Composite Weak Bosons: a Lattice Monte Carlo Analysis

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Abstract

We present a lattice Monte Carlo simulation for the evaluation of the spectrum of a confining Yang-Mills theory without Goldstone boson. We show that this theory is a very good candidate for describing composite weak bosons. In order to perform the spectrum analysis we have used standard lattice QCD Monte Carlo methods. We have also developed an efficient method to evaluate the mass of the pseudoscalar isosinglet which is present in our theory.
1 Introduction

The Standard Model (SM) describes the strong, weak and electromagnetic interactions by a gauge theory based on the group $G_{SM} = SU(3) \times SU(2) \times U(1)$ which is broken by the Higgs mechanism to $SU(3) \times U(1)$. The theory is essentially determined once the matter fields and their transformation under the local gauge transformations of $G_{SM}$ are specified. The matter fields (leptons and quarks) and the Higgs boson are considered to be elementary. They interact with each other by the exchange of gauge bosons which are also considered to be elementary. The structure of the SM has been phenomenologically confirmed to high accuracy.

In spite of the beautiful corroboration of the SM by experiments a natural question arises: How elementary are the leptons, the quarks, the Higgs bosons and the gauge bosons? The idea that the SM itself is an effective theory of another, more fundamental, where quarks, leptons and bosons are composites of more fundamental fields is almost as old as the SM itself. The idea of quark and lepton compositeness is motivated by the observed connection between quarks and leptons, by the generation puzzle and by the existence of too many parameters in the SM. The Higgs compositeness is motivated by the fine tuning problem. The W and Z compositeness is motivated by their relation to a composite Higgs and by the observation that all short-range interactions are residual interactions of a more fundamental long-range interaction.

The constituents, the new fundamental fields, are supposed to carry a new internal quantum number (which we denote as hypercolor) and the quarks, leptons and bosons are hypercolorless composite systems of them. The binding of the constituents due to hypercolor is viewed as an analogy to the color confinement mechanism of QCD. However, since the SM spectrum is different from the hadron spectrum, the hypercolor interaction has to be described by a strongly coupled Yang-Mills theory different from QCD.

Several models treat the quarks, leptons and bosons as composite systems. Today a conspicuous number of theorems exist which have ruled out most of the existing models and radically restricted the possibilities to construct realistic composite models. In principle there are two categories of models:

- **Three-fermion models**: In these models the leptons and quarks have three fermion spin 1/2 constituents. Most of these models are phenomenologically ruled out by the Weingarten, Nussinov and Witten constraints, by the the ’t Hooft’s anomaly-matching conditions, by the Weinberg-Witten theorem or by the Vafa-Witten theorem. Because most of the constraints dictated by these theorems can be avoided in a supersymmetric scenario of compositeness, some authors have proposed supersymmetric versions of these models.

- **Fermion-scalar models**: In these models the leptons and quarks are made out of two constituents, a fermion spin 1/2 and a scalar. The W, Z and the
Higgs bosons can have also two constituents. Two particular examples are the
"Strongly coupled SM" (SCSM) [11] and the Yang-Mills theory without Goldstone bosons [12].
The SCSM starts from the same lagrangian of the SM, however, with an unbroken
gauge group $SU(2) \otimes U(1)$. The aim of this theory is not to describe
a model more fundamental than the SM, but to propose an alternative to the
usual Higgs mechanism. The interesting feature arises that the W and Z are
only the lowest lying states of a whole spectrum of vector-mesons.
The Yang-Mills theory without Goldstone bosons is a strongly coupled non-
abelian gauge theory which can have a low lying spectrum equal to the spec-
trum of the SM. This is the model we want to study in this work. In particular
we will concentrate on the weak gauge bosons sector comparing the low lying
spectrum of this theory with the SM boson spectrum.

To be precise this last model considers the photon to remain elementary and switched
off. The weak gauge bosons $W^\pm$ and $Z^0$ then form a mass degenerate triplet. This
model is a usual confining Yang-Mills theory with $SU(2)$ local hypercolor gauge group,
$SU(2)$ global isospin group and generalized Majorana fermions in the fundamental
representation of the local and global symmetry groups. We note that the general-
ization of the Majorana fermions with non-trivial quantum numbers is possible only
if the symmetry groups are real.

To be viable a composite model of the weak bosons has to reproduce the known weak
boson spectrum: the lightest bound states have to be the W-bosons and heavier bound
states have to lie in an experimentally unexplored energy range. The only possibility
to have a Yang-Mills theory which reproduces the weak boson spectrum is to choose
the degrees of freedom in a way that they naturally avoid bound states lighter than
the vector isotriplet of the theory which characterizes the W-boson triplet. This is
possible if the unwanted light bound states which naturally show up as Goldstone
bosons or pseudo Goldstone bosons in many models (like, for example, a pseudoscalar
isomultiplet, which would be the pion analogue of QCD) are avoided. The choice of
Majorana fermions in this model avoids the $SU_A(2)$ global chiral symmetry of the
Yang-Mills Lagrangian because left- and right-handed degrees of freedom are not in-
dependent. The axial current (which would generate the $SU_A(2)$ chiral symmetry)
does not exist and it is not possible to have a breaking of $SU_A(2)$ with the related
low lying Goldstone bosons. In fact, the pseudoscalar isotriplet vanishes by the Pauli
principle (it is a symmetric combination of Grassmann variables).

Because of the strong coupling character of this theory, we need non-perturbative
methods to make predictions. It is important that the fermion theory under discus-

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\footnote{In [12] an unfortunate sign error has entered the calculations due to a wrong factor $i$ in the
fourth and fifth terms of eq. (3). To interpret the results correctly one has to interchange isosinglet
↔ isovector in [12]. I thank H.Schleret for a discussion of this point.}
à la Wilson [13] is possible because the choice of the isospin group $SU(2)$ allows us to replace the Dirac mass term and the Dirac-type Wilson term by a hypercolor gauge invariant Majorana type expression.

A strong coupling expansion analysis [14] of the spectrum of this theory has shown that the spin one isotriplet bound state (the right quantum number to represent the W-boson of the SM) could be the lightest state if the pseudoscalar isosinglet acquires a mass by the chiral anomaly in analogy to the $\eta'$ in QCD.

