The set of non-uniquely ergodic $d$-IETs has Hausdorff codimension $1/2$

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Dedicated to the memories of William Veech and Jean-Christophe Yoccoz.

Abstract We show that the set of not uniquely ergodic $d$-IETs with permutation in the Rauzy class of the hyperelliptic permutation has Hausdorff dimension $d - \frac{3}{2}$ [in the $(d - 1)$-dimension space of $d$-IETs] for $d \geq 5$. For $d = 4$ this was shown by Athreya–Chaika and for $d \in \{2, 3\}$ the set is known to have dimension $d - 2$. This provides lower bounds on the Hausdorff dimension of non-weakly mixing IETs and, with input from Al-Saqban et al. (Exceptional directions for the Teichmüller geodesic flow and Hausdorff dimension, 2017. arXiv:1711.10542), identifies the Hausdorff dimension of non-weakly mixing IETs with permutation $(d, d - 1, \ldots, 2, 1)$ when $d$ is odd.

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1 Introduction

**Definition 1.1** Given \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \) where \( x_i > 0 \), form the \( d \) sub-intervals of the interval \([0, \sum_i x_i] \):

\[
I_1 = [0, x_1), I_2 = [x_1, x_1 + x_2), \ldots, I_d = [x_1 + \cdots x_{d-1}, x_1 + \cdots + x_{d-1} + x_d).
\]

Given a permutation \( \pi \) on the set \( \{1, 2, \ldots, d\} \), we obtain a \( d \)-Interval Exchange Transformation (IET) \( T_{\pi,x} : [0, \sum_{i=1}^d x_i) \rightarrow [0, \sum_{i=1}^d x_i) \) which exchanges the intervals \( I_i \) according to \( \pi \). That is, if \( z \in I_j \) then

\[
T_{\pi,x}(z) = z - \sum_{k<j} x_k + \sum_{\pi(k')<\pi(j)} x_{k'}.
\]

Lebesgue measure is invariant under the action of \( T \).

**Definition 1.2** \( T := T_{\pi,x} \) is uniquely ergodic, if up to scalar multiple, Lebesgue is the only invariant measure.

The purpose of this paper is to prove the following Theorem. Let \( \Delta \subset \mathbb{R}^d \) the standard simplex of dimension \( d - 1 \). Let \( \pi_s \) be the hyperelliptic permutation.
on \( d \geq 4 \) letters (defined below) and let \( \mathcal{R}_d \) be the Rauzy class of \( \pi_s \). Let \( \pi \in \mathcal{R}_d \). Let \( \text{NUE}(\pi) \) denote the set of \( x \in \Delta \) such that \( T_{\pi,x} \) is not uniquely ergodic.

**Theorem 1.3** For \( \pi \in \mathcal{R}_d \) the Hausdorff dimension of \( \text{NUE}(\pi) \) is \( d - \frac{3}{2} \).

Note that the space is \((d - 1)\)-dimensional and so the Hausdorff codimension of this set is \( \frac{1}{2} \). It is easy to show that the set of IET that are not minimal (orbits are not dense) has Hausdorff codimension 1. So the main theorem says that for \( d \geq 4 \) the minimal non-uniquely ergodic IET have smaller codimension. In the case \( d = 2 \) the classical Kronecker–Weyl theorem says that \( T \) is not uniquely ergodic if and only if \( x_1, x_2 \in \mathbb{Q} \). (Here \( x_1 + x_2 = 1 \).) Since 3-IETs are first return maps of 2-IETs to intervals, minimality implies unique ergodicity for 3-IETs as well.

The upper bound \( \text{HDim}(\text{NUE}) \leq d - \frac{3}{2} \) follows from Masur [16]. (See Section 6 of [2]). In the case of \( d = 4 \) there is only one Rauzy class and Athreya–Chaika [2] proved the Theorem in that case. This paper is devoted to the proof of the lower bound for the given permutations \( \pi \). The main theorem has an interpretation in the context of translation surfaces. (A short discussion of translation surfaces will be in Sect. 1.1).

**Theorem 1.4** For \( g \geq 2 \), let \( \mathcal{H}_{\text{hyp}}(2g - 2) \) and \( \mathcal{H}_{\text{hyp}}(g - 1, g - 1) \) be the hyperelliptic components of the strata \( \mathcal{H}(2g - 2) \) and \( \mathcal{H}(g - 1, g - 1) \). Then the set of \((X, \omega)\) in these strata such that the vertical flow is not uniquely ergodic has Hausdorff codimension \( 1/2 \).

Given a translation surface \((X, \omega)\) denote by \( \text{NUE}(X, \omega) \) the set of directions \( \theta \in [0, 2\pi) \) such that the vertical flow of \( e^{i\theta}\omega \) is not uniquely ergodic.

**Theorem 1.5** For almost every \((X, \omega) \in \mathcal{H}_{\text{hyp}}(2g - 2) \) or \( \mathcal{H}_{\text{hyp}}(g - 1, g - 1) \) we have \( \text{HDim}(\text{NUE}(X, \omega)) = \frac{1}{2} \).

This Theorem follows from Theorem 1.3 with the exactly same proof as in the paper [2, Proposition 6.7] in the case of \( \mathcal{H}(2) \).

It is worth noting that for \( d > 4 \) there are Rauzy classes other than \( \pi_s \) so an open question is if the Theorem holds for all classes.

**Question 1.6** What is the Hausdorff dimension of the set of non-weakly mixing IETs.

Avila and Leguil proved that it has positive Hausdorff codimension [4]. Our result shows its Hausdorff codimension is at most \( \frac{1}{2} \). Boshernitzan and Nogueira [6] showed that an interval exchange with type \( W \) permutation that has the property that if the Teichmüller geodesic defined by a corresponding Abelian differential is recurrent in the stratum, then the IET is weak mixing. (Note that even though the assumption is on a related translation surface, the
conclusion is about the IET not the flow on the surface.) Al-Saqban, Apisa, Erchenko, Khalil, Mirzadeh and Uyanik [1] showed that the Hausdorff codimension of Teichmüller geodesics which are not recurrent in its stratum is at least one half, strengthening a result of [16]. (They prove more, treating the larger set of trajectories that are divergent on average.) As in [2, Sect. 6] this bounds the corresponding set of IETs, showing that the Hausdorff codimension of not weakly mixing type W IETs is at least $\frac{1}{2}$. Since when $d$ is odd the permutation $(d, d-1, \ldots, 2, 1)$ is type W, combining our results with [6] and [1] one obtains:

**Corollary 1.7** (Al-Saqban, Apisa, Erchenko, Khalil, Mirzadeh–Uyanik and Chaika–Masur) If $d$ is odd the set of non-weakly mixing IETs with permutation $(d, d-1, \ldots, 1)$ has Hausdorff codimension $\frac{1}{2}$.

It is natural to wonder if this result holds in all strata.

In the construction in this paper the non-uniquely ergodic IETs have exactly 2 ergodic measures. One can therefore ask

**Question 1.8** What is the Hausdorff dimension of $d$-IETs with $1 < k \leq \frac{d}{2}$ ergodic measures for each $k$?

### 1.1 History

Constructions of minimal non-uniquely ergodic IETs are due to Katok–Stepin [9], Sataev [19], Keane [10] and Keynes–Newton [13], based on examples of Veech [20]. Masur [15] and Veech [22] independently proved the Keane conjecture that with respect to Lebesgue measure on $\Delta$, and any irreducible permutation $\pi$, almost every IET is uniquely ergodic. There is a strong connection between the theory of IETs and translation surfaces. A genus $g$ translation surface $(X, \omega)$ is a compact, genus $g$ Riemann surface together with a nonzero holomorphic one-form $\omega$. This gives the structure of a flat metric away from a finite number of singular points, as integrating the one-form $\omega$ gives charts (away from zeros of $\omega$) to $\mathbb{C}$ where the transition functions between charts are translations. The zeros of $\omega$ are singular points of the metric, and have cone angles $2\pi(k + 1)$ at a zero of order $k$.

For any direction $\theta$ there is a flow in direction $\theta$ and the first return map to a transversal is an IET. For each translation surface, there is only a countable set of directions where the flow is not minimal (that is, there are non-dense infinite trajectories). Kerckhoff–Masur–Smillie [12] showed that the Lebesgue measure of NUE$(X, \omega)$ is 0. Masur [16] showed that $HDim NUE(X, \omega) \leq \frac{1}{2}$.

Moduli spaces of translation surfaces are stratified by their genus $g$ and the combinatorics of their singularities. We say a singularity has order $k$ if the angle is $2\pi(k + 1)$. The sum of orders of singularities on a genus $g$ surface is $2g - 2$. Given a partition $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, $\sum \alpha_i = 2g - 2$, define the
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stratum \( \mathcal{H} = \mathcal{H}(\alpha) \) to be the moduli space of (unit-area) translation surfaces with singularity pattern \( \alpha \). On each stratum \( \mathcal{H} \), there are coordinate charts to an appropriate Euclidean space, and pulling back Lebesgue measure yields a natural Lebesgue measure \( \mu_{MV} \) on \( \mathcal{H} \).

Masur–Smillie [17] showed that for every stratum of translation surfaces \( \mathcal{H}(\alpha) \) of surfaces of genus at least 2 there is a constant \( c = c(\alpha) > 0 \) such that for \( \mu_{MV} \)-almost every flat surface \((X, \omega) \in \mathcal{H}, \)

\[
\text{Hdim}(\text{NUE}(X, \omega)) = c.
\]

In terms of the simplex \( \Delta \) of IET, this implies the codimension of non uniquely \( \alpha \)-ergodic IETs is less than 1. The result in [16] referenced above said that \( c(\alpha) \leq 1/2 \) for all \( \alpha \).

Theorem 1.5 says that for almost every surface, in the strata we treat, the set of non-uniquely ergodic directions has Hausdorff dimension exactly \( \frac{1}{2} \). On the other hand for so-called lattice or Veech surfaces the dimension is 0. It is likely that Boshernitzan’s argument in the appendix of [7] can be applied to strata and show that a residual set of surfaces \((X, \omega) \) also have \( \text{Hdim}(\text{NUE}(X, \omega)) = 0 \).

This leads to a natural question:

**Question 1.9** Cheung, Hubert and Masur [8] establish that \( \text{Hdim}(\text{NUE}(X, \omega)) \in \{0, \frac{1}{2}\} \) for a certain locus of translation surfaces arising from billiards in rectangles with barriers. Is there a translation surface where the the set of non-uniquely ergodic directions has Hausdorff dimension \( c \notin \{0, \frac{1}{2}\} \)?

## 2 Plan of paper and background material and notation

Our plan is to build a subset of \( \Delta \) consisting of minimal and non-uniquely ergodic IETs that has Hausdorff codimension \( \frac{1}{2} \). The proof (and indeed many results on ergodic properties of IETs) uses in a crucial fashion the Rauzy induction renormalization procedures for IETs, involving induced maps on certain subintervals, and closely related to the Teichmüller geodesic flow. Our treatment of Rauzy induction will be the same as in [14]. For further details of the procedure (and much more on IETs) we refer the interested reader to [24] for an excellent survey.

The idea in using Rauzy induction to build non-uniquely ergodic IET is that one produces many sequences \( M_k = A_1 A_2 \ldots A_k \) of nonnegative matrices from Rauzy induction such that under the (projective) action the nested sequence of simplices \( M_k \Delta \) converges to a positive dimensional simplex. It is a classical result that the limiting points are non-uniquely ergodic IETs. There are many natural ways to find subsets of non-uniquely ergodic IETs as intersections of unions of simplices \( M_k \Delta \), which we will call *approximating sets* in this introduction. The strategy of proving a lower bound for the Haus-
Hausdorff dimension of the set of non-uniquely ergodic IETs is by building such sequences $M_k$ and measures $\mu_k$ on the approximating sets such that the weak-$*$ limit of these measures, which is a measure on a set of non-uniquely ergodic IETs, has the appropriate decay properties so we can apply Frostman’s lemma to get the desired lower bound for the Hausdorff dimension.

If our simplices could be constructed to have the property that they each contained a ball of volume comparable to the volume of the simplex, then with the independence results of Rauzy induction (like those in Sect. 5 and conditional versions in Sect. 6) one would likely be able to provide enough separated simplices to obtain the necessary decay properties of the limiting measure. Unfortunately our simplices are highly distorted. This motivates us to find a positive measure family of parallel planes so that our simplices intersected with the planes are not too distorted (Sect. 7) and to prove a lower bound of $\frac{3}{2}$ for the Hausdorff dimension for the intersection of the limiting simplices with this positive measure set of planes.

The proof of the main theorem has two parts: the abstract geometric framework, which is stated and proved to be sufficient in Sect. 3 using Frostman’s Lemma, and the much more involved part showing that the sets we build satisfy the abstract geometric framework. To do this we show that a positive measure set of planes intersect many approximating sets. This requires much of the work of the paper, because our approximating sets have rapidly decaying measure and so it is a priori unclear why the typical plane from a fixed pair of directions coming from Sect. 7 should intersect enough simplices and in big enough sets.

Section 2 recalls preliminaries (some of which are also in Sect. 5). We describe our paths, matrix sizes and approximating sets in Sect. 4 and indicate some properties these matrices need to satisfy. Our paths break up into stages. The assumption that the permutation is hyperelliptic allows one to break each stage into parts where we do Rauzy induction on $d - 2$ intervals which we call the left hand side (LHS) and a Rauzy induction on 2 intervals; the right hand side (RHS). Each side itself breaks up into a freedom part and a restricted part. In freedom on the left Rauzy induction is essentially arbitrary in that any of the first $d - 2$ intervals can win, and when one these intervals competes with one of the last two intervals it always wins. During restriction the first $d - 2$ intervals never compete with the last two intervals, and when the first interval competes with the other $d - 3$ it always loses. This cuts down the measure so that after infinitely many stages, the result has measure 0 as it must. However we keep enough simplices for the desired lower bound on the Hausdorff dimension. We analogously have freedom and restriction on the right.

We now outline some of the basic issues with the argument about controlling the intersection of parallel planes and simplices. At its heart the argument is probabilistic. For this we use probabilistic results of Rauzy induction due to
Kerckhoff and Veech and elementary probability theory to prove large deviations results (Sect. 5). However, since we will be doing Rauzy induction on $d - 2$ intervals and also 2 intervals, we will consistently need to use these probabilistic results to discuss the typical behavior of points lying in codimension 2, 3 and dimension 2 ‘faces’ of a simplex (Sects. 6, 10, 11) and indicate how these results on subsimplices apply to the entire simplex. Section 7 sets up the geometry of subsimplices intersecting planes, and is essentially a section on linear algebra.

Sections 8–11 are the most technically difficult parts of the paper. The goal in Sects. 8 and 9 is Theorem 8.2 which says that during each stage in freedom on LHS, if we throw out a small set of points, the intersection of our planes with the remaining simplices intersect a fixed designated face. This face corresponds to the first interval having length 0. It is necessary, because during restriction the first interval always loses so our simplices will be contained in a small neighborhood of this face. What this says is that we lose lots of measure. This loss has to be controlled to prove a lower bound for the Hausdorff dimension. We do this by showing that we keep most of the measure in this small neighborhood of the (codimension 1) face of our simplex. This is done in Sect. 10 (for the left hand side). On the right hand side we have a similar, perhaps easier picture since there are only 2 intervals. This is done in Sect. 11. In Sect. 12 we put these estimates together to prove that the abstract setting of Sect. 3 holds. Section 13 is an appendix that includes three results needed in the body of the paper.

2.1 Preliminaries on Rauzy induction

We follow the description of interval exchange transformations introduced in [14] and also explicated in [3]. We have the set $\mathcal{A}$ which consists of the first $d$ positive integers. Break an interval $I = [0, x)$ into intervals $\{I_i; 1 \leq i \leq d\}$ and rearrange in a new order by translations. Thus the interval exchange transformation is entirely defined by the following data:

1. The lengths of the intervals
2. Their orders before and after rearranging

The first are called length data, and are given by a vector $x \in \mathbb{R}^d$. The second are called combinatorial data, and are given by a pair of bijections $\pi = (\pi_t, \pi_b)$ from $\mathcal{A} \rightarrow \mathcal{A}$.

The bijections can be viewed as a pair of rows, the top corresponding to $\pi_t$ and the bottom corresponding to $\pi_b$.

Given an interval exchange $T$ defined by $(x, \pi)$ let $i, j \in \mathcal{A}$ be the last elements in the top and bottom. The operation of Rauzy induction is applied when $x_i \neq x_j$ to give a new IET $T'$ defined by $(x', \pi')$ where $x', \pi'$ are as
follows. If \( x_i > x_j \) then \( \pi' \) keeps the top row unchanged, and it changes the bottom row by moving \( j \) to the position immediately to the right of the position occupied by \( i \). We say \( i \) wins and \( j \) loses. For all \( k \neq i \) define \( x_k' = x_k \) and define

\[
x_i' = x_i - x_j.
\]

If \( x_j > x_i \) then to define \( \pi' \) we keep the bottom row the same and the top row is changed by moving \( i \) to the position to the right of the position occupied by \( j \). Then \( x_k' = x_k \) for all \( k \neq j \) and \( x_j' = x_j - x_i \). We say \( j \) wins and \( i \) loses.

In either case one has a new interval exchange \( T' \) determined by \( (x', \pi') \) and defined on an interval \( I' = [0, |x'|) \) where

\[
|x'| = \sum_i x_i'.
\]

The map \( T' : I' \to I' \) is the first return map to a subinterval of \( I \) obtained by cutting from \( I \) a subinterval with the same right endpoint and of length \( x_k \) where \( k \) is the loser of the process described above.

Let \( \Delta \) be the standard simplex in \( \mathbb{R}^d \) and let \( \mathcal{P} \) be the set of permutations on \( n \) letters. We can normalize so that all IET are defined on the unit interval. Let

\[
R : \Delta \times \mathcal{P} \to \Delta \times \mathcal{P}
\]

by \( R(x, \pi) = (\frac{x'}{|x'|}, \pi') \) denote (renormalized) Rauzy induction.

The set of permutations on \( d \) letters form the vertices of a directed graph that we now define. We have a directed edge joining \( \pi \) to \( \pi' \) if one of the two possibilities for Rauzy induction at \( \pi \) yields \( \pi' \). We say they are in the same Rauzy class if they are in the same connected component.

There is a corresponding visitation matrix \( M = M(T) \). Let \( \{e_i\} \) be the standard basis. If \( i \) is the winner and \( j \) the loser, then \( M(e_k) = e_k \) for \( k \neq j \) and \( M(e_j) = e_i + e_j \). We can view \( M \) as simply arising from the identity matrix by adding the \( i \) column to the \( j \) column. We can projectivize the matrix \( M \) and consider it as \( M : \Delta \to \Delta \).

When the interval exchange \( T \) is understood, and we perform Rauzy induction \( n \) times then define \( M(1) = M(T) \) and inductively

\[
M(n) = M(n - 1)M(R^{n-1}T).
\]

That is, the matrix \( M(n) \) comes from multiplying \( M(n - 1) \) on the right by the matrix of Rauzy induction applied to the IET after we have done Rauzy \( n - 1 \) times. We will also use the following notation. A vector \( x \in \Delta \) and permutation \( \pi \) determines an IET \( T \). The corresponding matrix after performing Rauzy
induction \(n\) times and suppressing the permutations is denoted \(M(x, n)\). For \(y \in M(x, n)\) denote by \(R^n y\) the \(x' \in \Delta\) such that \(M(x, n)x' = y\). Observe that if \(x, \eta\) satisfy \(\eta \in M(x, k)\), then

\[M(\eta, k) = M(x, k)\]

That is, the IETs determined by \(x\) and \(\eta\) have the same first \(k\) steps of Rauzy induction.

Given a matrix \(M\), we write

\[M\Delta = M\mathbb{R}^+_d \cap \Delta = \left\{ \frac{Mv}{|Mv|} : v \in \Delta \right\} \]

We also have the following notation.

- If \(x^1, \ldots, x^j \in \mathbb{R}^d\), let \(\text{span}_\Delta(x^1, \ldots, x^j) = \{\sum a_i x^i : a_i \geq 0 \text{ and } \sum a_i x^i \in \Delta\}\). In a mild abuse of notation we sometimes put subsets of \(\mathbb{R}^d\) in the argument as well.
- For \(M\) any matrix of Rauzy induction let \(C_{\text{max}}(M)\) be the column \(C_j(M)\) that maximizes \(|C_j(M)|\) over \(1 \leq j \leq d\). If there are two or such columns choose the one with the smallest index. Similarly with \(C_{\text{min}}(M)\).
- \(\lambda_s\) refers to Lebesgue measure on a \(s\) dimensional simplex. We will use this for \(s \in \{1, 2, d-4, d-3, d-2, d-1\}\).
- For \(0 \leq c \leq 1\) let \(\Delta_c = \{x \in \Delta : x_{d-1} + x_d = c\}\).
- If \(v, w \in \mathbb{R}^k\) let \(\Theta(v, w)\) denote the angle between \(v\) and \(w\). In a mild abuse of notation, if \(V, W \subset \mathbb{R}^k\) then \(\Theta(V, W) = \min\{\Theta(v, w) : v \in V, w \in W\}\).
- If \(M\) is a matrix let \(\|M\|\) denote the \(L^1\) operator norm of \(M\). If all the entries of \(M\) are non-negative, this is \(|C_{\text{max}}(M)|\).
- \(V(M) = \text{span}_\Delta(C_1(M), \ldots, C_{d-2}(M))\)
- \(W(M) = \text{span}_\Delta(C_{d-1}(M), C_d(M))\).
- \(F_i(M)\) the \(i\)th face is the span of all but \(C_i(M)\).
- If \(\mathcal{M}\) is a set of matrices so that \(\mathcal{M}\Delta\) is a simplex with labeled extreme points \(p_1, \ldots, p_d\), let \(F_i(\mathcal{M})\) be the convex hull of \(\{p_\ell\}_{\ell \neq i}\). In this setting we let \(V(\mathcal{M}) = \text{span}_\Delta(p_1, \ldots, p_{d-2})\) and \(W(\mathcal{M}) = \text{span}_\Delta(p_{d-1}, p_d)\).

A note on constants. On numerous occasions we will make use of a constant \(C\) in upper bounds. It is a local constant in that it will not depend on matrices or \(k\). Similarly we will use \(c > 0\) for lower bounds. We will also frequently have a constant \(\rho\) that appears in probabilistic statements.
3 Hausdorff dimension

We recall the definition of Hausdorff dimension. Let \( A \) a subset of a metric space \( X \) and \( s, \epsilon > 0 \). Define

\[
\mathcal{H}^s_\epsilon(A) = \inf_{\mathcal{U}} \left\{ \sum_{I \in \mathcal{U}} \text{diam}(I)^s : \mathcal{U} \text{ is a cover of } A \text{ by balls of diameter } < \epsilon \right\}.
\]

Then the \( s \) dimensional Hausdorff measure \( \mathcal{H}^s(A) \) is defined to be

\[
\mathcal{H}^s(A) = \lim_{\epsilon \to 0} \mathcal{H}^s_\epsilon(A).
\]

The Hausdorff dimension

\[
HDim(A) = \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\}.
\]

In our construction we will have a parallel family \( \mathcal{P} \) of 2 planes, parametrized by points in a \( d - 3 \) dimensional orthogonal subspace intersected with \( \Delta \). Using Lebesgue measure on the orthogonal complement gives a measure on the set of 2-planes.

The majority of the paper will be devoted to proving the following theorem.

**Theorem 3.1** There exists

- a positive measure set \( \hat{\mathcal{P}} \) of parallel 2-planes \( P \),
- for each \( k \in \mathbb{N} \) a set \( \mathcal{C}_k \) of disjoint simplices, \( \Delta^j_k \subset \Delta \) and
- for each \( P \in \hat{\mathcal{P}} \) and \( k \in \mathbb{N} \), a set \( \mathcal{C}_k(P) \subset \{ \Delta^j_k \cap P : \Delta^j_k \in \mathcal{C}_k \} \) of polygons

so that for each \( P \in \hat{\mathcal{P}} \), and \( J \in \mathcal{C}_k(P) \), when \( r_k(J) \) is the diameter of this polygon, we have

(a) each polygon in \( \mathcal{C}_{k+1}(P) \) is a subset of some element of \( \mathcal{C}_k(P) \) and is called a descendant (of that element).

(b) each point in an infinite nested sequence of polygons is not uniquely ergodic

(c) if we set \( \bar{r}_k = \max_{P \in \hat{\mathcal{P}}} \max_{J \in \mathcal{C}_k(P)} r_k(J) \) and \( \hat{r}_k = \min_{P \in \hat{\mathcal{P}}} \min_{J \in \mathcal{C}_k(P)} r_k(J) \),

\[
\text{then for each } \epsilon > 0 \text{ we have } \lim_{k \to \infty} \frac{\bar{r}_k^{1+\epsilon}}{\hat{r}_{k+1}} = 0.
\]

(d) There exists a sequence \( a_k \) with the two properties that first, for all \( \epsilon > 0 \),

\[
\lim_{k \to \infty} \frac{\hat{r}_k^{1+\epsilon}}{\left( \prod_{j=1}^k a_j \right)^{-1}} = 0,
\]
and second, letting $\mathcal{D}_{k+1}(J)$ be the set of all descendants $J'$ at level $k+1$ of a polygon $J$ at level $k$, then except for a set of polygons $\mathcal{B}_k(P) \subseteq \mathcal{C}_k(P)$ satisfying

$$\sum_{J \in \mathcal{B}_k(P)} \lambda_2(J) < \frac{1}{9^k} \sum_{J \in \mathcal{C}_k(P)} \lambda_2(J)$$

we have

$$\sum_{J' \in \mathcal{D}_{k+1}(J)} \lambda_2(J') > a_k \lambda_2(J).$$

(e) For each $P \in \hat{\mathcal{P}}$, $J \in \mathcal{C}_k(P)$ and $J' \in \mathcal{C}_{k+2}(P)$, where $J' \subseteq J$, then $\mathcal{N}(J', \tilde{r}_{k+2}) \cap P \subseteq J$. Here $\mathcal{N}(J', \tilde{r}_{k+2})$ refers to the $\tilde{r}_{k+2}$ neighborhood of $J'$

Assuming Theorem 3.1 in the rest of this section we show how Theorem 1.3 follows. We first recall Frostman’s Lemma and prove a useful Corollary.

**Lemma 3.2** (Frostman) Let $A \subseteq \mathbb{R}^k$ be Borel. The following are equivalent:
- $\mathcal{H}^s(A) > 0$ where $\mathcal{H}^s$ denotes $s$-dimensional Hausdorff measure.
- There exists a Borel measure $\mu$ satisfying $\mu(A) > 0$ and $\mu(B(x, r)) \leq r^s$ for all $x \in \mathbb{R}^k$ and $r > 0$.

**Corollary 3.3** Suppose $A \subseteq \Delta_{d-1} \subset \mathbb{R}^d$ is Borel and there exists a Borel measure $\mu$ so that $\mu(A) > 0$ and for all $\epsilon > 0$ there exists $r_0$ so that for all $0 < r < r_0$ and $x \in \mathbb{R}^d$ one has $\mu(B(x, r)) < r^{s-\epsilon}$. Then $\text{Hdim}(A) \geq s$.

**Proof** First observe that if $\mathcal{H}^t(A) > 0$ for all $0 \leq t < s$ then $Hdim(A) \geq s$. Now observe that if there exists a measure $\nu$ with $\nu(A) > 0$ and there exists $r_0$ so that $\nu(B(x, r)) < r^{s-\epsilon}$ for all $0 < r < r_0$ and $x \in \mathbb{R}^d$ then there exists a measure $\mu$ so that $\mu(A) > 0$ and $\mu(B(x, r)) < r^{s-\epsilon}$ for all $r$. Indeed, choose $r < r_0$ and $x$ so that $\nu(B(x, r) \cap A) > 0$ and choose $\mu$ to be $\nu$ restricted to $B(x, r)$. So it is clear that $\mathcal{H}^{s-\epsilon}(A) > 0$ for all $\epsilon > 0$ and by our previous observation we have the corollary. \qed

**Proposition 3.4** Each 2-plane $P$ occurring in Theorem 3.1 satisfies $Hdim(\cap_{k=1}^{\infty} \mathcal{C}_k(P)) \geq \frac{3}{2}$.

**Proof** We will put a non-zero Borel measure $\mu$ on $\cap_{k=1}^{\infty} \mathcal{C}_k(P)$ with the property that for all $\epsilon > 0$ there exists $r_0 > 0$ so that $\mu(B(x, r)) < r^{\frac{3}{2} - \epsilon}$ for all $r < r_0$. By Corollary 3.3 this will prove the proposition.

The measure $\mu$ will be the weak-$*$ limit of measures defined inductively. Let $\mu_1$ be Lebesgue measure restricted to $\mathcal{C}_1(P)$. Given $\mu_k$ defined on $\mathcal{C}_k(P)$, we inductively define $\mu_{k+1}$ in the following way. Define $\mu_{k+1}$ so that
• $\mu_{k+1}$ is supported on the set of $J \in \mathcal{C}_k(P)$ that satisfy (d).

• $\frac{\mu_{k+1}(J'_1)}{\mu_{k+1}(J'_2)} = \frac{\lambda_2(J'_1)}{\lambda_2(J'_2)}$ for all $J'_1, J'_2 \in \mathcal{C}_{k+1}(P)$, with $J'_1, J'_2 \subset J$.

• if $\mu_{k+1}(J) > 0$ and $J \in \mathcal{C}_k(P)$ then $\mu_{k+1}(J) = \mu_k(J)$.

• $\mu_{k+1}$ is a multiple of Lebesgue on each $J' \in \mathcal{C}_{k+1}(P)$ (the constant can depend on its immediate ancestor) and

• $\mu_{k+1}(J \setminus \mathcal{C}_{k+1}) = 0$.

