Asymptotic tails of massive scalar fields in a stationary axisymmetric EMDA black hole geometry

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The late-time tail behavior of massive scalar fields is studied analytically in a stationary axisymmetric EMDA black hole geometry. It is shown that the asymptotic behavior of massive perturbations is dominated by the oscillatory inverse power-law decaying tail \( t^{-(l+3/2)} \sin(\mu t) \) at the intermediate late times, and by the asymptotic tail \( t^{-5/6} \sin(\mu t) \) at asymptotically late times. Our result seems to suggest that the intermediate tails \( t^{-(l+3/2)} \sin(\mu t) \) and the asymptotically tails \( t^{-5/6} \sin(\mu t) \) may be quite general features for evolution of massive scalar fields in any four dimensional asymptotically flat rotating black hole backgrounds.

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The no-hair theorem, introduced by Wheeler in the early 1970s \cite{1, 2}, states that the external field of a black hole relaxes to a Kerr-Newman field characterized solely by the black-hole’s mass, charge and angular momentum. So it is interesting to analyze the dynamical mechanism by which perturbation fields outside a black hole are radiated away. Price first studied the massless neutral external perturbations in \cite{3} and found that the late-time behavior is dominated by the factor \( t^{-(2l+3)} \) for each multiple moment \( l \). Hod and Piran investigated the late-time evolution of a charged massless scalar field in Refs. \cite{4}-\cite{6} and their conclusion was that a charged scalar hair outside a charged black hole is dominated by a \( t^{-(2l+2)} \) tail which decay slower than a neutral one. The asymptotic late-time tail of massless perturbations outside realistic, rotating black hole had been considered in Refs. \cite{7}-\cite{9}. And recently the massless Dirac perturbations have been examined in Refs. \cite{10}-\cite{12}.

Although these works are mainly concerned with massless fields, the evolution of massive scalar hair is also important and it has attracted a lot of attention recently. The physical mechanism by which late-time tails of massive scalar fields are generated may be qualitatively different from that of massless ones. It has been shown \cite{5} that if the field mass \( \mu \) is small, namely \( \mu M \ll 1 \), the oscillatory inverse power-law behavior \( t^{-(l+3/2)} \sin(\mu t) \), dominates as the intermediate late-time tails. Recently the very late-time tails of the massive fields in the Schwarzschild or Reissner-Nordström background were studied in Refs. \cite{13, 14}, and it has been pointed out that the oscillatory inverse power-law behavior of the dominant asymptotic tail is approximately given by \( t^{-5/6} \sin(\mu t) \) which is slower than the intermediate ones. The late-time tails of a massive scalar field in the spacetime of black holes are studied numerically by Khanna \cite{15}. Yu \cite{16}.

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studied the decay of massive scalar hair in the background of a black hole with a global monopole.

Nowadays, it seems that the superstring theories are the most promising candidates for a consistent quantum theory of gravitation. So there have been many investigations concerning the spacetimes of dilaton black hole. For instance, the late-time tails of massive scalar fields in the background of dilaton black hole had been explored by Moderski and Rogatko, and it had been shown that this late-time evolution is identical with that in the Schwarzschild and Reissner-Nordström backgrounds. Obviously they only discussed the massive power-law tails in the static, spherically symmetric black hole geometry. Thus, it is worthwhile to investigate analytically the massive late-time behavior in the background of dilaton black hole being the stationary axisymmetric solution of the so-called low-energy string theory.

Now we begin to study analytically the asymptotic tails of massive scalar fields in the background of a stationary axisymmetric EMDA black hole. The metric for this considered black hole can be expressed as

\[
ds^2 = -\frac{\Sigma - a^2 \sin^2 \theta}{\Delta} dt^2 - \frac{2a \sin^2 \theta}{\Delta} \left[ \Xi - \Sigma \right] dt \, d\phi + \frac{\Delta}{\Sigma} dr^2 + \Delta d\theta^2 + \frac{\sin^2 \theta}{\Delta} \left[ \Sigma^2 - \Sigma a^2 \sin^2 \theta \right] d\phi^2,
\]

with \( \Sigma = r^2 - 2Mr + a^2, \Delta = r^2 - 2Dr + a^2 \cos^2 \theta, \Xi = r^2 - 2Dr + a^2, \) and \( D, a, \) and \( M \) represent the dilaton, rotational, and mass parameter.

