On the optimal analytic continuation from discrete data

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Abstract

We consider analytic functions from a reproducing kernel Hilbert space. Given that such a function is of order $\epsilon$ on a set of discrete data points, relative to its global size, we ask how large can it be at a fixed point outside of the data set. We obtain optimal bounds on this error of analytic continuation and describe its asymptotic behavior in $\epsilon$. We also describe the maximizer function attaining the optimal error in terms of the resolvent of a positive semidefinite, self-adjoint and finite rank operator.

1 Introduction

Analytic functions are of central importance in many applied problems. They appear in diverse areas, such as medical imaging [9], nuclear physics [3, 4] and optimal design problems [19, 20]. For example Fourier (or Laplace) transforms of real-valued functions vanishing on negative semi-axis are analytic in the upper half-plane. Such functions describe linear, time-invariant and causal systems. Some concrete examples are the complex magnetic permeability and complex dielectric permittivity functions [18, 10], complex impedance and admittance functions of electrical circuits [2]. Further examples include transfer functions of digital filters [14], the dependence of effective moduli of composites on the moduli of its constituents [1, 20] etc.

Typically these functions can be measured only on a subset $\Gamma$ of their domain of analyticity (or its boundary). During the measurement process unavoidable error occurs, as a result an analytic function is known on $\Gamma$ up to a certain precision of order $\epsilon > 0$. In order to predict the behavior of the system and to expand its application horizon, often times one is interested in extrapolating from the measured data to a given point $z$ lying outside of the data set $\Gamma$. On one hand, working in the class of analytic functions we expect rigidity (in the sense that values of an analytic function on $\Gamma$ affect its values elsewhere). On the other hand, recent work [7, 30, 16, 17] shows, that surprisingly there is also flexibility, meaning that the measured data can be matched up to the given precision $\epsilon$ by two analytic functions that are very different outside of the data set.

Let $S = S(\Omega)$ denote a class of physically admissible functions that are analytic in a domain $\Omega$ of the complex plane. Aside from analyticity, the set $S$ may also contain further physical restrictions (cf. [15]), such as certain symmetry constraints, asymptotic constraints at infinity or inequality constraints e.g. nonnegative imaginary parts (which can be interpreted as presence of energy loss in the system). Let $\Gamma \subset \Omega$ denote the data set
where the measurements are done with relative error $\epsilon$, with respect to some norm $\| \cdot \|_\Gamma$ on $\Gamma$. To quantify the flexibility of the class $S$ we ask the following question: given two functions from $S$ that are $\epsilon$-close on $\Gamma$ (relative to their total size on $\Omega$), how much can they differ at a point $z \in \Omega \setminus \Gamma$? Assume that the total size of a function on $\Omega$ is measured in some norm $\| \cdot \|_\Omega$, then we arrive at the quantity

$$\Delta_z(\epsilon) = \sup \left\{ \frac{|\phi(z) - \psi(z)|}{\max(\|\phi\|_\Omega, \|\psi\|_\Omega)} : \phi, \psi \in S \text{ with } \frac{\|\phi - \psi\|_\Gamma}{\max(\|\phi\|_\Omega, \|\psi\|_\Omega)} \leq \epsilon \right\}. \quad (1.1)$$

Another related quantity interesting in its own right, is the relative error of analytic continuation. Loosely speaking, consider the difference $f = \phi - \psi$ and rescale it (say $S$ is a cone) by the total norm of $\phi$ and $\psi$ on $\Omega$, assume also that $S - S$ can be approximated by functions from some normed space $H = H(\Omega)$ of analytic functions in $\Omega$. We arrive at an analogous question: given that $f \in H$ is of order $\epsilon$ on $\Gamma$ and is of order 1 on $\Omega$, how large can it be at the point $z$? So to quantify the stability of analytic continuation in the normed space $H$ we introduce

$$A_z(\epsilon) = \sup \{|f(z)| : f \in H \text{ with } \|f\|_H \leq 1, \|f\|_\Gamma \leq \epsilon \}. \quad (1.2)$$

