When Is Recoverable Consensus Harder Than Consensus?

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Abstract

We study the ability of different shared object types to solve recoverable consensus using non-volatile shared memory in a system with crashes and recoveries. In particular, we compare the difficulty of solving recoverable consensus to the difficulty of solving the standard wait-free consensus problem in a system with halting failures. We focus on the model where individual processes may crash and recover and the large class of object types that are equipped with a read operation. We characterize the readable object types that can solve recoverable consensus among a given number of processes. Using this characterization, we show that the number of processes that can solve consensus using a readable type can be larger than the number of processes that can solve recoverable consensus using that type, but only slightly larger.

1 Introduction

Recoverable consensus can play a key role in the study of asynchronous systems with non-volatile shared memory where processes can crash and recover, just as the standard consensus problem plays a central role in the study of asynchronous systems where processes may halt. In this paper, our goal is to leverage extensive research on the solvability of the standard consensus problem in systems equipped with different types of shared objects to gain knowledge about recoverable consensus in systems with non-volatile memory.

We consider an asynchronous model of computation, where processes communicate with one another by accessing shared memory. In particular, we are interested in studying how concurrent algorithms can take advantage of recent advances in non-volatile main memory, which maintains its stored values even when its power supply is turned off. This allows for algorithms that can carry on with a computation when processes crash and recover. We consider a standard theoretical model [3, 20, 22, 21] for this setting, where each process’s local memory is volatile, but shared memory is non-volatile, and processes may crash and recover individually in an asynchronous manner. After a process crashes, its local memory, including its programme counter, is reinitialized to its initial state when the process recovers. Process crashes do not affect the state of shared memory. At recovery time, the process begins to execute its code again from the beginning. We refer to the sequence of steps that a process takes between crashes as a run of its code.

1 Alternatively, it could execute a recovery function. Our results hold either way. We use the simpler assumption of re-starting upon recovery to prove our results.
The consensus problem, where each process gets an input and all processes must agree to output one of them, has been central to the study of shared-memory computation in asynchronous systems with process halting failures (but no recoveries). A shared object type is defined by a sequential specification, which specifies the set of possible states of the object, the operations that can be performed on it, and how the object changes state and returns a response when an operation is applied on it. Herlihy [25] defined the consensus number of a type \( T \), denoted \( \text{cons}(T) \), to be the maximum number of processes that can solve consensus using objects of type \( T \) and read/write registers, or \( \infty \) if there is no such maximum. The classification of types according to their consensus number is called the consensus hierarchy. This classification is particularly meaningful because of Herlihy’s universality result: a type \( T \) can be used (with registers) to obtain wait-free implementations of all object types in a system of \( n \) processes if and only if \( \text{cons}(T) \) is at least \( n \).

\[ \begin{align*}
\text{n-recording} & \quad \text{RC is solvable} & \quad \text{RC is solvable} \\
(\text{n-1})\text{-recording} & \quad \text{RC is solvable} & \quad \text{RC is solvable} \\
\downarrow & \quad \downarrow & \quad \downarrow \\
\text{n-process consensus} & \quad \text{n-discerning} & \quad \text{RC is solvable} \\
\text{is solvable} & & \\
\end{align*} \]

Figure 1: Relationships between conditions and solvability of consensus and recoverable consensus (with independent crashes) using a deterministic, readable type.

Golab [20] defined the recoverable consensus (RC) problem, where processes must agree on one of their input values, even if processes may crash and recover. An algorithm for RC defines a routine for each process to execute that takes an input value and eventually returns an output value, satisfying the following three properties.

- **Agreement**: no two output values produced are different. (This includes outputs by different processes and outputs of the same process when it performs multiple runs of the algorithm because it crashes and recovers.)

- **Validity**: each output value is the input value of some process.

- **Recoverable wait-freedom**: if a process executes its algorithm from the beginning, it either crashes or outputs a value after a finite number of its own steps.

Like Golab, we assume a process’s input value does not change, even across multiple runs, but this is not a crucial assumption. (If an RC algorithm requires this precondition, it can be transformed into one that does not using a register for each process’s input. When a process begins a run, it reads this register and, if it has not yet been written, the process writes its input value. It then uses the value in the register as its input, ensuring that all of the process’s runs of the original algorithm use the same input value.) Berryhill, Golab and Tripunitara [6] described how Herlihy’s universality result carries over to the model with crashes and recoveries, using RC in place of consensus. (See Section 4 for details.)

There are two common failure models for crashes and recoveries: simultaneous crashes [20], where all processes crash simultaneously, and independent crashes (introduced in [23] to study recoverable mutual exclusion), where processes can crash and recover individually in an asynchronous way. Golab [20] defined two recoverable consensus hierarchies. For an object type \( T \), the simultaneous RC number of \( T \) is the maximum number of processes that can solve RC using an unbounded number of shared objects of type \( T \) and read/write registers when simultaneous crashes may occur. Similarly, the independent RC number of \( T \), which we denote \( r\text{cons}(T) \), is the maximum number of processes that can solve RC using shared objects of type \( T \) and read/write registers when independent crashes may occur. In both cases, if no maximum exists we
say the RC number is ∞. This is a slight modification of Golab’s definition.\footnote{Golab’s definition of RC numbers required the RC algorithms to use a bounded number of objects. We permit an infinite number of objects. When Jayanti formalized Herlihy’s consensus hierarchy, he similarly allowed an unbounded number of objects to be used in solving consensus. (However, it follows from König’s Lemma that any wait-free algorithm for the standard consensus problem that uses objects with finite non-determinism will use finitely many objects.) Universal constructions, which are one of the main motivations for studying the hierarchy, require an infinite number of instances of consensus anyway, so even if each instance uses a finite number of objects, the overall construction would still use an infinite number.} As an example, we show in Appendix H that \( rcons(stack) = 1, rcons(stack) = 1 \)\footnote{See Appendix A for details. See the full version for details.}, whereas it is known that \( cons(stack) = 2 \)\footnote{See Appendix A for details. See the full version for details.}.

1.1 Our Results

We focus on independent crashes since a simple extension of Golab’s result\footnote{Golab’s definition of RC numbers required the RC algorithms to use a bounded number of objects. We permit an infinite number of objects. When Jayanti formalized Herlihy’s consensus hierarchy, he similarly allowed an unbounded number of objects to be used in solving consensus. (However, it follows from König’s Lemma that any wait-free algorithm for the standard consensus problem that uses objects with finite non-determinism will use finitely many objects.) Universal constructions, which are one of the main motivations for studying the hierarchy, require an infinite number of instances of consensus anyway, so even if each instance uses a finite number of objects, the overall construction would still use an infinite number.} described in Section 2 shows that RC has exactly the same difficulty as consensus in a system with simultaneous crashes.

Our main results are for deterministic shared object types that are readable, meaning that they are equipped with a read operation that returns the current state of the object without changing it. We define, for all \( n \geq 2 \), the \( n \)-recording property for shared object types. Roughly speaking, a readable type \( T \) is \( n \)-recording if \( n \) processes can be divided into two teams and use one object of type \( T \) to determine which of the two teams “wins”, even when processes crash and recover. The first team to perform an update operation on the object is the winning team, and this information is recorded in the object’s state, so that processes can determine which team wins by reading the object.

We show in Section 3.1 that being \( n \)-recording is sufficient for solving RC among \( n \) processes. We also show in Section 3.2 that the slightly weaker condition of being \( (n - 1) \)-recording is necessary for solving RC among \( n \) processes. Thus, we have a fairly simple way of determining the approximate value of \( rcons(T) \): if \( T \) is \( n \)-recording but not \( (n + 1) \)-recording, we know that \( rcons(T) \) is either \( n \) or \( n + 1 \). Our \( n \)-recording property is related to Ruppert’s \( n \)-discerning property\footnote{See Appendix A for details. See the full version for details.}, which was defined to characterize readable types that can solve \( n \)-process consensus. In Section 3.3 we prove relationships between these two properties. This allows us to prove that if a type has consensus number \( n \), then its RC number is between \( n - 2 \) and \( n \). We give examples of types \( T \) with \( rcons(T) = cons(T) \) and others with \( rcons(T) < cons(T) \). In Section 3.4 we also use our characterization to show that weak types do not become much stronger (in terms of their power to solve RC) when used together. Section 4 describes how Herlihy’s motivation for studying the consensus hierarchy carries over to the RC hierarchy for the setting of non-volatile memory. See Figure 1 for an overview of our results.

2 Simultaneous Crash Model

In the case of simultaneous crashes, the RC hierarchy is identical to the standard consensus hierarchy.

Theorem 1. Recoverable consensus is solvable among \( n \) processes using objects of type \( T \) and registers in the simultaneous crash model if and only if \( cons(T) \geq n \).

Golab\footnote{Golab’s definition of RC numbers required the RC algorithms to use a bounded number of objects. We permit an infinite number of objects. When Jayanti formalized Herlihy’s consensus hierarchy, he similarly allowed an unbounded number of objects to be used in solving consensus. (However, it follows from König’s Lemma that any wait-free algorithm for the standard consensus problem that uses objects with finite non-determinism will use finitely many objects.) Universal constructions, which are one of the main motivations for studying the hierarchy, require an infinite number of instances of consensus anyway, so even if each instance uses a finite number of objects, the overall construction would still use an infinite number.} showed how to transform a standard consensus algorithm into an algorithm for RC in the case of simultaneous crashes. His transformation required a bound on the number of crashes to ensure that the space used by the algorithm is bounded. Since we allow an unbounded number of objects to be used to solve RC, a simple modification of Golab’s algorithm can be used to prove Theorem 1. See Appendix A for details. See the full version for details. In
view of Theorem 1, we focus on determining RC numbers of types in the presence of independent crashes in the remainder of the paper.

3 Readable Objects

A deterministic object type has a sequential specification that specifies a unique response and state transition when a given operation is applied to an object of this type that is in a given state. An object is readable if it has a READ operation that returns the entire state of the object without altering it. Ruppert [33] provided a characterization of deterministic, readable types that can solve consensus among \( n \) processes. In this section, we develop a similar characterization for RC with independent crashes, and use this to compare the ability of types to solve the two problems.

The characterizations for consensus and for RC are linked to the team consensus problem, which is the problem of solving consensus when the set of processes are divided in advance into two non-empty teams and all processes on the same team get the same input. (This problem is also known as static consensus [31].)

We first review the characterization for standard consensus [33]. Suppose each process can perform a single update operation on an object \( O \) of type \( T \), and then read \( O \) at some later time, and, based only on the responses of these two steps, determine which team updated \( O \) first. If this is possible, we say \( T \) is \( n \)-discerning.

**Definition 2.** A deterministic type \( T \) is called \( n \)-discerning if there exist

- a state \( q_0 \),
- a partition of \( n \) processes \( p_1, \ldots, p_n \) into two non-empty teams \( A \) and \( B \), and
- operations \( op_1, op_2, \ldots, op_n \)

such that, for all \( j \in \{1, \ldots, n\} \), \( R_{A,j} \cap R_{B,j} = \emptyset \), where \( R_{X,j} \) is the set of pairs \( (r, q) \) for which there exist distinct process indices \( i_1, \ldots, i_\alpha \) including \( j \) with \( p_{i_1} \in X \) such that if \( op_{i_1}, \ldots, op_{i_\alpha} \) are performed in this order on an object of type \( T \) initially in state \( q_0 \), then \( op_j \) returns \( r \) and the object ends up in state \( q \).

