Mesoscopic admittance of a double quantum dot

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We calculate the mesoscopic admittance $G(\omega)$ of a double quantum dot (DQD), which can be measured directly using microwave techniques. This quantity reveals spectroscopic information on the DQD and is also directly sensitive to a Pauli spin blockade effect. We then discuss the problem of a DQD coupled to a high quality photonic resonator. When the photon correlation functions can be developed along a RPA-like scheme, the response of the resonator gives an access to $G(\omega)$.

The possibility to couple nanoconductors to capaci-
tive gates has been instrumental for exploring electronic transport in these systems. Applying DC gate volt-
ages allows one to tune the energies of localized electronic orbitals to perform the transport spectroscopy of a nanoconductor and reach various conduction regimes. Gates can also be coupled to AC electric fields, to obtain e.g. photo-assisted tunneling or charge pumping. Recently, the mesoscopic admittance $G(\omega)$ of a single quantum dot subject to an AC gate voltage has been investigated experimentally. The low frequency limit $G(\omega \to 0) \approx -i\omega C_{\text{meso}}$ can be interpreted in terms of a mesoscopic capacitance $C_{\text{meso}}$ determined by the circuit geometric capacitances but also by the dot energy spectrum, which sets the ability of the dot to absorb electrons. This problem has been discussed theoretically in the regimes of weak and strong Coulomb interactions. In a more quantum view, gates can mediate a coupling between the electrons of a nanocircuit and cavity photons. This is widely exploited in the context of Circuit-Quantum ElectroDynamics. Coupling superconducting qubits to a coplanar waveguide photonic resonator allows an efficient manipulation, coupling and readout of the qubits. In the dispersive regime where a qubit and a resonator are strongly detuned, the cavity photons experience a frequency shift which reveals the qubit state. This shift is sometimes discussed in terms of the qubit mesoscopic capacitance. The resonant regime leads to vacuum Rabi oscillations in which the nanocircuit alternatively emits and reabsorbs a single photon.

Double quantum dots (DQDs) are mesoscopic circuits which can be made out of e.g. submicronic two dimensional electron gas structures, or top-gated carbon nanotubes. These devices can be used to elaborate various types of qubits, and offer interesting possibilities in the context of Circuit-Quantum ElectroDynamics. The behavior of a photonic resonator coupled to a DQD has been recently studied experimentally. However, on the theoretical side, this problem has aroused little attention. Besides, the AC gate-biasing of DQDs has been studied in the context of spin and charge pumping (see e.g. and Refs. therein) and photo-assisted DC transport, but no theoretical study has been performed in the context of mesoscopic admittance measurements.

In the first part of this paper, we calculate the mesoscopic admittance $G(\omega)$ of a DQD. We show that this quantity displays a very rich behavior. In particular, it is directly sensitive to a Pauli spin-blockade effect. A measurement of $G(\omega)$ seems an interesting way to perform the spectroscopy of a DQD, in the context of e.g. a qubit use, which can forbid invasive DC probes. In the second part of this paper, we discuss the problem of a DQD weakly coupled to a high quality photonic resonator. The resonator could offer an alternative to direct AC gate biasing for measuring $G(\omega)$. When the photon correlation functions can be developed along a RPA-like scheme, both the dispersive and resonant behaviors of the resonator can be predicted from $G(\omega)$. We briefly discuss the range of validity of the RPA scheme in the non-interacting limit.

We first discuss the mesoscopic admittance measurement (Figs. 1a and b). We consider two single-orbital dots $L$ and $R$ with orbital energies $\xi_L$ and $\xi_R$, coupled together through a spin-conserving tunnel barrier with a hoping constant $t$ and a capacitance $C_m$. We note $c_{d\sigma}$...
the annihilation operator associated to an electron with spin \( \sigma \in \{\uparrow, \downarrow\} \) on dot \( d \in \{L, R\} \), \( \hat{n}_d = \hat{c}^\dagger_d \hat{c}_d \), and \( \hat{n}_d = \hat{n}_d^\uparrow + \hat{n}_d^\downarrow \). Dot \( d \) is connected through a tunnel contact to a grounded reservoir, and connected through a capacitance \( C_g^d [C_{ac}^d] \) to a DC [AC] bias generator with voltage \( V_g^d [V_{ac}(t)] \). The reservoir states are described by annihilation operators \( \hat{c}_d \). The full hamiltonian of the circuit writes (up to a term proportional to the identity operator), \( \hat{H}_1 = \hat{H}_{QQDD} + \hat{H}_1 + \hat{H}_{ac} \) with