In this work we calculate the spectrum of the lightest bound states by a quenched Monte Carlo simulation and we show that the vector isotriplet bound state of this theory is the lightest one. We have performed two different type of Monte Carlo simulations. In the first one we have calculated the masses of bound states without the contribution of the chiral anomaly. This simulation confirmed the results of the strong coupling expansion. In the second one we have developed an efficient method to evaluate the chiral anomaly contribution to the mass of the pseudoscalar isosinglet bound state. Its mass turned out to be heavier than the vector isotriplet mass.

Our work is organized as follows: in section 2 we introduce the model in question and because this paper is not addressed only to lattice specialists in section 3 we shortly present the technical methods of lattice spectroscopy that we will use. In section 4 we explain the method that we have developed to estimate the chiral anomaly contribution to the pseudoscalar isosinglet mass and in section 5 we analyse the results of the simulations.

## 2 Confining Gauge Theories without Goldstone Bosons

### 2.1 The Wilson action

We consider a gauge theory whose fermion content is represented by a Weyl spinor $F^A_{\alpha,a}(x)$. Here $\alpha$ denotes the (undotted) spinor index ($\alpha = 1, 2$), $A$ denotes the fundamental representation index of a global $SU(2)$ isospin group ($A = 1, 2$) and $a$ denotes the fundamental representation index of the local $SU(2)$ hypercolor gauge group ($a = 1, 2$). We introduce the generalized Majorana spinor $\psi$ starting from the Weyl spinors $F$ and its conjugate $F^\dagger$

$$\psi(x) = \begin{pmatrix} F(x) \\ QF^\dagger(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \varphi(x)$$

and its adjoint

$$\bar{\psi}(x) = (F^T(x)Q, F^\dagger(x)) = \varphi^T(x) \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}$$

The matrix $Q$ represents the antisymmetric matrix in spin, hypercolor and isospin space, which correspond to the Kronecker product of $i\sigma_2, i\tau_2, iT_2$ (the antisymmetric matrices in spin, hypercolor and isospin space, respectively). Of course the fields $\psi$
and $\bar{\psi}$ are not independent fields. The choice of the global isospin group $SU(2)$ and of the local hypercolor group $SU(2)$ allows us to write gauge invariant mass terms for the Majorana fermion fields
\[ \bar{\psi}\psi = FQF + F^\dagger QF^\dagger \] (3)

Note that this choice is unique if one deals with Majorana fermions. Because of the existence of the mass term we can define the Yang-Mills action on the lattice in Euclidean space in the form of a Wilson action.
\[ S = \beta \sum_{p \subset A} TrU(\partial p) - k \sum_{b=(xy)} \bar{\psi}(x)\Gamma(b)U(b)\psi(y) - \frac{1}{2} \sum_x \bar{\psi}(x)\psi(x) \equiv \] (4)

For further details we refer to ref. \[14\].
We will make repeated use of the fact that the fermion fields can be integrated out from the functional integral. Here, one has to be careful since $\bar{\psi}$ and $\psi$ are not independent fields and the fermionic functional integral is represented in terms of $\varphi$.
As a result we obtain.
\[ Z = \int [d\varphi][dU] \exp \{-S(\varphi,U)\} = \int [dU] \sqrt{\det M(U)} \exp \{-S_{Gauge}(U)\} \] (5)

2.2 CP eigenstates

In weak interactions CP is a good quantum number but not C and P separately. Therefore, we perform a classification of the composite operators according to the CP eigenvalues. The CP transformation of the Weyl spinor $F$ is defined by
\[ F^{CP}(x) = i\gamma_2^2\psi(x) \]
\[ \bar{\psi}^{CP}(x) = \bar{\psi}(x)\gamma_2^0 \] (6)

We can build CP odd and CP even eigenstates\[2\]. In a confining Yang-Mills theory the physical states are hypercolor singlets. Because in our model the hypercolor group is $SU(2)$ all physical states are described by bilinear forms in $\bar{\psi}$ and $\psi$.

\[ \text{Lorentz scalars } S, \text{ vectors } V^\mu \text{ and tensors } T^{\mu\nu} \text{ are CP even if } CP S = S, CP V^\mu = V_\mu \text{ and } CP T^{\mu\nu} = T_{\mu\nu} \text{ and are CP odd if } CP S = -S, CP V^\mu = -V_\mu \text{ and } CP T^{\mu\nu} = -T_{\mu\nu}, \text{ respectively.} \]

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**Scalar CP eigenstates:** The scalar CP **even** combination of scalars is

\[ S_+(x) = \bar{\psi}(x)\psi(x) \]  

(7)

The pseudoscalar CP **odd** combination is

\[ S_-(x) = i\bar{\psi}(x)\gamma_5\psi(x) \]  

(8)

**Vector CP eigenstates:** The axial vector isosinglet CP **even** combination is

\[ V^\mu(x) = \frac{i}{2}\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) \]  

(9)

The isotriplet vector states is

\[ V^{\mu I}(x) = \frac{1}{2}\bar{\psi}(x)\gamma^\mu T^I\psi(x) \]  

(10)

This state is CP **even** for \( I=1,3 \) and CP **odd** for \( I=2 \) and characterizes the W-boson triplet.

**Tensor CP eigenstates:** The tensors are

\[ \tilde{B}^{\mu\nu I} = \bar{\psi}(x)\sigma^{\mu\nu}T^I\psi(x) \]  

(11)

and their dual

\[ \tilde{B}^{*\mu\nu I} = \epsilon^{\mu\nu\sigma\rho}\tilde{B}_\sigma^I \]  

(12)

where \( \sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu,\gamma^\nu] \). \( \tilde{B}^{\mu\nu I} \) has CP properties opposite to the vector isotriplet \( V^I \) ones. Its dual \( \tilde{B}^{*\mu\nu I} \) has CP properties opposite to \( \tilde{B}^{\mu\nu I} \).

