A NOTE ON THE NEBENHÜLLE OF SMOOTH COMPLETE HARTOGS DOMAINS

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Abstract. It is shown that a smooth bounded pseudoconvex complete Hartogs domain in $\mathbb{C}^2$ has trivial Nebenhülle. The smoothness assumption is used to invoke a theorem of D. Catlin from [2].

1. Introduction

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$ and let $\psi(z)$ be a continuous and bounded function on $\mathbb{D}$. Let us consider the domain $\Omega$ in $\mathbb{C}^2$ defined by;

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \in \mathbb{D}; |z_2| < e^{-\psi(z_1)}\}.$$ 

The domain $\Omega$ is a bounded complete Hartogs domain. Moreover, it is known that (see [8, page 129]) $\Omega$ is a pseudoconvex domain if and only if $\psi(z)$ is a subharmonic function on $\mathbb{D}$. In order to focus on pseudoconvex domains; we assume that $\psi(z)$ is a subharmonic function for the rest of the note.

Definition 1 ([4]). The nebenhülle of $\Omega$, denoted by $N(\Omega)$, is the interior of the intersection of all pseudoconvex domains that compactly contain $\Omega$. We say $\Omega$ has nontrivial Nebenhülle if $N(\Omega) \setminus \Omega$ has interior points.

Let $\mathcal{F}$ be the set of functions $r(z)$ where $r(z)$ is a subharmonic function on a neighborhood of $\mathbb{D}$ such that $r(z) \leq \psi(z)$ on $\mathbb{D}$. We define the following two functions;

$$R(z) = \sup_{r \in \mathcal{F}} \{r(z)\},$$

$$R^*(z) = \limsup_{D \ni \zeta \to z} R(z).$$

Note that $R^*(z)$ is upper semicontinuous and subharmonic on $\mathbb{D}$.

The following proposition from [6, Theorem 1] gives the description of $N(\Omega)$ for $\Omega$ a complete Hartogs domain as above.

Proposition 2. $N(\Omega) = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \in \mathbb{D}; |z_2| < e^{-R^*(z_1)}\}$.

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This description does not give much information about the interior of the set difference $N(\Omega) \setminus \Omega$. When we drop the continuity assumption on $\psi(z)$; the well known Hartogs triangle gives an example of a domain for which $N(\Omega) \setminus \Omega$ has nonempty interior. On the other hand, the continuity assumption is not enough to avoid this phenomena as seen in the following example from [3].

**Example.** Let us take a sequence of points in $D$ that accumulates at every boundary point of $D$ and let us take a nonzero holomorphic function $f$ on $D$ that vanishes on this sequence. The function defined by $\psi(z) = |f(z)|$ is a subharmonic function and $\Omega$, defined as above for this particular $\psi$, is a pseudoconvex domain. On the other hand, any pseudoconvex domain that compactly contains $\Omega$ has to contain the closure of the unit polydisc $D \times D$. Therefore, $N(\Omega) \setminus \Omega$ has nonempty interior.

This example suggests to impose more conditions on $\psi$ or $\Omega$ to have trivial Nebenhülle. We prove the following theorem in this note.

**Theorem 3.** Suppose $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 | z_1 \in D; |z_2| < e^{-\psi(z_1)}\}$ is a smooth bounded pseudoconvex complete Hartogs domain. Then $N(\Omega) = \Omega$, in particular $\Omega$ does not have nontrivial Nebenhülle.

Note that the smoothness assumption on the domain $\Omega$ is a stronger condition than the smoothness assumption on the function $\psi(z)$.

For the rest of the note; $O(\Omega)$ denotes the set of functions that are holomorphic on $\Omega$, $C^\infty(\Omega)$ denotes the set of functions that are smooth up to the boundary of $\Omega$ and $A^\infty(\Omega)$ denotes the intersection of these two sets.

### 2. Proof of Theorem 3

Suppose $N(\Omega) \neq \Omega$ and take $p = (p_1, p_2) \in N(\Omega) \setminus \Omega$. By Proposition 2, we have $R^*(p_1) < \psi(p_1)$ and by semicontinuity of $R^*$ and continuity of $\psi$; there exists a neighborhood $\mathcal{U}$ of $p_1$ inside $D$ such that for all $q_1 \in \mathcal{U}$ we have $R^*(q_1) < \psi(q_1)$. The neighborhood $\mathcal{U}$ guarantees that $N(\Omega)$ contains a full neighborhood (in $\mathbb{C}^2$) of the the boundary point $(p_1, e^{-\psi(p_1)}) \in b\Omega$.