\[
\hat{H}_{QQDD} = \sum_{d,s}(\epsilon_d - \sigma|\mu_B B/2|)\hat{n}_d^\sigma + \sum_d \hat{n}_d(\hat{n}_d - 1)E_c^d + U_m\hat{n}_L\hat{n}_R + t\sum_d\left(\hat{c}_d^\dagger\hat{c}_R + h.c.\right),
\]

\[
\hat{H}_1 = \sum_{d,k,\sigma} \left( [t_d^\uparrow c^\dagger_{d\sigma} \hat{c}_{RS} + h.c.] + \epsilon_d k_{\sigma} \hat{c}^\dagger_{d\sigma} \hat{c}_{d\sigma} \right),
\]

\[
\hat{H}_{ac}(V_{ac}(t)) = \sum_d \epsilon_d \hat{n}_d V_{ac}(t) - \frac{\hbar}{2} \int_{-\infty}^{+\infty} d\varepsilon \rho(\varepsilon) g_{s,s'}(\varepsilon) \left[ |\varepsilon - E_s\|^2 + 2\gamma^2 \right] \quad (6)
\]

DQD response for totally closed quantum dots (i.e. \( t_d = t = 0 \)). In the low frequency limit, i.e. \( \omega \) much smaller than the characteristic energies involved in the DQD dynamics (including \( \Gamma_{L(R)} \)), we obtain \( C_m = 2\lambda_2 - (e^2/h)\Pi(0) \in \mathbb{R} \). One can calculate \( \Pi(0) \) from the definition of \( \chi_{d,d'}(t) \). Alternatively, assuming that \( \hat{n}_L \) and \( \hat{n}_R \) have finite correlation times, i.e. \( \lim_{t\to\infty} \chi_{d,d'}(t) = 0 \), one can use

\[
\Pi(0) = \hbar\sum_{d,d'} \alpha_d \alpha_{d'} \partial_t \langle \hat{n}_d \rangle_0 \langle \hat{n}_{d'} \rangle \quad (4)
\]

with, assuming that the eigenstates \( |\psi_i\rangle \) of \( \hat{H}_{DQD} \) (with energies \( E_i \)), are thermally populated \( \langle \psi_i | \psi_i \rangle \)

\[
\langle \hat{n}_d \rangle_0 = \sum_i \langle \psi_i | \hat{n}_d | \psi_i \rangle \exp(-\beta E_i) / \sum_i \exp(-\beta E_i) \quad (5)
\]
relevant anymore because they are too slow, and \( \Pi(\omega) \) shows a resonant behavior due to the internal dynamics of the DQD.

We now discuss the interacting case for \( 0 < \Gamma_{L(R)} \ll k_B T \ll \epsilon_{c} \), \( U_{m} \). The DQD stability diagram corresponds to the standard honeycomb pattern [21]. Figure 2b shows the variations of \( \Pi(0) \) with \( n_{g}^{L(R)} \), calculated from Eqs. (4) and (5) for \( B = 0 \). Different kinds of resonant lines occur in this graph. The first kind corresponds to electron transfers between the DQD and a lead, and has a width set by \( T \). For instance, line \( a \) corresponds to transitions between states \( (0,0) \) and \( (0,\sigma) \), with \( \sigma \in \{\uparrow, \downarrow\} \) [22]. This line can be approximated (away from triple points) as \( \Pi(0) \sim -\alpha_{L}^{2} / 4 \cos^{2}(\beta_{R}/2) \), which is reminiscent from Eq. (7). Similarly, line \( b \) corresponds to \( \Pi(0) \sim -\alpha_{L}^{2} / 4 \cos^{2}(\beta_{R}/2) \). The second kind of resonances corresponds to electron transfers between the two dots, in the same \( \hat{n}_{R} + \hat{n}_{L} \) subspace. For instance, line \( c \) involves resonances between DQD states \( (\sigma,0) \) and \( (0,\sigma) \), with \( \sigma \in \{\uparrow, \downarrow\} \). It can be approximated by \( \Pi(0) \sim -2\alpha_{L}^{2} / 4 \cos^{2}(\beta_{R}/2) \), which recalls Eq. (8). Line \( d \) corresponds to a resonance between \( (0,\uparrow,\downarrow) \), \( (\downarrow,\downarrow) \) and \( (\downarrow,\uparrow) \). It can be approximated by \( \Pi(0) \sim -4\alpha_{L}^{2} / 4 \cos^{2}(\beta_{R}/2) \) with \( \Delta_{d} = \sqrt{(E_{d0} - E_{d1})^{2} + 4\alpha_{L}^{2} / 4} \), and \( E_{d0} - E_{d1} = \epsilon_{R} - \epsilon_{L} + 2E_{c}^{R} - U_{m} \). The above expressions again do not involve the values of the tunnel rates due to \( \Gamma_{L(R)} \ll k_{B} T \).

