Further Results on the Structure of (Co)Ends in Finite Tensor Categories

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Abstract

Let $C$ be a finite tensor category, and let $M$ be an exact left $C$-module category. The action of $C$ on $M$ induces a functor $\rho : C \to \text{Rex}(M)$, where $\text{Rex}(M)$ is the category of $k$-linear right exact endofunctors on $M$. Our key observation is that $\rho$ has a right adjoint $\rho^{ra}$ given by the end

$$\rho^{ra}(F) = \int_{M \in M} \text{Hom}(M, F(M)) \quad (F \in \text{Rex}(M)).$$

As an application, we establish the following results: (1) We give a description of the composition of the induction functor $\mathcal{C}^M \to \mathcal{Z}(\mathcal{C}^M)$ and Schauenburg’s equivalence $\mathcal{Z}(\mathcal{C}^M) \approx \mathcal{Z}(C)$. (2) We introduce the space $\text{CF}(M)$ of ‘class functions’ of $M$ and initiate the character theory for pivotal module categories. (3) We introduce a filtration for $\text{CF}(M)$ and discuss its relation with some ring-theoretic notions, such as the Reynolds ideal and its generalizations. (4) We show that $\text{Ext}_{C}^*(1, \rho^{ra}(\text{id}_M))$ is isomorphic to the Hochschild cohomology of $M$. As an application, we show that the modular group acts projectively on the Hochschild cohomology of a modular tensor category.

Keywords Finite tensor category · Module category · Modular tensor category · Hochschild cohomology

1 Introduction

Let $C$ be a finite tensor category. In recent study of finite tensor categories and their applications, it is important to consider the end $A = \int_{X \in C} X \otimes X^*$ and the coend $L = \int^{X \in C} X^* \otimes X$. The end $A$ is a categorical counterpart of the adjoint representation of a Hopf algebra. By using the end $A$, we have established the character theory and the integral theory for finite tensor categories in [44,47], respectively. The coend $L$, which is isomorphic to $A^*$, plays a cen-
tral role in Lyubashenko’s work on ‘non-semisimple’ modular tensor categories [23,27–29]. These results are used in recent progress of topological quantum field theory and conformal field theories [18,21,22].

Since these objects are defined by the universal property, it is difficult to analyze its structure. The aim of this paper is to provide a general framework to deal with such (co)ends. Let \( \mathcal{M} \) be an indecomposable exact left \( \mathcal{C} \)-module category in the sense of [17]. We denote by \( \text{Rex}(\mathcal{M}) \) the category of \( k \)-linear right exact endofunctors on \( \mathcal{M} \). The action of \( \mathcal{C} \) on \( \mathcal{M} \) induces a functor \( \rho : \mathcal{C} \to \text{Rex}(\mathcal{M}) \) given by \( \rho(X)(M) = X \otimes M \). Our key observation is that a right adjoint of \( \rho \), say \( \rho^{ra} \), is a \( k \)-linear faithful exact functor such that

\[
\rho^{ra}(F) = \int_{M \in \mathcal{M}} \text{Hom}(M, F(M)) \quad (F \in \text{Rex}(\mathcal{M})),
\]

where \( \text{Hom} \) is the internal Hom functor (Theorem 3.4). The end \( A \) considered at the beginning of this paper is just the case where \( \mathcal{M} = \mathcal{C} \) and \( F = \text{id}_\mathcal{C} \). This result allows us to discuss interaction between several ends through \( \rho^{ra} \). As applications, we obtain several results on finite tensor categories and their module categories as summarized below:

1. Let \( \mathcal{C}^*_{\mathcal{M}} \) be the dual of \( \mathcal{C} \) with respect to \( \mathcal{M} \). We give an explicit description of the composition of the induction functor \( \mathcal{C}^*_{\mathcal{M}} \to \mathcal{Z}(\mathcal{C}^*_{\mathcal{M}}) \) and Schauenburg’s equivalence \( \mathcal{Z}(\mathcal{C}^*_{\mathcal{M}}) \approx \mathcal{Z}(\mathcal{C}) \). We note that this kind of method has been utilized to compute higher Frobenius–Schur indicators [41].

2. Generalizing [44], we introduce the space \( \text{CF}(\mathcal{M}) \) of class functions of \( \mathcal{M} \). We also introduce the notion of pivotal module category and develop the character theory for such a module category. Especially, we show that the characters of simple objects are linearly independent.

3. We introduce a filtration \( \text{CF}_1(\mathcal{M}) \subset \text{CF}_2(\mathcal{M}) \subset \cdots \subset \text{CF}(\mathcal{M}) \) for the space of class functions. If \( \mathcal{M} \) is pivotal, then the first term \( \text{CF}_1(\mathcal{M}) \) is spanned by the characters of simple objects of \( \mathcal{M} \) and the second term has the following expression:

\[
\text{CF}_2(\mathcal{M}) \cong \text{CF}_1(\mathcal{M}) \oplus \bigoplus_{L \in \text{Irr}(\mathcal{M})} \text{Ext}^1_{\mathcal{M}}(L, L).
\]

4. We show that \( \text{Ext}^*_{\mathcal{C}}(1, A_{\mathcal{M}}) \) is isomorphic to the Hochschild cohomology of \( \mathcal{M} \), where \( 1 \) is the unit object of \( \mathcal{C} \) and \( A_{\mathcal{M}} = \rho^{ra}(\text{id}_\mathcal{M}) \). As an application, we show that the modular group \( \text{SL}_2(\mathbb{Z}) \) acts projectively on the Hochschild cohomology of a modular tensor category, generalizing [26].

**Organization of this paper**

This paper is organized as follows: Section 2 collects several basic notions and facts on finite abelian categories, finite tensor categories and their module categories from [11,12,14,19,31].

In Sect. 3, we study adjoints of the action functor \( \rho : \mathcal{C} \to \text{Rex}(\mathcal{M}) \) for a finite tensor category \( \mathcal{C} \) and a finite left \( \mathcal{C} \)-module category \( \mathcal{M} \). We show that \( \rho \) is an exact functor, and thus has a left adjoint and a right adjoint. It turns out that a right adjoint \( \rho^{ra} \) of \( \rho \) is expressed as in (1.1). Moreover, \( \rho^{ra} \) is \( k \)-linear faithful exact functor if \( \mathcal{M} \) is indecomposable and exact (Theorem 3.4).

The functor \( \rho^{ra} \) has a natural structure of a monoidal functor and a \( \mathcal{C} \)-bimodule functor as a right adjoint of \( \rho \). The structure morphisms of \( \rho^{ra} \) are expressed in terms of the universal dinatural transformation of \( \rho^{ra} \) as an end (Lemmas 3.7 and 3.8). By using the structure morphisms of \( \rho^{ra} \), we can ‘lift’ the adjoint pair \((\rho, \rho^{ra})\) to an adjoint pair between the Drinfeld
center \( Z(C) \) and the category \( \text{Rex}_C(M) \) of \( k \)-linear right exact \( C \)-module endofunctors on \( M \) (Theorem 3.11). As an application, we give an explicit description of the composition
\[
\mathcal{C}_M^* := \text{Rex}_C(M)^{\text{rev}} \xrightarrow{\text{induction}} Z(C_M^*) \xrightarrow{\text{Schauenburg's equivalence}} Z(C)
\]
in terms of the structure morphisms of \( \rho^a \) (Theorem 3.14).

In Sect. 4, we consider an end of the form \( A_S := \int_{X \in S} \text{Hom}(X, X) \) for some topologizing full subcategory \( S \) of \( M \) in the sense of Rosenberg [37]. The end \( A_S \) has a natural structure of an algebra in \( C \). The main result of this section states that, if \( M \) is an indecomposable exact left \( C \)-module category, then \( A_S \) is a quotient algebra of \( A_M \) and the map
\[
\begin{cases}
\text{topologizing full subcategories of } M \\
\text{quotient algebras of } A_M \text{ in } C
\end{cases}
\]
\[\mapsto\]
\[
S \mapsto A_S
\]
preserves and reflects the order in a certain sense (Theorem 4.6). Another important result in Sect. 4 is that, if \( S \) is closed under the action of \( C \), then \( A_S \) lifts to a commutative algebra \( \mathbb{A}_S \) in \( Z(C) \) (Theorem 4.9).

In Sect. 5, we consider the space \( \text{CF}(M) := \text{Hom}_C(A_M, 1) \) of ‘class functions’ of \( M \). As we have seen in [44], \( \text{CF}(M) \) is an algebra if \( M = C \). We extend this result by constructing a map \(* : \text{CF}(C) \times \text{CF}(M) \to \text{CF}(M)\) making \( \text{CF}(M) \) a left \( \text{CF}(C) \)-module (Lemma 5.3). We also introduce the notion of pivotal structure of an exact module category over a pivotal finite tensor category (Definition 5.6) in terms of the relative Serre functor introduced in [19]. Let \( C \) be a pivotal finite tensor category, and let \( M \) be a pivotal exact \( C \)-module category. Then, for each object \( M \in M \), the internal character \( \text{ch}_M(M) \) is defined in an analogous way as [44] (Definition 5.8). Our main result in this section is the following generalization of [44]: The linear map
\[
\text{ch}_M : \text{Gr}_k(M) \to \text{CF}(M), \quad [M] \mapsto \text{ch}_M(M)
\]
is a well-defined injective map, where \( \text{Gr}_k(\cdot) = k \otimes \mathbb{Z} \text{Gr}(\cdot) \) is the coefficient extension of the Grothendieck group. Moreover, we have
\[
\text{ch}_M(X \otimes M) = \text{ch}_C(X) \ast \text{ch}_M(M)
\]
for all objects \( X \in C \) and \( M \in M \).

In Sect. 6, we introduce a filtration to the space of class functions. Let \( C \) be a finite tensor category, and let \( M \) be an exact left \( C \)-module category. There is the socle filtration \( M_1 \subset M_2 \subset \cdots \) of \( M \). By the result of Sect. 4, we have a series \( A_{M_1} \to \cdots \to A_{M_2} \to A_{M_1} \) of epimorphisms in \( C \). Thus, by applying the functor \( \text{Hom}_C(\cdot, 1) \) to this series, we have a filtration
\[
\text{CF}_1(M) \subset \text{CF}_2(M) \subset \text{CF}_3(M) \subset \cdots \subset \text{CF}(M),
\]
where \( \text{CF}_n(M) := \text{Hom}_C(A_{M_n}, 1) \). We investigate relations between this filtration and some ring-theoretic notions, such as the Jacobson radical, the Reynolds ideal and the space of symmetric linear forms. We see that \( \text{CF}_1(M) \) is spanned by the characters of simple objects of \( M \). The second term \( \text{CF}_2(M) \) is expressed in terms of \( \text{Ext}^1_{M_1}(L, L) \) for simple objects \( L \in M \). For \( \text{CF}_n(M) \) with \( n \geq 3 \), we have no general results but study some examples.

In Sect. 7, we discuss the Hochschild (co)homology of finite tensor categories and their module categories. One can define the Hochschild homology \( HH^\bullet(M) \) and the Hochschild cohomology \( HH^\bullet(M) \) of a finite abelian category \( M \) in terms of the Ext functor in \( \text{Rex}(M) \). We then show that, if \( M \) is an exact \( C \)-module category, then there is an isomorphism
\[
HH^\bullet(M) \cong \text{Ext}^\bullet_C(1, A_M).
\]
If, in addition, $M$ is pivotal, then there is also an isomorphism

$$\text{HH}_*(M) \cong \text{Ext}_*^*(A_M, 1)^*.$$ 

The isomorphism (1.2) is a generalization of the known fact that the Hochschild cohomology of a Hopf algebra can be computed by the cohomology of the adjoint representation. We use (1.2) to extend recent results of [26].

## 2 Preliminaries

### 2.1 Ends and coends

For basic theory on categories, we refer the reader to the book of Mac Lane [31]. Let $C$ and $D$ be categories, and let $S$ and $T$ be functors from $C^{\text{op}} \times C$ to $D$. A dinatural transformation [31, IX] from $S$ to $T$ is a family $\xi = \{\xi_X : S(X, X) \to T(X, X)\}_{X \in C}$ of morphisms in $D$ satisfying

$$T(id_X, f) \circ \xi_X \circ S(f, id_X) = T(f, id_Y) \circ \xi_Y \circ S(id_Y, f)$$

for all morphisms $f : X \to Y$ in $C$. An end of $S$ is an object $E \in D$ equipped with a dinatural transformation $\pi : E \to S$ that is universal in a certain sense (here the object $E$ is regarded as a constant functor from $C^{\text{op}} \times C$ to $D$). Dually, a coend of $T$ is an object $C \in D$ equipped with a ‘universal’ dinatural transformation from $T$ to $C$. An end of $S$ and a coend of $T$ are denoted by $\int_{X \in C} S(X, X)$ and $\int^X_{X \in C} T(X, X)$, respectively.

A (co)end does not exist in general. We note the following useful criteria for the existence of (co)ends. Suppose that $C$ is essentially small. Let $C$, $D$ and $S$ be as above. Since the category $\text{Set}$ of all sets is complete, the end $S^\flat(W) := \int_{X \in C} \text{Hom}_D(W, S(X, X))$ exists for each object $W \in D$. By the parameter theorem for ends [31, XI.7], we extend the assignment $W \mapsto S^\flat(W)$ to the contravariant functor $S^\flat : D \to \text{Set}$. The following lemma is the dual of [45, Lemma 3.1].

**Lemma 2.1** An end of $S$ exists if and only if $S^\flat$ is representable.

We also note the following lemma:

**Lemma 2.2** Let $A$, $B$ and $V$ be categories, and let $L : A \to B$, $R : B \to A$ and $H : A \times B^{\text{op}} \to V$ be functors. Suppose that $L$ is left adjoint to $R$. Then we have an isomorphism

$$\int_{V \in A} H(V, L(V)) \cong \int_{W \in B} H(R(W), W),$$

meaning that if either one of these ends exists, then both exist and they are canonically isomorphic.

This lemma is the dual of [9, Lemma 3.9]. For later use, we recall the construction of the canonical isomorphism (2.1). Let $E$ and $E'$ be the left and the right hand side of (2.1), respectively, and let

$$\pi(V) : E \to H(V, L(V)) \quad \text{and} \quad \pi'(W) : E' \to H(R(W), W)$$
be the respective universal dinatural transformations. We assume that \((L, R)\) is an adjoint pair with unit \(\eta : \text{id}_A \to RL\) and counit \(\varepsilon : LR \to \text{id}_B\). By the universal property of \(\mathcal{E}\), there is a unique morphism \(\alpha : \mathcal{E}' \to \mathcal{E}\) in \(\mathcal{V}\) satisfying
\[
\pi(V) \circ \alpha = H(\eta_V, \text{id}_{L(V)}) \circ \pi'(L(V))
\]
for all objects \(V \in \mathcal{A}\). Similarly, by the universal property of \(\mathcal{E}'\), there is a unique morphism \(\beta : \mathcal{E} \to \mathcal{E}'\) satisfying
\[
\pi'(W) \circ \beta = \pi(R(W)) \circ H(\text{id}_{R(W)}, \varepsilon_W)
\]
for all objects \(W \in \mathcal{B}\). By the zigzag identities and the dinaturality of \(\pi\) and \(\pi'\), one can verify that \(\alpha\) and \(\beta\) are mutually inverse.

### 2.2 Monoidal categories

A monoidal category \([31, VII]\) is a category \(\mathcal{C}\) equipped with a functor \(\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}\), an object \(1 \in \mathcal{C}\) and natural isomorphisms \((X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)\) and \(1 \otimes X \cong X \cong X \otimes 1\) \((X, Y, Z \in \mathcal{C})\) satisfying the pentagon and the triangle axiom. If these natural isomorphisms are identities, then \(\mathcal{C}\) is said to be strict. By the Mac Lane coherence theorem, we may assume that every monoidal category is strict.

We fix several conventions on monoidal categories and related notions: Let \(\mathcal{C}\) and \(\mathcal{D}\) be monoidal categories. A monoidal functor \([31, XI.2]\) from \(\mathcal{C}\) to \(\mathcal{D}\) is a functor \(F : \mathcal{C} \to \mathcal{D}\) equipped with a natural transformation
\[
f_{X,Y}^{(2)} : F(X) \otimes F(Y) \to F(X \otimes Y) \quad (X, Y \in \mathcal{C})
\]
and a morphism \(f^{(0)} : 1 \to F(1)\) in \(\mathcal{D}\) satisfying certain axioms. A monoidal functor \(F = (F, f^{(2)}, f^{(0)})\) is said to be strong if the structure morphisms \(f^{(2)}\) and \(f^{(0)}\) are invertible.

Let \(L\) and \(R\) be objects of a monoidal category \(\mathcal{C}\), and let \(\varepsilon : L \otimes R \to 1\) and \(\eta : 1 \to R \otimes L\) be morphisms in \(\mathcal{C}\). We say that \((L, \varepsilon, \eta)\) is a left dual object of \(R\) and \((R, \varepsilon, \eta)\) is a right dual object of \(L\) if the equations
\[
(\varepsilon \otimes \text{id}_L) \circ (\text{id}_L \otimes \eta) = \text{id}_L \quad \text{and} \quad (\text{id}_R \otimes \varepsilon) \circ (\eta \otimes \text{id}_R) = \text{id}_R
\]
hold. If this is the case, then the morphisms \(\varepsilon\) and \(\eta\) are called the evaluation and the coevaluation, respectively.

A monoidal category \(\mathcal{C}\) is said to be rigid if every object of \(\mathcal{C}\) has a left dual object and a right dual object. If \(\mathcal{C}\) is rigid, then we denote by \((X^*, \text{ev}_X, \text{coev}_X)\) the left dual object of \(X \in \mathcal{C}\). Let \(\mathcal{C}^{\text{ev}}\) denote the category \(\mathcal{C}\) equipped with the reversed tensor product \(X \otimes^{\text{ev}} Y = Y \otimes X\). The assignment \(X \mapsto X^*\) gives rise to a strong monoidal functor \((-)^* : \mathcal{C}^{\text{op}} \to \mathcal{C}^{\text{ev}}\) called the left duality functor of \(\mathcal{C}\). The right duality functor \(*(-)\) of \(\mathcal{C}\) is also defined by taking the right dual object. The left and the right duality functor are mutually quasi-inverse to each other.

### 2.3 Module categories

Let \(\mathcal{C}\) be a monoidal category. A left \(\mathcal{C}\)-module category \([14]\) is a category \(\mathcal{M}\) equipped with a functor \(\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}\), called the action of \(\mathcal{C}\), and natural isomorphisms
\[
(X \otimes Y) \otimes M \cong X \otimes (Y \otimes M) \quad \text{and} \quad 1 \otimes M \cong M \quad (X, Y \in \mathcal{C}, M \in \mathcal{M})
\]
(2.2)
satisfying certain axioms similar to those for monoidal categories. There is an analogue of the Mac Lane coherence theorem for $\mathcal{C}$-module categories. Thus, without loss of generality, we may assume that the natural isomorphisms (2.2) are the identity; see [14, Remark 7.2.4].

Let $\mathcal{M}$ and $\mathcal{N}$ be left $\mathcal{C}$-module categories. A lax left $\mathcal{C}$-module functor from $\mathcal{M}$ to $\mathcal{N}$ is a functor $F: \mathcal{M} \to \mathcal{N}$ equipped with a natural transformation

$$s_{X,M}: X \otimes F(M) \to F(X \otimes M) \quad (X \in \mathcal{C}, M \in \mathcal{M})$$

such that the equations

$$s_{1,M} = \text{id}_{F(M)} \quad \text{and} \quad s_{X \otimes Y,M} = s_{X,Y \otimes M} \circ (\text{id}_{X} \otimes s_{Y,M})$$

hold for all objects $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$. We omit the definition of morphisms of lax $\mathcal{C}$-module functors; see [11,12].

An oplax left $\mathcal{C}$-module functor from $\mathcal{M}$ to $\mathcal{N}$ is, in a word, a lax left $\mathcal{C}^\text{op}$-module functor from $\mathcal{M}^\text{op}$ to $\mathcal{N}^\text{op}$; see [12]. Now let $L: \mathcal{M} \to \mathcal{N}$ be a functor with right adjoint $R: \mathcal{N} \to \mathcal{M}$, and let $\eta: \text{id}_{\mathcal{M}} \to RL$ and $\varepsilon: LR \to \text{id}_{\mathcal{N}}$ be the unit and the counit of the adjunction $L \dashv R$. If $(L, v)$ is an oplax left $\mathcal{C}$-module functor, then $R$ is a lax left $\mathcal{C}$-module functor by the structure morphism defined by

$$X \otimes R(N) \xrightarrow{\eta_{X \otimes R(N)}} RL(X \otimes R(N)) \xrightarrow{\varepsilon_{X \otimes R(N)}} R(X \otimes LR(N)) \xrightarrow{R(\varepsilon_{X \otimes N})} R(X \otimes N) \quad (2.3)$$

for $X \in \mathcal{C}$ and $N \in \mathcal{N}$. Conversely, if $(R, w)$ is a lax $\mathcal{C}$-module functor, then $L$ is an oplax $\mathcal{C}$-module functor by

$$L(X \otimes M) \xrightarrow{\text{id}_{X} \otimes \eta_{M}} L(X \otimes RL(M)) \xrightarrow{\varepsilon_{X \otimes L(M)}} R(X \otimes L(M)) \xrightarrow{\varepsilon_{X \otimes N}} R(X \otimes N) \quad (2.4)$$

for $X \in \mathcal{C}$ and $M \in \mathcal{M}$ [12, Lemma 2.11].

We say that an (op)lax $\mathcal{C}$-module functor $(F, s)$ is strong if the natural transformation $s$ is invertible. If $\mathcal{C}$ is rigid, then every (op)lax $\mathcal{C}$-module functor is strong [12, Lemma 2.10] and thus we refer to an (op)lax $\mathcal{C}$-module functor simply as a $\mathcal{C}$-module functor.

### 2.4 Closed module categories

Let $\mathcal{C}$ be a monoidal category. A left $\mathcal{C}$-module category $\mathcal{M}$ is said to be closed if, for all objects $M \in \mathcal{M}$, the functor

$$\mathcal{C} \to \mathcal{M}, \quad X \mapsto X \otimes M \quad (5.5)$$

has a right adjoint (cf. the definition of a closed monoidal category). Suppose that $\mathcal{M}$ is closed. For each object $M \in \mathcal{M}$, we fix a right adjoint $\text{Hom}(M, -)$ of the functor (5.5). Thus, by definition, there is a natural isomorphism

$$\phi: \text{Hom}_{\mathcal{M}}(X \otimes M, N) \to \text{Hom}_{\mathcal{C}}(X, \text{Hom}(M, N)) \quad (6.6)$$

for $N \in \mathcal{M}$ and $X \in \mathcal{C}$. If we denote by

$$\text{coev}_{X,M}: X \to \text{Hom}(M, X \otimes M) \quad \text{and} \quad \text{ev}_{M,N}: \text{Hom}(M, N) \otimes M \to N$$

then $\text{coev}_{X,M}$ and $\text{ev}_{M,N}$ are the unit and the counit, respectively, of the adjunction $\text{Hom}(M, -) \dashv X \otimes -$.
the unit and the counit of the adjunction \((-) \otimes M \to \text{Hom}(M, -)\), respectively, then the isomorphism (2.5) is given by

$$\phi(f) = \text{Hom}(M, f) \circ \text{coev}_{M, X} \quad \text{and} \quad \phi^{-1}(g) = \text{ev}_{M,N} \circ (g \otimes \text{id}_M)$$

(2.7)

for morphisms \(f : X \otimes M \to N\) in \(\mathcal{M}\) and \(g : X \to \text{Hom}(M, N)\) in \(\mathcal{C}\).

By [31, IV.7], one can extend the assignment \((M, N) \mapsto \text{Hom}(M, N)\) to a functor from \(\mathcal{M}^{\text{op}} \times \mathcal{M}\) to \(\mathcal{C}\) in such a way that the isomorphism (2.6) is also natural in the variable \(M\). We call the functor \(\text{Hom}\) the internal Hom functor of \(\mathcal{M}\). This makes \(\mathcal{M}\) a \(C\)-enriched category: The composition

$$\text{comp}_{M_1, M_2, M_3} : \text{Hom}(M_2, M_3) \otimes \text{Hom}(M_1, M_2) \to \text{Hom}(M_1, M_3)$$

is defined to be the morphism corresponding to

$$\text{ev}_{M_1, M_2, M_3} := \text{ev}_{M_2, M_3} \circ (\text{id}_{\text{Hom}(M_2, M_3)} \otimes \text{ev}_{M_1, M_2})$$

(2.9)

via the isomorphism (2.6) with \(X = \text{Hom}(M_2, M_3) \otimes \text{Hom}(M_1, M_2)\), \(M = M_1\) and \(N = M_3\).

The identity on \(M \in \mathcal{M}\) is \(\text{coev}_{1, M}\).

We suppose that \(C\) is rigid. Let \(M \in \mathcal{M}\) be an object. Since the functor (2.5) is a \(C\)-module functor, so is its right adjoint \(\text{Hom}(M, -)\). We denote by

$$a_{X, M, N} : X \otimes \text{Hom}(M, N) \to \text{Hom}(M, X \otimes N) \quad (X \in C, N \in \mathcal{M}) \quad (2.10)$$

the left \(C\)-module structure of \(\text{Hom}(M, -)\).

There is also an isomorphism

$$b_{X, M, N} : \text{Hom}(X \otimes M, N) \to \text{Hom}(M, N) \otimes X^* \quad (X \in C, M, N \in \mathcal{M}) \quad (2.11)$$

induced by the natural isomorphisms

$$\text{Hom}_C(W, \text{Hom}(Y \otimes M, N)) \cong \text{Hom}_{\mathcal{M}}(W \otimes Y \otimes M, N) \cong \text{Hom}_C(W \otimes Y, \text{Hom}(M, N)) \cong \text{Hom}_C(W, \text{Hom}(M, N) \otimes Y^*)$$

for \(W, Y \in \mathcal{C}\) and \(M \in \mathcal{M}\). It is convenient to introduce the morphism

$$b_{X, M, N}^\natural : \text{Hom}(X \otimes M, N) \otimes X \to \text{Hom}(M, N) \quad (X \in C, M, N \in \mathcal{M}) \quad (2.12)$$

defined by \(b_{X, M, N}^\natural = (\text{id}_{\text{Hom}(M, N)} \otimes \text{ev}_X) \circ (b_{X, M, N} \otimes \text{id}_X)\). We note that \(b_{X, M, N}^\natural\) is natural in \(M\) and \(N\) and dinatural in \(X\).

