Two-point Ostrowski and Ostrowski–Grüss type inequalities with applications

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Abstract
In this work, an extension of two-point Ostrowski’s formula for \( n \)-times differentiable functions is proved. A generalization of Taylor formula is deduced. An identity of Fink type for this extension is provided. Error estimates for the considered formulas are also given. Two-point Ostrowski–Grüss type inequalities are pointed out. An expansion of Guessab–Schmeisser two points formula for \( n \)-times differentiable functions via Fink type identity is established. Generalization of the main result for harmonic sequence of polynomials is established. Several bounds of the presented results are proved.

Keywords Approximations · Expansions · Quadrature rule · Euler–Maclaurin formula · Ostrowski inequality

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1 Introduction

To approximate \( \int_a^b f(t) dt \), by a general one-point rule

\[
\int_a^b f(t) dt = (b - a)f(x) + E(f, x), \quad \forall x \in [a, b],
\]

Let us suppose \( f \) is differentiable on \([a, b]\) and \( f' \in L[a, b] \). If \( \|f'\|_{\infty} = \sup_{x \in [a, b]} |f'(x)| \leq \infty \).

Therefore, the Ostrowski estimates [49] reads:

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for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

In 1976 Milovanović and Pečarić [46] provided their famous generalization of (1.2) via Taylor series, where they proved that:

$$\int_a^b f(t)dt = \frac{b-a}{n} \left( f(x) + \sum_{k=1}^{n-1} F_k(x) \right) + E_n(f, x), \quad \forall x \in [a, b],$$

such that

$$F_k(x) = \frac{n-k}{n!} \cdot \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}.$$  (1.4)

with error estimates

$$|E_n(f, x)| \leq \frac{(x-a)^{n+1} + (b-x)^{n+1}}{n(n+1)!} \cdot \|f^{(n)}\|_\infty,$$  (1.5)

In 1992, Fink studied (1.4) in different presentation, he introduced a new representation of real $n$-times differentiable function whose $n$-th derivative ($n \geq 1$) is absolutely continuous by combining Taylor series and Peano kernel approach together. Namely, in [33] we find:

$$E_n(f, x) = \frac{1}{n!} \int_a^b (x - t)^{n-1} p(t, x) f^{(n)}(t)dt,$$  (1.6)

for all $x \in [a, b]$, where

$$p(t, x) = \begin{cases} t - a, & t \in [a, x] \\ t - b, & t \in [x, b] \end{cases}.$$  (1.7)

In the same work, Fink proved the following bound of (1.6).

$$|E_n(f, x)| \leq C(n, p, x) \left\| f^{(n)} \right\|_p,$$  (1.8)

where $\| \cdot \|_r$, $1 \leq r \leq \infty$ are the usual Lebesgue norms on $L_r[a, b]$, i.e.,

$$\|f\|_\infty := \text{ess sup}_{t \in [a, b]} |f(t)|,$$

and

$$\|f\|_r := \left( \int_a^b |f(t)|^r dt \right)^{1/r}, \quad 1 \leq r < \infty,$$

such that

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\[ C(n, p, x) = \frac{1}{n!} B^{1/q} ((n - 1)q + 1, q + 1) [(x - a)^{q+1} + (b - x)^{q+1}]^{1/q}, \]

for \(1 < p \leq \infty\), \(B(\cdot, \cdot)\) is the beta function, and for \(p = 1\)

\[ C(n, 1, x) = \frac{(n - 1)^{n-1}}{n^n n!} \max \{ (x - a)^n, (b - x)^n \}. \]

All previous bounds are sharp.

Indeed Fink representation can be considered as the first elegant work (after Darboux work [42], p. 49) that combines two different approaches together, so that Fink representation is not less important than Taylor expansion itself. So that, many authors were interested to study Fink representation approach, more detailed and related results can be found in [1, 2, 12, 13, 22].

In 2002 and the subsequent years after that, the Ostrowski’s inequality entered in a new phase of modifications and developments. A new inequality of Ostrowski’s type was born, where Guessab and Schmeisser in [40] discussed an inequality from algebraic and analytic points of view which is in connection with Ostrowski inequality; called ‘the companion of Ostrowski’s inequality’ as suggested later by Dragomir in [28]. The main part of Guessab–Schmeisser inequality reads the difference between symmetric values of a real function \(f\) defined on \([a, b]\) and its weighed value, i.e.,

\[
\frac{f(x) + f(a + b - x)}{2} - \frac{1}{b - a} \int_a^b f(t) dt, \quad x \in \left[ a, \frac{a + b}{2} \right].
\]

Namely, in the significant work [40] we find the first primary result is that:

**Theorem 1** Let \(f : [a, b] \to \mathbb{R}\) be satisfies the Hölder condition of order \(r \in (0, 1]\). Then for each \(x \in [a, \frac{a+b}{2}]\), the we have the inequality

\[
\left| \frac{f(x) + f(a + b - x)}{2} - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{M}{b - a} \frac{(2x - 2a)^{r+1} + (a + b - 2x)^{r+1}}{2^r (r+1)}.
\]

This inequality is sharp for each admissible \(x\). Equality is attained if and only if \(f = \pm M f_\ast + c\) with \(c \in \mathbb{R}\) and

\[
f_\ast(t) = \begin{cases} 
(x - t)^r, & \text{if } a \leq t \leq x \\
(t - x)^r, & \text{if } x \leq t \leq \frac{a+b}{2} \\
f_\ast(a + b - t), & \text{if } \frac{a+b}{2} \leq t \leq b 
\end{cases}.
\]

In the same work [40], the authors discussed and investigated (1.8) for other type of assumptions. Among others, a brilliant representation (or identity) of \(n\)-times differentiable functions whose \(n\)-th derivatives are piecewise continuous was established as follows:
Theorem 2 Let $f$ be a function defined on $[a, b]$ and having there a piecewise continuous $n$-th derivative. Let $Q_n$ be any monic polynomial of degree $n$ such that $Q_n(t) = (-1)^n Q_n(a + b - t)$. Define

$$K_n(t) = \begin{cases} (t - a)^n, & \text{if } a \leq t \leq x \\ Q_n(t), & \text{if } x \leq t \leq a + b - x \\ (t - b)^n, & \text{if } a + b - x \leq t \leq b \end{cases}$$

Then,

$$\int_a^b f(t) dt = (b - a) \frac{f(x) + f(a + b - x)}{2} + E(f;x)$$

(1.10)

where,

$$E(f;x) = \sum_{v=1}^{n-1} \left[ (x - a)^{v+1} - \frac{Q_n^{(v+1)}(x)}{(v + 1)!} \right] \left[ f^{(v)}(a + b - x) + (-1)^v f(x) \right]$$

$$+ \frac{(-1)}{n!} \int_a^b K_n(t)f^{(n)}(t)dt.$$

This generalization (1.10) can be considered as a companion type expansion of Euler–Maclaurin formula that expand symmetric values of real functions. In this way, families of various quadrature rules can be presented, as shown—for example—in [41]. Therefore, since 2002 and after the presentation of (1.9), several authors have studied, developed and established new presentations concerning (1.10) using several approaches and different tools, for this purpose see the recent survey [27].

Far away from this, in the last thirty years the concept of harmonic sequence of polynomials or Appell polynomials have been used at large in numerical integrations and expansions theory of real functions. Let us recall that, a sequence of polynomials $\{P_k(t, \cdot)\}_{k=0}^{\infty}$ satisfies the Appell condition (see [10]) if $\frac{\partial}{\partial t} P_k(t, \cdot) = P_{k-1}(t, \cdot)$ (\forall k \geq 1) with $P_0(t, \cdot) = 1$, for all well-defined order pair $(t, \cdot)$. A slightly different definition was considered in [45].

In 2003, motivated by work of Matić et al. [45], Dedić et al. in [22], introduced the following smart generalization of Ostrowski’s inequality via harmonic sequence of polynomials:

$$\frac{1}{n} \left[ f(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x)f^{(k)}(x) + \sum_{k=1}^{n-1} \bar{F}_k(a, b) \right]$$

$$= \frac{(-1)^{n-1}}{(b - a)n} \int_a^b P_{n-1}(t)p(t, x)f^{(n)}(t)dt,$$

(1.11)
where $P_k$ is a harmonic sequence of polynomials satisfies that $P'_k = P_{k-1}$ with $P_0 = 1$,

$$
\tilde{F}_k(a, b) = \frac{(-1)^k(n-k)}{b-a} \left[ P_k(a)f^{(k-1)}(a) - P_k(b)f^{(k-1)}(b) \right]
$$

(1.12)

and $p(t, x)$ is given in (1.5). In particular, if we take $P_k(t) = \frac{(t-x)^k}{k!}$ then we refer to Fink representation (1.5).

In 2005, Dragomir [28] proved the following bounds of the companion of Ostrowski’s inequality for absolutely continuous functions.

**Theorem 3** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then we have the inequalities

$$
\left| \frac{f(x) + f(a + b - x)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right|
\leq \begin{cases}
\left( \frac{1}{8} + 2 \left( \frac{x-3a+b}{b-a} \right)^2 \right) (b-a) \| f' \|_\infty, & f' \in L_\infty[a, b] \\
\frac{2^{1/q}}{(q+1)^{1/q}} \left[ \left( \frac{x-a}{b-a} \right)^{q+1} - \left( \frac{2b-x}{b-a} \right)^{q+1} \right]^{1/q} (b-a)^{1/q} \| f' \|_{[a,b],p}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \text{ and } f' \in L_p[a, b]
\end{cases}
$$

(1.13)

for all $x \in [a, \frac{a+b}{2}]$. The constants $\frac{1}{8}$ and $\frac{1}{4}$ are the best possible in (1.13) in the sense that it cannot be replaced by smaller constants.

In the last fifteen years, constructions of quadrature rules using expansion of an arbitrary function in Bernoulli polynomials and Euler–Maclaurin’s type formulae have been established, improved and investigated. These approaches permit many researchers to work effectively in the area of numerical integration where several error approximations of various quadrature rules presented with high degree of exactness. Mainly, works of Dedić et al. [22–26], Aljinović et al. [1, 2], Kovač et al. [41] and others, received positive responses and good interactions from other focused researchers. Among others, Franjić et al. in several works (such as [34–38]) constructed several Newton-Cotes and Gauss quadrature type rules using a certain expansion of real functions in Bernoulli polynomials or Euler–Maclaurin’s type formulae.
Unfortunately, the expansions (1.3), (1.10) and (1.11) have not been used to construct quadrature rules yet. It seems these expansions were abandoned or neglected in literature because most of authors are still use the classical Euler–Maclaurin’s formula and expansions in Bernoulli polynomials.

