THE WU-YAU THEOREM ON SASAKIAN MANIFOLDS

YONG CHEN

ABSTRACT. In this note, we proved that a compact Sasakian manifold
\((M, \xi, \eta, \Phi, g)\) with negative transverse holomorphic sectional curvature
must have a Sasakian structure \((\xi, \eta', \Phi', g')\) with negative transverse Ricci curvature. Similarly, a compact Sasakian manifold with nonpositive transverse holomorphic sectional curvature, then the negative first basic Chern class \(-c_1^B(M, \mathcal{F}_\xi)\) is transverse nef and we have the
Miyaoka-Yau type inequality. When transverse holomorphic sectional curvature is quasi-negative, we obtain a Chern number inequality.

1. INTRODUCTION

The study of holomorphic sectional curvature in Kähler geometry has been
a classical topic, and it attracted many attention in recent years. The recent
breakthrough of Wu and Yau \[22\] in which they proved that any projective
manifolds with negative holomorphic sectional curvature must have an ample
canonical line bundle. Tosatti and Yang \[21\] proved that any compact Kähler
manifold with nonpositive holomorphic sectional curvature must have a nef
canonical line bundle and then extended Wu-Yau’s work to the Kähler case.
More recently, Diverio and Trapani \[8\] further generalized the result by
assuming that the holomorphic sectional curvature is only quasi-negative.
In \[23\], Wu and Yau give a direct proof of the statement that any compact
Kähler manifold with quasi-negative holomorphic sectional curvature must
have an ample canonical line bundle.

Inspired by the Kobayashi conjecture and the above results, it was conjectured
that any Hermitian manifolds with quasi-negative (nonpositive) holomorphic sectional curvature must have an ample (nef) canonical line bundle.
To the best of author’s knowledge, the Wu-Yau’s theorem in Hermitian case
is still widely open. There are some related works about this problem, readers can refer to \[14\] \[27\].

An odd dimensional Riemannian manifold \((M, g)\) is said to be a Sasakian
manifold if the cone manifold \((C(M), \tilde{g}) = (M \times \mathbb{R}^+, dr^2 + r^2g)\) is Kähler.
Sasakian geometry was introduced by Sasaki \[19\] in 1960s and is often described as an odd dimensional counterparts of Kähler geometry. Sasakian
manifold has received a lot of attention because it is the natural intersection of CR, contact and Riemannian geometry, and plays a very important role in Riemannian, algebraic geometry and in physics. Sasakian manifolds first appeared in String theory in [15]. Sasaki-Einstein metric is useful in Ads/CFT correspondence. In this paper, we would consider the theorem of Wu-Yau type on Sasakian manifolds. Sasakian manifolds can be studies from many view points as they have many structures. They have a natural foliation structure $\mathcal{F}_\xi$, called Reeb foliation, which has a transverse Kähler structure; they also has a contact structure. Sasakian geometry is a special kind of contact metric geometry. There exists a unique transverse connection $\nabla^T$ corresponding to the Sasakian structure. A good reference on Sasakian geometry can be found in the monograph [2] by Boyer and Galicki. We can also define the transverse holomorphic sectional curvature (see section 2). In Kähler geometry, The Wu-Yau theorem tell us the negativity of holomorphic sectional curvature would lead to the positivity of canonical bundle $K_M$, which is equivalent to the positivity of the negative first Chern class $c_1(M)$. Similarly, We have the theorems on Sasakian manifold.

**Theorem 1.1.** Let $(M, \xi, \eta, \Phi, g)$ be a $2n + 1$ dimensional compact Sasakian manifold with negative transverse holomorphic sectional curvature, then the first basic Chern class $c^B_1(M, \mathcal{F}_\xi)$ is negative. By the transverse Calabi-Yau theorem, $M$ has a Sasakian structure $(\xi, \eta', \Phi', g')$ with negative transverse Ricci curvature, which is compatible with $(\xi, \eta, \Phi, g)$.

In complex geometry, a $(1, 1)$ class $\alpha \in H^{(1,1)}(M, \mathbb{R})$ on a Kähler manifold $(M, \omega)$ is called nef if for every $\epsilon > 0$ there exists a smooth $(1, 1)$-form $\theta_\epsilon \in \alpha$ on $M$ such that satisfies

$$
\theta_\epsilon \geq -\epsilon \omega.
$$

We will give the definition of nefness in Sasakian geometry in Section 2 following the definition of nefness in complex geometry.

**Theorem 1.2.** Let $(M, \xi, \eta, \Phi, g)$ be a $2n + 1$ dimensional compact Sasakian manifold with nonpositive transverse holomorphic sectional curvature, then the negative first basic Chern class $-c^B_1(M, \mathcal{F}_\xi)$ is transverse nef.

**Theorem 1.3.** Let $(M, \xi, \eta, \Phi, g)$ be a $2n + 1$ dimensional compact Sasakian manifold with quasi-positive transverse holomorphic sectional curvature, then we have the basic Chern number inequality

$$
\int_M (-c^B_1(M, \mathcal{F}_\xi))^n \wedge \eta > 0.
$$

There are many results about Miyaoka-Yau inequality for Chern numbers inequality in Kähler case; an incomplete list: $K_X$ is ample [26], minimal
manifolds of general type [30], minimal projective varieties [12], and compact\nKähler manifolds whose $c_1(K_X)$ admits a smooth semipositive representa-
tive [16], compact Kähler manifolds with almost nonpositive holomorphic\nsectional curvature [31]. In Sasakian case, Zhang, Xi proved the Miyaoka-
Yau inequality for basic Chern numbers holds on compact Sasakian-Einstein\nmanifolds [29]. We proved the Miyaoka-Yau inequality for basic Chern num-
bers holds on compact Sasakian manifolds with nonpositive transverse holomor-
phic sectional curvature.

