MEAN CURVATURE FLOW SOLITONS FROM SYMMETRY GROUP VIEWPOINT

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Abstract. The symmetry group of the mean curvature flow in general ambient Riemannian manifolds is determined, based on which we define generalized solitons to the mean curvature flow. We also provide examples of homothetic solitons in non-Euclidean surfaces and prove that all the affine solutions to the mean curvature flow are self-similar solutions.

1. Introduction

Since self-similar solutions to the mean curvature flow in Euclidean spaces are significant on several aspects, it is tempting to extend the notion to solitons in general ambient manifolds. Indeed, there have been several successful attempts: Hungerbuehler and Smoczyk [12] considered the group of isometries of the ambient manifold and constructed examples of rotating solitons; Smoczyk [16] further studied closed conformal vector fields on the ambient manifold; A. Futaki, K. Hattori and H. Yamamoto [6] studied mean curvature flow solitons in cone manifolds; J. Alias, J. de Lira, and M. Rigoli [2] introduced a notion of mean curvature flow soliton using vector fields on the ambient manifold including manifolds of constant sectional curvature, Riemannian products and warped product spaces, and particularly investigated the solitons induced by closed conformal vector fields.

In this paper, we take the symmetry group point of view to define solitons to the mean curvature flow. This viewpoint can be traced back to Richard Hamilton’s paper [10] on page 218, which in the authors’ opinion a natural perspective to extend the classical notion, and can be applied to other geometric flows. We can also use this viewpoint to get a refined understanding of certain aspects of the classical self-similar solutions such as Theorem 2 of this paper. Similarly, we can answer a question appears in [16] on page 176: in an ambient Riemannian manifold, are the conformal solutions in the sense of [16] able to move along the integral curve of the corresponding conformal vector field? The answer is generally no, unless the conformal vector field is a homothetic one.

We now give the definition of solitons to any geometric flow. The more precise definition of the mean curvature flow solitons is given in Section 5.

Definition 1.1. A smooth solution to a geometric flow is called a soliton, if it can be generated by a time-parameter subgroup of the symmetry group acting on the initial hypersurface.
Concerning the symmetry group of the mean curvature flow, Peter Olver [15] computed the symmetry group of the curve shortening flow in the plane. K. Chou and G. Li [4] obtained the symmetry group of the generalized curve shortening flow in the plane. In this paper, we determine the symmetry group of the mean curvature flow of codimension one in an ambient Riemannian manifold of any dimension.

Let $F : M^n \times I \rightarrow (N^{n+1}, \bar{g})$, where $I$ is an interval, be a smooth family of hypersurface immersions satisfying

$$
\left( \frac{\partial F}{\partial t} (x,t) \right)^\perp = H(x,t)\nu(x,t)
$$

(1.1)

where $H(x,t)$ is the mean curvature with respect to the unit normal $\nu(x,t)$, and $\perp$ denotes the projection along the normal direction.

**Theorem 1.** The infinitesimal symmetries and symmetry transformations of (1.1) are listed in the following table:

| Infinitesimal Symmetry | Symmetry Transformation |
|-----------------------|------------------------|
| $\partial_t$          | Translations in $t$    |
| $X(M)$                | Diffeomorphisms of $M$ |
| $K(N)$                | Isometries of $N$      |
| $X + 2\lambda t\partial_t$ | Parabolic Rescalings  |

where $X(M)$ and $K(N)$ denote all the smooth vector fields and Killing vector fields on $M$ respectively, and $X$ is any homothetic vector field on $N$ such that the Lie derivative of the metric $\bar{g}$ with respect to $X$ satisfies $L_X \bar{g} = 2\lambda \bar{g}$ for a non-zero constant $\lambda$.

As an application of Theorem 1, we can solve the problem concerning the classical self-similar solutions in Euclidean spaces. For the curve shortening flow in the plane, Halldorsson [9] combines the classical self-similar curves in the plane generated by motions such as rotation [3], scaling [1] and translation [5]. We can define the affine solutions in a similar manner for the mean curvature flow in Euclidean spaces of any dimension, and prove

**Theorem 2.** All the affine solutions to the mean curvature flow are self-similar solutions.

**Remark.** Here we regard the minimal hypersurface, i.e. the stationary solution to the mean curvature flow as a trivial self-similar solution.

Examples of the soliton solutions have been constructed and studied in a wide range of literature, among which we quote [1], [2], [5], [6], [9], [11] and [12]. However, the only homothetic solitons are those in Euclidean spaces which appear in [1] and [11]. In this paper, we provide examples of homothetic solitons in non-Euclidean ambient surfaces.

The organization of the paper is the following. In Section 2, we review some basic results on the symmetry group theory to which our main reference is [14]. In Section 3 and Section
4, we prove Theorem 1 in dimension two and higher dimensions respectively. In Section 5, we formulate the mean curvature flow solitons and construct examples of non-Euclidean homothetic solitons. In Section 6, we prove Theorem 2.

2. Preliminaries

Peter Olver [14] gives a beautiful presentation on the symmetry group theory. For the convenience, we include in this section the necessary concepts and results, and suggest the reader refer to [14] for the detailed accounts.