In the background geometry, let massive scalar field as \( \Phi = \Xi^{-1/2} \sum_{m=-\infty}^{\infty} \Psi^{m} e^{im \phi}, \) one obtains a wave-equation for each value of azimuthal number \( m: \)

\[
C_1(r, \theta) \frac{\partial^2 \Psi}{\partial t^2} + C_2(r, \theta) \frac{\partial \Psi}{\partial t} + C_3(r, \theta) \Psi - \frac{\partial^2 \Psi}{\partial y^2} = \frac{\Sigma}{\Xi^2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Psi}{\partial \theta} \right) = 0,
\]

where we use the tortoise radial coordinate \( y \) defined by \( dy = \Xi dr/\Sigma, \) and \( C_i(r, \theta) \) are given by

\[
C_1(r, \theta) = 1 - \frac{\Sigma a^2 \sin^2 \theta}{\Xi^2},
\]

\[
C_2(r, \theta) = \frac{4im(M - D)r}{\Xi^2},
\]

\[
C_3(r, \theta) = \frac{\Sigma}{\Xi^2} \left[ -m^2 \left( \frac{a^2}{\Sigma} - \frac{1}{\sin^2 \theta} \right) + \frac{2(r - M)(r - D) + \Sigma}{r^2 - 2Dr + a^2} - \frac{3(r - D)^2 \Sigma}{\Xi^2} + \Delta \mu^2 \right],
\]

where \( \mu \) is the mass of the scalar field.

The time evolution of a wave field described by Eq. (2) is given by

\[
\Psi(x, t) = 2\pi \int_0^\pi \int_0^\pi \left( \sum_{m=-\infty}^{\infty} \left[ C_1(x') \left[ G(x, x'; t) \Psi_t(x', 0) + G_t(x, x'; t) \Psi(x', 0) \right] + C_2(x') G(x, x'; t) \Psi(x', 0) \right] \sin \theta' d\theta' dy' \right) \sin \theta d\theta d\phi.
\]

for \( t > 0, \) where \( x \) stands for \( (y, \theta). \) The Green’s function \( G(x, x'; t) \) is defined by

\[
\left[ C_1(r, \theta) \frac{\partial^2}{\partial t^2} + C_2(r, \theta) \frac{\partial}{\partial t} + C_3(r, \theta) - \frac{\partial^2}{\partial y^2} \right] \frac{\Sigma}{\Xi^2 \sin \theta} \frac{1}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) G(x, x'; t) = \delta(t) \delta(y - y') \frac{\delta(\theta - \theta')}{2\pi \sin \theta},
\]

(4)
The causality condition gives us the initial condition $G(x, x'; t) = 0$ for $t \leq 0$. We express the Green’s function in terms of the Fourier transform
\[
G(x, x'; t) = \frac{1}{(2\pi)^2} \sum_{l} \int_{-\infty+ic}^{\infty+ic} \tilde{G}_l(y, y'; \omega) Y_l^m(\theta, a\omega) Y_l^m(\theta', a\omega)e^{-i\omega t}d\omega,
\]
where $c$ is some positive constant and $Y_l^m(\theta, a\omega)$ are the spheroidal harmonics.

The Fourier transform is analytic in the upper half $\omega-$plane and it satisfies
\[
\left[ \frac{d^2}{dy^2} + \omega^2 - R_l(r, \omega) \right] \tilde{G}_l(y, y'; \omega) = \delta(y - y'),
\]
where
\[
R_l(r, \omega) = \sum_{\Sigma} \left[ \frac{a^2 \omega^2 + \frac{4am\omega(M - D)r}{\Sigma}}{\Sigma} + l(l + 1) + \mu^2(r^2 - 2Dr) - \frac{2(r - M)(r - D) + \Sigma}{\Sigma} - \frac{3(r - D)^2\Sigma}{\Sigma^2} \right].
\]

We define two auxiliary functions $\tilde{\Psi}_1(y, \omega)$ and $\tilde{\Psi}_2(y, \omega)$ which are (linearly independent) solutions to the homogeneous equation
\[
\left[ \frac{d^2}{dy^2} + \omega^2 - R_l(r, \omega) \right] \tilde{\Psi}_i(y, \omega) = 0, \quad i = 1, 2.
\]

Then, the Green’s function can be given by
\[
\tilde{G}_l(y, y'; \omega) = -\frac{1}{W(\omega)} \begin{cases} 
\tilde{\Psi}_1(y, \omega)\tilde{\Psi}_2(y', \omega), & y < y'; \\
\tilde{\Psi}_1(y', \omega)\tilde{\Psi}_2(y, \omega), & y > y'.
\end{cases}
\]

Here $W(\omega) = W(\tilde{\Psi}_1, \tilde{\Psi}_2) = \tilde{\Psi}_1\tilde{\Psi}_{2,y} - \tilde{\Psi}_2\tilde{\Psi}_{1,y}$ is the Wronskian.

