Fluid analogs for rotating black holes

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Fluid analog models for gravity are based on the idea that any spacetime geometry admits a reinterpretation in which space is thought of as a fluid flowing with a prescribed velocity. This fluid picture is a restatement of the ADM decomposition of the metric. Most of the literature has focused on flat spatial geometries and physical fluid flows, with a view toward possible laboratory realizations. Here we relax these conditions and consider fluid flows on curved and time-dependent spatial geometries, as a way of understanding and visualizing solutions to general relativity. We illustrate the utility of the approach with rotating black holes. For the Kerr black hole we develop a fluid description based on Doran coordinates. For spinning BTZ black holes we develop two different fluid descriptions. One involves static conical spatial slices, with the fluid orbiting the tip of the cone. The other resembles a cosmology, with the fluid flowing on a time-dependent cylindrical geometry.

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1 Introduction

Consider the metric

\[ ds^2 = -c^2 dt^2 + h_{ij}(dx^i - v^i dt)(dx^j - v^j dt) \]  

(1)

where \( c(t, x), h_{ij}(t, x) \) and \( v^i(t, x) \) are functions of the spacetime coordinates. Null cones in this metric are determined by

\[ h_{ij}\left( \frac{dx^i}{dt} - v^i \right)\left( \frac{dx^j}{dt} - v^j \right) = c^2 \]  

(2)

This equation has a simple physical interpretation, in which space is regarded as a fluid moving on a spatial 3-geometry characterized by the line element

\[ ds_{\text{spatial}}^2 = h_{ij}dx^i dx^j \]  

(3)

In general this spatial geometry could be time-dependent. The space-fluid is taken to move on this 3-geometry with coordinate velocity \( v^i \), and light rays are assumed to propagate with speed \( c \) relative to the moving fluid. Such light rays obey the null cone condition (2), which justifies the fluid interpretation of the metric (1). We argue this in more detail, and discuss the condition under which fluid flow is geodesic, in appendix A.

This interpretation provides an example of analog gravity, a subject which has been explored in the literature beginning with [1]. More recent works include [2, 3, 4, 5, 6, 7]. The subject has been reviewed in [8] and gives a perspective which is closely related to the river model of black holes developed in [9]. We should however note an important distinction. Much previous work on the subject begins from the equations describing a non-relativistic fluid in flat space and shows that small disturbances propagate according to an effective curved pseudo-Riemannian metric.\footnote{Below in referring to the literature we will refer to such descriptions as realistic fluid flows.} In contrast to this previous work we do not impose physical constraints such as the continuity equation on the motion of the space-fluid, nor do we impose requirements such as a flat Euclidean spatial geometry \( h_{ij} = \delta_{ij} \). Thus our aim is not to develop fluid models for gravity which could be realized in the laboratory, even in
principle. Instead our aim is to accept the fluid motion which general rel-

ativity prescribes, and to use the analogy as a tool to help understand and
visualize properties of solutions.

We are able to do this in complete generality, since (by abandoning con-
siderations of physical fluids and laboratory realizations) any metric can be
put in the so-called acoustic form \(^1\). In fact \(^1\) is nothing but the ADM
decomposition of the metric, which is usually presented in the form \(^10\)

\[
\begin{align*}
g_{tt} &= -(N^2 - h_{ij}N^iN^j) \\
g_{ti} &= g_{it} = h_{ij}N^j \\
g_{ij} &= h_{ij}
\end{align*}
\]

Thus the speed of light \(c\) can be identified with the ADM lapse \(N\), and the
fluid velocity \(v^i = -N^i\) with the negative of the ADM shift. Incidentally
this provides a purely three-dimensional interpretation of general relativity:
rather than describing a foliation of a four-dimensional spacetime, one can
think of the lapse and shift as describing a fluid flow on a spatial three-
manifold.

