Non Uniform Weighted Extended B-Spline Finite Element Analysis of Non Linear Elliptic Partial Differential Equations

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Abstract
We propose a non uniform web spline based finite element analysis for elliptic partial differential equation with the gradient type nonlinearity in their principal coefficients like $p$-laplacian equation and Quasi-Newtonian fluid flow equations. We discuss the well-pos edness of the problems and also derive the apriori error estimates for the proposed finite element analysis and obtain convergence rate of $O(h^{\alpha})$ for $\alpha > 0$.

Keywords Finite element · Non uniform web-spline · Error estimates

Finite element method is one of the popular numerical techniques for solving partial differential equation modeling real life problems from science and engineering. Currently there is marked interest for meshless approach for solving boundary value problems as it significantly saves the cost and trouble of generating mesh, which are infinitesimal in many cases may turn out to be the computationally the most expensive job. Weighted extended B-splines is a finite element method (fem) in a infinitesimal costs mesh framework. The present work on nonlinear elliptic problems is based on non-uniform weighted extended B-splines (NUWEBS) fem which was originally proposed by Höllig et al. [1–3] on trivial mesh framework. The $p$-laplacian equation used into the design of shock free airfoil and non-Newtonian fluid flow model used in understanding seepage through coarse grained porous media in some geological problems etc. have gradient type non linearity in their principal coefficients. They also occur in the description of non linear diffusion [4, 5] and filtration [6], power law materials [7] and Quasi Newtonian flows [8]. Earlier in a grid based framework mixed finite element methods were developed and analyzed in [9, 10] for elliptic problems. Recently efforts have been made to solve non-linear fractional problems [11–14] analytically under certain assumptions and approximations leading to simplification of models to facilitate the analytical solution derivation. Such an approach may be of help in analytically solving the simplified version of the set of models under current

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consideration. Here, devoid of such simplifying assumptions, we are concerned with the finite element analysis in a gridless framework and provide the convergence analysis of weighted extended b-spline finite element analysis for \( p \)-laplacian equation and Quasi-Newtonian flow model.

An outline of the paper is as follows. We present some preliminary knowledge on the non-uniform weighted extended b-spline (WEB-S) space in “Non Uniform Weighted Extended B-Splines” and establish an optimal order a priori error estimates of \( p \)-Laplacian problem in “\( p \)-Laplace Problem”. We discuss Quasi-Newtonian problems in “Quasi-Newtonian Fluids”. Furthermore, throughout our discussion \( \Omega \) is a bounded, multiply connected domain, i.e., it may contain holes and we use the symbols \( \leq, \geq \) or \( \asymp \) instead of \( \leq, \geq \) or \( = \) whenever the constants are clear from the context and independent of the parameters.

**Non Uniform Weighted Extended B-Splines**

Let \( \Omega_h \) be a webs fem approximation to \( \Omega \) defined by \( \Omega_h = \bigcup \Omega_h \), where \( \Omega_h \) is a partitioning of \( \Omega_h \) into a finite number of disjoint open regular domains Fig 2, each of maximum diameter bounded above by \( h \). In addition, for any two distinct domains, their closures are either disjoint, or have a common boundary. Associated with \( \Omega_h \) is the finite-dimensional space \( B_h \) (see below)

We can approximate a function on a bounded domain \( \Omega \subset \mathbb{R}^n \) by forming a spline, i.e., a linear combination of all relevant B-splines

\[
b_k, k \in K = \text{set of all inner and outer B-splines}
\]

which have some support in \( \Omega \). Depending on the degree, this yields approximations of arbitrary order and smoothness. However, numerical instabilities may arise due to the outer B-splines

\[
b_j, j \in J
\]

for which no complete grid cell of their support lies in \( \Omega \). Here and in the sequel, a grid cell is an interval which in every coordinate direction is bounded by two consecutive, but different knots, and an inner grid cell is a grid cell whose interior is completely contained in \( \Omega \). A further difficulty is that, in general, splines do not conform to homogeneous boundary conditions, which is essential for standard finite element schemes [15] or for matching boundaries in data fitting problems. Fortunately, both problems can be resolved. A stable basis is obtained by forming appropriate extensions of the inner B-splines

\[
b_i, i \in I = K \sim J
\]

which have at least one inner grid cell in their support. Fig 1 and Fig 2 overall demonstrates the difference between uniform and non uniform web basis. Readers are referred to [3, 16, 17] for detailed description.

