Noncommutative Unification of General Relativity and Quantum Mechanics. A Finite Model.

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Abstract

We construct a model unifying general relativity and quantum mechanics in a broader structure of noncommutative geometry. The geometry in question is that of a transformation groupoid \( \Gamma \) given by the action of a finite group on a space \( E \). We define the algebra \( A \) of smooth complex valued functions on \( \Gamma \), with convolution as multiplication, in terms of which the groupoid geometry is developed. Owing to the fact that the group \( G \) is finite the model can be computed in full details. We show that by suitable averaging of noncommutative geometric quantities one recovers the standard space-time geometry. The quantum sector of the model is explored in terms of the regular representation of the algebra \( A \), and its correspondence with the standard quantum mechanics is established.

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1 Introduction

There are many attempts to create a quantum gravity theory, or at least some kind of unification of general relativity and quantum mechanics, based on noncommutative geometry (see, for example, [1, 14, 2, 9, 10]). In a series of works [4, 5, 6, 7] we have also proposed a model aimed at a unification of relativity and quanta which differs from other models of this type by the ample use of the groupoid concept. We consider a transformation groupoid $\Gamma = E \times G$, where $E$ is typically the frame bundle over space-time $M$ and $G$ a group acting on $E$, and the noncommutative algebra $A$ of compactly supported complex valued functions on $\Gamma$ with convolution as multiplication. As a next step we develop the geometry of the groupoid based on the algebra $A$ and its derivations. The $E$-component of this geometry reconstructs standard general relativity, and its $G$-component is interpreted as a quantum sector of the model. The natural choice for $G$ is the Lorentz group or some of its representations.

Preliminary results indicate that the model is worth exploring as a possible step in the right direction. However, it is involved in many mathematical intricacies that often overshadow interconceptual relations, and at this stage exactly these relations are especially important. It turns out that if $G$ is assumed to be a finite group, the model becomes “fully computable” and all conceptual issues are clarified. To construct such a model is the aim of the present work. Because of the finiteness of $G$ we call this model a finite model, although other its “components” remain infinite. In particular, $M$ can be any relativistic space-time.

We compute such a model in all its details. Some of our previous results have been confirmed, and some new have emerged. The most interesting aspects of the model concern the architecture of the groupoid geometry, the structure of Einstein equations, and dynamics in the quantum sector. The nice result is that the transition from the noncommutative geometry of our model to the classical space-time geometry can be done by averaging elements of the algebra $A$ in analogy to what is done in the usual quantum mechanics.

We begin our analysis with a brief reminder, in Section 2, of the transformation groupoid structure (mainly to fix notation). In Section 3, we discuss the noncommutative algebra $A$ on the transformation groupoid $\Gamma = E \times G$ where $G$ is a finite group and, in Section 4, we develop the geometry of the groupoid $\Gamma$ based on the algebra $A$ and the module of its derivations. As a
simple but instructive example we compute, in Section 5, the geometry of
the groupoid with $G = \mathbb{Z}_2$. Section 6 is devoted to establishing the corre-
spondence between the geometry of our model and the classical space-time
geometry via the above mentioned averaging procedure. In Section 7, we ex-
plore the quantum sector of our model in terms of the regular representation
of the algebra $\mathcal{A}$ and discuss its correspondence with quantum mechanics.
Main results are collected in Section 8.

2 A Transformation Groupoid

In this section we give a brief description of the groupoid structure mainly
to fix notation (for details see, for instance, [13, chapter 1]). Groupoid is a
set $\Gamma$ with a distinguished subset $\Gamma^2 \subset \Gamma \times \Gamma$, called the set of composable
elements, equipped with two mappings:

$\cdot : \Gamma^2 \rightarrow \Gamma$ defined by $(x, y) \mapsto x \cdot y$, called multiplication, and
$\cdot^{-1} : \Gamma \rightarrow \Gamma$ defined by $x \mapsto x^{-1}$ such that $(x^{-1})^{-1} = x$, called inversion.

These mappings have the following properties
(i) if $(x, y), (y, z) \in \Gamma^2$ then $(xy, z), (x, yz) \in \Gamma^2$ and $(xy)z = x(yz),
(ii) (y, y^{-1}) \in \Gamma^2$ for all $y \in \Gamma$, and if $(x, y) \in \Gamma^2$ then $x^{-1}(xy) = y$ and
$(xy)y^{-1} = x$.

One also defines the set of units $\Gamma^0 = \{ x^{-1}x : x \in \Gamma \} \subset \Gamma$, and the two
following mappings: $d, r : \Gamma \rightarrow \Gamma^0$ by $d(x) = x^{-1}x$, and $r(x) = xx^{-1}$, called the source mapping and the target mapping, respectively. Two elements $x$ and $y$ can be multiplied, i.e., $(x, y) \in \Gamma^2$, if and only if $d(x) = r(y)$. For each
$u \in \Gamma^0$ one defines the sets

$$\Gamma_u = \{ x \in \Gamma : d(x) = u \} = d^{-1}(u)$$

and

$$\Gamma''_u = \{ x \in \Gamma : r(x) = u \} = r^{-1}(u).$$

Both these sets give different fibrations of $\Gamma$.

The above purely algebraic construction can be equipped with the
smoothness structure. If this is the case, it is called a smooth or Lie groupoid
[13, chapter 2.3].

