Experimentally friendly bounds on non-Gaussian entanglement from second moments

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We demonstrate that the entanglement in a class of two-mode non-Gaussian states obtained by subtracting photons from Gaussian twin beams can be bounded from above and from below by functions of the second moments only. Knowledge of the covariance matrix thus suffices for an entanglement quantification with appreciable precision. The absolute error in the entanglement estimation scales with the non-Gaussianity of the considered states.

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I. INTRODUCTION

Interfaces between light and matter are key building blocks of a future quantum web, a global secure communication network where the manipulation and transmission of information are regulated by the quantum laws. The transfer of quantum states and the distribution of correlations across the interfaces are enabled by the common mathematical language of canonically conjugate observables with continuous spectrum, such as the quadratures of light and the collective spin components of atomic ensembles. It is thus very fascinating to witness how the second generation of quantum information research is focusing more and more, from both theoretical and experimental viewpoints, on the characterization of continuous variable (CV) entanglement and its applications for communication, computation and metrology.

Out of the infinite-dimensional Hilbert space of CV states, a special class of states has played a prominent role in recent years: Gaussian states. Their mathematical treatment is advantageous thanks to a compact formalism based on symplectic analysis, and a very accurate degree of control is reached in their experimental realizations with light and matter. There are however many tasks which are impossible by using only the Gaussian states and operations toolbox (e.g., entanglement distillation and universal quantum computation), and many other tasks which can be sharply improved by suitably resorting to some non-Gaussianity (e.g. CV teleportation and loss estimation). These premises have spurred astonishing progresses in the experimental engineering of non-Gaussian states, such as Fock states and states obtained by deGaussifying Gaussian resources via addition and/or subtraction of single photons. It has been in particular verified experimentally that a photon subtraction from a two-mode squeezed entangled Gaussian state leads to an enhancement of the entanglement (at fixed squeezing) and it is known that such resource might be used for a more performant quantum teleportation of classical and nonclassical states (a primitive of the quantum internet) and for yet-to-be-achieved demonstrations of loophole-free Bell tests of nonlocality.

The bottleneck for unleashing the power of non-Gaussian CV quantum technology has remained the quantitative characterization of entanglement in states which deviate from Gaussianity, a crucial step to evaluate and control their usefulness for applications and the bona-fides of their preparation. While for Gaussian states all the information is encoded in the second moments of the canonical operators (collected in the covariance matrix), for any other state an infinite hierarchy of moments is in principle needed for an exact entanglement quantification. This translates, in experimental terms, into the demand for a complete state tomography, a process which is time- and resource-consuming especially for two or more modes.

In this paper we provide an advance in the characterization of non-Gaussian entanglement which reduces the complexity of its experimental determination exactly to the same level of Gaussian states. This cannot be possible for any generic non-Gaussian state: hence here we focus on the important class of photon-subtracted states (PSS) under quite realistic conditions, which represent the preferred resources for most current and future applications requiring non-Gaussianity. Combining results from the extremality of Gaussian states with an analysis of quadrature correlations, we derive analytical lower and upper bounds which individuate the entropy of entanglement of pure two-mode PSS (obtained from Gaussian twin beams by conditional subtraction of \(k\) photons per beam) within a very narrow band, with relative error vanishing with increasing entanglement. The bandwidth is linked to the degree of non-Gaussianity of the analyzed PSS. The results are extended to bound the entanglement of formation of a class of mixed PSS obtained by means of realistic “on/off” type photodetectors. Crucially, all these bounds are only functions of the second moments of the non-Gaussian states. A novel method for the fast and reliable reconstruction of the complete covariance matrix of optical two-mode CV states has been recently demonstrated: our result proves that it can be applied to real deGaussified PSS as well and it is enough to quantitatively estimate entanglement with surprisingly high accuracy.

