CLOSED STRING FIELD THEORY: AN INTRODUCTION

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ABSTRACT

In these introductory notes I explain some basic ideas in string field theory. These include: the concept of a string field, the issue of background independence, the reason why minimal area metrics solve the problem of generating all Riemann surfaces with vertices and propagators, and how Batalin-Vilkovisky structures arise from the state spaces of conformal field theories including ghosts. More advanced topics and recent developments are summarized. (To appear in the proceedings of the 1992 Summer School at Les Houches.)

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1. The Origin of the String Field

Much of the work in string theory has been done in the context of first quantization. This means working with two dimensional field theories, and making use of some interpretation in order to relate observables in the two dimensional theory, the world-sheet theory, to observables in the physical spacetime theory, the target space theory. This two dimensional field theory, typically a conformal field theory, is the two dimensional analog of one-dimensional field theories that describe the classical mechanics of point particles. Most of particle physics, however, is not done in first quantization. Our description of Yang-Mills theories, or gravitation, is done in the context of relativistic quantum field theory, or second quantization. String field theory aims to formulate string theory as a spacetime field theory. It follows that string field theory is the unconventional approach to string theory based on the idea of extending and generalizing the conventional field theory approach to particle physics.

One must certainly keep in mind that generalizations of the usual particle field theory concepts are necessary when one formulates string field theory. For example, in classical mechanics one writes an action for the coordinates $x^\mu(\tau)$ of a point particle, and in the passage to field theory one defines the field $\phi(x^\mu)$, dropping the proper time variable $\tau$. For strings the analog is to pass from $X^\mu(\sigma, \tau)$ to functional fields $\Phi(X^\mu(\sigma))$. This is well known not to work, the string field must have extra arguments that include the ghost fields of reparametrization invariance (see [1]). Only for such string fields one can write suitable string field actions. Further surprises were found in writing closed string field theory (see [2]), and very likely additional ones are awaiting us in our way to a complete formulation. Thus, in the final formulation, string field theory may have little in common with standard formulations of particle field theory except for the use of fields as dynamical variables. We can also hope, since the physics of strings is different from that of particles, that nonperturbative string phenomena may be extracted from string field theory less painfully than is typically the case in particle field theory.

It is very important to emphasize that, in current approaches to string field theory, we work in the path integral approach. That is, we do not work with string field operators that create or destroy strings, but rather with string fields which are classical c-numbers (and classical anticommuting numbers, for ghost fields or fermi fields). Quantization is defined by doing path integrals with the string field action.
Let us consider in some detail the type of first-quantized actions used to describe string theory. This will allow us to understand the origin of the string field. One of the simplest actions used to describe strings in first quantization is the following

$$S \sim \int d^2 \xi \sqrt{h} h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \quad (1.1)$$

where $a, b = 1, 2$ are indices labeling the coordinates $\xi^a$ in the two dimensional surface, $h_{ab}$ is a two dimensional metric, the scalar fields $X^\mu(\xi)$ give the embedding of the worldsheet in spacetime, and $\eta_{\mu\nu}$ is the flat Minkowski spacetime metric. String theory is supposed to be a theory of gravity so one may ask, where is the graviton, or the metric tensor in the above action? It is not there. That is the case because this is a first quantized action. One finds that this action can be used in a very indirect way to describe gravitons.

There are two dimensional actions that provide a somewhat more explicit framework for the study of the dynamics of the target space fields. These are the sigma model actions, and typically read as follows

$$S' \sim \int d^2 \xi \left( \sqrt{h} h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) \right. \quad (1.2)$$

\begin{align*}
+ \sqrt{h} R^{(2)}(h) \Phi(X) + \cdots.
\end{align*}

Here we see that the Minkowski metric $\eta_{\mu\nu}$ was replaced by the arbitrary metric $G_{\mu\nu}(X)$ and extra interactions were added, including in particular one parametrized by an antisymmetric object $B_{\mu\nu}(X)$, and another by $\Phi(X)$. In a sigma model action, the objects $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$, and $\Phi(X)$, are prescribed nonlinear functions of the two dimensional field $X(\xi)$. The above action is therefore that of a nonlinear sigma model. The object $G_{\mu\nu}(X)$ should play the role of a metric tensor in spacetime, $B_{\mu\nu}(X)$ should correspond to an antisymmetric tensor in spacetime, and $\Phi(X)$ should correspond to a scalar field, a dilaton in spacetime. Now we have the metric tensor, but where are Einstein’s equations for $G_{\mu\nu}$, or the equations for the antisymmetric tensor $B_{\mu\nu}$, or the equations for the dilaton field $\Phi$? They are not found directly from the action, again because this is not a second quantized action. They arise, however, from a rather remarkable condition. The condition that the above action be conformal invariant yields Einstein’s equations for $G_{\mu\nu}$ along with equations for the other fields. While fascinating,
it seems very difficult to use this approach for a complete formulation of string theory. The complete classical equations for the spacetime fields involve calculations to all loops in the two dimensional field theory. The method is also very difficult to use for spacetime fields that are not massless (we did not include those in the above action) since they correspond to nonrenormalizable interactions in two dimensions.

Nevertheless we see a general pattern. Each possible two dimensional interaction, or local operator, is accompanied by a spacetime field which appears as a coupling "constant" multiplying the interaction. If we think of the string field as a collection of spacetime fields, the above action suggests that the string field simply encodes the data of a two dimensional field theory. Given a particular string field we can associate a particular two dimensional field theory. It seems very difficult at this stage to make precise this idea for a variety of technical and conceptual difficulties. The way we know how to proceed goes as follows.

If we have a conformal field theory, as is the case for the first action we wrote above, conformal invariance provides us the complete and explicit list of all possible local operators $\Phi_s$ of the two dimensional conformal theory. For each of these operators we associate a spacetime field $\psi_s$. The state space $\mathcal{H}_{\text{CFT}}$ of a conformal field is the space of states created by letting all possible local operators act on the vacuum. Such states are denoted by $|\Phi_s\rangle$. For example, to the familiar graviton vertex operator $\partial X^\mu \partial X^\nu e^{ipX}$ we must associate the graviton field $h_{\mu\nu}(p)$

$$\partial X^\mu \partial X^\nu e^{ipX} \leftrightarrow h_{\mu\nu}(p). \quad (1.3)$$

We are then naturally led to assemble the string field as a general vector in the state space $\mathcal{H}_{\text{CFT}}$

$$|\Psi\rangle = \sum_s |\Phi_s\rangle \psi_s. \quad (1.4)$$

Here each target space field $\psi_s$ is the component of the vector $|\Psi\rangle$ along the basis vector $|\Phi_s\rangle$. The target space fields are in general complex numbers, and may be Grassmann even or odd. The string field action $S(\Psi)$ is a function from $\mathcal{H}_{\text{CFT}}$ to the real numbers.

* We must include, however, the reparametrization ghosts.
Loosely speaking, the string field does indeed appear to encode the data of a two dimensional field theory. If $S_c$ denotes the action for the conformal field theory that is being used to define the string field, then associated to the string field in Eqn.(1.4) we may attempt to define the two dimensional action

$$S_c + \sum_s \psi^s \cdot \int d^2 \xi \Phi_s(\xi, \bar{\xi}),$$

(1.5)

using the components of the string field (the spacetime fields) to weight the various interactions.† Again, due to technical difficulties it is difficult to define precisely the meaning of the above expression, unless the operators $\Phi_s$ are primary fields of dimension (1,1). Therefore, at present, the precise formulation of closed string field theory does not attempt to define the string action as a function in the space of two dimensional theories, but rather as a function in the state space of a given conformal field theory.

Therefore, in order to write a string field theory, we must first choose a conformal field theory. A conformal field theory defines a consistent spacetime background for string propagation. This means that we are only able to write string field theory once we pick a background that must correspond to a classical solution of the theory we are aiming to write. An analogy is useful to understand the situation. In Einstein’s gravity the dynamical variable is the metric tensor $g_{\mu\nu}(x)$ on some manifold $M$. The Einstein action, given a metric $g_{\mu\nu}$ gives us a number, this action is therefore a function on the space $\mathcal{G}$ of metrics on $M$. There are some special metrics on $\mathcal{G}$, the Ricci flat ones. They solve the classical Einstein’s equations and therefore define consistent spacetime backgrounds. We can study physics around any Ricci flat background $\hat{g}_{\mu\nu}$ by expanding the metric tensor as $g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ represents a fluctuation. The gravity action becomes a function of the field $h_{\mu\nu}$. While this action for $h_{\mu\nu}$ does contain all of the physics of gravity, the action depends explicitly of the background metric $\hat{g}_{\mu\nu}$ in a complicated way. In string theory the analog of a given metric $g_{\mu\nu}$ is a two dimensional field theory, and the analog of the space $\mathcal{G}$ of metrics on $M$ is the space of two dimensional field theories (it is not clear how to think of a two-dimensional field theory as a structure in some space, thus there is no compelling analog for $M$). Corresponding to the Ricci-flat metrics we have the conformal field theories. The string field $|\Psi\rangle$ we have discussed above corresponds to

† I am ignoring ghost insertions that are necessary to go from the string field to the world sheet interactions.
the fluctuation field $h_{\mu\nu}$. Indeed the string field action has background dependence; it uses, for example, the BRST operator of the conformal field theory. This necessity to fix a conformal field theory to get started writing a string field action is usually referred to as the issue of background independence of string field theory. It is certainly the central question facing string field theory. A background independent string field theory would most likely be the formulation of string theory we are looking for.