The basic idea of determining the symmetry group of a PDE system is that we first extend the usual space of variables to the jet space to include partial derivatives, which transforms the original system into a system of algebraic equations, then apply the corresponding results on symmetries of algebraic equations, however, the symmetry of the algebraic system is generally less than that of the original system, so we have to add certain conditions to guarantee the equivalence.

Definition 2.1. Let \( M \) be a smooth manifold and \( G \) be a Lie group or a local Lie group. Suppose that \( \mathcal{U} \) is an open subset of \( G \times M \), such that

\[
\{e\} \times M \subset \mathcal{U}.
\]  

(2.1)

A local group of transformations acting on \( \mathcal{U} \) is defined as a smooth map \( \Omega : \mathcal{U} \rightarrow M \) with the following properties:

(a) If \((h, x) \in \mathcal{U}, (g, \Omega(h, x)) \in \mathcal{U} \) and \((g \cdot h, x) \in \mathcal{U}\), then

\[
\Omega(g, \Omega(h, x)) = \Omega(g \cdot h, x).
\]  

(2.2)

(b) For all \( x \in \mathcal{U} \),

\[
\Omega(e, x) = x.
\]  

(2.3)

(c) If \((g, x) \in \mathcal{U}, \) then \((g^{-1}, \Omega(g, x)) \in \mathcal{U} \) and

\[
\Omega(g^{-1}, \Omega(g, x)) = x.
\]  

(2.4)

Remark. In this paper, by a one-parameter family of transformations, we mean a one-parameter subgroup of a local group of transformations.

Definition 2.2. Let \( G \) be a local group of transformations defined on some open subset \( U \subset \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^q\} \), which is the space of independent and dependent variables of a PDE system. We call \( G \) is a symmetry group of the system, if \( u = f(x) \) is a solution to the system, then whenever \( g \cdot f \) is defined for \( g \in G \), we have \( u = g \cdot f(x) \) is also a solution.

Remark. We only consider the case \( q = 1 \) in the paper, so the variable space becomes \( \{(x, u) \in \mathbb{R}^{n+1}\} \) and the independent variable is also denoted by \( x = (x^1, ..., x^n) \).
Definition 2.3. The $m$-th prolongation of a smooth function $u = f(x)$, which is denoted by $u^{(m)} = \text{pr}^{(m)}f(x)$, is a vector-valued function $\{u_J\}$ for all multi-indices $J = (j_1, ..., j_k)$, and $0 \leq k \leq m$ defined by the following equations

$$u_J := \frac{\partial^k f(x)}{\partial x^{j_1} \cdots \partial x^{j_k}},$$

(2.5)

particularly, we define $u_J = u$ when $k = 0$.

Definition 2.4. Given a local group of transformations $G$ acting on an open subset $U$ of the variable space, and for any smooth function $u = f(x)$ such that $u_0 = f(x_0)$ and $(\tilde{x}_0, \tilde{u}_0) = g \cdot (x_0, u_0)$ is defined for $g \in G$, we define the $m$-th prolongation of $g$ at the point $(x_0, u_0)$ by

$$\text{pr}^{(m)}g \cdot (x_0, u^{(m)}_0) = (\tilde{x}_0, \tilde{u}^{(m)}_0),$$

(2.6)

where

$$\tilde{u}^{(m)}_0 := \text{pr}^{(m)}(g \cdot f)(\tilde{x}_0).$$

(2.7)

Definition 2.5. Suppose $v$ is a vector field on $U \subset \mathbb{R}^{n+1}$, with the corresponding (local) one-parameter transformations $\Omega_\varepsilon$. Then $m$-th prolongation of $v$ at the point $(x, u^{(m)})$, which is denoted by $\text{pr}^{(m)}v(x, u^{(m)})$, is defined by the following equation

$$\text{pr}^{(m)}v(x, u^{(m)}) = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \text{pr}^{(m)}\Omega_\varepsilon(x, u^{(m)})$$

(2.8)

Definition 2.6. A system of differential equations

$$\Phi^\alpha(x, u^{(m)}) = 0, \quad \alpha = 1, ..., l$$

(2.9)

is of maximal rank if the Jacobian matrix of $\Phi := \{\Phi^\alpha\}$

$$J_\Phi(x, u^{(m)}) = \left\{\frac{\partial \Phi^\alpha}{\partial x^i}, \frac{\partial \Phi^\alpha}{\partial u_J}\right\},$$

(2.10)

is of rank $l$ on $S := \{(x, u^{(m)}) : \Phi(x, u^{(m)}) = 0\}$.

Definition 2.7. A system of differential equations $\Phi(x, u^{(m)}) = 0$ is said to be locally solvable at a point $(x_0, u^{(m)}_0) \in S$, if there is a solution $u = f(x)$ of the system defined in a neighbourhood of $x_0$, such that $u^{(m)}_0 = \text{pr}^{(m)}f(x_0)$

Definition 2.8. A system of differential equations is called non-degenerate, if it is both locally solvable and of maximal rank at every point $(x_0, u^{(m)}_0) \in S$.

We have the following necessary and sufficient condition for a group to be the symmetry group.

