Almost Riemann Solitons and Gradient Almost Riemann Solitons on \( LP \)-Sasakian Manifolds

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Abstract. The upcoming article aims to investigate almost Riemann solitons and gradient almost Riemann solitons in a \( LP \)-Sasakian manifold \( M^3 \). At first, it is proved that if \((\gamma, Z, \lambda)\) be an almost Riemann soliton on a \( LP \)-Sasakian manifold \( M^3 \), then it reduces to a Riemann soliton, provided the soliton vector \( Z \) has constant divergence. Also, we show that if \( Z \) is pointwise collinear with the characteristic vector field \( \xi \), then \( Z \) is a constant multiple of \( \xi \), and the ARS reduces to a Riemann soliton. Furthermore, it is proved that if a \( LP \)-Sasakian manifold \( M^3 \) admits gradient almost Riemann soliton, then the manifold is a space form. Also, we consider a non-trivial example and validate a result of our paper.

1. Introduction

The idea of Ricci flow was introduced by Hamilton [5] and defined by \( \frac{\partial}{\partial t} g(t) = -2S(t) \), where \( S \) denotes the Ricci tensor.

As a natural generalization, the concept of Riemann flow ([14],[15]) is defined by \( \frac{\partial}{\partial t} G(t) = -2Rg(t) \), \( G = \frac{1}{2} g \otimes g \), where \( R \) is the Riemann curvature tensor and \( \otimes \) is Kulkarni-Nomizu product (executed as (see Besse [2], p. 47),

\[
(P \otimes Q)(X, Y, Z, W) = P(X, W)Q(Y, U) + P(Y, U)Q(X, W)
- P(X, U)Q(Y, W) - P(Y, W)Q(X, U).
\]

Similar to Ricci soliton, the interesting idea of Riemann soliton was introduced by Hirica and Udriste [6]. Analogous to Hirica and Udriste [6], a Lorentzian metric \( g \) on a Lorentzian manifold \( M \) is called a Riemann solitons if there exists a \( C^\infty \) vector field \( Z \) and a real scalar \( \lambda \) such that

\[
2R + \alpha g \otimes g + g \otimes \mathcal{L}_Z g = 0. \tag{1}
\]

On this occasion, we should mention that the space of constant sectional curvature is generalized by the Riemann soliton. If the vector field \( Z \) is the gradient of the potential function \( \gamma \), then the manifold is called gradient Riemann soliton. Then the foregoing equation can be written as

\[
2R + \alpha g \otimes g + g \otimes \nabla^2 \gamma = 0, \tag{2}
\]

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where $\nabla^2 f$ denotes the Hessian of $\gamma$. If we modified the equation (1) and (2) by fixing the condition on the parameter $\lambda$ to be a variable function, then it reduces to ARS and gradient ARS respectively. Here the terminology “almost Riemann solitons” is written as ARS which will be applied throughout the article.

A general idea of Lorentzian para-Sasakian (briefly LP-Sasakian) manifold has been introduced by K. Matsumoto [7], in 1989 and several geometers in different context ([1], [8], [9], [10]) have studied LP-Sasakian manifolds. Riemann solitons and gradient Riemann solitons on Sasakian manifolds have been discussed in detail by Hirica and Udriste (see, [6]). Moreover, Riemann’s soliton concerning infinitesimal harmonic transformation was investigated in [13]. Here it is appropriate to notice that Sharma in [11] investigated almost Ricci soliton in K-contact geometry and in [12], with divergence-free soliton vector field. Very recently in [4], the authors studied Riemann soliton within the framework of a contact manifold and proved various fascinating results.

The above studies motivate us to investigate an ARS and the gradient ARS in a 3-dimensional LP-Sasakian manifold.

The upcoming article is structured as follows: In section 2, we recall some fundamental facts and formulas of LP-Sasakian manifolds, which will be needed in later sections. Beginning from Section 3, after providing the proof, we will write our prime theorems. This article terminates with a concise bibliography which has been used during the formulation of the upcoming article.

