Isoperimetric and Universal Inequalities for Eigenvalues

Mark S. Ashbaugh*
Department of Mathematics
University of Missouri
Columbia, MO 65211-0001
e-mail: mark@math.missouri.edu

March 7, 2018

1991 Mathematics Subject Classification: Primary 35P15, Secondary 58G25, 49Rxx.

Keywords and phrases: eigenvalues of the Laplacian, isoperimetric inequalities for eigenvalues, Faber-Krahn inequality, Szegő-Weinberger inequality, Payne-Pólya-Weinberger conjecture, Sperner’s inequality, biharmonic operator, bi-Laplacian, clamped plate problem, Rayleigh’s conjecture, buckling problem, the Pólya-Szego conjecture, universal inequalities for eigenvalues, Hile-Protter inequality, H. C. Yang’s inequality.

Short title: Isoperimetric and Universal Inequalities

Abstract

This paper reviews many of the known inequalities for the eigenvalues of the Laplacian and bi-Laplacian on bounded domains in Euclidean space. In particular, we focus on isoperimetric inequalities for the low eigenvalues of the Dirichlet and Neumann Laplacians and of the vibrating clamped plate problem (i.e., the biharmonic operator with “Dirichlet” boundary conditions). We also discuss the known universal inequalities for the eigenvalues of the Dirichlet Laplacian and the vibrating clamped plate and buckling problems and go on to

*Partially supported by National Science Foundation (USA) grants DMS-9500968 and DMS-9870156.
present some new ones. Some of the names associated with these inequalities are Rayleigh, Faber-Krahn, Szegő-Weinberger, Payne-Pólya-Weinberger, Sperner, Hile-Protter, and H. C. Yang. Occasionally, we will also comment on extensions of some of our inequalities to bounded domains in other spaces, specifically, $S^n$ or $H^n$.

1 Introduction

1.1 The Eigenvalue Problems

The first eigenvalue problem we shall introduce is that of the fixed membrane, or Dirichlet Laplacian. We consider the eigenvalues and eigenfunctions of $-\Delta$ on a bounded domain (=connected open set) $\Omega$ in Euclidean space $\mathbb{R}^n$, i.e., the problem

\begin{align}
-\Delta u &= \lambda u \quad \text{in } \Omega \subset \mathbb{R}^n, \\
 u &= 0 \quad \text{on } \partial \Omega.
\end{align}

(1.1a)

(1.1b)

It is well-known that this problem has a real and purely discrete spectrum $\{\lambda_i\}_{i=1}^\infty$ where

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \rightarrow \infty.$$  

(1.2)

Here each eigenvalue is repeated according to its multiplicity. An associated orthonormal basis of real eigenfunctions will be denoted $u_1, u_2, u_3, \ldots$. In fact, throughout this paper we will assume that all functions we consider are real. This entails no loss of generality in the present context.

The next problem we introduce is that of the free membrane, or Neumann Laplacian. This is the problem

\begin{align}
-\Delta v &= \mu v \quad \text{in } \Omega \subset \mathbb{R}^n, \\
 \frac{\partial v}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{align}

(1.3a)

(1.3b)

Here $\partial/\partial n$ denotes the outward normal derivative on $\partial \Omega$, where we now assume that $\partial \Omega$ is sufficiently smooth. With this assumption, problem (1.3) has spectrum $\{\mu_i\}_{i=0}^\infty$ where

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots \rightarrow \infty,$$

(1.4)

with the eigenvalues again repeated according to their multiplicities. A corresponding orthonormal basis of real eigenfunctions will be denoted $\{v_i\}_{i=0}^\infty$.

Next we introduce the clamped plate problem, or eigenvalue problem for the Dirichlet biharmonic operator (for an explanation of this terminology see
or \([77]\)], which describes the characteristic vibrations of a clamped plate. This problem is given by

\[
\Delta^2 w = \Gamma w \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1.5a}
\]

\[
w = 0 = \frac{\partial w}{\partial n} \quad \text{on } \partial \Omega. \tag{1.5b}
\]

We will denote the eigenvalues and an associated orthonormal basis of real eigenfunctions by \(\{\Gamma_i\}_{i=1}^\infty\) and \(\{w_i\}_{i=1}^\infty\), respectively. The eigenvalues \(\Gamma_i\) satisfy

\[
0 < \Gamma_1 \leq \Gamma_2 \leq \Gamma_3 \leq \ldots \rightarrow \infty. \tag{1.6}
\]

Lastly, we introduce the \textit{buckling problem}, which determines the critical buckling load of a clamped plate subjected to a uniform compressive force around its boundary (this description applies to the \(n = 2\) case of the problem). This problem again involves the biharmonic operator and is formulated as

\[
\Delta^2 v = -\Lambda \Delta v \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1.7a}
\]

\[
v = 0 = \frac{\partial v}{\partial n} \quad \text{on } \partial \Omega. \tag{1.7b}
\]

It also has a discrete spectrum consisting of positive eigenvalues of finite multiplicity with infinity as their only accumulation point. We denote the eigenvalues by \(\{\Lambda_i\}_{i=1}^\infty\) and a corresponding orthonormal basis of real eigenfunctions by \(\{v_i\}_{i=1}^\infty\). Thus

\[
0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \ldots \rightarrow \infty. \tag{1.8}
\]

Good sources of information on many of these problems include the books of Bandle [24], Béard [26], Courant-Hilbert [38], Davies [39], Kesavan [58], Leis [67], Pólya and Szegő [54], Reed and Simon [87], and Safarov and Vasiliev [88]. The review papers of Davies [40], Payne [79], [80], [81], Protter [85], and Talenti [99], [100] are also quite useful, as is Leissa’s monograph [68] on the vibration of plates. Rayleigh’s classic \textit{The Theory of Sound} [86] is highly recommended as collateral reading.

### 1.2 Rearrangement

In this subsection we introduce the notion of \textit{spherically symmetric rearrangement} (or \textit{Schwarz symmetrization}) and some of its properties. Suppose that we have a bounded measurable function \(f\) on the bounded measurable set \(\Omega \subset \mathbb{R}^n\). We can consider its \textit{distribution function} \(\mu_f(t)\) defined by

\[
\mu_f(t) = |\{x \in \Omega| |f(x)| > t\}|
\]
where $|\cdot|$ denotes Lebesgue measure. The distribution function can be viewed as a function from $[0, \infty)$ to $[0, |\Omega|]$. It is clearly a nonincreasing function. The *decreasing rearrangement of* $f$, denoted $f^*$, is essentially the inverse of $\mu_f$ and is defined by

$$f^*(s) = \inf\{t \geq 0 | \mu_f(t) < s\}. \tag{1.10}$$

It is a nonincreasing function on $[0, |\Omega|]$. Before defining the spherically decreasing rearrangement of a function we define the *spherical* (or *symmetric*) rearrangement of a set. For a bounded measurable set $\Omega \subset \mathbb{R}^n$, we define its spherical rearrangement $\Omega^*$ as the ball centered at the origin having the same measure as $\Omega$, i.e., $|\Omega^*| = |\Omega|$. We can now define the *spherically (symmetric) decreasing rearrangement* $f^* : \Omega^* \to \mathbb{R}$ by

$$f^*(x) = f^*(C_n|x|^n) \quad \text{for} \quad x \in \Omega^* \tag{1.11}$$

where $C_n = \pi^{n/2}/\Gamma\left(\frac{n}{2} + 1\right)$ = volume of the unit ball in $\mathbb{R}^n$. A verbal description of $f^*$ runs as follows: $f^*$ is that function of $x \in \Omega^*$ which is spherically symmetric, radially decreasing (in the weak sense of nonincreasing) and equimeasurable with $|f|$, i.e., $f^*$ and $|f|$ share the same distribution function (see (1.9) above). One way to view this is that for each of the level sets $\Omega_t \equiv \{x \in \Omega | |f(x)| > t\}$ of $|f|$ we take its spherical rearrangement $\Omega_t^*$ and define this to be the corresponding level set of $f^*$ (which, we recall, is placed concentrically with $\Omega^*$). This viewpoint gives an interpretation of $f^*$ which avoids the intermediate function $f^*$ of a single (=volume) variable. Good sources of further information on rearrangements are [24], [45], [47], [48], [56], [71], [72], [73], [84], [90], [95], [96], [97], [99], [100].

A key fact, which is evident from the equimeasurability of $f$, $f^*$, and $f^*$, is that

$$\int_\Omega f^2 = \int_0^{\Omega} f^*(s)^2 ds = \int_{\Omega^*} (f^*)^2. \tag{1.12}$$

There is no reason for not putting $|f|$, $f^*$, and $f^*$ as integrands here, or even $|f|^p$, $|f^*|^p$, and $|f^*|^p$, except that (1.12) is all we need later in this paper.

We shall be particularly concerned with how spherical rearrangement affects the first eigenfunction $u_1$ of the Dirichlet Laplacian. The key property here is that spherical rearrangement typically decreases, and in any case cannot increase, the Dirichlet norm $\int_\Omega |\nabla u_1|^2$. In particular, it is known that for any function $f$ in the Sobolev space $H^1_0(\Omega)$ (for background and notation on Sobolev spaces we recommend [2], [29], [58], and [73]) $f^* \in H^1_0(\Omega^*)$ and

$$\int_{\Omega^*} |\nabla f^*|^2 \leq \int_{\Omega} |\nabla f|^2. \tag{1.13}$$
This inequality is crucial to the proof of the Faber-Krahn inequality given in Section 2. Note, too, that by (1.12) the $L^2$-norm of $u_1$ does not change when we replace it by $u_1^*$. For a discussion of (1.13), see Glaser, Martin, Grosse, and Thirring [13], Gunson [17], Kawohl [26], Lieb [71], [72], Lieb and Loss [73], Pólya-Szego [84], and/or Talenti [95], [96], [99], [100].

We also note a certain elementary property of rearrangement as it affects integrals of products of functions. This is that for two nonnegative measurable functions $f$ and $g$ on $\Omega$

$$\int_{\Omega} fg \leq \int_{\Omega^*} f^*g^*. \quad (1.14)$$

There is a corresponding lower bound if we rearrange $f$ and $g$ in opposite senses. For this we need the notion of \textit{spherically (symmetric) increasing rearrangement}, which we denote by a lower $\star$. The definition is almost identical to that of spherically decreasing rearrangement, except that $g_\star$ should be radially increasing (in the weak sense) on $\Omega^\star$. Then we have, for example,

$$\int_{\Omega} fg \geq \int_{\Omega^*} f^*g_\star. \quad (1.15)$$

In fact, these inequalities take their most elementary form if we use “signed” rearrangements of $f$ and $g$, that is, we define $f^*$, $g^*$, $f_\star$, $g_\star$, etc., in terms of the distribution function as given in (1.9) except that we use just $f(x)$ in place of $|f(x)|$ in that formula. With $f^*$, $g^*$, etc., defined in this alternative way, (1.14) and (1.15) hold without any need to restrict $f$ and $g$ to be nonnegative functions. All of this basic material is admirably presented in [48], where the essential features of the process (similarly ordered, oppositely ordered) are brought to the fore. In a sense the essence of the whole business is the simple algebraic inequality

$$(a - b)(c - d) \geq 0 \quad (1.16)$$

if $(a, b)$ and $(c, d)$ are similarly ordered (and the reverse if these vectors are oppositely ordered). Thus, if $\vec{v} = (a, b)$ and $\vec{w} = (c, d)$ are similarly ordered

$$\vec{v} \cdot \vec{w} = ac + bd \geq ad + bc = \vec{v} \cdot (d, c). \quad (1.17)$$

The relevance of this simple vector inequality for (1.14) (in its general “signed” version) is that we can approach (1.14) by approximating $f$ and $g$ by simple functions decomposed over sets of equal measures. In this setting the relevance of (1.17) is apparent (as a first approximation in the single-variable setting one might think of passing to Riemann sums over subintervals of equal length). See [48], and the more recent article by Baernstein [25], for further development of these ideas.
We come finally to a result on how rearrangement affects the solution to the Poisson equation on a bounded domain $\Omega$ with homogeneous Dirichlet boundary conditions

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^n, \quad (1.18a)$$

$$u = 0 \quad \text{on } \partial \Omega. \quad (1.18b)$$

Suppose we solve this problem for $u$ and then take its spherically decreasing rearrangement $u^\star$. We want to compare $u^\star$ to the solution $v$ of the symmetrized (=spherically rearranged) problem

$$-\Delta v = f^\star \quad \text{in } \Omega^\star, \quad (1.19a)$$

$$v = 0 \quad \text{on } \partial \Omega^\star. \quad (1.19b)$$

It turns out that if $u \geq 0$ on $\Omega$, then

$$0 \leq u^\star \leq v. \quad (1.20)$$

For $f^\star$ in (1.19a) we can even use a signed rearrangement of $f$ (still under the assumption that $f$ is sufficiently positive that $u \geq 0$ on $\Omega$, which implies, in particular, that $\int_\Omega f \geq 0$). For example, if $f$ is bounded below by $c$ we can take $f^\star = (f-c)^\star + c$; this gives a radially decreasing function that passes into negative values out near $\partial \Omega$ if $f$ itself is not always nonnegative.

The key to proving (1.20) is a differential inequality for $u^\star$ (or equivalently $u^\ast$) and the fact that (1.19) reduces to a one-dimensional problem that can be solved explicitly. In particular, we have (in the radial variable $r = |x|$)

$$-\frac{1}{r^{n-1}}(r^{n-1}v')' = f^\star \quad (1.21)$$

and hence

$$r^{n-1}v'(r) = -\int_0^r \tau^{n-1} f^\ast(\tau) d\tau \quad (1.22)$$

and finally

$$v(r) = \int_r^R t^{1-n} \int_0^t \tau^{n-1} f^\ast(\tau) d\tau dt, \quad (1.23)$$

where we have incorporated the conditions $v'(0) = 0$ (necessary for $v$ to be smooth at the origin in $\Omega^\star$) and $v(R) = 0$, with $R$ denoting the radius of $\Omega^\star$. Recalling the condition $\int_\Omega f \geq 0$, we see that this is precisely the condition needed to keep $v$ nonnegative, since this guarantees that $\int_0^t \tau^{n-1} f^\ast(\tau) d\tau \geq 0$ for all $t \in [0, R]$ (recall, too, that up to the constant
factor $nC_n$, $\int_0^s \tau^{n-1} f^*(\tau)d\tau$ is $\int_0^s f^*(\sigma)d\sigma$ where $\sigma$ is now the volume variable $\sigma = C_n \tau^n$ and similarly $s$ is related to $t$ by $s = C_n t^n$; in general, we shall reserve $s$ and $\sigma$ as “volume variables”, while $t$, $\tau$, $r$, and $\rho$ will be used as radial variables).