In order to generalize Guessab–Schmeisser formula (1.9) for symmetric and non-symmetric points, we have introduced the following general quadrature rule in the recent work [3].

\[ \int_a^b f(s)ds = Q(f; y, x, z) + E(f; y, x, z), \]  
(1.14)

where \( Q(f; y, x, z) \) is the general two-point formula

\[ Q(f; y, x, z) := (x - a)f(y) + (b - x)f(z), \]  
(1.15)

for all \( x \in [y, z] \) with \( a \leq y \leq z \leq b \), with error term

\[ E(f; y, x, z) := \int_a^b K(s; y, x, z)df(s), \]  
(1.16)

where

\[ K(s; y, x, z) = \begin{cases} 
  s - a, & a \leq s \leq y \\
  s - x, & y < s < z \\
  s - b, & z \leq s \leq b 
\end{cases} \]

for all \( x \in [y, z] \subseteq [a, b] \).

In the same work [3], we provided error estimates of \( E(f; y, x, z) \) involving functions possess at most first derivatives. Namely, we proved the following generalization of Ostrowski inequality which is considered as two-point inequality.

\[ |E(f; y, x, z)| \leq \frac{(x - a)^2 + (b - x)^2}{4} + \left( y - \frac{a + x}{2} \right)^2 + \left( z - \frac{x + b}{2} \right)^2 \cdot \|f''\|_{\infty, [a, b]}, \]  
(1.17)

for all \( a \leq y \leq x \leq z \leq b \). The constant \( \frac{1}{4} \) is the best possible.

In particular, we deduced a sharp error estimates for the average of general two point formula as follows:

\[ |E(f; y, \frac{a + b}{2}, z)| \leq \left[ \frac{(b - a)^2}{8} + \left( y - \frac{3a + b}{4} \right)^2 + \left( z - \frac{a + 3b}{4} \right)^2 \right] \cdot \|f''\|_{\infty, [a, b]}, \]  
(1.18)

The constant \( \frac{1}{8} \) is the best possible.

**Remark 1** If we choose \( y = x = z \), the we recapture the Ostrowski inequality (1.2). By setting \( z = a + b - y \) and choose \( x = \frac{a + b}{2} \), we get the Guessab–Schmeisser
formula (1.9) for \( n = 1 \). Also, for \( y = a \) and \( z = b \) we recapture the generalized trapezoid inequality that was introduced in [15].

The author of this paper has been given serious attention to Guessab–Schmeisser inequality in the works [3–9]. For other related results and generalizations concerning Ostrowski’s inequality and its applications we refer the reader to [12–19, 26, 28–30, 43, 50].

This work has several aims and goals, the first aim is to generalize Guessab–Schmeisser formula for symmetric and non-symmetric two points for \( n \)-times differentiable functions via Peano functional approach and Fink type identity and thus provide several type of bounds for the remainder formula. An extension of our result (1.14) for \( n \)-times differentiable functions is proved. A Fink type identity for this extension is provided. Error estimates for the considered formulas are also given. The second goal, is to highlight the importance of these expansions by giving a serious attention to their applicable usefulness in constructing various quadrature rules. The third aim, is to spotlight the role of Čebyšev functional in integral approximations.

This work is organized as follows: in the next section, a generalization of Two point formula for \( n \)-times differentiable is considered. A Taylor expansion would be then deduced as a special case. Indeed, this Two point formula can be read as an expansion of real analytic function near two point instead of one point. Remainder estimates for the constructed formula are provided. The constant in the obtained estimates are shown to be the best possible. A mean value theorem is also given. Bounds for functions possess Hölder continuity of order \( r \in (0, 1] \) are proved. In Sect. 3, A Fink type identity for the presented two point formula is established. For instance, a Guessab–Schmeisser two points formula for \( n \)-times differentiable functions via Fink type identity is also deduced. Bounds for the remainder term of the presented formula are proved. In Sect. 4, Inequalities of two-point Ostrowski–Grüss type are introduced. Bounds for the remainder of some previously obtained formulas via Chebyshev-Grüss type inequalities are presented. A practical and applicable example of the previous section; in fact, bounds for the Guessab–Schmeisser two points formula are given.

### 2 Two-point Ostrowski’s inequality

#### 2.1 Expansions

Let \( I \) be a real interval and \( a, b \) in \( I^\circ \) (the interior of \( I \)) with \( a < b \). Let \( H_{(n,i)} \) be a harmonic sequences of polynomials for \( i = 1, 2, 3 \), and let \( f : I \to \mathbb{R} \) be such that \( f^{(n-1)} \) \((n \geq 1)\) is of bounded variation on \( I \) for all \( n \geq 1 \).

For all \( y, z \in [a, b] \) with \( y \leq z \). Define the kernel \( S_n : [a, b]^3 \to \mathbb{R} \) given by

\[
S_n(t, y, z) = \begin{cases} 
\frac{(t-a)^n}{n!}, & t \in [a, y] \\
\frac{(t-y)^n}{n!}, & t \in (y, z) \\
\frac{(t-b)^n}{n!}, & t \in [z, b] 
\end{cases}
\]  

(2.1)
Integrating by parts

\[
(-1)^n \int_a^y \frac{(t-a)^n}{n!} df^{(n-1)}(t) = (-1)^n \frac{(y-a)^n}{n!} f^{(n-1)}(y)
\]

+ \(-1\)^{n-1} \int_a^y \frac{(t-a)^{n-1}}{(n-1)!} f^{(n-1)}(t) dt.

Successive integrations by parts yield

\[
(-1)^n \int_a^y \frac{(t-a)^n}{n!} df^{(n-1)}(t) = \sum_{k=1}^{n} (-1)^k \frac{(y-a)^k}{k!} f^{(k-1)}(y) + \int_a^y f(t) dt. \tag{2.2}
\]

Similarly,

\[
(-1)^n \int_y^z \frac{(t-x)^n}{n!} df^{(n-1)}(t)
= \sum_{k=1}^{n} (-1)^k \left[ \frac{(z-x)^k}{k!} f^{(k-1)}(z) - \frac{(y-x)^k}{k!} f^{(k-1)}(y) \right] + \int_y^z f(t) dt, \tag{2.3}
\]

and

\[
(-1)^n \int_z^b \frac{(t-b)^n}{n!} df^{(n-1)}(t) = - \sum_{k=1}^{n} (-1)^k \frac{(z-b)^k}{k!} f^{(k-1)}(z) + \int_z^b f(t) dt. \tag{2.4}
\]

Adding (2.2)–(2.4) we get the representation

\[
(-1)^n \int_a^b S_n(t,y,z) df^{(n-1)}(t)
= \sum_{k=1}^{n} \frac{(-1)^k}{k!} \left[ \left\{ (y-a)^k - (y-x)^k \right\} f^{(k-1)}(y) \right.
+ \left\{ (z-x)^k - (z-b)^k \right\} f^{(k-1)}(z) \bigg] + \int_a^b f(t) dt. \tag{2.5}
\]

A convenient presentation of (2.5) which gives an expansion of real function \( f \) near arbitrary two point \( y, z \in [a, b] \) with \( y \leq z \) can be written as:

\[
(x - a)f(y) + (b - x)f(z) = \int_a^b f(t) dt + Q_n(f; S_n,y,x,z) + R_n(f; S_n,y,x,z) \tag{2.6}
\]

for all \( a \leq y \leq x \leq z \leq b \), where \( Q_n(f; y, x, z) \) is the two-point Ostrowski’s formula given by
\[ Q_n \{ f, S_n; y, x, z \} = \sum_{k=2}^{n} \frac{(-1)^k}{k!} \left[ \{(y-a)^k - (y-x)^k\}f^{(k-1)}(y) + \{(z-x)^k - (z-b)^k\}f^{(k-1)}(z) \right], \]

(2.7)

and \( R_n \{ f, S_n; y, x, z \} \) is the remainder term given such as

\[ R_n \{ f, S_n; y, x, z \} := (-1)^{n+1} \int_a^b S_n(t; y, x, z)df^{(n-1)}(t), \]

(2.8)

for all \( a \leq y \leq x \leq z \leq b \). If \( n = 1 \), then the summation in (2.7) is defined to be 0.

**Remark 2** In (2.5) setting \( a = y, b = z \) and replace every \( f \) by \( f' \), rearranging the terms gives the generalized Taylor formula

\[ f(z) = f(y) + \sum_{k=1}^{n} \frac{1}{k!} [(x-y)^k f^{(k)}(y) - (x-z)^k f^{(k)}(z)] + \frac{1}{n!} \int_y^z (x-t)^n f^{(n+1)}(t)dt \]

for all \( x \in [y, z] \). This formula expand \( f \) near two point instead of one point. Moreover, if one chooses \( y = a \) and \( z = x \) then we recapture the celebrated Taylor formula. From different point of view, another Two-point formula was considered by Davis in ([21], p.37), which completely independent formula.

**Theorem 4** If \( f^{(2n)} (n \geq 1) \) is continuous on \( [a, b] \), then there exists \( \eta \in (a, b) \) such that

\[ R_{2n} \{ f, S_{2n}; y, x, z \} = -\frac{1}{(2n + 1)!} \left[ (y-a)^{2n+1} + (x-y)^{2n+1} + (z-x)^{2n+1} + (b-z)^{2n+1} \right] f^{(2n)}(\eta). \]

(2.9)

**Proof** From (2.8) since \( S_{2n}(t; y, x, z) \) does not change sign over all \( [a, b] \), the by Mean Value theorem there is an \( \eta \in (a, b) \) such that

\[ R_{2n} \{ f, S_{2n}; y, x, z \} = (-1)^{2n+1} \int_a^b S_{2n}(t; y, x, z)df^{(2n)}(t)dt \]

\[ = -f^{(2n)}(\eta) \int_a^b S_{2n}(t; y, x, z)dt \]

\[ = -\frac{1}{(2n + 1)!} \left[ (y-a)^{2n+1} + (x-y)^{2n+1} + (z-x)^{2n+1} + (b-z)^{2n+1} \right] f^{(2n)}(\eta), \]
which proves the result.

In the previous theorem, since the error term for the Two-point rule (2.9) involves \( f^{(2n)} \), the rule gives the exact result when applied to any function whose \((2n)\)-th derivative is identically zero, that is, any polynomial of degree \( \leq 2n - 1 \), \( n \in \mathbb{N} \).

2.2 Remainder estimates

Let \( f \) be defined on \([a, b]\), \( P := \{x_0, x_1, \ldots, x_n\} \) be a partition of \([a, b]\), and

\[
\Delta f_i = f(x_i) - f(x_{i-1}),
\]

for \( i = 1, 2, \ldots, n \). A function \( f \) is said to be of bounded \( p \)-variation if there exists a positive number \( M \) such that

\[
\sum_{i=1}^{n} \left| \Delta f_i \right|^p \leq M,
\]

for all partitions of \([a, b]\).

Let \( f \) be of bounded \( p \)-variation on \([a, b]\), and let \( \sum(P) \) denote the sum corresponding to the partition \( P \) of \([a, b]\). The number

\[
\int_a^b (f; p) = \sup \left\{ \sum(P) : P \in \mathcal{P}(a, b) \right\}, \quad 1 \leq p < \infty
\]

is called the total \( p \)-variation of \( f \) on the interval \([a, b]\), where \( \mathcal{P}(a, b) \) denotes the set of all partitions of \([a, b]\). It can be easily seen that for \( p = 1 \), \( p \)-variation reduces to ordinary variation or Jordan variation of functions.