The problem of setting up a background independent string field theory is exactly analogous as that of reconstructing Einstein’s theory if we only knew the expansion of the Einstein lagrangian around flat space. Of course, the Einstein-Hilbert action was written before the flat space expansion was known, but as a problem of principle, the issue of reconstructing the gravity action from the interactions of a spin two field $h_{\mu\nu}$ has received considerable attention. Our current problem, and in this case we do not know the answer in advance, is to find the action from which our string field action $S(\Psi)$ arises by expansion around a classical solution. We need to understand the geometrical meaning of the series of self-interactions of the string field $\Psi$.

There are various ideas that are probably going to be helpful in the near future. They will be discussed in §4. Some of them center around the Batalin-Vilkovisky (BV) theory of quantization. The string field action $S(\Psi)$ actually satisfies a second order, nonlinear partial differential equation called the BV master equation. A generalization of the concept of a Lie algebra, a structure called “homotopy Lie algebra” seems to be at the basis of classical closed string field theory. At the quantum level, the algebraic structure becomes what one could call a “quantum homotopy Lie algebra”. Understanding these algebraic structures seems essential. At a geometrical level it is probably necessary, at least in the short run, to develop more of the geometry in the space of two dimensional field theories. Here again BV seems to be of help; it is reasonable to assume that this theory space is actually a (super) symplectic manifold. Finally, geometry of two dimensional Riemann surfaces, or geometry on the space of Riemann surfaces (moduli spaces) plays a crucial role in writing string field theory. I would expect this geometry to tie in nicely with theory space geometry in the future formulations of string field theory.
2. String Diagrams from an Extremal Problem

If string theory is to be formulated in a natural way as a field theory there has to exist a natural way to generate the sum over all surfaces (necessary in the computation of any string amplitude) with the use of a propagator and a set of vertices. For a covariant string field theory, the vertices should be symmetric under the exchange of the various strings. It was widely believed that no natural way would be found to generate all Riemann surfaces once and only once. Indeed, experience suggested that any simple choice of vertices and propagator would either miss surfaces, produce some surfaces more than once (typically an infinite number of times), or both.

I would like to show here how one can build all the different Riemann surfaces efficiently by using vertices and propagators. I will prove explicitly two basic results and then explain why they guarantee that all Riemann surfaces can be produced without missing any surface and without overcounting any surface. In order to achieve this we use string diagrams. For original references the reader may consult [3–12].

A string diagram is the analog of a particle field theory Feynman diagram. One speaks of the string diagram corresponding to a given Riemann surface. The string diagram is nothing else than the Riemann surface equipped with some extra structure. This structure can be a analytic one-form (abelian differential), an analytic two-form (quadratic differential), or typically, a (conformal) metric. It is the extra structure on the Riemann surface that should tell us how the surface is built in terms of vertices and propagators. What we have found in the last few years is that, for covariant string field theory, a metric seems to be the right structure to put on a surface. That metric, moreover, is very special, it is the solution to an extremal problem. Not only for closed string field theory these metrics are relevant. Extremal metrics actually define the string diagrams of open string theory and of open-closed string field theory. Thus actually all string diagrams for covariant string field theory are defined by extremal metrics.

Let us first explain what is a (conformal) metric on a Riemann surface (see [13]). A two dimensional Riemannian manifold, that is, a two dimensional manifold with a Riemannian metric \( g_{\alpha\beta} \), actually defines a fixed Riemann surface \( \mathcal{R} \), since a metric defines a conformal structure. The metric can be put locally in the conformal form \( dl^2 = \rho^2(dx^2 + dy^2) \), with \( z = x + iy \) the local analytic coordinate, and, with \( \rho(x, y) \) the positive semidefinite Weyl
factor. \( \rho \) is said to be a conformal metric (metric, for short) on the Riemann surface \( \mathcal{R} \). In the extremal problem we wish to consider the complex structure will be kept fixed, and we will choose a (conformal) metric satisfying some extremal property. Varying the Riemannian metric would mean that we are also changing the conformal structure -this we do not want to do. A (conformal) metric \( \rho \) on a Riemann surface \( \mathcal{R} \) defines a length element \( dl = \rho|dz| \), and an area element \( \rho^2 dx \wedge dy \). Under analytic changes of coordinates the metric must transform as \( \rho(z)|dz| = \tilde{\rho}(w)|dw| \). This transformation guarantees the invariance of the length of a curve and the invariance of the area under analytic mappings.

In the standard minimal area problem one has a Riemann surface \( \mathcal{R} \), and one chooses a set \( \Gamma_i \) of free\(^*\) nontrivial homotopy classes of closed curves. Let \( \tilde{\gamma}_i \) be representative curves for the chosen homotopy classes. Associated to each homotopy class one chooses a constant \( A_i \geq 0 \). One then asks for the metric \( \rho \) of least possible area on \( \mathcal{R} \) under the condition that for any curve \( \gamma \) freely homotopic to \( \gamma_i \) (\( \gamma \sim \gamma_i \)) the length of \( \gamma \) on the metric \( \rho \) must be greater than or equal to \( A_i \):

\[
\int_{\gamma \sim \gamma_i} \rho|dz| \geq A_i. \tag{2.1}
\]

This requirement must hold for all \( i \) in the set. A metric is called \emph{admissible} if it satisfies the length conditions (2.1). Thus the minimal area problem asks for the admissible metric of least possible area. A useful property of the space of admissible metrics is that it is a convex space; if \( \rho_0 \) and \( \rho_1 \) are admissible metrics then

\[
\rho_t = (1-t)\rho_0 + t\rho_1, \tag{2.2}
\]

is an admissible metric for all \( t \in [0,1] \), as follows directly from (2.1). Let \( \mathcal{A}(\rho) \) denote the area of \( \mathcal{R} \) with the metric \( \rho \).

\(^*\) Basepoint free.
2.1. TWO USEFUL RESULTS

The area functional \( A \) has a very nice property: it is strictly convex. This means that for the above one-parameter family of metrics, with \( \rho_0 \neq \rho_1 \), one has

\[
A(\rho_t) < (1 - t)A(\rho_0) + tA(\rho_1),
\]

for \( t \in (0, 1) \). In words, the area is lower than the linear interpolation of the areas at the endpoints. The inequality is strict since we do not include the endpoints \( t = 0 \), and \( t = 1 \), corresponding to \( \rho_0 \) and \( \rho_1 \). It simply means that the area, as a function of \( t \), is a convex function. This relation is simple to derive, so we will do so now. By definition

\[
A(\rho_t) = \int \rho_t^2 \, dx \, dy = \int ( (1 - t)\rho_0 + t\rho_1 )^2 \, dx \, dy
\]

\[
= (1 - t)^2 A(\rho_0) + t^2 A(\rho_1) + 2t(1 - t) \int \rho_0 \rho_1 \, dx \, dy.
\]

(2.4)

Since \( \rho_0 \neq \rho_1 \) (by assumption) Schwarz’s inequality implies that

\[
\int \rho_0 \rho_1 \, dx \, dy < \left( \int \rho_0^2 \, dx \, dy \int \rho_1^2 \, dx \, dy \right)^{1/2} = (A(\rho_0)A(\rho_1))^{1/2}.
\]

(2.5)

Using this inequality for the last term in (2.4) we find

\[
\sqrt{A(\rho_t)} < (1 - t) \sqrt{A(\rho_0)} + t \sqrt{A(\rho_1)},
\]

(2.6)

which shows that \( f(t) = \sqrt{A(\rho_t)} \) is convex. But, if an everywhere positive function is convex, its square is also convex. This implies the desired result, namely, the area functional is convex.