Along line \( c \), \( \Pi(0) \) reaches a maximum which is \( \sqrt{2} \) higher than along line \( d \), because lines \( c \) and \( d \) involve resonances between a different number of states. For \( \omega \gg \Gamma_{L(R)} \), using a master equation approach, we find \( \Pi(\omega) \sim -\Pi(0)\Delta_{c(d)}^{2} / (\omega^{2} - \Delta_{c(d)}^{2}) \) along line \( c(d) \). The finite frequency behavior of \( \Pi(\omega) \) will be discussed in a more complete way elsewhere.

We now discuss the effect of a Zeeman field on \( \Pi(0) \) (see Fig. 3). We use \( B > 0 \) so that \( \uparrow \) spins have a lower energy. Lines of type \( a \) or \( b \) are shifted by \( B \) because they now correspond essentially to a transfer of \( \uparrow \) spins between the dots and leads. However, their height is almost not modified (except too close to triple points). For a magnetic field \( g\mu_{B}B \sim t \), \( \Pi(0) \) cancels in a region where line \( d \) was formerly extending (see Fig. 3b). This is because in this area, the state \( (\uparrow,\uparrow) \) becomes the most stable state, and therefore charge fluctuations between the two dots become impossible. This effect represents a near-equilibrium version of Pauli spin blockade [22]. As a result, line \( d \) is shifted to higher (lower) values of \( n_{g}^{R} \) (\( n_{g}^{L} \)), and it reaches a higher maximum which depends strongly on \( T \). Indeed, we obtain for \( g\mu_{B}B \ll E_{c}^{L(R)}, U_{m} \)

\[
-\Pi(0) \sim \frac{8t^{2} + e^{-\beta(\Lambda - g\mu_{B}B)}(\Delta_{c}^{2}\Lambda + 2t^{2}(2 - \Delta_{d}))}{\hbar(\alpha_{L} - \alpha_{R})^{2}} \sqrt{2} 
\]

with \( \Lambda = (E_{d1} - E_{d2} + \Delta_{d})/2 \). In contrast, line \( c \) is not affected by a magnetic field \( g\mu_{B}B \sim t \). Line \( c \) is affected by \( B \) once \( (\uparrow,\uparrow) \) becomes the most stable state near \( n_{g}^{R} = n_{g}^{L} = 0.5 \), which occurs only for higher values of magnetic field \( g\mu_{B}B \sim U_{m} \) (not shown).

To conclude this first part, mesoscopic admittance measurements appear as an interesting alternative to charge sensing [21, 22], for performing the spectroscopy of quasi-closed multi-quantum-dot systems. We have mainly discussed the \( \Gamma_{L(R)} \ll k_{B} T \) limit. The fronts between the different \( (n_{L},n_{R}) \) domains can be seen in \( \Pi(0) \). In the interacting case, the parity of the DQD total occupation number can be determined directly from the difference of amplitude between lines of type \( c \) and \( d \) obtained at \( B = 0 \), or from the spin blockade effect obtained for \( B \not= 0 \). The DQD mesoscopic admittance also gives a direct access to information on the DQD spin state, since spin singlet and triplet states can be discriminated using spin blockade. At high frequencies \( \omega \gg \Gamma, \Pi(\omega) \) shows resonances due to the internal dynamics of the DQD. We have disregarded spin and orbital relaxation effects (with rates denoted \( \Gamma_{s/o}^{r} \)), which can be due e.g. to magnetic impurities, spin-orbit coupling, or phonons. However, assuming \( \Gamma_{s/o}^{r} \ll \Gamma_{L(R)} \), the results presented here [Eqs. (7) to (9)] will not be affected for \( \omega \) much smaller or much larger than \( \Gamma_{s/o}^{r} \). For an intermediary value of \( \omega \), the expression of \( \Pi(\omega) \) can involve explicitly \( \Gamma_{s/o}^{r} \) and \( \Gamma_{L(R)} \).

