Hydrodynamical formulation of quantum mechanics, Kähler structure, and Fisher information

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Abstract

The Schrödinger equation can be derived using the minimum Fisher information principle. I discuss why such an approach should work, and also show that the Kähler and Hilbert space structures of quantum mechanics result from combining the symplectic structure of the hydrodynamical model with the Fisher information metric.

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I. INTRODUCTION

In a previous paper [1], it was shown that the hydrodynamical formulation of the Schrödinger equation can be derived using an information-theoretical approach that is based on the principle of minimum Fisher information. A derivation along similar lines is also possible for other non-relativistic quantum mechanical equations, such as the Pauli equation [2] and the equation for the quantum rotator [3]. The purpose of this paper is two-fold: to examine why such an information-theoretical approach should work, and to show that the Kähler and Hilbert space structures of quantum mechanics result from combining the symplectic structure of the hydrodynamical model with the Fisher information metric of information theory. The complex transformation of the hydrodynamical variables that puts this Kähler metric in its canonical form is the one that leads to the usual Schrödinger representation.

Frieden [4] was the first one to point out a connection between the principle of minimum Fisher information and the Schrödinger equation. Frieden and coworkers later developed and extended this work in a series of papers which made use of a new principle called the extreme physical information (EPI) principle. In this paper I will not discuss the EPI principle, which differs from the principle of minimum Fisher information in many ways (for
a review of the EPI approach, see the book by Frieden [5], but will concentrate instead on the information-theoretical approach used in [1]. In this approach, the emphasis is on using the principle of minimum Fisher information to complement a physical picture derived from a hydrodynamical model. Applying the principle under the assumption that one can describe the motion of particles in terms of a hydrodynamical model leads directly to Madelung’s hydrodynamical formulation of quantum mechanics [6].

II. CROSS-ENTROPY AND FISHER INFORMATION

Let $P(y^i)$ be a probability density which is a function of $n$ continuous coordinates $y^i$, and let $P(y^i + \Delta y^i)$ be the density that results from a small change in the $y^i$. Expand the $P(y^i + \Delta y^i)$ in a Taylor series, and calculate the cross-entropy $J$ up to the first non-vanishing term,

$$J(P(y^i + \Delta y^i) : P(y^i)) = \int P(y^i + \Delta y^i) \ln \frac{P(y^i + \Delta y^i)}{P(y^i)} d^n y$$

$$\approx \left[ \frac{1}{2} \int \frac{1}{P(y^i)} \frac{\partial P(y^i)}{\partial y^j} \frac{\partial P(y^i)}{\partial y^k} d^n y \right] \Delta y^j \Delta y^k$$

$$= I_{jk} \Delta y^j \Delta y^k$$

The $I_{jk}$ are the elements of the Fisher information matrix. This is not the most general expression for the Fisher information matrix, but the particular case that is of interest here. The general expression is of the form [7]

$$I_{jk}(\theta^i) = \frac{1}{2} \int \frac{1}{P(x^i|\theta^i)} \frac{\partial P(x^i|\theta^i)}{\partial \theta^j} \frac{\partial P(x^i|\theta^i)}{\partial \theta^k} d^n x$$

(2)

where $P(x^i|\theta^i)$ is a probability density that depends on a set of $n$ parameters $\theta^i$ in addition to the $n$ coordinates $x^i$. The expression for the $I_{jk}$ that appears in equation (1) can be derived from the general formula if

$$P(x^i|\theta^i) = P(x^i + \theta^i).$$

To see this, introduce a new set of parameters $y^i = x^i + \theta^i$. Then
\[ I_{jk}(\theta^i) \rightarrow \frac{1}{2} \int \frac{1}{P(y^i)} \frac{\partial P(y^i)}{\partial y^j} \frac{\partial P(y^i)}{\partial y^k} d^n y = I_{jk} \]

since \( d^n x \rightarrow d^n y \) as the integration over the \( x^i \) coordinates is for fixed values of \( \theta^i \).

