Kaon Condensation and Goldstone’s Theorem

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Abstract

We consider QCD at a nonzero chemical potential for strangeness. At a critical value of the chemical potential equal to the kaon mass, kaon condensation occurs through a continuous phase transition. We show that in the limit of exact isospin symmetry a Goldstone boson with the dispersion relation \( E \sim p^2 \) appears in the kaon condensed phase. At the same time, the number of the Goldstone bosons is less than the number of broken generators. Both phenomena are familiar in non-relativistic systems. We interpret our results in terms of a Goldstone boson counting rule found previously by Nielsen and Chadha. We also formulate a criterion sufficient for the equality between the number of Goldstone bosons and the number of broken generators.

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1 Introduction

Because of the spontaneous breaking of chiral symmetry, QCD at low energy reduces to a theory of weakly interacting Goldstone bosons. In the chiral limit the Lagrangian of this theory is completely determined by chiral symmetry and Lorentz invariance. Its predictions for the low-energy phenomenology of QCD have been very successful [1]. One might be tempted to extend this theory to describe QCD at nonzero chemical potentials. For a baryon chemical potential, the extension is rather trivial since the Goldstone bosons do not carry baryon charge. Thus a small baryon chemical potential does not have any effect on the chiral dynamics. The situation is more interesting for an isospin chemical potential. In this case a transition to a pion condensed phase takes place at a chemical potential equal to the pion mass. For light quarks this phase transition takes place within the domain of validity of the chiral Lagrangian and is completely described by means of a low energy effective theory [2] which also describes phase quenched QCD [3]. Similarly, the low energy limit of QCD at nonzero strange chemical potential, including its rich phase diagram at zero temperature can be completely described in terms of a chiral Lagrangian [4].

In this letter we analyze the general properties of the spectrum of Goldstone bosons in a symmetry-broken phase induced by a chemical potential. In the process, we will encounter a rather unusual manifestation of Goldstone’s theorem [5, 6], in which the number of Goldstone bosons is less than the number of broken generators. A comment on the subtleties of Goldstone’s theorem is thus in order.

According to the most frequently encountered version of Goldstone’s theorem, the number of Goldstone modes is equal to the number of independent broken symmetry generators [7]. It is worth noting, however, that the original formulation of Goldstone’s theorem is much weaker: it states that in the presence of broken symmetries, there exists at least one massless mode. In relativistically covariant theories the number of Goldstone bosons is always equal to the number of broken generators (we will give our version of the proof later in the paper). There are, however, well-known examples of nonrelativistic theories where the number of Goldstone modes is not equal to the number broken generators [8, 9, 10, 11]. The simplest case is the Heisenberg ferromagnet, which has only one magnon despite the fact that the symmetry breaking pattern is $O(3) \to O(2)$. [This is in contrast to the antiferromagnet where the symmetry breaking pattern is the same but there are two magnons].

Goldstone’s theorem has been refined in a little known article by Nielsen and Chadha [12]. They distinguish two types of Goldstone bosons: those with an energy proportional to an even power of the momentum and those with a dispersion relation that is an odd power of the momentum. They formulated a theorem, which states that one has to count each Goldstone mode of the first type twice. More precisely, the sum of twice the number of Goldstone modes of the first type and the number of Goldstone modes of the second type is at least equal to the number of independent broken symmetry generators. For relativistic invariant systems the dispersion relation for massless states is necessarily linear.
and the total number of Goldstone bosons is equal to the number of broken symmetry generators. This theorem explains why one (instead of two) Goldstone boson appears in the Heisenberg ferromagnet: the dispersion relation for this boson is quadratic in the momentum \( E \sim p^2 \). The so-called canted phase of ferromagnets carry one Goldstone boson of each type \( \mathbb{Z}_2 \), and require a complete breaking of \( O(3) \).

