EPW CUBES

ATANAS ILIEV, GRZEGORZ KAPUSTKA, MICHAL KAPUSTKA, AND KRISTIAN RANESTAD

Dedicated to Piotr Pragacz on the occasion of his 60th birthday.

Abstract. We construct a new 20-dimensional family of projective 6-dimensional irreducible holomorphic symplectic manifolds. The elements of this family are deformation equivalent with the Hilbert scheme of three points on a K3 surface and are constructed as natural double covers of special codimension 3 subvarieties of the Grassmanian $G(3, 6)$. These codimension 3 subvarieties are defined as Lagrangian degeneracy loci and their construction is parallel to that of EPW sextics, we call them the EPW cubes. As a consequence we prove that the moduli space of polarized IHS sixfolds of K3-type, Beauville-Bogomolov degree 4 and divisibility 2 is unirational.

1. Introduction

By an irreducible holomorphic symplectic (IHS) $2n$-fold we mean a $2n$-dimensional simply connected compact K"ahler manifold with trivial canonical bundle that admits a unique (up to a constant) closed non-degenerate holomorphic 2-form and is not a product of two manifolds (see [Bea83]). The IHS manifolds are also known as hyperk"ahler and irreducible symplectic manifolds, in dimension 2 they are called K3 surface.

Moduli spaces of polarized K3 surfaces are a historically old subject, studied by the classical Italian geometers. Mukai extended the classical constructions and proved unirationality results for the moduli spaces $\mathcal{M}_{2d}$ parametrizing polarized K3 surfaces of degree $2d$ for many cases with $d \leq 19$ see [Muk92], [Muk06], [Muk12]. On the other hand it was proven in [GHS07] that $\mathcal{M}_{2d}$ is of general type for $d > 61$ and some smaller values. Note that when the Kodaira dimension of such moduli space is positive the generic element of such moduli space is believed to be non-constructible.

There are only five known descriptions of the moduli space of higher dimensional IHS manifolds (all these examples are deformations equivalent to $K3^{[n]}$). In dimension four we have the following unirational moduli spaces:

- double EPW sextics with Beauville-Bogomolov degree $q = 2$ (see [O'G06]),
- Fano scheme of lines on four dimensional cubic hypersurfaces with $q = 6$ (see [BD85]),
- $\text{VSP}(F, 10)$ where $F$ define a cubic hypersurface of dimension 4 with $q = 38$ (see [IR01]),
- zero locus of a section of a vector bundle on $G(6, 10)$ with $q = 22$ described in [DV10].

Moreover, there is only one more known family in dimension 8 with $q = 2$ studied in [LLSvS15]. Analogously to the case of K3 surfaces there are results in [GHS10] about the Kodaira dimension of the moduli spaces of polarized IHS fourfolds of $K3^{[2]}$-type:

2000 Mathematics Subject Classification. 14J10, 14J40.

Key words and phrases. Irreducible symplectic manifolds, hyperkähler varieties, Lagrangian degeneracy loci.
In particular it is proven that such moduli spaces with split polarization of Beauville-Bogomolov degree \( q \geq 24 \) are of general type (and for \( q = 18, 22 \) are of positive Kodaira dimension). We expect that the number of constructible families in higher dimension becomes small.

According to O’Grady \cite{OG06}, the 20-dimensional family of natural double covers of special sextic hypersurfaces in \( \mathbb{P}^5 \) (called EPW sextics) gives a maximal dimensional family of polarized IHS fourfold deformation equivalent to the Hilbert scheme of two points on a \( K3 \)-surface (this is a maximal dimensional family since \( b_2(S^{[2]}) = 23 \) for \( S \) a \( K3 \)-surface). Our aim is to perform a construction parallel to that of O’Grady to obtain a unirational 20-dimensional family (also of maximal dimension) of polarized IHS sixfolds deformation equivalent to the Hilbert scheme of three points on a \( K3 \)-surface (i.e. of \( K3^{[3]} \) type). The elements of this family are natural double covers of special codimension 3 subvarieties of the Grassmannian \( G(3,6) \) that we call EPW cubes.

Let us be more precise. Let \( W \) be a complex 6-dimensional vector space. We fix an isomorphism \( j : \wedge^6 W \to \mathbb{C} \) and the skew symmetric form
\[
\eta : \wedge^3 W \times \wedge^3 W \to \mathbb{C}, \quad (u,v) \mapsto j(u \wedge v).
\]
We denote by \( LG_\eta(10, \wedge^3 W) \) the variety of 10-dimensional Lagrangian subspaces of \( \wedge^3 W \) with respect to \( \eta \). For any 3-dimensional subspace \( U \subset W \), the 10-dimensional subspace
\[
T_U := \wedge^2 U \wedge W \subset \wedge^3 W
\]
belongs to \( LG_\eta(10, \wedge^3 W) \), and \( \mathbb{P}(T_U) \) is the projective tangent space to
\[
G(3, W) \subset \mathbb{P}(\wedge^3 W)
\]
at \([U]\).

For any \([A] \in LG_\eta(10, \wedge^3 W) \) and \( k \in \mathbb{N} \), we consider the following Lagrangian degeneracy locus, with natural scheme structure (see \cite{PR97}),
\[
D^A_k = \{ [U] \in G(3, W) \mid \dim A \cap T_U \geq k \} \subset G(3, W).
\]
For the fixed \([A] \in LG_\eta(10, \wedge^3 W) \) we call the scheme \( D^A_3 \) an EPW cube. We prove that if \( A \) is generic then \( D^A_3 \) is a sixfold singular only along the threefold \( D^A_4 \) and that \( D^A_4 \) is empty. Moreover, \( D^A_3 \) is smooth such that the singularities of \( D^A_3 \) are transversal \( \frac{1}{2}(1,1,1) \) singularities along \( D^A_3 \).

Before we state our main theorem we shall need some more notation. The projectivized representation \( \wedge^3 \) of \( PGL(W) \) on \( \wedge^3 W \) splits \( \mathbb{P}^{19} = \mathbb{P}(\wedge^3 W) \) into a disjoint union of 4 orbits
\[
\mathbb{P}^{19} = ( \mathbb{P}^{19} \setminus W ) \cup ( F \setminus \Omega ) \cup ( \Omega \setminus G(3, W) ) \cup G(3, W),
\]
where \( G(3, W) \subset \Omega \subset F \subset \mathbb{P}^{19}, \dim(\Omega) = 14, \Sing(\Omega) = G(3, W), \dim(F) = 18, \Sing(F) = \Omega \), see \cite{Don77}. We call the invariant sets \( G, \Omega, F \) and \( \mathbb{P}^{19} \) the (projective) orbits of \( \wedge^3 \) for \( PGL(6) \). See \cite[Appendix]{Kap14} for some results about the geometry of \( \Omega \) and its relations with EPW sextics. For any nonzero vector \( w \in W \), denote by
\[
F_{[w]} = \langle w \rangle \wedge (\wedge^2 W)
\]
the 10-dimensional subspace of \( \wedge^3 W \), such that
\[
\bigcup_{[w] \in \mathbb{P}(W)} \mathbb{P}(F_{[w]}) = \Omega \subset \mathbb{P}(\wedge^3 W).
We denote, after O’Grady [O’G13],
\[ \Sigma = \{ [A] \in LG_{\eta}(10, \wedge^3 W) \mid \mathbb{P}(A) \cap G(3, W) \neq \emptyset \} \]
and
\[ \Delta = \{ [A] \in LG_{\eta}(10, \wedge^3 W) \mid \exists w \in W \colon \dim A \cap F[w] \geq 3 \}. \]
We also consider a third subset
\[ \Gamma = \{ A \in LG_{\eta}(10, \wedge^3 W) \mid \exists [U] \in G(3, W) \colon \dim A \cap T_U \geq 4 \}. \]
Denote by
\[ LG^1_{\eta}(10, \wedge^3 W) := LG_{\eta}(10, \wedge^3 W) \setminus (\Sigma \cup \Delta). \]
All three subsets \( \Sigma, \Delta, \Gamma \) are divisors (see [O’G13] and Lemma 3.6) and has trivial canonical class. The proof of the
Theorem 1.1. If \( [A] \in LG^1_{\eta}(10, \wedge^3 W) \), then there exists a natural double cover \( Y_A \) of the EPW cube \( D_2^A \) branched along its singular locus \( D_3^A \) such that \( Y_A \) is an IHS sixfold of \( K3^{[3]} \)-type with polarization of Beauville-Bogomolov degree \( q = 4 \) and divisibility 2. In particular, the moduli space of polarized IHS sixfolds of \( K3^{[3]} \)-type, Beauville-Bogomolov degree 4 and divisibility 2 is unirational.

We prove the theorem in Section 5 at the very end of the paper. The plan of the proof is the following: In Proposition 5.1 we prove that for \( [A] \in LG^1_{\eta}(10, \wedge^3 W) \), the variety \( D_2^A \) is singular only along the locus \( D_3^A \) and that it admits a smooth double cover \( Y_A \to D_2^A \) branched along \( D_3^A \) with a trivial canonical class. The proof of the Proposition is based on a general study of Lagrangian degeneracy loci contained in Section 2. By globalizing the construction of the double cover to the whole affine variety \( LG^1_{\eta}(10, \wedge^3 W) \) we obtain a smooth family
\[ \mathcal{Y} \to LG^1_{\eta}(10, \wedge^3 W) \]
with fibers \( \mathcal{Y}_{[A]} = Y_A \). Note that the family \( \mathcal{Y} \) is naturally a family of polarized varieties with the polarization given by the divisors defining the double cover.

In Lemma 3.7 we prove that \( \Delta \setminus (\Gamma \cup \Sigma) \) is nonempty. Following [O’G13] Section 4.1, we associate to a general \( [A_0] \in \Delta \setminus (\Gamma \cup \Sigma) \) a K3 surface \( S_{A_0} \). Then, in Proposition 4.1 we prove that there exists a rational 2 : 1 map from the Hilbert scheme \( S_{[3]} \) of length 3 subschemes on \( S_{A_0} \) to the EPW cube \( D_2^{A_0} \). We infer in Section 5 that in this case the sixfold \( Y_{A_0} \) is birational to \( S_{[3]}^{A_0} \). Together with the fact that \( Y_{A_0} \) is smooth, irreducible and has trivial canonical class, this proves that \( Y_{A_0} \) is IHS.

Since flat deformations of IHS manifolds are still IHS, the family \( \mathcal{Y} \) is a family of smooth IHS sixfolds. The fact that the obtained IHS manifolds are of \( K3^{[3]} \)-type is a straightforward consequence of Huybrechts theorem [Huy99] Thm. 4.6.

During the proof of Theorem 1.1 we retrieve also some information on the constructed varieties. We prove in Section 2.3 that the polarization \( \xi \) giving the double cover \( Y_A \to D_2^A \) has Beauville-Bogomolov degree \( q(\xi) = 4 \) and is primitive. Moreover, the degree of an EPW cube \( D_2^A \subset G(3, 6) \subset \mathbb{P}^{19} \) is 480.

Note that the coarse moduli space \( \mathcal{M} \) of polarized IHS sixfolds of \( K3^{[3]} \)-type and Beauville-Bogomolov degree 4 has two components distinguished by divisibility. We conclude the paper by proving that the image of the moduli map \( LG^1_{\eta}(10, \wedge^3 W) \to \mathcal{M} \) defined by \( \mathcal{Y} \) is a 20 dimensional open and dense subset of the component of \( \mathcal{M} \) corresponding to divisibility 2 (see Proposition 5.3).
Acknowledgements. We thank Olivier Debarre, Alexander Kuznetsov and Kieran O’Grady for useful comments, O’Grady in particular for pointing out a proof of Proposition 5.3. A. Iliev was supported by SNU grant 0450-20130016, G. Kapustka by NCN grant 2013/08/A/ST1/00312, M. Kapustka by NCN grant 2013/10/E/ST1/00688 and K. Ranestad by RCN grant 239015.

2. Lagrangian degeneracy loci

In this section we study resolutions of Lagrangian degeneracy loci. Let us start with fixing some notation and definitions. We fix a vector space $W_{2n}$ of dimension $2n$ and a symplectic form $\omega \in \wedge^2 W_{2n}^*$. Let $X$ be a smooth manifold and let $W = W_{2n} \times O_X$ be the trivial bundle with fiber $W_{2n}$ on $X$ equipped with a nondegenerate symplectic form $\tilde{\omega}$ induced on each fiber by $\omega$. Consider $J \subset W$ a Lagrangian vector subbundle, i.e. a subbundle of rank $n$ whose fibers are isotropic with respect to $\tilde{\omega}$. Let $A \subset W_{2n}$ be a Lagrangian vector subspace inducing a trivial subbundle $A \subset W$. For each $k \in \mathbb{N}$ we define the set

$$D^A_k = \{x \in X | \dim(J_x \cap A) \geq k\} \subset X$$

where $J_x$ and $A_x$ denote the fibers of the bundles $J$ and $A$ as subspaces in the fiber $W_x$. Let us now define $LG_{\omega}(n, W_{2n})$ to be the Lagrangian Grassmannian parametrizing all subspaces of $W_{2n}$ which are Lagrangian with respect to $\omega$. Then $J$ defines a map $\iota : X \rightarrow LG_{\omega}(n, W_{2n})$ in such a way that $J = \iota^*L$ where $L$ denotes the tautological bundle on the Lagrangian Grassmannian $LG_{\omega}(n, W_{2n})$. Moreover, similarly as on $X$, we can define

$$D^A_k = \{[L] \in LG_{\omega}(n, W_{2n}) | \dim(L \cap A_{[L]}) \geq k\} \subset LG_{\omega}(n, W_{2n}),$$

and $D^A_k$ admits a natural scheme structure as a degeneracy locus. We then have $D^A_k = \iota^{-1}D^A_k$, i.e. the scheme structure on $D^A_k$ is defined by the inverse image of the ideal sheaf of $D^A_k$ [Har77, p.163].