**Theorem 1.4.** Let $(M, \xi, \eta, \Phi, g)$ be a $2n + 1$ dimensional compact Sasakian\nmanifold with nonpositive transverse holomorphic sectional curvature, then\nwe have the following Miyaoka-Yau type inequality:

$$\int_M (2c_2^B(M, F_\xi) - \frac{n}{n+1}c_1^B(M, F_\xi)^2 \wedge (-c_1^B(M, F_\xi))^{n-2} \wedge \eta \geq 0$$

2. Preliminaries

2.1. **Preliminary results in Sasakian Geometry.** Sasakian manifolds\nhave many equivalent descriptions. It can be defined in terms of metric con-
tact geometry or transverse Kähler geometry. Boyer and his collaborators\n[3, 4, 5, 6] published a series of papers investigating various differential geo-
metric aspects of Sasakian manifolds. We can find transverse counterparts\non Sasakian manifolds of the famous results in Kähler manifolds, such as\nthe transverse Calabi-Yau theorem [13] (see also [3, 20]), the existence of\ncanonical metrics on Sasakian manifolds [10]. A good reference on Sasakian\ngeometry can be found in the monograph [2] by Boyer and Galicki.

A Sasakian manifold $(M, g)$ has a contact structure $(\xi, \eta, \Phi)$. $\eta \wedge (d\eta)^n$\ndefines a volume element on $M$. There is a canonical vector field $\xi$ defined\nby

$$\eta(\xi) = 1, d\eta(\xi, \ast) = 0$$

$\xi$ is called the Reeb vector field. The contact 1-form $\eta$ defines a $2n$-dimensional\nvector bundle $D$ over $M$, the fiber $D_p$ of $D$ is given by

$$D_p = \ker \eta_p.$$ 

There is a decomposition of the tangent bundle $TM$,

$$TM = D \oplus L_\xi$$

where $L$ is the trivial bundle generated by the Reeb vector $\xi$. The Riemann-
ian metric $g$ and a tensor field $\Phi$ of type $(1, 1)$ satisfy

$$\Phi^2 = -Id + \eta \otimes \xi$$

and

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$$
Now we exploit the transverse structure of Sasakian manifolds. On the subbundle $D$, it is naturally endowed with both a complex structure $\Phi|_D$ and a symplectic structure $d\eta$. Since both $g$ and $\xi$ are invariant under $\xi$, there is a well-defined Kähler structure $(g^T, \omega^T, J^T)$ on the local leaf space of Reeb foliation $F_\xi$. We call this a transverse Kähler structure. Clearly

$$g^T(X, Y) = \frac{1}{2} d\eta(X, \Phi Y).$$

$$J^T = \Phi|_D$$

The metric $g^T$ is related to the Sasakian metric $g$ by

$$g = g^T + \eta \otimes \eta.$$  

The upper script $T$ is used to denote both the transverse geometry quantity, and the corresponding quantity on the bundle $D$. For the transverse metric $g^T$, there is a unique, torsion-free connection on the subbundle $D$, which is called the transverse Levi-Civita connection

$$\nabla^T_X Y = \begin{cases} (\nabla_X Y)^p, & X \in D \\ [\xi, Y]^p, & X = \xi \end{cases}$$

where $Y$ is a section of $D$ and $X^p$ the projection of $X$ onto $D$, and it satisfies

$$\nabla^T_X Y - \nabla^T_Y X - [X, Y]^p = 0,$$

$$Xg^T(Z, W) = g^T(\nabla^T_X Z, W) + g^T(Z, \nabla^T_X W).$$

for any $X, Y \in TM$ and $Z, W \in D$. The transverse curvature relating with the above transverse connection is defined by

$$R^T(X, Y)Z = \nabla^T_X \nabla^T_Y Z - \nabla^T_Y \nabla^T_X Z - \nabla^T_{[X, Y]} Z,$$

where $X, Y \in TM$ and $Z \in D$. From the above transverse curvature operator we define the transverse Ricci curvature by

$$Ric^T(X, Y) = \sum_i \langle R^T(X, e_i) e_i, Y \rangle,$$

Where $\{e_i\}$ is an orthonormal basis of $D$ and $X, Y \in D$. One can easily check that

$$Ric^T(X, Y) = Ric(X, Y) + 2g^T(X, Y)$$

for any $X, Y \in D$. Let $\rho^T(X, Y) = Ric^T(\Phi X, Y)$. $\rho^T$ is called the transverse Ricci form, which is a representation of the first basic Chern class.

We consider the complexified bundle $D^C = D \otimes \mathbb{C}$. Using the structure $\Phi$ we decompose $D^C$ into two subbundles $D^{(1,0)}$ and $D^{(0,1)}$, where $D^{(1,0)} = \{X \in D^C|\Phi X = \sqrt{-1}X\}$ and $D^{(0,1)} = \{X \in D^C|\Phi X = -\sqrt{-1}X\}$. 
Definition 2.1. Given a $\Phi$-invariant planes $\sigma$ in $D_x \subseteq T_x M$, the transverse holomorphic sectional curvature $H^T(\sigma)$ is defined by
\[ H^T(\sigma) = \langle R^T(X, JX)JX, X \rangle, \]
where $X$ is a unit vector in $\sigma$. It is easy to check that $\langle R^T(X, JX)JX, X \rangle$ only depend on $\sigma$. We say that the transverse holomorphic sectional curvature is negative (nonpositive) if $H^T(\sigma) < 0 (\leq 0)$, for any $\sigma$ in $D_x$ and any $x$ in $M$. The transverse holomorphic sectional curvature is said to be quasi-negative if $H^T(\sigma) \leq 0$ and, moreover, there exists at least one point $p \in M$ such that $H^T(\sigma) < 0$ for every $\sigma \in D_p$.

Let $u$ be a unit vector on $D$, and a $\Phi$-invariant plane spanned by $u, \Phi u$ are denoted by $\sigma_u$. Setting $U = \frac{1}{2}(u - \sqrt{-1}\Phi u)$. Then, we have
\[ \langle R^T(U, \bar{U})U, \bar{U} \rangle = \frac{1}{4}H^T(\sigma_u) \]
By the above formula, we know that the positivity of of transverse holomorphic sectional curvature is equivalent to $\langle R^T(U, \bar{U})U, \bar{U} \rangle > 0$, for all $U \in D^{(1,0)}$.