Theorem 3. Let

$$\Phi^\alpha(x, u^{(m)}) = 0, \quad \alpha = 1, ..., l$$

(2.11)
be a non-degenerate system of differential equations defined on $U \subset \mathbb{R}^{n+1}$. If $G$ is a connected local group of transformations acting on $U$, then $G$ is a symmetry group of the system if and only if for every infinitesimal generator $v$ of $G$, the following are satisfied on $S$,

$$pr^{(m)}v[\Phi(\alpha \cdot x, u^{(m)})] = 0,$$

for $\alpha = 1, \ldots, l$.

We also need a prolongation formula.

**Definition 2.9.** The $i$-th total derivative of a given function $P(x, u^{(m)})$ is defined by the following equation

$$D_i P = \frac{\partial P}{\partial x^i} + \sum_J u_{J,i} \frac{\partial P}{\partial u_J},$$

(2.13)

where $J = (j_1, \ldots, j_k)$, and

$$u_{J,i} = \frac{\partial u_J}{\partial x^i} = \frac{\partial^{k+1} u}{\partial x^i \partial x^{j_1} \cdots \partial x^{j_k}}.$$

(2.14)

The $J$-th total derivative is defined by

$$D_J = D_{j_1}D_{j_2} \cdots D_{j_k}.$$  

(2.15)

**Theorem 4.** Let

$$v = \sum_{i=1}^n \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta(x, u) \frac{\partial}{\partial u}$$

(2.16)

be a vector field defined on the variable space. Suppose that the $m$-th prolongation of $v$ is

$$pr^{(m)}v = v + \sum_J \phi^J(x, u^{(m)}) \frac{\partial}{\partial u_J}.$$

(2.17)

Then the coefficient functions are given by

$$\phi^J(x, u^{(m)}) = D_J \left( \eta - \sum_{i=1}^n \xi^i u_i \right) + \sum_{i=1}^n \xi^i u_{J,i}.$$  

(2.18)

3. **Proof of Theorem 1 in Dimension Two**

In this section, we prove Theorem 1 when the ambient manifolds are surfaces. And in Section 4, we apply the same procedures to all dimensions. The result on surfaces is a little stronger than that of higher dimensions due to technical reasons that will be seen along the proof.

The outline of the proof is given as follows: firstly, we combine the mean curvature flow equation and the metric evolution equation to form a system, and use Theorem 4 and the sufficient part of Theorem 3 to calculate the determining equations; secondly, in order to apply the necessary part of Theorem 3, we have to check the non-degeneracy condition of the system. Since the symmetry group of the system is exactly that of the mean curvature flow, we complete the proof.
Let $N$ be a Riemannian surface with metric $\bar{g}$ and Levi-Civita connection $D$. Suppose that a smooth family of immersed curves $F : M \times I \rightarrow (N, \bar{g})$ is a solution to the mean curvature flow such that
\[
\left( \frac{\partial F}{\partial t} (x, t) \right)^\perp = k(x, t) \nu(x, t),
\]
where $k(x, t)$ is the geodesic curvature of the curves in $N$ with respect to the unit normal $\nu(x, t)$. The induced metric of $M$ is denoted by $g$ and the corresponding Levi-Civita connection is denoted by $\nabla$. We also define notations $F_t(x) := F(x, t)$ and $M_t = F_t(M)$.

**Lemma 3.1.** (Theorem 1.3.13 of [8]) Let $(N^{n+1}, \bar{g})$ be a Riemannian manifold and $M \subset N$ be a connected smooth hypersurface. Then for any point $p \in M$, there exists a neighbourhood $U \subset N$ and a coordinate system $\{x^\alpha\}$, $0 \leq \alpha \leq n$ such that
\[
M \cap U = \{x^0 = 0\}. \tag{3.2}
\]
and
\[
\bar{g}|_U = (dx^0)^2 + \sum_{i,j=1}^{n} \sigma_{ij}(dx^i)(dx^j). \tag{3.3}
\]
The local chart $(U, \{x^\alpha\})$ is usually called a normal Gaussian coordinate system.

By Lemma 3.1, we can choose a normal Gaussian coordinate system $\{x, y\}$ of a neighbourhood $U \subset N$ containing the initial curve locally, such that
\[
\bar{g} = A(x, y) dx^2 + dy^2. \tag{3.4}
\]
We can further choose $x$ to be the arc-length parameter of the initial curve such that $A(x, 0) = 1$. By the following lemma, we can represent $M_t$ locally as graphs. Since the computation of the symmetries is in local, no generality is lost under the assumptions.

**Lemma 3.2.** Let $F(x, t)$ be a solution to the mean curvature flow. During some time interval $I'$ short enough, for each $t \in I'$, $M_t$ is locally the graph of a function $u(x, t)$.

Proof. For any $t_0 \in I$, by Lemma 3.1 we can choose a normal Gaussian coordinates $\{x^\alpha\}$ of some neighbourhood $U \subset N$ such $M_{t_0} \cap U = \{x^0 = 0\}$ and $x = (x^1, ..., x^n)$ is the local coordinates of $M_{t_0}$. Since $\nabla x_0 = 0$ on $M_{t_0}$, there exists a time interval $I'$ short enough such that $\nabla x_0(x, t)$ is bounded on $M_t \cap U$ for any $t \in I'$. Thus $M_t \cap U$ is the graph of a function $x^0 = u(x, t)$.