We now consider an experiment where the DQD is con-
connected through $C_{ac}^L$ and $C_{ac}^R$ to an external $(L_r, C_r)$ circuit, which is a simple model for a photonic resonator. The full circuit hamiltonian $H_2$ includes terms in $(C_r/2)\hat{V}_r^2 + (1/2L_r)\hat{\phi}_r^2$ and $\lambda_1 \hat{V}_{ac} + \lambda_2 \hat{V}_{ac}^2$ due to the resonator and DQD respectively, with $\hat{\Phi}_{ac}$ the flux operator through the inductance $L_r$ and $\hat{V}_{ac}$ the operator associated to $V_{ac}$. We define the charge operator conjugated to $\hat{\Phi}_{ac}$ as $\hat{Q}_{ac} = \hat{V}_{ac}/C_r$, $\hat{C}_{ac}' = C_c + 2\lambda_2$ and the photon annihilation operator $\hat{a} = -i\sqrt{2\hbar Z_r} \hat{\Phi}_{ac} + \sqrt{Z_r/2\hbar} \hat{Q}_{ac}$, with $Z_r = \sqrt{L_r/C_r'}$. We assume that the photon resonators are coupled to an external photonic bath corresponding to the annihilation operator $\hat{b}$.[24] We finally have

$$\hat{H}_2 = \hat{H}_{DQD} + eV_{rms} \sum_a k \hat{a}^\dagger \hat{a} + \lambda_1 \hat{V}_{rms}(\hat{a} + \hat{a}^\dagger) + \hbar \omega_r' \hat{a}^\dagger \hat{a} + \sum_p \hbar \omega_p \hat{b}^\dagger \hat{b} + \sum_p (\tau \hat{b}^\dagger \hat{a} + \tau^* \hat{a}^\dagger \hat{b}) + (\kappa e^{i\omega_{ac}t} + \kappa^* e^{-i\omega_{ac}t}) + \hat{H}_I$$

(10)

with $\omega_r' = 1/\sqrt{L_r C_r'}$ and $V_{rms} = \sqrt{\hbar \omega_r'/2C_r'}$. The terms in $\kappa$ account for an external driving of the resonator at frequency $\omega_d/2\pi$. For simplicity, we study the response of the resonator through its mean value. The linear response theory gives $\Delta V_{ac}(t) = \mathbb{E}[G_{\hat{a}^\dagger \hat{a}}(\omega_d) \kappa e^{i\omega_{ac}t}]$ with $G_{\hat{a}^\dagger \hat{a}}(t) = -i\theta(t) \langle [\hat{A}(t), \hat{B}] \rangle_{\kappa=0}$. We can relate $G_{\hat{a}^\dagger \hat{a}}$ and $G_{\hat{a}^\dagger \hat{a}^\dagger} \hat{a}^\dagger$ to $\hat{X}_{\alpha d', t}$ by using equations of motion approach, which takes into account the stationarity of $\hat{X}_{\alpha d', t}$. We assume that the self-energy terms $\sum_p |\tau_p|^2/(\hbar \omega_d + \hbar \omega_r + i\Delta)$ due to the coupling to the outer photonic bath write $-i\hbar \Lambda$, with $\Lambda > 0$, to account simply for the finite quality factor of the resonator. We obtain the exact relation $G_{\hat{a}^\dagger \hat{a}}(\omega) = G_0 + G_0 \omega^2_{rms} \Pi(\omega) G_0$ with $G_0 = (\omega - \omega_r' + i\Delta)^{-1}$, $\Pi(\omega) = \sum_{d', \alpha} \alpha_d \delta_d' \hat{X}_{\alpha d', t}(\omega)$ and $\omega_{rms} = V_{rms}/\hbar$. Using an analogous expression for $G_{\hat{a}^\dagger \hat{a}}$ and assuming $\Lambda \ll \omega_r'$, one finds $G_{\hat{a}^\dagger \hat{a}^\dagger} \hat{a}^\dagger \simeq G_{\hat{a}^\dagger \hat{a}}$. To find the poles of $G_{\hat{a}^\dagger \hat{a}}$, a self-consistent approach is necessary[28][29]. We postulate a CPA-like approximation $G_{\hat{a}^\dagger \hat{a}}(\omega) = G_0 - G_0 \omega_{rms}^2 \Pi(\omega) G_0$, which yields

$$\Pi^{-1}(\omega) = G_0^{-1} - \omega^2_{rms} \Pi(\omega)$$

(11)

