The Boundary between Compact and Noncompact Complete Riemann Manifolds

D. Holcman* C. Pugh †

Abstract

In 1941 Sumner Myers proved that if the Ricci curvature of a complete Riemann manifold has a positive infimum then the manifold is compact and its diameter is bounded in terms of the infimum. Subsequently the curvature hypothesis has been weakened, and in this paper we weaken it further in an attempt to find the ultimate, sharp result.

1 Introduction

Myers’ Theorem [13] states that a complete Riemann manifold \((M, g)\) of dimension \(n \geq 2\) is compact if its Ricci curvature is uniformly positive, and furthermore it has diameter \(\leq \pi/\sqrt{C}\) if its Ricci curvature satisfies

\[
Ric_p \geq (n - 1)C
\]

everywhere on \(M\), \(C\) being a positive constant. (Here and below we adopt the shorthand that \(Ric_p \geq c\) means that for all \(X \in T_pM\),

\[
Ric_p(g, X, X) \geq c\langle X, X \rangle_p,
\]

where \(\langle \ , \ \rangle_p\) is the \(g\)-inner product on \(T_pM\). Later, asymptotic conditions on the curvature were found that still imply compactness, \([6]\), \([3]\), \([2]\).

*Weizmann Institute of Science, Department of Mathematics, Rehovot 76100, Israel. email:david.holcman@weizmann.ac.il

†Mathematics Department, University of California, Berkeley California, 94720, U.S.A. email:pugh@math.berkeley.edu
although definitive conditions remain unknown. The idea is to fix an origin $O \in M$ and study the curvature along geodesics emanating from $O$. One always assumes the curvature is positive, but permits it to decay to zero far from $O$.

To be more specific, we set

$$\text{Ric}(r) = \inf\{\text{Ric}_p : p = \exp_O(v) \text{ and } |v| = r\},$$

and assume throughout that $\text{Ric}(r) > 0$. Hypotheses that imply compactness and give a diameter estimate are:

(a) (Cheeger-Gromov-Taylor [5]) For some $\nu > 0$, some $r_0 > 0$, and all $r \geq r_0$,

$$\text{Ric}(r) \geq \frac{n - 1}{4} \left( \frac{1 + 4\nu^2}{r^2} \right).$$

(b) (Cheeger-Gromov-Taylor [5]) For some $\nu > 0$, some $r_0 > 1$, and all $r \geq r_0$,

$$\text{Ric}(r) \geq \frac{n - 1}{4} \left( \frac{1}{r^2} + \frac{1 + 4\nu^2}{(r \ln r)^2} \right).$$

(c) (Boju-Funar [3]) For some $\nu > 0$, some integer $k$, some $r_0 > e_k$, and all $r \geq r_0$,

$$\text{Ric}(r) \geq \frac{n - 1}{4} \left( \frac{1}{r^2} + \frac{1 + 4\nu^2}{(r \ln r)^2} + \cdots + \frac{1 + 4\nu^2}{(r \ln r \ln(r) \cdots \ln^k(r))^2} \right),$$

where $\ln^k$ is the $k$th iterated logarithm, $\ln^k(r) = \ln \circ \ln \circ \cdots \circ \ln(r)$, and $\ln^k(e_k) = 0$.

It is natural to set $\ln^0(r) = r$. Then (a) and (b) are (c) with $k = 0$ and $k = 1$. The diameter estimates on $M$ involve $\nu$, $r_0$, and $k$. When $k = 0$, one has $\text{diam}(M) \leq e^{\pi/\nu} r_0$. See [5] and [3].

Remark. In [6], Dekster and Kupka prove that the estimate in (a) is sharp.

Remark. In terms of decay rates, these results are nearly optimal. For example, there exist noncompact complete manifolds whose Ricci curvature satisfies a Boju-Funar equality with $\nu = 0$,

$$\text{Ric}(r) = \frac{n - 1}{4} \left( \frac{1}{r^2} + \cdots + \frac{1}{(r \ln(r) \ln(r) \cdots \ln^k(r))^2} \right),$$

(1.2)

See [3].
Remark. Dekster and Kupka consider also the sectional curvature $K$, and show that all noncompact complete Riemann manifolds of positive curvature satisfy
\[
\liminf_{r \to \infty} k(r)r^2 \leq \frac{1}{4},
\]
where $k(r) = \inf\{K_p : p = \exp_O(v) \text{ and } |v| = r\}$. The constant $1/4$ is sharp, [6].

Our first results, the Kick Theorems, state that asymptotic estimates are not the only way to guarantee compactness. Instead of curvature that decays to zero at a positive rate ($\nu > 0$) as the radius tends to infinity, it is enough that in addition to Ricci curvature obeying (1.2), there is a certain amount of extra curvature on a finite shell \( \{p = \exp_{O}(v) : a \leq |v| \leq b\} \). We refer to the extra curvature as a “kick.” See Sections 2 and 3 for details.

Remark. None of these conditions is truly optimal; in Section 4 we show that a surface approximating the capped cylinder has the property that every more curved surface is compact, but this is implied by none of the asymptotic or kick conditions.

Nevertheless, it is tempting to postulate some kind of a boundary in the space of positive Ricci curvature functions with all compact manifolds on one side and all non-compact complete manifolds on the other.\(^1\) A manifestation of such a boundary would be a topology on the space $\mathcal{R}$ of Ricci curvature functions which are defined on a fixed tangent space $T_{O}M$, and a closed subset $\mathcal{R}_{0} \subset \mathcal{R}$ such that through each $R_{0} \in \mathcal{R}_{0}$ there is a curve $\tilde{R}_{t}$ of Ricci curvature functions, and

(a) If $t > 0$ and $M$ has Ricci curvature $R_{t}$ then it is compact.

