QUIVER GIT FOR VARIETIES WITH TILTING BUNDLES

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Abstract. In the setting of a variety $X$ admitting a tilting bundle $T$ we consider the problem of constructing $X$ as a quiver GIT quotient of the algebra $A := \text{End}_X(T)^\text{op}$. We prove that if the tilting equivalence restricts to a bijection between the skyscraper sheaves of $X$ and the closed points of a quiver GIT moduli functor for $A = \text{End}_X(T)^\text{op}$ then $X$ is indeed a fine moduli space for this quiver GIT moduli functor, and we prove this result without any assumptions on the singularities of $X$.

As an application we consider varieties which are projective over an affine base, $\pi : X \to \text{Spec}(R)$, such that $\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R$ and $\pi$ has fibres of dimension $\leq 1$. In this situation there is a particular tilting bundle, $T_0$, on $X$ constructed by Van den Bergh [29], and our result allows us to reconstruct $X$ as a quiver GIT quotient of $A_0 = \text{End}_X(T_0)^\text{op}$ for an easy to describe stability condition and dimension vector. This result applies to flips and flops in the minimal model program, and in the situation of flops shows that both a variety and its flop appear as quiver GIT moduli spaces for algebras produced from different tilting bundles on the variety.

We also give an application to rational surface singularities, showing that their minimal resolutions can always be constructed as quiver GIT quotients for specific dimension vectors and stability conditions. This gives a construction of minimal resolutions as moduli spaces for all rational surface singularities, generalising the $\text{Hilb}^2(\mathbb{C}^2)$ moduli space construction which only exists for quotient singularities [16,17].

1. Introduction

1.1. Overview. Any variety $X$ equipped with a tilting bundle $T$ induces a derived equivalence between the bounded derived category of coherent sheaves on $X$ and the bounded derived category of finitely generated left modules for the algebra $A := \text{End}_X(T)^\text{op}$. This situation is similar to the case of an affine variety $\text{Spec}(R)$ where we can construct the commutative algebra $R = \text{End}_X(\mathcal{O}_X)^\text{op}$ and there is an abelian equivalence between coherent sheaves on $\text{Spec}(R)$ and finitely generated left $R$-modules. However, whereas in the affine case we can recover the variety $\text{Spec}(R)$ from the algebra $R$, it is not so clear how to recover the variety $X$ from the algebra $A$. One possibility is to present $A$ as the path algebra of a quiver with relations, construct the quiver GIT moduli space of $A$ for some dimension vector and stability condition, and attempt to relate this back to $X$.

While this approach may not work in general there are many examples where this is known to be successful, such as del Pezzo surfaces [11,19], minimal resolutions of Kleinian singularities [8,12,21], and crepant resolutions of Gorenstien quotient singularities in dimension 3 [5,10], which lead us to hope it may work in some other interesting settings.

In this paper we will determine conditions for $X$ to be a fine moduli space for the quiver GIT moduli functor $F_A$, (Section 2.6), and this will allow us to prove that $X$ is a quiver GIT quotient for a specific stability condition and dimension vector in a large class of examples. These examples include applications to the minimal model program and to resolutions of rational surface singularities.

This problem was also considered by Bergman and Proudfoot, [2], who study embeddings of closed points and tangent spaces to show that a smooth variety is a connected component of the quiver GIT quotient for ‘great’ stability condition and dimension vector. However, their approach cannot be extended to singular varieties and it can be difficult to identify which conditions are ‘great’. The methods developed in this paper have the advantages of applying to singular varieties, such as those occurring in the minimal model program, and allowing us to identify a specific stability condition and dimension vector in applications.

1.2. Comparing Moduli Functors. In developing methods to understand quiver GIT moduli functors we are inspired by the following result of Sekiya and Yamaura [28].
Theorem ([28, Theorem 4.20]). Let $B$ be an algebra with tilting module $T$. Define $A = \text{End}_B(T)^{\text{op}}$, suppose that both $A$ and $B$ are presented as path algebras of quivers with relations, and let $\mathcal{F}_A$ and $\mathcal{F}_B$ denote quiver moduli functors on $A$ and $B$ for some choice of stability conditions and dimension vectors. Then if the tilting equivalences

$$
\begin{array}{ccc}
\text{RHom}_B(T,-) & \text{RHom}_B(T,-) \\
D^b(B\text{-mod}) & D^b(A\text{-mod}) \\
T \otimes_A (-) & T \otimes_A (-)
\end{array}
$$

restrict to a bijection between $\mathcal{F}_B(\mathbb{C})$ and $\mathcal{F}_A(\mathbb{C})$ then $\mathcal{F}_B$ is naturally isomorphic to $\mathcal{F}_A$.

This leads us to the idea of working with a moduli functor for which $X$ is a fine moduli space instead of working with $X$ itself, and we then prove the following variant of Sekiya and Yamamura’s result.

Theorem (Theorem 4.0.1). Let $\pi : X \rightarrow \text{Spec}(R)$ be a projective morphism of varieties. Suppose $X$ is equipped with a tilting bundle $T$, define $A = \text{End}_X(T)^{\text{op}}$, and suppose that $A$ is presented as a quiver with relations. Let $\mathcal{F}_A$ be a quiver GIT moduli functor on $A$ for some stability condition and dimension vector. Then if the tilting equivalences

$$
\begin{array}{ccc}
\text{RHom}_X(T,-) & \text{RHom}_X(T,-) \\
D^b(\text{Coh } X) & D^b(A\text{-mod}) \\
T \otimes_A (-) & T \otimes_A (-)
\end{array}
$$

restrict to a bijection between $\mathcal{F}_X(\mathbb{C})$ and $\mathcal{F}_A(\mathbb{C})$ then $\mathcal{F}_X$ is naturally isomorphic to $\mathcal{F}_A$.

We recall the definitions of the moduli functors $\mathcal{F}_A$ and $\mathcal{F}_X$ in Sections 2.6 and 2.7. The moduli functor $\mathcal{F}_X$ is similar to the Hilbert functor of one point on a variety, which is well-known to be represented by $X$, but for lack of a reference in this setting we provide a proof.

Theorem (Theorem 4.0.3). Let $\pi : X \rightarrow \text{Spec}(R)$ be a projective morphism of varieties. Then there is an natural isomorphism between the functor of points $\text{Hom}_{\text{Sch}}(-, X)$ and the moduli functor $\mathcal{F}_X$. In particular $X$ is a fine moduli space for $\mathcal{F}_X$.

Combining these two results we have a method to show when a variety $X$ with tilting bundle $T$ can be recovered as a quiver GIT moduli quotient of the algebra $A = \text{End}_X(T)^{\text{op}}$.

Corollary 1.2.1. Let $\pi : X \rightarrow \text{Spec}(R)$ be a projective map of varieties and suppose $X$ has a tilting bundle $T$. Define $A = \text{End}_X(T)^{\text{op}}$, suppose that $A$ is presented as a quiver with relations, and let $\mathcal{F}_A$ be a quiver GIT moduli functor on $A$ for some stability condition $\theta$ and dimension vector $d$. Then if the tilting equivalences

$$
\begin{array}{ccc}
\text{RHom}_X(T,-) & \text{RHom}_X(T,-) \\
D^b(\text{Coh } X) & D^b(A\text{-mod}) \\
T \otimes_A (-) & T \otimes_A (-)
\end{array}
$$

restrict to a bijection between the skyscraper sheaves on $X$ and the $\theta$-semistable $A$-modules with dimension vector $d$ then $X$ is isomorphic to the quiver GIT quotient of $A$ for the stability condition $\theta$ and dimension vector $d$. 
1.3. Applications. To give an application of this theorem we need a class of varieties with well-understood tilting equivalences. We consider the situation arising in following theorem of Van den Bergh.

**Theorem 1.3.1** ([29, Theorem A]). Let $\pi : X \to \text{Spec}(R)$ be a projective morphism of Noetherian schemes such that $\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R$ and $\pi$ has fibres of dimension $\leq 1$. Then there are tilting bundles $T_0$ and $T_1 = T_0'$ on $X$ such that the derived equivalences $\mathbb{R}\text{Hom}_X(T_i, \ast) : D^b(\text{Coh} X) \to D^b(A_i\text{-mod})$ restrict to equivalences of abelian categories between $\mathbb{R}\text{Per}(X/R)$ and $A_i\text{-mod}$, where $A_i = \text{End}_X(T_i)^{\text{op}}$.

This gives us a large class of varieties with well-understood tilting equivalences. We recall the definition of $\mathbb{R}\text{Per}(X/R)$ for $i = 0, 1$ in Definition 5.2.1. We then show that in this situation there is a particular choice of dimension vector $d_{T_0}$ and stability condition $\theta_{T_0}$ such that $X$ occurs as the quiver GIT quotient of $A_0$.

**Corollary** (Corollary 5.2.5). Suppose we are in the situation of Theorem 1.3.1 and that $X$ and $\text{Spec}(R)$ are both varieties. Then $X$ is isomorphic to the quiver GIT quotient of $A_0 = \text{End}_X(T_0)^{\text{op}}$ for dimension vector $d_{T_0}$ and stability condition $\theta_{T_0}$.

See Section 5.1 for the definitions of $\theta_{T_0}$ and $d_{T_0}$. We note they are easy to define and depend only on a decomposition of $T$ into indecomposable summands.

1.4. Applications to the Minimal Model Program. The class of varieties in the above corollary includes flips and flops of dimension 3 in the minimal model program. In the setting of smooth, projective 3-folds flops were construct as components of moduli spaces and shown to be derived equivalent in the work of Bridgeland [4], and this work was extended to include projective 3-folds with Gorenstein terminal singularities by Chen [9]. These results were reinterpreted more generally via tilting bundles by Van den Bergh [29]. We can now reinterpret these results once again by combining Corollary 5.2.5 with Van den Bergh’s results.

It is immediate from Corollary 5.2.5 that if $\pi : X \to \text{Spec}(R)$ is either a flipping or flopping contraction with fibres of dimension $\leq 1$ then both $X$ and its flip/flop can be reconstructed as quiver GIT quotients. Further, in the case of flops, the following corollary shows that both $X$ and its flop can be constructed as quiver GIT quotients arising from tilting bundles on $X$.

**Corollary** (Corollary 5.3.2). Suppose we are in the situation of Corollary 5.2.5 and that $\pi : X \to \text{Spec}(R)$ is a flopping contraction with flop $\pi' : X' \to \text{Spec}(R)$. Then $X$ is the quiver GIT quotient of the algebra $A_0 = \text{End}_X(T_0)^{\text{op}}$ for dimension vector $d_{T_0}$ and stability condition $\theta_{T_0}$, and the flop $X'$ is the quiver GIT quotient of the algebra $A_1 = \text{End}_X(T_1)^{\text{op}}$ for dimension vector $d_{T_1}$ and stability condition $\theta_{T_1}$.

This fits into a general philosophy of having a preferred stability condition defined by a tilting bundle and realising all minimal models via quiver GIT by changing the tilting bundle rather than changing the stability condition.

1.5. Applications to Resolutions of Rational Surface Singularities. Minimal resolutions of affine rational surface singularities automatically satisfy the conditions of Corollary 5.2.5 hence provide another class of examples.

**Corollary** (Example 5.4.2). Suppose that $X$ is a variety and that $\pi : X \to \text{Spec}(R)$ is the minimal resolution of a rational surface singularity. Then there is a tilting bundle $T_0$ on $X$ such that $X$ is the quiver GIT quotient of $A_0 = \text{End}_X(T_0)^{\text{op}}$ for dimension vector $d_{T_0}$ and stability condition $\theta_{T_0}$.

For quotient surface singularities this result was already known when either $G < \text{SL}_2(\mathbb{C})$ [12], or when $G$ was a cyclic or dihedral subgroup of $\text{GL}_2(\mathbb{C})$ [31,33,34], but is new in other cases. In particular, for quotient surface singularities the minimal resolution is known to have moduli space interpretation as $\text{Hilb}_d^f(\mathbb{C}^2)$, see [16,17], and this corollary extends a similar moduli space interpretation to minimal resolutions of all rational surface singularities.
1.6. Outline. In Section 2 we recall a number of preliminary definitions and theorems relating to tilting bundles and quiver GIT which we will need in later sections. Section 3 consists of a collection of preliminary lemmas which form the bulk of the proofs of our main results. We then prove our main results in Section 4, and give an application to a class of examples motivated from the minimal model program, and also to resolutions of rational singularities, in Section 5.

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2. Preliminaries

2.1. Geometric and Notational Preliminaries. We begin by giving some geometric and notational preliminaries. Throughout this paper all schemes will be over \( \mathbb{C} \) and a variety will be a scheme which is separated, reduced, irreducible and of finite type over \( \mathbb{C} \). In the introduction we stated our results for varieties projective over an affine base, but in fact we will prove our results in the generality of schemes, \( X \), arising from projective morphisms \( \pi : X \to \text{Spec}(R) \) of finite type schemes over \( \mathbb{C} \). Such schemes are quasi-projective over \( \mathbb{C} \), and hence separated, so are a slight generalisation of varieties projective over an affine base in that they may not be reduced or irreducible. For an affine scheme \( \text{Spec}(R) \) we will let \( \mathcal{O}_R \) denote \( \mathcal{O}_{\text{Spec}(R)} \). We denote the category of coherent sheaves on a scheme \( X \) by \( \text{Coh} X \), we denote the skyscraper sheaf of a closed point \( x \in X \) by \( \mathcal{O}_x \), and for a locally free sheaf \( \mathcal{F} \in \text{Coh} X \) we let \( \mathcal{F}^\vee \) denote the dual \( \mathcal{H}\text{om}_X(\mathcal{F}, \mathcal{O}_X) \). For an algebra \( A \) we let \( A^{\text{op}} \) denote the opposite algebra of \( A \), and \( A\text{-mod} \) denote the category of finitely generated left \( A \)-modules.

