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Adiabatic limits of Seifert fibrations, Dedekind sums, and the diffeomorphism type of certain 7-manifolds

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Abstract. We extend the adiabatic limit formula for $\eta$-invariants by Bismut–Cheeger and Dai to Seifert fibrations. Our formula contains a new contribution from the singular fibres that takes the form of a generalised Dedekind sum.

As an application, we compute the Eells–Kuiper and $t$-invariants of certain cohomogeneity one manifolds that were studied by Dearricott, Grove, Verdiani, Wilking, and Ziller. In particular, we determine the diffeomorphism type of a new manifold of positive sectional curvature.

Keywords. Adiabatic limit, Seifert fibration, eta-invariant, cohomogeneity one manifold, Eells–Kuiper invariant, $t$-invariant

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Manifolds of positive sectional curvature are a rare phenomenon, and the differential topological conditions for the existence of positive sectional curvature metrics are not yet fully understood. For this reason, one is still interested in finding new examples of positive sectional curvature metrics. Most known examples are quotients or biquotients of compact Lie groups. Cohomogeneity one manifolds constitute another potential source of examples. By work of Grove, Wilking and Ziller [17], there are only two families $(P_k)$, $(Q_k)$ of seven-dimensional manifolds, which possibly allow cohomogeneity one metrics of positive sectional curvature and contain new examples. The space $R$ mentioned there does not admit a positive sectional curvature metric by [27]. Dearricott [10] and Grove, Verdiani, and Ziller [16] have succeeded in constructing a positive sectional curvature metric on $P_2$, the first nontrivial member of the family $(P_k)$. This manifold is homeomorphic to the unit tangent bundle $T^1S^4$ of the four-dimensional sphere. In this paper, we will specify among other things an exotic sphere $\Sigma$ such that $P_2$ is diffeomorphic to the connected sum of $T^1S^4$ and $\Sigma$.

The manifolds $P_k$ are highly connected with a finite cyclic cohomology group $H^4(P_k) \cong \pi_3(P_k) \cong \mathbb{Z}/k\mathbb{Z}$. By Crowley’s work [6], it suffices to compute the Eells–Kuiper invariant $\mu(P_k)$ and a certain quadratic form $q$ on $H^4(P_k)$ to determine their diffeomorphism types. These two invariants are classically defined using an oriented spin manifold $N$ bounding $P_k$, but it is not clear how to construct such a manifold $N$ directly. On the other hand, by results of Donnelly [11], Kreck and Stolz [21], and Crowley and the

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author [8], both invariants can equivalently be expressed as linear combinations of \( \eta \)-invariants of certain Dirac operators and Cheeger-Chern-Simons correction terms on \( P_k \) itself. Having computed these invariants, one can write the spaces \( P_k \) as connected sums of exotic spheres and \( S^3 \)-bundles over \( S^4 \) using the computations for these bundles in [7].

In order to determine the necessary \( \eta \)-invariants, we write the spaces \( P_k \) as Seifert fibrations as indicated in [17]. That is, the spaces \( P_k \) are fibered by compact manifolds over some base orbifold \( B \).

The process of blowing up the base space of a fibration \( M \to B \) by a factor \( \varepsilon^{-1} \) is called the adiabatic limit. It has been shown by Bismut and Cheeger [3] and Dai [9] that the \( \eta \)-invariants of a family of compatible Dirac operators \( D_{M,\varepsilon} \) converge in the adiabatic limit \( \varepsilon \to 0 \), if the kernels of the associated vertical Dirac operators \( D_B \) form a vector bundle \( H \to B \). This result can be generalised to Seifert fibrations \( M \to B \). Thus, we consider adiabatic families of Dirac operators \((D_{M,\varepsilon})_\varepsilon\) as in Definition 1.6. In particular, we assume that \( H = \ker(D_X) \) is a vector orbibundle on \( B \). Let \( \Lambda B \) be the inertia orbifold of \( B \) and let \( \hat{A}_{\Lambda B}(TB, \nabla^T B) \in \Omega^*(\Lambda B; \Lambda B \otimes o(\Lambda B)) \) denote the orbifold \( \hat{A} \)-form as in Kawasaki’s index theorem [20] (see Section 1.b). Let \( \hat{A} \) denote Bismut’s Levi-Civita superconnection associated with \( D_{M,\varepsilon} \). In Definition 1.7, we construct orbifold \( \eta \)-forms \( \eta_{\Lambda B}(\hat{A}) \in \Omega^*(\Lambda B; \Lambda B) \) as in [3] and [13]. In Definition 1.8, we define the effective horizontal operator \( D_{\varepsilon}^{\text{eff}} \) of the family \( D_{M,\varepsilon} \), which acts on sections of \( H \to B \).

Let \((\lambda_\varepsilon(\varepsilon))_\varepsilon\) denote the finite family of very small eigenvalues of \( D_{M,\varepsilon} \) (see Section 1.c). In [25, Theorem 8.6], Rochon proved a special case of the following theorem where \( B \) is a very good orbifold and the fibrewise operator is invertible.

**Theorem 0.1** (cf. [3], [9], [25]). Let \( p : M \to B \) be a Seifert fibration and \((D_{M,\varepsilon})_\varepsilon\) an adiabatic family of Dirac operators over \( M \) as in Definition 1.6. For \( \varepsilon_0 > 0 \) sufficiently small, we have

\[
\lim_{\varepsilon \to 0} \eta(D_{M,\varepsilon}) = \int_{\Lambda B} \hat{A}_{\Lambda B}(TB, \nabla^T B) 2\eta_{\Lambda B}(\hat{A}) + \eta(D_{\varepsilon}^{\text{eff}}) + \sum \text{sign}(\lambda_\varepsilon(\varepsilon_0)).
\]

With this result, one can compute the Eells–Kuiper invariant and the quadratic form \( q \) and hence determine the diffeomorphism type of each space \( P_k \).

**Theorem 0.2.** The Eells–Kuiper invariant of \( P_k \) is given by

\[
\mu(P_k) = -\frac{4k^3 - 7k + 3}{2^5 \cdot 3 \cdot 7} \in \Q / \Z.
\]

The quadratic form \( q \) on \( H^4(P_k) \cong \Z / k\Z \) is given by

\[
q(\ell) = \frac{\ell(\ell - k)}{2k} \in \Q / \Z.
\]

By comparing these values with the corresponding values for \( S^3 \)-bundles over \( S^4 \) in [7] and [8], one can construct manifolds that are diffeomorphic to \( P_k \).
Theorem 0.3. Let $E_{k,k} \to S^4$ denote the principal $S^3$-bundle with Euler class $k \in H^4(S^4) \cong \mathbb{Z}$, and let $\Sigma_7$ denote the exotic seven sphere with $\mu(\Sigma_7) = 1/28$. Then there exists an orientation preserving diffeomorphism

$$P_k \cong E_{k,k} \# \Sigma_7^{\#(k-k^3)/6}.$$ 

In particular, $P_k$ and $E_{k,k}$ are homeomorphic.

More generally, let $E_{p,n}$ denote the unit sphere bundle of a four-dimensional real spin vector bundle over $S^4$ with Euler class $n$ and half Pontryagin class $p \in H^4(S^4) \cong \mathbb{Z}$.

Corollary 0.4. For the space $P_k$, there exists an $S^3$-bundle $E_{ak,k} \to S^4$ that is

1. oriented diffeomorphic to $P_k$ if and only if $k$ is odd or $8 \mid k$, with

$$a^2 k \equiv \frac{7k - 4k^3}{3} \mod 224\mathbb{Z};$$

2. orientation reversing diffeomorphic to $P_k$ if and only if

   (a) $k$ is not divisible by 7,
   
   (b) $k \equiv 1 \mod 4$, or $k \equiv 2, 10 \mod 32$, and
   
   (c) $-1$ is a quadratic remainder $\mod k$,

   with

$$a^2 k \equiv 2 - \frac{7k - 4k^3}{3} \mod 224\mathbb{Z}.$$  

Some of the $P_k$ are discussed in greater detail in Example 3.12.

The article is organised as follows. In Section 1, we introduce Seifert fibrations and define all the ingredients of Theorem 0.1. Its proof is given in Section 2. In Section 3, we introduce the family $(P_k)$ as a subfamily of the larger family $(M(p,q_{-1}), (p,q_{+1}))$ that was also considered in [17]. The quadratic forms $q_{M(p,q_{-1}), (p,q_{+1})}$ for some of those manifolds are given in Theorem 3.3, and their Eells–Kuiper invariants in Theorem 3.7. For the spaces $P_k$, we obtain the simplified formulas of Theorem 0.2 and prove Theorem 0.3 and Corollary 0.4. Finally, Section 4 contains the computations of $\eta$-invariants needed to prove Theorems 3.3 and 3.7.

1. An adiabatic limit theorem for $\eta$-invariants of Seifert fibrations

A Seifert fibration is a map from a smooth manifold to an orbifold that becomes a proper fibration over the smooth covering of each orbifold chart. Each Seifert fibration is thus a Riemannian foliation with compact leaves. The leaves over singular points of the orbifolds are quotients of the generic leaf over a regular point.

We extend the adiabatic limit theorem of Bismut–Cheeger and Dai to Seifert fibrations. We have to take care of additional terms arising at the singular locus of the orbifold. In some special cases, these extra terms give rise to Dedekind sums.
1.a. Orbifolds, orbibundles and Seifert fibrations

We recall the definition of an orbifold. By Remark 1.3 below, we may assume that the base orbifold \( B \) is effective. This will be assumed for the rest of the paper and makes some constructions a lot easier.

**Definition 1.1.** Let \( G \) be a compact Lie group together with an action on \( \mathbb{R}^n \). An \( n \)-dimensional smooth \( G \)-orbifold is a second countable Hausdorff space \( B \) with the following additional structure.

1. For each point \( b \in B \) there exists a neighbourhood \( U \subset B \) of \( b \), an open subset \( V \subset \mathbb{R}^n \) invariant under the action of a finite group \( \Gamma \) via \( \rho : \Gamma \leftarrow G \rightarrow \text{GL}(n, \mathbb{R}) \), and a homeomorphism \( \psi : \rho(0) \setminus V \rightarrow U \) with \( \psi(0) = b \).

We call \( \psi \) an orbifold chart, and we call \( \rho \) the isotropy representation and \( \Gamma \) the isotropy group of \( b \) in \( B \). Let \( \tilde{\psi} : V \rightarrow U \) be the obvious map.

2. If \( b \in U \subset B \) and \( \psi : \rho(\Gamma) \setminus V \rightarrow U \) are as above, if \( b' \in U \), and if \( \psi' : \rho'(\Gamma') \setminus V' \rightarrow U' \) are chosen analogously for \( b' \), then there exists an open embedding \( \psi' \circ \rho^{-1}(U) \rightarrow V \) and a group homomorphism \( \vartheta : \Gamma' \rightarrow \Gamma \) such that

\[
\psi \circ \rho' = \rho \vartheta(\gamma') \circ \psi \quad \text{for all } \gamma' \in \Gamma', \quad \text{and} \quad \psi(\rho(\Gamma)\psi'(v')) = \psi'(\rho'(\Gamma')v').
\]

We call \( \psi \) a coordinate change and \( \vartheta \) an intertwining homomorphism.

An oriented orbifold is an \( \text{SO}(n) \)-orbifold where all coordinate changes are orientation preserving.

We will say \( n \)-orbifold briefly for \( \text{O}(n) \)-orbifold, and we will drop \( \rho \) from the notation when the action of \( \Gamma \) is clear from the context. If \( \psi \) is a coordinate change with intertwining homomorphism \( \vartheta \) as above, then \( \rho \vartheta \circ \psi \) is another coordinate change with intertwining homomorphism \( \vartheta' \mapsto \gamma \vartheta(\gamma') \gamma^{-1} \) for each \( \gamma \in \Gamma \). If \( B \) is effective, we do not have to impose any further condition (like a cocycle condition) on the choices of coordinate changes and intertwining homomorphisms. See [26, Section 13.2] for a general definition.

**Definition 1.2.** Let \( B \) be a \( G \)-orbifold and let \( X \) be a smooth manifold. An orbibundle with fibre \( X \) is a map \( p \) from a topological space \( M \) to \( B \) with the following extra structure.

1. For each \( b \in B \), there exists an orbifold chart \( \psi : \rho(\Gamma) \setminus V \rightarrow U \subset B \) around \( b \), a fibre-preserving action \( \sigma \) of \( \Gamma \) by diffeomorphisms on \( V \times X \) covering \( \rho \) and a homeomorphism \( \tilde{\psi} : \sigma(\Gamma)(V \times X) \rightarrow p^{-1}(U) \) such that the diagram

\[
\begin{array}{ccc}
V \times X & \longrightarrow & \sigma(\Gamma)(V \times X) \\
\downarrow & & \downarrow \tilde{\psi} \\
V & \longrightarrow & \rho(\Gamma) \setminus V
\end{array}
\]

\[
\begin{array}{ccc}
& & p^{-1}(U) \\
\downarrow & & \downarrow p \\
& & U
\end{array}
\]

commutes.
(2) If $\psi: \rho(\Gamma)V \to U$ and $\psi': \rho'(\Gamma')V' \to U'$ are orbifold charts as in Definition 1.1(2) with coordinate change $\varphi$ and intertwining homomorphism $\vartheta$, and $\sigma, \sigma'$ and $\tilde{\psi}, \tilde{\psi}'$ are as above, then there exists a diffeomorphism $\tilde{\psi}: \psi'^{-1}(U) \times X \to V \times X$ such that

$$\tilde{\psi} \circ \sigma'_{\psi'} = \sigma_{\vartheta(\gamma')} \circ \tilde{\psi}$$

for all $\gamma' \in \Gamma'$, and such that the diagram

$$(\psi'^{-1}(U) \times X) \xrightarrow{\phi} \sigma'(\Gamma')(\psi'^{-1}(U) \times X) \xrightarrow{\psi'} p^{-1}(U \cap U')$$

commutes.

If all actions $\sigma$ are free, then $M$ carries the structure of a smooth manifold, and we call $p: M \to \Sigma$ a Seifert fibration. If $X$ is a vector space and all actions $\sigma$ and all diffeomorphisms $\tilde{\psi}$ are fibrewise linear, then we call $p: M \to \Sigma$ a vector orbibundle. If $X = G$ is a Lie group and all $\sigma$ and all $\tilde{\psi}$ commute with the right action of $G$ on $X$, then $G$ acts on $M$, and we call $p: M \to \Sigma$ a $G$-principal orbibundle.

**Remark 1.3.** Alternatively, a Seifert fibration with compact fibres is a connected manifold $M$ with a Riemannian foliation $\mathcal{F}$ such that all leaves are compact. To see this, we pick a holonomy invariant metric on $M$ and let $\Sigma = M/\mathcal{F}$ denote the space of leaves and $p: M \to \Sigma$ the quotient map.

Let $L$ be a leaf with normal bundle $N_L \to L$. By compactness, there exists $r > 0$ such that the normal exponential map $\exp_L$ is an injective local diffeomorphism from the disc bundle $N_r$ to $M$. We use $\exp_L$ to construct an orbifold chart for $\Sigma$ and an orbibundle chart for $M$ around $L$. Fix $\ell \in L$ and let $\tilde{\rho}_{L,\ell}: \pi_1(L, \ell) \to O(N_{\ell})$ denote the holonomy representation. Then $\tilde{\rho}_{L,\ell}$ induces a representation $\rho_{L,\ell}: \Gamma_{L,\ell} = \pi_1(L, \ell)/\ker(\tilde{\rho}_{L,\ell}) \to O(N_{\ell})$, and $D_{r}N_{\ell}$ is a bundle chart for $\Sigma$ around $L$ with isotropy group $\Gamma_{L,\ell}$ and isotropy representation $\rho_{L,\ell}$. The transition maps and intertwining homomorphisms are not hard to construct either.

Moreover, let $\tilde{L}$ denote the universal covering space of $L$, so $L = \pi_1(L, \ell)\tilde{L}$. Then $\Gamma_{L,\ell}$ acts on $X_{L,\ell} = \ker(\tilde{\rho}_{L,\ell})\tilde{L}$, and because $M$ is connected, all $X_{L,\ell}$ are diffeomorphic. This way, we can construct orbibundle charts and transition maps as in Definition 1.2.

If $B$ is an orbifold, then there is a natural tangent orbibundle $TB \to B$. There is a natural notion of a Riemannian metric on $B$, and such metrics always exist.

If $p: M \to \Sigma$ is a Seifert fibration, then there exists a natural map $dp: TM \to TB$ and a well-defined vertical subbundle $TX = \ker dp \subset TM$. If $g^TM$ is a Riemannian
metric on $M$, let $T^h M = (TX)^\perp \to M$ denote the horizontal subbundle. Then $g^T M$ is a submersion metric if there exists a Riemannian metric $g^T B$ on $B$ such that $d|_{T^h M}$ is a fibrewise isometry.

Remark 1.4. Whitney sums, Whitney tensor products, dual bundles and exterior powers can be defined for vector orbibundles over a base orbifold $B$. However, because in general not all “fibres” of a vector orbibundle are vector spaces, one cannot apply these constructions fibrewise. Instead, one has to perform the respective constructions fibrewise with the bundle charts and transition maps of Definition 1.2. By functoriality, the resulting collection of bundle charts and transition maps define another vector orbibundle on $B$. Similarly, there is a natural notion of a Dirac orbibundle over an orbifold in analogy with the notion of a Dirac bundle (see Section 1.b).

If $W \to B$ is a vector orbibundle with fibre $k^r$, the space of sections is given locally in a chart $V \to \Gamma \setminus V \cong U$ as a space of $\Gamma$-invariant maps

$$\Gamma(W|_U) \cong C^\infty(V; k^r)^\Gamma.$$ 

After these preparations, we may now write

$$\Omega^*(B; W) = \Gamma(\Lambda^* T^* B \otimes W).$$

If $W$ is graded, the tensor product is understood in the graded sense.

Let $M \to B$ be a Seifert fibration with fibre $X$, let $T^h M$ denote a horizontal subbundle, and let $V \to M$ be a vector bundle. Let $\Omega^*(M/B; W) \to B$ denote the infinite-dimensional vector orbibundle with fibre $\Omega^*(X; W|_X)$. Then

$$\Omega^*(M; W) \cong \Omega^*(B; \Omega^*(M/B; W)),$$

and this isomorphism depends explicitly on the choice of $T^h M$. This follows by regarding the pullback of the local situation to bundle charts.

In particular, all constructions of local family index theory such as adiabatic limits and Getzler rescaling are still well-defined for Seifert fibrations.

1.b. The inertia bundle and characteristic classes

Kawasaki’s index theorem for orbifolds has been formulated for general elliptic differential operators. The topological index is formulated in terms of characteristic classes of symbols. For the task at hand, we need to specialise these classes to the case of twisted Dirac operators.

We recall the definition of the inertia orbifold $\Sigma B$ of $B$ in [20, p. 137], where it is called $\Sigma B = \Lambda B \setminus B$. Its points are given as pairs $(p, (\gamma))$, where $p \in B$ and $\gamma$ is the $\Gamma$-conjugacy class of an element of the isotropy group $\Gamma$ of $p$. If $\psi: \Gamma \setminus V \to U$ is an orbifold chart for $B$ around $p = \psi(0)$, we obtain an orbifold chart

$$\psi(\gamma): C_{\Gamma(\gamma)} \setminus V \to \psi(V \setminus (\gamma)) \subset \Lambda B$$

(1.1)
by restriction, where \( V^\gamma \) denotes the fixpoint set of \( \gamma \) and \( C_\Gamma(\gamma) \) is the centraliser of \( \gamma \) in \( \Gamma \). In general, the inertia orbifold is no longer effective. Hence, let

\[
m(\gamma) = \# \{ \vartheta \in C_0(\gamma) \mid \rho_\vartheta|_{V^\gamma} = \text{id}_{V^\gamma} \}
\]

(1.2)
denote the multiplicity of \((p, (\gamma)) \in \Lambda B \). Then \( m(\gamma) \) defines a locally constant function on \( \Lambda B \).

Let \( N_\gamma \to V^\gamma \) denote the normal bundle to \( V^\gamma \) in \( V \), and let \( R^{N_\gamma} \) be the curvature of the connection on \( N_\gamma \) induced by the pullback of the Levi-Civita connection. Let \( \tilde{\gamma} \) denote a lift of the action of \( \gamma \) on \( N_\gamma \) to the spin group under the natural projection \( \text{Spin}(N_\gamma) \to \text{SO}(N_\gamma) \). If \( B \) is a spin orbifold, such a lift is part of the orbifold spin structure. Otherwise, the lift \( \tilde{\gamma} \) is determined uniquely up to sign. Hence, the inertia orbifold has a natural double cover

\[
\tilde{\Lambda} B = \{ (p, (\tilde{\gamma})) \mid \tilde{\gamma} \text{ lifts } \gamma \} \to \Lambda B.
\]

(1.3)

As in (1.1), one constructs charts for \( \tilde{\Lambda} B \) by

\[
\psi(\tilde{\gamma}) : C_0(\gamma) \setminus V^\gamma \to \psi(V^\gamma) \times \{ (\tilde{\gamma}) \} \subset \tilde{\Lambda} B.
\]

(1.4)

The equivariant Chern character form of a Hermitian vector bundle \((E, \nabla^E)\) with connection, equipped with a parallel fibrewise automorphism \( g \), is classically defined as

\[
\text{ch}_g(E, \nabla^E) = \text{tr}(g e^{-\nabla^E^2/2\pi i}).
\]

(1.5)

There exists a local spinor bundle \( \text{SN}_\gamma \to V^\gamma \) for \( N_\gamma \). Given a local orientation of \( N_\gamma \), there is a natural local splitting \( \text{SN}_\gamma = S^+ N_\gamma \oplus S^- N_\gamma \). Using \( R^{N_\gamma} \) and a lift \( \tilde{\gamma} \) of \( \gamma \) as above, we can define the equivariant \( \hat{A} \)-form on \( V^\gamma \) by

\[
\hat{A}_{\tilde{\gamma}}(TV, \nabla^{TV}) = (-1)^{(rk N_\gamma)/2} \frac{\hat{A}(TV^\gamma, \nabla^{TV^\gamma})}{\text{ch}_\gamma(S^+ N_\gamma - S^- N_\gamma, \nabla^{SN_\gamma})}.
\]

(1.6)

**Remark 1.5.** This construction of \( \hat{A}_{\tilde{\gamma}}(TV, \nabla^{TV}) \) has the following properties.

1. Because 1 is not an eigenvalue of \( \gamma|_{N_\gamma} \), the denominator is invertible in \( \Omega^*(V^\gamma) \). In fact, as explained in [2, Section 6.4], one has

\[
\text{ch}_\gamma(S^+ N_\gamma - S^- N_\gamma, \nabla^{SN_\gamma}) = \pm i^{(rk N_\gamma)/2} \det_{N_\gamma}(\text{id} - e^{-\nabla N_\gamma^2/2\pi i})^{1/2}.
\]

2. The form \( \hat{A}_{\tilde{\gamma}}(TV, \nabla^{TV}) \) only depends on the conjugacy class of \( \tilde{\gamma} \).

3. Replacing the lift \( \tilde{\gamma} \) of \( \gamma \) by the lift \( -\tilde{\gamma} \) changes the sign of \( \hat{A}_{\tilde{\gamma}}(TV, \nabla^{TV}) \).

4. If one changes the orientation on \( V^\gamma \) but keeps the orientation of the total tangent space \( TV|_{V^\gamma} \), then the orientation of \( N_\gamma \) changes as well, and the subbundles \( S^+ N_\gamma \) and \( S^- N_\gamma \) are swapped. Hence the form \( \hat{A}_{\tilde{\gamma}}(TV, \nabla^{TV}) \) changes its sign. On the other hand, its integral over the corresponding stratum of \( \Lambda B \) then does not depend on the orientation chosen on \( V^\gamma \), only on the orientation of \( V \).
Let us introduce the notation

\[ \Omega^* (\Lambda B; \tilde{\Lambda B}) = \{ \alpha \in \Omega^* (\tilde{\Lambda B}) \mid \alpha|_{(p, (-\tilde{\gamma}))} = -\alpha|_{(p, (\tilde{\gamma}))} \}, \]

and let us denote by \( \Omega^* (\Lambda B; \Lambda B \otimes o(\Lambda B)) \) the space of forms that change sign depending on the choice of a local orientation of \( \Lambda B \). We assume that \( B \) is an oriented orbifold. By (2)–(4), the forms \( \hat{A}_\varphi (TV, \nabla^{TV}) \) in local coordinates can be used to construct a well-defined form \( \hat{A}_{\Lambda B} (TB, \nabla^{TB}) \in \Omega^* (\Lambda B; \Lambda B \otimes o(\Lambda B)) \) with

\[
\psi_{\tilde{\gamma}}^* \hat{A}_{\Lambda B} (TB, \nabla^{TB}) = \frac{1}{m(\gamma)} \hat{A}_\varphi (TV, \nabla^{TV}),
\]

where \( m(\gamma) \) is the multiplicity of (1.2). With the choice of \( \tilde{\gamma} \) given by a spin structure, this corresponds to the integrand in Kawasaki’s orbifold index theorem [20, p. 139] when specialised to untwisted Dirac operators.