Rather remarkably, several well-known solutions to general relativity have
simple fluid analogs. One prime example is the Schwarzschild metric in
Painlevé - Gullstrand coordinates \(^11\)\(^\text{12}\)\(^\text{13}\)

\[
ds^2 = -dt^2 + \left( dr + \sqrt{\frac{2GM}{r}} dt \right)^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2
\]

Here the spatial geometry is flat and the speed of light \(c = 1\). The fluid
moves radially inward, with a velocity

\[
v^r = -\sqrt{\frac{2GM}{r}}
\]

This fluid analog very naturally captures the image of space flowing into the
singularity at \(r = 0\). The radial velocity of the fluid reaches the speed of
light at the horizon \(r = 2GM\). The coordinates \(^5\) cover the right exterior
and future interior of the black hole, as shown for example in \(^13\). Note that
with respect to the Killing vector \(\frac{\partial}{\partial t}\) the 4-geometry is stationary but not
static due to the off-diagonal \(g_{tr}\) components of the metric, or equivalently
in our language due to the steady motion of the fluid.
Another example is a flat $k = 0$ FRW cosmology with

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2d\Omega^2)$$  \hspace{1cm} (6)

In terms of the proper radial distance $R = a(t)r$ we have

$$ds^2 = -dt^2 + (dR - HR\, dt)^2 + R^2d\Omega^2$$ \hspace{1cm} (7)

The spatial slices are static and flat, with metric $ds_{\text{spatial}}^2 = dR^2 + R^2d\Omega^2$, and the fluid moves radially outward with the Hubble velocity

$$v^R = HR, \quad H = \frac{\dot{a}}{a}$$

Again the speed of light $c = 1$. Note that the fluid velocity reaches the speed of light at the Hubble or apparent horizon $R = 1/H$.

One of the aims of the present work is to explore fluid analogs for other exact solutions of general relativity, in particular for rotating black holes. We explore fluid analogs for the Kerr solution in section 2 and for the BTZ solution in section 3. We conclude in section 4.

## 2 Kerr metric

In developing a fluid analog for the Kerr geometry [14] the first question which arises is the choice of coordinates. This is largely a matter of taste, as any metric can be decomposed in the ADM form (4). However particular choices may make the fluid analog particularly simple or appealing. This was certainly the case for the Schwarzschild and FRW metrics (5), (7). For Kerr the choice of coordinates is less compelling.

One obstacle is that (unlike Schwarzschild) there is no slicing of the Kerr geometry that is metrically or even conformally flat [15, 16]. This led some previous studies to focus on developing an analog model for the equatorial plane [2]. On the equatorial plane one can develop a fluid analog based on Boyer - Lindquist coordinates, as shown in [2]. However the resulting fluid motion has some rather counter-intuitive features: the motion is purely angular, and the horizon manifests itself as a locus where the speed of light $c \to 0$. 

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We will instead develop a fluid analog based on Doran coordinates [17], which were specifically developed to extend the Painlevé - Gullstrand coordinates (5) to rotating black holes. A similar description was developed in [3] using somewhat different coordinates. A realistic analog description for Kerr is an open problem; recent work can be found in [7, 18]. The general problem of constructing coordinates which are regular at the horizon has been considered in [19].

In Doran coordinates the Kerr metric is

\[ ds^2 = -dt^2 + \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} \left[ dr + \frac{\sqrt{2mr(r^2 + a^2)}}{r^2 + a^2 \cos^2 \theta} (dt - a \sin^2 \theta d\phi) \right]^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 \]

with horizons at \( r^2 + a^2 = 2mr \) and ergospheres at \( r^2 + a^2 \cos^2 \theta = 2mr \). The relation between Doran and Boyer - Lindquist coordinates is given in appendix [3].

It is straightforward to extract the analog fluid motion. Restricting to constant time slices we find that the spatial geometry is static,

\[ ds_{\text{spatial}}^2 = \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} \left[ dr - \frac{a \sin^2 \theta \sqrt{2mr(r^2 + a^2)}}{r^2 + a^2 \cos^2 \theta} d\phi \right]^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 \]

The fluid moves with coordinate velocity

\[ v^r = -\frac{\sqrt{2mr(r^2 + a^2)}}{r^2 + a^2 \cos^2 \theta} \]

\[ v^\theta = v^\phi = 0 \]

and the speed of light \( c = 1 \). Note that the coordinate velocity of the fluid is purely radial, but this is a bit deceptive because the spatial coordinates \( r \) and \( \phi \) aren’t orthogonal. An appealing feature of these coordinates is that

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2Our discussion of the geometry closely follows [17]. Also see [20, 21].
3We set \( G = 1 \) so the mass \( m \) and angular momentum per unit mass \( a = J/m \) have units of length. We restrict to \(-m < a < m\) so the singularity is inside the horizon.
as $a \to 0$ we smoothly recover the Schwarzschild expressions in Painlevé-Gullstrand coordinates.

