Quantization and coherent states for a time-dependent Landau problem

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The ordinary Landau problem consists of describing a charged particle in time-independent magnetic field. In the present case the problem is generalized onto time-dependent uniform electric fields with time-dependent mass and harmonic frequency. The spectrum of a Hamiltonian describing this system is obtained. The configuration space wave functions of the model is expressed in terms of the generalised Laguerre polynomials. To diagonalize the time-dependent Hamiltonian we employ the Lewis-Riesenfeld method of invariants. To this end, we introduce an unitary transformation in the framework of the algebraic formalism to construct the invariant operator of the system and then to obtain the exact solution of the Hamiltonian. We recover the solutions of the ordinary Landau problem in the absence of the electric and harmonic fields, for a constant particle mass. The quantization of this system exhibits many symmetries such as $U(1), SU(2), SU(1,1)$. We therefore construct the corresponding coherent states and the associated photon added and nonlinear coherent states.

1 The model

We consider the problem of a charged particle of charge $q$ with time-dependent mass $M(t)$ and frequency $\omega(t)$ moving in a two-dimensional plane and subjected to a static magnetic $B$ field perpendicular to the plane and a spatially uniform but time-dependent electric field $E(t)$ in lying position in the Euclidean plane. In the symmetric gauge, the vector potential and the time-dependent scalar potential are given by $A_i(x) = -\frac{1}{2}B\epsilon_{ij}x_j$
and \( \varphi(t) = E_i(t)x_i \), respectively (with \( i, j = 1, 2 \)). The Lagrangian of the system is written as follows

\[
L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) = \frac{1}{2} M(t)(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} M(t)\omega^2(t)(x_1^2 + x_2^2) + q[A_{x_1}(x_1, x_2)\dot{x}_1 + A_{x_2}(x_1, x_2)\dot{x}_2] - q\varphi(x_1, x_2, t).
\]  

(1)

The Euler-Lagrange equations of motion for the system are

\[
\begin{align*}
\ddot{x}_1 - \frac{qB}{M(t)}\dot{x}_2 + \omega(t)x_1 - \frac{q}{M(t)}E_{x_1}(t) &= 0, \\
\ddot{x}_2 + \frac{qB}{M(t)}\dot{x}_1 + \omega(t)x_2 - \frac{q}{M(t)}E_{x_2}(t) &= 0.
\end{align*}
\]  

(2)

Since we are in two dimensional configuration space, we can look for the solutions of the classical equations in the complex system by setting \( z = x_2 + ix_1 \). The classical equation of motion in term of the coordinate \( z \) is

\[
\ddot{z} + i\omega_c(t)\dot{z} + \omega(t)z = E_0(t),
\]  

(3)

where \( \omega_c(t) = \frac{qB}{M(t)} \) and \( E_0 = \frac{qE_{x_1}}{M(t)} + i\frac{qE_{x_2}}{M(t)} \). The general homogenous solution of this equation is

\[
z(t) = Ae^{-i\omega_+ t} + Be^{i\omega_- t} + \frac{E_0}{\omega_c},
\]  

(4)

where \( \omega_+ = \sqrt{\omega^2 + \frac{1}{4}\omega_c^2 + \frac{q^2}{2}} \) and \( \omega_- = \sqrt{\omega^2 + \frac{1}{4}\omega_c^2 - \frac{q^2}{2}} \).

The canonical momentum associated with the variables \( x_1 \) and \( x_2 \) are

\[
\begin{align*}
p_1 &= \frac{\partial L}{\partial \dot{x}_1} = M(t)\dot{x}_1 + qA_{x_1}, \\
p_2 &= \frac{\partial L}{\partial \dot{x}_2} = M(t)\dot{x}_2 + qA_{x_2}.
\end{align*}
\]  

(5)

The Hamiltonian is given by

\[
H(x_1, x_2, p_1, p_2, t) = \dot{x}_1p_1 + \dot{x}_2p_2 - L = \frac{1}{2M(t)}\left[p_1 + \frac{q}{2}Bx_2\right]^2 + \frac{1}{2M(t)}\left[p_2 - \frac{q}{2}Bx_1\right]^2 + \frac{1}{2}M(t)\omega^2(t)(x_1^2 + x_2^2) - q[E_1(t)x_1 + E_2(t)x_2].
\]  

(6)

Note that for \( \tilde{E}(t) = 0 \), \( \omega(t) = 0 \) and for a constant mass \( M \), we obtain a Hamiltonian operator describing the ordinary Landau problem [2,3,4].

We introduce the following changes of variables

\[
x = x_1 + \frac{qE_1(t)}{M(t)\omega^2(t)} \\
y = x_2 + \frac{qE_2(t)}{M(t)\omega^2(t)},
\]  

(7)

\[
p_x = p_1 - \frac{q^2BE_2(t)}{2M(t)\omega^2(t)} \\
p_y = p_2 - \frac{q^2BE_1(t)}{2M(t)\omega^2(t)}.
\]  

(8)
and get the transformed Hamiltonian in its simple form rewritten as follows

\[ H(t) = \frac{1}{2M(t)}(p_x^2 + p_y^2) + \frac{\Omega^2(t)M(t)}{2}(x^2 + y^2) - \frac{\omega_c(t)}{2}L_z - \frac{q^2E^2(t)}{2M(t)\omega(t)}, \tag{9} \]

where \( L_z = xp_y - yp_x \) is the angular momentum of the system, \( \omega_c(t) = \frac{eB}{M(t)} \) is the cyclotronic frequency of oscillations, \( \Omega(t) = \sqrt{\omega_1^2(t) + \frac{1}{4}\omega_2^2(t)} \) is the general frequency of oscillations and \( E(t) = \sqrt{E_1^2(t) + E_2^2(t)} \). From the above choice of the gauge symmetric, the system generated the angular momentum \( L_z \) that commutes with the Hamiltonian of the system

\[ [H(t), L_z] = 0. \tag{10} \]

Therefore, both operators admit a common basis in which they can be simultaneously diagonalized. Furthermore, \( L_z \) can be interpreted as the generator of \( SO(2) = U(1) \) group which conserve the invariance of the orientation and the rotation in the plane i.e the eigenstates of the Hamiltonian \( H(t) \) is invariant under this rotation. The group of rotation of the system around the \( z \)-axis is written as follows

\[ R_z(\alpha) = \exp \left(-\frac{i\alpha}{\hbar}L_z\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i\alpha}{\hbar}L_z\right)^n, \tag{11} \]

with \( \alpha \in \mathbb{R} \) is the angle of rotation. Since the magnetic field is constant and uniform along the \( z \)-axis this group satisfy the commutation relation

\[ R_z(\alpha)R_z(\beta) = R_z(\beta)R_z(\alpha), \quad \alpha, \beta \in \mathbb{R}. \tag{12} \]

Furthermore, the operator \( L_z \) is self-adjoint then the condition of unitarity is guaranteed. In fact the quantum rotational operator obeys the following important properties

\[ R_z(\alpha_3, \alpha_2)R_z(\alpha_2, \alpha_1) = R_z(\alpha_3, \alpha_1), \tag{13} \]
\[ R_z(\alpha_2, \alpha_1) = R_z^{-1}(\alpha_2, \alpha_1) = R_z(\alpha_1, \alpha_2), \tag{14} \]
\[ R_z(0) = \mathbb{I}. \tag{15} \]

Since the system of our study contained parameters which change in time, we consider suitable Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^3) \) of square integrable wavefunctions \( \psi(x, y, t) \) over \( \mathbb{R}^2 \) and the time-parameter \( t \in \mathbb{R} \). For simplicity we consider the wavefunction in the form \( \psi(x, y, t) \) defined over \( L^2(\mathbb{R}^3) \). The Schrödinger equation is explicitly written as follows

\[ i\partial_t \psi(x, y, t) = -\frac{1}{2M(t)}(\partial_x^2 + \partial_y^2)\psi(x, y, t) + \frac{\Omega^2(t)M(t)}{2}(x^2 + y^2)\psi(x, y, t) \]
\[ + i\frac{\omega_c(t)}{2}(x\partial_y - y\partial_x)\psi(x, y, t) - \frac{q^2E^2(t)}{2M(t)\omega(t)}\psi(x, y, t), \tag{16} \]

where we set \( \hbar = 1 \).
Since the Hamiltonian involves time-dependent parameters, the above differential equation cannot be easily solved. Then, we use the so-called Lewis and Riesenfeld method [16] which consists in constructing a Hermitian operator \( I(t) \) that fulfills the condition
\[
\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{i}[I, H] = 0. \tag{17}
\]
The Schrödinger equation for the invariant operator reads as
\[
I(t)\phi(x, y, t) = E\phi(x, y, t), \quad \text{with} \quad \frac{\partial E}{\partial t} = 0, \quad \phi(x, y, t) \in \mathcal{H}. \tag{18}
\]
The solutions of Schrödinger equation (18) is linked to the invariant’s eigenfunction by
\[
\psi(x, y, t) = e^{i\gamma(t)}\phi(x, y, t), \tag{19}
\]
where the time-dependent Lewis-Riesenfeld phase \( \gamma(t) \) satisfies the following equation
\[
\frac{d\gamma}{dt} = \langle \phi | i\frac{\partial}{\partial t} - H(t) | \phi \rangle. \tag{20}
\]
Therefore to determine the solution of the system (16), it found important to construct firstly the Hermitian invariant \( I(t) \) and secondly determine its eigensystems.

1.1 The invariant operator

We look for the Hermitian operator in the form
\[
I(t) = \frac{1}{2} \left[ \alpha(t)(x^2 + y^2) + \beta(t)(p_x^2 + p_y^2) + \delta(t)(xp_x + p_x x + yp_y + p_y y) \right], \tag{21}
\]
where \( \alpha, \beta, \) and \( \delta \) are real coefficient functions. Using (17) and the following equations
\[
[x^2, p_x^2] = 2i\{x, p_x\}, \quad [x^2, \{x, p_x\}] = 4ix^2, \quad [p_x^2, \{x, p_x\}] = -4ip_x^2, \tag{22}
\]
\[
[y^2, p_y^2] = 2i\{y, p_y\}, \quad [y^2, \{y, p_y\}] = 4iy^2, \quad [p_y^2, \{y, p_y\}] = -4ip_y^2. \tag{23}
\]
We obtain the first-order linear differential equations for the unknown coefficient functions
\[
\dot{\alpha} - 2M\Omega^2\delta = 0, \tag{24}
\]
\[
\dot{\beta} + \frac{2}{M}\delta = 0, \tag{25}
\]
\[
\dot{\delta} + \frac{1}{M}\alpha - M\Omega\beta = 0, \tag{26}
\]
\[
\dot{\delta} = 0. \tag{27}
\]
Equation (27) implies that \( \delta \) is a constant. In the rest of the text, we choose \( \delta = 0 \) for the other invariant. From (24)-(26), it follows:

\[
\frac{d}{dt}(\delta^2 - \alpha \beta) = 0,
\]

whence

\[
\delta^2 - \alpha \beta = -\kappa^2,
\]

where \( \kappa \) is either a real or an imaginary constant (so that \( \kappa^2 \) is real). It is convenient to introduce another real function \( \rho \) defined by

\[
\beta(t) = \rho^2(t).
\]

For an arbitrary positive constant \( \kappa \), the other coefficients are

\[
\delta(t) = -M \dot{\rho}, \quad \alpha(t) = \frac{\kappa^2}{\rho^2} + M^2 \dot{\rho}^2.
\]

The Hermitian invariant therefore acquires the form

\[
I(t) = \frac{1}{2} \left[ \frac{\kappa^2}{\rho^2} x^2 + \frac{\kappa^2}{\rho^2} y^2 + (\rho \dot{x} - M \dot{\rho} x)^2 + (\rho \dot{y} - M \dot{\rho} y)^2 \right],
\]

where the function \( \rho \) is the solution of the so-called nonlinear Ermakov-Pinney equation

\[
\ddot{\rho} + \frac{M}{M} \dot{\rho} + \Omega^2(t) \rho = \frac{\kappa^2}{M^2 \rho^3}.
\]

As matter of fact, for concrete computations of measurable quantities one needs to address the auxiliary problem and solve the equations explicitly for the time-dependent functions appearing in the Hamiltonian. Surprisingly little attention has been paid to this problem in the context of solving time-dependent Hamiltonian systems and therefore we will discuss the solutions of our auxiliary equation (33) in the next subsection.

1.1.1 Solutions of the Ermakov-Pinney equation

The simplest special solution arises when taking \( M(t) = \tau = \text{constant} \) (\( \dot{M} = 0 \)), consequently the friction coefficient vanishes and the solution found by Pinney [6] was

\[
\rho(t) = \sqrt{\nu_1^2 + \nu_2^2 W^{-2}};
\]

where \( \nu^2 = \frac{\kappa^2}{\tau^2} \); \( \nu_1, \nu_2 \) are linear independent solutions of the equation

\[
\ddot{v} + \omega^2 v = 0,
\]

and \( W = v_1 \dot{v}_2 - v_2 \dot{v}_1 \neq 0 \) is their Wronskian.
For still the non-dissipative solutions ($\dot{M} = 0$), we can simply pre-select any explicit form for $\Omega(t) = \tau e^{\alpha t}, \tau \in \mathbb{R}$, we solve \ref{eq:33} in terms of the Bessel functions and subsequently obtain the particular solution by means

$$\rho(t) = \sqrt{\frac{\pi^2 \nu^2}{\alpha^2 A_1^2} Y_0 \left( \frac{2\nu e^{\alpha t}}{\sqrt{\tau \alpha}} \right) + A_1^2 J_0^2 \left( \frac{2\nu e^{\alpha t}}{\sqrt{\tau \alpha}} \right)}$$ \hspace{1cm} (36)

with integration constant $A_1 \in \mathbb{R}$ and $J_0, Y_0$ denoting the Bessel functions of first and second kind, respectively.

Now, for special values of $M(t) = e^{-\alpha t}$ and $\Omega(t) = \sqrt{\frac{\nu}{\alpha}}$, the transformation $\rho = e^{-\frac{1}{2} \alpha t} y$ leads to the Yermakov’s equation \cite{7}

$$\ddot{y} + \alpha^2 y = \kappa^2 y^3.$$ \hspace{1cm} (37)

Using the solution of this equation, we obtain for $\rho$ the solutions:

$$\rho(t) = e^{-\frac{1}{2} \alpha t} \left[ \frac{\kappa s^2}{d_1} + \frac{s^2}{d_2} \left( d_2 + d_1 \int_{0}^{t} \frac{dt'}{s^2} \right)^2 \right]^{\frac{1}{2}},$$ \hspace{1cm} (38)

where $d_1, d_2$ arbitrary real constants and $s(t')$ is a function such a

$$s(t') = e_1 \sin \left( \alpha t' \right) + e_2 \cos \left( \alpha t' \right),$$ \hspace{1cm} (39)

where $e_1$ and $e_2$ are arbitrary real constants.