**Splines on Bounded Domains**

The Splines \( B^\theta_h(D) \) on a bounded domain \( D \subset \mathbb{R}^m \) consist of all linear combinations

\[
\sum_{k \in K} c_k b_{k,h}^n
\]

of relevant B-Splines; i.e., the set \( K \) of relevant indices contains all \( k \) with \( b_{k,h}^n(x) \neq 0 \) for some \( x \in D \), where \( b_{k,h}^n(x) = b^n(x/h - k) \) is the scaled translates.
Inner and Outer Splines

Grid cells \( \mathbb{Q} = h \{(0, 1)^m + I\} \) are partitioned into interior, exterior and boundary cells depending on whether \( \mathbb{Q} \subseteq \overline{D} \), the interior of \( \mathbb{Q} \) intersects \( \partial D \), or \( \mathbb{Q} \cap D = \emptyset \). Among the relevant B-Splines, \( b_k, k \in \mathbb{K} \), distinction made between inner B-Splines

\[
b_i, i \in \mathbb{I}
\]

which have at least one interior cell in their support, and outer B-Splines

\[
b_j, j \in \mathbb{J} = \mathbb{K} \setminus \mathbb{I}
\]

for which \( \text{supp} \, b_j \) consists entirely of boundary and exterior cells.

**Theorem 1** *The Spline*
is a polynomial of order $n$ on $D$ iff $q$ is a polynomial of order $n$ on $K$.

We assume that the boundaries are smooth so that smooth solution could exist. As usual, the solution is approximated by a linear combination

$$
\sum_{i} a_i B_i
$$

of basis functions $B_i$ which vanish outside a set with diameter $\simeq h$. Moreover, the basis functions are required to vanish on the boundary so we simply multiply by a fixed weight function $w$ which satisfy the criteria, and in addition $w$ is to be smooth and $\simeq \text{dist}(x, D)$. Readers are suggested [18, 19] for more details

**Definition 1.1 (Extended B-Splines)** For $i \in I(j), j \in J(i)$ and $Q_j$ we denote by $p_{i,j}$ the polynomial which agrees with $b_i$ on $Q_j$ and define the extension coefficients

$$
e_{i,j} = \lambda_j p_{i,j}.
$$

Then, the extended B-splines are

$$
B_i = b_i + \sum_{j \in J(i)} e_{i,j} b_j = b_i + \sum_{j \in J(i)} \lambda_j p_{i,j}.
$$

where, the set of related inner indices is defined by $I(j)$ for an outer index, and $Q_j$ is an inner grid cell which is closest to $\text{supp} b_j$ for $j \in J$ with respect to the Hausdorff metric, conversely for an inner index we define the set of related outer indices by $J(i)$.

**Theorem 2** $eb$-splines and de Boor–Fix functionals $\{\lambda_k\}$ are bi-orthogonal, i.e.

$$
\lambda_k B_{k'} = \delta_{k,k'}, \quad k, k' \in \mathbb{Z}
$$

In addition if $Q$ is an inner grid cell in the support of $B_k$ with length bounded by $|Q| \geq \alpha |\text{supp} B_k|$ for some constant $\alpha \in (0, 1]$, then

$$
|\lambda_k p| \leq \text{const}(n, \alpha) |p|_{\infty, Q}, \quad p \in \mathcal{P}_n
$$

where, $| \cdot |$ represents measure of a set and $\mathcal{P}_n$ is a linear space of polynomials of degree $\leq n$

Generalizing the univariate definitions and results for $n \geq 2$ variables are straightforward. The arguments are completely analogous. Merely the notation needs to be adapted to the multivariate setting. We consider a tensor product grid in $\mathbb{R}^n$ with knot sequences $t = [t^1, \ldots, t^n]$ and denote by,

$$
b_k = b_{k_1, t_1}^{m_1} (x_1) \ldots b_{k_n, t_n}^{m_n} (x_n), \quad k \in \mathbb{Z}^n
$$

the corresponding tensor product B-splines of degree $m = [m_1, \ldots, m_n]$. 