(Informally, we rescale $\lambda_2$ restricted to $\mathcal{C}_{k+1}(P) \cap J$ so that $\mu_{k+1}(J) = \mu_k(J)$.) Let $\mu_{k+1}$ be the zero measure everywhere else. Let $\mu$ be a weak-* limit of the $\mu_k$. Now the second bullet implies $\frac{\mu_{k+1}(J')}{\lambda_2(J')}$ is independent of the descendants $J'$ of $J$. The third bullet says if we sum up $\mu_{k+1}(J')$ over these descendants we get $\mu_{k+1}(J) \in \{0, \mu_k(J)\}$ with $\mu_{k+1}(J) = \mu_k(J)$ iff $J$ satisfies the assumptions of (d). Combined with Conclusion (d) of Theorem 3.1 we get

$$\mu_k(J) = \sum_{J'} \mu_{k+1}(J') = \frac{\mu_{k+1}(J')}{\lambda_2(J')} \sum_{J'} \lambda_2(J') \geq \frac{\mu_{k+1}(J')}{\lambda_2(J')} a_k \lambda_2(J)$$

and so passing to the weak-* limit

$$\frac{\mu(J')}{\lambda_2(J')} \leq \frac{\mu_k(J)}{a_k \lambda_2(J)}$$

for all $J'$ descendants of $J \in \mathcal{C}_{k+1}(P)$. We claim

$$\mu(\cap_{i=1}^{\infty} \mathcal{C}_i(P)) > 0.$$  \hspace{1cm} (2)

By Conclusion (d) of Theorem 3.1 we have

$$\mu_{k+1}(\cap_{i=1}^{k+1} \mathcal{C}_i(P)) \geq (1 - \frac{1}{9^k}) \mu_k(\cap_{i=1}^{k+1} \mathcal{C}_i(P)).$$

It follows that

$$\mu(\cap_{i=1}^{\infty} \mathcal{C}_i(P)) \geq \liminf_{k \to \infty} \mu_k(\cap_{i=1}^{k} \mathcal{C}_i(P)) \geq (1 - \sum_{j=1}^{\infty} \frac{1}{9^j}) \mu_1(P) > 0,$$

establishing Inequality (2).

Next we check that for each $k$, if $r < \bar{r}_{k+2}$, then for all $x$,

$$\mu(B(x, r)) \leq \lambda_2(B(x, \bar{r}_k)) \left( \prod_{j=1}^{k} a_j \right)^{-1}.$$  \hspace{1cm} (3)
By the construction of the measures we claim that if \( J \in \mathcal{C}_k(P) \), then \( \mu(J) \leq \lambda_2(J)(\prod_{j=1}^k a_j)^{-1} \). Indeed, by (1) and induction,

\[
\mu_k(J) \leq \lambda_2(J)(\prod_{j=1}^k a_j)^{-1} \leq \lambda_2(B(x, \hat{r}_k))(\prod_{j=1}^k a_j)^{-1}
\]

for every \( x \). Also, by construction \( \mu_\ell(J) \leq \mu_k(J) \) for any \( \ell > k \).

If \( r < \frac{\hat{r}_{k+2}}{2} \) and \( \mu(B(x,r)) > 0 \) then there exists \( J' = \Delta^k_{k+2} \cap P \) so that \( B(x,r) \cap J' \neq \emptyset \). Let \( J \) be the (unique) element of \( \mathcal{C}_k(P) \) so that \( J' \subset J \). By Conclusion (e) of Theorem 3.1 we have that \( B(x,r) \subset N(J',r) \subset J \) and so \( \mu(B(x,r)) \leq \mu(J) \), establishing Inequality (3).

We now finish the proof of Proposition 3.4 by showing that for all \( \epsilon > 0 \) sufficiently small there exists \( r' \) so that \( \mu(B(x,r)) < r^{\frac{3}{2} - \epsilon} \) for all \( x \) and \( 0 < r < r' \).

By Conclusion (d) of Theorem 3.1, for all \( 0 < \epsilon < \frac{1}{2} \) there exists \( k' \) so that for all \( k > k' \)

\[
(\prod_{i=1}^k a_i) > \hat{r}_k^{\frac{1}{2} + \frac{\epsilon}{4}}.
\]

Given \( r > 0 \) let

\[
k = \max\{i : \frac{\hat{r}_{i+2}}{2} > r\}.
\]

By Conclusion (c) of Theorem 3.1, we have that for all \( \epsilon > 0 \) sufficiently small there exists \( r' \) so that for \( r < r' \) the \( k \) defined above satisfies

\[
\hat{r}_k < r^{1 - \frac{\epsilon}{4}}.
\]

We can also assume \( r' \) small enough so \( k > k' \). Putting this together, we now have our claim:

\[
\mu(B(x,r)) < \lambda_2(B(x,r_k))(\prod_{i=1}^k a_i)^{-1} < \pi r_k^{\frac{3}{4} - \frac{\epsilon}{4}} < \pi r^{\frac{3}{2}(1 - \frac{\epsilon}{4})} r^{-(\frac{1}{2} + \frac{\epsilon}{4})} < r^{\frac{3}{2} - \epsilon}
\]

with the last inequality holding for all small enough \( r \). We now apply Corollary 3.3. This completes the proof of the Proposition. \( \square \)

The Main Theorem will now follow from the following standard result:

**Proposition 3.5** ([18, Proposition 6.6]) Let \( A \subset \mathbb{R}^d \). If \( \mathcal{H}^s(A) < \infty \) then for all \( m \) planes \( V \subset \mathbb{R}^d \), for a.e. \( a \in V \mathcal{H}^{s-m}(A \cap (V^\perp + a)) < \infty \).
We now prove the Main Theorem.

**Proof of Theorem 1.3 assuming Theorem 3.1** We prove the contrapositive. Let $V$ be the orthogonal subspace to $P$. Assume

$$Hdim(NUE) < d - 1 - \frac{1}{2}$$

so there exists $s < d - 1 - \frac{1}{2}$ so that $\mathcal{H}^s((NUE)) < \infty$. So by Proposition 3.5 for almost $\tau \in V$,

$$\mathcal{H}^{s-(d-3)}(P + \tau \cap NUE) < \infty.$$ 

Since $s < d - 1 - \frac{1}{2}$ this contradicts Proposition 3.4.

\[\square\]

### 4 Paths and matrices

In this section we define our paths of Rauzy induction, put conditions that the corresponding matrices must satisfy, and at the end of the section show that these matrices produce non-uniquely ergodic IETs. To begin again let

$$\pi_s = \begin{pmatrix} 1 & \ldots & d \\ d & \ldots & 1 \end{pmatrix}$$

be the hyperelliptic permutation and $R_d$ be its Rauzy class. Our set of non-uniquely ergodic IETs is obtained by producing large families of special paths in the graph. Our paths break into segments with five different types.

- Freedom on the left hand side.
- Restriction on the left hand side.
- Transition from the left side to the right side
- Freedom on the right hand side.
- Restriction on the right hand side.

We now describe these types, which reference two special permutations

$$\pi_L = \begin{pmatrix} 1 & d-1 & d & 2 & 3 & \ldots & d-2 \\ d & d-1 & d-2 & d-3 & d-4 & \ldots & 1 \end{pmatrix}$$

$$\pi_R = \begin{pmatrix} 1 & 2 & 3 & \ldots & d-1 & d \\ d & d-2 & d-3 & \ldots & 1 & d-1 \end{pmatrix}.$$ 

We define *freedom on the left side*. It starts and ends at $\pi_s$. We begin by moving to $\pi_L$ in two steps with 1 beating $d$ and then $d - 1$. We are now on the left hand side (LHS) at $\pi_L$. Then starting at $\pi_L$ after an arbitrary sequence of
The set of non-uniquely ergodic $d$-IETs

permutations, (including possibly 1 losing, but $d-1$, $d$ always losing whenever they are compared) eventually 1 beats $d - 3, \ldots, 2$ and we reach $\pi_s$. This loop may be repeated over and over.

We next define restriction on LHS. It starts at $\pi_s$ and ends at $\pi_L$. Again first 1 beats $d$ and then $d - 1$ to reach $\pi_L$. After that 1 will lose when matched with $i; 1 < i \leq d - 2$. Note that $d$ and $d - 1$ will not be compared to anything. So they will neither be added to a column nor have a column added to them. We call this restriction on LHS because 1 is losing. The first row of the corresponding matrix is $(1, 0, \ldots, 0)$ and the first column reflects that other columns are added to the first. Also the graph formed from anything that can be reached from $\pi_L$ without 1 winning is a copy of $\mathcal{R}_{d-3}$ where the symbols are $2, \ldots, d - 2$. Note there is an extra vertex at $\pi_L$ where 1 is compared to $d - 2$ (and loses). In the set we are describing this vertex has exactly one incoming edge and one outgoing edge and so we “collapse” this vertex and these two edges to obtain $\mathcal{R}_{d-3}$.

The next set of paths are called transition from left side to right side. Any such path starts at $\pi_L$ and ends at $\pi_s$. The path does not return to $\pi_L$ and $\pi_s$ is reached only at the end.

We now define freedom on RHS. We start at $\pi_s$ and end at $\pi_R$. Suppose starting at $\pi_s$, $d$ successively beats $1, \ldots, d - 2$ to reach $\pi_R$. We are now on the right hand side (RHS). Then starting at $\pi_R$ there is a loop which consists of an arbitrary sequence of $d - 1$ beating $d$ followed by $d$ beating $d - 1$, followed by $d$ beating $1, \ldots, d - 2$ before returning to $\pi_R$. We can repeat this loop an arbitrary number of times.

Finally we have restriction on RHS. It starts at $\pi_R$ and ends at $\pi_s$. At $\pi_R$, $d$ loses a number of times before beating $d - 1$. Then the permutation returns to $\pi_s$. As long as $d$ keeps losing to $d - 1$ the letters $1, \ldots, d - 2$ are not compared.

4.1 Choice of matrices

The letter $A$ denotes matrices for Rauzy induction on the LHS and $B$ denotes the matrices on the RHS. The matrices $T$ correspond to transition from left to right. We add $'$ for matrices $A$, $B$ during restriction and no prime denotes freedom. We will return infinitely often to each side and to freedom and restriction on each side.

If we are at freedom on LHS via a path with corresponding matrix $M(x, r)$, let $A(R'(y), m)$ the matrix of Rauzy induction done $m$ times at $R'y$.

Fix $k_0$ to be determined later. (We will have a (finite) collection of conditions on $k_0$ but they will all hold for all $k_0$ large enough.) For the $k$th return and $k \geq 1$, we will build matrices $A_k, A'_k, B_k, B'_k, T_k$. Our matrices will be products of these matrices and will be denoted with letters $M_k, \bar{M}_k$. 

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We will start with $A_1'$ and end with any of $A_k, A_k', T_k, B_k, B_k'$. For example after ending after freedom on LHS ($k \geq 2$) we will have the matrix $M_k = A_1'T_1B_1B_1'A_2A_2\ldots A_{k-1}A_{k-1}'T_{k-1}B_{k-1}B_{k-1}'A_k$. The simplices contained in $C_k$ in the statement of Theorem 3.1 are of form $M_k(\Delta)$ where $M_k$ is a matrix as above after freedom on LHS.

We choose

$$10^{(3+k_0)^6} \leq \|A_1'\| \leq 2 \cdot 10^{(3+k_0)^6}.$$  

For $k \geq 2$

$$\|A_k\| \in [10^{(2k+k_0)^6-(k+k_0)^4}, 10^{(2k+k_0)^6-(k+k_0)^4+(k+k_0)^2}]$$

$$\|A_k'\| \in [10^{(2k+1+k_0)^6+(k+k_0)^4}, 10^{(2k+1+k_0)^6+(k+k_0)^4+(k+k_0)^2}]$$

For $k \geq 1$,

$$\|B_k\| \in [10^{(2k+1+k_0)^6-(k+k_0)^4}, 10^{(2k+1+k_0)^6-(k+k_0)^4+(k+k_0)^2}]$$

$$\|B_k'\| \in [10^{(2k+2+k_0)^6+(k+k_0)^4}, 2 \cdot 10^{(2k+2+k_0)^6+(k+k_0)^4}]$$

$$\|T_k\| \leq 10^{(k+k_0)^2}.$$

We will impose the following conditions on these matrices, and in the course of the paper prove that there exists $\rho < 1$ so that they can be satisfied at each stage $k$ except for a set of proportion $\rho^{(k+k_0)^2}$ in each simplex. This will be sufficient for our purposes.

**Conditions**

1. $|C_i(A_1' \ldots A_k)| / |C_i'(A_1' \ldots A_k)| < \zeta$ for all $i, i' \leq d - 2$ and $k$.
2. $|C_i(A_k' T_k)| / |C_i'(A_k' T_k)| < 10^{2(k+k_0)^2}$ for all $i, i' \leq d - 2$ and $k$.
3. $|C_j(B_k)| / |C_j'(B_k)| < \zeta$ for all $j, j' \geq d - 1$ and $k$.
4. $|C_j(B_k')| / |C_j'(B_k')| < 2$ for all $j, j' \geq d - 1$ and $k$.

The last condition is automatic. The third condition follows from the bounds we will put on the matrices in $B_k$ (see Sect. 11). The first condition will be established in Corollary 8.6 condition 3 (up to renaming $\zeta'$ as $\zeta$). The second condition follows by combining Theorem 10.1 and Lemma 11.1.

**Condition**

1. if $M$ is a matrix at the end of freedom on LHS at stage $k$

$$\Theta(C_i(M), C_i'(M)) < 10^{-c(2k+k_0)^6}$$ (4)
for \( i, i' \leq d - 2 \) and

2. if \( M \) is a matrix at the end of freedom on LHS at stage \( k \),

\[
\Theta(C_{d-1}(M), C_d(M)) < 10^{-(2k+1+k_0)^6}. \tag{5}
\]

This is proven in Lemma 6.6.

Now we wish to find bounds on the size of columns of products of matrices. Let

\[
U_k = \max_{i \leq d-2} |C_i(A'_1, \ldots, A_k)|
\]

\[
u_k = \min_{i \leq d-2} |C_i(A'_1, \ldots, A'_k)|
\]

\[
V_k = \max_{i > d-2} |C_i(A'_1, \ldots, B_k)|
\]

\[
v_k = \min_{i > d-2} |C_i(A'_1, \ldots, B'_k)|.
\]

Note \( U_k \) is the maximum size of the first \( d - 2 \) columns after freedom on LHS, \( u_k \) the minimum size after restriction on LHS and so forth.

**Proposition 4.1** Under Conditions \(*\), if \( k_0 \) is large enough we have

1. \[
\frac{1}{\xi^{2(k-2)}} 10^{-(k+k_0)^4} 10^{- \sum_{i=1}^{k-1} 4(i+k_0)^2} 10^{\sum_{i=3}^{2k} (i+k_0)^6} \leq U_k
\]

\[
\leq 2^k 10^{-(k+k_0)^4} 10^{\sum_{i=3}^{2k} (i+k_0)^6} + 2 \sum_{i=2}^{k} (i+k_0)^2 + \sum_{i=1}^{k} (i+k_0)^2.3
\]

2. \[
\frac{1}{\xi^{2(k-1)}} 10^{- \sum_{i=1}^{k-1} 4(i+k_0)^2} 10^{\sum_{i=3}^{2k+1} (i+k_0)^6} \leq u_k
\]

\[
\leq 2^k 10^{\sum_{i=3}^{2k+1} (i+k_0)^6} + 2 \sum_{i=2}^{k} (i+k_0)^2 + \sum_{i=1}^{k} (i+k_0)^2.3
\]

3. \[
\frac{1}{(2\xi)^k} 10^{-(k+k_0)^4} 10^{\sum_{i=3}^{2k+1} (i+k_0)^6} \leq V_k \leq 2^k 10^{-(k+k_0)^4} 10^{\sum_{i=3}^{2k+1} (i+k_0)^6} + 2 \sum_{i=1}^{k} (i+k_0)^2
\]

4. \[
\frac{1}{(2\xi)^k} 10^{\sum_{i=3}^{2k+2} (i+k_0)^6} \leq v_k \leq 2^k 10^{\sum_{i=3}^{2k+2} (i+k_0)^6} + 2 \sum_{i=1}^{k} (i+k_0)^2
\]

**Proof** We will find the upper and lower bounds for \( U_k \). The proofs of the other inequalities are similar. It is straightforward to check that our conclusions are satisfied for \( u_1, U_1, v_1 \) and \( V_1 \). We now prove the claim by induction, assuming the claim on \( u_{k-1}, U_{k-1}, v_{k-1} \) and \( V_{k-1} \) and then establish it for \( U_k \).
We claim first that

\[ U_k \leq \left( U_{k-1}10^{(2(k-1)+1+k_0)^6+(k-1+k_0)^4+2(k-1+k_0)^2} \right. \]

\[ + V_{k-1} \left. \right) 10^{(2k+k_0)^6-(k+k_0)^4+(k+k_0)^2} \]. \tag{6} \]

\[ U_k \geq \left( 10^{-2(k+k_0)^2} U_{k-1}10^{(2(k-1)+1+k_0)^6+(k-1+k_0)^4} \right. \]

\[ + \frac{V_{k-1}}{\zeta} \left. \right) 10^{-2(k+k_0)^2} 10^{(2k+k_0)^6-(k+k_0)^4} \]. \tag{7} \]

To justify these estimates note first that going from the end of freedom on LHS at stage \( k - 1 \) to the end of freedom at stage \( k \) on LHS we have a series of steps. We first have restriction on the left followed by transition from left to right. During restriction we add a column \( C_i \) to a column \( C_{i'} \) where \( i, i' \leq d - 2 \). The total will increase the size of the first \( d - 2 \) columns by at most \( \| A'_{k-1} \| \). The upper bound on \( \| A'_{k-1} \| \) and the upper bound on \( \| T_{k-1} \| \) give the first term in the parentheses of (6) The fact that the first \( d - 2 \) columns of these matrices are \( 10^{2(k+k_0)^2} \) balanced (by Condition * (2)) means that each column is increased by a multiplicative factor which is the lower bound of \( \| A_{k-1} \| \) divided by \( 10^{2(k+k_0)^2} \). This gives the first term in the lower bound, (7).

Then we enter freedom on RHS. Now the first \( d - 2 \) columns are changed by adding the last two columns to the first \( d - 2 \). Thus the effect of freedom on RHS is that we add at most \( V_{k-1} \) and at least \( \frac{V_{k-1}}{\zeta} \) to these columns. The first \( d - 2 \) columns are not changed during restriction on RHS. Then finally at level \( k \) we have freedom on LHS. For an upper bound we multiply by an upper bound for \( \| A_k \| \) and for a lower bound we multiply by a lower bound for \( 10^{-2(k+k_0)^2} \| A_k \| \) (because the previous matrix had that the first \( d - 2 \) columns were \( 10^{2(k+k_0)^2} \) balanced). This proves (6) and the corresponding lower bound (7).

Notice during restriction on LHS the last two columns do not change, and so in going from \( U_{k-1} \) to \( V_{k-1} \), using the bound on \( \| B_k \| \), we find

\[ 0 < V_{k-1} < U_{k-1}10^{(2(k-1)+1+k_0)^6+(k-1+k_0)^2} \]

and so plugging this into (6) we get

\[ U_k \leq U_{k-1} \left( 10^{2(k-1)+1+k_0)^6+(k-1+k_0)^4+2(k-1+k_0)^2} \right. \]

\[ + 10^{(2k-1+k_0)^6+(k-1+k_0)^2} \left( 10^{(2k+k_0)^6-(k+k_0)^4+(k+k_0)^2} \right)^2 \]
\[
2U_{k-1} \cdot 10^{2k-1+k_0} \cdot (k-1+k_0)^2 \cdot 10^{(2k+k_0)6-(k+k_0)^4+(k+k_0)^2} \cdot \sum_{i=1}^{k-1} (i+k_0)^2 \cdot \sum_{i=2}^{k} (i+k_0)^2 \cdot \sum_{i=2}^{k} (i+k_0)^2.
\]

Now plugging in the upper bound
\[
U_{k-1} \leq 2^{k-1} \cdot 10^{-(k-1+k_0)^4} \cdot 10^{\sum_{i=3}^{2k-2} (i+k_0)^6 + 2 \sum_{i=2}^{k-1} (i+k_0)^2 + \sum_{i=2}^{k} (i+k_0)^2} \cdot \sum_{i=3}^{2k-2} (i+k_0)^6 + 2 \sum_{i=2}^{k-1} (i+k_0)^2 + \sum_{i=2}^{k} (i+k_0)^2.
\]

given by the induction hypothesis, we get the desired upper bound for \(U_k\). A similar calculation gives the lower bound. \(\square\)

### 4.2 Non-unique ergodicity

**Theorem 4.2** Under Conditions *, for \(k_0\) large enough for any sequence \(M_k\) we have that \(\cap_{k=1}^{\infty} M_k(\Delta)\) consists of non-uniquely ergodic IET.

**Proof** By [21, Proposition 3.22] we need to show \(\cap_{k=1}^{\infty} M_k(\Delta)\) has positive dimension. To show this, it is enough to show for any \(\epsilon > 0\), if \(k_0\) large enough then for all \(k\) we have

\[
\Theta(C_i(M_k), \text{span}(e_1, \ldots, e_{d-2})) \leq \epsilon
\]

for all \(i \leq d-2\) and

\[
\Theta(C_j(M_k), \text{span}(e_{d-1}, e_d)) < \epsilon
\]

for \(j = d-1, d\).

Notice that during freedom and restriction on LHS the only columns added to the first \(d-2\) columns are themselves. This implies that \(\text{span}_\Delta(C_1(M), \ldots, C_{d-2}(M))\) is a subset of what it was at the start of these phases. So \(\max_{i \leq d-2} \Theta(C_i(M), \text{span}(e_1, \ldots, e_{d-2}))\) is non-increasing when we are on LHS. Similarly, \(\max_{j \geq d-1} \Theta(C_j(M), e_{d-1} \oplus e_d)\) is non-increasing on RHS. The theorem therefore will follow by estimating how much the angles of the first \(d-2\) columns can change during RHS, and how much the angles of the last two columns can change on LHS.

During freedom on the RHS we add a vector of norm at most \(V_k\) to a vector of norm at least \(u_k\). Using the upper bound for \(V_k\) and the lower bound for \(u_k\) given by Proposition 4.1 we see that

\[
\frac{V_k}{u_k} < \zeta^{2(k-1)} 2^k \cdot 10^{[-(k_0+k)^4+\sum_{i=2}^{k} (i+k_0)^2]} \leq 10^{-(k+k_0)^3},
\]

the last inequality for \(k_0\) large enough.
Similarly, during freedom on the LHS we add a vector of norm at most $U_k$ to a vector of norm at least $v_{k-1}$. We have $\frac{U_k}{v_{k-1}} < C 10^{-(k+k_0)^3}$ for a constant $C$. These are the only times vectors in $C_1, \ldots, C_{d-2}$ are added to $C_{d-1}$ and $C_d$ and vice-versa. Notice we started with $A_1'$ so that the first $d-2$ columns are not added to the $d-1$ and $d$ columns. Thus after $A_1'$ by taking $k_0$ sufficiently large the first $d-2$ columns are arbitrarily close to the span of $e_1, \ldots, e_{d-2}$ and the last two columns are exactly $e_{d-1}$ and $e_d$. Similarly considering $B_1$ and $B_1'$, for any $\epsilon' > 0$, by choosing $k_0$ large enough we can ensure that $\sum_{i=1}^{\infty} \frac{V_i}{u_i} < \epsilon'$ and $\sum_{i=2}^{\infty} \frac{U_i}{u_{i-1}} < \epsilon'$. If $\epsilon'$ is small enough depending on $\epsilon$ this establishes our sufficient condition.\hfill\square

5 Distortion and probabilistic results

We start this section by recalling some known results due to Kerckhoff [11] and Veech [21].

Definition 5.1 We say a matrix is $\zeta$-balanced if the ratio of the $\ell^1$ norms of any two columns is bounded by $\zeta$.

The following Lemma says that given any matrix of Rauzy induction, a definite proportion of points with that matrix of Rauzy induction will have a balanced matrix of Rauzy induction before the norm increases by more than a fixed factor.

Lemma 5.2 ([11, Corollary 1.7]) There exists $\zeta, K', \rho' > 0$ so that if $M = M(x, r)$ is a matrix of Rauzy induction then

\[ \lambda_{d-1}(\{y \in M\Delta : \exists n \text{ so that } MA(R^r y, n) \text{ is } \zeta\text{-balanced and} \ 
\ |C_{\max}(MA(R^r y, n))| \in [\|C_{\max}(M)\|, K'|C_{\max}(M)|]) \} > \rho' \lambda_{d-1}(M\Delta). \]

Lemma 5.3 ([11, Corollary 1.2]) Let $M(x, r)$ be a $\zeta$-balanced matrix of Rauzy induction. Let $U \subset \Delta$ be measurable. Then $\frac{\lambda_{d-1}(MU)}{\lambda_{d-1}(M\Delta)} > \zeta^{-d} \lambda_{d-1}(U)$.

Lemma 5.4 ([21, Proposition 5.2]) Given a matrix of Rauzy induction $M(x, n)$ and $W \subset \Delta$ then

\[ \lambda_{d-1}(\{y \in M(x, n)\Delta : R^n y \in W\}) = \int_W \frac{1}{(\sum_{i=1}^{d} |C_i(M(x, n))|z_i)^d} dz. \]

We also have the following version on faces: Let $M$ be a matrix of Rauzy induction in $R_d$ and $J_{i_1, \ldots, i_k}$ be the Jacobian of the projective action of $M$ restricted to $\Delta\{e_{i_1}, \ldots, e_{i_k}\}$.
Lemma 5.5 For \( u \in \text{span}_\Delta(e_{i_1}, \ldots, e_{i_k}) \) then
\[
J_{i_1,\ldots,i_k}(M)(u) = \frac{c_M}{(\sum_{j=1}^{k} u_{ij}|C_{ij}(M)|)^k},
\]
where \( c_M \) is a constant depending on \( M \) and \( i_1, \ldots, i_k \).

We include a proof for completeness.

Proof We treat the case of \( k = d - 1 \). The general situation follows by repeating the proof below \( d - k \) times. For simplicity of notation assume that \( i_\ell = \ell \) so the Jacobian of interest is \( J_{1,\ldots,d-1} \) which we denote \( J_{d-1} \).

We have that if \( u \in \mathbb{F}_d(M) \)
\[
J(u) = \frac{1}{(\sum_{i=1}^{d} u_i|C_i(M)|)^d} = \frac{1}{(\sum_{i=1}^{d-1} u_i|C_i(M)|)^d},
\]
the Jacobian of \( M \) acting on the entire simplex \( \Delta \) evaluated at \( u \).

Let \( W \) be a small neighborhood of \( u \in \text{span}_\Delta(e_1, \ldots, e_{d-1}) \) restricted to \( \text{span}_\Delta(e_1, \ldots, e_{d-1}) \). Let \( W_\epsilon = \{ v \in \Delta : v = (1 - s)w + sv' \text{ with } w \in W, \ v' \in \Delta \text{ and } s \leq \epsilon \} \).

For small \( \epsilon \) and small \( W \), we now approximate \( \lambda_{d-1}(MW_\epsilon) \) in two different ways. First, since \( W \) is a small neighborhood of \( u \) and \( \epsilon \) small,
\[
\lambda_{d-1}(MW_\epsilon) \sim J(u)\lambda_{d-1}(W_\epsilon) \sim \epsilon\lambda_{d-2}(W)\frac{1}{(\sum_{i=1}^{d-1} u_i|C_i(M)|)^d}.
\]
The notation here \( \sim \) is that the ratio goes to 1 as \( \epsilon \) goes to 0 and the neighborhood shrinks to \( u \).

On the other hand let \( \gamma(u) \) be the line segment in \( MW_\epsilon \) orthogonal to \( MW \). Then
\[
\lambda_{d-1}(MW_\epsilon) \sim \lambda_{d-2}(MW) \cdot |\gamma(u)| \sim J_{d-1}(u)\lambda_{d-2}(W)\frac{\epsilon|C_d(M)|}{\sum_{i=1}^{d-1} u_i|C_i(M)|} d\left(\frac{C_d(M)}{|C_d(M)|}, \text{span}_\Delta(C_1(M), \ldots, C_{d-1}(M))\right).
\]
Taking the ratio of these two expressions for \( \lambda_{d-1}(MW_\epsilon) \) and letting \( \epsilon \to 0 \) and the neighborhood \( W \) converging down to \( u \), we get
\[
J_{d-1}(u)\frac{|C_d(M)|}{\sum_{i=1}^{d-1} u_i|C_i(M)|} d\left(\frac{C_d(M)}{|C_d(M)|}, \text{span}_\Delta(C_1(M), \ldots, C_{d-1}(M))\right)
\]
\[
= \frac{1}{(\sum_{i=1}^{d-1} u_i|C_i(M)|)^d}.
\]
which solving for $J_{d-1}$ gives the desired expression for $J_{d-1}$.

\[ \sum_{j=0}^{\infty} |j - 1| = \frac{1}{2} \]

**Lemma 5.6** ([21, Equation 5.5]) There exists a constant $c_d$ depending only on dimension, so that $\lambda_{d-1}(M \Delta) = c_d \prod |C_i(M)]^{-1}$.

We have a version for faces, with the same proof as the previous result in [21].

**Lemma 5.7** Let $M$ be a matrix of $\mathcal{R}_d$ and $A_1, A_2$ be a matrices of freedom on LHS then $\lambda_{d-3}(V(MA_1)) = \prod |C_i(M)|^{-1}$.

