Existence and uniqueness of the modified error function

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Abstract

This article is devoted to prove the existence and uniqueness of solution to the non-linear second order differential problem through which is defined the modified error function introduced in Cho-Sunderland, J. Heat Transfer, 96-2:214-217, 1974. We prove here that there exists a unique non-negative analytic solution for small positive values of the parameter on which the problem depends.

Key words Modified error function, error function, phase change problem, temperature-dependent thermal conductivity, nonlinear second order ordinary differential equation.

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1 Introduction

In 1974, Cho and Sunderland [2] studied a solidification process with temperature-dependent thermal conductivity and obtained an explicit similarity solution in terms of what they called a modified error
function. This function is defined as the solution to the following non-linear differential problem:

\[ [(1 + \delta y(x))y'(x)]' + 2xy'(x) = 0 \quad 0 < x < +\infty \quad (1a) \]

\[ y(0) = 0 \quad (1b) \]

\[ y(+\infty) = 1 \quad (1c) \]

where \( \delta \geq -1 \) is given. Graphics for numerical solutions of (1) for different values of \( \delta \) can be found in [2]. The classical error function is defined by:

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2)dz, \quad x > 0, \quad (2) \]

and it is a solution to (1) when \( \delta = 0 \). This makes meaningful the denomination modified error function given for the solution to problem (1).

The modified error function has also appeared in the context of diffusion problems before 1974 [4,12]. It was also used later in several opportunities to find similarity solutions to phase-change processes [13,14,15,16,17]. It was cited in [13], were several non-linear ordinary differential problems arise from a wide variety of fields are presented. Closed analytical solutions for Stefan problems with variable diffusivity is given in [11]. Temperature-dependent thermal coefficients are very important in thermal analysis, e.g. see [9]. Nevertheless, to the knowledge of the authors, the existence and uniqueness of the solution to problem (1) has not been yet proved. This article is devoted to prove it for small \( \delta > 0 \) using a fixed point strategy.

## 2 Existence and uniqueness of solution to problem (1)

The main idea developed in this Section is to study problem (1) through the linear problem given by the differential equation:

\[ [(1 + \delta \Psi_h(x))y'(x)]' + 2xy'(x) = 0, \quad 0 < x < +\infty, \quad (1a) \]

and conditions (1b), (1c). The function \( \Psi_h \) in (1a) is defined by:

\[ \Psi_h(x) = 1 + \delta h(x), \quad x > 0, \quad (3) \]

where \( \delta > 0, \ h \in K \subset X \) is given and:

\[ X = \{ h : \mathbb{R}^+_0 \to \mathbb{R} / \text{h is an analytic function, } ||h||_\infty < \infty \} \quad (4a) \]

\[ K = \{ h \in X / ||h||_\infty \leq 1, \ 0 \leq h, \ h(0) = 0, \ h(+\infty) = 1 \}. \quad (4b) \]
Hereinafter, we will refer to the problem given by (1a), (1b) and (1c) as problem (1). Let us observe that $K$ is non-empty closed subset of the Banach space $X$.

The advantage in considering the linear equation (1a) is that it can be easily solved through the substitution $v = y'$. Thus, we have the following result:

**Theorem 2.1.** Let $h \in K$ and $\delta > 0$. The solution $y$ to problem (1) is given by:

$$y(x) = C_h \int_0^x \frac{1}{\Psi_h(\eta)} \exp\left(-2 \int_0^\eta \frac{\xi}{\Psi_h(\xi)} \, d\xi\right) \, d\eta \quad x \geq 0,$$

where the constant $C_h$ is defined by:

$$C_h = \left(\int_0^{+\infty} \frac{1}{\Psi_h(\eta)} \exp\left(-2 \int_0^\eta \frac{\xi}{\Psi_h(\xi)} \, d\xi\right) \, d\eta\right)^{-1}.$$  

**Proof.** Let us first observe that the constant $C_h$ given by (6) is well defined, that is, that $C_h \in \mathbb{R}$. In fact, we have:

$$|C_h^{-1}| = \int_0^{+\infty} \frac{1}{\Psi_h(\eta)} \exp\left(-2 \int_0^\eta \frac{\xi}{\Psi_h(\xi)} \, d\xi\right) \, d\eta \geq \frac{1}{1 + \delta} \int_0^{+\infty} \exp(-\eta^2) \, d\eta = \frac{\sqrt{\pi}}{2(1 + \delta)}.$$  

(7)

Now the proof follows easily by checking that the function $y$ given by (5) satisfies problem (1). \hfill \blacksquare

The following result is an immediate consequence of Theorem 2.1.

**Corollary 2.1.** Let $y \in K$ and $\delta > 0$. Then $y$ is a solution to problem (1) if and only if $y$ is a fixed point of the operator $\tau$ from $K$ to $X$ defined by:

$$\tau(h)(x) = C_h \int_0^x \frac{1}{\Psi_h(\eta)} \exp\left(-2 \int_0^\eta \frac{\xi}{\Psi_h(\xi)} \, d\xi\right) \, d\eta \quad x > 0,$$

with $C_h$ given by (6).

**Remark 1.** Observe that $\tau(K) \subset K$.

We will now focus on analyzing when $\tau$ has only one fixed point. The estimations summarized next will be useful in the following.