In special case, if there exists a positive number \( M \) such that

\[
\sum_{i=1}^{n} \text{Osc} \left( f; \left[ x_{i-1}^{(n)}, x_i^{(n)} \right] \right) = \sum_{i=1}^{n} (\sup - \inf) f(t_i) \leq M, \quad t_i \in \left[ x_{i-1}^{(n)}, x_i^{(n)} \right],
\]

for all partitions of \([a, b]\), then \( f \) is said to have \( \infty \)-variation on \([a, b]\). The number

\[
\int_a^b (f; \infty) = \sup \left\{ \sum(P) : P \in \mathcal{P}[a, b] \right\} := \text{Osc} (f; [a, b]),
\]

is called the oscillation of \( f \) on \([a, b]\). Equivalently, we may define the oscillation of \( f \) such as, (see [32]):

\[
\int_a^b (f; \infty) = \lim_{p \to \infty} \int_a^b (f; p) = \sup_{x \in [a, b]} \{f(x)\} - \inf_{x \in [a, b]} \{f(x)\} = \text{Osc} (f; [a, b]).
\]

If \( f \) is a real function of bounded \( p \)-variation on an interval \([a, b]\), then ([3]):
• \( f \) is bounded, and

\[
\text{Osc} (f;[a,b]) \leq \sqrt[1-p]{(f;1)} \leq \sqrt[p]{(f;1)}.
\]

This fact follows by Jensen’s inequality applied for \( h(p) = \sqrt[p]{(f;1)} \) which is log-convex and decreasing for all \( p > 1 \). Moreover, the inclusions

\[
\mathcal{W}_\infty(f) \subset \mathcal{W}_q(f) \subset \mathcal{W}_p(f) \subset \mathcal{W}_1(f)
\]

are valid for all \( 1 < p < q < \infty \), where \( \mathcal{W}_p(\cdot) \) denotes the class of all functions of bounded \( p \)-variation \( (1 \leq p \leq \infty) \) (see [51]).

• \( f \) is continuous except at most on a countable set.

• \( f \) has one-sided limits everywhere (limits from the left everywhere in \((a,b]\), and from the right everywhere in \([a,b))\);

• The derivative \( f'(x) \) exists almost everywhere (i.e. except for a set of measure zero).

• If \( f(x) \) is differentiable on \([a,b]\), then

\[
\sqrt[p]{(f;1)} = \left( \int_a^b |f'(t)|^p dt \right)^{\frac{1}{p}} = \|f'\|_p, \quad 1 \leq p < \infty.
\]

In [3], we find the following lemma:

**Lemma 1** Fix \( 1 \leq p < \infty \). Let \( f, g : [a,b] \to \mathbb{R} \) be such that \( f \) is continuous on \([a,b]\) and \( g \) is of bounded \( p \)-variation on \([a,b]\). Then the Riemann–Stieltjes integral \( \int_a^b f(t)dg(t) \) exists and the inequality:

\[
\left| \int_a^b w(t)dv(t) \right| \leq \|w\|_\infty \cdot \text{Osc} (v;[a,b]) \leq \|w\|_\infty \cdot \sqrt[p]{(v;1)},
\]

holds. The constant ‘1’ in the both inequalities is the best possible.

In all next results, we need the following identities.

**Lemma 2** For the kernel \( S_n : [a,b]^3 \to \mathbb{R} \) (\( n \geq 1 \)) given in (2.1), and for all \( a \leq y \leq x \leq z \leq b \), we have the following computations:

\[
\int_a^b S_n(t;y,x,z)dt = \frac{1}{(n+1)!} \left[ (y-a)^{n+1} + (z-x)^{n+1} + (-1)^n(x-y)^{n+1} + (-1)^n(b-z)^{n+1} \right].
\]

(2.11)
\[
\int_{a}^{b} |S_n(t; y, x, z)| \, dt
= \frac{1}{(n+1)!} \left[ (y-a)^{n+1} + (x-y)^{n+1} + (z-x)^{n+1} + (b-z)^{n+1} \right].
\] (2.12)

\[
\int_{a}^{b} |S_n(t; y, x, z)|^q \, dt
= \frac{1}{(nq + 1)n!} \left[ (y-a)^{nq+1} + (x-y)^{nq+1} + (z-x)^{nq+1} + (b-z)^{nq+1} \right], \ \forall q \geq 1.
\] (2.13)

\[
\sup_{a \leq t \leq b} |S_n(t; y, x, z)| = \frac{1}{n!} \left[ \max \left\{ (y-a), \left( \frac{z-y}{2} + \left| \frac{y+z}{2} - x \right| \right), (b-z) \right\} \right]^n.
\] (2.14)

**Proof** The proof is straightforward. \(\square\)

Now we are ready to state our first result regarding the estimation of the error term \(R_n(f; y, x, z)\).

**Theorem 5** Let \(I\) be a real interval such that \(a, b \in I^o\); the interior of \(I\) with \(a < b\). Let \(f : I \to \mathbb{R}\) be such that \(f^{(n-1)}\) is of bounded \(p\)-variation \((1 \leq p \leq \infty)\) on \([a, b] \subset I^o\), \(\forall n \geq 1\). Then

\[
\left| R_n(f, S_n; y, x, z) \right|
\leq \frac{1}{n!} \left[ \max \left\{ (y-a), \left( \frac{z-y}{2} + \left| \frac{y+z}{2} - x \right| \right), (b-z) \right\} \right]^n \cdot \left\| f^{(n-1)}(p) \right\|,
\] (2.15)

for all \(a \leq y \leq x \leq z \leq b\).

Moreover, if \(f^{(n)}\) exists then

\[
\int_{a}^{b} (f^{(n-1)}; p) = \left( \int_{a}^{b} \left| f^{(n)}(t) \right|^p \, dt \right)^{\frac{1}{p}} = \left\| f^{(n)} \right\|_p, \quad 1 \leq p < \infty,
\]

and therefore

\[
\left| R_n(f, S_n; y, x, z) \right|
\leq \frac{1}{n!} \left[ \max \left\{ (y-a), \left( \frac{z-y}{2} + \left| \frac{y+z}{2} - x \right| \right), (b-z) \right\} \right]^n \cdot \left\| f^{(n)} \right\|_p.
\] (2.16)
Proof Since \( f^{(n-1)} \) is of bounded \( p \)-variation (\( 1 \leq p \leq \infty \)) on \([a, b] \subset I^\circ, \forall n \geq 1\), utilizing the triangle integral inequality on the identity (2.8) and employing Lemma 1, we get

\[
\left| (-1)^{n+1} \int_a^b S_n(t; y, x, z) df^{(n-1)}(t) \right| \leq \sup_{t \in [a, b]} |S_n(t; y, x, z)| \cdot \sqrt[p]{(f^{(n-1)}, p)}
\]

\[
\leq \frac{1}{n!} \left[ \max \left\{ (y - a), \left( \frac{z - y}{2} + \frac{y + z}{2} - x \right), (b - z) \right\} \right]^n \cdot \sqrt[p]{(f^{(n-1)}, p)},
\]

for all \( p \in [1, \infty] \) and all \( n \geq 1 \), which completes the proof. The moreover case follows from the properties of \( p \)-variation and this completes the proof. \( \square \)

Theorem 6 Let \( I \) be a real interval such that \( a, b \) in \( I^\circ \), the interior of \( I \) with \( a < b \). Let \( f : I \to \mathbb{R} \) be such that \( f^{(n-1)} \) is absolutely continuous on \([a, b] \subset I^\circ, \forall n \geq 1\). Then we have

\[
|R_n(f, S_n; y, x, z)| \leq \begin{cases} \frac{1}{n!} \left[ \max \left\{ (y - a), \left( \frac{z - y}{2} + \frac{y + z}{2} - x \right), (b - z) \right\} \right]^n \cdot \|f^{(n)}\|_1, & \text{If } f^{(n)} \in L^1([a, b]) \\ \frac{1}{(nq+1)^s \cdot n!} \left[ (y - a)^{nq+1} + (x - y)^{nq+1} + (z - x)^{nq+1} + (b - z)^{nq+1} \right]^\frac{s}{n} \cdot \|f^{(n)}\|_p, & \text{If } f^{(n)} \in L^p([a, b]) \\ \frac{1}{(n+1)!} \left[ (y - a)^{n+1} + (x - y)^{n+1} + (z - x)^{n+1} + (b - z)^{n+1} \right] \cdot \|f^{(n)}\|_\infty, & \text{If } f^{(n)} \in L^\infty([a, b]) \end{cases}
\]

(2.17)

where \( p > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). The inequality is the best possible for \( p = 1 \), and sharp for \( 1 < p \leq \infty \). The equality is attained for every function \( f(t) = Mf_0(t) + p_{n-1}(t) \), \( t \in [a, b] \), where \( M \) is real constant and \( p_{n-1} \) is an arbitrary polynomial of degree at most \( n - 1 \), and \( f_0 \) is a function defined on \([a, b] \) given by

\[
f_0(t) = \int_a^t \frac{(t - u)^{n-1}}{(n-1)!} \, \text{sgn} S_n(u; y, x, z) \cdot \left| S_n(u; y, x, z) \right|^{\frac{1}{n}} \, du, \quad t \in [a, b]
\]

for \( 1 < p < \infty \), and

\( \square \) Springer
\[ f_0(t) = \int_a^t \frac{(t-u)^{n-1}}{(n-1)!} \text{sgn} S_n(u; y, x, z) du \]

for \( p = \infty \).

**Proof** For \( p = 1 \). Utilizing the triangle integral inequality on the identity (2.8) then we have

\[
\left| (-1)^{n+1} \int_a^b S_n(t; y, x, z) f^{(n)}(t) dt \right|
\leq \sup_{t \in [a,b]} |S_n(t; y, x, z)| \cdot \int_a^b |f^{(n)}(t)| dt
= \frac{1}{n!} \max \left\{ (y-a), \left( \frac{z-y}{2} + \frac{y+z}{2} - x \right), (b-z) \right\} \cdot \|f^{(n)}\|_1.
\]

For \( 1 < p < \infty \), applying the Hölder integral inequality on the identity (2.8) we get

\[
\left| (-1)^{n+1} \int_a^b S_n(t; y, z) f^{(n)}(t) dt \right|
\leq \left( \int_a^b |f^{(n)}(t)|^p dt \right)^{\frac{1}{p}} \left( \int_a^b \left| S_n(t; y, x, z) \right|^q dt \right)^{\frac{1}{q}}
= \frac{1}{(nq+1)^{\frac{1}{q}} n!} \left[ (y-a)^{nq+1} + (x-y)^{nq+1} + (z-x)^{nq+1} + (b-z)^{nq+1} \right]^{\frac{1}{q}} \cdot \|f^{(n)}\|_p.
\]

For \( p = \infty \), we have

\[
\left| (-1)^{n+1} \int_a^b S_n(t; y, x, z) f^{(n)}(t) dt \right|
\leq \sup_{t \in [a,b]} |f^{(n)}(t)| \cdot \int_a^b |S_n(t; y, x, z)| dt
= \frac{1}{(n+1)!} \left[ (y-a)^{n+1} + (x-y)^{n+1} + (z-x)^{n+1} + (b-z)^{n+1} \right] \cdot \|f^{(n)}\|_\infty,
\]

which proves the desired result. In order to prove the sharpness, for \( p = 1 \) we need to prove the first inequality in (2.17) is the best possible. Suppose \( |S_n(t; y, x, z)| \) attains
its supremum at point \( t_0 \in [a, b] \) and let \( \sup_{t \in [a,b]} |S_n(t,y,x,z)| = S_n(t_0,y,x,z) \) for some \( k = 1, 2, 3 \).