Let $\tilde{E}$ be a differential manifold (or a differential space, see [3]) with a
group $\tilde{G}$ acting on it smoothly and freely to the right, $\tilde{E} \times \tilde{G} \rightarrow \tilde{E}$. This
action leads to the bundle \((\tilde{E}, \pi_M, M = \tilde{E}/\tilde{G})\). The special case of this construction is the frame bundle over \(M\) with the Lorentz group \(\tilde{G}\) as its structural group. Let now \(G\) be a finite subgroup of \(\tilde{G}\), and let \(S : M \to \tilde{E}\) be a cross section of the bundle \((\tilde{E}, \pi_M, M)\). We do not assume that this cross section must be continuous (we simply chose one element of \(E\) from each fibre). Now, we define \(E = \bigcup_{x \in M} S(x)G\). We understand \(E\) as a differential space \((E, C^\infty(\tilde{E}))\).

\(G\) acts freely (to the right) on \(E\), \(E \times G \to E\), which gives the groupoid structure to the Cartesian product \(\Gamma = E \times G\). It is a special case of transformation groupoids and will constitute the subject matter of the present study. The elements of \(\Gamma\) are pairs \(\gamma = (p, g)\) where \(p \in E\) and \(g \in G\). Two such pairs \(\gamma_1 = (p, g)\) and \(\gamma_2 = (pg, h)\) are composed in the following way

\[
\gamma_2 \circ \gamma_1 = (pg, h)(p, g) = (p, gh).
\]

The inverse of \((p, g)\) is \((pg, g^{-1})\). The set of units is

\[
\Gamma^0 = \{\gamma^{-1} \gamma : \gamma \in \Gamma\} = \{(p, e) : p \in E\}.
\]

We could think of \(\gamma = (p, g)\) as of an arrow beginning at \(p\) and ending at \(pg\). Two arrows \(\gamma_1\) and \(\gamma_2\) can be composed if and only if the beginning of \(\gamma_2\) coincides with the end of \(\gamma_1\).

Let us notice that if the cross section \(S : M \to \tilde{E}\) is smooth, the bundle \((\tilde{E}, \pi_M, M)\), where \(\pi_M\) is the canonical projection \(\pi_M : \tilde{E} \to M\), is a trivial \(\tilde{G}\)-bundle. Indeed, the trivializing diffeomorphism \(\phi : \tilde{E} \to M \times \tilde{G}\) is given by \(\phi(p) = (\pi_M(p), g_p)\) where \(g_p\) is the element of the group \(\tilde{G}\) such that \(p = S(\pi_M(p))g_p\).

Let us also notice that the source and range mappings for \(\gamma = (p, g)\) can now be written as

\[
d(\gamma) = p = S(x) \cdot g_1,
\]

\[
r(\gamma) = pg = S(x) \cdot g_2,
\]

\(x \in M\), for \(g_1, g_2 \in G\), respectively; of course, \(g_2 = g_1g\).

3 The Groupoid Algebra

We define the algebra \(\mathcal{A} = C^\infty(\Gamma, \mathbb{C})\) of smooth complex valued functions on the groupoid \(\Gamma = E \times G\) with the convolution as multiplication. If \(f, g \in \mathcal{A}\),
the convolution is defined as

\[(f * g)(\gamma) = \sum_{\gamma_1 \in \Gamma_d(\gamma)} f(\gamma \circ \gamma_1^{-1})g(\gamma_1).\]

Let us define the mapping \(\varphi : \Gamma \rightarrow \bigcup_{x \in M} E_x \times E_x\), where \(E_x = \pi_M^{-1}(x)\), by \(\varphi(\gamma) = (p_0, p_1)\) with \(p_0 = d(\gamma)\) and \(p_1 = r(\gamma)\). Here we identify \(E\) with the set \(\Gamma^0\) of units of the groupoid \(\Gamma\). It can be easily seen that \(\varphi\) is a bijection. If we introduce the abbreviation \(\tilde{f}(p_0, p_1) = f(\varphi^{-1}(p_0, p_1))\), the convolution is expressed by

\[(\tilde{f} * \tilde{g})(p_0, p_1) = \sum_{p \in E_x} \tilde{f}[(p_0, p_1) \circ (p, p_0)]\tilde{g}(p_0, p) = \sum_{p \in E_x} \tilde{f}(p, p_1)\tilde{g}(p_0, p).\]

Here \(x = \pi_M(p_0)\).

Now, with a function

\[\phi : M \times G \times G \rightarrow \mathbb{C}\]

we associate the matrix \(A_{\phi}\) given by

\[A_{\phi}(\cdot, i, j) = \phi(\cdot, g_i, g_j).\]

The function \(\phi\) allows us to define the mapping

\[A_{\phi} : M \times \{1, 2, \ldots, k\} \times \{1, 2, \ldots, k\} \rightarrow \mathbb{C}\]

with the help of the formula

\[A_{\phi}(x, i, j) = \phi(x, \sigma(i), \sigma(j))\]

where \(\sigma\) is a bijection given by \(\sigma(i) = g_i\), and \(k = |G|\).

It is easy to see that \(M \times G \times G\) is a groupoid with the multiplication

\[(x, g', \tilde{g}) \circ (x, g, g') = (x, g, \tilde{g}).\]

**Lemma.** The mappings

\[\varphi : \Gamma \rightarrow \bigcup_{x \in M} E_x \times E_x\]
defined above, and
\[ \Phi : \Gamma \to M \times G \times G \]
given by
\[ \Phi(\gamma) = (pr(\gamma), \lambda(d(\gamma)), \lambda(r(\gamma))) \]
are isomorphisms of groupoids. Here \( pr : \Gamma \to M \) is the natural projection and \( \lambda : E \to G \) is a mapping given by \( \lambda(p) = g \) such that \( p = S(x)g \).