2.1. Resolution of $D^A_k$. For each $k \in \mathbb{N}$, let $G(k, A)$ be the Grassmannian of $k$-dimensional subspaces of $A$ and let

$$\tilde{D}^A_k = \{([L], [U]) \in LG_{\omega}(n, W_{2n}) \times G(k, A) | L \supset U\}.$$  

By [PR97], $\tilde{D}^A_k$ is a resolution of $D^A_k$. We shall describe the above variety more precisely. First of all we have the following incidence described more generally in [PR97]:

$$\begin{array}{ccc}
\tilde{D}^A_k & \xrightarrow{\phi} & D^A_k \\
\downarrow \phi & & \downarrow \pi \\
G(k, A) & & 
\end{array}$$

The projection $\phi$ is clearly birational, whereas $\pi$ is a fibration with fibers isomorphic to a Lagrangian Grassmannian $LG(n - k, 2n - 2k)$. In particular $\tilde{D}^A_k$ is a smooth manifold of Picard number two with Picard group generated by $H$, the pullback of the hyperplane section of $LG(n, W_{2n})$ in its Plücker embedding, and $R$, the pullback of the hyperplane section of $G(k, A)$ in its Plücker embedding. Denote by $Q$ the tautological bundle on $G(k, A)$ seen as a subbundle of the trivial symplectic bundle $W_{2n} \otimes O_{G(k, A)}$. Consider the subbundle $Q^\perp \subset W_{2n} \otimes O_{G(k, A)}$ perpendicular to $Q$ with respect the symplectic form. The following was observed in [PR97].
Lemma 2.1. The variety $\mathbb{D}_k^A$ is isomorphic to the Lagrangian bundle 
\[ F := LG(n-k, Q^1/Q). \]

Of course the tautological Lagrangian subbundle on $LG(n-k, Q^1/Q)$ can be identified with the bundle $\phi^*L/\pi^*Q =: \mathcal{W}$. In particular, we have $c_1(\mathcal{W}) = \phi^*c_1(L) - \pi^*c_1(Q) = R - H$.

Lemma 2.2. The relative tangent bundle $T_\pi$ of $\pi: F \to G(k, A)$ is the bundle $S^2(\mathcal{W}^\vee)$.

Proof. This can be seen by globalizing the construction of the tangent space of the Lagrangian Grassmannian described for example in [Muk10]. \qed

Lemma 2.3. The canonical class of $\mathbb{D}_k^A$ is $-(n+1-k)H - (k-1)R$.

Proof. We use the exact sequence 
\[ 0 \to T_\pi \to T_F \to \pi^*T_{G(k,A)} \to 0. \]

Now $\mathcal{W}^\vee$ has rank $n-k$, so 
\[ c_1(T_\pi) = c_1(S^2(\mathcal{W}^\vee)) = (n+1-k)c_1(\mathcal{W}^\vee) = (n+1-k)(H - R) \]
while $\pi^*c_1(T_{G(k,A)}) = nR$. Hence $K_F = -c_1(T_F) = -(n+1-k)H - (k-1)R$. \qed

Lemma 2.4. The variety $\mathbb{D}_1^A$ is a hyperplane section of $LG_{\omega}(n,W_{2n})$.

Proof. Indeed $\mathbb{D}_1^A$ is the intersection of the codimension one Schubert cycle on the Grassmannian $G(n,2n)$ with the Lagrangian Grassmannian, hence a hyperplane section of the Lagrangian Grassmannian. \qed

Let us denote by $E$ the exceptional divisor of $\phi$.

Lemma 2.5. For $k = 2$ we have: \[ [E] = [H] - 2[R]. \]

Proof. It is clear that $[E] = a[H] + b[R]$ for some $a, b \in \mathbb{Z}$. Let us now consider the restriction of $E$ to a fiber of $\pi$ i.e. we fix $V_2 \subset A$ a vector space of dimension 2 and consider $LG(n-2,V_2^\perp/V_2)$. Since $E = \phi^{-1}\mathbb{D}_2^A$ we have 
\[ E \cap \pi^{-1}[V_2] = \{(L) \in LG(n-2,V_2^\perp/V_2) | \dim(L/V_2 \cap A/V_2) \geq 1 \}. \]

It is hence a divisor of type $\mathbb{D}^A_{1/V_2}$ which is a hyperplane section of the fiber by Lemma 2.4. It follows that $a = 1$.

To compute the coefficient at $[R]$ we fix a subspace $V_{n-2}$ of dimension $n-2$ in $A$ and consider the Schubert cycle 
\[ \sigma_{n-2} = \{[U] \in G(2,A) | \dim(U \cap V_{n-2}) \geq 1 \}. \]

The class $[\sigma_{n-2}]$ in the Chow group of $G(2,A)$ is then the class of a hyperplane section. We now describe $\phi_*\pi^*(\sigma_{n-2})$ as the class of the Schubert cycle $\sigma_{n-2,n}$ on $LG(n,2n)$ defined by 
\[ \sigma_{n-2,n} = \{[L] \in LG(n,2n) | \dim(L \cap V_{n-2}) \geq 1, \dim(L \cap A) \geq 2 \}. \]

By [PR97] Theorem 2.1 we have 
\[ [\sigma_{n-2,n}] = c_1(\mathcal{L}^\vee)c_3(\mathcal{L}^\vee) - 2c_4(\mathcal{L}^\vee). \]

Moreover, from the same formula [PR97] Theorem 2.1 we have: 
\[ [\mathbb{D}_2^A] = c_1(\mathcal{L}^\vee)c_2(\mathcal{L}^\vee) - 2c_3(\mathcal{L}^\vee). \]
In terms of intersection on $\tilde{D}_2^4$ this gives
\[
H^{-\frac{n(n+1)}{2} - 3} \cap [\tilde{D}_2^4] = c_1(L)\frac{n(n+1)}{2} c_2(L) - 2 c_1(L)\frac{n(n+1)}{2} - 3 c_3(L)
\]
and
\[
H^{-\frac{n(n+1)}{2} - 4} \cdot R \cap [\tilde{D}_2^4] = c_1(L)\frac{n(n+1)}{2} c_2(L) - 2 c_1(L)\frac{n(n+1)}{2} - 4 c_4(L).
\]

Since we know that $E$ is contracted by the resolution to $D_3^4$ we also have $E \cdot H^{-\frac{n(n+1)}{2} - 4} = 0$. We can now compute $b$:

(2.1) \quad 0 = E \cdot H^{-\frac{n(n+1)}{2} - 4} = (H + bR) \cdot H^{-\frac{n(n+1)}{2} - 4} = H^{-\frac{n(n+1)}{2} - 3} + bR^{-\frac{n(n+1)}{2} - 4} \cdot R =

(2.2) \quad c_1(L)\frac{n(n+1)}{2} c_2(L) + (b - 2)c_1(L)c_3(L) - 2bc_4(L).

Now, using the theorem of Hiller-Boe ([Pra91, Theorem 6.4]) on relations in the Chow ring of the Lagrangian Grassmannian we get
\[
c_1(L)^2 = 2c_2(L) \quad \text{and} \quad c_2(L)^2 = 2(c_3(L)c_1(L) - c_4(L)).
\]

Substituting in (2.1) we get:
\[
0 = (b + 2) \deg(c_1(L)c_3(L) - 2c_4(L)) = (b + 2) \deg \sigma_{n-2,n}.
\]

It follows that $b = -2$. 

2.2. The embedding of $G(3, W)$ into $LG_9(10, \wedge^3 W)$. Let $W$ be a 6-dimensional vector space. Let $G = G(3, W) \subset \mathbb{P}(\wedge^3 W)$ be the Grassmannian of 3-dimensional subspaces in $W$ in its Plücker embedding. Now, recall for each $[U] \in G$,
\[
T_U = \wedge^2 U \wedge W \subset \wedge^3 W.
\]

$\mathbb{P}(T_U)$ is tangent to $G(3, W)$ at $[U]$. Let $T$ be the corresponding vector subbundle of $\wedge^3 W \otimes O_G$. Let $A$ be a 10-dimensional subspace of $\wedge^3 W$ isotropic with respect to the symplectic form $\eta$ defined by $(1.1)$ and such that $\mathbb{P}(A) \cap G(3, W) = \emptyset$. Recall that for $k = 1, 2, 3, 4$ we defined
\[
D_k^4 = \{[U] \in G| \dim(T_U \cap A) \geq k\} \subset G.
\]

Observe that $T$ is a Lagrangian subbundle of $\wedge^3 W \otimes O_G$ with respect to the 2-form $\eta$. It follows that we are in the general situation described at the beginning of Section 2 with $n = 10$, $W_2 = \wedge^3 W$, $X = G$, $J = T$ and $A = A$. Then $T$ defines a map
\[
\iota : G(3, W) \to LG_9(10, \wedge^3 W), \quad [U] \mapsto [T_U].
\]

We denote by $C_U := \mathbb{P}(T_U) \cap G(3, W)$ the intersection of $G(3, W)$ with its projective tangent space $[U]$. Then $C_U$ is linearly isomorphic to a cone over $\mathbb{P}^2 \times \mathbb{P}^2$ with vertex $[U]$. The quadrics containing the cone $C_U$ plays in this situation a similar role in the local analyze of the singularities of $D_k^4$ as the Plücker quadrics containing the Grassmannian $\mathbb{P}(F_{[U]}) \cap G(3, W)$ in $O(G13)$; this will be made more precise in Lemma 2.7.

We aim at proving the following:

**Proposition 2.6.** Let $A \in LG_9(10, \wedge^3 W)$ such that $\mathbb{P}(A) \cap G(3, W) = \emptyset$.

The map $\iota$ is an embedding and $\iota(G(3, W))$ meets transversely all loci $D_k^4 \setminus D_{k+1}^4$ for $k = 1, 2, 3$. In particular each $D_k^4$ is of expected dimension.
For the proof we shall adapt the idea of \cite{OG13} to our context, that we first need to introduce. Let us describe \( \iota \) more precisely locally around a chosen point \([U_0] \in G(3, W)\). For this, we choose a basis \(v_1, \ldots, v_6\) for \(W\) such that \(U_0 = \langle v_1, v_2, v_3 \rangle\) and define \(U_\infty = \langle v_4, v_5, v_6 \rangle\). For any \([U] \in G(3, W)\) we have \(T_U = \wedge^2 U \wedge W\), so \(T_{U_0}, T_{U_\infty}\) are two Lagrangian spaces that intersect only at 0; \(T_{U_0} \cap T_{U_\infty} = 0\). By appropriate choice of \(v_4, v_5, v_6\) we can also assume that \(T_{U_\infty} \cap A = 0\).

Let

\[
\mathcal{V} = \{ [L] \in LG_\eta(10, \wedge^3 W) | L \cap T_{U_\infty} = 0 \}.
\]

The decomposition \(\wedge^3 W = T_{U_0} \oplus T_{U_\infty}\) into Lagrangian subspaces, and the isomorphism \(T_{U_\infty} \to T_{U_0}^{\vee}\) induced by \(\eta\), allows us to view a Lagrangian space \(L\) in \(\mathcal{V}\) as the graph of a symmetric linear map \(Q_L : T_{U_0} \to T_{U_\infty} = T_{U_0}^{\vee}\). Let \(q_L \in Sym^2 T_{U_0}^{\vee}\) be the quadratic form corresponding to \(Q_L\). The map \([L] \mapsto q_L\) defines an isomorphism \(\mathcal{V} \to Sym^2 T_{U_0}^{\vee}\).

Consider the open neighbourhood

\[
\mathcal{U} = \{ [U] \in G(3, W) | T_U \cap T_{U_\infty} = 0 \}
\]

of \([U_0] \in G(3, W)\). For \([U] \in \mathcal{U}\) we denote by \(Q_U := Q_{T_U}\) and \(q_U := q_{T_U}\) the symmetric linear map and the quadratic form corresponding to the Lagrangian space \(T_U\).

We shall describe \(q_U\) in local coordinates. Observe that for any \([U] \in G(3, W)\),

\[
T_U \cap T_{U_\infty} = 0 \iff U \cap U_\infty = 0
\]

and that any such subspace \(U\) is the graph of a linear map \(\beta_U : U_0 \to U_\infty\). In particular, there is an isomorphism:

\[
\rho : \mathcal{U} \to Hom(U_0, U_\infty); \quad [U] \mapsto \beta_U
\]

whose inverse is the map

\[
\alpha : [U_0] := [(v_1 + \alpha(v_1)) \wedge (v_2 + \alpha(v_2)) \wedge (v_3 + \alpha(v_3))].
\]

In the given basis \((v_1, v_2, v_3), (v_4, v_5, v_6)\) for \(U_0\) and \(U_\infty\) we let \(B_U = (b_{ij})_{i,j \in \{1, \ldots, 3\}}\) be the matrix of the linear map \(\beta_U\). In the dual basis we let \((m_0, M)\), with \(M = (m_{ij})_{i,j \in \{1, \ldots, 3\}}\), be the coordinates in

\[
T_{U_0}^{\vee} = (\wedge^3 U_0 \oplus \wedge^2 U_0 \oplus U_\infty)^\vee = (\wedge^3 U_0 \oplus Hom(U_0, U_\infty))^\vee.
\]

Note, that under our identification the map \(\iota : G(3, W) \to LG(10, \text{wedge}^3 W)\) restricted to \(\mathcal{U}\) is the map \([U] \mapsto q_U\), which justifies our slight abuse of notation in the following.

**Lemma 2.7.** In the above coordinates, the map

\[
\iota : \mathcal{U} \ni [U] \mapsto q_U := q_{T_U} \in Sym^2 T_{U_0}^{\vee}
\]

is defined by

\[
q_U(m_0, M) = \sum_{i,j \in \{1, \ldots, 3\}} b_{ij} M^{i,j} + m_0 \sum_{i,j \in \{1, \ldots, 3\}} B_{ij}^{i,j} m_{i,j} + m_0^2 \det B_U,
\]

where \(M^{i,j}, B_{ij}^{i,j}\) are the entries of the matrices adjoint to \(M\) and \(B_U\).