### 1.1.2 Eigensystems of the invariant operator

In order to solve easily the eigenvalue equation of the invariant operator, we consider the unitary operator $U$ which is written as follows

$$U = \exp \left[ -i \frac{M \dot{\rho}}{2 \rho} (x^2 + y^2) \right], \quad U^\dagger U = U U^\dagger = \mathbb{I}. \hspace{1cm} (40)$$

This operator transforms, the invariant operator’s eigenfunction into

$$\phi'(x, y, t) = U \phi(x, y, t), \hspace{1cm} (41)$$

reduces the invariant operator into the form

$$I'(t) = U I(t) U^\dagger = \frac{1}{2} \left[ \rho^2 (p_x^2 + p_y^2) + \frac{\kappa^2}{\rho^2} (x^2 + y^2) \right], \hspace{1cm} (42)$$

but keeps its eigenvalue invariant

$$I'(t) \phi'(x, y, t) = E' \phi'(x, y, t) \hspace{1cm} \text{and} \hspace{1cm} U I(t) U^\dagger \phi(x, y, t) = E \phi(x, y, t). \hspace{1cm} (43)$$
Multiplying the equation (43) with the operator $U^\dagger$ and using the unitarian relation $U^\dagger U = I$, we get

$$E_\alpha = E'_\alpha.$$  

Instead of using analytical method [13, 14, 15] to solve the eigenvalue equation of the invariant operator, we achieve its diagonalisation through algebraic method. We introduce the reduced lowering and raising operators given by

$$a'_x = \frac{1}{\sqrt{2\kappa}} \left( \frac{\kappa}{\rho} x + i \rho y \right), \quad a'_y = \frac{1}{\sqrt{2\kappa}} \left( \frac{\kappa}{\rho} y + i \rho x \right),$$

that fulfill the commutation relations

$$[a'_x, a'_x] = I = [a'_y, a'_y], \quad [a'_x, a'_y] = 0 = [a'_x, a'_y].$$

Let us consider any nonnegative integers $n_x, n_y$ and $|\phi'_{n_x,n_y}(t)\rangle$ the orthonormalized Fock space such as

$$|\phi'_{n_x,n_y}(t)\rangle = \frac{1}{\sqrt{n_x!n_y!}} (a'_x)^{n_x} (a'_y)^{n_y} |\phi'_{0,0}(t)\rangle,$$

$$\langle \phi'_{n_x,n_y}(t) | \phi'_{m_x,m_y}(t) \rangle = \delta_{n_x,m_x} \delta_{n_y,m_y},$$

with $|\phi'_{0,0}(t)\rangle$ is a normalized state annihilated by $a'_x, a'_y$.

In order to determine the exact solution $\psi_{n_x,n_y}(x,y,t)$ of the invariant operator $I(t)$, we first express the ground state $|\psi_{0,0}(t)\rangle$ in the configuration space base as follows

$$\phi_{0,0}(x,y,t) = U^\dagger \langle x | \phi'_0(t) \rangle \langle y | \phi'_0(t) \rangle$$

$$= \left( \frac{\kappa}{\pi \rho^2} \right)^{\frac{1}{2}} \exp \left[ \left( i M \rho - \kappa \rho^2 \right) \left( \frac{x^2 + y^2}{2} \right) \right].$$

Then, the $n$th eigenfunctions are obtained from (48) as

$$\phi_{n_x,n_y}(x,y,t) = U^\dagger \phi'_{n_x,n_y}(x,y,t)$$

$$= \frac{1}{\rho} \left( \frac{\kappa}{2^{n_x+n_y} \pi^{n_x+n_y}} \right)^{\frac{1}{2}} H_{n_x} \left( \frac{\kappa \rho}{2} \right) H_{n_y} \left( \frac{\sqrt{\kappa} \rho}{2} \right)$$

$$\times \exp \left[ \left( i M \rho - \kappa \rho^2 \right) \left( \frac{x^2 + y^2}{2} \right) \right],$$

where $H_{n_x}$ and $H_{n_y}$ are the Hermite polynomial of order $n_x$ and $n_y$. To obtain the eigenvalues $E_{n_x,n_y}$ of the invariant operator $I(t)$, let us introduce a new pair of raising and lowering operators define as

$$a_j = U^\dagger a'_j U = \frac{1}{\sqrt{2\kappa}} \left( M \rho x_j - \rho p_j + i \frac{\kappa}{\rho} x_j \right),$$

$$a'_j = U^\dagger a'_j U = \frac{1}{\sqrt{2\kappa}} \left( M \rho x_j - \rho p_j - i \frac{\kappa}{\rho} x_j \right).$$
with \( j = x, y \). In term of these operators the operator \( I(t) \) takes the form

\[
I(t) = \kappa \left( a_x^+ a_x + a_y^+ a_y + 1 \right),
\]

(54)

The action of \( a_j \) and \( a_j^\dagger \) on \( |\psi_{nj}(t)\rangle \) finds expression in

\[
\begin{align*}
    a_j^\dagger |\phi_{nj}(t)\rangle &= \sqrt{n_j + 1} |\phi_{nj+1}(t)\rangle, \\
    a_j |\phi_{nj}(t)\rangle &= \sqrt{n_j} |\phi_{nj-1}(t)\rangle, \\
    a_j^\dagger a_j |\phi_{nj}(t)\rangle &= n_j |\phi_{nj}(t)\rangle.
\end{align*}
\]

(55) \( (56) \) \( (57) \)

Basing on these definitions, the invariant is diagonalized as follows

\[
I(t) |\phi_{nx,ny}(t)\rangle = E_{nx,ny} |\phi_{nx,ny}(t)\rangle, \quad E_{nx,ny} = \kappa (n_x + n_y + 1).
\]

(58)

However, as we pointed out in the previous section, this system possesses a conserved angular-momentum

\[
L_z = i(a_y^+ a_x - a_y a_x),
\]

(59)

which commutes with the invariant operator \( I(t) \)

\[
[I(t), L_z] = 0.
\]

(60)

Although the operator \( L_z \) commutes with both \( I(t) \) and \( H(t) \), but the bases \( |\phi_{nx,ny}(t)\rangle \) does not diagonalize them simultaneously. It convenient to work in another bases of Hilbert space which take account of the invariance of rotation and diagonalizes these operators.

## 2 \( \text{U}(1) \) symmetry and its coherent states

### 2.1 \( \text{U}(1) \) symmetry

In order to make explicit the \( SO(2) = U(1) \) circular symmetry of the system and to determine the corresponding bases which can diagonalize simultaneously the invariant operator, the angular momentum and the Hamiltonian of the system. Let us consider the reduced helicity Fock algebra generators as follows

\[
\begin{align*}
    a_{\pm}' &= \frac{1}{\sqrt{2}} \left( a_x' \pm i a_y' \right), \quad a_{\pm}' &= \frac{1}{\sqrt{2}} \left( a_x^\dagger + i a_y^\dagger \right),
\end{align*}
\]

(61)

with

\[
[a_{\pm}', a_{\mp}'] = \mathbb{I}, \quad [a_{\pm}', a_{\pm}'] = 0.
\]

(62)
The inverse relations are,
\[
a'_x = \frac{1}{\sqrt{2}} (a'_+ + a'_-), \quad a''_x = \frac{1}{\sqrt{2}} (a''_+ + a''_-), \quad (63)
\]
\[
a'_y = -\frac{i}{\sqrt{2}} (a'_+ - a'_-), \quad a''_y = \frac{i}{\sqrt{2}} (a''_+ - a''_-). \quad (64)
\]
The associated helicity-like bases \(|\phi'_{n+, n-}(t)\rangle\) are defined as follows
\[
|\phi'_{n+, n-}(t)\rangle = \frac{1}{\sqrt{n_+! n_-!}} (a''_+)^{n_+} (a''_-)^{n_-} |\phi_{0, 0}(t)\rangle, \quad (65)
\]
\[
\langle \phi'_{m+, m-}(t)| \phi'_{n+, n-}(t) \rangle = \delta_{n_+, m_+} \delta_{n_-, m_-}, \quad n_+, n_- \in \mathbb{N}. \quad (66)
\]
With the intention of determining the corresponding exact solution \(\psi_{n+, n-}(x, y, t)\), we introduce the polar coordinates through the following canonical transformation \(r \cos \theta, \ y \sin \theta, \ p_x : -i(\cos \theta \partial_\rho - \frac{\sin \theta}{\rho} \partial_r)\) and \(p_y : -i(\sin \theta \partial_\rho + \frac{\cos \theta}{\rho} \partial_r)\). In terms of these representations the operators \((61)\) can be written as
\[
a'_{\pm} = \frac{1}{2} e^{\mp i \theta} \left[ \left( \frac{\kappa}{\rho} - \rho \partial_\rho \right) \pm i \frac{\rho}{r} \phi_0 \right], \quad (67)
\]
\[
a''_{\pm} = \frac{1}{2} e^{\pm i \theta} \left[ \left( \frac{\kappa}{\rho} + \rho \partial_\rho \right) \mp i \frac{\rho}{r} \phi_0 \right]. \quad (68)
\]
From the relation \((65)\) we construct the eigenfunctions for the invariant operator of the system \([19]\). One finds
\[
\psi_{n+, n-}(x, y, t) = U^\dagger \psi'_{n+, n-}(x, y, t) = (-)^n \frac{(\kappa)^{\frac{|\ell|}{2}}}{\rho^{|\ell|} \sqrt{\pi}} \sqrt{n_! n_-!} \Gamma(n + |\ell| + 1) \times
\]
\[
e^{i (\ell M - \frac{\phi_0}{\rho^2})} \frac{\kappa^{|\ell|} \rho^{|\ell|}}{\sqrt{\pi \rho^2}} \times L_n^{\ell}(x, y, t) \times e^{i \theta t}, \quad (69)
\]
where \(\ell = n_+ - n_-\), \(n = \min(n_+, n_-) = \frac{1}{2} (n_+ + n_- - |\ell|)\), \(\Gamma(u)\) and \(L_n^{\ell}(u)\) are the Gamma function and the generalised Laguerre polynomials of argument \(u\), respectively.

To obtain the expectative values \(E_{\ell, n_+}, l_{n_+}, E_{n_+}\) of the operators \(I(t), L_z, H(t)\) respectively, let us introduce a new pair of raising and lowing helicity operators define as
\[
a_{\pm} = U^\dagger a'_{\pm} U = \frac{1}{2 \sqrt{\kappa}} \left[ \left( \hat{M} + i \frac{\kappa}{\rho} \right) (x \pm iy) - \rho(p_x \pm ip_y) \right], \quad (70)
\]
\[
a''_{\pm} = U^\dagger a''_{\pm} U = \frac{1}{2 \sqrt{\kappa}} \left[ \left( \hat{M} - i \frac{\kappa}{\rho} \right) (x \mp iy) - \rho(p_x \mp ip_y) \right]. \quad (71)
\]
Conversely

\[
x = -\frac{i\rho}{2\sqrt{\kappa}} (a_- - a_+^\dagger + a_+ - a_-^\dagger),
\]

\[
x = \frac{i\hat{\rho}}{2\sqrt{\kappa}} (a_- - a_+^\dagger + a_+ - a_-^\dagger) - \frac{\sqrt{\kappa}}{2\rho} (a_- + a_+^\dagger + a_+ + a_-^\dagger),
\]

\[
y = \frac{\rho}{2\sqrt{\kappa}} (a_- - a_+^\dagger - a_+ + a_-^\dagger),
\]

\[
y = \frac{M\hat{\rho}}{2\sqrt{\kappa}} (a_- - a_+^\dagger + a_+ - a_-^\dagger) - i\frac{\sqrt{\kappa}}{2\rho} (a_- + a_+^\dagger - a_+ - a_-^\dagger).
\]

In particulary

\[
x - iy = \frac{i\rho}{\sqrt{\kappa}} (a_+^\dagger - a_-),
\]

\[
p_x + ip_y = \frac{iM\hat{\rho}}{\sqrt{\kappa}} (a_- - a_+) - \frac{\sqrt{\kappa}}{\rho} (a_-^\dagger + a_+),
\]

\[
x + iy = \frac{i\rho}{\sqrt{\kappa}} (a_-^\dagger - a_+),
\]

\[
p_x - ip_y = \frac{iM\hat{\rho}}{\sqrt{\kappa}} (a_- - a_-^\dagger) - \frac{\sqrt{\kappa}}{\rho} (a_-^\dagger + a_-).
\]

With these, we have

\[
I(t) = \kappa \left( a_+^\dagger a_+ + a_-^\dagger a_- + 1 \right),
\]

\[
L_z = - (a_-^\dagger a_-- a_+^\dagger a_+),
\]

\[
H(t) = \frac{1}{2\kappa} \left( M\hat{\rho}^2 + \frac{\kappa^2}{M\rho^2} + M\Omega^2 \rho^2 \right) \left( a_+^\dagger a_+ + a_-^\dagger a_- + 1 \right)
- \frac{\omega}{2} (a_-^\dagger a_- - a_+^\dagger a_+) - \frac{\rho^2 E^2}{2M\omega}
+ \left( -\frac{M\hat{\rho}^2}{2\kappa} + i\frac{\rho^2}{\rho} - \frac{\Omega^2 M\rho^2}{2\kappa} + \frac{\kappa}{2M\rho^2} \right) a_- a_+
+ \left( -\frac{M\hat{\rho}^2}{2\kappa} - i\frac{\rho^2}{\rho} - \frac{\Omega^2 M\rho^2}{2\kappa} + \frac{\kappa}{2M\rho^2} \right) a_+ a_-^\dagger,
\]

and the angular momentum particulary satisfy the following relations

\[
[L_z, a_\pm] = \mp a_\pm, \quad [L_z, a_\pm^\dagger] = \pm a_\pm^\dagger.
\]