**Lemma 2.3** For all objects \(X, Y \in \mathcal{C}\) and \(M, N \in \mathcal{M}\), we have the equation

$$b_{1, M, N} = \text{id}_{\text{Hom}(M, N)}$$

(2.13)

and the following commutative diagrams:

$$\begin{array}{ccc}
\text{Hom}(X \otimes Y \otimes M, N) & \xrightarrow{b_{X \otimes Y, M, N}} & \text{Hom}(M, N) \otimes (X \otimes Y)^* \\
\downarrow \text{id}_{X \otimes Y \otimes M, N} & & \downarrow \text{id}_{M, N} \\
\text{Hom}(Y \otimes M, N) \otimes X^* & \xrightarrow{b_{Y, M, N} \otimes \text{id}_X^*} & \text{Hom}(M, N) \otimes Y^* \otimes X^*
\end{array}$$

(2.14)

$$\begin{array}{ccc}
X \otimes \text{Hom}(Y \otimes M, N) & \xrightarrow{\text{id}_X \otimes b_{Y, M, N}} & X \otimes \text{Hom}(M, N) \otimes Y^* \\
\downarrow a_{X, Y, M, N} & & \downarrow a_{X, M, N} \otimes \text{id}_{Y^*} \\
\text{Hom}(Y \otimes M, X \otimes N) & \xrightarrow{b_{Y, M, X \otimes N}} & \text{Hom}(M, X \otimes N) \otimes Y^*
\end{array}$$

(2.15)
The category $\mathcal{M}^{\text{op}} \times \mathcal{M}$ is a $C$-bimodule category by the actions given by

$$X \otimes (M, N) = (M^{\text{op}}, X \otimes N) \quad \text{and} \quad (M, N) \otimes X = (X \otimes M, N)$$  \hspace{1cm} (2.16)

for $X \in \mathcal{C}$ and $M, N \in \mathcal{M}$. The above lemma means that the internal Hom functor of $\mathcal{M}$
is a $C$-bimodule functor from $\mathcal{M}^{\text{op}} \times \mathcal{M}$ to $\mathcal{C}$ with left $C$-module structure $a$ and the right $C$-module structure given by

$$(b_{Y,M,N})^{-1} : \text{Hom}(M, N) \otimes Y \rightarrow \text{Hom}(Y \otimes M, N) \quad (Y \in \mathcal{C}, M, N \in \mathcal{M}).$$

It is well-known among experts that $\text{Hom} : \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{C}$ is a $C$-bimodule functor, but no detailed proof seems to exist in the literature. We give a proof of Lemma 2.3 in Appendix A for the sake of completeness. We will also give some equations involving the natural isomorphisms $a$ and $b$ in Appendix A.

In view of Lemma 2.3, we define the isomorphism

$$c_{X,M,N,Y} : X \otimes \text{Hom}(M, N) \otimes Y^* \rightarrow \text{Hom}(Y \otimes M, X \otimes N)$$  \hspace{1cm} (2.17)

for $X, Y \in \mathcal{C}$ and $M, N \in \mathcal{M}$ by

$$c_{X,M,N,Y} = b_{Y,M,X \otimes N}^{-1} \circ (a_{X,M,N} \otimes \text{id}_{Y^*}) = a_{X,Y \otimes M,N} \circ (\text{id}_X \otimes b_{Y,M,N}^{-1}).$$  \hspace{1cm} (2.18)

### 2.5 Finite abelian categories

Throughout this paper, we work over an algebraically closed field $k$ of arbitrary characteristic. By an algebra (over $k$), we mean an associative unital algebra over the field $k$. Given two algebras $A$ and $B$, we denote by $A\text{-mod}$, $\text{mod}-B$ and $A\text{-mod}-B$ the category of finite-dimensional left $A$-modules, the category of finite-dimensional right $B$-modules, and the category of finite-dimensional $A$-$B$-bimodules, respectively.

A finite abelian category [14, Definition 1.8.5] is a $k$-linear category that is equivalent to $A\text{-mod}$ for some finite-dimensional algebra $A$. For finite abelian categories $\mathcal{M}$ and $\mathcal{N}$, we denote by $\text{Rex}(\mathcal{M}, \mathcal{N})$ the category of $k$-linear right exact functors from $\mathcal{M}$ to $\mathcal{N}$. If $A$ and $B$ are finite-dimensional algebras over $k$, then the Eilenberg-Watts theorem gives an equivalence

$$B\text{-mod}-A \xrightarrow{\sim} \text{Rex}(A\text{-mod}, B\text{-mod}), \quad M \mapsto M \otimes_A (-)$$  \hspace{1cm} (2.19)

of $k$-linear categories. Thus $\text{Rex}(\mathcal{M}, \mathcal{N})$ is a finite abelian category. The above equivalence also implies that a $k$-linear functor $F : \mathcal{M} \rightarrow \mathcal{N}$ is right exact if and only if $F$ has a right adjoint.

A $k$-linear category $\mathcal{M}$ is finite abelian if and only if $\mathcal{M}^{\text{op}}$ is. Thus, by the dual argument, we see that a $k$-linear functor $F : \mathcal{M} \rightarrow \mathcal{N}$ is left exact if and only if $F$ has a left adjoint. We denote by $\text{Lex}(\mathcal{M}, \mathcal{N})$ the category of $k$-linear left exact functors from $\mathcal{M}$ to $\mathcal{N}$. For a $k$-linear functor $F$, we denote by $F^{\text{la}}$ and $F^{\text{ra}}$ a left and a right adjoint of $F$, respectively, if it exists.

Now let $\mathcal{M}$ be a finite abelian category. Let $M$ be an object of $\mathcal{M}$. Then the functor $\text{Hom}_\mathcal{M}(M, -) : \mathcal{M} \rightarrow k\text{-mod}$ is left exact, and thus has a left adjoint. We denote it by $(-) \otimes_k M$. By definition, there is a natural isomorphism

$$\text{Hom}_\mathcal{M}(X \otimes_k M, N) \cong \text{Hom}_k(X, \text{Hom}_\mathcal{N}(M, N)) \quad (X \in k\text{-mod}, N \in \mathcal{M}).$$

For two finite abelian categories $\mathcal{M}$ and $\mathcal{N}$, we denote by $\mathcal{M} \otimes \mathcal{N}$ their Deligne tensor product [14, §1.11]. If $\mathcal{M} = A\text{-mod}$ and $\mathcal{N} = B\text{-mod}$ for some finite-dimensional algebras
A and B, then \( \mathcal{M} \boxtimes \mathcal{N} \) is identified with \((A \otimes_k B)\text{-mod}\). In view of the equivalence (2.19), one has:

**Lemma 2.4** ([46, Lemma 3.3]). The k-linear functor

\[
\Phi_{\mathcal{M}, \mathcal{N}} : \mathcal{M}^{\text{op}} \boxtimes \mathcal{N} \to \text{Rex}(\mathcal{M}, \mathcal{N}), \quad M^{\text{op}} \boxtimes N \mapsto \text{Hom}_{\mathcal{M}}(-, M)^* \otimes_k N \quad (2.20)
\]

is an equivalence. Moreover, the functor

\[
\text{Rex}(\mathcal{M}, \mathcal{N}) \to \mathcal{M}^{\text{op}} \boxtimes \mathcal{N}, \quad F \mapsto \int_{M \in \mathcal{M}} M^{\text{op}} \boxtimes F(M) \quad (2.21)
\]

is a quasi-inverse of (2.20).

The following lemma is proved by utilizing the equivalences (2.20) and (2.21).

**Lemma 2.5** ([47, Lemma 2.5]). Let \( \mathcal{M} \) and \( \mathcal{N} \) be finite abelian categories. For a k-linear right exact functor \( F : \mathcal{M} \to \mathcal{N} \), the following are equivalent:

1. \( F \) is a projective object of the abelian category \( \text{Rex}(\mathcal{M}, \mathcal{N}) \).
2. \( F(M) \) is a projective object of \( \mathcal{N} \) for all objects \( M \in \mathcal{M} \) and \( F^{\text{ra}}(N) \) is an injective object of \( \mathcal{M} \) for all objects \( N \in \mathcal{N} \).

For finite abelian categories \( \mathcal{M} \) and \( \mathcal{N} \), there is also an equivalence

\[
\Psi_{\mathcal{M}, \mathcal{N}} : \text{Lex}(\mathcal{M}, \mathcal{N}) \to \mathcal{M}^{\text{op}} \boxtimes \mathcal{N}, \quad F \mapsto \int_{M \in \mathcal{M}} M^{\text{op}} \boxtimes F(M)
\]

[45, Lemmas 3.2 and 3.3]. Fuchs, Schaumann and Schweigert [19] defined the Nakayama functor of \( \mathcal{M} \) by

\[
\mathbb{N}_{\mathcal{M}} := \Phi_{\mathcal{M}, \mathcal{M}} \Psi_{\mathcal{M}, \mathcal{M}}(\text{id}_{\mathcal{M}}) \in \text{Rex}(\mathcal{M}, \mathcal{M}).
\]

For later use, we recall from [19] the following results:

1. We say that \( \mathcal{M} \) is Frobenius if the class of injective objects of \( \mathcal{M} \) coincides with the class of projective objects of \( \mathcal{M} \) (or, equivalently, \( \mathcal{M} \approx A\text{-mod} \) for some Frobenius algebra \( A \)). The Nakayama functor \( \mathbb{N}_{\mathcal{M}} \) is an equivalence if and only if \( \mathcal{M} \) is Frobenius.
2. We say that \( \mathcal{M} \) is symmetric Frobenius if \( \mathcal{M} \approx A\text{-mod} \) for some symmetric Frobenius algebra \( A \). The Nakayama functor \( \mathbb{N}_{\mathcal{M}} \) is isomorphic to \( \text{id}_{\mathcal{M}} \) if and only if \( \mathcal{M} \) is symmetric Frobenius.
3. If \( F : \mathcal{M} \to \mathcal{N} \) is a k-linear exact functor between finite abelian categories \( \mathcal{M} \) and \( \mathcal{N} \), then there is an isomorphism \( \mathbb{N}_{\mathcal{M}} \circ F^{\text{ra}} \cong F^{\text{ra}} \circ \mathbb{N}_{\mathcal{N}} \).
4. The Nakayama functor of \( \text{Rex}(\mathcal{M}, \mathcal{N}) \) is given by

\[
\mathbb{N}_{\text{Rex}(\mathcal{M}, \mathcal{N})}(F) = \mathbb{N}_{\mathcal{N}} \circ F \circ \mathbb{N}_{\mathcal{M}} \quad (F \in \text{Rex}(\mathcal{M}, \mathcal{N})).
\]

### 2.6 Finite tensor categories and their modules

A finite tensor category [17] is a rigid monoidal category \( \mathcal{C} \) such that \( \mathcal{C} \) is a finite abelian category, the tensor product of \( \mathcal{C} \) is k-linear in each variable, and the unit object \( 1 \) of \( \mathcal{C} \) is a simple object. A finite tensor category is Frobenius. The tensor product of a finite tensor category is exact in each variable.

Let \( \mathcal{C} \) be a finite tensor category. A finite left \( \mathcal{C} \)-module category is a left \( \mathcal{C} \)-module category \( \mathcal{M} \) such that \( \mathcal{M} \) is a finite abelian category and the action of \( \mathcal{C} \) on \( \mathcal{M} \) is k-linear and right
exact in each variable. One can define a finite right $C$-module category and a finite $C$-bimodule category in a similar manner.

Given an algebra $A \in C (= a$ monoid in $C [31])$, we denote by $C_A$ the category of right $A$-modules in $C$. The category $C_A$ is a finite left $C$-module category in a natural way. Moreover, every finite left $C$-module category is equivalent to $C_A$ for some algebra $A \in C$ as a $C$-module category. This implies that the action of $C$ on a finite $C$-module category is exact in each variable [12, Corollary 2.26], although only the right exactness is assumed in our definition.

An exact left $C$-module category [17] is a finite left $C$-module category $\mathcal{M}$ such that $P \otimes M$ is a projective object of $\mathcal{M}$ for all projective objects $P \in C$ and for all objects $M \in \mathcal{M}$. It is known that exact module categories are Frobenius.

3 Adjoint of the action functor

3.1 The action functor

Let $C$ be a finite tensor category, and let $\mathcal{M}$ and $\mathcal{N}$ be two finite left $C$-module categories. Then $\text{Rex}(\mathcal{M}, \mathcal{N})$ is a $C$-bimodule category by the left action and the right action given by

$$(X \otimes F)(M) = X \otimes F(M) \quad \text{and} \quad (F \otimes X)(M) = F(X \otimes M),$$

(3.1)

respectively, for $F \in \text{Rex}(\mathcal{M}, \mathcal{N})$, $X \in C$ and $M \in \mathcal{M}$. The category $\mathcal{M}^{\text{op}} \boxtimes \mathcal{N}$ is also a $C$-bimodule category by the left and the right action determined by

$$X \otimes (M^{\text{op}} \boxtimes N) = (M^{\text{op}} \boxtimes (X \otimes N)) \quad \text{and} \quad (M^{\text{op}} \boxtimes N) \otimes X = (\ast X \otimes M)^{\text{op}} \boxtimes N,$$

respectively, for $M \in \mathcal{M}$, $N \in \mathcal{N}$ and $X \in C$. It is easy to see that the equivalence (2.20) is in fact an equivalence of $C$-bimodule categories. Since $\mathcal{M}^{\text{op}} \boxtimes \mathcal{N}$ is a finite $C$-bimodule category, so is $\text{Rex}(\mathcal{M}, \mathcal{N})$.

Now we define the functor $\rho_\mathcal{M} : C \rightarrow \text{Rex}(\mathcal{M}) : = \text{Rex}(\mathcal{M}, \mathcal{M})$ by $X \mapsto X \otimes \text{id}_\mathcal{M}$ and call $\rho_\mathcal{M}$ the action functor of $\mathcal{M}$. Since the action of $C$ on a finite $C$-module category is $k$-linear and exact in each variable, we have:

Lemma 3.1 The action functor $\rho_\mathcal{M}$ is $k$-linear and exact.

Thus the action functor $\rho_\mathcal{M}$ has a left adjoint and a right adjoint. The aim of this section is to study properties of adjoints of $\rho_\mathcal{M}$. Before doing so, we characterize some properties of $\mathcal{M}$ in terms of $\rho_\mathcal{M}$.

Lemma 3.2 A finite left $C$-module category $\mathcal{M}$ is exact if and only if the action functor $\rho_\mathcal{M}$ preserves projective objects.

Proof Suppose that $\mathcal{M}$ is an exact $C$-module category. We fix a projective object $P \in C$ and set $F = \rho_\mathcal{M}(P)$. By the definition of an exact module category, the object $F(M) = P \otimes M$ is projective for all $M \in \mathcal{M}$. Since $C$ and $\mathcal{M}$ are Frobenius, the object $F^{\text{ra}}(M) \equiv \ast P \otimes M$ is injective for all $M \in \mathcal{M}$. Thus, by Lemma 2.5, $F$ is a projective object of $\text{Rex}(\mathcal{M})$. Hence $\rho_\mathcal{M}$ preserves projective objects. The converse is easily proved by Lemma 2.5. □

Let $\mathcal{A}$ and $\mathcal{B}$ be finite abelian categories. A $k$-linear functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is said to be dominant if every object of $\mathcal{B}$ is a subobject of an object of the form $F(X)$, $X \in \mathcal{A}$. Suppose that $F$ is exact and $\mathcal{B}$ is Frobenius. Then, as remarked in [13, Lemma 2.3], the functor $F$ is dominant if and only if every object of $\mathcal{B}$ is a quotient of $F(X)$ for some $X \in \mathcal{A}$. Springer
Lemma 3.3 An exact left $C$-module category $\mathcal{M}$ is indecomposable if and only if the action functor $\rho_\mathcal{M}$ is dominant.

Proof Suppose that there are non-zero $C$-module full subcategories $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{M}$ such that $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Then we have the decomposition

$$\text{Rex}(\mathcal{M}) = E_{11} \oplus E_{12} \oplus E_{21} \oplus E_{22}, \quad E_{ij} = \text{Rex}(\mathcal{M}_i, \mathcal{M}_j),$$

into four non-zero full subcategories. Since the image of $\rho_\mathcal{M}$ is contained in the diagonal part $E_{11} \oplus E_{22}$, the action functor $\rho_\mathcal{M}$ cannot be dominant. Thus the ‘if’ part has been proved. The ‘only if’ part is [13, Proposition 2.6 (ii)].

3.2 Description of adjoints

For a while, we fix a finite tensor category $C$ and a finite left $C$-module category $\mathcal{M}$. We write $\rho = \rho_\mathcal{M}$ for simplicity. By Lemma 3.1, the functor $\rho$ has a right adjoint.

Theorem 3.4 For all $k$-linear right exact functor $F : \mathcal{M} \to \mathcal{M}$, the end of

$$\mathcal{M}^{\text{op}} \times \mathcal{M} \to C, \quad (M, M') \mapsto \text{Hom}(M, F(M'))$$

exists and a right adjoint of $\rho$ is given by

$$\rho^{\text{ra}} : \text{Rex}(\mathcal{M}) \to C, \quad F \mapsto \int_{M \in \mathcal{M}} \text{Hom}(M, F(M)).$$

We also have:

(a) If $\mathcal{M}$ is exact, then $\rho^{\text{ra}}$ is exact.

(b) If $\mathcal{M}$ is exact and indecomposable, then $\rho^{\text{ra}}$ is faithful.

Proof Let $\rho^{\text{ra}}$ be a right adjoint of $\rho$. Then we have

$$\text{Hom}_C(X, \rho^{\text{ra}}(F)) \cong \text{Nat}(\rho(X), F)$$

$$\cong \int_{M \in \mathcal{M}} \text{Hom}_\mathcal{M}(X \otimes M, F(M))$$

$$\cong \int_{M \in \mathcal{M}} \text{Hom}_\mathcal{M}(X, \text{Hom}_\mathcal{M}(M, F(M)))$$

for all $X \in C$ and $F \in \text{Rex}(\mathcal{M})$. Thus, by Lemma 2.1, we see that the end in question exists and $\rho^{\text{ra}}$ is given as stated.

(a) We suppose that $\mathcal{M}$ is exact. Let $P$ be a projective generator of $C$. Then, by Lemma 3.2, the object $\rho(P) \in \text{Rex}(\mathcal{M})$ is projective. Thus the functor

$$\text{Hom}_C(P, \rho^{\text{ra}}(-)) \cong \text{Hom}_C(\rho(P), -) : \text{Rex}(\mathcal{M}) \to k\text{-mod}$$

is exact. Since $P$ is a projective generator, we conclude that $\rho^{\text{ra}}$ is exact.

(b) We suppose that $\mathcal{M}$ is exact and indecomposable. Since the functor $\rho^{\text{ra}}$ is exact by Part (a), it is enough to show that $\rho^{\text{ra}}$ reflects zero objects. Let $F$ be an object of $\text{Rex}(\mathcal{M})$ such that $\rho^{\text{ra}}(F) = 0$. By Lemma 3.3, there is an object $X \in C$ such that $F$ is an epimorphic image of $\rho(X)$. If $F \neq 0$, then we have

$$0 = \text{Hom}_C(X, \rho^{\text{ra}}(F)) \cong \text{Nat}(\rho(X), F) \neq 0,$$

a contradiction. Thus $F = 0$. The proof is done.
For $M, M' \in \mathcal{M}$, we set $\text{coHom}(M, M') = \text{coHom}(M', M)$. It is easy to see that there is a natural isomorphism

$$\text{Hom}_\mathcal{M}(M, X \otimes M') \cong \text{Hom}_\mathcal{M}(\text{coHom}(M, M'), X)$$

for $X \in \mathcal{C}$ and $M, M' \in \mathcal{M}$. A left adjoint of $\rho$ is expressed as follows:

**Theorem 3.5** For all $k$-linear right exact functor $F: \mathcal{M} \to \mathcal{M}$, the coend of

$$\mathcal{M}^{\text{op}} \times \mathcal{M} \to \mathcal{C}, \quad (M^{\text{op}}, M') \mapsto \text{coHom}(M, F(M'))$$

exists. Moreover, a left adjoint of $\rho$ is given by

$$\rho^{\text{la}}: \text{Rex}(\mathcal{M}) \to \mathcal{C}, \quad F \mapsto \int^{M \in \mathcal{M}} \text{coHom}(M, F(M))$$

We also have:

(a) If $\mathcal{M}$ is exact, then $\rho^{\text{la}}$ is exact.

(b) If $\mathcal{M}$ is exact and indecomposable, then $\rho^{\text{la}}$ is faithful.

**Proof** Let $\rho^{\text{la}}$ be a left adjoint of $\rho$. Then we have

$$\text{Hom}_\mathcal{C}(\rho^{\text{la}}(F), X) \cong \text{Nat}(F, \rho(X))$$

$$\cong \int_{M \in \mathcal{M}} \text{Hom}_\mathcal{M}(F(M), X \otimes M)$$

$$\cong \int_{M \in \mathcal{M}} \text{Hom}_\mathcal{C}(\text{coHom}(M, F(M')), X)$$

for all $X \in \mathcal{C}$ and $F \in \text{Rex}(\mathcal{M})$. Thus, by the dual of Lemma 2.1, we see that the coend in question exists and $\rho^{\text{la}}$ is given as stated.

Suppose that $\mathcal{M}$ is exact. Then, since $\mathcal{M}$ is Frobenius, the Nakayama functor of $\text{Rex}(\mathcal{M}) \cong \mathcal{M}^{\text{op}} \boxtimes \mathcal{M}$ is an equivalence. Parts (a) and (b) of this theorem follow from Theorem 3.4 and $\rho^{\text{la}} \cong \text{Nat}^{-1} \circ \rho^{\text{la}} \circ \text{Nat}_{\text{Rex}(\mathcal{M})}$.

**Remark 3.6** In summary, for $F \in \text{Rex}(\mathcal{M})$, we have

$$\rho^{\text{la}}(F) = \int_{M \in \mathcal{M}} \text{Hom}(M, F(M))$$

and

$$\rho^{\text{la}}(F) = \int^{M \in \mathcal{M}} \text{coHom}(M, F(M)).$$

There is a left exact version of the action functor

$$\lambda_\mathcal{M}: \mathcal{C} \to \text{Lex}(\mathcal{M}):= \text{Lex}(\mathcal{M}, \mathcal{M}), \quad \lambda_\mathcal{M}(X)(M) = X \otimes M.$$

By the same way as above, one can prove that $\lambda = \lambda_\mathcal{M}$ is a $k$-linear exact functor and its adjoints are given by

$$\lambda^{\text{ra}}(F) = \int_{M \in \mathcal{C}} \text{Hom}(M, F(M))$$

and

$$\lambda^{\text{la}}(F) = \int^{M \in \mathcal{C}} \text{coHom}(M, F(M))$$

for $F \in \text{Lex}(\mathcal{M})$. Moreover, there are natural isomorphisms

$$\lambda^{\text{ra}}(F) \cong *\text{Nat}(\rho^{\text{la}}(F^{\text{ra}}))$$

and

$$\lambda^{\text{la}}(F) \cong *\text{Nat}(\rho^{\text{ra}}(F^{\text{la}}))$$

for $F \in \text{Lex}(\mathcal{M})$. Indeed, we have natural isomorphisms

$$\text{Hom}_\mathcal{C}(X, *\text{Nat}(\rho^{\text{la}}(F^{\text{ra}}))) \cong \text{Hom}_\mathcal{C}(\rho^{\text{la}}(F^{\text{ra}}), X^*) \cong \text{Nat}(F^{\text{la}}, \rho(X^*))$$

$$\cong \text{Nat}(F^{\text{la}}, \rho(X)^*) \cong \text{Nat}(\rho(X), F) = \text{Nat}(\lambda(X), F)$$

for $F \in \text{Lex}(\mathcal{M})$ and $X \in \mathcal{C}$. The second isomorphism of (3.2) is established in a similar way. Theorems 3.4 and 3.5 imply the following results:
(a) If $\mathcal{M}$ is exact, then $\lambda^a$ and $\lambda^r a$ are exact.
(b) If $\mathcal{M}$ is exact and indecomposable, then $\lambda^a$ and $\lambda^r a$ are faithful.

### 3.3 The unit and the counit of $(\rho, \rho^r a)$

In what follows, we concentrate to study the structures of the right adjoint of $\rho = \rho_\mathcal{M}$. For this purpose, it is useful to describe the unit and the counit of the adjunction $\rho \dashv \rho^r a$. For $F \in \text{Rex}(\mathcal{M})$ and $M \in \mathcal{M}$, we denote by

$$\pi_F(M) : \rho^r a(F) \to \text{Hom}(M, F(M))$$

(3.3)

the universal dinatural transformation and define

$$\varepsilon_{F,M} = \xi_{M, F(M)} \circ (\pi_F(M) \otimes \text{id}_M).$$

(3.4)

By the proof of Theorem 3.4, the adjunction isomorphism

$$\text{Hom}_C(X, \rho^r a(F)) \cong \text{Hom}_{\text{Rex}(\mathcal{M})}(\rho(X), F) = \text{Nat}(\rho(X), F)$$

(3.5)

sends $a \in \text{Hom}_C(X, \rho^r a(F))$ to the natural transformation $\tilde{a}$ given by

$$\tilde{a}_M = \varepsilon_{F,M} \circ (a \otimes \text{id}_M) \quad (M \in \mathcal{M}).$$

(3.6)

This implies that $\varepsilon = \{\varepsilon_{F,M}\}_{F,M}$ is the counit of (3.5). We also observe that the morphism $a$ is characterized by the property that the equation

$$\text{Hom}(M, \tilde{a}_M) \circ \text{coev}_{X,M} = \pi_F(M) \circ a$$

(3.7)

holds for all objects $M \in \mathcal{M}$. Let $\eta : \text{id}_C \to \rho^r a \circ \rho$ be the unit of the adjunction isomorphism (3.5). By substituting $a = \eta_X$ and $F = \rho(X)$ into (3.7), we see that $\eta$ is characterized by the property that the equation

$$\pi_{\rho(X)}(M) \circ \eta_X = \text{coev}_{X,M}$$

(3.8)

holds for all objects $X \in \mathcal{C}$ and $M \in \mathcal{M}$. We also have

$$\pi_F(M) = \text{Hom}(M, \varepsilon_{F,M}) \circ \text{coev}_{\rho^r a(F),M}$$

(3.9)

by substituting $X = \rho^r a(F)$ and $a = \text{id}$ into (3.7).

