Research Article

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**Modular forms of half-integral weight on \( \Gamma_0(4) \) with few nonvanishing coefficients modulo \( \ell \)**

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**Abstract:** Let \( k \) be a nonnegative integer. Let \( K \) be a number field and \( \mathcal{O}_K \) be the ring of integers of \( K \). Let \( \ell \geq 5 \) be a prime and \( v \) be a prime ideal of \( \mathcal{O}_K \) over \( \ell \). Let \( f \) be a modular form of weight \( k + \frac{1}{2} \) on \( \Gamma_0(4) \) such that its Fourier coefficients are in \( \mathcal{O}_K \). In this article, we study sufficient conditions that if \( f \) has the form

\[
 f(z) \equiv \sum_{n=1}^{\infty} \sum_{i=1}^{t} a_i(n^2)q^{s_i n^2} \pmod{v}
\]

with square-free integers \( s_i \), then \( f \) is congruent to a linear combination of iterated derivatives of a single theta function modulo \( v \).

**Keywords:** Fourier coefficients of modular forms, Galois representations, modular forms of half-integral weight, theta functions

**MSC 2020:** 11F33, 11F80

1 **Introduction**

The Fourier coefficients of modular forms of half-integral weight are related to various objects in number theory and combinatorics such as the algebraic parts of the central critical values of modular L-functions, orders of Tate-Shafarevich groups of elliptic curves, the number of partitions of a positive integer, and so on. With a lot of application to these objects, Bruinier [1], Bruinier and Ono [2], Ono and Skinner [3], Ahlgren and Boylan [4,5], and the others studied congruence properties modulo a power of a prime for Fourier coefficients of modular forms of half-integral weight. Many of them considered modular forms of half-integral weight whose the Fourier coefficients are supported on only finitely many square classes modulo \( \ell \).

Let \( f \) be a modular form of half-integral weight on \( \Gamma_0(4N) \). Vignéras [6] proved that if the \( q \)-expansion of \( f \) has the form

\[
 f(z) = a_i(0) + \sum_{n=1}^{\infty} \sum_{i=1}^{t} a_i(n^2)q^{s_i n^2}, \quad q = e^{2\pi iz}
\]

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Modular forms of half-integral weight on $\Gamma_0(4)$ with few nonvanishing coefficients modulo $\ell$

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with a positive integer $t$ and square-free integers $s_i$, then $f$ is a linear combination of single variable theta functions (a different proof of this result was given by Bruinier [1]). Many of the aforementioned results can be considered as positive characteristic extensions of Vignéras’ result on classification of modular forms of half-integral weight such that their nonvanishing Fourier coefficients lie in only finitely many square classes. Especially, Ahlgren et al. [7] obtained an explicit mod $\ell$ analog of the result of Vignéras for modular forms of half-integral weight on $\Gamma_0(4)$ satisfying the Kohnen-plus condition.

Let $K$ be a number field and $O_K$ be the ring of integers of $K$. Let $M_{k,1/2}(\Gamma_0(4); O_K)$ (resp. $S_{k,1/2}(\Gamma_0(4); O_K)$) be the space of modular forms (resp. cusp forms) of weight $k + 1/2$ on $\Gamma_0(4)$ such that their Fourier coefficients are in $O_K$ and $S_{k,1/2}(\Gamma_0(4); O_K)$ be the subspace of $S_{k,1/2}(\Gamma_0(4); O_K)$ consisting of $f \in S_{k,1/2}(\Gamma_0(4); O_K)$ satisfying the Kohnen-plus condition.

Let $\ell \geq 5$ be a prime and $v$ be a prime ideal of $O_K$ over $\ell$. For $f \in S_{k,1/2}(\Gamma_0(4); O_K)$, Ahlgren et al. [7] proved that if

$$k + \frac{1}{2} < \ell \left( \ell + \frac{3}{2} \right)$$

(1.1)

and

$$f(z) \equiv \sum_{n=1}^{\infty} \sum_{i=1}^{t} a_i(s_n m^2) q^{s_n m^2} \pmod{v}$$

(1.2)

with square-free integers $s_i$, then $k$ is even and

$$f(z) \equiv a_1(1) \sum_{n=1}^{\infty} n^k q^{s_n} \pmod{v}.$$  

In this article, we study sufficient conditions that if $f$ has the form (1.2), then $f$ is congruent to a linear combination of iterated derivatives of a single theta function modulo $v$.

For a positive number $\varepsilon$, let $P_\varepsilon$ be the set of primes $\ell$ such that for every $f \in S_{k,1/2}(\Gamma_0(4); O_K)$ with $k + \frac{1}{2} < \ell (\log \ell)^{3-\varepsilon}$, if

$$f(z) \equiv \sum_{n=1}^{\infty} \sum_{i=1}^{t} a_i(s_n m^2) q^{s_n m^2} \pmod{v}$$

with square-free integers $s_i$, then

$$f(z) \equiv a_1(1) \left( \sum_{n=1}^{\infty} n^k q^{s_n} \right) + a_\ell(1) \left( \sum_{n=1}^{\infty} n^{k+1/2} q^{n^m} \right) \pmod{v}.$$  

The following theorem proves that the portion of $P_\varepsilon$ in the set of primes is one.

**Theorem 1.1.** For a positive integer $X$, there is an absolute constant $C$ such that

$$\# \{ \ell : \ell \notin P_\varepsilon \text{ and } \ell \leq X \} \leq C_0 \frac{X}{(\log X)^{1+\varepsilon}} \left( 1 + C \frac{\log \log X}{\log X} \right),$$

where $C_0 := \frac{2^{2/3} \pi}{3} \prod_{P \geq 2} \frac{P^3}{P^2 - 1}$.

For a nonnegative real number $r$, we define an operator $\Theta^r$ on $C[[q]]$ by

$$\Theta^r \left( \sum_{n=0}^{\infty} a(n) q^n \right) = \begin{cases} \sum_{n=0}^{\infty} n^r a(n) q^n & \text{if } r \in \mathbb{Z}_{>0}, \\ 0 & \text{elsewhere.} \end{cases}$$

For convenience, we let $\Theta = \Theta^1$. As in Theorem 1.1, the previous results on modular forms of half-integral weight having the form (1.2) such as $[1,2,4,5,7]$ and so on imply that in many cases, if $f$ has the form (1.2),
then $\Theta(f)$ is congruent to a linear combination of iterated derivatives of a single theta function modulo $v$. These lead us to the following conjecture on modular forms $f$ of half-integral weight having the form (1.2).

**Conjecture 1.2.** Let $K$ be a number field and $O_K$ be the ring of integers of $K$. Let $\ell \geq 5$ be a prime and $v$ be a prime ideal of $O_K$ over $\ell$. Assume that $f \in S_{k,\frac{1}{2}}(\Gamma_0(4); O_K)$ has the form

$$\Theta(f)(z) \equiv \sum_{n=1}^{\infty} s_n^2 a_n(s_n^2)q^{sn^2} \pmod{v}$$

with a square-free integer $s$, then

$$\Theta(f)(z) \equiv \frac{1}{2} a_q(1) \left( \sum_{n \in \mathbb{Z} \bmod{\ell}} n^{k+2} q^{n^2} \right) \pmod{v}.$$ 

Assume that $\ell$ is a prime and $m$ is a nonnegative integer. Let $\eta(m)$ be the least positive integer such that

$$\eta(m) \equiv m \pmod{\ell - 1}.$$ 

Let $a(\ell, m)$ be the smallest nonnegative integer $i$ such that

$$m + \frac{1}{2} < \ell^2 \left( \eta(m) + \frac{\ell + 1}{2} + \frac{1}{2} \right),$$

and $\beta(\ell, m)$ be the smallest nonnegative integer $i$ such that

$$m + \frac{1}{2} < \ell^{2i+1} \left( \eta(m + \frac{\ell - 1}{2}) + \frac{\ell + 1}{2} + \frac{1}{2} \right).$$

Let

$$T(z) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2}.$$ 

For convenience, let

$$\sum_{n=a}^{b} a_n = \begin{cases} \sum_{n=a}^{b} a_n & \text{if } a \leq b, \\ 0 & \text{if } a > b. \end{cases}$$

By using Conjecture 1.2, we have an explicit formula for modular forms of half-integral weight having the form (1.2).