### 2.3 Correlators

Masses are computed in lattice simulation from the asymptotic behavior in the Euclidean-time direction of static correlation functions. Static here means that we have projected the correlation functions at zero momentum transfer. In our case we need to consider only two point functions. A typical static bound state propagator can be written as

\[ C^\Gamma(t) = \sum_{\vec{x}} \langle \left( \bar{\psi}(t,\vec{x})\Gamma\psi(t,\vec{x}) \right) \bar{\psi}(0)\Gamma\psi(0) \rangle \]  

(13)

where the \( \Gamma \) represent a matrix with hypercolor, isospin and spinor indices, for example \( \Gamma = \gamma^kT^I \) for the vector isotriplet. Here the contraction of the hypercolor, isospin and spinor indices is implicit. Contracting creation and annihilation operators into fermion propagators

\[ \langle T\psi(x)\bar{\psi}(y) \rangle = G(x,y) \]  

(14)
we obtain for the correlation function

\[ C_{\Gamma}(t) = \sum_{\vec{x}} \langle ((\bar{\psi}(x)\Gamma\psi(x))^\dagger \bar{\psi}(y)\Gamma\psi(y) \rangle = \]

\[ = \sum_{\vec{x}} \left\{ Tr \left[ \Gamma G(x,0)\Gamma G(0,x) - \Gamma G(x,0)AG^T AG(0,x) \right] - \ight. \]

\[ \left. \left( Tr[\Gamma G(0,0)] Tr[\Gamma G(x,x)] - Tr[AG^T AG(0,0)] Tr[AG^T AG(x,x)] \right) \right\} \]

where \( \Gamma^T \) indicates the transposition of \( \Gamma \) and \( A \) is the matrix \( A = Q \oplus Q \). Simplifying this expression we obtain the full propagator of the CP eigenstates.

\[ C_{\Gamma}(t) = 2 \sum_{\vec{x}} \left\{ Tr \left[ \Gamma G(x,0)\Gamma G(0,x) \right] - Tr[\Gamma G(0,0)] Tr[\Gamma G(x,x)] \right\} \quad (16) \]

To prove the formula (15) one has to contract properly all fermion fields in (13) rewriting them in terms of \( \varphi \) fields and remembering that they are not Dirac fields. The disconnected part of eq. (16) corresponds to the chiral anomaly contribution to the propagators of the CP eigenstates. This contribution vanishes for all isomultiplet bound states. It yields an important contributions for the pseudoscalar isosinglet bound states \( S_\pm \).

The anomaly contribution may be switched off by restricting to the sector of configurations of topological charge zero. We will make use of the fact that when the chiral anomaly contribution is switched off the pseudoscalar isosinglet bound state behaves like a massless Goldstone bosons and has a propagator of the form

\[ C_{\gamma_5}(t) = 2 \sum_{\vec{x}} Tr \left[ \gamma_5 G(x,0)\gamma_5 G(0,x) \right] \]

(17)

On the other hand, when we take into account the chiral anomaly contribution it is no more a massless Goldstone boson but it acquires a mass, like the \( \eta' \) in QCD. Its propagator is of the form

\[ C^{an}_{\gamma_5}(t) = 2 \sum_{\vec{x}} \left\{ Tr \left[ \gamma_5 G(x,0)\gamma_5 G(0,x) \right] - \xi Nf Tr[\gamma_5 G(0,0)] Tr[\gamma_5 G(x,x)] \right\} \quad (18) \]

Here \( N_f = 2 \) characterizes the \( SU(N_f) \) isospin group. Notice that we have introduced the parameters \( \xi \) in the propagator (18). This parameter is \( \xi = 1 \) if one computes with dynamical fermions. Unfortunately, present computational capabilities make necessary the quenched approximation and the parameter \( \xi \) can acquire a value different from unity \[15\].

We will evaluate the mass once without and once with anomaly term. In the case without the chiral anomaly contribution the mass of the operator \( S_- \) will be used to identify the chiral limit. The propagator \( C_{\gamma_5}(t) \) can be evaluated using standard Monte Carlo techniques. For the case where the chiral anomaly contribution is

\[ \text{To distinguish the two cases we denote by } S_{an}^- \text{ the pseudoscalar isosinglet when the chiral anomaly is switched on.} \]
switched on we have used an improved version of the method of ref. \cite{15} to evaluate the propagator $C^{an}_{\gamma_5}(t)$. 

Because we will evaluate static propagators we consider the operators being given by the following set of $\Gamma$’s:

| $\Gamma$ | flavor |
|----------|---------|
| $\gamma_5$ | $S_-$ |
| $\gamma_5 \gamma_0$ | $S^{an}_-$ |
| $\gamma_k T^I$ | $V^{kI}$ |
| $\sigma^{ik} T^I$ | $B^{kI}$ |
| $\gamma_5 \gamma_k$ | $V^k$ |
| $\sigma^{kj} T^I$ | $B^{kjI}$ |
| 1 | $S_+$ |

3 Lattice spectroscopy

The general procedure in lattice spectroscopy is to first calculate the fermion propagator $G(x, y)$ in the presence of an external gauge field and then to evaluate the hypercolor singlet propagators (16). The action (4) is quadratic in the fermion fields, these can be formally integrated out and the fermion propagator needed to evaluate the correlation functions (16) takes the form:

$$G(x, y) = \langle T\psi(x)\bar\psi(y) \rangle = \frac{1}{Z} \int [d\phi][dU] T\psi(x)\bar\psi(0) \exp\{-S(\varphi, U)\} =$$

$$= \frac{1}{Z} \int [dU] \left( M(U)_{x,y}^{-1} \right) \sqrt{\det M(U)} \exp\{-S_{\text{Gauge}}(U)\}$$

Just as a simple integral can be evaluated as a limit of a sum, we can evaluate the path integral (19) by discretizing it on a lattice in an Euclidian four dimensional space. We put our model on a $N_s^3 \times N_t$ lattice with spacing $a$. A gauge field configuration is defined as a set of $N \times N$ complex matrices defined on each oriented lattice bond $b$ with some boundary condition. Fermion fields are defined as $N \times N_f \times 4$ complex vectors on the lattice points with some boundary condition. The lattice spacing $a$ acts as ultra-violet cut-off and provides a regularization scheme necessary for any quantum field theory. The Wilson lattice action (4) provides a regularization which

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4Because of the relations $\partial_\mu V^\mu = C \times S_-$ (when the anomaly is switched off) and $\partial_\mu \tilde{B}^{\mu\nu A} = C' \times V^{\nu A}$ (where $C$ and $C'$ are constants) the bound states $V^0$ and $\tilde{B}^{0A}$ are not independent from $S_-$ and, respectively, $V^{iA}$. Therefore, after summing over all $\vec{x}$ in eq. (13) we expect that the correlators of $V^0$ and $\tilde{B}^{0A}$ receive contributions from their low-lying states (at large time separation) and their masses will be degenerated with the masses of $S_-$ and, respectively, $V^{iA}$.

