Research Article

A Decoupled Energy Stable Numerical Scheme for the Modified Cahn–Hilliard–Hele–Shaw System with Logarithmic Potential

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A decoupled unconditionally stable numerical scheme for the modified Cahn–Hilliard–Hele–Shaw system with logarithmic potential is proposed in this paper. Based on the convex-splitting of the associated energy functional, the temporal discretization of the scheme is given. The fractional step method is used to decouple the nonlinear modified Cahn–Hilliard equation from the pressure gradient. Then, at each time step, one only needs to solve Poisson’s equation which is obtained by using an incremental pressure-stabilization technique. In terms of logarithmic potential, using the regularization procedure can make the domain extended from \((-1, 1)\) to \((-\infty, \infty)\) for \(F(\phi)\). Finally, the proof of the energy stability and the calculation of the error estimate are presented. Numerical examples are recorded to illustrate the accuracy and performance for the proposed scheme.

1. Introduction

The Cahn–Hilliard–Hele–Shaw system (CH-HS) [1], which is used to describe the motion of a viscous fluid between two closely spaced parallel plates, is an important mathematical model. The applications of the CH-HS model and its variant are abundant. Aritomo Shinozaki and Yoshitsugu Oono performed the simulation of spinodal decomposition of a binary incompressible fluid in a Hele–Shaw cell [2] by a similar set of equations. Recently, the CH-HS system has been applied in the study of Saffman–Taylor instability [3] when a more viscous fluid is displaced by a less viscous one resulting in complex pattern formation. This system also serves as a phase field and diffusion interface models of tumor growth [4–6]. When given thought to permeability/ hydraulic conductivity, the CH-HS system, also known as Cahn–Hilliard–Darcy (CH-D) equation, is used to model multiphase flow in porous media. The CH-D system has been studied by many authors and one can refer to Refs. [7, 8].

Let \(\Omega \in \mathbb{R}^d (d = 2, 3)\) be an open polygonal or polyhedral domain with a Lipschitz continuous boundary \(\partial \Omega\). The modified Cahn–Hilliard–Hele–Shaw system can be studied in the article as follows:

\[
\begin{align*}
\partial_t \phi + \nabla \cdot (\phi \mathbf{u}) &= \epsilon \Delta \mu, \quad \text{in } \Omega_T := \Omega \times (0, T), \\
\mu &= \frac{1}{\epsilon} (F(\phi) - \epsilon \Delta \phi + \xi), \quad \text{in } \Omega_T, \\
-\Delta \xi &= \theta (\phi - \bar{\phi}_0), \quad \text{in } \Omega_T, \\
\mathbf{u} &= -\left(\nabla p + \gamma \phi \nabla \mu\right), \quad \text{in } \Omega_T, \\
\nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega_T, \\
\phi|_{t=0} &= \phi_0, \quad \forall x \in \Omega, \\
\mathbf{u} \cdot \mathbf{n} &= 0, \partial_n \phi &= \partial_n \mu &= 0, \quad \text{on } \partial \Omega \times (0, T),
\end{align*}
\]

where \(\phi\) is the concentration field, \(\mathbf{u}\) is the velocity, \(\mu\) is the chemical potential, \(\xi\) is an auxiliary variable, \(p\) is the pressure, \(\bar{\phi}_0 = 1/|\Omega| \int_{\Omega} \phi_0 (x) dx\), \(\mathbf{n}\) is the unit outer normal of the boundary \(\partial \Omega\), and \(\gamma > 0, \epsilon > 0, \theta \geq 0\) are physical constants. In particular, the system (1a)–1g is a classical CH-HS system when \(\theta = 0\) (and further \(\xi = 0, \mathbf{u} = 0\)) [9, 10]. Noting that the equations (1a) and (1b) in the system (1a)–1g can be seen as the modified Cahn–Hilliard(CH) equation...
with the convective term, the equations (1d) and (1e) are considered as the Darcy equation with the elastic forcing term, and the equations (1f) and (1g) are the initial and Neumann boundary conditions, respectively.

Numerical methods are a crucial tool to study the dynamics described by the CH-HS system, which have been extensively investigated. For instance, an unconditionally stable and solvable finite difference scheme for the CH-HS equations [11] was presented by Wise et al., which was based on a convex splitting of the discrete Cahn–Hilliard(CH) energy and was semiimplicit. Guo et al. put forward an efficient and stable energy fully discrete local discontinuous Galerkin (LDG) method for the CH-HS system in Ref. [12]. Also, in literature [13], the error estimates of the mixed finite element method of the CH-HS system was analyzed by Chen Wenbin et al. We can refer to references [14–17].

The modified Cahn–Hilliard equation was introduced by Shahriari [18] to suppress the phase coarsening. A discontinuous Galerkin method coupled with convex splitting for the modified Cahn–Hilliard equation was proposed by Aristotelous et al. [19]. Other related researches for the modified Cahn–Hilliard equation can be seen in Refs. [20–22]. When the modified Cahn–Hilliard equation is coupled with the Darcy equation, there is the modified Cahn–Hilliard–Hele–Shaw equation. For the modified Cahn–Hilliard–Hele–Shaw system, Jia presented a coupled finite element method for solving the modified CH-HS system [23], in which the time discretization was based on the convex splitting of the energy functional in the modified CH equation, and the high-order nonlinear term and the linear term of the chemical potential were treated explicitly and implicitly, respectively.

About the potential function, a modified CH-HS system with double well potential was solved in Ref. [23]. We can find that no theoretical and numerical analysis has been presented for the modified CH-HS system with logarithmic potential although the CH-HS equation with logarithmic potential had appeared, such as the nonlocal CH-HS system with logarithmic potential [24]. A decoupling numerical method for solving the CH-HS system with logarithmic potential was proposed by Guo et al. [25], and a fully discrete scheme was proposed for solving the CH-HS system with logarithmic potential and concentration dependent mobility by Guo et al. [25]. Therefore, we used the free energy density function of the logarithmic potential in this paper, which is defined as follows [26]:

\[ F(\phi) = \frac{a}{2} ((1 + \phi)ln(1 + \phi) + (1 - \phi)ln(1 - \phi)) \]

\[ + \frac{1}{2} (1 - \phi^2), \quad \phi \in (-1, 1), \]

\[ f(\phi) = F'(\phi) = \frac{a}{2} (ln(1 + \phi) + ln(1 - \phi)) - \phi. \] (3)

The logarithmic free energy functional is defined as

\[ E(\phi) = \int_{\Omega} \left( \frac{\epsilon^2}{2} |\nabla \phi|^2 + F(\phi) \right) dx, \] (4)

and the CH-HS system is mass conservative, i.e., \((\phi(\cdot, t), 1) = (\phi_0, 1)\) and energy dissipative:

\[ \frac{dE}{dt} = -\epsilon \int_{\Omega} |\nabla \mu|^2 dx - \frac{1}{\gamma} \int_{\Omega} |u|^2 dx \leq 0, \] (5)

where \((\cdot, \cdot)\) is the standard \(L^2\) inner product over \(\Omega\).