An immediate consequence is the uniqueness of the minimal area metric (if it exists). This is proven as follows. Assume there are two different metrics \( \rho_0 \) and \( \rho_1 \) on the surface \( \mathcal{R} \), both admissible and both having the (same) least possible area \( A \). It follows that the admissible metric \( \rho_t \) (for any \( t \in (0, 1) \)) must satisfy \( A(\rho_t) < (1 - t)A + tA = A \), in contradiction with the assumption that \( A \) was the least possible value for the area. The uniqueness of the minimal area metric is fundamental for our purposes, as will be explained later.
There is another result, which is also standard, easy to derive, and very powerful. Consider the rectangular region in the complex plane bounded by the points 0, $a$, $ib$, and $a+ib$, where $a$ and $b$ are real numbers greater than zero. On this region of the complex plane we would like to find the metric of least possible area such that any curve starting anywhere on the left vertical segment $[0, ib]$ and ending anywhere on the right vertical segment $[a, a+ib]$ should be longer than or equal to $a$. This problem is the canonical (and simplest) minimal area problem. We are looking for a metric $\rho^2(dx^2 + dy^2)$ with least possible area $A(\rho) = \int_0^b dy \int_0^a dx \rho^2$. It is clear that the metric $\rho(x, y) = 1$ is admissible, indeed, any curve crossing the rectangle from left to right must be longer than or equal to the length of a strictly horizontal line ($y$ constant). The horizontal lines are precisely of length $a$. The area of this metric is $ab$. It follows that for the metric $\hat{\rho}$ of least possible area we must have $A(\hat{\rho}) \leq ab$. We will show now that actually $\rho = 1$ is the metric of least possible area. To this end consider an admissible metric $\rho$. Since the length condition must be satisfied for a horizontal curve with fixed $y$

$$\int_0^a \rho(x, y) \, dx \geq a, \quad \forall y, \quad \rightarrow \quad \int_0^b \int_0^a \rho(x, y) \, dx \geq ab. \quad (2.7)$$

The Schwarz inequality

$$\int f^2 \, dx \, dy \int g^2 \, dx \, dy \geq \left( \int f \, g \, dx \, dy \right)^2, \quad (2.8)$$

with the choice $f = \rho$, and $g = 1$, gives us

$$A(\rho) = \int \rho^2 \, dx \, dy \geq \frac{1}{ab} \left( \int \rho \, dx \, dy \right)^2. \quad (2.9)$$

All integrals here are over the rectangular region. Using (2.7) we immediately see that for any admissible metric $\rho$ we have that $A(\rho) \geq ab$. Thus the least possible area is indeed $(ab)$. Since that was the area for the metric $\rho(x, y) = 1$, the uniqueness of the minimal area metric implies that the minimal area metric is indeed $\rho(x, y) = 1$. This is what we wanted to show.

We can get a further result as a simple consequence of the above. Identify the left edge and the right edge of the rectangle via the relation $iy \equiv a + iy$, for $0 \leq y \leq b$. We then get an annulus, or a cylinder. It is clear from the above derivation that the metric of least possible area under the condition that any closed curve, freely homotopic to a horizontal closed curve, be longer than or equal to $a$, is still the flat metric $\rho(x, y) = 1$. 

2.2. THE MINIMAL AREA PROBLEM FOR CLOSED STRING THEORY

We discussed at the beginning of §2 what is the general minimal area problem. One chooses a set (finite or infinite) of homotopy classes of closed curves and imposes length conditions on all curves belonging to the chosen homotopy classes. This problem has not yet been solved, in fact, very little is known. The solution is known for the case when we choose a set of homotopy classes having representative curves that can be chosen to be nonintersecting simple closed Jordan curves (no self intersections). Such a set of representative curves is called, in the mathematical literature, an admissible set. For any fixed genus the maximum number of curves in an admissible set is finite –if we try to add additional curves we must produce intersections. If we impose length conditions on curves homotopic to those on an admissible set the minimal area metric is known. It is a metric that arises from a Jenkins-Strebel quadratic differential [13]. Intuitively, this means that the surface, with the minimal area metric, is built by gluing together flat euclidean cylinders of circumferences equal to the length parameters $A_i$ (see equation (2.1)). If we put length conditions on curves homotopic to representatives that must intersect, even on two such representatives, the solution is not known.

The minimal area problem relevant for closed string field theory is the following. For any surface $\mathcal{R}$ (with or without punctures) we are interested in the metric of least possible area under the condition that all nontrivial closed curves on $\mathcal{R}$ be longer than or equal to $2\pi$ [10]. This minimal area problem, in contrast with the related extremal problems studied earlier [13], does not require specifying some homotopy classes of curves on the Riemann surface on which we impose length conditions. The length conditions are imposed on all homotopy classes. This is why this is a problem defined on moduli space, why the problem is modular invariant. Whenever we choose particular curves on a surface we are introducing extra data. For example, on a torus, choosing two curves without self intersections, and intersecting each other once, amounts to choosing a particular representative for the torus in Teichmuller space. Having a well defined problem which does not require extra structure is crucial for us. Since we have put conditions on all homotopy classes of curves, and clearly their representatives intersect, this problem does not fall within the class of problems whose solution is known.

The results established in §2.1 will now allow us to prove a result that gives us a fair amount of intuition about the minimal area problem for closed string theory. I claim that if a surface is built by gluing together flat cylinders of circumference $2\pi$, and moreover, no closed curve
on the resulting surface is shorter than $2\pi$, then the surface has a metric solving the minimal area problem. The argument (which is truly simple) goes as follows. The cylinders split the surface into a set of rectangular domains of the type considered in the previous section, each with $a = 2\pi$, and with the vertical edges identified. Since the core curve of each rectangle (a closed horizontal curve) is nontrivial, it is necessary that any curve homotopic to a core curve and contained completely on the corresponding rectangle be longer than or equal to $2\pi$. Let us call this condition \((\alpha)\). Condition \((\alpha)\) is necessary, although not sufficient for a metric to be admissible. Imagine there is an admissible metric with area lower than that of the original metric. Then it would have to have lower area at least on one of the rectangles. But this is impossible since, on each rectangle, the original flat $\rho = 1$ metric has already the least possible area under condition \((\alpha)\). This proves the result. We therefore have a simple way to construct many metrics of minimal area. We glue flat cylinders watching out that no closed curve is smaller than $2\pi$.

Let us now explain a very fundamental idea, the idea that shows the relevance of minimal area metrics. The origin of all the difficulties in constructing a field theory of closed strings lies on the fact that given two Riemann surfaces it is, in general, very hard to tell whether or not they are the same, that is, whether or not there is a conformal map from one to the other. This means that when we construct Riemann surfaces using vertices and propagators it is very hard to guarantee that no Riemann surface is produced more than once. The situation is very much improved if we put metrics on the surfaces. It is actually easy to see if two surfaces with metrics are the same. For example, if we build our surfaces using the cylinders discussed in the paragraph above two metrics are the same if they look the same. That is, they must have the same number of cylinders, and the gluing patterns must be the same. This is the case because the cylinders determine very special saturating geodesics on the surface, length $2\pi$ geodesics that saturate the length conditions. Two surfaces cannot have the same metric if their patterns of saturating geodesics are not the same. When we construct our surfaces with metrics, our problem is making sure that for any two such surfaces \((\mathcal{R}_1, \rho_1)\), and \((\mathcal{R}_2, \rho_2)\) the underlying Riemann surfaces \(\mathcal{R}_1\) and \(\mathcal{R}_2\) are not the same (otherwise we have overcounting). Suppose we can tell that the metrics are not the same, what does this buy for us? If the metrics are not the same and the Riemann surfaces are different that is no problem\(^*\). The problem

\(^*\) Precisely speaking, in this case, the metrics are different as Riemannian metrics on the underlying two
happens if the Riemann surfaces are the same despite the fact that the metrics are different. Here is where the minimal area principle helps; if we can assure that the metrics are of minimal area, their being different guarantees that the Riemann surfaces are different! The reason is uniqueness of the minimal area metric on a given Riemann surface. If the two Riemann surfaces were the same they could not have two different metrics solving our extremal problem. As one can imagine, different Feynman graphs in string theory correspond to different length of cylinders and different gluing patterns. Therefore different Feynman graphs produce different metrics, and if we guarantee that they are all of minimal area, we are free of the problem of overcounting. I have explained above one simple criterion to tell whether a metric is of minimal area (the surface is built with cylinders). This criterion is sufficient but not necessary, in fact not all minimal area metrics are of this type. In the next section I will sketch an argument why the rules of sewing, which allow us to build complicated surfaces starting from vertices and propagators, are compatible with minimal area.

A very interesting problem, an extremal problem for metrics on Riemannian manifolds, has been investigated by Gromov [14] and Calabi [15]. Their problem for the case of two dimensional surfaces is a particular case of our minimal area problem. In their case they consider a fixed two dimensional manifold $M$ and the space of Riemannian metrics on $M$ such that all nontrivial closed curves are longer than or equal to $2\pi$. Then they ask for the Riemannian metric of least possible area. Since they can change the complex structure of the surface, it seems clear that their “extremal isosystolic” metrics should correspond, at every genus, to a minimal area metric on the Riemann surface whose extremal metric has the least possible area. They can prove existence of the extremal isosystolic metric. Uniqueness is not clear, it could be that a finite number of different Riemann surfaces have extremal metrics whose area is the same, and at the same time lowest among all the areas of the extremal metrics on all other Riemann surfaces.

dimensional manifold, since as conformal metrics they are not defined on the same Riemann surface and any comparison is meaningless.
2.3. Why Minimal Area Metrics Work

Let us now sketch the logic that shows that minimal area metrics solve our problem of generating all Riemann surfaces once and only once. Here I will not be completely self-contained, but hope to give a reasonably clear idea of how things fit together. For complete details the reader should consult the original papers.

