RELATIVE RIGID OBJECTS IN TRIANGULATED CATEGORIES

CHANGJIAN FU, SHENGFEI GENG, AND PIN LIU

Abstract. Let $T$ be a Krull-Schmidt, Hom-finite triangulated category with suspension functor [1]. Let $R$ be a basic rigid object, $\Gamma$ the endomorphism algebra of $R$, and $\text{pr}(R) \subseteq T$ the subcategory of objects finitely presented by $R$. We investigate the relative rigid objects, i.e. $R[1]$-rigid objects of $T$.

Our main results show that the $R[1]$-rigid objects in $\text{pr}(R)$ are in bijection with $\tau$-rigid $\Gamma$-modules, and the maximal $R[1]$-rigid objects with respect to $\text{pr}(R)$ are in bijection with support $\tau$-tilting $\Gamma$-modules. We also show that various previously known bijections involving support $\tau$-tilting modules are recovered under respective assumptions.

1. Introduction

This note attempts to unify and generalize certain bijections involving support $\tau$-tilting modules. Support $\tau$-tilting module is the central notion in the $\tau$-tilting theory introduced by Adachi-Iyama-Reiten [AIR14], which can be regarded as a generalization of the classical tilting module. Since its appearance, support $\tau$-tilting module has been rapidly found to be linked up with various important objects in representation theory, such as torsion class, (co)-l-structure, cluster tilting object, silting object and so on, see [AIR14, IJY14, CZZ15, LX16, YZ15, YZZ17] for instance. Among others, Adachi-Iyama-Reiten [AIR14] proved that for a 2-Calabi-Yau triangulated category $T$ with a basic cluster tilting object $T$, there is a one-to-one correspondence between the set of basic cluster tilting objects of $T$ and the set of basic support $\tau$-tilting $\text{End}_T(T)$-modules. It is known that there exist 2-Calabi-Yau triangulated categories which have no cluster tilting objects but only maximal rigid objects. Then the correspondence was generalized to such kind of 2-Calabi-Yau triangulated categories by Chang-Zhang-Zhu [CZZ15] and Liu-Xie [LX16]. The Adachi-Iyama-Reiten’s correspondence has been further generalized by Yang-Zhu [YZ15]. More precisely, let $T$ be a Krull-Schmidt, Hom-finite triangulated category with suspension functor [1]. Assume that $T$ admits a Serre functor and a cluster tilting object $T$. By introducing the notion of $T[1]$-cluster tilting objects as a generalization of cluster tilting objects, Yang-Zhu [YZ15] established a one-to-one correspondence between the set of $T[1]$-cluster tilting objects of $T$ and the set of support $\tau$-tilting modules over $\text{End}_T(T)$. On the other hand, for a Krull-Schmidt, Hom-finite triangulated category $T$ with a silting object $S$, Iyama-Jørgensen-Yang [IJY14] proved that the two-term silting objects of $T$ with respect to $S$ which belong to the finitely presented subcategory $\text{pr}(S)$ are in bijection with support $\tau$-tilting modules over $\text{End}_T(S)$.

In this note, we work in the following general setting. Let $T$ be a Krull-Schmidt, Hom-finite triangulated category with suspension functor [1] and $R$ a basic rigid object of $T$ with endomorphism algebra $\Gamma = \text{End}_T(R)$. Denote by $\text{pr}(R)$ the subcategory of objects finitely presented by $R$. Following [YZ15], we introduce the $R[1]$-rigid object of $T$ and the maximal $R[1]$-rigid object with respect to $\text{pr}(R)$ (cf. Definition 2.2). We prove that the $R[1]$-rigid objects in $\text{pr}(R)$ are in bijection with $\tau$-rigid $\Gamma$-modules, and the maximal $R[1]$-rigid objects with respect to $\text{pr}(R)$ are in bijection with support $\tau$-tilting $\Gamma$-modules (cf. Theorem 2.5). When $R$ is a cluster tilting object of $T$, we show that the bijection reduces to the
bijection between the set of basic \( R[1] \)-cluster tilting objects of \( T \) and the set of basic support \( \tau \)-tilting \( \Gamma \)-modules obtained by Yang-Zhu [YZ15] (Corollary 2.8). We remark that, compare to [YZ15], we do not need the existence of a Serre functor for \( T \) (cf. also [YZZ17]). Since tilting modules are faithful support \( \tau \)-tilting modules, we also obtain a characterization of tilting \( \Gamma \)-modules via the bijection (cf. Theorem 2.9).

We apply the aforementioned bijection to the cases of silting objects, \( d \)-cluster tilting objects and maximal rigid objects in Section 3 and Section 4 respectively. When \( R \) is a silting object of a triangulated category \( T \), we proved that the maximal \( R[1] \)-rigid objects with respect to \( \text{pr}(R) \) coincide with the silting objects of \( T \) in \( \text{pr}(R) \) (cf. Theorem 3.4). As a consequence, Theorem 2.5 recovers the bijection between the set of basic silting objects of \( T \) in \( \text{pr}(R) \) and the set of basic support \( \tau \)-tilting \( \text{End}(R) \)-modules obtained by Iyama-Jørgensen-Yang [IJY14] (cf. Corollary 3.5). If \( T \) is a \( d(\geq 2) \)-cluster category and \( R \) is a \( d \)-cluster tilting object of \( T \), then Theorem 2.5 reduces to the bijection obtained by Liu-Qiu-Xie [LQX] (cf. Corollary 4.5). Assume that \( T \) is a 2-Calabi-Yau triangulated category and \( R \) is a basic maximal rigid object of \( T \). We show that Theorem 2.5 implies the bijection between the set of basic maximal rigid object of \( T \) and the set of basic support \( \tau \)-tilting \( \Gamma \)-modules obtained in [LX16, CZZ15] (cf. Corollary 4.7).

**Con convention.** Let \( k \) be an algebraically closed field. Throughout this paper, \( T \) will be a Krull-Schmidt, Hom-finite triangulated category over \( k \) unless stated otherwise. For an object \( M \in T \), denote by \( |M| \) the number of non-isomorphic indecomposable direct summands of \( M \). Denote by \( \text{add} M \) the subcategory of \( T \) consisting of objects which are finite direct sum of direct summands of \( M \).

