On singularities, mixing and non-Markovianity of Pauli dynamical maps

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Quantum non-Markovianity (even of the eternal or quasi-eternal kind) can be produced by mixing Markovian channels, as observed recently by various authors. This evokes the question of whether a dynamical map with a singular generator can be produced by mixing quantum channels whose generators lack a singularity. Here we answer the question in the negative in the context of Pauli channels. Interestingly, mixing channels with singular generators can lead to the elimination of singularities in the generator of the resultant map. These results place non-trivial restrictions on the experimental realization of quantum non-Markovianity with singularities.

I. Introduction

Open quantum systems assert that the evolution of a system is effected by its ambient environment [1]. The dynamics can be roughly categorized into Markovian [2, 3] and non-Markovian [4–8], depending upon whether the system dynamics satisfies certain memory criteria or whether the system-environment interaction is such that a clear separation between system and environment time scales exists or not. A plethora of work in quantum optics have made use of the Markovian evolution, epitomized by the Lindblad equation [9–11]. Over the last decade a lot of efforts have been made towards the understanding of non-Markovian evolution. This has had ramifications into various applications such as those in quantum thermodynamics [12–14], quantum cryptography [15, 16], quantum walks [17, 18], quantum correlations and coherence [19–21], among others (cf. [22]).

An important class of open system dynamics is that known as unital evolution, wherein there is decoherence without dissipation [23–26]. An important subclass of these channels is the non-Markovian dephasing and depolarizing channels obtained by non-Markovian versions of familiar Markovian maps, in specific, qubit Pauli channels [27] and is characterized by singularities in the generators of the channel.

It is interesting to note that phenomenological models like a two level system interacting with a bosonic dissipative reservoir [28], without Born-Markov approximation, may have a time-local generator that contains infinite singularities. Another example would be a random telegraph noise [29], found mostly in solid state devices, modeled by a time-local generator with infinite number of singularities. However, there are well-known dynamical noisy processes with non-singular generators that are CP-divisible [6], and indeed that are CP-divisible [7, 30].

The effect of mixing channels has been studied by various authors. Refs. [31, 32] show that a convex combination of semigroup dynamical maps can lead to a deviation from the semi-group structure. Recently various authors have shown that it is possible to obtain a non-Markovian (in the sense of CP-indivisible) Pauli dynamical map by mixing Markovian (in the sense of CP-divisible) Pauli channels [33–35]. This raises the question of whether channels with singular generators can also be produced by mixing channels with regular (i.e., lacking singularities) generators, and more generally what constraints can be imposed on the mixing Pauli channels (with or without a singular generator) to produce a resultant general Pauli dynamical map with a singular generator. In this work, we investigate this question in the context of mixing Pauli channels.

This work is organized as follows. In Section II, we show that it is not possible to produce singularities in the generator by mixing differentiable CP-divisible channels. In Sections III and IV, we discuss the results pertaining to mixing channels of two broad and physically motivated catagories. In all cases, the results are illustrated with examples. Finally, we conclude in Section V.

II. Mixing of CP-divisible Pauli channels

A general Pauli dynamical map is given by

\[
\mathcal{L}_t \rho = \sum_{i=0}^{3} k_i(t) \sigma_i \rho \sigma_i^\dagger,
\]

(1)

where \(\sigma_0 = I\), and \(\sigma_i, i \in \{1, 2, 3\}\) are Pauli X, Y, Z operators respectively, and \(\sum_{i=0}^{3} k_i(t) = 1\). The canonical form of master equation corresponding to the map (1) has the form

\[
\dot{\rho} = \mathcal{L}_t \rho = \sum_{j=1}^{3} \gamma_j(t) (\sigma_j \rho \sigma_j^\dagger - \rho)
\]

(2)

where \(\gamma_j(t)\) are the rates, which must be positive for a CP-divisible channel. The generator

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may feature singularities, which correspond to momentary non-invertibility of the map $E_i$ [27]. In the following, by “non-singular” we mean a channel whose generator lacks any singularities.

**Lemma 1** It is impossible to produce a general Pauli dynamical map with a singular generator by mixing only Pauli channels with regular generators.