We will prove at the end of the section:

**Proposition 5.8** There exists $K > 1$ and $\sigma < 1$ so that for all large enough $\zeta$, if $M = M(x, r)$ is a matrix of Rauzy induction then for all $m$,

\begin{equation}
\lambda_{d-1} \left( \{ y \in M \Delta : \exists n \text{ so that } MA(R^r y, n) \text{ is } \zeta \text{-balanced and } |C_{\text{max}}(MA(R^r y, n))| \in [ |C_{\text{max}}(M)|, K^m |C_{\text{max}}(M)| ] \right) > (1 - \sigma^m) \lambda_{d-1}(M' \Delta)
\end{equation}

(8)

In fact we will prove a stronger result:

**Proposition 5.9** If $M'' = M(w, s)$ is a fixed matrix of Rauzy induction, then there exists $K'' > 1, \delta'' < 1$ so that if $M' = M'(x, r)$ then for all $m$,

\begin{equation}
\lambda_{d-1} \left( \{ y \in M' \Delta : \exists n \text{ so that } R^r y + n \in M'' \text{ with } |C_{\text{max}}(M'A(R^r y, n))| \in [ |C_{\text{max}}(M')|, K''^m |C_{\text{max}}(M')| ] \right) > (1 - \delta''^m) \lambda_{d-1}(M' \Delta)
\end{equation}

(9)

This implies the following useful, weaker result:

**Proposition 5.10** There exists $K', \sigma'$ with $0 < \sigma' < 1$ so that if $M = M(x, r)$ is a matrix of Rauzy induction, then for all $m$,

\begin{equation}
\lambda_{d-1} \left( \{ y \in M(\Delta) : \exists n \text{ so that } A(R^r y, n) \text{ is positive and } |C_{\text{max}}(MA(R^r y, n))| \in [ |C_{\text{max}}(M)|, K'^m |C_{\text{max}}(M)| ] \right) > (1 - \sigma'^m) \lambda_{d-1}(M \Delta)
\end{equation}

(10)

**Proof** In Proposition 5.9 choose $M''$ to be a positive matrix.

We also have:
Proposition 5.11 There exist constants \( \tau < 1, \alpha < 1, \) and \( K \) so that for any matrix of Rauzy induction \( M = M(x, r) \) and \( j \in \mathbb{N} \) we have

\[
\lambda_{d-1}\{ y \in M \Delta : \exists m \text{ so that } |C_{\text{max}}(MA(R^i y, m))| < K^j |C_{\text{max}}(M)| \text{ and } \\
\text{diam}(V(MA(R^i y, m))) < \tau^j \text{diam}(V(M)) > (1 - \alpha^j)\lambda_{d-1}M \Delta
\]

In order to prove these propositions we first prove

Proposition 5.12 Let \((\Omega, \mu)\) be a measure space and \(F_i : (\Omega, \mu) \to \{0, 1\}\) be a sequence of random variables such that there exists \( 0 < \rho < \frac{1}{2} \) so that for any \( j \), the conditional probability that \( F_j \) is 1 given \( F_1, \ldots, F_{j-1} \) is at least \( \rho \). Then

- For all \( j > i \), \( \mu(\{ \omega : F_\ell(\omega) = 0 \text{ for all } i \leq \ell \leq j \}) \leq (1 - \rho)^{j-i} \).
- For all \( \epsilon > 0 \) there exists \( C \) and \( \tau < 1 \) depending on \( \epsilon \) and \( \rho \) so that for all \( N \),

\[
\mu(\{ \omega : \sum_{i=1}^{N} F_i(\omega) < N(1 - \epsilon)\rho \}) < C\tau^N \mu(\Omega).
\]

To prove Proposition 5.12 we make a comparison to a case of independent random variables:

Lemma 5.13 Let \((\Omega, \mu)\) be a probability space and \(F_i : (\Omega, \mu) \to \{0, 1\}\) be a sequence of random variables such that there exists \( 0 < \rho < 1 \) so that for any \( j \), the conditional probability that \( F_j \) is 1 given \( F_1, \ldots, F_{j-1} \) is at least \( \rho \). Let \( G_i : (\Omega, \mu) \to \{0, 1\} \) be independent and distributed according to \( \mu(G_i^{-1}(1)) = \rho \). Then for all \( \ell \) and \( r \),

\[
\mu(\{ \omega : \sum_{i=1}^{\ell} F_i(\omega) \leq r \}) \leq \mu(\{ \omega : \sum_{i=1}^{\ell} G_i(\omega) \leq r \}).
\]

Proof Let \( X = \{0, 1\}^\ell \times [0, 1]^\ell \). Let \( v \) be a measure defined on \( X \) by\(^1\)

\[
v((a_1, \ldots, a_\ell), A_1 \times \cdots \times A_\ell) = \mu(F_1^{-1}(a_1) \cap \cdots \cap F_\ell^{-1}(a_\ell))\lambda_\ell(A_1 \times \cdots \times A_\ell).
\]

Notice that

\[
\mu(\{ \omega : \sum_{i=1}^{\ell} F_i(\omega) = r \}) = v(\{ (\tilde{v}, \tilde{r}) : \sum_{i=1}^{\ell} v_i = r \})
\]

\(^1\) \( \lambda_\ell \) is Lebesgue measure on \([0, 1]^\ell \).
for all $r$. Let $\Phi : X \to \{0, 1, \ast\}^\ell \times [0, 1]^\ell$ by $\Phi((\vec{v}, \vec{t})) = (\vec{w}, \vec{r})$ where

$$w_i = \begin{cases} 0 & \text{if } v_i = 0 \\ 1 & \text{if } v_i = 1 \text{ and } \mu(F_{i-1}^{-1}(v_1) \cap \cdots \cap F_{i-1}^{-1}(v_{j-1})) \rho \geq t_i \mu(F_{i-1}^{-1}(v_1) \cap \cdots \cap F_{i-1}^{-1}(v_j)) . \\ \ast & \text{else} \end{cases}$$

By construction we have that the $(\Phi_* \nu)$ conditional probability that $w_i = 1$ given $w_1, \ldots, w_{i-1}$ is exactly $\rho$. To see this, first let $\Phi_{\vec{s}}(\vec{v}) = \vec{w}$ where $\Phi(\vec{v}, \vec{s}) = (\vec{w}, \vec{s})$. Notice that for any $a_1, \ldots, a_{i-1} \in \{0, 1\}^{i-1}$.

$$\nu(\{(\vec{v}, \vec{s}) : (v_1, \ldots, v_{i-1}) = (a_1, \ldots, a_{i-1}) \text{ and } \Phi_{\vec{s}}(\vec{v})_i = 1\} ) = \rho \nu(\{(\vec{v}, \vec{s}) : (v_1, \ldots, v_{i-1}) = (a_1, \ldots, a_{i-1})\} ).$$

Also, because $\Phi_{\vec{s}}(\vec{v})$ does not depend on $s_1, \ldots, s_{i-1}$, we have that the tuples $(a_1, \ldots, a_{i-1}), (b_1, \ldots, b_{i-1}) \in \{0, 1, \ast\}^{i-1}$ have the property that $a_j = 0$ iff $b_j = 0$ implies

$$\Phi_* \nu(\{(\vec{w}, \vec{t}) : w_1 = a_1, \ldots, w_{i-1} = a_{i-1} \text{ and } w_i = 1\}) \cdot \Phi_* \nu(\{(\vec{w}, \vec{t}) : w_1 = a_1, \ldots, w_{i-1} = a_{i-1}\}) = \Phi_* \nu(\{(\vec{w}, \vec{t}) : w_1 = b_1, \ldots, w_{i-1} = b_{i-1} \text{ and } w_i = \ast\} ) \cdot \Phi_* \nu(\{(\vec{w}, \vec{t}) : w_1 = b_1, \ldots, w_{i-1} = b_{i-1}\} ) = \rho. $$

That is to say, changing $\ast$ to 1 or vice-versa does not affect conditional probabilities. Therefore we have that

$$(\Phi_* \nu)(\{(\vec{w}, \vec{t}) : |i \leq \ell : w_i = 1| = r\}) = \mu(\omega : \sum_{i=1}^\ell G_i(\omega) = r)$$

for all $r$. However we also clearly have that

$$(\Phi_* \nu)(\{(\vec{w}, \vec{t}) : |i \leq \ell : w_i = 1| \leq r\}) \leq \nu(\{(\vec{v}, \vec{t}) : \sum_{i=1}^\ell v_i \leq r\}) = \mu(\omega : \sum_{i=1}^\ell F_i(\omega) \leq r).$$

This establishes the lemma.

Proposition 5.12 now follows from the following standard large deviations estimate whose proof is omitted.

Lemma 5.14 Let $F_i : (\Omega, \mu) \to \{0, 1\}$ be independent and distributed according to $\mu(F_i^{-1}(1)) = \rho$. Then for any $\epsilon > 0$ there exists $c_1$ and $c_2 < 1$ so that for all $\ell$,
Let $K$ be the constant given by Lemma 5.2. Define $F_j : \Delta \to \{0, 1\}$ by

$$F_j(y) = \begin{cases} 1 & \text{if there exists } n \text{ so that } |C_{\max}(M(R'y, n))| \in [2^j K'^j, 2^{j+1} K'^j+1] \text{ and } M(R'y, n) \text{ is } \zeta\text{-balanced} \\ 0 & \text{else} \end{cases}$$

We claim that by Lemma 5.2, the $F_i$ satisfy the assumption of Proposition 5.12. This is because if $M$ is a matrix of Rauzy induction so that $C_{\max}(M) \in [2^j K'^j, 2^{j+1} K'^j]$ we have

$$\mu(|x \in M\Delta : F_j(x) = 1|) > \rho' \mu(M\Delta). \quad (11)$$

Indeed, the chance $M$ becomes balanced before its norm increases by $K'$ is at least $\rho'$. This will give a matrix of norm at least $2^j K'^j$ and most $2^{j+1} K'^j K'$. This matrix will cause $F_j$ to be 1, establishing (11). Since this is true for every matrix, and the outcome of previous $F_i$ is about matrices with norm less than $2^j K'^j$ we have the assumption of Proposition 5.12. Indeed we apply (11) to the matrices $M(x, n)$ so that $|C_{\max}(M(x, n))| \in [2^j K'^j, 2^{j+1} K'^j]$ and $|C_{\max}(M(x, n-1))| < 2^j K'^j$. The proposition follows with $K = 2K'$ and $\sigma = \rho'$.

**Proof of Proposition 5.9** This is similar to the previous proof. Let $K = 2^{s+1} K'$ (where $s$ is as in the definition of $M''$). Define $F_j : \Delta \to \{0, 1\}$ by $F_j(y) = 1$ if there exists $n$ such that $R'y \in M''\Delta$ and $|C_{\max}(M'(R'y, n))| \in [2^j K'^j, 2^{j+1} K'^j+1]$ and 0 otherwise. We claim that the conditional probability $F_j = 1$ given $F_1, \ldots, F_{j-1} = 0$ is at least $\rho \xi^{-d} \lambda(M'(w, r)\Delta)$. Indeed the probability that there exists $n$ so that $M'(R'y, n)$ is $\zeta$-balanced and $|C_{\max}(M'(R'y, n))| \in [2^j K'^j, 2^{j+1} K'^j+1]$ is at least $\rho$. By Lemma 5.3 once this occurs, the conditional probability that $z \in M'(y, r + n)\Delta$ satisfies $z \in M''\Delta$ is at least $\xi^{-d} \lambda_{d-1}(M''\Delta)$. We may now apply Proposition 5.12.

**Proof of Proposition 5.11** Choose $M''$ to be a positive matrix and note that each occurrence of a fixed positive matrix contracts the simplex by a definite amount. Repeat the proof of Proposition 5.9 for this $M''$ and apply the second conclusion of Proposition 5.12 to obtain that off of a set of exponentially small measure in $n$ we have at least $\frac{d}{2} n$ occurrences of $M''$ by the time the matrix norm increases by a factor of $(2K)^n$. By the first sentence of the proof this establishes the proposition.
6 Remaining on left hand side, remaining on right hand side

The object of this section is to establish the two Theorems below and Lemma 6.6 at the end of this section. The first Theorem says if we are on the left hand side, a property that holds for most points on the face \( V(M) \) of a simplex \( M \Delta \) leads to saying the same about most points in the simplex \( M \Delta \) itself. The second makes the same statement on right hand side using the face \( W(M) \). We prove the first Theorem; the proof of the second is identical. We will need these statements because for example we wish to make statements about matrices for Rauzy induction just on LHS and have the estimates hold for the entire simplex.

We say that a matrix of Rauzy induction, \( M(x, n) \) is maximal for given \( N \), if \( \| M(x, n) \| \geq \frac{N}{2} \), and \( \| M(x, n - 1) \| < \frac{N}{2} \). The point of this definition is the set of simplices \( A \Delta \) for \( A \) maximal for given \( N \) will cover \( \Delta \) (minus the codimension 1 set where some power of Rauzy induction is not defined) and there is no redundancy.

Before stating the next theorem recall that on LHS \( d \) and \( d - 1 \) never win.

**Theorem 6.1** Given \( \xi \) there exists \( C \) so that if \( M := M(x, n) \) is a matrix of Rauzy induction, \( N \in \mathbb{R}^+ \), and \( 0 < \epsilon < 1 \), \( 0 < \delta < 1 \) are constants such that

1. \( \pi(R^n x) = \pi_L \)
2. \( \min_{j \in [d-1, d]} |C_j(M)| > \frac{N}{\epsilon^2} \max_{i \in \{1, \ldots, d-2\}} |C_i(M)| \)
3. \( \max_{j \in [d-1, d]} |C_j(M)| < \xi \) and \( \min_{i \leq d-2} |C_i(M)| < \xi \).

- if \( A_1, \ldots, A_r \) are a set of matrices on LHS such that
  \[
  \lambda_{d-3}(V(M) \setminus \bigcup_{i=1}^r V(MA_i)) > \delta \lambda_{d-3}(V(M)) \text{ and } \|A_i\| < N
  \]
  then
  \[
  \lambda_{d-1}(M \setminus \bigcup_{i=1}^r MA_i \Delta) > \frac{\delta}{C} \lambda_{d-1}(M \Delta).
  \]

- If \( A_1, \ldots, A_r \) are a set of matrices on LHS such that
  \[
  \lambda_{d-3}(V(M) \setminus \bigcup_{i=1}^r V(MA_i)) < \delta \lambda_{d-3}(V(M)) \text{ and } \|A_i\| < N
  \]
  then
  \[
  \lambda_{d-1}(M \setminus \bigcup_{i=1}^r MA_i \Delta) < C(\frac{\delta}{1-\delta} + \epsilon) \lambda_{d-1}(M \Delta).
  \]

We note that assumption (1) that \( \pi(R^n x) = \pi_L \) lets us assume that we can then consider matrices \( A_i \) on LHS as described in the bullets.

We also have a similar result about the right hand side.

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Theorem 6.2  Given \( \zeta \) there exists \( C \) so that if \( M := M(x, n) \) is a matrix of Rauzy induction, \( N \in \mathbb{R}_+ \), and \( 0 < \epsilon < 1 \), \( 0 < \delta < 1 \) are constants such that

1. \( \pi(R^n x) = \pi_R \)
2. \( \min_{i \in [1, \ldots, d-2]} |C_i(M)| > \frac{N}{\epsilon^2} \max_{j \in [d-1, d]} |C_j(M)| \)
3. \( \min_{j \in [d-1, d]} |C_j(M)| < \zeta \) and \( \max_{i \leq d-2} |C_i(M)| < \zeta \).

- if \( B_1, \ldots, B_r \) are a set of matrices on right hand side such that

\[
\lambda_1(W(M) \setminus \bigcup_{i=1}^r W(M B_i)) > \delta \lambda_1(W(M)) \quad \text{and} \quad \|B_i\| < N
\]

then

\[
\lambda_{d-1}(M \Delta \setminus \bigcup_{i=1}^r M B_i \Delta) > \frac{\delta}{C} \lambda_{d-1}(M \Delta).
\]

- If \( B_1, \ldots, B_r \) are a set of matrices on right hand side that satisfy

\[
\lambda_1(W(M) \setminus \bigcup_{i=1}^r W(M B_i)) < \delta \lambda_1(W(M)) \quad \text{and} \quad \|B_i\| < N
\]

then

\[
\lambda_{d-1}(M \Delta \setminus \bigcup_{i=1}^r M B_i \Delta) < C\left(\frac{\delta}{1 - \delta} + \epsilon\right) \lambda_{d-1}(M \Delta).
\]

We prove Theorem 6.1. The proof of Theorem 6.2 is essentially identical.

Proof of Theorem 6.1 modulo (13) below  Given \( M \) let \( M_N \) be the set of all maximal matrices \( A \) of size \( N \) where \( d - 1 \) and \( d \) have not won. The corresponding simplices \( MA \Delta \) cover \( V(M) \). The first conclusion says it cannot happen that a subset of \( V(M) \) whose complement has definite proportion of the measure of \( V(M) \) can produce matrices in \( M_N \) that almost cover \( M \Delta \). To show this it is enough to show for all \( A, A' \in M_N \) that

\[
\frac{\lambda_{d-3}(V(MA))}{\lambda_{d-3}(V(MA'))} \leq 4 \frac{\lambda_{d-1}(MA \Delta)}{\lambda_{d-1}(MA' \Delta)} \tag{12}
\]

We prove (12). First, notice that applying Lemma 5.7 we have

\[
\frac{\lambda_{d-3}(V(MA))}{\lambda_{d-3}(V(MA'))} = \frac{\prod_{i=1}^{d-2} |C_i(MA)|^{-1}}{\prod_{i=1}^{d-2} |C_i(MA')|^{-1}}.
\]

Now, for \( j \in [d-1, d] \) and \( A \in M_N \)

\[
|C_j(MA)| < |C_j(M)| + N \max\{|C_i(M)| : i \leq d - 2\} < 2|C_j(M)|
\]
by Assumption (2) of the Theorem. Therefore, for any \( A, A' \in \mathcal{M}_N \) we have
\[
\prod_{i=1}^{d-2} |C_i(MA)|^{-1} = \frac{|C_{d-1}(MA)| \cdot |C_d(MA)|}{|C_{d-1}(MA')| \cdot |C_d(MA')|} \prod_{i=1}^{d-2} |C_i(MA')|^{-1}
\leq 4 \prod_{i=1}^{d-2} |C_i(MA)|^{-1} \cdot \prod_{i=1}^{d-2} |C_i(MA')|^{-1}.
\]
So by Lemma 5.6 we have Inequality (12).

The heart of the proof of the second conclusion is (13) below which we will prove after we use it. First observe for any \( N \) that
\[
M \Delta = \bigcup_{A \in \mathcal{M}_N} MA \Delta
\]
modulo a set of codimension 1. Then there exists \( C \) such that given \( \epsilon > 0 \) and \( N \) and \( M \) a matrix of Rauzy induction satisfying the assumptions (1), (2), and (3) of Theorem 6.1 then
\[
\lambda_{d-1}(\bigcup_{A \in \mathcal{M}_N} MA \Delta) > (1 - C\epsilon)\lambda_{d-1}(M \Delta).
\]
(13)
The above inequality (which we have not proven) and (12), which we have, will establish the Theorem. Namely by (12) we have that
\[
\lambda_{d-1}(\bigcup_{A \in \mathcal{M}_N} MA \Delta \setminus \bigcup_{i=1}^{d-1} MA_i \Delta) < 4\lambda_{d-1}(\bigcup_{i=1}^{d-1} MA_i \Delta) \frac{\lambda_{d-3}(\bigcup_{A \in \mathcal{M}_N} V(MA) \setminus \bigcup_{i=1}^{d-1} V(MA_i))}{\lambda_{d-3}(V(M))} \frac{\lambda_{d-3}(V(M))}{\lambda_{d-3}(\bigcup_{i=1}^{d-1} V(MA_i))}.
\]
By the assumption of the second bullet point, the second term in the product on the right side is at most \( \delta \) and the last term is at most \( \frac{1}{1 - \delta} \). Thus the right hand side is at most
\[
4 \frac{\delta}{1 - \delta} \lambda_{d-1}(M \Delta).
\]
By (13) we have that \( \lambda_{d-1}(\bigcup_{A \in \mathcal{M}_N} MA \Delta) > (1 - C\epsilon)\lambda_{d-1}(M \Delta) \). Putting these two estimates together we have established the second bullet point and thus the theorem.

Now (13) follows from the next two Propositions.

**Proposition 6.3** There exists \( C \) such that for \( \epsilon \) sufficiently small and any \( N \), if \( M := M(x, n) \) is a matrix of Rauzy induction satisfying hypotheses (2) and (3) of Theorem 6.1 and the permutation of \( R^n x \) is \( \pi_L \), then
\[
\lambda_{d-1}(\{y \in M \Delta : \max_{j \in [d-1, d]} (R^n y)_j > \frac{\epsilon}{N}\}) < C\epsilon \lambda_{d-1}(M \Delta).
\]
Proposition 6.4 There exists $C$ such that for all small enough $\epsilon$ and all $N$ and permutation $\pi_L$
\[
\lambda_{d-1}(\{x \in \Delta : x_{d-1} + x_d < \frac{\epsilon}{N} \text{ and } d-1 \text{ or } d \text{ win and the corresponding matrix } \|A(x, m)\| < N\}) < C \epsilon \lambda_{d-1}(\{x : x_{d-1} + x_d < \frac{\epsilon}{N}\}).
\]

Proof of (13) assuming Propositions 6.3 and 6.4. Let $C$ be the maximum of the constants $C$ in Propositions 6.3 and 6.4.

Let $M$ in (13) be of the form $M(z, n)$. By Proposition 6.3 (and taking complements) there is $C$ such that
\[
\lambda_{d-1}(\{y \in M \Delta : R^n(y)_{d-1} + R^n(y)_d < \frac{\epsilon}{N}\}) > (1 - C \epsilon) \lambda_{d-1}(M \Delta).
\]

To prove (13) it then suffices to show that for $C$ large enough,
\[
\lambda_{d-1}(\{y \in \bigcup_{A \in \mathcal{M}_N} MA \Delta : R^n(y)_{d-1} + R^n(y)_d < \frac{\epsilon}{N}\}) \geq (1 - C \epsilon) \lambda_{d-1}(\{y \in M \Delta : R^n(y)_{d-1} + R^n(y)_d < \frac{\epsilon}{N}\}). \quad (14)
\]

For then we would combine these last two inequalities, giving for a new constant $C$
\[
\lambda_{d-1}(\{y \in \bigcup_{A \in \mathcal{M}_N} MA \Delta : R^n(y)_{d-1} + R^n(y)_d < \frac{\epsilon}{N}\}) > (1 - C \epsilon) \lambda_{d-1}(M \Delta).
\]

We observe then that (13) follows since the left hand side in that inequality contains the set that appears in the left hand side above.

We now prove (14). Denote by $T_0(\frac{\epsilon}{N})$ the set on the left. Set $x = R^n y$ so $x_{d-1} + x_d < \frac{\epsilon}{N}$. We now show that the set of $x$ such that $x_{d-1} + x_d < \frac{\epsilon}{N}$ and $d - 1$ or $d$ wins (so we leave the left side) with a matrix of size at most $N$ is $O(\epsilon)$. Then we will apply Jacobian estimates to conclude the same thing about the set of $y$. Then we take complements to establish (14).

To that end, define
\[
S_0(\frac{\epsilon}{N}) = R^n(T_0(\frac{\epsilon}{N})) = \{x : x_{d-1} + x_d < \frac{\epsilon}{N}\}
\]
where as always $R$ denotes (normalized) Rauzy induction. For $j \in \mathbb{N}$ set
\[
S_j(\frac{\epsilon}{N}) = \{x \in \Delta : \frac{\epsilon}{N^{2j+1}} \leq x_{d-1} + x_d < \frac{\epsilon}{N^{2j}}\}
\]
so $S_0\left(\frac{\epsilon}{N}\right)$ is a disjoint union of $S_j\left(\frac{\epsilon}{N}\right)$. There are corresponding sets

$$T_j\left(\frac{\epsilon}{N}\right) = R^{-n}(S_j\left(\frac{\epsilon}{N}\right)) \cap M\Delta$$

whose union over $j$ is $T_0\left(\frac{\epsilon}{N}\right)$.

Now for each $j$ apply Proposition 6.4 to $\cup_{i \geq j} S_i$ to find

$$\lambda_{d-1}\left(\{x \in \cup_{i \geq j} S_i\left(\frac{\epsilon}{N}\right) : d - 1 \text{ or } d \text{ wins within } n \text{ steps and } \|A(x, n)\| < 2^j N\}\right)$$

$$< C \epsilon \lambda_{d-1}\left(\cup_{i \geq j} S_i\left(\frac{\epsilon}{N}\right)\right).$$

Now $\{x \in S_j\left(\frac{\epsilon}{N}\right) : d - 1 \text{ or } d \text{ wins within } n \text{ steps and } \|A(x, n)\| < 2^j N\}$ is contained in the set on the left and the measure of the set on the right is proportional to the measure of $S_j$. We conclude for a new constant $\tilde{C}$,

$$\lambda_{d-1}\left(\{x \in S_j\left(\frac{\epsilon}{N}\right) : d - 1 \text{ or } d \text{ wins within } n \text{ steps and } \|A(x, n)\| < 2^j N\}\right)$$

$$\leq \tilde{C} \epsilon \lambda_{d-1}\left(S_j\left(\frac{\epsilon}{N}\right)\right).$$

The measure where $\|A\| \leq N$ is even smaller.

By Lemma 5.4, if $U \subset S_j\left(\frac{\epsilon}{N}\right)$

$$\lambda_{d-1}(U)[(1 - \frac{\epsilon}{N2^{j+1}}) \max_{i \leq d-2} |C_i(M)| + \frac{\epsilon}{N2^j} \max_{i \geq d-1} |C_i(M)|]^{-d} \leq \lambda(R^{-n}U \cap M\Delta)$$

$$\leq \lambda_{d-1}(U)[(1 - \frac{\epsilon}{N2^j}) \min_{i \leq d-2} |C_i(M)| + \frac{\epsilon}{N2^{j+1}} \min_{i \geq d-1} |C_i(M)|]^{-d}.$$

Therefore by assumption (3) of Theorem 6.1 there exists $\hat{C}$ depending on $\zeta$ such that for any $U, V \subset S_j\left(\frac{\epsilon}{N}\right)$ we have

$$\frac{\lambda_{d-1}(R^{-n}U \cap M\Delta)}{\lambda_{d-1}(R^{-n}V \cap M\Delta)} \leq \hat{C} \frac{\lambda_{d-1}(U)}{\lambda_{d-1}(V)}.$$

This says that inside $T_j\left(\frac{\epsilon}{N}\right)$ the proportion of $y$ such that $d - 1 \text{ or } d \text{ wins}$ is at most $\hat{C} \epsilon \lambda_{d-1}(T_j\left(\frac{\epsilon}{N}\right))$. Summing over $j$ and then taking complements we have proven (14).

**Proof of Proposition 6.3** As before, set

$$S_0\left(\frac{\epsilon^2}{N}\right) = \{x \in \Delta : x_{d-1} + x_d < \frac{\epsilon^2}{N}\}$$
and now for $j \in \mathbb{N}$, we define

$$\hat{S}_j(\frac{\epsilon}{N}) = \{x \in \Delta : \frac{2j\epsilon}{N} \leq x_{d-1} + x_d < \frac{2j+1\epsilon}{N}\}.$$ 

For some $c' > 0$ depending on $\zeta$, the Assumption (3) of Theorem 6.1 implies

$$\inf_{z \in \hat{S}_j(\frac{\epsilon}{N})} |C_{d-1}(M)|z_{d-1} + |C_d(M)|z_d > c'2^j \epsilon \sup_{y \in S_0(\frac{\epsilon^2}{N})} |C_{d-1}(M)|y_{d-1} + |C_d(M)|y_d.$$ 

This inequality together with Assumption (2) on the size of the columns which says

$$\frac{1}{\min_{i \in [d-1,d)} |C_i(M)|} \sum_{i=1}^{d-2} |C_i(M)|z_m \leq \frac{1}{\min_{i \in [d-1,d)} |C_i(M)|} \sum_{i=1}^{d-2} |C_i(M)| < d^2 \frac{\epsilon^2}{N}$$

implies there is a constant $C$ such that

$$\sup_{z \in \hat{S}_j(\frac{\epsilon}{N})} \frac{1}{(|C_1(M)|z_1 + \cdots + |C_d(M)|z_d)^d} < C 2^{-d} \epsilon^d \inf_{y \in S_0(\frac{\epsilon^2}{N})} \frac{1}{(|C_1(M)|y_1 + \cdots + |C_d(M)|y_d)^d}. \quad (15)$$

Moreover for some constant $C'$,

$$\lambda_{d-1}(\hat{S}_j(\frac{\epsilon}{N})) \leq C' \frac{2^j}{\epsilon^2} \lambda_{d-1}(S_0(\frac{\epsilon^2}{N})), \quad (16)$$

Recalling Veech’s Jacobian formula, Lemma 5.4, we see from (15) and (16) that there exists $c'' > 0$ so that for each $j$,

$$\lambda_{d-1}([y \in M\Delta : R^ny \in S_0(\frac{\epsilon^2}{N})]) \geq c'' 2^j \epsilon^2 \lambda_{d-1}([y \in M\Delta : R^ny \in \hat{S}_j(\frac{\epsilon}{N})])$$

$$\geq c'' 2^j \epsilon^{-2} \lambda_{d-1}([y \in M\Delta : R^ny \in \hat{S}_j(\frac{\epsilon}{N})]).$$

This uses that $d \geq 4$. Summing over $j$ from 1 to $[\log_2 \frac{N}{\epsilon}]$ we see for some $C'$ that
\[ \lambda_{d-1}(\{y \in M \Delta : \max_{i \in [d-1,d]} (R^n y)_i\}) \]
\[ \geq \frac{\epsilon}{N} \leq C' \epsilon^2 \lambda_{d-1}(\{y \in M \Delta : R^n y \in S_0(\frac{\epsilon^2}{N})\}) \leq C' \epsilon^2 \lambda_{d-1}(M \Delta). \]

We note that the result is slightly stronger than what is stated in the Proposition.

### 6.1 Proof of Proposition 6.4

Let \( \Lambda_{d-2} = \text{span}_\Delta(e_1, \ldots, e_{d-2}) \). Let \( \tilde{\Lambda} \) be a set of matrices where \( d \) and \( d-1 \) have not won and

- \( \|A\| \in [N, 2N] \) for all \( A \in \tilde{\Lambda} \)
- \( A \Delta \cap A' \Delta = \emptyset \) for \( A, A' \in \tilde{\Lambda} \) with \( A \neq A' \).

For example we could choose \( A \) maximal for \( 2N \) with the additional property that \( d \) and \( d-1 \) have not won.

The assumption that \( d-1 \) and \( d \) have not won implies \( \Lambda_{d-2} \subset \bigcup_{A \in \tilde{\Lambda}} A \Delta \).

We need the following lemma in the proof. In this lemma let \( \Lambda_{d-2} \) denote \( \text{span}_\Delta(e_1, \ldots, e_{d-2}) \).

#### Lemma 6.5

For all \( s, t \in [0, 1] \) and \( A \in \tilde{\Lambda} \) we have

\[ \lambda_{d-3}(A \Delta \cap \{x \in \Delta : x_{d-1} = t, x_d = s\}) \geq \lambda_{d-3}(V(A))(1 - 2(t + s)N)^{d-3}. \]

**Proof** Consider the simplex \( A \Delta \) as being made of codimension 2 slices parallel to \( V(A) \) (which is \( A \Delta \cap \Lambda_{d-2} \)). Let \( p_{d-1}(A) = \frac{C_{d-1}(A)}{|C_{d-1}(A)|} \), \( p_d(A) = \frac{C_d(A)}{|C_d(A)|} \) denote the two extreme points, of \( A \Delta \) that are disjoint from \( V(A) \). Every point of a slice parallel to \( V(A) \) has the form \( w + ap_{d-1} + bp_d \), where \( a + b \leq 1 \) are fixed and determine the slice and \( \frac{w}{|w|} \in V(A) \). Every side has length \( 1 - a - b \) times what it had in \( V(A) \). It follows that the volume of this slice is \( \lambda_{d-3}(V(A))(1 - a - b)^{d-3} \).

