Separable Dyson-Schwinger model at finite $T$\textsuperscript{*}

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Abstract

Theoretical understanding of experimental results from relativistic heavy-ion collisions requires a microscopic approach to the behavior of QCD n-point functions at finite temperatures, as given by the hierarchy of Dyson-Schwinger equations, properly generalized within the Matsubara formalism. The technical complexity of related finite-temperature calculations however mandates modeling. We present a model where the QCD interaction in the infrared, nonperturbative domain, is modeled by a separable form. Results for the mass spectrum of light quark flavors at finite temperature are presented.

1 Introduction

While the experiments at RHIC [1, 2] advanced the empirical knowledge of the hot QCD matter dramatically, the understanding of the state of matter that has been formed is still lacking. For example, the STAR collaboration’s assessment [2] of the evidence from RHIC experiments depicts a very intricate, difficult-to-understand picture of the hot QCD matter. Among the issues pointed out as important was the need to clarify the role of quark-antiquark ($q\bar{q}$) bound states continuing existence above the critical temperature $T_c$, as well as the role of the chiral phase transition.

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Both of these issues are consistently treated within the Dyson-Schwinger (DS) approach to quark-hadron physics. Dynamical chiral symmetry breaking (DChSB) as the crucial low-energy QCD phenomenon is well-understood in the rainbow-ladder approximation (RLA), a symmetry preserving truncation of the hierarchy of DS equations. Thanks to this, the QCD low energy theorems are fulfilled and the behavior of the chiral condensate and pion mass and decay constant are in accord with the Gell-Mann–Oakes–Renner relation, i.e. the Goldstone theorem.

For recent reviews of the DS approach, see, e.g., Refs. [3, 4], of which the first [3] also reviews the studies of QCD DS equations at finite temperature, started in [5]. Unfortunately, the extension of DS calculations to non-vanishing temperatures is technically quite difficult. This usage of separable model interactions greatly simplifies DS calculations at finite temperatures, while yielding equivalent results on a given level of truncation [6, 7]. A recent update of this covariant separable approach with application to the scalar σ meson at finite temperature can be found in [8]. Here, we present results for the quark mass spectrum at zero and finite temperature, extending previous work by including the strange flavor.

2 The separable model at zero temperature

The dressed quark propagator \( S_q(p) \) is the solution of its DS equation [3, 4],

\[
S_q(p)^{-1} = i\gamma \cdot p + \overline{m}_q + \frac{4}{3} \int \frac{d^4\ell}{(2\pi)^4} g^2 D^{\text{eff}}_{\mu\nu}(p - \ell) \gamma_\mu S_q(\ell) \gamma_\nu,
\]

while the \( q\bar{q} \) meson Bethe-Salpeter (BS) bound-state vertex \( \Gamma_{q\bar{q}}(p, P) \) is the solution of the BS equation (BSE)

\[
-\lambda(P^2)\Gamma_{q\bar{q}}(p, P) = \frac{4}{3} \int \frac{d^4\ell}{(2\pi)^4} g^2 D^{\text{eff}}_{\mu\nu}(p - \ell) \gamma_\mu S_q(\ell_+) \Gamma_{q\bar{q}}(\ell, P) S_q(\ell_-) \gamma_\nu,
\]

where \( D^{\text{eff}}_{\mu\nu}(p - \ell) \) is an effective gluon propagator modeling the nonperturbative QCD effects, \( \overline{m}_q \) is the current quark mass, the index \( q \) (or \( q' \)) stands for the quark flavor (\( u, d \) or \( s \)), \( P \) is the total momentum, and \( \ell_\pm = \ell \pm P/2 \). The chiral limit is obtained by setting \( \overline{m}_q = 0 \). The meson mass is identified from \( \lambda(P^2 = -M^2) = 1 \). Equations (1) and (2) are written in the Euclidean space, and in the consistent rainbow-ladder truncation.
The simplest separable Ansatz which reproduces in RLA a nonperturbative solution of $1$ for any effective gluon propagator in a Feynman-like gauge $g^2 D_{\mu\nu}^{\text{eff}}(p - \ell) \rightarrow \delta_{\mu\nu} D(p^2, \ell^2, p \cdot \ell)$ is $[6, 7]$\n
\[
D(p^2, \ell^2, p \cdot \ell) = D_0 F_0(p^2) F_0(\ell^2) + D_1 F_1(p^2)(p \cdot \ell) F_1(\ell^2). \tag{3}
\]