In the setting of Hilbert spaces and when $\Gamma \Subset \Omega$ is a curve with $\| \cdot \|_\Gamma$ denoting the $L^2(\Gamma)$-norm (with respect to the arclength measure) we analyzed (1.2) in [16, 17], where we derived optimal bounds for it and showed that it behaves like a power law: $A_z(\epsilon) \approx \epsilon^\gamma(z)$, where the exponent $\gamma(z) \in (0, 1)$ decreases to 0, as we move further away from the source of data. How fast $\gamma(z)$ decays depends strongly on the geometry of the domain and the data source. The most common setting, where (1.2) is analyzed in the literature is in the space of bounded analytic functions. The power law estimates are then derived from a maximum modulus principle, the Hadamard three-circles theorem is a classical example of such estimate. For related works we refer the reader to [6, 5, 25, 29, 12, 31, 13, 7, 30].

This paper is dedicated to the analysis of (1.2) in the Hilbert space setting, when $\Gamma = \{z_j\}_{j=1}^n$ represents a finite set of distinct points, where the function values are measured. In this case $\| \cdot \|_\Gamma$ is a seminorm, so we use the notation $[\cdot]_\Gamma$ instead, and treating all the points equally we consider the $l^2$-seminorm: $[f]_\Gamma^2 = \sum_j |f(z_j)|^2$. The first difference of the discrete setting vs. the continuum one is that in the former case an analytic function is not determined uniquely by its values on $\Gamma$, as a result $A_z(\epsilon)$ does not converge to zero as $\epsilon \to 0$. So then the questions are what is $A_z(0)$ and what is the next term in the asymptotic expansion of $A_z(\epsilon)$. The answer to the last question reveals the second key distinction of the discrete setting, showing that there is no fractional power of $\epsilon$ and the correction term is of order $\epsilon$. Namely, we will characterize $A_z(0)$ (in terms of the reproducing kernel of the space $H$, cf. Theorem 2.1) and show that

$$A_z(\epsilon) = (1 + \sigma \epsilon)A_z(0) + O(\epsilon^2), \quad (1.3)$$

\footnote{For a rigorous comparison of quantities (1.1) and (1.2) in the context of the complex electromagnetic permittivity function we refer to [15] (in this case $S$ is a cone of functions that is related to Herglotz-Nevanlinna functions).}
where \( \sigma = \sigma(z) > 0 \) will depend on the space \( \mathcal{H} \) and the data set \( \Gamma \). Note that the set of values \( V_\varepsilon(\varepsilon) = \{ f(z) : \| f \|_\mathcal{H} \leq 1, |f|_{\Gamma} \leq \varepsilon \} \) is a convex, centrally symmetric \( (c \in V_\varepsilon \iff -c \in V_\varepsilon) \) subset of the complex plane, and \( A_\varepsilon(\varepsilon) \) is its ”radius”, i.e. half of the diameter. The formula (1.3) then shows the relation between the radii \( A_\varepsilon(0), A_\varepsilon(\varepsilon) \) of the original and perturbed function value sets, respectively.

The quantity (1.2) is also related to the optimal estimation of the point evaluation functional \( f \mapsto f(z) \) (see [23][24] and references therein for the general theory of optimal estimations and optimal recovery). Following [24] let us formulate the question of optimal recovery. Let \( f_j' = f(z_j) + \delta_j \) represent the erroneous measurement of the function value at the point \( z_j \) for \( j = 1, \ldots, n \). Assume, that the error is of order \( \varepsilon \), namely let \( \delta = (\delta_1, \ldots, \delta_n) \) be the error vector and let \( |\delta| \leq \varepsilon \), where \( |\cdot| \) is the Euclidean length of a vector in \( \mathbb{C}^n \). The task is to approximate \( f(z) \) at a fixed point \( z \notin \Gamma = \{ z_j \}_n \). The error of a linear estimation algorithm (it is enough to restrict consideration only to linear algorithms [22]) is then defined as