The basic idea used to get (1.20) goes back at least to Weinberger [103]. The method is also given by Bandle in her book [24] and figures prominently in works of Talenti [93], [94], [97], and Chiti [34], [35], [36], [37]. The extended form presented above (allowing a signed rearrangement of $f$) occurs in Talenti [98], where it is instrumental in his treatment of the first eigenvalue $\Gamma_1$ of the clamped plate problem. As such, it also figures in the later papers on the subject by Nadirashvili [74], [75], [76] and Ashbaugh and Benguria [18] (see also [22]). A useful discussion of the extended form is also found in the papers of Kesavan [57], [59], [60], [61]. The interaction between rearrangements and partial differential equations encompasses a variety of topics by a wide range of authors. A small sampling of other works in this field includes [3], [4], [5], [6], and [25].

That one can allow the function $f$ to have variable sign while using a signed rearrangement of it so long as the condition $u \geq 0$ holds was perhaps first observed by Talenti. Earlier authors had assumed $f \geq 0$, apparently because this is a clean and easily checked condition from which $u \geq 0$ follows immediately via the maximum principle. At any rate, throughout this paper when we invoke the result (1.20) under the condition $u \geq 0$ while employing a signed rearrangement of $f$, we shall refer to it as Talenti’s theorem.

To show (1.20), one applies a fairly standard procedure to (1.18a) integrated over level sets of $u$. One then uses the classical isoperimetric inequality and arrives at an integral-differential inequality for $u^*$ in which $v'$ (for $v$ as given by (1.23)) appears on one side. Inequality (1.20) then follows by integration. For details, one might consult the papers of Talenti or Kesavan mentioned above, or books such as [24] or [60].

1.3 The Rayleigh-Ritz Inequality

Throughout this paper we shall have many occasions to use the Rayleigh-Ritz inequality, which gives a simple way to bound eigenvalues from above based on trial functions. For example, for the Dirichlet Laplacian on the bounded domain $\Omega$ one has

$$\lambda_1(\Omega) = \inf_{\begin{subarray}{c} \varphi \in D(-\Delta) \\ \varphi \neq 0 \end{subarray}} \frac{\int_\Omega \varphi(-\Delta \varphi)}{\int_\Omega \varphi^2}$$

(1.24)

where $\varphi$ is a real trial function in the domain of $-\Delta$ (denoted $D(-\Delta)$). One can also get at the higher eigenvalues by imposing orthogonality conditions.
on the class of trial functions used. For example,

\[
\lambda_{k+1}(\Omega) = \inf_{\varphi \in D(-\Delta) \setminus u_1, \ldots, u_k} \frac{\int_\Omega \varphi(-\Delta \varphi)}{\int_\Omega \varphi^2}
\]

where \(u_1, \ldots, u_k\) denote the first \(k\) eigenfunctions of \(-\Delta\) on \(\Omega\). Beyond this, and somewhat more useful for our purposes below, there is a quadratic form formulation of the Rayleigh-Ritz inequality. For the Dirichlet Laplacian it reads as follows:

\[
\lambda_{k+1}(\Omega) = \inf_{\varphi \in H^1_0(\Omega) \setminus u_1, \ldots, u_k} \frac{\int_\Omega |\nabla \varphi|^2}{\int_\Omega \varphi^2}
\]  

We note that the Sobolev space \(H^1_0(\Omega) = Q(-\Delta)\) = the form domain of \(-\Delta\) in this case (see, for example \([39], [58], \) and \([87]\)). This formulation has the advantage over (1.25) that the trial function \(\varphi\) can be chosen from a larger class of functions (and, in particular, it need have essentially only one square-integrable derivative, not two).

The Rayleigh-Ritz inequality (and the closely related Min-Max Principle) applies to any semi-bounded (from below) self-adjoint operator on a Hilbert space. For more details and discussion, the reader might consult Bande \([24]\), Bérard \([26]\), Chavel \([30], [31]\), Davies \([39]\), Kesavan \([58]\), and/or Reed and Simon, vol. 4 \([87]\).

## 2 Isoperimetric Inequalities for Eigenvalues

### 2.1 The Faber-Krahn Inequality

One of the earliest isoperimetric inequalities for an eigenvalue is certainly that for the first eigenvalue of the Dirichlet Laplacian (= fixed membrane problem) conjectured by Rayleigh \([50]\) in 1877:

\[
\lambda_1(\Omega) \geq \lambda_1(\Omega^*) \quad \text{for } \Omega \subset \mathbb{R}^n
\]

with equality if and only if \(\Omega\) is a ball, i.e., \(\Omega = \Omega^*\). This result was subsequently proved (independently) by Faber \([13]\) and Krahn \([33], [34]\) in the 1920’s using symmetrization. In terms of spherical symmetrization (=spherical rearrangement) the proof can now be reduced to a few lines. One uses the first eigenfunction \(u_1\) for \(\Omega\) and (1.12), (1.13) to conclude

\[
\lambda_1(\Omega) = \int_\Omega |\nabla u_1|^2 \\
\geq \int_{\Omega^*} |\nabla u_1^*|^2 \\
\geq \lambda_1(\Omega^*)
\]  

where the last line follows from the Rayleigh-Ritz inequality for \( \lambda_1 \) of \( -\Delta \) on \( \Omega^* \) and the fact (mentioned in connection with (1.13)) that \( u_1 \in H_0^1(\Omega) \) implies that \( u_1^* \in H_0^1(\Omega^*) \), and thus that \( u_1^* \) is a valid trial function in the Rayleigh-Ritz inequality for \( \lambda_1(\Omega^*) \). The characterization of the case of equality is somewhat technical, so we refer the reader to the literature for it. Good sources are Kawohl’s book [56], or Kesavan’s articles [59], [60].

2.2 The Szegő-Weinberger Inequality

We next turn to the isoperimetric result for \( \mu_1(\Omega) \), the first nonzero Neumann eigenvalue, due to Szegő [93] (for \( n = 2 \) and \( \Omega \) simply connected) and Weinberger [102] (in full generality). This is the next simplest (or even perhaps the simplest) isoperimetric result for eigenvalues. The result was first suggested by Kornhauser and Stakgold [62], who also obtained some results in support of it. The Szegő-Weinberger result states that

\[
\mu_1(\Omega) \leq \mu_1(\Omega^*) \quad \text{for } \Omega \subset \mathbb{R}^n,
\]

with equality if and only if \( \Omega \) is a ball.

We follow Weinberger’s method of proof, since it is very natural and lends itself to the problem we treat next, that of finding the optimal upper bound for \( \lambda_2/\lambda_1 \). First we recall that \( \mu_1(\Omega) \) may be characterized through the variational principle

\[
\mu_1(\Omega) = \min_{\varphi \in H^1(\Omega)} \frac{\int_\Omega |\nabla \varphi|^2}{\int_\Omega \varphi^2}.
\]

This is just the (quadratic form version of the) Rayleigh-Ritz inequality for \( \mu_1 \), since \( v_0 \) is a constant. Following Weinberger, we take as trial functions \( \varphi = P_i, i = 1, \ldots, n \), such that \( \int_\Omega P_i = 0 \) for \( i = 1, \ldots, n \) (this is proved via a topological argument, but given (2.5) below all it says is that we can choose to place our origin of coordinates at an appropriate generalized center of mass) with

\[
P_i(x) = g(r) \frac{x_i}{r},
\]

where the \( x_i \)'s are Cartesian coordinates, \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n, r = |x| \), and

\[
g(r) = \begin{cases} 
w(r) = \text{“right” radial function for a ball } B_R \text{ of radius } R & \text{for } 0 \leq r \leq R, \\
w(R) & \text{for } r \geq R.
\end{cases}
\]

where \( B_R = \Omega^* \).
The function \(w\) arises as a solution of the radial equation when one separates variables on the ball \(B_R\) and is therefore basically a Bessel function. We use only the facts that \(w(0) = 0\) and that \(w\) satisfies
\[
 w'' + \frac{n-1}{r}w' - \frac{n-1}{r^2}w + \mu_1(B_R)w = 0 \quad \text{for} \quad 0 < r < R. \tag{2.7}
\]
Since \(w'(R) = 0\) and \(\mu_1(B_R)\) is defined as the first eigenvalue of this boundary value problem, it follows (by an appropriate choice of sign) that \(w(r)\) is increasing on \([0, R]\) and hence that \(g\) is everywhere nondecreasing for \(r \geq 0\).

By substituting our trial functions \(P_i\) into the Rayleigh-Ritz inequality for \(\mu_1\), we find
\[
 \mu_1(\Omega) \int_\Omega P_i^2 \leq \int_\Omega |\nabla P_i|^2. \tag{2.8}
\]
Summing this in \(i\) for \(1 \leq i \leq n\), we arrive at
\[
 \mu_1(\Omega) \leq \frac{\int_\Omega \sum_{i=1}^n |\nabla P_i|^2}{\int_\Omega \sum_{i=1}^n P_i^2} = \frac{\int_\Omega \left[ g'(r)^2 + \frac{n-1}{r^2}g(r)^2 \right]}{\int_\Omega g(r)^2} \tag{2.9}
\]
where we have defined
\[
 B(r) \equiv g'(r)^2 + \frac{n-1}{r^2}g(r)^2. \tag{2.10}
\]
Now \(B(r)\) is easily seen to be decreasing for \(0 \leq r \leq R\) by differentiating and using the differential equation (2.7). One finds
\[
 B'(r) = -2[\mu_1(B_R)gg' + (n-1)(rg' - g)^2/r^3] < 0 \quad \text{for} \quad 0 < r < R. \tag{2.11}
\]
In addition, \(B(r) = (n-1)w(R)^2/r^2\) for \(r \geq R\) shows that \(B\) is decreasing for \(r \geq R\). Since \(B\) is continuous for all \(r \geq 0\), it is clearly also decreasing there. Now one has only to observe that
\[
 \int_\Omega B(r) \leq \int_{\Omega^*} B(r) \tag{2.12}
\]
since the volumes integrated over are the same in both cases, while in passing from the left- to right-hand sides you are exchanging integrating over \(\Omega \backslash \Omega^*\) for integrating over \(\Omega^* \backslash \Omega\) (which are sets of equal volume). Since \(B\) is (strictly) decreasing this clearly increases the value of the integral unless \(\Omega = \Omega^*\), when equality obtains. Similarly we find that
\[
 \int_\Omega g(r)^2 \geq \int_{\Omega^*} g(r)^2 \tag{2.13}
\]
since $g$ is nondecreasing. Thus we arrive at

$$\mu_1(\Omega) \leq \int_{\Omega^*} B(r) \int_{\Omega^*} g(r)^2 = \mu_1(\Omega^*),$$

(2.14)

since each $P_i$ is precisely an eigenfunction of $-\Delta$ with eigenvalue $\mu_1(B_R)$ for the domain $B_R = \Omega^*$. This completes the proof of the Szegő-Weinberger inequality, including the characterization of the case of equality.

### 2.3 The Payne-Pólya-Weinberger Conjecture

The next isoperimetric result that we consider is that for $\lambda_2/\lambda_1$ for the fixed membrane problem. In 1955 and 1956, Payne, Pólya, and Weinberger [82], [83] proved that

$$\frac{\lambda_2}{\lambda_1} \leq 3 \quad \text{for} \quad \Omega \subset \mathbb{R}^2$$

(2.15)

and conjectured that

$$\frac{\lambda_2}{\lambda_1} \leq \left. \frac{\lambda_2}{\lambda_1} \right|_{\text{disk}} = \frac{j_{2,1}^2}{j_{0,1}^2} \approx 2.5387$$

(2.16)

with equality if and only if $\Omega$ is a disk (i.e., $\Omega = \Omega^*$) and where $j_{p,k}$ denotes the $k^{th}$ positive zero of the Bessel function $J_p(t)$ (we follow the notation of Abramowitz and Stegun [4] here). For general dimension $n \geq 2$ the analogous statements are

$$\frac{\lambda_2}{\lambda_1} \leq 1 + \frac{4}{n} \quad \text{for} \quad \Omega \in \mathbb{R}^n,$$

(2.17)

and the PPW conjecture

$$\frac{\lambda_2}{\lambda_1} \leq \left. \frac{\lambda_2}{\lambda_1} \right|_{\text{n-ball}} = \frac{j_{n/2,1}^2}{j_{n/2-1,1}^2},$$

(2.18)

with equality if and only if $\Omega$ is an $n$-ball. This PPW conjecture was proved in 1990 by Rafael Benguria and the author (see [10], [11], [13]).

We proceed now with the proof of (2.18). This proof follows the main outline of Weinberger’s proof of the $\mu_1$ result given previously, but it is substantially more complicated due mainly to the fact that, unlike $v_0 = 1/\sqrt{|\Omega|}$, $u_1$ is not constant. We start from the variational principle for $\lambda_2$

$$\lambda_2(\Omega) = \min_{\varphi \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla \varphi|^2}{\int_{\Omega} \varphi^2},$$

(2.19)
or, better for our purposes here,

\[ \lambda_2(\Omega) - \lambda_1(\Omega) \leq \min_{P u_1 \in H^1_0(\Omega)} \frac{\int_{\Omega} |\nabla P|^2 u_1^2}{\int_{\Omega} P^2 u_1^2}, \]  

(2.20)

which follows from (2.19) via integration by parts. (Note that with \( \varphi = P u_1 \),

\[ \int_{\Omega} |\nabla \varphi|^2 = \int_{\Omega} [|\nabla P|^2 u_1^2 + 2P u_1 \nabla P \cdot \nabla u_1 + P^2 |\nabla u_1|^2] \]

\[ = \int_{\Omega} |\nabla P|^2 u_1^2 + \int_{\Omega} [\nabla(P^2 u_1)] \cdot \nabla u_1 \]

\[ = \int_{\Omega} |\nabla P|^2 u_1^2 + \int_{\Omega} P^2 u_1 (-\Delta u_1) \]

\[ = \int_{\Omega} |\nabla P|^2 u_1^2 + \lambda_1(\Omega) \int_{\Omega} P^2 u_1^2, \]

where we integrated by parts in the second-to-last step.) In (2.20) we shall use

\( n \) trial functions \( P = P_i, i = 1, \ldots, n \), such that \( \int_{\Omega} P_i^2 u_1 = 0 \) for \( i = 1, \ldots, n \) (again proved by a topological argument and interpretable as a generalized center of mass result) where

\[ P_i = g(r) \frac{x_i}{r} \]

(2.21)

and

\[ g(r) = \begin{cases} 
  w(r) = \text{“right” radial function for a ball } B_R & \text{for } 0 \leq r \leq R, \\
  w(R) & \text{for } r \geq R.
\end{cases} \]

(2.22)

The right \( R \) in this case turns out to be the unique \( R \) such that \( \lambda_1(B_R) = \lambda_1(\Omega) \). This is a key fact and explains why at bottom our proof of the PPW conjecture is a “fixed-\( \lambda_1 \) result”. A major motivation for this is the comparison result of Chiti, to be explained shortly. The right function \( w(r) \) for \( 0 \leq r \leq R \) is fairly complicated, being a ratio of Bessel functions.

