A one-channel conductor in an ohmic environment: mapping to a TLL and full counting statistics

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It is shown that a one-channel coherent conductor in an ohmic environment can be mapped to the problem of a backscattering impurity in a Tomonaga-Luttinger liquid (TLL). This allows to determine non perturbatively the effect of the environment on $I - V$ curves, and to find an exact relationship between dynamic Coulomb blockade (DCB) and shot noise. We investigate critically how this relationship compares to recent proposals in the literature. The full counting statistics is determined at zero temperature.

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A mesoscopic conductor embedded in an electrical circuit forms a quantum system violating Ohm’s laws. The transmission/reflection processes of electrons through the conductor excite the electromagnetic modes of the environment, rendering the scattering inelastic, and reducing the current at low voltage, an effect called environmental Coulomb blockade. This picture, valid in the limit of a weak conductance, changes in the opposite limit of a good conductance. The description of tunnelling via discrete charge states becomes then ill defined, raising the question of whether dynamic Coulomb blockade (DCB) survives or is completely washed out by quantum fluctuations. It is quite clear that DCB vanishes for a perfectly transmitting conductor. This property is shared by shot noise which results as well from the random current pulses due to tunneling events. Such similarity was concretized through a challenging relationship between the DCB reduction of the current in a one-channel conductor in series with a weak impedance and the noise without impedance (see Fig.1). More generally, the DCB variation of the n-1th cumulant of the current was related to the n-th cumulant without environment. The environmental effect on the third cumulant has been the subject of a recent intensive experimental and theoretical activity.

An ohmic environment could as well simulate the electronic interactions in the coherent conductor. In this view, one can wonder whether a one channel conductor in series with a resistance is equivalent to a one dimensional interacting system, described by the TLL model. This is already suggested by the power law behavior at small transmission with an exponent determined by $r = e^2 R/h$, the dimensionless environmental resistance, instead of the microscopic interaction parameter. Furthermore, Kindermann and Nazarov have shown recently that at low enough energy, a many-channel conductor in series with a weak resistance $r \ll 1$ behaves as a one-channel conductor with an effective energy dependent transmission $T(E, r)$ similar to that obtained in a weakly interacting one-dimensional wire in the presence of a backscattering center. In this framework, the variation of the current due to DCB is rather given by the shot noise computed through $T(E, r)$ instead of the bare transmission $T$.

In this Letter we fully extend the analogy to a TLL in order to explore the case of an arbitrary resistance $r$ in series with a coherent one channel conductor with good transmission. By performing a careful integration over the environmental degrees of freedom, it is shown that there is an entire low energy regime where the conductor embedded in its ohmic environment behaves exactly like a point scatterer in a TLL liquid, with an effective parameter $K' = 1/(1 + r)$ where $r = e^2 R/h$. It is mapped to a TLL with parameter $K' = 1/(1 + r)$ where $r = e^2 R/h$.

**FIG. 1:** A quantum circuit of a one-channel coherent conductor with transmission coefficient $T$ in series with an impedance $Z(\omega) = R$ for $\omega < \omega_R = 1/RC$. It is mapped to a TLL with parameter $K' = 1/(1 + r)$ where $r = e^2 R/h$. The strong (weak) backscattering limit corresponds to the tunneling (weak backscattering) regime with a dimensionless amplitude $\Gamma' (v_B')$. In contrast to a renormalization of the TLL exponent by (for instance) electron-phonon coupling, which tend to make e-e interactions more attractive, here the resistance will rather induce repulsive interactions, which corresponds to $K' < 1$ in the TLL model. This allows to use exact field theory results obtained in the TLL con-
text to propose a novel relationship between the DCB current and shot noise, and more generally between all cumulants.