**Theorem 1.3.** Let $K, O_K, \ell, v$ be as in Conjecture 1.2. Assume that $f \in M_{k,\frac{1}{2}}(\Gamma_0(4); O_K)$. Conjecture 1.2 implies that if $f$ has the form

$$f(z) \equiv a_q(0) + \sum_{n=1}^{\infty} \sum_{i=1}^{l} a_{n+i}(s_n q^i) q^{sn^2} \pmod{v}$$

(1.3)

with square-free integers $s_i$, then the following statements are true.

1. If $\eta(k) \neq \ell - 1$ and $\eta(k) \neq \frac{\ell - 1}{2}$, then

$$f(z) \equiv \frac{1}{2} \sum_{l=0}^{a(k, l) - \ell'} a_l(T^k(z)) \Theta(k, 2^l) + \frac{1}{2} \sum_{l=0}^{\beta(k, l) - \ell'} a_l(T^{2l+1}) \Theta(2k+2l+1) \Theta(T(z)) q^{2l+1} \pmod{v}.$$ 

2. If $\eta(k) = \ell - 1$, then

$$f(z) \equiv a_q(0) T(z) + \frac{1}{2} \sum_{l=0}^{a(k, l) - \ell'} a_l(T^k(z)) \Theta(k, 2^l) + \frac{1}{2} \sum_{l=0}^{\beta(k, l) - \ell'} a_l(T^{2l+1}) \Theta(2k+2l+1) \Theta(T(z)) q^{2l+1} \pmod{v}.$$
(3) If \( \eta(k) = \frac{\ell - 1}{2} \), then
\[
f(z) = a_0(T)T(tz) + \frac{1}{2} \sum_{i=0}^{a(t,k)-1} a_i(T^2)O^{k/2}(T)(T^2z) + \frac{1}{2} \sum_{i=0}^{\beta(t,k)-1} (a_i(T^{2i+1}) - 2a_i(0))\Theta^{(2k+1)(i+1)}(T)(T^{2i+1}z) \pmod{v}.
\]

To give numerical evidence for Conjecture 1.2, we consider a basis of the space of modular forms of weight \( k + \frac{1}{2} \) on \( \Gamma_0(4) \). Let \( F_i(z) = \sum_{n=0}^{\infty} \sigma(2n+1)q^{2n+1} \) be the modular form of weight 2 on \( \Gamma_0(4) \), where \( \sigma(n) \) is the sum of positive divisors of \( n \). Then
\[
\{F_i^jT^{2k-1-i}j\}_{0 \leq j \leq \lfloor \frac{1}{2} \rfloor}
\]
is a \( \mathbb{C} \)-basis of the space of modular forms of weight \( k + \frac{1}{2} \) on \( \Gamma_0(4) \). Let \( A_{k,m} \) be an \( m \times \left( \left\lfloor \frac{k}{2} \right\rfloor + 1 \right) \) matrix such that the \((i,j)\)-entry of \( A_{k,m} \) is the \((i-1)\)th Fourier coefficient of \( F_i^{j-1}T^{2k-1-i}j \) modulo \( \ell \). Let \( B_{k,m} \) be a submatrix of \( A_{k,m} \) obtained by removing \( n^2 + 1 \)th rows for all nonnegative integers \( n \) with \((\ell, n) = 1\). Let \( \text{Null}(B_{k,m}) \) be the null space of \( B_{k,m} \). With this notation, we give the following conjecture.

**Conjecture 1.4.** Let \( \ell \geq 5 \) be a prime. Let 1, be the characteristic function of the set of positive real numbers. Then, for a positive even integer \( k \), we have
\[
\lim_{m \to \infty} \dim \text{Null}(B_{k,m}) = \sigma_{1}(\ell, k).
\]

By comparing the intersection of the null spaces of \( B_{k,m} \) and the space of mod \( v \) modular forms of weight \( k + \frac{1}{2} \) on \( \Gamma_0(4) \) having the form
\[
f(z) = \sum_{n \equiv 0 \mod{\ell}} \sigma(n^2)q^{n^2} \pmod{v},
\]
we have the following theorem.

**Theorem 1.5.** Conjecture 1.2 is equivalent to Conjecture 1.4.

Let us note that \( \text{Null}(B_{k,m}) \) is stable for sufficiently large \( m \). In the proof of Theorem 1.5, we prove that \( \dim \text{Null}(B_{k,m}) \) is larger than or equal to \( \sigma_{1}(\ell, k) \) for all positive integers \( m \). Hence, if there is a positive integer \( m \) such that \( \dim \text{Null}(B_{k,m}) = \sigma_{1}(\ell, k) \), then Conjecture 1.2 is true. To compute \( \dim \text{Null}(B_{k,m}) \), we consider the row echelon form of \( B_{k,m} \). We use C++ in this process. Then we have the following theorem.

**Theorem 1.6.** Assume that \( k \leq 1,000 \), or that \( \ell \in \{5, 7, 11, 13, 17, 19\} \) and \( k \leq 10,000 \). Then, Conjecture 1.2 is true.

The remainder of this article is organized as follows. In Section 2, we review some properties of \( f \) having the form (1.3) and the filtration for modular forms. In Section 3, we prove Theorems 1.1, 1.3, 1.5, and 1.6.

## 2 Preliminaries

In this section, we review some notions and properties of the filtration for modular forms, and then we introduce some properties about modular forms of half-integral weight on \( \Gamma_0(4) \) such that their Fourier coefficients are supported on finitely many square classes modulo a prime \( \ell \). For further details, see [8].

Throughout the rest of this article, we fix the following notation. For a congruence subgroup \( \Gamma \) and \( w \in \frac{1}{2} \mathbb{Z} \), let \( M_w(\Gamma) \) (resp. \( S_w(\Gamma) \)) be the space of modular forms (resp. cusp forms) of weight \( w \) on \( \Gamma \).
For a Dirichlet character \( \chi \) modulo \( N \), let \( M_\omega(\Gamma_0(N), \chi) \) (resp. \( S_\omega(\Gamma_0(N), \chi) \)) be the space of modular forms (resp. cusp forms) of weight \( w \) on \( \Gamma_0(N) \) with character \( \chi \).

Let \( k \) be a nonnegative integer and \( \ell \geq 5 \) be a prime. Let \( K \) be a number field and \( O_K \) be the ring of integers of \( K \). Let \( \nu \) be a prime ideal of \( O_K \) over \( \ell \). Let \( M_{k, \nu}(\Gamma_0(4N); O_K) \) (resp. \( S_{k, \nu}(\Gamma_0(4N); O_K) \)) be the space of modular forms (resp. cusp forms) of weight \( k + \frac{1}{2} \) on \( \Gamma_0(4N) \) such that their Fourier coefficients are in \( O_K \) and \( S_{k, \nu}(\Gamma_0(4); O_K) \) be the subspace of \( S_{k, \nu}(\Gamma_0(4); O_K) \) consisting of \( f \in S_{k, \nu}(\Gamma_0(4); O_K) \) satisfying the Kohnen-plus condition.

