Stability between foliations in general relativity

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Abstract

The aim of this paper is to study foliations that remain invariable by parallel transports along the integral curves of vector fields of another foliations. According to this idea, we define a new concept of stability between foliations. A particular case of stability (called regular stability) is studied, giving a useful characterization in terms of the Riemann curvature tensor. This characterization allows us to prove that there are no regularly self-stable foliations of dimension greater than 1 in Schwarzschild and Robertson-Walker space-times. Finally, we study the existence of regularly self-stable foliations in other space-times, like pp-wave space-times.

1 Introduction

During the last decades, applications of foliations to theoretical physics have been considerably increased [1]. At the sixties, J. M. Souriau introduced foliations associated to elementary particles to study their evolutions in the Minkowski space-time [2]. Later, the use of foliated manifolds has provided very good results in relativity and quantum mechanics [3]. For instance, the symplectic bundle structure allows us to enclose a space-time, a dynamical system and its evolution space in the same mathematical structure. In this way, a foliation describes the evolution of the dynamical system [4, 5]. These facts have motivated us to study some general properties of foliations.

In this paper, we analyze distributions and foliations that remain invariable by parallel transports. If a foliation is conserved by parallel transports along the integral curves of its vector fields, then this foliation satisfies a motion law; in this case it can be proved that its leaves are totally geodesic [6, 7]. However, we can obtain more general properties using parallel transports along the integral curves of vector fields of another foliation. For example, if a foliation is conserved by parallel transports along world lines of a congruence of observers, then they observe the leaves of the foliation as invariable along their evolution, and it is interesting to study. According to this idea, we define a new concept of stability.
A particular case of stability (called **regular stability**) is studied in Section 3, giving a useful characterization in Theorem 6. This result allows us to prove that there are no regularly self-stable foliations of dimension greater than 1 in Schwarzschild and Robertson-Walker space-times, but there exist foliations of this kind in other space-times. Finally, in Section 4, we study the existence of regularly self-stable foliations in pp-wave space-times.

## 2 Stability

We work on an $n$-dimensional space-time manifold $\mathcal{M}$ (although all results and proofs can be generalized to any manifold with a torsion-free metric connection) and we denote the Levi-Civita connection by $\nabla$. We use the convention that $\text{span} (X_1, \ldots, X_p)$ denotes the subbundle spanned by the vector fields $X_1, \ldots, X_p$, and it is called **distribution**. Usually, a distribution of dimension $p$ is called a $p$-distribution. All bases of distributions are local. A distribution that has an integral submanifold (leaf) in every point is a **foliation**. We say that a foliation is a **flat foliation** if its leaves are flat submanifolds, and we say that a foliation is a **totally geodesic foliation** if its leaves are totally geodesic submanifolds.

In previous works [5, 6, 7], the concept of **motion law** was introduced using foliations: let $\Omega$ be a foliation, $X$ a vector field of $\Omega$, $c$ a maximal integral curve of $X$ and

$$\tau^c_t : T_{c(0)} \mathcal{M} \rightarrow T_{c(t)} \mathcal{M}$$

the parallel transport along $c(t)$, for all $t \in I$, where $I$ is the domain of $c$. Then, $\Omega$ verifies a **motion law** if

$$\tau^c_t \Omega (c(0)) = \Omega (c(t)), \quad t \in I.$$  

This motion law is equivalent to say that $\Omega$ is a totally geodesic foliation. Intuitively, the curvature of the leaves has to “adapt” to the curvature of the space-time. In Definition 1 we show how to generalize this intuitive idea.

**Definition 1** Let $\Omega, \Omega'$ be two distributions. We will say that $\Omega$ is **stable with respect to** $\Omega'$, and we will denote it by $\nabla_{\Omega'} \Omega \subset \Omega$, if

$$\nabla_Y X \in \Omega$$

for all vector fields $X \in \Omega, Y \in \Omega'$.