(b) If $t \leq 0$ and $M$ has Ricci curvature $R_{t}$ then it is non-compact.

(c) This transverse, single-point crossing from non-compact to compact persists for all nearby curves $\tilde{R}_{t}$.

\(^1\) Readers familiar with Walter Rudin’s text, *Principles of Mathematical Analysis*, will recognize this phrase, in which Rudin asserts that there is no such boundary dividing convergent and divergent series. A difference between series and curvature functions is that local perturbations have no global effect on series, while for curvature functions this is not so. Perturbation of a finite number of terms in a series does not change convergence, but a compactly supported perturbation of curvature can affect the manifold’s topology at a long distance from the perturbation’s support, and hence such a curvature boundary is not unreasonable.
In Section 6 we establish this kind of result for planar curves, finding a boundary between embedded curves and nonembedded immersed curves; in Section 7 we pass to surfaces embedded in 3-space.

Our second result partially identifies the boundary $\mathcal{R}_0$ postulated above. We call a function $b : [0, \infty) \to (0, \infty)$ an SL-bifurcator if the solution to the Sturm-Liouville equation

$$w'' + b(r)w = 0, \quad w(0) = 0, \quad w'(0) = 1$$

is monotone and bounded. An example is

$$b(r) = \frac{2r}{(1 + r^2)^2 \arctan r}.$$ 

See Sections 5 and 7 for more on SL-bifurcators.

**Definition.** A function $f(x)$ exceeds a function $g(x)$, if for all $x$, $f(x) \geq g(x)$, and for some $x$, $f(x) > g(x)$.

**Boundary Theorem.** Let $M$ be a complete Riemann manifold with positive Ricci curvature, and let $b(r)$ be an SL-bifurcator.

(a) $M$ is noncompact if for all $r \geq 0$,

$$\sup \{\operatorname{Ric}_p : p = \exp(v) \text{ and } |v| = r \} \leq b(r).$$

(b) $M$ is compact if $\operatorname{Ric}(r)$ exceeds $b(r)$. (As above, $\operatorname{Ric}(r)$ denotes the infimum of $\operatorname{Ric}_p$ such that $p = \exp_O(v)$ and $|v| = r$.)

**Corollary.** SL-bifurcators distinguish compact and noncompact complete Riemann manifolds with positive Ricci curvature.

See Section 5 for the simple proofs.

## 2 Linear Kick

In this section we deal with the kick condition when $k = 0$. Thus we assume that for all $r \geq r_0 > 0$,

$$\operatorname{Ric}(r) \geq \frac{n-1}{4} \left( \frac{1}{r^2} \right). \quad (2.3)$$
and we find a sufficient amount of extra curvature on a shell \( p = \exp_O(v) : a \leq |v| \leq b \) (with \( r_0 \leq a \)) that implies \( M \) is compact. Define \( \lambda \) to be the smallest positive root of the equation

\[
\cot(\lambda \ln(b/a)) = \lambda \ln(a/r_0).
\]  

(2.4)

Note that \( \lambda = \lambda(a, b, r_0) \) exists, lies in \((0, \pi/2 \ln(b/a)])\), and is unique. For, as \( \mu \ln(b/a) \) varies from 0 to \( \pi/2 \), its cotangent decreases monotonically from \( \infty \) to 0, while \( \mu \ln(a/r_0) \) is non-decreasing.

**Linear Kick Theorem.** In addition to (2.3), assume that for all \( r \in [a, b] \), we have

\[
\text{Ric}(r) > \frac{n-1}{4} \left(1 + 4\lambda^2 \right) \frac{1}{r^2},
\]

where \( \lambda = \lambda(a, b, r_0) \). Then \( M \) is compact.

**Remark.** As an example of the kick we can take \( a = e \) and \( b = e^2 \). Then \( \lambda \) is approximately .46. Also, if the interval \([a, b]\) is small, in the sense that \( b - a = \epsilon \), it is not hard to check that a kick sufficient for compactness increases like \( \epsilon^{-1/2} \) as \( \epsilon \to 0 \).

The proof is based on analyzing a kicked Sturm-Liouville equation

\[
y'' + \frac{1}{4} \left(1 + 4\mu^2 \chi_{[a,b]}(r) \right) \frac{1}{r^2} y = 0.
\]

(2.5)

**Lemma 2.1.** If \( \lambda = \lambda(a, b, r_0) \) is determined as in (2.4) and if \( \mu > \lambda \) then the solution \( y(r) \) of (2.2) with initial conditions \( y(r_0) = 0, y'(r_0) > 0 \), necessarily vanishes at some \( r > r_0 \).

**Proof.** For any constants \( c, k > 0 \), the function \( w(r) = cy(r/k) \) satisfies (2.5) and has initial conditions

\[
w(kr_0) = 0 \quad w'(kr_0) = \frac{cy'(kr_0)}{k} > 0.
\]

Taking \( k = 1/r_0 \) and \( c = k/y'(r_0) \), we can assume without loss of generality that \( r_0 = 1 \) and \( y'(1) = 1 \). We do so.