5$N$ is the number of hypercolors.
preserves the gauge invariance but breaks the chiral symmetry. The chiral properties of the theory can be restored in the continuum limit.

The calculation of (19) at present is not yet possible because the computation of \( \det M(U) \) at each Monte Carlo step is very CPU time and computer memory consuming. Like in QCD we assume that the quenched approximation \( \det M(U) \simeq 1 \) is a reasonable approximation also in our model. To evaluate the fermion propagator \( G(x, y) \) with standard Monte Carlo techniques in the quenched approximation we statistically integrate (19) by replacing the functional integral over the \( U \) variables by the measure \( \exp\{-S(U)\}[dU] \) using the standard algorithms.

Given a gauge field configuration \( U' \) one obtains the fermion propagator of this configuration \( U' \) by looking at the inverse fermion matrix \( M(U')^{-1} \) in eq. (19). The correlators of the different bound states will be computed by averaging eq. (16) over different gauge configurations.

### 3.1 Matrix inversion technique

Most of the computational effort computing the correlators goes into the construction of the fermion propagator \( G(x, y) \). We have to find the inverse of the fermion matrix \( M(x, y) \equiv M(U)_{x,y} \), which obeys

\[
\sum_y M(x, y) G(y, z) = \delta_{x,z} \tag{20}
\]

Since \( M \) is a complex matrix of dimension \( V_M = N_s^3 \times N_t \times N \times 4 \) it is not possible to store \( G(x, y) \) for all \( x \) and \( y \) since this involves arrays of the order of \( V_M^2 \) complex numbers. Fortunately, we can exploit translation invariance such that one only needs to know \( G(x, y) \) for all \( x \) and for one selected point \( y \). Generally, one constructs \( G(x, y) \) by solving

\[
\sum_y M(x, y) \tilde{G}(y, z) = S_z(x) \tag{21}
\]

where \( S_z(x) \) is some external source which can be different from \( \delta_{x,z} \). The propagator \( \tilde{G}(x, y) \) will be the vector \( M^{-1}S \).

Because we expect the fermion masses of the constituents to be much smaller than a typical bound energy we should invert the fermion matrix for small masses. However, as the fermion masses tend to the chiral limit, the matrix inversion of \( M \) becomes a hard problem because \( M \) tends to be singular. Therefore one usually considers a few heavy fermion masses and extrapolates to the chiral limit.

In our calculation the inversion of the fermion matrix is performed by the Minimal-Residual algorithm or by the Conjugate Gradient algorithm in the case that the first

\( ^6 \)M has lattice indices, hypercolor indices and spin indices. All isospin index inversion can be simply done by hand to economize memory and CPU time.
method does not converge. To economize computer memory we organize the inversion algorithms on a checkerboards even/odd-splitting.

3.2 Smearing technique

The standard method to determine the bound state masses is to calculate the two-point function

\[ C_\Gamma(t) = \sum_{\vec{x}} \langle O_\Gamma^\dagger(\vec{x}, t) O_\Gamma(0) \rangle \]

\[ = \sum_n \left\{ |\langle n|O_\Gamma(0)|0\rangle|^2 e^{-m_{\Gamma}^n t} + |\langle 0|O_\Gamma^\dagger(0)|n\rangle|^2 e^{-m_{\Gamma}^n (N_t - t)} \right\} \]

(22)

where \( O_\Gamma(x) \equiv \bar{\psi}(x)\Gamma\psi(x) \) is an operator with the quantum number of the bound state in question, \( N_t \) denotes the number of lattice points in the time direction and \( |n\rangle \) is an eigenstate of the transfer matrix. Here we consider a lattice with periodic boundary conditions.

Our goal is to evaluate the mass of the lowest lying state contributing to (22). Here the problem of isolating this term in the sum over all eigenstates occurs. A reliable value for the mass can be extracted only at large time \( t \). An efficient method to extract the ground state contribution to \( C_\Gamma \) is to construct an operator \( O_\Gamma \) which has a large overlap with the ground state \( |n_0\rangle \). A pointlike operator \( O_\Gamma(x) \) performs very poorly as an operator which generates bound states. One therefore constructs a new bound state operator by smearing [16] the fermion field \( \psi(x) \) with a suitable wave function \( F(\vec{x}, \vec{x}') \). One has to choose the wave function which has a physical extent over several lattice spacing. \( F \) has to be smooth and to vanish for large separation \( |\vec{x} - \vec{x}'| \).

We have chosen \( F \) dependent in a gauge invariant way on the link gauge variables \( U \). The smeared fermion field takes the form:

\[ \psi_s(\vec{x}, t) = \sum_{\vec{x}'} F(\vec{x}, \vec{x}', U, t) \psi(\vec{x}', t) \]

(23)

where the subscription "s" indicates smeared fields. In this work the wave function is chosen to be of the Gaussian type (characterized by two parameters \( \alpha \) and \( n \))

\[ F(\vec{x}, \vec{x}', U, t) = (1 + \alpha H(\vec{x}, \vec{x}', U, t))^{n} \]

(24)

with the hopping matrix

\[ H(\vec{x}, \vec{x}', U, t) = \left\{ U(\langle xx'\rangle) + U^\dagger(\langle x'x\rangle) \right\} |_{x=(t,\vec{x}), x'=(t, \vec{x}')} \]

(25)

\(^7\)We defined a lattice point as even (odd) if its coordinate sum \( (x + y + z + t) \) is even (odd). Eq. (21) can be written in a checkerboard basis and if one knows \( \tilde{G} \) on one checkerboard (\( \tilde{G}_{e} \), say) one can reconstruct \( \tilde{G}_o \).
This wave function is defined in local time which means that it involves gauge fields only in one time slice and is gauge covariant.

The smeared fermion propagator \( \tilde{G}_\nu(x,x') \) is related to the local fermion propagator \( \tilde{G}(x,x') \) by

\[
\tilde{G}_\nu(x,x') = \sum_{\vec{y}} F^\dagger(\vec{x},\vec{y},U,t) \tilde{G}(y,z) F(\vec{z},\vec{x}',U,t) \tag{26}
\]

where \( x = (t,\vec{x}) \) and \( x' = (t',\vec{x}') \). We mention that the construction of the smeared fermion propagator does not take much more computer time than the calculation of the local propagator.

