CONTROL OF SQUEEZED LIGHT PULSE SPECTRUM IN THE KERR MEDIUM WITH AN INERTIAL NONLINEARITY

A.S. Chirkin and F. Popescu
Moscow State University,
119899, Moscow, Russia
chirkin@foton.ilc.msu.su  florentin.p@hotmail.com

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Abstract The consistent quantum theory of self-phase modulation (SPM) and cross-phase modulation (XPM) for ultrashort light pulses (USPs) in medium with electronic Kerr-nonlinearity are developed. The approach makes use of momentum operator of electrical field which takes account of the inertial behaviour of the nonlinearity. The spectrum of quantum fluctuations of squeezed-quadrature component as a function of response time of nonlinearity and values of nonlinear phase shifts due to the SPM and XPM effects, is investigated.

1. INTRODUCTION

There are some methods to obtain USPs in a nonclassical state. One is the degenerate three-frequency parametric amplification which has high phase sensitivity. Other is SPM in a Kerr-nonlinear medium which in comparison with parametrical processes, does not require phase-matching. This property is a real advantage for the pulsed nonclassical state formation. The both participation of SPM and dispersion in the nonlinear medium leads to the optical soliton formation [1]. The quantum theory of SPM and XPM for USPs developed during last 15 years have difficulties produced, in particular, by the multi-frequency structure of USPs and nonlinear kind of their interaction.

The aim of our study is to develop the consistent quantum theory of nonlinear propagation of USPs. An account is taken of the role of a response time of the nonlinearity. The first attempt has been made in [2] to give quantum analysis of SPM of USPs for the case of relaxation nonlinearity of medium, and has been noted that the correct quantum
theory should take account of additional noise sources connected with nonlinear absorption. However, the theory developed in [2] does not take account of noise sources, so that the commutation relation for annihilation and creation photon operators is not fulfilled. This was carried out in the developed in [3] quantum theory of SPM, where noise sources are considered as a fluctuation addition to the relaxation nonlinearity of medium. Therefore, the results of [3] are connected with the range of carrier frequencies of the pulse, where the nonlinear absorption is important. Actually the authors [3] have developed the quantum theory of USP propagation in a Raman active medium. In [3] the electronic Kerr nonlinearity with finite response time is modeled as the Raman active medium. Note, if we deal with the USP's propagation, for example, through fused-silica fibres only about 0.2 of the Kerr effect is attributable to the Raman oscillators and about 0.8 of the Kerr effect is due to electronic motion [4]. The case when the USP carrier frequency is far enough from some resonances (one- and two-photon and the Raman resonances), and therefore absorption is absent, was investigated by us in [5] where the finite response time is considered in the interaction Hamiltonian, and the commutation relation is exactly fulfilled.

In the present work the results of the consistent quantum theory of SPM and XPM of USPs based on the use of the momentum operator (quantity of movement) are presented. The developed approach can be used when the response time is much shorter than the pulse duration and dispersion in nonlinear medium is neglected. There are no restrictions on the pulse intensity in our theory.

2. QUANTUM THEORY OF SPM OF USPs

The traditional way to describe the SPM of USP is based on the interaction Hamiltonian use, at that usually we solve the time-evolution equation. The transition to the spatial-evolution equation is realized substituting \( t \) with \( z/u \), where \( z \) is the distance passed in medium and \( u \) is the group velocity. This approach seems to be good enough for single-mode radiation. If we deal with nonlinear propagation of USP then both \( t \) and \( z \) are present in analytical description. Thus, we use the momentum operator of pulse field related to the space-evolution [6].