**Proof.** We write in coordinates the map \(\wedge^3 U_0 \oplus \wedge^2 U_0 \oplus U_\infty \to \wedge^3 U_\infty \oplus \wedge^2 U_\infty \oplus U_0\) whose graph is \(\wedge^3 U \oplus \wedge^2 U \oplus U_\infty\) where \(U\) is the graph of the map \(U_0 \to U_\infty\) given by the matrix \(B_U\). \(\square\)

Let now \(Q_A\) be the symmetric map \(T_{U_0} \to T_{U_\infty} = T_{U_0}^{\vee}\) whose graph is \(A\) and \(q_A\) the corresponding quadratic form. In this way

\[
D^A_1 \cap \mathcal{U} = \{ [U] \in \mathcal{U} | \dim T_U \cap A \geq 1 \} = \{ [U] \in \mathcal{U} | \text{rk}(Q_U - Q_A) \leq 10 - l \},
\]
hence $D^A_t$ is locally defined by the vanishing of the $(11 - l) \times (11 - l)$ minors of the $10 \times 10$ matrix with entries being polynomials in $b_{i,j}$.

First we show that the space of quadrics that define $C_U$, surjects onto the space of quadrics on linear subspaces in $\mathbb{P}(T_U)$.

**Lemma 2.8.** If $P \subset \mathbb{P}(T_U) \setminus G(3, 6)$ is a linear subspace of dimension at most 2, then the restriction map $r_P : H^0(\mathbb{P}(T_U), I_{C_U}(2)) \to H^0(P, O_P(2))$ is surjective.

**Proof.** We may restrict to the case when $P$ is a plane. Since $C_U \subset \mathbb{P}(T_U) \cap G(3, 6)$ is projectively equivalent to the cone over $\mathbb{P}^r$ in its Segre embedding, it suffices to show that if $P \subset \mathbb{P}^8$ is a plane that do not intersect $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, then the Cremona transformation $Cr$ on $\mathbb{P}^8$ defined by the quadrics containing $\mathbb{P}^2 \times \mathbb{P}^2$ maps $P$ to a linearly normal Veronese surface. Note that the ideal of $\mathbb{P}^2 \times \mathbb{P}^2$ is a plane. Since the first syzygies between the generators of this ideal are generated by linear ones

$$\mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$$

are linear, $\mathbb{P}^2 \times \mathbb{P}^2$ is a regular, hence finite, morphism. Since the fibers of the $Cr$ emona transformation contracts the secant determinantal cubic hypersurface $V_3$, to a $\mathbb{P}^2 \times \mathbb{P}^2$, so the the inverse Cremona is of the same kind. Furthermore, the fibers of the map $V_3 \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$ are 3-dimensional linear spaces spanned by quadric surfaces in $\mathbb{P}^2 \times \mathbb{P}^2$. Now, by assumption, $P$ does not intersect $\mathbb{P}^2 \times \mathbb{P}^2$, so the restriction $Cr|_P$ is a regular, hence finite, morphism. Since the fibers of the Cremona transformation are linear, $P$ intersects each fiber in at most a single point, so the restriction $Cr|_P$ is an isomorphism. Thus, if $Cr(P)$ is not linearly normal, the linear span $\langle Cr(P) \rangle$ is a $\mathbb{P}^4$, being a smooth projected Veronese surface. Assume this is the case. Then $Cr(P)$ is not contained in any quadric. Since the quadrics that define the inverse Cremona, map $Cr(P)$ to the plane $P$, these quadrics form only a net, when restricted to the 4-dimensional space $\langle Cr(P) \rangle$. In fact the complement of $\mathbb{P}^2 \times \mathbb{P}^2 \cap \langle Cr(P) \rangle$ in $\langle Cr(P) \rangle$ is mapped to $P$ by the inverse Cremona transformation. Therefore $\langle Cr(P) \rangle$ must be contained in the cubic hypersurface that is contracted by this inverse Cremona. Since this hypersurface is contracted to the original $\mathbb{P}^2 \times \mathbb{P}^2$, we infer that $P$ is contained in $\mathbb{P}^2 \times \mathbb{P}^2$. This contradicts our assumption and concludes our proof. □

**Lemma 2.9.** Let $K = A \cap T_{U_0} = \ker Q_A \subset T_{U_0}$ and assume that $k = \dim K \leq 3$. Then for any $l \leq k$ the tangent cone $C_{A,U_0}$ of $D^A_t \cap \mathbb{U}$ at $U_0$ is linearly isomorphic to a cone over the corank $l$ locus of quadrics in $\mathbb{P}(H^0(\mathbb{P}(K), O_{\mathbb{P}(K)}(2)))$.

**Proof.** We follow the idea of [O’G10, Proposition 1.9]. If we choose a basis $\Lambda$ of $T_{U_0}$, the symmetric linear map $Q_U$ is defined by a symmetric matrix $M^A(B_U)$ with entries being polynomials in $(b_{i,j})_{i,j \in \{1...3\}}$.

The linear summands of each entry in $M^A(B_U)$ form a matrix that we denote by $N^A(B_U)$. Since $Q_A = 0$, the entries of $M^A(B_U)$ have no nonzero constant terms. Moreover, by using Lemma 2.7 and $A_0 = (m_0, M)$, we see that the map $\mathbb{U} \ni U \mapsto q_U' \in \text{Sym}^2 T_{U_0}$, where $q_U$ is the quadratic form corresponding to the symmetric map defined by the matrix $N^A_0(B_U)$, maps $\mathbb{U}$ linearly onto the linear system of quadrics containing the cone $C_{U_0}$. Of course, this surjection is independent of the choice of basis.

We now choose a basis $\Lambda$ in $T_{U_0}$ in which $Q_A$ is represented by a diagonal matrix $R_k = \text{diag}\{0...0, 1...1\}$ with $k$ zeros in the diagonal. Then

$$D^A_t \cap \mathbb{U} = \{[U] \in \mathbb{U} | \dim(T_U \cap A) \geq l\} = \{[U] \in \mathbb{U} | \dim \ker(Q_U - Q_A) \geq l\}$$

$$= \{[U] \in \mathbb{U} | \text{rank}(M^A(B_U) - R_k) \leq 10 - l\}.$$
Hence $D^A_I$ is defined in coordinates $(b_{i,j})_{i,j \in \{1, \ldots, 3\}}$ on $\mathfrak{U}$ by $(11 - l) \times (11 - l)$ minors of the matrix $M^\Lambda(B_U) - R_k$. Furthermore, since $[U_0]$ is the point 0 in our coordinates $(b_{i,j})_{i,j \in \{1, \ldots, 3\}}$, the tangent cone to $D^A_I \cap \mathfrak{U}$ at $[U_0]$ is defined by the initial terms of the $(11 - l) \times (11 - l)$ minors of $M^\Lambda(B_U) - R_k$. Note that we can write

$$M^\Lambda(B_U) - R_k = -R_k + N^\Lambda(B_U) + Z(B_U),$$

where the entries of the matrix $Z(B_U)$ are polynomials with no linear or constant terms. We illustrate this decomposition as follows.

$$
\begin{array}{cccc}
N^\Lambda_{k,k} + Z_k & N^\Lambda_{k+1,k} + Z_{k+1} & \cdots & N^\Lambda_{10,10} + Z_{10,10} \\
N^\Lambda_{k+1,k+1} + Z_{k+1} & N^\Lambda_{k+1,k+1} + Z_{k+1} & \cdots & N^\Lambda_{10,10} + Z_{10,10} \\
N^\Lambda_{10,10} + Z_{10,10} & N^\Lambda_{10,10} + Z_{10,10} & \cdots & N^\Lambda_{10,10} + Z_{10,10} \\
\end{array}
$$

Let $\Phi$ be an $(11 - l) \times (11 - l)$ minor of $M^\Lambda(B_U) - R_k$ and consider its decomposition $\Phi = \Phi_0 + \cdots + \Phi_r$ into homogeneous parts $\Phi_d$ of degree $d$. Observe that $\Phi_d = 0$ for $d \leq k-1$, moreover $\Phi_{k-l+1}$ can be nonzero only if the sub matrix associated to the minor $\Phi$ contains all nonzero entries of $R_k$. In the latter case $\Phi_{k-l+1}$ is a $(k+1-l) \times (k+1-l)$ minor of the $k \times k$ upper left corner sub matrix $N^\Lambda_k(B_U)$ of the matrix $N^\Lambda(B_U)$.

In the proof we note that $q'_{U,U}$ the quadric corresponding to the matrix $N^\Lambda(B_U)$ and by $\iota^N$ the map $U \mapsto q'_{U,U}$. Then, by changing $\Phi$ we get that the tangent cone of $D^A_I \cap \mathfrak{U}$ is contained in:

$$\mathfrak{C}^I_{A,U_0} := \{[U] \in \mathfrak{U} \mid \text{rank}(N^\Lambda_k(B_U)) \leq k - l\} = \{[U] \in \mathfrak{U} \mid \text{rank}(q'_{U,U}) \leq k - l\}.$$

The latter is the preimage by $r_K \circ \iota^N$ of the corank $l$ locus in the projective space of quadrics $\mathbb{P}(H^0(\mathcal{O}_C(2)))$. By Lemma [2.8] we have seen that $r_K \circ \iota^N$ is a linear surjection. So we conclude that $\mathfrak{C}^I_{A,U_0}$ is a cone over the corank $l$ locus of quadrics in $\mathbb{P}(H^0(\mathcal{O}_C(2)))$ with vertex a linear space of dimension $10 - \frac{k(k+1)}{2}$. It follows that $\mathfrak{C}^I_{A,U_0}$ is an irreducible variety of codimension $\frac{l(l+1)}{2}$ equal to the codimension of $D^A_I$. Thus we have equality $\mathfrak{C}^I_{A,U_0} = \mathfrak{C}^I_{A,U_0}$ which ends the proof.

**Corollary 2.10.** If $A$ is a Lagrangian space in $\wedge^3 W$, such that $\mathbb{P}(A)$ does not meet $G(3,W)$, then the variety $D^A_I$ is smooth of the expected codimension $\frac{l(l+1)}{2}$ outside $D^A_{I+1}$. Moreover, if $l = 2$ and $\dim A \cap T_{U_0} = 3$, i.e. $[U_0]$ is a point in $D^A_I \setminus D^A_{I+1}$ then the tangent cone $\mathfrak{C}^I_{A,U_0}$ is a cone over the Veronese surface in $\mathbb{P}^5$ centered in the tangent space of $D^3_A$.

**Proof of Proposition 2.6.** It is clear from Lemma 2.7 that $\iota$ is a local isomorphism into its image, and by Corollary 2.10 the subscheme $D^A_I = \iota^{-1}(\iota(G(3,W))) \cap \mathbb{P}^k_A$ is smooth outside $D^A_{I+1}$, so $\iota(G(3,W))$ meets the degeneracy loci transversally.

2.3. **Invariants.** We shall compute the classes of the Lagrangian degeneracy loci $D^A_k \subset G(3,W)$ in the Chow ring of $G(3,W)$. We consider the embedding $\iota : G(3,W) \to LG_q(10, \wedge^3 W)$ defined by the bundle of Lagrangian subspaces $T$ on $G(3,W)$. According to [PR97, Theorem 2.1] the fundamental classes of the Lagrangian degeneracy loci $D^A_k$ are

$$[D^A_1] = [c_1(T^\vee) \cap G(3,W)], \quad [D^A_2] = [(c_2c_1 - 2c_3)(T^\vee) \cap G(3,W)]$$

and

$$[D^A_3] = [(c_1c_2c_3 - 2c_1^2c_4 + 2c_2c_4 + 2c_1c_5 - 2c_3^2)(T^\vee) \cap G(3,W)].$$
The $\mathbb{P}^9$-bundle $\mathbb{P} (\mathcal{T})$ is the projective tangent bundle on $G (3, W)$. So $\mathcal{T}^\vee$ fits into an exact sequence

$$0 \to \Omega_{G (3, W)} (1) \to \mathcal{T}^\vee \to \mathcal{O}_{G (3, W)} (1) \to 0$$

and we get

$$\deg D_1^A = 168, \quad \deg D_2^A = 480, \quad \deg D_3^A = 720$$

**Remark 2.11.** This may be compared with the degree of the line bundle $2H - 3E$ on $S^{[3]}$, where $S$ is a K3 surface of degree 10, $H$ is the pullback of the line bundle of degree 10 on $S$, and $E$ is the unique divisor class such that the divisor of non-reduced subschemes in $S^{[3]}$ is equivalent to $2E$. The degree, i.e. the value of the Beauville Bogomolov form is $q (2H - 3E) = 4$, and the degree and the Euler-Poincaré characteristic of the line bundle is

$$(2H - 3E)^6 = 15q (2H - 3E)^3 = 960 \quad \text{and} \quad \chi (2H - 3E) = 10.$$ 

So if the map defined by $|2H - 3E|$ is a morphism of degree 2, the image would have degree 480, like $D_2^A$.

In the section 4, we show that $S^{[3]}$ for a general K3-surface $S$ of degree 10, admits a rational double cover of a degeneracy locus $D_2^A$. However that double cover is not a morphism.

### 3. The Double Cover of an EPW Cube

**Proposition 3.1.** Let $[A] \in LG_{3} (10, \wedge^3 W)$. If $\mathbb{P} (A) \cap G (3, W) = \emptyset$ and $D^A_2 = \emptyset$, then $D_2^A$ admits a double cover $f : Y_A \to D_2^A$ branched over $D_3^A$ with $Y_A$ a smooth irreducible manifold having trivial canonical class.

Before we pass to the construction of the double cover let us observe the following.