To describe the spatial geometry it’s convenient to expand

$$d^2 s_{\text{spatial}} = \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} \, dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2$$

$$- \left( \frac{8mr}{r^2 + a^2} \right)^{1/2} \sin^2 \theta dr d\phi + \frac{2mra^2 \sin^4 \theta}{r^2 + a^2 \cos^2 \theta} d\phi^2$$

(11)

Here

$$0 < r < \infty \quad 0 < \theta < \pi \quad \phi \approx \phi + 2\pi$$

(12)

The first line in (11) is flat space in oblate spheroidal coordinates, related to the usual Cartesian coordinates by

$$x = \sqrt{r^2 + a^2 \sin \theta \cos \phi}$$

$$y = \sqrt{r^2 + a^2 \sin \theta \sin \phi}$$

$$z = r \cos \theta$$

(13)

This is drawn in the $(x, z)$ plane in Fig. 1. Note that the intrinsic geometry of the $(x, z)$ plane is flat. However the orbits of the Killing vector $\frac{\partial}{\partial \phi}$ aren’t orthogonal to the $(x, z)$ plane due to the off-diagonal metric component

$$h_{r\phi} = - \left( \frac{2mr}{r^2 + a^2} \right)^{1/2} \sin^2 \theta$$

Also the radius $R$ of the Killing orbits is distorted, from the value it would have in flat space

$$R^2 = x^2 + y^2 = (r^2 + a^2) \sin^2 \theta$$

to

$$R^2 = x^2 + y^2 + \frac{2mra^2 \sin^4 \theta}{r^2 + a^2 \cos^2 \theta}$$

The singularity is a ring at $r = 0, \theta = \pi/2$ or equivalently $x^2 + y^2 = a^2, z = 0$. Note that the proper distance around the singularity depends on how it is approached. If one approaches from the inside in the equatorial plane ($r \to 0$ then $\theta \to \pi/2$) ring has radius $a$, but if one approaches from the outside ($\theta \to \pi/2$ then $r \to 0$) the radius diverges. Also note that the fluid travels towards $r = 0$ – that is, towards the disk $x^2 + y^2 < a^2, z = 0$ – along
Figure 1: The \((x, z)\) plane has a flat geometry. The singularity is at \(x = \pm a, z = 0\). Surfaces of constant \(r\) are ellipses \(x^2/(r^2 + a^2) + z^2/r^2 = 1\) with foci at the singularity. The fluid travels toward \(r = 0\) along lines of constant \(\theta\) which are hyperbolas \(x^2/\sin^2 \theta - z^2/\cos^2 \theta = a^2\).

lines of constant \(\theta, \phi\). We will not consider the extension of the solution to \(r < 0\).

One property of the fluid motion is easy to understand: the fluid reaches the speed of light at the ergosphere.

\[
\|v\|^2 = h_{ij}v^iv^j = \frac{2mr}{r^2 + a^2 \cos^2 \theta} \quad (14)
\]

\[
r^2 + a^2 \cos^2 \theta = 2mr \quad \Rightarrow \quad \|v\| = 1
\]

Since the fluid is moving at the speed of light at this radius, it’s clear this locus defines the stationary limit surface. To understand the horizon in these terms we decompose the fluid velocity

\[
v^i = v^i_{\text{radial}} + v^i_{\text{angular}} \quad (15)
\]

into orthogonal components in the radial and angular directions.

\[
v_{\text{radial}} = v^r \partial_r - u^\phi \partial_\phi
\]

\[
v_{\text{angular}} = u^\phi \partial_\phi \quad (16)
\]
Here
\[ u^\phi = \frac{2mra}{(r^2 + a^2)(r^2 + a^2 \cos^2 \theta) + 2mra^2 \sin^2 \theta} \] (17)
is chosen so that \( \langle v_{\text{radial}}, v_{\text{angular}} \rangle = 0 \). Then the magnitude of the radial velocity satisfies
\[ \|v_{\text{radial}}\|^2 = \frac{2mr(r^2 + a^2)}{(r^2 + a^2)(r^2 + a^2 \cos^2 \theta) + 2mra^2 \sin^2 \theta} \] (18)
and the radial velocity reaches the speed of light at the horizon \( r^2 + a^2 = 2mr \). Since the radial velocity reaches the speed of light, it’s clear this locus defines a trapped surface.