2.2. Derived Categories and Tilting. We recall the definitions of tilting bundles on schemes and several notions related to derived categories that we will make use of later.

Consider a triangulated \( \mathbb{C} \)-linear category \( \mathcal{C} \) with small direct sums. A subcategory is localising if it is triangulated and also closed under all small direct sums. A localising subcategory is necessarily closed under direct summands [26, Proposition 1.6.8]. An object \( T \in \mathcal{C} \) generates if the smallest localising category containing \( T \) is \( \mathcal{C} \).

**Definitions 2.2.1.** Let \( \mathcal{C} \) be a triangulated category closed under small direct sums. An object \( T \in \mathcal{C} \) is tilting if:

i) \( \text{Ext}_\mathcal{C}^k(T,T) = 0 \) for \( k \neq 0 \).

ii) \( T \) generates \( \mathcal{C} \).

iii) The functor \( \text{Hom}_\mathcal{C}(T, -) \) commutes with small direct sums.

For \( X \) a quasi-projective scheme let \( D(X) \) denote the derived category of quasicoherent sheaves on \( X \), and \( D^b(X) \) denote the bounded derived category of coherent sheaves. For \( X \) a Noetherian quasi-projective scheme \( D(X) \) is closed under small direct sums [25, Example 1.3], and \( D(X) \) is compactly generated with compact objects the perfect complexes [25, Proposition 2.5]. We let \( \text{Perf}(X) \) denote the category of perfect complexes on \( X \). When \( X \) is smooth the category of perfect complexes equals \( D^b(X) \).

For an algebra \( A \) we let \( D(A) \) be the derived category of left modules over \( A \), and \( D^b(A) \) the bounded derived category of finitely generated left \( A \)-modules. When \( D(X) \) has tilting object a sheaf, \( T \), then define \( A := \text{End}_X(T)^{\text{op}} \). When \( T \) is a locally free coherent sheaf on \( X \) then \( T \) is a tilting bundle and this gives a derived equivalence between \( D(X) \) and \( D(A) \).

**Theorem 2.2.2** ([15, Theorem 7.6], [6, Remark 1.9]). Let \( X \) be a scheme that is projective over an affine scheme of finite type, \( \pi : X \to \text{Spec}(R) \), with tilting bundle \( T \) on \( X \) and define \( A = \text{End}_X(T)^{\text{op}} \). Then:
ideal in arrows by the trivial paths, and define definitions required for quiver geometric invariant theory, following the definitions of King [20].

Moreover the equivalence $T_s$ is $R$-linear, and $A$ is a finite $R$-algebra.

2.3. Quivers and Quiver GIT. We set our notation for quivers and then recall the definitions required for quiver geometric invariant theory, following the definitions of King [20].

A quiver is a directed multigraph. We will denote a quiver $Q = (Q_0, Q_1)$, with $Q_0$ the set of vertices and $Q_1$ the set of arrows. The set of arrows is equipped with head and tail maps $h, t : Q_1 \to Q_0$ which take an arrow to the vertices that are its head and tail respectively. We compose arrows from right to left, that is

$$b \circ a = \begin{cases} b(a) & \text{if } h(a) = t(b); \\ 0 & \text{otherwise;} \end{cases}$$

and we extend this definition to paths. We recall that there is a trivial path $e_i$ for each vertex $i \in Q_0$ and that these form a set of orthogonal idempotents.

We denote the path algebra by $CQ$, define $S$ to be the subalgebra of $CQ$ generated by the trivial paths, and define $V$ to be the $C$-vector subspace of $CQ$ spanned by the arrows $a \in Q_1$. Then $S$ is a semisimple $C$-algebra, $V$ is an $S$-module, and $CQ = T_S(V) = \bigoplus_{i \geq 0} V^{\otimes i}$. Given $\Lambda$ an $S$-module we define $I(\Lambda)$ to be the two sided ideal in $CQ$ generated by $\Lambda$. We then define

$$\frac{CQ}{I(\Lambda)} = \frac{CQ}{\Lambda}$$

and refer to it as the path algebra with relations $\Lambda$.

We can now recall the definitions required for quiver GIT.

Definitions 2.3.1. Let $Q = (Q_1, Q_0)$ be a quiver.

i) A dimension vector for $Q$ is defined to be an element $d \in \mathbb{N}^{Q_0}$ assigning a non-negative integer to each vertex.

ii) A dimension $d$ representation of $Q$ is given by assigning to each vertex $i$ the vector space $V_i = \mathbb{C}^{d(i)}$, to each arrow $a$ a linear map $\phi_a : V_{t(a)} \to V_{h(a)}$, and to each trivial path $e_i$ the linear map $id_{V_i}$.

iii) A morphism, $\psi$, between two finite dimensional representations $(V_i, \rho_a)$ and $(W_i, \chi_a)$ is given by a linear map $\psi_i : V_i \to W_i$ for each vertex $i$ such that for every arrow $a$ we have $\chi_a \circ \psi_{t(a)} = \psi_{h(a)} \circ \rho_a$.

iv) The representation variety, $Rep_d(Q)$, is defined to be the set of all representations of $Q$ of dimension $d$, and we note that this is an affine variety.

We then suppose that the quiver has relations $\Lambda$ defining the algebra $A = CQ/\Lambda$.

v) A representation of the quiver with relations, $(Q, \Lambda)$, is a representation of $Q$ such that the linear maps assigned to the arrows satisfy the relations among the paths in the quiver. We recall that a representation of a quiver with relations corresponds to a left $CQ/\Lambda$-module.

vi) The representation scheme $Rep_d(Q, \Lambda)$ is the closed subscheme of the affine variety $Rep_d(Q)$ cut out by the ideal corresponding to the relations $\Lambda$. Closed points of $Rep_d(Q, \Lambda)$ correspond to representations of $(Q, \Lambda)$.

We can now define the action of a reductive group on the affine scheme $Rep_d(Q, \Lambda)$. For $\{\phi_a : a \in Q_1\}$, a dimension $d$ representation, there is an action of $GL_{d(i)}(\mathbb{C})$ at vertex $i$ by base change;

$$g \circ \phi_a = \begin{cases} g \circ \phi_a & \text{if } t(a) = i; \\ \phi_a \circ g^{-1} & \text{if } h(a) = i; \\ 0 & \text{otherwise.} \end{cases}$$

Then $G := GL_d(\mathbb{C}) = \prod_{i \in Q_0} GL_{d(i)}(\mathbb{C})$ acts on $Rep_d(Q, \Lambda)$ with kernel $\mathbb{C}^* = \Delta$. We note that orbits of $G$ correspond to isomorphism classes of representations.
Definition 2.3.2. The affine quotient with dimension vector $d$ is defined to be
$$\text{Rep}_d(Q, \Lambda)/G := \text{Spec}(\mathbb{C}[\text{Rep}_d(Q, \Lambda)]^G).$$

We now recall the definition of stability conditions in order to consider more general
GIT quotients.

Definitions 2.3.3.

i) A stability condition is defined to be a $\theta \in \mathbb{Z}^{Q_0}$ assigning an integer to each vertex
of $Q$. For a finite dimensional representation $M$ let $d_M$ be the dimension vector
of $M$, and define $\theta(M) = \sum_{i \in Q_0} \theta(i)d_M(i)$.

ii) A finite dimensional representation, $M$, is $\theta$-semistable if $\theta(M) = 0$ and any
subrepresentation $N \subset M$ satisfies $\theta(N) \geq 0$.

iii) A $\theta$-semistable representation $M$ is $\theta$-stable if the only subrepresentations $N \subset M$
with $\theta(N) = 0$ are $M$ and $0$. A stability $\theta$ is generic if all $\theta$-semistable
representations are stable.

iv) For a stability condition $\theta$ define $\text{Rep}_d(Q, \Lambda)_\theta^{ss}$ to be the set of $\theta$-stable representations,
and $\text{Rep}_d(Q, \Lambda)_\theta^{ss}$ to be the set of $\theta$-semistable representations.

Definition 2.3.4. Every finite dimensional $\theta$-semistable representation $M$ has a Jordan-
Holder filtration
$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$
such that each $M_i$ is $\theta$-semistable and each quotient is $\theta$-stable. Two $\theta$-semistable represen-
tations are defined to be $S$-equivalent if their Jordan-Holder filtrations have matching composition factors.

We note that $\theta$-stable objects have length one filtrations hence are $S$-equivalent if and only if they are isomorphic.

Any character of $G$ is given by powers of the determinant character and is of the form
$$\chi^\theta(g) := \prod_{i \in Q_0} \det(g_i)^{\theta_i}$$
for some collection of integers $\theta_i$. We will restrict our attention to characters which are
trivial on the kernel of the action, $\Delta$, which translates to the condition $\sum \theta(i)d(i) = 0$.
Hence these characters are in correspondence with stabilities.

We recall that $\text{Rep}_d(Q, \Lambda)$ is affine, and that $f \in \mathbb{C}[\text{Rep}_d(Q, \Lambda)]$ is a semi-invariant of
weight $\chi$ if $f(g.x) = \chi(g)f(x)$ for all $g \in G$ and all $x \in \text{Rep}_d(Q, \Lambda)$. We denote the set of
such $f$ as $\mathbb{C}[\text{Rep}_d(Q, \Lambda)]^{G,\chi}$.

Definition 2.3.5 ([20]). The quiver GIT quotient, for dimension vector $d$ and stability
condition $\theta$, is defined to be the scheme
$$M^{ss}_{d,\theta} := \text{Proj} \left( \bigoplus_{n \geq 0} \mathbb{C}[\text{Rep}_d(Q, \Lambda)]^{G,\chi} \right).$$

It is immediate from this definition that for any stability condition $\theta$ the quiver GIT
quotient $M^{ss}_{d,\theta}$ is projective over the affine quotient $M^{ss}_{d,0} = \text{Spec}(\mathbb{C}[\text{Rep}_d(Q, \Lambda)]^G)$.

2.4. Quivers and Tilting Bundles. We recall how quivers can be constructed from
tilting bundles.

Let $X \to \text{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Given
a tilting bundle $T'$ on $X$ and a decomposition into indecomposable summands $T' = \bigoplus_{i=0}^n E_i^{op}$,
with $E_i$ and $E_j$ non-isomorphic for $i \neq j$, then $T = \bigoplus_{i=0}^n E_i$ is also a tilting
bundle on $X$ and $\text{End}_X(T'^{op})$ is Morita equivalent to $\text{End}_X(T)^{op}$. Hence we will always
assume, without loss of generality, that our tilting bundles have a given multiplicity free
decomposition into indecomposables, $T = \bigoplus_{i=0}^n E_i$.

We then recall from Theorem 2.2.2 that $A = \text{End}_X(T)^{op}$ is a finite $R$-algebra for
$R$ a finite type commutative $\mathbb{C}$-algebra, and we wish to present $A$ as the path algebra
of a quiver with relations such that each indecomposable $E_i$ corresponds to the unique
idempotent $e_i = \text{id}_{E_i} \in \text{Hom}_X(E_i, E_i) \subset A = \text{End}_X(T)^{op}$ that is the trivial path at vertex
$i$. In particular $1 = \sum e_i$ and we have a diagonal inclusion $\bigoplus_{i=0}^n e_i R \subset A$. 

Indeed, we can construct a quiver by creating a vertex $i$ corresponding to each idempotent $e_i$. We then choose a finite set of generators of $e_i A e_j$ as an $R$-module, which is possible as $A$ is finite $R$-module, and create corresponding arrows from vertex $j$ to $i$ for all $0 \leq i, j \leq n$. We then consider a presentation of $R$ over $\mathbb{C}$ with finitely many generators, possible as it has finite type, and at each vertex add arrows corresponding to each generator of $R$. If we call this quiver $Q$ then by this construction there is a surjection of $R$-algebras $\mathbb{C} Q \to A$ given by mapping each trivial path to the corresponding idempotent, and each arrow to the corresponding generator. We then take the kernel of this map, $I$, and $\mathbb{C} Q/I \cong A$ as an $R$-algebra.

We note that this presentation has many unpleasant properties, for example it may be the case that the ideal of relations $I$ is not a subset of the paths of length greater than 1. In nice situations it is possible to simplify the presentation, see for example the situation considered in [2, Section 1].

We also note that there is a decomposition into projective modules $A = \bigoplus_{i=0}^n \text{Hom}_X(T, E_i)$ where the module $\text{Hom}_X(T, E_i)$ corresponds to paths in the quiver starting at vertex $i$.

2.5. Functor of Points. We recall the definition of the functor of points and the definitions of fine and coarse moduli spaces. Let $\mathsf{Sch}$ denote the category of finite type schemes over $\mathbb{C}$, let $\mathsf{Sets}$ denote the category of sets, and let $\mathsf{R}$ denote the category of finite type commutative $\mathbb{C}$-algebras. Suppose $X \in \mathsf{Sch}$, then the functor of points for $X$ is defined to be the functor

$$\text{Hom}_{\mathsf{Sch}}(-, X) : \mathsf{R} \to \mathsf{Sets}$$

and by Yoneda’s lemma this gives an embedding of $\mathsf{Sch}$ into the category of functors from $\mathsf{R}$ to $\mathsf{Sets}$.

