Superintegrable Hamiltonian Systems: an algebraic approach

J. A. Calzada 1, J. Negro 2 and M. A. del Olmo 2

1 Departamento de Matemática Aplicada,
2 Departamento de Física Teórica,
Universidad de Valladolid, E-47011, Valladolid, Spain

E-mail: juacal@eis.uva.es, jnegro@fta.uva.es, olmo@fta.uva.es

Abstract. Using an algebraic approach in terms of a complete set of shape-invariant intertwining operators we study a family of quantum superintegrable hamiltonian systems. These intertwining operators close a certain Lie algebra in such a way that the eigenstates of these Hamiltonians belong to some of its unitary representations.

1. Introduction

Let us consider a Hamiltonian system living in a $n$-dimensional configuration space. The notion of integrability of this system is related with the existence of constants of motion allowing to solve or reduce the equations of motion by quadratures. In this way, a Hamiltonian system is said to be integrable (or completely integrable) if there are $n$ constants of motion including the Hamiltonian $H$, which are independent and in involution. Incidentally, the maximal number of constants of motion, being the Hamiltonian one of them, functionally independents in involution is $n$ for a Hamiltonian system whose (symplectic) phase space is $2n$-dimensional. But, if it has $0 < k \leq n - 1$ additional constants of motion the system is called superintegrable and when there exist $2n-1$ invariants (well defined in phase-space) we say that the system is maximally superintegrable. In this case there exist more than one set of $n$ motion constants, including the Hamiltonian itself, independent and in involution. The constants of motion are related with (partially hidden) symmetries of the corresponding physical system. Normally these properties of the superintegrable Hamiltonian systems (SHS) are exhibited by the possibility of separation of variables in more than one coordinate system for the Hamilton-Jacobi equation in the classical case and for the Schrödinger equation in the quantum case, and by the fact that the finite classical trajectories are closed (periodic) while the discrete energy levels are degenerate in the quantum case.

Obviously, in the real world the rule is chaotic systems and physical systems exhibiting complete integrability are exceptional. A list of SHS can be found in the classification works by Evans [1]. In the last years superintegrability in constant curvature configuration spaces has been studied in a systematic way [2, 3, 4]. However, we can also mention two former references, among a long list of contributions in relation with some generalizations of SHS to constant curvature configuration spaces. Lakshmanan and Eswaran [5] analyzed the isotropic oscillator on a 3-sphere and Higgs [6] studied versions of the Coulomb potential and of the harmonic
oscillator living in the $N$-dimensional sphere and having $SO(N + 1)$ and $SU(N)$ symmetry in classical and in quantum mechanics, respectively.

From an algebraic point of view, we can also mention the works by Daskaloyannis and collaborators [7, 8] where they present an approach to superintegrability using deformed algebras. Also Ballesteros and collaborators have obtained new completely integrable systems and recover others well-known from a coalgebra approach [9] (see [10] in this volume for a review).

Another case of algebraic approach to integrable systems is the construction of a family of SHS from a group-theoretical method based on the (Marsden–Weinstein) symmetry reduction [11]. These systems were obtained from free systems in $\mathbb{C}^{p,q}$ with an original $u(p,q)$-symmetry [12]

$$
H = \frac{c}{4} g^{\mu\nu} \bar{p}_\mu p_\nu \xrightarrow{\text{MW reduction}} H' = \frac{c}{4} g^{\mu\nu} p_\mu s_\nu + V(s),
$$

(1)

(the bar stands for the complex conjugate) with $V(s)$ a potential in terms of the real coordinates $(s^\mu)$. The constant curvature configuration spaces of these SHS are $SO(p,q)$-homogeneous spaces. Many aspects of these SHS have been studied using standard procedures [12]–[16] and their quantum versions are well known (for instance, the Morse potential [17] and the Pöschl–Teller potential [18]). In this work we present a review of a new framework based on intertwining operators (IO), a form of Darboux transformations [19], that will allow us to study them from an algebraic point of view. The interest is that the IO’s obtained by this procedure close certain Lie algebras that show the symmetry properties, some times hidden, of the system under consideration and permit to describe these SHS in terms of representations of such “intertwining symmetry” algebras (or “dynamical algebras” [20]).

The intertwining operators (first order differential operators) $A$, connect different Hamiltonians, $H, H'$, belonging to the same hierarchy

$$
AH = H'A.
$$

From this family of SHS a complete set of such IO’s, in the sense that any of the Hamiltonians of the hierarchy can be expressed in terms of these operators, may be obtained. The IO’s are associated to separable coordinate systems for the Hamiltonians. The algebraic analysis of the IO’s associated to integrable Hamiltonians has been made, for instance, in Refs. [21, 22, 23]. The quantum physical systems factorized in terms of IO’s have been till now one-dimensional systems. Recently, we introduced a natural extension to higher dimensions of the intertwining (Darboux) transformations of the Schrödinger equation for one-dimensional quantum systems [24], since the Hamiltonians of this family of SHS originate a coupled set of $n$–differential equations, when a system of separable coordinates is used, which can be factorized one by one. We have used this approach to study two particular, but well representative cases, of this SHS family: a system [25] with “$u(3)$-symmetry” and other one with “$u(2,1)$-symmetry” [26] (see also for both cases [27]). As the reader will see later the generalization to systems with higher “$u(p,q)$-symmetry” is evident. This “$u(p,q)$-symmetry” is interpreted in the sense of (1), i.e., the original system before symmetry reduction has $u(p,q)$-symmetry. The case presented here is a $u(2,1)$-system, i.e., a quantum system with configuration space a $SO(2,1)$-homogeneous space (2D hyperboloid of two-sheets). We will be able to obtain a wide set of IO’s closing a $u(2,1)$ dynamical Lie algebras. However, there is also a “hidden symmetry” since considering discrete symmetry operators we can enlarge $u(2,1)$ to $so(4,2)$. Note that we have recovered the algebraic structure of the system that was involved in the Marsden-Weinstein reduction procedure

$$
su(p,q) \xrightarrow{\text{symmetry reduction}} so(p,q) \xrightarrow{\text{IO factorization}} u(p,q) \xrightarrow{\text{discrete symmetries}} so(2p,2q).
$$
This algebraic approach gives a simple explanation of the main features of these SHS. It allows us to characterize the discrete spectrum and the corresponding eigenfunctions of the system in terms of irreducible unitary representations (IUR) of the intertwining symmetry algebras.

We have shown the relation of eigenstates and eigenvalues with unitary representations of the $su(2)$, $su(2,1)$ and $so(4,2)$ Lie algebras. In particular, we have studied the degeneration problem as well as the number of bound states. Here, we remark that such a detailed study for a ‘non-compact’ superintegrable system had not been realized till now, up to our knowledge. We have restricted to IUR’s, but a wider analysis can be done for hierarchies associated to representations with a not well defined unitary character.

