1. Introduction

A widely studied class of heterotic string [1] vacua is provided by the (2,2)-
superconformal theories with central charge $c = 9$ [2–6] that correspond to space-time
compactifications on Calabi-Yau manifolds [7–11]. Of major interest in this context
is the special Kähler geometry [12–16] of the associated Calabi-Yau moduli spaces
[17–20], that to a large extent determines all the relevant features of the low-energy
effective supergravity action [21].

Recently it has become evident that there is a deep relation between this special
Kähler geometry and the flat geometry of the topological field theories [22–29]
obtained by twisting and deforming N=2 superconformal models [30–32]. So far the
analysis of these issues has been based mainly on the use of the Landau-Ginzburg for-
mulation of topological models and of its relation with singularity theory [23-26,33-36].
In such a formulation the parameters entering the Landau-Ginzburg superpotential
are interpreted as coordinates of some (moduli) space; however these are not the
flat coordinates one is interested in, since they do not correspond to deformations
around the conformal point; rather they are related to the latter by the solution of a
uniformization problem, which in general involves higher transcendental functions.

In this paper we present an alternative approach to topological models where
the relation to singularity theory is directly obtained in a natural system of flat
coordinates. At the same time, these are the parameters of a Landau-Ginzburg super-
potential as well as the deformations around the conformal point. The first step
of our construction is the use of free first-order $(b,c,\beta,\gamma)$-systems to describe N=2
superconformal theories as proposed in [37]. We then show that an arbitrary interac-
tion of the Landau-Ginzburg type – i.e. characterized by a polynomial potential $V$
– can be added to the free Lagrangian without spoiling the superconformal invariance
if $V$ is a quasi-homogeneous function. The deformation parameters of the potential
are then shown to be the flat coordinates.

The paper is organized as follows. In Section 2 we discuss the classical $(b,c,\beta,\gamma)$
realization of N=2 superconformal models and their topological twisting, and compare
our approach with the conventional topological Landau-Ginzburg description. In
Section 3 we discuss the quantum properties of our theories and show that the classical
(2,2)-superconformal invariance extends trivially to the quantum level due to the
absence of loop corrections. In Section 4 we bosonize the first-order systems and
make contact with the usual Coulomb gas representation of N=2 minimal models
[38,39]. Section 5 contains explicit calculations of topological correlation functions
and a direct verification that our coordinates are indeed the flat ones. We retrieve
established results for the minimal models with $c < 3$ and for the torus with $c = 3$.
In Appendix A we give a short derivation of the Landau- Ginzburg action using
rheonomic approach, and in Appendix B we show how our first order lagrangian can
be seen as a BRST commutator, with a suitable choice of BRST charge.
2. Lagrangian formulation of N=2 theories with \((b,c,\beta,\gamma)\)-systems and comparison with the Landau-Ginzburg approach

In this section we consider the realization of (2,2)-supersymmetric models in terms of free \((b,c,\beta,\gamma)\)-systems recently introduced in [37], and generalize it to include interactions of the Landau-Ginzburg (LG) type. As stressed in the introduction, N=2 superconformal theories with \(c = 9\) are of primary interest in connection with superstring compactifications on six-dimensional Calabi-Yau spaces. More generally, a (2,2)-superconformal theory with \(c = 3d\) corresponds to the critical point of an N=2 \(\sigma\)-model on a target space with complex dimension \(d\) and vanishing first Chern class.

We will call these spaces Calabi-Yau \(d\)-folds.

In the LG formulation of (2,2)-supersymmetric models, the superconformal theory is viewed as the infrared fixed point of a two-dimensional N=2 Wess-Zumino model with a polynomial superpotential \(W\). In particular, when \(W\) is an analytic quasi-homogeneous function of the chiral superfields \(X_i\), one can recover the complete ADE classification of the N=2 minimal models from the ADE classification of quasi-homogeneous potentials with zero modality [5,33-36]. Furthermore, the polynomials \(W\)'s can be identified with those used in the construction of Calabi-Yau \(d\)-folds. Indeed, it can be shown [5,35] that a superconformal model with \(c = 3d\), corresponding to a LG potential \(W\), is the same as that associated to a \(\sigma\)-model on the Calabi-Yau \(d\)-fold defined by the polynomial constraint \(W(X_i) = 0\) in a suitable projective or weighted projective space.\(^1\)

In this section we show that it is possible to add a polynomial interaction \(V\) of the LG type to a collection of free first-order \((b,c,\beta,\gamma)\)-systems in such a way that, if \(V\) is a quasi-homogeneous function, the theory possesses an N=2 superconformal symmetry already at the classical level. We also show that the interaction potential unambiguously fixes the weights of the pseudo-ghost fields. As in the standard LG case, also here we can recover the ADE classification of the N=2 minimal models from ADE classification of the interaction potential [35,36]; however in our case the theory is always manifestly superconformal invariant. Our formulation allows us to add all relevant perturbations (versal deformations of the potential) and to study the renormalization group flows in a very simple way. Whenever we use a quasi-homogeneous potential with modality different from zero, we can study marginal deformations and eventually Zamolodchikov’s metric on the associated moduli space. Alternatively, we can consider topological models by “twisting” the generators of the superconformal algebra and compute topological correlation functions. Our formulation provides valuable methods to evaluate these latter.

We start this program by defining our model. We consider a collection of pseudo-ghost fields \(\{b_\ell, c_\ell, \beta_\ell, \gamma_\ell; \tilde{b}_r, \tilde{c}_r, \tilde{\beta}_r, \tilde{\gamma}_r\}\) where \(\ell = 1, \ldots, N_L\) and \(r = 1, \ldots, N_R\). \(\beta_\ell\) and \(\gamma_\ell\) form a bosonic first-order system with weights \(\lambda_\ell\) and \(1 - \lambda_\ell\) respectively whereas \(b_\ell\)

---

\(^1\) In general \(W(X_i)\) is actually the sum of several terms \(W(X_i) = \sum_\alpha W_\alpha(X_i)\) and the Calabi-Yau \(d\)-fold is given by the complete intersection \(W_1 = W_2 = \cdots = W_n = 0\) [5].
and $c_\ell$ form a fermionic first-order system with weights $\lambda_\ell + \frac{i}{2}$ and $\frac{i}{2} - \lambda_\ell$ respectively. The same can be said for the tilded fields with $\lambda_\ell$ replaced by $\check{\lambda}_r$. The action is

$$S = \int d^2z \ L = \int d^2z \ (L_0 + \Delta L)$$

where

$$L_0 = \sum_\ell \left[ -\lambda_\ell \beta_\ell \bar{\partial} \gamma_\ell + (1 - \lambda_\ell) \gamma_\ell \bar{\partial} \check{\beta}_\ell - (\lambda_\ell + \frac{1}{2}) b_\ell \bar{\partial} c_\ell + (\lambda_\ell - \frac{1}{2}) c_\ell \bar{\partial} b_\ell \right]$$

$$+ \sum_r \left[ -\check{\lambda}_r \check{\beta}_r \partial \check{\gamma}_r + (1 - \check{\lambda}_r) \check{\gamma}_r \partial \check{\beta}_r - (\check{\lambda}_r + \frac{1}{2}) \check{b}_r \partial \check{c}_r + (\check{\lambda}_r - \frac{1}{2}) \check{c}_r \partial \check{b}_r \right]$$

and

$$\Delta L = \sum_{\ell, r} b_\ell \check{b}_r \partial V(\beta) \check{\partial} V(\check{\beta}) .$$

Here and in the following we use the short-hand notations $\partial_\ell \equiv \partial / \partial \beta_\ell$ and $\check{\partial}_r \equiv \partial / \partial \check{\beta}_r$. $L_0$ in (2.2a) represents the standard free Lagrangian for first-order systems of the given weights and $\Delta L$ in (2.2b) defines an interaction of the LG type when $V$ and $\check{V}$ are polynomial functions of $\beta_\ell$ and $\check{\beta}_r$ respectively.

From (2.1) one can derive the following equations of motion

$$\bar{\partial} \beta_\ell = 0 , \quad \bar{\partial} b_\ell = 0 ,$$

$$\bar{\partial} \check{\beta}_r = 0 , \quad \bar{\partial} \check{b}_r = 0 ,$$

$$\bar{\partial} c_\ell = \sum_r \check{b}_r \partial V(\beta) \check{\partial} \check{V}(\check{\beta}) ,$$

$$\bar{\partial} \gamma_\ell = \sum_m b_m \check{b}_r \partial_\ell \partial_m V(\beta) \check{\partial} \check{V}(\check{\beta}) ,$$

$$\bar{\partial} \check{c}_r = - \sum_\ell b_\ell \partial V(\beta) \check{\partial} \check{V}(\check{\beta}) ,$$

$$\bar{\partial} \check{\gamma}_r = \sum_{\ell, s} b_\ell \partial_\ell V(\beta) \check{b}_s \check{\partial} \check{\partial} \check{\partial} \check{V}(\check{\beta}) .$$

The first two lines of (2.3a) show that $\beta_\ell$, $b_\ell$, $\check{\beta}_r$ and $\check{b}_r$ satisfy the same equations as in the free case, whereas $c_\ell$, $\gamma_\ell$, $\check{c}_r$ and $\check{\gamma}_r$ have no longer a definite holomorphic or anti-holomorphic character in the presence of the interaction. We can write formal solutions to the equations for $c_\ell$ and $\gamma_\ell$ as follows [40]:

$$c_\ell(z, \bar{z}) = c_\ell^0(z) + \int_\Delta \frac{d^2w}{2\pi i} \frac{1}{w - z} \sum_r \check{b}_r(\check{w}) \partial_\ell V(\beta(w)) \check{\partial} V(\check{\beta}(\check{w})) ,$$

$$\gamma_\ell(z, \bar{z}) = \gamma_\ell^0(z) + \int_\Delta \frac{d^2w}{2\pi i} \frac{1}{w - z} \sum_{m, r} b_m(w) \check{b}_r(\check{w}) \partial_\ell \partial_m V(\beta(w)) \check{\partial} \check{V}(\check{\beta}(\check{w})) ,$$

(2.3b)
where $\Delta$ is a disk containing $w$, and $c^0_\ell$ and $\gamma^0_\ell$ are arbitrary holomorphic fields. Similar formal expressions (with the obvious changes) hold also for $\tilde{c}_r$ and $\tilde{\gamma}_r$. It is fairly easy to realize that under canonical quantization of (2.1) the fundamental operator product expansions are the same as in the free case. Indeed, even in the presence of the interaction, we have

$$
\beta_\ell(z) \gamma_m(w, \bar{w}) = -\frac{\delta_{\ell m}}{z - w} + \cdots ,
$$

$$
b_\ell(z) c_m(w, \bar{w}) = \frac{\delta_{\ell m}}{z - w} + \cdots ,
$$

and similarly for the tilded fields. Of course, the interaction is not immaterial and it has to be carefully analyzed in a complete quantum treatment as we will do in Section 3.

It is well-known that $L_0$ in (2.2a) describes a (2,2)-superconformal field theory with central charges

$$
c_L = \sum_\ell (3 - 12\lambda_\ell) , \quad c_R = \sum_r (3 - 12\tilde{\lambda}_r) ,
$$

for the left and the right sectors respectively. We will now show that the addition of the interaction $\Delta L$ does not destroy this (2,2)-superconformal invariance if $V$ and $\tilde{V}$ are quasi-homogeneous functions, i.e. if for any $a \in \mathbb{R}^+$,

$$
V(a^{\omega_\ell} \beta_\ell) = a V(\beta_\ell) , \quad \tilde{V}(a^{\tilde{\omega}_r} \tilde{\beta}_r) = a \tilde{V}(\tilde{\beta}_r) .
$$

The parameters $\omega_\ell$ and $\tilde{\omega}_r$ are called the homogeneous weights of $\beta_\ell$ and $\tilde{\beta}_r$ respectively. By enforcing the requirement that the interaction Lagrangian $\Delta L$ have the correct dimensions, one can see that

$$
\omega_\ell = 2\lambda_\ell , \quad \tilde{\omega}_r = 2\tilde{\lambda}_r ;
$$

the parameters $\lambda_\ell$ and $\tilde{\lambda}_r$ of the free Lagrangian (2.2a) are therefore fixed by the interaction terms. When (2.6) and (2.7) are satisfied, the action $S$ in (2.1) is invariant under the following N=2 holomorphic supersymmetry transformations

$$
\delta \beta_\ell = 2\sqrt{2} \epsilon^{-} b_\ell ,
$$

$$
\delta b_\ell = \frac{1}{\sqrt{2}} \epsilon^+ \partial \beta_\ell + \sqrt{2} \lambda_\ell \partial \epsilon^+ \beta_\ell ,
$$

$$
\delta c_\ell = 2\sqrt{2} \epsilon^{-} \gamma_\ell ,
$$

$$
\delta \gamma_\ell = \frac{1}{\sqrt{2}} \epsilon^+ \partial c_\ell - \sqrt{2} (\lambda_\ell - \frac{1}{4}) \partial \epsilon^+ c_\ell ,
$$

$$
\delta \tilde{\beta}_r = 0 ,
$$

$$
\delta \tilde{b}_r = 0 ,
$$

$$
\delta \tilde{c}_r = -\frac{1}{\sqrt{2}} \epsilon^+ V(\beta) \partial_r \tilde{V}(\tilde{\beta}) ,
$$

$$
\delta \tilde{\gamma}_r = \frac{1}{\sqrt{2}} \epsilon^+ V(\beta) \sum_s \tilde{\partial}_r \tilde{\partial}_s \tilde{V}(\tilde{\beta}) \tilde{b}_s ,
$$

4
where $\epsilon^\pm$ are arbitrary holomorphic functions ($\bar{\partial}\epsilon^\pm = 0$). The action $S$ is also invariant under N=2 anti-holomorphic symmetries which are similar to the ones defined in (2.8), with the exchange of the tilded and untilded quantities, and the replacement of $\epsilon^\pm$ with arbitrary anti-holomorphic functions $\bar{\epsilon}^\pm$ ($\bar{\partial}\bar{\epsilon}^\pm = 0$). Moreover, if we relax the hypothesis that $V$ and $\tilde{V}$ are quasi-homogeneous, the transformations (2.8) and their $\tilde{\epsilon}$-analogues remain symmetries of (2.1) provided $\epsilon^\pm$ and $\bar{\epsilon}^\pm$ are constant parameters. This means that our model has a global N=2 supersymmetry for any choice of $V$ and $\tilde{V}$, and an N=2 superconformal invariance for quasi-homogeneous potentials.