Now, we review the basic notions and properties about the Shimura correspondence. Assume that \( f \) is a cusp form of weight \( k + \frac{1}{2} \) on \( \Gamma_0(4) \). For a square-free integer \( t \), we define \( A_t(n) \) by

\[
\sum_{n=1}^{\infty} A_t(n) \frac{n^s}{n^s} = \sum_{n=1}^{\infty} \left( \frac{(-1)^n}{n} \right) \frac{1}{n^{s-k+1}} \sum_{n=1}^{\infty} a_{\nu^s}(f) \frac{n^s}{n^s},
\]

Then, the Shimura lift \( \text{Sh}_t(f) \) of \( f \) is defined by

\[
\text{Sh}_t(f)(x) = \sum_{n=1}^{\infty} A_t(n)q^n.
\]

Note that \( \text{Sh}_t(f) \in S_{2k}(\Gamma_0(2)) \). In particular, if \( f \in S_{k, \nu}(\Gamma_0(4)) \), then \( \text{Sh}_t(f) \in S_{2k}(\Gamma_0(1)) \). For each odd prime \( p \) with \( p \mid t \), we have

\[
\text{Sh}_t(f|T_{p^t,k} \frac{1}{2}) = \text{Sh}_t(f)|T_{p,2k},
\]

where \( T_{n,w} \) denotes the \( n \)th Hecke operator on the space of modular forms of weight \( w \). For each prime \( \ell \), operators \( U_\ell \) and \( V_\ell \) on formal power series are defined by

\[
\left( \sum_{n=0}^{\infty} a(n)q^n \right) U_\ell = \sum_{n=0}^{\infty} a(\ell n)q^n
\]

and

\[
\left( \sum_{n=0}^{\infty} a(n)q^n \right) V_\ell = \sum_{n=0}^{\infty} a(n)q^{\ell n}.
\]

### 2.1 Filtration for modular forms of half integral weight modulo a prime \( \ell \)

The theory of filtration for modular forms of integral weight was developed by Serre [9], Swinnerton-Dyer [10], Katz [11], and Gross [12]. From this, the theory of filtration for modular forms of half-integral weight on \( \Gamma_0(4) \) was studied. In this section, we review some properties of filtration for modular forms of half-integral weight on \( \Gamma_0(4) \). For the details, we refer to [13, Section 2].

We say that \( \sum_{n=0}^{\infty} a(n)q^n \) is congruent to \( \sum_{n=0}^{\infty} b(n)q^n \) modulo \( \nu \), i.e.,

\[
\sum_{n=0}^{\infty} a(n)q^n \equiv \sum_{n=0}^{\infty} b(n)q^n \pmod{\nu},
\]

if \( a(n) \equiv b(n) \pmod{\nu} \) for all nonnegative integers \( n \). For \( f \in M_{k, \nu}(\Gamma_0(4); O_K) \), we define a filtration \( \omega(f) \) of \( f \) modulo \( \nu \) by

\[
\omega(f) = \inf \left\{ k' + \frac{1}{2} : \text{there is } f' \in M_{k', \nu}(\Gamma_0(4); O_K) \text{ such that } f' \equiv f \pmod{\nu} \right\}.
\]

For convenience, if \( f \equiv 0 \pmod{\nu} \), then let \( \omega(f) = -\infty \). We summarize the properties of \( \omega(f) \) in the following lemma.
Lemma 2.1. Let \( f \in \mathbb{M}_{k + \frac{1}{2}}(\Gamma_0(4); \mathcal{O}_\ell) \). Then, the following statements are true.

1. \( k \equiv \omega(f) - \frac{1}{2} \) (mod \( \ell - 1 \)).
2. \( a(f) = \ell \cdot a(f) \).
3. There is a nonnegative integer \( k' \) such that

\[
k' \equiv k + \frac{\ell - 1}{2} \quad \text{(mod } \ell - 1) ,
\]

and there is \( g \in \mathbb{M}_{k + \frac{1}{2}}(\Gamma_0(4); \mathcal{O}_\ell) \) such that \( g \equiv f|U_\ell \) (mod \( \mathcal{O}_\ell \)). Moreover, if \( f(z) \equiv \sum_{n = 0}^{\infty} a_n(n)q^n \) (mod \( \mathcal{O}_\ell \)), then there is a nonnegative integer \( k' \) such that

\[
k' \equiv k + \frac{\ell - 1}{2} \text{ (mod } \ell - 1) \text{ and } k' + \frac{1}{2} \leq \frac{1}{\ell} \left( k + \frac{1}{2} \right) ,
\]

and there is \( g \in \mathbb{M}_{k + \frac{1}{2}}(\Gamma_0(4); \mathcal{O}_\ell) \) such that \( g \equiv f|U_\ell \) (mod \( \mathcal{O}_\ell \)).

4. There is \( h \in S_{k + \frac{1}{2}}(\Gamma_0(4)) \) such that \( h \equiv \Theta(f) \) (mod \( \mathcal{O}_\ell \)). In particular, if \( f \in S_{k + \frac{1}{2}}^+(\Gamma_0(4)) \), then \( h \in S_{k + \frac{1}{2}}(\Gamma_0(4)) \).

Proof. The proofs of (1) and (2) are in [13, Proposition 2.2]. The proof of (3) is obtained by combining [7, Lemma 4.2] and [13, Proposition 2.2]. To prove (4), let

\[
h = \left( k + \frac{1}{2} \right) \Theta(E_{\ell - 1}) f - (\ell - 1) E_{\ell - 1} \Theta(f),
\]

where \( E_{\ell - 1} \) denotes the Eisenstein series of weight \( \ell - 1 \). Since \( E_{\ell - 1} \equiv 1 \) (mod \( \mathcal{O}_\ell \)), we have \( h \equiv \Theta(f) \) (mod \( \mathcal{O}_\ell \)). By [14, Corollary 7.2], we obtain \( h \in S_{k + \frac{1}{2}}(\Gamma_0(4)) \). When \( f \) satisfies the Kohnen-plus condition, the proof of (4) is in [7, Lemma 4.1].

2.2 Modular forms of half-integral weight such that their Fourier coefficients are supported on finitely many square classes modulo \( \ell \)

In this section, we introduce some properties of modular forms of half-integral weight on \( \Gamma_0(4) \) such that their Fourier coefficients are supported on finitely many square classes modulo \( \ell \).

Ahlgren and Boylan [4] obtained the necessary conditions for the weight of \( f \in \mathbb{M}_{k + \frac{1}{2}}(\Gamma_0(4)) \) such that their Fourier coefficients are supported on finitely many square classes modulo \( \ell \) by using the theory of Galois representations. This was reproved in [15] by using only the theory of filtration for modular forms of integral weight. The Choi and Kilbourn [16] improved the necessary conditions for the weight by using only the theory of filtration for modular forms of integral weight. We review the results [4,16] in the following theorem.

Theorem 2.2. Let \( N \) be a positive integer and \( \ell \geq 5 \) be a prime with \( (\ell, N) = 1 \). Assume that \( f(z) \in \mathbb{M}_{k + \frac{1}{2}}(\Gamma_0(4N)) \cap \mathcal{O}_\ell[q] \) has the form

\[
f(z) \equiv a_0(0) + \sum_{n=1}^{\infty} \sum_{s(n^2) = 1} a_s(n^2) q^{sn^2} \equiv a_0(0) + \sum_{n=1}^{\infty} \sum_{s(n^2) = 1} a_s(n^2) q^{sn^2} \quad \text{(mod } \mathcal{O}_\ell \),
\]

with square-free integers \( s_i \). Let \( \overline{\ell} \) and \( \ell_k \) be nonnegative integers, which satisfy \( k = (\ell - 1)\ell_k + \overline{\ell} \) and \( \ell_k < \ell - 1 \). Then, the following statements are true.

1. If \( \ell \mid n_i \) for some \( i \), then

\[
\overline{\ell} \leq 2\ell_k + 1.
\]
(2) If $\ell | n_i$ for all $i$ and $\overline{k} \leq \frac{\ell - 3}{2}$, then
\[ \overline{k} \leq \ell - \frac{1}{2}. \]

(3) If $\ell | n_i$ for all $i$ and $\overline{k} > \frac{\ell - 3}{2}$, then
\[ \overline{k} \leq \ell + \frac{1}{2}. \]

Bruinier and Ono [2, Theorem 3.1] proved the following theorem by using an argument in [1].

