Electroweak Knot

Y. M. Cho
School of Physics, College of Natural Sciences, Seoul National University, Seoul 151-742, Korea
and
C.N. Yang Institute for Theoretical Physics, State University of New York, Stony Brook, NY 11790, USA

We demonstrate the existence of stable knot solitons in the standard electroweak theory whose topological quantum number \( \pi_3(\mathbb{S}^2) \) is fixed by the Chern-Simon index of the Z boson. The electroweak knots are made of the helical magnetic flux tube of Z boson which has a non-trivial dressing of the Higgs field, which could also be viewed as two quantized flux rings linked together whose linking number becomes the knot quantum number. We estimate the mass of the lightest knot to be around 21 TeV.

PACS numbers: 12.15.-y, 14.80.-j, 11.27.+d, 13.90.+i

Keywords: electroweak knot, topological knot

Ever since Dirac proposed his theory of monopoles the topological objects in physics have been the subject of intensive studies \([1,2]\). In particular the finite energy topological solitons have been widely studied in theoretical physics \([2,3]\). A remarkable type of solitons is the knots which have recently appeared almost everywhere, in nuclear physics in Skyrme theory \([4,5,6]\), in plasma physics in coronal loops \([7]\), in condensed matter physics in one-gap as well as multi-gap superconductors \([8-10]\), and in atomic physics in two-component Bose-Einstein condensates \([11,12]\).

Of these, the Faddeev-Niemi knot in Skyrme theory and the superconducting knot in ordinary superconductor are particularly important. The Faddeev-Niemi knot is the prototype of all the knots, which comes from the Skyrme-Faddeev Lagrangian

\[
L_{SF} = \frac{\mu^2}{2} (\partial_\mu \hat{n})^2 - \frac{1}{4} (\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2. \tag{1}
\]

The Lagrangian has the following equation \([5]\)

\[
\hat{n} \times \partial^2 \hat{n} - \frac{g}{\mu^2} (\partial_\mu H_{\mu\nu}) \partial_\nu \hat{n} = 0,
\]

\[
H_{\mu\nu} = \frac{1}{g} \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}), \tag{2}
\]

which admits not only the knot but also the helical baby skyrmion, a twisted baby skyrmion which is periodic in \(z\)-coordinate. The importance of the helical baby skyrmion is that the Faddeev-Niemi knot is nothing but a vortex ring made of the helical baby skyrmion with two periodic ends smoothly connected together. This tells that the Faddeev-Niemi knot originates from the helical baby skyrmion \([6]\). The knot is non-Abelian, because the knot topology \(\pi_3(\mathbb{S}^2)\) comes from the \(SU(2)\) symmetry of the theory.

On the other hand, the superconducting knot in ordinary superconductor is an Abelian knot whose knot topology is fixed by the Chern-Simon index of the electromagnetic potential \([5]\). Nevertheless this knot is closely related to the other knots in a fundamental way. It can also be viewed as a vortex ring made of a helical magnetic vortex, the twisted Abrikosov vortex \([5]\). This tells that the existence of a helical vortex is an essential condition for a knot. It guarantees the existence of the knot.

The purpose of this Letter is to demonstrate the existence of an electroweak knot in Weinberg-Salam theory. We show that the electroweak knot is made of two quantized neutral magnetic flux rings linked together, the first one winding the second \(m\) times and the second one winding the first \(n\) times, whose linking number \(mn\) becomes the knot quantum number. Furthermore we predict that the lightest knot has mass around 21 TeV and size of \(3.5 \times 10^{-18}\) \(m\). We also show that the knot has both topological and dynamical stability, which comes from the twisted topology of the neutral magnetic field. If confirmed by experiments, the knot could constitute the first topological particle in high energy physics.