The tangent vector field to the curve is
\[
F_\ast \left( \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial y}, \tag{3.5}
\]
and the induced metric $g$ on $M$ is then
\[
g = L^2 dx^2, \tag{3.6}
\]
where $L^2 := A + u_x^2$. We choose the unit normal to be
\[ \nu = \frac{1}{\sqrt{AL}} (-u_x \partial x + A \partial y), \] (3.7)
and the corresponding geodesic curvature is given by
\[ k = \frac{\sqrt{A}}{L^3} \left( u_{xx} - \frac{A_y u_x^2}{A^2} - \frac{A_x u_x}{2A} u_x - \frac{1}{2} A_y \right), \] (3.8)
where $A_x$ and $A_y$ are partial derivatives of $A$. The mean curvature flow equation under the graph representation can be written as
\[ \bar{g}(\frac{\partial F}{\partial t}, \nu) = \frac{\sqrt{A}}{L} u_t = k, \] (3.9)
which is equivalent to
\[ u_t = \frac{1}{L^2} \left( u_{xx} - \frac{A_y u_x^2}{A^2} - \frac{A_x u_x}{2A} u_x - \frac{1}{2} A_y \right), \] (3.10)
and we set
\[ \Phi^1 := L^2 u_t - u_{xx} + \frac{A_y}{A} u_x^2 + \frac{A_x}{2A} u_x + \frac{1}{2} A_y. \] (3.11)

The evolution equation of the induced metric under graph representation is given by (cf. [7])
\[ \frac{\partial L^2}{\partial t} = -2L^2 k^2, \] (3.12)
that is
\[ u_x u_{xt} = -L^2 k^2, \] (3.13)
and we set
\[ \Phi^2 := u_x u_{xt} + L^2 k^2. \] (3.14)
Combining the two equations (3.10) and (3.13) into a system
\[ \begin{cases} 
\Phi^1 = 0 \\
\Phi^2 = 0.
\end{cases} \] (3.15)
Generally, we need to find the infinitesimal symmetries of (3.15) in the form of
\[ v := \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}, \] (3.16)
but for the geometric flow, which is invariant under all diffeomorphisms of $M$, we can choose an arbitrary fixed coordinates of $M$ such that $\xi = \xi(x, u)$ and $\tau = \tau(t)$. In fact, given any fixed coordinates $(x, u)$ of $U \subset N$ containing $M$ locally, and suppose $g_\epsilon$ preserves the solution $u(x, t)$ as graphs for some short time interval, then the vector field of
\[ g_\epsilon(t, x, u(x, t)) := (t_\epsilon, x_\epsilon, u_\epsilon(x_\epsilon, t_\epsilon)), \] (3.17)
is
\[ \frac{d}{d\epsilon} \bigg|_{\epsilon=0} g_\epsilon(t, x, u(x, t)) = \frac{dt_\epsilon}{d\epsilon} \bigg|_{\epsilon=0} \frac{\partial}{\partial t} + \frac{dx_\epsilon}{d\epsilon} \bigg|_{\epsilon=0} \frac{\partial}{\partial x} + \frac{du_\epsilon}{d\epsilon} \bigg|_{\epsilon=0} \frac{\partial}{\partial u}. \] (3.18)
Since $x$ and $t$ are independent variables, we prove that $\xi = \xi(x, u)$ and $\tau = \tau(t)$. In dimension two this can also be easily derived from the determining equations, which can be seen along the proof of Theorem 1.

Proof of Theorem 1:

**Step 1.** Applying Theorem 4 to $v$, we obtain the prolongation of the vector $v$

$$pr^{(2)}v = v + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}}$$

(3.19)

where the coefficients are given by the the formula

$$\phi^x = \eta_x + (\eta_u - \xi_u)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t$$

$$\phi^t = \eta_t - \xi_t u_x + (\eta_u - \tau_u)u_t - \xi_u u_x u_t - \tau_u u_t^2$$

$$\phi^{xx} = \eta_{xx} + 2\eta_{xu} u_x - \xi_{xx} u_x - \tau_{xx} u_t + \eta_{uu} u_x^2 - 2\xi_{xu} u_x^2 - 2\tau_{xu} u_x u_t - \xi_{uu} u_x^3$$

$$\quad - \tau_{uu} u_x^2 u_t + \eta_{uu} u_{xx} - 2\xi_{xu} u_{xx} - 2\tau_{xu} u_{xt} - 3\xi_{uu} u_{xx} u_t - \tau_{uu} u_{xx} u_t - 2\tau_{uu} u_x u_{xt}$$

(3.20)

$$\phi^{xt} = \eta_{xt} + \eta_{ux} u_t + \eta_{ux} u_t - \xi_{xt} u_x - \tau_{xt} u_t + \eta_{uu} u_x u_t - \xi_{uu} u_x^2 - \xi_{uu} u_x u_x$$

$$\quad - \tau_{uu} u_x u_t + \eta_{uu} u_{xxt} - \xi_{xu} u_{xxt} - \tau_{uu} u_t u_t^2 - \xi_{uu} u_t u_t^2 - \tau_{uu} u_t$$

$$\quad - 2\xi_{xu} u_{xxt} - \xi_{uu} u_{xxt} - \tau_{uu} u_t u_t^2 - 2\tau_{uu} u_{xt} u_t - \tau_{uu} u_{xt} - \tau_{uu} u_{tt} u_t$$

By the sufficient condition of Theorem 3, we apply $pr^{(2)}v$ to (3.15) to obtain

$$\begin{cases}
pr^{(2)}v[\Phi^1(x, u^{(2)})] = 0 \\
pr^{(2)}v[\Phi^2(x, u^{(2)})] = 0.
\end{cases}$$

(3.21)

which is satisfied whenever $\Phi^1 = 0$ and $\Phi^2 = 0$. This can be regarded as a system of algebraic equations where the dependence of monomials is further restricted to the condition $\Phi^1 = 0$ and $\Phi^2 = 0$.

