QUANTUM AND BRAIDED LINEAR ALGEBRA

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ABSTRACT Quantum matrices $A(R)$ are known for every $R$ matrix obeying the Quantum Yang-Baxter Equations. It is also known that these act on 'vectors' given by the corresponding Zamalodchikov algebra. We develop this interpretation in detail, distinguishing between two forms of this algebra, $V(R)$ (vectors) and $V^*(R)$ (covectors). $A(R) \rightarrow V(R_{21}) \otimes V^*(R)$ is an algebra homomorphism (i.e. quantum matrices are realized by the tensor product of a quantum vector with a quantum covector), while the inner product of a quantum covector with a quantum vector transforms as a scaler. We show that if $V(R)$ and $V^*(R)$ are endowed with the necessary braid statistics $\Psi$ then their braided tensor-product $V(R) \otimes V^*(R)$ is a realization of the braided matrices $B(R)$ introduced previously, while their inner product leads to an invariant quantum trace. Introducing braid statistics in this way leads to a fully covariant quantum (braided) linear algebra. The braided groups obtained from $B(R)$ act on themselves by conjugation in a way impossible for the quantum groups obtained from $A(R)$.

RESUMÉ Les matrices quantiques $A(R)$ sont connus pour chaque matrice $R$ qui satisfait les équations de Yang-Baxter. Il est encore connu qu'ils agissent sur les 'vecteurs' donnés par l’algèbre de Zamalodchikov correspondant. Nous prolongons cette interprétation, distinguant deux versions de cette algèbre, $V(R)$ (vecteurs) et $V^*(R)$ (covecteurs). $A(R) \rightarrow V(R_{21}) \otimes V^*(R)$ est une homomorphisme des algèbres, et le produit intérieur d’un covecteur quantique avec un vecteur quantique se transforme comme un scalaire. Nous montrons que si $V(R)$ et $V^*(R)$ sont munis des statistiques tressées $\Psi$, alors leur produit tensoriel-tressé $V(R) \otimes V^*(R)$ est une réalisation des matrices tressées $B(R)$ introduites déjà, et leur produit intérieur s'amène à une trace invariante. Par introduisant les statistiques tressées dans cette façon nous obtenons un algèbre linéaire quantique (tressé) et totalement covariant. Les groupes tressés obtenus de $B(R)$ s'agissent sur eux-même par conjugaison dans une manière qui est impossible pour les groupes quantiques obtenus de $A(R)$.

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1 Introduction

Quantum matrices and groups have arisen in physics and it is well established that they play an important role in certain physical theories. They also suggest a new kind of quantum calculus (within the context of non-commutative geometry) describing such physics. One aspect of the physics which is not, however, so well covered by quantum groups is the braid-statistics of the quantum fields. Here the non-commutativity arises not due to quantum effects but due to non-trivial statistics (such as fermionic or anyonic statistics) and suggests a kind of braided calculus as well as a quantum one.

In this paper we study the quantum and braided linear algebra associated to a regular matrix \( R \in M_n \otimes M_n \) obeying the Quantum Yang-Baxter Equations (QYBE). We begin with the quantum case, where quantum matrices of type \( R \) are defined as the bialgebra \( A(\mathbb{R}) \) of [3], and clarify the precise way that this ‘acts’ on quantum vectors and quantum covectors. These are given by variants of the Zamalodchikov algebra associated to \( R \) as explained in [14, Sec. 6.3.2], but more care is needed now to distinguish their transformation properties. It can be expected that these considerations of quantum linear algebra will ultimately shed some light also on more complex constructions in the matrix form of quantum differential calculus as in [25]. For example, the quantum traces needed there arise in a particularly obvious way from our considerations.

After this warm-up with quantum linear algebra we proceed to ‘braided linear algebra’. Here the role of braided matrices is played by \( B(\mathbb{R}) \) introduced by the author in [18]. The only difference between the \( A(\mathbb{R}) \) and the \( B(\mathbb{R}) \), i.e. between quantum and braided linear algebra is that the latter is fully covariant under an underlying quantum group (which induces on it a braiding). Thus braided linear algebra means nothing other than covariant quantum linear algebra. Let \( H \) be a fixed quantum group (with universal \( R \)-matrix in the sense of [2]). Then an object is \( H \)-covariant if \( H \) acts on it in a way that preserves all its structure. For example, an algebra \( V \) is \( H \)-covariant if \( H \) acts on \( V \) (and hence on \( V \otimes V \) via the comultiplication \( \Delta(H) \subset H \otimes H \)) and the multiplication is an intertwiner

\[
V \otimes V \to V, \quad h \triangleright (a \cdot b) = (h \triangleright a \otimes b), \quad h \triangleright 1 = \epsilon(h)1
\]
where \(\triangleright\) is the relevant action. One says that \(V\) is an \(H\)-module algebra. Not only algebras but all quantum group constructions can be done fully \(H\)-covariantly (one says that the constructions take place in the braided category of \(H\)-representations.) This is the setting behind \[16\]. For example, a super-quantum group is nothing other than a \(\mathbb{Z}_2\)-covariant quantum group (where \(\mathbb{Z}_2\) denotes the group algebra of the group with two elements, equipped with a certain non-standard quantum group structure, and its action just corresponds to the grading).

The relevance of this notion to the present paper is that behind the bialgebras \(A(R)\) there is a quantum group (with universal \(R\)-matrix). For the standard \(R\) matrices this is \(U_q(g)\), but note that we will not be tied to the standard case below. The quantum group acts on \(A(R)\) by a quantum coadjoint action \[13, \text{Theorem 3.2}\] \[18, \text{Sec. 6}\] but this action does not leave the defining relations

\[ R t_1 t_2 = t_2 t_1 R \tag{2} \]

invariant, i.e. \(A(R)\) is not covariant in the way explained, even as an algebra unless it is commutative (here \(t_1, t_2\) denote the matrix of generators \(t\) viewed in \(M_n \otimes M_n\) in the standard way). The idea behind \(B(R)\) is that the relations \[3\] must be modified in a certain way to restore covariance. There is a canonical way to do this, namely a process of transmutation that turns a suitable quantum group into one that is covariant (for example it can be used to superize or anyonize suitable quantum groups) \[17\]. In our case the transmutation of \(A(R)\) gives \(B(R)\) as generated by \(n^2\) elements \(u^i_j\) (and 1) with relations and coalgebra \[18\]

\[ R_{21} u_1 R_{12} u_2 = u_2 R_{21} u_1 R_{12}, \quad \Delta u^i_j = u^i_k \otimes u^k_j, \quad \varepsilon u^i_j = \delta^i_j. \tag{3} \]

The relations were written with all the four \(R\)’s on the right in \[18\] but a close inspection of the indices will show that \[3\] is just the same. Note also that these relations \[3\] are known in quite another context, namely for the ordinary quantum groups \(U_q(g)\) in the form with generators \(L = t^+ S t^-\). The reason for this is in fact an accident of the particular ‘self-dual’ structure of \(U_q(g)\) as we explain in detail in \[20\]. In general \(B(R)\) is quite different as an algebra from any quantum group, especially in the triangular case when \(R_{21} R_{12} = 1\), and
its conceptual meaning is also quite different because its role is as a braided or covariant version of the quantum groups of function algebra type, not at all of enveloping algebra type. Nevertheless for $U_q(g)$ we can certainly exploit this accident to apply some of our results below about $B(R)$ to obtain information about the covariance properties of $U_q(g)$ also.

These covariance properties of $B(R)$ were explained in detail in [18] where we gave the coadjoint actions of the underlying quantum group etc. On the other hand, this underlying quantum group can be hard to compute in practice when $R$ is not a standard one. This can be avoided if we speak of $[\mathbb{1}]$ not in terms of a quantum group $H$ acting but in a dual form, in terms of the coaction of a dual quantum group $A$. A coaction is just like an action but with arrows reversed. Thus it means a map $V \to V \otimes A$ instead of $H \otimes V \to V$ (left actions correspond to right coactions of the dual). For the case of $B(R)$ above the underlying dual quantum group with respect to which everything is covariant, is nothing other than $A(R)$ itself (modulo ‘determinant-type’ relations to provide an honest antipode). Moreover, coactions might seem a little unfamiliar but when it comes to coactions of matrix dual quantum groups such as $A(R)$, they take a very simple ‘matrix’ form. This point of view has been stressed by Manin in [22] and has also become popular in physics, e.g. [25]. For this reason one of our first goals, in Section 2, will be to carefully convert the $H$-covariance conditions $[1]$ as used in [18], into this ‘matrix’ form.

A careful study of this will lead also to our notion of quantum vectors $V(R)$ and covectors $V^*(R)$ of $R$ type, based on the Zamalodchikov algebra and both fully covariant. We show how they can be used to realize the algebra $A(R)$ itself in the same way that a matrix can be decomposed into the rank one matrices $|i><j|$, while $\sum <i|i>$ transforms as a scaler. This has some similarities with Manin’s realization [22], see also [23], but represents in fact a different and ‘orthogonal’ formalism to that. This is evident from the simplest example where $R$ is the $SL_q(2)$ $R$-matrix, for then we find $V(R) = C_q^{2|0}$ and $V^*(R) = C_{q^{-1}}^{2|0}$ in Manin’s notation: they are both bosonic quantum planes. For another choice of normalization one can have both fermionic, but we do not mix bosonic and fermionic quantum planes as in
Manin’s approach. This is even more evident when $R$ has several (not only two) relevant normalizations, which in our approach are not mixed (there is a complete quantum linear algebra with both quantum vectors and quantum covectors for each choice of normalization).

Finally, we will be ready in Section 3 to give braided (i.e. covariant) versions of all these considerations with $B(R)$ recovered when $V(R)$ and $V^*(R)$ are no longer mutually commutative but treated instead with braid statistics $\Psi$. $B(R)$ acts on them, as well as on itself by conjugation. As a spin-off we will recover from the above scalar a general formula for the quantum trace, useful in other contexts. Section 4 is devoted to computing some of the simplest examples of the theory, including a non-standard one related to the 8-vertex model. Another example makes partial contact with some formulae of recent interest in physics\cite{6}. The paper concludes in Section 5 with some details of the relationship between $A(R)$ and $B(R)$ (i.e. of transmutation), interpreting it as a kind of partition function with prescribed boundary conditions. In addition, an appendix provides an abstract (diagrammatic) proof of one of our main theorems, based on the braided-commutativity of braided groups.

