Pressure filtration of the Newtonian fluid in the Darcy-Brinkman approximation through the horizontal porous rectangular channel with orthotropic anisotropy

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Abstract. We propose a mathematical model of the pressure flow of a viscous incompressible fluid in a porous horizontal channel with a rectangular cross section having an anisotropic structure described by an orthotropic tensor. The motion of the Newtonian fluid was assumed to be laminar (internal Reynolds number \(Re<50\)) and inertialess, as well as unidirectional along the axial direction of the porous channel, which allowed us to use the Darcy-Brinkman phenomenological equation, for which we formulated an initial-boundary-value problem, the solution of which we obtained analytically using a one-sided integral Laplace transformation and the finite integral Fourier transformation. A comparative analysis with known experimental data showed the correctness of the physical linearization of the Darcy-Brinkman equations. We show that mathematically filtering in a porous isotropic and anisotropic layer is described identically. The difference is in the values of the Darcy numbers, which characterize the internal structure of the granular layers.

1. Introduction

The use of porous media of various internal structures is an important component in the design and analysis of new-generation compact heat exchangers [1], in which heat transfer is significantly intensified due to the high specific surface area per unit volume and intensive mixing of the heat-transfer agent [2]. The use of such heat exchangers plays a special role in electronic systems since the failure rate increases almost exponentially with increasing temperature [3]. To intensify the heat transfer, it is necessary to reduce porosity, but in this case, the pressure loss to overcome the viscous forces of the heat-transfer agent increases significantly. A compromise was found and confirmed for isotropic porous media [4], however, for anisotropic porous structures, this problem has not yet been solved because the concept of anisotropy itself is not specified [5]. The solution of such a problem in the conjugate format is still unavailable due to the significant nonlinearity of the initial transfer equations, the large dimension, and the problem of incorrect closing [6], so almost the majority of studies focused on isotropic porous structures [7,8] and the intensification of heat transfer in them using nanofluids as heat carriers [9]. Nevertheless, attempts to conduct such an analysis for anisotropic media were made, but for the unsuitable 2-D format [10].

The mathematical description of anisotropy in porous materials is based on an analogy with the determinations of crystallophysics, i.e. on the linking physical properties to the peculiarities of the
internal structure and symmetry, which leads to the necessity of applying methods for describing anisotropic properties based on the theory of symmetry and groups [11]. The main tensor parameter characterizing the anisotropy of porous media is permeability [12], whose structure has a triclinic form, but integration, in this case, is difficult. Therefore, in practice, the hypothesis of the orthotropic structure of the permeability tensor is used [13].

In this article, we attempt to synthesize a mathematical model for estimating the initial hydrodynamic section in an anisotropic porous channel with a rectangular cross section under the assumption that the flow is unidirectional, which was successfully used for an isotropic porous medium [14].

2. Mathematical model
The Darcy-Brinkman-Forchheimer system of equations for describing the hydrodynamics of a Newtonian fluid in a stationary granular layer has the form [15]:

$$\nabla \cdot \mathbf{g} = 0;$$  \hspace{1cm} (1)

$$\frac{\rho_i}{\varepsilon} \left[ \frac{\partial \mathbf{g}}{\partial \tau} + \frac{1}{\varepsilon} (\mathbf{g} \cdot \nabla) \mathbf{g} \right] = -\nabla p + \mu_i \nabla^2 \mathbf{g} - \left( \frac{\mu_i}{K} + \rho_i - \frac{b \mathbf{g}}{\sqrt{K}} \right),$$  \hspace{1cm} (2)

where \( \tau \) - the current time; \( \rho_i, \mu_i \) - the density and dynamic viscosity of the fluid; \( \varepsilon \) - granular porosity; \( \mathbf{g} \) - the velocity vector of viscous incompressible fluid; \( p \) - the pressure; \( K \) - the granular scalar porosity; \( b \) - the Forchheimer factor. \( K \) is the orthotropic permeability tensor and let us denote it \( \overrightarrow{K} \), which is constructed using the rotation matrix \( A \) and its transposed analog \( A^T \)