A functor $F : \mathsf{R} \to \mathsf{Sets}$ is representable if there is some $Y \in \mathsf{Sch}$ such that $F$ is naturally isomorphic to $\text{Hom}_{\mathsf{Sch}}(-, Y)$. Then $Y$ is said to be a fine moduli space for $F$. A scheme $Y$ is said to be a coarse moduli space for $F$ if there is a natural transformation $\nu : F \to \text{Hom}_{\mathsf{Sch}}(-, Y)$ such that $\nu_{\mathbb{C}} : F(\mathbb{C}) \to \text{Hom}_{\mathsf{sch}}(\text{Spec}(\mathbb{C}), Y)$ is a bijection and for any scheme $Y'$ with a natural transformation $\nu' : F \to \text{Hom}_{\mathsf{Sch}}(-, Y')$ there is a unique morphism $Y \to Y'$ factoring $\nu'$ through $\nu$.

2.6. Quiver GIT moduli functors. We recall the definition of a moduli functor for quiver GIT. Let $A$ be a $\mathbb{C}$-algebra of finite type. Suppose that $A$ is presented as a quiver with relations, and for $B \in \mathsf{R}$ define $A^B := A \otimes_{\mathbb{C}} B$. We recall that left $A$-modules correspond to quiver representations. For a stability condition $\theta$ and dimension vector $d$ the quiver GIT moduli functor is defined as in [28, Definition 4.1],

$$F_{s, d, \theta} : \mathsf{R} \to \mathsf{Sets}$$

$$B \mapsto \left\{ M \in A^B \text{-mod} \begin{array}{l}
\bullet M \text{ is a finitely generated and flat } B\text{-module.} \\
\bullet The A\text{-module } B/m \otimes_B M \text{ has dimension vector } d \\
\text{and is } \theta\text{-}(semi)\text{stable for all maximal ideals } m \text{ of } B.
\end{array} \right\} / \text{Equivalence}$$

where equivalence is defined by $S$-equivalence at fibres; $M$ and $N$ are equivalent if $B/m \otimes_B M$ and $B/m \otimes_B N$ are $S$-equivalent $A$-modules for all maximal ideal $m$ of $B$. By [28, Remark 4.4] this functor coincides with the definition of King in [20], and hence this functor has a coarse moduli space.

**Theorem 2.6.1** ([20, Proposition 5.2]). The scheme $M_{s, d, \theta}^{ss}$ is a coarse moduli space for $F_{s, d, \theta}$. If we restrict to stable representations then the functor has a fine moduli space.

**Theorem 2.6.2** ([20, Proposition 5.3]). Suppose $d$ is indivisible, and let $M_{d, \theta}^{ss}$ be the open subscheme of $M_{s, d, \theta}^{ss}$ corresponding to the stable points. Then $M_{d, \theta}^{ss}$ is a fine moduli space for $F_{s, d, \theta}$. We note that when $d$ is indivisible and all semistable points are stable the two functors coincide and $M_{d, \theta}^{ss} = M_{d, \theta}^{ss}$ is a fine moduli space. We will often just refer to either functor as $F_A$, recalling the choices of $\theta, d$ and semistability/stability only when necessary.
2.7. Geometric Moduli Functors. We define a similar functor for a scheme, $X$, arising in a projective morphism, $\pi : X \rightarrow \text{Spec}(R)$, of finite type schemes over $\mathbb{C}$.

We first introduce several pieces of notation which we will frequently use. Let $\rho : X \rightarrow \text{Spec}(\mathbb{C})$ denote the structure morphism. For $B \in \mathcal{R}$ we define $X^B := X \times_{\text{Spec}(\mathbb{C})} \text{Spec}(B)$ and consider the following pullback diagram

\[
\begin{array}{ccc}
X^B & \xrightarrow{\rho^X} & X \\
\rho^B & \downarrow & \downarrow \rho \\
\text{Spec}(B) & \xrightarrow{i_p} & \text{Spec}(\mathbb{C})
\end{array}
\]

which defines the morphisms $\rho^B$ and $\rho^X$ from the structure morphism $\rho : X \rightarrow \text{Spec}(\mathbb{C})$. We note that $X^B$ is also of finite type over $\mathbb{C}$, and has a projective morphism $\pi^B : X^B \rightarrow \text{Spec}(R \otimes_{\mathbb{C}} B)$, see [6, Remark 1.7]. Also if $X$ has a tilting bundle $T$ the following result, which is a particular case of the result [6, Proposition 2.9] of Buchweitz and Hille, defines a tilting bundle $T^B$ on $X^B$.

**Proposition 2.7.1** ([6, Proposition 2.9]). If $T$ is a tilting bundle on $X$ and $A = \text{End}_X(T)^{\text{op}}$ then $T^B := \mathbb{L}\rho^X^* T$ is a tilting bundle on $X^B$, and $A^B := \text{End}_{X^B}(T^B)^{\text{op}} = A \otimes_{\mathbb{C}} B$.

We introduce a further piece of notation. For any $B \in \mathcal{R}$ we let $\text{MaxSpec}(B)$ denote the closed points of $\text{Spec}(B)$, and any $p \in \text{MaxSpec}(B)$ there is a closed immersion $i_p : \text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(B)$ and a pullback diagram

\[
\begin{array}{ccc}
X & \xrightarrow{j_p} & X^B \\
\rho & \downarrow & \downarrow \rho^B \\
\text{Spec}(\mathbb{C}) & \xleftarrow{i_p} & \text{Spec}(B)
\end{array}
\]

which we later refer to as the diagram $(i_p/j_p)$.

We can now define the geometric moduli functor. We define $\mathcal{F}_X(\mathbb{C})$ to be the set of skyscraper sheaves of $X$ considered up to isomorphism, and define the moduli functor $\mathcal{F}_X : \mathcal{R} \rightarrow \text{Sets}$

\[
B \mapsto \left\{ \mathcal{E} \in D^b(X^B) \mid \begin{array}{ll}
\bullet & \mathcal{E} \in \mathcal{F}_X(\mathbb{C}) \text{ for all } p \in \text{MaxSpec}(B), \\
\bullet & \mathbb{R}\rho^B_* \mathcal{H}om_{X^B}(\mathbb{L}\rho^X_* \mathcal{F}, \mathcal{E}) \in \text{Perf}(B) \text{ for all } \mathcal{F} \in \text{Perf}(X).
\end{array} \right\}/\text{Equivalence}
\]

where the equivalence is defined by equivalence at fibres; $\mathcal{E}_1$ is equivalent to $\mathcal{E}_2$ if $\mathbb{L}j_p^* \mathcal{E}_1$ is equivalent to $\mathbb{L}j_p^* \mathcal{E}_2$ in $\mathcal{F}_X(\mathbb{C})$ for all $p \in \text{MaxSpec}(B)$. We later prove in Theorem 4.0.3 that $X$ is a fine moduli space for this functor, and in Lemma 4.0.2 iii) we show that the definition of fibrewise equivalence is the same as defining $\mathcal{E}_1$ and $\mathcal{E}_2$ to be equivalent if there exists a line bundle $L$ on $\text{Spec}(B)$ such that $\rho^B_* L \otimes_{\mathcal{X}^B} \mathcal{E}_1 \cong \mathcal{E}_2$.

**Remark 2.7.2.** It follows immediately from Lemmas 2.7.3 and 2.7.4, which we state below, that if $X$ has a tilting bundle $T$ the set $\mathcal{F}_X(B)$ is equivalent to the set

\[
\left\{ \mathcal{E} \in \text{Coh}(X^B) \mid \begin{array}{ll}
\bullet & \mathcal{E} \text{ is flat as a } B\text{-module.} \\
\bullet & j_p^\ast \mathcal{E} \in \mathcal{F}_X(\mathbb{C}) \text{ for all } p \in \text{MaxSpec}(B). \\
\bullet & \mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E}) \in \text{Perf}(B).
\end{array} \right\}.
\]

**Lemma 2.7.3.** Suppose $X$ has a tilting bundle $T$. Then for $\mathcal{E} \in D^b(X^B)$ the condition

- $\mathbb{R}\rho^B_* \mathcal{H}om_{X^B}(\mathbb{L}\rho^X_* \mathcal{F}, \mathcal{E}) \in \text{Perf}(B)$ for any $\mathcal{F} \in \text{Perf}(X)$

is equivalent to the condition

- $\mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E}) \in \text{Perf}(B)$.

**Proof.** Define $\mathcal{T}$ to be the subset of $\text{Perf}(X)$ consisting of objects $\mathcal{G}$ such that $\mathbb{R}\text{Hom}_{X^B}(\mathbb{L}\rho^B_* \mathcal{G}, \mathcal{E}) \in \text{Perf}(B)$. Then $\mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E}) \in \text{Perf}(B)$ if and only if $T \in \mathcal{T}$. By [24, Lemma 2.2] as $T$ is a tilting bundle and $\mathcal{T}$ is closed under shifts, triangles, and direct summands $\mathcal{T}$ contains $T$ if and only if $\mathcal{T} = \text{Perf}(X)$.

$\square$
Lemma 2.7.4 ([3, Lemma 4.3]). Let \( f : X \to Y \) be a morphism of finite type schemes over \( \mathbb{C} \), and for each closed point \( y \in Y \) let \( j_y \) denote the inclusion of the fibre \( f^{-1}(y) \). Suppose \( E \in D^b(X) \) is such that \( Lj_y^*E \) is a sheaf for all \( y \). Then \( E \) is a coherent sheaf on \( X \) which is flat over \( Y \).

Remark 2.7.5. In the definition of the moduli functor \( \mathcal{F}_X \) we could change the set \( \mathcal{F}_X(\mathbb{C}) \) of skyscraper sheaves up to isomorphism to, for example, the set of perverse point sheaves as defined by Bridgeland, [4, Section 3], to obtain a functor mirroring Bridgeland’s perverse point sheaf moduli functor. Indeed, the results of Section 3 and Theorem 4.0.1 do not rely on the fact that \( \mathcal{F}_X(\mathbb{C}) \) consists of skyscraper sheaves up to isomorphism, but Theorem 4.0.3 and our applications in Section 5 do.

3. Preliminary Lemmas

In this section we give a series of lemmas required to prove the main results in the next section.

3.1. Derived Base Change. We first recall the following property, which we will make use of several times.

Lemma 3.1.1. Let \( f : X \to Y \) be a quasi-compact, separated morphism of Noetherian schemes over \( \mathbb{C} \). Then if \( T \in \Perf(Y) \)

\[
\mathbb{L}f^*\mathbb{R}\text{Hom}_Y(T,E) \cong \mathbb{R}\text{Hom}_X(\mathbb{L}f^*T,\mathbb{L}f^*E).
\]

for any \( E \in D^b(Y) \).

Proof. We consider the two functors

\[
\begin{align*}
\text{Hom}_{D^b(X)}(\mathbb{L}f^*\mathbb{R}\text{Hom}_Y(T,E),-) : D^b(X) &\to \text{Sets}, \\
\text{Hom}_{D^b(X)}(\mathbb{R}\text{Hom}_X(\mathbb{L}f^*T,\mathbb{L}f^*E),-) : D^b(X) &\to \text{Sets}.
\end{align*}
\]

We will show these are naturally isomorphic, hence that \( \mathbb{L}f^*\mathbb{R}\text{Hom}_Y(T,E) \cong \mathbb{R}\text{Hom}_X(\mathbb{L}f^*T,\mathbb{L}f^*E) \) as they represent the same functor under the Yoneda embedding. This follows from the chain of natural isomorphisms

\[
\begin{align*}
\text{Hom}_{D^b(X)}(\mathbb{L}f^*\mathbb{R}\text{Hom}_Y(T,E),-) &\cong \text{Hom}_{D^b(Y)}(\mathbb{R}\text{Hom}_Y(T,E),\mathbb{R}f_*(-)) \quad \text{(adjunction)} \\
&\cong \text{Hom}_{D^b(Y)}(\mathbb{E},\mathbb{R}f_*(\mathbb{L}f^*T \otimes_Y^L (-))) \quad \text{(projection)} \\
&\cong \text{Hom}_{D^b(Y)}(\mathbb{R}f_*(\mathbb{L}f^*T \otimes_Y^L (-)),-) \quad \text{(adjunction)} \\
&\cong \text{Hom}_{D^b(X)}(\mathbb{R}\text{Hom}_X(\mathbb{L}f^*T,\mathbb{L}f^*E),-) \quad \text{(adjunction)} \\
&\quad \text{\( (\mathbb{L}f^*T \text{ perfect}) \)}
\end{align*}
\]

We then recall the following derived base change results.

Lemma 3.1.2. Let \( \pi : X \to \text{Spec}(R) \) be a projective morphism of finite type schemes over \( \mathbb{C} \), and let \( B,C \in \mathbb{R} \). Consider the following pullback diagram for a morphism \( u : \text{Spec}(B) \to \text{Spec}(C) \), where we use the notation of Section 2.7.

\[
\begin{array}{ccc}
X^B & \to & X^C \\
\downarrow \rho^B & & \downarrow \rho^C \\
\text{Spec}(B) & \to & \text{Spec}(C)
\end{array}
\]

Suppose \( E \in D^b(X^C) \). Then

\[
\mathbb{L}u^*\mathbb{R}\rho^C_*E \cong \mathbb{R}\rho^C_*\mathbb{L}u^*E.
\]

Suppose further that \( X \) has a tilting bundle \( T \) and define \( A = \text{End}_X(T)^{\text{op}} \). If \( \mathbb{R}\text{Hom}_{X^C}(T^C,E) \) is an \( A^C \)-module which is flat as a \( C \)-module then

\[
B \otimes_C \mathbb{R}\text{Hom}_{X^C}(T^C,E) \cong \mathbb{R}\text{Hom}_{X^B}(T^B,\mathbb{L}u^*E)
\]

as \( A^B \)-modules.
Proof. As $X^C$ is flat over $\text{Spec}(C)$, for any $x \in X^C$ and any $b \in \text{Spec}(B)$ such that $\rho^C(x) = u(b) = c$ we have that $\text{Tor}^{D^C}_i(\mathcal{O}_{B,b}, O_{X^C,x}) = 0$ for all $i \neq 0$. Hence $X^C$ and $\text{Spec}(B)$ are Tor independent over $\text{Spec}(C)$, and so the first result follows from [1, Lemma 35.16.3(Tag 08IB)].