**Proof.** Let us prove this for \( \Phi \). The mapping \( \Phi^{-1}(x,g_1,g_2) = (S(x)g_1,g_1^{-1}g_2) \) determines the bijection between \( \Gamma \) and \( M \times G \times G \). \( \Phi \) is also a homomorphism. Indeed,
\[ \Phi(\gamma_1 \circ \gamma_2) = (pr(\gamma_1 \circ \gamma_2), \lambda(d(\gamma_1 \circ \gamma_2)), \lambda(r(\gamma_1 \circ \gamma_2))) \]
\[ = (x, g', \bar{g}) \circ (x, g, g') = \Phi(\gamma_1) \circ \Phi(\gamma_2) \]
where we have introduced the following abbreviations: \( \bar{g} = \lambda(r(\gamma_1)), g = \lambda(d(\gamma_2)), g' = \lambda(d(\gamma_1)) \).

By using the mapping \( \Phi \) one readily shows that the set of elements composable in \( \Gamma \) is bijective with the set of elements composable in \( M \times G \times G \). It remains to check the invertibility
\[ \Phi(\gamma^{-1}) = (x, \lambda(r(\gamma)), \lambda(d(\gamma))) \]
\[ = (x, \lambda(d(\gamma)), \lambda(r(\gamma))^{-1}) \]
\[ = [\Phi(\gamma)]^{-1}. \]

The proof for \( \varphi \) is analogous. \( \square \)

**Lemma.** The mapping \( \Phi^* : C^\infty(M \times G \times G) \to C^\infty(\Gamma) \) is an isomorphism of algebras.

**Proof.** Since \( \Phi^* \) is a bijection it is enough to show that it is a homomorphism
\[ (\phi \ast \psi)(x,g,\bar{g}) = \sum_{g' \in G} \phi(x,g',\bar{g}) \cdot \psi(x,g,g') \]
\[ = \sum_{g' \in G} \phi[(x,g,\bar{g}) \circ (x,g',g)]\psi(x,g,g') \]
\[ = \sum_{g' \in G} \phi[(x,g,\bar{g}) \circ (x,g,g')^{-1}]\psi(x,g,g') \]
\[ = \sum_{\gamma_1 \in \Gamma,M_{\gamma_1}} \phi[\Phi(\gamma \circ \gamma_1^{-1})]\psi(\Phi(\gamma_1)) \]
\[
\sum_{\gamma_1 \in \Gamma_{d(\gamma)}} (\Phi^* \phi)(\gamma \circ \gamma_1^{-1})(\Phi^* \psi)(\gamma_1)
\]
\[
= \sum_{\gamma_1 \in \Gamma_{d(\gamma)}} a(\gamma \circ \gamma_1^{-1})b(\gamma_1).
\]

In the last line the obvious abbreviations are introduced. □

We can interpret the algebra \( \mathcal{A} \) as the matrix algebra by defining the following mapping

\[
a \mapsto A_a = [a(x, g_i, g_j)]_{i,j=1}^k.
\]

The indices \( i \) and \( j \) label rows and columns of the respective matrix. In this representation the convolution becomes the usual matrix multiplication

\[
A_f * g = A_g \cdot A_f.
\]

If we remember that the center \( \mathcal{Z}(\mathcal{A}) \) of the algebra \( \mathcal{A} \) is

\[
\mathcal{Z}(\mathcal{A}) = \{ \overline{\pi^* f} : f \in C^\infty(M) \},
\]

where

\[
\overline{\pi^* f} = \begin{cases} 
0 & \text{if } \gamma \in E_x \times \{g\}, \ x \in M, \ g \neq e \\
f(x) & \text{if } \gamma \in E_x \times \{g\}, \ x \in M, \ g = e
\end{cases},
\]

we have the isomorphism of algebras \( \zeta : C^\infty(M) \to \mathcal{Z}(\mathcal{A}) \) given by

\[
\zeta(f : \mathcal{I}) = \overline{\pi^* f}.
\]

4 Geometry of the Groupoid

Let us consider the \( \mathcal{Z}(\mathcal{A}) \)-module of derivations of the algebra \( \mathcal{A} \)

\[
V \equiv \text{Der} \mathcal{A} = \text{Out} \mathcal{A} \oplus \text{Inn} \mathcal{A}
\]

where

\[
V_1 \equiv \text{Out} \mathcal{A} := \{ \bar{X} \in V : \bar{X}(a) = \Phi^* (X(\Phi^*)^{-1}(a)), \forall X \in \mathcal{X}(M) \};
\]

\[
V_2 \equiv \text{Inn} \mathcal{A} := \{ \text{ad}_a : a \in \mathcal{A} \}
\]

and \( \text{ad}_a(b) = [a, b] \) for \( b \in \mathcal{A} \). We have

\[
[\bar{X}, \text{ad}_a] = \text{ad}\bar{X}(a), \ [\bar{X}, \bar{Y}] = \overline{[X, Y]}, \ [\text{ad}_a, \text{ad}_b] = \text{ad}[a, b].
\]
This allows us to define the following metric on $V$

$$\mathcal{G}(u, v) = \bar{g}(u_1, v_1) + h(u_2, v_2)$$

where $\bar{g} : V_1 \times V_1 \rightarrow \mathcal{Z}(\mathcal{A})$ is a “lifting” of the metric $g : \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow C^\infty(M)$ on $M$

$$\bar{g}(\bar{X}, \bar{Y}) = \bar{g}(X, Y) = \zeta(g(X, Y)),$$

and $h : V_2 \times V_2 \rightarrow \mathcal{Z}(\mathcal{A})$ is a metric on the “noncommutative part” of the model.