From here we can proceed much as before finding

\[ \lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_{\Omega} B(r) u_1^2}{\int_{\Omega} g(r)^2 u_1^2} \]  

(2.23)

where

\[ B(r) \equiv g'(r)^2 + \frac{n - 1}{r^2} g(r)^2. \]  

(2.24)
Again it can be confirmed (though it is harder this time around) that $B$ is decreasing (and positive) and $g$ is increasing (and positive). For the proof of these facts, see [10], [11], [13], or [16]. The proof found in [10] (for $n = 2$ and 3) and [11] (for all $n \geq 2$) is based on the product representation for the Bessel functions involved and ultimately comes down to certain inequalities between Bessel function zeros (see also Section 4 of [13] for a brief further discussion of these issues). On the other hand, the proof found in [13] (see also [16]) is somewhat simpler and lends itself to generalization to a version of the PPW conjecture for domains in $S^n$. These topics are dealt with further in [20], [21], and also to some extent below.

To continue from (2.23) we use rearrangement as follows:

$$
\lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_{\Omega^*} B(r)^* u_1^2}{\int_{\Omega^*} g(r)^2 u_1^2} \leq \frac{\int_{\Omega^*} B(r) u_1^2}{\int_{\Omega^*} g(r)^2 u_1^2}.
$$

(2.25)

The basic ideas here are that the integral of a product is increased if we similarly rearrange both functions (see (1.14)), while it is decreased if we oppositely rearrange them (see (1.15)). Note that in (2.25) we similarly rearrange in the numerator, while we oppositely rearrange in the denominator. The removal of the $\star$’s from $g(r)^2$ and $B(r)$ in the last step is allowed due to their respective monotonicity properties.

Finally, we need to invoke a comparison result due to Chiti [34], [35], [36], [37] to replace the $u_1^*$’s in (2.24) by something more tractable. This result says that if we take $z(r)$ as the normalized first eigenfunction of $B_R$ (recall that $R$ is so chosen that $\lambda_1(\Omega) = \lambda_1(B_R)$) then $B_R \subset \Omega^*$ and $z$ will be larger than $u_1^*$ (thought of as a function of $r$) for $r$ near 0, say for $0 < r < r_1$, and smaller farther out, in this case for $r > r_1$. Under this condition one finds that if $u_1^*$ is replaced by $z$ in the final member of (2.24) the numerator can only go up and the denominator can only go down. Thus

$$
\lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_{B_R} B(r) z^2}{\int_{B_R} g(r)^2 z^2} = \lambda_2(B_R) - \lambda_1(B_R),
$$

(2.26)

the final equality holding because our choices of $g$ and $w$ were made precisely to make things come back together in this way. Since $\lambda_1(B_R) = \lambda_1(\Omega)$, (2.26) implies

$$
\lambda_2(\Omega) \leq \lambda_2(B_R)
$$

(2.27)

and hence also

$$
\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(B_R)}{\lambda_1(B_R)} = \frac{\lambda_2}{\lambda_1} \bigg|_{\text{any } n\text{-ball}} = \frac{\lambda_2(\Omega^*)}{\lambda_1(\Omega^*)}.
$$

(2.28)
We note that the PPW inequality (2.28) was really proved as a subsidiary result to (2.27), which should be regarded as a “fixed-λ₁ result”. It is this result that is in a certain sense more fundamental and which is most easily extended to other settings, for example, domains contained in a hemisphere of $S^n$.

Indeed, in subsequent work [16], [20], [21] we have established the fixed-λ₁ result (2.27) for domains contained in a hemisphere of $S^n$. In addition, from this result we can then derive the result

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(\Omega^*)}{\lambda_1(\Omega^*)}$$

for a domain $\Omega$ contained in a hemisphere of $S^n$. Here $\Omega^*$ denotes the geodesic ball (=polar cap) having the same volume as $\Omega$ and $B_R$ must be taken as the (unique) polar cap having the same $\lambda_1$ as $\Omega$. Furthermore, there is a corresponding result for $\mu_1$ (that is, $\mu_1(\Omega) \leq \mu_1(\Omega^*)$) for a domain $\Omega$ contained in a hemisphere of $S^n$ and for $\Omega$ a bounded domain in $H^n$. See [16], [17] for more on this aspect of the $\mu_1$ problem (in particular, the result for $H^n$ is due to Chavel, while both Chavel and Bandle had earlier variants of the $S^n$ result). The Faber-Krahn inequality also extends in sharp form to the spaces $S^n$ and $H^n$. See, for example, Sperner [21] or Friedland and Hayman [14]. In particular, Sperner’s bound $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$ for $\Omega \subset S^n$ is one of the elements necessary to our proof of (2.27) and (2.29) for domains contained in a hemisphere of $S^n$.

### 2.4 Rayleigh’s Conjecture for the Vibrating Clamped Plate

We turn now to Rayleigh’s conjecture for the vibration of a clamped plate. Rayleigh made this conjecture in 1877 in the first edition of his book *The Theory of Sound* [86] (see p. 382 of volume 1 of the second edition). In terms of the notation introduced in Section 1, Rayleigh’s conjecture states that

$$\Gamma_1(\Omega) \geq \Gamma_1(\Omega^*) \quad \text{for } \Omega \subset \mathbb{R}^2,$$

with equality if and only if $\Omega$ is itself a disk. It is natural to conjecture that this inequality might apply equally well in $\mathbb{R}^n$.

This conjecture seems to have lain dormant until around 1950, when Szegő made some progress on it (see [24], [14], and also the treatment in Pólya and Szegő’s book [44]). In fact, Szegő was able to prove (2.30) for simply connected domains $\Omega$ having a nonnegative first eigenfunction, $w_1 \geq 0$. At the time it seems to have been thought possible that $w_1 \geq 0$ for all domains.
However, it soon developed that this cannot be expected to hold in general. Results of Duffin [41] and Duffin and Shaffer [42] were enough to disabuse people of the notion that $w_1 \geq 0$ should always hold (see also the much more recent article of Kozlov, Kondrat’ev, and Maz’ya [65], as well as further references given in [22] and [23]).

The next advance toward the proof of Rayleigh’s conjecture (2.30) came in 1981 when Talenti [98] developed an approach to the sign-indeterminate case using separate rearrangements on the sets $\Omega_+ = \{ x \in \Omega | w_1 > 0 \}$ and $\Omega_- = \{ x \in \Omega | w < 0 \}$. This procedure led Talenti to two radial subproblems (tied to the two separate balls $(\Omega_+)^* \text{ and } (\Omega_-)^*$ and also to the full ball $\Omega^*$) which he decoupled and considered as variational problems in their own right. By this means, he obtained a lower bound to $\Gamma_1(\Omega)$ depending on the parameter $t \equiv |\Omega_+|/|\Omega|$ for $t \in [0,1]$. If this problem had been minimized at $t = 0$ (and therefore also at $t = 1$, by symmetry) then Talenti’s approach would have proved Rayleigh’s conjecture. But unfortunately Talenti’s minimum occurred at $t = 1/2$ (for each $n = 2, 3, \ldots$) and he was only able to obtain inequalities

$$\Gamma_1(\Omega) \geq d'_n \Gamma_1(\Omega^*) \quad \text{for } \Omega \subset \mathbb{R}^n. \quad (2.31)$$

These bounds fall short of Rayleigh’s conjecture, since the constants $d'_n$ are less than 1. However, Talenti was able to show that $d'_n \in (1/2, 1)$, and, in particular, he found $d'_2 \approx 0.9777$, $d'_3 \approx 0.7391$, $d'_4 \approx 0.6524$, \ldots In fact, the $d'_n$’s seem to decrease monotonically to 1/2 as $n$ goes to infinity (that $\lim_{n \to \infty} d'_n = 1/2$ is proved in [23]).

In 1992 Nadirashvili (see [74], [75], [76]) saw how to improve upon the approach of Talenti by using slightly different radial subproblems, each living only on $(\Omega_+)^* \text{ or } (\Omega_-)^*$, but coupled together via boundary conditions. With his approach and a geometric rearrangement argument, Nadirashvili was able to give a proof of (2.30) for $n = 2$. However, it was not clear that the same approach could handle cases with $n \geq 3$.

To that end, Rafael Benguria and the author introduced a more analytical variant of Nadirashvili’s approach, wherein the final analysis comes down to understanding a certain function defined explicitly in terms of Bessel functions. (Bessel functions, while certainly in the background in the clamped plate problem, were not in evidence in any part of Nadirashvili’s proof.) By studying the behavior of this function we were able to show [18] that for $n = 2$ and 3 the minimizer of our parametrized two-ball variational problem occurs when $t = |\Omega_+|/|\Omega|$ is 0 (or 1), thereby proving Rayleigh’s conjecture in dimensions 2 and 3. In subsequent work with Richard Laugesen [23] (see also [24]) it was shown that for all $n \geq 4$ things go the other way, i.e., the parametrized two-ball minimizer occurs at $t = 1/2$. This yields bounds of the
where our dimension-dependent constants \(d_n\) turn out always to be better than the corresponding \(d'_n\)'s of Talenti, and, in fact, go to 1 as \(n\) goes to infinity. For example, we have \(d_4 \approx 0.9537\), \(d_5 \approx 0.9218\), \(d_6 \approx 0.9077\), ... (\(d_8 \approx 0.8998\) appears to be the minimum over all dimensions \(n\)).

We now give the proof of Rayleigh’s conjecture (2.30) for dimensions 2 and 3. As a warm-up, we first treat the case where \(w_1 \geq 0\) on \(\Omega\). This is the case first successfully handled by Szegő [92], [94] (see also Pólya-Szego [84] and Talenti [98]). We recall the variational principle for \(\Gamma_1(\Omega) = \min_{\varphi \in H^1_0(\Omega)} \frac{\int_\Omega (\Delta \varphi)^2}{\int_\Omega \varphi^2}\) for \(\varphi \not\equiv 0\).

In particular, this gives equality if we take \(\varphi = w_1\). If we now decreasingly rearrange \(f = -\Delta w_1\) to \(f^\star = (-\Delta w_1)^\star\) where our rearrangement respects signs, we can invoke Talenti’s rearrangement result (1.20) for the problem \(-\Delta w_1 = f\) in \(\Omega\), \(w_1 = 0\) on \(\partial \Omega\), to get

\[
\Gamma_1(\Omega) = \frac{\int_\Omega (\Delta w_1)^2}{\int_\Omega w_1^2} \geq \frac{\int_{\Omega^\star} [(-\Delta w_1)^\star]^2}{\int_{\Omega^\star} v^2} = \frac{\int_{\Omega^\star} (\Delta v)^2}{\int_{\Omega^\star} v^2} \geq \Gamma_1(\Omega^\star),
\]

which is the result we want, assuming that \(v\) is truly an admissible trial function for the Rayleigh-Ritz inequality (= variational principle) for \(\Gamma_1\) on \(\Omega^\star\). To verify this, we note that \(v\) is a radial \(C^2\) function on the ball \(\Omega^\star\) and satisfies \(v(R) = 0\) where \(R\) is the radius of \(\Omega^\star\). To see that \(v\) also satisfies \(\frac{\partial v}{\partial \nu} = 0\) on \(\partial \Omega^\star\) we compute (since \(\frac{\partial w_1}{\partial \nu} = 0\) on \(\partial \Omega\))

\[
0 = \int_{\partial \Omega} \frac{\partial w_1}{\partial \nu} = -\int_{\Omega} (-\Delta w_1) = -\int_{\Omega} f^\star = -\int_{\Omega^\star} (-\Delta v) = \int_{\partial \Omega^\star} \frac{\partial v}{\partial \nu},
\]

and this shows that \(\frac{\partial v}{\partial \nu} = 0\) on \(\partial \Omega^\star\) since \(v\) is radial and therefore \(\int_{\partial \Omega^\star} \frac{\partial v}{\partial \nu} = |\partial \Omega^\star| v'(R) = nC_n R^{n-1} v'(R)\). This proves Szegő’s clamped plate result, i.e., Rayleigh’s conjecture for the clamped plate in the case of a first eigenfunction of fixed sign. This proof holds for all dimensions \(n\).

We now proceed to the proof of the general case for \(n = 2\) and 3. It turns out that the proof does not work for \(n \geq 4\) (though Rayleigh’s conjecture may well still be true there). As mentioned earlier, in the general case we proceed by decomposing the problem into two (coupled) subproblems, each on a separate ball (on \((\Omega_+)\) and \((\Omega_-)\) in the notation introduced above). Following this decomposition procedure, we have
\[
\Gamma_1(\Omega) = \frac{\int_\Omega (\Delta w_1)^2}{\int_\Omega w_1^2} = \frac{\int_{\Omega^+} (\Delta w_1)^2 + \int_{\Omega^-} (\Delta w_1)^2}{\int_{\Omega^+} w_1^2 + \int_{\Omega^-} w_1^2}
\geq \frac{\int_{B_a} (\Delta u)^2 + \int_{B_b} (\Delta v)^2}{\int_{B_a} u^2 + \int_{B_b} v^2}
\]

where \( B_a = \text{ball of radius } a = (\Omega_+)^* \) and \( B_b = \text{ball of radius } b = (\Omega_-)^* \) and \( u \) and \( v \) satisfy symmetrized Dirichlet problems on \( B_a \) and \( B_b \), respectively.