The expectative values of the above operators read as

\[
E_{n_z} = \langle \phi_{n_+,n_-}(t)|I(t)|\phi_{n_+,n_-}(t) \rangle = \kappa (n_+ + n_- + 1),
\]

\[
l_{n_z} = \langle \phi_{n_+,n_-}(t)|L_z|\phi_{n_+,n_-}(t) \rangle = n_- - n_+,
\]

\[
E_{n_\pm} = \langle \phi_{n_+,n_-}(t)|H(t)|\phi_{n_+,n_-}(t) \rangle = \frac{1}{2\kappa} \left( M\hat{\rho}^2 + \frac{\kappa^2}{M\rho^2} + M\Omega^2 \rho^2 \right) (n_+ + n_- + 1)
\]

10
\[-\frac{\omega_c}{2} (n_- - n_+) - \frac{q^2 E^2}{2M \omega}, \tag{86}\]

where the action of \(a_+\) and \(a_+^\dagger\) on \(|\phi_{n\pm}(t)\rangle\) finds expression in

\[a_+^\dagger |\phi_{n\pm, n\mp}(t)\rangle = \sqrt{n_{\mp} + 1} |\phi_{n_{\pm} + 1, n_{\mp}}(t)\rangle, \tag{87}\]

\[a_\pm |\phi_{n\pm, n\mp}(t)\rangle = \sqrt{n_{\mp} |\phi_{n_{\pm} - 1, n_{\mp}}(t)\rangle, \tag{88}\]

\[a_+^\dagger a_\pm |\phi_{n\pm, n\mp}(t)\rangle = n_{\pm} |\phi_{n\pm, n\mp}(t)\rangle. \tag{89}\]

To determine the exact solution of the Schrödinger equation (16), we have to find the exact expression of the phase function in equation (20) so that

\[\frac{d\gamma(t)}{dt} = \langle \phi_{n_+, n_-}(t) | i \frac{\partial}{\partial t} - H(t) | \phi_{n_+, n_-}(t) \rangle = \langle \phi_{n_+, n_-}(t) | i \frac{\partial}{\partial t} | \phi_{n_+, n_-}(t) \rangle - \langle \phi_{n_+, n_-}(t) | H(t) | \phi_{n_+, n_-}(t) \rangle. \tag{90}\]

Let us evaluate the following expression

\[\langle \phi_{n_+, n_-}(t) | \frac{\partial}{\partial t} | \phi_{n_+, n_-}(t) \rangle = \frac{1}{\sqrt{n_+! n_-!}} \langle \phi_{n_+, n_-}(t) | \frac{\partial}{\partial t} \left[ \left( a_+^\dagger \right)^{n_+} \left( a_- \right)^{n_-} \right] | \phi_{0,0}(t) \rangle \]

\[= \langle \phi_{0,0}(t) | \frac{\partial}{\partial t} \left( a_+^\dagger \right)^{n_+} \left( a_- \right)^{n_-} | \phi_{0,0}(t) \rangle \tag{91}\]

On the one hand, we have

\[\langle \phi_{0,0}(t) | \left[ \frac{\partial}{\partial t} \phi_{0,0}(t) \right] \tag{92}\]

\[= \int dxdy |\phi_{0,0}(x, y)\rangle \frac{\partial \phi_{0,0}(x, y)}{\partial t} \]

\[= \int dxdy |\phi_{0,0}(t)\rangle^2 \]

\[\times \left[ \left( iM \frac{\hat{p}}{\rho} + iM \frac{\hat{\rho}}{\rho} - iM \frac{\hat{\dot{\rho}}}{\rho^2} + 2\kappa \frac{\hat{\rho}}{\rho^2} \right) \frac{x^2 + y^2}{2} - \frac{\hat{\rho}}{\rho} \right]. \tag{92}\]

By the Stokes theorem such as

\[\int dxdy |\phi_{0,0}(t)\rangle^2 \frac{x^2 + y^2}{2} = \frac{e_0^2}{2} \int dxdy |\phi_{0,0}(x, y)\rangle^2 = \frac{e_0^2}{2}, \tag{93}\]

and some straightforward computations we obtain

\[\langle \phi_{0,0}(t) | \left[ \frac{\partial}{\partial t} \phi_{0,0}(t) \right] = \frac{iM}{2\kappa} \left( \hat{\rho} + \hat{\rho} \rho - \hat{\rho}^2 \right). \tag{94}\]

On the other hand,

\[\frac{1}{\sqrt{n_+! n_-!}} \langle \psi_{n_+, n_-}(t) | \frac{\partial}{\partial t} \left[ \left( a_+^\dagger \right)^{n_+} \left( a_- \right)^{n_-} \right] | \psi_{0,0}(t) \rangle = \frac{iM}{2\kappa} \left( \hat{\rho} + \frac{\hat{M} \rho}{M} \hat{\rho} - \hat{\rho}^2 \right) \]

11
where the expressions of $\frac{\partial a_+^\dagger}{\partial t}$ and $\frac{\partial a_-^\dagger}{\partial t}$ in terms of $a_\pm$ and $a_\pm^\dagger$ are given by

\[
\frac{\partial a_+^\dagger}{\partial t} = \frac{1}{2\sqrt{\kappa}} \left[ \left( \bar{M} \dot{\rho} + \frac{\bar{M}}{M} \rho \dot{\rho} - \dot{\rho}^2 \right) (x - iy) - \dot{\rho}(p_x + ip_y) \right] \\
= \frac{iM}{2\kappa} \left( \bar{M} \dot{\rho} + \frac{\bar{M}}{M} \rho \dot{\rho} - \dot{\rho}^2 \right) a_+^\dagger + \left[ \frac{\dot{\rho}}{\rho} - \frac{iM}{2\kappa} \left( \rho \ddot{\rho} + \frac{\bar{M}}{M} \rho - \dot{\rho}^2 \right) \right] a_-, \quad (96)
\]

\[
\frac{\partial a_-^\dagger}{\partial t} = \frac{1}{2\sqrt{\kappa}} \left[ \left( \bar{M} \dot{\rho} + \frac{\bar{M}}{M} \rho \dot{\rho} - \dot{\rho}^2 \right) (x + iy) - \dot{\rho}(p_x + ip_y) \right] \\
= \frac{iM}{2\kappa} \left( \bar{M} \dot{\rho} + \frac{\bar{M}}{M} \rho \dot{\rho} - \dot{\rho}^2 \right) a_-^\dagger + \left[ \frac{\dot{\rho}}{\rho} - \frac{iM}{2\kappa} \left( \rho \ddot{\rho} + \frac{\bar{M}}{M} \rho - \dot{\rho}^2 \right) \right] a_+. \quad (97)
\]

We obtain

\[
\langle \phi_{n_+,n_-} (t) | \frac{\partial}{\partial t} | \phi_{n_+,n_-} (t) \rangle = \frac{iM}{2\kappa} \left( \bar{M} \dot{\rho} + \frac{\bar{M}}{M} \rho \dot{\rho} - \dot{\rho}^2 \right) (n_+ + n_- + 1) \\
= \frac{iM}{2\kappa} \left( \frac{\kappa^2}{M^2 \rho^2} - \Omega^2 \rho^2 - \bar{\rho}^2 \right) \\
\times (n_+ + n_- + 1). \quad (98)
\]

Finally, taking into account (95), (96) and (97), we find that the phase function is given by

\[
\gamma(t) = -\frac{\kappa}{2} (n_+ + n_- + 1) \int_0^t dt' \frac{dt'}{M(t') \rho^2(t')} \\
+ \frac{1}{2} (n_- - n_+) \int_0^t dt' \omega_c(t') + \frac{q^2}{2} \int_0^t \omega_c(t') dt'. \quad (99)
\]

Our result for $\gamma(t)$ differs slightly from the one calculated in [9] as the term $\frac{q^2}{2} \int_0^t \frac{E^2(t')}{M(t') \omega_c(t')}$ doesn’t appear in their result. This is due to their choice of gauge. Concerning the results obtained in Refs [10] [11] [12], the comparison shows that the difference is more large. Indeed, in addition to time dependent electric field contribution $\frac{q^2}{2} \int_0^t \frac{E^2(t')}{M(t') \omega_c(t')}$, the angular momentum contribution $\frac{1}{2} (n_- - n_+) \int_0^t dt' \omega_c(t')$ doesn’t also appear. This is due to the analytical method used. Finally this result for $\gamma(t)$ is reduced to our recently result [7] in absence of electromagnetism field.

The solution of the Schrödinger equation is given by

\[
\psi_{n_+,n_-} (x, y, t) = (-)^n \frac{(\kappa)^{n+|\ell|}}{\rho^3 + \rho^3} \sqrt{\frac{n!}{\Gamma(n + |\ell| + 1)}} e^{-\left(\frac{r}{\rho^2}\right)^2} L_n^{|\ell|} \left( \frac{\kappa}{\rho^2} r_0^2 \right) e^{i\theta} e^{i\gamma(t)}. \quad (100)
\]
Moreover, this work arouses in our mind the following question: what happens in the case of the three dimensional confinement of the charged particle? In this way, the time-dependent Hamiltonian of the system in this new context reads as

\[
\mathcal{H}(t) = \frac{1}{2M(t)} \left[\left(p_1 + \frac{q}{2}B(t)x_2\right)^2 + \left(p_2 - \frac{q}{2}B(t)x_1\right)^2 + p_3^2\right] + \frac{1}{2}M(t)\omega(t)(x_1^2 + x_2^2 + x_3^2) - q[E_1(t)x_1 + E_2(t)x_2].
\]

(101)

Performing the same transformations (7), (8) and for being in the condition where the magnetic field is weak (Ω(t) ≃ ω(t)) for technical reason, the Hamiltonian of the system becomes

\[
\mathcal{H}(t) = \frac{1}{2M(t)}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}\Omega^2(t)M(t)(x^2 + y^2 + z^2) - \frac{\omega(t)}{2}L_z - \frac{q^2E^2(t)}{2M(t)\Omega(t)}.
\]

(102)

By proceeding as in the previous section, the 3D invariant operator is written in the form

\[
\mathcal{I}(t) = \frac{1}{2}\left[\kappa_0^2(\rho^2)(x^2 + y^2 + z^2) + (\rho p_x - M\dot{\rho}x)^2 + (\rho p_y - M\dot{\rho}y)^2 + (\rho p_z - M\dot{\rho}z)^2\right].
\]

(103)

with \( \rho \) satisfying the same equation (33).

Applying the unitary transformation (40), one reduces the invariant operator (103) as follows

\[
\mathcal{I}'(t) = W^\dagger\mathcal{I}(t)W = \frac{1}{2}\left[\kappa_0^2(p_x^2 + p_y^2 + p_z^2) + \frac{\kappa_0^2}{\rho^2}(x^2 + y^2 + z^2)\right],
\]

(104)

with

\[
W = \exp[-i\frac{M\dot{\rho}}{2\rho}(x^2 + y^2 + z^2)].
\]

(105)

Instead of choosing to diagonalize the invariant operator through analytical method proceeding by the cylinder parametrization (r, \( \theta, z \)) of the degrees of freedom of the system, we rather proceed by algebraic method to solve the eigenvalue equation for the invariant operator. The invariant operator and the angular momentum read

\[
\mathcal{I}'(t) = \frac{1}{2}\rho^2 p_z^2 + \kappa\left(a_+^{1}\bar{a}_+^{1} + a_-^{1}\bar{a}_-^{1} + I\right), \quad L_z = a_-^{1}\bar{a}_-^{1} - a_+^{1}\bar{a}_+^{1}.
\]

(106)

The eigenvalues of both operators are given by

\[
\mathcal{I}'(t)|\phi_{n_{\pm},p}(t)\rangle = \alpha_{n_{\pm},p}|\phi_{n_{\pm},p}(t)\rangle, \quad \alpha_{n_{\pm},p} = \frac{1}{2}\rho^2 p^2 + \kappa(n_+ + n_- + 1),
\]

(107)

\[
L_z|\phi_{n_{\pm},p}(t)\rangle = l_{n_{\pm},p}|\phi_{n_{\pm},p}(t)\rangle, \quad l_{n_{\pm},p} = n_- - n_+.
\]

(108)

with \(-\infty < p < \infty\).

One can remark that, the eigenvalues of the invariant operator (107) are time dependent.
2.2 Standard canonical coherent states

Given any complex values \( z_\pm \in \mathbb{C} \) (related to the operator \( a_\pm, a_\pm^\dagger \) ), such as

\[
\begin{align*}
    z_\pm &= \frac{1}{2\sqrt{\kappa}} \left( (M\dot{\rho} + i\frac{\kappa}{\rho})(x \pm iy) - \rho(p_x \pm ip_y) \right), \\
    z_\pm^* &= \frac{1}{2\sqrt{\kappa}} \left( (M\dot{\rho} - i\frac{\kappa}{\rho})(x \mp iy) - \rho(p_x \mp ip_y) \right).
\end{align*}
\]

(109)

(110)

The corresponding helicity (holomorphic) normalised coherent states are defined by

\[
|\psi_{z_-, z_+}\rangle = |\psi_{z_-}\rangle \otimes |\psi_{z_+}\rangle, \quad \text{with} \quad e^{z_+ a_\downarrow^\dagger - z_\downarrow^* a_\downarrow + z_\downarrow a_\downarrow^\dagger - z_\downarrow^* a_\downarrow} |\psi_{00}\rangle,
\]

\[
e^{-\frac{i}{2} |z_-|^2 - \frac{1}{2} |z_+|^2} \sum_{n_-, n_+ = 0}^\infty \frac{(z_-)^n (z_+)^{n_+}}{\sqrt{n_-! n_+!}} e^{-\frac{1}{2} |z_-|^2 - \frac{1}{2} |z_+|^2} \sqrt{\frac{n_-!}{\Gamma(n_-+1)}} \sqrt{\frac{n_+!}{\Gamma(n_++1)}} |\psi_{n_-, n_+}\rangle,
\]

(111)

In coordinate representation for \( \ell > 0 \), the wavefunction for the coherent states is given by

\[
\psi_{z_\pm}(x, y, t) = \sum_{n_-, n_+ = 0}^\infty (-)^{n_-} \frac{(\kappa)^{\frac{1+n_-+n_+}{2}}}{\rho^{1+n_-+n_+} \sqrt{\pi}} e^{-\frac{1}{2} |z_-|^2 - \frac{1}{2} |z_+|^2} \sqrt{\frac{n_-!}{\Gamma(n_-+1)}} \sqrt{\frac{n_+!}{\Gamma(n_++1)}}
\]

\[
e^{(iM\dot{\rho} - \frac{i\kappa}{\rho})^2} \sum_{n_-, n_+ = 0}^\infty \frac{(z_-)^n (z_+)^{n_+}}{\sqrt{n_-! n_+!}} e^{im_\theta} e^{i\gamma(t)}.
\]

(112)

The non-orthogonality of these states are given as follows

\[
\langle \psi_{n_-, n_+} | \psi_{z_-, z_+} \rangle = e^{-\frac{1}{2} |z_-|^2 - \frac{1}{2} |z_+|^2} e^{-\frac{1}{2} |z_-|^2 - \frac{1}{2} |z_+|^2}.
\]

(113)

In terms of these generating vectors, the resolution of the identity is expressed as

\[
\int \frac{dz_+ dz_-^*}{\pi^2} |\psi_{z_+, z_-}\rangle \langle \psi_{z_+, z_-}| = \mathbb{I} = \sum_{n_-, n_+ = 0}^\infty |\psi_{n_+, n_-}\rangle \langle \psi_{n_+, n_-}|,
\]

(114)

with the basic matrix elements for the change of bases given by

\[
\langle \psi_{n_+, n_-} | \psi_{z_-, z_+} \rangle = e^{-\frac{1}{2} |z_-|^2 - \frac{1}{2} |z_+|^2} \sum_{n_-, n_+ = 0}^\infty \frac{(z_-)^n (z_+)^{n_+}}{\sqrt{n_-! n_+!}}
\]

(115)

\[
\langle \psi_{z_+, z_-} | \psi_{n_-, n_+} \rangle = e^{-\frac{1}{2} |z_-|^2 - \frac{1}{2} |z_+|^2} \sum_{n_-, n_+ = 0}^\infty \frac{(z_-^*)^n (z_+^*)^{n_+}}{\sqrt{n_-! n_+!}}.
\]

(116)
From a straightforward analysis using (83), the action of the \( SO(2) = U(1) \) rotation generator \( L_z \) on the coherent states is
\[
e^{i\alpha L_z} |\psi_{z_{\pm}}\rangle = |e^{\pm i\alpha} \psi_{z_{\pm}}\rangle,
\]
where \( \alpha \in \mathbb{R} \).