We have the dual module $V^* = \text{Hom}_{\mathcal{Z}(\mathcal{A})}(V, \mathcal{C})$, and since $V$ is a locally free $\mathcal{Z}(\mathcal{A})$-module, there is the isomorphism $\Phi_\mathcal{G} : V \rightarrow V^*$ given by

$$\Phi_\mathcal{G}(u)(v) = \mathcal{G}(u, v) = \Phi_{\bar{g}}(u_1)(v_1) + \Phi_h(u_2)(v_2).$$

Now, we can define the preconnection $\nabla^* : V \times V \rightarrow V^*$ with the help of the Koszul formula

$$(\nabla^*_u v)(x) = \frac{1}{2}[u(\mathcal{G}(v, x)) + v(\mathcal{G}(u, x)) - x(\mathcal{G}(u, v))$$

$$+ \mathcal{G}(x, [u, v]) + \mathcal{G}(v, [x, u]) - \mathcal{G}(u, [v, x]),$$

and then the Levi-Civita connection by

$$\nabla = \Phi_\mathcal{G}^{-1} \circ \nabla^*.$$

Now, let us introduce the basis $(\bar{\partial}_\mu, e_i)$, $\mu = 0, 1, \ldots, m$, $i = 1, \ldots, n$, in the $\mathcal{Z}(\mathcal{A})$-module $V = V_1 \oplus V_2$. We have

$$[\bar{\partial}_\mu, \bar{\partial}_\nu] = 0, [e_i, e_j] = c_{ij}^k e_k, [\bar{\partial}_\mu, e_i] = 0$$

with $c_{ij}^k \in \mathcal{C}$ (indeed, if we put $e_i = \text{ad}E_i$, we have $[e_i, e_j] = \text{ad}[E_i, E_j] = \text{ad}(c_{ij}^k E_k)$).

Connection $\nabla$ determines the curvature tensor

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w.$$ 

Let us notice that $R(u, v)w = 0$ if $u, v, w \in \{\bar{\partial}_0, \ldots, \bar{\partial}_m, e_1, \ldots, e_n\}$ and $u, v, w$ do not belong simultaneously to the sets $\{\bar{\partial}_0, \ldots, \bar{\partial}_m\}$ or $\{e_1, \ldots, e_n\}$. Consequently, we have

$$R(u, v)w = R(u_1 + u_2, v_1 + v_2)(w_1 + w_2)$$

$$= R((u^\alpha \bar{\partial}_\alpha + u^i e_i), (v^\beta \bar{\partial}_\beta + v^j e_j))(w^\gamma \bar{\partial}_\gamma + w^k e_k)$$

$$= R^\mu_{\alpha \beta \gamma} u^\alpha v^\beta w^\gamma \bar{\partial}_\mu + R^k_{ijk} u^i v^j w^k e_l$$

8
where $R_{\alpha\beta\gamma}^\mu \in C^\infty(M)$ are the components of the curvature tensor $\bar{\nabla}$ of the connection $\bar{\nabla}$ and $R'_{ijkl}$ are the components of the curvature tensor $h\nabla$ of the connection $h\nabla$. Therefore,

$$R(u, v)w = \bar{\nabla} (u_1, v_1)w_1 + h\nabla (u_2, v_2)w_2.$$ 

This decomposition is, in general, valid for other geometric magnitudes. We should notice that the moduli $X(M)$ and $\text{Out}\mathcal{A}$ are isomorphic which means that the geometry of $\bar{g}$ is a copy of that of $g$ on $M$. Therefore, in the $\bar{g}$-sector the situation is the same as in the usual general relativity: we can formulate Einstein equations that are to be solved for the metric. In the $h$-sector of our model the situation is different; let us analyze it in the more detailed way.

Having a basis the trace of a $hZ(A)$-endomorphism $A : V_2 \to V_2$ is defined in the usual way: $\text{tr}A = \sum_{i=1}^n A_i^i \in hZ(A)$. For a fixed pair $x, y \in V_2$ one defines the family of operators $hR_{xy} : V_2 \to V_2$ by

$$hR_{xy} (v) = hR (v, x)y,$$

and the Ricci 2-form $\text{ric}_h : V_2 \times V_2 \to hZ(A)$ by

$$\text{ric}_h(x, y) = \text{tr} hR_{xy},$$

or in the local basis

$$\text{ric}_h(u, v) = hR_{ij} u^i v^j$$

where $u = u^i e_i$, $v = v^j e_j$. There exists uniquely defined operator $hR : V_2 \to V_2$ given by

$$\text{ric}_h(u, v) = hR (u), v).$$

And the scalar curvature is defined as

$$h\frac{1}{4} = \text{tr} hR.$$

Now, we have all quantities required to write the counterpart of the usual Einstein equation

$$hR - \frac{1}{2} h\text{id}_{V_2} + \Lambda\text{id}_{V_2} = \kappa T.$$
Here $\Lambda$ and $\kappa$ are counterparts of the cosmological constant and Einstein’s gravitational constant, respectively, and $T$ is a counterpart of the energy-momentum tensor. Since, however, our philosophy is that matter should be generated out of “purely noncommutative geometry”, we prefer to consider the above equation with $T = 0$, i.e.,

$$
\mathcal{h} \mathcal{R} + \Lambda \text{id}_{V_2} = 0,
$$

or, if we write it with the argument and omit the cumbersome superscript $h$,

$$
\mathbf{G}_h \equiv \mathcal{R}(u) + \Lambda u = 0. \tag{1}
$$

Now, it is clear that (1) is an eigenvalue equation, and it should be solved with respect to $u \in V_2$. If we assume that $u = u^i e_i$, $i = 1, \ldots, n$, with $u^i \in \mathcal{Z}(\mathcal{A})$, this equation takes the form

$$
u^i \mathcal{R}^j_i + \Lambda u^j = 0.
$$

It has nontrivial solutions if

$$\det(\mathcal{R}^j_i + \Lambda I) = 0.
$$

This implies that $\Lambda \in \mathcal{Z}(\mathcal{A})$ which means that $\Lambda$ is a function on $M$ (it is constant only at $x \in M$).

