at the end of round \( j + 1 \) of the while-loop. Recall that any agent \( a_i \in Tr \), she selects two pieces \( X^{(i)}_0 \) and \( X^{(i)}_1 \) in the SELECTION phase.\(^7\) Let us denote \( t = \max_{a_i \in Tr} \left( v_r(R^i) - v_r(X^{(i)}_1) \right) \) be the maximum trimmed value in round \( j \). Let \( a^*_i \in Tr \) be the agent who did the trim for the maximum trimmed value, i.e., after the trim, we have \( v_r(R^i) = v_r(X^{(i)}_1) = t \). Since a piece of value \( v_r(R^i) \) is added to the bundle \( A^{(i)}_1 \) and \( X^{(i)}_1 \) is added to the bundle \( A^{(i)}_1 \), therefore we have that the difference \( v_r(A_r) - v_r(A^{(i)}_1) \geq t \).

Next, we will prove that the residue at the beginning of round \( j + 1 \) is \( v_r(R^{j+1}) \leq \frac{n}{2} \cdot t \). Note that there are at most \( n/2 \) leaf agents, therefore at most \( n/2 \) pieces get trimmed and each trimmed part is of value at most \( t \) (according to \( a_r \)). Therefore, we obtain \( v_r(R^{j+1}) \leq \frac{n}{2} \cdot t \).

Finally, let us take a look at what happens in rounds \( j \) and \( j + 1 \). At the end of the round \( j \), we have \( v_r(A_r) - v_r(A^{(i)}_1) \geq t \) and \( v_r(R^{j+1}) \leq \frac{n}{2} \cdot t \). We also know that (by Lemma 2), at the end of round \( j + 1 \), the residue \( R^{j+2} \) has value \( v_r(R^{j+2}) \leq \frac{1}{2} v_r(R^{j+1}) \leq \frac{n}{4} t \). The domination condition for a trimmer agent to get removed from the set \( Tr \) in round \( j + 2 \) is

\[
v_r(A_r) - v_r(A^{(i)}_1) \geq \frac{\ell_i}{|D| + 1} v_r(R^{j+2}) \geq \frac{4}{n} v_r(R^{j+2})
\]

Note that, the difference on the left hand side is \( v_r(A_r) - v_r(A^{(i)}_1) \geq t \) and the value in the right hand side is \( \frac{4}{n} v_r(R^{j+1}) \leq t \). Therefore, the domination condition for agent \( a^*_i \) is satisfied at the end of round \( j + 1 \) of while-loop, and hence she is removed from the set \( Tr \) of trimmer agents. \( \square \)

Therefore, Lemma 5 proves that the while-loop will end in \( 2n \) rounds, and each round requires \( O(n) \) cuts (by Lemma 1). Therefore, we have \( O(n^2) \) cuts in total. For the evaluation queries, the worst case is that each agent evaluate all the contiguous pieces. Because there are \( O(n^2) \) cuts, there are \( O(n^2) \) contiguous pieces. We have at most \( O(n^3) \) eval queries. \( \square \)

5.1 Lower Bound for 1-STAR Graphs

In this section, we prove a lower bound on the query complexity of finding locally envy-free allocation among \( n \) agents on a 1-STAR graph. For the complete graph, the best known lower bound for finding envy-free allocation is \( \Omega(n^2) \) [Pro09]. This is an unconditional lower bound, and for a star

\(^7\)In Step 10, we rename these pieces such that \( v_r(X^{(i)}_0) \leq v_r(X^{(i)}_1) \).
Theorem 5. Consider a cake-division instance with $n$ agents on a 1-STAR graph. When the root agent divides the cake $[0, 1]$ into $n$ equally-valued connected pieces, it requires $\Omega(n^2)$ eval queries to find a locally envy-free allocation.

Proof Consider a set $\{a_1, a_2, \ldots, a_{n-1}, a_r\}$ of $n$ agents on a star graph with $a_r$ as the root agent. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be an $n$-partition of the cake made by the root agent. We will construct an adversary to respond to eval queries in a specific manner that will prove the stated lower bound.

To begin with, the adversary pairs pieces in the following manner: if $n$ is even, the adversary sets $K_1 = \{P_1, P_2\}$, $K_2 = \{P_3, P_4\}, \ldots, K_{n/2} = \{P_{n-1}, P_n\}$, and if $n$ is odd, the adversary sets $K_1 = \{P_1, P_2\}, \ldots, K_{n/2} = \{P_{n-2}, P_{n-1}, P_n\}$. We will say that a point $s \in [0, 1]$ belongs to a set $K_t$, for $1 \leq t \leq \lfloor n/2 \rfloor$, if $\alpha < s < \beta$, where $\cup_{P \in K_t} P = [\alpha, \beta]$. Given an eval query $[a, b] \subseteq [0, 1]$, we define the informed set of $[a, b]$, denoted by $F_{[a,b]} := \{P \in K_t : \text{ either } a \text{ or } b \text{ belongs to } K_t \}$ for $t \in \lfloor [n/2] \rfloor$, as the union of the pieces in those $K_t$’s where either $a$ or $b$ belongs to. By our construction, we have $|F_{[a,b]}| \leq 5$ for all $[a, b] \subseteq [0, 1]$.