The ground state dominance of the correlation functions \( C_\Gamma \) is signalled by the occurrence of a plateau in the local mass

\[
\mu_\Gamma(t) = \log \left( \frac{C_\Gamma(t)}{C_\Gamma(t-1)} \right) \tag{27}
\]

In this case the local mass is time independent for a range of \( t \). It was impossible to attain a plateau without smearing the fermion fields. In our calculation we have tuned the parameters \( \alpha \) and \( n \) of the wave function (24) to obtain early plateaux in the local masses of the smeared operators. Defining

\[
r^2 = \left\langle \frac{\sum_{\vec{x}} \bar{x}^2 T^r [ F(\vec{x},0) F^\dagger(\vec{x},0) ]}{\sum_{\vec{x}} T^r [ F(\vec{x},0) F^\dagger(\vec{x},0) ]} \rightangle \tag{28}
\]

the optimisation of the wave function coincides in our calculation with the radius \( r \) approximately of \( 3 \times a \), where \( a \) is the lattice spacing. This is a resonable size for a bound state of a typical size less or equal the spacing \( a \).

### 3.3 Setting the scale and the hopping parameter

From lattice calculation we can evaluate only dimensionless quantities. To connect these quantities to physics we have to set a scale. On the lattice we choose the scale to be the lattice spacing \( a \) and we express all quantities in unit of \( a \). For small enough \( a \) the lattice results up to a renormalization of the parameters and the fields yield the continuum theory.

In our model the experimental value of the W-boson mass will be used to setting the \( a \) scale. For setting the space lattice \( a \) we identify the experimental value \( M_W = 80.22(26) \text{ GeV} \) with the value of the mass of the \( V^{kA} \) resonance in the dimensionless quantity \( m_{\gamma kT^A} \times a \) determined from the simulation.

The lattice value of \( m_{\gamma kT^A} \times a \) has to be determined for heavy fermion masses to avoid the singularity of the fermion matrix. However, we expect that the constituent fermion masses are much smaller than a typical binding energy. Therefore we may proceed in analogy to QCD and extrapolate the quantities to the chiral limit.

In QCD the critical value \( k_c \) of the hopping parameter follows from the determination of the pion mass

\[
a^2 \times m_\pi^2 = A \left( \frac{1}{k} - \frac{1}{k_c} \right) \tag{29}
\]
suggested by the lowest order chiral perturbation theory \cite{17}. The pion is the Goldstone boson predicted by the spontaneously broken chiral symmetry and should be a massless particle in the chiral limit. The non-vanishing experimentally measured mass of the pion is the result of the breaking of flavor SU(2) symmetry.

In our model there is no Goldstone boson which can play the role of the pion in QCD. However, when the chiral anomaly is switched off the pseudo scalar bound state behaves like a Goldstone boson. We emphasise that switching on the chiral anomaly by choosing topological non-trivial gauge configurations we expect that the pseudoscalar bound state $S_{\pi}$ acquires a mass by the chiral anomaly in analogy to the $\eta'$ in QCD. The critical value $k_c$ of the hopping parameter can be evaluated by assuming that the pseudo scalar bound state $S_{-}$ behaves like a Goldstone boson and therefore it should be massless. In analogy to QCD one makes the following ansatz

$$a^2 \times m_{\pi}^2 = A \left( \frac{1}{k} - \frac{1}{k_c} \right)$$

In our fits the linear dependence of $m_{\pi}^2$ on the inverse hopping parameter $1/k$ is very well satisfied. All other bound state masses can be evaluated by linearly extrapolating their lattice predictions $m_{\Gamma} \times a$ to the critical hopping parameter $k_c$ as suggested by our strong coupling expansion \cite{14}. In particular, for the vector isotriplet $V^{kA}$ mass, the dimensionless quantity $m_{\gamma kT_A} \times a$ which is used to set the scale can be extrapolated into the chiral regime by the linear ansatz

$$a \times m_{\gamma kT_A} = B + C \left( \frac{1}{k} - \frac{1}{k_c} \right)$$

in analogy to the linear ansatz for the $\rho$ mass in QCD.

### 4 The pseudoscalar isosinglet mass.

We have shown in the ref. \cite{14} that the vector isotriplet bound state can be the lightest bound state provided that the pseudoscalar isosinglet acquires a mass from the chiral anomaly. In order that our model may be considered a viable theory of electroweak compositeness it has to turn out that this bound state is heavier than the vector isotriplet one.

A non-perturbative estimate of the contribution of the chiral anomaly to the pseudoscalar isosinglet mass is one of the most challenging problems of the lattice Monte Carlo simulations. In the past there were a few attempts to make a quantitative Monte Carlo analysis of that problem in QCD \cite{15,18,20}. In this section we will discuss the calculation of the mass of the pseudoscalar isosinglet bound state by applying and improving, using the smearing technique, some methods developed in \cite{15,18}.

In QCD, in the limit of zero bare masses of the u and d quarks, the pseudoscalar mesons ($\pi$ and $\eta'$) are Goldstone bosons due to the spontaneous symmetry breaking.
of the SU(2) axial symmetry. Because the quarks possess small but nonzero bare masses, the \( \pi \) has a small mass and only approximates a Goldstone boson. However the \( \eta' \) meson is too heavy for approximating a Goldstone boson. This is the \( U(1) \) problem \[21\]. t’Hooft removed the puzzle by showing that the existence of topologically non-trivial gauge configurations such as instantons resolves the \( U(1) \) problem.

In any confining Yang-Mills theory with fermion carrying non-trivial isospin quantum number the same problem arises. The pseudoscalar isosinglet bound states (the analogous to the \( \eta' \) in QCD) acquire heavy masses due to instanton effects. Their masses can be evaluated using the method of lattice gauge theories. There are, however, a number of technical difficulties which restrict the feasibility of the study.

The propagator of the pseudoscalar isosinglet \( S^m \) bound state is given by equation (18). From the numerical evaluation of this propagator we can determine the mass of the \( S^m \) bound state using the lattice spectroscopy methods presented in section 3. The major difficulty comes from the large amount of computer time needed to evaluate the disconnected contribution to \( C^m_{\gamma_5} \). For this purpose we need the fermion propagator \( G(x,x) \) for all points \( x \), which requires \( N^3_s \times N_t \) inversions of the fermion matrix using the method of the point source on a lattice of size \( N^3_s \times N_t \). This is a hopeless task because it needs too much computer time.

