An Asymptotic Form of the Generating Function
\[ \prod_{k=1}^{\infty} \left(1 + \frac{x^k}{k}\right) \]

Andreas B. G. Blobel
andreas.blobel@kabelmail.de
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Abstract

It is shown that the sequence of rational numbers \( r(k) \) generated by the ordinary generating function \( \prod_{k=1}^{\infty} \left(1 + \frac{x^k}{k}\right) \) converges to a limit \( C > 0 \). \( C \) can be expressed as

\[ C = \exp \left( - \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \zeta(k) \right) \]

where \( \zeta() \) denotes the Riemann zeta function.

The ordinary generating function (OGF)

\[ R(x) := \prod_{k=1}^{\infty} \left(1 + \frac{x^k}{k}\right) = \sum_{k=0}^{\infty} r(k) x^k \quad (1a) \]

is closely related to the well known OGF

\[ Q(x) := \prod_{k=1}^{\infty} \left(1 + x^k\right) = \sum_{k=0}^{\infty} q(k) x^k \quad (1b) \]

\( Q(x) \) generates the sequence of counters for the number of integer partitions with distinct parts \([\text{Wil}]. \ q(k) \) is equal to the number of partitions of \( k \) into distinct parts for each \( k \geq 0 \) \([\text{Int}]. \)

A partition with distinct parts of integer \( k \) can be regarded as a finite set \( S \) of (distinct) positive integers \( i \geq 1 \) whose sum equals \( k \). Let \( \mathcal{P}(k) \) denote the set of all such partitions of \( k \) and let \( S \in \mathcal{P}(k) \). We then have

\[ \sum_{i \in S} i = k \quad (2) \]
With each partition \( S \in \mathcal{P}(k) \) we can associate the inverse of the product of its (distinct) elements
\[
ip(S) := \frac{1}{\prod_{i \in S} i} \quad (3)
\]
With this in mind \( r(k) \) can be written as
\[
r(k) = \sum_{S \in \mathcal{P}(k)} \nip(S) = \sum_{S \in \mathcal{P}(k)} \frac{1}{\prod_{i \in S} i} \tag{4a} \quad : \quad r = 0
\]
\[
r(k) = \sum_{S \in \mathcal{P}(k)} \frac{1}{\prod_{i \in S} i} \tag{4b} \quad : \quad r \geq 1
\]
In other words, \( r(k) \) is equal to the sum over all partitions \( S \in \mathcal{P}(k) \) of the reciprocal of the product of the elements of \( S \).

How does the sequence \( r(k) \) given in (4a) and (4b) behave? Does it converge to some limit \( C > 0 \)? Taking the logarithm of (1a), applying the Mercator series expansion \([\text{Wol}]\), and summing up columns first gives
\[
\ln R(x) = \sum_{k \geq 1} \ln \left( 1 + \frac{x^k}{k} \right) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \cdots
\]
\[
+ \frac{x^2}{2} - \frac{1}{2} \left[ \frac{x^2}{2} \right]^2 + \frac{1}{3} \left[ \frac{x^2}{2} \right]^3 - \cdots
\]
\[
+ \frac{x^3}{3} - \frac{1}{2} \left[ \frac{x^3}{3} \right]^2 + \frac{1}{3} \left[ \frac{x^3}{3} \right]^3 - \cdots
\]
\[
\vdots
\]
\[
= \text{Li}_1(x) - \frac{1}{2} \text{Li}_2(x^2) + \frac{1}{3} \text{Li}_3(x^3) - \cdots \tag{5}
\]
Here \( \text{Li}_s(x) \) denotes the so-called polylogarithm \([\text{Wikc}]\), a Dirichlet type series \([\text{Wika}]\).

We are looking for an asymptotic relation of the form
\[
R(x) \xrightarrow{x \to 1^-} \frac{C}{1 - x} \tag{6}
\]
for some constant \( C > 0 \). This is equivalent to the existence of the limit
\[
C = \lim_{x \to 1^-} (1 - x)R(x) \tag{7}
\]
Taking the logarithm of (7) gives
\[
\ln C = \lim_{x \to 1^-} \left( \ln (1 - x) + \ln R(x) \right) \tag{8}
\]
If we insert (5), observe the identity
\[
\text{Li}_1(x) = -\ln(1 - x)
\] (9)
and finally set \( x = 1 \), we arrive at the condition
\[
\ln C = -\frac{1}{2} \text{Li}_2(1) + \frac{1}{3} \text{Li}_3(1) - \frac{1}{4} \text{Li}_4(1) + \cdots
\]
\[
= -\frac{1}{2} \zeta(2) + \frac{1}{3} \zeta(3) - \frac{1}{4} \zeta(4) + \cdots
\] (10)
where \( \zeta(s) \) denotes the Riemann Zeta function. We therefore have
\[
C = \exp \left( -\sum_{k=2}^{\infty} \frac{(-1)^k}{k} \zeta(k) \right)
\] (11)
We observe that \( \zeta(k) \) converges rapidly towards 1:

