Gaussian estimates for heat kernels of higher order Schrödinger operators with potentials in generalized Schechter classes

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Abstract
Let $m \in \mathbb{N}$, $P(D) := \sum_{|\alpha|=2m} (-1)^{m} a_{\alpha} D^{\alpha}$ be a $2m$-order homogeneous elliptic operator with real constant coefficients on $\mathbb{R}^{n}$, and $V$ a real-valued measurable function on $\mathbb{R}^{n}$. In this article, the authors introduce a new generalized Schechter class concerning $V$ and show that the higher order Schrödinger operator $\mathcal{L} := P(D) + V$ possesses a heat kernel that satisfies the Gaussian upper bound and the Hölder regularity when $V$ belongs to this new class. The Davies–Gaffney estimates for the associated semigroup and their local versions are also given. These results pave the way for many further studies on the analysis of $\mathcal{L}$.

MSC (2020)
35K08 (primary), 35J10, 35J30, 47G40 (secondary)

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INTRODUCTION

The analysis of the Schrödinger operator is an important topic in various fields of mathematics and physics (see, for instance, [45, 54, 58, 59]). Many aspects of this analysis focus on the estimates of the corresponding heat kernel, because the latter encodes plenty of information related to the operator, such as the structure of the parabolic equation and the geometry of the underlying space (see, for instance, [9, 14, 31, 32, 47]), and hence has wide applications; see, for instance, [14, 46, 52] for the spectral properties of differential operators, [13, 20, 21, 42, 49, 60] for the boundedness of some singular integral operators such as Riesz transforms, and [24–26, 37, 38, 43, 65, 66] for the function spaces associated with the Schrödinger operator.

In what follows, we use \( L^1_{\text{loc}}(\mathbb{R}^n) \) to denote the set of all locally integrable functions on \( \mathbb{R}^n \). To estimate the heat kernel of the Schrödinger operator, various conditions on the potential are proposed. Let \( -\Delta + V \) be the time-independent Schrödinger operator on the Euclidean space \( \mathbb{R}^n \) with \( \Delta := \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \) being the Laplace operator and the potential \( V \) a real-valued measurable function on \( \mathbb{R}^n \). If \( V \in L^1_{\text{loc}}(\mathbb{R}^n) \) is nonnegative, then, by the Feynman–Kac formula (see, for instance, [58]), it is well known that \( -\Delta + V \) possesses a heat kernel \( p_t \) on \( (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \) that satisfies the Gaussian upper bound: there exist some positive constants \( C \) and \( c_1 \) such that, for any \( t \in (0, \infty) \) and \( x, y \in \mathbb{R}^n \),

\[
0 \leq p_t(x, y) \leq \frac{C}{t^{n/2}} \exp \left\{ -c_1 \frac{|x - y|^2}{t} \right\}.
\]

Note that the condition \( 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n) \) is satisfied when \( V(x) := |x|^a \) for any \( x \in \mathbb{R}^n \setminus \{0\} \), with \( a \in (-n, \infty) \). Here and thereafter, \( \mathbf{0}_n \) denotes the origin of \( \mathbb{R}^n \). If \( V \) has a negative part, then the Gaussian upper bound (1.1) still holds true as long as \( V \in L^p(\mathbb{R}^n) \) with \( p \in (n/2, \infty) \) (see [67]). It is known that the Lebesgue space \( L^p(\mathbb{R}^n), p \in (n/2, \infty) \), is contained in a larger Kato class.
\( K_2(\mathbb{R}^n) \) (see [58] and (2.4) below for the definition of \( K_2(\mathbb{R}^n) \)). In the same article, Simon showed that the heat kernel of \(-\Delta + V\) satisfies the local Gaussian upper bound: there exist positive constants \( C, c_1, \) and \( w \) such that, for any \( t \in (0, \infty) \) and \( x, y \in \mathbb{R}^n \),

\[
0 \leq p_t(x, y) \leq \frac{C}{t^{n/2}} \exp \left\{ -c_1 \frac{|x - y|^2}{t} \right\} \tag{1.2}
\]

if \( V \) is in \( K_2(\mathbb{R}^n) \) (see also [48]). The nonnegative constant \( w \) in (1.2) can be zero if \( V \) satisfies some additional conditions (see \([11, 18, 67]\)). Recall that, if \( V(x) := -|x|^a \) for any \( x \in \mathbb{R}^n \setminus \{0\} \), then \( V \in K_2(\mathbb{R}^n) \) if and only if \( a \in (-2, 0) \) (see [58] or Remark 2.6 below). For the critical case \( a = -2 \), it is known that the Gaussian upper bounds, (1.1) and (1.2), may break down (see \([41, 50]\)).

In what follows, let \( \mathbb{N} := \{1, 2, \ldots\}, \mathbb{Z}_+ := \mathbb{N} \cup \{0\}, D := (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) \), and, for any \( \alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n := (\mathbb{Z}_+)^n \), \( |\alpha| := \alpha_1 + \cdots + \alpha_n \) and \( D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \). Let \( \mathcal{L} := P(D) + V \) be a higher order Schrödinger operator on \( \mathbb{R}^n \) with \( P(D) \) being a higher order elliptic operator. The techniques that can be used to estimate the heat kernel of \( \mathcal{L} \) are limited, due to the failure of many good properties of the Laplacian \( \Delta \) when the order of the unperturbed operator becomes higher. For instance, it is known that the biharmonic heat kernel of the operator \( \Delta^2 \) may change sign infinitely many times (see [29]). This indicates that the associated semigroup no longer preserves the positivity and hence we lose the effect of the Feynman–Kac formula which is a fundamental tool in the analysis of the second order Schrödinger operator (see \([8, 16]\) for some excellent expositions on the state of art of higher order elliptic operators and their perturbations).

Let \( m \in \mathbb{N} \) satisfy \( n < 2m \). If the unperturbed operator

\[
P(D) := \sum_{|\alpha| = m = |\beta|} (-1)^m D^\alpha (a_{\alpha, \beta}(x)D^\beta)
\]

is a homogeneous uniformly elliptic operator of order \( 2m \) on \( \mathbb{R}^n \) and the potential \( V \in L^1_{\text{loc}}(\mathbb{R}^n) \) is nonnegative locally integrable on \( \mathbb{R}^n \), Barbatis and Davies [7] showed that \( \mathcal{L} \) possesses a heat kernel \( p_t \) on \((0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \) that satisfies the following higher order Gaussian upper bound: there exist some positive constants \( C \) and \( c_2 \) such that, for any \( t \in (0, \infty) \) and \( x, y \in \mathbb{R}^n \),

\[
|p_t(x, y)| \leq \frac{C}{t^{n/(2m)}} \exp \left\{ -c_2 \frac{|x - y|^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\}. \tag{1.3}
\]

The key idea used to prove (1.3) in [7] is an exponential perturbation argument. To be precise, let \( Q(f, f) \) be the quadratic form associated with \( \mathcal{L} \) for any suitable function \( f \) in its domain. Barbatis and Davies [7] considered the exponential perturbation

\[
Q_{\lambda, \phi}(f, f) := Q(e^{\lambda \phi} f, e^{-\lambda \phi} f)
\]

of \( Q(f, f) \) for some \( \lambda \in \mathbb{R} \) and \( \phi \) being some smooth function on \( \mathbb{R}^n \). They proved an \( L^2(\mathbb{R}^n) \) form perturbation estimate of the type that, for any \( \varepsilon \in (0, \infty) \), there exists a positive constant \( C_{(\varepsilon)} \), independent of \( f \), such that, for any \( f \) in the domain of \( Q \) on \( L^2(\mathbb{R}^n) \),

\[
|Q_{\lambda, \phi}(f, f) - Q(f, f)| \leq \varepsilon Q(f, f) + C_{(\varepsilon)} \lambda^{2m} \|f\|_{L^2(\mathbb{R}^n)}^2. \tag{1.4}
\]
Here and thereafter, for any $q \in [1, \infty]$, we use $L^q(\mathbb{R}^n)$ to denote the set of all measurable functions $f$ on $\mathbb{R}^n$ such that

$$
\|f\|_{L^q(\mathbb{R}^n)} := \left[ \int_{\mathbb{R}^n} |f(x)|^q \, dx \right]^{1/q} < \infty
$$

with the usual modification made when $q = \infty$. This, together with the Sobolev embedding

$$
W^{m,2}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n), \quad (1.5)
$$

indicates the following ultracontractivity of the associated semigroup $\{e^{-tL_{\lambda,\phi}}\}_{t>0}$ that there exists a positive constant $C$ such that, for any $g \in L^2(\mathbb{R}^n)$,

$$
\left\|e^{-tL_{\lambda,\phi}} g\right\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-n/(2m)} \|g\|_{L^2(\mathbb{R}^n)}; \quad (1.6)
$$

from this and an optimizing argument, it follows that (1.3) holds true. Note that, since the Sobolev embedding (1.5) holds true only in the case $n < 2m$, the above argument works also only in this case.

To extend the estimates of the heat kernel to the general case $n \in \mathbb{N}$, Deng et al. [19] studied the situation that the unperturbed operator

$$
P(D) := \sum_{|\alpha|=2m} (-1)^m a_\alpha D^\alpha
$$

is a nonnegative homogeneous elliptic operator of order $2m$, which has real constant coefficients $\{a_\alpha\}_{|\alpha|=2m}$, and $V$ is some Kato perturbation of $P(D)$, where the latter means that, for any $\epsilon \in (0, \infty)$, there exists a positive constant $C_\epsilon$ such that, for any $f \in C_\epsilon^\infty(\mathbb{R}^n)$ (the set of all infinitely differentiable functions with compact support),

$$
\|Vf\|_{L^1(\mathbb{R}^n)} \leq \epsilon \|P(D)f\|_{L^1(\mathbb{R}^n)} + C_\epsilon \|f\|_{L^1(\mathbb{R}^n)}. \quad (1.7)
$$

Under this condition, Deng et al. [19] proved the following $L^1(\mathbb{R}^n)$ endpoint operator perturbation estimates of the type that: for any $\epsilon \in (0, \infty)$, there exists a positive constant $C_\epsilon$ such that, for any $f$ in the domain of $L$ on $L^1(\mathbb{R}^n)$,

$$
\left\|L_{\lambda,\phi} f - P(D)f\right\|_{L^1(\mathbb{R}^n)} \leq \epsilon \|P(D)f\|_{L^1(\mathbb{R}^n)} + C_\epsilon \|f\|_{L^1(\mathbb{R}^n)}, \quad (1.8)
$$

where $L_{\lambda,\phi}$ denotes the exponential perturbation of $L$ associated with the form $Q_{\lambda,\phi}$ as in (1.4). Since $[L^1(\mathbb{R}^n)]^* = L^\infty(\mathbb{R}^n)$, one can avoid to use the Sobolev inequality (1.5), but apply the dual and some iteration arguments, in order to derive (1.6). In view of this, Deng et al. [19] proved the following local version of (1.3) under the case $n \in \mathbb{N}$ and $P(D)$ being a nonnegative homogeneous elliptic operator of order $2m$ with real constant coefficients, that is, there exist positive constants $C$, $c_2$, and $w$ such that, for any $t \in (0, \infty)$ and $x, y \in \mathbb{R}^n$,

$$
|p_t(x,y)| \leq \frac{C}{t^{n/(2m)}} \exp \left\{ w t - c_2 \frac{|x - y|^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\} \quad (1.9)
$$
if $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a Kato type perturbation of $P(D)$. Recall that, if $V$ is in the higher order Kato class $K_{2m}(\mathbb{R}^n)$ [see (2.4) below for its definition], then $V$ is a Kato type perturbation of $P(D)$ (see [19, 68]). Further developments on the local estimates of the type (1.9) can be found in [5, 6, 28, 39].

As was pointed out in [19], it is still unknown whether or not the local positive constant $w$ in (1.9) can be zero even when $V \geq 0$. This is because the Kato perturbation (1.7) can only produce estimates of the type (1.8) with constant $1 + \lambda^{2m}$, rather than $\lambda^{2m}$ as in (1.4). Note that, if $w = 0$ in (1.9), then the local Gaussian upper bound (1.9) becomes the global Gaussian upper bound (1.3).

In the latter case, much better properties related to $L$ can be obtained (see, for instance, [9, 14, 16, 31, 32, 36]), because the Gaussian exponential term still works even when $t \in [1, \infty)$. Motivated by the aforementioned results, it is natural to ask the following question.

**Question.** Under what general conditions on $V$, can the local constant $w$ in (1.9) be zero?

In this article, we give an affirmative answer to this question by introducing a new general Schechter class on $V$ (see Definition 2.3 below for its definition), motivated by the now called Schechter class introduced in [55]. This new potential class, which coincides with the aforementioned Kato class in some special cases (see Proposition 2.5 below), enables us to make $w = 0$ in (1.9) and, therefore, to obtain a global Gaussian upper bound for the heat kernel of the type (1.3) for any $n, m \in \mathbb{N}$. Moreover, the Hölder regularity of the heat kernel is also established in the case $n \geq 2m$. To be precise, the main result of this article is as follows.

**Theorem 1.1.** Let $m \in \mathbb{N}, V$ be a real-valued measurable function on $\mathbb{R}^n$, and $L := P(D) + V$ the $2m$-order Schrödinger operator on $\mathbb{R}^n$ as in (2.3) with $P(D)$ being the $2m$-order homogeneous real constant coefficient elliptic operator as in (2.2). If (5.1) and one of (5.2) through (5.5) hold true for any $q \in (1, 2]$ or $[2, \infty)$, and

$$\sup_{|\lambda| \in (0, \infty)} M_{|\lambda|}(V) < 1$$

with $M_{|\lambda|}(V)$ as in (5.6), then the operator $L$ possesses a heat kernel $p_t$ on $(0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$ that satisfies the following estimates.

(a) There exist positive constants $C$ and $c_3$ such that, for any $t \in (0, \infty)$ and $x, y \in \mathbb{R}^n$,

$$|p_t(x, y)| \leq \frac{C}{t^{n/(2m)}} \exp \left\{ -c_3 \frac{|x - y|^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\}. \tag{1.10}$$

(b) If, in addition, $n \geq 2m$, then there exist a $\gamma \in (0, 1)$ and positive constants $C$ and $c_4$ such that, for any $t \in (0, \infty)$ and $x, y, h \in \mathbb{R}^n$ satisfying $|h| < t^{1/2m}$,

$$|p_t(x + h, y) - p_t(x, y)| \leq \frac{C}{t^{n/(2m)}} \exp \left\{ -c_4 \frac{|x - y|^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\} \left[ \frac{|h|}{t^{1/(2m)}} \right]^{\gamma}. \tag{1.11}$$

Theorem 1.1 and its local version (see Corollary 5.9 below) are proved in Section 5.3. Recall that the method used in [7] can obtain a global Gaussian upper bound (1.3), but has the dimension restriction $n < 2m$. The method used in [19] works in the general case $n \in \mathbb{N}$, but can only obtain
the local Gaussian upper bound (1.9). Thus, Theorem 1.1 makes the first effort to obtain the global Gaussian upper bound for any dimension $n \in \mathbb{N}$.

To prove Theorem 1.1, we develop a systematic treatment on the estimates of the heat kernel of the higher order Schrödinger operator $\mathcal{L} = P(D) + V$ from the condition of the potential $V$, via the spectral perturbation. More precisely, this treatment is divided into the following three steps.

(i) From the generalized Schechter condition on $V$, we first deduce a series of boundedness of the $T$-operator $T_{s,\delta}$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for any given $p, q \in (1, \infty)$, $\delta \in (0, \infty)$, and $s \in (0, 2m]$, where

$$T_{s,\delta} := V(\delta^2 - \Delta)^{-s/2} \tag{1.12}$$

and $(\delta^2 - \Delta)^{-s/2}$ is the Bessel potential of order $s$ (see Propositions 3.3, 3.6, 3.9, and 3.10).

(ii) Using the boundedness of the $T$-operator in (1.12), we establish the exponential perturbed resolvent estimate [see (5.17) below] that there exists a positive constant $C$ such that, for any $\lambda \in \rho(\mathcal{L})$ (the resolvent set of $\mathcal{L}$) and $f \in L^2(\mathbb{R}^n)$,

$$\left\| (\lambda - \mathcal{L}_\eta)^{-(l+1)} f \right\|_{L^\infty(\mathbb{R}^n)} \leq C |\lambda|^{\frac{n}{2m} - (l+1)} \left\| f \right\|_{L^2(\mathbb{R}^n)} \tag{1.13}$$

uniformly for certain $l \in \mathbb{N}$ and $\eta \in \mathbb{C}^n$, where $\mathcal{L}_\eta$ denotes the exponential perturbation of $\mathcal{L}$ [see (4.15) below]. This exponential perturbed resolvent estimate replaces the ultracontractivity (1.6) of the semigroup used in [7, 19].

(iii) From the resolvent estimate (1.13), we finally deduce the Gaussian estimates, (1.10) and (1.11), of $\mathcal{L}$. An essential tool used in this step is the following functional calculus identity, originally appearing in [3], that

$$e^{-t\mathcal{L}} = \frac{[2(l+1) - 1]!}{2\pi i(-t)^{2l(l+1)-1}} \int_\Gamma e^{-t\lambda}(\lambda - \mathcal{L})^{-2(l+1)} d\lambda \tag{1.14}$$

with $l \in \mathbb{N}$ and $\Gamma$ being a path in $\rho(\mathcal{L})$ (see Figure 2 below). The exponential terms in the Gaussian estimates, (1.10) and (1.11), then come from the exponential perturbed resolvent identity [see (4.29)].

We point out that the parameters $\delta$ and $\lambda$ in (1.12) and (1.13) are connected by the relation $\delta = |\lambda|^{1/(2m)}$. This, in view of (1.14), indicates that we need all values of $\lambda$ in the path $\Gamma$ and hence all $\delta \in (0, \infty)$ in the proof of Theorem 1.1. That is why all $\delta \in (0, \infty)$ are taken into consideration in the definition of the generalized Schechter class [see (2.9)]. Recall that, in the case of the Kato class [see (2.4) below], only the information of $\delta$ near 0 is considered. Also, since the functions in the generalized Schechter class may have negative parts, Theorem 1.1 is new even when $n < 2m$.

The proof of Theorem 1.1(b) depends on the following Sobolev embedding

$$W^{2m,q}(\mathbb{R}^n) \subset C^\gamma(\mathbb{R}^n)$$

with $C^\gamma(\mathbb{R}^n)$ being the Lipschitz space of order $\gamma := 2m - n/q \in (0, 1)$, when $q \in (1, \infty)$ and $(2m - 1)q < n < 2mq$ [see Lemma 5.8(ii)], where the latter condition implies the dimension condition $n \geq 2m$.

Using an approach similar to that used in the proof of Theorem 1.1, we are able to establish the following Davies–Gaffney estimates of the heat semigroup generated by $-\mathcal{L}$. Recall that, for many
Schrödinger operators, they may even not possess a heat kernel, not to mention the (local) Gaussian upper bound (see [10, 41]). In such a case, the Davies–Gaffney estimates are good substitutes. The following Theorem 1.2 and its local version (see Corollary 5.5) are proved in Section 5.2.

**Theorem 1.2.** Let $m \in \mathbb{N}$, $V$ be a real-valued measurable function on $\mathbb{R}^n$, and $\mathcal{L} := P(D) + V$ the $2m$-order Schrödinger operator on $\mathbb{R}^n$ as in (2.3) with $P(D)$ being the $2m$-order homogeneous real constant coefficient elliptic operator as in (2.2). Suppose that (5.1) and one of (5.2) through (5.5) hold true with $q = 2$, and

$$\sup_{|\lambda| \in (0, \infty)} M_{|\lambda|}(V) < 1$$

with $M_{|\lambda|}(V)$ as in (5.6). Assuming further that $\mathcal{L}$ satisfies (5.7) for any $\lambda \in \Sigma_{\partial_0}^C$ with $\partial_0 \in (0, \pi/2)$ [see (4.1) below for the definition of $\Sigma_{\partial_0}^C$], then there exist positive constants $C$ and $c_5$ such that, for any disjoint compact convex subsets $E$ and $F$, $t \in (0, \infty)$, and $f \in L^2(E)$ with supp $f := \{x \in \mathbb{R}^n : f(x) \neq 0\} \subset E$,}

$$\left\| e^{-t\mathcal{L}} f \right\|_{L^2(F)} \leq C \exp \left\{ -c_5 \frac{[d(E,F)]^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\} \|f\|_{L^2(E)},$$

(1.15)

here and thereafter, $d(E,F) := \inf_{x \in E, y \in F} |x - y|$ and

$$\|f\|_{L^2(E)} := \left[ \int_E |f(x)|^2 \, dx \right]^{1/2}.$$

Now, let $\mathcal{L}$ be the higher order Schrödinger operator satisfying the assumptions of Theorems 1.1 and 1.2. Applying these both theorems, we immediately obtain the following conclusions.

(i) The spectrum $\sigma_p(\mathcal{L})$ of $\mathcal{L}$ in $L^p(\mathbb{R}^n)$ is independent of $p$ for any given $p \in [1, \infty)$ (see [46, Theorem 1.1]).

(ii) The operator $\mathcal{L}$ has a bounded $H_\infty$-functional calculus on $L^p(\mathbb{R}^n)$ for any given $p \in (1, \infty)$ (see [23, Theorem 3.1]).

(iii) When $n > 2m$, the integral kernel $K_\lambda$ of the resolvent $(\lambda - \mathcal{L})^{-1}$ satisfies that, for any $x, y \in \mathbb{R}^n$ with $x \neq y$,

$$|K_\lambda(x, y)| \leq \frac{1}{|x - y|^{n-2m}} e^{-c|\lambda|^{1/(2m)}|x - y|},$$

where the implicit positive constant and the positive constant $c$ are independent of $x, y$ and $\lambda$ (see [35, Theorem 2.2]).

(iv) For any given $p, q \in (1, \infty)$, let $f \in L^p([0, \infty); L^q(\mathbb{R}^n))$ with

$$L^p([0, \infty); L^q(\mathbb{R}^n)) := \left\{ f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} : \|f\|_{L^p([0, \infty); L^q(\mathbb{R}^n))} < \infty \right\},$$

where

$$\|f\|_{L^p([0, \infty); L^q(\mathbb{R}^n))} := \left[ \int_0^\infty \|f(\cdot, t)\|_{L^q(\mathbb{R}^n)}^p \, dt \right]^{1/p}.$$
The following inhomogeneous initial value problem
\[
\begin{aligned}
\frac{\partial}{\partial t} u(x,t) &= \mathcal{L}u(x,t) + f(x,t), \quad (x,t) \in \mathbb{R}^n \times (0, \infty), \\
u(x,0) &= 0, \quad x \in \mathbb{R}^n
\end{aligned}
\]
has a unique solution that is of maximal $L^p(\mathbb{R}^n)$-$L^q(\mathbb{R}^n)$ regularity (see [36, Theorem 3.1]).

We also point out that, since the estimate of the heat kernel is the start point of many studies on the analysis of the Schrödinger operator, our main results pave the way for further studies such as the Sobolev inequalities (in particular, see [12, 52] for the Nash and the Gagliardo–Nirenberg inequalities), the boundedness of some singular integral operators, and the real-variable theory of function spaces associated with $\mathcal{L}$. We do not pursue these problems here, in order to limit the length of this article.

The remainder of this article is organized as follows. In Section 2, we provide some basic facts on the higher order Schrödinger operator $\mathcal{L} := P(D) + V$. We first review the definition of $\mathcal{L}$ in Section 2.1; then, in Section 2.2, we introduce the definition of the generalized Schechter class concerning the potential $V$. Some basic properties of the Schechter class are also presented in this section. Section 3 is devoted to the boundedness of the $T$-operator defined in (3.1). We obtain four kinds of boundedness in this section. In Section 4, we establish two perturbation estimates for the resolvent related to $\mathcal{L}$: the summation perturbations in Section 4.1 and the exponential perturbations in Section 4.2. Finally, we prove our main results, Theorems 1.1 and 1.2, in Section 5.

We end this section by making some conventions on the notation. Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, and $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$. For any $s \in \mathbb{R}$, let $|s|$ be the largest integer not greater than $s$. For any set $E \subset \mathbb{R}^n$, we use $1_E$ to denote its characteristic function. We use $C$ to denote a positive constant that is independent of the main parameters involved, whose value may differ from line to line. Constants with subscripts, such as $C_1$ and $c_1$, do not change in different occurrences. We also use $C_{(\alpha, \beta, \ldots)}$ to denote a positive constant depending on the indicated parameters $\alpha, \beta, \ldots$. If $f \leq C g$, we write $f \lesssim g$ and, if $f \lesssim g \sim f$, we then write $f \sim g$. If $f \lesssim C g$ and $g = h$ or $g \lesssim h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, rather than $f \lesssim g = h$ or $f \lesssim g \leq h$. We use $\bar{0}_n$ to denote the origin of $\mathbb{R}^n$.

# 2 | Higher Order Schrödinger Operators

In this section, we provide some basic facts on the higher order Schrödinger operator $\mathcal{L}$ and its potential. We begin with a review of the definition of $\mathcal{L}$.