Recall that by [22, Definition II.5.2], a Dirac bundle over a Riemannian manifold \((M, g)\) consists of a complex vector bundle \( E \to M \) with Hermitian metric \( g_E \), compatible connection \( \nabla_E \), and a Clifford action of \( TM \) on \( E \) that is skew-symmetric with respect to \( g_E \) and satisfies a Leibniz rule with respect to \( \nabla_E \) and the Levi-Civita connection on \( TM \). There is an analogous natural notion of a Dirac orbibundle over a Riemannian orbifold.

Let \((E, \nabla^E, g^E, c)\) denote a Dirac orbibundle over \( V \), and let \( \gamma^E \) be a compatible action of \( \gamma \) on \( E \). Then \( \gamma^{E/S} = \gamma^E \cdot \tilde{\gamma}^{-1} \) commutes with Clifford multiplication and has the same sign ambiguity as \( \tilde{\gamma} \). If we write \( E = SM \otimes W \) locally, then \( W \) carries a natural connection \( \nabla_W \) with curvature \( R^W = R^{E/S} \), called the twisting curvature in [2, Proposition 3.43]. We can regard \( \tilde{\gamma}^{E/S} \) as acting only on \( W \), and we can define the equivariant twist Chern character form

\[ \text{ch}_{\tilde{\gamma}^{E/S}} (E/S, \nabla^E) = \text{ch}_{\tilde{\gamma}^{E/S}} (W, \nabla^W). \]

Then \( \tilde{\gamma}^{E/S} \text{ch}_{\tilde{\gamma}^{E/S}} (E/S, \nabla^E) \) is the integrand in the local Atiyah–Segal–Singer equivariant index theorem. Berline, Getzler and Vergne propose a particular choice of the lift \( \tilde{\gamma} \) in [2, Section 6.4].

Because \( \text{ch}_{\tilde{\gamma}^{E/S}} (E/S, \nabla^E) \) only depends on the conjugacy class and the sign of the lift \( \tilde{\gamma} \), there exists a well-defined class \( \text{ch}_{\Lambda B} (E/S, \nabla^E) \in \Omega^* (\Lambda B; \tilde{\Lambda B}) \) such that

\[ \psi_{\tilde{\gamma}}^* \text{ch}_{\Lambda B} (E/S, \nabla^E) = \text{ch}_{\varphi} (E/S, \nabla^E). \]

Then \( \hat{A}_{\Lambda B} (TB, \nabla^{TB}) \text{ch}_{\Lambda B} (E/S, \nabla^E) \in \Omega^* (\Lambda B; o(\Lambda B)) \) is the integrand in Kawasaki’s index theorem for orbifolds when specialised to twisted Dirac operators. In particular, the integral over \( \Lambda B \) does not depend on the choices above. In the special case of an untwisted Dirac operator, we have \( E = S \), and \( \gamma^E \) is itself a lift of \( \gamma \). In this case, we simply have \( \text{ch}_{\varphi} (E/S, \nabla^E) = 1 \) if we take \( \tilde{\gamma} = \gamma^E \), and \( \text{ch}_{\varphi} (E/S, \nabla^E) = -1 \) otherwise.
1.c. Adiabatic limits

If $M \to B$ is a Seifert fibration and $g^{TM}$ is a submersion metric as in Section 1.a., we obtain a family $(g^{TM}_\varepsilon)_{\varepsilon > 0}$ of submersion metrics with the same horizontal bundle $T^H M \to M$ such that $g^{TM}_\varepsilon|_{TX} = g^{TM}|_{TX}$ and $g^{TM}_\varepsilon|_{T^HM} = \varepsilon^{-2} g^{TM}|_{T^HM}$. The limit $\varepsilon \to 0$ is called the adiabatic limit.

Let $e_1, \ldots, e_n$ and $f_1, \ldots, f_{m-n}$ be local orthonormal frames of $TX$ and $TB$. The horizontal lift of a vector field $v$ on $B$ will be denoted by $\bar{v}$. Then a local orthonormal frame of $TM$ for $g^\varepsilon$ is given by

$$
e_1^\varepsilon = e_1, \ldots, e_n^\varepsilon = e_n, \quad e_{n+1}^\varepsilon = \varepsilon \bar{f}_1, \ldots, \quad e_m^\varepsilon = \varepsilon \bar{f}_{m-n}.
$$

(1.8)

**Definition 1.6.** An adiabatic family of Dirac bundles for $p: M \to B$ consists of a Hermitian vector bundle $(E, g^E)$, a Clifford multiplication $c: TM \to \text{End} E$, and a family $(\nabla^{E,\varepsilon})_{\varepsilon \geq 0}$ of connections such that

1. The quadruple $(E, \nabla^{E,\varepsilon}, g^E, c^\varepsilon)$ is a Dirac bundle on $(M, g^{TM})$ for all $\varepsilon > 0$, where the Clifford multiplication $c^\varepsilon$ is given by $c^\varepsilon(e^\varepsilon_I) = c(e_I)$.
2. The connection $\nabla^{E,\varepsilon}$ is analytic in $\varepsilon$ around $\varepsilon = 0$.
3. The kernels of the fibrewise Dirac operators $D_X = \sum_{i=1}^n c(e_i) \nabla^{E,0}_e$ acting on $E|_{p^{-1}(b)}$ form a vector orbibundle $H \to B$.

We will call the associated family $(D_{M,\varepsilon})_{\varepsilon > 0}$ with

$$D_{M,\varepsilon} = \sum_{I=1}^m c^\varepsilon(e_I^\varepsilon) \nabla^{E,\varepsilon}_{e_I^\varepsilon}$$

an adiabatic family of Dirac operators for $p$.

We consider the infinite-dimensional vector orbibundle $p_*E \to B$ with

$$p_*E|_b = \Gamma(E|_{p^{-1}(b)})$$

for all regular points $b \in B_0$. It carries a fibrewise $L^2$-metric $g^{p_*E}$ that is independent of $\varepsilon$.

Associated to $(E, \nabla^{E,0}, g^E, c)$ is Bismut’s Levi-Civita superconnection

$$\mathcal{A}_t = t^{1/2} \mathcal{A}_0 + \mathcal{A}_1 + t^{-1/2} \mathcal{A}_2,$$

(1.9)

on $p_*E \to B$ for $t > 0$ (see [2, Proposition 10.15], [3, Definition 4.29]). Here, $\mathcal{A}_0 = D_X$ is the fibrewise Dirac operator of (3) above. The part $\mathcal{A}_1 = \nabla p_*E,0$ is the unitary connection on $p_*E$ that is induced by

$$\nabla^{E,0} = \frac{1}{2} h.$$

(1.10)
where $h: T^HM \to \mathbb{R}$ is the mean curvature of the fibres of $p$, and $\mathfrak{h}_2$ is an endomorphism of $E \to M$ with coefficients in $\Lambda^2T^*B$.

Let $\gamma \in \Gamma$ be an element of the isotropy group of $b \in B$; then $\gamma$ acts on $p_\ast E$. If $SB \to V'$ denotes a local spinor bundle on $B$, then there exists a fibrewise Dirac bundle $W \to M$ such that as vector bundles, locally,

$$E \cong p^\ast SB \otimes W \to p^{-1}(V') \quad \text{and} \quad p_\ast E \cong SB \otimes p_\ast W \to V'. \quad (1.11)$$

After choosing a lift $\tilde{\gamma} \in \text{Spin}(N_f)$ of the action of $\gamma$ on $N_f$, as in (1.3), we can split $\gamma = \tilde{\gamma}^W \circ \tilde{\gamma}$ (see (2.20) below). Over $V'$, we consider the equivariant $\eta$-form

$$\eta_{\tilde{\gamma}}(\mathfrak{h}) = \int_0^\infty \frac{1}{\sqrt{t}} (2\pi i)^{-N/V'/2} \text{tr}_{p_\ast W} \left( \tilde{\gamma}^W \frac{\partial \mathfrak{h}_t}{\partial t} e^{-\mathfrak{h}_t^2} \right) dt \in \Omega^\bullet(V') \quad (1.12)$$

as in [13, Remark 3.12], where $N^{V'}$ denotes the number operator on $\Omega^\bullet(V')$. Again, the sign of $\eta_{\tilde{\gamma}}(\mathfrak{h})$ depends on the choice of $\tilde{\gamma}$; if $B$ is a spin orbifold, then this choice is natural. Note that in contrast to [13], we already eliminate $2\pi i$-factors inside the differential form $\eta_{\tilde{\gamma}}(\mathfrak{h})$ and not after integration. By assumption (3) above, the integral converges uniformly near $t = \infty$ because the operators $D_X$ have a uniform spectral gap around the possible eigenvalue 0. If $\tilde{\gamma} = e$ is the neutral element, then $\eta_{\tilde{\gamma}}(\mathfrak{h}) = \eta(\mathfrak{h})$ is the $\eta$-form of Bismut and Cheeger, and the integral in (1.12) also converges near $t = 0$ by [3, (4.32)]. Otherwise, $\gamma$ acts freely on the fibres, and small time convergence is not an issue.

**Definition 1.7.** The orbifold $\eta$-form $\eta_{\Lambda B}(\mathfrak{h}) \in \Omega^\bullet(\Lambda B; \tilde{\Lambda}B)$ is defined such that in the orbifold charts of (1.4),

$$\psi_{\tilde{\gamma}}^{\ast} \eta_{\Lambda B}(\mathfrak{h}) = \eta_{\tilde{\gamma}}(\mathfrak{h}). \quad (1.13)$$

This is well-defined because $\eta_{\tilde{\gamma}}(\mathfrak{h})$ only depends on the conjugacy class and the sign of $\tilde{\gamma}$. Moreover, the integrand $\hat{A}_{\Lambda B}(TB, \nabla^{TB})2\eta_{\Lambda B}(\mathfrak{h}) \in \Omega^\bullet(\Lambda B; o(\Lambda B))$ in the first term on the right hand side of Theorem 0.1 depends only on the orientation of the fibres of $p: M \to B$; in particular, the integral over $\Lambda B$ only depends on the global orientation of $M$ by Remark 1.5(4). Note the different normalisation of $\eta$-forms and $\eta$-invariants. The component of $2\eta_{\tilde{\gamma}}(\mathfrak{h})$ of degree 0 in $\Omega^\bullet(\Lambda B)$ is the equivariant $\eta$-invariant of the fibre. This explains the additional factor 2 in the integrand in Theorem 0.1.

Locally, there exists a unique family of spinor bundles $(\mathcal{S}_M, \nabla^{SM, \varepsilon}, g^{SM}, c)_\varepsilon$ on $(M, g^TM)$, and the Dirac bundle $E$ splits as $E \cong SM \otimes W$. There exists a locally uniquely defined family of connections $\nabla^{E/S, \varepsilon} = \nabla^{W, \varepsilon}$ such that $\nabla^{E, \varepsilon}$ is the tensor product connection induced by $\nabla^{SM, \varepsilon}$ and $\nabla^{W, \varepsilon}$. Note that for the family of odd signature operators $B_M, \varepsilon$, we cannot assume that $\nabla^{E/S, \varepsilon}$ is independent of $\varepsilon$. We obtain globally well-defined endomorphism-valued differential forms

$$\nabla^{E/S, \varepsilon} - \nabla^{E/S, 0} \in \Omega^1(M; \text{End } E) \quad \text{and} \quad R^{E/S, \varepsilon} \in \Omega^2(M; \text{End } E) \quad (1.14)$$

that commute with Clifford multiplication. Both $\nabla^{E/S, \varepsilon} = \nabla^{E/S, 0}$ and $R^{E/S, \varepsilon}$ depend analytically on $\varepsilon$ around $\varepsilon = 0$ by assumption (2) above.

Let $P_X: p_\ast E \to H$ denote the $L^2$-orthogonal projection onto $H = \ker D_X$. 
**Definition 1.8.** The effective horizontal operator of an adiabatic family of Dirac bundles \( (E, \nabla^E, \nu^E, \epsilon) \) is defined as

\[
D_{eff}^B = P_X \circ \left( \sum_{\alpha=1}^{m-n} c(f_{\alpha}) \nabla^E_{\nu_{\alpha}} + \sum_{i=1}^{n} c(e_i) \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \nabla^E_{e_i/S,\epsilon} \right) \circ P_X.
\]

The operator \( D_{eff}^B \) is selfadjoint. Its \( \eta \)-invariant is further investigated in Proposition 2.4. If the base space is even-dimensional, we will see in Section 1.d that in important special cases, the \( \eta \)-invariant of the effective horizontal operator vanishes.

Again by assumption (2) above, there exists \( \epsilon_0 > 0 \) such that the kernel of \( D^B_{M,\epsilon} \) has constant dimension for all \( \epsilon \in (0,\epsilon_0) \). By [9, Theorem 1.5] and Section 2.g below, there are finitely many eigenvalues \( \lambda_\nu(\epsilon) \) of \( D^B_{M,\epsilon} \) (counted with multiplicity), called the “very small eigenvalues”, such that

\[
\lambda_\nu(\epsilon) = O(\epsilon^2) \quad \text{and} \quad 0 \neq \lambda_\nu(\epsilon) \quad \text{for all} \quad \epsilon \in (0,\epsilon_0).
\]

(1.15)

We have now defined all ingredients of Theorem 0.1. Its proof is deferred to Section 2.

**1.d. Special cases of the adiabatic limit theorem**

We consider fibres of positive scalar curvature, and the signature operator.

We assume first that \( M \) and \( B \) are spin. Then the vertical tangent bundle of \( p: M \to B \) also carries a spin structure. By abuse of notation, we write \( \eta_{\Lambda B}(D^B_{M/B}) \) instead of \( \eta_{\Lambda B}(\bar{\kappa}) \) for the orbifold \( \eta \)-form associated to the untwisted Dirac operator.

If the fibres of \( M \to B \) have positive scalar curvature, then for the untwisted Dirac operator \( D^B_{M,\epsilon} \), the fibrewise operator is invertible, hence \( H = 0 \) and there is neither an effective horizontal operator nor are there very small eigenvalues. In particular, \( D^B_{M,\epsilon} \) satisfies the conditions of Dai’s theorem. The same still holds for the Dirac operator \( D^W_{M,\epsilon} \) that is twisted by the pullback of an orbibundle \( W \to B \) with connection \( \nabla^W \).

**Corollary 1.9.** If the fibration \( M \to B \) and \( B \) are spin, the fibres of \( M \to B \) have positive scalar curvature, and if \( W \to B \) is an orbibundle, then

\[
\lim_{\epsilon \to 0} \eta(D^W_{M,\epsilon}) = \int_{\Lambda B} \hat{A}_{\Lambda B}(TB, \nabla^{TB}) \text{ch}_{\Lambda B}(W, \nabla^W) 2\eta_{\Lambda B}(D^B_{M/B}).
\]

**Proof.** If \( W \) is trivial, then the corollary follows from Theorem 0.1 by the considerations above. If \( W \to B \) is an orbibundle, then the result follows from Remark 2.13. \( \square \)

The odd signature operator \( B^B_{M,\epsilon} \) on \( M \) also satisfies the conditions of Dai’s theorem. Here, the bundle \( H \to B \) corresponds to the fibrewise cohomology, regarded as a \( Z_2 \)-graded vector bundle. In contrast to [9], we regard \( B^B_{M,\epsilon} \) as an operator on \( \Omega^{even}(M) \), not on all forms. Note that the twisting curvature depends on \( \epsilon \).

There is a natural notion of a differentiable Leray–Serre spectral sequence of \( M \to B \), and by Dai [9, Theorem 0.2] (see also [24]) the very small eigenvalues \( (\lambda_\nu(\epsilon))_\nu \) of \( B^B_{M,\epsilon} \) are related to its higher differentials. The effective horizontal operator \( B^B_{eff} \) is related to...
the $E_1$-term of this sequence by results of Dai [9, Section 4.1]. Dai also constructs a signature $\tau_r \in \mathbb{Z}$ on the $r$-th term $E_r$ of this spectral sequence for all $r \geq 2$.

Let $N^B$ denote the number operator on $\Omega^*(B)$, then the rescaled $L$-class

$$\hat{L}(TB, \nabla^{TB}) = \hat{A}(TB, \nabla^{TB}) \text{ch}(SB, \nabla^{SB}) = 2^{(\dim B - N^B) / 2} L(TB, \nabla^{TB})$$

has a natural general equivariant generalisation leading to $\hat{L}_{\Lambda B}(TB, \nabla^{TB}) \in \Omega^*(\Lambda B)$. Finally, let us write $\eta_{\Lambda B}(B_{M/B})$ instead of $\eta_{\Lambda B}(A)$ in this setting.

**Corollary 1.10** (see Dai [9, Theorem 0.3]). If the fibration $M \to B$ is oriented, then

$$\lim_{\varepsilon \to 0} \eta(B_{M,\varepsilon}) = \int_{\Lambda B} \hat{L}_{\Lambda B}(TB, \nabla^{TB}) 2 \eta_{\Lambda B}(B_{M/B}) + \sum_{r=2}^{\infty} \tau_r.$$

Moreover, if $\dim B$ is even, then $\eta(B_{\text{eff}}^B) = 0$.

**Proof.** This follows from Theorem 0.1 as in Dai’s paper [9]. The first term again arises because of Remark 2.13. The vanishing of $\eta(B_{\text{eff}}^B)$ for even-dimensional base orbifolds follows from Proposition 2.5. $\square$

**1.e. Seifert fibrations with compact structure group**

We assume that the fibres of the map $p : M \to B$ are totally geodesic submanifolds of $M$.

Assume for the moment that $B$ is a connected Riemannian manifold and that $p : M \to B$ is an ordinary Riemannian submersion. Each path in $B$ induces a parallel transport between the fibres over its endpoints. By a result of Hermann [19, Theorem 1], all these parallel translations are isometries if and only if the fibres of $p$ are totally geodesic. In this case, let $X$ be isometric to a fibre of $p$ and let $G$ denote the isometry group of $X$ acting from the left. Then there is a natural $G$-principal bundle

$$P = \{ f : X \to M \mid f \text{ is an isometry onto a fibre of } p \}$$

with a natural right $G$-action, and we have $M = P \times_G X$.

If we are given an adiabatic family $(E, \nabla^{E,\varepsilon}, g^E, c)$ of Dirac bundles as in Definition 1.6, then we assume further that the parallel transport between fibres lifts to isomorphisms between the restrictions of $(E, \nabla^{E,\varepsilon}, g^E, c)$ to the fibres of $p$. In this case, let $G$ denote the automorphism group of

$$(\langle E, \nabla^{E,\varepsilon}, g^E, c \rangle|_X) \to (X, g^X).$$

Then the family $p : M \to B$ is still associated to a $G$-principal bundle $P \to B$. In this case, we say that the adiabatic family $(E, \nabla^{E,\varepsilon}, g^E, c)$ has compact structure group $G$.

Let $b \in B$ and identify $p^{-1}(b)$ with $X$ and $E_{p^{-1}(b)}$ with $E|_X \to X$. Then for $v, w \in T_b B$, the fibre bundle curvature $[v, w] = [\tilde{v}, \tilde{w}]$ together with its natural action on $E$ is described by an element $\Omega(v, w) \in g$. Different identifications of $E \to X$ with $E|_{p^{-1}(b)}$ give elements of $g$ in the same $\text{Ad}_G$-orbit.
By [13, Lemma 1.14], there exists an $\Ad_G$-invariant formal power series $\eta_g(D_X)$ in $C[[g]]$, the infinitesimally equivariant $\eta$-invariant, such that
\[ 2\eta(\mathcal{A}) = \eta_{\mathcal{A}/2\pi i}(D_X). \] (1.16)

Here, $D_X$ is the component of $\mathcal{A}$ of degree 0, regarded as an operator on a bundle $W \to X$ such that locally, $E \cong W \otimes p^*SB$. This invariant has been computed for the untwisted Dirac operator and the signature operator on $S^3$ in [13, Theorem 3.9]. A more general formula for quotients of compact Lie groups with normal metrics can be found in [14, Section 2.4].

Now, let $p : M \to B$ be a Seifert fibration with generic fibre $X$ and assume again that all fibres are totally geodesic. Then the construction above still applies to bundle charts as in Definition 1.2. If we trivialise $p$ over $V$ by parallel translation along radial geodesics in $V$, then the isotropy group $\Gamma$ acts on $V \times X$ by
\[ \sigma_{\gamma}(v, x) = (\rho_{\gamma}(v), \sigma_{\gamma}(x)) \]
with $\sigma_{\gamma} \in G$ for all $\gamma \in \Gamma$. Thus, we obtain a $G$-principal orbibundle
\[ P = \{ f : X \to M \mid f \text{ is a local isometry onto a fibre of } p \}, \]
and again we have $M = P \times_G X \to B$. Moreover, for $\gamma \in \Gamma$, the restricted curvature $\Omega|_{V^\gamma}$ takes values in the $\Ad_{\gamma}$-invariant part of $\mathfrak{g}$. Thus, if $(p, (\gamma)) \in \Lambda B \setminus B$, let $\psi_{(\gamma)} : \mathcal{C}_G(\gamma)|_{V^\gamma} \to \Lambda B$ be an orbifold chart for $\Lambda B$ around $(p, (\gamma))$ as in (1.1). We regard the pullback of $M \to B$ restricted to $V^\gamma$ and identify $\gamma$ with $\sigma_{\gamma} \in G$ acting on $X$ and $E$. Then $\Omega|_{V^\gamma}$ takes values in the Lie algebra $\mathfrak{c}(\sigma_{\gamma})$ of the centraliser $C_G(\sigma_{\gamma})$ of $\sigma_{\gamma}$ in $G$.

**Theorem 1.11.** Let $p : M \to B$ be a Seifert fibration, and let $(E, \nabla^E, g^E, c)$ be an adiabatic family with compact structure group $G$. For each $(p, (\gamma)) \in \Lambda B$, there exists a formal power series
\[ \eta_{\mathfrak{c}(\sigma_{\gamma})}(D_X) \in R[[\mathfrak{c}(\sigma_{\gamma})]] \]
such that the orbifold $\eta$-form is given in an orbifold chart $\psi_{(\gamma)}$ around $(p, (\gamma))$ as
\[ \psi_{(\gamma)}^* \eta_{\Lambda B}(\mathcal{A}) = \eta_{\mathfrak{c}(\sigma_{\gamma})}(D_X). \]
If $\gamma = \text{id}$, then $\eta_{\mathfrak{c}(\sigma_{\gamma})}(D_X) = \eta_0(D_X)$ is the infinitesimally equivariant $\eta$-invariant. If $\gamma$ acts freely on the typical fibre $X$, then $\eta_{\mathfrak{c}(\sigma_{\gamma})}(D_X)$ is the formal power series expansion of the classical equivariant $\eta$-invariant $\eta_{\mathfrak{c}(\sigma_{\gamma})}(D_X)$ at $\mathbb{S} = 0 \in \mathfrak{c}(\sigma_{\gamma})$.

**Proof.** If $\gamma = \text{id}$, this is just [13, Lemma 1.14]. If $\gamma \neq \text{id}$, then $\gamma$ acts freely on the fibre $X$ because $p : M \to B$ is a Seifert fibration, and the result is explained and proved in [13, Remark 3.12]. \qed

Note that over each singular stratum of $B$, the fibres of $p$ are finite quotients of $X$, so that we are in a situation similar to [13, Lemma 3.11].
2. A proof of the adiabatic limit theorem

In this section, we sketch a proof of Theorem 0.1. We will omit most of the details, in particular those explained by Bismut and Cheeger [3] and Dai [9]. The proof is based on the well-known formula

\[ \eta(D_{M,\varepsilon}) = \int_0^\infty \frac{1}{\sqrt{\pi t}} \text{tr}(D_{M,\varepsilon}e^{-tD_{M,\varepsilon}^2}) \, dt. \]

We define a spectral projection \( P_\varepsilon \) onto the sum of the eigenspaces for the very small eigenvalues in Section 2.f, which commutes with \( D_{M,\varepsilon} \) for each \( \varepsilon > 0 \). We also find a small constant \( \alpha > 0 \) and write

\[ \eta(D_{M,\varepsilon}) = \int_{\varepsilon}^{\alpha-2} \frac{1}{\sqrt{\pi t}} \text{tr}(D_{M,\varepsilon}e^{-tD_{M,\varepsilon}^2}) \, dt + \int_{\varepsilon}^{\alpha-2} \frac{1}{\sqrt{\pi t}} \text{tr}((I - P_\varepsilon)D_{M,\varepsilon}e^{-tD_{M,\varepsilon}^2}) \, dt + \int_{\varepsilon}^{\alpha-2} \frac{1}{\sqrt{\pi t}} \text{tr}(P_\varepsilon D_{M,\varepsilon}e^{-tD_{M,\varepsilon}^2}) \, dt. \]

The three terms on the right hand side give rise to the three expressions on the right hand side of Theorem 0.1 by Propositions 2.12, 2.10 and 2.11, respectively, which will be stated and proved below. Thus, Theorem 0.1 follows from the results of this section.

2.a. Local computations

We will use small Roman indices in \([1, \ldots, n]\) referring to coordinates of the fibres, small Greek indices in \([n + 1, \ldots, m]\) referring to coordinates of the base, and capital indices in \([1, \ldots, m]\). Let \( \nabla^{TM,\varepsilon} \) denote the Levi-Civita connection with respect to the bundle-like metric \( g_\varepsilon = g_{TX} \oplus \varepsilon^{-2} p^*g_{TB} \).