**Lemma 3.2.** Under the assumptions of Proposition 3.1, the variety $D_2^A$ is integral.

**Proof.** We know that $D_2^A$ is of expected dimension. Observe now that by Corollary 2.10 the variety $D_2^A$ is irreducible if and only if it is connected. To prove connectedness we perform a computation in the Chow ring of the Grassmannian $G (3, W)$ showing that the class $[D_2^A]$ does not decompose into a sum of nontrivial effective classes in the Chow group $A^3 (G (3, W))$ whose intersection is the zero class in $A^0 (G (3, W))$. More precisely we compute:

$$[D_2^A] = 16h^3 - 12hs_2 + 12s_3$$

where $h$ is the hyperplane class on $G (3, W)$, $s_2$ and $s_3$ are the Chern classes of the tautological bundle on $G (3, W)$. We then solve in integer coordinates $a, b, c \in \mathbb{Z}$ the equation

$$(ah^3 - bs_2 + cs_3)((16 - a)h^3 - (12 - b)s_2 + (12 - c)s_3) = 0$$

in the Chow group $A^0 (G (3, W))$ which is generated by: $s_2^3, h^3 s_1 s_2, s_3^2$. Multiplying out the equation in the Chow ring and extracting coefficients at the generators we get a system of three quadratic diophantine equations in $a, b, c$:

$$
\begin{align*}
-5a^2 + 4ab - b^2 + 56a &- 20b = 0 \\
-6a^2 + 8ab - 2b^2 - 4ac + 2bc + 72a - 52b + 20c &- 0 \\
6a^2 - 6ab + b^2 + 2ac - c^2 &- 72a + 36b - 4c = 0
\end{align*}
$$

The only integer solutions are: $(0, 0, 0)$ and $(16, 12, 12)$. This ends the proof. \(\square\)
The plan of the construction of the double cover in Proposition 3.1 is the following. We consider the resolution \(\tilde{D}_2^A \to D_2^A\) with exceptional divisor \(E\). We prove that \(E\) is a smooth even divisor, and hence that there is a smooth double cover \(Y \to \tilde{D}_2^A\) branched over \(E\). Finally, we contract the branch divisor of the double cover using a suitable multiple of the pullback of a hyperplane class on \(D_2^A\) by the resolution and the double cover.

Thus, we start by defining the incidences

\[
\tilde{D}_2^A = \{ ([L], [U']) \in G(3, W) \times G(2, A) \mid T_U \supset U' \},
\]

and

\[
\tilde{D}_2^A = \{ ([L], [U']) \in LG_\omega(10, A^3 W) \times G(2, A) \mid L \supset U' \}.
\]

They fit in the following diagram:

\[
\begin{array}{ccc}
G(3, W) & \xrightarrow{\epsilon} & LG_\omega(10, A^3 W) \\
\cup\downarrow & & \cup\downarrow \\
D_2^A & \xrightarrow{\iota_\tilde{D}_2^A} & \tilde{D}_2^A \\
\alpha\downarrow & & \phi\downarrow \\
D_2^A & \xrightarrow{\iota} & \tilde{D}_2^A
\end{array}
\]

**Lemma 3.3.** Under the assumptions of Proposition 3.1, the variety \(\tilde{D}_2^A\) as well as the exceptional locus \(E\) of the map \(\alpha\) are smooth. In particular \(\alpha\) is a resolution of singularities of \(D_2^A\).

**Proof.** Since we know that \(D_2^A = \emptyset\), the resolution \(\alpha : \tilde{D}_2^A \to D_2^A\) is just the blow up of \(D_2^A\) along \(D_3^A\). Now, \(\tilde{D}_2^A \setminus E\) is isomorphic to \(D_2^A \setminus D_3^A\), so, by Corollary 2.10 we deduce that \(\tilde{D}_2^A\) is smooth outside \(E\). Let \(p \in E \subset \tilde{D}_2^A\). Then \(\alpha(p) \in D_3^A\). Take \(P_1, P_2, P_3\) to be three general hyperplanes passing through \(\alpha(p)\). Consider \(Z_P = D_2^A \cap P_1 \cap P_2 \cap P_3\) and its strict transform \(\tilde{Z}_P \subset \tilde{D}_2^A\). We have the following diagram:

\[
\begin{array}{ccc}
\tilde{Z}_P & \xrightarrow{\alpha_P} & \tilde{D}_2^A \\
\downarrow & & \downarrow \alpha \\
Z_P & \xrightarrow{\iota} & D_2^A
\end{array}
\]

The map \(\alpha_P : \tilde{Z}_P \to Z_P\) is the blow up of \(Z_P\) in \(D_3^A \cap P_1 \cap P_2 \cap P_3\), which by Corollary 2.10 is a finite set of isolated points. By the assumption on \(P_1, P_2, P_3\), the strict transform \(\tilde{Z}_P\) contains the whole fiber \((\alpha_P)^{-1}(p)) and hence also \(p \in \tilde{Z}_P\). Let \(P_i\) be the strict transform of \(P_i\) for \(i = 1, 2, 3\). Then \(P_i\) is a Cartier divisor on \(D_2^A\) and \(\tilde{Z}_P = \tilde{P}_1 \cap \tilde{P}_2 \cap \tilde{P}_3\) is a complete intersection of Cartier divisors on \(D_2^A\). Now, from Corollary 2.10 the exceptional divisor \(E_P = E \cap \tilde{Z}_P\) of \(\alpha_P\) is isomorphic to a finite union of disjoint \((\mathbb{P}^2)'s, one for each point in \(D_3^A \cap P_1 \cap P_2 \cap P_3\). But \(E_P\) is itself a Cartier divisor on \(\tilde{Z}_P\) by general properties of blow up. Therefore \(\tilde{Z}_P\) is smooth. We conclude that \(\tilde{D}_2^A\) is smooth at \(p\) and similarly, that \(E\) is smooth at \(p\). \(\square\)

We compute the first Chern class of the normal bundle of the embedding \(\iota : \tilde{D}_2^A \to \tilde{D}_2^A\). 

11
Lemma 3.4.

\[ c_1(\tilde{\nu} N_{i(D_2^3)}|\tilde{D}_2^3) = c_1(\alpha^* \tilde{\nu}^* N_{i(G(3,W))}|L_{G(10,3^3 W)}) = 38h, \]

where \( h \) is the pullback via the resolution \( \alpha \) of the restriction of the hyperplane class on \( G(3, W) \) to \( D_2^3 \).

**Proof.** From the transversality (Proposition 2.6) we have

\[ \tilde{\nu} N_{i(D_2^3)}|\tilde{D}_2^3) = \alpha^* \tilde{\nu} N_{i(G(3, W))}|L_{G(10,3^3 W)}, \]

which gives the first equality.

To get the second, consider the exact sequence:

\[ 0 \to T_{G(3, W)} \to \tilde{\nu}^*(T_{L_{G(10,3^3 W)})} \to \tilde{\nu}^*(N_{i(G(3, W))}|L_{G(10,3^3 W)}) \to 0, \]

and observe that \( \tilde{\nu}^*(T_{L_{G(10,3^3 W)})} = \tilde{\nu}^* (S^2 \mathcal{L}) = S^2 (\tilde{\nu}^* \mathcal{L}) = S^2 \mathcal{T}, \) where \( \mathcal{L} \) denotes, as before, the tautological bundle on the Lagrangian Grassmannian \( L_{G(10,3^3 W)} \). We obtain

\[ c_1(\alpha^* \tilde{\nu}^* N_{i(G(3, W))}|L_{G(10,3^3 W)}) = -11 \alpha^* c_1(T) - 6h. \]

Now, from

\[ 0 \to O_{G(3, W)}(-1) \to \mathcal{T} \to T_{G(3, W)}(-1) \to 0 \]

we obtain \( \alpha^* c_1(T) = -4h \), which proves the lemma. \( \square \)

Note that in our notation we have \( \tilde{\nu}^* H = \tilde{\nu}^* \phi^* c_1(\mathcal{L}) = \alpha^* \tilde{\nu}^* c_1(\mathcal{L}) = \alpha^* c_1(T) = 4h. \)

We aim now at constructing a double covering of \( D_2^3 \) branched along \( E \). It is enough to prove that \( E \) is an even divisor. This follows from the exact sequence:

\[ 0 \to T_{D_2^3} \to \tilde{\nu}^* T_{\tilde{D}_2^3) \to \tilde{\nu}^* N_{i(D_2^3)}|\tilde{D}_2^3) \to 0, \]

and Lemma 2.3. Indeed, from them we infer

\[ c_1(T_{D_2^3}) = \tilde{\nu}^* (9H + R) - 38h = \tilde{\nu}^* (R) - 2h, \]

which, by Lemma 2.3, means \( E = E \cap \tilde{D}_2^3 = \tilde{\nu}^*(H - 2R) = 2K_{D_2^3} \). By Lemma 3.3 there hence exists a smooth double cover \( \tilde{f} : \tilde{Y} \to \tilde{D}_2^3 \) branched along the exceptional locus \( E \) of the resolution \( \alpha \). Moreover, from the adjunction formula for double covers we get \( K_{\tilde{Y}} = \tilde{f}^{-1}(E) =: \tilde{E}. \)

We now need to contract \( \tilde{E} = \tilde{f}^{-1}(E) \) on \( \tilde{Y} \). For that, with slight abuse of notation, we denote by \( h \) the class of the hyperplane section on \( D_2^3 \subset G(3, W) \). Then \( |\tilde{f}^* \alpha^* h| \) is a globally generated linear system whose associated morphism defines \( \alpha \circ \tilde{f} \) and hence contracts \( \tilde{E} \) to a threefold and is 2:1 on \( \tilde{Y} \setminus \tilde{f}^{-1}(E) \). It follows by standard arguments (for example applying Stein factorization and [Har70 Proposition 4.4]) that there exists a number \( n \) such that the system \( n \tilde{f}^* \alpha^* h \) defines a morphism \( \tilde{\alpha} : \tilde{Y} \to \tilde{Y} \) which is a birational morphism contracting exactly \( \tilde{E} \) to a threefold \( Z \) and such that its image \( Y \) is normal. We then have the following diagram

\[ \begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{D}_2^3 \\
\downarrow{\tilde{\alpha}} & & \downarrow{\alpha} \\
Y & \xrightarrow{f} & D_2^3
\end{array} \]

in which \( Y \) admits a 2:1 map \( f : Y \to D_2^3 \) branched along \( D_3^3 \).
Proof of Proposition 3.1. We have constructed $Y$, a normal variety admitting a 2:1 map $f : Y \to D_2^3$ branched along $D_3^3$. Clearly $K_Y = \tilde{E}$ implies $K_Y = 0$. It hence remains to prove that $Y$ is smooth. Since $\alpha$ is a contraction that contracts only $\tilde{E}$ it is clear that $Y$ is smooth outside of $\mathcal{Z} = \alpha(\tilde{E})$. Let now $p \in \mathcal{Z}$ and let $p' = f(p)$. We then choose three general hypersurfaces $P_1, P_2, P_3$ of degree $n$ in $\mathbb{P}(\wedge^3 W)$ passing through $p'$. Consider $Z_P = D_2^3 \cap P_1 \cap P_2 \cap P_3$ and $Z'_P = D_2^3 \cap P_1 \cap P_2 \cap P_3$. Then $Z'_P$ is a finite set of points that includes $p'$. Consider the following natural restriction of the above diagram:

\[
\begin{array}{ccc}
\hat{Y}_P & \xrightarrow{f_P} & \hat{Z}_P \\
\downarrow^{\hat{\alpha}_P} & & \downarrow^{\hat{\alpha}_P} \\
Y_P & \xrightarrow{f_P} & Z_P
\end{array}
\]

Here $\alpha_P = \alpha|_{\alpha^{-1}(Z_P)} : \hat{Z}_P \to Z_P$ is just the blow up of $Z_P$ along $Z_P$. The exceptional divisor $E_P$ is then, by Corollary 2.10, isomorphic to a finite set of disjoint $(\mathbb{P}^2)'$s that each have normal bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$ in $\hat{Z}_P$. Taking the double cover of $\hat{Z}_P$ branched along the exceptional divisor $E_P$, the preimage of these $(\mathbb{P}^2)'$s are the components of $\hat{E}_P \subset \hat{Y}_P$, each component a $\mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)$. The contraction $\hat{\alpha}_P$ contracts the divisor $\hat{E}_P$ to a finite set of points in $Y_P$. It contracts one of its $(\mathbb{P}^2)'$s, denote it by $E^0_P$, to the point $p$. Note also that from the construction, $Y_P$ is the intersection of three Cartier divisors on $Y$ which is smooth outside the finite set of points $Z'_P$. Thus, since we constructed $Y$ to be normal, we deduce that $Y_P$ is also normal. We claim that $p$ must be a smooth point of $Y_P$. Indeed, we know that $\hat{\alpha}_P$ is a birational morphism onto the normal variety $Y_P$. Moreover, all lines $l \subset E^0_P = \mathbb{P}^2$ are numerically equivalent on $\hat{Y}_P$ and satisfy $1 \cdot K_{\hat{Y}_P} = -1 < 0$. It follows from [Mor82, Corollary 3.6], that there exists an extremal ray $r$ for $\hat{Y}_P$ whose associated contraction $\operatorname{cont}_r : \hat{Y}_P \to Y_P$ contracts $\hat{E}_P$ to a point $\hat{p}$ and that $\hat{\alpha}_P$ factorizes through $\operatorname{cont}_r$. By [Mor82, Theorem 3.3] we have that $\operatorname{cont}_r$ is the blow down of $\hat{E}_P$ and $\hat{p}$ is a smooth point of $Y_P$. Let us now denote by $\sigma : \hat{Y}_P \to Y_P$ the morphism satisfying $\hat{\alpha}_P = \sigma \circ \operatorname{cont}_r$. Consider $\sigma_r$, the restriction of $\sigma$ to small open neighborhoods of $\hat{p}$ and $p$. Then $\sigma_r$ is a birational proper morphism which is bijective to an open subset of the normal variety $Y_P$. It follows by Zariski Main Theorem that $\sigma_r$ is an isomorphism and in consequence, $p$ is a smooth point on $Y_P$.