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**Definition 1.2** Let \( w \) be a positive weight function which is smooth on \( \Omega \) and equivalent to some power \( r \geq 0 \) of the boundary distance function
\[
w(x) \asymp \text{dist}(x, \partial \Omega)^r.
\]
and denote by \( x_i \) the center of an inner grid cell in \( \text{supp } b_i \). Then, the weighted extended B-splines (web splines) are defined by
\[
B_i = \frac{w}{w(x_i)} \left( b_i + \sum_{j \in j(i)} e_{ij} b_j \right), \quad i \in I(j)
\]

**Theorem 3** (Jackson’s Inequality) Let \( u \in W^{1,p}(\Omega) \). Then
\[
|| u - P_h u ||_1 \leq O(h).
\]

**Remark**

- Theorem 2 remains valid for this new class of Splines.
- The linear span of web-splines is the web-space \( B \)
- The canonical projector \( P_h \) onto the spline space \( B \) is defined as
\[
P_h f = \sum_{i \in I} (A_i f) B_i.
\]
where, the weight functional \( A_i f = w(x_i) \lambda_i (f/w) \), \( i \in I \), with \( x_i \) as in Definition 1.2
- It satisfies,
\[
\int_{\Omega} \nabla \cdot (P_h u - u_h) v_h \, dx = 0.
\]

**Elliptic Partial Differential Equation Analysis in NUWEBSFEA Framework**

As NUWEBSFEA of variable coefficient of Poisson equation (VCPEA) is not available in literature, for simplicity we begin with NUWEBSFEA of VCPEA.

The computational domain is denoted by \( \Omega \) and the VCPE model is given by,
\[
- \nabla . a(x) \nabla u(x) = f(x), \quad \text{on } \Omega
\]
\[
u = 0, \quad \text{on } \partial \Omega
\]
where \( u \) is the scalar potential and \( f \) is the source term. In case of EEG imaging this equation can often be used under the quasi static approximation of the Maxwell’s equation. Moreover in EEG the source term is like the form \( f(x) = \nabla \cdot d(x) \) where \( d : \Omega \rightarrow \mathbb{R}^m \), \( m = 2, 3 \) is a vector field that describes the neural sources as idealized electric dipoles.

As usual the solution is approximated by a linear combination
which vanishes outside a set of diameter proportional to grid width $h$. The coefficients $a_i$’s are determined from Galerkin system

$$\int_\Omega a(x)\nabla B_j \nabla u_h \, dx = \sum_i \left( \int_\Omega a(x)\nabla B_j \nabla B_i \, dx \right) a_i = \int_\Omega B f. \quad (5)$$

**Theorem 4** Let $u \in H^n$ be the solution of the problem (3) and $u_h = \sum_i a_i B_i$ a finite element approximation obtained by solving the Galerkin system (5). If there exists a $\kappa > 0$ such that $a(x) \geq \kappa$ then

$$||u - u_h||_1 \leq h^{n-1} ||u||_1.$$  

**Proof** The proof relies on results and techniques from [2, 17] and the theory of weighted approximations. Moreover, the standard error estimates for splines are crucial for our arguments. We begin by noting that,

$$||u|| \leq ||a(x)u||, \quad ||wu|| \leq ||a(x)wu||.$$  

We refer also to [15, 18] where a weaker version of theorem was obtained, for some of the preliminary arguments. Finally by using Cea’s Lemma, the error of $u_h$ can be bounded, up to a constant factor, by the error of the best approximation from the finite element subspace. $\square$

**p-Laplace Problem**

We consider a $p$-Laplacian system: Given $p \in (0, \infty)$, $f \in L^2(\Omega)$, $g \in W^{1-1/p,p}(\Omega)$ and $\inf\{a(x), b(x)\} > 0$ find $u$ such that,

$$-\nabla \cdot (|a(x)\nabla u|^{p-2} \nabla u) + b(x)u = f, \quad \Omega \subset \mathbb{R}^2 \ u = g \text{ on } \partial \Omega \quad (6)$$

For the convenience sake we assume $a(x) = b(x) = 1$. The weak formulation is given by:

Find $u \in W^{1,p}_g(\Omega) \equiv \{ v \in W^{1,p}(\Omega) : v = g \text{ on } \partial \Omega \}$ such that,

$$\mathcal{L}(u, v) = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx = \int_\Omega fv \, dx, \quad \forall v \in W^{1,p}_0(\Omega), \quad (7)$$

where, $|v|^2 = \langle v, v \rangle_{\mathbb{R}^2}$.