**Proof.** In order to obtain a general dynamical Pauli map by mixing Pauli channels, let us denote $E_i(\rho) \equiv (1-p(t))\rho(0) + p(t)\sigma_1\rho\sigma_1$, $E_2(\rho) \equiv (1-q(t))\rho(0) + q(t)\sigma_2\rho\sigma_2$, and $E_3(\rho) \equiv (1-r(t))\rho(0) + r(t)\sigma_3\rho\sigma_3$. The corresponding individual Lindblad rates are

$$\gamma_x = \frac{-\dot{p}}{1 - 2p}, \quad (3)$$

etc. Let the three be mixed with proportion $a, b$ and $c$, where $0 \leq a, b, c \leq 1$ and $a + b + c = 1$. This gives rise to the Pauli dynamical map:

$$\mathcal{E}^*(\rho) = aE_1(\rho) + bE_2(\rho) + cE_3(\rho)$$

$$= (1 - ap - bq - cr)\rho + ap\sigma_1\rho\sigma_1 + bq\sigma_2\rho\sigma_2 + cr\sigma_3\rho\sigma_3 \quad (4)$$

for which we obtain the following rates:

$$\gamma_1(t) = \frac{1}{4}\left(\frac{\dot{\lambda}_1(t)}{\lambda_1(t)} - \frac{\dot{\lambda}_2(t)}{\lambda_2(t)} - \frac{\dot{\lambda}_3(t)}{\lambda_3(t)}\right),$$

$$\gamma_2(t) = \frac{1}{4}\left(\frac{\dot{\lambda}_2(t)}{\lambda_2(t)} - \frac{\dot{\lambda}_1(t)}{\lambda_1(t)} - \frac{\dot{\lambda}_3(t)}{\lambda_3(t)}\right),$$

$$\gamma_3(t) = \frac{1}{4}\left(\frac{\dot{\lambda}_3(t)}{\lambda_3(t)} - \frac{\dot{\lambda}_1(t)}{\lambda_1(t)} - \frac{\dot{\lambda}_2(t)}{\lambda_2(t)}\right), \quad (5)$$

where

$$\lambda_1(t) = 1 - 2(bq + cr), \quad (6a)$$

$$\lambda_2(t) = 1 - 2(ap + cr), \quad (6b)$$

$$\lambda_3(t) = 1 - 2(ap + bq), \quad (6c)$$

which are in fact eigenvalues of the map $\mathcal{E}^*$ corresponding to its eigenoperators given by the Pauli operators $\sigma_j$ [37].

The condition for a singularity is that one or more of $\lambda_j$ in Eq. (6) should vanish at a certain finite time(s) $t_s$. For example, consider $\lambda_1$ in Eq. (6a). In view of Eq. (3), if the mixing maps $E_i$ corresponds to a singular generator, this implies that $q$ and $r$ are restricted to take values within the range $[0, \frac{1}{2})$ for finite time $t$. Therefore, the smallest possible value that it can attain in finite time is $\lambda^\text{min}_1 > 1 - (b + c) \geq 0$. Repeating the argument for $\lambda_2$ and $\lambda_3$, we conclude that there can be no singularity in the mixed channel. 

It follows that to produce a singularity in the generator of the mixed map, the generators of the mixing Pauli channels must possess a singularity. It is convenient to distinguish two types of mixing channels characterized by singular generators.

**Channels of type I:** Those in which the decoherence fraction $p(t)$ of the Pauli operator is bounded from above by $\frac{1}{2}$. In this case, $p(t)$ is necessarily non-monotonic. The singularity corresponds to the point where $p(t) = \frac{1}{2}$.

**Channels of type II:** Those in which the decoherence function $p(t)$ of the Pauli channel exceeds $\frac{1}{2}$. In this case, $p(t)$ can be monotonic. Again, the singularity corresponds to the point where $p(t) = \frac{1}{2}$.

In both cases, the occurrence of non-Markovianity can be attributed to the fact that the function $p(t)$ leads to an unmixing, by non-monotonical behavior and/or exceeding $\frac{1}{2}$.

We may note that Type I is a more usual occurrence, and can for example result when a qubit system and its qubit environment evolve according to a Hamiltonian given by $\omega(|01\rangle \langle 01| + |10\rangle \langle 10|)$ acting on the initial state $|01\rangle$. The joint system remains in the subspace spanned by $\{|01\rangle, |10\rangle\}$, and the reduced state of the system is $\cos^2(\omega t)|0\rangle \langle 0| + \sin^2(\omega t)|1\rangle \langle 1|$. 

