Radiative Corrections to $\pi l_2$ and $K l_2$ Decays

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Abstract

We reexamine radiative corrections to $\pi l_2$ and $K l_2$ decays. We perform a matching calculation, including vector and axial vector resonances as explicit degrees of freedom in the long distance part. By considering the dependence on the matching scale and on the hadronic parameters, and by comparing with model independent estimates, we scrutinize the model dependence of the results. For the pseudoscalar meson decay constants, we extract the values $f_\pi = (92.1 \pm 0.3)$ MeV and $f_K = (112.4 \pm 0.9)$ MeV. For the ratios $R_\pi$ and $R_K$ of the electronic and muonic decay modes, we predict $R_\pi = (1.2354 \pm 0.0002) \cdot 10^{-4}$ and $R_K = (2.472 \pm 0.001) \cdot 10^{-5}$.

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1. Radiative corrections to $\pi\ell_2$ and $K\ell_2$ decays are interesting for two separate reasons. On the one hand, measurements of the decay rates for $\Gamma(\pi \rightarrow \mu \nu_\mu)$ and $\Gamma(K \rightarrow \mu \nu_\mu)$ are used to extract the decay constants $f_\pi$ and $f_K$, which are important input parameters for chiral perturbation theory [1]. Therefore it is important to understand how radiative corrections affect these parameters [2]. On the other hand, in the ratios $R_\pi = \Gamma(\pi \rightarrow e \nu_e)/\Gamma(\pi \rightarrow \mu \nu_\mu)$ and $R_K$, strong interaction uncertainties cancel to a large degree. Therefore they can be predicted very precisely [3, 4, 5, 6, 7, 8] and allow for low energy precision tests of the standard model.

The most recent discussion of radiative corrections to $\pi\ell_2$ decays can be found in [8]. These authors separate the radiative corrections into a long and a short distance part, matched at a scale $m_\rho$. In the long distance part, only pions are considered as active degrees of freedom. QED corrections to the decay of a pointlike pion [4] give rise to the leading, model independent contribution. Hadronic structure effects of the order $m_\pi^2/m_\rho^2 \ln(m_\rho^2/m_\pi^2)$ are also model independent [5] and have been included. For the remaining hadronic structure effects, order-of-magnitude estimates are given.

Regarding the short distance part, the running of the effective semileptonic four fermion interaction is used to evolve $G_F$ down from $m_Z$ to $m_\rho$.

This matching procedure is somewhat simplified, because in the long distance part there are missing hadronic structure effects, which become important for energy scales approaching $m_\rho$, and in the short distance part, the effective quark-antiquark-lepton-neutrino operator has been evolved down to the rather low scale $m_\rho$. We will use a different approach. We include vector and axial vector resonances as explicit degrees of freedom in the long distance part, which allows us to use a larger matching scale. We show that this leads to a drastic reduction of the matching scale dependence of the radiative correction. The inclusion of the hadronic resonances, however, unavoidably introduces model dependence. Our main goal therefore will be to study the size and the uncertainties of the model dependent contributions in detail, replacing the order-of-magnitude estimates in [8].

2. We separate the loop integration over the Euclideanized momentum $k^2$ of the virtual photon into long distances $k^2 = 0 \cdots m_\text{cut}^2$ and short distances $k^2 = \mu_\text{cut}^2 \cdots m_Z^2$. To calculate the long distance part, we construct an effective model by starting with the low energy theorems of QCD and adding resonance degrees of freedom along the lines of vector meson dominance.

The amplitude for the radiative decay $\pi \rightarrow \ell \nu_\ell \gamma$ consists of the model independent internal bremsstrahlung part (IB) and the hadronic structure dependent (SD) part [3, 10]. The latter is parametrized by two form factors $F_V(s)$ and $F_A(s)$. $F_V(0)$ and $F_A(0)$ can be determined from chiral perturbation theory [1, 10]

$$F_V^{(\pi)}(0) = \frac{m_\pi}{4\sqrt{2}\pi^2 f_\pi}.$$
We extrapolate from $s = 0$ to $s \leq m_\pi^2$ by assuming dominance by low lying resonances with the correct quantum numbers. In the case of $F_V$, we include small admixtures of the first two higher radials, with $\lambda = 0.136$ and $\mu = -0.051$ [11]. Thus

$$F_V(s) = \frac{F_V(0)}{1 + \lambda + \mu} [\text{BW}_\rho(s) + \lambda \text{BW}_{\rho'}(s) + \mu \text{BW}_{\rho''}(s)]$$

$$F_A(s) = F_A(0) \text{BW}_{a_1}(s)$$

(2)

$\text{BW}_X(t)$ denotes a Breit-Wigner propagator amplitude with energy dependent widths [12, 13].