The second result follows by applying the first result and the previous lemma:

$$B \otimes_C \mathbb{R}\text{Hom}_{X^C}(T^C, \mathcal{E}) \cong \text{Lu}^* \mathbb{R}\rho^C_\ast \mathbb{R}\text{Hom}_{X^C}(T^C, \mathcal{E})$$

$$\cong \mathbb{R}\rho^B_\ast \text{Lu}^* \mathbb{R}\text{Hom}_{X^C}(T^C, \mathcal{E}) \cong \mathbb{R}\rho^B_\ast \mathbb{R}\text{Hom}_{X^B}(\text{Lu}^* T^C, \text{Lu}^* \mathcal{E}) \quad \text{(Lemma 3.1.1)}$$

$$\cong \mathbb{R}\text{Hom}_{X^B}(T^B, \text{Lu}^* \mathcal{E}).$$

The following corollary is also useful.

Corollary 3.1.3. Let $X$ be a scheme of finite type over $\mathbb{C}$, and let $B \in \mathfrak{R}$. Suppose that $\mathcal{E} \in D^b(X^B)$ is such that $\mathbb{R}\rho^B_\ast \mathcal{E} \in D^b(\text{Spec}(B))$, and that for any $p \in \text{MaxSpec}(B)$ with diagram $(i_p/j_p)$ we have that $\mathbb{R}\rho^B_\ast j_p^- \mathcal{E}$ is a coherent sheaf on $\text{Spec}(\mathbb{C})$. Then $\mathbb{R}\rho^B_\ast \mathcal{E}$ is a flat coherent sheaf on $\text{Spec}(B)$.

Proof. By assumption $\mathbb{R}\rho^B_\ast \mathcal{E} \in D^b(\text{Spec}(B))$, hence by Lemma 2.7.4 if $Li^*_p \mathbb{R}\rho^B_\ast \mathcal{E}$ is a sheaf for all embeddings of closed points, $i_p : \text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(B)$, then $\mathbb{R}\rho^B_\ast \mathcal{E}$ is a flat coherent sheaf on $\text{Spec}(B)$. For any such $p \in \text{MaxSpec}(B)$ by Theorem 3.1.2 $Li^*_p \mathbb{R}\rho^B_\ast \mathcal{E} \cong \mathbb{R}\rho_\ast j_p^- \mathcal{E}$ which is a sheaf by the hypothesis. □

3.2. Natural Transformations. In this section let $\pi : X \rightarrow \text{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Suppose that $X$ has a tilting bundle $T$ and that $A = \text{End}_X(T)^{\text{op}}$ is presented as a quiver with relations. Choose some some stability condition $\theta$ and dimension vector $d$ in order to define $\mathcal{F}_A$. We aim to define a natural transformation, $\eta$, between the moduli functors $\mathcal{F}_X$ and $\mathcal{F}_A$ defined in sections 2.7 and 2.6. We define $\eta : \mathcal{F}_A \rightarrow \mathcal{F}_X$ by

$$\eta_B : \mathcal{F}_A(B) \rightarrow \mathcal{F}_X(B)$$

$$\mathcal{E} \mapsto \mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E})$$

for any $B \in \mathfrak{R}$, and we must check when this is well defined.

Lemma 3.2.1. Suppose $\eta_C$ is well defined. Then $\eta$ is well defined and is a natural transformation.

Proof. To prove that $\eta$ is well defined we must check the following for any $B \in \mathfrak{R}$ and any $\mathcal{E} \in \mathcal{F}_X(B)$:

i) $\mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E})$ is a $B$-module which is flat and finitely generated.

ii) For all maximal ideals $m$ of $B$ the $A$-module $B/m \otimes_B \mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E})$ is in $\mathcal{F}_A(C)$.

iii) If $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent in $\mathcal{F}_X(B)$ then $\mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E}_1)$ and $\mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E}_2)$ are equivalent in $\mathcal{F}_A(B)$.

Firstly we check i). It follows from the definition of $\mathcal{F}_X(B)$ that $\mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E}) \in \text{Perf}(B) \subset D^b(B)$. Then by Lemma 3.1.3 if $\mathbb{R}\rho_\ast j_p^\ast \mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E})$ is a sheaf on $\text{Spec}(\mathbb{C})$ for all $p \in \text{MaxSpec}(B)$ with diagrams $(i_p/j_p)$ then

$$\mathbb{R}\rho^B_\ast \mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E}) \cong \mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E})$$

is a flat and finitely generated $B$-module. For all $p \in \text{MaxSpec}(B)$ with diagrams $(i_p/j_p)$

$$\mathbb{R}\rho_\ast j_p^\ast \mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E}) \cong \mathbb{R}\text{Hom}_{X}(T, Lj_p^\ast \mathcal{E}),$$

by Lemma 3.1.1 and $\mathbb{R}\text{Hom}_{X}(T, Lj_p^\ast \mathcal{E}) \in \mathcal{F}_A(C)$ as $Lj_p^\ast \mathcal{E} \in \mathcal{F}_X(C)$ by the definition of $\mathcal{F}_X(B)$ and $\eta_C$ is well defined. Hence $\mathbb{R}\rho_\ast j_p^\ast \mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E}) \cong \mathbb{R}\text{Hom}_{X}(T, Lj_p^\ast \mathcal{E})$ is a coherent sheaf on $\text{Spec}(\mathbb{C})$, so we have proved i).

Secondly, to prove ii), we note for any maximal ideal $m$ of $B$ we have a corresponding closed point $p \in \text{MaxSpec}(B)$ and diagram $(i_p/j_p)$. Then we assume that $\mathcal{E} \in \mathcal{F}_X(B)$, and for each maximal ideal we have $B/m \otimes_B \mathbb{R}\text{Hom}_{X^B}(T^B, \mathcal{E}) \cong \mathbb{R}\text{Hom}_{X}(T, Lj_p^\ast \mathcal{E})$ by Lemma
3.1.2 as $\mathbb{R}\text{Hom}_X^a(T^B, \mathcal{E})$ is a flat $B$ module. Hence $B/m \otimes_B \mathbb{R}\text{Hom}_X^a(T^B, \mathcal{E}) \in \mathcal{F}_A(C)$ as $\eta_C$ is well defined and $Lj^*_p \mathcal{E} \in \mathcal{F}_X(C)$ by the definition of $\mathcal{F}_X(B)$.

Similarly, any maximal ideal $m$ of $B$ defines a closed point $p \in \text{MaxSpec}(B)$ and a diagram $(i_p/j_p)$. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be equivalent elements of $\mathcal{F}_X(B)$, then as $\mathbb{R}\text{Hom}_X^a(T^B, \mathcal{E}_i)$ are flat $B$-modules

$$B/m \otimes_B \mathbb{R}\text{Hom}_X^a(T^B, \mathcal{E}_i) \cong \mathbb{R}\text{Hom}_X(T, Lj^*_p \mathcal{E}_i)$$

by Lemma 3.1.2. As $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent in $\mathcal{F}_X(B)$ we know that $Lj^*_p \mathcal{E}_1 \cong Lj^*_p \mathcal{E}_2$, and hence the two $A$-modules $B/m \otimes_B \mathbb{R}\text{Hom}_X^a(T^B, \mathcal{E}_1)$ and $B/m \otimes_B \mathbb{R}\text{Hom}_X^a(T^B, \mathcal{E}_2)$ are $S$-equivalent as $\eta_C$ is well defined. This shows that $\eta_B(\mathcal{E}_1)$ and $\eta_B(\mathcal{E}_2)$ are equivalent in $\mathcal{F}_X(B)$ and proves part iii).

We now show that $\eta$ is a natural transformation. Suppose that $B, C \in \mathfrak{R}$ and $u : \text{Spec}(B) \to \text{Spec}(C)$, then we have the base change diagram

$$
\begin{array}{c}
\xymatrix{
X^B & X^C \\
\rho^B & \rho^C \\
\text{Spec}(B) & \text{Spec}(C) \\
& u
}
\end{array}
$$

and we consider the diagram

$$
\begin{array}{c}
\xymatrix{
\mathcal{F}_X(C) & \mathbb{R}\text{Hom}_X^c(T^C, -) & \mathcal{F}_A(C) \\
\mathcal{F}_X(B) & \mathbb{R}\text{Hom}_X^a(T^B, -) & \mathcal{F}_A(B) \\
\mathcal{F}_X^l & \mathcal{F}_A^l
}
\end{array}
$$

and to show that $\eta$ is natural we must check that this commutes. For $\mathcal{E} \in \mathcal{F}_X(C)$ as $\mathbb{R}\text{Hom}_X^c(T^C, \mathcal{E})$ is a flat $C$-module

$$B \otimes_C \mathbb{R}\text{Hom}_X^c(T^C, \mathcal{E}) \cong \mathbb{R}\text{Hom}_X^a(T^B, L\rho^* \mathcal{E})$$

as $A^B$-modules by Lemma 3.1.2. Hence $\eta$ is natural. \hfill \Box

**Lemma 3.2.2.** With the assumptions as in Lemma 3.2.1 if $\eta_C$ is also injective then $\eta_B$ is injective for all $B \in \mathfrak{R}$. If $\eta_C$ is also bijective with inverse $T \otimes^L_A (-)$ then $\eta_B$ is bijective for all $B \in \mathfrak{R}$.

**Proof.** Let $B \in \mathfrak{R}$. We first assume that $\eta_C$ is injective and show this implies that $\eta_B$ is injective. We suppose that $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{F}_X(B)$ and $\mathbb{R}\text{Hom}_X^a(T^B, \mathcal{E}_1)$ is equivalent to $\mathbb{R}\text{Hom}_X^a(T^B, \mathcal{E}_2)$, hence for all maximal ideals $m$ of $B$ the $A$-modules $B/m \otimes_B \mathbb{R}\text{Hom}_X^a(T^B, \mathcal{E}_1)$ and $B/m \otimes_B \mathbb{R}\text{Hom}_X^a(T^B, \mathcal{E}_2)$ are $S$-equivalent. We then note that each $\mathbb{R}\text{Hom}_X^a(T^B, \mathcal{E}_i)$ is a flat $B$-module, and that any maximal ideal $\mathfrak{m}$ of $B$ defines a closed point $p \in \text{MaxSpec}(B)$ by the injectivity of $\eta_C$, so $\mathfrak{m}$ is equivalent to $\mathcal{E}_2$ in $\mathcal{F}_X(B)$.

We now suppose that $\eta_C$ is bijective with inverse $T \otimes^L_A (-)$, and we show that $\eta_B$ is surjective. We consider $M \in \mathcal{F}_A(B)$ and note that as $T^B$ is a tilting bundle there exists some $\mathcal{E} \in \mathfrak{D}(X^B)$ such that $\mathbb{R}\text{Hom}_X^a(T^B, \mathcal{E}) \cong M$. Then if we can show that $\mathcal{E} \in \mathcal{F}_X(B)$ then we have proved that $\eta_B$ is surjective. We check first that $Lj^*_p \mathcal{E} \in \mathcal{F}_X(C)$ for any $p \in \text{MaxSpec}(B)$ and diagram $(i_p/j_p)$, and then we check that $\mathbb{R}\text{Hom}_X^a(Lj^*_p \mathcal{G}, \mathcal{E}) \in \text{Perf}(B)$ for any $\mathcal{G} \in \text{Perf}(X)$.

Firstly, for any maximal ideal $\mathfrak{m}$ of $B$ there is a corresponding closed point $p \in \text{MaxSpec}(B)$ and diagram $(i_p/j_p)$, and by Lemma 3.1.2

$$B/\mathfrak{m} \otimes_B M \cong \mathbb{R}\text{Hom}_X(T, Lj^*_p \mathcal{E})$$

as $M$ is flat over $B$. As $B/\mathfrak{m} \otimes_B M \cong \mathbb{R}\text{Hom}_X(T, Lj^*_p \mathcal{E}) \in \mathcal{F}_A(C)$ and $\eta_C$ is bijective with inverse $T \otimes^L_A (-)$ it follows that $Lj^*_p \mathcal{E} \cong T \otimes^L_A \mathbb{R}\text{Hom}_X(T, Lj^*_p \mathcal{E}) \in \mathcal{F}_X(C)$. 




The second condition holds by Lemma 2.7.3 as $M$ is a flat and finitely generated $B$-module so $\mathcal{R} \text{Hom}_X(T, \mathcal{E}) \cong M \in \text{Perf}(B)$. Hence $\mathcal{E} \in \mathcal{F}_X(B)$ and $\eta_B$ is surjective. □

4. Results

In this section we state our main result, which follows from the previous lemmas, and we also show that the moduli functor $\mathcal{F}_X$ is represented by $X$. We will find several applications of these results in the next section.

**Theorem 4.0.1.** Let $\pi : X \rightarrow \text{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Suppose $X$ is equipped with a tilting bundle $T$, define $A = \text{End}_X(T)^0$, and suppose that $A$ is presented as a quiver with relations. If there exists a stability condition $\theta$ and dimension vector $d$ defining the moduli functor $\mathcal{F}_A := \mathcal{F}_{A, \theta, d}$ such that the tilting equivalence

$$
\mathcal{R} \text{Hom}_X(T, -) \quad \text{restricts to a bijection between } \mathcal{F}_X(C) \text{ and } \mathcal{F}_A(C) \text{ then the map } \eta : \mathcal{F}_X \rightarrow \mathcal{F}_A \text{ defined by } \eta_B : \mathcal{E} \mapsto \mathcal{R} \text{Hom}_X(T, \mathcal{E}) \text{ is a natural isomorphism.}
$$

**Proof.** This follows from Lemmas 3.2.1 and 3.2.2. □

We now prove that the moduli functor $\mathcal{F}_X$ has $X$ as a fine moduli space. This closely follows the proof of the more general result [7, Theorem 2.10] of Calabrese and Groechenig, which we split into the following lemma and theorem in our setting.

