Entanglement beyond tensor product structure: algebraic aspects of quantum non-separability

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Abstract
An algebraic approach to quantum non-separability is applied to the case of two qubits. It is based on the partition of the algebra of observables into independent subalgebras and the tensor product structure of the Hilbert space is not exploited. Even in this simple case, such a general formulation has some advantages. Using algebraic formalism, we can explicitly show the relativity of the notion of entanglement to the observables measured in the system and characterize separable and non-separable pure states. As a universal measure of non-separability of pure states, we propose to take the so-called total correlation. This quantity depends on the state as well as on the algebraic partition. Its numerical value is given by the norm of the corresponding correlation matrix.

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1. Introduction
The standard approach to the entanglement of states of distinguishable particles is strictly related to the tensor product structure of the underlying Hilbert space. In the case of two parties, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and the state of the system, represented by a density matrix $\rho$, is separable if it can be written as

$$\rho = \sum_j p_j \rho_j^{(1)} \otimes \rho_j^{(2)}, \quad p_j \geq 0, \quad \sum_j p_j = 1,$$

where $\rho_j^{(i)}$, $i = 1, 2$, is the state of $i$th part. Otherwise the state $\rho$ is non-separable or entangled. This simple and natural definition does not work in the case of indistinguishable particles. A more general notion of quantum non-separability is needed. This problem was clearly formulated in [1], where the factorization of the corresponding algebra of observables into subalgebras describing subsystems is proposed as a basis for unambiguous discussion of quantum non-separability. The main idea is that the questions about entanglement or separability (in the case of indistinguishable particles) are meaningful
only when we specify which statistically independent (i.e. commuting) subalgebras of the total algebra are considered. In this formulation, entanglement of quantum states is nothing but the existence of non-vanishing correlations between such chosen independent observables. This general definition was earlier used in mathematically rigorous discussion of quantum correlations and entanglement in relativistic quantum field theory (see e.g. [2–4]). In such an approach, one can for example show that the usual vacuum state is maximally entangled with respect to the observables localized in complementary wedge-shaped regions in spacetime [2].

In this paper, we reconsider the simple case of distinguishable two qubits using this general algebraic perspective. This gives us a deeper insight into the fundamental aspects of quantum non-separability of this system than by considering just a formal structure of the underlying Hilbert space. In particular, we find that the discussion of relativity of the notion of entanglement to a set of observables (see e.g. [5, 6]) is natural and straightforward in the algebraic language. In this setting, the theory is defined by the total algebra of observables \( \mathcal{A}_{\text{tot}} \) and the set of states, given by the linear functionals \( \omega \) on \( \mathcal{A}_{\text{tot}} \) which attribute to the observables their expectation values. The subsystems are identified by the specific choice of subalgebras of \( \mathcal{A}_{\text{tot}} \), and the tensor product structure of the underlying Hilbert space is not used. The main novelty of our approach is that using the abstract algebraic characterization of non-separability of states, we are able to consider the question how the choice of subalgebras influences the entanglement of a given state \( \omega \). To answer this question, we need a ‘universal’ measure of quantum non-separability, not linked to the tensor product structure. In the case of pure states, we propose to use the total correlation in the state \( \omega \) (see the definition below) as such a measure. This measure depends on the state and the couple of subalgebras, and as we show is given by the norm of the corresponding correlation matrix. The last quantity can be computed by finding the maximal singular value of this matrix. Moreover, this notion generalizes the Wootters concurrence [7] in the sense that for the ‘canonical’ choice of subalgebras directly given by the tensor product structure, the total correlation and concurrence coincide. We also find that in the algebraic framework, the general characterization of separability of pure states is simple and is given in terms of restrictions of the state to the subsystems. The notion of restriction of the state to subalgebra is general and replaces the notion of partial trace of the density matrix, strictly connected to the tensor product structure. We show that separability is equivalent to the purity of restrictions. This gives also the algebraic characterization of quantum non-separability. In particular, maximal entanglement follows from the maximal non-purity or trace property of restrictions. Using this property, we are able to construct in an explicit way the states which are maximally non-separable with respect to a given choice of subalgebras.

We finish our study by showing in our framework the result previously established by many authors (see e.g. [6, 8]): any pure state can be separable or entangled to a given degree of non-separability, for the appropriate choice of subsystems.