A metric solving the minimal area problem is expected to give rise to closed geodesics of length $2\pi$ that foliate the surface completely. If there is a point through which there is no saturating geodesic the metric could be lowered at this point without destroying admissability. We can show that such curves must foliate the surface since they cannot intersect whenever they are of the same homotopy type. In fact, generically, two saturating geodesics can at most intersect in one point. For the case of Riemann spheres with punctures, since any Jordan closed curve must cut the surface into two separate pieces, two saturating geodesics that cross they must do so at least in two points. Since this cannot happen, the surface will be foliated by bands of geodesics that do not intersect. This is precisely what happens with Jenkins-Strebel (JS) quadratic differentials, the horizontal trajectories of the quadratic differential are the saturating geodesics. The surface is then foliated by bands of geodesics, the so-called ring-domains of the quadratic differential. The horizontal trajectories can intersect in the critical graph of the quadratic differential, but this graph is of zero measure on the surface. Thus all minimal area metrics on Riemann spheres (that define the classical closed string field theory) arise from JS quadratic differentials. They are now known explicitly and can be described in terms of polyhedra. In higher genus Riemann surfaces one can have saturating bands of geodesics that cross. Thus higher genus minimal area metrics do not always arise from JS quadratic differentials. One concrete example showing crossing of foliations was given in [12].

In a punctured Riemann surface, the minimal area metric must have all closed curves homotopic to the punctures satisfying the length conditions. This actually allows one to show that a minimal area metric is isometric to a semiinfinite cylinder of circumference $2\pi$ for some neighborhood of each puncture. This semiinfinite cylinder must end somewhere on the surface; let $C_0$ denote the boundary curve of the semiinfinite cylinder. Let $C_l$ denote the saturating geodesic in the cylinder a distance $l$ away from $C_0$. The minimal area metrics satisfy an amputation property; if we amputate the semiinfinite cylinder along $C_l$ for $l > 0$, the resulting surface still has a minimal area metric. This property allows us to show that the plumbing of
surfaces with minimal area metrics gives a surface with a minimal area metric [7,12]. The basic idea is simple, given two surfaces to be sewn together (or a single surface with two legs to be sewn), one first amputates the semiinfinite cylinders corresponding to the relevant legs. Given the amputation property the resulting surfaces with boundaries have minimal area metrics. If we glue together the open boundaries, and, by doing so we do not create closed curves that are smaller than $2\pi$, (this is the reason for stubs, as we will see), then the resulting surface inherits a minimal area metric. This follows because any candidate metric with less area would have to have less area in at least one of the surfaces that were glued, but this is impossible given that the amputated surfaces have minimal area metrics. The reader interested in the complete detailed argument should consult [12].

We define the height $h$ of a foliation (a band of geodesics making up an annulus) to be the shortest distance, along the annulus, between the two boundary components. It can be shown that a foliation with height $h$ greater than $2\pi$ is isometric to a flat cylinder of circumference $2\pi$ and height $h$. This suggests that on a string diagram we can identify propagators with the cylinders that have heights greater than or equal to $2\pi$. One can prove a cutting theorem (closely related to amputation); if we cut open a foliation of height greater than $2\pi$ on a metric of minimal area, the resulting (cut) surface still has a minimal area metric.

When building the string field theory we must choose vertices, we call $V_{g,n}$ the vertex that must be introduced at genus $g$ for processes with $n$ punctures. This vertex corresponds to the surfaces that are missed by the Feynman rules using lower dimensional vertices. There is a simple criterion to tell whether or not a surface $\mathcal{R}$ belongs to the vertex:

The String Vertex $V_{g,n}$. $\mathcal{R} \in V_{g,n}$ if and only if the heights of all internal foliations are less than or equal to $2\pi$. If $\mathcal{R}$ is in $V_{g,n}$ we define the coordinate curves to be the $C_\pi$ curves around each puncture.

In the above, internal foliation refers to a foliation that does not correspond to one of the semiinfinite cylinders. We have placed coordinate curves leaving "stubs" of length $\pi$ for each puncture. The coordinate curves define the amputated vertices, that is the vertices as surfaces with holes ready to be joined to each other with propagators. The short cylinder of length $\pi$ left from each semiinfinite cylinder is necessary in order to make sure that the plumbing procedure produces admissible metrics; if we had no stubs the plumbing of two holes on a single surface, with a propagator of small length, could introduce curves shorter than $2\pi$, and
the resulting surface would not inherit a minimal area metric. The stubs guarantee that upon plumbing no curve shorter than $2\pi$ is generated. Then plumbing is guaranteed to preserve the minimal area property.

Two things must be checked. First, no surface in $\mathcal{V}$ must be produced by sewing. Indeed, due to the stubs of length $\pi$, sewing must create foliations of height greater than to $2\pi$, and, by definition, the surface cannot belong in $\mathcal{V}$. The second point that must be checked is that all surfaces which are not in $\mathcal{V}$ are produced by sewing, and produced only once. The first part is clear because for any surface which is not in $\mathcal{V}$ there is at least one foliation whose height is greater than $2\pi$. By the cutting theorem we can cut all such foliations open and obtain a surface(s) with a minimal area metric(s). Restoring the semiinfinite cylinders around all boundaries one obtains the Riemann surface(s) whose plumbing must give us the original surface. Therefore no surface is missed. Finally, no surface could have been produced more than once, since different Feynman diagrams correspond to different metrics, which by uniqueness of the minimal area metric, cannot correspond to the same Riemann surface.

There are some important open questions left about the extremal metrics solving the minimal area problem at higher genus. We do not have a mathematical proof of the existence of such metrics. While I am not concerned about the possibility that the metrics might not exist, a proof of existence is likely to be helpful in understanding further properties of the extremal metrics. We would like to know how the metrics look in general, and what type of curvature singularities they have. It would also be interesting to know how to parametrize the general metrics of minimal area. Explicit knowledge of the minimal area metrics is likely to be useful in future calculations using string field theory.

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* At higher genus a fraction of each moduli space is made of Riemann surfaces whose minimal area metrics arise from quadratic differentials. We also know now of explicit examples of minimal area metrics that do not arise from quadratic differentials.
3. Batalin-Vilkovisky Structures

Batalin-Vilkovisky (BV) quantization is the complete solution of the problem of quantizing a general gauge theory (having no second class constraints). In the usual applications one starts with a classical gauge invariant theory with an action $S_0(\phi^i)$ defined on a set of fields $\phi^i$. As a first step one extends the set of fields into a larger set comprised of fields $\psi^i$, and antifields $\psi^*_i$. One then finds a master action $S_M(\psi_i, \psi^*_i)$ which reduces to the original action $S_0$ upon setting the antifields to zero, and that solves the classical master equation $\{S_M, S_M\} = 0$. The antibracket appearing on the left hand side in the analog of the Poisson bracket between coordinates and momenta in classical mechanics. Here it is defined using the pairing between fields and antifields. To define a quantum theory, however, one must make sure that the master action actually satisfies the complete master equation $\hbar \Delta S_M + \frac{1}{2} \{S_M, S_M\} = 0$. This typically requires extra work, since $\Delta S_M$ need not vanish in general. The final solution $S_M(\hbar)$ is the quantum master action.

BV quantization is extremely efficient in closed string field theory because, once we have a consistent set of string vertices $\mathcal{V}_{g,n}$, we can write directly the full quantum master action $S_M(\hbar)$ solving the quantum master equation [7]. The full spectrum of fields and antifields also arises naturally from the conformal field theory. In string field theory there is no need to go through the steps listed in the above paragraph.

In the present section I will explain in detail why the string field can be broken naturally into fields and antifields. This happens because in the state space $\mathcal{H}_{CFT}$ one can introduce a symplectic structure. In understanding this point we will have to set up the kinetic term of closed string field theory.

Batalin-Vilkovisky quantization in covariant open string field theory was developed by Thorn [16,17] and Bochicchio [18]. Hata found BV quantization relevant to the study of the unitarity of the HIKKO (light-cone-style) closed string field theory [19].
3.1. Symplectic Vector Spaces

In ordinary symplectic geometry, a symplectic vector space is a real vector space $V$ equipped with a nondegenerate bilinear two form $\omega$ (taking $V \otimes V$ to $\mathbb{R}$). The antisymmetry property of the form implies that given two vectors $X, Y \in V$, $\omega(X, Y) = -\omega(Y, X)$. The non-degeneracy property requires that whenever $\omega(X, Y) = 0, \forall Y$, this must imply that $X = 0$. Using a basis we can write explicitly $\omega(X, Y) = \omega_{IJ} X^I Y^J$, and nondegeneracy implies that the matrix $\omega_{IJ}$ is invertible. We let $\omega^{IJ}$ denote the inverse matrix. A symplectic vector space must be even dimensional, and one can always find a symplectic basis $(X_1, \cdots, X_n; Y_1, \cdots, Y^n)$ such that $\omega(X_i, Y_j) = \delta^j_i$, and, $\omega(X_i, X_j) = \omega(Y_i, Y_j) = 0$. This is how the symplectic structure can be used to provide a pairing between basis vectors.