## 2.1 | Preliminaries on Schrödinger operators

Let $P(x) := \sum_{|\alpha| = 2m} a_\alpha x^\alpha$ be a homogeneous polynomial of degree $2m$ on $x \in \mathbb{R}^n$ with real constant coefficients $\{a_\alpha\}_{|\alpha| = 2m}$ that satisfy the uniform ellipticity condition: there exists a positive constant $\lambda \in (0, \infty)$ such that, for any $x \in \mathbb{R}^n$,
\[
\sum_{|\alpha| = 2m} a_\alpha x^\alpha \geq \lambda |x|^{2m}.
\]
For any \( f \in C_c^\infty(\mathbb{R}^n) \), define the differential operator \( P(D) \) of order \( 2m \) on \( f \) by setting
\[
P(D)f := \sum_{|\alpha| = 2m} (-1)^{|\alpha|} a_\alpha D^\alpha f.
\] (2.2)

It is known that \( P(D) \) can be extended to a nonnegative self-adjoint operator in \( L^2(\mathbb{R}^n) \) with domain \( \text{dom}(P(D)) = W^{2m, 2}(\mathbb{R}^n) \) being the Sobolev space (see [55, p. 62, Corollary 2.2]). We call this self-adjoint extension the \( 2m \)-order homogeneous elliptic operator with real constant coefficients on \( \mathbb{R}^n \) and we still use the same notation \( P(D) \) to denote it.

Recall the following properties of \( P(D) \) from [4, Proposition 45], [45, p. 177, Problem III-6.16], and [55, p. 65, Corollary 3.4]. In what follows, for any given linear normed spaces \( \mathcal{X} \) and \( \mathcal{Y} \), and any linear operator \( T \) mapping \( \mathcal{X} \) into \( \mathcal{Y} \), we use \( \|T\|_{\mathcal{X} \to \mathcal{Y}} \) to denote its operator norm.

**Lemma 2.1.** Let \( P(D) \) be a \( 2m \)-order homogeneous elliptic operator with real constant coefficients on \( \mathbb{R}^n \) as in (2.2), and let \( p_t(x, y) \) defined on \( (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \) be the heat kernel of the semigroup generated by \(-P(D)\). The following assertions hold true.

(i) The resolvent set of \( P(D) \), \( \rho(P(D)) = \mathbb{C} \setminus [0, \infty) \) and, for any \( \lambda \in \rho(P(D)) \),
\[
\left\| (\lambda - P(D))^{-1} \right\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq \frac{1}{d(\lambda, [0, \infty))},
\]
where \( d(\lambda, [0, \infty)) := \inf_{s \in [0, \infty]} |\lambda - s| \).

(ii) There exist positive constants \( C \) and \( c_6 \) such that, for any \( l \in \{0, \ldots, m-1\} \), \( t \in (0, \infty) \), and \( x, y \in \mathbb{R}^n \),
\[
\left| D_l x p_t(x, y) \right| \leq \frac{C}{t^{(n+1)/(2m)}} \exp \left\{ -c_6 \frac{|x-y|^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\}.
\]

(iii) For any given \( \gamma \in (0, 1) \), there exist positive constants \( C \) and \( c_7 \) such that, for any \( l \in \{0, \ldots, m-1\} \), \( t \in (0, \infty) \), and \( x, y, h \in \mathbb{R}^n \),
\[
\left| D_l x p_t(x + h, y) - D_l x p_t(x, y) \right| \leq \frac{C}{t^{(n+1)/(2m)}} \left[ \frac{|h|}{t^{1/(2m)}} \right]^\gamma.
\]

Now, let \( V : \mathbb{R}^n \to \mathbb{R} \) be a real-valued measurable function on \( \mathbb{R}^n \). It induces a multiplication operator \( f \mapsto Vf \) in \( L^2(\mathbb{R}^n) \) with the domain
\[
\text{dom}(V) := \{ f \in L^2(\mathbb{R}^n) : Vf \in L^2(\mathbb{R}^n) \}.
\]
From [55, p.72, Lemma 6.1], it follows that \( V \) is a closed symmetric operator in \( L^2(\mathbb{R}^n) \). The operator \( V \) is said to be relatively \( P(D) \)-bounded if \( \text{dom}(P(D)) \subset \text{dom}(V) \) and there exist constants \( a, b \in (0, \infty) \) such that, for any \( g \in \text{dom}(P(D)) \subset L^2(\mathbb{R}^n) \),
\[
\|Vg\|_{L^2(\mathbb{R}^n)} \leq a \|P(D)g\|_{L^2(\mathbb{R}^n)} + b \|g\|_{L^2(\mathbb{R}^n)},
\]
where the infimum of all such \( a \) is called the relative bound of \( V \) with respect to \( P(D) \). Recall that the following Wüst theorem is just [54, Theorem X.14].
Lemma 2.2. Let $A$ be a self-adjoint operator and $B$ a symmetric operator in a Hilbert space $H$. Assume that $B$ is relatively $A$-bounded with relative bound $a \leq 1$. Then the sum $A + B$ of the operators $A$ and $B$ is essential self-adjoint on $\text{dom}(A)$.

By Lemma 2.2, we know that, if $V$ is relatively $P(D)$-bounded with relative bound $a \leq 1$, then the operator $P(D) + V$ is essentially self-adjoint on $W^{2m,2}(\mathbb{R}^n)$. Denote by

$$\mathcal{L} := P(D) + V,$$

(2.3)

the nonnegative self-adjoint extension of $P(D) + V$ in $L^2(\mathbb{R}^n)$. We call $\mathcal{L}$ the $2m$-order Schrödinger operator on $\mathbb{R}^n$ (see also Proposition 5.2 below for an extension of $\mathcal{L}$ to any space $L^p(\mathbb{R}^n)$ under some Schechter-type conditions). Note that, if $P(D) = -\Delta$ is the Laplace operator, then $\mathcal{L} := -\Delta + V$ is the usual second order Schrödinger operator on $\mathbb{R}^n$.

To study the operator $\mathcal{L}$, various conditions on the potential $V$ were introduced in literatures. For instance, Kato first introduced the Kato class $K_2(\mathbb{R}^n)$ in [44], which is useful in the study of many problems related to the Schrödinger operator $\mathcal{L}$ (see [17, 58, 68]). In what follows, the symbol $\delta \to 0^+$ means $\delta \in (0, \infty)$ and $\delta \to 0$. Recall that a real-valued measurable function $V$ on $\mathbb{R}^n$ is said to be in the Kato class $K_\alpha(\mathbb{R}^n)$, for any given $\alpha \in (0, \infty)$, if

$$\lim_{\delta \to 0^+} \sup_{x \in \mathbb{R}^n} \int_{|y-x| < \delta} |V(y)| w_\alpha(x-y) \, dy = 0,$$

(2.4)

where, for any $x \in \mathbb{R}^n \setminus \{0\}$,

$$w_\alpha(x) := \begin{cases} |x|^{\alpha - n} & \text{if } \alpha \in (0, n), \\ \log(|x|^{-1}) & \text{if } \alpha = n, \\ 1 & \text{if } \alpha \in (n, \infty). \end{cases}$$

(2.5)

It is known that $K_\alpha(\mathbb{R}^n)$ is a closed subspace of a larger Banach space $\tilde{K}_\alpha(\mathbb{R}^n)$, which is defined to be the set of all $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|V\|_{\tilde{K}_\alpha(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} \int_{|y-x| < 1/2} |V(y)| w_\alpha(x-y) \, dy < \infty.$$ 

Shen considered a number of Schrödinger operators with potentials, respectively, in the reverse Hölder class ([56]) and the Morrey space ([57]). Recall that, for any given $p \in (1, \infty)$, a nonnegative measurable function $V$ is said to belong to the reverse Hölder class $\text{RH}_p(\mathbb{R}^n)$ if there exists a positive constant $C$ such that, for any ball $B$ of $\mathbb{R}^n$,

$$\left( \frac{1}{|B|} \int_B [V(x)]^p \, dx \right)^{\frac{1}{p}} \leq \frac{C}{|B|} \int_B V(x) \, dx.$$

(2.6)

Also, for any given $p \in (1, \infty)$ and $\lambda \in [0, n]$, a measurable function $f$ is said to be in the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{L_{p,\lambda}(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} \left[ \sup_{r \in (0, \infty)} \int_{B(x,r)} |f(y)|^p \, dy \right]^{\frac{1}{p}} < \infty,$$

(2.7)
here and thereafter, for any \( x \in \mathbb{R}^n \) and \( r \in (0, \infty) \), \( B(x, r) := \{ y \in \mathbb{R}^n : |y - x| < r \} \); see also \([55, 58, 62]\) for many other kinds of potential classes.

### 2.2 Generalized Schechter classes

In this subsection, we introduce the generalized Schechter class which has close relations with the aforementioned potential classes. In what follows, for any given \( r \in (1, \infty) \), we use \( L^r_{\text{loc}}(\mathbb{R}^n) \) to denote the set of all measurable functions \( f \) such that \( |f|^r \in L^1_{\text{loc}}(\mathbb{R}^n) \).

**Definition 2.3.** Suppose \( V \) is a real-valued measurable function on \( \mathbb{R}^n \). For any given \( \alpha \in (0, n) \), \( r \in [1, \infty) \), \( t \in [1, \infty) \), and \( \delta \in (0, \infty) \), let

\[
M_{\alpha, r, t, \delta}(V) := \left\| \left[ \int_{|y - \cdot| < \delta} |V(y)|^r w_\alpha(\cdot - y) \, dy \right]^{1/r} \right\|_{L^t(\mathbb{R}^n)}
\]

(2.8)

with \( w_\alpha \) as in (2.5). Assume that \( S \in \mathbb{R} \) is a real number. The **generalized Schechter classes** \( M_{\alpha, r, t, S}(\mathbb{R}^n) \) and \( \tilde{M}_{\alpha, r, t, S}(\mathbb{R}^n) \) are defined, respectively, by setting

\[
M_{\alpha, r, t, S}(\mathbb{R}^n) := \left\{ V \in L^r_{\text{loc}}(\mathbb{R}^n) : \left\| V \right\|_{M_{\alpha, r, t, S}(\mathbb{R}^n)} := \sup_{\delta \in (0, \infty)} \delta^S M_{\alpha, r, t, \delta}(V) < \infty \right\}
\]

(2.9)

and

\[
\tilde{M}_{\alpha, r, t, S}(\mathbb{R}^n) := \left\{ V \in L^r_{\text{loc}}(\mathbb{R}^n) : \lim_{\delta \to 0^+} \delta^S M_{\alpha, r, t, \delta}(V) = 0 \right\}.
\]

(2.10)

**Remark 2.4.** The spaces \( M_{\alpha, r, t, S}(V) \) and \( \tilde{M}_{\alpha, r, t, S}(\mathbb{R}^n) \) are motivated by the definitions of the now called Schechter class in \([55, \text{Chapter 6.4}]\), where Schechter originally introduced the following **Schechter class**

\[
M_{\alpha, r, t}(\mathbb{R}^n) := \left\{ V \in L^r_{\text{loc}}(\mathbb{R}^n) : M_{\alpha, r, t, \delta}(V) < \infty \text{ for some } \delta > 0 \right\}
\]

with \( M_{\alpha, r, t, \delta}(V) \) as in (2.8). The parameter \( S \) in (2.9) and (2.10) comes from the scaling of \( \delta \in (0, \infty) \).

The following proposition establishes the relations between the generalized Schechter class and some other known potential classes.

**Proposition 2.5.** Let \( \alpha \in (0, n) \), \( r \in [1, \infty) \), \( t \in [1, \infty) \), and \( S \in \mathbb{R} \), and let \( V \) be a real-valued measurable function on \( \mathbb{R}^n \).

(i) Then \( \tilde{M}_{\alpha, 1, \infty, 0}(\mathbb{R}^n) = K_\alpha(\mathbb{R}^n) \) with \( K_\alpha(\mathbb{R}^n) \) being the Kato class as in (2.4).

(ii) Let \( p \in (nr/\alpha, \infty) \) and \( 0 \leq V \in RH_p(\mathbb{R}^n) \) be as in (2.6). For any given \( \delta \in (0, \infty) \), let

\[
m(V, \delta) := \sup_{x \in \mathbb{R}^n} \int_{B(x, \delta)} V(y) \, dy.
\]

If \( \lim_{\delta \to 0^+} \delta^{S-n+\alpha/r} m(V, \delta) = 0 \), then \( V \in \tilde{M}_{\alpha, r, \infty, S}(\mathbb{R}^n) \).
In particular, the above assertion holds true when \( \alpha \in (0, n) \), \( S - n + \alpha / r \in [0, \infty) \), and \( V \in RH_p(\mathbb{R}^n) \cap L^1_{\text{unif}}(\mathbb{R}^n) \) with \( p \in (nr / \alpha, \infty) \), where

\[
L^1_{\text{unif}}(\mathbb{R}^n) := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \| f \|_{L^1_{\text{unif}}(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} \int_{|y-x| < 1} |f(y)| \, dy < \infty \right\}.
\]

(iii) Suppose \( S \in (-\infty, 0) \) and \( Sr + \alpha \in [0, n] \), and let \( L_{r, \lambda}(\mathbb{R}^n) \) be the Morrey space as in (2.7) with \( \lambda = Sr + \alpha \). Then

\[
L_{r, \lambda}(\mathbb{R}^n) = M_{\alpha, r, \infty, S}(\mathbb{R}^n).
\]

Proof. Note that (i) is an easy consequence of (2.4) and (2.10).

To prove (ii), by assumptions \( \alpha \in (0, n) \), \( p \in (nr / \alpha, \infty) \), and \( 0 \leq V \in RH_p(\mathbb{R}^n) \), we obtain \((\alpha - n)(p/r)' + n > 0\) and

\[
\lim_{\delta \to 0^+} \delta^S M_{\alpha, r, \infty, \delta}(V) \leq \lim_{\delta \to 0^+} \delta^S \sup_{x \in \mathbb{R}^n} \left\{ \int_{|y-x| < \delta} |V(y)| \left| x - y \right|^\alpha \, dy \right\}^{1/r} \lesssim \lim_{\delta \to 0^+} \delta^S \frac{1}{\delta^n} \left\{ \int_{|y-x| < \delta} |V(y)|^p \, dy \right\}^{1/p} \left\{ \int_{|y-x| < \delta} \left| x - y \right| \left( \alpha - n \right)(p/r)' \, dy \right\}^{1/r(p/r)'} \sim \lim_{\delta \to 0^+} \delta^S \frac{1}{\delta^\alpha} m(V, \delta),
\]

which turns to be 0 when \( \lim_{\delta \to 0^+} \delta^{S - n + \alpha / r} m(V, \delta) = 0 \). This, combined with (2.10), shows that \( V \in \tilde{M}_{\alpha, r, \infty, S}(\mathbb{R}^n) \) and hence (ii) holds true.

To prove (iii), for any \( x \in \mathbb{R}^n \) being fixed, let \( d\mu_x(y) := 1_{B(x, \delta)}(y) |V(y)|' \, dy \) be a nonnegative measure on \( \mathbb{R}^n \). Then we obtain

\[
N_\delta(x) := \delta^S \left[ \int_{|y-x| < \delta} |V(y)|' \left| x - y \right|^\alpha \, dy \right]^{\frac{1}{r}} = \delta^S \left[ \int_{\mathbb{R}^n} \left| x - y \right|^\alpha \, d\mu_x(y) \right]^{\frac{1}{r}} = (n - \alpha)^{1/r} \delta^S \left[ \int_0^\infty s^{\alpha - \alpha} \mu_x \left( \{ y : \left| y - x \right| \leq s \} \right) \frac{ds}{s} \right]^{\frac{1}{r}} \sim \delta^S \left[ \int_0^\infty s^{\alpha - \alpha} \mu_x(B(x, s)) \frac{ds}{s} \right]^{\frac{1}{r}} \sim \delta^S \left[ \int_0^\infty s^{\alpha - \lambda} \left\{ \int_{B(x, s) \cap B(x, \delta)} |V(y)|' \, dy \right\} \frac{ds}{s} \right]^{\frac{1}{r}} \sim \delta^S \left[ \int_0^\infty \mu_x \left( \left\{ y : \left| y - x \right| > \delta \right\} \right) \frac{ds}{s} \right]^{\frac{1}{r}}
\]
\[ + \delta^{n-\lambda} \int_{\delta}^{\infty} s^{\alpha-n} \left\{ \int_{B(x,\delta)} |V(y)|^{r} \, dy \right\} \frac{ds}{s} \frac{1}{r} \]
\[ \lesssim \delta^{S+(\alpha-\lambda)/r} \| V \|_{L_{r,\lambda}(\mathbb{R}^{n})}, \]

where we used (2.7) and the assumption \( \lambda = Sr + \alpha < \alpha \) in the last inequality. This, together with (2.9) and the assumption \( Sr + \alpha - \lambda = 0 \) when \( \alpha \in (0, n) \), shows that

\[ M_{\alpha, r, \infty, \delta}(V) = \sup_{\delta \in (0, \infty)} \sup_{x \in \mathbb{R}^{n}} N_{\delta}(x) \lesssim \| V \|_{L_{r,\lambda}(\mathbb{R}^{n})}. \]

On the other hand, by (2.7) and the assumptions that \( \alpha \in (0, n) \) and \( Sr + \alpha - \lambda = 0 \) again, we have

\[ \| f \|_{L_{r,\lambda}(\mathbb{R}^{n})} = \sup_{\delta \in (0, \infty)} \sup_{x \in \mathbb{R}^{n}} \left[ \delta^{\lambda-n} \int_{B(x,\delta)} |f(y)|^{r} \, dy \right]^{1/r} \]
\[ \lesssim \sup_{\delta \in (0, \infty)} \sup_{x \in \mathbb{R}^{n}} \left[ \delta^{\lambda-\alpha} \int_{B(x,\delta)} |f(y)|^{r} |x - y|^{|\alpha-n|} \, dy \right]^{1/r} \]
\[ \lesssim \sup_{\delta \in (0, \infty)} \delta^{S} \sup_{x \in \mathbb{R}^{n}} \left[ \int_{B(x,\delta)} |f(y)|^{r} |x - y|^{|\alpha-n|} \, dy \right]^{1/r}, \]

which, combined with (2.9), indicates \( \| f \|_{L_{r,\lambda}(\mathbb{R}^{n})} \lesssim M_{\alpha, r, \infty, \delta}(f) \). This shows (iii) and hence finishes the proof of Proposition 2.5.

Remark 2.6. For any given \( \alpha \in (0, n) \), \( r \in [1, \infty) \), \( a \in (-\alpha/r, \infty) \), and \( \delta \in (0, \infty) \), and for any \( x \in \mathbb{R}^{n} \setminus \{0\} \), let \( V(x) := \pm |x|^{a} \) and

\[ N_{\alpha, r, \delta}(x) := \left[ \int_{|y-x|<\delta} |V(y)|^{r} |x - y|^{|\alpha-n|} \, dy \right]^{1/r}. \]

(2.11)

By an elementary calculation, we find that there exists a positive constant \( C \), independent of \( \delta \), such that, for any \( x \in \mathbb{R}^{n} \),

\[ N_{\alpha, r, \delta}(x) \leq C \begin{cases} \delta^{\alpha/r} |x|^{a} & \text{if } |x| \geq 2\delta, \\ \delta^{\alpha/r+a} & \text{if } |x| < 2\delta. \end{cases} \]

If \( a \) further satisfies \( a \in (-\infty, -n/t) \), then

\[ M_{\alpha, r, t, \delta}(V) = \| N_{\alpha, r, \delta} \|_{L_{1}(\mathbb{R}^{n})} \lesssim \delta^{\alpha/r+a+n/t} \]

with the implicit positive constant independent of \( \delta \). Thus, we conclude that, for any given \( \alpha \in (0, n) \), \( r \in [1, \infty) \), \( t \in [1, \infty] \), \( S \in \mathbb{R} \), and \( a \in (-\alpha/r, -n/t) \),

\[ \pm |x|^{a} \in \begin{cases} M_{\alpha, r, t, \delta}(\mathbb{R}^{n}) & \text{if } S + \frac{\alpha}{r} + a + \frac{n}{t} = 0, \\ \tilde{M}_{\alpha, r, t, \delta}(\mathbb{R}^{n}) & \text{if } S + \frac{\alpha}{r} + a + \frac{n}{t} > 0. \end{cases} \]
In particular, applying Proposition 2.5(i), we find that $\pm |x|^a$ is in the Kato class $K_\alpha(\mathbb{R}^n)$ as in (2.4) if $a \in (-\alpha, 0)$.

Similarly, for any given $a \in (-\infty, -n/t)$ and any $x \in \mathbb{R}^n$, let $V(x) := (1 + |x|)^a$ and $N_{\alpha, r, \delta}(x)$ be as in (2.11). By an elementary calculation, we have

$$N_{\alpha, r, \delta}(x) \lesssim \left\{ \begin{array}{l l}
\delta^{\alpha/r} \frac{|x|^a}{\delta} & \text{if } |x| \geq 2\delta, \\
\delta^{\alpha/r} & \text{if } |x| < 2\delta
\end{array} \right.$$  

with the implicit positive constant independent of $\delta$ and $x$. This implies that

$$\pm (1 + |x|)^a \in \tilde{M}_{\alpha, r, t, S}(\mathbb{R}^n)$$

if $S + \frac{\alpha}{r} + a + \frac{n}{t} > 0$. In particular, $\pm (1 + |x|)^a$ is in the Kato class $K_\alpha(\mathbb{R}^n)$ as in (2.4) when $a \in (-\alpha, 0)$.

We end this section by giving some embedding properties of generalized Schechter classes. Another embedding property, based on the properties of the Bessel potential, is established in Proposition 3.7.

**Proposition 2.7.** Let $\alpha \in (0, n)$, $r \in [1, \infty)$, $t \in [1, \infty]$, and $S \in \mathbb{R}$.

(i) Then $M_{\alpha, r, \infty, S}(\mathbb{R}^n) \subset L^r_{\text{unif}}(\mathbb{R}^n)$, where

$$L^r_{\text{unif}}(\mathbb{R}^n) := \left\{ f \in L^r_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^r_{\text{unif}}(\mathbb{R}^n)} < \infty \right\} \quad (2.12)$$

with

$$\|f\|_{L^r_{\text{unif}}(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} \left[ \int_{|y-x|<1} |f(y)|^r \, dy \right]^{1/r}.$$  

(ii) For any $\alpha_1 \in (0, \alpha]$,

$$M_{\alpha, r, \alpha_1, S}(\mathbb{R}^n) \subset M_{\alpha, r, t, S}(\mathbb{R}^n).$$

(iii) For any $a \in (1, \infty)$ and $\theta \in (0, 1)$ satisfying $1 - \frac{n}{(n-\alpha) a'} < \theta < \frac{n}{(n-\alpha) a}$ with $a' := \frac{a}{a-1}$,

$$M_{\alpha, r, a_1, S}(\mathbb{R}^n) \subset M_{\alpha, r, t, S}(\mathbb{R}^n),$$

where $\alpha_1 := n + (n-\alpha)\theta a$ and $\bar{S} := \frac{(\alpha-n)(1-\theta) a' + n}{a't}$.

**Proof.** Note that (i) is an easy consequence of (2.12) and (2.9) by taking $\delta = 1$.

To prove (ii), let $V$ be a real-valued measurable function on $\mathbb{R}^n$. It follows, from (2.8) and (2.5), that

$$\delta^S M_{\alpha, r, t, \delta}(V) = \delta^S \left\| \int_{|y-x|<\delta} |V(y)|^r |y - x|^{\alpha-n} \, dy \right\|_{L^t(\mathbb{R}^n)}.$$
\[ \leq \delta^{S+\alpha_1/r} \| \int_{|y - \cdot| < \delta} |V(y)|^{r|y - \cdot|^\alpha n} \, dy \|^{1/r}_{L^1(\mathbb{R}^n)} \]
\[ = \delta^{S+\alpha_1/r} M_{\alpha-\alpha_1, r, t, \delta}(V), \]

which, together with (2.9), shows (ii).

To prove (iii), by (2.8), we write
\[ \delta^S M_{\alpha, r, t, \delta}(V) = \delta^S \| \int_{|y - \cdot| < \delta} |V(y)|^{r|y - \cdot|^\alpha n} \, dy \|^{1/r}_{L^1(\mathbb{R}^n)} \]
\[ \leq \delta^S \| \int_{|y - \cdot| < \delta} |V(y)|^{r(\alpha - n)|y - \cdot|^\alpha a} \, dy \|^{1/(ar)}_{L^1(\mathbb{R}^n)} \times \| \int_{|y - \cdot| < \delta} |y - \cdot|^\alpha (\alpha - n)(1-\theta) a' \, dy \|^{1/(a'r)}_{L^1(\mathbb{R}^n)} \]

Since \( 1 - \frac{n}{(n-\alpha)d} < \theta < \frac{n}{(n-\alpha)d} \) and \( \theta > 0 \), we deduce that \( n + (\alpha - n)(1-\theta) a' > 0 \) and \( n + (\alpha - n)\theta a \in (0, n) \), which further implies that
\[ \delta^S M_{\alpha, r, t, \delta}(V) \lesssim \delta^{S+\tilde{S}} M_{\alpha_1, r, a, t, \delta}(V), \]
where we used the definitions of both \( \alpha_1 \) and \( \tilde{S} \) in the last equality. This, combined with (2.9), shows (iii) and hence finishes the proof of Proposition 2.7. \( \square \)

3 | BOUNDEDNESS OF T-OPERATORS

Let \( s, \delta \in (0, \infty) \) and \( V \) be a real-valued measurable function on \( \mathbb{R}^n \). The \( T \)-operator \( T_{s, \delta} \) is defined by setting, for any \( f \in S(\mathbb{R}^n) \) (the set of all Schwartz functions) and \( x \in \mathbb{R}^n \),
\[ T_{s, \delta}(f)(x) := V(G_{s, \delta} * f)(x), \]
where
\[ G_{s, \delta}(x) := \mathcal{F}^{-1} \left[ (2\pi)^{-n/2} \left( \delta^2 + |\xi|^2 \right)^{-s/2} \right](x) \]

denotes the \( s \)-order Bessel potential function and \( \mathcal{F}^{-1} \) the inverse Fourier transform. If \( \delta = 1 \), we remove the subscript \( \delta \) and write \( G_s(x) \) simply. Recall that, for any \( f \in S(\mathbb{R}^n) \) and \( \xi, x \in \mathbb{R}^n \),
\[ \mathcal{F}(f)(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx \]
and \( F^{-1}(f(x)) := F(f)(-x) \). Moreover,

\[
G_{s,\delta} \ast f = (\delta^2 - \Delta)^{-s/2} f.
\] (3.3)

The main purpose of this section is to study the boundedness of the \( T \)-operator as in (3.1), which plays an important role in the perturbation estimates for the resolvent of the higher order Schrödinger operator \( L \). We obtain four kinds of the boundedness of the \( T \)-operator by following the ideas used in [55, Chapter 6].