For \( r \in [a, b] \), the solution of (2.3) is of the form

\[
y(r) = Ar^{1/2} \cos(\mu \ln r) + Br^{1/2} \sin(\mu \ln r)
\]

(2.6)
where $A, B$ are constants. For $\mu = 0$, the solution degenerates as

$$y(r) = Ar^{1/2} + Br^{1/2} \ln r,$$

(2.7)

which can be seen by replacing $B$ with $B/\mu$ in (2.6) and letting $\mu$ tend to zero. Matching initial conditions at $r = 1$, $r = a$, and $r = b$ gives

$$y(r) = \begin{cases} 
  r^{1/2} \ln(r) & \text{if } 1 \leq r \leq a \\
  r^{1/2} \left( \ln(a) \cos(\mu \ln(r/a)) + \frac{1}{\mu} \sin(\mu \ln(r/a)) \right) & \text{if } a \leq r \leq b \\
  (r/b)^{1/2} (\alpha + \beta \ln(r/b)) & \text{if } b \leq r < \infty 
\end{cases}$$

where $\alpha, \beta$ are constants

$$\alpha = b^{1/2} \left( \ln(a) \cos(\mu \ln(b/a)) + \frac{1}{\mu} \sin(\mu \ln(b/a)) \right)$$

(2.8)

$$\beta = b^{1/2} \left( \cos(\mu \ln(b/a)) - \mu \ln(a) \sin(\mu \ln(b/a)) \right)$$

(2.9)

By the Sturm Comparison Theorem, a second zero of $y(r)$, if it exists, is a monotone decreasing function of $\mu$. Thus it is no loss of generality to assume that $\mu - \lambda$ is small. Since $\lambda$ is the smallest positive root of

$$\cot(\lambda \ln(b/a)) = \lambda \ln a \geq 0,$$

and since $\lambda \ln(b/a) \leq \pi/2$, we can assume $\mu - \lambda$ so small that $\mu \ln(b/a) < \pi$ and

$$\cot(\mu \ln(b/a)) > \frac{-1}{\mu \ln a}.$$ 

Since the cotangent is monotone decreasing on $(0, \pi)$, this implies that

$$\cot(\mu \ln(r/a)) > \frac{-1}{\mu \ln a}$$

for $a \leq r \leq b$, and hence that $y(r) > 0$ on $[a, b]$. For the same reasons,

$$\cot(\mu \ln(b/a)) < \cot(\lambda \ln(b/a)) = \lambda \ln a \leq \mu \ln a,$$

which implies that $\beta < 0$. But $y(b) > 0$ and $\beta < 0$ implies that $y(r) = 0$ for some $r > b$. \qed
Proof of the Linear Kick Theorem. We must show that $M$ is compact. By the Hopf-Rinow Theorem, it suffices to show that every geodesic through $O$ contains a pair of conjugate points. For if $M$ is not compact then it contains an everywhere distance minimizing geodesic $\gamma$ from $O$ to infinity ($M$ is complete), and this is contrary to conjugate pairs on $\gamma$. See [12].

By the assumption on the Ricci curvature and compactness of $[a, b]$, there exists $\mu$ such that

$$\text{Ric} (\gamma'(r), \gamma'(r)) > \mu > \lambda = \lambda(a, b, r_0)$$

for $r \in [a, b]$. Fix such a $\mu$, and let $y(r)$ be a solution of the kicked Sturm-Liouville equation (2.5) with initial conditions $y(r_0) = 0$, $y'(r_0) > 0$. By Lemma 2.1, $y(r_1) = 0$ for some $r_1 > r_0$.

Following Myers’ use of the Index Theorem, this gives a pair of conjugate points on $\gamma$. We recapitulate his proof.

Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_O M$ with $e_n = \gamma'(0)$, and let $E_1(r), \ldots, E_{n-1}(r)$ be the corresponding parallel vector fields along $\gamma$. Define vector fields along $\gamma$,

$$X_j(r) = y(r)E_j(r).$$

We will check that

$$\sum_{j=1}^{n-1} I(X_j, X_j) < 0 \quad (2.10)$$

where

$$I(X, X) = \int_{r_0}^{r_1} \left( |\nabla_t X|^2 - \langle R(X, \gamma'), X \rangle \right) dr$$

is the index of a vector field $X$ along $\gamma$. ($R$ is the curvature tensor.) To verify (2.10) we evaluate the Ricci curvature hypothesis on $(\gamma'(r), \gamma'(r))$ as

$$\sum \langle R(E_j(r), \gamma'(r))\gamma'(r), E_j(r) \rangle$$

$$= \text{Ric} (\gamma'(r), \gamma'(r)) > \frac{n-1}{4} \left( \frac{1 + 4\mu^2\chi_{[a,b]}(r)}{r^2} \right).$$
Then we write

\[ \sum I_j = \sum \int \langle X_j', X_j' \rangle - \langle R(X_j, \gamma') \gamma', X_j \rangle \, dr \]
\[ = \sum \int (y')^2 - \langle R(E_j, \gamma') \gamma', E_j \rangle y^2 \, dr \]
\[ = (n-1) \int (y')^2 \, dr - \int \sum \langle R(E_j, \gamma') \gamma', E_j \rangle y^2 \, dr \]
\[ < (n-1) \int (y')^2 \, dr - (n-1) \int \frac{1}{4} \left( \frac{1 + 4\mu^2 \chi_{[a,b]}(r^2)}{1} \right) y^2 \, dr \]
\[ = (n-1) \int (y')^2 + y'' y \, dr \]
\[ = (n-1)(y'y)|_{r_0}^{r_1} = 0, \]

where \( I_j = I(X_j, X_j) \), all integrands are evaluated at \( r \), all sums range from \( j = 1 \) to \( j = n-1 \), and all integrals are taken from \( r_0 \) to \( r_1 \). This verifies (2.10).