From this model for the amplitude $\pi \to l\nu_l\gamma$ we can derive amplitudes for one-loop virtual corrections by contracting the emitted photon back to the diagram in all possible ways, using the same Feynman rules for the couplings of the photon again. This will be a good model for the virtual corrections for very small $k^2$ (where $k$ is the momentum flowing through the photon), if the model for the radiative decay (where $k^2 = 0$) was a realistic one in the first place. To extrapolate these one-loop amplitudes from $k^2 \approx 0$ up to $k^2 = \mu_{cut}^2$, we again use vector meson dominance. For the coupling $\gamma\pi\pi$, we use the parameterization of [12] for the electromagnetic form factor of the pion (which includes the $\rho$ and the $\rho'$. For the couplings of the photon to $\rho\pi$ and to $a_1\pi$, we assume $\omega$ and $\rho$ dominance, respectively.

In the case of the kaon $K \to l\nu_l(\gamma)$, we proceed very similarly. $F_V^{(K)}(0)$ and $F_A^{(K)}(0)$ are obtained from chiral perturbation theory and extrapolated to higher $s$ assuming dominance by the $K^*$ and the $K_1$, respectively. For the electromagnetic form factor of the kaon we use a $1/2 : 1/6 : 1/3$ coherent superposition of $\rho$, $\omega$ and $\Phi \sim (s\bar{s})$.

For the short distance correction, arising from virtual photons with $k^2 = \mu_{cut}^2 \cdots m_Z^2$, we consider the one-loop running of the effective four-fermion weak interaction $[\bar{u}_l\gamma^\mu\gamma\gamma\nu_l][\bar{u}_d\gamma^\nu\gamma\gamma\nu_d]$ from $m_Z$ down to $\mu_{cut}$. The leading $m_l$ and $\mu_{cut}$ dependence is given by [14]

$$\left(\frac{\delta\Gamma}{\Gamma_0}\right)_{\text{short dist.}} \approx \frac{2\alpha}{\pi} \frac{1}{m_l^2 - \mu_{cut}^2} \left(m_l^2 \ln \frac{m_Z}{m_l} - \mu_{cut}^2 \ln \frac{m_Z}{\mu_{cut}}\right)$$

(4)

3. In Fig. [4], we present the numerical result for the radiative correction $\delta\Gamma/\Gamma_0$
Figure 1: Radiative correction to $\Gamma(\pi \rightarrow \mu \nu_\mu)$, from our evaluation (solid) and from Ref. [8] (dashed)

\[
\frac{\delta \Gamma}{\Gamma_0}(\pi \rightarrow \mu \nu_\mu(\gamma))[\%]
\]

for the decay $\pi \rightarrow \mu \nu_\mu(\gamma)$ in variation with the matching scale $\mu_{\text{cut}}$. (We include all photons in the radiative decay, without a cut on the photon energy.) We also compare to the corresponding result from [8]. It is seen clearly that the inclusion of the meson resonances as explicit degrees of freedom drastically reduces the matching scale dependence. Our numerical result is stable for $\mu_{\text{cut}}$ from below 0.5 GeV to well above 4 GeV.

From Fig. 1, we obtain

\[
\frac{\delta \Gamma}{\Gamma_0}(\pi \rightarrow \mu \nu_\mu(\gamma)) = (1.88 \pm 0.04 \pm 0.08\%) + O(\alpha^2) + O(\alpha \alpha_s) \quad (5)
\]

The central value 1.88% has been obtained using $\mu_{\text{cut}} = 1.5$ GeV. We use a somewhat high central value for $\mu_{\text{cut}}$, because we have included the radial excitations ($\rho'$, $\rho''$) in the long distance part. The first error quoted ($\pm 0.04\%$) is the matching uncertainty, estimated by varying $\mu_{\text{cut}}$ by a factor of two ($0.75 \ldots 3$ GeV). The second error ($\pm 0.08\%$) estimate is the uncertainty from the hadronic parameters, obtained by varying $F_{V,A}(0)$, the relative contributions of the higher resonances and the resonance widths over reasonable ranges.

Leading higher order short distance corrections have been estimated in [8], which increase the short distance correction by 0.10%. There exist no estimates of $O(\alpha^2)$
corrections in the long distance part.

Therefore we will use
\[
\frac{\delta \Gamma}{\Gamma_0}(\pi \rightarrow \mu \nu_\mu(\gamma)) = (2.0 \pm 0.2)\% 
\]
(6)
to extract \( f_\pi \). With \(|V_{ud}| = 0.9744 \pm 0.0010\) \cite{16}, we obtain
\[
\begin{align*}
\frac{\delta \Gamma}{\Gamma_0}(\pi \rightarrow \mu \nu_\mu(\gamma)) &= (2.0 \pm 0.2)\% \\
\end{align*}
\]
(6)
to extract \( f_\pi \). With \(|V_{ud}| = 0.9744 \pm 0.0010\) \cite{16}, we obtain
\[
f_\pi = (92.14 \pm 0.09 \pm 0.09) \text{ MeV} 
\]
(7)
where the first error, \( \pm 0.09 \), is due to \( V_{ud} \), and the second one to the radiative corrections.

It should be emphasized that the definition of \( f_\pi \) is not unambiguous at \( O(\alpha) \) \cite{8}. By convention, one could absorb part of the radiative correction in \( f_\pi \). We define \( f_\pi \) by factoring out all radiative corrections from \( f_\pi \). This convention is identical to the one used in \cite{17, 8}, but not to the one used in \cite{2}.