### 3.1 The first and the second boundedness

Recall the following basic properties of the Bessel potential function \( G_{s,\delta} \) from [55, 61].

**Lemma 3.1.** Let \( s, \delta \in (0, \infty) \) and \( G_{s,\delta} \) be as in (3.2). The following assertions hold true.

(i) \( G_{s,\delta}(x) = \delta^{n-s} G_s(\delta x) \) for any \( x \in \mathbb{R}^n \).

(ii) \( \omega_s(x) \leq C(s) G_s(x) \) for any \( x \in \mathbb{R}^n \) satisfying \(|x| < 1\), where \( \omega_s \) is as in (2.5) and the positive constant \( C(s) \) depends only on \( s \) and \( n \). The converse inequality also holds true for any \( x \in \mathbb{R}^n \) satisfying \(|x| \geq 1\), when \( s \neq n \).

(iii) For any given \( a \in (0, 1) \) and \( b \in (0, \infty) \), there exists a positive constant \( C_{(s,a,b)} \), depending on \( s, a, b, \) and \( n \), such that, for any \( |x| > b \),

\[
G_s(x) \leq C_{(s,a,b)} e^{-a|x|}.
\]

(iv) If, in addition, \( s \neq n \), then, for any given \( q \in (1, \infty) \) satisfying \((s-n)q + n > 0\) and any \( x \in \mathbb{R}^n \),

\[
[G_s(x)]^q \leq C_{(s,q)} G_{q(s-n)+n}(x),
\]

where the positive constant \( C_{(s,q)} \) depends only on \( s, q, \) and \( n \).

**Proof.** We refer the reader to [55, p. 118, (3.8)] for a proof of (i); [55, p. 135, (8.1) and p. 120, (3.14)] for (ii); [55, p. 120, (3.15)] for (iii); and [55, pp. 134-135, (7.26) and (7.27)] for (iv). This finishes the proof of Lemma 3.1. \( \square \)

The following lemma is useful to the first boundedness of the \( T \)-operator.

**Lemma 3.2.** Let \( p \in (0, \infty), q \in [1, \infty), t \in [1, \infty), s \in (0, \infty), \) and \( \alpha \in (0, n) \). Then there exists a positive constant \( C \), depending on \( n, s, p \) and \( q \), such that, for any \( \delta \in (0, \infty) \) and any real-valued measurable function \( V \) on \( \mathbb{R}^n \),

\[
\left\| \left\{ \int_{|y| > 1/\delta} |V(y)|^q [G_{s,\delta}(\cdot - y)]^p \, dy \right\}^{\frac{1}{q}} \right\|_{L^t(\mathbb{R}^n)} \leq C \delta^{(n-s)p+\alpha-n}/q M_{\alpha,q,t,1/\delta}(V).
\]
Proof. For any given $\delta \in (0, \infty)$ and $k \in \mathbb{N}$, let

$$S_{k,\delta} := \{ x \in \mathbb{R}^n : k/\delta \leq |x| < (k+1)/\delta \}$$

be the annulus with center at the origin, $Q_k$ the inscribed cube of $B(0_n, k/\delta)$, and $\tilde{Q}_{k+1}$ the externally tangent cube of $B(0_n, (k+1)/\delta)$. Let $Q_1$ be the inscribed cube of $B(0_n, 1/\delta)$ (see Figure 1 as below).

Since the volume $|\tilde{Q}_{k+1} \setminus Q_k| = \left(\frac{2}{\delta}\right)^n [(k+1)^n - \left(\frac{k}{\sqrt{n}}\right)^n]$ and $|Q_1| = \left(\frac{2}{\delta}\right)^n (\frac{1}{\sqrt{n}})^n$, we deduce that $S_{k,\delta}$ can be covered by $N(k) \leq c k^n$ balls $\{B(z_{k,j}, 1/\delta)\}_{j=1}^{N(k)}$ which consist of translations of $B(0_n, 1/\delta)$ with the implicit positive constant $c$ depending only on $n$, that is,

$$S_{k,\delta} \subset \bigcup_{j=1}^{N(k)} B(z_{k,j}, 1/\delta). \quad (3.4)$$

Note that, if $y - x \in B(z_{k,j}, 1/\delta)$, we then have $y \in B(z_{k,j} + x, 1/\delta)$. 

FIGURE 1  The covering of the annulus $S_{k,\delta}$
Now, we write by the assumption $q \in [1, \infty)$ that

$$\left\| \left\{ \int_{|y-| \geq 1/\delta} |V(y)|^q [G_{s,\delta}(\cdot - y)]^p \, dy \right\} \right\|_{L^1(\mathbb{R}^n)}^{1/q} \leq \sum_{k=1}^\infty \left\| \left\{ \int_{k/\delta \leq |y| < (k+1)/\delta} |V(y)|^q [G_{s,\delta}(\cdot - y)]^p \, dy \right\} \right\|_{L^1(\mathbb{R}^n)}^{1/q} =: \sum_{k=1}^\infty I_k.$$ 

Since $\delta |x - y| \geq k \geq 1$, from (i) and (iii) of Lemma 3.1, we infer that

$$[G_{s,\delta}(x - y)]^p = \delta^{(n-s)p} [G_s(\delta(x - y))]^p \lesssim \delta^{(n-s)p} e^{-akp}.$$ 

This, together with (3.4), (2.8), and the assumption $\alpha \in (0, n)$, implies that

$$\sum_{k=1}^\infty I_k \lesssim \sum_{k=1}^\infty \delta^{(n-s)p/q} e^{-akp/q} \left\| \sum_{j=1}^{N(k)} \int_{B(z_k, \cdot + 1/\delta)} |V(y)|^q \, dy \right\|_{L^1(\mathbb{R}^n)}^{1/q} \lesssim \sum_{k=1}^\infty \delta^{(n-s)p/q} e^{-akp/q} \sum_{j=1}^{N(k)} \left\| \int_{B(z_k, \cdot + 1/\delta)} |V(y)|^q \, dy \right\|_{L^1(\mathbb{R}^n)}^{1/q} \lesssim \sum_{k=1}^\infty \delta^{(n-s)p/q} e^{-akp/q} N(k) \left\| \int_{|y-| < 1/\delta} |V(y)|^q (\delta|\cdot - y|)^{\alpha-n} \, dy \right\|_{L^1(\mathbb{R}^n)}^{1/q} \sim \delta^{(n-s)p/q+(\alpha-n)/q} M_{\alpha,q,i,1/\delta}(V),$$

which completes the proof of Lemma 3.2. \boxed{}

The following proposition establishes the first boundedness of the $T$-operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

**Proposition 3.3.** Let $p \in [1, \infty]$, $s \in (0, n)$, $q \in [1, \frac{n}{(n-s)}]$, $\alpha \in (0, (s-n)q + n)$, and $\delta \in (0, \infty)$. Assume that $V$ is a real-valued measurable function on $\mathbb{R}^n$ satisfying $M_{\alpha,q,p',1/\delta}(V) < \infty$. Then the operator $T_{s,\delta}$ can be extended to a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Moreover, there exists a positive constant $C$, independent of $\delta$, such that, for any $f \in L^p(\mathbb{R}^n)$,

$$\|T_{s,\delta}(f)\|_{L^q(\mathbb{R}^n)} \leq C \delta^{(n-s)q+\alpha-n}/q M_{\alpha,q,p',1/\delta}(V) \|f\|_{L^p(\mathbb{R}^n)}. \quad (3.5)$$

**Proof.** For any $f \in S(\mathbb{R}^n)$, by (3.1) and the Minkowski inequality, we have

$$\|T_{s,\delta}(f)\|_{L^q(\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} V(x)G_{s,\delta}(x - y)f(y) \, dy \right)^q \, dx \right\}^{1/q} \leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |V(x)|^q |G_{s,\delta}(x - y)|^q \, dx \right)^{1/q} |f(y)| \, dy$$
\[
C_{s,q,p',\delta}(V) \leq \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} |V(x)|^q |G_{s,\delta}(x-y)|^q \, dx \right]^{p'/q} \, dy \right\}^{1/p'} \|f\|_{L^p(\mathbb{R}^n)}
= : C_{s,q,p',\delta}(V)\|f\|_{L^p(\mathbb{R}^n)}.
\]

To estimate \(C_{s,q,p',\delta}(V)\), we write
\[
C_{s,q,p',\delta}(V) \leq \left\{ \int_{\mathbb{R}^n} \left[ \int_{|x-y|<1/\delta} |V(x)|^q |G_{s,\delta}(x-y)|^q \, dx \right]^{p'/q} \, dy \right\}^{1/p'}
+ \left\{ \int_{\mathbb{R}^n} \left[ \int_{|x-y|\geq1/\delta} \ldots \, dx \right]^{p'/q} \, dy \right\}^{1/p'}
= : I_1 + I_2.
\]

For \(I_2\), using Lemma 3.2, we obtain
\[
I_2 \leq \delta^{(n-s)q+\alpha-n}/q M_{\alpha,q,p',1/\delta}(V).
\]

To bound \(I_1\), for any \(x, y \in \mathbb{R}^n\) satisfying \(|x - y| < 1/\delta\), by (i) and (ii) of Lemma 3.1 and the assumptions that \(s \in (0, n)\), \(q \in [1, n/(n-s))\), and \(\alpha \in (0, (s-n)q+n]\), we find that
\[
|G_{s,\delta}(x-y)|^q \leq C_{(s)} \delta^{(n-s)q}[w_{\alpha}(\delta(x-y))]^q \leq C_{(s)} \delta^{(n-s)q+\alpha-n} w_{\alpha}(x-y),
\]
where the last inequality follows from (2.5), \((s-n)q \geq \alpha - n\), and \(\delta|x-y| < 1\). Thus, we have
\[
I_1 \leq \delta^{(n-s)+\alpha-n}/q \left\| \left( \int_{|x-y|<1/\delta} |V(x)|^q w_{\alpha}(x-y) \, dx \right)^{1/q} \right\|_{L^{p'}(\mathbb{R}^n)}
\sim \delta^{(n-s)+\alpha-n}/q M_{\alpha,q,p',1/\delta}(V).
\]

Altogether, we conclude that \(C_{s,q,p',\delta}(V) \leq \delta^{(n-s)q+\alpha-n}/q M_{\alpha,q,p',1/\delta}(V)\), which completes the proof of Proposition 3.3.

To show the second boundedness of the \(T\)-operator, we need the following Gagliardo–Nirenberg inequality which is just [34, Corollary 2.4].

**Lemma 3.4.** Let \(p, p_0, p_1 \in (1, \infty)\), \(s, s_1 \in [0, \infty)\), and \(\theta \in [0, 1]\). Then there exists a positive constant \(C\) such that, for any \(f \in S(\mathbb{R}^n)\),
\[
\left\| \Delta^{s/2} f \right\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^{p_0}(\mathbb{R}^n)}^{1-\theta} \|\Delta^{s_1/2} f\|_{L^{p_1}(\mathbb{R}^n)}^\theta
\]
if and only if \(n/p - s = (1-\theta) n/p_0 + \theta(n/p_1 - s_1)\) and \(s \leq \theta s_1\).

The following result gives the \(L^p(\mathbb{R}^n)\)-boundedness of the resolvent of the Laplace operator \(\Delta\).
Lemma 3.5. Let $p \in (1, \infty)$ and $\delta, \alpha \in (0, \infty)$. Then there exists a positive constant $C$, independent of $\delta$, such that, for any $f \in L^p(\mathbb{R}^n)$:

(i) $\|\delta^2 - \Delta\|^\alpha f\|_{L^p(\mathbb{R}^n)} \leq \frac{C}{\delta^{2\alpha}} \|f\|_{L^p(\mathbb{R}^n)}$,

(ii) $\|\Delta^\alpha (\delta^2 - \Delta)^{-\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}$.

Proof. Let $m(\xi) := \frac{1}{(1 + |\xi|^{2})^{\alpha}}$ for any $\xi \in \mathbb{R}^n$ and $T_m$ be the Fourier multiplier associated with $m$. By the Hörmander–Mihlin multiplier theorem (see [30, Theorem 5.2.7]), we conclude that $T_m$ is bounded on $L^p(\mathbb{R}^n)$, which, combined with the dilation invariant property of the $L^p(\mathbb{R}^n)$ multiplier, shows that (i) holds true. Using an argument similar to that used in the proof of (i) and letting $\tilde{m}(\xi) = \frac{|\xi|^{2\alpha}}{(1 + |\xi|^{2})^{\alpha}}$ for any $\xi \in \mathbb{R}^n$, we know that (ii) also holds true. This finishes the proof of Lemma 3.5.

Based on Lemmas 3.4 and 3.5, we now establish the second boundedness of the $T$-operator as follows.

Proposition 3.6. Let $p, q \in (1, \infty)$, $t, \sigma \in (1, \infty]$, and $s \in (0, n)$ satisfy

$$\frac{1}{q} = \frac{1}{t} + \frac{1}{\sigma} \quad \text{and} \quad \frac{1}{\sigma} \leq \frac{1}{p} \leq \frac{s}{n} + \frac{1}{\sigma}.$$  

Assume $V \in L^t(\mathbb{R}^n)$. Then $T$ can be extended to a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Moreover, there exists a positive constant $C$ such that, for any $\delta \in (0, \infty)$ and $f \in L^p(\mathbb{R}^n)$,

$$\|T_{s, \delta}(f)\|_{L^q(\mathbb{R}^n)} \leq C \delta^{-s(1-\theta)} \|V\|_{L^t(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}$$  \hspace{1cm} (3.6)

with $\theta := \frac{n}{s} \left(\frac{1}{p} - \frac{1}{\sigma}\right) \in [0, 1]$.

Proof. By $\frac{1}{q} = \frac{1}{t} + \frac{1}{\sigma}$, (3.1), (3.3), Lemmas 3.4 and 3.5, we conclude that, for any $f \in S(\mathbb{R}^n)$,

$$\|T_{s, \delta}(f)\|_{L^q(\mathbb{R}^n)} \leq \|V\|_{L^t(\mathbb{R}^n)} \left\|\delta^2 - \Delta\right\|^{s/2} \left\|f\right\|_{L^p(\mathbb{R}^n)}$$

$$\leq \|V\|_{L^t(\mathbb{R}^n)} \left\|\delta^2 - \Delta\right\|^{s/2} \left\|f\right\|_{L^p(\mathbb{R}^n)} \left\|\Delta^{s/2} \left(\delta^2 - \Delta\right)^{-s/2} f\right\|_{L^p(\mathbb{R}^n)}$$

$$\leq \|V\|_{L^t(\mathbb{R}^n)} \delta^{-s(1-\theta)} \|f\|_{L^p(\mathbb{R}^n)},$$

which completes the proof of Proposition 3.6.

We end this subsection by giving another embedding property of the generalized Schechter class, as a byproduct of Lemma 3.2.

Proposition 3.7. Let $\alpha \in (0, n)$, $r \in [1, \infty)$, $t \in [1, \infty)$, and $S \in \mathbb{R}$. If $1 \leq r \leq t < \tau < \infty$ and $\frac{2}{nr} + \frac{1}{t} \leq \frac{\beta}{nr} + \frac{1}{t}$, then

$$M_{\alpha, r, t, S + S_1}(\mathbb{R}^n) \subset M_{\beta, r, \tau, S}(\mathbb{R}^n)$$

with $S_1 := \frac{1}{r}nr(\frac{1}{t} - \frac{1}{r}) + \beta - \alpha$. 

Proof. Let \( s := \beta - \alpha \). Since \( t < \tau \) and \( \frac{\alpha}{n} + \frac{1}{t} \leq \frac{\beta}{n} + \frac{1}{\tau} \), it follows that \( s > 0 \) and \( \frac{s}{n} \leq \frac{s}{n} \). Thus, if we let \( p := \frac{t}{r} \) and \( \sigma := \frac{\tau - r}{\tau - r} \), then \( \sigma' > 1 \) and

\[
\frac{1}{p} + \frac{1}{\sigma} \leq 1 + \frac{s}{n}. \tag{3.7}
\]

Now, using the Gaussian upper bound for the heat kernel of the Laplacian \( \Delta \), we obtain

\[
\|e^{t\Delta}\|_{L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)} \lesssim t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{\sigma'})}.
\]

This, together with (3.7) and the formula

\[
(\delta^2 - \Delta)^{-s/2} = \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2-1} e^{-\delta^2 t} e^{t\Delta} dt,
\]

shows that \((\delta^2 - \Delta)^{-s/2}\) is bounded from \(L^p(\mathbb{R}^n)\) to \(L^{p'}(\mathbb{R}^n)\) and

\[
\left\| (\delta^2 - \Delta)^{-s/2} \right\|_{L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)} \lesssim \delta^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{\sigma'}) + s} \sim \delta^{-S_1r}. \tag{3.8}
\]

Moreover, let \( G_{s,\delta} \) be the Bessel potential function as in (3.3). For any \(|x - y| < 1/\delta\), by (i) and (ii) of Lemma 3.1 and (2.5), we easily find that

\[
|x - y|^\beta - n = w_\beta(\delta(x - y)) \delta^{n-\beta} \lesssim G_\beta(\delta(x - y)) \delta^{n-\beta} \sim G_{\beta,\delta}(x - y).
\]

Combining this with (3.8), we obtain

\[
\begin{align*}
\delta^{-Q}M_{\beta,r,1/\delta}(V) &= \delta^{-Q} \left\| \int_{|x - y| < 1/\delta} |V(x)|' |x - y|^\beta - n \, dx \right\|_{L^r(\mathbb{R}^n)}^{1/r} \\
&\lesssim \delta^{-Q} \left\| G_{\beta,\delta} \ast V' \right\|_{L^{r/2}(\mathbb{R}^n)}^{1/r} \lesssim \delta^{-Q} \left\| G_{\beta,\delta} \ast (G_{\alpha,\delta} \ast V') \right\|_{L^{r/2}(\mathbb{R}^n)}^{1/r} \\
&\lesssim \delta^{-Q} \left\| (\delta^2 - \Delta)^{-s/2} (G_{\alpha,\delta} \ast V') \right\|_{L^{p'}(\mathbb{R}^n)}^{1/r} \\
&\lesssim \delta^{-Q - S_1} \left\| G_{\alpha,\delta} \ast V' \right\|_{L^p(\mathbb{R}^n)}^{1/r}. \tag{3.9}
\end{align*}
\]

Now, using the assumption \( p = \frac{t}{r} \), we further write

\[
\left\| G_{\alpha,\delta} \ast V' \right\|_{L^p(\mathbb{R}^n)}^{1/r} = \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} |V(y)|' |y - x|^\alpha - n \, dy \right]^{t/r} \, dx \right\}^{1/t} \lesssim \left\{ \int_{\mathbb{R}^n} \left[ \int_{|y - x| < 1/\delta} |V(y)|' |y - x|^\alpha - n \, dy \right]^{t/r} \, dx \right\}^{1/t}
\]
\[ + \left\{ \int_{\mathbb{R}^n} \left[ \int_{|y-x| \geq 1/\delta} \ldots dy \right]^{t/r} dx \right\}^{1/t} =: I_1 + I_2. \]  

By (2.8), it is easy to show that \( I_1 \sim M_{\alpha,r,t,1/\delta}(V) \). For \( I_2 \), using Lemma 3.2 with \( p = 1 \) and \( s = \alpha \) therein, we also obtain \( I_2 \sim M_{\alpha,r,t,1/\delta}(V) \). This, combined with (3.9) and (3.10), implies that

\[ \delta^{-Q} M_{\beta,r,t,1/\delta}(V) \lesssim \delta^{-Q-S} M_{\alpha,r,t,1/\delta}(V), \]

which, together with (2.9), then completes the proof of Proposition 3.7.

**3.2 The third and the fourth boundedness**

In this subsection, we consider the third and the fourth boundedness of the \( T \)-operator. First, we need some additional notation. To be precise, for any given \( s \in (0,n/2) \), \( r \in [1,\infty) \), \( t \in [1,\infty) \), and \( \delta \in (0,\infty) \), let

\[ \tilde{C}_{s,r,t,\delta}(V) := \left\| \left\{ \int_{\mathbb{R}^n} |V(x)| \left[ G_{2s,\delta}(x-\cdot) \right]^{r/2} dx \right\} \right\|_{L^t(\mathbb{R}^n)}^{1/r} \]  

with \( G_{2s,\delta} \) as in (3.2).

**Lemma 3.8.** Let \( t \in [1,\infty) \), \( \delta \in (0,\infty) \), \( s \in (0,n/2) \), \( r \in \left[ 1, \frac{2n}{n-2s} \right) \), and \( \alpha \in (0,n+(2s-n)r/2] \). Assume that \( V \) is a real-valued measurable function on \( \mathbb{R}^n \). Then

\[ \tilde{C}_{s,r,t,\delta}(V) \lesssim \delta^{[(n-2s)r/2+(\alpha-n)]/r} M_{\alpha,r,t,1/\delta}(V) \]

with the implicit positive constant independent of \( \delta \) and \( V \).

**Proof.** The proof of this lemma is similar to that of (3.10). By (3.11), we first write

\[ \tilde{C}_{s,r,t,\delta}(V) \lesssim \left\| \left\{ \int_{|x-\cdot| < 1/\delta} |V(x)| \left[ G_{2s,\delta}(x-\cdot) \right]^{r/2} dx \right\} \right\|_{L^t(\mathbb{R}^n)}^{1/r} + \left\| \left\{ \int_{|x-\cdot| \geq 1/\delta} \ldots dx \right\} \right\|_{L^t(\mathbb{R}^n)}^{1/r} =: I_1 + I_2. \]

For \( I_2 \), applying Lemma 3.2 with \( q = r \), \( p = r/2 \), and \( s \) therein replaced by \( 2s \), we have

\[ I_2 \sim \left\| \left\{ \int_{|x-\cdot| \geq 1/\delta} |V(x)| \left[ G_{2s,\delta}(x-\cdot) \right]^{r/2} dx \right\} \right\|_{L^t(\mathbb{R}^n)}^{1/r} \lesssim \delta^{[(n-2s)r/2+(\alpha-n)]/r} M_{\alpha,r,t,1/\delta}(V). \]
For $I_1$, since $\delta |x - y| < 1$, we deduce, from (i) and (ii) of Lemma 3.1, (2.5), and the assumption $2s < n$, that
\[
\left[ G_{2s, \delta}(x - y) \right]^{r/2} \lesssim \delta^{(n-2s)r/2} \left[ w_{2s}(\delta(x - y)) \right]^{r/2} \sim |x - y|^{(2s-n)r/2}.
\]
This, together with the assumption $\alpha \leq n + (2s - n)r/2$, shows that
\[
\left[ G_{2s, \delta}(x - y) \right]^{r/2} \lesssim \delta^{(n-2s)r/2} |\delta(x - y)\|_{\alpha-n} \sim \delta^{(n-2s)r/2+\alpha-n} w_{\alpha}(x - y),
\]
which then implies
\[
I_1 \lesssim \delta^{[(n-2s)r/2+\alpha-n]/r} \left\| \int_{|x - \cdot| < 1/\delta} |V(x)|^{r} w_{\alpha}(x - \cdot) \, dx \right\|_{L^{1}(\mathbb{R}^{n})}^{1/r} \lesssim \delta^{[(n-2s)r/2+\alpha-n]/r} M_{\alpha,r,t,1/\delta}(V).
\]
Combining the estimates for $I_1$ and $I_2$, we conclude that
\[
\tilde{C}_{s,r,t,\delta}(V) \lesssim \delta^{[(n-2s)r/2+\alpha-n]/r} M_{\alpha,r,t,1/\delta}(V),
\]
which completes the proof of Proposition 3.9. \qed

Based on Lemma 3.8, we now state the third boundedness of the $T$-operator.

**Proposition 3.9.** Let $s \in (0, n/2)$, $q \in [2, \infty)$, $t \in [q, \infty]$, and $r \in [q, \frac{2n}{n-2s})$ satisfy
\[
\frac{1}{t} + \frac{1}{r} = \frac{1}{q}.
\]
Assume that $V$ is a real-valued measurable function on $\mathbb{R}^n$ satisfying $M_{\alpha,r,t,1/\delta}(V) < \infty$ for some $\delta \in (0, \infty)$ and $\alpha \in (0, n + (2s - n)r/2)$. Then $T_{s,\delta}$ can be extended to a bounded linear operator from $L^2(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Moreover, there exists a positive constant $C$, independent of $\delta$, such that, for any $f \in L^2(\mathbb{R}^n)$,
\[
\|T_{s,\delta}(f)\|_{L^q(\mathbb{R}^n)} \leq C \delta^{[(n-2s)r/2+\alpha-n]/r} M_{\alpha,r,t,1/\delta}(V) \| f \|_{L^2(\mathbb{R}^n)}.
\]

**Proof.** Let $\tilde{C}_{s,r,t,\delta}(V)$ be as in (3.11). From [55, p. 129, Theorem 6.3], it follows that
\[
\|T_{s,\delta}(f)\|_{L^q(\mathbb{R}^n)} \lesssim \tilde{C}_{s,r,t,\delta}(V) \| f \|_{L^2(\mathbb{R}^n)},
\]
which, together with Lemma 3.8, then completes the proof of Proposition 3.9. \qed

The following proposition gives the fourth boundedness of the $T$-operator.