Quantum field theory at nonzero chemical potential is not Lorentz invariant. This fact opens up the possibility of having both Goldstone modes with a linear dispersion relation and a quadratic dispersion relation, and the character of the Goldstone modes could change across a phase transition. The number of Goldstone bosons and broken generators do not need to coincide. In this letter we illustrate this possibility for the example of QCD at a nonzero strangeness chemical potential.

In section 2 we study a simple field-theoretical model that describes the interaction of the strange Goldstone bosons in the neighborhood of the kaon-condensation point. This model has been recently used to describe kaon condensation \[13, 14, 15, 16\] in the color-flavor locked phase of QCD \[17\] at very high densities. Goldstone’s theorem will be discussed in section 3. In section 4 we analyze the low-energy limit of QCD at nonzero strangeness chemical potential which was constructed in \[\mathbb{I}\]. Concluding remarks are made in section 5.

## 2 Spectrum of Goldstone modes in a simple model

In this section we consider the simple (Euclidean) Lagrangian,

\[
\mathcal{L} = (\partial_0 + \mu)\phi^\dagger(\partial_0 - \mu)\phi + \partial_i\phi^\dagger\partial_i\phi + M^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2, \tag{1}
\]

where \( \phi \) is a complex scalar doublet,

\[
\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{2}
\]

The Lagrangian \( \mathbb{I} \) possesses a \( U(2) = SU(2) \times U(1) \) symmetry, and \( \mu \) is the chemical potential with respect to the \( U(1) \) charge. The \( U(2) \) symmetry is essentially that of the Higgs sector of the standard model. The motivation for considering Eq. \( \mathbb{I} \) at finite \( \mu \) is two-fold. First, from the physics of kaon condensation in the color-flavor locked phase of QCD at high baryon density. In this phase, the lightest modes are the four charged and neutral kaons, which are degenerate if up and down quarks have the same mass. These kaons are described by the fields \( \phi_1 \) and \( \phi_2 \), while other mesons are neglected. The role of \( \mu \) can be played by the strangeness chemical potential or effectively by a strange quark mass \[13\]. Though somewhat simplified, Eq. \( \mathbb{I} \) captures the essential physics of kaon condensation. Second, near the phase transition point, the low-energy limit of QCD at
nonzero strangeness chemical potential given by the effective theory constructed in \[4\] can be reduced to the Lagrangian one involving only the strange degrees of freedom. The reason is that, at the phase transition point, only strange degrees of freedom become massless.

The chemical potential contributes \(-\mu^2 \phi^\dagger \phi\) to the potential energy. Here we consider only the case when \(M^2 > 0\) so the symmetry is unbroken at zero chemical potential. Turning on \(\mu\), we start to favor quanta of, say, positive \(U(1)\) charge ("strangeness"), but if \(|\mu| < \mu_c \equiv M\) we expect no change since the chemical potential is not sufficient to excite any quanta. Only when \(\mu > \mu_c\) do we expect any condensation. This is exactly the regime where the potential energy has a nontrivial minimum. The vacuum can be chosen to have

\[
\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \text{with} \quad v^2 = \frac{\mu^2 - M^2}{\lambda}.
\]  

(3)

The symmetry is thus broken from \(U(2) \to U(1)\), resulting in three broken generators.

The particle spectrum is obtained by expanding about the minimum of the potential. In the normal phase \((\mu < \mu_c)\), with \(\langle \phi \rangle = 0\), we find four modes, two with strangeness \(S = 1\) and two with \(S = -1\), and the dispersion relations are given by

\[
(E + S\mu)^2 = p^2 + M^2.
\]  

(4)

At the transition point \(\mu = M\), two modes become gapless (those with \(S = +1\)). By continuity, these modes become the Goldstone bosons in the broken phase, therefore there are only two Goldstone modes. As we will show, this is possible because one of the Goldstone bosons has a quadratic dispersion relation.