2. Algebraic framework

We start with a short review of the algebraic formulation of quantum theory (see e.g. [9]). Let \( \mathcal{A}_{\text{tot}} \) be the \( \sigma \)-algebra of all observables of the quantum system. Since we consider the system with finite number of levels, \( \mathcal{A}_{\text{tot}} \) is isomorphic to the full matrix algebra. The most general state on \( \mathcal{A}_{\text{tot}} \) is given by the linear functional

\[
\omega : \mathcal{A}_{\text{tot}} \rightarrow \mathbb{C},
\]

which is positive, i.e. for all \( A \in \mathcal{A}_{\text{tot}} \),

\[
\omega (A^*A) \geq 0.
\]
and normalized, i.e.

\[ \omega(1) = 1. \]

If \( \Psi \) is a normalized vector in the Hilbert space of the system, then

\[ \omega_\Psi(A) = \langle \Psi, A\Psi \rangle \tag{2} \]

gives the state on \( \mathcal{A}_{\text{tot}} \) which is called a vector state. On the other hand, for any density matrix \( \varrho \), the formula

\[ \omega_\varrho(A) = \text{tr}(\varrho A) \tag{3} \]

defines also the state on \( \mathcal{A}_{\text{tot}} \) which is in general mixed. By the GNS construction [9, 10], every state \( \omega \) on \( \mathcal{A}_{\text{tot}} \) is a vector state on the appropriate Hilbert space \( \mathcal{H}_\omega \). These set \( E \) of all states is convex and compact, and the extremal points of \( E \) are identified with the pure states of the system. In the algebraic language, the pure states are characterized by the properties of the GNS representation: \( \omega \) is pure if and only if the corresponding representation is irreducible [10]. Let \( \alpha \) be \( * \)-automorphism of \( \mathcal{A}_{\text{tot}} \), i.e. \( \alpha \) is such isomorphism of \( \mathcal{A}_{\text{tot}} \) that \( \alpha(A^*) = \alpha(A)^* \), \( A \in \mathcal{A}_{\text{tot}} \). For every state \( \omega \) such that \( \omega \circ \alpha \neq \omega \), \( \omega_\alpha \) gives a new state on \( \mathcal{A}_{\text{tot}} \). Moreover, if \( \omega \) is pure, then \( \omega_\alpha \) is also pure.

To describe subsystems of the quantum system, we choose some subalgebras of the total algebra \( \mathcal{A}_{\text{tot}} \). In the context of separability of quantum states, we consider a pair \((\mathcal{A}, \mathcal{B})\) of isomorphic subalgebras of \( \mathcal{A}_{\text{tot}} \) with the following properties:

- the subalgebras \( \mathcal{A} \) and \( \mathcal{B} \) are statistically independent, in the sense that for all \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \), \( [A, B] = 0 \);
- the subalgebras \( \mathcal{A} \) and \( \mathcal{B} \) generate the total algebra \( \mathcal{A}_{\text{tot}} \).

Any pair of subalgebras satisfying the above conditions will be called a Bell pair of subalgebras of the total algebra \( \mathcal{A}_{\text{tot}} \).

**Definition 1.** Let \((\mathcal{A}, \mathcal{B})\) be a Bell pair of subalgebras of \( \mathcal{A}_{\text{tot}} \). The pure state \( \omega \) on \( \mathcal{A}_{\text{tot}} \) is \((\mathcal{A}, \mathcal{B})\)-separable if

\[ \omega(AB) = \omega(A)\omega(B), \quad A \in \mathcal{A}, \quad B \in \mathcal{B}. \]

A mixed state is \((\mathcal{A}, \mathcal{B})\)-separable if it can be expressed as a convex combination of pure \((\mathcal{A}, \mathcal{B})\)-separable states.

The state \( \omega \) is \((\mathcal{A}, \mathcal{B})\)-correlated or non-separable if it is not \((\mathcal{A}, \mathcal{B})\)-separable. To indicate how much a given pure state \( \omega \) differs from the separable one for a fixed choice of Bell pair of subalgebras, we introduce the quantity which we call total correlation in the state \( \omega \). It is defined as

\[ C_\omega(\mathcal{A}, \mathcal{B}) = \sup |\omega(AB) - \omega(A)\omega(B)|. \tag{4} \]

In formula (4), the supremum is taken over all normalized elements \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \). It follows that

\[ 0 \leq C_\omega(\mathcal{A}, \mathcal{B}) \leq 1. \]

In the following section, we apply the idea of algebraic non-separability to the case of two qubits.
3. Two qubits