Let \( A_- = \{(a, y), (y, z), (z, b)\} \) and assume that \( S_n(t_0,y,x,z) > 0 \). For \( \epsilon \) small enough define \( f^{(n-1)}_\epsilon(t) \) by

\[
f^{(n-1)}_\epsilon(t) = \begin{cases} 
0, & t \leq t_0 - \epsilon \\
\frac{t - t_0 + \epsilon}{\epsilon}, & t \in [t_0 - \epsilon, t_0] \\
1, & t < t_0 
\end{cases}
\]

If \( t_0 \in (c, d] \in A_- \). Then, for \( \epsilon \) small enough,

\[
\left| \int_a^b S_n(t; y, x, z)f^{(n)}_\epsilon(t) dt \right| = \frac{1}{\epsilon} \left| \int_{t_0-\epsilon}^{t_0} S_n(t; y, x, z) dt \right| \leq \frac{1}{\epsilon} \int_{t_0-\epsilon}^{t_0} S_n(t; y, x, z) dt \leq \sup_{t_0-\epsilon \leq t \leq t_0} |S_n(t; y, x, z)| \cdot \frac{1}{\epsilon} \int_{t_0-\epsilon}^{t_0} dt = S_n(t_0, y, x, z),
\]

also since \( \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_0-\epsilon}^{t_0} S_n(t; y, x, z) dt = S_n(t_0, y, x, z) \), the statement follows.

For \( t_0 = c \in \{a, y, z\} \), then we define, for \( \epsilon > 0 \) small enough, function \( f^{(n-1)}_\epsilon(t) \) by

\[
f^{(n-1)}_\epsilon(t) = \begin{cases} 
0, & t \leq t_0 \\
\frac{t - t_0}{\epsilon}, & t \in [t_0, t_0 + \epsilon] \\
1, & t > t_0 + \epsilon 
\end{cases}
\]

and we continue as above. In similar manner we can show the sharpness holds when \( S_n(t_0, y, x, z) < 0 \).

Finally, for \( 1 < p \leq \infty \), the function \( f_0(t) \) given in the Theorem 6 is \( n \)-times differentiable, and its \( n \)-th derivative is piecewise continuous function. Further, \( f_0 \) is a solution of the differential equation

\[
S_n(t; y, x, z)f^{(n)}(t) = \left| S_n(t; y, x, z) \right|^q, \quad q = \frac{p}{p - 1}, \forall p > 1,
\]

so the equality in (2.17) holds for \( 1 < p \leq \infty \). \(\square\)

To treat bounds for functions possess Hölder continuity of order \( r \in (0, 1] \). Let \( t_0 \in [a, b] \) be fixed point. From (2.8), we have
\[ R_n(f, S_n; y, x, z) \]
\[ = (-1)^{n+1} \int_a^b S_n(t; y, x, z)df^{(n-1)}(t) \]
\[ = (-1)^{n+1} \int_a^b S_n(t; y, x, z)d[f^{(n-1)}(t) - f^{(n-1)}(t_0)] \]
\[ = \frac{(-1)^a}{n!} (y - a)^n [f^{(n-1)}(y) - f^{(n-1)}(t_0)] + \frac{(-1)^b}{n!} (z - x)^n [f^{(n-1)}(z) - f^{(n-1)}(t_0)] \]
\[ - \frac{(-1)^a}{n!} (y - x)^n [f^{(n-1)}(y) - f^{(n-1)}(t_0)] - \frac{(-1)^b}{n!} (z - b)^n [f^{(n-1)}(z) - f^{(n-1)}(t_0)] \]
\[ + (-1)^n \int_a^b [f^{(n-1)}(t) - f^{(n-1)}(t_0)] dS_n(t; y, x, z) \]
\[ = \frac{(-1)^n}{n!} [(y - a)^n - (y - x)^n] [f^{(n-1)}(y) - f^{(n-1)}(t_0)] \]
\[ + \frac{(-1)^n}{n!} [(z - x)^n - (z - b)^n] [f^{(n-1)}(z) - f^{(n-1)}(t_0)] \]
\[ + (-1)^n \int_a^b [f^{(n-1)}(t) - f^{(n-1)}(t_0)] dS_n(t; y, x, z). \]
\[ (2.18) \]

Now, let us setting
\[ \tilde{R}_n(f, S_n; y, x, z) = (-1)^n \int_a^b [f^{(n-1)}(t) - f^{(n-1)}(t_0)] dS_n(t; y, x, z). \]
\[ (2.19) \]

**Theorem 7** Let \( I \) be a real interval such that \( a, b \) in \( I^p; \) the interior of \( I \) with \( a < b. \) Let \( f : I \to \mathbb{R} \) be such that \( f^{(n-1)} \) satisfy the Hölder condition with exponent \( r \in (0, 1] \) and constant \( H > 0 \) on \( [a, b] \subset I^p, \forall n \geq 1, \) then we have
\[ \left| \tilde{R}_n(f, S_n; y, x, z) \right| \leq H \left( \frac{(b-a)^{r+1} + (t_0-a)^{r+1}}{r+1} \cdot \left\| S_{n-1}(\cdot; y, x, z) \right\|_\infty \right) \]
\[ \left( \frac{(b-a)^{r+1} + (t_0-a)^{r+1}}{p^{r+1}} \right)^{1/p} \left\| S_{n-1}(\cdot; y, x, z) \right\|_q \]
\[ \left[ \frac{b-a}{2} + \left| t_0 - \frac{a+b}{2} \right| \right] \cdot \left\| S_{n-1}(\cdot; y, x, z) \right\|_1 \]
\[ (2.20) \]

for all \( x \in [a, b] \) and \( p > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1. \)

**Proof** Since \( f^{(n-1)} \) is \( r \)-Hölder continuous on \( [a, b], \) then
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\[ \left| \tilde{R}_n (f, S_n ; y, x, z) \right| \]

\[ = \left| (-1)^n \int_a^b [f^{(n)} (t) - f^{(n)} (t_0)] dS_n (t; y, x, z) \right| \]

\[ \leq \int_a^b \left\{ |f^{(n)} (t) - f^{(n)} (t_0)| \right\} |S_n (t; y, x, z)| dt \]

\[ \leq H \int_a^b |t - t_0|^{\gamma} |S_n (t; y, x, z)| dt \]

\[ \leq H \cdot \min \left\{ \left( \int_a^b |t - t_0|^{\gamma p} dt \right) \frac{1}{p} \right\} \frac{1}{q} \left( \int_a^b |S_n (t; y, x, z)|^q dt \right) \frac{1}{q} \]

\[ = H \cdot \left( \frac{(b-a)^r + (b-a)^{r+1}}{r+1} \right) \cdot \left( \frac{(b-a)^{r+1} + (a-b)^{r+1}}{p r+1} \right) \cdot \left[ \frac{b-a}{2} + |t_0 - \frac{a+b}{2}| \right]^r \cdot \left| S_n (\cdot; y, x, z) \right|_q \]

for all \( n \geq 1 \) and for every \( a \leq y \leq x \leq z \leq b \) with all \( t_0 \in [a, b] \). □

**Remark 3** In very interesting case, one may choose \( t_0 = x \) in (2.18)–(2.19), so that (2.20) becomes

\[ \left| \tilde{R}_n (f, S_n ; y, x, z) \right| \leq H \cdot \left[ \frac{(b-x)^{r+1}}{r+1} \cdot \left( \frac{(b-x)^{r+1} + (x-a)^{r+1}}{pr+1} \right) \cdot \left[ \frac{b-a}{2} + |x - \frac{a+b}{2}| \right]^r \cdot \left| S_n (\cdot; y, x, z) \right|_q \right] (2.21) \]

for all \( a \leq y \leq x \leq z \leq b \), \( r \in (0, 1) \), and all \( p > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Corollary 1** Let \( I \) be a real interval such that \( a, b \) in \( I \); the interior of \( I \) with \( a < b \). Let \( f : I \to \mathbb{R} \) be such that \( f^{(n)} \) satisfy the Hölder condition with exponent \( r \in (0, 1] \) and constant \( H > 0 \) on \( [a, b] \subset I \), \( \forall n \geq 1 \), then we have
\[ \left| \mathcal{R}_n(f, S_n; h, \frac{a + b}{2}, a + b - h) \right| \leq H \left\{ \frac{(b-a)^{(r+1)} \cdot \left\| S_{n-1} \left( \cdot, \frac{a+b}{2}, a + b - h \right) \right\|_\infty^{r+1}}{2^{(r+1)}} \cdot \left\| S_{n-1} \left( \cdot, \frac{a+b}{2}, a + b - h \right) \right\|_q \right\} \] \quad (2.22)

for all \( h \in \left[ a, \frac{a+b}{2} \right] \), \( r \in (0, 1) \), and all \( p > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).

Also, we have

\[ \left| \mathcal{R}_n(f, S_n; a, x, b) \right| \leq H \left\{ \frac{(b-a)^{r+1} \cdot \left\| S_{n-1}(\cdot; a, x, b) \right\|_\infty^{r+1}}{2^{(r+1)}} \cdot \left\| S_{n-1}(\cdot; a, x, b) \right\|_q \right\} \] \quad (2.23)

for all \( x \in [a, b] \), \( r \in (0, 1) \), and all \( p > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). Similarly, for \( \left| \mathcal{R}_n(f, S_n; x, x, x) \right| \) the bound in (2.23) holds.

Clearly, \( \mathcal{R}_n(f, S_n; y, x, z) = \mathcal{R}_n(f, S_n; y, x, y) \), when \( y = z = x = t_0 \). Therefore we may state the following bound for mappings \( f^{(n-1)} \) satisfy the Hölder condition.