Let \( \nabla^{TB} \) denote the Levi-Civita connection on the orbibundle \( TB \to B \). By [3, Section 4(a)], there exists a connection \( \nabla^T \) on \( TX \to M \), a symmetric tensor \( S: TX \otimes TX \to T^HM \) and an antisymmetric tensor \( T: T^HM \otimes T^HM \to TX \), with coefficients \( s_{ij\gamma} \) and \( t_{\alpha\beta\gamma k} \), such that

\[
\begin{align*}
\nabla_{e_i}^{TM,\varepsilon} e_j & = \nabla_{e_i}^{TX} e_j + \varepsilon \sum_{\gamma} s_{ij\gamma} e_\gamma, \\
\nabla_{e_\alpha}^{TM,\varepsilon} e_j & = \nabla_{e_\alpha}^{TX} e_j - \varepsilon \sum_{\beta} t_{\alpha\beta\gamma} e_\gamma, \\
\nabla_{e_i}^{TM,\varepsilon} e_\beta & = -\varepsilon \sum_k s_{ik\beta} e_k - \varepsilon^2 \sum_{\gamma} t_{\beta\gamma\gamma} e_\gamma, \\
\nabla_{e_\alpha}^{TM,\varepsilon} e_\beta & = (p^*\nabla^{TB}) e_\alpha e_\beta + \varepsilon \sum_k t_{\alpha\beta\gamma k} e_\gamma.
\end{align*}
\]
with \( e_\alpha = e_\alpha^1 \). We identify the tangent bundles \((TM, g_\varepsilon)\) orthogonally for different \( \varepsilon > 0 \) by sending \((e_1, \ldots, e_m)\) at \( \varepsilon = 1 \) to the \( g_\varepsilon \)-orthonormal frame of (1.8). With respect to this identification, we obtain the limit connection
\[
\nabla^{TM,0} = \lim_{\varepsilon \to 0} \nabla^{TM,\varepsilon} = \nabla^{TX} \oplus p^* \nabla^{TB}.
\]

(2.1)

Note that this differs from the geometric limit of Levi-Civita connections described for example in [2, Section 10.1].

Let us assume for the moment that the base \( B \) and the map \( p \) are spin, which is always true locally on \( M \), and let \( SX \to M \) be a spinor bundle for \( TX \to M \). Then we have an isomorphism of vector bundles
\[
SM \cong SX \otimes p^* SB
\]

independent of \( \varepsilon \), and the connection \( \nabla^{SM,\varepsilon} \) induced by \( \nabla^{TM,\varepsilon} \) is the tensor product connection. To define Clifford multiplication \( c_I \) by \( e_\varepsilon^I \) on \( SM \) for all \( I = 1, \ldots, m \), the tensor product is understood in a \( \mathbb{Z}_2 \)-graded sense. The connections \( \nabla^{TM,\varepsilon} \) induce connections \( \nabla^{SM,\varepsilon} \) on the spinor bundle \( SM \to M \) for all \( \varepsilon \geq 0 \). We have
\[
\nabla^{SM,\varepsilon}_{e_i} = \nabla^{SM,0}_{e_i} + \frac{\varepsilon}{2} \sum_{i,j} s_{ij \gamma} c_j c_\gamma - \frac{\varepsilon^2}{4} \sum_{\alpha,\beta} t_{\alpha \beta i} c_\alpha c_\beta,
\]
\[
\nabla^{SM,\varepsilon}_{e_\alpha} = \nabla^{SM,0}_{e_\alpha} - \frac{\varepsilon}{2} \sum_{i,\beta} t_{\alpha \beta i} c_i c_\beta.
\]

(2.2)

Let \((E, \nabla^{E,\varepsilon}, g^E, c)_{\varepsilon > 0}\) be an adiabatic family of Dirac bundles on \( M \) as in Definition 1.6. We can now define a vertical and a horizontal Dirac operator by
\[
D_X = \sum_i c_i \nabla^{E,0}_{e_i} \quad \text{and} \quad D_{B,\varepsilon} = \frac{1}{\varepsilon} (D_{M,\varepsilon} - D_X).
\]

(2.3)

Let \( \nabla^{E/S,\varepsilon} - \nabla^{E/S,0} \) denote the one-form of (1.14). Then
\[
D_{B,\varepsilon} = \sum_{\alpha} c_\alpha \left( \nabla^{E,0}_{e_\alpha} + \frac{1}{2} \sum_{i,j} s_{ij \alpha} c_i c_j - \frac{\varepsilon}{4} \sum_{i,\beta} t_{\alpha \beta i} c_\beta \right)
\]
\[
+ \sum_i c_i \frac{1}{\varepsilon} \left( \nabla^{E/S,\varepsilon} - \nabla^{E/S,0} \right)_{e_i} + \sum_{\alpha} c_\alpha \left( \nabla^{E/S,\varepsilon} - \nabla^{E/S,0} \right)_{e_\alpha}
\]
\[
+ \varepsilon \sum_{\alpha} c_\alpha \left( \frac{1}{\varepsilon} \left( \nabla^{E/S,\varepsilon} - \nabla^{E/S,0} \right)_{e_\alpha} - \frac{1}{4} \sum_{i,\beta} t_{\alpha \beta i} c_\beta \right).
\]

(2.4)

where \( h \in T^H M \) denotes the mean curvature vector of the fibres in \((M, g)\), and \( h_\alpha \) denotes its component in the direction of \( \alpha \). Note that the connection \( \nabla^{E,0} = \frac{1}{2} (h, \cdot) \)
for $\varepsilon = 0$ in the above expression for $D_{B,\varepsilon}$ is not unitary on $E \to M$ in general, but it induces a unitary connection $\nabla p_* E, 0$ on the infinite-dimensional vector orbibundle $p_* E \to B$.

**Lemma 2.1.** Let $(E, \nabla E, \varepsilon^E, g^E)_{\varepsilon > 0}$ be family of Dirac bundles on the family $(M, g^T M)_{\varepsilon > 0}$ of Riemannian manifolds. Decompose the associated family of Dirac operators $D_{M,\varepsilon} = D_X + \varepsilon D_{B,\varepsilon}$ as above. Then the anticommutator of $D_X$ and $D_{B,\varepsilon}$ is the sum of a fibrewise differential operator of order one and an endomorphism of $E$.

We write supercommutators as $[\cdot, \cdot]$.

**Proof.** Because $D_X$ is of order one and involves only fibrewise differentiation, supercommutators of $D_X$ with a zero order operator satisfy the assertion above. Hence, it suffices to consider

$$
\sum_u [D_X, \nabla E, 0] = \sum_{i,\alpha} c_i c(\nabla E, 0, e_i) \nabla E, 0 e_{\alpha} + c_i c_{\alpha} (\nabla E, 0)_{e_i, e_{\alpha}} + c_i c_{\alpha} (\nabla E, 0)_{e_i, e_i}.
$$

Because $e_\alpha$ is the horizontal lift of a vector field on $B$, we have $\nabla E, 0 e_i e_{\alpha} = 0$, and $[e_i, e_{\alpha}]$ is a vertical vector field. Our claim follows.

2.b. The effective horizontal operator

We regard the infinite-dimensional bundle $p_* E \to B$. Together with the connection $\nabla p_* E, 0$ of (1.10), it becomes an infinite-dimensional Dirac orbibundle on $B$.

Let $P_X \in \text{End}(p_* E)$ denote the fibrewise $L^2$-projection on ker $D_X$. By assumption (3) in Definition 1.6, $H = \ker D_X = \text{im} P_X$ is a finite rank vector orbibundle over $B$. Note that $P_X$ does not necessarily commute with the connection $\nabla p_* E, 0$. We define a connection $\nabla H$ on $H$ by

$$
\nabla H = P_X \circ \nabla p_* E, 0 \circ P_X = P_X \circ (\nabla E, 0 - \frac{1}{2} \langle h, \cdot \rangle) \circ P_X.
$$

**Proposition 2.2.** Let $P_X$ and $H$ be as above.

1. The operator $P_X$ is a fibrewise smoothing operator of finite rank that commutes with $D_X$ and with Clifford multiplication with horizontal vectors.

2. The orbibundle $H \to B$, equipped with the restriction of the fibrewise $L^2$-metric and the connection $\nabla H$, becomes a finite-dimensional Dirac orbibundle on $B$.

**Proof.** The projection $P_X$ commutes with $D_X$ by construction, and with $c_\alpha$ because $D_X$ anticommutes with $c_\alpha$.

The connection $\nabla p_* E, 0$ respects the $L^2$-scalar product, so its contraction $\nabla H$ onto $H$ respects the induced scalar product. Because $P_X$ commutes with $c_\alpha$, we obtain a Dirac orbibundle.
Remark 2.3. By Definition 1.8, the effective horizontal operator is a Dirac operator if the family of local twist connections $\nabla^{W,\varepsilon}$ considered in the previous section is constant in $\varepsilon$. This is not the case for the odd signature operator $B_{M,\varepsilon}$ on $(M, g_{TM})$, as explained in [9, Section 4.1]. The local twist bundle $W$ is now given by $(SM, \nabla^{SM,\varepsilon})$. Hence, by equation (2.2), we have

$$\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \nabla^{E/S,\varepsilon}_{c_i} = \frac{1}{2} \sum_{j,\gamma} s_{ij\gamma} c_{j\gamma}.$$ 

This term only depends on the second fundamental form of the fibres, in particular, for totally geodesic fibrations the effective horizontal operator is in fact the Dirac operator on the Dirac bundle $(H, g^H, \nabla^H)$ of Proposition 2.2(2) above.

Proposition 2.4. The $\eta$-invariant of $D^\text{eff}_B$ is given by a convergent integral,

$$\eta(D^\text{eff}_B) = \int_0^\infty \frac{1}{\sqrt{\pi t}} \text{tr}(D^\text{eff}_B e^{-t(D^\text{eff}_B)^2}) \, dt.$$ 

Proof. Convergence for $t \to \infty$ is clear because we assumed that $B$ is compact, and hence $D^\text{eff}_B$ has discrete spectrum.

For small-time convergence, we adapt the proof of [4, Section II]. We put

$$A = D^\text{eff}_B - D^H_B = \sum_i P_X \circ \left( c_i \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \nabla^{E/S,\varepsilon}_{c_i} \right) \circ P_X.$$ 

Because $c_\alpha$ commutes with $P_X$, we find that $A$ anticommutes with Clifford multiplication,

$$[c_\alpha, A] = P_X \circ \sum_i \left[ c_\alpha, c_i \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \nabla^{E/S,\varepsilon}_{c_i} \right] \circ P_X = 0.$$ 

In particular,

$$(D^\text{eff}_B)^2 = (D^H_B)^2 + \sum_a c_a [\nabla^H_{f\alpha}, A] + A^2.$$ 

We introduce an exterior variable $z$ that anticommutes with the Clifford multiplication $c_\alpha$ and is parallel with respect to $\nabla^H$. Consider the connection

$$\nabla^{H,z} = \nabla^H - \frac{z}{2} c(\cdot)$$ 

on the Dirac bundle $H$ of Proposition 2.2. Then instead of the usual Bochner–Lichnerowicz–Weitzenböck formula, one has

$$(D^\text{eff}_B)^2 + z D^\text{eff}_B = \nabla^{H,z} \ast \nabla^{H,z} + \frac{\kappa}{4} + \sum_{\alpha,\beta} c_\alpha c_\beta F^H_{f\alpha, f\beta}
+ \sum_a c_a [\nabla^H_{f\alpha}, A] + A^2 + z A.$$ 

If \( P, Q \) are endomorphisms of a vector space, define \( \text{tr}_z(P + zQ) = \text{tr}(Q) \), then
\[
\text{tr}(D_B^{\text{eff}} e^{-t(D_B^{\text{eff}})^2}) = \frac{1}{t} \text{tr}_z(e^{-t((D_B^{\text{eff}})^2 - zD_B^{\text{eff}})}).
\] (2.7)

We want to compute the heat kernel of \( e^{-t((D_B^{\text{eff}})^2 - zD_B^{\text{eff}})} \) using Getzler rescaling. To see that this is possible, we have to distinguish two cases.

If \( \dim B \) is even, only even elements of the Clifford algebra contribute to the trace. Hence, in the asymptotic expansion of the heat kernel, only terms involving the operator \( A \) an odd number of times will contribute. But \( A \) acts as \( \tau \otimes A' \), where \( \tau \) denotes the Clifford volume element and \( A' \) commutes with Clifford multiplication. Hence, we may replace \( A \) formally by \( A' \) and the trace by a supertrace, so Getzler rescaling is appropriate.

On the other hand, let \( \dim B \) be odd. Then \( \dim X \) is even, and the bundle \( H \) splits as \( H^+ \oplus H^- \). The splitting is preserved by \( D_B^{\text{H}} \), but \( A \) exchanges the summands. Hence, in the asymptotic expansion of the heat kernel, only terms involving the operator \( A \) an even number of times will contribute, so we have to take the trace on the odd part of the Clifford algebra, and Getzler rescaling is again appropriate.

Either way, we perform Getzler rescaling of the Clifford variables \( c_\alpha \), and \( A \) is not affected. Then the additional terms in the second line of (2.6) cause no trouble because \( A \) and \( [\nabla_B, A] \) do not involve Clifford multiplication at all. Hence, small time convergence follows as in [4].

\[ \square \]

**Proposition 2.5.** If \( \dim B \) is even, the effective horizontal operator of the adiabatic family of odd signature operators \((B_M, \varepsilon)\) on \( M \) has vanishing \( \eta \)-invariant.

**Proof.** The effective horizontal operator \( B_B^{\text{eff}} \) acts on \( \Omega^*(B; H) \) and exchanges even and odd forms by \([9, \text{Section 4.1}]\). Thus, the odd heat kernel \( B_B^{\text{eff}} e^{-t(B_B^{\text{eff}})^2} \) also exchanges even and odd forms, and so its trace is zero. Hence, the integrand in Proposition 2.4 vanishes.

\[ \square \]

### 2.c. The Dirac operator as a matrix

The following sections are inspired by work of Bismut and Lebeau [5, Chapter 9] and Ma [23, Chapter 5]. We will write operators acting on \( p_\ast E = \ker D_X \oplus \text{im} D_X \) as matrices of the form
\[
Y = \begin{pmatrix}
P_X Y P_X & P_X Y (1 - P_X) \\
(1 - P_X) Y P_X & (1 - P_X) Y (1 - P_X)
\end{pmatrix} = \begin{pmatrix}
Y_1 & Y_2 \\
Y_3 & Y_4
\end{pmatrix},
\]
in particular
\[
\frac{1}{\varepsilon} D_{M, \varepsilon} = \varepsilon^{-1} \begin{pmatrix}
D_{M, \varepsilon, 1} & D_{M, \varepsilon, 2} \\
D_{M, \varepsilon, 3} & D_{M, \varepsilon, 4}
\end{pmatrix} = \begin{pmatrix}
D_{B, \varepsilon, 1} & D_{B, \varepsilon, 2} \\
D_{B, \varepsilon, 3} & \varepsilon^{-1} D_X + D_{B, \varepsilon, 4}
\end{pmatrix}.
\]

**Proposition 2.6.** As \( \varepsilon \to 0 \),

1. the operator \( D_{B, \varepsilon, 1} - D_B^{\text{eff}} \) is an endomorphism of \( H \to B \) of magnitude \( O(\varepsilon) \), and
2. the operators \( D_{B, \varepsilon, 2} \) and \( D_{B, \varepsilon, 3} \) are uniformly bounded fibrewise smoothing operators of finite rank.
Proof. The first claim follows from the Definition 1.8 of the effective horizontal operator and equation (2.4).

The projection $P_X$ is a fibrewise smoothing operator of finite rank. It commutes with Clifford multiplication $c_\alpha$ by horizontal vectors. We conclude from (2.4) that the commutator $[D_{B,\varepsilon}, P_X]$ is again a fibrewise smoothing operator of finite rank. Now (2) follows because

$$D_{B,\varepsilon,2} = P_X \circ D_{B,\varepsilon} \circ (1 - P_X) = -[D_{B,\varepsilon}, P_X] \circ (1 - P_X),$$

$$D_{B,\varepsilon,3} = (1 - P_X) \circ D_{B,\varepsilon} \circ P_X = (1 - P_X) \circ [D_{B,\varepsilon}, P_X]$$

are uniformly bounded fibrewise smoothing operators of finite rank. $\square$

2.d. A resolvent estimate

Let $\lambda_B$ denote the smallest absolute value of a nonzero eigenvalue of the effective horizontal operator $D_{M,\varepsilon}^H$, and let $0 < c < \lambda_B/2$. Let $\Gamma = \Gamma_+ \cup \Gamma_0 \cup \Gamma_-$ denote a contour in $\mathbb{C}$, where $\Gamma_\pm$ goes around $\pm[\lambda_B, +\infty]$ at distance $c$, and $\Gamma_0$ is a circle around 0 with radius $c$. We choose $\varepsilon_0 > 0$ such that Proposition 2.7 is satisfied and such that all eigenvalues of $\varepsilon^{-1}D_{M,\varepsilon}$ lie inside the area enclosed by $\Gamma$ for all $\varepsilon > 0$.

For $\lambda \notin \text{spec}(D_{M,\varepsilon,4})$, we consider the resolvent

$$R_\epsilon(\lambda) = \frac{1 - P_X}{\lambda - \varepsilon^{-1}D_{M,\varepsilon,4}}.$$

We regard the family of Schatten norms on operators acting on $L^2(E)$, given by

$$\|A\|_p = \text{tr}((A^*A)^{p/2})^{1/p}$$

for $1 \leq p < \infty$, and let $\|A\|_\infty$ denote the operator norm.

Proposition 2.7. There exist constants $C, \varepsilon_0 > 0$ such that for all $p > \dim M$, all $\varepsilon$ in $(0, \varepsilon_0)$ and all $\lambda \in \Gamma$, one has

$$\|R_\varepsilon(\lambda)\|_\infty \leq C,$$

$$\|R_\varepsilon(\lambda)\|_\infty \leq C\varepsilon|\lambda|,$$

$$\|R_\varepsilon(\lambda)\|_p \leq C|\lambda|.$$
Proof. For $\sigma \in \text{im}(1-P_X)$, we have

$$
\| (i-\varepsilon^{-1} D_{M,\varepsilon,4}) \sigma \|_{L^2}^2 = \langle (1+\varepsilon^{-2} D_X^2 + \varepsilon^{-1}[D_X, D_{B,\varepsilon,4}] + D_{B,\varepsilon,4}^2) \sigma, \sigma \rangle.
$$

(2.8)

The operator $D_{B,\varepsilon,4}^2$ is selfadjoint with nonnegative spectrum.

The operator $D_{X}^2 |_{\text{im}(1-P_X)}$ is a fibrewise differential operator of order 2, hence its spectrum is contained in $[\lambda_0, \infty)$ for some $\lambda_0 > 0$. Let $\Delta_X$ denote the fibrewise connection Laplacian acting on $E \to M$, and let $\mathcal{R}^X$ denote the curvature term in the classical Bochner–Lichnerowicz–Weitzenböck formula for $D_X$. Write $A \geq B$ if $A - B$ is a nonnegative selfadjoint operator. Because $D_X^2 \geq \lambda_0^2 > 0$, we find a parameter $s > 0$ such that

$$
D_X^2 - s \Delta_X = (1-s)D_X^2 + s \left( \frac{\kappa X}{4} + \mathcal{R}^X \right)
$$

$$
\geq \frac{1-s}{2} D_X^2 + \frac{(1-s)^2}{2} \kappa_0^2 - s \left\| \frac{\kappa X}{4} + \mathcal{R}^X \right\|_{\infty}
$$

(2.9)

By Lemma 2.1, the anticommutator

$$
[D_X, D_{B,\varepsilon,4}] = (1-P_X)(D_X D_{B,\varepsilon} + D_{B,\varepsilon} D_X)(1-P_X)
$$

is the projection of a fibrewise differential operator of order 1. Write

$$
[D_X, D_{B,\varepsilon}] = \sum_v a_v \nabla V_v + b,
$$

where the $V_v$ are vertical vector fields, and $b$ and the $a_v$ are endomorphisms of $E \to M$ depending on $\varepsilon$. Note that $V_v$, $a_v$ and $b$ are uniformly bounded as $\varepsilon \to 0$. Because $[D_X, D_{B,\varepsilon}]$ is selfadjoint,

$$
\sum_v a_v \nabla V_v = \left( \sum_v a_v \nabla V_v \right)^* = - \sum_v (a_v^* \nabla V_v + [\nabla V_v, a_v^*] + (\text{div} V_v) a_v^*).
$$

Consider the nonnegative generalised fibrewise Laplace operator

$$
0 \leq s \left( \varepsilon^{-1} \nabla + \frac{1}{2s} \sum_v (V_v, \cdot) a_v^* \right) \left( \varepsilon^{-1} \nabla + \frac{1}{2s} \sum_v (V_v, \cdot) a_v^* \right)
$$

$$
= s \varepsilon^{-2} \Delta_X + \frac{1}{4s} \sum_{\mu, v} (V_{\mu}, V_v) a_\mu a_v^* + \frac{1}{2s} \sum_v \left( (a_v^* - a_v) \nabla V_v + (\text{div} V_v) a_v^* + [\nabla V_v, a_v^*] \right)
$$

$$
= s \varepsilon^{-2} \Delta_X + \varepsilon^{-1} [D_X, D_{B,\varepsilon}] - b + \frac{1}{4s} \sum_{\mu, v} (V_{\mu}, V_v) a_\mu a_v^*.
$$
Because $b$ acts as a fibrewise endomorphism on $E \to M$, we conclude that

$$s\varepsilon^{-2}\Lambda_X + \varepsilon^{-1}[D_X, D_{B,\varepsilon}] \geq -\varepsilon^{-1}C.$$  \hfill (2.10)

A similar conclusion still holds if we replace $D_{B,\varepsilon}$ by $D_{B,\varepsilon,4}$ because

$$[D_X, D_{B,\varepsilon}] - [D_X, D_{B,\varepsilon,4}] = D_X D_{B,\varepsilon,3} + D_{B,\varepsilon,2} D_X$$

is a fibrewise smoothing operator of finite rank by Proposition 2.6(2).

If we put (2.8)–(2.10) together, we see that

$$1 + \varepsilon^{-2}D_M^{3,\varepsilon,4} \geq 1 + \frac{1-s}{2\varepsilon^2} D_X^2 + \frac{\lambda_0^2}{4\varepsilon^2} - \frac{C}{\varepsilon} + D_{B,\varepsilon,4}^2.$$  \hfill (2.11)

We immediately find that

$$\|R_\varepsilon(i)\|_\infty \leq C \varepsilon.$$  

Hence there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the spectrum of $\varepsilon^{-1}D_M^{3,\varepsilon,4}$ is contained in $\mathbb{R} \setminus \varepsilon^{-1}(-c', c')$ for some constant $c' > 0$. The first estimate (1) follows from our choice of $\Gamma$.

We obtain (2) from (1) and

$$\|R_\varepsilon(\lambda)\| \leq \|R_\varepsilon(i) - R_\varepsilon(i)(\lambda - i)R_\varepsilon(\lambda)\| \leq C \varepsilon(1 + |\lambda - i| C).$$

Moreover, (2.11) implies that there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the operator $1 + (\varepsilon^{-1}D_X + D_{B,\varepsilon,4})^2$ differs from a fixed selfadjoint second order elliptic operator by some selfadjoint operator with nonnegative eigenvalues. By the variational characterisation of eigenvalues and the definition of the $p$-norm, we conclude that

$$\|R_\varepsilon(i)\|_p = \|(i - (\varepsilon^{-1}D_X + D_{B,\varepsilon,4}))^{-1}\|_p \leq C$$

for all $\varepsilon \in (0, \varepsilon_0)$ and all $p > \dim M$. By a similar argument, the proposition follows for all $\lambda \in \Gamma$. \hfill \Box

In particular, the resolvent $R_\varepsilon(\lambda)$ is uniformly bounded and of order $O(\varepsilon|\lambda|)$ for all $\lambda \in \Gamma$ as $\varepsilon \to 0$. We write $R_\varepsilon(\lambda) = O(1, \varepsilon|\lambda|)$. In particular, we may extend this operator by 0 for $\varepsilon = 0$.

2.e. The Schur complement

To compute the full resolvent of $\varepsilon^{-1}D_M$, we consider the Schur complement $M_\varepsilon(\lambda)$ of $\lambda - \varepsilon^{-1}D_M^{3,\varepsilon,4}$ in the matrix representation of Section 2.e. The Schur complement is given by

$$M_\varepsilon(\lambda) = \lambda - D_{B,\varepsilon,1} - D_{B,\varepsilon,2} \circ R_\varepsilon(\lambda) \circ D_{B,\varepsilon,3}.$$
Proposition 2.8. There exists $\varepsilon_0 > 0$ small such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\lambda \in \Gamma$, the operator $M_\varepsilon(\lambda)$ is invertible. Moreover, there exists $C > 0$ such that for all $p > \dim M$,

\[
\|M_\varepsilon(\lambda)^{-1}\|_\infty \leq C, \quad (1)
\]

\[
\|M_\varepsilon(\lambda)^{-1} - (\lambda - D_{\eff}^{\varepsilon})^{-1}\|_\infty \leq C \min(1, \varepsilon |\lambda|), \quad (2)
\]

\[
\|M_\varepsilon(\lambda)^{-1}\|_p \leq C|\lambda|, \quad (3)
\]

\[
\|\lambda - D_{\eff}^{\varepsilon}\|_p \leq C|\lambda|, \quad (4)
\]

Proof. By Propositions 2.6 and 2.7,

\[M_\varepsilon(\lambda) = \lambda - D_{\eff}^{\varepsilon} + O(1, \varepsilon |\lambda|).
\]

As $D_{B,\varepsilon,1} + D_{B,\varepsilon,2} \circ R_\varepsilon(\lambda) \circ D_{B,\varepsilon,3}$ is a selfadjoint operator, its spectrum is contained in $\mathbb{R}$, and $M_\varepsilon(\lambda)$ is invertible with $\|M_\varepsilon(\lambda)\| \leq 1/c$ for all $\lambda \in \Gamma$ with $\text{Im} \lambda = \pm \iota c$.