II. PRELIMINARIES

We deal with a CV system of \(N = 2\) bosonic modes, associated to an infinite-dimensional Hilbert space \(\mathcal{H} = F_1 \otimes F_2\). Here \(F_i\) is the Fock space of each individual mode \(i\), described by the ladder operators \(\hat{a}_i, \hat{a}_i^\dagger\) satisfying the canonical commutation relations \([\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}\). We can collect the field
quadrature operators into the vector \( \hat{X} = \{ \hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2 \} \), with \( \hat{q}_i = \hat{a}_i + \hat{a}_i^\dagger \) and \( \hat{p}_i = \hat{a}_i - \hat{a}_i^\dagger \). The exact description of a generic CV (non-Gaussian) state requires arbitrary-order moments of the canonical operators. In general, the first moments \( \bar{X} \equiv \langle \langle \hat{X}_1, \hat{X}_1 \rangle, \langle \hat{X}_2, \hat{X}_2 \rangle \rangle \) can be adjusted by local displacements without affecting entanglement: they will be set to zero without loss of generality. Two-mode Gaussian states are henceforth completely specified by the \( 4 \times 4 \) real symmetric covariance matrix (CM) \( \sigma \) of the second moments \( \sigma_{ij} = \langle \langle \hat{X}_i \hat{X}_j + \hat{X}_j \hat{X}_i \rangle \rangle / 2 - \langle \langle \hat{X}_i \rangle \rangle \langle \langle \hat{X}_j \rangle \rangle \). The CM elements for an arbitrary two-mode CV state can be efficiently reconstructed in the lab by means of homodyne detections \([17, 19]\).

We recall that any physical two-mode CM constructed in the lab by means of homodyne detections \([17, 19]\).

1. The optimal convex-roof extension of pure two-mode squeezed Gaussian state (or ‘twin beam’ in the continuous and strongly superadditive measure) than \( F_1 \) is known that for arbitrary two-mode Gaussian states the same CM has smaller entanglement (quantified by a continuous and strongly superadditive measure) than \( F_1 \) \([15]\). It is also known that for arbitrary two-mode Gaussian states \( F_E \) is computable, additive, and strongly superadditive \([24, 25]\).

### III. Covariance Matrix and Entanglement of Pure Photon Subtracted States

The starting point for the definition of “ideal” PSS is a pure two-mode squeezed Gaussian state or ‘twin beam’ in the optical language, \( |\psi_0(r_0)\rangle = \sum_{n} \lambda^{n} \sqrt{T - \lambda^2}^n |n, n\rangle \), where \( \lambda = \tanh r_0 \) and the positive \( r_0 \) is the squeezing degree. The CM \( \sigma_0 \) for this state is in standard form with \( a_1 = a_2 = \cosh 2r_0 \), \( \gamma_x = -\gamma_p = \sinh 2r_0 \). The beam 1 (2) of \( |\psi_0\rangle \) is let to interfere, via a beam splitter with transmittivity \( T \) (preferably \( T \) close to unity), with a vacuum mode 1’ (2’). The output is a four-mode Gaussian state of modes 1, 1’, 2, 2’. A photon-number-resolving detection of exactly \( k \) photons in each of the two beams 1’ and 2’, conditionally projects the state of modes 1, 2 into a pure symmetric non-Gaussian state \([26]\), given in the Fock basis by

\[
|\psi_k\rangle = \sum_{n=0}^{\infty} c_n^{(k)} |n - k, n - k\rangle, \text{ with } [11]
\]

\[
c_n^{(k)} = (T\lambda)^{n-k} \frac{\Gamma(n+k+1)\Gamma(n-k+1)}{\Gamma(k+1)\Gamma(n+1)(n-k+1)!}.
\]

where \( \frac{n!}{2F_1\left(-k,-k;1;T^2\lambda^2\right)} \),

\[
E_N(\psi_k) = 2\log[\sum_{n=0}^{\infty} c_n^{(k)}] \text{ and reads in our notation}
\]

\[
E_N(\psi_k) = -\log[\left(1 - z^{2k+2}\right) F_1 \left(k + 1, k + 1; 1; z^2\right)].
\]

The logarithmic negativity of the PSS states for arbitrary \( k \) can be computed in simple closed form via \( E_N(\psi_k) = 2\log[\sum_{n=0}^{\infty} c_n^{(k)}] \) and reads in our notation

\[
E_N(\psi_k) = -\log[\left(1 - z^{2k+2}\right) F_1 \left(k + 1, k + 1; 1; z^2\right)]
\]

Notice that \( E_N(\psi_k) \) increases with \( k \): iteration of the photon subtraction process further enhances the entanglement compared to the original Gaussian instance. We aim at an estimate of the entropy of entanglement of PSS for any \( k \), and specifically at accurate bounds on this universal entanglement quantifier which can be measured experimentally with high efficiency. We will now derive the CM of PSS states in closed form and show that it contains enough information, straightforwardly accessible in the lab, for the desired task.

