PAPER

Observations on the Bethe ansatz solutions of the spin-1/2 isotropic anti-ferromagnetic Heisenberg chain: the chiral string conjecture

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Keywords: heisenberg spin chains, bethe ansatz, bethe strings, chiral limit, wess-zumino-witten model

Abstract

Calculations of low lying excited states in even length \( L \rightarrow \infty \) chains begun in 2019 J. Phys. Commun. 3 025007 in the total spin \( S = 0 \) sector are here extended to \( S > 0 \). Bethe string configurations that were observed to transition from their definition at small \( L \) to different ‘apparent’ configurations at large \( L \) are now understood to be a manifestation of the approximate chiral symmetry that develops with increasing chain length. It is conjectured that when chiral symmetry is built into the configuration definitions a complete explicit labelling of all low lying excited states by strings is obtained. The structure is of the form of a product of Bethe strings defining left and right chiral states of spin \( s_L \) and \( s_R \) and a central complex ‘string’ necessary to bind \( s_L \) and \( s_R \) into total \( S \). Rules based on the string content of a state yield its momentum and asymptotic energy, including the amplitude of the leading \( \ln(L) \) correction.

1. Introduction and main results

Bethe [1] (for an English translation see [2]) already showed in his now classic paper that the approximate string configurations defining the eigenstates of modest length \( L \), spin \( s = \frac{1}{2} \), Heisenberg chains with periodic boundary conditions did not necessarily apply for large \( L \) and by inference in the limit \( L \rightarrow \infty \). This result has been rediscovered many times since (cf. [3, 4]) but has never been resolved in terms of a proposal for the configurations valid for \( L \rightarrow \infty \). Indeed, resolution by purely analytical methods is probably impossible; what is needed is guidance from a well chosen set of numerical solutions on very long chains. The impediment to the latter is that the set must contain complex solutions and, because of the very large parameter space involved, has always been understood to be a hard problem. The impasse of finding a significant number of such eigenstates was partially broken in [5] but because it was mostly restricted to the total spin \( S = 0 \) sector it was too limited to make many credible inferences for general \( S \). The present paper, while mostly restricted to \( S = 1 \), when combined with [5] contains sufficient information to lead to a conjecture for the configurations for all low lying exited states in the large \( L \) limit. The result has a simple intuitive structure. Excitations at the left and right Fermi ‘surface’ points bind separately into chiral spin states \( s_L \) and \( s_R \). These excitations are in general 1-strings from the filled Dirac sea that have moved into available left and right holes together with what are clearly recognizable Bethe \((n > 1)\)-strings. If no other excitation is present, the chiral states combine into a stretched state \( S = s_L + s_R \). Otherwise a central \( n \)-complex, \( n > 1 \), provides the coupling necessary to bind the chiral spins into \( S = s_L + s_R - n + 1 \). While an \( n \)-complex might be similar to a Bethe \( n \)-string, it is more commonly a very distorted \( n \)-string and may not even be recognizable as such \( ^1 \). The chiral state picture is not exclusively limited to large \( L \). As a simple example, the first excited singlet state made from \( L/2 \) overturned spins from the

\(^1\) Woynarovich [5, 7] provided examples of non-string complex solutions for the Heisenberg anti-ferromagnet and the present paper provides some context for their presence in low energy states for large \( L \). Many non-string configurations such as the Woynarovich quartets appear to be present only at intermediate lengths in a transition region before chiral symmetry has fully developed. In the chiral regime it seems that only the central \( n \)-complex can be of non-string or highly distorted string form.
ferromagnetic ground state can be considered as \( s_L = s_R = \frac{1}{2} \) bound into \( S = 0 \) by a central 2-complex that is exactly the Bethe 2-string \( \lambda = \pm i \). The first excited triplet state made from \( L/2 - 1 \) overturned spins requires no further excitation to form \( s_L = s_R = \frac{1}{2} \) into the stretched configuration \( S = s_L + s_R = 1 \).

To describe the conjectured solution I begin by establishing the notation in this paper which follows closely that in [5]. The states of (even) length \( L \) periodic chains of \( s = \frac{1}{2} \) spins anti-ferromagnetically coupled are defined by the Hamiltonian

\[
H = \frac{1}{2} \sum_{i=1}^{L} \sigma_i \cdot \sigma_{i+1} - \sum_{i=1}^{L} \sigma_i
\]

where \( i \) labels both the sites and distance along the chain and the components of \( \sigma_i \) are the Pauli spin matrices. Symmetry dictates that eigenstates of (1) can be labelled by total \( S \) and \( S_z \), and (quasi) momentum \( K \). Each stretched state \( (S_z = S) \) is constructed from \( N = L/2 - S \) overturned spins from the totally aligned spin configuration. Any eigenstate of (1) is defined by (quasi) momentum eigenvalues \( k_n = -\pi < k_n < \pi \), each associated with an overturned spin. These eigenvalues satisfy the Bethe ansatz equations (BAE)

\[
\left( \frac{\lambda_n + i}{\lambda_n - i} \right)^L = \Pi_{m<n} \left( \frac{\lambda_n - \lambda_m + 2i}{\lambda_n - \lambda_m - 2i} \right) \lambda_n = \cot \left( \frac{k_n}{2} \right), \quad n = 1, 2, \ldots \frac{L}{2} - S.
\]

The (scaled) sum of the \( k_n \) is the total momentum

\[
K = \left( \frac{L}{2\pi} \right) \sum_{n=1}^{N} k_n, \quad N = \frac{L}{2} - S,
\]

an integer which, with (symmetric) modulo \( L \) understood, satisfies \(-L/2 < K \leq L/2\). Solutions for negative \( K \), \(-L/2 < K < 0\), are obtained from those at positive \( K \) by sign reversal of all \( k_n \). The energy of any state that is a solution of the BAE (2) is given by

\[
E = \frac{L}{2} - 2 \sum_{n=1}^{N} (1 - \cos (k_n)) = \frac{L}{2} - \sum_{n=1}^{N} \frac{4}{1 + \frac{\lambda_n}{2}}
\]

and is clearly unaffected by \( k_n \) sign reversal.

For \( L \) not too large, the \( \{ \lambda \} \) in a BAE solution group into approximate strings; an ideal \( n \)-string being the list of \( n \) numbers \( \lambda = \lambda_1 + i(n-1), \lambda_2 + i(n-2), \ldots, \lambda_n + i(1-n) \) with common real part \( \lambda_n \). If we denote the number of \( n \)-strings or ‘particles’ in a particular solution with given \( L \) and \( S = S_y \) by \( p_n \) then the set \( \{ p_n \} \) is a partition of \( N = L/2 - S \) with the total number of particles \( P = \sum p_n \) and \( N = \sum n p_n \). Bethe [1] concluded the number of eigenstates with configuration \( \{ p_n \} \) is the product

\[
D(L, S, \{ p_n \}) = \Pi_n \left( \frac{p_n + h_n}{p_n} \right), \quad h_n = 2S + 2 \sum_{m>n} (m-n) p_m
\]

with each binomial the number of ways \( p_n \) particles and \( h_n \) holes can be arranged in \( p_n + h_n \) slots. He also confirmed that (5) summed over \( p(N) \), the unrestricted partitions of \( N \), gives the correct total number of states. The product (5) and its sum over \( p(N) \) is easily generalized (cf. [5] leading to equation (45)) into a generating function

\[
Z(L, S, q) = \sum_{p(N)} q^{(\text{\(p\)) mod 2}} \Pi_n \left( \frac{p_n + h_n}{p_n} \right) = \sum_{K=1-L}^{L} D_K (L, S) q^K
\]

for the number of states \( D_K (L, S) \) at momentum \( K \). The key feature in (6) is the Gaussian binomial \([\cdot]_q \) that replaces the binomial \((\cdot) \) in (5). It is defined here as the symmetric form

\[
\left[ \frac{p + h}{p} \right]_q = q^{-hp/2} \frac{(q)_{p+h}}{(q)_h}, \quad (q)_0 = 1, \quad (q)_m = \Pi_{k=1}^{m} (1 - q^k), \quad m > 0
\]

invariant under the replacement \( q \to 1/q \). The factor \( q^{(\text{\(p\)) mod 2}} \) in (6) accounts for the empirical observation that the Gaussian binomial product is centered at \( K = 0(L/2) \) if the number of particles \( P \) is even(odd). It is to be understood that periodicity will be used to shift terms following the first equality in (6) into the interval (first Brillouin zone) defined following the second equality. That is, appropriate integer multiples of \( L \) will be added to \( n \) in any term \( \times q^n \). An explicit result for the sum over \( p(N) \) in (6),

\[
Z(L, S, q) = \frac{(q)_L}{(q)_{L-N}(q)_N} \frac{1 - q^{2S+1}}{1 - q^{N+2S+1}} q^N, \quad 2S = L - 2N,
\]

has been determined and by taking the limit \( q \to 1 \) confirms Bethe’s result \( \frac{L - 2N + 1}{L - N + 1} \binom{L}{N} \) for the sum of (5) over \( p(N) \).
It is known [5] that with increasing L a chiral symmetry develops in which the total excitation energy, momentum and spin of the chain reduces to the sum of left and right energy, momentum and spin of (approximately) independent excited states at the two Fermi ‘surface’ points above a filled Dirac sea. The main conclusion of the present paper is that the left and right chiral towers of excitation have a natural Bethe string labelling parameterized by spin s and available 1-string holes h1. An additional important point is that the Dirac sea is a packed configuration of 1-strings only in the case that the total spin S is combined from sL and sR in the stretched arrangement, i.e. S = sL + sR. Otherwise the Dirac sea contains an n > 1 excitation complex with S = sL + sR − n + 1. These excitations do not change the total momentum from its left plus right chiral sum; nor do they change the energy in the L → ∞ limit. Rather they are responsible for the coupling of the (approximately) independent sL and sR into total S and result in the leading correction to scaling by an energy shift ∝ − sL · sR / ln(L).

The concept of cusp states introduced in [3] for the S = 0, L → ∞ state counting can be applied to the description of the chiral tower states and remains useful as a way of separating 1-string excitations from those of (n > 1)-strings. The (left or right) tower generating functions for cusp states are conjectured to be

\[
T^{(2s)}_{2\nu}(q) = q^{a_s} \sum_{\tilde{p}(n-1)} \prod_{n=1}^{p_n} q^{a_{\nu}} \left[ p_n + h_n \right]_{q} \begin{cases} h_0 \equiv 0, & h_1 = 2\nu \\ h_{n>0} = 2s + 2m > n(m - n)p_m \end{cases}
\]

for non-negative integral 2s and ν = s, s + 1, s + 2, … . The [1]s factors in the product are symmetrized Gaussian binomials defined in (7). The sum in (9) is over modified partitions \( \tilde{p}(N) \) obtained by adding one to all non-zero elements of the parts of every partition in p(N). A useful notation for any specific partition is the product n\( k \) over distinct n; examples of partition lists and associated \( \tilde{p}(N) \) are given by (10). Using this notation the modified sum \( \tilde{p}(N) \) is evaluated as the limit \( \tilde{p}(N) = \lim_{n \to \infty} \tilde{p}(N) \).

(10) that follows directly from (7) for q < 1. Consequently, the tower generator for all states is obtained from the cusp state generator (9) by the simple replacement

\[
T^{(2s)}_{2\nu}(q)\{\text{all states}\} = T^{(2s)}_{2\nu}(q) / (q)_{2\nu}.
\]

Summing (11) over ν ≥ s is shown in section 2 to produce the generator

\[
T^{(2s)}(q) = \sum_{s'=0}^{\infty} T^{(2s')}_{2\nu}(q) / (q)_{2\nu} = (q^{s^2} - q^{(s+1)^2}) / (q)_{\infty}
\]

for all s = s\( g \) states, while a further sum over s greater than some minimum s\( g \) yields

\[
U^{(2s)}(q) = \sum_{s'=s}^{\infty} T^{(2s')}(q) = q^{s^2} / (q)_{\infty} = q^{s^2} \sum_{n=0}^{\infty} p(n) q^n
\]

where the last equality is the known result that 1 / (q)\( s_g \) is the generator for the partitions p(n). We can identify \( U^{(2s)}(q) = q^{as} \sum_{s'=s}^{\infty} m(s', n) q^n \) as the generator for the multiplicity m(s\( g \), n) of tower states at s\( g \) and level n with the condition that minimum energy states at s\( g = 0 \) or s\( g = \pm \frac{1}{2} \) are level n = 0. Comparing this with (13) yields m(s\( g \), n) = p(n − s\( g \) + s\( g \)) in agreement with the recursion relations given in [8] based on the Kac-Moody algebra for the Wess-Zumino-Witten (WZW) model. Since (13) is based on (11), this agreement is a step towards confirmation that conjecture (9) provides a consistent labelling of all states in terms of Bethe’s strings.