Strategy of the adversary: If an eval query is made to agent $a_i$ for the interval $[a, b]$, then the adversary fixes the valuations of agent $a_i$ on all pieces in the set $F_{[a,b]}$ and answer the query in any way that is consistent with its previous responses. For every piece in $F_{[a,b]}$, the adversary sets the valuations in the range $[1/n - \epsilon, 1/n + \epsilon]$, where $\epsilon = o(1/n)$ is a small constant. In particular, we will show that after sufficient eval queries to an agent, the adversary will fix exactly one piece in $\mathcal{P}$ to be of value $1/n + \epsilon$ for the agent. We will refer to such a piece as her large piece.

We write $F_i := \{P \in \mathcal{P} : P \in F_{[a,b]} \text{ and eval query is made} \}$ to be the set of union of the pieces in the informed sets formed via eval queries to $a_i$. Additionally, we denote $D_j = \{a_i \mid P_j \in F_i\}$ to consist of the set of agents such that piece $P_j$ is in her informed set.

Now, we will explain the rules according to which the adversary responds to the eval queries. When an eval query is made to agent $a_i$, we denote $N_i = F_{[a,b]} \setminus F_i$ to be the set of newly added informed pieces. We update $F_i$ to include $F_{[a,b]}$ and $D_j$ to include $a_i$ if $P_j \in F_i$. The adversary will answer the query according to the following rules:

1. If the adversary has not revealed a large piece of agent $a_i$ yet, and either of the following two conditions is true: (1) $|F_i| \geq \frac{n}{2}$ or, (2) $|D_j| \geq \frac{n}{4}$ for some $P_j \in N_i$, then set some piece $P_k \in N_i$ as $a_i$’s large piece. That is, fix $v_i(P_k) = \frac{1}{n} + \epsilon$. (here, $k = j$ if (2) is true). Set the value for the piece that is paired up with $P_k$ at $\frac{1}{n} - \epsilon$. Set the value for the remaining pieces in $N_i$ at $\frac{1}{n}$.

2. Otherwise, set value $v_i(P) = \frac{1}{n}$ for all pieces $P \in N_i$.

Next, we establish the following two lemmas that would be crucial to analyse the query complexity of finding a locally envy-free allocation on a star graph.

Lemma 6. With the above-described adversary, consider a locally envy-free allocation $\{P_1, \ldots, P_{n-1}, P_r\}$ where the root agent is assigned piece $P_r$ and agent $a_i$ gets piece $P_i$ for $i \in [n-1]$. Then, for any agent $a_i$, we have

1. The piece $P_i \in F_i$
2. The piece $P_r \in F_i$

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8Without loss of generality, we assume that $P_1, \ldots, P_n$ are aligned from left to right.

9For example, say for an interval $[a, b]$ if $a$ is inside $\{P_3, P_4\}$ and $b$ does not belong to any $K_t$, then $F_{[a,b]} = \{P_3, P_4\}$. 

---
3. The valuation, \( v_i(P_r) < 1/n + \epsilon \).

**Proof** Fix a non-root agent \( a_i \) for \( i \in [n-1] \). To prove the first part of the claim, we begin by observing that for a piece \( P \notin F_i \) (i.e., \( P \) does not lie in the informed set of any eval queries made to agent \( a_i \)), we can set \( v_i(P) = \frac{1}{n} + 2\epsilon \) or \( \frac{1}{n} - 2\epsilon \), without incurring any inconsistency (with previous responses of the adversary). Suppose, for contradiction, agent \( a_i \) is assigned a piece that does not belong to \( F_i \), i.e., \( P \notin F_i \). We can then set \( v_i(P) = \frac{1}{n} - 2\epsilon \), and \( v_i(P') = \frac{1}{n} + 2\epsilon \) for all the remaining pieces \( P' \notin F_i \). Recall that, the adversary responses in a way that ensures \( v_i(P) \geq \frac{1}{n} - \epsilon \) for all \( P \in F_i \). Therefore, we obtain \( v_i(P) < v_i(P') \) for all pieces \( P \neq P' \), creating envy for agent \( a_i \) (towards the root agent). Hence, we must have \( P_r \notin F_i \).

Next, we prove the second part of the claim. For contradiction, let us assume that the piece \( P_r \notin F_i \). We can therefore set \( v_i(P_r) = \frac{1}{n} + 2\epsilon \). Since we have \( P_r \in F_i \), we know that \( v_i(P_r) \leq \frac{1}{n} + \epsilon \). Thus, \( a_i \) will envy the root agent, contradicting the local envy-freeness.