The latter implies that $Y$ must also be smooth at $p$ as it admits a smooth complete intersection subvariety which is smooth at $p$. $\square$

Corollary 3.5. Let $[\alpha] \in LG_9(10, \wedge^3 W)$ be a general Lagrangian subspace with a 3-dimensional intersection with some space $F_{[\alpha]} = \{ w \wedge \alpha \mid \alpha \in \wedge^2 W \}$, then there exists a double cover $f_A : Y_A \to D_2^3$ branched over $D_3^3$, where $Y_A$ is a smooth irreducible sixfold with trivial canonical class.

Proof. It is enough to make a dimension count to prove that the general Lagrangian space $A$ satisfying the assumptions of the Corollary also satisfies the assumptions of Proposition 3.1. Indeed, let as in the introduction

$$\Delta = \{ [A] \in LG_9(10, \wedge^3 W) \mid \forall w \in W : \dim(A \cap F_{[w]}) \geq 3 \},$$

and

$$\Gamma = \{ [A] \in LG_9(10, \wedge^3 W) \mid \exists U \in G(3, W) : \dim(A \cap T_U) \geq 4 \}.$$
We show:

**Lemma 3.6.** The set $\Gamma \subset LG_\eta(10, \wedge^3W)$ is a divisor.

**Proof.** Let us consider the incidence

$$\Xi = \{([U], [A]) \in G(3, W) \times LG_\eta(10, \wedge^3W) : \dim(T_U \cap A) \geq 4\}.$$

The dimension of $\Xi$ can be computed by looking at the projection $\Xi \to G(3, 6)$. For a fixed tangent plane we choose first a $\mathbb{P}^3$ inside: this choice has 24 parameters. Then for a fixed $\mathbb{P}^3$ we have $\dim(LG(6, 12)) = 21$ parameters for the choice of $A$. Thus the dimension of $\Xi$ is $9 + 24 + 21 = 54$. It remains to observe that the projection $\Xi \to LG_\eta(10, \wedge^3W)$ is finite, and that $\dim(LG_\eta(10, \wedge^3W)) = 55$. □

Note that in [OG13 Proposition 2.2] it is proven that $\Delta$ is irreducible and not contained in $\Sigma = \{[A] \in LG(10, 20)|\mathbb{P}(A) \cap G(3, W) \neq \emptyset\}$. Our corollary is now a consequence of Proposition 3.1 and the following lemma.

**Lemma 3.7.** The divisors $\Delta, \Gamma \subset LG_\eta(10, \wedge^3W)$ have no common components.

**Proof.** We need to prove $\dim(\Delta \cap \Gamma) < 54$ which, by the fact that $\Delta$ is irreducible and not contained in $\Sigma$, is equivalent to $\dim((\Delta \cap \Gamma) \setminus \Sigma) < 54$. For this, observe that if $[A] \in (\Delta \cap \Gamma) \setminus \Sigma$ then there exist $[U] \in G(3, W)$ and $[w] \in \mathbb{P}(W)$ with $\dim(A \cap T_U) = 4$ and $\dim(A \cap F_{[w]}) = 3$. We can hence consider the incidence:

$$\Theta = \{([A], [W_3], [W_4], [w], [U]) | W_3 = A \cap F_{[w]}, W_4 = A \cap T_U \} \subset LG_\eta(10, \wedge^3W) \times G(3, \wedge^3W) \times G(4, \wedge^3W) \times \mathbb{P}(W) \times G(3, W)$$

such that its projection to $LG_\eta(10, \wedge^3W)$ contains $(\Delta \cap \Gamma) \setminus \Sigma$. Note also that if we take $([A], [W_3], [W_4], [w], [U]) \in \Theta$ then $W_4 \cap W_3 = W_4 \cap F_{[w]} = W_3 \cap T_U$.

We shall now compute the dimension of $\Theta$ by considering fibers under subsequent projections:

$$\begin{align*}
&LG_\eta(10, \wedge^3W) \times G(3, \wedge^3W) \times G(4, \wedge^3W) \times \mathbb{P}(W) \times G(3, W) \\
&\xrightarrow{\pi_1} G(3, \wedge^3W) \times G(4, \wedge^3W) \times \mathbb{P}(W) \times G(3, W) \xrightarrow{\pi_2} G(4, \wedge^3W) \times \mathbb{P}(W) \times G(3, W) \\
&\xrightarrow{\pi_3} \mathbb{P}(W) \times G(3, W)
\end{align*}$$

We have two possibilities for pairs $([w], [U])$ which give us two types of points to consider:

1. $w \notin U$, then $\dim(T_U \cap F_{[w]}) = 3$.
2. $w \in U$, then $\dim(T_U \cap F_{[w]}) = 7$.

We then have different types of elements in the intersection $\pi_3^{-1}([w], [U]) \cap \pi_2(\pi_1(\Theta))$, depending on the number $d_1 := \dim(W_4 \cap F_{[w]}) = \dim(W_3 \cap W_4) \leq 3$. If $W_3^+ \cap F_{[w]}$ denotes the orthogonal to $W_4$ w.r.t. $\eta$ in $\wedge^3W$, then $\dim(W_3^+ \cap F_{[w]}) = 6 + d_1$. Now, in order for $[W_3]$ to be an element of $\pi_2^{-1}([W_4], [w], [U]) \cap \pi_1(\Theta)$ we must have $W_3 \subset W_4^+ \cap F_{[w]}$.

The fiber $\pi_1^{-1}([W_3], [W_4], [w], [U]) \cap \Theta$ is of dimension $(3 + d_1)(4 + d_1)$. Hence to compute the dimension of each component of $\Theta$ it is enough to compute the dimensions of the spaces $F_i d_i$ of elements $([W_3], [W_4], [w], [U])$ of types $(i, d_1)$, where $i = 1$ if $w \notin U$ and $i = 2$ if $w \in U$.

1. For $i = 1$ we start with a choice of $[U] \in G(3, W)$. Then $[w]$ belongs to an open subset of $\mathbb{P}^5$. We have $d_1 \leq 3$ and $[W_4]$ belongs to the Schubert cycle consisting of 4-spaces in the 10-dimensional space $T_U$ that meet the fixed 3-space $T_U \cap F_{[w]}$ in dimension $d_1$. And $[W_3]$ belongs to the Schubert cycle of
3-spaces in the \((6 + d_1)\text{-dimensional space} \ W^+_4 \cap F_{[w]}\) that contains the space \(W_4 \cap F_{[w]}\) of dimension \(d_1\).

(2) For \(i = 2\) we again start with a choice of \([U] \in G(3, W)\). In this case \([w]\) belongs to \(\mathbb{P}(U)\). We have \(d_1 \leq 3\) and \([W^+_4]\) belongs to the Schubert cycle of 4-spaces in the 10-dimensional space \(T_U\) that meet the fixed 7-space \(T_U \cap F_{[w]}\) in dimension \(d_1\). Then \([W^+_3]\) belongs to the Schubert cycle of 3-spaces in the \((6 + d_1)\text{-dimensional space} \ W^+_4 \cap F_{[w]}\) that contains the space \(W_4 \cap F_{[w]}\) of dimension \(d_1\).

We have:

\[
\dim F_{i,d_1} = \begin{cases} 
9 + 5 + d_1(3 - d_1) + (4 - d_1)6 + (d_1 + 3)(3 - d_1) = 47 - 3d_1 - 2d_1^2 & \text{for } i = 1, \\
9 + 2 + d_1(7 - d_1) + (4 - d_1)6 + (d_1 + 3)(3 - d_1) = 44 + d_1 - 2d_1^2 & \text{for } i = 2.
\end{cases}
\]

In each case we have \(\dim F_{i,d_1} \leq \frac{(3+d_1)(4+d_1)}{2} \leq 53\). It follows that \(\dim \Theta \leq 53\) which implies \(\dim (\Delta \cap \Gamma) \leq 53\). Hence \(\Delta\) and \(\Gamma\) have no common components. \(\square\)

This concludes the proof also of Corollary 4.5. \(\square\)

4. Special \(A\)

Let us recall from [O’G13] the following construction. Let \(V\) and \(V_0\) be two vector spaces of dimensions 5 and 1 respectively. Let \(W = V \oplus V_0\). Consider the space \(\wedge^3 W\) equipped with the symplectic form \(\eta\) given by the wedge product as above. Let \(v_0 \in V_0\), choose \(A\) a general Lagrangian subspace of \(\wedge^3 W\) such that \(A \cap F_{[v_0]}\) is a vector space of dimension 3 i.e. \([A]\) is a general element of the divisor \(\Delta \subset LG_{\eta}(10, \wedge^3 W)\). In particular, we assume \([A] \in \Delta \setminus \Sigma\). Note that, by [O’G13] Proposition 2.2 (2)), for a general \([A] \in \Delta\) there is a unique \([v_0]\) such that \(F_{[v_0]} \cap A\) is of dimension 3.

Let \(\tilde{K} = A \cap F_{[v_0]}\) and denote by \(K \subset \wedge^3 V\) the 3-dimensional subspace such that \(\tilde{K} = v_0 \wedge K\). Observe that there is a natural isomorphism \(\wedge^2 V \rightarrow F_{[v_0]}\) given by wedge product with \(v_0\). The latter induces an isomorphism \(\wedge^3 V \rightarrow F_{[v_0]}^{\vee}\).

Let \([B] \in LG_{\eta}(10, \wedge^3 W)\) be a Lagrangian subspace such that \(B \cap F_{[v_0]} = \{0\}\) and \(B \cap A = \{0\}\). Then the symplectic form \(\eta\) defines a canonical isomorphism \(B \rightarrow F_{[v_0]}^{\vee}\) by which \(A\) appears as the graph of a symmetric map \(\tilde{Q}_A : F_{[v_0]} \rightarrow B = F_{[v_0]}^{\vee}\). Composed with the isomorphisms \(\wedge^2 V \rightarrow F_{[v_0]}\) and \(\wedge^3 V \rightarrow F_{[v_0]}^{\vee}\) we get a symmetric map \(Q_A : \wedge^2 V \rightarrow \wedge^3 V \cong (\wedge^2 V)^\vee\).

Clearly \(\ker Q_A = K\). Let \(q_A\) be the quadric on \(\wedge^2 V\) given by \(Q_A\), then \(q_A\) is a quadric of rank 7; it is a cone over \(K\). The map \(Q_A\) defines an isomorphism \(\wedge^2 V / K \rightarrow K^\perp\) and hence the quadric \(q_A\) defines a quadric \(K^\perp \subset \wedge^3 V:\)

\[q_A^* : \beta \mapsto \text{vol}(\alpha \wedge \beta), \quad \text{where} \quad Q_A(\alpha) = \beta.\]

Moreover, to each \(v^* \in V^\vee\) we associate the quadric:

\[q_{v^*} : \wedge^3 V \ni \omega \mapsto \text{vol}(\omega(v^*) \wedge \omega) \in \mathbb{C}.\]

The quadrics \(q_{v^*}\) are the Plücker quadrics defining the Grassmannian \(G(3, V) \subset \mathbb{P}(\wedge^3 V)\). We denote by \(S^3_A\) the smooth K3 surface (see [O’G13] Corollary 4.9) of genus 6 defined on \(\mathbb{P}(K^\perp)\) by the restrictions of the quadrics \(q_{v^*}\) and the quadric \(q_A^*\). Let \(S^3_A[2]^{\text{vir}}\) and
$S_A^{[3]}$ denote the appropriate Hilbert schemes of points on $S_A$. Observe that we have a natural isomorphism:

$$W \cong V^* \oplus V_0^* \in v^* + cv_0^* \to q_v + c q_A \in H^0(I_{S_A}(2))$$

We then have a rational two to one map:

$$\varphi: S_A^{[2]} \to \mathbb{P}(W)$$

well defined on the open subset consisting of reduced subschemes whose span is not contained in $G$ consisting of quadrics containing the line $\langle \beta_1, \beta_2 \rangle$. Let us describe this map more precisely. Since $\kappa$ also for $\beta_i$ we have a rational two to one map:

$$\varphi(\{ \beta_1, \beta_2 \}) = [c_{12}v_0 + v_1] \in \mathbb{P}(W).$$

It is proven in [O'G13] that $\varphi(\{ \beta_1, \beta_2 \})$ lies on the EPW sextic associated to $A$. Let us present the proof in a way that we will be able to further generalize. It suffices to show that there are nonzero scalars $x_1, x_2$ and an element $\kappa \in K$, such that

$$(x_1(\beta_1 + v_0 \wedge \alpha_1) + x_2(\beta_2 + v_0 \wedge \alpha_2) + v_0 \wedge \kappa) \wedge (c_{12}v_0 + v_1) = 0.$$

Indeed, this implies $|x_1(\beta_1 + v_0 \wedge \alpha_1) + x_2(\beta_2 + v_0 \wedge \alpha_2) + v_0 \wedge \kappa| \in \mathbb{P}(F_{c_{12}v_0 + v_1}) \cap \mathbb{P}(A)$. Let us denote by $\kappa_1, \kappa_2, \kappa_3$ a basis of $K$, then we consider the equation

$$(x_1(\beta_1 + v_0 \wedge \alpha_1) + x_2(\beta_2 + v_0 \wedge \alpha_2) + \sum_{j=1}^3 y_j v_0 \wedge \kappa_j) \wedge (c_{12}v_0 + v_1) = 0.$$

i.e.

$$(-x_1c_{12}v_0 \wedge \beta_1 - x_2c_{12}v_0 \wedge \beta_2 + x_1v_0 \wedge \alpha_1 + x_2v_0 \wedge \alpha_2 + \sum_{j=1}^3 y_j v_0 \wedge \kappa_j \wedge v_1) = 0.$$