As noted in [3] the low lying cusp states in the L → ∞ chain are characterized by left and right 1-string holes h\( l \) and h\( r \) in addition to total spin S and (scaled) momentum K(3). The empirical evidence from many examples is that the generator for the multiplicity m\( g \), of cusp states at κ = K − K\( g \), momentum relative to a central K\( g \), is a sum of products of the cusp tower generators (9) given by
\[ C^{(S)}_{h_L, h_R}(q) = \sum_{n=0}^{\Delta} T_{h_L}^{(2s_L)} \left( \frac{1}{q} \right) T_{h_R}^{(2s_R)}(q) = \sum_{\kappa} m_{\kappa} q^{c^{(1)}} \]
\[ = \sum \left\{ \Pi_L n^R \right\}, \quad \left\{ \begin{array}{l} 0, \quad S = s_L + s_R \\ (s_L + s_R - S + 1)^s, \quad S < s_L + s_R \end{array} \right. \]
\[ \Pi_R n^R \}
\]

The replacement \( q \rightarrow 1/q \) in the left tower accounts for the negative momenta of left moving excitations while the sum \( \sum \) restricts \( s_L \) and \( s_R \) to the triangle condition \( \Delta(S, s_L, s_R) \), i.e. \( s_L - s_R \leq S \leq s_L + s_R \). Since \( S \) is integer, \( 2s_L \) and \( 2s_R \) together with \( h_L \) and \( h_R \) are either odd or all even. If \( h_L, h_R, S \) and \( L/2 \) are both odd or both even the central momentum \( K_c = 0 \); otherwise \( K_c = L/2 \). This rule replaces the particle \( P \) based rule in (6) for determining \( K_c \). The last equality in (14) is a schematic specifying the labelling convention adopted in this paper for the chiral tower states. It displays the string content of \( C^{(S)}_{h_L, h_R} \), which is that of the contributing towers \( T_{h_L}^{(2s_L)} \) supplemented by a central \( n \)-complex, \( n = s_L + s_R - S + 1 \) if \( S < s_L + s_R \). This \( n \)-complex acts as a single 'particle' like an \( n \)-string but does not contribute to the momentum \( K_c \) and therefore is not explicit in the \( m_{\kappa} \) in (14). The existence of an \( n \)-complex separate from the tower strings making up the products in (14) was not recognized in [5] and consequently no general consistent product labelling could be found. That (14) is such a consistent labelling scheme cannot of course be proved by just numerical work on finite chains but to date no counter example has been found.

An alternative to the chiral representation \( C^{(S)}_{h_L, h_R} \) is the generator \( B^{(S)}_{h_L, h_R}(q) = \sum_{\kappa} m_{\kappa} q^{c^{(1)}} \) which is just (6) restricted to cusp states and generalizes equation (86) in [5] to include all \( S \). Explicitly,
\[ B^{(S)}_{h_L, h_R}(q) = \sum_{n=1}^{\Delta} \prod_{\mu=1}^{\delta} q^{(\frac{2\mu-1}{2})h_n} \left( \frac{p_n + h_n}{p_n} \right), \quad \left\{ \begin{array}{l} h_n > 0, \quad 2S + 2m_{>}(m - n)p_m = 0 \\ h_0 \equiv 0 \end{array} \right. \]

and contains no reference to chirality. In particular (15) is not supplemented with any central \( n \)-complex excitation. As in (14) the momentum \( \kappa \) is relative to a central \( K_c = ((h_L + 1) \mod 2)L/2 \) and the equality of the \( m_{\kappa} \) determined from (14) and (15) has been confirmed numerically to high order. The labelling of a configuration is in general different in (14) and (15) and where such differences exist, the numerical evidence is that configurations that are described by (15) at small \( L \) morph into configurations described by (14) with increasing \( L \). The chain length beyond which the chiral representation (14) is better approximation is state dependent and varies widely. How widely was not appreciated in [5] and requires a correction; eqn. (105) in [3] is now plausibly eliminated as a counter example by additional numerical work to \( L = 3200 \) and extrapolation leading to \( L \approx 10^8 \) as the estimate for crossover to chiral behaviour (cf. figure 6). Indeed, I conjecture that for \( L \rightarrow \infty \) the chiral labelling (9) correctly describes all strings explicitly contained in \( C^{(S)}_{h_L, h_R} \) in (14). This does not preclude the central \( n \)-complex being possibly unrecognizable as an ideal Bethe \( n \)-string.

A clear advantage of (14) is that it is easily generalized to include energy in part by introducing left and right tower arguments \( e^2/q \) and \( e^2q \) respectively for a generator in which tower energies and (signed) momenta are additive. Based on the work of [8], the energy for any state in a long chain is expected to take the form
\[ E = -L \left( \ln(2) - 1 \right) + \frac{\pi^2}{L} \omega + \alpha \left( \frac{1}{L \ln(L)} \right), \quad \omega = -\frac{1}{6} + \varepsilon + \frac{\delta}{\ln(L)} \]
\[ \left( \Delta h_L, h_R \right) \left[ e^{(S)} \right] \left[ \text{all states} \right] = e^{-\chi} \sum_{\Delta h_L, h_R} T_{h_L}^{(2s_L)}(e^2/q) T_{h_R}^{(2s_R)}(e^2q) e^{\chi} \]
\[ e^{\chi} = \sum m_{\omega,\kappa} e^{\omega,\kappa, q^{c^{(1)}}} \]

which provides the multiplicity \( m_{\omega,\kappa} \) at momentum \( \kappa \) and energy \( \omega \), including the leading logarithmic correction to scaling. The denominators in the middle term in (17) are the 1-string contributions as in (11) and are to be understood as an expansion in \( e^2 \). As in (14), the explicit strings from the tower generators are supplemented by a central \( n \)-complex which here associates with the correction to scaling factor \( e^{\chi}/\ln(L) \). A final sum of (17) over both \( h_L \) and \( h_R \) from zero to infinity yields the generator for all low lying states in any sector \( S \).

The remainder of the paper elaborates on the results given above. Section 2 begins with the tower conjecture (9) and provides the proof for a number of sum rules including (13) which shows that (9) is consistent with what is known from the WZW model. The conjectures such as (9) and (17) rely heavily on the \( S = 0 \) sector calculations in [5] and on new \( S = 1 \) calculations here. For the latter it was useful to derive a complete list of \( S = 1, L = 16 \) BAE solutions to serve as initial conditions for larger \( L \) and that data is available in the supplementary data file 'L16_triplet.txt' as described at the end of section 2. Section 3 describes some of the properties of the central \( n \)-complex states and in particular outlines the resolution of the apparent paradox that adding excitations to the Dirac sea still leaves the chiral states at the Fermi surfaces essentially unchanged.

Sections 4 and 5 give examples of the transitions from Bethe labelled states \( B^{(S)}_{h_L, h_R} \) in (15) to chiral labelled \( C^{(S)}_{h_L, h_R} \). Labels for states in (15) are understood to be the single product \( \Pi n^R \).
in (14) with increasing L. It is these examples and many others that are the basis for the conjecture that (9), (14) and (17) define the correct labelling for all states in the \( L \to \infty \) or WZW model limit. Section 4 is devoted mostly to a comparison of the present ‘chiral’ string interpretation with the ‘apparent’ string results in [5]; section 5 treats mostly the new \( S = 1 \) results. Conclusions appear in section 6.

2. Chiral tower generator properties

It is useful to rewrite the conjecture (9) to emphasize the separate contributions \( f^{\nu-s}_{\mu}(2s; q) \) from different numbers \( \mu \) of \( (n > 1) \)-strings. The structure that follows from (9) is

\[
T^{(2s)}_{2s}(q) = \sum_{\mu=0}^{\nu-s} q^{\nu-s+\mu} f^{\nu-s}_{\mu}(2s; q), \quad f^{0}(2s; q) = 1,
\]

where the sum \( \Sigma \) is constrained to those modified partitions satisfying \( \Sigma_n > \mu \). The factor \( q^{\nu-s+\mu} \) in (18) is of intrinsic interest as the state count generator in the \( S_z \) representation without the stretched \( S_z \) constraint. The \( f^{0}_0 \) in (18) is the only \( \mu = 0 \) contribution while the \( \mu \neq 0 \) extremes are \( f^{n}_n(2s; q) = \left[ \frac{2s+1}{1} \right]_q \) from the single \( (n+1) \)-string and \( f^{n}_n(2s; q) = \left[ \frac{2s+n}{n} \right]_q \) from the \( 2^n \) configuration of \( n \) distinct \( 2 \)-strings. The configuration \( 2^{n-3}1 \) is the only one contributing to \( f^{n-1}_n(2s; q) = \left[ \frac{2s+n}{n} \right]_q \) ; all remaining \( f^{n}_n \) contain sums of configurations. The lowest order example of the latter is the sum \( 2^4 3^1 \) resulting in

\[
f^3(2s; q) = \left[ \frac{2s+5}{1} \right]_q \left[ \frac{2s+1}{2} \right]_q \left[ \frac{3}{1} \right]_q \left[ -q^{2s+5} \right]_q^2.
\]

The final expression in (19) is a special case of a general product to which all sums can be reduced. Its derivation below is greatly facilitated by first summing (18) over all \( s \) in the interval \( s_s \leq s \leq \nu \) with \( s-s_z \) and \( s-\nu \) integral.

If we define

\[
U^{(2s)}_{2s}(q) = \sum_{\nu-s_z}^{\nu} \left[ \sum_{\mu=0}^{\nu-s_z} q^{\nu-s+\mu} f^{\nu-s}_{\mu}(2s; q) \right]
\]

then \( f^{\nu-s}_{\mu} \) can be retrieved by the subtraction \( f^{\nu-s}_{\mu}(2s; q) = g^{\nu-s}_{\mu}(2s; q) - g^{\nu-s-1}_{\mu}(2s+2; q) \). Note that \( U^{(2s)}_{2s} \) is of intrinsic interest as the state count generator in the \( S_z \) representation without the stretched \( s_z = s \) constraint.

Many examples of the middle term in (20) suggest the general simple result

\[
g^{\nu-s}_{\mu}(2s; q) = \left[ \frac{\nu+s_z}{\mu} \right]_q \left[ \frac{\nu-s_z}{\mu} \right]_q
\]

which will be confirmed analytically below. Assuming its validity, we get from (20) in the notation introduced in (7) that

\[
U^{(2s)}_{2s}(q) = \sum_{\nu-s_z}^{\nu} \left[ \sum_{\mu=0}^{\nu-s_z} q^{\nu-s+\mu} \left[ \nu-s_z \right]_q \right] = q^{\nu-s_z} \sum_{\mu=0}^{\nu-s_z} \left( \frac{q_0}{q} \right)^{\nu-s-z} \left( \frac{q}{q_0} \right)^{s-z} \left( \frac{q}{q_0} \right)^{s-z-\mu} \left( \frac{q_0}{q} \right)^{\mu}.
\]

With use of the notation, cf. (17.2.1) in NIST Handbook of Mathematical Functions [9], hereafter NIST,

\[
\left( a; q \right)_n = 1; \quad \left( a; q \right)_0 = (1-a)(1-aq) \cdots (1-aq^{n-1}), \quad n > 0
\]

we note \( \left( q \right)_n = (q; q)_n \) and can express (22) as

\[
U^{(2s)}_{2s}(q) = q^{\nu-s_z} \sum_{\mu=0}^{\nu-s_z} \left( \frac{q^{\nu-s-z}}{q} \right)_n \left( \frac{q^{\nu-s_z}}{q} \right)_\mu \left( \frac{q^{\nu-s_z-\mu}}{q} \right)^{s-z} \left( q \right)^{\mu} \left( q \right)^{2\nu+1} \left( \frac{q_0}{q} \right)^{2\nu},
\]

a \( q \)-hypergeometric function (17.4.1) NIST. This in turn is the \( q \)-Chu-Vandermonde sum (17.6.2) NIST,

\[
U^{(2s)}_{2s}(q) = q^{\nu-s_z} \left( \frac{q^{\nu-s-z}+1}{q} \right)_n \left( \frac{q^{\nu-s_z}}{q} \right)_\mu = q^{\nu-s_z} \left( \frac{q_0}{q} \right)^{2\nu} \left( \frac{q}{q_0} \right)^{s-z} \left( \frac{q_0}{q} \right)^{s-z-\mu} \left( \frac{q^2}{q} \right)^{\mu} \left( \frac{q}{q_0} \right)^{2\nu+1} \left( \frac{q_0}{q} \right)^{2\nu}.
\]

Care must be taken throughout this paper that the Gaussian binomials \( [n]_q \) are defined in (7) as the symmetrized form. Thus all \( [n]_q \) and \( f^{n}_n \) are invariant under the replacement \( q \to 1/q \).
The generating function at fixed $s_z$ for all chiral tower states, including the 1-string excitations as in (11), follows from (25) by a sum over $\nu$. Specifically,

$$U^{(2s_z)}(q) \equiv \sum_{\nu=s_z}^{\infty} \frac{U^{(2s_z)}(q)}{(q)_{2\nu}} = \sum_{\nu=s_z}^{\infty} \frac{q^{\nu^2}}{(q)_{2\nu}^{(2s_z)}}. \quad (26)$$

To evaluate the final sum in (26) we start with the identity (2.2.8) in Andrews [10] multiplied by $(z; q)_{2s_z}$,

$$(z; q)_{2s_z} \sum_{n=0}^{\infty} \frac{q^{n^2-nz^2}}{(q)_n} = \frac{1}{(z; q)_{\infty}} = \frac{1}{(z; q^{2s_z}) q^{s_z}} \left( \sum_{n=0}^{s_z} q^{n^2-nz^2} (q)_n \right) + \sum_{n=-s_z}^{\infty} \frac{q^{n^2-nz^2}}{(q)_n (zq^{2s_z}; q)_{n-s_z}} = \frac{1}{(zq^{2s_z}; q)_{\infty}}. \quad (27)$$

If we now set $z = q^{1-2s_z}$ every term in the first sum following the equality vanishes and we are left with the identity

$$\sum_{n=-s_z}^{\infty} \frac{q^{n(n-2s_z)}}{(q)_n (q)_{n-s_z}} = \frac{1}{(q)_{\infty}} \quad (28)$$

which, with the replacement $n \to \nu + s_0$, shows

$$U^{(2s_z)}(q) = \sum_{\nu=s_z}^{\infty} \frac{q^{\nu^2}}{(q)_{2\nu}^{(2s_z)}} = \frac{q^{s_z^2}}{(q)_{\infty}} = q^{s_z^2} \sum_{n=0}^{\nu} p(n) q^n = q^{s_z^2} \sum_{n=-s_z}^{\nu} p(n - s_z^2 + s_0^2) q^n. \quad (29)$$

The reversion to a sum in (29), recorded in (13), is the known result that $1/(q)_{\infty}$ is the generator for partitions $p(n)$ while the shifted summation after the last equality is to be interpreted as $U^{(2s_z)}(q) = q^{s_z^2} \sum_{n=s_z}^{\infty} m(s_z, n) q^n$, thus defining the multiplicity of states at $s_z$ and energy 'level' $n$.