The conserved Noether’s currents associated to the symmetries (2.8) are

$$
G^+_z = \sqrt{2} \sum_\ell \left[ (\frac{3}{2} - \lambda_\ell) c_\ell \beta_\ell - \lambda_\ell \beta_\ell \partial c_\ell \right],
$$

$$
G^+_z = \sqrt{2} \sum_\ell \left[ \lambda_\ell \epsilon_\ell \bar{\partial} c_\ell + (\lambda_\ell - \frac{1}{4}) \partial \beta_\ell c_\ell \right] - \frac{1}{\sqrt{2}} \sum_r V(\beta) \tilde{b}_r \partial \tilde{\epsilon}_r \tilde{V}(\tilde{\beta}),
$$

$$
G^-_z = 2\sqrt{2} \sum_\ell \gamma_\ell b_\ell,
$$

$$
G^-_z = 0.
$$

If we use the equations of motion (2.3a) for quasi-homogeneous potentials, we see that $G^+_z$ vanishes on-shell; thus from the conservation laws we deduce that $G^+_z$ and $G^-_z$ are holomorphic currents even if they contain the non-holomorphic fields $c_\ell$ and $\gamma_\ell$. We denote these currents by $G^\pm(z)$.

The action (2.1) is also invariant under holomorphic conformal reparametrizations and $U(1)$-rescalings of the fields; the conserved Noether’s currents associated to such symmetries are the stress-energy tensor $T_{\mu\nu}$ and the $U(1)$-current $J_\mu$. For homogeneous potentials it is not difficult to see that the trace of $T_{\mu\nu}$ and the $\bar{z}$-component of $J_\mu$ are zero on-shell (see also Section 3). Therefore, from the conservation laws, we deduce that

$$
T_{zz} = \sum_\ell \left[ -\lambda_\ell \beta_\ell \partial \gamma_\ell + (1 - \lambda_\ell) \gamma_\ell \partial \beta_\ell - (\lambda_\ell + \frac{1}{4}) b_\ell \partial c_\ell + (\frac{1}{2} - \lambda_\ell) c_\ell \partial b_\ell \right],
$$

and

$$
J_z = \sum_\ell \left[ (2\lambda_\ell - 1) b_\ell c_\ell + 2\lambda_\ell b_\ell \gamma_\ell \right]
$$

are holomorphic currents. We denote them by $T(z)$ and $J(z)$ respectively.

Using the OPE’s in (2.4), it is straightforward to check that $T(z)$, $G^\pm(z)$ and $J(z)$ close an N=2 superconformal algebra with central charge

$$
c_L = \sum_\ell (3 - 12\lambda_\ell).
$$

\(^2\) From now on, to avoid repetitions we will discuss only the left sector and understand that similar considerations can be made in the right sector, with some obvious change of signs.
Thus, we have shown that the interaction $\Delta L$ with homogeneous polynomials $V$ and $\tilde{V}$ does not spoil the superconformal properties of $L_0$.

In our formulation the ADE classification of N=2 superconformal models is an immediate consequence of ADE classification of homogeneous polynomials of zero modality [5,35,36]. The latter are

\begin{align*}
A_n : & \quad V = \frac{1}{n+1} \beta^{n+1} \quad \Rightarrow \lambda = \frac{1}{2n+2} \quad n \geq 1 , \\
D_n : & \quad V = \frac{1}{n-1} \beta^{n-1}_1 + \frac{1}{2} \beta_1 \beta^2_2 \quad \Rightarrow \lambda_1 = \frac{1}{2n-2} , \quad \lambda_2 = \frac{n-2}{4n-4} \quad n \geq 2 , \\
E_6 : & \quad V = \frac{1}{3} \beta^3_1 + \frac{1}{4} \beta^4_2 \quad \Rightarrow \lambda_1 = \frac{1}{6} , \quad \lambda_2 = \frac{1}{8} , \\
E_7 : & \quad V = \frac{1}{3} \beta^3_1 + \frac{1}{3} \beta_1 \beta^3_2 \quad \Rightarrow \lambda_1 = \frac{1}{6} , \quad \lambda_2 = \frac{1}{9} , \\
E_8 : & \quad V = \frac{1}{3} \beta^3_1 + \frac{1}{5} \beta^5_2 \quad \Rightarrow \lambda_1 = \frac{1}{6} , \quad \lambda_2 = \frac{1}{10} .
\end{align*}

We remark that the values of $\lambda_\ell$’s listed in (2.13) are fixed by the homogeneous weights of $\beta_\ell$’s according to (2.7). If we now insert such values into (2.12) we obtain the correct central charges for the N=2 minimal models in the ADE classification, namely

$$c(A_n) = \frac{3n-3}{n+1} , \quad c(D_n) = \frac{3n-6}{n-1} , \quad c(E_6) = \frac{5}{2} , \quad c(E_7) = \frac{8}{3} , \quad c(E_8) = \frac{14}{5} . \quad (2.14)$$

It is also interesting to observe that the ring determined by the potential $V$, which contains all polynomials in $\beta_\ell$’s modulo the vanishing relations $\partial_\ell V = 0$, coincides with the ring of chiral primary operators of the N=2 minimal model associated to $V$. Indeed, using (2.10) and (2.11), one can easily check that the $U(1)$-charge of $(\beta_\ell)^n$ is twice its conformal dimension.

In order to compare our formulation of N=2 supersymmetric models with the standard LG approach and to establish a clear correspondence with the topological conformal field theories, it is convenient to specialize our system to the case of a complete symmetry between the left and the right sectors ($N_L = N_R = N$). We shall then consider interactions of the form

$$\Delta L = \sum_{i,j=1}^N b_i \partial_\ell \partial_{\tilde{\ell}} W \quad (2.15)$$

where $W$ is a quasi-homogeneous function of the variables

$$X_i = \beta_i \tilde{\beta}_i \quad , \quad i = 1, \ldots, N \quad . \quad (2.16)$$

This is clearly a very special case of (2.2b). Under these conditions, the Lagrangian (2.1) describes the infrared fixed point of an ordinary N=2 LG model with superpotential $W$. This equivalence will be fully illustrated in the next sections.
instead, we discuss the topological formulation of our models. It is well known that given an N=2 superconformal algebra generated by $T(z), G^\pm(z)$ and $J(z)$, one obtains a topological conformal algebra by “twisting” the currents according to

$$\hat{T}_\pm(z) = T(z) \pm \frac{1}{2} \partial J(z),$$

$$\hat{J}_\pm(z) = \pm J(z),$$

$$Q_\pm(z) = G^\pm(z),$$

$$G_\pm(z) = G^{\mp}(z).$$

(2.17)

We refer the reader to the original literature [30-32] for a discussion of the properties of the topological conformal algebra generated by $\hat{T}_\pm, \hat{J}_\pm, Q_\pm, G_\pm$; here we simply mention that $Q_\pm(z)$ is interpreted as a BRST current and the cohomology classes of the BRST charge

$$Q_\pm^{BRST} = \oint dz \, Q_\pm(z)$$

are identified with the physical fields of the topological theory. The two choices of signs in (2.17) lead to two different sets of BRST invariant states: the chiral primary fields of the original N=2 superconformal algebra for the $+$ sign, and the anti-chiral primary fields for the $-$ sign.

It is now interesting to study the consequences of the twist (2.17) on our $(b,c,\beta,\gamma)$ systems. We first analyze the $+$ case. From (2.10) and (2.11) we simply get

$$\hat{T}_+ = T + \frac{1}{2} \partial J = \sum_{i=1}^{N} (\partial c_i b_i + \gamma_i \partial \beta_i).$$

(2.19)

This is the canonical stress-energy tensor for a collection of $N$ commuting $(\beta,\gamma)$-systems of weight $\lambda = 0$, and $N$ anticommuting $(b,c)$-systems of weight $\lambda = 1$. $\hat{T}$ in (2.19) closes a Virasoro algebra with vanishing central charge. Indeed, the central charge of a first-order system of weight $\lambda$ is

$$c_\lambda = \varepsilon (1 - 3Q^2)$$

(2.20)

where

$$Q = \varepsilon (1 - 2\lambda)$$

(2.21)

is a “background charge” and $\varepsilon = 1$ or $-1$, depending on whether the system is anticommuting or commuting. In our case both the $(\beta,\gamma)$-systems and the $(b,c)$-systems have $Q = -1$, but since their statistics is different, their central charges exactly cancel.

To fully appreciate the effects of this topological twist on our models, we now write the topological Lagrangian and its BRST symmetries. The Lagrangian is

$$\mathcal{L}_{top} = \sum_{i=1}^{N} \left[ \gamma_i \partial^{\beta}\beta^i + \bar{\gamma}_i \partial^{\bar{\beta}}\bar{\beta}^i - b_i \partial c_i^i - \bar{b}_i \partial \bar{c}_i^i \right] + \sum_{i,j=1}^{N} \left[ b_i b_j \partial_i \partial_j W(X) \right]$$

(2.22)
The BRST transformations which leave (2.22) invariant, can be obtained from (2.8) and their analogues by identifying the BRST parameter $\theta$ with $\frac{\epsilon^+}{\sqrt{2}} = \bar{\epsilon}^+ + \sqrt{2} = \bar{\epsilon}^+ + \sqrt{2}$ (the factor of $1/\sqrt{2}$ is introduced for convenience). These transformations are most conveniently exhibited as the action of the nilpotent Slavnov operator $s$ on all fields, namely

\[
\begin{align*}
    s \beta_i &= 0 , \\
    s \bar{\beta}_i &= 0 , \\
    s b_i &= \partial \beta_i , \\
    s \bar{b}_i &= \bar{\partial} \bar{\beta}_i , \\
    s \gamma_i &= \partial c_i - \sum_{j=1}^{N} b_j \partial_i \partial_j W , \\
    s \bar{\gamma}_i &= \bar{\partial} \bar{c}_i + \sum_{j=1}^{N} \bar{b}_j \partial_i \bar{\partial}_j W , \\
    s c_i &= + \partial_i W , \\
    s \bar{c}_i &= - \bar{\partial}_i W .
\end{align*}
\]

Using (2.23) it is quite easy to construct the representatives of the BRST-cohomology classes and the corresponding integrated invariants. According to the general theory, we have to consider multiplets composed by a 0-form $\Phi_P$, a 1-form $\Phi_P^{(1)}$ and a 2-form $\Phi_P^{(2)}$ which satisfy the following descent equations

\[
\begin{align*}
    s \Phi_P &= 0 , \\
    s \Phi_P^{(1)} &= - d \Phi_P , \\
    s \Phi_P^{(2)} &= - d \Phi_P^{(1)} , \\
    d \Phi_P^{(2)} &= 0 .
\end{align*}
\]

Moreover the 0-form $\Phi_P$ must belong to a non-trivial BRST-cohomology class, i.e. it should not be BRST-exact. The solutions of the descent equations (2.23) provide the local physical observables $\Phi_P$ appearing in correlation functions as well as the integrated invariants $\Phi_P^{(2)}$ which can be used to deform the theory. Thus, the general form of a perturbed topological correlation function is

\[
c_{P_1,\ldots,P_m}(t_1,\ldots,t_M) = \left\langle \Phi_{P_1}(z_1) \cdots \Phi_{P_m}(z_m) \exp \left[ \sum_{k=1}^{M} t_k \int \Phi_P^{(2)} \right] \right\rangle_{\text{top}}
\]

where $\langle \cdots \rangle_{\text{top}}$ means functional integration with the measure provided by the unperturbed Lagrangian $L_{\text{top}}$ and $t_k$ are coupling constants parametrizing its deformations $\int \Phi_P^{(2)}$. 

8
In our \((b, c, \beta, \gamma)\) formulation the general solution of the descent equations (2.24) is

\[
\Phi_P = P(X), \\
\Phi^{(1)}_P = -\sum_{i=1}^N \left[ b_i \partial_i P \, dz + \tilde{b}_i \tilde{\partial}_i P \, d\tilde{z} \right], \\
\Phi^{(2)}_P = \sum_{i,j=1}^N \left[ b_i \tilde{b}_j \partial_i \tilde{\partial}_j P \right] \, dz \wedge d\tilde{z},
\]

(2.26)

where \(P(X)\) is any polynomial in the variables \(X_i = \beta_i \tilde{\beta}_i\) corresponding to a non-trivial element of the local ring determined by the superpotential \(W\) of the Lagrangian (2.22). Indeed if the polynomial \(P(X)\) is proportional to the vanishing relations (i.e. if \(P(X) = \sum_i p^i(X) \frac{\partial W}{\partial X^i}\)), then using the BRST transformations (2.23), we easily see that \(P(X) = s K\) and so \(\Phi_P\) would be exact. (For the proof it suffices to set \(K = p^i(X) \frac{\partial \beta_i}{\partial X^i} c_j\).) Thus, the physical observables in the topological theory are simply local polynomials of \(\beta_i\) and \(\tilde{\beta}_i\), which correspond to chiral primary fields of the original \(N=2\) superconformal theory.