To make this equation into a system of linear equations we multiply with the elements of basis in $\wedge^2 V$ and compose with the volume map $\text{vol}: \wedge^3 W \to \mathbb{C}$. We obtain trivial equations when multiplying by $v_1 \wedge v_i, i = 2, 3, 4, 5$. Multiplying with $v_2 \wedge v_3$ we get

$$\kappa_1 \wedge v_1 \wedge v_2 \wedge v_3 = \kappa_i \wedge \beta_1 = 0, i = 1, 2, 3,$$

$$\beta_1 \wedge v_2 \wedge v_3 = 0, \alpha_1 \wedge v_1 \wedge v_2 \wedge v_3 = \alpha_1 \wedge \beta_1 = 0.$$
\[ \omega_2 \land v_1 \land v_2 \land v_3 = \omega_2 \land \beta_1 = c_{12} = c_{12} \text{vol}(v_0 \land \beta_2 \land v_2 \land v_3). \]

So the equation multiplied with \( v_2 \land v_3 \) is also trivial. Similarly, the equation multiplied with \( v_4 \land v_5 \) is trivial. So the only nontrivial linear equations are obtained by multiplying by forms in \((v_2 \land v_4, v_2 \land v_5, v_3 \land v_4, v_3 \land v_5)\). Each of these 2-vectors annihilates \( \beta_1 \) and \( \beta_2 \), so we get the following four independent equations in 5 variables, with a unique solution up to scalars:

\[
\begin{align*}
(x_1 \alpha_1 + x_2 \alpha_2 + \sum_{j=1}^{3} y_j k_j) \land v_0 \land v_1 \land v_2 & = 0. \\
(x_1 \alpha_1 + x_2 \alpha_2 + \sum_{j=1}^{3} y_j k_j) \land v_0 \land v_1 \land v_2 \land v_4 & = 0. \\
(x_1 \alpha_1 + x_2 \alpha_2 + \sum_{j=1}^{3} y_j k_j) \land v_0 \land v_1 \land v_2 \land v_5 & = 0. \\
(x_1 \alpha_1 + x_2 \alpha_2 + \sum_{j=1}^{3} y_j k_j) \land v_0 \land v_1 \land v_3 \land v_4 & = 0. \\
(x_1 \alpha_1 + x_2 \alpha_2 + \sum_{j=1}^{3} y_j k_j) \land v_0 \land v_1 \land v_3 \land v_5 & = 0.
\end{align*}
\]

Let us now consider the rational map \( \psi : S_A^3 \to G(3, W) \) defined on general sub-schemes \( s \subset S_A \) of length 3 as the 3-codimensional space in \( W = H^0(I_S(2)) \) consisting of those quadrics which contain the plane spanned by \( s \). It is clear that for a subscheme corresponding to a general triple of points \( \{\beta_1, \beta_2, \beta_3\} \) we have

\[
(4.2) \quad \psi(\{\beta_1, \beta_2, \beta_3\}) = [(\varphi(\{\beta_1, \beta_2\}) \land \varphi(\{\beta_1, \beta_3\}) \land \varphi(\{\beta_2, \beta_3\})].
\]

**Proposition 4.1.** The map \( \psi \) is a generically 2:1 rational map onto \( D^3_A \).

**Proof.** Let \( \beta_1, \beta_2, \beta_3 \) be three general points on \( S_A \). The proof then amounts to two lemmas:

**Lemma 4.2.** The fiber of \( \psi \),

\[
\psi^{-1}(\psi(\{\beta_1, \beta_2, \beta_3\})) = \{\{\beta_1, \beta_2, \beta_3\}, \{\gamma_1, \gamma_2, \gamma_3\}\}
\]

is two triples of points on \( S_A \) whose union is a set of six distinct points on a twisted cubic contained in \( G(3, V) \).

**Proof.** Let \( U_\beta_1, U_\beta_2, U_\beta_3 \subset V \) be the subspaces corresponding to \( \beta_1, \beta_2, \beta_3 \). Then there exists a unique 3-dimensional subspace \( U_{\beta_1, \beta_2, \beta_3} \) meeting each \( U_{\beta_i} \) in a 2-dimensional space. It follows that \( U_{\beta_1, \beta_2, \beta_3} \) is contained in the intersection \( C_{\beta_1, \beta_2, \beta_3} \) of \( F^6 \) with the Schubert cycle \( S_{\beta_1, \beta_2, \beta_3} \) in \( G(3, V) \) of three-spaces meeting \( U_{\beta_1, \beta_2, \beta_3} \) in a 2-dimensional space. Since \( S_{\beta_1, \beta_2, \beta_3} \) is a cone over \( \mathbb{P}^1 \times \mathbb{P}^2 \) the considered intersection \( C_{\beta_1, \beta_2, \beta_3} \) is, in general, a twisted cubic. Moreover, under the generality assumption \( C_{\beta_1, \beta_2, \beta_3} \cap S_A = C_{\beta_1, \beta_2, \beta_3} \cap q_A^* \) consists of six points. Three of them are \( \beta_1, \beta_2, \beta_3 \) and the residual three will be denoted by \( \gamma_1, \gamma_2, \gamma_3 \). The linear span of \( C_{\beta_1, \beta_2, \beta_3} \) is \( \mathbb{P}^3 \), we denote it by \( P \), and its intersection with \( G(3, V) \) is \( P \cap G(3, V) = C_{\beta_1, \beta_2, \beta_3} \). We denote by \( \Pi \) the plane \( \langle \beta_1, \beta_2, \beta_3 \rangle \). Now, every quadric containing \( S_A \) and \( \Pi \), when restricted to \( P \), decomposes into \( \Pi \) and another plane \( \Pi' \). Since, in general, \( \Pi \) does not pass through \( \gamma_i \) for \( i = 1, 2, 3 \), the plane \( \Pi' \) must pass through the points \( \gamma_i \) for \( i = 1, 2, 3 \). This means that \( \Pi' = \langle \gamma_1, \gamma_2, \gamma_3 \rangle \). It is then clear that \( \psi(\{\beta_1, \beta_2, \beta_3\}) = \psi(\{\gamma_1, \gamma_2, \gamma_3\}) \).
Lemma 4.3.

Assume on the other hand that \( \psi(\{\beta_1, \beta_2, \beta_3\}) = \psi(\{\gamma_1', \gamma_2', \gamma_3'\}) \). Then, by the equations [4.2] and [4.1] we deduce that \( U_{\beta_1, \beta_2, \beta_3} = U_{\gamma_1', \gamma_2', \gamma_3'} \) hence \( C_{\beta_1, \beta_2, \beta_3} = C_{\gamma_1', \gamma_2', \gamma_3'} \). It follows that \( \{\gamma_1', \gamma_2', \gamma_3'\} \subset P \). But the net of quadrics corresponding to \( \psi(\{\beta_1, \beta_2, \beta_3\}) = \psi(\{\gamma_1', \gamma_2', \gamma_3'\}) \) define on \( P \) two planes \( (\gamma_1, \gamma_2, \gamma_3) \) and \( (\beta_1, \beta_2, \beta_3) \). It follows that \( \{\gamma_1, \gamma_2, \gamma_3\} = \{\beta_1, \beta_2, \beta_3\} \) or \( \{\gamma_1', \gamma_2', \gamma_3'\} = \{\gamma_1, \gamma_2, \gamma_3\} \). Which ends the proof.

**Proof.** By appropriate choice of basis of \( V \) we can assume, without loss of generality, that \( \beta_1 = v_1 \wedge v_2 \wedge v_3, \beta_2 = v_1 \wedge v_4 \wedge v_5, \) and \( \beta_3 = v_2 \wedge v_4 \wedge (v_3 + v_5) \). Observe as above that \( \beta_i \wedge \kappa = 0 \) for \( i = 1, 2 \) and \( \kappa \in K \), hence \( \beta_i \) is contained in the space spanned by \( A \) and \( F_{[\kappa]} \). It follows that there exist \( \alpha_i \in \wedge^2 V \) such that \( \beta_i = v_0 \wedge \alpha_i \in A \). We fix such \( \alpha_i \) (determined modulo \( K \)). Since \( A \) is Lagrangian we have:

\[
\alpha_i \wedge \beta_i = 0, \quad i = 1, 2, 3, \quad \alpha_i \wedge \beta_2 = \alpha_2 \wedge \beta_1 := c_{12},
\]
\[
\alpha_1 \wedge \beta_3 = \alpha_3 \wedge \beta_1 := c_{13} \quad \text{and} \quad \alpha_2 \wedge \beta_3 = \alpha_3 \wedge \beta_2 := c_{23}.
\]

As above, a direct computation gives

\[
\varphi(\{\beta_1, \beta_2\}) = c_{12}v_0 + v_1, \quad \varphi(\{\beta_1, \beta_3\}) = c_{13}v_0 + v_2, \quad \text{and} \quad \varphi(\{\beta_2, \beta_3\}) = -c_{23}v_0 + v_4.
\]

It follows that:

\[
T_{\psi(\{\beta_1, \beta_2, \beta_3\})} = \{ \omega \in \wedge^3 W | \omega \wedge (c_{12}v_0 + v_1) \wedge (c_{13}v_0 + v_2) = 0 \wedge (c_{12}v_0 + v_1) \wedge (-c_{23}v_0 + v_4) = \omega \wedge (c_{13}v_0 + v_2) \wedge (-c_{23}v_0 + v_4) = 0 \}.
\]

Again we denote by \( \kappa_1, \kappa_2, \kappa_3 \) a basis of \( K \). Now, \( \beta_i + v_0 \wedge \alpha_i \in A \) and \( K \wedge v_0 \subset A \), so to prove the lemma it is enough to prove that the system of equations

\[
\begin{align*}
(\sum_{i=1}^3 x_i(\beta_i + v_0 \wedge \alpha_i) & \wedge (c_{12}v_0 + v_1) \wedge (c_{13}v_0 + v_2) = 0 \\
(\sum_{i=1}^3 x_i(\beta_i + v_0 \wedge \alpha_i) & \wedge (c_{12}v_0 + v_1) \wedge (-c_{23}v_0 + v_4) = 0 \\
(\sum_{i=1}^3 x_i(\beta_i + v_0 \wedge \alpha_i) & \wedge (c_{13}v_0 + v_2) \wedge (-c_{23}v_0 + v_4) = 0
\end{align*}
\]

in variables \( x = (x_1, x_2, x_3), \ y = (y_1, y_2, y_3) \) has a 2-dimensional set of solutions satisfying \( x = (x_1, x_2, x_3) \neq 0 \). By reductions as above and rearranging we get the system

\[
\begin{align*}
\left\{ { \begin{array}{l}
v_0 \wedge (c_{12}x_2\beta_2 \wedge v_2 + c_{13}x_3\beta_3 \wedge v_1 + (\sum_{i=1}^3 x_i\alpha_i + y_i\kappa_i) \wedge v_1 \wedge v_2 = 0 \\
v_0 \wedge (-c_{12}x_1\beta_1 \wedge v_4 - c_{23}x_3\beta_3 \wedge v_1 + (\sum_{i=1}^3 x_i\alpha_i + y_i\kappa_i) \wedge v_1 \wedge v_4 = 0 \\
v_0 \wedge (-c_{13}x_1\beta_1 \wedge v_4 - c_{23}x_3\beta_2 \wedge v_2 + (\sum_{i=1}^3 x_i\alpha_i + y_i\kappa_i) \wedge v_2 \wedge v_4 = 0
\end{array} } \right.
\end{align*}
\]

(4.3)

To make the system of equations (4.3) into a system of linear equations we multiply each of the equation by the coordinate vectors and obtain a system of 18 linear equations in 6 coordinates. If we now denote the three left hand side expressions dependent on \( (x, y) \) in the equations from (4.3) by \( u_1(x, y), u_2(x, y), u_3(x, y) \in \wedge^3 W \), a straightforward computation, as above, shows that the following equations are trivial:

\[
\begin{align*}
u_1(x, y) \wedge v_0 & = u_1(x, y) \wedge v_1 = u_1(x, y) \wedge v_2 = u_1(x, y) \wedge v_3 = 0, \\
u_2(x, y) \wedge v_0 & = u_2(x, y) \wedge v_1 = u_2(x, y) \wedge v_4 = u_2(x, y) \wedge v_5 = 0, \\
u_3(x, y) \wedge v_0 & = u_3(x, y) \wedge v_2 = u_3(x, y) \wedge v_4 = u_3(x, y) \wedge (v_3 + v_5) = 0.
\end{align*}
\]

The following products are equal

\[
u_1(x, y) \wedge v_4 = -u_2(x, y) \wedge v_2 = u_3(x, y) \wedge v_1 = \left( \sum_{i=1}^3 x_i\alpha_i + y_i\kappa_i \right) \wedge v_0 \wedge v_1 \wedge v_2 \wedge v_4.
\]

18
three quadrics containing $S$ the point corresponding to $F$ double points spans a by $B$ must therefore be three of the six ordinary double points. Since the 3-space spanned $\psi$ moreover, the ramification locus of $\psi$ corresponds to a point of the EPW cube when the intersection of this reducible quadric.