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### 2 Transformation of Vectors and Covectors

In this preliminary section we begin by establishing the matrix description of the transformation properties that we will need. Since there seems to be a gap between the standard mathematical way of describing adjoint coactions etc and the matrix notations preferred by physicists, we will explain their equivalence carefully (with proofs). Most probably this is well-known to experts, but I didn’t find an adequate treatment elsewhere.

Firstly, recall that a bialgebra means an algebra $A$ over a field or commutative ring $k$ and a map $\Delta : A \to A \otimes A$ which is an algebra homomorphism and coassociative. There also needs to be a counit $\epsilon : A \to k$. The matrix notation stems from the following well-known and innocent observation. Let $A$ be an algebra with $n^2$ matrix generators $t = (t_{ij})$. 

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Suppose that \( \epsilon(t_{ij}) = \delta_{ij} \) extends multiplicatively to a map \( \epsilon : A \to k \). Let \( \Delta t_{ij} = t^i_k \otimes t^j_k \) and \( \Delta(1) = 1 \otimes 1 \). Then \( (A, \epsilon, \Delta) \) is a bialgebra if and only if the following holds: If \( t, t' \) are two identical sets of generators of \( A \), mutually commuting elementwise, then \( t'' = tt' \) is also a realization of \( A \) (i.e. \( t''_{ij} = t^i_k t^j_k \) obey its relations also). In truth, this observation is just saying that the \( \Delta \) that we desire is an algebra homomorphism to \( A \otimes A \) (i.e. \( A \otimes A \) is a realization of \( A \)), where \( A \otimes A \) as an algebra is of course given by two mutually commuting copies of \( A \), i.e. generated by \( t = t \otimes 1 \) and \( t' = 1 \otimes t \) in \( A \otimes A \). The main content of the notation is to omit writing the tensor product, distinguishing the elements of the second factor instead by the prime.

We have gone through the rationale in detail because the same method of omitting tensor products works for also comodule algebras. Thus, if \( A \) is a bialgebra and \( \beta : V \to V \otimes A \) a comodule (the dual notion of an action) and \( V \) is an algebra then \( V \otimes V \to V \) is \( A \)-covariant (in the sense of \([1]\)) but in our dual language) if \( \beta \) is an algebra homomorphism. One says that \( V \) is an \( A \)-comodule algebra. The reader can easily see that this is just the condition in \([1]\) with arrows reversed and left-right interchanged. The following observation was probably first stressed by Manin in \([22]\) in connection with the quantum plane. It is surely also well known to others.

**Lemma 2.1** Let \( A \) be a matrix bialgebra (as above) and \( V \) an algebra with \( n \) generators \( x = (x_i) \) (written as a row vector) and \( 1 \). Define \( \beta(1) = 1 \otimes 1 \) and \( \beta(x_j) = x_i \otimes t_{ij} \). Then \( \beta \) makes \( V \) a comodule algebra if and only if the following holds: whenever \( t \) is a copy of the generators of \( A \), \( x \) a copy of those of \( V \), commuting elementwise with the \( t \), then \( x' = xt \) is a realization of \( V \).

**Proof** Here \( t_{ij} = 1 \otimes t_{ij} \) and \( x_i = x_i \otimes 1 \) are the generators of the tensor product algebra \( V \otimes A \) built from mutually commuting copies of \( A \) and \( V \). The condition is just that the products \( x'_{ij} = x_i t^i_j \) are a realization of \( V \), i.e. that \( \beta : V \to V \otimes A \) as defined is an algebra map. On the other hand, \( \beta \) as defined is already a right coaction from the form of its definition. This is because to be a coaction one needs \((\beta \otimes \text{id}) \beta = (\text{id} \otimes \Delta) \beta \) and \((\text{id} \otimes \epsilon) \beta = \text{id} \), which we see automatically as \( \beta(x_i) \otimes t^i_j = x_i \otimes t^i_j \otimes t^i_j = x_i \otimes \Delta(t^i_j) \).
and $x_i \epsilon(t_{i,j}) = x_j$ due to the matrix form of $\Delta, \epsilon$. 

So far we have only said carefully what is well-known. But the same method also gives

**Lemma 2.2** Let $A$ be a matrix Hopf algebra (as above but with an antipode $S$) and $V$ an algebra with $n$ generators $v = (v^i)$ (written as a column vector) and $1$. Define $\beta(1) = 1 \otimes 1$ and $\beta(v^i) = v^i \otimes St^i$. Then $\beta$ makes $V$ a comodule algebra if and only if the following holds: whenever $t$ is a copy of the generators of $A$, $v$ a copy of those of $V$, commuting elementwise with the $t$, then $v' = t^{-1}v$ is a realization of $V$. Here $t^{-1} = St$, i.e., the matrix with entries $(St^i_j)$.

**Proof** Here $t^i_j = 1 \otimes t^i_j$ (as before) and $v^i = v^i \otimes 1$ are the generators of the tensor product algebra $V \otimes A$ built from mutually commuting copies of $A$ and $V$. We use the fact that they mutually commute in the tensor product to write the $St$ on the left even though it lives in the second factor of $V \otimes A$. The condition is just that the $v'^i = St^i_j v^j$ is a realization of $V$, i.e. that $\beta : V \to V \otimes A$ as defined is an algebra map. Once again, $\beta$ as defined is already a right coaction from the form of its definition. This is because $\beta(v^i) \otimes St^i_j = v'^i \otimes St^i_j \otimes St^i_j = v'^i \otimes \Delta(St^i_j)$ and $v^i \epsilon(St^i_j) = v^j$ due to the matrix form of $\Delta, \epsilon$ and that $S$ is an anti-coalgebra map while $\epsilon \circ S = \epsilon$. 

We have, combining these,

**Lemma 2.3** Let $A$ be a matrix Hopf algebra (as above) and $V$ an algebra with $n^2$ generators $b = (b^i_j)$ and $1$. Define $\beta(1) = 1 \otimes 1$ and $\beta(b^i_j) = b^m_n \otimes (St^i_m)t^m_j$. Then $\beta$ makes $V$ a comodule algebra if and only if the following holds: whenever $t$ is a copy of the generators of $A$, $b$ a copy of those of $V$, commuting with the $t$, then $b' = t^{-1}bt$ is a realization of $V$.

**Proof** Here $t^i_j = 1 \otimes t^i_j$ (as before) and $b^i_j = b^i_j \otimes 1$ are the generators of the tensor product algebra $V \otimes A$ built from mutually commuting copies of $A$ and $V$. We again use the fact that they mutually commute in the tensor product to write the $St^i_m$ part on the left even though it lives in the second factor of $V \otimes A$ along with the $t^m_j$ part. The condition is just that $b'^i_j = (St^i_m)b^m_n t^m_j$ is a realization of $V$, i.e. that $\beta : V \to V \otimes A$ as defined is an
algebra map. The map $\beta$ as defined is already a right coaction because $\beta(b^m_n) \otimes (St^t_m)t^n_j = b^{m'}_{n'} \otimes (St^{t'}_{m'})t'^n_j \otimes (St^t_m)t^n_j = b^{m'}_{n'} \otimes \Delta((St^{t'}_{m'})t'^n_j)$ and $b^m_n \epsilon((St^t_m)t^n_j) = b^j_j$ due to $\Delta, \epsilon$ being algebra homomorphisms and the arguments already given in the proofs of the two lemmas above. $\Box$

We now give some important (but not the only) examples of comodule algebras of such type. Let $R \in M_n \otimes M_n$ be a matrix solution of the QYBE and $A = A(R)$ the FRT bialgebra (which is of the matrix type above). The following two examples are variants of the Zamalodchikov algebra on $n$ generators and the known coaction of $A(R)$ on it as shown for general $R$ in [14, Sec. 6.3.2]. The new part lies in our careful and matching selection of conventions for our present purposes. We fix a single invertible constant $\lambda$ throughout (and a fixed normalization of $R$ which we do not change further).

**Example 2.4** We define $V^\ast(R)$ to be the algebra with $n$ generators $x_i$ and 1, and relations $x_i x_k = x_n x_m \lambda R^{m'}_i {n'}_k$. Writing $x = (x_i)$ as a row vector and $t = (t^i_j)$ as a matrix (with values in their respective algebras), the assignment $x' = xt$ makes $V^\ast(R)$ into a right $A(R)$-comodule algebra. We call it the algebra of quantum covectors of $R$-type.

**Proof** A proof in conventional comodule notation is in [14, Sec. 6.3.2]. In our matrix notation it is simply as follows. The relations of $V^\ast(R)$ are $x_1 x_2 = x_2 x_1 \lambda R$ where $x_1 = x \otimes 1$ and $x_2 = 1 \otimes x$. We check $x'_1 x'_2 = x_1 t_1 x_2 t_2 = x_2 x_1 \lambda R t_1 t_2 = x_2 x_1 \lambda t_2 t_1 R = x'_2 x'_1 \lambda R$ so the transformed covectors obey the same relations. We used that the $x, t$ commute, the relations in $V^\ast(R)$ and the relations of $A(R)$. $\Box$

**Example 2.5** We define $V(R)$ to be the algebra with $n$ generators $v^i$ and 1, and relations $v^i v^k = \lambda R^i_j k^i_j v^j$. Writing $v = (v^i)$ as a column vector and $t^{-1} = (St^t_j)$ for the matrix inverse of $t$ (with values in the respective algebras), the assignment $v' = t^{-1}v$ makes $V(R)$ into a right $A$-comodule algebra, where $A$ is a suitable dual quantum group obtained from $A(R)$. We call $V(R)$ the algebra of quantum vectors of $R$-type.
Proof The proof is similar to the preceding example. In the matrix notation it is as follows. The relations of $V(R)$ are $v_1v_2 = \lambda R v_2v_1$. Then $v_1'v_2' = t_1^{-1} t_2^{-1} v_1v_2 = t_1^{-1} t_2^{-1} \lambda R v_2v_1 = \lambda R t_2^{-1} t_1^{-1} v_2v_1 = \lambda R v_2v_1$. We used the relations for $A(R)$ in a form obtained by applying the antipode $S$ to the relations (3). We assume that $A$ can be obtained from $A(R)$ (by imposing determinant-type relations or by inverting a determinant etc) in a way consistent with the coaction. This is true, for example, for $R$ matrices that are regular in the sense of Section 3 below. □

The (standard) rationale behind these examples is from non-commutative geometry, as explained in detail in [14, Sec. 6.3.2] specifically for examples of the above type. The point is that entries of $t$ are non-commutative versions of the tautological functions $t^i_j(A) = A^i_j$ for actual matrices $A$, while similarly $v^i$ and $x_i$ are non-commutative versions of $v^i(V) = V^i$ and $x_i(X) = X_i$ for actual column and row vectors $V, X$. Thus $A(R), V(R), V^*(R)$ are non-commutative versions of $C(M_n), C(\mathbb{R}^n), C(\mathbb{R}^n)$. The usual action $\mathbb{R}^n \times M_n \rightarrow \mathbb{R}^n$ for example appears in terms of these as a right comodule algebra structure $C(\mathbb{R}^n) \rightarrow C(\mathbb{R}^n) \otimes C(M_n)$. It is just this structure which we keep in the quantum setting (of type $R$).