Here, we use the Elliott and Luckhaus regularization for problem with the logarithmic free energy function \(F(\phi)\) replaced by the twice continuously differentiable function \(\hat{F}(\phi)[27, 28]\), where \(\forall \kappa \in (0, 1)\):

\[
F(\phi) = \begin{cases}
\frac{a}{2} (1 + \phi)ln(1 + \phi) + \frac{a}{4\kappa} (1 - \phi)^2 + \frac{a}{2} (1 - \phi)lnx - \frac{a\kappa}{4} + \frac{1}{2} (1 - \phi^2), & \phi \geq 1 - \kappa, \\
\frac{a}{2} ((1 + \phi)ln(1 + \phi) + (1 - \phi)ln(1 - \phi)) + \frac{1}{2} (1 - \phi^2), & |\phi| < 1 - \kappa, \\
\frac{a}{2} (1 - \phi)ln(1 - \phi) + \frac{a}{4\kappa} (1 + \phi)^2 + \frac{a}{2} (1 + \phi)lnx - \frac{a\kappa}{4} + \frac{1}{2} (1 - \phi^2), & \phi \leq -1 + \kappa,
\end{cases}
\] (6)

and the monotone function is given by

\[
\hat{f}(\phi) = F'(\phi) = \begin{cases}
\frac{a}{2} ln(1 + \phi) + \frac{a}{2} (1 - \phi) - \frac{a}{2} lnx - \phi, & \phi \geq 1 - \kappa, \\
\frac{a}{2} (ln(1 + \phi) - ln(1 - \phi)) - \phi, & |\phi| < 1 - \kappa, \\
-\frac{a}{2} ln(1 - \phi) - \frac{a}{2} (1 + \phi) + \frac{a}{2} lnx - \phi, & \phi \leq -1 + \kappa.
\end{cases}
\] (7)
For $\kappa \leq 1/2$ and $\forall \phi$, $\bar{f}(\phi) \leq L$ holds true, where $L$ is a positive constant.

Hereinafter, $f$ and $F$ will be substituted by $\bar{f}$ and $\bar{f}$ in our analysis. However, for convenience, the will be omitted.

We note that the equations in (1a)–1g are coupled, nonlinear, and numerically stiff with large spatial derivative over a small transition layer. Thus, solving the modified CH-HS system numerically is challenging. In reference [23], Jia presented a coupled finite element method for solving the modified CH-HS system. In fact, compared with the method of coupling, the decoupling is more simple. Daozhi Han introduced a decoupled numerical scheme for the CH-HS system [29], an operator-splitting/fractional-step method is used to decouple the nonlinear Cahn–Hilliard equation from the pressure in the Darcy equation by using an operatorsplitting Cahn–Hilliard–Hele–Shaw system with logarithmic potential, the use of the regularization procedure to solve Possion’s equation at each time step. For the log-pressure in the Darcy equation by using an operator-splitting/fractional-step method is used in the update of pressure so that only Poisson’s equation decouple the nonlinear Cahn–Hilliard equation from the pressure in the Darcy equation by using an operator-splitting.

Daozhi Han introduced a decoupled numerical scheme for the CH-HS system [29], an operator-splitting/fractional-step method is used to decouple the nonlinear Cahn–Hilliard–Hele–Shaw system with logarithmic potential is a strong nonlinear system. Based on this reason, it is extremely difficult to find the exact solution of the system. To overcome this difficulty, we proposed a numerical method. The first-order scheme based on the convex-splitting method is introduced for the CHHS system. The fully discrete scheme exploits the first-order backward Euler method for time discretization and the mixed finite element method for spatial discretization. It is uniquely solvable and stable in energy.

The rest of the article is organized as follows. The semidiscrete and fully discrete finite element schemes are reviewed in Section 2. Then, we show that the fully discrete scheme is unconditionally stable in energy in Section 3. In Section 4, the detailed convergence analysis is given. Finally, some numerical examples show that the proposed scheme is convergent and efficient in Section 5. Some conclusions are drawn in Section 6.

2. The Discrete Numerical Scheme

2.1. The Simi-Discrete Scheme. $L^2(\Omega)$ is a square integrable function space, and $H^1(\Omega)$ is the Sobolev space. Let us denote $(u, v) = \int_\Omega u(x)v(x)dx$,

$\|\phi\|_{L^2(\Omega)} = \|\phi\|^{1/2}$, $\|\phi\|_{H^1(\Omega)} = \|\phi\|^{1/2}$,

and $f_2(\phi) = -\bar{f}(\phi)$. Let $\tau > 0$, $\tau = t_{n+1} - t_{n}$, $n = 0, 1, \cdots, N$, where $\tau$ is the time step size. For the modified Cahn–Hilliard–Hele–Shaw equation, we proposed the following numerical scheme which is discrete in time and continuous in space, i.e., seek $\{\phi^{n+1}, q^{n+1}, \xi^{n+1}, p^{n+1}\}$, such that

$$
\begin{aligned}
(\bar{f}_2(\phi) - f_2(\phi)) = 0,
\end{aligned}
$$

with the initial condition $\phi(t = 0) = \phi_0$, where $f(\phi) = f_1(\phi) + f_2(\phi)$.

$$
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{1}{2} \ln(1 + \phi) + \frac{a}{2} (1 - \phi) - \frac{a}{2} \ln \phi, \quad \phi \geq 1 - \kappa, \\
\frac{1}{2} \ln(1 + \phi) - \frac{a}{2} \ln \phi, \quad |\phi| < 1 - \kappa, \\
\frac{1}{2} \ln(1 + \phi) - \frac{a}{2} \ln(1 - \phi), \quad |\phi| < 1 - \kappa, \\
-\frac{a}{2} \ln(1 + \phi) + \frac{a}{2} \ln(1 - \phi), \quad |\phi| < 1 - \kappa, \\
\end{array} \right.
\end{aligned}
$$

and $f_2(\phi) = \bar{f}(\phi)$.