On the Sasakian manifold $(M, \xi, \eta, \Phi, g)$, the basic Laplacian is defined by
\[ \Delta_B u = \frac{4n}{\sqrt{-1}} \partial_B \bar{\partial}_B u \wedge (d\eta)^{n-1} \wedge \eta \]
for any basic function $u$. It is well-known that the basic Laplacian is equal to the Riemannian Laplacian $\Delta_g$ on basic function, i.e. $\Delta_B u = \Delta_g u$ for any basic function $u$.

Fixing a transverse holomorphic structure on $F_\xi$, there is also a notion of transverse cohomology on Sasakian manifolds. A $p$-form $\alpha$ on $(M, \xi, \eta, \Phi, g)$ is called basic if $\iota_\xi \alpha = 0$, and $\mathcal{L}_\xi \alpha = 0$. We let $\Lambda^p_B$ be the sheaf of basic $p$-forms, and $\Omega^p_B = \Gamma(M, \Lambda^p_B)$ the global sections. It is easy to see that the de Rham differential $d$ preserves basic forms, and hence restricts to a well defined operator $d_B : \Lambda^p_B \to \Lambda^{p+1}_B$. We thus get a complex
\[ 0 \to C^\infty_B(M) \to \Omega^1_B \xrightarrow{d_B} \cdots \xrightarrow{d_B} \Omega^{2n}_B \xrightarrow{d_B} 0. \]
whose cohomology groups, denoted by $H^p_B(M, F_\xi)$, are the basic de Rham cohomology groups. Moreover, the transverse complex structure $\Phi$ allows us to decompose
\[ \Lambda^r_B \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{p,q}_B \]
We can then decompose $d_B = \partial_B + \bar{\partial}_B$, where
\[ \partial_B : \Lambda^{p,q}_B \to \Lambda^{p+1,q}_B, \quad \text{and} \quad \bar{\partial}_B : \Lambda^{p,q}_B \to \Lambda^{p,q+1}_B. \]
So we can define the basic Dolbeault cohomology groups $H^{p,*}_{B}(M,F_\xi)$. We also have the transverse Chern-Weil theory and can define the basic Chern classes $c_k^B(M,F_\xi)$. For detail, see [2].

**Definition 2.2.** Let $(M,\xi,\eta,\Phi,g)$ be a compact Sasakian manifold, a $(1,1)$ basic class $\alpha \in H^{1,1}_{B}(M,F_\xi)$ is called transverse nef. If every $\epsilon > 0$ there exist basic $(1,1)$ forms $\theta_\epsilon \in \alpha$ on $M$ such that satisfies
\begin{equation}
\theta_\epsilon + \epsilon \eta \text{ is a basic transverse positive } (1,1) \text{ form.}
\end{equation}

On Sasakian manifolds, the $\partial \bar{\partial}$-lemma holds for basic forms.

**Proposition 2.3.** Let $\theta$ and $\theta'$ be two real closed basic forms of type $(1,1)$ on a compact Sasakian manifold $(M,\xi,\eta,\Phi,g)$. If $[\theta]_{B} = [\theta']_{B} \in H^{1,1}_{B}(M,F_\xi)$, then there is a basic real function $\phi$ such that
\begin{equation}
\theta - \theta' = \sqrt{-1} \partial B \bar{\partial} \phi.
\end{equation}

We fix a canonical orientation $\eta \wedge (d\eta)^n$ and introduce the concepts of transverse positivity on Sasakian manifolds corresponding to the complex case. A basic $(p,p)$ form $\sigma(\geq 0)$ is said to be transverse positive if for any basic $(1,0)$ forms $\gamma_j, 1 \leq j \leq p$, then
\begin{equation}
\sigma \wedge \sqrt{-1} \gamma_1 \wedge \bar{\gamma}_1 \wedge \cdots \wedge \sqrt{-1} \gamma_{n-p} \wedge \bar{\gamma}_{n-p} \wedge \eta
\end{equation}
is a positive volume form.

Any transverse positive basic $(p,p)$ form $\sigma$ is real, i.e., $\sigma = \bar{\sigma}$. In particular, in the local coordinates, a real basic $(1,1)$ form
\begin{equation}
\sigma = \sqrt{-1} \sigma_{ij} dz_i \wedge d\bar{z}_j
\end{equation}
is transverse positive if and only if $(\sigma_{ij})$ is a semipositive Hermitian matrix with $\xi(\sigma_{ij}) = 0$. We call a real basic $(1,1)$ form $\sigma$ strictly transverse positive if the Hermitian matrix $(\sigma_{ij})$ is positive definite. Given another real basic $(1,1)$ form $\beta$. We can define
\begin{equation}
\text{tr}_\sigma \beta := \frac{n \beta \wedge \sigma^{n-1}}{\sigma^n} = \sigma^{ij} \beta_{ij}
\end{equation}
where $(\sigma^{ij})$ is the inverse matrix of $\sigma_{ij}$. A basis $(1,1)$ class $\alpha \in H^{1,1}_{B}(M,F_\xi)$ is called positive, if there exist a real transverse positive basic $(1,1)$ form $\sigma \in \alpha$. We call a basic $(1,1)$ class $\alpha$ negative, if and only if $-\alpha$ is positive.