Note that this could for example happen as a result of actual quantization of a commutative algebra of observables of systems on $\mathbb{R}^n$ and $M_n$. In this case the $t^i_j$ etc are quantum observables and become operators. Thus we can think of the $t, x, v$ as operator-valued matrices, covectors, vectors, in spite of their origin as quantized tautological functions (this is a standard point of view in quantum mechanics).

This all seems very reasonable, but let us note that

Lemma 2.6 The subalgebra of $V(R) \otimes V^*(R)$ with generators 1 and

$$v x = \begin{pmatrix} v^1 x_1 & \cdots & v^1 x_n \\ \vdots & \vdots \\ v^n x_1 & \cdots & v^n x_n \end{pmatrix}$$

(i.e. with a matrix of generators $v^i x_j$) is a right $A$-comodule under the assignment $(v x)'' = t^{-1} v x t$ but not in general a right $A$-comodule algebra (they do not obey the right relations). Likewise, $A(R)$ itself is a right $A$-comodule under the assignment $t'' = t^{-1} t t'$ where $t'$
denotes the copy of the generators lying in the coacting dual quantum group $A$. But it is not in general a right $A$-comodule algebra.

**Proof** For the first part, since $A$ coacts on $V(R)$ and on $V^*(R)$ separately as above, it must have a tensor product coaction on their tensor product algebra. The problem is that this does not in general give a comodule algebra. The fundamental reason for this is that $x, v$ commute in the tensor product algebra, but $x' = xt, v' = t^{-1}v$ do not generally commute because the matrix entries of $t, t^{-1}$ do not generally commute. For the second part, let us note that every dual quantum group $A$ (here $A(R)$ modulo determinant-type relations to make it a Hopf algebra) coacts on itself by the adjoint coaction. This is the dual notion to the action of any quantum group on itself by the adjoint action. Just as the latter always respects its own multiplication (in the sense of (3)) so the adjoint coaction always respects its own comultiplication (it is a comodule coalgebra). This is true also for $A(R)$ as a comodule coalgebra. However, again because the matrix entries are generally non-commuting, we do not have in general a comodule algebra. 

Thus the situation is not quite as we would hope. One has nevertheless

**Proposition 2.7** The assignment $t = vx$ ($t^i_j = v^ix_j$) is a realization of $A(R)$ in the algebra $V(R_{21}) \otimes V^*(R)$, i.e. gives an algebra homomorphism $A(R) \rightarrow V(R_{21}) \otimes V^*(R)$. Here $R_{21}$ denotes $R$ transposed in the usual way.

**Proof** The relations for $V(R_{21})$ are $v_2v_1 = \lambda Rv_1v_2$. The $v$ commute with the $x$ in the tensor product algebra, so we have $Rv_1x_1v_2x_2 = Rv_1v_2x_1x_2 = \lambda^{-1}v_2v_1x_1x_2 = \lambda^{-1}v_2v_1x_2x_1\lambda R = v_2v_1x_1x_2$ so that the $vx$ realise (3). 

Finally, we note that the element $xv = \sum_i x_ix^i$ in $V^*(R) \otimes V(R)$ is clearly invariant under these transformations of the $x,v$ by the usual computation (even though the entries may be non-commuting). In other words, under the tensor product coaction on $V^*(R) \otimes V(R)$, this element maps to $xv \otimes 1$ in $V^*(R) \otimes V(R) \otimes A$ (it is a fixed point). Again this is what we would like, although let us remark that this $xv$ need not be central in the algebra $V^*(R) \otimes V(R)$ (nor in $V^*(R) \otimes V(R_{21})$) which is a little worrying.
In summary we see that (with a little care because the matrix entries are non-commuting),
the set-up above has some of the usual features of linear algebra. We say that an algebra $V$
transforms as a quantum covector if it is an example of Lemma 2.1, transforms as a quantum
vector if an example of Lemma 2.2 and transforms as a quantum matrix if an example of
Lemma 2.3. On the other hand, there are a couple of alarming features, where the most
naive expectations do not hold. In particular, $A(R)$ itself as well as the ‘quantum rank-one
matrices’ $vx$ do not exactly fulfill the conditions of Lemma 2.3, while the obvious scaler
element is not central.

The problems here are all attributable to the fact that $A(R)$, while it serves well (in a
quotient) as the quantum symmetry of the system, is not covariant under itself. It seems
that in the quantum universe, the role of quantum symmetry (played by the dual quantum
group) and quantum matrix (in the sense of non-commutative geometry) living in that
universe, become disassociated. As explained in the introduction, the braided matrices
$B(R)$ have been introduced in $[18]$ precisely as a covariantized version of $A(R)$ and better
serve the latter role. We see this now in the next section. For classical groups these two
objects coincide.

3 Braided Linear Algebra

In this section we develop quantum linear algebra in a way that is fully covariant under the
dual quantum group $A$ given by $A(R)$ modulo determinant-type relations. As explained in
the last section, this plays the hidden role of a symmetry but the role of matrices itself are
played by $B(R)$. We have

Example 3.1 The algebra $B(R)$ with generators $u = (u'_{ij})$ and relations in 3 forms an
$A$-comodule algebra under the assignment $u' = t^{-1}ut$ as in Lemma 2.3.

Proof This is the raison d’être of the theory of braided groups. The coaction origi-
nates as the adjoint coaction in Lemma 2.6 on $A(R)$, but the relations of the latter were
converted in 11 by a process of transmutation to derive those of $B(R)$ as explained in
13. For a direct proof we have easily $R_{21}t_1^{-1}u_1t_1R_{12}t_2^{-1}u_2t_2 = R_{21}t_1^{-1}u_1t_2^{-1}R_{12}t_1u_2t_2 =$
Here we used (2) in various forms and freely commuted $u_1$ with $t_2$ etc (they live in different algebras and in different matrix spaces). Applying (2) to the result we have similarly $t_2^{-1}t_1^{-1}u_2R_{21}u_1R_{12}t_1t_2 = t_2^{-1}u_2t_2R_{21}t_1^{-1}u_1t_1R_{12}$ so that the transformed $u$ obey the same relations (3). □

This $B(R)$, however, is not a bialgebra in an ordinary sense. With the matrix comultiplication in (3) it becomes one with braid statistics, i.e. we call it the *braided matrices* of type $R$ (to distinguish it from $A(R)$). It nevertheless transforms as a quantum matrix in the sense of Lemma 2.3. The reason for the necessity of a braiding here is not an accident but a fundamental feature of doing quantum linear algebra in a fully covariant way. To explain this let us note that there are situations in linear algebra where we have to make transpositions $V \otimes W \to W \otimes V$, yet when $V, W$ are quantum vectors or covectors such as above, the ordinary transposition map is *not* covariant. For example, under the usual transposition, $x \otimes v \mapsto v \otimes x$ but after a transformation $x' \otimes v' = xt \otimes t^{-1}v \neq t^{-1}v \otimes xt = v' \otimes x'$ precisely because the matrix entries of $t, t^{-1}$ need not commute when they are multiplied together (according to the definition of the tensor product coaction). In covariant quantum linear algebra we are forced to introduce a non-standard ‘transposition’ $\Psi_{V,W} : V \otimes W \to W \otimes V$ which is covariant and still obeys the rules

$$\Psi_{V,W} \otimes Z = \Psi_{V,Z} \Psi_{V,W}, \quad \Psi_{V} \otimes W, Z = \Psi_{V,Z} \Psi_{W,Z}$$

(4)

for any three covariant objects. This means that it does truly behave like transposition. Moreover, this $\Psi$ is required to be functorial, meaning that it must commute in an obvious way with any other covariant linear operations between objects. For example, if we multiply elements in one of our covariant algebras, and then ‘transpose’ the resulting element with an element in another covariant algebra, the result is the same as first ‘transposing’ the factors with the third element, and then multiplying. See [8] for more discussion. The main difference with ordinary transposition is that for general $R$ we are forced to drop $\Psi_{W,V} \Psi_{V,W} = \text{id}_{V,W}$. This means that mathematically all our objects live in a braided (or quasitensor) category and $\Psi$ is called the *braiding* or quasisymmetry.
The category in our case is the category of $A$-comodules and the braiding $\Psi$ exists if $A$ is dual quasitriangular (roughly speaking, the dual of a quantum group with universal $R$-matrix). It is easy to see that $A(R)$ is dual quasitriangular as a bialgebra for any $R$ obeying the QYBE (the essential computations for this were first given in [14, Sec. 3.2.3]). It is however, not automatic that this dual quasitriangular structure projects to the quotients that may be needed to obtain an honest Hopf algebra $A$ (or even that the latter really exists at all). We call $R$ regular if indeed a quotient of $A(R)$ becomes a Hopf algebra $A$ and $R$ extends to a dual quasitriangular structure $R: A \otimes A \to k$ with $R(t_1 \otimes t_2) = R$. We showed in [14, Sec. 3.2.3] (in some form) that this is formally speaking always true, but sometimes these formal expressions can fail. One needs $R$ and various matrices built from $R$ to be invertible. Needless to say, all the standard $R$ matrices are regular in this way, but we do not limit ourselves to the standard case, requiring only that $R$ is regular. Note that in this set-up based on [14, Sec. 3], the normalization of $R$ is determined.

The braiding between quantum matrices and themselves is the same as for the example of $B(R)$ and was already given in [18]. Likewise for vectors with vectors. We summarise these and the other combinations as follows.