Consider first a coherent one-channel conductor coupled to its environment, and described by the total Hamiltonian, restricting for simplicity to spinless electrons:

$$H = H_1 + H_2 + H_{env} + \Gamma_T \psi_1^\dagger \psi_2 e^{-i [e V t - \varphi(t)]} + h.c. \quad (1)$$

Here $H_{1,2}$ is the electronic Hamiltonian for the right and left electrodes, $H_{env}$ is a quadratic Hamiltonian describing the electromagnetic modes of the environment and including the capacitance of the junction, and $V$ is the potential imposed by the voltage generator. The last term couples the phase across the environmental impedance $Z(\omega)$ to the local electronic fields $\Psi_{1,2}(0)$ at the end points of the left/right electrodes, the momentum dispersion of the tunneling amplitude $\Gamma_T$ being ignored. In the following, we consider an ideal Ohmic resistor $R = h\tau^2/e^2$. Thus one is restricted to energies below $\omega_R = 1/RC$ since the capacitance of the conductor $C$ is included in the total impedance $Z(\omega) = R/(1 + i\omega/\omega_R) \sim R$ for $\omega < \omega_R$. At zero temperature, the large time behavior of the phase fluctuations becomes:

$$J(t) = \langle \varphi(t) \varphi(0) \rangle - \langle \varphi^2 \rangle = -2R \ln(i\omega_R t). \quad (2)$$

The differential dimensionless conductance has been computed to lowest order in the tunneling amplitude:

$$G_1 = \frac{\hbar}{e^2} \frac{dI}{dV} \approx \frac{1}{\Gamma(2r + 1)} \left( \frac{\Gamma_T}{h\nu_T} \right)^2 \left[ \frac{e|V|}{\hbar \nu_R} \right]^{2r}, \quad (3)$$

where $\Gamma$ is the gamma function. The similarity with the power-law behaviors familiar in TLL is striking. Now it will be shown that it is more than a coincidence.

Since tunnelling is punctual one can use bosonisation for the electronic part: performing a spherical wave decomposition of the modes in the electrodes, tunnelling affects only the s waves, whose dynamics is the same as that of one dimensional leads. One introduces the bosonic field $\theta$ with respect to which the electronic Hamiltonian $H_1 + H_2 = H^{el}_0$ in Eq. (1) is quadratic, thus its propagator is similar to Eq. (2):

$$\langle \theta(t) \theta(0) \rangle = -\ln(i\omega_F t). \quad (4)$$

The tunneling term becomes $\psi_1^\dagger(0) \psi_2(0) = e^{2\theta^2/2\pi a}$, with $a$ a distance cutoff, thus the total Hamiltonian Eq. (1) reads $H = H_0^{el} + H_{env} + \Gamma_T \varphi^2 e^{-i [e V t - \varphi(t)]} + h.c.$ Since $\theta$ and $\varphi$ commute, and both $H_0^{el}$ and $H_{env}$ are quadratic respectively with respect to $\theta$ and $\varphi$, their sum $H^{el}$ is quadratic with respect to the auxiliary field $\theta' = \theta + \varphi/2$, and $H = H_0^{el} + \frac{\Gamma_T}{2\pi a} e^{-2\theta'(t) - i \frac{e V t}{\hbar} - \varphi(t)} + h.c.$ Using Eqs. (4), one gets $\langle \theta'(t) \theta'(0) \rangle = -\frac{1}{2\pi a} \ln(i\omega_F t)$ up to a constant absorbed in $\Gamma_T$, the effective tunneling amplitude. The auxiliary parameter obeys $\frac{1}{\hbar \nu^2} \equiv 1 + r$ and the effective cutoff is $\omega_F' = \min(\omega_R, \omega_T)$. The problem then is formally equivalent to the strong back scattering limit through an impurity in a TLL with an interaction parameter $K' < 1$ and a cutoff energy $\omega_F'$. This equivalence holds not only for the Hamiltonian, but also for all the cumulants of the current. As a quick check, the standard first-order perturbative computation of the average current in the TLL problem Eq. (4) yields Eq. (3). In particular, it vanishes at zero voltage, which is a consequence of the irrelevance of tunneling. Thus any other neglected scattering process, depending on the realistic setup, could dominate the contribution of tunneling processes to $I$ at low enough energy, making the prediction Eq. (3) non-universal.