To establish the existence of the electroweak knot we first need to understand the deep connection which exists between the Skyrme theory and the non-Abelian gauge theory. Let \((\hat{n}_1, \hat{n}_2, \hat{n})\) be a right-handed orthonormal basis in \(SU(2)\) space, and consider the following decomposition of the potential \(\hat{A}_\mu\) into the restricted potential \(\hat{A}_\mu\) and the valence potential \(\hat{X}_\mu\) \([13,14]\).

\[
\hat{A}_\mu = A_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n}, \quad \hat{X}_\mu = A_\mu + \hat{X}_\mu,
\]

\[
A_\mu = \hat{n} \cdot \hat{A}_\mu, \quad \hat{X}_\mu = X_\mu^1 \hat{n}_1 + X_\mu^2 \hat{n}_2. \tag{3}
\]

Notice that \(\hat{A}_\mu\) is precisely the connection which leaves
invariant under parallel transport,
\[ \hat{D}_\mu \hat{n} = \partial_\mu \hat{n} + g \hat{A}_\mu \times \hat{n} = 0. \]

Under the infinitesimal gauge transformation
\[ \delta \hat{n} = -\hat{\alpha} \times \hat{n}, \quad \delta \hat{A}_\mu = \frac{1}{g} \hat{D}_\mu \hat{\alpha}, \]
\[ \delta \hat{X}_\mu = -\hat{\alpha} \times \hat{X}_\mu. \]

This tells that \( \hat{A}_\mu \) by itself describes an \( SU(2) \) connection which enjoys the full gauge degrees of freedom. Furthermore \( \hat{X}_\mu \) forms a gauge covariant vector field under the gauge transformation. But what is really remarkable is that the decomposition is gauge-independent. Once \( \hat{n} \) is given, the decomposition follows automatically independent of the choice of a gauge [13].

Notice that \( A_\mu \) retains the full topological characteristics of the original non-Abelian potential. Clearly, \( \hat{n} \) defines \( \pi_2(S^3) \) which describes the non-Abelian monopole [13]. Besides, with the \( S^3 \) compactification of \( R^3 \), \( \hat{n} \) describes the Hopf invariant \( \pi_3(S^3) \approx \pi_3(S^3) \) which characterizes both topologically distinct vacua and instanton number [14]. Furthermore \( A_\mu \) has a dual structure,
\[ F_{\mu \nu} = (F_{\mu \nu} + H_{\mu \nu}) \hat{n}, \]
\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]
\[ H_{\mu \nu} = -\frac{1}{g} \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu C_\nu - \partial_\nu C_\mu, \]

where \( A_\mu \) and \( C_\mu \) are the (chromo)electric and (chromo)magnetic potential [13] [14]. Notice that \( H_{\mu \nu} \) here is exactly the same two-form appeared in [2], which admits the potential \( C_\mu \) because it is closed.

The decomposition [3] reveals the deep connection between the Yang-Mills theory and the Skyrme theory. To see this notice that with
\[ H_{\mu \nu} = \partial_\mu C_\nu - \partial_\nu C_\mu + g C_\mu \times C_\nu = H_{\mu \nu} \hat{n}, \]
\[ C_\mu = -\frac{1}{g} \hat{n} \times \partial_\mu \hat{n}, \]
we have [13]
\[ \mathcal{L}_{SF} = -\frac{1}{4} H_{\mu \nu}^2 - \frac{\mu^2}{2} C_\mu^2. \]

This tells that the Skyrme-Faddeev theory can be interpreted as a massive Yang-Mills theory where the gauge potential has the special form [3]. This is a first indication that the Weinberg-Salam theory could admit a knot similar to Faddeev-Niemeijer knot. Our decomposition [3], which has recently been referred to as the “Cho decomposition” [19] [20] [21], plays a crucial role in QCD, in particular in the calculation of the effective action of QCD [18].

With these preliminaries we now demonstrate the existence of an electroweak knot. Consider the Weinberg-Salam Lagrangian
\[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu}^2 - \frac{1}{4} G_{\mu \nu}^2 - |\hat{D}_\mu \phi|^2 + m^2 \phi^* \phi - \frac{1}{2} (\phi^* \phi)^2, \]
\[ \hat{D}_\mu \phi = (\partial_\mu + \frac{g}{2i} \hat{A}_\mu + \frac{g'}{2i} B_\mu) \phi. \]