Negativity of a sum implies negativity of at least one term, so (2.10) implies that for some \( j_0 \), \( I(X_{j_0}, X_{j_0}) < 0 \), and so by Jacobi’s Theorem there exists a point \( \gamma(r) \) with \( r_0 < r < r_1 \) which is conjugate to \( \gamma(r_0) \).

**Remark.** It is straightforward to estimate the diameter of \( M \) as follows. All points of \( M \) lie inside the geodesic ball at \( O \) of radius \( r_1 \), such that the Sturm-Liouville solution \( y(r) \) above has its second zero at \( r_1 \). Here is how this reads in two cases.

**Case 1.** \( r_1 \in (a, b) \), a trivial situation. The root \( r_1 \) occurs when \( \tan(\mu \ln(r/a)) = -\mu \ln a \). If \( a = 1 \) then this gives

\[ D \leq 2e^{\pi/\mu}. \]

**Case 2.** \( y(r) > 0 \) for \( a \leq r \leq b \). The second zero of \( y(r) \) occurs at the first root of

\[ \alpha + \beta \ln(r/b) = 0 \]

beyond \( b \). (As in the theorem, we have \( \beta < 0 \).) This is equivalent to \( r = be^F \) where

\[ F = \frac{\alpha}{-\beta} = \frac{\ln a \cos \theta + (\sin \theta)/\mu}{\mu \ln a \sin \theta - \cos \theta} \quad \theta = \mu \ln(b/a), \]
and thus, \[ D \leq 2be^F. \]

When \( a = 1 \) we get
\[ D \leq 2be^{-\tan(\mu \ln b)/\mu}. \]

If the curvature hypotheses are valid at all origins \( O \) then the factors 2 in these diameter estimates are superfluous. Note that as \( \mu \) decreases to \( \lambda \), the second diameter estimate tends to \( +\infty \).

### 3 Logarithmic Kick

Let \( e_k < r_0 \leq a < b \) be given, where \( e_k \) is the \( k^{th} \) superpower of \( e \), \( \ln^k(e_k) = 0 \).

Define \( \lambda = \lambda_k(r_0, a, b) \) as the smallest positive solution of the equation
\[
\cot(\lambda(\ln^k b - \ln^k a)) = \lambda(\ln^k a - \ln^k r_0).
\]

Define
\[
F_k(r, \mu) = \frac{1}{4} \left( \frac{1}{r^2} + \frac{1}{(r \ln r)^2} + \cdots + \frac{1 + 4\mu^2}{(r \ln(r) \cdots \ln^k(r))^2} \right).
\]

**Logarithmic Kick Theorem.** Assume that for all \( r \geq r_0 \) we have \( \text{Ric}(r) \geq (n - 1)F_k(r, 0) \) and that for all \( r \in [a, b] \) we have
\[
\text{Ric}(r) > (n - 1)F_k(r, \lambda),
\]
where \( \lambda = \lambda_k(r_0, a, b) \) as above. Then \( M \) is compact.

When \( \mu > 0 \), the general solution to the Boju-Funar equation
\[
y'' + F_k(r, \mu)y = 0 \tag{3.11}
\]
is of the form
\[
\Phi_k(r)(A \cos(\mu \ln^k r) + B \sin(\mu \ln^k r)),
\]
where \( A, B \) are constants and
\[
\Phi_k(r) = (r \ln(r) \cdots \ln^{k-1}(r))^{1/2}.
\]

When \( \mu = 0 \) the solution degenerates to
\[
\Phi_k(r)(A + B \ln^k(r)).
\]

See [3].
Lemma 3.1. If \( y(r) \) solves the Boju-Funar equation (3.11) with initial conditions \( y(r_0) = 0 \) and \( y'(r_0) > 0 \) and if \( \mu > \lambda_k(r_0, a, b) \) then \( y(r) = 0 \) for some \( r > r_0 \).

Proof. Linear rescaling of the \( r \)-variable is invalid in the logarithmic context, but still the proof is similar to that of Lemma 2.1. The solution is

\[
y(r) = \Phi_k(r) \begin{cases} 
\ln^k(r) - \ln^k(r_0) & \text{if } r_0 \leq r \leq a \\
A \cos(\mu \ln^k r) + B \sin(\mu \ln^k r) & \text{if } a \leq r \leq b \\
\alpha + \beta \ln^k r & \text{if } b \leq r < \infty.
\end{cases}
\]

Matching values at \( a \), gives

\[
\ln^k(a) - \ln^k(r_0) = A \cos(\mu \ln^k a) + B \sin(\mu \ln^k a) \tag{3.12}
\]

after canceling the common factor \( \Phi_k(r) \). Matching derivative values gives

\[
\Phi_k'(a)(\ln^k(a) - \ln^k(r_0)) + \Phi_k(a)(\ln^k)'(a) \\
= \Phi_k'(a)(A \cos(\mu \ln^k a) + B \sin(\mu \ln^k a)) \\
+ \Phi_k(a)(-A \sin(\mu \ln^k a) + B \cos(\mu \ln^k a)) \mu(\ln^k)'(a). \tag{3.13}
\]

Plugging in (3.12), discarding the equal terms, and then canceling the common factor \( \Phi_k(a)(\ln^k)'(a) \) gives

\[
\frac{1}{\mu} = -A \sin(\mu \ln^k a) + B \cos(\mu \ln^k a).
\]

From this and (3.12) it follows that

\[
A = \cos(\mu \ln^k a)[\ln^k a - \ln^k r_0] - \frac{1}{\mu} \sin(\mu \ln^k a) \\
B = \sin(\mu \ln^k a)[\ln^k a - \ln^k r_0] + \frac{1}{\mu} \cos(\mu \ln^k a). \tag{3.14}
\]