The IO’s can also be used to find second order integrals of motion for a Hamiltonian and their algebraic relations, which is the usual approach to (super)-integrable systems. As a general consideration we can say that it is easier to deal directly with the IO’s, because they are more elementary and simpler, than with constants of motion.

2. Superintegrable classical systems

Let us define the free Hamiltonian

$$ H = 4 g_{\mu \nu} p_\mu \bar{p}_\nu, \quad \mu, \nu = 0, \ldots, n = p + q - 1, \quad (2) $$

on the (complex) configuration space

$$ SU(p,q) \over SU(p-1,q) \times U(1). $$

This space is an Hermitian hyperbolic space with metric $g_{\mu \nu}$ and coordinates $y^\mu \in \mathbb{C}$ [28, 14] verifying that $g_{\mu \nu} \bar{y}^\mu \bar{y}^\nu = 1$.

A symmetry reduction procedure by means of a maximal abelian subalgebra (MASA) of $su(p,q)$ [29] originates a (non free) real Hamiltonian on a homogeneous $SO(p,q)$-space [12, 13]

$$ H = \frac{1}{2} g^{\mu \nu} p_\mu p_\nu + V(s) $$

from the original one (2). Now $V(s)$ is a potential depending on the real coordinates $s^\mu$ which satisfy $g_{\mu \nu} s^\mu s^\nu = 1$.

The symmetry reduction procedure transforms the set of complex coordinates $y^\mu$ into a set of ignorable variables $x^\mu$ and the actual real coordinates $s^\mu$. Let us consider a basis of a MASA of $u(p,q)$ constituted only by pure imaginary matrices $Y_\mu (\mu = 0, \ldots, n)$. The relation between old $y^\mu$ and new coordinates $x^\mu, s^\mu$ is given by

$$ y^\mu = B(x)^\mu_{\nu} s^\nu, \quad B(x) = \exp \left( x^\mu Y_\mu \right). $$

The ignorability of the $x$ coordinates (i.e., the vector fields corresponding to the MASA are straightened out in these coordinates) is due to the fact that the $x^\mu$ are the parameters of the transformation associated to the MASA of $u(p,q)$ used in the reduction. The Jacobian matrix, $J$, of the coordinate transformation $((y, \bar{y}) \rightarrow (x, s))$ is

$$ J = \frac{\partial (y, \bar{y})}{\partial (x, s)} = \left( \begin{array}{cc} A & B \\ \bar{A} & \bar{B} \end{array} \right), $$

where

$$ A^\mu_{\nu} = \frac{\partial y^\mu}{\partial x^\nu} = (Y_\nu)^\mu_{\rho} y^\rho. $$
Then, the original Hamiltonian (2) reduces to the Hamiltonian

$$H = c \left( \frac{1}{2} g^{\mu \nu} p_\mu p_\nu + V(s) \right), \quad V(s) = p_x^T (A^\dagger K A)^{-1} p_x,$$

which depends only in the new real coordinates $s$, with $p_x$ the constant momenta associated to the ignorable coordinates $x$ and $K$ is the matrix defined by the metric $g$.

3. A superintegrable $u(2,1)$-Hamiltonian system

A classical superintegrable Hamiltonian associated, for instance, to $su(2,1)$ can be constructed using the symmetry reduction procedure presented before. A suitable basis of $su(2,1)$ in terms of $3 \times 3$ matrices $X_1, \ldots, X_8$ (with metric $K = \text{diag}(-1, -1, +1)$) is

$$X_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad X_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad X_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}.$$  

There are four MASAs for $su(2,1)$ [29]: the compact Cartan subalgebra, the noncompact Cartan subalgebra, the orthogonally decomposable subalgebra and the nilpotent subalgebra. In this example we will consider the symmetry reduction by the compact Cartan subalgebra, but other three SHS can be obtained with the other MASAs. The compact Cartan subalgebra can be generated by the matrices

$$\text{diag}(i, -i, 0) \quad \text{diag}(0, i, -i).$$

However, in order to simplify the computations we use the following matrices

$$Y_0 = \text{diag}(i, 0, 0), \quad Y_1 = \text{diag}(0, i, 0), \quad Y_2 = \text{diag}(0, 0, i),$$

that determine a basis for the corresponding MASA in $u(2,1)$.

The old complex coordinates $y$ are related to the real ones $s$ by

$$y^\mu = e^{i x^\mu Y_\mu} s^\mu, \quad \mu = 0, 1, 2,$$

and the Hamiltonian can be written as

$$H = \frac{1}{2} c \left( -p_0^2 - p_1^2 + p_2^2 \right) + \frac{m_0^2}{s_0} + \frac{m_1^2}{s_1} - \frac{m_2^2}{s_2},$$

being $c$ a constant. It lives in a 2-dimensional two-sheet hyperboloid $-s_0^2 - s_1^2 + s_2^2 = 1$. Non-negative real numbers $m_i$ can be chosen for the potential constants.

The system has three independent invariants of motion

$$R_{01} = (s_0 p_1 - s_1 p_0)^2 + \left( m_0 \frac{s_1}{s_0} + m_1 \frac{s_0}{s_1} \right)^2,$$

$$R_{02} = (s_0 p_2 + s_2 p_0)^2 + \left( m_0 \frac{s_2}{s_0} - m_2 \frac{s_0}{s_2} \right)^2,$$

$$R_{12} = (s_1 p_2 + s_2 p_1)^2 + \left( m_1 \frac{s_2}{s_1} - m_2 \frac{s_1}{s_2} \right)^2,$$
that can be written in terms of the basis of $su(2, 1)$ (in the realization as function of $s_\mu$ and $p_\mu$)
\[ Q_1 = R_{01} = X_1^2 + X_4^2, \quad Q_2 = R_{02} = X_3^2 + X_6^2, \quad Q_3 = R_{12} = X_7^2 + X_8^2. \]
The sum of these three invariants is the Hamiltonian (3) up to an additive constant
\[ H = -Q_1 + Q_2 + Q_3 + \text{cnt.} \quad (4) \]
The quadratic Casimir of $su(2, 1)$ can be also written in terms of the constants of motion and the second order operators in the enveloping algebra of the compact Cartan subalgebra of $su(2, 1)$
\[ C_{su(2,1)} = 3Q_1 - 3Q_2 - 3Q_3 + 4X_1^2 + 2[X_1, X_2]_+ + 4X_2^2. \]
The Hamiltonian is in involution with all the three constants of motion (i.e. $[H, Q_i] = 0$, $i = 1, 2, 3$), but the $Q_i$’s do not commute among themselves
\[
\begin{align*}
\{Q_1, Q_2\} &= \{Q_3, Q_1\} = \{Q_3, Q_2\} \\
&= -[X_3, [X_5, X_7]_+]_+ - [X_3, [X_6, X_8]_+]_+ + [X_4, [X_5, X_8]_+]_+ - [X_4, [X_6, X_7]_+]_+.
\end{align*}
\]
So, the system (3) is superintegrable.