2. \( R[1] \)-rigid objects and \( \tau \)-rigid modules

2.1. **Recollection on \( \tau \)-tilting theory.** We follow [AIR14]. Let \( A \) be a finite dimensional algebra over \( k \). Denote by \( \text{mod} A \) the category of finitely generated right \( A \)-modules and \( \text{proj} A \) the category of finitely generated right projective \( A \)-modules. For a module \( M \in \text{mod} A \), denote by \( |M| \) the number of non-isomorphic indecomposable direct summands of \( M \). Let \( \tau \) be the Auslander-Reiten translation of \( \text{mod} A \).

An \( A \)-module \( M \) is \( \tau \)-rigid if \( \text{Hom}_A(M, \tau M) = 0 \). A \( \tau \)-rigid pair is a pair of \( A \)-modules \((M, P)\) with \( M \in \text{mod} A \) and \( P \in \text{proj} A \), such that \( M \) is \( \tau \)-rigid and \( \text{Hom}_A(P, M) = 0 \). A basic \( \tau \)-rigid pair \((M, P)\) is a basic support \( \tau \)-tilting pair if \(|M| + |P| = |A|\). In this case, \( M \) is a support \( \tau \)-tilting \( A \)-module and \( P \) is uniquely determined by \( M \). It has been proved in [AIR14] that for each \( \tau \)-rigid pair \((M, P)\), we always have \(|M| + |P| \leq |A|\) and each \( \tau \)-rigid pair can be completed into a support \( \tau \)-tilting pair.

The following criterion for \( \tau \)-rigid modules has been proved in [AIR14].

**Lemma 2.1.** For \( M \in \text{mod} A \), denote by \( P_1^M \xrightarrow{f} P_0^M \rightarrow M \rightarrow 0 \) a minimal projective presentation of \( M \). Then \( M \) is \( \tau \)-rigid if and only if
\[
\text{Hom}_A(f, M): \text{Hom}_A(P_0^M, M) \rightarrow \text{Hom}_A(P_1^M, M)
\]
is surjective.

2.2. **\( R[1] \)-rigid objects.** Let \( T \) be a Krull-Schmidt, Hom-finite triangulated category with shift functor \([1] \). For \( X, Y, Z \in T \), we denote by \( Z(X, Y) \) the subgroup of \( \text{Hom}_T(X, Y) \) consisting of morphisms which factor through \( \text{add} Z \). An object \( X \in T \) is called rigid if \( \text{Hom}_T(X, X[1]) = 0 \). It is maximal rigid if it is rigid and \( \text{Hom}_T(X \oplus Z, X[1] \oplus Z[1]) = 0 \) implies \( Z \in \text{add} X \) for any \( Z \in T \). Let \( \mathcal{C} \subseteq T \) be a full subcategory of \( T \). An object \( X \in \mathcal{C} \subseteq T \) is called maximal rigid with respect to \( \mathcal{C} \) provided that it
is rigid and for any object \( Z \in \mathcal{C} \) such that \( \text{Hom}_\mathcal{T}(X \oplus Z, X[1] \oplus Z[1]) = 0 \), we have \( Z \in \text{add} \, X \). It is clear that a maximal rigid object of \( \mathcal{T} \) is just a maximal rigid object with respect to \( \mathcal{T} \).

Let \( R \) be a basic rigid object of \( \mathcal{T} \). An object \( X \) is \textit{finitely presented} by \( R \) if there is a triangle \( R_1^X \to R_0^X \to X \to R_1^X[1] \) with \( R_0^X, R_1^X \in \text{add} \, R \). Denote by \( \text{pr}(R) \) the subcategory of \( \mathcal{T} \) consisting of objects which are finitely presented by \( R \). Throughout this section, \( R \) will be a basic rigid object of \( \mathcal{T} \).

We introduce the relative rigid objects with respect to \( R \) (cf. \cite{YZ15, CZZ15}).

\textbf{Definition 2.2.} Let \( R \in \mathcal{T} \) be a basic rigid object.

1. An object \( X \in \mathcal{T} \) is called \( R[1] \)-rigid if \( R[1](X, X[1]) = 0 \).
2. An object \( X \in \text{pr}(R) \subseteq \mathcal{T} \) is called maximal \( R[1] \)-rigid with respect to \( \text{pr}(R) \) if \( X \) is \( R[1] \)-rigid and for any object \( Z \in \text{pr}(R) \) such that \( R[1](X \oplus Z, X[1] \oplus Z[1]) = 0 \), then \( Z \in \text{add} \, X \).

By definition, it is clear that rigid objects are \( R[1] \)-rigid, but the converse is not true in general. We are interested in \( R[1] \)-rigid objects of \( \mathcal{T} \) which belong to the subcategory \( \text{pr}(R) \). We have the following observation.

\textbf{Lemma 2.3.} Let \( R_1 \xrightarrow{f} R_0 \xrightarrow{g} X \to R_1[1] \) be a triangle in \( \mathcal{T} \) with \( R_0, R_1 \in \text{add} \, R \). Then \( X \) is \( R[1] \)-rigid if and only if
\[
\text{Hom}_\mathcal{T}(f, X) : \text{Hom}_\mathcal{T}(R_0, X) \to \text{Hom}_\mathcal{T}(R_1, X)
\]
is surjective.

\textbf{Proof.} Applying the functor \( \text{Hom}_\mathcal{T}(\ast, X) \) to the triangle \( X[-1] \xrightarrow{h} R_1 \xrightarrow{f} R_0 \xrightarrow{g} X \) yields a long exact sequence
\[
\text{Hom}_\mathcal{T}(R_0, X) \xrightarrow{f^*} \text{Hom}_\mathcal{T}(R_1, X) \xrightarrow{h^*} \text{Hom}_\mathcal{T}(X[-1], X) \to \text{Hom}_\mathcal{T}(R_0[-1], X),
\]
where \( f^* = \text{Hom}_\mathcal{T}(f, X) \) and \( h^* = \text{Hom}_\mathcal{T}(h, X) \).