On the other hand, comparing the expression of the 2-form \(\Phi^{(2)}_P\) in (2.26) with the topological Lagrangian (2.22), it is easy to see that a deformation of the potential with some element \(P(X)\) of the local ring, i.e.

\[
W(X) \rightarrow W(X) - t_P \, P(X),
\]

corresponds to a perturbation of the action with \(\int \Phi^{(2)}_P\), i.e.

\[
\int d^2z \, \mathcal{L}_{\text{top}} \rightarrow \int d^2z \, \mathcal{L}_{\text{top}} - t_P \int \Phi^{(2)}_P.
\]

(2.28)

Thus, the possible perturbations of the theory are in one-to-one correspondence with the possible deformations of the potential. As we are going to see, something similar happens also in the ordinary LG models, but only up to BRST-exact terms.

For the sake of comparison we now write the general form of the Lagrangian, of the supersymmetry transformations and, after twisting, of the topological BRST-transformations of an ordinary \(N=2\) LG model [33,34,41]. A short rheonomic derivation of the results hereinafter reported is given in Appendix A. Let \(X^i(z, \bar{z})\) be \(N\) complex scalar fields, \(X^i(z, \bar{z})\) their complex conjugates, \(\psi^i\) and \(\psi^i\) their left-moving anticommuting superpartners, and \(\tilde{\psi}^i\) and \(\tilde{\psi}^i\) their right-moving anticommuting superpartners. The Lagrangian for a LG model with superpotential \(W\) is

\[
\mathcal{L} = - \left[ \partial X^i \bar{\partial} X^{i*} + \partial X^i \partial X^{i*} \right] \eta_{ij} + 8 \partial_i W \partial_j \bar{W} \eta^{ij*} \\
+ 4i \left[ \psi^j \bar{\partial} \psi^{j*} + \tilde{\psi}^j \partial \tilde{\psi}^{j*} \right] \eta_{ij} \\
+ 8 \left[ \partial_i \partial_j W \psi^i \tilde{\psi}^j - \partial_i \psi^i \partial_j \bar{W} \psi^j \right].
\]

(2.29)
where $\eta_{ij}$ is the flat Kählerian metric of $\mathbb{C}^n$. Here we have understood summations over repeated indices, and used the short-hand notations $\partial_i \equiv \partial/\partial X^i$ and $\partial_i^* \equiv \partial/\partial X^{i*}$. The Lagrangian above is invariant against the following global N=2 supersymmetry transformations

$$
\begin{align*}
\delta X^i & = -\varepsilon^- \psi^i - \bar{\varepsilon}^- \bar{\psi}^i , \\
\delta X^{i*} & = +\varepsilon^+ \psi^{i*} + \bar{\varepsilon}^+ \bar{\psi}^{i*} , \\
\delta \psi^i & = -\frac{i}{2} \partial X^i \varepsilon^+ + \eta^{ij} \partial_j \bar{W} \bar{\varepsilon}^- , \\
\delta \bar{\psi}^i & = -\frac{i}{2} \partial X^i \bar{\varepsilon}^+ - \eta^{ij} \partial_j W \varepsilon^- , \\
\delta \psi^{i*} & = \frac{i}{2} \partial X^{i*} \varepsilon^- + \eta^{ij} \partial_j \bar{W} \bar{\varepsilon}^+ , \\
\delta \bar{\psi}^{i*} & = \frac{i}{2} \partial X^{i*} \bar{\varepsilon}^- - \eta^{ij} \partial_j W \varepsilon^+ .
\end{align*}
$$

Contrary to our $(b,c,\beta,\gamma)$ formulation, the global supersymmetries (2.30) do not extend to classical superconformal symmetries of the action (2.29), even when the superpotential $W(X)$ is a quasi-homogeneous function. Indeed, it is only after quantization that one can argue the equivalence of (2.29) at its infrared fixed point with a (2,2) superconformal model. Our theory in (2.2) instead, is superconformal already at the classical level whenever the potentials $V$ and $\bar{V}$ are quasi-homogeneous. Of course, this applies in particular to the left-right symmetric case we are discussing where we have a single potential $W(\beta \bar{\beta})$ that can be identified with the superpotential $W(X)$ of the LG theory.

Performing the topological twist does not modify the Lagrangian (2.29) but merely changes the spin of the fields [41]. If we choose as BRST-parameter $\theta = \varepsilon^+ = \bar{\varepsilon}^+$ (as is appropriate for the $+$ twist), the action of the topological Slavnov operator on the LG fields turns out to be

$$
\begin{align*}
\mathcal{s} X^i & = 0 , \\
\mathcal{s} X^{i*} & = \psi^{i*} + \bar{\psi}^{i*} , \\
\mathcal{s} \psi^i & = -\frac{i}{2} \partial X^i , \\
\mathcal{s} \bar{\psi}^i & = -\frac{i}{2} \partial X^i , \\
\mathcal{s} \psi^{i*} & = \eta^{ij} \partial_j W , \\
\mathcal{s} \bar{\psi}^{i*} & = -\eta^{ij} \partial_j W .
\end{align*}
$$

Using (2.31), we can easily solve the descent equations (2.23) and find

$$
\begin{align*}
\Phi_P & = P(X) , \\
\Phi_P^{(1)} & = -2i \partial_i P \left( \psi^i dz + \bar{\psi}^i d\bar{z} \right) , \\
\Phi_P^{(2)} & = -4 \left[ \partial_i \partial_j P \psi^i \bar{\psi}^j + \partial_k P \partial_k \bar{W} \eta^{kl} \right] dz \wedge d\bar{z} ,
\end{align*}
$$

10
where $P(X)$ is a polynomial corresponding to some non trivial element of the local ring determined by the superpotential $W(X)$. Indeed, if $P(X)$ is proportional to the vanishing relations (i.e. if $P(X) = \sum_i p^i(X) \frac{\partial W}{\partial X_i}$), then using the BRST transformations (2.31), one can see that $P(X) = s K$ and so $\Phi_P$ would be exact. (For the proof it suffices to set $K = p^i(X) \psi^j \eta_{ij}$.)

It is interesting to observe that under the deformation $W \rightarrow W - \frac{1}{2} t_P P(X)$, where $P(X)$ is some element of the local ring and $t_P$ is the corresponding coupling constant, the (topological) LG action changes as follows

$$\int d^2z \mathcal{L} \rightarrow \int d^2z \mathcal{L} - t_P \int \Phi_P^{(2)} - \bar{t}_P \int \bar{\Phi}_P^{(2)}$$

where $\mathcal{L}$ is given in (2.29), $\Phi_P^{(2)}$ in (2.32) and $\bar{\Phi}_P^{(2)}$ is the complex conjugate 2-form. These equations have to be compared with the analogous ones (2.26) and (2.27) of the $(b,c,\beta,\gamma)$ theory. At first sight, in the LG models there seem to be a problem in identifying the topological perturbations of the Lagrangian with the deformations of the superpotential because of the last term in (2.34). However, this problem does not exist because the 2-form $\bar{\Phi}_P^{(2)}$ is BRST-exact, and so adding or not its integral to the action is completely irrelevant. In fact, using the BRST-transformations (2.31), one can check that

$$\bar{\Phi}_P^{(2)} = s \left(-4 \partial_j \bar{P} \psi^{j^*}\right)$$

We want to emphasize that in the $(b,c,\beta,\gamma)$ formulation instead, there is no counterpart of this BRST-trivial part and deformations of the superpotential identically coincide with topological deformations of the Lagrangian.

We conclude this section by briefly commenting on the other choice of sign in the topological twist for our $(b,c,\beta,\gamma)$-system. If one chooses in (2.17) the $-$ sign, from (2.10) and (2.11) one obtains

$$\hat{T}_- = T - \frac{1}{2} \partial J = \sum_{i=1}^{N} (1 - 2\lambda_i) \partial c_i b_i - 2\lambda_i b_i \partial c_i + (1 - 2\lambda_i) \gamma_i \partial \beta_i - 2\lambda_i \beta_i \partial \gamma_i$$

This is the canonical stress-energy tensor for $N$ commuting $(\beta,\gamma)$-systems with weight $2\lambda_i$ and $N$ anticommuting $(b,c)$-systems also with weight $2\lambda_i$. It is straighforward to check that $T_-$ closes a Virasoro algebra with zero central charge; indeed the bosonic and fermionic contributions to the central charge exactly cancel each other. However, the cohomology classes of the BRST charge $Q_-^{\text{BRST}}$ correspond to anti-chiral primary fields of the original $N=2$ algebra and these do not have a simple and local
representation in terms of the elementary fields appearing in the Lagrangian: indeed, to describe the anti-chiral operators one has to resort to the bosonization of the \((b,c,\beta,\gamma)\)-systems (see Section 4). On the other hand, as we explicitly show in appendix B, after performing the topological twist, the Lagrangian is BRST-exact, i.e. it is of the form \( \mathcal{L}_{\text{top}} = [Q_{-}^{\text{BRST}}, \mathcal{L}'] \) for some local functional \( \mathcal{L}' \). Using the terminology of [22], this means that the \( - \) twist defines a topological field theory of the Witten-type. On the contrary, the \( + \) twist leads to the Lagrangian (2.22) which is not BRST exact with respect to \( Q_{+}^{\text{BRST}} \); thus the \( + \) twist defines a topological field theory of the Schwarz-type. As pointed out in [41], also the ordinary topological LG models are theories of the Schwarz-type.

In conclusion, we have shown that N=2 LG models admit a \((b,c,\beta,\gamma)\)-formulation which is already superconformal at the classical level. After topological twisting, there is a natural correspondence between the deformations of the LG potential and the abstract topological deformations. In the next sections, after discussing the renormalization group properties of our theory, we shall illustrate how one can use this explicit formulation to calculate (perturbed) topological correlation functions in LG models.

### 3. On the quantum properties of the \((b,c,\beta,\gamma)\) system

In the previous section we discussed the classical properties of the action (2.1) and showed that with a suitable choice of the interaction potential, the theory exhibits a non trivial \((2,2)\)-superconformal invariance. However, the presence of interactions can in principle spoil this invariance at the quantum level and one has eventually to restore it after a suitable renormalization [42]. In this section we are going to show that no loop corrections are present in our model so that the classical results automatically extend to the quantum theory.

For the sake of clarity, we begin by considering a single left-right symmetric \((b,c,\beta,\gamma)\)-system of weight \( \lambda = \tilde{\lambda} \) with potential

\[
W = \frac{1}{n+1} (\beta \bar{\beta})^{n+1} .
\]  

(3.1)

This corresponds to the \( A_n \) minimal model of the N=2 discrete series if \( \lambda = 1/(2n + 2) \) (see (2.13)). However, for the time being, we leave the weight \( \lambda \) unfixed. The Lagrangian for this system is \( \mathcal{L} = \mathcal{L}_0 + \Delta \mathcal{L} \) where \( \mathcal{L}_0 \) is as in (2.2a) and the interaction term is

\[
\Delta \mathcal{L} = g \, b \bar{b} \beta^n \bar{\beta}^n
\]  

(3.2)

where \( g \) is a coupling constant. Since the weight of \( \beta \) and \( \bar{\beta} \) is arbitrary, \( g \) is a quantity with dimension

\[
[g] = (1 - 2\lambda(n + 1)) .
\]  

(3.3)
To study the scaling properties of this system, we compute the trace of the stress-energy tensor which turns out to be

\[ T_{\bar{z}z} = \left[ -\lambda \beta \partial \gamma + (1 - \lambda) \gamma \bar{\partial} \beta - (\lambda + \frac{1}{2}) b \partial c - (\frac{1}{2} - \lambda) c \bar{\partial} b + g \, \bar{b} \beta^n \bar{\beta}^n \right] + \text{c.c.} \quad (3.4) \]

After using the equations of motion (2.3a), we have

\[ \Theta \equiv -T_{\bar{z}z} = g(2(n + 1)\lambda - 1) \, \bar{b} \beta^n \bar{\beta}^n , \quad (3.5) \]

so that our system is classically invariant under scale transformations (i.e. \( \Theta = 0 \)) either if

\[ g = 0 \quad \text{for any} \quad \lambda , \quad (3.6a) \]

or if

\[ \lambda = \frac{1}{2(n + 1)} \quad \text{for} \quad g \neq 0 . \quad (3.6b) \]

Discarding the case (3.6a) which corresponds to a free theory, we see from (3.6b) that \( \lambda \) must be fixed by the homogeneous weight of the potential (cf. (2.7)); when (3.6b) is satisfied of course \( g \) becomes dimensionless and the operator \( \bar{b} \beta \beta \) becomes marginal, so that no dimensionful parameters are left in the model.

Let us now quantize this system by using perturbation theory in \( g \). From the explicit expression of the Lagrangian \( L \), we see that the propagators are

\[ \langle \gamma(z, \bar{z}) \beta(w, \bar{w}) \rangle = \langle b(z, \bar{z}) c(w, \bar{w}) \rangle = \frac{1}{z - w} , \quad (3.7) \]

so that it is obvious that even when the interaction (3.2) is present, it is impossible to form loops. Therefore we conclude that there are no (perturbative) quantum corrections to the classical results simply because there are no loops! These considerations imply in particular that \( \Theta \) in (3.5) is also the quantum trace of the stress-energy tensor and hence the coefficient of the spinless operator \( \bar{b} \beta \beta^n \bar{\beta}^n \) appearing in (3.5) can be interpreted as a renormalization group \( \beta \)-function [43], namely

\[ \beta(g) = g \left( 2(n + 1)\lambda - 1 \right) . \quad (3.8) \]

The zeroes of \( \beta(g) \) identify the conformal fixed points and these are given precisely by (3.6a) and (3.6b).