The problem can be solved requiring \( N_t \) fermion matrix inversions by calculating the ratio

\[
R(t) = \frac{C^m_{\gamma_5}(t)}{C'_{\gamma_5}(t)}
\]

for \( N_t \) walls. Here, \( C'_{\gamma_5} \) and \( C^m_{\gamma_5} \) denote the propagators computed at a suitable chosen value of \( \vec{x} \) in eq. (17) and (18). This means that one does not perform the sum over \( \vec{x} \) in (17) and (18). The long distance behaviour of the propagators (17-18) for a fixed \( \vec{x} \) is then \[19\]

\[
C^m_{\gamma_5}(t) = A \left( e^{-m^m_{\gamma_5} t} \frac{1}{tD} + e^{-m^m_{\gamma_5}(N_t-t)} \frac{(N_t-t)^D}{(N_t-t)^D} \right)
\]

\[
C'_{\gamma_5}(t) = B \left( e^{-m_{\gamma_5} t} \frac{1}{tD'} + e^{-m_{\gamma_5}(N_t-t)} \frac{(N_t-t)^D'}{(N_t-t)^D'} \right)
\]

where \( A, B \) and \( D, D' \) are constants, \( m^m_{\gamma_5} \) and \( m_{\gamma_5} \) denote the masses of \( S^m \) and \( S_\gamma \), respectively. The smearing of the source over the space-like lattice for each time wall has the effect to partially sum over the space-like point over each time wall and thus reduces the \( tD \) and \( tD' \) dependence of the propagators (33). To a good approximation we expect that with smearing the exponents \( D \) and \( D' \) are \( D \approx D' \ll 1 \). The ratio \( R(t) \) eliminates the \( tD \) and \( tD' \) dependence of the propagators \( C'_{\gamma_5} \) and \( C^m_{\gamma_5} \) and can be expressed for sufficiently large time \( t \) by

\[
R(t) \simeq E \left( e^{-\Delta m_{\gamma_5} t} + e^{-\Delta m_{\gamma_5}(N_t-t)} \right)
\]

where \( E = A/B \) is a constant and \( \Delta m_{\gamma_5} = m^m_{\gamma_5} - m_{\gamma_5} \) represents the mass difference between the \( S^m \) and the \( S_\gamma \) bound states for any \( k \leq k_c \). Because, in the quenched
approximation, this mass difference is a pure hypergluonic effect, it does not depend on the mass of the Majorana fermion fields and also on the hopping parameter $k$. At the chiral limit the mass of $S_-$ is equal to zero and $\Delta m_{\gamma_5}$ correspond to the mass of $S_{\gamma_5}^m$. Checking that the ratio $R(t)$ is independent of the hopping parameter $k$ one can extrapolate $\Delta m_{\gamma_5}$ to the chiral limit by a constant function in $k$ obtaining then $m_{\gamma_5}^{an}$.

The contribution of the disconnected term to the propagator of $S_{\gamma_5}$ is present only if topological nontrivial gauge configurations are used. $C_{\gamma_5}^{an} = C_{\gamma_5}'$ (statistically) for topological trivial configurations. Topological nontrivial configurations can be obtained thermalizing an instanton on the lattice using the heat bath updating. The cooling algorithm is also used to determine the topological property of the configurations. Cooling [22] works as described next: we use the standard plaquette action which will be locally minimized during the iterations sweeps of the heat bath algorithm. The effect is to locally smoothen the lattice gauge configuration, removing the ultraviolet fluctuations which are responsible for destroying the topological charge. The starting instanton which will be thermalized can be defined on the lattice by discretizing the instanton solution proposed by t’Hooft [21]. To control the topological properties of the gauge configurations we have used Peskin’s lattice definition of the topological charge [23] in the symmetrized version [24]

$$Q = -\sum_{x \in \Lambda} \sum_{(\mu, \nu, \rho, \sigma) = \pm 1}^{\pm 4} \frac{\tilde{\epsilon}_{\mu\nu\rho\sigma}}{2432\pi^2} Tr [U(x)_{\mu\nu} U(x)_{\rho\sigma}]$$

which has the right continuum limit. Here $\tilde{\epsilon}_{\mu\nu\rho\sigma}$ is the generalized total antisymmetric tensor\(^8\) in any direction (positive and negative) of the Euclidean space and $U(x)_{\mu\nu}$ is the product of four links $U(b)$ around a plaquette lying in the $\mu\nu$ plane and with starting point $x$. There are other definitions of the topological charge on the lattice [20, 25]. We have chosen Peskin’s one because it is the simplest to be programmed on the computer. We emphasize that a less naive (and more complicate) definition of the topological charge would be needed if one would like to know the topological charge density $Q(x)$ or the topological susceptibility like in [20]. However, in our work we only need a naive control over the global topological charge $Q = \sum_{x \in \Lambda} Q(x)$ and the simple definition of Peskin is sufficient for our purpose.

In order to perform a Monte Carlo integration with topological non-trivial configurations one has to generate a set of configurations having integer topological charges. Unfortunately, because the lattice definition (35) of the topological charge is a discretization of the continuum definition which allows non integer values of $Q$, we can’t generate configurations with an exact integer topological charge, but only with a $Q$ lying near an integer value. We accepted only configurations with $Q$ lying in an interval $\pm 0.1$ around an integer number.

An other difficult is to generate configurations with $|Q| > 1$ because the small volume of the lattice does not allow a proper thermalization of such instantons: they loose their topological charge during the iterations and in configurations with $|Q| = 1$

\[^8\]For example: $1 = \tilde{\epsilon}_{1234} = -\tilde{\epsilon}_{-1234}$.
5 Results

5.1 Computational details

The simulation was performed on the Cray YMP at the ETH in Zürich and on the NEC SX-3 at the CSCS in Manno. The simulation was done on different lattices in order to understand the finite volume and a effects (see Table 1). The \( SU(2) \) configurations were generated by the combination of heat bath and over-relaxed updating (1 heat bath for 6 over-relaxed sweep). The cooling updating was also used for topological non-trivial configurations to determine the topological charge. The plateaux of the local masses \((27)\) were obtained from the fit to local masses of smeared-smeared correlators. They were independent on the parameters \( \alpha \) and \( n \) of the applied smearing function \((24)\). Having established the wave function independence, for the rest of the calculation we performed the estimation of the spectrum using only the optimized wave function with \( \alpha = 3 \) and \( n = 25 \).