Now the \( d-1 \) and \( d \) entries of \( p_{d-1} \) and \( p_d \) are respectively at least \( \|A\|^{-1} = (2N)^{-1} \). Thus \( A \Delta \cap \{x \in \Delta : x_{d-1} = t, x_d = s\} \) is the set of points in \( A \Delta \) that have the form \( (1 - a - b)w + ap_{d-1} + bp_d \), where \( a \leq 2tN, b \leq 2sN \) and \( w \in V(A) \). The lemma follows. \( \square \)

**Proof of Proposition 6.4** We will consider the set

\[ \tilde{\Lambda} \Delta \cap (\cup_{c \leq \frac{\epsilon}{N}} \Delta_c). \]

The set in the proposition
\[ \{ x \in \Delta : x_{d-1} + x_d < \frac{\epsilon}{N} \text{ and } d - 1 \text{ or } d \text{ win and the corresponding matrix } \| A(x, m) \| < N \} \]

that we would like to show has small measure is contained in the complement of the above set.

We view \( \bar{\Delta} \cap (\cup_{c \leq \frac{\epsilon}{N}} \Delta_c) \) as being cut by codimension 2 planes parallel to \( \Lambda_{d-2} \). By the previous lemma \( \bar{\Delta} \) intersected with any such slice has measure at least \((1 - 2N \frac{\epsilon}{N})^{d-3} \lambda_{d-3}(\Lambda_{d-2})\). Since the volume of a slice in \( \Delta \) parallel to \( \Lambda_{d-2} \) is at most the volume of \( \Lambda_{d-2} \), we have that \( \bar{\Delta} \) occupies at least a \((1 - 2\epsilon)^{d-3}\) proportion of \( \cup_{c \leq \frac{\epsilon}{N}} \Delta_c \). Because for small \( s \) we have \((1 - s)^r = 1 - rs + O(s^2)\) the proposition follows with \( C = 2(d - 3) \) (since we may choose \( \epsilon \) small enough). \( \square \)

Later in the paper, we will need one additional result.

**Lemma 6.6** There exist \( c > 0 \) so that for \( k_0 \) large enough, given a matrix \( M = M(x, r) \) at beginning of freedom on LHS, for all \( y \) except for a subset of \( M \Delta \) of measure at most \( 10^{-c(k+k_0)^4} \lambda_{d-1}(M \Delta) \), there exists \( A(R^r y, m) \), a matrix of freedom on LHS such that

\[
\Theta(C_i(MA(R^r y, m), C_1(MA(R^r y, m)))) < 10^{-c(2k+k_0)^6} \quad (17)
\]

for \( i \leq d - 2 \) and

\[
\Theta(C_{d-1}(M), C_d(M)) < 10^{-(2k+1+k_0)^6}. \quad (18)
\]

**Proof** By Proposition 5.11 applied to \( \mathcal{R}_{d-2} \) we have that there exists \( \tau < 1 \) and \( c' > 0 \) so that for any positive matrix \( M \), for all but a set of measure \( 10^{-c'(k+k_0)^6} \lambda_{d-3}(V(M)) \) set of points \( y \in V(M) \) we have that there exists \( A(R^r y, m) \) with \( \| A(R^r y, m) \| < 10^{(k+k_0)^6-(k+k_0)^4+(k+k_0)^2} \) such that

\[
diam(V(MA(R^r y, m))) < \tau^{c'[((k+k_0)^6-(k+k_0)^4)]diam(V(M))}
\]

\[
< \tau^{c''(k+k_0)^6} diam((V(M)),
\]

with the last inequality holding for some \( c'' > 0 \). This implies that for all but a \( \tau^{-c''(k+k_0)^6} \) proportion of \( V(M) \) we have that the matrix given by freedom on the left hand side \( A \) has

\[
\max\{\Theta(C_i(MA), C_{i'}(MA)) : i, i' \leq d - 2\} < \tau^{c''(k+k_0)^6} \max\{\Theta(C_i(M), C_{i'}(M)) : i, i' \leq d - 2\}.
\]
Now set
\[ N = 10^{(k+k_0)^6-(k+k_0)^4+(k+k_0)^2} \]
and
\[ \frac{1}{\epsilon^2} = 10^{(k+k_0)^4-\sum_{i=1}^{k} 6(i+k_0)^2(2\xi)^{-k}}. \]

The ratio \( \frac{N}{\epsilon^2} \) gives a lower bound for ratio of column sizes as in (2) of Theorem 6.1. We now apply that Theorem. For a constant \( c \) we obtain inequality (17) for matrices of freedom that cover all but a proportion \( 10^c(k+k_0)^4 \) of \( M\Delta \). During restriction on the LHS, the angle between these columns can only get smaller and so we obtain inequality (17) for the remainder of LHS.

We now prove the bound on \( \Theta(C_{d-1}(M), C_d(M)). \) Let \( M' \) be the ancestor of \( M \) at the end of freedom on the right hand side. We have \( C_{d-1}(M) = C_d(M') + (b + 1)C_{d-1}(M') \) and \( C_d(M) = C_d(M') + bC_{d-1}(M') \) where \( b \geq 10^{(2k+1+k_0)^6+(k+k_0)^4} \). So we have (18).

\[ \square \]

7 Input and output singular direction

We will need to control the size of singular values and directions for the matrices arising from Rauzy induction. This will be necessary to control the geometry of simplices and of the intersection of planes with these simplices. This section is devoted to this endeavor and to define our family of parallel planes. Bounds on large singular values give bounds on small singular values, because our matrix preserves a (possibly degenerate) symplectic form. Before we begin our estimates we briefly describe this.

7.1 Symplectic

Let \( \Omega_\pi \) denote the (possibly degenerate) symplectic form, preserved by matrices of Rauzy induction from \( \pi \) to \( \pi \). The preserved subspace of the symplectic form is its image and is the orthogonal complement of its kernel. (See for example [23, Section 1.9].) Thus \( Im(\Omega_\pi) = \Omega_\pi(\mathbb{R}^k) \) is the preserved subspace of the symplectic form and we have the isomorphism
\[ \Omega_\pi|_{Im(\Omega_\pi)} : Im(\Omega_\pi) \rightarrow Im(\Omega_\pi). \]

Let \( \Omega_\pi^{-1} \) be defined as the inverse of the restricted map.
We use the following formula. For any path of Rauzy induction joining \( \pi \) to \( \pi' \) with matrix \( M \)

\[
M^T \Omega_\pi M = \Omega_{\pi'},
\]

(19)

where \( M^T \) denotes \( M \) transpose.

We now discuss the polar decomposition \( M = UP \) where \( U \) is unitary and \( P \) is positive definite. By the largest (resp. second largest) singular input vector \( v \) (resp. \( v' \)) we mean the vector \( v \) (resp. \( v' \) in the orthocomplement of \( v \)) expanded most by \( P \). The singular values are the expansion factors of these vectors. An output vector will denote the image under \( U \) of a singular input vector of \( M \).

Since \( P = U_1 DU_1^{-1} \) for some unitary \( U_1 \) and diagonal matrix \( D \) we have that \( M = UDV' \) with \( U' \) and \( V' \) unitary. Note that if \((v, a)\) is a pair consisting of a singular input direction and value of \( M \), then \((v, \frac{1}{a})\) is a pair consisting of singular output direction and value of \( M^{-1} \), and vice-versa. Also the singular values of \( M \) and \( M^T \) are the same. By the symplectic property we can relate singular input or output directions and values of \( M^{-1} \) to corresponding directions and values of \( M^T \). We obtain:

**Lemma 7.1** (i) If \((v, a)\) is a pair of singular input direction and singular values of \( M^{-1} \) then \((v, \frac{1}{a})\) is a pair consisting of singular output direction and value of \( M \).

(ii) If \((v, a)\) is a pair of singular input direction and singular value of \( M^{-1} \) and \( v \in \text{Im}(\Omega_\pi) \) then \((\Omega_\pi v, a)\) is such a pair for \( M^T \).

(iii) If \((\Omega_\pi v, a)\) is a pair of singular input direction and value of \( M^T \), then \((v, \frac{1}{a})\) is a pair of singular output direction and value of \( M \).

(iv) The singular values of \( M \) in the invariant subspace preserved by the symplectic form come in pairs \( a \) and \( \frac{1}{a} \).

7.2 Largest singular input and output vectors of \( M^T \)

By the previous section this will also tell us about the singular directions of \( M \). In this section we consider the singular decomposition of \( M^T = UP \). Let \( w \) the largest input vector.

Let \( W \) be the ortho-complement of \( w \), so \( w' \in W \) and \( \text{Proj}_W \) the orthogonal projection onto \( W \).

**Proposition 7.2** At the end of freedom on LHS, the largest and second largest singular input vectors \( w, w' \) of \( M^T \) satisfy

1. For all \( \epsilon > 0 \), if \( k_0 \) is large enough (depending on \( \epsilon \) ) then for every \( k \) we have \( \Theta(C_d(M), w) < 10^{-\left(\frac{3}{2} - \epsilon\right)(k+k_0)^4} \).
2. For all $\epsilon > 0$, if $k_0$ is large enough (depending on $\epsilon$) then for every $k$ we have $\Theta(\text{Proj}_W C_1(M), w') < 10^{-(\frac{1}{2} - \epsilon)(k+k_0)^4}$.

**Proof** We first note that the estimates on $U_k$ and $v_k$ in Proposition 4.1 imply that for all $\epsilon > 0$, for $k_0$ large enough, then for any matrix $M$ during freedom on LHS at stage $k$ we have that for all $i \leq d - 2$

$$10^{-(1+\epsilon)(k+k_0)^4} < \frac{|C_i(M)|}{|C_d(M)|} < 10^{-(1-\epsilon)(k+k_0)^4}. \quad (20)$$

We prove the first conclusion. It suffices to show that if $v$ is a unit vector so that $\Theta(v, C_d(M)) = 10^{-(\frac{3}{2} - 3\epsilon)(k+k_0)^4}$ then

$$|M^T v|_2 < |M^T \frac{C_d(M)}{|C_d(M)|_2}|_2 \quad (21)$$

Indeed, this establishes that there is a local maximum of the function $f : S^{d-1} \subset \mathbb{R}^d \to \mathbb{R}$ defined by $f(v) = |M^T v|$ within angle $10^{-(\frac{3}{2} - 3\epsilon)(k+k_0)^4}$ of $\frac{C_d(M)}{|C_d(M)|_2}$. However a local maximum is a global maximum. The reason is that at if $B$ is a basis of the subspace spanned by the vectors that have the maximal singular input vector then the local and global maxima are exactly points in $\text{span}(B)$.

Showing (21) is equivalent to showing that

$$\sum_{i=1}^{d} (C_i(M) \cdot v)^2 < \sum_{i=1}^{d} (C_i(M) \cdot \frac{C_d(M)}{|C_d(M)|_2})^2. \quad (22)$$

Now from the assumed equation for $\Theta(v, C_d(M))$ and the bound $\Theta(C_{d-1}(M), C_d(M)) \leq 10^{-(k+k_0)^6}$ (Lemma 6.6), we get, writing inner products in terms of cos and using Taylor’s expansion for cos $\theta$ that for $j = d - 1, d$ and $k_0$ large enough

$$(C_j(M) \cdot v) - (C_j(M) \cdot \frac{C_d(M)}{|C_d(M)|_2}) < -|C_j(M)|_1 (10^{-(\frac{3}{2} - 3\epsilon)(k+k_0)^4})^2.\quad \text{for } j \geq d - 1$$

Then for $j \geq d - 1$ and $i \leq d - 2$

$$|C_j(M) \cdot v|^2 - (C_j(M) \cdot \frac{C_d(M)}{|C_d(M)|_2})^2 < -|C_j(M)| \cdot |C_j(M)|_1 (10^{-(\frac{3}{2} - 3\epsilon)(k+k_0)^4})^2$$
there is a local maximum close to \( \text{Proj}_W \) by \( f \) is big enough, then for all \( i \)

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Let \( \hat{w} \) be the vector \( w \) be the vector 

\[
\hat{w} = \langle w_1 \cdot w_1, \ldots, w_1 \cdot w_{d-2}, 0, 0 \rangle.
\]
By Inequality (18) of Lemma 6.6

\[ |w_{d-1}| \leq 10^{-(2k+k_0)^6}|C_{d-1}|. \]  

(26)

Because \( w' \in W^\perp \), by the first conclusion of Proposition 7.2, for \( \epsilon > 0 \), for \( k_0 \) large enough,

\[ \Theta(w', W_d) \leq 10^{-\left(\frac{3}{2}-\epsilon\right)(k+k_0)^4}. \]  

(27)

By the second conclusion of Proposition 7.2, for all \( \epsilon > 0 \), for \( k_0 \) large enough,

\[ \Theta(w', w_1) \leq 10^{-\left(\frac{1}{2}-\epsilon\right)(k+k_0)^4}. \]  

(28)

These two angle bounds, the bounds \( \frac{|C_j(M)|}{|C_i(M)|} \leq 10^{(1+\epsilon)(k+k_0)^4} \) for \( j \geq d - 1 \) and \( i \leq d - 2 \) (inequality (20)) and the bound on \( |w_{d-1}| \) imply that

\[ |w_i \cdot w_1| \geq 10^{2(k+k_0)}|w_j \cdot w_1| \]

for all \( i \leq d - 2, j \geq d - 1 \). Now

\[ M^T w' = (C_1(M) \cdot w', \ldots, C_d(M) \cdot w'). \]

Putting this together we see that \( |M^T w' - \hat{w}| \leq 10^{-2(1-\epsilon)(k+k_0)|\hat{w}|} \). This implies

\[ \Theta(M^T w', \hat{w}) \leq 10^{-(1-\epsilon)(k+k_0)}. \]  

(29)

Using Lemma 6.6 which says the \( C_i(M) \) are exponentially close in angle to each other, we see that for all \( i, i' \leq d - 2 \),

\[ \left| \frac{|w_i \cdot w_1|}{|w_{i'} \cdot w_1|} - \frac{|C_i(M)|}{|C_{i'}(M)|} \right| \leq 10^{-\tilde{c}(k+k_0)^6}. \]

This estimate together with (29) implies the statement of the lemma. \( \square \)

In the next Lemma, by proportional we mean the ratio of two terms is bounded above and below by uniform constants.

**Lemma 7.4** At the end of freedom on LHS, the second largest singular value of \( M^T \) is proportional to \( |C_{\min}(M)| \).
Proof By Proposition 7.2 the second largest input singular direction \( w' \) is exponentially close to \( C_1(M) \). The proof of Lemma 7.3 says that \( M^T w' \) is exponentially close to \((w_1 \cdot w_1, \ldots, w_{d-2} \cdot w_1, 0, 0)\). By Theorem 4.2, \(|M^T w'|\) is proportional to \( \sum_{i=1}^{d-2} C_i(M) \cdot C_1(M) \). By Condition (1)* this quantity is proportional to \( (d-2)|C_i(M)| \cdot |C_1(M)| \) for any \( i \leq d-2 \). Dividing by \( |w'| \) this is proportional to \( |C_{\min}| \). (Because this in turn is proportional to \( |C_i(M)| \) for all \( i \leq d-2 \).) □

The next result is used in Sect. 11.

**Proposition 7.5** At the start of freedom on RHS

1. the largest singular value of \( M^T \) is proportional to \(|C_{\max}(M)|\).
2. The second largest is smaller than \( 10^{-(k+k_0)^5} \|M\| \).

At the end of freedom on the RHS,

3. the largest is proportional to \(|C_{\max}(M)|\) and
4. the second largest is at least a constant multiple of \(|C_{\min}(M)|\).

Proof The claims (1) and (3) are trivial. We now prove (2). Inequality (17) of Lemma 6.6, implies that for a constant \( c > 0, \Theta(C_i(M), C_{i'}(M)) < 10^{-c(2k+k_0)^6} \) for all \( i, i' \leq d-2 \). Also by the bounds in Sect. 4 we see we have that

\[
\max\left\{ \frac{|C_j(M)|}{|C_i(M)|} : i \leq d-2, \ j \geq d-1 \right\} \leq \frac{2v_{k-1}}{u_k} \leq 10^{-\frac{1}{2}(k+k_0)^6},
\]

for \( k_0 \) big enough. (Indeed, it is easy to see the much worse than optimal bound that the norm of \( C_{d-1} \) and \( C_d \) increase by less than a factor of 2 from the end of restriction on the RHS to the start of freedom on the RHS.) So analogously to the first conclusion of Proposition 7.2 for some \( c' \), we have that the angle the largest singular input vector makes with \( C_1(M) \) is less than \( 10^{-c'(k+k_0)^6} \). It follows that \( M^T \) restricted to the ortho-complement of the direction of largest singular vector is less than \( 10^{-(k+k_0)^5} \|M\| \) (if \( k_0 \) is large enough). The claim of the second largest singular value follows.

The last conclusion (4) follows analogously to above and we sketch it. Analogously to Proposition 7.2 we have that the angle the top singular input vector makes with \(|C_1(M)|\) is smaller than \( 10^{-(\frac{3}{2}-\epsilon)(k+k_0)^4} \) (if \( k_0 \) is large enough). From this it follows that the operator norm of \( M^T \) on the orthocomplement of the largest input singular vector is at least \( \max\{10^{-(\frac{3}{2}-\epsilon)(k+k_0)^4} |C_i(M)|, |C_j(M)| : i \leq d-2, \ j \geq d-1 \} \geq |C_{\min}(M)| \). □
7.3 Small singular input and output directions of $M$ and choice of planes

For this section let $\Delta' = (\{x \in \Delta : x_{d-1} = x_d = 0\}, \pi_L)$.

We can view $\Delta'$ as the $d - 3$ dimensional simplex parametrizing interval exchange transformations on the first $d - 2$ letters. As a result we can let $\tilde{R} : \Delta' \rightarrow \Delta'$ be the first return of Rauzy induction to this set. Note that almost every point in $\Delta'$ has that $R^k$ is defined on it for all $k$. Indeed, $d$ and $d - 1$ will always lose when compared to other symbols and not change the length of those symbols. Also $d$ and $d - 1$ will never be compared (they will only be compared to the symbol 1. Let $\tilde{A}(x, 1)$ be the corresponding incidence matrix. Let $\tilde{A}(x, n + 1) = \tilde{A}(x, n)\tilde{A}(R^n x, 1)$. This is a transpose cocycle.

This section is devoted to the proof of the following Proposition. It says that we can arrange things so that a certain face always has volume at least comparable to the volumes of other faces. Its proof will be at the end of the section after several preliminaries.

**Proposition 7.6** There exists $c > 0$, $N$ so that for any $M$ there exists $x \in \Delta'$, and $n$ so that $\|A(x, n)\| < N$ and

$$\lambda_{d-2}(F_1(MA(x, n))) > c \max_{1 \leq d-2} \lambda_{d-2}(F_i(MA(x, n))).$$

**Lemma 7.7** If $M$ is a matrix of Rauzy induction during freedom on LHS, $w$ is the smallest singular input direction of $M$, $\omega$ the corresponding singular value and $A$ is a matrix of LHS with $\|A\| < 10^{(k + k_0)^3}$ then, if $k_0$ is large enough, the smallest singular direction of $MA$ makes angle at most $10^{1/9(k + k_0)^4}$ with $A^{-1}w$.

**Proof** For any vector $v$ express it as $v = v' + cw$ where $v' \perp w$. Because $w$ is an input singular direction, $(Mw)^\perp = M(w^\perp)$ and so $|Mv|^2 = |Mv'|^2 + c^2|Mw|^2$. Now, if the angle between $v$ and $w$ is at least $10^{1/9(k + k_0)^4}$ then

$$|Mv| > 10^{1/2(k + k_0)^4}\omega|v|.$$ (30)

This follows from Lemma 7.4 and the fact that the smallest singular value of $M$ has size proportional to $\frac{1}{|c_{max}(M)|}$ and so we have that for any singular value $\sigma \neq \omega$ of $M$, we have

$$\sigma \geq 10^{1/2(k + k_0)^4}\omega.$$
Let $u$ be a smallest unit input singular vector of $MA$, and so

$$|MAu| \leq |MA A^{-1}w| |A^{-1}w|.$$ 

We claim then

$$\Theta(w, Au) < 10^{-\frac{1}{8}(k+k_0)^4}. \quad (31)$$

To prove (31) notice first that our matrix is symplectic so $|Au| \geq \|A\|^{-1}$. Now if (31) is false, then by by the bound on $\|A\|$ and (30)

$$|MAu| > \|A\|^{-1} 10^{\frac{1}{8}(k+k_0)^4} \omega > \|A\| \omega \geq |MA A^{-1}w| |A^{-1}w|,$$

a contradiction proving (31). We finish the proof of the Lemma. If $\Theta(u, A^{-1}w) > 10^{-\frac{1}{8}(k+k_0)^4}$, then $\Theta(Au, w) > 2\|A\|^{-2} 10^{-\frac{1}{8}(k+k_0)^4} > 10^{-\frac{1}{8}(k+k_0)^4}$ (for all $k_0$ large enough), a contradiction to (31) and so we have the lemma. \(\square\)

**Definition 7.8** Given a matrix valued cocycle or transpose cocycle $A(x, n)$ we say a subspace $W$ is right invariant if $AW = W$ for all $A$ that can occur as matrices of the cocycle.

A key tool in the proof of Proposition 7.6 is the following theorem which shows that the Rauzy cocycle has few invariant subspaces.

**Theorem 7.9** (Avila–Viana [5, Corollary 5.2]) For any permutation $\pi \in R_{d-2}$, $v \in \mathbb{R}$ a one dimensional subspace of $\text{Im}(\Omega_\pi)$ and $W$ a codimension 1 subspace of $\text{Im}(\Omega_\pi)$ we have that there exists a matrix $M$ corresponding to a path from $\pi$ to $\pi$ so that $Mv \notin W$.

This directly follows from Avila–Viana’s Theorem that the Rauzy monoid twists subspaces of the preserved subspace of the symplectic form (which is part of the statement that its action on this subspace is simple).

We wish to apply this result to matrices of freedom on the LHS in $R_d$ and subspaces of $e_1 \oplus \cdots \oplus e_{d-2}$. To do this, note that these matrices have corresponding matrices of $R_{d-2}$ and the action on the symbols $1, \ldots, d-2$ are the same for both of them. Let $\pi_0$ be the symmetric permutation for $R_{d-2}$. Let $K_{d-2}$ be the subspace of $e_1 \oplus \cdots \oplus e_{d-2} \subset \mathbb{R}^d$ that corresponds to the kernel of $\Omega_{\pi_0}$ in $R_{d-2}$. (This is $\{0\}$ if $d-2$ is even and $(1, -1, \ldots, -1, 1, 0, 0)$ if $d-2$ is odd.) Similarly let $I_d$ be the image of the symplectic form preserved by Rauzy induction from $\pi_L$ to $\pi_L$ and $I_{d-2}$ be the subspace of $e_1 \oplus \cdots \oplus e_{d-2} \subset \mathbb{R}^d$ that corresponds to the image of $\Omega_{\pi_0}$ in $R_{d-2}$.
Lemma 7.10 W is right invariant for the cocycle \( A(x, n)^{-1} \) if and only if \( W \) is right invariant for the transpose cocycle \( A(x, n) \).

Proof W is right invariant for \( A(\cdot, \cdot)^{-1} \) if and only if \( A(x, n)^{-1} W = W \) for all \( x, n \). This is if and only if \( A(x, n) W = W \) for all \( x, n \), so \( W \) is right invariant for \( A(\cdot, \cdot) \).

\[ \]

Lemma 7.11 The only nontrivial right \( \tilde{A}(\cdot, \cdot) \)-invariant subspace contained in \( e_2 \oplus \cdots \oplus e_d \) is \((e_{d-1} - e_d)\mathbb{R}\).

Note that we will treat more general right invariant subspaces in the proof.

Proof First, we have that \( I_{d-2}, K_{d-2} \) and \((e_{d-1} - e_d)\mathbb{R}\) are invariant subspaces. Let \( B_1 \) be a basis for \( I_{d-2} \), \( B_2 \) be a basis for \( K_{d-2} \) and write \( A(x, n) \) in the basis \( B_1 \cup B_2 \cup \{e_{d-1} - e_d\} \cup \{e_{d-1} + e_d\} \). It suffices to show that all of these matrices have the following form:

- The columns corresponding to \( B_1 \) have that the entries not corresponding to elements of \( B_1 \) are zero. Moreover, there are no proper right invariant subspaces for the cocycle in this block.
- The cocycle is diagonal on \((e_{d-1} - e_d)\) and \( K_{d-2} \).
- \( A(x, n)(e_{d-1} + e_d) = v_n + e_{d-1} + e_d \) where \( v_n \in e_1 \oplus \cdots \oplus e_{d-2} \) and can have arbitrarily large norm.

To see why the bullets suffice, the first and second bullets imply that any invariant subspace non-trivially intersecting \( e_1 \oplus \cdots \oplus e_{d-2} \) is either \( K_{d-2} \), \( I_{d-2} \) or \( e_1 \oplus \cdots \oplus e_{d-2} \). The second and third bullets say that \((e_{d-1} - e_d)\mathbb{R}\) is the only non-trivial subspace that only trivially intersects \( e_1 \oplus \cdots \oplus e_{d-2} \). The intersection of an invariant subspace and \( e_1 \oplus \cdots \oplus e_{d-2} \) is an invariant subspace contained in \( e_1 \oplus \cdots \oplus e_{d-2} \). If it is non-trivial, it falls in the previous list of such subspaces, none of which are contained in \( e_2 \oplus \cdots \oplus e_d \).

The first bullet follows by Avila–Viana’s Theorem 7.9. The second is by how our cocycle acts on \( K_{d-2} \) and the fact that the last two columns of \( A(x, n) \) in the standard basis are always the same. The final bullet is because (in the standard basis), \( C_j(\tilde{A}(x, 1)) = C_1(\tilde{A}(x, 1)) + e_j \) for \( j \geq d - 1 \).

Lemma 7.12 If there exists a subspace \( W \) and a vector \( v \) so that \( \tilde{A}(x, n)^{-1} v \in W \) for all \( x, n \) then there is an \( \tilde{A}(\cdot, \cdot)^{-1} \) invariant subspace \( V \subset W \) such that \( \tilde{A}(x, n)^{-1} v \in V \) for all \( x, n \)

Proof First note that by how our cocycle acts for any \( x, y, n, m \) there exists \( z \) so that \( \tilde{A}(z, n + m)^{-1} = \tilde{A}(x, n)^{-1} \tilde{A}(y, m)^{-1} \). This implies that \( \tilde{A}(x, n)^{-1} \tilde{A}(y, m)^{-1} v \in W \) for all \( x, y, n, m \). This implies that \( \tilde{A}(x, n)^{-1} v \in \text{span}\{\tilde{A}(y_1, m_1)^{-1} v, \ldots, \tilde{A}(y_k, m_k)^{-1} v\} \subset W \) for all \( x, n, y_1, \ldots, y_k, m_1, \ldots, m_k \). The result follows with \( V \) being the span of the images of \( v \) under the cocycle.
**Corollary 7.13** There exists a finite set \( \tilde{A}(x_1, n_1), \ldots, \tilde{A}(x_k, n_k) \) so that for any vector \( v \) and subspace \( W \) if \( v \) is not contained in any invariant subspace of \( W \) there exists \( \tilde{A}(x_i, n_i) \) such that \( \tilde{A}(x_i, n_i)^{-1}v \notin W \). Moreover by compactness of the space of subspaces minus an \( \epsilon \)-neighborhood of the invariant subspaces.

**Proof of Proposition 7.6** Again let \( w(M) \) be the smallest singular input direction of \( M \) and let \( w'(M) \) be the second smallest. Fix \( \epsilon_0 > 0 \) small. Let \( c' \) be the constant given by Corollary 7.13, \( A(x_1, n_1), \ldots, A(x_k, n_k) \) be the matrices and \( N = \max_{\ell \leq k} |A(x_\ell, n_\ell)| \). The proof is split into 2 cases:

*Case 1:* \( \Theta(w(M), e_d - e_d) > \epsilon_0 \). We assume that \( k_0 \) is large enough so that \( c' > 9 \cdot 10^{-k_0} \) and \( N < 9 \cdot 10^{k_0} \).

By Corollary 7.13 there exists \( x, n \) with \( \|A(x, n)\| < N \ll 10^{k+k_0} \) (if \( k_0 \) is large enough) so that

\[
\Theta(A^{-1}(x, n)w(M), e_2 \oplus \cdots \oplus e_d) > c'
\]

where \( \|A(x, n)\| < N \). Indeed, if \( w(M) \) is within \( \epsilon_0 \) of an invariant subspace \( U \), we apply the corollary to \( W = U \cap e_2 \oplus \cdots \oplus e_d \). Note that by Lemma 7.7 this implies that

\[
\Theta(w(MA(x, n)), e_2 \oplus \cdots \oplus e_d) > c' - 10^{-\frac{1}{2}(k+k_0)^4} > \frac{1}{2}c'. \tag{32}
\]

Now let \( Conv_i \) be the convex hull of

\[
\frac{C_1(MA(x, n))}{|C_1(MA(x, n))|}, \ldots, \frac{C_{i-1}(MA(x, n))}{|C_{i-1}(MA(x, n))|}, \frac{C_{i+1}(MA(x, n))}{|C_{i+1}(MA(x, n))|}, \ldots, \frac{C_d(MA(x, n))}{|C_d(MA(x, n))|},
\]

which is span of \( \lambda_i |MA(x, n)|, \ldots, C_{i+1}|MA(x, n)|, C_d(MA(x, n)) \). So if \( \sigma_1 \geq \cdots \geq \sigma_d \) are the singular values of \( MA(x, n) \) we have by (32) that \( \lambda_{d-2}(Conv_1) \) is at least proportional (in terms of \( c' \)) to

\[
\sigma_1 \cdot \sigma_2 \cdot \cdots \cdot \sigma_{d-1} \xi^{d-1} |C_1(MA(x, n))|^{-d+3} |C_d(MA(x, n))|^{-2} \tag{33}
\]

which is the largest the volume of a face can be. Indeed, \( Conv_i \) is the image of the convex hull of \( e_i \) under the linear action of \( MA(x, n) \). The largest it can be is if this set contains a vector parallel to the \( d - 1 \) largest singular directions which gives the estimate in (33).
Case 2: $\Theta(w(M), e_{d-1} - e_d) < \epsilon_0$.

Since the largest singular input direction of $M$ is very close to $e_{d-1} + e_d$, it follows that

$$
\Theta(w'(M), e_{d-1} \oplus e_d) = \frac{\pi}{2} - C\epsilon_0
$$

for some $C$. Indeed, $\omega'(M)$ is in the orthocomplement of $\omega(M) \oplus v$ where $v$ is the largest input singular direction. Since the last two columns are much larger than the first two, $v$ makes a small angle with $e_{d-1} \oplus e_d$, establishing (34).