Let us now consider the full Einstein equation on the groupoid

$$
\mathbf{G}_g + \mathbf{G}_h = 0 \tag{2}
$$

where $\mathbf{G}_g = 0$ are the usual Einstein equations on space-time $M$ suitably lifted to the groupoid. It is, therefore, evident that $\bar{g}$ solves $\mathbf{G}_{\bar{g}} = 0$ if and only if the corresponding metric $g$ solves the usual Einstein equations on $M$. Let us notice that the generalized Einstein equation (2) determines the pair $(V, \mathcal{G})$, i.e., the module of derivations and the metric on it. In the case of the standard geometry on space-time $M$, the module of derivations is unique and we are looking for the metric. This is also true for equation $\mathbf{G}_{\bar{g}} = 0$, but for equation (1) it could be that $h$ is unique (see [8, p. 75], and in this case we should solve this equation for derivations.
5 A Simple Example

In this section we test our approach by considering a simple example in which \( G = \mathbb{Z}_2 \), where \( \mathbb{Z}_2 = \{1, \epsilon\}, \epsilon^2 = 1 \), and \( E = M \times \mathbb{Z}_2 \). Therefore, we have the groupoid \( \Gamma = E \times G \). Its elements are:

\[
\begin{align*}
\gamma_1 &= \gamma_{1,x} = ((x, 1), 1) \xrightarrow{\Phi} (x, 1, 1) \in M \times G \times G, \\
\gamma_2 &= \gamma_{2,x} = ((x, 1), \epsilon) \xrightarrow{\Phi} (x, 1, \epsilon) \in M \times G \times G, \\
\gamma_3 &= \gamma_{3,x} = ((x, \epsilon), \epsilon) \xrightarrow{\Phi} (x, \epsilon, 1) \in M \times G \times G, \\
\gamma_4 &= \gamma_{4,x} = ((x, \epsilon), 1) \xrightarrow{\Phi} (x, \epsilon, \epsilon) \in M \times G \times G.
\end{align*}
\]

We remember that \( \Phi \) is an isomorphism of groupoids. In fact, we have here a family of groupoids (a groupoid over each \( x \in M \)) which is also a groupoid.

Let us now consider the algebra \( A = (C^\infty(\Gamma, \mathbb{C}), \ast) \). If \( f \in A \), we have \( f_{11} = f_{11,x} = f(\gamma_{1,x}) \), and similarly for other elements. There is the correspondence

\[
A \ni f \mapsto M_f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in C^\infty(M) \otimes M_{2 \times 2}(\mathbb{C}).
\]

For fixed \( x \in M \) it is a matrix with numerical entries.

The convolution is antiisomorphism. We have

\[
(f \ast g)(\gamma_1) = (f \ast g)_{11} = f(\gamma_1 \circ \gamma_1^{-1})g(\gamma_1) + f(\gamma_1 \circ \gamma_2^{-1})g(\gamma_2)
\]

\[= f_{11} \cdot g_{11} + f_{21} \cdot g_{12}\]

which is the matrix multiplication rule. And similarly for other matrix elements.

The \( \mathcal{Z}(A) \)-module of inner derivations \( V_2 \) is isomorphic with \( \mathfrak{sl}_2(\mathbb{C}) \otimes C^\infty(M) \). Let us choose the basis in \( \mathfrak{sl}_2(\mathbb{C}) \)

\[
H_0 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]

which leads to the following commutation relations

\[
[H_0, X_1] = X_1, \quad [H_0, X_2] = -X_2, \quad [X_1, X_2] = 2H_0.
\]

The natural choice for the metric is the Killing form

\[
h(X, Y) = \langle X, Y \rangle = \text{Tr}(\text{ad}X \circ \text{ad}Y).
\]
We can easily compute that the only nonvanishing components of this metric are
\[ \langle H_0, H_0 \rangle = 2 \] and \[ \langle X_1, X_2 \rangle = 4. \]
Now, it is convenient to change to the new basis
\[
H = \frac{1}{\sqrt{2}} H_0, \quad Y_1 = \frac{1}{2\sqrt{2}} (X_1 + X_2), \quad Y_2 = \frac{1}{2\sqrt{2}} (X_1 - X_2)
\]
in which
\[ \langle H, H \rangle = \langle Y_1, Y_1 \rangle = 1, \quad \langle Y_2, Y_2 \rangle = -1 \]
and
\[ \langle H, Y_1 \rangle = \langle H, Y_2 \rangle = \langle Y_1, Y_2 \rangle = 0. \]

Let us introduce the following notation: \( \partial_x = \text{ad} x \) where \( x \) is a traceless matrix, and \( \partial_i = \text{ad} E_i \). It can be shown that the connection
\[
\nabla_{\partial_x} \partial_y = \alpha [\partial_x, \partial_y],
\]
for any \( \alpha \in \mathcal{Z}(\mathcal{A}) \), is compatible with the Killing metric, i.e.
\[
\nabla_{\partial_x} \langle \partial_x, \partial_y \rangle = \langle \nabla_{\partial_x} \partial_x, \partial_y \rangle + \langle \partial_x, \nabla_{\partial_x} \partial_y \rangle.
\]

We now readily compute the curvature tensor
\[
R(\partial_x, \partial_y) \partial_z = (\alpha^2 - \alpha) [[\partial_x, \partial_y], \partial_z],
\]
and the torsion tensor
\[
T(\partial_x, \partial_y) = (2\alpha - 1)[\partial_x, \partial_y].
\]
To have \( T = 0 \) we must assume \( \alpha = 1/2 \). Hence, for the Ricci tensor we have
\[
\text{ric}(\partial_k, \partial_l) = \frac{1}{4} \sum_{j=1}^{N^2-1} \langle [\partial_k, \partial_j], [\partial_l, \partial_j] \rangle.
\]