Our result for \( f_\pi \) has to be compared to the one in \cite{8}, which is also quoted by the particle data group \cite{16}. Transcribing their result to our convention and to \(|V_{ud}| = 0.9744 \pm 0.0010\), their result reads
\[
f_\pi = (92.47 \pm 0.09 \pm 0.26) \text{ MeV} 
\]
(8)
where the first error \( \pm 0.09 \) is due to \( V_{ud} \), and the second error is estimated from the matching scale dependence. This is compatible with our result (7).

In applications of \( f_\pi \), one should use an error estimate which includes the full model dependence. Therefore we quote
\[
f_\pi = (92.1 \pm 0.3) \text{ MeV} 
\]
(9)
as our final result for \( f_\pi \).

In Fig. 2, we show our results for the radiative correction to the ratio \( R_\pi \). By convention, we have included all radiative decays \( \pi \rightarrow l\nu_l\gamma \) (IB + SD) in calculating \( R_\pi \) (no cut on the photon energy). We obtain
\[
\delta R_\pi = -(3.793 \pm 0.019 \pm 0.007)\% + O(\alpha^2) 
\]
(10)
where the first error (0.019\%) is the matching uncertainty, estimated by varying \( \mu_{\text{cut}} \) from 0.75 up to 3 GeV, and the second error (0.007\%) arises from the uncertainties in the hadronic parameters.

To further study the model independence of the result, we have analyzed in detail which scales contribute to the the loop integrals. We find that the contribution to the corrections to the decay rates themselves remain sizable at large \( k^2 \). However, the results for the electronic and muonic modes become approximately equal for large \( k^2 \),
Figure 2: Radiative correction to the ratio $R_\pi$. Solid: central values for the hadronic parameters. Dashed and dotted: Reasonable variations of these parameters.

$$\delta R_\pi [%]$$

and so the ratio $R_\pi$ is dominated by very small scales. The total contribution within the range $\sqrt{k^2} = 0.5 \cdots 3.0$ GeV, where the theoretical uncertainties are largest, is found to be 0.026% (we have added absolute values to take care of cancellations).

We have also compared our result to the leading model independent contributions [5] in detail. The hadronic structure dependent correction can be separated into three parts, corresponding to three gauge invariant sets of diagrams. Adding separately the absolute values of the differences of our full results minus the model independent contributions for these three parts, we obtain 0.011%. In view of this, we consider the error estimate ±0.020% in (10) as reliable.

We again have to consider higher order radiative corrections. In [8], the leading logarithms in $\ln(m_\mu/m_e)$ have been summed up to all orders in $\alpha$, increasing $R_\pi$ by $5.5 \cdot 10^{-4}$.

And so our prediction for $\delta R_\pi$ is

$$\delta R_\pi = (-3.793 \pm 0.020 + 0.055 \pm 0.01)\% = (-3.74 \pm 0.03)\%$$ (11)

In the sum, the first number is the central value and the second number the uncertainty of the $O(\alpha)$ correction. The third number is the leading higher order correction and ±0.01% is our estimate of the next-to-leading correction.

For the ratio $R_\pi$, this implies

$$R_\pi = R_\pi^{(0)} \left(1 + \delta R_\pi \right) = (1.2354 \pm 0.0002) \cdot 10^{-4}$$ (12)
This agrees with the prediction \( R_\pi = (1.2352 \pm 0.0005) \cdot 10^{-4} \) in [8] within their error estimate.

From a similar analysis for kaon decays, we obtain \( \frac{\delta \Gamma}{\Gamma_0} (K \to \mu \nu_\mu(\gamma)) = (1.3 \pm 0.2)\% \), which using \(|V_{us}| = 0.2205 \pm 0.0018\) [16] results in

\[
f_K = (112.4 \pm 0.9 \pm 0.1)\text{MeV}
\]  

(13)

The first error is due to \( V_{us} \) and the second one to the radiative correction.

In calculating \( R_K \), we include only the (soft) internal bremsstrahlung (IB) part of \( K \to e\nu_e\gamma \) and exclude the (hard) structure dependent (SD) part. Experimental results can be corrected to comply with this convention using the theoretical results for the differential distributions for the IB and the SD radiation [9, 10]. We obtain

\[
R_K = \frac{\Gamma(K \to e\nu_e(\gamma))}{\Gamma(K \to \mu\nu_\mu(\gamma))} = R_K^{(0)} \left( 1 + \delta R_K \right) = 2.569 \cdot 10^{-5} \times \left( 1 - 0.0378 \pm 0.0004 \right)
\]

\[
= (2.472 \pm 0.001) \cdot 10^{-5}
\]

(14)

4. We have calculated \( f_\pi, f_K, R_\pi \) and \( R_K \) in an improved matching calculation, which includes vector and axial vector resonances as explicit degrees of freedom. The central values we quote include small model dependent contributions, but their error bars are based on the full size of these model dependent contributions, and in this sense our final predictions can be considered as model independent.

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