### III. Mixing of Type I Pauli channels

In Eq. (6), for $\lambda_1$ to vanish at some finite time $t_c$, observe that this maximum for $q$ and $r$ should occur at the same time $t_c$, i.e., $q(t_c) = r(t_c) = \frac{1}{2}$. In other words, the generators of the mixing $\mathcal{E}_x$ and $\mathcal{E}_y$ channels must each possess a singularity and these singularities must occur simultaneously. Moreover, we require that the mixing parameter $a = 0$. Thus, only two of the three channels should be non-vanishing. A similar argument holds for $\lambda_2$ and $\lambda_3$. That is, the mixing should involve precisely two Pauli channels which are synchronized in the occurrence of their generators’ singularity.

If the singularities of the generators of the two mixing type I CP-indivisible channels are not simultaneous, then the resulting combination will lack a singularity. The fact that the mixing of two channels whose generator has a singularity can produce a non-singular channel turns out to be another rather non-intuitive property of channel mixing, and could be compared to the situation that mixing non-Markovian channels can produce Markovianity [33, 34].

**Example 1:** As an example, consider the instance of mixing channels $\mathcal{E}_x$ and $\mathcal{E}_y$, with $p(t) = \frac{1}{2}[1 - \cos^2(\mu t)]$ and $q(t) = \frac{1}{2}[1 - \cos^2(\nu t)]$, with $a, b > 0$ and $a + b = 1$ in Eq. (4). In view of Eq. (5), the eigenvalues $\lambda_i$ read

$$\lambda_1(t) = 1 - b \sin^2(\mu t),$$

$$\lambda_2(t) = 1 - a \sin^2(\nu t),$$

$$\lambda_3(t) = 1 - a \sin^2(\mu t) - b \sin^2(\nu t), \quad (7)$$

showing that there is a singularity only from the zeros of $\lambda_3(t)$, and furthermore this happens if and only if $\mu = \nu$. That is, if the two mixing Pauli channels are synchro-
For the present purpose, let all three type I Pauli channels are mixed, then the generator singularity occurring when \( \Lambda_t = \frac{3}{2} \), \( \sigma_x = \sigma_1, \sigma_y = \sigma_2, \sigma_z = \sigma_3 \) are Pauli operators. Here,

\[
\begin{align*}
    P_0 &= \frac{1}{4}[1 + \Lambda_1 + \Lambda_2 + \Lambda_3], \\
    P_1 &= \frac{1}{4}[1 + \Lambda_1 - \Lambda_2 - \Lambda_3], \\
    P_2 &= \frac{1}{4}[1 - \Lambda_1 + \Lambda_2 - \Lambda_3], \\
    P_3 &= \frac{1}{4}[1 - \Lambda_1 - \Lambda_2 + \Lambda_3],
\end{align*}
\]

(8)

where

\[
\Lambda_i = \exp(-rt) \left( \frac{\sin(rt\mu_i)}{\mu_i} + \cos(rt\mu_i) \right),
\]

(9)

The quantity \( r = \frac{1}{\sqrt{2\pi}} \) the spectral bandwidth while \( \tau \) is the rate of fluctuation of the environment affecting the qubit, and \( \mu_i = \sqrt{\frac{2a_i}{r}} \), with \( a_i \) representing the coupling strengths corresponding to \( i \)-th Pauli channel. For the present purpose, let all \( a_i \)’s in Eq. (9) be taken to be equal, given by \( a \). This corresponds to equal mixing of the \( X, Y \) and \( Z \) Pauli RTN channels, as a result of which we obtain an isotropic depolarizing RTN channel. For \( a \gg r \), the process is CP-indivisible and for \( a < r \) it is CP-divisible. It is important for our purpose to note that \( \Lambda \) is bounded in the range \((-1, +1]\).

Now consider individual RTN Pauli channels of type I being mixed with a view to reproducing the above RTN depolarizing channel, with their individual decoherence function given by

\[
\begin{align*}
    p(t) &= q(t) = r(t) = \frac{1 - \Lambda(t)}{2},
\end{align*}
\]

(10)

where \( \Lambda(t) \) is given by Eq. (9) with \( \mu_1 = \mu_2 = \mu_3 \). To conform to type I, the parameters \( a \) and \( r \) must be so chosen that \( \Lambda(t) \) is confined in the range \([0, 1]\), with the singularity occurring when \( \Lambda(t) = 0 \).