In the limit where $\hbar \omega_r$ and $\hbar \omega^2_{rms} \Pi(0)$ are both much smaller than the energy scales involved in the DQD dynamics, Eq. (11) gives a dispersive shift of the photonic resonance frequency, i.e. $\omega'_r \approx \omega' + \omega^2_{rms} \Pi(0)$. This result can be recovered by considering a classical parallel $(L_r, C_r)$ circuit in parallel with a capacitance $2\lambda_2 - (e^2/\hbar)\Pi(0)$ following from Eq. (9). Indeed, assuming $\omega^2_{rms} \Pi(0) \ll \omega_r$, we expect free oscillations with a frequency $(L_r/C_r - (e^2/\hbar)\Pi(0))^{-1/2} \approx \omega'_r$. For larger values of $\omega_r$, in the general case, the response of the resonator is not simply given by $\Pi(\omega)$ but by the functional form of $\Pi(\omega)$ [and thus $G(\omega)$]. For instance, let us use the resonant form $\Pi(\omega) \approx \Omega/(\omega^2 - \Delta^2)$ obtained previously. One expects an anticrossing effect when the photonic resonator becomes resonant with the DQD. From Eq. (11), we indeed obtain $\omega'^{\text{tot}}_{r, \pm} = (\Delta + \omega'_r/2) \pm \sqrt{A + (\Delta - \omega'_r)^2/4}$ with $A = (\Omega^2_{rms})/(\Delta + \omega'_r)$.

In the non-interacting case, the CPA-like approximation of $G_{\hat{a}^\dagger \hat{a}}$ can be justified by using a standard diagrammatic perturbation theory in $\alpha_{L(R)}$. For each order in $\alpha_{L(R)}$, the contribution to $G_{\hat{a}^\dagger \hat{a}}$ corresponding to a series of “bubble” diagrams must be dominant. In principle, an estimation of diagrams at fourth order in $\alpha_{L(R)}$ already provides a good indication on the validity of the CPA scheme[30]. From a dimensional analysis, the CPA-like development of $G_{\hat{a}^\dagger \hat{a}}$ is valid at least in the regime $T = 0$ with $\Lambda, E_\pm, \hbar \omega_r', \hbar \omega^\text{tot}_r - \hbar \omega'_r \ll \Gamma$. Considering the relevance of the results given by Eq. (11), the CPA scheme is probably valid in a much wider range of parameters. However, from the fourth order diagrams, it seems crucial to have $\omega'^{\text{tot}}_{r} - \omega'_r$ and $\Lambda$ small, and $\Gamma$ finite, this assertion being difficult to define quantitatively in the general case[31].

As a conclusion for this second part, we have discussed the behavior of a high finesse photonic resonator coupled to a DQD. When photonic correlation functions can be developed along a CPA-like scheme, both the dispersive and resonant behaviors of the resonator reveal information on the DQD admittance.