To find particle dispersion in the broken phase, we expand the fields about the minimum as follows

\[
\phi = \frac{1}{\sqrt{2}} e^{i\pi_3 \tau_k / v} \begin{pmatrix} 0 \\ v + \varphi \end{pmatrix}.
\]  

(5)

The quadratic part of the Lagrangian has the form

\[
\mathcal{L} = \frac{1}{2} \partial_\nu \varphi \partial^\nu \varphi + \frac{1}{2} \partial_\nu \pi_k \partial^\nu \pi_k - i \mu (\varphi \partial_0 \pi_3 - \pi_3 \partial_0 \varphi) - i \mu (\pi_1 \partial_0 \pi_2 - \pi_2 \partial_0 \pi_1) + v^2 \lambda \varphi^2.
\]  

(6)

The particle spectrum and dispersion relations thus follow from diagonalizing the \(2 \times 2\) matrix that couples the \(\varphi\) and the \(\pi_3\) modes,

\[
D_1 = \begin{pmatrix} p^2 - E^2 + 2v^2 \lambda & 2i\mu E \\ -2i\mu E & p^2 - E^2 \end{pmatrix}.
\]  

(7)
and the $2 \times 2$ matrix that couples the $\pi_1$ and the $\pi_2$ modes,

$$D_2 = \begin{pmatrix} p^2 - E^2 & 2i\mu E \\ -2i\mu E & p^2 - E^2 \end{pmatrix}. \quad (8)$$

First let us consider the sector of $\varphi$ and $\pi_3$. The dispersion relations are obtained from

$$\text{det} \ D_1 = (p^2 - E^2 - \lambda_1)(p^2 - E^2 - \lambda_2) = 0. \quad (9)$$

The eigenvalues $\lambda_1$ and $\lambda_2$ satisfy $\lambda_1\lambda_2 = -4\mu^2 E^2$. With $\lambda_1 = -2v^2\lambda + O(E)$ we thus have that $\lambda_2 = 2\mu^2 E^2 / \lambda v^2 + O(E^3)$. In this case with only one broken generator we thus find a quadratic $E$-dependence of $\lambda_2$ resulting in a linear dispersion relation. The small momentum expansion of the dispersion relation is given by

$$E^2 = \frac{\mu^2}{3} - \frac{M^2}{M^2} p^2 + O(p^4). \quad (10)$$

The other mode remains massive for $p \to 0$ with dispersion relation

$$E^2 = 6\mu^2 - 2M^2 + O(p^2). \quad (11)$$

Notice that the coefficient of $p^2$ in (10) vanishes at $\mu = M$. As we will see below, this is a necessary consequence of the continuity of the dispersion relation across the phase transition point.

In the sector of $\pi_1$, $\pi_2$ the curvature matrix ($D_2$ at $E = p = 0$) has two zero eigenvalues. However, only one of the pole masses vanish. This follows immediately from the dispersion relations

$$E = 2\mu + O(p^2), \quad E = \frac{p^2}{2\mu} + O(p^4). \quad (12)$$

which are obtained from

$$\text{det} \ D_2 = (p^2 - E^2 - \lambda_1)(p^2 - E^2 - \lambda_2) = 0, \quad (13)$$
Figure 1: Spectrum of the excitations of (1) as a function of $\mu$. The critical chemical potential is at $\mu = M$, and it is depicted by a vertical line. The degeneracies of each of the branches is given by the number along the curves.

with

$$\lambda_1 = 2\mu E, \quad \lambda_2 = -2\mu E.$$  \hspace{1cm} (14)

We observe that in our theory the total number of Goldstone modes is consistent with the Nielsen-Chadha theorem \[12\]. The behavior of the gaps of different modes in our theory as a function of $\mu$ is illustrated in Fig. 1.