$$\overrightarrow{K} = A\overrightarrow{K}_0 A^T,$$

where \( \overrightarrow{K}_0 \) - canonical form (diagonal view) of \( \overrightarrow{K} \). In this case, equation (2) takes the form

$$\frac{\partial \mathbf{g}}{\partial \tau} + \left( \frac{\mathbf{g} \cdot \nabla}{\varepsilon} \right) \mathbf{g} = \frac{\varepsilon}{\rho_i} \left( -\nabla p + \frac{\mu_i}{\varepsilon} \nabla^2 \mathbf{g} - \frac{b \mathbf{g}}{\sqrt{K}} \right),$$  \hspace{1cm} (3)

where \( \overrightarrow{K}^{-1} \) - the inverse tensor for \( \overrightarrow{K} \). With the selected Cartesian coordinate system, the axis \( z \) is directed in the axial direction, the axes \( x, y \) characterize the cross section, the origin is located at the lower corner point of the input section of the channel with a rectangular cross section.

In the selected coordinate system, system (1), (3), assuming unidirectional flow in the axial direction, takes the form

$$\frac{\partial \mathbf{g}_x}{\partial \tau} = -\frac{\varepsilon}{\rho_i} \frac{\partial p}{\partial y} + \frac{\mu_i}{\rho_i} \left( \frac{\partial^2 \mathbf{g}_y}{\partial x^2} + \frac{\partial^2 \mathbf{g}_y}{\partial z^2} \right) - \varepsilon \frac{\mu_i}{\rho_i} K_{yy}^* \mathbf{g}_y,$$  \hspace{1cm} (4)

where \( \mathbf{g}_x \) - the axial component of the velocity vector \( \mathbf{g} \);

$$K_{yy}^* = \left[ F_x (\alpha, \beta, \gamma) K_{yy}^* + F_x (\alpha, \beta, \gamma) K_{yy}^* + F_x (\alpha, \beta, \gamma) \right]/K_x; \hspace{0.5cm} K_{yy}^* = K_x/K_y; \hspace{0.5cm} K_y^* = K_y/K_z; \hspace{0.5cm} K_x^* = K_z/K_x;$$

$$F_x (\alpha, \beta, \gamma) = -\cos^2 \alpha \cos^2 \beta \cos^2 \gamma + 2 \cos^2 \alpha \cos^2 \beta + \cos^2 \alpha \cos^2 \gamma + \cos^2 \beta \cos^2 \gamma + \cos^2 \alpha \cos^2 \beta - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 1 - 2 \sin \alpha \cos \alpha \sin \beta \sin \gamma \cos \gamma; \hspace{0.5cm} F_x (\alpha, \beta, \gamma) = -\cos^2 \beta \cos^2 \gamma + \cos^2 \alpha + \cos^2 \gamma - 2 \cos^2 \alpha \gamma + \cos^2 \alpha \gamma + 2 \sin \alpha \cos \alpha \sin \beta \sin \gamma \cos \gamma; \hspace{0.5cm} F_x (\alpha, \beta) = \cos^2 \beta - \cos^2 \alpha \beta;$$

\( K_x, K_y, K_z \) - the diagonal elements of the tensor \( \overrightarrow{K}_0 \). Adding initial and boundary conditions

$$\mathbf{g}_y (x, z, 0) = 0;$$  \hspace{1cm} (5)
\( \partial_y(0,z,\tau) = \partial_y(h_x,z,\tau) = \partial_y(x,0,z) = \partial_y(y,h_x,\tau) = 0, \)  