Similarly at \( r = b \) we have

\[
A \cos(\mu \ln^k b) + B \sin(\mu \ln^k b) = \alpha + \beta \ln^k b,
\]

and through more canceling we get

\[
-A \sin(\mu \ln^k b) + B \cos(\mu \ln^k b) = \frac{\beta}{\mu}.
\]
Combined with (3.14), this gives

\[ \beta = \mu \left( -c(a)s(b) + s(a)c(b) \right)[\ln^k a - \ln^k r_0] + s(a)s(b) + c(a)c(b) \]

where \( c(a) = \cos(\mu \ln^k a), \quad s(a) = \sin(\mu \ln^k a) \), etc. But then

\[ \beta = \mu \sin(\mu (\ln^k a - \ln^k b))[\ln^k a - \ln^k r_0] + \cos(\mu (\ln^k a + \ln^k b)). \]

As in the proof of Lemma 2.1 it is fair to assume that \( \mu - \lambda \) is small. This ensures that \( \beta < 0 \), and therefore that \( y(r) \) vanishes at some \( r > r_0 \).

**Proof of the Logarithmic Kick Theorem.** Using Lemma 3.1 in place of Lemma 2.1, the proof is the same as in the linear case.

4 Comparison of Results

How does our kick assumption compare to that in [5]? It is different and slightly weaker. Take, for instance \( a = e \) and \( b = e^{\ell+1} \). The corresponding \( \lambda = \lambda(\ell) \) tends to 0 as \( \ell \to \infty \). The curvature assumption in [5] is

\[ \text{Ric}(r) > n - 1 \frac{1 + 4\nu^2}{4r^2} \]

for some \( \nu > 0 \) and all \( r \geq 1 \). If \( \ell \) is large then \( \lambda(\ell) < \nu \), and for all \( r \in [e, e^{\ell+1}] \) we have

\[ \text{Ric}(r) > n - 1 \frac{1 + 4\lambda(\ell)^2}{4r^2}, \]

which is the hypothesis of the Linear Kick Theorem with \( a = e, \ b = e^{\ell+1} \). Similar remarks are valid in the logarithmic context.

5 The Boundary Theorem

In this section we prove our second main result: manifolds with more curvature than a Sturm-Liouville bifurcator are compact, while those with less curvature are noncompact. More precisely, we assume that \( M \) is a complete \( n \)-dimensional Riemann manifold with positive Ricci curvature, and
that $O \in M$ is a fixed origin. We fix an SL-bifurcator $b(r)$, i.e., a continuous function $b : [0, \infty) \to (0, \infty)$ such that the solution of
\[ w'' + b(r)w = 0 \quad w(0) = 0 \quad w'(0) = 1 \]
is monotone and bounded. Then we assert
\begin{enumerate}[(a)]
  \item $M$ is noncompact if for all $r \geq 0$, 
\[ \sup \{ \text{Ric}_p : p = \exp_O(v) \text{ and } |v| = r \} \leq b(r). \]
  \item $M$ is compact if $\text{Ric}(r)$ exceeds $b(r)$.
\end{enumerate}

**Proof of (a).** This is trivial. An SL-bifurcator has
\[ \lim \inf_{r \to \infty} b(r) = 0, \]
whereas, the infimum of the Ricci curvature on a compact manifold of positive Ricci curvature is positive. \hfill \square

**Proof of (b).** As in the proof of the Kick Theorem, it suffices to show that if $c(r)$ exceeds the SL-bifurcator $b(r)$ then the solution $y(r)$ of the ODE
\[ y'' + c(r)y = 0 \quad y(0) = 0 \quad y'(0) = 1 \]
has a second zero. We use Picone’s Formula from [10], page 226, to compare $y(r)$ and the corresponding SL-bifurcator’s solution $w(r)$. In our case, the formula reads:
\[ \frac{w'(x)}{y(x)} [w'(x)y(x) - w(x)y'(x)] \bigg|_0^r = \int_0^r (c(x) - b(x))w(x)^2 \, dx + \int_0^r \left( \frac{w'(x)y(x) - w(x)y'(x)}{y(x)} \right)^2 \, dx, \]
provided that $y(x) > 0$ on the interval $0 < x \leq r$. At $x = 0$, the contribution $[w'y - wy']$ drops out since $w(x)/y(x) \to 1$ as $x \to 0$. Thus,
\[ \frac{w(r)}{y(r)} [w'(r)y(r) - w(r)y'(r)] = I(r) \]
where $I(r)$ is the sum of the two integrals. Rearranging the formula gives
\[ \frac{y'(r)}{y(r)} = \frac{w'(r)}{w(r)} - \frac{I(r)}{w^2(r)}. \quad (5.15) \]
The r.h.s. of (5.15) converges to a negative number or to $-\infty$ as $r \to \infty$. For

$$\lim_{r \to \infty} w(r) > 0 \text{ and } \lim_{r \to \infty} w'(r) = 0.$$ 

Therefore there exists an $r$ such that

$$\frac{w'(r)}{w(r)} - I(r) < 0.$$

It follows that $y(x)$ has a second zero. For if $y(x)$ remains positive on $(0, r]$ then $y'(r) < 0$, and concavity implies that the graph $y = y(x)$ subsequently crosses the $x$-axis.

**Remark.** In [1] it is shown that an SL-bifurcator satisfies

(a) $\int_0^\infty rb(r) \, dr < \infty$

(b) All solutions of $y'' + b(r)y = 0$ have limit derivatives as $r \to \infty$.