3.1. The total algebra

Consider the four-level quantum system. It is given by the Hilbert space $\mathcal{H} = \mathbb{C}^4$ with the canonical bases $e_1, e_2, e_3,$ and $e_4$. The total algebra $\mathcal{A}_\text{tot}$ can be considered as generated by matrix unit $I$ and elements $\lambda_1, \ldots, \lambda_{15}$, where

$$\lambda_i = I \otimes \sigma_i, \quad \lambda_{3+i} = \sigma_i \otimes I, \quad i = 1, 2, 3,$$

and $\lambda_j$, $j = 7, \ldots, 15$, are given by Kronecker products of the Pauli matrices $\sigma_i$ taken in the lexicographical order. In the following, we will write

$$\mathcal{A}_\text{tot} = [I, \lambda_1, \ldots, \lambda_{15}].$$

An arbitrary element $A \in \mathcal{A}_\text{tot}$ has the form

$$A = c_0 I + \sum_{j=1}^{15} c_j \lambda_j, \quad c_0, c_j \in \mathbb{C},$$

so every state is defined by the formula

$$\omega(A) = c_0 + \sum_{j=1}^{15} c_j w_j,$$

where

$$w_j = \omega(\lambda_j), \quad j = 1, \ldots, 15,$$

are the real numbers. In a particular case of pure state $\omega_\Psi$, where

$$\Psi = z_1 e_1 + \cdots + z_4 e_4, \quad |z_1|^2 + \cdots + |z_4|^2 = 1,$$

the sequence $\{w_j\}$ is given by

$$w_1 = 2 \text{Re} (\xi_1 z_2 + \xi_3 z_4), \quad w_2 = 2 \text{Im} (\xi_1 z_2 + \xi_3 z_4),$$
$$w_3 = |z_1|^2 - |z_2|^2 + |z_3|^2 - |z_4|^2, \quad w_4 = 2 \text{Re} (\xi_1 z_3 + \xi_2 z_4),$$
$$w_5 = -2 \text{Im} (\xi_1 z_3 + \xi_2 z_4), \quad w_6 = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2,$$
$$w_7 = 2 \text{Re} (\xi_2 z_3 + \xi_1 z_4), \quad w_8 = 2 \text{Im} (\xi_2 z_3 - \xi_1 z_4),$$
$$w_9 = 2 \text{Re} (\xi_2 z_4 + \xi_1 z_3), \quad w_{10} = -2 \text{Im} (\xi_2 z_4 + \xi_1 z_3),$$
$$w_{11} = 2 \text{Re} (\xi_3 z_3 - \xi_2 z_4), \quad w_{12} = 2 \text{Im} (\xi_3 z_4 - \xi_2 z_3),$$
$$w_{13} = 2 \text{Re} (\xi_3 z_2 - \xi_1 z_4), \quad w_{14} = 2 \text{Im} (\xi_3 z_4 - \xi_1 z_2),$$
$$w_{15} = |z_1|^2 - |z_2|^2 - |z_3|^2 + |z_4|^2.$$

3.2. Bell pair of subalgebras and correlation matrix

In the case of two qubits, it is convenient to take the subalgebras $\mathcal{A}$ and $\mathcal{B}$ defined in the following way. Let $A_1$, $A_2$, $A_3$ and $B_1$, $B_2$, $B_3$ be the linearly independent Hermitian elements of $\mathcal{A}_\text{tot}$, which satisfy

$$A_i^2 = B_i^2 = I, \quad [A_i, B_j] = 0, \quad i, j = 1, 2, 3,$$

and

$$A_i A_j + A_j A_i = B_i B_j + B_j B_i = 0, \quad i \neq j, \quad i, j = 1, 2, 3.$$
We put
\[ A = [1, A_1, A_2, A_3], \quad B = [1, B_1, B_2, B_3]. \] (13)
Consider the elements \( A \in A, B \in B \) defined as
\[ A = a_1A_1 + a_2A_2 + a_3A_3, \quad B = b_1B_1 + b_2B_2 + b_3B_3, \] (14)
where \( \vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3) \) are the real vectors. By conditions (11) and (12),
\[ A^2 = ||\vec{a}||^2 1, \quad B^2 = ||\vec{b}||^2 1, \]
so if the vectors \( \vec{a} \) and \( \vec{b} \) are normalized, \( A^2 = 1 \) and \( B^2 = 1 \). From now on, we will always assume that \( ||\vec{a}|| = ||\vec{b}|| = 1 \). Let \( \omega \) be an arbitrary pure state on \( \mathcal{A}_{\text{tot}} \). Note that for \( A \) and \( B \)
defined by (14),
\[ \omega(AB) - \omega(A)\omega(B) = \langle \vec{a}, \vec{b} \rangle, \] (15)
where the correlation matrix \( Q = (q_{ij}) \) has the matrix elements
\[ q_{ij} = \omega(A_i B_j) - \omega(A_i)\omega(B_j). \]
Thus,
\[ C_\omega(A, B) = \sup_{\vec{a}, \vec{b}} ||\langle \vec{a}, \vec{b} \rangle|| = ||Q||, \] (16)
where the supremum is taken over all normalized vectors \( \vec{a}, \vec{b} \in \mathbb{R}^3 \). Thus, in the case of two qubits, the total correlation in the pure state \( \omega \) can be computed by finding the matrix norm of the corresponding correlation matrix \( Q \), i.e. the largest singular value of \( Q \).