We describe the SPM of USP using the momentum operator (cf. [5])

\[
\hat{G}_{spm}(z) = \frac{1}{2} \hbar \beta \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} H(t - t_1) \hat{N} [\hat{n}(t, z)\hat{n}(t_1, z)] \, dt_1, \tag{1.1}
\]

where \( \hat{n}(t, z) = \hat{A}^+(t, z)\hat{A}(t, z) \) is the Bose operator creating (annihilating) photons in a given cross-section \( z \) of the medium.
at a given time \( t \), \( \hbar \) is the Planck’s constant and \( \hat{N} \) is the operator of normal ordering. The coefficient \( \beta \) is defined by the Kerr nonlinearity of a medium at stationary conditions. \( H(t) \) is the function of the nonlinear response of a medium; \( H(t) \neq 0 \) at \( t \geq 0 \) and \( H(t) = 0 \) at \( t < 0 \). The expression under the first integral in (1.1) can be interpreted as a generalized force acting in a defined cross-section \( z \), which at the moment of time \( t \) depends only on the previous ones \( (t_1 \leq t) \), i.e. it satisfies the causality principle. The nonlinear response function of a medium should be introduced as:

\[
H(t) = \left(1/\tau_r\right)e^{-t/\tau_r} \quad (t \geq 0),
\]

where \( \tau_r \) is the response time of nonlinear medium, that we assume to be much shorter than the pulse duration \( \tau_p \). This nonlinearity takes place in absence of the one- and two-photons absorption and Raman resonances [1]. Besides, as it will be shown below, also in this limit case the account of finite nonlinear response time plays an important role.

The space evolution equation for \( \hat{A}(t,z) \) follows from (see [6])

\[
-i\hbar \frac{\partial \hat{A}(t,z)}{\partial z} = \left[ \hat{A}(t,z), \hat{G}_{spm}(z) \right],
\]

and making use of (1.1), we finally get

\[
\frac{\partial \hat{A}(t,z)}{\partial z} - i\frac{1}{2} \beta q[\hat{n}(t,z)] \hat{A}(t,z) = 0,
\]

where

\[
q[\hat{n}(t,z)] = \int_{-\infty}^{\infty} H(t_1) [\hat{n}(t-t_1,z) + \hat{n}(t+t_1,z)] \, dt_1.
\]

Here \( \hat{n}(t,z) = \hat{A}^+(t,z)\hat{A}(t,z) \) is the photon number “density”. The Eq. (1.4) describes SPM in the moving coordinate frame: \( z = z', t = t' - z/u \), where \( t' \) is the running time. According to (1.4) the operator \( \hat{n}(t,z) \) does not depend on \( z \), i.e. \( \hat{n}(t,z) = \hat{n}(t,0) = \hat{n}_0(t) = \hat{A}_0^+(t)\hat{A}_0(t) \) \( (z = 0 \) corresponds to the entrance into the medium). Taking account of this fact \( q[\hat{n}(t,z)] = q[\hat{n}_0(t)] \), and (1.5) can be rewritten as:

\[
q[\hat{n}_0(t)] = \int_{-\infty}^{\infty} h(t_1)\hat{n}_0(t-t_1) \, dt_1, \quad (h(t) = H(|t|)).
\]

Taking account of (1.6), we get the solution of (1.4)

\[
\hat{A}(t,z) = \exp \{i\gamma q[\hat{n}_0(t)]\} \hat{A}_0(t).
\]
For hermitian conjugate operator of $\hat{A}(t, z)$ we have
\[ \hat{A}^+(t, z) = \hat{A}^+_0(t) \exp \{ -i\gamma q [\hat{n}_0(t)] \}. \]  
(1.8)

In expressions (1.7), (1.8) $\gamma = \beta z/2$. For time-independent operator $\hat{n}_0$, Eqs. (1.7), (1.8) give us the results for single mode radiation [7]. In the case of the instantaneous response of the nonlinearity $H(t) = \delta(t)$ we get [2]
\[ \hat{A}(t, z) = e^{i2\gamma \hat{n}_0(t)} \hat{A}_0(t), \quad \hat{A}^+(t, z) = \hat{A}^+_0(t) e^{-i2\gamma \hat{n}_0(t)}. \]  
(1.9)

At the input to the medium Bose operators satisfy the commutation relation $[\hat{A}_0(t_1), \hat{A}^+_0(t_2)] = \delta(t_1 - t_2)$. In a correct quantum methodology the analogical commutation relation must also be satisfied in nonlinear medium, i.e. for any $z$
\[ [\hat{A}(t_1, z), \hat{A}^+(t_2, z)] = \delta(t_1 - t_2). \]  
(1.10)