| \( k \) | \( \zeta(k) - 1 \) |
|---|---|
| 2 | \( \frac{\pi^2}{6} - 1 \) | 0.644934 |
| 3 | - | 0.202057 |
| 4 | \( \frac{\pi^4}{90} - 1 \) | 0.082323 |
| 5 | - | 0.036928 |
| 6 | \( \frac{\pi^6}{945} - 1 \) | 0.017343 |
| 7 | - | 0.008349 |
| 8 | \( \frac{\pi^8}{9450} - 1 \) | 0.004077 |
| 9 | - | 0.002008 |
| 10 | \( \frac{\pi^{10}}{93555} - 1 \) | 0.000995 |
| 11 | - | 0.000494 |

\( \zeta(k) \xrightarrow{k \to \infty} 1 \)

This motivates the decomposition of (10)
\[
\ln C = -\frac{1}{2} \zeta(2) + \frac{1}{3} \zeta(3) - \frac{1}{4} \zeta(4) + \cdots
\]
\[
= -\frac{1}{2} \left[ \zeta(2) - 1 \right] + \frac{1}{3} \left[ \zeta(3) - 1 \right] - \frac{1}{4} \left[ \zeta(4) - 1 \right] + \cdots
\]
\[
- \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots
\]
\[
= -\Delta + \ln 2 - 1
\] (12)
where $\Delta$ is defined as

$$\Delta := + \frac{1}{2} \left[ \zeta(2) - 1 \right] - \frac{1}{3} \left[ \zeta(3) - 1 \right] + \frac{1}{4} \left[ \zeta(4) - 1 \right] - \cdots$$  \hspace{1cm} (13)

We therefore have from (12)

$$C = \frac{2}{e^{1+\Delta}}$$  \hspace{1cm} (14)

From (13) we derive the sequence of corrections $\Delta_m$ as follows

$$\Delta_m = \begin{cases} 
0 & : m = 1 \\
\sum_{k=2}^{m} \frac{(-1)^k}{k} (\zeta(k) - 1) & : m \geq 2 
\end{cases}$$  \hspace{1cm} (15)

This creates the sequence

$$C_m = \frac{2}{\exp(1 + \Delta_m)} : m \geq 1$$  \hspace{1cm} (16)

of approximations of $C$ whose first elements are listed in Table 1

| $m$ | $\Delta_m$ | $\frac{2}{\exp(1+\Delta_m)}$ |
|-----|-------------|--------------------------------|
| 1   | 0.0         | 0.7357589                      |
| 2   | 0.3224670   | 0.5329542                      |
| 3   | 0.2551147   | 0.5700863                      |
| 4   | 0.2756955   | 0.5584734                      |
| 5   | 0.2683100   | 0.5626133                      |
| 6   | 0.2712005   | 0.5609894                      |
| 7   | 0.2700078   | 0.5616589                      |
| 8   | 0.2705174   | 0.5613727                      |
| 9   | 0.2702943   | 0.5614980                      |
| 10  | 0.2703937   | 0.5614421                      |
| 11  | 0.2703488   | 0.5614674                      |
| 12  | 0.2703693   | 0.5614559                      |
| 13  | 0.2703599   | 0.5614612                      |

Table 1: Approximation of $C$
Useful recurrence relations for computation

For $n > 0$ we define the finite products

\[
R_n(x) := \prod_{k=1}^{n} \left(1 + \frac{x^k}{k}\right) = \sum_{k=0}^{\infty} r_n(k) \ x^k
\]

(17a)

\[
Q_n(x) := \prod_{k=1}^{n} (1 + x^k) = \sum_{k=0}^{\infty} q_n(k) \ x^k
\]

(17b)