Suppose that \( X \) is \( R[1] \)-rigid, that is \( R[1](X, X[1]) = 0 \). It follows that \( h^* = 0 \) and hence \( f^* \) is surjective. Now assume that \( f^* \) is surjective. To show \( X \) is \( R[1] \)-rigid, it suffices to prove that \( R(X[-1], X) = 0 \).

Let \( a \in R(X[-1], X) \) and \( a = b \circ c \), where \( c : X[-1] \to R \) and \( b : R \to X \). As \( R \) is rigid, we know that each morphism from \( X[-1] \) to \( R \) factors through \( h \). Hence there is a morphism \( c' : R_1 \to R \) such that \( c = c' \circ h \). Since \( f^* \) is surjective, there is a morphism \( b' : R_0 \to X \) such that \( b \circ c' = b' \circ f \circ h = 0 \) (cf. the following diagram).

\[
\begin{array}{ccc}
X[-1] & \xrightarrow{h} & R_1 \\
\downarrow{c} & & \downarrow{f} \\
R & \xrightarrow{h} & X \\
\end{array}
\qquad
\begin{array}{ccc}
R_0 & \xrightarrow{g} & X \\
\downarrow{b} & & \downarrow{b'} \\
R_1 & \xrightarrow{f} & R_1[1]
\end{array}
\]

\[\square\]

2.3. \textbf{From \( R[1] \)-rigid objects to }\( \tau \)-rigid modules. Recall that \( R \) is a basic rigid object of \( \mathcal{T} \). Denote by \( \Gamma := \text{End}_\mathcal{T}(R) \) the endomorphism algebra of \( R \) and \( \text{mod} \, \Gamma \) the category of finitely generated right \( \Gamma \)-modules. Let \( \tau \) be the Auslander-Reiten translation of \( \text{mod} \, \Gamma \). It is known that the functor \( \text{Hom}_\mathcal{T}(R, \ast) : \mathcal{T} \to \text{mod} \, \Gamma \) induces an equivalence of categories
\[
\text{Hom}_\mathcal{T}(R, \ast) : \text{pr}(R)/(R[1]) \to \text{mod} \, \Gamma,
\] (2.1)
where \( \text{pr}(R)/(R[1]) \) is the additive quotient of \( \text{pr}(R) \) by morphisms factorizing through \( \text{add}(R[1]) \) (cf. \cite{LY08}). Moreover, the restriction of \( \text{Hom}_\mathcal{T}(R, \ast) \) to the subcategory \( \text{add} \, R \) yields an equivalence between \( \text{add} \, R \) and the category \( \text{proj} \, \Gamma \) of finitely generated projective \( \Gamma \)-modules. The following result is a direct consequence of the equivalence (2.1) and the fact that \( R \) is rigid.
Lemma 2.4. For any $R' \in \text{add} R$ and $Z \in \text{pr}(R)$, we have
\[
\text{Hom}_T(\text{Hom}_T(R, R'), \text{Hom}_T(R, Z)) \cong \text{Hom}_T(R', Z).
\]

Now we are in position to state the main result of this note.

Theorem 2.5.

(a) Let $X$ be an object in $\text{pr}(R)$ satisfying that $\text{add} X \cap \text{add}(R[1]) = \{0\}$. Then $X$ is $R[1]$-rigid if and only if $\text{Hom}_T(R, X)$ is $\tau$-rigid.

(b) The functor $\text{Hom}_T(R, -)$ yields a bijection between the set of isomorphism classes of basic $R[1]$-rigid objects in $\text{pr}(R)$ and the set of isomorphism classes of basic $\tau$-rigid pairs of $\Gamma$-modules.

(c) The functor $\text{Hom}_T(R, -)$ induces a bijection between the set of isomorphism classes of basic maximal $R[1]$-rigid objects with respect to $\text{pr}(R)$ and the set of isomorphism classes of basic support $\tau$-tilting $\Gamma$-modules.

Proof. Let $R_1 \xrightarrow{f} R_0 \xrightarrow{g} X \to R_1[1]$ be a triangle in $\mathcal{T}$ with $R_1, R_0 \in \text{add} R$ such that $g$ is a minimal right $\text{add} R$-approximation of $X$. As $R$ is rigid and $\text{add} X \cap \text{add}(R[1]) = \{0\}$, applying the functor $\text{Hom}_T(R, -)$, we obtain a minimal projective resolution of $\text{Hom}_T(R, X)$
\[
\text{Hom}_T(R, R_1) \xrightarrow{\text{Hom}_T(R,f)} \text{Hom}_T(R, R_0) \xrightarrow{\text{Hom}_T(R,g)} \text{Hom}_T(R, X) \to 0.
\]

According to Lemma 2.4, we have the following commutative diagram
\[
\begin{array}{ccc}
\text{Hom}_T(R_0, X) & \cong & \text{Hom}_T(\text{Hom}_T(R, R_0), \text{Hom}_T(R, X)) \\
\text{Hom}_T(f, X) & & \text{Hom}_T(\text{Hom}_T(R,f), \text{Hom}_T(R,X)) \\
\text{Hom}_T(R_1, X) & \cong & \text{Hom}_T(\text{Hom}_T(R, R_1), \text{Hom}_T(R, X)).
\end{array}
\]

Now it follows from Lemma 2.1 and Lemma 2.3 that $X$ is $R[1]$-rigid if and only if $\text{Hom}_T(R, X)$ is $\tau$-rigid. This finishes the proof of (a).