2.1 Equatorial Kerr

The spatial 3-geometry of the Kerr metric (9) is somewhat complicated by the presence of the off-diagonal \( h_{r\phi} \) components of the metric. As a simple setting where this can be avoided we focus on the equatorial plane of the Kerr geometry, following [2]. After setting \( \theta = \pi/2 \) in [8] the spatial coordinates can be “straightened” by setting
\[ \phi = \tilde{\phi} + f(r) \] (19)
with
\[ \left( \frac{r^2 + a^2}{a} + \frac{2ma}{r} \right) f'(r) = \sqrt{\frac{2mr}{r^2 + a^2}} \] (20)
This leads to
\[ ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{\sqrt{8mr(r^2 + a^2)}}{r^2 + a^2 + \frac{2ma^2}{r}} dt dr - \frac{4ma}{r} dt d\tilde{\phi} \]
\[ + \frac{r^2 dr^2}{r^2 + a^2 + \frac{2ma^2}{r}} + \left( r^2 + a^2 + \frac{2ma^2}{r} \right) d\tilde{\phi}^2 \] (21)
One then reads off
\[ \text{spatial metric } ds^2_{\text{spatial}} = \frac{r^2 dr^2}{r^2 + a^2 + \frac{2ma^2}{r}} + \left( r^2 + a^2 + \frac{2ma^2}{r} \right) d\tilde{\phi}^2 \]
Some fluid streamlines in the equatorial plane are shown in Fig. 2. As the fluid moves from \( r = \infty \) to the singularity it rotates by an angle

\[
\Delta \tilde{\phi} = \int_0^\infty dr \frac{v^\phi}{v^r}
\]  

(23)

For a non-rotating black hole of course \( \Delta \tilde{\phi} = 0 \), and in the extremal limit \( a = \pm m \) this gives \( \Delta \tilde{\phi} \approx \pm 36^\circ \).
3 BTZ black hole

In this section we consider the BTZ black hole [22, 23]. In Schwarzschild coordinates the metric is

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\phi - \frac{r^+r^-}{\ell r^2}dt\right)^2 \quad (24) \]

\[ f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} \]

The coordinate \( \phi \) is periodically identified with period \( 2\pi \). The geometry is asymptotic to AdS_3 with radius of curvature \( \ell \). Inner and outer horizons are located at \( r = r_- \) and \( r = r_+ \), respectively. These parameters are related to the mass and angular momentum of the black hole by\(^4\)

\[ M = \frac{1}{\ell^2}(r_+^2 + r_-^2) \quad J = \frac{2r_+r_-}{\ell} \quad (25) \]

The Penrose diagram for a spinning black hole is shown in Fig. 3 while the spinless case is shown in Fig. 4. A realistic fluid description of BTZ was developed in [7]. In this section we consider some instructive generalizations.

3.1 Conical acoustic metric

We first consider “conical coordinates” which lead to simple static spatial slices.

The BTZ metric is invariant under shifts of \( t \) and \( \phi \), so it’s natural to consider reparametrizing

\[ t \rightarrow t + g(r) \quad \phi \rightarrow \phi + h(r) \quad (26) \]

in terms of two arbitrary functions \( g(r) \) and \( h(r) \). Setting \( h' = \frac{r_+r_-}{\ell^2}g' \) eliminates the off-diagonal components of the resulting spatial metric and puts the BTZ metric in the form

\[ ds^2 = -f(r)(dt + g'(r)dr)^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\phi - \frac{r^+r^-}{\ell r^2}dt\right)^2 \quad (27) \]

\(^4\)in units where \( 8G = 1 \)
At this stage it’s convenient to choose a constant $c$ which is positive and dimensionless but otherwise arbitrary. (The name is not misleading as $c$ will shortly be identified with the speed of light.) Setting $\frac{1}{f} - fg'^2 = \frac{1}{c^2}$ brings the metric to a conical acoustic form with

\begin{equation}
\text{spatial metric} \quad d{s_{\text{spatial}}}^2 = \frac{1}{c^2} \left( dr^2 + c^2 r^2 d\phi^2 \right)
\end{equation}

\begin{equation}
\text{fluid velocity} \quad v^r = \pm c \sqrt{c^2 - f(r)} \quad v^\phi = \frac{r_+ r_-}{\ell r^2}
\end{equation}

speed of light $= c$

This generalizes the results of [7] who considered the case $c = 1$. Note that the choice of $\pm$ in (29) corresponds to an outgoing or ingoing fluid, that is, to a white or black hole patch of the geometry.
Figure 4: Penrose diagram for a BTZ black hole with $J = 0$. The cosmological coordinates (36) cover the shaded diamond.