To calculate $G(x, x'; t)$ using Eq. (5), we can close the contour of integration into the lower half of the complex frequency plane. It has been argued that the asymptotic massive tail is associated with the existence of a branch cut (in $\tilde{\Psi}_2$) placed along the interval $-\mu \leq \omega \leq \mu \left[1, 2\right]$. As will be shown in this Letter, this tail arises from the integral of $\tilde{G}(y, y'; \omega)$ around the branch cut (denoted by $G^\ast$) which gives rise to oscillatory inverse power-law behavior of the field. So our goal is to evaluate $G^\ast(x, x'; t)$.

Assume that both the observer and the initial data are situated far away from the black hole so that $r \gg M - D$, we may expand the wave-equation as a power series in $\frac{\omega}{\sqrt{\Omega}}$, neglecting terms of order $O\left(\frac{\omega}{\sqrt{\Omega}}\right)$. And if we define $\varpi = \sqrt{\mu^2 - \omega^2}$ and $z = 2\varpi r$, we obtain
\[
\left\{ \frac{d^2}{dz^2} + \left[ -\frac{1}{4} + \frac{\beta}{z} + \frac{1/4 - (l + 1/2)^2}{z^2} \right] \right\} \xi(r, \omega) = 0,
\]
where $\xi(r, \omega) = |\Sigma/\Xi|^{1/2}\tilde{\Psi}(y, \omega)$ and $\beta = (M - D)\mu^2/\varpi - 2(M - D)\varpi$. This equation is the Whittaker’s Equation and it has two basic solutions needed to construct the Green’s function (for $|\omega| \leq \mu$)
\[
\tilde{\Psi}_1 = M_{\beta, l+1/2}(2\varpi r) = e^{-\varpi r}(2\varpi r)^{l+1}M(l + 1 - \beta, 2l + 2, 2\varpi r),
\]
\[
\tilde{\Psi}_2 = W_{\beta, l+1/2}(2\varpi r) = e^{-\varpi r}(2\varpi r)^{l+1}U(l + 1 - \beta, 2l + 2, 2\varpi r),
\]
\[
\tilde{\Psi}_1 = M_{\beta, l+1/2}(2\varpi r) = e^{-\varpi r}(2\varpi r)^{l+1}M(l + 1 - \beta, 2l + 2, 2\varpi r),
\]
\[
\tilde{\Psi}_2 = W_{\beta, l+1/2}(2\varpi r) = e^{-\varpi r}(2\varpi r)^{l+1}U(l + 1 - \beta, 2l + 2, 2\varpi r),
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\]
\[
\tilde{\Psi}_2 = W_{\beta, l+1/2}(2\varpi r) = e^{-\varpi r}(2\varpi r)^{l+1}U(l + 1 - \beta, 2l + 2, 2\varpi r),
\]
where $M(a, b, z)$ and $U(a, b, z)$ are the two standard solutions to the confluent hypergeometric equation [21]. $U(a, b, z)$ is a many-valued function; i.e., there is a cut in $\tilde{\Psi}_2$.

From Eq. (5), we find that the branch cut contribution to the Green’s function is

$$G^c(x, x'; t) = \frac{1}{(2\pi)^2} \sum_l \int_{-\mu}^{\mu} f(\omega) Y_l(\theta, a\omega) Y_l(\theta', a\omega) e^{-i\omega t} d\omega,$$

where

$$f(\omega) = \frac{\tilde{\Psi}_1(y', \omega e^{\pi i}) \tilde{\Psi}_2(y, \omega e^{\pi i})}{W(\omega e^{\pi i})} - \frac{\tilde{\Psi}_1(y', \omega) \tilde{\Psi}_2(y, \omega)}{W(\omega)}.$$

For simplicity we take $y > y'$. Of course, this doesn’t change the late-time behavior. One can easily verify that in the large $t$ limit the effective contribution to the integral in Eq. (13) arises from $|\omega| = 0(\mu - 1/t)$ or equivalently $\omega = 0(\sqrt{\mu} t)$. This is due to the rapidly oscillating term $e^{-i\omega t}$ which leads to a mutual cancellation between the positive and the negative parts of the integrand.

