Boundary feedback stabilization of a Rao-Nakra sandwich beam

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Abstract. In this paper, we study the boundary feedback stabilization of a three-layer Rao-Nakra sandwich beam. Since we found a proper state space so that the energy of the system is dissipative. We then convert the beam system to an abstract first-order evolution equation with operator $A$. Afterwards, we obtain $A$ is the infinitesimal generator of a $C_0$—semigroup of contractions by using semigroup theory. The boundary damping makes the system is stable of order 1/2, which is proved by using the frequency domain characterization of polynomial stability.

1. Introduction
The classical sandwich beam is an engineering model for a three-layer Rao-Nakra sandwich beam. Several three-layer sandwich beam and plate models were first proposed in the late 1960’s and early 1970’s [1-3]. Later, the following general model was developed in [4],

$$\begin{align*}
\rho_i h_i u_i' &= E_i h_i u_i' + \tau, \\
\rho_i h_i u_i' &= E_i h_i u_i' - \tau, \\
\rho h w_i'' &= G_i h_i (w_i + \phi_i) + G_j h_j (w_j + \phi_j) + h_j \phi_j, \\
\rho_i I_j \phi_i'' &= E_i h_i \phi_i'' + \frac{h_i}{2} \tau - G_i h_i (w_i + \phi_i), \\
\rho_j I_j \phi_j'' &= E_j h_j \phi_j'' + \frac{h_j}{2} \tau - G_j h_j (w_j + \phi_j).
\end{align*}$$

(1.1) (1.2) (1.3) (1.4) (1.5)

Here, $u'_i, \phi_i, i=1,3$ are the longitudinal displacement and shear angle of the ith layer (bottom and top layers); $w_i$ is the transverse displacement of the beam; $\tau$ is the shear stress in the core layer ($i = 2$). The physical parameters represent the material properties. $h_i, \rho_i, E_i, G_i, I_i$ are the thickness, density, Young’s modulus, shear modulus, and moments of inertia of the ith layer for $i = 1,2,3$, respectively, and $\rho h = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3$.

When the rotatory inertia and the transverse shear of the bottom and top layers are neglected, equations (1.4) and (1.5) reduce to the familiar Euler-Bernoulli hypothesis $\phi_1 - w_1 = \phi_3 - w_3 = 0$. If we consider the core material to be linearly elastic, i.e., $\tau = 2G_2 \phi$,

$$\begin{align*}
\rho h w_i'' &= G_i h_i (w_i + \phi_i) + G_j h_j (w_j + \phi_j), \\
\rho_i I_j \phi_i'' &= E_i h_i \phi_i'' + \frac{h_i}{2} \tau - G_i h_i (w_i + \phi_i), \\
\rho_j I_j \phi_j'' &= E_j h_j \phi_j'' + \frac{h_j}{2} \tau - G_j h_j (w_j + \phi_j).
\end{align*}$$

(1.6) (1.7) (1.8)
with the shear strain \( \theta = \frac{1}{2h_2^2}(-u'^4 + u'^3 + \alpha w_x) \), and \( \alpha = h_2 + (h_1 + h_3)/2 \), we then obtained the Rao-Nakra sandwich beam [1],

\[
\begin{align*}
\rho_1h_1u''_n &= E_1h_1u''_{xx} + \frac{G_2}{h_2}(-u'^4 + u'^3 + \alpha w_x), \\
\rho_3h_3u''_m &= E_3h_3u''_{xx} - \frac{G_2}{h_2}(-u'^4 + u'^3 + \alpha w_x), \\
\rho h w''_n &= -EIw'''_{xxx} - \frac{G_2\alpha}{h_2}(-u'^4 + u'^3 + \alpha w_x),
\end{align*}
\]

where the new coefficients \( EI = E_1I_1 + E_3I_3 \).