**Proposition 3.2** Let $x, v, u$ be any $A$-comodule algebras of covector, vector and matrix type in the sense of Lemmas 2.1-2.3. Their mutual braiding is given by

\[
\Psi(x_i \otimes x_j) = x_n \otimes x_m R^{m}_{n} R^{m}_{j} R^{n}_{j}, \quad \Psi(v^i \otimes v^j) = R^{i}_{m} R^{j}_{n} v^m \otimes v^n \\
= R^{i}_{m} R^{j}_{n} v^m \otimes v^n
\]

\[
\Psi(x_i \otimes v^j) = \hat{R}^{m}_{i} R^{m}_{j} v^m \otimes x_m, \quad \Psi(v^i \otimes x_j) = x_n \otimes v^m R^{-1i}_{m} R^{n}_{j}
\]

\[
\Psi(u_{ij} \otimes x_k) = x_m \otimes u^a_{b} R^{-1i}_{a} R^{m}_{n} R^{b}_{j} R^{n}_{k}, \quad \Psi(x_k \otimes u_{ij}) = u^a_{b} \otimes x_m R^{m}_{n} R^{i}_{j} R^{n}_{k}
\]

\[
\Psi(u_{ij} \otimes v^k) = v^m \otimes u^a_{b} R^{i}_{m} R^{n}_{j} R^{a}_{b} R^{k}_{n}, \quad \Psi(v^k \otimes u_{ij}) = u^a_{b} \otimes v^m R^{k}_{n} R^{i}_{j} R^{n}_{b} R^{a}_{b}
\]

\[
\Psi(u_{ij} \otimes u_{kl}) = u^p \otimes u^m R^{i}_{p} R^{d}_{a} R^{d}_{c} R^{b}_{m} R^{a}_{c} R^{b}_{l} R^{k}_{d}
\]

where $\hat{R} = ((R^{t_2})^{-1})^{t_2}$ and $t_2$ denotes transposition in the last two indices.
Proof The braidings in the proposition are special cases of the braiding in the category of $A$-comodules. The braiding $\Psi_{V,W}$ in general is given by applying the comodule maps to each of $V, W$, transposing the resulting $V, W$ in the usual way and applying the dual quasitriangular structure $\mathcal{R} : A \otimes A \to k$ to the two $A$ factors. Thus $\Psi(x_i \otimes x_j) = \mathcal{R}(t^{m_i} \otimes t^{n_j})x_n \otimes x_m$ which evaluates to the matrix $R$. Likewise $\Psi(x_i \otimes v^j) = \mathcal{R}(t^{m_i} \otimes S t^j u^j) v^m \otimes x_m$, which evaluates to the matrix $\tilde{R}$. The others are similar. This is the method by which $\Psi(u^i_j \otimes u^k_l)$ was initially obtained and then verified directly in [18]. Likewise, we can verify directly that all the above extend to products of the generators and to tensor products according to the desired properties of a braiding. □

As in Section 2 we can adopt a more compact notation in which $\otimes$ is omitted, so that it looks like an algebra product. The rationale behind this is precisely the formation of braided tensor product algebras.

Lemma 3.3 If $V, W$ are two $A$-comodule algebras then $V \otimes W$ defined with multiplication

$$(v \otimes w)(u \otimes z) = v\Psi(w \otimes u)z, \quad v, u \in V, \quad w, z \in W$$

is also an $A$-comodule algebra. Writing $v = v \otimes 1, w = 1 \otimes w$ (so that $v \otimes w = vw$) the braided tensor product algebra structure has the relations of $V$, the relations of $W$ and the cross relations $wu := \Psi(w \otimes u)$. We call these cross relations (expressing $\Psi$) the statistics relations between $V, W$.

Proof This is an elementary first step in the theory of algebras and Hopf algebras in braided categories as in [18] [12]. Since the multiplications of $V, W$ are covariant and $\Psi$ is also covariant, the multiplication of $V \otimes W$ must also be covariant, i.e. it is an $A$-comodule algebra. That this multiplication is associative follows from functoriality of $\Psi$ and its braid relations [18]. □

Thus when we use such statistics, the algebra structure on $B(R) \otimes B(R)$ is different from the usual one, including now the effects of the statistics $\Psi$. Only with respect to this is the comultiplication $\Delta : B(R) \to B(R) \otimes B(R)$ an algebra homomorphism [18]. In this respect
then, $B(R)$ resembles more a super-quantum group than an ordinary one, with even more complicated statistics than in the super case. For other examples we note,

**Lemma 3.4** The statistics relations between quantum spaces of covector, vector and matrix type (with generators $x, v, u$ respectively) are

$$
x_1 x_2 := x_2 x_1 R, \quad v_1 v_2 := R v_2 v_1, \quad x_1 R v_2 := v_2 x_1, \quad v_1 x_2 := x_2 R^{-1} v_1
$$

$$
u_1 x_2 := x_2 R^{-1} u_1 R, \quad v_1 u_2 := R u_2 R^{-1} v_1, \quad R^{-1} u_1 R v_2 := v_2 u_1, \quad x_1 R u_2 R^{-1} := u_2 x_1
$$

$$R^{-1} u_1 R v_2 := u_2 R^{-1} u_1 R$$

The use of $:=$ is to stress that the right hand side is the definition of the left hand side in the tensor product algebra (and not vice-versa if $\Psi^2 \neq \text{id}$).

**Proof** This is simply Proposition 3.2 written in a compact form with the symbol $\Psi$ omitted on the left of each equation and tensor products omitted. The $1, 2$ induced refer to matrix indices. Also, we have to be careful not to use the above relations in the wrong way. The reverse ones, given by $\Psi^{-1}$ are generally different. Thus, $v_1 u_2 := R u_2 R^{-1} v_1$ should not be confused with $v_2 u_1 := R^{-1} u_1 R v_2$. Another way to distinguish them is to label the elements of the second algebra with a $'$ as explained in [18, Sec. 2].

Let us note the formal similarity between these statistics relations and the algebra defining relations in the examples of $V(R), V^*(R), B(R)$. This similarity reflects the sense in which these algebras are all braided-commutative [11, 12, 18]. We are now ready to give braided analogs of the results of Section 2. From Lemma 3.3 we know that $V(R) \otimes V^*(R)$ is an $A$-comodule algebra (i.e. transforms as a quantum matrix) – so long as we use the braided tensor product algebra there is no problem such as in Lemma 2.6.

**Proposition 3.5** The assignment $u = vx = \left( \begin{array}{cccc} v^1 x_1 & \cdots & v^1 x_n \\ \vdots & \ddots & \vdots \\ v^n x_1 & \cdots & v^n x_n \end{array} \right)$ is a realization of $B(R)$ in $V(R) \otimes V^*(R)$, i.e. gives a (covariant) algebra homomorphism $B(R) \to V(R) \otimes V^*(R)$, where the latter is the braided tensor product algebra.
Proof We compute $R_{21}v_1 x_1 R_{12}v_2 x_2 := R_{21}v_1 v_2 x_1 x_2 = v_2 v_1 \lambda^{-1} x_1 x_2 = v_2 v_1 x_2 R_{12} =: v_2 x_2 R_{21}v_1 x_1 R_{12}$. The first and last equalities use the third statistics relation displayed in the preceding lemma (of the form $\Psi(x_i \otimes v^j)$). The middle equalities use the defining relations in the algebras $V(R), V^*(R)$. Hence $v x$ is a realization of the braided matrices $B(R)$. Moreover, this realization is manifestly covariant, so that (by functoriality) it must be fully consistent with the braiding of $u$ with other objects in comparison to the braiding of $v x$ computed from Lemma 3.4. □

This says that the tensor product of a quantum covector with a quantum vector, when treated with the correct braid statistics (i.e. as ‘braided covectors’ and ‘braided vectors’), is a braided matrix. Also,

**Theorem 3.6** The invariant element $x v \in V^*(R) \otimes V(R)$ maps under

$$V^*(R) \otimes V(R) \xrightarrow{\Psi} V(R) \otimes V^*(R)$$

to the invariant element $\Psi(x v) = \text{Tr} v x \vartheta$ where $\text{Tr}$ is the ordinary matrix trace and $\vartheta^i_j = \tilde{R}^i_{k \cdot j}$. This element $\Psi(x v)$ is central in the algebra $V(R) \otimes V^*(R)$. Likewise, $\text{Tr} u \vartheta$ and more generally $\text{Tr} u^n \vartheta$ are invariant and central in $B(R)$.

Proof Firstly, $\Psi(x v)$ must be bosonic (i.e. $A$-invariant) since $\Psi$ is covariant so it must take invariant elements to invariant elements. Computing it from Proposition 3.2 we have $\Psi(x v) = \tilde{R}^i_{k \cdot j} v^j x_i = \text{Tr} \vartheta v x$. In view of the preceding proposition we are led also to propose $\text{Tr} \vartheta u$ as an invariant element. We prove this and that $\text{Tr} \vartheta u$ is central in $B(R)$, which also implies this for its image in $V(R) \otimes V^*(R)$ (and similarly for higher powers of $u$). The proof depends on the theory of dual quasitriangular Hopf algebras, for in any such Hopf algebra there is a linear functional $\vartheta : A \rightarrow k$ defined by $\vartheta(a) = R(a_{(1)} \otimes S a_{(2)})$ and obeying $a_{(1)} \vartheta(a_{(2)}) = \vartheta(a_{(1)}) S^2 a_{(2)}$ where $\Delta a = a_{(1)} \otimes a_{(2)}$ (formal sum). For proof see [12 Appendix], or argue by duality with a well-known result for quasitriangular Hopf algebras. To apply this to $t$ we define the matrix $\vartheta = \vartheta(t)$ so that the above well-known result becomes

$$t \vartheta = \vartheta S^2 t.$$  (5)
We can now compute that \( \text{Tr} \, u^a \vartheta \) transforms to \( \text{Tr} \, (St)u^a t \vartheta = \text{Tr} \, u^a \vartheta (S^2 t) \cdot \text{op}, \) \( St = \text{Tr} \, u^a \vartheta S(tSt) = \text{Tr} \, u^a \vartheta S(1) = \text{Tr} \, u^a \vartheta. \) Here \( \cdot \text{op} \) denotes the reverse multiplication in \( A \) and we used that \( S \) is an antialgebra map. Although it plays the role of inverse, we were careful not to suppose that \( S^2 = \text{id}. \) Clearly \( u^a \) here can be any matrix of generators transforming as a quantum matrix. To prove centrality let us note two other useful properties of \( \vartheta \) as defined above, namely

\[
\vartheta_2 = R \vartheta_2 \tilde{R}, \quad \vartheta_1 = \tilde{R} \vartheta_1 R.
\]