The integer numbers $q_n(k)$ in (17b) count the number of partitions of $k$ with distinct parts where no part exceeds $n$. The coefficients $q_n(k)$ clearly have 3 basic properties:

\[
q_n(k) = q(k) \quad \text{if} \quad k \leq n
\]

(18a)

\[
q_n(k) = 0 \quad \text{if} \quad k > \frac{n(n+1)}{2}
\]

(18b)

\[
\sum_{k \geq 0} q_n(k) = 2^n
\]

(18c)

where (18c) follows from evaluation of $Q_n(1)$. The $q_n(k)$ obey the recurrence relations

\[
q_0(k) = \begin{cases} 1 & : k = 0 \\ 0 & : k \geq 1 \end{cases}
\]

(19a)

\[
q_n(k) = q_{n-1}(k) \quad : 0 \leq k < n
\]

(19b)

\[
q_n(k) = q_{n-1}(k-n) + q_{n-2}(k-n+1) + q_{n-3}(k-n+2) + \cdots + q_1(k-2) + q_0(k-1) \quad : k \geq n > 0
\]

(19c)

Initial values are prescribed in row $n = 0$ (19a). The values in any subsequent row $n \geq 1$ are determined by values in previous rows $m < n$. 

| 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 5 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 2 | 1 | 1 | 1 | 1 | 0 |

Table 2: Upper left section of the $q_n(k)$ field $[0 \leq n \leq 5, \ 0 \leq k \leq 16]$
Analogous properties and relations hold for the rational numbers \( r_n(k) \) in (17a):

\[
\begin{align*}
    r_n(k) &= r(k) & \text{if } k \leq n & \quad (20a) \\
    r_n(k) &= 0 & \text{if } k > \frac{n(n+1)}{2} & \quad (20b) \\
    \sum_{k \geq 0} r_n(k) &= n + 1 & \quad (20c)
\end{align*}
\]

\[
\begin{align*}
    r_0(k) &= \begin{cases} 
        1 & : k = 0 \\
        0 & : k \geq 1
    \end{cases} & \quad (21a) \\
    r_n(k) &= r_{n-1}(k) : 0 \leq k < n & \quad (21b) \\
    r_n(k) &= \frac{1}{n} r_{n-1}(k-n) + \frac{1}{n-1} r_{n-2}(k-n+1) + \frac{1}{n-2} r_{n-3}(k-n+2) + \cdots \\
    & \quad + \frac{1}{n} r_1(k-2) + \frac{1}{n} r_0(k-1) : k \geq n > 0 & \quad (21c)
\end{align*}
\]

|     | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 11  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1   | 11  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2   | 1   | 1  | \frac{1}{2} | \frac{1}{2} | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3   | 1   | 1  | \frac{1}{2} | \frac{5}{6} | \frac{1}{3} | \frac{1}{6} | \frac{1}{6} | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 4   | 1   | 1  | \frac{1}{2} | \frac{5}{6} | \frac{7}{12} | \frac{1}{2} | \frac{5}{24} | \frac{1}{24} | \frac{1}{24} | \frac{1}{2} | 0   | 0   | 0   | 0   | 0   | 0   |
| 5   | 1   | 1  | \frac{1}{2} | \frac{5}{6} | \frac{7}{12} | \frac{37}{60} | \frac{37}{120} | \frac{1}{4} | \frac{19}{120} | \frac{1}{8} | \frac{7}{120} | \frac{1}{24} | \frac{1}{60} | \frac{1}{120} | \frac{1}{120} | 0   |

Table 3: Upper left section of the \( r_n(k) \) field \([ 0 \leq n \leq 5 , \ 0 \leq k \leq 16 ]\)

Figure [1] assembles some instances of \( r(k) \) which have been computed on the R platform for statistical computing \([\text{RPr}]\) using recurrence relations \((21a), (21b), \) and \((21c)\). The plot shows that the \( r(k) \) approach the asymptotic value

\[
C = 0.56146 \ldots
\]

from above as \( k \) increases. The constant \( C \) is determined by \((11)\) and \((14)\) and is marked by a dashed horizontal line.
Figure 1: Some computed instances of $r(k)$
Conclusion

It has been shown that the function

\[ f(x) = \frac{C}{1 - x} \]

is an asymptotic form of the generating function \( (1a) \) in the sense that the sequence of rational numbers \( r(k) \) generated by \( (1a) \) converges towards \( C > 0 \) which is determined by \( (11) \) and \( (14) \).
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