Let us consider the statement (b). For each object $X \in \text{pr}(R)$, $X$ admits a unique decomposition as $X \cong X_0 \oplus R_X[1]$, where $R_X \in \text{add} R$ and $X_0$ has no direct summands in $\text{add} R[1]$. We then define
\[
F(X) := (\text{Hom}_T(R, X_0), \text{Hom}_T(R, R_X)) \in \text{mod} \Gamma \times \text{proj} \Gamma.
\]

If $X$ is $R[1]$-rigid, according to (a), we deduce that $\text{Hom}_T(R, X_0)$ is a $\tau$-rigid $\Gamma$-module. And by Lemma 2.4, we know $\text{Hom}_T(\text{Hom}_T(R, R_X), \text{Hom}_T(R, X_0)) = 0$. That is, $F$ maps a basic $R[1]$-rigid object to a basic $\tau$-rigid pair of $\Gamma$-modules. We claim that $F$ is the desired bijection.

Since $\text{Hom}_T(R, -) : \text{pr}(R)/(R[1]) \to \text{mod} \Gamma$ is an equivalence, we clearly know that $F$ is injective. It remains to show that $F$ is surjective. For each basic $\tau$-rigid pair $(M, P)$ of $\Gamma$-modules, denote by $\hat{P} \in \text{add} R$ the object in $\text{pr}(R)$ corresponding to $P$ and similarly by $\hat{M} \in \text{pr}(R)$ the object corresponding to $M$, which has no direct summands in $\text{add} R[1]$. By definition, we clearly have $F(\hat{M} \oplus \hat{P}[1]) = (M, P)$. It remains to show that $\hat{M} \oplus \hat{P}[1]$ is $R[1]$-rigid, which is a consequence of (a), Lemma 2.4 and the fact that $R$ is rigid. This completes the proof of (b).

For (c), let $X = X_0 \oplus R_X[1]$ be a basic maximal $R[1]$-rigid object with respect to $\text{pr}(R)$, where $R_X \in \text{add} R$ and $X_0$ has no direct summands in $\text{add} R[1]$. We claim that $F(X)$ is a support $\tau$-tilting pair. Otherwise, at least one of the following two situations happen:

(i) there is an indecomposable object $R_X' \in \text{add} R$ such that $R_X' \not\in \text{add} R_X$ and $\text{Hom}_T(R, X_0), \text{Hom}_T(R, R_X \oplus R_X')$ is a basic $\tau$-rigid pair;
(ii) there is an indecomposable object $X_1 \in \text{pr}(R) \setminus \text{add } R[1]$ such that $X_1 \not\in \text{add } X_0$ and $(\text{Hom}_\tau(R, X_0 \oplus X_1), \text{Hom}_\tau(R, R_X))$ is a basic $\tau$-rigid pair.

Let us consider the case (i). By definition, we have

$$\text{Hom}_\tau(\text{Hom}_\tau(R, R_X \oplus R_X^\tau), \text{Hom}_\tau(R, X_0)) = 0.$$ 

According to Lemma 2.4, we clearly have $\text{Hom}_\tau(R_X \oplus R_X^\tau, X_0) = 0$. Now it is straightforward to check that $X \oplus R_X[-1] \in \text{pr}(R)$ is $R[1]$-rigid. Note that we have $R_X[-1] \not\in \text{add } X$, which contradicts to the assumption that $X$ is a basic maximal $R[1]$-rigid object with respect to $\text{pr}(R)$. Similarly, one can obtain a contradiction for the case (ii).

Now assume that $(M, P)$ is a basic support $\tau$-tilting pair of $\Gamma$-modules. According to (b), let $\hat{M} \oplus \hat{P}[1]$ be the basic $R[1]$-rigid object in $\text{pr}(R)$ corresponding to $(M, P)$. We need to prove that $\hat{M} \oplus \hat{P}[1]$ is maximal with respect to $\text{pr}(R)$. By definition, we show that if $Z \in \text{pr}(R)$ is an object such that $R[1](\hat{M} \oplus \hat{P}[1] \oplus Z, \hat{M}[1] \oplus \hat{P}[2] \oplus Z[1]) = 0$, then $Z \in \text{add}(\hat{M} \oplus \hat{P}[1])$. Without loss of generality, we assume that $Z$ is indecomposable. We separate the remaining proof by considering whether the object $Z$ belongs to $\text{add } R[1]$ or not.

If $Z \not\in \text{add } R[1]$, then $M \oplus \text{Hom}_\tau(R, Z)$ is a $\tau$-rigid $\Gamma$-module by (a). Moreover, according to $R[1](\hat{M} \oplus \hat{P}[1] \oplus Z, \hat{M}[1] \oplus \hat{P}[2] \oplus Z[1]) = 0$, we have

$$\text{Hom}_\Gamma(P, \text{Hom}_\tau(R, Z)) = \text{Hom}_\tau(\hat{P}, Z) = 0.$$ 

Consequently, $(M \oplus \text{Hom}_\tau(R, Z), P)$ is a $\tau$-rigid pair. By the assumption that $(M, P)$ is a basic support $\tau$-tilting pair, we conclude that $\text{Hom}_\tau(R, Z) \in \text{add } M$ and hence $Z \in \text{add } \hat{M} \subseteq \text{add}(\hat{M} \oplus \hat{P}[1])$.

Similarly, for $Z \in \text{add } R[1]$, one can show that $(M, P \oplus \text{Hom}_\tau(R, Z[-1]))$ is a $\tau$-rigid pair of $\Gamma$-modules. Consequently, we have $Z \in \text{add } \hat{P}[1] \subseteq \text{add}(\hat{M} \oplus \hat{P}[1])$. This completes the proof of (c). \qed

Since all basic support $\tau$-tilting pairs of $\Gamma$-modules have the same number of non-isomorphic indecomposable direct summands [AIR14]. As a byproduct of the proof, we have

**Corollary 2.6.**

1. Each $R[1]$-rigid object in $\text{pr}(R)$ can be completed to a maximal $R[1]$-rigid object with respect to $\text{pr}(R)$.
2. All basic maximal $R[1]$-rigid objects with respect to $\text{pr}(R)$ have the same number of non-isomorphic indecomposable direct summands.