### 3.4 Bimodule structure of $\rho^r a$

Since $\rho : \mathcal{C} \to \text{Rex}(\mathcal{M})$ is a $\mathcal{C}$-bimodule functor, its right adjoint $\rho^r a$ is also a $\mathcal{C}$-bimodule functor such that the unit and the counit are $\mathcal{C}$-bimodule transformations. We denote by

$$\xi^{(l)}_{X,F} : X \otimes \rho^r a(F) \to \rho^r a(X \otimes F)$$

and

$$\xi^{(r)}_{F,X} : \rho^r a(F) \otimes X \to \rho^r a(F \otimes X)$$

the left and the right $\mathcal{C}$-module structure of $\rho^r a$. These morphisms are expressed in terms of the universal dinatural transformation $\pi$ as follows:

**Lemma 3.7** For all objects $F \in \text{Rex}(\mathcal{M})$, $X \in \mathcal{C}$ and $M \in \mathcal{M}$, we have

$$\pi_{X \otimes F}(M) \circ \xi^{(l)}_{X,F} = a_{X,M,F,M} \circ (\text{id}_X \otimes \pi_F(M)),$$

(3.10)

$$\pi_{F \otimes X}(M) \circ \xi^{(r)}_{F,X} = b^*_X \circ (\pi_F(X \otimes M) \otimes \text{id}_X).$$

(3.11)
See Sect. 2.4 for definitions of a and b. By the universal property of $\rho^{ra}(F)$ as an end, the isomorphisms $\xi^{(l)}_{X,F}$ and $\xi^{(r)}_{F,X}$ are characterized by Eqs. (3.10) and (3.11), respectively. We postpone the proof of this lemma to Appendix A since it is straightforward but lengthy.

3.5 Monoidal structure of $\rho^{ra}$

Since the action functor $\rho : C \to \text{Rex}(\mathcal{M})$ is a strong monoidal functor, its right adjoint $\rho^{ra}$ has a canonical structure of a (lax) monoidal functor. We denote the structure morphisms of $\rho^{ra}$ by

$$\mu^{(2)}_{F,G} : \rho^{ra}(F) \otimes \rho^{ra}(G) \to \rho^{ra}(F \circ G) \quad \text{and} \quad \mu^{(0)} : 1 \to \rho^{ra}(\text{id}_M)$$

for $F, G \in \text{Rex}(\mathcal{M})$. They are expressed in terms of the universal dinatural transformation $\pi$ as follows:

**Lemma 3.8** For all objects $F, G \in \text{Rex}(\mathcal{M})$ and $M \in \mathcal{M}$, we have

$$\pi_{FG}(M) \circ \mu^{(2)}_{F,G} = \text{comp}_{M,G(M),FG(M)}(\pi_F(G(M)) \otimes \pi_G(M)),$$

$$\pi_{\text{id}_M}(M) \circ \mu^{(0)} = \text{coev}_{1,M}.$$ (3.12) (3.13)

By the universal property, $\mu^{(2)}$ and $\mu^{(0)}$ are characterized by Eqs. (3.12) and (3.13), respectively. The proof is postponed to Appendix A.

3.6 Lifting the adjunction to the Drinfeld center

Given two finite left $C$-module categories $\mathcal{M}$ and $\mathcal{N}$, we denote by $\text{Rex}_C(\mathcal{M}, \mathcal{N})$ the category of $k$-linear right exact $C$-module functors from $\mathcal{M}$ to $\mathcal{N}$. The aim of this subsection is to show that the adjoint pair $(\rho, \rho^{ra})$ can be ‘lifted’ to an adjoint pair between the Drinfeld center of $C$ and $\text{Rex}_C(\mathcal{M}, \mathcal{M})$.

We first introduce the following generalization of the Drinfeld center construction: For a $C$-bimodule category $\mathcal{M}$, we define the category $\mathcal{Z}(\mathcal{M})$ as follows: An object of this category is a pair $(M, \sigma)$ consisting of an object $M \in \mathcal{M}$ and a natural isomorphism $\sigma_X : M \otimes X \to X \otimes M$ $(X \in C)$ satisfying the equations

$$\sigma_1 = \text{id}_M \quad \text{and} \quad \sigma_X \otimes Y = (\text{id}_X \otimes \sigma_Y) \circ (\sigma_X \otimes \text{id}_Y)$$

for all objects $X, Y \in C$. If $M = (M, \sigma_M)$ and $N = (N, \sigma_N)$ are objects of $\mathcal{Z}(\mathcal{M})$, then a morphism $f : M \to N$ is a morphism $f : M \to N$ satisfying

$$(\text{id}_X \otimes f) \circ \sigma_{M;X} = \sigma_{N;X} \circ (f \otimes \text{id}_X)$$

for all objects $X \in C$. The composition of morphisms in $\mathcal{Z}(\mathcal{M})$ is defined by the composition of morphisms in $\mathcal{M}$.

**Example 3.9** The category $\mathcal{C}$ is a finite $C$-bimodule category by the tensor product of $C$. The category $\mathcal{Z}(\mathcal{C})$ is the Drinfeld center of $\mathcal{C}$. If this is the case, then $\mathcal{Z}(\mathcal{C})$ is not only a category but a braided finite tensor category [17].

**Example 3.10** If $\mathcal{M}$ and $\mathcal{N}$ are finite left $C$-module categories, then $\mathcal{F} := \text{Rex}(\mathcal{M}, \mathcal{N})$ is a finite $C$-bimodule category by the actions given by (3.1). The category $\mathcal{Z}(\mathcal{F})$ can be identified with $\text{Rex}_C(\mathcal{M}, \mathcal{N})$. 

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Now let $\mathcal{C}$-bimod be the 2-category of finite $\mathcal{C}$-bimodule categories, $k$-linear right exact $\mathcal{C}$-bimodule functors and $\mathcal{C}$-bimodule natural transformations. Given a 1-cell $F : \mathcal{M} \to \mathcal{N}$ in $\mathcal{C}$-bimod with structure morphisms
\[ \ell_{X,M} : X \otimes F(M) \to F(X \otimes M) \quad \text{and} \quad \rho_{M,X} : F(M) \otimes X \to F(M \otimes X), \]
we define the $k$-linear functor $Z(F) : Z(\mathcal{M}) \to Z(\mathcal{N})$ by
\[ Z(F)(M) = (F(M), \ell^{-1} \circ F(\sigma) \circ \rho) \]
for $M = (M, \sigma)$ in $Z(\mathcal{M})$. It is routine to check that these constructions extends to a 2-functor $Z : \mathcal{C}$-bimod $\to k$-Cat, where $k$-Cat is the 2-category of essentially small $k$-linear categories, $k$-linear functors and natural transformations.

We apply the 2-functor $Z$ to the action functor and its adjoint. Let $\mathcal{M}$ be a finite left $\mathcal{C}$-module category. Since $\rho = \rho_{\mathcal{M}}$ is a $\mathcal{C}$-bimodule functor, its right adjoint $\rho_{\mathcal{M}}$ is a $\mathcal{C}$-bimodule functor in such a way that the unit and the counit of the adjunction are bimodule natural transformations. Namely, there is an adjoint pair $(\rho, \rho_{\mathcal{M}})$ in the 2-category $\mathcal{C}$-bimod. By applying the 2-functor $Z$, we obtain:

**Theorem 3.11** There is an adjoint pair
\[ (Z(\rho) : Z(\mathcal{C}) \to Rex_{\mathcal{C}}(\mathcal{M}), \ Z(\rho_{\mathcal{M}}) : Rex_{\mathcal{C}}(\mathcal{M}) \to Z(\mathcal{C})), \]
where we have identified $Z(\text{Rex}(\mathcal{M}))$ with $\text{Rex}_{\mathcal{C}}(\mathcal{M})$.

It is instructive to describe the functors $Z(\rho)$ and $Z(\rho_{\mathcal{M}})$ explicitly. Given an object $X = (X, \sigma) \in Z(\mathcal{C})$, we have $Z(\rho)(X) = \rho(X)$. The left $\mathcal{C}$-module structure of $X := Z(\rho)(X)$ is given by
\[ (\sigma_W)^{-1} \otimes \text{id}_M : W \otimes X(M) = W \otimes X \otimes M \to X \otimes W \otimes M = X(W \otimes M) \]
for $W \in \mathcal{C}$ and $M \in \mathcal{M}$. For an object $F = (F, s) \in \text{Rex}_{\mathcal{C}}(\mathcal{M})$, we have
\[ Z(\rho_{\mathcal{M}})(F) = (\rho_{\mathcal{M}}(F), \sigma^F), \quad \text{where} \quad \sigma^F_X = (\xi^{(r)}_{X,F})^{-1} \circ \rho_{\mathcal{M}}(s^{-1}) \circ \xi^{(r)}_{F,X} \]
for $X \in \mathcal{C}$. More explicitly:

**Lemma 3.12** The half-braiding $\sigma^F$ is a unique natural transformation such that the following diagram commutes for all objects $X \in \mathcal{C}$ and $M \in \mathcal{M}$.

\[
\begin{array}{ccc}
X \otimes \rho_{\mathcal{M}}(F) & \xrightarrow{\text{id} \otimes \rho_{\mathcal{M}}(X)} & X \otimes \text{Hom}(M, F(M)) \\
\downarrow \sigma^F_X & & \downarrow s_{X,M,F(X \otimes M)}^{-1} \\
\text{Hom}(X \otimes M, F(X \otimes M)) & \xrightarrow{b_{X,M,F(X \otimes M)}} & \text{Hom}(M, F(X \otimes M)) \\
\downarrow \text{Hom}(M, F(X \otimes M)) & & \downarrow \text{Hom}(M, F(M)) \\
\text{Hom}(M, X \otimes F(M)) & \xrightarrow{\text{id} \otimes \rho_{\mathcal{M}}(X)} & \text{Hom}(M, F(M)).
\end{array}
\]

**Proof** The commutativity of this diagram follows from Lemma 3.7. By the Fubini theorem for ends, we see that $X \otimes \rho_{\mathcal{M}}(F)$ is an end of the functor
\[ \mathcal{M}^{\text{op}} \times \mathcal{M} \to \mathcal{C}, \quad (M^{\text{op}}, M') \mapsto X \otimes \text{Hom}(M, F(M')). \]
The universal property proves the ‘uniqueness’ part of this lemma. \[ \square \]
3.7 Induction to the Drinfeld center

Let \( \mathcal{C} \) be a finite tensor category, and let \( \mathcal{M} \) be a finite left \( \mathcal{C} \)-module category. The \( k \)-linear monoidal category \( \mathcal{C}^*_\mathcal{M} := \text{Rex}(\mathcal{C}, \mathcal{M})^\text{rev} \) is called the dual of \( \mathcal{C} \) with respect to \( \mathcal{M} \). Schauenburg’s result [40] yields an equivalence \( \mathcal{Z}(\mathcal{C}) \approx \mathcal{Z}(\mathcal{C}^*_\mathcal{M}) \) of \( k \)-linear braided monoidal categories. In this subsection, we show that \( \mathcal{Z}(\rho^\text{ra}_\mathcal{M}) \) is right adjoint to the composition

\[
\mathcal{Z}(\mathcal{C}) \xrightarrow{\text{Schauenburg’s equivalence}} \mathcal{Z}(\mathcal{C}^*_\mathcal{M}) \xrightarrow{\text{the forgetful functor}} \mathcal{C}^*_\mathcal{M}.
\]

We first describe the equivalence \( \mathcal{Z}(\mathcal{C}) \approx \mathcal{Z}(\mathcal{C}^*_\mathcal{M}) \). As a first step, we consider the case where \( \mathcal{M} = \mathcal{C}_A \) for some algebra \( A \) in \( \mathcal{C} \). The category \( \mathcal{A} \mathcal{C}_A \) of \( A \)-bimodules in \( \mathcal{C} \) is a \( k \)-linear monoidal category with respect to the tensor product over \( A \). The dual tensor category \( \mathcal{C}^*_\mathcal{M} \) is identified with \( \mathcal{A} \mathcal{C}_A \) by the equivalence

\[
\mathcal{A} \mathcal{C}_A \rightarrow \text{Rex}(\mathcal{C}_A, \mathcal{C}_A)^\text{rev}, \hspace{1em} M \mapsto (-) \otimes_A M.
\]

Now we define a \( k \)-linear monoidal functor \( \theta_A : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{Z}(\mathcal{A} \mathcal{C}_A) \) as follows: Let \( m : A \otimes A \rightarrow A \) denote the multiplication of \( A \). For an object \( V = (V, \sigma) \in \mathcal{Z}(\mathcal{C}) \), we set \( \theta_A(V) = (A \otimes V, \tilde{\sigma}) \), where the left action of \( A \) on \( A \otimes V \) is given by \( m \otimes \text{id}_V \), the right action is given by the composition

\[
(A \otimes V) \otimes A \xrightarrow{\text{id}_A \otimes \sigma} A \otimes A \otimes V \xrightarrow{m \otimes \text{id}_V} A \otimes V,
\]

and the half-braiding \( \tilde{\sigma} \) is determined by the commutative diagram

\[
\begin{array}{ccc}
A \otimes V \otimes M & \xrightarrow{\text{id}_A \otimes \sigma_M} & X \otimes A \otimes M \\
\downarrow & & \downarrow \cong \\
A \otimes V & \xrightarrow{\text{id}_A \otimes \sigma} & M \otimes A \otimes (A \otimes M)
\end{array}
\]

for an \( A \)-bimodule \( M \) in \( \mathcal{C} \) with left action \( \triangleright_M : A \otimes M \rightarrow M \). For a morphism \( f \) in \( \mathcal{Z}(\mathcal{C}) \), we set \( \theta_A(f) = \text{id}_A \otimes f \). The monoidal structure of \( \theta_A \) is given by the canonical isomorphism

\[
\theta_A(V) \otimes_A \theta_A(W) = (A \otimes V) \otimes_A (A \otimes W) \cong A \otimes (V \otimes W) = \theta_A(V \otimes W)
\]

for \( V = (V, \sigma), W = (W, \tau) \in \mathcal{Z}(\mathcal{C}) \). The functor \( \theta_A \) is in fact an equivalence of \( k \)-linear braided monoidal categories [40].

Now, let, in general, \( \mathcal{M} \) be a finite left \( \mathcal{C} \)-module category. Then there is an algebra \( A \) in \( \mathcal{C} \) such that \( \mathcal{M} \approx \mathcal{C}_A \) and hence we have equivalences \( \mathcal{Z}(\mathcal{C}^*_\mathcal{M}) \approx \mathcal{Z}(\mathcal{A} \mathcal{C}_A) \approx \mathcal{Z}(\mathcal{C}) \) of \( k \)-linear braided monoidal categories. Since we are interested in the general theory of finite tensor categories and their module categories, it is preferable to describe the equivalence \( \mathcal{Z}(\mathcal{C}) \approx \mathcal{Z}(\mathcal{C}^*_\mathcal{M}) \) without referencing the algebra \( A \) such that \( \mathcal{M} \approx \mathcal{C}_A \). Thus, for a finite left \( \mathcal{C} \)-module category \( \mathcal{M} \), we define the functor \( \theta_\mathcal{M} : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{Z}(\mathcal{C}^*_\mathcal{M}) \) as follows: For an object \( V = (V, \sigma) \in \mathcal{Z}(\mathcal{C}) \), we set \( \theta_\mathcal{M}(V) = \rho(V) \) as an object of \( \text{Rex}(\mathcal{M}) \). We make \( \rho(V) \) into a left \( \mathcal{C} \)-module functor by the structure morphism given by

\[
(\sigma_X)^{-1} \otimes \text{id}_M : X \otimes \rho(V)(M) = X \otimes V \otimes M \rightarrow V \otimes X \otimes M = \rho(V)(X \otimes M)
\]

for \( X \in \mathcal{C} \) and \( M \in \mathcal{M} \). The half-braiding of \( \theta_\mathcal{M}(V) \) is given by

\[
s_{V, M} : (\theta_\mathcal{M}(V) \circ F)(M) = V \otimes F(M) \rightarrow F(V \otimes M) = (F \circ \theta_\mathcal{M}(V))(M)
\]

for \( F = (F, s) \in \mathcal{C}^*_\mathcal{M} \) and \( M \in \mathcal{M} \). The following theorem is obtained by rephrasing Schauenburg’s result.

\[\square\]
Theorem 3.13 The functor $\theta_{\mathcal{M}} : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{C}_{\mathcal{M}}^*)$ is an equivalence of $k$-linear braided monoidal categories.

Proof If finite left $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$ are equivalent, then there is an equivalence $F : \mathcal{C}_{\mathcal{M}}^* \to \mathcal{C}_{\mathcal{N}}^*$ of $k$-linear monoidal categories. It is easy to check $\tilde{F} \circ \theta_{\mathcal{M}} = \theta_{\mathcal{N}}$, where $\tilde{F} : \mathcal{Z}(\mathcal{C}_{\mathcal{M}}^*) \to \mathcal{Z}(\mathcal{C}_{\mathcal{N}}^*)$ is the braided monoidal equivalence induced by the monoidal equivalence $F$. Thus, to show that $\theta_{\mathcal{M}}$ is an equivalence, we may assume that $\mathcal{M} = \mathcal{C}_A$ for some algebra $A$ in $\mathcal{C}$. We consider the equivalence 

$$\theta'_{\mathcal{M}} := \left( \mathcal{Z}(\mathcal{C}) \xrightarrow{\theta_A} \mathcal{Z}(\mathcal{A}) \xrightarrow{\text{by (3.15)}} \mathcal{Z}(\mathcal{C}_A^*) \right)$$

of $k$-linear braided monoidal categories. One can check that $\theta_{\mathcal{M}} \cong \theta'_{\mathcal{M}}$ as monoidal functors via the isomorphism given by

$$\theta'_{\mathcal{M}}(V) = V \otimes M \xrightarrow{\sigma_{\mathcal{M}}} M \otimes V \xrightarrow{\cong} M \otimes_A (A \otimes V) = \theta'_{\mathcal{M}}(V)(M)$$

for $V = (V, \sigma) \in \mathcal{Z}(\mathcal{C})$ and $M \in \mathcal{M}$. Thus $\theta_{\mathcal{M}}$ is also an equivalence of $k$-linear braided monoidal categories.

Now we prove the result mentioned at the beginning of this subsection:

Theorem 3.14 Let $U : \mathcal{Z}((\mathcal{C})^*_{\mathcal{M}}) \to \mathcal{C}_A^*$ be the forgetful functor. Then

$$(U \circ \theta_{\mathcal{M}} : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}_{\mathcal{M}}^*, \mathcal{Z}(\rho_{\mathcal{C}}^\mathcal{M}) : \mathcal{C}_{\mathcal{M}}^* \to \mathcal{Z}(\mathcal{C}))$$

is an adjoint pair.

Proof By Theorem 3.13, the functor $U \circ \theta_{\mathcal{M}}$ is identical to $\mathcal{Z}(\rho_{\mathcal{M}})$ and therefore it is left adjoint to $\mathcal{Z}(\rho_{\mathcal{M}})$.

Corollary 3.15 Let $U_{\mathcal{C}} : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ and $U_{\mathcal{D}} : \mathcal{Z}(\mathcal{D}) \to \mathcal{D}$ be the forgetful functors, where $\mathcal{D} = \mathcal{C}_{\mathcal{M}}^*$. Then $U_{\mathcal{D}}$ has a right adjoint. The composition

$$\mathcal{D} \xrightarrow{U_{\mathcal{D}}} \mathcal{Z}(\mathcal{D}) \xrightarrow{\theta_{\mathcal{M}}^{-1}} \mathcal{Z}(\mathcal{C}) \xrightarrow{U_{\mathcal{C}}} \mathcal{C}$$

sends an object $F = (F, s) \in \mathcal{D}$ to the end $\int_{M \in \mathcal{M}} \text{Hom}(M, F(M))$.

Proof Theorem 3.14 implies that $\theta_{\mathcal{M}} \circ \mathcal{Z}(\rho_{\mathcal{C}}^\mathcal{M})$ is right adjoint to $U_{\mathcal{D}}$. Thus $U_{\mathcal{D}}$ exists and is isomorphic to $\theta_{\mathcal{M}} \circ \mathcal{Z}(\rho_{\mathcal{C}}^\mathcal{M})$. Hence the composition in question is isomorphic to $U_{\mathcal{C}} \circ \mathcal{Z}(\rho_{\mathcal{C}}^\mathcal{M})$. Now the result follows from the explicit description of $\mathcal{Z}(\rho_{\mathcal{C}}^\mathcal{M})$ given in the previous subsection.

4 Integral over a topologizing full subcategory

4.1 Topologizing full subcategory

We first introduce the following terminology and notation: A full subcategory of an abelian category is said to be topologizing [37] if it is closed under finite direct sums and subquotients. We denote by $\text{Top}(\mathcal{A})$ the class of topologizing full subcategories of an abelian category $\mathcal{A}$. 

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Let \( \mathcal{M} \) be a finite module category over a finite tensor category. In Section 3, we have considered several ‘integrals’ over the category \( \mathcal{M} \). In this section, based on our results on adjoints of the action functor, we extend techniques used in [44] and provide a framework to deal with ‘integrals’ of the form \( \int_{X \in \mathcal{S}} \text{Hom}(X, X) \) for some \( \mathcal{S} \in \text{Top}(\mathcal{M}) \).

We first summarize basic results on topologizing full subcategories of a finite abelian category. Let \( \mathcal{M} \) be a finite abelian category, and let \( \mathcal{S} \) be a topologizing full subcategory of \( \mathcal{M} \) with inclusion functor \( i : \mathcal{S} \to \mathcal{M} \). For \( M \in \mathcal{M} \), we set

\[
i^\sharp(M) = (\text{the largest subobject of } M \text{ belonging to } \mathcal{S}). \quad (4.1)
\]

By the assumption that \( \mathcal{S} \) is a topologizing full subcategory, one can extend the assignment \( M \mapsto i^\sharp(M) \) to a \( k \)-linear functor from \( \mathcal{M} \) to \( \mathcal{S} \). Dually, we set

\[
\kappa_\mathcal{S}(M) = \bigcap \{ X \subset M \mid M/X \in \mathcal{S} \} \quad \text{and} \quad \iota^\flat(M) = M/\kappa_\mathcal{S}(M) \quad (4.2)
\]

for \( M \in \mathcal{M} \). One can also extend the assignment \( M \mapsto \iota^\flat(M) \) to a \( k \)-linear functor from \( \mathcal{M} \) to \( \mathcal{S} \). It is easy to see that \( i^\sharp \) and \( \iota^\flat \) are a right and a left adjoint of \( i \), respectively. We now define

\[
\tau_\mathcal{S} := i \circ \iota^\flat \quad \text{and} \quad \tau'_\mathcal{S} := i \circ i^\sharp. \quad (4.3)
\]

Since \( \iota^\flat \rightharpoonup i \rightharpoonup i^\sharp \), we have natural isomorphisms

\[
\text{Hom}_{\mathcal{M}}(\tau_\mathcal{S}(M), N) \cong \text{Hom}_{\mathcal{S}}(\iota^\flat(M), \iota^\flat(N)) \cong \text{Hom}_{\mathcal{M}}(M, \tau'_\mathcal{S}(N)) \quad (4.4)
\]

for \( M, N \in \mathcal{M} \). Thus \( \tau_\mathcal{S} \in \text{Rex}(\mathcal{M}) \) and \( \tau'_\mathcal{S} = \tau_\mathcal{S}^\flat \). Moreover, since \( \iota^\flat \circ i = \text{id}_\mathcal{S} \), the endofunctor \( \tau_\mathcal{S} \) is an idempotent monad on \( \mathcal{M} \) whose category of modules coincides with \( \mathcal{S} \).

By this observation, we have the following consequence:

**Lemma 4.1** A topologizing full subcategory of a finite abelian category is a finite abelian category such that the inclusion functor preserves and reflects exact sequences.

Now we choose a finite-dimensional algebra \( A \) such that \( \mathcal{M} \cong A\text{-mod} \). If we identify \( \text{Rex}(\mathcal{M}) \) with \( A\text{-mod}-A \) by the equivalence

\[
A\text{-mod}-A \to \text{Rex}(\mathcal{M}), \quad M \mapsto M \otimes_A (-),
\]

then \( \text{id}_\mathcal{M} \in \text{Rex}(\mathcal{M}) \) corresponds to the \( A \)-bimodule \( A \). Thus a subobject of \( \text{id}_\mathcal{M} \) in \( \text{Rex}(\mathcal{M}) \) corresponds to a (two-sided) ideal of \( A \). By abuse of terminology, we call a subobject of \( \text{id}_\mathcal{M} \) in \( \text{Rex}(\mathcal{M}) \) an **ideal** of \( \mathcal{M} \). Then we have the following correspondence (cf. Rosenberg [37, Chapter III]):

**Lemma 4.2** For a finite abelian category \( \mathcal{M} \), there is a one-to-one correspondence between the class \( \text{Top}(\mathcal{M}) \) and the set of ideals of \( \mathcal{M} \).