Let $A(x, n)$ be given by Corollary 7.13 for $w'(M)$ and $W = U \cap e_2 \oplus \cdots \oplus e_d$, where $U$ is the smallest invariant subspace $w'(M)$ is contained in. We may apply the corollary because $w'(M)$ makes a definite angle with $e_{d-1} - e_d$ (since it is perpendicular to $w(M)$ which makes a small angle with $e_{d-1} - e_d$) and so by Lemma 7.11, $U \cap e_2 \oplus \cdots \oplus e_d$ is not invariant. We now control the smallest singular direction: Because $A(x, n)$ acts as the identity on $e_{d-1} - e_d$ and $\|A\| \leq N$, we have

$$
\Theta(w(MA(x, n)), e_{d-1} - e_d) \in \left[ \frac{\Theta(w(M), e_{d-1} - e_d)}{CN}, CN\Theta(w(M), e_{d-1} - e_d) \right]
$$

and so for all $i \leq d - 2$ we have

$$
\Theta(w(MA(x, n)), e_1 \oplus \cdots \oplus e_{i-1} \oplus e_{i+1} \oplus \cdots \oplus e_d)
\in \left[ \frac{\Theta(w(M), e_{d-1} - e_d)}{CN}, CN\Theta(w(M), e_{d-1} - e_d) \right].
$$

From this we have Proposition 7.6 with the constant comparable to $(CN)^2(c')^{-1}$. Indeed, as before

$$
\lambda_{d-2}(Conv_1) \geq \tilde{c}\sigma_1 \cdots \sigma_{d-2} \left( \frac{\Theta(w(M), e_{d-1} \oplus e_d)}{CN} \sigma_{d-1} + \sigma_d \right)
\zeta^{-d+1}|C_1(MA(x, n))|^{-d+3}|C_d(MA(x, n))|^{-2}
$$

which is at least comparable to

$$
\lambda_{d-2}(Conv_i) \leq \sigma_1 \cdots \sigma_{d-2} (\Theta(w(M), e_{d-1} \oplus e_d)CN\sigma_{d-1} + \sigma_d)
\zeta^{d-1}|C_1(MA(x, n))|^{-d+3}|C_d(MA(x, n))|^{-2}
$$

for $i \leq d - 2$. \qed
7.4 Diameters and choice of planes

In this section we will define the family of parallel planes which we will intersect with simplices to compute diameters, areas and so forth. We now explain our choices of parallel planes. The regions we are interested in are projective images of simplices under matrices with very different singular values. We have good control of the top two (and by symplecticity) and bottom two singular values. We wish to cut our projective images by planes so that the dimension of the intersection of the projective images and the planes are governed by the bottom two singular values. This holds, unless the planes contain a vector that makes small angle with the span of the codimension 2 hyperplane spanned by all but the two smallest singular output vectors. It is easier to control large singular values than small singular values and so we find our small singular output values as large singular input values of the inverse matrix. These can be found from large singular input values of the transpose matrix via the symplectic form. We will need to discuss the inverse of a possibly degenerate symplectic matrix \( \Omega_\pi L \). When we talk about the inverse it is the inverse of \( \Omega_\pi L \) restricted to \( \text{Im}(\Omega_\pi L) \) (the orthocomplement of its kernel). Let \( P_{\pi L} = \Omega_\pi^{-1} L \Omega_\pi L \) denote orthogonal projection to \( \text{Im}(\Omega_\pi L) \).

Recall \( \Delta_c \) is the subset of \( \Delta \) defined by \( x_{d-1} + x_d = c \). Fix \( c \). Let \( A_1', B_1 \) be the first matrices of restriction on LHS and freedom on the RHS. Let \( u \) be the projection of \( \Omega_\pi^{-1} L P_{\pi L} (C_1(A_1'B_1)) \) to \( \Delta_c \cap \Omega_\pi^{-1} L P_{\pi L} (A_1'B_1)^\perp \).

Let \( v \) be the projection of \( \Omega_\pi^{-1} L P_{\pi L} (C_d(A_1'B_1)) \) to \( \Delta_c \cap \Omega_\pi^{-1} L P_{\pi L} (A_1'B_1)^\perp \). These are the \( \Omega_\pi^{-1} L \) images of vectors not too close to the orthocomplement of the top two singular input vectors of \( (A_1'B_1)^T \) and we will see are therefore not too close to the orthocomplement of the bottom two singular output vectors of the matrices referenced in Theorem 7.14 below. Consider the plane \( P_0 \) defined by

\[
P_0 = u \oplus v.
\]

Our family of planes \( \mathcal{P} \) is the family parallel to \( P_0 \). Since the \( \Delta_c \) are parallel for different \( c \), \( \mathcal{P} \) does not depend on \( c \).

In the next theorem \( \omega' \) denotes the second smallest singular value of \( M \).

**Theorem 7.14** There are positive constants \( c_1, c_2 \) such that if \( k_0 \) is large enough, and \( M \) is at the end of freedom on the left hand side or at the end of freedom on right side then there is a plane \( P_0 \in \mathcal{P} \) which slices \( M \Delta \) into a polygon \( Q \) so that

\[
\frac{c_1 \omega'(M)}{|C_{\max}(M)|} \leq \text{diameter}(Q) \leq \frac{c_2}{|C_{\min}(M)|^2},
\]

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We will need a couple of preliminary lemmas.

**Lemma 7.15** Let $M$ be as in Theorem 7.14. Then for any $w \in P_0$ \[|M^T \Omega_{\pi L} w| \geq \frac{|C_{\min}(M)|}{100} |w| \]

**Proof** Let

\[x = \sum_{i=1}^{d-2} C_i(M) \cdot \Omega_{\pi L}(u) \text{ and } y = \sum_{j=d-1}^{d} C_j(M) \cdot \Omega_{\pi L}(v).\]

Similarly let

\[x' = \sum_{j=d-1}^{d} C_j(M) \cdot \Omega_{\pi L}(u) \text{ and } y' = \sum_{i=1}^{d-2} C_i(M) \cdot \Omega_{\pi L}(v).\]

Now by Theorem 4.2 we may assume that $\Omega_{\pi L} u$ is as close as we want to $e_1 \oplus \cdots \oplus e_{d-2}$ and similarly for $\Omega_{\pi L} v$ and $e_{d-1} \oplus e_d$. Moreover, by Condition ** the angles of $C_i(M)$ are close to each other for $i \leq d - 2$ and $C_{d-1}(M)$ and $C_d(M)$ are close to each other. It follows that for $k_0$ big enough

\[\frac{x}{y'}, \frac{y}{x'} > 100.\]

It follows that for any $w = \alpha u + \beta v \in P_0$ that

\[|M^T \Omega_{\pi L}(\alpha u + \beta v)| \geq \max\{|\alpha x + \beta y'|, |\alpha x' + \beta y'|\} > \frac{1}{2} \max\{\alpha x, \beta y\}.\]

The lemma follows from the fact that $x$ and $y$ are at least proportional to $C_{\min}(M)$. Indeed, $\Omega_{\pi L} u$ is not close to being perpendicular to $C_i(M)$ for $i \leq d - 2$ and $\Omega_{\pi L} v$ is not close to being perpendicular to $C_j(M)$. \(\square\)

We next prove a result on relating the projective action and the linear action. For the purposes of clarity in the next lemma, let $\hat{M}$ denote the projective action of $M$ and $\tilde{M}$ denote its linear action. This is local notation that is only used in this lemma and its proof. Let the pairs (singular value, direction) of $M$ be $(\gamma_1, \theta_1), \ldots, (\gamma_d, \theta_d)$ ordered so that $\gamma_i \geq \gamma_{i+1}$.

**Lemma 7.16** If $v, w \in \Delta$ are such that $v - w$ is in the direction $\theta_k$, then

\[\frac{|v - w|}{|C_{\max} M|} \leq d(\hat{M} v, \hat{M} w) \leq \frac{2\pi \gamma_k}{|C_{\min}(M)|}.\]
The set of non-uniquely ergodic $d$-IETs

**Proof** First the lower bound. Let $u, v, w \in \mathbb{R}^d_+$ so that $\tilde{M}u = \tilde{v}$ and $\tilde{M}u = \tilde{w}$. Now $|u|, |u| \geq \frac{1}{|C_{\max}(M)|}$ and $|u - u| \leq \gamma_d$ (by the definition of the smallest singular value). Then

$$|\hat{M}(v) - \hat{M}(u)| = |\tilde{M}(u) - \tilde{M}(w)| \geq \gamma_d |u - u| \geq \gamma_d |v - w|/|C_{\max}(M)|.$$

Now we prove the upper bound. Consider the line $\ell$ through the origin and $\tilde{M}w$. Take the closest point, denoted $t\tilde{M}w$ on $\ell$ to $\tilde{M}v$. Then

$$|t\tilde{M}w - \tilde{M}v| \leq |\tilde{M}v - \tilde{M}w| = \gamma_k |v - w| \leq 2\gamma_k.$$

We consider the right triangle with vertices at the origin, $t\tilde{M}w$ and $\tilde{M}v$ (the hypotenuse is the line segment from the origin to $\tilde{M}v$). Now we want to know the angle $\psi$ the hypotenuse makes with the line from the origin to $t\tilde{M}w$. We have $|\tilde{M}v| \geq |C_{\min}(M)|$ so $\sin(\psi) \leq \frac{2\gamma_k}{|C_{\min}(M)|}$. Since $\psi \leq \frac{\pi}{2}$ we have $\psi \leq \frac{\pi\gamma_k}{|C_{\min}(M)|}$. Combining these inequalities we have $\psi < \frac{\pi\gamma_k}{|C_{\min}(M)|}$ and changing the angle between two vectors to the distance between the corresponding unit vectors gives us the result after multiplying by an additional factor of 2.

**Proof of Theorem 7.14** By Lemma 7.15 and the fact that $M$ is symplectic, it follows that for $w \in \mathcal{P}$ we have

$$|M^{-1}(w)| = |\Omega^{-1}_{\pi_L} M^T \Omega_{\pi_L}(w)| \geq |C_{\min}(M)| \cdot |w|/100.$$

This says that $w$ makes a definite angle with the space perpendicular to the singular input vectors for $M^{-1}$ with singular value at least $|C_{\min}(M)|/200$ and so makes definite angle with the perpendicular to the singular output vectors of $M$ with singular value at most $|C_{\min}(M)|/200$.

Now the directions of the plane are fixed and therefore by the above remark the diameter of the image under the linear action is bounded above by a multiple of $\frac{1}{|C_{\min}(M)|}$. By Lemma 7.16 the image under the projective action has diameter bounded above by a constant multiple of $\frac{|C_{\min}(M)|}{|C_{\max}(M)|}$. By Lemma 7.16 the contraction for the projective action is at most the second smallest singular value $\omega'$ multiplied by $\frac{1}{|C_{\max}(M)|}$. Thus the major axis of the ellipse is at least proportional to $\frac{\omega'}{|C_{\max}(M)|}$. \qed
8 Geometry of slices on LHS and illumination

The next two sections are interconnected. The results used outside of these two sections are Theorem 8.2, Corollary 8.6 [which is used to prove Condition * (1)] and Remark 9.8 which is quoted in Sect. 12.1.

The next two sections control the geometry of simplices during freedom on LHS. This is used in Sect. 10 to show that even though we lose most of the measure during restriction, we keep enough (in all but an exponentially small proportion of planes) to verify the assumptions of Theorem 3.1. The current section shows that as a first step, if we have a simplex at the beginning of freedom on LHS we can find a fixed finite collection of subsimplices which intersect the planes in $\mathcal{P}$ nicely. What this means is that for every point in the subsimplex, the plane through it intersects the face $F_1$, which is the image under the ancestor matrix of the face $\{x_1 = 0\} \subset \Delta$. We say the point is illuminated. We do this, because under restriction on LHS, the interval $I_1$ always loses so our future simplices lie in a neighborhood of this face. The following section, Sect. 9, shows that we may iterate this argument so that all but an exponentially small proportion of the points that we still have at the end of restriction on RHS have this illumination property. Then Sect. 10 deals with restriction.

Given any freedom LHS matrix $A$ at stage $k + 1$, and sequence $A_1', T_1, B_1, B_1', A_2, A_2', \ldots, B_k$ through freedom on RHS at stage $k$, then for each matrix of restriction on RHS $B_k'$ at stage $k$, set

$$M_{B_k'} = A_1'T_1B_1B_1' \ldots B_kB_k'A,$$

to be the product of matrices and set

$$\mathcal{M}_A = \{M_{B_k'}\}$$

to be the collection as $B_k'$ varies. Observe that one can order $\mathcal{M}_A$ by ordering the $C_{d-1}(M_{B_k'})$ for $M_{B_k'} \in \mathcal{M}_A$ (they all lie on a line). In a mild abuse of notation set

$$\mathcal{M}_A \Delta = \bigcup_{M_{B_k'} \in \mathcal{M}_A} M_{B_k'} \Delta \cap (\bigcup_{c \in (1,9)} \Delta_c).$$

**Lemma 8.1** For any two matrices $M_1, M_2 \in \mathcal{M}_A$, the column lengths $|C_d(M_1)|$ and $|C_d(M_2)|$ are uniformly comparable. The same holds for $|C_{d-1}(M_1)|$ and $|C_{d-1}(M_2)|$.

**Proof** After finishing freedom on RHS we have columns $C_{d-1}, C_d$. Then during restriction on RHS columns $d - 1$ and $d$ are of the form $C_{d-1}$ and $C_d$. Observe that $C_{d-1}$ and $C_d$ are uniformly comparable, and that $|C_{d-1}(M_1)|$ and $|C_d(M_2)|$ are also uniformly comparable. Therefore, $|C_{d-1}(M_1)|$ and $|C_d(M_2)|$ are uniformly comparable.
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\[ C_d + sC_{d-1}, \text{ where } s \in [s_k, 2s_k], \text{ (and } s_k = 10^{(2k+2+k_0)^6}+(k+k_0)^4 \). Thus any pair of columns \( C_d \) have comparable size. This holds even after restriction ends on RHS when \( C_d \) is added to \( C_{d-1} \). This is also true of columns \( C_{d-1} \).

Then during freedom on LHS, the same vectors are added to both \( C_{d-1} \) and \( C_d \). Thus they remain comparable in size.

⊓⊔

Now let \( \phi \) be the direction of the vector \( u \) defined in Sect. 7.4 as one of the pair of vectors defining the plane \( P \). For any freedom LHS matrix \( A \), and \( M \in \mathcal{M}_A \) let

\[ S_\phi(M) = \{ y \in M\Delta \text{ such that there exists a line in direction } \phi \text{ joining } y \text{ to a point in } F_1(M) \}
\]

and

\[ S_\phi(\mathcal{M}_A) = \bigcup_{M \in \mathcal{M}_A} S_\phi(M). \]

We call \( S_\phi(\mathcal{M}_A) \) the set illuminated by \( \mathcal{M}_A \). The point of this definition is that when we move to restriction on LHS, the first entry does not win and so our simplices will have a face contained in \( F_1 \). We want the lines to intersect those simplices; hence the face \( F_1 \). The main Theorem below says that except for an exponentially small set, at each stage \( k \) we can insure all points are illuminated by some \( \mathcal{M}_A \) for

\[ \|A\| \in [10^{(2(k+1)+k_0)^6}-(k+k_0)^4, \ 10^{(2(k+1)+k_0)^6}-(k+k_0)^4+(k+k_0)^2]. \]

\textbf{Theorem 8.2} There is \( \rho < 1 \) so that for all \( k_0 \) large enough and \( c \in [.1, .9] \), given \( M \) and \( \hat{A} \) in freedom on LHS at stage \( k+1 \), there are matrices \( A_1, \ldots, A_\ell \) in freedom on LHS such that

\[ \lambda_{d-2} \left( \bigcup_{j=1}^\ell S_\phi(\mathcal{M}_{\hat{A}A_j}) \cap \Delta_c \right) > (1 - \rho^{(k+k_0)^{1.1}})\lambda_{d-2}(\mathcal{M}_{\hat{A}}(\Delta) \cap \Delta_c). \]

The Theorem will be proved in two stages; the first step in this section and the second in the next. The remainder of this section is devoted to proving (with preliminaries) Proposition 8.5 which says that we obtain a definite proportion of \( \mathcal{M}_{\hat{A}}\Delta \) in the illuminated set.

\textbf{Definition 8.3} Given \( C_0, \theta_0 > 0 \) we say a set of matrices \( \mathcal{M} \) is LHS \((\phi, C_0, \theta_0)\)-ready for illumination if

1. \( \bigcup_{M \in \mathcal{M}} M\Delta \) is a single simplex.
2. \( \lambda_{d-2} \bigcup_{M \in \mathcal{M}} F_1(M) \) is at least \( C_0 \) times the volume of the largest face of \( \bigcup_{M \in \mathcal{M}} M \Delta \).

3. \( \Theta(\bigcup_{M \in \mathcal{M}} F_1(M), \phi) > \theta_0 \).

Let \( N \) be the maximum of the constant in Proposition 7.6 and the norms of the matrices \( \tilde{A}(x_i, n_i) \) in Corollary 7.13.

**Proposition 8.4** There exists \( \theta_0, C_0, > 0 \) so that for any matrix \( \hat{A} \) of freedom on LHS with \( \| \hat{A} \| < \frac{1}{N} 10^{(2k+k_0)^6-(k+k_0)^4+(k+k_0)^{2.3}} \) there exists some \( \tilde{A} \) with \( \| \tilde{A} \| \leq N \) such that the family \( \mathcal{M}_{\hat{A} \tilde{A}} \) is \((\phi, C_0, \theta_0)\) ready for illumination.

**Proof** We first show that for each matrix of freedom on LHS \( \tilde{A} \), \( \mathcal{M}_{\hat{A} \tilde{A}} \) satisfies condition (1).

Note that since \( \hat{A} \) is a LHS matrix and \( B'_k \) is a matrix of restriction on RHS, the first \( d - 2 \) columns are not effected by \( B'_k \). This implies that for \( i \leq d - 2 \), these columns \( C_i(A'_1, \ldots, B_k B'_k \hat{A} \tilde{A}) \) do not depend on \( B'_k \). The next observation is that for fixed \( A'_1, \ldots, B_k \),

\[
\bigcup_{B'_k} \text{span}_\Delta \left( C_{d-1}(A'_1, \ldots, B_k B'_k), C_d(A'_1, \ldots, B_k B'_k) \right)
\]

is a line segment. Indeed setting

\[
s_k = 10^{(2k+2+k_0)^6+(k+k_0)^4}
\]

applying the matrix \( B'_k \) means that the \( C_{d-1} \) column is added to the \( C_d \) column between \( s_k \) and \( 2s_k \) times. Taking the union over all these possible times gives a line segment

\[
\text{span}_\Delta \left( s_k C_{d-1}(A'_1, \ldots, B_k) + C_d(A'_1, \ldots, B_k), 2s_k C_{d-1}(A'_1, \ldots, B_k) + C_d(A'_1, \ldots, B_k) \right) + C_d(A'_1, \ldots, B_k).
\]

Now since \( \hat{A} \tilde{A} \) is a LHS matrix, each time it adds to \( C_j(A_1, \ldots, B_k B'_k) \) for \( j \in \{d - 1, d\} \) it adds the same vector to both. So \( \bigcup_{B'_k} \text{span}_\Delta \left( C_{d-1}(A'_1, \ldots, B_k B'_k \hat{A} \tilde{A}), C_d(A'_1, \ldots, B_k B'_k \hat{A} \tilde{A}) \right) \) is the result of adding the same vector to each point on a line segment. That is, it is a line segment

\[
\text{span}_\Delta \left( s_k C_{d-1}(A'_1, \ldots, B_k) + C_d(A'_1, \ldots, B_k) + v, (2s_k + 1) C_{d-1}(A'_1, \ldots, B_k) + v \right)
\]

for some \( v \). This proves Condition (1).
We now verify (2). We apply Proposition 7.6 which says we may choose $\tilde{A}$, with $\|\tilde{A}\| < N$ so that $\lambda_{d-2}(F_1(\mathcal{M}_{\tilde{A}\tilde{A}}))$ is at least comparable to $\lambda_{d-2}(F_1(\mathcal{M}_{\tilde{A}\tilde{A}}))$ for all $i \leq d - 2$.

We now wish to make the comparison for $F_j$ when $j \geq d - 1$. If the smallest singular direction is not close to $e_{d-1} - e_d$ then this case is covered as in Proposition 7.6 Case 1 and we see that $\lambda_{d-2}(F_1(\mathcal{M}_{\tilde{A}\tilde{A}}))$ is at least comparable to $\lambda_{d-2}(F_j(\mathcal{M}_{\tilde{A}\tilde{A}}))$. If it does make a small angle with $e_{d-1} - e_d$, then since the largest singular input direction is close to $e_{d-1} - e_d$, as in the proof of (34) the second smallest input singular direction makes small angle with $e_1 \oplus \cdots \oplus e_{d-2}$.

So by Corollary 7.13 and Lemma 7.11 (as in the proof of Proposition 7.6) we may assume the second smallest input direction of $MA(x, n)$ makes a definite angle with either the smallest or second smallest output direction of $\tilde{A}$ and singular output directions for $F_j(\mathcal{M}_{A(x, n)}) = F_j(MA(x, n))$ for some $M \in \mathcal{M}$, while $F_1$ is formed by the union of more than $s_k = 10^{(k+k_0)^6}$ such simplices. Since the ratio of second smallest to smallest satisfies for some $C$,

$$\frac{\omega'(M)}{\omega(M)} \leq C \frac{|C_{\min}(M)|}{|C_{\max}(M)|} \leq 10^{-2(2k+k_0)^4},$$

for all $M \in \mathcal{M}$, and $j \geq d - 1$ we have that

$$\lambda_{d-2}(F_1(M\tilde{A}A_i)) > 10^{-2(2k+k_0)^4} \lambda_{d-2}(F_j(M\tilde{A}A_i)).$$

Since $F_1(\mathcal{M}_{\tilde{A}A_i})$ is made of $s_k \gg 10^{2(2k+k_0)^4}$ such subsimplices and $F_j(\mathcal{M}_{\tilde{A}A_i}) = F_j(M\tilde{A}A_i)$ for $j \in \{d - 1, d\}$ and each $M \in \mathcal{M}$ we have that

$$\lambda_{d-2}(F_1(M\tilde{A}A_i)) > s_k 10^{-2(2k+k_0)^4} \lambda_{d-2}(F_j(M\tilde{A}A_i)) > \lambda_{d-2}(F_j(M\tilde{A}A_i)).$$

We verify the third condition of ready for illumination. We chose $\phi$ to be the vector $u$ which was defined to be the projection of $\Omega_{\pi_1(L)}^{-1}(C_1(A_1'B_1))$ to $\Delta_c \cap \Omega_{\pi_1(L)}^{-1}(C_d(A_1'B_1))$. For $k_0$ large enough, by Theorem 4.2, under Condition ** $u$ is close in angle to the projection of $\Omega_{\pi_1(L)}^{-1}(C_1(M))$. By Proposition 7.2 (which controls the large singular input directions of $M^T$), (19) (which uses $\Omega$ to relate $M^T$ and $M^{-1}$) and Lemma 7.1 (i) (which relates the singular input directions for $M^{-1}$ and singular output directions for $M$) this direction itself makes small angle with $w'$, the second smallest output vector of $M$. Since by Corollary 7.13, we may assume $F_1(\cup_{i=1}^l \mathcal{M}_{\tilde{A}A}) = M(e_2 \oplus \cdots \oplus e_d) \cap \Delta$ makes a definite angle with either the smallest or second smallest output direction of $M$, we conclude that it makes a definite angle with $u$. 

\[ \text{Springer} \]
Lastly, we need that the product, $\hat{A}\tilde{A}$, is a matrix of freedom on LHS at step $k + 1$. By our choice of $N$ we have that $\|\hat{A}\tilde{A}\| \leq \|\hat{A}\| \cdot \|\tilde{A}\| \leq N\|\hat{A}\|$ and so by our bound on $\|\hat{A}\|$ the product is a matrix of freedom on LHS.

**Proposition 8.5** For all large enough $\zeta > 0$, there exists $C > 0$ so that given matrix $\hat{A}$ of freedom on LHS satisfying the bound assumption in Proposition 8.4 and also satisfies $\frac{1}{\zeta} < \frac{|C_i(\hat{M}\hat{A})|}{|C'_{i'}(M\hat{A})|} < \zeta$ for $1 \leq i, i' \leq d - 2$, then

$$\lambda_{d-1}S_{\phi}(\hat{M}\hat{A}) > C\lambda_{d-1}(\hat{M}\hat{A}\Delta),$$

where $\hat{A}$ is given by Proposition 8.4.

**Proof of Proposition 8.5** Proposition 8.4 says $\lambda_{d-2}(F_1(\hat{M}\hat{A}))$ is comparable to $\lambda_{d-3}(V(\hat{M}\hat{A}))$.

Since $\|\hat{A}\| \leq N$ and $\frac{1}{\zeta} < \frac{|C_i(\hat{M}\hat{A})|}{|C'_{i'}(M\hat{A})|} < \zeta$ for all $1 \leq i, i' \leq d - 2$, Lemma 5.5 implies $\lambda_{d-3}(V(\hat{M}\hat{A}))$ is comparable to $\lambda_{d-3}(V(\hat{M}_1))$.²

Now the first conclusion of Theorem 6.1 says that there is a constant $C$ such that $\lambda_{d-3}(V(\hat{M}\hat{A})) > \delta\lambda_{d-3}(V(\hat{M}_1))$ implies

$$\lambda_{d-1}(\hat{M}\hat{A}\Delta) > \frac{\delta}{C}\lambda_{d-1}(\hat{M}\hat{A}\Delta).$$

Since the direction $\phi$ makes an angle bounded away from the face $F_1$, we conclude that the measure of $S(\hat{M}\hat{A}\Delta)$ is a definite proportion of the measure of $\hat{M}\hat{A}\Delta$ and hence of the measure of $\hat{M}\hat{A}\Delta$ which finishes the proof of Proposition 8.5.

By Theorem 4.2 we may assume that $V(\hat{M}\hat{A}) \subset \cup_{c<.05}\Delta_c$ and $W(\hat{M}\hat{A}) \subset \cup_{c>.95}\Delta_c$. Since $F_1(\hat{M}\hat{A})$ is a simplex, it follows that the volume of $\Delta_c \cap F_1(M)$ is comparable for all $c \in [.1, .9]$. Since our directions are parallel to $\Delta_c$ we obtain analogously to above:

**Corollary 8.6** There exists $\zeta', K$ and $\hat{c} > 0$ so that for all $c \in [.1, .9]$ and large enough $k_0$, for any matrix $\hat{A}$ of freedom on LHS, with $\frac{1}{\zeta} < \frac{|C_i(\hat{M}\hat{A})|}{|C'_{i'}(M\hat{A})|} < \zeta$

for $1 \leq i, i' \leq d - 2$ and $\|\hat{A}\| < \frac{1}{K}10^{(2k+k_0)^6-(k+k_0)^4+(k+k_0)2^3}$ there exists matrix $\hat{A}$ so that

1. $\hat{A}\tilde{A}$ is a matrix of freedom on LHS
2. $\lambda_{d-2}(S_{\phi}(\hat{M}\hat{A}\Delta) \cap \Delta_c) > \hat{c}\lambda_{d-2}(\hat{M}\hat{A}\Delta \cap \Delta_c)$
3. $\frac{|C_i(M\hat{A}\Delta)|}{|C'_{i'}(M\hat{A}\Delta)|} < \zeta'$ for all $i, i' \leq d - 2$.

² Indeed the conditional probability that $\tilde{A}$ occurs given $\hat{M}\hat{A}$ is proportional to $\frac{\lambda_{d-1}(V(\hat{A}))}{\lambda_{d-3}(\hat{A}\Delta)}$, where the proportionality depends only on $\zeta$. Also $\frac{\lambda_{d-1}(V(\hat{A}))}{\lambda_{d-3}(\hat{A}\Delta)}$ can be bounded by a constant only depending on $N$. 

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The third condition is the only condition that is not immediate. This follows because $\hat{A}$ is $\zeta$-balanced and the matrices in $\mathcal{A}$ have norm at most $N$. (In particular, we can let $\zeta'$ be $N\zeta$ with $N$ as before Definition 8.3.)

9 Repeating to illuminate

The result in the last section says a definite proportion of $\mathcal{M}_{\hat{A}} \Delta$ is illuminated. We have to improve this to all but $\rho(k+\kappa_0)\frac{1}{2}$ of our simplex is illuminated for some $\rho < 1$. We will actually prove a stronger result, that we can cover most of $\mathcal{S}_\phi(MA)$ by matrices $M' = A'_1 \ldots B'_k A \hat{A}$ so that $A \hat{A}$ is a matrix of freedom on LHS and $M' \Delta \cap \Delta_c \subset \mathcal{S}_\phi(MA)$ for all $c \in [1, .9]$. See Remark 9.8.

We begin with a lemma.

**Lemma 9.1** There is $b = b(d) \in \mathbb{N}$ so that $\mathcal{S}(\mathcal{M}_{\hat{A}})$ is a convex set bounded by at most $b$ faces.

**Proof** The simplex $\mathcal{M}_{\hat{A}} \Delta$ is a convex set bounded by $d$ faces. The illuminated set is formed by intersecting it with the codimension 1 in $\mathcal{M}_{\hat{A}} \Delta$ affine subspaces that contain lines in direction $\phi$ that intersect the faces of $F_1$. $\square$

This lemma implies that $\mathcal{S}(\mathcal{M}_{\hat{A}})$ is the intersection of at most $b$ half spaces each of which is bounded by a hyperplane $H$. This motivates us to study such regions intersected with $\mathcal{M}_{\hat{A}}(\Delta)$ and $\mathcal{M}_{\hat{A}} \Delta \cap \Delta_c$. For $H$ a bounding hyperplane let $H_c = H \cap \Delta_c$. Let

$$W_{\hat{A}} = \cup_{M \in \mathcal{M}_{\hat{A}}} \text{span}_\Delta(C_{d-1}(M), C_d(M)).$$

In other words $W_{\hat{A}}$ is a union of line segments such that the span of it with $V(\mathcal{M}_{\hat{A}})$ is $\mathcal{M}_{\hat{A}} \Delta$. Consequently for each $p \in W_{\hat{A}}$, with $p \neq C_j(M)$ for all $j \geq d - 1$ and $M \in \mathcal{M}_{\hat{A}}$, there is a unique $M \in \mathcal{M}_{\hat{A}}$ such that $p \in \text{span}_\Delta(C_{d-1}(M), C_d(M))$.

We say $M \in \mathcal{M}_{\hat{A}}$ is conflicted if there is $p \in \text{span}_\Delta(C_{d-1}(M), C_d(M))$ and $x \in V(\mathcal{M}_{\hat{A}})$ such that $\text{span}_\Delta(x, p) \cap H_c \neq \emptyset$. Let $\mathcal{W}_c(\mathcal{M}_{\hat{A}})$ be the set of conflicted $M$.