Denoting by \( \hat{O} \) an observable involving normally ordered combinations of ladder operators on modes 1 and 2, we observe that \( \langle \psi_k | \hat{O} | \psi_0 \rangle = N_k^{-1} |\psi_0\rangle |\tilde{a}_1 | \tilde{a}_2 | \hat{O} | \tilde{a}_1 | \tilde{a}_2 | \psi_0 \rangle \), that is, we can evaluate expectation values of relevant operators in terms of normally ordered higher moments computed on the unperturbed Gaussian twin beams. Recalling that, for \( |\psi(\tau)\rangle \), the normally ordered characteristic function reads \([2]\)

\[
\chi^{(N)}_0(\alpha_1, \alpha_1^*, \alpha_2, \alpha_2^*) = \exp\{\sinh r_1[(\alpha_1 \alpha_2 + \alpha_1^* \alpha_2^*) \cosh r - (|\alpha_1|^2 + |\alpha_2|^2) \sinh r]\},
\]

\[
M_{il}^{jm} = \psi_0((\hat{a}_1^\dagger)^j (\hat{a}_2^\dagger)^m (\hat{a}_1)^l (\hat{a}_2)^p) |\psi_0\rangle
\]

\[
= \frac{(-1)^{j+m} \partial^{j+m} \chi^{(N)}_0}{\partial \alpha_1^j \partial \alpha_2^m \partial \alpha_1^l \partial \alpha_2^p} |\alpha_1, \alpha_1^*, \alpha_2, \alpha_2^*\rangle_{\alpha_1, \alpha_2 = 0}
\]

The formula Eq. (3) can be readily applied to compute the second moments of the canonical operators on our non-Gaussian states. After some algebra, the CM \( \sigma_k \) of PSS of the form \( |\psi_k\rangle \) turns out to be that of a symmetric two-mode squeezed thermal state, automatically in standard form, with \( a_1 = a_2 = a(k) = N_k^{-1} (M_{k+k, k+k} + 2M_{k+k+1, k+k} + M_{k+k, k+k}) \), and \( \gamma_x = -\gamma_p = \gamma(k) = N_k^{-1} (M_{k+k, k+k+1} + M_{k+k+1, k+k}) + \)

\[
= (T^2\lambda^2)^{n-k} \frac{\Gamma(n+k+1)\Gamma(n-k+1)}{\Gamma(k+1)\Gamma(n+1)(n-k+1)!}
\]
We will now show how to extract from the CM elements a lower bound on the entanglement of PSS obtained from second moments, depicted as functions of the squeezing \( r \). The entanglement of formation is individuated within the narrow shaded region between the bounds. Panels [(a)–(c)] depict ideal PSS \( |\psi_k\rangle \), whose logarithmic negativity is shown as well for reference (thin dashed line). Panel (d) depicts a more realistic PSS \( g_{k=1} \) modeled as a binomial mixture of the Gaussian twin beam and the first four pure PSS, \( \bar{r}_{k=1} = \sum_{n=0}^{n=p^{-1} = 4} (1-p)^{n-k} \binom{n}{k} |\psi_k\rangle \langle \psi_k| \), with \( n = p^{-1} = 4 \); notice how only slight modifications occur to the bounds compared to the corresponding ideal case (a).

\[
M^{k,k+1} + M^{k+1,k}.
\]

\[
a^{(k)} = (cosh r)^{2+4k} [2F_1(k + 1, k + 1; tanh^2 r)]^{-1} \times [2(k + 1)^2 sinh^2 r 2F_1(-k, -k; 2; tanh^2 r) + 2F_1(-k, -k; 1; tanh^2 r)];
\]

\[
\gamma^{(k)} = \frac{2 tanh(r(k+1)) 2F_1(k + 1, k + 1; 2; tanh^2 r)}{2F_1(k + 1, k + 1; 1; tanh^2 r)}.
\]

We will now show how to extract from the CM elements a lower and an upper bound on \( E(\psi_k) \).