\[
E_\varepsilon(\varepsilon, c) = \sup \left\{ |f(z) - \sum_{j=1}^{n} c_j(f(z_j) + \delta_j)| : \| f \|_\mathcal{H} \leq 1, |\delta| \leq \varepsilon \right\},
\]

where \( c = (c_1, \ldots, c_n) \in \mathbb{C}^n \) is a given vector defining the linear algorithm and the supremum goes over all \( f \in \mathcal{H} \) and \( \delta \in \mathbb{C}^n \) satisfying the above-mentioned constraints. The intrinsic error of the estimation problem is

\[
E_\varepsilon(\varepsilon) = \inf_c E_\varepsilon(\varepsilon, c).
\]

Any algorithm achieving this infimum yields an optimal procedure for estimating \( f(z) \). Theorem 1 of [22] implies that

\[
A_\varepsilon(\varepsilon) = E_\varepsilon(\varepsilon).
\]

Let us actually prove this equality using an idea from [11] (Section 7.5). The constraints in (1.4) are invariant under multiplying \( f \) and \( \delta \) with a constant phase factor, so instead of maximizing the absolute value in (1.4) we can equivalently maximize the real part. Next, applying von Neumann’s minimax theorem [27] we obtain

\[
E_\varepsilon(\varepsilon, c) = \inf_c \sup_{f, \delta} \Re \left\{ f(z) - \sum_{j=1}^{n} c_j(f(z_j) + \delta_j) \right\} = \sup_{f, \delta} \inf_c \Re \left\{ f(z) - \sum_{j=1}^{n} c_j(f(z_j) + \delta_j) \right\}.
\]

It remains to note that the inner infimum will be \( -\infty \) unless \( f(z_j) + \delta_j = 0 \) for all \( j \). This implies that the supremum can be restricted to considering those \( f \in \mathcal{H} \) with \( \| f \|_\mathcal{H} \leq 1 \) for which the choice \( \delta_j = -f(z_j) \) can be made, which means that \( f \) must also satisfy the second constraint \( |\delta| = |f|_{\Gamma} \leq \varepsilon \). This concludes the proof of (1.6).

In [21] the authors analyze a quantity related to \( E_\varepsilon(\varepsilon) \), namely in order to obtain constructive results in (1.4) they replace the target functional with the square root of \( |f(z) - \sum_j c_j f(z_j)|^2 + |\sum_j c_j \delta_j|^2 \). The square of the replaced quantity is comparable to \( E_\varepsilon(\varepsilon) \) and hence also to \( A_\varepsilon(\varepsilon) \). In this work we take an alternative approach and analyze
\( A_z(\epsilon) \) directly using variational methods and derive the asymptotic expansion result (1.3) that is analogous to that of [21] (see Theorem 4 therein). Further, we do not assume linear independence of the point evaluation functionals \( f \mapsto f(z_j) \) for \( j = 1, \ldots, n \). Moreover, we describe the maximizer function attaining the supremum in (1.2) via the resolvent of a positive semidefinite, self-adjoint and finite rank operator, which (by taking limits as \( \epsilon \to 0 \)) also allows us to obtain a characterization for \( A_z(0) \).

2 The Main Result

Let \( \mathcal{H} = \mathcal{H}(\Omega) \) be a Hilbert space of analytic functions in a domain \( \Omega \subset \mathbb{C} \). Assume that the point evaluation functional \( f \mapsto f(z) \) is continuous for any point \( z \in \Omega \), then by the Riesz representation theorem, there exists an element \( p_z \in \mathcal{H} \) such that \( f(z) = (f, p_z)_\mathcal{H} \). So inner products with the function \( p(\zeta, z) := p_z(\zeta) \) reproduce values of a function in \( \mathcal{H} \). In this case \( \mathcal{H} \) is called a a reproducing kernel Hilbert space (RKHS) (cf. [28]). Examples of such spaces include the Hardy spaces \( H^2 \), the Bergman spaces \( A^2 \) etc. From now on let us drop the subscript \( H \) from the notation of the norm and the inner product of \( H \).