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[1] G. Platero and R. Aguado, Phys. Rep. 395, 1 (2004).
[2] J. Gabelli, et al., Science 313, 499 (2006). G. Fèvre, et al., Science 316, 1169 (2007).
[3] M. Büttiker, H. Thomas, A. Prêtre, Phys. Lett. A 180, 364 (1993); A. Prêtre, H. Thomas, M. Büttiker, Phys. Rev. B 54, 8130 (1996).
[4] S. E. Nigg, R. Lopez, and M. Büttiker, Phys. Rev. Lett. 97, 206804 (2006); Z. Ringel, Y. Inny, and O. Entin-Wohlman, Phys. Rev. B 78, 165304 (2008).
[5] Y. Hamamoto et al. Phys. Rev. B 81, 153305 (2010); C. Mora and K. Le Hur, Nature Physics 6, 697 (2010).
[6] J. Splettstoesser et al., Phys. Rev. B 81, 165318 (2010).
[7] A. Blais et al. Phys. Rev. A 69, 062320 (2004).
[8] A. Wallraff et al. Nature 431, 162 (2004).
[9] M. A. Sillanpää et al., Phys. Rev. Lett. 95, 206806 (2005), T. Duty et al., Phys. Rev. Lett. 95, 206807 (2005).
[10] A. Wallraff et al., Nature 431, 162 (2004).
[11] G. Burkard et al., Phys. Rev. B 59, 2070 (1999).
[12] J.R. Petta et al., Science 309, 2180 (2005).
[13] T. Hayashi et al. Phys. Rev. Lett. 91, 226804 (2003).
[14] L. Childress, A. S. Sørensen, and M. D. Lukin, Phys. Rev. A 69, 042302 (2004).
[15] A. Cottet and T. Kontos, Phys. Rev. Lett. 105, 160502 (2010).
[16] K. D. Petersson et al., Nano Lett., 10, 2789 (2010).
[17] R.-P. Rwar and J. Splettstoesser, Phys. Rev. B 82, 205308 (2010).
[18] R. Ziegler, C. Bruder, and Herbert Schoeller, Phys. Rev.
B 62, 1961 (2000).
[19] K. Ono et al., Science 297, 1313 (2002).
[20] H. O. H. Churchill et al., Phys. Rev. Lett. 102, 166802 (2009).
[21] W. G. van der Wiel et al., Rev. Mod. Phys. 75, 1 (2002).
[22] We use \( E_{c}^{L(R)} = \frac{C_{c}^{R(L)} e^2}{2D} \), \( U_m = \frac{C_{m} e^2}{D} \), \( D = C_{\Sigma}^{L} C_{\Sigma}^{R} - C_{m}^{2} \), \( \alpha_{L(R)} = \frac{-C_{c}^{L(R)} C_{\Sigma}^{R(L)} + C_{c}^{R(L)} C_{m}}{D} \), \( \lambda_2 = \sum_d (1 + \alpha_d) C_{uc}^{d}/2 \), \( C_{uc}^{d} = C_{uc}^{d} + C_{uc}^{d} + C_{uc}^{d} + C_{m} \). We use \( e > 0 \) and \( A(\omega) = \int_{-\infty}^{+\infty} A(t) \exp(i\omega t) dt \).
[23] In the non-interacting limit, one must assume that geometric capacitances have a negligible contribution to \( G(\omega) \approx i\omega \left( \frac{e^2}{\hbar} \right) \Pi(\omega) \). In principle, one can account for geometric capacitances by treating \( \hat{H}_{DQD} \) at the Hartree level. In the Coulomb-blockade limit, our treatment fully takes into account geometric capacitances.
[24] We note \((s_L, s_R)\) a DQD charge state with dot \( d \in \{ L, R \} \) in the occupation state \( s_d \in \{ 0, 1, 2 \} \), or \( \{ 0, \uparrow, \downarrow, \uparrow\downarrow \} \) if the spin state is specified.
[25] J. M. Elzerman et al., Nature 430, 431 (2004); Y. Hu et al., Nature Nanotech. 2, 622 (2007).
[26] A. A. Clerk et al., Rev. Mod. Phys. 82, 1155 (2010).
[27] A. Blais et al., Phys. Rev. A 75, 032329 (2007).
[28] A perturbative treatment at lowest order in \( \alpha_{L(R)} \) would give the absurd result \( \omega = \omega'_{r} \).
[29] J. Skoldberg et al., Phys. Rev. Lett. 101, 087002 (2008).
[30] D. F. Urban, R. Avriller, A. Levy Yeyati, Phys. Rev. B 82, 121414 (2010).
[31] If the photon and electron linewidths \( \Lambda \) and \( \Gamma \) both vanish, all fourth order diagrams diverge like \( (\omega - \omega'_{r})^{-3} \). If \( \Gamma \) remains finite while \( \Lambda \) vanishes, the double bubble diagram (DBD) keeps a divergence in \( (\omega - \omega'_{r})^{-3} \), while the others diverge like \( (\omega - \omega'_{r})^{-2} \). Therefore, we expect that for \( \Gamma \) sufficiently large, and \( \omega'_{r} - \omega'_{r} \) sufficiently small, the DBD will be the dominant fourth order contribution to \( G_{a\alpha}^{-1}(\omega'_{r}) \).