3.3. Canonical Bell pair and concurrence

The most natural choice of Bell pair is obtained by considering the subalgebras
\[ \mathcal{A}_0 = [1, \lambda_1, \lambda_2, \lambda_3], \quad \mathcal{B}_0 = [1, \lambda_4, \lambda_5, \lambda_6]. \] (17)
All conditions defining a Bell pair are trivially satisfied. Note also that \((\mathcal{A}_0, \mathcal{B}_0)\)-correlated states can be identified with standard entangled states with respect to the partition \( \mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2 \); therefore, the Bell pair given by (17) will be called the canonical Bell pair of subalgebras. An interesting link between the algebraic theory of non-separability and standard theory of entanglement is given by the following result (established in the another context by Verstraete et al [11]).

**Theorem 1.** Let \((\mathcal{A}_0, \mathcal{B}_0)\) be the canonical Bell pair of subalgebras of the total algebra of the two-qubit system. For an arbitrary pure state \( \omega \),
\[ C_\omega(\mathcal{A}_0, \mathcal{B}_0) = C(\omega), \]
where \( C(\omega) \) is the concurrence of \( \omega \).

**Proof.** For an arbitrary vector state \( \omega \) defined by vector (8), its Wootters concurrence [7] reads
\[ C(\omega) = 2|z_1z_3 - z_1z_4|. \] (18)
On the other hand, by elementary but tedious computation, one verifies that the correlation matrix \( Q \) of \( \omega \) with respect to the canonical Bell pair has the norm
\[ ||Q|| = 2|z_2z_3\overline{z}_4 - z_1z_2z_3\overline{z}_4 - z_2z_1z_3\overline{z}_4 + z_1z_2z_3z_4|^{1/2} \]
\[ = 2|(z_2z_3 - z_1z_4)(z_2z_3 - z_1z_4)|^{1/2} \]
\[ = 2|z_2z_3 - z_1z_4|, \] (19)
which is exactly equal to concurrence (18).
3.4. Non-canonical Bell pairs and relativity of entanglement

We show by considering the explicit examples that the notion of entanglement is highly non-unique. Non-separability of a state is always relative to the measurement setup, which is fixed by the specific choice of observables, forming the statistically independent subalgebras \(A\) and \(B\).

We start the discussion of this point considering the states which are obviously separable with respect to the canonical subalgebras \(A_0\) and \(B_0\). Take the vector states defined by the basic vectors \(e_1, e_2, e_3\), but consider observables belonging to another subalgebras of \(A_{\text{tot}}\). In this case, let \(A\) and \(B\) be defined as follows:

\[
A = \left[ 1, \frac{1}{\sqrt{2}}(\lambda_4 + \lambda_{11}), \frac{1}{\sqrt{2}}(\lambda_{10} - \lambda_{12}), -\frac{1}{2}(\lambda_1 + \lambda_3 - \lambda_{13} + \lambda_{15}) \right],
\]

\[
B = \left[ 1, \frac{1}{\sqrt{2}}(\lambda_7 + \lambda_9), -\frac{1}{\sqrt{2}}(\lambda_5 + \lambda_8), \frac{1}{2}(\lambda_1 - \lambda_3 - \lambda_{13} - \lambda_{15}) \right].
\]  (20)

Using the relations between the generators \(\lambda_j\), one can easily check that \((A, B)\) is a Bell pair. To find the correlation matrices corresponding to the considered states, we can utilize formulas (9) and (10), and we obtain the following result: the states \(e_1\) and \(e_2\) have correlation matrices with all zero elements, but the correlation matrices corresponding to \(e_3\) and \(e_4\) are given by

\[
Q_{e_1} = \text{diag}(1, 1, -1), \quad Q_{e_2} = \text{diag}(-1, -1, -1),
\]

and

\[
||Q_{e_3}|| = ||Q_{e_4}|| = 1.
\]

Thus, the states \(e_1\) and \(e_2\) are \((A, B)\)-separable, whereas the states \(e_3\) and \(e_4\) are maximally \((A, B)\)-entangled.