The solutions of the motion problem for this system can be obtained solving the corresponding Hamilton–Jacobi (HJ) equation in an appropriate coordinate system, such that the HJ equation separates into a system of ordinary differential equations.

The 2-dimensional two-sheet hyperboloid can be parametrized, for instance, on pseudo-spherical coordinates $(\theta, \xi)$
\[ s_0 = \sinh \xi \cos \theta, \quad s_1 = \sinh \xi \sin \theta, \quad s_2 = \cosh \xi, \quad (5) \]
with $0 \leq \theta < \pi/2$ and $0 \leq \xi < \infty$. Now in these coordinates and specializing $c = -1$ the Hamiltonian takes the form
\[ H = \frac{1}{2} \left( p_\theta^2 + \frac{p_\xi^2}{\sinh^2 \xi} \right) + \frac{1}{\sinh^2 \xi} \left( \frac{m_0^2}{\cos^2 \theta} + \frac{m_1^2}{\sin^2 \theta} \right) - \frac{m_2^2}{\cosh^2 \xi}. \]
The potential (Fig. 1) is regular inside the domain of the variables and there is a saddle point for the values $\theta = \arctan \sqrt{m_1/m_0}$ and $\xi = \arg \tanh \sqrt{m_2(m_0 + m_1)}$ if $m_0 + m_1 > m_2$.

The explicit form of the invariants of motion, $Q_i$, in terms of the coordinates $(\xi, \theta)$ is
\[
\begin{align*}
Q_1 &= \frac{1}{2} p_\theta^2 + \frac{m_0^2}{\cos^2 \theta} + \frac{m_1^2}{\sin^2 \theta} \\
Q_2 &= \coth^2 \xi \left( \frac{1}{2} p_\theta^2 \sin^2 \theta + \frac{m_0^2}{\cos^2 \theta} \right) + \cos^2 \theta \left( \frac{1}{2} p_\xi^2 + \frac{m_2^2}{\coth^2 \xi} \right) \\
&+ \frac{1}{2} p_\theta p_\xi \sin 2 \theta \coth \xi, \\
Q_3 &= \coth^2 \xi \left( \frac{1}{2} p_\theta^2 \cos^2 \theta + \frac{m_0^2}{\sin^2 \theta} \right) + \sin^2 \theta \left( \frac{1}{2} p_\xi^2 + \frac{m_2^2}{\coth^2 \xi} \right) \\
&- \frac{1}{2} p_\theta p_\xi \sin 2 \theta \coth \xi.
\end{align*}
\]
Consideremos el hamiltoniano dado por
\[ H = \frac{1}{2} \left( p^2 \xi + p^2 \theta \sinh^2 \xi \right) + \frac{1}{\sinh^2 \xi} \left( m^2_0 \cos^2 \theta + m^2_1 \sin^2 \theta \right) - m^2_2 \cosh^2 \xi. \]

El potencial asociado es
\[ V(\xi, \theta) = \frac{1}{\sinh^2 \xi} \left( m^2_0 \cos^2 \theta + m^2_1 \sin^2 \theta \right) - m^2_2 \cosh^2 \xi. \]

La representación para \( m_0 = 1, m_1 = 2 \) y \( m_2 = 2.5 \) es

\[ \text{Figure 1.} \text{ Graphical representation of the potential for } m_0 = 1, m_1 = 2, m_2 = 2.5. \]

La representación para \( m_0 = 1, m_1 = 2, m_2 = 5/2 \) es

The Hamilton-Jacobi equation for this system is
\[ \frac{1}{2} \left( \frac{\partial S}{\partial \xi} \right)^2 - \frac{m^2_2}{\cosh^2 \xi} + \frac{1}{\sinh^2 \xi} \frac{1}{2} \left( \frac{\partial S}{\partial \theta} \right)^2 + \frac{m^2_0}{\cos^2 \theta} + \frac{m^2_1}{\sin^2 \theta} = E, \]

which separates into two ordinary differential equations, if one takes into account that its solution can be written as \( S(\theta, \xi) = S_1(\theta) + S_2(\xi) - Et \),
\[ \frac{1}{2} \left( \frac{\partial S_1}{\partial \theta} \right)^2 + \frac{m^2_0}{\cos^2 \theta} + \frac{m^2_1}{\sin^2 \theta} = \alpha_1, \]
\[ \frac{1}{2} \left( \frac{\partial S_2}{\partial \xi} \right)^2 - \frac{m^2_2}{\cosh^2 \xi} + \frac{\alpha_1}{\sinh^2 \xi} = \alpha_2, \]

where \( \alpha_2 = E \) and \( \alpha_1 \) are the separation constants being \( \alpha_1 \) always positive. Both equations are formally similar to those of the corresponding one-dimensional problem [15]. Their solutions can be found as particular cases of a more general approach [14]. The solutions depend on the value of the energy. So, we can distinguish three cases according with \( E < 0, E = 0 \) and \( E > 0 \) for the second Hamilton-Jacobi equation

\[ E < 0 \quad \cosh^2 \xi = \frac{1}{2E} \left[ -b_2 - \sqrt{b_2^2 + 4m_2^2E \sin \left( 2\sqrt{2|E|}(\beta_2 + t) \right)} \right], \]
\[ E = 0 \quad \cosh^2 \xi = \frac{m_2^2}{m_2^2 - \alpha_1} + 2(m_2^2 - \alpha_1)(\beta_2 + t)^2, \]
\[ E > 0 \quad \cosh^2 \xi = \frac{1}{2E} \left[ -b_2 + \sqrt{b_2^2 + 4m_2^2E \sinh \left( 2\sqrt{2E}(\beta_2 + t) \right)} \right], \]

where \( b_2 = -E - \alpha_1 + m_2^2 \) and \( \beta_2 = \partial S/\partial \alpha_2 - t \). The remaining Hamilton-Jacobi equation has the following solution
\[ \cos^2 \theta = \frac{1}{2\alpha_1} \left[ b_1 - \sqrt{b_1^2 - 4\alpha_1 m_0^2 \sin \left( 2\sqrt{2\alpha_1}(\beta_1) \right)} \right], \]

with \( b_1 = \alpha_1 + m_0^2 - m_2^2 \) and \( \beta_1 = \partial S/\partial \alpha_1 \).
4. Superintegrable quantum systems