Particularly, if $\Omega = \Omega'$ we will say that $\Omega$ is **self-stable**.

Clearly, a distribution $\Omega$ is self-stable if and only if it is a totally geodesic foliation. Note that if $\Omega$ is a self-stable distribution, then $[X, Y] = \nabla_X Y - \nabla_Y X \in \Omega$ for all $X, Y \in \Omega$. So, $\Omega$ is involutive and hence, by Frobenius’ Theorem, it is integrable. In consequence, a self-stable distribution is in fact a totally geodesic foliation.
In order to know if $\Omega$ is stable with respect to $\Omega'$ it is sufficient to check that, given $\{X_i\}_{i=1}^p, \{Y_j\}_{j=1}^q$ some arbitrary bases of $\Omega$ and $\Omega'$ respectively, the following conditions hold:

$$\nabla_{Y_j} X_i \in \Omega, \quad \begin{cases} i = 1, \ldots, p, \\ j = 1, \ldots, q. \end{cases}$$ (1)

Besides, conditions (1) show that any span of vector fields of $\Omega$ is conserved by parallel transports along the integral curves of vector fields of $\Omega'$.

Sometimes it is easier to deal with the orthogonal distribution of $\Omega$ (denoted $\Omega^\perp$) instead of dealing with $\Omega$. In these cases, Proposition 2 is very useful.

**Proposition 2** Let $\Omega, \Omega'$ be two distributions. Then $\Omega$ is stable with respect to $\Omega'$ if and only if $\Omega^\perp$ is stable with respect to $\Omega'$; i.e.

$$\nabla_{\Omega'} \Omega \subset \Omega \iff \nabla_{\Omega'} \Omega^\perp \subset \Omega^\perp.$$ (2)

**Proof.** It is known [1] that for all triplet of vector fields $X, Y, Z$ in a pseudo-Riemannian manifold $M$ with metric $g$ and connection $\nabla$, we have

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$ (2)

Necessary condition: let $\Omega, \Omega'$ be two distributions such that $\nabla_{\Omega'} \Omega \subset \Omega$. Given three arbitrary vector fields $X, Y, Z \in \Omega$, $Y \in \Omega^\perp$, $Z \in \Omega'$, by (2) we have

$$0 = g(X, \nabla_Z Y).$$

So $\nabla_Z Y \in \Omega^\perp$, and then $\nabla_{\Omega'} \Omega^\perp \subset \Omega^\perp$.

The proof of the sufficient condition is analogous. $\blacksquare$

Proposition 2 says that $\Omega$ and $\Omega^\perp$ have the same behaviour in relation to stability. So, given a distribution $\Omega$, we can study the stability of $\Omega$ through the stability of $\Omega^\perp$. This is very useful when $\Omega$ is a $(n-1)$-distribution, since $\Omega^\perp$ is a 1-distribution and the study of the stability becomes easier. Moreover, if $\Omega$ is a lightlike $(n-1)$-distribution, then $\Omega^\perp$ is the span of a lightlike vector field of $\Omega$. In this particular case, the leaves of $\Omega$ are interpreted as wave fronts and the integral curves of $\Omega^\perp$ represent the world lines of a congruence of massless particles. Hence, Proposition 2 can be regarded as a “wave-particle duality” result.

### 3 Regular stability

We are going to introduce a special kind of stability, called regular stability.

**Definition 3** Let $\Omega, \Omega'$ be two distributions. We will say that $\Omega$ is regularly stable with respect to $\Omega'$, and we will denote it by $\nabla_{\Omega'} \Omega = 0$, if there exists a basis $\{X_i\}_{i=1}^p$ of $\Omega$ such that

$$\nabla_Y X_i = 0 \quad i = 1, \ldots, p,$$

for all vector field $Y \in \Omega'$. In this case we will say that $\{X_i\}_{i=1}^p$ is a regularly stable basis of $\Omega$ with respect to $\Omega'$.
Particularly, if \( \Omega = \Omega' \) we will say that \( \Omega \) is *regularly self-stable* and \( \{X_i\}_{i=1}^p \) is a *regularly self-stable basis* of \( \Omega \).