Let $N$ be a positive integer and $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of $[0, T]$. Let us denote by $\tau_n = t_{n+1} - t_{n}$, $n = 0, 1, \cdots, N$, where $\tau$ is the time step size. For the modified Cahn–Hilliard–Hele–Shaw equation, we proposed the following numerical scheme which is discrete in time and continuous in space, i.e., seek $\{\phi^{n+1}, q^{n+1}, \xi^{n+1}, p^{n+1}\}$, such that

$$
\begin{aligned}
(\bar{f}_2(\phi) - f_2(\phi)) = 0,
\end{aligned}
$$

with the initial condition $\phi(t = 0) = \phi_0$, where $f(\phi) = f_1(\phi) + f_2(\phi)$.
where the velocity is given by

$$u^{n+1} = (\nabla p^n + \gamma \phi^n \nabla \mu^{n+1})$$  \hspace{1cm} (11)

In fact, the velocity $u^{n+1}$ in equation (10) is an intermediate velocity, and the correction of velocity is the same as the incremental pressure projection method for Navier–Stokes equation (31). So the velocity can be obtained by using the intermediate velocity and satisfies

$$\nabla \cdot u^{n+1} = 0.$$  \hspace{1cm} (12)

Combining (11) with (12), the Darcy equation will be obtained. By using the divergence operator to (12), the true velocity $u^{n+1}$ can be eliminated. So, we have

$$\nabla \cdot u^{n+1} = \nabla \cdot (\nabla p^{n+1} - \nabla p^n).$$  \hspace{1cm} (13)

2.2. The Fully Discrete Scheme. Let $T_h = \{K\}$ be a conforming, shape-regular, globally quasi-uniform family of triangulations of $\Omega$. $h = \max h_i$, where $h_i$ is the grid size. $S_h$ is a piecewise continuous finite element space, which is defined as

$$S_h = \{v_h \in C^0(\Omega) | v_h|_K \in P_k(x, y), K \in T_h \subset H^1(\Omega),$$  \hspace{1cm} (14)

where $P_k(x, y)$ is the space of polynomials of degree at most $k$ in $Z$. In addition, we define $L^2_h = \{\phi \in L^2(\Omega) | (\phi, 1) = 0\}$ and $S_h = S_h \cap L^2_h(\Omega)$. The fully discrete finite element formulation for the modified Cahn–Hilliard–Hele–Shaw equation is introduced: for any $\{\omega_h, v_h, \phi_h, q_h\} \in S_h \times S_h \times S_h$, find $\{\phi_h^{n+1}, p_h^{n+1}\} \in S_h \times S_h$ and $\{\xi_h^{n+1}, p_h^{n+1}\} \in S_h \times S_h$ such that

$$W := \{u \in L^2(\Omega) | (u, \nabla q) = 0, \forall q \in H^1(\Omega)\},$$  \hspace{1cm} (17)

and the projection $P := L^2(\Omega) \rightarrow W$ is defined via

$$P(w) = \nabla p + w,$$  \hspace{1cm} (18)

where $p \in H^1(\Omega) := \{\phi \in H^1(\Omega) | (\phi, 1) = 0\}$ is the unique solution to

$$(\nabla p + w, \nabla q) = 0, \forall q \in H^1(\Omega).$$  \hspace{1cm} (19)

Clearly, $P(w) \in W$ for any $w \in L^2(\Omega)$. Furthermore, we have what as follows.

\[ \text{Definition 1 (see Ref. [13]). Define} \]

$$\begin{align*}
\left\{ \frac{\phi^{n+1} - \phi^n}{\tau}, \omega \right\} - (\phi^n u^{n+1}, \nabla \omega) + \varepsilon (\nabla \mu^{n+1}, \nabla \omega) = 0, \quad &\forall \omega \in H^1(\Omega), \\
(\mu^{n+1}, v) - \frac{1}{\varepsilon} (f_1(\phi^{n+1}) - f_2(\phi^n), v) - \varepsilon (\nabla \phi^{n+1}, \nabla v) - (\xi^{n+1}, v) = 0, \quad &\forall v \in H^1(\Omega), \\
(\nabla \xi^{n+1}, \nabla \phi_h) - \theta (\phi^{n+1} - \bar{\phi}_0, \phi_h) = 0, \\
(\nabla p^{n+1} + \gamma \phi_h \nabla \mu^{n+1}, \nabla q_h) = 0,
\end{align*}$$

where $\phi_0 := R_h \phi_0$.

The operator $R_h: \phi \in H^1(\Omega) \rightarrow S_h$ is the Ritz projection and satisfies

$$(\nabla (R_h \phi - \phi), \nabla \chi) = 0, \forall \chi \in S_h, \ (R_h \phi - \phi, 1) = 0.$$  \hspace{1cm} (16)

To analyze our scheme, we make the following projection.

To define
Lemma 1 (see Ref. [13]). $P$ is linear, and given $w \in L^2(\Omega)$, it follows that

$$
(P(w) - w, v) = 0, \forall v \in W. \quad (20)
$$

In particular, since $P(w) \in W$,

$$
(P(w) - w, P(w)) = 0. \quad (21)
$$

Consequently,

$$
\|P(w)\| \leq \|w\|, \forall w \in L^2(\Omega). \quad (22)
$$

Definition 2 (see Ref. [13]). Defining

$$
\mathbf{W}_h : = \{u_h \in L^2(\Omega) \mid (u_h, \nabla q_h) = 0, \forall q_h \in S_h\},
$$

we observe that $\mathbf{W} \subset \mathbf{W}_h$. The projection $P_h : L^2(\Omega) \longrightarrow \mathbf{W}_h$ is defined via

$$
P_h(w) = \nabla p_h + w, \quad (24)
$$

where $p_h \in S_h$ is the unique solution to

$$
(\nabla p_h + w, \nabla q_h) = 0, \forall q_h \in S_h. \quad (25)
$$

Clearly $P_h \in \mathbf{W}_h$. Furthermore, we have what as follows.

Lemma 2 (see Ref. [13]). $P_h$ is linear, and given any $w \in L^2(\Omega)$, it follows that

$$
(P_h(w) - w, v) = 0, \forall v \in \mathbf{W}_h. \quad (26)
$$

In particular, since $P_h(w) \in \mathbf{W}_h$,

$$
(P_h(w) - w, P_h(w)) = 0. \quad (27)
$$

Consequently,

$$
\|P_h(w)\| \leq \|w\|, \forall w \in L^2(\Omega). \quad (28)
$$

There is an estimate for the difference between the projections $P$ and $P_h$.