It is now interesting to see what happens when a second interaction

\[ \Delta L' = g' \bar{b} b \beta^m \bar{\beta}^m \quad (m < n) \quad (3.9) \]

\(^3\) Here and in the following “c.c.” means exchanging the untilded fields with the tilded ones and \( \partial \) with \( \bar{\partial} \).
is added to the original system. We now assume that \( \lambda = 1/(2n + 2) \) so that (3.9) can be considered as a perturbation around a conformal theory. Following the same procedure as above, we compute the \( \beta \)–function \( \beta(g') \) and find

\[
\beta(g') = g' \left( \frac{m + 1}{n + 1} - 1 \right) = \frac{m - n}{n + 1} g' .
\]  

(3.10)

It is clear from (3.10) that the new model does not have any non-trivial fixed point; indeed the only solution to \( \beta(g') = 0 \) is \( g' = 0 \) which is achieved in the ultraviolet regime for \( m < n \). Hence we cannot have a renormalization group flow to another \( N=2 \) superconformal field theory, in agreement with the conclusions of [44].

The extension of these results to the generic case of quasi-homogeneous potentials is an easy task. To this end let us first recall that if \( f \) is a quasi–homogeneous polynomial in \( N \) variables with weights \( (\omega_1, \cdots, \omega_N) \), \( \omega_i \in \mathbb{Q} \), \( \omega_i > 0 \) and

\[
f = \sum_\rho a_\rho x^\rho
\]  

(3.11a)

where \( \rho \equiv (\rho_1, \cdots, \rho_N) \), \( X^\rho \equiv X_1^{\rho_1} \cdots X_N^{\rho_N} \), \( \rho_i \in \mathbb{Z}^+ \) and \( a_\rho \neq 0 \), then

\[
\rho_1 \omega_1 + \cdots + \rho_N \omega_N = 1 .
\]  

(3.11b)

Let us now consider the following interaction term

\[
\Delta L = g \sum_{i,j} \tilde{b}_i \tilde{b}_j \partial_i V \partial_j \tilde{V}
\]  

(3.12)

where \( V(\beta) \) and \( \tilde{V}(\tilde{\beta}) \) are quasi–homogeneous potentials satisfying (3.11). For simplicity, we take \( V(\beta) = \tilde{V}(\tilde{\beta}) \) and assume that the weights \( \lambda_i = \tilde{\lambda}_i \) are unconstrained. Then, the trace of the stress–energy tensor, upon using the equations of motion (2.3a), turns out to be

\[
\Theta = -T_{zz} = -g \sum_{i,j} \left( \tilde{b}_i \tilde{b}_j \partial_i V \partial_j \tilde{V} - 2 \lambda_i \tilde{b}_i \tilde{b}_j \partial_i V \partial_j \tilde{V} \right) \\
- \sum_{i,j,l} \left( \lambda_i \tilde{\beta}_i \tilde{b}_i \partial_i \partial_l V \partial_j \tilde{V} - \lambda_i \tilde{\beta}_i \tilde{b}_i \partial_i \partial_l \tilde{V} \partial_j V \right) .
\]  

(3.13)

Using (3.11), after some algebra, the trace (3.13) can be rewritten as

\[
\Theta = g \left( 2 \sum_k \lambda_k \rho_k - 1 \right) \left( \sum_{i,\rho} a_{\rho} b_i \rho_i \beta_1^{\rho_1} \cdots \beta_N^{\rho_N} \right) \left( \sum_{j,\rho} \tilde{a}_{\rho} \tilde{b}_j \rho_j \tilde{\beta}_1^{\rho_1} \cdots \tilde{\beta}_N^{\rho_N} \right) .
\]  

(3.14)

If \( g \neq 0 \), the system is invariant under scale transformations only if

\[
\sum_i \lambda_i \rho_i = \frac{1}{2}
\]  

(3.15)
Comparing (3.15) with (3.11) we see that the weights $\lambda_i$ must be one half of the homogeneous weights of the potential $\omega_i$. Since there are no loop corrections, this result extends automatically to the quantum theory. Furthermore, we point out that the same conclusion is obtained in a similar way when $V(\beta)$ and $\tilde{V}(\tilde{\beta})$ are different, or when the interaction depends on a single quasi–homogeneous function $W$ in the variables $X_i = \beta_i \tilde{\beta}_i$ with weight $\omega_i$.

Finally, in our formulation it is easy to realize that the potential

$$\hat{V}(\beta^{(i,A)}) = V(\beta^i) + \sum_{A=n+1}^{m+n} (\beta^A)^2$$

(3.16)

defines the same conformal theory as the potential $V$ (as one should expect from the notion of stable singularity [35,36]). Indeed, a $(b,c,\beta,\gamma)$-system with $\lambda_A = \frac{1}{4}$ gives a $c = 0$ conformal field theory.

4. Bosonization of the $(b,c,\beta,\gamma)$-system
before and after topological twisting

The result of our previous analysis is that for any value $\lambda = (2n + 2)^{-1}$ with $n = 2, 3, \ldots$, we have a realization of the N=2 superconformal algebra with central charge

$$c = 3 - 12\lambda = \frac{3(n-1)}{n+1}.$$ (4.1)

The conformal weights $h$ and the $U(1)$-charges $q$ of the pseudo-ghost fields that define such a realization are given by

$$h(\beta) = \frac{1}{2(n+1)}, \quad q(\beta) = \frac{1}{n+1}; \quad (4.2a)$$
$$h(\gamma) = \frac{2n+1}{2(n+1)}, \quad q(\gamma) = -\frac{1}{n+1}; \quad (4.2b)$$
$$h(b) = \frac{n+2}{2(n+1)}, \quad q(b) = -\frac{n}{n+1}; \quad (4.2c)$$
$$h(c) = \frac{n}{2(n+1)}, \quad q(c) = \frac{n}{n+1}. \quad (4.2d)$$

The Fock space generated by the modes of the almost-free fields $b, c, \beta, \gamma$ and their spin fields, contains the irreducible representations of the N=2 minimal models. Such representations can be obtained from the Fock space through a suitable projection like in the case of the standard free-field realization of the minimal models as given for instance in [38]. In this section our aim is to make contact with this Coulomb gas formalism, which, as we will see in the sequel, enables us to calculate explicitly (perturbed) topological correlation functions in the presence of a LG interaction.
Adopting the conventions of [38], an N=2 minimal model with central charge as in (4.1), can be described in terms of three scalar fields \( \phi_0, \phi_1 \) and \( \phi_2 \) with mode expansions

\[
\phi_i(z) = \hat{q}_i - \hat{p}_i \ln z + \sum_{k \neq 0} \frac{\hat{a}_k^i}{k} z^{-k}, \quad i = 0, 1, 2, \tag{4.3}
\]

where

\[
[\hat{q}_i, \hat{p}_j] = \delta_{ij}, \quad [\hat{a}_k^i, \hat{a}_\ell^j] = k \delta_{k+\ell,0} \delta^{ij}. \tag{4.4}
\]

While \( \phi_0 \) and \( \phi_2 \) are really free fields, \( \phi_1 \) is coupled to a background charge

\[
Q_1^{(n)} = \sqrt{\frac{2}{n+1}}. \tag{4.5}
\]

In this realization the holomorphic currents of the N=2 superconformal algebra are

\[
T = \frac{1}{2} \left[ (\partial \phi_0)^2 + (\partial \phi_1)^2 + (\partial \phi_2)^2 \right] - \frac{1}{2} Q_1^{(n)} \partial^2 \phi_1, \tag{4.6a}
\]

\[
J = \sqrt{\frac{n-1}{n+1}} \partial \phi_0, \tag{4.6b}
\]

\[
G^\pm = \sqrt{\frac{2n-2}{n+1}} \Psi^\pm \exp \left[ \pm \sqrt{\frac{n+1}{n-1}} \phi_0 \right], \tag{4.6c}
\]

\[
\Psi^\pm = \sqrt{\frac{1}{2} \sqrt{n+1}} \exp \left[ \pm i \sqrt{\frac{2}{n-1}} \phi_2 \right] \left( \sqrt{\frac{n+1}{n-1}} \partial \phi_1 \pm i \partial \phi_2 \right). \tag{4.6d}
\]

The field \( \phi_0 \) bosonizes the U(1)-current and its exponentials realize the well known N=2 spectral flow [5]. The operators \( \Psi^\pm \) in (4.6d) are, instead, parafermionic currents and generate the non trivial part of the N=2 algebra. The complete Fock space which embeds the N=2 irreducible modules is generated by the vertex operators

\[
V_{q,\ell,m} = \exp \left[ \frac{q}{\sqrt{n^2-1}} \phi_0 + \frac{\ell}{\sqrt{2(n+1)}} \phi_1 + \frac{m}{\sqrt{2(n-1)}} \phi_2 \right] \tag{4.7}
\]

and their derivatives.

In particular the N=2 primary fields are given by

\[
\Lambda_{\ell,m,s}^{(n)} = \frac{1}{\sqrt{2}} \exp \left[ \frac{m+sn-s}{\sqrt{n^2-1}} \phi_0 + \frac{\ell}{\sqrt{2(n+1)}} \phi_1 + \frac{m}{\sqrt{2(n-1)}} \phi_2 \right] \tag{4.8}
\]

where \( \ell \) takes the integer values \( 0 \leq \ell \leq n-1 \) and \( m \) takes the integer values \( m = -l, -l+2, ..., l \). The quantum number \( s \) represents the sector and is 0 in the

\[4\] Here and in the following, any exponential of free fields is understood as normal ordered.
Neveu-Schwarz sector and $\pm 1/2$ in the Ramond sector. The conformal weight $h$ and the $U(1)$-charge $q$ of $\Lambda^{(n)}_{\ell,m;s}$ are given by the standard formulas

$$h(\ell, m; s) = \frac{\ell(\ell + 2)}{4(n + 1)} - \frac{m^2}{4n - 4} + \frac{(m + sn - s)^2}{2(n^2 - 1)},$$

$$q(m; s) = \frac{m + sn - s}{n + 1}.$$  

(4.9)

As we will see later, it is convenient to factor out the $\phi_0$ contribution and rewrite the primary fields (4.8) as

$$\Lambda^{(n)}_{\ell,m;s} = \exp \left[ \sqrt{\frac{3}{c}} q(m, s) \phi_0 \right] \varphi^\ell_m$$

(4.10)

where

$$\varphi^\ell_m = \exp \left[ \frac{\ell}{\sqrt{2(n + 1)}} \phi_1 + i \frac{m}{\sqrt{2(n - 1)}} \phi_2 \right].$$

(4.11)

The operators $\varphi^\ell_m$ are the principal primary fields of the $\mathbb{Z}_{n-1}$ parafermion algebra and must be identified according to

$$\varphi^{n-1-\ell}_m \sim \varphi^{m+\ell}_{n+1}.$$ 

(4.12)

In fact the Hilbert space created by $\varphi^\ell_m$ is isomorphic to the one created by $\varphi^{n-1-\ell}_m$ due to the existence of a map between the two that commutes with all generators of the algebra [38].

In order to relate this realization of the N=2 minimal models to the one provided by our $(b,c,\beta,\gamma)$-system, we bosonize the latter according to the standard rules and write\footnote{Notice that the bosonization rules we are giving are actually true for the “free” holomorphic part of the $c, \gamma$ fields. However, as we are going to see, we only need of the bosonized $b, \beta$ fields (which are correctly expressed by (4.13) (4.19a) and (4.19b)) in application to topological correlation functions}

$$b = e^{-\pi_1}, \quad \beta = e^{i\pi_2-\pi_3}, \quad c = e^{\pi_1}, \quad \gamma = e^{-i\pi_2+\pi_3}\partial \pi_3$$

(4.13)

where the $\pi_i$’s are scalar fields coupled to the following background charges

$$\tilde{Q}^{(n)}_1 = -\frac{1}{n+1}, \quad \tilde{Q}^{(n)}_2 = i \frac{n}{n+1}, \quad \tilde{Q}^{(n)}_3 = -1.$$  

(4.14)

These numbers are explained as follows: $\pi_1$ bosonizes the anticommuting $(b,c)$-system whose weight is $\lambda + \frac{1}{2} = -\frac{1}{n+1}$. Insertion of this value in the general formula (2.21) yields $\tilde{Q}^{(n)}_1$ as listed in (4.14). The field $i\pi_2$ bosonizes the commuting $(\beta, \gamma)$-system according to the rule

$$\gamma = e^{-i\pi_2}\partial \xi, \quad \beta = e^{i\pi_2}\eta$$  

(4.15)
where $\xi$ and $\eta$ form an anticommuting first-order system of weight $\lambda_{\xi\eta} = 1$. The background charge $\tilde{Q}_2^{(n)}$ of the field $\pi_2$ follows from (2.21) with $\lambda_{\beta\gamma} = \lambda = \frac{1}{2n+2}$.

Finally $\pi_3$ is the scalar field that bosonizes the $(\xi, \eta)$-system and its background charge $\tilde{Q}_3^{(n)}$ also follows from (2.21) upon use of the value $\lambda_{\xi\eta} = 1$.