If \( P \) is defined over an \( n \)-dimensional manifold \( M \) with (positive) inverse metric \( g^{ik} \), there is a natural definition of the amount of information \( I \) associated with \( P \), which is obtained by contracting \( g^{ik} \) with the elements of the Fisher information matrix,

\[ I = g^{ik} I_{ik} = g^{ik} \frac{1}{2} \int \frac{1}{P} \frac{\partial P}{\partial y^j} \frac{\partial P}{\partial y^k} d^n y. \]  

(3)

The case of interest here is the one where \( M \) is the \( n + 1 \) dimensional extended configuration space \( QT \) (with coordinates \( \{ t, x^1, ..., x^n \} \)) of a non-relativistic particle of mass \( m \). Then, the inverse metric is the one used to define the kinematical line element in configuration space, which is of the form \( g^{ik} = diag(0, 1/m, ..., 1/m) \). Sometimes it will be convenient to use quantities defined over the configuration space \( Q \) (with coordinates \( \{ x^1, ..., x^n \} \)) rather than \( QT \), and I will do so if it simplifies the notation.

**III. DERIVATION OF THE SCHRÖDINGER EQUATION**

In the Hamilton-Jacobi formulation of classical mechanics, the equation of motion takes the form

\[ \frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu \nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + V = 0 \]  

(4)

where \( g^{\mu \nu} = diag(1/m, ..., 1/m) \) is the inverse metric used to define the kinematical line element in the configuration space \( Q \) parametrized by coordinates \( \{ x^\mu \} \). The velocity field \( u^\mu \) is derived from \( S \) according to

\[ u^\mu = g^{\mu \nu} \frac{\partial S}{\partial x^\nu}. \]  

(5)

When the exact coordinates that describe the state of the classical system are unknown, one usually describes the system by means of a probability density \( P(t, x^\mu) \). The probability density must satisfy the following two conditions: it must be normalized,
\[ \int P \, d^n x = 1, \]

and it must satisfy a continuity equation,

\[ \frac{\partial}{\partial t} P + \frac{\partial}{\partial x^\mu} \left( P g^{\mu\nu} \frac{\partial S}{\partial x^\nu} \right) = 0. \] (6)

Equations (4) and (6), together with (5), completely determine the motion of the classical ensemble. Equations (4) and (6) can be derived from the Lagrangian

\[ L_{CL} = \int P \left\{ \frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + V \right\} \, dt \, d^n x \] (7)

by fixed end-point variation (\( \delta P = \delta S = 0 \) at the boundaries) with respect to \( S \) and \( P \).

Quantization of the classical ensemble is achieved by adding to the classical Lagrangian (7) a term proportional to the information \( I \) defined by equation (3) [1]. This leads to the Lagrangian for the Schrödinger equation,

\[ L_{QM} = L_{CL} + \lambda I \]

\[ = \int P \left\{ \frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \left[ \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + \lambda \frac{1}{2} \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} \right] + V \right\} \, dt \, d^n x. \] (8)

Fixed end-point variation with respect to \( S \) leads again to (4), while fixed end-point variation with respect to \( P \) leads to

\[ \frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \left[ \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + \lambda \left( \frac{1}{P^2} \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} - \frac{2}{P} \frac{\partial^2 P}{\partial x^\mu \partial x^\nu} \right) \right] + V = 0 \] (9)

Equations (6) and (9) are identical to the Schrödinger equation provided the wave function \( \psi(t, x^\mu) \) is written in terms of \( S \) and \( P \) by

\[ \psi = \sqrt{P} \exp(iS/\hbar) \]

and the parameter \( \lambda \) is set equal to

\[ \lambda = \left( \frac{\hbar}{2} \right)^2. \]

Note that the classical limit of the Schrödinger theory is not the Hamilton-Jacobi equation for a classical particle, but the equations (4) and (6) which describe a classical ensemble.
It can be shown (see Appendix) that the Fisher information $I$ increases when $P$ is varied while $S$ is kept fixed. Therefore, the solution derived here is the one that minimizes the Fisher information for a given $S$.

The approach followed here is of interest in that it provides a way of distinguishing between physical and information-theoretical assumptions (for a very clear account of the importance of making this type of distinction in quantum mechanics see the paper by Jaynes [9]). In general terms, the information-theoretical content of the theory lies in the prescription to minimize the Fisher information associated with the probability distribution that describes the position of particles, while the physical content of the theory is contained in the assumption that one can describe the motion of particles in terms of a hydrodynamical model.