The spatial geometry (28) is a cone with a total angle $2\pi c$ about the tip. The fluid flow (29) is rather curious as there are radial turning points where $f(r) \to c^2$ and $v^r \to 0$. So the flow is bounded between the radii $r_{\text{min}}$ and $r_{\text{max}}$ where

$$ (r_{\text{max}})_{\text{min}}^2 = \frac{1}{2} \left( c^2 \ell^2 + r_+^2 + r_-^2 \pm \sqrt{(c^2 \ell^2 + r_+^2 + r_-^2)^2 - 4r_+^2 r_-^2} \right) $$

Note that $r_{\text{min}} < r_- < r_+ < r_{\text{max}}$. The intuitive picture is that the asymptotic AdS geometry acts as a gravitational potential well which pushes the fluid back toward the black hole. This accounts for the outer turning point. But the singularity of a spinning BTZ black hole is repulsive, so the fluid is also pushed away from $r = 0$ which gives rise to the inner turning point. Thus the fluid spirals up the Penrose diagram of Fig. 3 emerging from a white hole and falling back into the next black hole. The case $J = 0$ is special in that there is no inner turning point. The fluid emerges from the white hole, reaches a maximum radius, then falls back into the black hole.

From a spacetime perspective the arbitrary constant $c$ we have introduced controls the way that constant-time slices are embedded in the (2 + 1)-dimensional geometry. In the fluid interpretation $c$ plays the role of the

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5Note that $g_{tt}$ diverges both as $r \to 0$ and as $r \to \infty$.  

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speed of light with respect to the fluid. Somewhat curiously $c$ also controls the deficit angle of the spatial cone, with $c = 1$ corresponding to flat spatial slices. Finally the radial fluid velocity $v^r$ increases with $c$, which means the range of radii covered by these coordinates $r_{\text{min}} < r < r_{\text{max}}$ grows with $c$. The limit $c \to \infty$ is however singular.

The equation $\frac{dr}{d\phi} = \frac{v^r}{v^\phi}$ can be integrated to find the fluid streamlines, which up to a shift of $\phi$ are given by

$$r^2 = \frac{r_{\text{max}}^2 r_{\text{min}}^2}{r_{\text{max}}^2 \cos^2(c\phi) + r_{\text{min}}^2 \sin^2(c\phi)}$$

(31)

In general the fluid flows along a precessing ellipse on a cone, as shown in the left panel of Fig. 5. When $c = 1$ the flow is along a closed ellipse on a plane, as shown in the right panel of Fig. 5.

### 3.2 Cosmological acoustic metric for $J = 0$

Cosmological coordinates lead to a different fluid description of the BTZ black hole with some instructive new features. Here we develop this for a
non-rotating black hole.

To describe a non-rotating black hole we set \( r_- = 0 \) so that
\[
\begin{align*}
    ds_{\text{BTZ}}^2 &= -\frac{r^2 - r_+^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2 - r_+^2}dr^2 + r^2d\phi^2
\end{align*}
\]
(32)

To motivate the introduction of cosmological coordinates we reduce along \( \phi \) and consider the \( rt \) plane, described by
\[
\begin{align*}
    ds_{\text{AdS}_2}^2 &= -\frac{r^2 - r_+^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2 - r_+^2}dr^2
\end{align*}
\]
(33)

In fact this is AdS_2 in Rindler coordinates [24], which can also be described as an open FRW cosmology with metric
\[
\begin{align*}
    ds_{\text{AdS}_2}^2 &= -d\tau^2 + \ell^2 \cos^2 \frac{\tau}{\ell} \, d\chi^2
\end{align*}
\]
(34)

To see that both (33) and (34) describe patches of AdS_2, recall that AdS_2 is a hyperboloid \(-U^2 - V^2 + X^2 = -\ell^2\) in \( \mathbb{R}^{2,1} \) with metric \(-dU^2 - dV^2 + dX^2\).