Consider now the family of states which are maximally entangled with respect to the canonical subalgebras \(A_0\) and \(B_0\). As is known [12], such a property has the states \(\omega_{\varphi, \theta}\), defined by the vectors

\[
\Psi_{\varphi, \theta} = Ae_1 + Be^{i\varphi}e_2 + Be^{i\theta}e_3 - Ae^{i(\varphi + \theta)}e_4,
\]  (21)

where

\[
A = \frac{a}{\sqrt{2}}, \quad B = \sqrt{1 - a^2},
\]

and \(a \in [0, 1], \varphi, \theta \in [0, 2\pi]\). This time we ask about the entanglement properties of \(\omega_{\varphi, \theta}\) but with respect to the experimental setup given by the pair \((A', B')\) defined below:

\[
A' = \left[ 1, -\frac{1}{2}(\lambda_3 - \lambda_6 - \lambda_7 + \lambda_{11}), -\lambda_{10}, -\frac{1}{2}(\lambda_3 + \lambda_6 - \lambda_7 - \lambda_{11}) \right],
\]

\[
B' = \left[ 1, \frac{1}{\sqrt{2}}(\lambda_1 - \lambda_9), -\frac{1}{\sqrt{2}}(\lambda_5 - \lambda_{14}), \lambda_{15} \right].
\]  (22)

Again, using (9) and (10), we are able to find the corresponding correlation matrix. This matrix has the elements

\[
q_{11} = \sqrt{2a}\sqrt{1 - a^2}(\cos \varphi - a^2 \cos \theta \cos(\varphi + \theta)),
\]

\[
q_{12} = \sqrt{2a}\sqrt{1 - a^2}(\sin \theta + a^2 \sin \varphi \cos(\varphi + \theta)),
\]

\[
q_{13} = 2a^2(1 - a^2) \cos(\varphi + \theta),
\]

\[
q_{21} = \sqrt{2a}\sqrt{1 - a^2} \sin \theta (2a^2 \cos \varphi \cos \theta - \cos(\varphi - \theta)),
\]

\[
q_{22} = 2a\sqrt{1 - a^2} \cos \varphi (\cos(\varphi - \theta) - 2a^2 \sin \varphi \cos \theta),
\]

\[
q_{23} = 4a^2(1 - a^2) \cos \varphi \sin \theta.
\]
The elements of the subalgebra $A$ let

and we see that all states $\omega$ restriction of a state to the subalgebra partial trace of the state. In the algebraic setting, the partial trace can be replaced by the more product structure of the Hilbert space, such a characterization is given by the notion of the restriction $C$. If a given state is separable or entangled, we should always specify which statistically independent subalgebras of the total algebra of observables are considered. In this general setting, we propose to take the total correlation $C_{\omega}$ in a given state $\omega$ as a universal measure of non-separability of pure states. This quantity does not depend on the tensor product structure of the underlying Hilbert space, and in the case of two qubits and canonical pair of subalgebras, it coincides with the Wootters concurrence. Thus, the state defined by the vector

$\Psi = \frac{1}{\sqrt{2}}(e_1 + e_2 + ie_3 - ie_4)$

gives the maximal value of norm (23), so it is not only maximally correlated with respect to the canonical Bell pair $(A_0, B_0)$ but also with respect to the pair $(A', B')$. So we see that depending on the experimental setup, separable states can be maximally entangled and vice versa: maximally entangled states can be separable.

4. Algebraic non-separability of two qubits

4.1. Characterization of pure separable states

The algebraic approach to quantum non-separability consequently takes into account the relativity of this notion shown by the above discussion. To avoid ambiguities in deciding if a given state is separable or entangled, we should always specify which statistically independent subalgebras of the total algebra of observables are considered. In this general setting, we propose to take the total correlation $C_{\omega}(A, B)$ in a given state $\omega$ as a universal measure of non-separability of pure states. This quantity does not depend on the tensor product structure of the underlying Hilbert space, and in the case of two qubits and canonical pair of subalgebras, it coincides with the Wootters concurrence. Thus, the state $\omega$ is $(A, B)$-separable if $C_{\omega}(A, B) = 0$ and it is $(A, B)$-correlated (or entangled) if $C_{\omega}(A, B) = c > 0$. Although the total correlation can be expressed by the norm of the corresponding correlation matrix, in general it is not easy to find the states for which this norm vanishes or is greater than zero. We need another characterization of separability. In the standard approach based on the tensor product structure of the Hilbert space, such a characterization is given by the notion of the partial trace of the state. In the algebraic setting, the partial trace can be replaced by the more general notion of restriction of a state to the subalgebra. The functional $\omega$ considered only for the elements of the subalgebra $A$ is the restriction $\omega_A$ of $\omega$ to $A$. Analogously, we define the restriction $\omega_B$. Using this notion, we obtain the following result.

**Theorem 2.** Let $\omega$ be the pure state on $A_{tot}$ and let $(A, B)$ be a Bell pair of subalgebras of $A_{tot}$. $\omega$ is $(A, B)$-separable if and only if the restrictions $\omega_A$ and $\omega_B$ of $\omega$ to $A$ and $B$, respectively, are also pure.