**Theorem 2.3.** Let $N$ be a positive integer and $\ell \geq 5$ be a prime with $(\ell, N) = 1$. Let $\chi$ be a real Dirichlet character modulo $4N$ and $f(z) \in S_{k, \frac{1}{2}}(\Gamma_0(4N), \chi) \cap O_K[[q]]$. For each prime $p$ with $(p, 4N\ell) = 1$, if there exists $\varepsilon_p \in \{\pm 1\}$ such that
\[ f(z) \equiv \sum_{n=1}^{\infty} a_n(n)q^n \pmod{\nu}, \]
then
\[ (p - 1)f|T_{p^k, k+\frac{1}{2}} \equiv \varepsilon_p \left( \frac{(-1)^k}{p} \right) \chi(p)(p^k + p^{k-1})(p - 1)f \pmod{\nu}. \]

Ahlgren et al. [7] proved that if $f \in S_{k, \frac{1}{2}}(\Gamma_0(4); O_K)$ and the Fourier coefficients of $f$ are supported on finitely many square classes modulo $\nu$, then $f$ has the form
\[ f(z) \equiv \sum_{n=1}^{\infty} a_f(n^2)q^{n^2} + \sum_{n=1}^{\infty} a_f(\ell n^2)q^{\ell n^2} \pmod{\nu}. \]
By using the theory of Galois representations, we extend the result [7] to cusp forms of half-integral weight on $\Gamma_0(4)$ without the Kohnen-plus condition.

**Proposition 2.4.** Assume that $f \in S_{k, \frac{1}{2}}(\Gamma_0(4); O_K)$ has the form
\[ f(z) \equiv \sum_{n=1}^{\infty} \sum_{i=1}^{f} a_f(s_i n^2)q^{s_i n^2} \pmod{\nu} \quad (2.1) \]
with square-free integers $s_i$. Then, the following statements are true.

(1) If $2 | k$ and $\ell \equiv 1 \pmod{4}$, then
\[ f(z) \equiv \sum_{n=1}^{\infty} a_f(n^2)q^{n^2} + \sum_{n=1}^{\infty} a_f(\ell n^2)q^{\ell n^2} \pmod{\nu}. \]

(2) If $2 | k$ and $\ell \equiv 3 \pmod{4}$, then
\[ f(z) \equiv \sum_{n=1}^{\infty} a_f(n^2)q^{n^2} \pmod{\nu}. \]

(3) If $2 | k$ and $\ell \equiv 3 \pmod{4}$, then
\[ f(z) \equiv \sum_{n=1}^{\infty} a_f(\ell n^2)q^{\ell n^2} \pmod{\nu}. \]

(4) If $2 | k$ and $\ell \equiv 1 \pmod{4}$, then
\[ f(z) \equiv 0 \pmod{\nu}. \]
Proof. Assume that for each $i \in \{1, \ldots, t\}$, there is a positive integer $n_i$ such that $a_i(s_i n_i^2) \neq 0 \mod v$. Following the proof of Lemma 4.1 in [4], there exist distinct odd primes $p_i, \ldots, p_r$, each relatively to $n_i \ell$, and a modular form $f_i \in S_k, \ell \Gamma_0(\prod_{j \neq i} p_j^{e_j}); O_k)$ such that

$$f_i(z) = \sum_{n=1}^{\infty} a_i(s_i n^2) q^{n^2} \equiv 0 \mod v.$$  

By Theorem 2.3, for each prime $p$ with $p|2s_i \ell \prod_{j \neq i} p_j$ and $p \neq 1 \mod \ell$, we have

$$f_i|T_p^{k-1}f_i = \left(\frac{(-1)^k s_i}{p}\right)(p^k + p^{k-1})f_i \mod v.$$

Since $S_2(\Gamma_0(4)) = S_2(\Gamma_0(4)) = \{0\}$, we may assume that $k \geq 2$. Let $f_i \in S_{2k}(\Gamma_0(2 \prod_{j \neq i} p_j^{e_j}))$ be the Shimura lift of $f_i$. Since the Shimura correspondence commutes with the Hecke operators, for each prime $p$ with $p|2s_i \ell \prod_{j \neq i} p_j$ and $p \neq 1 \mod \ell$, we obtain

$$f_i|T_p^{2k} = \left(\frac{(-1)^k s_i}{p}\right)(p^k + p^{k-1})f_i \mod v.$$

Then, there is an integer $N_i$ such that $N_i|2 \prod_{j \neq i} p_j^{e_j}$, and there is a newform $G_i \in S_{2k}(\Gamma_0(N_i))$ such that for each prime $p$ with $p|2s_i \ell \prod_{j \neq i} p_j$ and $p \neq 1 \mod \ell$, we have

$$\lambda_i(p) = \left(\frac{(-1)^k s_i}{p}\right)(p^k + p^{k-1}) \mod v.$$

Here, $\lambda_i(p)$ denotes the $p$th Hecke eigenvalue of $G_i$. Let $\mathbb{F}_v = O_v/v$. Note that there is a semi-simple Galois representation

$$\rho_i : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{F}_v),$$

such that for each prime $p$ with $p|N_i \ell$,

$$\text{tr}(\rho_i(\text{Frob}_p)) = \lambda_i(p) \mod v \quad \text{and} \quad \det(\rho_i(\text{Frob}_p)) = p^{2k-1} \mod v,$$

where $\text{Frob}_p$ denotes any Frobenius element at $p$. Let $\chi_i : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{F}_v^*$ be the mod-$\ell$ cyclotomic character. Following the argument of the proof of [5, Proposition 4.3], we have

$$\rho_i \equiv \begin{cases} \left(\begin{array}{cc} \left(\frac{-1)^k s_i}{\ell}\right) \chi_i^k & 0 \\ 0 & \left(\frac{-1)^k s_i}{\ell}\right) \chi_i^{k-1} \end{array}\right) & \text{if } \ell | s_i, \\ \left(\begin{array}{cc} \left(\frac{-1)^k s_i}{\ell}\right) \chi_i^k \cdot \frac{\ell-1}{2} & 0 \\ 0 & \left(\frac{-1)^k s_i}{\ell}\right) \chi_i^{k-1} \cdot \frac{\ell-1}{2} \end{array}\right) & \text{if } \ell | s_i, \end{cases} \quad (2.2)$$

where $\ell s_i' = s_i$.

By the result of Carayol [17], the conductor of $\rho_i$ divides $N_i$. By (2.2), we obtain that if $\ell | s_i$, then $s_i^2$ divides the conductor of $\rho_i$, and if $\ell | s_i$, then $(s_i')^2$ divides the conductor of $\rho_i$. Since $N_i|2 \prod_{j \neq i} p_j^{e_j}$ and $\gcd(s_i, \prod_{j \neq i} p_j) = 1$, we have $s_i \in \{1, \ell\}$. Moreover, the conductor of $\rho_i$ is not divided by 4. Therefore, we conclude that if $k$ is odd, then $s_i \neq 1$ and if $k + \frac{\ell-1}{2}$ is odd, then $s_i \neq \ell$.