Define a strictly convex functional, $J : W^{1,p}(\Omega) \rightarrow \mathbb{R}$

$$J(v) := \frac{1}{p} \int_\Omega |\nabla v|^p \, dx + \frac{1}{2} \int_\Omega v^2 \, dx - \int_\Omega fv \, dx.$$  

Assuming, $J' : W^{1,p}(\Omega) \rightarrow (W^{1,p}(\Omega))^\prime$ we have,

$$\langle J'(u), v \rangle = a(u, v) - \langle f, v \rangle.$$  

Define quasi-norm for $u \in W^{1,p}(\Omega)$

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\[ |u|_{(u,p,2)}^2 = \int_{\Omega} (|\nabla u_s| + |\nabla u|)^{p-2} |\nabla u|^2 \, dx. \]

where \( u_s \) is the solution of the problem.

**Theorem 5** We have for \( p \in (1, 2] \)

\[ |u|_{(u,p,2)}^2 \leq |u|^p_{W^{1,p}(\Omega)} \leq C \left[ |u_s|_{W^{1,p}(\Omega)} + |u|^2_{W^{1,p}(\Omega)} \right]^{2-p} |u|_{(u,p,2)}^{1/2}. \]

**Proof** see [20] \( \square \)

**Discretization Using Non Uniform WEB-Splines Basis**

Let, \( \mathcal{X}_h \) be a quadrangulation of the \( \Omega \). Let,

\[ \mathcal{Y}_h^{(n)} := \left\{ v_h \in C^0(\Omega) \mid v_h \big|_{\partial \Omega} = 0 \land v_h \big|_s \in Q_n(\tau) \forall \tau \in \mathcal{X}_h \right\}. \]

where \( Q_n(\tau) \) is the WEB-Spline space of degree \( n \) defined on each cell \( \tau \). The approximation is then to seek \( u_h \in \mathcal{Y}_h^{(n)} \) such that

\[ \mathcal{L}(u_h, v_h) = (f, v_h), \quad \forall v_h \in \mathcal{Y}_h^{(n)}. \quad (8) \]

We write,

\[ u_h = \sum_{i=1}^{N} c_i B_i, \quad c_i \in \mathbb{R}. \quad (9) \]

The coefficients \( c_i \) can be obtained from the following system after linearization. Consequently from the Eqs. (7 , 8) and (14) we have,

\[ \sum_{i=1}^{N} \left[ \sum_{\delta \in \mathcal{X}_h} \int_{\delta} |\nabla u_h|^p \left( \frac{\partial B_i}{\partial x} \frac{\partial B_j}{\partial y} + \frac{\partial B_i}{\partial y} \frac{\partial B_j}{\partial x} \right) \right] + \int_{\Omega} B_i B_j \, dx \]

\[ = \int_{\Omega} f B_j, \quad j = 1, 2, \ldots, N \]

**Error Bounds**

The finite element approximation of (7) that we wish to consider is: Find \( u_h \in \mathcal{B}^g_h \) and \( v_h \in \mathcal{B}^0_h \) such that

\[ \mathcal{L}(u_h, v_h) = \int_{\Omega} |\nabla u_h|^p \nabla u_h \cdot \nabla v_h \, dx + \int_{\Omega} u_h v_h \, dx = \int_{\Omega} f v_h \, dx. \]

where,

\[ \mathcal{B}^g_h := \{ u_h \in \mathcal{B}_h : u_h = g \text{ on } \partial \Omega \}. \]

\[ \mathcal{B}^0_h := \{ u_h \in \mathcal{B}_h : u_h = 0 \text{ on } \partial \Omega \}. \]
The following error bounds

\[
||\nabla(u - u_h)||_{L_p} \leq \begin{cases} 
C h^{1/(3-p)} & \text{if } p \leq 2 \\
C h^{1/(p-1)} & \text{if } p > 2 
\end{cases}
\]

\[
\mathcal{L}(u, w) - \mathcal{L}(v, w) \leq \begin{cases} 
v||\nabla(u - v)||_{L_{p,0}} ||\nabla w||_{L_{p,0}} , & \text{if } 1 < p \leq 2 \\
v[C_1 + C_2(||\nabla u||_{L_{p,0}} + ||\nabla v||_{L_{p,0}})^{2-p}][||\nabla(u - v)||_{L_{p,0}} ||\nabla w||_{L_{p,0}} , & \text{if } 2 < p < \infty 
\end{cases}
\]

\(C\) depends on the domain \(\Omega\) and the degree of the polynomials were proved in Glowinski and Marrocco [21] for the case \(\Omega_h = \Omega\) and \(g = 0\). In this paper we are improving the error bound by employing an approach of Chow [22] and Tyukhtin [23] in the framework of NUWEBS.