The actions of the operators \( a_{\pm} \) on \( |\psi_{z_{-},z_{+}}\rangle \) satisfy the following properties
\[
a_{\pm} |\psi_{z_{-},z_{+}}\rangle = z_{\pm} |\psi_{z_{-},z_{+}}\rangle, \quad \langle \psi_{z_{-},z_{+}} | a_{\pm} = \langle \psi_{z_{-},z_{+}} | \partial_{z_{\pm}},
\]
and for the creator operators we have
\[
a_{\mp}^\dagger |\psi_{z_{-},z_{+}}\rangle = \partial_{z_{\pm}} |\psi_{z_{-},z_{+}}\rangle, \quad \langle \psi_{z_{-},z_{+}} | a_{\mp}^\dagger = \langle \psi_{z_{-},z_{+}} | z_{\mp}^*
\]
The actions of the operators \( a_{\pm} \) on \( |\psi_{z_{-},z_{+}}\rangle \) satisfy the following properties
\[
l(z_+,z_+,z_+^*,z_-^*) = |z_+|^2 - |z_+|^2,
\]
\[
h(z_+,z_+,z_+^*,z_-^*) = \frac{1}{\kappa} \left[ \langle \psi_{z_{-},z_{+}} | H(t) |\psi_{z_{-},z_{+}}\rangle \right] = \frac{1}{\kappa} \left[ \langle \psi_{z_{-},z_{+}} | H(t) |\psi_{z_{-},z_{+}}\rangle \right]
\]
\[
= \frac{1}{2\kappa} \left[ M\rho^2 + \frac{\kappa^2}{M\rho^2} + M\Omega^2 \right] \left( |z_+|^2 + |z_-|^2 + 1 \right) - \frac{\omega_c}{2} \left( |z_-|^2 - |z_+|^2 \right) - \frac{q^2 E^2}{2M}\rho^2 + \frac{\Omega^2 M\rho^2}{2M\rho^2} \left( z_- z_+ + z_+ z_- \right)
\]
\[
= \frac{1}{2\kappa} \left[ M\rho^2 + \frac{\kappa^2}{M\rho^2} + M\Omega^2 \right] \left( |z_+|^2 + |z_-|^2 + 1 \right) - \frac{\omega_c}{2} \left( |z_-|^2 - |z_+|^2 \right) - \frac{q^2 E^2}{2M}\rho^2 + \frac{\Omega^2 M\rho^2}{2M\rho^2} \left( z_- z_+ + z_+ z_- \right)
\]

The time evolution of the system reads as
\[
|\psi_{z_{-},z_{+}}(\tau)\rangle = e^{-iH\tau} |\psi_{z_{-},z_{+}}\rangle,
\]
\[
= e^{-\frac{i}{2} |z_-|^2 - \frac{i}{2} |z_+|^2} \sum_{n_{-},n_+=0}^{\infty} \frac{(z_-)^{n_-} (z_+)^{n_+}}{\sqrt{n_-! n_+!}} e^{-i\tau E_{n_+,n_-}} |\psi_{n_{-},n_+}\rangle,
\]
\[
= e^{-\frac{i}{2} |z_-|^2 - \frac{i}{2} |z_+|^2} e^{-i(T_1 - \lambda)\tau} \times \sum_{n_{-},n_+=0}^{\infty} \frac{[a_{\pm}^\dagger e^{-i(T_1 - T_2)\tau} z_-]^n_- [a_{\pm} e^{-i(T_1 + T_2)\tau} z_+]^n_+]}{n_! n_+!} |\psi_{00}\rangle
\]
\[
= e^{-i(T_1 - \lambda)\tau} |e^{-i(T_1 - T_2)\tau} e^{-i(T_1 + T_2)\tau} |\psi_{z_{-},z_{+}}\rangle,
\]
where \( T_1 = \frac{1}{2\kappa} \left( M\rho^2 + \frac{\kappa^2}{M\rho^2} + M\Omega^2 \right) \), \( T_2 = \frac{q^2 E^2}{2M\rho^2} \) and \( \lambda = \frac{q^2 E^2}{2M\rho^2} \).

The Heisenberg uncertainty relations for the simultaneous measurement of the observables \( A \) and \( B \) in the state \( \{ |\psi_{n_{-},n_+}\rangle \} \) has to obey the inequality
\[
\Delta A = \sqrt{\langle \psi_{n_{-},n_+} | A^2 |\psi_{n_{-},n_+}\rangle - \langle \psi_{n_{-},n_+} | A |\psi_{n_{-},n_+}\rangle^2},
\]
\[
\Delta B = \sqrt{\langle \psi_{n_{-},n_+} | B^2 |\psi_{n_{-},n_+}\rangle - \langle \psi_{n_{-},n_+} | B |\psi_{n_{-},n_+}\rangle^2}.
\]
As far as our context is concerned, the standard expectation values of operators $x, y, p_x, p_y$ are evaluating as follows the Heisenberg uncertainty relations can be inferred

$$\Delta x \Delta p_x = \Delta y \Delta p_y = \frac{1}{2} (2n + |\ell| + 1) \sqrt{1 + \frac{M^2 \rho^2 \rho^2}{\kappa^2}} \geq \frac{1}{2}. \quad (126)$$

We can conclude that from this relation the helicity Fock states do not minimize the Heisenberg uncertainty. But for $n = 0$ and $\ell = 0$ we have the following relation

$$\Delta x \Delta p_x = \Delta y \Delta p_y = \frac{1}{2} \sqrt{1 + \frac{M^2 \rho^2 \rho^2}{\kappa^2}} \geq \frac{1}{2}. \quad (127)$$

which minimizes the Heisenberg uncertainty.

The particle distribution is given by the Poissonian function

$$P_{z \pm} (n_+, n_-) = |\langle \psi_{n \pm} | \psi_{z \pm} \rangle|^2 = e^{-(z_-)^2 - (z_+)^2 [z_-^{2n_+} - z_+^{2n_-}]} n_-! n_+!.$$

$$\quad (128)$$

### 2.3 Nonlinear canonical coherent states

We introduce two nonlinear Fock algebras of the type constructed by Jannussis et al.\[7\]. These algebras are factorized with respect to helicity and are defined by two real functions $f_{\pm}(N_{\pm} \neq 0)$ and the generators $A_{\pm}, A_{\pm}^{\dagger}$ such that

$$A_{\pm} = a_{\pm} f_{\pm}(N_{\pm}), \quad A_{\pm}^{\dagger} = f_{\pm}(N_{\pm}) a_{\pm}^{\dagger}, \quad (129)$$

$$[A_{\pm}, A_{\pm}^{\dagger}] = (N_{\pm} + 1)f_{\pm}^2(N_{\pm} + 1) - (N_{\pm})f_{\pm}^2(N_{\pm}), \quad (130)$$

where $N_{\pm} = a_{\pm}^{\dagger} a_{\pm}$. The $f_{\pm}$-coherent states $|\psi_{\alpha \pm, f_{\pm}}\rangle$ are defined by the following equation

$$A_{\pm} |\psi_{\alpha \pm, f_{\pm}}\rangle = \alpha_{\pm} |\psi_{\alpha \pm, f_{\pm}}\rangle. \quad (131)$$

Referring to the previous chapter in the case of the construction of one mode nonlinear coherent states, we deduce

$$|\psi_{\alpha \pm, f_{\pm}}\rangle = N_{\alpha \pm, f_{\pm}} \sum_{n_+ = 0}^{\infty} \sum_{n_- = 0}^{\infty} \frac{(\alpha_+)^{n_+} (\alpha_-)^{n_-}}{\sqrt{n_-! n_+! [f_{\pm}(n_-)][f_{\pm}(n_+)]!}} |\psi_{n \pm}\rangle \quad (132)$$

where

$$N_{\alpha \pm, f_{\pm}} = \sum_{n_+ = 0}^{\infty} \sum_{n_- = 0}^{\infty} \left(|\alpha_+|^{2n_+} |\alpha_-|^{2n_-} n_-! n_+! ([f_{\pm}(n_-)][f_{\pm}(n_+)]!)^{-2}\right)^{-\frac{1}{2}}. \quad (133)$$

In the case $f_1 = f_2 = 1$, we recover the above two-mode coherent states, namely,

$$|\psi_{\alpha \pm, 1}\rangle = e^{-|\alpha_+|^2 |\alpha_-|^2/2} \sum_{n_+ = 0}^{\infty} \sum_{n_- = 0}^{\infty} \frac{(\alpha_+)^{n_+} (\alpha_-)^{n_-}}{\sqrt{n_-! n_+!}} |\psi_{n \pm}\rangle. \quad (134)$$
As we can see, the states (131) are not orthogonal

\[
\langle \psi_{\sigma_{\pm},g_{\pm}} | \psi_{\alpha_{\pm},f_{\pm}} \rangle = N_{\alpha_{\pm},f_{\pm}}N_{\sigma_{\pm},g_{\pm}} \sum_{n_{\pm},m_{\pm}=0}^{\infty} \sum_{m_{\pm}=0}^{\infty} \delta_{n_{\pm},m_{\pm}} \frac{(\alpha_{\pm})^{n_{\pm}}(\sigma_{\pm}^{+})^{m_{\pm}}}{\sqrt{n_{\pm}!m_{\pm}!f_{\pm}(n_{\pm})!g_{\pm}(m_{\pm})!}} \times \frac{(\alpha_{\pm})^{n_{\pm}}(\sigma_{\pm}^{\flat})^{m_{\pm}}}{\sqrt{n_{\pm}!m_{\pm}!f_{\pm}(n_{\pm})!g_{\pm}(m_{\pm})!}} 
\]

The completeness holds for the defined position weight function \( \Omega(t) \) such as

\[
\int_{\mathbb{C}} \frac{\mathcal{N}_{\alpha_{\pm},f_{\pm}}}{\pi^2} |\alpha_{\pm},f_{\pm}\rangle \langle \alpha_{\pm},f_{\pm}| \Omega(|\alpha_{\pm}|^2)d^2\alpha_{\pm} = \mathbb{I}. \tag{136}
\]

This completeness holds for the defined position weight function \( \Omega(t) \) such as

\[
\int_{0}^{\infty} t^{n_{\pm}+n-\Omega^2(t)}d^2t = n_{\pm}!n_{\pm}![f_{\pm}(n_{\pm})]^2[f_{\pm}(n_{\pm})]^2. \tag{137}
\]

Since the state \(|\alpha_{\pm},f_{\pm}\rangle\) is given as series of Fock helicity states, we can easily write the wave function of these states in different representations explicitly. In coordinate representation, the wave function is

\[
\psi_{\alpha_{\pm},f_{\pm}}^{\ell,n}(x,y,t) = (-)^{n_{\pm}}(\kappa)^{\frac{\ell+|\ell|}{2\pi}}(\rho+|\ell|)^{\frac{n_{\pm}!}{\sqrt{\Gamma(n+|\ell|+1)}}} \times \left( e^{iM_{\ell}^{2}t} \right) \frac{r^{\ell}}{\sqrt{n_{\pm}!m_{\pm}!f_{\pm}(n_{\pm})!g_{\pm}(m_{\pm})!}} \sum_{n_{\pm}=0}^{\infty} \sum_{m_{\pm}=0}^{\infty} \left( \frac{\alpha_{\pm}^{n_{\pm}}}{\sigma_{\pm}^{(\flat)}} \right)^{m_{\pm}} e^{i\theta e^{i\gamma}(t)}. \tag{138}
\]

For the Bargmann representation (the usual coherent states), the wave function \( \langle \psi_{z_{\pm},z_{\pm}} | \psi_{\alpha_{\pm},f_{\pm}} \rangle \), where we use the bases \(|\psi_{z_{\pm}}\rangle \) \((z_{\pm} \in \mathbb{C})\) takes the form

\[
\psi_{\alpha_{\pm},f_{\pm}}(z_{\pm}) = \mathcal{N}_{z_{\pm}}^{\frac{1}{2}} |z_{\pm}|^{\frac{1}{2}n_{\pm}} \sum_{n_{\pm}=0}^{\infty} \sum_{m_{\pm}=0}^{\infty} \frac{(z_{\pm}^{\ast}\alpha_{\pm}^{(\flat)})^{n_{\pm}}(z_{\pm}^{\ast}\alpha_{\pm})^{m_{\pm}}}{n_{\pm}!m_{\pm}!f_{\pm}(n_{\pm})!g_{\pm}(m_{\pm})!}. \tag{139}
\]