We extend Proposition 2.4 to general modular forms of half-integral weight including noncusp forms in the following proposition.
Proposition 2.5. Assume that \( f \in M_{k+\frac{1}{2}}(\Gamma_0(4); O_K) \) has the form
\[
f(z) \equiv a_f(0) + \sum_{n=1}^{\infty} \sum_{i=1}^{l} a_f(s_i n^2)q^{bn^2} \pmod{\nu}
\]
with square-free integers \( s_i \). Then,
\[
f(z) \equiv a_f(0) + \sum_{n=1}^{\infty} a_f(n^2)q^{n^2} + \sum_{n=1}^{\infty} a_f(tn^2)q^{tn^2} \pmod{\nu}.
\]

Proof. Without loss of generality, we assume that there is a positive integer \( n \) such that \( a_f(s_i n^2) \neq 0 \) (mod \( \nu \)). Let \( a \) be the exponent of \( \ell \) in \( s_i n^2 \). Then, there is a unique square-free integer \( s'_i \) such that \( s'_i n^2 = \ell^a s_i' m_i^2 \) for some positive integer \( m_i \). By Lemma 2.1 (3), there is an integer \( k' \) and a modular form \( g \in M_{k+\frac{1}{2}}(\Gamma_0(4)) \) such that \( g \equiv f|U_{k'} \pmod{\nu} \). By Lemma 2.1 (4), there is \( h \in S_{k+\ell+\frac{1}{2}}(\Gamma_0(4)) \) such that \( h \equiv \Theta(g) \pmod{\nu} \). Since \( a_f(s_i n^2) \neq 0 \) (mod \( \nu \)), we have \( a_h(s_i' m_i^2) \neq 0 \) (mod \( \nu \)) and then \( h \) has the form (2.1). Then, \( s'_i = 1 \) by Proposition 2.5. This implies that \( s_i \in \{1, \ell\} \). Therefore, Proposition 2.5 is proved.

Combining Theorem 2.2 and Proposition 2.5, we obtain an explicit formula of \( f \in M_{k+\frac{1}{2}}(\Gamma_0(4)) \) having the form (2.3) when \( k < \ell - 1 \).

Lemma 2.6. Assume that \( f \in M_{k+\frac{1}{2}}(\Gamma_0(4); O_K) \) has the form (2.3) and \( f \not\equiv 0 \pmod{\nu} \). If \( k < \ell - 1 \), then \( k \in \{0, \frac{\ell-1}{2}\} \). Moreover,
\[
f(z) \equiv a_f(0) \left(1 + 2 \sum_{n=1}^{\infty} q^n\right) \pmod{\nu} \text{ if } k = 0
\]
and
\[
f(z) \equiv a_f(0) \left(1 + 2 \sum_{n=1}^{\infty} q^{tn^2}\right) \pmod{\nu} \text{ if } k = \frac{\ell - 1}{2}.
\]

Proof. We assume that \( k < \ell - 1 \). By Theorem 2.2, we have \( k \in \{0, 1, \frac{\ell-1}{2}\} \). Note that \( M_{\frac{1}{2}}(\Gamma_0(4)) \) is generated by \( T \). Thus, when \( k = 0 \), we obtain that \( f \) is a constant multiple of \( T \). If \( f \) has the form (2.3), then \( a_f(2) \equiv 0 \pmod{\nu} \) by Proposition 2.5. Note that \( M_{\frac{1}{2}}(\Gamma_0(4)) \) is generated by \( T^3 \) and \( a_f(2) = 3 \). Thus, when \( k = 1 \), we have \( f \equiv 0 \pmod{\nu} \). When \( k = \frac{\ell-1}{2} \), we have by Theorem 2.2
\[
f(z) \equiv \sum_{n=0}^{\infty} a_f(tn)q^n \pmod{\nu}.
\]
By Lemma 2.1 (3), there is \( g \in M_{\frac{1}{2}}(\Gamma_0(4)) \) such that \( g \equiv f|U_k \pmod{\nu} \). Since \( g \) is a constant multiple of \( T \), \( f \) is congruent to a constant multiple of \( T|V_k \pmod{\nu} \).

3 Proof of Theorems

In this section, we prove Theorems 1.1, 1.3, 1.5, and 1.6. First, we prove Theorem 1.3.

Proof of Theorem 1.3. We fix a prime \( \ell \geq 5 \). We prove Theorem 1.3 by induction on \( k \). When \( k < \ell - 1 \), Theorem 1.3 is true by Lemma 2.6. Thus, we assume that Theorem 1.3 is true when \( k < k_0 \) with a fixed positive integer \( k_0 \), where \( k_0 \) is a positive integer larger than \( \ell - 1 \).

To prove Theorem 1.3, it is enough to show that Theorem 1.3 is true when \( k = k_0 \) by induction on \( k \). Assume that \( f \in M_{k_0+\frac{1}{2}}(\Gamma_0(4); O_K) \) has the form (1.3). Then by Lemma 2.5, \( f \) has the form
\[ f(z) = a_0(0) + \sum_{n=1}^{\infty} a_n(n^2)q^{n^2} + \sum_{n=1}^{\infty} a_{\ell n^2}(\ell n^2)q^{\ell n^2} \pmod{\nu}, \]

and

\[ \Theta^{(t-1)/2}(f)(z) = \frac{1}{2} \left( \sum_{n \in \mathbb{Z}} a_n(n^2)q^{n^2} \right) \pmod{\nu}. \]

By Lemma 2.1 (4), there is \( g_0 \in \mathcal{S}_{k_0+\frac{1}{2}}(\Gamma_0(4)) \) such that

\[ g_0 \equiv \Theta^{(t-1)/2}(f) \pmod{\nu}. \]

Let \( k_1 = \text{max}\left( k_0 + \frac{1}{2}, \omega(g_0) \right) - \frac{1}{2} \). Then, there is \( g_1 \in M_{k_1+\frac{1}{2}}(\Gamma_0(4); \mathcal{O}_K) \) such that

\[ g_1(z) \equiv (f - \Theta^{(t-1)/2}(f))(z) \equiv a_0(0) + \sum_{n=1}^{\infty} a_n(\ell n^2)q^{n^2} + \sum_{n=1}^{\infty} a_{\ell n^2}(\ell^2 n^2)q^{\ell n^2} \pmod{\nu}. \]

Let \( k_2 \) be the largest integer satisfying

\[ k_2 + \frac{1}{2} \leq \frac{1}{\ell} \left( k_1 + \frac{1}{2} \right) \quad \text{and} \quad k_2 \equiv \frac{\ell - 1}{2} + k_0 \pmod{\ell - 1}. \]  

(3.1)

By Lemma 2.1 (3), there is \( g_2 \in M_{k_2+\frac{1}{2}}(\Gamma_0(4); \mathcal{O}_K) \) such that

\[ g_2(z) \equiv g_1|U(z) \equiv a_0(0) + \sum_{n=1}^{\infty} a_n(\ell n^2)q^{n^2} + \sum_{n=1}^{\infty} a_{\ell n^2}(\ell^2 n^2)q^{\ell n^2} \pmod{\nu}. \]

Since \( k_0 > \frac{\ell}{2} \), we have

\[ k_0 + 1 < \frac{1}{\ell} \left( k_0 + \frac{1}{2} \right) \leq \frac{1}{\ell} \left( k_0 + \frac{\ell^2}{2} \right) < k_0 + \frac{1}{2}. \]

For a nonnegative integer \( k \), we define a subset \( \mathcal{B}_k \) of \( M_{k+\frac{1}{2}}(\Gamma_0(4)) \) by

\[ \mathcal{B}_k = \begin{cases} \{ \Theta^{k/2}(V_{\ell^2}) \}_{0 \leq i \leq \text{att}(k, \ell)} \cup \{ \Theta^{(2k+t-1)/4}(V_{\ell^2}) \}_{0 \leq i \leq \text{att}(k, \ell)} \cup \{ T \} & \text{if } \eta(k) = \ell - 1, \\ \{ \Theta^{k/2}(V_{\ell^2}) \}_{0 \leq i \leq \text{att}(k, \ell)} \cup \{ \Theta^{(2k+t-1)/4}(V_{\ell^2}) \}_{0 \leq i \leq \text{att}(k, \ell)} \cup \{ T | V_i \} & \text{if } \eta(k) = \frac{\ell - 1}{2}, \\ \{ \Theta^{k/2}(V_{\ell^2}) \}_{0 \leq i \leq \text{att}(k, \ell)} \cup \{ \Theta^{(2k+t-1)/4}(V_{\ell^2}) \}_{0 \leq i \leq \text{att}(k, \ell)} \cup \{ T | V_i \} & \text{otherwise.} \end{cases} \]

To prove Theorem 1.3, it is enough to show that if \( f \in M_{k_0+\frac{1}{2}}(\Gamma_0(4); \mathcal{O}_K) \) has the form (1.3), then \( f \) is congruent to a linear combination of \( \mathcal{B}_{k_0} \) modulo \( \nu \).