For the cases of vector spaces whose vectors can be either even or odd objects, as is the case for the conformal field theories relevant to us, we must consider the super case. We then have a super vector space with an odd, bilinear, nondegenerate two form $\omega$. The form being odd means that $\omega(A, B)$, for vectors $A$ and $B$ of definite statistics, is nonvanishing only if $A$ and $B$ have opposite statistics. Bilinearity and nondegeneracy have exactly the same meaning as in the commuting case. Finally, the exchange symmetry of a (super) two form requires that

$$\omega(A, B) = -(-)^{AB} \omega(B, A),$$

where $A, B$ appearing in the exponent refer to the Grassmanality of the object (0[mod 2] for even objects, and, 1[mod 2] for odd objects). For even vectors this gives the expected minus sign. A (super) vector space equipped with such (super) symplectic structure, as in the bosonic case, admits a symplectic basis, which determines a pairing of basis vectors. We want to exhibit explicitly this pairing for $\mathcal{H}_{\text{CFT}}$.

I want to show why (3.1) is compatible with the exchange properties of the antibracket. The antibracket is defined on a (super)symplectic manifold, that is, a manifold equipped with a odd nondegenerate closed two form $\omega$. At every point on the manifold the tangent space is a symplectic vector space. Given two functions $\mathcal{A}$ and $\mathcal{B}$ on the manifold, the antibracket is defined as follows

$$\{\mathcal{A}, \mathcal{B}\} = \omega(\mathcal{A}, \mathcal{B}),$$

where $\mathcal{A}, \mathcal{B}$ are the Hamiltonian vector fields associated to the functions $\mathcal{A}, \mathcal{B}$. Given the
standard relation $i_x \omega = -dA$, between functions and their corresponding Hamiltonian vectors, we note that they have opposite statistics. Therefore

$$\{A, B\} = \omega(A, B) = -(-)^{AB} \omega(B, A) = -(-)^{(A+1)(B+1)} \{B, A\}, \quad (3.3)$$

which is the correct exchange property of the antibracket.

3.2. Ghost Conformal Field Theory

The conformal field theories relevant for string theory are those with total central charge zero. These conformal theories must include the conformal field theory of the reparametrization ghosts, having central charge $(-26)$, together with some other conformal theories adding up to a central charge of $(+26)$. We need to know the basics of this ever present ghost conformal field theory. We consider the conformal field theory formulated in the $z$-plane with $z = \exp(\tau + i\sigma)$. We have ghost fields $c(z)$ and $\bar{c}(\bar{z})$ of dimensions $(-1, 0)$ and $(0, -1)$ respectively, and antighost fields $b(z)$ and $\bar{b}(\bar{z})$ of dimensions $(2, 0)$ and $(0, 2)$ respectively:

$$c(z) = \sum_n c_n \frac{z^n}{z^{n-1}}, \quad \bar{c}(\bar{z}) = \sum_n \bar{c}_n \frac{\bar{z}^n}{\bar{z}^{n-1}}; \quad (3.4)$$

$$b(z) = \sum_n b_n \frac{z^n}{z^{n+2}}, \quad \bar{b}(\bar{z}) = \sum_n \bar{b}_n \frac{\bar{z}^n}{\bar{z}^{n+2}}. \quad (3.5)$$

The stress tensor corresponding to this conformal theory is given by

$$T_g(z) = -2b(z) \cdot \partial c(z) - \partial b(z) \cdot c(z), \quad (3.6)$$

with a similar relation for the antiholomorphic piece. The basic operator product expansion is

$$b(z)c(w) \sim \frac{1}{z-w}. \quad (3.7)$$

The modes satisfy the anticommutation relations

$$\{b_n, c_m\} = \{\bar{b}_n, \bar{c}_m\} = \delta_{m+n,0}, \quad (3.8)$$

with all other anticommutators equal to zero. It is convenient to define new zero modes by
linear combinations of the old ones

\[ c_0^\pm = \frac{1}{2}(c_0 \pm \bar{c}_0), \quad b_0^\pm = b_0 \pm \bar{b}_0. \]  

(3.9)

One defines an in-vacuum \(|0\rangle \in \mathcal{H}_{\text{CFT}}\), corresponding to \(z = 0\), with no operator inserted there, and an out-vacuum \langle 0| \in \mathcal{H}_{\text{CFT}}^*\), corresponding to \(z = \infty\), with no operator inserted there. It follows from the regularity of the conformal fields \(c\) and \(\bar{c}\) at \(z = 0\), and \(z = \infty\), that the oscillators \((c_{-1}, \bar{c}_{-1}, c_0^+, c_0^-, c_1, \bar{c}_1)\) do not annihilate the in-vacuum nor the out-vacuum. This requires that the basic overlap be of the form

\[ \langle 0|c_{-1}\bar{c}_{-1}c_0^+c_0^-c_1\bar{c}_1|0\rangle \sim 1. \]  

(3.10)

We define the first quantized ghost number operator \(G\) by

\[ G = 3 + \left[ \frac{1}{2}(c_0b_0 - b_0c_0) + \sum_{n=1}^{\infty} (c_{-n}b_n - b_{-n}c_n) + \text{a.h.} \right]. \]  

(3.11)

The reader should verify that \(G|0\rangle = 0\). This operator satisfies the following commutation relations

\[ [G, c_n] = c_n, \quad [G, b_n] = -b_n, \quad [G, Q] = Q, \]  

(3.12)

and analogous relations for the antiholomorphic objects.

### 3.3. Symplectic Structure on \(H'_{\text{CFT}}\)

Once the ghost conformal field theory is combined with a matter conformal field theory, the total stress tensor is the sum of the matter stress \((T_m(z), \overline{T}_m(\bar{z}))\), and the ghost stress tensor \((T_g(z), \overline{T}_g(\bar{z}))\). It has central charge zero and dimension two

\[ T(z) = \sum_n \frac{L_n}{z^{n+2}}, \quad \overline{T}(z) = \sum_n \frac{\overline{T}_n}{\overline{z}^{n+2}}. \]  

(3.13)

The BRST operator of the combined conformal theory is given by

\[ Q = \int \frac{dz}{2\pi i} c(z)(T_m(z) + \frac{1}{2}T_g(z)) + \int \frac{d\bar{z}}{2\pi i} \overline{c}(\bar{z})(\overline{T}_m(\bar{z}) + \frac{1}{2}\overline{T}_g(\bar{z})), \]  

(3.14)

where the operators in the integrand are normal ordered. We can verify that

\[ \{Q, b(z)\} = T_m(z) + T_g(z) = T(z), \quad \{Q, \bar{b}(z)\} = \overline{T}_m(\bar{z}) + \overline{T}_g(\bar{z}) = \overline{T}(\bar{z}). \]  

(3.15)
Reflector State  Let $|\Phi_i\rangle \in \mathcal{H}_{\text{CFT}}$ be a basis for states, and $\langle \Phi^i | \in \mathcal{H}^*_{\text{CFT}}$ be the dual basis: $\langle \Phi^i | \Phi_i \rangle = \delta^i_i$. In any conformal theory there is an association of surfaces to states. Consider the two punctured sphere with uniformizing coordinate $z$, with a puncture at $z = 0$, and local coordinate $z_1 = z$ at that point, and a puncture at $z = \infty$, with local coordinate $z_2 = 1/z$ at that point. The state $\langle R_{12} | \in \mathcal{H}^*_{\text{CFT}} \otimes \mathcal{H}^*_{\text{CFT}}$ corresponding to this surface is defined via the relation

$$
\langle R_{12} | \Phi_i \rangle_{(1)} | \Phi_j \rangle_{(2)} \equiv \langle \Phi^i (z_1 = 0) \Phi^j (z_2 = 0) \rangle \equiv G_{ij},
$$

namely, the components of the state $\langle R_{12} |$ are given by correlators on the above two punctured sphere. The local coordinates we have chosen are necessary to be able to define the correlators of local operators which are not dimension zero primaries. The metric $G_{ij}$ must be nondegenerate. We also have the useful relations

$$
\langle R_{12} | (c_n^{(1)} + c_{-n}^{(2)}) = 0,
\langle R_{12} | (b_n^{(1)} - b_{-n}^{(2)}) = 0,
\langle R_{12} | (Q^{(1)} + Q^{(2)}) = 0,
\langle R_{12} | (G^{(1)} + G^{(2)} - 6) = 0.
$$

The first two identities, which hold for all $n$, can be obtained by expressing the oscillators in terms of contour integrals of the ghosts (or antighost) conformal fields, and using the definition (3.16). The last two identities also follow from contour deformation, both the BRST charge and the ghost charge are contour integrals of holomorphic currents. The last identity also follows directly from the first two ones, and Eqn.(3.11); it implies that a nonvanishing overlap $\langle R_{12} | A \rangle_{(1)} | B \rangle_{(2)} \neq 0$ requires that the ghost numbers of $A$ and $B$ add up to six: $G(A) + G(B) = 6$.