For each $M \in \mathcal{M}_{\hat{A}}$ we can define a function on $V(M)$ to subsets of $W_{\hat{A}}$ by

$$\beta_{c,\hat{A}}(x) = \{ p \in W_{\hat{A}} : \text{span}_\Delta(x, p) \cap H_c \neq \emptyset \}$$

if this set is nonempty. Otherwise $\beta_{c,\hat{A}}(x)$ is not defined. Now for each $p \in W_{\hat{A}}$ let

$$X_c(p) = \{ x \in V(M_{\hat{A}}) : \beta_c(x) = p \}.$$
Lemma 9.2 For any subinterval $I \subset W_\hat{A}$ we have $\bigcup_{p \in I} X_c(p)$ is connected.

Proof It suffices to prove that for each line segment $\vec{\ell}$ it is the case that $\vec{\ell} \cap \bigcup_{p \in I} X_c(p)$ is connected. This follows from the fact that on line segments $\beta_{c, \hat{A}}$ changes monotonically. \qed

For the next definition, recall that $V(M \hat{A})$ is independent of $M \in M$. Now given $\hat{A}, K, \zeta, \tau$ and $\ell$, let $G(\hat{A}, K, \zeta, \ell, \tau)$ be the set of $y \in V(M \hat{A})$ such that there exists $m$ so that for some $M \hat{A} \in M \hat{A}$, with $M \hat{A} = M(x, r)$

- $\|A(R^r y, m)\| < K^\ell,$
- $|W_c(M \hat{A}(R^r y, m))| < \max\{\frac{1}{3} - \frac{\zeta}{2}, |M \hat{A}|, 2\}$
- $\frac{|C_{\ell}(M \hat{A}(R^r y, m))|}{|C_{\ell}(M \hat{A}(R^r y, m))|} < \zeta$ for all $1 \leq i, i' \leq d - 2$
- the permutation of $R^r y$ is $\pi_L$

Proposition 9.3 There exists $K, \zeta, C > 1, \tau$ and $\rho < 1$ so that for all $\ell$ and $\hat{A}$

$$\lambda_{d-3}(G(\hat{A}, K, \zeta, \ell, \tau)) > (1 - C \rho^\ell)\lambda_{d-3}(V(M \hat{A}))$$

Remark 9.4 This proposition is improved on in fairly straightforward ways to prove Theorem 8.2. Corollary 9.5 and (40), prove measure bounds on sets that can be covered by simplices entirely contained in the illuminated set. Lemma 9.7 translates results about the entire simplex to its intersection with $\Delta_c$ for $c \in [0.1, 0.9]$. This gives that we can cover most of the illuminated set intersected with a slice by simplices entirely contained in the illuminated set. See also Remark 9.8.

We briefly explain the proof of this technical proposition. Its proof has three basic steps. The first step is setting up definitions culminating in the definition of $E_n$. The second step is showing that the conditional probability that $x \in E_n$ given whether or not $x \in E_i$ for $i < n$ satisfies the axioms of Proposition 5.12 and thus we have good lower bounds on the measure of the set of points that are in many $E_i$. The third step is that if a point is in many $E_i$ it is in $G(\hat{A}, K, \zeta, \ell, \tau)$. This follows from repeating (37), which shows that the conflicted set decays by a definite fraction each time $x \in E_i$.

Proof Step one: We will need the following family of paths.

Let $\pi'$ be the permutation on the LHS

$$\begin{pmatrix} 1 & 3 & \ldots & d & 2 \\ d & d - 1 & \ldots & 2 & 1 \end{pmatrix},$$

one step before $\pi_s$. We consider paths $\gamma(x, n)$ of permutations of length $n$ starting at $x$ with permutation $\pi_L$ that go through $\pi'$ and return to $\pi_s$ in one
step with 1 beating 2, and this is the only time of going from \( \pi' \) to \( \pi_s \). We call these \( \pi_s \) via \( \pi' \) isolated. The point of this definition is that no columns are added to the last two columns. Thus the interval \( W_{\^A} \) does not change. Let \( M_{mid}(\^A) \in W_c(M_{\^A}) \) be chosen so that

\[
||\{M \in W_c(M_{\^A}) : M \geq M_{mid}(\^A)\}|| - 1 \leq ||\{M \in W_c(M_{\^A}) : M < M_{mid}(\^A)\}|| \leq ||\{M \in W_c(M_{\^A}) : M \geq M_{mid}(\^A)\}||.
\]

(36)

Note \( \beta_c^{-1}(C_d(M_{mid}(\^A))) \subset V(M_{\^A}) \) is a hyperplane.

Fix \( K' \) to be determined later. For any \( n \), let \( E_n \) be the set of \( y \in V(M_{\^A}) : \exists p \) so that

- The permutation \( R^{p+r-2}y \) is \( \pi_s \) and \( R^{p+r}y \) is \( \pi_L \)
- \( \frac{|C_i(\^A A(R^r y, p))|}{|C_i(\^A A(R^r y, p))|} < \zeta \) for \( 1 \leq i, j \leq d - 2 \)
- \( \|A(R^r y, p)\| \in [K'^{2n}, K'^{2n+1}] \)

Now given \( E_n \), let \( \hat{E}_n \) be the set of \( x \in E_n : \exists m \) so that \( x \in V(M_{\^A}(y, p)) \) with \( y, p \) as in the definition of \( E_n \) and so that

(i) the path \( y(m + 2, R^{p+r}x) \) is \( \pi_s \) via \( \pi' \) isolated.
(ii) \( \|A(R^{p+r}x, m)\| < K' \)
(iii) \( \beta_c^{-1}(C_d(M_{mid}(\^A A(R^r x, p))) \) and \( V(M_{\^A}(R^r x, p + m)) \) are disjoint

If \( K' \) is large enough, then by Lemma 5.2 there exists \( \tau_1 > 0 \) so that given any outcome of \( \hat{E}_1, \ldots, \hat{E}_{i-1} \), the conditional probability is at least \( \tau_1 \) that \( x \in E_i \).

**Step two:**

We now show that there exists \( \tau_2 > 0 \) so that for \( K'' \) large enough, given the hyperplane \( \beta_c^{-1}(C_d(M_{mid}(\^A A(R^r x, p))) \), the conditional probability that \( y \in \hat{E}_i \), given any outcomes of \( \hat{E}_1, \ldots, \hat{E}_{i-1} \) and \( x \in E_i \) is at least \( \tau_2 > 0 \). Indeed, we apply Corollary 13.2 in the appendix to find the matrices that avoid \( \beta_c^{-1}(C_d(M_{mid}(\^A A(R^r x, p))) \). Now because \( x \in \hat{E}_i \), Lemma 5.5 says that this is a definite proportion of \( V(\^A A(R^r x, p))) \).

We now let

\[
F_i(x) = \begin{cases} 1 & \text{if } x \in \hat{E}_i \\ 0 & \text{otherwise} \end{cases}.
\]

Let

\[
\tau = \tau_1 \tau_2
\]
and let
\[ \tilde{K} = \max\{K', K''\}. \]

Finally let \( K = \tilde{K}^3 \). We apply Proposition 5.12 with \( \epsilon = \frac{\tau}{2} \), and so there exists \( \rho < 1 \) so for each \( \ell \), all but a proportion of at most \( C\rho^\ell \) of the points \( x \in V(MAˆ) \) have the property that they belong to at least \( \ell \frac{\tau}{2} \) of the sets \( \hat{E}_n \) with \( n \leq \ell \).

**Step three:**

Now suppose \( x \in \hat{E}_n \). Then by definition \( \max j \leq d - 2 |C_i^*| < 2^{2n+1} \) and there exists \( m \) so that \( \max j \leq d - 2 |C_i^*| < \tilde{K} \). We claim
\[
|\mathcal{W}_c(\mathcal{M}_{\hat{A}A(R^r x, p+m)})| \leq \left\lceil \frac{1}{2} |\mathcal{W}_c(\mathcal{M}_{\hat{A}A(R^r x, p)})| \right\rceil \leq \frac{1}{2} |\mathcal{W}_c(\mathcal{M}_{\hat{A}A(R^r x, p)})| + 1. \tag{37}
\]

To see this, by Conclusion (iii) we have that
\[
\beta^{-1}_{c,\hat{A}}(C_d(M_{mid}(\hat{A}A(R^r x, p)))) \cap V(A(R^r x, p + m)) = \emptyset.
\]

By (36) all the \( C_d \) on one side of \( C_d(M_{mid}(\hat{A}A(R^r x, p))) \) have the property that
\[
\beta^{-1}_{c,\hat{A}}(C_d) \cap V(A(R^r x, p + m)) = \emptyset.
\]

By our choice of \( M_{mid} \), (37) follows. Now for any \( A \), if \( M \notin \mathcal{W}_c(M_{\hat{A}}) \) then \( M \notin \mathcal{W}_c(M_{\hat{A}A}) \). Given any \( \ell \) we have shown that except for a set of \( x \in V(M \hat{A}) \) of measure \( C\rho^\ell \lambda_{d-3} V(M \hat{A}) \), for at least \( \frac{\tau}{2} \ell \) values of \( n \leq \ell \) we have \( x \in \hat{E}_n \). For such \( x \) the corresponding matrix satisfies \( |C_{\max}(A(R^r + p x, m))| < K'^{2n+1} \leq K^\ell \). Furthermore
\[
|\mathcal{W}_c(\mathcal{M}_{\hat{A}A(R^r x, n)})| \leq 2^{-\ell \frac{\tau}{2}} |\mathcal{W}_c(\mathcal{M})| + 2 \tag{38}
\]

This finishes the proof. \( \square \)

Based on the last result we make the following definition. Given \( \hat{A}, k, k_0, N_0 \) let \( H_{k+k_0}(\hat{A}, N_0) \) be the set of \( y \in V(M \hat{A}) \) such that there exists \( A(R^r y, m) \) so that

(i) \( \frac{|C_i^*(M_{\hat{A}A(R^r y, m))}|}{|C_i(M_{\hat{A}A(R^r y, m))}|} < \xi \) for \( 1 \leq i, i' \leq d - 2 \)

(ii) \( |\mathcal{W}_c(\mathcal{M}_{\hat{A}A(R^r y, m))}| < (\frac{2}{3})^{(k+k_0)^{1.1}} |\mathcal{M}_{\hat{A}}| + 2 \)
(iii) \( \|A(R^r y, m)\| < N_0^{(k+k_0)^{1.1}} \).

(iv) the permutation of \( R^r y \) is \( \pi_L \).

**Corollary 9.5** There is \( C, N_0, \rho' \) such that for all \( k, k_0 \)

\[
\lambda_{d-3}(H_{k+k_0}(\hat{\Lambda}, N_0)) \geq (1 - C \rho'^{(k+k_0)^{1.1}}) \lambda_{d-3}(V(M \hat{\Lambda})).
\]  

(39)

**Proof** Let \( K, \tau, \rho \) be a triple so that Proposition 9.3 is satisfied with this triple and some \( C, \zeta \). Choose \( \rho' = \rho^{\frac{2}{\tau}} \). This choice says that for any \( k, k_0 \), if we set \( \ell = \frac{2}{\tau}(k + k_0)^{1.1} \), then \( \rho^{\ell} = \rho'^{(k+k_0)^{1.1}} \).

Similarly, choose \( N_0 \) so that if \( \ell = \frac{2}{\tau}(k + k_0)^{1.1} \), then \( N_0^{(k+k_0)^{1.1}} > K \ell \) (so \( N_0 = K^{\frac{2}{\tau}} \)). Apply Proposition 9.3 with \( \ell = \frac{2}{\tau}(k + k_0)^{1.1} \) and we obtain the Corollary.

Next let \( \mathcal{H}_{k+k_0}(\hat{A}) \) be the set of matrices of the form \( A(R^r y, m) \) where \( y \in H_{k+k_0}(\hat{\Lambda}, N_0) \) and \( m \) is as above. These are matrices that give a small conflicted set of simplices.

The next lemma says that \( \mathcal{H}_{k+k_0}(\hat{A})\Delta \) covers most of \( \mathcal{M}_{\hat{\Lambda}} \Delta \) and that for matrices in \( \mathcal{H}_{k+k_0} \) the corresponding conflicted matrices only cover a set of small measure.

**Lemma 9.6** There is \( \hat{C}' \) such that for all \( k \)

\[
\lambda_{d-1}(\mathcal{M}_{\hat{\Lambda}} \Delta \setminus \mathcal{H}_{k+k_0}(\hat{A})\Delta) + \lambda_{d-1}(\bigcup_{A \in \mathcal{H}_{k+k_0}(\hat{A})}(\mathcal{W}_c(\mathcal{M}_{\hat{\Lambda}}\Delta))) < \hat{C}' \rho'^{(k+k_0)^{1.1}} + 10^{-(k+k_0)^{1.1}} + \left(\frac{2}{3}\right)^{(k+k_0)^{1.1}} \lambda_{d-1}(\mathcal{M}_{\hat{\Lambda}} \Delta).
\]  

(40)

**Proof** First of all applying (39) we find \( \rho', C, N_0 \) such that the simplices corresponding to matrices \( A \in \mathcal{H}_{k+k_0}(\hat{A}) \) cover a subset of \( V(M \hat{\Lambda}) \) whose complement has measure at most \( C \rho'^{(k+k_0)^{1.1}} \). Each such \( A \) satisfies \( \|A\| \leq N_0^{(k+k_0)^{1.1}} \). Moreover for each such \( A \) and for \( i \leq d - 2, j \geq d - 1 \), and \( k_0 \) large enough, we have

\[
\frac{N_0^{(k+k_0)^{1.1}}}{10^{-(k+k_0)^{1.1}}} \leq 10^{(k+k_0)^{2.3}} \leq \frac{|C_j(A)|}{|C_i(A)|}
\]

so we can apply the second conclusion of Theorem 6.1 with \( N = N_0^{(k+k_0)^{1.1}} \), \( \epsilon = 10^{-(k+k_0)^{1.1}} \) and \( \delta = \rho'^{(k+k_0)^{1.1}} \). This bounds the first term on the left by the first two terms on the right in (40).
We now bound the second term on the left. First we note that the bound in (ii) on the cardinality of \( \mathcal{W}_c(\mathcal{M}_{\hat{A}}) \) in the definition of \( \mathcal{H} \) says that our conflicted set has a small number of simplices compared to the non-conflicted set. To obtain a measure estimate, we apply Lemma 8.1 and Veech’s volume estimate (Lemma 5.6) which together say that the \( \lambda_{d-1} \) volume of the different span\(\Delta(V(M), C_{d-1}(M), C_d(M)) \) as \( M \) varies in \( \mathcal{M}_{\hat{A}} \) are uniformly comparable,\(^3\) to conclude that

\[
\lambda_{d-1}\left(\bigcup_{A \in \mathcal{H}_{k+k_0}(\hat{A})}(\mathcal{W}_c(\mathcal{M}_{\hat{A}}))\Delta\right) \leq \left(\frac{2}{3}\right)^{(k+k_0)1.1}
\]

\( \square \)

In the next lemma we take the estimates of the last lemma and intersect with the sets \( \Delta_s \). It is a straightforward consequence of the geometry of simplices.

**Lemma 9.7** There are \( C, \rho'' \), so that for \( k_0 \) large enough, for all \( k \) and for each \( c \) satisfying \( .1 \leq c \leq .9 \)

\[
\lambda_{d-2}\left(\mathcal{M}_{\hat{A}}\Delta \cap \Delta_c \setminus \mathcal{H}_{k+k_0}(\hat{A})\Delta \cap \Delta_c\right) + \lambda_{d-2}\left(\bigcup_{A \in \mathcal{H}_{k+k_0}(\hat{A})}(\mathcal{W}_c(\mathcal{M}_{\hat{A}}))\Delta \cap \Delta_c\right)
\]

\[
< C(\rho''(k+k_0)^{1.1} + 10^{-(k+k_0)^{1.1}} + \left(\frac{2}{3}\right)^{(k+k_0)1.1})\lambda_{d-2}(\mathcal{M}_{\hat{A}}\Delta \cap \Delta_c).
\]

(41)

**Proof** By Theorem 4.2, if \( k_0 \) is large enough then for all \( M \in \mathcal{M}_{\hat{A}} \) and for any \( 1 \leq i \leq d - 2 \), the columns \( C_i(M) \) lie within \(.05\) of \( e_1 \oplus \cdots \oplus e_{d-2} \) and the last two columns lie within \(.05\) of \( e_{d-1} \oplus e_d \). So, \( V(\mathcal{M}_{\hat{A}}) \subset \bigcup_{s \in [0,.05]} \Delta_s \) and \( \mathcal{W}_{\hat{A}} \subset \bigcup_{s \in [.95,1]} \Delta_s \). Thus, for any \( s, s' \in [1,.9] \) for each \( M \in \mathcal{M}_{\hat{A}} \) we have that

\[
\frac{\lambda_{d-3}(\text{span}_\Delta(V(M), p) \cap \Delta_s)}{\lambda_{d-3}(\text{span}_\Delta(V(M), p) \cap \Delta_{s'})} < C,
\]

where \( C \) depends only on the dimension. Indeed, \( \text{span}_\Delta(V(M), p) \) is a simplex and \( \Delta_s, \Delta_{s'} \) are parallel planes that are at least \( \frac{5}{100} \) of the diameter of the simplex away from any of the extreme points of the simplex. Recalling (35), we now have the bound

\[
\frac{\lambda_{d-2}(M \Delta \cap \Delta_s)}{\lambda_{d-2}(M \Delta \cap \Delta_{s'})} = \frac{\lambda_{d-2}(\bigcup_{p \in \mathcal{W}_{\hat{A}}} \text{span}_\Delta(V(M), p) \cap \Delta_s)}{\lambda_{d-2}(\bigcup_{p \in \mathcal{W}_{\hat{A}}} \text{span}_\Delta(V(M), p) \cap \Delta_{s'})} < C' \quad (42)
\]

by Fubini’s theorem. Indeed, by Fubini’s theorem, there exists \( c_s \) so that \( \lambda_{d-2}(M \Delta \cap \Delta_s) = c_s \int_{\mathcal{W}_{\hat{A}}} \lambda_{d-3}(\text{span}_\Delta(V(M), p) \cap \Delta_s) dp \). Moreover, since

\(^3\) Notice that \( C_i(M) = C_i(M') \) for all \( M, M' \in \mathcal{M}_{\hat{A}} \) and \( i \leq d - 2 \).

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\[ W_A, V(M) \subset \bigcup_{t \in [0.05, 0.95]} \Delta_t, \] we have that \( c_s \) changes polynomially as \( s \) ranges from 0.95 to 0.05 and so \( c_s, c_s' \) are comparable for all \( s, s' \in [0.1, 0.9] \).

Applying (42) to each simplex in (40) we obtain the lemma. \( \square \)

Applying Eq. (41) to each hyperplane that cuts out the illuminated setLemma 9.1 gives

\[
\lambda_{d-2} \left( \mathcal{M}_{\tilde{A}} \Delta \cap \Delta_c \backslash (S_{\phi}(\mathcal{M}_{\tilde{A}}) \cup_{A \in \mathcal{H}_{k+k_0}(\tilde{A})} W_c(\tilde{A}A)) \right) \\
< Cb(d)\left(\frac{2}{3}\right)^{k+k_0} + \rho^{n(k+k_0)} + 10^{-(k+k_0)} \lambda_{d-2}(\mathcal{M}_{\tilde{A}} \Delta \cap \Delta_c).
\]

(43)

**Remark 9.8** We now explain the proof of Theorem 8.2. At this point in the proof, we can cover all but an exponentially small amount of the illuminated set by simplices entirely contained in the illuminated set. (These have the form \( M' = A'_1, \ldots, B_k B' \tilde{A} \tilde{A} \) where \( \tilde{A} \in \mathcal{H}_{k+k_0} \) for some \( B' \), a matrix of restriction on the RHS so that \( M \Delta \cap S_{\phi}(M) \neq \emptyset \) and \( M' \notin W_c(M \tilde{A} \tilde{A}) \).) Theorem 8.2 is then proved by applying Corollary 8.6 many times to show that the illuminated sets of successive matrices cover most of \( M \Delta \). Indeed, each time we apply Corollary 8.6 we obtain that a definite proportion of the part that is not illuminated is in the next illuminated set. There is a subtlety here: Because we are covering (most of) the illuminated set by simplices entirely contained in the illuminated set, when we apply Corollary 8.6 again we do it on the compliment of these simplices and in this way get an additional fraction at each step and avoid worrying about repeatedly counting covering the same set. This is the reason for showing that the measure of the conflicted set is (more than) exponentially small.

**Proof of Theorem 8.2** We are at stage \( k \) but suppress it in the definition of matrices. We restrict our attention to \( \alpha \in [0.1, 0.9] \) and we are given the constants \( \hat{c} \) and \( K \) from Corollary 8.6.

Choose \( N \) so that

\[
(1 - \hat{c})^N < \frac{1}{10}.
\]

We choose \( k_0 \) large enough to guarantee that for all \( k \) and \( N_0 \) as in Corollary 9.5,

\[
(N_0^{(k+k_0)^{1.1}} + K)^{N(k+k_0)^{1.1}} < \frac{1}{2} 10^{(k+k_0)^2.3}.
\]

(44)
The proof is by an inductive procedure. For the first step we are given $\hat{\mathcal{A}}$ a LHS matrix with $\|\hat{\mathcal{A}}\| \in [10^{(2k+2+k_0)^6-(2k+1+k_0)^4}, 2\cdot 10^{(2k+2+k_0)^6-(2k+1+k_0)^4}]$, and a set of matrices $\mathcal{M}_{\hat{\mathcal{A}}}$. We apply Corollary 8.6 to produce matrix $A_1$ so that $\mathcal{M}_{\hat{\mathcal{A}}A_1}$ is ready for illumination, and for a constant $\hat{c} > 0$, the simplices $\mathcal{M}_{\hat{\mathcal{A}}A_1} \Delta \cap \Delta_{\alpha}$ satisfy that $X_1 := \cup_{\alpha} \mathcal{S}(\mathcal{M}_{\hat{\mathcal{A}}A_1}) \cap \Delta_{\alpha}$ is a set of measure at least $\hat{c}\lambda_{\alpha-1}(\mathcal{M}_{\hat{\mathcal{A}}} \Delta \cap \Delta_{\alpha})$.

Now let

$$W_1 := \left( (\mathcal{M}_{\hat{\mathcal{A}}A_1} \Delta \cap \Delta_{\alpha}) \setminus \cup_{A \in \mathcal{H}_{k+k_0}(\hat{\mathcal{A}}A_1)} \mathcal{M}_{\hat{\mathcal{A}}A_1A} \Delta \right) \cup \cup_{A \in \mathcal{H}_{k+k_0}(\hat{\mathcal{A}}A_1, j)} W_\alpha(\mathcal{M}_{\hat{\mathcal{A}}A_1A} \Delta \cap \Delta_{\alpha}).$$

This is the set not covered by simplices of matrices $A$ with a small conflicted set union the conflicted subset of those $A$ that do have a small conflicted set.

By Lemma 9.7, $X_1 \setminus W_1$ has measure at least

$$\hat{c}(1 - C(\rho''(k+k_0)^{1.1} + 10^{-(k+k_0)^{1.1}} + (\frac{2}{3})(k+k_0)^{1.1}))\lambda_{\alpha-1}(\mathcal{M}_{\hat{\mathcal{A}}} \Delta \cap \Delta_{\alpha}).$$

We next consider $\mathcal{M}_{\hat{\mathcal{A}}} \Delta \setminus (X_1 \cup W_1)$. Notice that it is disjoint from the illuminated set, and it can be partitioned into simplices of the form $\mathcal{M}_{\hat{\mathcal{A}}A_{\alpha, \beta}} \Delta$ where $\|\hat{\mathcal{A}}_{\beta}\| < (N_0^{(k+k_0)^{1.1}} + K)$.

We now again have a collection of families $\mathcal{M}_{\hat{\mathcal{A}}A_{\alpha, \beta}}$. Using Corollary 8.6 we first make each of them ready for illumination by a matrix $A_2$. In this way we obtain an illuminated subset

$$X_2 \subset \mathcal{M}_{\hat{\mathcal{A}}} \Delta \cap \Delta_{\alpha} \setminus (X_1 \cup W_1)$$

of measure at least $\hat{c}\lambda_{\alpha-2}(\mathcal{M}_{\hat{\mathcal{A}}} \Delta \cap \Delta_{\alpha} \setminus (X_1 \cup W_1))$. As before, to each $A_2$ we obtain the set $W_2$ not covered by simplices of matrices $A$ with small conflicted set union the conflicted subset of those $A$ that do have a small conflicted set. We remove $X_2, W_2$ and repeat this procedure $N(k+k_0)^{1.1}$ total times, constructing disjoint illuminated sets $X_j$ and removed sets $W_j$. We now show that

$$\cup_{i=1}^{N(k+k_0)^{1.1}} X_i \setminus W_i = \cup_{i=1}^{N(k+k_0)^{1.1}} X_i \setminus W_i$$

is covered by simplices from matrices of freedom on LHS. First note that since we performed a procedure $N(k+k_0)^{1.1}$ times that increased the norm by at most $N_0^{(k+k_0)^{1.1}} + K$, our assumption on $\|\hat{\mathcal{A}}\|$ and Inequality (44) implies that our set is contained in the matrices of freedom on the left hand side (assuming $k_0$ is large enough) at step $k$. 

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We now bound $\lambda_{d-2}(\bigcup_{i=1}^{N(k+k_0)} X_i \setminus W_i)$ from below by first bounding $\lambda_{d-2}(\bigcup_{i=1}^{N(k+k_0)} W_i)$ from above and then bounding $\lambda_{d-2}(\mathcal{M}_{\tilde{A}} \Delta \cap \Delta_\alpha \setminus (\bigcup_{i=1}^{N(k+k_0)} X_i \cup W_i))$ from above. Now

$$
\sum_{i=1}^{N(k+k_0)} \lambda_{d-2}(W_i)
\leq (\rho'^{(k+k_0)} + \frac{2}{3}(k+k_0) + 10^{-(k+k_0)})
$$

$$
\sum_{i=1}^{N(k+k_0)} \lambda_{d-2}(\mathcal{M}_A \Delta \cap \Delta_\alpha \setminus \bigcup_{j=1}^{j-1} X_j \cup W_j)
\leq (\rho'^{(k+k_0)} + \frac{2}{3}(k+k_0) + 10^{-(k+k_0)})
$$

$$
\sum_{i=1}^{N(k+k_0)} (1 - \hat{c})^{i-1} \lambda_{d-2}(\mathcal{M}_A \Delta \cap \Delta_\alpha)
\leq C(\rho'^{(k+k_0)} + \frac{2}{3}(k+k_0) + 10^{-(k+k_0)}) \lambda_{d-2}(\mathcal{M}_A \Delta \cap \Delta_\alpha)
$$

On the other hand inductively we see that

$$
\lambda_{d-2}(\mathcal{M}_A \Delta \cap \Delta_\alpha \setminus (\bigcup_{i=1}^{j} X_i \cup W_i)) \leq (1 - \hat{c})^{j} \lambda_{d-2}(\mathcal{M}_A \Delta \cap \Delta_\alpha).
$$

By our choice of $N$ and taking $j = N(k + k_0)$ this implies that

$$
\lambda_{d-2}(\bigcup_{i=1}^{N(k+k_0)} X_i \cup W_i) \geq (1 - \left(\frac{1}{10}\right)^{(k+k_0)}) \lambda_{d-2}(\mathcal{M}_A \Delta \cap \Delta_\alpha).
$$

Combining this with Inequality (*) proves the Theorem (if $k_0$ is large enough so that there exists $\rho < 1$ so that $C(\rho'^{(k+k_0)} + \frac{2}{3}(k+k_0) + 10^{-(k+k_0)}) < \rho^{(k+k_0)}$ for all $k \geq 1$).

\begin{flushright}
\square
\end{flushright}

10 \hspace{1cm} \textbf{Restriction on left side}

The point of this section is to prove Theorem 10.1 below. It says that during restriction on LHS we can define a neighborhood $\mathcal{N}$ of a face and show that we can almost fill up this neighborhood with simplicies. This will let us show, in Sect. 12.1, that after restriction on the LHS, most of our planes still intersect our simplicies in large enough measure to satisfy the assumptions of Theorem
3.1. In this section \( R \) refers to normalized Rauzy induction and \( \hat{R} \) for non-normalized.

Recall \( \pi_L = \begin{pmatrix} 1 & d - 1 & d & 2 & \ldots & d - 3 & d - 2 \\ d & d - 1 & d - 2 & d - 3 & \ldots & 2 & 1 \end{pmatrix} \), the matrix sizes at stage \( k \) during restriction are

\[ \|A'_k\| \in [10^{(2k+1+k_0)^6+(k+k_0)^4}, 10^{(2k+1+k_0)^6+(k+k_0)^4+(k+k_0)^2}], \]

and are given by 1 losing to \( d - 2 \) and 1 not winning until returning to \( \pi_L \) with norm in the range given above. (This is followed by transition, a path from \( \pi_L \) to \( \pi_s \).) Let \( A'_k \) be the set of these matrices satisfying Condition **.

Now after finishing freedom on LHS we have families \( \mathcal{M}_{\hat{A}} \) for some LHS matrix \( \hat{A} \).

In this section balanced and positive refer to \( d - 3 \times d - 3 \) submatrices \( A \) consisting of columns \( C_2, \ldots, C_{d-2} \) and the entries 2 through \( d - 2 \) in these columns.

Given \( B, L \) let \( \mathcal{A}(B, L) \) be the set of \( A \in A'_k \) such that

- \( A \) is of form \( A = A_1 PA_2 \) where \( \|P\| < B \)
- the upper \( d - 2 \times d - 2 \) submatrix of \( P \) has all but the 1st row positive
- \( \|A_2\| < 10^L \).

The idea is that \( A \) has a fixed bounded size matrix \( P \) not long before the end of Rauzy induction.

Given \( M \in \mathcal{M}_{\hat{A}} \), let

\[
U_{d-2}(M) = \text{span}_\Delta(C_2(M), \ldots, C_{d-2}(M)) \subset V(M),
\]

\[
U_d(M) = \text{span}_\Delta(C_2(M), \ldots, C_{d-2}(M), C_{d-1}(M), C_d(M))
\]

\[
d_M = d(C_1(M), U_{d-2}),
\]

\[
t_k = 10^{-[(2k+1+k_0)^6+(k+k_0)^4+\frac{1}{2}(k+k_0)^2]}, \quad (45)
\]

\[
\mathcal{N}' = \mathcal{N}_{t_k d_M} U_{d-2} \cap V(M)
\]

and let

\[
\mathcal{N} = \text{span}_\Delta(\mathcal{N}', C_{d-1}(M), C_d(M)).
\]

Note this neighborhood depends on \( M \) as well as \( k \). We denote it \( \mathcal{N}(M) \) when the matrix is unclear.
Theorem 10.1  For any $\zeta$ there exist constants $\alpha > 0$ and $B$ so that for all large enough $k_0$ if $M$ is $\zeta$-balanced then for all $c \in (0.1, 0.9)$

$$\lambda_{d-2}(MA(B, \frac{1}{16}(k + k_0)^2)\Delta \cap \mathcal{N} \cap \Delta_c) \geq (1 - \alpha^{(k+k_0)^2})\lambda_{d-1}(\mathcal{N} \cap \Delta_c)$$

(46)

Note that $B$ can be chosen to only depend on the Rauzy class.