**Lower bound.** The extremality of Gaussian states [15] entails that [27]

\[
E^\text{low} \equiv E_F(\phi_k^G) \leq E(\psi_k),
\]

where \( \phi_k^G \) is the mixed Gaussian state with CM \( \sigma_k \). Explicitly,

\[
E^\text{low} = g[a^{(k)} - \gamma^{(k)}] \quad \text{where} \quad [24] \quad g(x) = \frac{(1+x)^2}{4x} \log \frac{(1+x)^2}{4x} - \frac{(1-x)^2}{4x} \log \frac{(1-x)^2}{4x}.
\]

**Upper bound.** The logarithmic negativity of pure PSS is already an upper bound for \( E \), however it is defined in terms of all the moments of the state, and its experimental determination (in real conditions requiring a complete state tomography) becomes rather demanding even when the states take very special forms [3]. The strength of our investigation is to derive a slightly looser upper bound but which is a function of the second moments only of the PSS, i.e. of \( \sigma_k \). We simply observe that for \( k = 0 \), namely for Gaussian twin beams, \( E_N(\psi_0) = \text{arcsinh}[\gamma^{(0)}] \). The standard-form parameter \( \gamma^{(0)} \) quantifies the maximum quadrature correlation between the two modes [23]. It is tempting to postulate that, for PSS \( |\psi_k\rangle \) with an arbitrary degree \( k \) of deGaussification, a function of \( \gamma^{(k)} \) may yield an overestimate of the actual entanglement (which, we remark, is nontrivially encoded in higher-order correlations too). We can turn this blurry bit of intuition into the following

**Theorem 1.** For all \( k \geq 0 \),

\[
E^\text{up} \equiv \log[1 + 2\gamma^{(k)}] \geq E_N(\psi_k) \geq E(\psi_k).
\]

**Proof.** The rightmost inequality holds by definition. Here we sketch the (quite technical) proof of the leftmost one, which is one of the main results of this paper. The simplest cases \( k = 0, 1 \) can be proven by inspection, hence we specify here to arbitrary \( k \) \( \geq 2 \). Using Eq. (2) and Eq. (5), and exponentiating both sides of the inequality, the problem reduces to proving that \( F^{(k)}(z) \equiv 2F_1(k + 1, k + 1; z^2) + 4z(k + 1) 2F_1(k + 2, k + 1; 1; z^2) - (1 - z)^{-2k-2} \geq 0 \) where we recall that \( tanh r \equiv z \in (0, 1) \). We can write \( F^{(k)}(z) \) as a power series, \( F^{(k)}(z) = \sum_{m=1}^{\infty} f^{(k)}_m z^m \), where \( f^{(k)}_m = \left[ 1 - (-1)^m \right] (2k + m + \frac{3}{2}) + 1 \left[ 1 + \left( \frac{1}{2} + \left( -1 \right)^m + k + \frac{3}{2} \right)^2 \right] \left( 2k+2m+1 \right) \). We observe that \( f^{(k)}_{2j} \geq 0 \) and \( f^{(k)}_{2j} \leq 0 \) \( \forall j \geq 1 \), and moreover \( f^{(k)}_{2j+i} \geq -f^{(k)}_{2j} \) \( \forall j > k + 1 \). This entails that by truncating the power series at \( m = 2k + 2 \) we discard a positive remainder: \( F^{(k)}(z) \geq \tilde{F}^{(k)}(z) \equiv \sum_{m=1}^{2k+2} f^{(k)}_m z^m \). Let us now define a parametric class of hypergeometric sums, \( S^{(k)}_l \equiv \sum_{m=1}^{2k+2} \binom{2k+1}{2k+1-l} f^{(k)}_m \). By means of Zeilberger’s
algorithm \footnote{29} one can verify that $S_{\ell}^{(k)} > 0 \ \forall \ell \geq 2, \ \ell \geq 0$. We will now show that $\tilde{F}(k)(z) \geq S_{0}^{(k)} z^{2k+2} \geq 0$ to conclude the proof. We take the ratio $R^{(k)}(z) = [\tilde{F}(k)(z)/S_{0}^{(k)} z^{2k+2}]$ and expand it in power series around $z = 1^{-}$: $R^{(k)}(z) = 1 + \sum_{l=1}^{\infty} [-1]^{l} [S_{l}^{(k)}/S_{0}^{(k)}](z - 1)^{l}$. But trivially $(z - 1)^{l} = (-1)^{l} (1 - z)^{l}$, and being $z \leq 1$ the alternating sign is cancelled to yield $R^{(k)}(z) \geq 1$. \hfill \Box