Let \( z, z_1, \ldots, z_n \in \Omega \) be distinct points and set \( [f]^2 = \sum_j |f(z_j)|^2 \). Consider the problem

\[
A_z(\epsilon) = \sup \{ |f(z)| : f \in \mathcal{H} \text{ with } \|f\| \leq 1, \ [f] \leq \epsilon \}.
\]

Introduce the restriction operator \( R : \mathcal{H} \to \mathbb{C}^n \) given by \( Rf = f := (f(z_1), \ldots, f(z_n)) \) and let \( K = R^*R \) (cf. [6]), then it is easy to see that

\[
Kf = \sum_{j=1}^n f(z_j)p_{z_j}
\]

and \((Kf, g) = (f, g)_{\mathbb{C}^n}\), in particular the second constraint of (2.1) can be rewritten via the quadratic form of \( K \):

\[
(Kf, f) = [f]^2.
\]

Clearly \( K \) is a self-adjoint, positive semidefinite and compact (in fact, finite rank) operator on \( \mathcal{H} \). Set \( U_0 = \ker K \), let \( \{\lambda_1, \ldots, \lambda_m\} \) be all distinct nonzero eigenvalues of \( K \) and set \( U_j = \ker(K - \lambda_j I), \ j = 1, \ldots, m \). Hence, \( \mathcal{H} = \bigoplus_{j=0}^m U_j \), and let \( P_j : \mathcal{H} \to U_j \) denote the orthogonal projection onto the closed subspace \( U_j \) for \( j = 0, \ldots, m \).

**Theorem 2.1.** With the notation introduced above and assuming \( p_z \notin K(\mathcal{H}) \), we have

\[
A_z(\epsilon) = (1 + \sigma \epsilon)A_z(0) + O(\epsilon^2),
\]

where, in fact \( A_z(0) = \|P_0 p_z\| \) and

\[
\sigma^2 = \frac{1}{\|P_0 p_z\|^2} \sum_{j=1}^m \frac{\|P_j p_z\|^2}{\lambda_j}.
\]

\(^2\)when \( \{p_{z_j}\} \) are linearly independent \( U_0 = \{f \in \mathcal{H} : f(z_1) = \ldots = f(z_n) = 0\} \)
Moreover, the maximizer function for $A_z(\epsilon)$ is given via the resolvent operator

$$A_z(\epsilon) = \frac{u_\epsilon(z)}{\|u_\epsilon\|}, \quad u_\epsilon = (\mathcal{K} + \eta(\epsilon))^{-1}p_z, \quad (2.6)$$

where $\eta = \eta(\epsilon) > 0$ is the unique solution of the equation

$$\eta^2 \sum_{j=1}^{m} \frac{(\lambda_j - \epsilon^2)\|Pzp_z\|^2}{(\lambda_j + \eta)^2} = \epsilon^2\|P_0p_z\|^2.$$ 

In particular, $\eta(\epsilon) \sim \epsilon/\sigma$ as $\epsilon \to 0$.

The proof of this theorem is based on [16, 15] with a few technical differences, so for completeness we will present the full argument here. Let us make a few remarks before presenting the proof:

1. In the formula (2.6) for $A_z(\epsilon)$ we can take limits as $\epsilon \to 0$ and obtain (see Section 3)

$$A_z(0) = \frac{P_0p_z(z)}{\|P_0p_z\|} = \|P_0p_z\|.$$ 

In particular, the maximizer function for $A_z(0)$ is $P_0p_z/\|P_0p_z\|$ (note that the assumption $p_z \notin \mathcal{K}(\mathcal{H})$ implies $P_0p_z \neq 0$).

2. Let $G \in \mathbb{C}^{n \times n}$ be the Gram matrix with $G_{jk} = (p_zj, p_zk) = p_zj(z_k)$, then $\mathcal{K}f = \lambda f$ implies that $G^Tf = \lambda f$. So $\{\lambda_j\}_{j=1}^m$ are also eigenvalues of $G$, and after finding the corresponding eigenvectors $\{f_j\}$, from (2.2) we see that the eigenfunctions of $\mathcal{K}$ are linear combinations of the functions $\{p_zj\}$ with coefficients given by the eigenvectors.