### 2.2. Foliated local coordinate and local computations.

In this section, we first review local coordinates on a Sasakian manifold. In [11], it has been proven that every Sasakian manifold have a good local coordinates $(x,z^1,z^2,\cdots,z^n)$ on a small neighborhood $U$ such that
\begin{equation}
\xi = \frac{\partial}{\partial x},
\end{equation}
where $h : U \to \mathbb{R}$ is a local basic function, i.e., $\frac{\partial h}{\partial x^i} = 0$, $h_i = \frac{\partial h}{\partial z^i}$, and $X_j = \frac{\partial}{\partial z^j} + \sqrt{-1}h_{j\bar{k}}\frac{\partial}{\partial x^k}$, we denote $2dz^i\bar{dz}^j = dz^i \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^i$. In such local coordinates, $D \otimes \mathbb{C}$ is spanned by $X_i$ and $\bar{X}_i$, it is clear that
\[
\Phi X_i = \sqrt{-1}X_i, \quad \Phi \bar{X}_i = -\sqrt{-1}\bar{X}_i,
\]
and $\{\eta, dz^1, d\bar{z}^1\}$ is the dual basis of $\{\frac{\partial}{\partial x}, X_i, \bar{X}_j\}$, and
\[
\omega^T = \frac{1}{2}d\eta = \sqrt{-1}h_{ij}dz^i\bar{d}z^j,
\]
the transverse metric
\[
g^T = 2g^T_{ij}dz^i\bar{d}z^j = 2h_{ij}dz^i\bar{d}z^j,
\]
where $g^T_{ij} = g^T(X_i, 
\bar{X}_j) = h_{ij}$. From the formula above, we know that
\[
\nabla^T_{\partial x} X_i = \nabla^T_{\partial \bar{x}} \bar{X}_j = 0. \text{ Define } \Gamma^A_{BC} \text{ by }
\]
\[
\nabla^T_{\partial x} X_C = \Gamma^A_{BC} X_A.
\]
for $A, B, C = 1, 2, \cdots, n, \bar{1}, \bar{2}, \ldots, \bar{n}$, where $X_j = \bar{X}_j$. It is easy to check that only $\Gamma^i_{jk}$ and $\Gamma^i_{jk}$ may not vanish as in the Kähler case. Moreover,
\[
\Gamma^i_{jk} = \Gamma^i_{kj} = h^{ij} \frac{\partial h_{j\bar{l}}}{\partial z^k}.
\]
This local coordinates are also called by a normal coordinates on Sasakian manifold. We can cover $M$ by finite foliated local coordinate charts $\{U_\alpha\}$ which is diffeomorphism to $(-\epsilon_0, \epsilon_0) \times B_2(0)$ with $\epsilon_0 > 0$, where $B_2(0)$ is the ball in $\mathbb{C}^n$ centered at origin with radius 2, and on $B_2(0)$, there holds
\[
(2.3) \quad C^{-1} \delta_{ij} \leq d\eta \leq C \delta_{ij}
\]
for a uniform constant $C$. Moreover, $\{\frac{1}{2}U_\alpha\}$ is diffeomorphism to $(-\frac{1}{2}\epsilon_0, \frac{1}{2}\epsilon_0) \times B_1(0)$ still cover $M$. We have basic Sobolev space
\[
(2.4) \quad C^k_{\alpha}(M) = \{u \mid \xi u = 0, ||u||_{C^k_{\alpha}(M)} < \infty\},
\]
where we use the following notation : in finite foliated local coordinate charts $\{U_i\}_{i=1}^m$
\[
||u||_{C^k_{\alpha}(M)} := \sum_{1 \leq i \leq m} ( \sum_{0 \leq j \leq k} \sup_{U_i} |D^j u| + \sup_{x,y \in U_i, x \neq y} \frac{|D^k u(x) - D^k u(y)|^\alpha}{||x - y||} )
\]
Remark 2.4. For a fixed point \( P \in M \), we can always choose the above coordinates \( (x, z^1, z^2, \cdots, z^n) \) centered at \( P \) satisfying additionally that \( \{ (\frac{\partial}{\partial z^i})_P \} \in D^{(1,0)} \) or equivalently \( h_i(P) = 0 \) for all \( i \). Furthermore, in the same way as that in Kähler case, one can choose a normal coordinates \( (x, z^1, z^2, \cdots, z^n) \), such that \( h_i = 0, h_{i\bar{j}}(P) = \delta_{ij} \), and \( dh_{i\bar{j}}|_P = 0 \), i.e., \( \Gamma_{i\bar{j}k}|_P = 0 \) for all \( i, j, k \).

One can easily check that the transverse Ricci curvature can be expressed by

\[
R^T_{ij} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g^T_{mn}).
\]

and \( \rho^T = \sqrt{-1} R^T_{ij} dz^i \wedge d\bar{z}^j \).

Suppose that \( (\xi, \eta, \Phi, g) \) defines a Sasakian structure on \( M \). Let \( \varphi \) be a basic function satisfying

\[
\eta_\varphi \wedge (d\eta_\varphi)^n \neq 0.
\]

set

\[
\eta_\varphi = \eta + \frac{1}{2} \sqrt{-1} (\partial_B - \partial_B) \varphi.
\]

\[
\Phi_\varphi = \Phi - \xi \otimes (d\Phi_\varphi) \circ \Phi, \quad g_\varphi = \frac{1}{2} d\eta_\varphi \circ (\text{Id} \otimes \Phi_\varphi) + \eta_\varphi \otimes \eta_\varphi.
\]

It is clear that

\[
d\eta_\varphi = d\eta + \sqrt{-1} \partial_B \bar{\partial}_B \varphi.
\]

and \( (\xi, \eta_\varphi, \Phi_\varphi, g_\varphi) \) is also a Sasakian structure on \( M \). Furthermore, \( (\xi, \eta_\varphi, \Phi_\varphi, g_\varphi) \) and \( (\xi, \eta, \Phi, g) \) have the same Reeb field and the same transversely holomorphic structure, i.e. the Sasakian structures compatible with \( (\xi, \eta, \Phi, g) \). By El-Kacimi’s \cite{13} generalization of Yau’s estimates for transverse Monge-Ampère equations. For any \( \alpha \in c_{\mathbb{R}}^B(M, F_\xi) \), there exists a compatible Sasakian structure \( (\xi, \eta_\varphi, \Phi_\varphi, g_\varphi) \) such that the transverse Ricci form satisfying

\[
\rho^T_\varphi = \alpha.
\]

3. Proof of Theorem 1.1

In this section, We will follow Wu and Yau’s method \cite{22, 23} to prove that a compact Sasakian manifold with negative transverse holomorphic sectional curvature has the negative first basic Chern class. We will prove Theorem 1.1 by solve a family of transverse Monge-Ampère equations parameterized by \( t \) and consider the limit of the family of transverse basic positive \((1, 1)\)-form

\[
(3.1) \quad t d\eta - \rho^T + \sqrt{-1} \partial_B \bar{\partial}_B u_t.
\]