The remaining $\lambda \in \Gamma$ satisfy $|\lambda| \leq 2\lambda_B$, so the remainder term $M_\varepsilon(\lambda) - \lambda + D_{\eff}^{\varepsilon}$ is a bounded endomorphism of $H$ with operator norm uniformly of order $O(\varepsilon)$. Then in particular, the series

\[M_\varepsilon(\lambda)^{-1} = \frac{1}{\lambda - D_{\eff}^{\varepsilon}} \sum_{k=0}^\infty \left((\lambda - D_{\eff}^{\varepsilon}) - M_\varepsilon(\lambda)\right) \frac{1}{\lambda - D_{\eff}^{\varepsilon}}
\]

converges if $\varepsilon > 0$ is small enough. This proves invertibility of $M_\varepsilon(\lambda)$. Together with the above, we obtain (1).

We deduce (2) from (1) and our choice of $\Gamma$ in Section 2.d because

\[M_\varepsilon(\lambda)^{-1} - (\lambda - D_{\eff}^{\varepsilon})^{-1} = (\lambda - D_{\eff}^{\varepsilon})^{-1}((\lambda - D_{\eff}^{\varepsilon}) - M_\varepsilon(\lambda))M_\varepsilon(\lambda)^{-1} = O(1, \varepsilon |\lambda|).
\]

For (3), we use that $\|(i - D_{\eff}^{\varepsilon})^{-1}\|_p \leq C$. Moreover

\[\|M_\varepsilon(\lambda)^{-1}\|_p = \|(i - D_{\eff}^{\varepsilon})^{-1} - (i - D_{\eff}^{\varepsilon})^{-1}(M_\varepsilon(\lambda) - (i - D_{\eff}^{\varepsilon}))M_\varepsilon(\lambda)^{-1}\|_p \leq \|(i - D_{\eff}^{\varepsilon})^{-1}\|_p + \|(i - D_{\eff}^{\varepsilon})^{-1}\|_p \|M_\varepsilon(\lambda) - (i - D_{\eff}^{\varepsilon})\|_\infty \|M_\varepsilon(\lambda)^{-1}\|_\infty \leq C(1 + (|\lambda - i| + O(1, \varepsilon |\lambda|)))C).
\]

The last estimate (4) is similar. \(\square\)

We can now write the resolvent of $\varepsilon^{-1}D_{M,\varepsilon}$ as

\[
\frac{1}{\lambda - \varepsilon^{-1}D_{M,\varepsilon}} = \begin{pmatrix}
M_\varepsilon(\lambda)^{-1} & M_\varepsilon(\lambda)^{-1} D_{B,\varepsilon,2} R_\varepsilon(\lambda) \\
R_\varepsilon(\lambda) + D_{B,\varepsilon,2} & R_\varepsilon(\lambda) D_{B,\varepsilon,3} M_\varepsilon(\lambda)^{-1} D_{B,\varepsilon,2} R_\varepsilon(\lambda)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{\lambda - D_{\eff}^{\varepsilon}} & 0 \\
0 & R_\varepsilon(\lambda)
\end{pmatrix} + O(1, \varepsilon |\lambda|).
\]

The remainder terms consist of the resolvent of $D_{\eff}^{\varepsilon}$ and one or more of the following finite-rank endomorphisms of $p_*E$:

\[((\lambda - D_{\eff}^{\varepsilon}) - M_\varepsilon(\lambda)) \frac{1}{\lambda - D_{\eff}^{\varepsilon}}, \quad D_{B,\varepsilon,2} R_\varepsilon(\lambda), \quad R_\varepsilon(\lambda) D_{B,\varepsilon,3},
\]

the behaviour of which is described in Propositions 2.6–2.8.
We summarise the results of this section.

**Proposition 2.9.** There exist constants $C, \varepsilon_0 > 0$ such that for all $p > \dim M$, all $\varepsilon \in (0, \varepsilon_0)$, and all $\lambda \in \Gamma$, one has

$$\|\lambda - \varepsilon^{-1} D_{M,\varepsilon}^{-1}\|_\infty \leq C,$$

$$\|\lambda - \varepsilon^{-1} D_{M,\varepsilon}^{-1}\|_p \leq C|\lambda|,$$

$$\|\lambda - \varepsilon^{-1} D_{M,\varepsilon}^{-1} - (\lambda - D_{B}^{\text{eff}})^{-1}\|_\infty \leq C|\lambda|.$$  

In particular, the resolvent of $\varepsilon^{-1} D_{M,\varepsilon}$ converges to the resolvent of the effective horizontal operator $D_{B}^{\text{eff}}$ in a certain precise sense.

### 2.f. Long time convergence

Define a spectral projection $P_\varepsilon$ on $\Gamma(p^*, E)$ by

$$P_\varepsilon = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{dz}{z - \varepsilon^{-1} D_{M,\varepsilon}}.$$ 

Then $P_\varepsilon$ obviously commutes with $D_{M,\varepsilon}$. By Proposition 2.9(3) and our choice of $c$ and $\Gamma_0$, we find that $P_0 = \lim_{\varepsilon \to 0} P_\varepsilon$ is the projection onto the kernel of the effective horizontal operator $D_{B}^{\text{eff}}$. In particular, $P_\varepsilon$ is of constant finite dimension for all $\varepsilon > 0$ sufficiently small.

**Proposition 2.10.** There exists $\alpha > 0$ such that

$$\lim_{\varepsilon \to 0} \int_0^\infty \frac{1}{\sqrt{\pi t}} \tr((1 - P_\varepsilon) \circ (D_{M,\varepsilon} e^{-t D_{M,\varepsilon}^2}) \circ (1 - P_\varepsilon)) dt = \eta(D_{B}^{\text{eff}}).$$

**Proof.** By Proposition 2.4, we may write

$$\eta(D_{B}^{\text{eff}}) = \int_0^\infty \frac{1}{\sqrt{\pi t}} \tr((1 - P_0) \circ D_{B}^{\text{eff}} e^{-t D_{B}^{\text{eff}}^2}) \circ (1 - P_0)) dt,$$

because $P_0$ projects onto the kernel of $D_{B}^{\text{eff}}$.

We rewrite the integral on the left hand side in the proposition as

$$\int_0^\infty \frac{1}{\sqrt{\pi t}} \tr((1 - P_\varepsilon) \circ (D_{M,\varepsilon} e^{-t D_{M,\varepsilon}^2}) \circ (1 - P_\varepsilon)) dt$$

$$= \int_0^\infty \frac{1}{\sqrt{\pi t}} \tr((1 - P_\varepsilon) \circ (\varepsilon^{-1} D_{M,\varepsilon} e^{-t \varepsilon^{-2} D_{M,\varepsilon}^2}) \circ (1 - P_\varepsilon)) dt.$$ 

Using dominated convergence, we will show that this integral converges to $\eta(D_{B}^{\text{eff}})$ as $\varepsilon \to 0$. 

For $t > 0$ and each integer $k \geq 0$, we define two holomorphic functions $F_{k,t}^+, F_{k,t}^- : \mathbb{C} \to \mathbb{C}$ with
\[
\frac{d^k}{dz^k} F_{k,t}^\pm(z) = z e^{-t|z|^2}
\]
and
\[
\lim_{z \to \pm \infty} F_{k,t}^\pm(z) = 0.
\]
Then obviously
\[
F_{k,t}^\pm(z) = i^{-(k+1)/2} F_{k,1}^\pm(\sqrt{t} z).
\]
(2.13)

By holomorphic functional calculus,
\[
(1 - P_\epsilon) \circ (\epsilon^{-1} D_{M,\epsilon} e^{-t e^{-2D_{M,\epsilon}^2}}) \circ (1 - P_\epsilon) = \frac{1}{2\pi i} \int_{\Gamma_+} \frac{z e^{-t|z|^2}}{z - \epsilon^{-1} D_{M,\epsilon}} \, dz
\]
\[
= \frac{1}{2\pi i k!} \int_{\Gamma_\pm} F_{k,t}^+(z)(z - \epsilon^{-1} D_{M,\epsilon})^{-k-1} \, dz + \frac{1}{2\pi i k!} \int_{\Gamma_-} F_{k,t}^-(z)(z - \epsilon^{-1} D_{M,\epsilon})^{-k-1} \, dz.
\]
(2.14)

A similar expression holds for
\[
D_{B,\epsilon}^{\text{eff}} e^{-t(D_{B,\epsilon}^{\text{eff}})^2} = (1 - P_0) \circ (D_{B}^{\text{eff}} e^{-t(D_{B}^{\text{eff}})^2}) \circ (1 - P_0).
\]

By the Hölder inequality, $\|X^p\|_1 \leq \|X\|_p^p$. We choose $k > \dim M + 1$. Let $\mu$ denote the arc length measure on $\Gamma$. By Proposition 2.9(2) & (3), there exist constants $C$ varying from line to line such that
\[
\left\| \frac{1}{2\pi i k!} \int_{\Gamma_+} F_{k,t}^\pm(z)((z - \epsilon^{-1} D_{M,\epsilon})^{-k-1} - (z - D_{B}^{\text{eff}})^{-k-1}) \, dz \right\|_1
\]
\[
\leq C \int_{\Gamma_+} |F_{k,t}^\pm(z)| \sum_{j=0}^k \| (z - \epsilon^{-1} D_{M,\epsilon})^{-1} \|_k^j
\]
\[
\cdot \| (z - \epsilon^{-1} D_{M,\epsilon})^{-1} - (z - D_{B}^{\text{eff}})^{-1} \|_{\infty} \| (z - D_{B}^{\text{eff}})^{-1} \|_k^{j-1} \, d\mu(z)
\]
\[
\leq C \epsilon \int_{\Gamma_+} |F_{k,t}^\pm(z)| |z|^{k+1} \, d\mu(z).
\]
(2.15)

This clearly implies
\[
\lim_{\epsilon \to 0} \text{tr} ((1 - P_\epsilon) \circ (\epsilon^{-1} D_{M,\epsilon} e^{-t e^{-2D_{M,\epsilon}^2}}) \circ (1 - P_\epsilon)) = \text{tr} (D_{B}^{\text{eff}} e^{-t(D_{B}^{\text{eff}})^2}).
\]
(2.16)

Using (2.13) and (2.15), we estimate
\[
\left\| (1 - P_\epsilon) \circ (\epsilon^{-1} D_{M,\epsilon} e^{-t e^{-2D_{M,\epsilon}^2}}) \circ (1 - P_\epsilon) - (D_{B}^{\text{eff}} e^{-t(D_{B}^{\text{eff}})^2}) \right\|_1
\]
\[
\leq C \epsilon \int_{\Gamma_+} |F_{k,t}^+(z)| |z|^{k+1} \, d\mu(z)
\]
\[
\leq C \epsilon t^{-k-1} \int_{\Gamma_+} |F_{k,t}^+(\sqrt{t} z)| |(\sqrt{t} z)^{k+1} | \, d\mu(z)
\]
\[
\leq C \epsilon t^{-k-3/2} \int_{\sqrt{t} \Gamma_+} |F_{k,t}^+(z)| |z|^{k+1} \, d\mu(z) \leq C \epsilon t^{-k-3/2} e^{-ct}.
\]
(2.17)
Choose $0 < \alpha < 1/(k+2)$. For $\varepsilon^\alpha \leq t$, (2.17) implies
\[
\left\| (1 - P_\varepsilon) \circ (\varepsilon^{-1} D_{M,\varepsilon} e^{-t\varepsilon^{-2} D_{M,\varepsilon}^2}) \circ (1 - P_\varepsilon) - (D^\text{eff}_B e^{-t(D^\text{eff}_B)^2}) \right\|_1 
\leq C t^{1/\alpha - 1/2} e^{-c t}.
\]

Because $t$ occurs with positive exponent in the last line, the integral of the right hand side above over $(0, \infty)$ converges, and we may apply dominated convergence and (2.16) to complete the proof. 

2.g. The very small eigenvalues

We now want to estimate the contribution of the finite-dimensional vector space $\text{im } P_\varepsilon$. The operator $P_\varepsilon \circ \varepsilon^{-1} D_{M,\varepsilon} \circ P_\varepsilon$ depends holomorphically on $\varepsilon$, so its eigenvalues are given by analytic functions $\lambda_\nu$ in $\varepsilon$. In particular, we may choose $\varepsilon_0$ in Section 2.c such that for all $\varepsilon \in (0, \varepsilon_0]$, we have $\lambda_\nu(\varepsilon) = O(\varepsilon)$, and by the above, the sign of $\lambda_\nu(\varepsilon)$ does not change on $(0, \varepsilon_0]$. 

**Proposition 2.11.** For $0 < \varepsilon < \varepsilon_0$, we have
\[
\lim_{\varepsilon \to 0} \int_{\varepsilon^\alpha - 2}^{\varepsilon^\alpha} \frac{1}{\sqrt{\pi t}} \text{tr}(P_\varepsilon \circ (D_{M,\varepsilon} e^{-tD_{M,\varepsilon}^2}) \circ P_\varepsilon) dt \equiv \sum_{\nu=1}^{\dim \ker D^\text{eff}_B} \text{sign}(\lambda_\nu(\varepsilon)).
\]

**Proof.** We have
\[
\int_{\varepsilon^\alpha - 2}^{\varepsilon^\alpha} \frac{1}{\sqrt{\pi t}} \text{tr}(P_\varepsilon \circ (D_{M,\varepsilon} e^{-tD_{M,\varepsilon}^2}) \circ P_\varepsilon) dt 
= \int_{\varepsilon^\alpha}^{\varepsilon^\alpha} \frac{1}{\sqrt{\pi t}} \text{tr}(P_\varepsilon \circ (\varepsilon^{-1} D_{M,\varepsilon} e^{-t\varepsilon^{-2} D_{M,\varepsilon}^2}) \circ P_\varepsilon) dt 
= \sum_{\nu=1}^{\dim \ker D^\text{eff}_B} \int_{\varepsilon^\alpha}^{\varepsilon^\alpha} \frac{1}{\sqrt{\pi t}} \varepsilon^{-t\lambda_\nu(\varepsilon)} dt 
= \sum_{\nu=1}^{\dim \ker D^\text{eff}_B} \text{sign}(\lambda_\nu(\varepsilon)) + O(\varepsilon^{3/2}).
\]

2.h. Short time convergence

Let $\alpha > 0$ denote the constant introduced in Proposition 2.10 and consider
\[
\int_0^{\varepsilon^\alpha - 2} \frac{1}{\sqrt{\pi t}} \text{tr}(D_{M,\varepsilon} e^{-tD_{M,\varepsilon}^2}) dt.
\]

We treat the limit of this integral as $\varepsilon \to 0$ as in [3] and [9]. Over the singular strata of $B$, we get additional contributions involving equivariant $\eta$-forms (see Definition 1.7 of the orbifold $\eta$-forms).
Proposition 2.12. For $\alpha > 0$ sufficiently small, we have
\[
\lim_{\varepsilon \to 0} \int_0^{\varepsilon^{-2}} \frac{1}{\sqrt{4\pi t}} \text{tr}(D_{M,\varepsilon} e^{-t D_{M,\varepsilon}^2}) \, dt = \int_{\Lambda B} \hat{A}_{\Lambda B}(TB, \nabla^TB) 2\eta_{\Lambda B}(h).
\]

Proof. We introduce an exterior variable $z$ that anticommutes with the Clifford multiplication $c$ and is parallel with respect to $\nabla^{E,\varepsilon}$ for all $\varepsilon$. In analogy with (2.5), consider the connection
\[
\nabla^{E,\varepsilon,z} = \nabla^{E,\varepsilon} - z c(\cdots).
\]
We will use the $g^{TM}$-orthonormal frame $e^I_\varepsilon$ of (1.8). Then as in (2.6), the Bochner–Lichnerowicz–Weitzenböck formula implies
\[
D^2_{M,\varepsilon} + 2zD_{M,\varepsilon} = \nabla^{E,\varepsilon,z} \nabla^{E,\varepsilon,z} + \frac{\kappa}{4} + \frac{1}{4} \sum_{I,J} c_I c_J F^{E/S}_{e^I_\varepsilon e^J_\varepsilon}.
\]

Define $\text{tr}_z$ as in the proof of Proposition 2.4. Then as in (2.7),
\[
\text{tr}(D_{M,\varepsilon} e^{-t D_{M,\varepsilon}^2}) = \frac{1}{t} \text{tr}_z\left( e^{-t(D^2_{M,\varepsilon}-z D_{M,\varepsilon})} \right).
\]

From now on, we assume that $t \leq \varepsilon^{\alpha-2}$ for some small $\alpha > 0$. We fix $q \in B$ and choose an orbifold chart $\psi: \rho(0) \setminus V \to U \subset B$ with $q = \psi(0)$ and a local trivialisation $\hat{\psi}: \Gamma \backslash (V \times X) \to p^{-1}(U)$ as in Definition 1.2. We assume that $\hat{\psi}$ defines geodesic coordinates, and that $\hat{\psi}$ is the trivialisation by horizontal lifts of radial geodesics. By parallel transport along these geodesics with respect to $\nabla^{E,\varepsilon}$, we also identify the pullback of $E|_p^{-1}(U)$ to $V \times X$ with $E|_X \times V$.

As explained in [9, Section 3.1], to compute the $z$-trace of the heat kernel over $q$, we may assume that $V = \mathbb{R}^{m-n}$, and that outside a suitably large compact subset, the metric on $V$ is flat and the geometry of the fibration is of product type. It is possible to perform all these modifications in a $\Gamma$-invariant way.

Let $v$ denote the $V$-coordinates of a point in $V \times X$. As in [3, (4.58), (4.64)], we consider the operator
\[
H_{e,t} = \left( 1 + \frac{zc(v)}{2\varepsilon \sqrt{t}} \right) \left( t D^2_{M,\varepsilon} + 2z \sqrt{t} D_{M,\varepsilon} \right) \left( 1 - \frac{zc(v)}{2\varepsilon \sqrt{t}} \right).
\]
In the trivialisations above, let $\tilde{k}_{e,t}((v, x), (v', x')): E_{x'} \to E_x$ denote the heat kernel of the operator $e^{-H_{e,t}}$ on $V \times X$. The corresponding heat kernel $k_{e,t}$ on $\Gamma \backslash (V \times X)$ then lifts to $k_{e,t}([v, x], [v', x']) = \sum_{\gamma \in \Gamma} \tilde{k}_{e,t}((v, x), \gamma(v', x')) \circ \gamma: E_{x'} \to E_x$.

Thus, we have
\[
\text{tr}_z(k_{e,t}([v, x], [v, x])) = \sum_{\gamma \in \Gamma} \text{tr}_z(\tilde{k}_{e,t}((v, x), \gamma(v, x)) \circ \gamma).
\]
We will consider the contribution of each \( \gamma \in \Gamma \) over \( V \) to the overall trace of \( e^{-t(D_{M,\varepsilon}^2 + \varepsilon D_{M,\varepsilon})} \) separately in the limit \( \varepsilon \to 0 \). Moreover,

\[
\int_{\Gamma \backslash (V \times X)} \text{tr}_z(k_{\varepsilon,t}((v, x), [v, x])) d(v, x) = \frac{1}{\#\Gamma} \int_{V \times X} \sum_{\gamma \in \Gamma} \text{tr}_z(k_{\varepsilon,t}((v, x), \gamma(v, x)) \circ \gamma) d(v, x)
\]

(2.19)

because each point \([v, x] \in \Gamma \backslash (V \times X)\) has \( \#\Gamma \) different preimages in \( V \times X \).

For a fixed \( \gamma \in \Gamma \), let \( V_\gamma \subset V \) denote the fixpoint set of \( \gamma \), which is a linear subspace of \( V \). Let \( N_\gamma \) denote its orthogonal complement. Because we have assumed that \( B \) is orientable, \( \dim N_\gamma \) is even. Put

\[ m_\gamma = m - \dim N_\gamma. \]

The action of \( \gamma \) on \( E|_X \) can be decomposed as

\[ \gamma = \tilde{\gamma}^{E/SB} \circ \tilde{\gamma}^{SB} \]

(2.20)
such that \( \tilde{\gamma}^{SB} \) is an element in the Clifford algebra of \( N_\gamma \) and \( \gamma^{E/SB} \) commutes with Clifford multiplication by horizontal vectors, and this decomposition is unique up to sign.

As \( \varepsilon \to 0 \), we will rescale \( v \in V \) by a factor \( \varepsilon \sqrt{t} \). We will apply Getzler rescaling by \( \varepsilon \sqrt{t} \) only to Clifford multiplication with elements of \( V_\gamma \), whereas Clifford multiplication with elements of \( N_\gamma \) and \( TX \) will not be rescaled. Let us denote the complete rescaling by \( G_{\gamma,\varepsilon} \). In particular, the action of \( \gamma \) commutes with \( G_{\gamma,\varepsilon} \).

We choose the basis in Section 2.a such that \( f_{n+1}, \ldots, f_{m_\gamma} \) are tangent to \( V_\gamma \). Let \( \varepsilon_{\alpha} \) denote exterior multiplication with \( dv_{\alpha} \). For \( I \in \{1, \ldots, m_\gamma\} \), define

\[ \mu_I = \begin{cases} \epsilon_I & \text{if } 1 \leq I < n \text{ or } m_\gamma < I \leq m, \\ t^{-1/2} \epsilon_I & \text{if } n < I \leq m. \end{cases} \]

Bismut’s Levi-Civita superconnection can be defined as the operator

\[ \hat{A}_t = \sqrt{t} D_X + \nabla_{E,0} - \frac{\sqrt{t}}{4} \sum_{i\alpha\beta} t_{i\alpha\beta} \mu_I \mu_I \mu_I. \]

Then as in [3, (4.68), (4.70)], the limit of the rescaled operator \( H_{\gamma, t} \) becomes

\[
\lim_{\varepsilon \to 0} G_{\gamma,\varepsilon}(H_{\gamma, t}) = -t \left( \varepsilon_{\gamma} + \frac{1}{4} \sum_{J,K=1}^{m_\gamma} s_{iJK} \mu_I \mu_J - t^{-1/2} \varepsilon_I \right)^2 \]

\[
- \left( \frac{\partial}{\partial a} + \frac{1}{8} (R^B|_{V_\gamma} e_{\alpha}, v) \right)^2 + \sum_{i=1}^{m_\gamma} t R_{i\alpha\beta}^{E/SB} \mu_I \mu_I + \frac{t}{4} \kappa_X.
\]

(2.21)
Both operators on the right hand side have coefficients in $\Lambda^\bullet(V^*)$. The operator $\tilde{h}_t^2 + t\frac{dh_t}{dt}$ acts on $\Gamma(E \to X)$ and commutes with Clifford multiplication by horizontal vectors, while $(\tilde{h}_t^2 + \frac{1}{8}(R^B|_V, e_\alpha, v))^2$ acts on $\Omega^\bullet(V)$.