The particle distribution given by the Poissonian function in the case \( f_{\pm} \)-coherent states becomes

\[
P_{\alpha_{\pm},f_{\pm}} = \left( \sum_{j_{\pm}=0}^{\infty} \sum_{j_{\pm}=0}^{\infty} \frac{|\alpha_{\pm}|^{2j_{\pm}}|\alpha_{\pm}|^{2j_{\pm}}}{j_{\pm}!f_{\pm}(j_{\pm})!^2[f_{\pm}(j_{\pm})]!^2} \right)^{-1} \frac{|\alpha_{\pm}|^{2n_{\pm}}|\alpha_{\pm}|^{2m_{\pm}}}{n_{\pm}!m_{\pm}!f_{\pm}(n_{\pm})!^2[f_{\pm}(n_{\pm})]!^2}. \tag{140}
\]
2.4 Photon added coherent states

The two-mode linear Photon added coherent states of the one mode we defined in the above section are defined as follows

$$|\psi_{\alpha_\pm,m_\pm}\rangle = N_{\alpha_\pm,m_\pm}(a_+^\dagger)^{m^-}(a_-^\dagger)^{m^+}|\psi_{\alpha_\pm}\rangle,$$  \hspace{1cm} (141)

where $m_\pm$ are positive integers being the numbers of added quanta (or added photons) and

$$N_{\alpha_\pm,m_\pm} = \left[ \langle \psi_{\alpha_\pm} | a_- a_+^\dagger (a_-^\dagger)^{m^-} (a_+^\dagger)^{m^+} | \psi_{\alpha_\pm} \rangle \right]^{-\frac{1}{2}}.$$  

These states in terms of Fock states can be written as

$$|\psi_{\alpha_\pm,m_\pm}\rangle = N_{\alpha_\pm,m_\pm}(|\alpha_\pm\rangle \sum_{n_+,n_-=0}^{\infty} \frac{(\alpha_-)^{n_-}(\alpha_+)^{n_+} \sqrt{(n_-+m_-)!/(n_++m_+)!}}{n_-!n_+!} \times |\psi_{n_-+m_-,n_++m_+}\rangle),$$ \hspace{1cm} (142)

where $N_{\alpha_\pm,m_\pm}(|\alpha_\pm\rangle) = N_{\alpha_\pm,m_\pm} e^{-\frac{|\alpha_-|^2}{2} - \frac{|\alpha_+|^2}{2}}$. The non-orthogonality of these states reads as

$$\langle \psi_{\sigma_\pm,m_\pm} | \psi_{\alpha_\pm,m_\pm}\rangle = N_{\alpha_\pm,m_\pm}(|\alpha_\pm\rangle)N_{\alpha_\pm,m_\pm}(|\sigma_\pm\rangle) \sum_{n_+,n_-=0}^{\infty} \frac{(\alpha_- \sigma_-^\dagger)^{n_-}(\alpha_+ \sigma_+^\dagger)^{n_+}(n_-+m_-)!/(n_++m_+)!}{[n_-!]^2[n_+]^2}.$$ \hspace{1cm} (143)

The two-modes photon coherent states are eigenstates of the annihilation operators $a_+, a_-$ given by the following equation

$$a_+ a_- |\psi_{\alpha_\pm,m_\pm}\rangle = \alpha_\pm |\psi_{\alpha_\pm,m_\pm}\rangle.$$ \hspace{1cm} (144)

Multiplying both sides by $(a_-^\dagger)^{m^+} (a_+^\dagger)^{m^-}$

$$(a_-^\dagger)^{m^+} (a_+^\dagger)^{m^-} a_+ a_- |\psi_{\alpha_\pm,m_\pm}\rangle = \alpha_\pm (a_-^\dagger)^{m^+} (a_+^\dagger)^{m^-} |\psi_{\alpha_\pm,m_\pm}\rangle,$$ \hspace{1cm} (145)

which, by making use of the commutation relations $[a_+, (a_+^\dagger)^{m^+}] = m^+(a_+^\dagger)^{m^+-1}$, $[a_-, (a_-^\dagger)^{m^-}] = m^- (a_-^\dagger)^{m^-}$, and the identity

$$(a_+^\dagger)^{m^+} a_+ = a_+(a_+^\dagger)^{m^+} - m^+(a_+^\dagger)^{m^+-1},$$ \hspace{1cm} (146)

$$(a_-^\dagger)^{m^-} a_- = a_-(a_-^\dagger)^{m^-} - m^- (a_-^\dagger)^{m^-},$$ \hspace{1cm} (147)

it leads to

$$\left(1 - \frac{m^+}{a_+ a_+ + 1}\right) \left(1 - \frac{m^-}{a_- a_- + 1}\right) a_+ a_- |\psi_{\alpha_\pm,m_\pm}\rangle = \alpha_\pm |\psi_{\alpha_\pm,m_\pm}\rangle.$$ \hspace{1cm} (148)
From (148) we can see that the two-mode photon-added coherent states arise two-mode class of nonlinear coherent with the corresponding nonlinear function

$$f(N_+, N_-) = \left( 1 - \frac{m_+}{a_+ a_+ + 1} \right) \left( 1 - \frac{m_-}{a_- a_- + 1} \right),$$  \hspace{1cm} (149)

with $N_\pm = a_\pm^\dagger a_\pm$. Another aspect of these states are revealed by considering their photon number distributions $P_{\alpha \pm}(n_\pm, m_\pm)$ which is the probability of finding an oscillator in the state $|\psi_{n_\pm, m_\pm}\rangle$ defined as follows

$$P_{\alpha \pm}(n_\pm, m_\pm) = N_{\pm}^2 \frac{\left| \langle \alpha_+^{2n_+} | \alpha_-^{2n_-} (n_+ + m_+)! (n_+ + m_-)! \right|^2}{[n_-!]^2 [m_+!]^2}.$$  \hspace{1cm} (150)

3 SU(2) symmetry and its coherent states for the model

3.1 SU(2) symmetry of the model

As matter of fact, one observed the degeneracy of energy $E_{n_x}$ or $E_{n_x, n_y}$ of the invariant operator at each levels in term of a fixed value $n = n_x + n_y = n_+ + n_-$ except for the ground state. We have as many states as there are partitions of the natural number $n$ in two natural numbers $n_1$, $n_2$ or $n_+, n_-$ namely $(n + 1)$-dimension states. Therefore, this fact cannot be just a mere numerical coincidence, the must exist a solid explanation for this fact. Indeed even though, the system possesses a $SO(2) = U(1)$ symmetry, this degeneracy cannot be related to it, since the angular-momentum operator does not map states belonging to a same level into one another.

The first thought that comes to one’s mind is that of a symmetry. As a matter of fact, the system possesses a global symmetry larger than the $SO(2) = U(1)$ rotational symmetry considered which explains the degeneracies of the energy spectrum $E(n_x, n_y) = E(n_+, n_-) = E(n) = \kappa(n + 1)$. This larger dynamical symmetry is the $SU(2)$ symmetry or the $SO(3)$ symmetry which are identical algebras but not as groups since $SO(3) = SU(2)/\mathbb{Z}_2$ as a quotient of groups. This $SU(2)$ symmetry is related to the possibility of performing arbitrary $SU(2)$ rotations among the creation (or the annihilation) operators $a_i$ (or $a_i^\dagger$) ($i = 1, 2$), or $a_\pm^\dagger$ (or $a_\pm$). The generators of this algebra

$$J_k = \frac{\sigma_k}{2}, \quad k = 1, 2, 3$$  \hspace{1cm} (151)

where $\sigma_k$ are the usual Pauli matrices such that

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \hspace{1cm} (152)$$

These operators generate the element $SU(2)$ Lie group in the same way that $L_z$ generate $U(1)$ group

$$\mathcal{R}(\theta) = \exp (-i \theta_k J_k), \quad \theta_k \in \mathbb{R}. \hspace{1cm} (153)$$
The commutation relation satisfied by the generators $J_k$ is written

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \quad i, j, k = 1, 2, 3. \quad (154)$$

Let us now see how we may introduce, based on the helicity Fock algebra generators $a_\pm$ and $a_\mp^\dagger$ operators which described the degeneracies of the energy level of the invariant operator $I(t)$. Indeed, we introduce the raising $J_+$ and lowering $J_-$ operators defined as follows

$$J_\pm = J_1 \pm iJ_2. \quad (155)$$

Therefore, we have

$$J_+ = a_+^\dagger a_-, \quad J_- = a_-^\dagger a_+, \quad J_3 = \frac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) = \frac{L_z}{2}. \quad (156)$$

It takes but only a little calculation to obtain

$$[J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm. \quad (157)$$

The action of the operators $J_\pm$ on the states $|\psi_{n\pm}\rangle$ turns to

$$J_+|\phi_{n+, n_-}\rangle = \sqrt{(n_+ + 1)n_-}|\phi_{n+, n_- + 1}\rangle, \quad (158)$$

$$J_-|\phi_{n+, n_-}\rangle = \sqrt{n_+(n_- + 1)}|\phi_{n_, n_- + 1}\rangle, \quad (159)$$

$$J_3|\phi_{n+, n_-}\rangle = \frac{1}{2}(n_+ - n_-)|\phi_{n+, n_-}\rangle. \quad (160)$$

From the helicity occupation numbers $n_\pm$ representation, we introduce new set of representation label by the pair of values $(j, m)$, which we define here as follows

$$n_+ + n_- = n = 2j, \quad n_+ - n_- = 2m, \quad (161)$$

such as

$$n_+ = j + m, \quad n_- = j - m, \quad (162)$$

where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$ for $n = 0, 1, 2, 3, ...$ and $-j \leq m \leq j$. This representation is characterised by the single integer or half-integer number $j$, known as the spin of that representation, whereas the states within that representation of given spin $j$ are distinguished by their $J_3$ eigenvalue $m$ lying between $(-j)$ and $j$ in integer steps, with the above matrix elements for the action of the two other operators $J_\pm$ of the $SU(2)$ algebra.

Therefore we define a certain $SU(2)$ representation as a finite $n + 1 = 2j + 1$ dimensional subspace of the complete Hilbert space defined as follows

$$\mathcal{H}_j = \{|\phi_{n\pm}\rangle = |\phi_{j, m}\rangle : 2j \in \mathbb{N}, \quad m = -j, \ldots, j\}. \quad (163)$$

These states satisfy the properties of orthogonality and completeness

$$\langle \phi_{j, m} | \phi_{j, m} \rangle = \delta_{mm}, \quad \sum_{m=0}^{\infty} |\phi_{j, m}\rangle \langle \phi_{j, m}| = \mathbb{I}. \quad (164)$$
The above action of the operators $J_\pm$ and $J_3$ then read as
\begin{align}
J_+|\phi_{j,m}\rangle &= \sqrt{(j-m)(j+m+1)}|\phi_{j,m+1}\rangle, \\
J_-|\phi_{j,m}\rangle &= \sqrt{(j+m)(j-m+1)}|\phi_{j,m-1}\rangle, \\
J_3|\phi_{j,m}\rangle &= m|\phi_{j,m}\rangle.
\end{align}
(165) (166) (167)

In conclusion, we have in this manner recovered all finite dimensional representations of the symmetry algebra SU(2), which is also the algebra of the symmetry group SO(3) of rotations in three dimensional Euclidean space. The above formula for the action of $J_\pm$ and $J_3$ in a given spin $j$ representation are valid as such in full generality, independently of the system in which these symmetries may be realised. Any rotationally invariant system in three dimensions will find its quantum states organised according to these spin representations of SU(2). But it is matter of experiment to determine which spin values are realised for a specific physical system. For example, that the electron has spin $\frac{1}{2}$ may only be determined experimentally. As an illustration, consider the value $n = 1$ or $j = \frac{1}{2}$, namely the first excited state of the present system. It is thus doubly degenerated, with
\begin{align}
J_3|\phi_{\frac{1}{2},\frac{1}{2}}\rangle &= \frac{1}{2}|\phi_{\frac{1}{2},\frac{1}{2}}\rangle, ~ J_+|\phi_{\frac{1}{2},\frac{1}{2}}\rangle = 0, ~ J_-|\phi_{\frac{1}{2},\frac{1}{2}}\rangle = \frac{1}{2}|\phi_{\frac{1}{2},-\frac{1}{2}}\rangle, \\
J_3|\phi_{\frac{1}{2},-\frac{1}{2}}\rangle &= \frac{1}{2}|\phi_{\frac{1}{2},-\frac{1}{2}}\rangle, ~ J_+|\phi_{\frac{1}{2},-\frac{1}{2}}\rangle = \frac{1}{2}|\phi_{\frac{1}{2},\frac{1}{2}}\rangle, ~ J_-|\phi_{\frac{1}{2},-\frac{1}{2}}\rangle = 0,
\end{align}
(168) (169)
and the associations $|\phi_{\frac{1}{2},m=\frac{1}{2}}\rangle = |\phi_{n=1,n_-=0}\rangle$, $|\phi_{\frac{1}{2},m=-\frac{1}{2}}\rangle = |\phi_{n=0,n_-=1}\rangle$. In other words, in this two dimensional bases, the matrix representation of the operators $J_3$ and $J_\pm$ is given precisely by the Pauli matrices. Clearly, the ground state of the system, $|\phi_{j=0,m=0}\rangle = |\phi_{n=0,n_-=0}\rangle$, corresponds to a trivial representation of the SU(2) algebra for which $J_3 = 0$ and $J_\pm = 0$.

Based on what followed, the action of operators $I(t), L_z$ are given by
\begin{align}
I(t)|\phi_{j,m}\rangle &= 2\kappa(j+1)|\phi_{j,m}\rangle, \\
L_z|\phi_{j,m}\rangle &= \frac{m}{2}\phi_{j,m}\rangle,
\end{align}
(170) (171)
and the expectation values of the Hamiltonian is given by
\begin{equation}
\langle\phi_{j,m}|H(t)|\phi_{j,m}\rangle = \frac{(j+1)}{\kappa} \left( M\rho^2 + \frac{\kappa^2}{M\rho^2} + M\Omega^2\rho^2 \right) - m\omega_c - \frac{q^2E^2}{2M\omega}. \tag{172}
\end{equation}

By successively applying $J_+$ on the ground state $|\phi_{j,-j}\rangle$, we generate the state $|\phi_{j,m}\rangle$ of the system as follows
\begin{equation}
|\phi_{j,m}\rangle = \sqrt{\frac{(j-m)!}{(j+m)!(2j)!}}(J_+)^{j+m}|\phi_{j,-j}\rangle, \tag{173}
\end{equation}
where
\begin{equation}
J_-|\phi_{j,-j}\rangle = 0. \tag{174}
\end{equation}
\text{21}
3.2 SU(2) coherent states

The SU(2) coherent states for this model denoted by $|\phi_{j,\zeta}\rangle$ are obtained from action of rotational group Lie on the Lowest weight state $|\psi_{j,-j}\rangle$.

$$|\phi_{j,\zeta}\rangle = e^{\zeta J_+ + \zeta^* J_-} |\phi_{j,-j}\rangle = (1 + |\zeta|^2)^{-j} \sum_{m=0}^{2j} \left[ \frac{(2j)!}{(m)!(2j-m)!} \right]^{\frac{1}{2}} \zeta^m |\phi_{j,m}\rangle,$$  

(175)

where $\zeta = e^{i\phi} \tan \frac{\theta}{2}$ defined the complex plane which can been seen as a stereographic projection of sphere ($S^2$) onto $\mathbb{C}$ of unit vectors $(\theta, \phi)$.