Consequently, in terms of the fields $\pi_i$'s, the stress-energy tensor of the N=2 model is

$$T = \frac{1}{2} \left[ (\partial \pi_1)^2 + (\partial \pi_2)^2 + (\partial \pi_3)^2 \right] - \frac{1}{2} \left( -\frac{1}{n+1} \partial^2 \pi_1 + i n \frac{1}{n+1} \partial^2 \pi_2 - \partial^2 \pi_3 \right).$$

(4.16)

Similarly, using (4.13) in (2.9) and (2.11) we obtain

$$J = \frac{1}{n+1} (n \partial \pi_1 - i \partial \pi_2),$$

(4.17a)

$$G^- = 2\sqrt{2} \exp \left[ -\pi_1 - i \pi_2 + \pi_3 \right] \partial \pi_3.$$

(4.17b)

Comparing (4.16) and (4.17) with (4.6), we obtain the relation between the $\pi$'s and the $\phi$'s, namely

$$\pi_1 = \frac{n}{\sqrt{n^2-1}} \phi_0 - \frac{1}{\sqrt{2(n+1)}} \phi_1 + \frac{1}{\sqrt{2(n-1)}} \phi_2,$$

(4.18a)

$$i \pi_2 = \frac{1}{\sqrt{n^2-1}} \phi_0 - \frac{n}{\sqrt{2(n+1)}} \phi_1 + \frac{n}{\sqrt{2(n-1)}} \phi_2,$$

(4.18b)

$$\pi_3 = -\sqrt{\frac{n+1}{2}} \phi_1 + i \sqrt{\frac{n-1}{2}} \phi_2.$$

(4.18c)

One can proceed even further and use (4.13) and (4.18) to identify the pseudo-ghost fields with the operators of the abstract N=2 superconformal model. Explicitly one finds

$$\beta = \exp \left[ \frac{1}{\sqrt{n^2-1}} \phi_0 + \frac{1}{\sqrt{2(n+1)}} \phi_1 + i \frac{1}{\sqrt{2(n-1)}} \phi_2 \right],$$

(4.19a)

$$b = \exp \left[ \frac{-1}{\sqrt{n^2-1}} \phi_0 + \frac{1}{\sqrt{2(n+1)}} \phi_1 + i \frac{-1}{\sqrt{2(n-1)}} \phi_2 \right],$$

(4.19b)

$$c = \exp \left[ \frac{-1}{\sqrt{n^2-1}} \phi_0 + \frac{-1}{\sqrt{2(n+1)}} \phi_1 + i \frac{-1}{\sqrt{2(n-1)}} \phi_2 \right],$$

(4.19c)

$$\gamma = \exp \left[ \frac{-1}{\sqrt{n^2-1}} \phi_0 + \frac{-1}{\sqrt{2(n+1)}} \phi_1 + i \frac{-1}{\sqrt{2(n-1)}} \phi_2 \right] \times$$

$$\times \sqrt{\frac{n-1}{2}} \left( -\sqrt{\frac{n+1}{n-1}} \partial \phi_1 + i \partial \phi_2 \right).$$

(4.19d)
From (4.19a) one realizes that $\beta$ is a chiral primary field and is given by

$$\beta = \Lambda_{1,1,0}^{(n)} \ .$$

(4.20a)

More generally one can write

$$\beta^\ell = \Lambda_{\ell,\ell,0}^{(n)} \quad \text{for} \quad \ell = 0, \ldots, n-1 \ ,$$

(4.20b)

which shows that at the quantum level the general chiral primary field is simply the $\ell$-th power of $\beta$ and the vanishing relation is recovered by enforcing the bound $\ell \leq n-1$. Moreover $b$ is the first component of a chiral primary superfield and can be explicitly obtained by

$$\frac{1}{2\sqrt{2}} \oint_z G^-(w) \beta(w) = b(z) ;$$

the same is true for fields of the form $b\beta^{\ell-1}$. On the contrary $\gamma$ and $c$ are in the Fock space of the three scalar fields, but not in the N=2 irreducible module.

We now consider the case when the theory is topologically twisted with

$$Q_+(z) = G^+(z) \ .$$

(4.21)

As mentioned in Section 2, we have a new $(b, c, \beta, \gamma)$-system with $\lambda_\beta = 0, \lambda_b = 1$ whose Lagrangian is given in (2.22). These new pseudo-ghost fields are still bosonized as in (4.13), but now the background charges of the $\pi_i$’s become

$$\tilde{Q}_1^{(n)} = -1 \ , \quad \tilde{Q}_2^{(n)} = i \ , \quad \tilde{Q}_3^{(n)} = -1 \ .$$

(4.22)

The new stress-energy tensor is then given by

$$\hat{T} = \frac{1}{2} \left[ (\partial \pi_1)^2 + (\partial \pi_2)^2 + (\partial \pi_3)^2 \right] - \frac{1}{2} \left( -\partial^2 \pi_1 + i \partial^2 \pi_2 - \partial^2 \pi_3 \right) \ ,$$

(4.23)

whereas the $U(1)$-current $J$ is the same as in (4.17a). On the other hand, the twist of the stress-energy tensor is given by $\hat{T} = T + \frac{1}{2} J$, and hence, using (4.6a,b) we get

$$\hat{T} = \frac{1}{2} \left[ (\partial \phi_0)^2 + (\partial \phi_1)^2 + (\partial \phi_2)^2 \right] - \frac{1}{2} \sqrt{\frac{n}{n+1}} \partial^2 \phi_1 + \frac{1}{2} \sqrt{\frac{n-1}{n+1}} \partial^2 \phi_0 \ .$$

(4.24)

If we now compare (4.23) and (4.24) and observe that $J$ is the same before and after the twist, so that

$$J = \sqrt{\frac{n-1}{n+1}} \partial \phi_0 = \frac{n}{n+1} \partial \pi_1 - \frac{i}{n+1} \partial \pi_2 \ ,$$

(4.25)

we can realize that the relations (4.18) and the identifications (4.19) and (4.20) hold true also in the topological field theory, giving us a complete characterization of the fields $b, c, \beta$ and $\gamma$ at the quantum level. Moreover, using (4.19) and taking into account both left and right movers, we can easily check the descent equations (2.24) in the complete bosonized formalism. As is clear from (4.24) in comparison with
(4.6a), the net effect of the topological twist is simply to switch on a background charge for \( \phi_0 \) given by
\[
Q_0^{(n)} = \frac{1 - n}{\sqrt{n^2 - 1}}.
\] (4.26)

Therefore, even if the bosonized expressions for the topological \( b, c, \beta \) and \( \gamma \) are still given by (4.19), their conformal dimensions change with the twist. In particular the chiral primary fields \( \beta^\ell \) lose their conformal weight and become dimensionless, as is appropriate for the physical operators of a topological field theory. Furthermore, the \( U(1) \) current \( J \) acquires an anomaly proportional to \( Q_0^{(n)} \).

Let us briefly mention that if we perform the topological twist with \( Q_- = G^- \) instead of \( Q_+ = G^+ \), not only the stress energy tensor but also the lagrangian becomes a BRST commutator (see Appendix B for details). In this case, however, there is no identification of the chiral fields in terms of local expressions of \( b, c, \beta \) or \( \gamma \): they can only be written in a bosonized form. As is well known, chiral fields with respect to the BRST charge \( Q_- = G^- \) are antichiral fields with respect to the BRST charge \( Q_+ = G^+ \). This means that the bosonized expression of these fields could be obtained via spectral flow from (4.20).

The complete bosonization of the \((b, c, \beta, \gamma)\)-system we have just presented is the technical tool which enables us to make explicit calculations of (perturbed) correlation functions for single minimal models as well as for tensor products thereof. It is also very useful in establishing the precise relationship between the correlation functions of topological minimal models and the chiral Green functions of LG theories as computed in [34].

To this end, let us first introduce the following notation
\[
|q, \ell, m\rangle \equiv \lim_{z \to 0} V_{q, \ell, m}(z) \ |0, 0, 0\rangle
\] (4.27)
where \( V_{q, \ell, m} \) is defined in (4.7) and \(|0, 0, 0\rangle\) is the \( Sl(2, \mathbb{C}) \) invariant vacuum of \( \phi_0, \phi_1 \) and \( \phi_2 \). Before the topological twist only \( \phi_1 \) has a background charge and the dual conjugate of \(|q, \ell, m\rangle\) is \((-q, -2 - \ell, -m]\). After the twist also \( \phi_0 \) acquires a background charge and so the dual conjugate of \(|q, \ell, m\rangle\) becomes \(|n - 1 - q, -2 - \ell, -m]\).

In the LG theory with superpotential \( W \sim X^{n+1} \), the supersymmetric vacua \(|m\rangle\) \((m = 0, \ldots, n - 1)\) are identified at the conformal point with the Ramond vacua of the minimal model \( A_n \). According to (4.8), the Ramond chiral primary fields of such a model are
\[
R_m(z) = \mathcal{N}_m \Lambda^{(n)}_{m; -\frac{2}{4}}(z)
\] (4.28)
where \( m = 0, \ldots, n - 1 \) and the normalization factor \( \mathcal{N}_m \) is introduced to enforce the standard structure constants of the \( N=2 \) operator algebra \(^6\). This normalization can

\(^6\) Notice that the Ramond fields \( R_m \) do not have a local expression in terms of the fields \( b, c, \beta, \gamma \) of our model, contrary to the Neveu-Schwarz chiral primaries which are simply powers of the bosonic field \( \beta \).
be computed using different techniques [34,39] and is given by

\[ \mathcal{N}_m^2 = \frac{1}{(2n+2)^{\frac{n+1}{n+2}}} \sqrt{\frac{\sin \left( \frac{\pi}{n+1} \right)}{\sin \left( \frac{\pi(n+1)}{n+1} \right)}} \Gamma \left( \frac{1}{n+1} \right) \Gamma \left( \frac{m+1}{n+1} \right). \] (4.29)

Therefore at the conformal point, the supersymmetric vacua of the LG theory are

\[ |m\rangle = \lim_{z \to 0} R_m(z)|0,0,0\rangle_L \times \lim_{\bar{z} \to 0} R_m(\bar{z})|0,0,0\rangle_R \]
\[ = \mathcal{N}_m^2 |m - (n-1)/2, m, m\rangle_L \times |m - (n-1)/2, m, m\rangle_R \] (4.30)

where the subscripts L and R refer to the holomorphic and anti-holomorphic components respectively. The vacua \(|m\rangle\) are obtained by taking the dual conjugate of (4.30) and remembering that only \(\phi_1\) has a background charge (indeed the topological twist has not been performed yet).

It is now straightforward to compute the correlation functions of a string of chiral primary fields between two supersymmetric vacua. From Eqs. (4.20b) and (4.30) we have

\[ \langle m_1 | \prod_{i=1}^N (\beta(z_i) \bar{\beta}(\bar{z}_i))^{\ell_i} | m_2 \rangle = \frac{\mathcal{N}_m^2}{\mathcal{N}_m} \prod_{i=1}^N (z_i \bar{z}_i)^{\frac{-\ell_i}{2n+2}} \delta \left( \sum_i \ell_i + m_2 - m_1 \right) \] (4.31)

where the \(\delta\)-function arises from charge conservation. Apart from the \(z\)-dependent factor, this result coincides with the LG chiral Green functions computed in [34] using quantum field theory techniques. To make the precise comparison, however, one has to remember that in [34] the LG theory was defined on a cylinder, whereas our formula (4.31) applies to the plane. This difference is easily eliminated by mapping the plane to the cylinder, under which the chiral primary fields transform as

\[ \beta^{\ell_i}(z_i) \to \beta^{\ell_i}(w_i) z_i^{\frac{\ell_i}{2n+2}}. \] (4.32)

Here \(w_i\) are the cylinder coordinates. An analogous expression holds also for the \(\bar{\beta}\) fields. The \(z\)-dependent factors of (4.31) are therefore cancelled in going from the plane to the cylinder and we can conclude that

\[ \langle m_1 | \prod_{i=1}^N (\beta \bar{\beta})^{\ell_i} | m_2 \rangle \bigg|_{\text{cyl}} = \frac{\mathcal{N}_m^2}{\mathcal{N}_m^2} \delta \left( \sum_i \ell_i + m_2 - m_1 \right) \] (4.33)

exactly coincides (normalization factors included) with the chiral Green function of the LG theory of [34]. Once more we see that the L.G. field \(X\) has to be identified with the product \((\beta \bar{\beta})\) (see also Eq. (2.16)).
We now proceed to establish the relationship with the topological conformal field theories. To this end let us consider a particular case of (4.31), namely the correlation functions between the lowest vacuum $|0\rangle$ and the highest one $⟨n−1|$, 

\[ \langle n−1 | \prod_{i=1}^{N} (\beta(z_i)\tilde{β}(\bar{z}_i))^{\ell_i} |0\rangle = \]

\[ \frac{N_0}{N_{n−1}} L_{\langle(1−n)/2,−1−n,−1−n|} \prod_{i=1}^{N} \beta^{\ell_i}(z_i) |(1−n)/2,0,0\rangle_L \times (\text{c.c.}) . \]  

Since 

\[ |(1−n)/2,0,0\rangle = \exp \left[ \frac{(1−n)/2}{\sqrt{n^2−1}} \hat{q}_0 \right] |0,0,0\rangle \]  

and 

\[ \beta^\ell(z) \exp \left[ \frac{(1−n)/2}{\sqrt{n^2−1}} \hat{q}_0 \right] = \exp \left[ \frac{(1−n)/2}{\sqrt{n^2−1}} \hat{q}_0 \right] \beta^\ell(z) z^{\frac{-\ell}{m+2}} , \]

from (4.34) and (4.31), we get 

\[ L_{\langle 1−n,−1−n,−1−n|} \prod_{i=1}^{N} \beta^{\ell_i}(z_i) |0,0,0\rangle_L \times (\text{c.c.}) = \delta \left( \sum_{i} \ell_i − n + 1 \right) . \]

This is the natural candidate for a topological correlation function. Notice that (4.37) is independent of $z_i$ as any topological correlator should be. Indeed all the $z$-dependent factors are canceled in flowing from the lowest vacuum $|0\rangle$ of the Ramond sector to the $SL(2,\mathbb{C})$ invariant vacuum $|0,0,0\rangle$ of the Neveu-Schwarz sector. However, the fields $φ_0$, $φ_1$ and $φ_2$ which implicitly appear in (4.37) are those which bosonize the original $(b,c,β,γ)$-system before the topological twist. To obtain a more adequate characterization of topological correlation functions in the bosonized formalism, it is more appropriate to use the fields which bosonize a twisted $(b,c,β,γ)$-system. As we have seen, only very few things change; most notably the field $φ_0$ acquires the background charge (4.26). Thus, in (4.37) instead of the state $⟨1−n,−1−n,−1−n|$, which is the dual conjugate of $|n−1,n−1,n−1\rangle$ before the twist, we should have the state $⟨0,−1−n,−1−n|$, which is the conjugate of $|n−1,n−1,n−1\rangle$ after the twist. If we define the topological vacuum $|0\rangle_{\text{top}}$ as 

\[ |0\rangle_{\text{top}} \equiv |0,0,0\rangle \]  

and its dual conjugate $\langle\text{top}\rangle\langle∞|$ as 

\[ \langle\text{top}\rangle\langle∞\rangle \equiv ⟨n−1,−2,0| , \]

22
we can rewrite the holomorphic part of (4.37) and define the topological correlator as follows

\[ \langle N \prod_{i=1}^{N} \beta^{\ell_i}(z_i) \rangle_{\text{top}} \equiv \langle 0, -1 - n, 1 - n | \prod_{i=1}^{N} \beta^{\ell_i}(z_i) | 0, 0, 0 \rangle \]

\[ = \text{top} \langle \infty | \hat{\Omega}^\dagger \prod_{i=1}^{N} \beta^{\ell_i}(z_i) | 0 \rangle_{\text{top}} \]

\[ = \delta \left( \sum_{i} \ell_i - n + 1 \right) \]

where

\[ \Omega = \exp \left[ \frac{n-1}{\sqrt{n^2-1}} \hat{q}_0 + \frac{n-1}{\sqrt{2(n+1)}} \hat{q}_1 + i \frac{n-1}{\sqrt{2(n-1)}} \hat{q}_2 \right] \]

and the \( \dagger \) operation is defined as \( (e^{\alpha \hat{q}_i})^\dagger = e^{-\alpha \hat{q}_i} \) (see e.g. [38]). Notice that \( \Omega \) is simply the zero-mode part (the only one which survives on the left vacuum) of the top chiral primary field \( \beta^{n-1} \).