Let $SB$ be a local spinor bundle on $V$. Then there exists a fibrewise Dirac bundle $W \to M$ as in (1.11). We continue as in [3], using the heat kernel proof of the equivariant index theorem in order to conclude that on $V \times X$,

$$\lim_{\varepsilon \to 0} \int_{V \times X} \text{tr}(\tilde{k}_\varepsilon, ((v, x), (v, x)) \circ \gamma) \, d(v, x)$$

$$= \int_V \text{tr}_{SB}(k_V(v, y) \circ \tilde{\gamma}^{SB})(2\pi i)^{-N^{V^*/2}} \text{tr}_{p_*W} \left( 2t \frac{dh_t}{dt} e^{\tilde{h}_t^2 \tilde{\gamma}^{E/SB}} \right) dV$$

$$= \int_{V^*} \tilde{A}_{SB}(TV, \nabla^{TV})(2\pi i)^{-N^{V^*/2}} \text{tr}_{p_*W} \left( 2t \frac{dh_t}{dt} e^{\tilde{h}_t^2 \tilde{\gamma}^{E/SB}} \right).$$

From (2.19) and the above, we obtain

$$\lim_{\varepsilon \to 0} \int_{\Gamma(V \times X)} \text{tr}(k_\varepsilon, ([v], [v])) \, d(v, x)$$

$$= \frac{1}{\# \gamma \in \Gamma} \sum_{\gamma \in \Gamma} \int_{V^*} \tilde{A}_{SB}(TV, \nabla^{TV})(2\pi i)^{-N^{V^*/2}} \text{tr}_{p_*W} \left( 2t \frac{dh_t}{dt} e^{\tilde{h}_t^2 \tilde{\gamma}^{E/SB}} \right)$$

$$= \sum_{\gamma \in \Gamma} \int_{\text{C}^2(\gamma \setminus V^*)} \psi_{(v, \gamma)} \tilde{A}_{SB}(TB, \nabla^{TB})(2\pi i)^{-N^{V^*/2}} \text{tr}_{p_*W} \left( 2t \frac{dh_t}{dt} e^{\tilde{h}_t^2 \tilde{\gamma}^{E/SB}} \right)$$

in analogy with the index computations in [20]. By (2.18) and the above, we have the global formula

$$\lim_{\varepsilon \to 0} \text{tr}\left( \sqrt{t} D_{M,\varepsilon} e^{-tD_{M,\varepsilon}} \right)$$

$$= \int_{A_\Lambda} \hat{A}_{\Lambda,B}(TB, \nabla^{TB})(2\pi i)^{-N^{V^*/2}} \text{tr}_{p_*W} \left( 2t \frac{dh_t}{dt} e^{\tilde{h}_t^2 \tilde{\gamma}^{E/SB}} \right).$$

By [9, Theorem 3.1], we have uniform convergence as $\varepsilon \to 0$. By (1.12) and Definition 1.7,

$$\lim_{\varepsilon \to 0} \int_0^{\varepsilon^2} \frac{1}{\sqrt{t^2}} \text{tr}(D_{M,\varepsilon} e^{-tD_{M,\varepsilon}}) \, dt = \int_{A_\Lambda} \hat{A}_{\Lambda,B}(TB, \nabla^{TB})(2\eta_{\Lambda,B}(h)). \quad \Box$$

Remark 2.13. We replace the vector bundle $E$ by $E \otimes p^*W$, where $W \to B$ is a vector orbibundle. We also assume that the twist connection $\nabla(E \otimes p^*W)/S, 0$ splits as the tensor
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product connection of $\nabla_{E/S,0}$ and $p^*\nabla_W$ in the limit $\varepsilon \to 0$. A relevant special case is the case of the signature operator on $M$, where the (local) spinor bundle of $B$ plays the role of $W$. In this case, equation (2.21) becomes

$$\lim_{\varepsilon \to 0} G_{\gamma,\varepsilon}(H_{\varepsilon,t}) = \left( A^2_t + tz \frac{dA_t}{dt} \right)^2 - \left( \frac{\partial}{\partial a} + \frac{1}{8} (R^B|_{\nabla_V e_\alpha, v}) \right)^2 + p^*R^W.$$ 

This implies that now,

$$\lim_{\varepsilon \to 0} \int_0^{\varepsilon^{-2}} \frac{1}{\sqrt{\pi t}} \tr(DM_\varepsilon e^{-1}D^2_{M_\varepsilon} \gamma) \, dt = \int_{\Lambda_B} \hat{A}_{\Lambda B}(TB, \nabla^TB) \, 2\eta_{\Lambda B}(\hat{a}) \, ch_\Lambda(W, \nabla^W).$$

3. The spaces $P_k$ and $M_{(p_-,q_-),(p_+,q_+)}$

We consider the family $M_{(p_-,q_-),(p_+,q_+)}$ of manifolds with a cohomogeneity one action of $G = \Sp(1) \times \Sp(1)$ that are described in [17, Chapter 13]. This family contains the spaces $P_k = M_{(1,1),(2k-1,2k+1)}$ as well as the Berger space $\SO(5)/\SO(3) = M_{(3,1),(1,3)}$. The manifolds $M_{(p_-,q_-),(p_+,q_+)}$ are two-connected with finite cyclic third homotopy group, so by [6, Theorem A] and [8, Theorem 2.2], it suffices to compute the Eells–Kuiper invariants and the modified Kreck-Stolz invariants for quaternionic line bundles of [8, Definition 1.4] to determine the diffeomorphism type.

3.a. Construction as manifolds of cohomogeneity one

Let $(p_+,q_+)$ and $(p_-,q_-)$ be two pairs of relatively prime positive odd integers. We consider the subgroup

$$H = \{ \pm(1,1), \pm(i, (-1)^{(q_-+p_-)2/2} j), \pm(j, (-1)^{(q_+-p_+)2/2} j), \pm(k, (-1)^{(q_+q_-+p_-p+)2/2} k) \} \subset G = \Sp(1) \times \Sp(1).$$

which is isomorphic (in fact conjugate) to the diagonal subgroup $\Delta Q$, with

$$Q = \{ \pm 1, \pm i, \pm j, \pm k \} \subset \Sp(1).$$

If $a \in S^2 \subset \HH$ is an imaginary unit quaternion and $p, q$ are relatively prime odd integers as above, we consider the subgroup

$$C^{a}_{(p,q)} = \{ e^{ap\theta}, e^{aq\theta} \mid \theta \in \mathbb{R} \} \subset G = \Sp(1) \times \Sp(1),$$

which is isomorphic to $S^1$. For an odd integer $2l + 1$, we have $e^{a(2l+1)\pi/2} = (-1)^l a$. This implies that

$$\{ \pm(a, (-1)^{(p-q)/2}a), \pm(1, 1) \} \subset C^{a}_{(p,q)}.$$
We put
\[ K_+ = C_i(p_-,q_-) \cdot H \quad \text{and} \quad K_- = C_j(p_+,q_+) \cdot H \subset G. \tag{3.2} \]
Then in particular \( H = K_- \cap K_+ \), and we have isomorphisms
\[ K_- = C_i(p_-,q_-) \cup (j, (-1)(q_+ - p_+)^2 j) C_i(p_-,q_-) \cong \text{Pin}(2), \]
\[ K_+ = C_j(p_+,q_+) \cup (i, (-1)(q_- - p_-)^2 i) C_j(p_+,q_+) \cong \text{Pin}(2). \]
The actions of \( K_\pm \) on \( S^1 \cong K_+ / H \cong K_- / H \) are \( \mathbb{R} \)-linear.

We now consider the cohomogeneity one manifolds \( M_{(p_-,q_-),(p_+,q_+)} \) with group diagram
\[ G \]
\[ K_- \cong \text{Pin}(2) \cong K_+ \] \( H \tag{3.3} \)
Thus, the generic \( G \)-orbit takes the form \( G / H \cong S^3 \times \mathbb{R}P^2 / (\mathbb{Z}/2\mathbb{Z})^2 \), and the two singular orbits are of the form \( M_\pm = G / K_\pm \). We will study the geometry of \( M_{(p_-,q_-),(p_+,q_+)} \) in Section 4.

**Theorem 3.1** ([17, Theorem 13.1]). The manifolds \( M = M_{(p_-,q_-),(p_+,q_+)} \) are two-connected. If \( p_- q_+ = \pm p_+ q_- \), then \( H^3(M) = H^4(M) = \mathbb{Z} \), otherwise \( H^3(M) = 0 \) and \( H^4(M) = \mathbb{Z}/k\mathbb{Z} \) with \( k = (p_2 q_2^2 - p_2^2 q_2) / 8 \).

3.b. The \( t \)-invariant
In this section, we want to determine the homeomorphism type of the spaces \( P_k \).

In [6], Crowley has constructed a quadratic form \( q_M : H^4(M) \to \mathbb{Q}/\mathbb{Z} \) for all two-connected closed topological seven-manifolds with finite \( H^4(M) \) satisfying
\[ \text{lk}_M(a, b) = q_M(a + b) - q_M(a) - q_M(b), \quad \text{lk}_M \left( a, \frac{p_1}{2}(TM) \right) = q_M(a) - q_M(-a), \]
for all \( a, b \in H^4(M) \), where \( p_1 / 2 \) denotes the natural refinement of the first Pontryagin class \( p_1 \) for spin manifolds. Note that two quadratic forms with the properties above differ by the pairing with an element of \( H_4(M; \mathbb{Z}/2\mathbb{Z}) \). Crowley has then proved that two such manifolds \( M_0, M_1 \) are homeomorphic (in fact almost diffeomorphic) if and only if \( (H^4(M_0), q_{M_0}) \) and \( (H^4(M_1), q_{M_1}) \) are isomorphic.

In analogy with the Kreck–Stolz invariants \( s_2 \) and \( s_3 \) of [21], Crowley and the author have defined an invariant \( t_M(E) \in \mathbb{Q}/\mathbb{Z} \) for a two-connected smooth closed seven-manifold \( M \) and a quaternionic line bundle \( E \to M \), such that
\[ q_M(c_2(E)) = 12 t_M(E). \]
For each cohomology class $a \in H^4(M)$ of such a manifold $M$, there exist quaternionic line bundles $E \to M$ with $c_2(E) = a$. We will thus compute $t_M(E)$ for sufficiently many quaternionic line bundles $E \to M$ in order to determine the diffeomorphism type of the spaces $M = P_k$.

Let us recall the intrinsic definition of $t_M(E)$ in [8]. We assume that $E$ carries a quaternionic Hermitian metric $g^E$ and a quaternionic Hermitian connection $\nabla^E$. Then there is a natural representative $c_2(E, \nabla^E) \in \Omega^4(M)$ of the class $c_2(E)$. If $H^4_{dR}(M) = H^4_{dR}(M) = 0$, there exists a differential form $\hat{c}_2(E, \nabla^E) \in \Omega^2(M)$ such that

$$d\hat{c}_2(E, \nabla^E) = c_2(E, \nabla^E) \in \Omega^4(M),$$

and $\hat{c}_2(E, \nabla^E)$ is unique up to an exact form. Let $D$ and $D^E$ denote the untwisted Dirac operator on $M$ and the Dirac operator twisted with $(E, g^E, \nabla^E)$, and let $h(D) = \dim \ker D$.

**Definition 3.2** ([8, Definition 1.4]). For a quaternionic line bundle $E \to M$ on a compact oriented seven-dimensional spin manifold $M$ with $H^4_{dR}(M) = 0$, put

$$t_M(E) = \frac{\eta + \hat{h}}{4}(D^E_M) - \frac{\eta + \hat{h}}{2}(D_M)$$

$$- \frac{1}{24} \int_M \left( \frac{p_1}{2}(TM, \nabla^TM) + c_2(E, \nabla^E) \right) \hat{c}_2(E, \nabla^E) \in \mathbb{Q}/\mathbb{Z}.$$

**Theorem 3.3.** Assume that $p_-$ and $p_+$ are relatively prime. Then there exists an isomorphism $H^4(M\langle p_-, q_-, (p_+, q_+) \rangle) \cong \mathbb{Z}/k\mathbb{Z}$ and a family of quaternionic line bundles $E_\ell \to M\langle p_-, q_-, (p_+, q_+) \rangle$ with $c_2(E_\ell) = \ell \in \mathbb{Z}/k\mathbb{Z}$ such that

$$t_M(E_\ell) = \ell \frac{p_-^2 - p_+^2 + \ell p_-^2 p_+^2}{24k} + \frac{\ell}{24} \in \mathbb{Q}/\mathbb{Z}.$$ 

This theorem will be proved in Section 4.1.

**Remark 3.4.** More generally, suppose that $a = (p_-^2, p_+^2)$ and $b = (q_-^2, q_+^2)$ are the greatest common divisors. Because $p_-$ and $q_-$ are relatively prime, so are $a$ and $b$. Moreover, clearly $a \mid k$ and $b \mid k$.

The proof of Theorem 3.3 gives a formula for $q_M(\ell)$ if $a \mid \ell \in H^4(M)$ by identifying the class $\ell$ with a class pulled back from the base of the Seifert fibration $p: M \to B$ considered in Proposition 4.1. Swapping the roles of the $p$s and $q$s gives an analogous formula for $q_M(\ell)$ if $b \mid \ell \in H^4(M)$.

To see that these two formulas determine $q_M$ uniquely, for each $\ell \in H^4(M) \cong \mathbb{Z}/n\mathbb{Z}$, we find $x, y \in \mathbb{Z}$ such that

$$\ell = xa^2 + yb^2.$$ 

Because $q$ refines the linking form, we have

$$q_M(\ell) = q_M(xa^2 + yb^2) = q_M(xa^2) + q_M(yb^2) + a^2b^2 \operatorname{lk}(x, y)$$

$$= q_M(xa^2) + q_M(yb^2) + q_M(ab(x + y)) - q_M(abx) - q_M(aby),$$

and each of the terms on the right hand side is computable. The main difficulty consists in determining the respective classes in the two base orbifolds.
Example 3.5. We consider the special case \( P_k = M_{(1,1),(2k-1,2k+1)} \) and obtain

\[
q_M(\ell) = \ell \frac{4k(1 - k) + \ell(2k - 1)^2}{2k} + \frac{\ell}{2} = \ell(\ell + k) \mod \mathbb{Z}. \tag{3.4}
\]

We compute

\[
\text{lk}(i,j) = q_M(\ell + j) - q_M(\ell) - q_M(j) = \frac{i}{k}, \tag{3.5}
\]

which proves that the linking form on \( H^4(P_k) \) is standard and that the class represented by \( \ell = 1 \) is a generator. We also see that \( \frac{\eta}{2}(TP_k) = 0 \) because

\[
\text{lk}\left(\frac{p_1}{2}(TM), \ell\right) = q_M(-\ell) - q_M(\ell) = 0 \mod \mathbb{Z}. \tag{3.6}
\]

3.c. The Eells–Kuiper invariant

The Eells–Kuiper invariant \( \mu \) has first been defined in [12] for certain manifolds using zero bordisms. It distinguishes all exotic spheres in dimension 7. Crowley [6] has shown that two homeomorphic two-connected closed smooth seven-manifolds with finite \( H^4 \) are diffeomorphic if and only if their Eells–Kuiper invariants agree.

We will use the intrinsic description of \( \mu(M) \) by Donnelly [11] and Kreck and Stolz [21]. Let \( M \) be an oriented spin Riemannian seven-manifold with \( H^3_{\text{dR}}(M) = H^4_{\text{dR}}(M) = 0 \), and let \( D_M \) denote the untwisted Dirac operator on \( M \). Let \( B_M \) denote the odd signature operator, acting on \( \Omega^{\text{even}}M \). Let \( p_1(TM, \nabla^{TM}) \) denote the first Pontryagin form of \( M \). Then there exists a form \( \hat{p}_1(TM, \nabla^{TM}) \in \Omega^3(M) \) such that

\[
d\hat{p}_1(TM, \nabla^{TM}) = p_1(TM, \nabla^{TM}),
\]

and \( \hat{p}_1(TM, \nabla^{TM}) \) is uniquely determined up to an exact form. Following [21], the Eells–Kuiper invariant of \( M \) can be computed as

\[
\mu(M) = \frac{\eta + \mathcal{h}}{2}(D_M) + \frac{\eta}{2^{5/7} \cdot 7}(B_M) - \frac{1}{2^{5/7} \cdot 7} \int_M (p_1 \wedge \hat{p}_1)(TM, \nabla^{TM}) \in \mathbb{Q}/\mathbb{Z}. \tag{3.7}
\]

We will use Theorem 0.1 to compute the \( \eta \)-invariants in (3.7). Again, we make use of the Seifert fibration \( M \to B \) discussed in Proposition 4.1 below. In analogy with the classical Dedekind sums occurring in the study of quadratic forms, we consider a particular family of sums over rational functions in sines and cosines. These sums represent the contribution of the twisted sectors of \( B \) to \( \mu(M_{(p,q),\ell}) \).

Definition 3.6. If \( p, q \in \mathbb{N} \) are odd and relatively prime, define the generalised Dedekind sums

\[
D(p, q) = \sum_{a=1}^{p-1} \left( \frac{14 \cos \frac{4\pi a}{p} + \cos^2 \frac{2\pi a}{p}}{2^4 \cdot 7^2 \sin^2 \frac{4\pi a}{p} \sin^3 \frac{2\pi a}{p}} + q \cos \frac{\pi a}{p} \left( \frac{14 + \cos \frac{4\pi a}{p}}{p^2} \right) \right).
\]

We will give explicit formulas for some of these sums in the next subsection.
We have
\[\mu(M_{(p_- q_-), (p_+ q_+)}) = \frac{\text{sign}(q^2 p^2 - q^2 p^2)}{2^5 \cdot 7} + D(p_-, q_-) - D(p_+, q_+)\]
\[- \frac{(p_+^2 - p_-^2)^2}{2^2 \cdot 7 p_+^2 p_-^2 (q^2 p_+^2 - q^2 p_-^2)} - \frac{2^4 (p_+^2 - p_-^2) + (q^2 p_+^2 - q^2 p_-^2)}{2^8 \cdot 7 p_+^2 p_-^2}.
\]

This theorem will be proved in Section 4.h.

Corollary 3.8. Assume that \(p\) and \(q\) are odd and relatively prime. Then there is a duality of generalised Dedekind sums
\[D(p, q) + D(q, p) = \frac{2^6 + 2^4 (p^2 + q^2) + (p^4 + q^4)}{2^8 \cdot 7 p^2 q^2} + \frac{7}{2^7} \in \mathbb{Z}.
\]

Proof. Swapping the \(ps\) and \(qs\) in both pairs \((p_\pm, q_\pm)\) corresponds to changing the orientation on \(M\), hence the Eells–Kuiper invariant changes its sign. Let \(A(p, q)\) denote the expression in the corollary. Then by Theorem 3.7,
\[A(p_-, q_-) = A(q_+, p_+) = \mu(M_{(p_- q_-), (p_+ q_+)}) + \mu(M_{(q_- p_-), (q_+ p_+)}) \in \mathbb{Z}.
\]

Because we can choose the pairs \((p_-, q_-)\) and \((p_+, q_+)\) independent of each other, subject only to the relation \(p_-/q_- \neq p_+/q_+\), it is enough to check that \(A(1, 1) = 0 \in \mathbb{Z}\). \(\square\)

3.d. Some examples

Some of the manifolds \(M_{(p_- q_-), (p_+ q_+)}\) are diffeomorphic to well-known spaces by Grove, Wilking and Ziller [17]. In this subsection, we make sure that our computations above agree with other computations of the invariants.

Let us denote by \(E_{p,n}\) the unit sphere bundle of a four-dimensional real vector bundle \(V \to S^4\) with Euler class \(n = e(V)\) and half Pontryagin class \(p = \frac{p_1}{2}(V) \in \mathbb{Z} \cong H^4(S^4)\). Such a bundle exists if and only if \(n\) and \(p\) are of the same parity, and is unique up to isomorphism in this case. It is known that \(E_{p,n}(T E_{p,n}) \equiv p / n \mathbb{Z} \cong H^4(E_{p,n}).\) The bundles \(E_{p,n}\) and \(E_{q,n}\) are oriented diffeomorphic, and \(E_{p, n}\) and \(E_{p, -n}\) are orientation reversing diffeomorphic. By [8, Proposition 2.6] and [7, Section 3.1], we know that
\[q_{E_{p,\ell}}(\ell) = \frac{\ell(p + \ell)}{2k} \quad \text{and} \quad \mu(E_{p,k}) = \frac{p^2 - k}{2^5 \cdot 7k}.
\]

Note that Crowley and Escher use the parameters \(n = k\) and \(m = (p - k)/2\).

Example 3.9. If \(p_+ = p_- = 1\), then the base \(B\) is the manifold \(S^4\), represented as the unit sphere in the space of real trace-free symmetric endomorphisms of \(\mathbb{R}^4\) with its natural \(SO(3)\)-action by conjugation. The manifold \(M\) is a principal \(S^3\)-bundle over \(S^3\), and the induced \(\mathbb{R}^4\)-bundle \(V \to B\) has Euler number
\[e(V)[B] = k = \frac{q_+^2}{} \in \mathbb{Z}
\]
by (4.21) below.
Because $M$ is a principal bundle, we also have $p = \frac{p_2}{2} = \pm k$. By [8, Proposition 2.6], the $q$-invariant is given by

$$q_M(\ell) = \frac{\ell(k + \ell)}{2k},$$

which agrees with Theorem 3.3.

The formula for the Eells–Kuiper invariant reduces to

$$\mu(M_{(1,q_-),(1,q_+)}(\ell)) = \text{sign}(q_2^2 - q_1^2) \frac{25}{28 \cdot 7} - \text{sign} e(V)[B] - e(V)[B] \frac{25}{28 \cdot 7},$$

which agrees with the computations by Crowley and Escher [7].

**Example 3.10.** By [17], the space $M_{(3,1),(1,3)}$ is diffeomorphic to the Berger space $B^7 = SO(5)/SO(3)$. Kitchloo, Shankar and the author [15] computed the Eells–Kuiper invariant of this space and obtained

$$\mu(B^7) = -\frac{27}{1120},$$

which agrees with Theorem 3.7.

In [15], we concluded that $M$ with reversed orientation is diffeomorphic to an $S^3$-bundle over $S^4$ with Euler number 10 and half Pontryagin number 8. By Theorem 3.3, we find

$$q_{M_{(3,1),(1,3)}}(\ell) = \frac{\ell(2 + 9\ell)}{20} = \frac{\ell(2 + 9\ell)}{20} - \frac{\ell(8 + \ell)}{20} \mod \mathbb{Z},$$

which agrees with [8, Proposition 2.6] (see (3.8) above).

3.e. The spaces $P_k$

It is shown in [17, Theorem A] that among the various manifolds $M_{(p,-q_-),(p,+q_+)}$, only the Berger space $M_{(3,1),(1,3)}$ and the spaces

$$P_k = M_{(1,1),(2k-1,2k+1)}$$

can carry a cohomogeneity one metric of positive sectional curvature. So far, such metrics have been found on $P_1 \cong S^7$, the manifold $P_2$, and the Berger space. In this section, we determine the diffeomorphism type of the $P_k$. In particular, we prove Theorems 0.2, 0.3 and Corollary 0.4.

We start by evaluating the Dedekind sums of Definition 3.6. For $q = p + 2$, we have $\cos \frac{2\pi a}{p} = (-1)^a \cos \frac{2\pi a}{p}$ and similarly $\sin \frac{q\pi a}{p} = (-1)^a \sin \frac{2\pi a}{p}$. Hence these sums
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simplify as follows:

\[
D(p, p + 2) = \sum_{a=1}^{p-1} \left( \frac{14 \cos \frac{2\pi a}{p} + \cos^2 \frac{2\pi a}{p}}{2^4 \cdot 7p^2 \sin^2 \frac{4\pi a}{p}} + \frac{(p + 2) \cos \frac{2\pi a}{p} \left( 14 + \cos \frac{4\pi a}{p} \right)}{2^5 \cdot 7p^2 \sin^2 \frac{4\pi a}{p} \sin^2 \frac{2\pi a}{p}} \right)
\]

\[
= \frac{1}{2^4 \cdot 7p^2} \sum_{a=1}^{p-1} \left( \frac{15}{4 \sin \frac{4\pi a}{p}} - \frac{14}{\sin^2 \frac{4\pi a}{p}} \right) + \frac{p + 2}{2^5 \cdot 7p^2} \sum_{a=1}^{p-1} \left( \frac{15}{2 \sin \frac{4\pi a}{p}} - \frac{2}{2 \sin^2 \frac{2\pi a}{p}} \right)
\]

\[
= \frac{15(p + 3)}{2^6 \cdot 7p^2} \sum_{a=1}^{p-1} \frac{1}{\sin \frac{4\pi a}{p}} - \frac{p + 30}{2^5 \cdot 7p^2} \sum_{a=1}^{p-1} \frac{1}{\sin \frac{2\pi a}{p}}.
\]

For the second equation, we have used the double-angle formulae for sine and cosine. For the last equation, we have substituted \( \frac{2\pi a}{p} \) for \( \frac{4\pi a}{p} \) in one of the sums; this is possible because \( p \) is odd.

The following route of computation was suggested by Zagier. Let \( \mu_p \subset \mathbb{C} \) denote the group of \( p \)-th roots of unity, let \( z = e^{\frac{2\pi i a}{p}} \in \mu_p \setminus \{1\} \), and consider

\[
f(z) = \frac{(-4z)^\ell}{(z - 1)^{2\ell}} = \frac{(2i)^{2\ell}}{(\sqrt{z} - 1/\sqrt{z})^{2\ell}} = \frac{1}{\sin^{2\ell} \frac{2\pi a}{p}}.
\]

The rational function

\[
g(z) = \frac{p}{z^p - 1} \cdot \frac{1}{z}
\]

has simple poles at \( \mu_p \cup \{0\} \). The residue at each \( \xi \in \mu_p \) equals 1 because

\[
\text{Res}_{z=\xi} g(z) = \frac{p}{\xi^p - 1} \prod_{\xi' \in \mu_p \setminus \{\xi\}} \frac{1}{\xi - \xi'} = \frac{p}{\xi^p} \prod_{\xi' \in \mu_p \setminus \{1\}} \frac{1}{1 - \xi'}
\]

moreover

\[
\prod_{\xi' \in \mu_p \setminus \{1\}} (1 - \xi') = \frac{z^p - 1}{z - 1} \bigg|_{z=1} = (z^{p-1} + \cdots + z + 1)_{z=1} = p.
\]

Hence

\[
\text{Res}_{z=\xi} (f(z)g(z)) = f(\xi)
\]

for all \( \xi \in \mu_p \setminus \{1\} \).