In this definition and in Definition 4, an operation \( op_i \) includes the name of the operation and any arguments to it. For example, WRITE(42) is an operation on a read/write register. Operations \( op_1, \ldots, op_n \) need not be distinct. Ruppert used a valency argument to show that any deterministic, readable type that can solve consensus among \( n \) processes must be \( n \)-discerning. Conversely, team consensus can be solved using a readable \( n \)-discerning object \( O \) and one register per team as follows. Each process \( p_i \) writes its input in its team’s register, performs its operation \( op_i \) on \( O \) and then reads \( O \)'s state. The process determines which team updated \( O \) first and outputs the value in that team’s register. A tournament then solves consensus: processes within each team agree on an input value recursively and then run team consensus to choose the final output value. The argument sketched here yields the following characterization.

**Theorem 3** ([33]). A deterministic, readable type can be used, together with registers, to solve \( n \)-process wait-free consensus if and only if it is \( n \)-discerning.

We now consider how to characterize readable types that can solve recoverable consensus, with independent process crashes. Recoverable team consensus is the RC problem where the processes are partitioned in advance into two non-empty teams and inputs are constrained so

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3 We use this definition for simplicity, but our results would apply equally well to the original, more general definition of readable objects in [33], which allows the state of the object to be read piece-by-piece. For example, an array of registers is also readable under the more general definition.
that all processes on the same team have the same input value. We shall show that RC is solvable if and only if recoverable team consensus is solvable: the only if direction is trivial, and the converse will be shown using the same tournament algorithm outlined above. So, it suffices to characterize types that can solve recoverable team consensus for \( n \) processes.

We shall define a property called \( n \)-recording such that a type \( T \) satisfying the property will allow \( n \) processes to solve recoverable team consensus in a simple way. A shared object \( O \) of type \( T \) is initialized to some state \( q_0 \). To solve team consensus using an \( n \)-discerning type, each process performs a single operation on \( O \) and then reads \( O \), and is able to conclude from the responses to these two steps which team updated \( O \) first. There are two key difficulties when we consider processes that may crash and recover: (1) if a process crashes after performing its update, thereby losing the response of that update, the process cannot use the response to determine which team won, and (2) a process that recovers should try to avoid performing its update on \( O \) a second time so that it does not obliterate the evidence of which team updated \( O \) first.

To cope with (1), our new \( n \)-recording property should allow a process to determine which team updated \( O \) first based simply on the state of \( O \), which can be read at any time. Thus, two sequences of update operations that start with processes on opposite teams must not take \( O \) to the same state. This is formalized in condition 1 of Definition 4, below.

We now consider how to cope with (2). If \( O \) could never return to its initial state \( q_0 \), checking that \( O \)'s state is \( q_0 \) before updating \( O \) would ensure that no process ever updates \( O \) twice. (See the code for processes on team \( A \) in Figure 2.) However, we can solve team consensus under a weaker condition: \( O \)'s state can return to \( q_0 \) after a process from team \( A \) updates \( O \) first, provided that team \( B \) has only one process. In this case, condition 2 of Definition 4 implies that the state cannot return to \( q_0 \) if a process on team \( B \) updates \( O \) first. Processes on team \( A \) behave as before, updating \( O \) if they find it in state \( q_0 \). If \( |B| > 1 \), processes on team \( B \) do likewise. However, if \( |B| = 1 \), the lone process on team \( B \) updates \( O \) if it finds \( O \) in state \( q_0 \) and sees that no process on team \( A \) has started its algorithm; in this case it knows that no operation has been performed on \( O \), since \( O \) can return to \( q_0 \) only if a process on team \( A \) updated it first. If the lone process on team \( B \) finds that a process on team \( A \) has already started, it simply outputs team \( A \)'s input value. (See the code for processes on team \( B \) in Figure 2.) This motivates condition 2 of Definition 4 below. A symmetric scenario motivates condition 3.

The approach of having processes on team \( B \) defer to team \( A \) if they see that a process on team \( A \) has started running works only if \( |B| = 1 \): if the algorithm used this approach with \( |B| > 1 \), one process on team \( B \) might start running before any process on team \( A \) and later go on to be the first process to update \( O \), while another process on team \( B \) might start after a process on team \( A \) has taken steps and defer to team \( A \). In this case, the latter process on team \( B \) would conclude that team \( A \) won, while others would conclude that team \( B \) won, violating agreement.

These considerations lead us to formulate the \( n \)-recording property in Definition 4 which uses the following notation. Fix a deterministic, readable type \( T \). Let \( X \) be a subset of the set of all processes \( \{p_1, \ldots, p_n\} \) and let \( op_1, \ldots, op_n \) be operations. Let \( q_0 \) be a state of type \( T \). Define \( Q_X(q_0, op_1, \ldots, op_n) \) to be the set of all states \( q \) for which there exist distinct process indices \( i_1, \ldots, i_{\alpha} \) with \( p_{i_1} \in X \) such that the sequence of operations \( op_{p_{i_1}}, \ldots, op_{p_{i_{\alpha}}} \) applied to an object of type \( T \) initially in state \( q_0 \) leaves the object in state \( q \). We omit the parameters of \( Q_X \) when they are clear from context.

**Definition 4.** A deterministic type \( T \) is \( n \)-recording if there exist

- a state \( q_0 \),
- a partition of \( n \) processes \( p_1, \ldots, p_n \) into two non-empty teams \( A \) and \( B \), and

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• operations op₁, . . . , opₙ,
satisfying the following three conditions.

1. \( Q_A(q₀, op₁, . . . , opₙ) \cap Q_B(q₀, op₁, . . . , opₙ) = \emptyset \).
2. \( q₀ \notin Q_A(q₀, op₁, . . . , opₙ) \text{ or } |B| = 1 \).
3. \( q₀ \notin Q_B(q₀, op₁, . . . , opₙ) \text{ or } |A| = 1 \).

We call a type that satisfies this property \( n \)-recording because it records its state information about the team that first updates the team, if it is initialized to state \( q₀ \).

We first prove some simple consequences of Definition 4.

Observation 5. For \( n \geq 2 \), if a deterministic type is \( n \)-recording, then it is \( n \)-discerning.

To see why this is true, we can use the same choice of \( A, B, q₀, op₁, . . . , opₙ \) for both definitions. If, for some \( j \), there were an \((r, q) \in R_{A,j} \cap R_{B,j}\) then \( q \) would also be in \( Q_A \cap Q_B \), which would violate property 4 of the definition of \( n \)-recording. So we can conclude that \( R_{A,j} \cap R_{B,j} \) must be empty, as required for the definition of \( n \)-discerning.

Observation 6. For \( n \geq 3 \), if a deterministic type is \( n \)-recording, then it is \((n-1)\)-recording.

If a type satisfies the definition of \( n \)-recording with teams \( A \) and \( B \), we can omit one process from the larger team to get a division of \( n - 1 \) processes into non-empty teams \( A' \) and \( B' \). We use the same initial state \( q₀ \) and assign the same operation to each process to satisfy the definition \((n-1)\)-recording.

We now summarize the results about deterministic, readable types that we prove in the remainder of this section. Theorem 8 shows that any readable type that is \( n \)-recording is capable of solving RC among \( n \) processes. We prove in Theorem 14 that all types that can solve RC among \( n \) processes satisfy the \((n-1)\)-recording property. (This is true even if the type is not readable.) Given a specification of a shared type, it is fairly straightforward to check whether it is \( n \)-recording. By determining the maximum \( n \) for which a given readable object type \( T \) is \( n \)-recording, we can conclude that \( rcons(T) \) is either \( n \) or \( n + 1 \).

We also prove that an \( n \)-discerning type must be \((n-2)\)-recording (Theorem 16), but not necessarily \((n-1)\)-recording (Proposition 19). As a corollary of these results, we show that \( cons(T) - 2 \leq rcons(T) \leq cons(T) \). These relationships are summarized in Figure 1. In Theorem 22 we also show how the power of a collection of readable types to solve RC is related to the power of each type when used in isolation.

3.1 Sufficient Condition

We use the algorithm in Figure 2 to show that recoverable team consensus can be solved using a deterministic, readable object \( O \) whose type is \( n \)-recording. The intuition for the algorithm has already been described above, but we now describe the code in more detail. The code assumes \( q₀ \notin Q_B \): if \( q₀ \in Q_B \), then \( q₀ \notin Q_A \) and we would reverse the roles of \( A \) and \( B \) in the code. Each process first writes its input in its team’s register. It then reads \( O \). If \( O \) is not in the initial state \( q₀ \), then the process determines which team went first based on the state of \( O \) and returns the value written in that team’s register (lines 11, 12, and lines 27, 28). Otherwise, it updates \( O \) before reading the state again (lines 9, 10) and 23, 24) to determine which team updated \( O \) first. There is one exception: if team \( B \) has only one process, it yields to team \( A \) (line 21) if it sees that some process on team \( A \) has already written its input value. This allows for the case where \( q₀ \in Q_A \) and \( |B| = 1 \): it could be that a process on team \( A \) updated \( O \) first, and then other processes (including the process on team \( B \), in a previous run) performed updates that returned \( O \) to state \( q₀ \). In this case, those processes would have output team \( A \)’s input value,
shared variables
Object $O$ of type $T$, initially in state $q_0$

Registers $R_A$ and $R_B$, initially in state $\bot$

**Decide**$(v)$ // code for process $p_i$ on team $A$

1. $R_A \leftarrow v$
2. $q \leftarrow O$
3. if $q = q_0$ then
   4. apply $op_i$ to $O$
   5. $q \leftarrow O$
4. end if
5. if $q \in Q_A$ then return $R_A$
6. else return $R_B$
7. end if

**Decide**$(v)$ // code for process $p_i$ on team $B$

1. $R_B \leftarrow v$
2. $q \leftarrow O$
3. if $q = q_0$ then
   4. if $|B| = 1$ and $R_A \neq \bot$ then
      5. return $R_A$
   6. else
      7. apply $op_i$ to $O$
      8. $q \leftarrow O$
   9. end if
4. end if
5. if $q \in Q_A$ then return $R_A$
6. else return $R_B$
7. end if

Figure 2: Algorithm for recoverable team consensus (assuming $q_0 \notin Q_B$).
so we must ensure that the process on team $B$ does not perform its update again, since that could cause processes to output team $B$’s input value, violating agreement.

The next lemma will help us argue that the algorithm behaves correctly in the tricky case where $q_0 \in Q_A$ and $|B| = 1$.

**Lemma 7.** Suppose $q_0, A, B, op_1, \ldots, op_n$ satisfy the definition of $n$-recording for a deterministic type $T$. Let $X \in \{A,B\}$. If $q_0 \notin Q_X$ and $i_1, \ldots, i_n$ is a sequence of distinct process indices such that the sequence of operations $op_{i_1}, \ldots, op_{i_n}$ takes an object of type $T$ from state $q_0$ to state $q_0$, then the indices of all processes of team $X$ appear in the sequence.

**Proof.** To derive a contradiction, suppose the claim is false, i.e., $j \notin \{i_1, \ldots, i_n\}$ for some process $p_j$ on team $X$. If $p_{i_1}$ were on team $X$, then the fact that the sequence of operations $op_{i_1}, \ldots, op_{i_n}$ take the state of an object from $q_0$ to $q_0$ would imply that $q_0 \in Q_X$, contrary to our assumption. Thus, $p_{i_1}$ must be on the opposite team $X$. Let $q_j$ be the state that results when $op_j$ is applied to an object in state $q_0$. We have $q_j \in Q_X$ since the sequence $op_j$ takes an object from state $q_0$ to $q_j$. We also have $q_j \in Q_X^X$ since the sequence $op_{i_1}, \ldots, op_{i_n}, op_j$ takes an object of type $T$ from state $q_0$ back to state $q_0$ and then to state $q_j$. Thus, $q_j \in Q_X \cap Q_X^X$, which violates condition [1] in the definition of $n$-recording.