We can introduce coordinates on the hyperboloid in two ways.
\[
\begin{align*}
    U &= \frac{\ell r}{r_+} = \ell \cos \frac{\tau}{\ell} \cosh \chi \\
    V &= \ell \left( \frac{r^2}{r_+^2} - 1 \right)^{1/2} \sinh \frac{r_+ t}{\ell^2} = \ell \sin \frac{\tau}{\ell} \\
    X &= \ell \left( \frac{r^2}{r_+^2} - 1 \right)^{1/2} \cosh \frac{r_+ t}{\ell^2} = \ell \cos \frac{\tau}{\ell} \sinh \chi
\end{align*}
\]
(35)

The induced metrics are (33) and (34) respectively.

Motivated by this we apply the change of coordinates (35) to the BTZ metric (32), obtaining
\[
\begin{align*}
    ds_{\text{BTZ}}^2 &= -d\tau^2 + \ell^2 \cos^2 \frac{\tau}{\ell} \left( d\chi^2 + \frac{r_+^2}{\ell^2} \cosh^2 \chi d\phi^2 \right)
\end{align*}
\]
(36)

Here \(-\frac{\pi \ell}{2} < \tau < \frac{\pi \ell}{2}\), \(-\infty < \chi < \infty\) and \(\phi \approx \phi + 2\pi\). These cosmological coordinates cover the shaded region of the BTZ Penrose diagram shown in Fig. 4.
Figure 6: The wormhole geometry, conformally compactified and embedded in $\mathbb{R}^3$.

As we now show, this metric can be interpreted as describing a comoving fluid on a cosmological Einstein-Rosen bridge. First note that the spatial slices in (36) are cosmological wormholes. That is, the spatial metric has the form

$$ds^2_{\text{spatial}} = a^2(\tau)ds^2_{\text{wormhole}}$$

where the wormhole metric and scale factor are

$$ds^2_{\text{wormhole}} = d\chi^2 + \frac{r^2}{\ell^2} \cosh^2 \chi d\phi^2$$
$$a(\tau) = \ell \cos \frac{\tau}{\ell}$$

The wormhole geometry approaches a hyperbolic plane as $\chi \to \pm \infty$. Thus it describes a pair of Poincaré disks connected by a throat. As $r_+ \to 0$ the throat pinches off, so the $M = 0$ geometry is singular. Empty AdS corresponds to $M = -1$.

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6 Up to a shift in $\chi$ the metric approaches $ds^2_{\text{H}_2} = d\chi^2 + \sinh^2 \chi d\phi^2$.

7 As $r_+ \to 0$ the throat pinches off, so the $M = 0$ geometry is singular. Empty AdS corresponds to $M = -1$. 

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To complete the fluid description note that the fluid velocity vanishes,

\[ v^\chi = v^\phi = 0 \quad (40) \]

and the speed of light \( c = 1 \). The vanishing fluid velocity just means the fluid is comoving with the spatial coordinates as the wormhole expands and contracts. As in (7) we could introduce \( R = a(\tau)\chi, \Phi = a(\tau)\phi \) to make this motion explicit. But unlike the case of flat FRW, in the present context this change of coordinates would leave us with a time-dependent spatial metric. So it seems more natural to leave the metric in the comoving form (36).

### 3.3 Cosmological spinning BTZ

Finally we introduce cosmological coordinates for a spinning BTZ black hole. The construction can be motivated by recalling that BTZ is a quotient of \( \text{AdS}_3 \) by a particular element of the \( \text{SO}(2,2) \) isometry group [25]. So we begin by introducing coordinates on \( \text{AdS}_3 \), regarded as a hyperboloid

\[ -U^2 - V^2 + X^2 + Y^2 = -\ell^2 \quad (41) \]

embedded in \( \mathbb{R}^{2,2} \) with metric \(-dU^2 - dV^2 + dX^2 + dY^2\). A convenient set of coordinates is

\[
\begin{align*}
U &= \ell \cos \frac{\tau}{\ell} \cosh \frac{r + \phi}{\ell} \\
V &= \ell \sin \frac{\tau}{\ell} \cosh \frac{r + \chi + r - \phi}{\ell} \\
X &= -\ell \sin \frac{\tau}{\ell} \sinh \frac{r + \chi + r - \phi}{\ell} \\
Y &= \ell \cos \frac{\tau}{\ell} \sinh \frac{r + \phi}{\ell}
\end{align*} \quad (42)
\]