(6)

where \( h_x, h_z \) - the geometric parameters of the cross section of the channel. Let \( \partial_0 \) be the cross section with hydraulic diameter \( d_h = 2h_x h_z / (h_x + h_z) \) average carrier velocity, then, entering relative parameters \( \Theta = \tau \partial_0 / d_h; \) \( X = x / d_h; \) \( Z = z / d_h; \) \( V = \partial_1 / \partial_0; \) \( Re = \rho_1 d_h \partial_0 / \mu_1 \) - the Reynolds number; \( Da = \left( K / d_h^2 \right)^{-2} \) - the Darcy number, we represent problem (4)-(6) in the dimensionless form:

\[
\frac{\partial V}{\partial \Theta} = -\varepsilon \frac{\partial P}{\partial Y} + \frac{1}{\varepsilon Re} \left( \frac{\partial^3 V}{\partial X^2} + \frac{\partial^3 V}{\partial Z^2} \right) - \frac{V}{\varepsilon Re Da};
\]

(7)

\[
V(X,Z,0) = 0;
\]

(8)

\[
V(0,Z,\Theta) = V(H_x,Z,\Theta) = V(X,0,\Theta) = V(X,H_z,\Theta) = 0.
\]

(9)

3. Problem solution.

Physically, system (7)-(9) describes the acceleration flow of the liquid medium in the considered flat channel from the rest state. If instead of (8) we write

\[
V(X,Z,0) = 1,
\]

(10)

then the initial problem is transformed into the acceleration flow problem with a certain initial velocity. This shows the mathematical identity of the problem of the initial hydrodynamic section in an isotropic porous channel with a rectangular cross section and an accelerating flow with a uniform non-zero initial velocity corresponding to its average consumption value. Therefore, the solution of the original problem can be represented using the one-sided integral Laplace transform [15] and the finite-integral sine of the Fourier transform [16]

\[
V(X,Z,\Theta) = \frac{4}{H_x H_z} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \lambda_m \mu_n \right)^{-1} \left[ 1 - (-1)^m \right] \left[ 1 - (-1)^n \right] \times
\]

\[
\left[ 1 + \frac{C}{a_{mn}} \right] \exp\left( a_{mn} \Theta \right) - \frac{C}{a_{mn}} \sin(\lambda_m X) \sin(\mu_n Z),
\]

(11)

where \( \lambda_m = m\pi / H_x; \) \( \mu_n = n\pi / H_z; \) \( a_{mn} = -\left( \lambda_m^2 + \mu_n^2 + Da^{-1} \right) / (\varepsilon Re); \) \( C = -\varepsilon^{-1} dP/dX. \)

The parameter \( C \) is found from the condition

\[
\frac{1}{H_x H_z} \int_0^{H_x} \int_0^{H_z} V(X,Z,\infty) dXdZ = 1
\]

and its value is determined by the expression

\[
C = \left\{ \frac{4}{H_x H_z} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \lambda_m \mu_n \right)^2 \left[ 1 - (-1)^m \right] \left[ 1 - (-1)^n \right]^2 / a_{mn} \right\},
\]

which determines the coefficient of hydraulic resistance, which correlates with the classical formula of Ergun [18]. As \( \Theta = Y, \) the length of the hydrodynamic initial section is

\[
Y = \frac{\varepsilon Re}{F_1} \ln \left[ \frac{F_1 C}{(C - F_1 \varepsilon Re)} \right].
\]

(12)
where \( F_1 = (\pi/H_z)^2 + (\pi/H_y)^2 + 1/Da \); \( F_2 = 1 - V(H_y/2, H_z/2, y)/V(H_y/2, H_z/2, x) \).

4. Results analysis
We compared the calculation results according to the proposed model with the data [19] obtained for a flat microchannel filled with an isotropic porous medium, in which a laminar flow of an incompressible viscous fluid is assumed at constant thermophysical parameters (Figure 1).

![Figure 1. The velocity profile in a flat isotropic porous microchannel at \( Da = 10^{-2} \); \( \varepsilon = 0.4 \); \( Re = 10 \): ● – the data from [19]; —— - the calculation according to the proposed model.](image)

5. Conclusion
The constructed mathematical model is an actual toolkit for assessing the initial hydrodynamic section, and the internal structure of the porous medium is determined by the value of the Darcy number \( Da \), which reflects the continuous cohesion between isotropic and anisotropic porous media from a single phenomenological position.

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**Acknowledgments**

The reported study was funded by RFBR, project number 19-38-90114.