To gain some intuition, we describe why the following bad scenario cannot occur when $|B| = 1$ and $q_0 \in Q_A$. Suppose a process $p_1$ on team $B$ begins, sees $R_A = ⊥$, and is poised to update $O$ at line 23. Then, a process $p_2$ on team $A$ runs to completion, updating $O$ and deciding $R_A$. Then, other processes update $O$, returning $O$’s state to $q_0$. If $p_1$ were still poised to update $O$ at line 23, then it would decide $R_B$, violating agreement. But this cannot happen: Lemma 7 ensures that $p_1$ must have been among the processes that already applied their operations on $O$ to return $O$’s state to $q_0$.

We also describe why the condition $|B| = 1$ on line 20 is necessary. If this test were missing, consider an execution where one process $p_1$ on team $B$ begins, sees $R_A = ⊥$ and is about to update $O$ at line 23. Then, a process $p_2$ on team $A$ writes to $R_A$. Next, another process $p_3$ on team $B$ sees that $R_A ≠ ⊥$ and decides $R_A$ (at line 21). Finally, process $p_1$ resumes and updates $O$. Since it is the first process to update $O$, $O$’s state would then be in $Q_B$, so $p_1$ would then read $O$ and decide $R_B$, violating agreement. We avoid this scenario by the test $|B| = 1$ of line 20: line 21 is executed only if $B$ contains just one process (whereas two processes on team $B$ are needed for the bad scenario described above).

**Theorem 8.** If a deterministic, readable type $T$ is $n$-recording, then objects of type $T$, together with registers, can be used to solve recoverable consensus for $n$ processes.

**Proof.** If team recoverable consensus can be solved, then RC can be solved. Processes on each team agree recursively on an input value for their team, and then use team consensus to determine the final output. See Appendix B for details.

Thus, it suffices to show that the algorithm in Figure 2 solves recoverable team consensus using a type $T$ that satisfies the condition of the theorem. Since $Q_A \cap Q_B = \emptyset$, we know that either $q_0 \notin Q_A$ or $q_0 \notin Q_B$. Without loss of generality, assume $q_0 \notin Q_B$. (If this is not the case, just swap the names of the two teams.)

Recoverable wait-freedom is clearly satisfied, since there are no loops in the code. It remains to show that every execution of the algorithm satisfies validity and agreement.

**Lemma 9.** Validity and agreement are satisfied in executions where no process ever performs an update on $O$.

**Proof.** In this case, $O$ remains in state $q_0$ forever. Thus, no process can reach line 11 or 27 since it would first have to update $O$ at line 8 or 23 respectively. So, processes output only at line 21. By the test on line 20, $R_A$ is written before a process outputs its value on line 21. Thus, all outputs are the input value of team $A$. □
For the remainder of the proof of the theorem, consider executions where at least one update is performed on $O$. Let $s$ be the first step in the execution that performs an update on $O$.

**Lemma 10.** For $X \in \{A, B\}$, if a process on team $X$ performs $s$ and $q_0 \notin Q_X$, then $O$’s state is in $Q_X$ at all times after $s$.

**Proof.** We first show that no process performs more than one update on $O$. To derive a contradiction, suppose some process performs two updates on $O$. Let $s'$ be the first step in the execution when a process performs its second update on $O$ and let $p_1$ be the process that performs $s'$. Let $r'$ be $p_1$’s run of the code that performs $s'$. Since $r'$ begins after $p_1$’s first update on $O$, $r'$ begins after $s$. By definition of $s'$, each process does at most one update on $O$ before $s'$. Thus, the state of $O$ is in $Q_X$ at all times between $s$ and $s'$. Since $q_0 \notin Q_X$, the state of $O$ is never $q_0$ between $s$ and $s'$. This contradicts the fact that $r'$ must read the state of $O$ to be $q_0$ between $s$ and $s'$; otherwise $r'$ would not perform $s'$.

Thus, each process performs at most one update on $O$. By the definition of $Q_X$, the state of $O$ is in $Q_X$ at all times after $s$. \hfill \square

We next prove a similar lemma for the case where $q_0 \in Q_A$. In this case, the situation is a little more complicated. The state of $O$ might return to $q_0$. If this happens, we show that each process updates $O$ at most once before the state returns to $q_0$, and that only processes of team $A$ can update $O$ after the state returns to $q_0$ and each process does so at most once. This is enough to ensure that $O$’s state remains in $Q_A$ at all times.

**Lemma 11.** If $s$ is performed by a process of team $A$ and $q_0 \in Q_A$, then $O$’s state is in $Q_A$ at all times.

**Proof.** Since $q_0 \in Q_A$, there is a unique process $p_j$ on team $B$, by condition 2 of the definition of $n$-recording. $O$’s state is $q_0 \in Q_A$ at all times before $s$. It remains to show that $O$’s state is in $Q_A$ at all times after $s$. We consider two cases.

First, suppose $O$ is never in state $q_0$ after $s$. Consider any process $p_i$ that performs an update on $O$. Let $s_i$ be $p_i$’s first update on $O$. By definition, $s_i$ is either equal to $s$ or after $s$. Any run by $p_i$ that begins after $s_i$ (and hence after $s$) that reads $O$ on line 6 or 18 sees a value different from $q_0$, so it does not perform an update on $O$. Thus, $O$ does not perform more than one update on $O$. It follows from the definition of $Q_A$ that $O$’s state is in $Q_A$ at all times after $s$.

Now, suppose $O$’s state is equal to $q_0$ at some time after $s$. Let $s''$ be the first step at or after $s$ that changes $O$’s state back to $q_0$. We next prove that no process performs two updates on $O$ between $s$ and $s''$ (inclusive). To derive a contradiction, suppose some process performs two such updates. Let $s'$ be the first step when any process performs its second update on $O$. By definition, $s'$ is between $s$ and $s''$ (inclusive). Let $p_i$ be the process that performs $s'$ and let $r'$ be the run by $p_i$ that performs $s'$. Since $r'$ begins after $p_i$’s first update on $O$, $r'$ begins after $s$. Thus, $r'$ reads $O$’s state to be different from $q_0$ at line 6 or 18 and therefore fails the test on line 7 or 19. This contradicts the fact that $r'$ updates $O$. Hence, each process performs at most one update on $O$ between $s$ and $s''$ (inclusive).

It follows from the definition of $Q_A$ that the state of $O$ is in $Q_A$ at all times between $s$ and $s''$. By Lemma 7, the unique process $p_j$ on team $B$ updates $O$ between $s$ and $s''$ (inclusive).

Next, we argue that the process $p_j$ on team $B$ updates $O$ exactly once in the entire execution. We have already seen that $p_j$ updates $O$ exactly once between $s$ and $s''$ (inclusive). Any run by process $p_j$ that begins after that first update to $O$ by $p_j$ (and therefore after $s$) would see that $R_A \neq \perp$, since the process on team $A$ that performs $s$ writes to $R_A$ before $s$. That run by $p_j$ would therefore pass the test on line 20 and could not update $O$ on line 23.

Thus, any updates to $O$ after $s''$ are by processes in $A$. If there are no updates to $O$ after $s''$, then $O$ remains in state $q_0 \in Q_A$ at all times after $s''$. If there is some update to $O$ after $s''$,
let $s''$ be the first one. Since $q_0 \notin Q_B$ and no process on team $B$ updates $O$ after $s''$, the state of $O$ can never be $q_0$ after $s''$, by Lemma 7. Consider any process $p_i$ on team $A$ that performs an update on $O$ after $s''$. Let $s_i$ be $p_i$’s first update on $O$ after $s''$. By the definition of $s''$, $s_i$ is either $s''$ or after $s''$. Any run by $p_i$ that begins after $s_i$ (and therefore after $s''$) that reads $O$ on line 6 will see a value different from $q_0$, so it does not perform an update on $O$. Thus, no process performs more than one update on $O$ after $s''$. It follows from the definition of $Q_A$ that $O$’s state is in $Q_A$ at all times after $s''$.

Lemma 12. Any output produced by a process on team $A$ is the input value of the team that first updated $O$.

Proof. Consider a run $r$ of the code by a process in $A$ that produces an output. If $r$ reads $O$ at line 6 before $s$, then it will read the value $q_0$ and read $O$ again at line 9, which is after $s$. Thus, the value tested at line 11 is read from $O$ after $s$.

If the first update to $O$ is by a process on team $A$, the value tested is in $Q_A$, by Lemma 10 and 11. So, $r$ outputs the value of $R_A$.

If the first update to $O$ is by a process on team $B$, the value tested is in $Q_B$, by Lemma 10 and the fact that $q_0 \notin Q_B$. Since $Q_A \cap Q_B = \emptyset$, the value tested will not be in $Q_A$. So, $r$ outputs the value of $R_B$.

In both cases, the relevant register is written before $s$, so $r$ outputs the input value of the team that first updates $O$.

Lemma 13. Any output produced by a process on team $B$ is the input value of the team that first updated $O$.

Proof. Consider any run $r$ of the code by a process in $B$ that produces an output. We consider three cases.

Case 1: a process from team $A$ performs $s$.

We first show $r$ returns a value read from $R_A$ by considering two subcases.

(a) $q_0 \in Q_A$. In this case $|B| = 1$, by condition 2 of the definition of $n$-recording. By Lemma 11, $O$’s state is in $Q_A$ at all times, so $r$ cannot return at line 28. Therefore, $r$ outputs the value it reads from $R_A$ at line 21 or 27.

(b) $q_0 \notin Q_A$. By Lemma 10, $O$’s state is in $Q_A$ at all times after $s$. If $r$ reads $O$ at line 18 before $s$, it will see $q_0$ and execute the test at line 20. Then, it will either return the value in $R_A$ at line 21 or read $O$ again at line 24 after $s$, getting a value in $Q_A$ and returning the value in $R_A$ at line 27.

To derive a contradiction, suppose $R_A$ is still $\perp$ when $r$ reads it at line 21 or 27. Then, $r$ returns before $s$, since $R_A$ must be written before $s$. So $r$ must have read $q_0$ from $O$ at line 18. Thus, the test at line 19 is true and the test at line 20 is false, so $r$ performs an update on $O$ before $s$, contradicting the definition of $s$.

Therefore, $r$ outputs team $A$’s input value, as required.

Case 2: A process from team $B$ performs $s$ and $|B| > 1$. Since $q_0 \notin Q_B$, it follows from Lemma 10 that $O$’s state is in $Q_B$ at all times after $s$. If $r$ reads $O$ at line 18 before $s$, it will see $q_0$ and execute the test at line 20, which fails because $|B| > 1$. Then, it will read $O$ again at line 24 after $s$, getting a value in $Q_B$ and return the value in $R_B$ at line 28.

Since $r$ wrote $R_B$ at line 17, $r$ outputs team $B$’s input value, as required.
Case 3: A process from team \( B \) performs \( s \) and \( |B| = 1 \). Let \( p_j \) be the unique process on team \( B \). By Lemma 10 and the fact that \( q_0 \notin B \), the state of \( O \) is in \( Q_B \) at all times after \( s \).

If \( r \) is the run of \( p_j \) that performs \( s \), then \( r \) sees \( R_A = \bot \) on line 20; otherwise it would not execute line 23. So, if \( r \) returns a value, it reads \( O \) at line 24 after \( s \) and gets a value in \( Q_B \). It must then return a value at line 28.