For example, in the trivial case when there is only one data point $z_1$ we have $\lambda_1 = \|p_{z_1}\|^2$, $P_0 = Id - P_1$ and

$$P_1f = \frac{f(z_1)}{\|p_{z_1}\|^2}P_{z_1}.$$ 

3. If $p_z \in \mathcal{K}(\mathcal{H})$, then $\mathcal{K}u = p_z$ for some $u \in \mathcal{H}$. This relation means that the value of any function at $z$ is determined by its values at $\{z_j\}$, indeed $f(z) = \sum_j c_jf(z_j)$ for any $f \in \mathcal{H}$ where we set $c_j = u(z_j)$. So in this case we have complete stability: order $\epsilon$ smallness on $\{z_j\}$ implies order $\epsilon$ smallness at $z$. Indeed, applying the Cauchy-Schwartz inequality to $f(z)$ and using that $|f| \leq \epsilon$ we obtain

$$A_z(\epsilon) \leq \epsilon|c|, \quad c = (c_1, ..., c_n).$$

Such situations arise if we consider spaces that contain ”boundary conditions”. For example, $S = \{f \in \mathcal{H} : f(z) = f(z_1)\}$ is a closed subspace of $\mathcal{H}$ and hence is a RKHS in its own right, to which the above discussion applies.
3 Proof of Theorem 2.1

First consider the case \( p_z \in \ker \mathcal{K} \). Let us show that \( A_z(\epsilon) = A_z(0) \) for any \( \epsilon \), and so (2.4) is satisfied with \( \sigma = 0 \). Indeed, using the Cauchy-Schwartz inequality

\[
|(f, p_z)| \leq \|f\|\|p_z\| \leq \|p_z\|,
\]

giving the trivial bound \( A_z(\epsilon) \leq \|p_z\| \). But this yields an optimal bound in this case, since the function \( f_* = p_z/\|p_z\| \) satisfies both of the constraints and \( |f_*(z)| = \|p_z\| \), attains the upper bound.

So let us concentrate on the non-trivial case \( p_z \notin \ker \mathcal{K} \) (this assumption is used later in the proof, namely in (3.8)).

The two constraints of (2.1) are invariant under multiplying \( f \) with a constant phase factor: if \( f \) satisfies the constraints, then so does \( \lambda f \) for any \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \). So instead of maximizing \( |f(z)| \) we can equivalently maximize \( \Re f(z) \). Using the reproducing kernel property and (2.3) we rewrite (2.1) as a convex maximization problem with a linear target functional and two quadratic constraints:

\[
\begin{aligned}
\begin{cases}
\Re(f, p_z) \to \max \\
(f, f) \leq 1 \\
(Kf, f) \leq \epsilon^2
\end{cases}
\end{aligned}
\tag{3.1}
\]

Introduce Lagrange multiplies: nonnegative numbers \( \mu \) and \( \nu \) such that \( \mu + \nu \neq 0 \). Multiply the first constraint of (3.1) by \( \mu \), the second one by \( \nu \) and add the two inequalities to obtain

\[
((\mu + \nu K)f, f) \leq \mu + \nu \epsilon^2.
\tag{3.2}
\]

Now, if \( M \) is a uniformly positive definite self-adjoint operator on \( \mathcal{H} \), expanding \( (M(M^{-1}g - f), (M^{-1}g - f)) \geq 0 \), we obtain that for any \( f, g \in \mathcal{H} \)

\[
2\Re(f, g) - (M^{-1}g, g) \leq (Mf, f).
\]