For the Rao-Nakra model (1.6)-(1.8), not only for the three-layer, but also for the multi-layer models, boundary controllability was proved for a variety of boundary conditions: clamped, hinged, clamped-hinged, and hinged-clamped, etc., see [5]-[10]. Recently, some other good results about linear and nonlinear sandwich beam with time-dependent velocity have been obtained, see [11]-[13]. In [14], the authors showed the eigenfunctions of the system from a Riesz basis. The exponential stability of the system was obtained when standard boundary damping is imposed on one end of the beam for all three displacements.

In this paper, we will focus on the stability of Rao-Nakra model (1.6)-(1.8) with boundary damping only on one end for two displacements which is different from the conditions given in [14]. Our main tool is the frequency domain characterization of polynomial stability by Borichev and Tomilov [15]. This paper is organized as following. In section 2, we give a semigroup formulation of the system. We prove that the semigroup is a \( C_0 \)-semigroup of contractions on an appropriate Hilbert space and give the main result. In Section 3, we prove our main stabilization result in Theorem 2.2. Section 4 is the conclusion part.

2. Preliminary and main results

We will consider the following Rao-Nakra beam system:

\[
\begin{align*}
\rho_1h_1u''_n &= E_1h_1u''_{xx} - G_2\phi, \\
\rho_3h_3u''_m &= E_3h_3u''_{xx} + G_2\phi, \\
\rho hw'' + EIw'''_{xxx} &= -G_2\alpha\phi_x, \\
\phi &= \frac{1}{h_2}(-u'^4 + u'^3 + \alpha w_x),
\end{align*}
\]

with the boundary and initial conditions

\[
\begin{align*}
u''(0) &= u''(0) = w(0) = w_x(0) = 0, \\
E_1h_1u''_n(L) &= 0, \\
E_3h_3u''_m(L) &= -\gamma u''_m(L), \\
EIw'''_{xx}(L) &= 0, \\
-EIw'''_{xx}(L) &= -\delta w''_{xx}(L),
\end{align*}
\]

\[
(u',u'',u''',w,w',w''',w''',\dot{w})(x,0) = (u'_0,u''_0,w_0,u'_1,u''_1,w_1,0,0),
\]

where \( \frac{\rho_1h_1}{E_1h_1} = \frac{\rho_3h_3}{E_3h_3} \) and parameters \( \gamma, \delta > 0 \) are damping coefficients.
Firstly, it is important to identify a proper state space so that the "energy" of the system (2.1)-(2.10) is dissipative. For this purpose, we take the inner product of $u^t_1, u^t_3, w^t$ with (2.1), (2.2) and (2.3) in $L^2(\Omega)$, respectively. We can get

$$\frac{1}{2} \frac{d\varepsilon(t)}{dt} = -\gamma \|u^t_1(L)\|^2 - \delta \|w^t_1(L)\|^2,$$

(2.11)

where the "energy" of the system (2.1)-(2.10) is

$$\varepsilon(t) = \rho_1 h_1 \|u^t_1\|^2 + E_1 h_1 \|u^t_1\|^2 + \rho_3 h_3 \|u^t_3\|^2 + E_3 h_3 \|u^t_3\|^2 + \rho h \|w^t_1\|^2 + EI \|w^t_{xx}\|^2 + G_2 h_2 \|\phi\|^2.$$

(2.12)

Let $H^0 = \{ X \in H^1([0, L]) : X(0) = 0 \}$, $H^2 = \{ X \in H^3([0, L]) : X(0) = X_x(0) = 0 \}$ and hence

$$H := H^0 \times H^0 \times H^0 \times H^2 \times H^2 \times L^2 \times L^2 \times L^2 \times L^2$$

and

$$D(A) = \left\{ \begin{array}{l}
(u^1, u^3, V^1, V^3, w, W)^T \in H : \\
\text{if} u^1, u^3 \in H^2([0, L]), V^1, V^3 \in H^1([0, L]), w \in H^4([0, L]), \\
E_1 h_1 u^1_x(L) = 0, E_3 h_3 u^3_x(L) = 0, E_1 h_1 u^1_x(0) = 0 = w(0) = w_x(0) = 0,
\end{array}\right\}$$

Then, we will use semigroup theory to analyse the operator $A$.