Given a regularly stable basis of \( \Omega \) with respect to \( \Omega' \), its vector fields are conserved by parallel transports along the integral curves of vector fields of \( \Omega' \). But only some bases of \( \Omega \) have this property.

It is clear that any subset of vector fields of a regularly self-stable basis spans a regularly self-stable foliation and, obviously, it is a regularly self-stable basis of this foliation. Particularly, for dimension 1, we obtain that the vector fields of a regularly self-stable basis are geodesic.

To illustrate the concept of regular stability, let us see the following example.

**Example 4** In spherical coordinates \((t, r, \theta, \varphi)\) the metrics of Schwarzschild and Robertson-Walker are given by

\[
ds^2 = \frac{1}{a_S}dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) - a_Sdt^2,
\]

\[
ds^2 = \frac{F^2}{a_{RW}^2} \left( dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right) - dt^2,
\]

respectively, where \(a_S := 1 - \frac{2m}{r}, F = F(t) \geq 0, a_{RW} := (1 + \frac{1}{4}kr^2)\) and \(k = -1, 0, 1\).

Let us consider the 2-foliations

\[
\Omega := \text{span} \left( \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi} \right), \quad \Omega' := \text{span} \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial t} \right).
\]

The leaves of \( \Omega \) are surfaces with \( r \) and \( t \) constant (i.e. spatial 2-spheres centered on the origin), and the leaves of \( \Omega' \) are surfaces with \( \theta \) and \( \varphi \) constant. It is easy to prove that \( \nabla_{\Omega'} \Omega = 0 \) in both space-times. The following bases of \( \Omega \)

\[
\left\{ \begin{array}{c}
1 \\
\frac{1}{r} \frac{\partial}{\partial \theta} \\
\frac{1}{r} \frac{\partial}{\partial \varphi}
\end{array} \right\}, \quad \left\{ \begin{array}{c}
a_{RW} \frac{\partial}{\partial \theta} \\
a_{RW} \frac{\partial}{\partial t}
\end{array} \right\},
\]

are regularly stable with respect to \( \Omega' \) in Schwarzschild and Robertson-Walker space-times respectively.

Next, we are going to study the relationships between two regularly stable bases of the same distribution \( \Omega \).

**Proposition 5** Let \( \Omega, \Omega' \) be two distributions such that \( \nabla_{\Omega'} \Omega = 0 \), and let \( \{X_i\}_{i=1}^p \) be a regularly stable basis of \( \Omega \) with respect to \( \Omega' \). Then, \( \{X_i\}_{i=1}^p \) is another regularly stable basis of \( \Omega \) with respect to \( \Omega' \) if and only if there exists a family of functions \( \left\{ \alpha_i^j \right\}_{i,j=1}^p \) such that

- \( \det \alpha_i^j \neq 0 \),
- \( X_i = \alpha_i^j X_j \) for all \( i = 1, \ldots, p \),
\[ Y \left( \alpha_j^i \right) = 0, \text{ for all } i, j = 1, \ldots, p, \text{ and for all } Y \in \Omega' \text{ (i.e. } \{ \alpha_j^i \}_{i,j=1}^p \text{ is a family of constant functions for } \Omega') \].

**Proof.** Necessary condition: let us suppose that \( \{X_i \}_{i=1}^p \) is a regularly stable basis of \( \Omega \) with respect to \( \Omega' \). Then, it is clear that there exists a family of functions \( \{ \alpha_j^i \}_{i,j=1}^p \) such that \( \det \alpha_j^i \neq 0 \), and \( X_i = \alpha_j^i X_j \) for all \( i = 1, \ldots, p \).