By the transverse Aubin-Yau theorem \cite{10, 13, 20}, we have
Theorem 3.1. Let \((M, \xi, \eta, \Phi, g)\) be a compact Sasakian manifold with \(\dim_\mathbb{R} M = 2n + 1\). \(\frac{1}{2}d\eta\) is the transverse Kähler form corresponding to the transverse metric \(g^T\) defined by \(g^T(X, Y) = \frac{1}{2}d\eta(X, \Phi Y)\). Let \(\sigma\) be a strictly transverse positive basic real \((1, 1)\)-form on \(M\). There exists a unique basic function \(u \in C^\infty_B(M, \mathbb{R})\) solves the equation
\[
(\sigma + \sqrt{-1}\partial_B \bar{\partial} Bu)^n \wedge \eta = e^u (d\eta)^n \wedge \eta.
\]
where \(\sigma + \sqrt{-1}\partial_B \bar{\partial} Bu\) is a strictly transverse positive basic real \((1, 1)\) form.

Proof of Theorem 1.1. For \(t > 0\), We consider a family of transverse Monge-Ampère equations
\[
(td\eta - \rho^T + \sqrt{-1}\partial_B \bar{\partial} Bu_t)^n \wedge \eta = e^{u_t} (d\eta)^n \wedge \eta.
\]
Since \(d\eta\) is a transverse Kähler form and \(M\) is compact, there exists a sufficiently large constant \(t_1 > 0\) such that \(t_1 d\eta - \rho^T + \sqrt{-1}\partial_B \bar{\partial} Bu_t > 0\).

Fix a nonnegative integer \(k\) and \(0 < \alpha < 1\). We use the notation that
\[
C_B^{k+2,\alpha}(M) = \{ u \in C^{k+2,\alpha}(M) | \xi u = 0 \}.
\]
Define
\[
I = \{ t \in [0, t_1] | \text{there is a solution } u_t \in C_B^{k+2,\alpha}(M) \text{ satisfying (3.3)} \}.
\]
First, by Theorem 1.1, we know \(I \neq \emptyset\), since \(t_1 \in I\).

That \(I\) is open in \([0, t_1]\) follows from the implicit function theorem. Let \(t_0 \in I\) with corresponding function \(u_{t_0} \in C_B^{k+2,\alpha}(M)\). Then, there exists a small neighborhood \(J\) of \(t_0\) in \([0, t_1]\) and a small neighborhood \(U\) of \(u_{t_0}\) in \(C_B^{k+2,\alpha}(M)\) such that
\[
\sigma_t = t d\eta - \rho^T + \sqrt{-1}\partial_B \bar{\partial} Bu > 0.
\]
for all \(t \in J\) and \(u \in U\). Define a map \(\Phi : J \times U \to C_B^{k,\alpha}(M)\).

\[
\Phi(t, u) = \log \frac{(td\eta - \rho^T + \sqrt{-1}\partial_B \bar{\partial} Bu)^n \wedge \eta}{(d\eta)^n \wedge \eta} - u.
\]
with \(\Phi(t_0, u_{t_0}) = 0\). It suffices to prove the invertibility of the linearization
\[
(D\Phi)_{u}(t_0, u_{t_0}) : C_B^{k+2,\alpha}(M) \to C_B^{k,\alpha}(M)
\]
where the linearization is
\[
(D\Phi)_{u}(t_0, u_{t_0})h = \frac{d}{ds}\Phi(t_0, u_{t_0} + sh)|_{s=0} = \sigma_{t_0}^{ij} h_{ij} - h.
\]
By maximum principle, we know that the linearization is injective. From the implicit function theorem, we know \(I\) is open. The estimate (3.28) shows the closeness of \(I\). In particular, \(0 \in I\) with corresponding \(u_0 \in C_B^{\infty}(M)\). This gives us the desired strictly transverse positive basic \((1, 1)\) form \(\cdot - \rho^T +
\( \sqrt{-1} \partial_B \bar{\partial}_B u_0 \in -c^i_t(M, F) \). By the transverse Calabi-Yau theorem, \( M \) has a Sasakian structure \((\xi, \eta', \Phi', g')\)
\[
\rho^T(g') = \rho^T - \sqrt{-1} \partial_B \bar{\partial}_B u_0
\]
which is transverse negative and compatible with \((\xi, \eta, \Phi, g)\). \( \square \)

**Proposition 3.2.** Let \((M, \xi, \eta, \Phi, g)\) be a compact Sasakian manifold with the upper bounded transverse holomorphic sectional curvature by \(-\kappa\) \((\kappa > 0)\), assuming that \(\sigma(t)\) is a solution of \((3.3)\), where \(t \in I\). We have the uniformly second estimate
\[
\text{tr}_{\sigma(t)} d\eta \leq \frac{2n\kappa}{n + 1},
\]
for all \(t \in I\).