**Lemma 4.0.2.** Let $\pi : X \rightarrow \text{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Suppose that $B \in \mathfrak{R}$ and that $\mathcal{E} \in \mathcal{F}_X(B)$. Then:

i) $\mathcal{E}$ is a coherent sheaf on $X^B$ that is flat over $\text{Spec}(B)$, and $\mathcal{R} \rho^B \mathcal{E}$ is a line bundle on $\text{Spec}(B)$.

ii) Let $\iota : Z \rightarrow X^B$ be the schematic support of $\mathcal{E}$. Then $\rho^B \circ \iota : Z \rightarrow \text{Spec}(B)$ is an isomorphism.

iii) If $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent objects in $\mathcal{F}_X(B)$ then there exists a line bundle $L$ on $\text{Spec}(B)$ such that $\mathcal{E}_1 \otimes \rho^B L \cong \mathcal{E}_2$.

**Proof.** Firstly, as $\mathcal{E} \in \mathcal{F}_X(B)$ it is a coherent sheaf on $X^B$ which is flat over $\text{Spec}(B)$ by Remark 2.7.2. Then $\mathcal{O}_X \in \text{Perf}(X)$ hence by the definition of $\mathcal{F}_X(B)$ we know that $\mathcal{R} \rho^B \mathcal{E} = \mathcal{R} \text{Hom}_X(\mathcal{O}_X, \mathcal{E}) \in \text{Perf}(B) \subset D^b(B)$. It follows that $\mathcal{R} \rho^B \mathcal{E}$ is a flat coherent sheaf on $\text{Spec}(B)$ by Corollary 3.1.3 as for all $p \in \text{MaxSpec}(B)$ with diagrams $(i_p/j_p)$ $\mathcal{R} \rho^B \mathcal{E} = \mathcal{C}$ as $j^*_p \mathcal{E}$ is a skyscraper sheaf. As $\mathcal{L} \iota^* \mathcal{R} \rho^B \mathcal{E} = \mathcal{C}$ the flat coherent sheaf $\mathcal{R} \rho^B \mathcal{E}$ has rank 1 and is a line bundle on $\text{Spec}(B)$.

To prove ii) let $Z$ denote the schematic support of $\mathcal{E}$ with closed immersion $\iota : Z \rightarrow X^B$, and let $\mathcal{G} := \iota^* \mathcal{E}$ denote the sheaf on $Z$ such that $\iota_* \mathcal{G} \cong \mathcal{E}$. We then have the diagram

```
\begin{array}{c}
Z \quad \iota \quad X^B \quad \rho^X \quad X \\
\psi \quad \quad \quad \rho^n \quad \rho \\
\text{Spec}(B) \quad \rightarrow \quad \text{Spec}(\mathbb{C})
\end{array}
```

where we define $\psi = \rho^B \circ \iota$. We recall that $X^B$ is projective over affine and $\rho^B$ can be factored into

$$
X^B \xrightarrow{\pi^B} \text{Spec}(R \otimes_{\mathbb{C}} B) \xrightarrow{\alpha^B} \text{Spec}(B)
$$

where $\pi^B$ is projective, and $\alpha^B$ is affine. We then see that as $\iota$ is a closed immersion, hence proper, $\pi^B \circ \iota$ is a proper map and it has affine fibres, as the fibres are all empty or points,
so is an affine morphism by [27, Theorem 8.5]. We then conclude that \( \psi = \alpha^B \circ (\pi^B \circ \iota) \) is an affine morphism as it is the composition of two affine morphisms, in particular \( \psi_* \) is exact.

We recall that \( \psi_* \mathcal{G} \) is defined as an \( \mathcal{O}_B \)-module via its definition as an \( \psi_* \mathcal{O}_Z \)-module by the map of rings

\[
\mathcal{O}_B \to \psi_* \mathcal{O}_Z \to \text{End}_{\psi_* \mathcal{O}_Z}(\psi_* \mathcal{G}) \to \text{End}_{\mathcal{O}_B}(\psi_* \mathcal{G}).
\]

Then as \( \psi_* \mathcal{G} \cong \mathbb{R}\rho^B \mathcal{E} \) is a line bundle this series of maps composes to an isomorphism, hence the first map is injective and the last surjective. We also note that the last map is the forgetful map so is also injective, thus is an isomorphism. Hence the middle map is surjective. Then as the support of \( \mathcal{G} \) is \( Z \), the middle map is also injective, hence is an isomorphism, so in fact the first map must also be an isomorphism. In particular this implies \( \mathcal{O}_B \cong \psi_* \mathcal{O}_Z \) and as \( \psi \) is affine it follows that \( Z \cong \text{Spec}(B) \) and \( \psi \) is an isomorphism.

To prove iii) we begin by noting that as \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are equivalent in \( \mathcal{F}_X(B) \) they share the same support \( \iota : Z \cong \text{Spec}(B) \to X^B \) and there exists \( \mathcal{G}_i := \iota^* \mathcal{E}_i \) such that \( i_* \mathcal{G}_i \cong \mathcal{E}_i \). Hence, using the isomorphism of part ii), we see that the \( \mathcal{G}_i \) are line bundles on \( Z \cong \text{Spec}(B) \), and we define a line bundle \( L = \psi_*(\mathcal{G}_1 \otimes \mathcal{G}_2) \) on \( \text{Spec}(B) \). Then

\[
\mathcal{E}_2 \cong i_* \mathcal{G}_2 \\
\cong i_* (\mathcal{G}_1 \otimes (\mathcal{G}_1 \otimes \mathcal{G}_2)) \\
\cong i_* (\mathcal{G}_1 \otimes \iota^* \rho^B \mathcal{L}) \\
\cong \mathcal{E}_1 \otimes \rho^B \mathcal{L}.
\]

(projection formula)

\( \square \)

**Theorem 4.0.3.** Let \( \pi : X \to \text{Spec}(R) \) be a projective morphism of finite type schemes over \( \mathbb{C} \). Then there is a natural isomorphism between the functor of points \( \text{Hom}_{\mathcal{O}_B}(-, X) \) and the moduli functor \( \mathcal{F}_X \). In particular \( X \) is a fine moduli space for \( \mathcal{F}_X \).

**Proof.** Consider

\[
\mu : \text{Hom}_{\mathcal{O}_B}(-, X) \to \mathcal{F}_X
\]

defined by

\[
\mu_C : (g : \text{Spec}(C) \to X) \mapsto (\Gamma_g)_* \mathcal{O}_C
\]

for \( C \in \mathfrak{A} \), where \( \Gamma_g : \text{Spec}(C) \to X^C \) is the graph of \( g \). The graph is a closed immersion as \( X \) is separated, and hence \( \Gamma_g \) is affine and \( (\Gamma_g)_* \) is exact.

We now show this is a well defined natural transformation. To show that it is well defined we consider a morphism \( g : \text{Spec}(C) \to X \) and check that \( (\Gamma_g)_* \mathcal{O}_C \in \mathcal{F}_X(C) \).

Firstly, as \( \Gamma_g \) is a closed immersion it is proper, hence \( (\Gamma_g)_* \mathcal{O}_C \) is a coherent sheaf by [1, Lemma 29.17.2 (Tag 0205)]. Further, as \( \Gamma_g \) is a closed immersion and \( \mathcal{O}_C \) is flat over \( \text{Spec}(C) \) it follows by considering stalks that \( (\Gamma_g)_* \mathcal{O}_C \) is also flat over \( \text{Spec}(C) \). Then as \( \Gamma_g \) is affine \( j^*_p(\Gamma_g)_* \mathcal{O}_C \cong (\Gamma_{g_{(p)}})_* j^*_p \mathcal{O}_C \) for all \( p \in \text{MaxSpec}(C) \) with diagrams (\( i_p/j_p \)) by [1, Lemma 29.5.1 (Tag 02KE)], hence

\[
Lj^*_p(\Gamma_g)_* \mathcal{O}_C \cong j^*_p(\Gamma_g)_* \mathcal{O}_C \cong (\Gamma_{g_{(p)}})_* j^*_p \mathcal{O}_C \cong \mathcal{O}_{g_{(p)}}.
\]

Secondly, for any \( \mathcal{F} \in \mathcal{F}_X \) both \( Lg^* \mathcal{F} \) and its derived dual \( \mathbb{R}\text{Hom}_C(Lg^* \mathcal{F}, \mathcal{O}_C) \) are in \( \mathcal{F}(\mathcal{O}_C) \) so \( \mathbb{R}\text{Hom}_C(Lg^* \mathcal{F}, \mathcal{O}_C) \cong \mathbb{R}\text{Hom}_C(Lg^* \mathcal{F}, \mathcal{O}_C) \in \mathcal{F}(\mathcal{O}_C) \). Hence \( \mu_C \) is well defined as \( (\Gamma_g)_* \mathcal{O}_{\text{Spec}(C)} \in \mathcal{F}_A(C) \) for any \( g \in \text{Hom}_{\mathcal{O}_B}(\text{Spec}(C), X) \). It is natural as if \( B \in \mathfrak{A} \) with a morphism \( u : \text{Spec}(B) \to \text{Spec}(C) \) and \( g : \text{Spec}(C) \to X \in \text{Hom}_{\mathcal{O}_B}(\text{Spec}(C), X) \) we have the diagram

\[
\begin{array}{ccc}
X^B & \xrightarrow{v} & X^C \\
\downarrow \Gamma_{g \circ u} & & \downarrow \Gamma_g \\
\text{Spec}(B) & \xrightarrow{u} & \text{Spec}(C)
\end{array}
\]

and check that (projection formula)
where \( g = \rho^X \circ \Gamma_g, g \circ u = \rho^X \circ \psi \circ \Gamma_{g_{OU}} \) and the squares can be seen to be pullback squares using the universal property of pullback squares and the fact that \( \rho_B \circ \Gamma_{g_{OU}} \) is the identity. As above, as \( \Gamma_g \) and \( \Gamma_{g_{OU}} \) are closed immersions

\[
(\Gamma_{g_{OU}})_* u^* \mathcal{E} \cong v^*(\Gamma_g)_* \mathcal{E}
\]

for any \( \mathcal{E} \in \text{Coh}(\text{Spec}(C)) \) by [1, Lemma 29.5.1 (Tag 02KE)]. Hence

\[
\mu_B(g \circ u) \cong \Gamma_{(g_{OU})_* \mathcal{O}_B} \cong \Gamma_{(g_{OU})_* u^* \mathcal{O}_C} \cong v^*(\Gamma_g)_* \mathcal{O}_C \cong v^* \mu_B(g).
\]

To show it is a natural isomorphism we need to check that \( \mu_B \) is bijective for all \( B \in \mathcal{R} \). We do this now by constructing an inverse \( \nu_B \). For \( B \in \mathcal{R} \), given \( \mathcal{E} \in \mathcal{F}_X(B) \) we consider its support \( Z \), and we then have the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\iota} & X^B \\
\downarrow \psi & & \downarrow \rho^X \\
\text{Spec}(B) & \xrightarrow{\nu_B} & \text{Spec}(C)
\end{array}
\]

where we define \( \psi = \rho_B \circ \iota \). We recall that \( \psi \) is an isomorphism from Lemma 4.0.2 ii), and we then consider the map \( \rho^X \circ \iota \circ \psi^{-1} : \text{Spec}(B) \to \text{Hom}_{\mathcal{O}_B}(\text{Spec}(B), X) \), and our inverse is defined by sending \( \mathcal{E} \in \mathcal{F}_X(B) \) to this element of \( \text{Hom}_{\mathcal{O}_B}(\text{Spec}(B), X) \):

\[
\nu_B : \mathcal{F}_X(B) \to \text{Hom}_{\mathcal{O}_B}(\text{Spec}(B), X)
\]

\[
\mathcal{E} \mapsto (\rho_X \circ \iota \circ \psi^{-1} : \text{Spec}(B) \to X).
\]

Finally we note that this is an inverse, as

\[
\nu_B \circ \mu_B(g) = \nu_B(\Gamma_g)_* \mathcal{O}_B) = g
\]

and

\[
\mu_B \circ \nu_B(\mathcal{E}) = \mu_B((\rho_X \circ \iota \circ \psi^{-1}) : \text{Spec}(B) \to X^B) = \Gamma_{(\rho_X \circ \iota \circ \psi^{-1})_*}(\mathcal{O}_B)
\]

where we note that \( \Gamma_{(\rho_X \circ \iota \circ \psi^{-1})_*}(\mathcal{O}_B) \) is equivalent to \( \mathcal{E} \) in \( \mathcal{F}_X(B) \) as they agree at all fibres. Hence \( \text{Hom}_{\mathcal{O}_B}(\rho_X, X) \) is naturally isomorphic \( \mathcal{F}_X \). \( \square \)

5. Applications

Let \( \pi : X \to \text{Spec}(R) \) be a projective morphism of finite type schemes over \( \mathbb{C} \), suppose \( X \) has a tilting bundle \( T \), and suppose that \( A = \text{End}_X(T)^{\text{op}} \) is presented as a quiver with relations. In this section we will introduce a dimension vector \( d_T \) and stability condition \( \theta_T \) defined by a decomposition of the tilting bundle and give general conditions for the map \( \eta : \mathcal{F}_X \to \mathcal{F}_A \) introduced in the previous sections to be a natural isomorphism for this stability condition and dimension vector. We will then use these general conditions to produce the applications outlined in the introduction.