Using Eqs. (11) and (12), with the help of Eqs. 13.5.1 and 13.5.2 in Ref. [21], we have

$$W(\omega e^{\pi i}) = -W(\omega) = 2(2l + 1)\omega \frac{\Gamma(2l + 1)}{\Gamma(l + 1 - \beta)}.$$

In order to evaluate $G^c(x, x'; t)$ using Eq. (13), we only need to study asymptotical form of $\tilde{\Psi}_i(y, \omega)(i = 1, 2)$ both at the intermediate and very late times.

**[Intermediate late-time tail]:** It is by now well known that the intermediate asymptotic behavior of a massive scalar field on the EMDA background is dominated by flat spacetime effects [4, 22]. That is the tail in the range $M - D \ll r \ll t \ll (M - D)/[\mu(M - D)]^2$. In this time scale, the frequency range $\omega = 0(\sqrt{\mu} t)$ implies $\beta \ll 1$. Notice that $\beta$ describes the effect of backscattering from asymptotically far regions because it originates from the $1/r$ term in the massive scalar field equation. If the relation $\beta \ll 1$ is satisfied, the backscattering from asymptotically far regions is negligible.

Using the following relations

$$W_{\beta, l+1/2}(2\omega r) = \frac{\Gamma(-2l - 1)}{\Gamma(1 - l - \beta)} M_{\beta, l+1/2}(2\omega r) + \frac{\Gamma(2l + 1)}{\Gamma(l + 1 - \beta)} M_{\beta, -(l+1/2)}(2\omega r),$$

$$M_{\beta, l+1/2}(e^{\pi i}2\omega r) = e^{(l+1)i\pi} M_{\beta, l+1/2}(2\omega r),$$

and conditions $\beta \ll 1, M(a, b, z) \simeq 1$ as $z \to 0$, Eq. (14) can be expressed as

$$f(\omega) = \frac{2^{2l+2} \Gamma(-2l - 1) \Gamma(l + 1)(yy')^{l+1} \omega^{2l+1}}{(2l + 1) \Gamma(2l + 1) \Gamma(-l)}.$$

Substituting it into Eq. (13) and performing the integration, we obtain

$$G^c(x, x'; t) = \frac{1}{(2\pi)^2} \sum_l 2^{3l+\delta} \frac{\Gamma(-2l - 1) \Gamma(l + \frac{3}{2}) \Gamma(l + 1)}{(2l + 1) \Gamma(2l + 1) \Gamma(-l)} \mu^{l+\delta} Y_l(\theta, a\mu) Y_l^*(\theta', a\mu)$$

$$\times (yy')^{l+1} t^{-(l+\frac{3}{2})} \cos[\mu t - \frac{1}{2}(l + \frac{3}{4})\pi].$$
Thus, the intermediate behavior is dominated by an oscillatory inverse power-law decaying tail $t^{-(l+3/2)} \sin(\mu t)$ for each multiple moment $l$.

**[Asymptotically late-time tail]**: In the above discussion we have used the approximation of $\beta \ll 1$, which only holds when $\mu t \ll 1/|\mu(M-D)|^2$. But at very late times $\mu t \gg 1/|\mu(M-D)|^2$ the inverse power law decay is replaced by another pattern of decay, which is slower than any power law. This asymptotic tail behavior is caused by a resonance backscattering due to spacetime curvature. In this case $\beta$ is not negligibly small, namely, the backscattering from asymptotically far regions is important.