Let \( Y \) be an arbitrary vector field in \( \Omega' \). Then
\[
0 = \nabla_Y X_i = \nabla_Y \left( \alpha_j^i X_j \right) = Y \left( \alpha_j^i \right) X_j + \alpha_j^i \nabla_Y X_j, \quad i = 1, \ldots, p. \tag{3}
\]
Since \( \nabla_Y X_j = 0 \) for all \( j = 1, \ldots, p \), by (3) we have \( Y \left( \alpha_j^i \right) X_j = 0 \) for all \( i = 1, \ldots, p \) and then \( Y \left( \alpha_j^i \right) = 0 \) for all \( i, j = 1, \ldots, p \).

Sufficient condition: it is clear that \( \{X_i \}_{i=1}^p \) is another basis of \( \Omega \). Moreover, given \( Y \in \Omega' \) we have
\[
\nabla_Y X_i = \nabla_Y \left( \alpha_j^i X_j \right) = Y \left( \alpha_j^i \right) X_j + \alpha_j^i \nabla_Y X_j, \quad i = 1, \ldots, p. \tag{4}
\]
Since \( \nabla_Y X_j = 0 \) for all \( j = 1, \ldots, p \), and \( Y \left( \alpha_j^i \right) = 0 \) for all \( i, j = 1, \ldots, p \), by (4) we have \( \nabla_Y X_i = 0 \) for all \( i = 1, \ldots, p \), concluding the proof. \( \blacksquare \)

Proposition \( \text{\#} \) assures us uniqueness, up to constant functions for \( \Omega' \), of regularly stable bases of \( \Omega \) with respect to \( \Omega' \). Moreover, using this result, given a regularly stable basis of \( \Omega \) with respect to \( \Omega' \), we can construct all the regularly stable bases of \( \Omega \) with respect to \( \Omega' \).

The main result of this paper is given in the next theorem, showing an operational condition for the equivalence between stability and regular stability in terms of the Riemann curvature tensor \( R \). This condition is very useful because the study of regular stability is easier than the study of stability in general.

**Theorem 6** Let \( \Omega \) and \( \Omega' \) be a \( p \)-distribution and a \( q \)-foliation respectively such that \( \nabla_\Omega \Omega \subset \Omega \). Then, \( \nabla_\Omega \Omega = 0 \) if and only if \( R(Y, Z) X = 0 \) for all \( X \in \Omega \) and for all \( Y, Z \in \Omega' \).

**Proof.** Let \( \{X_i \}_{i=1}^p, \{Y_j \}_{j=1}^q \) be two bases of \( \Omega \) and \( \Omega' \) respectively, where \( p = \dim \Omega \) and \( q = \dim \Omega' \). Then, there exist some functions \( h_{jk}^i \), where \( i, k = 1, \ldots, p \) and \( j = 1, \ldots, q \) such that
\[
\nabla_{Y_j} X_k = h_{jk}^i X_i, \quad \left\{ \begin{array}{l}
  k = 1, \ldots, p, \\
  j = 1, \ldots, q.
\end{array} \right. \tag{5}
\]
Since \( \Omega' \) is a foliation, we can suppose that \( Y_j = \frac{\partial}{\partial x_j} \) for \( j = 1, \ldots, q \), where \( (x^1, \ldots, x^n) \) is a flat chart for \( \Omega' \).
Let us state the eqs. \( \nabla_j (y^i X_i) = 0 \) for \( j = 1, \ldots, q \), where \( \nabla_j \) denotes \( \frac{\partial}{\partial x^j} \) and \( y^i \) are unknown functions for \( i = 1, \ldots, p \). By using (5), we have

\[
\left( \frac{\partial y^i}{\partial x^j} + y^k h^i_{jk} \right) X_i = 0, \quad j = 1, \ldots, q.
\tag{6}
\]

Since \( \{X_i\}_{i=1}^p \) is a linearly independent family of vector fields, expression (6) becomes

\[
\frac{\partial y^i}{\partial x^j} + y^k h^i_{jk} = 0, \quad \begin{cases} i = 1, \ldots, p, \\ j = 1, \ldots, q. \end{cases}
\tag{7}
\]