In order to obtain a quantum version of the classical system described on the previous section, we can proceed as follows. Taking into account the group theoretical method of construction of the Hamiltonian (3) and the constants of motions together with the inspection of all these objects we can relate the terms like \( s_\mu p_\mu \pm v_\mu p_\mu \) with generators of "rotations" when \( p_\mu \to \partial_\mu \) in the plane \( X_\mu X_\nu \). Since the Hamiltonian is the sum of the three constants of motion up to constants (4) we can make the following Ansatz for a quantum Hamiltonian as a linear combination of the infinitesimal generators of "rotations" in the plane \( X_1 X_2 \), \( X_0 X_2 \) and \( X_0 X_1 \) around the axis \( X_0 \), \( X_1 \) and \( X_2 \), respectively (i.e. \( J_3^2 \), \( J_1^2 \) and \( J_2^2 \)). The signature of the metric of the configuration space imposes that the rotations be compact or noncompact. Hence, the generators will span \( so(3) \) or \( so(2,1) \) and it is enough to take a differential realization of them. In this way, the Casimir operator of \( so(2,1) \) \( (C = J_0^2 + J_1^2 - J_2^2) \) will give the "kinetic" part of the Hamiltonian. The "potential" term will be obtain directly from the potential of (3) making the change \( m_\nu^2 \to \ell_\nu^2 - 1/4 \) in order to simplify the computations. Thus, we obtain the Hamiltonian

\[
H_\ell = -J_0^2 - J_1^2 + J_2^2 + \frac{l_0^2 - \frac{1}{2}}{s_0^2} + \frac{l_1^2 - \frac{1}{2}}{s_1^2} - \frac{l_2^2 - \frac{1}{2}}{s_2^2},
\]

defined in the 2-dimensional two-sheet hyperboloid \(-s_0^2 - s_1^2 + s_2^2 = 1\), with \( \ell = (l_0, l_1, l_2) \in \mathbb{R}^3 \), \( 2 \, m = 1 \) and \( h = 1 \). The differential operators

\[
J_0 = s_1 \partial_2 + s_2 \partial_1, \quad J_1 = s_2 \partial_0 + s_0 \partial_2, \quad J_2 = s_0 \partial_1 - s_1 \partial_0,
\]

determine a differential realization of \( so(2,1) \) with Lie commutators

\[
[J_0, J_1] = -J_2, \quad [J_2, J_0] = J_1, \quad [J_1, J_2] = J_0.
\]

In the coordinates (5) the expressions of the \( J_i \) are

\[
J_0 = \sin \theta \partial_\xi + \cos \theta \coth \xi \partial_\theta, \quad J_1 = \cos \theta \partial_\xi - \sin \theta \coth \xi \partial_\theta, \quad J_2 = \partial_\theta,
\]

and constitute a set of anti-Hermitian operators acting on the space of square-integrable functions with invariant measure \( d\mu(\theta, \xi) = \sinh \xi \, d\theta \, d\xi \).

Now the Hamiltonian \( H_\ell \) (4) has the expression

\[
H_\ell = -\partial_\xi^2 - \coth \xi \partial_\xi - \frac{l_0^2 - \frac{1}{2}}{\cosh^2 \xi} + \frac{1}{\sinh^2 \xi} \left[ -\partial_\theta^2 + \frac{l_0^2 - \frac{1}{2}}{\sin^2 \theta} + \frac{l_0^2 - \frac{1}{2}}{\cos^2 \theta} \right].
\]

We see that it can be separated in the variables \( \xi \) and \( \theta \), by choosing its eigenfunctions \( \Phi_\ell \) as \( \Phi_\ell(\theta, \xi) = f(\theta) \, g(\xi) \). Thus, we get a pair of equations

\[
H_{0, l_1}^0 f(\theta) = \left[ -\partial_\theta^2 + \frac{l_1^2 - \frac{1}{2}}{\sin^2 \theta} + \frac{l_0^2 - \frac{1}{2}}{\cos^2 \theta} \right] f(\theta) = \alpha f(\theta),
\]
\[
\left[ -\partial_\xi^2 - \coth \xi \partial_\xi - \frac{l_0^2 - \frac{1}{2}}{\cosh^2 \xi} + \frac{\alpha}{\sinh^2 \xi} \right] g(\xi) = E g(\xi),
\]

with \( \alpha > 0 \) a separation constant.

5. Algebraic approach to superintegrable quantum systems

The algebraic analysis of this quantum system involves the construction of a complete set of intertwining operators that determine algebraic structures (Lie algebras) that are closely related with the symmetry of the system.
5.1. Intertwining symmetry algebras

We can consider the differential operator associated to the first equation (6) as the one-dimensional Hamiltonian $H^0_{l_0,l_1}$. It can be factorized as a product of first order operators $A^\pm$ and a constant $\lambda$

$$H^0_{l_0,l_1} = A^+_{l_0,l_1} A^-_{l_0,l_1} + \lambda_{l_0,l_1},$$

$$A^\pm_{l_0,l_1} = \pm \partial_\theta - (l_0 + 1/2) \tan \theta + (l_1 + 1/2) \cot \theta,$$

$$\lambda_{l_0,l_1} = (l_0 + l_1 + 1)^2.$$

A hierarchy of Hamiltonians

$$\ldots, H^0_{l_0-1,l_1-1}, H^0_{l_0,l_1}, H^0_{l_0+1,l_1+1}, \ldots, H^0_{l_0+n,l_1+n}, \ldots,$$

satisfying the recurrence relations

$$A^-_{l_0,l_1} H^0_{l_0,l_1} = H^0_{l_0+1,l_1+1} A^-_{l_0,l_1}, \quad A^+_{l_0,l_1} H^0_{l_0+1,l_1+1} = H^0_{l_0,l_1} A^+_{l_0,l_1}$$

(7) is constructed using the fundamental relation between contiguous pairs of operators $A^\pm$

$$H^0_{l_0,l_1} = A^+_{l_0,l_1} A^-_{l_0,l_1} + \lambda_{l_0,l_1} = A^-_{l_0+1,l_1+1} A^+_{l_0+1,l_1+1} + \lambda_{l_0+1,l_1+1}.$$

Relations (7) show that $A^\pm_{l_0,l_1}$ act as shape invariant intertwining operators and that $A^-_{l_0,l_1}$ transforms eigenfunctions of $H^0_{l_0,l_1}$ on eigenfunctions of $H^0_{l_0+1,l_1+1}$ and viceversa for $A^+_{l_0,l_1}$, such that the original and the transformed eigenfunctions have the same eigenvalue.