(c) Each solution of $y'' + b(r)y = 0$ independent from $w(r)$ diverges to $\pm \infty$ as $r \to \infty$.

6 Planar Curves

It should be easy to descend from higher dimensions to the simple one dimensional case. Unfortunately the Ricci condition does not make much sense. To overcome this we use the extrinsic curvature of curves. The corollary to the following result is a classification of smooth, complete, planar curves with non-vanishing curvature.

We say that a smooth function $h : [0, \infty) \to \mathbb{R}^2$ is a hook if $|h'(s)| = 1$ for all $s \in [0, \infty)$, $h$ in an embedding (i.e., it is a homeomorphism from $[0, \infty)$ to the $h$-image equipped with the inherited topology), and the curvature $\kappa(s)$ vanishes nowhere. Examples of hooks are a half parabola and a half spiral of infinite arclength.

**Hook Theorem.** Under ambient homeomorphisms of $\mathbb{R}^2$, every hook is equivalent to one of the following seven curves, the first four of which are bounded:
1. An inward spiral to a point. (This is ambiently homeomorphic to a half closed segment.)

2. An inward spiral to a segment.

3. An inward spiral to a circle.

4. An outward spiral to a circle.

5. An outward spiral to infinity. (This is ambiently homeomorphic to a ray.)

6. An outward spiral to a line. (A line is a circle through infinity in the 2-sphere.)

7. An outward spiral to the union of two parallel lines.

**Corollary 6.1.** Under ambient homeomorphisms, every complete curve $C$ smoothly embedded in $\mathbb{R}^2$ with non-vanishing curvature is equivalent to one the following fourteen curves: the circle, the straight line, the twelve combinations of the bounded and unbounded hooks.

The proof of the Hook Theorem and its corollary are left to the reader.

**Remark.** For curves other than the line, no change in the previous corollary occurs if we further require that the curvature of $C$ exceed a given positive continuous function $\kappa_0(s)$ with $\lim_{|s| \to \infty} \kappa_0(s) = 0$. In particular, the curvature of $C$ can be made to exceed the curvature of a parabola.

Despite the preceding remark, we want to say that the parabola lies at the boundary between embedded and non-embedded curves. We express this in two ways.

**Theorem 6.2.** There is an arc $\kappa_t$ in the space of curvature functions such that if $P_t$ is a planar curve with curvature $\kappa_t$ then

(a) If $t_2 > t_1$ then $\kappa_{t_2}$ exceeds $\kappa_{t_1}$.

(b) If $t < 0$ then $P_t$ is embedded.

(c) If $t > 0$ then $P_t$ is immersed, not embedded.

(d) $P_0$ is a parabola.
If $t \mapsto \tilde{\kappa}_t$ is an arc of curvature functions that approximates $t \mapsto \kappa_t$ and $\tilde{P}_t$ has curvature $\tilde{\kappa}_t$ then the arc $t \mapsto \tilde{P}_t$ continues to pass from embedded curves to non-embedded immersed curves at a single time-parameter near $t = 0$.

**Theorem 6.3.** A parabola is the most curved planar curve that is complete and embedded as a closed subset of the plane.

**Proof of Theorem 6.2.** This is trivial. Fix a parabola $P_0$. Its curvature function is $\kappa_0$. Fix a smooth bump function $\beta : \mathbb{R} \to [0, 1]$ with support $[-1, 1]$, and set

$$\kappa_t(s) = \kappa_0(s) + t\beta(s).$$

It is easy to check that this arc satisfies our assertions. ☐

To prove Theorem 6.3 we use the following lemma from sophomore calculus.

**Lemma 6.4.** Is $C$ a smooth curve in the plane then the angle its tangent turns is the integral of its curvature.

**Proof.** Parameterize $C$ by arclength, $C(s) = (x(s), y(s))$. The angle turned by the tangent is

$$\theta(s) = \angle(C'(0), C'(s)) = \arctan \left( \frac{y'(s)}{x'(s)} \right).$$

Its derivative is

$$\theta'(s) = \frac{(y''x' - y'x'')/(x')^2}{1 + (y'/x')^2} = (x'', y'')(y' - x'),$$

since $|C'| = 1$. For the same reason, $C''$ is perpendicular to $C'$, and so $|\theta'| = |(x'', y'')| = \kappa$, and the result follows. ☐

**Proof of Theorem 6.3.** The strategy resembles that of the Poincaré-Bendixson Theorem. Let $P$ be a parabola $y = kx^2$ with $k > 0$. We parameterize $P$ by arclength with $P(0) = (0, 0)$. Let $C$ be a more curved curve. Its curvature $\kappa(s)$ exceeds the curvature $\kappa_P(s)$ of the parabola, $s$ being arclength.

By Corollary 6.1 $C$ is ambiently homeomorphic to a straight line. It has no spirals. After translation and rotation, $C(0) = (0, 0)$ and $C'(0) = (1, 0)$. Since $\kappa$ exceeds $\kappa_P$, there is an $s_0$ such that

$$\kappa(s_0) > \kappa_P(s_0).$$
We can assume $s_0 > 0$. By Lemma 6.4, $\int_0^\infty \kappa_P(s) \, ds = \pi/2$, so
\[
\int_0^\infty \kappa(s) \, ds > \pi/2.
\]

It follows that for some first $s_1 > 0$ we have the turning angle $\theta(s_1) = \pi/2$. This means that $x(s_1)$ is a local maximum of $x(s)$ and $C'(s_1)$ points vertically upward. For some slightly greater $s_2$, $C'(s_2)$ points up and leftward. The line $L$ through $C(s_2)$ in the direction $C''(s_2)$ therefore meets the positive $y$-axis.