We easily compute that the only nonvanishing component of the Ricci tensor, in the basis \((H, Y_1, Y_2)\), is
\[
\text{ric}(Y_2, Y_2) = \frac{1}{4}.
\]
Finally, the Einstein equation assumes the form of the eigenvalue equation
\[ \mathcal{R}(u) + \Lambda(u) = 0. \]
For the eigenvalue \( \Lambda = -1/4 \), the space of its solutions is
\[ W = \{ u \in V_2 : u = f \cdot Y_2, f \in C^\infty(M) \}. \]
Of course, \( W \) is a \( \mathcal{Z}([\mathcal{A}]) \)-submodule of the module \( V_2 \). The modular dimension of \( W \) is one. If \( \Lambda = 0 \), the space of solutions is spanned by \( H \) and \( Y_1 \), and is of modular dimension two.

6 Correspondence with Classical Theory

It is interesting to notice that the transition from the noncommutative geometry on the groupoid \( \Gamma \) to the classical geometry on the manifold \( M \) can be done with the help of an averaging procedure where the averaging of a functional matrix \( A \) is given by
\[ \langle A \rangle = \frac{1}{|G|} \text{Tr} A. \]
This averaging kills noncommutativity; indeed
\[ \langle AB \rangle = \frac{1}{|G|} \text{Tr}(AB) = \frac{1}{|G|} \text{Tr}(BA) = \langle BA \rangle. \]
Let \( f \in C^\infty(M) \) be a function on \( M \); it can be expressed as \( A_x = f(x) \cdot I \), \( x \in M \), and its averaging gives \( \langle A_x \rangle = \frac{1}{|G|} \text{Tr}(A) = f(x) \). In this way, we have demonstrated that functions \( f(x) \) on \( M \), interpreted as \( A_x = f(x) \cdot I \), have the property that the average of \( A_x \) is equal to \( f(x) \).
Moreover, there exists the mapping \( \text{tr} : \mathcal{A} \to C^\infty(M) \) given by
\[ \text{tra} := \text{Tr}((\Phi^{-1})^* a) \]
for every \( a \in \mathcal{A} \). Indeed, we have \((\Phi^{-1})^* a \in C^\infty(M \times G \times G)\), i.e., \((\Phi^{-1})^* a = \varphi(x, g_1, g_2)\), and its trace \( \text{Tr} : C^\infty(M \times G \times G) \to \mathbb{C} \) is given by
\[ (\text{Tr}\varphi)(x) = \sum_{g \in G} \varphi(x, g, g). \]
It is easy to check that, besides the usual properties of trace, one has
\[ \text{tr}(\varphi \ast \psi) = \text{tr}(\psi \ast \varphi) \]
for \( \varphi, \psi \in C^\infty(M \times G \times G) \). Let us also notice that
\[ \frac{1}{|G|} \text{tr}|_Z(\mathcal{A}) = \zeta^{-1}. \]

Lemma. There exists the canonical projection \( P : \mathcal{A} \to \mathcal{Z}(\mathcal{A}), P = P^2 \)
and \( P \) is \( \mathbf{C} \)-linear, such that \( P|_{\mathcal{Z}(\mathcal{A})} = \text{id}_{\mathcal{Z}(\mathcal{A})} \).

Proof. We define \( P : \mathcal{A} \to \mathcal{Z}(\mathcal{A}) \) by
\[ P = \zeta \circ \frac{1}{|G|} \text{tr}, \]
and easily check its properties formulated in the Lemma. ☐.

It can be easily seen that for any \( u \in \text{Der}\mathcal{A} \) and any element \( a \in (\mathcal{Z}(\mathcal{A})) \)
we have \( u(a) \in \mathcal{Z}(\mathcal{A}) \). For every \( u \in \mathcal{Z}(\mathcal{A}) \) we define the projection \( u^# : C^\infty(M) \to C^\infty(M) \) by
\[ u^#(f) = \zeta^{-1}(u(\zeta(f))). \]
If \( u \) is an inner derivation, then \((\text{ad}b)^# = 0\) for any \( b \in \mathcal{A} \), and for any \( X \in \mathcal{X}(M) \) one has \( \hat{X}^# = X \).

We see that \( \text{Der}\mathcal{A} \ni u \mapsto u^# \in \text{Der}(C^\infty(M)) \) is a homomorphism of Lie algebras, and its restriction to the center \( u|_{\mathcal{Z}(\mathcal{A})} \mapsto u^# \) is an isomorphism of Lie algebras.

Let \( \omega \) be a \( k \)-form
\[ \omega : \underbrace{\text{Der}\mathcal{A} \times \cdots \times \text{Der}\mathcal{A}}_{k \text{ times}} \to \mathcal{Z}(\mathcal{A}), \]
then \( \omega^# = \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to C^\infty(M) \) is given by
\[ \omega^#(X_1, \ldots, X_k) = \zeta^{-1}(\omega(\hat{X}_1, \ldots, \hat{X}_k)) = (\omega(\hat{X}_1, \ldots, \hat{X}_k))^#. \]