Lemma 3 (see Ref. [13]). Suppose that $w \in H^q(\Omega)$ with the compatible boundary conditions $w \cdot n = 0$ on $\partial \Omega$ and $p \in H^{q+1}(\Omega)$, where

$$
\nabla p = P(w) - w. \quad (29)
$$

Then

$$
\|P_h(w) - P(w)\| \leq C h^q |p|_{H^{q+1}}. \quad (30)
$$

Lemma 4 (see Ref. [32]). Define the following variational problem, given $\zeta \in \hat{S}_h$, find $T_h(\zeta) \in S_h$ such that

$$
(\nabla T_h(\zeta), \nabla \chi) = (\zeta, \chi), \quad (31)
$$

where $T_h : S_h \longrightarrow \hat{S}_h$ is an invertible linear operator. Suppose $\zeta, \psi \in \hat{S}_h$ and set

$$
(\zeta, \psi)_{-1,h} = (\nabla T_h(\zeta), \nabla T_h(\psi)) \quad (32)
$$

and

$$
(\zeta, (T_h(\zeta), \psi)), \quad (33)
$$

where $(\cdot, \cdot)_{-1,h}$ defines an inner product on $\hat{S}_h$. Hence, for $\forall \zeta \in \hat{S}_h$ and $\forall g \in S_h$,

$$
|\zeta, g| \leq \|\zeta\|_{-1,h} \|\nabla g\|. \quad (34)
$$

Moreover, the following Poincar e-type estimate holds true: for some constants $c > 0$, independent of $h$, where

$$
\|\zeta\|_{-1,h} \leq c \|\zeta\|, \forall \zeta \in \hat{S}_h. \quad (35)
$$

3. The Energy Stability

Theorem 1. Let $\{\phi_{h,1}, \mu_{h,1}, \nu_{h,1}, P_{h,1}\}$ be the unique solution of scheme (15), and define

$$
\Xi(\phi_{h,1}) = E(\phi_{h,1}) + \frac{\theta}{2} \|\mu_{h,1} - \theta\|_{-1,h}^2 + \frac{\tau_1}{2} \|\nu_{h,1} - \theta\|_{-1,h}^2. \quad (36)
$$

Then, the following energy law holds true for any $\tau, h, \epsilon > 0$, where

$$
\Xi(\phi_{h,1}^n) + \frac{\tau}{2} \|\nabla(\phi_{h,1}^n - \phi_{h,1}^n)^2\| + \frac{\theta}{2} \|\mu_{h,1} - \phi_{h,1}^n\|_{-1,h}^2 + \frac{\tau}{2} \|\nu_{h,1} - \phi_{h,1}^n\|_{-1,h}^2 \leq \Xi(\phi_{h,1}^n). \quad (37)
$$

Proof. Utilizing the definition, we get

$$
u_{h,1} = -(\nabla p_{h,1} + \gamma \psi \nabla \omega_{h,1}), \quad (38)
$$

one sees that the scheme (15) can be reformulated as

$$
\left(\frac{\phi_{h,1}^n - \phi_{h,1}^n}{\tau}, \omega_h\right) - \left(\phi_{h,1}^n \nabla \omega_h, \nabla \omega_h\right) \in \epsilon(\nu_{h,1} \nabla \omega_h) = 0, \quad (39)
$$

$$
\left(\mu_{h,1}^n, v_h\right) - \left(\phi_{h,1}^n \phi_{h,1}^n, v_h\right) - \epsilon(\nabla \phi_{h,1}^n, \nabla v_h) = 0, \quad (40)
$$

$$
\left(\nabla \nabla \phi_{h,1}^n, \nabla \phi_{h,1}^n\right) - \left(\phi_{h,1}^n \phi_{h,1}^n, v_h\right) \in \epsilon(\nabla \phi_{h,1}^n, \nabla v_h) = 0. \quad (41)
$$

Taking the test function $\omega_h = \tau \phi_{h,1}^n, v_h = -\phi_{h,1}^n - \phi_{h,1}^n, \phi_{h,1}^n = -\tau_h (\phi_{h,1}^n - \phi_{h,1}^n)$ in equations (38) and (39), respectively, and using the identity $2(a, a - b) = a^2 - b^2 + (a - b)^2$, we obtain
\begin{align}
\phi^{n+1}_h - \phi^n_h &= \frac{\tau}{\epsilon} \left( f \left( \phi^{n+1}_h \right) - f \left( \phi^n_h \right) \right), \\
+ \left( \mu^{n+1}_h \cdot \phi^{n+1}_h - \mu^n_h \cdot \phi^n_h \right) + \frac{1}{\epsilon} \left( f_1 \left( \phi^{n+1}_h \right) - f_2 \left( \phi^n_h \right) \right), \\
+ \epsilon \left( \| \nabla \phi^{n+1}_h \|_2^2 - \| \nabla \phi^n_h \|_2^2 + \| \nabla \phi^{n+1}_h - \phi^n_h \|_2^2 \right), \\
+ \frac{\theta}{\epsilon} \left( \phi^{n+1}_h \cdot \nabla \phi^n_h - \phi^n_h \cdot \nabla \phi^n_h \right) = 0.
\end{align}

Taking the inner product of equation (36) with \( \tau/\epsilon \mathbf{u}_h^{n+1} \), we get

\begin{align}
\frac{\tau}{\epsilon} \| \mathbf{u}_h^{n+1} \|^2 + \frac{\tau}{\epsilon} \left( \nabla \mathbf{P}_h^n, \mathbf{u}_h^{n+1} \right) &= -\left( \phi^n_h \nabla \mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1} \right).
\end{align}

Then taking the test function \( q_h = \tau/\epsilon \mathbf{p}_h^n \) in equation (40) and using the identity \( (a - b, b) = a^2 - b^2 - (a - b)^2 \), we get

\begin{align}
\frac{\tau}{\epsilon} \left( \| \nabla \mathbf{p}_h^{n+1} \|^2 - \| \nabla \mathbf{p}_h^n \|^2 - \| \nabla \mathbf{p}_h^{n+1} - \nabla \mathbf{p}_h^n \|^2 \right)
= \frac{\tau}{\epsilon} \left( \nabla \mathbf{p}_h^n, \mathbf{u}_h^{n+1} \right).
\end{align}

Summing up equations (49)–(51), one can conclude that

\begin{align}
\frac{\tau}{\epsilon} \| \nabla \mathbf{u}_h^{n+1} \|^2 + \frac{1}{\epsilon} \left( F \left( \phi^{n+1}_h \right) - F \left( \phi^n_h \right) \right), 1
+ \frac{\epsilon}{2} \left( \| \nabla \phi^{n+1}_h \|^2 - \| \nabla \phi^n_h \|^2 + \| \nabla \left( \phi^{n+1}_h - \phi^n_h \right) \|^2 \right),
+ \frac{\theta}{\epsilon} \left( \| \phi^{n+1}_h - \phi^n_h \|^2 - \| \phi^{n+1}_h - \phi^n_h \|^2 \right)
+ \frac{\tau}{\epsilon} \| \mathbf{u}_h^{n+1} \|^2,
+ \frac{\tau}{\epsilon} \left( \| \nabla \mathbf{p}_h^{n+1} \|^2 - \| \nabla \mathbf{p}_h^n \|^2 - \| \nabla \mathbf{p}_h^{n+1} - \nabla \mathbf{p}_h^n \|^2 \right) \leq 0.
\end{align}