The masses were calculated at different \( k < k_c \)'s from a two parameter fit to eq. \((22)\) or \((34)\) on a time interval \([t_{\text{min}}, t_{\text{max}}]\) determined by the plateaux and by studying the stability of the fits under changes to the fitting range. The typical best fitting ranges for the different lattices A,B,C (see Table 1) are \([t_{\text{min}}, t_{\text{max}}] = [6,10] , [4,7] \) and \([10,16]\), respectively.

All needed fits were performed by minimizing the correlated \( \chi^2 \) and the statistical errors were calculated using the jackknife method \([26]\) (with binning to control the autocorrelation).

5.2 Phase diagram and \( a^{-1}(\beta) \) dependence

The critical hopping parameter \( k_c \) was determined by linearly extrapolating \( m_{\gamma_5}^2 \) to the chiral limit as explained in section 3.3. For the different lattices the phase diagram in the \((k, \beta)\)-plane is plotted in Fig. 1. In this range of the parameter space there are no evident finite volume and a effect for these data. Data obtained from simulations with topological trivial and topological non-trivial configurations are consistent with each other.

In a previous work \([14]\) we have computed the mass \( m_{\gamma_5} \) by a strong coupling expansion to high orders. In Fig. 2 we plot in the \((k, \beta)\)-plane the critical line determined by the chiral limit \( m_{\gamma_5}(k, \beta) = 0 \) over a large range of \( \beta \) values (from 0.0 to 2.8). In the same plot we compare the critical line with the Monte Carlo data. Considering that the convergence radius of the strong coupling expansion can be estimated to be \( \beta_{\text{conv.}} \approx 1.7 \) the agreement between the two calculation is surprising.

\[9\text{Although the t-dependent local masses depend on the underlying wave function, they finally end up with the same plateau!}\]
The lattice spacing $a$ was determined from the vector isotriplet mass extrapolated to the chiral limit and normalized with the experimental value of the $W$-boson mass. In Fig. 3 we have plotted the Monte Carlo data for the different lattices. We clearly see in the plot that the values of $a^{-1}(\beta)$ for a fixed $\beta$ are affected by finite volume effects, as expected. The prediction of the strong coupling expansion and the Monte Carlo data are plotted in Fig. 4. For values of $\beta$ less than the estimated convergence radius $\beta_{\text{conv.}} \approx 1.7$ we obtain the function $a^{-1}(\beta)$ (solid line). For values of $\beta$ bigger than $\beta_{\text{conv.}}$ the behaviour of this function can be interpolated by the Monte Carlo data (dashed line).

5.3 Spectrum

5.3.1 The axial vector and tensor bound state masses

The masses of the axial vector $V^k$ and the tensor $\tilde{B}^{kjA}$ bound states are evaluated on the lattices $A_i (i=1,..3)$, $B_i (i=0,..3)$ and $C_i (i=1,..3)$ (see Table 1) using topological trivial gauge configurations. After a thermalization by 5000 iterations sweeps we use gauge configurations separated by 200 sweeps. For 1 heat bath sweeps we do 6 over-relaxed sweep.

For each $\beta$ we have fitted the masses using eq. (22) for eight different hopping parameters and extrapolated them to the chiral limit.

In Fig. 5 and Fig. 6 we present the result of the lattices $B_1$, $B_2$, $B_3$ and $C_1$, $C_2$, $C_3$ for which the finite volume effects are small. The same results are presented also in Table 2. We plot the masses as a function of the lattice spacing $a$ normalized with the $W$-boson mass at 80 GeV. All data points with $m_T \times a > 1$ are excluded from the analysis of the finite $a$ effects. These correspond to masses evaluated on the lattice $B_0$ with the smallest $\beta$. The continuum limit of the masses is evaluated by linearly extrapolating to $a = 0$ for the different lattices. The results of the extrapolations are presented in Fig. 5 and Fig. 6. Finite volume effects at the continuum limit are negligible between the lattices $B_1$, $B_2$, $B_3$ and $C_1$, $C_2$, $C_3$ in comparison to the corresponding statistical errors.

5.3.2 The pseudoscalar isosinglet mass

The mass of the pseudoscalar isosinglet $S_{\pi}$ is evaluated on the lattices $B_0^*$, $B_1^*$ and $B_2^*$. We used topological non-trivial gauge configurations with topological charge $Q = 1$. After a thermalization by 5000 iterations sweeps we used gauge configurations separated by 1000 sweeps. The evolution of the topological charge during the iterations is plotted in Fig. 7.

For each $\beta$ we have fitted the mass difference $\Delta m_{\gamma_5}$ with the ratio $R(t)$ of eq. (34) for five different hopping parameters $k$. In Fig. 8 we plot the different values of $\Delta m_{\gamma_5} \times a$ as a function of the $m_{\gamma_5}^2 \times a^2$ (which is proportional to the inverse hopping parameter). The same result is presented also in Table 3. We note that it is independent of $m_{\gamma_5}^2$, as it should be. After checking this mass independence the extrapolation to the
chiral limit at $m_{\gamma_5}^2 = 0$ was done by a constant function.
In Fig. 9 we plot the chiral limit of $\Delta m_{\gamma_5}$ as a function of the lattice spacing $a$.
At the chiral limit the mass difference coincides with the mass of the pseudoscalar isosinglet $m_{\gamma_5}$. We clearly see that decreasing the lattice spacing $a$ the pseudoscalar isosinglet mass increase and it is bigger than the W-boson mass.

6 Conclusion

We have discussed a composite model for the weak bosons of the SM. The model is based on a Yang-Mills theory without Goldstone bosons. In this theory the W-boson of the SM is represented by the vector isotriplet bound state. As a main result our lattice Monte Carlo simulation has shown that the vector isotriplet bound state is the lightest bound state in our model as it should be for any viable candidate of electroweak composite model.
We have developed an efficient method to compute the mass of the pseudoscalar isosinglet bound states. This computation requires the evaluation of disconnected fermion loops and the generation of topological non-trivial gauge configurations.
We have also predicted the mass of the first bound states heavier than the vector isotriplet. These bound states are a vector isotriplet and a vector isosinglet and a pseudoscalar with masses in the range of a few hundred GeV (see Fig. 5, Fig. 6 and Fig. 9). These predictions open new experimental perspective at LEPII and LHC.