This is a rank-2 separable interaction with two strength parameters $D_i$ and corresponding form factors $F_i(p^2), i = 1, 2$. The choice for these quantities is constrained to the solution of the DSE for the quark propagator $1$\n
\[
S_q(p)^{-1} = i\gamma \cdot p A_q(p^2) + B_q(p^2) \equiv Z_q^{-1}(p^2)[i\gamma \cdot p + m_q(p^2)], \tag{4}
\]

where $m_q(p^2) = B_q(p^2)/A_q(p^2)$ is the dynamical mass function and $Z_q(p^2) = A_q^{-1}(p^2)$ the wave function renormalization. Using the separable Ansatz in $1$, the gap equations for the quark amplitudes $A_q(p^2)$ and $B_q(p^2)$ read\n
\[
B_q(p^2) - \tilde{m}_q = \frac{16}{3} \int \frac{d^4 \ell}{(2\pi)^4} D(p^2, \ell^2, p \cdot \ell) \frac{B_q(\ell^2)}{\ell^2 A_q^2(\ell^2) + B_q^2(\ell^2)} = b_q F_0(p^2), \tag{5}
\]

\[
\left[A_q(p^2) - 1\right] = \frac{8}{3p^2} \int \frac{d^4 \ell}{(2\pi)^4} D(p^2, \ell^2, p \cdot \ell) \frac{(p \cdot \ell) A_q(\ell^2)}{\ell^2 A_q^2(\ell^2) + B_q^2(\ell^2)} = a_q F_1(p^2). \tag{6}
\]

Once the coefficients $a_q$ and $b_q$ are obtained by solving the gap equations and $1$, the only model parameters remaining are $\tilde{m}_q$ and the parameters of the gluon propagator, to be fixed by meson phenomenology.

3 Extension to finite temperature

The extension of the separable model studies to the finite temperature case, $T \neq 0$, is systematically accomplished by a transcription of the Euclidean quark 4-momentum via $p \rightarrow p_n = (\omega_n, \vec{p})$, where $\omega_n = (2n + 1)\pi T$ are the discrete Matsubara frequencies. In the Matsubara formalism, the number of coupled equations represented by $1$ and $2$ scales up with the number of fermion Matsubara modes included. For studies near and above the transition, $T \geq 100$ MeV, using only 10 such modes appears adequate. Nevertheless, the appropriate number can be more than $10^3$ if the continuity with $T = 0$ results is to be verified. The effective $\bar{q}q$ interaction will automatically decrease with increasing $T$ without the introduction of an explicit $T$-dependence which would require new parameters.
The solution of the DS equation for the dressed quark propagator now takes the form

\[ S_q^{-1}(p_n, T) = i\gamma \cdot \vec{p} A_q(p_n^2, T) + i\gamma_4 \omega_n C_q(p_n^2, T) + B_q(p_n^2, T), \]  

(7)

where \( p_n^2 = \omega_n^2 + \vec{p}^2 \) and the quark amplitudes

\[ B_q(p_n^2, T) = \tilde{m}_q + b_q(T)F_0(p_n^2), \]
\[ A_q(p_n^2, T) = 1 + a_q(T)F_1(p_n^2), \]
\[ C_q(p_n^2, T) = 1 + c_q(T)F_1(p_n^2) \]

are defined by the temperature-dependent coefficients \( a_q(T), b_q(T), \) and \( c_q(T) \) to be determined from the set of three coupled non-linear equations

\[ a_q(T) = \frac{8D_1}{9} T \sum_n \int \frac{d^3p}{(2\pi)^3} F_1(p_n^2) \tilde{p}^2 [1 + a_q(T)F_1(p_n^2)] d^{-1}(p_n^2, T), \]  