Any run \( r \) of \( p_j \) that ends before \( s \) evaluates the test at line 19 to true and the test at line 20 to false, so it must crash before reaching line 23 and does not produce an output.

If \( r \) is a run of \( p_j \) that starts after \( s \), it reads a value in \( Q_B \) at line 18. Since \( q_0 \notin Q_B \), it would return at line 28.

Thus, all outputs by \( p_j \) are read from \( R_B \) at line 28, which contains team \( B \)'s input value written at line 17.

Lemmas 12 and 13 prove validity and agreement when some process updates \( O \), completing the proof of Theorem 8.

\[ \square \]

### 3.2 Necessary Condition

In this section, we show that being \((n-1)\)-recording is a necessary condition for a deterministic type to be capable of solving \( n \)-process RC. This result holds whether the type is readable or not. The proof uses a valency argument [17]. Assuming an algorithm exists, the valency argument constructs an infinite execution in which no process ever returns a value. Unfortunately, in the case of RC, it is possible to have an infinite execution where no process returns a value (if infinitely many crashes occur). Thus, the proof considers a restricted set of executions where each execution must produce an output value for some process within a finite number of steps, and uses this restricted set to define valency. This technique was used by Golab [20] to prove a necessary condition (weaker than the 2-recording property) for solving 2-process RC. Lo and Hadzilacos [30] had previously used a similar technique of defining valency using a pruned execution tree. Attiya, Ben-Baruch and Hendler [3] also used a valency argument in the context of non-volatile memory in their proof that a recoverable test-and-set object cannot be built from ordinary test-and-set objects (and registers).

**Theorem 14.** For \( n \geq 3 \), if a deterministic type \( T \) can be used, together with registers, to solve recoverable consensus among \( n \) processes, then \( T \) is \((n-1)\)-recording.

**Proof.** Assume there is an algorithm \( A \) for RC among \( n \) processes \( p_1, \ldots, p_n \) using objects of type \( T \) and registers. Let \( \mathcal{E}_A \) be the set of all executions of \( A \) where \( p_2, \ldots, p_n \) never crash, and in any prefix of the execution, the number of crashes of \( p_1 \) is less than or equal to the total number of steps of \( p_2, \ldots, p_n \).

Consider a finite execution \( \gamma \) in \( \mathcal{E}_A \). Define \( \gamma \) to be \( v \)-valent if there is no decision different from \( v \) in any extension of \( \gamma \) in \( \mathcal{E}_A \). An execution \( \gamma \) cannot be both \( v \)-valent and \( v' \)-valent if \( v \neq v' \), since a failure-free extension of \( \gamma \) must eventually produce a decision. We call \( \gamma \) univalent if it is \( v \)-valent for some \( v \), or multivalent otherwise.

To see that a multivalent execution exists, consider an execution with no steps where processes \( p_1 \) and \( p_2 \) have inputs 0 and 1. If \( p_1 \) runs by itself, it must output 0; if \( p_2 \) runs by itself it must output 1.

Next, we argue that there is a critical execution \( \gamma \), i.e., a multivalent execution in \( \mathcal{E}_A \) such that every extension of \( \gamma \) in \( \mathcal{E}_A \) is univalent. If there were not, we could construct an infinite execution of \( \mathcal{E}_A \) in which every prefix is multivalent, meaning that no process ever returns a value. Such an execution could be constructed inductively by starting with a multivalent execution and, at each step of the induction, extending it to a longer multivalent execution.
This would violate the termination property of RC, since some process takes an infinite number of steps without crashing.

For $1 \leq i \leq n$, let $v_i$ be the value such that $\gamma$ followed by the next step of $p_i$’s algorithm is $v_i$-valent. We show not all of $v_1, \ldots, v_n$ are the same. To derive a contradiction, suppose they are all equal. Since $\gamma$ is multivalent, some extension of $\gamma$ in $\mathcal{E}_A$ is $v'$-valent for some $v' \neq v_2$. By assumption, the next step of each process’s algorithm produces a $v_2$-valent execution, so the $v'$-valent extension must begin with a crash of $p_1$. But the extensions of $\gamma$ shown in Figure 3(a) are indistinguishable to $p_2$. Thus, $p_2$ returns the same value in both, contradicting the fact that one extends a $v_2$-valent execution and the other extends a $v'$-valent execution, where $v_2 \neq v'$.

![Figure 3: Proof of Theorem 14. Circles represent states of the system. Squares represent the state of $O$.](image)

A standard argument shows that at the end of $\gamma$, each process is about to perform an operation on the same object $O$ of type $T$, and that step cannot be a read operation. For $i \in \{1, \ldots, n\}$, let $op_i$ be the update operation that $p_i$ is poised to perform on $O$ after $\gamma$. Let $q_0$ be the state of $O$ at the end of $\gamma$.

We next prove a technical lemma that will be used several times to complete the theorem’s
proof. It captures a valency argument we use: if two sequences of steps by distinct processes chosen from \( p_1, \ldots, p_n \) after \( \gamma \) can take \( O \) to the same state and process \( p_1 \) can crash after both of them, then the two extensions must have the same valency. To ensure that \( p_1 \) can crash, the hypothesis of the lemma requires that neither sequence consists of a single step by \( p_1 \).

**Lemma 15.** Suppose there is a sequence of distinct process ids \( i_1, \ldots, i_\alpha \) and another sequence of distinct ids \( j_1, \ldots, j_\beta \) such that each sequence contains an element of \( \{2, \ldots, n\} \) and the sequences of operations \( op_{i_1}, \ldots, op_{i_\alpha} \) and \( op_{j_1}, \ldots, op_{j_\beta} \) both take object \( O \) from state \( q_0 \) to the same state \( q \). Then, \( v_{i_1} = v_{j_1} \).

**Proof.** The two executions in Figure 3(b) are in \( E_A \) since one of \( p_2, \ldots, p_n \) takes a step in each extension of \( \gamma \) before \( p_1 \) crashes. \( O \) is in state \( q \) before \( p_1 \) crashes in both extensions, and no other shared object changes between the end of \( \gamma \) and the crash of \( p_1 \). Thus, these two extensions are indistinguishable to the last run \( \phi \) of \( A \) by \( p_1 \). Since the left extension is \( v_{i_1} \)-valent and the right extension is \( v_{j_1} \)-valent, we must have \( v_{i_1} = v_{j_1} \). □

We now describe how to split \( n - 1 \) of the processes into two teams \( A \) and \( B \) according to their valency to satisfy the definition of \((n - 1)\)-recording. The following two cases describe how to relabel the processes (if necessary) so that we can split processes \( p_1, \ldots, p_{n-1} \) into the two required teams.

**Case 1:** Suppose there is an \( i \) such that, for all \( j \neq i \), \( v_i \neq v_j \). Without loss of generality, assume that \( i < n \). (If \( i = n \), we can swap the ids of \( p_2 \) and \( p_n \) to ensure \( i < n \), since \( n \geq 3 \).) Let \( A = \{p_i\} \) and \( B = \{p_1, \ldots, p_{n-1}\} - \{p_i\} \).

**Case 2:** Suppose that for every \( i \), there is a \( j \neq i \) such that \( v_i = v_j \). If there is a sequence of distinct ids \( i_1, \ldots, i_\alpha \) chosen from \( \{1, \ldots, n\} \) such that the sequence of operations \( op_{i_1}, \ldots, op_{i_\alpha} \) take the object \( O \) from state \( q_0 \) back to \( q_0 \), then let \( \ell = i_1 \). Otherwise, let \( \ell \) be any id. Without loss of generality, assume \( \ell < n \). (If this is not the case, swap the labels of processes \( n - 1 \) and \( n \) to make it true.) Again, without loss of generality, assume \( v_n \neq v_\ell \). (Since not all of \( v_1, \ldots, v_n \) are the same, there is some \( \ell' \) such that \( v_{\ell'} \neq v_\ell \). By the assumption of Case 2, we can choose such an \( \ell' > 1 \). If \( \ell' < n \), swap the ids of \( p_{\ell'} \) and \( p_n \). This ensures that \( v_n \neq v_{\ell'} \).

Then, define \( A \) to be \( \{p_i : 1 \leq i \leq n - 1 \text{ and } v_i = v_1\} \) and \( B \) to be \( \{p_i : 1 \leq i \leq n - 1 \text{ and } v_i \neq v_1\} \). It follows from the fact that not all of \( v_1, \ldots, v_n \) are the same and the assumption of Case 2 that both teams are non-empty.

It follows from the definitions of \( A \) and \( B \) that, in either case, they form a partition of the processes \( p_1, \ldots, p_{n-1} \) into two non-empty teams satisfying the following properties:

**P1:** \( v_i \neq v_j \) for all \( p_i \in A \) and \( p_j \in B \), and

**P2:** \( v_i \neq v_n \) for all \( p_i \in A \).

We check that \( Q_A(q_0, op_1, \ldots, op_{n-1}) \) and \( Q_B(q_0, op_1, \ldots, op_{n-1}) \) satisfy the definition of \((n - 1)\)-recording.

To derive a contradiction, suppose there is a state \( q \in Q_A \cap Q_B \). This means there is a sequence of distinct process ids \( i_1, \ldots, i_\alpha \) chosen from \( \{1, \ldots, n-1\} \) with \( p_{i_1} \in A \) and another sequence of distinct process ids \( j_1, \ldots, j_\beta \) chosen from \( \{1, \ldots, n-1\} \) with \( p_{j_1} \in B \) such that the sequences \( op_{i_1}, \ldots, op_{i_\alpha} \) and \( op_{j_1}, \ldots, op_{j_\beta} \) both take object \( O \) from state \( q_0 \) to state \( q \). Adding one more operation \( op_n \) to the end of these sequences would leave \( O \) in the same state \( q' \). (See Figure 3(c).) By Lemma 15, \( v_{i_1} = v_{j_1} \). This contradicts property P1. Thus, condition 1 of the definition of \((n - 1)\)-recording holds.
To derive a contradiction, suppose \( q_0 \in Q_A \). Then, there is a sequence of distinct process ids \( i_1, \ldots, i_\alpha \) chosen from \( \{1, \ldots, n-1\} \) with \( p_{i_1} \in A \) such that the sequence of operations \( op_{i_1}, \ldots, op_{\alpha} \) takes object \( O \) from state \( q_0 \) back to state \( q_0 \). The two sequences of operations on \( O \) shown in Figure 3(d) both leave \( O \) in the same state. Thus, \( v_{i_1} = v_{n} \), by Lemma 15 contradicting property \( P_2 \). Thus, condition 2 of the definition of \( (n-1) \)-recording is satisfied.

To derive a contradiction, suppose \( q_0 \in Q_B \) and \( |A| > 1 \). Since \( |A| > 1 \), the teams must have been defined according to Case 2. Since \( q_0 \in Q_B \), there is a sequence of distinct process ids \( j_1, \ldots, j_\beta \) chosen from \( \{1, \ldots, n-1\} \) with \( p_{j_1} \in B \) such that \( op_{j_1}, \ldots, op_{j_\beta} \) takes object \( O \) from state \( q_0 \) back to \( q_0 \). So, in Case 2 of the definition of the teams, we chose \( \ell = i_1 \), where \( i_1, \ldots, i_\alpha \) is some sequence of distinct process ids chosen from \( \{1, \ldots, n\} \) such that \( op_{i_1}, \ldots, op_{\alpha} \) also takes object \( O \) from state \( q_0 \) back to \( q_0 \). Since \( i_1 = \ell \leq n-1 \), we have \( p_{i_1} \in A \). (We remark that this sequence’s existence does not contradict the fact proved above that \( q_0 \notin Q_A(q_0, op_1, \ldots, op_{n-1}) \), since this sequence may include the index \( n \).)