IV. ON THE USE OF THE MINIMUM FISHER INFORMATION PRINCIPLE IN QUANTUM MECHANICS

The cross-entropy $J$,

$$J(Q : P) = \int Q(y^i) \ln \left( \frac{Q(y^i)}{P(y^i)} \right) d^n y,$$

where $P, Q$ are two probability densities, plays a central role in information theory and in the theory of inference. It has properties that are desirable for an information measure [7], and it can be argued that it measures the amount of information needed to change a prior probability density $P$ into the posterior $Q$ [10]. Maximization of the relative entropy (which is defined as the negative of the cross-entropy[1]) is the basis of the maximum entropy

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1 A note on terminology: due to the connection between relative entropy and cross-entropy, the maximum entropy principle is also known as the minimum cross-entropy principle, which can lead to some confusion. The cross-entropy (or its negative) may found in the literature under various names: Kullback-Leibler information, directed divergence, discrimination information, Renyi’s
principle, a method for inductive inference that leads to a posterior distribution given a prior distribution and new information in the form of expected values. The maximum entropy principle asserts that of all the probability densities that are consistent with the new information, the one which has the maximum relative entropy is the one that provides the most unbiased representation of our knowledge of the state of the system. There are several approaches that lead to the maximum entropy principle. In the original derivation by Jaynes [11], the use of the maximum entropy principle was justified on the basis of the relative entropy’s unique properties as an uncertainty measure. An independent justification based on consistency arguments was later given by Shore and Johnson [12]. Jaynes had already remarked that inferences made using any other information measure than the entropy may lead to contradictions. Shore and Johnson considered the consequences of requiring that methods of inference be self-consistent. They introduced a set of axioms that were all based on one fundamental principle: if a problem can be solved in more than one way, the results should be consistent. They showed that given information in the form of a set of constraints on expected values, there is only one distribution satisfying the set of constraints which can be chosen using a procedure that satisfies their axioms, and this unique distribution can be obtained by maximizing the relative entropy. Therefore, they concluded that if a method of inference is based on a variational principle, maximizing any function but the relative entropy will lead to inconsistencies unless that function and the relative entropy have identical maxima (any monotonic function of the relative entropy will work, for example).

It is tempting to argue by analogy that the minimum Fisher information derivation of the Schrödinger equation is in essence nothing but a variation on maximum entropy, one in which maximization of relative entropy is simply replaced by minimization of the Fisher information (some similarities and differences of the two approaches were discussed briefly information gain, expected weight of evidence, entropy, entropy distance.)
in [4]). But if we take into consideration the unique properties that make cross-entropy 
the fundamental measure of information together with the result of Shore and Johnson, it 
becomes difficult to justify a principle of inference based on information theory that would 
operate along the same lines as maximum entropy but using the principle of minimum 
Fisher information instead. To understand the use of the minimum Fisher information 
principle in the context of quantum mechanics, it is crucial to take into consideration that 
here one is selecting those probability distributions $P(y^i)$ for which a perturbation that 
leads to $P(y^i + \Delta y^i)$ will result in the smallest increase of the cross-entropy for a given 
$S(y^i)$. In other words, the method of choosing $P(y^i)$ is based on the idea that a solution 
should be stable under perturbations in the very precise sense that the amount of additional 
information needed to describe the change in the solution should be as small as possible. 
We have then a new principle: choose the probability densities that describe the quantum 
system on the basis of the stability of those solutions, where the measure of the stability is 
given by the amount of information needed to change $P(y^i)$ into $P(y^i + \Delta y^i)$. Why should 
restricting the choice of $\{P, S\}$ to those that are stable in this sense lead to the excellent 
predictions of quantum mechanics? Such an approach should work for physical systems that 
can be represented by models in which the probability density $P$ describes the equilibrium 
density of an underlying stochastic process (see for example the derivation of the diffusion 
equation using the minimum Fisher information principle in [13]). Such models of quantum 
mechanics do exist: a formulation along these lines was first proposed by Bohm and Vigier 
[14], and later a different but related formulation was given by Nelson [15] (for a review of the 
stochastic formulation of the quantum theory that compares these two approaches, see [16]). 
Whether the additional assumptions needed to build these particular models are sound, and 
whether they provide a correct description of quantum mechanics will depend of course on 
the experimental predictions that they make. The minimum Fisher information approach 
can be of no help here, since it is only concerned with making inferences about probability 
distributions and operates therefore at the epistemological level.
V. KÄHLER AND HILBERT SPACE STRUCTURES OF QUANTUM MECHANICS

I now want to examine the assumptions that are needed to construct the Kähler and Hilbert space structures of quantum mechanics. My aim is not to give a mathematically rigorous derivation of these results, but to give arguments that justify introducing the Kähler space structure on the basis of mathematical structures that arise naturally in the hydrodynamical model and in information theory. In particular, I want to show that the Kähler structure of quantum mechanics results from combining the symplectic structure of the hydrodynamical model with the Fisher information metric of information theory. The complex transformation of the hydrodynamical variables that puts this Kähler metric in its canonical form is the one that leads to the usual Schrödinger representation. Good descriptions of the geometrical formulation of quantum mechanics covering the case of infinite-dimensional Kähler manifolds are available in the literature; see for example Cirelli et. al. [17], Ashtekar and Schilling [18] and Brody and Hughston [19]. The approach of Brody and Hughston is of special interest in that they make explicit use of the Fisher information metric, although without making reference to the hydrodynamical formulation.