Affleck, et al [8] provided a set of rules satisfied by the chiral towers in the $s_z$ representation based on the Kac-Moody algebra for the WZW model understood to be the chiral limit of the Heisenberg spin-$\frac{1}{2}$ antiferromagnetic chain. Their rules included one for the recursive generation of the multiplicity of states, $m(s_z, n)$, where $n = 0, 1, 2, \ldots$ is the 'level' with $n = 0$ the lowest energy state at $s_z = s_0 = 0$ for the integer $s_z$ tower and at $s_z = s_0 = \pm \frac{1}{2}$ for the half-(odd) integral $s_z$ tower. Here I show the multiplicity as given by (29),

$$m(s_z, n) = p(n - s_z^2 + s_0^2), \quad (30)$$

does indeed satisfy the Affleck et al [8] recursion constraint equation

$$\sum_{j=\infty}^{\infty} (-1)^j m(s_z - j, n - j(j + 1)/2) = 0. \quad (31)$$

The substitution of (30) into (31) simplifies if we shift the dummy summation variable by $2s_z/3$ rounded to the nearest integer and results in one of the three possible equations

$$0 = \sum_{j=\infty}^{\infty} (-1)^j \begin{cases} p(n - n' - j(3j - 1)/2), & n - n' > 0, \\ p(n - n' - j(3j + 1)/2), & n - n' > 0, \\ p(n - n' - 3j(j + 1)/2) & \end{cases} \quad (32)$$

where $n'$ is an integer depending on $s_z$ and $s_0$, but independent of $j$. The third equation is automatically satisfied by symmetry while the first two differ only in order of summation, i.e. $j \to -j$. They are satisfied since these are Euler's recursion for $p(n)$ based on his pentagonal number theorem.

The generator corresponding to (29) in the $s$ representation is the subtraction

$$T^{(2s_z)}(q) = U^{(2s_z)}(q) - U^{(2s_z+1)}(q) = (q^{z^2} - q^{(s_z+1)^2})/(q)_{\infty} \quad (33)$$

as recorded in (12). Similarly, from (25) we get

$$T^{(2s_z)}(q) = U^{(2s_z)}(q) - U^{(2s_z+1)}(q) = q^{s_z^2} \frac{(q)_{2\nu}}{(q)_{2\nu+1} (q)_{s_z-1}} \frac{1 - q^{2s_z+1}}{1 - q^{s_z+1}} q^{(s_z-1)^2}. \quad (34)$$

Finally, a corresponding subtraction provides an explicit $f^{\nu-s}$ from the known $g^{\nu-s}_\mu$ in (21). We have, provided $\nu - s > 0$.

---

4 Subject to certain conditions specified in [8].
\[ f_\mu^{-\nu}(2s; q) = \sum_{\mu'=0}^{\mu} \binom{\nu+s}{\mu} \binom{\nu-s}{\mu} \binom{\nu-s-1}{\mu} q^{(\nu-s)/2}. \]  

(35)

It is to be understood that (35) is supplemented by the isolated case \( f_0^{-\nu}(2s; q) = 1 \) as in (18).

If we can prove (35) analytically then the validity of \( g_0^{-\nu-} \) in (21) and all formulae derived from it are confirmed. To do this I rely on an analogy with Bethe’s [1] solution to the completeness problem by first showing that \( f_\mu^{-\nu} \) satisfies the recursion relation

\[ f_\mu^{-\nu}(2s; q) = \sum_{\mu'=0}^{\mu} \binom{\nu-\mu+s+1}{\mu'} \binom{\nu-\mu-s-1}{\mu'-1} q^{(\nu-\mu-\nu)/2}. \]  

(36)

where the \([\cdot]_q \) factor in (36) is the factor in (18) associated with \( n = 2 \) in any configuration with \( \mu' = p_2 \), the number of 2-string 'holes' from (18)

\[ h_2 = 2s + 2\Sigma_{n>1}(n-2)p_2 = 2s + 2\Sigma_{n>1}(n-1)p_2 = 2\Sigma_{n>1}p_n = h_1 - 2\mu = 2(\nu - \mu). \]  

(37)

The \( f \) factor on the right hand side of (36) is understood to have the same form as that on the left while accounting for the remaining \((n > 2)\)-strings. In particular the lower index \( \mu = \Sigma_{n>1}p_n \) on the left is replaced by \( \Sigma_{n>2}p_n = \mu - p_2 = \mu - \mu' \) on the right. Similarly the upper index \( \nu-s = h_1/2 - s \) on the left becomes \( h_2/2 - s = \nu - \nu - s \) from (37) on the right. This argument applies to every configuration and thus to the exhaustive sum of configurations in (36). Finally, having eliminated all 2-strings we perform a relabelling

\[ n \rightarrow n - 1, \ p_2 \rightarrow p_{n-1}, \ h_n \rightarrow h_{n-1}, \ f_{\mu'-\nu}(2s; q) \rightarrow f_\mu^{-\nu}(2s; q) \]  

(38)

and return to a new (36), identical in form, and iterate until all strings have been eliminated.

To complete the proof of (35) it only remains to evaluate the sum in (36) and confirm the equality since \( f_\mu^{-\nu} \)

in (35) agrees with examples listed after (18) which provide the initial conditions for the recursion (36). Now the right hand side of (36) with assumed (35) has the explicit form

\[ Rhs = \sum_{\mu'=0}^{\mu} \binom{\nu-\mu+s+1}{\mu'} \binom{\nu-\mu-s-1}{\mu'-1} q^{(\nu-\mu-\nu)/2} \]  

(39)

which, in terms of new independent variables

\[ n = \nu - \mu - s - 1, \ k = \mu - \mu' - 1 \]  

(40)

replacing \( \nu \) and \( \mu' \), is

\[ Rhs = \sum_{k=0}^{n} \binom{2n+2s+\mu+1}{\mu+1-k} \binom{n+2s+2}{k} \binom{n}{k} q^{(n+1)/2} \]  

\[ = (q)_n, k \]  

\[ = \frac{(q)_{2n+2s+\mu+1}}{(q)_{2n+2s+\mu+1-(k+1)}} \frac{(q)_{n+1}}{(q)_{n+1-(k+1)}} \frac{(q)_{n+2s+1}}{(q)_{n+2s+1-(k+1)}} \frac{(q)_{n+k}}{(q)_{n+k-(k+1)}} \]  

(41)

The prefactor to the sum after the equality is the \( k = 0 \) value of \( Rhs \) leaving ratios in the sum that are of one of the two forms

\[ \frac{(q)_{n+k}}{(q)_{n}} = (-1)^{k} q^{(k+1)/2} (1 - q^{-n}) \]  

\[ \frac{(q)_{n-k}}{(q)_{n-k}} = (-1)^{k} q^{(k+1)/2} (1 - q^{-n}) \]  

(42)

The sum term in (41) is now expressible as

\[ \sum_{k} \binom{q^{\mu}}{(q^{-n-k-1}; q^{n-k}; q^{n-k}; q^{n-k})_{k}} q^{k} = s_{2} \]  

(43)

which is a q-hypergeometric function (17.4.1) NIST. Since it is Saalschützian it has an explicit representation (17.7.4) NIST,
and on combining this with the prefactor to the sum in (41) gives

$$\text{Rhs} = \frac{(q)_{n+\mu}}{(q)_{n+2\mu+2}} \frac{1 - q^{2\mu+1}}{1 - q^{2\mu+1}(1 - \mu) - \mu} q^{(n+1)(1 - \mu) - \mu}$$

$$= \left[ \frac{n + 2\mu + \mu + 2\mu}{\mu} \right] (n + \mu) \frac{1 - q^{2\mu+1}}{1 - q^{2\mu+1}(1 - \mu) - \mu} q^{(n+1)(1 - \mu) - \mu}.$$  (45)

Substituting $n = \nu - \mu - s - 1$ from (40) into the final equality for Rhs in (45) yields $f_{\mu}^{n-s}(2\nu; q)$ in (35) and completes the proof.

I close this section to show the close relation between the chiral cusp generator (18) for length $L \rightarrow \infty$ and the all state generator for finite (even) $L$ and total spin $S$ with $0 \leq S \leq L/2$. The generalization of Bethe’s state count to counts at specified momenta is the $q$-generator

$$Z(L, S) = \sum_{\|P\|} q^{L_P} f_{\nu}^{L_S} (2S; q), \quad f_{\nu}^{L_S} (2S; q) = 1,$$

$$f_{\nu}^{L_S} (2S; q) = \Sigma'_{\mu \in \{\nu - s - 1\}} \prod_{\Pi_{P_n}} P_n = H_n,$$  (46)

where the sum $\Sigma'$ is constrained to partitions satisfying $\sum_{\mu} P_{\mu} = P$. The factor $q^{L_P}$, which because of periodicity could equally well be written in the form $q^{(P \mod 2L)}$ appearing in (5) equation (45), accounts for the empirical observation that the (symmetric) $f$ factors are centered at $K = 0(L/2)$ for even (odd) $P$. The main difference of (46) from (18) is the return from modified partitions to partitions, i.e. $\tilde{P} \rightarrow P$, so that the product in (46) contains 1-string terms that were excluded in (18). The structure of the equations however is such that the $f_{\nu}^{L_S}$ are the identically same polynomials with the identifications chiral spin $s \leftrightarrow S$, finite chain integer (chiral) spin; $2\nu = h_1 \leftrightarrow L$, (even) length; chiral (n > 1)-string count $\sum_{\mu>0} P_{\mu} = \mu \leftrightarrow P = \sum_{\mu>P} P$, total string count. We also have $\nu \rightarrow S \leftrightarrow L/2 - S = N$, the number of overturned spins from the fermion magnetic state. The generators differ by a factor, i.e. $T_{2L}^{(2\nu)} (q)/q^{2\nu} \rightarrow Z(L, S)$. Additionally, because of periodicity, the $Z$ polynomial can always be reduced to terms in the first Brillouin zone by the replacement $q^{m} \rightarrow q^{m}$ with $m$ an integer multiple of $L$ and $-L/2 < k \leq L/2$.

On applying the above correspondence to (35) we arrive at

$$f_{\nu}^{L_S} (2S; q) \equiv \left[ L - N + \frac{1}{P} \right] \frac{N - 1}{P - 1} \frac{1 - q^{2S+1}}{1 - q^{2S+1} q^{N/2}} q^{N}, \quad 2S = L - 2N.$$  (47)

and from (34)

$$Z(L, S) = \frac{(q)_L}{(q)_{L-N}} \frac{1 - q^{2S+1}}{1 - q^{2S+1} q^{N}} q^{N}, \quad 2S = L - 2N.$$  (48)

As an example of (48), the $L = 16$ singlet generator after reduction to the first Brillouin zone is

$$Z(16, 0) = \left( 85q^{7} + 93q^{5} + 85q^{5} + 94q^{4} + 85q^{3} + 93q^{2} + 85q + 95q \right) (1 + q^4)$$  (49)

in agreement with that found in [5] equation (84) by a direct number theoretic method. The triplet generator is

$$Z(16, 1) = \left( 218q^{7} + 211q^{6} \right) \left( 1 + \sum_{n=1}^{7} q^{2n} \right).$$  (50)

where, based on [5] equation (58), the 211 count at $K = 0(8)$ is the sum of 21(35) non-degenerate and 95(88) doubly degenerate states. All Bethe ansatz solutions for $L = 16$ triplet states have been generated and are listed in supplementary data as ’L16_triplet.txt’ in the notation and format described following [5] equation (84) for singlets. A few necessary changes are as follows. The triplet non-degenerate states at $K = 0$ are $[\pi, \pi/2 + i\infty, \pi/2 - i\infty, k_0 = k_i = 1..2]$ so that $kh^0$ is now a list of $[n, i = 1..2]$ while at $K = 0(8)$ the non-degenerate states are $[n, k_0 = k_i = 1..3]$ and $kh^0$ remains a list of $[n, i = 1..3]$. Clearly all $k3, 0 \leq S \leq 8$, are lists $[n, i = 1..7]$.

3. The central $n$-complex

An important observation is that in the $L \rightarrow \infty$ limit cusp states of chiral spin $s_L$ and $s_R$ coupled to total spin $S = s_L + s_R$ are formed with the Dirac sea a filled configuration of 1-strings. Furthermore, for the same chiral spin $s_L$ and $s_R$ configuration, each reduction of $S$ by one unit from the stretched $s_L + s_R$ value requires the Dirac sea to accommodate one extra overturned spin because of the connection $N = L/2 - S$. For an $S$ reduction of $2n$, $n$ complex conjugate $\lambda$ pairs are added to the Dirac sea and there is a rearrangement of the real $\lambda$ such that the net
result can be interpreted as a \((2n + 1)\)-complex similar to a \((2n + 1)\)-string with one real \(\lambda\) taken from the existing Dirac sea list. The total number of real \(\lambda\) does not change. For an \(S\) reduction of \(2n - 1\), again \(n\) complex conjugate pairs are added but now one real \(\lambda\) is lost from the existing 1-string list. This lost 1-string does not form a hole in the Dirac sea—rather the final configuration is a \((2n)\)-complex similar to a \((2n)\)-string superimposed on a smoothly reduced real \(\lambda\) density. In all cases the new configurations do not change the asymptotic \(L \to \infty\) energy or momentum from their stretched spin configuration values.

To illustrate these features of the central complex in the BAE solutions I consider the symmetric case of cusp states \(s_L = s_R = 2, 0 \leq S \leq 4\) generated from the chiral towers given in (51). Crosses mark calculated values at lengths from \(L = 16\) to \(3200\) (64 to \(5120\) for \(S = 4\)); the straight lines are the known asymptotes \(16 + (12 - S(S + 1))/\ln(L)\).