These are normalized but are non-orthogonal states and satisfy the completeness relations

$$\langle \phi_{j,\zeta_1}|\phi_{j,\zeta_2}\rangle = (1 + |\zeta_1|^{-j}(1 + |\zeta_2|)^{-j}(1 + \zeta_1^* \zeta_2)^{2j},$$  

(176)

$$\int d\mu_j(\phi_{j,\zeta}^*) \langle \phi_{j,\zeta}| = \sum_{m=0}^{2j} |\phi_{j,m}\rangle \langle \phi_{j,m}| = I.$$

(177)

$d\mu_j(j, \zeta) = \frac{2^{j+1}}{\pi} \frac{d^2\zeta}{(1 + |\zeta|^2)^{2j+1}}$.

For an arbitrary states $|\Phi\rangle = \sum_{m=0}^{2j} c_m |\psi_{j,m}\rangle$ of $\mathcal{H}_j$, one can use the above states to construct in this space an analytical function $f(\zeta)$ such as

$$f(\zeta) = (1 + |\zeta|^2)^j \langle \phi_{j,\zeta}^*|\Phi\rangle,$$  

(178)

$$\sum_{m=0}^{2j} \left[ \frac{(2j)!}{(m)!(2j-m)!} \right]^{\frac{1}{2}} \zeta^m |\phi_{j,m}\rangle.$$  

(179)

The expansion of $|\Phi\rangle$ on the SU(2) coherent states bases is

$$|\Phi\rangle = \int d\mu_j(j, \zeta) (1 + |\zeta|^2)^{-j} f(\zeta^*) |\phi_{j,\zeta}\rangle,$$

(180)

$$\langle \Phi|\Phi\rangle = \int d\mu_j(j, \zeta) (1 + |\zeta|^2)^{-j} |f(\zeta^*)|^2 < \infty.$$

(181)

3.3 SU(2) photon added coherent states

The Photon-added SU(2) coherent states are obtained by successive application of the raising operator $J_+$ on SU(2) coherent states

$$|\phi_{j,\zeta, p}\rangle = \mathcal{W}_p(|\zeta|)(J_+^p|\phi_{j,\zeta}\rangle$$

(182)

Making use of the expression

$$(J_+^p|\phi_{j,m}\rangle = \left[ \frac{(m+p)!(2j-m)!}{(m)!(2j-m-p)!} \right]^{\frac{1}{2}} |\phi_{j,m+p}\rangle,$$

(183)
we finally obtain

\[ |\phi_{j\zeta,p}\rangle = \mathcal{W}_p^j(|\zeta|) \sum_{m=0}^{2j-p} \frac{(2j)! (m+p)!}{m! (2j-m-p)!} \zeta^m |\phi_{j,m+p}\rangle \]  

(184)

Using the above expression, we obtain the normalization

\[ \langle \phi_{j\zeta,p}|\phi_{j\zeta,p}\rangle = 1, \]  

(185)

from which results the normalization constant

\[ \mathcal{W}_p^j(|\zeta|) = \left( \frac{\Gamma(1+2j) \Gamma(1+p) \mathcal{F}_1(1+p,-2j+p;1;-|z|^2)}{\Gamma(1+2j-p)} \right)^{-1/2} \]  

(186)

The completeness relation is given as

\[ \int d\mu(\zeta,\zeta^*) |\phi_{j,\zeta}\rangle \langle \phi_{j,\zeta}| = \Pi_p^j = \sum_{m=0}^{2j} |\phi_{j,m+p}\rangle \langle \phi_{j,m+p}|. \]  

(187)

where the integration measure \( d\mu(\zeta,\zeta^*) \) is given by

\[ d\mu(\zeta,\zeta^*) = V_p^j(|\zeta|) \frac{d^2\zeta}{\pi}. \]  

(188)

In case of \( p = 0 \) the weight function is reduced to \( V_0^j(|\zeta|) = \frac{2j+1}{\pi (1+|\zeta|^2)}. \) Thus, in order to resolve the identity operator in (187), one should find the general form of the weight function \( V_p^j(|\zeta|) \). By means of a change of the complex variables in terms of polar coordinates \( \eta = re^{i\rho} \), using the completeness of the states \( |\phi_{j,m+p}\rangle \) and integrating on the angular variable \( \rho \), we can use the resolution of the identity operator (187) to solve the following integral equation:

\[ \int_0^\infty x^m \tilde{V}_j^p(x) dx = \frac{(m!)^2 (2j-m-p)!}{(2j)!(m+p)!} \]  

(189)

where \( x = r^2 \) and \( \tilde{V}_j^p(x) = |\mathcal{W}_p^j(x)|^2 V_p^j(x) \). This is a Stieltjes moment problem and the function \( V_j^p(x) \) may be found using the substitution \( m = s - 1 \)

\[ \int_0^\infty x^{s-1} \tilde{V}_j^p(x) dx = \frac{\Gamma(s)^2 \Gamma(2j+2-p-s)}{\Gamma(2j+1) \Gamma(m+p)} \]  

(190)

It is useful to express the unknown function through the Meijer’s G-functions and the Mellin inversion theorem. Therefore using the definition of Meijer’s G-function, it follows that

\[ \int_0^\infty dxx^{s-1} G^{m,m}_{p,q} \left( \alpha x | a_1, \cdots, a_n, a_{n+1}, \cdots, a_p \right. \left| b_1, \cdots, b_n, b_{n+1}, \cdots, b_q \right) \]
\[ V_p(\xi) = \frac{1}{[\mathcal{W}_p(\xi)]^2 \Gamma(2j + 1)} G_{2,2}^{2,1}(x_{0,0}^{-2j-1+p}, p) \]  

(193)

The weight function \( V_p(\xi) \) becomes

\[ V_j(\xi) = \frac{1}{[\mathcal{W}_j(\xi)]^2 \Gamma(2j + 1)} G_{2,2}^{2,1}(x_{0,0}^{-2j-1+p}, p) \]  

(193)

For an arbitrary states \( |\Phi\rangle = \sum_{m=0}^{2j} c_m |\phi_{k,m+n}\rangle \) in this space one may use the above states to construct in this space an analytical function \( f(\xi) \) such as

\[ f(\xi) = (\mathcal{W}_j)^{-1} \langle \xi | \Phi \rangle = \sum_{m=0}^{2j} c_m \left[ \frac{(2j)!(m+p)!}{m!(2j-m-p)!} \right]^{\frac{1}{2}} \xi^m \]  

(194)

The photon count probability is given by

\[ P_{\xi}(p) = |\langle \phi_{m+p} | \phi_{j-p} \rangle|^2, \]

\[ = (\mathcal{W}_j)^2(\xi) \sum_{m=0}^{2j-p} |\xi|^{2m} \frac{(2j)!(m+p)!}{m!(2j-m-p)!}. \]  

(195)

4 \ SU(1,1) symmetry and its coherent states for the model

As we pointed out in the previous section, the eigenvalues of the invariant operator \( I(t) \) which describes the stationary dynamic of the system is degenerated. This degeneracy is related to the presence of the \( SU(2) \) symmetry in the system. Whereas, the eigenvalues of the Hamiltonian \( H(t) \) of the system is not degenerated. This nondegeneracy can be described by the presence of another symmetry which is the \( SU(1,1) \) symmetry.

Therefore, the \( SU(1,1) \) algebra realization of the system may be spanned by the following generators

\[ K_0 = \frac{1}{2} \left( a_+^\dagger a_+ + a_-^\dagger a_- + 1 \right), \quad K_+ = a_+^\dagger a_-^\dagger, \quad K_- = a_+a_- \]  

(196)

which satisfy

\[ [K_0, \pm K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0. \]  

(197)

The action of these operators on the Fock space states \( \{|\psi_{k,n}\rangle\} \) is

\[ K_+ |\psi_{n+,n_-}\rangle = \sqrt{(n_++1)(n_-+1)} |\psi_{n+1,n_-+1}\rangle, \]  

(198)

\[ K_- |\psi_{n+,n_-}\rangle = \sqrt{n_+n_-} |\psi_{n-,n_-+1}\rangle, \]  

(199)

\[ K_0 |\psi_{n+,n_-}\rangle = \frac{1}{2} (n_+ + n_- + 1) |\psi_{n+1,n_-+1}\rangle, \]  

(200)

\[ K_+ K_- |\psi_{n+,n_-}\rangle = n_+n_- |\psi_{n+,n_-}\rangle \]  

(201)
From the helicity occupation numbers $n_{\pm}$ representation, we introduce new set of representation label by the pair of values $(k, m)$, which we define here as follows

$$k = \frac{1}{2}(n_+ - n_+ + 1), \quad m = n_+, \quad (202)$$

where the parameters $k$ is the Bargmann index which determine the representation of $m$.

The action of operators $K_{\pm}$ and $K_0$ on the space span by $\{|k, m\rangle, m = 0, 1, 2, \ldots\}$ is

$$K_+ |\psi_{k,m}\rangle = \sqrt{(m + 1)(2k + m)}|\psi_{k,m+1}\rangle, \quad (203)$$

$$K_- |\psi_{k,m}\rangle = \sqrt{m(2k + m - 1)}|\psi_{k,m-1}\rangle, \quad (204)$$

$$K_0 |\psi_{k,m}\rangle = (k + m)|\psi_{k,m}\rangle. \quad (205)$$

We recover from these actions of operators the standard $SU(1,1)$ irreducible representation

$$|k, m\rangle = \sqrt{\frac{\Gamma(2k)}{m!\Gamma(2k + m)}}(K_+)^m |k, 0\rangle, \quad (206)$$

with

$$K_- |k, 0\rangle = 0. \quad (207)$$

The corresponding Barut-Giraldello and the Perelomov coherent states are given by

$$|\psi_{k,z}\rangle = \sqrt{|z|^{2k-1}} \frac{\sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!\Gamma(2k + 2m)}} |\psi_{k,m}\rangle}{I_{2k-1}(2|z|)}, \quad (208)$$

$$|\psi_{k,\eta}\rangle = (1 - |\eta|^2)^k \frac{\sum_{m=0}^{\infty} \frac{\Gamma(2k + m)}{m!\Gamma(2k)} \eta^m |\psi_{k,m}\rangle}{\sqrt{\Gamma(2k + m)}}, \quad (209)$$

From these coherent states, we aim to construct the corresponding the photon added coherent and the nonlinear coherent states

4.1 Photon added coherent states

4.1.1 Perelomov photon added coherent states

The photon added coherent states associated to Perelomov $SU(1,1)$ coherent states are obtained by successive action of the raising operator $K_+$ on the Perelomov $SU(1,1)$ coherent states

$$|\psi_{k,\eta}\rangle = \mathcal{N}_i(K_+)^m |\psi_{k,\eta}\rangle,$$

$$= \mathcal{N}_i(1 - |\eta|^2)^k \sum_{m=0}^{\infty} \frac{\Gamma(2k + m)}{m!\Gamma(2k)} \eta^m (K_+)^m |\psi_{k,m}\rangle. \quad (210)$$
Making use of the expression

\[
(K_+)^l |\psi_{kn}\rangle = \sqrt{\frac{(m + l)! (m + 2k + l - 1)!}{m! (2k + m - 1)!}} |\psi_{k,m+l}\rangle = \sqrt{\frac{\Gamma(m + l + 1) \Gamma(m + 2k + l)}{m! \Gamma(2k + m)}} |\psi_{k,m+l}\rangle, \tag{211}
\]

we then obtain

\[
|\psi_{kn,l}\rangle = \mathcal{N}_l(\eta) \sum_{m=0}^{\infty} \frac{\eta^m}{\sqrt{F_l(k, m)}} |\psi_{k,m+l}\rangle, \tag{212}
\]

where

\[
F_l(k, m) = \left[ \frac{\Gamma(m + l + 1) \Gamma(m + 2k + l)}{(m!)^2 \Gamma(2k)} \right] (1 - |\eta|^2)^k
\]

\[
\mathcal{N}_l(\eta) = \frac{1}{\sqrt{\langle \psi_{kn} | (K_-)^l (K_+)^l | \psi_{kn}\rangle}} \tag{214}
\]

The non-orthogonality is expressed as

\[
\langle \psi_{kn',l'} | \psi_{kn,l}\rangle = \mathcal{N}_l(\eta) \mathcal{N}_{l'}(\eta') \sum_{m=0}^{\infty} \frac{\eta^m \eta'^{m'}}{\sqrt{F_l(k, m) F_{l'}(k, m')}} \langle \psi_{k,m+l} | \psi_{k,m+l}\rangle
\]

\[
\times \sum_{m=0}^{\infty} \frac{\Gamma(m + 2l - l' + 1) \Gamma(m + 2l - l' + 2k)}{m! \Gamma(2k)} \frac{1}{\sqrt{m! \Gamma(2k)}}
\]

\[
\times \frac{\Gamma(m + l + 1) \Gamma(m + 2k + l)}{m! \Gamma(2k)} (\eta' \ast \eta)^m \tag{215}
\]

The overcompleteness property of these states is

\[
\int_{\mathbb{C}} d\nu(\eta, \eta^*) |\psi_{kn,l}\rangle \langle \psi_{kn,l}| = \mathbb{I}_k^n = \sum_{m=0}^{\infty} |\psi_{k,m+l}\rangle \langle \psi_{k,m+l}|, \tag{216}
\]

where the determination of the integration measure \(d\nu(\eta, \eta^*)\) guarantees the resolution of the unity. To this end, we assume the existence of a positive weight function \(W_k^l(\eta)\) such that

\[
d\nu(\eta, \eta^*) = \frac{d^2\eta}{\pi} W_k^l(|\eta|), \tag{217}
\]

where \(d^2\eta\) the usual Lebesgue measure. For \(l = 0\) this weight function is reduced to the ordinary Perelomov one,

\[
W_k^0(|\eta|) = \frac{2k - 1}{\pi (1 - |\eta|^2)^2}, \tag{218}
\]