5.1. Dimension Vectors and Stability. We introduce a certain dimension vector and stability condition defined from a decomposition of a tilting bundle and then, using Theorem 4.0.1, we give criterion for \( \eta \) to be a natural isomorphism with respect to this stability condition and dimension vector. In order to do this we make the following assumption on \( T \), a tilting bundle on a scheme \( X \).

Assumption 5.1.1. The tilting bundle \( T \) has a decomposition into non-isomorphic indecomposables \( T = \bigoplus_{i=0}^n E_i \) such that there is a unique indecomposable, \( E_0 \), isomorphic to \( \mathcal{O}_X \).

We then consider a presentation of \( A = \text{End}_X(T)^{\text{op}} \) as the path algebra of a quiver with relations such that each indecomposable \( E_i \) corresponds to a vertex \( i \) of the quiver, as in Section 2.4. In particular the 0 vertex in the quiver corresponds to the summand \( \mathcal{O}_X \).

Definitions 5.1.2. Suppose \( T \) is a tilting bundle \( T \) with decomposition \( T = \bigoplus_{i=0}^n E_i \).
i) The dimension vector $d_T$ is defined by

$$d_T(i) = \text{rk } E_i.$$ 

In particular $d_T(0) = 1$ as $E_0$ is assumed to be isomorphic to $\mathcal{O}_X$.

ii) The stability condition $\theta_T$ is defined by

$$\theta_T(i) = \begin{cases} - \sum_{i \neq 0} \text{rk } E_i & \text{if } i = 0; \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 5.1.3. The stability condition $\theta_T$ has the following properties:

i) Let $P_0 := \mathbb{R}\text{Hom}_X(T, \mathcal{O}_X)$ and $M$ be an $A$-module with dimension vector $d_T$. Then $\text{Hom}_A(P_0, M)$ is one dimensional, and $M$ is $\theta_T$-stable if and only if there is a surjection $P_0 \to M \to 0$.

ii) The stability $\theta_T$ is generic for $A$-modules of dimension $d_T$.

Proof. The $A$-module $P_0$ is the projective module consisting of paths in the quiver starting at the vertex 0. For any representation $M$ with dimension vector $d_T$ a homomorphism from $P_0$ to $M$ is determined by the image of the trivial path $e_0 \in P_0$ in the vector space $C \subset M$ at vertex 0, which we denote by $1_0$. This is as any path $p$ starting at 0 must be sent to the evaluation in $M$ of the linear map corresponding to $p$ on the element $1_0$. Hence $\text{Hom}_A(P_0, M) \cong C$, and any nonzero element of $\text{Hom}_A(P_0, M)$ is surjective precisely when the linear maps in $M$ corresponding to paths starting at 0 form a surjection from the vector space at the zero vertex onto $M$. By the definition of $\theta_T$ the module $M$ is $\theta_T$-semistable if and only if any proper submodule $N$ has $d_N(0) = 0$, and this property is equivalent to the linear maps in $M$ corresponding to paths starting at 0 forming a surjection. This proves part i).

We now prove ii). It is clear by the definitions of $\theta_T$ and $d_T$ that any dimension $d_T$ module $M$ can have no proper submodules $N \subset M$ such that $\theta_T(N) = 0$ as if $N$ is a nontrivial submodule, either $d_N(0) = 0$ and $\theta_T(N) > 0$, or $d_N(0) = 1$ and $N = M$. □

We now give conditions for $\eta : \mathcal{F}_X \to \mathcal{F}_A$ to be a natural isomorphism for this stability condition and dimension vector. We note that there is an abelian category $A$ corresponding to the abelian category $A$-mod under the tilting equivalence between $D^b(X)$ and $D^b(A)$ such that $T$ is a projective generator of $A$. Then $\mathbb{R}\text{Hom}_X(T, -)$ and $T \otimes_A^\mathbb{L}(-)$ define an equivalence of abelian categories between $A$ and $A$-mod. Our conditions are defined on this category $A$.

Lemma 5.1.4. Take the dimension vector $d_T$ and stability condition $\theta_T$ as above. Suppose the following conditions hold:

i) The structure sheaf $\mathcal{O}_X$ is in $A$, and for all closed points $x \in X$ the skyscraper sheaf $\mathcal{O}_x$ is in $A$.

ii) For all closed points $x \in X$ there are surjections $\mathcal{O}_X \to \mathcal{O}_x \to 0$ in $A$, and $\text{Hom}_A(\mathcal{O}_X, \mathcal{O}_x) = 0$.

Then $\eta$ is a well defined natural transformation and $\eta_B$ is injective for all $B \in \mathfrak{R}$. Suppose further that the following condition also holds:

iii) The set

$$S := \left\{ E \in A \mid \begin{array}{l} \mathbb{R}\text{Hom}_X(T, E) \text{ has dimension vector } d_T, \\ \text{Hom}_A(E, \mathcal{O}_x) = 0 \text{ for all closed points } x \in X. \end{array} \right\}$$

is empty.

Then $\eta$ is a natural isomorphism.

Proof. We first assume that conditions i) and ii) hold and prove that $\eta_C$ is well defined and injective. Then it follows from Lemmas 3.2.1 and 3.2.2 that $\eta$ is a natural transformation and $\eta_B$ is injective for all $B \in \mathfrak{R}$.

Any element of $\mathcal{F}_X(\mathbb{C})$ is a skyscraper sheaf on $X$ up to isomorphism. For any closed point $x \in X$ the $A$-module $\mathbb{R}\text{Hom}_X(T, \mathcal{O}_x)$ has dimension vector $d_T$, hence the map $\eta_C$ is well defined if and only if all $\mathbb{R}\text{Hom}_X(T, \mathcal{O}_x)$ are $\theta_T$-semistable $A$-modules. By condition i) they are $A$-modules. By considering the surjections of condition ii), $\mathcal{O}_X \to \mathcal{O}_x \to 0$
in \( \mathcal{A} \), and applying the abelian equivalence \( \mathbb{R}\text{Hom}_X(T, -) \) we see that all \( \mathbb{R}\text{Hom}_X(T, \mathcal{O}_x) \) are \( \theta_T \)-stable by Lemma 5.1.3 i). Hence \( \eta_C \) is well defined.

By Lemma 5.1.3 ii) \( \theta_T \) is generic so \( \mathbb{R}\text{Hom}_X(T, \mathcal{O}_x) \) and \( \mathbb{R}\text{Hom}_X(T, \mathcal{O}_y) \) are \( S \)-equivalent if and only if they are isomorphic, then as \( \mathbb{R}\text{Hom}_X(T, -) \) is an equivalence \( \mathbb{R}\text{Hom}_X(T, \mathcal{O}_x) \cong \mathbb{R}\text{Hom}_X(T, \mathcal{O}_y) \) implies \( \mathcal{O}_x \) and \( \mathcal{O}_y \) are isomorphic, so \( \eta_C \) is injective.

We now also assume that condition iii) holds and prove that \( \eta_C \) is also surjective with inverse \( T \circ \eta_A^{-1}(-) \). It then follows from Theorem 4.0.1 that \( \eta \) is a natural isomorphism. Take an \( A \)-module, \( M \), with dimension vector \( d_T \) and which is \( \theta_T \)-stable. As \( M \) is \( \theta_T \)-stable by Lemma 5.1.3 ii) there is a surjection

\[
P_0 \rightarrow M \rightarrow 0
\]

which under the abelian equivalence gives an exact sequence in \( \mathcal{A} \)

\[
\mathcal{O}_X \rightarrow \mathcal{E} \rightarrow 0
\]

where \( \mathcal{E} \equiv M \otimes^\mathbb{L}_A T \in D^b(X) \). Then by condition iii) there must be some closed point \( x \in X \) such that \( \text{Hom}_A(\mathcal{E}, \mathcal{O}_x) \neq 0 \). We then apply \( \text{Hom}_A(-, \mathcal{O}_x) \) to the surjection \( \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow 0 \) to obtain an injection

\[
0 \rightarrow \text{Hom}_A(\mathcal{E}, \mathcal{O}_x) \rightarrow \text{Hom}_A(\mathcal{O}_X, \mathcal{O}_x) = \mathbb{C}
\]

and hence the surjection \( \mathcal{O}_X \rightarrow \mathcal{O}_x \rightarrow 0 \) factors through \( \mathcal{E} \), and there is a surjection \( \mathcal{E} \rightarrow \mathcal{O}_x \rightarrow 0 \). We then apply the abelian equivalence functor \( \mathbb{R}\text{Hom}_X(T, -) \) to obtain a surjection of finite dimensional \( A \)-modules

\[
M \rightarrow \mathbb{R}\text{Hom}_X(T, \mathcal{O}_x) \rightarrow 0
\]

and by comparing dimension vectors we see that the map is an isomorphism, hence that \( \mathbb{R}\text{Hom}_X(T, \mathcal{O}_x) \cong M \) and \( T \otimes^\mathbb{L}_A M \cong \mathcal{O}_x \). \( \square \)

**Corollary 5.1.5.** Let \( \pi : X \rightarrow \text{Spec}(R) \) be a projective morphism of finite type schemes over \( \mathbb{C} \). Let \( T \) be a tilting bundle on \( X \) which defines an equivalence of an abelian category \( \mathcal{A} \) with \( A \)-mod, where \( A = \text{End}_X(T)^{op} \). Choose the stability condition \( \theta_T \) and dimension vector \( d_T \) as above, define \( F_A = F^A_{ss, d_T, \theta_T} \), and assume that conditions i) and ii) of Lemma 5.1.4 hold for \( \mathcal{A} \). Then:

i) The map \( \eta : F_X \rightarrow F_A \) defined in Section 3.2 is a natural transformation and induces a morphism \( f : X \rightarrow M^A_{ss, \theta_T} \) between \( X \) and the quiver GIT quotient of \( A \) for stability condition \( \theta_T \) and dimension vector \( d_T \). This morphism is a monomorphism in the sense of \([1, \text{Definition 25.23.1 (Tag 01L2)}]\).

ii) If condition iii) of Lemma 5.1.4 also holds for \( \mathcal{A} \) then the morphism \( f \) is an isomorphism.

**Proof.** We note that \( M^A_{ss, \theta_T} = M^A_{ss, \theta_T} \) as \( \theta_T \) is generic by Lemma 5.1.3 ii) and that \( M^A_{ss, \theta_T} \) is a fine moduli space for \( F_A \) by Theorem 2.6.2 as the dimension vector \( d_T \) is indecomposable. The map \( \eta : F_X \rightarrow F_A \) is a natural transformation as conditions i) and ii) of Lemma 5.1.4 hold for \( \mathcal{A} \). It then follows that there is a corresponding morphism \( f : X \rightarrow M^A_{ss, \theta_T} \) as \( F_A \) is represented by \( M^A_{ss, \theta_T} \) and \( F_X \) is represented by \( X \) by Theorem 4.0.3. For all \( B \in \mathfrak{B} \) the map \( \eta_B \) is injective by Lemma 5.1.4, hence the corresponding morphism, \( f \), is a monomorphism.

If condition iii) of Lemma 5.1.4 also holds for \( \mathcal{A} \) then \( \eta \) is actually a natural isomorphism by Lemma 5.1.4. Hence \( f \) is an isomorphism, proving ii). \( \square \)

**Remark 5.1.6.** While we make no further use of the monomorphism property we note that it can be a useful notion as proper monomorphisms are exactly closed immersions, \([1, \text{Lemma 40.7.2 (Tag 04XY)}]\), and étale monomorphisms are exactly open immersions, \([1, \text{Theorem 40.14.1 (Tag 025G)}]\).

### 5.2. One Dimensional Fibres

To apply Lemma 5.1.4 and Corollary 5.1.5 we need a class of varieties with tilting bundles such that we understand the abelian categories \( \mathcal{A} \). Such a class was introduced in Theorem 1.3.1; if \( \pi : X \rightarrow \text{Spec}(R) \) is a projective morphism of Noetherian schemes such that \( \mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R \) and the fibres of \( \pi \) have dimension \( \leq 1 \) then there exist tilting bundles \( T_i \) on \( X \) such that the abelian category \( \mathcal{A} \) is \( \mathbb{R}\pi\text{-Per}(X/R) \), defined as follows.
Definition 5.2.1 ([29, Section 3]). Let $\pi : X \to \text{Spec}(R)$ be a projective morphism of Noetherian schemes such that $\mathbb{R}\pi_* \mathcal{O}_X \cong \mathcal{O}_R$ and $\pi$ has fibres of dimension $\leq 1$. Define $\mathcal{C}$ to be the abelian subcategory of $\text{Coh} \ X$ consisting of $\mathcal{F} \in \text{Coh} \ X$ such that $\mathbb{R}\pi_* \mathcal{F} \cong 0$. For $i = 0, 1$ the abelian category $^{-i}\text{Per}(X/R)$ is defined to contain $\mathcal{E} \in D^b(X)$ which satisfy the following conditions:

i) The only non-vanishing cohomology of $\mathcal{E}$ lies in degrees $-1$ and $0$.

ii) $\pi_* \mathcal{H}^{-1}(\mathcal{E}) = 0$ and $\mathbb{R}^1 \pi_* \mathcal{H}^0(\mathcal{E}) = 0$, where $\mathcal{H}^j$ denotes taking the $j$th cohomology sheaf.

iii) For $i = 0$, $\text{Hom}(X, \mathcal{E}) = 0$ for all $\mathcal{C} \in \mathcal{C}$.

iv) For $i = 1$, $\text{Hom}(X, \mathcal{H}^0(\mathcal{E}), C) = 0$ for all $\mathcal{C} \in \mathcal{C}$.

We note that the abelian categories $^{-i}\text{Per}(X/R)$ are hearts of $t$-structures on $D^b(X)$ so short exact sequences in $^{-i}\text{Per}(X/R)$ correspond to triangles in $D^b(X)$ whose vertices are in $^{-i}\text{Per}(X/R)$.