A Kinetic Term for Closed String Fields  There is no unique symplectic structure that can be introduced in $\mathcal{H}_{\text{CFT}}$. We need to understand which is the physically relevant one. The answer emerges clearly, as I will show now, from an analysis of equations of motion for closed string fields.

It was long felt that a satisfactory linearized closed string field equation would be $Q | \Psi \rangle = 0$, possibly supplemented with a ghost number constraint for the string field. This would have
been the analog of the open string linearized field equation. It is very clear now that this
could not be the correct answer. It is interesting that the difficulty and its resolution can be
understood by attempting to write a kinetic term for string fields that would give \( Q|\Psi\rangle = 0 \) as
an equation of motion. Due to the nondegeneracy of the reflector an obvious choice for kinetic
term would seem to be
\[
S_0^2 \sim \langle R_{12}|\Psi \rangle (1) Q^{(2)} |\Psi \rangle (2) \tag{3.18}
\]
The problem now is ghost number. Since the vacuum was assigned ghost number zero, and we
only have operators of integer ghost number, the ghost number of the string field should be
an integer. Moreover, the ghost numbers of \(|\Psi\rangle\) and that of \(Q|\Psi\rangle\) must add up to six. This is
clearly impossible, as it would require a string field of ghost number \(5/2\). Therefore the above
candidate action does not work. The fact that closed string vertex operators are usually of the
form \(c(z)\bar{c}(\bar{z})V(z, \bar{z})\), with \(V\) a matter operator, suggests that the string field ought to be of
ghost number \(+2\). In that case the kinetic operator \(Q\) of ghost number \(+1\) should be replaced
by an operator of ghost number \(+2\).

A clue emerges when we note that we can restrict ourselves to work with the subspace of
the state space consisting of states that are annihilated by \(L_0 - \overline{L}_0\). We do not lose anything
because any physical state (a state annihilated by \(Q\)) which is not annihilated by \(L_0 - \overline{L}_0\)
is actually trivial (can be written as \(Q\) acting on something). Indeed, let \(h - \overline{h}\) denote the
eigenvalue of \(L_0 - \overline{L}_0\) on \(|\Psi\rangle\), we then have
\[
|\Psi\rangle = (L_0 - \overline{L}_0) \cdot \frac{1}{h - \overline{h}} |\Psi\rangle = \{Q, b_0 - \overline{b}_0\} \cdot \frac{1}{h - \overline{h}} |\Psi\rangle = Q \cdot \frac{b_0 - \overline{b}_0}{h - \overline{h}} |\Psi\rangle, \tag{3.19}
\]
showing that indeed, such states are trivial. As a subsidiary condition \(L_0 - \overline{L}_0 = 0\) is not a
strong condition, even with this condition the string field still can be off the mass shell. The
same would not be true for the condition \(L_0 + \overline{L}_0 = 0\), such a condition would impose a familiar
field equation. Our ability to impose an \(L_0 - \overline{L}_0 = 0\) condition suggests that we could impose
a further condition on the off shell string field; we could demand that \((b_0 - \overline{b}_0)|\Psi\rangle = 0\). This
condition cannot be justified as we justified the \((L_0 - \overline{L}_0) = 0\) condition. We cannot prove that
a physical state which is not annihilated by \(b_0 - \overline{b}_0\) must be trivial. Therefore this condition
has a nontrivial effect on the definition of physical states. We may reasonably expect not to
get in trouble since on physical states \(Q|\Psi\rangle = 0\), the condition \((b_0 - \overline{b}_0)|\Psi\rangle = 0\) does not lead
to further constraints, as we have already required that $L_0 - \overline{L}_0 = \{Q, b_0 - \overline{b}_0\}$ annihilate all states. The cohomology of $Q$ on the space of states annihilated by $b_0 - \overline{b}_0$ is called semirelative cohomology.

All of the pieces of the puzzle are now in place. If we impose a $b_0^- (= b_0 - \overline{b}_0)$ condition on all states, the overlap $\langle R_{12} | A \rangle |B\rangle$ would always vanish. This is not hard to see. A state annihilated by $b_0^-$ can always be written as $b_0^-$ acting on something $(b_0^- |\alpha\rangle = 0 \rightarrow |\alpha\rangle = b_0^- c_0^- |\alpha\rangle$ since no state can be simultaneously annihilated by $b_0^-$ and $c_0^-$. Since $b_0^-$ acting on the reflector gives us another $b_0^-$ (Eqn.(3.17)) the relation $b_0^- b_0^- = 0$ implies that we always get zero. The solution is clear, a nondegenerate inner product must include a $c_0^-$. We are therefore led to define

$$\langle A, B \rangle \equiv \langle R_{12} | A \rangle (1) c_0^{-}(2) |B\rangle (2). \quad (3.20)$$

This bilinear inner product has the following exchange property

$$\langle A, B \rangle = (-)^{(A+1)(B+1)} \langle B, A \rangle, \quad (3.21)$$

which follows from the symmetry of the reflector $\langle R_{12} |$ under the exchange of state spaces (see [2] for details). Restricting ourselves to $\mathcal{H}'_{\text{CFT}}$, the subspace of $\mathcal{H}_{\text{CFT}}$ where all states are annihilated by $b_0^-$ and $L_0^-$, we have that

Claim: $\langle , \rangle$ is nondegenerate on $\mathcal{H}'_{\text{CFT}}$. Here is a proof. Assume $\langle A, B \rangle = 0$, for all $|A\rangle \in \mathcal{H}'_{\text{CFT}}$. It follows from

$$\langle A, B \rangle = \langle R_{12} | A \rangle (1) c_0^{-}(2) |B\rangle (2) = 0, \quad (3.22)$$

that the inner product actually vanishes for all $|A\rangle \in \mathcal{H}_{\text{CFT}}$. This is the case because any state that is not annihilated by $b_0^-$ can be written as $c_0^-$ acting on a state. The $c_0^-$ acting on the reflector gives us another $c_0^-$ which kills the $c_0^- |B\rangle$ state. The nondegeneracy of the metric arising from $\langle R_{12} |$ therefore implies that $c_0^- |B\rangle = 0$. Multiplying by $b_0^-$ we find that $|B\rangle = 0$, thus establishing the claim.
Another basic property of this inner product is that, on $\mathcal{H}'_{\text{CFT}}$ the BRST operator satisfies

$$\langle QA, B \rangle = (-)^A \langle A, QB \rangle. \quad (3.23)$$

This is readily proven:

$$\begin{align*}
\langle QA, B \rangle &= \langle R_{12}|Q^{(1)}|A \rangle_{(1)} c_0^{-2} |B \rangle_{(2)} \\
&= (-)^{A+1} \langle R_{12}|A \rangle_{(1)} Q^{(2)} c_0^{-2} |B \rangle_{(2)} \\
&= (-)^{A+1} \langle R_{12}|A \rangle_{(1)} (b_0^- c_0^- + c_0^- b_0^-)^{(2)} Q^{(2)} c_0^{-2} |B \rangle_{(2)} \\
&= (-)^{A+1} \langle R_{12}|A \rangle_{(1)} c_0^{-2} b_0^{-2} Q^{(2)} c_0^{-2} |B \rangle_{(2)},
\end{align*}$$

where we introduced $1 = \{b_0^-, c_0^-\}$, and in the next step the $b_0^- c_0^-$ term was seen to vanish upon taking $b_0^-$ into the reflector. Since the anticommutator of $b_0^-$ with $Q$ gives $L_0^-$, which kills $c_0^- |B \rangle$, we find

$$\begin{align*}
\langle QA, B \rangle &= (-)^A \langle R_{12}|A \rangle_{(1)} c_0^{-2} Q^{(2)} b_0^{-2} c_0^{-2} |B \rangle_{(2)}, \\
&= (-)^A \langle R_{12}|A \rangle_{(1)} c_0^{-2} Q^{(2)} |B \rangle_{(2)}, \\
&= (-)^A \langle A, QB \rangle,
\end{align*}$$

as we wanted to show.