Therefore our conclusion is that in the bosonized formalism a topological correlation function of a string of fields is obtained by taking the expectation value between the topological vacua \( | 0, 0, 0 \rangle \) and \( \langle 0, -1 - n, 1 - n | \).

5. Explicit calculations of topological correlation functions

In this section we show in a few examples how to use the \((b,c,\beta,\gamma)\) representation to compute explicitly some (perturbed) topological correlation functions. We also verify that the parameters of the deformed LG potential for the \((b,c,\beta,\gamma)\)-system are the flat coordinates of the topological field theories. We stress that our techniques can be applied to single minimal models as well as to their tensor products, since there is practically no difference between the two cases. Even though the final goal is to use our methods in the interesting case of the Calabi-Yau 3-fold, for the sake of clarity here we will limit ourselves to the simpler cases of the minimal models and the torus.

We start by considering the simplest possible situation: the \(A_2\) minimal model, which corresponds to the potential

\[ W = \frac{1}{3}(\beta \bar{\beta})^3. \]

In this model, besides the identity \( \Phi_0 = 1 \), there is only one other chiral primary field: \( \Phi_1 = (\beta \bar{\beta}) \) with \( U(1) \)-charge \( q = 1/3 \). As one can see from (2.26), the component of the 2-form operator associated to \((\beta \bar{\beta})\) is simply \((b \bar{b})\). Therefore, \( \int d^2w b(w) \bar{b}(\bar{w}) \) is the only relevant deformation which can be used to perturb the minimal model \(A_2\).
The resulting Lagrangian is then
\[ \mathcal{L} = \mathcal{L}_{\text{top}} - t \int d^2 w \ b(w) \tilde{b}(\bar{w}) \] (5.2)

where \( \mathcal{L}_{\text{top}} \) is the Lagrangian for a topological \((b, c, \beta, \gamma)\)-system as given in (2.22), and \( t \) is a dimensionful coupling constant parametrizing its perturbation. Using the rules explained in the previous section and in particular enforcing the anomalous \( U(1) \) charge conservation, it is not difficult to realize that at \( t = 0 \) the only non-vanishing topological 3-point function for this model is
\[ c_{001} = \langle \Phi_0 \Phi_0 \Phi_1(z, \bar{z}) \rangle_{\text{top}} = 1 . \] (5.3)

However, things change when \( t \neq 0 \). The perturbed topological 3-point functions (see (2.25)) are in fact given by
\[ c_{\ell_1 \ell_2 \ell_3}(t) \equiv \left\langle \Phi_{\ell_1}(z_1, \bar{z}_1) \Phi_{\ell_2}(z_2, \bar{z}_2) \Phi_{\ell_3}(z_3, \bar{z}_3) \right\rangle_{\text{top}} \int d^2 w \ b(w) \tilde{b}(\bar{w}), \] (5.4)

where \( \ell_1, \ell_2, \ell_3 \) can be either 0 or 1. A simple analysis reveals that the only interesting case is the correlation \( c_{111}(t) \); all other correlators are indeed zero because of charge conservation. To compute \( c_{111}(t) \) we expand the exponential and evaluate each term using the bosonization rules of Section 4. In particular, once again because of charge conservation, all terms in this expansion vanish except for the first-order one. Thus we obtain
\[ c_{111}(t) = t \left\langle \Phi_1(z_1, \bar{z}_1) \Phi_1(z_2, \bar{z}_2) \Phi_1(z_3, \bar{z}_3) \right\rangle_{\text{top}} \int d^2 w \ b(w) \tilde{b}(\bar{w}). \] (5.5)

Let us now turn to the calculation of the conformal blocks appearing in the integrand of (5.5). We will focus just on the holomorphic piece, since the anti-holomorphic one is simply obtained by complex conjugation. First of all let us use (4.19) and (4.11) for \( n = 2 \) and write
\[ \beta = \exp \left[ \frac{1}{\sqrt{3}} \phi_0 \right] \varphi_1^1, \] \[ b = \exp \left[ -\frac{2}{\sqrt{3}} \phi_0 \right] \varphi_{-1}^1. \] (5.6)

Then, using (5.6) together with the definition of topological correlation functions given at the end of Section 4, we have
\[ \langle \beta(z_1) \beta(z_2) \beta(z_3) b(w) \rangle_{\text{top}} \]
\[ \equiv \langle 0, -3, -1 | \beta(z_1) \beta(z_2) \beta(z_3) b(w) | 0, 0, 0 \rangle \]
\[ = \langle 0 | e^{\frac{1}{\sqrt{3}} \phi_0(z_1)} e^{\frac{1}{\sqrt{3}} \phi_0(z_2)} e^{\frac{1}{\sqrt{3}} \phi_0(z_3)} | 0 \rangle \times \]
\[ \times \langle -3, -1 | \varphi_1^1(z_1) \varphi_1^1(z_2) \varphi_1^1(z_3) \varphi_{-1}^1(w) | 0, 0 \rangle. \] (5.7)
The $\phi_0$-contribution in (5.7) is immediate: the charges exactly soak up the background anomaly $Q_0^{(2)} = -1/\sqrt{3}$ and so we get

$$
\langle 0 | e^{i \frac{1}{\sqrt{3}} \phi_0(z_1)} e^{i \frac{1}{\sqrt{3}} \phi_0(z_2)} e^{i \frac{1}{\sqrt{3}} \phi_0(z_3)} | 0 \rangle
= \left[ (z_1 - z_2)(z_1 - z_3)(z_2 - z_3) \right]^{\frac{1}{3}} \left[ (z_1 - w)(z_2 - w)(z_3 - w) \right]^{-\frac{2}{3}}.
$$

(5.8)

The parafermion contribution

$$
\langle -3, -1 | \varphi_1^1(z_1) \varphi_1^1(z_2) \varphi_1^1(z_3) \varphi_{-1}^1(w) | 0, 0 \rangle
$$

(5.9)

is also easily computed. The most efficient way is perhaps the following: if we take into account the identification (4.12), we see that the fields $\varphi_1^1$ and $\varphi_{-1}^1$ are both proportional to $\varphi_0^0 = 1$. Moreover, the vacuum $\langle -3, -1 \rangle$ is proportional to $\langle -2, 0 \rangle$ since their dual conjugates $| 1, 1 \rangle$ and $| 0, 0 \rangle$ are equivalent because of (4.12). Thus, (5.9) is simply the vacuum expectation value of the identity and so it is a constant. One can also verify this result by explicitly computing (5.9) using for example the method of the screening charges [38,45]. Putting everything together, we obtain

$$
\langle \beta(z_1) \beta(z_2) \beta(z_3) b(w) \rangle_{\text{top}}
\sim \left[ (z_1 - z_2)(z_1 - z_3)(z_2 - z_3) \right]^{\frac{1}{3}} \left[ (z_1 - w)(z_2 - w)(z_3 - w) \right]^{-\frac{2}{3}}.
$$

(5.10)

Notice that this topological correlation function does depend on the coordinates $z_i$ where the chiral fields $\beta$ are inserted. This is not at all a surprise because (5.10) is not a correlator of only physical fields.

The full topological correlation function $c_{111}(t)$, which of course should be independent of $z_i$, can now be easily computed. If we substitute (5.10) and its complex conjugate into (5.5), we get

$$
c_{111}(t) \sim t \left( |z_1 - z_2| |z_1 - z_3| |z_2 - z_3| \right)^\frac{2}{3} \int d^2w \left( |z_1 - w| |z_2 - w| |z_3 - w| \right)^{-\frac{2}{3}}.
$$

(5.11)

The integral $I$ in (5.11) is evaluated using elementary techniques and the result is

$$
I \sim \left( |z_1 - z_2| |z_1 - z_3| |z_2 - z_3| \right)^{-\frac{2}{3}}.
$$

(5.12)

Absorbing all numerical factors of (5.10) and (5.11) into a rescaling of $t$, we can conclude that

$$
c_{111}(t) = t
$$

(5.13)

Notice that only after doing the integral over $d^2w$, the correlation function (5.11) becomes independent of the coordinates $z_i$ of the physical fields, as it should. In the minimal model $A_2$, $c_{111}(t) = t$ and $c_{001}(t) = 1$ are the only non vanishing topological 3-point functions.

As it is well known, the 2-point function $\eta_{\ell_1 \ell_2}(t) \equiv c_{0\ell_1 \ell_2}(t)$ serves as a metric in the space of coupling constants. It can be proven on general grounds [23] that this
metric is flat and hence there exist special flat coordinates in which it is constant. For the minimal model $A_2$ we simply have

$$\eta_{00}(t) = \eta_{11}(t) = 0, \quad \eta_{01}(t) = 1.$$  \hfill (5.14)

Since $\eta(t)$ is not only flat but also constant, the parameter $t$ in (5.2) is a flat coordinate.

This example is however too trivial to let us reach any conclusion, and one should test our methods in more complicated cases. This can be easily done and in several non-trivial examples for single minimal models we have checked that the parameters entering the deformed LG potential in our formulation are indeed flat coordinates. This is to be contrasted with the usual formulation where the parameters of the deformed LG potential are not flat coordinates but are related to these by more or less complicated transformations. The origin of this difference is that we compute the perturbed correlation functions without using the residue pairing defined by the perturbed potential [23] and stay in the context of perturbed conformal field theory. Thus we can always maintain in a natural way a frame of flat coordinates.

We have verified these properties also in the case of the torus, which can be described by the tensor product of three minimal models $A_2$ deformed by its marginal operator. Therefore, the potential we should consider is

$$W = \frac{1}{3}(\beta_x \beta_x)^3 + \frac{1}{3}(\beta_y \beta_y)^3 + \frac{1}{3}(\beta_z \beta_z)^3 - t(\beta_x \beta_x)(\beta_y \beta_y)(\beta_z \beta_z)$$ \hfill (5.15)

The dimensionless parameter $t$ in (5.15) is (related to) the modulus of the torus. The chiral ring of the tensor product of three $A_2$ minimal models is generated by $\Phi_0 = 1$, $(\Phi_1, \Phi_2, \Phi_3) = (\beta_x \beta_x, \beta_y \beta_y, \beta_z \beta_z)$, $\Phi_4 = \beta_x \beta_x \beta_y \beta_y$, $\Phi_5 = \beta_y \beta_y \beta_z \beta_z$, $\Phi_6 = \beta_x \beta_x \beta_z \beta_z$, and $\Phi_7 = \beta_x \beta_x \beta_y \beta_y \beta_z \beta_z$. $\Phi_7$ is a marginal operator and is the one appearing in (5.15) as a deformation. The Lagrangian corresponding to (5.15) is

$$\mathcal{L} = \mathcal{L}_{\text{top}} - t \int \Phi_7^{(2)}$$ \hfill (5.16)

where $\mathcal{L}_{\text{top}}$ is the Lagrangian for three topological $(b,c,\beta,\gamma)$-systems as in (2.22), and $\Phi_7^{(2)}$ is the 2-form associated to $\Phi_7$. According to (2.29), we have

$$\int \Phi_7^{(2)} = \int d^2w \left[(b_x \beta_y \beta_z + b_y \beta_x \beta_z + b_z \beta_x \beta_y)(w) \times (c.c)\right].$$ \hfill (5.17)

The simplest way to check whether the parameter $t$ in (5.16) is a flat coordinate or not, is to compute the topological metric

$$\eta_{ij}(t) = \left< \Phi_i \Phi_j e^{t \int \Phi_7^{(2)}} \right>_{\text{top}}$$ \hfill (5.18)

and see whether it is constant or not. To this end it is enough to consider one component, for example

$$\eta_{07}(t) = \left< \Phi_0 \Phi_7 e^{t \int \Phi_7^{(2)}} \right>_{\text{top}}.$$ \hfill (5.19)
By looking at the $U(1)$-charges of the operators in (5.19), it is easy to realize that when expanding the exponential, only terms with $3n$ insertions of $\int \Phi_7^{(2)}$ will satisfy charge conservation. Thus, (5.19) can be rewritten as

$$\eta_{07}(t) = \sum_{n=0}^{\infty} \frac{1}{(3n)!} a_{3n} t^{3n}$$

(5.20)

where

$$a_{3n} = \left\langle \Phi_0 \Phi_7 \left( \int \Phi_7^{(2)} \right)^{3n} \right\rangle_{\text{top}}$$

(5.21)

It is immediate to see that $a_0 = 1$. The first non-trivial contribution is $a_3$ which, spelled out in detail, is

$$a_3 = \int d^2w_1 \int d^2w_2 \int d^2w_3 \, f(u, w_1, w_2, w_3) \bar{f}(\bar{u}, \bar{w}_1, \bar{w}_2, \bar{w}_3)$$

(5.22)

where

$$f(u, w_1, w_2, w_3) = \left\langle (\beta_x \beta_y \beta_z)(u) \prod_{i=1}^{3} [(b_x \beta_y \beta_z + b_y \beta_x \beta_z + b_z \beta_x \beta_y)(w_i)] \right\rangle_{\text{top}}$$

(5.23)

and $\bar{f}$ is its complex conjugate. To compute $f$, we split the r.h.s. of (5.23) into a sum of factorized correlation functions for each of the three minimal models and enforce on them (anomalous) conservation of the $U(1)$ charges. During this process the $b$ fields in (5.23) must be suitably rearranged and proper minus signs arise from their anticommutation relations. After some straightforward algebra, it turns out that $f = 0$, which implies

$$a_3 = 0$$

(5.24)

Actually it is very easy to generalize this result to all higher-order coefficients, and eventually conclude that

$$\eta_{07}(t) = 1$$

(5.25)

The other entries of the metric $\eta_{ij}(t)$ can be computed in a similar way and all of them turn out to be constants. Thus the parameter $t$ in (5.16) is a flat coordinate.