**Lemma 2.1.** Let $h, h_1, h_2 \in K$, $\delta > 0$ and $x \geq 0$. We have:

\[ a) \int_0^x \left| \frac{\exp\left(-2 \int_0^\eta \frac{\xi}{\Psi_{h_1}(\xi)} \, d\xi\right)}{\Psi_{h_1}(\eta)} \right| - \left| \frac{\exp\left(-2 \int_0^\eta \frac{\xi}{\Psi_{h_2}(\xi)} \, d\xi\right)}{\Psi_{h_2}(\eta)} \right| \, d\eta \]
\[ \leq \frac{\sqrt{C}}{\Psi} \delta \sqrt{1 + \delta(3 + \delta)} ||h_1 - h_2||_\infty, \]

b) \(|C_{h_1} - C_{h_2}| \leq \frac{1}{\sqrt{\Psi}} \delta \sqrt{1 + \delta(1 + \delta)^2(3 + \delta)} ||h_1 - h_2||_\infty, \]

c) \[ \int_0^x \frac{1}{\Psi h(\eta)} \exp \left(-2 \int_0^\eta \frac{\xi}{\Psi h_1(\xi)} d\xi \right) d\eta \leq \frac{\sqrt{\pi(1 + \delta)}}{2}. \]

Proof. Let \( f \) be the real function defined on \( \mathbb{R}^+_0 \) by \( f(x) = \exp(-2x) \). If \( h_1 \leq h_2 \), it follows from the Mean Value Theorem applied to function \( f \) that:

\[ \exp \left(-2 \int_0^\eta \frac{\xi}{\Psi h_1(\xi)} d\xi \right) - \exp \left(-2 \int_0^\eta \frac{\xi}{\Psi h_2(\xi)} d\xi \right) = 2 \exp \left(-2 \int_0^\eta \frac{\xi}{\Psi h_2(\xi)} d\xi \right) \left[ \int_0^\eta \frac{\xi}{\Psi h_1(\xi)} d\xi - \int_0^\eta \frac{\xi}{\Psi h_2(\xi)} d\xi \right] \leq \delta ||h_2 - h_1||_\infty \eta^2 \exp \left(-\frac{\eta^2}{1 + \delta} \right), \]

where \( h_1 \leq h_3 \leq h_2 \). Now a) follows from regular computations. When \( h_1 \not\leq h_2 \), as the LHS in a) can be bounded for the same expression but applied to \( h_m = \min\{h_1, h_2\} \) and \( h_M = \max\{h_1, h_2\} \), the proof runs as before and it is completed having into consideration that \( ||h_1 - h_2||_\infty = ||h_M - h_m||_\infty \).

The proof of b) follows from a), and c) can be obtained from regular computations.

Theorem 2.2. Let \( \delta > 0 \) be the only one positive solution to the equation:

\[ \frac{x}{2} (1 + x)^{3/2}(3 + x)[1 + (1 + x)^{3/2}] = 1. \]

If \( 0 < \delta \leq \delta_1 \), then \( \tau \) is a contraction.

Proof. Let \( g \) be the real function defined by:

\[ g(x) = \frac{x}{2} (1 + x)^{3/2}(3 + x)[1 + (1 + x)^{3/2}], \quad x \geq 0. \]

Since \( g \) is an increasing function from 0 to \(+\infty\), we have that equation (10) admits only one positive solution \( \delta_1 \).

Let be now \( h_1, h_2 \in K \) and \( x \geq 0 \). From Lemma [2.1] and:

\[ |\tau(h_1)(x) - \tau(h_2)(x)| \leq C_{h_1} \int_0^x \left| \frac{\exp \left(-2 \int_0^\eta \frac{\xi}{\Psi h_1(\xi)} d\xi \right)}{\Psi h_1(\eta)} - \frac{\exp \left(-2 \int_0^\eta \frac{\xi}{\Psi h_2(\xi)} d\xi \right)}{\Psi h_2(\eta)} \right| d\eta \]

\[ + |C_{h_1} - C_{h_2}| \int_0^x \frac{1}{\Psi h_2(\eta)} \exp \left(-2 \int_0^\eta \frac{\xi}{\Psi h_2(\xi)} d\xi \right) d\eta, \]

\[ \leq C_{h_1} \int_0^x \left| \frac{\exp \left(-2 \int_0^\eta \frac{\xi}{\Psi h_1(\xi)} d\xi \right)}{\Psi h_1(\eta)} - \frac{\exp \left(-2 \int_0^\eta \frac{\xi}{\Psi h_2(\xi)} d\xi \right)}{\Psi h_2(\eta)} \right| d\eta \]

\[ + |C_{h_1} - C_{h_2}| \int_0^x \frac{1}{\Psi h_2(\eta)} \exp \left(-2 \int_0^\eta \frac{\xi}{\Psi h_2(\xi)} d\xi \right) d\eta. \]
it follows that \( \| \tau(h_1) - \tau(h_2) \|_\infty \leq \gamma \| h_1 - h_2 \|_\infty \), where \( \gamma = g(\delta) \). Recalling that \( g \) is an increasing function, it follows that \( \tau \) is a contraction when \( 0 < \delta < \delta_1 \).

\[ \tau(h_1) - \tau(h_2) \leq \gamma |h_1 - h_2|, \]

\( \gamma = g(\delta) \).

\( \tau \) is a contraction when \( 0 < \delta < \delta_1 \).

\[ \gamma = g(\delta) \]

Remark 2. From a numerical computation, it can be found that:

\[ 0.203701 < \delta_1 < 0.203702. \]

We are now in the position to formulate our main result:

**Corollary 2.2.** Let \( \delta_1 \) be as in Theorem 2.2. If \( 0 < \delta < \delta_1 \), then problem 4 has a unique non-negative analytic solution.

**Proof.** It is a direct consequence of Corollary 2.1, Theorem 2.2 and the Banach Fixed Point Theorem.

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