**Corollary 2** Let \( I \) be a real interval such that \( a, b \) in \( I^c \); the interior of \( I \) with \( a < b \). Let \( f : I \to \mathbb{R} \) be such that \( f^{(n-1)} \) satisfy the Hölder condition with exponent \( r \in (0, 1] \) and constant \( K > 0 \) on \( [a, b] \subset I, \forall n \geq 1 \), then we have

\[ \left| \mathcal{R}_n(f, S_n; x, x, x) \right| \leq H \frac{\Gamma(1 + r)}{\Gamma(1 + n + r)} \left[ (x - a)^{r+n} + (b - x)^{r+n} \right], \]

for all \( x \in [a, b] \) and \( r \in (0, 1] \). Moreover, if \( f^{(n-1)} \) satisfies the Lipschitz condition with constant \( L, \) i.e., \( r = 1 \), then we have

\[ \left| \mathcal{R}_n(f, S_n; x, x) \right| \leq \frac{L}{(n + 1)!} \left[ (x - a)^{n+1} + (b - x)^{n+1} \right]. \]

**Proof** Now, since \( t_0 \) is arbitrarily chosen in \([a, b]\), we give the following detailed estimates of \( \mathcal{R}_n(f, S_n; y, x, z) \). From (2.19), we can write
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and this ends the proof.

3 Two-point Ostrowski formula via Fink approach

In this sections we present a Fink type identity for two-point formula (1.14) and then we give some estimates of the remainder.

3.1 General Fink type identity

An identity which express two-point formula of Ostrowski’s type via Fink approach and using harmonic sequence of polynomials can be given by the representation:

**Theorem 8** Let $\mathcal{I}$ be a real interval, $a, b \in \mathcal{I}$ ($a < b$). Let $Q_k$ be a harmonic sequence of polynomials and let $f : \mathcal{I} \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous on $\mathcal{I}$ for $n \geq 1$ with $Q_{n-1}(t)S(t, \cdot f^{(n)}(t)$ is integrable. Then we have the representation

$$
\left[\tilde{R}_n(f, S_n;x, x, x)\right] \\
\leq \frac{1}{(n-1)!} \int_a^x \left| f^{(n-1)}(t) - f^{(n-1)}(t_0) \right| (t-a)^{n-1} \, dt \\
+ \frac{1}{(n-1)!} \int_x^b \left| f^{(n-1)}(t) - f^{(n-1)}(t_0) \right| (b-t)^{n-1} \, dt \\
\leq \frac{H}{(n-1)!} \left[ \int_a^x |t-x|^r (t-a)^{n-1} \, dt + \int_x^b |t-x|^r (b-t)^{n-1} \, dt \right] \\
= \frac{H}{(r+n)\Gamma(r+n)} \left[ (x-a)^{r+n} + (b-x)^{r+n} \right],
$$

and this ends the proof.

$$
\left(3.1\right)
$$

$$
T_k(y, x, z) := \frac{(-1)^k}{b-a} \left\{ (x-a)Q_k(y)f^{(k)}(y) + (b-x)Q_k(z)f^{(k)}(z) \right\},
$$

$$
\left(3.2\right)
$$
\[ F_k(a, b) = \frac{(-1)^k(n-k)}{(b-a)} \left[ Q_k(a)f^{(k-1)}(a) - Q_k(b)f^{(k-1)}(b) \right], \] (3.3)

and

\[ K(t; y, x, z) = \begin{cases} 
  t - a, & a \leq t \leq y \\
  t - x, & y < t < z \\
  t - b, & z \leq t \leq b
\end{cases}, \] (3.4)

for all \( a \leq y \leq x \leq z \leq b. \)

**Proof** From (1.11), we find the formula

\[
\frac{1}{n} \left[ f(x) + \sum_{k=1}^{n-1} (-1)^k Q_k(x)f^{(k)}(x) + \sum_{k=1}^{n-1} F_k(a, b) \right] - \frac{1}{b-a} \int_a^b f(y) dy = \frac{(-1)^{n-1}}{(b-a)n} \int_a^b \left( \int_s^x Q_{n-1}(t)f^{(n)}(t) dt \right) ds
\] (3.5)

where

\[ F_k(a, b) = \frac{(-1)^k(n-k)}{b-a} \left[ Q_k(a)f^{(k-1)}(a) - Q_k(b)f^{(k-1)}(b) \right]. \] (3.6)

Fix \( y, z \in [a, b]. \) In the representation (3.5), replace \( x \) by \( y \) and \( b \) by \( x \) we get

\[
\frac{1}{n} \left[ f(y) + \sum_{k=1}^{n-1} (-1)^k Q_k(y)f^{(k)}(y) + \sum_{k=1}^{n-1} F_k(a, x) \right] - \frac{1}{x-a} \int_a^x f(s) ds = \frac{(-1)^{n-1}}{(x-a)n} \int_a^x \left( \int_s^y Q_{n-1}(t)f^{(n)}(t) dt \right) ds
\] (3.7)

Multiplying both sides of (3.7) by \((x-a)\), we get

\[
\frac{(x-a)}{n} \left[ f(y) + \sum_{k=1}^{n-1} (-1)^k Q_k(y)f^{(k)}(y) + \sum_{k=1}^{n-1} F_k(a, x) \right] - \int_a^x f(s) ds s = \frac{(-1)^{n-1}}{n} \int_a^x \left( \int_s^y Q_{n-1}(t)f^{(n)}(t) dt \right) ds
\] (3.8)

The second step, in the same formula (3.5) we replace every \( x \) by \( z \) and \( a \) by \( x \), then \( f \) has the representation

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\[ \frac{1}{n} \left[ f(z) + \sum_{k=1}^{n-1} (-1)^k Q_k(z)f^{(k)}(z) + \sum_{k=1}^{n-1} F_k(x, b) \right] - \frac{1}{b-x} \int_x^b f(s)ds \]

(3.9)

Multiplying both sides of (3.9) by \((b-x)\) we get

\[ \frac{(b-x)}{n} \left[ f(z) + \sum_{k=1}^{n-1} (-1)^k Q_k(z)f^{(k)}(z) + \sum_{k=1}^{n-1} F_k(x, b) \right] - \int_x^b f(s)ds \]

(3.10)

Adding the equations (3.8) and (3.10) we get

\[
\frac{1}{n} \left[ (x-a)f(y) + (b-x)f(z) + \sum_{k=1}^{n-1} (-1)^k \left\{ (x-a)Q_k(y)f^{(k)}(y) + (b-x)Q_k(z)f^{(k)}(z) \right\} \right] \\
+ \left[ (x-a) \sum_{k=1}^{n-1} F_k(a,x) + (b-x) \sum_{k=1}^{n-1} F_k(x,b) \right] - \int_a^b f(s)ds \\
= \frac{(-1)^{n-1}}{n} \left[ \int_a^x \left( \int_y^x Q_{n-1}(t)f^{(n)}(t)dt \right) ds \\
+ \int_x^b \left( \int_y^z Q_{n-1}(t)f^{(n)}(t)dt \right) ds \right] \\
\]

(3.11)

To simplify the right hand side, we write

\[
\int_a^x ds \int_s^y dt = \int_a^y ds \int_s^y dt + \int_y^x ds \int_s^y dt = \int_a^y dt \int_a^t ds - \int_y^x ds \int_y^s dt \\
= \int_a^y dt \int_a^t ds - \int_y^x dt \int_t^x ds, \\
\]

(3.12)
\[
\int_x^b ds \int_s^z dt = \int_x^z ds \int_s^t dt + \int_s^b ds \int_s^t dt = \int_x^z dt \int_x^t ds - \int_x^s ds \int_x^t dt
= \int_x^z dt \int_x^t ds - \int_z^t dt \int_t^b ds.
\]

(3.13)

Adding (3.12) and (3.13), we get

\[
\int_a^x dy \int_y^z dx = \int_a^y dy \int_y^x dx + \int_x^b dy \int_y^z dx
= \int_a^y dy \int_y^x dx - \int_x^z dy \int_x^t dt + \int_x^t dx \int_x^y dx - \int_t^b dx \int_t^b dy.
\]

(3.14)

In viewing of (3.14), the right hand side of (3.11) is simplified to be

\[
\frac{(-1)^{n-1}}{n} \left[ \int_a^x \left( \int_s^y Q_{n-1}(t)f^{(n)}(t)dt \right) ds + \int_x^b \left( \int_s^z Q_{n-1}(t)f^{(n)}(t)dt \right) ds \right]
= \frac{(-1)^{n-1}}{n} \int_a^b Q_{n-1}(t)K(t;y,x,z)f^{(n)}(t)dt,
\]

where

\[
K(t;y,x,z) = \begin{cases} 
  t - a, & a \leq t \leq y \\
  t - x, & y < t < z \\
  t - b, & z \leq t \leq b 
\end{cases}
\]

for all \( a \leq y \leq x \leq z \leq b \).

Also, we note that in the right hand side

\[
(x - a)F_k(a; x) + (b - x)F_k(x; b) = (-1)^k(n - k) \left[ Q_k(a)f^{(k-1)}(a) - Q_k(b)f^{(k-1)}(b) \right]
= (b - a)F_k(a; b).
\]

Hence, the identity (3.3) is obtained by combining the last two equalities with (3.6), and this ends the proof.

\[\square\]

### 3.2 Error estimates

To approximate \( \int_a^b f(t)dt \), let us define the general quadrature rule
\[
\int_a^b f(t)\,dt = G_n(f, Q_n; y, x, z) + E_n(f, Q_n; y, x, z), \tag{3.15}
\]
for all \(a \leq y \leq x \leq z \leq b\), where \(G_n(f, Q_n; y, x, z)\) is the quadrature formula given by
\[
G_n(f, Q_n; y, x, z) = \frac{b - a}{n} \left[ \frac{(x - a)f(y) + (b - x)f(z)}{b - a} + \sum_{k=1}^{n-1} \{ T_k(y, x, z) + F_k(a, b) \} \right], \tag{3.16}
\]
and \(E_n(f, Q_n; y, x, z)\) is the error term given by
\[
E_n(f, Q_n; y, x, z) = \frac{(-1)^n}{n} \int_a^b Q_{n-1}(t)K(t; y, x, z)f^{(n)}(t)\,dt. \tag{3.17}
\]

**Theorem 9** Under the assumptions of Theorem 8, we have
\[
\left| E_n(f, Q_n; y, x, z) \right| \leq N(f; y, x, z) \cdot \left\| f^{(n)} \right\|_p, \tag{3.18}
\]
\(\forall p \in [1, \infty)\) and all \(a \leq y \leq x \leq z \leq b\), where
\[
N(f; x, a, b) := \begin{cases} 
\frac{1}{n} \left( \sup_{k\leq t \leq b} \left\{ |Q_{n-1}(t)||K(t; y, x, z)| \right\} \right)^{1/p}, & p = 1 \\
\left( \frac{1}{n} \int_a^b |Q_{n-1}(t)|^q |K(t; y, x, z)|^q \,dt \right)^{1/q}, & 1 < p < \infty \\
\int_a^b |Q_{n-1}(t)||K(t; y, x, z)| \,dt, & p = \infty
\end{cases} \tag{3.19}
\]

**Proof** Utilizing the triangle integral inequality on the identity (3.17) and employing some known norm inequalities we get
\[
\left| E_n(f, Q_n; y, x, z) \right| = \left| \frac{(-1)^{n-1}}{n} \int_a^b Q_{n-1}(t)K(t; y, x, z)f^{(n)}(t)\,dt \right|
\leq \frac{1}{n} \int_a^b |Q_{n-1}(t)||K(t; y, x, z)| \left\| f^{(n)} \right\|_p \,dt
\leq \frac{1}{n} \left\| f^{(n)} \right\|_1 \sup_{a \leq t \leq b} \left\{ |Q_{n-1}(t)||K(t; y, x, z)| \right\}, \quad p = 1
\leq \frac{1}{n} \left\| f^{(n)} \right\|_p \left( \int_a^b |Q_{n-1}(t)|^q |K(t; y, x, z)|^q \,dt \right)^{1/q}, \quad 1 < p < \infty
\leq \frac{1}{n} \left\| f^{(n)} \right\|_\infty \int_a^b |Q_{n-1}(t)||K(t; y, x, z)| \,dt, \quad p = \infty
\]
\[
= N(f; x, a, b) \left\| f^{(n)} \right\|_p, \quad \forall p \in [1, \infty].
\]
and this completes the proof.