The proof of the first of these is \((R^{-1} \vartheta_2)^a_{b^i_k} = \mathcal{R}(St_a^b \otimes t^i_j) \vartheta_{i k} = \vartheta_{i j} \mathcal{R}(St_a^b \otimes S^2 t^j_k) = \vartheta_{j} \mathcal{R}(t^a_b \otimes St^i_k) = (\vartheta_2 \tilde{R})^a_{b^i_k} \) by (5) and invariance of \( \mathcal{R} \) under \( S \otimes S. \) The proof of the second is similar, \( \vartheta_2 \mathcal{R}(t^i_j \otimes St^a_b) \vartheta_{i k} = \vartheta_{i j} \mathcal{R}(S^2 t^j_k \otimes St^a_b) = \vartheta_{i j} R^{-1} a_b. \) We note also that (3) implies by iteration that

\[
R_{21} u_1 R_{12}^n u_2 = u_2 R_{21} u_1 R_{12} u_2^{-1} = \cdots = u_2^n R_{21} u_1 R_{12}
\]

and applying \( \text{Tr} \) to this we have \( \text{Tr}_2 u_1 R_{12} u_2^n R_{12}^{-1} \vartheta_2 = \text{Tr}_2 R_{21}^{-1} u_2^n R_{21} u_1 \vartheta_2. \) Computing the left hand side with the aid of the first of (6) we have \( u_1 \text{Tr}_2 R_{12} u_2^n \vartheta_2 \tilde{R}_{12} = u_1 \text{Tr}_2 u_2^n \vartheta_2 \tilde{R}_{12} \cdot \text{op}, \) \( R_{12} = u_1 \text{Tr}_2 u_2^n \vartheta_2 \) where \((\tilde{R}_{12} \cdot \text{op}, R_{12})^i_j k_l = \tilde{R}^a_{j b} R^b_{i a} = \delta^i_j \delta^k_l. \) Similarly on the right hand side we move \( \vartheta_2 \) and apply the second of (6) (with permuted indices) to obtain \( \text{Tr}_2 \tilde{R}_{21} \vartheta_2 u_2^n R_{21} u_1 = (\text{Tr}_2 \vartheta_2 u_2^n R_{21} \cdot \text{op}, \tilde{R}_{21}) u_1 = \text{Tr}_2 u_2^n \vartheta_2 u_1. \)

Let us note that this ‘quantum trace’ \( \text{Tr}( ) \vartheta \) is nothing other than a version of the abstract category theoretic trace for any braided category with dual objects. This has been studied previously in, for example, \([13]\) where we gave the anyonic trace as a generalization of the super-trace. The main difference between that setting and the one above is that previously we worked with quantum groups not dual quantum groups, and hence with an element \((SR^{(2)})^R^{(1)} \) rather than \( \vartheta \) above. There is also a change from left-handed to right-handed conventions. The present form is particularly useful because in some cases \( B(R) \) is also isomorphic to important algebras (such as the Sklyanin algebra\([20]\)) as well as, in a quotient, the algebra of \( U_q(g) \). In these cases the quantum trace in the theorem maps to central elements in the algebra, a fact that is already quite well-known in these cases\([3]\).
Our derivation of the quantum trace as the element corresponding to $\Psi(xv)$ from the point of view of braided linear algebra, is a novel interpretation even in these cases.

Related to the invariant trace, should be a braided determinant. The braided determinant $\text{BDET}(u)$ should be bosonic (i.e. have trivial braid statistics) central and group-like according to $\Delta \text{BDET}(u) = \text{BDET}(u) \otimes \text{BDET}(u)$, so that $\text{BDET}(uu') = \text{BDET}(u)\text{BDET}(u')$ when $u, u'$ are treated with the braid statistics from Lemma 3.4. In addition, we can expect that the braided-determinant of a rank one quantum matrix should be zero, i.e.

$$\text{BDET}(vx) = 0. \quad (8)$$

A general treatment of this topic must surely await a treatment of braided exterior algebras, but we shall at least see these properties in some explicit examples in the next section.

Also, now that we have introduced our braided matrices $B(R)$ we can use it to act on covectors and vectors, as well as itself, in a fully covariant way with respect to the hidden dual quantum group symmetry $A$. Thus we have analogs of Examples 2.4,2.5 and Lemma 2.6 as follows.

**Proposition 3.7** The assignment $x' = xu$ makes $V^*(R)$ into a right braided $B(R)$-comodule algebra, i.e. gives a (covariant) algebra homomorphism $B(R) \to V^*(R) \otimes B(R)$. Thus, provided $x, u$ are treated with the correct braid statistics, $x'$ is also a realization of $V^*(R)$.

**Proof** We use the braid statistics $u_1x_2 := \Psi(u_1x_2) = x_2R_1^{-1}u_1R$ from Lemma 3.4, and associativity of the braided tensor product algebra. Thus $x'_1x'_2 = x_1u_1x_2u_2 := x_1x_2R_1^{-1}u_1R_1u_2 = \lambda x_2x_1u_1R_1u_2$ from the relations in $V^*(R)$. Meanwhile, we also have $\lambda x'_2x'_1 = \lambda x_2u_2x_1u_1R_{12} := \lambda x_2x_1R_2^{-1}u_2R_2u_1R_{12}$ using the braid statistics again (with indices permuted). These expressions are equal after using (3). Hence $x'$ is also a realization of $V^*(R)$. The construction is manifestly covariant under the background dual quantum group $A$. $\square$

**Proposition 3.8** Let $B$ be the braided group obtained by quotienting $B(R)$, with braided-antipode $S$. We write $u^{-1} = Su$. The assignment $v' = u^{-1}v$ makes $V(R)$ into a right
braided $B$-comodule algebra, i.e. gives a (covariant) algebra homomorphism $B \rightarrow V(R) \otimes B$. Thus, provided $x, u$ are treated with the correct braid statistics, $v'$ is also a realization of $V(R)$.

**Proof** Firstly, it is necessary to quotient $B(R)$ by suitable ‘braided determinant-type’ relations (or invert suitable elements) such that the braided antipode $S$ exists and is compatible with the braiding (this is possible whenever it is possible for the corresponding ordinary dual quantum group). This is the content of our regularity assumption on $R$.

We denote the resulting braided matrix group by $B$ and the result of the braided antipode by $u^{-1}$. The axioms for it are the same as the usual ones (but with respect to the braided comultiplication), so $u^{-1}u = 1 = uu^{-1}$. Most importantly for us, this map $S$ (like all the braided group maps) is covariant so that it commutes with $\Psi$. This means that the braid statistics of $u^{-1}$ transforming as a quantum matrix (in place of $u$ there). These statistics are essential because the meaning of $u^{-1}v$ is precisely $u^{-1}v := \Psi(u^{-1}v)$ by definition as an element of the braided tensor product algebra $V(R) \otimes B$. We write $u^{-1}v$ with $u^{-1}$ on the left for convenience with regard to its matrix structure, but it officially belongs on the right of the $v$ after using the cross relations. In practice, it is convenient to write the statistics relations in the implicit form $R^{-1}u_1^{-1}Rv_2 := v_2u_1^{-1}$ (i.e. $\Psi(R^{-1}u_1^{-1}Rv_2) = v_2u_1^{-1}$) from Lemma 3.4. Then $v_1v_2' = u_1^{-1}v_1u_2^{-1}v_2 =: u_1^{-1}R_2^{-1}u_2^{-1}R_1v_1v_2 = R_{12}^{-1}u_2^{-1}R_{12}^{-1}u_1^{-1}v_1v_2 = \lambda R_{12}^{-1}u_2^{-1}R_{12}^{-1}u_1^{-1}R_{12}v_2v_1 := \lambda R_{12}^{-1}v_2u_1^{-1}v_1 = \lambda \lambda v_2'v_1'$. We used the relations (3) and the defining relations of $V(R)$, as well as the statistics relations as explained. Thus the $v'$ also realise $V(R)$, and in a manifestly covariant way. □

**Theorem 3.9** $B$ obtained from $B(R)$ acts on itself in the sense that the assignment $u'' = u^{-1}uu'$ makes $B$ into a right braided $B$-comodule algebra, where $u'$ denotes the second (coacting) copy of $B$. Thus, (provided $u, u'$ are treated with the correct braid statistics) $u''$ is also a realization of $B$ and so provides a (covariant) algebra homomorphism $B \rightarrow B \otimes B$. We call it the braided adjoint coaction of $B$ on itself.
Proof As in the previous proposition, the expression $u^{-1}uu'$ means $\Psi(u^{-1}u)u'$ where the statistics relations between $u^{-1}$ and $u$ must be used if we want to exhibit this as an element of $B \otimes B$ with the $u'$ parts living in the second factor of $B$. It is essential to keep these statistics in mind. Again, we need to be careful not to confuse $\Psi$ with $\Psi^{-1}$. In the present computation there is no danger of this because all elements living in the second (coacting) factor of $B \otimes B$ are labeled with a prime. Thus $g'h$ always means $\Psi(g'h)$ for $h$ in the first factor and $g'$ in the second factor of the resulting expression. The prime means there is no danger of confusion with $hg' = h \otimes g'$ in $B \otimes B$, so we will suppress the $:=$ distinction (we could have used a similar device in the proofs of the preceding two propositions). Thus we just work with the associative algebra $B \otimes B$ generated by the relations of $B$ on primed and unprimed variables from (3) and the cross relations

$$R^{-1}u'_1Ru_2 = u_2R^{-1}u'_2R,$$

from Lemma 3.4. Since the braiding is functorial, the same cross relations hold for $u'^{-1}$ in place of $u'$, which is the form that we will use. Then

$$R_{21}u''_1R_{12}u''_2 = R_{21}u''_1R_{12}u''_2 = (R_{21}u''_1R_{12}u''_2)(R_{21}u''_1R_{12}u''_2) = u''_2(R_{12}u''_1R_{12}u''_2)u_1u_1' = u''_2u''_2R_{21}u''_1R_{12}u''_2 = u''_2u''_2R_{21}u''_1R_{12}u''_2.$$