After this observation, we prove the following uniform estimate.

**Lemma 4.** Suppose $p \in N(\Omega)$ and $f$ is a function that is holomorphic in a neighborhood of $\Omega$. Then $|f(p)| \leq \sup_{q \in \Omega} |f(q)|$.

**Proof.** Assume otherwise, then $g(z_1, z_2) = \frac{1}{f(z_1, z_2) - f(p)}$ is a holomorphic function on some complete Hartogs domain $\Omega_1$ that compactly contains $\Omega$. 

The domain $\Omega_1$ is not necessarily pseudoconvex but its envelope of holomorphy $\tilde{\Omega}_1$ (which is a single-sheeted (schlicht) and complete Hartogs domain) is pseudoconvex (see [8, page 183]). Moreover, any function holomorphic on $\Omega_1$ extends to a holomorphic function on $\tilde{\Omega}_1$.

In particular, $g(z_1, z_2)$ is holomorphic on $\tilde{\Omega}_1$ and therefore $p \notin \tilde{\Omega}_1$. But this is impossible since $p \in N(\Omega)$. This contradiction finishes the proof of the lemma.

Next, we state an approximation result that is a simpler version of the one in [1]. Let us take a holomorphic function $f$ on $\Omega$. We can expand $f$ as follows:

$$f(z_1, z_2) = \sum_{k=0}^{\infty} a_k(z_1) z_2^k,$$

where $a_k(z_1)$ is a holomorphic function on $\mathbb{D}$ for all $k \in \mathbb{N}$. Let us define the following functions, for any $N \in \mathbb{N}$,

$$(5) \quad P_N(z_1, z_2) = \sum_{k=0}^{N} a_k \left( \frac{z_1}{1 + \frac{1}{N}} \right) z_2^k.$$

It is clear that, each $P_N$ is a holomorphic function in a neighborhood of $\Omega$.

**Lemma 6.** Suppose $f \in A^\infty(\Omega)$. Then the sequence of functions $\{P_N\}$ converges uniformly to $f$ on $\tilde{\Omega}$.

**Proof.** For $(z_1, z_2) \in \Omega$ and $k \geq 2$, we have;

$$|a_k(z_1) z_2^k| = \left| \frac{1}{k!} \left( \frac{(k-2)!}{2\pi i} \int_{|\zeta| = e^{-\psi(z_1)}} \frac{\partial^2}{\partial \zeta^2} f(z_1, \zeta) \zeta^{-k} d\zeta \right) \right| \leq \frac{1}{2\pi k(k-1)} \left( e^{-\psi(z_1)} \right)^k 2\pi e^{-\psi(z_1)} \left( \sup_{\Omega} \left| \frac{\partial^2}{\partial z_2^2} f \right| \right) \frac{1}{(e^{-\psi(z_1)})^{k-1}} \leq \frac{C}{k^2}$$

for some global constant $C$. This gives the uniform convergence. □

Since each $P_N$ is holomorphic on a neighborhood of $\Omega$; in particular it is holomorphic on $N(\Omega)$. By Lemma 4, the uniform convergence percolates onto $N(\Omega)$ and therefore we get a holomorphic extension of any function in $A^\infty(\Omega)$ to $N(\Omega)$. On the other hand, let us remember the following theorem from [2].
Theorem 7 (Catlin, [2]). On any smooth bounded pseudoconvex domain there exists a function in $A^\infty(\Omega)$ that does not extend holomorphically to a neighborhood of any boundary point.

In the first paragraph of this section we showed if $N(\Omega) \neq \Omega$ then $N(\Omega)$ contains a full neighborhood of a boundary point. This observation with the one in the previous paragraph contradict Theorem 7. Therefore we conclude the proof of Theorem 3.

Remark. In the description of $\Omega$, the base domain is assumed to be the unit disc $\mathbb{D}$. However, the result is true when the base is any other planar domain $D$. The description in Proposition 2 and the remark about the envelope of holomorphies, in the proof of Lemma 4, are also valid for any base $D$. The approximation statement in Lemma 6 can be modified for any base $D$, see [1].

Remark. Note that $N(\Omega) = \Omega$ does not imply that $\Omega$ has a Stein neighborhood basis; see [3] Proposition 1] for a false proof and [7] for a counterexample.

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