For proving the last part, observe that there is at most one piece in \( F_i \) that agent \( a_i \) values at \( \frac{1}{n} + \epsilon \). From the first and the second part, we know that both \( P_i, P_r \in F_i \). If \( v_i(P_r) = \frac{1}{n} + \epsilon \), then we must have \( v_i(P_i) < \frac{1}{n} + \epsilon \). This creates envy for agent \( a_i \), therefore, we have \( v_i(P_i) < \frac{1}{n} + \epsilon \). \( \square \)

Next, we prove a lower bound on the number of agents for which the adversary must have found the large piece for, before one computes a locally envy-free allocation.

**Lemma 7.** For achieving local envy-freeness, the adversary must have found large pieces for at least \( n/2 \) agents.

**Proof** Consider a locally envy-free allocation \( \{ P_1, \ldots, P_{n-1}, P_r \} \) where the root agent is assigned piece \( P_r \) and agent \( a_i \) gets piece \( P_i \) for \( i \in [n-1] \). Using Lemma 6, we know that \( P_r \in F_i \) for all agents \( a_i \) for \( i \in [n-1] \). Therefore, we have \( |D_r| = n - 1 \). Consider all agents \( a_{r'} \) for whom after adding \( P_r \in F_{r'} \), we obtain \( |D_{r'}| \geq n/2 \). Note that, there are at least \( n/2 \) such agents.

Again, by Lemma 6, we know that the adversary sets the value \( v_{r'}(P_r) < \frac{1}{n} + \epsilon \). Following the rules according to which the adversary responds to eval queries, the time \( P_r \) is added to \( F_{r'} \), either agent \( a_{r'} \) had already found her large piece or the adversary will set some piece to have value \( \frac{1}{n} + \epsilon \) for her. Therefore, at least \( n/2 \) agents must have found their large pieces, before an algorithm can output an allocation that is locally envy-free. \( \square \)

Finally, we will establish the stated lower bound on the number of eval queries required to find a locally envy-free allocation in the \( n \)-partition \( P \). Note that it is sufficient to find a lower bound for the sum \( \sum |F_i| \). Recall that, with each query, the sum \( \sum |F_i| \) will increase at most by 5. Therefore, the total number of eval queries required is \( \Omega(\sum |F_i|) \). Observe that, we can write \( \sum_i |F_i| = \sum_j |D_j| \).

We write \( f := \{i \in [n] : |F_i| \geq n/2\} \) to denote the number of agents such that \( |F_i| \geq n/2 \) and \( d := \{j \in [n] : |D_j| \geq n/2\} \) to denote the number of pieces such that \( |D_j| \geq n/2 \).

To prove the lower bound, we will prove that either \( f = \Omega(n) \) or \( d = \Omega(n) \). Consider the pair \( (i,j) \) such that the adversary sets \( v_i(P_j) = \frac{1}{n} + \epsilon \) for agent \( a_i \). Claim 7 proves that there are at least \( n/2 \) such pairs. Combining it with the rules of the adversary, we know that for such a pair \( (i,j) \), either \( |F_i| \geq n/2 \) or \( |D_j| \geq n/2 \) must be true. Therefore, we obtain that the sum \( f + d \geq n/2 \). This implies that, either \( f \geq n/4 \) or \( d \geq n/4 \). Observe that, we can now write \( \sum_i |F_i| \geq \max\{f \cdot \frac{n}{2}, d \cdot \frac{n}{2}\} = \Omega(n^2) \). This establishes the stated lower bound. \( \square \)

6 Discussion and Future Directions

This work explores the classic fair cake division problem when agents lie on a graph and its edges correspond to envy-constraints among the agents. We develop a novel algorithm that outputs a
locally envy-free allocation—using $O(n^3 \log(n))$ cuts—among $n$ agents on a tree with depth at most two. Using the exact same algorithm, we show a significantly lower query complexity of $O(n^2)$ cuts for achieving local envy-freeness on 2-STAR graphs. Additionally, we establish that where agents lie on a star graph and the root agent cuts the cake into $n$ pieces, it requires $\Omega(n^2)$ evaluation queries to find a locally envy-free allocation.

To the best of our knowledge, we are the first to develop a discrete and efficient protocol for finding locally envy-free allocations among $n$ agents wherein the envy-constraints are specified via a non-trivial class of graphs. The techniques developed in our algorithms give interesting insights. First, they indicate that the maximum length of a path in the network graph is a bottleneck for determining the complexity of finding a locally envy-free allocation. Furthermore, the fact that we extend our ideas for five agents on a LINE graph to DEPTH2TREE shows that for finding locally envy-free allocation for tree graphs having depth $k$, it should be enough to solve the case of $2k + 1$ agents on a LINE graph.