**Proof.** Choose a normal coordinate system \((x, z^1, z^2, \ldots, z^n)\) about Sasakian metric \(g\) near a point \(P\) of \(M\) such that \(\{\sigma_{ij}\}\) is diagonal, we have
\[
\sigma^{kl}(t) \partial_k \partial_l \text{tr}_{\sigma(t)} d\eta = \sigma^{ij}(t) \partial_i \partial_j (\sigma^{kl}(t) g_{kl}^T) = \sigma^{ij}(t) \sigma^{kl}(t) \partial_i \partial_j g_{kl}^T + \sigma^{ij}(t) \partial_i \partial_j \sigma^{kl}(t) g_{kl}^T
\]
where
\[
R_{ijkl}^T = -\partial_i \partial_j g_{kl}^T.
\]
By Royden’s lemma [13], we have
\[
\sigma^{ij}(t) \sigma^{kl}(t) R_{ijkl}^T \leq -\frac{n + 1}{2n} \kappa (\text{tr}_{\sigma(t)} d\eta)^2
\]
By some calculations, we have
\[
\sigma^{ij}(t) \partial_i \partial_j \sigma^{kl}(t) g_{kl}^T = \sigma^{ij}(t) (-\partial_i \partial_j \sigma_{kk}(t) \sigma^{kl}(t) \sigma^{kk}(t) + \sigma^{kk}(t) \sigma^{qk}(t) \partial_i \sigma_{qk}(t) \partial_j \sigma_{kp}(t) + \sigma^{kk}(t) \sigma^{p}(t) \partial_i \sigma_{pk}(t) \partial_j \sigma_{kp}(t))
\]
\[
-\partial_i \partial_j \log \det \sigma(t) = -\sigma^{kk}(t) \partial_i \partial_j \sigma_{kk}(t) + \sigma^{kk}(t) \partial_j \sigma_{pk}(t) \sigma^{pp}(t) \partial_i \sigma_{kp}(t)
\]
By \((3.12)\) and \((3.13)\), we have
\[
\sigma^{ij}(t) \partial_i \partial_j \sigma^{kl}(t) g_{kl}^T = -\partial_k \partial_k \log \det \sigma(t) \sigma^{kk}(t) \sigma^{kl}(t) + \sigma^{ij}(t) \sigma^{kk}(t) \sigma^{qk}(t) \sigma^{kk}(t) \partial_i \sigma_{qk}(t) \partial_j \sigma_{kl}(t)
\]
Lemma 3.1. Let $\sigma(t)$ be a compact Sasakian manifold with the upper bounded transverse holomorphic sectional curvature by $-\kappa$ ($\kappa > 0$), assuming that $\sigma(t)$ is a solution of (3.3), where $t \in I$. There exist a uniform constant $C$ which is independent of $\epsilon$ such that

\[ \sup_M |u_t| \leq C. \]
Proof. Choose a point $x_t \in M$ where $u(t)$ achieves its maximum, we know that $td\eta - \rho^T$ is a strictly transverse positive at $x_t$ and

\begin{equation}
(3.22) \quad e^{\sup_M u_t} = e^{u_t(x_t)} \leq \frac{(td\eta - \rho^T)^n \wedge \eta}{(d\eta)^n \wedge \eta}(x_t) \leq C.
\end{equation}

so we have

\begin{equation}
(3.23) \quad \sup_M \sigma(t)^n \wedge \eta \leq C.
\end{equation}

Combining (3.20) and (3.23) and the elementary inequality

\begin{equation}
(3.24) \quad \operatorname{tr} d\eta \sigma(t) \leq \operatorname{tr} \sigma(t) d\eta \frac{\sigma(t)^n}{(d\eta)^n \wedge \eta}
\end{equation}

we have

\begin{equation}
(3.25) \quad \operatorname{tr} d\eta \sigma(t) \leq C,
\end{equation}

and (3.20) and (3.25) together give

\begin{equation}
(3.26) \quad \frac{1}{C} d\eta \leq \sigma(t) \leq Cd\eta.
\end{equation}

Combining (3.3) and (3.26) we have the lower uniform bound estimate about $u(t)$

\begin{equation}
(3.27) \quad \inf_M u(t) \geq -C.
\end{equation}

The $C^{2,\alpha}$-estimate about (3.3) follows Blocki’s theorem.

**Theorem 3.3** ([1], Theorem 3.1). Let $u$ be a $C^4$-psh function in an open $\Omega \subseteq \mathbb{C}^n$. Assume that for some positive $K_0, K_1, K_2, b, B_0$ and $B_1$, we have

$$|u| \leq K_0, |Du| \leq K_1, \Delta u \leq K_2$$

and

$$b \leq \Phi \leq B_0, |D\Phi|^2 \leq B_1$$

in $\Omega$, where $\Phi = \det(u_{ij})$. Let $\Omega' \Subset \Omega$. Then there exist $\alpha \in (0, 1)$ depending, besides these constants, on $\operatorname{dis}(\Omega', \partial \Omega)$ such that

$$||D^2 u||_{C^0(\Omega')} \leq C.$$

In the foliated local coordinate patch $(-\epsilon_0, \epsilon_0) \times B_2(0)$, we work in $B_2(0)$. Combining (3.21) and (3.26), the $C^{2,\alpha}$ estimate follows from Theorem 3.3. The transverse elliptic Schauder estimates give the higher order estimates.

\begin{equation}
(3.28) \quad ||u(t)||_{C^k_b(M)} \leq C_k
\end{equation}

where $C_k$ is independent of $\epsilon$, for all $k \geq 0$ and $t \in I$. 
4. Proof of Theorem 1.2