Then, resetting \( q_h = \tau/\epsilon \left( \mathbf{p}_h^{n+1} - \mathbf{p}_h^n \right) \) in (40), we get

\begin{align}
\frac{\tau}{\epsilon} \left( \| \nabla \mathbf{p}_h^{n+1} - \mathbf{p}_h^n \|^2 = \frac{\tau}{\epsilon} \left( \nabla \mathbf{p}_h^{n+1} - \mathbf{p}_h^n \right), \mathbf{u}_h^{n+1} \right).
\end{align}

and using the Cauchy–Schwarz inequality, it can be written that

\begin{align}
\frac{\tau}{\epsilon} \| \nabla \mathbf{p}_h^{n+1} - \mathbf{p}_h^n \|^2 \leq \frac{\tau}{\epsilon} \| \mathbf{u}_h^{n+1} \|^2.
\end{align}

Adding (53) into (51), one obtains

\begin{align}
\frac{\tau}{\epsilon} \| \nabla \mathbf{u}_h^{n+1} \|^2 + \frac{1}{\epsilon} \left( F \left( \phi^{n+1}_h \right) - F \left( \phi^n_h \right) \right), 1
+ \frac{\epsilon}{2} \left( \| \nabla \phi^{n+1}_h \|^2 - \| \nabla \phi^n_h \|^2 + \| \nabla \left( \phi^{n+1}_h - \phi^n_h \right) \|^2 \right),
+ \frac{\theta}{\epsilon} \left( \| \phi^{n+1}_h - \phi^n_h \|^2 - \| \phi^{n+1}_h - \phi^n_h \|^2 \right)
+ \frac{\tau}{\epsilon} \| \mathbf{u}_h^{n+1} \|^2,
+ \frac{\tau}{\epsilon} \left( \| \nabla \mathbf{p}_h^{n+1} \|^2 - \| \nabla \mathbf{p}_h^n \|^2 \right) \leq 0.
\end{align}

Thus, the proof is completed. \( \square \)
Corollary 1. Let \( \{ \phi_{n+1}^i, \mu_{n+1}^i, \xi_{n+1}^i, p_{n+1}^i \} \) be the unique solution of scheme (15). Assume that \( \Xi(\phi_{n+1}^i) \leq C_0\), we have
\[
\max_{0 < t < h} \left( \| \nabla \phi_{n+1}^i \|^2 + \| \phi_{n+1}^i - \phi_{0}^i \|^2 - h \right) \leq C,
\]
(56)
\[
\sum_{i=0}^{n} \left( \| \nabla \phi_{n+1}^i \|^2 + \| \phi_{n+1}^i - \phi_{0}^i \|^2 - h \right) \leq C,
\]
(57)
\[
\tau \sum_{i=0}^{n} \left( \| \nabla \mu_{n+1}^i \|^2 + \| \mu_{n+1}^i - \mu_{0}^i \|^2 - h \right) \leq C,
\]
(58)
for constant \( C > 0 \) that is independent of \( h \) and \( \tau \).

**Proof.** Using the definition of free energy (35), we easily see that (56) holds true.

Summing equation (36) from \( i = 1 \) to \( i = n \), we get
\[
\Xi(\phi_{n+1}^i) + \frac{\theta}{2} \sum_{i=0}^{n} \| \phi_{n+1}^i - \phi_{0}^i \|^2 + \frac{\theta}{2} \sum_{i=0}^{n} \| \phi_{n+1}^i - \phi_{0}^i \|^2 - h \leq \Xi(\phi_{0}^i) \leq C,
\]
(59)
which leads to (56) and (57) immediately.

4. The Error Estimate

For \( \{ \phi, \mu, \xi, u \} \), we make the following regularization assumption:
\[
\phi \in H^1(0, T; H^q(\Omega)) \cap L^\infty(0, T; H^1(\Omega))
\]
\[
\cap L^\infty(0, T; H^q(\Omega)),
\]
\[
\mu \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^{q+1}(\Omega)),
\]
\[
\xi \in L^2(0, T; H^q(\Omega)),
\]
\[
u \in L^\infty(0, T; H^q(\Omega)),
\]
\[
\phi \nabla \mu \in L^\infty(0, T; H^q(\Omega)),
\]
where \( q \geq 1 \) is the spatial approximation order given in the space \( S_h \).

Define the backward difference operator \( \delta_{i+} \phi_{n+1}^i = \phi_{n+1}^i - \phi_{n}^i / \tau \). For simplicity, the following notation is introduced:
\[
\begin{align*}
\bar{\varepsilon}_\phi^i & = R\phi^i - \phi_{n+1}^i, \\
\bar{\varepsilon}_\mu^i & = R\mu^i - \mu_{n+1}^i, \\
\bar{\varepsilon}_\xi^i & = R\xi^i - \xi_{n+1}^i, \\
\bar{\varepsilon}_p^i & = R\rho^i - \rho_{n+1}^i, \\
\sigma(\phi_{n+1}^i) & = \delta_h R_h \phi_{n+1}^i - \partial_t \phi_{n+1}^i.
\end{align*}
\]
(60)

**Definition 3** (see Ref. [32]). The Ritz projection satisfies the following estimate
\[
\| \phi - R_h \phi \| + h \| \phi - R_h \phi \|_{H^1(\Omega)} \leq C h^{q+1} \| \phi \|_{H^{q+1}(\Omega)}.
\]
(62)

**Lemma 5.** Assuming that \( (\phi, \mu) \) is weak solution to equation (8), satisfying the regularity assumptions (62), we have the following estimate
\[
\| \sigma(\phi_{n+1}^i) \| \leq C h^{q+1} + C \tau, \quad \forall t \in (\tau, T],
\]
(63)
where \( C > 0 \) is a constant independent of \( h \) and \( \tau \).