3 Goldstone’s theorem

One might still be puzzled by the fact that both $\pi_1$ and $\pi_2$ in (3) correspond to two flat directions of the potential energy, but describe only one Goldstone mode. One way to understand this fact is to consider the limit of very low frequencies, much lower than $\mu$. In this case the terms proportional to $(\partial_0 \pi_1)^2$ and $(\partial_0 \pi_2)^2$ in Eq. (3) are negligible compared to the $\pi_1 \partial_0 \pi_2 - \pi_2 \partial_0 \pi_1$. Once these terms are ignored one immediately sees that $\pi_1$ and $\pi_2$ are two canonically conjugate variables, and hence corresponds to a Goldstone field and its time derivative, but not two different Goldstone fields.
As mentioned above, Lorentz invariant theories can have only Goldstone modes with \( \omega = p \), i.e., with a linear dispersion relation. The inverse is, however, not true: there are many theories which are not relativistically invariant but have only Goldstone modes with linear dispersion relations. The most familiar example is the antiferromagnet; there are also other examples \cite{18}. Since this class of theories seems to be rather wide, a more refined criterion to count the Goldstone bosons beside Lorentz invariance could be useful.

To distinguish the different possibilities we have formulated the following theorem:

**THEOREM.** If \( Q_i, i = 1, \ldots, n \) is the full set of broken generators, and if \( \langle 0 | [Q_i, Q_j] | 0 \rangle = 0 \) for any pair \((i, j)\), then the number of Goldstone bosons is equal to \( n \), i.e., the number of broken generators.

Indeed, assume that there are less than \( n \) Goldstone bosons. Since the zero-momentum Goldstone bosons are obtained by acting the broken generators on the ground state, there exist a linear combination of \( Q_i \) which annihilates the ground state,

\[
\sum_i a_i Q_i |0\rangle = 0.
\]  

Equation (15) does not necessarily contradict the assumption that there are \( n \) broken generators, since the coefficients \( a_i \) are, in general, complex, so \( \sum_i a_i Q_i \) may be non-Hermitian and hence does not need to be a generator of the symmetry algebra. However, the following objects are generators,

\[
Q_a = \sum_i \text{Re} a_i Q_i, \quad \text{and} \quad Q_b = \sum_i \text{Im} a_i Q_i,
\]

and hence \( Q_a |0\rangle \neq 0 \) and \( Q_b |0\rangle \neq 0 \). We will show that \( \langle 0 | [Q_a, Q_b] | 0 \rangle \neq 0 \). Indeed, Eq. (15) reads \( (Q_a + iQ_b) |0\rangle = 0 \), hence we can write

\[
Q_a |0\rangle = |b\rangle, \quad Q_b |0\rangle = -i|b\rangle,
\]

which is the definition of the state \( |b\rangle \). It is now trivial to see that

\[
\langle 0 | [Q_a, Q_b] | 0 \rangle = -2i\langle b|b\rangle \neq 0,
\]

which contradicts the original assumption. QED.

Recalling that \([Q_i, Q_j]\) is a linear combination of the symmetry generators (in general, both broken and unbroken), one can immediately conclude that if the densities of all conserved charges are equal zero, the number of Goldstone bosons is equal to the number of broken generators. To have a mismatch between the number of Goldstone bosons and the broken generators, it is necessary (but not sufficient) to have finite charge density.
For example, in ferromagnets the density of the total spin is nonzero, so it is possible to have only one magnon for two broken generators, while in antiferromagnets all densities vanish, and hence there should be two magnons. In particular, if the ground state $|0\rangle$ is Lorentz-invariant, all charge densities should vanish (since they are zeroth components of the corresponding four-currents), and there should be no mismatch.

In our example, there are three broken generators, $\tau_1, \tau_2$ and $\tau_3$. One can easily check that with the choice of the ground state (3) the expectation value of $[\tau_1, \tau_2] = i\tau_3$ is non-vanishing so that the usual counting of Goldstone modes cannot be used.$^1$ (Our theorem however does not require the number of Goldstone modes to be different from three).