These coordinates are induced by starting from a reference point \((\ell, 0, 0, 0)\) and (i) rotating by \( \tau/\ell \) in the \( UV \) plane, (ii) boosting by \( r+\phi/\ell \) in the \( UY \) plane, and (iii) boosting by \( -(r+\chi + r - \phi)/\ell \) in the \( VX \) plane.

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8 This is set up to make the description of the spinning black hole as simple as possible. The coordinates (35) we introduced in the spinless case are induced by a different sequence of \( \text{SO}(2,2) \) transformations.
coordinate $\phi$ has infinite range. To make a BTZ black hole the appropriate quotient is to identify $\phi$ with $\phi + 2\pi$. This gives

$$ds_{\text{BTZ}}^2 = -d\tau^2 + \sin^2 \frac{\tau}{\ell} (r_+ d\chi + r_- d\phi)^2 + r_+^2 \cos^2 \frac{\tau}{\ell} d\phi^2$$

(43)

$$0 < \tau < \frac{\pi\ell}{2}, \quad -\infty < \chi < \infty, \quad \phi \approx \phi + 2\pi$$

Rather amusingly this describes a cosmology with two scale factors whose spatial slices are infinite flat cylinders.

To simplify the description it’s convenient to reparametrize

$$\phi \to \phi - q(\tau) \chi$$

(44)

where

$$q(\tau) = \frac{r_+ r_- \sin^2 \frac{\tau}{\ell}}{r_-^2 + (r_+^2 - r_-^2) \cos^2 \frac{\tau}{\ell}}$$

(45)

is chosen to make the spatial metric diagonal. This brings the metric to the form (overdot $= \frac{d}{d\tau}$)

$$ds_{\text{BTZ}}^2 = -d\tau^2 + a_\chi^2 d\chi^2 + a_\phi^2 (d\phi - \dot{q}\chi d\tau)^2$$

(46)

where the scale factors are

$$a_\chi^2 = \frac{r_+^4 \cos^2 \frac{\tau}{\ell} \sin^2 \frac{\tau}{\ell}}{r_-^2 + (r_+^2 - r_-^2) \cos^2 \frac{\tau}{\ell}}$$

(47)

$$a_\phi^2 = \frac{r_-^2 + (r_+^2 - r_-^2) \cos^2 \frac{\tau}{\ell}}{r_-^2 + (r_+^2 - r_-^2) \cos^2 \frac{\tau}{\ell}}$$

From this we can read off the fluid description. The spatial geometry is an infinite cylinder.

$$ds_{\text{spatial}}^2 = a_\chi^2 d\chi^2 + a_\phi^2 d\phi^2$$

(48)

The fluid is not quite comoving with the cylinder coordinates, rather it has a peculiar velocity

$$v^\chi = 0, \quad v^\phi = \dot{q}\chi$$

(49)

The speed of light $c = 1$.

Note that the radius of the spatial cylinder decreases monotonically from $r_+$ at $\tau = 0$ to $r_-$ at $\tau = \pi\ell/2$, so these coordinates cover region II of the Penrose diagram shown in Fig. 3. One can send $r_- \to 0$ to describe a non-rotating black hole. In this limit these coordinates cover the future interior of the black hole, that is, the region $0 < r < r_+$. 

16
4 Conclusions

In this paper we developed fluid descriptions for various black hole geometries. A fluid description, or equivalently an ADM decomposition, lacks manifest covariance and is tied to a particular choice of coordinates. Our goal was not to identify coordinate systems which led to realistic fluid flows that could be realized in the laboratory. Rather we developed fluid descriptions that helped us to understand and visualize properties of the spacetime geometry. Thus our approach should be distinguished from much work in the literature which is based on obtaining the effective Lorentzian geometry that governs small perturbations in realistic non-relativistic fluids.