![Figure 1](https://example.com/figure1.png)

**Figure 1.** The reduced energy \(E_{\text{red}} = L(\varepsilon + (2 \ln(2) - 1/2))/\pi^2 + 1/6\) versus \(1/\ln(L)\) for cusp states \(s_L = s_R = 2, 0 \leq S \leq 4\) generated from the chiral towers given in (51). Crosses mark calculated values at lengths from \(L = 16\) to \(3200\) (64 to \(5120\) for \(S = 4\)); the straight lines are the known asymptotes \(16 + (12 - S(S + 1))/\ln(L)\).

That contributes to \(C_{2n-1}^{(1)}(\varepsilon, \lambda)\) in (17). The states in (51) are particularly simple in that the chiral spin \(s = 2\) at each end is associated with a pair of spins from the ferromagnetic state which have not been overturned and hence carry no string labels. If we adopt as our labelling convention the triad (left chiral string [central complex] right chiral string) as indicated in (14) then the label for \(S = 4\) in (51) is \((0|0)0\) and the labels for \(S < 4\) are \((0|(S-S)^1)|0\). For comparison, the Bethe cusp state labels are simply \(0\) (no \(n > 1\)-string) for \(S = 4\) and \((S-S)^1\) for \(S < 4\) resulting in the explicit identification central \(n\)-complex \(\leftrightarrow n\)-string.

In figure 1 the calculated energies \(E\) for all \(0 \leq S \leq 4\) in (51) are displayed in the reduced form

\[
E_{\text{red}} = L(\varepsilon + (2 \ln(2) - 1/2))/\pi^2 + 1/6
\]

as motivated by (16). The expected asymptotic form of \(E_{\text{red}}\) which is \(\varepsilon - 2s_L \cdot s_R/\ln(L)\) is confirmed as shown by the straight lines converging to \(\varepsilon = 16\) from (51). The associated BAE solutions at each \(L\) are characterized by the density \(\rho(\lambda) = -\partial n/\partial \lambda\) of the real \(\lambda\) Bethe roots for each \(S\) and by the central complex pairs \(\lambda = \pm \Lambda i\) for \(S < 4\). A practical definition for the density based on the numerical BAE real ordered solution arrays \(\lambda_n\) with \(\lambda_{n+1} < \lambda_n\) is

\[
\rho(\lambda) = \frac{1}{\lambda_n - \lambda_{n+1}}, \quad \lambda = \frac{\lambda_n + \lambda_{n+1}}{2}, \quad n = 1, 2, \ldots
\]

(52)

together with an interpolation/extrapolation scheme as needed. The densities (52) are even functions of \(\lambda\) and are shown in figure 2 for \(L = 3200\). There is clear evidence for resonance/anti-resonance like structure for all \(S = 4\) which can be associated with the complex conjugate pairs \(\lambda = \pm \Lambda i\). For \(S\) odd with corresponding \((2n)\)-complex configuration \((0|(2n)^1)|0\) there is one complex pair with \(\Lambda = 1\) and an associated anti-resonance

\[
\delta \rho_{n=1}(\lambda) = -\frac{1}{\pi} \frac{1}{\lambda^2 + 1}.
\]

(53)
the solutions in Figure 2 at \( L \) show the effect of a change in \( L \) and for the \( L \)
is seen in the ground state approximation. The curves for \( \Lambda > 0 \) and \( \Lambda < 0 \) curves carry the colours of the \( S \) labelled \( \Lambda > 0 \) curves and have the sequential order of \( E_{\text{red}} \) in Figure 1. Crosses mark the 10 largest \(|\lambda|\) from (52). Two extra curves on the left are for \( L = 5120, S = 4 \) to show the effect of a change in \( L \) and for the \( L = 3200, S = 0 \) ground state to indicate the \( \rho_{\Lambda}(\lambda) \) continuum approximation error.

Note that \( \int_{-\infty}^{\infty} d \lambda \delta \rho_{\Lambda}(\lambda) = -1 \) and that this anti-resonance accounts for the lost 1-string in the Dirac sea. In spite of this there is no change in the energy (4) in the continuum limit—the \( \lambda = \pm i \) pair contributes \(-2\) while the associated \( \delta \rho \) in (53) contributes \(-\int_{-\infty}^{\infty} d \lambda \delta \rho_{\Lambda}(\lambda) 4/ (\lambda^2 + 1) = 2\). For all other complex pairs, including those for \( S \) even with corresponding \((2n-1)\)-complex configuration \((0)(2n-1)^{\pm 0}\), the associated \( \delta \rho_{\Lambda}(\lambda) \) is the resonance/anti-resonance combination

\[
\delta \rho_{\Lambda}(\lambda) = \frac{1}{\pi} \frac{\Lambda - 2}{\lambda^2 + (\Lambda - 2)^2} = \frac{1}{\pi} \frac{\Lambda}{\lambda^2 + \Lambda^2}, \quad \Lambda \neq 1
\]

for which \( \int_{-\infty}^{\infty} d \lambda \delta \rho_{\Lambda}(\lambda) = 0 \) and so no induced change in the 1-string Dirac sea count. Here too there is no change in the energy (4) in the continuum limit—the contribution from each \( \lambda = \pm a i \) is \( \pm 4/(\Lambda^2 - 1) \) and from \( \delta \rho \) in (54) is \(-\int_{-\infty}^{\infty} d \lambda \delta \rho_{\Lambda}(\lambda) 4/ (\lambda^2 + 1) = -8/ (\Lambda^2 - 1)\). To get the \( O(1/ (\ln L)) \) corrections to \( E \) seen in Figure 1 would require more extensive calculation that has not been attempted. A summary of the relations between densities displayed in figure 2 for \( \Lambda > 0 \) is

\[
\delta \rho^{(S)}(\lambda) = \rho^{(S)}(\lambda) - \rho^{(0)}(\lambda) \approx \begin{cases} \\
\delta \rho_{\lambda}(\lambda), & S = 3 \\
\delta \rho_{p_{2.9266\ldots}}(\lambda), & S = 2 \\
\delta \rho_{p_{1}}(\lambda) + \delta \rho_{p_{4.9705\ldots}}(\lambda), & S = 1 \\
\delta \rho_{p_{2.6775\ldots}}(\lambda) + \delta \rho_{p_{8.1100\ldots}}(\lambda), & S = 0
\end{cases}
\]

The empirical evidence confirming (55) is seen in the \( \lambda < 0 \) curves \( \rho^{(S)}(\lambda) - \delta \rho^{(S)}(\lambda) \) in figure 2 that are almost indistinguishable from \( \rho^{(0)}(\lambda) \). Except for \( \Lambda = 1 \), the numerical \( \Lambda \) values in (55) differ substantially from the \( n-1, n-3, \ldots, 1, n \) for an ideal n-string. Indeed, each \( \Lambda \neq 1 \) is well approximated by \( \Lambda = a + b \ln(L) \), \( b > 0 \) at large \( L \) with no indication of saturation as \( L \to \infty \). An additional observation based on (55) and which appears to be general is that any central complex consisting of \( n \) complex \( \lambda \) pairs is either a \((2n)\)-complex or a \((2n + 1)\)-complex depending on the presence or absence of a pair \( \lambda \approx a \pm i \).

The \( \delta \rho \) in (53) and (54) have an analytic basis, subject to certain approximations, in the BAE. To show this I use the BAE in the form of the continuous function.
\[ n(\lambda) = \frac{1}{2\pi i} \left\{ L \ln \left( \frac{\lambda + i}{\lambda - i} \right) - \sum_m \ln \left( \frac{\lambda - \lambda_m + 2i}{\lambda - \lambda_m - 2i} \right) \right\} \]  
(56)

where the ‘phase’ \( n(\lambda) \) takes on half-(odd)integer values whenever \( \lambda = \lambda_m \), one of the BAE solutions \( \lambda_m \). For \( S = 4 \) with only real \( \lambda_m \) approximate the sum in (56) by an integral to get

\[ n^{(4)}(\lambda) = \frac{1}{2\pi i} \left\{ L \ln \left( \frac{\lambda + i}{\lambda - i} \right) - \int_{-\infty}^{\infty} d\mu \rho^{(4)}(\mu) \ln \left( \frac{\lambda - \mu + 2i}{\lambda - \mu - 2i} \right) \right\} \]  
(57)

and for \( S = 3 \) with the single complex pair \( \lambda = \pm i \),

\[ n^{(3)}(\lambda) = \frac{1}{2\pi i} \left\{ L \ln \left( \frac{\lambda + i}{\lambda - i} \right) - \int_{-\infty}^{\infty} d\mu \rho^{(3)}(\mu) \ln \left( \frac{\lambda - \mu + 2i}{\lambda - \mu - 2i} \right) \right\} \]  
(58)

Combining (57) and (58) gives the BAE result for the difference

\[ n^{(3)}(\lambda) - n^{(4)}(\lambda) = \frac{1}{2\pi i} \left\{ L \ln \left( \frac{\lambda + i \pm i/2}{\lambda - i \mp i/2} \right) - \int_{-\infty}^{\infty} d\mu \delta \rho(\mu) \ln \left( \frac{\lambda - \mu + 2i}{\lambda - \mu - 2i} \right) \right\} \]  
(59)

on using (55) and (53) followed by straightforward evaluation by residue methods. Because the same result

\[ n^{(3)}(\lambda) - n^{(4)}(\lambda) = \int_{-\infty}^{\infty} d\mu \delta \rho(\mu) = -\frac{1}{\pi} \arccot(\lambda) \]  
(60)

follows directly by integrating \( \partial n / \partial \lambda = -\rho(\lambda) \) with the boundary condition \( n^{(3)}(\infty) = n^{(4)}(\infty) \), we conclude \( \delta \rho(\lambda) \) in (53) is the correct analytic approximation. Errors can arise both because of the continuum approximation and because the Dirac sea of occupied 1-strings does not extend to the full \( \lambda \) interval \((-\infty, \infty)\) as assumed in the integrations in (57)–(59). The latter can be more serious when the integrand involves \( \delta \rho(\lambda) \) with a \( \Lambda \) that increases with increasing \( L \). An important practical consequence of (53) is that once solutions \( \lambda_n^{(3)} \) have been obtained for \( S = 4 \), approximate values for \( \lambda_n^{(3)} \) follow trivially. Corresponding BAE solutions \( \lambda_n^{(4)} \) and \( \lambda_n^{(4)} \) have the same phase and so satisfy \( n^{(3)}(\lambda_n^{(3)}) = n^{(4)}(\lambda_n^{(4)}) \). Combining this with (60) with \( \lambda = \lambda_n^{(3)} \) yields

\[ n^{(3)}(\lambda_n^{(3)}) - n^{(4)}(\lambda_n^{(4)}) = -\arccot(\lambda_n^{(3)}) / \pi. \]  

To first order in the \( \Lambda \) difference,

\[ \lambda_n^{(3)} = \lambda_n^{(4)} \]  
(61)

which can be improved by Newton–Raphson iteration of the BAE for \( S = 3 \).

A similar argument can be used for \( S = 2 \) for the effect of a complex pair \( \lambda = \pm \Lambda i, \Lambda \neq 1 \). We have

\[ n^{(2)}(\lambda) - n^{(4)}(\lambda) = \frac{1}{2\pi i} \left\{ - \ln \left( \frac{\lambda - (\Lambda - 2)i}{\lambda - (\Lambda + 2)i} \right) \ln \left( \frac{\lambda - \mu + 2i}{\lambda - \mu - 2i} \right) \right\} \]  
(62)

\[ = \frac{1}{2\pi i} \ln \left( \frac{\lambda - (\Lambda - 2)i}{\lambda - (\Lambda + 2)i} \right) = \frac{1}{\pi} \left\{ \arccot \left( \frac{\lambda}{\Lambda - 2} \right) - \arccot \left( \frac{\lambda}{\Lambda} \right) \right\} \]

by use of (55) and (54) and evaluation by residue methods. This equals the directly determined

\[ n^{(2)}(\lambda) - n^{(4)}(\lambda) = \int_{-\infty}^{\infty} d\mu \delta \rho(\mu) = \frac{1}{\pi} \left\{ \arccot \left( \frac{\lambda}{\Lambda - 2} \right) - \arccot \left( \frac{\lambda}{\Lambda} \right) \right\} \]  
(63)

confirming \( \delta \rho(\lambda) \) in (54) as the analytic (approximate) result. Just as (60) implied (61), here (63) implies

\[ \lambda_n^{(2)} = \lambda_n^{(4)} + \frac{1}{\pi \rho^{(4)}(\lambda_n)} \left\{ \arccot \left( \frac{\lambda_n}{\Lambda - 2} \right) - \arccot \left( \frac{\lambda_n}{\Lambda} \right) \right\} \]  
(64)

to first order in the \( \Lambda \) difference. The effects of multiple complex pairs are additive so that the above calculations contain all the necessary ingredients for \( S = 1 \) and \( S = 0 \) also.