Free-index operators $H^\theta, \hat{A}^\pm, \hat{A}$ are defined from the set of index-depending operators $\{H^\theta_{l_0+n,l_1+n}, A^\pm_{l_0+n,l_1+n}\}_{n \in \mathbb{Z}}$ as follows:

$$\hat{H}^\theta f_{l_0,l_1} := H^\theta_{l_0,l_1} f_{l_0,l_1},$$

$$\hat{A}^- f_{l_0,l_1} := \frac{1}{2} A^-_{l_0,l_1} f_{l_0,l_1},$$

$$\hat{A}^+ f_{l_0+1,l_1+1} := \frac{1}{2} A^+_{l_0,l_1} f_{l_0+1,l_1+1},$$

$$\hat{A} f_{l_0,l_1} := -\frac{1}{2} (l_0 + l_1) f_{l_0,l_1},$$

where $f_{l_0+n,l_1+n}$ are the eigenfunctions of $H^\theta_{l_0+n,l_1+n}$. These free-index operators close a $su(2)$-algebra

$$[\hat{A}, \hat{A}^\pm] = \pm A^\pm, \quad [\hat{A}^+, \hat{A}^-] = 2 \hat{A},$$

(8) and together with the operator $D f_{l_0,l_1} := (l_0 - l_1) f_{l_0,l_1}$, that commutes with the other three ones, determine a $u(2)$-algebra.

The fundamental states of some of these Hamiltonians are related with the IUR’s of $su(2)$. The Hamiltonian $H^\theta$ can be written in terms of the Casimir of $su(2)$, $C = A^+ A^- + A(A - 1)$

$$H^\theta = 4(C + 1/4).$$

An eigenstate $f^0_{l_0+n,l_1+n}$ of $H^\theta_{l_0+n,l_1+n}$ will be a fundamental (highest or lowest weight) vector if it verifies the equation

$$A^- f^0_{l_0+n,l_1+n} \equiv A^-_{l_0+n,l_1+n} f^0_{l_0+n,l_1+n} = 0.$$
Its solution is
\[ f^0_{(l_0+n,l_1+n)}(\theta) = N \cos^{l_0+1/2+n}(\theta) \sin^{l_1+1/2+n}(\theta), \]

(being \(N\) a normalization constant) with eigenvalue
\[ E^0_{(l_0+n,l_1+n)} = \lambda_{(l_0+n,l_1+n)} = (l_0 + l_1 + 1 + 2n)^2. \]

This solution is regular and square-integrable if \(l_0, l_1 \geq -1/2\). Moreover, it is also eigenfunction of the free-index operator \(A\)
\[ A f^0_{(l_0+n,l_1+n)} \equiv A_{(l_0+n,l_1+n)} f^0_{(l_0+n,l_1+n)} = -\frac{1}{2} (l_0 + l_1 + 2n) f^0_{(l_0+n,l_1+n)}. \]  

From (9) we can made the following identification
\[ f^0_{(l_0+n,l_1+n)} \simeq |j_n, -j_n\rangle, \]  

with \(j_n = \frac{1}{2} (l_0 + l_1 + 2n), n = 0, 1, 2, \ldots\) Hence, the representation, \(D^{l_0}\), fixed by (10), will be a IUR of \(su(2)\) of dimension \(2j_n + 1 = l_0 + l_1 + 2n + 1\) if \(l_0 + l_1 \in \mathbb{Z}^+\), \(n \in \mathbb{Z}^+\). The other eigenstates belonging to \(D^{l_0}\) are obtained by recursion with \(A^+\). Thus,
\[ f^n_{(l_0,l_1)} = (A^+)^n f^0_{(l_0+n,l_1+n)} = A^+_{(l_0,l_1)} A^+_{(l_0+1,l_1+1)} \cdots A^+_{(l_0+n-1,l_1+n-1)} f^0_{(l_0+n,l_1+n)}, \]

and
\[ f^n_{(l_0,l_1)} \simeq |j_n, -j_n + n\rangle. \]

The explicit form of \(f^n_{(l_0,l_1)}\) is
\[ f^n_{(l_0,l_1)} = \sin^{l_1+1/2}(\theta) \cos^{l_0+1/2}(\phi_1) P^m_{(l_1)}[\cos(2 \theta)], \]

being \(P_m\) a Jacobi polynomial, with eigenvalue
\[ E^n_{(l_0,l_1)} = (l_0 + l_1 + 1 + 2n)^2, \ n \in \mathbb{Z}^+. \]

However, different fundamental states with values of \(l_0\) and \(l_1\), such that \(j_0 = (l_0 + l_1)/2\) is fixed, would lead to the same \(j\)-IUR of \(su(2)\), and different \(u(2)\)-IUR’s (since \(Df_{(l_0,l_1)} = (l_0 - l_1)f_{(l_0,l_1)}\)) may correspond to states with the same energy. This fact suggests the existence of a larger algebra of operators such that all the eigenstates with the same energy belong to only one of its IUR’s.

Another interesting fact is that although \(A^\pm_{(l_0,l_1)}\) depend only on the \(\theta\)-variable, they can act also as IO’s of the complete Hamiltonians \(H_\ell\) (4) and its global eigenfunctions \(\Phi_\ell\), leaving the parameter \(l_2\) unchanged
\[ A^-_{\ell'} H_{\ell'} = H_{\ell'} A^-_{\ell'}, \quad A^+_{\ell'} H_{\ell'} = H_{\ell'} A^+_{\ell'}, \]
\[ \text{where } \ell = (l_0, l_1, l_2) \text{ and } \ell' = (l_0 - 1, l_1 - 1, l_2). \] In this sense, many of the above relations can be straightforwardly extended under this global point of view.
5.2. A larger symmetry algebra: \( su(2, 1) \)

The construction of a larger algebra starts with the recognition of other coordinate sets that allow us to obtain new sets of IO's whose associated free-index operators close new \( su(2) \) or \( su(1, 1) \) Lie algebras.

A second coordinate set that parametrizes the hyperboloid and separates the Hamiltonian is

\[
s_0 = \cosh \psi \sinh \chi, \quad s_1 = \sinh \psi, \quad s_2 = \cosh \psi \cosh \chi,
\]

with \( -\infty < \psi < +\infty \) and \( 0 \leq \chi < +\infty \). In these coordinates the \( so(2, 1) \)-generators take the expressions

\[
J_0 = -\tanh \psi \sinh \chi \partial_\chi + \cosh \chi \partial_\psi, \quad J_1 = \partial_\chi, \quad J_2 = \sinh \chi \partial_\psi - \tanh \psi \cosh \chi \partial_\chi,
\]

and the Hamiltonian

\[
H_\ell = -\partial_\psi^2 - \tanh \psi \partial_\psi + \frac{\ell_1^2 - \frac{1}{4}}{\sinh^2 \psi} + \frac{1}{\cosh^2 \psi} \left[ -\partial_\chi^2 + \frac{\ell_0^2 - \frac{1}{4}}{\sinh^2 \chi} - \frac{\ell_2^2 - \frac{1}{4}}{\cosh^2 \chi} \right]
\]

separates in the variables \( \psi \) and \( \chi \) when the eigenfunctions \( H_\ell \) are like \( \Phi(\chi, \psi) = f(\chi) g(\psi) \). So,

\[
H_{l_0, l_2}^\chi f(\chi) = \left[ -\partial_\chi^2 + \frac{\ell_0^2 - \frac{1}{4}}{\sinh^2 \chi} - \frac{\ell_2^2 - \frac{1}{4}}{\cosh^2 \chi} \right] f(\chi) = \alpha f(\chi), \tag{11}
\]

\[
\left[ -\partial_\psi^2 - \tanh \psi \partial_\psi + \frac{\ell_1^2 - \frac{1}{4}}{\sinh^2 \psi} + \frac{\alpha}{\cosh^2 \psi} \right] g(\psi) = E g(\psi),
\]

with \( \alpha \) a separation constant.