For \( \mathcal{S} \in \text{Top}(\mathcal{M}) \), we define \( \kappa_\mathcal{S} \) by (4.2). The correspondence of the above lemma assigns \( \kappa_\mathcal{S} \subset \text{id}_\mathcal{S} \) to \( \mathcal{S} \). Conversely, given an ideal \( I \) of \( \mathcal{M} \), we consider the quotient \( \tau := \text{id}_\mathcal{M}/I \). If we identify \( \mathcal{M} \) with \( A\text{-mod} \) as above, then \( I \) can be regarded as an ideal of the algebra \( A \) and the functor \( \tau \) is identified with \( (A/I) \otimes_A (-) \). Thus \( \tau \) is a \( k \)-linear right exact idempotent monad on \( \mathcal{M} \). The correspondence of Lemma 4.2 assigns the category of \( \tau \)-modules to the ideal \( I \).
4.2 Integral over a full subcategory

Let $C$ be a finite tensor category, and let $M$ be a finite left $C$-module category. Given $S \in \mathcal{Top}(M)$, we consider the end

$$A'_S := \rho^{RA}_M(\tau_S) = \int_{M \in M} \operatorname{Hom}(M, \tau_S(M)),$$

where $\tau_S$ is defined by (4.3). Let $i : S \to M$ be the inclusion functor, and define $i^\circ$ by (4.2). By applying Lemma 2.2 to the adjunction $i^\circ \dashv i$, we see that the end of the functor

$$S^\circ \times S \to C, \quad (X, X') \mapsto \operatorname{Hom}(i(X), i(X'))$$

exists and is canonically isomorphic to $A'_S$. We denote the end of (4.5) by

$$A_S = \int_{X \in S} \operatorname{Hom}(X, X)$$

with omitting the inclusion functor. Let $\beta_S : A'_S \to A_S$ be the canonical isomorphism given by Lemma 2.2. If we denote by

$$\pi_S(X) : A_S \to \operatorname{Hom}(X, X) \quad \text{and} \quad \pi'_S(M) : A'_S \to \operatorname{Hom}(M, \tau_S(M))$$

the respective universal dinatural transformations, then the isomorphism $\beta_S$ is characterized as a unique morphism in $C$ such that the equation

$$\pi_S(X) \circ \beta_S = \pi'_S(X)$$

holds for all $X \in S$.

We recall that $\tau_S$ is an idempotent monad on $M$. Thus $A'_S$ is an algebra in $C$ as the image of an algebra under the monoidal functor $\rho^{RA}_M$. On the other hand, by the universal property of the end $A_S$, we can define

$$m_S : A_S \otimes A_S \to A_S \quad \text{and} \quad u_S : 1 \to A_S$$

(4.7)

to be unique morphisms such that the equations

$$\pi_S(X) \circ m_S = \text{comp}^{M}_{X,X,X,X} \circ (\pi_S(X) \otimes \pi_S(X)) \quad \text{and} \quad \pi_S(X) \circ u_S = \text{coev}^{1,X}_{1}$$

hold for all objects $X \in S$. It is easy to see that $A_S$ is an algebra in $C$ with multiplication $m_S$ and unit $u_S$.

**Lemma 4.3** The morphism $\beta_S$ is an isomorphism of algebras in $C$.

**Proof** Noting $\tau_S(X) = X$ for all $X \in S$, we easily verify that the equations

$$\pi_S(X) \circ \beta_S \circ \mu^{(2)}_{\tau_S, \tau_S} = \text{comp}^{M}_{X,X,X} = \pi_S(X) \circ m_S \circ (\beta_S \otimes \beta_S),$$

$$\pi_S(X) \circ \beta_S \circ \mu^{(0)} = \text{coev}^{1,X}_{1} = \pi_S(X) \circ u_S$$

hold for all objects $X \in S$. By the universal property of the end $A_S$, we conclude that $\beta_S$ is a morphism of algebras. $\square$

For $S \in \mathcal{Top}(M)$, we denote by $q_S : \text{id}_M \to \tau_S$ the quotient morphism. We recall that the kernel of $q_S$ is $\kappa_S$. For $S_1, S_2 \in \mathcal{Top}(M)$ with $S_1 \supset S_2$, we have $\kappa_{S_1} \subset \kappa_{S_2}$ as subobjects of $\text{id}_M$. Thus there is a unique morphism $q_{S_1|S_2} : \tau_{S_1} \to \tau_{S_2}$ such that $q_{S_1|S_2} \circ q_{S_1} = q_{S_2}$.
For \( S_1, S_2 \) with \( S_1 \supset S_2 \), we also define a morphism \( \phi_{S_1 \mid S_2} : A_{S_1} \to A_{S_2} \) to be a unique morphism such that the equation

\[
\pi_{S_2}(X) \circ \phi_{S_1 \mid S_2} = \pi_{S_1}(X)
\]

holds for all objects \( X \in S_2 \).

**Lemma 4.4** With the above notation, we have

\[
\phi_{S_1 \mid S_2} \circ \beta_{S_1} = \beta_{S_2} \circ \rho_{\mathcal{M}}^a(q_{S_1 \mid S_2}).
\]

**Proof** For all objects \( X \in S_2 \), we have

\[
\pi_{S_2}(X) \circ \phi_{S_1 \mid S_2} \circ \beta_{S_1} = \pi_{S_1}(X) \circ \beta_{S_1} = \pi_{S_1}'(X)
\]

by (4.6) and (4.8). Noting \( \tau_{S_1}(X) = X \) and \( (q_{S_1 \mid S_2})_X = \text{id}_X \), we also have

\[
\pi_{S_2}(X) \circ \beta_{S_2} \circ \rho_{\mathcal{M}}^a(q_{S_1 \mid S_2}) = \pi_{S_2}'(X) \circ \rho_{\mathcal{M}}^a(q_{S_1 \mid S_2})
\]

\[
= \text{Hom}(\text{id}_X, (q_{S_1 \mid S_2})_X) \circ \pi_{S_2}'(X) = \pi_{S_2}'(X).
\]

The claim follows from the universal property of \( A_{S_2} \). \( \square \)

For \( S_1, S_2, S_3 \in \text{T}_{\mathcal{M}} \) with \( S_1 \supset S_2 \supset S_3 \), we have

\[
q_{S_2 \mid S_3} \circ q_{S_1 \mid S_2} = q_{S_1 \mid S_3} \quad \text{and} \quad \phi_{S_2 \mid S_3} \circ \phi_{S_1 \mid S_2} = \phi_{S_1 \mid S_3}.
\]  

(4.9)

**Lemma 4.4** says that the inverse system \( ([A_S], \{\phi_{S_1 \mid S_2}\}) \) in \( \mathcal{C} \) is obtained from the inverse system \( ([\tau_{S_1}], \{q_{S_1 \mid S_2}\}) \) in \( \text{Rex}(\mathcal{M}) \) by applying \( \rho_{\mathcal{M}}^a \). We note that an exact functor preserves epimorphisms. By Theorem 3.4 and Lemma 4.4, we have:

**Lemma 4.5** If \( \mathcal{M} \) is an exact \( \mathcal{C} \)-module category, then \( ([A_S], \{\phi_{S_1 \mid S_2}\}) \) is an inverse system of epimorphisms in \( \mathcal{C} \).

We use the above observation to state the main result of this section. For an object \( X \) of an essentially small category \( \mathcal{E} \), we denote by \( \Omega_{\mathcal{E}}(X) \) and \( \text{Sub}(X) \) the set of quotient objects of \( X \) and the set of subobjects of \( X \), respectively. We introduce partial orders on these sets as follows: For \( Q_1, Q_2 \in \Omega_{\mathcal{E}}(X) \), we write \( Q_1 \geq Q_2 \) if there is a morphism \( Q_1 \to Q_2 \) in \( \mathcal{E} \) compatible with the quotient morphisms from \( X \). Dually, for \( S_1, S_2 \in \text{Sub}(X) \), we write \( S_1 \geq S_2 \) if there is a morphism \( S_2 \to S_1 \) in \( \mathcal{E} \) compatible with the inclusion morphisms to \( X \).

**Theorem 4.6** Let \( \mathcal{M} \) be an exact \( \mathcal{C} \)-module category. Then the map

\[
\tau_{\mathcal{M}}(X) \to \Omega_{\mathcal{M}}(A_{\mathcal{M}}), \quad S \mapsto A_S = \bigcap_{X \in S} \text{Hom}(X, X)
\]

preserves the order. If, moreover, \( \mathcal{M} \) is indecomposable, then this map reflects the order.

**Proof** **Lemma 4.5** means that the map in question preserves the order. To complete the proof, we suppose that \( \mathcal{M} \) is indecomposable. Let \( S_1 \) and \( S_2 \) be topologizing full subcategory of \( \mathcal{M} \) such that \( A_{S_1} \geq A_{S_2} \) in \( \Omega_{\mathcal{M}}(A_{\mathcal{M}}) \). Then we have \( \rho_{\mathcal{M}}^a(\kappa_{S_1}) \leq \rho_{\mathcal{M}}^a(\kappa_{S_2}) \) in \( \text{Sub}(A_{\mathcal{M}}) \).

Since \( \rho_{\mathcal{M}}^a \) is exact, we have

\[
\rho_{\mathcal{M}}^a(\kappa_{S_2}) = \rho_{\mathcal{M}}^a(\kappa_{S_2} \cap \kappa_{S_1}) = \rho_{\mathcal{M}}^a(\kappa_{S_2} \cap \kappa_{S_1}) = 0.
\]

Since \( \rho_{\mathcal{M}}^a \) is faithful by Theorem 3.4, we have \( \kappa_{S_2}/(\kappa_{S_2} \cap \kappa_{S_1}) = 0 \). This implies that \( \kappa_{S_1} \subset \kappa_{S_2} \). Hence \( S_1 \subset S_2 \). The proof is done. \( \square \)
We also state the dual of Theorem 4.6. Let $\lambda_M : C \to \text{Lex}(M)$ be the left exact version of the action functor. By Remark 3.6 and the dual of Lemma 2.2 (see [9, Lemma 3.9]), the coend of the functor

$$S^{op} \times S \to C, \quad (X, X') \mapsto \coHom(X, X')$$

exists for all $S \in \text{Top}(M)$ and is canonically isomorphic to the coend

$$\lambda_M^{la}(\tau_S^{ra}(M)) = \int_{M \in M} \coHom(M, \tau_S^{ra}(M)).$$

We denote the coend of (4.10) by $L_S = \int_{X \in S} \coHom(X, X)$. Since the duality functor is an anti-equivalence, the object $^*A_S$ is also a coend of the functor (4.10) with universal dinatural transformation

$$^*\pi_S : ^*A_S \to ^* \text{Hom}(X, X) = \coHom(X, X) \quad (X \in S).$$

Thus there is an isomorphism $^*A_S \cong L_S$ respecting the universal dinatural transformations. By the above observation, we now obtain the following theorem:

**Theorem 4.7** Let $M$ be an exact $C$-module category. Then the map

$$\text{Top}(M) \to \text{Sub}(L_M), \quad S \mapsto L_S$$

preserves the order. If, moreover, $M$ is indecomposable, then this map reflects the order.

### 4.3 Integral over a module full subcategory

Let $C$ be a finite tensor category, and let $M$ be a finite left $C$-module category. We introduce the following terminology:

**Definition 4.8** A $C$-module full subcategory of $M$ is a topologizing full subcategory of $M$ closed under the action of $C$.

Let $S$ be a $C$-module full subcategory of $M$, and let $\kappa_S$ and $\tau_S$ be the endofunctors on $M$ defined by (4.2) and (4.3), respectively. Then we have

$$(V \otimes M)(V \otimes \kappa_S(M)) \cong V \otimes (M/\kappa_S(M)) = V \otimes \tau_S(M) \in S$$

for all $V \in C$ and $M \in M$. Thus we have a natural transformation

$$\tau_S(V \otimes M) \to V \otimes \tau_S(M) \quad (V \in C, M \in M)$$

making $\tau_S \in \text{Rex}(M)$ an oplax $C$-module endofunctor on $M$. Since $C$ is rigid, it is actually a strong $C$-module functor.

By Theorem 3.11, we endow the algebra $A'_S = \rho^{ra}(\tau_S)$ with a half-braiding $\sigma'_S$ such that $(A'_S, \sigma'_S)$ is an algebra in $Z(C)$. Since $A_S$ is isomorphic to $A'_S$, the algebra $A_S$ also give rise to an algebra in $Z(C)$. By Lemma 3.12, the half-braiding

$$\sigma_S(V) : A_S \otimes V \to V \otimes A_S \quad (V \in C)$$

of $A_S$ inherited from $\rho^{ra}(\tau_S)$ is the unique morphism such that the diagram

$$
\begin{array}{ccc}
A_S \otimes V & \xrightarrow{\pi_S(V \otimes X) \otimes \text{id}_V} & \text{Hom}(V \otimes X, V \otimes X) \otimes V \\
\downarrow \sigma_S(V) & & \downarrow \text{by}_{V \otimes X, \text{Hom}(X, X)} \\
V \otimes A_S & \xrightarrow{\text{id}_V \otimes \pi_S} & V \otimes \text{Hom}(X, X) \xrightarrow{a_{V \otimes X, X}} \text{Hom}(V \otimes X, X)
\end{array}
$$

is commutative.
commutes for all objects \( X \in \mathcal{S} \). We write \( A_{\mathcal{S}} := (A_{\mathcal{S}}, \sigma_{\mathcal{S}}) \in \mathcal{Z}(\mathcal{C}) \). The following result is well-known in the case where \( \mathcal{M} = \mathcal{S} = \mathcal{C} \).

**Theorem 4.9** The algebra \( A_{\mathcal{S}} \in \mathcal{Z}(\mathcal{C}) \) is commutative.

**Proof** We postpone the proof of this theorem to Appendix A since it requires some technical results on the natural isomorphisms \( \alpha \) and \( \beta \).

Let \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) be \( \mathcal{C} \)-module full subcategories of \( \mathcal{M} \) such that \( \mathcal{S}_1 \supset \mathcal{S}_2 \). We have introduced the morphism \( \phi_{\mathcal{S}_1 \mid \mathcal{S}_2} : A_{\mathcal{S}_1} \to A_{\mathcal{S}_2} \) in \( \mathcal{C} \) in the previous subsection. By the definition of \( \phi_{\mathcal{S}_1 \mid \mathcal{S}_2} \) and the above explicit description of the half-braiding, we have the following result:

**Theorem 4.10** \( \phi_{\mathcal{S}_1 \mid \mathcal{S}_2} : A_{\mathcal{S}_1} \to A_{\mathcal{S}_2} \) is a morphism in \( \mathcal{Z}(\mathcal{C}) \).

Thus, if \( \mathcal{M} \) is exact, then we have an order-preserving map

\[
\{ \text{\( \mathcal{C} \)-module full subcategories of} \ \mathcal{M} \} \to \{ \text{quotient algebras of} \ A_M \ \text{in} \ \mathcal{Z}(\mathcal{C}) \}
\]

defined by \( S \mapsto A_S \). If, moreover, \( \mathcal{M} \) is indecomposable, then this map reflects the order.

**5 Class functions and characters**

**5.1 The space of class functions**

Let \( \mathcal{C} \) be a finite tensor category. For each finite left \( \mathcal{C} \)-module category \( \mathcal{M} \), we have introduced an algebra \( A_M \in \mathcal{C} \) in the last section. The vector space \( \text{Hom}_\mathcal{C}(A_M, 1) \) with \( \mathcal{M} = \mathcal{C} \) is called the space of class functions in [44] as it generalizes the usual notion of class functions on a finite group. The aim of this section is to explore the structure of the space of class functions and its generalization to module categories. We first introduce the following notation:

**Definition 5.1** For a finite left \( \mathcal{C} \)-module category \( \mathcal{M} \), we define the space \( \text{CF}(\mathcal{M}) \) of class functions of \( \mathcal{M} \) by \( \text{CF}(\mathcal{M}) = \text{Hom}_\mathcal{C}(A_M, 1) \).

Let \( U : Z(\mathcal{C}) \to \mathcal{C} \) be the forgetful functor. To study class functions, we consider the functor \( Z := U^\alpha \circ U \). There is an equivalence \( \rho' : \mathcal{C} \to \mathcal{C}^e \) given by \( \rho'(V) = \text{id}_\mathcal{C} \otimes V \). By applying Theorem 3.14 to \( \mathcal{M} = \mathcal{C} \), we have

\[
Z(V) = \rho'^{\otimes 2} \rho'(V) = \int_{X \in \mathcal{C}} X \otimes V \otimes X^* \ (V \in \mathcal{C}).
\]

Now let \( \pi_Z^X(V) : Z(V) \to X \otimes V \otimes X^* \ (V, X \in \mathcal{C}) \) be the universal dinatural transformation for the end \( Z(V) \). The assignment \( V \mapsto Z(V) \) extends to an endofunctor on \( \mathcal{C} \) in such a way that \( \pi_Z^X(V) \) is natural in \( V \) and dinatural in \( X \). By the universal property, we define natural transformations \( \Delta_Z^X : Z \to Z^2 \) and \( \epsilon_Z^X : Z \to \text{id}_\mathcal{C} \) by

\[
(id_X \otimes \pi_Z^X(Y) \otimes \text{id}_{X^*}) \circ \pi_Z^{Z(V)}(X) \circ \Delta_Z^X = \pi_Y^X(X \otimes Y) \quad \text{and} \quad \epsilon_Z^X = \pi_Z^X(1)
\]

for all objects \( V, X, Y \in \mathcal{C} \). The functor \( Z \) is a comonad on \( \mathcal{C} \) with comultiplication \( \Delta_Z^X \) and counit \( \epsilon_Z^X \).

Given an object \( V = (V, \sigma) \in Z(\mathcal{C}) \), we define the morphism \( \delta : V \to Z(V) \) in \( \mathcal{C} \) to be the unique morphism such that the equation

\[
\pi_Z^X(V) \circ \delta = (\sigma_X \otimes \text{id}_{X^*}) \circ (\text{id}_V \otimes \coev_X)
\]
holds for all objects $X \in \mathcal{C}$. The assignment $(V, \sigma) \mapsto (V, \delta)$ allows us to identify $\mathcal{Z}(\mathcal{C})$ with the category of $\mathcal{Z}$-comodules. If we identify them, then a right adjoint of $U$ is given by the free $\mathcal{Z}$-comodule functor

$$U^{\text{ra}}: \mathcal{C} \to \text{(the category of } \mathcal{Z}\text{-comodules)}, \quad V \mapsto (Z(V), \Delta_V^Z).$$

By Theorem 4.9, for each finite left $\mathcal{C}$-module category $\mathcal{M}$, there is a commutative algebra $A_\mathcal{M} = (A_\mathcal{M}, \sigma_\mathcal{M})$ in $\mathcal{Z}(\mathcal{C})$ such that $A_\mathcal{M} = U(A_\mathcal{M})$.

**Definition 5.2** Let $\mathcal{M}$ be as above, and let $\delta_\mathcal{M}: A_\mathcal{M} \to Z(A_\mathcal{M})$ be the coaction of $Z$ associated to the half-braiding $\sigma_\mathcal{M}$. For $f \in CF(\mathcal{C})$ and $g \in CF(\mathcal{M})$, we define their product $f \ast g \in CF(\mathcal{M})$ by

$$f \ast g = f \circ Z(g) \circ \delta_\mathcal{M}. \quad (5.1)$$

In particular, we have a binary operation on $CF(\mathcal{C})$ by considering the case where $\mathcal{M} = \mathcal{C}$ in the above definition. As we have observed in [44], $CF(\mathcal{C})$ is an associative unital algebra with respect $\ast$. Moreover, we have:

**Lemma 5.3** $CF(\mathcal{M})$ is a left $CF(\mathcal{C})$-module by $\ast$.

**Proof** We remark $A_\mathcal{C} = U^{\text{ra}}(1)$. Thus there is an isomorphism

$$\Phi_\mathcal{M}: CF(\mathcal{M}) \to \text{Hom}_{\mathcal{Z}(\mathcal{C})}(A_\mathcal{M}, A_\mathcal{C}), \quad f \mapsto Z(f) \circ \delta_\mathcal{M}.$$  

Then, noting $\delta_\mathcal{C} = \Delta_\mathcal{Z}$, we compute

$$\Phi_\mathcal{C}(f) \circ \Phi_\mathcal{M}(g) = Z(f) \circ \Delta_\mathcal{Z} \circ Z(g) \circ \delta_\mathcal{M} = Z(f) \circ Z^2(g) \circ \Delta^Z_{A_\mathcal{M}} \circ \delta_\mathcal{M} = Z(f) \circ Z^2(g) \circ Z(\delta_\mathcal{M}) \circ \delta_\mathcal{M} = Z(f \ast g) \circ \delta_\mathcal{M} = \Phi_\mathcal{M}(f \ast g)$$

for all elements $f \in CF(\mathcal{C})$ and $g \in CF(\mathcal{M})$. Since the composition of morphisms is unital and associative, the action $\ast: CF(\mathcal{C}) \times CF(\mathcal{M}) \to CF(\mathcal{M})$ is also unital and associative.

The proof is done. \(\square\)

We set $F = CF(\mathcal{C})$ and $E = \text{End}_Z(A_\mathcal{C})$ for simplicity. The proof of the above lemma implies the following interesting consequence:

**Theorem 5.4** There is an isomorphism $E \cong F$ of algebras. Moreover, the left $F$-module $CF(\mathcal{M})$ corresponds to the left $E$-module $\text{Hom}_{\mathcal{Z}(\mathcal{C})}(A_\mathcal{M}, A_\mathcal{C})$ under the isomorphism $E \cong F$.

### 5.2 Pivotal module category

We recall that a *pivotal monoidal category* is a rigid monoidal category $\mathcal{C}$ equipped with a pivotal structure, that is, an isomorphism $X \to X^{**}$ ($X \in \mathcal{C}$) of monoidal functors. Let $\mathcal{C}$ be a pivotal finite tensor category with pivotal structure $j$. For an object $X \in \mathcal{C}$, we set

$$\text{tr}_\mathcal{C}(X) = \text{ev}_{X^*} \circ (j_X \otimes \text{id}_{X^*})$$

and define the *internal character* [44] of $X$ by

$$\text{ch}(X) = \text{tr}_\mathcal{C}(X) \circ \pi_C(X) \in CF(\mathcal{C}).$$

Some applications of this notion are given in [44]. It is interesting to extend results of [44] to module categories. We first introduce the notion of *pivotal module category*. To give its precise definition, we recall the following notion:
Definition 5.5 ([19]). For an exact $C$-module category $\mathcal{M}$, there is a unique functor $S_\mathcal{M} : \mathcal{M} \to \mathcal{M}$ equipped with a natural isomorphism
\[
\text{Hom}(M, N)^* \cong \text{Hom}(N, S_\mathcal{M}(M)) \tag{5.2}
\]
for $M, N \in \mathcal{M}$. We call $S_\mathcal{M}$ the relative Serre functor of $\mathcal{M}$.

Let $\mathcal{M}$ be an exact left $C$-module category. We make $\mathcal{M}^{\text{op}} \times \mathcal{M}$ a $C$-bimodule category by (2.16). Then $\text{Hom} : \mathcal{M}^{\text{op}} \times \mathcal{M} \to C$ is a $C$-bimodule functor. Given a strong monoidal functor $T : C \to C$ and a left $C$-module category $\mathcal{N}$, we denote by $\mathcal{N}^T$ the left $C$-module category whose underlying category is $\mathcal{N}$ but the action of $C$ on $\mathcal{N}$ is twisted by $T$. The functor
\[
\mathcal{M}^{\text{op}} \times \mathcal{M} \to (\_)^{\ast\ast} C, \quad (M, N) \mapsto \text{Hom}(N, M)^*
\]
is a $C$-bimodule functor by (2.10) and (2.11). By [19, Lemma 4.22], there is a unique natural isomorphism
\[
X^{\ast\ast} \otimes S_\mathcal{M}(M) \to S_\mathcal{M}(X \otimes M) \tag{5.3}
\]
making $S_\mathcal{M}$ a left $C$-module functor $S_\mathcal{M} : \mathcal{M} \to (\_)^{\ast\ast} \mathcal{M}$ such that (5.2) is an isomorphism of $C$-bimodule functors.

Definition 5.6 Let $C$ be a pivotal finite tensor category with pivotal structure $j$, and let $\mathcal{M}$ be an exact $C$-module category. A pivotal structure of $\mathcal{M}$ is a natural isomorphism $j' : \text{id}_\mathcal{M} \to S_\mathcal{M}$ such that the equation
\[
j_X \otimes M = \left( X \otimes M \xrightarrow{j_X \otimes j_M} X^{\ast\ast} \otimes S_\mathcal{M}(M) \xrightarrow{(5.3) \cong} S_\mathcal{M}(X \otimes M) \right) \tag{5.4}
\]
holds for all $X \in C$ and $M \in \mathcal{M}$. A pivotal left $C$-module category is an exact left $C$-module category equipped with a pivotal structure. Let $\mathcal{M}$ be such a category, and let $j'$ be the pivotal structure of $\mathcal{M}$. Then we define the trace
\[
\text{tr}_\mathcal{M}(M) : \text{Hom}(M, M) \to 1 \quad (M \in \mathcal{M})
\]
to be the morphism corresponding to $j'_M : M \to S_\mathcal{M}(M)$ via
\[
\text{Hom}_\mathcal{M}(M, S_\mathcal{M}(M)) \cong \text{Hom}_\mathcal{M}(1, \text{Hom}(M, S_\mathcal{M}(M))) \cong \text{Hom}_\mathcal{M}(1, \text{Hom}(M, M)^* ) \cong \text{Hom}_\mathcal{M}(\text{Hom}(M, M), 1).
\]

Remark 5.7 Let $C$ and $\mathcal{M}$ be as above. Then, for a morphism $f : M \to M$ in $\mathcal{M}$, the pivotal trace $\text{ptr}(f) \in k$ is defined by
\[
\text{ptr}(f) \cdot \text{id}_1 = \text{tr}_\mathcal{M}(M) \circ \text{Hom}(\text{id}_M, f) \circ \text{coev}_{1,M}.
\]
As in the ordinary trace, the pivotal trace is cyclic, multiplicative with respect to $\otimes$ and additive with respect to exact sequences; see Propositions B.2 and B.4 in Appendix B.