Proof of Theorem 1.2. If \(-c^B_1(M,F_\xi)\) is not transverse nef, then
\[
(4.1) \quad \epsilon_0 = \inf \{\epsilon > 0 | \exists u_\epsilon \in C^\infty(M) \text{ s.t. } \epsilon d\eta - \rho^T + \sqrt{-1}\partial_B\bar{\partial}_Bu_\epsilon > 0\}
\]
is positive. For \(\epsilon > 0\), We consider a family of transverse Monge-Ampère equations
\[
(4.2) \quad ((\epsilon + \epsilon_0)d\eta - \rho^T + \sqrt{-1}\partial_B\bar{\partial}_Bu_\epsilon)^n \wedge \eta = e^{u_\epsilon}(d\eta)^n \wedge \eta.
\]
Denoting \(\sigma(\epsilon) = (\epsilon + \epsilon_0)d\eta - \rho^T + \sqrt{-1}\partial_B\bar{\partial}_Bu_\epsilon\). We have uniformly \(C^2\)-estimate as Proposition 3.2
\[
\sigma^{kl}(\epsilon)\partial_k\partial_l \log \text{tr}_{\sigma(\epsilon)} d\eta \geq \frac{1}{\text{tr}_{\sigma(\epsilon)} d\eta} \left(\frac{n+1}{2n}\kappa(\text{tr}_{\sigma(\epsilon)} d\eta)^2 - \text{tr}_{\sigma(\epsilon)} d\eta + \frac{\epsilon_0}{n}(\text{tr}_{\sigma(\epsilon)} d\eta)^2\right)
\]
\[
\geq -1 + \frac{\epsilon_0}{n} \text{tr}_{\sigma(\epsilon)} d\eta,
\]
so we will have
\[
(4.4) \quad \frac{1}{C} d\eta \leq \sigma(\epsilon) \leq Cd\eta.
\]
where \(C\) is independent of \(\epsilon\). The higher order estimates is the same argument as Theorem 1.1.
\[
(4.5) \quad \|u_\epsilon\|_{C^k(M)} \leq C_k.
\]
where \(C_k\) is independent of \(\epsilon\), for all \(k \geq 0\). By Ascoli-Arzelà theorem and a diagonal argument with (4.3) and (4.4), we obtain that there exists a sequence \(\epsilon_i \to 0\) such that \(u_\epsilon\) converges smoothly to a basic function \(u_0\) and \(\sigma_{\epsilon_i}\) converges smoothly to a strictly positive basic \((1,1)\) form
\[
(4.6) \quad \sigma_0 = \epsilon_0 d\eta - \rho^T + \sqrt{-1}\partial_B\bar{\partial}_Bu_0.
\]
Since \(\sigma_0\) is a strictly transverse positive basis \((1,1)\) form and \(M\) is compact, we know
\[
(4.7) \quad \sigma_0 = (\epsilon_0 - \epsilon) d\eta_B - \rho^T + \sqrt{-1}\partial_B\bar{\partial}_Bu_0.
\]
is also a strictly transverse positive basis \((1,1)\) form, when \(\epsilon\) is enough small. This is contradictory to the assumption in (4.1). So \(\epsilon_0\) is equal to 0, then
\[-c^B_1(M,F_\xi)\] is transverse nef. \(\square\)

5. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3 following the method in [23]. Zhang ([28]) generalized the exponential estimates of plurisubharmonic functions in Kähler geometry to the Sasakian situation.
Lemma 5.1 ([23], Proposition 3.3). Let $\sigma$ be a basic $(1, 1)$ form on a compact Sasakian manifold $(M, \xi, \eta, \Phi, g)$, and $\mathcal{H}(M, \xi, \eta, \Phi, g, \sigma) = \{ u \in C^\infty_B(M) | \sigma + \sqrt{-1} \partial_B \bar{\partial} u \geq 0 \}$. Then, there exist two positive constants $\alpha$ and $C$, where $C$ depends only on $\alpha$ and the geometry of $(M, \xi, \eta, \Phi, g)$ and $\sigma$, such that

$$\int_M e^{-\alpha(u - \max_M u)} (d\eta)^n \wedge \eta \leq C.$$  

The next lemma is from [23].

Lemma 5.2. Let $u$ be a negative basic $C^2$ function on a compact Sasakian manifold $(M, \xi, \eta, \Phi, g)$. Suppose the basic Laplacian $\Delta_B u \geq -v$ for some continuous basic function $v$ on $M$. Then

$$\int_M |\nabla \log(-u)|^2 (d\eta)^n \wedge \eta \leq \frac{1}{\min_M (-u)} \int_M |v|(d\eta)^n \wedge \eta.$$  

where $\nabla$ is the Levi-Civita connection corresponding to the Riemannian metric $g$.

Because the basic Laplacian is equal to the Riemannian Laplacian $\Delta_g$ on basic functions, so the proof is the same as the argument in Lemma 4 ([23]).

Lemma 5.3. Let $\sigma$ be a basic $(1, 1)$ form on a compact Sasakian manifold $(M, \xi, \eta, \Phi, g)$, and $\mathcal{H}(M, \xi, \eta, \Phi, g, \sigma) = \{ u \in C^\infty_B(M) | \sigma + \sqrt{-1} \partial_B \bar{\partial} u \geq 0 \}$. Then $v \equiv u - \max_M u - 1$ satisfies

$$\int_M |\log(-v)|^2 (d\eta)^n \wedge \eta + \int_M |\nabla \log(-v)|^2 (d\eta)^n \wedge \eta \leq C.$$  

where $C > 0$ is a constant depending only on the geometry of $(M, \xi, \eta, \Phi, g)$ and $\sigma$. So any sequence

$$\log(1 + \max_M u_k - u_k)$$  

is relatively compact in $L^2_B(M)$, where $u_k \in (M, \xi, \eta, \Phi, g)$.

Proof of Theorem 1.3. By Theorem 1.2 we know that the transverse Monge-Ampère equations

$$(td\eta - \rho^T + \sqrt{-1} \partial_B \bar{\partial} u_t)^n \wedge \eta = e^{u_t}(d\eta)^n \wedge \eta.$$  

(5.3)$$
$$

have solutions for any $t > 0$. We denote $\sigma(t) = td\eta - \rho^T + \sqrt{-1} \partial_B \bar{\partial} u_t$. Clearly,

$$\int_M (c_1^B(M, \mathcal{F}_\xi))^n \wedge \eta = \lim_{t \to 0} \int_M \sigma(t)^n \wedge \eta = \lim_{t \to 0} \int_M e^{u_t}(d\eta)^n \wedge \eta.$$

(5.4)
By inequality (3.19) and Cauchy-Schwarz inequality we have

\[ \text{tr}_{\sigma(t)} \sqrt{-1} \partial_B \bar{\partial}_B \log \text{tr}_{\sigma(t)} \ d\eta \geq \frac{(n+1)\kappa}{2} \exp\left(-\frac{\max_M u_t}{n}\right) - 1. \]

Integrating inequality (3.19) w.r.t volume element \( \sigma(t)^n \wedge \eta \) we have

\[ \exp\left(-\frac{\max_M u_t}{n}\right) \leq \frac{\int_M \sigma(t)^n \wedge \eta}{\frac{n+1}{2} \int_M \kappa \sigma(t)^n \wedge \eta} \leq \frac{\int_M \exp(u_t - \max_M u_t - 1)(d\eta)^n \wedge \eta}{\frac{n+1}{2} \int_M \kappa \exp(u_t - \max_M u_t - 1)(d\eta)^n \wedge \eta} \]