I first look at the symplectic structure of the hydrodynamical formulation. Introduce as basic variables the hydrodynamical fields \( \{P, S\} \). The symplectic structure is given by the two form

\[
\omega(\delta P(x^\mu), \delta S(x^\mu); \delta' P(x^\mu), \delta' S(x^\mu)) = \int \left\{ \begin{pmatrix} \delta P(x^\mu) & \delta S(x^\mu) \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta' P(x^\mu) \\ \delta' S(x^\mu) \end{pmatrix} \right\} \Omega \cdot \begin{pmatrix} \delta' P(x^\mu) \\ \delta' S(x^\mu) \end{pmatrix} d^n x
\]

where \( \delta \) and \( \delta' \) are two generic systems of increments for the phase-space variables. The Poisson brackets for two functions \( \mathcal{F}^1(P, S), \mathcal{F}^2(P, S) \) take the form

\[
\{ \mathcal{F}^1(P, S), \mathcal{F}^2(P, S) \} = \int \left\{ [\delta \mathcal{F}^1/\delta P] [\delta \mathcal{F}^2/\delta S] - [\delta \mathcal{F}^1/\delta S] [\delta \mathcal{F}^2/\delta P] \right\} d^n x.
\]
The equations of motion (6), (9) can be written as
\[
\frac{\partial P}{\partial t} = \{P, \mathcal{H}\} = \frac{\delta \mathcal{H}}{\delta S} \frac{\partial S}{\partial t}
\]
\[
\frac{\partial S}{\partial t} = \{S, \mathcal{H}\} = -\frac{\delta \mathcal{H}}{\delta P}
\]
with the Hamiltonian \(\mathcal{H}\) given by
\[
\mathcal{H} = \int P \left\{ \frac{1}{2} g^{\mu\nu} \left[ \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + \frac{\hbar}{2} \frac{1}{P^2} \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} \right] + V \right\} d^3 x.
\]
\(\mathcal{H}\) acts as the generator of time translations.

To introduce the Fisher information metric, let \(\theta^\mu\) be a set of real continuous parameters, and consider the parametric family of positive distributions defined by
\[
P(x^\mu | \theta^\mu) = P(x^\mu + \theta^\mu)
\]
where the probability densities \(P\) are solutions of the Schrödinger equation (at time \(t = 0\)). Then there is a natural metric over the space of parameters \(\theta^\mu\) given by the Fisher information matrix [21], and it leads to a concept of distance defined by
\[
d s^2(\theta^\mu) = \frac{1}{2} \left[ \int \frac{1}{P(x^\mu | \theta^\mu)} \frac{\partial P(x^\mu | \theta^\mu)}{\partial \theta^\rho} \frac{\partial P(x^\mu | \theta^\mu)}{\partial \theta^\sigma} d^3 x \right] \delta \theta^\rho \delta \theta^\sigma
\]
(10)

Using
\[
\delta P = \frac{\partial P}{\partial \theta^\mu} \delta \theta^\mu
\]
one can write equation (10) as
\[
d s^2(\theta^\mu) = \frac{1}{2} \left[ \int \frac{1}{P(x^\mu | \theta^\mu)} \delta P(x^\mu | \theta^\mu) \delta P(x^\mu | \theta^\mu) d^3 x \right]
\]
(11)

We use equation (11) to introduce a metric over the space of solutions of the Schrödinger equation (i.e., \(P(x^\mu | \theta^\mu)\) with \(\theta^\mu = 0\)) by setting
\[
d s^2(\delta P, \delta' P) = \frac{1}{2} \left[ \int \frac{1}{P(x^\mu)} \delta P(x^\mu) \delta' P(x^\mu) d^3 x \right] = \int g^{(P)}(x) \delta P(x^\mu) \delta' P(x^\mu) d^3 x
\]
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where

\[ P(x^\mu) = P(x^\mu | \theta^\mu = 0), \]

\[ \delta P(x^\mu) = \delta P(x^\mu | \theta^\mu | \theta^\nu = 0) \]

\[ g^{(P)} = \frac{1}{2P(x^\mu)} \]