Note that the numerator in (2.36) is unchanged by the introduction of \( u \) and \( v \), while the denominator is (typically) increased (and certainly does not decrease) by virtue of inequality (1.20) as applied to \( u \) and \( v \) (and \((w_1^+)^*\) and \((w_1^-)^*\)), respectively. Now, not only do \( u \) and \( v \) satisfy
\[
u(a) = 0 = v(b)
\]

but \( u'(a) \) and \( v'(b) \) are coupled by the equation
\[
a^{n-1}u'(a) = b^{n-1}v'(b).
\]

This comes about by an argument similar to that in (2.35). We have
\[
0 = \int_{\partial \Omega} \frac{\partial w_1}{\partial n} - \int_\Omega \Delta w_1 = -\int_{B_a} (-\Delta w_1)^* + \int_{B_b} (\Delta w_1)^*
\]
\[
= \int_{B_a} \Delta u - \int_{B_b} \Delta v = \int_{\partial B_a} \frac{\partial u}{\partial n} - \int_{\partial B_b} \frac{\partial v}{\partial n}
\]
\[
= nC_n[a^{n-1}u'(a) - b^{n-1}v'(b)].
\]

The seemingly odd changes of sign in certain terms come about because we introduce a sign change for \( \Omega_- \) so that we will still be comparing positive functions (and thus \( u \) and \( v \) are both positive). As a curiosity we mention that in fact (2.39) (and also (2.37)) uses only the fact that \( \partial w_1 / \partial n \) has average value 0 on the boundary of \( \Omega \). Thus our results all hold under the weaker assumption that our plate has its edge fixed and “clamped on average”, a somewhat curious result first observed by Richard Laugesen.

To finish the argument we view the final member of (2.36) as a variational problem in its own right where we now treat \( u \) and \( v \) as trial functions subject to (2.37), (2.38), and the condition that they be radial functions on \( B_a \) and \( B_b \), respectively. Thus we consider the minimization problem
\[
J(a) \equiv \inf_{\varphi, \psi} \frac{\int_{B_a} (\Delta \varphi)^2 + \int_{B_b} (\Delta \psi)^2}{\int_{B_a} \varphi^2 + \int_{B_b} \psi^2}
\]

(2.40)
with \( \varphi \) and \( \psi \) radial and satisfying the boundary conditions (2.37) and (2.38) (with \( \varphi \) and \( \psi \) replacing \( u \) and \( v \) respectively). This is what we refer to as our parametrized two-ball variational problem. Note that \( a \) now appears as a final parameter (which we will eventually need to minimize over, since we have no control on it). We observe that, due to scaling freedom, we can assume that \( a^n + b^n = 1 \), which is equivalent to normalizing \( |\Omega| \) at \( C_n \). In (2.40), and henceforth, we regard \( b \) as defined implicitly in terms of \( a \) by this relation. We also remark that in terms of our earlier parameter \( t = |\Omega_+|/|\Omega| \) we have \( t = a^n \). In a sense \( t \) is the preferred variable here, since as a function of \( t \in [0,1] \) \( J \) is symmetric about \( 1/2 \). We compromise by speaking in terms of \( a^n \) at times as we proceed.

Leaving \( a \in (0,1) \) fixed for the time being, standard variational theory can be applied to (2.40) with the result that we find the Euler equations

\[
\begin{align*}
\Delta^2 \varphi &= \mu \varphi \quad \text{in } B_a, \quad (2.41a) \\
\Delta^2 \psi &= \mu \psi \quad \text{in } B_b, \quad (2.41b)
\end{align*}
\]

and boundary conditions

\[
\begin{align*}
\varphi(a) &= 0 = \psi(b), \quad (2.41c) \\
a^{n-1} \varphi'(a) &= b^{n-1} \psi'(b), \quad (2.41d)
\end{align*}
\]

and

\[
\Delta \varphi(a) + \Delta \psi(b) = 0. \quad (2.41e)
\]

The expressions in the last formula make sense because \( \varphi \) and \( \psi \) are radial. This formula comes about as a product of the variational theory, and is what is known as a “natural” boundary condition (see, for example, [46] or [104]). See [18] for more on its derivation and for more details on this material in general (see also [22] for a general overview of the topic).

The boundary value problem (2.41) can now be solved explicitly in terms of Bessel functions and modified Bessel functions. The result is an implicit relation for the eigenvalues (and in particular the first eigenvalue \( \mu_1 = J(a) \)) which we can analyze to determine how \( J(a) \) behaves for \( 0 \leq a \leq 1 \). Since

\[
\Gamma_1(\Omega) \geq \min_a J(a) \quad (2.42)
\]

and \( J(0) = J(0^+) = J(1) = J(1^-) = \Gamma(\Omega^*) \) (this requires some work to see, though it certainly should already appear plausible; essentially one wants to confirm that taking \( a = 0 \) gives the same result as the limit \( a \rightarrow 0^+ \) and similarly for \( a = 1 \)), we will be done if we can show that \( \min J(a) \) occurs at \( a = 0 \) (and \( a = 1 \)). For dimensions 2 and 3 this can be shown (see [18]) and
this completes the proof of the general Rayleigh conjecture for the clamped plate in these cases.

For \( n \geq 4 \), the analysis given above holds all the way until the final step, where it is found that \( \min_a J(a) \) occurs not at \( a = 0 \) and 1, but at \( a^n = 1/2 \). Following this outcome to its logical conclusion leads to the nonoptimal bounds (2.32). The detailed arguments and related material appear in [23].

### 2.5 The Pólya-Szegő Conjecture for the Buckling of a Clamped Plate

Much the same strategy as was used for the vibrating clamped plate can be used for the first eigenvalue \( \Lambda_1(\Omega) \) of the buckling problem. For \( n = 2 \), this eigenvalue determines the critical buckling load of the clamped plate of the shape of \( \Omega \) under uniform compressive loading around its boundary. The analog of the Rayleigh conjecture in this setting,

\[
\Lambda_1(\Omega) \geq \Lambda_1(\Omega^*) \quad \text{for } \Omega \subset \mathbb{R}^n,
\]

with equality if and only if \( \Omega \) is a ball, was conjectured by Pólya and Szegő around 1950. It is discussed in their book [84]. At around the same time Szegő proved this conjecture [92], [94] under the assumption that \( \Lambda_1(\Omega) \) has a first eigenfunction of fixed sign (which we can take to be nonnegative, i.e., \( v_1 \geq 0 \) on \( \Omega \)). Again this assumption turns out not to hold in all cases (see the article by Kozlov, Kondrat’ev, and Maz’ya [65]).

The proof of the fixed-sign case (we assume \( v_1 \geq 0 \)) can be accomplished much as was the corresponding case of the vibrating clamped plate above. One simply starts from the variational principle

\[
\Lambda_1(\Omega) = \min_{\varphi \in H^2_0(\Omega) \setminus \{0\}} \frac{\int_\Omega (\Delta \varphi)^2}{\int_\Omega |\nabla \varphi|^2} = \frac{\int_\Omega (\Delta v_1)^2}{\int_\Omega |\nabla v_1|^2}
\]

and proceeds as before. One rearranges \( -\Delta v_1 \) to obtain a symmetrized comparison problem. Since Talenti’s version of the comparison argument (for (1.18) and (1.19)) can also be used to compare derivatives (indeed, this can be done directly from the integral-differential inequality for \( u^* \)) we find easily that

\[
\Lambda_1(\Omega) \geq \frac{\int_{\Omega^*} (\Delta \varphi)^2}{\int_{\Omega^*} |\nabla \varphi|^2}
\]

where \( \varphi \) is now a radial function on \( \Omega^* \) satisfying \( \varphi(R) = 0 = \varphi'(R) \) (the proof that \( \varphi'(R) = 0 \) is exactly the same as that given for \( v \) in (2.35)). Since this means that \( \varphi \) is a valid trial function for \( \Lambda_1(\Omega^*) \), the proof of \( \Lambda_1(\Omega) \geq \Lambda_1(\Omega^*) \) is complete.
For the general case (where $v_1$ is not necessarily of fixed sign) we can also proceed as in the vibrating clamped plate problem. That is, we break $\Omega$ into the two parts $\Omega_+ \pm \Delta v_1$ on each part separately to obtain two symmetrized subproblems. The hitch again comes at the very last step, where the minimum over $a \in [0,1]$ turns out to occur at $a^n = 1/2$ for all dimensions $n \geq 2$. Thus we do not obtain the general result (2.43) for any $n \geq 2$, but we can again get nonoptimal lower bounds for $\Lambda_1(\Omega)$ in the form

$$\Lambda_1(\Omega) > c_n \Lambda_1(\Omega^*) \quad \text{for } \Omega \subset \mathbb{R}^n,$$

where the dimension-dependent constants $c_n$ are found to tend to 1 as $n$ goes to infinity. Some values of $c_n$ for low dimensions are $c_2 \approx 0.7877$, $c_3 \approx 0.7759$, $c_4 \approx 0.7872$, $c_5 \approx 0.8020$, $c_6 \approx 0.8163$. These results and related material can be found in [23] (see also [22]).

It is a rather remarkable fact that the bounds (2.46) with precisely the same constants $c_n$ can also be obtained by combining two classical eigenvalue inequalities. These are the inequality of Payne [78]

$$\Lambda_1(\Omega) \geq \lambda_2(\Omega) \quad \text{for } \Omega \subset \mathbb{R}^n,$$

with equality if and only if $\Omega$ is a ball, and Krahn’s $\lambda_2$ inequality [41]

$$\lambda_2(\Omega) > 2^{2/n} \lambda_1(\Omega^*) \quad \text{for } \Omega \subset \mathbb{R}^n$$

(this saturates as $\Omega$ disconnects into two equal disjoint balls). Together these yield

$$\Lambda_1(\Omega) > 2^{2/n} \lambda_1(\Omega^*) = 2^{2/n} (C_n/|\Omega|)^{2/n} j_{n/2-1,1}^2$$

$$= 2^{2/n} \left( \frac{j_{n/2-1,1}}{j_{n/2,1}} \right)^2 \left( \frac{C_n}{|\Omega|} \right)^{2/n} j_{n/2,1}^2 \quad \text{(2.49)}$$

since $\Lambda_1(\Omega^*) = (C_n/|\Omega|)^{2/n} j_{n/2,1}^2$ and hence the constants $c_n$ are given by

$$c_n = 2^{2/n} \left( \frac{j_{n/2-1,1}}{j_{n/2,1}} \right)^2. \quad \text{(2.50)}$$

The basic observation here was made by Bramble and Payne [27], though they only gave the inequality for $n = 2$ and hence did not investigate the behavior of the $c_n$’s with varying $n$. Neither did they mention the connection between (2.49) and the conjecture (2.43) of Pólya and Szegö. For further comments and details, one should consult [23] and/or [22].
3 Universal Inequalities for Eigenvalues

3.1 The General Inequalities of Payne-Pólya-Weinberger for the Fixed Membrane Eigenvalues and their Extensions

In this section we turn our attention to universal eigenvalue inequalities. We begin with a study of the Dirichlet eigenvalues of $-\Delta$ on a bounded domain $\Omega \subset \mathbb{R}^n$. This subject began in 1955 with the work of Payne, Pólya, and Weinberger [82], [83], who proved (among other things) the bound

$$\lambda_{m+1} - \lambda_m \leq \frac{2}{m} \sum_{i=1}^{m} \lambda_i, \quad m = 1, 2, \ldots$$

(3.1)

for $\Omega \subset \mathbb{R}^2$. This result easily extends to $\Omega \subset \mathbb{R}^n$ as

$$\lambda_{m+1} - \lambda_m \leq \frac{4}{mn} \sum_{i=1}^{m} \lambda_i, \quad m = 1, 2, \ldots$$

(3.2)

Since Payne, Pólya, and Weinberger’s paper [83], (3.2) has been extended in several ways by a number of authors. For the Euclidean case there have been two main advances: that of Hile and Protter [53] in 1980 and that of H. C. Yang [105] in 1991 (this paper is yet to be published, as far as this author knows). It turns out that, even though the proofs given by Hile-Protter and Yang are dissimilar in several respects and are rather involved, the main approach is still that of Payne-Pólya-Weinberger. Moreover, we have succeeded in streamlining these proofs to the point where, with the addition of one key idea, they can be unified into a single overall approach that is no harder than the original proof of (3.1) by Payne, Pólya, and Weinberger. The process of reducing these proofs to their essence was begun in [16], [19] and is concluded in [9]. Since our approach is unified we head directly for the result of H. C. Yang, remarking only at the end on how the results of PPW and Hile-Protter follow “by simplification”.

To give an idea of Payne, Pólya, and Weinberger’s basic method, we begin by proving their bound

$$\frac{\lambda_2}{\lambda_1} \leq 1 + \frac{4}{n} \quad \text{for } \Omega \subset \mathbb{R}^n$$

(3.3)

as a warm-up exercise (note that this is the $m = 1$ case of (3.2)). The $n$-dimensional inequality (3.3) was first given explicitly by Thompson [101] in 1969, but certainly it (and more) is implicit in the work of Payne, Pólya,
and Weinberger, as the generalization of their results to \( \Omega \subset \mathbb{R}^n \) is entirely straightforward. To prove (3.3), we introduce the trial function

\[
\varphi = xu_1
\]

(3.4)

where \( x \) represents any Cartesian coordinate \( x_\ell (1 \leq \ell \leq n) \) for \( \mathbb{R}^n \). By an appropriate choice of origin we can arrange that \( \int_\Omega x_\ell u_1^2 = 0 \) for \( 1 \leq \ell \leq n \) (i.e., we put the origin at the center of mass of \( \Omega \) for the mass distribution defined by \( u_1^2 \)). This guarantees \( \varphi \perp u_1 \) (also \( \varphi^{(\ell)} \perp u_1 \) for \( 1 \leq \ell \leq n \) where \( \varphi^{(\ell)} \equiv x_\ell u_1 \)) and hence, by the Rayleigh-Ritz inequality (for \( \lambda_2 \)),

\[
\lambda_2 \leq \frac{\int_{\Omega} \varphi (-\Delta \varphi)}{\int_{\Omega} \varphi^2}.
\]

(3.5)

Since

\[
-\Delta \varphi = x(-\Delta u_1) - 2u_{1x} = \lambda_1 xu_1 - 2u_{1x} = \lambda_1 \varphi - 2u_{1x},
\]

(3.6)

(3.5) gives

\[
\lambda_2 - \lambda_1 \leq -\frac{2\int_{\Omega} \varphi u_{1x}}{\int_{\Omega} \varphi^2}.
\]

(3.7)

Now

\[
0 \leq -2 \int_{\Omega} \varphi u_{1x} = -2 \int_{\Omega} xu_1 u_{1x} = - \int_{\Omega} x(u_1^2)_x = \int_{\Omega} u_1^2 = 1
\]

(3.8)

where in the second-to-last step we integrated by parts, and also by the Cauchy-Schwarz inequality,

\[
(-2 \int_{\Omega} \varphi u_{1x})^2 \leq 4(\int_{\Omega} \varphi^2)(\int_{\Omega} u_{1x}^2),
\]

(3.9)

implying (since \(-2 \int_{\Omega} \varphi u_{1x} > 0\))

\[
\lambda_2 - \lambda_1 \leq -\frac{2\int_{\Omega} \varphi u_{1x}}{\int_{\Omega} \varphi^2} \leq \frac{4\int_{\Omega} u_{1x}^2}{-2\int_{\Omega} \varphi u_{1x}} = 4 \int_{\Omega} u_{1x}^2.
\]

(3.10)

Obviously this same argument applies to \( \varphi^{(\ell)} \equiv x_\ell u_1 \) for \( 1 \leq \ell \leq n \), allowing us to arrive at

\[
\lambda_2 - \lambda_1 \leq 4 \int_{\Omega} u_{1x\ell}^2 \text{ for } 1 \leq \ell \leq n.
\]

(3.11)

If we now average these inequalities over \( \ell \) we find

\[
\lambda_2 - \lambda_1 \leq \frac{4}{n} \int_{\Omega} |\nabla u_1|^2 = \frac{4}{n} \lambda_1
\]

(3.12)
and hence (3.3) follows.