Since $C$ has no spirals, $C[s_2, \infty)$ cannot be confined to the compact part of the plane bounded by the $y$-axis, $L$, and $C[0, s_2]$. Since $C$ does not cross itself, it must cross the positive $y$-axis, say at $C(s_+) = (0, y_+)$. See Figure 1

**Case 1.** For some $s \leq 0$, $\kappa(s) > \kappa_P(s)$. Then $C(-\infty, 0)$ also crosses the positive $y$-axis, say at $C(s_-) = (0, y_-)$ with $0 < y_+ < y_-$. Then $C(s_+, \infty)$ is trapped inside the curve $C[s_-, s_+] \cup 0 \times [y_+, y_-]$, giving a spiral, contrary to
the hypothesis on $C$. See Figure 2. (If $y_- < y_+$ the roles of $C(s_+, \infty)$ and $C(-\infty, s_-)$ are reversed.)

**Case 2.** For all $s \leq 0$, we have $\kappa(s) = \kappa_P(s)$. Then $C(-\infty, 0)$ is the parabolic arc $y = kx^2$ with $x < 0$. The line $L$ crosses it, say at $C(s_-)$, and the curve $C(s_2, \infty)$ is confined by $L \cup C[s_-, s_2]$, contrary to the fact that $C$ is ambiently homeomorphic to a line. See Figure 3.

## 7 Embedded Surfaces

In this section we analyze surfaces embedded in $\mathbb{R}^3$. We prove two things:

**Theorem 7.1.** Among surfaces of revolution which are complete and embedded, the paraboloid is most curved.

**Theorem 7.2.** A surface of revolution that approximates the capped cylinder has curvature which is an SL-bifurcator.
Figure 3: $C(s_2, \infty)$ is trapped.
Proof of Theorem 7.1. Let $M$ be a surface of revolution whose curvature exceeds the curvature of the paraboloid. The curvature of its profile curve exceeds the curvature of a parabola, so by Theorem 6.3 it is not embedded.

Remark. It seems probable that there is a better theorem along these lines. It would remove the hypothesis that $M$ is a surface of revolution. The reason is that there are no surfaces embedded in $\mathbb{R}^3$ whose curvature decays like that of a logarithmic (or worse) spiral.

Proof of Theorem 7.2. Consider the surface of revolution $M$ generated by the profile curve

$$z = \frac{1}{1 - \rho}$$

in cylindrical $(\rho, \theta, z)$-coordinates. The curvature of the profile curve is $\kappa = z''/(1 + (z')^2)^{3/2}$, which is

$$\kappa = \frac{2}{((1 - \rho)^2 + (1 - \rho)^{-2})^{3/2}}.$$  

The curvature of $M$ at the point $p = (\rho, \theta, z(\rho))$ is the product of the profile curvature and the circular curvature, namely

$$K(p) = \frac{2/\rho}{(z^{-2} + z^2)^{3/2}}.$$  

As before, $r$ denotes the geodesic distance from the origin to $p$, and for large $r$, we have $r \approx z$. Thus

$$K(p) \approx \frac{2}{(0 + r^2)^{3/2}} = \frac{2}{r^3}$$

as $r \to \infty$.

The geodesics that emanate from the origin are profile curves, and since they are asymptotically parallel as $r \to \infty$, we see that the solution to the Jacobi equation

$$w'' + b(r)w = 0 \quad w(0) = 0 \quad w'(0) = 1$$

is monotone and bounded, where $b(r)$ is the curvature at a point $p$ whose distance from the origin is $r$. In other words, the curvature is an SL-bifurcator, and by the Boundary Theorem, any surface with more curvature is compact.
Remark. The surface $M$ resembles a capped cylinder. It is asymptotic to the cylinder $\rho = 1$ as $r \to \infty$.

Corollary 7.3. Arbitrarily small perturbations of the curvature of $M$ exist that correspond to the topological sphere and others that correspond to a paraboloid-like surface.

Proof. The curvature is an SL-bifurcator.

Remark. The preceding surface $M$ is not unique. Corresponding to any SL-bifurcator $b(r)$ there is a surface of revolution whose curvature is $b(r)$. Its properties are the same as those of $M$.

8 Discussion

We have investigated in this article the minimum amount of energy or curvature necessary for a complete manifold with positive Ricci curvature to collapse into a compact one. The minimum amount of energy depends on the manifold being embedded or not. In any case, the set of such manifolds, for which the minimum amount of extra curvature imposes compactness is considered to be the boundary between compact and noncompact.

The following are some questions and ideas that extend our investigation.

1. Decay versus kick. Above, we concentrated on the effect of a curvature kick. This amounts to putting the Whitney topology on the space of curvature functions, and seeking the corresponding boundary between compact and noncompact manifolds. On the other hand, one could seek such a boundary in terms of decay rates. The decay rate of curvature for the paraboloid is $r^{-2}$ as $r \to \infty$, while that of the smoothly capped cylinder is $r^{-3}$. With more work, a capped cylinder's decay rate can be made to be $r^{-(2+\epsilon)}$ for any given $\epsilon > 0$. (The geodesics from the origin still are asymptotically parallel.) Thus, in terms of decay rates, the paraboloid is at the boundary of the SL-bifurcators, although this is not the case in terms of Whitney (kick) perturbations.