We used only the relations for the braided tensor product $B \otimes B$ in the form described, applied in each expression to the parts in parentheses to obtain the next expression. Thus $u''$ is a (manifestly covariant) realization of $B$. □

The abstract picture behind the theorem is as follows. Just as any dual quantum group coacts on itself by the adjoint coaction, so every Hopf algebra $B$ in a braided category coacts on itself by the braided adjoint coaction. In the quantum group case we saw, as in Lemma 2.6 that this coaction of a dual quantum group on itself does not respect its own algebra structure (unless the dual quantum group is commutative). The same is true in
general in the braided setting, the braided adjoint coaction of $B$ on itself does not in general respect the algebra of $B$ unless $B$ is `braided-commutative' in a certain precise sense. See the appendix. But the braided groups $B$ obtained by transmutation are commutative in precisely the right sense, see \[\text{[1]}\] where we show this (in the form of the braided adjoint actions and braided-cocommutativity rather than coactions as here). The $B$ in Theorem 3.9 is just of this type (it is formally the transmutation of the dual quantum group $A$), and this is the abstract reason behind the result. Thus, dual quantum groups are not full covariant under their own adjoint coaction, but the process of transmutation turns the dual quantum group into (the functions on) an actual group (a braided-commutative ring of functions, like the super-commutative ring of functions on a super-group). The non-commutativity of the dual quantum group is placed now in the braided category (i.e. in the braid statistics) and after allowing for these, the resulting object behaves like a classical (not quantum) group. Because of this, it acts on itself by conjugation just as ordinary (not quantum) groups do. This is the rationale (apart from covariance) behind the introduction of braided groups in \[\text{[1]}\text{ [2]}\text{ [8]}\]. Theorem 3.9 confirms the usefulness of this picture.

4 Examples

In this section we develop several examples of the general covariant-quantum (braided) linear algebra above. Before describing these, we need to make a note about the normalization of the $R$ matrices. In the above we have assumed that $R$ is regular in the sense that it is the restriction to the generators $t$ of a dual quasitriangular Hopf algebra $A$ obtained from $A(R)$. In general such dual-quasitriangular structures cannot be rescaled (the axioms are not linear). However, if we concentrate on the bialgebra $A(R)$ rather than any special quotient $A$, then we are free to rescale. This is because $A(R)$ is a quadratic algebra (in particular, with homogeneous relations) so that every element has a well-defined degree (the number of generators making up the element). Moreover, the matrix form of the comultiplication means that each factor of $\Delta a$ has the same degree as $a$. Hence if $\mathcal{R}$ is a dual quasitriangular structure then so is $\mathcal{R}'(a \otimes b) = \lambda^{\deg(a) \deg(b)} \mathcal{R}(a \otimes b)$. This is the dual quasitriangular
structure corresponding to the rescaled matrix $R' = \lambda R$.

Thus, at the general bialgebra level we are free to rescale $R$. We have given direct matrix proofs of all the main results above and it is clear that these results hold even at this general bialgebra level (where we do not worry about the existence of $A$ as an honest dual quantum group) provided $R^{-1}, \tilde{R}, \vartheta$ exist with various matrix properties. This is the level at which we will work in the present section. We note that the relations of $B(R)$ and its statistics, as well as the statistics between $u$ and $x, v$ are in any case independent of the normalization of $R$. Meanwhile, the relations of $V(R), V^*(R)$ already have a specific parameter $\lambda$ to accommodate different normalizations, so that only the statistics relations of $V(R), V^*(R)$ are affected. These do depend on the normalization of $R$, but let us note that $V(R), V^*(R)$ are again quadratic algebras (with homogeneous relations). Clearly, if $\Psi$ is a braiding on them then $\Psi'(x \otimes y) = \lambda^{\deg(x) \deg(y)} \Psi(x \otimes y)$ on homogeneous elements $x, y$ is also a braiding. This is all that happens when we change the normalization of $R$, and in the examples below we can exploit it to put the braidings on $V(R), V^*(R)$ in the simplest form.

Our examples are as follows. We begin with the obligatory example of the standard $SL_q(2)$ $R$-matrix. It demonstrates the features of the general standard $R$ matrices also. We next give its two-parameter variant, the $GL_{p,q}$ $R$-matrix studied in [1][24] and elsewhere. This is followed by the non-standard variant related to the Alexander-Conway knot polynomial (where the braided linear algebra reduces to super-linear algebra as $q \to 1$). Finally, we study an $R$-matrix connected with the 8-vertex model. Its dual quantum group $A(R)$ was recently studied in [4]. Each of these examples demonstrates a different aspect of the theory. All the examples have $4 \times 4$ $R$-matrices and we denote the four braided matrix generators by $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We denote the two covector generators by $x = (x \ y)$ and the two vector generators by $v = \begin{pmatrix} u \\ v \end{pmatrix}$. Some of the computations have been done with the assistance of the computer package REDUCE. For simplicity we state the results over a field of characteristic zero, such as $k = \mathbb{C}$. 
4.1 Standard R-Matrix

Here we note how the constructions of braided (i.e., covariant-quantum) linear algebra look for the standard solution of the QYBE corresponding to the dual quantum group $SL_q(2)$ and to the Jones knot polynomial. This provides orientation for the non-standard examples that follow. We take the normalization that gives (with $\lambda = 1$) the standard (not fermionic) quantum planes for the vectors and covectors. This is

$$ R(q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 1 - q^{-2} & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & q^{-2} - 1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \vartheta = \begin{pmatrix} q^{-2} & 0 \\ 0 & 1 \end{pmatrix} $$

and $R^{-1} = R(q^{-1})$. The braided matrices $B(R) = BM_q(2)$ were given in [18] along with their statistics $\Psi$. We do not repeat its details here, but note only that all the algebra relations and braiding depend on $q^2$ and not directly on $q$ itself. This is also true for the invariant trace element $\text{Tr} \ u \vartheta = q^{-2}a + d$ and for the braided determinant

$$ \text{BDET}(u) = ad - q^2 cb \quad (10) $$

found in [18]. In [19] we show that the algebra $BM_q(2)$ (with some elements taken invertible) is isomorphic to the degenerate Sklyanin algebra and $\text{Tr} \ u \vartheta$ and $\text{BDET}(u)$ become its two Casimirs. After setting $\text{BDET}(u) = 1$ we also obtain $BSL_q(2) \cong U_q(sl(2))$ as an algebra and its quadratic Casimir as usual.

The braided algebra of covectors $V^*(R)$ is

$$ xy = q^{-1}yx \quad (11) $$

$$ \Psi(x \otimes x) = x \otimes x, \quad \Psi(x \otimes y) = y \otimes xq^{-1} $$

$$ \Psi(y \otimes x) = x \otimes yq^{-1} + (1 - q^{-2})y \otimes x, \quad \Psi(y \otimes y) = y \otimes y \quad (12) $$

Recall that there are various notations for the braiding. For example, written as the statistics relations (the cross relations in the algebra $V^*(R) \otimes V^*(R)$) it is $x'x = xx'$, $x'y = q^{-1}yx'$, $y'x = xy'q^{-1} + (1 - q^{-2})yx'$ and $y'y = yy'$ as explained above. We see that over $\mathbb{C}$ the algebra is that of the standard quantum plane $\mathbb{C}^2_{q^{-1}}$. The vectors $V(R)$ are similar, namely

$$ vw = qvw \quad (13) $$
\[ \Psi(v \otimes v) = v \otimes v, \quad \Psi(v \otimes w) = w \otimes v q^{-1} + (1 - q^{-2}) v \otimes w \]
\[ \Psi(w \otimes v) = v \otimes w q^{-1}, \quad \Psi(w \otimes w) = w \otimes w. \] (14)

This is isomorphic to the covectors with \( w, v \) in the role of \( x, y \) (note the reversal).

The cross relations in \( V(R) \otimes V^*(R) \) (i.e. the braidings \( \Psi(x \otimes v) \) etc) are

\[ xv := vx, \quad xw := qwx, \quad yv := qvy, \quad yw := wy + (q^{-2} - 1) vx \] (15)

These relations together with the algebra relations (11)(13) give the algebra \( V(R) \otimes V^*(R) \).

Our general theory says that \( BM_q(2) \) (i.e. the degenerate Sklyanin algebra) is realized in this braided tensor product. We compute BDET(\( u \)) in this realization, i.e. BDET(\( vx \)) as an element of \( V(R) \otimes V^*(R) \). We have \( a = vx, b = vy \) etc so that BDET(\( vx \)) = \( vxvy - q^2 wxvy := qvwxy - q^2 wvxy = 0 \) using the relations in \( V(R) \otimes V^*(R) \).

Thus the BDET on \( BM_q(2) \), in addition to being (central) bosonic and group-like, vanishes on rank-one matrices as in (8).

The braiding \( \Psi(v \otimes x) \) etc in the other direction (the cross relations in \( V^*(R) \otimes V(R) \)) are similar but different and easily computed in the same way, as are the braidings \( \Psi(u \otimes v) \) and \( \Psi(v \otimes u) \) from Lemma 3.4. With these braid statistics, one can verify the action of \( BSL_q(2) \) on the braided vectors and covectors, and on itself as in Theorem 3.9. In another normalization we have ‘fermionic’ versions of the above. Some similar results apply for all the standard \( R \)-matrixes corresponding to simple Lie algebras \( g \).

### 4.2 2-Parameter Solution

Here we note the results for the 2-parameter solution of the QYBE leading to the quantum matrices \( M_{p,q}(2) \) and dual quantum group \( GL_{p,q}(2) \) after inverting some elements. This dual quantum group has been studied by several authors, notably [1][2]. In the normalization that we use we have,

\[
R(p, q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 1 - pq & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p^{-1} & pq - 1 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \varphi = \begin{pmatrix} pq & 0 \\ 0 & 1 \end{pmatrix}
\]
and $R^{-1} = R(p^{-1}, q^{-1})$. Our first task is to compute the braided matrices $B(R) = BM_{p,q}(2)$, say. Using the relations (3) one finds

$$BM_{p,q}(2) = BM_{(pq)^{-1}}(2), \quad BDET(u) = ad - p^{-1}q^{-1}cb, \quad Tr u \vartheta = pqa + d$$

Thus, although the dual quantum group for this solution is different from the $SL_q(2)$ or $GL_q(2)$ case above, the braided group comes out the same. Recall above that $BM_q(2)$, its invariant trace element and braided-determinant etc depended only on $q^2$ (not on $q$ itself). That $q^2$ is factorizing now into $p^{-1}, q^{-1}$. Note also that whereas the ordinary quantum determinant is not central[1], the braided determinant is central in $BM_{p,q}(2)$.