5.3 Internal characters for module categories

Let $C$ be a pivotal finite tensor category, and let $\mathcal{M}$ be a pivotal exact left $C$-module category. We now define:
Definition 5.8 The internal character of \( M \in \mathcal{M} \) is defined by
\[
\text{ch}_\mathcal{M}(M) = \text{tr}_\mathcal{M}(M) \circ \pi_\mathcal{M}(M) \in \text{CF}(\mathcal{M}).
\]

We give basic properties of internal characters:

Lemma 5.9 For all \( X \in \mathcal{C} \) and \( M \in \mathcal{M} \), we have
\[
\text{ch}_\mathcal{C}(X) \ast \text{ch}_\mathcal{M}(M) = \text{ch}_\mathcal{M}(X \otimes M).
\]

Proof Straightforward. See Appendix B for the detail.

Lemma 5.10 The internal character is additive in exact sequences: For any exact sequence \( 0 \to M_1 \to M_2 \to M_3 \to 0 \) in \( \mathcal{M} \), we have
\[
\text{ch}_\mathcal{M}(M_2) = \text{ch}_\mathcal{M}(M_1) + \text{ch}_\mathcal{M}(M_2).
\]

Proof It is well-known that the pivotal trace is additive in exact sequences. One can find a detailed proof of this fact in [20, Lemma 2.5.1]. The proof of this lemma goes along the same line; see Appendix B for the detail.

For a finite abelian category \( \mathcal{A} \), we denote by \( \text{Gr}(\mathcal{A}) \) the Grothendieck group of \( \mathcal{A} \), that is, the quotient of the additive group generated by the isomorphism classes of objects of \( \mathcal{A} \) by the relation \( [M_2] = [M_1] + [M_3] \) for all exact sequences \( 0 \to M_1 \to M_2 \to M_3 \to 0 \) in \( \mathcal{A} \). We set \( \text{Gr}_k(\mathcal{A}) = k \otimes_{\mathbb{Z}} \text{Gr}(\mathcal{A}) \).

Now let \( \{L_1, \ldots, L_n\} \) be a complete set of representatives of isomorphism classes of simple objects of \( \mathcal{M} \). As a generalization of the main result of [44], we prove the following theorem:

Theorem 5.11 The set \( \{\text{ch}(L_i)\}_{i=1}^n \subset \text{CF}(\mathcal{M}) \) is linearly independent.

Proof The proof goes along the same way as [44]. Let \( \mathcal{S} \) be the full subcategory of \( \mathcal{M} \) consisting of all semisimple objects of \( \mathcal{M} \). Then, since \( \mathcal{S} \) is semisimple, we may assume \( \text{AS} = \bigoplus_{i=1}^n \text{Hom}(L_i, L_i) \) and \( \pi_\mathcal{S}(L_i) \) is the projection to \( \text{Hom}(L_i, L_i) \) for \( i = 1, \ldots, n \). Let \( \phi_{\mathcal{M}|\mathcal{S}} : A_\mathcal{M} \to A_\mathcal{S} \) be the morphism defined by (4.8). Since \( \phi_{\mathcal{M}|\mathcal{S}} \) is an epimorphism, the map
\[
\bigoplus_{i=1}^n \text{Hom}_\mathcal{C}(\text{Hom}(L_i, L_i), \mathbb{I}) = \text{CF}(\mathcal{S}) \xrightarrow{\text{Hom}_\mathcal{C}(\phi_{\mathcal{M}|\mathcal{S}}, \mathbb{I})} \text{CF}(\mathcal{M})
\]
is injective. Since the morphism \( \text{ch}_\mathcal{M}(L_i) \) is the image of the morphism \( \text{tr}_\mathcal{M}(L_i) \) under this map, the set \( \{\text{ch}(L_i)\}_{i=1}^n \) is linearly independent in \( \text{CF}(\mathcal{M}) \).

Let \( \mathcal{C} \) and \( \mathcal{M} \) be as above. By Lemma 5.10 and Theorem 5.11, the linear map
\[
\text{ch}_\mathcal{M} : \text{Gr}_k(\mathcal{M}) \to \text{CF}(\mathcal{M}), \quad [M] \mapsto \text{ch}_\mathcal{M}(M) \quad (M \in \mathcal{M})
\]
is well-defined and injective. Lemma 5.9 implies that \( \text{ch}_\mathcal{C} : \text{Gr}_k(\mathcal{C}) \to \text{CF}(\mathcal{C}) \) is an algebra map and \( \text{ch}_\mathcal{M} : \text{Gr}_k(\mathcal{M}) \to \text{CF}(\mathcal{M}) \) is \( \text{Gr}_k(\mathcal{C}) \)-linear if we view \( \text{CF}(\mathcal{M}) \) as a left \( \text{Gr}_k(\mathcal{C}) \)-module through the algebra map \( \text{ch}_\mathcal{C} \).

By the proof of the above lemma, we see that the linear map \( \text{ch}_\mathcal{M} \) is bijective if \( \mathcal{M} \) is semisimple. We have proved that, under the assumption that \( \mathcal{C} \) is unimodular in the sense of [17], the map \( \text{ch}_\mathcal{C} : \text{Gr}_k(\mathcal{C}) \to \text{CF}(\mathcal{C}) \) is bijective if and only if \( \mathcal{C} \) is semisimple [44]. It would be interesting to establish an analogous result for module categories. The unimodularity of module categories, introduced in [19], may be useful to formulate such a result.
5.4 Class functions of the dual tensor category

Let \( \mathcal{C} \) be a finite tensor category, and let \( \mathcal{M} \) be an indecomposable exact left \( \mathcal{C} \)-module category. As an application of our results, we give the following description of the algebra of class functions of the dual tensor category:

**Theorem 5.12** \( \text{CF}(\mathcal{C}_\mathcal{M}^*) \cong \text{End}_{\mathbb{Z}(\mathcal{C})}(\mathbf{A}_\mathcal{M}) \) as algebras.

**Proof** Set \( \mathcal{D} = \mathcal{C}_\mathcal{M}^* \). Let \( U : \mathcal{Z}(\mathcal{D}) \to \mathcal{D} \) be the forgetful functor. By Theorems 3.13, 3.14 and 5.4, we have isomorphisms

\[
\text{CF}(\mathcal{D}) \cong \text{End}_{\mathcal{Z}(\mathcal{D})}(\mathcal{U}_{\mathcal{D}}^\mathcal{D}(1_{\mathcal{D}})) \cong \text{End}_{\mathbb{Z}(\mathcal{C})}(\theta_{\mathcal{M}}^{-1}\mathcal{U}_{\mathcal{D}}^\mathcal{D}(1_{\mathcal{D}})) \cong \text{End}_{\mathbb{Z}(\mathcal{C})}(\mathbf{A}_\mathcal{M})
\]

of algebras. The proof is done. \( \square \)

A semisimple finite tensor category is called a fusion category [16]. Our results give some new results on fusion categories. For example:

**Corollary 5.13** Suppose that the base field \( k \) is of characteristic zero. Let \( \mathcal{C} \) be a fusion category, and let \( \mathcal{M} \) be an indecomposable exact left \( \mathcal{C} \)-module category such that \( \mathcal{C}_\mathcal{M}^* \) admits a pivotal structure. Then there is an isomorphism

\[
\text{Gr}_k(\mathcal{C}_\mathcal{M}^*) \cong \text{End}_{\mathbb{Z}(\mathcal{C})}(\mathbf{A}_\mathcal{M})
\]

of algebras.

**Proof** \( \mathcal{C}_\mathcal{M}^* \) is a pivotal fusion categories by the assumption [16]. Thus the result follows from the results of the previous subsection. \( \square \)

The following result generalizes [35, Example 2.18]:

**Corollary 5.14** Under the same assumption on the above corollary, the following two assertions are equivalent:

1. The Grothendieck ring of \( \mathcal{C}_\mathcal{M}^* \) is commutative.
2. The object \( \mathbf{A}_\mathcal{M} \in \mathcal{Z}(\mathcal{C}) \) is multiplicity-free.

**Proof** Since \( k \) is of characteristic zero, \( \mathcal{Z}(\mathcal{C}) \) is a fusion category [16]. Moreover, the ring \( \text{Gr}(\mathcal{D}) \) is commutative if and only if the algebra \( \text{Gr}_k(\mathcal{D}) \) is. Now the claim follows from the above corollary. \( \square \)

6 A filtration on the space of class functions

6.1 A filtration on the space of class functions

Let \( \mathcal{M} \) be a finite abelian category. For an object \( M \in \mathcal{M} \), we denote by \( \text{soc}(M) \) the socle of \( M \). Every object \( M \in \mathcal{M} \) has a canonical filtration

\[
0 = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots \subset M
\]

such that \( M_{i+1}/M_i = \text{soc}(M/M_i) \). We denote \( M_n \) by \( \text{soc}_n(M) \). Then the assignment \( M \mapsto \text{soc}_n(M) \) extends to a \( k \)-linear left exact endofunctor on \( \mathcal{M} \), which we call the \( n \)-th socle functor. The number

\[
\text{Lw}(M) = \min\{n = 0, 1, 2, \ldots \mid \text{soc}_n(M) = M\}
\]
is called the \textit{Loewy length} of $M$. We define $\mathcal{M}_n$ to be the full subcategory of $\mathcal{M}$ consisting of all objects $M$ with $\text{Lw}(M) \leq n$. Since $\mathcal{M}$ is finite, the number

$$\text{Lw}(\mathcal{M}) := \min\{n = 0, 1, 2, \ldots \mid \mathcal{M}_n = \mathcal{M}\} = \max\{\text{Lw}(M) \mid M \in \mathcal{M}\}$$

is finite. We call $\text{Lw}(\mathcal{M})$ the \textit{Loewy length} of $\mathcal{M}$ and the filtration

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_w = \mathcal{M} \quad (w = \text{Lw}(\mathcal{M})) \quad (6.1)$$

the \textit{socle filtration} of $\mathcal{M}$.

It is easy to see that each $\mathcal{M}_n$ is a topologizing full subcategory of $\mathcal{M}$. Thus, if $\mathcal{C}$ is a finite tensor category and $\mathcal{M}$ is an exact left $\mathcal{C}$-module category with Loewy length $w$, then we have a series

$$A_{\mathcal{M}} = A_{\mathcal{M}_w} \to A_{\mathcal{M}_{w-1}} \to \cdots \to A_{\mathcal{M}_2} \to A_{\mathcal{M}_1} \quad (6.2)$$

of epimorphisms of algebras in $\mathcal{C}$ by Theorem 4.6. Applying $\text{Hom}_\mathcal{C}(-, 1)$ to this series, we obtain the filtration of the space of class functions

$$\mathcal{CF}_1(\mathcal{M}) \subset \mathcal{CF}_2(\mathcal{M}) \subset \cdots \subset \mathcal{CF}_{w-1}(\mathcal{M}) \subset \mathcal{CF}_w(\mathcal{M}) = \mathcal{CF}(\mathcal{M}), \quad (6.3)$$

where $\mathcal{CF}_n(\mathcal{M}) = \text{Hom}_\mathcal{C}(A_{\mathcal{M}_n}, 1)$. In this section, we investigate how this filtration relates to representation-theoretic properties of $\mathcal{M}$.

\section*{6.2 Jacobson radical functor}

For further study of the series (6.2) and the filtration (6.3), we introduce the following abstract definition of the Jacobson radical: Let $\mathcal{M}$ be a finite abelian category. For an object $M \in \mathcal{M}$, we define the subobject $\text{rad}(M)$ of $M$ to be the intersection of all maximal subobjects of $M$. It is easy to see that $M \mapsto \text{rad}(M)$ extends to a $k$-linear right exact endofunctor on $\mathcal{M}$. We call $\text{rad} \in \text{Rex}(\mathcal{M})$ the \textit{Jacobson radical functor} of $\mathcal{M}$.

We rephrase several known results in the representation theory in terms of the Jacobson radical functor. Let $A$ be a finite-dimensional algebra such that $\mathcal{M} \cong A\text{-mod}$, and let $J$ be the Jacobson radical of $A$. Then the Jacobson radical functor may be identified with $J \otimes_A (-)$. Thus we have the series

$$\text{id}_\mathcal{M} \mapsto \text{rad} \supset \text{rad}\text{-}\text{rad} \supset \cdots \supset \text{rad}^{w-1} \supset \text{rad}^w = 0 \quad (w = \text{Lw}(\mathcal{M})) \quad (6.4)$$

of subobjects in $\text{Rex}(\mathcal{M})$. We have $\text{rad}^i_{\mathcal{M}} \neq \text{rad}^{i+1}_{\mathcal{M}}$ for all $i = 0, \ldots, w-1$ by the Nakayama lemma.

For a positive integer $n$, we define the \textit{$n$th capital functor} $\text{cap}_n \in \text{Rex}(\mathcal{M})$ as the quotient object $\text{id}_\mathcal{M} / \text{rad}^n$. If we identify $\text{Rex}(\mathcal{M})$ with $A\text{-mod}_A$, then this functor corresponds to the bimodule $A/J^n$ and therefore

$$\text{cap}_n(M) = (A/J^n) \otimes_A M \cong M/J^nM \quad (6.5)$$

for all $M \in \mathcal{M}$. By Sakurai [39, Lemma 2.3], there is an adjunction

$$\text{Hom}_\mathcal{M}(\text{cap}_n(M), M') \cong \text{Hom}_\mathcal{M}(M, \text{soc}_n(M')) \quad (M, M' \in \mathcal{M}). \quad (6.6)$$

The $n$-th term $\mathcal{M}_n$ of the socle filtration (6.1) coincides with the full subcategory of $\mathcal{M}$ consisting of all objects $M$ such that $\text{soc}_n(M) = M$. Comparing (6.6) with (4.4), we have $\text{cap}_n = \tau_{\mathcal{M}_n}$, with the notation in Sect. 4.1. In other words, $\mathcal{M}_n$ corresponds to the ideal $\text{rad}^n$ via the correspondence of Lemma 4.2.
Now we consider the case where \( C \) is a finite tensor category and \( M \) is an exact left \( C \)-module category with Loewy length \( w \). There is a series
\[
\text{id}_M = \text{cap}_w \twoheadrightarrow \text{cap}_{w-1} \twoheadrightarrow \cdots \twoheadrightarrow \text{cap}_2 \twoheadrightarrow \text{cap}_1
\] (6.7)
of epimorphisms in \( \text{Rex}(\mathcal{M}) \). We have a canonical isomorphism
\[
\rho^{ra}(\text{cap}_n) \cong \int_{X \in M_n} \text{Hom}(X, X),
\]
and the series (6.2) is obtained by applying \( \rho^{ra} \) to (6.7).

### 6.3 Reynolds ideal and its generalization

Let \( A \) be a finite-dimensional algebra. For \( n \in \mathbb{Z}_+ \), we define the \( n \)-th Reynolds ideal \([38] \)
of \( A \) by
\[
\text{Rey}_n(A) = \text{soc}_n(A) \cap Z(A),
\] (6.8)
where \( \text{soc}_n(A) \) is the \( n \)-th socle of the left \( A \)-module \( A \). As \( \text{Rey}_n(A) \) is a Morita invariant \([38] \), it is natural to expect that the \( n \)-th Reynolds ideal of a finite abelian category is defined in an intrinsic way. For \( n = 1 \), this was achieved by Gainutdinov and Runkel in [22]. By using the Jacobson radical functor, we propose the following definition, which is different to [22]:

**Definition 6.1** Let \( \mathcal{M} \) be a finite abelian category. For a non-negative positive integer \( n \), we define the \( n \)-th Reynolds ideal of \( \mathcal{M} \) by
\[
\text{Rey}_n(\mathcal{M}) = \{ \xi \in \text{End}(\text{id}_\mathcal{M}) \mid \xi \circ i_n = 0 \},
\]
where \( i_n : \text{rad}^n \to \text{id}_\mathcal{M} \) is the inclusion morphism.

Let \( A \) be a finite-dimensional algebra. We explain that \( \text{Rey}_n(\mathcal{M}) \) can be identified with the \( n \)-th Reynolds ideal of \( A \) when \( \mathcal{M} = A \text{-mod} \). Let \( J \) be the Jacobson radical of \( A \). Then the \( n \)-th socle of \( M \in A \text{-mod} \) is given by
\[
\text{soc}_n(M) = \{ m \in M \mid rm = 0 \text{ for all } r \in J^n \},
\]
and hence the \( n \)-th Reynolds ideal of \( A \) is expressed as follows:
\[
\text{Rey}_n(A) = \{ z \in Z(A) \mid rz = 0 \text{ for all } r \in J^n \}. \tag{6.9}
\]
If \( \mathcal{M} = A \text{-mod} \), then \( \text{Rex}(\mathcal{M}) \) can be identified with \( A \text{-mod}\text{-}A \). Under this identification, \( \text{id}_\mathcal{M} \) and \( \text{rad}^n \) correspond to the \( A \)-bimodule \( A \) and its submodule \( J^n \), respectively. By (6.9), it is easy to check that the isomorphism \( Z(A) \cong \text{End}(\text{id}_\mathcal{M}) \) restricts to an isomorphism \( \text{Rey}_n(A) \cong \text{Rey}_n(A \text{-mod}) \) for each \( n \).

We consider the case where \( C \) is a finite tensor category and \( \mathcal{M} \) is an indecomposable exact left \( C \)-module category with action functor \( \rho = \rho_{\mathcal{M}} \). Then we have an adjunction isomorphism
\[
\text{Hom}_C(1, A \mathcal{M}) = \text{Hom}_C(\mathcal{1}, \rho^{ra}(\text{id}_\mathcal{M})) \cong \text{Nat}(\rho(\mathcal{1}), \text{id}_\mathcal{M}) = \text{End}(\text{id}_C). \tag{6.10}
\]
Moreover, since \( \rho^{ra} \) is exact by Theorem 3.4, the object \( J^n_{\mathcal{M}} := \rho^{ra}(\text{rad}^n) \) is a subobject of \( A \mathcal{M} \). The following description of \( \text{Rey}_n(\mathcal{M}) \) may be regarded as a generalization of (6.9).
Lemma 6.2 For an indecomposable exact C-module category $\mathcal{M}$, we define

$$R_n(\mathcal{M}) = \{a \in \text{Hom}_C(1, A_M) \mid m \circ (a \otimes i) = 0\},$$

where $m$ is the multiplication of $A_M$ and $i : J^n_M \to A_M$ is the inclusion morphism. Then (6.10) restricts to an isomorphism between $R_n(\mathcal{M})$ and $\text{Rey}_n(\mathcal{M})$.

Proof We use the monoidal structure of $\rho^{ra}$ described in Lemma 3.8. Let $a : 1 \to A_M$ be a morphism in $\mathcal{C}$, and let $\tilde{a} \in \text{End}(\text{id}_C)$ be the natural transformation corresponding to $a$ via (6.10). By the definition of $\mu^{(0)}$, we have $a = \rho^{ra}(\tilde{a}) \circ \mu^{(0)}$. Let $i_n : \text{rad}^n \to \text{id}_M$ be the inclusion morphism. Since $i = \rho^{ra}(i_n)$, we have

$$m \circ (a \otimes i) = \mu^{(2)}_{id_M, id_M} \circ (\rho^{ra}(\tilde{a}) \otimes \rho^{ra}(i_n)) \circ (\mu^{(0)} \otimes \text{id}_{f^n_M}) = \rho^{ra}(\tilde{a} \circ i_n).$$

Thus, by the faithfulness of $\rho^{ra}$ (Theorem 3.4), the morphism $a$ belongs to $R_n(\mathcal{M})$ if and only if $\tilde{a} \in \text{Rey}_n(\mathcal{M})$. The proof is done.

We recall that an algebra $A$ in $\mathcal{C}$ is said to be Frobenius if there is an isomorphism $\phi : A \to A^*$ of right $A$-modules in $\mathcal{C}$. Given such an isomorphism $\phi$, we define

$$e_\phi = \text{ev}_A \circ (\phi \otimes \text{id}_A) \text{ and } d_\phi = (\text{id}_A \otimes \phi^{-1}) \circ \text{coev}_A.$$ 

Then the triple $(A, e_\phi, d_\phi)$ is a left dual object of $A$. Thus the map

$$\text{Hom}_C(A, 1) \to \text{Hom}_C(1, A), \quad \xi \mapsto (\xi \otimes \text{id}_A) \circ d_\phi$$ (6.11)

is an isomorphism of vector spaces with inverse

$$\text{Hom}_C(1, A) \to \text{Hom}_C(A, 1), \quad a \mapsto e_\phi \circ (a \otimes \text{id}_A).$$ (6.12)

The $A$-linearity of $\phi$ imply

$$e_\phi \circ (m \otimes \text{id}_A) = e_\phi \circ (\text{id}_A \otimes m),$$ (6.13)

$$(\text{id}_A \otimes m) \circ (d_\phi \otimes \text{id}_A) = (m \otimes \text{id}_A) \circ (\text{id}_A \otimes d_\phi),$$ (6.14)

where $m$ is the multiplication of $A$. The following lemma may be well-known:

Lemma 6.3 Let $J$ be an ideal of $A$ with inclusion morphism $i : J \to A$. Then the isomorphisms (6.11) and (6.12) restricts to an isomorphism

$$\text{Hom}_C(A/J, 1) \cong \{a \in \text{Hom}_C(1, A) \mid m \circ (a \otimes i) = 0\}.$$ 

Proof Let $\xi : A \to 1$ be a morphism in $\mathcal{C}$, and let $a : 1 \to A$ be the morphism corresponding to $\xi$ by (6.11) and (6.12). We first suppose that $\xi$ belongs to $\text{Hom}_C(A/J, 1)$, that is, $\xi \circ i = 0$. Then we compute

$$m \circ (a \otimes i) = (\xi \otimes \text{id}_A) \circ (\text{id}_A \otimes m) \circ (d_\phi \otimes \text{id}_A) \circ i$$

$$= (\xi \otimes \text{id}_A) \circ (m \otimes \text{id}_A) \circ (\text{id}_A \otimes d_\phi) \circ i$$

$$= ((\xi \circ m \circ (i \otimes \text{id}_A)) \otimes \text{id}_A) \circ (\text{id}_A \otimes d_\phi)$$

by (6.14). Since $J$ is an ideal of $A$, the image of $m \circ (i \otimes \text{id}_A)$ is contained in $J$. Thus we have $\xi \circ m \circ (i \otimes \text{id}_A) = 0$. Therefore $m \circ (a \otimes i) = 0$. If, conversely, this equation holds, then we have
\[ \xi \circ i = e_\phi \circ (i \otimes a) = e_\phi \circ (m \otimes \text{id}_A) \circ (u \otimes i \otimes a) = 0 \]

by (6.13), where \( u : 1 \to A \) is the unit of \( A \). Thus \( \xi \in \text{Hom}_\mathcal{C}(A/J, 1) \). The proof is done. \( \square \)

Now we have the following representation-theoretic description of \( \text{CF}_n \).

**Theorem 6.4** Let \( M \) be an indecomposable exact \( \mathcal{C} \)-module category. If \( A_M \) is a Frobenius algebra, then the isomorphism

\[
\text{CF}(M) = \text{Hom}_\mathcal{C}(A_M, 1) \overset{(6.11)}{\cong} \text{Hom}_\mathcal{C}(1, A_M) \overset{(6.10)}{\cong} \text{End}(\text{id}_M) \quad (6.15)
\]

restricts to isomorphisms

\[
\text{CF}_n(M) \cong \text{Rey}_n(M) \quad (n = 1, 2, 3, \ldots).
\]

**Proof** The subobject \( \rho^{na}(\text{rad}^n) \) is an ideal of \( A_M = \rho^{na}(\text{id}_M) \). The proof is done by applying the above two lemmas to this ideal. \( \square \)

A finite tensor category \( D \) is said to be unimodular [17] if the projective cover of the unit object \( 1 \in D \) is also an injective hull of \( 1 \). Following [45], a finite tensor category \( D \) is unimodular if and only if the algebra \( R(1) \in \mathcal{Z}(D) \) is Frobenius, where \( R : D \to \mathcal{Z}(D) \) is a right adjoint of the forgetful functor.

Let \( M \) be an indecomposable exact left \( \mathcal{C} \)-module category. Then \( D:=\mathcal{C}^n_M \) is a finite tensor category. By Theorem 3.14 and the above-mentioned fact, the algebra \( A_M \in \mathcal{Z}(\mathcal{C}) \) is Frobenius if and only if \( D \) is unimodular. Thus the algebra \( A_M \in \mathcal{C} \) is Frobenius if \( D \) is unimodular. By the above theorem, we have:

**Corollary 6.5** Let \( M \) be an indecomposable exact \( \mathcal{C} \)-module category. If \( \mathcal{C}^n_M \) is unimodular, then we have \( \text{CF}_n(M) \cong \text{Rey}_n(M) \).

In particular, if \( \mathcal{C} \) is unimodular, then \( \text{CF}_n(\mathcal{C}) \cong \text{Rey}_n(\mathcal{C}) \).

### 6.4 Symmetric linear forms on an algebra

For a finite-dimensional algebra \( A \) with Jacobson radical \( J \), we set

\[
\text{SLF}(A) = \{ f \in A^* \mid f(ab) = f(ba) \text{ for all } a, b \in A \},
\]

\[
\text{SLF}_n(A) = \{ f \in \text{SLF}(A) \mid f(J^n) = 0 \} \quad (n \in \mathbb{Z}_+)\).
\]

If \( G \) is a finite group, then \( \text{SLF}(kG) \) is the space of class functions on \( G \). Thus, for a finite left \( \mathcal{C} \)-module category \( \mathcal{M} \) such that \( \mathcal{M} \cong A \text{-mod} \) (as mere \( k \)-linear categories), it is natural to ask how \( \text{CF}(\mathcal{M}) \) relates to \( \text{SLF}(A) \). To consider this problem, we first introduce the following categorical definition of the space of symmetric linear forms:

**Definition 6.6** For a finite abelian category \( \mathcal{M} \) and \( n \in \mathbb{Z}_+ \), we set

\[
\text{SLF}(\mathcal{M}) := \text{Nat}(\text{id}_\mathcal{M}, N_{\mathcal{M}}) \quad \text{and} \quad \text{SLF}_n(\mathcal{M}) = \{ f \in \text{SLF}(\mathcal{M}) \mid f \circ i_n = 0 \},
\]

where \( i_n : \text{rad}^n \to \text{id}_\mathcal{M} \) is the inclusion morphism.