We want to emphasize that instead, in the standard LG formulation of the torus described by the potential

$$W = \frac{1}{3} X^3 + \frac{1}{3} Y^3 + \frac{1}{3} Z^3 - sX Y Z$$

(5.26)

the topological correlation $\langle \Phi_0 \Phi_7 \rangle(s)$ is a non-trivial function of the LG parameter $s$, which therefore cannot be a flat coordinate.

7 It turns out that $\langle \Phi_0 \Phi_7 \rangle(s) = [F(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; s^3)]^{-2}$, where $F$ is the hypergeometric function.
It is interesting now to investigate the relation between \( s \) and \( t \). On general grounds [25], it is possible to show that

\[
s(t) = \frac{c_{111}(t)}{c_{123}(t)}
\]  

(5.27)

where

\[
c_{ijk}(t) = \left\langle \Phi_i \Phi_j \Phi_k e^{t \int \Phi_i(t)} \right\rangle_{\text{top}}
\]  

(5.28)

In our formulation it is easy to compute these perturbed 3-point functions as a power series in the flat coordinate \( t \). Let us briefly see how \( c_{123}(t) \) is evaluated. Expanding the exponential and looking for terms which satisfy charge conservation, we get

\[
c_{123}(t) = \sum_{n=0}^{\infty} \frac{1}{(3n)!} \tilde{a}_{3n} t^{3n}
\]  

(5.29)

where

\[
\tilde{a}_{3n} = \left\langle \Phi_1 \Phi_2 \Phi_3 \left( \int \Phi_i^2(t) \right)^{3n} \right\rangle_{\text{top}}
\]  

(5.30)

It is immediate to see that \( \tilde{a}_0 = 1 \). The next contribution is

\[
\tilde{a}_3 = \int d^2 w_1 \int d^2 w_2 \int d^2 w_3 g(x, y, z, w_1, w_2, w_3) \bar{g}(\bar{x}, \bar{y}, \bar{z}, \bar{w_1}, \bar{w_2}, \bar{w_3})
\]  

(5.31)

where

\[
g(x, y, z, w_1, w_2, w_3) = \left\langle \beta_x(x) \beta_y(y) \beta_z(z) \prod_{i=1}^{3} [(b_x \beta_y \beta_z + b_y \beta_x \beta_z + b_z \beta_x \beta_y)(w_i)] \right\rangle_{\text{top}}
\]  

(5.32)

By splitting the r.h.s. of (5.32) into a sum of factorized terms and using the explicit results derived earlier for each of the three minimal models, one can prove that the integrand of (5.31) is

\[
|x - y|^2 |x - z|^2 |y - z|^2 \prod_{i=1}^{3} (|x - w_i| |y - w_i| |z - w_i|)^{-\frac{4}{3}}
\]  

(5.33)

Using (5.12) and rescaling \( t \) to absorb all numerical constants, we conclude that \( \tilde{a}_3 = 1 \), and hence

\[
c_{123}(t) = 1 + \frac{1}{6} t^3 + O(t^6)
\]  

(5.34)

Similarly one can check that

\[
c_{111}(t) = t + O(t^7)
\]  

(5.35)
so that from (5.27) it follows

\[ s(t) = t - \frac{1}{6} t^4 + O(t^7) \]  

(5.36)

These are precisely the first terms in the power series expansion of the solution of the schwwarzian differential equation

\[ \{s; t\} = \frac{1}{2} \frac{(8 + s^3)}{(1 - s^3)^2} (s')^2 s \]  

(5.37)

where

\[ \{s; t\} = \frac{s'''}{s'} - \frac{3}{2} \left( \frac{s''}{s'} \right)^2 \]

and the primes denote \( t \)-derivatives. It has been recently shown [25,26,29] that (5.37) is equivalent to the requirement that \( t \) be a flat coordinate. Our methods provide automatically the solution to (5.37) as a power series. As pointed out at the beginning of this section there is no obstruction to extend our techniques to the calculation of topological correlators in \( c > 3 \) models. The actual calculation of these correlators is postponed to future work.
Appendix A

Landau-Ginzburg action and transformation rules in component formalism

In this appendix we present the explicit form of the Landau-Ginzburg lagrangian in component formalism and use the rheonomic approach to find the N=2 supersymmetry transformations. In the notations of [21], we write the following curvatures

\[ T^a = \mathcal{D}V^a - \frac{i}{2}\bar{\xi} \wedge \gamma^a \xi , \]
\[ \rho = \mathcal{D}\xi , \]
\[ F = dA - i\bar{\xi} \wedge \xi , \]
\[ R^{ab} = d\omega^{ab} , \]

where \( V^a \) is the zweibein, \( \xi \) is the gravitino one-form, \( A \) is the U(1) connection, \( \omega^{ab} \) is the spin connection and \( \mathcal{D} \) is the Lorentz covariant derivative. The gravitino \( \xi \) is a Dirac spinor. In general we can write

\[ \xi = e^{-i\pi/4} \left( \begin{array}{c} \zeta \\ \bar{\zeta} \end{array} \right) \]  

(A.2)

with \( \zeta \neq \zeta^* \), \( \bar{\zeta}^* \neq \bar{\zeta} \). More precisely, if we set

\[ e^\pm = \frac{1}{2}(V^0 \pm V^1) , \]
\[ \omega^{ab} = e^{ab}\omega , \]

(A.3)

we obtain

\[ T^a = dV^a - \omega^{ab} \wedge V^b - \frac{i}{2}\bar{\xi} \wedge \gamma^a \xi , \]  

(A.4)

or

\[ T^\pm = de^\pm \wedge e^\pm - \frac{i}{2}\bar{\xi} \wedge \gamma^\pm \xi , \]  

(A.5)

where \( \gamma^\pm \equiv \frac{1}{2}(1 \pm \gamma_3) \). Using (A.2), we have

\[ T^+ = de^+ + \omega \wedge e^+ - \frac{i}{2}\bar{\zeta}^* \wedge \zeta , \]
\[ T^- = de^- + \omega \wedge e^- - \frac{i}{2}\bar{\zeta}^* \wedge \bar{\zeta} . \]  

(A.6)

Similarly, we get

\[ F = dA - \zeta^* \wedge \bar{\zeta} + \bar{\zeta}^* \wedge \zeta . \]  

(A.7)
Following the general recipes of the rheonomic procedure \cite{21}, we write the background Maurer-Cartan equations

\begin{align}
\text{de}^+ + \omega \wedge e^+ - \frac{i}{2} \zeta^+ \wedge \zeta^- &= 0 , \\
\text{de}^- - \omega \wedge e^- - \frac{i}{2} \zeta^+ \wedge \bar{\zeta}^- &= 0 , \\
d\zeta^+ + \frac{1}{2} \omega \wedge \zeta^+ &= 0 , \\
d\bar{\zeta}^+ - \frac{1}{2} \omega \wedge \bar{\zeta}^+ &= 0 , \\
d\omega &= 0 , \\
dA - \zeta^- \wedge \bar{\zeta}^+ + \bar{\zeta}^+ \wedge \zeta^- &= 0 ,
\end{align}

where we have set \( \zeta^- = \zeta \) and \( \zeta^+ = \zeta^* \). Using these notations, we can write the general form of the LG lagrangian in components. From the Bianchi identities \( d^2 X^i = d^2 \psi^i = d^2 \bar{\psi}^i = 0 \) and from (A.8) one derives the following rheonomic parametrizations

\begin{align}
\text{d}X^i &= \partial_+ X^i e^+ + \partial_- X^i e^- + \psi^i \zeta^- + \bar{\psi}^i \bar{\zeta}^- , \\
\text{d}X^i^* &= \partial_+ X^i^* e^+ + \partial_- X^i^* e^- - \psi^i^* \zeta^+ - \bar{\psi}^i^* \bar{\zeta}^+ , \\
\text{d}\psi^i &= \partial_+ \psi^i e^+ + \partial_- \psi^i e^- - \frac{i}{2} \partial_+ X^i \zeta^- + \eta^i j^* \partial_j W \bar{\zeta}^- , \\
\text{d}\bar{\psi}^i &= \partial_+ \bar{\psi}^i e^+ + \partial_- \bar{\psi}^i e^- - \frac{i}{2} \partial_- X^i \bar{\zeta}^- + \eta^i j^* \partial_j W \zeta^- ,
\end{align}

where \( X^i, X^i^* = (X^i)^* \) are complex coordinates in a flat Kähler manifold, \( \eta^i j^* \) is the flat metric and \( \psi^i, \bar{\psi}^i \) are the complex spin-\( \frac{1}{2} \) fermionic partners of \( X^i \)'s. The parametrizations of \( d\psi^i^* \) and \( d\bar{\psi}^i^* \) are obtained by complex conjugation. Using standard techniques, one finds that the action, from which (A.9) follow as field equations in the vertical directions, is

\[ S = \int \mathcal{L} \]

(A.10)
where

\[
\mathcal{L} = \eta_{ij}^*(dX^i - \psi^i \zeta^- - \tilde{\psi}^i \tilde{\zeta}^-) \wedge (\Pi_{i}^- e^+ - \Pi_{i}^+ e^-) \\
+ \eta_{ij}^*(dX^j + \psi^j \zeta^+ + \tilde{\psi}^j \tilde{\zeta}^+) \wedge (\Pi_{i}^+ e^+ - \Pi_{i}^- e^-) \\
+ \eta_{ij}^*(\Pi_{i}^+ \Pi_{j}^- + \Pi_{i}^- \Pi_{j}^+) e^+ \wedge e^- - (4i\eta_{ij}^* \psi^i d\psi^j + \frac{i}{2} \psi^k \partial_k W \tilde{\zeta}^+) \wedge e^+ \\
+ (4i\eta_{ij}^* \tilde{\psi}^j d\tilde{\psi}^i + \frac{i}{2} \tilde{\psi}^k \partial_k W \zeta^+ \wedge e^- \\
+ \frac{i}{2} \psi^k \partial_k W \zeta^- \wedge e^+ - \frac{i}{2} \tilde{\psi}^k \partial_k W \zeta^- \wedge e^- \\
+ (8\psi^i \psi^j \partial_i \partial_j W + 8\tilde{\psi}^i \tilde{\psi}^j \partial_i \partial_j \tilde{W} + 8\eta_{ij}^* \partial_i W \partial_j \tilde{W}) e^+ \wedge e^- \\
+ dX^j \wedge \psi^j \tilde{\zeta}^- - dX^j \wedge \tilde{\psi}^j \zeta^- \eta_{ij}^* \\
+ dZ^i \wedge \psi^j \zeta^+ \eta_{ij}^* - dZ^i \wedge \tilde{\psi}^j \zeta^+ \eta_{ij}^* .
\]

The lagrangian (A.11) is written in first-order formalism and the auxiliary fields \( \pi_i^\pm, \pi_i^{\pm*} \) can be eliminated through their equations of motion: \( \Pi_i^\pm = \partial_\pm X^i \) and \( \Pi_i^{\pm*} = \partial_\pm X^{i*} \). The lagrangian (2.29) of Section 2 is obtained from (A.11) by restricting it to the bosonic surface (namely discarding all terms containing the gravitino forms \( \zeta^\pm, \tilde{\zeta}^\pm \)) and substituting back the above mentioned equations for \( \Pi_i^{\pm*} \).

Form the curvature parametrization (A.9), we easily recover the N=2 supersymmetry transformations

\[
\begin{align*}
\delta X^i &= -\varepsilon^- \psi^i - \bar{\varepsilon}^- \tilde{\psi}^i , \\
\delta X^{i*} &= \varepsilon^+ \psi^{i*} + \bar{\varepsilon}^+ \tilde{\psi}^{i*} , \\
\delta \psi^i &= \frac{-i}{2} \partial X^i \varepsilon^+ + \eta^{ij*} \partial_j \tilde{W} \bar{\varepsilon}^- , \\
\delta \tilde{\psi}^i &= \frac{-i}{2} \partial X^i \bar{\varepsilon}^- + \eta^{ij*} \partial_j W \varepsilon^- , \\
\delta \psi^{i*} &= \frac{i}{2} \partial X^{i*} \varepsilon^- + \eta^{j*} \partial_j W \bar{\varepsilon}^+ , \\
\delta \tilde{\psi}^{i*} &= \frac{i}{2} \partial X^{i*} \bar{\varepsilon}^+ - \eta^{j*} \partial_j W \varepsilon^+ ,
\end{align*}
\]

which coincide precisely with the ones in (2.31) of Section 2.