The system (7) is formed by \( q \) first order homogeneous linear sub-systems with \( p \) differential equations and \( p \) unknown functions each one. In each sub-system, it appears only one differential operator \( \frac{\partial}{\partial x^j} \) for \( j = 1, \ldots, q \). By a Frobenius’ Theorem (see [10]), we have that (7) has non-zero solutions if and only if some compatibility conditions (between the \( q \) sub-systems that form (7)) are satisfied. These conditions are known as the “cross-derivatives conditions” and are built imposing that the cross-derivatives of the functions \( y^i \) are equal:

\[
\begin{align*}
\frac{\partial}{\partial x^j} \left( \frac{\partial y^i}{\partial x^l} \right) &= \frac{\partial}{\partial x^j} \left( -y^k h^i_{jk} \right) = -\frac{\partial y^k}{\partial x^j} h^i_{jk} - y^k \frac{\partial h^i_{jk}}{\partial x^j}, \\
\frac{\partial}{\partial x^l} \left( \frac{\partial y^i}{\partial x^j} \right) &= \frac{\partial}{\partial x^l} \left( -y^k h^i_{lk} \right) = -\frac{\partial y^k}{\partial x^l} h^i_{lk} - y^k \frac{\partial h^i_{lk}}{\partial x^l}.
\end{align*}
\]

\[
\Rightarrow \frac{\partial}{\partial x^j} h^i_{jk} + y^k h^i_{jk} = \frac{\partial}{\partial x^l} h^i_{lk} + y^k \frac{\partial h^i_{lk}}{\partial x^j}, \quad \begin{cases} i = 1, \ldots, p, \\ j, l = 1, \ldots, q. \end{cases}
\tag{8}
\]

Taking into account (7) and changing indexes, from (8) we obtain

\[
\left( h^m_{ik} h^i_{jm} - h^m_{jk} h^i_{lm} + \frac{\partial h^i_{lk}}{\partial x^j} - \frac{\partial h^i_{jk}}{\partial x^l} \right) y^k = 0, \quad \begin{cases} i = 1, \ldots, p, \\ j, l = 1, \ldots, q. \end{cases}
\tag{9}
\]

So, a necessary and sufficient condition for the system (7) to have non-zero solutions is

\[
h^m_{ik} h^i_{jm} - h^m_{jk} h^i_{lm} + \frac{\partial h^i_{lk}}{\partial x^j} - \frac{\partial h^i_{jk}}{\partial x^l} = 0, \quad \begin{cases} i, k = 1, \ldots, p, \\ j, l = 1, \ldots, q. \end{cases}
\tag{9}
\]

Moreover, if (9) is satisfied, the set of solutions of (7) form a vector space of dimension \( p \) (see [10]), i.e. there exists a family of differentiable functions \( \{f^k_i\}_{i,k=1}^p \) such that \( \det f^k_i \neq 0 \) and any solution of (7) has the form

\[
y^k = C^i f^k_i, \quad k = 1, \ldots, p,
\]

where \( \{C^i\}_{i=1}^p \) are parameter functions (i.e. \( \frac{\partial C^i}{\partial x^j} = 0 \) for \( i = 1, \ldots, p \) and \( j = 1, \ldots, q \)). Hence, \( \{f^k_i X_k\}_{i=1}^p \) is a regularly stable basis of \( \Omega \) with respect to \( \Omega' \), and so \( \nabla_\Omega \Omega = 0 \) if and only if (9) is satisfied.