The function \( f(z)g(z) \) has poles only along \( \mu_p \) for \( \ell \geq 1 \), and it vanishes at \( z = \infty \). Using once more that \( p \) is odd to substitute \( \xi \in \mu_p \setminus \{1\} \) for \( e^{\frac{2\pi i a}{p}} \) in the sum below, we
obtain
\[
\sum_{a=1}^{p-1} \frac{1}{\sin \frac{2\pi a}{p}} = \sum_{\zeta \in \mu_p \setminus \{1\}} \text{Res}_{z=\zeta} \left( \frac{(-4\zeta)^{\ell}}{(\zeta-1)^{2\ell+1}} \cdot \frac{p}{z \cdot p - 1} \cdot \frac{1}{z} \right) = -\text{Res}_{z=1} \left( \frac{(-4)^{\ell}}{(z-1)^{2\ell+1}} \cdot \frac{p^{\ell-1}(z-1)}{z \cdot p - 1} \right).
\]

For \( \ell = 1 \) and 2, we obtain
\[
\sum_{a=1}^{p-1} \frac{1}{\sin \frac{2\pi a}{p}} = \frac{p^2 - 1}{3}, \quad \sum_{a=1}^{p-1} \frac{1}{\sin^2 \frac{2\pi a}{p}} = \frac{p^4 + 10p^2 - 11}{45}.
\]

We combine the above and find that
\[
D(p, p+2) = \frac{15(p+3)}{2^5 \cdot 7} \cdot \frac{p^4 + 10p^2 - 11}{45} - \frac{p + 30}{2^5 \cdot 7} \cdot \frac{p^2 - 1}{3} = \frac{(p^2 - 1)(p^3 + 3p^2 + 9p - 27)}{2^5 \cdot 3 \cdot 7p^2}.
\]

**Proof of Theorem 0.2.** Using Theorem 3.7 and the above, we compute
\[
\mu(M(1,1),(p,p+2)) = \frac{\text{sign}(p^2 - (p + 2)^2)}{2^5 \cdot 7} + D(1, 1) - D(p, p + 2)
= \frac{(p^2 - 1)^2}{2^5 \cdot 7p^2(p^2 - (p + 2)^2)} - \frac{2^4(p^2 - 1) + (p^2 - (p + 2)^2)}{2^8 \cdot 7p^2}
= -\frac{1}{2^5 \cdot 7} \cdot \frac{(p^2 - 1)(p^3 + 3p^2 + 9p - 27)}{2^6 \cdot 3 \cdot 7p^2}
+ \frac{(p^2 - 1)(p - 1)}{2^4 \cdot 7p^2} = \frac{4(p^2 - 1) - (p + 1)}{2^6 \cdot 7p^2}
= -\frac{p^3 + 3p^2 - 4p}{2^6 \cdot 3 \cdot 7} \in \mathbb{Q}/\mathbb{Z}.
\]

With \( p = 2k - 1 \), we have
\[
\mu(P_k) = \mu(M(1,1),(2k-1,2k+1)) = -\frac{4k^3 - 7k + 3}{2^5 \cdot 3 \cdot 7} \in \mathbb{Q}/\mathbb{Z}.
\]

We also compute \( t_{P_k} \) and \( q_{P_k} \) using Theorem 3.3 as
\[
t_{P_k}(E_\ell) = \frac{\ell(k + \ell)}{24k} + \frac{\ell(\ell - 1)(k - 1)}{6} \quad \text{and} \quad q_{P_k}(\ell) = \frac{\ell(k + \ell)}{2k} \in \mathbb{Q}/\mathbb{Z}.
\]

Having computed the Eells–Kuiper invariant and Crowley’s quadratic form \( q \), we can now compare the spaces \( P_k \) with the principal \( S^3 \)-bundles \( E_{k,k} \) over \( S^3 \).
Proof of Theorem 0.3. By [6, Theorem A], highly connected seven-manifolds are classified up to oriented diffeomorphism by their Eells–Kuiper invariants and the quadratic function $q$ on $H^4$. These invariants have been computed for $E_{k,k}$ in [7] and [8] (see (3.8)).

By Theorem 0.2(2) and (3.8), the quadratic forms $q_{P_k}$ and $q_{E_{k,k}}$ are isomorphic. Hence, the spaces $P_k$ and $E_{k,k}$ are homeomorphic, and even almost diffeomorphic, which means that there is a homeomorphism that is a diffeomorphism outside a single point.

Comparing the value of $\mu(P_k)$ from Theorem 0.2(1) with (3.8), we find

$$\mu(P_k) - \mu(E_{k,k}) = -\frac{4k^3 - 7k}{2^5\cdot 7} + \frac{1}{2^5\cdot 7} - \frac{k - 1}{2^5\cdot 7} = \frac{k - k^3}{6} \cdot \frac{1}{28}$$

with $(k - k^3)/6 \in \mathbb{Z}$. Because both $q$ and $\mu$ are additive under connected sums and $q_{\Sigma_7}$ is trivial whereas $\mu(\Sigma_7) = 1/28$, we conclude again by [6] that $P_k$ and $E_{k,k} \# \Sigma_7^{(k-k^3)/6}$ are oriented diffeomorphic. \qed

Remark 3.11. Grove, Verdiani and Ziller [16] have already observed that $P_2$ is homeomorphic to the unit tangent bundle $T_1S^4$ of $S^4$. The group $Sp(1)$ acts with isolated fixpoints on $S^4$, so this action induces a free action on $T_1S^4$, hence $T_1S^4$ is diffeomorphic to $E_{2,2}$. Of course, this fits with our result above.

Proof of Corollary 0.4. We start with case (1). We already know that Crowley’s form $q$ for $P_k$ is the quadratic form of the principal sphere bundle $E_{k,k} \to S^4$. This implies that the Pontryagin number of a sphere bundle $E_{p,k}$ homeomorphic to $P_k$ must be of the form $p = ak$ with $a$ odd if $k$ is even (see (3.8)).

It thus remains to solve $\mu(P_k) = \mu(E_{ak,k}) \in \mathbb{Q}/\mathbb{Z}$ depending on $p = ak$. By [7], we know that

$$\mu(E_{ak,k}) - \mu(P_k) \equiv a^2k^2 - k - \frac{7k - 4k^3}{3\cdot 2^5\cdot 7} - \frac{1}{2^5\cdot 7} = \frac{a^2k^2 - 7k - 4k^3}{2^5\cdot 7} \mod \mathbb{Z}.$$ 

In other words,

$$a^2k \equiv \frac{7k - 4k^3}{3} \mod 224\mathbb{Z}.$$ 

It suffices to solve this equation modulo 7 and 32.

Modulo 7, the equation is trivial if $7 | k$. Otherwise, we can clearly solve

$$a^2 \equiv \frac{7 - 4k^2}{3} \equiv k^2 \mod 7\mathbb{Z}.$$ 

Modulo 32, we start with the case that $k$ is odd. Because $3 \cdot 11 \equiv 1$, we have to solve

$$a^2 \equiv \frac{7 - 4k^2}{3} \equiv 77 - 44k^2 \equiv 13 - 12k^2 \mod 32\mathbb{Z}.$$ 

The right hand side equals 1 modulo 8 and hence is a quadratic remainder modulo 32. Next, if $k$ is even but not divisible by 8, then we would have at least

$$a^2 \equiv 77 \equiv 5 \mod 8\mathbb{Z},$$
but 5 is not a quadratic remainder modulo 8. Finally, if $8 \mid k$, we can clearly solve
\[ a^2 \equiv 1 \mod 4\mathbb{Z}. \]

In particular, if $k$ is even then $a$ will be odd, so the quadratic forms $q$ agree as well. This settles (1).

In case (2), let $n = k$ be as above. Then $p = ak$ because we still have
\[ \text{lk}_{P_k}(b, p) = q_{P_k}(b) - q_{P_k}(-b) = 0 \in \mathbb{Q}/\mathbb{Z} \]
by Theorem 0.2(1). We will first try to solve $\mu(P_k) + \mu(E_{ak,k}) = 0 \in \mathbb{Q}/\mathbb{Z}$. We find that
\[ 224(\mu(E_{ak,k}) + \mu(P_k)) = a^2 k + \frac{7k - 4k^3}{3} - 2 \mod 224\mathbb{Z}. \]

Modulo 7, there is no solution if $7 \mid k$. On the other hand, a case-by-case check reveals that
\[ \frac{2}{k} - \frac{7 - 4k^2}{3} \equiv \frac{2}{k} - k^2 \mod 7\mathbb{Z} \]
is a quadratic remainder for $k \in \{1, \ldots, 6\}$ modulo 7. Thus, a solution modulo 7 exists if and only if (2) is satisfied.

Modulo 32, if $k$ is odd, we have $k^3 \equiv k \mod 8$. Hence we have to solve
\[ a^2 k \equiv 2 - 13k + 12k^3 \equiv 2 - k \mod 32\mathbb{Z}. \]
The inverse of $k = 4\ell \equiv 1 \mod 16$ is $-4\ell \equiv 1$, hence we obtain
\[ a^2 \equiv -8\ell \pm 2 \equiv -1 \mod 32\mathbb{Z}, \]
which is a quadratic remainder if and only if $k = 4\ell + 1 \equiv 1 \mod 4$. If $k = 2\ell$ is even, then $12k^3 \equiv 0 \mod 32$, and we are left with
\[ a^2 \ell \equiv 1 - 13\ell \equiv 1 + 3\ell \mod 16\mathbb{Z}, \]
and $\ell$ has to be odd. The inverse of $\ell = \pm 1 + 4m$ is $\pm 1 - 4m$, and
\[ a^2 \equiv \pm 1 - 4m + 3 \]
is a square if and only if $\ell = 1 + 4m \in \{1, 5\}$ modulo 16, hence $k \in \{2, 10\}$ modulo 32. This gives (2).

Finally, we have
\[ -\text{lk}_{E_{ak,k}} = \text{lk}_{P_k}(b \cdot, b \cdot) \]
for some $b \in \mathbb{Z}/k\mathbb{Z}$ if and only if $b^2 \equiv -1 \mod k$. Because the half Pontryagin forms vanish and the topological Eells–Kuiper invariants satisfy $28\mu(P_k) + 28\mu(E_{ak,k}) \in \mathbb{Z}$ by the above, it follows from [6] that then the quadratic forms $q$ are isomorphic as well. Thus by [6], there exists an orientation reversing diffeomorphism $E_{ak,k} \to P_k$ if and only if the conditions (2) hold. \(\square\)
Example 3.12. (1) We have $P_1 = S^7$, which of course fibres over $S^4$, independent of the orientation. More precisely, $P_1$ is diffeomorphic to $E_{a,1}$ if and only if

$$\frac{a^2 - 1}{224} \equiv 0 \in \mathbb{Q}/\mathbb{Z},$$

that is, if and only if $a \in \{\pm 1, \pm 15\}$ mod 112.

(2) There is an orientation reversing diffeomorphism from $P_2$ to $E_{4,2}$. Indeed,

$$q_{P_2}(1) = \frac{3}{4} = -q_{E_{4,2}}(1) \in \mathbb{Q}/\mathbb{Z} \quad \text{and} \quad \mu(P_2) = -\frac{1}{32} = -\mu(E_{4,2}) \in \mathbb{Q}/\mathbb{Z}.$$ More generally, $P_2$ is orientation reversing diffeomorphic to $E_{2a,2}$ if and only if $a \equiv \pm 2$ mod 28.

(3) There is an orientation preserving diffeomorphism of $P_3$ with $E_{51,3}$ because

$$\mu(P_3) = -\frac{15}{112} = \frac{433}{112} = \mu(E_{51,3}) \in \mathbb{Q}/\mathbb{Z}.$$ More generally, $P_3$ is oriented diffeomorphic to $E_{3a,3}$ if and only if $a \equiv \pm 17, \pm 31$ mod 112.

(4) For $k = 4$, there exists no diffeomorphic sphere bundle, regardless of the orientation.

(5) For $k = 5$, we have oriented diffeomorphisms with $E_{5a,5}$ if and only if $a \equiv \pm 33, \pm 47$ mod 112. We also have orientation reversing diffeomorphisms with $E_{5a,5}$ if and only if $a \equiv \pm 11, \pm 53$ mod 112. For example,

$$\mu(P_5) = -\frac{156}{224} = \frac{5444}{224} = \mu(E_{165,5})$$

$$= -\frac{604}{224} = -\mu(E_{55,5}) \mod \mathbb{Z}.$$ Because $-1 \equiv 2^2$ is a quadratic remainder modulo 5, we can compare the quadratic forms in the latter case and find that

$$q_{P_5}(2\ell) = \frac{2\ell(2\ell - 5)}{10} \equiv -\frac{\ell(\ell - 55)}{10} = -q_{E_{55,5}}(\ell) \mod \mathbb{Z}.$$

4. Computation of the invariants

We write the spaces $M(p_-,q_-),(q_-,q_+)$ as Seifert fibrations so that we can apply Theorem 0.1 to compute their Eells–Kuiper invariants and $t$-invariants.

4.a. Description as a Seifert fibration

Recall the construction of the spaces $M = M(p_-,q_-),(p_+,q_+)$ as manifolds of cohomogeneity one with group diagram (3.3), with the groups $H$ and $K \subset G = \text{Sp}(1) \times \text{Sp}(1)$ described in (3.1) and (3.2).
The subgroups $\text{Sp}(1) \times \{e\}$ and $\{e\} \times \text{Sp}(1) \subset G$ act freely from the left on the generic orbit $G/H$. We focus on the group $L = \{e\} \times \text{Sp}(1)$ and consider the quotient map

$$M \to B = L \setminus M.$$ 

The group $L$ acts on the singular orbits $G/K_\pm$ with finite stabilizer

$$L_{gK_\pm} = L \cap gK_\pm g^{-1} = g\Gamma_\pm g^{-1} \subset L$$

at the point $gK_\pm \in G/K_\pm$, where

$$\Gamma_- = (\gamma_-) \cong \mathbb{Z}/p_-\mathbb{Z} \quad \text{and} \quad \gamma_- = (1, e^{2\pi i q_-/p_-}) \in K_-,$$

$$\Gamma_+ = (\gamma_+) \cong \mathbb{Z}/p_+\mathbb{Z} \quad \text{and} \quad \gamma_+ = (1, e^{2\pi i q_+/p_+}) \in K_+.$$ 

The quotient $L \setminus M$ has a cohomogeneity one action by the group $\text{SO}(3) \cong \text{Sp}(1)/\{\pm 1\}$. It is induced by the action of $\text{Sp}(1) \times \{e\} \subset G$ on $M$, with group diagram

$$\text{SO}(3) \begin{scope}[shift={(1cm,0cm)}] \draw[->] (0,0) -- (1,1); \draw[->] (1,0) -- (0,1); \end{scope}$$

$$p_1K_- \cong O(2) \cong p_1K_+ \begin{scope}[shift={(1cm,0cm)}] \draw[->] (0,0) -- (1,1); \draw[->] (1,0) -- (0,1); \end{scope}$$

$$p_1H$$

Here $p_1$ denotes the projection

$$G = \text{Sp}(1) \times \text{Sp}(1) \to (\text{Sp}(1)/\{\pm 1\}) \times \{e\} \cong \text{SO}(3).$$

In particular, $p_1H \cong Q/\{\pm 1\} \cong (\mathbb{Z}/2\mathbb{Z})^2$ is the subgroup of diagonal matrices in $\text{SO}(3)$.

If $a$ is an imaginary unit quaternion, let $S^1_a \subset \text{Sp}(1)$ denote the one-parameter subgroup generated by $a$. Because

$$p_1K_- = (S^1 \cup jS^1)/\{\pm 1\} \cong O(2) \quad \text{and} \quad p_1K_+ = (S^1 \cup iS^1)/\{\pm 1\} \cong O(2),$$

the singular orbits of $B$ are given by

$$B_\pm = L \setminus M_\pm \cong \text{SO}(3)/O(2) \cong \mathbb{R}P^2.$$

We want to understand the geometry of $p: M \to B$ near the singular orbits. The action of $K_-$ on $S^1 \cong K_-/H$ extends to $\mathbb{C} \supset S^1$ by

$$(e^{ip_-.\theta}, e^{jq_-\theta})z = e^{4\theta}z \quad \text{and} \quad (e^{ip_-\theta}j, (-1)^{jq_-\theta}/2 e^{jq_-\theta}j)z = e^{4\theta}z,$$ 

and there is a similar action of $K_+$ on $\mathbb{C}$. Thus, the singular orbits $M_\pm = G/K_\pm$ have neighbourhoods $M \setminus M_\pm$ diffeomorphic to the normal bundles

$$N_\pm = G \times_{K_\pm} \mathbb{C} \to G/K_\pm.$$ 

For the generator $\gamma_\pm \in \Gamma_\pm$ of (4.1), we have the angle $\vartheta = 2\pi/p_\pm$ in (4.3). So $\gamma_\pm$ acts on the fibre of $N_\pm$ by multiplication with $e^{8\pi i/p_\pm} \in \mu_{p_\pm}$, where $\mu_{p_\pm}$ denotes the group of $p_\pm$-th roots of unity.
fibration

We want to study the metric structure on $M$. The geometry of the Seifert fibration

$\gamma$ of $\mathbb{C}$ diffeomorphic to $\mathbb{C}$

Proposition 4.1.

The map $p : M \to B = L\setminus M$ is a Seifert fibration and a left $\text{Sp}(1)$-principal orbibundle. The inertia orbifold $\Lambda B$ of the base orbifold $B$ is diffeomorphic to

$$\Lambda B = B \cup \left( \hat{B}_- \times \left\{ 1, \ldots, \frac{p_- - 1}{2} \right\} \right) \cup \left( \hat{B}_+ \times \left\{ 1, \ldots, \frac{p_+ - 1}{2} \right\} \right).$$  \hspace{1cm} (1)

Elements $(p, (y_{\pm})_0) \in \Lambda B \setminus B$ are represented by $(p, \ell) \in \hat{B}_\pm \times \{ k \}$ with $\pm k \equiv \ell \mod p_\pm$ and $\ell \in \{ 1, \ldots, (p_\pm - 1)/2 \}$. The components $\hat{B}_\pm \times \{ k \}$ have multiplicity

$$n(y_{\pm}^k) = \#\Gamma_\pm = p_\pm.$$  \hspace{1cm} (2)

In a suitable orbifold chart around $p$, the element $y_{\pm}^k$ acts by

$$\rho(y_{\pm}^k) = \begin{pmatrix} 1 & 0 \\ 0 & e^{8\pi i k/p_\pm} \end{pmatrix} \in \text{U}(2),$$  \hspace{1cm} (3)

and the fibrewise action on $S^3$ is conjugate to

$$e^{2\pi i k_{y_{\pm}^k}/p_\pm} \in \text{Sp}(1).$$  \hspace{1cm} (4)

Proof. From the discussion above, it is clear that $p$ is both a Seifert fibration and an $\text{Sp}(1)$-principal orbibundle, where the group $L \cong \text{Sp}(1)$ acts from the left. Assertion (1) follows from the considerations above, and (2) follows from the definition of multiplicity in (1.2).

We construct an orbifold chart by taking a neighbourhood of $p$ in $B_\pm \cong \mathbb{R}P^2$ that is diffeomorphic to $\mathbb{C}$ with trivial action of $y_{\pm}^k$. The normal bundle of $B_\pm$ in $B$ is represented by another copy of $\mathbb{C}$ on which $y_{\pm}^k$ acts as in (4.6). This proves (3). Finally, the action of $y_{\pm}^k$ on $S^3$ follows from (4.1) and is conjugate to the expression in (4). \hfill $\square$

4.b. The geometry of the Seifert fibration

We want to study the metric structure on $M$ and $B$. In particular, we want to derive formulas for the curvature of the horizontal and vertical tangent bundles of the Seifert fibration $M \to B$. 
Recall that as a cohomogeneity one manifold, we may write

\[ M = ([−1, 1] \times G/H)/\sim. \]

Let \( \tau : M \to [−1, 1] \) denote the natural projection, and define a curve \( c : [−1, 1] \to M \) joining \( G/K_- \) to \( G/K_+ \) by

\[ c(t) = [t, eH]. \]

We define \( G \)-invariant vector fields \( e_1, e_2, e_3 \) and \( f_0, \ldots, f_3 \) on \( M \setminus (M_+ \cup M_-) = \tau^{-1}((-1, 1)) \) by specifying them along \( c \). Therefore put \( f_0(c(t)) = \dot{c}(t) \) and

\[
\begin{align*}
  f_1(c(t)) &= \frac{d}{dt} \bigg|_{t=0} (e^{it}, 1)(c(t)), \\
  e_1(c(t)) &= \frac{d}{dt} \bigg|_{t=0} (1, e^{it})(c(t)), \\
  f_2(c(t)) &= \frac{d}{dt} \bigg|_{t=0} (e^{i\tau}, 1)(c(t)), \\
  e_2(c(t)) &= \frac{d}{dt} \bigg|_{t=0} (1, e^{i\tau})(c(t)), \\
  f_3(c(t)) &= \frac{d}{dt} \bigg|_{t=0} (e^{it}, 1)(c(t)), \\
  e_3(c(t)) &= \frac{d}{dt} \bigg|_{t=0} (1, e^{it})(c(t)).
\end{align*}
\]

We regard the vector fields \( e_1, e_2, e_3 \) as vertical and \( f_0, \ldots, f_3 \) as horizontal fields with respect to the Seifert fibration \( M \to B \). All Lie brackets between these vector fields vanish except

\[
[e_1, e_2] = 2e_3, \quad [e_2, e_3] = 2e_1, \quad [e_3, e_1] = 2e_2, \quad [f_1, f_2] = 2f_3, \quad [f_2, f_3] = 2f_1, \quad [f_3, f_1] = 2f_2. \tag{4.8}
\]

Let \( \psi : [0, 1] \to \mathbb{R} \) denote a cutoff function with \( \psi|_{[0, \varepsilon)} = 0 \) and \( \psi|_{(1-\varepsilon, 1]} = 1 \) for some small \( \varepsilon > 0 \). For \( x \in M \), let \( t = \tau(x) \), and define functions \( h, k : \tau^{-1}((-1, 0]) \to \mathbb{R} \) by

\[
h = \frac{4 + 4\tau \cdot \psi(-\tau)}{4 + (p_- - 4) \cdot \psi(-\tau)} \quad \text{and} \quad k = \frac{q_-}{4} h'. \tag{4.9}
\]

These functions satisfy

\[
\begin{align*}
  h|_{(−1, ε−1)} &= \frac{4}{p_-}(1 + \tau), \quad h|_{(−ε, 0)} = 1, \\
  k|_{(−1, ε−1)} &= \frac{q_-}{p_-}, \quad k|_{(−ε, 0)} = 0. \tag{4.10}
  \end{align*}
\]

Let \( g^{TM} \) be a \( G \)-invariant metric such that for \( t \leq 0 \), the vectors \( e_2, e_3, f_0, f_2 \) and \( f_3 \) are orthonormal and perpendicular to the subspace spanned by \( e_1, f_1 \), and such that on this subspace, \( g^{TM} \) is given by the matrix

\[
g^{TM}|_{\text{span}(e_1, f_1)} = \begin{pmatrix}
  1 & -k \\
  -k & h^2 + k^2
\end{pmatrix}. \tag{4.11}
\]

This metric extends to a smooth metric around \( G/K_- = \tau^{-1}(-1) \) by \[16, \text{Theorem 6.1}\]. The orbits of \( L = \{e\} \times \text{Sp}(1) \) are all quotients of round spheres with the standard metric.
A $g^{TM}$-orthonormal frame on $\tau^{-1}(-1, 0]$ is given by $(\vec{e}_1, \ldots, \vec{e}_3)$, where $\vec{e}_i = e_i$ and $\vec{f}_i = f_i$ except

$$
\vec{f}_1 = \frac{k}{h} e_1 + \frac{1}{h} f_1.
$$

(4.12)

By (4.8), the Lie brackets between the vector fields $\vec{f}_0, \ldots, \vec{e}_3$ vanish except

$$
[\vec{f}_0, \vec{f}_1] = \frac{k'}{h} \vec{e}_1 - \frac{h'}{h} \vec{f}_1, \quad [\vec{f}_1, \vec{e}_2] = \frac{2k}{h} \vec{e}_3, \quad [\vec{f}_1, \vec{e}_3] = -\frac{2k}{h} \vec{e}_2, \\
[\vec{f}_1, \vec{f}_2] = \frac{2}{h} \vec{f}_3, \quad [\vec{f}_2, \vec{f}_3] = 2h\vec{f}_1 - 2k\vec{e}_1, \quad [\vec{f}_3, \vec{f}_1] = \frac{2}{h} \vec{f}_2, \\
[\vec{e}_1, \vec{e}_2] = 2\vec{e}_3, \quad [\vec{e}_2, \vec{e}_3] = 2\vec{e}_1, \quad [\vec{e}_3, \vec{e}_1] = 2\vec{e}_2.
$$

(4.13)

For $t \geq 0$, we can proceed similarly. We extend $h$ and $k$ to $\tau^{-1}[0, 1)$ by

$$
h = \frac{4 - 4\tau \cdot \psi(\tau)}{4 + (p_+ - 4) \cdot \psi(\tau)} \quad \text{and} \quad k = -\frac{q_+}{4} h',
$$

so that

$$
h|_{0, \epsilon} = 1, \quad h|_{1-\epsilon, 1} = \frac{4}{p_+} (1 - \tau), \\
k|_{0, \epsilon} = 0, \quad k|_{1-\epsilon, 1} = \frac{q_+}{p_+}.
$$

(4.15)

We then modify the metric similarly, to obtain a $g^{TM}$-orthonormal frame $\vec{f}_0, \ldots, \vec{e}_3$ that differs from $f_0, \ldots, e_3$ only by

$$
\vec{f}_2 = \frac{k}{h} e_2 + \frac{1}{h} f_2.
$$

Again, this metric extends smoothly over $\tau^{-1}[0, 1]$, and it is also compatible along $\tau^{-1}(0)$ with the metric chosen above on $\tau^{-1}[-1, 0]$.