**Proposition 3.10.** Let $s \in (0, n)$, $\delta \in (0, \infty)$, and $T_{s,\delta}$ be as in (3.1).
(i) For any given $p, q \in [1, \infty)$, let $r \in [q, \infty)$, $t \in [1, \infty)$, $\alpha \in (0, (1 - \vartheta_1)r(s - n) + n)$, and $\tilde{\alpha} \in (0, \tilde{\vartheta}_1 r'(s - n) + n)$ for some $\vartheta_1 \in (0, \min\{1, \frac{n}{r'(n-s)}\})$. If

\[
\frac{1}{q} < \frac{1}{p} + \frac{1}{t} < \min\left\{1, 1 + \vartheta_1 \left(\frac{s}{n} - 1\right) + \frac{1}{q} - \frac{1}{r}\right\}
\]  

(3.13)

and $M_{\alpha, r, 1/\delta}(V) < \infty$, then $T_{s, \delta}$ can be extended to a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Moreover, there exists a positive constant $C$, independent of $\delta$ and $V$, such that, for any $f \in L^p(\mathbb{R}^n)$,

\[
\|T_{s, \delta}(f)\|_{L^q(\mathbb{R}^n)} \leq C\delta^{(n-s) + (\alpha-n)/r + (\tilde{\alpha}-n)/r'} M_{\alpha, r, 1/\delta}(V) \|f\|_{L^p(\mathbb{R}^n)}.
\]

(3.14)

(ii) Let $r \in (1, 2]$ and $\alpha \in (0, n)$ satisfy $\alpha - n \leq r(s - n) + nr/r'$ and $2n > r'(n - s)$. If $M_{\alpha, r, \infty, 1/\delta}(V) < \infty$, then $T_{s, \delta}$ can be extended to a bounded linear operator on $L^r(\mathbb{R}^n)$. Moreover, there exists a positive constant $C$, independent of $\delta$ and $V$, such that, for any $f \in L^r(\mathbb{R}^n)$,

\[
\|T_{s, \delta}(f)\|_{L^r(\mathbb{R}^n)} \leq C\delta^{\alpha/r - s} M_{\alpha, r, \infty, 1/\delta}(V) \|f\|_{L^r(\mathbb{R}^n)}.
\]

(3.14)

To prove Proposition 3.10, we first need to study two terms $h_{s, \delta}$ and $W_{s, \delta}$. To be precise, for any given $s \in (0, n)$, $\delta \in (0, \infty)$, $r \in [1, \infty)$, and $\vartheta_1 \in [0, 1]$, and any $y \in \mathbb{R}^n$, let

\[
h_{s, \delta}(y) := \left\{ \int_{\mathbb{R}^n} |V(x)|^r |G_{s, \delta}(x - y)|^{(1 - \vartheta_1)r} \, dx \right\}^{1/r}.
\]

(3.15)

Similarly to the proof of Lemma 3.8, we find that, for any given $t \in [1, \infty)$, and $\alpha \in (0, n)$ satisfying $\alpha - n \leq (1 - \vartheta_1)r(s - n),

\[
\|h_{s, \delta}\|_{L^t(\mathbb{R}^n)} \leq \delta^{(n-s)(1-\vartheta_1)+(\alpha-n)/r} M_{\alpha, r, t, 1/\delta}(V),
\]

(3.16)

where the implicit positive constant is independent of $\delta$ and $V$.

On the other hand, since $1 \leq q \leq r < \infty$ and $\frac{1}{q} < \frac{1}{p} + \frac{1}{t} < 1$, it follows that

\[
0 < \frac{1}{p} + \frac{1}{t} - \frac{1}{q} \leq \frac{1}{p} + \frac{1}{t} - \frac{1}{r} < 1 - \frac{1}{r} = \frac{1}{r'}.
\]

Moreover, let

\[
\left\{ \begin{array}{l}
\rho_1 := \frac{r'}{q'}, \\
\rho_2 := \frac{1}{p} + \frac{1}{t} - \frac{1}{r'}, \\
\nu := \frac{1}{p} + \frac{1}{t}, \\
\vartheta_2 := \frac{p}{r \nu'}, \\
\gamma := 1 - \frac{t}{r \nu}.
\end{array} \right.
\]

(3.17)

Clearly, $\rho_1 \in [1, \infty)$ and $\rho_2, \nu \in (1, \infty)$. By $\nu' = 1 + \frac{p}{t}$, we conclude that $\vartheta_2, \gamma \in (0, 1)$. 

Now, let $h_{s,\delta}$ be as in (3.15). For any $f, g \in \mathcal{S}(\mathbb{R}^n)$, define
\[
W_{s,\delta} := \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^n} |G_{s,\delta}(x-y)|^{\rho_1 r'} |g(x)|^{r'} |f(y)|^{(1-\partial_2) r'} [h_{s,\delta}(y)]^{\gamma r'} \, dx \, dy \right\}^{1/r'}.
\] (3.18)

The following lemma provides an estimate for $W_{s,\delta}$.

**Lemma 3.11.** Let $s, r, \rho_1, \rho_2, \theta_1, \theta_2, \tilde{\alpha}$, and $\gamma$ be as in (3.13) and (3.17). If
\[
\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{\tilde{\alpha}}{n} + 1,
\]
then
\[
W_{s,\delta} \lesssim \delta^{(n-s)\theta_1 + (\tilde{\alpha}-n) / r'} \left\| \left\| g \right\|_{L^\rho_1(\mathbb{R}^n)} \right\|^{1/r'} \left\| f \right\|_{L^\rho_2(\mathbb{R}^n)}^{(1-\partial_2) r'} h_{s,\delta}^{\gamma r'} \right\|^{1/r'}_{L^\rho_2(\mathbb{R}^n)}
\] with the implicit positive constant independent of $f$, $g$, and $\delta$.

**Proof.** By (3.18), we write
\[
W_{s,\delta} = \int_{\mathbb{R}^n} \left( \int_{|x-y| < 1/\delta} + \int_{|x-y| \geq 1/\delta} \right) |G_{s,\delta}(x-y)|^{\rho_1 r'} |g(x)|^{r'} |f(y)|^{(1-\partial_2) r'} [h_{s,\delta}(y)]^{\gamma r'} \, dx \, dy
\] =: I_1 + I_2. \tag{3.19}

For $I_1$, since $0 < \tilde{\alpha} \leq \theta_1 r'(s-n) + n < n$ and $\delta |x-y| < 1$, it follows, from (i) and (ii) of Lemma 3.1 and (2.5), that
\[
|G_{s,\delta}(x-y)|^{\rho_1 r'} \lesssim \delta^{(n-s)\theta_1 r'} \left\| g \right\|_{L^\rho_1(\mathbb{R}^n)} \left\| f \right\|_{L^\rho_2(\mathbb{R}^n)}^{(1-\partial_2) r'} [h_{s,\delta}(y)]^{\gamma r'} \lesssim \delta^{(n-s)\theta_1 r' + (\tilde{\alpha}-n)} |x-y|^\tilde{\alpha}-n.
\]
This implies
\[
I_1 \lesssim \delta^{(n-s)\theta_1 r' + \tilde{\alpha}-n} \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^\tilde{\alpha}-n |g(x)|^{r'} |f(y)|^{(1-\partial_2) r'} [h_{s,\delta}(y)]^{\gamma r'} \, dx \, dy \right\}.
\]
By the assumption $\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{\tilde{\alpha}}{n} + 1$ and the boundedness of the Riesz potential of order $\tilde{\alpha}$ from $L^{\rho_1}(\mathbb{R}^n)$ to $L^{\rho_2}(\mathbb{R}^n)$ (see, for instance, [61, p. 119, Theorem 1]), we obtain
\[
I_1 \lesssim \delta^{(n-s)\theta_1 r' + (\tilde{\alpha}-n)} \left\| g \right\|_{L^\rho_1(\mathbb{R}^n)} \left\| f \right\|_{L^\rho_2(\mathbb{R}^n)}^{(1-\partial_2) r'} h_{s,\delta}^{\gamma r'} \right\|_{L^\rho_2(\mathbb{R}^n)}
\]
which is the desired estimate.

For $I_2$, by Lemma 3.1(iii), we know that, for any $\delta |x-y| > 1$,
\[
|G_{s}(\delta(x-y))|^{\theta_1 r'} \lesssim e^{-a \delta r' |x-y|} \lesssim (\delta |x-y|)^{\tilde{\alpha}-n}.
\]
From this, we further deduce that $I_2$ satisfies the same estimate as that of $I_1$, which, combined with (3.19), then completes the proof of Lemma 3.11.

Based on Lemma 3.11, we now show Proposition 3.10(i).

**Proof of Proposition 3.10(i).** From (3.15) and (3.18), we first deduce that, for any $s \in (0, n)$, $\delta \in (0, \infty)$, and $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\| (T_{s, \delta}(f), g) \| = \| (V(G_{s, \delta} * f), g) \|$$

$$\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |V(x)| G_{s, \delta}(x-y) |f(y)| |g(x)| \, dx \, dy$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} |V(x)||G_{s, \delta}(x-y)||f(y)|^{\frac{\gamma}{2}} \left[ G_{s, \delta}(x-y)^{\beta_1} |g(x)| |f(y)|^{1-\beta_2} [h_{s, \delta}(y)]^{\gamma} \right] \, dx \, dy$$

$$\leq W_{s, \delta} \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^n} |V(x)||G_{s, \delta}(x-y)||f(y)|^{1-\beta_1} [h_{s, \delta}(y)]^{-\gamma} |f(y)|^{\frac{\gamma}{2} r} \, dx \, dy \right\}^{1/r}$$

$$= W_{s, \delta} \left\{ \int_{\mathbb{R}^n} [h_{s, \delta}(y)]^{1-\gamma} |f(y)|^{\frac{\gamma}{2 r} r} \, dy \right\}^{1/r}$$

$$\leq W_{s, \delta} \left\| [h_{s, \delta}]^{1-\gamma} \right\|_{L^r(\mathbb{R}^n)} \left\| |f|^{\frac{\gamma}{2 r}} \right\|_{L^{r'}(\mathbb{R}^n)}$$

(3.20)

where $W_{s, \delta}$ is as in (3.18). Moreover, by (3.17), the assumption $\frac{1}{q} < \frac{1}{p'} + \frac{1}{i} < \min\{1, 1 + \beta_1 \left( \frac{s}{n} - 1 \right) + \frac{1}{q'} - \frac{1}{r'} \}$, and an elementary calculation, we conclude that

$$\left\{ \begin{array}{l}
\frac{1}{r'} \rho_1 + \frac{1}{r} \rho_2 = \frac{1}{p}, \\
\frac{1}{r'} \rho_2 + \frac{1}{r} \rho_1 = \frac{1}{i'}, \\
1 < \frac{1}{\rho_1} + \frac{1}{\rho_2} \leq 1 + \frac{\beta_1}{n} (s-n) + n, \\
(r') \rho_1 = q', (1-\theta_2) r' \rho_2 = p, \gamma r' \rho_2 = t, (1-\gamma) r v = t', \theta_2 r v' = p.
\end{array} \right.$$

Now, applying (3.20), (3.16), and Lemma 3.11 with $\bar{\alpha} \leq \theta_1 r'(s-n) + n$ therein, we obtain

$$\| (V G_{s, \delta} * f, g) \|$$

$$\leq W_{s, \delta} \left\| [h_{s, \delta}]^{1-\gamma} \right\|_{L^r(\mathbb{R}^n)} \left\| |f|^{\frac{\gamma}{2 r}} \right\|_{L^{r'}(\mathbb{R}^n)}$$

$$\leq \delta^{(n-s)\beta_1 + (\bar{\alpha} - n)/r'} \left\| |g| \right\|_{L^p(\mathbb{R}^n)}^{1/r'} \left\| |f|^{1-\beta_2} \right\|_{L^{r'}(\mathbb{R}^n)} \left\| [h_{s, \delta}]^{1-\gamma} \right\|_{L^r(\mathbb{R}^n)} \left\| |f|^{\frac{\gamma}{2 r}} \right\|_{L^{r'}(\mathbb{R}^n)}$$

$$\leq \delta^{(n-s)\beta_1 + (\bar{\alpha} - n)/r'} \left\| |g| \right\|_{L^p(\mathbb{R}^n)} \left\| |f|^{1-\beta_2} \right\|_{L^{r'}(\mathbb{R}^n)} \left\| [h_{s, \delta}]^{1-\gamma} \right\|_{L^r(\mathbb{R}^n)} \left\| |f|^{\frac{\gamma}{2 r}} \right\|_{L^{r'}(\mathbb{R}^n)}$$
\[ \times \left\| f^2 \right\|_{L^{r'}(\mathbb{R}^n)} \]
\[ \sim \delta^{(n-s)\frac{1}{2} + (\alpha - n)/r'} \left\| g \right\|_{L^{q'}(\mathbb{R}^n)} \left\| f \right\|_{L^p(\mathbb{R}^n)} \left\| h_{s,\delta} \right\|_{L^r(\mathbb{R}^n)} \]
\[ \lesssim \delta^{(n-s) + (\alpha - n)/r + (\alpha - n)/r'} M_{\alpha, r, t, 1/\delta}(V) \left\| g \right\|_{L^{q'}(\mathbb{R}^n)} \left\| f \right\|_{L^p(\mathbb{R}^n)}, \]

which completes the proof of Proposition 3.10(i). \qed

To prove Proposition 3.10(ii), for any given \( s \in (0, n) \) and \( \delta \in (0, \infty) \), and, for any \( g \in S(\mathbb{R}^n) \), define

\[ \widetilde{W}_{s, \delta} := \left\{ \int_{\mathbb{R}^n} \left[ G_{s, \delta}(x - y) \right]^{r'/2} \left[ h_{s, \delta}(y) \right]^{r'} dy \mid g(x) \right\}^{1/r'}, \]

where \( G_{s, \delta} \) and \( h_{s, \delta} \) are, respectively, as in (3.2) and (3.15) with \( \theta_1 = 1/2 \) therein. Similarly to Lemma 3.11, we have the following estimate for \( \widetilde{W}_{s, \delta} \).

**Lemma 3.12.** Let \( s \in (0, n) \), \( \delta \in (0, \infty) \), \( r \in (1, 2] \), and \( \alpha \in (0, n) \) satisfy \( \alpha - n < r(s - n) + \frac{n r}{r'} \) and \( 2n > r'(n - s) \). Then

\[ \widetilde{W}_{s, \delta} \lesssim \delta^{\alpha/r - s} M_{\alpha, r, \infty, 1/\delta}(V) \left\| g \right\|_{L^{r'}(\mathbb{R}^n)} \]

with the implicit positive constant independent of \( \delta \).

**Proof.** Write

\[ \widetilde{W}_{s, \delta} = \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \left[ G_{s, \delta}(x - y) \right]^{r'/2} \left[ h_{s, \delta}(y) \right]^{r'} dy \right]^{1/r'} dx \right\}^{1/r}. \]

By (3.15), \( r'/r \gtrless 1 \), and the Minkowski integral inequality, we conclude that, for any \( x \in \mathbb{R}^n \),

\[ Q_{s, \delta}(x) = \left\{ \int_{\mathbb{R}^n} \left[ G_{s, \delta}(x - y) \right]^{r'/2} \left[ V(z) \right]^{r/2} \left\{ G_{s, \delta}(z - y) \right\}^{r'/2} dz \right\}^{1/r}, \]

\[ \lesssim \left[ \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \left[ G_{s, \delta}(z - y) \right]^{r'/2} \left[ G_{s, \delta}(x - y) \right]^{r'/2} dy \right\}^{r'/r'} |V(z)|^{r'} dz \right]^{r'/r}, \]

\[ \lesssim \left[ \int_{\mathbb{R}^n} \left\{ [G_{s, \delta}]^{r'/2} * [G_{s, \delta}]^{r'/2}(x - z) \right\}^{r'/r'} |V(z)|^{r'} dz \right]^{r'/r}. \]
Using Lemma 3.1(i), we obtain
\[
\left( [G_{s,\delta}]^{r'}/2 \ast [G_{s,\delta}]^{r'}/2 \right) = \delta^{(n-s)r'/2} [G_{s}(\delta \cdot)]^{r'}/2 \ast [G_{s}(\delta \cdot)]^{r'}/2.
\]
Moreover, by the assumptions \( s \in (0, n) \) and \( 2n > r'(n - s) \), we know that \( r'(s - n)/2 + n > 0 \), which, combined with Lemma 3.1(iv), implies
\[
[G_{s}(\delta \cdot)]^{r'}/2 \ast [G_{s}(\delta \cdot)]^{r'}/2 = G_{r'(s-n)/2+n,\delta} \ast G_{r'(s-n)/2+n,\delta}.
\]
Thus, by Lemma 3.1(i) again, we further obtain
\[
G_{s,\delta}^{r'}/2 \ast [G_{s,\delta}]^{r'}/2 = \left( \delta^{(n-s)r'/2} G_{r'(s-n)/2+n,\delta} \right) \ast \left( \delta^{(n-s)r'/2} G_{r'(s-n)/2+n,\delta} \right)
\]
\[
= G_{r'(s-n)/2+n,\delta} \ast G_{r'(s-n)/2+n,\delta} = G_{r'(s-n)+2n,\delta}.
\]
This implies
\[
\|Q_{s,\delta}\|_{L^{1/r'}(\mathbb{R}^n)} \lesssim \left\{ \int_{\mathbb{R}^n} \left| G_{r'(s-n)+2n,\delta}(\cdot - z) \right|^{r'/r} |V(z)|^r \, dz \right\}^{1/r} \leq \delta^{\alpha/r-s} M_{\alpha, r, \infty, 1/\delta}(V),
\]
which implies that
\[
\bar{W}_{s,\delta} = \left\{ \int_{\mathbb{R}^n} Q_{s,\delta}(x) |g(x)|^{r'} \, dx \right\}^{1/r'} \lesssim \delta^{\alpha/r-s} M_{\alpha, r, \infty, 1/\delta}(V) \|g\|_{L^{r'}(\mathbb{R}^n)}
\]
and hence completes the proof of Lemma 3.12.

Applying Lemma 3.12, we now show Proposition 3.10(ii).

\textbf{Proof of Proposition 3.10(ii).} For any given \( s \in (0, n) \) and \( \delta \in (0, \infty) \), and any \( f, g \in S(\mathbb{R}^n) \), by Lemma 3.12, we write
\[
\left| (T_{s,\delta}(f), g) \right| = \left| (VG_{s,\delta} \ast f, g) \right| \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} \left| V(x) |G_{s,\delta}(x-y)| f(y) |g(x)| \right| \, dx \, dy
\]
\[
= \int_{\mathbb{R}^n} \left\{ |V(x)||G_{s,\delta}(x-y)|^{1/2} |h_{s,\delta}(y)|^{-1} |f(y)| \right\} \, dx.
\]
\[
\times \left\{ [G_{s,\delta}(x-y)]^{1/2} |h_{s,\delta}(y)||g(x)| \right\} \, dx \, dy
\]

\[
\leq \overline{W}_{s,\delta} \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^n} |V(x)|^{1/r} [G_{s,\delta}(x-y)]^{r/2} |h_{s,\delta}(y)|^{-r} |f(y)|^r \, dx \, dy \right\}^{1/r}
\]

\[
= \overline{W}_{s,\delta} \|f\|_{L^r(\mathbb{R}^n)} \leq \delta^{\alpha/s} M_{s,\alpha, r, \infty, 1/\delta}(V) \|f\|_{L^r(\mathbb{R}^n)} \|g\|_{L^r(\mathbb{R}^n)}.
\]

This finishes the proof of Proposition 3.10(ii). \(\square\)

## 4 PERTURBATION ESTIMATES FOR RESOLVENTS

This section is devoted to the perturbation estimate for the resolvent related to the Schrödinger operator \(\mathcal{L} = P(D) + V\), which is essential to the estimate of the heat kernel. We consider two kinds of perturbations: the summation and the exponential perturbations.

### 4.1 Summation perturbation estimates

We first need some estimates for the resolvent of the homogeneous elliptic operator \(P(D)\). For the sake of consistency, we always use \((\lambda^{2m} - T)^{-1}\), where \(\lambda \in \mathbb{C}\), to denote the resolvent of a 2m-order differential operator \(T\) in this section. For any \(\vartheta_0 \in (0, \pi/2)\), let

\[
\Sigma_{\vartheta_0}^C := \{ z \in \mathbb{C} : |\arg z| \geq \vartheta_0 \}.
\]  

**Lemma 4.1.** Let \(m \in \mathbb{N}, q \in (1, \infty), \vartheta_0 \in (0, \pi/2), \) and \(P(D)\) be a homogeneous elliptic operator of order 2m with real constant coefficients as in (2.2).

(i) Then there exists a positive constant \(C\) such that, for any \(\lambda^{2m} \in \Sigma_{\vartheta_0}^C\), \(f \in L^q(\mathbb{R}^n)\),

\[
\left\| (\lambda^{2m} - P(D))^{-1} f \right\|_{L^q(\mathbb{R}^n)} \leq \frac{C}{|\lambda|^{2m}} \|f\|_{L^q(\mathbb{R}^n)}.
\]

(ii) For any \(s \in (0, 2m)\) and \(p \in [q, \infty)\) satisfying \(\frac{n}{p} - \frac{1}{q} - \frac{1}{s} < m - s/2\), there exists a positive constant \(C_2\) such that, for any \(\lambda^{2m} \in \Sigma_{\vartheta_0}^C\) and \(f \in L^q(\mathbb{R}^n)\),

\[
\left\| (|\lambda|^2 - \Delta)^{s/2} (\lambda^{2m} - P(D))^{-1} f \right\|_{L^p(\mathbb{R}^n)} \leq C_2 |\lambda|^{-[2m-s+n(\frac{1}{p} - \frac{1}{q})]} \|f\|_{L^q(\mathbb{R}^n)}.
\]  

**Proof.** We first prove (i) by following the idea used in the proof of [53, Theorem 2.6]. For any given \(\lambda^{2m} \in \Sigma_{\vartheta_0}^C\), let

\[
\mu^{2m} := -\lambda^{2m} = |\lambda|^{2m} e^{i\theta} \in \Sigma_{\pi - \vartheta_0} := \{ z \in \mathbb{C} : |\arg z| \leq \pi - \vartheta_0 \},
\]

\[
\Sigma_{\pi - \vartheta_0} := \{ z \in \mathbb{C} : |\arg z| \leq \pi - \vartheta_0 \}.
\]
and $\tilde{\mu}^{2m} := |\lambda|^{2m} e^{i\theta}/2$. It is easy to show $|\theta|/2 \in [0, (\pi - \theta_0)/2]$ and

$$(\lambda^{2m} - P(D))^{-1} = - (\mu^{2m} + P(D))^{-1}.$$ 

Since $P(D)$ is a sectorial operator of angle 0, we deduce that $-P(D)$ generates a bounded holomorphic semigroup in the right half complex plane (see, for instance, [33, Proposition 3.4.4]). Moreover, by the higher order Gaussian upper bound of the heat kernel $p_t$ of $P(D)$ (see Lemma 2.1) and an argument similar to that used in the proof of [52, Theorem 6.16], we conclude that, for any given such a $\theta$, and for any $t \in (0, \infty)$ and $x, y \in \mathbb{R}^n$,

$$\left| p_{t e^{-i\theta/2}}(x, y) \right| \lesssim \frac{1}{\cos(\theta/2)t^{n/(2m)}} \exp \left\{ - \frac{\bar{c}_0 |x - y|^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\},$$

$$\lesssim \frac{1}{\sin(\theta_0/2)t^{n/(2m)}} \exp \left\{ - \frac{\bar{c}_0 |x - y|^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\}. \quad (4.3)$$

Now, let $k \in \mathbb{N}$ satisfy $kn > 2m$. Since the semigroup generated by $-P(D)$ is holomorphic in the right half complex plane, we deduce, from the Phillips calculus for semigroup generators (see [33, Proposition 3.3.5]), $\tilde{\mu}^{2m} = \mu^{2m} e^{-i\theta/2}$, and the change of variables ($t = e^{-i\theta/2} \tilde{t}$), that

$$e^{i\theta/2} (\mu^{2m} + P(D))^{-1/k} = e^{i\theta/2} \frac{e^{i\theta(1-1/k)/2}}{\Gamma(1/k)} \int_0^\infty \tilde{t}^{1/k-1} e^{-\tilde{t} \tilde{\mu}^{2m}} e^{-\tilde{t}e^{-i\theta/2} P(D)} d\tilde{t},$$

which, together with (4.3), implies that the integral kernel $K_{\mu,k}(x, y)$ of $(\mu^{2m} + P(D))^{-1/k}$ satisfies that, for any $x, y \in \mathbb{R}^n$,

$$\left| K_{\mu,k}(x, y) \right| \lesssim \int_0^\infty e^{-t|\lambda|^{2m} \cos(\theta/2)} t^{1/k-n/(2m)} \exp \left\{ - \frac{\bar{c}_0 |x - y|^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\} dt,$$

$$\sim |x - y|^{2m/k-n} \int_0^\infty s^{1/k-n/(2m)} \exp \left\{ - \cos(\theta/2)|\lambda|^{2m} |x - y|^{2ms} - \bar{c}_0 s^{-1/(2m-1)} \right\} ds,$$

$$\lesssim |x - y|^{2m/k-n} \int_0^\infty s^{1/k-n/(2m)} \exp \left\{ - \epsilon s^{-1/(2m-1)} \right\} \times \exp \left\{ - \sin(\theta_0/2)|\lambda|^{2m} |x - y|^{2ms} - \left( \bar{c}_0 - \epsilon \right) s^{-1/(2m-1)} \right\} ds,$$

$$\lesssim |x - y|^{2m/k-n} \int_0^\infty s^{1/k-n/(2m)} \exp \left\{ - \epsilon s^{-1/(2m-1)} \right\} ds \lesssim 1. \quad (4.4)$$

where we made the change of variables $t = s|x - y|^{2m}$ in the second equality. Using $kn > 2m$, we find that

$$\int_0^\infty s^{1/k-n/(2m)} \exp \left\{ - \epsilon s^{-1/(2m-1)} \right\} ds \lesssim 1.$$
Next, for any $s \in (0, \infty)$, let

$$ F(s) := \exp \left\{ -\sin(\theta_0/2) |\lambda|^{2m} |x - y|^{2m} s - (\bar{c}_0 - \epsilon) s^{-\frac{1}{2m-1}} \right\}. $$

It is easy to show that $F(s) \lesssim \exp\{-c_8|\lambda||x - y|\}$ for some positive constant $c_8$ which depends on $\theta_0$, but is independent of $s$, $x$, $y$, and $\lambda$. By this and (4.4), we obtain

$$ |K_{\mu,k}(x,y)| \lesssim |x - y|^{2m/k-n} \exp \{-c_8|\lambda||x - y|\} =: P_{\lambda,k}(x-y). $$

Now, applying the Young inequality, we conclude that, for any $f \in L^q(\mathbb{R}^n)$,

$$ \left\| \left( \mu^{2m} + P(D) \right)^{-1/k} f \right\|_{L^q(\mathbb{R}^n)} \lesssim \|P_{\lambda,k} \ast |f|\|_{L^q(\mathbb{R}^n)} \leq \|P_{\lambda,k}\|_{L^1(\mathbb{R}^n)} \|f\|_{L^q(\mathbb{R}^n)} \lesssim \frac{1}{|\lambda|^{2m/k}} \|f\|_{L^q(\mathbb{R}^n)}, $$

which immediately shows (i).