We now state an important theorem which is relevant in providing a sharper error estimate.

**Theorem 6** If \(u\) and \(u_h\) be the weak and approximate solutions then for some \(C > 0\) we have

\[
|u - u_h|_{(u,p)} \leq C \inf_{v_h \in B_h} |u - v_h|_{(u,p)}.
\]

**Proof** see [24]

**Theorem 7** Let, \(u\) be the weak solution and \(u_h\) be the NU-WEBS based solution of the equation. For, \(1 < p < 2\)

\[
|u - u_h|_{(u,p)} \leq \begin{cases} 
O(h^{p/2}) & \text{whenever } u \in W^{2,p}(\Omega) \\
O(h) & \text{whenever } u \in C^{2,2/p-1}(\Omega) \cap W^{3,1}(\Omega)
\end{cases}
\]

Again for \(2 < p < \infty\)

\[
|u - u_h|_{(u,p)} \leq O(h^{a/2}) \text{ whenever } u \in W^{1,\infty}(\Omega) \cap W^{2,a}(\Omega), \ 1 \leq a \leq 2
\]

**Proof** For the case \(1 < p < 2\). We have for \(u \in W^{2,p}(\Omega)\)

\[
|u - P_h u|_{(u,p)}^2 = \int_{\Omega} (|\nabla u_s| + |\nabla (u - P_h u)|)^{p-2} |\nabla (u - P_h u)|^2 \, dx, \\
\leq \int_{\Omega} |\nabla (u - P_h u)|^p \, dx, \quad \text{from Theorem (5)}
\]

\[
\leq O(h^p), \quad \text{from Theorem (3)}
\]

Again for \(u \in C^{2,2/(p-1)}(\Omega) \cap W^{3,1}(\Omega)\)

\[
|\nabla (u - P_h u)| \leq Ch[\mathcal{H}[u]]_{0,\infty,B_h} \leq Ch[\mathcal{H}[u]] + O(h^{2/p}) \text{ where } \mathcal{H}[u] = |u_{x_1}| + |u_{y_1}| + |u_{x_2}|.
\]

We know for non negative \(x\), \(\psi_\lambda(x) = (\lambda + x)^{p-2}x^2\) is non decreasing.

Therefore,

\[
|u - P_h u|_{(u,p)}^2 = \int_{\Omega} (|\nabla u_s| + |\nabla (u - P_h u)|)^{p-2} |\nabla (u - P_h u)|^2 \, dx,
\]

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Clearly, right hand side is in the form of \( \psi_\alpha(x) \) using the additional property 
\[
\frac{\psi_\alpha(x)}{2} \leq \psi_\alpha(x) + \psi_\beta(x),
\]
\[
\leq C h^2 \int_{\Omega} (|\nabla u_s| + ChH(u))^{p-2} H(u)^2\, dx + \mathcal{O}(h^{4/p}),
\]
\[
\leq \mathcal{O}(h^2).
\]
Now for the case \( 2 < p < \infty \) we proceed in this way,
\[
|u - \mathcal{P}_h u|_{(u_p)} = \int_{\Omega} (|\nabla u_s| + |\nabla (u - \mathcal{P}_h u)|)^{p-2}
\]
\[
|\nabla (u - \mathcal{P}_h u)|^2\, dx; u \in W^{1,\infty}(\Omega) \cap W^{2,\alpha}(\Omega).
\]
\[
\leq \int_{\Omega} (|\nabla u_s| + |\nabla (u - \mathcal{P}_h u)|)^{p-\alpha}|\nabla (u - \mathcal{P}_h u)|^\alpha\, dx, \quad \text{for } \alpha \in [1, 2]
\]
\[
\leq C |\nabla (u - \mathcal{P}_h u)|_{0,\alpha} \leq \mathcal{O}(h^{\alpha}).
\]

\[ \square \]

**Quasi-Newtonian Fluids**

We now consider the following stationary non-Newtonian problem:

Find \((u, p)\) such that

\[
- \sum_{j=1}^d \partial(a|D(u)|^2)D_{ij}(u)\partial \chi_j + \partial p/\partial \chi_i = \phi_i , \quad \text{in } \Omega \quad i = 1, 2, \ldots, d
\]

(10)

\[
\nabla \cdot u = 0 , \quad \text{in } \Omega.
\]

(11)

\[
u = 0 \text{ on } , \partial \Omega
\]

(12)

where, \(D(u)\) is the rate of deformation tensor with entries

\[
D_{ij}(u) := \frac{1}{2} \left( \frac{\partial u_i}{\partial \chi_j} + \frac{\partial u_j}{\partial \chi_i} \right).
\]

and

\[
|D(u)|^2 := \sum_{i,j=1}^d \left[ D_{ij}(u) \right]^2.
\]

\(\Omega\) is bounded and connected with Lipschitz boundary and \(a \in C[0, \infty)\) is a positive function satisfying

\[
|a_0 - a_\infty| \leq |a(x) - a_\infty| \leq |a_0 - a_\infty| (1 + |x|^{1/2})^{-c}, \quad c \geq 0
\]

(13)

where, \(a_0 = a(0)\) and \(a_\infty = \sup_{x \in [0, \infty)} a(x)\).

Below we introduce a non linear functional,
\[ \mathcal{L}(v) \equiv \int_{\Omega} \left[ \int_{0}^{r}[D(v)]^2 \, ds \right] - \sum_{i=1}^{d} \langle \phi_i, v_i \rangle. \]

\( \mathcal{L} \) has the following properties:

- It is Gateaux differentiable,
- strictly convex,
- \( \mathcal{L}'(\cdot) \) is strictly monotone,
- it is coercive,

for more details readers are suggested [25] and [26].

Let, \( u \in (H^r_0(\Omega))^d = X \) and \( L'^0(\Omega) = Y \) where, \( 1/r + 1/r' = 1 \). (\( H^r_0 \) is the space of trace zero elements of \( H^r \) and \( L'^0 \) is the mean zero \( r' \) integrable functions). Consequently we have to find \( X \) such that,

\[ A(u, v) := \int_{\Omega} \sum_{i,j=1}^{d} a \left( [D(u)]^2 \right) D_{ij}(u) D_{ij}(v) = \sum_{i=1}^{d} \langle \phi_i, v_i \rangle. \]

\[ b(u, q) = - \int_{\Omega} q \nabla \cdot u. \]

A finite element discretization of (10) is based on the mixed weak formulation which seeks \((u, p) \in X \times Y \) (\( = (H^r_0(\Omega))^d \times L'^0(\Omega) \)) such that

\[ \int_{\Omega} \sum_{i,j=1}^{d} a \left( [D(u)]^2 \right) D_{ij}(u) D_{ij}(v) - \int_{\Omega} p \nabla \cdot v := \langle \phi, v \rangle, \quad \forall v \in X. \]  

\[ A(u, v) + b(v, p) = \langle \phi, v \rangle \quad \forall v \in X. \]  

\[ b(u, q) = 0 \quad \forall q \in Y. \]  

We state the LBB condition for the existence-uniquenes of the solution.

**Theorem 8** [27] If for any \( r \in (1, \infty) \) there exists a positive \( c(q) \) such that

\[ \inf_{q \in L'^0(\Omega)} \sup_{w \in (H^r_0(\Omega))^d} \frac{(q, \nabla \cdot w)}{[u]\|_{L'^0(\Omega)} \|w\|_{H^r(\Omega)}} \geq c(q) > 0. \]

then there exists a unique solution to (14)

**Error Bounds for NUWEBS Approximation**

Let,

\[ B_h \subset X \cap (W^{1,\infty}(\Omega))^d, \quad Y_h \subset Y \cap L^\infty(\Omega) \]

and \( B_h^0 = \{ v_h \in B_h : b(v_h, q) = 0 \forall q_h \in Y_h \} \)
be finite dimensional subspaces. So the corresponding approximation problems: finding 
\((u_h, p_h) \in \mathcal{B}_h \times \mathcal{Y}_h\) such that
\[
\mathcal{A}(u_h, v_h) + b(v_h, p_h) = \langle \phi, v_h \rangle, \quad \forall v_h \in \mathcal{B}_h.
\]
\[
b(u_h, q_h) = 0, \quad \forall q_h \in \mathcal{Y}_h.
\]
- Analogously we have the discrete version of the LBB condition.
- Approximation Property on \(\mathcal{Y}_h\):