2. Geometry at infinity. Does the decay of the Ricci curvature impose geometry of the manifold at infinity? (Topologically, $M$ is diffeomorphic to $\mathbb{R}^n$, $n \leq 3$ since its Ricci curvature is positive. See and )
example, if the decay of curvature for a complete, noncompact surface $M$ is asymptotically $r^{-3}$, are its geodesics emanating from the origin like those of the capped cylinder? That is, are they asymptotically parallel in the sense that they stay a bounded distance apart? This contrasts with the geometry of the paraboloid, where the curvature is asymptotically $r^{-2}$ and the geodesics diverge. So in general, we ask: is the asymptotic geometry of $M$ governed by the asymptotic decay of curvature?

3. **Classification of the geometry at infinity.** For noncompact, complete surfaces of positive curvature there are two extreme possibilities for the geometry at infinity. The geodesics from a fixed origin can be asymptotically parallel, as for the capped cylinder, or they can diverge, as for the paraboloid. Naturally, the behavior can also be of mixed type with some sectors of geodesics becoming asymptotically parallel, and others diverging. What happens under kick perturbation in the mixed case? See also Question 8 below.

In higher dimensions the situation becomes more complicated. For example, the three dimensional pure capped cylinder has asymptotic geometry at infinity $\mathbb{R} \times S^2$ and is at the boundary between compact and noncompact manifolds, while the pure three dimensional paraboloid is not at this boundary. What other behavior is there?

Passing to dimension four, we could consider products in which we take the product Riemann structure. (It is useful to remember that since we are dealing with Ricci curvature, the product of manifolds of positive Ricci curvature also has positive Ricci curvature.) For example, let $M$ be the product of a capped cylinder and a paraboloid. Kick perturbations do not produce compact manifolds, but they can change the geometry of $M$ at infinity from $(\mathbb{R} \times S^1) \times \mathbb{R}^2$ to $S^2 \times \mathbb{R}^2$. What is the general geometry at infinity and how does it change under kick (Whitney) perturbations?

4. **Tensors.** On a complete noncompact manifold $M$ with Riemann structure $g$, the positive Ricci curvature condition $\text{Ric} \geq (1/4r^2)g$ can be re-written as

$$\text{Ric} = \frac{1}{4r^2}g + A$$

where $A$ is a non-negative symmetric tensor. In dimension $n \geq 3$, the
second Bianchi identity \[2\] implies that the tensor \(A\) is not proportional to \(g\). For if \(\text{Ric} = fg\) then \(f\) is constant. Positivity of the Ricci curvature implies that the constant is positive, but Myers’ Theorem then implies that \(M\) is compact. What does the space of tensors \(A\) that decay faster than \(r^{-2}\) look like?

5. **Convergence.** In what sense does a sequence of noncompact complete manifolds converge to a manifold that lies on the boundary of compact manifolds?

6. **Singularities.** Consider a \(C^1\) \(g\) such that

\[
g = dr^2 + \alpha^2 r \sin^2(\nu \ln(r) + \phi) d\theta^2 \quad \text{for} \quad a < r < b
\]

\[
g = dr^2 + r d\theta^2 \quad \text{for} \quad b \leq r
\]

where \(\phi\) and \(\alpha\) are constants. How can the metric be continued smoothly in \(0 < r \leq a\) so the curvature stays positive and the singularity at \(r = 0\) is minimal? Can the singularities be classified?

7. **SL-bifurcators in higher dimensions.** There are other possibilities to define manifolds at the boundary, using the Jacobi fields. The generalization of SL-bifurcators in dimension \(n \geq 3\) deals with the matrix ODE

\[
Y'' + R(s)Y = 0 \tag{8.16}
\]

along geodesics \(\gamma\), where \(R(s)\) is the Ricci tensor \(R(s) = \text{Ric}(\dot{\gamma}(s), \cdot \dot{\gamma}(s))\). Consider a complete noncompact manifold, with positive Ricci curvature. Once an origin is fixed, if the exponential map at the origin has no conjugate points then the matrix ODE has a solution \(Y(t)\) such that \(Y(0) = 0\), \(Y'(0) = I\), and \(Y(s)\) is nonsingular for \(s > 0\). When \(Y(s)\) has a finite limit as \(s \to \infty\) (see [7], and [4] p.250), for any \(y\) orthogonal to \(\dot{\gamma}(0)\), \(Y(s)y\) converges to a finite vector as \(s\) tends to infinity.

This situation corresponds to the case where \(R(s)\) has positive eigenvalues. In addition, the existence of a limit at infinity imposes some conditions on the decay of the positive eigenfunctions.

Manifolds with such a property could be called SL-manifolds. When the matrix \(R(s)\) can be diagonalized by a matrix \(P(s)\) and \(P(s)\) converges
to an invertible matrix at infinity, the matrix equation reduces to \( n - 1 \) SL-bifurcators. In that case, a small increase of curvature leads to conjugate points and hence implies compactness. But in general, is it true that every small increase of curvature \( (R(s) + \epsilon Id) \) of an SL-manifold gives conjugate points and hence implies compactness?

8. **Conjecture.** Finally, we offer the following simple conjecture. Suppose that \( M \) is two dimensional, complete, and, judged from a fixed origin \( O \), its curvature is greater than or equal to an SL-bifurcator. If there is a geodesic from \( O \) that has a conjugate point then \( M \) is compact.

**Remark.** Clearly, if \( M \) is a surface of revolution, the conjecture is true. But imagine a capped cylinder, and increase its curvature on a small open set away from the origin. (The perturbation depends on \( \theta \).) Does this force compactness?

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