We note the following property of morphisms in $^{-i}\text{Per}(X/R)$.

Lemma 5.2.2. Let $\pi : X \to \text{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$, and suppose that $X$ has a tilting bundle $T$ that induces an abelian equivalence between $0\text{Per}(X/R)$ and $A\text{-mod}$ where $A = \text{End}_X(T)^{\text{op}}$. Then $\text{Hom}_{\text{Per}(X/R)}(\mathcal{E}_1, \mathcal{E}_2) \cong \text{Hom}_{D^b(X)}(\mathcal{E}_1, \mathcal{E}_2)$ for $\mathcal{E}_1, \mathcal{E}_2 \in 0\text{Per}(X/R)$.

Proof. Let $\mathcal{E}_1, \mathcal{E}_2 \in 0\text{Per}(X/R)$. Then $M_i = \mathbb{R}\text{Hom}(X, \mathcal{E}_i)$ is an $A$-module for $i = 1, 2$ and $\text{Hom}_A(\mathcal{E}_1, \mathcal{E}_2) \cong \text{Hom}_A(M_1, M_2) \cong \text{Hom}_{D^b(A)}(M_1, M_2) \cong \text{Hom}_{D^b(X)}(\mathcal{E}_1, \mathcal{E}_2)$ by the abelian and then derived equivalence. \hfill \square

Any projective generator of the abelian category $^{-1}\text{Per}(X/R)$ gives a tilting bundle $T_i$ with the properties defined in Theorem 1.3.1, and we can assume that such a tilting bundle contains $\mathcal{O}_X$ as a summand by the following proposition.

Proposition 5.2.3 ([29, Proposition 3.2.7]). Define $\mathfrak{M}_X$ to be the category of vector bundles $\mathcal{M}$ on $X$ which are generated by global sections and such that $\mathcal{H}^1(X, \mathcal{M}^\vee) = 0$, and define $\mathfrak{M}_X^\nu := \{\mathcal{M}^\nu : \mathcal{M} \in \mathfrak{M}_X\}$. The projective generators of $^{-1}\text{Per}(X/R)$ are the $\mathcal{M} \in \mathfrak{M}_X$ such that $\dim_{\mathbb{K}} \mathcal{M}$ is ample and $\mathcal{O}_X$ is a summand of $\mathcal{M}^{\oplus a}$ for some $a \in \mathbb{N}$. The projective generators of $0\text{Per}(X/R)$ are the elements of $\mathfrak{M}_X^\nu$ which are dual to projective generators of $^{-1}\text{Per}(X/R)$.

Hence we let $T_i$ be a projective generator of $0\text{Per}(X/R)$ with a decomposition as required in Assumption 5.1.1. Then the algebra $A_i = \text{End}_X(T_i)^{\text{op}}$ can be presented as a quiver with relations with vertex $0$ corresponding to $\mathcal{O}_X$ and the stability condition $\theta_{T_i}$ and dimension vector $d_{T_i}$ are well defined.

We now check that the conditions of Lemma 5.1.4 hold for $0\text{Per}(X/R)$.

Theorem 5.2.4. Let $\pi : X \to \text{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$ such that $\pi$ has fibres of dimension $\leq 1$ and $\mathbb{R}\pi_* \mathcal{O}_X \cong \mathcal{O}_R$. Then the abelian category $0\text{Per}(X/R)$ satisfies conditions i), ii) and iii) of Lemma 5.1.4.

Proof. We begin by checking $\mathcal{A}$ satisfies conditions i) and ii) of Lemma 5.1.4. All skyscraper sheaves $\mathcal{O}_x$, and the structure sheaf $\mathcal{O}_X$ are in $\mathcal{A}$ as they satisfy the conditions of Definition 5.2.1. Then, for any $x \in X$, the short exact sequence of sheaves $0 \to I \to \mathcal{O}_X \to \mathcal{O}_x \to 0$ corresponds to a triangle in $D^b(X)$, and the ideal sheaf $I$ is also in $\mathcal{A}$ as $\mathbb{R}\pi_* I = 0$ due to the exact sequence $0 \to \pi_* I \to \pi_* \mathcal{O}_X \to \pi_* \mathcal{O}_x \to \mathbb{R}^1 \pi_* I \to 0$ where $\pi_* \mathcal{O}_x \cong \mathcal{O}_R$ and the third arrow is a surjection. Hence the map $\mathcal{O}_X \to \mathcal{O}_x \to 0$ is in fact a surjection in $\mathcal{A}$. We then note, for all $x \in X$, that $\text{Hom}(\mathcal{O}_X, \mathcal{O}_x) \cong \text{Hom}_{D^b(X)}(\mathcal{O}_X, \mathcal{O}_x) \cong \text{Hom}(X, \mathcal{O}_x) \cong \mathbb{C}$, hence $\text{Hom}(\mathcal{O}_X, \mathcal{O}_x) \cong \mathbb{C}$ corresponding to the map of sheaves $\mathcal{O}_X \to \mathcal{O}_x \to 0$ which is surjective in $\mathcal{A}$.

To check condition iii) suppose $S$ is not empty and so there exists some $\mathcal{E} \in S$. In particular, $M \cong \text{Hom}_{D^b(X)}(T_0, \mathcal{E})$ has dimension vector $d_{T_0}$ so $\mathbb{R}\pi_* \mathcal{E} \cong \mathcal{O}_y$ for some $y \in \text{Spec}(R)$. As $\mathcal{E} \in \mathcal{A}$ there is a short exact sequence in $\mathcal{A}$

$$0 \to \mathcal{H}^{-1}(\mathcal{E})[1] \to \mathcal{E} \to \mathcal{H}^0(\mathcal{E}) \to 0$$
where \([1]\) is the shift in \(D^b(X)\). Hence, for all closed points \(x \in X\), there is an injection
\[
0 \to \text{Hom}_A(\mathcal{H}^0(\mathcal{E}), O_x) \to \text{Hom}_A(\mathcal{E}, O_x).
\]

Then, from the assumption \(\text{Hom}_A(\mathcal{H}^0(\mathcal{E}), O_x) = 0\), it follows that \(\text{Hom}_A(\mathcal{H}^0(\mathcal{E}), O_x) = \text{Hom}_{D^b(X)}(\mathcal{H}^0(\mathcal{E}), O_x) = 0\) for all \(x \in X\) and hence \(\mathcal{H}^0(\mathcal{E}) = 0\) as a nonzero coherent sheaf must be supported somewhere. So \(\mathcal{E} = \mathcal{H}^{-1}(\mathcal{E})[1]\) and we note in particular that \(\pi_* \mathcal{H}^{-1}(\mathcal{E}) = 0\) and \(\mathbb{R}^1 \pi_* \mathcal{H}^{-1}(\mathcal{E}) = \mathcal{O}_y\). By [29, Lemma 3.1.3] there is an injection of sheaves
\[
0 \to \mathcal{H}^{-1}(\mathcal{E}) \to \mathcal{H}^{-1}(\pi^* \mathcal{O}_y)
\]
and hence \(\mathcal{H}^{-1}(\mathcal{E})\) is set-theoretically supported on \(\pi^{-1}(y)\). In particular \(y\) corresponds to a maximal ideal \(m_y\) of \(R\) and we consider the completion \(R \to \hat{R} = \varprojlim(R/m^n_y)\). This produces the following pullback diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{j} & X \\
\downarrow{\pi} & & \downarrow{\pi} \\
\text{Spec}(\hat{R}) & \xrightarrow{i} & \text{Spec}(R)
\end{array}
\]

where \(Y\) is the formal fibre \(Y := \varprojlim(\text{Spec}(R/m^n_y) \times_{\text{Spec}(R)} X)\), the morphisms \(i\) and \(j\) are both flat and affine, and the morphism \(\pi\) is projective. Then we have the following isomorphism, where we recall that the morphisms \(i\) and \(j\) are both flat and affine so we need not derive them,
\[
\begin{align*}
\mathbb{R}\text{Hom}_X(T_0, j_* j^* \mathcal{E}) & \cong i_* \mathbb{R}\text{Hom}_Y(j^* T_0, j^* \mathcal{E}) & (j_*, j^* \text{ adjoint pair}) \\
& \cong i_* \hat{\pi}_* j^* j^* \mathbb{R}\text{Hom}_X(T_0, \mathcal{E}) & (\text{Lemma 3.1.1}) \\
& \cong i_* \hat{\pi}^* \mathbb{R}\text{Hom}_X(T_0, \mathcal{E}) & (\text{Flat base change})
\end{align*}
\]

Then as \(M \cong \mathbb{R}\text{Hom}_X(T_0, \mathcal{E})\) is finite dimensional and supported on \(m_y\) it follows that completion in \(m_y\) followed by restriction of scalars acts as the identity, see [13, Theorem 2.13] and [22, Lemma 2.5], hence \(i_* i^* M := \hat{R} \otimes_R M \cong M\). We deduce that \(\mathbb{R}\text{Hom}_X(T_0, j_* j^* \mathcal{E}) \cong \mathbb{R}\text{Hom}_X(T_0, \mathcal{E})\), and so \(\mathcal{E} \cong j_* j^* \mathcal{E}\) as \(T_0\) is a tilting bundle. Finally we can define \(\mathcal{G} := j^* \mathcal{H}^{-1}(\mathcal{E})\) with the property that \(j_* \mathcal{G}[1] \cong \mathcal{E}\).

We now note that by Lemma 5.2.3 there exists \(P \in \mathfrak{W}_X\) such that \(T_0 = P^\vee\). We then note that as \(P\) is a vector bundle generated by global sections so is \(j^* P\), hence as \(\hat{R}\) is a complete local ring there exists a short exact sequence
\[
0 \to \mathcal{O}^{\oplus d-1}_Y \to j^* P \to \wedge^d j^* P \to 0
\]
by [29, Lemma 3.5.1], where \(d = \text{rk} P = \text{rk} j^* P\). Also, as \(P \in \mathfrak{W}_X\), the line bundle \(\wedge^d P\) is ample and so the line bundle \(\mathcal{L} := \wedge^d j^* P \cong j^* \wedge^d P\) is also ample as \(j\) is affine. Then by Serre vanishing, [14, III Theorem 5.2], there exists some \(N > 0\) such that \(\text{Hom}_{D^b(Y)}(\mathcal{L}^{\otimes -N}, \mathcal{G}[1]) \cong \text{Ext}_{Y}(\mathcal{O}_Y, \mathcal{L}^{\otimes N} \otimes \mathcal{G}) = 0\). As \(j^* P\) is generated by global sections the vector bundle \(j^* P^{\oplus N}\) is also generated by global sections so again there exists a short exact sequence
\[
0 \to \mathcal{O}^{\oplus N d-1}_Y \to (j^* P)^{\oplus N} \to \mathcal{L}^{\otimes N} \to 0
\]
by [29, Lemma 3.5.1]. Dualising this we obtain the short exact sequence
\[
0 \to \mathcal{L}^{\otimes -N} \to (j^* T_0)^{\oplus N} \to \mathcal{O}^{\oplus N d-1}_Y \to 0,
\]
where \((j^* P)^\vee = j^*(P^\vee)\) by Lemma 3.1.1. As \(\text{Hom}_{D^b(Y)}(\mathcal{L}^{\otimes -N}, \mathcal{G}[1]) = 0\) applying \(\text{Hom}_{D^b(Y)}(\cdot, \mathcal{G}[1])\) to this sequence produces an exact sequence
\[
\text{Hom}_{D^b(Y)}(\mathcal{O}_Y, \mathcal{G}[1])^{\oplus N d-1} \to \text{Hom}_{D^b(Y)}(j^* T_0, \mathcal{G}[1])^{\oplus N} \to 0.
\]

Then
\[
\dim_{\mathbb{C}} \text{Hom}_{D^b(Y)}(\mathcal{O}_Y, \mathcal{G}[1]) = \dim_{\mathbb{C}} \text{Hom}_{D^b(X)}(\mathbb{L}j^*\mathcal{O}_X, \mathcal{G}[1]) = (\mathcal{O}_Y \cong \mathbb{L}j^*\mathcal{O}_X)
\]
\[
= \dim_{\mathbb{C}} \text{Hom}_{D^b(X)}(\mathcal{O}_X, \mathbb{R}j_*\mathcal{G}[1]) = (\mathbb{L}j^*, \mathbb{R}j_* \text{ adjoint pair})
\]
\[
= \dim_{\mathbb{C}} \text{Hom}_{D^b(X)}(\mathcal{O}_X, j_*\mathcal{G}[1]) = (\text{As } j \text{ affine } \mathbb{R}j_* = j_*)
\]
\[
= \dim_{\mathbb{C}} \text{Hom}_{D^b(X)}(\mathcal{O}_X, \mathcal{E}) = 1
\]
and
\[
\dim_{\mathbb{C}} \text{Hom}_{D^b(Y)}(j^*T_0, \mathcal{G}[1]) = \dim_{\mathbb{C}} \text{Hom}_{D^b(X)}(\mathbb{L}j^*T_0, \mathcal{G}[1]) = (T_0 \text{ locally free})
\]
\[
= \dim_{\mathbb{C}} \text{Hom}_{D^b(X)}(T_0, \mathbb{R}j_*\mathcal{G}[1]) = (\mathbb{L}j^*, \mathbb{R}j_* \text{ adjoint pair})
\]
\[
= \dim_{\mathbb{C}} \text{Hom}_{D^b(X)}(T_0, j_*\mathcal{G}[1]) = (\text{As } j \text{ affine } \mathbb{R}j_* = j_*)
\]
\[
= \dim_{\mathbb{C}} \text{Hom}_{D^b(X)}(T_0, \mathcal{E})
\]
\[
= \dim_{\mathbb{C}} M = d
\]
as \(M \cong \text{Hom}_{D^b(X)}(T_0, \mathcal{E})\) has dimension vector \(d_{T_0}\) and \(d = \text{rk}T_0\). Comparing the dimensions in the sequence (1) we find a contradiction since a \(Nd - 1\) dimensional space cannot surject onto an \(Nd\) dimensional space. Hence such an \(\mathcal{E}\) cannot exist and so \(S\) is empty.