It is much more useful to think instead of the "dual limit" of weak backscattering with amplitude $v_B$, thus $T = (1 + \omega^2 / v_B^2)^{-1}$ close to one. In the absence of coupling to the environment, the problem is nothing but free electrons in the presence of a potential scatterer, whose locality allows to use again bosonisation. A bosonic field $\Phi(x)$ determines the electronic density through $\rho = -\partial_x \Phi / \pi$, thus the current $j = e \rho \Phi / \pi$. For pedagogical reasons, here we only present arguments at the level of the Euclidian action. It is convenient in this limit to integrate the bulk degrees of freedom and formulate the problem purely in terms of the local field at the impurity $\Phi = \Phi(x = 0)$ [14]. If $r$ is the imaginary time and $\omega_n$ are the Matsubara frequencies, one has $hS_{el} = \frac{1}{\beta} \sum_n |\omega_n| \langle \phi(\omega_n) \rangle^2 + \frac{e^2}{\pi a} \int_0^\beta \frac{d\tau}{2\pi} \cos 2\phi(\tau)$. The coupling to the impedance with a fluctuating potential drop $eV_{env} = h\nu_T \varphi$ is described by a term $Qu_{env}$ where the transferred charge $Q$ can be identified as $e\varphi$. Thus the action acquires an additional part $\delta S = \int d\tau \phi(\tau)(eV/h - \partial_\tau \varphi)/\pi \tau$ with $V$ the applied voltage. Performing a partial integration over the field $\varphi$ whose corresponding truncated action is [15]:

$$S_{env} = \sum_{|\omega_n| < \omega_R} |\omega_n| \langle \phi(\omega_n) \rangle^2/(2Re^2 \pi \beta)$$

leads to a renormalization of the kinetic term, $|\omega_n| \rightarrow |\omega_n| (1 + r)$. There is a formal equivalence to one impurity problem in a TLL, this time in the weak backscattering limit and at low energy compared to $\omega_F' \equiv \min(\omega_R, \omega_T)$. Remarkably, one gets the same auxiliary parameter as that found in the strong backscattering regime, $\frac{1}{\hbar \nu^2} \equiv 1 + r$. The auxiliary amplitude $v_B'$ will be taken as dimensionless in the following: it is proportional to $v_B$ but depends non-trivially on the cutoffs. The advantage of this limit is that the cosine term now defines a relevant perturbation. Thus the predictions of the field theory are universal as long as $v_B'$ is small enough. The generating Keldysh functional for $\phi$ turns out to be identical to that in the auxiliary TLL model. Thus one can exploit known results both for average current [14] [16] and higher cumulants defined by [17]:

$$I_n = \int \langle \langle j(t_1) \ldots j(t_n) \rangle \rangle_e dt_1 \ldots dt_{n-1}, \quad (5)$$

c indicating the connected part of the $n - \theta$ sym-
metrized current correlator. In the following, we will introduce the differential dimensionless cumulants:

\[ G_n = \frac{h}{e^{n+1} dV}. \]  

(6)

Let us first discuss the differential dimensionless conductance \( G_1 \) as inferred from lowest order perturbation with respect to \( v_B' \), in the limits where \( k_B T/eV \) is either small or large [14][10]:

\[ G_1 = \frac{h}{e^2} \frac{dI}{dV} = K' - c(K')v_B'^2 \left( \frac{\omega}{\omega_K} \right)^2(1+K') \]  

(7)

where \( h\omega = \text{max}(k_B T, eV) \), and \( c(K') \) a constant depending on \( K' \). First, observe that for \( v_B' = 0 \), one has a purely linear regime with Ohm’s law restored. The relation \( I = \frac{e}{\hbar} K' V \) is obtained, which translates into \( V = (R + R_q)/I \). This is nothing but the series resistance of a perfect point contact with resistance \( R_q = \frac{h}{e^2} \) and the resistor \( R \). Second, a bare amplitude \( v_B' \) is modified into an effective larger amplitude \( v_B'^{-1+K'} \) which diverges at low \( \omega \) because \( K' < 1 \), thus the above perturbative result is valid above a voltage scale \( V_B \propto \omega^{-1/(1-K')} \). Increasing \( \omega \) up to \( \omega_K \) drives \( G_1 \) to its maximum value, \( G_{\text{max}} = K' - c(K')v_B'^2 \) which is still smaller than the conductance without environment, \( G = 1 - c(1)v_B'^2 \). Notice that linearity can be maintained only at \( k_B T \gg eV \gg eV_B \), but breaks at \( k_B T \ll eV \). On the other hand, decreasing \( \omega \) below \( V_B \) increases the effective barrier height, and the conductance drops to zero at zero \( \omega \). The low-energy behavior of an almost transparent junction coupled to the environment is thus qualitatively similar to the one of a very poorly transmitting junction. In this limit, one can do perturbation with respect to a dimensionless tunnelling amplitude \( \Gamma_T \) related to \( v_B' \) in a non-universal way. Thus one gets a similar result to Eq. [4] at \( eV_B \gg eV \gg k_B T \), while \( eV \) has to be replaced by \( k_B T \) if \( eV_B \gg k_B T \gg eV \). All these considerations can be made non-perturbative using the exact solution of [18]. Thus increasing the bare transparency of the conductor does not wash out DCB but reduces its domain to \( V < V_B' \).