The calculation of average values of (1.9) and the normal ordering are followed by the non-integrable singularities appearance (for instance, $\delta(t)$ appears in the exponent). These shortcomings are absent using (1.7) and (1.8). Besides, their use requires knowledge of an algebra of time-dependent Bose operators. The later has been developed in [2, 8]. The following permutation relations [8]
\[ \hat{A}_0(t_1) e^{\hat{O}(t_2)} = e^{\hat{O}(t_2)+i\gamma h(t_2-t_1)} \hat{A}_0(t_1), \]  
(1.11)
\[ e^{\hat{O}(t_2)} \hat{A}^+_0(t_1) = \hat{A}^+_0(t_1) e^{\hat{O}(t_2)+i\gamma h(t_2-t_1)}, \]  
(1.12)

(where $\hat{O}(t) = i\gamma q[\hat{n}_0(t)]$) and the theorem of normal ordering [9]
\[ e^{\hat{O}(t)} = N \exp \left\{ \int_{-\infty}^{\infty} \left[ e^{i\gamma h(\theta)} - 1 \right] \hat{n}_0(t - \tau, \theta) d\theta \right\}, \]  
(1.13)

are valid, where $\tilde{h}(\theta) = e^{-|\theta|}$, $\theta = t/\tau$. As $[\hat{O}(t_1), \hat{O}(t_2)] = 0$, then $e^{\hat{O}(t_1)} e^{\hat{O}(t_2)} = e^{\hat{O}(t_1)+\hat{O}(t_2)}$. Making use of relations (1.11) and (1.12), one can verify that the commutation relation (1.10) is exactly fulfilled.

As we analyse the SPM of an initial coherent USP, the equation on eigenvalues $\hat{A}_0(t) |\alpha(t)\rangle = \alpha(t) |\alpha(t)\rangle$ is satisfied by $\hat{A}_0(t)$ (see [1]), where $|\alpha(t)\rangle$ is the initial coherent state, $\alpha(t)$ is the eigenvalue of $\hat{A}_0(t)$, and $|\alpha(t)|^2 = \hat{n}_0(t)$ is the average photon density. Particular attention is given to the quantum fluctuations behaviour of $X$-quadrature; $\hat{X}(t, z) = [\hat{A}(t, z) + \hat{A}^+(t, z)]/2$. The behaviour of another quadrature component is shifted in phase with $\pi/2$. For correlation function (see [5])
\[ R(t, \tau) = \langle \hat{X}(t, z) \hat{X}(t+\tau, z) \rangle - \langle \hat{X}(t, z) \rangle \langle \hat{X}(t+\tau, z) \rangle, \]  
(1.14)
where the brackets denote averaging over the initial coherent state of the pulse, making use of (1.11)-(1.13) we get

\[ R(t, \tau) = \frac{1}{4} \left[ \delta(\tau) - \psi(t) h(\tau) \sin 2\Phi(t) + \psi^2(t) g(\tau) \sin^2 \Phi(t) \right]. \]  

In (1.15) the following notations have been introduced: \( \Phi(t) = \psi(t) + \varphi(t) \), \( \varphi(t) = \arg \alpha(t) \) - the phase of USP or the heterodyne pulse’s phase at balanced homodyne detection, \( \psi(t) = 2\gamma \bar{n}_0(t) = \psi_0 \rho^2(t) \) - the nonlinear phase shift resulted from SPM, \( \psi_0 = \psi(0) = 2\gamma \bar{n}_0 \) - the maximum nonlinear phase shift, \( \rho(t) \) - the envelope of USP (\( |\alpha(t)| = \alpha_0 \rho(t) \), \( \rho(0) = 1 \)), and \( g(\tau) = \int_{-\infty}^{\infty} \tilde{h}(\theta) \tilde{h}(\theta + \tau) d\theta \). The derivation of (1.15) took into consideration the fact that, in many experimental situations nonlinear phase shift per photon \( \gamma \ll 1 \).