The positive results developed in this work motivate various generalizations. One immediate research direction is to find a locally envy-free protocol for the case when six or more agents lie on a LINE graph. Another interesting direction for future work is to develop upper and lower bounds for the query complexity of finding locally envy-free allocations for graph structures such as cycles.

Our approach designates a single agent as the cutter. However, interestingly this approach does not seem to generalize to graphs having a path of length more than five. The study of graphs with larger path length may require new techniques with possibly multiple cutter agents and remains a compelling open problem. Finally, we believe that the study of envy-free protocols for individual graph structures will provide the necessary building blocks to improve our understanding of the general envy-free cake cutting problem.

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References

[ABKR19] Eshwar Ram Arunachaleswaran, Siddharth Barman, Rachitesh Kumar, and Nidhi Rathi. Fair and efficient cake division with connected pieces. In International Conference on Web and Internet Economics, pages 57–70. Springer, 2019.

[ACF+18] Georgios Amanatidis, George Christodoulou, John Fearnley, Evangelos Markakis, Christos-Alexandros Psomas, and Eftychia Vakaliou. An improved envy-free cake cutting protocol for four agents. In Algorithmic Game Theory - 11th International Symposium, SAGT’18, volume 11059, pages 87–99. Springer, 2018.

[Adj] Adjusted winner. http://www.nyu.edu/projects/adjustedwinner/. Accessed: 2019-07-07.

[AKP17] Rediet Abebe, Jon Kleinberg, and David C. Parkes. Fair division via social comparison. In Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems, AAMAS ’17, page 281–289, 2017.

[AM16a] Haris Aziz and Simon Mackenzie. A discrete and bounded envy-free cake cutting protocol for any number of agents. In 2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS), pages 416–427. IEEE, 2016.
[AM16b] Haris Aziz and Simon Mackenzie. A discrete and bounded envy-free cake cutting protocol for four agents. In Proceedings of the Forty-Eighth Annual ACM Symposium on Theory of Computing, STOC ’16, page 454–464, 2016.

[BKN22] Robert Bredereck, Andrzej Kaczmarczyk, and Rolf Niedermeier. Envy-free allocations respecting social networks. Artificial Intelligence, 305:103664, 2022.

[BQZ17] Xiaohui Bei, Youming Qiao, and Shengyu Zhang. Networked fairness in cake cutting. In Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI-17, pages 3632–3638, 2017.

[BR21] Siddharth Barman and Nidhi Rathi. Fair cake division under monotone likelihood ratios. Mathematics of Operations Research, 2021.

[BSW+20] Xiaohui Bei, Xiaoming Sun, Hao Wu, Jialin Zhang, Zhijie Zhang, and Wei Zi. Cake cutting on graphs: a discrete and bounded proportional protocol. In Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’20, pages 2114–2123, 2020.

[BT96] Steven J. Brams and Alan D. Taylor. Fair Division: From Cake-Cutting to Dispute Resolution. Cambridge University Press, 1996.

[CKM+19] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. The unreasonable fairness of maximum nash welfare. ACM Transactions on Economics and Computation, ACM TEAC ’19, 7(3), 2019.

[DQS12] Xiaotie Deng, Qi Qi, and Amin Saberi. Algorithmic solutions for envy-free cake cutting. Operations Research, 60(6):1461–1476, 2012.

[ES99] Francis Edward Su. Rental harmony: Sperner’s lemma in fair division. The American Mathematical Monthly, 106(10):930–942, 1999.

[Fol67] Duncan K Foley. Resource allocation and the public sector. 1967.

[KLP13] David Kurokawa, John K Lai, and Ariel D Procaccia. How to cut a cake before the party ends. In Twenty-Seventh AAAI Conference on Artificial Intelligence, 2013.

[Mou04] Hervé Moulin. Fair division and collective welfare. MIT press, 2004.

[PM16] Ariel D. Procaccia and Hervé Moulin. Cake Cutting Algorithms, page 311–330. Cambridge University Press, 2016.

[Pro09] Ariel D. Procaccia. Thou shalt covet thy neighbor’s cake. In Proceedings of the 21st International Joint Conference on Artificial Intelligence, IJCAI’09, page 239–244, 2009.

[RW98] Jack Robertson and William Webb. Cake-cutting algorithms: Be fair if you can. AK Peters/CRC Press, 1998.

[Str80] Walter Stromquist. How to cut a cake fairly. The American Mathematical Monthly, 87(8):640–644, 1980.

[Str08] Walter Stromquist. Envy-free cake divisions cannot be found by finite protocols. Electronic Journal of Combinatorics, 15(1), 2008.

[Tuc21] Jamie Tucker-Foltz. Thou shalt covet the average of thy neighbors’ cakes. CoRR, abs/2106.11178, 2021.