Motivated by the recurrence relation between cumulants suggested in previous works with a restriction to a weak resistance \( r \ll 1 \) [3],[4],[6], we now establish a more general relation holding for an arbitrary \( r \), starting by a comparison of \( G_1 \) to the (dimensionless) differential noise \( G_2 \) (Eqs. [5],[9]). Let us stick first to the two perturbative regimes so far discussed, and to \( k_B T \ll eV \), such that the noise is poissonian [13]. More precisely \( G_2 \approx 2G_1 \) for \( V \ll V_B \), while \( G_2 \approx 2K'(K' - G_1) \) for \( V \gg V_B \). Together with the expressions of \( G_1 \) in Eqs. [5] and [7] respectively at low and high voltages, one can check the relation, for an arbitrary \( r \), and for \( n = 2 \):

\[ \frac{\partial G_{n-1}(V,r)}{\partial \log V} = -2r G_n(V,r). \]  

(8)

Notice that the left hand side would vanish at \( r = 0 \), because \( G_1 \) becomes voltage-independent in this limit, thus this quantity is purely related to the presence of the environmental resistance. This relation expresses that the dynamical Coulomb blockade contribution to the conductance is related to the total noise in the presence of the environment. It holds not only at leading order in \( V/V_B' \) or \( V_B'/V \), but to any order. This truly non perturbative observation was dubbed a “generalized fluctuation dissipation theorem” in [13]. In order to compare this relation to the recent related works dealing with a small resistance \( r \), we now restrict to \( K' \) close to one. At strictly vanishing \( r \), \( K' = 1 \), and both the low and high energy series of the exact differential conductance [13] can be trivially re-summed to give the transmission probability \( \mathcal{T} \equiv G_1 = \frac{1}{1 + \frac{eV_B}{eV}} = \frac{eV}{eV + eV_B} \). To lowest order in \( r \), it is tempting to replace \( G_1 \), on the r.h.s of Eq. [5] by its value at \( r = 0 \), here \( G_2(V,r) \approx G_2(V,0) \): doing this would suggest that the DCB contribution of a small resistance \( r \) to the current would be proportional to the shot noise without environment as argued in [3],[6]. But one has to be careful with the limit \( r \ll 1 \), as can be seen already in the two dual limits where the noise is poissonian [3],[7], \( G_2 \approx \Gamma_T^2 V^{-2r/(1+r)} \) or \( G_2 \approx \Gamma_T^2 V^{2r} \); even if \( r \ll 1 \), \( G_2 \) can be replaced by its noninteracting value only if \( V \) is not too small. A more quantitative comparison of \( G_2(V,r) \), inferred from the exact solution of [13], to its noninteracting value is given by the continuous curve of Fig. [2]: the ratio \( x = G_2(V,r)/G_2(r=0) \) is plotted as a function of \( \log V \) at \( r = 0.05 \). Here, \( G_2(r=0) = \mathcal{T}(1-\mathcal{T}) \) is obviously voltage-independent. \( x \) is close to one for an intermediate values of voltages, but deviates from one in the limit of small or large voltages, a manifestation of the breakdown of perturbation theory with respect to \( r \). But the quality of this agreement depends on the value of the transmission coefficient, and will be discussed in more details elsewhere.