\( \square \) Springer
If $\mathcal{M}$ is a finite abelian category such that $\mathcal{M} \cong A\text{-mod}$, then $\text{Rex}(\mathcal{M})$ can be identified with $A\text{-mod}$-$A$. Under this identification, $\text{id}_A$ and $N_A$ correspond to the $A$-bimodules $A$ and $A^*$, respectively. Thus we have
\[
\text{SLF}(\mathcal{M}) \cong \text{Hom}_{A\text{-mod}}(A, A^*) \cong \text{SLF}(A),
\]
where the second isomorphism is given by $f \mapsto f(1)$. If we identify $\text{SLF}(\mathcal{M})$ with $\text{SLF}(A)$ by this isomorphism, then $\text{SLF}_n(\mathcal{M})$ is identified with $\text{SLF}_n(A)$.

**Remark 6.7** Let $\mathcal{M}$ be a finite abelian category. We suppose that $\mathcal{M}$ is symmetric Frobenius and choose an isomorphism $\lambda : \text{id}_\mathcal{M} \to N_\mathcal{M}$. Then the map
\[
\text{End}(\text{id}_\mathcal{M}) \to \text{SLF}(\mathcal{M}), \quad \ z \mapsto \lambda \circ z
\]
is an isomorphism. By Definitions 6.1 and 6.6, we also have isomorphisms
\[
\text{Rey}_n(\mathcal{M}) \to \text{SLF}_n(\mathcal{M}), \quad \ z \mapsto \lambda \circ z \quad (n \in \mathbb{Z}_+).
\]
In ring-theoretic terms, this means: Let $A$ be a symmetric Frobenius algebra, and let $\lambda : A \to A^*$ be an isomorphism of $A$-bimodules. For each $n \in \mathbb{Z}_+$, the isomorphism $\lambda$ restricts to an isomorphism between $\text{Rey}_n(A)$ and $\text{SLF}_n(A)$.

Now we consider the case where $\mathcal{M}$ is an exact module category over a finite tensor category $C$. Although $\text{CF}(\mathcal{M})$ is an analogue of the space of class functions, it is not isomorphic to $\text{SLF}(\mathcal{M})$ in general. To see when they are isomorphic, we provide the following lemma:

**Lemma 6.8** There is a natural isomorphism
\[
\text{Hom}_C(\rho^{ra}(S_\mathcal{M} \circ F), X^{**}) \cong \text{Nat}(F, X \otimes N_\mathcal{M}) \quad (F \in \text{Rex}(\mathcal{M}), V \in C).
\]

**Proof** Let $D$ be the distinguished invertible object of $C$ introduced in [15]. Then there are natural isomorphisms
\[
N_C(X) \cong D^* \otimes X^{**} \quad \text{and} \quad N_{\mathcal{M}}(M) \cong D^* \otimes S_\mathcal{M}(M)
\]
for $X \in C$ and $M \in \mathcal{M}$ [19]. Since $S_\mathcal{M} : \mathcal{M} \to (-)^{**} \mathcal{M}$ is a $\mathcal{C}$-module functor, and since $D$ is an invertible object, we have natural isomorphisms
\[
(N^{-1}_{\mathcal{M}} \circ S_\mathcal{M})(M) \cong S^{-1}_\mathcal{M}(D \otimes S_\mathcal{M}(M)) \cong D^{**} \otimes S^{-1}_\mathcal{M}S_\mathcal{M}(M) \cong D \otimes M
\]
for $M \in \mathcal{M}$. By using these isomorphisms and basic results on the Nakayama functor recalled in Sect. 2.5, we have natural isomorphisms
\[
\text{Hom}_C(\rho^{ra}(S_\mathcal{M} \circ F), X^{**}) \cong \text{Hom}_C(\rho^{la}(N^{-1}_{\mathcal{M}} \circ S_\mathcal{M} \circ F \circ N^{-1}_{\mathcal{M}}), N^{-1}_C(X^{**})) \cong \text{Nat}(N^{-1}_{\mathcal{M}} \circ S_\mathcal{M} \circ F \circ N^{-1}_{\mathcal{M}}, N^{-1}_C(X^{**}) \otimes \text{id}_\mathcal{M}) \cong \text{Nat}(N^{-1}_{\mathcal{M}} \circ S_\mathcal{M} \circ F, N^{-1}_C(X^{**}) \otimes N_\mathcal{M}) \cong \text{Nat}(D \otimes F, D \otimes X \otimes N_\mathcal{M}) \cong \text{Nat}(F, X \otimes N_\mathcal{M})
\]
for $F \in \text{Rex}(\mathcal{M})$ and $X \in C$. The proof is done. \hfill \Box

The following theorem is an immediate consequence of the above lemma.
Theorem 6.9  If $\mathcal{M}$ is an exact $C$-module category whose relative Serre functor is isomorphic to the identity functor, then there is a natural isomorphism
\[ \text{Hom}_C(p^\mathcal{M}(F), X^{**}) \cong \text{Nat}(F, X \otimes N_\mathcal{M}) \]
for $F \in \text{Rex}(\mathcal{M})$ and $X \in \mathcal{C}$. In particular, we have an isomorphism
\[ \text{CF}(\mathcal{M}) \cong \text{SLF}(\mathcal{M}), \]
which restricts to isomorphisms
\[ \text{CF}_n(\mathcal{M}) \cong \text{SLF}_n(\mathcal{M}) \quad (n \in \mathbb{Z}_+). \]

6.5 Dimension of $\text{CF}_1$

For a finite abelian category $\mathcal{A}$, we denote by $\text{Irr}(\mathcal{A})$ the set of isomorphism classes of simple objects of $\mathcal{M}$. Let $\mathcal{C}$ be a finite tensor category, and let $\mathcal{M}$ be an exact $\mathcal{C}$-module category. Then, by the proof of Theorem 5.11, we have isomorphisms
\[ \text{CF}_1(\mathcal{M}) \cong \bigoplus_{L \in \text{Irr}(\mathcal{M})} \text{Hom}_\mathcal{C}(\text{Hom}(L, L), 1) \cong \bigoplus_{L \in \text{Irr}(\mathcal{M})} \text{Hom}_\mathcal{C}(L, S_\mathcal{M}(L)). \quad (6.17) \]
Thus, by Schur’s lemma, we have
\[ \dim_k \text{CF}_1(\mathcal{M}) = \#{\{L \in \text{Irr}(\mathcal{C}) \mid S_\mathcal{M}(L) \cong L\}}. \]

We suppose, moreover, that $\mathcal{C}$ is a pivotal finite tensor category and $\mathcal{M}$ is a pivotal $\mathcal{C}$-module category with pivotal structure $j'^\ast$. Again by the proof of Theorem 5.11, the internal character of $L \in \text{Irr}(\mathcal{M})$ corresponds to $j'^{\ast}_L$ via (6.17). Thus the set $\{\text{ch}(L) \mid L \in \text{Irr}(\mathcal{M})\}$ of ‘irreducible characters’ is a basis of $\text{CF}_1(\mathcal{M})$.

6.6 Dimension of $\text{CF}_2$

As we have seen in the above, the dimension of $\text{CF}_1$ is expressed in representation-theoretic terms. It is interesting to give such an expression for the dimension of $\text{CF}_n$ for $n \geq 2$. Here we give the following result:

Theorem 6.10  Let $\mathcal{C}$ be a finite tensor category. For an exact $\mathcal{C}$-module category $\mathcal{M}$ such that $S_\mathcal{M} \cong \text{id}_\mathcal{M}$, there is an isomorphism
\[ \text{CF}_2(\mathcal{M}) = \text{CF}_1(\mathcal{M}) \oplus \bigoplus_{L \in \text{Irr}(\mathcal{M})} \text{Ext}_{\mathcal{M}}^1(L, L). \]
To prove Theorem 6.10, we recall the following expression of $\text{Ext}^1$: Let $A$ be a finite-dimensional algebra. Given $X \in A\text{-mod}$, we denote by $g_X : A \to \text{End}_k(X)$ the algebra map induced by the action of $A$ on $X$. For $V, W \in A\text{-mod}$, the vector space $\text{Ext}_A^1(V, W)$ is identified with the set of equivalence classes of short exact sequences of the form $0 \to W \to X \to V \to 0$ in $A\text{-mod}$. If $X \in A\text{-mod}$ fits into such an exact sequence, then we may assume that $X = V \oplus W$ as a vector space and the algebra map $g_X$ is given by
\[ g_X(a) = \begin{pmatrix} g_V(a) & 0 \\ \xi(a) & g_W(a) \end{pmatrix} \in \text{End}_k(X) \quad (a \in A) \]
for some $\xi \in \text{Hom}_k(A, \text{Hom}_k(V, W))$. Since $g_X$ is an algebra map, we have
\[ \xi(ab) = \xi(a) \circ g_V(b) + g_W(a) \circ \xi(b) \quad (a, b \in A). \]  
\text{(6.18)}

We define $\partial : \text{Hom}_k(V, W) \to \text{Hom}_k(A, \text{Hom}_k(V, W))$ by
\[ \partial(f)(a) = f \circ g_V(a) - g_W(a) \circ f \quad (f \in \text{Hom}_k(V, W), a \in A). \]

For two linear maps $\xi_i : A \to \text{Hom}_k(V, W)$ ($i = 1, 2$) satisfying (6.18), the corresponding short exact sequences are equivalent if and only if $\xi_1 - \xi_2 \in \text{Im}(\partial)$. Thus the vector space $\text{Ext}_A^1(V, W)$ is identified with
\[ E_A^1(V, W):=\{\xi \in \text{Hom}_k(A, \text{Hom}_k(V, W)) \text{ satisfying (6.18)}/ \text{Im}(\partial). \]  
\text{(6.19)}

Theorem 6.10 is in fact an immediate consequence of Theorem 6.9 and the following theorem:

**Theorem 6.11** For $M \in A\text{-mod}$, we define
\[ \text{Tr}_{A,M}^* : \text{Ext}_A^1(M, M) = E_A^1(M, M) \to \text{SLF}(A), \quad \xi \mapsto \text{Tr} \circ \xi. \]

The map $\text{Tr}_{A,L}^*$ is injective for all $L \in \text{Irr}(A)$. Moreover, we have
\[ \text{SLF}_2(A) = \text{SLF}_1(A) \oplus \bigoplus_{L \in \text{Irr}(A)} \text{Im}(\text{Tr}_{A,L}^*). \]

**Remark 6.12** Our construction of the map $\text{Tr}_{A,M}^*$ is inspired from the construction of pseudo-trace functions introduced by Miyamoto [30] and further studied in [2–4] in relation with conformal field theory and vertex operator algebras.

**Remark 6.13** Theorem 6.11 is inspired by the following Okuyama’s result: For a symmetric Frobenius algebra $A$, Okuyama [34] showed
\[ \dim_k \text{Rey}_2(A) = |\text{Irr}(A)| + \sum_{L \in \text{Irr}(A)} \dim_k \text{Ext}_A^1(L, L) \]
(see also Koshitani’s review [24, Section 2]). This formula follows from the above theorem and Remark 6.7. We note that Theorem 6.11 does not require $A$ to be a symmetric Frobenius algebra.

We give a proof of Theorem 6.11. Let $A$ be a finite-dimensional algebra, and write $\text{Irr}(A) = \{S_1, \ldots, S_m\}$. For each $i = 1, \ldots, m$, we fix a primitive idempotent $e_i \in A$ such that $Ae_i$ is a projective cover of $S_i$. Set $e = e_1 + \cdots + e_m$. Then $A^b:=eAe$ is a basic algebra and the functor
\[ A\text{-mod} \to A^b\text{-mod}, \quad X \mapsto eX \]  
\text{(6.20)}

is an equivalence. The following lemma is well-known [33], but we give a proof from the viewpoint of Definition 6.6.

**Lemma 6.14** The following map is bijective:
\[ \text{SLF}(A) \to \text{SLF}(A^b), \quad f \mapsto f|_{A^b} \]  
\text{(6.21)}

**Proof** The equivalence (6.20) induces an equivalence $A\text{-mod}-A \cong A^b\text{-mod}-A^b$ sending $M \in A\text{-mod}-A$ to $eMe$. By (6.16) and this equivalence, we have
\[ \text{SLF}(A) \cong \text{Hom}_{A\text{-mod}-A}(A, A^b) \cong \text{Hom}_{A^b\text{-mod}-A^b}(A^b, (A^b)^*) \cong \text{SLF}(A^b). \]

The composition yields the map (6.21). \qed
The equivalence (6.20) also induces an isomorphism
\[ E^1_A(V, W) = \operatorname{Ext}^1_A(V, W) \cong \operatorname{Ext}^1_{Ab}(eV, eW) = E^1_{Ab}(eV, eW) \] (6.22)
for \( V, W \in \mathcal{M} \), which sends \( \xi \in E^1_A(V, W) \) to
\[ \xi^b : A^b \to \operatorname{Hom}_k(eV, eW), \quad eae \mapsto \left( ev \mapsto e\xi(\lambda)(e)e \right) \] \( (a \in A, v \in V) \).

Lemma 6.15 For \( V \in A\text{-mod} \), the following diagram commutes:
\[
\begin{array}{ccc}
E^1_A(V, V) & \xrightarrow{\operatorname{Tr}_{A,V}} & \operatorname{SLF}(A) \\
\downarrow\text{(6.22)} & & \\
E^1_{Ab}(eV, eV) & \xrightarrow{\operatorname{Tr}^*_A(eV)} & \operatorname{SLF}(A^b)
\end{array}
\]

Proof Let \( g : A \to \operatorname{End}_k(V) \) be the algebra map defined by the action of \( A \). For \( \xi \in E^1_A(V, V) \) and \( a \in A \), we have \( \xi(e) = \xi(e^2) = \xi(e)g(e) + g(e)e\xi(e) \) by (6.18). By multiplying \( g(e) \) to both sides, we obtain \( g(e)\xi(e)g(e) = 0 \). Again by (6.18),
\[
\operatorname{Tr}(\xi(\lambda)(e)) = \operatorname{Tr}(\xi(e \cdot \lambda(e \cdot e))) = \operatorname{Tr}(\xi(e)g(\lambda)(e))g(e) + g(e)\xi(e)g(e) + g(e)g(e)\xi(e) = \operatorname{Tr}(\xi(\lambda)(e)g(e)) + \operatorname{Tr}(g(e)\xi(\lambda)(e)g(e)) + \operatorname{Tr}(g(e)g(e)\xi(e)).
\]
The first term is zero, since \( \operatorname{Tr}(\xi(e)g(\lambda)(e))g(e) = 0 \) and \( \operatorname{Tr}(\xi(e)g(\lambda)(e))g(e) = \operatorname{Tr}(0) = 0 \). By a similar computation, the third term is also zero. Thus we have
\[
\operatorname{Tr}(\xi(\lambda)(e)) = \operatorname{Tr}(g(e)\xi(e)g(e)) = \operatorname{Tr}(\xi^b(\lambda)(e)).
\]
This means that the diagram in question commutes. \( \square \)

Proof of Theorem 6.11 Let \( A \) be a finite-dimensional algebra, and let \( J \) be the Jacobson radical of \( A \). Since the \( n \)-th power of the Jacobson radical of \( A^b \) is \( eJ^n e \), we see that (6.21) restricts to isomorphisms
\[
\operatorname{SLF}_n(A) \to \operatorname{SLF}_n(A^b), \quad f \mapsto f|_{A^b} \quad (n \in \mathbb{Z}_+)
\] (6.23)
Thus, by Lemma 6.15, it is sufficient to consider the case where \( A \) is basic.

We assume that \( A \) is basic. Then \( C := A^* \) is a pointed coalgebra. Let \( \Delta \) and \( \epsilon \) denote the comultiplication and the counit of \( C \), respectively. We define:
\[
G = \{ c \in C \mid \Delta(c) = c \otimes c \text{ and } \epsilon(c) = 1 \},
\]
\[
P_{g,h} = \{ x \in C \mid \Delta(x) = x \otimes g + h \otimes x \} \quad (g, h \in G).
\]
Then the set \( \operatorname{Irr}(A) \) is identified with the set \( G \). Let \( g, h \in G = \operatorname{Irr}(A) \). By (6.19), the vector space \( \operatorname{Ext}_A^1(g, h) \) is identified with \( P_{g,h}/k(g - h) \), and the map \( \operatorname{Tr}^*_A \) is just the inclusion map \( P_{g,g} \to C \). Thus, to prove this theorem, it is enough to show the following equation:
\[
\operatorname{SLF}_2(A) = C_0 \oplus \bigoplus_{g \in G} P_{g,g}.
\] (6.24)
Let \( \{ C_n \}_{n \geq 0} \) be the coradical filtration of \( C \). By [32, Proposition 5.2.9], we have \( C_n = (A/J^{n+1})^* \) for all \( n \geq 0 \). Thus we have
\[
\operatorname{SLF}_n(A) = C_{n-1} \cap \operatorname{SLF}(A) \quad (n \in \mathbb{Z}_+).
\] (6.25)
For each $g, h \in G$, we choose a subspace $P'_{g,h}$ of $P_{g,h}$ such that $P_{g,h} = P'_{g,h} \oplus k(g - h)$. The Taft-Wilson theorem [32, Theorem 5.4.1] states

$$C_1 = C_0 \oplus \bigoplus_{g,h \in G} P'_{g,h}. \quad (6.26)$$

Thus $C_1 \otimes C_1$ is decomposed as follows:

$$C_1 \otimes C_1 = (C_0 \otimes C_0) \oplus \bigoplus_{f,g,h \in G} \left( (f \otimes P'_{g,h}) \oplus (P'_{g,h} \otimes f) \right) \oplus \bigoplus_{e,f,g,h \in G} P'_{e,f} \otimes P'_{g,h}. \quad (6.27)$$

Now let $x \in \text{SLF}_2(A)$. By (6.25) and (6.26), we have $x = x_0 + \sum_{g,h \in G} x_{g,h}$ for some $x_0 \in C_0$ and $x_{g,h} \in P'_{g,h}$. Let $\pi_{f,g,h}$ be the projection from $C_1 \otimes C_1$ to $f \otimes P'_{g,h}$ along the direct sum decomposition (6.27). Then we have

$$\pi_{f,g,h} \Delta(x) = \sum_{g,h \in G} \pi_{f,g,h}(x_{g,h} \otimes g + h \otimes x_{g,h}) = \delta_{f,h} h \otimes x_{g,h}.$$

Let $\Delta^\text{cop} : C \to C \otimes C$ be the opposite comultiplication of the coalgebra $C$. Then, since $x \in \text{SLF}(A) = \{ c \in C \mid \Delta(x) = \Delta^\text{cop}(x) \}$, we also have

$$\pi_{f,g,h} \Delta(x) = \pi_{f,g,h} \Delta^\text{cop}(x) = \sum_{g,h \in G} \pi_{f,g,h}(g \otimes x_{g,h} + x_{g,h} \otimes h) = \delta_{f,g} g \otimes x_{g,h}.$$

This implies that $x_{g,h} = 0$ unless $g = h$. Hence,

$$\text{SLF}_2(A) \subset C_0 \oplus \bigoplus_{g \in G} P'_{g,g} = C_0 \oplus \bigoplus_{g \in G} P_{g,g}.$$

Thus the left-hand side of (6.24) is contained in the right-hand side. It is easy to show the converse inclusion. The proof is done.

### 6.7 Examples

We have no general results on $\text{CF}_n$ for $n \geq 3$. Here we give some computational results on $\text{CF}_n(C)$ for the case where $C$ is the category of modules over a finite-dimensional Hopf algebra.

Let $H$ be a finite-dimensional Hopf algebra with comultiplication $\Delta$, counit $\varepsilon$ and antipode $S$. We use the Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ to express the comultiplication of $h \in H$. Set $C = H\text{-mod}$. If we identify $\text{Rex}(C)$ with $H\text{-mod}\text{-}H$, then the action functor $\rho : C \to \text{Rex}(C)$ is given by $\rho(X) = \otimes_k H$, where the left and the right action of $H$ on $\rho(X)$ are given by

$$h \cdot (x \otimes h') = h_{(1)} x \otimes h_{(2)} h' \quad \text{and} \quad (x \otimes h') \cdot h = x \otimes h' h,$$

respectively, for $x \in X$ and $h, h' \in H$. A right adjoint of $\rho$ is given as follows: As a vector space, $\rho^\text{ra}(M) = M$. The action of $H$ on $\rho^\text{ra}(M)$ is given by

$$h \cdot m = h_{(1)} m S(h_{(2)}) \quad (h \in H, m \in M).$$

Indeed, one can check that the map

$$\text{Hom}_H(\rho(X), M) \to \text{Hom}_{H\text{-mod}\text{-}H}(X, \rho^\text{ra}(M)), \quad f \mapsto f(1_H \otimes -)$$
is a natural isomorphism for $X \in H\text{-mod}$ and $M \in H\text{-mod-}H$.

In particular, as remarked in [44], the algebra $A_C = \rho^a(\text{id}_C) \in C$ is the adjoint representation of $H$. Thus the space of class functions is given by

$$CF(H\text{-mod}) = \{ f \in H^* \mid f(h_1xS(h_2)) = \varepsilon(h)f(x) \text{ for all } h, x \in H \}$$

$$= \{ f \in H^* \mid f(ab) = f(bs^2(a)) \text{ for all } a, b \in H \}.$$ 

By the above description of $\rho^a$, we also have

$$CF_n(H\text{-mod}) = \{ f \in CF(H\text{-mod}) \mid f(J^n) = 0 \}$$

for all positive integer $n$, where $J$ is the Jacobson radical of $H$. As these expressions show, if $S^2$ is inner, then there are isomorphisms

$$CF(H\text{-mod}) \cong SLF(H) \quad \text{and} \quad CF_n(H\text{-mod}) \cong SLF_n(H).$$

**Example 6.16** Suppose that the base field $k$ is of characteristic $p > 0$. We consider the cyclic group $G = \langle g \mid g^p = 1 \rangle$ of order $p$. It is easy to see that the Jacobson radical of $kG$ is generated by $x := g - 1$. We note that the set $\{1, x, \ldots, x^{p-1}\}$ is a basis of $kG$. Since the square of the antipode of $kG$ is the identity, we have

$$CF_n(kG\text{-mod}) = SLF_n(kG) = \{ f \in (kG)^* \mid f(x^r) = 0 \text{ for all } r \geq n \}.$$ 

Hence the dimension of $CF_n := CF_n(kG\text{-mod})$ is given by

$$\dim_k CF_n = n \quad (n = 1, 2, \ldots, p) \quad \text{and} \quad \dim_k CF_n = p \quad (n > p).$$

A basis of $CF := CF(kG\text{-mod})$ can be constructed in the following roundabout but interesting way: There is a matrix representation

$$\rho : kG \to \text{Mat}_p(k), \quad g \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}.$$ 

Let $\rho_{ij} \in (kG)^*$ be the $(i, j)$-entry of $\rho$. Then the set $\{\rho_{n1}\}_{n=1,\ldots,p}$ is a basis of $CF$ such that $\rho_{n1} \in CF_n$ and $\rho_{n1} \notin CF_{n-1}$ for all $n = 1, \ldots, p$ (with convention $CF_0 = \{0\}$).

**Example 6.17** Suppose that the base field $k$ is of characteristic zero. Let $p \geq 2$ be an integer, and let $q \in k$ be a primitive $2p$-th root of unity. The algebra $U_q := U_q(sl_2)$ is generated by $E, F$ and $K$ subject to the relations

$$E^p = F^p = 0, \quad K^{2p} = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$ 

The algebra $U_q$ has the Hopf algebra structure determined by

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K.$$ 

The antipode is given by $S(E) = -EK^{-1}, S(F) = -KF$ and $S(K) = K^{-1}$ on the generators. Thus $S^2$ is the inner automorphism implemented by $K$. In view of (6.28), we consider $SLF_n(U_q)$ instead of $CF_n(U_q\text{-mod})$.

An explicit basis of $SLF(U_q)$ is given by Arike [3]. We recall his construction: For $\alpha \in \{+, -\}$ and $s \in \{1, \ldots, p\}$, there is an $s$-dimensional simple left $U_q$-module $\lambda_\alpha^s$ (see [3, Subsection 3.4 for notations). The module $\lambda_\alpha^s$ is projective. For $s < p$, the module $\lambda_\alpha^s$ is
not projective. Let $P_{s}^{\alpha}$ be the projective cover of $A_{s}^{\alpha}$. Arike [3, Subsection 5.1] showed that $P_{s}^{\alpha}$ has a matrix presentation of the form

$$
\rho_{s}^{\alpha} : U_{q} \rightarrow \text{Mat}_{2p}(k), \quad \rho_{s}^{\alpha}(x) = \begin{pmatrix}
    g^{\alpha}_{s}(x) & 0 & 0 & 0 \\
    a^{\alpha}_{s}(x) & g^{-\alpha}_{p-s}(x) & 0 & 0 \\
    b^{\alpha}_{s}(x) & 0 & g^{-\alpha}_{p-s}(x) & 0 \\
    h^{\alpha}_{s}(x) & a^{-\alpha}_{p-s}(x) & b^{-\alpha}_{p-s}(x) & g^{\alpha}_{s}(x)
\end{pmatrix},
$$

where $g^{\alpha}_{s} : U_{q} \rightarrow \text{Mat}_{s}(k)$ is a matrix presentation of $A_{s}^{\alpha}$ and $a^{\alpha}_{s}$, $b^{\alpha}_{s}$ and $h^{\alpha}_{s}$ are certain matrix-valued linear functions on $U_{q}$ (given by $a^{\pm}_{s} = A_{p-s,s}, a^{-}_{s} = C_{s,p-s}, b^{+}_{s} = B_{p-s,s}, b^{-}_{s} = D_{s,p-s} h^{+}_{s} = H_{s}$ and $h^{-}_{s} = \tilde{H}_{s}$ with Arike’s original notation). Now we define linear forms $\chi^{\alpha}_{s}(\alpha \in \{+,-\}, s = 1, \ldots, p)$ and $\varphi_{s'}(s' = 1, \ldots, p - 1)$ on $U_{q}$ by

$$\chi^{\alpha}_{s}(x) = \text{Tr}(g^{\alpha}_{s}(x)) \quad \text{and} \quad \varphi_{s'}(x) = \text{Tr}(h^{+}_{s}(x)) + \text{Tr}(h^{-}_{p-s'}(x)) \quad (x \in U_{q}).$$

Then the following set is a basis of $SLF(U_{q})$ [3, Theorem 5.5]:

$$\{ \chi^{+}_{s}, \chi^{-}_{s} \mid s = 1, \ldots, p \} \cup \{ \varphi_{s} \mid s = 1, \ldots, p - 1 \}.$$

Arike’s basis respects the filtration of $SLF(U_{q})$. More precisely, we have:

$$\begin{align*}
    SLF_{1}(U_{q}) &= \text{span}\{ \chi^{+}_{s}, \chi^{-}_{s} \mid 1 \leq s \leq p \}, \quad (6.29) \\
    SLF_{2}(U_{q}) &= SLF_{1}(U_{q}), \quad (6.30) \\
    SLF_{3}(U_{q}) &= SLF_{2}(U_{q}) \oplus \text{span}\{ \varphi_{s} \mid 1 \leq s \leq p - 1 \}. \quad (6.31)
\end{align*}$$

Indeed, (6.29) follows from the fact that $SLF_{1}(U_{q})$ is spanned by the characters of simple modules. Equation (6.30) follows from Theorem 6.11 and the fact that the self-extension vanishes for every simple $U_{q}$-module [48, p.379]. To show (6.31), we note $Lw(U_{q}) = 3$ [48, p.367]. Thus we have $SLF_{n}(U_{q}) = SLF(U_{q})$ for $n \geq 3$. This implies (6.31).