(8)

\[ c_q(T) = \frac{8D_1}{3} T \sum_n \int \frac{d^3p}{(2\pi)^3} F_1(p_n^2) \omega_n^2 [1 + c_q(T)F_1(p_n^2)] d^{-1}(p_n^2, T), \]  

(9)

\[ b_q(T) = \frac{16D_0}{3} T \sum_n \int \frac{d^3p}{(2\pi)^3} F_0(p_n^2) [\tilde{m}_q + b_q(T)F_0(p_n^2)] d^{-1}(p_n^2, T). \]  

(10)

The function \( d_q(p_n^2, T) \) is the denominator of the quark propagator \( S_q(p_n, T) \), and is given by

\[ d_q(p_n^2, T) = \tilde{p}^2 A_q^2(p_n^2, T) + \omega_n^2 C_q^2(p_n^2, T) + B_q^2(p_n^2, T). \]  

(11)

The procedure for solving gap equations for a given temperature \( T \) is the same as in \( T = 0 \) case, but one has to control the appropriate number of Matsubara modes as mentioned above.

4 Confinement and Dynamical Chiral Symmetry Breaking

If there are no poles in the quark propagator \( S_q(p) \) for real timelike \( \vec{p}^2 \) then there is no physical quark mass shell. This entails that quarks cannot propagate freely, and the description of hadronic processes will not be hindered by unphysical quark production thresholds. This sufficient condition is a viable possibility for realizing quark confinement [7]. A nontrivial solution for \( B_q(p^2) \) in the chiral limit \((\tilde{m}_0 = 0)\) signals DChSB. There is a connection between quark confinement realized as the lack of a quark mass shell and the existence of a strong quark mass function in the infrared through DChSB.
The propagator is confining if \( m_q^2(p^2) \neq -p^2 \) for real \( p^2 \), where the quark mass function is \( m_q(p^2) = B_q(p^2)/A_q(p^2) \). In the present separable model, the strength \( b_q = B_q(0) \), which is generated by solving Eqs. (5) and (6), controls both confinement and DChSB. At finite temperature, the strength \( b_q(T) \) for the quark mass function will decrease with \( T \), until the denominator (11) of the quark propagator can vanish for some timelike momentum, and the quark can come on the free mass shell. The connection between deconfinement and disappearance of DChSB is thus clear in the DS approach. Also the present model is therefore expected to have a deconfinement transition at or a little before the chiral restoration transition associated with \( b_0(T) \rightarrow 0 \).

The following simple choice of the separable interaction form factors,\[
\mathcal{F}_0(p^2) = \exp\left(-p^2/\Lambda_0^2\right), \quad \mathcal{F}_1(p^2) = \frac{1 + \exp(-p_0^2/\Lambda_1^2)}{1 + \exp((p^2 - p_0^2)/\Lambda_1^2)},
\]
is used to obtain numerical solutions which reproduce very well the phenomenology of the light pseudoscalar mesons and generate an acceptable value for the chiral condensate.

The resulting quark propagator is found to be confining and the infrared strength and shape of quark amplitudes \( A_q(p^2) \) and \( B_q(p^2) \) are in quantitative agreement with the typical DS studies. Gaussian-type form factors are used as a minimal way to preserve these properties while realizing that the ultraviolet suppression is much stronger than the asymptotic fall off (with logarithmic corrections) known from perturbative QCD and numerical studies on the lattice [9].