**Theorem 2.1.** $A$ is the infinitesimal generator of a $C_0$-semigroup of contractions on the Hilbert space $H$.

**Proof.**

$$\text{Re} \langle AZ, Z \rangle_u = \frac{1}{2} \frac{d}{dt} \|Z\|^2_u = -\gamma \|u^1_x(L)\|^2 - \delta \|w^t_1(L)\|^2 \leq 0.$$

(2.14)

Thus, $A$ is dissipative. It is easily to show that $D(A)$ is dense in $H$. Suppose that $\lambda = 0$ is an eigenvalue and $Z$ be the normalized eigenfunction,
\[ V^1 = 0, H^1([0, L]), \]  
(2.15)  
\[ \frac{E_1}{\rho_1} u_1 + \frac{G_2}{\rho_1} \phi = 0, L^2([0, L]) \]  
(2.16)  
\[ V^3 = 0, H^1([0, L]) \]  
(2.17)  
\[ \frac{E_3}{\rho_3} u_3^3 - \frac{G_3}{\rho_3} \phi = 0, L^2([0, L]) \]  
(2.18)  
\[ W = 0, H^2([0, L]) \]  
(2.19)  
\[ -\frac{EI}{\rho h} w_{xxx} + \frac{G_\alpha}{\rho h} \phi = 0, L^2([0, L]) \]  
(2.20)

From dissipation, we have
\[ -\gamma \left\| V^3(L) \right\|^2 = -\delta \left\| W(L) \right\|^2 = 0. \]  
(2.21)

Multiplying (2.16), (2.18) and (2.20) by \( \rho_1 h u^1, \rho_3 h u^3 \) and \( \rho hw \) in \( L^2([0, L]) \), respectively.

We can obtain
\[ -E_1 h_1 \left\| u^1 \right\|^2 + G_2 \langle \phi, u^1 \rangle = 0, \]  
(2.22)  
\[ E_3 h_3 u^3(L) u^3(L) - E_2 h_2 \left\| u^2 \right\|^2 + G_2 \langle \phi, u^1 \rangle = 0, \]  
(2.23)  
and
\[ -EI w_{xxx} (L) w(L) - EI \left\| w_{xx} \right\|^2 + G_\alpha \langle \phi, w \rangle = 0. \]  
(2.24)

Take the sum of (2.22), (2.23) and (2.24), by (2.21), we can obtain
\[ E_1 h_1 \left\| u^1 \right\|^2 + E_3 h_3 \left\| u^3 \right\|^2 + EI \left\| w_{xx} \right\|^2 + G_\alpha \left\| \phi \right\|^2 = 0, \]  
(2.25)

which together with (2.15), (2.17) and (2.19), we can conclude that \( \left\| u \right\|_{H^3} = 0 \). Thus \( 0 \in \rho(A) \).

Then, thanks to fixed Lumer-Phillips Theorem, we conclude that \( A \) generates a \( C_0 \)-semigroup of contractions on \( H \).

Finally, the main result for system (2.1)-(2.10) is stated in the following theorem.

**Theorem 2.2.** Let \( e^{At} \) be the semigroup associated with system (2.1)-(2.10), the semigroup \( e^{At} \) is polynomially stable of order 1/2.

Our main tool is the frequency domain characterization of polynomial stability by Borichev and Tomilov which is stated as follow.

**Theorem 2.3.** [15] Let \( H \) be a Hilbert space and \( A \) generates a bounded \( C_0 \)-semigroup in \( H \). Assume that
\[ \text{span} \subset \rho(A), \]  
(2.26)  
\[ \sup_{|\beta| \leq k} \left\| (i \beta - A)^{-1} \right\| < +\infty, \text{ for some } k > 0. \]  
(2.27)

Then, there exists a positive constant \( C > 0 \) such that
\[ \left\| e^{At} U_0 \right\| \leq C \left( \frac{1}{t} \right)^{\frac{1}{2}} \left\| U_0 \right\|_{D(A)}, \forall t > 0, \]  
(2.28)

for all \( U_0 \in D(A) \).