**Theorem 2.** Assume that \( \{ \phi, \mu, \xi, p \} \) and \( \{ \phi_{n+1}^i, \mu_{n+1}^i, \xi_{n+1}^i, p_{n+1}^i \} \) are the solutions of (8) and (15), respectively. Then, for any \( h, \tau > 0 \), there exists \( C > 0 \) independent of \( h \) and \( \tau \), such that
\[
\sum_{i=0}^{n} 2 \epsilon \| \nabla \phi_{n+1}^i \|^2 + \epsilon \| \nabla \phi_{n+1}^i \|^2 \leq C \tau + C h^{q+q},
\]
(64)

**Proof.** Subtracting equation (15) from (8) at \( t = n + 1 \), we have
\[
\begin{align*}
(\delta_\tau \bar{\varepsilon}_{\phi_{n+1}^i}, \omega_h) + \epsilon (\nabla \bar{\varepsilon}_{\phi_{n+1}^i}, \nabla \omega_h) - (\sigma(\phi_{n+1}^i), \omega_h), \\
+ (\bar{\varepsilon}_{\mu_{n+1}^i} \nabla \bar{\varepsilon}_{p_{n+1}^i} + \gamma \bar{\varepsilon}_{\xi_{n+1}^i} \nabla \bar{\varepsilon}_{\mu_{n+1}^i}, \nabla \omega_h)
\end{align*}
\]
(65)
\[
- (\bar{\varepsilon}_{\mu_{n+1}^i} \nabla \bar{\varepsilon}_{\mu_{n+1}^i} + \gamma \bar{\varepsilon}_{\xi_{n+1}^i} \nabla \bar{\varepsilon}_{\mu_{n+1}^i}, \nabla \omega_h) = 0,
\]
\[
(\bar{\varepsilon}_{\phi_{n+1}^i}, v_h) + (\bar{\varepsilon}_{\mu_{n+1}^i}, v_h) - \frac{1}{\epsilon} (f_1(\phi_{n+1}^i) - f_2(\phi_{n+1}^i), v_h)
\]
(66)
\[
+ \frac{1}{\epsilon} (f_1(\phi_{n+1}^i) - f_2(\phi_{n+1}^i), v_h),
\]
\[
- \epsilon (\nabla \bar{\varepsilon}_{\phi_{n+1}^i}, \nabla v_h) - (\bar{\varepsilon}_{\xi_{n+1}^i}, v_h) - (\bar{\varepsilon}_{\xi_{n+1}^i}, v_h) = 0,
\]
\[
(\nabla \bar{\varepsilon}_{\phi_{n+1}^i}, \nabla v_h) + (\nabla \bar{\varepsilon}_{\phi_{n+1}^i}, \nabla v_h) - (\nabla \bar{\varepsilon}_{\phi_{n+1}^i}, \nabla v_h) = 0,
\]
(67)
\[
(\nabla \bar{\varepsilon}_{\phi_{n+1}^i}, \nabla v_h) + (\nabla \bar{\varepsilon}_{\phi_{n+1}^i}, \nabla v_h) - (\nabla \bar{\varepsilon}_{\phi_{n+1}^i}, \nabla v_h) = 0.
\]
(68)

Setting \( \omega_h = \bar{\varepsilon}_{\phi_{n+1}^i}, v_h = -\bar{\varepsilon}_{\phi_{n+1}^i}, \phi_h = -T_h(\delta_\tau \bar{\varepsilon}_{\phi_{n+1}^i}) \), and \( q_h = \bar{\varepsilon}_{\phi_{n+1}^i} \) in equations (65)–(68), respectively, and adding them together and using Lemma 4, we get
\begin{align*}
\epsilon \|\nabla v_{\mu}^{n+1}\|^2 & + \frac{\epsilon}{27} \left( \|v_{\mu}^{n+1}\| - \|v_{\mu}^{n}\| + \|v_{\mu}^{n+1} - v_{\mu}^{n}\|^2 \right), \\
+ \epsilon \|v_{\mu}^{n+1}\|^2 & + \frac{\theta}{27} \left( \|v_{\mu}^{n}\|^2 - \|v_{\mu}^{n+1}\|^2 + \|v_{\mu}^{n+1} - v_{\mu}^{n}\|^2 \right), \\
= & \left( \phi_{\mu}^{n+1}, \phi_{\mu}^{n+1} \right) + \left( e_{\mu}^{n+1}, e_{\mu}^{n+1} \right) - \left( \phi_{\mu}^{n+1}, T_h(\delta \phi_{\mu}^{n+1}) \right), \\
- & \left( \phi_{\mu}^{n+1}(\nabla p_{\mu}^n + \nabla \phi_{\mu}^{n+1}), \nabla v_{\mu}^{n+1} \right), \\
+ & \left( \phi_{\mu}^{n+1}(\nabla p_{\mu}^n + \nabla \phi_{\mu}^{n+1}), \nabla v_{\mu}^{n+1} \right), \\
- & \frac{1}{\epsilon} \left( f_1(\phi_{\mu}^{n+1}) - f_2(\phi_{\mu}^{n+1}), \delta \phi_{\mu}^{n+1} \right), \\
+ & \frac{1}{\epsilon} \left( f_1(\phi_{\mu}^{n+1}) - f_2(\phi_{\mu}^{n+1}), \delta \phi_{\mu}^{n+1} \right), \\
- & e(\phi_{\mu}^{n+1}, \phi_{\mu}^{n+1}) - \phi_{\mu}^{n+1}(\nabla \phi_{\mu}^{n+1}, \nabla v_{\mu}^{n+1})), \\
= & M_1 + M_2 + M_3 + M_4 + M_5 + M_6,
\end{align*}

where we denote

\begin{align*}
M_1 & = \left( \phi_{\mu}^{n+1}, \phi_{\mu}^{n+1} \right), \\
M_2 & = \left( \phi_{\mu}^{n+1}, \delta \phi_{\mu}^{n+1} \right), \\
M_3 & = -\left( \phi_{\mu}^{n+1}, T_h(\delta \phi_{\mu}^{n+1}) \right), \\
M_4 & = -\left( \phi_{\mu}^{n+1}(\nabla p_{\mu}^n + \nabla \phi_{\mu}^{n+1}), \nabla v_{\mu}^{n+1} \right), \\
M_5 & = \frac{1}{\epsilon} \left( f_1(\phi_{\mu}^{n+1}) - f_2(\phi_{\mu}^{n+1}), \delta \phi_{\mu}^{n+1} \right), \\
M_6 & = \frac{1}{\epsilon} \left( f_1(\phi_{\mu}^{n+1}) - f_2(\phi_{\mu}^{n+1}), \delta \phi_{\mu}^{n+1} \right), \\
M_7 & = -e(\phi_{\mu}^{n+1}(\nabla \phi_{\mu}^{n+1}, \nabla v_{\mu}^{n+1})),
\end{align*}