There is a continuous operator \(\Pi_h : L^s(\Omega) \rightarrow \mathcal{Y}_h\) such that ,
\[
\|p - \Pi_h p\|_s \leq h^r \|p\|_s, \quad p \in H^r(\Omega).
\]

**Theorem 9** Let, \(a\) satisfies (13), \((u, p) \in X \times Y\) be the unique solution of (14) and 
\((u_h, p_h) \in X_h \times Y_h\) be the unique solution of (17). If the approximation property holds on 
\(Y_h\) then,
\[
\|u - u_h\|_X + \|p - p_h\|_2 \leq \mathcal{O}(h).
\]

**Proof** We follow the technique of (28)
\[
\|u - u_h\|_X^2 \leq \mathcal{L}(u_h) - \mathcal{L}(u) - \langle \mathcal{L}'(u), u_h - u \rangle.
\]
\[
\leq \mathcal{L}(w_h) - \mathcal{L}(u) - \langle \mathcal{L}'(u), u_h - u \rangle.
\]
\[
\leq \int_0^1 \left[ \int_{\Omega} a \left( |D(u + \tau(\mathcal{P}_h u - u))|^2 D(u + \tau(\mathcal{P}_h u - u)) - a(|D(u)|^2) D(u) \right) - \|D(u - \mathcal{P}_hu)| d\tau.
\]
\[
+ \langle \nabla(\mathcal{P}_h u - u_h), p - \Pi_h p \rangle.
\]
\[
\leq \|u - \mathcal{P}_hu\|_X^2 + \|\mathcal{P}_hu - u_h\|_X \|p - \Pi_h p\|_2.
\]
\[
\leq \|u - \mathcal{P}_hu\|_X^2 + \|u - u_h\|_X \|p - \Pi_h p\|_2.
\]
\[
\leq \|u - \mathcal{P}_hu\|_X^2 + \frac{1}{2} \left( \|u - u_h\|_X^2 + \|u - \mathcal{P}_hu\|_X^2 + 2\|p - \Pi_h p\|_2^2 \right).
\]
Simplifying ,
\[
\leq 3 \|u - \mathcal{P}_hu\|_X^2 + 2\|p - \Pi_h p\|_2^2.
\]
Again we have ,
Simplifying by using Jackson’s Inequality and Approximation property we obtain the desired results.

\[ \langle p - p_h, \nabla P_h u \rangle = \langle A u - A u_h, P_h u \rangle. \]
\[ \Rightarrow \langle p - \Pi h p, \nabla P_h u \rangle + \langle \Pi h p - p_h, \nabla P_h u \rangle = \langle A u - A u_h, P_h u \rangle. \]
\[ \Rightarrow \langle \Pi h p - p_h, \nabla P_h u \rangle = \langle A u - A u_h, P_h u \rangle - \langle p - \Pi h p, \nabla P_h u \rangle. \]
\[ \Rightarrow ||p_h - \Pi h p||_2 ||P_h u|| \leq c_h^{-1} \sup(p_h - \Pi h p, \nabla P_h u) \]
\[ \leq c_h^{-1} \left[ \langle A u - A u_h, P_h u \rangle \right] + \sup \left[ \langle p - \Pi h p, \nabla P_h u \rangle \right]. \]
\[ \Rightarrow ||p - p_h||_2 \leq (1 + c_h^{-1}) ||p - \Pi h p||_2 \]
\[ + c_h^{-1} \left[ \int_{\Omega} \left| a \left( |D(u)|^2 \right) D(u) - a \left( |D(u_h)|^2 \right) D(u_h) \right|^2 \right]^{1/2}. \]
\[ \leq (1 + c_h^{-1}) ||p - \Pi h p||_2 + c_h^{-1} ||u - u_h||_X. \]

\[ \square \]

Conclusion

We propose non uniform web spline based mesh free finite element method for $\rho$-Laplacian problems and quasi -Newtonian problem. We provide a priori error bounds in this context. In future, analysis of this method will be extended to Navier Stokes problem, or, second order wave equations and miscible displacement problems in porous media by using weighted isogeometric method with NURBS basis.

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