The uniform positivity of \( M \) ensures that \( M^{-1} \) is defined on all of \( \mathcal{H} \). This is an example of convex duality (cf. [8]) applied to the convex function \( f \mapsto (Mf, f)/2 \). Then we also have for \( \mu > 0 \), taking \( M = \mu + \nu K \) and \( g = p_z \) in the above inequality we get

\[
2\Re(f, p_z) - ((\mu + \nu K)^{-1}p_z, p_z) \leq ((\mu + \nu K)f, f) \leq \mu + \nu \epsilon^2,
\tag{3.3}
\]

so that

\[
\Re(f, p_z) \leq \frac{1}{2} \left( ((\mu + \nu K)^{-1}p_z, p_z) + \frac{1}{2} (\mu + \nu \epsilon^2) \right),
\tag{3.4}
\]

which is valid for every \( f \), satisfying the constraints of (3.1) and all \( \mu > 0, \nu \geq 0 \). In order for the bound to be optimal we must have equality in (3.3), which holds iff \( p_z = (\mu + \nu K)f \) giving the formula for optimal function \( f \):

\[
f = (\mu + \nu K)^{-1}p_z.
\tag{3.5}
\]
The goal is to choose the Lagrange multipliers $\mu$ and $\nu$ so that the constraints in (3.1) are satisfied by $f$, given by (3.5).

1. if $\nu = 0$, then $f = \frac{p_z}{\|p_z\|}$ does not depend on the small parameter $\epsilon$, which leads to a contradiction, because $p_z \notin \ker K$ implies that $(Kf, f) > 0$ and hence the second constraint $(Kf, f) \leq \epsilon^2$ is violated if $\epsilon$ is small enough.

2. if $\mu = 0$, then $Kf = \frac{1}{\nu}p_z$, contradicting to our assumption $p_z \notin K(H)$.

Thus we are looking for $\mu > 0$, $\nu > 0$, so that equalities hold at both of the constraints for the function (3.5) (these are the complementary slackness relations in Karush-Kuhn-Tucker conditions.), i.e.

$$\begin{align*}
\|(\mu + \nu K)^{-1}p_z\| &= 1, \\
[(\mu + \nu K)^{-1}]p_z &= \epsilon.
\end{align*}$$

(3.6)

Let $\eta = \xi$, solving the first equation of (3.6) for $\nu$ we find $\nu = \|(K+\eta)^{-1}p_z\|$. Then the square of the second equation of (3.6) reads

$$\Phi(\eta) := \left[\left(\frac{p_z}{\eta p_z}\right)\right]^2 = \epsilon^2.$$

(3.7)

Let us now use the spectral decomposition of $K$. Recall that $P_j P_k = 0$ if $j \neq k$ and

$$Id = \sum_{j=0}^{m} P_j, \quad \text{and} \quad K = \sum_{j=1}^{m} \lambda_j P_j,$$

further we also have

$$K + \eta = \eta P_0 + \sum_{j=1}^{m} (\lambda_j + \eta) P_j, \quad (K + \eta)^{-1} = \eta^{-1} P_0 + \sum_{j=1}^{m} P_j \frac{P_j}{\lambda_j + \eta}.$$ 

Then writing the numerator of $\Phi$ as the quadratic form of $K$ using (2.3) we get

$$\Phi(\eta) = \frac{\sum_{j=1}^{m} \lambda_j \|P_j p_z\|^2}{\eta^2 \|P_0 p_z\|^2 + \sum_{j=1}^{m} \|P_j p_z\|^2 \left(\frac{1}{\lambda_j + \eta}\right)^2}.$$

Next our goal is to show that the equation (3.7) has a unique solution $\eta = \eta(\epsilon) > 0$. Clearly, $\Phi(0^+) = 0$ and

$$\Phi(+\infty) = \frac{\|p_z\|^2}{\|p_z\|} = \frac{(K p_z, p_z)}{\|p_z\|^2} > 0,$$