The Hamiltonian \( H_{l_0, l_2}^\chi \) (11) is factorized as a product of first order operators \( B^\pm \)

\[
H_{l_0, l_2}^\chi = B_{l_0, l_2}^+ B_{l_0, l_2}^- + \lambda_{l_0, l_2} = B_{l_0-1, l_2-1}^+ B_{l_0-1, l_2-1}^- + \lambda_{l_0-1, l_2-1},
\]

such that

\[
B_{l_0, l_2}^\pm = \pm \partial_\chi + (l_2 + 1/2) \tanh \chi + (l_0 + 1/2) \coth \chi,
\]

\[
\lambda_{l_0, l_2} = -(1 + l_0 + l_2)^2.
\]

The intertwining relations are now

\[
B_{l_0-1, l_2-1}^- H_{l_0, l_2}^\chi B_{l_0-1, l_2-1}^- = H_{l_0, l_2}^\chi, \quad B_{l_0-1, l_2-1}^+ H_{l_0, l_2}^\chi = H_{l_0-1, l_2-1}^\chi B_{l_0-1, l_2-1}^+.
\]

Hence, the operators \( B^\pm \) connect eigenfunctions of \( H_{l_0, l_2}^\chi \) as follows

\[
B_{l_0-1, l_2-1}^- : f_{l_0-1, l_2-1} \rightarrow f_{l_0, l_2}, \quad B_{l_0-1, l_2-1}^+ : f_{l_0, l_2} \rightarrow f_{l_0-1, l_2-1}.
\]

It is worthy noticing that \( B_{l_0, l_2}^\pm \) can be also expressed in terms of the original coordinates \((\xi, \theta)\)

\[
B_{l_0, l_2}^\pm = \pm (\cos \theta \partial_\xi - \sin \theta \cosh \xi \partial_\theta) + (l_2 + 1/2) \tanh \xi \cos \theta + (l_0 + 1/2) \coth \xi \sec \theta.
\]

In this case the free-index operators

\[
\tilde{B}^- f_{l_0, l_2} := \frac{1}{2} B_{l_0, l_2}^- f_{l_0, l_2}, \quad \tilde{B}^+ f_{l_0, l_2} := \frac{1}{2} B_{l_0, l_2}^+ f_{l_0, l_2}, \quad \tilde{B} f_{l_0, l_2} := -\frac{1}{2} (l_0 + l_2) f_{l_0, l_2},
\]
close a \( su(1, 1) \) Lie algebra
\[
[\hat{B}^+, \hat{B}^-] = -2 \hat{B}, \quad [\hat{B}, \hat{B}^\pm] = \pm \hat{B}^\pm.
\] (12)

Like in the previous case we will relate the (infinite dimensional) IUR’s of \( su(1, 1) \) with the states of some Hamiltonians of the hierarchy. The appropriate IUR’s of \( su(1, 1) \) are the discrete series having a fundamental state annihilated by the lowering operator
\[
B^- f^0_{l_0, l_2} = 0.
\]

Hence
\[
f^0_{l_0, l_2}(\lambda) = N (\cosh \lambda)^{l_2+1/2} (\sinh \lambda)^{l_0+1/2},
\]
with \( N \) is a normalization constant. Imposing
\[
l_0 \geq -1/2, \quad -k_1 \equiv l_0 + l_2 < -1.
\]
we get a regular and square-integrable function \( f^0_{l_0, l_2}(\chi) \). The lowest weight of the IUR is characterized by
\[
j'_1 = k_1/2 > 1/2.
\]

because \( \hat{B} f^0_{l_0, l_2} = -\frac{1}{2} (l_0 + l_2) f^0_{l_0, l_2} \).

On the other hand, the IO’s \( \hat{B}^\pm \) can also be considered as intertwining operators of the Hamiltonians \( H \) linking their eigenfunctions \( \Phi \), similarly to the \( \hat{A}^\pm \), described before, but now with \( l_1 \) remaining unchanged.

A third set of coordinates,
\[
s_0 = \sinh \phi, \quad s_1 = \cosh \phi \sinh \beta, \quad s_2 = \cosh \phi \cosh \beta,
\]
with \( 0 \leq \psi < +\infty \) and \( -\infty < \beta < +\infty \), allows us to obtain a new \( su(1, 1) \) algebra following the same procedure like in the previous cases. The expression of the Hamiltonian is now
\[
H_\ell = -\partial_\phi^2 - \tanh \phi \partial_\phi + \frac{l_0^2 - \frac{1}{4}}{\sinh^2 \phi} + \frac{1}{\cosh^2 \phi} \left[ -\partial_\beta^2 + \frac{l_0^2 - \frac{1}{4}}{\sinh^2 \beta} - \frac{l_0^2 - \frac{1}{4}}{\cosh^2 \beta} \right].
\]

It separates in the variables \( \phi, \beta \) in terms of its eigenfunctions \( \Phi(\gamma, \phi) = f(\beta) g(\phi) \) as
\[
H_{1_1, 1_2}^\beta f(\beta) \equiv \left[ -\partial_\beta^2 + \frac{l_0^2 - \frac{1}{4}}{\sinh^2 \beta} - \frac{l_0^2 - \frac{1}{4}}{\cosh^2 \beta} \right] f(\beta) = \alpha f(\beta),
\]
\[
\left[ -\partial_\phi^2 - \tanh \phi \partial_\phi + \frac{l_0^2 - \frac{1}{4}}{\sinh^2 \phi} + \frac{\alpha}{\cosh^2 \phi} \right] g(\phi) = E g(\phi),
\]
with the separation constant \( \alpha \).