The mapping to a TLL at an arbitrary \( r \) and the subsequent exact solution can be used as well to shed some light on Ref. [11] which are in the spirit of Ref.[11] where an effective energy-dependent transmission coefficient \( T(E,r) \) is introduced, and argued to satisfy the following RG equation in the limit of small \( r \): \( \partial T(E,r)/\partial \log E = -2r(T(E,r) )/(1 - T(E,r)) \). This formula suggests approximating the differential noise on the r.h.s of Eq. [5], for \( r \ll 1 \), as \( G_2(V,r) \approx G_1(V,r) (1 - G_1(V,r)) \). However this is not satisfactory at high voltages, when \( G_1(V,r) \rightarrow K' = 1/(1+r) \). Rather, a better approximation is obtained by defining \( T(V,r) = (1+r) G_1(V,r) \) such that \( G_2(V,r) \approx T(V,r) (1 - T(V,r)) \), as shown through their ratio \( x \) in Fig.[2] (the dashed curve). This approximation is good up to an accuracy of \( r \) in intermediate to high voltage regimes.

Remarkably, for an arbitrary resistance \( r \), Eq. [5] can be extended to all cumulants, i.e. to \( n > 2 \) where \( G_n \)
in Eqs. (40) is computed with the environmental resistance in series, i.e. at $K' < 1$. Again, the limit $r \ll 1$ requires care: replacing $G_n$ on the r. h. s. by its value without environment, $G_n(V,r) \approx G_n(r=0)$ yields the prediction of $\langle \Sigma \rangle$, but with a restricted validity domain. Rather, a better fit to $\langle \Sigma \rangle$ is expected if one replaces $G_n(V,r \ll 1)$ by that expressed in the scattering approach through the effective transmission $T(V,r) = (1+r)G_1(V,r)$, which needs to be checked.

A study of the various properties for arbitrary $K'$ and temperature requires complex Bethe ansatz calculations, and is identical to the examples carried out in $\text{[18]}$. The case $K' = 1/2$ is particularly simple, especially to introduce the finite temperature. This corresponds to a crossover value, $R$ being a quantum resistance: $R = \hbar/e^2$. While the average current and noise have been expressed analytically, it would be interesting to compute the higher cumulants $\text{[21]}$.

An interesting extension of these results can be done for a point scatterer in a TLL of parameter $K$ coupled to an Ohmic environment: the auxiliary parameter becomes $1/K' = 1/K + r$, which increases the effective interactions by making $K$ smaller, thus the power law exponent is a combination of effects of the environment and the microscopic interactions. Note however that the role of the reservoirs, like in ordinary quantum wires, will have to be carefully understood.

In conclusion, we have first seen in this Letter how the coupling to an Ohmic environment induces effective repulsive e-e interactions. While this idea is not entirely new $\text{[21]}$, the setup of a well transmitting element coupled to an arbitrary resistance provides a concrete realization, which seems very amenable to experimental study, especially in view of the recent progress in good transmitting atomic contacts $\text{[22]}$. It is particularly exciting to have a potential new way of seeing TLL physics $\text{[23]}$, and the dramatic effect of a weak backscattering barrier at low energy. Conceptually, the relationship between TLL and dissipation is not that surprising: starting inversely from a TLL, an electron can view the surrounding electrons with which it interacts as an effective electromagnetic environment $\text{[8]}$. Thus a TLL can as well be viewed as the simplest one-channel conductor coupled to a resistor $\text{[21]}$.

Beyond its qualitative interest, the mapping has allowed us to make contact with exact field theoretic calculations, yielding the full counting statistics for the current at zero temperature. We have then been able to propose a more general link between the dynamic Coulomb blockade and the shot noise, embodied in the exact relation (8), a "non-equilibrium fluctuation dissipation theorem", whose deep origin remains somewhat mysterious, and which extends to higher cumulants. In particular, the mapping yields the third cumulant at arbitrary environmental resistance and transmission, opening the perspective to include the finite temperature, especially feasible at $r = 1$.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Limit of a weak resistance, $r = 0.05$: the ratio $x$ of the exact differential noise $G_2(V,r)$ to the noise at $r = 0$ (continuous curve), and to a "renormalized" noise given by $T(V,r)(1-T(V,r))$ (dashed curve) is plotted against log $V$ where $V$ is implicitly divided by an arbitrary voltage scale. The bare transmission is taken to be $T = 0.65$, while $T(V,r) = (\hbar/e^2)(1+r)dI/dV$.}
\end{figure}
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