(3.8)

because $p_z \notin \ker K$. Therefore, showing that $\Phi$ is strictly increasing will imply that (3.7) has a unique solution for $\epsilon$ small enough, namely for any $\epsilon^2 < \Phi(+\infty)$. So let us prove that
\(\Phi'(\eta) > 0\). Set \(a_j = \|P_j p_z\|^2\) for \(j = 0, \ldots, m\), then the numerator of \(\Phi'(\eta)\) (up to a factor of 2) can be simplified to

\[
a_0 \sum_j \frac{\lambda_j^2 a_j}{(\lambda_j + \eta)^2} + \sum_{j,k} \lambda_j b_{jk}(\lambda_j - \lambda_k),
\]

where \(b_{jk} = a_j a_k / (\lambda_j + \eta)^3 (\lambda_k + \eta)^3\). In particular \(b_{jk} = b_{kj}\), so splitting the second sum of (3.9) into parts where \(j > k, j < k\) and swapping the indices \(j, k\) in the second part we find

\[
\sum_{j,k} \lambda_j b_{jk}(\lambda_j - \lambda_k) = \sum_{j>k} \lambda_j b_{jk}(\lambda_j - \lambda_k) + \sum_{j>k} \lambda_k b_{kj}(\lambda_k - \lambda_j) = \sum_{j>k} b_{jk}(\lambda_j - \lambda_k)^2.
\]

Recalling that all \(\lambda_j\) are distinct and positive we conclude that (3.9) is strictly positive (the value 0 is excluded since otherwise \(a_j = 0\) for all \(j = 1, \ldots, m\) implying that \(p_z \in \ker \mathcal{K}\)) and hence \(\Phi'(\eta) > 0\).

Observe that, \(\Phi(\eta) \sim \sigma^2 \eta^2\) as \(\eta \to 0\), where \(\sigma\) is given by (2.5). Hence for the solution of the equation (3.7) we have \(\eta(\epsilon) \sim \epsilon / \sigma\) as \(\epsilon \to 0\).

Setting \(u = (\mathcal{K} + \eta(\epsilon))^{-1} p_z\), from (3.4) we obtain

\[
\Re(f, p_z) \leq \frac{(u, p_z)}{2\|u\|} + \frac{\|u\|}{2}(\epsilon^2 + \eta(\epsilon)).
\]

Definitions of \(u\) and \(\eta(\epsilon)\) imply that \((\mathcal{K}u, u) / \|u\|^2 = \epsilon^2\), on the other hand

\[
u(z) = (u, p_z) = (u, \mathcal{K}u + \eta(\epsilon) u) = (\mathcal{K}u, u) + \eta(\epsilon) \|u\|^2 = (\epsilon^2 + \eta(\epsilon)) \|u\|^2,
\]

which implies the optimal bound

\[
|f(z)| = \Re(f, p_z) \leq \frac{u(z)}{2\|u\|} + \frac{u(z)}{2\|u\|} = \frac{u(z)}{\|u\|}.
\]

Thus

\[
A_z(\epsilon) = \frac{u(z)}{\|u\|} = [\epsilon^2 + \eta(\epsilon)] \|u\|.
\]

It remains to analyze the asymptotic behavior of \(A_z(\epsilon)\). To that end, using the spectral decomposition of \(\mathcal{K}\), note that

\[
\|u\|^2 = \eta(\epsilon)^{-2} a_0 + \sum_{j=1}^m \frac{a_j}{(\lambda_j + \eta(\epsilon))^2},
\]

but then

\[
A_z^2(\epsilon) = \left(1 + \frac{\epsilon^2}{\eta(\epsilon)}\right)^2 a_0 + [\epsilon^2 + \eta(\epsilon)]^2 \sum_{j=1}^m \frac{a_j}{(\lambda_j + \eta(\epsilon))^2}.
\]
Letting $\epsilon \to 0$ in the last formula gives $A_2^2(0) = a_0$. Finally, using the asymptotics of $\eta(\epsilon)$ we conclude that

$$A_2^2(\epsilon) = (1 + 2\sigma \epsilon)A_2^2(0) + O(\epsilon^2),$$

which, upon using the expansion $\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2)$ for small $x$, implies the relation (2.4) and concludes the proof.

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