### 7 Hochschild (co)homology

#### 7.1 Hochschild (co)homology of a finite abelian category

For an algebra $A$, the Hochschild homology and the Hochschild cohomology of $A$ are defined by

$$HH_{\bullet}(A) = \text{Tor}_{\bullet}^{A^{e}}(A, A) \quad \text{and} \quad HH^{\bullet}(A) = \text{Ext}_{A^{e}}^{\bullet}(A, A),$$

respectively, where $A^{e} = A \otimes k A^{\text{op}}$. We note that the 0-th Hochschild cohomology $HH^{0}(A) = \text{Hom}_{A^{e}}(A, A)$ is the center of $A$. It has been known that the modular group $SL_{2}(\mathbb{Z})$ acts projectively on the center of a ribbon factorizable Hopf algebra [49]. Recently, Lentner, Mierach, Schweigert and Sommerhäuser [26] showed that $SL_{2}(\mathbb{Z})$ also acts projectively on the higher Hochschild cohomology of such a Hopf algebra.

A modular tensor category (in the sense of Kerler–Lyubashenko [23]) is a category-theoretical counterpart of a ribbon factorizable Hopf algebra. The aim of this section is to extend the construction of [26] to modular tensor categories. To accomplish this, we first need to discuss what the Hochschild cohomology of a finite abelian category is. Our proposal is the following definition:
Definition 7.1 For a finite abelian category $\mathcal{M}$, we define the Hochschild cohomology $\text{HH}^*(\mathcal{M})$ of $\mathcal{M}$ by $\text{HH}^*(\mathcal{M}) = \text{Ext}^*_\text{Rex}(\mathcal{M})(\text{id}_\mathcal{M}, \text{id}_\mathcal{M})$.

If $\mathcal{M} \approx A\text{-mod}$ for some finite-dimensional algebra $A$, then $\text{Rex}(\mathcal{M})$ is equivalent to $A\text{-mod}-A$ and the identity functor $\text{id}_\mathcal{M} \in \text{Rex}(\mathcal{M})$ corresponds to $A$ via the equivalence. Since a category equivalence preserves Ext, we have

$$\text{HH}^*(\mathcal{M}) = \text{Ext}^*_\text{Rex}(\mathcal{M})(\text{id}_\mathcal{M}, \text{id}_\mathcal{M}) \cong \text{Ext}^*_{A\text{-mod}-A}(A, A) = \text{HH}^*(A),$$

which justifies the definition.

Although it is not directly related to our main purpose of this section, it is also interesting to give a definition of the Hochschild homology of a finite abelian category. Our proposal is:

Definition 7.2 For a finite abelian category $\mathcal{M}$, we define the Hochschild homology $\text{HH}_*(\mathcal{M})$ of $\mathcal{M}$ by $\text{HH}_*(\mathcal{M}) = \text{Ext}^*_\text{Rex}(\mathcal{M})(\text{id}_\mathcal{M}, \mathcal{N}_\mathcal{M})^*$, where $\mathcal{N}_\mathcal{M}$ is the Nakayama functor on $\mathcal{M}$.

This definition is justified as follows: If $M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$ is a projective resolution of $M \in A\text{-mod}-A$, then $\text{Tor}^*_A(A, M)$ is the homology of

$$0 \leftarrow A \otimes_A^e P_0 \leftarrow A \otimes_A^e P_1 \leftarrow \cdots.$$  

(7.1)

By the tensor-hom adjunction, the dual of this chain complex is:

$$A \otimes_A^e P^*_e = \text{Hom}_A(A \otimes_A^e P^*_e, k) \cong \text{Hom}_{A^e}(P^*_e, \text{Hom}_A(A, k)) \cong \text{Hom}_{A^e}(P^*_e, A^*).$$

Thus we have $\text{Tor}^*_A(A, M)^* \cong \text{Ext}^*_{A^e}(M, A^*)$ by taking the cohomology of the dual of (7.1). If $\mathcal{M} = A\text{-mod}$, then $A$ and $A^*$ corresponds to $\text{id}_\mathcal{M}$ and $\mathcal{N}_\mathcal{M}$, respectively, via the equivalence $A\text{-mod}-A \cong \text{Rex}(\mathcal{M})$. Hence we have

$$\text{HH}_*(\mathcal{M}) = \text{Ext}^*_\text{Rex}(\mathcal{M})(\text{id}_\mathcal{M}, \mathcal{N}_\mathcal{M})^* \cong \text{Ext}^*_{A^e}(A, A^*)^* \cong \text{Tor}^*_A(A, A) = \text{HH}_*(A).$$

7.2 Formulas of $\text{HH}^*$ and $\text{HH}_*$ by the adjoint algebra

Let $H$ be a finite-dimensional Hopf algebra, and let $A$ be the adjoint representation of $H$. It is known that there is an isomorphism

$$\text{Ext}^*_H(k, A) \cong \text{HH}^*(H),$$  

(7.2)

where $k$ is the trivial $H$-module. We now generalize this result to exact module categories. Let $\mathcal{C}$ be a finite tensor category, and let $\mathcal{M}$ be a finite left $\mathcal{C}$-module category over $\mathcal{C}$ with action functor $\rho : \mathcal{C} \to \text{Rex}(\mathcal{M})$.

Theorem 7.3 If $\mathcal{M}$ is exact, then there is a natural isomorphism

$$\text{Ext}^*_\mathcal{C}(V, \rho^\text{ra}(F)) \cong \text{Ext}^*_\text{Rex}(\mathcal{M})(\rho(V), F)$$

for $V \in \mathcal{C}$, $F \in \text{Rex}(\mathcal{M})$.

Proof We set $\mathcal{E} = \text{Rex}(\mathcal{M})$ for simplicity. Let $V \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$ be a projective resolution of $V$ in $\mathcal{C}$. By applying $\text{Hom}_\mathcal{C}(-, \rho^\text{ra}(F))$ to this resolution, we have the following commutative diagram:

$$
\begin{array}{cccccccc}
0 & \longrightarrow & \text{Hom}_\mathcal{C}(P_0, \rho^\text{ra}(F)) & \longrightarrow & \text{Hom}_\mathcal{C}(P_1, \rho^\text{ra}(F)) & \longrightarrow & \text{Hom}_\mathcal{C}(P_2, \rho^\text{ra}(F)) & \longrightarrow & \cdots \\
\cong & & \cong & & \cong & & \\
0 & \longrightarrow & \text{Hom}_\mathcal{E}(\rho(P_0), F) & \longrightarrow & \text{Hom}_\mathcal{E}(\rho(P_1), F) & \longrightarrow & \text{Hom}_\mathcal{E}(\rho(P_2), F) & \longrightarrow & \cdots
\end{array}
$$
By Lemmas 3.1 and 3.2, the sequence $0 \leftarrow \rho(V) \leftarrow \rho(P^0) \leftarrow \rho(P^1) \leftarrow \cdots$ is a projective resolution of $\rho(V)$. Now the claim is proved by taking the cohomology of the rows of the above commutative diagram. □

**Theorem 7.4** If $\mathcal{M}$ is an exact $C$-module category and the relative Serre functor of $\mathcal{M}$ is isomorphic to $\text{id}_{\mathcal{M}}$, then there is a natural isomorphism

$$\text{Ext}_C^*(\rho^A(F), V^{**}) \cong \text{Ext}_{\text{Rex}(\mathcal{M})}^*(F, V \otimes N_{\mathcal{M}})$$

for $V \in C$ and $F \in \text{Rex}(\mathcal{M})$.

**Proof** We set $\mathcal{E} = \text{Rex}(\mathcal{M})$ for simplicity. Let $F \leftarrow P^0 \leftarrow P^1 \leftarrow \cdots$ be a projective resolution of $F$ in $\mathcal{E}$. By applying $\text{Hom}_C(\cdot, V \otimes N_{\mathcal{M}})$ to this resolution and using Lemma 6.8, we obtain the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Hom}_\mathcal{E}(P^0, V \otimes N_{\mathcal{M}}) & \rightarrow & \text{Hom}_\mathcal{E}(P^1, V \otimes N_{\mathcal{M}}) & \rightarrow & \text{Hom}_\mathcal{E}(P^2, V \otimes N_{\mathcal{M}}) & \rightarrow & \cdots \\
\text{\cong} & & \text{\cong} & & \text{\cong} & & \text{\cong} & \\
0 & \rightarrow & \text{Hom}_\mathcal{E}(\rho^A(P^0), V^{**}) & \rightarrow & \text{Hom}_\mathcal{E}(\rho^A(P^1), V^{**}) & \rightarrow & \text{Hom}_\mathcal{E}(\rho^A(P^2), V^{**}) & \rightarrow & \cdots
\end{array}
$$

The claim is proved by taking the cohomology of the rows of this commutative diagram. □

Specializing the above theorems, we obtain:

**Corollary 7.5** For an exact left $C$-module category $\mathcal{M}$, we have

$$\text{HH}^*(\mathcal{M}) \cong \text{Ext}^*_C(1, A_M). \quad (7.3)$$

If $S_M \cong \text{id}_M$, then we also have an isomorphism

$$\text{HH}_*(\mathcal{M}) \cong \text{Ext}^*_C(A_M, 1)^* \quad (7.4)$$

We consider the case where $C = H\text{-mod}$ for some finite-dimensional Hopf algebra $H$ and $\mathcal{M} = C$. Let $A$ be the adjoint representation of $H$. Since $A_M \cong A$ in this case, the isomorphism (7.3) specializes to (7.2) in this case. Since the relative Serre functor of $\mathcal{M}$ is the double dual functor, $S_M \cong \text{id}_M$ if and only if the square of the antipode of $H$ is inner. If this is the case, then we have an isomorphism $\text{HH}^*_*(H) \cong \text{Ext}^*_A(A, k)^*$ by (7.4).

### 7.3 Modular group action on the Hochschild cohomology

Let $C$ be a ribbon finite tensor category with braiding $\sigma$ and twist $\theta$. Then the coend $L := \int_{X \in C} X^* \otimes X$ has a natural structure of a Hopf algebra in $C$. We note that the algebra $A := (A_C, m_C, u_C)$ is dual to the coalgebra $L$, and thus $A$ is also a Hopf algebra (see (4.7) for the definition of $m_C$ and $u_C$). By using the universal property, we define $Q : 1 \rightarrow A \otimes A$ to be the unique morphism such that the equation

$$(\pi_C(X) \otimes \pi_C(Y)) \circ Q = (\text{id}_X \otimes \sigma_{X,Y} \circ \sigma_{Y,X}^* \otimes \text{id}_{Y^*}^*) \circ (\text{coev}_X \otimes \text{coev}_Y)$$

holds for all $X, Y \in C$. The morphism $Q$ is dual to the Hopf pairing $\omega : L \otimes L \rightarrow 1$ used in [27–29] to define the modularity of $C$. Thus we say that $C$ is a modular tensor category if there is a morphism $e : A \otimes A \rightarrow 1$ such that $(A, e, Q)$ is a left dual object of $A$ [23, Definition 5.2.7].

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Now we suppose that $\mathcal{C}$ is a modular tensor category. Then the Hopf algebra $A$ has a morphism $\lambda : A \to 1$, unique up to sign, such that
\[
(id_A \otimes \lambda) \circ \Delta = u_C \circ \lambda = (\lambda \otimes id_A) \circ \Delta \quad \text{and} \quad (\lambda \otimes \lambda) \circ Q = id_1,
\]
where $\Delta$ is the comultiplication of $A$. We fix such a morphism $\lambda$ and then define two morphisms $\mathcal{G}, \mathcal{T} : A \to A$ by
\[
\mathcal{G} = (\lambda \otimes id_A) \circ (m_C \otimes id_A) \circ (id_A \otimes Q) \quad \text{and} \quad \mathcal{T} = \int_{x \in C} \theta_X \otimes id_{x^*}.
\]
The morphisms $\mathcal{G}$ and $\mathcal{T}$ are the dual of the morphisms $\mathcal{S}$ and $\mathcal{T}$, respectively, given in [27, Definition 6.3]. Thus $\mathcal{G}$ and $\mathcal{T}$ are invertible and there is an element $c \in k^\times$ such that the following ‘modular relation’ hold:
\[
(\mathcal{G} \mathcal{T})^3 = c \cdot \mathcal{G}^2 \quad \text{and} \quad \mathcal{G}^4 = \theta_A^{-1}.
\]

**Theorem 7.6** With the above notation, we set
\[
\tilde{\mathcal{G}} = Ext^*_C(\mathcal{G}) \quad \text{and} \quad \tilde{\mathcal{T}} = Ext^*_C(\mathcal{T}).
\]
Then we have a well-defined projective representation
\[
SL_2(\mathbb{Z}) \to PGL\left(\text{HH}^*(\mathcal{C})\right), \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \tilde{\mathcal{G}}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \tilde{\mathcal{T}}.
\]

**Proof.** It is enough to show that $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{T}}$ satisfy $(\tilde{\mathcal{G}} \tilde{\mathcal{T}})^3 = \tilde{\mathcal{G}}^2$ and $\tilde{\mathcal{G}}^4 = id$ up to scalar multiple. By the funtorial property of Ext and (7.6), we have
\[
(\tilde{\mathcal{G}} \tilde{\mathcal{T}})^3 = c \cdot \mathcal{G}^2 \quad \text{and} \quad \mathcal{G}^4 = Ext^*_C(1, \theta_A^{-1}) = Ext^*_C(\theta_A^{-1}, A) = id. \quad \square
\]

Let $H$ be a finite-dimensional ribbon Hopf algebra with universal R-matrix $R$ and ribbon element $v$. We consider the case where $\mathcal{C} = H$-$\text{mod}$. Then $A$ is the adjoint representation of $H$. For $X \in H$-$\text{mod}$, the composition
\[
A \otimes X \xrightarrow{\pi_C(X) \otimes id_X} X \otimes X^* \otimes X \xrightarrow{id_X \otimes \text{ev}_X} X
\]
sends $a \otimes x \in A \otimes X$ to $ax$ [44, Subsection 3.7]. From this, we see that $\pi_C(X)$ is given as follows: Let $\{x_i\}$ be a basis of $X$, and let $\{x^i\}$ be the dual basis of $\{x_i\}$. With the Einstein notation, we have
\[
\pi_C(X)(a) = ax_i \otimes x^i \quad (a \in A).
\]
The morphism $\lambda$ is in fact a suitably normalized left integral on $H$. The morphism $Q$ can be regarded as an element of $A \otimes_k A$. For simplicity, we express $R$ and $Q$ as $R = R_1 \otimes R_2$ and $Q = Q_1 \otimes Q_2$, respectively. Then the braiding is given by
\[
\sigma_{X,Y}(x \otimes y) = R_{2}y \otimes R_{1}x \quad (x \in X, \ y \in Y)
\]
for $X, Y \in H$-$\text{mod}$. Let $\{h_i\}$ and $\{h^i\}$ be a basis of $H$ and the dual basis of $H^*$, respectively. Then, by (7.7) and the definition of $Q$, we have
\[
Q_1 h_i \otimes h^i \otimes Q_2 h_i \otimes h^i = (\pi_C(H) \otimes \pi_C(H))(Q)
\]
\[
= h_i \otimes \sigma_{H,H^*}(h^i \otimes h_i) \otimes h^i
\]
\[
= h_i \otimes R'^{i}_{2}R_{1}h_i \otimes R'_{1}R_{2}h_i \otimes h^i
\]
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\[ S(R'_2 R_1) h_i \otimes h^i \otimes R'_1 R_2 h_i \otimes h^i, \]

where \( R' = R'_1 \otimes R'_2 \) is a copy of \( R \). Thus we have \( Q = S(R'_2 R_1) \otimes R'_1 R_2 \). By using the element \( Q \), the morphisms \( \varnothing, \Sigma : A \to A \) are given by

\[ \varnothing(a) = \lambda(aQ_1)Q_2 \quad \text{and} \quad \Sigma(a) = va \quad (a \in A), \]

respectively. Thus our \( \varnothing \) and \( \Sigma \) coincide with those in [25, Theorem 4.4]. If we replace \((H, R, v)\) with \((H^{\text{op}}, R', v)\), then the morphisms \( \varnothing \) and \( \Sigma \) coincide with the morphisms considered in [26].

In [26], the action \( SL_2(\mathbb{Z}) \to \text{PGL}(HH^2(H)) \) is defined as follows: First, they extend \( \varnothing \) and \( \Sigma \) to cochain maps \( \varnothing^\bullet \) and \( \Sigma^\bullet \) of a cochain complex \( C^\bullet_1 \) computing the cohomology \( \text{Ext}^n_{\mathbb{Z}}(k, A) \). They also established an explicit isomorphism between the complex \( C^\bullet_1 \) and the Hochschild complex \( C^\bullet_2 \) computing the Hochschild cohomology of \( H \). The isomorphism \( C^\bullet_1 \cong C^\bullet_2 \) induces (7.2). The projective action of \( SL_2(\mathbb{Z}) \) on \( HH^\bullet(H) \) is then given by \( \varnothing^\bullet \) and \( \Sigma^\bullet \) through (7.2). By the definition of \( \text{Ext} \) functor, we see that their action is expressed as in Theorem 7.6. Thus, in conclusion, we have obtained a generalization of [26].

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Appendix A. Computation of structure morphisms of \( \rho^\text{ra} \)

A.1 Bimodule structure of \( \text{Hom} \)

Let \( C \) be a rigid monoidal category, and let \( \mathcal{M} \) be a closed left \( C \)-module category in the sense of Sect. 2.4. We establish some results on the natural transformations \( \alpha, \beta \) and \( \beta_i \) introduced in that subsection. For simplicity, we write \( \text{Hom}(M, N) = [M, N] \). We recall that \([M, -]\) is defined to be a right adjoint of the functor \( C \to \mathcal{M} \) given by \( X \mapsto X \otimes M \). As before, we denote by \( \text{coev}_{\text{ra}}(\_), M \) and \( \text{ev}_{\text{ra}}(\_), M \) the unit and the counit of this adjunction, respectively. Then, by (2.3), we have

\[ a_{X, M, N} = \text{Hom}(\text{id}_M, \text{id}_X \otimes \text{ev}_{Y, M, N}) \circ \text{coev}_X \otimes \text{Hom}(M, N), M \]  

(A.1)

for \( X \in \mathcal{C} \) and \( M, N \in \mathcal{M} \). By (2.7), we have

\[ b_{Y, M, N} = (b^\Sigma_{Y, M, N} \otimes \text{id}_Y) \circ (\text{id}_{\text{Hom}(M, N)} \otimes \text{coev}_Y) \]  

(A.2)

for \( Y \in \mathcal{C} \) and \( M, N \in \mathcal{M} \), where

\[ b^\Sigma_{Y, M, N} = \text{Hom}(M, \text{ev}_{Y \otimes M, N}) \circ \text{coev}_{\text{Hom}(Y \otimes M, N) \otimes Y, M} \]  

(A.3)

By the zig-zag identities for the adjunction \( (\_ \otimes M) \dashv \text{Hom}(\_ \otimes M, \_), \) we have

\[ \text{ev}_{M, X \otimes M} \circ (\text{coev}_{X, M} \otimes \text{id}_M) = \text{id}_{X \otimes M}, \]  

(A.4)

\[ [\text{id}_M, \text{ev}_{M, N}] \circ \text{coev}_{[M, N], M} = \text{id}_{[M, N]} \]  

(A.5)

for \( X \in \mathcal{C} \) and \( N \in \mathcal{M} \). By (A.4), (A.5) and the naturality of \( \text{ev}_{M,(\_)} \), we have

\[ \text{ev}_{M, N} \circ (a_{X, M, N} \otimes \text{id}_M) = \text{id}_X \otimes \text{ev}_{M, N}, \]  

(A.6)

\[ \text{ev}_{M, N} \circ (b^\Sigma_{Y, M, N} \otimes \text{id}_M) = \text{ev}_{Y \otimes M, N} \]  

(A.7)

for all objects \( X, Y \in \mathcal{C} \) and \( M, N \in \mathcal{M} \). We now prove:
Lemma A.1 (= Lemma 2.3). The equations
\begin{align}
    b_{1,M,N} &= \text{id}_{[M,N]}, \quad (2.13) \\
    b_{X \otimes Y,M,N} &= (b_{Y,M,N} \otimes \text{id}_X) \circ b_{X,Y \otimes M,N}. \quad (2.14) \\
    (a_{X,M,N} \otimes \text{id}_Y) \circ (\text{id}_X \otimes b_{Y,M,N}) &= b_{Y,M,X \otimes N} \circ a_{X,Y \otimes M,N} \quad (2.15)
\end{align}
hold for all objects \( X, Y \in \mathcal{C} \) and \( M, N \in \mathcal{M} \).

**Proof** Equation (2.13) is trivial. By the canonical isomorphisms
\[
    \text{Hom}_\mathcal{C}([X \otimes Y \otimes M, N], [M, N] \otimes Y^* \otimes X^*) \\
    \cong \text{Hom}_\mathcal{C}([X \otimes YM, N] \otimes X \otimes Y, [M, N]) \\
    \cong \text{Hom}_\mathcal{C}([X \otimes YM, N] \otimes X \otimes Y \otimes M, N),
\]
we see that (2.14) is equivalent to the equation
\[
    \text{ev}_{M,N} \circ (b_{X \otimes Y,M,N}^\otimes \otimes \text{id}_M) = \text{ev}_{M,N} \circ (b_{Y,M,N}^\otimes \otimes \text{id}_M) \circ (b_{X,Y \otimes M,N}^\otimes \otimes \text{id}_Y \otimes \text{id}_M).
\]
By (A.7), the both sides are equal to \( \text{ev}_{X \otimes Y \otimes M,N} \). Thus (2.14) is verified. In a similar way, we see that (2.15) is equivalent to the equation
\[
    \text{ev}_{M,X \otimes N} \circ (a_{X,M,N} \otimes \text{id}_M) \circ (\text{id}_X \otimes b_{Y,M,N}^\otimes) \circ (a_{X,Y \otimes M,N} \otimes \text{id}_Y \otimes \text{id}_M) = \text{ev}_{M,X \otimes N} \circ (b_{Y,M,X \otimes N}^\otimes \otimes \text{id}_M) \circ (a_{X,Y \otimes M,N} \otimes \text{id}_Y \otimes \text{id}_M).
\]
By (A.4)–(A.7), the both sides are equal to \( \text{id}_X \otimes \text{ev}_{Y \otimes M,N} \). The proof is done. \( \square \)

In view of this lemma, we have defined the natural isomorphism
\[
c_{X,M,N,Y} : X \otimes [M,N] \otimes Y^* \to [Y \otimes M, X \otimes N] \quad (X, Y \in \mathcal{C}, M, N \in \mathcal{M})
\]
by (2.18). The following Lemmas A.2 and A.3 will be used in later:

Lemma A.2 For all \( X \in \mathcal{C} \) and \( M \in \mathcal{M} \), the following equation holds:
\[
    \text{coev}_{1,X \otimes M} = c_{X,M,M,X} \circ (\text{id}_X \otimes \text{coev}_{1,M} \otimes \text{id}_X^*) \circ \text{coev}_X. \quad (A.8)
\]

**Proof.** By the definition of \( c \), the claim is equivalent to that the equation
\[
    b_{X,M,X \otimes M}^\otimes \circ (\text{coev}_{1,X \otimes M} \otimes \text{id}_X) = a_{X,M,M} \circ (\text{id}_X \otimes \text{coev}_{1,M})
\]
holds for all \( X \in \mathcal{C} \) and \( M \in \mathcal{M} \). By (A.4)–(A.7), the both sides correspond to the identity morphism under the canonical isomorphism
\[
    \text{Hom}_\mathcal{C}(X, \text{Hom}(X \otimes M, M)) \cong \text{Hom}_\mathcal{M}(X \otimes M, X \otimes M). \quad \square
\]

Lemma A.3 For all \( M_1, M_2, M_3 \in \mathcal{M} \), the following diagram commutes:
\[
\begin{array}{ccc}
    [M_1, [M_2, M_3] \otimes M_2] & \xrightarrow{\text{comp}} & [[M_1, M_2] \otimes M_1, M_3] \otimes [M_1, M_2] \\
    \text{id} \circ \text{ev}_{M_2, M_3} & \xrightarrow{b_{[M_1,M_2],M_1,M_3}^\otimes} & [M_1, M_3]
\end{array}
\]
Proof By the naturality of $\text{ev}_{\rho_C}$ and (A.6), we compute

$$\text{ev}_{\rho_{M_1}} \circ ((\text{id}_{M_1} \cdot \text{ev}_{\rho_{M_2}}) \cdot \text{id}_{M_1}) = \text{ev}_{\rho_{M_1}} \circ \text{ev}_{\rho_{M_2}} \circ \text{id}_{M_1} \circ \text{id}_{M_1} \circ \text{id}_{M_1} \circ \text{id}_{M_1}.$$  

This shows the commutativity of the left triangle of the diagram. Similarly, by (A.7) and the dinaturality of $\text{ev}_{\rho_C}$, we compute

$$\text{ev}_{\rho_{M_1}} \circ ((\text{id}_{M_1} \cdot \text{ev}_{\rho_{M_2}}) \cdot \text{id}_{M_1}) = \text{ev}_{\rho_{M_1}} \circ \text{ev}_{\rho_{M_2}} \circ \text{id}_{M_1} \circ \text{id}_{M_1} \circ \text{id}_{M_1} \circ \text{id}_{M_1}.$$  

Thus the left triangle of the diagram also commutes. □

A.2 Structure morphisms of $\rho^{\mathbb{1}}$

Now we consider the case where $\mathcal{C}$ is a finite tensor category and $\mathcal{M}$ is a finite left $\mathcal{C}$-module category. To save space, we set $\rho^{\mathbb{1}} = \rho^{\mathbb{1}}_{\mathcal{M}}$. Let $\xi^{(l)}$ and $\xi^{(r)}$ be the left and the right $\mathbb{C}$-module structure of $\rho$, respectively. By (2.3) and its right module version, we have

$$\xi^{(l)}_{X,F} = \rho(\text{id}_X \otimes \varepsilon_F) \circ \eta_X \otimes \varphi(F) \quad \text{and} \quad \xi^{(r)}_{X,F} = \rho(\varepsilon_F \otimes \text{id}_X) \circ \eta_{\varphi(F) \otimes X}$$

for $F \in \text{Rex}(\mathcal{M})$ and $X \in \mathcal{C}$. By using the universal dinatural transformation $\pi_F$ of the end $\varphi(F)$, these structure morphisms are given as follows:

Lemma A.4 (= Lemma 3.7). The equations

$$\pi_{X \otimes F}(M) \circ \xi^{(l)}_{X,F} = a_{X,M,F(M)} \circ (\text{id}_X \otimes \pi_F(M)),$$

$$\pi_{F \otimes X}(M) \circ \xi^{(r)}_{X,F} = b_{X,M,F(X \otimes M)} \circ (\pi_F(X \otimes M) \otimes \text{id}_X)$$

hold for all $F \in \text{Rex}(\mathcal{M})$ and $X \in \mathcal{C}$.