Our goal is to show that \( v_{i_1} = v_{j_1} \), which will contradict property \( P_1 \). We use a case argument, showing that it is possible to apply Lemma 15 in each case. Let \( I = \{k: 2 \leq k \leq n \text{ and } v_k = v_{i_1}\} \) and let \( J = \{k: 2 \leq k \leq n \text{ and } v_k = v_{j_1}\} \). A step by a process whose index is in \( I \) or \( J \) extends the critical execution \( \gamma \) to a \( v_{i_1} \)- or \( v_{j_1} \)-valent execution, respectively. Moreover, a step by any process in \( I \) or \( J \) allows us to invoke Lemma 15 since the sets \( I \) and \( J \) do not include 1.

Case a: Suppose some \( k \in I \) does not appear in \( i_1, \ldots, i_\alpha \). Then, the two sequences of operations on \( O \) in Figure 3(e) leave \( O \) in the same state. Since \( k \geq 2 \), Lemma 15 implies that \( v_{i_1} = v_k \). By definition of \( J \), \( v_k = v_{j_1} \). Thus, \( v_{i_1} = v_{j_1} \).

Case b: Suppose there is some \( k \in I \) that does not appear in \( j_1, \ldots, j_\beta \). By an argument symmetric to Case a, \( v_{i_1} = v_{j_1} \).

Case c: Suppose \( J \subseteq \{i_1, \ldots, i_\alpha\} \) and \( I \subseteq \{j_1, \ldots, j_\beta\} \). We first argue that \( I \) is non-empty. If \( i_1 > 1 \), then \( i_1 \in I \). Otherwise, \( i_1 = 1 \) and by the assumption of Case 2, there is some other process id \( k \) such that \( v_k = v_{i_1} \) and this \( k \) is in \( I \). A symmetric argument can be used to show that \( J \) is non-empty. Thus, both of the sequences \( i_1, \ldots, i_\alpha \) and \( j_1, \ldots, j_\beta \) contain at least one of the ids in \( \{2, \ldots, n\} \). Since both sequences of operations shown in Figure 3(f) leave \( O \) in the same state \( q_0 \), it follows from Lemma 15 that \( v_{i_1} = v_{j_1} \).

In all three cases, \( v_{i_1} = v_{j_1} \), contradicting Property \( P_1 \). Thus, condition 3 of the definition of \( (n-1) \)-recording holds.

In proving that \( T \) is \( (n-1) \)-recording, we split \( n-1 \) of the processes into two teams according to the valency induced by their next step after the critical execution and assigned each process the operation they perform in this step. To show that these choices satisfy the definition of \( (n-1) \)-recording, it was essential to have one process \( p_n \) “in reserve” that we could use to take one step in Figures 3(c) and 3(d). This step enables the crash of \( p_1 \) needed to prove Lemma 15 which shows that the two executions in those figures lead to the same outcome, thereby deriving the necessary contradiction. This is the reason we show that being \( (n-1) \)-recording (rather than \( n \)-recording) is necessary for solving RC.

### 3.3 Relationship Between Consensus and Recoverable Consensus

Next, we prove a relationship between the characterizations of types that solve consensus and those that solve RC.

**Theorem 16.** For \( n \geq 4 \), if a deterministic type \( T \) is \( n \)-discerning, then it is \( (n-2) \)-recording.
Proof. Let \( q_0, A, B, op_1, \ldots, op_n \) be chosen to satisfy the definition of \( n \)-discerning. Without loss of generality, assume that \( \{p_1, \ldots, p_{n-2}\} \) includes at least one process from each of \( A \) and \( B \), and that \( \{p_{n-1}, p_n\} \) includes at least one process from each team that contains more than one process. (The ids of the processes can be permuted to make this true.) We partition the processes \( \{p_1, \ldots, p_{n-2}\} \) into two non-empty teams \( A' = A \cap \{p_1, \ldots, p_{n-2}\} \) and \( B' = B \cap \{p_1, \ldots, p_{n-2}\} \).

We show that \( Q_{A'}(q_0, op_1, \ldots, op_{n-2}) \) and \( Q_{B'}(q_0, op_1, \ldots, op_{n-2}) \) satisfy the definition of \((n - 2)\)-recording.

To derive a contradiction, assume \( Q_{A'} \cap Q_{B'} \) contains some state \( q \). Then, there are sequences \( i_1, \ldots, i_n \) and \( j_1, \ldots, j_n \), each of distinct ids from \( \{1, \ldots, n - 2\} \), such that \( p_{i_j} \in A, p_{j_i} \in B \) and the sequences \( op_{i_1}, \ldots, op_{i_n} \) and \( op_{j_1}, \ldots, op_{j_n} \) both take an object of type \( T \) from state \( q_0 \) to \( q \). Operation \( op_i \) takes the object from state \( q \) to some state \( q' \) and returns some response \( r \). By adding \( op_n \) to the end of each of the two sequences, we see the pair \((r, q')\) is in both \( R_{A,n} \) and \( R_{B,n} \) in the definition of \( n \)-discerning, a contradiction. Thus, condition 4 of the definition of \((n - 2)\)-recording is satisfied.

To derive a contradiction, assume \( q_0 \in Q_{A'} \) and \( |B'| > 1 \). Since \( |B| \geq |B'| > 1 \), some process \( p_j \) is in \( B \cap \{p_{n-1}, p_n\} \). Operation \( op_j \) takes an object of type \( T \) from \( q_0 \) to some state \( q \) and returns some response \( r \). Thus, \((r, q)\) is in the set \( R_{B,j} \) of the definition of \( n \)-discerning. Since \( q_0 \in Q_{A'} \), there is a sequence \( i_1, \ldots, i_n \) of distinct ids chosen from \( \{1, \ldots, n - 2\} \) such that \( p_{i_1} \in A \) and the sequence \( op_{i_1}, \ldots, op_{i_n} \) takes an object of type \( T \) from state \( q_0 \) back to the state \( q_0 \). By adding \( op_j \) to the end of this sequence, we see that the pair \((r, q)\) is also in \( R_{A,j} \), contradicting the fact that \( R_{A,j} \cap R_{B,j} \) must be empty, according to the definition of \( n \)-discerning. Thus, condition 5 of the definition of \((n - 2)\)-recording is satisfied.

The proof of condition 3 is symmetric. \( \square \)

**Corollary 17.** A deterministic, readable object type \( T \) with consensus number at least \( n \) can solve recoverable consensus among \( n - 2 \) processes. Thus, cons\((T) - 2 \leq rcons(T) \leq cons(T)\).

The first inequality in the corollary is a consequence of Theorem 8 and 16. The second inequality follows from the fact that any algorithm that solves RC is also an algorithm that solves consensus.

For \( n = 3 \), we can strengthen Theorem 16 and Corollary 17 as follows. See Appendix D for the full version 13 for the proof.

**Proposition 18.** If a deterministic, readable type is \( 3 \)-discerning, then it is \( 2 \)-recording. Thus, if cons\((T) = 3 \) then \( 2 \leq rcons(T) \leq 3 \).

The following example shows that Theorem 16 cannot be strengthened when \( n > 3 \).

**Proposition 19.** For all \( n \geq 4 \), there is a type that is \( n \)-discerning, but is not \((n - 1)\)-recording.

A complete proof is in Appendix 14 and 13. We sketch it here. We define a type \( T_n \) whose set of states is \( \{(\text{winner}, \text{row}, \text{col}) : \text{winner} \in \{A, B\}, 0 \leq \text{row} < \lceil n/2 \rceil, 0 \leq \text{col} < \lfloor n/2 \rfloor\} \cup \{(\perp, 0, 0)\} \). \( T_n \) has two operations \( op_A \) and \( op_B \), and a read operation. Intuitively, if the object is initialized to \( (\perp, 0, 0) \), \( \text{winner} \) keeps track of whether the first update was \( op_A \) or \( op_B \), while \( \text{col} \) and \( \text{row} \) store the number of times \( op_A \) and \( op_B \) have been applied. If \( op_A \) is performed more than \( \lceil n/2 \rceil \) times or \( op_B \) is performed more than \( \lfloor n/2 \rfloor \) times, the object “forgets” all the information it has stored by going back to state \( (\perp, 0, 0) \). It is easy to verify that \( T_n \) is \( n \)-discerning but not \((n - 1)\)-recording.

It follows easily from Proposition 19 combined with Theorems 3 and 13 that there are readable types whose RC numbers are strictly smaller than their consensus numbers.

**Corollary 20.** For all \( n \geq 4 \), there is a deterministic, readable type \( T_n \) such that \( rcons(T_n) < cons(T_n) = n \).
On the other hand, there are also types whose RC numbers are equal to their consensus numbers. The next proposition also shows that every level of the RC hierarchy is populated, since there are types with consensus number $n$ for all $n$.

**Proposition 21.** For all $n$, there is a deterministic, readable type $S_n$ such that $rcons(S_n) = cons(S_n) = n$.

A complete proof is in Appendix E [13]. We sketch it here. We define a type $S_n$ whose set of possible states is $\{(\text{winner, row}) : \text{winner} \in \{A, B\}, 0 \leq \text{row} < n\}$. $S_n$ has two operations $op_A$ and $op_B$, and a read operation. Intuitively, if the object is initialized to $(B, 0)$, and then accessed by update operations, $\text{winner}$ records whether the first update was $op_A$ or $op_B$ and $\text{row}$ counts the number of times $op_B$ has been applied. If $op_A$ is performed more than once or if $op_B$ is performed more than $n - 1$ times, then the object “forgets” all the information it has stored by going back to state $(B, 0)$. It is fairly straightforward to check that $S_n$ is $n$-recording, but is not $(n + 1)$-discerning. Thus, $n \leq rcons(S_n) \leq cons(S_n) \leq n$.

### 3.4 Recoverable Consensus Using Several Types

The (recoverable) consensus number of a set $T$ of object types is the maximum number of processes that can solve (recoverable) consensus using objects of those types, together with registers (or $\infty$ if there is no such maximum). A classic open question, originally formulated by Jayanti [27], is whether the standard consensus hierarchy is robust for deterministic types, i.e., whether $\text{cons}(T) = \max\{\text{cons}(T) : T \in T\}$. If this equation holds so, it is possible to study the power of a system equipped with multiple types by studying the power of each type individually. See [16, Section 9] for some history of the robustness question. Ruppert’s characterization (Theorem [3]) was used to show the consensus hierarchy is robust for the class of deterministic, readable types. Similarly, our characterization allows us to show how the power of a set of deterministic, readable types to solve RC is related to the power of the individual types.

**Theorem 22.** Let $T$ be a non-empty set of deterministic, readable types and suppose $n = \max\{rcons(T) : T \in T\}$ exists. Then, $n \leq rcons(T) \leq n + 1$. (If $\max\{rcons(T) : T \in T\}$ does not exist, then $rcons(T) = \infty$.)

**Proof.** If $\max\{rcons(T) : T \in T\}$ does not exist, then for any $n$ there is an algorithm that solves RC using some type $T_n \in T$. It follows that $rcons(T) = \infty$. So for the remainder of the proof, assume the maximum does exist.

It follows from the definition that $rcons(T) \geq rcons(T)$ for all $T \in T$. Thus, $rcons(T) \geq \max\{rcons(T) : T \in T\}$.