The braided algebra of covectors $V^*(R)$ is

$$xy = pyx$$

$$\Psi(x \otimes x) = x \otimes x, \quad \Psi(x \otimes y) = y \otimes xp$$

$$\Psi(y \otimes x) = x \otimes yq + (1 - pq)y \otimes x, \quad \Psi(y \otimes y) = y \otimes y$$

Let us call this braided algebra $C^2_{p,q}$ if we work over $C$. The $p$ is the quantum parameter controlling the non-commutativity of the algebra, and the additional $q$ (along with $p$) is a parameter appearing in the statistics relations. The vectors $V(R)$ are similar, namely

$$vw = q^{-1}wv$$

$$\Psi(v \otimes v) = v \otimes v, \quad \Psi(v \otimes w) = w \otimes vp + (1 - pq)v \otimes w$$

$$\Psi(w \otimes v) = v \otimes wq, \quad \Psi(w \otimes w) = w \otimes w.$$  

Thus, as braided algebras we have that $w, v$ in place of $x, y$ (note the reversal) generate $C^2_{q,p}$.

The cross relations in $V(R) \otimes V^*(R)$ are

$$xv := vx, \quad xw := p^{-1}wx, \quad yv := q^{-1}vy, \quad yw := wy + (pq - 1)vx$$

These relations together with the algebra relations (16)(18) give the algebra $V(R) \otimes V^*(R)$. The other statistics relations can be obtained similarly. As before, these statistics relations for $V(R) \otimes V^*(R)$ and a computation similar to the preceding example gives that $BDET(vx) = 0$ as it should.
4.3  Alexander-Conway Solution

Here we mention the analogous results for the non-standard $R$-matrix studied by various authors and known to be connected with the Alexander-Conway polynomial. A recent work is [21] (where the full quantum group structure, including the quasitriangular structure, is found). The $R$-matrix in one suitable normalization is

$$R(q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 1 - q^{-2} & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & -q^{-2} \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & q^2 - 1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & -q^2 \end{pmatrix}, \quad \vartheta = \begin{pmatrix} q^2 & 0 \\ 0 & -q^2 \end{pmatrix}$$

and $R^{-1} = R(q^{-1})$. The corresponding braided matrices $B(R)$ (as well as the invariant trace element in this case) were computed in [18]. We assume that $q^2 \neq -1$.

The covectors and vectors as braided algebras are

$$xy = q^{-1}yx, \quad y^2 = 0, \quad vw = qvw, \quad w^2 = 0 \quad (21)$$

$$\Psi(y \otimes y) = -q^{-2}y \otimes y, \quad \Psi(w \otimes w) = -q^{-2}w \otimes w \quad (22)$$

with the other $\Psi$ as in (12)-(14). The statistical cross relations in $V(R) \otimes V^*(R)$ are also modified to

$$xv := vx, \quad xv := qwx, \quad yv := qvy, \quad yw := -q^2wy + (q^2 - 1)vx. \quad (23)$$

In these equations the main difference from the standard example in Section 4.1 is that $y$ and $w$ become $q$-deformations of fermionic variables in terms of their various relations and statistics (as do the elements $b$, $c$ of the braided matrices). In the limit $q \rightarrow 1$, the braided matrices $B(R)$ become the super-matrices $M_{1|1}$ [18], and $V(R), V^*(R)$ become 1|1-super planes. In another choice of normalization, it is the $x, v$ rather than the $y, w$ that become ‘fermionic’. Note that a connection between this Yang-Baxter matrix and super-symmetry is well established in a physical way in [8], but here we see the connection at the level of elementary $q$-deformed super-linear algebra.

4.4  8-Vertex Solution

Here we give the details for a less-well known $R$-matrix related to the 8-Vertex model in statistical mechanics. Its bialgebra $A(R)$ was studied recently in [4] and is non-commutative.
The $R$-matrix for our purposes is

$$R(q) = (q + 1)^{-1} \begin{pmatrix} 1 & 0 & 0 & nq \\ 0 & m & q & 0 \\ 0 & q & m & 0 \\ nq & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{R} = R^{-1} = R(-q), \quad \vartheta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $q^2 \neq 1$ and $m^2 = 1 = n^2$. Our first goal is to compute $B(R)$ from (3). After some computation one finds

$$\{a, b, c, d\} \text{ commute, } \quad b^2 = c^2, \ a^2 = d^2, \ ac = mnbd, \ cd = mnba$$

where $\vartheta = (1, 0)$ and $m = 1$. Our first goal is to compute $B(R)$ from (3). After some computation one finds

$$\{a, b, c, d\} \text{ commute, } \quad b^2 = c^2, \ a^2 = d^2, \ ac = mnbd, \ cd = mnba$$

so that the braided matrices, in addition to being ‘braided commutative’ are actually commutative! The braiding from Proposition 3.2 is however, non-trivial. To describe it, it is convenient to choose new generators

$$D = d - a, \quad B = b - mnc, \quad C_1 = d + a, \quad C_2 = b + mnc.$$ 

In these variables, the relations of $B(R)$ are

$$BC_i = 0, \quad DC_i = 0; \quad B(R) = k[B, D] \oplus k[C_1, C_2]$$

where we mean that $B(R)$ is generated by polynomials in $B, D$ and by polynomials in $C_1, C_2$, and apart from the identity, the product of an element in one polynomial algebra with an element from the other is zero. This means that (apart from the identity element, which is common to both), the algebra splits as a direct sum. The underlying variety can be thought of as the union of two planes, one at $C_1 = C_2 = 0$ and the other at $B = D = 0$. The element $C_1$ is the invariant trace element and so is necessarily bosonic (i.e. has trivial braiding with everything else), but it turns out that $C_2$ is also bosonic. The remaining statistics between $B, D$ take the form

$$\Psi(B \otimes B) = \alpha B \otimes B + \beta mn D \otimes D, \quad \Psi(B \otimes D) = \alpha D \otimes B + \beta B \otimes D$$

$$\Psi(D \otimes B) = \alpha B \otimes D + \beta D \otimes B, \quad \Psi(D \otimes D) = \alpha D \otimes D + \beta mn B \otimes B$$

$$\alpha = \frac{q^4 + 6q^2 + 1}{(q^2 - 1)^2}, \quad \beta = \frac{4q(q^2 + 1)}{(q^2 - 1)^2}.$$ 

In fact, the matrix describing these braid statistics is of the same type as $R$ itself, with new values of parameters $n' = nm, q' = \beta/\alpha$ and $m' = 1$. Using the relation $\alpha^2 - \beta^2 = 1$ between
the rational functions \( \alpha, \beta \), it is easy to see that \( D^2 - mnB^2 \) and \( C_1^2 - mnC_2^2 \) are bosonic. The matrix comultiplication \( \Delta \) means that these elements are not themselves group-like but noting that \( \alpha + \beta = (\frac{q+1}{q-1})^4 \), one finds that the combination

\[
\text{BDET}(u) = \frac{q^2 + 1}{(q-1)^2} \left( \frac{q^2 + 1}{q-1} \right) (D^2 - mnB^2) = \frac{q^2 + 1}{(q-1)^2} (ad - bc) - \frac{2q}{(q-1)^2} (a^2 - mnb^2)
\]

is group-like. Recall that \( \Delta \) extends to products as an algebra homomorphism to the braided tensor product. This (27) is the braided-determinant for \( B(R) \). If we set \( \text{BDET}(u) = 1 \) we obtain a braided group with braided-antipode

\[
S \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = (q-1)^{-2} \left( \begin{array}{cc} (q^2 + 1)d - 2qa & -(q^2 + 1)b + 2mnqc \\ -(q^2 + 1)c + 2mnb & (q^2 + 1)a - 2qd \end{array} \right).
\]

Thus completes our description of the braided-matrices and braided group for this \( R \)-matrix. Its classical limit is at \( q = 0 \), with another classical limit (with the same braided matrices and braided group) at \( q = \infty \) in a suitable sense.

The covectors \( V^*(R) \) and vectors \( V(R) \) for the above normalization and for \( q \neq 0 \) are given by

\[
xy = myx, \quad x^2 = ny^2, \quad vw = muv, \quad v^2 = nw^2
\]

\[
\Psi(x \otimes x) = x \otimes x + nqy \otimes y, \quad \Psi(x \otimes y) = y \otimes xm + qx \otimes y
\]

\[
\Psi(y \otimes x) = x \otimes ym + qy \otimes x, \quad \Psi(y \otimes y) = y \otimes y + nqx \otimes x
\]

with braiding on \( V(R) \) given by the same formulae with \( v, w \) in the role of \( x, y \). We have suppressed an overall factor \( (q + 1)^{-1} \) on the right hand sides. The statistics relations between these two braided algebras, i.e. the cross relations in \( V(R) \otimes V^*(R) \) are

\[
xv := vx - qwy, \quad xv := mwx - nqvy, \quad yv := mvy - nqw, \quad yw := wy - qvx
\]

with an overall factor \( (1 - q)^{-1} \) suppressed on the right hand sides. Using these and the relations (29) in each algebra we can easily verify that \( a, b, c, d \) when realized in \( V(R) \otimes V^*(R) \) really are mutually commutative as they must be by Proposition 3.5. For example, \( ab = vxvy := v^2xy - qvwy^2 = v^2myx - qvwx^2 =: vyvx = ba \) etc. Also, we can compute

\[
D^2/2 = a^2 - ad = vxvx - wyvx := v^2x^2 - qvwx - mwx + nqx^2 = (1 + q)(v^2x^2 - wvx)
\]
while similarly $B^2/2 = b^2 - mnb = vwy - mnvywx := v^2g^2m - nqvwxy - mnvyx + mnqv^2x^2 = mnD^2/2$. Thus $D^2 - mnB^2 = 0$ in this realization of $B(R)$. Likewise, one computes that $C_i^2 - mnC_j^2 = 0$ in this realization. Hence we have $BDET(vx) = 0$ as it should on our braided rank-one matrices.

5 Transmutation by Sewing

In Section 2 we have described quantum linear algebra and in Section 3 developed a covariant braided version based on the braided matrixes $B(R)$ acting rather than the dual quantum group $A(R)$. By way of concluding remarks we now study further the process of transmutation that relates the two. The situation here for general dual quantum groups is given in [11][12], but we want to note the form that it takes in the matrix case.