Proposition 4.5. The rational map $\psi$ is well defined outside a set of codimension 2. Moreover, the ramification locus of $\psi$ is of codimension $\geq 2$. Before we pass to the proof of the Proposition we introduce some more notation. Recall first that, by the assumption on generality of $A$, we know that $S_A$ does not contain any line, conic or twisted cubic. Let $F_A$ be the Fano threefold obtained as the intersection $G(3, V) \cap \langle S_A \rangle$. By the generality of $A$, it follows that $F_A$ is smooth. Let

\[ u_1(x, y) \wedge v_5 = c_{13} x_3 v_0 \wedge \ldots \wedge v_5 + \left( \sum_{i=1}^{3} x_i \alpha_i + y_i \kappa_i \right) \wedge v_0 \wedge v_1 \wedge v_2 \wedge v_5 \]

\[ u_2(x, y) \wedge v_3 = c_{23} x_3 v_0 \wedge \ldots \wedge v_5 - \left( \sum_{i=1}^{3} x_i \alpha_i + y_i \kappa_i \right) \wedge v_0 \wedge v_1 \wedge v_3 \wedge v_4, \]

\[ u_3(x, y) \wedge (v_3 - v_5) = (c_{13} x_1 - c_{23} x_2) v_0 \wedge \ldots \wedge v_5 - \left( \sum_{i=1}^{3} x_i \alpha_i + y_i \kappa_i \right) \wedge v_0 \wedge v_2 \wedge v_4 \wedge (v_3 - v_5). \]

So the 18 linear equations are reduced the following four independent ones:

\[ u_1(x, y) \wedge v_4 = 0, \quad u_1(x, y) \wedge v_5 = 0, \]

\[ u_2(x, y) \wedge v_3 = 0, \quad u_3(x, y) \wedge (v_3 - v_5) = 0. \]

It follows that the system of linear equations admits a 2-dimensional system of solutions. To prove that nonzero solutions satisfy $x \neq 0$ it is enough to observe that a solution with $x = 0$ is a 3-vector $v_0 \wedge \kappa$ with $\kappa \in K$ such that

\[ \kappa \wedge v_1 \wedge v_2 = \kappa \wedge v_1 \wedge v_4 = \kappa \wedge v_2 \wedge v_4 = 0. \]

But any such $\kappa$ lies in the space $\langle v_1 \wedge v_2, v_1 \wedge v_4, v_2 \wedge v_4 \rangle = \wedge^2 \langle v_1, v_2, v_4 \rangle$. By assumption, $\mathbb{P}(v_0 \wedge K) \subset \mathbb{P}(A)$ does not intersect $G(3, W)$, so this is impossible. Therefore the only solution of the system \([4.3]\) satisfying $x = 0$ is $(x, y) = (0, 0)$.

Remark 4.4. There is an alternative approach to the Proposition 4.1. We consider the intersection $F$ of the quadrics containing $S_A$ together with a generic plane $B = \langle \beta_1, \beta_2, \beta_3 \rangle$. This is a complete intersection of degree 8 with six ordinary double points that span a 3-space. Three of them are the points of intersection of $B \cap S_A$ and the residual three points span another plane $B'$ contained in our Fano threefold $F$. Since $S_A$ does not contain any plane curve, if a plane passes through three points of $S_A$ these points are isolated in the intersection. If the plane is contained in $F$, the three points must therefore be three of the six ordinary double points. Since the 3-space spanned by $B$ and $B'$ cuts $F$ along the sum of $B \cup B'$ it follows that the degree of $\psi$ is two at the point corresponding to $F$. On the other hand the generic complete intersection of three quadrics containing $S_A$ have also six ordinary double points. The six ordinary double points spans a $\mathbb{P}^3$ and a complete intersection $F$ of degree 8 that contain $S_A$ corresponds to a point of the EPW cube when the intersection of this $\mathbb{P}^3$ with $F$ is a reducible quadric.

Next, we compute the codimension of the indeterminacy locus and the ramification locus of $\psi$.
Consider the Schubert cycle \( S_U = \{ U' \in G(3, V) \mid \dim(U \cap U') \geq 2 \} \). It is clear that in the Plücker embedding of \( G(3, V) \subset \mathbb{P}(\wedge^3 V) \) the variety \( S_U \) is the tangent cone of \( G(3, V) \) in \([U]\). It spans the projective tangent space and is a cone over \( \mathbb{P}^1 \times \mathbb{P}^2 \) with vertex \([U]\). We are interested in intersections \( S_U \cap F_A \). Note that \( F_A \) is of degree 5 and has Picard group of rank 1 generated by the hyperplane class. Hence \( F_A \) does not contain any surface of degree \( \leq 4 \). It follows that \( \mathbb{P}^6 \cap S_U = F_A \cap S_U \) is a cubic curve, a possibly reducible or nonreduced degeneration of a twisted cubic curve.

We denote the corresponding subscheme of the Hilbert scheme of twisted cubics in \( F_A \) by \( \mathcal{H}_A \).

Let \( \mathcal{B}_1 \) be the subset of \( S_A^{[3]} \) consisting of those subschemes that are contained in a conic in \( F_A \subset G(3, V) \). Since \( F_A \) is a linear section of \( G(3, V) \) and contains no planes, the Hilbert scheme of conics in \( F_A \) admits a birational map to \( \mathbb{P}(V) \) associating to a conic \( \mathcal{C} \) the intersection of three-spaces parametrized by points on \( \mathcal{C} \). It is hence of dimension 4 and we get that \( \mathcal{B}_1 \) is of dimension 4. Let \( \mathcal{B}_2 \) be the subset of \( S_A^{[3]} \) consisting of those subschemes that meet some line contained in \( G(3, V) \) in a scheme of length two. Then \( \mathcal{B}_2 \) is also of dimension 4, since the Hilbert scheme of lines in \( F_A \) isomorphic to \( \mathbb{P}^2 \) (cf. \cite[Proposition 5.2]{OG13}, \cite{Isk77}).

**Lemma 4.6.** Let \( \mathfrak{s} \) be a subscheme of length 3 in \( S_A \) corresponding to a point from \( S_A^{[3]} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2) \). Then there is a unique, possibly degenerate, twisted cubic from \( \mathcal{H}_A \) that contains \( \mathfrak{s} \). Furthermore, the induced map \( S_A^{[3]} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2) \to \mathcal{H}_A \) is dominant.

**Proof.** Since \( S_A \subset G(3, V) \cong G(2, V^\vee) \), we may characterize the elements of \( \sigma \in S_A^{[3]} \) via the incidence of curves \( C_\sigma \) of degree 3 in \( \mathbb{P}(V^\vee) \) supported on lines. For a general \( \sigma \), the curve \( C_\sigma \) is the union of three lines and has a unique transversal line, a line that meet all three lines. If \( \sigma \in S_A^{[3]} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2) \) the curve \( C_\sigma \) spans \( \mathbb{P}(V^\vee) \) and contains no conic. It follows that \( C_\sigma \) admits a unique transversal line hence \( \mathfrak{s}_\sigma \) is contained in \( S_U \) for a unique \( U \). We conclude by the definition of \( \mathcal{H}_A \). For dominancy of the map we observe that if \( \mathcal{C} \in \mathcal{H}_A \) then \( \mathcal{C} \cap q_A^* \subset S_A \) and clearly contains a subscheme in \( S_A^{[3]} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2) \).

We can now pass to the proof of Proposition 4.5.

**Proof of Proposition 4.5.** Any subscheme \( \mathfrak{s} \) of length 3 in \( S_A \) spans a plane \( \Pi_\mathfrak{s} \). The map \( \psi \) associates to \( \mathfrak{s} \) the space \( V^\vee_\mathfrak{s} \) of quadrics containing \( S_A \cap \Pi_\mathfrak{s} \). For general \( \mathfrak{s} \) the latter is a space of dimension 3. Now, \( \psi \) is well defined exactly on those \( \mathfrak{s} \) for which \( \dim V^\vee_\mathfrak{s} = 3 \). But \( V^\vee_\mathfrak{s} \) is the kernel of the restriction map \( H^0(S_A, \mathcal{I}_{S_A}(2)) \to H^0(\Pi_\mathfrak{s}, \mathcal{I}_{S_A \cap \Pi_\mathfrak{s}}(2)) \). The latter kernel is 3-dimensional unless \( \dim H^0(\Pi_\mathfrak{s}, \mathcal{I}_{S_A \cap \Pi_\mathfrak{s}}(2)) \leq 2 \). Hence \( \psi \) is not defined only if \( S_A \cap \Pi_\mathfrak{s} \) has length at least 4. Then the intersection \( \Pi_\mathfrak{s} \cap G(3, V) \) contains a scheme of length 4. As \( S_A \) contains no conics, \( \Pi_\mathfrak{s} \) cannot be contained in \( G(3, V) \). We infer by \cite[proof of Lemma 2.2]{Muk93} that \( \Pi_\mathfrak{s} \cap G(3, V) \) contains a line or a unique conic. If \( \Pi_\mathfrak{s} \cap G(3, V) \) contains a line, then it is either a reducible conic or the union of this line with a point. In the latter case, since \( S_A \) contains no lines, the intersection \( \Pi_\mathfrak{s} \cap S_A \) does not contain any subscheme of length 4. It follows that there is a map with finite fibers from the indeterminacy locus of \( \psi \) to the Hilbert scheme of conics in \( G(3, V) \cap \mathbb{P}^6 \) which is of dimension 4. We conclude that the indeterminacy locus is of dimension at most 4. In fact it is equal to 4 since a general \( V_4 \subset V \) defines a conic in \( G(3, V) \cap \mathbb{P}^6 \) which meets \( S_A \) in four points.

Finally to bound the dimension of the ramification locus, we again let \( \mathfrak{s} \) be a subscheme of length 3 in \( S_A \) corresponding to a point from \( S_A^{[3]} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2) \). Then by Lemma
there is a possibly degenerate twisted cubic from $H_s$ spanning a $\mathbb{P}^3$ and containing $s$. Now, from the proof of Proposition \ref{prop:existence}, we know that a point from $S_A^{[3]} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ can be in the ramification locus of $\psi$ only if the quadric $Q_A$ is totally tangent to the twisted cubic. The latter is a codimension 3 condition on twisted cubics in $G(3, V) \cap \mathbb{P}^6$, hence by Lemma \ref{lem:existence} a codimension 3 condition for the ramification locus. To be more precise we have an incidence:

$$\mathcal{X} = \{(C, Q) \in H_A \times H^0(O_{\mathbb{P}^6}(2))| \ Q|_C \text{ is totally non reduced}\}.$$  

We compute its dimension from the projection onto $H_A$. Indeed, fixing $C$ we get a codimension 3 space of quadrics totally tangent to it. The dimension of the general fiber of the second projection follows giving codimension 3 in $H_A$. \hfill $\square$

5. the proof of Theorem 1.1

Let us choose a generic Lagrangian space $A_0$ satisfying $[A_0] \in \Delta \setminus (\Gamma \cup \Sigma) \subset LG_9(10, \wedge^3 W)$. Note that from Lemma \ref{lem:existence} we can choose $A_0$ such that $K$ is generic in $F_{[v_0]}$. From Proposition \ref{prop:existence} there is a rational 2 : 1 map $\psi : S_A^{[3]} \rightarrow D_A^{A_0}$. On the other hand from Proposition \ref{prop:double_cover} there exists a double cover $Y_{A_0} \rightarrow D_A^{A_0}$ such that $Y_{A_0}$ is a smooth sixfold with trivial canonical bundle. Our aim is to construct a birational map

$$S_A^{[3]} \rightarrow X_{A_0}.$$  

We consider the subset $B$ in $S_A^{[3]}$, the union of the indeterminacy locus and the ramification locus of the rational 2 : 1 map $\psi : S_A^{[3]} \rightarrow D_A^{A_0}$. Clearly the restriction of the map $\psi$ to $S_A^{[3]} \setminus B$ is an étale covering of degree 2 onto a smooth open subset $D \subset D_A^{A_0}$. In particular $D \cap D_A^{A_0} = \emptyset$. Note that $S_A^{[3]}$ is simply connected and by Proposition \ref{prop:existence} the subset $B$ is of codimension 2. This implies that $S_A^{[3]} \setminus B$ is also simply connected. It follows that $\pi_1(D) = \mathbb{Z}_2$ and $\psi|_{S_A^{[3]} \setminus B}$ is a universal covering.

Since $D$ is disjoint from $D_A^{A_0}$, the restriction of the double cover $f_{A_0} : Y_{A_0} \rightarrow D_A^{A_0}$ to $f_{A_0}^{-1}(D)$ is also an étale covering.

By Proposition \ref{prop:double_cover} the variety $Y_{A_0}$ is smooth and irreducible. It follows that the étale covering $f_{A_0}|_{f_{A_0}^{-1}(D)}$ is not trivial. We infer that $f_{A_0}|_{f_{A_0}^{-1}(D)}$ is also the universal covering, and deduce that $Y_{A_0}$ is birational to $S_A^{[3]}$.

Note that the fact that $f_{A_0}|_{f_{A_0}^{-1}(D)}$ is the universal covering implies $f_{A_0}^{-1}(D)$ is simply connected. It follows that $Y_{A_0}$ is also simply connected because $f_{A_0}^{-1}(D)$ is obtained from the smooth variety $Y_{A_0}$ by removing a subset of real codimension 2. Moreover, since both $Y_{A_0}$ and $S_A^{[3]}$ have trivial canonical bundle, by \cite[Theorem 1.1]{Ito03} they have equal Hodge numbers. Thus

$$h^2(O_{Y_{A_0}}) = h^2(O_{S_A^{[3]}}) = 1.$$  

From the Beauville classification theorem \cite[Theorem 2]{Bea83} we infer that $Y_{A_0}$ is IHS.

Recall the notation

$$LG^1_q(10, \wedge^3 W) := \{[A] \in LG_q(10, \wedge^3 W)|\mathbb{P}(A) \cap G(3, W) = \emptyset, \forall [U] \in G(3, W) : \dim(A \cap T_U) \leq 3\}.$$  

Consider now the varieties:

$$D_k = \{([A], [U]) \in LG^1_q(10, \wedge^3 W) \times G(3, W)|[U] \in D_{k}^1\},$$

where $k$ is a natural number.
for \( k = 2,3 \). By globalizing the construction in Proposition 5.1 to the affine variety \( LG^1_\eta(10, \wedge^3 W) \) we construct a variety \( \mathcal{Y} \) which is a double cover of \( D_2 \) branched in \( D_3 \). We get a smooth family 
\[
\mathcal{Y} \to LG^1_\eta(10, \wedge^3 W)
\]
with fibers \( \mathcal{Y}_{|A} = Y_A \) polarized by the divisor defining the double cover. In particular a special fiber \( \mathcal{Y}_{|A_0} = Y_{A_0} \) is an IHS manifold. Since a smooth deformation of an IHS manifold is still IHS we obtain that \( Y_A \) is IHS for every \( A \in LG^1_\eta(10, \wedge^3 W) \). So \( \mathcal{Y} \to LG^1_\eta(10, \wedge^3 W) \) is a family of IHS manifolds.