We prove the other inequality by contradiction. Suppose $(n + 2)$-process RC can be solved using types in $T$. As in the proof of Theorem [14] there is a critical execution $\gamma$ at the end of which each process is about to update the same object $O$ of some type $T \in T$. As in that proof, $T$ is $(n + 1)$-recording. By Theorem [5], there is an $(n + 1)$-process RC algorithm using objects of type $T$ and registers. So, $rcons(T) \geq n + 1 > n \geq rcons(T)$, a contradiction.

### 4 The Significance of Recoverable Consensus

Herlihy’s universal construction [25] builds a linearizable, wait-free implementation of any shared object using a consensus algorithm as a subroutine. It creates a linked list of all operations performed on the implemented object, and this list defines the linearization ordering. Berryhill, Golab and Tripunitara [6] observed that this result extends to the model with simultaneous crashes, simply by placing the linked list in non-volatile memory and using RC in place of consensus. Their model allows a part of shared memory to be volatile. Using that volatile
memory, their universal construction provides strictly linearizable implementations. (Strict linearity [1] is similar to linearizability, with the requirement that an operation in progress when a process crashes is either linearized before the crash or not at all.) Without volatile shared memory, the history satisfies only the weaker property of recoverable linearizability (proposed in [3], with a correction to the definition in [29]).

Similarly, we observe that Herlihy’s universal construction also extends to the independent crash model. To execute an operation \( op \), a process creates a node \( nd \) containing \( op \) (including its parameters). Then, it announces \( op \) by storing a pointer to \( nd \) in an announcement array. Other processes can then help add \( op \) to the list, ensuring wait-freedom. Processes use an instance of consensus to agree on the next pointer of each node in the list. A process executes a routine \texttt{Perform} that traverses the list. At each visited node, it proposes a value from the announcement array to the consensus algorithm for the node’s next pointer, until it discovers its own operation’s node \( nd \) has been appended. Processes choose which announced value to propose so that each process’s announced value is given priority in a round-robin fashion. This ensures each announced node is appended within a finite number of steps.

In our setting, all shared variables are non-volatile, and we use an algorithm for RC (such as the one in Section 3.1) in place of consensus. For simplicity, we use a standard assumption (as in, e.g., [8, 10, 11, 18, 19]): when a process recovers from a crash, it executes a recovery function. This assumption is not restrictive; we could, alternatively, add the code of the recovery function at the beginning of the universal algorithm, thus forcing every process to execute this code before it actually starts executing a new operation. When a process \( p \) crashes and recovers, the recovery function checks if the last operation that \( p \) announced before crashing has been appended in the list and if not, it executes the code to append it. Specifically, the recovery function simply calls \texttt{Perform} for the last announced node of \( p \). See Appendix F for pseudocode of the recoverable universal construction \texttt{RUniversal}.

As in Herlihy’s construction, the helping mechanism of \texttt{RUniversal} ensures wait-freedom. The recoverable implementations obtained using \texttt{RUniversal} satisfy nesting-safe recoverable linearizability (NRL) [3], which requires that a crashed operation is linearized within an interval that includes its crashes and recovery attempts. NRL implies detectability [3] which ensures that a process can discover upon recovery whether or not its last operation took effect, and guarantees that if it did, its response value was made persistent. Other well-known safety conditions for the crash-recovery setting include durable linearizability [26], which has been proposed for the system-crash failures model and requires that the effects of all operations that have completed before a crash are reflected in the object’s state upon recovery, and persistent linearizability [24], which has been proposed for a model where no recovery function is provided and requires that an operation interrupted by a crash can be linearized up until the invocation of the next operation by the same process. With minor adjustments these conditions are meaningful in our setting and \texttt{RUniversal} satisfies both of them.

Moreover, \texttt{RUniversal} has the following desirable property. Suppose an implementation \( I \) uses a linearizable object \( X \) in a system with halting failures, but no crash-recovery failures. We can transform \( I \) to an implementation \( I’ \) by replacing every instance of \( X \) in \( I \) with an invocation of \texttt{RUniversal} (that implements \( X \)). Then, every trace produced by \( I’ \) in a system with crash and recovery failures is also a trace of \( I \) using a linearizable object \( X \) in a system with halting failures. In this way, any algorithm designed for the standard asynchronous model with halting failures can be automatically transformed to another algorithm to run in the independent crash-recovery model, as long as we can solve RC.

The traditional consensus hierarchy gives us information about which implementations are possible (via universality), but also tells us some implementations are impossible. This is another reason to study the consensus hierarchy. Specifically, if \( \text{cons}(T_1) < \text{cons}(T_2) \), then there is no wait-free implementation of object type \( T_2 \) from objects of type \( T_1 \) for more than \( \text{cons}(T_1) \) processes [25]. We give an analogous result for the RC hierarchy. The proof is in
Appendix G. For the proof, see [13].

Theorem 23. Let \( n \leq \text{rcons}(T_2) \). If there is a wait-free, persistently linearizable implementation of \( T_2 \) from atomic objects of type \( T_1 \) (and registers) in a system of \( n \) processes with independent crashes, then \( \text{rcons}(T_1) \geq n \).

Corollary 24. If \( \text{rcons}(T_1) < \text{rcons}(T_2) \) then there is no wait-free, persistently linearizable implementation of \( T_2 \) from atomic objects of type \( T_1 \) and registers in a system of more than \( \text{rcons}(T_1) \) processes with independent crashes.

5 Discussion

In this paper, we studied solvability, without considering efficiency. A lot of research has focused on designing efficient recoverable transactional memory systems [34, 8, 7, 32, 9, 4] and recoverable universal constructions [11, 15]. Wait-free solutions appear in [32, 11, 14]. Some [11, 15] are based on existing wait-free universal constructions [12, 14] for the standard shared-memory model with halting failures. All except [15], satisfy weaker consistency conditions than nesting-safe recoverable linearizability. Attiya et al. [3] gave a recoverable implementation of a Compare&Swap (CAS) object. Any concurrent algorithm from read/write and CAS objects can become recoverable by replacing its CAS objects with their recoverable implementation [3]. Capsules [5] can also be used to transform concurrent algorithms that use only read and CAS primitives to their recoverable versions. Many other general techniques [2, 19, 18] have been proposed for deriving recoverable lock-free data structures from their concurrent implementations.

Our work leaves open several questions. Is there a deterministic, readable type \( T \) with \( \text{rcons}(T) = \text{cons}(T) - 2 \)? We saw in Corollary 17 that \( \text{cons}(T) – \text{rcons}(T) \) can be at most 2 for deterministic, readable types. How big can this difference be for non-readable types?

It would be nice to close the gap between the necessary condition of being \((n-1)\)-recording and the sufficient condition of being \(n\)-recording for the solvability of RC using deterministic, readable types. Perhaps a good starting point is to determine whether being 2-recording is actually necessary for solving 2-process RC. Finally, it would be interesting to characterize read-modify-write types capable of solving \( n \)-process RC (as was done in [33] for the standard consensus problem), and see whether the RC hierarchy is robust for deterministic, readable types (or for all deterministic types).

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A Proof of Theorem 1

31 shared variables:
32 array $Round[1..n]$ of registers, initially 0
33 array $D[1..\infty]$ of registers, initially \bot

34 $\text{Decide}(v)$
35 \hspace{1em} $\text{pref} \leftarrow v$
36 \hspace{1em} $r \leftarrow 1$
37 \hspace{1em} loop
38 \hspace{2em} if $Round[j] < r$ then
39 \hspace{3em} $Round[j] \leftarrow r$
40 \hspace{3em} if $r > 1$ and $D[r - 1] \neq \bot$ then
41 \hspace{4em} $\text{pref} \leftarrow D[r - 1]$
42 \hspace{3em} end if
43 \hspace{3em} $\text{pref} \leftarrow C_r.\text{Decide}(\text{pref})$
44 \hspace{3em} $D[r] \leftarrow \text{pref}$
45 \hspace{3em} if $\forall k, \text{Round}[k] \leq r$ then
46 \hspace{4em} return $\text{pref}$
47 \hspace{3em} end if
48 \hspace{3em} else if $r > 1$ and $D[r - 1] \neq \bot$ then
49 \hspace{4em} $\text{pref} \leftarrow D[r - 1]$
50 \hspace{3em} end if
51 \hspace{2em} $r \leftarrow r + 1$
52 \hspace{1em} end loop
53 end $\text{Decide}$

Figure 4: Algorithm for process $p_j$ to solve recoverable consensus with simultaneous crashes using instances of standard consensus $C_1, C_2, \ldots$

We must show that RC is solvable for $n$ processes with simultaneous crashes if and only if the standard consensus problem is solvable for $n$ processes. The “only if” direction is trivial, since any algorithm for RC also solves consensus: just consider executions in which there are no crashes.

We prove the converse using the algorithm shown in Figure 4 which is similar to the algorithm in [20] for a bounded number of crashes. The RC algorithm uses multiple instances of the consensus algorithm, denoted $C_1, C_2, \ldots$. Each process attempts to access $C_1, C_2, \ldots$ in turn, until it receives a result from one of them and sees that no process has yet moved on to the next object. The register $D[r]$ is used to record the output of $C_r$. An iteration of the loop with $r = i$ is called a run of round $i$. A process may run round $i$ multiple times if it crashes and recovers. Process $p_j$ records in register $Round[j]$ the largest $r$ for which $p_j$ has started to run round $r$. This variable is used to ensure that $p_j$ does not access $C_r$ a second time if it crashes during a run of round $r$.

Lemma 25. The algorithm in Figure 4 satisfies the recoverable wait-freedom property.

Proof. To derive a contradiction, suppose there is a time $t$ after which there are no more crashes and processes continue to take steps without any process terminating. Let $i$ be greater than the maximum entry in $Round[1..n]$ at time $t$. Any process that executes enough steps after $t$ will either terminate, crash or reach round $i$. The processes that reach round $i$ without crashing will satisfy the test at line 38. Among those processes, the first to complete line 45 will satisfy the test and terminate, a contradiction.

Observation 26. For any $j$, the value in $Round[j]$ only increases.
Proof. Only process $p_j$ writes to $Round[j]$. Moreover, $p_j$ writes a value $r$ to $Round[j]$ (at line 39) only after seeing (at line 38) that the current value of $Round[j]$ is less than $r$. \hfill \Box

Lemma 27. For each $i$, no process invokes \textsc{Decide} on $C_i$ more than once.

Proof. Before $p_j$ invokes \textsc{Decide} on $C_i$ for the first time, $p_j$ writes $i$ to $Round[j]$. By Observation 26, $Round[j] \geq i$ at all times after that invocation. Thus, any subsequent test by $p_j$ at line 38 of a run of round $i$ will fail, and $p_j$ will never invoke \textsc{Decide} on $C_i$ again. \hfill \Box

Since $C_i$ is accessed correctly, even if processes crash and recover, it follows that calls to $C_i$, \textsc{Decide} cannot return different values and that the common output value is one of the input values to one of the calls to $C_i$.\textsc{Decide}. In particular, this means that all values written to $D[i]$ at line 44 are identical.

Lemma 28. Let $P_0$ be the set of input values to the RC algorithm. For $r \geq 1$, let $P_r$ be the set of all values that are either written into $D[r]$, or stored in the local variable $pref$ of some process when it completes a run of round $r$ by terminating at line 46 or reaching the end of the iteration of the loop. For $r \geq 1$, $P_r \subseteq P_{r-1}$.