The result of mixing these channels produces the generalized Pauli dynamical map Eq. (4), such that the time-dependent eigenvalues in the equation \( \tilde{\mathcal{E}}_\tau^\ast[\sigma_i] = \lambda_i(t)\sigma_i \) read, in view of Eq. (6):

\[
\begin{align*}
    \lambda_1(t) &= 1 - (b + c)(1 - \Lambda), \\
    \lambda_2(t) &= 1 - (a + c)(1 - \Lambda), \\
    \lambda_3(t) &= 1 - (a + b)(1 - \Lambda).
\end{align*}
\]

(11)

If \( a, b, c > 0 \), then the sum of any two of them is strictly less than 1. It follows from Eq. (11) and the type I restriction that each \( \lambda_i \) in Eq. (11) never vanishes.

We now consider an analogous result when type II channels are mixed, and in this case, we will find that the above fine-tuning is not required.

IV. Mixing of type II Pauli channels

In Eq. (6), if we relax the requirement that \( p, q \) and \( r \) are bounded by \( \frac{1}{2} \), then neither the restriction that only two mixing Pauli channels should be involved to produce a singularity, nor that these two channels should be singularity-synchronized holds.

**Example 3:** In Example 2, suppose on the other hand that the mixing channels are allowed to be type II, then the above RTN depolarizing channel can indeed be obtained by their convex combination. For simplicity, let \( a = b = c = \frac{1}{4} \), in which case, the eigenvalues become

\[
\lambda_{1,2,3} = \frac{1}{3}(1 + 2\Lambda(t)),
\]

(12)

so that the singularity in the generator of the RTN depolarizing channel occurs when the choice of parameters \( a \) and \( r \) allow that \( \Lambda(t) = -\frac{a}{2} \), from Eq. (10), we find that this singularity in the respective generators of the mixing channels happens when \( p(t) = q(t) = r(t) = \frac{3}{4} \), which is permissible for mixing channels of type II.

As a final example, consider the three mixing channels to be of type II, having different functional forms, but all reaching an asymptotic value greater than (say) \( \frac{1}{4} \).

Now, at some time \( t_d \), let \( q(t_d) = \frac{3}{4} \) and \( r(t_d) = \frac{1}{4} \), and furthermore let the mixing fraction \( b = \frac{1}{6} \) and \( c = \frac{1}{2} \), so that \( a = 1 - b - c = \frac{1}{3} \), implying that there is a finite fraction of the channel \( \tilde{\mathcal{E}}_X \) in the mixing. It follows from the first equation in Eq. (6) that \( \lambda_1 = 0 \) at \( t_d \), meaning that this is a singularity of the resultant channel.

Note that we have not required the singularities of the generators of the mixing channels to be synchronized. That is, let \( t_q \) and \( t_r \) be such that \( q(t_q) = r(t_r) = \frac{1}{2} \), then \( t_q \neq t_r \) in general.

V. Discussions and Conclusions

This work discusses the problem of generating a general Pauli dynamical map with singularities in its generator by the mixing Pauli channels whose generators may or may not feature singularities. We point out that it is impossible to produce a channel characterized by a sin-
Regular generator as a result of mixing channels whose generators are regular. Different conditions on the classes of mixing channels that feature singular generators are considered in order to guarantee that the resultant channel’s generator is singular.

In the examples we considered, the singularity signals non-Markovianity, essentially because the sign of the Lindblad rate flips sign from positive to negative just after the singularity. For illustration, consider a decay rate of the form \( \gamma(t) = \tan(\omega t) \), where \( \omega \) is some real number. The generator has a singularity at \( t = \frac{\pi}{2\omega} \) and gives rise to a CP-indivisible process. A similar sign-flipping behavior just after the singularity is seen in naturally occurring processes such as non-Markovian amplitude damping [5, 11, 28], dephasing [38] and quantum depolarizing [29]. However, this behavior is not necessary, as may be verified in the Lindblad rate of the form \( \gamma(t) = \tan^2(\omega t) \). The generator now contains singularities at the same instants as the above channel, but gives rise to a CP-divisible process.

A further question that may be considered here is the power of mixing weaker forms of non-Markovianity than CP-indivisibility (cf. Ref. [7] and references therein), in terms of generating singularities and/or stronger forms of non-Markovianity. An aspect of this for the measure of CP-indivisible maps produced by mixing a class of Pauli maps was considered in Ref. [35]. From an experimental perspective, our result means that for the production of singularities, one cannot have recourse to mixing simpler channels. Although we have shown this for qubit Pauli channels, we conjecture that this is true quite generally.

Finally, it should be noted that all the arguments and results presented in this paper pertain only to Pauli dynamical maps. It is an interesting question how they generalize to non-Pauli maps.

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