To prove (ii), for any $f \in L^q(\mathbb{R}^n)$, write

$$ (|\lambda|^2 - \Delta)^{s/2} \left( \lambda^{2m} - P(D) \right)^{-1} f = (|\lambda|^2 - \Delta)^{-(m-s/2)} \circ (|\lambda|^2 - \Delta)^{m} \left( \lambda^{2m} - P(D) \right)^{-1} f. \quad (4.5) $$

Using the uniform ellipticity condition (2.1) of $P(D)$ and the Hörmander–Mihlin multiplier theorem (see [30, Theorem 5.2.7]), we conclude that the operator $\Delta^{m} P(D)^{-1}$ is bounded on $L^q(\mathbb{R}^n)$ for any given $q \in (1, \infty)$, that is, for any $g \in L^q(\mathbb{R}^n)$,

$$ \left\| \Delta^{m} P(D)^{-1} g \right\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|_{L^q(\mathbb{R}^n)}. \quad (4.6) $$

From this and (i), it follows that

$$ \left\| \Delta^{m} \left( \lambda^{2m} - P(D) \right)^{-1} f \right\|_{L^q(\mathbb{R}^n)} \lesssim \left\| P(D) \left( \lambda^{2m} - P(D) \right)^{-1} f \right\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\mathbb{R}^n)} + \left\| |\lambda|^{2m} \left( \lambda^{2m} - P(D) \right)^{-1} f \right\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\mathbb{R}^n)} $$

which, combined with the Gagliardo–Nirenberg inequality (see Lemma 3.4), implies

$$ \left\| (|\lambda|^2 - \Delta)^m \left( \lambda^{2m} - P(D) \right)^{-1} f \right\|_{L^q(\mathbb{R}^n)} \lesssim \sum_{k=0}^m |\lambda|^{2k} \left\| \Delta^{m-k} \left( \lambda^{2m} - P(D) \right)^{-1} f \right\|_{L^q(\mathbb{R}^n)} \lesssim \sum_{k=0}^m |\lambda|^{2k} \|f\|_{L^q(\mathbb{R}^n)} \left\| \Delta^{m} \left( \lambda^{2m} - P(D) \right)^{-1} f \right\|_{L^q(\mathbb{R}^n)}^{(m-k)/m}. $$
\[
\sum_{k=0}^{m} |\lambda|^{2m} \left( |\lambda|^{2m} - P(D) \right)^{-1} f \right\|_{L^q(\mathbb{R}^n)}^{k/m} \sum_{k=0}^{m} \left\| \Delta^m \left( |\lambda|^{2m} - P(D) \right)^{-1} f \right\|_{L^q(\mathbb{R}^n)}^{(m-k)/m} \\
\lesssim \|f\|_{L^q(\mathbb{R}^n)}.
\] (4.7)

On the other hand, since the kernel of the semigroup \( \{e^t\Delta\} \) satisfies the Gaussian upper bound, we deduce that, for any given \( 1 \leq q \leq p \leq \infty \), and for any \( t \in (0, \infty) \) and \( g \in L^q(\mathbb{R}^n) \),

\[
\left\| e^t\Delta g \right\|_{L^p(\mathbb{R}^n)} \lesssim t^{\frac{n}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \|g\|_{L^q(\mathbb{R}^n)}.
\]

Thus, by the Phillips calculus for semigroup generators again (see [33, Proposition 3.3.5]) and the assumption \( \frac{n}{2} \left( \frac{1}{q} - \frac{1}{p} \right) < m - s/2 \), we find that

\[
\left\| (|\lambda|^2 - \Delta)^{-\left( m-s/2 \right)} g \right\|_{L^p(\mathbb{R}^n)} = \left\| \frac{1}{\Gamma(m-s/2)} \int_0^\infty t^{m-s/2-1} e^{-|\lambda|^2 t} e^{t\Delta} g \, dt \right\|_{L^p(\mathbb{R}^n)} \\
\lesssim \int_0^\infty t^{m-s/2-1} e^{-|\lambda|^2 t} \left\| e^{t\Delta} g \right\|_{L^p(\mathbb{R}^n)} \, dt \\
\lesssim |\lambda|^{-\left[ 2m-s+n\left( \frac{1}{p} - \frac{1}{q} \right) \right]} \|g\|_{L^q(\mathbb{R}^n)},
\]

which, together with (4.5) and (4.7), shows (ii). This finishes the proof of Lemma 4.1. \( \square \)

Based on Lemma 4.1, we now give the first perturbation result as follows.

**Proposition 4.2.** Let \( m \in \mathbb{N} \), \( s \in (0,2m) \), \( \Theta_0 \in (0, \pi/2) \), \( \lambda^{2m} \in \Sigma^\circ_{\Theta_0} \) be as in (4.1), \( q \in (1, \infty) \), and \( p \in [q, \infty) \) satisfy \( \frac{n}{2} \left( \frac{1}{q} - \frac{1}{p} \right) < m - s/2 \). Assume that \( V \) is a real-valued measurable function on \( \mathbb{R}^n \) and the operator \( T_{s,|\lambda|} \), defined as in (3.1), is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) satisfying

\[
C_{(|\lambda|)} := C_2 |\lambda|^{-\left[ 2m-s+n\left( \frac{1}{p} - \frac{1}{q} \right) \right]} \left\| T_{s,|\lambda|} \right\|_{L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} < 1,
\] (4.8)

where the constant \( C_2 \) is as in (4.2). Then there exists a positive constant \( C \) such that, for any \( f \in L^q(\mathbb{R}^n) \),

\[
\left\| (|\lambda|^2 - \Delta)^{-s/2} \circ (|\lambda|^2 - \Delta)^{s/2} \left( |\lambda|^2 - P(D) \right)^{-1} f \right\|_{L^q(\mathbb{R}^n)} \leq \frac{C}{1-C_{(|\lambda|)}} \frac{1}{|\lambda|^{2m}} \|f\|_{L^q(\mathbb{R}^n)}.
\]

**Proof.** By Lemma 4.1 and (4.8), we obtain, for any \( f \in L^q(\mathbb{R}^n) \),

\[
\left\| V (|\lambda|^2 - P(D))^{-1} f \right\|_{L^q(\mathbb{R}^n)} = \left\| V (|\lambda|^2 - \Delta)^{-s/2} \circ (|\lambda|^2 - \Delta)^{s/2} \left( |\lambda|^2 - P(D) \right)^{-1} f \right\|_{L^q(\mathbb{R}^n)} \\
\leq \left\| T_{s,|\lambda|} \right\|_{L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} \left\| (|\lambda|^2 - \Delta)^{s/2} \left( |\lambda|^2 - P(D) \right)^{-1} f \right\|_{L^p(\mathbb{R}^n)}
\]

where...
\[ \leq C_{2} |\lambda|^{-2m-s+n\left(\frac{1}{p} \frac{1}{q} \right)} \left\| T_{s,|\lambda|} \right\|_{L^{p}(\mathbb{R}^{n}) \rightarrow L^{q}(\mathbb{R}^{n})} \left\| f \right\|_{L^{q}(\mathbb{R}^{n})} \]
\[ = C_{\langle |\lambda| \rangle} \left\| f \right\|_{L^{q}(\mathbb{R}^{n})} < \left\| f \right\|_{L^{q}(\mathbb{R}^{n})}. \] (4.9)

This implies that the operator \( I - V(\lambda^{2m} - P(D))^{-1} \) has a bounded inverse given by the Neumann series
\[ \left[ I - V(\lambda^{2m} - P(D))^{-1} \right]^{-1} = \sum_{k=0}^{\infty} \left[ V(\lambda^{2m} - P(D))^{-1} \right]^{k} \]
on \( L^{q}(\mathbb{R}^{n}) \) (see, for instance, [27, p. 162]) that satisfies
\[ \left\| \left[ I - V(\lambda^{2m} - P(D))^{-1} \right]^{-1} \right\|_{L^{q}(\mathbb{R}^{n}) \rightarrow L^{q}(\mathbb{R}^{n})} \leq \frac{1}{1 - C_{\langle |\lambda| \rangle}} < \infty. \] (4.10)

Thus, the following perturbed resolvent identity (see, for instance, [27, p. 172, Lemma 2.5])
\[ (\lambda^{2m} - P(D) - V)^{-1} = (\lambda^{2m} - P(D))^{-1} \left[ I - V(\lambda^{2m} - P(D))^{-1} \right]^{-1} \] (4.11)
holds true. Then (4.11), combined with (4.10) and Lemma 4.1, implies
\[ \left\| (\lambda^{2m} - P(D) - V)^{-1} \right\|_{L^{q}(\mathbb{R}^{n}) \rightarrow L^{q}(\mathbb{R}^{n})} \leq \frac{1}{|\lambda|^{2m}} \frac{1}{1 - C_{\langle |\lambda| \rangle}}. \]

This finishes the proof of Proposition 4.2. \( \square \)

### 4.2 Exponential perturbation estimates

Let \( \mathcal{L} : = P(D) + V \) be the higher order Schrödinger operator as in (2.3). In this subsection, we establish the exponential perturbation estimate for the resolvent of \( \mathcal{L} \). To this end, for any two closed subsets \( E \) and \( F \) in \( \mathbb{R}^{n} \), the intrinsic distance \( d_{\mathcal{E}} \) is defined by setting
\[ d_{\mathcal{E}}(E, F) := \sup_{\phi \in \mathcal{E}_{2m}(\mathbb{R}^{n})} \left\{ \inf \left\{ \phi(x) - \phi(y) : \forall x \in E, y \in F \right\} \right\}, \]
where
\[ \mathcal{E}_{2m}(\mathbb{R}^{n}) := \left\{ \phi \in C^{\infty}(\mathbb{R}^{n}) : \left\| D^{\alpha} \phi \right\|_{L^{\infty}(\mathbb{R}^{n})} \leq 1, \forall 0 \leq |\alpha| \leq 2m \right\} \] (4.12)
and \( C^{\infty}(\mathbb{R}^{n}) \) denotes the set of all infinitely differentiable functions on \( \mathbb{R}^{n} \).

Let us recall the following property of the intrinsic distance, which is just [15, Lemma 4].
Lemma 4.3. Let $E$ and $F$ be two disjoint compact convex subsets in $\mathbb{R}^n$. Then

$$d_{\mathcal{G}}(E, F) \sim d(E, F)$$

with the positive equivalence constants independent of $E$ and $F$, where

$$d(E, F) := \inf\{|x - y| : \forall x \in E, y \in F\}$$

denotes the Euclidean distance between $E$ and $F$.

Now, for any given $\eta \in \mathbb{C}$ and $\phi \in \mathcal{E}_{2m}(\mathbb{R}^n)$, the exponential perturbed operator $\mathcal{L}_{\eta, \phi}$ of $\mathcal{L}$ is defined by setting

$$\text{dom}(\mathcal{L}_{\eta, \phi}) := \{f \in L^2(\mathbb{R}^n) : e^{\eta \phi} f \in \text{dom}(\mathcal{L})\}$$

and, for any $f \in \text{dom}(\mathcal{L}_{\eta, \phi})$,

$$\mathcal{L}_{\eta, \phi} f := e^{-\eta \phi} \mathcal{L}(e^{\eta \phi} f). \quad (4.13)$$

Remark 4.4.

(i) For any $\eta \in \mathbb{R}$, let $L^2_{\eta, \phi}(\mathbb{R}^n)$ be the weighted Lebesgue space equipped with the norm

$$\|f\|_{L^2_{\eta, \phi}(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} |f(x)|^2 e^{2\eta \phi(x)} \, dx \right\}^{1/2}.$$

Since $e^{\eta \phi}$ is an isometry from $L^2_{\eta, \phi}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, it follows, from [27, p. 59], that the operator $-\mathcal{L}_{\eta, \phi}$ generates a semigroup $\{e^{-\eta \phi} e^{-t \mathcal{L}} e^{\eta \phi}\}_{t \geq 0}$ on the weighted space $L^2_{\eta, \phi}(\mathbb{R}^n)$ that is similar to $\{e^{-t \mathcal{L}}\}_{t \geq 0}$ on $L^2(\mathbb{R}^n)$. Moreover, $\sigma(\mathcal{L}) = \sigma(\mathcal{L}_{\eta, \phi})$ and the following perturbation identity

$$(\lambda^{2m} - \mathcal{L}_{\eta, \phi})^{-1} = e^{-\eta \phi} \left(\lambda^{2m} - \mathcal{L}\right)^{-1} e^{\eta \phi} \quad \text{on} \quad L^2_{\eta, \phi}(\mathbb{R}^n) \quad (4.14)$$

holds true for any given $\lambda^{2m} \in \rho(\mathcal{L})$.

(ii) If $\eta \in \mathbb{C}^n$ is a $n$-dimensional complex vector, then, similarly to $\mathcal{L}_{\eta, \phi}$ in (4.13), another exponential perturbed operator $\mathcal{L}_{\eta}$ of $\mathcal{L}$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$\mathcal{L}_{\eta} f(x) := e^{-\eta x} \mathcal{L}(e^{\eta} f)(x) \quad (4.15)$$

with $\text{dom}(\mathcal{L}_{\eta}) := \{f \in L^2(\mathbb{R}^n) : e^{\eta x} f \in \text{dom}(\mathcal{L})\}$ (see [63, Chapter 5.3] for more details). As we show later in this subsection, the perturbed operators $\mathcal{L}_{\eta}$ and $\mathcal{L}_{\eta, \phi}$ share many properties.

In what follows, we establish some resolvent estimates for both $\mathcal{L}_{\eta, \phi}$ and $\mathcal{L}_{\eta}$, and show that the perturbation identity (4.14) holds true also for some $\eta$ that are complex. To this end, we use...
$C^m(\mathbb{R}^n)$ for any $m \in \mathbb{N}$ to denote the set of all functions having continuous derivatives up to order $2m$ and we need the following technical lemma.

**Lemma 4.5.** Let $\eta \in \mathbb{C}$ and $V$ be a real-valued measurable function on $\mathbb{R}^n$. Assume that $\mathcal{L}$ and $\mathcal{L}_{\eta,\phi}$ are, respectively, as in (2.3) and (4.13). Then, for any $f \in C^{2m}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{L}_{\eta,\phi} f(x) = (P(D + \eta D\phi) + V)f(x),$$

(4.16)

where $P(D + \eta D\phi) = \sum_{|\alpha|=2m} (-1)^m a_\alpha (D + \eta D\phi)^\alpha$.

**Proof.** For any given $j \in \{1, \ldots, n\}$, and for any $f \in C^{2m}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, it is easy to see that

$$e^{-\eta \phi} \frac{\partial}{\partial x_j} (e^{\eta \phi} f)(x) = \left( \frac{\partial}{\partial x_j} + \eta \frac{\partial \phi}{\partial x_j} \right) f(x),$$

which, together with (4.13), shows (4.16) holds true. This finishes the proof of Lemma 4.5. \(\Box\)

The following proposition establishes the resolvent estimate for $\mathcal{L}_{\eta,\phi}$.

**Proposition 4.6.** Let $p \in (1, \infty)$, $\theta_0 \in (0, \pi/2)$, and $\lambda^{2m} \in \Sigma^\mathbb{C}_{\theta_0}$ be as in (4.1). Assume that $\mathcal{L}$ and $\mathcal{L}_{\eta,\phi}$ are defined, respectively, as in (2.3) and (4.13). If

$$\left\| (1 + |\lambda|^{2m}) (\lambda^{2m} - \mathcal{L})^{-1} \right\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} + \left\| V (\lambda^{2m} - \mathcal{L})^{-1} \right\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} < \infty,$$

(4.17)

then there exist positive constants $C$ and $\delta \in (0, 1)$ such that, for any $\eta \in \mathbb{C}$ satisfying $|\eta| < \delta |\lambda|$, and any $f \in L^p(\mathbb{R}^n)$,

$$\left\| |\lambda|^{2m} (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} f \right\|_{L^p(\mathbb{R}^n)} + \left\| \Delta^m (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} f \right\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.$$

(4.18)

**Proof.** For any given $p \in (1, \infty)$ and $\lambda^{2m} \in \Sigma^\mathbb{C}_{\theta_0}$, and for any $f \in L^p(\mathbb{R}^n)$, let

$$g := (\lambda^{2m} - \mathcal{L})^{-1} f.$$  

(4.19)

From (2.3), we deduce that

$$(P(D) + V - \lambda^{2m}) g = -(\lambda^{2m} - \mathcal{L}) g = -f.$$  

(4.20)

Moreover, by Lemma 4.5 and (4.12), we obtain, for any $x \in \mathbb{R}^n$,

$$\left| (\mathcal{L}_{\eta,\phi} - \mathcal{L}) g(x) \right| \leq \sum_{|\alpha| = 2m} \left| (D + \eta D\phi)^\alpha g(x) - D^\alpha g(x) \right|$$

$$\leq \sum_{k=0}^{2m-1} \sum_{l=1}^{2m-k} \left| \nabla^k g(x) \right| |\eta|^l \leq \sum_{k=0}^{2m-1} \left( |\eta| + |\eta|^{2m-k} \right) \left| \nabla^k g(x) \right|.$$  

(4.21)
which, combined with the Gagliardo–Nirenberg inequality (see Lemma 3.4) and the assumptions $|\eta| < \delta |\lambda|$ with $\delta < 1$, implies

\[
\left\| (\mathcal{L}_\eta - \mathcal{L}) g \right\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{k=0}^{2m-1} \left( |\eta| + |\eta|^{2m-k} \right) \left\| \nabla^k g \right\|_{L^p(\mathbb{R}^n)}
\]

\[
\lesssim \sum_{k=0}^{2m-1} \left( |\eta|^{2m-k} \right) \left\| g \right\|_{L^p(\mathbb{R}^n)} \left\| \Delta^{m/2} g \right\|_{L^p(\mathbb{R}^n)} + |\eta| \left\| g \right\|_{L^p(\mathbb{R}^n)} \left\| \Delta^{m/2} g \right\|_{L^p(\mathbb{R}^n)}
\]

\[
\lesssim \delta \sum_{k=0}^{2m-1} \left[ |\lambda|^{2m/2k} + |\lambda|^{2m} \right] \left\| g \right\|_{L^p(\mathbb{R}^n)} \left\| \Delta^{m/2} g \right\|_{L^p(\mathbb{R}^n)}
\]

\[
\lesssim \delta \left[ (1 + |\lambda|^{2m}) \left\| g \right\|_{L^p(\mathbb{R}^n)} + \left\| \Delta^m g \right\|_{L^p(\mathbb{R}^n)} \right].
\]

On the other hand, from (4.6), it follows that the operator $\Delta^m P(D)^{-1}$ is bounded on $L^p(\mathbb{R}^n)$. By this, (4.17), (4.19), and (4.20), we obtain

\[
(1 + |\lambda|^{2m}) \left\| g \right\|_{L^p(\mathbb{R}^n)} + \left\| \Delta^m g \right\|_{L^p(\mathbb{R}^n)}
\]

\[
\lesssim (1 + |\lambda|^{2m}) \left\| g \right\|_{L^p(\mathbb{R}^n)} + \left\| P(D) g \right\|_{L^p(\mathbb{R}^n)}
\]

\[
\lesssim (1 + |\lambda|^{2m}) \left\| g \right\|_{L^p(\mathbb{R}^n)} + \left\| (P(D) + V - \lambda^{2m}) g \right\|_{L^p(\mathbb{R}^n)} + \left\| V g \right\|_{L^p(\mathbb{R}^n)} \lesssim \left\| f \right\|_{L^p(\mathbb{R}^n)}.
\]

(4.23)

Thus, if the constant $\delta$ in (4.22) is sufficiently small, we then have

\[
\left\| (\mathcal{L}_\eta, \phi - \mathcal{L})(\lambda^{2m} - \mathcal{L})^{-1} \right\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} < 1,
\]

which implies that the operator $I - (\mathcal{L}_\eta, \phi - \mathcal{L})(\lambda^{2m} - \mathcal{L})^{-1}$ has a bounded inverse on $L^p(\mathbb{R}^n)$. By the identity

\[
\lambda^{2m} - \mathcal{L}_\eta, \phi = \left[ I - (\mathcal{L}_\eta, \phi - \mathcal{L})(\lambda^{2m} - \mathcal{L})^{-1} \right] (\lambda^{2m} - \mathcal{L})
\]

and (4.17), we conclude that $(\lambda^{2m} - \mathcal{L}_\eta, \phi)^{-1}$ exists and

\[
(\lambda^{2m} - \mathcal{L}_\eta, \phi)^{-1} = (\lambda^{2m} - \mathcal{L})^{-1} \left[ I - (\mathcal{L}_\eta, \phi - \mathcal{L})(\lambda^{2m} - \mathcal{L})^{-1} \right]^{-1}.
\]

Moreover, using (4.19) and (4.23), we find that both the operators $|\lambda|^{2m}(\lambda^{2m} - \mathcal{L})^{-1}$ and $\Delta^m(\lambda^{2m} - \mathcal{L})^{-1}$ are bounded on $L^p(\mathbb{R}^n)$. This shows that (4.18) holds true and hence finishes the proof of Proposition 4.6.

\[\square\]

Remark 4.7. Let $\mathcal{L}_\eta$ be the exponential perturbed operator as in (4.15) with $\eta \in \mathbb{C}^n$. Since, for any $j \in \{1, \ldots, n\}$, $f \in C^{2m}(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$,

\[
e^{-\eta x} \frac{\partial}{\partial x_j} (e^{\eta x} f)(x) = \left( \frac{\partial}{\partial x_j} + \eta_j \right) f(x),
\]
it follows that, for any \( f \in C^2m(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \),

\[
\mathcal{L}_\eta f(x) = (P(D + \eta) + V)f(x),
\]

where \( P(D + \eta) = \sum_{|\alpha|=2m}(-1)^m a_\alpha (D + \eta)^\alpha \) (see also [63, (5.92)]). Similarly to (4.21), we also obtain, for \( g \) as in (4.19) and any \( x \in \mathbb{R}^n \),

\[
\|\mathcal{L}_\eta g(x) - (D + \eta)^\alpha g(x) - D^\alpha g(x)\|_p \lesssim \sum_{k=0}^{2m-1} |\eta|^{2m-k} \|\nabla^k g(x)\|.
\]

Here, since \( \eta \) is a constant vector, we only have the term \( |\eta|^{2m-k} \) in the last formula, while in (4.21) we need \( |\eta| + |\eta|^{2m-k} \). This is the main difference for applications of the exponential perturbed operators \( \mathcal{L}_{\eta,\phi} \) and \( \mathcal{L}_\eta \).

Thus, following the argument used in the estimation of (4.22) in the proof of Proposition 4.6, we further obtain

\[
\|\lambda^{2m}(\lambda^{2m} - \mathcal{L}_\eta)^{-1} f\|_{L^p(\mathbb{R}^n)} + \|\Delta^m(\lambda^{2m} - \mathcal{L}_\eta)^{-1} f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}
\]

with the implicit positive constant independent of \( \lambda, \eta, \) and \( f \), under the assumption that

\[
\|\lambda^{2m}(\lambda^{2m} - \mathcal{L})^{-1}\|_{L^p(\mathbb{R}^n)\rightarrow L^p(\mathbb{R}^n)} + \|\nabla(\lambda^{2m} - \mathcal{L})^{-1}\|_{L^p(\mathbb{R}^n)\rightarrow L^p(\mathbb{R}^n)} < \infty
\]

which is slightly different from (4.17).