As already mentioned, the numerical \( \Lambda \) listed in (55) and used for the \( \lambda < 0 \) curves in figure 2 differ substantially from the values for ideal n-strings. To understand the origin of this difference consider again the

5 Half-(odd)integer phase in (56) rather than integer is the result of dropping the constraint \( m = n \) seen in (2).

6 If instead of \( \delta \rho(\mu) \) in (59) and (60) we had used \(-\Lambda / \sqrt{\pi^2 + \Lambda^2} / \pi\) the two results would have differed by \( (\arccot(\lambda) - \arccot(\lambda/\Lambda) + \arccot(\lambda/3 - \arccot(\lambda/(\Lambda + 2i))) / \pi \) thus requiring \( \Lambda = 1 \) for consistency.
BAE (56) for \( S = 2 \) but now for \( \lambda = \Lambda i, \Lambda > 2 \). In this case it is essential to introduce a cutoff \( \lambda_c \) defined by
\[
\int_0^{\lambda_c} d\lambda \rho^{(i)}(\lambda) = (L/4) - 2 \text{ where the two accounts for the two 1-string holes beyond each end of the Dirac sea.}
\]
Then with \( \delta \rho_{\lambda}(\lambda) \) given by (54),
\[
\eta^{(i)}(\Lambda i) = \frac{1}{2} + \frac{1}{2\pi i} \left\{ (L - 1) \ln \left( \frac{\Lambda + 1}{\Lambda - 1} \right) - \int_0^{\lambda_c} d\lambda \rho_{\lambda}(\lambda) \ln \left( \frac{\lambda^2 + (\Lambda + 2\lambda)^2}{\lambda^2 + (\Lambda - 2\lambda)^2} \right) \right\}
\]
(65)
and for \( \eta^{(i)}(\Lambda i) \) to be real the expression in braces \{ \} must vanish. It is useful to write \( \rho^{(i)}(\lambda) = \rho_{\lambda}(\lambda) + \Delta \rho^{(i)}(\lambda) \) where \( \rho_{\lambda}(\lambda) = (L/4) / \cosh(\pi\lambda/2) \) is the Hulthén [11] ground state function; \( L = 3200 \) and 5120 examples of \( \Delta \rho^{(i)}(\lambda) \) can be seen in figure 2. Now the \( \rho_{\lambda}(\lambda) + \Delta \rho_{\lambda}(\lambda) \) part of the integral in (65) with \( \lambda_c \) replaced by infinity is
\[
\int_0^\infty d\lambda \left( \frac{L/4}{\cosh(\pi\lambda/2)} + \frac{(\Lambda - 2\lambda)/\pi}{\lambda^2 + (\Lambda - 2\lambda)^2} - \frac{\Lambda/\pi}{\lambda^2 + \Lambda^2} \right) \ln \left( \frac{\lambda^2 + (\Lambda + 2\lambda)^2}{\lambda^2 + (\Lambda - 2\lambda)^2} \right)
\]
(66)
so that this contribution cancels exactly the explicit \( O(L) \) term in \{ \} in (65). The remaining \( O(1) \) terms that determine \( \Lambda \) are
\[
\ln \left( \frac{\Lambda}{\Lambda - 2} \right) = \int_0^\lambda d\lambda \rho_{\lambda}(\lambda) + \Delta \rho_{\lambda}(\lambda) \ln \left( \frac{\lambda^2 + (\Lambda + 2\lambda)^2}{\lambda^2 + (\Lambda - 2\lambda)^2} \right)
\]
(67)
Both integrals in (67) receive their largest contributions from the immediate vicinity of \( \lambda_c \) and so with little error we can replace \( \lambda \) by \( \lambda_c \) in the integrand logarithms. Since the logarithms are decreasing functions of \( \lambda \) we get the bound
\[
\ln \left( \frac{\Lambda}{\Lambda - 2} \right) < \left\{ \int_0^{\lambda_c} d\lambda \rho_{\lambda}(\lambda) + \Delta \rho_{\lambda}(\lambda) \right\} - \int_0^\lambda \Delta \rho_{\lambda}(\lambda) \ln \left( \frac{\lambda^2 + (\Lambda + 2\lambda)^2}{\lambda^2 + (\Lambda - 2\lambda)^2} \right)
\]
\[
= 2 \ln \left( \frac{\lambda^2 + (\Lambda + 2\lambda)^2}{\lambda^2 + (\Lambda - 2\lambda)^2} \right)
\]
(68)
where \( \Lambda < \lambda_c \) has been assumed and the equality is the sum rule \( \int_0^\lambda d\lambda \rho^{(i)}(\lambda) = (L/4) - 2 \). Had the explicit \( O(L) \) term in (65) not exactly cancelled, the divergence in \( L \) would have driven \( \Lambda \) on the left hand side of (67) towards 2, the ideal 3-string value. As it is, in the limit \( L \to \infty \), (68) reduces to
\[
\frac{2}{\Lambda} < 2 \frac{8\Lambda}{\lambda_c^2 + \Lambda^2} \to \Lambda > \frac{\lambda_c}{\sqrt{2}}
\]
(69)
and \( \lambda_c \), determined from the sum rule, is
\[
\lambda_c = \frac{2}{\pi} \ln \left( 2 + \int_0^{\lambda_c} d\lambda \Delta \rho^{(i)}(\lambda) \right), \quad \lambda_c = \frac{2}{\pi} \ln \left( \frac{L}{\pi} \right).
\]
(70)
The integral in (70) is estimated as \( \approx 0.6 \) with almost no \( L \) dependence for large \( L \). Thus, within the continuum approximation, we confirm the logarithmic divergence \( \Lambda > 2/(\pi\sqrt{7}) \ln(L) = 0.2406 \ln(L) \). For comparison, an estimate by extrapolation of the exact data from \( L \leq 3200 \) to \( L \to \infty \) gives the divergent part of \( \Lambda \) as \( (0.243 \pm 0.001) \ln(L) \), in reasonable agreement.

4. Transition to chiral states: \( S = 0 \)

The many examples of ‘apparent’ string labelling replacing Bethe string labelling in [5] can now be understood as a necessary requirement of the chiral symmetry that develops with increasing chain length \( L \). Consider first the symmetric examples of table 1 in [5] beginning with the transition Bethe 2 3 1 4 1 -> apparent 3 1 3 shown in figure 2 in [5]. The \( s = s_l = s_h = 1 \) inferred in [5] from the observed \( \delta/\ln(L) \) correction to scaling implies a central 3-complex in the chiral representation as described in section 3. The \( h_1 = 2 \sum_{m=1}^{m}(m-1)p_m = 12 \) holes in [5] splits into \( h_1 = h_2 = 6 \) 1-string holes in the chiral representation. This cusp state and all others with the same \( s \) and \( h \) values arise from the tower generator.
\[ T^{(2)}_0(q) = \frac{q^{11} + q^{12} + q^{13} + q^{14} + 2q^{15} + q^{16} + q^{17}}{z^2} \]  

(71)

constructed according to the rules described in (9) and labelled by the contributing string configurations. The contributions of (71) to the cusp states in table 1 in [5] are the diagonal elements of the product

\[ T^{(2)}_0(e^{2}/q)T^{(2)}_0(e^{2}q)\big|_{\text{diag}} = \frac{3y^5}{(3^3 3^3 3)} + \frac{2y^6}{e^{24} + e^{52} + 2} + \frac{3^3}{2^3} + \frac{2y^4}{e^{56} + e^{60} + e^{64} + e^{68}} = \Sigma e^{y^5} \]  

(72)

with chiral labels (=apparent labels 3\(^3\) and 2\(^4\)3\(^5\) below) and the Bethe labels above taken from lines 11, 14, 15 and 16 in table 1 in [5]. Missing in (72) is a Bethe label 2\(^3\)3\(^5\) for the final cluster of 3 states but because this is a label for 11 overturned spins it cannot appear in a state list generated from \(L = 20\). One can carry out a similar analysis for all other cusp states in table 1 in [5] and find agreement between chiral and apparent string labels in every case.

Another comparison with [5] is with the states on the special energy versus momentum lines \(e = 4s^2 + 2|\kappa|\) where \(s = s_k = s_8\) is the uniquely contributing spin. In terms of the chiral tower products contributing to (17) they are the sum

\[ f_c = T^{(2)}_2(e^{2}/q) \sum_{\mu=0}^{2s} T^{(2)}_2(e^{2}q) = e^{e^{y^{5}} \left\{ 1 + \sum_{\mu=1}^{y^{5}} \sum_{p(\mu)} \prod_{n \neq y^{5}} Y^{(\mu+1)p} \left[ \frac{P_n + h_n}{P_n} \right] \right\} } = \Sigma e^{e^{y^{5}}} \]  

(73)

for \(\kappa \geq 0\); \(0 < \kappa \leq 0\) is obtained by left \(\leftrightarrow\) right, \(q \leftrightarrow 1/q\) reflection symmetry. For any given partition, \(h_n = 2s + 2\sum_{m>n}(m-n)p_n\) is understood as in (9). The configuration label list associated with (73) is the chiral

\[ \begin{pmatrix} 0/ \cr (2s + 1) \end{pmatrix} \sum_{\mu=0}^{2s} \hat{q}(\mu) = \begin{pmatrix} 0/ \cr (2s + 1) \end{pmatrix} + \begin{pmatrix} 0/ \cr (2s + 1) \end{pmatrix} + \begin{pmatrix} 3/ \cr (2s + 1) \end{pmatrix} + \begin{pmatrix} 4/ \cr (2s + 1) \end{pmatrix} + \begin{pmatrix} 3^2/ \cr (2s + 1) \end{pmatrix} + \begin{pmatrix} 3^2/ \cr (2s + 1) \end{pmatrix} + \ldots \]  

(74)

where the central complex is the \(S = 0\) specialization of that given in (14) and is understood to be \((2s + 1)^3\) only if \(s \geq 0\). Because chiral \(s\) is fixed in (73) the corresponding expression for the Bethe string labelling of configurations cannot be a chiral independent form such as (15). The correct formula was conjectured to be eqn. (100) in [5]; in the present notation it is

\[ f_B = e^{e^{y^{5}}} \sum_{\mu=0}^{2s} \sum_{n \neq} \prod_{n \neq} Y^{(\mu+1)n} \left[ \frac{P_n + h_n}{P_n} \right] \bigg[ \begin{array}{c} p_n + h_n - 2s + n - 1 \\ p_n \end{array} \bigg] \bigg] , \quad n \leq 2s \]  

(75)

where because \(S = 0\), \(h_n = 2\sum_{m>n}(m-n)p_n\) for any given partition. In general the \(f_c\) and \(f_B\) will yield different labels for states but the number of terms in each \(\mu\) block in (73) and (75) must be the same; this is

\[ C^{(2\omega)}_\mu = \left( \begin{array}{c} 2s + 2\mu \\ \mu \end{array} \right) \frac{2s + 1}{2s + \mu + 1} \]  

(76)

given by the \(q \rightarrow 1\) limit of (34) with \(\nu = s + \mu\). The totality of terms is conveniently summarized by the generator

\[ C^{(2\omega)}_\mu = \sum_{\mu=0}^{\infty} C^{(2\omega)}_\mu x^\mu = \left( \frac{2}{1 + \sqrt{1 - 4x}} \right)^{2s+1} \]  

(77)

where the final equality is the eqn. (101) in [5] result that was based on (75). To confirm this equality note that \(C^{(0)}_\mu = \sum_{\mu=0}^{\infty} \frac{2\mu+1}{2\mu!} x^\mu\) which is the series expansion of \(2/(1 + \sqrt{(1-4x)})\) and the generating function of the Catalan numbers. It is easy to show starting from (76) and the first equality in (77) that \(x C^{(1)} = C^{(0)} + 1\) and \(x C^{(2\omega+1)} = C^{(2\omega)}(2\omega+1)\) for \(2s \geq 0\). Straightforward algebra starting from these relations and the explicit \(C^{(0)}\) yields \(C^{(1)} = (C^{(0)})^2\) and proves the general power \(C^{(2\omega)} = (C^{(0)})^{2\omega+1}\) in (77).

The simplest example of (73) and (75) is for \(s = 0\) so that \(e = 2|\kappa|\) for general \(\kappa\). The \(\kappa \geq 0\) generators \(f_c\) and \(f_B\) are identical in this case giving

7 The ground state contribution of 1 is to be added to \(f_B\) in (75) in the case that \(s = \mu = 0\).
interpolation replaced by Fourier series

8 Solutions for \( L = 4096 \) have been obtained by a method similar to Newton-Raphson/Lagrange described in [12] but with Lagrange interpolation replaced by Fourier series fitting.
The mean additional transitions beyond extensive numerical calculations that I have not pursued this analysis further. I have also not pursued any path. The magnitude of the imaginary part identiﬁes the chiral 3-string. More interesting is the transition $2141$ as expected for a central 3-complex. The two components of the Bethe 4-string that approaches a $\kappa$ component of the Bethe 4-string which has the Bethe 2-string $\lambda$ and a small sample of Dirac 1-string $\lambda_0$. The solutions at the end points $L = 16$ and $L = 320$ are $\lambda_2 \approx 0.665 \pm 3i$; $\lambda_4 \approx -0.579 \pm 3.059i$ (red); central 3-complex $\lambda_{3c} \approx -0.373 \pm 2.691i$; 3-string $\lambda_3 \approx -2.129 \pm 2.00006i$ and $-2.129 \pm 2.00006i$. Crosses mark calculated values in the vicinity of the inferred $9$ root collision point $\lambda \approx -0.740 \pm i$ at $L \approx 104.76$. If we accept the chiral conjecture leading to an approximate quartet $a_i, a, a_2 \pm i$ at small $L$ collide just beyond $L = 104$ to form an approximate quartet $a + bi, a + (2-b)i$ in the interval $108 \leq L \leq 168$ which at $L = 172$ has separated into a 3-string and a 1-string that becomes part of the Dirac sea. Note that this sequence is essentially just the reverse of the sequence for the transition $3 \rightarrow (0|3^3|3^1)$ at $\kappa = 11$ as described following (79). Figure 4. The transition $2^41 \rightarrow (0|3^3|3^1)$ at $\kappa = 11$. Only the real parts of the complex BAE solutions are shown together with the $\lambda_F$ indicated. The curves labeled $\lambda$ is $\lambda_2, \lambda_4, \lambda_3c, \lambda_3, \lambda_4\lambda_2$ and $\lambda_4\lambda_3c$. The curves labeled $\lambda_4\lambda_2$ and $\lambda_4\lambda_3c$ are marked with crosses in the vicinity of the inferred root collision point $\lambda \approx -0.740 \pm i$ at $L \approx 104.76$.