2. LP-Sasakian manifolds

Let $\eta, \xi, \phi$ are tensor fields on a smooth manifold $M^n$ of types $(0,1), (1,0)$ and $(1,1)$ respectively, such that

$$\eta(\xi) = -1, \quad \phi^2 E = E + \eta(E)\xi. \quad (3)$$

The foregoing equations imply that

$$\phi \xi = 0, \quad \eta \circ \phi = 0. \quad (4)$$

Then $M^n$ admits a Lorentzian metric $g$ of type $(0,2)$ such that

$$g(E, \xi) = \eta(E), \quad \eta(\phi E, \phi F) = g(E, F) + \eta(E)\eta(F) \quad (5)$$

for any vector fields $E, F$. Then the structure $(\eta, \xi, \phi, g)$ is called Lorentzian almost para-contact structure. The manifold $M^n$ equipped with a Lorentzian almost para-contact structure $(\eta, \xi, \phi, g)$ is called a Lorentzian almost para-contact manifold (briefly LAP-manifold).

If we denote $\Phi(E, F) = g(E, \phi F)$, then we obtain [7]

$$\Phi(E, F) = g(E, \phi F) = g(\phi E, F) = \Phi(F, E), \quad (6)$$

where $E, F$ are any vector fields.

An LAP-manifold $M^n$ equipped with the structure $(\eta, \xi, \phi, g)$ is said to be a Lorentzian para-contact manifold (briefly LP-manifold) if

$$\Phi(E, F) = \frac{1}{2} (\nabla_E \eta) F + (\nabla_F \eta) E, \quad (7)$$

where $\Phi$ is defined by (6) and $\nabla$ indicates the covariant differentiation operator with respect to the Lorentzian metric $g$. A Lorentzian almost para-contact manifold $M^n$ is said to be a LP-Sasakian manifold if it satisfies

$$(\nabla_E \phi) F = \eta(F) E + g(E, F) \xi + 2\eta(E)\eta(F)\xi. \quad (8)$$

Also since the vector field, $\eta$ is closed in an LP-Sasakian manifold we have

$$(\nabla_E \eta) F = \Phi(E, F) = g(E, \phi F), \quad \Phi(E, \xi) = 0, \quad \nabla E \xi = \phi E. \quad (9)$$

Furthermore, we find that the eigen values of $\phi$ are -1, 0 and 1. Here the multiplicity of 0 is one. Let us assume that the multiplicities of -1 and 1 are $k$ and $l$ respectively. Then we get, $\text{trace}(\phi) = l - k$. Hence, if
(trace(\(\phi\)))^2 = (n - 1)$, then either \(l = 0\) or \(k = 0\). Then the structure is called a trivial \(LP\)-Sasakian structure. Throughout this article we presume that \(\text{trace}(\phi) \neq 0\), i.e., \(\xi\) is not harmonic.

Let us presume that \(\{e_i\}\) be an orthonormal basis such that \(e_1 = \xi\). Then the well-known Ricci tensor \(S\) and the scalar curvature \(r\) are defined by

\[
S(E, F) = \sum_{i=1}^{n} e_i g(R(e_i, E), e_i)
\]

and

\[
r = \sum_{i=1}^{n} e_i S(e_i, e_i),
\]

where we put \(e_i = g(e_i, \xi)\), that is, \(e_1 = -1, e_2 = \cdots = e_n = 1\).

Also in an \(LP\)-Sasakian manifold \(M^n\), the subsequent relations hold ([1], [7], [10]):

\[
\eta(R(E, F)Z) = g(F, Z)\eta(E) - g(E, Z)\eta(F),
\]

\[
R(E, F)\xi = \eta(F)E - \eta(E)F,
\]

\[
R(\xi, E)F = g(E, F)\xi - \eta(F)E,
\]

\[
S(E, \xi) = (n - 1)\eta(E),
\]

\[
\nabla_\xi \eta = 0,
\]

for any vector fields \(E, F, Z\) where \(R\) is the Riemannian curvature tensor, \(S\) is the Ricci tensor and \(\nabla\) is the Levi-Civita connection associated to the metric \(g\).