Now, for any given \( \lambda^{2m} \in \rho(\mathcal{L}_{\eta,\phi}) \), it is well known that the resolvent \((\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1}\) is holomorphic in \( \lambda^{2m} \) due to the resolvent identity

\[
(\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} - (\xi^{2m} - \mathcal{L}_{\eta,\phi})^{-1} = (\xi^{2m} - \lambda^{2m})(\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1}(\xi^{2m} - \mathcal{L}_{\eta,\phi})^{-1}.
\]

In what follows, we show that \((\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1}\) is also holomorphic in \( \eta \) for any fixed \( \lambda^{2m} \in \rho(\mathcal{L}_{\eta,\phi}) \). That is, for any \( f \in L^p(\mathbb{R}^n) \), the limit

\[
\lim_{h \in C, h \rightarrow 0} \frac{(\lambda^{2m} - \mathcal{L}_{\eta+h,\phi})^{-1} f - (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} f}{h}
\]

exists in \( L^p(\mathbb{R}^n) \) (see [40, Section B.3] for some backgrounds about the vector-valued holomorphic functions). To this end, we need the following resolvent identity.

**Lemma 4.8.** Let \( \eta, h \in C, \theta_0 \in (0, \pi/2), \lambda^{2m} \in \Sigma^C_{\theta_0} \), and \( \mathcal{L} \) and \( \mathcal{L}_{\eta,\phi} \) be defined, respectively, as in (2.3) and (4.13). Assume that \( \mathcal{L} \) satisfies (4.17), and \( \eta \) and \( h \) satisfy \(|\eta| + |h| < \delta |\lambda| \) with \( \delta \) as in Proposition 4.6. Then it holds true that

\[
(\lambda^{2m} - \mathcal{L}_{\eta+h,\phi})^{-1} - (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} = (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1}(\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi})(\lambda^{2m} - \mathcal{L}_{\eta+h,\phi})^{-1}.
\]
Proof. Since \( \mathcal{L} \) satisfies (4.17) and \( |\eta| + |h| < \delta|\lambda| \), we deduce, from Proposition 4.6, that both \( (\lambda^{2m} - \mathcal{L}_{\eta+h,\phi})^{-1} \) and \( (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} \) exist. Moreover, we have
\[
(\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} + (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} (\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi}) (\lambda^{2m} - \mathcal{L}_{\eta+h,\phi})^{-1}
= (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} [(\lambda^{2m} - \mathcal{L}_{\eta+h,\phi}) + (\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi})] (\lambda^{2m} - \mathcal{L}_{\eta+h,\phi})^{-1}
= (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} (\lambda^{2m} - \mathcal{L}_{\eta,\phi}) (\lambda^{2m} - \mathcal{L}_{\eta+h,\phi})^{-1} = (\lambda^{2m} - \mathcal{L}_{\eta+h,\phi})^{-1},
\]
which completes the proof of Lemma 4.8. \( \square \)

In what follows, for any \( m \in \mathbb{N} \) and \( q \in (1, \infty) \), the Sobolev space \( W^{2m,q}(\mathbb{R}^n) \) is defined by setting
\[
W^{2m,q}(\mathbb{R}^n) := \left\{ f \in L^p(\mathbb{R}^n) : \|f\|_{W^{2m,q}(\mathbb{R}^n)} := \left[ \sum_{k=0}^{2m} \sum_{|\alpha|=k} \|D^\alpha f\|_{L^p(\mathbb{R}^n)}^p \right]^{1/p} < \infty \right\}.
\]

Proposition 4.9. Let \( \theta_0 \in (0, \pi/2) \), \( \lambda^{2m} \in \Sigma^C_{\theta_0} \) be as in (4.1), \( p \in (1, \infty) \), and \( \mathcal{L} \) and \( \mathcal{L}_{\eta,\phi} \) with \( \eta \in \mathbb{C} \) and \( \phi \in \mathcal{E}_{2m}(\mathbb{R}^n) \) be defined, respectively, as in (2.3) and (4.13). Assume that \( \mathcal{L} \) satisfies (4.17) and \( \eta \) satisfies \( |\eta| < \delta|\lambda| \) with \( \delta \) as in Proposition 4.6. Then:

(i) \( (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} \) is holomorphic in \( \eta \);
(ii) for any \( f \in L^p(\mathbb{R}^n) \), \( (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} f = e^{-\eta \phi}(\lambda^{2m} - \mathcal{L})^{-1}(e^{\eta \phi} f) \) in \( L^p(\mathbb{R}^n) \).

Proof. We first prove (i). Let \( \eta \in \mathbb{C} \) and \( \phi \in \mathcal{E}_{2m}(\mathbb{R}^n) \). For any \( f \in L^p(\mathbb{R}^n) \), let
\[
u_\eta := (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} f.
\]
We need to show the limit of (4.24) exists in \( L^p(\mathbb{R}^n) \). By Proposition 4.6, it is easy to show that \( u_\eta \in W^{2m,p}(\mathbb{R}^n) \) and
\[
(\lambda^{2m} - \mathcal{L}_{\eta,\phi}) u_\eta = f.
\]
Differentiating both sides of the identity with respect to \( \eta \), we conclude that
\[
(\lambda^{2m} - \mathcal{L}_{\eta,\phi}) \frac{\partial u_\eta}{\partial \eta} = \frac{\partial \mathcal{L}_{\eta,\phi}}{\partial \eta} u_\eta =: g,
\]
where
\[
\frac{\partial \mathcal{L}_{\eta,\phi}}{\partial \eta} = \sum_{|\alpha|=2m} (-1)^m a_\alpha \frac{\partial}{\partial \eta} (D + \eta D\phi)^\alpha.
\]
Since \( u_\eta \in W^{2m,p}(\mathbb{R}^n) \), we deduce that \( g \in L^p(\mathbb{R}^n) \). Now, let \( \nu := (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} g \in L^p(\mathbb{R}^n) \) and, for any given \( h \in \mathbb{C} \) and \( |h| \ll 1 \) small enough,

\[
    u_{\eta,h} := \frac{1}{h} (u_{\eta+h} - u_\eta).
\]

Then, using Lemma 4.8, we obtain

\[
(\lambda^{2m} - \mathcal{L}_{\eta,\phi})(u_{\eta,h} - \nu)
= \frac{1}{h} \left[ (\lambda^{2m} - \mathcal{L}_{\eta,\phi}) \left( (\lambda^{2m} - \mathcal{L}_{\eta+h,\phi})^{-1} - (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} \right) \right] f - g
= \frac{1}{h} (\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi})(\lambda^{2m} - \mathcal{L}_{\eta+h,\phi})^{-1} f - g
= \frac{1}{h} (\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi})u_{\eta+h} - g. \tag{4.27}
\]

Let

\[
F_h := \frac{1}{h} (\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi})(u_{\eta+h} - u_\eta).
\]

By (4.25) and Lemma 4.8, we easily conclude that

\[
F_h = \frac{1}{h} (\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi}) \left( (\lambda^{2m} - \mathcal{L}_{\eta+h,\phi})^{-1} - (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} \right) f
= \frac{1}{h} (\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi})(\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} \circ (\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi})(\lambda^{2m} - \mathcal{L}_{\eta+h,\phi})^{-1} f. \tag{4.28}
\]

From Lemma 4.5, (4.21), and the assumptions \( |\eta| < \delta |\lambda| \), \( |\delta| < 1 \), and \( |h| < 1 \), we deduce that, for any \( u \in W^{2m,p}(\mathbb{R}^n) \),

\[
\left| (\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi}) u \right| \lesssim \sum_{|\alpha|=2m} \left|\right( (D + \eta D\phi) + hD\phi \right|^\alpha u - (D + \eta D\phi)^\alpha u \right|
\lesssim \sum_{k=0}^{2m-1} \sum_{l=1}^{2m-k} \sum_{|\beta|=k} \left| (D + \eta D\phi)^\beta u \right| |h|^l \leq C_{(\lambda)} |h| \left[ \sum_{k=0}^{2m-1} \left| \nabla^k u \right| \right],
\]

where the positive constant \( C_{(\lambda)} \) depends on \( \lambda \), but is independent of \( h \) and \( u \). Applying Proposition 4.6 and letting \( |h| \ll 1 \) small enough, we obtain

\[
\left\| (\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi})(\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} \right\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)}
+ \left\| (\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi})(\lambda^{2m} - \mathcal{L}_{\eta+h,\phi})^{-1} \right\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)}
\lesssim |h|,
\]

\[
||| (\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi})(\lambda^{2m} - \mathcal{L}_{\eta,\phi}) u - (\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi})(\lambda^{2m} - \mathcal{L}_{\eta+h,\phi}) u |||_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)}
\lesssim |h|.
\]

\[
||| (\mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi}) u |||_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)}
\lesssim |h|.
\]
which, together with (4.28), implies that

$$
\| F_h \|_{L^p(\mathbb{R}^n)} = \left\| \frac{1}{h} \left( \mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi} \right) \left( \lambda^{2m} - \mathcal{L}_{\eta,\phi} \right)^{-1} \circ \left( \mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi} \right) \left( \lambda^{2m} - \mathcal{L}_{\eta+h,\phi} \right)^{-1} f \right\|_{L^p(\mathbb{R}^n)}
\lesssim |h| \| f \|_{L^p(\mathbb{R}^n)},
$$

and hence \( \lim_{h \to \overrightarrow{0}_2} \| F_h \|_{L^p(\mathbb{R}^n)} = 0 \). By this, (4.27), and Proposition 4.6 again, we conclude that

$$
\| u_{\eta,h} - \nu \|_{L^p(\mathbb{R}^n)} \lesssim \left\| \left( \lambda^{2m} - \mathcal{L}_{\eta,\phi} \right) (u_{\eta,h} - \nu) \right\|_{L^p(\mathbb{R}^n)}
\lesssim \left\| \frac{1}{h} \left( \mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi} \right) (u_{\eta+h} - u_{\eta}) \right\|_{L^p(\mathbb{R}^n)} + \left\| \frac{1}{h} \left( \mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi} \right) u_{\eta} - g \right\|_{L^p(\mathbb{R}^n)}
\lesssim \| F_h \|_{L^p(\mathbb{R}^n)} + \left\| \frac{1}{h} \left( \mathcal{L}_{\eta+h,\phi} - \mathcal{L}_{\eta,\phi} \right) u_{\eta} - g \right\|_{L^p(\mathbb{R}^n)},
$$

which, together with (4.26), shows that the above norm turns to 0 as \( h \to \overrightarrow{0}_2 \). This shows that

$$
\lim_{h \in \mathbb{C}, h \to \overrightarrow{0}_2} \frac{(\lambda^{2m} - \mathcal{L}_{\eta+h,\phi})^{-1} f - (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} f}{h} = \nu
$$

in \( L^p(\mathbb{R}^n) \). Thus, the limit of (4.24) exists in \( L^p(\mathbb{R}^n) \), which shows (i).

For (ii), it is easy to see that (ii) holds true when \( \eta \) is a pure imaginary number, because, in this case, \( e^{\eta \phi} \) is an isometry on \( L^2(\mathbb{R}^n) \) for any \( \phi \in L^2_{2m}(\mathbb{R}^n) \) (see also Remark 4.4 for a similar case). Then, by (i), we conclude that \( (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} \) is holomorphic in \( \eta \) for any \( \eta \in \mathbb{C} \) satisfying \( |\eta| < \delta |\lambda| \). This, combined with the Morera theorem, shows that (ii) holds true for any such \( \eta \), which shows (ii) and hence completes the proof of Proposition 4.9.

\[ \square \]

Remark 4.10. Let \( \mathcal{L}_\eta \) be the exponential perturbed operator as in (4.15) with \( \eta \in \mathbb{C}^n \). Following the proof of Proposition 4.9, we obtain that \( (\lambda^{2m} - \mathcal{L}_\eta)^{-1} \) is also holomorphic in \( \eta \). Moreover, for any given \( p \in (1, \infty) \), and for any \( f \in L^p(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \)

$$
(\lambda^{2m} - \mathcal{L}_\eta)^{-1} f(x) = e^{-\eta x} (\lambda^{2m} - \mathcal{L})^{-1} (e^{\eta \cdot} f)(x),
$$

in \( L^p(\mathbb{R}^n) \) (see [63, Lemma 5.15 and (5.103)] for similar results).

5  \quad \textbf{ESTIMATES FOR HEAT KERNELS}

In this section, we prove the main results of this article, based on the perturbation estimates established in Section 4. To begin with, we first summarize some boundedness results of the resolvent of \( \mathcal{L}_{\eta,\phi} \) and \( \mathcal{L}_\eta \), respectively, as in (4.13) and (4.15).
5.1 Preliminaries on parameters

For any given \( m \in \mathbb{N}, q \in (1, \infty), p \in [q, \infty), \) and \( s \in (0, 2m]\) satisfying

\[
0 \leq n \left( \frac{1}{q} - \frac{1}{p} \right) \leq 2m - s ,
\]

(5.1)

let \( V \) be a real-valued measurable function on \( \mathbb{R}^n \). Combining (4.8) of Proposition 4.2, respectively, with Propositions 3.3, 3.6, 3.9, and 3.10, we summarize the following four groups of Schechter-type conditions.

(i) The parameters \( p, q, s, \alpha, \) and \( S_1 \) satisfy

\[
\begin{align*}
& s \in (0, n), \\
& q \in \left[ 1, \frac{n}{n - s} \right), \\
& \alpha \in (0, (s - n)q + n], \\
& S_1 := s - 2m + n \left( \frac{1}{q} - \frac{1}{p} \right) + n - s + \frac{\alpha - n}{q}, \\
& C_2 |\lambda|^{S_1} M_{\alpha, q, p', 1/|\lambda|}(V) < 1
\end{align*}
\]

with \( C_2 \) and \( C \), respectively, as in (4.2) and (3.5).

(ii) The parameters \( p, q, s, \alpha, t, \sigma, \) and \( S_2 \) satisfy

\[
\begin{align*}
& s \in (0, n), \\
& t, \sigma \in (1, \infty], \\
& \frac{1}{q} = \frac{1}{t} + \frac{1}{\sigma}, \\
& \frac{1}{\sigma} \leq \frac{1}{p} \leq \frac{1}{\sigma} + \frac{s}{n}, \\
& S_2 := s - 2m + n \left( \frac{1}{q} - \frac{1}{p} \right) - s \left[ 1 - n - \left( \frac{1}{p} - \frac{1}{\sigma} \right) \right], \\
& C_2 |\lambda|^{S_2} \|V\|_{L^t(\mathbb{R}^n)} < 1
\end{align*}
\]

with \( C_2 \) and \( C \), respectively, as in (4.2) and (3.6).

(iii) The parameters \( p, q, s, \alpha, t, \) and \( S_3 \) satisfy

\[
\begin{align*}
& s \in (0, n/2), q \in [2, \infty), t \in (q, \infty), r \in \left[ q, \frac{2n}{n - 2s} \right), \\
& \alpha \in \left( 0, n + \frac{(2s - n)r}{2} \right], \\
& \frac{1}{t} + \frac{1}{r} = \frac{1}{q}, \\
& S_3 := s - 2m + n \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{1}{r} \left[ \frac{(n - 2s)r}{2} + \alpha - n \right], \\
& C_2 |\lambda|^{S_3} M_{\alpha, r, t, 1/|\lambda|}(V) < 1
\end{align*}
\]

with \( C_2 \) and \( C \), respectively, as in (4.2) and (3.12).
(iv) The parameters $p$, $q$, $s$, $\alpha$, and $S_4$ satisfy

\[
\begin{align*}
\alpha - n &\leq p(s - n) + \frac{np}{p'}, \\
2n &> p'(n - s), \\
S_4 &:= \alpha / p - 2m, \\
C_2 |\lambda|^{s_4} M_{\alpha, p, \infty, 1/|\lambda|}(V) &< 1
\end{align*}
\]  
(5.5)

with $C_2$ and $C$, respectively, as in (4.2) and (3.14).

For the sake of simplicity, we use the same notation $M_{|\lambda|}(V)$ to denote the last quantity in (5.2) through (5.5), respectively. By (2.8), it is easy to find that, for any $c \in (0, \infty)$,

\[
M_{|\lambda|}(cV) = cM_{|\lambda|}(V).
\]
(5.6)

**Remark 5.1.** Let parameters $p$, $q$, $s$, $\alpha$, $C_2$, and $\{S_4\}_{i=1}^4$ satisfy (5.2) through (5.5). It is easy to see that

\[
C_{(|\lambda|)} := C_2 |\lambda|^{-[2m - s - n (\frac{1}{q} - \frac{1}{p})]} \left\| T_{s, |\lambda|} \right\|_{L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} < 1.
\]

We first show that, for parameters satisfying (5.2) through (5.5), the higher order Schrödinger operator $\mathcal{L}$ defined in (2.3) can be extended to a closed operator on $W^{2m,q}(\mathbb{R}^n)$ for any given $q \in (1, \infty)$.

**Proposition 5.2.** Let $\theta_0 \in (0, \pi/2)$, $\lambda^{2m} \in \Sigma^\circ_{\theta_0}$ be as in (4.1), $q \in (1, \infty)$, $p \in [q, \infty)$, and $s \in (0, 2m]$ satisfy (5.1). Assume that $\mathcal{L}$ is as in (2.3) and one of (5.2) through (5.5) holds true. Then $\mathcal{L}$ is a closed linear operator on $W^{2m,q}(\mathbb{R}^n)$.

**Proof.** For any given $q \in (1, \infty)$, it is known that $P(D)$ can be extended to a closed linear operator on $L^q(\mathbb{R}^n)$ with the domain

\[
\text{dom}_q(P(D)) = W^{2m,q}(\mathbb{R}^n).
\]

If one of (5.2) through (5.5) holds true, then, applying Remark 5.1 and Proposition 4.2 [see, in particular, (4.9)], we know that, for any $f \in L^q(\mathbb{R}^n)$,

\[
\| V f \|_{L^q(\mathbb{R}^n)} = \left\| V (\lambda^{2m} - P(D))^{-1} \circ (\lambda^{2m} - P(D))^\prime f \right\|_{L^q(\mathbb{R}^n)} < \| P(D)f \|_{L^q(\mathbb{R}^n)} + |\lambda|^{2m} \| f \|_{L^q(\mathbb{R}^n)},
\]

which implies that $V$ is relatively bounded with respect to $P(D)$, with bound constant strictly less than 1. Thus, by [27, p. 171, Lemma 2.4], we conclude that $P(D) + V$ is closed on $W^{2m,q}(\mathbb{R}^n)$. This finishes the proof of Proposition 5.2. □
The next result summarizes the boundedness of the resolvent \((\lambda - \mathcal{L}_{\eta,\phi})^{-1}\) based on the conditions (5.2) through (5.5).

**Proposition 5.3.** Let \(m \in \mathbb{N}, \theta_0 \in (0, \pi/2), \lambda^{2m} \in \Sigma^C_{\theta_0}\) be as in (4.1), \(q \in (1, \infty), p \in [q, \infty),\) and \(s \in (0, 2m]\) satisfy (5.1). Assume that \(\mathcal{L}\) and \(\mathcal{L}_{\eta,\phi}\) are defined, respectively, as in (2.3) and (4.13). If one of (5.2) through (5.5) holds true and
\[
\left\| (\lambda^{2m} - \mathcal{L})^{-1} \right\|_{L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} < \infty, \tag{5.7}
\]
then there exist positive constants \(C\) and \(\delta \in (0, 1)\) such that, for any \(\eta \in \mathbb{C}\) satisfying \(|\eta| < \delta |\lambda|\), and any \(f \in L^q(\mathbb{R}^n),\)
\[
\left\| |\lambda|^{2m} (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} f \right\|_{L^q(\mathbb{R}^n)} + \left\| \Delta^m (\lambda^{2m} - \mathcal{L}_{\eta,\phi})^{-1} f \right\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^q(\mathbb{R}^n)}. \tag{5.8}
\]

**Proof.** Let parameters \(p, q, s, \alpha, C_2,\) and \(\{S_i\}_{i=1}^{14}\) satisfy (5.2) through (5.5). Using Remark 5.1, we obtain
\[
C_{(|\lambda|)} = C_2 |\lambda|^{-[2m-s-n(1-q^{-1}_p)]} \|T_{s,|\lambda|}\|_{L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} < 1.
\]
This, combined with Proposition 4.2, implies that, for any \(f \in L^p(\mathbb{R}^n),\)
\[
\left\| |\lambda|^{2m} (\lambda^{2m} - \mathcal{L})^{-1} f \right\|_{L^q(\mathbb{R}^n)} \lesssim \frac{1}{1 - C_{(|\lambda|)}} \|f\|_{L^q(\mathbb{R}^n)}.
\]
Moreover, by (4.9) through (4.11), we have
\[
\left\| V (\lambda^{2m} - \mathcal{L})^{-1} f \right\|_{L^q(\mathbb{R}^n)} \lesssim \left\| V (\lambda^{2m} - P(D))^{-1} \right\|_{L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} \left\| [I - V (\lambda^{2m} - P(D))^{-1}]^{-1} f \right\|_{L^q(\mathbb{R}^n)} \lesssim \frac{1}{1 - C_{(|\lambda|)}} \|f\|_{L^q(\mathbb{R}^n)},
\]
which, together with (5.7), shows (4.17) with \(p\) replaced by \(q\). By Proposition 4.6, we find that (5.8) holds true. This finishes the proof of Proposition 5.3.

**Remark 5.4.** Let \(\mathcal{L}_{\eta}\) be the exponential perturbed operator as in (4.15) with \(\eta \in \mathbb{C}^n\). By Remark 4.7, we know that (5.8) holds true with \(\mathcal{L}_{\eta,\phi}\) replaced by \(\mathcal{L}_{\eta}\), under the same assumptions of Proposition 5.3 but without the condition (5.7).

### 5.2 Davies–Gaffney estimates

In this subsection, we establish the Davies–Gaffney estimates for the semigroup \(\{e^{-t\mathcal{L}}\}_{t>0}\) generated by \(-\mathcal{L}\) by proving Theorem 1.2.
Proof of Theorem 1.2. To begin with, we first claim that $-\mathcal{L}$ generates a bounded holomorphic semigroup $\{e^{-t\mathcal{L}}\}_{t>0}$ on $L^2(\mathbb{R}^n)$. Indeed, by Remark 5.1 and Proposition 4.2, we conclude that, for any given $\lambda \in \Sigma_{\delta_0}$ as in (4.1) with $\delta_0 \in (0, \pi/2)$, and for any $f \in L^2(\mathbb{R}^n)$,

$$\left\| (\lambda - \mathcal{L})^{-1} f \right\|_{L^2(\mathbb{R}^n)} \lesssim \frac{1}{|\lambda|} \|f\|_{L^2(\mathbb{R}^n)},$$

which implies that $\mathcal{L}$ is a sectorial operator on $L^2(\mathbb{R}^n)$. This shows the above claim by applying [27, Chapter II, Theorem 4.6].

Based on the $L^2(\mathbb{R}^n)$ boundedness of $\{e^{-t\mathcal{L}}\}_{t>0}$, we know that, to prove (1.15), it suffices to consider the case $t < [d(E, F)]^{2m}$. Using Proposition 4.9(ii), we find that, for some $\delta \in (0, 1)$ sufficiently small, any given $\lambda \in \Sigma_{\delta_0}$, $\phi \in \mathcal{E}_{2m}(\mathbb{R}^n)$, and $\eta \in \mathbb{R}^+$ satisfying $\eta < \delta |\lambda|^{1/(2m)}$, and any $f \in L^2(\mathbb{R}^n)$,

$$(\lambda - \mathcal{L}_{\eta, \phi})^{-1} f = e^{-\eta \phi}(\lambda - \mathcal{L})^{-1}(e^{\eta \phi} f)$$

holds true in $L^2(\mathbb{R}^n)$. Then, by (5.8) and Lemma 4.3, we obtain, for any given $\lambda \in \Sigma_{\delta_0}$ and any disjoint compact convex subsets $E$ and $F$,

$$\left\| 1_E (\lambda - \mathcal{L})^{-1} 1_F \right\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq \left\| e^{\eta \phi} \circ \left( \left( \lambda - \mathcal{L}_{\eta, \phi} \right)^{-1} \right) \right\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq \frac{1}{|\lambda|} \exp \left\{ -\eta \left[ \inf_{y \in F} \phi(y) - \sup_{x \in E} \phi(x) \right] \right\} \lesssim \frac{1}{|\lambda|} \exp \{-c\eta d(E, F)\} \quad (5.9)$$

for some $c \in (1, \infty)$. Using the following identity based on functional calculus (see, for instance, [3, (2.25)])

$$e^{-t\mathcal{L}} = \frac{1}{2\pi i} \int_{\Gamma} e^{-\lambda} (\lambda - \mathcal{L})^{-1} d\lambda,$$  \hspace{1cm} (5.10)

where $\Gamma$ is a path in the complex plane that consists of three parts:

$$\Gamma_0 := \left\{ z = Re^{i\theta} : |\theta| \geq \theta_0 \right\}, \quad \Gamma_+ := \left\{ z = re^{i\theta_0} : r \geq R \right\}, \quad \text{and}$$

$$\Gamma_- := \left\{ z = re^{-i\theta_0} : r \geq R \right\} \quad (5.11)$$

with $R \in (0, \infty)$ to be determined later (see Figure 2).