Proof. Let $P_r'$ be the set of all values that are in the local variable $pref$ of some process at some time during a run of round $r$ by that process. Since $P_r \subseteq P_r'$, we must simply show that $P_r' \subseteq P_{r-1}$. Let $v \in P_r'$. Setting $pref$ at line 43 does not add any new value to $P_r'$, by the validity property of the consensus algorithm $C_r$. Thus, some process's $pref$ was either equal to $v$ at the beginning of a run of round $r$, or set to $v$ at line 41 or 49. We show that in each of these cases, $v \in P_{r-1}$. If $r = 1$ and $pref$ is $v$ at the start of a run of round 1, then $pref$ was set to $v$ at line 38 so $v \in P_0$. If $r > 1$ and $pref$ is $v$ at the start of a run of round $r$, then $pref$ was $v$ at the end of $p_j$'s run of round $r-1$, so $v \in P_{r-1}$. If $pref$ is set to $v$ at line 41 or 49, then $r > 1$ and $v$ was stored in $D[r-1]$, so $v \in P_{r-1}$. \hfill \Box

The validity property follows from Lemma 28 if a process returns $x$ at line 46 in a run of round $r$, then $x \in P_r \subseteq P_0$. Thus, all output values are in the set $P_0$ of input values.

Lemma 29. The algorithm in Figure 4 satisfies the agreement property.

Proof. Consider an execution in which some processes return a value. Let $i$ be the minimum number such that some process returns a value at line 46 of a run of round $i$. Any process that returns a value at line 46 during a run of round $i$ returns the value produced by $C_i$ at line 43. By Lemma 27 and the agreement property of $C_i$, all such processes return the same value $x$, and $x$ is the only non-$\bot$ value that can ever be stored in $D[i]$.

The process that returns a value $x$ at line 46 in its run of round $i$ first writes $x$ into $D[i]$ at line 44 and then reads values less than or equal to $x$ from each entry of $Round[1..n]$ at line 45. By Observation 26, no entry of $Round$ is greater than $i$ at any time before $x$ is first written into $D[i]$.

We show that $P_{i+1}$ (as defined in Lemma 28) can only contain $x$. Consider a process $p_j$ that either writes to $D[i+1]$ or completes a run of round $i+1$. We consider two cases. First, suppose $Round[j] \geq r$ at line 38 of $p_j$'s run of round $i+1$. By the argument of the previous paragraph, $x$ has already been written into $D[i]$, so $p_j$ updates its $pref$ to $x$ at line 49. Otherwise, $Round[j] < r$ at line 38 of a run of round $i+1$. In this case, line 39 must be performed after $x$ has been written to $D[i]$, as argued in the previous paragraph. Thus, $p_j$ updates its $pref$ to $x$ at line 41.

Since all processes $p_j$ that execute round $i+1$ update $pref$ to $x$, it follows that all inputs to $C_{i+1}$ are $x$. By the validity property of $C_{i+1}$, processes can only update their $pref$ to $x$ at line 43 of a run of round $i+1$, and can only write the value $x$ into $D[i+1]$.

Thus, $P_{i+1} \subseteq \{x\}$. Consider any round $i' \geq i+1$. By Lemma 28, $P_{i'} \subseteq P_{i+1} \subseteq \{x\}$. In particular, this means that any value that is returned in any round $i' \geq i+1$ must also be $x$. \hfill \Box
This completes the proof of Theorem 1. The algorithm in Figure 4 uses an unbounded number of instances of consensus. In the full version of [20], Golab showed that this is indeed necessary for such a construction.

B Using Recoverable Team Consensus to Solve Recoverable Consensus

Suppose a collection of processes is partitioned into two non-empty teams. The recoverable team consensus problem is the same as the RC problem, except with the precondition that all input values for processes on the same team must be the same. The following proposition, used in the proof of Theorem 8, can be proved in the same way as the analogous claim in [31, 33] for standard consensus, but we include the proof here for the sake of completeness.

**Proposition 30.** If objects of type $T$ and registers can be used to solve recoverable team consensus among $n$ processes, then objects of type $T$ and registers can be used to solve recoverable consensus among $n$ processes.

**Proof.** Assume we have a recoverable team consensus for $n$ processes divided into two non-empty teams $A$ and $B$. We use induction on $k$ show that $k$ processes can solve RC using objects of type $T$ and registers.

For $k = 1$, this is trivial: a process can simply return its own input value.

Let $1 < k \leq n$. Assume the claim holds for fewer than $k$ processes. We construct an algorithm to solve RC for $k$ processes. Split the $k$ processes into two non-empty teams $A'$ and $B'$ such that $|A'| \leq |A|$ and $|B'| \leq |B|$. We use two RC algorithms $R_{A'}$ and $R_{B'}$ for $|A'|$ and $|B'|$ processes, respectively. These algorithms can be built from objects of type $T$ and registers, by the induction hypothesis since $|A'|$ and $|B'|$ are less than $k$. Each process first runs the RC algorithm for its team, and uses the output from it as the input to a recoverable team consensus algorithm $T_C$ to produce the final output. (Note that the $n$-process recoverable team consensus algorithm still works if only $k$ processes uses it; we think of the other $n - k$ processes simply taking no steps.)

The agreement property of $R_{A'}$ and $R_{B'}$ ensure that the precondition of the recoverable team consensus algorithm is satisfied. The agreement property of the $k$-process RC algorithm follows from the agreement property of $T_C$. The recoverable wait-freedom and validity properties follow from the corresponding properties of $R_{A'}, R_{B'}$ and $T_C$. □

C Proof of Proposition 18

**Proof.** Let $A, B, q_0, op_1, op_2, op_3$ be chosen to satisfy the definition of 3-discerning for the type $T$. Without loss of generality, assume $A = \{p_1\}$ and $B = \{p_2, p_3\}$. Let $A' = \{p_1\}$ and $B' = \{p_2\}$. We show that $A', B', q_0, op_1, op_2$ satisfy the definition of 2-recording. Since $|A'| = |B'| = 1$, conditions 2 and 3 of the definition of 2-recording are trivially satisfied. The argument that condition 1 is satisfied is similar to the proof of Theorem 16 if some state $q$ can be reached by two sequences starting with operations on opposite teams, we can append $op_3$ to the end of each to show that $R_{A,3} \cap R_{B,3} \neq \emptyset$, contradicting the definition of 3-discerning.

Since $T$ is 2-recording, Theorem 8 implies that $2 \leq r_{\text{cons}}(T)$. Moreover $r_{\text{cons}}(T) \leq \text{cons}(T) = 3$, since any algorithm for RC also solves consensus. □

D Proof of Proposition 19

We use the following definitions of Herlihy [25]. Operations $op_i$ and $op_j$ commute from state $q_0$ if the sequences $op_i, op_j$ and $op_j, op_i$ take the object from $q_0$ to the same state $q$. Operation
opA
if winner = ⊥ then
    winner ← A
return A
else
    result ← winner
    col ← (col + 1) mod ⌊n/2⌋
    if col = 0 then
        winner ← ⊥
        row = 0
end if
return result
end if
end opA

opB
if winner = ⊥ then
    winner ← B
return B
else
    result ← winner
    row ← (row + 1) mod ⌈n/2⌉
    if row = 0 then
        winner ← ⊥
        col = 0
end if
return result
end if
end opB

Figure 5: Behaviour of type $T_n$ used in the proof of Proposition 19. A transition diagram for
the object when $n = 6$ is shown, where the $opA$ and $opB$ operations are shown by the horizontal
and vertical arrows leaving each state, respectively. Each transition is labelled by the output
value of the operation that causes the transition.

$op_i$ overwrites $op_j$ from $q_0$ if the sequences $op_i$ and $op_j, op_i$ take the object from $q_0$ to the same
state $q$. It is easy to check that if $q_0, op_1, op_2$ satisfy the definition of 2-recording, then $op_1$ and
$op_2$ cannot commute from $q_0$ nor can one overwrite the other from $q_0$.

Proof. We define the type $T_n$ as follows. The set of possible states is $\{(winner, row, col) :$ winner $\in \{A, B\}, 0 \leq row < \lfloor n/2 \rfloor, 0 \leq col < \lfloor n/2 \rfloor \} \cup \{(⊥, 0, 0)\}$. $T_n$ supports two operations $opA$ and $opB$, as well as a read operation. If an update operation is applied to an object in state (winner, row, col), it executes the code in Figure 5 atomically to update the state and return a
result.

To see that $T_n$ is $n$-discerning, let $q_0 = (⊥, 0, 0)$ and partition processes into team $A$ of size
$\lfloor n/2 \rfloor$ and team $B$ of size $\lceil n/2 \rceil$. Assign $opA$ to all processes on team $A$ and $opB$ to all processes
on team $B$. Then, if any sequence of operations assigned to distinct processes is applied to an
object starting in state $q_0$, every operation’s result will be the name of the team of the process
that performed the first operation in the sequence. This is because the state of the object can
return to $q_0$ after a process on one team takes the first step only after all processes on the other
team have taken a step. Thus, for any $j$, all pairs in $R_{A,j}$ will be of the form $(\{A, \ast\})$ and all pairs
in $R_{B,j}$ will be of the form $(\{B, \ast\})$, so $R_{A,j} \cap R_{B,j} = \emptyset$.

It remains to show that $T_n$ is not $(n-1)$-recording. To derive a contradiction, suppose there
is some $q_0, A, B, op_1, \ldots, op_{n-1}$ that satisfy the definition of $(n-1)$-recording.
If \( q_0 \neq (\bot, 0, 0) \), then any pair of operations either commute or overwrite, so even the definition of 2-recording is not satisfied. So \( q_0 \) must be \((\bot, 0, 0)\). If two processes \( p_i \) and \( p_j \) on opposite teams are assigned the same operation, then \( op_i \) and \( op_j \) would both take an object from state \( q_0 \) to the same state, violating condition \( \text{I} \) of the definition of \((n - 1)\)-recording.

Thus, without loss of generality, all processes on team \( A \) are assigned \( op_A \) and all processes on team \( B \) are assigned \( op_B \).

If \( |A| \geq \lfloor n/2 \rfloor \), then allowing one process on team \( B \) to take a step followed by \( \lceil n/2 \rceil \) processes on team \( A \) would take the object from state \( q_0 \) back to \( q_0 \), so \( q_0 \in Q_B \). This violates condition \( \text{I} \) of the definition of \((n - 1)\)-recording since \( |A| \geq \lfloor 4/2 \rfloor = 2 \). Thus, \( |A| \leq \lfloor n/2 \rfloor - 1 \).

Similarly, if \( |B| \geq \lfloor n/2 \rfloor \), then allowing one process on team \( A \) to take a step followed by \( \lceil n/2 \rceil \) processes on team \( B \) would take the object from state \( q_0 \) back to \( q_0 \), so \( q_0 \in Q_A \). This violates condition \( \text{II} \) of the definition of \((n - 1)\)-recording, since \( |B| \geq \lfloor 4/2 \rfloor = 2 \). Thus, \( |B| \leq \lfloor n/2 \rfloor - 1 \).

Hence, \( n - 1 = |A| + |B| \leq (\lfloor n/2 \rfloor - 1) + (\lceil n/2 \rceil - 1) = n - 2 \), a contradiction. \( \square \)

### E Proof of Proposition 21

**Proof.** For \( n \geq 2 \), let \( S_n \) be a type that provides only a read operation.

For \( n \geq 2 \), we define type \( S_n \) as follows. The set of possible states is \( \{(\text{winner}, \text{row}) : \text{winner} \in \{A, B\}, 0 \leq \text{row} < n\} \). \( S_n \) supports two operations \( op_A \) and \( op_B \), as well as a read operation. If an update operation is applied to an object in state \((\text{winner}, \text{row})\), it executes the code in Figure 8 atomically to update the state and return a result.