The instantaneous spectral density of the quantum fluctuations of the X-quadrature component \( S_X(\omega, t) = \int_{-\infty}^{\infty} R_X(t, \tau) e^{i\omega \tau} d\tau \), according to (1.15), takes the form

\[ S_X(\Omega, t) = \frac{1}{4} \left[ 1 - 2\psi(t) L(\Omega) \sin 2\Phi(t) + 4\psi^2(t) L^2(\Omega) \sin^2 \Phi(t) \right], \]  

where \( L(\Omega) = 1/(1 + \Omega^2) \), \( \Omega = \omega \tau_r \). From (1.16) it follows that the quantum fluctuations level below the short noise one depends on the nonlinear phase shift \( \psi(t) \) and \( \Phi(t) \). At the initial phase of the pulse chosen optimal for a frequency \( \Omega_0 = \omega_0 \tau_r \)

\[ \varphi_0(t) = \frac{1}{2} \arctan \left( \frac{1}{\psi(t)L(\Omega_0)} \right) - \psi(t), \]  

the spectral density (1.16) is

\[ S_X(\Omega_0, t) = \frac{1}{4} \left\{ \left[ 1 + \psi^2(t) L^2(\Omega_0) \right]^{1/2} - \psi(t) L(\Omega_0) \right\}^2. \]  

From (1.18) it follows that the spectral density adiabatically changes itself with the changing pulse’s envelope and it is lower than 1/4 which corresponds to the coherent state of the pulse. From (1.18) we can also see that at the optimal phase \( \varphi_0(t) \) with the increasing of the nonlinear phase shift \( \psi(t) \), the spectral density \( S_X(\Omega_0, t) \) monotonously decreases.

We point out that at \( \tau_r = 0 \) from (1.16) the results for single mode radiation can be obtained, at that the quantum fluctuation spectrum’s level does not depend on the frequency. According to (1.18), increasing \( \tau_r \) (increasing \( \Omega_0 \)) at fixed frequency \( \omega_0 \), the quadrature fluctuations level increases.

**3. QUANTUM THEORY OF XPM OF USPs**

We analyse now two-pulse propagation in an inertial nonlinear medium. We will consider that pulses have orthogonal polarizations or/and different frequencies. Then, besides SPM of each pulse, the XPM effect takes
Here we assume that the parametrical interaction of pulses can be neglected. In this case, the analysed process can be depicted making use of the following momentum operator \([10]\).

\[
\hat{G}(z) = \sum_{j=1}^{2} \hat{G}_{spm}^{(j)}(z) + \hat{G}_{xpm}^{(1,2)}(z). \tag{1.19}
\]

Here \(\hat{G}_{spm}^{(j)}(z)\) is due to the SPM of \(j\)-pulse \((j = 1, 2)\) (see 1.1) and operator \(\hat{G}_{xpm}^{(1,2)}(z)\) is connected with the XPM of pulses

\[
\hat{G}_{xpm}^{(1,2)}(z) = \hbar \bar{\beta} \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} H(t-t_1)[\hat{n}_1(t,z)\hat{n}_2(t_1,z) + \hat{n}_2(t,z)\hat{n}_1(t_1,z)]dt_1. \tag{1.20}
\]

In (1.20) \(\hat{n}_j(t,z)\) is the photon number operator of \(j\)-pulse and coefficients \(\beta_j, \bar{\beta}\) are responsible for the SPM and XPM respectively. As stated earlier, the operator \(\hat{n}_j(t,z)\) commutes with \(\hat{G}(z)\). In consequence \(\hat{n}_j(t,z) = \hat{n}_j(t,z = 0) = \hat{n}_{0,j}(t)\), which means that the photons statistics of each pulse remains unchanged in the nonlinear medium. In accordance with (1.3), the evolution of \(\hat{A}_1(t,z)\) for the first pulse is given by (cf. (1.4))

\[
\frac{\partial \hat{A}_1(t,z)}{\partial z} - i \left\{ \frac{1}{2} \beta_1 q [\hat{n}_{0,1}(t)] + \bar{\beta} q [\hat{n}_{0,2}(t)] \right\} \hat{A}_1(t,z) = 0. \tag{1.21}
\]