Next we address the general results for $\lambda_{m+1}$. The basic strategy is the same as above in that we base our trial functions $\varphi$ on $xu_i$ where $x$ is a Cartesian coordinate and $u_i$ is a lower eigenfunction (i.e., $1 \leq i \leq m$), but now to enforce orthogonality to $u_1, \ldots, u_m$ we no longer rely on the device of locating the origin at the center of mass but rather we subtract away counterterms which are just the projections of $xu_i$ along the eigenfunctions $u_j$ for $1 \leq j \leq m$ (when $m = 1$, this amounts to shifting to the center of mass as before). Thus as our trial functions we take

$$\varphi_i = xu_i - \sum_{j=1}^{m} a_{ij} u_j \quad \text{for } 1 \leq i \leq m$$

(3.13)

where

$$a_{ij} \equiv \int_{\Omega} xu_i u_j = a_{ji}$$

(3.14)

are the components of $xu_i$ along $u_j$ for $1 \leq j \leq m$ and thus clearly $\varphi_i \perp u_j$ for $1 \leq i, j \leq m$. Also, it is straightforward to compute

$$\int_{\Omega} \varphi_i^2 = \int_{\Omega} \varphi_i xu_i = \int_{\Omega} x^2 u_i^2 - \sum_{j=1}^{m} a_{ij}^2$$

(3.15)

and

$$-\Delta \varphi_i = \lambda_i xu_i - 2u_ix - \sum_{j=1}^{m} a_{ij} \lambda_j u_j$$

(3.16)

so that (since $\varphi_i \perp u_j$ for $1 \leq j \leq m$)

$$\int_{\Omega} \varphi_i (-\Delta \varphi_i) = \lambda_i \int_{\Omega} \varphi_i^2 - 2 \int_{\Omega} \varphi_i u_ix.$$ 

(3.17)

It therefore follows from the Rayleigh-Ritz inequality for $\lambda_{m+1}$ that

$$(\lambda_{m+1} - \lambda_i) \int_{\Omega} \varphi_i^2 \leq -2 \int_{\Omega} \varphi_i u_ix.$$ 

(3.18)

Proceeding much as before, we find

$$0 \leq -2 \int_{\Omega} \varphi_i u_ix = -2 \int_{\Omega} u_ix \left[ xu_i - \sum_{j=1}^{m} a_{ij} u_j \right]$$

$$= - \int_{\Omega} x(u_i^2)_x + 2 \sum_{j=1}^{m} a_{ij} \int_{\Omega} u_i u_j$$

$$= 1 + 2 \sum_{i=1}^{m} a_{ij} b_{ij}$$

(3.19)
where we have defined $b_{ij}$ by

$$b_{ij} = \int_{\Omega} u_{ix} u_j. \quad (3.20)$$

At this point, we can both recognize the $b_{ij}$’s as the components of the $u_{ix}$’s along the $u_j$’s and compute them explicitly in terms of the $a_{ij}$’s. It turns out that these observations are both important for us. (Also, $b_{ij}$ is antisymmetric in $i$ and $j$, as can be recognized from (3.20) immediately via integration by parts.) We first relate $b_{ij}$ to $a_{ij}$:

$$\lambda_i a_{ij} = \int_{\Omega} (-\Delta u_i) x u_j$$

$$= \int_{\Omega} u_i[-\Delta (x u_j)]$$

$$= \int_{\Omega} u_i[\lambda_j x u_j - 2u_{jx}] \quad (3.21)$$

$$= \lambda_j a_{ij} + 2 \int_{\Omega} u_{ix} u_j$$

$$= \lambda_j a_{ij} + 2b_{ij}$$

or

$$2b_{ij} = (\lambda_i - \lambda_j)a_{ij}. \quad (3.22)$$

Thus, from (3.19),

$$0 \leq -2 \int_{\Omega} \varphi_i u_{ix} = 1 + \sum_{j=1}^{m} (\lambda_i - \lambda_j)a^2_{ij}. \quad (3.23)$$

Furthermore, by the Cauchy-Schwarz inequality,

$$(-2 \int_{\Omega} \varphi_i u_{ix})^2 = (-2 \int_{\Omega} \varphi_i [u_{ix} - \sum_{j=1}^{m} b_{ij} u_j])^2$$

$$\leq 4(\int_{\Omega} \varphi_i^2) (\int_{\Omega} [u_{ix} - \sum_{j=1}^{m} b_{ij} u_j]^2) \quad (3.24)$$

$$= 4(\int_{\Omega} \varphi_i^2) [\int_{\Omega} u^2_{ix} - \sum_{j=1}^{m} b^2_{ij}].$$

From (3.18) we now find

$$(\lambda_{m+1} - \lambda_i)(\int_{\Omega} \varphi_i^2)(-2 \int_{\Omega} \varphi_i u_{ix}) \leq (-2 \int_{\Omega} \varphi_i u_{ix})^2$$

$$\leq 4(\int_{\Omega} \varphi_i^2) [\int_{\Omega} u^2_{ix} - \sum_{j=1}^{m} b^2_{ij}] \quad (3.25)$$
and hence, dividing by $\int_\Omega \varphi_i^2$,

$$(\lambda_{m+1} - \lambda_i)(-2 \int_\Omega \varphi_i u_{ix}) \leq 4\left[ \int_\Omega u_{ix}^2 - \sum_{j=1}^m b_{ij}^2 \right].$$  \hspace{1cm} (3.26)

Inequality (3.26) holds even in the (unlikely) event that $\varphi_i \equiv 0$, since in that case clearly its left-hand side is 0 while its right-hand side is nonnegative (it is 4 times the square of the norm of $u_{ix} - \sum_{j=1}^m b_{ij} u_j$). Now, using (3.22) and (3.23), (3.26) can be put in the form

$$(\lambda_{m+1} - \lambda_i)[1 + \sum_{j=1}^m (\lambda_i - \lambda_j) a_{ij}^2] \leq 4 \int_\Omega u_{ix}^2 - \sum_{j=1}^m (\lambda_i - \lambda_j)^2 a_{ij}^2$$  \hspace{1cm} (3.27)

or

$$\lambda_{m+1} - \lambda_i + \sum_{j=1}^m (\lambda_{m+1} - \lambda_j)(\lambda_i - \lambda_j) a_{ij}^2 \leq 4 \int_\Omega u_{ix}^2.$$  \hspace{1cm} (3.28)

From here it is clear how to finish our argument: we want to introduce a factor of $(\lambda_{m+1} - \lambda_i)$ to make the term involving $a_{ij}^2$ antisymmetric in $i$ and $j$, and then sum on $i$ from 1 to $m$ to make this term drop out. We thus arrive at

$$\sum_{i=1}^m (\lambda_{m+1} - \lambda_i)^2 \leq 4 \sum_{i=1}^m (\lambda_{m+1} - \lambda_i) \int_\Omega u_{ix}^2.$$  \hspace{1cm} (3.29)

If we now recall that $x$ here could be any $x_\ell (1 \leq \ell \leq n)$ and sum over $\ell$ from 1 to $n$ we obtain

$$n \sum_{i=1}^m (\lambda_{m+1} - \lambda_i)^2 \leq 4 \sum_{i=1}^m \lambda_i (\lambda_{m+1} - \lambda_i)$$  \hspace{1cm} (3.30)

or

$$\sum_{i=1}^m (\lambda_{m+1} - \lambda_i) \left( \lambda_{m+1} - \left(1 + \frac{4}{n}\right) \lambda_i \right) \leq 0,$$  \hspace{1cm} (3.31)

which is the main (or “first”) inequality of Hong Cang Yang \[105\]. This completes our derivation of Hong Cang Yang’s first inequality.

We now make a variety of comments about this inequality and the inequalities of Payne-Pólya-Weinberger and Hile-Protter. First, noting that the left-hand side of (3.31) is a quadratic in $\lambda_{m+1}$, we write it as

$$m \lambda_{m+1}^2 - 2 \left(1 + \frac{2}{n}\right) \left(\sum_{i=1}^m \lambda_i\right) \lambda_{m+1} + \left(1 + \frac{4}{n}\right) \sum_{i=1}^m \lambda_i^2 \leq 0$$  \hspace{1cm} (3.32)
and derive the explicit upper bound

\[ \lambda_{m+1} \leq \left[ \text{larger root of the quadratic} \right] \]

\[
= \frac{1}{m} \left[ \left( 1 + \frac{2}{n} \right) \sum_{i=1}^{m} \lambda_i + \left\{ \left( 1 + \frac{2}{n} \right)^2 \left( \sum_{i=1}^{m} \lambda_i \right)^2 - m \left( 1 + \frac{4}{n} \right) \sum_{i=1}^{m} \lambda_i^2 \right\}^{1/2} \right].
\] (3.33)

This bound is the best general upper bound yet derived by the methods of Payne, Pólya, and Weinberger (or any other methods, for that matter). As was already observed by H. C. Yang [105], the bound (3.33) is much sharper than previously known bounds for large \( m \), since it comes much closer to incorporating the Weyl asymptotic behavior of the eigenvalues \( \lambda_i \). For further comments and observations in this regard, see [52], [19], [9].

A simpler inequality due to H. C. Yang, **Yang’s second inequality**, follows readily if we use the Cauchy-Schwarz inequality to replace \( m \sum_{i=1}^{m} \lambda_i^2 \) by \( \left( \sum_{i=1}^{m} \lambda_i \right)^2 \) on the right-hand side of (3.33):

\[
\lambda_{m+1} \leq \left( 1 + \frac{4}{n} \right) \frac{1}{m} \sum_{i=1}^{m} \lambda_i \quad \text{for} \quad m = 1, 2, \ldots.
\] (3.34)

This weaker inequality already implies the **Payne-Pólya-Weinberger inequality**, (3.2), since we can obtain (3.34) from (3.2) by replacing the \( \lambda_m \) occurring on the left-hand side of (3.2) by the average \( \frac{1}{m} \sum_{i=1}^{m} \lambda_i \), which is clearly less than \( \lambda_m \) (and is, in fact, strictly less for all \( m \geq 2 \) since \( \lambda_1 < \lambda_i \) for all \( i \geq 2 \)).

To obtain **Hile and Protter’s inequality** we go back into our proof of H. C. Yang’s inequality and make one modification. In inequality (3.24), where we made use of the Cauchy-Schwarz inequality, we use it, but without incorporating the counterterms involving \( b_{ij} \)’s, arriving at

\[
(-2 \int_{\Omega} \varphi_i u_{ix})^2 \leq 4 \left( \int_{\Omega} \varphi_i^2 \right) \left( \int_{\Omega} u_{ix}^2 \right).
\] (3.35)

This is how everyone had proceeded prior to H. C. Yang. No one had realized previously that one could make better (=“optimal”) use of the Cauchy-Schwarz inequality in this way (taking advantage of the known orthogonalities \( \varphi_i \perp u_j \) for \( 1 \leq i, j \leq m \) and the fact that the \( b_{ij} \)’s have a simple relation to the \( a_{ij} \)’s). If our argument above is now carried through with (3.33) replacing (3.24), we arrive at

\[
(\lambda_{m+1} - \lambda_i) \left[ 1 + \sum_{j=1}^{m} (\lambda_i - \lambda_j) a_{ij}^2 \right] \leq 4 \int_{\Omega} u_{ix}^2 \] (3.36)
3 UNIVERSAL INEQUALITIES FOR EIGENVALUES

From here the clear thing to do to eliminate the unwanted (because they’re not easily controlled) terms in the $a_{ij}^2$’s is to divide through by $\lambda_{m+1} - \lambda_i$ and sum on $i$ from 1 to $m$, yielding

$$\sum_{i=1}^{m} \frac{\int_{\Omega} u_{ix}^2}{\lambda_{m+1} - \lambda_i} \geq \frac{m}{4}. \quad (3.37)$$

All the terms in $a_{ij}^2$ have dropped out due to antisymmetry. Finally, recalling that $x$ could be any $x_\ell$, $1 \leq \ell \leq n$, and summing on $\ell$ we arrive at

$$\sum_{i=1}^{m} \frac{\lambda_i}{\lambda_{m+1} - \lambda_i} \geq \frac{mn}{4}, \quad (3.38)$$

which is the Hile-Protter inequality. This inequality is stronger than the Payne-Pólya-Weinberger inequality (3.2), since if we replace the $\lambda_i$ appearing in the denominator of the left-hand side of (3.38) by $\lambda_m$ we obtain (3.2). It is also weaker than either of Yang’s inequalities, (3.33) or (3.34). It therefore follows that (for each $m = 1, 2, \ldots$)

$$\text{Yang 1} \Rightarrow \text{Yang 2} \Rightarrow \text{Hile-Protter} \Rightarrow \text{PPW}. \quad (3.39)$$

While our derivation above of the Hile-Protter inequality certainly suggests that Yang’s inequalities are stronger, it is not altogether straightforward to show the middle implication in (3.39). This is done in our longer paper [4], which is devoted to the topic of universal inequalities for the eigenvalues of the Dirichlet Laplacian. That paper also contains a discussion of when the various inequalities given above are known to hold strictly, as well as various other extensions and generalizations. A more complete set of references and some further comments on them will be found there as well.