### 4.c. Orbifold characteristic numbers of the base space

The base orbifold $B = L\backslash M$ has a principal cohomogeneity one action by $\text{SO}(3)$ (see (4.2)). We define $\tau: B \to [-1, 1]$ similarly to the above. Then we can describe $\tau^{-1}(-1, 1)$ as a product $(-1, 1) \times \text{Sp}(1)/Q$. The projection $M \to B$ becomes a Riemannian submersion with respect to an invariant Riemannian metric $g^{TB}$ on $(-1, 1) \times \text{Sp}(1)/Q$ that degenerates over $[-1, 1]$.

By abuse of notation, let $\vec{f}_0, \ldots, \vec{f}_3$ also denote the projection of the vector fields above to $B$. Then these vector fields form a $g^{TB}$-orthonormal frame everywhere, and their nonzero Lie brackets on $\tau^{-1}[-1, 0]$ are completely described by

$$
[\vec{f}_0, \vec{f}_1] = -\frac{h'}{h} \vec{f}_1, \quad [\vec{f}_1, \vec{f}_2] = \frac{2}{h} \vec{f}_3, \quad [\vec{f}_2, \vec{f}_3] = 2h\vec{f}_1, \quad [\vec{f}_3, \vec{f}_1] = \frac{2}{h} \vec{f}_2.
$$
The Christoffel symbols of the Levi-Civita connection on $B$ with respect to these fields over $\tau^{-1}((-1, 0)) \subset B$ are given by
\[
\begin{align*}
\Gamma^1_{10} &= -\Gamma^0_{11} = \frac{h'}{h}, & \Gamma^3_{12} &= -\Gamma^2_{13} = \frac{2}{h} - h, \\
\Gamma^1_{23} &= -\Gamma^3_{21} = h, & \Gamma^2_{31} &= -\Gamma^1_{32} = h;
\end{align*}
\]

those $\Gamma^k_{ij}$ not listed above vanish. The Riemannian curvature tensor as a $4 \times 4$-matrix is given by
\[
R = \begin{pmatrix}
\frac{k'}{\pi} f^{01} - 2h' f^{23} & \ldots & 0 \\
0 & \ldots & 0 \\
\ldots & \ldots & \ldots \\
0 & 0 & 0
\end{pmatrix},
\]
with $\bar{f}^{ij}$ shorthand for $\bar{f}^{i} \wedge \bar{f}^{j}$. Over $\tau^{-1}[0, 1)$, the matrix looks similar, but with the matrix indices and the form indices 1, 2, 3 permuted cyclically.

The Pontryagin and Euler forms are thus given by
\[
\begin{align*}
p_1(TB, \nabla^{TB}) &= \frac{1}{8\pi^2} \mathrm{tr}(R^2) = \frac{1}{\pi^2} \left( \frac{k''}{h} h' + 4h^2 - 4h' \right) f^{0123}, \\
e(TB, \nabla^{TB}) &= \frac{1}{4\pi^2} \mathrm{Pr}(R) = \frac{1}{4\pi^2} \left( 6h'^2 + 3h''h - \frac{4h^3}{h} \right) f^{0123}. 
\end{align*}
\]

The fibre $\mathbb{RP}^3 / \mathbb{Z}/2\mathbb{Z}$ over $t$ has volume $\left( \frac{2}{\pi^2} \right) h(t)$. By \eqref{eq:vol1}, \eqref{eq:vol2}, and \eqref{eq:vol3}, we get the orbifold characteristic numbers
\[
\begin{align*}
\left. \int_B p_1(TB, \nabla^{TB}) \right|_{t=1} &= \left. \frac{h'(t)^2 + 2h(t)^4 - 4h(t)^2}{8} \right|_{t=1} = \frac{2}{p_+} - \frac{2}{p_+^2}, \\
\left. \int_B e(TB, \nabla^{TB}) \right|_{t=1} &= \left. \frac{3h'(t)^2 - 4h(t)^2}{16} \right|_{t=1} = \frac{1}{p_-} + \frac{1}{p_+}. 
\end{align*}
\]

4.d. Characteristic numbers of the Seifert $S^3$-fibration

The Seifert fibration $M \rightarrow B$ is a principal orbibundle with structure group $\text{Sp}(1)$. Let $TX = \ker p_\ast$ denote the vertical tangent bundle.

A connection $\omega \in \text{Hom}(TM, TX)$ acts as the identity on $TX$ and is $\text{Sp}(1)$-invariant. It is uniquely described by its horizontal bundle $T^H M = \ker \omega$. We define $\omega$ such that
\[
T^H M = \text{span}\{ \bar{f}_0, \bar{f}_1, \bar{f}_2, \bar{f}_3 \}.
\]

Then by \eqref{eq:curv1}, its curvature $\Omega$ is given by
\[
\Omega = \begin{cases}
\left( -\frac{k'}{\pi} f^{01} + 2k f^{23} \right) \varepsilon_1 & \text{for } t \in [-1, 0], \\
\left( -\frac{k'}{\pi} f^{02} - 2k f^{13} \right) \varepsilon_2 & \text{for } t \in [0, 1].
\end{cases}
\]

The Seifert fibration $M \rightarrow B$ is the unit sphere orbibundle of the vector orbibundle
\[
V = M \times_{\text{Sp}(1)} \mathbb{H} \rightarrow B.
\]
The vectors $\tilde{e}_1, \tilde{e}_2$ correspond to the elements $i, j \in \mathfrak{sp}(1) \subset \mathbb{H}$. These act on $\mathbb{H} \cong \mathbb{R}^4$ with Pfaffian $\text{Pf}(i) = \text{Pf}(j) = 1$, hence the Euler form of the connection $\nabla^V$ on $V$ induced by $\omega$ is given by

$$e(V, \nabla^V) = -\frac{k'k}{\pi^3h} f^{0123}. \quad (4.20)$$

As in (4.18), integration over $B$ gives the characteristic number

$$\int_B e(V, \nabla^V) = -\int_{-1}^1 \frac{k'(t)k(t)}{4} \, dt = -\frac{1}{8} \left|\frac{k(t)^2}{t=-1}\right| = \frac{q_+^2}{8p_+^2} - \frac{q_-^2}{8p_-^2}. \quad (4.21)$$

With these computations, we can now compute the first term in the adiabatic limit of formula (3.7) for the Eells–Kuiper invariant. Recall that $B \cong B \times \{e\} \subset \Delta B$.

**Proposition 4.2.** For the Seifert fibration $p : M_{(p_-,q_-),(p_+,q_+)} \to B$, we have

$$\frac{1}{2} \int_B \hat{A}_{\Delta B}(TB, \nabla^{TB}) 2\eta_{\Delta B}(D_{S^3}) + \frac{1}{25 \cdot 7} \int_B \hat{L}_{\Delta B}(TB, \nabla^{TB}) 2\eta_{\Delta B}(B_{S^3})$$

$$= -\frac{1}{22 \cdot 5 \cdot 7} \int_B e(V, \nabla^V) = \frac{1}{210 \cdot 7} \left( \frac{q_+^2}{p_+^2} - \frac{q_-^2}{p_-^2} \right).$$

**Proof.** Let $V \to B$ be the induced vector bundle with connection $\nabla^V$ as above. The $\eta$-form of the untwisted fibrewise Dirac operator and the fibrewise signature operator are given by

$$2\eta_{\Delta B}(D_{S^3})|_B = \eta_{\Omega/2\pi i}(D_{S^3}) = -\frac{1}{960} e(V, \nabla^V),$$

$$2\eta_{\Delta B}(B_{S^3})|_B = \eta_{\Omega/2\pi i}(B_{S^3}) = -\frac{1}{30} e(V, \nabla^V) \in \Omega^4(B),$$

by Theorem 1.11 and [13, Theorem 3.9].

Because both $\eta$-forms are homogeneous of degree 4, we only need the degree zero components of $\hat{A}$ and $\hat{L}$, which are given by

$$\hat{A}(TB, \nabla^{TB})^{[0]} = 1 \quad \text{and} \quad \hat{L}(TB, \nabla^{TB})^{[0]} = 2^{(\dim TB)/2} = 4.$$ From the above and (4.21), we obtain our result because

$$\frac{1}{2} \int_B \hat{A}(TB, \nabla^{TB}) \eta_{\Omega/2\pi i}(D_{S^3}) = \frac{1}{210 \cdot 3 \cdot 5} \left( \frac{q_+^2}{p_+^2} - \frac{q_-^2}{p_-^2} \right),$$

$$\frac{1}{25 \cdot 7} \int_B \hat{L}(TB, \nabla^{TB}) \eta_{\Omega/2\pi i}(B_{S^3}) = \frac{1}{22 \cdot 5 \cdot 7 \cdot 3} \left( \frac{q_+^2}{p_+^2} - \frac{q_-^2}{p_-^2} \right). \quad \Box$$

4.e. The contributions from the twisted sectors

To compute the contribution from $\Delta B \setminus B$, we need some equivariant characteristic numbers and the equivariant $\eta$-forms of the pullback of $M$ to $\tilde{B}_\pm$. Let $(p, (y^\alpha_\delta)) \in \Delta B \setminus B$, let $N_\pm \to B_\pm$ be the normal bundle of $B_\pm \cong \mathbb{R}P^2$ in $B$, and let $\tilde{N}_\pm$ denote its pullback to $\tilde{B}_\pm \cong S^2$. 
In an orbifold chart, the elements $\gamma_a^\pm$ for $a = 1, \ldots, (p_\pm - 1)/2$ act on $\tilde{N}_\pm$ by multiplication with $e^{8\pi ia/p_\pm} \in S^1 \cong \text{SO}(2)$ (see Proposition 4.1(3)). Because $\Gamma_\pm \cong \mathbb{Z}/p_\pm \mathbb{Z}$ is an odd cyclic group, this action has a unique lift to $\text{Spin}(2)$, represented by

$$\tilde{\gamma}_a^\pm = e^{4\pi ia/p_\pm} \in S^1 \cong \text{Spin}(2).$$

This lift provides us with a unique section of the bundle $\tilde{\Lambda} \to \Lambda B$ of (1.3). All forms in $\Omega^2(\Lambda B; \tilde{\Lambda} \otimes o(\Lambda B))$ will be computed with respect to this lift and with respect to the orientation of $\tilde{B}_\pm \cong S^2$ with volume form $f^{23}$ or $f^{31}$, respectively.

The curvature $R_{\tilde{N}_-}$ can be computed as the limit of $\langle R_{\tilde{f}_0}, \tilde{f}_1 \rangle |_{\text{span} \{\tilde{f}_2, \tilde{f}_3\}}$ as $t \to -1$, so by (4.16) and (4.10) we have

$$R_{\tilde{N}_-} = \begin{pmatrix} 0 & 2h' f^{23} \\ -2h' f^{23} & 0 \end{pmatrix} = -\frac{8i}{p_-} f^{23}$$

with $\tilde{f}_1 = i \tilde{f}_0$. The induced curvature of the spinor bundle at the origin is

$$R_{\tilde{S}_-} = \frac{4i}{p_-} f^{23}$$

and similarly for $\tilde{N}_+$. By (1.5)–(1.7), the orbifold $\tilde{A}$-form on $\tilde{B}_- \times \{a\} \subset \Lambda B$ is represented by

$$\tilde{A}_{\Lambda B}(TB, \nabla^{TB}) = -\frac{\tilde{A}(TB_-, \nabla^{TB_-})}{m(\tilde{\gamma}_a^\pm) \text{ch}_{\tilde{\gamma}_a^\pm}(S^+ \tilde{N}_--S^- \tilde{N}_-, \nabla^{S \tilde{N}_-})} = -\frac{1}{p_- \cdot 2i \sin \left( \frac{4}{p_-} \left( \pi a + \frac{f^{23}}{2\pi i} \right) \right)} \in \Omega^*(\tilde{B}_-). \quad (4.22)$$

A similar computation gives the orbifold $\tilde{L}$-form

$$\tilde{L}_{\Lambda B}(TB, \nabla^{TB}) = \tilde{A}_{\Lambda B}(TB, \nabla^{TB}) \text{ch}_{\Lambda B}(S^+ B_- + S^- B_-, \nabla S B_-) = \frac{2i}{p_-} \cot \left( \frac{4}{p_-} \left( \pi a + \frac{f^{23}}{2\pi i} \right) \right) \in \Omega^*(\tilde{B}_-). \quad (4.23)$$

We also need the equivariant $\eta$-forms of $G|_{B_\pm} \to B_\pm$. We know by Proposition 4.1(4) that $\gamma_a^\pm$ act on the fibres $S^3$ as

$$\gamma_a^\pm = e^{2\pi i a q_- / p_-} \quad \text{and} \quad \gamma_a^\pm = e^{2\pi j a q_+ / p_+},$$

and the curvatures at $B_\pm$ are given by (4.10) and (4.19) as

$$\Omega_- = -\frac{2q_-}{p_-} f^{23} e_1 \quad \text{and} \quad \Omega_+ = \frac{2q_+}{p_+} f^{31} e_2.$$
We compute the mixed equivariant $\eta$-invariant, from which we derive the orbifold $\eta$-form of Definition 1.7 using Theorem 1.11. We use the formulas for the equivariant $\eta$-invariants of the untwisted Dirac operator $D_{S^3}$ in [18] and of the odd signature operator $B_{S^3}$ in [1, proof of Proposition 2.12]. On $\check{B}_- \times \{a\} \subset \Lambda B$, we obtain in particular

$$2\eta_{AB}(D_{S^3}) = \eta^{\varphi_0 e^{-\Omega_{/2\pi i}}}(D_{S^3}) = -\frac{1}{2\sin^2\left(\frac{q_-}{p_-}(\pi a + \frac{23}{2\pi i})\right)},$$

$$2\eta_{AB}(B_{S^3}) = \eta^{\varphi_0 e^{-\Omega_{/2\pi i}}}(B_{S^3}) = -\cot^2\left(\frac{q_-}{p_-}(\pi a + \frac{23}{2\pi i})\right).$$  \hspace{1cm} (4.24)

We can now compute the contribution from the singular orbits $M_{\pm}$ to the adiabatic limit of the $\eta$-invariants and the Eells–Kuiper invariant and relate it to the generalised Dedekind sums $D(p, q)$ of Definition 3.6.

**Proposition 4.3.** The singular orbits $M_{\pm}$ contribute to the Eells–Kuiper invariant by the generalised Dedekind sums

$$\int_{\check{B}_- \times \{1, \ldots, (p_--1)/2\}} \left(\frac{1}{2} \hat{A}_{AB}(TB, \nabla^TB)2\eta_{AB}(D_{S^3}) + \frac{1}{2\pi \cdot 7} \hat{L}_{AB}(TB, \nabla^TB)2\eta_{AB}(B_{S^3})\right) = D(p_-, q_-),$$

$$\int_{\check{B}_+ \times \{1, \ldots, (p_+-1)/2\}} \left(\frac{1}{2} \hat{A}_{AB}(TB, \nabla^TB)2\eta_{AB}(D_{S^3}) + \frac{1}{2\pi \cdot 7} \hat{L}_{AB}(TB, \nabla^TB)2\eta_{AB}(B_{S^3})\right) = -D(p_+, q_+).$$

**Proof.** The twisted sectors $\check{B}_- \times \{a\}$ are spheres of sectional curvature 4 by (4.16), in particular their volume is $\pi$. We combine (1.7) and (1.13) with (4.22)–(4.24) and find that

$$\int_{\check{B}_- \times \{a\}} \left(\frac{1}{2} \hat{A}_{AB}(TB, \nabla^TB)2\eta_{AB}(D_{S^3}) + \frac{1}{2\pi \cdot 7} \hat{L}_{AB}(TB, \nabla^TB)2\eta_{AB}(B_{S^3})\right)$$

$$= \int_{\check{B}_-} \left(\frac{1}{8i p_- \sin\frac{4\pi a}{p_-} \sin\frac{q_-}{p_-} \sin\frac{q_- x}{p_-}} \right) \cdot \frac{i}{24 \cdot 7 p_-} \cdot \cot\left(\frac{4x}{p_-}(\pi a + \frac{23}{2\pi i})\right) \cdot \cot\left(\frac{q_- x}{p_-}(\pi a + \frac{23}{2\pi i})\right)\right)$$

$$= d \left|_{x=\pi a} \right. \left(\frac{1}{8i p_- \sin\frac{4\pi a}{p_-} \sin\frac{q_-}{p_-} \sin\frac{q_- x}{p_-}} \right) \cdot \frac{i}{24 \cdot 7 p_-} \cdot \cot\left(\frac{4x}{p_-}(\pi a + \frac{23}{2\pi i})\right) \cdot \cot\left(\frac{q_- x}{p_-}(\pi a + \frac{23}{2\pi i})\right)\right)$$

$$= \frac{14 \cos\frac{4\pi a}{p_-} + \cos^2\frac{q_-}{p_-}}{24 \cdot 7 p_- \sin^2\frac{4\pi a}{p_-} \sin^2\frac{q_-}{p_-}} + \frac{q_- \cos\frac{2\pi a}{p_-} (14 + \cos\frac{4\pi a}{p_-})}{24 \cdot 7 p_- \sin^2\frac{4\pi a}{p_-} \sin^2\frac{q_-}{p_-}}.$$

To obtain the first equation above, we note that the summands for $a$ and $p_- - a$ in Definition 3.6 are identical. The second equation is proved similarly. \qed

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{adiabatic limitations}
\caption{Adiabatic limits of Seifert fibrations}
\end{figure}
4.f. The secondary Pontryagin term

For the computation of the Eells–Kuiper invariant \( \mu(M(p-q,-),(p-q+,+)) \) using formula (3.7), it remains to compute the correction term in the adiabatic limit.

If we regard the limit of the Levi-Civita connections on \((M,g_\varepsilon)\) as in (2.1), we find that

\[
\lim_{\varepsilon \to 0} \rho_1(TM, \nabla^{TM, \varepsilon}) = \rho_1(TX, \nabla^{TX}) + p^* \rho_1(TB, \nabla^{TB}).
\]

The form \( \rho_1(TB, \nabla^{TB}) \) has already been determined in (4.17). By the variation formula for Chern–Weil classes, it is clear that

\[
\lim_{\varepsilon \to 0} \int_M (\rho_1 \wedge \hat{\rho}_1)(TM, \nabla^{TM, \varepsilon}) = \int_M (\rho_1(TX, \nabla^{TX}) + p^* \rho_1(TB, \nabla^{TB}))
\]

\[
\wedge (\hat{\rho}_1(TX, \nabla^{TX}) + \hat{\rho}_1(p^* TB, \nabla^{p^* TB})),
\]

(4.25)

where again

\[
d\hat{\rho}_1(TX, \nabla^{TX}) = \rho_1(TX, \nabla^{TX}), \quad d\hat{\rho}_1(p^* TB, \nabla^{p^* TB}) = p^* \rho_1(TB, \nabla^{TB}).
\]

Note that since \( H^3_{dR}(B) \neq 0 \), we cannot expect to construct \( \hat{\rho}_1(TB, \nabla^{TB}) \in \Omega^3(B) \).

We start by computing \( \rho_1(TX, \nabla^{TX}) \). The connection \( \nabla^{TX} \) is defined as the compression of the Levi-Civita connection \( \nabla^{TM} \) on \( M \) to \( TX \). Hence, we can compute it with respect to the basis \( \bar{e}_1, \bar{e}_2, \bar{e}_3 \) of \( TX \) using (4.13). Its connection one-form is given by

\[
\omega^{TX} = \begin{pmatrix}
0 & -\bar{e}_3 & -\bar{e}_2 \\
-\bar{e}_3 & 0 & -2k \bar{f}_2 - \bar{e}_3 \\
-\bar{e}_2 & 2k \bar{f}_1 + \bar{e}_1 & 0
\end{pmatrix}.
\]

The corresponding curvature is then given by

\[
\Omega^{TX} = d\omega^{TX} + \omega^{TX} \wedge \omega^{TX} = \begin{pmatrix}
0 & \bar{e}^{12} & \bar{e}^{13} \\
-\bar{e}^{12} & 0 & -k \bar{f} + \bar{e}^{23} \\
-\bar{e}^{13} & -k \bar{f} + \bar{e}^{23} & 0
\end{pmatrix}.
\]

Note that since the group \( G = SO(3) \times SO(3) \) does not act freely on \( \tau^{-1}[-1, 1] \), the basis \( (\bar{e}_1, \bar{e}_2, \bar{e}_3) \) does not extend over \( M_{\pm} \). Hence, the form \( \omega^{TX} \) and its curvature \( \Omega^{TX} \) are not necessarily smooth at \( \tau = \pm 1 \). Nevertheless, the Pontryagin form \( \rho_1(TX, \nabla^{TX}) \) will be smooth. It is given by

\[
\rho_1(TX, \nabla^{TX}) = \frac{1}{8\pi^2} \text{tr}((\Omega^{TX})^2) = \frac{1}{4\pi^2} \left( \frac{4kk'}{h} f_0^{0123} + \frac{2k'}{h} f^{01} e^{23} - 4k f^{23} e^{23} \right).
\]

(4.26)
The forms $p_1(TX, \nabla TX)$ and $p^* p_1(TB, \nabla TB)$ are clearly $G$-invariant. We will now construct $G$-invariant forms $\hat{p}_1(TX, \nabla TX)$ and $\hat{p}_1(p^* TB, \nabla p^* TB)$. The complex of smooth $G$-invariant forms on $M$ can be described as

$$\Omega^\bullet(M)^G = (C^\infty([-1, 1]) \otimes \Lambda^\bullet \mathbb{R}^7) \cap \Omega^\bullet(M),$$

where $\mathbb{R}^7$ is spanned by the dual basis $\bar{e}^1, \ldots, \bar{e}^3$ to the basis $\bar{e}_1, \ldots, \bar{e}_3$ of Section 4.b. Smoothness at the singular orbits gives boundary conditions. In particular, functions $\bar{e}_1, \ldots, \bar{e}_3$ of smooth $G$-invariant forms $G$ extend to smooth $G$-invariant $G$-invariant functions if and only if they are even at $\pm 1$, and among others, the monomials $h f^0, h f^1, f^{01}, f^{23}, \bar{e}^3, \bar{e}^{23}$ are smooth at $M_-$, and $h f^0, h f^2, f^{02}, f^{31}, \bar{e}^2, \bar{e}^{31}$ are smooth at $M_+$. From (4.13) and Cartan’s formula for the exterior derivative, we deduce that on $\tau^{-1}(-1, 0]$,

$$dg = g' f^0, \quad d f^0 = 0,$$

$$df^1 = \frac{h'}{h} f^{01} - 2h f^{23}, \quad d\bar{e}^1 = -\frac{k'}{h} f^{01} + 2k f^{23} - 2\bar{e}^{23},$$

$$df^2 = \frac{2}{h} f^{13}, \quad d\bar{e}^2 = \left(\frac{2k}{h} f^1 + 2\bar{e}^1\right) \bar{e}^3,$$

$$df^3 = -\frac{2}{h} f^{12}, \quad d\bar{e}^3 = -\left(\frac{2k}{h} f^1 + 2\bar{e}^1\right) \bar{e}^2,$$

for functions $g$ of $\tau$. Similar formulas with the indices 1, 2, 3 rotated hold over $\tau^{-1}[0, 1)$. From this we conclude that

$$d\left(\frac{1}{h} f^{123}\right) = 0 \quad \text{and} \quad d\bar{e}^{123} = -\left(\frac{k'}{h} f^{01} + 2k f^{23}\right) \bar{e}^{23}. \quad (4.27)$$

We also find that over $[-1, 0]$,

$$d\left(\frac{k'}{h} f^{01} - 2k f^{23} - 2\bar{e}^{23}\right) = d(-d\bar{e}^1 - 4\bar{e}^{23}) = 0,$$

$$d\left(\left(\frac{k'}{h} f^{01} - 2k f^{23} - 2\bar{e}^{23}\right) \bar{e}^1\right) = \left(\frac{k'}{h} f^{01} - 2k f^{23} - 2\bar{e}^{23}\right) \bar{e}^1 + \left(-\frac{k'}{h} f^{01} + 2k f^{23} - 2\bar{e}^{23}\right) \bar{e}^{23} = 4kk' f^{0123}. \quad (4.28)$$

Similarly over $[0, 1]$, we have

$$d\left(\left(\frac{k'}{h} f^{02} - 2k f^{31} - 2\bar{e}^{31}\right) \bar{e}^2\right) = 4kk' f^{0123}. \quad (4.29)$$
Thus, if we put
\[
\hat{p}_1(TX, \nabla TX) = \frac{1}{4\pi^2} \begin{cases} 
\frac{k'}{q} \tilde{f}^{01} - 2k \tilde{f}^{23} & \text{on } [-1, 0], \\
\frac{k'}{q} \tilde{f}^{02} - 2k \tilde{f}^{31} & \text{on } [0, 1],
\end{cases}
\]
then the form \( \hat{p}_1(TX, \nabla TX) \) is smooth on \( M \) because near \( \tau^{-1}(0) \), only the term \(-4\varepsilon_{123} \) is present, and because \( k' \) vanishes near \( M_{\pm} \). From (4.26)–(4.29), we immediately find that
\[
d \hat{p}_1(TX, \nabla TX) = p_1(TX, \nabla TX). \tag{4.30}
\]

For the next step, we assume that \( q_+ p_+ \neq q_- p_- \), because \( H^4(M; \mathbb{R}) = 0 \) in this case by [17, Theorem 13.1]. Recall that by (4.9) and (4.14), we have
\[
h'(t)h''(t) = \begin{cases} 
\frac{16k(t)k'(t)}{q_-} & \text{if } t \in [-1, 0], \\
\frac{16k(t)k'(t)}{q_+} & \text{if } t \in [0, 1].
\end{cases}
\]

We now consider the form
\[
\hat{p}_1(p^*TB, \nabla p^*TB) = \frac{4}{\pi^2} \frac{p_2^2 - p_2^-}{q_-^2 p_2^2 + q_+^2 p_-^2} \left( \frac{k'}{h} \tilde{f}^{01} - 2k \tilde{f}^{23} - 2\varepsilon^{23} \right) \varepsilon^1 + \frac{1}{2\pi^2} \left( h'^2 - 16 \frac{p_2^2 - p_2^-}{q_-^2 p_2^2 + q_+^2 p_-^2} k^2 - 16 \frac{q_2^2 - q_2^-}{q_-^2 p_2^2 + q_+^2 p_-^2} + 4h^4 - 4h^2 \right) \frac{1}{h} \tilde{f}^{123}
\]
over \([-1, 0]\) and similarly over \([0, 1]\) using (4.29). Using (4.9), (4.10), (4.14) and (4.15), we can check that the coefficient of \( \tilde{f}^{123} \) vanishes to first order near \( \pm 1 \), so the form above is indeed smooth. By (4.17) and (4.27)–(4.29), we conclude that
\[
d \hat{p}_1(p^*TB, \nabla p^*TB) = \frac{1}{2\pi^2} \left( \frac{h'}{h} h'' + 4h^2 h' - 4h' \right) \tilde{f}^{013} = p^* p_1(TB, \nabla TB). \tag{4.32}
\]

We can now compute the correction term in the Eells–Kuiper invariant.