Now, for any $f \in L^2(E)$ with supp $f \subset E$, we estimate $\|e^{-t\mathcal{L}} f\|_{L^2(F)}$ by considering the corresponding integrals, respectively, over $\Gamma_+$, $\Gamma_-$, and $\Gamma_0$. To be precise, let $\eta := \frac{\delta}{c} |\lambda|^{1/(2m)}$ and $R := \varepsilon \left( \frac{d(E, F)}{t} \right)^{2m/(2m-1)}$ with $c$ as in (5.9) and $\varepsilon \in (0, 1)$ satisfying $\delta \varepsilon^{1/(2m)} > \varepsilon$. Then, from (5.9) and the assumptions $\theta_0 \in (0, \pi/2)$ and $t < [d(E, F)]^{2m}$, we deduce that
FIGURE 2 The path $\Gamma$

$$I_+ := \int_{\Gamma_+} e^{-t \Re \lambda} \left\| (\lambda - \mathcal{L})^{-1} f \right\|_{L^2(F)} \, |d\lambda|$$

$$\lesssim \int_{R} \frac{1}{r} e^{-t r \cos \theta_0} e^{-t \eta d(E,F)} \, |d\lambda| \lesssim \int_{R} \frac{1}{r} e^{-t r \cos \theta_0} e^{-\delta \epsilon^{1/(2m)} d(E,F)} \, |d\lambda| \lesssim \exp \left\{ -\bar{c}_5 \frac{[d(E,F)]^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\} \left\| f \right\|_{L^2(E)}$$

by choosing a suitable constant $\bar{c}_5 \in (0, \infty)$, which is the desired estimate. Similarly, we have

$$I_- := \int_{\Gamma_-} e^{-t \Re \lambda} \left\| (\lambda - \mathcal{L})^{-1} f \right\|_{L^2(F)} \, |d\lambda| \lesssim \exp \left\{ -\bar{c}_5 \frac{[d(E,F)]^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\} \left\| f \right\|_{L^2(E)},$$

For the integral over $\Gamma_0$, using the fact that $\int_0^{2\pi} e^{-s \cos \theta} \, d\theta = 2\pi I_0(s) \lesssim \frac{e^s}{s^{3/2}}$ with $I_0(\cdot)$ being the modified Bessel function as in [1, p. 376-377, 9.6.16 and 9.7.1] and the assumption $\delta \epsilon^{1/(2m)} > \epsilon$, we find that

$$I_0 := \int_{\Gamma_0} e^{-t \Re \lambda} \left\| (\lambda - \mathcal{L})^{-1} f \right\|_{L^2(F)} \, |d\lambda|$$

$$\lesssim \int_{-\delta_0}^{\delta_0} e^{-t \cos \theta} \, d\theta \exp \left\{ -\bar{c}_5 \frac{[d(E,F)]^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\} \left\| f \right\|_{L^2(E)}$$

by choosing a suitable constant $\bar{c}_5 \in (0, \infty)$. 
Combining the estimates for $I_+$, $I_-$, and $I_0$, we conclude that (1.15) holds true. This finishes the proof of Theorem 1.2.

□

**Corollary 5.5.** Let $m \in \mathbb{N}$, $\mathcal{L} = P(D) + V$ be the $2m$-order Schrödinger operator on $\mathbb{R}^n$ as in (2.3), and $V$ a real-valued measurable function on $\mathbb{R}^n$. If one of (5.2) through (5.5) holds true with $\sup_{|\lambda| \in [w/2, \infty)} M_{|\lambda|}(V) < 1$ for some $w \in (0, \infty)$ and $M_{|\lambda|}(V)$ as in (5.6), then there exist positive constants $C$ and $c_5$ such that, for any $t \in (0, \infty)$, any disjoint compact convex subsets $E$ and $F$, and any $f \in L^2(E)$ with $\sup f \subset E$,

$$
\left\| e^{-t\mathcal{L}} f \right\|_{L^2(F)} \leq C e^{c_5 t} \exp \left\{ -c_5 \frac{[d(E,F)]^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\} \|f\|_{L^2(E)}. \tag{5.12}
$$

**Proof.** Since one of (5.2) through (5.5) holds true, from Remark 5.1, we deduce that

$$
\sup_{|\lambda| \in [w/2, \infty)} C_{(|\lambda|^{1/(2m)})} < 1,
$$

where $C_{(|\lambda|^{1/(2m)})}$ is the same constant as in (4.8). Thus, following the proof of Proposition 4.2, we conclude that, for any $\theta_0 \in (0, \pi/2)$ and any $\lambda \in \Sigma^C_{\theta_0}$, with $\Sigma^C_{\theta_0}$ as in (4.1), satisfying $|\lambda| > w/2$,

$$
\left\| (\lambda - \mathcal{L})^{-1} \right\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \lesssim \frac{1}{1 - a |\lambda|}.
$$

(5.13)

On the other hand, let $\mathcal{L}_w := \mathcal{L} + w$. It is easy to show, if $\theta_0 \in (0, \pi/2)$ is large enough, then, for any $\lambda \in \Sigma^C_{\theta_0}$ satisfying $|\arg \lambda| \geq \theta_0$,

$$
|\lambda - w| > w/2,
$$

which, together with (5.13), shows that, for any $\lambda \in \Sigma^C_{\theta_0}$ with $|\arg \lambda| \geq \theta_0$,

$$
\left\| (\lambda - \mathcal{L}_w)^{-1} \right\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} = \left\| (\lambda - w) - \mathcal{L})^{-1} \right\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \lesssim \frac{1}{1 - a |\lambda - w|} \lesssim \frac{1}{1 - a w}.
$$

Thus, following the proof of Proposition 5.3, we obtain

$$
\left\| |\lambda| (\lambda - (\mathcal{L}_w)_{\eta, \phi})^{-1} f \right\|_{L^q(\mathbb{R}^n)} + \left\| \Delta^m (\lambda - (\mathcal{L}_w)_{\eta, \phi})^{-1} f \right\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\mathbb{R}^n)}.
$$

Observing that the points $\lambda$ in the integral path $\Gamma$ of (5.10) (see also Figure 2) satisfy $|\arg \lambda| \geq \theta_0$, following the proof of Theorem 1.2, we conclude that there exists a positive constant $c_5$ such that, for any $t \in (0, \infty)$, any disjoint compact convex subsets $E$ and $F$, and any $f \in L^2(E)$ with $\sup f \subset E$,

$$
\left\| e^{-t\mathcal{L}_w} f \right\|_{L^2(F)} \leq \exp \left\{ -c_5 \frac{[d(E,F)]^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\} \|f\|_{L^2(E)}.
$$
This, combined with the identity 

\[ e^{-t \mathcal{L}} = e^{-t \mathcal{L} e^{\omega t}}, \]

shows that (5.12) holds true. This finishes the proof of Corollary 5.5. \qed

**Remark 5.6.**

(i) For any given \( m \in \mathbb{N} \) and \( a \in (-2m, 0) \), let \( V(x) = \pm |x|^a \) for any \( x \in \mathbb{R}^n \setminus \{0\} \). By taking the parameters in (5.1) and (5.5) with \( q = p = 2 \), \( s \in (0, 2m) \), \( \alpha \in (0, n) \), and \( \alpha \leq 2s \), and using Remark 2.6, if further assuming \( a \in (-\min\{2s, n/2\}, 0) \), we then have \( M_{\alpha, 2, \infty, 1/|\lambda|}(V) \sim |\lambda|^{-\alpha/2-a} \) with the positive equivalence constants independent of \( \lambda \). This implies that

\[ M_{|\lambda|}(V) = C_2 C |\lambda|^{S_2} M_{\alpha, 2, \infty, 1/|\lambda|}(V) \sim |\lambda|^{S_2 - (\frac{\alpha}{2} + a)} \]

with the positive equivalence constants independent of \( \lambda \). Since \( S_2 - (\frac{\alpha}{2} + a) = -a - 2m < 0 \), it follows that

\[ \sup_{|\lambda| > \epsilon_0} M_{|\lambda|}(V) < 1 \]

for some \( \epsilon_0 \in (0, \infty) \). Applying Corollary 5.5, we conclude that the semigroup generated by \(- (P(D) + V)\) satisfies the local Davies–Gaffney estimate (5.12).

(ii) For any given \( m \in \mathbb{N} \) and \( a \in (-\infty, 0) \), let \( V(x) = \pm (1 + |x|)^a \) for any \( x \in \mathbb{R}^n \). It is easy to show that \( ||V||_{L^t(\mathbb{R}^n)} \leq 1 \) for any given \( t \in (-n/a, \infty) \), where the implicit positive constant depends on \( a \) and \( t \). Thus, by taking the parameters in (5.1) and (5.3) with \( q = p = 2 \), \( s = 2m \), and \( t \in (n/(2m), \infty) \), we obtain

\[ S_2 = -2m \left( 1 - \frac{n}{2m} \frac{1}{t} \right) \leq 0. \]

This implies \( \sup_{|\lambda| > \epsilon_0} M_{|\lambda|}(V) = C_2 C |\lambda|^{S_2} ||V||_{L^t(\mathbb{R}^n)} < 1 \) for some \( \epsilon_0 \in (0, \infty) \) and hence the semigroup generated by \(- (P(D) + V)\) satisfies the local Davies–Gaffney estimate (5.12).

### 5.3 Gaussian estimates

In this subsection, we establish the Gaussian estimate for the heat kernel of \( \mathcal{L} \) by proving Theorem 1.1. We begin with the following lemma concerning the exponential perturbed operator \( \mathcal{L}_\eta f(x) := e^{-\eta x} \mathcal{L}(e^{\eta} f)(x) \) defined in (4.15) for any \( x \in \mathbb{R}^n \).

**Lemma 5.7.** Let \( p \in (1, \infty) \). If, for any \( \theta_0 \in (0, \pi/2) \), and \( \lambda \in \Sigma_{\theta_0}^C \), with \( \Sigma_{\theta_0}^C \) as in (4.1), and \( \eta \in \mathbb{C}^n \) satisfying \( |\eta| < \delta |\lambda|^{1/(2m)} \) with \( \delta \in (0, 1) \),

\[ \left\| (\lambda - \mathcal{L}_\eta)^{-1} \right\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} < \infty, \]

then \( \left\| (\lambda - \mathcal{L}_\eta)^{-1} \right\|_{L^{p'}(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)} < \infty \) also holds true for any such \( \lambda \) and \( \eta \).
Proof. For any $\lambda \in \Sigma^C_{\theta_0}$ and $\eta \in \mathbb{C}^n$ satisfying $|\eta| < \delta|\lambda|^{1/(2m)}$ with $\delta \in (0, 1)$, it easily follows, from the assumption that $V$ is real-valued, that

$$\left[(\lambda - \mathcal{L}_\eta)^{-1}\right]^* = (\overline{\lambda} - \overline{\mathcal{L}_\eta})^{-1}.$$ 

Since $\overline{\lambda} \in \Sigma^C_{\theta_0}$ and $-\overline{\eta} \in \mathbb{C}^n$ also satisfy $| - \overline{\eta}| < \delta|\overline{\lambda}|^{1/(2m)}$ with $\delta \in (0, 1)$, we deduce that

$$\left\| (\lambda - \mathcal{L}_\eta)^{-1} \right\|_{L^p'(\mathbb{R}^n) \to L^p'(\mathbb{R}^n)} = \left\| (\overline{\lambda} - \overline{\mathcal{L}_\eta})^{-1} \right\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} < \infty.$$

This finishes the proof of Lemma 5.7. \hfill $\Box$

The following Gagliardo–Nirenberg inequality and Sobolev embedding can be deduced from [51, p. 125, Theorem] (see also [64, (1.2)]) and [2, Theorem 4.12, PART II], respectively.

Lemma 5.8.

(i) Let $1 \leq p \leq \sigma \leq \infty$ and $m \in \mathbb{N}$ satisfy $0 \leq \frac{n}{2m} \left(\frac{1}{p} - \frac{1}{\sigma}\right) \leq 1$. Then there exists a positive constant $C$ such that, for any $f \in S(\mathbb{R}^n)$,

$$\|f\|_{L^\sigma(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}^{1-\theta} \|\Delta^m f\|_{L^p(\mathbb{R}^n)}^\theta$$

with $\theta := \frac{n}{2m} \left(\frac{1}{p} - \frac{1}{\sigma}\right)$.

(ii) Let $m \in \mathbb{N}$ and $q \in (1, \infty)$ satisfy $(2m-1)q < n < 2mq$. Then $W^{2m,q}(\mathbb{R}^n) \subset C^\gamma(\mathbb{R}^n)$ with $\gamma := 2m - n/q \in (0, 1)$, where $C^\gamma(\mathbb{R}^n)$ denotes the Lipschitz space of order $\gamma$ on $\mathbb{R}^n$ equipped with the norm

$$\|f\|_{C^\gamma(\mathbb{R}^n)} := \sup_{x, y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma}.$$

Based on the aforementioned lemmas, we now turn to the proof of Theorem 1.1

Proof of Theorem 1.1. We first prove (a). Suppose that (5.1) and one of (5.2) through (5.5) hold true for any $q \in (1, 2]$, and

$$\sup_{|\lambda| \in (0, \infty)} M_{|\lambda|}(V) < 1$$

with $M_{|\lambda|}(V)$ as in (5.6). In this case, applying Proposition 5.3 and Remark 5.4, we obtain

$$\left\| |\lambda| (\lambda - \mathcal{L}_\eta)^{-1} \right\|_{L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} + \left\| \Delta^m (\lambda - \mathcal{L}_\eta)^{-1} \right\|_{L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} \lesssim 1,$$

which, together with Lemma 5.7, implies that (5.14) holds true also for any given $q \in [2, \infty)$ and $\theta_0 \in (0, \pi/2)$, and for any $\lambda \in \Sigma^C_{\theta_0}$ and all $\eta \in \mathbb{C}^n$ satisfying $|\eta| < \delta|\lambda|^{1/(2m)}$. 

Now, take \( l \in \mathbb{Z}_+ \) and choose the numbers \( 2 = q_0 < q_1 < \cdots < q_{l-1} < q_l < q_{l+1} = \infty \) as follows. If \( n < 4m \), then take \( l = 0 \); if \( n \geq 4m \), we take \( l \) large enough such that \( 2(l + 1) > \frac{n}{2m} \) and

\[
\begin{aligned}
q_l &\in \left( \frac{n}{2m}, \frac{n}{2m - 1} \right), \\
\frac{n}{q_j} &\not\in \mathbb{N} \quad \text{for any } j \in \{1, \ldots, l\}, \\
\frac{1}{q_j} - \frac{1}{q_{j+1}} &< \frac{2m}{n} \quad \text{for any } j \in \{0, \ldots, l-1\}.
\end{aligned}
\]

(5.15)

Then, for any \( j \in \{0, \ldots, l\} \), let

\[
a_j := \frac{n}{2m} \left( \frac{1}{q_j} - \frac{1}{q_{j+1}} \right) \in (0, 1)
\]

with the usual convention made when \( q_{l+1} = \infty \). Then, by (5.14) and Lemma 5.8, we find that, for any \( f \in \mathcal{S}(\mathbb{R}^n) \),

\[
\left\| (\lambda - \mathcal{L}_\eta)^{-(l+1)} f \right\|_{L^\infty(\mathbb{R}^n)} \lesssim \left\| \Delta^m (\lambda - \mathcal{L}_\eta)^{-(l+1)} f \right\|_{L^q(\mathbb{R}^n)}^{a_j} \left\| (\lambda - \mathcal{L}_\eta)^{-(l+1)} f \right\|_{L^q(\mathbb{R}^n)}^{1-a_j}
\]

\[
\lesssim |\lambda|^{a_l-1} \left\| (\lambda - \mathcal{L}_\eta)^{-l} f \right\|_{L^q(\mathbb{R}^n)}.
\]

(5.16)

Repeating the above argument, we then obtain

\[
\left\| (\lambda - \mathcal{L}_\eta)^{-(l+1)} f \right\|_{L^\infty(\mathbb{R}^n)} \lesssim |\lambda| \sum_{j=0}^{l} (a_j-1) \left\| f \right\|_{L^2(\mathbb{R}^n)} \sim |\lambda|^{\frac{n}{2m} - (l+1)} \left\| f \right\|_{L^2(\mathbb{R}^n)},
\]

(5.17)

which, combined with a duality argument, shows that, for any \( g \in L^1(\mathbb{R}^n) \),

\[
\left\| (\lambda - \mathcal{L}_\eta)^{-(2l+1)} g \right\|_{L^\infty(\mathbb{R}^n)} \lesssim |\lambda|^{\frac{n}{2m} - 2(l+1)} \left\| g \right\|_{L^1(\mathbb{R}^n)}.
\]

(5.18)

By the well-known [22, p. 503, Theorem 6], we know that the operator \((\lambda - \mathcal{L}_\eta)^{-(2l+1)}\) has an integral kernel \(K_{2(l+1),\eta}(x,y)\) on \(\mathbb{R}^n \times \mathbb{R}^n\) that satisfies, for any \((x,y) \in \mathbb{R}^n \times \mathbb{R}^n\),

\[
|K_{2(l+1),\eta}(x,y)| \lesssim |\lambda|^{\frac{n}{2m} - 2(l+1)}.
\]

(5.19)

Moreover, using (4.29), we find that the operator \((\lambda - \mathcal{L})^{-(2l+1)}\) also has an integral kernel \(K_{2(l+1)}(x,y)\) on \(\mathbb{R}^n \times \mathbb{R}^n\) that satisfies, for any \((x,y) \in \mathbb{R}^n \times \mathbb{R}^n\),

\[
|K_{2(l+1)}(x,y)| \lesssim |\lambda|^{\frac{n}{2m} - 2(l+1)} e^{\langle x-y \rangle \eta}.
\]

Now, taking \( \eta := -\text{sgn}(x-y)\delta |\lambda|^{1/(2m)} \), we conclude that, for any \((x,y) \in \mathbb{R}^n \times \mathbb{R}^n\),

\[
|K_{2(l+1)}(x,y)| \lesssim |\lambda|^{\frac{n}{2m} - 2(l+1)} \exp \left\{ -\delta |x-y| |\lambda|^{1/(2m)} \right\}.
\]
By the formula
\[ e^{-t\mathcal{L}} = \frac{[2(l + 1) - 1]!}{2\pi i(-t)^{2(l+1)-1}} \int_{\Gamma} e^{-t\lambda} (\lambda - \mathcal{L})^{-2(l+1)} d\lambda \]
(see [3, (3.8)]), where \( \Gamma \) is the curve as in (5.11) (see also Figure 2), we know that \( e^{-t\mathcal{L}} \) has an integral kernel \( p_t \) on \((0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \) that satisfies, for any \( t \in (0, \infty) \) and \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n \),
\[ p_t(x, y) = \frac{[2(l + 1) - 1]!}{2\pi i(-t)^{2(l+1)-1}} \int_{\Gamma} e^{-t\lambda} K_{2(l+1)}(x, y) d\lambda. \] (5.20)

From this, we then deduce that
\[ |p_t(x, y)| \lesssim \frac{1}{t^{2(l+1)-1}} \int_{\Gamma} e^{-t \text{Re}\lambda |\lambda|^{\frac{n}{2m}} - 2(l+1)} \exp \left\{ -\delta |x - y| |\lambda|^{1/(2m)} \right\} |d\lambda| \]
\[ \sim \frac{1}{t^{2(l+1)-1}} \left( \int_{\Gamma_0} + \int_{\Gamma_+} + \int_{\Gamma_-} \right) |d\lambda| =: I_0 + I_+ + I_. \]

For \( I_+ \), by the assumptions that \( \theta_0 \in (0, \pi/2) \) and \( 2(l + 1) > \frac{n}{2m} \), we find that
\[ I_+ \lesssim \frac{1}{t^{2(l+1)-1}} \int_R e^{-tr \cos \theta_0 r \frac{n}{2m} - 2(l+1)} dr \exp \left\{ -\delta |x - y| R^{\frac{1}{2m}} \right\} \]
\[ \lesssim \frac{(Rt)^{n/(2m)-2(l+1)}}{t^{n/(2m)}} \int_R e^{-tr \cos \theta_0} d(t r) \exp \left\{ -\delta |x - y| R^{\frac{1}{2m}} \right\} \]
\[ \lesssim \frac{(Rt)^{n/(2m)-2(l+1)}}{t^{n/(2m)}} \exp \left\{ -\delta |x - y| R^{\frac{1}{2m}} \right\}. \]

Similarly, we also obtain
\[ I_- \lesssim \frac{(Rt)^{n/(2m)-2(l+1)}}{t^{n/(2m)}} \exp \left\{ -\delta |x - y| R^{\frac{1}{2m}} \right\}. \]

For \( I_0 \), we have
\[ I_0 \lesssim \frac{1}{t^{2(l+1)-1}} \int_{|\theta| \geq \theta_0} \exp \{-tR \cos \theta\} d\theta R^{\frac{n}{2m} - 2(l+1)+1} \exp \left\{ -\delta |x - y| R^{\frac{1}{2m}} \right\}. \]

Using the fact that \( \int_0^{2\pi} e^{-s \cos \theta} d\theta \lesssim \frac{e^s}{s^{3/2}} \) for any \( s \in (0, \infty) \) (see also [3, (2.30)]), we know that
\[ I_0 \lesssim \frac{1}{t^{n/(2m)}} (tR)^{n/(2m)-2(l+1)+1/2} \exp \left\{ tR - \delta |x - y| R^{\frac{1}{2m}} \right\}. \]

Now, taking \( R := \epsilon \frac{|x-y|^{2m/(2m-1)} |t|^{2m/(2m-1)}}{t^{1/(2m-1)}} \), we then have \( tR = \epsilon \frac{|x-y|^{2m/(2m-1)} |t|^{1/(2m-1)}}{t^{1/(2m-1)}} \) and hence
\[ \delta |x - y| R^{\frac{1}{2m}} = \delta |x - y| \epsilon^{\frac{1}{2m}} \frac{|x - y|^{1/(2m-1)}}{t^{1/(2m-1)}} = \delta \epsilon^{\frac{1}{2m}} \frac{|x - y|^{2m/(2m-1)}}{t^{1/(2m-1)}}. \]
Taking $\varepsilon \in (0, \infty)$ small enough such that $\delta \varepsilon^{1/(2m)} > \varepsilon$ and using the assumption $2(l + 1) > \frac{n}{2m}$, we conclude that

$$I_0 \lesssim \frac{1}{t^{n/(2m)}} \left[ |x - y|^{2m/(2m - 1)} \right]^{n/(2m) - 2(l + 1) + 1/2} \exp \left\{ - \left( \frac{\delta \varepsilon^{1/(2m)} - \varepsilon}{t^{1/(2m - 1)}} \right) \frac{|x - y|^{2m/(2m - 1)}}{t^{1/(2m - 1)}} \right\} \lesssim \frac{1}{t^{n/(2m)}} \exp \left( - c_3 \frac{|x - y|^{2m/(2m - 1)}}{t^{1/(2m - 1)}} \right)$$

by choosing a suitable constant $c_3 \in (0, \infty)$. Combining the estimates of $I_0$, $I_+$, and $I_-$, we find that (1.10) holds true in the case $q \in (1, 2]$. The proof of the case that $q \in [2, \infty)$ is similar; we omit the details. This finishes the proof of (a).

We are now in a position to prove (b). We use the same notation as in the proof of (a). Assume first that $n \geq 4m$. In this case, using Lemma 5.8(ii) and Proposition 4.2, we know that, for any $f \in L^2(\mathbb{R}^n)$ and $x, h \in \mathbb{R}^n$,

$$\left| (\lambda - L)^{-2(l+1)} f(x + h) - (\lambda - L)^{-2(l+1)} f(x) \right| \lesssim \| \Delta^n (\lambda - L)^{-2(l+1)} f \|_{L^q(\mathbb{R}^n)} |h|^\gamma$$

where $\gamma := 2m - \frac{n}{2m} \in (0, 1)$ by (5.15). Following the argument used in the proof of (5.18), we find that, for any $g \in L^1(\mathbb{R}^n)$ and $x, h \in \mathbb{R}^n$,

$$\left| (\lambda - L)^{-2(l+1)} g(x + h) - (\lambda - L)^{-2(l+1)} g(x) \right| \lesssim |\lambda|^{-a_l} |\lambda|^{\frac{n}{2m} - 2(l+1)} \| g \|_{L^1(\mathbb{R}^n)} |h|^\gamma,$$

which implies that, for any $x, y \in \mathbb{R}^n$ and any $\tilde{h}_n \neq h \in \mathbb{R}^n$,

$$\left| K_{2(l+1)}(x + h, y) - K_{2(l+1)}(x, y) \right| \lesssim |\lambda|^{-a_l} |\lambda|^{\frac{n}{2m} - 2(l+1)} \| h \|_{L^1(\mathbb{R}^n)} |h|^\gamma,$$  \hfill (5.21)

where $K_{2(l+1)}(x, y)$ denotes the integral kernel of $(\lambda - L)^{-2(l+1)}$. Thus, using (5.20), we conclude that, for any $x, y, h \in \mathbb{R}^n$ satisfying $0 < |h| < t^{1/2m}$,

$$| p_l(x + h, y) - p_l(x, y) | \lesssim \frac{1}{t^{2(l+1) - 1}} \int_{\Gamma} e^{-t\lambda} \left| K_{2(l+1)}(x + h, y) - K_{2(l+1)}(x, y) \right| d\lambda$$

with $\Gamma$ being the curve as in (5.11).