Now, we estimate \( M_1 \) separately. Using Cauchy–Schwarz inequality, Poincaré inequality, Young’s inequality, and Lemma 5, we get

\begin{align*}
M_1 & \leq \left( \phi_{\mu}^{n+1}, \phi_{\mu}^{n+1} \right), \\
& = \left( \phi_{\mu}^{n+1}, \phi_{\mu}^{n+1} - \phi_{\mu}^{n+1} \right), \\
& \leq \|\phi_{\mu}^{n+1}\| \|\phi_{\mu}^{n+1} - \phi_{\mu}^{n+1}\|, \\
& \leq \|\phi_{\mu}^{n+1}\| \|\nabla v_{\mu}^{n+1}\|, \\
& \leq 2 \|\phi_{\mu}^{n+1}\|^2 + \frac{1}{8} \|\nabla v_{\mu}^{n+1}\|^2, \\
& \leq Ch^2 \|v_{\mu}^{n+1}\|^2 + Cr^2 + C \|\nabla v_{\mu}^{n+1}\|^2.
\end{align*}

Using (33) in Lemma 4, Definition 3, and Young’s inequality, we get the following estimate:

\begin{align*}
M_2 & \leq \left[ \phi_{\mu}^{n+1}, \delta \phi_{\mu}^{n+1} \right], \\
& \leq \|\nabla v_{\mu}^{n+1}\| \|\delta \phi_{\mu}^{n+1}\|_{-1,h}, \\
& \leq Ch^2 + C \|\delta \phi_{\mu}^{n+1}\|_{-1,h}.
\end{align*}
where we note \( f'(\zeta) = L \).

Using Definitions 1 and 2, Lemma 3, Taylor extension \( \| \nabla \delta \| p^{n+1} \leq C r^2 \), the Cauchy–Schwarz inequality, and Young's inequality, one obtains

\[
M_{\delta} = -\epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n} \mu^{n+1} + \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}),
\]

\[
= -\epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n} \mu^{n+1} + \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}),
\]

\[
+ \epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n}, \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}),
\]

\[
+ \epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n+1}, \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}),
\]

\[
+ \epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n+1} \mu^{n+1} + \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}),
\]

\[
= \epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n}, \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}),
\]

\[
+ \epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n}, \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}),
\]

\[
\leq C \| \psi (p^{n+1} - p^n) \|^2 + \frac{1}{4} \| \mathcal{V}_p^{n+1} \|^2,
\]

\[
+ C \| \psi (p^{n+1} - p^n) \|^2 + \frac{1}{4} \| \mathcal{V}_p^{n+1} \|^2,
\]

\[
\leq C r^2 + C h^2 + C \| \mathcal{V}_p^{n+1} \|^2.
\]

Combining equations (69)–(75), we have

\[
\| \delta c^{n+1}_\phi \|_{-1,h}^2 = -\epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n} \mu^{n+1} + \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}),
\]

\[
+ b(k^{n+1} + \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}, T_h(\delta c^{n+1}_\phi)),
\]

\[
+ b(-\epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n}, \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}), T_h(\delta c^{n+1}_\phi)),
\]

\[
+ b(-\epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n+1}, \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}), T_h(\delta c^{n+1}_\phi)),
\]

\[
+ b(-\epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n+1} \mu^{n+1} + \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}), T_h(\delta c^{n+1}_\phi)),
\]

\[
\leq \epsilon \| \mathcal{V}_p^{n+1} \|_h \| (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n}, \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}) \|_h + \| \phi^{n+1} \|_h.
\]

\[
+ C d^2 + C h^2 + C \| \mathcal{V}_p^{n+1} \|^2 + C \| \mathcal{V}_p^{n+1} \|^2 + C \| \delta c^{n+1}_\phi \|_{-1,h}^2.
\]

(79)

where \( b(\phi, u, \omega) = (\phi u, V u) \).

Following the same idea as in Ref. [13], we know \( D_t = \| \mathcal{V}_p^{n+1} \|_{-1,h} + 1 \leq C \). Then, substituting (76) into equation (75) and multiplying the result by \( 2T \), we get

\[
2 \epsilon \| \mathcal{V}_p^{n+1} \|^2 + \epsilon (\| \mathcal{V}_p^{n+1} \|^2 - \| \mathcal{V}_p^{n} \|^2 + \| \mathcal{V}_p^{n+1} - \mathcal{V}_p^{n} \|^2)
\]

\[
+ 2 \epsilon \| \mathcal{V}_p^{n+1} \|^2
\]

\[
+ \frac{\Theta}{2T} (\| \mathcal{V}_p^{n+1} \|^2 - \| \mathcal{V}_p^n \|^2 + \| \mathcal{V}_p^{n+1} - \mathcal{V}_p^n \|^2),
\]

\[
\leq C r^2 + C h^2 + C D_t^2 + C \| \mathcal{V}_p^{n+1} \|^2 + C \| \mathcal{V}_p^{n+1} \|^2
\]

\[
+ C \| \mathcal{V}_p^{n+1} \|^2 + C D_h \| \mathcal{V}_p^{n+1} \|^2 + C \| \delta c^{n+1}_\phi \|_{-1,h}^2.
\]

(80)

The next step is to estimate the norm \( \| \delta c^{n+1}_\phi \|_{-1,h} \). Setting \( \omega_h = T_h(\delta c^{n+1}_\phi) \) in equation (63) and using the same concept as \( M_o \), we obtain

\[
\| \delta c^{n+1}_\phi \|_{-1,h}^2 = -\epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n} \mu^{n+1} + \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}),
\]

\[
+ b(k^{n+1} + \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}, T_h(\delta c^{n+1}_\phi)),
\]

\[
+ b(-\epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n}, \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}), T_h(\delta c^{n+1}_\phi)),
\]

\[
+ b(-\epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n+1}, \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}), T_h(\delta c^{n+1}_\phi)),
\]

\[
+ b(-\epsilon \gamma (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n+1} \mu^{n+1} + \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}), T_h(\delta c^{n+1}_\phi)),
\]

\[
\leq \epsilon \| \mathcal{V}_p^{n+1} \|_h \| (\phi_0^{n+1} \mu^{n+1} - \phi_0^{n}, \mathcal{V}_p^{n+1} + \mathcal{V}_p^{n}) \|_h + \| \phi^{n+1} \|_h.
\]

\[
+ C d^2 + C h^2 + C \| \mathcal{V}_p^{n+1} \|^2 + C \| \mathcal{V}_p^{n+1} \|^2 + C \| \delta c^{n+1}_\phi \|_{-1,h}^2.
\]

Summing the above estimate (77) from \( i = 1 \) to \( i = n \) and applying the discrete Gronwall inequality, one can conclude that
randomly perturbed concentration fields as follows:
\[ \sum _{i=0}^{n} 2 \text{Re} \left| \nabla \psi _{\mu}^{i} \right|^2 + \epsilon \left| \nabla \psi _{\phi}^{n+1} \right|^2 + \theta \left| \nabla \phi _{\mu}^{n+1} \right|^2 < O(h^4). \] (81)

Thus, the theorem is proved. \( \square \)

5. Numerical Results

In the part of numerical experiment, some numerical examples are given to verify the accuracy of the above theoretical analysis. In the following tests, for the phase field \( \phi \), chemical potential \( \mu \), velocity \( u \), and the pressure \( p \), we use the P1 finite element space. All examples are carried out by using the Freefem++ package [33].