5 Results

Parameters of the model are completely fixed by meson phenomenology calculated from the model as discussed in [7, 8]. In the nonstrange sector, we work in the isosymmetric limit and adopt bare quark masses \( \tilde{m}_u = \tilde{m}_d = 5.5 \) MeV and in strange sector we adopt \( \tilde{m}_s = 115 \) MeV. Then the parameter values

\[
\Lambda_0 = 758 \text{ MeV}, \quad \Lambda_1 = 961 \text{ MeV}, \quad p_0 = 600 \text{ MeV}, \quad (12) \\
D_0\Lambda_0^2 = 219, \quad D_1\Lambda_1^4 = 40, \quad (13)
\]

lead, through the gap equation, to \( a_{u,d} = 0.672, b_{u,d} = 660 \) MeV, \( a_s = 0.657 \) and \( b_s = 998 \) MeV i.e., to the dynamically generated momentum-dependent
mass functions $m_q(p^2)$ shown in Fig. 1. In the limit of high $p^2$, they converge to $\bar{m}_u$ and $\bar{m}_s$. However, at low $p^2$, the values of $m_u(p^2)$ are close to the typical constituent quark mass scale $\sim m_\rho/2 \sim m_N/3$ with the maximum value at $p^2 = 0$, $m_u(0) = 398$ MeV. The corresponding value for the strange quark is $m_s(0) = 672$ MeV. Thus, Fig. 1 shows that in the domain of low and intermediate $p^2 \lesssim 1$ GeV$^2$, the dynamically generated quark masses have typical constituent quark mass values. Thus, the DS approach provides a derivation of the constituent quark model [10] from a more fundamental level, with the correct chiral behavior of QCD.

Obtaining such dynamically generated constituent quark masses, as previous experience with the DS approach shows (see, e.g., Refs. such as [3, 4, 10]), is essential for reproducing the static and other low-energy properties of hadrons, including decays. (We would have to turn to less simplified DS models for incorporating the correct perturbative behaviors, including that of the quark masses. Such models are amply reviewed or used in, e.g., Refs. [3, 4, 10, 11], but addressing them is beyond the present scope, where the perturbative regime is not important.)

The extension of results to finite temperatures is given in Figs. 2, 3. Very important is the temperature dependence of the chiral-limit quantities.

Figure 1: The $p^2$ dependence (at $T = 0$) of the dynamically generated quark masses $m_s(p^2), m_u(p^2)$ for, respectively, the strange and the (isosymmetric) nonstrange case.
$B_0(0)_T$ and $\langle q\bar{q}\rangle_0(T)$, whose vanishing with $T$ determines the chiral restoration temperature $T_{Ch}$. We find $T_{Ch} = 128$ MeV in the present model.

![Figure 2](image)

Figure 2: The temperature dependence of $B_s(0)$, $B_u(0)$ and $B_0(0)$, the scalar propagator functions at $p^2 = 0$, for the strange, the nonstrange and the chiral-limit cases, respectively. The temperature dependence of the chiral quark-antiquark condensate, $-\langle q\bar{q}\rangle_0^{1/3}$, is also shown (by the lowest curve). Both chiral-limit quantities, $B_0(0)$ and $-\langle q\bar{q}\rangle_0^{1/3}$, vanish at the chiral-symmetry restoration temperature $T_{Ch} = 128$ MeV.

The temperature dependences of the functions giving the vector part of the quark propagator, $A_{u,s}(0)_T$ and $C_{u,s}(0)_T$, are depicted in Fig. 4. Their difference is a measure of the O(4) symmetry breaking with the temperature $T$.

The presented model, when applied in the framework of the Bethe-Salpeter approach to mesons as quark-antiquark bound states, produces a very satisfactory description of the whole light pseudoscalar nonet, both at zero and finite temperatures [12].

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Figure 3: The temperature dependence of $m_s(0)$, $m_u(0)$ and $m_0(0)$, the dynamically generated quark masses at $p^2 = 0$ for the strange, the nonstrange and the chiral-limit cases, respectively.

Figure 4: The violation of $O(4)$ symmetry with $T$ is exhibited on the example of $A_{u,s}(0)_T$ and $C_{u,s}(0)_T$.

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