It is well-known that a 3-dimensional Riemannian manifold \(M\) assumes the following curvature form

\[
R(E, F)Z = g(F, Z)QE - g(E, Z)QF + S(F, Z)E - S(E, Z)F - \frac{r}{2} [g(F, Z)E - g(E, Z)F],
\]

for any vector fields \(E, F, Z\) where \(Q\) is the Ricci operator, i.e., \(g(QE, F) = S(E, F)\) and \(r\) is the scalar curvature of the manifold. Replacing \(F = Z = \xi\) in the previous equation and utilizing (11) and (13) we get (see [10])

\[
QE = \frac{1}{2} [(r - 2)E + (r - 6)\eta(E)\xi].
\]

In view of (16) the Ricci tensor is written as

\[
S(E, F) = \frac{1}{2} [(r - 2)g(E, F) + (r - 6)\eta(E)\eta(F)].
\]

Using (17) and (16) in (15), we deduce

\[
R(E, F)Z = \frac{(r - 4)}{2} [g(F, Z)E - g(E, Z)F] + \frac{(r - 6)}{2} [g(F, Z)\eta(E)\xi - g(E, Z)\eta(F)\xi + \eta(F)\eta(Z)E - \eta(E)\eta(Z)F].
\]

We first prove the following Lemma:
Lemma 2.1. Let \( M^3 \) be a LP-Sasakian manifold. Then we have
\[
\xi r = -2(r - 6)\text{trace}(\phi).
\]

Proof. The equation (16) can be rewritten as:
\[
QF = \frac{1}{2}[r - 2F + (r - 6)\eta(F)\xi].
\]
Taking covariant derivative along \( E \) and recalling (9) we write
\[
(V_Q E)F = \frac{(Er)F}{2} + \frac{(Er)}{2} \eta(F)\xi + \frac{(r - 6)}{2} g(E, \phi F)\xi
+ \frac{(r - 6)}{2} \eta(F)\phi E.
\]
Taking inner product operation with respect to \( Z \) in the foregoing equation, we obtain
\[
g((V_Q E)F, Z) = \frac{(Er)F}{2} g(F, Z) + \frac{(Er)F}{2} \eta(F)\eta(Z) + \frac{(r - 6)}{2} g(E, \phi F)\eta(Z)
+ \frac{(r - 6)}{2} \eta(F)g(\phi E, Z).
\]
Putting \( E = Z = e_i \) (where \( \{e_i\} \) is an orthonormal basis for the tangent space of \( M^3 \) and taking \( \sum_i i, 1 \leq i \leq 3 \) ) in the above equation and utilizing the formula of Riemannian manifolds \( \text{div} Q = \frac{1}{2}\text{grad} r \), we obtain
\[
(\xi r)\eta(F) = -2(r - 6)\eta(F)\text{trace}(\phi).
\]
Substituting \( F = \xi \) in the above equation we get the desired result. This finishes the proof. \( \square \)

If an LP-Sasakian manifold \( M^3 \) is a space of constant curvature, then the manifold is said to be a space form.

Lemma 2.2. (Lemma 1.1 of [10]) A 3-dimensional LP-Sasakian manifold is a space form if and only if the scalar curvature \( r \) is 6.

Lemma 2.3. (Lemma 3.8 of [4]) For any vector fields \( E, F \) on \( M^3 \), for a gradient ARS \( (M, g, \gamma, m, \lambda) \), we have
\[
R(E, F)D\gamma = (V_Q E)E - (V_Q E)F
+ \{F(2\lambda + \Delta)E - E(2\lambda + \Delta)E\},
\]

where \( \Delta \gamma = \text{div} D\gamma \), \( \Delta \) is the Laplacian operator.