Firstly, we recall that a dual quantum group $A$ (in the strict sense) comes equipped with a dual quasitriangular structure $\mathcal{R} : A \otimes A \to k$ obeying some obvious axioms dual to those of Drinfeld for a quasitriangular Hopf algebra. For $A(R)$ it is given by the matrix $R$ in the generators and extended in such a way that $\mathcal{R}(( ) \otimes t^i_j)$ is a matrix representation of $A$, and $\mathcal{R}(t^i_j \otimes ( ))$ is a matrix anti-representation. We showed this in some form in [14, Sec. 3.2.3] and called it the bimultiplicativity property of $\mathcal{R}$ in [10, Sec. 4.1]. See also [9] and others.

Now according to [12] the structure of $B(R)$ can be realised in the linear space of $A(R)$ but with a modified product, which we will denote explicitly by $\cdot$ to distinguish it. It is not necessary here for $A(R)$ itself to be a Hopf algebra, as long as $R$ is regular so that $A(R)$ has a quotient which becomes a dual quasitriangular Hopf algebra $A$. We transmute with respect to the bialgebra map $A(R) \to A$. In our case it comes out from [12] as

$$u^i_j = t^i_j, \quad u^i_j\cdot u^{k}_l = t^a_i t^d_j R^i_a c\tilde{R}^b_j c \quad (i.e. \quad u_1 \cdot R u_2 = R t_1 t_2).$$

(32)

Thus, the generators can identified but not their products. This is why the $u$ transform in the same way as the $t$ under the quantum adjoint coaction, but only the $\cdot$ multiplication is covariant. What does $\cdot$ look like on general elements, viewed as a modified multiplication on $A(R)$?
To explain this we define the ‘partition function’ \( Z_{R(A, B)} \) as in [14, Sec. 5.2.1] by

\[
Z_{R(A, B)} = R_{m_1 n_1}^{a_1 b_1} R_{m_2 n_2}^{a_2 b_2} \cdots R_{m_{N-1} n_{N-1}}^{a_{N-1} b_{N-1}}
\]

\[
R_{m_1 n_1}^{a_1 b_1} R_{m_2 n_2}^{a_2 b_2} \cdots R_{m_{N-1} n_{N-1}}^{a_{N-1} b_{N-1}}
\]

\[
R_{m_N n_N}^{a_N b_N}
\]

where the \( A = (a_1, a_2, \cdots, a_M) \) etc are multi-indices (arranged consecutively following any marked orientation of the edge of the lattice). We also write \( t_{IJ} = t_{i_1 j_1} t_{i_2 j_2} \cdots t_{i_M j_M} \) etc as a typical element of \( A(R) \). The general transmutation formula in [12] involves computing such expressions as \( R(t_{IJ} \otimes t_{KL}) \), \( R(t_{IJ} \otimes S t_{KL}) \). Using the bimultiplicativity property of \( R \) explained above, we can factorise such expressions into products of \( R \) and \( \tilde{R} \) respectively.

Computing in this way, we obtain

\[
t_{IJ} t_{KL} = t_{AB} t_{DL} Z_{R(A, B)} Z_{\tilde{R}(C, F)}
\]

We gave a similar ‘partition function’ description of the category-theoretic rank or ‘quantum dimension’ of \( A(R) \) in [11]. The present expression suggests a possible interpretation of the transmutation of the usual \( A(R) \) to the braided matrices \( B(R) \) in terms of a statistical transfer matrix with the input and output states appearing on the boundary of the lattice. This is a little like the definition of vertex operators in string field theory, as is perhaps the factorization into \( Z_R \) and \( \tilde{Z}_R \). Such a physical interpretation is an interesting direction for further work.

**A Diagrammatic Proof of Braided Adjoint Coaction**

In this section we develop some of the abstract picture underlying Theorem 3.9. We have given a direct matrix proof in the text but mentioned that the underlying reason why it works is that \( B(R) \), unlike \( A(R) \), is a (braided)-commutative object in a certain sense, much as a super-group is super-commutative. The general setting for developing this remark is that of braided monoidal categories and allows us to give a diagrammatic proof of the result for any braided group (not just of matrix type). Since the tensor products here will always
be braided, we will write simply $\otimes$ rather than any special notation such as $\otimes$. For a formal treatment of braided monoidal categories, see [7].

There is a standard diagrammatic notation for working with structures in braided categories, which we will use here also. We write all morphisms pointing downwards and write $\Psi, \Psi^{-1}$ as braids, $\Psi = \downarrow, \Psi^{-1} = \uparrow$. Other morphisms, such as the multiplication $B \otimes B \to B$ of an algebra $B$ living in the category, are written as vertices with inputs and outputs according to the valency of the map. The functoriality of $\Psi, \Psi^{-1}$ means that we can translate these vertices through the braid crossings (without cutting any paths). For example, using this notation, it is easy to see that if $B$ is an algebra in the category then the multiplication on $B \otimes B$ defined with the braid statistics $\Psi$ is associative. We will use such diagrammatic notation freely below. For details, and for the axioms of Hopf algebras in braided categories written out in this way, we refer to [13] [17] where the notation is used extensively. The braided-comultiplication $\Delta : B \to B \otimes B$ and the braided-antipode $S : B \to B$ of a Hopf algebra in the braided category, are of course required to be morphisms (and so represented by 3- and 2-vertices). Again, since all structures are braided, we do not explicitly underline them.

In this notation, a braided group means a pair $(B, \mathcal{O})$ where $B$ is a Hopf algebra in the braided category and $\mathcal{O}$ is a class of right $B$-comodules (also living in the braided category) such that

\[
\begin{align*}
V \otimes B & \quad V \otimes B \\
\beta & = \beta
\end{align*}
\]

for all $(V, \beta)$ in $\mathcal{O}$. One says that $B$ is braided-commutative with respect to a comodule if it obeys this condition. Thus a braided group means a Hopf algebra in the braided category equipped with a class $\mathcal{O}$ of comodules with respect to which it is braided-commutative.

**Proposition A.1** Let $B$ be a Hopf algebra in a braided category. Then $B$ coacts on itself by the braided adjoint coaction defined by $\text{Ad} = (\text{id} \otimes \cdot)(\text{id} \otimes S \otimes \text{id})(\Psi_{B,B} \otimes \text{id})(\Delta \otimes \text{id})\Delta$. 
Figure 1: Proof of braided adjoint coaction

Proof This is depicted in Figure 1. We have to show that \((\text{Ad} \otimes \text{id})\text{Ad} = (\text{id} \otimes \Delta)\text{Ad}\) (and \(\text{id} = (\text{id} \otimes \epsilon)\text{Ad}\), which is easy and left to the reader). The first diagram on the left in Figure 1 depicts \((\text{Ad} \otimes \text{id})\text{Ad}\) according to the diagrammatic notation. The upper and lower parts are each \(\text{Ad}\) as stated in the proposition. Coassociativity of \(\Delta\) means that we could combine \((\Delta \otimes \text{id})\Delta\) as a single vertex with one input and three outputs (but we should be careful to keep their horizontal order). The first equality is this coassociativity again and functoriality of \(\Psi\) to translate the top \(S\) to the left. The second equality is the fact that \(S\) is an anti-coalgebra homomorphism in the sense \(\Delta S = \Psi(S \otimes S)\Delta\) (see [19] for a similar fact with regard to the algebra structure). The last equality is the Hopf algebra axiom that \(\Delta\) is an algebra homomorphism to the braided tensor product algebra, and gives us \((\text{id} \otimes \Delta)\text{Ad}\) as required. ⊓⊔

For the braided groups \((B, \mathcal{O})\) of interest, this canonical braided adjoint coaction does lie in the class \(\mathcal{O}\), i.e. the Hopf algebra \(B\) in the braided category is braided-commutative with respect to its own braided adjoint coaction. One could even require this as an axiom, though we have not done this since the point of view in [12] is more general.

This is a general fact for all braided groups obtained by transmutation of dual quantum groups \(A\), such as of interest in the main text. This process assigns to a dual quantum group \(A\) (with dual quasitriangular structure) a braided group \(B = B(A, A)\) and also to any right \(A\)-comodule a transmuted right \(B\)-comodule in the braided category. Moreover, \(B\) is always braided-commutative with respect to these comodules that arise by transmutation. They constitute a canonical class \(\mathcal{O}\) in this case. Recall that transmutation does not change the
underlying coalgebra, i.e. the coalgebra of $B$ coincides (when the linear spaces are identified) with the coalgebra of $A$, and the transmutation of comodules is simply to view that same linear map which is an $A$-comodule, as a $B$-comodule. Noting this, it is not hard to see that the braided-adjoint coaction in this case is simply the transmutation of the ordinary quantum adjoint coaction of $A$ on itself. A similar computation was made for adjoint actions in [17]. This is the fundamental reason that the braided groups that arise by transmutation are braided-commutative with respect to their own braided adjoint coaction.

**Proposition A.2** Let $B$ be a Hopf algebra in a braided category and assume that it is braided-commutative with respect to its own adjoint coaction (e.g. the braided groups that arise by transmutation). Then Ad is a comodule algebra structure in the braided category, i.e. Ad : $B \rightarrow B \otimes B$ is an algebra homomorphism to the braided tensor product algebra.

**Proof** The proof is shown in Figure 2. The left-most diagram is Ad $\circ \cdot$ where $\cdot$ is the multiplication in $B$. The first and second equality use coassociativity and the Hopf algebra
axiom that $\Delta$ is a homomorphism. The third equality uses the fact that $S$ is an anti-algebra homomorphism (see $[13]$ for a proof). The fourth uses associativity of the multiplication in $B$ and functoriality to rearrange the diagram so that we can recognise a part that is Ad, which we write explicitly in the fifth. The sixth equality uses associativity of the multiplication to write in a form suitable for applying the braided-commutativity condition. Finally, the last equality uses that $B$ is braided-commutative with respect to Ad to obtain precisely $\text{Ad} \otimes \text{Ad}$ followed by the multiplication in $B \otimes B$, as required. \( \square \)

This is the abstract reason for Theorem 3.9. The $B$-comodule algebras in Propositions 3.7, 3.8 are also obtained by transmutation and hence also lie in the class with respect to which $B$ there is braided-commutative.

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