Since \( t_1 d\eta - \rho^T + \sqrt{-1} \partial_B \bar{\partial}_B u_t > \sigma_t \geq 0 \) for any \( 0 < t \leq t_1 \), by Lemma 5.3, we know the set

\[ \log(1 + \max_M u_t - u_t); 0 < t \leq t_1 \]

is uniformly bounded in \( W^{1,2}_B \) and \( \log(1 + \max_M u_t - u_t) \) converges to \( w \) almost everywhere on \( M \). By Lebesgue dominated convergence theorem the right side of (5.6) converges to \( C > 0 \). So we have the lower bound of \( \max_M u_t \), the upper bound get from (5.3) by maximum principle. Up to a sequence we can assume \( u_t \) converges to \( -e^w + c \). Plugging these back to (5.4), we prove Theorem 1.3. \( \square \)

6. Proof of Theorem 1.4

In this section we will prove the Miyaoka-Yau inequality for basic Chern numbers holds on compact Sasakian manifolds with nonpositive transverse holomorphic sectional curvature.

Proof of Theorem 1.4. Let \( (M, \xi, \eta, \Phi, g) \) be a \( 2n + 1 \) dimensional compact Sasakian manifold with nonpositive transverse holomorphic sectional curvature. We can cover \( M \) by finite foliated local coordinate charts \( \{U_a\} \). Assume \( \sigma \) is a transverse strictly positive basic \((1, 1)\) form. Then we have the induced metric on the contact bundle \( D \).

\[ h^T(X, Y) = \sigma(X, \Phi Y). \]

Where \( X, Y \in \Gamma(D) \). So we have a Riemann metric \( g \) on \( M \)

\[ g = h^T + \eta \otimes \eta. \]

We can define a transverse connection \( \nabla^T \)

\[ \nabla^T_X Y = \begin{cases} (\nabla_X Y)^p, & X \in D \\ [\xi, Y]^p, & X = \xi \end{cases} \]
where \( Y \) is a section of \( D \) and \( X^p \) the projection of \( X \) onto \( D \). Let \( X_i \) be local foliate transverse frame on the contact bundle \( D \), by argument of section (2.2), we have
\[
\nabla^T_{X_i} X_j = \Gamma^k_{ij} X_k,
\]
\[
\Gamma^i_{jk} = \Gamma^i_{kj} = h^{ji} \partial h_{ij} \partial z^k,
\]
Where \( X_i, X_j \in D^{(1,0)} \) and \( X \in D \). By some computation, we know the transverse curvature of \( \nabla^T \) is
\[
R^T_{ijkl} = -\partial_i \partial_j h^T_{kl} + (h^T)^p d \partial_i h^T_{kp} \partial_j h^T_{pl}.
\]
From the proof of Theorem 1.1, we have a family of transverse strictly positive basic \((1, 1)\) form \( \sigma(t) \), \( t > 0 \). By direct calculation, we have
\[
(2\pi)^2 \int_M (2c^B_2(M, F_\xi) - \frac{n}{n + 1} c^B_1(M, F_\xi)^2) \wedge \frac{\sigma(t)^{n-2}}{(n-2)!} \wedge \eta
\]
(6.2)
\[
= \int_M \{\text{tr}(R^T(t) \wedge R^T(t)) - \frac{1}{n+1} \text{tr} R^T(t) \wedge \text{tr} R^T(t)\} \wedge \frac{\sigma(t)^{n-2}}{(n-2)!} \wedge \eta
\]
\[
= \int_M |R^T(t)|^2 - (S^T(t))^2 - \frac{n+2}{n+1} (|\rho^T(t)|^2 - (S^T(t))^2) \frac{\sigma(t)^n}{n!} \wedge \eta
\]
where \( S^T = (h^T)_{ij} (h^T)_{kl} R^T_{ijkl} \). Locally, set
\[
Q(t)_{ijkl} = R^T(t)_{ijkl} - \frac{S^T(t)}{n(n+1)} (h^T(t)_{ij} h^T(t)_{kl} + h^T(t)_{ij} h^T(t)_{kj}).
\]
By direct calculation, we have
\[
|Q(t)|^2 = |R^T(t)|^2 - \frac{2(S^T(t))^2}{n(n+1)}
\]
(6.3)
Combining (6.2) and (6.3), we have
\[
(2\pi)^2 \int_M (2c^B_2(M, F_\xi) - \frac{n}{n + 1} c^B_1(M, F_\xi)^2) \wedge \frac{\sigma(t)^{n-2}}{(n-2)!} \wedge \eta
\]
\[
= \int_M (|Q(t)|^2 + \frac{n+2}{n+1} (S^T(t) + n)^2 - \frac{n+2}{n+1} |\rho^T(t) + \sigma(t)|^2) \frac{\sigma(t)^n}{n!} \wedge \eta
\]
\[
\geq \int_M (|Q(t)|^2 - \frac{n+2}{n+1} |\rho^T(t) + \sigma(t)|^2) \frac{\sigma(t)^n}{n!} \wedge \eta
\]
By the equation (6.3), We know
\[
\rho^T(t) = td\eta - \sigma(t).
\]
(6.5)
Combining (6.4) and (6.5), let $t \to 0$ we have
\[
(2\pi)^2 \int_M (2c^B(M,F_\xi) - \frac{n}{n+1}c^B_1(M,F_\xi)^2) \wedge \frac{(-c^B(M,F_\xi))^{n-2}}{(n-2)!} \wedge \eta \\
= \lim_{t \to 0} (2\pi)^2 \int_M (2c^B(M,F_\xi) - \frac{n}{n+1}c^B_1(M,F_\xi)^2) \wedge \frac{\sigma(t)^{n-2}}{(n-2)!} \wedge \eta \\
\geq \lim_{t \to 0} \int_M (|Q|^2 - \frac{n+2}{n+1}\rho^T(t) + \sigma(t)^2)\frac{\sigma(t)^n}{n!} \wedge \eta \geq 0.
\]

\[\square\]

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School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, P. R. China

Email address: cybwv9880163.com