26
By means of a change of the complex variables in terms of polar coordinates $\eta = re^{i\varphi}$ where $r \in \mathbb{R}_+$, $\varphi \in [0, 2\pi)$, and $d^2(\eta) = rdrd\varphi$, the equation (210) becomes

$$\sum_{m'm=0}^{\infty} \frac{1}{\sqrt{F_l(k,m)F_l(k,m')}} \int_0^\infty d\tau r^{1+m'+m} W_l^+(r^2) J_l^2(r^2) \int_0^{2\pi} \frac{d\varphi}{\pi} e^{i(m-m')\varphi}$$

$$\times |\psi_{k,m+r}\rangle \langle \psi_{k,m'+r}| = I_l^k$$  \hspace{1cm} (219)

By performing the angular integration, i.e

$$\int_0^{2\pi} \frac{d\varphi}{\pi} e^{i(m-m')\varphi} = 2\delta_{mm'},$$  \hspace{1cm} (220)

the resolution of the identity operator (221) is

$$2 \sum_{m=0}^{\infty} \frac{1}{F_l(k,m)} \int_0^\infty d\tau r^{1+2m} W_l^+(r^2) J_l^2(r^2)$$

$$|\psi_{k,m+r}\rangle \langle \psi_{k,m'+r}| = I_l^k.$$  \hspace{1cm} (221)

Setting the weight function such as

$$W_l^+(r^2) = \frac{1}{N_l^2(r^2)} g_k^l(r^2),$$  \hspace{1cm} (222)

and using the completeness of the states $|\psi_{k,m+r}\rangle$ and taking $x = r^2$, we have the following integral to solve

$$\int_0^\infty dx r^m g_k^l(x) = F_l(k, m) = \frac{(m!)^2 \Gamma(2k)}{\Gamma(m + l + 1)\Gamma(m + 2k + l)}.$$  \hspace{1cm} (223)

To solve this integral, one should find the function $g_k^l(x)$. To do so, we proceed as follows. Instead of solving this integral for $g_k^l(x)$, we shall study its existence of solutions. The equation (222) is the well-known Stieltjes moment problem, and this integral equation does not admit a general solution. In fact, whether a solution exists or not and the form of this solution in the positive case depends on the form of the box function. For solving this integral equation, we use the Fourier transforms method by multiplying Eq. (222) by $\frac{(iy)^m}{m!}$ and summing over $m$ yields

$$\int_0^\infty dx e^{iyx} g_k^l(x) = \sum_{m=0}^{\infty} F_l(k, m) \frac{(iy)^m}{m!} = \bar{g}_k^l(y).$$  \hspace{1cm} (224)

In the case where the series defining the function $\bar{g}_k^l(y)$ above converges, the inverse Fourier transforms reads

$$g_k^l(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixy} \bar{g}_k^l(y) dy.$$  \hspace{1cm} (225)
Finally, the weight function $W^l_k(x)$ allowing for a resolution of the identity operator, is written as

$$W^l_k(x) = \frac{1}{2\pi N_l^2(x)} \int_{-\infty}^{+\infty} e^{-ix\bar{g}_l^k(y)}dy.$$  \hspace{1cm} (226)

For an arbitrary states $|\Phi\rangle = \sum_{n=0}^{\infty} c_n|\psi_{k,m+n}\rangle$ in this space one can use the above states to construct in this space an analytical function $f(\eta)$ such as

$$f(\eta) = \mathcal{N}_l^{-1}(|\eta||\psi_{k,\eta}|\Phi),$$  \hspace{1cm} (227)

$$= c_m \sum_{m=0}^{\infty} \frac{(\eta^*)^m}{\sqrt{F_n(k,m)}}.$$  \hspace{1cm} (228)

The photon distribution is given by

$$\rho_{k\eta}(n) = |\langle \psi_{l+m,\eta} | \psi_{k\eta} \rangle|^2,$$

$$= \mathcal{N}_l^2(|\eta|) \sum_{m=0}^{\infty} \frac{|\eta|^{2m}}{F_l(k,m)}.$$  \hspace{1cm} (229)

4.1.2 Barut-Giraldello photon added coherent states

Photon added of Barut-Giraldello coherent states can be obtained by repeated application of the raising operator $K_+$ on the Barut-Giraldello coherent states

$$|\psi_{kz,n}\rangle = \mathcal{M}_n(K_+)^n|\psi_{kz}\rangle,$$

$$= \mathcal{M}_n \sqrt{\frac{|z|^{2k-1}}{I_{2k-1}(2|z|)}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m! F_m(k,m)}} (K_+)^n|\psi_{k,m}\rangle,$$

$$= \mathcal{M}_n(|z|) \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{\rho_n(k,m)}} |\psi_{k,m+n}\rangle,$$  \hspace{1cm} (230)

where we have used the notation

$$\rho_n(k,m) = \frac{[\Gamma(m+1)]^2[\Gamma(m+2k)]^2}{\Gamma(m+n+1)\Gamma(m+n+2k)},$$  \hspace{1cm} (231)

$$\mathcal{M}_n(|z|) = \frac{\sqrt{\frac{|z|^{2k-1}}{I_{2k-1}(2|z|)}}}{\sqrt{\langle \psi_{kz} | (K_-)^n(K_+)^n | \psi_{kz} \rangle}}$$  \hspace{1cm} (232)

The non-orthogonality condition is realised as follows

$$\langle \psi_{kz,n'|\psi_{kz,n}\rangle = \mathcal{M}'_n(|z|) \mathcal{M}_n(|z|) \frac{z^n(z^*)^{n'}}{\sqrt{\rho_n(k,m)\rho_{n'}(k,m')}} |\psi_{k,m+n'}|\psi_{k,m+n}\rangle.$$

(233)

Due to the orthogonality relation of the number vectors (206), it follows that

$$\langle \psi_{kz,n'}|\psi_{kz,n}\rangle = \mathcal{M}'_n(|z|) \mathcal{M}_n(|z|) \frac{[\Gamma(n+1)]^2[\Gamma(n+2k)]^2}{\Gamma(n-n'+1)\Gamma(n-n'+2k)\Gamma(2k)} \times$$

$$\times$$

28
\[ 2F_3(n + 1, n + 2k; n - n' + 1, n - n' + 2k; z_2^2 z_1), \]  

(234)

where \( n \) and \( n' \) are positive integers, \( n \geq n' \) and \( 2F_3(\cdots; z_2^2 z_1) \) is the generalized hypergeometric series (see appendix)

The resolution of unity operator in this space is

\[
\int_C \frac{d^2 z}{\pi} Y^n_k(|z|) |\psi_{kz,n}\rangle \langle \psi_{kz,n}| = 1^n_k = \sum_{m=0}^{\infty} |\psi_{k,m+n}\rangle \langle \psi_{k,m+n}|. \tag{235}
\]

The overcompleteness is satisfy if one can find a function \( Y^n_k(|z|) \). At the limit \( n = 0 \), this weight function must lead to the weight function of the ordinary Barut-Giraldello coherent states,

\[ Y^0_k(|z|) = 2K_{2k-1}(2|z|)I_{2k-1}(2|z|). \tag{236} \]

For \( n \neq 0 \), by substituting equation (230) into equation (235) we obtain

\[
\int_C \frac{d^2 z}{\pi} Y^n_k(|z|)[\mathcal{M}_n(|z|)] \sum_{m,m'=0}^{\infty} \frac{(z^*)^{m'} z^m}{\rho_n(k, m') \rho_n(k, m)} |\psi_{k,m+n}\rangle \langle \psi_{k,m'+n}| = 1^n_k. \tag{237}
\]

Then, it is obvious that the weight function \( Y^n_k(|z|) \) must have the following structure

\[ Y^n_k(|z|) = \frac{1}{[\mathcal{M}_n(|z|)]^2} |z|^{2m} g^m_k(|z|) \]  

(238)

By considering the following transformation \( z = re^{i\theta}, \ r \in \mathbb{R}^+, \ \theta \in [0, 2\pi) \), \( d^2 z = rd\theta d\theta \)

and after performing the angular integration, i.e

\[
\int_0^{2\pi} \frac{d\theta}{\pi} e^{i(m-m')\theta} = 2\delta_{mm'}, \tag{239}
\]

equation (237) becomes

\[
2 \sum_{m=0}^{\infty} \frac{1}{\rho_n(k, m)} \int_0^{\infty} dr r^{2m+2n+1} g^m_k(r^2) |\psi_{k,m+n}\rangle \langle \psi_{k,m+n}| = 1^n_k. \tag{240}
\]

When we perform the variable change \( r^2 = x \) and \( n + m = s - 1 \), the integral from the above equation is called the Mellin transform (see Appendix)

\[
\int_0^{\infty} dx x^{s-1} g^m_k(x) = \rho_n(k, s - n - 1) = \frac{[\Gamma(s-n)]^2 [\Gamma(s-n+2k-1)]^2}{\Gamma(s)\Gamma(s+2k-1)} \tag{241}
\]

Using the definition of Meijer’s G-function, it follows that

\[
\int_0^{\infty} dx x^{s-1} G_{p,q}^{m,n} \left( \alpha_{a_1}, \ldots, a_m, a_{m+1}, \ldots, a_p \left| \beta_1, \ldots, b_m, b_{m+1}, \ldots, b_q \right. \right)
\]

29
\[ \frac{1}{\alpha^s} \prod_{j=m+1}^{p} \Gamma(1 - b_j - s) \prod_{j=1}^{m} \Gamma(b_j + s) \prod_{j=1}^{n} \Gamma(1 - a_j - s) \prod_{j=1}^{q} \Gamma(a_j + s). \]  

Comparing equations (241) and (242), we obtain that
\[ g_k^n(x) = G_{2,4}^{4,0} \left( x |^{0, 2k-1}_{-n, 2k-1-n} \right). \]

Then, the weight function becomes
\[ Y_k^n(|z|) = \frac{1}{|M_n(|z|)|^2} G_{2,4}^{4,0} \left( |z|^2 |^{n, 2k-1+n}_{0, 0} \right). \]

Finally, the resolution of unity can be written in an exhaustive manner
\[ \int_{\mathbb{C}} \frac{d^2 z}{\pi} \frac{1}{|M_n(|z|)|^2} G_{2,4}^{4,0} \left( |z|^2 |^{n, 2k-1+n}_{0, 0} \right) |\psi_{k,m+n}\rangle \langle \psi_{k,m+n}| = \mathbb{I}^n_k. \]

For an arbitrary states \(|\Psi\rangle = \sum_{m=0}^{\infty} c_m |\psi_{k,m+n}\rangle\) in this space one can use the above states to construct in this space an analytical function \(f(z)\) such as
\[ f(z) = M_n^{-1}(|z|) \langle \psi_z^k | \Psi \rangle = \sum_{m=0}^{\infty} c_m \frac{z^m}{\rho_n(k, m)}. \]

The photon distribution is given by
\[ P_{kz}(n) = |\langle \psi_{n+m} | \psi_{kz,n} \rangle|^2, \]
\[ = M_n^2(|z|) \sum_{m=0}^{\infty} \frac{|z|^{2m}}{\rho_n(k, m)}. \]

5 \ SU(1,1) coherent states from factorisation of \(U(1)\) wavefunction

To construct the \(SU(1,1)\) coherent states for this system whose eigenfunction is expressed in terms of the generalized Laguerre functions, we firstly factorise this eigenfunction to discover the hidden \(su(1,1)\) symmetry of the system. Secondly, we construct the corresponding coherent states and deduce the related properties. Finally, from the \(SU(1,1)\)-coherent states constructed we constructed the associated nonlinear and photon added coherent states.

5.1 The hidden dynamical \(su(1,1)\) algebra

We construct here the raising and lowering operators from the eigenfunctions of this system which generate the hidden \(su(1,1)\) algebra. Since the eigenfunctions of the invariant operator and the Hamiltonian are expressed in terms of the generalized Laguerre functions \(L^n_\ell(u)\) with \(\ell > 0\). It is important to review some useful properties
related to this special function that will be used to generate the symmetry operators. Thus, the generalised Laguerre polynomials \( L_n^\ell(u) \) are defined as

\[
L_n^\ell(u) = \frac{1}{n!} u^n e^{-u} \frac{d^n}{du^n} (e^{-u} u^{n+\ell}). \tag{248}
\]

For \( \ell = 0 \), \( L_n^0(u) = L_n(u) \), and for \( n = 0 \), \( L_0^\ell(u) = 1 \). The generating functions corresponding to the associated Laguerre polynomials are

\[
\frac{e^{z} u^{-\ell} - 1}{(1-z)^{\ell+1}} = \sum_{n=0}^{\infty} L_n^\ell(u) z^n, \quad |z| < 1, \tag{249}
\]

\[
J_{\ell} \left( 2 \sqrt{u z} \right) e^{z (u z)} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+\ell+1)} L_n^\ell(u), \tag{250}
\]

where the \( J_\kappa(x) \) is the ordinary Bessel function of \( \kappa \)-order.

The orthogonality relation is

\[
\int_0^\infty du e^{-u} L_n^\ell(u) L_m^\ell(u) = \frac{\Gamma(\ell + n + 1)}{n!} \delta_{nm}, \tag{251}
\]

The generalised Laguerre polynomials satisfy the following differential equation

\[
\left[ u \frac{d^2}{du^2} + (\ell - u + 1) \frac{d}{du} + n \right] L_n^\ell(u) = 0, \tag{252}
\]

and the recurrence relations

\[
(n + 1) L_{n+1}^\ell(u) - (2n + \ell + 1 - u) L_n^\ell(u) + (n + \ell) L_{n-1}^\ell(u) = 0, \tag{253}
\]

\[
u \frac{d}{du} L_n^\ell(u) - n L_n^\ell(u) + (n + \ell)L_{n-1}^\ell(u) = 0. \tag{254}
\]

In view of these, we rewrite the eigenfunction of the invariant operator \( (69) \) in the form

\[
\phi_n^\ell(u) = N(\rho, \alpha) \sqrt{\frac{n!}{\Gamma(n+\ell+1)}} e^{-\frac{\beta}{2} u} L_n^\ell(u), \tag{255}
\]

where \( u = \frac{2}{\rho^2} r^2 \), \( N(\rho, \theta) = (-)^n \sqrt{\frac{\kappa}{2 \pi \rho^2}} e^{i \theta} \), \( \beta = 1 - i M \frac{\phi_k}{\kappa} \), and \( \Gamma(n) = (n-1)! \).