All the above makes it clear that the correct kinetic term for closed string field theory is

$$S_0^2 = \frac{1}{2} \langle \Psi , Q \Psi \rangle , \quad |\Psi \rangle \in \mathcal{H}'_{\text{CFT}}. \quad (3.26)$$

The equation of motion, which follows upon variation, use of the nondegeneracy of the inner product, and Eqn. (3.23), is $Q|\Psi \rangle = 0$. This action is gauge invariant under $\delta|\Psi \rangle = Q|\Lambda \rangle$, with $|\Lambda \rangle \in \mathcal{H}'_{\text{CFT}}$. For the classical string field theory one must restrict the sum over states in $|\Psi \rangle$ to states of ghost number (+2). It turns out that the kinetic term of the master action is given by the same formula, the only difference being that the string field satisfies no ghost number condition.
Symplectic Structure in $\mathcal{H}'_{\text{CFT}}$. The physically relevant bilinear nondegenerate two-form we must choose is therefore

$$\omega(A, B) \equiv (-)^A \langle A, B \rangle.$$ \hspace{1cm} (3.27)

Indeed this is a bilinear, nondegenerate pairing. The sign factor in front was introduced to get the correct exchange property

$$\omega(A, B) = (-)^A \langle A, B \rangle$$

$$= (-)^{A+(A+1)(B+1)} \langle B, A \rangle$$

$$= (-)^{A+(A+1)(B+1)+B} \omega(B, A)$$

$$= -(-)^{AB} \omega(B, A),$$ \hspace{1cm} (3.28)

as we wanted to show (Eqn.(3.1))

BV structure in Spacetime. We can now use the pairing of states in $\mathcal{H}'_{\text{CFT}}$ to pair up the spacetime fields, which are nothing else but the expansion coefficients of the string field $|\Psi\rangle$. In all generality we may choose a basis where we pair up the states

$$\{ |\Phi_1\rangle, |\Phi_2\rangle, \cdots \} ; \{ |\tilde{\Phi}^1\rangle, |\tilde{\Phi}^2\rangle \cdots \}$$ \hspace{1cm} (3.29)

with the condition

$$\omega (|\Phi_i\rangle, |\tilde{\Phi}^j\rangle) = \delta_i^j.$$ \hspace{1cm} (3.30)

Let us denote the first group of states by $|\Phi_s\rangle$. It is convenient to have those fields be of ghost number less than or equal to +2. It then follows that the states $|\tilde{\Phi}^s\rangle$ are all of ghost number greater than or equal to +3 (with our inner product with $c_0^\dagger$, the nonvanishing condition demands that the ghost numbers should add up to five). Then the expansion of the string field reads

$$|\Psi\rangle = \sum_s \left( |\Phi_s\rangle \psi^s + |\tilde{\Phi}^s\rangle \psi^*_s \right).$$ \hspace{1cm} (3.31)

The spacetime fields $\psi^s$, and spacetime antifields $\psi^*_s$ are paired. This is how spacetime fields and antifields arise. The string field is defined to be Grassmann even. Since the statistics of $|\Phi_s\rangle$ and $|\tilde{\Phi}^s\rangle$ are opposite, the statistics of $\psi^s$ and that of $\psi^*_s$ must also be opposite.
We define the spacetime ghost number $g^t$ of a spacetime field to be equal to $2 - G$, with $G$ the ghost number of the corresponding first quantized state. Then, we readily find that $g^t(\psi^s) + g^t(\psi_s^*) = 4 - (G_s + (5 - G_s)) = -1$. These are standard properties of the BV pairing. The reader may wish to verify that one can define explicitly the tilde states as $|\tilde{\Phi}^j\rangle = b_0^- G^{ji} |\Phi_i\rangle$.

In some sense this is the beginning of all the algebraic work in setting up closed string field theory. I have tried in the above to be very explicit about all of the basic and fundamental points. The discussion of the construction of the complete string field action still requires further work. In the next section I will give a brief discussion of some of the remaining issues and of recent developments.

4. Recent Developments

We discussed in §2 and §3 how an extremal problem in Riemann surface theory allows us to find string vertices, and how we go about setting up the Batalin-Vilkovisky field-antifield structure of string field theory. In order to write the string field theory explicitly one needs to develop the differential geometry on an extended space $\hat{P}_{g,n}$ fibered over the space $M_{g,n}$ of Riemann surfaces with $n$ punctures and genus $g$ (for details consult [20,21,2,22]). The fiber over a surface $R$ corresponds to all possible choices of local coordinates, up to a phase, at every puncture of $R$. The string vertex $V_{g,n}$ is properly thought as a subspace of $\hat{P}_{g,n}$. One can define differential forms $\Omega^{(k)}_{\Psi}$ of degree $k$ on $\hat{P}_{g,n}$. The forms are labeled by $\Psi$, which denotes the $n$ states in $H_C^\prime$ to be inserted on the punctures of the surface. These forms satisfy very nice relations that tie up the differential geometry of $\hat{P}_{g,n}$ and the algebraic structures in the conformal theory. For example, the role of the exterior derivative $d$ is played by the BRST operator: $d \Omega^{(k)}_{\Psi} \sim \Omega^{(k+1)}_{Q\Psi}$. The Lie derivative along some vector field $U$ in $\hat{P}_{g,n}$ is represented by a stress energy insertion: $L_U \Omega_{\Psi} = \Omega_{T(u)\Psi}$, with $u$ a ‘Schiffer’ vector on the surface representing the vector $U$. Finally, the role of the contraction operator on forms is played by an antighost insertion: $i_u \Omega^{(k+1)}_{\Psi} = \Omega^{(k)}_{b(u)\Psi}$. In addition to providing conceptual understanding, such relations simplify the verification that the string action satisfies the master equation.

The understanding of the geometry of BV quantization has improved considerably thanks to the work of A. Schwarz [23]. His work allows a covariant description, in the sense of
symplectic geometry, of the BV formalism. The BV master equation, as originally formulated, reads

$$\hbar \Delta S + \frac{1}{2} \{S, S\} = 0,$$

where the antibracket $\{\cdot, \cdot\}$ and the $\Delta$ operator are given by

$$\{G, H\} = \frac{\partial_r G}{\partial \psi^s} \frac{\partial_l H}{\partial \psi^*_s} - \frac{\partial_r G}{\partial \psi^*_s} \frac{\partial_l H}{\partial \psi^s}, \quad \Delta = (-)^\phi \frac{\partial}{\partial \psi^s} \frac{\partial}{\partial \psi^*_s}. \quad (4.2)$$

While the antibracket has a covariant expression using the symplectic two form of the manifold:

$$\{G, H\} \sim \partial_I G \omega^{IJ} \partial_J H,$$

the second order differential operator $\Delta$ is manifestly noncovariant. The solution [23] was to interpret $\Delta$ as an operator which acting on functions gave the divergence of the corresponding Hamiltonian vector field. In order to define a divergence, however, one must introduce a volume element $d\mu = \rho \prod_I dz^I$ on the symplectic manifold. Consequently the delta operator $\Delta_\rho$ acquires a $\rho$ dependence. The main result of [23] is that $\rho$ must be chosen so that $\Delta^2_\rho = 0$. If this is satisfied one can prove the existence of a ‘Darboux-Schwarz’ system of coordinates where the symplectic form $\omega$ becomes a constant and $\rho = 1$. Such system of coordinates seems necessary to carry out the standard BV argument for the gauge independence of the observables. In summary, a BV manifold is the object $(M, \omega, d\mu)$, with $M$ a supermanifold, $\omega$ an odd, symplectic, closed two-form, and $\mu$ a volume element that leads to a nilpotent operator $\Delta$.

As we discussed in §1 there are indications that string field theory may be thought as a function on the space of two dimensional field theories. Nobody has been able to understand this space concretely. I believe it is likely that string field theory will be eventually formulated on some abstract space closely related to a theory space, or to theory space with some extra conditions. As we have seen the relevant state space of conformal theories including ghosts have a symplectic structure. It seems clear that this state space corresponds to the tangent space to ‘theory space’. This suggests that theory space may be a symplectic manifold. This idea was advocated by Witten [24] who also proposed a concrete construction for the case of open string field theory. In this case, theory space is the space of one dimensional lagrangians defined on the boundary of a disk. Rather than finding an action $S$ (which would be a function in this theory space) satisfying the master equation, one looks for the corresponding Hamiltonian vector field $V_S$. This vector must be odd and should satisfy the condition $\{V_S, V_S\} = 0$ where
the bracket is the graded Lie bracket. This equation, which can also be written as $V_S^2 = 0$, must hold everywhere, as it guarantees that the master equation is satisfied everywhere. At the points where $V_S = 0$, we have conformal field theories. The setup is very attractive, although it seems likely that the explicit construction needs further work [25,26]. In order to formulate a complete quantum theory we need to obtain the full master equation. This would mean that theory space should be a BV manifold in the full sense, that is, an object $(M, \omega, d\mu)$, with $d\mu$ a consistent volume element. This volume element has not been found yet. Nevertheless for a complete formulation the equation we must solve is not $V_S^2 = 0$ but rather $V_S^2 = -\hbar V_{\text{div}S}$ [22].