Recall that an object $T \in \mathcal{T}$ is a cluster tilting object provided that

$$\text{add } T = \{X \in \mathcal{T} \mid \text{Hom}_\tau(T, X[1]) = 0\} = \{X \in \mathcal{T} \mid \text{Hom}_\tau(X, T[1]) = 0\}.$$ 

It is clear that cluster tilting objects are maximal rigid. Let $R$ be a cluster tilting object of $\mathcal{T}$. In this case, we have $\text{pr}(R) = \mathcal{T}$ (cf. [LY08, KZ08]). An object $X \in \mathcal{T}$ is called $R[1]$-cluster tilting if $X$ is $R[1]$-rigid and $|X| = |R|$ (cf. [YZ15]). As a direct consequence of Corollary 2.6, we have

**Lemma 2.7.** Let $R$ be a cluster tilting object of $\mathcal{T}$. Then an object $T \in \mathcal{T}$ is maximal $R[1]$-rigid with respect to $\mathcal{T}$ if and only if $T$ is $R[1]$-cluster tilting.

Combining Lemma 2.7 with Theorem 2.5, we obtain the following result of Yang-Zhu [YZ15, Theorem 1.2].

**Corollary 2.8.** Let $R$ be a cluster tilting object of $\mathcal{T}$ with endomorphism algebra $\Gamma = \text{End}_\tau(R)$. There is a bijection between the set of isomorphism classes of basic $R[1]$-cluster tilting objects and the set of isomorphism classes of basic support $\tau$-tilting $\Gamma$-modules.
2.4. A characterization of tilting modules. Recall that a basic \( \Gamma \)-module \( M \) is a tilting module provided that

- \( \text{pd}_R M \leq 1 \);
- \( \text{Ext}^1_{\Gamma}(M, M) = 0 \);
- \( |M| = |\Gamma| \).

It has been observed in [AIR14] that tilting \( \Gamma \)-modules are precisely faithful support \( \tau \)-tilting \( \Gamma \)-modules. As in [LX14, BBT14], we consider the projective dimension of \( \Gamma \)-modules and give a characterization of tilting \( \Gamma \)-modules via \( \text{pr}(R) \).

**Theorem 2.9.** For an object \( X \in \text{pr}(R) \) without direct summands in \( \text{add} R[1] \), we have

\[
\text{pd}_R \text{Hom}_\Gamma(R, X) \leq 1 \quad \text{if and only if} \quad X(R[1], R[1]) = 0.
\]

In particular, for a basic object \( X \in \text{pr}(R) \) which has no direct summands in \( \text{add} R[1] \), \( \text{Hom}_\Gamma(R, X) \) is a tilting \( \Gamma \)-module if and only if \( X(R[1], R[1]) = 0 \) and \( X \) is maximal \( R[1] \)-rigid with respect to \( \text{pr}(R) \).

**Proof.** Since \( X \in \text{pr}(R) \), we have a triangle \( R_1 \xrightarrow{f} R_0 \xrightarrow{g} X \xrightarrow{h} R_1 \] in \( T \) such that \( R_0, R_1 \in \text{add} R \) and \( g \) is a minimal right \( \text{add} R \)-approximation of \( X \). Applying the functor \( \text{Hom}_\Gamma(R, -) \), we obtain a long exact sequence

\[
\text{Hom}_\Gamma(R, X[-1]) \xrightarrow{\text{Hom}_\Gamma(R,h[-1])} \text{Hom}_\Gamma(R, R_1) \xrightarrow{\text{Hom}_\Gamma(R,f)} \text{Hom}_\Gamma(R, R_0) \rightarrow \text{Hom}_\Gamma(R, X) \rightarrow 0.
\]

Assume that \( X(R[1], R[1]) = 0 \). It follows that \( \text{Hom}_\Gamma(R, h[-1]) = 0 \) and \( \text{Hom}_\Gamma(R, f) \) is injective. That is, \( \text{pd}_R \text{Hom}_\Gamma(R, X) \leq 1 \).

Suppose that \( \text{pd}_R \text{Hom}_\Gamma(R, X) \leq 1 \). Then \( \text{Hom}_\Gamma(R, f) \) is injective and \( \text{Hom}_\Gamma(R, h[-1]) = 0 \). It suffices to prove that \( X[-1](R, R) = 0 \). Let \( s : R \rightarrow X[-1] \) be a morphism from \( R \) to \( X[-1] \) and \( t : X[-1] \rightarrow R \) a morphism from \( X[-1] \) to \( R \). We need to show that \( t \circ s = 0 \). Since \( R \) is rigid, we know that the morphism \( t : X[-1] \rightarrow R \) factors through the morphism \( h[-1] \). In particular, there is a morphism \( t' : R_1 \rightarrow R \) such that \( t = t' \circ h[-1] \). On the other hand, by \( \text{Hom}_\Gamma(R, h[-1]) = 0 \), we deduce that \( h[-1] \circ s = 0 \). Consequently, \( t \circ s = t' \circ h[-1] \circ s = 0 \).

Now we assume that \( X \) is a maximal \( R[1] \)-rigid object with respect to \( \text{pr}(R) \) such that \( X(R[1], R[1]) = 0 \). By Theorem 2.5, \( \text{Hom}_\Gamma(R, X) \) is a support \( \tau \)-tilting \( \Gamma \)-module. Since \( X \) does not admit an indecomposable direct summand in \( \text{add} R[1] \), we have \( |\text{Hom}_\Gamma(R, X)| = |X| = |R| = |\Gamma| \) by Corollary 2.6. The condition \( X(R[1], R[1]) = 0 \) implies that \( \text{pd}_R \text{Hom}_\Gamma(R, X) \leq 1 \). Putting all of these together, we conclude that \( \text{Hom}_\Gamma(R, X) \) is a tilting \( \Gamma \)-module.

Conversely, let us assume that \( \text{Hom}_\Gamma(R, X) \) is a tilting \( \Gamma \)-module. Since tilting modules are support \( \tau \)-tilting modules, we know that \( X \) is a maximal \( R[1] \)-rigid object with respect to \( \text{pr}(R) \) by Theorem 2.5 (c). By definition of tilting modules, we have \( \text{pd}_R \text{Hom}_\Gamma(R, X) \leq 1 \). Consequently, \( X(R[1], R[1]) = 0 \) and we are done.