With the decomposition [4] we can identify the Higgs field \( \rho \) and W boson \( W_\mu \) by,
\[ \phi = \frac{\rho}{\sqrt{2}} \xi (\xi^\dagger \xi = 1), \quad W_\mu = \frac{1}{\sqrt{2}} (X_\mu^1 + i X_\mu^2), \]
and express [10] in terms of the physical fields alone
\[ \mathcal{L} = -\frac{1}{2} (D_{\mu \nu}^{(em)} W_\mu - D_{\nu \mu}^{(em)} W_\nu) + i e g (Z_\mu W_\nu - Z_\nu W_\mu)^2 \]
\[ + e F_{\mu \nu}^{(em)} W_\mu W_\nu + e^2 \rho g (Z_\mu W_\mu W_\nu), \]
\[ + \frac{g^2}{4} (W^* W_\mu - W_\mu W^* W_\mu)^2, \]

where \( \rho_0^2 = 2 m^2 / \lambda, \) \( D_{\mu \nu}^{(em)} = \partial_\mu + i e A_\mu^{(em)}, \) \( A_\mu^{(em)} \) and \( Z_\mu \) are the electromagnetic potential and the Z boson. We emphasize that this expression is obtained without any gauge fixing, which is made possible with the gauge independent decomposition [3].

Now, with the ansatz
\[ A_\mu^{(em)} = 0, \quad W_\mu = 0, \]
we have the following equation
\[ \partial^2 \rho_{\mu} - g_0^2 Z_{\mu}^2 \rho_{\mu} = \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho, \]
\[ \partial_\mu Z_{\mu \nu} = j_\nu = g_0^2 \rho^2 Z_\nu, \]

where \( g_0 = \sqrt{g^2 + \rho^2 / 2} \) and \( Z_{\mu \nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu. \) This admits not only a twisted magnetic field of the Z boson.

We construct the helical vortex first. Choose the cylindrical coordinates \( (\varphi, \varphi, z) \) and the following ansatz
\[ \rho = \rho(\varphi), \]
\[ Z_\mu = \frac{1}{g_0} (n Z_1(\varphi) \partial_\mu \varphi + mk Z_2(\varphi) \partial_\mu z). \]

With this we have
\[ Z_{\varphi \varphi} = \frac{n}{g_0} \partial_1, \quad Z_{\varphi z} = \frac{mk}{g_0} \partial_2, \quad Z_{\varphi z} = 0, \]
\[ j_\mu = g_0 \rho^2 \left( n Z_1(\partial_\mu \varphi + mk Z_2(\partial_\mu z)\right). \]
describes the well-known Z boson vortex. Notice that, when m is neutral, not electromagnetic. FIG. 1: The helical electroweak vortex with m = n = 1 in Weinberg-Salam theory. Here we have put g0 = 1, λ = 2, k = ρ0/10, and ϕ is in the unit of 1/ρ0.

which clearly shows that both the magnetic flux and the supercurrent have a helical structure.

With the ansatz (14) is reduced to
\[ \frac{1}{\rho} - \frac{n^2}{\rho^2}Z_1^2 + m^2k^2Z_2^2 = \frac{\lambda}{2}(\rho^2 - \rho_0^2)\rho, \]
\[ \frac{1}{\rho}Z_1 - \frac{1}{\rho}Z_1 - \frac{n^2}{\rho_0^2}Z_1 = 0, \]
\[ \frac{1}{\rho}Z_2 + \frac{1}{\rho}Z_2 - \frac{n^2}{\rho_0^2}Z_2 = 0. \]
\[ (17) \]

Now, we impose the following boundary condition
\[ \rho(0) = 0, \rho(\infty) = \rho_0, \quad Z_1(0) = 1, \quad Z_1(\infty) = 0. \]
\[ (18) \]

As for Z2 we require that the vortex carries a non-vanishing supercurrent. This uniquely fixes the boundary condition for Z2, which dictates that Z2(\infty) = 0 but Z2(0) must have a logarithmic divergence.

With this we obtain the helical vortex shown in Fig.1. Notice that, when m = 0, the solution (with Z2 = 0) describes the well-known Z boson vortex [22]. But when m is not zero, it describes a helical vortex which has a non-vanishing supercurrent (not only around the z-axis but also) along the z-axis,
\[ i_z = mk\rho_0 \int \rho^2Z_2d\rho d\varphi = \frac{2\pi nk}{g_0} (gZ_2) \bigg|_{\varphi=0}. \]
\[ (19) \]

This confirms that the logarithmic divergence of Z2 at the origin is what we need to make the vortex a superconducting string. But here, of course, the supercurrent is neutral, not electromagnetic.