Hata and the author [22] have found the BV approach useful to understand the significance of the freedom to choose different sets of string vertices $V_{g,n}$ in order to reproduce the sum over surfaces. More concretely, one wished to know what is the relation between string field theories that use different ways of breaking up the sum over surfaces. The answer is simple: two such string field master actions differ by terms induced by a field-antifield transformation canonical with respect to the antibracket. It is possible to use the differential geometry language described above in order to write explicitly the generator of the canonical transformation in terms of the vector field $U$ in $\hat{P}_{g,n}$ generating the variation of the string vertices $V_{g,n}$. Using the covariant description of gauge fixing one can then show that two theories using different decompositions of moduli space yield the same gauge fixed action upon use of different gauge fixing conditions.

During the last year the algebraic structures that arise in closed string field theory have been brought into the open [27, 28, 2]. The first object to emerge was a homotopy Lie algebra. In some sense homotopy Lie algebras (and their more familiar relatives, the homotopy associative algebras [29]) are the answer to a longstanding question. We have long suspected that string theory on-shell amplitudes have a group theoretical meaning, indeed, special three point functions have been related to structure constants of Lie algebras. The three point amplitudes, together will all the higher $n$-point functions (at the classical level) define the structure constants of a homotopy Lie algebra! This structure can be readily explained. Suppose we have a graded Lie algebra $\{T_a, T_b\} = f_{ab}^c T_c$. Introduce for each generator $T_a$ an object $\eta^a$ of opposite statistics. Then consider the following anticommuting vector field

$$V = \left(f_{a_1a_2}^b \eta^{a_1}\eta^{a_2} + f_{a_1a_2a_3}^b \eta^{a_1}\eta^{a_2}\eta^{a_3} + \cdots \right) \frac{\partial}{\partial \eta^b}.$$  (4.3)
where the $f$'s with more than three indices are the higher structure constants of the algebra. The condition $V^2 = 0$ is then imposed. This condition gives, at lowest order, the Jacobi identity for the original structure constants of the graded Lie algebra. At higher orders it gives an infinite set of quadratic constraints on the structure constants. If the condition $V^2 = 0$ is satisfied, we have a homotopy Lie algebra. The physical interpretation arises when we show that, for string theory, the structure constants are nothing else than on-shell scattering amplitudes, and the quadratic relations are the Ward identities of the theory. In particular, if we label the physical states (BRST semirelative cohomology classes) as $|\Phi_a\rangle$, the structure constant $f_{a_1a_2\ldots a_n}^b$ is simply given by the correlation function $\langle \langle \Phi_{a_1}\Phi_{a_2}\ldots\Phi_{a_n}\tilde{\Phi}^b \rangle \rangle$ where the double bracket indicates that one integrates over the positions of the punctures (this is a string amplitude). While this homotopy Lie algebra exists on the cohomology of $Q$ in $H'_\text{CFT}$, there is a homotopy Lie algebra defined on the full state space $H'_\text{CFT}$. This is the homotopy Lie algebra that underlies classical closed string field theory, and it corresponds to a simple modification of the previous equation

$$V = \left( f_{a_1}^b \eta^{a_1} + f_{a_1a_2}^b \eta^{a_1} \eta^{a_2} + f_{a_1a_2a_3}^b \eta^{a_1} \eta^{a_2} \eta^{a_3} + \ldots \right) \frac{\partial}{\partial \eta^b}. \quad (4.4)$$

Here we have added a structure constant with two indices that can be thought as the matrix elements of some operator: $f_{a_1}^b = (Q)_{a_1}^b$. This time $V^2 = 0$ requires, as the lowest consistency condition, that $Q$ be a nilpotent matrix. The next consistency condition demands that $Q$ be a derivation of a product defined by the $f_{a_1a_2}^b$'s. The third consistency condition is quite amusing. It expresses the fact that structure constants $f_{a_1a_2}^b$ need not satisfy anymore the Jacobi identity; the Jacobi identity must only be satisfied weakly, that is, it differs from zero by terms having to do with $Q$ acting on the structure constants representing four point correlators. Therefore the $f_{a_1a_2}^b$'s of this homotopy Lie algebra do not define a Lie algebra. The reader may have guessed what is the physical relevance of this algebra. This time the index $a$ labels a general state $|\Phi_a\rangle$ in $H'_\text{CFT}$. $Q$ is nothing else but the BRST operator, and the higher structure constant $f_{a_1a_2\ldots a_n}^b$ is given by the off shell correlation $\langle \langle \Phi_{a_1}\Phi_{a_2}\ldots\Phi_{a_n}\tilde{\Phi}^b \rangle \rangle$ integrated over the set $V_{0,n+1}$ defining the string vertex coupling $n + 1$ strings at genus zero. Thus the off-shell amplitudes of string field theory, together with the BRST operator form a homotopy Lie algebra. I would like to emphasize that closed string field theory seems to be necessarily nonpolynomial. It was shown by Sonoda and the author [30], under very general conditions,
that no cubic interaction alone could give a consistent four point amplitude. This actually means that for covariant closed string field theory there is no clever choice of $f_{a_1 a_2}^b$'s that satisfies a strict Jacobi identity. It would seem that there is no Lie algebra underlying closed string field theory, just a homotopy Lie algebra. I believe this point should be investigated more closely. Finally, consider the most general homotopy Lie algebra:

\[ V = \left( f^b + f_{a_1}^b \eta^{a_1} + f_{a_1 a_2}^b \eta^{a_1} \eta^{a_2} + \cdots \right) \frac{\partial}{\partial \eta^b}. \]  

(4.5)

If we view the lowest structure constant $f^b$ as a vector $F$, and $f_{a_1}^b$ as a matrix $Q$, this time the lowest identities following from $V^2 = 0$ can be written as $QF = 0$, and $Q^2 B + F \star B = 0$, with $B$ arbitrary, and where $\star$ denotes the product defined with the structure constants $f_{a_1 a_2}^b$. The reader may note that we have lost the nilpotency of $Q$! This is the algebraic setup necessary for background independence. Indeed, only when $f^b = 0$ we recover the BRST operator, and therefore, a CFT. This corresponds to a zero of the vector $V$ at $\eta^a = 0$. We recall that $V = 0$ is indeed the equation of motion in the setup of [24]. The homotopy Lie algebra of (4.5) can be constructed indirectly (and in a background dependent way) by shifting closed string field theory with a string field which is not a classical solution [2].

If we wish to consider the quantum theory this all gets generalized. One can write a master equation for the on-shell action [28] and this structure can be seen to arise from the master equation for the complete off shell action [2]. If we wish to use the description using an anticommuting vector field, then, following [22], the equation $V^2 = 0$ must be turned into $V^2 = -\hbar v_{\text{div}}V$ (with $v_f$ denoting the Hamiltonian vector corresponding to the function $f$). Quantum closed string field theory defines a concrete realization of this algebra (constructed in a Darboux-Schwarz frame of fields and antifields, with a density $\rho = 1$). This structure may be called a ‘quantum homotopy Lie algebra’. The more desirable name of BV algebra is now being used by mathematicians to describe a differential graded commutative algebra with a nilpotent second order operator $\Delta$ [31,32,33,34]. The antibracket can be defined as the failure of $\Delta$ to be a derivation.

Back to theory space and background independence. The problem of background independence in the simplest setting is that of proving that closed string field theories formulated around nearby conformal field theories are actually equivalent. This is not so simple to prove,
but has been achieved to a large degree through the work of A. Sen [35]. I believe a better geometrical understanding of his result is very much needed to appreciate well the issues involved in background independence. A host of problems arise when one realizes that in comparing string field theories formulated on state spaces $\mathcal{H}$ and $\mathcal{H}'$ corresponding to different conformal field theories many natural identifications of the two spaces become meaningless as soon as the theories are not infinitesimally away. Some of these difficulties, whose origin lies in the fact that the state spaces are infinite dimensional, were found in [36]. Together with K. Ranganathan and H. Sonoda we have investigated geometry of the vector bundle whose base manifold is a space of conformal field theories and whose vector space at each point is the infinite dimensional state space of the theory [37]. We gave a characterization of the connections that can be introduced in this vector bundle, and studied special connections that allow the construction of a conformal theory using the state space of another theory a finite distance away. I would expect connections to be a useful ingredient in future analysis of background independence.

As we have reviewed in this section, from the algebraic viewpoint homotopy Lie algebras and their generalizations seem to play a prominent role in string field theory. From the geometrical viewpoint we have seen the role of geometry on moduli spaces of Riemann surfaces, geometry of BV quantization, and theory space geometry. I look forward to see the fascinating relations that are likely to be uncovered between the different geometries. Such developments should pave the way to a complete formulation of string theory.

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