5.1. Convergence and Energy Dissipation. We consider the modified Cahn–Hilliard–Hele–Shaw system given in domain \([0, 1] \times [0, 1]\) with the initial condition
\[ \phi_0 = 0.24 \cos(2\pi x) \cos(2\pi y) + 0.4 \cos(\pi x) \cos(3\pi y). \] (82)

5.1.1. The Order of Spatial Convergence. Here, we discuss the spatial convergence order of \( \phi \). The fixed test parameters \( \epsilon = 0.1, a = 0.01, \kappa = 0.01, \tau = 0.01, T = 0.1 \), and the varying grid step size \( h = 1/16, 1/32, 1/64, 1/128 \) are used here. The errors and convergence rates in the \( H^1 \) norm are presented in Tables 1 and 2 with \( \theta = 0.00005 \) and \( \gamma = 0.1, 0.001 \), respectively, which confirm the theoretical convergence of \( O(h) \).

5.1.2. The Order of Temporal Convergence. Similarly, the temporal convergence order of \( \phi \) is discussed as follows. The test parameters are chosen as \( \epsilon = 0.01, a = 0.01, \kappa = 0.01, \tau = 0.01, T = 0.1 \), and time step size \( \tau = h = 1/64, 1/128, 1/256 \). The errors and convergence rates in the \( H^1 \) norm are presented in Tables 3 and 4 with \( \theta = 0, 0.0001 \) and \( \gamma = 0.01, 0.001 \), respectively, which confirm the theoretical convergence of \( O(h) \).

5.2. Spinodal Decomposition. Now, we study the phase separation dynamics, which is called spinodal decomposition in the modified Cahn–Hilliard–Hele–Shaw system with logarithmic potential. We select test parameters \( \epsilon = 0.02, \kappa = 0.01, \gamma = 0.01, \) and \( h = 1/32 \) for simulation in the domain \([0, 1] \times [0, 1]\). The initial condition is taken as the randomly perturbed concentration fields as follows:
\[ \phi_0 = 2 \ast \text{rand} - 1, \] (83)
where \( \text{rand} \in [0, 1] \). We show the following coarsening process.

We demonstrate evolutions of coarsening dynamics for \( \theta = 0, \theta = 10, \) and \( \theta = 200 \) with \( \tau = 0.0001 \) in Figures 1–18. It can be seen that the particles obtained by different \( \theta \) are similar when \( T = 0.0001 \). When \( \theta = 0 \), the coarsening process gradually become clearer with the increase of \( T \). We can obtain similar results for \( \theta = 10 \) and \( \theta = 200 \), so we do not report them here for the sake of brevity.

5.2.1. Energy Dissipation. Now, we test the energy dissipation for our proposed scheme. The energy function (4) of the modified CH–HS system (1a)–1g can be discreteized as
\[ E(\phi_h^{n+1}) = \int _{\Omega} \left( \frac{\epsilon}{2} \left| \nabla \phi_h^{n+1} \right|^2 + F(\phi_h^{n+1}) \right) dx, \] (84)
and the modified energy of the fully discrete scheme (15) is defined as
\[ \mathcal{E}(\phi_h^{n+1}) = E(\phi_h^{n+1}) + \frac{\theta}{2} \left| \nabla \phi_h^{n+1} - \nabla \phi_h^{n} \right|_{L^2}^2 + \frac{\tau}{2} \left| \nabla P_h^{n+1} \right|_{L^2}^2. \] (85)

We choose the final time \( T = 0.1, \tau = 0.01, \) and \( \epsilon = 0.1, a = 0.5, \gamma = 0.01, \kappa = 0.01 \). The evolution of energy is shown in Figure 19, and it is nonincreasing apparently.

In Figure 19(a), we conclude that the evolution of energy is dissipative with \( \theta = 0 \). Comparison of the energy with different \( \theta \) is shown in Figure 19(b). We can easily obtain that the energy decreases faster with larger \( \theta \) when time step \( \tau = 0.02, \) which is consistent with the fact that the rate of coarsening dynamics can be improved largely.
Figure 1: $T = 0.0001$, $\theta = 0$.

Figure 2: $T = 0.0001$, $\theta = 10$. 
Figure 3: $T = 0.0001$, $\theta = 200$.

Figure 4: $T = 0.0005$, $\theta = 0$. 
Figure 5: $T = 0.0005$, $\theta = 10$.

Figure 6: $T = 0.0005$, $\theta = 200$. 
Figure 7: $T = 0.001, \theta = 0$. 

Figure 8: $T = 0.001, \theta = 10$. 
Figure 9: $T = 0.001$, $\theta = 200$.

Figure 10: $T = 0.005$, $\theta = 0$. 
Figure 11: $T = 0.005$, $\theta = 10$.

Figure 12: $T = 0.005$, $\theta = 200$. 
Figure 13: $T = 0.01$, $\theta = 0$.

Figure 14: $T = 0.01$, $\theta = 10$. 
Figure 15: $T=0.01$, $\theta=200$.

Figure 16: $T=0.02$, $\theta=0$. 
Figure 17: $T = 0.02$, $\theta = 10$.

Figure 18: $T = 0.02$, $\theta = 200$. 
6. Conclusion

This paper presented an efficient and unconditionally stable method for studying the modified Cahn–Hilliard–Hele–Shaw system with logarithmic potential, which incorporated with the first-order backward differentiation formula in time and the mixed finite element method in space. In order to update the pressure, it is significant to solve the linear system at each time step by using an incremental pressure-stabilization technique. By exploiting the convex splitting method, we could treat the nonlinear term free energy for the logarithmic potential, and the use of the regularization procedure makes the domain for the functional $F(\phi)$ extended from $(-1,1)$ to $(-\infty,\infty)$ so that the problem of the overflow on the boundary can be avoided. Then, we not only proved that the first-order discrete scheme is unconditionally stable in energy but also carried out a rigorous error analysis. Various numerical experiments were given to demonstrate the first-order temporal accuracy of numerical algorithm and stability of discrete energy consistent with theoretical analysis. Moreover, we illustrated the capability to capture the realistic phenomena, such as the spinodal decomposition, which revealed the status of internal phase separation.

Data Availability

All data generated or analyzed during this study are included in this published article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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