This says the image of $\Delta$ under elements of $\mathcal{A}(B, \frac{1}{16}(k + k_0)^2)$ takes up most of $\mathcal{N}$.

To prove this result consider the following three sets of matrices whose dependence on $k$ is suppressed.

$\mathcal{E}_1 = \mathcal{A}_k \setminus \mathcal{A}(B, \frac{1}{16}(k + k_0)^2)$,

$\mathcal{E}_2$ be the set of matrices of restriction of the LHS with norm at least $10^{(k+k_0)^2 + (k+k_0)^3 + \frac{3}{2}(k+k_0)^2}$ and

$\mathcal{E}_3$ the set of matrices $A$ that leave restriction on the LHS with $\|A\| \leq 10^{(k+k_0)^2 + (k+k_0)^3}$.

The Theorem will follow from establishing the following results:

Lemma 10.2  There exists $\rho < 1$ such that $\lambda_{d-3}(ME_1 \Delta \cap \mathcal{N}' \cap \Delta_c) < \rho^{(k+k_0)^2}\lambda_{d-3} \mathcal{N}'$.

Proposition 10.3  There exists $\rho < 1$ such that $\lambda_{d-3}(ME_2 \Delta \cap \mathcal{N}' \cap \Delta_c) < \rho^{(k+k_0)^2}\lambda_{d-3} \mathcal{N}'$.

Proposition 10.4  There exists $\rho < 1$ such that $\lambda_{d-1}(ME_3 \Delta \cap \mathcal{N}) < \rho^{\frac{1}{2}(k+k_0)^2}\lambda_{d-1}(\mathcal{N})$.

Proof of Theorem 10.1 assuming previous 3 results  First observe that if $y \in \mathcal{N}$ and $y \notin MA(B, (k + k_0)^2)\Delta \cap \mathcal{N}$ then $y \in ME_i \Delta$ for some $i \in \{1, 2, 3\}$. So it suffices to prove that Lemma 10.2 and Proposition 10.3 imply analogous bounds for $\lambda_{d-1}(ME_i \Delta \cap \mathcal{N})$ for $i \in \{1, 2\}$. Now $C_{d-1}$ and $C_d$ are unchanged during restriction on LHS and so our sets $\mathcal{N}$, $ME_1 \Delta \cap \mathcal{N}$ and $ME_2 \Delta \cap \mathcal{N}$ are obtained from $\mathcal{N}'$, $ME_1 \Delta \cap \mathcal{N}'$ and $ME_2 \Delta \cap \mathcal{N}'$ by taking the convex combinations with the same line. So the analogous estimates hold.

Proof of Lemma 10.2  Let $\Delta_{d-4} = \{x \in \mathbb{R}^{d-4} : x_i \geq 0$ for all $i$ and $\sum x_i = 1\}$. For $x \in \Delta_{d-4}$, let $M_{d-3}(x, r)$ denote the corresponding matrix of Rauzy induction and let $R_{d-3}$ denote Rauzy induction on $d - 3$ letters.

We apply Proposition 5.10 to $R_{d-3}$ with $P_{d-3} = M_{d-3}(x, n)$ a fixed positive matrix of Rauzy induction on $d - 3$ letters starting at

$$\begin{pmatrix} 1 & 2 & \ldots & d - 4 & d - 3 \\ d - 3 & d - 4 & \ldots & 2 & 1 \end{pmatrix}$$

to get that there exists $\rho < 1$ such that
\[ \lambda_{d-4}(\{y \in \Delta_{d-4} : \frac{3}{2} r : \| M_{d-3}(y, r) \| < 10 \frac{1}{\pi}(k+k_0)^2 \text{ and } M_{d-3}(R^y, n) = P_{d-3} \}) < C \rho^{(k+k_0)^2} \lambda_{d-4}(\Delta_{d-4}). \quad (47) \]

Let \( P \) be the matrix of Rauzy induction on \( d \) letters corresponding to \( P_{d-3} \) that starts at \( \pi_L \). That is, \( P = M(x', n') \) where \( x_i' = x_{i-1} \) for \( 2 \leq i \leq d - 2 \) and \( x_1' = x_{d-1}' = x_d = 0 \). We have

\[ \lambda_{d-4}(\{y \in U_{d-2} : \frac{3}{2} r : \| M(y, r) \| < 10 \frac{1}{\pi}(k+k_0)^2 \text{ and } M(R^y, n') = P \}) < C \rho^{(k+k_0)^2} \lambda_{d-4}(U_{d-2}). \quad (48) \]

Let \( \mathcal{D} \) denote a set of matrices so that \( \mathcal{D} \Delta = \mathcal{E}_1 \). To complete the lemma, we use a similar argument to the proof of Proposition 6.4, that because \( \mathcal{D} \Delta \) is a negligible proportion of \( U_{d-2} \), it is also a negligible part of dimension \((d-4)\) ‘hyperplanes’ parallel to \( U_{d-2} \) close to \( U_{d-2} \). Let \( H \) be a \((d-4)\)-dimension hyperplane in \( V(M) \) parallel to \( U_{d-2} \). Then if \( A \in \mathcal{D} \)

\[ \lambda_{d-4}(H \cap M A \Delta) \leq \lambda_{d-4}(\text{span}_\Delta(C_2(MA), \ldots, C_{d-2}(MA))). \quad (49) \]

We now take an orthogonal transversal in \( V(M) \) to \( U_{d-2} \) and exhaust \( M \mathcal{D} \Delta \cap \mathcal{N}' \) by taking a hyperplane, \( H \), through each point of the transversal. Applying Inequalities (49) and (48) and observing that for every \( H \subset \mathcal{N}' \) we have \( \lambda_{d-4}(H \cap \mathcal{N}') \) is proportional to \( \lambda_{d-4}(U_{d-2}) \) we obtain that the complement of \( M \mathcal{D} \Delta \) is all but an exponentially small proportion of \( \mathcal{N}' \), completing the lemma.

\[ \square \]

### 10.1 Proof of Proposition 10.3

The proof of Proposition 10.3 is similar to the previous lemma (reducing to \( \Lambda_{d-3} \) and using Proposition 5.10) but requires a couple of preliminaries.

**Lemma 10.5** Let \( M = M(\xi, n) \) be a matrix of Rauzy induction. If \(|C_j(M)| > N \) for some \( j \) then the unnormalized length \( \lambda_j \) of the \( j \)th interval of \( \hat{R}^n \xi \) is at most \( \frac{1}{N} \).

**Proof** The \( a_{i,j} \) entry of the \( C_j \) column of \( M \) is the number of visits of points of the \( j \)th interval of \( R^n \xi \) to the \( i \)th interval \( I_i \) of the original IET before these points return to \( R^n \xi \). By assumption \( \sum_i a_{i,j} \geq N \). We conclude \( N \lambda_j \leq \sum_i a_{i,j} \lambda_j \leq 1 \), the length of our initial interval. \[ \square \]

**Lemma 10.6** Let \( M(x, r) \) be a matrix of Rauzy induction. For any path \( \gamma \) of Rauzy induction of some length \( p \) where at the end \( d-2 \) beats 1 and such that

- \( A(R^y, n) \) is a matrix of restriction on the LHS,
The set of non-uniquely ergodic $d$-IETs

$|C_{d-2}(A(R^r, y, n))| > 2N\zeta$ and

$R^{r+n} y$ follows $\gamma$

then $MA(R^{r+n+p} y) \Delta \subset N_{dM}^{U_d}$.

**Proof** Because 1 loses during restriction and $d - 2$ beats 1 on the last step of $\gamma$, it follows that every $z \in MA(R^r, y, n + p - 1)\Delta$ satisfies

$$(R^r z)_1 = (R^{p+n-1} R^r z)_{d-2}. \quad (\hat{R^n R^r} y)_{d-2} < \frac{1}{2N\zeta}.$$ 

However by our assumptions we have $|C_{d-2}(A(R^r, y, n))| > 2N\zeta$ and so by Lemma 10.5 we have $(\hat{R^n R^r} y)_{d-2} < \frac{1}{2N\zeta}$. So if $z \in MA(R^r, y, n + p)$ then $(R^r z)_1 < \frac{1}{2N\zeta}$ and

$$d(z, U_d) < d_M \frac{|z_1 C_1(M)|}{\sum_{i=2}^d |z_i C_i(M)|} \leq \frac{d_M}{2N(1 - z_1)}. \quad \square$$

**Proof of Proposition 10.3** Let

$$N'' = N_{10^{-4}(k+k_0)^2 t_k dM} U_{d-2} \cap V(M) \subset N' = N_{t_k dM} U_{d-2} \cap V(M).$$

We partition $E_2$ into two parts. The first subset consists of those matrices $A$ such that $MA \Delta \subset N''$. This gives an exponentially small part of $N'$ by simple geometry. The second subset consists of those $A \in E_2$ such that $MA \Delta \subset N''' \neq \emptyset$ which we now treat. That is, let $\tilde{A}$ be the set of matrices $A$ of restriction on the LHS with

$$\|A\| \geq 10^{(k+k_0)^6 + (k+k_0)^4 + \frac{3}{4}(k+k_0)^2}$$

and such that

$$MA \Delta \cap N''' \neq \emptyset.$$

We now show there exists $\rho < 1$ such that

$$\lambda_{d-4}(MA \Delta \cap U_{d-2}) < \rho^{(k+k_0)^2} \lambda_{d-4}(U_{d-2}) \quad (50)$$

by showing that $\lambda_{d-4}(MA \Delta \cap U_{d-2})$ is an exponentially small amount in $\frac{1}{8} (k + k_0)^4$ multiplied by the measure of $U_{d-2}$. To do this we show that its measure is smaller than the volume of $N''$. Set

$$N = \frac{1}{2\zeta} 10^{(k+k_0)^6 + (k+k_0)^4 + \frac{5}{8}(k+k_0)^2}.$$
By Lemma 10.6, once a matrix $A$ of norm at least $2\zeta N$ satisfies

$$\frac{|C_i(A)|}{|C_{i'}(A)|} \leq \zeta \text{ for all } i, i' \leq d - 2 \quad (51)$$

it can not be in $\tilde{A}$ because

$$MA\Delta \subset N_{\frac{dM}{N}} U_d \cap V(M) \subset N'' \quad (52)$$

the last inclusion is by the choice of $t_k$. We now show (51) for all but an exponentially small proportion of the points in $N'$. Let $M''_{d-2}$ be a matrix in $R_{d-2}$ that arises from a positive matrix of $R_{d-3}$ followed by a fixed path where at the end $d-2$ beats 1, and where $MM''$ is $\zeta$-balanced for every (non-negative) matrix $M$. We now apply Proposition 5.9 (for $R_{d-2}$) with this $M''$ and obtain that regardless of our past, off of an exponentially small proportion of $\frac{dM}{N}$ we produce matrices with ratio of columns at most $\zeta$ before the norm has increased by more than $10^{\frac{1}{8}(k+k_0)^2}$. Re-interpreting this we produce matrices of restriction in $R_d$ coming from points in $V(W)$ that satisfy (51) and so their intersections with $V(M)$ are contained in $N''$ (via (52)) obtaining (50). With our initial remark we have that $\lambda_{d-3}(E_2 \Delta \cap N''')$ is an exponentially small proportion of $N'$.

**10.2 Proof of Proposition 10.4**

Let $\hat{A}$ be a set of matrices of restriction on the LHS so that

1. $\|A\| \in [10^{(k+k_0)^6+(k+k_0)^4}, 2 \cdot 10^{(k+k_0)^6+(k+k_0)^4}]$ for all $A \in \hat{A}$.
2. $U_d \subset \bigcup_{A \in \hat{A}} MA\Delta$
3. $A \neq A' \in \hat{A}$ then $A\Delta \cap A'\Delta = \emptyset$.

Parametrize affine hyperplanes parallel to $U_d$ and non-trivially intersecting $M\Delta$ by their distance from $U_d$. So $H_\alpha$ is the affine hyperplane parallel to $U_d$ whose distance from $U_d$ is $\alpha$. To prove Proposition 10.4 it will suffice to show there exists $\rho < 1$ so that for all $A \in \hat{A}$ and $\alpha \leq t_kdM$

$$\lambda_{d-2}(H_\alpha \cap MA\Delta) > (1 - \rho^{\frac{1}{2}(k+k_0)^2})\lambda_{d-2}(MA\Delta \cap U_d) \quad (53)$$

We prove this inequality. For each $A$ let $\alpha_0 = d(C_1(MA), U_d)$ so $H_{\alpha_0} \cap MA\Delta$ is a single point.

Notice that because $\frac{|C_1(M)|}{|C_{i'}(M)|} > \frac{1}{\zeta}$ we have

$$\alpha_0 \geq \frac{1}{2\zeta}dM 10^{-(k+k_0)^6+(k+k_0)^4}.$$
Now the side lengths of parallel hyperplanes $H_\alpha$ intersecting a simplex vary linearly and angles are constant so $\lambda_{d-2}(H_\alpha \cap M A \Delta) = (1 - \frac{\alpha}{\alpha_0})^{d-2}\lambda_{d-2}(M A \Delta \cap U_d)$.

By the bound $\alpha < t_k d_M$ and the definition of $t_k$ we have

$$\frac{\alpha}{\alpha_0} < 2\xi \cdot 10^{-\frac{1}{2}(k+k_0)^2}.$$

We have established (53) for an appropriate $\rho$.

We now finish the proof of Proposition 10.4. Because $\cup_{A \in \hat{A}} M A \Delta \supset U_d$, inequality (53) implies that

$$\lambda_{d-2}(H_\alpha \cap \cup_{A \in \hat{A}} M A \Delta) > (1 - \rho_3^{\frac{1}{2}(k+k_0)^2})\lambda_{d-2}(U_d)$$

$$> (1 - \rho_3^{\frac{1}{2}(k+k_0)^2})\lambda_{d-2}(H_\alpha \cap M \Delta).$$

This establishes that all but an exponentially small portion of $N$ is not in $M E_3 \Delta$.

11 Transition, freedom and restriction on the right hand side

Theorem 11.3 is used in the proof of Theorem 3.1. The other results used outside of this section are Lemmas 11.1 and 11.2, which are used to establish Condition * (2) and (3). We now establish the second condition of Condition *.

Lemma 11.1 There exists $C > 1$ and $\rho < 1$ so that if $M(x, r)$ is at end of restriction on LHS, then the measure of the set of $y \in M \Delta$ such that there exists $n$ with $A(R^r y, n)$ reaches $\pi_s$ and $\|A\| \leq 10^{(k+k_0)^2}$ is at least $(1 - C\rho^{(k+k_0)^2})\lambda_{d-1}(\hat{M} \Delta)$

Proof By Proposition 5.8 we have that there exists $\zeta, C, \rho$ so that

$$\lambda_{d-1}(\{y \in \hat{M} \Delta : \exists n \text{ with } |C_{\text{max}}(M(R^r y, n))| < 10^{(k+k_0)^2}|C_{\text{max}}(\hat{M})|$$

and $M(R^r y, n)$ is $\zeta$-balanced) $> (1 - C\rho^{(k+k_0)^2})\lambda_{d-1}(\hat{M} \Delta)$.}

For the matrix to become balanced (as $d \times d$ matrix), $d$ has to be compared to $i$ for $i \leq d - 2$, because these columns are much larger than $C_d$. For this to occur, we have to reach $\pi_s$. \qed

With Theorem 10.1 this establishes Condition* (2). Indeed if $A$ satisfies the conclusions of Theorem 10.1 and Lemma 11.1, $MA$ has the form $MA_1 PA_2 A_3$
where $P$ is positive of bounded size and so $|C_i(MA_1P)|/|C_{i'}(MA_1P)| \leq \|P\|$ for all $i, i' \leq d - 2$, $\|A_2\| \leq 10^{1+k(k_0)^2}$ and $\|A_3\| \leq 10^{(k+k_0)^2}$.

Given $\xi$ let $B_k$ be a collection of matrices $B$ so that

- $B$ is a matrix of Rauzy induction corresponding to a path starting at $\pi_s$ and ending at $\pi_R$. Anytime it returns to $\pi_s$ we have $d$ beating 1, $d_1 - 2$.
- For $B_1, B_2 \in B_k, B_1 \Delta \cap B_2 \Delta = \emptyset$.
- $\|B\| \in [10^{(2k+1+k_0)^6-(k+k_0)^4}, 10^{(2k+1+k_0)^6-(k+k_0)^4+(k+k_0)^2}]$
- $\frac{1}{\xi} < \frac{|C_{d-1}(B)|}{|C_d(B)|} < \xi$.

**Lemma 11.2** Let $M$ be a matrix at the end of transition. There exists $c, \rho < 1$, $C$ and $\xi$ so that for $k_0$ sufficiently large

1. $\lambda_{d-1}(\cup_{B \in B_k} MB \Delta) > (1 - C\rho^{(k+k_0)^2})\lambda_{d-1}(M \Delta)$
2. For all but a set of planes $P$ of measure $\rho^{c(k+k_0)^2}$

$$\lambda_2(P \cap (M \Delta \setminus \cup_{B \in B_k} MB \Delta) < \rho^{\frac{1}{\xi}(k+k_0)^2} \lambda_{d-1}(P \cap M \Delta)$$

**Proof** We first show that the subset of $M \Delta$ for which there is no $B \in B_k$ is exponentially small part of $M \Delta$. We restrict to $W(M)$ where the first bullet of $B_k$ automatically holds. By applying Proposition 5.8 to $R_2$ (which is justified by Lemma 5.5) we obtain that the last three bullets hold off of an exponentially small in $(k+k_0)^2$ subset. These matrices give us paths in $R_d$ corresponding to points in $W(M)$. So we obtain

$$\lambda_1(\cup_{B \in B_k} W(MB)) > (1 - C\rho^{(k+k_0)^2})\lambda_1(W(M)).$$

By the second conclusion of Theorem 6.2 we have (1) of the lemma. We now prove Conclusion (2). By Proposition 13.4, there is a constant $c'$ so that for $k_0$ large enough, the set of $P$ with

$$\lambda_2(P \cap M \Delta) < \rho^{\frac{1}{\xi}(k+k_0)^2} \lambda_{d-1}(M \Delta) \quad (54)$$

is at most $\rho^{c'(k+k_0)^2}$ proportion of the planes intersecting $M \Delta$. Now for a plane $P$ not satisfying (54) to fail the conclusion of the lemma we have that

$$\lambda_2(P \cap (M \Delta \setminus \cup_{B \in B_k} MB \Delta)) > 2\rho^{\frac{3}{\xi}(k+k_0)^2} \lambda_{d-1}(M \Delta).$$

The set of such planes is at most $2\rho^{\frac{3}{\xi}(k+k_0)^2} \lambda_{d-1}(\Delta)$ by Fubini’s Theorem. We have the lemma for $c$ any number smaller than $\min\{c', \frac{3}{\xi}\}$ (provided $k_0$ is large enough). 

\[ \square \]
Let $B'_k$ be matrices $B'$ of restriction on RHS starting at $\pi_R$ and ending at $\pi_s$. Let

$$s_k = 10(2k+2+k_0)^6+(k+k_0)^4.$$  \hfill (55)

Recall from our choice of matrix sizes

$$s_k \leq \|B'\| \leq 2s_k.$$ 

and that $d-1$ beats $d$ between $s_k$ and $2s_k$ times, and then $d$ beats $d-1$. The purpose of this section is to prove the following:

**Theorem 11.3** There exists $c > 0$, $\rho < 1$ and $C$ such that for all but a proportion at most $C\rho^{(k+k_0)^2}$, of planes $P \in \mathcal{P}$ that intersect $M\Delta$,

$$\lambda_2(\bigcup_{B \in B_k, B' \in B'_k} MBB' \Delta \cap P) \geq \frac{c}{s_k} \lambda_2(M \Delta \cap P).$$

11.1 Freedom on RHS

**Lemma 11.4** Let $B$ be a matrix on the right hand side. Then $F_i(MB) \subset F_i(M)$ for all $i \leq d-2$.

**Proof** Since every entry that wins is either $d-1$ or $d$, for all $i \leq d-2$, $C_i$ is not added to another column. So

$$\text{span}_\Delta(C_{i_1}(MB), \ldots, C_{i_r}(MB), C_{d-1}(MB), C_d(MB)) \subset \text{span}_\Delta(C_{i_1}(M), \ldots, C_{i_r}(M), C_{d-1}(M), C_d(M))$$

for any $i_1, \ldots, i_r \subset \{1, \ldots, d-2\}$. This implies the result. \qed

Consider the partition of $M\Delta$ into $MB\Delta$ where $B$ are matrices of the RHS and let $E$ be the complement of these partition elements.

**Lemma 11.5** The boundary of these partition elements are subsets of $F_i(M)$ for $i \leq d-2$ or $F_d(MB)$ or $F_{d-1}(MB)$.

**Proof** Since we are on the right hand side, either $d$ beats some other letter or $d-1$ beats $d$. When $d$ beats some other letter the new boundary is the face $F_d$. When $d-1$ beats $d$ the new face is $F_{d-1}$. \qed

Given a plane $P$ and a matrix $B \in B_k$, we say $P$ is standard for $MB$, if $P \cap F_{d-1}(MB) \neq \emptyset$ and $P \cap F_d(MB) \neq \emptyset$.

Let $\hat{B}_k$ be the set of $B$ for which $P$ is standard for $MB$. 

\[\text{Springer}\]
Lemma 11.6 For any \( P \), there are at most 2 different \( B \in B_k \) so that \( P \cap MB \Delta \neq \emptyset \) and \( P \) is not standard for \( MB \).

Proof If there is \( B \) such that \( MB \cap P = M \cap P \), then it is unique and the lemma holds. So we assume that this is not the case. Notice

\[
\{ x \in \text{span}_\Delta (C_{d-1}(M), C_d(M)) : \text{span}_\Delta (V(M), x) \cap P \neq \emptyset \}
\]

is an interval. Let \( p_a \) and \( p_b \) be the two extreme points of this interval and \( q_a \) and \( q_b \) be points in \( \text{span}_\Delta (V(M), p_a) \cap P \) and \( \text{span}_\Delta (V(M), p_b) \cap P \) respectively. If \( B \in B_k \) and \( MB \Delta \cap P \neq \emptyset \) choose \( q \in MB \Delta \cap P \) and consider the line segments connecting \( q_a \) to \( q \) and \( q_b \) to \( q \). If \( q_a \) or \( q_b \) are not in \( MB \Delta \), then the line segments cross the boundary of \( MB \Delta \) at \( F_j(MB) \) for \( j \geq d-1 \). To see this, if the line segment crossed \( F_i(MB) \subset F_j(M) \) for \( i \leq d-2 \) it would be leaving \( M \Delta \) (by Lemma 11.4) and so can not be entering some other simplex, \( MB_2 \Delta \cap P \). Also the two line segments have to cross at different faces (so one at \( F_d-1(MB) \) and one at \( F_d(MB) \)). It follows that all but possibly the two \( B \in B_k \) whose projection to \( \text{span}_\Delta (C_{d-1}(M), C_d(M)) \) is most extreme are standard.

\(\square\)

Corollary 11.7 For any line \( \ell \) contained in \( P \), we have there are at most 2 different \( B \) so that \( \ell \cap MB \Delta \neq \emptyset \) and \( B \) is not standard for \( P \).

Proposition 11.8 There are constants \( C \) and \( \rho < 1 \) so that for \( k_0 \) large enough, if \( M \) is a matrix at start of freedom on RHS, then for all \( B \in B_k \), except for a proportion at most \( C \rho^{(k+k_0)^2} \) of planes \( P \in \mathcal{P} \) intersecting \( M \Delta \), we have \( \lambda_2(P \cap MB \Delta) < 10^{-\frac{1}{2} k} \rho^{(k+k_0)^5} \lambda_2(P \cap M \Delta) \).

Proof We show that the second smallest singular value of \( MB \) is at most \( 10^{-\frac{1}{2} (k+k_0)^5} \) times the second smallest singular value of \( M \). By Proposition 7.5 at the start of RHS the second smallest singular value is at least

\[
\|M\|^{-1} 10^{(k+k_0)^5}
\]

So by Proposition 13.4, for a constant \( c > 0 \), off of a proportion \( \rho^{c(k+k_0)^2} \) of planes intersecting the simplex the intersection is at least

\[
\|M\|^{-1} 10^{(k+k_0)^5} \rho^{(k+k_0)^2}.
\]

Now by Proposition 4.1 and Condition *

\[
\|M\| \leq 10^{2(k+k_0)^2} u_k \leq 10^{2(k+k_0)^2} 2^k 10^{\sum_{i=3}^{2k+1} (i+k_0)^6 + 2 \sum_{i=2}^{k} (i+k_0)^2 + \sum_{i=2}^{k} (i+k_0)^2 3}.
\]

This gives us that the second smallest singular value of \( M \) is at least

\[
2^{-k} 10^{-\sum_{i=3}^{2k+1} (i+k_0)^6 - 2 \sum_{i=2}^{k} (i+k_0)^2 - \sum_{i=2}^{k} (i+k_0)^2 3} 10^{-2(k+k_0)+(k+k_0)^5}. \tag{56}
\]
At the end of freedom on the right hand side by Lemma 7.15 we have that the second smallest singular value of \( MB \) is at most \( \tilde{C} \frac{(k+1)^3}{\log(MB)} \) for some \( \tilde{C} \), which by the lower bound for \( V_k \) in Proposition 4.1 is bounded by

\[
\tilde{C}(2\zeta)^{-k} 10^{-\sum_{i=3}^{2k+1}(i+k_0)^6} 10^{-(k+k_0)^4}
\]

(because \( C_d \) and \( C_{d-1} \) have ratio at most \( \zeta \)). Comparing this with Inequality (56), we see that for \( k_0 \) large enough, the second smallest singular value of \( MB \) is at most \( 10^{-\frac{2}{3}(k+k_0)^5} \) multiplied by the second smallest singular value of \( M \).

If \( P \) is a plane so that \( \lambda_2(P \cap M\Delta) \geq 10^{-\frac{3}{2}(k+k_0)^2} \lambda_{d-1}(M\Delta) \), then since the smallest singular value is nonincreasing, the area has decayed proportionally to at least the decay in the second smallest singular value (that is, \( 10^{-\frac{2}{3}(k+k_0)^5} \)) multiplied by \( 10^{3(k+k_0)^2} \) giving a decay of at least

\[
10^{-\frac{2}{3}(k+k_0)^5} 10^{3(k+k_0)^2} \geq 10^{-\frac{1}{2}(k+k_0)^5},
\]

for \( k_0 \) large enough. By Proposition 13.4 off of a set of planes of proportion \( \rho^{(k+k_0)^2} \) of the planes intersecting \( M\Delta \) we have \( \lambda_2(P \cap M\Delta) \geq 10^{-3(k+k_0)^2} \lambda_{d-1}(M\Delta) \), establishing the proposition.

\[\square\]

### 11.2 End of restriction

Let \( B \in B_k \) and \( M = A'_1 B_1, \ldots, A'_k T_k \) be its ancestor. Consider the subinterval \( W_{MB} \subset [0, 1] \) defined by

\[
W_{MB} = \left[ s_k C_{d-1}(MB) + C_d(MB) \right], \frac{2s_k C_{d-1}(MB) + C_d(MB)}{[s_k C_{d-1}(MB) + C_d(MB)]}, \frac{2s_k C_{d-1}(MB) + C_d(MB)}{[2s_k C_{d-1}(MB) + C_d(MB)]}.
\]

Now fix \( k \) and for \( \ell \) such that \( s_k \leq \ell \leq 2s_k \), let \( B'_\ell \) be the matrix in restriction given by \( d \) beating \( d \) exactly \( \ell \) times and then \( d \) beating \( d-1 \) (to return to \( \pi_s \)). The following statements follow immediately from the definition.

\[
\bigcup_{\ell \in [s_k, 2s_k]} MB'B'_\ell \Delta = \{ y \in MB\Delta : s_k < \frac{(R^n y)_{d-1}}{(R^n y)_d} < 2 \cdot s_k + 1 \}. \tag{57}
\]

At the end of restriction, the set that is left is \( \text{span}_\Delta(C_1(MB), \ldots, C_{d-2}(MB), W_{MB}) \).
Lemma 11.9 Suppose $B$ is a matrix after freedom on RHS such that $\frac{1}{\xi} < \frac{|C_{d-1}(MB)|}{|C_d(MB)|} < \xi$. Then

$$\frac{1}{9\xi^2s_k} \lambda_{d-1}(MB\Delta) \leq \lambda_{d-1}(\text{span}_\Delta(C_1(MB), \ldots, C_{d-2}(MB), W_{MB})) \leq \frac{\xi^2}{s_k} \lambda_{d-1}(MB\Delta)$$

Proof By the fourth condition on matrices in $B_k$ we have that if $B \in B_k$ then for each matrix in $B' \in B'_k$ we have

$$|C_j(MBB')| \in \left[ \frac{1}{\xi s_k} |C_j(MB)|, (2s_k + 1)\xi |C_j(MB)| \right]$$

for $j \geq d - 1$. Moreover, $C_i(MBB') = C_i(MB)$ for $i \leq d - 2$. So by Lemma 5.6 we have that

$$(3s_k\xi)^{-2}\lambda_{d-1}(MB\Delta) \leq \lambda_{d-1}(MBB'\Delta) \leq (\frac{\xi}{s_k})^2 \lambda_{d-1}(MB\Delta).$$

There are $s_k$ such disjoint simplices, giving the lemma. $\square$

Lemma 11.10 There exists $\rho < 1$ so that off of a set of planes of measure at most $C\rho^{(k+k_0)^2}$ we have

$$\lambda_2(\bigcup_{B \in B_k, B' \in B'_k} MB\Delta \cap P)$$

is proportional to $\frac{1}{s_k} \lambda_2(\bigcup_{B \in B_k} MB\Delta \cap P)$.