The second order operator \( H_{1_1, 1_2}^\beta \) factorizes as
\[
H_{1_1, 1_2}^\beta = C_{1_1, 1_2}^1 C_{1_1, 1_2}^{-1} + \lambda_{1_1, 1_2} = C_{1_1+1, 1_2-1}^+ C_{1_1+1, 1_2-1}^- + \lambda_{1_1+1, 1_2-1},
\]
where
\[
C_{1_1, 1_2}^{\pm} = \pm \partial_\beta + (l_2 + 1/2) \tanh \beta + (-l_1 + 1/2) \coth \beta,
\]
\[
\lambda_{1_1, 1_2} = -(1 - l_1 + l_2)^2.
\]
The operators $C_{l_1,l_2}^{\pm}$ verify

$$C_{l_1+1,l_2}^{+} H_{l_1,l_2}^{\beta} = H_{l_1+1,l_2}^{\beta} C_{l_1+1,l_2}^{+}, \quad C_{l_1+1,l_2}^{-} H_{l_1,l_2}^{\beta} = H_{l_1+1,l_2}^{\beta} C_{l_1+1,l_2}^{-}.$$ 

and they can also be rewritten in terms of the first set of coordinates $(\xi, \theta)$

$$C_{l_1,l_2}^{\pm} = \pm (\sin \theta \, \partial_\xi + \cos \theta \, \coth \xi \, \partial_\theta) + (l_2 + 1/2) \, \tanh \xi \, \sin \theta + (-l_1 + 1/2) \, \coth \xi \, \csc \theta.$$ 

Free-index operators are defined by

$$\hat{C}^- f_{l_1,l_2} := \frac{1}{2} C_{l_1,l_2}^- f_{l_1,l_2}, \quad \hat{C}^+ f_{l_1,l_2} := \frac{1}{2} C_{l_1,l_2}^+ f_{l_1,l_2}, \quad \hat{C} f_{l_1,l_2} := -\frac{1}{2} (l_2 - l_1) f_{l_1,l_2}.$$ 

They satisfy the commutation relations of the $su(1,1)$ algebra

$$[\hat{C}^-, \hat{C}^+] = 2 \hat{C}, \quad [\hat{C}, \hat{C}^\pm] = \pm \hat{C}^\pm.$$ 

The equation $\hat{C}^- f_{l_1,l_2}^0 = 0$ gives the fundamental state for a representation of $su(1,1)$

$$f_{l_1,l_2}^0(\beta) = N (\cosh \beta)^{l_2+1/2} (\sinh \beta)^{-l_1+1/2},$$ 

with $N$ a normalization constant. If besides

$$l_1 \leq 1/2, \quad -k_2 \equiv l_2 - l_1 < -1$$

we obtain a (discrete series) IUR labeled by

$$j_2^2 = k_2/2 > 1/2.$$

Also in this case the $C^\pm$ can be considered as IO’s connecting global Hamiltonians $H_\ell$ and their eigenfunctions, having in mind that now the parameter $l_0$ is unaltered.

The set of IO’s $\{\hat{A}^\pm, \hat{A}, \hat{B}^\pm, \hat{B}, \hat{C}^\pm, \hat{C}\}$ obtained before generates a $su(2,1)$ Lie algebra with Lie commutators displayed in (8), (12) and (13) together with the crossed commutators that the interested reader can compute himself using, for instance, the differential expression of these operators in the coordinates $(\xi, \theta)$. There is a constraint in the differential realization:

$$\hat{A} - \hat{B} + \hat{C} = 0.$$ 

The second order Casimir operator of $su(2,1)$ is

$$\mathcal{C} = \hat{A}^+ \hat{A}^- - \hat{B}^+ \hat{B}^- - \hat{C}^+ \hat{C}^- + \frac{2}{3} \left( \hat{A}^2 + \hat{B}^2 + \hat{C}^2 \right) - (\hat{A} + \hat{B} + \hat{C}).$$ 

The generator,

$$\mathcal{C} = l_1 + l_2 - l_0,$$ 

commutes with the generators of $su(2,1)$ and we have

$$\langle \hat{A}^\pm, \hat{A}, \hat{B}^\pm, \hat{B}, \hat{C}^\pm, \hat{C} \rangle \oplus \langle \mathcal{C} \rangle \approx u(2,1).$$

The Hamiltonian (4) is a linear combination of of $\mathcal{C}$ and $\mathcal{C}^\prime$ up an additive constant

$$H_\ell = -4 \mathcal{C} + \frac{5}{3} \mathcal{C}^2 - \frac{15}{4}$$

$$= -4 \left( \hat{A}^+ \hat{A}^- - \hat{B}^+ \hat{B}^- - \hat{C}^+ \hat{C}^- + \frac{2}{3} \left( \hat{A}^2 + \hat{B}^2 + \hat{C}^2 \right) - (\hat{A} + \hat{B} + \hat{C}) \right) + \frac{1}{3} \mathcal{C}^2 - \frac{15}{4}.$$ 

$$= -4 \hat{A}^+ \hat{A}^- + 4 \hat{B}^+ \hat{B}^- + 4 \hat{C}^+ \hat{C}^- + \text{constant}.$$
The quadratic operators $\hat{A}^+\hat{A}^-$, $\hat{B}^+\hat{B}^-$ and $\hat{C}^+\hat{C}^-$ are constants of motion since they commute with the Hamiltonian, but they do not commute among themselves giving cubic expressions

$$[\hat{A}^+\hat{A}^-, \hat{B}^+\hat{B}^-] = [-\hat{A}^+\hat{C}^-, \hat{B}^+\hat{C}^-] = \hat{A}^+\hat{C}^+\hat{B}^- - \hat{B}^+\hat{C}^-\hat{A}^-.$$ 

Obviously, we can implement a quadratic algebra by means of the generators

$$X_1 := \hat{A}^+\hat{A}^-, \quad X_2 := \hat{B}^+\hat{B}^-, \quad X_3 := \hat{C}^+\hat{C}^-, \quad Y_1 := \hat{A}^+\hat{C}^+\hat{B}^-, \quad Y_2 := \hat{B}^+\hat{C}^-\hat{A}^-.$$ 

The eigenfunctions of the Hamiltonians $H_\ell$, that have the same energy, support an IUR of $su(2, 1)$ characterized by a value of $\ell$ and other of $\ell'$. The fundamental state has to be simultaneously annihilated by $\hat{A}^-, \hat{C}^-$ and $\hat{B}^-$ ($A^-\Phi^0_\ell = C^-\Phi^0_\ell = B^-\Phi^0_\ell = 0$). So,

$$\Phi^0_\ell(\xi, \theta) = N(cos \theta)^{l_0+1/2}(sin \theta)^{l_2+1/2}(cosh \xi)^{l_0+1},$$

where $\ell = (l_0, 0, l_2)$ and $N$ is a normalization constant. The parameters $l_0$ and $l_2$ have to verify the following conditions

$$(l_0 + l_2) < -3/2, \quad l_0 \geq -1/2, \quad (l_0 + l_2) < -5/2.$$ 

This state $\Phi^0_\ell$ also supports IUR’s of the subalgebras $su(2)$ ($j = l_0/2$) and $su(1, 1)$ (generated by $\hat{C}^\pm$ with $j'_2 = -l_2/2$). The energy of $\Phi^0_\ell(\xi, \theta)$ is

$$H_\ell\Phi^0_\ell = -(l_0 + l_2 + 3/2)(l_0 + l_2 + 5/2)\Phi^0_\ell \equiv E^0_\ell\Phi^0_\ell.$$ 

The other states of the $su(2, 1)$ representation can be obtained applying the raising operators $\hat{A}^+, \hat{B}^+, \hat{C}^+$ over $\Phi^0_\ell$. Obviously, all of them share the same energy eigenvalue $E^0_\ell$. All the states in the family of IUR’s derived from fundamental states $\Phi^0_\ell(\xi, \theta)$, with the same value of $l_0 + l_2$, shall have the same energy. The energy of bound states is negative and the set of such bound states for each Hamiltonian $H_\ell$ is finite.