**Proof.** This theorem is a particular case of a general result given by Takesaki [13], but for the reader’s convenience, we sketch the proof. Obviously, if $\omega$ is separable and pure, then $\omega_A$ and $\omega_B$ are pure. Assume now that the restriction $\omega_A$ is pure. Let $A \in A$ and $B \in B$. Without the loss of generality, we can consider only such $B \in B$ that $0 \leq B \leq 1$. Assume also that $0 < \omega(B) < 1$; then for $A \in A$, we have

$$\omega_A(A) = \omega(B) \frac{1}{\omega(B)} \omega(AB) + (1 - \omega(B)) \frac{1}{1 - \omega(B)} \omega(A(1 - B)).$$

(24)
Using condition (26), we can find all separable states for a fixed pair of subalgebras, i.e.,

\[ \omega_1(A) = \frac{1}{\omega(B)} \omega(AB), \quad \omega_2(A) = \frac{1}{1 - \omega(B)} \omega(A(1 - B)) \]

are both states of \( \mathcal{A} \). Indeed, \( \omega_1(1) = \omega(1) = 1 \) and

\[ \omega_1(A^*A) = \frac{1}{\omega(B)} \omega(A^*AB) = \frac{1}{\omega(B)} \omega((B^{1/2}A)^*B^{1/2}B)) \geq 0 \]

and similarly for \( \omega_2 \). So equality (24) means that \( \omega_A \) is a convex combination of other states of \( \mathcal{A} \), but by the assumption that \( \omega_A \) is pure; hence,

\[ \omega_A(A) = \omega_1(A) = \omega_2(A), \]

i.e.

\[ \omega_A(A) = \frac{\omega(AB)}{\omega(B)} \]

and \( \omega \) is \( (A, B) \)-separable. \( \square \)

In the case of two qubits, the condition of separability given by the above theorem can be reformulated as follows. Let

\[ \mathcal{A} = [A_1, A_2, A_3], \quad \mathcal{B} = [B_1, B_2, B_3] \]

be any Bell pair of subalgebras. The restrictions \( \omega_A \) and \( \omega_B \) are equivalent to one-qubit states, so the states \( \omega_A \) and \( \omega_B \) are pure if the real vectors \( \vec{r}_\omega \) and \( \vec{s}_\omega \), defined by

\[ \vec{r}_\omega = (\omega(A_1), \omega(A_2), \omega(A_3)), \quad \vec{s}_\omega = (\omega(B_1), \omega(B_2), \omega(B_3)), \] (25)

satisfy

\[ ||\vec{r}_\omega|| = 1, \quad ||\vec{s}_\omega|| = 1. \] (26)

Using condition (26), we can find all separable states for a fixed pair of subalgebras \( \mathcal{A} \) and \( \mathcal{B} \). To clarify this point and give an example, take the Bell pair \( (A', B') \) defined by (22) and consider the most general pure state \( \omega_{A'} \). By (9) and (10), we have that the values of the functional \( \omega_{A'} \) on the generators \( A_i \) of \( \mathcal{A}' \) are given by

\[ \omega_{A'}(A_1) = |z_2|^2 - |z_3|^2 + 2 \text{Re} \bar{z}_1 z_4, \]
\[ \omega_{A'}(A_2) = -2 \text{Im} (\bar{z}_2 z_3 + \bar{z}_1 z_4), \]
\[ \omega_{A'}(A_3) = |z_4|^2 - |z_3|^2 + 2 \text{Re} \bar{z}_2 z_3. \] (27)

Analogously, for the generators \( B_i \) of \( \mathcal{B}' \),

\[ \omega_{A'}(B_1) = \sqrt{2} \text{Re} (\bar{z}_1 (z_2 - z_3) + (\bar{z}_2 + \bar{z}_3) z_4), \]
\[ \omega_{A'}(B_2) = \sqrt{2} \text{Im} (\bar{z}_1 (z_3 - z_2) + (\bar{z}_2 + \bar{z}_3) z_4), \]
\[ \omega_{A'}(B_3) = |z_1|^2 - |z_2|^2 - |z_3|^2 + |z_4|^2. \] (28)

It is not an easy task to obtain a general solution of conditions (26), so we look for the ‘basic’ separable states \( \omega_0 \) for which the vectors \( \vec{r}_\omega \) and \( \vec{s}_\omega \) have only one non-zero component which is equal to +1 or −1. Analyzing expressions (27) and (28), we find the simple solutions of conditions (26) which correspond to the canonical basis of \( \mathbb{C}^4 \). In particular, we obtain that

\[ \vec{r}_\omega = (0, 0, -1), \quad \vec{s}_\omega = (0, 0, 1) \]

give the state \( e_1 \),

\[ \vec{r}_\omega = (1, 0, 0), \quad \vec{s}_\omega = (0, 0, -1) \]

give the state \( e_2 \),

\[ \vec{r}_\omega = (-1, 0, 0), \quad \vec{s}_\omega = (0, 0, 1) \]

give the state \( e_3 \),

\[ \vec{r}_\omega = (0, 0, 1), \quad \vec{s}_\omega = (0, 0, -1) \]

give the state \( e_4 \).