**Corollary 3** Under the assumptions of Theorem 8, we have

\[
\left| E_n(f, Q_n; y, x, z) \right| \\
\leq \frac{1}{n} \max \left\{ (y - a), \left( \frac{y - x}{2} + \left| x - \frac{y + z}{2} \right| \right), (b - z) \right\} \cdot \| Q_{n-1} \|_q \cdot \| f^{(n)} \|_p,
\]

\(\forall p, q \geq 1\) with \(\frac{1}{p} + \frac{1}{q} = 1\) and all \(a \leq y \leq x \leq z \leq b\), where

\(N(f; x, y, z) \leq \frac{1}{n} \sup \{|K(t; y, x, z)|\} \cdot \| Q_{n-1} \|_q \)

\[= \frac{1}{n} \max \left\{ (y - a), \left( \frac{y - x}{2} + \left| x - \frac{y + z}{2} \right| \right), (b - z) \right\} \cdot \| Q_{n-1} \|_q,
\]

\(\forall q \in [1, \infty]\) and all \(a \leq y \leq x \leq z \leq b\). \(\square\)

In particular case we have

**Corollary 4** Under the assumptions of Theorem 8, we have

\[
\left| \frac{1}{n} \left[ \frac{(x - a)f(y) + (b - x)f(z)}{b - a} + \sum_{k=1}^{n-1} \left( \tilde{T}_k(y, x, z) + \tilde{F}_k(a, b) \right) \right] - \frac{1}{b - a} \int_a^b f(y)dy \right| \\
\leq \frac{1}{n(b - a)} \max \left\{ (y - a), \left( \frac{y - x}{2} + \left| x - \frac{y + z}{2} \right| \right), (b - z) \right\} \cdot \| S_{n-1} \|_q \cdot \| f^{(n)} \|_p,
\]

\(\forall p, q \geq 1\) with \(\frac{1}{p} + \frac{1}{q} = 1\) and all \(a \leq y \leq x \leq z \leq b\), where
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\[ \|S_{n-1}\|_q = \begin{cases} \frac{1}{n!} \left[ \max \left\{ (y-a), \left( \frac{z-y}{2} + \frac{y+z}{2} - x \right) \right\}, (b-z) \right]\right] \right)^n, & q = \infty \\
\frac{(y-a)^{(n-1)}}{(n-1)!} + (z-y)^{(n-1)} + (z-x)^{(n-1)} + (b-z)^{(n-1)}, & 1 < q < \infty \\
\frac{(y-a)^n + (z-y)^n + (z-x)^n + (b-z)^n}{n!}, & q = 1 \end{cases} \]

**Proof** The proof is an immediate consequence of Corollary 3, by setting \( Q_k(t) = \frac{1}{k!} (t-a)^k \).

**Remark 4** In all above estimates, if one assumes that \( f^{(n)} \) is convex, \( r \)-convex, quasi-convex, \( s \)-convex, \( P \)-convex, or \( Q \)-convex; we can obtain other new bounds involving convexity.

### 3.3 Fink representation for Guessab–Schmeisser formula

In this part, we give some special formulas of the previous expansion via Fink approach with some error estimates.

Seeking Taylor like expansion of (3.1), we set \( Q_k(t) = \frac{(t-a)^k}{k!} \), \( a \leq x \leq b \), then we have the expansion

\[
\frac{1}{n} \left[ \frac{(x-a)f(y) + (b-x)f(z)}{b-a} \right] + \sum_{k=1}^{n-1} \left\{ \widetilde{T}_k(y,x,z) + \widetilde{F}_k(\alpha;a,b) \right\} - \frac{1}{b-a} \int_a^b f(s)ds
\]

\[
= \frac{1}{n!(b-a)} \int_a^b (\alpha - t)^{n-1} K(t; y, x, z) f^{(n)}(t) dt,
\]

(3.22)

where

\( \widetilde{T}_k(y,x,z) := \frac{1}{(b-a)^k} \left\{ (x-a)(a-y)^k f^{(k)}(y) + (-1)^{(k-1)} (b-x)(z-a)^k f^{(k)}(z) \right\} \)

and

\( \widetilde{F}_k(\alpha;a,b) = \frac{(n-k)}{(b-a)^k} \left[ (\alpha - a)^{(k-1)} + (-1)^{(k+1)} (b - a)^k \right] \),

for all \( a \leq y \leq x \leq z \leq b \), which gives Fink representation of general two-point Ostrowski’s formula. One could get more informative representation by choosing \( \alpha = x \) in (3.22)

**Remark 5** By setting \( x = y = z = \alpha \) in (3.22), we refer to Fink representation (1.6).

Furthermore, the Fink representation for Guessab–Schmeisser formula is deduced by setting \( y = h, z = a + b - h \) and \( x = \frac{a+b}{2} \) in (3.1), so we get:
where

\[ T_k \left( h, \frac{a + b}{2}, a + b - h \right) := \frac{(-1)^k}{2} \left( Q_k(h) f^{(k)}(h) + Q_k(a + b - h) f^{(k)}(a + b - h) \right) \]

for all \( a \leq h \leq \frac{a+b}{2} \). In particular, for \( Q_k(a + b - t) = (-1)^k Q_k(t) \) \( a \leq t \leq \frac{a+b}{2} \), we have

\[
\frac{1}{n} \left[ \frac{f(h) + f(a + b - h)}{2} + \sum_{k=1}^{n-1} \left\{ T_k \left( h, \frac{a + b}{2}, a + b - h \right) + F_k(a, b) \right\} \right] 
- \frac{1}{b - a} \int_a^b f(s)ds 
= \frac{(-1)^{n-1}}{n(b-a)} \int_a^b Q_{n-1}(t)K \left( t; h, \frac{a + b}{2}, a + b - h \right) f^{(n)}(t) dt, \tag{3.23}
\]

where

\[ T_k \left( h, \frac{a + b}{2}, a + b - h \right) := \frac{(-1)^k}{2} Q_k(h) f^{(k)}(h) + (-1)^k f^{(k)}(a + b - h) \]

for all \( a \leq h \leq \frac{a+b}{2} \).

Now, by substituting \( Q_k(t) = \frac{(t-h)^k}{k!} \) in (3.23), so that we get

\[
\frac{1}{n} \left[ \frac{f(h) + f(a + b - h)}{2} + \sum_{k=1}^{n-1} \widetilde{F}_k(a, b) \right] - \frac{1}{b - a} \int_a^b f(y)dy 
= \frac{1}{n!(b-a)} \int_a^b (h - t)^{n-1} S(t, h) f^{(n)}(t) dt. \tag{3.24}
\]

Since \( Q_k(b) = (-1)^k Q_k(a) \), then

\[
\widetilde{F}_k(a, b) = \frac{(-1)^k(n-k)}{b-a} Q_k(a) f^{(k-1)}(a) - (-1)^k f^{(k-1)}(b) 
= \frac{(-1)^k(n-k)}{b-a} \frac{(a-h)^k}{k!} \left[ f^{(k-1)}(a) - (-1)^k f^{(k-1)}(b) \right] 
= \frac{(n-k)(h-a)^k}{b-a} \frac{1}{k!} \left[ f^{(k-1)}(a) + (-1)^k f^{(k-1)}(b) \right].
\]
Also, we note that
\[
Q_k\left(\frac{a + b}{2}\right) = \frac{(a+b-h)^k}{k!} = (-1)^k \frac{(h-a)^k}{k!} = (-1)^k Q_k\left(\frac{a + b}{2}\right) = (-1)^k Q_k\left(\frac{a + b - a + b}{2}\right),
\]
this gives that
\[
0 = \frac{(n-k)}{(b-a)k!} \left[ Q_k\left(\frac{a + b}{2}\right) - (-1)^k Q_k\left(\frac{a + b}{2}\right) \right]
= \frac{(n-k)}{(b-a)k!} \cdot (1 + (-1)^{k+1}) Q_k\left(\frac{a + b}{2}\right).
\]
By our choice of \(Q_k\); we have \(Q_k\left(\frac{a+b}{2}\right) = \frac{(a+b)^k}{k!}\), for all \(x \in \left[a, \frac{a+b}{2}\right]\), therefore we can write
\[
\tilde{F}_k(a, b) + 0 = \tilde{F}_k(a, b) + \frac{(n-k)}{(b-a)k!} \left[ (h-a)^k (f^{(k-1)}(a) + (-1)^{k+1} f^{(k-1)}(b)) 
+ (1 + (-1)^{k+1}) \left( \frac{a + b}{2} - h \right)^k \right]
= : G_k(h;a,b).
\]
for all \(h \in \left[a, \frac{a+b}{2}\right]\). Hence, we just proved that

**Corollary 5** Let \(I \) be a real interval, \(a, b \in I^o (a < b)\). Let \(f : I \to \mathbb{R}\) be such that \(f^{(n)}\) is absolutely continuous on \(I\) for \(n \geq 1\) with \((\cdot - t)^{n-1}S(\cdot , t)f^{(n)}(t)\) is integrable. Then we have the representation
\[
\frac{1}{n} \left[ \frac{f(x) + f(a + b - x)}{2} + \sum_{k=1}^{n-1} G(x; a, b) \right] - \frac{1}{b-a} \int_a^b f(y)dy
= \frac{1}{n!(b-a)} \int_a^b (x - t)^{n-1}S(t,x)f^{(n)}(t)dt. 
\]
(3.25)
for all \(x \in \left[a, \frac{a+b}{2}\right]\) where
for all $x \in \left[a, \frac{a+b}{2}\right]$, and

$$S(t, h) = \begin{cases} 
  t - a, & t \in [a, x] \\
  t - \frac{a+b}{2}, & t \in (x, a+b-x) \\
  t - b, & t \in [a+b-x, b]
\end{cases}$$

(3.27)

\[ G_k(x) \triangleq G_k(x; a, b) = \frac{(n-k)}{k!(b-a)} \cdot \{ (x-a)^k \left[ f^{(k-1)}(a) + (-1)^{k+1} f^{(k-1)}(b) \right] \\
+ (1 + (-1)^{k+1}) \left( \frac{a+b}{2} - x \right)^k f^{(k-1)} \left( \frac{a+b}{2} \right) \}, \quad (3.26) \]

\textbf{Theorem 10} Under the assumptions of Theorem 5. We have

$$\left| \frac{1}{n} \left( f(x) + f(a+b-x) \right) \right| + \frac{1}{b-a} \int_a^b f(y) dy \leq K(n, p, x) \left\| f^{(n)} \right\|_p$$

(3.28)

holds for all $x \in \left[ a, \frac{a+b}{2} \right]$, where

$$K(n, p, x) = \begin{cases} 
  \frac{1}{n^n (b-a)} \left( \frac{b-a}{n} \right)^{n-1} \left[ \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^n, & \text{if } p = 1 \\
  \frac{2^{1/q}}{n^n (b-a)} \left[ (x-a)^{nq+1} + \left( \frac{a+b}{2} - x \right)^{nq+1} \right]^{1/q}, & \text{if } 1 < p \leq \infty, \quad q = \frac{p}{p-1}
\end{cases}$$

(3.29)

The constant $K(n, p, x)$ is the best possible in the sense that it cannot be replaced by a smaller one.