For anyone worried by our statement following inequality (3.26) admitting the possibility that our trial functions $\varphi_i$ might vanish identically and hence worried that our derivation might result in a triviality, we make two remarks to allay such fears. The first is that our proof as given takes account of these matters, and does indeed lead to a nontrivial inequality. The second, and probably more useful, observation is that it can never happen that all the $\varphi_i$’s vanish identically (if indeed even one of them can vanish). To see this directly, it suffices to consider (3.23) and sum it on $i$ from 1 to $m$. By antisymmetry the sum in $a_{ij}^2$ drops out, yielding $m = -2 \sum_{i=1}^{m} \int_{\Omega} \varphi_i u_{ix}$ and showing that not all the $\varphi_i$’s can vanish identically (or even each be orthogonal to $u_{ix}$). Thus at least some of the inequalities represented by (3.26) will be nontrivial and will lead to a nontrivial inequality (3.30) as the outcome of our final summations.
3.2 Universal Inequalities for Low Fixed Membrane Eigenvalues

We next turn to some universal inequalities for low Dirichlet eigenvalues, which again stem from the original work of Payne, Pólya, and Weinberger [83]. In 1956 they proved that for $\Omega \subset \mathbb{R}^2$

$$\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 6.$$  \hfill (3.40)

This easily extends to $\Omega \subset \mathbb{R}^n$ as

$$\frac{\lambda_2 + \lambda_3 + \ldots + \lambda_{n+1}}{\lambda_1} \leq n \left(1 + \frac{4}{n}\right) = n + 4.$$ \hfill (3.41)

These inequalities are proved by using the Rayleigh-Ritz inequality in “trace form” or as usually applied, using rotations and translations to enforce the further orthogonalities needed. No $a_{ij}$’s appear (or can be tolerated) in these arguments. There are also a variety of extensions of results of this type, the simplest being that of Brands [28] from 1964

$$\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 5 + \frac{\lambda_1}{\lambda_2} \quad \text{for } \Omega \subset \mathbb{R}^2$$ \hfill (3.42)

and its extension to $\mathbb{R}^n$

$$\frac{\lambda_2 + \lambda_3 + \ldots + \lambda_{n+1}}{\lambda_1} \leq n + 3 + \frac{\lambda_1}{\lambda_2} \quad \text{for } \Omega \subset \mathbb{R}^n$$ \hfill (3.43)

(see [12] for a derivation, though this extension was certainly known to Hile and Protter earlier [53]).

Beyond this, much work has been done toward bounding the range of values of $(\lambda_2/\lambda_1, \lambda_3/\lambda_1)$ for an arbitrary domain $\Omega \subset \mathbb{R}^2$ (with corresponding, but less extensive, work for $\Omega \subset \mathbb{R}^n$). In particular, it would be very interesting to know the best bounds for $\lambda_3/\lambda_1$ and $(\lambda_2 + \lambda_3)/\lambda_1$ (the best bound for $\lambda_2/\lambda_1$ is, of course, its value for a disk, given by (2.14)). These quantities can be gotten at by looking at the range of possible values of $(\lambda_2/\lambda_1, \lambda_3/\lambda_1)$, which is one motivation for its study.

The current state of knowledge regarding the range of values of $(\lambda_2/\lambda_1, \lambda_3/\lambda_1)$ is summarized in the paper of Ashbaugh and Benguria [19] (see also its precursors [12] and [15]). In particular, one should consult the graph given as Figure 1 on p. 38 of [19]. It is shown in [19] that

$$5.077^+ \leq \sup_{\Omega} \frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 5.50661^-$$ \hfill (3.44)
and that
\[ 3.1818^+ \leq \sup_{\Omega} \frac{\lambda_3}{\lambda_1} \leq 3.83103^- . \] (3.45)

The lower bound in (3.44) is simply the value for a disk. It is a conjecture of Payne, Pólya, and Weinberger that this is the actual maximum value of \((\lambda_2 + \lambda_3)/\lambda_1\) as well. The lower bound in (3.43) is \(35/11\), the value taken by \(\lambda_3/\lambda_1\) for a \(\sqrt{8}\) by \(\sqrt{3}\) rectangle (and is certainly the maximum of \(\lambda_3/\lambda_1\) among rectangles). However, there is no current guess as to the precise shape of domain that will maximize \(\lambda_3/\lambda_1\) (if a maximizer even exists). The best current thinking would have it be an elongated convex figure, roughly in the shape of an oval or ellipse. (Among ellipses the largest value of \(\lambda_3/\lambda_1\) seems to be very near to, but slightly less than, 3.1818.)

### 3.3 Universal Eigenvalue Inequalities in Other Spaces

There are also versions of the Payne-Pólya-Weinberger inequality (3.2) and its extensions for bounded domains in the constant curvature spaces \(S^n\) and \(H^n\). For example, for \(\Omega \subset S^2\) in 1975 Cheng \[33\] derived the bound
\[ \frac{\lambda_2}{\lambda_1} \leq 1 + 2 \left( \frac{2}{1 + \cos \Theta} \right)^4 \quad \text{for } 0 < \Theta < \pi \] (3.46)
where \(\Theta\) is defined as the outradius (=geodesic radius of the circumscribing circle) of \(\Omega \subset S^2\). This was improved to
\[ \frac{\lambda_2}{\lambda_1} \leq 1 + 2 \left( \frac{2}{1 + \cos \Theta} \right)^2 \quad \text{for } 0 < \Theta < \pi \] (3.47)
by Harrell \[19\] in 1993. Both of these results have extensions to higher eigenvalues and to \(S^n\).

On a slightly different front, by an extension of the method they used to prove the Payne-Pólya-Weinberger conjecture for \(\lambda_2/\lambda_1\), Ashbaugh and Benguria proved \[11, 16\] (see Section 5) that for \(\Omega \subset S^2\)
\[ \frac{\lambda_2}{\lambda_1} \leq 1 + 1.5387 \left( \frac{2}{1 + \cos \Theta} \right)^2 \quad \text{for } 0 < \Theta < \pi. \] (3.48)
The constant here is \(j_{1,1}^2/j_{0,1}^2 - 1\) and comes from the proof of the Euclidean PPW conjecture as extended to general second-order elliptic operators in Section 4 of \[11\].

Beyond this, there are the sharp PPW results for \(\Omega\) contained in a hemisphere of \(S^n\),
\[ \lambda_2(\Omega) \leq \lambda_2(B_{\lambda_1}) \] (3.49)
where $B_{\lambda_1}$ is the geodesic ball in $S^n$ having the same value of $\lambda_1$ as $\Omega$, and
\[
\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(\Omega^*)}{\lambda_1(\Omega^*)}
\] (3.50)
where $\Omega^*$ is the geodesic ball in $S^n$ having the same measure as $\Omega$. These
last two results are due to Ashbaugh and Benguria (see [16], [20], [21]). Note
that (3.49) and (3.50) are much sharper than the bounds in terms of $\Theta$ listed
previously, since the outradius of $\Omega$ is a much cruder measure of the size of
$\Omega$ than either $\lambda_1(\Omega)$ or $|\Omega|$. On the other hand, the bounds in terms of $\Theta$
apply to all domains $\Omega$, not just to those contained in a hemisphere. While it
may be possible that (3.49) and/or (3.50) hold beyond the hemisphere, this is
not proved as yet. It might also be mentioned that (3.50) follows easily from
(3.49) since it can be shown that $\lambda_2/\lambda_1$ for a geodesic ball is an increasing
function of its radius. See [16], [20], and [21] for more details and discussion.

Further results for domains $\Omega$ contained in $S^n$, $H^n$, or other more general
manifolds are due to Cheng [33] in 1975, Li [70] in 1980, P. C. Yang and S.-T.
Yau [106] in 1980, Lee [66] and Leung [69] in 1991, Harrell [49] in 1993, and
Harrell and Michel [50], [51] in 1994 and 1995. Some of these papers estab-
lish analogs of the Hile-Protter inequality in a more general setting. More
discussion of this literature will be found in [1]. We mention, in particular,
the Hile-Protter-type bound for a domain in $S^n$
\[
\sum_{i=1}^{m} \frac{4\lambda_i + n^2}{\lambda_{m+1} - \lambda_i} \geq mn. 
\] (3.51)
For $m = 1$ this bound gives us
\[
\lambda_2 \leq (1 + \frac{4}{n})\lambda_1 + n. 
\] (3.52)
(Indeed, one has, more generally, the PPW-type bound
\[
\lambda_{m+1} \leq \lambda_m + \frac{4}{mn} \sum_{i=1}^{m} \lambda_i + n \leq (1 + \frac{4}{n})\lambda_m + n. 
\] (3.53)
This bound can be regarded as an alternative and quite natural bound for
$\lambda_2$ in terms of $\lambda_1$ for domains $\Omega \subset S^n$ (to be compared with, say, (3.48), at
least when $n = 2$). Note that this bound “takes account of” the blow-up of
$\lambda_2/\lambda_1$ that we expect when we approach the whole sphere $S^n$ by the presence
of the constant term $n$ on its right-hand side. Thus, even though $\lambda_1 \to 0$ as
we approach the whole sphere while $\lambda_2 \to n$, this new bound gives us control
of $\lambda_2$ without the need of a coefficient that blows up in this limit. Moreover,
the new bound $\lambda_2 \leq (1 + \frac{4}{n})\lambda_1 + n$ is seen to be sharp in this limit (viz., for
the full sphere $\lambda_0 = n$, while the right-hand side becomes $n$ since $\lambda_1 = 0$).

The bound (3.51) is actually the “HP-weakening” of a stronger Yang-style bound for $\Omega \subset S^n$ first given in this context by H. C. Yang [103] (in the 1995 version of this paper). For a full discussion of Yang’s bounds and related inequalities (such as (3.51)) in this context, see [9]. Weaker, but generally more complex, precursors may be found in [33], [50], [69], and [106] (most of these are directed at compact minimal hypersurfaces in $S^n$, but the ideas and calculations are much the same).

3.4 Universal Eigenvalue Inequalities for the Vibrating Clamped Plate

In their 1956 paper [83], Payne, Pólya, and Weinberger also established the bound
\[ \Gamma_{m+1} - \Gamma_m \leq \frac{8}{m} (\Gamma_1 + \ldots + \Gamma_m), \quad m = 1, 2, \ldots, \] (3.54)
for the eigenvalues $\Gamma_i$ of the clamped plate problem (1.5) on a domain $\Omega \subset \mathbb{R}^2$. This is the analog for the clamped plate of their bound (3.1) for the fixed membrane. For $\Omega \subset \mathbb{R}^n$ this bound becomes
\[ \Gamma_{m+1} - \Gamma_m \leq \frac{8(n+2)}{n^2 m} (\Gamma_1 + \ldots + \Gamma_m), \quad m = 1, 2, \ldots, \] (3.55)
and, even better,
\[ \Gamma_{m+1} - \Gamma_m \leq \frac{8(n+2)}{n^2 m^2} (\Gamma_1^{1/2} + \ldots + \Gamma_m^{1/2})^2, \quad m = 1, 2, \ldots. \] (3.56)

This latter inequality was not given by Payne, Pólya, and Weinberger, even for $n = 2$, but it certainly could have been, as it is in a sense implicit in their work. Beyond this, in 1984 Hile and Yeh [54], extending the approach of Hile and Protter to the clamped plate problem, established the bound
\[ \sum_{i=1}^m \frac{\Gamma_i^{1/2}}{\Gamma_{m+1} - \Gamma_i} \geq \frac{n^2 m^{3/2}}{8(n+2)} \left( \sum_{i=1}^m \Gamma_i \right)^{-1/2}, \quad m = 1, 2, \ldots. \] (3.57)

Again, implicit in their work is the better bound
\[ \frac{n^2 m^2}{8(n+2)} \leq \left( \sum_{i=1}^m \frac{\Gamma_i^{1/2}}{\Gamma_{m+1} - \Gamma_i} \right) \left( \sum_{i=1}^m \Gamma_i^{1/2} \right), \quad m = 1, 2, \ldots, \] (3.58)
which was exhibited explicitly only later. In fact Hook [55] in 1990 established (3.58) as a strict inequality. Also in 1990, Chen and Qian [32] independently stated and proved (3.58).
Possibly (3.58) could be improved to
\[
\left( \sum_{i=1}^{m} \sqrt{\frac{\Gamma_i}{\Gamma_{m+1} - \Gamma_i}} \right)^2 \geq \frac{n^2 m^2}{8(n + 2)}, \quad m = 1, 2, \ldots .
\] (3.59)

This inequality would imply all the previous ones in this subsection ((3.58), for example, would follow easily using the Cauchy-Schwarz inequality).

It might be noted that the quantity \(8(n + 2)/n^2\) that appears in the foregoing inequalities is indeed natural from the PPW/HP point of view, just as \(4/n\) is in the case of the fixed membrane. This is because \(8(n + 2)/n^2\) arises as \((1 + 4/n)^2 - 1\) just as \(4/n\) arises as \((1 + 4/n) - 1\).

Yet another inequality in this vein is
\[
\sum_{i=1}^{m} \frac{\Gamma_i}{\Gamma_{m+1} - \Gamma_i} \geq \frac{n^2 m}{8(n + 2)}, \quad m = 1, 2, \ldots .
\] (3.60)

This inequality is an easy consequence of (3.58) via Chebyshev’s inequality (it would also follow directly from (3.59) via the Cauchy-Schwarz inequality). Its chief appeal is its simplicity. It should be mentioned that as yet no one has established an analog of H. C. Yang’s bound (3.31) (or equivalently (3.33)) in this setting, i.e., for the eigenvalues of the vibrating clamped plate. So far (3.58), which is best regarded as an analog of the Hile-Protter inequality (3.38) in this setting, is the closest anyone has come.