An observation is that the second equation of (3.21) is immaterial for calculating the infinitesimal symmetries. By straightforward calculation, we obtain that $pr^{(2)}v[\Phi^1(x, u^{(2)})] = 0$ is equivalent to

$$0 = u_{xt} \left( 2u_x \tau_u + 2\tau_u \frac{A}{L^3} + 3u_x^2 \tau_u + 2u_x \tau_x \right)$$

$$+ u_{xx} \left( 3u_x \xi_u + 2\xi_x - \eta_u \right) + u_t F_1(u_x) + F_2(u_x).$$

(3.22)

where $F_1$ and $F_2$ are both polynomials with respect to $u_x$. It can be seen that $\Phi^1 = 0$ and $\Phi^2 = 0$ only provide extra dependence of $u_t$ and $u_{xt}$ as follows

$$\begin{cases}
u_t = \frac{L}{\sqrt{A}} k \\
u_{xu_{xt}} = -L^2 k^2.
\end{cases}$$

(3.23)

Inserting (3.23) into (3.22), we find that among all independent monomials containing $u_{xx}$ there are two terms

$$-3\tau_u \frac{A}{L^3} u_{xx}^2 \quad \text{and} \quad -2\tau_x \frac{u_{xx} u_{xt}^2}{L^4}$$

(3.24)
This implies $\tau_u = 0$ and $\tau_x = 0$ due to $A > 0$. Thus (3.22) contains no terms of $u_xt$, which implies that $\Phi^2 = 0$ provides no further restrictions on (3.22).

**Step 2.** We now focus on $pr^{(2)}v[\Phi^1(x, u^{(2)})] = 0$ to calculate the infinitesimal symmetries. By expanding (3.22) and considering $\tau = \tau(t)$, we get the following PDE system of determining equations by some simple reductions

$$
\begin{align*}
2A\xi_x + A_x\xi + A_u\eta &= A\tau_t \tag{3.25a} \\
A\xi_u + \eta_x &= 0 \tag{3.25b} \\
2\eta_u &= \tau_t \tag{3.25c} \\
(A\xi_u)_u &= A\xi_t \tag{3.25d} \\
\left(\frac{A_u}{A}\right)_u\eta + \left(\frac{A_u}{A}\right)_{x}\xi + \frac{A_u}{A}\xi_u + 2\xi_{xu} + \eta + \frac{A_u}{A}\eta_u &= 0 \tag{3.25e} \\
\left(\frac{A_x}{A}\right)_u\eta + \left(\frac{A_x}{A}\right)_{x}\xi - 2A\xi_t + \frac{A_u}{A}\xi_x + 2\xi_{xx} &= 0 \tag{3.25f} \\
A_uu\eta + A_{ux}\xi + 2A_u\xi_x + 2A\eta_t - A_u\eta_u + \frac{A_x}{A}\eta_x - 2\eta_{xx} &= 0 \tag{3.25g} \\
\tau &= \tau(t). \tag{3.25h}
\end{align*}
$$

We claim that $\eta$ and $\xi$ are independent on $t$. By (3.25c) and (3.25h), we have $\eta_u$ is independent on $u$ and $x$, thus $\eta = \frac{\tau}{2}u + \beta(x, t)$ for some function $\beta(x, t)$. And by (3.25b), we have $A\xi_u$ is independent on $u$, thus $A\xi_u = \alpha(x, t)$ for some function $\alpha(x, t)$, then by (3.25d), we have $\xi = \xi(x, u)$. Differentiating (3.25a) with respect to $u$ and using (3.25e), we can finally obtain $\eta = \eta(x, u)$.

Since the left hand side of (3.25c) does not depend on $t$ and its right hand side depends only on $t$, we know that $\tau_u$ must be a constant, so $\tau = 2\lambda t + c_1$ for some constants $\lambda$ and $c_1$. Therefore, (3.25a),(3.25b) and (3.25c) can be further reduced to

$$
\begin{align*}
2A\xi_x + A_x\xi + A_u\eta &= 2A\lambda \\
A\xi_u + \eta_x &= 0 \tag{3.26} \\
\eta_u &= \lambda.
\end{align*}
$$

In fact, the above equations are exactly the characteristic equations of homothetic vector fields on $N$. Suppose that

$$
X := \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u} \tag{3.27}
$$

is a conformal vector field on $N$ such that $L_X\bar{g} = 2\lambda\bar{g}$, then we can check that $\xi$ and $\eta$ satisfy (3.26) by the formula $\bar{g}(D_WX, Y) + \bar{g}(DY, X) = L_X\bar{g}(W, Y)$ for arbitrary vector fields $W$ and $Y$ on $N$.