**Proposition 4.4.** The adiabatic limit of the secondary Pontryagin term is given by
\[
\frac{1}{2^7 \cdot 7} \lim_{\varepsilon \to 0} \int_M (p_1 \wedge \hat{p}_1)(TM, \nabla TM, \varepsilon)
\]
\[
= \frac{(p_2^2 - p_2^-)^2}{2^2 \cdot 7p_2^2 p_2^2 (q_2^2 p_2^2 - q_2^2 p_-^2) + 2^6 (p_2^2 - p_2^-)^2 + 3(q_2^2 p_2^2 - q_2^2 p_-^2)}{210 \cdot 7 p_2^2 p_2^2}.
\]
Proof. The forms \( p_1(TX, \nabla^{TX})_{\tau^{-1}[-1, 0]} \) and \( p^* p_1(TB, \nabla^{TB}) \) do not contain the exterior variable \( e^1 \) by (4.17) and (4.26). Hence only the terms in \( \hat{p}_1(p^* TB, \nabla^{p^* TB}) \) containing the exterior variable \( e^1 \) contribute to the integral over \( \tau^{-1}[-1, 0] \subset M \). Using (4.25), we find that

\[
\lim_{\varepsilon \to 0} \int_{\tau^{-1}[-1, 0]} p_1(TM, \nabla^{TM, \varepsilon}) \hat{p}_1(TM, \nabla^{TM, \varepsilon}) = \int_{\tau^{-1}[-1, 0]} \frac{1}{4\pi^2} \left( \left( \frac{4h'h''}{h} + 16h^2h' - 16h' \right) \right)^{0123}
\]

and a similar formula gives the integral over \( \tau^{-1}[0, 1] \). Recall that the generic fibres of \( p \) have volume \( \text{vol}(S^3) = 2\pi^2 \), and that the slices \( \tau^{-1}(t) \subset B \) have volume \( h(t) \text{vol}(\mathbb{R}P^3/(\mathbb{Z}/2\mathbb{Z})^2) = h(t)\pi^2/4 \). Hence we have

\[
\text{vol}(\tau^{-1}(t)) = h(t)\pi^2/2. \tag{4.33}
\]

Combining this with the above, we obtain

\[
\lim_{\varepsilon \to 0} \int_M p_1(TM, \nabla^{TM, \varepsilon}) \hat{p}_1(TM, \nabla^{TM, \varepsilon}) = -\int_{-1}^{1} \left( \left( \frac{p_+^2 - p_-^2}{q_-^2 p_+^2 - q_+^2 p_-^2} + \frac{1}{8} \right) \left( 4h'h'' + 16h^2h' - 16hh' + 4kk' \right)(t)
\]

\[
+ \left( \frac{p_+^2 - p_-^2}{q_-^2 p_+^2 - q_+^2 p_-^2} + \frac{1}{16} \right) \left( 4kk' \right)(t) \right) dt
\]

\[
= 32 \left( \frac{(p_+^2 - p_-^2)^2}{p_-^2 p_+^2 q_-^2 p_+^2 - q_+^2 p_-^2 p_+^2} \right) + \frac{8}{p_-^2} - \frac{8}{p_+^2} + \frac{8 q_+^2}{p_-^2} - \frac{8 q_-^2}{p_+^2}.
\]

\( \square \)

4.g. The Leray–Serre spectral sequence

The adiabatic limit of the \( \eta \)-invariant of the odd signature operator consists of terms that correspond to the various terms in the Leray spectral sequence. The \( E_0 \)-term gives the integral of the \( \eta \)-form of the fibre against a characteristic form on the base. The \( E_1 \)-term contributes by an \( \eta \)-invariant of the base orbifold. This invariant vanishes here because the base is even-dimensional. The higher terms contribute by the signs of the corresponding eigenvalues. There are no similar contributions for \( \eta(D) \) because the fibres have positive scalar curvature and hence the fibrewise operator does not admit harmonic spinors.
To see that the Leray spectral sequence does not degenerate at $E_2$, we note that the fibrewise cohomology forms a trivial bundle over $B$ with generators $1$ and $\bar{e}^{123}$, so we have

$$E_2^{p,q} = E_3^{p,q} = E_4^{p,q} \cong \begin{cases} \mathbb{R} & \text{if } p \in \{0, 4\} \text{ and } q \in \{0, 3\}, \\ 0 & \text{otherwise}, \end{cases}$$

whereas $E_n^{0,3} = E_n^{4,0} = 0$ for $n \geq 5$ if the Euler class of (4.20) does not vanish.

**Proposition 4.5.** In the adiabatic limit, we have

$$\lim_{\varepsilon \to 0} \frac{1}{2^5 \cdot 7} \sum_{\lambda_0 = \lambda_1 = 0} \text{sign} \lambda_{\varepsilon} \frac{\text{sign}(q^2 p^2_+ - q^2_+ p^2)}{2^5 \cdot 7}.$$

**Proof.** From [9, Theorem 0.3], we know that it is sufficient to study the signature of the quadratic form

$$\langle \alpha, \beta \rangle = (\alpha \wedge d_4 \beta)[M]$$

on $E_4^{0,3}$. Since $\dim E_4^{0,3} = 1$, we only have to compute the sign of $(\alpha \wedge d_4 \alpha)[M]$ for one $\alpha \in E_4^{0,3} \setminus \{0\}$. As a representative of $\alpha$, we may choose

$$\alpha = \begin{cases} \frac{k}{h} \tilde{f}^{01} \tilde{e}^1 - 2k \tilde{f}^{23} \tilde{e}^1 - 2\tilde{e}^{123} & \text{on } [-1, 0], \\ \frac{k}{h} \tilde{f}^{02} \tilde{e}^2 - 2k \tilde{f}^{31} \tilde{e}^2 - 2\tilde{e}^{123} & \text{on } [0, 1]. \end{cases}$$

By (4.28), we know that

$$d\alpha = \frac{4kk'}{h} \tilde{f}^{0123}$$

is of horizontal degree 4 as required. Moreover, integration over the generic fibre of $p: M \to B$ shows that $\alpha$ represents a nontrivial class in $E_2^{0,3} \cong E_4^{0,3}$.

The proof is completed by the computation of the sign of

$$\int_M \alpha \wedge d\alpha = - \int_M \frac{8kk'}{h} \tilde{f}^{0123} \tilde{e}^{123} = - \int_{-1}^1 4\pi^4 k(t)k'(t) dt$$

$$= -2\pi^4 k(t)^2 \big|_{t=-1}^1 = 2\pi^4 \left( \frac{q^2}{p^2} - \frac{q^2_+}{p^2_+} \right). \quad \Box$$

**4.h. The Eells–Kuiper invariant**

We combine Propositions 4.2–4.5 and prove Theorem 3.7 by computing the Eells–Kuiper invariants of the spaces $M = M_{(p_-,q_-),(p_+,q_+)}$. 

Proof of Theorem 3.7. Using Donnelly’s formula for the Eells–Kuiper invariant and the formulas of Bismut–Cheeger and Dai for the adiabatic limit of $\eta$-invariants, we find that

$$\mu(M(p_-, q_-), (p_+, q_+))$$

\[
\begin{align*}
\frac{\eta(B)}{25 \cdot 7} + \frac{3 + h}{2} (D) - \frac{1}{210 \cdot 7} \int_M (p_1 \wedge \hat{p}_1) (TM, \nabla^{TM,0}) \\
= \frac{1}{210 \cdot 7} \left( \frac{q_2^2 - p_2^2}{p^2} - \frac{q_3^2}{p^2} \right) + \frac{\text{sign}(q_2^2 p_2^2 - q_3^2 p_1^2)}{25 \cdot 7} + D(p_-, q_-) - D(p_+, q_+)
\end{align*}
\]

$$= \frac{(p_2^2 - p_2^2)^2}{2 \cdot 7 \cdot p_2^2 p_3^2 (q_2^2 p_2^2 - q_3^2 p_1^2)} - \frac{2^5 (p_2^2 - p_2^2) + 3(q_2^2 p_2^2 - q_3^2 p_1^2)}{210 \cdot 7 \cdot p_2^2 p_3^2}.
\]

\[\square\]

4.i. Quaternionic line bundles

In this subsection, we will prove Theorem 3.3. We will compute the $t$-invariant of [8] for sufficiently many vector bundles on $M$ to determine Crowley’s quadratic form $q: H^4(M) \to \mathbb{Q}/\mathbb{Z}$. To keep computations simple, we only consider bundles $p^* E \to M$ where $E$ is an honest vector bundle over the base orbifold $B$, which becomes trivial after restriction to $B_-$ and $B_+$. This will turn out to be sufficient if $p_-$ and $p_+$ are relatively prime.

To construct $E$, we regard a map $B \to S^4$ of degree one, where the coordinate $\tau$ introduced in Section 4.b is mapped to the height function $\mathbb{R}^5 \supset S^4 \to \mathbb{R}$, and where $B_0 \cong S^3/Q$ is mapped to the equator $S^3 \subset S^4$ by a map of degree one. In particular, for each $\ell \in \mathbb{Z}$, there is a quaternionic bundle $E \to B$, pulled back from $S^4$ by the map above, such that

$c_2(E)[B] = \ell$.

We choose a connection $\nabla^E$ on $E$ that is flat near the singular strata of $B$.

To compute the class $c_2(p^* E)$, we have to study the map

$$p^*: \mathbb{Z} \cong H^4(B) \to H^4(M) \cong \mathbb{Z}/k\mathbb{Z}.$$

We consider the following commutative diagram:

$$\begin{array}{ccc}
H^3(S^3 \times S^3) & \xleftarrow{\eta} & H^3(M_0) \\
\uparrow (\text{id}, 0) & & \uparrow p_0^* \\
H^3(S^3) & \xleftarrow{\eta} & H^3(B_0) \\
\uparrow \delta & & \uparrow \delta \\
H^3(S^3) & \xleftarrow{\delta} & H^4(B)
\end{array}$$

where $\eta: S^3 \times S^3 \to (S^3 \times S^3)/H \cong M_0$ and $\tilde{\eta}: S^3 \to S^3/Q \cong B_0$ are quotient maps, and $\delta$ and $\tilde{\delta}$ are the connecting homomorphisms from the Mayer–Vietoris sequences for the decompositions

$$M = (M \setminus M_+) \cup (M \setminus M_-) \quad \text{with} \quad (M \setminus M_+) \cup (M \setminus M_-) \sim M_0,$$

$$B = (B \setminus B_+) \cup (B \setminus B_-) \quad \text{with} \quad (B \setminus B_+) \cup (B \setminus B_-) \sim B_0.$$
From [17, Section 13], we know that $\eta^*$ is injective with

$$\text{im} \eta^* = \{(a, b) \mid a + b \equiv 0 \mod 8\} \subset \mathbb{Z}^2 \cong H^3(S^3 \times S^3).$$

Similarly,

$$\tilde{\eta}^* = 8 \cdot \mathbb{Z} \cong H^3(B_0) \to H^3(S^3) \cong \mathbb{Z}.$$

It follows that $\eta^*$ maps $\text{im} \eta^0$ to $\langle (8^2, 0) \rangle$. By [17], we also know that $\delta$ is surjective with

$$\eta^* \ker \delta = \langle (-q_2^2, p_2^2), (-q_2^2, p_2^2) \rangle \subset \mathbb{Z}^2 \cong H^3(S^3 \times S^3).$$

Similarly, $\tilde{\delta}$ is an isomorphism.

Let us determine the subset $\text{im} p^* \subset H^4(M)$. All our computations will be done in the standard coordinates on $H^3(S^3 \times S^3) \cong \mathbb{Z} \times \mathbb{Z}$. Then $p^*$ maps a generator of $H^4(B)$ to the image of $(8, 0)$, and $\delta(8\ell, 0) = 0 \in H^4(M)$ if and only if

$$(8\ell, 0) = a(-q_2^2, p_2^2) + b(-q_2^2, p_2^2).$$

If $c$ denotes the greatest common divisor of $p_-$ and $p_+$, then $(8\ell, 0) \in \ker \delta$ if and only if we can choose $a = np_2^2/c^2$ and $b = -np_2^2/c^2$, so

$$\ell = n\frac{p_2^2 - q_2^2}{8c^2} = \pm \frac{nk}{c^2}.$$

Note that $c^2$ divides $k$. In particular, the image of $p^*$ has index $c^2$ in $H^4(M) \cong \mathbb{Z}/k\mathbb{Z}$. If $p_-$ and $p_+$ are relatively prime, then $p^*$ is the isomorphism referred to in Theorem 3.3.

By [8] (see Definition 3.2),

$$t(p^*E) = \frac{\eta + h}{4}(D_{p^*E}^*E) - \frac{\eta + h}{2}(D_M)$$

$$- \frac{1}{24} \int_M \left( \frac{p_1}{2}(TM, \nabla^TM) + c_2(p^*E, \nabla^{p^*E}) \right) \wedge \hat{c}_2(p^*E, \nabla^{p^*E}).$$

Because the fibres are of positive scalar curvature, we can apply Corollary 1.9. Hence,

$$\lim_{\epsilon \to 0} \left( \frac{\eta + h}{4}(D_{p^*E, \epsilon}^*E) - \frac{\eta + h}{2}(D_{M, \epsilon}) \right)$$

$$= \frac{1}{4} \int_{\Lambda B} \hat{A}_{AB}(TB, \nabla^{TB})\eta_{AB}(\hat{k})(\text{ch}(E, \nabla^E) - 2) = 0.$$

Here, the singular strata do not contribute because $\text{ch}(E, \nabla^E) - 2$ vanishes near the singular strata. Over the regular stratum, the degree 0 part of the $\eta$-form is the $\eta$-invariant of the untwisted Dirac operator on the fibre, which vanishes because the fibre is a spin symmetric space. Hence both $\eta(\hat{k})$ and $\text{ch}(E, \nabla^E) - 2$ are of degree 4, so the whole integrand vanishes for degree reasons.
In order to determine $\hat{c}_2(p^*E, \nabla p^*E)$, we first check that
\[
\int_B \frac{4\ell}{\pi^2} \frac{p_-^2}{p_-^2 q_+^2 - p_+^2 q_-^2} \frac{2kk'}{h} \hat{f}^{0123} = \ell \frac{p_-^2}{p_-^2 q_+^2 - p_+^2 q_-^2} \frac{2k(k')}{h} \int_{-1}^{1} 2k(t)k'(t) \, dt
\]
\[
= \ell \frac{p_-^2}{p_-^2 q_+^2 - p_+^2 q_-^2} \frac{2k}{h} \left( \frac{q_+^2}{p_+^2} - \frac{q_-^2}{p_-^2} \right) = \ell
\]
because $\text{vol}(\tau^{-1}(t)) = h(t)\pi^2/4$. Since we have chosen $\nabla^E$ flat near the singular strata, we conclude that
\[
c_2(E, \nabla^E) = \frac{4\ell}{\pi^2} \frac{p_-^2}{p_-^2 q_+^2 - p_+^2 q_-^2} \frac{2kk'}{h} \hat{f}^{0123} + d\gamma
\]
for some form $\gamma \in \Omega^2(B)$ that is supported away from the singular set $B_- \cup B_+$. By (4.28), we may put
\[
\hat{c}_2(p^*E, \nabla p^*E)|_{\tau^{-1}[-1,0]} = \frac{2\ell}{\pi^2} \frac{p_-^2}{p_-^2 q_+^2 - p_+^2 q_-^2} \left( \frac{k'}{h} \hat{f}^{0123} - 2k \hat{f}^{23} - \hat{e}^{23} \right) \hat{e}^1 + p^*\gamma,
\]
and similarly on $\tau^{-1}[0,1]$.

As in the proof of Proposition 4.4, we compute the correction term in the adiabatic limit $\varepsilon \to 0$. We have computed the Pontryagin forms of $TB$ and $TX$ in (4.17) and (4.26). Over $\tau^{-1}(-1, 0)$, we have
\[
\left( \frac{p_1}{2} (TX, \nabla TX) + p^* \frac{p_1}{2} (TB, \nabla TB) + p^* c_2(E, \nabla E) \right) \cdot \hat{c}_2(p^*E, \nabla p^*E) = \frac{1}{4\pi^2} \left( \frac{2h' h''}{h} + 8h^2 h' - 8h' \right) \hat{f}^{0123} + \frac{2kk'}{h} \hat{f}^{0123} + \frac{k'}{h} \hat{f}^{0123} - 2k \hat{f}^{23} \hat{e}^{23}
\]
\[
+ 16\ell \frac{p_-^2}{p_-^2 q_+^2 - p_+^2 q_-^2} \frac{2kk'}{h} \hat{f}^{0123} + 4\pi^2 p^* d\gamma
\]
\[
\cdot \left( \frac{2\ell}{\pi^2} \frac{p_-^2}{p_-^2 q_+^2 - p_+^2 q_-^2} \left( \frac{k'}{h} \hat{f}^{0123} - 2k \hat{f}^{23} - \hat{e}^{23} \right) \hat{e}^1 + p^*\gamma \right)
\]
\[
= - \frac{2\ell}{\pi^4} \frac{p_-^2}{p_-^2 q_+^2 - p_+^2 q_-^2} \left( \frac{h' h''}{h} + 4h^2 h' - 4h' + \frac{2kk'}{h} \right)
\]
\[
+ 16\ell \frac{p_-^2}{p_-^2 q_+^2 - p_+^2 q_-^2} \frac{kk'}{h} \hat{f}^{0123} + 2\pi^2 p^* d\gamma \hat{e}^{123}.
\]
Over $\tau^{-1}(0, 1)$, we obtain the same right hand side. By partial integration and (4.27), we see that there is no contribution from $p^*\gamma$ and $p^* d\gamma$. 

By (4.33), we compute

\[ t(p^*E) = -\lim_{\varepsilon \to 0} \frac{1}{24} \int_M \left( \frac{p_1}{2} (TM, \nabla^{TM, \varepsilon}) + p^* c_2(E, \nabla^E) \right) \cdot \hat{c}_2(p^*E, \nabla^E) \]

\[ = \frac{\ell}{24} \int_{-1}^1 \left( h''h + 4h'h^3 - 4h'h + 2kk' \right. \]

\[ + 16\ell \frac{p_2^2 p_1^2}{p_-^2 q_+^2 - p_+^2 q_-^2} kk' \)

\[ \left. \left( t \right) dt \right) = \frac{\ell}{24} \left( \frac{p_2^2}{p_-^2 q_+^2} - \frac{p_2^2}{p_+^2 q_-^2} \right) + \frac{\ell}{24} \]

\[ = \frac{\ell}{24} \left( p_2^2 - p_+^2 + \ell p_2^2 p_1^2 \frac{p_2^2}{p_-^2 q_+^2} - \frac{\ell}{24} \right)

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References

[1] Atiyah, M. F., Patodi, V. K., Singer, I. M.: Spectral asymmetry and Riemannian geometry. II. Math. Proc. Cambridge Philos. Soc. 78, 405–432 (1975) Zbl 0314.58016 MR 0397798
[2] Berline, M., Getzler, E., Vergne, N.: Heat Kernels and Dirac Operators. Springer, Berlin (1992) Zbl 0744.58001 MR 2273508
[3] Bismut, J.-M., Cheeger, J.: \( \eta \)-invariants and their adiabatic limits. J. Amer. Math. Soc. 2, 33–70 (1989) Zbl 0671.58037 MR 0966608
[4] Bismut, J.-M., Freed, D. S.: The analysis of elliptic families II, Dirac operators, eta invariants, and the holonomy theorem. Comm. Math. Phys. 107, 103–163 (1986) Zbl 0657.58038 MR 0861886
[5] Bismut, J.-M., Lebeau, G.: Complex immersions and Quillen metrics, Inst. Hautes Études Sci. Publ. Math. 74, 1–298 (1991) Zbl 0784.32010 MR 1188532
[6] Crowley, D.: The classification of highly connected manifolds in dimensions 7 and 15. Thesis, Indiana Univ. (2001); arXiv:math.AT/0203253 Zbl 0999.57027 MR 2703475
[7] Crowley, D., Escher, C.: Classification of \( S^3 \)-bundles over \( S^4 \). Differential Geom. Appl. 18, 363–380 (2003) Zbl 1027.55014 MR 1975035
[8] Crowley, D., Goette, S.: Kreck–Stolz invariants for quaternionic line bundles. Trans. Amer. Math. Soc. 365, 3193–3225 (2013) Zbl 1281.58014 MR 3034463
[9] Dai, X.: Adiabatic limits, nonmultiplicativity of signature, and Leray spectral sequence. J. Amer. Math. Soc. 4, 265–321 (1991) Zbl 0736.58039 MR 1088332

[10] Dearricott, O.: A 7-manifold with positive curvature. Duke Math. J. 158, 307–346 (2011) Zbl 1221.53069 MR 2805071

[11] Donnelly, H.: Spectral geometry and invariants from differential topology. Bull. London Math. Soc. 7, 147–150 (1975) Zbl 0306.58019 MR 0372929

[12] Eells, J., Kuiper, N.: An invariant for certain smooth manifolds. Ann. Mat. Pura Appl. 60, 93–110 (1962) Zbl 0119.18704 MR 0156356

[13] Goette, S.: Equivariant $\eta$-invariants and $\eta$-forms. J. Reine Angew. Math. 526, 181–236 (2000) Zbl 0974.58021 MR 1778304

[14] Goette, S.: Eta invariants of homogeneous spaces. Pure Appl. Math. Quart. 5, 915–946 (2009) Zbl 1185.58010 MR 2532710

[15] Goette, S., Kitchloo, N., Shankar, K.: Diffeomorphism type of the Berger space SO(5)/SO(3). Amer. J. Math. 126, 395–416 (2004) Zbl 1066.53069 MR 2045506

[16] Grove, K., Verdiani, L., Ziller, W.: An exotic $T_1S^4$ with positive curvature. Geom. Funct. Anal. 21, 499–524 (2011) Zbl 1230.53032 MR 2810857

[17] Grove, K., Wilking, B., Ziller, W.: Positively curved cohomogeneity one manifolds and 3-Sasakian geometry. J. Differential Geom. 78, 33–111 (2008) Zbl 1145.53023 MR 2406265

[18] Hanson, A. J., Römer, H.: Gravitational instanton contribution to Spin 3/2 axial anomaly. Phys. Lett. 80B, 58–60 (1978)

[19] Hermann, R.: A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle. Proc. Amer. Math. Soc. 11, 236–242 (1960) Zbl 0112.13701 MR 0112151

[20] Kawasaki, T.: The index of elliptic operators over V-manifolds. Nagoya Math. J. 84, 135–157 (1981) Zbl 0437.58020 MR 0641150

[21] Kreck, M., Stolz, S.: A diffeomorphism classification of 7-dimensional homogeneous Einstein manifolds with SU(3)$\times$SU(2)$\times$U(1)-symmetry. Ann. of Math. 127, 373–388 (1988) Zbl 0649.53029 MR 0932303

[22] Lawson, H. B., Michelsohn, L.: Spin Geometry. Princeton Univ. Press, Princeton, NJ (1989) Zbl 0688.57001 MR 1031992

[23] Ma, X.: Functoriality of real analytic torsion forms. Israel J. Math. 131, 1–50 (2002) Zbl 1042.58019 MR 1942300

[24] Mazzeo, R., B. Melrose, R.: The adiabatic limit, Hodge cohomology and Leray’s spectral sequence for a fibration. J. Differential Geom. 31, 185–213 (1990) Zbl 0702.58007 MR 1030670

[25] Rochon, F.: Pseudodifferential operators on manifolds with foliated boundaries. J. Funct. Anal. 262, 1309–1362 (2012) Zbl 1238.58018 MR 2863864

[26] Thurston, W. P.: The Geometry and Topology of Three-Manifolds. http://www.msri.org/publications/books/gt3m/ (2012)

[27] Verdiani, L., Ziller, W.: Concavity and rigidity in non-negative curvature. J. Differential Geom. 97, 349–375 (2014) Zbl 06322520 MR 3263509