### 3.5 Universal Eigenvalue Inequalities for the Buckling Problem

For the buckling problem for a clamped plate (problem (1.7)) even less is known, largely owing to the fact that the inner product one must employ for this problem, \(\langle f, g \rangle \equiv \int_{\Omega} \nabla f \cdot \nabla g\), does not induce a symmetric \(a_{ij}\) matrix when one attempts the usual PPW approach and sets \(a_{ij} = \langle x v_i, v_j \rangle\). This leads to all sorts of extra complications, and thus far no one has been able to bring the general case (for \(\Lambda_{m+1} - \Lambda_i\) or even just \(\Lambda_{m+1} - \Lambda_m\)) to a satisfactory conclusion. Payne, Pólya, and Weinberger [83] were able to establish the low eigenvalue result

\[
\frac{\Lambda_2}{\Lambda_1} < 3 \quad \text{for } \Omega \subset \mathbb{R}^2.
\] (3.61)

For \(\Omega \subset \mathbb{R}^n\) this reads

\[
\frac{\Lambda_2}{\Lambda_1} < 1 + 4/n.
\] (3.62)
Subsequently Hile and Yeh \[54\] reconsidered this problem obtaining the improved bound

$$\frac{\Lambda_2}{\Lambda_1} \leq 2.5 \quad \text{for } \Omega \subset \mathbb{R}^2$$

(3.63)

and, in general,

$$\frac{\Lambda_2}{\Lambda_1} \leq \frac{n^2 + 8n + 20}{(n + 2)^2} \quad \text{for } \Omega \subset \mathbb{R}^n.$$  

(3.64)

Both of these inequalities can actually be shown to hold as strict inequalities. It is of note that $\Lambda_2/\Lambda_1$ for a disk in 2 dimensions is 1.796. If the analog of the PPW conjecture held for this problem, then this would be the best possible upper bound for $\Lambda_2/\Lambda_1$ in 2 dimensions.

### 3.6 More Inequalities for the Low Eigenvalues of the Clamped Plate and Buckling Problems

One can also derive inequalities analogous to (3.41) for the clamped plate and buckling problems. These read

$$\frac{\Lambda_2 + \ldots + \Lambda_{n+1}}{\Lambda_1} \leq n + 4 \quad \text{for } \Omega \subset \mathbb{R}^n,$$  

(3.65)

$$\frac{\Gamma_2^{1/2} + \ldots + \Gamma_{n+1}^{1/2}}{\Gamma_1^{1/2}} \leq n + 4 \quad \text{for } \Omega \subset \mathbb{R}^n,$$  

(3.66)

and

$$\frac{\Gamma_2 + \ldots + \Gamma_{n+1}}{\Gamma_1} \leq n + 24 \quad \text{for } \Omega \subset \mathbb{R}^n.$$  

(3.67)

Of the last two inequalities, (3.66) is the natural “PPW-analog” and (3.67) is a weaker inequality that derives from it. Note the presence of the PPW factor $(1 + \frac{4}{n})$ in (3.65) and (3.66) (in the form $n + 4 = n \left(1 + \frac{4}{n}\right)$). The constant $n + 24$ in (3.67) comes from the extreme case of (3.66) where $\Gamma_i^{1/2}/\Gamma_1^{1/2} = 1$ for $2 \leq i \leq n$ and $\Gamma_{n+1}^{1/2}/\Gamma_1^{1/2} = 5$.

Finally, we mention the known results for $\Gamma_2/\Gamma_1$. Obviously (from (3.55)), Payne, Pólya, and Weinberger had

$$\Gamma_2/\Gamma_1 \leq (1 + 4/n)^2 \quad \text{for } \Omega \subset \mathbb{R}^n$$

(3.68)

(and even $\Gamma_{m+1}/\Gamma_m \leq (1 + 4/n)^2$ for all $m$). This was improved upon more recently by fairly elaborate means by Hile and Yeh \[54\], who, for example, obtained the bounds 7.103 and 4.792 for dimensions 2 and 3 respectively. In
general, these upper bounds are determined as the unique root larger than 1 of the cubic
\[(x - 1)^3 = \frac{512}{n^2(n + 2)}x\]  

(though Hile and Yeh formulated their result in rather different terms). These values might be compared to those of the ball in 2 and 3 dimensions: 4.3311 and 3.2390, respectively. Again the analog of the PPW conjecture for this problem would project these values as the optimal upper bounds for $\Gamma_2/\Gamma_1$ in 2 and 3 dimensions.

As a final remark, we note that any result mentioned in this section (Section 3) without specific attribution is due to the author and that fuller details will be presented in forthcoming papers.

## 4 Concluding Remarks and Open Problems

In this concluding section we mention a few open problems and give some further hints to the literature.

Of the original conjectures of Payne, Pólya, and Weinberger [3], only two remain open. These are to show that
\[
\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq \frac{\lambda_2 + \lambda_3}{\lambda_1} \bigg|_{\text{disk}} \approx 5.077 \text{ for } \Omega \subset \mathbb{R}^2, 
\]
and the analogous result for $(\lambda_2 + \ldots + \lambda_{n+1})/\lambda_1$ for $\Omega \subset \mathbb{R}^n$, and that
\[
\frac{\lambda_{m+1}}{\lambda_m} \leq \frac{\lambda_2}{\lambda_1} \bigg|_{\text{ball}} \text{ for all } \Omega \subset \mathbb{R}^n. 
\]

The latter inequality is known for $m = 1, 2$, and 3 (for $m = 2$ and 3 it follows from the stronger inequality $\lambda_4/\lambda_2 \leq (\lambda_2/\lambda_1)|_{\text{ball}}$ proved in [14]), but as yet all higher cases remain open. For further discussion of these problems see [12], [13], and [14] (as well as [14]).

There are also the well-known Pólya conjectures for the Dirichlet and Neumann eigenvalues of the Laplacian on a domain $\Omega \subset \mathbb{R}^n$. For the case of dimension 2, and with notation as in Section 1, these read
\[
\lambda_k \geq \frac{4\pi k}{A} \text{ for } k = 1, 2, \ldots 
\]

and
\[
\mu_k \leq \frac{4\pi k}{A} \text{ for } k = 0, 1, 2, \ldots, 
\]
where $A = |\Omega|$. There are analogous conjectures for all dimensions $n > 2$. See [16] for their statements and for further discussion.

For the vibrating clamped plate problem there remains the Rayleigh conjecture

$$\Gamma_1(\Omega) \geq \Gamma_1(\Omega^*) \quad \text{for } \Omega \subset \mathbb{R}^n$$

(4.5)

for all $n \geq 4$, and, if one could prove (4.3) (or in any event for $n = 2, 3$), the PPW-type conjecture

$$\frac{\Gamma_2}{\Gamma_1} \leq \frac{\Gamma_2}{\Gamma_1}_{\text{ball}} \quad \text{for all } \Omega \subset \mathbb{R}^n.$$ (4.6)

Similarly, for the buckling problem for the clamped plate there remains the Pólya-Szegö conjecture

$$\Lambda_1(\Omega) \geq \Lambda_1(\Omega^*) \quad \text{for } \Omega \subset \mathbb{R}^n$$

(4.7)

for all $n \geq 2$. If this conjecture could be proved then one could also consider the following conjecture for ratios:

$$\frac{\Lambda_2}{\Lambda_1} \leq \frac{\Lambda_2}{\Lambda_1}_{\text{ball}} \quad \text{for all } \Omega \subset \mathbb{R}^n.$$ (4.8)

Obviously many other problems could be formulated and investigated. For example, one could consider the ratios $\Lambda_{m+1}/\Lambda_m$, $\Gamma_{m+1}/\Gamma_m$, $(\Lambda_2 + \Lambda_3)/\Lambda_1$, $(\Gamma_2 + \Gamma_3)/\Gamma_1$, and many other combinations analogous to those that have been considered for the eigenvalues of the Dirichlet Laplacian. One could also consider much of what has been discussed in this paper in the more general setting of domains in Riemannian manifolds or for general second-order elliptic operators.

Other more extensive problem lists occur in the review papers of Payne [79], [80], [81], and in Yau’s recent problem lists [107], [108] (reprinted in [89]). In addition, one could consult the final section of [16] and also [8].

**Acknowledgements**

The author is grateful to Brian Davies and Yuri Safarov for the opportunity to participate in the Instructional Conference on Spectral Theory and Geometry in Edinburgh (March 29-April 9, 1998) and to the International Centre for Mathematical Sciences for its generous support. In addition, he gratefully acknowledges his collaborators, Rafael Benguria and Richard Laugesen, with whom a number of the results summarized here were obtained.
References

[1] Abramowitz, M., and I. A. Stegun, editors, Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series, vol. 55, U.S. Government Printing Office, Washington, D.C., 1964.

[2] Adams, R. A., Sobolev Spaces, Academic Press, New York, 1975.

[3] Alvino, A., J. I. Diaz, P.-L. Lions, and G. Trombetti, Equations elliptiques et symétrisation de Steiner, Comptes Rendus Acad. Sci. Paris (Ser. I) 314 (1992), 1015-1020.

[4] Alvino, A., P.-L. Lions, and G. Trombetti, A remark on comparison results via symmetrization, Proc. Roy. Soc. Edinburgh 102A (1986), 37-48.

[5] Alvino, A., P.-L. Lions, and G. Trombetti, Comparison results for elliptic and parabolic equations via Schwarz symmetrization, Ann. Inst. H. Poincaré 7 (1990), 37-65.

[6] Alvino, A., P.-L. Lions, and G. Trombetti, Comparison results for elliptic and parabolic equations via symmetrization: A new approach, Diff. and Integral Eqs. 4 (1991), 25-50.

[7] Alvino, A., G. Trombetti, J. I. Diaz, and P.-L. Lions, Elliptic equations and Steiner symmetrization, Commun. Pure Appl. Math. 49 (1996), 217-236.

[8] Ashbaugh, M. S., Open problems on eigenvalues of the Laplacian, Analytic and Geometric Inequalities and Applications, Th. M. Rassias and H. M. Srivastava, editors, Kluwer Academic Publishers, Dordrecht, The Netherlands, to appear.

[9] Ashbaugh, M. S., The universal eigenvalue bounds of Payne-Pólya-Weinberger, Hile-Protter, and H. C. Yang, in preparation.

[10] Ashbaugh, M. S., and R. D. Benguria, Proof of the Payne-Pólya-Weinberger conjecture, Bull. Amer. Math. Soc. 25 (1991), 19-29.

[11] Ashbaugh, M. S., and R. D. Benguria, A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions, Ann. Math. 135 (1992), 601-628.

[12] Ashbaugh, M. S., and R. D. Benguria, More bounds on eigenvalue ratios for Dirichlet Laplacians in $n$ dimensions, SIAM J. Math. Anal. 24 (1993), 1622-1651.
REFERENCES

[13] Ashbaugh, M. S., and R. D. Benguria, A second proof of the Payne-Pólya-Weinberger conjecture, Commun. Math. Phys. 147 (1992), 181-190.

[14] Ashbaugh, M. S., and R. D. Benguria, Isoperimetric bounds for higher eigenvalue ratios for the n-dimensional fixed membrane problem, Proc. Roy. Soc. Edinburgh 123A (1993), 977-985.

[15] Ashbaugh, M. S., and R. D. Benguria, The range of values of $\lambda_2/\lambda_1$ and $\lambda_3/\lambda_1$ for the fixed membrane problem, Rev. Math. Phys. 6 (1994), 999-1009 (in a special issue dedicated to Elliott H. Lieb). [Also in: The State of Matter: A Volume Dedicated to E. H. Lieb (Copenhagen, 1992), M. Aizenman and H. Araki, editors, Advanced Series in Mathematical Physics, vol. 20, World Scientific, Singapore, 1994, pp. 167-181.]

[16] Ashbaugh, M. S., and R. D. Benguria, Isoperimetric inequalities for eigenvalue ratios, Partial Differential Equations of Elliptic Type, Cortona, 1992, A. Alvino, E. Fabes, and G. Talenti, editors, Symposia Mathematica, vol. 35, Cambridge University Press, Cambridge, 1994, pp. 1-36.

[17] Ashbaugh, M. S., and R. D. Benguria, Sharp upper bound to the first nonzero Neumann eigenvalue for bounded domains in spaces of constant curvature, J. London Math. Soc. (2) 52 (1995), 402-416.

[18] Ashbaugh, M. S., and R. D. Benguria, On Rayleigh’s conjecture for the clamped plate and its generalization to three dimensions, Duke Math. J. 78 (1995), 1-17.

[19] Ashbaugh, M. S., and R. D. Benguria, Bounds for ratios of the first, second, and third membrane eigenvalues, Nonlinear Problems in Applied Mathematics, in Honor of Ivar Stakgold on his Seventieth Birthday, T. S. Angell, L. Pamela Cook, R. E. Kleinman, and W. E. Olmstead, editors, Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1996, pp. 30-42.

[20] Ashbaugh, M. S., and R. D. Benguria, On the Payne-Pólya-Weinberger conjecture on the n-dimensional sphere, General Inequalities 7 (Oberwolfach, 1995), C. Bandle, W. N. Everitt, L. Losonczi, and W. Walter, editors, International Series of Numerical Mathematics, vol. 123, Birkhäuser, Basel, 1997, pp. 111-128.
REFERENCES

[21] Ashbaugh, M. S., and R. D. Benguria, A sharp bound for the ratio of the first two Dirichlet eigenvalues of a domain in a hemisphere of $S^n$, Trans. Amer. Math. Soc., to appear.

[22] Ashbaugh, M. S., R. D. Benguria, and R. S. Laugesen, Inequalities for the first eigenvalues of the clamped plate and buckling problems, General Inequalities 7 (Oberwolfach 1995), C. Bandle, W. N. Everitt, L. Losonczi, and W. Walter, editors, International Series of Numerical Mathematics, vol. 123, Birkhäuser, Basel, 1997, pp. 95-110.

[23] Ashbaugh, M. S., and R. S. Laugesen, Fundamental tones and buckling loads of clamped plates, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 23 (1996), 383-402.

[24] Bandle, C., Isoperimetric Inequalities and Applications, Pitman Monographs and Studies in Mathematics, vol. 7, Pitman, Boston, 1980.

[25] Baernstein, A., II, A unified approach to symmetrization, Partial Differential Equations of Elliptic Type, Cortona, 1992, A. Alvino, E. Fabes, and G. Talenti, editors, Symposia Mathematics, vol. 35, Cambridge University Press, Cambridge, 1994, pp. 47-91.

[26] Bérard, P. H., Spectral Geometry: Direct and Inverse Problems (with an Appendix by G. Besson), Lect. Notes in Math., vol. 1207, Springer-Verlag, Berlin, 1986.

[27] Bramble, J. H., and L. E. Payne, Pointwise bounds in the first biharmonic boundary value problem, J. Math. and Phys. 42 (1963), 278-286.

[28] Brands, J. J. A. M., Bounds for the ratios of the first three membrane eigenvalues, Arch. Rational Mech. Anal. 16 (1964), 265-268.

[29] Brezis, H., Analyse Fonctionnelle: Théorie et applications, Masson, Paris, 1983.