So, we have to prove that (9) is equivalent to \( R(Y, Z) X = 0 \) for all \( X \in \Omega \) and for all \( Y, Z \in \Omega' \). In fact, from the linearity of the Riemann curvature
tensor, we have to prove that (9) is equivalent to
\[ R \left( \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^l} \right) X_i = 0 \]
for all \( i = 1, \ldots, p \) and \( j, l = 1, \ldots, q \):

\[ R \left( \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^l} \right) X_i = 0 \iff \nabla_j \nabla_l X_i - \nabla_l \nabla_j X_i = 0, \quad \{ i = 1, \ldots, p, j, l = 1, \ldots, q \}. \]

Applying (5) we have
\[ \nabla_j \left( h^k_{li} X_k \right) - \nabla_l \left( h^k_{ji} X_k \right) = 0 \]
\[ \iff h^k_{li} \nabla_j X_k + \frac{\partial h^k_{li}}{\partial x^j} X_k - h^k_{ji} \nabla_l X_k - \frac{\partial h^k_{ji}}{\partial x^l} X_k = 0, \quad \{ i = 1, \ldots, p, j, l = 1, \ldots, q \}. \]

Changing indexes,
\[ \iff \left( h^m_{ik} h^i_{jm} - h^m_{jk} h^i_{lm} + \frac{\partial h^i_{lk}}{\partial x^j} - \frac{\partial h^i_{lj}}{\partial x^k} \right) X_i = 0 \]
\[ \iff h^m_{ik} h^i_{jm} - h^m_{jk} h^i_{lm} + \frac{\partial h^i_{lk}}{\partial x^j} - \frac{\partial h^i_{lj}}{\partial x^k} = 0, \quad \{ i, k = 1, \ldots, p, j, l = 1, \ldots, q \}. \] (10)

Since expressions (9) and (10) are the same, we conclude the proof. ■

Next, we are going to give some useful corollaries of Theorem 6 related to some interesting cases.

**Corollary 7** Let \( \Omega \) and \( \Omega' \) be a \( p \)-distribution and a \( q \)-foliation respectively such that \( \nabla_{\Omega'} \Omega \subset \Omega \).

(i) In a flat space-time (Minkowski) we have that \( \nabla_{\Omega'} \Omega \subset \Omega \) if and only if \( \nabla_{\Omega'} \Omega = 0 \).

(ii) If \( q = 1 \) we have that \( \nabla_{\Omega'} \Omega \subset \Omega \) if and only if \( \nabla_{\Omega'} \Omega = 0 \).

In these cases, the study of stability becomes the study of regular stability. This fact simplifies remarkably the problem.

Let us suppose that \( \Omega' = \text{span}(Y) \). According to Corollary (ii), there exist regularly stable bases of \( \Omega \) with respect to \( \Omega' \), i.e. bases of \( \Omega \) whose vector fields are conserved by parallel transports along the integral curves of \( Y \). If \( \Omega \) is a \((n-1)\)-foliation and \( Y \) is a vector field which is not contained in \( \Omega \), then it is possible to reconstruct the entire foliation from only one leaf of \( \Omega \), by means of parallel transports of a regularly stable basis of \( \Omega \) with respect to \( \Omega' \) along the integral curves of \( Y \) (see fig. 1). Moreover, if \( Y \) is a future-pointing timelike vector field, then its integral curves represent observers and therefore, each observer detects \( \Omega \) as invariable along its evolution (i.e. along its world line), as we show in the next example.
Figure 1: If $\Omega$ is a $(n-1)$-foliation and $\Omega' = \text{span}(Y)$ is a 1-foliation such that $\nabla_{\Omega'} \Omega \subset \Omega$, then $\nabla_{\Omega'} \Omega = 0$ by Corollary 7 (ii). Moreover, if $Y$ is not in $\Omega$, then it is possible to reconstruct the entire foliation $\Omega$ from only one leaf, by means of parallel transports of a regularly stable basis of $\Omega$ with respect to $\Omega'$ along the integral curves of $Y$.