This phenomenon was not observed in earlier studies of meson condensation in QCD $^{18,1,19}$. For example, in the case of a chemical potential for isospin in QCD with $N_c = 3$ and $N_f = 2$ meson condensation only leads to the breakdown of a global $U(1)$, so there is only one broken generator and one Goldstone mode. Also the case of QCD with $N_c = 2$ colors and $N_f = 2$ flavors at finite baryon density we have only one Goldstone boson due to the spontaneous breaking of the $U(1)$ baryon symmetry by the diquark condensate.

In general we can write the determinant of the inverse propagator as

$$\det D = \prod_k (p^2 - E^2 - \lambda_k). \quad (19)$$

The dispersion relations are obtained from $\det D = 0$. For a charge conjugation invariant system, $\det D$ is an even function of $\mu$, and, for finite quark masses, a smooth function of $\mu$ in the neighborhood of $\mu = 0$. Since $\mu$ only occurs in the combination $\mu E$ in the coupling matrix elements, we conclude that $\det(D)$ is a polynomial in $E^2$. Suppose that we have a curvature matrix with $n$ zero eigenvalues and $n_1$ ($n_2$) of these eigenvalues vanish linearly (quadratically) for small $E$. For each eigenvalue that approaches zero linearly in $E$ we find one mode with a quadratic dispersion relation and one mode with a nonzero mass, and, for each eigenvalue that approaches zero quadratically in $E$, we find two modes with a linear dispersion relation. Using charge conjugation invariance properties of our theory we find that the Nielsen-Chadha theorem $^{12}$ in this case reduces to the simple statement that $n_1 + n_2 = n$.

### 4 Kaon Condensation in QCD

Let us finally consider the low-energy limit of QCD for $N_f$ light quarks and a nonzero chemical potential $\mu$ for one of these quarks. For convenience, this quark will be identified as a strange quark. It is now well understood how to extend a chiral Lagrangian to nonzero chemical potential $^{20,18,3,4,19,21}$: Requiring QCD and the low-energy effective Lagrangian to have the same transformation properties of a local external vector

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$^1$The ground state has nonzero isospin density because the condensed kaon has a definite isospin.
source uniquely determines the dependence of the chiral Lagrangian on this vector source and thus on the chemical potential. No additional parameters are necessary. The low-energy limit of QCD with a nonzero strangeness chemical potential is thus given by the Lagrangian,

$$\mathcal{L} = \frac{F^2}{4} \text{Tr} \nabla_\nu \Sigma \nabla^{\nu} \Sigma^\dagger - \frac{1}{2} G \text{Tr}(\mathcal{M}^\dagger \Sigma + \Sigma \Sigma^\dagger),$$

(20)

where the $SU(N_f)$ valued field $\Sigma$ contains the pseudo-Goldstone modes and $\mathcal{M}$ is the mass matrix which is taken proportional to the identity in this note. This theory contains two parameters, the pion decay constant $F$ and the chiral condensate $G$. The chemical potential enters through the covariant derivative

$$\nabla_\nu \Sigma = \partial_\nu \Sigma - i[B_\nu, \Sigma],$$

(21)

where $B_\nu$ only contains the chemical potential for the strange quark and is given by

$$B_\nu = \delta_{0\nu} \text{diag}(0, \ldots, 0, -i\mu).$$

(22)

Kaon-condensation in QCD at very high density is described by the same chiral Lagrangian with the chemical potential replaced by an induced chemical potential $\mu_{\text{eff}} \simeq m_s^2/2p_F$. The covariant derivative of the chiral field takes the form

$$\nabla_\nu \Sigma = \partial_\nu \Sigma + \delta_{0\nu} \left(\frac{\mathcal{M} \mathcal{M}^\dagger}{2p_F} \Sigma - \Sigma^\dagger \frac{\mathcal{M}^\dagger \mathcal{M}}{2p_F}\right).$$

(23)

In the CFL phase, the structure of the mass term is also different. Because of color-flavor-locking, we have to replace $\mathcal{M} \rightarrow \tilde{\mathcal{M}} = \det(\mathcal{M}) \mathcal{M}^{-1}$ [22].