Clearly the vortex has the magnetic field \( H_z \) and the quantized magnetic flux \( \Phi_z \) along the z-axis
\[ H_z = \frac{n}{g_0} \frac{\dot{Z}_1}{\rho}, \quad \Phi_z = \int H_z d\rho d\varphi = \frac{2\pi n}{g_0} \]
\[ (20) \]

But it also has the magnetic field \( H_\varphi \) around the vortex
\[ H_\varphi = \frac{mk}{g_0} \dot{Z}_2. \]
\[ (21) \]

Unfortunately this produces an infinite magnetic flux \( \Phi_\varphi \) around the vortex, because of the singularity at the origin. So one might conclude that the helical vortex is unphysical which does not exist, because one need an infinite energy to create it.

The importance of the helical vortex, however, is not in that it is physical but in that it ensures the existence of a knot. To see this notice that with the vortex we can make a vortex ring by smoothly bending and connecting two periodic ends. In this vortex ring the infinite magnetic flux \( Z_2 \) becomes finite because the finite supercurrent in the vortex ring, should produce a finite magnetic flux passing through the disk of the ring. This tells that the infinite magnetic flux \( \Phi_\varphi \) of the vortex is an artifact of a straight vortex, which disappears in the vortex ring. Furthermore, we can certainly make the finite magnetic flux passing through the knot disk to have the quantized value \( 2\pi m/g_0 \) by adjusting the supercurrent of the ring with k. Remarkably, this vortex ring now becomes a topologically stable knot. To see this notice that the magnetic flux of the knot is made of two flux rings, a \( 2\pi m/g_0 \) flux around the knot tube and a \( 2\pi n/g_0 \) flux along the knot. Moreover the two flux rings are linked together, whose linking number becomes mn. This is precisely the mathematical description of a knot, two rings linked together. This assures that the vortex ring does indeed become a topological knot. The knot quantum number is described by the Chern-Simon index of \( Z_\mu \) field,
\[ Q = \frac{g_0^2}{32\pi^2} \int \epsilon_{ijk}Z_iZ_{jk}d^3x = mn, \]
\[ (22) \]

which describes the non-trivial topology \( \pi_3(S^2) \) of the magnetic field \( Z_\mu \). Notice that this is formally identical to the quantum number of the Faddeev-Niemi knot [3, 4]. The only difference is that here the (chromo)magnetic potential \( C_\mu \) is replaced by \( Z_\mu \). Furthermore, just as in the Faddeev-Niemi case, the Chern-Simon index is given by the linking number of two magnetic fluxes.

Obviously the two flux rings linked together can not be separated with a continuous deformation of the field configuration. This provides the topological stability of the knot.

Furthermore, this topological stability of the knot is backed up by a dynamical stability. This is because the supercurrent along the knot now generates a net angular momentum around the knot which naturally provides the centrifugal repulsive force to prevent the collapse of the knot. This tells that the knot is dynamically stable. Another way to understand the dynamical stability is to notice that, when the knot tries to shrink, the energy density of magnetic field trapped in the knot disk increases.
inevitably. This generates a repulsive force against the collapse, which makes the knot stable.

We emphasize that the stability of the knot crucially depends on the helical structure of the supercurrent and magnetic field. Without this there is neither the topological stability nor the dynamical stability.

We can estimate the mass of the electroweak knot. To do this notice that in the absence of the Higgs field the knot becomes almost identical to the Faddeev-Niemi knot in Skyrme theory. From this observation we may estimate the energy density (per length) and the energy, thus the radius of the knot as

\[ E = \pi n \rho_0^2, \quad E \geq 16\pi^2 \times 3^{3/8} \left( \frac{m}{n} \right)^{3/4} g_0 \rho_0, \]
\[ R = \frac{E}{2\pi n} \simeq \frac{8g_0 \times 3^{3/8} m^{3/4}}{\rho_0 n^{1/4}}. \]  

So, with the experimental values \( g_0 \approx 0.36 \) and \( \rho_0 \approx 246 \text{ GeV} \) we expect the lightest electroweak knot to have mass around \( 21 \text{ TeV} \) and radius of about \( 3.5 \times 10^{-18} \text{ m} \), with tube size \( 0.7 \times 10^{-18} \text{ m} \).

Clearly the electroweak knot is closely related to the Faddeev-Niemi knot. But one can not overemphasize the striking similarity between the electroweak knot and superconducting knot in ordinary superconductor [3]. Mathematically they are identical. The only difference is that the superconducting knot exists at the atomic scale and carries the real electric supercurrent, whereas the knot here exists at the electroweak scale and carries the neutral current. It is really remarkable that mathematically identical knots can exist in totally different physical environments, in \( e \text{V} \) scale and in \( T \text{eV} \) scale.