**Example 8** In the Schwarzschild space-time with spherical coordinates, $U := \frac{\partial}{\partial t}$ is a future-pointing timelike vector field, whose integral curves represent stationary observers. We are going to find all the lightlike 3-foliations that are stable with respect to $\text{span}(U)$, i.e. all the light waves that are observed as invariable by any stationary observer. If we don’t take into account Corollary 7 (ii), this becomes a hard work. But, applying this result, we only have to find the lightlike 3-foliations that are regularly stable with respect to $\text{span}(U)$. We obtain only two lightlike 3-foliations:

$$\text{span}\left( \pm \frac{\partial}{\partial t} + \frac{\partial}{\partial r} a, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \right).$$

The leaves of these foliations are spheres expanding and contracting respectively at the speed of light. So, all the observers represented by the integral curves of $U$ detect these foliations as invariable along their evolutions. But the most remarkable fact is that there are no other foliations with this property.

**Corollary 9** Let $\Omega$ be a self-stable $p$-foliation. Then $\Omega$ is regularly self-stable if and only if

$$R(Y, Z)X = 0,$$

for all $X, Y, Z \in \Omega$.

It is important to remark that condition (11) does not imply that $\Omega$ is a flat foliation. For example, in the Minkowski space-time, any foliation satisfies (11) but it is not necessarily flat. However, in general, if $\Omega$ is totally geodesic (i.e. it is self-stable) then $R \equiv R$ and vice versa, in the sense that $R(Y, Z)X = R(Y, Z)X$ where $R$ is the Riemann curvature tensor of the metric $\gamma$ induced in
the leaves of $\Omega$, $i : \mathcal{M}(\Omega) \rightarrow \mathcal{M}$ is the canonical inclusion and $X = i_*X, Y = i_*Y, Z = i_*Z$. So, if $\Omega$ is self-stable, then it is flat if and only if (11) is satisfied.

A foliation is regularly self-stable if and only if it is totally geodesic and flat, and hence, the regular self-stability generalizes the concept of flat wave fronts, introduced by J.M. Souriau in the Minkowski space-time [1]. We will discuss this fact deeply in Section 5.

Example 10 In the Schwarzschild and Robertson-Walker space-times, we can prove easily that there are not any distribution $\Omega$ of dimension greater than 1 such that $R(Y, Z)X = 0$ for all $X, Y, Z \in \Omega$. So, by Corollary 9 there are not any regularly self-stable foliations of dimension greater than 1. But, of course, there exist self-stable foliations, for example, in spherical coordinates, the timelike 2-foliation

$$\text{span}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right)$$

is a self-stable foliation whose leaves are surfaces with $\theta$ and $\varphi$ constant.

We will show, in Section 4, that there exist regularly self-stable foliations of dimension greater than 1 in pp-wave space-times. Moreover, we can find these kinds of foliations in other space-times, as we show in the next example.

Example 11 If we consider the metric $ds^2 = -\frac{1}{z}dt^2 + dx^2 + dy^2 + dz^2$ in the open set $\{ (t, x, y, z) : z > 0 \}$, then the Einstein tensor is positive definite. So it is a valid non-flat space-time. The spacelike 2-foliation given by

$$\text{span}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

is self-stable and satisfies (11). So, by Corollary 9 we obtain that this foliation is a regularly self-stable foliation. A regularly self-stable basis is given by $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$.

4 Examples of regularly self-stable foliations in pp-wave space-times

It is known [12] that, in standard coordinates $(u, v, y, z)$, a pp-wave metric can be expressed by $ds^2 = dy^2 + dz^2 - 2Hdu^2 - 2dudv$, where $u, v$ are the retarded and the advanced time coordinates respectively, and $H = H(u, y, z)$. According to [12], in a pp-wave space-time, the lightlike hypersurfaces with $u$ constant are leaves of a lightlike 3-foliation $\Omega$ given by

$$\Omega := \text{span}\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right),$$

and its leaves are called plane-fronted gravitational waves with parallel rays. The foliation $\Omega$ is self-stable and flat, i.e., $R(Y, Z)X = 0$ for all $X, Y, Z \in \Omega$. By
applying Corollary 2, we obtain that \( \Omega \) is a regularly self-stable foliation. Then there exists a basis \( \{ X_i \}^3_{i=1} \) of \( \Omega \) such that \( \nabla_X X_i = 0, i = 1, 2, 3 \), for all vector field \( X \in \Omega \).