3. \( R[1] \)-rigid objects and presilting objects

3.1. (Pre)silting objects. Recall that \( T \) is a Krull-Schmidt, Hom-finite triangulated category with shift functor \( [1] \). Following [AI12], for \( X, Y \in T \) and \( m \in \mathbb{Z} \), we write the vanishing condition \( \text{Hom}_T(X, Y[i]) = 0 \) for \( i > m \) by \( \text{Hom}_T(X, Y[> m]) = 0 \). An object \( X \in T \) is called presilting if \( \text{Hom}_T(X, X[> 0]) = 0 \); \( X \) is called silting if \( X \) is presilting and the thick subcategory of \( T \) containing \( X \) is \( T \); \( X \) is called partial silting if \( X \) is a direct summand of some silting objects.

It is clear that (pre)silting objects are rigid. The following result has been proved in [AI12].

**Lemma 3.1.** All silting objects in \( T \) have the same number of non-isomorphic indecomposable summands.
In general, it is not known that whether a presilting object is partial silting. The following is proved in [A13].

**Lemma 3.2.** Let $R$ be a silting object and $X$ a presilting object of $\mathcal{T}$. If $X \in \text{pr}(R)$, then there is a presilting object $Y \in \text{pr}(R)$ such that $X \oplus Y$ is a silting object of $\mathcal{T}$.

### 3.2. From $R[1]$-rigid objects to (pre)silting objects.

**Lemma 3.3.** Let $R$ be a presilting object and $X \in \text{pr}(R)$. Then $\text{Hom}_{\mathcal{T}}(X, X[>1]) = 0$.

**Proof.** As $X \in \text{pr}(R)$, we have the following triangle
\[
R_1 \xrightarrow{f} R_0 \xrightarrow{g} X \xrightarrow{h} R_1[1],
\] (3.2)
where $R_0, R_1 \in \text{add} R$. Applying the functor $\text{Hom}_{\mathcal{T}}(R, -)$ to the triangle yields a long exact sequence
\[
\cdots \to \text{Hom}_{\mathcal{T}}(R, R_0[i]) \to \text{Hom}_{\mathcal{T}}(R, X[i]) \to \text{Hom}_{\mathcal{T}}(R, R_1[i+1]) \cdots.
\]
Then the assumption that $R$ is presilting implies that
\[
\text{Hom}_{\mathcal{T}}(R, X[>0]) = 0.
\] (3.3)
On the other hand, applying the functor $\text{Hom}_{\mathcal{T}}(-, X[i])$ to the triangle (3.2), we obtain a long exact sequence
\[
\cdots \to \text{Hom}_{\mathcal{T}}(R_1[1], X[i]) \to \text{Hom}_{\mathcal{T}}(X, X[i]) \to \text{Hom}_{\mathcal{T}}(R_0, X[i]) \to \cdots.
\]
Then (3.3) implies that $\text{Hom}_{\mathcal{T}}(X, X[>1]) = 0$. \hfill \Box

**Theorem 3.4.** Let $X$ be an object in $\text{pr}(R)$.

1. If $R$ is a presilting object, then the followings are equivalent.
   a. $X$ is an $R[1]$-rigid object;
   b. $X$ is a rigid object;
   c. $X$ is a presilting object.

2. If $R$ is a silting object, then $X$ is a maximal $R[1]$-rigid object with respect to $\text{pr}(R)$ if and only if $X$ is a silting object.

**Proof.** For (1), according to Lemma 3.3, it suffices to prove that each $R[1]$-rigid object is rigid.

Let us assume that $X$ is an $R[1]$-rigid object in $\text{pr}(R)$. Then there exists a triangle
\[
R_1 \xrightarrow{f} R_0 \xrightarrow{g} X \xrightarrow{h} R_1[1],
\] (3.4)
with $R_0, R_1 \in \text{add} R$. By applying the functor $\text{Hom}_{\mathcal{T}}(R, -)$ to the triangle (3.4), we obtain an exact sequence
\[
\cdots \to \text{Hom}_{\mathcal{T}}(R, R_0[1]) \to \text{Hom}_{\mathcal{T}}(R, X[1]) \to \text{Hom}_{\mathcal{T}}(R, R_1[2]) \to \cdots.
\]
Since $R$ is a presilting object and $R_0, R_1 \in \text{add} R$, we have
\[
\text{Hom}_{\mathcal{T}}(R, R_0[1]) = 0 = \text{Hom}_{\mathcal{T}}(R, R_1[2]).
\]
Consequently, $\text{Hom}_{\mathcal{T}}(R, X[1]) = 0$. Now applying the functor $\text{Hom}_{\mathcal{T}}(-, X[1])$ to (3.4), we obtain an exact sequence
\[
\text{Hom}_{\mathcal{T}}(R_1[1], X[1]) \to \text{Hom}_{\mathcal{T}}(X, X[1]) \to 0.
\]
In other words, each morphism from $X$ to $X[1]$ factors through the morphism $h : X \to R_1[1]$. However, we have $R[1](X, X[1]) = 0$, which implies that $\text{Hom}_{\mathcal{T}}(X, X[1]) = 0$ and hence $X$ is rigid. This completes the proof of (1).
Now suppose that \( R \) is a silting object. If \( X \) is a silting object, then \( X \) is an \( R[1] \)-rigid object by (1). By Lemma 3.1, we have \( |X| = |R| \). Hence, \( X \) is a maximal \( R[1] \)-rigid object with respect to \( \text{pr}(R) \) by Corollary 2.6.

On the other hand, if \( X \) is a maximal \( R[1] \)-rigid object, then \( X \) is a presilting object by (1). Since \( X \in \text{pr}(R) \), \( X \) is a partial silting object by Lemma 3.2. According to Corollary 2.6, we know that \( |X| = |R| \). Therefore, \( X \) must be a silting object by Lemma 3.1.

Combining Theorem 2.5 with Theorem 3.4, we obtain the following bijection, which is due to Iyama-Jørgensen-Yang (cf. [IY14, Theorem 0.2]).