Proof Consider the direction orthogonal to the smallest singular direction of $M$ that is in the direction $P$. Because $P \cap MB\Delta$ is convex, for all but an exponentially small proportion of these lines $\ell$ we have that

$$\lambda_1(\ell \cap M\Delta) > 10^{-(k+k_0)^3} \text{diam}(M\Delta \cap P).$$

By Proposition 11.8 we have that all but an exponentially small proportion of the planes, have that at least half of its length is in segments of size at most $10^{-\frac{1}{2}(k+k_0)^2} \text{diam}(M\Delta \cap P)$. By Corollary 11.7 all but at most 2 such segments are cut by $F_j(MB_1)$ and $F_{j'}(MB_2)$ where $j, j' \geq d - 1$. We have that on any such segment the part that survives restriction has length at least $\frac{1}{3s_k}$ times the length of the segment. The lemma follows. $\square$

11.3 Proof of Theorem 11.3

The Theorem follows by combining the estimates in Lemmas 11.2 and 11.10.
12 Proof of Theorem 3.1

Recall for each $k$ we have a set of simplices $C_k = M_k(\Delta)$ for a collection of matrices $M_k$. The simplices $C_k$ are measured while just starting restriction on the left. The matrices satisfy Proposition 4.1. We will have a decreasing nested set of planes $P_{k-1}$ with the property that $P_{k-1} \setminus P_k$ has exponentially small measure. The set $\hat{P} = \cap P_k$ is the set of planes in the statement of Theorem 3.1. Note that (a) is satisfied by construction and (b) is satisfied by Theorem 4.2.

The goal now is to verify (c) of Theorem 3.1. Suppose we have just finished freedom on the left side at stage $k+1$, having finished restriction on the right hand side. We apply Theorem 7.14. The second smallest singular value $\omega'(M) \geq \frac{1}{|C_{\min}(M)|}$. Combined with Theorem 7.14 we have constants $c_1, c_2 > 0$ and a plane $P$ with the property that the diameter $r_k(P)$ of its intersection $M \Delta \cap P$ is a maximum among all planes and which satisfies

$$\frac{c_1}{|C_{\min}(M)||C_{\max}(M)|} \leq r_k(P) \leq \frac{c_2}{|C_{\min}(M)|^2}.$$  

Now by Condition (1) and (4) * this gives

$$\frac{c'_1}{|C_d(M)|^2} \leq r_k(P) \leq \frac{c'_2}{|C_1(M)|^2}.$$  

Now by our choice of sizes of matrices (Proposition 4.1) we have

$$\hat{r}_k \leq r_k(P) \leq \frac{c_2}{|C_1(M)|^2} \leq c_2 \left(\frac{1}{\xi} U_{k+1}\right)^{-2} \leq \frac{c_2}{\xi^k} 10^{-2 \sum_{j=3}^{2k+1} (j+k)^6 + (j+k)^4 + \sum_{j=1}^{k} (j+k)^2}.$$  

(58)

Also, except for an exponentially small proportion of the planes intersecting the simplex (Proposition 13.4) we have,

$$\hat{r}_k \geq \frac{c_1}{|C_d(M)|^2} 10^{-(k+k+o)^2} \geq c_1 (2v_k + U_{k+1})^{-2} 10^{-(k+k+o)^2} \geq c_1 2^{-2k} 10^{-2 \sum_{j=3}^{2k+2} (j+k)^6} 10^{-(k+k+o)^2}.$$  

(59)

Recall $\hat{r}_k$ is the maximum value for the diameters at stage $k$, and $\hat{r}_k$ is the minimum at stage $k$.

From these bounds it is easy to see that (c) holds: for all $\epsilon > 0$ that

$$\lim_{k \to \infty} \frac{\hat{r}_k^{1+\epsilon}}{\hat{r}_{k+1}^{1+\epsilon}} = 0.$$  

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12.1 Proof of (d)

Let $M_k$ be the set of matrices we keep after freedom on LHS at stage $k$ and $M'_k$ after restriction on LHS. Let $M''_k$ be the set of all matrices we keep at the end of restriction on RHS at the end of stage $k$. Let $C_k$ denote the union of the illuminated sets at the end of freedom on LHS at stage $k$. Note that by Sect. 9 (Remark 9.8) we may assume that it is a union of simplices.

Recall the definition

$$t_k = 10^{-[(2k+1+k_0)^6+(k+k_0)^4+\frac{1}{2}(k+k_0)^2]}$$

given by (45) and

$$s_k = 10^{(2k+2+k_0)^6+(k+k_0)^4}$$

given by (55).

For a constant $c > 0$ to be determined, set

$$a_k = \frac{c t_k}{s_k}$$

Inductive Assumption: There exists a constant $c > 0$ so at the end of freedom on LHS at stage $k$ a set of planes $P \in \mathcal{P}$ whose Lebesgue measure is at least half of the total measure of planes satisfies

$$\lambda_2(P \cap C_k) \geq \prod_{i=1}^{k-1} a_i$$  \hspace{1cm} (Y_{k-1})$$

Call the planes that satisfy this inequality $P_{k-1}$. Also, there is a constant $C$ such that

$$\lambda_{d-1}(C_k) \leq C^{k-1} \prod_{i=1}^{k-1} a_i$$  \hspace{1cm} (Z_{k-1})$$

Note that we will show that $(Y_{k-1})$ and $(Z_{k-1})$ implies that the set of planes that satisfy $(Y_{k-1})$ and not $(Y_k)$ is exponentially small which will imply that more than half of the set of planes satisfy $(Y_i)$ for all $i$.

To show that these values of $a_k$ (for a given $c$) allow us to prove (d) we use the following theorem.

**Theorem 12.1** There exists $\rho < 1$ and $c > 0$ so that for each plane $P \in \mathcal{P}_{k-1}$ except for a subset of $\mathcal{P}_{k-1}$ of measure at most $\rho^{k+k_0}$, has the property that
for a set of connected components $J \subset P \cap C_k$ whose union has measure at least $(1 - \frac{1}{q^r})\lambda_2(C_k \cap P)$,

$$\lambda_2(M_{k+1} \cap J) \geq a_k \lambda_2(J) \quad (60)$$

where $M_{k+1}$ are the matrices $M_{k+1}$ at the start of restriction on LHS at stage $k + 1$ so that

$$\text{diam}(P \cap M_{k+1} \Delta) > \hat{r}_k. \quad (61)$$

Moreover the inductive assumptions $(Y_k)$ and $(Z_k)$ hold. For some $C$

$$\lambda_2(P \cap C_{k+1}(\Delta)) \geq \prod_{j=1}^{k} a_k$$

and

$$\lambda_{d-1}(C_{k+1}(\Delta)) \leq C^k \prod_{j=1}^{k} a_k.$$

We begin the proof. In proving Theorem 12.1 in going from $C_k$ to $C_{k+1}$ we proceed in 3 steps going through restriction on LHS at stage $k$ where we apply Theorem 10.1; to going through freedom and restriction on RHS where we will apply Theorem 11.3 at stage $k$; and finally to going through freedom on LHS (at stage $k + 1$) where we apply Theorem 8.2.

**Proposition 12.2** There exists $\rho < 1$ and $c' > 0$ so that except for a set of $P \in \mathcal{P}_{k-1}$ of measure at most $\frac{1}{4} \rho^{(k+0)}$ we have that for a set of connected components $J \subset P \cap C_k$ of total measure at least $(1 - \frac{1}{4^r})\lambda_2(P \cap C_k)$ satisfies:

$$\lambda_2\left( \bigcup_{M \in \mathcal{M}_k} (M \Delta \cap J) \right) \geq c' t_k \lambda_2(J). \quad (62)$$

To prove this Proposition we will need the following lemmas.

**Lemma 12.3** Let $\mathcal{N}(M)$ be as defined just before Theorem 10.1. There exists $c' > 0$ so that if $\ell$ is a line in direction $\phi$ intersecting $M(\Delta)$ then

$$\lambda_1(\ell \cap \mathcal{N}(M)) \geq c' t_k \lambda_1(\ell \cap M(\Delta)).$$

Proof $\mathcal{N}(M)$ is a slab about $F_1(M)$ with width $t_k d_M$. If the line segment crosses $\mathcal{N}(M)$ the length of the intersection with $\mathcal{N}(M)$ is $\frac{t_k d_M}{\sin(\gamma)}$ where $\gamma$ is the angle $\ell$ makes with $F_1(M)$. Note that $\lambda_1(\ell \cap M \Delta) \leq \frac{d_M}{\sin(\gamma)}$. □

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From the definition of $\mathcal{N}(M)$ we have:

**Lemma 12.4** There exists a constant $C'$ so that $\lambda_{d-1}(\mathcal{N}(M)) < C't_k\lambda_{d-1}M(\Delta)$.

**Proof of Proposition 12.2** By Lemma 12.3, and $(Y_{k-1})$, for all of the planes in $\mathcal{P}_{k-1}$ we have that

$$\lambda_2(\bigcup_{M \in \mathcal{M}_k} \mathcal{N}(M) \cap P) \geq c't_k \prod_{j=1}^{k-1} a_j. \quad (63)$$

By applying Theorem 10.1 to each simplex in $\mathcal{M}_k(\Delta)$ we have that

$$\lambda_{d-1}\left( \bigcup_{M \in \mathcal{M}_k} \mathcal{N}(M) \setminus (\bigcup_{M \in \mathcal{M}'_k} M \Delta) \right) \leq \alpha^{(k+k_0)^2} \lambda_{d-1}\left( \bigcup_{M \in \mathcal{M}_k} \mathcal{N}(M) \right) \leq \alpha^{(k+k_0)^2} C't_k \lambda_{d-1}(C_k).$$

Lemma 12.4 gives the second inequality. By a straightforward estimate using Fubini’s Theorem the measure of planes $P \in \mathcal{P}_{k-1}$ so that

$$\lambda_2 \left( P \cap \left( \bigcup_{M \in \mathcal{M}_k} \mathcal{N}(M) \setminus (\bigcup_{M \in \mathcal{M}'_k} M \Delta) \right) \right) > \alpha^{1/2} \lambda_{d-1}(C_k) \quad (64)$$

is at most $C'' \alpha^{1/2} C''(k+k_0)^2$ for some $C''$. Appealing to Assumption $(Y_{k-1})$ and $(Z_{k-1})$ which say that $\lambda_{d-1}(C_k) < \left( \frac{C}{c} \right)^{k-1} \lambda_2(P \cap C_k)$ we have the measure of the set of planes such that

$$\lambda_2 \left( P \cap \left( \bigcup_{M \in \mathcal{M}_k} \mathcal{N}(M) \setminus (\bigcup_{M \in \mathcal{M}'_k} M \Delta) \right) \right) > \alpha^{1/2} \lambda_{d-1}(\bigcup_{M \in \mathcal{M}_k} \mathcal{N}(M) \cap P) \quad (65)$$

is at most $C'' \alpha^{1/2} C''(k+k_0)^2$ (if $k_0$ is large enough). Choosing $k_0$ large enough so that $C'' \alpha^{1/2} C''(k+k_0)^2 > \frac{1}{9}$ for all $k$, the set of planes for which (65) holds is contained in the set of planes (62) holds for. \qed

Analogously to the previous proposition we have the following:

**Proposition 12.5** There exists $\rho < 1$, $c' > 0$ so that except for a set of $P \in \mathcal{P}_{k-1}$ of measure at most $\frac{1}{4} \rho^{k+k_0}$, for a set of connected components $J = \mathcal{M}'_k \Delta \cap P$ whose union has measure at least $(1 - \frac{1}{4} \frac{1}{9k}) \lambda_2(P \cap C_k)$ we have

$$\lambda_2(\bigcup_{M \in \mathcal{M}'_k} M(\Delta) \cap J) \geq \frac{c'}{s_k} \lambda_2(J).$$
We sketch the proof. By Theorem 11.3 there exists \( \rho < 1, c' > 0 \) so that for each simplex \( M \Delta \) (with \( M \in \mathcal{M}_k' \)) there exists a subset \( \mathcal{E}(M) \subset M \Delta \) of measure at most \( \rho^{(k+k_0)^2} \lambda_{d-1}(M \Delta) \), so that if \( P \cap M \Delta \setminus \mathcal{E}(M) \neq \emptyset \) then

\[
\lambda_2(P \cap M \Delta \cap \mathcal{M}_k') > \frac{c'}{s_k} \lambda_2(P \cap M \Delta).
\]

So we want to show that for all planes \( P \), except for a set of planes of measure at most \( \frac{1}{4} \rho^{k+k_0} \), we have

\[
\lambda_2\left( P \cap (\bigcup_{M \in \mathcal{M}_k'} \mathcal{E}(M)) \right) < \frac{1}{4} \frac{1}{9^k} \lambda(P \cap \mathcal{M}_k' \Delta).
\]

Now by Lemma 12.4 and Assumptions \((Z_{k-1})\) we have that \( \lambda_{d-1}(\mathcal{M}_k' \Delta) \leq C' C^{k-1} t_k \prod_{i=1}^{k-1} \frac{t_i}{s_i} \) and so

\[
\lambda_{d-1}(\bigcup_{M \in \mathcal{M}_k'} \mathcal{E}(M)) < \rho^{(k+k_0)^2} C' C^{k-1} t_k \frac{1}{c^{k-1}} \prod_{j=1}^{k-1} a_j.
\]

The remainder of the proof is as in Proposition 12.2.

Theorem 8.2 together with a proof analogous to Proposition 12.2 implies:

**Proposition 12.6** There exists \( \rho < 1 \) and \( c' > 0 \), so that except for a set of planes \( P \in \mathcal{P}_{k-1} \) of measure at most \( \frac{1}{4} \rho^{k+k_0} \), for a set of connected components \( J \subset P \cap \mathcal{M}_k' \Delta \) of total measure at least \( (1 - \frac{1}{4 \rho^k}) \lambda_2(P \cap C_k) \) we have

\[
\lambda_2(J \cap C_{k+1}) \geq c' \lambda_2(J).
\]

**Proof of Theorem 12.1** The last three propositions establish the desired result except for (61) on the diameters on the planes. (In particular (60) is established by the Propositions). We prove (61). By Theorem 7.14 each simplex is cut by some plane whose intersection has diameter at least proportional to \( \frac{\omega'}{|C_{\max}(M)|} \), where by Proposition 7.5, \( \omega' \) is at least proportional to \( |C_{\min}(M)|^{-1} \). This is at least proportional to \( U_k^{-1} \) in Proposition 4.1, which is at least

\[
(2^k 10^{2k+1} (i+k_0)^6 + 2 \sum_{i=2}^{k} (i+k_0)^2 + \sum_{i=1}^{k} (i+k_0)^2 (i+k_0)^3 -(k+k_0)^4)^{-1}
\]

By Proposition 13.7, since a simplex is a convex set, the proportion of the measure of a simplex \( \Delta_k \) that is cut with diameter smaller that \( 10^{-(k+k_0)^2} U_k^{-1} \) is at most \( c 10^{-(k+k_0)^2} \lambda_{d-1}(\Delta_k) \). By the above discussion, the measure of a
simplex that lies in the complement of these small intersections has diameter at least $\hat{r}_k$. Analogously to the proof of Proposition 12.2 we obtain the theorem off of a subset of planes in $\mathcal{P}_{k-1}$ of measure at most $\frac{1}{4}\rho^{k+k_0}$. The total loss of measures of planes using this comment and the three propositions is then at most $\rho^{k+k_0}$. Then we take $\mathcal{P}_k$ to be $\mathcal{P}_{k-1}$ minus this exponentially small set of planes. The total removed has measure at most $\sum_{k=1}^{\infty} \rho^{k+k_0}$. We choose $k_0$ large enough, so this sum is at most $1/2$ the total measure of the set of planes, which establishes our condition on the measure of $\mathcal{P}_k$.

Inequality (60) implies Assumption $(Y_k)$. Assumption $(Z_k)$ follows from Lemmas 12.4 and 11.9. Indeed, we apply these estimates to each simplex. Summing over the simplices we obtain $(Z_k)$. $\square$

Note that $\hat{\mathcal{P}}$ in the statement of Theorem 3.1 is $\cap_{k=1}^{\infty} \mathcal{P}_k$. We call $P \in \mathcal{P}$ good.

We finish the proof of (d) by showing that for every $\epsilon > 0$ and good plane

$$\lim_{k \to \infty} \frac{1}{2} + \epsilon \prod_{j=1}^{k} a_j^{-1} = 0. \quad (66)$$

Recall $a_j = c \frac{t_j}{s_j}$. There exists $q$ a degree 5 polynomial so that both $t_j$ and $\frac{1}{s_j}$ are at least $10^{-(2j)^6 + q(j)}$. From this it follows that there exists $p$ a degree 6 polynomial so that

$$\prod_{j=1}^{k} \langle a_j \rangle^{-1} < 10^{\frac{1}{2} (2k)^7 + p(k)}.$$

Now by (58) for every good plane we have that there exists a degree at most 6 polynomial $\tilde{p}$ so that $\hat{r}_k < 10^{-\frac{2}{2} (2k)^7 + \tilde{p}(k)}$. This establishes (66) and so (d) holds.

### 12.2 Proof of (e)

For any $j$ let $F_j = \text{span}(e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_d)$ and for any matrix $M_k$ after finishing freedom on LHS recall $F_j(M_k) = M_k(F_j)$ the $j$ face. For any $i, j$ and matrix $D_{k+1} = A_k' T_k B_k B_k' A_{k+1}$ built from that point to finishing freedom on LHS at stage $k+1$, since $D_{k+1}$ is positive, we have for all $i, j$ the projective distances satisfy,

$$d\left(\frac{C_i(D_{k+1})}{|C_i(D_{k+1})|}, F_j\right) = d\left(\frac{D_{k+1}(e_i)}{|D_{k+1}e_i|}, F_j\right) \geq \frac{1}{\|D_{k+1}\|}. $$

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Then since $M$ projectively contracts distances by at most $\frac{1}{\|M_k\|}$ we have
\begin{align*}
\frac{d(C_i(M_{k+1})}{|C_i(M_k)|}, F_j(M_k)) &= d\left(\frac{C_i(M_{k}D_{k+1})}{|C_i(M_kD_{k+1})|}, F_j(M_k)\right) = d\left(\frac{M_kD_{k+1}(e_i)}{|M_kD_{k+1}e_i|}, F_j(M_k)\right) \\
&\geq \frac{1}{\|M_k\|^2}d(D_{k+1}(e_i), F_j) \geq \frac{2}{|C_d(M_k)|}10^{-[2(2k+k_0)^6+(2k+1+k_0)^2]+2(k+k_0)^2}.
\end{align*}

This gives a lower bound on distance between intersection of descendant simplices at stage $k + 1$ and the complement of their stage $k$ ancestors. The distance between descendants at stage $k + 2$ and the complement of their stage $k$ ancestors is even larger. On the other hand from (58) we have that
\[ \tilde{r}_{k+2} \leq \frac{c_2}{|C_1(M_{k+2})|^2} \leq c_2 10^{-2(2k+k_0)^6} 10^{-2(2k+1-k_0)^6} \frac{1}{|C_d(M_k)|^2}, \]

by our choice of matrix sizes. For large enough $k_0$, for all $k$ this diameter is smaller than the above distance to the complement.

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13 Appendix

We denote by $A$ matrices of freedom on LHS. Recall from the proof of Proposition 9.3, the definition of a path of Rauzy induction $\gamma(x, m)$ being $\pi_s$ via $\pi'$ isolated. We wish to prove

**Lemma 13.1** There exists $\rho > 0$ and $m_0$ so that for any hyperplane $H$

\[ \lambda_\Delta(\{x \in \Delta \times \pi_L : \exists \gamma(x, m) \text{ } \pi_s \text{ via } \pi' \text{ isolated } m \leq m_0 \text{ with } A(x, m) \Delta \cap H = \emptyset\}) > \rho. \]

With the same notations as above, via usual balanced estimates we have:

**Corollary 13.2** Given $\zeta$ there exists $\rho_1$ such that if $M = M(x, r)$ a matrix of Rauzy induction so that

- $|C_i(M)| < |C_i'(M)|$ for all $1 \leq i, i' \leq d - 2$,
- $\pi(R'y) = \pi_L$.
and if $H$ is any hyperplane contained in $\text{span}_\Delta(C_1(M), \ldots, C_{d-2}(M))$ then

$$\lambda_{d-3}(\{y \in \text{span}_\Delta(C_1(M), \ldots, C_{d-2}(M)) : \exists m \leq m_0 \text{ with } \gamma(R^s y, m) \pi_s \text{ via } \pi' \text{ isolated and } \text{MA}(R^s y, m) \Delta \cap H = \emptyset\}) > \rho_1 \lambda_{d-3}(\text{span}_\Delta(C_1(M), \ldots, C_{d-2}(M))).$$

In order to prove Lemma 13.1 we need the following simple lemma first.

**Lemma 13.3** There is constant $\epsilon_0$ such that if a hyperplane in $\Delta$ intersects the radius $\frac{1}{2}$ ball about the $e_1$ vertex it cannot intersect the $\epsilon_0$ ball about every other vertex.

*Proof* The proof is by contradiction. If the lemma is false there is a sequence of hyperplanes $H_n$ defined by $a_{1,n}x_1 + a_{2,n}x_2 + \cdots + a_{d,n}x_n = 1$ which intersect the $\frac{1}{n}$ neighborhoods of $e_j$ for $j > 1$ and the $\frac{1}{2}$ neighborhood of $e_1$. This forces the coefficients $a_{j,n}$ to be bounded. Passing to a subsequence and taking a limit we find that the limiting hyperplane $H$ would contain $e_j$ for $j > 1$, and intersect the $\frac{1}{2}$ neighborhood of $e_1$. Then $H$ must be of form

$$a_1x_1 + x_2 + \cdots + x_d = 1.$$ 

But this is not a hyperplane subset of $\Delta$. \hfill \Box

*Proof of Lemma 13.1* Let $\epsilon_0$ be from the last lemma. It suffices to show that for any $i > 1$ there exists a bounded length path of Rauzy induction whose corresponding subsimplices are contained in $B(e_i, \epsilon_0)$, and there exists a path whose corresponding subsimplex of Rauzy induction is contained in $B(e_1, \frac{1}{2})$.

In the second case consider the path where 1 wins $d-3$ consecutive times. It reaches $\pi'$ and then after 1 beats 2 it returns to $\pi_s$. So the first interval is longer than the sum of the other intervals, so its length is at least $\frac{1}{2}$.

In the first case ($i > 1$) starting at $\pi_L$ have $d-2$ beat 1 then 2, $\ldots$, $i-1$. Then have $i$ beat $d-2$, $d-3$, $\ldots$ $i+1$ for $\frac{4}{\epsilon_0}$ consecutive times. Then have $d-2$ beat $i$, $i+1 \ldots 2$ then have it beat 1, $\ldots$, $d-3$. This implies that $x_{d-2} > 2(x_1 + \cdots x_{i-1} + x_{i+1} \ldots + x_{d-3})$. So,

$$x_i > \frac{4}{\epsilon_0}(x_{d-2} - (x_1 + \cdots + x_{i-1})) > \frac{4}{\epsilon_0}\frac{1}{2}x_{d-2}$$

$$> \frac{4}{\epsilon_0}\frac{1}{4}(x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_{d-2}),$$

establishing that the hyperplane intersects $B(e_i, \epsilon_0)$. \hfill \Box
13.1 Measure of parallel planes

Let $D$ be a compact, convex set in $\mathbb{R}^k$. Let $P_0$ be a plane and $\mathcal{P}$ be the set of planes parallel to $P_0$ that intersect $D$. The orthocomplement of $P_0$ is a copy of $\mathbb{R}^{k-2}$ and has Lebesgue measure. We identify $\mathcal{P}$ with a (convex) set $E \subset \mathbb{R}^{k-2}$, by identifying a point in $\mathcal{P}$ with the point in the orthocomplement it intersects. This induces a measure $\nu$ on $\mathcal{P}$. Let

$$A = \max\{\lambda_2(P \cap D) : P \in \mathcal{P}\}.$$

**Proposition 13.4** There exists a constant $C$, depending only on dimension, such that for all $\epsilon > 0$,

$$\nu(\{P \in \mathcal{P} : \lambda_2(P \cap D) < \epsilon A\}) \leq C\sqrt{\epsilon} \nu(\mathcal{P}).$$

We first prove

**Lemma 13.5** Let $f : E \to [0, \infty)$ by $f(e) = \lambda_2(P_e \cap D)$ where $P_e$ is the element of $\mathcal{P}$ that contains $e$. Then on each line, $f$ can be represented as $u + v$ where $u$ is a concave function and $v$ is the square of a concave function.

**Proof** Let $e_0, e_1 \in E$, identify $e_0$ with 0, $e_1$ with 1 and $(1-t)e_0 + te_1$ with $t$. Consider a direction $w$ in $P_0$ and parametrize the lines parallel to $w$ in each $P \in \mathcal{P}$ by the orthogonal direction to $w$ in $P_0$. So on each $P$ we have a family of lines $\ell_x$ parallel to $w$, for $x \in \mathbb{R}$. Let $h(t, x)$ be the length of the line $\ell_x$ on $P((1-t)e_0 + te_1)$ intersected with $D$. We assume our parametrization of orthogonal direction to $w$ is chosen so that

$$\inf\{x : h(1, x) > 0\} = \inf\{x : h(0, x) > 0\}$$

and that these infimum are 0. Let us assume that

$$b = \sup\{x : h(1, x) > 0\} \geq \sup\{x : h(0, x) > 0\} = c.$$  

Now since the simplex is convex, for all $x$ with $h(0, x) > 0$ and for all $0 \leq t \leq 1$

$$h(t, x) \geq (1-t)h(0, x) + th(1, x).$$

So we have that

$$u(t) = \int_0^c h(t, x) dx.$$  

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is a concave function. Now we wish to show that if we set

$$v(t) = \int_c^b h(t, x) \, dx$$

then $\sqrt{v}$ is concave. To do that it suffices to show that for $0 \leq s \leq 1$

$$v(s) \geq s^2 v(1) \quad (67)$$

(because $v(0) = 0$). Let $\ell_x(0)$ be the line $\ell_x$ on $P_{e_0}$ and $\ell_x(1)$ be the line $\ell_x$ on $P_{e_1}$. Let $E$ be the convex hull of $\bigcup_{x \in [c, b]} \ell_x(1) \cap D$ and any point in $\ell_x(0) \cap \mathcal{D}$. Note that $\ell_x(0) \neq \emptyset$ because $D$ is closed. Since $D$ is convex, $E \subset D$ and so

$$v(s) \geq \lambda_2(P_{(1-s)e_0+se_1} \cap E). \quad (68)$$

Now $E$ is a convex cone so

$$\lambda_2(P_{(1-a)e_0+se_1} \cap E) = s^2 \lambda_2(E \cap P_{e_1}). \quad (69)$$

Combining (68) and (69) verifies the sufficient condition (67), completing the proof.

**Lemma 13.6** Let $\ell \subset \mathbb{R}^1$ be a segment If $g : \ell \to [0, \infty)$ is concave then for all $\epsilon > 0$, $\lambda(\{x \in \ell : g(x) < \epsilon \max g\}) < \epsilon \lambda(\ell)$.

**Proof** Let $p$ be a point that maximizes $g$. For any $q \in \Omega$, let $\ell(q, p)$ be the line connecting $q$ and $p$ with unit speed parametrization of $\ell(q, p)$. Let its length be $r$. If $g(q) < \epsilon g(p)$ then since $g$ concave, $q \in [0, \epsilon r)$. The lemma follows. \qed

We now prove Proposition 13.4.

**Proof** We fix a point $p \in \mathcal{D}$ with $\lambda_2(P_p \cap \mathcal{D}) = A$, where $P_p$ is the plane in $\mathcal{P}$ going through $p$. We compute

$$v(\{P \in \mathcal{P} : \lambda_2(P \cap \mathcal{D}) < \epsilon A\})$$

via integrating with respect to polar coordinates. Indeed, let $E$ be the $k - 2$ dimensional subspace of $\mathbb{R}^k$ containing the directions orthogonal to the directions in the planes of $\mathcal{P}$. Let $S^{k-3}$ denote the unit sphere in $E$, $\ell_{\theta}$ denote the line in direction $\theta$ through $p$ for each $\theta \in S^{k-3}$. Let $v_{\theta}$ denote $\ell_{\theta} \cap \mathcal{D}$. Let $\tau_{\theta}$ be the set of $q$ at least half of the way from $p$ to one of the endpoints of $v_{\theta}$. It is the union of two line segments, $\tau_{\theta,1}$ and $\tau_{\theta,2}$. Let $\sigma_{\theta}$ be the set of points on $v_{\theta}$ between a quarter and $\frac{3}{4}$ of the way from $p$ to the endpoints of $v_{\theta}$. Let $\sigma_{\theta,1} = \sigma_{\theta} \cap \tau_{\theta,1}$ and $\sigma_{\theta,2} = \sigma_{\theta} \cap \tau_{\theta,2}$. Observe that if $\epsilon$ is small enough
then for any $\theta, i$ we have that if $q \in \sigma_{\theta, i}$ then $\lambda_2(P_q \cap D) > \epsilon A$. Let $\hat{\phi}$ be normalized Lebesgue measure on $S^{k-3}$. Let $P_q$ denote the plane in $\mathcal{P}$ through $q$. By Lemma 13.5 the area of $P_q \cap D$ as a function of $q \in \ell_\theta$ is the sum of a concave and the square of a concave function. Therefore by Lemma 13.6

\[ |\{q \in \tau_{\theta, i} : \text{area}(P_q \cap D) < \epsilon A\}| \leq 2\epsilon |\tau_{\theta, i}|. \]  

(70)

Observe that for $q \in \sigma_{\theta, i}$, $q' \in \tau_{\theta, i}$ we have that $d(q', p) < 2^{k-3}$. From this we obtain that for any measurable sets $B, B'$ using polar coordinates

\[ \int_{S^{k-3}} \int_{\sigma_{\theta, 1} \cup \sigma_{\theta, 2}} d(q, p)k^{-3} \chi_B(q)dq \hat{\phi}(\theta) \geq 2^{-3(k-3)} \int_{S^{k-3}} \int_{\tau_{\theta, 1} \cup \tau_{\theta, 2}} d(q, p)k^{-3} \chi_{B'}(q)dq \hat{\phi}(\theta). \]

Considering $B = D$ and $B' = \{q \in E : \lambda_2(P_q \cap D) < \epsilon A\}$, we obtain the Proposition from (70) applied to each line.

Proposition 13.7 Let $A' = \max\{\text{diam}(P \cap D)\}$. There exists a constant $C$, depending only on dimension, such that for all $\epsilon > 0$,

\[ \nu(\{P \in \mathcal{P} : \text{diam}(P \cap D) < \epsilon A'\}) \leq C\epsilon \nu(\mathcal{P}). \]

Noticing that on each line in $E$, the function $f(\epsilon) = \text{diam}(P_\epsilon \cap D)$ is a concave function we obtain the proposition analogously to Proposition 13.4.

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