By Proposition 2.4, if \( k_0 \) is odd, then \( g_0 \equiv 0 \pmod{\nu} \). Combining the assumption that Conjecture 1.2 is true, we have

\[ g_0 \equiv \frac{a_{\ell}(1)}{2} \Theta^{k_0/2}(T) \pmod{\nu}. \]

Since \( k_2 \equiv k_0 + \frac{\ell - 1}{2} \pmod{\ell - 1} \), it follows that \( \Theta^{k_2/2}(T) \equiv \Theta^{(2k_2+t-1)/4}(T) \pmod{\nu} \). By the induction hypothesis, \( g_2 \) is congruent to a linear combination of \( \mathcal{B}_{k_2} \). Since

\[ f \equiv (f - \Theta^{(t-1)/2}(f)) + \Theta^{(t-1)/2}(f) \equiv g_1 | V_i + g_0 \pmod{\nu}, \]

we deduce that \( f \) is congruent to a linear combination of

\[ \begin{cases} \{ \Theta^{k/2}(V_{\ell^2}) \}_{0 \leq i \leq \text{att}(k, \ell)} \cup \{ \Theta^{(2k+t-1)/4}(V_{\ell^2}) \}_{0 \leq i \leq \text{att}(k, \ell)+1} \cup \{ T | V_i \} & \text{if } \eta(k) = \ell - 1, \\ \{ \Theta^{k/2}(V_{\ell^2}) \}_{0 \leq i \leq \text{att}(k, \ell)} \cup \{ \Theta^{(2k+t-1)/4}(V_{\ell^2}) \}_{0 \leq i \leq \text{att}(k, \ell)+1} \cup \{ T | V_i \} & \text{if } \eta(k) = \frac{\ell - 1}{2}, \\ \{ \Theta^{k/2}(V_{\ell^2}) \}_{0 \leq i \leq \text{att}(k, \ell)} \cup \{ \Theta^{(2k+t-1)/4}(V_{\ell^2}) \}_{0 \leq i \leq \text{att}(k, \ell)+1} & \text{otherwise.} \end{cases} \]
If $r(k_2) = \ell - 1$, then
\[
T|V_2^2 \equiv T - \Theta^{(t-1/2)}(T) \equiv T - \Theta^{(2k_2 + t - 1/2)}(T) \pmod{v}.
\]
Thus, $f$ is congruent to a linear combination of $TV_{2\ell+1}$ and $TV_{2\ell+2}$ if $r(k_2) = \ell - 1$, otherwise.

To complete the proof, it is sufficient to show that $\alpha(\ell, k_2) \leq \beta(\ell, k_2) + 1 \leq \alpha(\ell, k_0)$. (3.2)

First, we assume that $k_0 + \frac{1}{2} \geq \frac{t}{T}$. Since $\Theta^m(T) \equiv \Theta^{2m + t - 1/2}(T)$ for any positive integer $m$, we have $\omega(g_0) \leq \omega(\Theta^{k_0/2}(T)) \leq \frac{t}{2}$. This implies that
\[
k_2 = \max\left(k_0, \omega(g_0) - \frac{1}{2}\right) = k_0.
\]
Then by (3.1), we obtain (3.2).

Now, we assume that $k_0 + \frac{1}{2} < \frac{t}{2}$. In this case, we have
\[
k_2 + \frac{1}{2} \leq \frac{1}{\ell}\left(k_1 + \frac{1}{2}\right) \leq \frac{1}{\ell} \cdot \max\left(k_0 + \frac{1}{2}, \omega(g_0)\right) \leq \frac{t}{2}.
\]
Further, assume that $k_2 \neq 0$ and $k_2 \neq \frac{t - 1}{2}$. Then $\alpha(\ell, k_2) = \beta(\ell, k_2) = \beta(\ell, k_0) = 0$. By Lemma 2.6, we have $g_2 \equiv 0 \pmod{v}$, and then
\[
f \equiv \Theta^{k_2/2}(f) \equiv \frac{a_1(1)}{2} - \Theta^{k_0/2}(T) \pmod{v}.
\]
Note that $\Theta^{(t-1/2)}(T) \equiv T - T^{\ell/2} \pmod{v}$, we have $\omega(\Theta^{(t-1/2)}(T)) = \frac{t}{2}$. Then, for a positive integer $m$ with $m \leq \frac{t - 1}{2}$, we have
\[
\omega(\Theta^m(T)) = (\ell + 1)m + \frac{1}{2}.
\]
By (3.3), we have
\[
\omega(\Theta^{k_0/2}(T)) = \eta(k_0) \leq \frac{t}{2} + \frac{1}{2} \leq k_0 + \frac{1}{2}.
\]
It implies that $a(\ell, k_0) = 1$. Hence, $a(\ell, k_2) = \beta(\ell, k_0)$ and $\beta(\ell, k_0) + 1 = a(\ell, k_0)$. For the cases when $k_0 = 0$ and $k_0 = \frac{t - 1}{2}$, we obtain (3.2) by direct computation. Thus, we conclude that if $f \in M_{k_0 + \ell/2}(\Gamma_0(k); O_K)$ has the form (1.3), then $f$ is congruent to a linear combination of $B_{k_0}$ modulo $v$. Therefore, Theorem 1.3 is proved by induction on $k$.

To prove Theorem 1.1, we use the following theorem which gives a sufficient condition for the weight $k + \frac{1}{2}$ that Conjecture 1.2 holds for $f \in S_{k+1/2}^+(\Gamma_0(k); O_K)$. It was proved in the proof of [7, Theorem 5.2].

**Theorem 3.1.** Assume that $f \in S_{k+1/2}^+(\Gamma_0(A); O_K)$ has the form
\[
f(z) \equiv \frac{1}{2} \sum_{n \in \ell} a_\ell(n^2)q^{n^2} \pmod{v} \tag{3.4}
\]
and $f \not\equiv 0 \pmod{v}$. Let $p_\ell$ be the smallest positive prime $p$ such that $p \equiv 1 \pmod{\ell}$. If $2k + 1 < p_\ell^2$, then $k$ is even and
\[ f \equiv \frac{1}{2} a_{f}(1) \Theta^{k/2}(T) \pmod{\nu}. \]

**Proof.** We follow the proof of [7, Theorem 5.2]. By Proposition 2.4, we obtain that \( k \) is even. By Theorem 2.3, for each odd prime \( p \) with \( p \not\equiv 0, 1 \pmod{\ell} \), we have
\[
 f|_{p^2} \equiv (p^k + p^{k-1})f \pmod{\nu}.
\]
Hence, for any positive odd integer \( m \) which is not divisible by any prime \( p \) with \( p \equiv 0, 1 \pmod{\ell} \), we have
\[
 a_{f}(m^2) \equiv a_{f}(1)m^k \pmod{\nu}.
\]
Let \( k_i = \max \left(k, \frac{n(k_i)(\ell + 1)}{2} \right) \). Then, there is \( g_i \in \mathcal{S}_{k_i+2}^{+}(\Gamma_0(4); \mathcal{O}_{K}) \) such that
\[
 g_i \equiv f - \frac{1}{2} a_{f}(1) \Theta^{n(k_i)/2}(T) \pmod{\nu}.
\]
Let \( h = g_i - g_j|U|V_\ell \in \mathcal{S}_{k_i+2}^{+}(\Gamma_0(16)) \). Then, \( a_{h}(n) \equiv 0 \pmod{\nu} \) for \( n < p_i^2 \). Since
\[
 \frac{1}{12} \left(k_i + 1\right) [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(16)] = 2k_i + 1 < p_i^2,
\]
we have \( h \equiv 0 \pmod{\nu} \) by the result of Sturm [18] called the Sturm bound. Then,
\[
 g_i(z) \equiv g_i|U|V_\ell(z) \equiv \sum_{m=1}^{\infty} a_{g_i}(4m^2)q^{4m^2} \pmod{\nu}.
\]
From the proof of [7, Theorem 5.2], we have \( g_i \equiv 0 \pmod{\nu} \). Then,
\[
 f(z) \equiv \frac{1}{2} a_{f}(1) \Theta^{k/2}(T)(z) \equiv \frac{1}{2} a_{f}(1) \left( \sum_{n \in \mathbb{Z}} n^k q^n \right) \pmod{\nu}. \]

The following proposition is a refinement of Theorem 1.1.