**Corollary 3.5.** Let \( R \) be a basic silting object of \( \mathcal{T} \) with endomorphism algebra \( \Gamma = \text{End}_\mathcal{T}(R) \). There is a bijection between the set of presilting objects which belong to \( \text{pr}(R) \) and the set of \( \tau \)-rigid pair of \( \Gamma \)-modules, which induces a one-to-one correspondence between the set of silting objects in \( \text{pr}(R) \) and the set of support \( \tau \)-tilting \( \Gamma \)-modules.

4. \( R[1] \)-rigid objects and \( d \)-rigid objects in \( (d + 1) \)-Calabi-Yau category

Let \( d \) be a positive integer. Throughout this section, we assume that \( \mathcal{T} \) is \( (d + 1) \)-Calabi-Yau, i.e. we are given bifunctorial isomorphisms

\[
\text{Hom}_\mathcal{T}(X, Y) \cong \mathbb{D} \text{Hom}_\mathcal{T}(Y, X[d + 1]) \quad \text{for } X, Y \in \mathcal{T},
\]

where \( \mathbb{D} = \text{Hom}_k(-, k) \) is the usual duality over \( k \).

4.1. From \( R[1] \)-rigid objects to \( d \)-rigid objects. An object \( T \in \mathcal{T} \) is called \( d \)-rigid if \( \text{Hom}_\mathcal{T}(T, T[i]) = 0 \) for \( i = 1, 2, \ldots, d \). It is maximal \( d \)-rigid if \( T \) is \( d \)-rigid and for \( i = 1, \ldots, d \), \( \text{Hom}_\mathcal{T}(T \oplus X, (T \oplus X)[i]) = 0 \) implies that \( X \in \text{add} T \). An object \( T \in \mathcal{T} \) is a \( d \)-cluster-tilting object if \( T \) is \( d \)-rigid and for \( i = 1, \ldots, d \), \( \text{Hom}_\mathcal{T}(T, X[i]) = 0 \) implies that \( X \in \text{add} T \).

**Lemma 4.1.** Let \( R \) be a \( d \)-rigid object of \( \mathcal{T} \) and \( X \in \text{pr}(R) \). Then \( X \) is rigid if and only if \( X \) is \( d \)-rigid.

**Proof.** It is obvious that a \( d \)-rigid object is rigid.

Now suppose that \( X \) is rigid. As \( X \in \text{pr}(R) \), we have a triangle

\[
R_1 \xrightarrow{f} R_0 \xrightarrow{g} X \xrightarrow{h} R_1[1]
\]

with \( R_0, R_1 \in \text{add} R \). Note that we have \( \text{Hom}_\mathcal{T}(R, R[i]) = 0 \) for \( i = 1, 2, \ldots, d \). Applying the functor \( \text{Hom}_\mathcal{T}(R, -) \) to the triangle (4.5), we obtain a long exact sequence

\[
\cdots \rightarrow \text{Hom}_\mathcal{T}(R, R_0[i]) \rightarrow \text{Hom}_\mathcal{T}(R, X[i]) \rightarrow \text{Hom}_\mathcal{T}(R, R_1[i + 1]) \rightarrow \cdots.
\]

Consequently,

\[
\text{Hom}_\mathcal{T}(R, X[i]) = 0, \quad i = 1, \ldots, d - 1.
\]

On the other hand, applying the functor \( \text{Hom}_\mathcal{T}(-, X[i]) \) to the triangle (4.5) yields a long exact sequence

\[
\cdots \rightarrow \text{Hom}_\mathcal{T}(R_1[1], X[i]) \rightarrow \text{Hom}_\mathcal{T}(X, X[i]) \rightarrow \text{Hom}_\mathcal{T}(R_0, X[i]) \rightarrow \cdots.
\]

Then (4.6) implies that \( \text{Hom}_\mathcal{T}(X, X[i]) = 0 \) for \( i = 2, \ldots, d - 1 \).

Recall that \( X \) is rigid and \( \mathcal{T} \) is \( (d + 1) \)-Calabi-Yau, we have

\[
\text{Hom}_\mathcal{T}(X, X[d]) \cong \mathbb{D} \text{Hom}_\mathcal{T}(X, X[1]) = 0.
\]

Hence \( X \) is a \( d \)-rigid object of \( \mathcal{T} \).

**Theorem 4.2.** Let \( R \in \mathcal{T} \) be a \( d \)-rigid object and \( X \in \text{pr}(R) \). Then the followings are equivalent.

1. \( X \) is an \( R[1] \)-rigid object.
(2) $X$ is a rigid object.
(3) $X$ is a $d$-rigid object.

Proof. According to Lemma 4.1, it suffices to prove that each $R[1]$-rigid object is rigid.

Let us first consider the case that $d \geq 2$. Applying the functor $\text{Hom}_T(R, -)$ to the triangle (4.5) yields a long exact sequence

$$
\cdots \rightarrow \text{Hom}_T(R, R_0[1]) \rightarrow \text{Hom}_T(R, X[1]) \rightarrow \text{Hom}_T(R, R_1[2]) \rightarrow \cdots .
$$

As $R$ is $d$-rigid and $R_0, R_1 \in \text{add } R$, we conclude that $\text{Hom}_T(R, X[1]) = 0$. Applying the functor $\text{Hom}_T(-, X[1])$ to the triangle (4.5), we obtain an exact sequence

$$
\text{Hom}_T(R_1[1], X[1]) \xrightarrow{\text{Hom}_T(h, X[1])} \text{Hom}_T(X, X[1]) \rightarrow 0.
$$

In particular, each morphism from $X$ to $X[1]$ factors through the morphism $h : X \rightarrow R_1[1]$. Hence the assumption that $X$ is an $R[1]$-rigid object implies that $X$ is rigid.