One can get for \(\hat{A}_2(t,z)\) the similar equation changing the index 1 \(\mapsto\) 2 in (1.21). The functions \(q[\hat{n}_{0,j}(t)]\) is similar to (1.6). We remark that (1.4), (1.21), are written in the moving frame. The solution of (1.21) is

\[
\hat{A}_1(t,z) = e^{i \gamma_1 q[\hat{n}_{0,1}(t)] + i \bar{\gamma} q[\hat{n}_{0,2}(t)]} \hat{A}_{0,1}(t), \tag{1.22}
\]

where \(\gamma_1 = \beta_1 z/2, \bar{\gamma} = \bar{\beta} z\). We define the correlation function of the investigated light pulse as (1.14). For the spectra of quantum fluctuations of \(X_1\)-quadrature we get

\[
S_{X_1}(\Omega, t) = \frac{1}{4} \left\{ 1 - 2\psi_1(t) L(\Omega) \sin 2\Phi_{1,2}(t) + 4[\psi_1^2(t) + \bar{\psi}_1(t) \bar{\psi}_2(t)] L^2(\Omega) \sin^2 \Phi_{1,2}(t) \right\}. \tag{1.23}
\]

Here we denoted: \(\Phi_{1,2}(t) = \Phi_1(t) + \bar{\psi}_2(t), \Phi_1(t) = \psi_1(t) + \varphi_1(t), \psi_1(t) = 2\gamma_1 n_{0,1}(t), \bar{\psi}_j(t) = 2\bar{\gamma} n_{0,j}(t) (j = 1, 2)\). As already mentioned, the phase \(\Phi_1(t)\) is connected with own parameters of the investigated pulse, and the phase \(\bar{\psi}_1(t)\) with the XPM. From comparison of (1.23) and (1.16)
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one can see that the XPM adds new terms in the multiplier and in the phase in (1.23). This circumstance allows us to control the fluctuations spectrum of the investigated pulse. At the initial phase

$$\varphi_{0,1}(t) = \frac{1}{2} \arctan \left[ \frac{\psi_1(t)}{\psi_{1,2}(t) L(\Omega_0)} \right] - \psi_1(t) - \bar{\psi}_2(t),$$

(1.24)

chosen optimal for frequency $$\Omega_0 = \omega_0 \tau_r$$, the spectral density (1.23) is

$$S_{X1}(\Omega_0, t) = \frac{1}{4} \left\{ 1 - 2 L(\Omega_0) \left[ \psi_1^2(t) + \psi_{1,2}^2(t) L^2(\Omega_0) \right] \right\}^{1/2} \psi_{1,2}(t) L(\Omega_0),$$

(1.25)

where $$\psi_{1,2}^2(t) = \psi_1^2(t) + \bar{\psi}_1(t) \bar{\psi}_2(t)$$. The influence of the second pulse to the spectrum of the first one is depicted in Figure 1. Increasing the photon number (intensity) of the second pulse, their quantum fluctuations can substantially increase the level of this ones for the investigated pulse. Changing the intensity of the second pulse, one can control the spectrum of quantum fluctuations of the the investigated pulse’s quadrature.

4. CONCLUSIONS

The main result of our work is represented by the development of the consistent quantum theory of SPM and XPM of USPs in non-absorption nonlinear media, when the carrier frequencies of the pulse is far enough from any resonances. However, the consideration of the finite response
time of nonlinearity is necessary to get correct solutions ((1.7), (1.8), (1.22)). Our results are valid for response time of nonlinearity much shorter than duration of pulses but for unspecified intensities of USPs. In some sense, our results are complementary with the ones presented in [3] which treats the situation when the Raman resonance is important. The quantum theory of XPM of USPs in the developed approach here is presented for the first time. We have shown that the form and the level of the fluctuation spectrum can be controlled with the change of the pulse’s phase and the another pulse’s intensity in the presence of XPM. This fact can be used for quantum non-demolition measurements of parameters of the pulse [11].

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