Note on Extended Coherent Operators and Some Basic Properties

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Abstract

This is a continuation of the paper (quant-ph/0009012). In this letter we extend coherent operators and study some basic properties (the disentangling formula, resolution of unity, commutation relation, etc). We also propose a perspective of our work.

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1 Introduction

This is a continuation of the paper [1]. For the aim of this letter see [1].

We extend coherent operators and study some basic properties such as

(a) the disentangling formula,

(b) the resolution of unity,

(c) the commutation relation,

(d) the matrix elements,

(e) the trace,

(f) the Glauber formula.

These properties are very well known for coherent operators, see [1]. We give explicit forms to (a) ∼ (f) for extended version of coherent operators (extended coherent operators in our terminology ... easy and bad naming!).

Since our extension is in a certain sense very natural, our results must have been known. But the author could not find such references in spite of his efforts. At any rate let us list our results.

The details and further developments of this letter will be included in [1].

2 Coherent Operators and Basic Properties

We make a brief review of some basic properties of coherent operators within our necessity.

Let \( a(a^\dagger) \) be the annihilation (creation) operator of the harmonic oscillator. If we set \( N \equiv a^\dagger a \) (: number operator), then we have

\[
[N, a^\dagger] = a^\dagger , \quad [N, a] = -a , \quad [a^\dagger, a] = -1 .
\] (1)
Let $\mathcal{H}$ be a Fock space generated by $a$ and $a^\dagger$, and $\{|n\rangle \mid n \in \mathbb{N} \cup \{0\}\}$ be its basis. The actions of $a$ and $a^\dagger$ on $\mathcal{H}$ are given by

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad N|n\rangle = n|n\rangle$$

(2)

where $|0\rangle$ is a normalized vacuum ($a|0\rangle = 0$ and $\langle 0|0\rangle = 1$). From (2) state $|n\rangle$ for $n \geq 1$ are given by

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle.$$  

(3)

These states satisfy the orthogonality and completeness conditions

$$\langle m|n\rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1.$$  

(4)

**Definition** The coherent operators and coherent states are defined as

$$U(z) = e^{za^\dagger - \bar{z}a}$$  

for $z \in \mathbb{C}$,  

(5)

$$|z\rangle = U(z)|0\rangle$$  

for $z \in \mathbb{C}$.  

(6)

Let us list the three basic properties of coherent operators and coherent states, see [2] and [3] for the proofs.

(a) **Disentangling Formula** We have

$$U(z) = e^{-|z|^2/2} e^{za^\dagger} e^{-\bar{z}a} = e^{z^2/2} e^{-\bar{z}a} e^{za^\dagger}.$$  

(7)

In the proof we use the elementary Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^{A} e^{B}$$  

(8)

whenever $[A,[A,B]] = [B,[A,B]] = 0$, see [4]. This is the key formula in the theory of coherent states.

The coherent states are usually defined as eigenvectors of annihilation operator $a|z\rangle = z|z\rangle$ for $z \in \mathbb{C}$. The important feature of coherent states is the following resolution (partition) of unity.
(b) Resolution of Unity  We have

\[
\int_C \frac{[d^2 z]}{\pi} |z\rangle \langle z| = \sum_{n=0}^{\infty} |n\rangle \langle n| = 1,
\]

where we have set \([d^2 z] = d(\text{Re} z)d(\text{Im} z)\) for simplicity.

c) Commutation Relation  We have

\[
U(z)U(w) = e^{z\bar{w} - \bar{z}w} U(w)U(z)
\]

for \(z, w \in \mathbb{C}\).

d) Matrix Elements  The matrix elements of \(U(z)\) are

(i)  \(n \leq m\) \(\langle n|U(z)|m\rangle = e^{-\frac{1}{2}|z|^2} \sqrt{\frac{n!}{m!}} (-z)^{m-n} L_n^{(m-n)}(|z|^2),\)


(ii) \(n \geq m\) \(\langle n|U(z)|m\rangle = e^{-\frac{1}{2}|z|^2} \sqrt{\frac{m!}{n!}} z^{n-m} L_m^{(n-m)}(|z|^2),\)

where \(L_n^{(\alpha)}\) is the associated Laguerre’s polynomial defined by

\[
L_n^{(\alpha)}(x) = \sum_{j=0}^{n} (-1)^j \binom{n+\alpha}{n-j} \frac{x^j}{j!}.
\]

e) Trace  We have

\[
\text{Tr}U(z) = \pi \delta^2(z) \equiv \pi \delta(x)\delta(y) \quad \text{if} \ z = x + iy.
\]

(f) Glauber Formula  Let \(A\) be any observable. Then we have

\[
A = \int_C \frac{[d^2 z]}{\pi} \text{Tr}[AU^\dagger(z)]U(z)
\]

This formula plays an important role in the field of homodyne tomography.
3 Extended Coherent Operators and Basic Properties

We in this section define extended coherent operators and study the basic properties corresponding to ones of coherent operators.

**Definition** The extended coherent operators and extended coherent states are defined as follows:

\[
U(z, t) = e^{za^\dagger - \bar{z}a + itN} \quad \text{for} \quad z \in \mathbb{C}, \ t \in \mathbb{R},
\]

(16)

\[
|z, t\rangle = U(z, t)|0\rangle \quad \text{for} \quad z \in \mathbb{C}, \ t \in \mathbb{R}.
\]

(17)

In this definition at first sight we would want to transform (16) as

\[
z a^\dagger - \bar{z}a + itN = it \left( a + \frac{z}{it} \right)^\dagger \left( a + \frac{z}{it} \right) - \frac{i|z|^2}{t}.
\]

(18)

But with this form it is impossible to take a limit \( t \to 0 \), so we don’t use this one in this paper.