Now we have $\beta \gg 1$. Using Eq. 13.5.13 of Ref. [21], we obtain

$$M_{\beta,\pm(l+1/2)}(2\omega r) = \Gamma(1 \pm (2l + 1))(2\omega r)^{1/2} \beta^{l+1/2} J_{\pm(2l+1)}(\sqrt{8\beta \omega r}),$$

where $J_{\pm(2l+1)}(z)$ is the Bessel function. Thus, we have

$$f(\omega) = \frac{\Gamma(2l + 2)\Gamma(-2l)}{(2l + 1)} (yy')^{1/2} [J_{-l}(\sqrt{8\beta \omega y})J_{l}(\sqrt{8\beta \omega y}) - I_{l}(\sqrt{8\beta \omega y})]$$

$$\times I_{-l}(\sqrt{8\beta \omega y}) + \frac{\Gamma(2l + 2)\Gamma(-2l + 1)\Gamma(l + 1 - \beta)}{(2l + 1)\Gamma(2l + 1)\Gamma(l - \beta)} (yy')^{1/2} \beta^{-(2l+1)}$$

$$\times [J_{l}(\sqrt{8\beta \omega y})J_{-l}(\sqrt{8\beta \omega y}) + I_{l}(\sqrt{8\beta \omega y})I_{-l}(\sqrt{8\beta \omega y})],$$

where $I_{\pm l}(z)$ is the modified Bessel functions. Notice that $8\beta \omega \approx 8(M-D) \mu^2$ is independent of $\omega$. Thus, the late time tail arising from the first term will be $t^{-1}$. Now let us discuss the second term in the limit $\mu t \rightarrow \infty$ and $|\omega| \rightarrow \mu$

$$G^c(x,x';t) = \frac{1}{(2\pi)^2} \sum_l Q(l) \int_{-\mu}^{\mu} \frac{\Gamma(l + 1 - \beta)}{\Gamma(-l - \beta)} \beta^{-(2l+1)} e^{-i\omega t} d\omega,$$

where

$$Q(l) = \frac{\Gamma(2l + 1)\Gamma(-2l + 1)}{(2l + 1)\Gamma(2l + 1)} (yy')^{1/2} Y_l(\theta, a\mu) Y_l^*(\theta', a\mu)$$

$$\times [J_{l}(\sqrt{8\beta \omega y})J_{-l}(\sqrt{8\beta \omega y}) + I_{l}(\sqrt{8\beta \omega y})I_{-l}(\sqrt{8\beta \omega y})].$$

Since $\beta \gg 1$, Eq. (21) can be further approximated to give

$$G^c(x,x';t) = \frac{1}{(2\pi)^2} \sum_l Q(l) \int_{-\mu}^{\mu} e^{i(2\pi \beta - \omega t)} e^{i\phi} d\omega.$$

Here the phase $\phi$ is defined by $e^{i\phi} = [1 - e^{-i(2\pi \beta)}]/[1 - e^{i(2\pi \beta)}]$. The integral Eq. (23) can be evaluated by the method of the saddle-point integration. So we get

$$G^c(x,x';t) \sim \frac{1}{(2\pi)^2} \sum_l Q(l) t^{-5/6} \sin(\mu t).$$

Obviously the very late-time tail behavior is dominated by an oscillatory inverse power-law decaying tail $t^{-(5/6)} \sin(\mu t)$. 
In summary, we have studied analytically both the intermediate and asymptotically late-time behavior of massive scalar fields in the background of a stationary axisymmetric EMDA dilaton black hole. In the massive perturbations fields we find that if the field’s mass is small, namely \( \mu(M - D) \ll 1 \), the intermediate tails \( \Phi \sim t^{-\left(l + 3/2\right)} \sin(\mu t) \) have been shown to dominate at the intermediate late-time \( \mu(M - D) \ll \mu t \ll 1/\left[\mu(M - D)\right]^2 \). These oscillatory inverse power-law decaying tails originate from the flat spacetime effects. Obviously, these tails depend not only on the multiple moment \( l \) but also on the field’s mass \( \mu \), but they are independent of the rotational parameter \( a \) of the black hole. However, the intermediate late-time tails are not the final pattern and a transition to the oscillatory asymptotically tails \( \Phi \sim t^{-5/6} \sin(\mu t) \) is to occur when \( \mu t \gg 1/\left[\mu(M - D)\right]^2 \). The origin of these tails may be attributed to the resonance backscattering due to spacetime curvature. It is interesting to mention that these behaviors depend only on \( \mu \), but they are independent of \( l \) and \( a \). Obviously, they are qualitatively similar to those found in the Schwarzschild and nearly extreme Reissner-Nordström backgrounds. Our result seems to suggest that the intermediate tails \( \Phi \sim t^{-\left(l + 3/2\right)} \sin(\mu t) \) and the asymptotically tails \( \Phi \sim t^{-5/6} \sin(\mu t) \) may be a quite general feature for the evolution of massive scalar fields in any four dimensional asymptotically flat rotating black hole backgrounds.

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