3. ARS on 3-dimensional LP-Sasakian manifolds

We consider a 3-dimensional para-Sasakian manifold \( M \) admitting an ARS defined by (1). Using Kulkarni-Nomizu product in (1) we write
\[
2R(E, F, W, X) + 2\lambda(g(E, X)g(F, W) - g(E, W)g(F, X))
+ \{g(E, X)(E_M g)(F, W) + g(F, W)(E_M g)(E, X)
- g(E, W)(E_M g)(F, X) - g(F, X)(E_M g)(E, W)\} = 0.
\]
Contracting (25) over \( E \) and \( X \), we get
\[
(E_M g)(F, W) + 2S(F, W) + (4\lambda + 2\text{div}Z)g(F, W) = 0.
\]
Utilizing (17) in the above equation we obtain
\[
(E_M g)(F, W) = -(r - 2 + 4\lambda + 2\text{div}Z)g(F, W)
- (r - 6)\eta(F)\eta(W) = 0.
\]
Applying $Z$ has constant divergence and executing covariant derivative along $E$, we lead

\[
(V_EZg)(F, W) = -[(Er) + 4(E\lambda)]g(F, W) - (Er)\eta(F)\eta(W) - (r - 6)[g(\phi E, F)\eta(W) + g(\phi E, W)\eta(F)] = 0. \tag{28}
\]

Now we recall the formula by Yano (see, [16]):

\[
(E_ZV g - V_EZg - V(Z,E)g)(F, W) = -g((E_ZV)(E, F), W) - g((E_ZV)(E, W), F).
\]

Hence by a straightforward calculation, we infer

\[
(V_EZg)(F, W) = g((E_ZV)(E, F), W) + g((E_ZV)(E, W), F). \tag{29}
\]

Using symmetric property of $E_i\nabla$, it reveals from (29) that

\[
g((E_ZV)(E, F), W) = \frac{1}{2}(V_EZg)(F, W) + \frac{1}{2}(V_EZg)(E, W) - \frac{1}{2}(V_WZg)(E, F). \tag{30}
\]

Utilizing (28) in (30) we obtain

\[
2g((E_ZV)(E, F), W) = -[(Er) + 4(E\lambda)]g(F, W) - (Er)\eta(F)\eta(W) - (r - 6)[g(\phi E, F)\eta(W) + g(\phi E, W)\eta(F)] - [(Fr) + 4(F\lambda)]g(E, W) - (Fr)\eta(E)\eta(W) - (r - 6)[g(\phi F, E)\eta(W) + g(\phi F, W)\eta(E)] + [(Wr) + 4(W\lambda)]g(E, F) + (W\eta)(E)\eta(F) - (r - 6)[g(\phi W, E)\eta(F) + g(\phi W, F)\eta(E)]. \tag{31}
\]

After substituting $E = F = e_i$ in the foregoing equation and removing $Z$ from both sides, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking $\sum_i 1 \leq i \leq 3$, we have

\[
(E_ZV)(e_i, e_i) = 2D\lambda - (\xi r)\xi - 2(r - 6)\text{trace}(\phi)\xi, \tag{32}
\]

where $E\xi = g(D\xi, E)$, $D$ denotes the gradient operator with respect to $g$. Now differentiating (1) and utilizing it in (29) we can easily determine

\[
g((E_ZV)(E, F), W) = (V_WS)(E, F) - (V_ES)(F, W) - (V_FS)(E, W). \tag{33}
\]

Taking $E = F = e_i$ (where $\{e_i\}$ is an orthonormal frame) in (33) and summing over $i$ we obtain

\[
(E_ZV)(e_i, e_i) = 0, \tag{34}
\]

for all vector fields $Z$. Combining (32) and (34) gives

\[
-2D\lambda + (\xi r)\xi + 2(r - 6)\text{trace}(\phi)\xi = 0. \tag{35}
\]

Utilizing (19) in the previous equation, we get

\[
D\lambda = 0. \tag{36}
\]

This implies that $\lambda$ is constant. This leads to the following theorem:

**Theorem 3.1.** If the soliton vector $Z$ has constant divergence in a LP-Sasakian manifold $M^3$, then an ARS reduces to a Riemann soliton.
Now let the potential vector field $Z$ be point-wise collinear with the characteristic vector field $\xi$ (i.e., $Z = b \xi$, where $b$ is a function on $M^3$) and has constant divergence. Therefore from (26) we lead

\[ g(\nabla_{\xi} b \xi, F) + g(\nabla_{b} \xi, E) + 2S(E, F) + 4\lambda g(E, F) = 0. \] (37)