**Proposition 3.2.** Let \( g : \mathbb{R} \to \mathbb{R} \) be a function such that \( \sqrt{g(x)} \log x \) is an increasing function and \( \lim_{x \to \infty} g(x) = 0 \). Let \( P \) be a set of primes \( \ell \) such that for every \( f \in \mathcal{S}_{k_i+2}^{+}(\Gamma_0(4); \mathcal{O}_{K}) \) with \( k + \frac{1}{2} < g(\ell)(\log \ell)^2 \), if \( f \) has the form (1.2), then
\[
 f(z) \equiv \frac{1}{2} \sum_{l=0}^{a(k-1)} a_{f}(\ell^{2l})\Theta^{k/2}(T)(\ell^{2l}z) + \frac{1}{2} \sum_{l=0}^{b(k-1)} a_{f}(\ell^{2l+1})\Theta^{2k+1-1/2}(T)(\ell^{2l+1}z) \pmod{\nu}.
\]
Then, there is an absolute constant \( C \) such that
\[
 \# \{ \ell : \ell \not\in P \text{ and } \ell \leq X \} \leq C \sqrt{g(X)} \frac{X}{\log X} \left(1 + C \frac{\log \log X}{\log X} \right),
\]
where \( C_0 = \frac{2^{2/3} \pi^2}{3} \prod_{p>2} p^{-1} \).

**Proof.** Let \( p_i \) be the smallest positive prime \( p \) with \( p \equiv 1 \pmod{\ell} \). By using Theorem 3.1 to follow the proof of Theorem 1.3, we deduce that if \( p_i^2 > 2g(\ell)(\log \ell)^2 \), then \( \ell \in P \). From this, for a positive number \( X \), we have
\[
 \# \{ \ell : \ell \not\in P \text{ and } \ell \leq X \} \leq \# \{ \ell : p_i^2 \leq 2g(\ell)(\log \ell)^2 \text{ and } \ell \leq X \}.
\]
For convenience, let \( h(x) = \sqrt{g(x)} \). Then, we have
\[ \#\{\ell : p_{\ell}^2 \leq 2g(\ell)\ell \log \ell \text{ and } \ell \leq X\} = \#\{\ell : p_{\ell} \leq 2h(\ell)\ell \log \ell \text{ and } \ell \leq X\} \]
\[ \leq \sum_{n=1}^{\infty} \#\{\ell : p_{\ell} = 2n\ell + 1, n < h(\ell)\log \ell \text{ and } \ell \leq X\} \]
\[ \leq \sum_{n=1}^{[h(X)\log X]} \#\{\ell : p_{\ell} = 2n\ell + 1, n < h(X)\log X \text{ and } \ell \leq X\} \]
\[ \leq \sum_{n=1}^{[h(X)\log X]} \#\{\ell : 2n\ell + 1 \text{ is a prime and } \ell \leq X\}. \quad (3.5) \]

By [19, Theorem 3.12], for any positive integer \( n \), there is an absolute constant \( C \) such that
\[ \#\{\ell : 2n\ell + 1 \text{ is a prime and } \ell \leq X\} \leq A \left( \prod_{p|n} \frac{p-1}{p-2} \right) \frac{X}{(\log X)^2} \left( 1 + \frac{\log \log X}{\log X} \right), \]

where
\[ A := 8 \prod_{2<p} \left( 1 - \frac{1}{(p-1)^2} \right). \]

Note that for any positive integer \( n \), we have
\[ \prod_{2<p|n} \frac{p-1}{p-2} \leq \prod_{2<p} \frac{p(p-1)}{(p+1)(p-2)} \prod_{p|n} \frac{p+1}{p} \leq \left( \prod_{2<p} \frac{p(p-1)}{(p+1)(p-2)} \right) \frac{\sigma(n)}{n}. \]

From this, we have
\[ \frac{\sigma(n)}{n} = \sum_{n=1}^{[h(X)\log X]} \prod_{2<p|n} \frac{p-1}{p-2} \leq \prod_{2<p} \frac{p(p-1)}{(p+1)(p-2)} \sum_{n=1}^{[h(X)\log X]} \frac{\sigma(n)}{n} \]
\[ = \prod_{2<p} \frac{p(p-1)}{(p+1)(p-2)} \sum_{n=1}^{[h(X)\log X]} \frac{\sigma(n)}{n} \sum_{d|n} \frac{1}{d} \]
\[ \leq \prod_{2<p} \frac{p(p-1)}{(p+1)(p-2)} \sum_{d=1}^{[h(X)\log X]} \frac{h(X)\log X}{d} \frac{1}{d} \]
\[ \leq \prod_{2<p} \frac{p(p-1)}{(p+1)(p-2)} \frac{h(X)\log X}{6}. \]

Thus, (3.5) becomes
\[ \#\{\ell : p_{\ell} \leq 2h(\ell)\ell \log \ell \text{ and } \ell \leq X\} \leq \frac{4\pi^2}{3} \prod_{2<p} \frac{p^2}{p^2-1} \cdot \frac{h(X)}{\log X} \left( 1 + \frac{\log \log X}{\log X} \right). \]

Therefore, we conclude that
\[ \#\{\ell : \ell \notin P \text{ and } \ell \leq X\} \leq \left( \frac{2\sqrt{2\pi^2}}{3} \prod_{2<p} \frac{p^2}{p^2-1} \right) \cdot \sqrt{g(X)} \frac{X}{\log X} \left( 1 + \frac{\log \log X}{\log X} \right). \]

By using Proposition 3.2, we prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( g(x) = (\log x)^{-\epsilon} \). When \( 0 \leq \epsilon \leq 2 \), we obtain Theorem 1.1 by Proposition 3.2. If \( \epsilon > 2 \), then there is no prime \( \ell \) satisfying \( p_{\ell}^2 \leq 2g(\ell)\ell \log \ell \). Therefore, Theorem 1.1 is proved.
Now, we prove Theorem 1.5.

**Proof of Theorem 1.5.** To prove Theorem 1.5, first, we prove that if $a(\ell, k) \geq 1$ and $k$ is even, then $\dim \text{Null}(B_{k,m}) \geq 1$ for any positive integer $m$. Since $a(\ell, k) \geq 1$, we have

$$\omega(\Theta^{(k)}(T)) = \frac{\eta(k)}{2} - (\ell + 1) + \frac{1}{2} \leq k + \frac{1}{2}.$$ 

Then, there is $h \in M_{k,\frac{1}{2}}(\Gamma_0(4); \mathbb{C})$ such that $h \equiv \Theta^{(k/2)}(T) \pmod{\nu}$. Let $(c(0), \ldots, c(k/2)) \in \mathbb{Z}^{(k/2)+1}$ such that

$$h = \sum_{j=0}^{k/2} c(j)F_j^2\ell^{k-4j}.$$ 

Then, $(c(0), \ldots, c(k/2)) \in \text{Null}(B_{k,m})$ for any positive integer $m$ since $h$ has the form

$$h(z) \equiv \frac{1}{2} \sum_{m \in \mathbb{Z}} a_h(n)q^n \pmod{\nu}.$$ 

Here, $c(j)$ is the reduction of $c(j)$ modulo $\ell$. Thus, we conclude that $\dim \text{Null}(B_{k,m}) \geq 1$ for any positive integer $m$, when $a(\ell, k) \geq 1$ and $k$ is even.