This fact gives us a new geometrical perspective of the plane-fronted gravitational waves with parallel rays, because it ensures us explicitly the existence of this kind of bases of \( \Omega \) that remain invariant under parallel transports along the integral curves of vector fields of \( \Omega \).

For example, a regularly self-stable basis of \( \Omega \) is given by \( \{ \frac{\partial}{\partial v}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \} \), and we can use Proposition 2 to find all the regularly self-stable bases of \( \Omega \).

On the other hand, the subfoliations \( \text{span} \left( \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \), \( \text{span} \left( \frac{\partial}{\partial v}, \frac{\partial}{\partial y} \right) \), and \( \text{span} \left( \frac{\partial}{\partial v}, \frac{\partial}{\partial z} \right) \) are regularly self-stable 2-foliations. The first one is spacelike (its leaves are called wave surfaces [12]) and the others are lightlike. Regularly self-stable bases of these subfoliations are given by \( \left\{ \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} \), \( \left\{ \frac{\partial}{\partial v}, \frac{\partial}{\partial y} \right\} \), and \( \left\{ \frac{\partial}{\partial v}, \frac{\partial}{\partial z} \right\} \) respectively.

Moreover, the timelike 2-foliation \( \text{span} \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) \) is regularly self-stable too. A regularly self-stable basis is now given by \( \left\{ \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\} \).

5 Discussion and comments

We have introduced some new properties for foliations: stability and regular stability. Theorem 6 provides a relationship between both concepts in terms of the curvature. As particular cases, self-stability and regular self-stability are two interesting properties: a self-stable foliation is conserved by parallel transports along the integral curves of vector fields of the foliation, and a regularly self-stable foliation has a set of bases (characterized by Proposition 2) whose vector fields are conserved by parallel transports along the integral curves of vector fields of the foliation, i.e. the curvature of the leaves is “adapted” to the curvature of the space-time. From Corollary 9 it follows that regular self-stability is a motion law for flat foliations, in contrast to self-stability, that it is a motion law for foliations in general.

Finally, we show a direct interpretation of the leaves of a regularly self-stable lightlike \( p \)-foliation \( \Omega \) with \( p = n - 1 \), extending some properties of flat wave fronts given in special relativity (see [2] [11]) to general relativity: let \( \{ X_1, \ldots, X_p \} \) be a basis of \( \Omega \), where \( X_1, \ldots, X_{p-1} \) are spacelike and \( X_p \) is lightlike. Given a world line of a future-pointing timelike vector field \( U \) (i.e. an observer), the wave fronts of \( \Omega \) relative to \( U \) are the leaves of the intersection of \( \Omega \) and the Landau foliation \( \mathcal{L}_U \) associated to \( U \) (see [3] [6] [13] [14]). Let \( U \) be a future-pointing timelike vector field such that the wave fronts of \( \Omega \) relative to \( U \) are the leaves of the foliation \( \Omega \cap \mathcal{L}_U = \text{span} (X_1, \ldots, X_{p-1}) \), i.e.

\[
\Omega \cap \mathcal{L}_U = \text{span} (X_1, \ldots, X_{p-1}).
\] (12)

Since \( \{ X_1, \ldots, X_{p-1} \} \) is a regularly self-stable basis, the leaves of \( \text{span} (X_1, \ldots, X_{p-1}) \) are totally geodesic and flat. So \( U \) observes the wave fronts of \( \Omega \) as
spacelike totally geodesic and flat \((n - 2)\)-planes moving in the relative direction of \(X_p\) \((i.e. X_p \text{ projected onto the leaves of } L_U)\) at the speed of light. But we cannot ensure that the wave fronts of \(\Omega\) relative to any observer are totally geodesic and flat \((n - 2)\)-planes.

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