Hence, the vectors \( e_1, e_2, e_3 \) and \( e_4 \) are not only \( (\mathcal{A}_0, \mathcal{B}_0) \)-separable, but they are also \( (\mathcal{A}', \mathcal{B}') \)-separable.

Other solutions of (26) allow us to construct interesting examples of ‘non-standard’ separable vector states. In particular, we find that
• \( \vec{r}_0 = (-1, 0, 0) \), \( \vec{s}_0 = (0, 0, 1) \) give the state
  \[
  \Psi_\pm = \frac{1}{\sqrt{2}}(e_1 - e_2),
  \]
• \( \vec{r}_0 = (1, 0, 0) \), \( \vec{s}_0 = (0, 0, 1) \) give the state
  \[
  \Psi_+ = \frac{1}{\sqrt{2}}(e_1 + e_2),
  \]
• \( \vec{r}_0 = (0, 0, 1) \), \( \vec{s}_0 = (0, 0, 1) \) give the state
  \[
  \Phi_- = \frac{1}{\sqrt{2}}(e_2 - e_3),
  \]
• \( \vec{r}_0 = (0, 0, 1) \), \( \vec{s}_0 = (0, 0, 1) \) give the state
  \[
  \Phi_+ = \frac{1}{\sqrt{2}}(e_2 + e_3).
  \]

Note that \( \Psi_- \), \( \Psi_+ \), \( \Phi_- \) and \( \Phi_+ \) are standard Bell states which are maximally \( (A_0, B_0) \)-entangled.

### 4.2. Algebraic non-separability

In the algebraic theory, we can also simply characterize pure non-separable states. As follows from theorem 2, this property is shared by the states \( \omega \) whose restrictions \( \omega_A \) and \( \omega_B \) are non-pure. In this context, the maximal non-purity of \( \omega_A \) and \( \omega_B \) corresponds to the maximal non-separability of the state \( \omega \). The state \( \omega_A \) is maximally non-pure if it is a trace state, i.e. if it satisfies

\[
\omega_A(AA') = \omega_A(A'A)
\]

for all \( A, A' \in \mathcal{A} \). When we apply this condition to the generators of the subalgebra \( \mathcal{A} \), we obtain

\[
\omega_A(A_jA_k) = \omega_A(A_jA_k) = -\omega_A(A_jA_k), \quad j \neq k.
\]  

(29)

Hence, \( \omega_A(A_jA_k) = 0 \), and \( \omega_A(A_i) = 0 \) for \( i = 1, 2, 3 \). The same conclusion also follows for the subalgebra \( \mathcal{B} \). Thus, the pure state \( \omega \) on \( \mathcal{A}_\text{tot} \) is maximally \( (A, B) \)-correlated if

\[
\omega_A(A) = 0, \quad \omega_B(B) = 0
\]

(30)

for all \( A \in \mathcal{A} \), \( A \neq 1 \) and \( B \in \mathcal{B} \), \( B \neq 1 \).

We can use condition (30) to find maximally entangled states for a given choice of subalgebras \( \mathcal{A} \) and \( \mathcal{B} \). Let us give an example. Take the Bell pair \( (A, B) \), defined by (20), and consider the general pure state \( \omega_{\Phi} \). Condition (30), applied to generators of \( \mathcal{A} \) and \( \mathcal{B} \), gives the following equations for the parameters \( z_i \):

\[
\Re z_1 (\bar{z}_3 - \bar{z}_4) + \Re z_2 (\bar{z}_3 + \bar{z}_4) = 0,
\]
\[
\Im z_1 (\bar{z}_4 - \bar{z}_3) + \Im z_2 (\bar{z}_4 + \bar{z}_3) = 0,
\]
\[
|z_2|^2 - |z_1|^2 - 2 \Re z_3 \overline{z}_4 = 0,
\]

(31)

and

\[
\Re z_1 (\bar{z}_3 + \bar{z}_4) + \Re z_2 (\bar{z}_3 - \bar{z}_4) = 0,
\]
\[
\Im z_1 (z_3 + z_4) + \Im z_2 (z_3 - z_4) = 0,
\]
\[
|z_2|^2 - |z_1|^2 + 2 \Re z_3 \overline{z}_4 = 0.
\]