Similarly, for the connection \( \nabla : \text{Der}\mathcal{A} \times \text{Der}\mathcal{A} \to \text{Der}\mathcal{A} \) we have \( \nabla^# : \text{Der}(C^\infty(M)) \times \text{Der}(C^\infty(M)) \to \text{Der}(C^\infty(M)) \) given by
\[ \nabla^#_X Y = (\nabla_X Y)^#. \]
And generally for a tensor \( A : \text{Der}\mathcal{A} \times \cdots \times \text{Der}\mathcal{A} \to \text{Der}\mathcal{A} \) we obtain \( A^\# : \text{Der}(C^\infty(M)) \times \cdots \times \text{Der}(C^\infty(M)) \to \text{Der}(C^\infty(M)) \) which is given by
\[
A^\#(X_1, \ldots, X_k) = (A(\bar{X}_1, \ldots, \bar{X}_k))^\#.
\]

Consequently, for our metric we have
\[
(\bar{g} + h)^\#(X_1, X_2) = (\bar{g}(\bar{X}_1, \bar{X}_2))^\# = (g(X_1, X_2))^\# = g(X_1, X_2)
\]
for all \( X_1, X_2 \in \mathcal{X}(M) \), as it should be. This is obvious if one remembers that \( h(\bar{X}_1, \bar{X}_2) = 0 \).

Therefore, we can say that the usual differential geometry on the base manifold \( M \) is the averaging of the differential geometry developed in Section 4. This averaging corresponds to the averaging with respect to units of the groupoid (which, in the matrix representation of the groupoid, is equivalent to the averaging of the diagonal elements of a given matrix).

## 7 Regular Representation and Quantum Sector of the Model

Let us consider the regular representation of the groupoid algebra
\[
\pi_p : \mathcal{A} \to \mathcal{B}(\mathcal{H}_p),
\]
where \( \mathcal{H}_p = L^2(\Gamma_{d(\gamma)}), \gamma \in \Gamma, d(\gamma) = p \in E \), defined by
\[
\pi_p(a)(\xi) = \frac{i}{\hbar} \xi^T \cdot M_a
\]
where \( \xi \in \mathbb{C}^n, n = |G| \), and the coefficient \( i/\hbar \) is added to have the correspondence with quantum mechanics. To specify \( \xi \) we should remember that
\[
\Gamma_{d(\gamma)} = \{(\pi_M(d(\gamma)), \lambda(d(\gamma)), \lambda(r(\gamma))) = \{(x, g_0, g) : g \in G\}
\]
where the first equality should be understood as the bijection. Then \( \xi : \Gamma_{d(\gamma)} \to \mathbb{C} \) is given by
\[
\xi(x, g_0, g) = (\xi_g)_{g \in G}.
\]

Let us now consider how do derivations behave under the above representation. Let \( v = v_1 + v_2 \) where \( v_1 \in \text{Out}\mathcal{A} \) and \( v_2 \in \text{Inn}\mathcal{A} \). If we assume
that $a \in \mathcal{Z}(\mathcal{A})$ then $\pi_p(v(a)) \in \pi_p(\mathcal{Z}(\mathcal{A})) \subset \mathcal{Z}(\mathcal{B}(\mathcal{H}_p))$ which means that we have

$$\pi_p(v(a)) = k \cdot I.$$ 

If $a$ is any element of $\mathcal{A}$, we can decompose it

$$a = a_1 + a_2$$

where

$$a_1 = \zeta(\langle a \rangle) \in \mathcal{Z}(\mathcal{A}).$$

and

$$a_2 = a - a_1 \notin \mathcal{Z}(\mathcal{A}).$$

Then

$$\pi_p((v_1 + v_2)(a_1 + a_2))\xi = \xi^T \cdot M_{(v_1 + v_2)(a_1 + a_2)} = \xi^T(M_{v_1a_1} + M_{v_1a_2} + M_{v_2a_2}).$$

Let us now consider the $v_2(a_2)$-terms of the above equation

$$\pi_p(v_2(a_2)) = \frac{i}{\hbar} M_{v_2a_2}.$$ 

Since $v_2 = adb$ for a certain $b \in \mathcal{A}_2 := \{b \in \mathcal{A} : \text{tr}b = 0\}$ (in such a case the choice of $b$ is unique), one has

$$M_{v_2a_2} = M_{[b,a_2]} = [M_b,M_{a_2}],$$

and

$$\pi_p(v_2(a_2)) = \frac{i}{\hbar} [M_b,M_{a_2}].$$

By taking into account that $\xi^T \tilde{X}M_{a_2} = \pi_p(v_1a_2)$, where $\tilde{X}$ is an outer derivation, we finally obtain

$$\pi_p((v_1 + v_2)(a_1 + a_2))\xi = \xi^T(fI + \tilde{X}M_{a_2} + [M_b,M_{a_2}]). \quad (3)$$

By analogy with quantum mechanics we could say that if $a_2$ is a self-adjoint element of $\mathcal{A}$, equation (3) describes the evolution of the “observable” $a_2$. This dynamical equation can be coupled with generalized Einstein equation
(2) by postulating that \( v \) solves equation (2) (i.e., \( v_2 \in \ker G_h \) and \( v_1 \in \mathcal{X}(M) \)).

To go from the above generalized dynamics of our model to the usual dynamics of quantum mechanics we no longer postulate that \( v_2 \in \ker G_h \), i.e., that equation (3) is coupled to the generalized Einstein equation (2), and we assume that there exist a one-parameter family of unitary operators \( U(t) = e^{iM_b t} \). The existence of one-parameter operator families is guaranteed by the Tomita-Takesaki theorem but, in general, such a family depends on a state on a given algebra. The above postulate of the existence of \( U(t) \) (independent of state) amounts to imposing on the algebra \( \mathcal{A} \) some further conditions (see [5]).