**Remark 2.1** (i). Whether the order 1/2 is optimal still open, it will be studied in the further work.
We are also interested in the system (2.1)-(2.4) with the boundary damping (2.7) is imposed on one end for only one displacement. Unfortunately, we failed to get any stability property of the system, it may need to develop new thoughts or methods.

3. Proof of the main result
We will apply the Theorem 2.3 to prove Theorem 2.2.

Proof. We first verify condition (2.26). Assume that it is false, i.e., there is a \( \lambda = i \beta \in \sigma(A) \). Then there exist \( \lambda_n = i \beta_n \to \lambda \) and normalized \( U_n = (u_{n1}, V_{n1}, u_{n3}, V_{n3}, w_n, W_n)^T \) such that
\[
\|i \beta_n - A\| \|U_n\|_H = 0, \tag{3.1}
\]
which implies
\[
\begin{aligned}
&\beta_1 u_1^1 - V_1^1 = 0, H_1^1([0, L]), \\
&\beta V_1^1 - E_1 u_{1x}^1 - G_2 \frac{\phi}{\rho h_1} = 0, L^2([0, L]) \\
&\beta u_3^3 - V_3^3 = 0, H_3^3([0, L]) \\
&\beta V_3^3 - E_3 u_{3x}^3 + G_2 \frac{\phi}{\rho h_3} = 0, L^2([0, L]) \\
&\beta w - W = 0, H^1_0([0, L]) \\
&\beta w + \frac{E_l}{\rho h} w_{xxx} - \phi = 0, L^2([0, L]). \\
\end{aligned} \tag{3.2}
\]
Substituting (3.2) into (3.3), (3.4) into (3.5) and (3.6) into (3.7), respectively, we get
\[
\begin{aligned}
- \beta^2 u_1^1 - \frac{E_1}{\rho_1} u_{1x}^1 - \frac{G_2}{\rho_1 h_1} \phi &= 0, \\
- \beta^2 u_3^3 - \frac{E_3}{\rho_3} u_{3x}^3 + \frac{G_2}{\rho_3 h_3} \phi &= 0, \\
- \beta^2 w + \frac{E_l}{\rho h} w_{xxx} - \frac{G_2 \alpha}{\rho h} \phi &= 0. \tag{3.9}
\end{aligned}
\]
Multiplying (3.10) and (3.11) by \( x u_3^3 \) and \( x w_x \), respectively,
\[
\begin{aligned}
&\left\langle - \beta^2 u_3^3, x u_3^3 \right\rangle + \left\langle \frac{E_3}{\rho_3} u_{3x}^3, x u_3^3 \right\rangle + \left\langle \frac{G_2}{\rho_3 h_3} \phi, x u_3^3 \right\rangle = 0, \tag{3.12}
\end{aligned}
\]
\[
\begin{aligned}
&\left\langle - \beta^2 w, x w_x \right\rangle + \left\langle \frac{E_l}{\rho h} w_{xxx}, x w_x \right\rangle + \left\langle \frac{G_2 \alpha}{\rho h} \phi, x w_x \right\rangle = 0. \tag{3.13}
\end{aligned}
\]
By dissipation (3.8), (3.4) and (3.6), we have
\[
\| \beta u^1 (L) \|^2 = \| \beta w (L) \|^2 = 0, \beta \text { is finite}. \]
Integrating by parts, (3.12) and (3.13) become
\[
\begin{aligned}
\| \beta u^1 \|^2 + \frac{E_3}{\rho_3} \| u_3^3 \|^2 + 2 \text{Re} \left\langle \frac{G_2}{\rho_3 h_3} \phi, x u_3^3 \right\rangle &= 0, \tag{3.14}
\end{aligned}
\]
By Hölder inequality, we have
\[ 2 \Re \left( \frac{G_2}{\rho_j h_j} \phi, xu_i^j \right) \leq G_2 C \left( \|u_i^j\| \|u_i^j\| + \|w_i^j\| \|u_i^j\| \right). \] (3.16)