**Figure 2.** States of IUR’s of $su(2, 1)$ with the same energy and represented by points in the three dark planes associated to the ground state $\Phi^0_\ell$ with $\ell = (0, 0, -3)$, $\ell = (1, 0, -4)$ and $\ell = (2, 0, -5)$.
As we see in Fig. 2 the states of the IUR’s of $su(2,1)$ are represented by points in parallel 2-dimensional planes placed inside a tetrahedral unbounded pyramid whose basis extends towards $-\infty$ along the axis $l_2$.

There are some points (in the parameter space $(l_0, l_1, l_2)$) which are degenerated because they correspond to an eigenspace with dimension bigger than 1. For instance, the states of the IUR whose fundamental state $\Psi^0_\ell$ ($\ell = (0,0,-3)$) lie in a triangle and are nondegenerated.

The $su$ associated to the ground state with $\ell' = (1,0,-4)$ has states with the same energy, $E = -(-3+3/2)(-3+5/2)$, as the previous one, since in both cases $l_0 + l_2 = -3$. The states characterized by to $\ell'' = (0,0,-5)$, inside this IUR, are:

$$\Phi^2_{(0,0,-5)} = \tilde{C}^+ \tilde{A}^+ \Phi^0_{(0,0,-3)}, \quad \Phi^2_{(0,0,-5)} = \tilde{A}^+ \tilde{C}^+ \Phi^0_{(0,0,-3)}.$$ 

Since $[A^+, C^+] = B^+$ both states are independent and span a 2-dimensional eigenspace of the Hamiltonian $H_{(0,0,-5)}$ with eigenvalue $E = -(-3+3/2)(-3+5/2)$. The ground state for $H_{(0,0,-5)}$ is the wavefunction $\Phi^0_{(0,0,-5)}$ with energy $E^0_{(0,0,-5)} = -(-5 + 3/2)(-5 + 5/2)$.

### 5.3. The complete symmetry algebra $so(4,2)$

It is trivial to see the Hamiltonian $H_\ell$ (4) is invariant under reflections in the space of parameters $(l_0, l_1, l_2)$

$$I_0 : (l_0, l_1, l_2) \rightarrow (-l_0, l_1, l_2), \quad I_1 : (l_0, l_1, l_2) \rightarrow (l_0, -l_1, l_2), \quad I_2 : (l_0, l_1, l_2) \rightarrow (l_0, l_1, -l_2),$$

however, the implications of this property are very interesting as we show now.

These reflections generate by conjugation other sets of intertwining operators from the ones already defined. Thus, for instance,

$$I_0 : \{ \tilde{A}_\pm, \tilde{A} \} \rightarrow \{ \tilde{A}_\pm = I_0 \tilde{A}_\pm I_0, \tilde{A} = I_0 \tilde{A} I_0 \},$$

where

$$\tilde{A}_{\ell_0,\ell_1} = \pm \partial_\theta - (-l_0 + 1/2) \tan \theta + (l_1 + 1/2) \cot \theta, \quad \tilde{\lambda}_{\ell_0,\ell_1} = (1 - l_0 + l_1)^2.$$ 

They act on the eigenfunctions of the Hamiltonians (6)

$$\tilde{A}_{\ell_0,\ell_1}^\pm : F_{\ell_0,\ell_1} \rightarrow F_{\ell_0,-\ell_1+1}, \quad \tilde{A}_{\ell_0,\ell_1}^\pm : F_{\ell_0,-\ell_1+1} \rightarrow F_{\ell_0,\ell_1}.$$ 

We can consider global operators $\tilde{A}_\pm$ as before. Then, $\tilde{A}_\pm$ together with $\tilde{A}$ ($\tilde{A} f_{\ell_0,\ell_1} := -\frac{1}{2} (-l_0 + l_1) f_{\ell_0,\ell_1}$) close a new $\tilde{su}(2)$.

In a similar procedure new sets of operators $\{ \tilde{B}_\pm, \tilde{B} \} \{ \tilde{C}_\pm, \tilde{C} \}$ closing $\tilde{su}(1,1)$ algebras, can also be constructed under the action of these reflections. The whole set of the operators obtained in this way $\{ A^\pm, \tilde{A}_\pm, B^\pm, \tilde{B}_\pm, C^\pm, \tilde{C}_\pm \}$ together with the set of diagonal operators $\{ L_0, L_1, L_2 \}$, defined by

$$L_\ell \Psi_\ell = l_\ell \Psi_\ell,$$

span a Lie algebra of rank three: $o(4,2)$. All these generators link eigenstates of Hamiltonians $H_\ell$ with the same eigenvalue.

The fundamental state $\Psi^0_\ell$ for $so(4,2)$ will be characterized by

$$A^- \Psi^0_\ell = \tilde{A}^- \Psi^0_\ell = C^- \Psi^0_\ell = \tilde{C}^- \Psi^0_\ell = B^- \Psi^0_\ell = \tilde{B}^- \Psi^0_\ell = 0.$$ 

Thus,

$$\Phi^0_{(l_0=0, l_1=0, l_2)}(\xi, \theta) = N(\cos \theta)^{1/2}(\sin \theta)^{1/2}(\cosh \xi)^{1/2+1/2} \sinh \xi,$$

where $l_2 < -5/2$. In Fig. 2 the point $(l_0 = 0, l_1 = 0, l_2)$ for the cases plotted corresponds to the top vertex of the pyramid $(0,0,-3)$. All the other points can be obtained from this point. All these points correspond to an IUR of $so(4,2)$ that includes IUR’s of $su(2,1)$. 

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Figure 3. Two pyramids associated to the same IUR of \( so(4, 2) \). The points of the faces of the exterior pyramid (with vertex \((0, 0, -3)\)) represent non-degenerated levels. The exterior faces of the inner pyramid (vertex \((0, 0, -5)\)) are first excited double-degenerated levels.

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