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Figure Caption

1. Lattice Monte Carlo data for the phase diagram in the $(\beta, k)$-plane.

2. Lattice Monte Carlo data and strong coupling expansion prediction for the phase diagram in the $(\beta, k)$-plane. The dotted line represents the limit of $1/8$ for $\beta \to \infty$.

3. Lattice Monte Carlo data for the inverse lattice spacing $a^{-1}$ as a function of $\beta$.

4. Lattice Monte Carlo data and strong coupling expansion calculation for the inverse lattice space $a^{-1}$ as a function of $\beta$. The dotted line represents the estimate convergence radius of the strong coupling expansion. The dashed line is a qualitative estimation of $a^{-1}(\beta)$ based on an interpolation of the Monte Carlo data of the biggest lattice.

5. Lattice Monte Carlo prediction of the triplet $\tilde{B}^A_{jk}$ bound state mass as a function of the lattice spacing. Dashed and dotted lines are the extrapolation to the continuum limit.

6. Lattice Monte Carlo prediction of the axial vector $V_k$ bound state mass as a function of the lattice spacing. Dashed and dotted lines are the extrapolation to the continuum limit.

7. Evolution of the topological charge during the iterations. The topological charge is measured at each 50 iterations.

8. The mass difference $\Delta m_{\gamma_5}$ as a function of the $m_{\gamma_5}$ mass in unit of the lattice spacing $a$. The dashed lines represent the extrapolations to the chiral limit.

9. The pseudoscalar isosinglet mass as a function of the lattice spacing $a$. 

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| lattice | $\beta$ | $N_s$ | $N_t$ | no. config’s. | $a^{-1}$/GeV | $k_c$  |
|---------|--------|------|------|---------------|---------------|--------|
| A1      | 2.3    | 6,   | 24   | 44           | 120(10)       | 0.164(6) |
| A2      | 2.4    | 6,   | 24   | 40           | 145(12)       | 0.156(6) |
| A3      | 2.5    | 6,   | 24   | 40           | 156(24)       | 0.153(4) |
| B0*     | 2.2    | 8,   | 16   | 40           | 129(7)        | 0.173(1) |
| B0      | 2.2    | 8,   | 16   | 40           | 119(9)        | 0.173(1) |
| B1*     | 2.3    | 8,   | 16   | 40           | 156(4)        | 0.170(2) |
| B1      | 2.3    | 8,   | 16   | 40           | 141(8)        | 0.166(2) |
| B2*     | 2.4    | 8,   | 16   | 40           | 181(17)       | 0.154(2) |
| B2      | 2.4    | 8,   | 16   | 42           | 174(16)       | 0.154(1) |
| B3      | 2.5    | 8,   | 16   | 40           | 198(17)       | 0.153(2) |
| B4      | 2.7    | 8,   | 16   | 40           | *****         | 0.152(8) |
| C1      | 2.3    | 12,  | 36   | 40           | 158(6)        | 0.1672(4) |
| C2      | 2.4    | 12,  | 36   | 40           | 177(4)        | 0.1558(3) |
| C3      | 2.5    | 12,  | 36   | 40           | 225(12)       | 0.1515(4) |

Table 1: Parameters of the lattices used for this work. The inverse lattice spacing is obtained by fixing the vector isotriplet mass to 80 GeV. The critical hopping parameter is obtained by extrapolating $m_{\gamma_0}$ to the chiral limit. Asterisks (*****) indicate that the fit was not accepted due to a large $\chi^2$ or the missing of a plateau in the local masses in the region of the fit. The lattices B0*, B1* and B2* indicate simulations performed with topological non-trivial configurations.

| lattice | $a^{-1}$/GeV | $m_{\alpha i \gamma i}$ GeV | $m_{\gamma_0 \gamma_0}$ GeV |
|---------|--------------|------------------------------|-------------------------------|
| A1      | 120(10)      | 73(47)                      | 83(48)                       |
| A2      | 145(12)      | 76(21)                      | 104(27)                      |
| A3      | 156(24)      | 111(40)                     | 98(39)                       |
| B0      | 119(9)       | 202(36)                     | 152(37)                      |
| B1      | 141(8)       | 108(27)                     | 125(24)                      |
| B2      | 174(16)      | 135(25)                     | 130(31)                      |
| B3      | 198(17)      | 122(26)                     | 128(29)                      |
| C1      | 158(6)       | 97(14)                      | 128(27)                      |
| C2      | 177(4)       | 96(10)                      | 163(34)                      |
| C3      | 225(12)      | 113(20)                     | 133(20)                      |

Table 2: Lattice Monte Carlo predictions of the triplet $\bar{B}_{jk}^I$ and singlet $V^k$ bound states as a function of the inverse lattice spacing $a^{-1}$. 
Table 3: Lattice Monte Carlo predictions of the mass difference $\Delta m_{35}^{an}$ in unit of the lattice spacing.

| lattice | $a^{-1}$/GeV | $m_{35}^2 \times a^2$ | $\Delta m_{35}^{an} \times a$ |
|---------|-------------|-----------------|------------------|
| B0*     | 129(7)      | 0.270(27)       | 0.71(15)         |
|         |              | 0.359(20)       | 0.71(14)         |
|         |              | 0.442(18)       | 0.70(14)         |
|         |              | 0.562(16)       | 0.69(12)         |
|         |              | 0.719(17)       | 0.68(11)         |
| B1*     | 156(4)      | 0.489(6)        | 0.98(15)         |
|         |              | 0.543(7)        | 1.00(16)         |
|         |              | 0.596(7)        | 1.02(17)         |
|         |              | 0.632(8)        | 1.03(17)         |
|         |              | 0.702(9)        | 1.05(18)         |
| B2*     | 181(17)     | 0.535(14)       | 0.97(9)          |
|         |              | 0.610(14)       | 1.00(10)         |
|         |              | 0.712(13)       | 1.06(12)         |
|         |              | 0.776(12)       | 1.10(13)         |
|         |              | 0.830(11)       | 1.14(14)         |
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