\textbf{Proof} Utilizing the triangle integral inequality on the identity (3.23) and employing some known norm inequalities we get
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It is easy to find that for \( p = 1 \), we have

\[
\sup_{a \leq t \leq b} \left\{ |x - t|^{n-1} |S(t, x)| \right\} = \frac{1}{n} \left( \frac{n-1}{n} \right)^{n-1} \max \left\{ (x - a)^n, \left( \frac{a + b}{2} - x \right)^n \right\} = \frac{1}{n} \left( \frac{n-1}{n} \right)^{n-1} \left[ \frac{b - a}{4} + \left| x - \frac{3a + b}{4} \right| \right]^n,
\]

and for \( 1 < p < \infty \), we have

\[
\int_a^b |x - t|^{(n-1)q} |S(t, x)|^q dt = \int_a^x |x - t|^{(n-1)q} (t - a)^q dt + \int_x^{a+b-x} |x - t|^{(n-1)q} \left| t - \frac{a + b}{2} \right|^q dt + \int_{a+b-x}^b |x - t|^{(n-1)q} (b - t)^q dt = 2 \left[ (x - a)^{pq+1} + \left( \frac{a + b}{2} - x \right)^{pq+1} \right] \left( \int_0^1 (1 - s)^{(n-1)q} s^q ds \right) = 2 \left[ (x - a)^{pq+1} + \left( \frac{a + b}{2} - x \right)^{pq+1} \right] B((n-1)q + 1, q + 1)
\]
where, we use the substitutions \( t = (1 - s)a + sx \), \( t = (1 - s)x + s(a + b - x) \) and \( t = (1 - s)(a + b - x) + sb \); respectively. The third case, \( p = \infty \) holds by setting \( p = \infty \) and \( q = 1 \), i.e.,

\[
\int_a^b |x - t|^{(n-1)}|S(t, x)|dt = 2 \left[ (x - a)^{n+1} + \left( \frac{a + b}{2} - x \right)^{n+1} \right] B(n, 2),
\]

where \( B(\cdot, \cdot) \) is the Euler beta function. To argue the sharpness, we consider first when \( 1 < p \leq \infty \), so that the equality in (3.28) holds when

\[
f^{(n)}(t) = |x - t|^{(n-1)q-1}|S(t, x)|^{q-1} \text{sgn}\{ (x - t)^{n-1}S(t, x) \},
\]

thus the inequality (3.28) holds for \( 1 < p \leq \infty \). In case that \( p = 1 \), setting

\[
g(t, x) = (x - t)^{n-1}S(t, x) \quad \forall x \in [a, \frac{a+b}{2}],
\]

let \( t_0 \) be the point that gives the supremum. If \( t_0 = \frac{x+(n-1)a}{n} \), we take

\[
f^{(n)}_\varepsilon(t) = \begin{cases} \varepsilon^{-1}, & t \in (t_0 - \varepsilon, t_0) \\ 0, & \text{otherwise} \end{cases}
\]

Since

\[
\left| \int_a^b g(t, x)f^{(n)}(t)dt \right| = \frac{1}{\varepsilon} \left| \int_{t_0-\varepsilon}^{t_0} g(t, x)dt \right| \leq \frac{1}{\varepsilon} \int_{t_0-\varepsilon}^{t_0} |g(t, x)|dt \\
\leq \sup_{t_0-\varepsilon \leq t \leq t_0} |g(t, x)| \cdot \frac{1}{\varepsilon} \int_{t_0-\varepsilon}^{t_0} dt \\
= \left| g(t_0, x) \right|,
\]

also, we have

\[
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{t_0-\varepsilon}^{t_0} |g(t, x)|dt = \left| g(t_0, x) \right| = C(n, 1, x)
\]

proving that \( C(n, 1, x) \) is the best possible. \qed
A representation of Cerone-Dragomir formula [16] (see also [17]) via Fink approach can be also deduced from (3.1) by setting $y = a$, $z = b$ and $x \in [a, b]$. Taylor type expansion can be also given using the representation (3.15).

### 4 Error bounds via Chebyshev–Grüss type inequalities

In this section, we highlight the role of Čebyšev functional in integral approximations by using the Čebyšev–Grüss type inequalities (3.25).

The famous Čebyšev functional

$$
\mathcal{T}(h_1, h_1) = \frac{1}{d-c} \int_c^d h_1(t)h_2(t)dt - \frac{1}{d-c} \int_c^d h_1(t)dt \cdot \frac{1}{d-c} \int_c^d h_2(t)dt.
$$

(4.1)

has multiple applications in several area of mathematical sciences specially in Integral Approximations of real functions. For more detailed history see [47].

It is well known that the pre–Grüss inequality reads:

$$
\left| \mathcal{T}(h_1, h_2) \right| \leq \sqrt{\mathcal{T}(h_1, h_1)} \sqrt{\mathcal{T}(h_2, h_2)},
$$

(4.2)

for all measurable functions $h_1, h_2$ defined on $[a, b]$. This inequality was used by Grüss to prove the second inequality in (4.1). A ramified inequality could be deduced as follows:

$$
\left| \mathcal{T}(h_1, h_2) \right| \leq \frac{1}{2}(\Phi - \phi)\sqrt{\mathcal{T}(h_2, h_2)},
$$

(4.3)

where $h_1, h_2 : [a, b] \to \mathbb{R}$ are assumed to be such that $h_2$ is integrable and $h_2$ is measurable bounded on $[a, b]$, i.e., there exist constants $\phi, \Phi$ such that $\phi \leq h_2(t) \leq \Phi$, for $t \in [a, b]$.

The most famous bounds of the Čebyšev functional are incorporated in the following theorem:

**Theorem 11** Let $f, g : [c, d] \to \mathbb{R}$ be two absolutely continuous functions, then

$$
\left| \mathcal{T}(h_1, h_2) \right| \leq
\begin{cases}
\frac{(d-c)^2}{12} \left\| h'_1 \right\|_\infty \left\| h'_2 \right\|_\infty, & \text{if } h'_1, h'_2 \in L_\infty([c, d]), \text{ proved in } \text{[21]} \\
\frac{1}{4} (M_1 - m_1)(M_2 - m_2), & \text{if } m_1 \leq h_1 \leq M_1, \ m_2 \leq h_2 \leq M_2, \text{ proved in } \text{[40]} \\
\frac{(d-c)^2}{\pi^2} \left\| h'_1 \right\|_2 \left\| h'_2 \right\|_2, & \text{if } h'_1, h'_2 \in L_2([c, d]), \text{ proved in } \text{[45]} \\
\frac{1}{8} (d-c)(M - m) \left\| h'_2 \right\|_\infty, & \text{if } m \leq h_1 \leq M, \ h'_2 \in L_\infty([c, d]), \text{ proved in } \text{[49]}
\end{cases}
$$

(4.4)
The constants $\frac{1}{12}, \frac{1}{4}, \frac{1}{2}$ and $\frac{1}{8}$ are the best possible.

Setting $h_1(t) = \frac{1}{n!} f^{(n)}(t)$ and $h_2(t) = (x - t)^{n-1} S(t, x)$, we have

$$C(h_1, h_2) = \frac{1}{n!(b-a)} \int_a^b (x - t)^{n-1} S(t, x) f^{(n)}(t) dt$$

$$- \frac{1}{n!} \cdot \frac{1}{b-a} \int_a^b (x - t)^{n-1} S(t, x) dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t) dt$$

$$= \frac{1}{n!(b-a)} \int_a^b (x - t)^{n-1} S(t, x) f^{(n)}(t) dt$$

$$- \frac{2}{n!(b-a)} \left[ (x-a)^{n+1} + \left( \frac{a+b}{2} - x \right)^{n+1} \right] \frac{B(n, 2)}{b-a}$$

which means

$$C(h_1, h_2) = \frac{1}{n} \left( \frac{f(x) + f(a + b - x)}{2} + \sum_{k=1}^{n-1} G_k(x) \right)$$

$$- \frac{1}{b-a} \int_a^b f(y) dy$$

$$- \frac{2}{(n+1)!n(b-a)} \left[ (x-a)^{n+1} + \left( \frac{a+b}{2} - x \right)^{n+1} \right] \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}$$

$$:= P(f; x, n).$$

for all $x \in \left[ a, \frac{a+b}{2} \right]$.

**Theorem 12** Let $I$ be a real interval, $a, b \in I^\circ$ ($a < b$). Let $f : I \to \mathbb{R}$ be $(n+1)$-times differentiable on $I^\circ$ such that $f^{(n+1)}$ is absolutely continuous on $I^\circ$ with $(\cdot - t)^{n-1} k(t, \cdot) f^{(n)}(t)$ is integrable. Then, for all $n \geq 2$ we have

$$|P(f; x, n)| \leq \left\{ \begin{array}{ll}
(b-a)^2 \left( \frac{n-2}{n} \right)^{n-2} \left( \frac{n^2-2n+2}{12n(n^2)} \right)^{n-1} + \left[ x - \frac{3a+b}{4} \right]^{n-1} \cdot \|f^{(n+1)}\|_{\infty}, & \text{if } f^{(n+1)} \in L_\infty([a, b]) \\
(b-a)^2 \left( \frac{n-2}{n} \right)^{n-2} \left( \frac{n^2-2n+2}{4n(n^2)} \right)^{n-2} \left( \frac{n^2-2n+2}{2n(n^2)} \right)^{n-2} (b-a)^{-2n} \cdot (M - m), & \text{if } m \leq f^{(n)} \leq M,
\end{array} \right.$$
holds for all \( x \in \left[ a, \frac{a+b}{2} \right] \), where
\[
A(n) = \frac{2(n-1)^2}{(2n-1)(2n-2)(2n-3)}
\]
and
\[
B(n) = \frac{2^{2n-3}(2n-1)(2n-2) + 4n(2n-1) + 2n^2}{(2n-1)(2n-2)(2n-3)}
\]
\( \forall n \geq 2 \).