We believe that our analysis has established the existence of a electroweak knot beyond reasonable doubt. Of course, one might like to see an analytic solution of the knot. Unfortunately this is impossible. At present even the simplest knot does not allow an analytic solution [14]. But one can construct an approximate solution analytically which has all the characteristic features of the electroweak knot, using the toroidal coordinates and adopting an educated ansatz [23,24]. This provides another evidence which endorses the fact that the electroweak knot is for real. A challenging task now would be to confirm the existence of the electroweak knot by high energy experiments. We hope that the LHC at CERN could confirm the existence of the electroweak knot.

The physical implications of the electroweak knot and the details of our analysis will be published separately [24].

ACKNOWLEDGEMENT

We thank Professor C. N. Yang for the illuminating discussions. The work is supported by the Basic Research Program of Korea Science and Engineering Foundation (Grant R02-2003-000-10043-0) and by the BK21 project of the Ministry of Education.

[1] P. A. M. Dirac, Proc. Roy. Soc. A113, 60 (1931); Phys. Rev. 74, 817 (1948).
[2] T. H. R. Skyrme, Proc. Roy. Soc. (London) 260, 127 (1961); 262, 237 (1961). Nucl. Phys. 31, 556 (1962).
[3] G. ’t Hooft, Nucl. Phys. 79, 276 (1974); A. Polyakov, JETP Lett. 20, 194 (1974); M. Prasad and C. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).
[4] L. Faddeev and A. Niemi, Nature 387, 58 (1997); R. Battye and P. Sutcliffe, Phys. Rev. Lett. 81, 4798 (1998).
[5] Y. M. Cho, Phys. Rev. Lett. 87, 252001 (2001).
[6] Y. M. Cho, cond-mat/0309905.
[7] L. Faddeev and A. Niemi, Phys. Rev. Lett. 85, 3416 (2000).
[8] Y. M. Cho, cond-mat/0000000.
[9] Y. M. Cho, cond-mat/0112408.
[10] E. Babaev, Phys. Rev. Lett. 88, 177002 (2002); E. Babaev, L. Faddeev, and A. Niemi, Phys. Rev. B65, 100512 (2002).
[11] Y. M. Cho, cond-mat/0112325; Y. M. Cho and H. J. Khim, cond-mat/0308182.
[12] J. Ruostekoski and J. Anglin, Phys. Rev. Lett. 86, 3934 (2001); H. Stoof at al., Phys. Rev. Lett. 87, 120407 (2001); R. Battye, N. Cooper, and P. Sutcliffe, Phys. Rev. Lett. 88, 080401 (2002); C. Savage and J. Ruostekoski, Phys. Rev. Lett. 91, 010403 (2003).
[13] Y. M. Cho, Phys. Rev. D21, 1080 (1980); Y. M. Cho, Phys. Rev. D62, 074009 (2000).
[14] Y. M. Cho, Phys. Rev. Lett. 46, 302 (1981); Phys. Rev. D23, 2415 (1981); W. S. Bae, Y. M. Cho, and S. W. Kimm, Phys. Rev. D65, 025005 (2002).
[15] Y. M. Cho, Phys. Rev. Lett. 44, 1115 (1980); Phys. Lett. B115, 125 (1982).
[16] Y. M. Cho, Phys. Lett. B81, 25 (1979).
[17] W. S. Bae, Y. M. Cho, and S. W. Kimm, Phys. Rev. D65, 025005 (2002).
[18] Y. M. Cho, H. W. Lee, and D. G. Pak, Phys. Lett. B525, 347 (2002); Y. M. Cho and D. G. Pak, Phys. Rev. D65, 074027 (2002).
[19] L. Faddeev and A. Niemi, Phys. Rev. Lett. 82, 1624 (1999); Phys. Lett. B449, 214 (1999); B464, 90 (1999); S. Shabanov, Phys. Lett. B458, 322 (1999); B463, 263 (1999).
[20] H. Gies, Phys. Rev. D63, 125023 (2001).
[21] R. Zucchini, hep-th/0303027.
[22] T. Vachaspati and M. Barriola, Phys. Rev. Lett. 69, 1867 (1992).
[23] L. Kapitansky and A. Vakulenko, Sov. Phys. Doklady 24, 433 (1979).
[24] Y. M. Cho and N. S. Yong, to be published.