(32)
We solve equations (31) and (32) and find that

\[ z_1 = \frac{r}{\sqrt{2}} \cos \varphi, \]
\[ z_2 = -\frac{r}{\sqrt{2}} \cos \varphi e^{2i \theta}, \]
\[ z_3 = r \sin \varphi e^{i \theta}, \]
\[ z_4 = i \sqrt{1 - r^2} e^{i \theta}, \]

where \( r \in [0, 1] \), \( \varphi \in [0, \pi/2] \), and \( \theta \in [0, 2\pi] \). So the maximally \((A, B)\)-correlated states form a three-parameter family of vector states,

\[ \Psi_{r, \varphi, \theta} = \frac{r}{\sqrt{2}} \cos \varphi e_1 - \frac{r}{\sqrt{2}} \cos \varphi e^{2i \theta} e_2 + r \sin \varphi e^{i \theta} e_3 + i \sqrt{1 - r^2} e^{i \theta} e_4. \] (33)

As we see, none of the standard Bell states belong to this family.

4.3. Any pure state can be separable (or entangled)

As we have shown in the previous sections, there are examples of two-qubit states which are separable and at the same time entangled, depending on the choice of Bell pair of subalgebras. Now we show the general result: for any vector state \( \omega \), there exists a Bell pair \((A, B)\) such that the corresponding total correlation \( C_\omega(A, B) = c \), where \( 0 \leq c \leq 1 \), is a given number. It means that the same state can be separable or entangled to the given degree of non-separability. Although this fact was already demonstrated (see e.g. [6]), our proof is simpler and we are not using the notion of tensor product.

Let \( \alpha \) be the \( * \)-automorphism of \( A_{\text{tot}} \) such that it does not leave invariant the subalgebras \( A_0 \) and \( B_0 \). Obviously \( (\alpha(A_0), \alpha(B_0)) \) as well as \( (\alpha^{-1}(A_0), \alpha^{-1}(B_0)) \) are the new Bell pairs. We start with the following simple observation: the total correlations in the states \( \omega_\alpha = \omega \circ \alpha \) and \( \omega \) are related by

\[ C_\omega(A_\alpha, B_\alpha) = C_\omega(A_0, B_0). \] (34)

where

\[ A_\alpha = \alpha^{-1}(A_0), \quad B_\alpha = \alpha^{-1}(B_0). \]

Let \( \omega = \omega_\alpha \) be an arbitrary pure state and let \( \omega_0 \) be given by the vector \( \Phi \) defined by

\[ \Phi = \frac{1 - d}{2} e_2 + \frac{1 + d}{2} e_1, \quad d = \sqrt{1 - c^2}, \] (35)

where \( c \in [0, 1] \). One easily verifies that the correlation matrix of \( \omega_0 \) with respect to the canonical Bell pair is given by \( Q_0 = \text{diag}(c, c - c) \), so \( ||Q_0|| = c \). Thus, the state \( \omega_0 \) is \((A_0, B_0)\)-correlated, with the total correlation equal to the given number \( c \), where \( 0 \leq c \leq 1 \).

As is well known, all pure states of two qubits form the complex projective space \( \mathbb{C}P^3 \), on which unitary matrices act transitively (see e.g. [14]). Let \( U \) be the unitary matrix such that \( \Psi = U \Phi \) and define \( * \)-automorphism \( \alpha \),

\[ \alpha(A) = U^{-1}AU. \]

Then \( \omega = \omega_0 \circ \alpha \) and if we take \( A = A_\alpha, B = B_\alpha \), then by (34),

\[ C_\omega(A, B) = C_{\omega_0}(A_0, B_0) = c. \]

So we have
Theorem 3. Let \( \omega \) be an arbitrary pure state of two qubits. We can always choose a Bell pair of subalgebras in such a way that the total correlation in \( \omega \) is equal to a given number between \( 0 \) and \( 1 \).

It is worth stressing that the Bell pair realizing non-separability of a fixed state is not unique. There are pure states which are separable with respect to different choices of subalgebras. There are also maximally entangled states for at least two different Bell pairs (see the examples discussed above).

5. Conclusions

We have studied entanglement properties of two qubits, using the ideas of algebraic quantum mechanics. General theory of entanglement is based on the properties of the algebra of physical observables and its partitions representing the subsystems. The states, defined as linear functionals on observables, are entangled if they give non-vanishing correlations between independent subsystems. So the entanglement is always relative to a particular set of physical observables. We have shown that the universal measure of pure-state entanglement can be obtained by considering the total correlation between subsystems in a given state. The value of this quantity can be simply computed in terms of the norm of the corresponding correlation matrix. The abstract algebraic characterization of separability or non-separability of any pure state can be obtained by considering its restrictions to subsystems. Purity (non-purity) of restrictions are equivalent to separability (non-separability) of a state.

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