I now want to extend the metric \( g^{(P)} \) over the probability densities to a metric \( g_{ab} \) over the whole space \( \{P, S\} \) of solutions of the Schrödinger equation, in such a way that the metric structure is compatible with the symplectic structure. To do this, introduce a complex structure \( J^a_b \) and impose the following conditions,

\[ \Omega_{ab} = g_{ac}J^c_b \quad (12) \]

\[ J^a_c g_{ab}J^b_d = g_{cd} \quad (13) \]

\[ J^a_b J^b_c = -\delta^a_c \quad (14) \]

A set of \( \{\Omega_{ab}, g_{ab}, J^a_b\} \) that satisfy equations (12), (13) and (14) defines a Kähler structure. Equation (12) is a compatibility equation between \( \Omega_{ab} \) and \( g_{ab} \), equation (13) is the condition that the metric should be Hermitian, and equation (14) is the condition that \( J^a_b \) should be a complex structure. Let

\[ \Omega_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

and require that \( g_{ab} \) be a real, symmetric matrix of the form

\[ g_{ab} = \begin{pmatrix} \hbar g^{(P)} & \cdot \\ \cdot & \cdot \end{pmatrix} \]

Then the solutions \( g_{ab} \) and \( J^a_b \) to equations (12), (13) and (14) depend on an arbitrary real function \( A \) and are of the form
\[
g_{ab}(A) = \begin{pmatrix}
\hbar g(P) & A \\
A & \left(\hbar g(P)\right)^{-1} \left(1 + A^2\right)
\end{pmatrix},
\]

\[
J^a_b(A) = \begin{pmatrix}
A & \left(\hbar g(P)\right)^{-1} \left(1 + A^2\right) \\
-\hbar g(P) & -A
\end{pmatrix}.
\]

The choice of \(A\) that leads to the simplest Kähler structure is \(A = 0\), which is a unique choice in that it leads to the flat Kähler metric. I will show this by carrying out the complex transformation that leads to the canonical form for the flat Kähler metric. I set \(A = 0\), and work with the Kähler structure given by

\[
\Omega_{ab} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

(15)

\[
g_{ab} = \begin{pmatrix}
\hbar g(P) & 0 \\
0 & \left(\hbar g(P)\right)^{-1}
\end{pmatrix}
\]

(16)

\[
J^a_b = \begin{pmatrix}
0 & \left(\hbar g(P)\right)^{-1} \\
-\hbar g(P) & 0
\end{pmatrix}
\]

(17)

The complex coordinate transformation is nothing but the Madelung transformation

\[
\psi = \sqrt{P} \exp(iS/\hbar)
\]

\[
\psi^* = \sqrt{P} \exp(-iS/\hbar)
\]

In terms of the new variables, (15), (16) and (17) take the canonical form

\[
\Omega_{ab} = \begin{pmatrix}
0 & \imath \hbar \\
-\imath \hbar & 0
\end{pmatrix}
\]
\[
g_{ab} = \begin{pmatrix} 0 & \hbar \\ \hbar & 0 \end{pmatrix}
\]

\[
J^a_b = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}
\]

The Madelung transformation is remarkable in that the Hamiltonian takes the very simple form

\[
\mathcal{H} = \int \left\{ \frac{\hbar^2}{2} g^{\mu \nu} \frac{\partial \psi^*}{\partial x^\mu} \frac{\partial \psi}{\partial x^\nu} + V \psi^* \psi \right\} d^n x,
\]

and the equations of motion become linear.

Finally, one introduces a Hilbert space structure using \( g_{ab}, \Omega_{ab} \) to define the Dirac product. For two wave functions \( \phi, \varphi \), define the Dirac product by

\[
< \phi | \varphi > = \frac{1}{2\hbar} \int \begin{pmatrix} \phi(x^\mu), \varphi^*(x^\mu) \end{pmatrix} \cdot \begin{pmatrix} [g + i\Omega] \cdot \begin{pmatrix} \varphi(x^\mu) \\ \varphi^*(x^\mu) \end{pmatrix} \end{pmatrix} d^n x
\]

\[
= \frac{1}{2\hbar} \int \begin{pmatrix} \phi(x^\mu), \varphi^*(x^\mu) \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & \hbar \\ \hbar & 0 \end{pmatrix} + i \begin{pmatrix} 0 & i\hbar \\ -i\hbar & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \varphi(x^\mu) \\ \varphi^*(x^\mu) \end{pmatrix} d^n x
\]

\[
= \int \phi^*(x^\mu)\varphi(x^\mu) d^n x
\]

In this way the Hilbert space structure of quantum mechanics results from combining the symplectic structure of the hydrodynamical model with the Fisher information metric of information theory.