Combining this theorem with Corollary 5.1.5 gives us the following result, showing that in this situation schemes can be reconstructed by quiver GIT.

**Corollary 5.2.5.** Let \(\pi : X \to \text{Spec}(R)\) be a projective morphism of finite type schemes over \(\mathbb{C}\) such that \(\pi\) has fibres of dimension \(\leq 1\) and \(\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R\). Let \(T_0\) be a tilting bundle which is a projective generator of \(^0\text{Per}(X/R)\) as defined by Theorem 1.3.1, define \(A_0 = \text{End}_X(T_0)^{\text{op}}\), and choose the stability condition \(\theta_{T_0}\) and dimension vector \(d_{T_0}\) as above. Then \(X\) is isomorphic to the quiver GIT quotient of \(A_0 = \text{End}_X(T_0)^{\text{op}}\) for dimension vector \(d_{T_0}\) and stability condition \(\theta_{T_0}\).

### 5.3. Example: Flops

The class of varieties considered in Section 5.2 were originally motivated by flops in the minimal model program. In the paper [4] Bridgeland proves that smooth varieties in dimension three which are related by a flop are derived equivalent, and in the process constructs the flop of such a variety as a moduli space of perverse point sheaves. In this section we show that this moduli space construction can in fact be done using quiver GIT. Recall the following theorem.

**Theorem 5.3.1** ([29, Theorems 4.4.1, 4.4.2]). Suppose \(\pi : X \to \text{Spec}(R)\) is a projective birational map of quasiprojective Gorenstein varieties of dimension \(\geq 3\), with \(\pi\) having fibres of dimension \(\leq 1\), the exceptional locus of \(\pi\) having codimension \(\geq 2\), and \(Y\) having canonical hypersurface singularities of multiplicity \(\leq 2\). Then the flop \(\pi' : X' \to \text{Spec}(R)\) exists and is unique. Further \(X\) and \(X'\) are derived equivalent such that \(^{-1}\text{Per}(X/R)\) corresponds to \(^0\text{Per}(X'/R)\). In particular, for a tilting bundle \(T_1\) on \(X\) which is a projective generator of \(^{-1}\text{Per}(X/R)\) there is a tilting bundle \(T'_0\) on \(X'\) which is a projective generator of \(^0\text{Per}(X'/R)\) such that \(A_1 = \text{End}_X(T_1)^{\text{op}} \cong \text{End}_{X'}(T'_0)^{\text{op}} = A'_0\) and \(\pi_*T_1 \cong \pi'_*T'_0\).

We refer the reader to [29, Theorem 4.4.1] for the definition of a flop in this setting. The results from the previous sections now imply the following corollary, showing that the variety \(X\) and its flop \(X'\) can both be constructed as quiver GIT quotients from tilting bundles on \(X\).

**Corollary 5.3.2.** Suppose we are in the situation of Theorem 5.3.1. Then \(X\) is the quiver GIT quotient of \(A_0 = \text{End}_X(T_0)^{\text{op}}\) for stability condition \(\theta_{T_0}\) and dimension vector \(d_{T_0}\), and \(X'\) is the quiver GIT quotient of \(A_1 = \text{End}_X(T_1)^{\text{op}}\) for stability condition \(\theta_{T_1}\) and dimension vector \(d_{T_1}\).

**Proof.** Corollary 5.2.5 tells us both that \(X\) is the quiver GIT quotient of \(A_0\) for stability condition \(\theta_{T_0}\) and dimension vector \(d_{T_0}\), and that \(X'\) is the quiver GIT quotient of \(A'_0 = \text{End}_{X'}(T'_0)^{\text{op}}\) for stability condition \(\theta_{T_0}\) and dimension vector \(d_{T'_0}\). We now relate \(A'_0\), \(\theta_{T_0}\) and \(d_{T'_0}\) to \(A_1\), \(\theta_{T_1}\) and \(d_{T_1}\).
We note that by Theorem 5.3.1 $A_0 \cong A_1$, and we choose a presentation of $A_1$ as a quiver with relations matching that of $A_0$ in order to identify the stability condition and dimension vector matching $\theta_{T_0}$ and $d_{T_0}$. In particular there is a decomposition of $T_1 = \bigoplus_{i=0}^{n} E_i$ and $T_0 = \bigoplus_{i=0}^{n} E'_i$ such that $\pi_* E_i \cong \pi'_* E'_i$. We note that under this correspondence the vertices corresponding to $\mathcal{O}_X$ and $\mathcal{O}_X$ correspond by [29, Lemma 4.2.1] as $\pi_* \mathcal{O}_X \cong \pi'_* \mathcal{O}_X \cong \mathcal{O}_R$, and since $\pi$ and $\pi'$ are birational $rk E_i = rk_R \pi_* E_i = rk_R \pi'_* E'_i = rk_Y E'_i$. Hence $A_0 \cong A_1$, $d_{T_0} = d_{T_1}$, and $\theta_{T_0} = \theta_{T_1}$, so $X'$ is the quiver GIT quotient of $A_1 = \text{End}_X(T_1)^{op}$ for stability condition $\theta_{T_1}$ and dimension vector $d_{T_1}$. \qed

5.4. Example: Resolutions of Rational Singularities. We give a further application of Theorem 5.2.5 to the case of rational singularities, extending and recapturing several well-known examples.

Definitions 5.4.1. Let $Y$ be a (possibly singular) variety. A smooth variety $X$ with a projective birational map $\pi : X \to Y$ that is bijective over the smooth locus of $Y$ is called a resolution of $Y$. A resolution, $X$, is a minimal resolution of $Y$ if any other resolution factors through it. In general minimal resolutions do not exist, but they always exist for surfaces, [23, Corollary 27.3]. A resolution, $X$, is a crepant resolution of $Y$ if $\pi^* \omega_Y = \omega_X$, where $\omega_X$ and $\omega_Y$ are the canonical classes of $X$ and $Y$ which we assume are normal. In general crepant resolutions do not exist. A singularity, $Y$, is rational if for any resolution $\pi : X \to Y$

$$\mathbb{R}\pi_* \mathcal{O}_X \cong \mathcal{O}_Y.$$

If this holds for one resolution it holds for all resolutions, [30, Lemma 1].

Minimal resolutions of rational affine singularities $\pi : X \to \text{Spec}(R)$ satisfy the condition $\mathbb{R}\pi_* \mathcal{O}_X \cong \mathcal{O}_R$ by definition, and in the case of surface singularities it is immediate that the dimensions of the fibres of $\pi$ are $\leq 1$. Hence the following corollary is immediate from Corollary 5.2.5 ii).

Corollary 5.4.2. Suppose that $\pi : X \to \text{Spec}(R)$ is the minimal resolution of a rational surface singularity. Then there is a tilting bundle $T_0$ on $X$ as in Theorem 1.3.1, and by Corollary 5.2.5 ii) $X$ is the quiver GIT quotient of $A_0 = \text{End}_X(T_0)^{op}$ for dimension vector $d_{T_0}$ and stability condition $\theta_{T_0}$.

This gives a moduli interpretation of minimal resolutions for all rational surface singularities. In certain examples the tilting bundles and algebras are well-understood and this corollary recovers previously known examples.

Example 5.4.3 (Kleinian Singularities). Kleinian singularities are quotient singularities $\mathbb{C}^2/G$ for $G$ a non-trivial finite subgroup of $\text{SL}_2(\mathbb{C})$. These have crepant resolutions, and in particular $\text{Hilb}^G(\mathbb{C}^2) = X \to \mathbb{C}^2/G$ is a crepant resolution, [17]. There is a tilting bundle $T$ on $X$ constructed by Kapranov and Vasserot [18], which, if we take the multiplicity free version, matches the $T_0$ of Theorem 1.3.1. Then $A = \text{End}_X(T)^{op}$ is presentable as the McKay quiver with relations, the preprojective algebra, and $\text{Hilb}^G(\mathbb{C}^2)$ is the quiver GIT quotient of the preprojective algebra for stability condition $\theta_T$ and dimension vector $d_T$. The crepant resolutions were previously constructed as hyper-Kähler quotients by Kronheimer [21], this approach was interpreted as a GIT quotient construction by Cassens and Slodowy [8], and as a quiver GIT quotient by Crawley-Boevey [12].

Example 5.4.4 (Surface Quotient Singularities). As an expansion of the previous example we consider $G$ a non-trivial, pseudo-reflection-free, finite subgroup of $\text{GL}_2(\mathbb{C})$. Then $\mathbb{C}^2/G$ is a rational singularity with a minimal resolution $\pi : \text{Hilb}^G(\mathbb{C}) = X \to \mathbb{C}^2/G$ by [16]. The variety $X$ has the tilting bundle $T_0$, and the algebras $A = \text{End}_X(T_0)^{op}$ can be presented as the path algebras of quivers with relations, the reconstruction algebras, which are defined and explicitly calculated in [31–34]. If $G < \text{SL}_2(\mathbb{C})$ then this example falls into the case of Kleinian singularities above, otherwise these fall into a classification in types $A, D, T, I$, and $O$, [32, Section 5]. It was shown by explicit calculation in [31,33,34] that in types $A$ and $D$ the minimal resolutions $X$ are quiver GIT quotients of $A$ with stability condition $\theta_{T_0}$ and dimension vector $d_{T_0}$. Corollary 5.4.2 recovers these cases without needing to perform explicit calculations, and also includes the same result for the remaining cases $T, I, L$, and $O$. 
Corollary 5.4.5. Suppose $G < \text{GL}_2(\mathbb{C})$ is a finite, non-trivial, pseudo-reflection-free group. Then the minimal resolution of the quotient singularity $\mathbb{C}^2/G$ can be constructed as the quiver GIT quotient of the corresponding reconstruction algebra for stability condition $\theta_{T_0}$ and dimension vector $d_{T_0}$.

Proof. We note that in Theorem 1.3.1 $T_1 = T_0^\vee$ and that $\text{End}_X(T_0^\vee) \cong \text{End}_X(T_0)^{\text{op}}$. Hence our definition of $A = \text{End}_X(T_0)^{\text{op}}$ as the reconstruction algebra matches that given in [31–34] as $A = \text{End}_X(T_1)$. Then the result is an immediate corollary of Corollary 5.4.2. \hfill \Box

Example 5.4.6 (Determinantal Singularities). We give one higher dimensional example. Let $R$ be the $\mathbb{C}$-algebra $\mathbb{C}[X_0, \ldots , X_l, Y_1, \ldots , Y_{l+1}]$ subject to the relations generated by all two by two minors of the matrix

$$
\begin{pmatrix}
X_0 & X_1 & \cdots & X_i & \cdots & X_l \\
Y_1 & Y_2 & \cdots & Y_{i+1} & \cdots & Y_{l+1}
\end{pmatrix}
$$

Then $\text{Spec}(R)$ is a $l + 2$ dimensional rational singularity and has an isolated singularity at the origin. This has a resolution given by $\pi: X = \text{Tot} \left( \bigoplus_{i=1}^l \mathcal{O}_{\mathbb{P}^1}(-1) \right) \to \text{Spec}(R)$, the total space of the locally free sheaf $\bigoplus_{i=1}^l \mathcal{O}_{\mathbb{P}^1}(-1)$ mapping onto its affinisation. The variety $X$ has a tilting bundle $T_0$ by Theorem 1.3.1, which, considering the bundle map $f: X \to \mathbb{P}^1$, we can identify as $T_0 = \mathcal{O}_X \oplus f^* \mathcal{O}_{\mathbb{P}^1}(-1)$. We can then present $A_0 = \text{End}_X(T_0)^{\text{op}}$ as the following quiver with relations, $(Q, \Lambda)$.

![Quiver diagram](diagram.png)

By Theorem 5.2.5 we know that $X$ can be reconstructed as the quiver GIT quotient of $A_0$ with dimension vector $d_{T_0} = (1, 1)$ and stability condition $\theta_{T_0} = (-1, 1)$. In this example we will explicitly verify this. A dimension $d_{T_0}$ representation is defined by assigning a value $\lambda_i \in \mathbb{C}$ to each $k_i$ and $(\alpha, \gamma) \in \mathbb{C}^2$ to $(a, c)$. The relations are all automatically satisfied so $\text{Rep}_{d_{T_0}}(Q, \Lambda) = \mathbb{C}^{l+1} \times \mathbb{C}^2$. Then a representation is $\theta_{T_0}$ stable if it has no dimension $(1, 0)$ submodules, so these correspond to the subvariety with $(\alpha, \gamma) \in \mathbb{C}^2/(0,0)$, hence $\text{Rep}_{\theta_{T_0}}(Q, \Lambda)^{ss} = \mathbb{C}^{l+1} \times \mathbb{C}^2/(0,0)$. We then find that the corresponding quiver GIT quotient is given by the action of $\mathbb{C}^*$ on $\mathbb{C}^{l+1} \times \mathbb{C}^2/(0,0)$ with weights $-1$ on $\mathbb{C}^{l+1}$ and $1$ on $\mathbb{C}^2$. This produces the total bundle $X$.

When $l = 2$ this is the motivating example of the Atiyah flop given as the opening example of [29] and $A_0$ is the conifold quiver. In this case, by Theorem 5.3.2, we can calculate the flop as the quiver GIT quotient of $A_1 \cong A_0^{\text{op}}$.

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