In [4] this theory was studied in detail for $N_f = 3$. It was found that a phase transition to a kaon condensed phase takes place at $\mu = M$. For $\mu < M$ the saddle point of the static part of the effective Lagrangian is given by the identity matrix. The pole masses of the pseudo-Goldstone modes are modified by the chemical potential according to

$$M \rightarrow M - S\mu, \quad \mu < M,\quad (24)$$

where $S$ is the strangeness of the Goldstone bosons. These pseudo-Goldstone modes satisfy standard dispersion relations with $E^2 - (\sqrt{p^2 + M^2} - S\mu)^2 = 0$. For $N_f = 3$ two pseudo-Goldstone bosons become massless at $\mu = M$. For general value of $N_f$, $N_f - 1$ Goldstone bosons become massless at the critical point.
In the kaon condensed phase both the quark-antiquark condensate and the kaon condensate are nonzero. The saddle point manifold has the structure

$$
\hat{\Sigma} = \begin{pmatrix}
1 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 1 & 0 & 0 \\
0 & \cdots & 0 & \cos \alpha & e^{i\theta} \sin \alpha \\
0 & \cdots & 0 & -e^{-i\theta} \sin \alpha & \cos \alpha
\end{pmatrix},
$$

(25)

where \( \cos \alpha = M^2/\mu^2 \) for \( \mu > M \). The iso-vector symmetry of the QCD Lagrangian with non-zero quark mass and non-zero strange chemical potential is thus broken spontaneously according to

$$
SU(N_f - 1) \times U(1) \rightarrow SU(N_f - 2) \times U(1).
$$

(26)

According to the naive version of Goldstone’s theorem the total number of massless states is equal to \((N_f - 1)^2 - (N_f - 2)^2 = 2N_f - 3\). We thus find a mismatch in the total number of massless states on either sides of the phase transition point. In the same way as was discussed before this mismatch is resolved because of the presence of Goldstone modes with a quadratic dispersion relation.

However, this leads to another puzzle. Our model describes a continuous phase transition with a saddle point that is a continuous function of the chemical potential. The eigenvalues of the second derivative matrix at the saddle point and the corresponding poles masses should be continuous at the phase transition point as well. However, while above the critical point there is a Goldstone mode with linear dispersion, right at the transition point all gapless degrees of freedom have quadratic dispersion. Indeed, at that point the two poles with non-zero strangeness quantum number obey the dispersion relation

$$
E^2 = [\sqrt{M^2 + p^2} \pm M]^2 = 2M^2 + p^2 \pm 2M \sqrt{M^2 + p^2}.
$$

(27)

The dispersion relation of the would-be Goldstone modes is thus given by

$$
E^2 = \frac{p^4}{4M^2} + \cdots,
$$

(28)

which is quadratic.

The solution to this puzzle is simple: continuity in \( \mu \) requires that a linear dispersion relation for \( \mu > \mu_c \) is given by

$$
E^2 = \gamma(\mu)p^2 + \frac{p^4}{4M^2} + \cdots \quad \text{for} \quad \mu > \mu_c,
$$

(29)
where $\gamma(\mu_c) = 0$. Indeed, this general result is in agreement with the dispersion relations (10,13) of our model Lagrangian (1).

Above the phase transition point in the superfluid phase, the matrix of second derivatives factorizes into two $4 \times 4$ matrices given by [1]

$$
M_1 = \frac{1}{2} \begin{pmatrix}
E^2 - p^2 - \mu^2 & 0 & 0 & 0 \\
0 & E^2 - p^2 - \frac{2M^2 + \mu^2}{3\mu} & -\frac{2}{\sqrt{3}}\mu \sin \alpha & 0 \\
0 & -\frac{2}{\sqrt{3}}\mu \sin \alpha & E^2 - p^2 & -2iE\mu \cos \alpha \\
0 & 0 & 2iE\mu \cos \alpha & E^2 - p^2 - \mu^2 \sin^2 \alpha
\end{pmatrix},
$$