**Proof** • If \( f^{(n+1)} \in L^\infty([a,b]) \): Applying the first inequality in (4.4), it is not difficult to observe that
\[
\sup_{a \leq t \leq b} \left\{ \left| h_1'(t) \right| \right\} = \frac{1}{n!} \| f^{(n+1)} \|_{\infty} \quad \text{and}
\]
\[
\sup_{a \leq t \leq b} \left\{ \left| h_2'(t) \right| \right\} = \left( \frac{n-2}{n} \right)^{n-2} \frac{n^2 - 2n + 2}{n} \left[ \frac{b-a}{4} + \left| x - \frac{3a + b}{4} \right| \right]^{n-1}, \quad \forall n \geq 2.
\]
So that
\[
|P(f; x, n)| \leq (b-a)^2 \left( \frac{n-2}{n} \right)^{n-2} \frac{n^2 - 2n + 2}{12n \cdot n!} \left[ \frac{b-a}{4} + \left| x - \frac{3a + b}{4} \right| \right]^{n-1} \cdot \frac{1}{n!} \| f^{(n+1)} \|_{\infty}.
\]
• If \( m \leq f^{(n)}(t) \leq M \), for some \( m, M > 0 \): Applying the second inequality in (4.4), we get
\[
|P(f; x, n)| \leq \frac{n^2 - 2n + 2}{4n \cdot n!} \left( \frac{n-2}{n} \right)^{n-2} \left( 2^{n-2} - 2^{2n-2} \right) (b-a)^{n-2} \cdot \frac{1}{n!} (M - m).
\]
• If \( f^{(n+1)} \in L^2([a,b]) \): Applying the third inequality in (4.4), we get
\[
|P(f; x, n)| \leq \frac{(b-a)}{n! \pi^2} \cdot \sqrt{A(n)(x-a)^{2n-1} + B(n)\left( \frac{a+b}{2} - x \right)^{2n-1}} \cdot \frac{1}{n!} \| f^{(n+1)} \|_2.
\]
\( \forall n \geq 2 \), where \( A(n) \) and \( B(n) \) are defined above.
• If \( m \leq f^{(n)}(t) \leq M \), for some \( m, M > 0 \): Applying the forth inequality in (4.4), we get
\[
|P(f; x, n)| \leq (b-a) \left( \frac{n-2}{n} \right)^{n-2} \frac{n^2 - 2n + 2}{8n \cdot n!} \left[ \frac{b-a}{4} + \left| x - \frac{3a + b}{4} \right| \right]^{n-1} \cdot \frac{1}{n!} (M - m).
\]
By applying the forth inequality again the with dual assumptions, i.e., \( f^{(n+1)} \in L^\infty([a,b]) \), we have

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\[ |P(f;x,n)| \leq \frac{n^2 - 2n + 2}{8n \cdot n!} \left( \frac{n - 2}{n} \right)^{n-2} (2^{-n-2} - 2^{-2n-2}) (b - a)^n \cdot \frac{1}{n!} \left\| f^{(n+1)} \right\|_\infty. \]

Hence the proof is completely established. \( \square \)

**Corollary 6** Let assumptions of Theorem 12 hold. If moreover, \( f^{(a-1)}(a) = f^{(a-1)}(b) \) \( (n \geq 2) \), then the inequality

\[
\left| \frac{1}{n} \left( f(x) + f(a + b - x) \right) - \frac{1}{b - a} \int_a^b f(y) dy \right| \leq \begin{cases} 
(b - a)^2 \left( \frac{n - 2}{n} \right)^{n-2} 2^{n-1} & \text{if } f^{(n+1)} \in L_\infty([a,b]) \\
\frac{b - a}{(n!)^2 \pi^2} A(n)(x - a)^{2n-1} + B(n) \left( \frac{a + b}{2} - x \right)^{2n-1} & \text{if } f^{(n+1)} \in L_2([a,b]), \\
(b - a)^2 \left( \frac{n - 2}{n} \right)^{n-2} 2^{n-1} & \text{if } f^{(n+1)} \in L_\infty([a,b]), \\
\frac{b - a}{(n!)^2 \pi^2} A(n)(x - a)^{2n-1} + B(n) \left( \frac{a + b}{2} - x \right)^{2n-1} & \text{if } f^{(n+1)} \in L_2([a,b]), \\
\end{cases}
\]

holds for all \( x \in \left[ a, \frac{a+b}{2} \right] \), where

\[
A(n) = \frac{2(n - 1)^2}{(2n - 1)(2n - 2)(2n - 3)}
\]

and

\[
B(n) = \frac{2^{2n-3}(2n - 1)(2n - 2) + 4n(2n - 1) + 2n^2}{(2n - 1)(2n - 2)(2n - 3)}
\]

\( \forall n \geq 2. \)

**Remark 7** By setting \( h_1(t) = \frac{1}{n} f^{(n)}(t) k(t,x) \) and \( h_2(t) = (x - t)^{n-1} \), we obtain that

\[
C(h_1, h_2) = \frac{1}{n} \left( f(x) + f(a + b - x) \right) + \sum_{k=1}^{n-1} G_k(x) - \frac{1}{b - a} \int_a^b f(y) dy \\
- \frac{1}{n!} \frac{(x - a)^n - (x - b)^n}{n(b - a)} \cdot \frac{f^{(n)}(x) + f^{(n)}(a + b - x)}{2}
\]

\( \equiv Q(f;x,n). \)

Applying Theorem 11 Chebyshev type bounds for \( Q(f;x,n) \) can be proved. We shall omit the details.

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**Remark 8** Bounds for the generalized formula (3.23) via Chebyshev-Grüss type inequalities can be done by setting $h_1(t) = \frac{(-1)^{n-1}}{n} f^{(n)}(t)$ and $h_2(t) = Q_{n-1}(t) S(t, x)$, therefore we have

$$C(h_1, h_2)$$

$$= \frac{(-1)^{n-1}}{n(b-a)} \int_a^b Q_{n-1}(t) S(t, x) f^{(n)}(t) dt$$

$$- \frac{1}{b-a} \int_a^b Q_{n-1}(t) S(t, x) dt \times \frac{(-1)^{n-1}}{n(b-a)} \int_a^b f^{(n)}(t) dt$$

$$= \frac{1}{n(b-a)} \int_a^b Q_{n-1}(t) S(t, x) f^{(n)}(t) dt$$

$$- \frac{1}{b-a} \int_a^b Q'(n) S(t, x) dt \times \frac{(-1)^{n-1}}{n} \cdot \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}$$

$$= \frac{(-1)^{n-1}}{n(b-a)} \int_a^b Q_{n-1}(t) S(t, x) f^{(n)}(t) dt$$

$$- \left[ \frac{Q_n(x) + Q_n(a + b - x)}{2} - \frac{Q_{n+1}(b) - Q_{n+1}(a)}{b-a} \right]$$

$$\times \frac{(-1)^{n-1}}{n} \cdot \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}$$

$$= \mathcal{L}(f, Q_n, x)$$

for all $x \in \left[ a, \frac{a+b}{2} \right]$. We left the representations to the reader.

**Theorem 13** Let $I$ be a real interval, $a, b \in I^0$ $(a < b)$. Let $f : I \to \mathbb{R}$ be $(n+1)$-times differentiable on $I^0$ such that $f^{(n+1)}$ is absolutely continuous on $I^0$ with $(-t)^{n}k(t, \cdot)f^{(n)}(t)$ is integrable. Then, for all $n \geq 2$ we have

$$|\mathcal{L}(f, Q_n, x)|$$

$$\leq \begin{cases} \frac{(b-a)^2}{12n} \|Q_{n-1} + Q_{n-2}S(\cdot, x)\|_\infty \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]) \\ \frac{1}{4n} (M_1 - m_1) (M_2 - m_2), & \text{if } m_1 \leq f^{(n)} \leq M_1, \\ \frac{b-a}{8n} D(n, x) \cdot \|f^{(n+1)}\|_2, & \text{if } f^{(n+1)} \in L_2([a, b]), \\ \frac{b-a}{8n} Q_{n-1} + Q_{n-2}S(\cdot, x) \|\infty \cdot (M_1 - m_1), & \text{if } m_1 \leq f^{(n)} \leq M_1, \\ \frac{b-a}{8n} (M_2 - m_2) \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]), \end{cases}$$

(4.9)
holds for all \( x \in \left[a, \frac{a+b}{2}\right] \), where

\[
M_2 := \max_{a \leq t \leq b} \{Q_{n-1}(t)S(t,x)\}, \quad m_2 := \min_{a \leq t \leq b} \{Q_{n-1}(t)S(t,x)\}
\]

and

\[
D(n, x) = \left( \int_a^b \left| Q_{n-1}(t) + Q_{n-2}(t)S(t,x) \right|^2 dt \right)^{1/2} \quad \forall n \geq 2.
\]

**Proof** The proof of the result follows directly by applying Theorem 11 to the functions \( h_1(t) = \left(-1\right)^{n-1} f^{(n)}(t) \) and \( h_2(t) = P_{n-1}(t)S(t,x) \) as shown previously in Remark 8 and the rest of the proof done using Theorem 12. \( \square \)

**Corollary 7** Let assumptions of Theorem 13 hold. If moreover, \( f^{(n-1)}(a) = f^{(n-1)}(b) \) \((n \geq 2)\), then the inequality

\[
\left| \frac{1}{n} \left( f(x) + f(a + b - x) \right) \left(\frac{1}{2} + \sum_{k=1}^{n-1} \left\{ T_k(x) + \widetilde{T}_k(a, b) \right\} \right) - \frac{1}{b-a} \int_a^b f(y)dy \right| \leq \begin{cases} 
\frac{(b-a)^2}{12n} \left\{ Q_{n-1} + Q_{n-2}S(\cdot, x) \right\} \|f^{(n+1)}\|_2, & \text{if } f^{(n+1)} \in L_2([a, b]), \\
\frac{1}{4n} \left( M_1 - m_1 \right) \left( M_2 - m_2 \right), & \text{if } m_1 \leq f^{(n)} \leq M_1, \\
\frac{b-a}{8n} \left\{ Q_{n-1} + Q_{n-2}S(\cdot, x) \right\} \|f^{(n+1)}\|_\infty \cdot \left( M_1 - m_1 \right), & \text{if } m_1 \leq f^{(n)} \leq M_1, \\
\frac{b-a}{8n} \left( M_2 - m_2 \right) \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]).
\end{cases}
\]

holds for all \( x \in \left[a, \frac{a+b}{2}\right] \).

**Remark 9** In all above estimates, if one assumes that \( f^{(n)} \) is convex, \( r \)-convex, quasi-convex, \( s \)-convex, \( P \)-convex, or \( Q \)-convex; we can obtain other new bounds involving convexity.

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Compliance with ethical standards

Ethical approval This article does not contain any studies with human participants performed by any of the authors.

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