Now, we assume that Conjecture 1.2 is true. Let $v = (\nu(0), \ldots, \nu(\frac{1}{2} \ell)) \in \text{Null}(B_{k,m})$ for all positive integers $m$, and let $v(j)$ be an integer such that the reduction of $v(j)$ modulo $\ell$ is equal to $\overline{v(j)}$. Let

$$f_v = \sum_{j=0}^{k/2} v(j)F_j^2\ell^{k-4j} \in M_{k,\frac{1}{2}}(\Gamma_0(4)).$$

Then $f_v$ has the form

$$f_v(z) \equiv \frac{1}{2} \sum_{m \in \mathbb{Z}} a_v(n^2)q^n \pmod{\nu}.$$ 

Note that $f_v \equiv \Theta^{(\ell-1)2}(f_v) \pmod{\nu}$. We assume that $k$ is even. By the assumption that Conjecture 1.2 is true, we have

$$f_v \equiv \frac{a_v(1)}{2} \Theta^{(k)}(T) \pmod{\nu}.$$ 

Thus, $\lim_{m \to \infty} \dim \text{Null}(B_{k,m})$ is less than or equal to 1. If $\lim_{m \to \infty} \dim \text{Null}(B_{k,m}) = 1$, then there is $f \in S_{k,\frac{1}{2}}(\Gamma_0(4); \mathbb{C})$ such that

$$f \equiv \Theta^{(k)}(T) \pmod{\nu}.$$ 

This implies that

$$\eta(k) \cdot \frac{\ell}{2} + 1 + \frac{1}{2} = \omega(\Theta^{(k)}(T)) \leq k + \frac{1}{2}.$$ 

By the definition of $a(\ell, k)$, we have $a(\ell, k) \geq 1$. Hence, we conclude that Conjecture 1.4 is true.

To complete the proof of Theorem 1.5, we assume that Conjecture 1.4 is true. Further, assume that $f \in S_{k,\frac{1}{2}}(\Gamma_0(4); \mathbb{O})$ has the form

$$\Theta(f) \equiv \frac{1}{2} \sum_{n \in \mathbb{Z}} s n^2 a_f(sn^2)q^{sn^2} \pmod{\nu}$$

with a square-free integer $s$ and $\Theta(f) \not\equiv 0 \pmod{\nu}$. Then, $k$ is even and $s = 1$ by Proposition 2.4. By Lemma 2.1, there is $f_0 \in S_{k+\ell,\frac{1}{2}}(\Gamma_0(4))$ such that $f_0 \equiv \Theta(f) \pmod{\nu}$. Let $(d(0), \ldots, d((k + \ell + 1)/2)) \in \mathbb{O}_k^{(k+\ell+1)/2}$ satisfying
Let $F_v := O_R^v$. Then, for any positive integer $m$, we have
\[
(d(0), \ldots, d((k + \ell + 1)/2)) \in \text{Null}(B_{k+\ell+1,m}) \otimes_{F_v} F_v,
\]
where $d(f)$ is the reduction of $d(j)$ modulo $v$. By the assumption that Conjecture 1.4 is true, the dimension of $\text{Null}(B_{k+\ell+1,m})$ is 1 for a sufficiently large $m$. Hence, $f_0$ is congruent to a constant multiple of $\Theta^{(k+\ell+1)/2}(T)$ modulo $v$. Since $\eta(k + \ell + 1) = \eta(k + 2)$, we conclude that $\Theta(f)$ is congruent to a constant multiple of $\Theta^{(k+\ell+2)/2}(T)$ modulo $v$. \hfill \Box

We confirm Conjecture 1.2 under the assumption that $k \leq 1000$, or that $\ell \in \{5, 7, 11, 13, 17, 19\}$ and $k \leq 10,000$.

**Proof of Theorem 1.6.** Note that if $\Theta(f) \equiv 0 \pmod{v}$, then Conjecture 1.2 is true since $a_v(1) \equiv 0 \pmod{v}$. Thus, we may assume that $\Theta(f) \not\equiv 0 \pmod{v}$. By Proposition 2.4, $s = 1$ and $k$ is even. Then, $f$ has the form
\[
f(z) \equiv \frac{1}{2} \sum_{n \in \mathbb{Z}} a(n^2)q^n + \sum_{n=1}^{\infty} a(\ell n)q^{\ell n} \pmod{v}.
\]
From this, we have
\[
(f - \Theta^{(\ell-1)/2}(f))(z) \equiv \sum_{n=1}^{\infty} a(\ell n)q^{\ell n} \pmod{v}.
\]
Assume that $k < \frac{\ell - 1}{2}$. By Lemma 2.1 (3), if $f \not\equiv \Theta^{(\ell-1)/2}(f) \pmod{v}$, then there is a nonnegative integer $k_0$ such that
\[
k_0 \equiv k + \frac{\ell - 1}{2} \pmod{\ell - 1} \quad \text{and} \quad k_0 + \frac{1}{2} \leq \frac{1}{\ell} \left(k + \frac{\ell^2}{2}\right),
\]
and there is $g_0 \in S_{k_0^2}(\Gamma(4))$ such that
\[
g_0(z) \equiv (f - \Theta^{(\ell-1)/2}(f))U(z) \equiv \sum_{n=1}^{\infty} a(\ell n)q^{\ell n} \pmod{v}.
\]
Since $k < \frac{\ell - 1}{2}$, we have $k_0 = 0$ and then $g_0 = 0$. Thus, $f \equiv \Theta^{(\ell-1)/2}(f) \pmod{v}$ when $k < \frac{\ell - 1}{2}$. This implies that $f \equiv 0 \pmod{v}$ by Lemma 2.6. Hence, we conclude that Conjecture 1.2 is true when $k < \frac{\ell - 1}{2}$.

We fix a prime $\ell$ with $5 \leq \ell \leq 2001$. Assume that there is $f \in S_{k^2}(\Gamma(4); O_K)$ having the form
\[
\Theta(f)(z) \equiv \frac{1}{2} \sum_{n \in \mathbb{Z}} n^2 a(n^2)q^n \pmod{v}
\]
such that
\[
\Theta(f) \not\equiv \frac{a_v(1)}{2} \Theta^{(k^2+1)/2}(T) \pmod{v}.
\]
Then, $f \cdot E_{\ell-1} \in S_{k+\ell-1}(\Gamma(4); O_K)$ satisfies
\[
\Theta(f \cdot E_{\ell-1}) \not\equiv \frac{a_v(1)}{2} \Theta^{(k+\ell+1)/2}(T) \pmod{v}.
\]
Thus, for a positive integer $m_0$, confirming Conjecture 1.2 for positive integers $k$ such that $k \leq m_0$ reduces to confirming Conjecture 1.2 for positive integers $k$ such that $m_0 + 2 - \ell \leq k \leq m_0$. 

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When \( \max(0, 1002 - \ell) \leq k \leq 1000 \) and \( k \) is even, we obtain by numerical method
\[
\text{dimNull}(B_{k, 1000}) = 1, (\alpha(\ell, k)).
\]
In the proof of Theorem 1.5, we have \( \text{dimNull}(B_{k, m}) \geq 1, (\alpha(\ell, k)) \) for any positive integer \( m \). Since \( \text{dimNull}(B_{k, m}) \leq \text{dimNull}(B_{k, 1000}) \) for \( m \geq 1000 \), we have
\[
\lim_{m \to \infty} \text{dimNull}(B_{k, m}) = 1, (\alpha(\ell, k))
\]
when \( \max(0, 1002 - \ell) \leq k \leq 1000 \). By Theorem 1.5, we conclude that Conjecture 1.2 is true when \( k \leq 1000 \).

The proofs for the cases when \( \ell \in \{5, 7, 11, 13, 17, 19\} \) and \( k \leq 10,000 \) are similar to the proof of the previous case. So, we skip it. \( \square \)

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