Proof Equation (3.10) is proved as follows:

$$\pi_{X \otimes F}(M) \circ \xi^{(l)}_{X,F} = [\text{id}_M, \text{id}_X \otimes \varepsilon_{F,M}] \circ \pi_{\rho_X(\varphi(F))}(M) \circ \eta_X \otimes \varphi(F)$$

$$= [\text{id}_M, \text{id}_X \otimes \text{ev}_{\rho_{M,F(M)}}] \circ (\text{id}_X \otimes \pi_F(M) \otimes \text{id}_M) \circ \text{coev}_{X \otimes \varphi(F),M}$$

$$= [\text{id}_M, \text{id}_X \otimes \text{ev}_{\rho_{M,F(M)}}] \circ \text{coev}_{X \otimes [M,F(M)],M} \circ (\text{id}_X \otimes \pi_F(M))$$

$$= a_{X,M,F(M)} \circ (\text{id}_X \otimes \pi_F(M)).$$

Here, the first equality follows from (A.9) and the naturality of $\pi_{\rho_{X}}(M)$, the second from (3.4) and (3.8), and the third from the naturality of $\text{coev}_{\rho_{X},M}$. To prove Eq. (3.11), we note that the symbol $\varepsilon_F \otimes \text{id}_X$ in (A.9) means the natural transformation whose component is given by $(\varepsilon_F \otimes \text{id}_X)_M = \varepsilon_{F,X \otimes M}$ for $M \in \mathcal{M}$. Thus we compute:

$$\pi_{F \otimes X}(M) \circ \xi^{(r)}_{X,F}$$
The right-hand side of (3.12) corresponds to

\[ \text{hold for all } F \in \text{Rex}(\mathcal{M}). \]

Thus (3.12) is verified. The proof is done.

We note that there is an isomorphism

\[ F \mapsto \rho = \rho_M \text{ is a strict monoidal functor. Hence its right adjoint } \overline{\rho} \text{ has a structure of a monoidal functor. We denote the structure morphisms by} \]

\[ \mu^{(2)}_{F,G} : \overline{\rho}(F) \otimes \overline{\rho}(G) \rightarrow \overline{\rho}(FG) \quad \text{and} \quad \mu^{(0)} : 1 \rightarrow \overline{\rho}(\text{id}_M) \]

for \( F, G \in \text{Rex}(\mathcal{M}) \). With the use of \( \eta \) and \( \varepsilon \), they are expressed by

\[ \mu^{(2)}_{F,G} = \overline{\rho}(\varepsilon_F \otimes \varepsilon_G) \circ \eta_{\overline{\rho}(F) \otimes \overline{\rho}(G)} \quad \text{and} \quad \mu^{(0)} = \eta_{1}, \quad (A.10) \]

where \( \varepsilon_F \otimes \varepsilon_G \) means the tensor product of \( \varepsilon_F \) and \( \varepsilon_G \) in \( \text{Rex}(\mathcal{M}) \), or, equivalently, the horizontal composition of \( \varepsilon_F \) and \( \varepsilon_G \).

**Lemma A.5** (\( = \text{Lemma 3.8} \)). The equations

\[ \pi_{FG}(M) \circ \mu^{(2)}_{F,G} = \text{comp}_{M,G(M),FG(M)} \circ (\pi_F(G(M)) \otimes \pi_G(M)), \quad (3.12) \]

\[ \pi_{\text{id}_M}(M) \circ \mu^{(0)} = \text{coev}_{1,M} \quad (3.13) \]

hold for all \( F, G \in \text{Rex}(\mathcal{M}) \) and \( M \in \mathcal{M} \).

**Proof** Equation (3.13) follows from (3.8) and (A.10). To prove (3.12), we set

\[ w = \text{ev}^{(3)}_{M,G(M),FG(M)} \circ (\pi_{FG}(M) \otimes \pi_G(M) \otimes \text{id}_M). \]

We note that there is an isomorphism

\[ \text{Hom}_C(\overline{\rho}(F) \otimes \overline{\rho}(G), [M, FG(M)]) \cong \text{Hom}_C(\overline{\rho}(F) \otimes \overline{\rho}(G) \otimes M, FG(M)). \quad (A.11) \]

The right-hand side of (3.12) corresponds to \( w \) via (A.11). On the other hand, the left-hand side of (3.12) corresponds to \( (\varepsilon_F \otimes \varepsilon_G)_M \) via (A.11). By (3.4) and the definition of the horizontal composition, we have

\[ (\varepsilon_F \otimes \varepsilon_G)_M = \varepsilon_{FG}(M) \circ (\text{id}_\overline{\rho(F)} \otimes \varepsilon_{G,M}) = w. \]

Thus (3.12) is verified. The proof is done.

**A.3 Commutativity of \( A_S \)**

For a \( C \)-module full subcategory \( S \subset \mathcal{M} \), we have proved that the end \( A_S := \int_{S \in S} \text{Hom}(S, S) \) has the half-braiding \( \sigma_S \) given by the commutative diagram (4.11). Namely, the equation

\[ \alpha_X,W,W \circ (\text{id}_X \otimes \pi_S(W)) \circ \sigma_S(X) = b^S_{X,W,X \otimes W} \circ (\pi_S(X \otimes W) \otimes \text{id}_X) \quad (A.12) \]

holds for all \( X \in C \) and \( W \in S \), where \( \pi_S(X) : A_S \rightarrow \text{Hom}(X, X) \) is the universal dinatural transformation. Let \( m_S : A_S \otimes A_S \rightarrow A_S \) be the multiplication of \( A_S \), and set \( m^{op}_S = m_S \circ \sigma_S(A_S). \)
**Proof of Theorem 4.9** The claim of this theorem is that \( A_S = (A_S, \sigma_S) \) is a commutative algebra in \( Z(\mathcal{C}) \). Thus it is sufficient to show \( m_S^{op} = m_S \). We fix \( W \in S \) and set \( E = [W, W] \) for simplicity. Then we compute

\[
\pi_S(W) \circ m_S^{op} = \text{comp}_{W, W} \circ (\pi_S(W) \circ \pi_S(W)) \circ \sigma_S(A_S)
\]

\[
= [\text{id}_W, \text{ev}_{W, W}] \circ \text{ae}_{E, W} \circ (\pi_S(W) \otimes \text{id}_X) \circ \sigma_S(E) \circ (\text{id}_{A_S} \otimes \pi_S(W))
\]

\[
= [\text{id}_W, \text{ev}_{W, W}] \circ b^{\pi}_{E, W, E \otimes W} \circ (\pi_S(E \otimes W) \otimes \pi_S(W))
\]

\[
= b^{\pi}_{E, W, W} \circ ((\text{id}_E \otimes \text{ev}_{W, W}) \otimes \text{id}_E) \circ (\pi_S(E \otimes W) \otimes \pi_S(W))
\]

\[
= b^{\pi}_{E, W, W} \circ (\text{ev}_{W, W}, \text{id}_W) \circ (\pi_S(W) \otimes \pi_S(W))
\]

\[
= \text{comp}_{W, W} \circ (\pi_S(W) \otimes \pi_S(W)) = (\pi_S(W) \otimes \pi_S(W)) \circ m_S.
\]

Here, the first and the last equalities follow from the definition of \( m_S \), the second and the sixth from Lemma A.3, the third from (A.12), the fourth from the naturality of \( \zeta \), and the fifth from the dinaturality of \( \pi_S \). We now obtain \( m_S^{op} = m_S \) by the universal property of \( A_S \). The proof is done. \( \Box \)

**Appendix B. On the properties of the pivotal trace**

We complement properties of the trace in a pivotal module category. Let \( \mathcal{C} \) be a finite tensor category, and let \( \mathcal{M} \) be an exact left \( \mathcal{C} \)-module category. For simplicity, we write \( \mathcal{M} = \text{Hom}(M, N) \). By the definition of the relative Serre functor, there is a natural isomorphism

\[
\delta_{M, N} : [M, N]^* \to [N, \mathcal{S}(M)] \quad (M, N \in \mathcal{M}),
\]

where \( \mathcal{S} = \mathcal{S}_\mathcal{M} \). There is also a natural isomorphism

\[
\zeta_{X, M} : X^{**} \otimes \mathcal{S}(M) \to \mathcal{S}(X \otimes M)) \quad (X \in \mathcal{C}, M \in \mathcal{M})
\]

such that \( \delta \) is an isomorphism of \( \mathcal{C} \)-bimodule functors from \( \mathcal{M}^{op} \times \mathcal{M} \) to \((-) \otimes \mathcal{C} \), that is, the equations \( \zeta_{M, M} = \text{id}_{\mathcal{S}(M)} \) and

\[
[\text{id}_X \otimes \zeta_{X, M} \otimes \zeta_{M, M}] \circ \delta_{M, X \otimes M, N} \circ (\zeta_{X, M, N}) = \delta_{M, X \otimes M, N} \circ (\zeta_{X, M, N})^{-1}
\]

\[
t (\text{id}_X \otimes \zeta_{X, M} \otimes \zeta_{M, M}) \circ \delta_{M, X \otimes M, N} \circ (\zeta_{X, M, N}) = \delta_{M, X \otimes M, N} \circ (\zeta_{X, M, N})^{-1}
\]

hold for all objects \( X, Y \in \mathcal{C} \) and \( M, N \in \mathcal{M} \).

Now we suppose that \( \mathcal{C} \) and \( \mathcal{M} \) are pivotal with pivotal structures \( j \) and \( j' \), respectively. By Definition 5.6, we have

\[
\zeta_{X, M} \circ j'_{X \otimes M} = j_X \otimes j'_M \quad (X \in \mathcal{C}, M \in \mathcal{M}).
\]

The trace \( \text{tr}_\mathcal{M} \), defined in Sect. 5.2, is characterized by

\[
\delta_{M, M} \circ \text{tr}_\mathcal{M}(M)^* = [\text{id}_M, j'_M] \circ \text{coev}_{1, M} \quad (M \in \mathcal{M}).
\]

We recall that \( \text{tr}_\mathcal{C} \) is defined by \( \text{tr}_\mathcal{C}(X) = \text{ev}_{X^*} \circ (j_X \otimes \text{id}_X) \) for \( X \in \mathcal{C} \). Thus,

\[
\text{tr}_\mathcal{C}(X)^* = (\text{id}_X \otimes j_X^*) \circ \text{coev}_{X^*} = (j_X \otimes \text{id}_X^*) \circ \text{coev}_X.
\]

**Lemma B.1** The trace \( \text{tr}_\mathcal{M} \) is dinatural, and the equation

\[
\text{tr}_\mathcal{M}(X \otimes M) \circ \zeta_{X, M, M, X} = \text{tr}_\mathcal{C}(X) \circ (\text{id}_X \otimes \text{tr}_\mathcal{M}(M) \otimes \text{id}_{X^*})
\]

\[\text{Springer}\]
holds for all objects \(M \in \mathcal{M}\) and \(X \in \mathcal{C}\).

**Proof.** The dinaturality of \(\text{tr}_M\) follows from the naturality of \(j'\) and the dinaturality of \(\text{coev}_{1,(-)}\). Equation \((\text{B.5})\) is proved as follows:

\[
(id_{X^{**}} \otimes \delta_{M,M} \otimes id_{X^*}) \circ (\text{tr}_C(X) \circ (id_X \otimes \text{tr}_M(M) \otimes id_{X^*}))^*
= (id_{X^{**}} \otimes [id_M, j_M'] \text{coev}_{1,M} \otimes id_{X^*}) \circ (j_X \otimes id_{X^*}) \circ \text{coev}_X
\]

(by \((\text{B.3}), (\text{B.4})\)\)

\[
= (j_X \otimes [id_M, j_M'] \otimes id_{X^*}) \circ (id_X \otimes \text{coev}_{1,M} \otimes id_{X^*}) \circ \text{coev}_X
\]

(by \((\text{A.8})\)\)

\[
= c^{-1}_{X^{**},M,S(M),X} \circ [id_X \otimes \xi X, j_X \otimes j_M'] \circ \text{coev}_{X \otimes M}
\]

(by \((\text{B.2})\)\)

\[
= c^{-1}_{X^{**},M,S(M),X} \circ [id_X \otimes \xi X, \delta X \otimes \delta M] \circ \text{tr}_M(X \otimes M)^*
\]

(by \((\text{B.3})\)\)

\[
= (id_{X^{**}} \otimes \delta_{M,M} \otimes id_{X^*}) \circ c^{-1}_{X,M,M,X} \circ \text{tr}_M(X \otimes M)^*
\]

(by \((\text{B.1})\)\).

For a morphism \(f : M \to M\) in \(\mathcal{M}\), we have defined \(\text{ptr}_M(f) \in k\) of \(f\) by

\[
\text{tr}(M) \circ [id_M, f] \circ \text{coev}_{1,M} = \text{ptr}(f) \cdot id_1.
\]

\((\text{B.5})\)

**Proposition B.2** For morphisms \(f : M \to N\) and \(g : N \to M\) in \(\mathcal{M}\), we have

\[
\text{ptr}(fg) = \text{ptr}(gf).
\]

\((\text{B.6})\)

**Proof** Equation \((\text{B.6})\) follows from the dinaturality of \(\text{tr}_M\) and \(\text{coev}_{1,(-)}\). Equation \((\text{B.7})\) follows from \((\text{B.5})\).

For \(M \in \mathcal{M}\), we have defined the internal character \(\text{ch}_M(M) \in \text{CF}(\mathcal{M})\) by

\[
\text{ch}_M(M) = \text{tr}_M(M) \circ \pi_M(M),
\]

\((\text{B.6})\)

where \(\pi_M : A_M \to [M, M]\) is the universal dinatural transformation.

**Proposition B.3** \((\text{= Lemma 5.9})\). For all \(X \in \mathcal{C}\) and \(M \in \mathcal{M}\), we have

\[
\text{ch}_M(X \otimes M) = \text{ch}_C(X) \ast \text{ch}_M(M),
\]

where \(\ast\) is the action \((\text{5.1})\) of \(\text{CF}(\mathcal{C})\) on \(\text{CF}(\mathcal{M})\).

**Proof.** We recall that \(A_M\) has the \(Z\)-coaction \(\delta_M : A_M \to Z(A_M)\) induced from the half-braiding of \(A_M\). By definition,

\[
(id_X \otimes \pi_M(M) \otimes id_{X^*}) \circ \pi^Z(A_M) \circ \delta_M = c_{X,M,M,X}^{-1} \circ \pi_M(X \otimes M)
\]

for all objects \(X \in \mathcal{C}\) and \(M \in \mathcal{M}\). Thus, by Lemma B.1, we have

\[
\text{ch}_C(X) \ast \text{ch}_M(M) = \text{ch}_C(X) \circ Z(\text{ch}_M(M)) \circ \delta_M
\]

\[
= \text{tr}_C(X) \circ (id_X \otimes (\text{tr}_M(M) \circ \pi_M(M)) \otimes id_{X^*}) \circ \pi^Z(A_M) \circ \delta_M
\]

\[
= \text{tr}_C(X) \circ (id_X \otimes \text{tr}_M(M) \otimes id_{X^*}) \circ c_{X,M,M,X}^{-1} \circ \pi_M(X \otimes M)
\]

\[
= \text{tr}_C(X \otimes M) \circ \pi_M(X \otimes M) = \text{ch}_M(X \otimes M).
\]

\(\square\)
Finally, we give a proof of Lemma 5.10 in the following general form: Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $(\langle - , - \rangle : \mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathcal{B}$ be a functor that is additive and exact in each variable. Let $X$ and $Y$ be objects of $\mathcal{B}$. Suppose that there are two dinatural transformations $d(M) : X \to \langle M, M \rangle$ and $e(M) : \langle M, M \rangle \to Y$ ($M \in \mathcal{A}$). For a morphism $f : M \to M$ in $\mathcal{A}$, we define

$$t(f) = e(M) \circ (\text{id}_M, f) \circ d(M) \in \text{Hom}_B(X, Y).$$

**Proposition B.4** Suppose that

$$
\begin{array}{c}
0 \\
\downarrow \\
0
\end{array}
\xrightarrow{r} 
\begin{array}{c}
M_1 \\
\downarrow f_1 \\
M_1
\end{array} 
\xrightarrow{r} 
\begin{array}{c}
M_2 \\
\downarrow f_2 \\
M_2
\end{array} 
\xrightarrow{r} 
\begin{array}{c}
M_3 \\
\downarrow f_3 \\
M_3
\end{array} 
\xrightarrow{r} 
\begin{array}{c}
0 \\
\downarrow \\
0
\end{array}
$$

is a commutative diagram in $\mathcal{A}$ with exact rows. Then we have

$$t(f_2) = t(f_1) + t(f_3).$$

We note that the internal Hom functor of an exact module category is exact in each variable. Lemma 5.10 is the case where $\mathcal{A} = \mathcal{M}, \mathcal{B} = \mathcal{C}, \langle - , - \rangle = [-, -], d = \pi_{\mathcal{M}}, e = \text{tr}_{\mathcal{M}}$ and $f_i = \text{id}_M$ for $i = 1, 2, 3$. If we consider $d = \text{coev}_{1, (-)}$ instead of $d = \pi_{\mathcal{M}}$, then we obtain the additivity of the pivotal trace with respect to exact sequences.

**Proof** By the assumption on $(\langle - , - \rangle$, we obtain the following commutative diagram with exact rows and exact columns:

$$
\begin{array}{c}
0 \\
\downarrow \\
0
\end{array}
\xrightarrow{(\text{id}, r)} 
\begin{array}{c}
\langle M_3, M_1 \rangle \\
\downarrow (s, \text{id}) \\
\langle M_2, M_1 \rangle
\end{array} 
\xrightarrow{(\text{id}, s)} 
\begin{array}{c}
\langle M_3, M_2 \rangle \\
\downarrow (s, \text{id}) \\
\langle M_2, M_3 \rangle
\end{array} 
\xrightarrow{(\text{id}, s)} 
\begin{array}{c}
\langle M_3, M_3 \rangle \\
\downarrow (s, \text{id}) \\
\langle M_3, M_3 \rangle
\end{array} 
\xrightarrow{(\text{id}, s)} 
\begin{array}{c}
\langle M_1, M_1 \rangle \\
\downarrow (r, \text{id}) \\
\langle M_1, M_1 \rangle
\end{array} 
\xrightarrow{(\text{id}, s)} 
\begin{array}{c}
\langle M_1, M_2 \rangle \\
\downarrow (r, \text{id}) \\
\langle M_1, M_3 \rangle
\end{array} 
\xrightarrow{(\text{id}, s)} 
\begin{array}{c}
\langle M_1, M_3 \rangle \\
\downarrow (r, \text{id}) \\
\langle M_1, M_3 \rangle
\end{array} 
\xrightarrow{(\text{id}, s)} 
\begin{array}{c}
\langle M_1, M_3 \rangle \\
\downarrow (r, \text{id}) \\
\langle M_1, M_3 \rangle
\end{array} 
\xrightarrow{(\text{id}, s)} 
\begin{array}{c}
0 \\
\downarrow \\
0
\end{array}
$$

We set $K = \text{Ker}(\langle r, s \rangle : \langle M_2, M_2 \rangle \to \langle M_1, M_3 \rangle)$. Then we have $K = I_1 + I_2$, where

$$I_1 = \text{Im}(\langle \text{id}_{M_2}, r \rangle : \langle M_2, M_1 \rangle \to \langle M_2, M_2 \rangle),$$

$$I_2 = \text{Im}(\langle s, \text{id}_{M_2} \rangle : \langle M_3, M_2 \rangle \to \langle M_2, M_2 \rangle).$$

Moreover, there are morphisms $p_i : K \to \langle M_i, M_i \rangle$ ($i = 1, 3$) such that

$$\langle \text{id}_{M_1}, r \rangle \circ p_1 = \langle r, \text{id}_{M_2} \rangle \quad \text{and} \quad \langle s, \text{id}_{M_2} \rangle \circ p_3 = \langle \text{id}_{M_2}, s \rangle. \tag{B.8}$$

These claims are checked by chasing the diagram. See [20, Lemma 2.5.1] for the detail of the verification, since the proof up to here is completely same.

For simplicity of notation, we set $d_i = \langle \text{id}_{M_i}, f_i \rangle \circ d(M_i)$. By $f_2 s = s f_3$ and the dinaturality of $d$, we have

$$\langle \text{id}_{M_2}, s \rangle \circ d_2 = \langle s, \text{id}_{M_3} \rangle \circ d_3. \tag{B.9}$$
\[ \langle r, \id_{M_2} \rangle \circ d_2 = \langle \id_{M_1}, r \rangle \circ d_3. \]  
(B.10)

Thus \( \langle r, s \rangle \circ d_2 = \langle r, \id \rangle \circ \langle \id, s \rangle \circ d_2 = \langle r, \id \rangle \circ \langle s, \id \rangle \circ d_3 = 0 \), that is, \( \Im(d_2) \subset K \).

Hence the following morphism is defined:

\[ \Gamma := \langle e(M_1) \rangle \circ p_1 \circ d_2 + \langle e(M_3) \rangle \circ p_3 \circ d_2. \]  
(B.11)

We first show that \( \Gamma = t(f_1) + t(f_3) \). By (B.8) and (B.10), we have

\[ \langle \id_{M_1}, r \rangle \circ p_1 \circ d_2 = \langle r, \id_{M_2} \rangle \circ d_2 = \langle \id_{M_1}, r \rangle \circ d_1. \]

Since \( r \) is monic, so is \( \langle \id_{M_1}, r \rangle \). Thus we have \( p_1 \circ d_2 = d_1 \). Therefore the first term of (B.11) is \( t(f_1) \). Similarly, we have

\[ \langle s, \id_{M_3} \rangle \circ p_3 \circ d_2 = \langle \id_{M_2}, s \rangle \circ d_2 = \langle s, \id_{M_1} \rangle \circ d_3 \]

by (B.8) and (B.9), and thus \( p_3 \circ d_2 = d_3 \). From this, we see that the second term of (B.11) is \( t(f_3) \). Thus the claim follows.

To complete the proof, we show \( \Gamma = t(f_2) \). To see this, we remark

\[ \langle \id_{M_1}, r \rangle \circ p_1 \circ \langle s, \id_{M_2} \rangle = \langle r, \id_{M_2} \rangle \circ \langle s, \id_{M_2} \rangle = 0, \]
\[ \langle s, \id_{M_3} \rangle \circ p_3 \circ \langle \id_{M_2}, r \rangle = \langle \id_{M_2}, s \rangle \circ \langle \id_{M_2}, r \rangle = 0 \]

by (B.8). Since both \( \langle \id_{M_1}, r \rangle \) and \( \langle s, \id_{M_2} \rangle \) are monic, \( p_1 \circ \langle s, \id_{M_2} \rangle \) and \( p_3 \circ \langle \id_{M_2}, r \rangle \) are zero morphisms. Set \( \Gamma' = \langle e(M_1) \rangle \circ p_1 + \langle e(M_3) \rangle \circ p_3 \). We have

\[ \Gamma' \circ \langle s, \id_{M_2} \rangle = \langle e(M_2) \rangle \circ \langle s, \id_{M_2} \rangle \quad \text{and} \quad \Gamma' \circ \langle \id_{M_2}, r \rangle = \langle e(M_2) \rangle \circ \langle \id_{M_2}, r \rangle \]

by the dinaturality of \( e \). These equation imply that \( \Gamma' = \langle e(M_2) \rangle \) on \( K = I_1 + I_2 \). Since \( \Im(d_3) \subset K \), we conclude that \( \Gamma = t(f_2) \). The proof is done. \( \square \)

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