Using (9) in (37), we get

\[ (Eb)\eta(F) + (Fb)\eta(E) + 2S(E, F) + (4\lambda + 2\text{div}Z)g(E, F) = 0. \] (38)

Putting $F = \xi$ in (38) yields

\[ -(Eb)\eta(E) + 4\eta(E) + (4\lambda + 2\text{div}Z)\eta(E) = 0. \] (39)

Putting $E = \xi$ in (39) we have

\[ (\xi b) = (2\lambda + \text{div}Z - 2). \] (40)

Putting the value of $\xi b$ in (39) gives

\[ db = -(6\lambda + 3\text{div}Z + 2)\eta. \] (41)

Operating (41) by $d$ and utilizing Poincare lemma $d^2 \equiv 0$, we infer

\[ 0 = d^2 b = -(6\lambda + 3\text{div}Z + 2)d\eta - 6d\lambda\eta. \] (42)

Executing wedge product of (42) with $\eta$, we have

\[ -(6\lambda + 3\text{div}Z + 2)\eta \land d\eta = 0. \] (43)

Since $\eta \land d\eta \neq 0$ in a $LP$-Sasakian manifold $M^3$, therefore

\[ \lambda = -(\frac{1}{2}\text{div}Z + \frac{1}{3}). \] (44)

Using (44) in (41) gives $db = 0$ i.e., $b =$constant. Also from (32) we obtain

\[ \lambda = -(\frac{1}{2}\text{div}Z + \frac{1}{3}) = \text{constant}. \] (45)

Hence we write the following:

**Theorem 3.2.** If the metric of a $LP$-Sasakian manifold $M^3$ is ARS and $Z$ is pointwise collinear with $\xi$ and has constant divergence, then $Z$ is a constant multiple of $\xi$ and the ARS reduces to a Riemann soliton.

**Corollary 3.3.** If a $LP$-Sasakian manifold $M^3$ admits an ARS of type $(g, \xi)$, then the ARS reduces to a Riemann soliton.

4. Gradient Almost Riemann soliton

This section is devoted to investigate a $LP$-Sasakian manifold $M^3$ admitting gradient ARS. Now before producing the detailed proof of our main theorems, we first write the following results without proof (Since the result can be obtained directly from (21)):

**Lemma 4.1.** For a $LP$-Sasakian manifold $M^3$, we have

\[ (\nabla_{E}Q)\xi = -(\frac{r}{2} - 3)\phi E, (\nabla_{E}Q)E = -2(r - 6)\text{trace}\phi[E + \eta(E)\xi]. \] (46)
Replacing $F$ by $\xi$ in (24) and utilizing the foregoing Lemma, we obtain
\[
R(E, \xi)D\gamma = \left(\frac{r}{2} - 3\right)\phi E - 2(r - 6)\text{trace}\phi[E + \eta(E)\xi] + \xi[(2\lambda + \Delta\gamma)E - E(2\lambda + \Delta\gamma)\xi].
\]

Then using (8), we infer
\[
g(E, D\gamma + D(2\lambda + \Delta\gamma))\xi = \left(\frac{r}{2} - 3\right)\phi E - 2(r - 6)\text{trace}\phi[E + \eta(E)\xi] + \xi[(\xi\gamma) + \xi(2\lambda + \Delta\gamma)]E.
\]

Executing the inner product of the previous equation with $\xi$ gives
\[
E(\gamma + (2\lambda + \Delta\gamma)) = [(\xi\gamma) + \xi(2\lambda + \Delta\gamma)]\eta(E),
\]
from which easily we obtain
\[
d(\gamma + (2\lambda + \Delta\gamma)) = [(\xi\gamma) + \xi(2\lambda + \Delta\gamma)]\eta,
\]
where $d$ indicates the exterior derivative. From the previous equation we see that $\gamma + (2\lambda + \Delta\gamma)$ is invariant along the distribution $\mathcal{D}$. In other terms, $E(\gamma + (2\lambda + \Delta\gamma)) = 0$ for any $E \in \mathcal{D}$. Using (49) in (48), we lead
\[
[(\xi\gamma) + \xi(2\lambda + \Delta\gamma)][\eta(E)\xi - E] = \left(\frac{r}{2} - 3\right)\phi E - 2(r - 6)\text{trace}\phi[E + \eta(E)\xi].
\]