(30)

and

$$
M_2 = \frac{1}{2} \begin{pmatrix}
E^2 - p^2 - M^2 & iE\mu(\cos \alpha - 1) & E\mu \sin \alpha & 0 \\
-iE\mu(\cos \alpha - 1) & E^2 - p^2 - M^2 & 0 & -E\mu \sin \alpha \\
E\mu \sin \alpha & 0 & E^2 - p^2 & iE\mu(\cos \alpha + 1) \\
0 & -E\mu \sin \alpha & -iE\mu(\cos \alpha + 1) & E^2 - p^2
\end{pmatrix}.
$$

(31)

At the transition point the two matrices are the same and the determinant agrees with the determinant obtained from the determinant of the would be zero modes in the normal phase. The dispersion relation of the modes is obtained by calculating the zeroes of the determinant of $M_1$ and $M_2$ which can be obtained perturbatively for $\mu$ approaching $\mu_c$.

The dispersion relations obtained from the determinant of $M_1$ are given by

$$
E^2 = \bar{\mu}p^2,
$$

(32)

$$
E^2 = M^2 + p^2 + 2\bar{\mu}M^2,
$$

(33)

$$
E^2 = M^2 + p^2 + \bar{\mu}\left(-\frac{22}{9}M^2 + \frac{16}{27}p^2\right),
$$

(34)

$$
E^2 = 4M^2 + 2p^2 + \bar{\mu}\left(\frac{28}{9}M^2 - \frac{43}{27}p^2\right),
$$

(35)

where $\bar{\mu} = (\mu - \mu_c)/M$. For $M_2$ we find the dispersion relations

$$
E^2 = \frac{1}{4}M^2(1 - \bar{\mu}^2)p^4,
$$

(36)

$$
E^2 = M^2 + p^2 + \bar{\mu}\left(-2M^2 + p^2\right),
$$

(37)

$$
E^2 = M^2 + p^2 + \bar{\mu}\left(-\frac{2}{3}M^2 - \frac{1}{9}p^2\right),
$$

(38)

$$
E^2 = 4M^2 + 2p^2 + \bar{\mu}\left(\frac{32}{3}M^2 - \frac{8}{9}p^2\right).
$$

(39)
In both cases the modes match up to Goldstone modes in the normal phase with strangeness equal to +1, 0, 0 and −1, respectively. At the transition point we observe one Goldstone boson with a linear dispersion relation and one with a quadratic dispersion relation. This requires that the total number of broken generators is equal to three which is indeed the case. In agreement with the above continuity argument, the coefficient of the linear dispersion relation vanishes at the critical point.

As a side remark we next answer the question of what is the simplest effective Lagrangian that described the massless states of the theory. General arguments show that an effective Lagrangian for a particle with a quadratic dispersion relation and a kinetic term that is quadratic in the momenta requires at least two degrees of freedom. The simplest effective Lagrangian describing our massless state thus has to contain at least three degrees of freedom. We thus conclude that the simplest effective Lagrangian for the massless modes at the transition point is given by

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \Sigma \partial^{\mu} \Sigma^\dagger + \mu \text{Tr} \Sigma^\dagger V \partial_0 \Sigma,
\]

where \( \Sigma \in SU(2) \) and \( V = \text{diag}(0, 1) \). This Lagrangian might be of interest for phenomenological applications of the results obtained in this paper.

5 Conclusion

In conclusion, in the kaon condensed phase we have found one Goldstone mode with a linear dispersion relation and all others with a quadratic dispersion relation. A mismatch in the naive counting of the Goldstone modes is resolved by the observation that in the latter case the number of Goldstone modes is only half the number of broken generators.

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After this work was completed we became aware of Ref. [23] in which the dispersion relations of the model (1) were derived as well.

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