An important result that comes out of this analysis concerns the issue of suitable boundary conditions for the fields \( P \) and \( S \). It has been pointed out \[21\] that the Schrödinger theory is not strictly equivalent to some of the other formulations (i.e., the hydrodynamical formulation and stochastic mechanics) because features such as the quantization of angular momentum, which are natural when the theory is formulated in terms of wave functions, require an additional constraint in a theory formulated in terms of hydrodynamical variables.
For example, in the case of the hydrogen atom, the quantization of angular momentum results from requiring that the wave function be single-valued in configuration space. But the derivation of the Kähler structure and Hilbert space structure presented here shows that the Schrödinger representation follows naturally from the hydrodynamical formulation provided we take into account the role of the Fisher information metric, and furthermore that this representation is unique in that it is the coordinate system in which the Kähler structure takes the simplest form. From a purely mathematical point of view, it is not surprising that the correct boundary conditions are those that are simplest when formulated in the simplest coordinate system, i.e. single-valuedness of the canonically conjugate fields $\psi, \psi^*$.

VI. APPENDIX

I want to examine the extremum obtained from the fixed end-point variation of the Lagrangian $L_{QM}$, equation (8). In particular, I wish to show the following: given $P$ and $S$ that satisfy equations (6) and (9), a small variation of the probability density $P(x^\mu, t) \to P(x^\mu, t)' = P(x^\mu, t) + \epsilon \delta P(x^\mu, t)$ for fixed $\sigma$ will lead to an increase in $L_{QM}$, as well as an increase in the Fisher information $I$.

I assume fixed end-point variations, and variations $\epsilon \delta P$ that are well defined in the sense that $P'$ will have the usual properties required of a probability density (such as $P' > 0$ and normalization).

Let $P \to P' = P + \epsilon \delta P$. Since $P$ and $S$ are solutions of the variational problem, the terms linear in $\epsilon$ vanish. If one keeps terms up to order $\epsilon^2$, the change in $L_{QM}$ is given by

\[ \Delta L_{QM} = L_{QM}(P', S) - L_{QM}(P, S) \]

\[ = \frac{\epsilon^2 \lambda}{2} \int g^{\mu\nu} \left\{ \left( \frac{\delta P}{P} \right)^2 \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} - \frac{2(\delta P)}{P^2} \frac{\partial P}{\partial x^\mu} \frac{\partial (\delta P)}{\partial x^\nu} + \frac{1}{P} \frac{\partial (\delta P)}{\partial x^\mu} \frac{\partial (\delta P)}{\partial x^\nu} \right\} dt d^n x + O(\epsilon^3). \]

Using the relation

\[ P g^{\mu\nu} \frac{\partial}{\partial x^\mu} \left( \frac{\delta P}{P} \right) \frac{\partial}{\partial x^\nu} \left( \frac{\delta P}{P} \right) = g^{\mu\nu} \left\{ \frac{\delta P^2}{P^3} \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} - \frac{2\delta P}{P^2} \frac{\partial P}{\partial x^\mu} \frac{\partial \delta P}{\partial x^\nu} + \frac{1}{P} \frac{\partial \delta P}{\partial x^\mu} \frac{\partial \delta P}{\partial x^\nu} \right\}, \]

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one can write $\Delta L_{QM}$ as

$$
\Delta L_{QM} = \frac{\epsilon^2 \lambda}{2} \int P \left\{ g^{\mu\nu} \frac{\partial}{\partial x^\mu} \left( \frac{\delta P}{P} \right) \frac{\partial}{\partial x^\nu} \left( \frac{\delta P}{P} \right) \right\} dt d^n x + O(\epsilon^3),
$$

which shows that $\Delta L_{QM} > 0$ for small variations, and therefore that the extremum of $\Delta L_{QM}$ is a minimum. Furthermore, since $\Delta L_{QM} \sim \lambda$, it is the Fisher information term $I$ in the Lagrangian $\Delta L_{QM}$ that increases, and the extremum is also a minimum of the Fisher information.
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