Since \( \|U\|_{H^2} = 1 \), we can get \( \|u_i^j\| \) and \( \|w_i^j\| \) are bounded. By the Poincare inequality, we can obtain \( \|u_i^j\| \) and \( \|w_i^j\| \) are bounded. Then (3.16) becomes
\[ \frac{E_j}{\rho_j} \|u_i^j\|^2 \leq G_2 C \|u_i^j\|^2. \] (3.17)

It implies that \( \|u_i^j\| = 0 \) for \( G_2 \) small enough. Similarly, we can get \( \|w_i^j\| = 0 \). By the Poincare inequality, we have \( \|u_i^j\| = 0 \) and \( \|w_i^j\| = 0 \). Multiplying (3.9) and (3.10) by \( u_i^j \) and \( u_i^j \), respectively.

\[ G_2 \frac{\rho_j h_j}{\rho_j h_j} \left( G_2 - \frac{\rho_j h_j}{\rho_j h_j} \right) \frac{\|u_i^j\|^2}{\|u_i^j\|^2 + \Re \left( \frac{G_2}{\rho_j h_j} (u_i^j + w_i^j), u_i^j \right) + \Re \left( \frac{G_2}{\rho_j h_j} (-u_i^j + w_i^j), u_i^j \right)} \] (3.18)

which yields \( \|u_i^j\| = 0 \). It follows from (3.2) that \( \|V^i\| = 0 \). Multiplying (3.3) by \( u_i^j \), we can get \( \|u_i^j\| = 0 \).

We conclude that \( \|U\|_{H^2} = 0 \). This is a contradiction with the assumption \( \|U\|_{H^2} = 1 \). Thus, \( \bar{I} \subseteq \rho(A) \).

To verify condition (2.27), we assume that (2.27) is false. Then by the uniform boundedness theorem, there exist a sequence \( \beta \to \infty \) and a unit sequence \( U = (u_i^1, V^1, u_i^1, V^1, w, W)^T \subseteq D(A) \) such that
\[ \beta^k \|i \beta I - A\|_{H^2} \to 0, \] (3.19)
which implies that
\[ \beta^k \|i \beta u_i^1 - V^1\| = f_1, H^1_0([0, L]), \] (3.20)
\[ \beta^k \|i \beta V^1 - \frac{E_j}{\rho_j} u_i^1 - \frac{\rho_j h_j}{\rho_j h_j} \phi = f_2, L^2([0, L]) \] (3.21)
\[ \beta^k \|i \beta u_i^3 - V^1\| = f_3, H^1_0([0, L]) \] (3.22)
\[ \beta^k \|i \beta V^3 - \frac{E_j}{\rho_j} u_i^3 + \frac{G_2}{\rho_j h_j} \phi = f_4, L^2([0, L]) \] (3.23)
\[ \beta^k \|i \beta w - W\| = f_5, H^2_0([0, L]) \] (3.24)
\[ \beta^k \|i \beta W^3 - \frac{E_j}{\rho_j} w_i^3 - \frac{G_2}{\rho_j h_j} \phi = f_6, L^2([0, L]) \] (3.25)

From dissipation, we have
\[ -\gamma \beta^k \|V^2(L)\|^2 = o(1) \quad \text{and} \quad -\delta \beta^k \|W(L)\|^2 = o(1). \] (3.26)