We now further assume that $|x - y| < t^{1/(2m)}$. In this case, for the integral over $\Gamma_+$ (the corresponding term is denoted by $\widetilde{I}_+$), by $2(l + 1) > n/(2m)$, $R = \varepsilon \frac{|x - y|^{2m/(2m - 1)}}{t^{n/(2m - 1)}}$, and $|x - y| < t^{1/(2m)}$, we obtain

$$\widetilde{I}_+ \lesssim \frac{|h|^\gamma}{t^{n/(2m)}} \int_R e^{-t\cos \theta_0 r^1 - a_l + \frac{n}{2m} - 2(l+1)} dr \lesssim \frac{|h|^\gamma}{t^{n/(2m)}} \left( R_t^n / (2m - 2(l+1)) \right) \frac{1}{t^{1-a_l}} \int_0^\infty e^{-t\cos \theta_0 (rt)^{1-a_l}} d(rt) \lesssim \frac{|h|^\gamma}{t^{1-a_l}} \frac{1}{t^{n/(2m)}}.$$
which implies (1.11) by taking $\gamma := 2m - \frac{n}{q_l}$ and using $1 - a_l = 1 - \frac{n}{2m q_l} = \frac{1}{2m} (2m - \frac{n}{q_l})$. The estimations for the integrals, respectively, over $\Gamma_-$ and $\Gamma_0$ are similar; we omit the details.

If $|x - y| \geq t^{1/2m}$, then, by (4.29), we first have, for any $h \in \mathbb{R}^n$ satisfying $|h| < t^{2m}$,

$$|K_{2(l+1)}(x + h, y) - K_{2(l+1)}(x, y)|$$

$$= \left| e^{(x+h)\eta} e^{-\eta y} K_{2(l+1),\eta}(x + h, y) - e^{\eta x} e^{-\eta y} K_{2(l+1),\eta}(x, y) \right|$$

$$\leq e^{(x-y)\eta} |K_{2(l+1),\eta}(x + h, y) - K_{2(l+1),\eta}(x, y)| + e^{h\eta} - 1 \left| e^{(x-y)\eta} K_{2(l+1),\eta}(x + h, y) \right|$$

$$= : H_h(x, y) + J_h(x, y).$$

From this and (5.20), we then deduce

$$|p_t(x + h, y) - p_t(x, y)| \lesssim \frac{1}{t^{2(l+1)-1}} \int_\Gamma |e^{-t\lambda}| [H_h(x, y) + J_h(x, y)] d\lambda | = : H + J$$

with $\Gamma$ as in (5.11).

To estimate $H$, similarly to (5.21), we obtain, for any $t \in (0, \infty)$, and any $x, y, h \in \mathbb{R}^n$ satisfying $|x - y| \geq t^{1/2m}$ and $|h| < t^{2m}$,

$$|K_{2(l+1),\eta}(x + h, y) - K_{2(l+1),\eta}(x, y)| \lesssim |\lambda|^{1-a_l}|\lambda|^\frac{n}{2m-2(l+1)}|h|^\gamma$$

with $\gamma := 2m - \frac{n}{q_l}$. Thus, by letting $\eta := -\text{sgn}(x - y) |\lambda|^{1/(2m)}$ and $R := \epsilon \frac{|x-y|^{2m/(2m-1)}}{t^{2m/(2m-1)}}$, we know that the integral of $H_h$ over $\Gamma_+$ satisfies

$$\frac{1}{t^{2(l+1)-1}} \int_{\Gamma_+} |e^{-t\lambda}| H_h(x, y) d\lambda |$$

$$\lesssim \frac{|h|^\gamma}{t^{2(l+1)-1}} \int_R^\infty \exp \left\{ -\delta |x - y|r^{1/(2m)} \right\} e^{-tr \cos \delta} r^{1-a_l + \frac{n}{2m} - 2(l+1)} dr$$

$$\lesssim \frac{|h|^\gamma}{t^{a_l} t^{n/(2m)}} \exp \left\{ - c_4 |x - y|^{2m/(2m-1)} \right\},$$

which implies (1.11) with $\gamma := 2m - \frac{n}{q_l}$. The estimations for the integrals of $H_h$, respectively, over $\Gamma_-$ and $\Gamma_0$ are similar; we omit the details.

On the other hand, for the integral of $J_h$ over $\Gamma_+$, using the Taylor theorem, (5.19), and the assumptions that $|x - y| \geq t^{1/2m}$ and $|h| < t^{1/(2m)}$, we know that there exists a positive constant $\theta \in (0, 1)$ such that

$$\frac{1}{t^{2(l+1)-1}} \int_{\Gamma_+} |e^{-t\lambda}| J_h(x, y) d\lambda |$$

$$\lesssim \frac{1}{t^{2(l+1)-1}} \int_R^\infty \exp \left\{ -\delta |x - y|r^{1/(2m)} \right\} e^{-tr \cos \delta_0 r^{\frac{n}{2m} - 2(l+1)} e^{\delta_1/(2m)} r^{1/(2m)} |h\eta|} dr$$

$$\lesssim \frac{|h|}{t^{2(l+1)-1}} \int_R^\infty \exp \left\{ -\delta (1 - \theta) |x - y|r^{1/(2m)} \right\} e^{-tr \cos \delta_0 r^{\frac{n+1}{2m} - 2(l+1)}} dr.$$
\[
\frac{|h|}{t^{1/(2m)}} \frac{1}{t^{n/(2m)}} \exp \left\{ -c_4 |x - y|^{2m/(2m-1)} t^{1/(2m-1)} \right\},
\]

which implies that (1.11) holds true. The estimations for the integrals of \( J_h \), respectively, over \( \Gamma_- \) and \( \Gamma_0 \) are similar; we omit the details.

If \( 2m \leq n < 4m \), then, by the argument used in (5.15), we have

\[
1 = q_2' < q_1' < q_0 = 2 < q_1 < q_2 = \infty.
\]

Without loss of generality, we may take \( q_1' \in (1, 2) \cap \left( \frac{n}{2m}, \frac{n}{2m-1} \right) \). Since \( (2m - 1)q_1' < n < 2mq_1' \), we deduce, from Lemma 5.8(ii) and Proposition 4.2, that, for any \( f \in L^1(\mathbb{R}^n) \) and \( x, h \in \mathbb{R}^n \),

\[
\left| (\lambda - \mathcal{L})^{-2} f(x + h) - (\lambda - \mathcal{L})^{-2} f(x) \right| \lesssim \left\| \Delta^{n/2} (\lambda - \mathcal{L})^{-2} f \right\|_{L^{q'_1}(\mathbb{R}^n)} |h|^\gamma \lesssim \left\| (\lambda - \mathcal{L})^{-1} f \right\|_{L^{q'_1}(\mathbb{R}^n)} |h|^\gamma,
\]

where \( \gamma := 2m - \frac{n}{q'_1} \in (0, 1) \). Moreover, using (5.16) and a dual argument, we find that

\[
\left\| (\lambda - \mathcal{L})^{-1} f \right\|_{L^{q'_1}(\mathbb{R}^n)} \lesssim |\lambda|^{-1} \| f \|_{L^1(\mathbb{R}^n)}
\]

with \( a_1 := \frac{n}{2mq_1'} \). This, together with \( \frac{\gamma}{2m} = 1 - \frac{n}{2mq_1'} \), implies that, for any \( f \in L^1(\mathbb{R}^n) \) and \( x, h \in \mathbb{R}^n \),

\[
\left| (\lambda - \mathcal{L})^{-2} f(x + h) - (\lambda - \mathcal{L})^{-2} f(x) \right| \lesssim |h|^\gamma |\lambda|^{\gamma/(2m)+n/(2m)-2} \| f \|_{L^1(\mathbb{R}^n)}
\]

and the integral kernel \( K_2 \) of \( (\lambda - \mathcal{L})^{-2} \) satisfies that, for any \( x, y, h \in \mathbb{R}^n \) satisfying \( \tilde{\theta}_n \neq h \in \mathbb{R}^n \),

\[
|K_2(x + h, y) - K_2(x, y)| \lesssim |\lambda|^{\gamma/(2m)+n/(2m)-2} |h|^\gamma.
\]

Since this estimation is similar to (5.21) [with \( 2(l + 1) \) therein replaced by \( 2 \)], the remainder of the proof is similar to the case \( n \geq 4m \); we omit the details. This shows that (b) holds true, and hence finishes the proof of Theorem 1.1.

Similarly to Corollary 5.5, we have the following local Gaussian estimate for the heat kernel of \( e^{-t \mathcal{L}} \).

**Corollary 5.9.** Let \( m \in \mathbb{N} \), \( \mathcal{L} = P(D) + V \) be the 2m-order Schrödinger operator on \( \mathbb{R}^n \) as in (2.3), and \( V \) a real-valued measurable function on \( \mathbb{R}^n \). If (5.1) and one of (5.2) through (5.5) hold true for any \( q \in (1, 2] \) or \( [2, \infty) \), and \( \sup_{|\lambda| \in (w/2, \infty)} M_{|\lambda|}(V) < 1 \) for some \( w \in (0, \infty) \) and \( M_{|\lambda|}(V) \) as in (5.6), then the operator \( \mathcal{L} \) possesses a heat kernel \( p_t \) on \( (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \) that satisfies the following assertions.
(i) There exist positive constants $C$ and $c_3$ such that, for any $t \in (0, \infty)$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$|p_t(x, y)| \leq \frac{C}{t^{n/(2m)}} \exp \left\{ wt - c_3 \frac{|x - y|^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\}. \tag{5.22}$$

(ii) If, in addition, $n \geq 2m$, then there exist a $\gamma \in (0, 1)$ and positive constants $C$ and $c_4$ such that, for any $t \in (0, \infty)$, $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, and $h \in \mathbb{R}^n$ satisfying $|h| < t^{1/2m}$,

$$|p_t(x + h, y) - p_t(x, y)| \leq \frac{C}{t^{n/(2m)}} \exp \left\{ wt - c_4 \frac{|x - y|^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\} \left[ \frac{|h|}{t^{1/(2m)}} \right]^{\gamma}. \tag{5.23}$$

Remark 5.10.

(i) For any given $m \in \mathbb{N}$ and $a \in (-2m, 0)$, let $V(x) = \pm |x|^a$ for any $x \in \mathbb{R}^n \setminus \{0\}$. By taking the parameters in (5.1) and (5.5) with $q = p = 2$, $s \in (0, 2m]$, $\alpha \in (0, n)$, and $\alpha \leq 2s$, as in Remark 5.6(i), if further assume $a \in (-\min\{2s, n\}/2, 0)$, then

$$M_{|\lambda|}(V) = C_2 C |\lambda|^{\frac{n}{2m}} M_{\alpha, 2, 1/|\lambda|}(V) \sim |\lambda|^{\frac{n}{2} - 2m - \left(\frac{\alpha}{2} + a\right)} \sim |\lambda|^{2m - a}$$

and $\sup_{|\lambda| > \epsilon_0} M_{|\lambda|}(V) < 1$ for some $\epsilon_0 > 0$ with the positive equivalence constants independent of $\lambda$.

On the other hand, in the case $q = 1$, by Remark 2.6, we know that $V$ is in the Kato class $K_{2m}(\mathbb{R}^n)$. Using [19, Proposition 2.2], we find that the operator $T_{2m, \delta}$ defined as in (3.1) is bounded on $L^1(\mathbb{R}^n)$, and

$$\lim_{\delta \to \infty} \|T_{2m, \delta}\|_{L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)} = 0.$$

Following the arguments used in the proof of Proposition 5.3 and using an interpolation of linear operators between $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, we conclude that, for any given $q \in (1, 2)$, (5.8) holds true when $|\lambda|$ is large enough. Thus, by Corollary 5.9, we know that the semigroup generated by $-(P(D) + V)$ has a heat kernel satisfying the local Gaussian estimates (5.22) and (5.23).

(ii) For any given $m \in \mathbb{N}$ and $a \in (-\infty, 0)$, let $V(x) = \pm (1 + |x|)^a$ for any $x \in \mathbb{R}^n$. It is easy to show that $\|V\|_{L^1(\mathbb{R}^n)} \leq 1$ for any given $t \in (-n/a, \infty)$ with the implicit positive constant depending only on $a$, $t$, and $n$. Thus, by taking the parameters in (5.1) and (5.5) with $q = p \in (1, \infty)$, $s := 2m$, and $t \in \left[ \frac{n}{2m}, \infty \right)$, we obtain

$$S_2 = -2m \left( 1 - \frac{n}{2mt} \right) \leq 0.$$

This indicates that the semigroup generated by $-(P(D) + V)$ has a heat kernel satisfying the local Gaussian estimates (5.22) and (5.23) when $n \geq 2m$.

Moreover, if $a \in (-\infty, -2m)$ and $V(x) = \pm c(1 + |x|)^a$ for any $x \in \mathbb{R}^n$ with $c \in (0, \infty)$, then $S_2 = 0$ by taking $t := n/(2m)$. Thus, by taking $c$ sufficiently small, we have

$$\sup_{|\lambda| \in (0, \infty)} M_{|\lambda|}(V) < 1.$$
This, combined with Theorem 1.1, implies that the semigroup generated by $-(P(D) + V)$ has a heat kernel satisfying the Gaussian estimates (1.10) and (1.11) when $n \geq 2m$.

**ACKNOWLEDGEMENTS**

The authors would like to thank Professor Jacek Dziubański for a motivating discussion on a related subject of this article which led us to consider the problem of this article. The authors would also like to thank Professor Xiaohua Yao for some helpful discussions on the subject of this article. Finally, the authors are very grateful to the anonymous referee for her/his very careful reading and so many valuable comments which essentially improve the quality of the article.

This project is supported by the National Natural Science Foundation of China (Grant Nos. 12071431, 11671031, 11971058, 12071197 and 11971431), the National Key Research and Development Program of China (Grant No. 2020YFA0712900) and the Zhejiang Provincial Natural Science Foundation of China (Grant Nos. LR22A010006 and LY22A010011).

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**REFERENCES**

1. M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, vol. 55, U.S. Government Printing Office, Washington D.C., 1964.
2. R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, 2nd ed., Pure Appl. Math., vol. 140, Elsevier/Academic Press, Amsterdam, 2003.
3. P. Auscher, A. McIntosh, and P. Tchamitchian, *Heat kernels of second order complex elliptic operators and applications*, J. Funct. Anal. **152** (1998), 22–73.
4. P. Auscher and M. Qafsaoui, *Equivalence between regularity theorems and heat kernel estimates for higher order elliptic operators and systems under divergence form*, J. Funct. Anal. **177** (2000), 310–364.
5. G. Barbatis, *Gaussian estimates with best constants for higher-order Schrödinger operators with Kato potentials*, Proc. Amer. Math. Soc. **145** (2017), 191–200.
6. G. Barbatis and B. Braniakas, *On the heat kernel of a class of fourth order operators in two dimensions: sharp Gaussian estimates and short time asymptotics*, J. Differential Equations **265** (2018), 5237–5261.
7. G. Barbatis and E. B. Davies, *Sharp bounds on heat kernels of higher order uniformly elliptic operators*, J. Operator Theory **36** (1996), 179–198.
8. G. Barbatis and F. Gazzola, *Higher order linear parabolic equations*, Contemp. Math. **594** (2013), 77–97.
9. M. T. Barlow, A. Grigor’yan, and T. Kumagai, *On the equivalence of parabolic Harnack inequalities and heat kernel estimates*, J. Math. Soc. Japan **64** (2012), 1091–1146.
10. A. G. Belyi and Y. A. Semenov, *On the $L^p$-theory of Schrödinger semigroups. II*, Siberian Math. Zh. **31** (1990), 16–26, 220; translation in Siberian Math. J. **31** (1990), 540–549.
11. K. Bogdan, J. Dziubański, and K. Szczypkowski, *Sharp Gaussian estimates for heat kernels of Schrödinger operators*, Integral Equations Operator Theory **91** (2019), 3.
12. S. Boutayeb, T. Coulhon, and A. Sikora, *A new approach to pointwise heat kernel upper bounds on doubling metric measure spaces*, Adv. Math. **270** (2015), 302–374.
13. J. Cao, D.-C. Chang, D. Yang, and S. Yang, *Boundedness of second order Riesz transforms associated to Schrödinger operators on Musielak-Orlicz-Hardy spaces*, Commun. Pure Appl. Anal. **13** (2014), 1435–1463.
14. E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts Math., vol. 92, Cambridge Univ. Press, Cambridge, 1990.
15. E. B. Davies, Uniformly elliptic operators with measurable coefficients, J. Funct. Anal. 132 (1995), 141–169.
16. E. B. Davies, $L^p$ spectral theory of higher-order elliptic differential operators, Bull. Lond. Math. Soc. 29 (1997), 513–546.
17. E. B. Davies and A. M. Hinz, Kato class potentials for higher order elliptic operators, J. Lond. Math. Soc. (2) 58 (1998), 669–678.
18. E. B. Davies and B. Simon, $L^p$ norms of noncritical Schrödinger semigroups, J. Funct. Anal. 102 (1991), 95–115.
19. Q. Deng, Y. Ding, and X. Yao, Gaussian bounds for higher-order elliptic differential operators with Kato type potentials, J. Funct. Anal. 266 (2014), 5377–5397.
20. Q. Deng, Y. Ding, and X. Yao, Riesz transforms associated with higher-order Schrödinger type operators, Potential Anal. 49 (2018), 381–410.
21. B. Devyver, Heat kernel and Riesz transform of Schrödinger operators, Ann. Inst. Fourier (Grenoble) 69 (2019), 457–513.
22. N. Dunford and J. T. Schwartz, Linear operators. Part I. General theory, Reprint of the 1958 original, Wiley Classics Library, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, N.Y., 1988.
23. X. T. Duong and D. W. Robinson, Semigroup kernels, Poisson bounds, and holomorphic functional calculus, J. Funct. Anal. 142 (1996), 89–128.
24. X. T. Duong, L. Yan, and C. Zhang, On characterization of Poisson integrals of Schrödinger operators with BMO traces, J. Funct. Anal. 266 (2014), 2053–2085.
25. J. Dziubański and J. Zienkiewicz, Hardy space $H^1$ associated to Schrödinger operator with potential satisfying reverse Hölder inequality, Rev. Mat. Iberoam. 15 (1999), 279–296.
26. J. Dziubański and J. Zienkiewicz, Hardy spaces $H^1$ for Schrödinger operators with certain potentials, Studia Math. 164 (2004), 39–53.
27. K.-J. Engel and R. Nagel, One-parameter semigroups for linear evolution equations, Grad. Texts Math., vol. 194, Springer-Verlag, New York, N.Y., 2000.
28. H. Feng, A. Soffer, Z. Wu, and X. Yao, Decay estimates for higher-order elliptic operators, Trans. Amer. Math. Soc. 373 (2020), 2805–2859.
29. A. Ferrero, F. Gazzola, and H.-C. Grunau, Decay and eventual local positivity for biharmonic parabolic equations, Discrete Contin. Dyn. Syst. 21 (2008), 1129–1157.
30. L. Grafakos, Classical Fourier analysis, 3rd ed., Grad. Texts Math., vol. 249, Springer, New York, N.Y., 2014.
31. A. Grigor’yan, Heat kernel and analysis on manifolds, AMS/IP Stud. Adv. Math., vol. 47, Amer. Math. Soc., Providence, R.I.; International Press, Boston, M.A., 2009.
32. A. Grigor’yan, J. Hu, and K.-S. Lau, Heat kernels on metric measure spaces, Geometry and analysis of fractals, Springer Proc. Math. Stat., 88, Springer, Heidelberg, 2014, pp. 147–207.
33. M. Haase, The functional calculus for sectorial operators, Operator Theory: Adv. Appl., vol. 169, Birkhäuser Verlag, Basel, 2006.
34. H. Hajłasz, L. Molinett, T. Ozawa, and B. Wang, Necessary and sufficient conditions for the fractional Gagliardo–Nirenberg inequalities and applications to Navier-Stokes and generalized boson equations, Harmonic analysis and nonlinear partial differential equations, RIMS Kôkyûroku Bessatsu, B26, Res. Inst. Math. Sci. (RIMS), Kyoto, 2011, pp. 159–175.
35. M. Hieber, Gaussian estimates and holomorphy of semigroups on $L^p$ spaces, J. Lond. Math. Soc. (2) 54 (1996), 148–160.
36. M. Hieber and H. Prüss, Heat kernels and maximal $L^p$–$L^q$ estimates for parabolic evolution equations, Comm. Partial Differential Equations 22 (1997), 1647–1669.
37. S. Hofmann, G. Lu, D. Mitrea, M. Mitrea, and L. Yan, Hardy spaces associated to non-negative self-adjoint operators satisfying Davies–Gaffney estimates, Mem. Amer. Math. Soc. 214 (2011), no. 1007, vi+78 pp.
38. J. Huang, P. Li, and Y. Liu, Regularity properties of the heat kernel and area integral characterization of Hardy space $H^1$ related to degenerate Schrödinger operators, J. Math. Anal. Appl. 466 (2018), 447–470.
39. S. Huang, M. Wang, Q. Zheng, and Z. Duan, $L^p$ estimates for fractional Schrödinger operators with Kato class potentials, J. Differential Equations 265 (2018), 4181–4212.
40. T. Hytönen, J. van Neerven, M. Veraar, and L. Weis, Analysis in Banach spaces. vol. I, Martingales and Littlewood-Paley theory, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas, 3rd Series, A Series of Modern Surveys in Mathematics], vol. 63, Springer, Cham, 2016.
41. K. Ishige, Y. Kabeya, and E. M. Ouhabaz, *The heat kernel of a Schrödinger operator with inverse square potential*, Proc. Lond. Math. Soc. (3) **115** (2017), 381–410.

42. R. Jiang and F. Lin, *Riesz transform under perturbations via heat kernel regularity*, J. Math. Pures Appl. (9) **133** (2020), 39–65.

43. R. Jiang, D. Yang, and Y. Zhou, *Localized Hardy spaces associated with operators*, Appl. Anal. **88** (2009), 1409–1427.

44. T. Kato, *Schrödinger operators with singular potentials*, Israel J. Math. **13** (1972), 135–148 (1973).

45. T. Kato, *Perturbation theory for linear operators*, Reprint of the 1980 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

46. P. C. Kunstmann, *Heat kernel estimates and \( L^p \) spectral independence of elliptic operators*, Bull. Lond. Math. Soc. **31** (1999), 345–353.

47. P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 153–201.

48. V. Liskevich and Y. Semenov, *Two-sided estimates of the heat kernel of the Schrödinger operator*, Bull. Lond. Math. Soc. **30** (1998), 596–602.

49. Y. Liu and J. Dong, *Some estimates of higher order Riesz transform related to Schrödinger type operators*, Potential Anal. **32** (2010), 41–55.

50. L. Moschini and A. Tesei, *Harnack inequality and heat kernel estimates for the Schrödinger operator with Hardy potential*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **16** (2005), 171–180.

51. L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **13** (1959), 115–162.

52. E. M. Ouhabaz, *Analysis of heat equations on domains*, Lond. Math. Soc. Monographs Series, vol. 31, Princeton Univ. Press, Princeton, N.J., 2005.

53. M. M. H. Pang, Resolvent estimates for Schrödinger operators in \( L^p(\mathbb{R}^N) \) and the theory of exponentially bounded \( C \)-semigroups, Semigroup Forum **41** (1990), 97–114.

54. M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York–London, 1975.

55. M. Schechter, *Spectra of partial differential operators*, 2nd ed., North-Holland Ser. Appl. Math. Mech., vol. 14, North-Holland Publishing Co., Amsterdam, 1986.

56. Z. Shen, *\( L^p \) estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble) **45** (1995), 513–546.

57. Z. Shen, *The periodic Schrödinger operators with potentials in the Morrey class*, J. Funct. Anal. **193** (2002), 314–345.

58. B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 447–526.

59. B. Simon, *Schrödinger operators in the twentieth century*, J. Math. Phys. **41** (2000), 3523–3555.

60. L. Song and L. Yan, *Riesz transforms associated to Schrödinger operators on weighted Hardy spaces*, J. Funct. Anal. **259** (2010), 1466–1490.

61. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math. Series, vol. 30, Princeton Univ. Press, Princeton, N.J., 1970.

62. F. Stummel, *Singular\-äre elliptische differential-operatoren in Hilbertschen Räumen*, Math. Ann. **132** (1956), 150–176.

63. H. Tanabe, *Functional analytic methods for partial differential equations*, Monographs Textbooks Pure Appl. Math., vol. 204, Marcel Dekker, Inc., New York, N.Y., 1997.

64. H. Triebel, *Gagliardo–Nirenberg inequalities*, Reprint of Tr. Mat. Inst. Steklova 284 (2014), 271–287, Proc. Steklov Inst. Math. **284** (2014), 263-279.

65. L. Wu and L. Yan, *Heat kernels, upper bounds and Hardy spaces associated to the generalized Schrödinger operators*, J. Funct. Anal. **270** (2016), 3709–3749.

66. D. Yang and Y. Zhou, *Localized Hardy spaces \( H^1 \) related to admissible functions on RD-spaces and applications to Schrödinger operators*, Trans. Amer. Math. Soc. **363** (2011), 1197–1239.

67. Q. S. Zhang, *A sharp comparison result concerning Schrödinger heat kernels*, Bull. Lond. Math. Soc. **35** (2003), 461–472.

68. Q. Zheng and X. Yao, *Higher-order Kato class potentials for Schrödinger operators*, Bull. Lond. Math. Soc. **41** (2009), 293–301.