Appendix B

The lagrangian for the topological \((b, c, \beta, \gamma)\)-system

In Section 2 we gave the explicit form of the descent equations (2.26), using (2.23) as BRST transformations. We know that this case corresponds to the twisted
stress-energy tensor
\[ \hat{T}_+ = T + \frac{1}{2} \partial J = \partial cb + \gamma \partial \beta \]  
(B.1)

which is associated to a \((b, c, \beta, \gamma)\)-system with \(\lambda_\beta = 0\), \(\lambda_b = 0\). For simplicity we consider the case of only one collection of pseudo-ghost fields and as usual, we understand the expressions for the tilded operators. The main advantage of this approach is that we can write the chiral primary fields of the N=2 theory in terms of the local fields appearing in the lagrangian, and hence we can construct the representatives of the BRST cohomology in a rather simple way. The lagrangian, however, is not a BRST commutator: we are dealing with a topological theory of the Schwartz-type [22].

As pointed out in Section 2, there is another possibility and one can take as twisted stress-energy tensor the following expression
\[ \hat{T}_- = T - \frac{1}{2} \partial J = (1 - 2\lambda) \partial cb - 2\lambda b \partial c + (1 - 2\lambda) \gamma \partial \beta - 2\lambda \beta \partial \gamma . \]  
(B.2)

This corresponds to a commuting \((\beta, \gamma)\)-system with weights \(\lambda_\beta = 2\lambda\), \(\lambda_\gamma = 1 - 2\lambda\), and to an anticommuting \((b, c)\)-system with weights \(\lambda_b = 2\lambda\) and \(\lambda_c = 1 - 2\lambda\). In this case the BRST transformations are
\[ 
s_\beta = 2b \quad , \quad s_\tilde{\beta} = 2\tilde{b} , \]
\[ 
s_c = 2\gamma \quad , \quad s_\tilde{c} = 2\tilde{\gamma} , \]
\[ 
s_b = 0 \quad , \quad s_\tilde{b} = 0 , \]  
(B.3)

Eq.s (B.3) are obtained from the supersymmetry transformations (2.8) by setting \(\epsilon^+ = \tilde{\epsilon}^+ = 0\) and choosing as BRST parameter \(\theta = \sqrt{2} \epsilon^- = \sqrt{2} \tilde{\epsilon}^-\). This corresponds to taking
\[ Q = \frac{1}{\sqrt{2}} \left( \oint G^- (z) dz + \oint \tilde{G}^- (\tilde{z}) d\tilde{z} \right) \]  
(B.4)

as BRST charge. The stress-energy tensor (B.2) is a BRST commutator, namely
\[ \hat{T} = s \left[ - (\lambda - \frac{1}{2}) c \partial \beta - \lambda \beta \partial c \right] \]  
(B.5)

where, as usual, we have defined \(s\phi = [Q, \phi]\) for a generic field \(\phi\). Now we want to show that the twist (B.2) can be seen directly at the lagrangian level, or equivalently that we can write the lagrangian (including the interaction term) as a BRST commutator. If this is the case, then we have a topological theory of Witten-type [22]. To understand this point, we first recall that the supercurrent \(G^+\), as shown in (2.9), is actually composed of a \(G^+_z\) term and a \(G^+_\tilde{z}\) term. The latter is zero on shell, but it has to be
taken into account in defining the superconformal transformations \[37\]. Keeping this in mind, we recognize that

\[ \mathcal{L} = s(-G_\bar{z}^+ - \bar{G}_z^+) \]
\[ = s[-(\lambda - \frac{1}{2})c\check{d}\bar{\beta} - \lambda\beta\check{d}c + \frac{1}{2}V\check{b}\check{d}\bar{V} - (\lambda - \frac{1}{2})\check{c}\check{d}\bar{\beta} - \lambda\bar{\beta}\check{d}\check{c} - \frac{1}{2}\check{V}b\check{d}V] . \]  

(B.6)

Using (B.5), we get

\[ \mathcal{L} = \left[ -2\lambda\beta\check{d}\gamma + (1 - 2\lambda)\gamma\check{d}\bar{\beta} + (1 - 2\lambda)\check{d}bc - 2\lambda b\check{d}\check{c} + \check{b}\check{d}V\check{d}\bar{V} \right] + c.c \]  

(B.7)

which is the expected lagrangian for the twisted \( \lambda \)'s. It is suggestive to point out that, if we define

\[ G^+ = -(\lambda - \frac{1}{2})c\check{d}\beta - \lambda\beta\check{d}c + \frac{1}{2}V\check{b}\check{d}\bar{V}d\bar{z} , \]
\[ \bar{G}^+ = -(\lambda - \frac{1}{2})\check{c}\check{d}\bar{\beta} - \lambda\bar{\beta}\check{d}\check{c} - \frac{1}{2}\check{V}b\check{d}Vdz , \]  

(B.8)

where for a generic pseudo-ghost field \( \phi \)

\[ d\phi = \partial\phi dz + \bar{\partial}\phi d\bar{z} \]  

(B.9)

denotes the space-time part of the rheonomic parametrizations (that is we disregard the gravitino contributions), we get

\[ S = \int \mathcal{L}_{\text{top}} dz \wedge d\bar{z} = \int dz \wedge s(G^+) + \int s(\bar{G}^+) \wedge d\bar{z} . \]  

(B.10)

Finally, if we regard (B.7) as a topological lagrangian without special requirements on the interaction term, we can ask when this model defines a conformal field theory. The answer to this question is almost obvious, even if the calculation is a little different: only for quasi-homogeneous potential \( V(\beta) \) with \( 2\lambda = \omega \) we get a conformal field theory.
References

[1] For a review see M.B. Green, J. Schwarz and E. Witten, “Superstring theory”, Cambridge University Press, 1987.
[2] M. Ademollo, L. Brink, A. D’Adda, R. D’Auria, E. Napolitano, S. Sciuto, E. del Giudice, P. di Vecchia, S. Ferrara, F. Gliozzi, R. Musto and R. Pettorino, Phys. Lett. 62B (1976) 105.
[3] A. Sen, Nucl. Phys. B278 (1986) 289 and Nucl. Phys. B284 (1987) 423; T. Banks, L.J. Dixon, D. Friedan and E. Martinec, Nucl. Phys. B299 (1988) 613 and Nucl. Phys. B307 (88) 93.
[4] D. Gepner, Nucl. Phys. B296 (1988) 757 and Phys. Lett. 199B (1987) 380.
[5] W. Lerche, C. Vafa and N.P. Warner, Nucl. Phys. B324 (1989) 427; B. Greene, C. Vafa and N.P. Warner, Nucl. Phys. B324 (1989) 371.
[6] T. Eguchi, H. Ouguri, A. Taormina and S. K. Yang, Nucl. Phys. B315 (1989) 192.
[7] P. Candelas, C.T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B298 (1988) 493.
[8] P. Candelas, A.M. Dale, C.A. Lutken and R. Schimmrick, Nucl. Phys. B298 (1988) 493.
[9] M. Linker and R. Schimmrick, Phys. Lett. 208B (1988) 216 and Phys. Lett. 215B (1988) 681.
[10] C.A. Lutken and G.C. Ross, Phys. Lett. 213B (1988) 152.
[11] P. Zoglin, Phys. Lett. 218B (1989) 444.
[12] B. de Wit and A. Van Proeyen, Nucl. Phys. B245 (1984) 89.
[13] E. Cremmer, C. Kounnas, A. Van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, Nucl. Phys. B250 (1985) 385.
[14] B. de Wit, P.G. Lauwers and A. Van Proeyen, Nucl. Phys. B255 (1985) 560.
[15] L. Castellani, R. D’Auria and S. Ferrara, Phys. Lett. 241B (1990) 57; Class. and Quantum Grav. 1 (1990) 163; R. D’Auria, S. Ferrara and P. Frè, Nucl. Phys. B359 (1991) 705.
[16] A. Strominger, Comm. Math. Phys. 133 (1990) 163.
[17] S. Cecotti, S. Ferrara and L. Girardello, Int. Jour. Mod. Phys. A10 (1989) 2475 and Phys. Lett. 213B (1988) 443.
[18] S. Ferrara and A. Strominger, Strings 89, Eds. R. Arnowitt, R. Bryan, M. Duff, D. Nanopulos and C. Pope, World Scientific, Singapore, 1989.
[19] P. Candelas and X. de la Ossa, Nucl. Phys. B342 (1990) 246.
[20] L. J. Dixon, V. Kaplunowskij and J. Louis, Nucl. Phys. B329 (1990) 27.
[21] For a review see L. Castellani, R. D’ Auria and P. Frè, “Supergravity and superstrings: a geometric prospective”, World Scientific, Singapore (1991).
[22] For a review see D. Birmingham, M. Blau and M. Rakowski, Phys. Rep. 209 (1991) 129.
[23] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B352 (1991) 59.
[24] B. Block and A. Varchenko, Prepr. IASSNS-HEP-91/5.
[25] E. Verlinde and N.P. Warner, Phys. Lett. B269 (1991) 96.
[26] A. Klemm, S. Theisen and M. Schmidt, Prepr. TUM-TP-129/91, KA-THEP-91-00, HD-THEP-91-32.
[27] S. Ferrara, J. Louis, Prepr. CERN-TH-6334/91.
[28] A. Ceresole, R. D’Auria, S. Ferrara, W. Lerche and J. Louis, CERN-TH-6441/92.
[29] Z. Maassarani, Prepr. USC-91/023.
[30] E. Witten, Comm. Math. Phys. 118 (1988) 411 and Nucl. Phys. B340 (1990) 281;
T. Eguchi and S.K. Yang Mod Phys Lett A5 (1990) 1693
[31] L. Baulieu and E. M. Singer, Comm. Math. Phys. 125 (1989) 125.
[32] E. Witten, Prepr. IASSNS-HEP-91/83.
[33] For a review see A. Pasquinucci, Ph.D. Thesis SISSA/EP 1990.
[34] S. Cecotti, L. Girardello and A. Pasquinucci, Nucl. Phys. B328 (1989) 701 and
Int. J. Mod. Phys. A6 (1991) 2427.
[35] N.P. Warner, Lectures at Trieste Spring school 1988, World Scientific, Singapore;
E. Martinec, Phys. Lett. 217B (1989) 431;
For a review see also “Criticality, Catastrophe and Compactification”, V.G.
Knizhnik memorial volume, 1989 .
[36] V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko, “Singularities of differentiable
maps”, Vol I, II Birkäuser, Boston.
[37] P. Frè, F. Gliozzi, R. Monteiro and A. Piras, Class. and Quantum Grav 8 (1991) 1455.
[38] M. Frau, J.G. McCarthy, A. Lerda, S. Sciuto and J. Sidenius, Phys. Lett. 254B (1991) 381 and
Phys. Lett. 245B (1990) 453.
[39] G. Mussardo, G. Sotkov and M. Stanishkhov, Int. J. Mod. Phys. A4 (1986) 1135.
[40] P. Griffiths and J. Harris, “Principles of algebraic geometry” ed.s John Wiley
and sons.
[41] C. Vafa, Mod. Phys. Lett. A6 (1991) 337.
[42] M.T. Grisaru, A. Lerda, S. Penati and D. Zanon, Nucl. Phys. B342 (1990) 564
and Phys. Lett. 234B (1990) 88.
[43] A. B. Zamolodchikov, JEPT Lett. 43 (1986) 730 and Sov. J. Nucl. Phys. 46 (1987) 1090.
[44] E. Gava and M. Stanishkov, Mod. Phys. Lett. 27 (1990) 2261.
[45] G. Felder, Nucl. Phys. B317 (1989) 215 and Nucl. Phys. B324 (1989) 548E;
V.S. Dotsenko and V.A. Fateev, Nucl. Phys. B240 (1984) 312.
TOPOLOGICAL FIRST-ORDER SYSTEMS WITH LANDAU-GINZBURG INTERACTIONS∗

Pietro Frè

SISSA - International School for Advanced Studies
Via Beirut 2, I-34100 Trieste, Italy
and I.N.F.N. sezione di Trieste

Luciano Girardello

Dipartimento di Fisica, Università di Milano
Via Celoria 16, I-20133 Milano, Italy
and I.N.F.N. sezione di Milano

Alberto Lerda

Institute for Theoretical Physics, S.U.N.Y. at Stony Brook
Stony Brook, N.Y. 11794, U.S.A.

Paolo Soriani

SISSA - International School for Advanced Studies
Via Beirut 2, I-34100 Trieste, Italy
and I.N.F.N. sezione di Trieste

Abstract

We consider the realization of N=2 superconformal models in terms of free first-order \((b,c,\beta,\gamma)\)-systems, and show that an arbitrary Landau-Ginzburg interaction with quasi-homogeneous potential can be introduced without spoiling the \((2,2)\)-superconformal invariance. We discuss the topological twisting and the renormalization group properties of these theories, and compare them to the conventional topological Landau-Ginzburg models. We show that in our formulation the parameters multiplying deformation terms in the potential are flat coordinates. After properly bosonizing the first-order systems, we are able to make explicit calculations of topological correlation functions as power series in these flat coordinates by using standard Coulomb gas techniques. We retrieve known results for the minimal models and for the torus.

* Work supported in part by Ministero dell’Università e della Ricerca Scientifica e Tecnologica, and by NSF grants PHY 90-08936

1 Also at Dipartimento di Fisica Teorica, Università di Torino, Via P. Giuria 1, I-10125 Torino, Italy and I.N.F.N. sezione di Torino.