We have shown that, quite remarkably, the simple measurement of the CM of a pure PSS enables to pin down the entropy of entanglement quantitatively within analytical a priori bounds. In fact, one can appreciate how close the lower and upper bounds (both functions of the second moments only) are to each other for various values of $k$ in Fig. 1(a)-(c). A crucial fact is that the absolute error $\Delta_{k} = (E^{\text{up}} - E^{\text{low}})/(E^{\text{up}} + E^{\text{low}})$ on the estimate of $E(\psi_{k})$ from the CM 	extit{vanishes} for $r \rightarrow \infty$, rendering our method rigorously accurate. We notice that, in general, the error $\Delta_{k}$ increases with $k$, although it stays of the order of few units – on a scale ranging to infinity – even for big $k$ (e.g., $\Delta_{k} \leq 4$ for up to $k = 1000$ photon subtractions per beam), thus scarcely affecting the quality of the estimate [see Fig. 1]. We believe that a physical explanation for the scaling of $\Delta_{k}$ is rooted in the fact that with increasing $k$ the PSS $|\psi_{k}\rangle$ are increasingly more non-Gaussian, hence there is more information not retrievable from second moments only. This argument can be made quantitative by evaluating the entropic non-Gaussianity \footnote{16} $\Upsilon_{k}$ of $|\psi_{k}\rangle$, which simply amounts in this case to the Von Neumann entropy of the associated Gaussian state with CM $\sigma_{k}$. We obtain, for $r \gg 0$,

$$\Upsilon_{k}^{\text{max}} = \frac{1}{2} \left[ \log \left( \frac{k}{2} \right) + \sqrt{2k + 1} \log \left( \frac{k + \sqrt{2k + 1} + 1}{k} \right) \right].$$

For any $r$, the non-Gaussianity and the absolute error are very close to each other, with $\Upsilon_{k} \geq \Delta_{k}$, and exhibit the same scaling with $k$ (see Fig. 2). This fascinating connection adds insight to our analysis and leads us to sum up the results achieved so far as follows. \textit{Entanglement in ideal photon-subtracted states can be measured from the covariance matrix up to a narrow error that scales with the states’ non-Gaussianity.}

\section{IV. Generalization to Mixed States and Further Remarks}

In most practical implementations, efficient photon-number-resolving detectors are not available and the conditional generation of PSS is achieved by means of “on/off” type detectors which are only able to discriminate the vacuum from a bunch of an undefined number of photons \footnote{11}. This means that a more appropriate description of a class of PSS must be in terms of statistical mixtures of the form $\tilde{\psi}_{k} = \sum_{p} p_{k}|\psi_{k}\rangle/|\psi_{k}\rangle$, where we can define $k = \sum_{p} p_{k} k$ as the ‘average’ number of photons subtracted per beam \footnote{26}. For these mixed states even the logarithmic negativity is available in closed form (and its numerical evaluation on a computer requires several days for any given squeezing degree \footnote{11}), let alone the entanglement of formation. Remarkably, our bounds can be immediately extended to pinpoint the entanglement of formation of mixed symmetric PSS from the sole knowledge of second moments. We first observe that (having zero first moments) the CM transforms linearly: $\sigma_{k} = \sum_{p} p_{k} \sigma_{k}$, hence it can be computed analytically for any probability distribution $\{p_{k}\}$ from Eqs. \footnote{15}. The $E_{F}$ of the corresponding Gaussian state with CM $\sigma_{k}$, according to the extremality theorem \footnote{15}, stands as a lower bound for the $E_{F}$ of the non-Gaussian mixed PSS $\tilde{\psi}_{k}$ \footnote{27}. On the other hand, denoting by $\gamma_{F}^{(k)}$ the $(1, 3)$ element of the mixed-state CM $\sigma_{k}$, $\gamma_{F}^{(k)} = \sum_{p} p_{k} \gamma_{F}^{(k)}$, the upper bound Eq. \footnote{7} is immediately extended to the mixed case: $\log[1 + 2 \gamma_{F}^{(k)}] \geq E_{F}(\tilde{\psi}_{k})$. The proof follows from the concavity of the log function, the convexity of the entanglement of formation, and obviously Theorem 1. Namely, $\log[1 + 2 \gamma_{F}^{(k)}] \geq \sum_{p} p_{k} \log(1 + 2 \gamma_{F}^{(k)}) \geq \sum_{p} p_{k} E(\gamma_{F}^{(k)}) \geq E_{F}(\tilde{\psi}_{k})$. The behavior of the bounds in such more realistic conditions is shown in Fig. \footnote{11}(d) for an instance with $k = 1$. We observe, in general, that for reasonable modeling of the mixture (e.g., $p_{k}$ following a binomial distribution) the loosening of the bounds compared to the ideal cases with $k = \lfloor k \rfloor$ is negligible: our scheme is efficient and robust against the specific source of imperfection considered here (the usage of non-phon-number-resolving detectors). We plan to deepen our investigation in the future following the experimental progresses in the generation of (generally non-symmetric) PSS states \footnote{24}, thus properly modeling other sources of imperfections (e.g. mismatches or dark counts in the photon conditioning) that arise in practical demonstrations \footnote{30}, in order to test the robustness and reliability of our techniques for estimating entanglement in fully realistic situations.