Let us list the corresponding properties of extended coherent operators (16) and extended coherent states (17). Before stating our result let us prepare some notations.

\[
f(t) = \frac{e^{it} - 1}{it},
\]

(19)

\[
g(t) = \frac{e^{it} - (1 + it)}{t^2}.
\]

(20)

We here note that

\[
|f(t)| = \frac{\sin(t/2)}{t/2},
\]

(21)

\[
|f(t)|^2 = - \{g(t) + \overline{g(t)}\}.
\]

(22)

Let us state our result:

(a) **Disentangling Formula** We have

\[
U(z, t) = e^{g(t)|z|^2} e^{\langle f(t)za^\dagger \rangle} e^{itN} e^{-f(t)\bar{z}a}, \quad \text{or}
\]

\[
= e^{-\overline{g(t)}|z|^2} e^{-\overline{f(t)}\bar{z}a} e^{itN} e^{f(t)za^\dagger}.
\]

(23)

(24)
This becomes the key formula in the theory of extended coherent states.

(b) Resolution of Unity  We have

\[ \int_{\mathbb{C}} \frac{|f(t)|^2 |d^2 z|}{\pi} |z, t\rangle \langle z, t| = \sum_{n=0}^{\infty} |n\rangle \langle n| = 1. \]  

(25)

We note that the measure to satisfy resolution of unity is not unique, so the following measure is not so bad:

\[ \int_{\mathbb{R}} \frac{|f(t)|^2 e^{-t|t|}}{2} dt \int_{\mathbb{C}} \frac{|d^2 z|}{\pi} |z, t\rangle \langle z, t| = \sum_{n=0}^{\infty} |n\rangle \langle n| = 1 . \]  

(26)

(c) Commutation Relation  We have

\[ U(z, t)U(w, s) = e^{\{f(t)f(s)z \bar{w} - f(t)\bar{f}(s)zw\}} U(we^{it}, s)U(z e^{-is}, t) \]  

(27)

for \( z, w \in \mathbb{C} \) and \( t, s \in \mathbb{R} \).

Let us here make a change of variables \( z \mapsto w = f(t)z \). Then matrix elements of \( U(z, t) \) are written down by those of \( U(w) \).

(d) Matrix Elements  The matrix elements of \( U(z, t) \) are

\[ \langle n|U(z, t)|m\rangle = e^{-\left\{ \frac{1}{2} + \frac{g(t)}{|f(t)|^2} \right\}|w|^2 + itm} \langle n|U(w)|m\rangle. \]  

(28)

(e) Trace  For \( U(z, t) \) we have another decomposition:

\[ U(z, t) = e^{\frac{-t}{4}|z|^2} e^{\frac{1}{4}(za^\dagger + \bar{z}a)} e^{itN} e^{-\frac{1}{4}(za^\dagger + \bar{z}a)}, \]  

(29)

so we have

\[ \text{Tr} U(z, t) = e^{\frac{-t}{4}|z|^2} \text{Tr e}^{itN} = \frac{e^{-|z|^2/t}}{1 - e^{it}} \text{ (as Abel sum)}, \]  

(30)
see also (18).

We here note $\text{Tr}U(z,0) = \text{Tr}U(z) = \pi \delta^{(2)}(z)$, (14). Then we have a natural question:

**Question** Is the following correct?

$$\lim_{t \to 0} \text{Tr}U(z,t) = \text{Tr}U(z,0) \quad \text{for} \quad z \in \mathbb{C}. \quad (31)$$

At first sight the answer of this question seems to be no, but is yes. This has been solved by S. Sakoda.

Making use of (28) and (22) we have the Glauber formula.

(f) **Glauber Formula** Let $A$ be any observable. Then we have

$$A = \int_{\mathbb{C}} \frac{|f(t)|^2[d^2z]}{\pi} \text{Tr}[AU^\dagger(z,t)]U(z,t). \quad (32)$$

A comment is in order. When taking the limit $t \to 0$ these properties reduce to corresponding ones in the preceding section because $f(t) \to 1$ and $g(t) \to -1/2$ from (19) and (20).

4 Discussion

In this paper we listed the six basic properties of coherent operators and investigated these ones for extended coherent operators. We had a natural extension. The details of several calculations performed in Section 3 will be published in [4].

Here let us give a perspective (or application) of our work. We can also extend the squeezed operator in [3] as follows.

$$V(z, t) = e^{zK_+ - \bar{z}K_- + itK_3} \quad \text{for} \quad z \in \mathbb{C} \text{ and } t \in \mathbb{R} \quad (33)$$

where
\[ K_+ \equiv \frac{1}{2} (a^\dagger)^2, \quad K_- \equiv \frac{1}{2} a^2, \quad K_3 \equiv \frac{1}{2} \left( a^\dagger a + \frac{1}{2} \right). \] (34)

We have studied some basic properties of this operator in [4], so that we can consider the product of two unitary operators

\[ U(z, t)V(w, s) \quad \text{for} \quad z, w \in \mathbb{C} \text{ and } t, s \in \mathbb{R}. \] (35)

It may be possible to search the same lines as in [5] and [6], [7]. These will be published in an another paper, [8].

Acknowledgment.

The author wishes to thank S. Sakoda for his helpful comments and suggestions.

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