Contracting the above equation yields
\[
[(\xi\gamma) + \xi(2\lambda + \Delta\gamma)] = 0.
\]
Utilizing (52) in (51), we get
\[
(r - 6)(\phi E - 4\text{trace}\phi[E + \eta(E)\xi)] = 0.
\]
If $[\phi E - 4\text{trace}\phi[E + \eta(E)\xi]] = 0$, operating $\phi$ we can easily obtain $\phi^2 E = 4\text{trace}\phi(\phi E)$, which is obviously a contradiction. Thus we have $r = 6$. Hence by Lemma 2.2, the manifold is a space form.

Hence we write the following:

**Theorem 4.2.** If a LP-Sasakian manifold $M^3$ admits a gradient ARS, then the manifold is a space form.

5. Example

Here we consider a known example of our paper [3]. In this article, we considers a 3-dimensional manifold $\mathcal{M} = \{(u, v, w) \in \mathbb{R}^3, w \neq 0\}$ and The vector fields
\[
e_1 = e^v \frac{\partial}{\partial v}, \quad e_2 = e^v(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}), \quad e_3 = \frac{\partial}{\partial w}
\]
are linearly independent at each point of $\mathcal{M}$ and shows that the manifold is a LP-Sasakian manifold. Further, the well-known Koszul’s formula gives
\[
\nabla_{\delta_1} \delta_1 = -\delta_3, \quad \nabla_{\delta_1} \delta_2 = 0, \quad \nabla_{\delta_1} \delta_3 = -\delta_1,
\]
\[
\nabla_{\delta_2} \delta_1 = 0, \quad \nabla_{\delta_2} \delta_2 = -\delta_3, \quad \nabla_{\delta_2} \delta_3 = -\delta_2,
\]
\[
\nabla_{\delta_3} \delta_1 = 0, \quad \nabla_{\delta_3} \delta_2 = 0, \quad \nabla_{\delta_3} \delta_3 = 0.
\]
Also, we have obtained the expressions of the curvature tensor and the Ricci tensor, respectively, as follows:
Using (54) we get

\[ R(\delta_1, \delta_2)\delta_3 = 0, \quad R(\delta_2, \delta_3)\delta_3 = -\delta_2, \quad R(\delta_1, \delta_3)\delta_3 = -\delta_1, \]
\[ R(\delta_1, \delta_2)\delta_2 = \delta_1, \quad R(\delta_2, \delta_3)\delta_2 = -\delta_3, \quad R(\delta_1, \delta_3)\delta_2 = 0, \]
\[ R(\delta_1, \delta_2)\delta_1 = -\delta_2, \quad R(\delta_2, \delta_3)\delta_1 = 0, \quad R(\delta_1, \delta_3)\delta_1 = -\delta_3, \]

and

\[
S(\delta_1, \delta_1) = g(R(\delta_1, \delta_2)\delta_2, \delta_1) - g(R(\delta_1, \delta_3)\delta_3, \delta_1) = 2.
\]

Similarly we have

\[ S(\delta_2, \delta_2) = 2, \quad S(\delta_3, \delta_3) = -2 \]

and

\[ S(\delta_i, \delta_j) = 0 (i \neq j). \]

Therefore,

\[ r = S(\delta_1, \delta_1) + S(\delta_2, \delta_2) - S(\delta_3, \delta_3) = 6. \]

From the expressions of the Ricci tensor, we find that \( M \) is an Einstein manifold.

Suppose \( f: M^3 \to \mathbb{R} \) be a smooth function such that \( f = w \). Then we can obtain

\[ Df = \frac{\partial}{\partial w} = \delta_3. \]

Using (54) we get

\[ \text{Hess} f(\delta_1, \delta_1) = 0. \]

Thus from (2) we can easily see that \( g \) is a gradient Riemann soliton with \( f = w \) and \( \lambda = -1 \). Hence the Theorem 4.2. is verified.

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