In order to show that the IHS sixfolds in the family \( \mathcal{Y} \) are of \( K3^{[3]} \)-type we use the fact proved above that \( S^{[3]}_{\eta} \) and \( Y_{A_0} \) are birational. Indeed, two birational IHS manifolds are deformation equivalent from \([\text{Huy99, Theorem 4.6}]\). The Beauville-Bogomolov degree \( q = 4 \) of our polarization follows from our computation of degree in Section 2.3.

We end the proof of Theorem 1.1 by performing a study of the moduli map defined by the family \( \mathcal{Y} \).

**Proposition 5.1.** Let \( \mathcal{M} \) be the coarse moduli space of polarized IHS sixfolds of \( K3^{[3]} \)-type and Beauville-Bogomolov degree 4. Let
\[
\mathfrak{M}_\mathcal{Y} : LG^1_\eta(10, \wedge^3 W) \to \mathcal{M}, \quad A \mapsto [Y_A]
\]
be the map given by \( \mathcal{Y} \). The image of \( \mathfrak{M}_\mathcal{Y} \) is a dense open subset of a component of dimension 20 in \( \mathcal{M} \).

For the proof we will need the following lemma.

**Lemma 5.2.** Let \( A \in LG^1_\eta(10, \wedge^3 W) \). If \( g \in PGL(\wedge^3 W) \) is such that \( D_2^4 \subset G(3, W) \cap g(G(3, W)) \), then \( G(3, W) = g(G(3, W)) \).

**Proof.** Let us denote by \( G_1, G_2 \) the varieties \( G(3, W) \) and \( g(G(3, W)) \) respectively.

Let \( X \subset G_1 \cap G_2 \) be an irreducible component of the intersection that contains \( D_2^4 \). Then \( X \) has codimension at most 3 in both \( G_1 \) and \( G_2 \) and spans \( \mathbb{P}^{19} \). Furthermore it is contained in a complete intersection of quadric hypersurfaces on each \( G_i \). If \( X \) has codimension 3, then \( X = D_2^4 \) and lies in a complete intersection of three quadrics. But the complete intersection has degree \( 8 \cdot 42 = 336 \), while \( D_2^4 \) has degree 480, so this is impossible.

For lower codimension of \( X \) we first note that \( D_2^4 \subset D_2^4 \). Since \( [D_2^4] = [c_1(T^\vee) \cap G(3, W)] \) and \( c_1(T^\vee) = 4h \), the divisor \( D_2^4 \) is a quartic hypersurface section of \( G_1 \) and \( G_2 \). So we may assume that \( D_2^4 \) is contained in a quartic hypersurface section of \( X \).

Consider the following subvariety in \( G_1 \): Let \( V_5 \subset W \) be a general 5-dimensional subspace, and let \( V_1 \) be a general 1-dimensional subspace of \( V_5 \). Let \( F(1, 5) = \{ [U] \in G_1 | V_1 \subset U \subset V_5 \} \subset G_1 \) and denote by \( P(1, 5) \) the span of \( F(1, 5) \). Then \( F(1, 5) \) is a 4-dimensional smooth quadric and the span, \( P(1, 5) \), is a \( \mathbb{P}^5 \).

If \( X \) has codimension 2, then \( X_{(1, 5)} := X \cap F(1, 5) \) is an irreducible surface. Furthermore, \( X_{(1, 5)} \) is contained in at least 2 quadric sections of \( F(1, 5) \). So \( X_{(1, 5)} \) has degree at most 8. On the other hand
\[
D_{(1, 5)} := D_2^4 \cap F(1, 5) \subset X_{(1, 5)}
\]
is a curve of degree 56, contained in a quartic hypersurface section of \( X_{(1, 5)} \), which has degree at most 32. Since this is absurd, we may assume that \( X \) has codimension one, i.e. is a divisor in the \( G_i \).

Since \( D_2^4 \) spans \( \mathbb{P}^9 \), the divisor \( X \) must be a quartic hypersurface section of each \( G_i \). Then \( P(1, 5) \cap X \) is complete intersection of two quadrics, and through every point of
P(1, 5) there are infinitely many secant lines to X. The union of the spaces P(1, 5) as V₅ and V₁ varies is a variety Ω₁ ⊂ P^{19}, characterized in [Don77] Lemma 3.3 as the locus of points in P^{19} that lies on more than one secant line to G₁. Furthermore G₁ is the singular locus of Ω₁. Similarly, Ω₂ is defined with respect to G₂. By the above argument each P(1, 5) in Ω₁ is also contained in Ω₂. Thus Ω₁ ⊂ Ω₂. But then they coincide, and since G₁ = Sing(Ω₁), the two grassmannians G₁ and G₂ coincide.

\[\square\]

**Proof.** We claim that \(\mathfrak{M}_Y([A₁]) = \mathfrak{M}_Y([A₂])\) if and only if there exists a linear automorphism \(g ∈ Aut(G(3, W)) \simeq \mathbb{Z}/2 \times PGL(W)\) such that \(g(A₁) = A₂\). Indeed, assume that \(\mathfrak{M}_Y([A₁]) = \mathfrak{M}_Y([A₂])\). Then \(A₁\) and \(A₂\), polarized by ample classes defining double covers to \(D_{A₁}^2\) and \(D_{A₂}^2\) respectively, are isomorphic. It follows that there is a linear automorphism \(g ∈ PGL(\wedge³W)\) such that \(g(D_{A₁}^2) = D_{A₂}^2\). It follows that \(D_{A₂}^2 \subset G(3, W) \cap g(G(3, W))\). By Lemma [5.2] we deduce that \(G(3, W) = g(G(3, W))\). It follows that \(g ∈ Aut(G(3, W))\).

By [O’G06] the locus \(LG₁^{11}(10, \wedge³W)\) is contained in the stable locus of the natural linearized \(PGL(W)\) action on \(LG₁^{11}(10, \wedge³W)\). From our claim we hence infer that

\[\dim(\mathfrak{M}_Y(LG₁^{11}(10, \wedge³W))) ≥ \dim LG₁^{11}(10, \wedge³W) - \dim(PGL(W)) = 55 - 35 = 20.\]

But 20 is the dimension of \(\mathcal{M}\), so our map is surjective onto an (also by stability) open subset of a component of \(\mathcal{M}\) of dimension 20. \(\square\)

We conclude by determining the component of the moduli space that is filled by our family.

Recall that for \(v ∈ H²((K3)^{[3]}, \mathbb{Z})\) the divisibility of \(v\) is defined as the generator of the subgroup \((v, H²((K3)^{[3]}, \mathbb{Z})) ⊂ \mathbb{Z}\) where \((., .)\) is the scalar product induced by the Beauville-Bogomolov form. Note that for Beauville-Bogomolov degree 4 there are two possible divisibilities for \(H\) either \(l = 1\) or \(2\) (see [GHS10] Proposition 3.6]). It follows from [Apo11] Proposition 2.1(3) and Corollary 2.4] that there are exactly two components, distinguished by the divisibility, of the coarse moduli space of polarized IHS sixfolds of \(K³^{[3]}\)-type and Beauville-Bogomolov degree 4. Which one is determined by the following proposition, whose proof was pointed out to us by Kieran O’Grady.

**Proposition 5.3.** The image of \(\mathfrak{M}_Y\) is open and dense in the connected component of the coarse moduli space of IHS sixfolds of \(K³^{[3]}\)-type, Beauville-Bogomolov degree 4 and divisibility 2.

**Proof.** By the above, it remains to compute the divisibility of our polarization. For this, fix \(A\) general and denote the polarization by \(P\). Observe that the involution of the double cover \(Y_A → D_A^2\) defined by the polarization is anti-symplectic. Indeed as an involution on an IHS manifold it is either symplectic or anti-symplectic, but the fixed point locus of a symplectic involution is a symplectic manifold (see [Cam12] Proposition 3]) whereas the fixed locus of our involution is of dimension 3. This means that the involution must be anti-symplectic. Moreover, since we proved that our family is of maximal dimension, we may assume that \(Y_A\) has Picard group spanned by the polarization \(P\). It follows that the action of the involution on \(H²(Y_A)\) has an invariant subspace spanned by the class \([P]\). Furthermore, the involution respects the Beauville-Bogomolov bilinear form \((., .)\). Thus, since \(([P], [P]) = 4\), the involution on \(H²(Y_A)\) is of the form

\[v ↦ -v + \frac{1}{2}(v, [P])[P].\]
Since the involution must map integral cohomology to integral cohomology, it follows
that \((v, [P])\) is even for all integral classes \(v\). This implies that the divisibility of \([P]\) is
not equal to 1. We infer that it is equal to 2. □

REFERENCES

[Apo11] Apostol Apostolov. Moduli spaces of polarized irreducible symplectic manifolds are not
necessarily connected. arXiv:1109.0175, 2011.

[AR04] Alberto Alzati and Francesco Russo. Some extremal contractions between smooth varieties
arising from projective geometry. Proc. London Math. Soc. (3), 89(1):25–53, 2004.

[BD85] Arnaud Beauville and Ron Donagi. La variété des droites d’une hypersurface cubique de
dimension 4. C. R. Acad. Sci. Paris Sér. I Math., 301(14):703–706, 1985.

[Bea83] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. J. Differential Geom., 18(4):755–782 (1984), 1983.

[Cam12] Chiara Camere. Symplectic involutions of holomorphic symplectic four-folds. Bull. Lond. Math. Soc., 44(4):687–702, 2012.

[Don77] Ron Y. Donagi. On the geometry of Grassmannians. Duke Math. J., 44(4):795–837, 1977.

[DV10] Olivier Debarre and Claire Voisin. Hyper-kähler fourfolds and Grassmann geometry. J. Reine Angew. Math., 649:63–87, 2010.

[GHS07] V. A. Gritsenko, K. Hulek, and G. K. Sankaran. The Kodaira dimension of the moduli of
K3 surfaces. Invent. Math., 169(3):519–567, 2007.

[GS10] V. Gritsenko, K. Hulek, and G. K. Sankaran. Moduli spaces of irreducible symplectic manifolds. Compos. Math., 146(2):430–434, 2010.

[Har70] Robin Hartshorne. Ample subvarieties of algebraic varieties. Lecture Notes in Mathematics, Vol. 156. Springer-Verlag, Berlin-New York, 1970. Notes written in collaboration with C. Musili.

[Har77] Robin Hartshorne. Algebraic geometry. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Huy99] Daniel Huybrechts. Compact hyper-Kähler manifolds: basic results. Invent. Math., 135(1):63–113, 1999.

[IR01] Atanas Iliev and Kristian Ranestad. K3 surfaces of genus 8 and varieties of sums of powers of
cubic fourfolds. Trans. Amer. Math. Soc., 353(4):1455–1468, 2001.

[Iso77] V. A. Iskovskih. Fano threefolds. I. Izv. Akad. Nauk SSSR Ser. Mat., 41(3):516–562, 717, 1977.

[Ito03] Tetsushi Ito. Birational smooth minimal models have equal Hodge numbers in all dimen-
sions. In Calabi-Yau varieties and mirror symmetry (Toronto, ON, 2001), volume 38 of Fields Inst. Commun., pages 183–194. Amer. Math. Soc., Providence, RI, 2003.

[Kap14] Grzegorz Kapustka. On ihs fourfolds with \(b_2 = 23\). arXiv:1403.1074, The Michigan Mathematical Journal 65 (1), 3–33, 2014.

[LLSvS15] Christian Lehnh, Manfred Lehn, Christoph Sorger, and Duco van Straten. Twisted cubics
on cubic fourfolds. J. reine angew. Math., 2015.

[Mor82] Shigefumi Mori. Threefolds whose canonical bundles are not numerically
effective. Ann. of Math., 116:133–176, 1982.

[Muk92] Shigeru Mukai. Polarized K3 surfaces of genus 18 and 20. In Complex projective geometry
(Trieste, 1989/Bergen, 1989), volume 179 of London Math. Soc. Lecture Note Ser., pages
264–276. Cambridge Univ. Press, Cambridge, 1992.

[Muk93] Shigeru Mukai. Curves and Grassmannians. In Algebraic geometry and related topics (In-
cheon, 1992), Conf. Proc. Lecture Notes Algebraic Geom., I, pages 19–40. Int. Press, Cam-
bridge, MA, 1993.

[Muk06] Shigeru Mukai. Polarized K3 surfaces of genus thirteen. In Moduli spaces and arithmetic
gometry, volume 45 of Adv. Stud. Pure Math., pages 315–326. Math. Soc. Japan, Tokyo,
2006.

[Muk10] Shigeru Mukai. Curves and symmetric spaces, II. Ann. of Math. (2), 172(3):1539–1558,
2010.

[Muk12] Shigeru Mukai. K3 surfaces of genus 16. RIMS 1743, 2012.

[O’G06] Kieran G. O’Grady. Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics.
Duke Math. J., 134(1):99–137, 2006.

[O’G10] Kieran G. O’Grady. Epw-sextics: taxonomy. arXiv:1007.3882 [math.AG], 2010.
Kieran G. O’Grady. Double covers of EPW-sextics. *Michigan Math. J.*, 62:143–184, 2013.

P. Pragacz and J. Ratajski. Formulas for Lagrangian and orthogonal degeneracy loci; Q-polynomial approach. *Compositio Math.*, 107(1):11–87, 1997.

Piotr Pragacz. Algebro-geometric applications of Schur S- and Q-polynomials. In *Topics in invariant theory (Paris, 1989/1990)*, volume 1478 of *Lecture Notes in Math.*, pages 130–191. Springer, Berlin, 1991.

Seoul National University, Department of Mathematics, Gwanak Campus, Bldg. 27, Seoul 151-747, Korea  
*E-mail address*: ailiev@snu.ac.kr

Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, P.O. Box 21, 00-956 Warszawa, Poland  
*E-mail address*: grzegorz.kapustka@uj.edu.pl

Jagiellonian University in Kraków, ul. Łojasiewicza 6, 30-348 Kraków, Poland  
*E-mail address*: michal.kapustka@uis.no

University of Stavanger, Department of Mathematics and Natural Sciences, NO-4036 Stavanger, Norway  
*E-mail address*: ranestad@math.uio.no