[30] Chavel, I., Eigenvalues in Riemannian Geometry, Academic Press, New York, 1984.

[31] Chavel, I., Riemannian Geometry: A Modern Introduction, Cambridge University Press, Cambridge, 1993.

[32] Chen, Z.-C., and C. L. Qian, Estimates for discrete spectrum of Laplacian operator with any order, J. China Univ. Sci. Tech. 20 (1990), 259-266.
REFERENCES

[33] Cheng, S.-Y., Eigenfunctions and eigenvalues of Laplacian, Proc. Symp. Pure Math., vol. 27, part 2, Differential Geometry, S. S. Chern and R. Osserman, editors, American Mathematical Society, Providence, Rhode Island, 1975, pp. 185-193.

[34] Chiti, G., Norme di Orlicz delle soluzioni di una classe di equazioni ellittiche, Boll. Un. Mat. Ital. (5) 16-A (1979), 178-185.

[35] Chiti, G., A reverse Hölder inequality for the eigenfunctions of linear second order elliptic operators, J. Appl. Math. and Phys. (ZAMP) 33 (1982), 143-148.

[36] Chiti, G., An isoperimetric inequality for the eigenfunctions of linear second order elliptic operators, Boll. Un. Mat. Ital. (6) 1-A (1982), 145-151.

[37] Chiti, G., A bound for the ratio of the first two eigenvalues of a membrane, SIAM J. Math. Anal. 14 (1983), 1163-1167.

[38] Courant, R., and D. Hilbert, Methods of Mathematical Physics, vol. 1, Interscience Publishers, Wiley, New York, 1953.

[39] Davies, E. B., Spectral Theory and Differential Operators, Cambridge Studies in Advanced Mathematics, vol. 42, Cambridge University Press, Cambridge, 1995.

[40] Davies, E. B., $L^p$ spectral theory of higher-order elliptic differential operators, Bull. London Math. Soc. 29 (1997), 513-546.

[41] Duffin, R.J., Nodal lines of a vibrating plate, J. Math. and Phys. 31 (1953), 294-299.

[42] Duffin, R. J., and D. H. Shaffer, On the modes of vibration of a ring-shaped plate, Bull. Amer. Math. Soc. 58 (1952), 652.

[43] Faber, G., Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt, Sitzungsber. Bayr. Akad. Wiss. München, Math.-Phys. Kl. 1923, 169-172.

[44] Friedland, S., and W. K. Hayman, Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions, Comment. Math. Helvetici 51 (1976), 133-161.
[45] Glaser, V., A. Martin, H. Grosse, and W. Thirring, A family of optimal conditions for the absence of bound states in a potential, Studies in Mathematical Physics: Essays in Honor of Valentine Bargmann, E. H. Lieb, B. Simon, and A. S. Wightman, editors, Princeton University Press, Princeton, New Jersey, 1976, pp. 169-194.

[46] Gould, S. H., Variational Methods for Eigenvalue Problems: An Introduction to the Weinstein Method of Intermediate Problems, second edition, revised and enlarged, University of Toronto Press, Mathematical Expositions, Number 10, Toronto, 1966.

[47] Gunson, J., Inequalities in mathematical physics, Inequalities: Fifty Years On from Hardy, Littlewood, and Pólya, W. N. Everitt, editor, Marcel Dekker, New York, 1991, pp. 53-79 (see especially Section 5, pp. 70-74).

[48] Hardy, G. H., J. E. Littlewood, and G. Pólya, Inequalities, second edition, Cambridge University Press, Cambridge, 1952.

[49] Harrell, E. M., II, Some geometric bounds on eigenvalue gaps, Commun. Partial Diff. Eqs. 18 (1993), 179-198.

[50] Harrell, E. M., II, and P. L. Michel, Commutator bounds for eigenvalues, with applications to spectral geometry, Commun. Partial Diff. Eqs. 19 (1994), 2037-2055.

[51] Harrell, E. M., II, and P. L. Michel, Commutator bounds for eigenvalues of some differential operators, Evolution Equations, G. Ferreyra, G. R. Goldstein, and F. Neubrander, editors, Lecture Notes in Pure and Applied Mathematics, vol. 168, Marcel Dekker, New York, 1995, pp. 235-244.

[52] Harrell, E. M., II, and J. Stubbe, On trace identities and universal eigenvalue estimates for some partial differential operators, Trans. Amer. Math. Soc. 349 (1997), 1797-1809.

[53] Hile, G. N., and M. H. Protter, Inequalities for eigenvalues of the Laplacian, Indiana Univ. Math. J. 29 (1980), 523-538.

[54] Hile, G. N., and R. Z. Yeh, Inequalities for eigenvalues of the biharmonic operator, Pac. J. Math. 112 (1984), 115-133.

[55] Hook, S.M., Domain-independent upper bounds for eigenvalues of elliptic operators, Trans. Amer. Math. Soc. 318 (1990), 615-642.
[56] Kawohl, B., *Rearrangements and Convexity of Level Sets in PDE*, Lect. Notes in Math., vol. 1150, Springer-Verlag, Berlin, 1985.

[57] Kesavan, S., Some remarks on a result of Talenti, Ann. Scuola Norm. Sup. Pisa (4) 15 (1988), 453-465.

[58] Kesavan, S., *Topics in Functional Analysis and Applications*, Wiley, New York, 1989.

[59] Kesavan, S., On a comparison theorem via Schwarz symmetrization, Proc. Roy. Soc. Edinburgh 119A (1991), 159-167.

[60] Kesavan, S., Comparison theorems via Schwarz symmetrization—a survey, *Partial Differential Equations of Elliptic Type, Cortona, 1992*, A. Alvino, E. Fabes, and G. Talenti, editors, Symposia Mathematica, vol. 35, Cambridge University Press, Cambridge, 1994, pp. 185-196.

[61] Kesavan, S., Comparison theorems via symmetrization: revisited, Boll. Un. Mat. Ital. (7) 11-A (1997), 163-172.

[62] Kornhauser, E. T., and I. Stakgold, A variational theorem for $\nabla^2 u + \lambda u = 0$ and its application, J. Math. and Phys. 31 (1952), 45-54.

[63] Krahn, E., Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises, Math. Ann. 94 (1925), 97-100.

[64] Krahn, E., Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen, Acta Comm. Univ. Tartu (Dorpat) A9 (1926), 1-44. [English translation: Minimal properties of the sphere in three and more dimensions, *Edgar Krahn 1894-1961: A Centenary Volume*, Ü. Lumiste and J. Peetre, editors, IOS Press, Amsterdam, 1994, Chapter 11, pp. 139-174.]

[65] Kozlov, V. A., V. A. Kondrat’ev, and V. G. Maz’ya, On sign variation and the absence of “strong” zeros of solutions of elliptic equations, Izv. Akad. Nauk SSSR, Ser. Mat. 53 (1989), 328-344 (in Russian) [English translation in Math. USSR-Izv. 34 (1990), 337-353].

[66] Lee, J. M., The gaps in the spectrum of the Laplace-Beltrami operator, Houston J. Math. 17 (1991), 1-24.

[67] Leis, R., *Initial Boundary Value Problems in Mathematical Physics*, B. G. Teubner, Stuttgart, and John Wiley and Sons, Chichester, 1986.
[68] Leissa, A. W., Vibration of Plates, National Aeronautics and Space Administration (NASA SP-160), U.S. Government Printing Office, Washington, D.C., 1969.

[69] Leung, P.-F., On the consecutive eigenvalues of the Laplacian of a compact minimal submanifold in a sphere, J. Austral. Math. Soc. (Series A) 50 (1991), 409-416.

[70] Li, P., Eigenvalue estimates on homogeneous manifolds, Comment. Math. Helvetici 55 (1980), 347-363.

[71] Lieb, E. H., Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation, Stud. Appl. Math. 57 (1977), 93-105.

[72] Lieb, E. H., Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. Math. 118 (1993), 349-374.

[73] Lieb, E. H., and M. Loss, Analysis, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, Rhode Island, 1997.

[74] Nadirashvili, N. S., An isoperimetric inequality for the principal frequency of a clamped plate, Dokl. Akad. Nauk 332 (1993), 436-439 (in Russian) [English translation in Phys. Dokl. 38 (1993), 419-421].

[75] Nadirashvili, N. S., New isoperimetric inequalities in mathematical physics, Partial Differential Equations of Elliptic Type, Cortona, 1992, A. Alvino, E. Fabes, and G. Talenti, editors, Symposia Mathematica, vol. 35, Cambridge University Press, Cambridge, 1994, pp. 197-203.

[76] Nadirashvili, N. S., Rayleigh’s conjecture on the principal frequency of the clamped plate, Arch. Rational Mech. Anal. 129 (1995), 1-10.

[77] Owen, M. P., Asymptotic first eigenvalue estimates for the biharmonic operator on a rectangle, J. Diff. Eqs. 136 (1997), 166-190.

[78] Payne, L. E., Inequalities for eigenvalues of membranes and plates, J. Rational Mech. Anal. 4 (1955), 517-529.

[79] Payne, L. E., Isoperimetric inequalities for eigenvalues and their applications, Autovalori e autosoluzioni, Centro Internazionale Matematico Estivo (C.I.M.E.) 2° Ciclo, Chieti, 1962, pp. 1-58.

[80] Payne, L. E., Isoperimetric inequalities and their applications, SIAM Review 9 (1967), 453-488.
[81] Payne, L. E., Some comments on the past fifty years of isoperimetric inequalities, *Inequalities: Fifty Years On from Hardy, Littlewood, and Pólya*, W. N. Everitt, editor, Marcel Dekker, New York, 1991, pp. 143-161.

[82] Payne, L. E., G. Pólya, and H. F. Weinberger, Sur le quotient de deux fréquences propres consécutives, Comptes Rendus Acad. Sci. Paris 241 (1955), 917-919.

[83] Payne, L. E., G. Pólya, and H. F. Weinberger, On the ratio of consecutive eigenvalues, J. Math. and Phys. 35 (1956), 289-298.

[84] Pólya, G., and G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, Annals of Mathematics Studies, Number 27, Princeton University Press, Princeton, New Jersey, 1951.

[85] Protter, M. H., Can one hear the shape of a drum? revisited, SIAM Review 29 (1987), 185-197.

[86] Rayleigh, J. W. S., *The Theory of Sound*, second edition revised and enlarged (in 2 volumes), Dover Publications, New York, 1945 (republication of the 1894/96 edition).

[87] Reed, M., and B. Simon, *Methods of Modern Mathematical Physics, vol. IV: Analysis of Operators*, Academic Press, New York, 1978.

[88] Safarov, Yu., and D. Vassiliev, *The Asymptotic Distribution of Eigenvalues of Partial Differential Equations*, Translations of Mathematical Monographs, vol. 155, American Mathematical Society, Providence, Rhode Island, 1997.

[89] Schoen, R., and S.-T. Yau, *Lectures on Differential Geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, vol. 1, International Press, Boston, 1994.

[90] Simon, B., *Functional Integration and Quantum Physics*, Academic Press, New York, 1979 (see especially pp. 142-143).

[91] Sperner, E., Zur Symmetrisierung von Funktionen auf Sphären, Math. Z. 134 (1973), 317-327.

[92] Szegő, G., On membranes and plates, Proc. Nat. Acad. Sci. (USA) 36 (1950), 210-216.
[93] Szegő, G., Inequalities for certain eigenvalues of a membrane of given area, J. Rational Mech. Anal. 3 (1954), 343-356.

[94] Szegő, G., Note to my paper “On membranes and plates”, Proc. Nat. Acad. Sci. (USA) 44 (1958), 314-316.

[95] Talenti, G., Best constant in Sobolev inequality, Ann. Mat. Pura Appl. (Ser. 4) 110 (1976), 353-372.

[96] Talenti, G., Elliptic equations and rearrangements, Ann. Scuola Norm. Sup. Pisa (4) 3 (1976), 697-718.

[97] Talenti, G., Linear elliptic P.D.E.’s: Level sets, rearrangements and a priori estimates of solutions, Boll. Un. Mat. Ital. (6) 4-B (1985), 917-949.

[98] Talenti, G., On the first eigenvalue of the clamped plate, Ann. Mat. Pura Appl. (Ser. 4) 129 (1981), 265-280.

[99] Talenti, G., Rearrangements and PDE, Inequalities: Fifty Years On from Hardy, Littlewood, and Pólya, W. N. Everitt, editor, Marcel Dekker, New York, 1991, pp. 211-230.

[100] Talenti, G., On isoperimetric theorems of mathematical physics, Chapter 4.4 of Handbook of Convex Geometry, vol. B, P. M. Gruber and J. M. Wills, editors, North-Holland, Amsterdam, The Netherlands, 1993, pp. 1131-1147.

[101] Thompson, C. J., On the ratio of consecutive eigenvalues in n-dimensions, Stud. Appl. Math. 48 (1969), 281-283.

[102] Weinberger, H. F., An isoperimetric inequality for the n-dimensional free membrane problem, J. Rational Mech. Anal. 5 (1956), 633-636.

[103] Weinberger, H. F., Symmetrization in uniformly elliptic problems, Chapter 58 of Studies in Mathematical Analysis and Related Topics: Essays in Honor of George Pólya, G. Szegő, C. Loewner, S. Bergman, M. M. Schiffer, J. Neyman, D. Gilbarg, and H. Solomon, editors, Stanford University Press, Stanford, California, 1962, pp. 424-428.

[104] Weinstein, A., and W. Stenger, Methods of Intermediate Problems for Eigenvalues: Theory and Ramifications, Academic Press, New York, 1972.
[105] Yang, H. C., Estimates of the difference between consecutive eigenvalues, 1995 preprint (revision of International Centre for Theoretical Physics preprint IC/91/60, Trieste, Italy, April, 1991).

[106] Yang, P. C., and S.-T. Yau, Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 7 (1980), 55-63.

[107] Yau, S.-T., Problem section, Seminar on Differential Geometry, S.-T. Yau, editor, Annals of Mathematics Studies, Number 102, Princeton University Press, Princeton, New Jersey, 1982, pp. 669-706 [reprinted as pp. 277-314 of [89]].

[108] Yau, S.-T., Open problems in geometry, Differential Geometry: Partial Differential Equations on Manifolds, Proc. Symp. Pure Math., vol. 54, part 1, R. Greene and S.-T. Yau, editors, American Mathematical Society, Providence, Rhode Island, 1993, pp. 1-28 [reprinted as pp. 365-409 of [89]].