Since $\xi(x, u)$ and $\eta(x, u)$) satisfying (3.26) and $\tau = 2\lambda t + c_1$, when $\lambda = 0$, $c_1 = 1$ and $\xi(x, u) = \eta(x, u) = 0$, the infinitesimal symmetry is $\partial_t$ corresponding to a translation in $t$. 


When \( \lambda = c_1 = 0 \), we obtain the Killing vector fields corresponding to the isometries of \( N \). And when \( \lambda \neq 0 \) and \( c_1 = 0 \), the infinitesimal symmetry is \( X + 2\lambda t \partial_t \) corresponding to the parabolic rescalings. It is easy to check that all these transformations indeed preserve the mean curvature flow except maybe the parabolic rescalings. Let \( \Omega_\varepsilon \) be the corresponding transformation acting on \((t, x, u)\) such that

\[
\left. \frac{d\Omega_\varepsilon(t, x, u)}{d\varepsilon} \right|_{\varepsilon=0} = X + 2\lambda t \partial_t. \tag{3.28}
\]

If we assume

\[
\begin{cases}
  t(\varepsilon) := \varphi_\varepsilon(t) \\
  (x(\varepsilon), u(\varepsilon)) := \omega_\varepsilon(x, u)
\end{cases}
\tag{3.29}
\]

then

\[
\frac{dt(\varepsilon)}{d\varepsilon} = 2\lambda t(\varepsilon). \tag{3.30}
\]

Noting that \( \varphi_0(t) = t \), we obtain

\[
\varphi_\varepsilon(t) = e^{2\lambda \varepsilon} t. \tag{3.31}
\]

Thus

\[
\Omega_\varepsilon(t, x, u) = (e^{2\lambda \varepsilon} t, \omega_\varepsilon(x, u)), \tag{3.32}
\]

where

\[
\left. \frac{d\omega_\varepsilon(x, u)}{d\varepsilon} \right|_{\varepsilon=0} = X. \tag{3.33}
\]

Since \( X \) is a homothetic vector field, we see that \( \omega_\varepsilon \) acting on \( N \) is a smooth family of homothetic transformations. We can calculate the conformal factor as follows.

**Lemma 3.3.** Suppose \( \omega^*_\varepsilon \bar{g} = c^2(\varepsilon) \bar{g} \) and \( c(\varepsilon) > 0 \), then \( c(\varepsilon) = e^{\lambda \varepsilon} \).

Proof. This is directly from the definition of Lie derivative. Since

\[
(L_X)^k \bar{g} = \lim_{\varepsilon \to 0} \frac{(\omega^*_\varepsilon - Id)^k \bar{g}}{\varepsilon^k} = (2\lambda)^k \bar{g}, \tag{3.34}
\]

and by induction on \( k \) we have

\[
\left. \frac{d^k(\omega^*_\varepsilon \bar{g})}{d\varepsilon^k} \right|_{\varepsilon=0}. \tag{3.35}
\]

We complete the proof using Taylor’s expansion of \( \omega^*_\varepsilon \bar{g} \)

\[
\omega^*_\varepsilon \bar{g} = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \left. \frac{d^k(\omega^*_\varepsilon \bar{g})}{d\varepsilon^k} \right|_{\varepsilon=0} \bar{g} = e^{2\lambda \varepsilon} \bar{g}. \tag{3.36}
\]
Since the homothetic transformations $\omega_\varepsilon$ of $N$ can be regarded as equipping $N$ with a smooth family of metric $\omega_\varepsilon^*g$, we obtain by Lemma 3.3

\begin{align}
\nu|_{\omega_\varepsilon^*g}(x,u) &= e^{-\lambda_\varepsilon}\nu|_g(x,u), \\
H|_{\omega_\varepsilon^*g}(x,u) &= e^{-\lambda_\varepsilon}H|_g(x,u),
\end{align}

that is

\begin{align}
(H\nu)|_{\omega_\varepsilon^*g}(x,u) &= e^{-2\lambda_\varepsilon}(H\nu)|_g.
\end{align}

Therefore, considering (3.32), $\Omega_\varepsilon$ indeed preserve the mean curvature flow. In summary, besides the diffeomorphisms we have four types of infinitesimal symmetries which span the Lie algebra of the symmetry group.

**Step 3.** In order to apply Theorem 3 to complete the proof, it remains to check the non-degeneracy condition. Since

\begin{align}
\Phi^1 &= L^2(u_t - \frac{L}{\sqrt{A}}k) \\
\Phi^2 &= u_xu_{xt} + L^2k^2,
\end{align}

we have

\begin{align}
\frac{\partial\Phi^1}{\partial u_t} = L^2.
\end{align}

and

\begin{align}
J_{\Phi^2}(x,t,u^{(2)}) = 0
\end{align}

on $\{(x,t,u^{(2)}) : \Phi^1 = 0, \Phi^2 = 0\}$ if and only if $u_x \equiv 0$, i.e. $k \equiv 0$. By Definition 2.6, the system is of maximal rank whenever the solution is not a geodesic.