An advantage of our approach is that, as a restatement of the ADM decomposition, it applies to any solution to general relativity and can be used to obtain useful intuition about the solution. For example the fluid description of the Kerr metric in Doran coordinates captures the intuitive notion of space rotating as it is pulled in to a spinning black hole. For BTZ we developed several fluid descriptions which captured different aspects of the geometry. Conical acoustic coordinates have simple static spatial slices, and the fluid flow captures the property that both the AdS boundary and the BTZ singularity are repulsive. Cosmological coordinates for $J = 0$ capture a complete spatial slice of the geometry, including the wormhole which the other coordinate systems miss. Finally cosmological coordinates for a spinning BTZ black hole provide a simple description of the region between the inner and outer horizon, in a form that resembles a toy cosmology.

It would be interesting to explore the insights that can be gained by developing fluid descriptions for other solutions to general relativity. It would also be interesting to develop a better physical understanding of the space-fluid itself. Can other fluid parameters be usefully introduced, beyond the fluid velocity and speed of light?

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A  Fluid interpretation and geodesic flow

Consider the metric
\[ ds^2 = -c^2 dt^2 + h_{ij}(dx^i - v^i dt)(dx^j - v^j dt) \] (50)
which leads to the null cones
\[ h_{ij}\left(\frac{dx^i}{dt} - v^i\right)\left(\frac{dx^j}{dt} - v^j\right) = c^2 \] (51)

We generally assign \( t \) units of length, so the velocities \( v^i \) and \( c \) are dimensionless.

In the fluid picture we view space as moving with coordinate velocity \( v^i \) and we imagine that light rays move with speed \( c \) – meaning proper distance per unit coordinate time – relative to the moving fluid. The coordinate velocity of such light rays \( \frac{dx^i}{dt} \) satisfies (51). An easy way to see this is to note that locally one can make a Galilean transformation \( dx^i' = dx^i - v^i dt, \quad dt' = dt \) to a frame in which the fluid is at rest, and in this frame the condition (51) becomes \( h_{ij}\frac{dx^i}{dt'}\frac{dx^j}{dt'} = c^2 \). So the fluid picture leads to the null cones (51) associated with the metric (50).

It’s worth noting that, as pointed out in [8], the fluid moves along geodesics if and only if \( c \) is independent of position. To see this note that proper time along a trajectory \( x^i(t) \) in the metric (1) is
\[ \tau = \int dt \sqrt{c^2 - h_{ij}(\dot{x}^i - v^i)(\dot{x}^j - v^j)} \] (52)
where an overdot denotes a time derivative. Varying with respect to the trajectory and making the ansatz \( \dot{x}^i = v^i \) one finds that the geodesic equation is satisfied if and only if \( \partial_i c = 0 \). That is, the fluid moves along geodesics if and only if the speed of light is independent of position. Note that this is the case for all geometries considered in this paper.
\section*{B Doran coordinates}

The Kerr metric is frequently described in Boyer-Lindquist coordinates

\begin{equation}
\begin{aligned}
ds^2 &= -dt^2 + \frac{2mr}{r^2 + a^2 \cos^2 \theta} \left( dt - a \sin^2 \theta d\phi \right)^2 \\
&\quad + \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2 - 2mr} \left( dr + \left( r^2 + a^2 \cos^2 \theta \right) d\theta \right)^2 \\
&\quad + \left( r^2 + a^2 \cos^2 \theta \right) \sin^2 \theta d\phi^2
\end{aligned}
\end{equation}

(53)

The metric is invariant under shifts of $t$ and $\phi$. To pass to Doran coordinates we translate these coordinates by an amount which depends on $r$.

\begin{align}
t &\rightarrow t + f(r) \\
\phi &\rightarrow \phi + g(r)
\end{align}

(54)

With

\begin{align}
f' &= -\frac{\sqrt{2mr(r^2 + a^2)}}{r^2 + a^2 - 2mr} \\
g' &= -\frac{a}{r^2 + a^2 - 2mr} \sqrt{\frac{2mr}{r^2 + a^2}}
\end{align}

(55)

we obtain

\begin{align}
ds^2 &= -dt^2 + \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} \left[ dr + \sqrt{\frac{2mr(r^2 + a^2)}{r^2 + a^2 \cos^2 \theta}} \left( dt - a \sin^2 \theta d\phi \right) \right]^2 \\
&\quad + (r^2 + a^2 \cos^2 \theta) d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2
\end{align}

(56)

This is the Kerr metric in Doran coordinates.

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