Taking \( k = 2 \), by (3.26), (3.22) and (3.24), we get
\[ \|\beta u_i^3(L)\| = o(1) \quad \text{and} \quad \|\beta w(L)\| = o(1). \] (3.27)
Substituting (3.20), (3.22) and (3.24) into (3.21), (3.23) and (3.25), respectively. Thus,

\[
\begin{align*}
-\beta^2 u^4 - \frac{E_1}{\rho_1} u_{xx}^4 - \frac{G_z}{\rho_1 h_1} \phi &= f_2 + i \beta f_3, \\
-\beta^2 u^4 - \frac{E_2}{\rho_2} u_{xx}^4 + \frac{G_z}{\rho_2 h_2} \phi &= f_4 + i \beta f_5, \\
-\beta^2 W + \frac{E I}{\rho h} w_{xxx}^4 + \frac{G_z \alpha}{\rho h} \phi &= f_6 + i \beta f_8.
\end{align*}
\]

(3.28) (3.29) (3.30)

Multiplying (3.28) by \(xu_x^4\), (3.29) by \(xu_x^4\) and (3.30) by \(xw_x^4\). Integrating by parts, together with boundary conditions, we have

\[
\begin{align*}
\|\beta u^2\|^2 + \frac{E_1}{\rho_1} \|u_{xx}^4\|^2 &= 2 \text{Re} \left\langle \frac{G_z}{\rho_1 h_1} \phi, xu_x^4 \right\rangle + L \|\beta u^4(L)\|^2 + 2 \text{Re} \left\langle \frac{f_2 + i \beta f_3}{\beta^2}, xu_x^4 \right\rangle, \\
\|\beta u^2\|^2 + \frac{E_2}{\rho_2} \|u_{xx}^4\|^2 &= 2 \text{Re} \left\langle \frac{G_z}{\rho_2 h_2} \phi, xu_x^4 \right\rangle + L \|\beta u^4(L)\|^2 + 2 \text{Re} \left\langle \frac{f_4 + i \beta f_5}{\beta^2}, xu_x^4 \right\rangle, \\
\|\beta w^2\|^2 + \frac{E I}{\rho h} \|w_{xxx}^4\|^2 &= 2 \text{Re} \left\langle \frac{E I}{\rho h} w_{xxx}, w_x + xu_{xx}^4 \right\rangle + L \|\beta w(L)\|^2 + \frac{2L}{\rho h} \text{Re}(-EIw_{xxx}(L)) + G_z \phi(L)w_x(L) + 2 \text{Re} \left\langle \frac{f_6 + i \beta f_8}{\beta^2}, xw_x^4 \right\rangle.
\end{align*}
\]

(3.31) (3.32) (3.33)

Since \(\|U\|_{H^1} = 1\), so \(\|V^4\|\|V^4\|\) and \(\|W^4\|\) are bounded, then by (3.20), (3.22) and (3.24), we obtain

\[
\|u_x^4\| = o(1), \|u_x^4\| = o(1), \|w_x^4\| = o(1), \quad \text{as} \quad \beta \to \infty
\]

(3.34)

From dissipation, (3.27), (3.37), and (3.38), then (3.31), (3.32) and (3.33) can be written as

\[
\|\beta u^2\|^2 + \frac{E_1}{\rho_1} \|u_{xx}^4\|^2 = L \|\beta u^4(L)\|^2 + o(1), \quad \text{as} \quad \|u_x^4\| \text{ is bounded,}
\]

(3.36)

\[
\|\beta u^2\|^2 + \frac{E_2}{\rho_2} \|u_{xx}^4\|^2 = o(1), \quad \text{as} \quad \|u_x^4\| \text{ is bounded,}
\]

(3.37)

\[
\|\beta w^2\|^2 + \frac{3E I}{\rho h} \|w_{xxx}^4\|^2 = o(1).
\]

(3.38)

Hence,

\[
\|\beta u^2\|^2 = \|u_x^4\|^2 = o(1) \text{ and } \|\beta w^2\|^2 = \|w_x^4\|^2 = o(1).
\]

(3.39)