Following (78) and (79) in the sequence of increasing $s$ is $2s = 2, \epsilon = 4 + 2[\kappa]$. The $\kappa \neq 0$ generator from (73) and (75) is

$$\frac{\sqrt{3} \sqrt{3^{\ast} 3^3}}{(0|3^3)} + \frac{\sqrt{3^{\ast} 3^4}}{(0|3^3|3^1)} + \frac{\sqrt{3^{\ast} 3^5}}{(0|3^3|3^2)} + \frac{\sqrt{3^{\ast} 3^6}}{(0|3^3|3^3)} + \frac{\sqrt{3^{\ast} 3^7}}{(0|3^3|3^4)} + \frac{\sqrt{3^{\ast} 3^8}}{(0|3^3|3^5)} + \frac{\sqrt{3^{\ast} 3^9}}{(0|3^3|3^6)} + \frac{\sqrt{3^{\ast} 3^{10}}}{(0|3^3|3^7)} + \frac{\sqrt{3^{\ast} 3^{11}}}{(0|3^3|3^8)} + \frac{\sqrt{3^{\ast} 3^{12}}}{(0|3^3|3^9)} + \frac{\sqrt{3^{\ast} 3^{13}}}{(0|3^3|3^{10})} + \frac{\sqrt{3^{\ast} 3^{14}}}{(0|3^3|3^{11})} + \frac{\sqrt{3^{\ast} 3^{15}}}{(0|3^3|3^{12})} + \frac{\sqrt{3^{\ast} 3^{16}}}{(0|3^3|3^{13})} + \ldots$$

(80)

where asterisks mark Bethe labels that will transition to chiral as in (79). The first transition $4^1 \rightarrow (0|3^3|2^1)$ at $\kappa = 4$ is similar to that seen in figure 3 in that only smooth drifts in the complex pair components of the Bethe 4-string occur. However the clear distinction $(0|2^3|3^1) \approx (0|3^3|2^1)$ is manifest in that here it is the $\approx a_1 \pm i$ component of the Bethe 4-string that approaches a fixed distance from $\lambda_F$ indicative of a 2-string. The $\approx a_2 \pm 3i$ component on the other hand remains deep within the Dirac sea with the imaginary part diverging $\propto \ln(L)$ as expected for a central 3-complex as discussed in section 3. Both asymptotic behaviours are already well established by $L = 256$. The same drift pattern occurs in the transitions $2^41 \rightarrow (0|3^3|2^1)$ for $\kappa = 12, 13, 14$ where the Bethe 2-string $\lambda$ in all cases is well separated from the 4-string $\lambda$ and acts only as spectator with little direct influence on the transition. The transition $5^1 \rightarrow (0|3^3|3^1)$ at $\kappa = 10$ is also similar in that the complex component with largest imaginary part smoothly separates and drifts deep into the Dirac sea and can be identified as the central 3-complex. The remaining parts of the 5-string remains close to $\lambda_F$ and so identiﬁes as the chiral 3-string. More interesting is the transition $2^41 \rightarrow (0|3^3|3^1)$ at $\kappa = 11$ illustrated in figure 4. The complex component of $\lambda_3$ with largest imaginary part also drifts deep into the Dirac sea but follows a more convoluted path. The magnitude of the imaginary part ﬁrst decreases to a minimum of 2.366 at $L = 164$ and then increases as expected for a central 3-complex. The two $\lambda \approx a_1 \pm i, a_2 \pm i$ at small $L$ collide just beyond $L = 104$ to form an approximate quartet $a + bi, a + (2-b)i$ in the interval $108 \leq L \leq 168$ which at $L = 172$ has separated into a 3-string and a 1-string that becomes part of the Dirac sea. Note that this sequence is essentially just the reverse of the sequence for the transition $3 \rightarrow (0|2^4|2^1)$ at $\kappa = 3$ described following (79).

For $2s = 1$ we observe from (79) that the fraction of Bethe labelled states that transition to a different chiral label is 1/2 and 3/5 for $\mu = 1$ and 2 respectively. For $2s = 2$ these fractions obtained from (80) are 1/3 and 5/9. If we accept the chiral conjecture leading to (73) and the independent conjecture (75) we can determine these

$9$ The mean $\lambda$ of the colliding roots as well as the square of their difference are smooth functions of $L$ and are ﬁtted to polynomials. The curves in the vicinity of the collision point are reconstructed from these ﬁts.
Table 1. The fraction of states that transition from Bethe labels from (75) at small L to different chiral labels (74) at large L for given 2s and μ. The zero fraction s = 0 row and μ = 0 column are not shown.

| 2s \ μ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------|---|---|---|---|---|---|---|---|---|
| 1     | .5 | .6 | .643 | .595 | .568 | .545 | .520 | .498 | .479 |
| 2     | .333 | .556 | .679 | .678 | .663 | .635 | .608 | .585 | .566 |
| 3     | .25 | .429 | .563 | .566 | .710 | .731 | .734 | .723 | .708 |
| 4     | .2 | .35 | .467 | .56 | .635 | .691 | .731 | .758 | .773 |
| 5     | .167 | .296 | .4 | .485 | .556 | .615 | .665 | .706 | .739 |
| 6     | .143 | .257 | .351 | .429 | .495 | .551 | .600 | .643 | .680 |
| 7     | .125 | .227 | .313 | .385 | .446 | .5 | .547 | .588 | .625 |

Table 2. A lower bound to the fraction of states that transition from \( B_{hL,hR}^{(0)} \) (15) labels at small L to different chiral \( c_{hL,hR}^{(0)} \) (14) labels at large L for blocks specified by \( h = h_L + h_R \) and \( h_c = \min(h_L, h_R) \). The zero fraction \( h_c = 0 \) column is not shown.

| h \ h_c | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---------|---|---|---|---|---|---|---|---|
| 2       | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4       | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6       | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8       | .6428 | .3571 | .5 | .3371 | 0 | 0 | 0 | 0 |
| 10      | .5952 | .4523 | .5 | .4523 | .4285 | 0 | 0 | 0 |
| 12      | .5681 | .4621 | .4924 | .4924 | .4318 | .5151 | 0 | 0 |
| 14      | .5454 | .4592 | .5081 | .4825 | .4638 | .5198 | .4568 | 0 |
| 16      | .5202 | .4447 | .5265 | .4657 | .5006 | .4909 | .4790 | .5069 |

fractions for general 2s and \( μ \) directly by label comparison in the analogs of (79) and (80) without the need for any explicit BAE solutions. Some of the results so obtained are summarized in table 1 with the specific values above emphasized in bold type. An explicit formula for the fractions for 2s \( < \) \( μ \) is not known. On the other hand the data is consistent with the fraction

\[
 f = \frac{\mu}{2s + \mu + 2} \left( \frac{2s + 3}{2s + 1} \right), \quad 2s \geq \mu
\]

and shows, for example, that \( f \to 1/2 \) for 2s = \( μ \to \infty \). I conclude that, at least for the limited number of states represented by (73) and (75), the breakdown of Bethe labelling is a significant feature.

The equivalence between chiral (14) and Bethe (15) differs from that between (73) and (75) because in general a Bethe configuration label from (15) provides only momentum \( κ \) information and not \( L \to \infty \) energy \( ε \). As an illustration of the general case consider \( B_{4,6}^{(0)} \) from (15) restricted to \( κ = 7 \). Direct evaluation provides the cusp state configuration list \( B_{4,6}^{(0)} \mid \kappa = 7 = \{2^{5+1}, 3^4, 2 (2^{2+4}), 2^{3^2}, 2^3 \} \) and explicit BAE solutions to \( L \approx 256 \) are sufficient to infer the asymptotic \( ε \) and \( δ \) defined in (16) together with the chiral configurations. This completed list is

\[
 B_{4,6}^{(0)}(\varepsilon, δ) \mid \kappa = 7 = \left\{ \frac{315}{2} \right\}_{0}^{3} \left\{ \frac{30}{12} \right\}_{2+3} \left\{ \frac{34}{4} \right\}_{2^3} \left\{ \frac{42}{4} \right\}_{3^2} \left\{ \frac{38}{4} \right\}_{2^2} \left\{ \frac{213}{6} \right\}_{3^3} \left\{ \frac{293}{6} \right\}_{2^2} \left\{ \frac{46}{0} \right\}_{2^3} \right. \]

with asterisks marking those states for which Bethe and chiral labels differ. These all involve a 4-string in which the real parts of the two complex pairs drift apart with one pair becoming the central 3-complex at large L. The chiral labels in (82) have been confirmed by direct evaluation of chiral (14) generalized to include energy as in (17). It is the sum

\[
 C_{4,6}^{(0)} \mid \kappa = 7 = \sum_{n=0}^{\infty} T_{4}^{(2n)}(\varepsilon) q^{n} \left[ T_{6}^{(2n+1)}(\varepsilon) q^{n+1} / L \right] \mid \kappa = 7 = \sum_{n=0}^{\infty} \varepsilon^{n} / \Gamma(n+1) q^{n+1} \]

with \( T_{2}^{(2n)} \) given by (9) and a relevant example in (71).
I conclude this section with an elaboration and correction of two results in [5]. Equation (104) in [5] is identified in the current notation as the specific product elements

$$T_1(\lambda) (e^2/q) T_1(\lambda) (e^2/q) = (e^2/q)^{3/4} (e^2/q)^{3/4} (e^2/q)^{3/4} (e^2/q)^{3/4}$$

arising from the indicated Bethe labelled configurations in the final equality. BAE solutions to $L = 640$ in [5] have been extended to larger $L$ and these together with extrapolation are shown in figure 5 and confirm the assignments made in [5]. The 5-string complex pair with imaginary part of largest magnitude is a smooth function of $L$ whereas the remainder undergoes transitions often seen for a 3-string, namely 3-string $\rightarrow$ quartet $\rightarrow$ 2-string pair. In this case one of these 2-strings together with the other 5-string complex identifies as the chiral central 4-complex $\lambda_0$ labelled in figure 5. Note that each chiral 3-string appears to asymptote to the same 3-string or 2-string configuration as $L \rightarrow \infty$ for all states in (84) thus confirming the expected independence of chiral configurations at the two Fermi surfaces.

The states in equation (105) in [5] identify with the chiral product terms

$$T_2(\lambda) (e^2/q) T_2(\lambda) (e^2/q) = (e^2/q)^{3/4} (e^2/q)^{3/4} (e^2/q)^{3/4} (e^2/q)^{3/4}$$

but unlike the agreement between (104) in [5] and (84), the doubly degenerate states in (105) in [5] were labelled as ‘apparent’ $3^1 [2^1 4^1]$ and not the chiral $(2^3 3^1) [2^3 3^1] (2^3 3^1)$ in (85). This case can no longer be considered as a counter-example to the chiral hypothesis of this paper because the discrepancy has been resolved by extending the BAE solution from $L = 640$ in [5] to $L = 3200$ and extrapolating the data to $L \approx 500 000$. The results are displayed in figure 6 where one can see that with data only at $L = 640$ as given in (105) in [5] that ‘apparent’

10 One of the complex pairs contributing to the central complex is $\lambda \approx a \pm i$ and this unambiguously excludes the central complex label $\lambda_0$. See the discussion following (53).
subject to the triangle condition $\Delta, |s_L - s_R| \leq S \leq s_L + s_R$. In addition to the explicit n-strings provided as labels in (86) a chiral label includes a central n-complex, $n = s_L + s_R - S + 1$, for states with $s_L + s_R > S$. The product terms in (86) that contribute for $S = 1$ are listed in table 3 together with the corresponding Bethe labelled states based on the generator (15). The placement of the Bethe cusp states is fixed by calculation at different $\kappa$ with $L$ large enough to determine asymptotic $\epsilon$ and correction to scaling $\delta = -2s_L \cdot s_R$. Reversion of $\epsilon = 2(\kappa_L + \kappa_R)$ and $\kappa = \kappa_R - \kappa_L$ determines the required $\kappa_L$ and $\kappa_R$.

The evolution of the three Bethe states $4^1$ in table 3 with $L$ is shown in figure 7. The short segments in the evolution of the four Bethe $2^3 3^1 \to$ chiral $(0)^3 (2)^1$ have been added to emphasize that the combination of Bethe $4^1$ at $\kappa = 9$ and the four Bethe $2^3 3^1$ at $\kappa = 10, \ldots, 13$ all at $\delta = 6$ merge to form the common chiral $(0)^3 (2)^1$ sequence at $\kappa = 9, \ldots, 13$. Similarly, the Bethe $4^1$ at $\kappa = 10$ and two Bethe $2^1 3^3$ at $\kappa = 11, 12$ all at $\delta = 2$ merge to

$$(2^1 4^1 | 2^1 3^1 )$$ was a reasonable assignment. By continuous graphing to $L = 3200$ it is clear this assignment is wrong but it takes the extrapolation shown in figure 6 to identify the central complex in a chiral representation as a 2-string. Further evidence for the assignment in (85) comes from a comparison of figures 3 and 6. Except for the spectator 3-string and two 2-strings in figure 6, the qualitative similarity between the behaviour in figure 5 for $12 < L < 2^{14}$ and that in figure 6 for $640 < L < 2^{14}$ is striking and supports the central complex as a 2-string. This qualitative similarity extends to the imaginary parts of $\lambda_i$ which at $L = 640$ are $\approx \pm 3.25i$ but by $2^{14}$ have dropped to $\approx \pm 2.9i$ and by $2^{16}$ to $\approx \pm 2.5i$ and would not be typical of a central 3-complex. Finally, just as for the states in (84), the chiral $2^3 3^1_6$ and $2^3 3^1_6$ in the states in (85) plausibly asymptote to the same common values as $L \to \infty$.