By Definition 2.7, in order to check the local solvability condition, we have to find a solution $u = f(x,t)$ of the flow such that $u_0^{(2)} = pr^{(2)}f(x_0,t_0)$, for any given data

\begin{align}
(x_0,t_0,u_0, (u_t)_0, (u_x)_0, (u_{xx})_0, (u_{xt})_0, (u_{tt})_0),
\end{align}

satisfying

\begin{align}
\begin{cases}
(u_t)_0 = \frac{L_0}{\sqrt{A_0}}k_0 \\
(u_x)_0(u_{xt})_0 = -L_0^2k_0^2,
\end{cases}
\end{align}

where $(u_t)_0 := u_t(x_0,t_0), A_0 := A(x_0,u_0), L_0 := L(x_0,u_0)$ and $k_0 := k(x_0,u_0)$. Since the flow is invariant under translations in $t$, we can assume $t = 0$. And it suffices to construct a initial curve $f(x,0) = f_0(x)$ such that $u_0^{(2)} = pr^{(2)}f_0(x_0)$. Then we can involve the initial curve to construct a solution $f(x,t)$ to the mean curvature flow due to short-time existence of the solution.

We assume that

\begin{align}
\begin{split}
f_0(x) &= u_0 + (u_x)_0(x-x_0) + \frac{1}{2!}(u_{xx})_0(x-x_0)^2 + \frac{1}{4!}C(x-x_0)^4,
\end{split}
\end{align}

where $C$ is a constant to be determined, and $u = f(x,t)$ satisfies
\[
\begin{align*}
\begin{cases}
  f_t - \frac{L}{\sqrt{A}} k = 0 \\
  f_x f_{xt} + L^2 k^2 = 0 \\
  f(x, 0) = f_0(x)
\end{cases}
\end{align*}
\] (3.46)

It can be easily seen from the definition of \(f_0(x)\) that
\[
f(x_0, 0) = u_0, \quad f_x(x_0, 0) = (u_x)_0, \quad f_{xx}(x_0, 0) = (u_{xx})_0.
\] (3.47)

Since \(f_t - \frac{L}{\sqrt{A}} k = 0\), we see that \(f_t(x_0, 0)\) is completely determined by \(x_0, u_0, (u_x)_0\) and \((u_{xx})_0\), and by the first equation of (3.44), it is easy to check that
\[
f_t(x_0, 0) = (u_t)_0.
\] (3.48)

For the initial value \((u_{xt})_0\), we have to consider the evolution of the metric
\[
f_x f_{xt} + L^2 k^2 = 0.
\] (3.49)

Similarly we see that \(f_{xt}(x_0, 0)\) is also determined by \(x_0, u_0, (u_x)_0\) and \((u_{xx})_0\) and by the second equation of (3.44), we have
\[
f_{xt}(x_0, 0) = (u_{xt})_0.
\] (3.50)

For the initial value \((u_{tt})_0\), we differentiate \(u_t - \frac{L}{\sqrt{A}} k\) with respect to \(t\)
\[
u_{tt} = \left(\frac{L}{\sqrt{A}}\right)_t t + \frac{L}{\sqrt{A}} k_t.
\] (3.51)

and use the evolution equation of the curvature (cf. Lemma 10.7 of [17])
\[
\frac{\partial k}{\partial t} = \frac{1}{L^2} \frac{\partial^2 k}{\partial x^2} + (k^2 + R)k,
\] (3.52)

where \(R\) is the Gaussian curvature of \(N\). By the expression of \(k\), i.e. (3.8), we obtain
\[
\frac{\partial k}{\partial t} = \frac{\sqrt{A}}{L^5} u_{xxxx} + \alpha(u_{xxx}, u_{xx}, u_x, x, u) + \beta(u_{xx}, u_x, x, u),
\] (3.53)

for some function \(\alpha\) and \(\beta\). Inserting (3.53) into (3.51), and noting that
\[
\left(\frac{L}{\sqrt{A}}\right)_t k = \gamma(u_{xx}, u_x, x, u),
\] (3.54)

for some function \(\gamma\), we finally obtain
\[
f_{tt}(x_0, 0) = \gamma((u_{xx})_0, (u_x)_0, x_0, u_0)
\]
\[
+ \frac{C}{L_0^2} + \frac{L_0}{\sqrt{A_0}} \left(\frac{\alpha(0, (u_{xx})_0, (u_x)_0, x_0, u_0) + \beta((u_{xx})_0, (u_x)_0, x_0, u_0)}{L_0}ight)
\] (3.55)

and we can solve \(C\) such that \(f_{tt}(x_0, 0) = (u_{tt})_0\) due to \(L_0 \neq 0\).
4. **Proof of Theorem 1 in any dimensions**

In this section, we set the index notations: $0 \leq \alpha, \beta, \gamma, \ldots \leq n$ and $1 \leq i, j, k, \ldots \leq n$ and use Einstein summation convention unless otherwise stated.