Multiplying (3.28) by \(u^4\) and (3.29) by \(u^4\). Since \(\frac{E h_1}{E h_1} = \frac{E h_2}{E h_2}\), \(\|u_x^4\|\) is bounded, together with (3.39), we can obtain

\[
\begin{align*}
\frac{G_z}{\rho h h_2} \|\beta u^2\|^2 - \frac{G_z}{\rho h h_2} \|\beta u^4\|^2 + \text{Re} \left\langle \frac{G_z}{\rho h h_2} \phi, \beta u^4 \right\rangle + \text{Re} \left\langle \frac{G_z \alpha}{\rho h h_2} \beta^2 w_x^4, u^4 \right\rangle \\
+ \beta^2 u_x^4(L) u_x^4(L) + \text{Re} \left\langle \frac{G_z}{\rho h h_2} \phi, \beta u^4 \right\rangle + \text{Re} \left\langle \frac{G_z \alpha}{\rho h h_2} \beta^2 w_x^4, u^4 \right\rangle
\end{align*}
\]

(3.40)

Multiplying (3.28) and (3.29) by \(w_x^4\), we have
\[ \langle -\beta^2 u^1, w_x \rangle - \left( \frac{E_2}{\rho_1} u_{xx}^1, w_x \right) + \left( \frac{G_2}{\rho_1 h_3} \phi, w_x \right) = \left( f_2 + i\beta f_1, w_x \right) = o(1), \] \tag{3.41}

\[ \langle -\beta^2 u^3, w_x \rangle - \left( \frac{E_2}{\rho_2} u_{xx}^3, w_x \right) + \left( \frac{G_2}{\rho_2 h_3} \phi, w_x \right) = \left( f_3 + i\beta f_3, w_x \right) = o(1). \] \tag{3.42}

Integrating by parts, since \( \| u_x^1 \| \) is bounded, \( \| w_x \| = o(1), \| u^1 \| = o(1), \| u^1 \| = o(1), \)
\( \| w_{xx} \| = o(1) \) and \( \| u_{xx}^1(L) w_x(L) \| = o(1) \), we can get \( \langle -\beta^2 u^1, w_x \rangle = o(1) \) and \( \langle -\beta^2 u^3, w_x \rangle = o(1) \). From (3.36), \( \| \beta u^1 \| \) and \( \| u^1 \| \) are bounded, we can obtain \( \| \beta u^1(L) \| \) is bounded. Then \( \beta u^1(L) u^1(L) = o(1) \). Thus, (3.40) can be written as

\[ \frac{G_2}{\rho_1 h_x h_2} \| u^1 \|^2 = \frac{G_2}{\rho_2 h_x h_2} \| \beta u^1 \|^2 + o(1). \] \tag{3.43}

As \( \| \beta u^1 \|^2 = o(1) \), we can obtain \( \| \beta u^1 \|^2 = o(1) \). Multiplying (3.28) by \( u^1 \), integrating by parts, we can easily get

\[ -\| \beta u^1 \|^2 + \frac{E_1}{\rho_1} \| u^1 \|^2 + o(1), \] \tag{3.44}

it yields \( \| u^1 \|^2 = o(1) \). From all the above, we have \( \| U \|^2_{u_{x}^1} = o(1) \). This is a contradiction with the assumption \( \| U \|^2_{u_{x}^1} = 1 \).

4. Conclusions
In this paper, polynomial stability for Rao-Nakra sandwich beam model with boundary damping only on one end for two displacements have been investigated. Firstly, we found a proper state space so that the energy of the system is dissipative. Secondly, we convert the beam system to an abstract first-order evolution equation with operator \( A \). We obtain operator \( A \) in the dissipative system is the infinitesimal generator of a \( C_0 \) - semigroup of contractions by using semigroup operator theory. Finally, we prove the system is polynomial stable of order \( 1/2 \) by using the frequency domain characterization of polynomial stability. Among the proof process, the most difficult part is we adopt the standard multiplier method to pass the boundary approximation to the whole system.

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