In this context, let us briefly comment on the direct implementation of our results in experiments. Once a two-mode PSS is prepared, one needs to measure the full CM by homodyne detections as in \footnote{17, 19}, transform it in standard form (i.e. extract the symplectic invariants $a_{1}, a_{2}, \gamma_{x}, \gamma_{p}$) \footnote{3} [20].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{(color online). Scaling as a function of $k$ of the asymptotic absolute error $\Delta_{k}^{\text{max}}$ on the estimate of entanglement via second moments (dark bars) and of the asymptotic entropic non-Gaussianity $\Upsilon_{k}^{\text{max}}$ (light bars) for pure $k$-photon-subtracted states $|\psi_{k}\rangle$.}
\end{figure}
and then (upon verification that the standard-form CM has the structure predicted here: a benchmark for the state engineering) readily evaluate our lower and upper bounds to ensure an accurate estimate of the entanglement of formation of the produced non-Gaussian state.

V. CONCLUSION

In this paper, in the spirit of [13, 28], we have gone beyond the conventional belief that out of Gaussian states the covariance matrix plays a marginal role in CV entanglement quantification. On the contrary, we demonstrated that clever exploitation of such an easily accessible component of the state, bears extremely useful and precise information on the quantification of non-Gaussian entanglement produced in experiments, specifically for the important class of (realistic) photon-subtracted states [10, 11]. This is the start of a program which will continue with the systematic investigation of quantitative entanglement witnesses for other classes of non-Gaussian states in terms of low-order moments. We hope with our result to stimulate advances in the engineering and characterization of non-Gaussian resources, and their exploitation for demonstrations impossible to achieve with Gaussian states only, in order to explore the actual limits that quantum mechanics poses on the access and manipulation of information.