Let us notice that

\[
\frac{i}{\hbar}[M_b, M_{a_2}] = \frac{d}{dt}(M_{a_2}(t))
\]

where

\[
M_{a_2}(t) = U(t)M_{a_2}U(t)^{-1},
\]

and \( M_{a_2} \) satisfies the equation

\[
\frac{d}{dt}M_{a_2}(t + s)|_{t=0} = i[M_b, M_{a_2}].
\]

Since \( v_1 \) is any vector of \( V_1 \) we can choose it to be \( t \)-directed; in such a case

\[
M_{v_1a_2} = \frac{\partial}{\partial t}(M_{a_2}(t)).
\]

If we assume that \( M_{a_2} \) is self-adjoint and denote if by \( \hat{A} \), and \( M_b \) is the Hamiltonian of the system and denote it by \( H \), ten the \( a_2 \)-components of equation (3) give

\[
\frac{d}{dt}\hat{A} = \frac{\partial}{\partial t}\hat{A} + [H, \hat{A}]
\]

where we have assumed \( \hbar = 1 \). It is the Heisenberg equation of motion well known from quantum mechanics.

8 Concluding Remarks

The model constructed in this work is too simple to be a candidate for even a step towards the final unification of general relativity and quantum mechanics. However, it shows the consistency of the idea that the noncommutative
generalization of the standard geometry, when combined with the groupoid generalization of the symmetry concept, leads to an interesting mathematical structure having a remarkable unifying power. Many typically relativistic and quantum concepts smoothly cooperate with each other within this structure (at least for a finite group $G$, and produce a handful of valuable results. The most important of them seem to be the following.

1. Noncommutative geometry of the transformation groupoid $\Gamma = E \times G$ is reach enough to accommodate for the standard space-time geometry with a nontrivial contribution coming from the group $G$ which, through its regular representation, can be interpreted as describing the quantum sector of our model.

2. The model contributes to the understanding of the structure of the Einstein equations. The metric is always defined on the module of derivations, and in a more general setting these equations are to be solved with respect to both metric and derivations. In a usual space-time geometry, the module of derivations is unique, and one looks for the metric. In our model this fact is preserved in its space-time sector, but in its quantum sector one looks for the derivations. This fact was also signalled in one of our previous works [6]. It was Madore who first demonstrated that in some derivation based noncommutative geometries the metric could be unique [8, p. 75].

3. It is also interesting that in the quantum sector of our model the Einstein equation has the form of the eigenvalue equation with the cosmological constant as an eigenvalue.

4. The new result is that the transition from the noncommutative geometry of our model to the classical geometry of space-time can be done by the averaging procedure of the elements of the algebra $\mathcal{A}$. This procedure is analogous to that typically used in quantum mechanics. The same procedure is valid for other geometric magnitudes, such as: derivations, differential forms, connection, metric. One can say shortly that “averaging kills noncommutativity”.

5. The transition from the dynamics of our model to the dynamics of the usual quantum mechanics is done by restricting the model to its quantum sector, and enforcing upon the algebra $\mathcal{A}$ (more strictly: upon its representation on a Hilbert space) the existence of a one-parameter family of unitary operators.

Although our model is too simple to serve as a realistic physical model, it shows some further perspectives. It would be interesting to explore the
geometry of the dual object to the transformation groupoid considered in the present work. If the geometry of the groupoid is to be interpreted as giving the “position representation” of our model, the geometry of its dual object could be regarded as describing its “momentum representation”. It seems that the natural way to construct such a “dual geometry” is via making the algebra \( \mathcal{A} \) a Hopf algebra. This approach, by making contact with the theory of quantum groups, and especially with the Majid program [11, 12], would pave the way for constructing a more realistic physical model.

**References**

[1] Chamseddine, A. H., Felder, G. and Fröhlich, J. “Gravity in Non-Commutative Geometry”, *Commun. Math. Phys.* **155**, 1993, 205-217.

[2] Connes, A. “Gravity Coupled with Matter and the Foundations of Non-commutative Geometry”, *Comm. Math. Phys.* **182**, 1996, 155-176.

[3] Heller, M. and Sasin, W. “Structured Spaces and Their Application to Relativistic Physics”, *J. Math. Phys.* **36**, 1995, 3644-3662.

[4] Heller, M., Sasin, W. and Lambert, D. “Groupoid Approach to Noncommutative Quantization of Gravity”, *J. Math. Phys.* **38**, 1997, 5840-5853.

[5] Heller, M. and Sasin, W. “Emergence of Time”, *Phys. Lett. A* **250**, 1998, 48-54.

[6] Heller, M. and Sasin, W. “Noncommutative Unification of General Relativity and Quantum Mechanics”, *Intern. J. Theor. Phys.* **38**, 1999,1619-1642.

[7] Heller, M., Sasin, W. and Odrzygódź, Z. “State Vector Reduction as a Shadow of a Noncommutative Dynamics”, *J. Math. Phys.* **41**, 2000, 5168-5179.

[8] Madore, J. *An Introduction to Noncommutative Differential Geometry and Its Physical Applications*, second edition, Cambridge University Press, Cambridge, 1999.
[9] Madore, J. and Mourad, J. “Quantum Space-Time and Classical Gravity”, *J. Math. Phys.* **39**, 1998, 424-442.

[10] Madore, J. and Saeger, L. A. “Topology at the Planck Length”, *Class. Quantum Grav* **15**, 1998, 811-826.

[11] Majid, S., *Foundations of Quantum Group Theory*, Cambridge University Press, Cambridge, 1995.

[12] Majid, S. “Quantum Groups and Noncommutative Geometry”, *J. Math. Phys.* **41**, 2000, 3892-3942.

[13] Paterson, A. L. *Groupoids, Inverse Semigroups and Their Operator Algebras*, Birkhäuser, Boston 1999.

[14] Sitartz, A. “Gravity from Non-Commutative Geometry”, *Class. Quantum Grav.* **11**, 1994, 2127-2134.