5. Transition to chiral states: $S = 1$

A significant feature when total $S > 0$ is the increase in the number of chiral spin, $s_L$ and $s_R$, combinations that are allowed in $C_{h_L h_R}^{(S)}$. As illustration consider $h_L = 2, h_R = 6$ where the tower product contributing to the cusp states in $C_{2,6}^{(S)}(\epsilon, q)$ in (17) is

\[
\sum_{\kappa_2} \sum_{\kappa_6} T_2^{(u)} T_6^{(u)} = \prod_{i=1,3} \left( y_i^{2 \kappa - 2} + y_i^{2 \kappa - 2} \right) + \prod_{i=1,3} y_i^{2 \kappa - 6} \left( y_i^{2 \kappa - 6} + y_i^{2 \kappa - 6} + y_i^{10} + y_i^{12} + y_i^{13} + y_i^{14} \right) + \prod_{i=1,3} \left( y_i^{2 \kappa - 4} + y_i^{2 \kappa - 4} + y_i^{12} + y_i^{14} + y_i^{15} + y_i^{16} + y_i^{17} \right) + \prod_{i=1,3} \left( y_i^{2 \kappa - 0} + y_i^{2 \kappa - 0} + y_i^{12} + y_i^{14} + y_i^{15} + y_i^{16} + y_i^{17} \right) \left( y_i^{2 \kappa - 0} + y_i^{2 \kappa - 0} + y_i^{12} + y_i^{14} + y_i^{15} + y_i^{16} + y_i^{17} \right) \right) (86)
\]

Figure 6. Data for the states in (85) in the same symmetry reduced form as those for (84) in figure 5. Crosses here too mark the largest $L$ at which the BAE have been solved. Extrapolation to $L = 2^{19}$ has been by 10 point fits of the form $\sum a_n \ln(L)^n$ to data for $400 \leq L \leq 3200$. The triplet of curves for $\lambda_2$, (and neighbouring $\lambda_3$) employed $n$ ranges in the fit sum $n_m - 9$, $n_m$, $n_m = 1, 2, 3$. 
states at the chiral boundary. Calculations of all 28 3-string transitions at a given k that is closer to symmetrical because of the hole count change. The real parts of the two complex pairs at k = 2,4, 5, 6 are also within the Dirac sea, but asymptotically drift to the quartet sea as seen in Figure 7 for k = 9, 10, 11 respectively. Also shown are trajectory segments of Bethe 231 → chiral (0|23|22)(crosses at L = 256, k = 10,...,13) and (0|23|21)(crosses at L = 512, k = 11, 12).

Table 3. Labels for S = 1 cusp states for hL = 2, hR = 6. Chiral labels are determined by the product (86) understood as $\Sigma_j\delta_j\Sigma_j\rho_j$,

$$\kappa_i^\delta_j \kappa_j$$

with each individual term $(\kappa_i^j/\kappa_j^\rho_j) = \kappa_i^\rho_j$ providing the momentum $\kappa$ and asymptotic energy $\kappa$. Separate rows are used if there are differences between Bethe and chiral labels. Each asterisk indicates one state that transitions from a Bethe label at small L to a different chiral label as L → ∞.

| $\kappa_i$ | $\delta$ | Bethe | Chiral |
|------------|----------|--------|--------|
| 1          | 0        | 41     | (0|0|41) 2,0 |
| "          | 231      | 11     | (0|231|231) 2,0 |
| "          | 231      | 12     | (0|231|231) 2,0 |
| "          | 231      | 13     | (0|231|231) 2,0 |
| "          | 231      | 14     | (0|231|231) 2,0 |
| "          | 231      | 15     | (0|231|231) 2,0 |
| "          | 231      | 16     | (0|231|231) 2,0 |
| "          | 231      | 17     | (0|231|231) 2,0 |
| "          | 231      | 18     | (0|231|231) 2,0 |

the chiral (0|231|22) sequence at $\kappa = 10, 11, 12$. Three Bethe 231 at $\kappa = 12, 13, 14$ and $\delta = 2$ evolve by the sequence 3-string → quartet → 2-string pair. The root collision for the final step in this sequence that gives Bethe 231 → chiral (0|231|22) is estimated to occur at L ≈ 1400, 720, 490 for $\kappa = 12, 13, 14$ respectively. Calculations of all 28 $B_{2,6}^{(1)}$ cusp states from (15) has confirmed the chiral conjecture for this $h_L = 2, h_R = 6$ case.

As another example consider $h_L = 3, h_R = 5$ which is summarized in table 4. The triplet of Bethe labelled 41 states at $\kappa = 4, 5, 6$ at small L are qualitatively like those seen in figure 7 at $\kappa = 9, 10, 11$ but with an overall pattern that is closer to symmetrical because of the hole count change $h_L/h_R = (2,6) → (3,5)$. The evolution to large L is very different. The real parts of the two complex pairs at $\kappa = 4$ do not split to opposite sides of the Dirac sea as seen in figure 7 for $\kappa = 9$ but drift apart only slightly and by L = 1000 are clearly identifiable as the chiral central 4-complex (0|41|0). For the state at $\kappa = 5$, the real part of the complex pair with largest imaginary part appears to approach a constant while the other also remains within the Dirac sea, it asymptotically drifts.
with a Dirac sea root to form a complex conjugate pair with small imaginary part. With increasing $L$ the now two parallel to $\lambda_6$. They are still outside the Dirac sea at $L = 25,600$ which is the largest length for which solutions have been obtained. Extrapolation shows qualitative behaviour like that for Bethe $4^1$ in figure $3$, $\kappa = 8$ or figure $7$, $\kappa = 10$ but stretched to longer lengths. Specifically, the real parts of the complex pair with largest imaginary part cross $\lambda_6$ at $L \approx 6.5 \times 10^5$; those for the other pair cross at $L \approx 5 \times 10^6$. The real parts of the two pairs cross each other at $L$ the order of $10^7$. This length is also what is estimated from the imaginary parts as the crossover point where the complex roots become clearly identifiable as $\lambda_{2c}$ and $\lambda_3$ and thus components of the chiral $(0|2|3^3)$ in table $4$.

Another illustration of a slow drift from outside the Dirac sea from table $4$ is that for the Bethe labelled $2^3$ state that transitions to chiral $(0|2|3^2)$ at $\kappa_L = 21/4$, $\kappa_R = 101/4$ and $\delta = 5/2$. This state is part of a triplet at $\kappa = 8$, 9 and 10 which associate with the three possible locations of two 2-strings in three possible $h_2$ holes and are shown in figure $8$ with the hole positions labelled 1, 2 and 3. The empty $h_2$ hole moves from 3, maximally outside the Dirac sea at $\kappa = 8$, to 1, inside the Dirac sea at $\kappa = 10$. The latter identifies with the Bethe labelled $2^3$ state for all $L$ and which by the maximum $L = 400$ calculated is clearly seen to be the chiral $(0|2|3^2)$. For empty $h_2$ hole at $\kappa = 9$ the Bethe 3-string evolves by the ‘standard’ route in which the real root first joins with a Dirac sea root to form a complex conjugate pair with small imaginary part. With increasing $L$ the now two complex pairs of the Bethe $3$-string form an identifiable and then increasingly accurate quartet $\lambda_{3c}$. At $L \approx 49,000$ a root collision occurs and two 2-strings emerge—completing the transition $2^3 \rightarrow (0|2|3^2)$. For empty hole 3 at $\kappa = 8$ the Bethe 3-string drifts parallel to that at $\kappa = 9$ and it is plausible that the same sequence 3-string $\rightarrow$ quartet $\rightarrow$ 2-string pair will occur at some large $L = \mathcal{O}(10^7)$ and confirm the predicted transition Bethe $2^3 \rightarrow$ chiral $(0|2|3^2)$.

The chiral states $(2^3|2^1|2^1)$ at $2\kappa_L = 1$, $2\kappa_R = 3$ and $\delta = 5/2$ in table $4$ associate with a 2-string at two locations for $2\kappa_L = 1$ and a 2-string at four locations for $2\kappa_R = 3$. For the four cases in which $\kappa_L = 3/4$ the Bethe labels are $2^3$ and at small $L$ all 3-strings lie outside the Dirac sea and drift towards $\lambda_6$ with increasing $L$. The first 3-string real root to collide with and absorb $\lambda_6$ is that for $\kappa = 14$; at $L = 1024$ a 3-string real root is still observed but by $L = 1280$ the 3-string is two pairs of complex roots and by $L = 4096$ the 3-string is an identifiable quartet with the real parts of the pairs having passed the new $\lambda_6$ and moved into the Dirac sea. Extrapolation suggests that a quartet root collision to form a $\lambda_2$ and $\lambda_{2c}$ pair, which combine with the remaining spectator 2-string to complete the $2^3 \rightarrow (2^1|2^2|2^1)$ transition, will occur at $L \approx 200,000$. For $\kappa = 5$ a 3-string real root is observed at $L = 3200$ but this has become a complex conjugate pair by $L = 4096$. This increase in $L$ by approximately factor 3 from the preceding in figure $3$ case before the appearance of the quartet configuration very likely applies to its demise in the transition to $(2^1|2^1|2^1)$ as well. Similar results are expected at even longer lengths for $\kappa = 6$ and 7 but no calculations have been done at these longer lengths. In only one of the four cases of chiral $(2^1|2^2|2^2)$ in which $\kappa_L = 4/3$ is the Bethe label $2^3$. Here the Bethe 3-string has a real root for $L \leq 28$ and is an approximate quartet for $L \geq 32$. This transitions to a 2-string pair $\lambda_2$ and $\lambda_{2c}$ at $L = 252.8$ inferred from numerical fits to

| $\kappa_{L/4}$ | $\delta$ | $\beta_{L/4}$ | $T_{\kappa_L} | (T_{\kappa_R})$ | $2\kappa_L, 2\kappa_R$ |
|-------------|-------|----------|------------------|-------------------|
| 2           | 5/2   | $4^1$    | 1                |                  |
| $''$         | $2^3$ | 1        | 1''              |                  |
| $''$         | $2^3$ | 1        |                  | (0|2|3^3)         |
| $''$         | $11/2$| $4^1$    | 1                |                  |
| $''$         | $2^3$ | 1        | 1''              |                  |
| $''$         | $1$   | 1        | 1                | (0|2|3^3)         |

Table 4. Labels for $S = 1$ cusp states for $h_2 = 3$, $h_3 = 5$ in the same format as table 3.
roots at both shorter and longer lengths. This last example together with the preceding ones illustrates the tremendous range of lengths at which Bethe to chiral labelling transitions can occur. It needs to be emphasized however that how the energy approaches its asymptotic form seems to be quite independent of the labelling transitions and that the placement of Bethe labelled states within tables such as tables 3 and 4 requires calculations for chains of moderate length only.

Calculations have been carried out also for all other combinations $h_L + h_R = 8$ as well as the smaller $h_L + h_R = 6$ and 4. No counter examples to the conjectured chiral string labelling transitions have been found although for confirmation, just as noted above, some resort to extrapolation was necessary. For instance for $h_L = h_R = 3$ at $\kappa_L = 2\frac{1}{4}$, $\kappa_R = 3\frac{1}{4}$ the transition $\text{Bethe } 3^1 \rightarrow \text{chiral } (0^2|12^1)$ is estimated from calculations to $L = 5120$ to culminate via quartet root collision at $L \approx 400 000$. The chiral conjecture has also been confirmed for a limited number of total $S > 1$ spin states.

6. Summary and conclusions

A conjecture for the string labelling of all low lying states in any sector of total spin $S$ in the length $L \rightarrow \infty$ chiral limit of the spin-$\frac{1}{2}$ Heisenberg anti-ferromagnetic chain has been presented. A central role is played by cusp states consisting of a filled Dirac sea of 1-strings between the $h_L$ (left) and $h_R$ (right) 1-string holes and a product of left and right combinations of $(n > 1)$-string excitations forming chiral spins $s_L$ and $s_R$ that couple to $S$. This coupling requires an additional central $(n > 1)$-complex ‘string’ if $S < s_L + s_R$. The complete picture is summarized by the generator (14), $C^{(S)}_{h_L,h_R}(q)$, for the number of cusp states at any momentum. This in turn is easily generalized to the generator (17), $C^{(S)}_{h_L,h_R}(g, q)$ in both energy and momentum and which includes 1-string excitations from the Dirac sea into the available $h_L$ and $h_R$ holes. The tower generators (9), $T^{(2)}_{2g}(q)$, that are the essential ingredients of (14) and (17) are proved to be consistent with the towers for the WZW model that is understood to be the chiral limit of the spin-$\frac{1}{2}$ Heisenberg anti-ferromagnetic chain (cf. (29)–(34)). This is important evidence for the chiral string conjecture of the present paper.

The chiral cusp state generator (14) has been verified numerically to high order to equal the Bethe cusp state generator (15). By starting from known solutions of the BAE at small $L$ for states in (15) and extending these solutions to large $L$ by ‘continuity’ one may arrive at the chiral representations in (14). It is this study of well over

\[ \text{Figure 8. Evolution of cusp states from table 4 that at large } L \text{ are the three chiral } (0^2|12^1) \text{ with } 2\kappa_L = 3, 2\kappa_R = 1 \text{ and } \delta = 3/2. \] Solutions at $\kappa = 8, 9$ (Bethe 2$^3$) and 10 (Bethe 2$^4$) have been determined to maximum $L = 5120, 2560$ and 400 respectively as indicated by crosses. For more details see text.