We first argue that \( S_n \) is \( n \)-recording. Let \( q_0 = (B, 0) \), \( A = \{p_1\} \), \( B = \{p_2, \ldots, p_n\} \), \( op_1 = op_A \), \( op_2 = op_B = \cdots = op_n = op_B \). Then, \( Q_A = \{(A, \text{row}) : 0 \leq \text{row} < n\} \) and \( Q_B = \{(B, \text{row}) : 0 \leq \text{row} < n\} \) satisfy all the conditions of the definition of \( n \)-recording. It follows from Theorem 8 that \( rcons(S_n) \geq n \).

Next, we argue that \( S_n \) is not \((n + 1)\)-discerning, in order to prove that \( cons(S_n) \leq n \). To derive a contradiction, suppose \( S_n \) is \((n + 1)\)-discerning. Operations assigned to processes on opposite teams cannot commute or overwrite if they are performed on an object initially in state \( q_0 \). (If two operations \( op_1 \) and \( op_j \) assigned to opposite teams commuted, then for the two sequences \( op_1, op_j \) and \( op_j, op_1 \), the operation \( op_1 \) gets the same result ack in both sequences, and both sequences leave the object in the same state. This would violate the definition of \((n + 1)\)-discerning. A similar argument applies for overwriting operations.) It is easy to check that \( q_0 \) must therefore be \((\bot, 0)\) and processes on one team (without loss of generality, team \( A \)) must be assigned the operation \( op_A \) and processes on the other team \( B \) must be assigned the operation \( op_B \). If \( |A| \geq 2 \), the sequences \( op_A, op_A, op_B \) and \( op_B \) performed on an object would both take an object from state \((B, 0)\) to state \((B, 1)\) and would return the same result for \( op_B \), violating the definition of \((n + 1)\) discerning. Thus, there must be just one process on team \( A \). So, \( |B| = n \). Consider the following two sequences of operations.

- All processes on team \( B \) perform \( op_B \) followed by the process on team \( A \) performing \( op_A \).
- The process on team \( A \) performs \( op_A \).

Both sequences take the object from state \( q_0 \) to \((A, 0)\) and return the same result to \( op_A \), violating the definition of \((n + 1)\)-discerning. This contradiction completes the proof. \( \square \)

### F Universal Construction

The pseudocode for our slightly modified version of Herlihy’s universal construction is given in Figure 7. Each list node contains the following fields.
82 $\text{op}_A$
83 if $(\text{winner}, \text{row}) = (B, 0)$ then
84 \hspace{1em} \text{winner} \leftarrow A
85 else
86 \hspace{1em} \text{winner} \leftarrow B
87 \hspace{1em} \text{row} \leftarrow 0
88 end if
89 return ack
90 end $\text{op}_A$

91 $\text{op}_B$
92 \hspace{1em} \text{row} \leftarrow \text{(row} + 1) \mod n
93 if \text{row} = 0 then
94 \hspace{1em} \text{winner} \leftarrow B
95 end if
96 return ack
97 end $\text{op}_B$

Figure 6: Behaviour of type $S_n$ used in proof of Proposition 21. A transition diagram is shown, where the $\text{op}_A$ and $\text{op}_B$ operations are shown by the horizontal and vertical arrows leaving each state, respectively. All operations return ack.
shared variables:

\[ \text{Announce}[1..n] \text{ of registers, each entry initially points to the dummy node at the beginning of the list} \]

\[ \text{Head}[1..n], \text{ each entry initially points to a dummy node at the beginning of the list} \]

**ApplyOperation** // ensures that node Announce[i] is added to the list and returns result of that node’s operation

while Announce[i]->seq = 0 // keep trying until my operation has been added to the list

\[ \text{priority} \leftarrow (\text{Head}[i]->seq + 1) \mod n \text{ / id of process who has priority for next list position} \]

if Announce[priority]->seq = 0 then // check if process with id priority needs help

\[ \text{pointer} \leftarrow \text{Announce}[\text{priority}] \text{ / try to add operation of process with id priority} \]

else

\[ \text{pointer} \leftarrow \text{Announce}[i] \text{ / try to add my own operation} \]

end if

\[ \text{winner} \leftarrow \text{Decide}(\text{Head}[i]->\text{next}, \text{pointer}) \text{ / propose pointer to RC instance associated with next pointer of node} \]

\[ \langle \text{winner}->\text{seq}, \text{winner}->\text{response} \rangle \leftarrow \text{Apply}(\text{winner}->\text{op}, \text{Head}[i]->\text{newState}) \]

\[ \text{Head}[i]->\text{seq} \leftarrow \text{winner} \text{ / advance to next node} \]

end while

return Announce[i]->response

end ApplyOperation

**Universal**(op) // perform op on implemented object and return result

\[ \text{nd} \leftarrow \text{pointer to new list node} \]

\[ \text{nd}->\text{op} \leftarrow \text{op} \text{ / op includes name of operation to apply and its arguments} \]

\[ \text{nd}->\text{seq} \leftarrow 0 \]

\[ \text{Announce}[i] \leftarrow \text{nd} \]

for j ← 0..n - 1 // make sure Head[i] pointer is not too out of date

if Head[j]->seq > Head[i]->seq then

\[ \text{Head}[i] \leftarrow \text{Head}[j] \]

end if

end for

return ApplyOperation

end Universal

**Recover**

return ApplyOperation

end Recover

Figure 7: Universal construction pseudocode for process \( p_i \).

- \( \text{seq} \) is initially 0 but is changed to the node’s position in the list once the node is added to the list.

- \( \text{op} \) is the operation on the implemented object represented by the node; this includes the name of the operation and any arguments to it.

- \( \text{newState} \) is the state of the implemented object after the operations on the list up to and including this node have been applied to it.

- \( \text{response} \) is the result of the operation represented by the node.

- \( \text{next} \) is an instance of RC that will be used to agree upon the next node in the list.

Initially, the list contains a single dummy node whose sequence number is 1 and whose \( \text{newState} \) field stores the initial state of the implemented object.

We remark that a process that crashes and recovers might access the RC instance associated with the \( \text{next} \) pointer of a node multiple times with different input values. So, we should use the
mechanism described in the introduction to mask this behaviour and ensure that the process’s inputs to the RC instance are identical.

G Proof of Theorem 23

Proof. Let \( A \) be an \( n \)-process algorithm for RC that uses atomic objects of type \( T_2 \) and registers. Construct an algorithm \( A' \) by replacing each object of type \( T_2 \) by a persistent linearizable implementation from atomic objects of type \( T_1 \) and registers.

We first argue that \( A' \) satisfies the recoverable wait-freedom property. If a process continues to take steps without crashing, it will eventually complete each operation it calls on a simulated object of type \( T_2 \), since the implementation of \( T_2 \) is wait-free. Thus, it will eventually produce an output, since \( A \) satisfies recoverable wait-freedom.

It remains to show that \( A' \) satisfies the agreement and validity properties of RC. Let \( \alpha' \) be any execution of \( A' \). We construct a corresponding execution \( \alpha \) of \( A \) as follows. Remove all internal steps of the implementation of \( T_2 \) (i.e., all steps of a process between an invocation step on a \( T_2 \) object and its subsequent response or process crash, or to the end of the execution if there is no such response or crash). Each simulated operation on an object of type \( T_2 \) in \( \alpha' \) that is not linearized must not have a response in \( \alpha' \). We also remove its invocation when forming \( \alpha \).

For each remaining operation on a \( T_2 \) object that has a response in \( \alpha' \), we “contract” the operation so that its invocation and response occur immediately after each other at the linearization point of the operation. Finally, we consider operations on \( T_2 \) objects that are linearized but have no response in \( \alpha' \), either because the process executing the operation crashes or does not take enough steps to complete the simulated operation. We move the invocation step to the linearization point and add a response step immediately afterwards. If the linearization point is after the crash that occurred while the operation was pending, then we also shift this crash step immediately after the response step.

It is easy to check that the constructed execution \( \alpha \) is a legal execution of \( A \) with atomic objects. In particular, the sequence of steps taken by any process is the same in \( \alpha \) as it is in \( \alpha' \) (except for the removal of some invocations of operations on \( T_2 \) objects that do not terminate, either because they occur immediately before a crash of the process or because the process ceases taking steps). Thus, it will satisfy agreement and validity. The execution \( \alpha \) contains the same output steps as \( \alpha' \), so \( \alpha' \) also satisfies agreement and validity.

H \( rcons(stack) = 1 \)

We use a valency argument to show that \( rcons(stack) = 1 \), i.e., that two processes cannot solve RC using stacks and registers. To derive a contradiction, assume there is an algorithm \( A \) for two processes to solve RC using stacks and registers. As in the proof of Theorem 14, we define valency with respect to a set \( E_A \) of executions of \( A \) in which \( p_2 \) never crashes and, in any prefix of an execution, the number of crashes by \( p_1 \) is less than or equal to the number of steps taken by \( p_2 \). By the same argument as in Theorem 14, there is a critical execution \( \gamma \), and the next step specified by the algorithm for every process \( p_i \) is an operation \( op_i \) on a single stack \( O \). Let \( v_1 \) be the valency of the execution that is obtained by allowing \( p_1 \) to perform its next step \( op_1 \) after \( \gamma \). As argued in Theorem 14, \( v_1 \) must be different from \( v_2 \). The remainder of the proof is a case analysis, similar to Herlihy’s proof that \( cons(stack) = 2 \) [23]. See Figure 8 in which the sequence of elements on the stack are shown from bottom to top in the order they are pushed, and \( \alpha \) represents a (possibly empty) sequence of elements.

If both \( op_1 \) and \( op_2 \) are both \( POP \), then the steps commute (Figure 8(a)). If \( op_1 \) is a \( PUSH(v) \) and \( op_2 \) is a \( POP \), and the stack is empty at the end of \( \gamma \), then \( op_2 \) overwrites \( op_1 \) (Figure 8(b)). In either of these cases, we have \( v_1 = v_2 \) by Lemma 15 (which applies even when \( O \) is not readable). This is the desired contradiction.
Figure 8: Impossibility of 2-process recoverable consensus using a stack.
If \( op_1 \) is a \texttt{Push}(v) and \( op_2 \) is a \texttt{Pop} and the stack is non-empty at the end of \( \gamma \), then consider the two extensions of \( \gamma \) shown in Figure 8(c). After the operations \( op_1 \) and \( op_2 \) are done (in opposite order in the two extensions), the only differences between the two resulting states of the system are the local state of \( p_2 \) and the element that is on the top of the stack. Thus, if \( p_1 \) continues to run, it must run until it pops that top element; otherwise it would output the same value in both extensions, contradicting the fact that one is \( v_1 \)-valent and the other is \( v_2 \)-valent. During this solo execution by \( p_1 \) it takes the same steps in both extensions. Since \( p_2 \) has taken a step in both extensions, we can then crash \( p_1 \). After \( p_1 \) crashes, the states of the system in the two extensions are identical except for \( p_2 \)'s state. Thus, if \( p_1 \) recovers and executes \( A \) to completion, it must output the same value in the two extensions, contradicting the fact that the two extensions have different valencies.

The other cases shown in Figure 8(d) to 8(f) are argued similarly to the preceding case, and this completes the proof.

A similar argument could be used to show that \( rcons(queue) = 1 \).