Basing on the recurrence relation \( (253) \) and \( (254) \), we obtain the following equations

\[
\left( -u \frac{d}{du} + \frac{\ell}{2} + n - \frac{\beta}{2} u \right) \phi_n^\ell(u) = \sqrt{n(n+\ell)} \phi_{n-1}^\ell(u), \tag{256}
\]

\[
\left( u \frac{d}{du} + \frac{\ell}{2} + n - \frac{\beta}{2} u + 1 \right) \phi_n^\ell(u) = \sqrt{(n+1)(n+\ell+1)} \phi_{n+1}^\ell(u), \tag{257}
\]

31
where $\tilde{\beta} = 2 - \beta$. For the sake of simplicity we define the raising operator $K_+$ and the lowering operator $K_-$ for the generalised Laguerre functions as

$$
K_- = \left(-u \frac{d}{du} + \frac{\ell}{2} + n - \frac{\beta}{2} u\right), \quad (258)
$$

$$
K_+ = \left(u \frac{d}{du} + \frac{\ell}{2} + n - \frac{\beta}{2} u + 1\right), \quad (259)
$$

and hence obtain

$$
K_- \phi_n^\ell(u) = \sqrt{n(n + \ell)} \phi_{n-1}^\ell(u), \quad (260)
$$

$$
K_+ \phi_n^\ell(u) = \sqrt{(n+1)(n+\ell+1)} \phi_{n+1}^\ell(u). \quad (261)
$$

By multiplying both side of the latter equations by the factor $e^{i\gamma_n,\ell(t)}$ we have

$$
K_- \psi_n^\ell(u) = \sqrt{n(n + \ell)} \psi_{n-1}^\ell(u), \quad (262)
$$

$$
K_+ \psi_n^\ell(u) = \sqrt{(n+1)(n+\ell+1)} \psi_{n+1}^\ell(u). \quad (263)
$$

By successively applying $K_+$ on the ground state $\psi_0^\ell(u)$, we generate the eigenfunction $\psi_n^\ell(u)$ of the system as follows

$$
\psi_n^\ell(u) = \sqrt{\frac{\Gamma(1+\ell)}{n!\Gamma(n+\ell+1)}} (K_+)^n \psi_0^\ell(u), \quad (264)
$$

where,

$$
\psi_0^\ell(u) = \frac{N(\rho,\theta)}{\sqrt{\Gamma(\ell+1)}} u^{\frac{\ell}{2}} e^{-u/2} e^{i\theta_{n,\ell(t)}}, \quad (266)
$$

$$
K_- \psi_0^\ell(u) = 0. \quad (267)
$$

One can also observe that the following relations are satisfied

$$
K_+ K_- \psi_n^\ell(u) = n(n+\ell) \psi_n^\ell(u), \quad (268)
$$

$$
K_- K_+ \psi_n^\ell(u) = (n+1)(n+\ell+1) \psi_n^\ell(u). \quad (269)
$$

Now, to establish the dynamic group associated with the ladder operators $K_\pm$, we calculate the commutator

$$
[K_-, K_+] \psi_n^\ell(u) = (2n + \ell + 1) \psi_n^\ell(u). \quad (270)
$$

As a consequence, we can introduce the operator $K_0$ defined to satisfy

$$
K_0 \psi_n^\ell(u) = \frac{1}{2} (2n + \ell + 1) \psi_n^\ell(u). \quad (271)
$$
The operators $K_\pm$ and $K_0$ satisfy the following commutation relations

$$[K_-, K_+] = 2K_0, \quad [K_0, K_\pm] = \pm K_\pm,$$

which can be recognized as commutation relation of the generators of a non-compact Lie algebra $su(1,1)$. The corresponding Casimir operator for any irreducible representation is the identity times a number

$$K^2 = K_0^2 - \frac{1}{2}(K_+ K_- + K_- K_+) = \frac{1}{4}(\ell + 1)(\ell - 1).$$

It satisfies

$$[K^2, K_\pm] = [K^2, K_0] = 0.$$ (274)

Thus, a representation of $su(1,1)$ algebra is determined by the single real positive number $\ell$, called the Bargmann index.

Now, with the properties of the generators $K_\pm$ and $K_0$ of the $su(1,1)$ algebra, we are in the position to construct the corresponding coherent states to this system.

We investigate in this section the $SU(1,1)$ coherent states by adopting Barut-Girardello [18] and Perelomov [20] approaches. We examin for each approach the resolution of unity and overlapping properties.

5.2 Barut-Girardello coherent states

5.2.1 Construction

Following the Barut and Girardello approach [18], $SU(1,1)$ coherent states are defined to be the eigenstates of the lowering generator $K_-$

$$K_- |\psi^\ell_z\rangle = z |\psi^\ell_z\rangle,$$ (275)

where $z$ is an arbitrary complex number. Based on the completeness of the wavefunction such that $|\psi_n^\ell\rangle \langle \psi_n^\ell | = I$, on can represent the coherent states $|z, \ell\rangle$ as follows

$$|\psi^\ell_z\rangle = \sum_{n=0}^{\infty} \langle \psi_n^\ell| \psi^\ell_z\rangle |\psi_n^\ell\rangle.$$ (276)

Acting the operator $K_-$ on the equation (276) and then, using the equations (275) and (262) we have the following result

$$\langle \psi_n^\ell| \psi^\ell_z\rangle = \frac{z}{\sqrt{(n+1)(n+\ell+1)}} \langle \psi_{n-1}^\ell| \psi^\ell_z\rangle.$$ (277)

After the recurrence procedure, the formal equation becomes

$$\langle \psi_n^\ell| \psi^\ell_z\rangle = z^n \frac{\Gamma(1+\ell)}{n!\Gamma(n+\ell+1)} \langle \psi_0^\ell| \psi^\ell_z\rangle.$$ (278)
Referring to [19], the Gamma function is linked to the modified Bessel function $I_\mu(x)$ of order $\mu$ through the relation

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!\Gamma(n + \mu + 1)} = \frac{I_\mu(2x)}{x^\mu}. \quad (279)$$

Therefrom, by setting $x = z$ and $\mu = \ell$, we deduce the Barut-Girardello coherent states as follows

$$|\psi^\ell_z\rangle = \sqrt{|z|^\ell} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\Gamma(n + \ell + 1)}} |\psi^\ell_n\rangle, \quad (280)$$

$$\psi^\ell_z(u) = |z|^{\frac{\ell}{2}} \frac{N(\rho, \alpha)}{\sqrt{I_\ell(2|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n + \ell + 1)} u^{\frac{\ell}{2}} e^{-\frac{u}{2} z} L_n^\ell(u) e^{i\gamma_n(t)}. \quad (281)$$

Taking the limit $t \to 0$ the phase factor $\gamma_n(t) \to 0$ and $\psi^\ell_z(u) \to \phi^\ell_z(u)$.

However, in term of the generating function (250), the Barut-Girardello coherent states can be written as follows

$$\psi^\ell_z(u) = \left(\frac{z}{|z|}\right)^{\frac{\ell}{2}} N(\rho, \theta) e^{\frac{z^2}{2} - \frac{u}{2}} \sqrt{I_\ell(2|z|)} J_\ell \left(2 \sqrt{|z|} u\right) e^{i\gamma(t)}. \quad (282)$$

### 5.2.2 Properties

It is well-known that the states (280) are normalized but not orthogonal and satisfy the resolution of identity. Thus, we can see that the scalar product of two coherent states does not vanish

$$\langle \psi^\ell_{z_1} | \psi^\ell_{z_2} \rangle = \frac{I_\ell(2\sqrt{|z_1 z_2|})}{\sqrt{I_\ell(2|z_1|)I_\ell(2|z_2|)}}. \quad (283)$$

The overcompleteness relation reads as follows

$$\int d\mu(z, \ell) |\psi^\ell_z\rangle \langle \psi^\ell_z| = \sum_{n=0}^{\infty} |\psi^\ell_n\rangle \langle \psi^\ell_n| = \mathbb{I}, \quad (284)$$

with the measure

$$d\mu(z, \ell) = \frac{2}{\pi} K_\ell(2|z|) I_\ell(2|z|) d^2z, \quad (285)$$

where $d^2z = d(Rez) d(Imz)$ and $K_\nu(x)$ is the $\nu$-order modified Bessel function of the second kind.

For arbitrary state $|\Phi\rangle = \sum_{n=0}^{\infty} c_n |\psi^\ell_n\rangle$ in the Hilbert space, one can construct the analytic function $f(z)$ such that

$$f(z) = \sqrt{\frac{I_\ell(2|z|)}{|z|^{\ell}}} \langle \psi^\ell_z | \Phi \rangle = \sum_{m=0}^{\infty} \frac{c_m}{\sqrt{m!\Gamma(m + \ell + 1)}} z^m. \quad (286)$$
On the Barut-Girardello coherent states \( |\Phi\rangle \) one can explicitly express the state \( |\Phi\rangle \) as follows
\[
|\Phi\rangle = \int \mathcal{D}\nu(z,\ell) \left( \frac{z^*}{\sqrt{I_\ell(z)}} \right) f(z) |\psi_\ell^\ell\rangle,
\]
and we have
\[
\langle \Phi | \Phi \rangle = \int \mathcal{D}\mu(z,\ell) \left( \frac{|z|}{I_\ell(z)} \right) |f(z)|^2 < \infty.
\]

5.3 Perelomov coherent states

5.3.1 Construction

In analogy to canonical coherent states construction, Perelomov \( SU(1,1) \) coherent states \( |\psi_\ell^\ell\rangle \) are obtained by acting the displacement operator \( S(\xi) \) on the ground state \( |\psi_\ell^0\rangle \)
\[
|\psi_\ell^\ell\rangle = S(\xi) |\psi_\ell^0\rangle, = \exp (\xi K_+ - \xi^* K_-) |\psi_\ell^0\rangle,
\]
where \( \xi \in \mathbb{C} \), such that \( \xi = -\frac{\theta}{2} e^{-i\varphi} \), with \( -\infty < \theta < +\infty \) and \( 0 \leq \varphi \leq 2\pi \).

Using Baker-Campbell-Hausdorff relation we explicit the displacement as follows
\[
S(\xi) = \exp(\eta K_+) \exp(\zeta K_0) \exp(-\eta^* K_-),
\]
where \( \eta = -\tanh(\frac{\theta}{2}) e^{-i\varphi} \) and \( \zeta = -2 \ln \cosh |\xi| = \ln(1 - |\eta|^2) \). By using this normal form of the displacement operator \( S(\xi) \), the standard Perelomov \( SU(1,1) \) coherent states are found to be
\[
|\psi_\ell^\ell\rangle = (1 - |\eta|^2)^\ell + 1 \sum_{n=0}^{\infty} \left( \frac{\Gamma(n + \ell + 1)}{n!\Gamma(\ell + 1)} \right) \eta^n |\psi_\ell^n\rangle,
\]
\[
\psi_\ell^\ell(u) = N(\rho,\theta) (1 - |\eta|^2)^\ell + 1 \sum_{n=0}^{\infty} \eta^n L_n^\ell(u) e^{\gamma_n,\ell(t)}.
\]
Taking the limit \( t \to 0 \) the phase factor \( \gamma_n,\ell(t) \to 0 \) and \( \psi_\ell^\ell(u) \to \phi_\ell^\ell(u) \).

In term of the generating function \( \psi_\ell^\ell(u) \), the Perelomov coherent states can be written as follows
\[
\psi_\ell^\ell(u) = N(\rho,\theta) (1 - |\eta|^2)^\ell + 1 \sum_{n=0}^{\infty} \frac{\eta^n}{\Gamma(\ell + 1)} u^\frac{n}{2} e^{-\frac{\eta^2}{2u}} e^{\frac{\eta^2}{2u}} (1 - \eta)^{1+\ell} e^{\gamma_n,\ell(t)}.
\]

5.4 Properties

The Perelomov \( SU(1,1) \) coherent states as the Barut-Girardello coherent states are normalized states but not orthogonal
\[
\langle \psi_\ell^\ell_n | \psi_\ell^\ell_m \rangle = (1 - |\eta_1|^2)(1 - |\eta_2|^2)^{\ell + 1} (1 - \eta_1 \eta_2^* \eta^{*-\ell})^{-1}.
\]
and satisfy the completeness relation

\[ \int |\psi_{\ell}^\eta\rangle\langle\psi_{\ell}^\eta| d\mu(\eta, \ell) = \sum_{n=0}^{\infty} |\psi_{n}^\ell\rangle\langle\psi_{n}^\ell| = \mathbb{I} \]  

where the measure \( d\mu(\eta, \ell) = \frac{\ell}{\pi} \frac{d^2\eta}{(1-|\eta|^2)^2} \).

As we noted for the Barut-Girardello coherent states, for any \(|\Psi\rangle = \sum_{n=0}^{\infty} c_n|\psi_{n}^\ell\rangle \) in the Hilbert space, one can construct an analytic function

\[ f(\eta) = (1 - |\eta|^2)^{-\ell+1} \langle \psi_{\ell}^\eta | \psi_{\ell} \rangle = \sum_{n=0}^{\infty} c_n \sqrt{\frac{\Gamma(n+\ell+1)}{n!\Gamma(\ell+1)}} (\eta^*)^n. \]  

The expansion of \(|\Psi\rangle\) on the bases of coherent states \((291)\) can be written as

\[ |\Psi\rangle = \int d\mu(\eta, \ell)(1 - |\eta|^2)^{-\ell+1} |\psi_{\ell}^\eta\rangle, \]  

\[ \langle \Psi | \Psi \rangle = \int d\mu(\eta, \ell)(1 - |\eta|^2)^{\ell+1} |f(\eta)|^2 < \infty. \]  

We have reported in this chapter a Landau particle in time-dependent background electric field with time-dependent mass and frequency. We have studied this system at the classical level and have formulated the corresponding quantum system. At the classical level we solved the equations of motion and at the quantum level, we used the Lewis-Riesenfeld’s method to construct the spectra of the invariant operator \( I(t) \) and the Hamiltonian \( H(t) \) on the helicity-like bases \(|\phi_{n\pm}(t)\rangle\). The configuration space wave functions of both operators are expressed in terms of the generalised Laguerre polynomials. This quantization has shown that the system possesses \( U(1), SU(2) \) and \( SU(1,1) \) symmetries. Consequently, a system of coherent states associated to those symmetries is constructed and some usual related properties to those states are examined.

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