11 The complete set of states for $S = 1$ at $L = 16$ determined from the polynomials listed in the supplementary data file ‘L16_triplet.txt’ were very useful for the calculations in section 5.
Table 5. Data for the proof of (87) for combinations \( h_L + h_R = 8 \) at total \( S = 1 \). The large type entries are the right end counts \( c(R) = C_{h_R/2-\sigma}^{(2s)} \) and the small type (subscript) entries are the left end summed counts \( \Sigma c(L) = \sum_{h_L=0}^{S} C_{h_L/2-\sigma}^{(2s)} \) for fixed \( s_R \). Bold type indicates \( \Delta \) allowed \( 2s_R \) when \( 2h_L = h_R \). A breakdown of terms for \( h_R = 6 \) and 5 appears in tables 3 and 4 respectively.

| \( h_R, h_L \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 2s_R |
|-------------|---|---|---|---|---|---|---|------|
| 0, 8        | 1_2^8 | 0_6^2 | 0_35 | 0_2^8 | 8,6,4,2,0 |
| 1, 7        | 1_2^8 | 1_2^8 | 0_14 | 0_21 | 7,5,3,1 |
| 2, 6        | 1_y | 1_y | 0_15 | 0_3 | 6,4,2,0 |
| 3, 5        | 2_y | 2_y | 1_10 | 0_3 | 5,3,1 |
| 4, 4        | 2_5 | 2_5 | 1_3 | 0_3 | 4,2,0 |
| 5, 3        | 5_3 | 5_3 | 1_1 | 0_3 | 3,1 |
| 6, 2        | 5_1 | 9_2 | 4_3 | 1_1 | 1_0 | 2,0 |
| 7, 1        | 14_1 | 14_1 | 6_0 | 1_0 | 1 |
| 8, 0        | 14_0 | 28_1 | 20_0 | 7_0 | 0 |

one hundred such numerical cases without ever finding a counter example that provides additional convincing evidence and justification for the chiral labelling conjectures (14) and (17). The equality of (15) and (14) in the \( q \to 1 \) limit is a total count sum rule and an interesting binomial identity, namely

\[
C_{h_L/2-\sigma}^{(2s)} = \sum_{h_R} C_{h_R/2-\sigma}^{(2s)} C_{h_L/2-\sigma}^{(2s)} - \Delta = \left( \begin{array}{c} 2s_1 + 2\mu \mu \end{array} \right) \frac{2s_1 + 1}{2s_1 + \mu + 1} \tag{87}
\]

where the sum is subject to the triangle condition \( \Delta, |s_I - s_R| \leq S \leq s_L + s_R \). In addition to the explicit \( C_{\mu}^{(2s)} \) in (87) taken from (76), the \( C_{\mu}^{(2s)} \) are derivable from the generating function (77). An elementary proof of (87) is sketched in the appendix.

There remains the problem of going beyond the Bethe (15) and chiral (14) generator equality to a general rule for the chiral \( \kappa_L, \kappa_R, s_I, s_R \) state labels given any particular Bethe label combination \( \kappa, S \). In the present paper this was done by explicit numerical computation of BAE solutions starting from small \( L \) and extending these to large \( L \); as seen from the many examples treated the latter could be very large \( L \) indeed. It is not known whether a general rule that circumvents the numerical computation exists although the special Bethe label formulae (75) for singlet states on the lines \( \varepsilon = 4s^2 + 2|\kappa| \) were already conjectured in [5].

It is also unknown whether an analog of the \( L \to \infty \) string labelling found here for the anti-ferromagnetic chain exists in the case of a ferromagnetic chain. It was of course for the latter that Bethe [1] derived the ‘string hypothesis’ but also presented an example of its breakdown.

Appendix

This appendix is an outline of an analytical proof of the total count sum rule (87). I begin by generalizing the generator (77) by an extra summation to

\[
G = \frac{2}{1 - 2xy + \sqrt{1 - 4x^2}} = \sum_{2s=0}^{\infty} \left( 1 + \sqrt{1 - 4x^2} \right)^{2s+1} \frac{2}{2s+1} \frac{1}{2s+1} \frac{x^{2s+1}}{2s+1} \frac{1}{2s+1} \frac{y^{2s+1}}{2s+1} \frac{1}{2s+1} \frac{(xy)^{2s}}{2s+1} = \sum_{2s=0}^{\infty} \sum_{2s=0}^{\infty} C_{2s}^{(2s)} x^{2s} (xy)^{2s} \tag{A.1}
\]

where the last equality is the series expansion of the first and conveniently provides \( C_{2s}^{(2s)} \) as the coefficient of \( y^{2s} \) in the polynomial multiplying \( x^h \). Table 5 provides an example list for the right end chiral counts \( c(R) \) taken directly from (A.1) for \( 0 \leq h_R \leq 8 \). An important observation is that these coefficients arrange into a (truncated) Pascal triangle. The left end chiral sum counts \( \Sigma c(L) \) are sums of those terms in (A.1) allowed by \( \Delta \) for fixed \( s_R \) and \( S \). For example, for \( 2s_R = 0 \), only \( 2s_L = 28 = 2 \) is allowed so we record the coefficients of \( y^{2s} \) in (A.1), namely 1, 3, 9 and 28 in the rows \( h_L = 2, 4, 6, 8 \) respectively. For \( 2s_R = 1 \) both \( 2s_L = 1 \) and 3 are allowed; we read off from (A.1) the sum of the coefficients of \( y \) and \( y^3 \) to obtain 1, 3, 9 and 28 for \( h_L = 1, 3, 5, \) and 7 respectively. The entries for all \( 2s_R \geq 2S = 2 \) are the sum of three coefficients, those for \( y^{2s-1}, y^{2s} \) and \( y^{2s+1} \). Here too we make
the important observation that these coefficients arrange in a (truncated) Pascal triangle. Because of this structure the product \( c(R) \times \Sigma c(L) \) in any element in some \( h_L, h_R \) row is the product of the ‘walks’ from the \( h_L = 0 \) start to that row times the ‘walks’ from that row to the \( h_L = 0 \) finish. The sum of the product of ‘walks’ in any row is the totality of ‘walks’ from start to finish and thus independent of the row. In the \( S = 1 \) example above this is 28, the single element in the \( h_L = 0, h_R = 8 \) row and also the value of the left hand side of (87). The proof of (87) thus hinges on the proof of the Pascal triangle recursion for both \( c(R) \) and \( \Sigma c(L) \) to which I now turn.

The Pascal triangle generator \( T = y^2/(1-x(y^{-1} + y)) \) satisfies the recursion \( y^0 + x(y^{-1} + y)T = T \) with the factor \( y^0 \) defining the position of the triangle vertex. The analog Pascal recursion for \( c(R) \) follows directly from the first equality in (A.1) which satisfies
\[
1 + x \left( \frac{1}{y} + y \right) G = G + \frac{g}{y}, \quad g = \frac{2}{1 + \sqrt{1 - 4x^2}} \tag{A.2}
\]
where the unphysical \( s < 0 \) term following \( G \), i.e. \( xg/y \), is what defines \( G \) as a ‘truncated’ Pascal triangle. Its source is \( x/y \) times \( g \), the \( y \) independent part of \( G \) and has no compensating \( xy \) times a term \(-g/y^2\) from \( G \) to cancel.

For the Pascal recursion for \( \Sigma c(L) \) I start with the sums discussed above for the \( S = 1 \) example. These can be expressed as the product \((y^{-2} + 1 + y^2)G\) with subtractions to eliminate the misremembered terms—these are \((y^{-2} + 1)\) times the coefficient independent of \( y \) and \( y^{-2} \) times the coefficient of \( y \). The result is the \( \Sigma c(L) \) generator
\[
H_{S+1} = \left( \frac{1}{y^2} + 1 + y^2 \right) G - \left( \frac{1}{y^2} + 1 \right) g - \frac{x}{y^2} \tag{A.3}
\]
\[
= y^2 + (y^2 + y^3)x + (1 + 2y^2 + y^4)x^2 + (3y + 3y^3 + y^4)x^3 + (3 + 6y^2 + 4y^4 + y^6)x^4 + \ldots
\]
\[
\text{in agreement with entries labelled by} \ h_1 \text{ and } 2s_1 \text{ in table 5. The Pascal recursion for } \Sigma c(L), \text{ as in the case of } c(R), \text{ now follows directly from (A.2) and the first equality in (A.3) which satisfies}
\]
\[
y^{2S} + x \left( \frac{1}{y} + y \right) H_S = H_S + \frac{1}{y} (gx)^{2S+1} \tag{A.4}
\]
for \( S = 1 \). Note that for \( S = 0 \), the \( \Delta \) condition requires \( s_L = s_R \) so there is no summation in \( \Sigma c(L) \) and \( H_{S=0} \) simply reduces to \( G \) (except of course for \( h_L \leftrightarrow h_R \) interchange). Thus (A.4) is applicable also to \( S = 0 \) and suggests it is a general result. This can indeed be shown to be the case by using the ‘walk’ equivalence to derive explicit expressions for the relevant generating functions.

Since \( G = H_0 \) we can restrict our analysis to \( H_S \). The examples above suggest the (truncated) Pascal ‘walk’ generator for \( \Sigma c(L) \) might be formed from
\[
H_S^\Delta = \frac{1}{1 - x(y^{-1} + y)} \left( y^{2S} - \frac{1}{y^{2S+1}} \right) = \sum_{h_0=0}^S x^h \sum_{n=0}^h \binom{n}{h} \left( \left( \frac{y^{-2n+2S}}{h} - \frac{y^{-2n-2S-2}}{h} \right) \right) \tag{A.5}
\]
which is a difference of two Pascal binomial towers with vertices \( h = h_L = 0 \) at \( s = s_R = S \) and \(-S-1\). The change in dummy sum variable from \( n \) to \( 2s \) that leads to the final equality in (A.5) also imposes the restriction that \( h \) and \( 2s \) are either both even or both odd. The form of (A.5) as the difference of two towers guarantees that all \( 1/y \) terms vanish; this provides a picture of the ‘truncation’ process to be the result of perfect absorption of Pascal ‘walks’ on the line \( 2s = -1 \). A tentative physical \( H_S \) is the last equality in (A.5) to which the constraint \( s \geq 0 \) is applied. With the identification \( h = h_L, S = s_R \) now made explicit we have
\[
H_S = \sum_{h_L=0}^{h_L} x^h \sum_{2s_R \geq 0} \left\{ \left( \frac{h_L/2 - s_R + S}{h_L/2 - s_R - S - 1} \right) - \left( \frac{h_L/2 - s_R - S}{h_L/2 - s_R - S - 1} \right) \right\} \tag{A.6}
\]
Since all coefficients of \( y^h, p \geq -1 \), \( H_S^\Delta \) and \( H_S^\Delta \) are identical we confirm (A.4) except possibly for the \((gx)^{2S+1}/y \) term which is sourced by \( x/y \) times the \( y \) independent part of \( H_S \). The latter, from (A.6), is
\[
H_S(s_R = 0) = \sum_{h_L} x^h \left\{ \left( \frac{h_L}{h_L/2 - S} \right) - \left( \frac{h_L}{h_L/2 - S - 1} \right) \right\} = \sum_{h_L} x^h \frac{h_L}{h_L/2 - S} \frac{2S + 1}{h_L/2 + S + 1} \tag{A.7}
\]
\[
= x^{2S} \sum_{\mu=0} x^{2\mu} \left( \frac{2S + 2\mu}{2S + \mu + 1} \right) = x^{2S} (S+1)
\]
where we have used \( \frac{h_L}{h_L / 2 + S} = \left( \frac{h_L}{h_L / 2 - S} \right) \) and set \( h_L / 2 - S = \mu \), thus leading to the final equality that follows from (77). With this result we confirm (A.4) for all \( S \geq 0 \).

It only remains to show the tentative \( H_S \) in (A.6) is the generator for \( \Sigma c(L) \). For that we need to show the coefficient in braces in (A.6) equals the corresponding term in the \( \Delta \) allowed sum

\[
\Sigma c(L) = \sum_{\alpha = |S - L|}^{S + L} C^{(2)}_{h_L / 2 \alpha} = \sum_{\alpha = |S - L|}^{S + L} \left( \frac{h_L}{h_L / 2 - s_L} \right) \frac{2s_L + 1}{h_L / 2 + s_L + 1}
\]

at fixed \( S \) and \( S_R \). Because the summand in the last equality in (A.8) is the difference of two terms differing only in \( s_L \rightarrow s_L + 1 \), every term in the sum except first and last cancels. Explicitly,

\[
\Sigma c(L) = \left( \frac{h_L}{h_L / 2 - S - s_L} \right) - \left( \frac{h_L}{h_L / 2 - S - s_R - 1} \right)
\]

which is the coefficient in braces in (A.6) and shows \( H_S \) in (A.6) is the (no longer tentative) generator for \( \Sigma c(L) \). Together with \( G \) in (A.1) as generator for \( c(R) \) and (A.2) and (A.4) showing \( G \) and \( H_S \) to be (truncated) Pascal 'walks', the proof of (87) is complete.

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**References**

[1] Bethe H 1931 Z. Physik 71 205–26
[2] Mattis D C (ed) 1993 The Many-Body Problem (Singapore: World Scientific) pp 689–716
[3] Eisler F H L, Korepin V E and Schoutens K 1992 J. Phys. A: Math. Gen. 25 4113–26
[4] Vladimirov A A 1984 Phys. Lett. 105A 418–20
[5] Nickel B 2019 J. Phys. Commun. 3 025007
[6] Woynarovich F 1982 J. Phys. A: Math. Gen. 15 2985–96
[7] Woynarovich F 1982 J. Phys. C: Solid State Phys. 15 6397–401
[8] Affleck I, Gepner D, Schulz H J and Ziman T 1989 J. Phys. A: Math. Gen. 22 511–29
[9] Olver F W J, Lozier D W, Boisvert R F and Clarke C W 2010 NIST Handbook of Mathematical Functions (New York NY: Cambridge University Press)
[10] Andrews G E 1976 The Theory of Partitions (Addison-Wesley, Reading Mass)
[11] Hultenh L 1938 Ark. Mat. Astron. Fyz. 26A 1–106
[12] Nickel B 2017 J. Phys. Commun. 1 035021