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[1] H. J. Kimble, Nature 453, 1023 (2008).
[2] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. 77, 513 (2005); N. Cerf, G. Leuchs, and E. S. Polzik (eds.), Quantum Information with Continuous Variables of Atoms and Light (Imperial College Press, London, 2007).
[3] G. Adesso and F. Illuminati, J. Phys. A 40, 7821 (2007).
[4] J. Eisert, S. Scheel, and M. B. Plenio, Phys. Rev. Lett. 89, 137903 (2002); G. Giedke and J. I. Cirac, Phys. Rev. A 66, 032316 (2002).
[5] S. Lloyd and S. L. Braunstein, Phys. Rev. Lett. 82, 1784 (1999); N. C. Menicucci, P. van Loock, M. Gu, C. Weedbrook, T. C. Ralph, and M. A. Nielsen, Phys. Rev. Lett. 97, 110501 (2006).
[6] F. Dell’Ano, S. De Siena, L. Albano, and F. Illuminati, Phys. Rev. A 76, 022301 (2007).
[7] G. Adesso, F. Dell’Ano, S. De Siena, F. Illuminati, and L. A. M. Souza, arXiv:0807.3958.
[8] A. I. Lvovsky, H. Hansen, T. Aichele, O. Benson, J. Mlynek, and S. Schiller, Phys. Rev. Lett. 87, 050402 (2001); A. Zavatta, S. Viciani, and M. Bellini, Science 306, 660 (2004); J. Wenger, R. Tuille-Broui, and P. Grangier, Phys. Rev. Lett. 92, 153601 (2004); J. S. Neergaard-Nielsen, B. Melholt Nielsen, C. Hettich, K. Mølmer, and E. S. Polzik, Phys. Rev. Lett. 97, 083604 (2006); K. Wakuhi, H. Takahashi, A. Furusawa, and M. Sasaki, Opt. Express 15, 3568 (2007); A. Ourjoumtsev, H. Jeong, R. Tuille-Broui, and P. Grangier, Nature 448, 784 (2007).
[9] A. Ourjoumtsev, A. Dantan, R. Tuille-Broui, and P. Grangier, Phys. Rev. Lett. 98, 030502 (2007).
[10] T. Opatrny, G. Kurizki, and D.-G. Welsch, Phys. Rev. A 61, 032302 (2000); P. T. Chou, T. C. Ralph, and G. J. Milburn, ibid. 65, 062306 (2002); S. Olivares, M. G. A. Paris, and R. Bonifacio, ibid. 67, 032314 (2003).
[11] A. Kitagawa, M. Takeoka, M. Sasaki, and A. Chefles, Phys. Rev. A 73, 042310 (2006).
[12] H. Nha and H. J. Carmichael, Phys. Rev. Lett. 93, 020401 (2004); R. Garcia-Patron, J. Fiurasek, N. J. Cerf, J. Wenger, R. Tuille-Broui, and P. Grangier, ibid. 93, 130409 (2004).
[13] E. Schubkin and W. Vogel, Phys. Rev. Lett. 95, 230502 (2005).
[14] A. I. Lvovsky and M. G. Raymer, Rev. Mod. Phys. (in press).
[15] M. M. Wolf, G. Giedke, and J. I. Cirac, Phys. Rev. Lett. 96, 080502 (2006).
[16] M. G. Genoni, M. G. A. Paris, and K. Banaszek, Phys. Rev. A 78, 060303(R) (2008).
[17] V. D’Auria, S. Fornaro, A. Porzio, S. Solimeno, S. Olivares, and M. G. A. Paris, Phys. Rev. Lett. 102, 020502 (2009).
[18] A similar method had been demonstrated for the detection of continuous variable entanglement in cold atomic ensembles by V. Josse, A. Dantan, A. Bramati, M. Pinard, and E. Giacoboni, Phys. Rev. Lett. 92, 123601 (2004).
[19] J. Laurat, G. Keller, J. A. Oliveira-Huguenin, C. Fabre, T. Coudreau, A. Serafini, G. Adesso, and F. Illuminati, J. Opt. B 7, S577 (2005); J. DiGuglielmo, B. Hage, A. Franzen, J. Fiurasek, and R. Schnabel, Phys. Rev. A 76, 012323 (2007).
[20] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. 84, 2722 (2000).
[21] M. B. Plenio and S. Virmani, Quant. Inf. Comp. 7, 1 (2007).
[22] G. Vidal and R. F. Werner, Phys. Rev. A 65, 032314 (2002); M. B. Plenio, Phys. Rev. Lett. 95, 090503 (2005).
[23] Logarithms are taken in natural basis throughout the paper.
[24] G. Giedke, M. M. Wolf, O. Krüger, R. F. Werner, and J. I. Cirac, Phys. Rev. Lett. 91, 107901 (2003).
[25] P. Marian and T. A. Marian, Phys. Rev. Lett. 101, 220403 (2008).
[26] Symmetry between the parties is typically required by quantum protocols and should be fulfilled by experimental realizations, hence we do not consider here non-symmetric states where a different number of photons is subtracted from each beam. However, since these instances can more closely resemble realistic implementations, we commit ourselves to extend our methods to encompass such a more general framework in further work.
[27] The rigorous validity of this lower bound relies on the widespread conjecture on the additivity of $F_{ij}$, which holds true for Gaussian states [24, 25], but no proof is known for PSS.
[28] C. Rodó, G. Adesso, and A. Sanpera, Phys. Rev. Lett. 100, 110505 (2008).
[29] D. Zeilberger, Discrete Math. 80, 207 (1990); M. Petkovsek, H. S. Wilf, and D. Zeilberger, “A=B” (A. K. Peters Publishers, Wellesley, MA, 1996); P. Paule and M. Schorn, J. Symb. Comput. 20, 673, 1995.
[30] A. Ourjoumtsev, R. Tuille-Broui, J. Laurat, and P. Grangier, Science 312, 83 (2006).