Now suppose that $d = 1$. In this case, $\mathcal{T}$ is a 2-Calabi-Yau triangulated category. Applying the functor $\text{Hom}_T(-, X[1])$ to (4.5) yields a long exact sequence

$$
\cdots \rightarrow \text{Hom}_T(R_1[1], X[1]) \xrightarrow{\text{Hom}_T(h, X[1])} \text{Hom}_T(X, X[1]) \xrightarrow{\text{Hom}_T(g, X[1])} \text{Hom}_T(R_0, X[1]) \rightarrow \cdots .
$$

Then the assumption that $X$ is $R[1]$-rigid implies that $\text{Hom}_T(g, X[1])$ is injective. Consequently, the morphism

$$
\mathbb{D} \text{Hom}_T(g, X[1]) : \mathbb{D} \text{Hom}_T(R_0, X[1]) \rightarrow \mathbb{D} \text{Hom}_T(X, X[1])
$$

is surjective. Thanks to the 2-Calabi-Yau property, we have the following commutative diagram

$$
\begin{array}{ccc}
\text{Hom}_T(R_0, X[1]) & \xrightarrow{\text{D Hom}_T(g, X[1])} & \text{Hom}_T(X, X[1]) \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}_T(X, R_0[1]) & \xrightarrow{\text{Hom}_T(g, 0[1])} & \text{Hom}_T(X, X[1]).
\end{array}
$$

In particular, $\text{Hom}_T(X, g[1]) : \text{Hom}_T(X, R_0[1]) \rightarrow \text{Hom}_T(X, X[1])$ is surjective. Again, $R[1](X, X[1]) = 0$ implies that $\text{Hom}_T(X, g[1]) = 0$ and then $\text{Hom}_T(X, X[1]) = 0$. 

\end{proof}

4.2. $d$-cluster-tilting objects in $d$-cluster categories. This subsection concentrates on $d$-cluster categories, a special class of $(d + 1)$-Calabi-Yau triangulated categories. We refer to [K05, T07] for definitions and [ZZ09, W09] for basic properties of $d$-cluster categories. Among others, the following result proved in [ZZ09, W09] is useful.

Lemma 4.3. Let $\mathcal{T}$ be a $d$-cluster category. Then an object $T$ is a $d$-cluster tilting object if and only if $T$ is a maximal $d$-rigid objects. Moreover, all $d$-cluster tilting objects have the same number of non-isomorphic indecomposable direct summands.

Then for relative rigid objects, we have the following.

Proposition 4.4. Let $\mathcal{T}$ be a $d$-cluster category and $R$ be a $d$-cluster tilting object in $\mathcal{T}$. Assume $X \in \text{pr}(R)$, then $X$ is a maximal $R[1]$-rigid object with respect to $\text{pr}(R)$ if and only if $X$ is a $d$-cluster tilting object.

Proof. Let $X$ be a maximal $R[1]$-rigid object with respect to $\text{pr}(R)$. By Corollary 2.6, we have $|X| = |R|$. According to Theorem 4.2, $X$ is $d$-rigid and hence a maximal $d$-rigid object in $\mathcal{T}$. It follows from Lemma 4.3 that $X$ is a $d$-cluster tilting object in $\mathcal{T}$.

Conversely, assume that $X$ is a $d$-cluster tilting object. According to Lemma 4.3, $X$ is a maximal $d$-rigid object with $|X| = |R|$ and hence a maximal $R[1]$-rigid object respect to $\text{pr}(R)$ by Theorem 4.2. 

\end{proof}
Combining Theorem 2.5, Thereom 4.2 with Proposition 4.4, we obtain the following main result of [LQX].

**Corollary 4.5.** Assume that $d \geq 2$. Let $\mathcal{T}$ be a $d$-cluster category with a $d$-cluster-tilting object $R$. Denote by $\Gamma = \text{End}_{\mathcal{T}}(R)$ the endomorphism algebra of $R$. The functor $\text{Hom}_{\mathcal{T}}(R, -)$ yields a bijection between the set of isomorphism classes of $d$-rigid objects of $\mathcal{T}$ which belong to $\text{pr}(R)$ and the set of isomorphism classes of $\tau$-rigid $\Gamma$-modules. The bijection induces a bijection between the set of isomorphism classes of $d$-cluster tilting objects of $\mathcal{T}$ which belong to $\text{pr}(R)$ and the set of isomorphism classes of support $\tau$-tilting $\Gamma$-modules.

4.3. Maximal rigid objects in 2-Calabi-Yau categories. In this subsection, we assume that $\mathcal{T}$ is a 2-Calabi-Yau category and $R$ a basic maximal rigid object of $\mathcal{T}$. It has been proved in [BIRS09, ZZ11] that each rigid object of $\mathcal{T}$ belongs to $\text{pr}(R)$.

The following proposition is a direct consequence of Theorem 4.2 and Corollary 2.6.

**Proposition 4.6.** Let $\mathcal{T}$ be a 2-Calabi-Yau triangulated category and $R$ a basic maximal rigid object of $\mathcal{T}$. Let $X$ be an object of $\mathcal{T}$, then $X$ is a maximal $R[1]$-rigid object with respect to $\text{pr}(R)$ if and only if $X$ is a maximal rigid object of $\mathcal{T}$.

Combining Theorem 2.5 with Proposition 4.6, we obtain the main result of [CZZ15, LX16].

**Corollary 4.7.** Let $\mathcal{T}$ be a 2-Calabi-Yau category with a basic maximal rigid object $R$. Denote by $\Gamma = \text{End}_{\mathcal{T}}(R)$ the endomorphism algebra of $R$. Then there is a bijection between the set of isomorphism classes of rigid objects of $\mathcal{T}$ and the set of isomorphism classes of $\tau$-rigid $\Gamma$-modules, which induces a bijection between the set of isomorphism classes of maximal rigid objects of $\mathcal{T}$ and the set of isomorphism classes of support $\tau$-tilting $\Gamma$-modules.

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Changjian Fu, Department of Mathematics, SiChuan University, 610064 Chengdu, P.R.China
E-mail address: changjianfu@scu.edu.cn

Shengfei Geng, Department of Mathematics, SiChuan University, 610064 Chengdu, P.R.China
E-mail address: genshengfei@scu.edu.cn

Pin Liu, Department of Mathematics, Southwest Jiaotong University, 610031 Chengdu, P.R.China
E-mail address: pinliu@swjtu.edu.cn