Suppose that a smooth family of hypersurface immersions $F : M^n \times I \to (N^{n+1}, \bar{g})$ satisfies the mean curvature flow equation

$$
\left( \frac{\partial F}{\partial t} (x, t) \right)_\perp = H(x, t) \nu(x, t),
$$

(4.1)

By the same arguments as those in Section 3, we can choose a normal Gaussian coordinate system $\{ x^\alpha \}$ of $U \subset N$ containing $F_0(M)$ locally such that

$$
\bar{g} = (dx^0)^2 + \sigma_{ij}(x^0, x)dx^idx^j,
$$

(4.2)

and the family of hypersurfaces can be represented as graphs $x^0 = u(x^1, \ldots, x^n, t)$ during some short time interval. The induced metric on $M$ can be written as

$$
g_{ij} = u^i u^j + \sigma_{ij},
$$

(4.3)

where $u_i$ is the partial derivative of $u$ with respect to $x^i$. And its inverse is

$$
g^{ij} = \sigma^{ij} - \frac{u^i u^j}{L^2},
$$

(4.4)

where $\sigma^{ij}$ is the inverse of $\sigma_{ij}$, $u^i = \sigma^{ij} u_j$ and $L^2 = 1 + \sigma^{ij} u_i u_j$. The unit normal of the graph is

$$
\nu = (1, -u^1, \ldots, -u^n) \frac{1}{L}.
$$

(4.5)

By the Gauss formula

$$
D \frac{\partial}{\partial x^j} = \nabla \frac{\partial}{\partial x^j} + h(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^j}) \nu,
$$

(4.6)

the second fundamental form $h_{ij} = h(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ of $M$ satisfies

$$
\frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} + \Gamma^{\beta \gamma}_{\beta i} \frac{\partial F^\beta}{\partial x^j} - \Gamma^{\beta \gamma}_{\gamma j} \frac{\partial F^\beta}{\partial x^i} = h_{ij} \nu^\alpha.
$$

(4.7)

Then

$$
h_{ij} = \frac{1}{L} \{ u_{ij} - u_t (\Gamma^i_{0j} u^i + \Gamma^i_{i0} u_j + \Gamma^i_{ij}) + \Gamma^0_{ij} \}.
$$

(4.8)

The mean curvature follows by taking the trace

$$
H = \frac{1}{L} g^{ij} \{ u_{ij} - u_t (\Gamma^i_{0j} u^i + \Gamma^i_{i0} u_j + \Gamma^i_{ij}) + \Gamma^0_{ij} \}.
$$

(4.9)

Since

$$
\bar{g}(\frac{\partial F}{\partial t}, \nu) = \frac{u_t}{L} = H,
$$

(4.10)

we obtain

$$
\Psi^1 := u_t L^2 - (L^2 \sigma^{ij} - u^i u^j) (u_{ij} - \Gamma^i_{0j} u^i u_0 - \Gamma^i_{i0} u_j u_l - \Gamma^0_{ij} u_l + \Gamma^0_{ij}) = 0.
$$

(4.11)
Under the graph representation, the evolution equation of the induced metric
\[ \frac{\partial g_{kl}}{\partial t} = -2H h_{kl} \] (4.12)
becomes
\[ \Psi^{kl} := L^2(u_{klt} + u_{lt}u_k) + 2g^{ij}(u_{ij} - \Gamma^s_{ij}u_su_s - \Gamma^s_{0ij}u_0 u_s - \Gamma^s_{ij}u_0 + \Gamma^0_{ij}) (u_{kl}) 
- \Gamma^s_{0l}u_k u_s - \Gamma^s_{k0}u_l u_s - \Gamma^s_{kl}u_s + \Gamma^0_{kl} = 0. \] (4.13)

Proof of Theorem 1. The principles of calculation are exactly the same as those in the two dimensional case. And by the arguments in Section 3, it suffices to find the infinitesimal symmetries of the form
\[ v := \tau(t) \frac{\partial}{\partial t} + \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta(t, x, u) \frac{\partial}{\partial u}. \] (4.14)

**Step 1.** By Theorem 4, the prolongation of X is
\[ pr^{(2)}v = v + \eta^i \frac{\partial}{\partial u_i} + \eta^j \frac{\partial}{\partial u_j} + \eta^{ij} \frac{\partial}{\partial u_{ij}} + \eta^{it} \frac{\partial}{\partial u_{it}}, \] (4.15)
where the coefficients are computed similarly by the formula in Theorem 4. By Theorem 3, we apply \( pr^{(2)}v \) to \( \Psi^1 \) and \( \Psi^{kl} \)
\[ \begin{cases} pr^{(2)}v[\Psi^1(x, t, u^{(2)})] = 0 \\ pr^{(2)}v[\Psi^{kl}(x, t, u^{(2)})] = 0 \end{cases} \] (4.16)
whenever \( \Psi^1 = 0 \) and \( \Psi^{kl} = 0 \). Inserting \( \Psi^1 = 0 \) into \( pr^{(2)}v[\Psi^1(x, t, u^{(2)})] = 0 \), all the terms containing \( u_{jt} \) are \( L^2g^{ij}u_{ij}(\tau_i + u_i \tau_u) \), which equal zero due to \( \tau = \tau(t) \). Thus \( \Psi^{kl} = 0 \) provides no further dependence.

**Step 2.** To calculate the determining equations, we only have to consider the first equation \( pr^{(2)}v[\Psi^1(x, t, u^{(2)})] = 0 \). We need not find all the independent monomials and their coefficients. Actually in order to prove Theorem 1, it suffices to focus on the terms containing \( u_t \) and the second order derivatives of \( u \). This is the main trick that makes the tedious calculation in higher dimensions relatively easy to handle.

We denote \( L^2g^{ij} \) by \( G^{ij} \), then the expansion of \( pr^{(2)}v[\Psi^1(x, t, u^{(2)})] = 0 \) can be written as
\[ 0 = 2G^{kl}u_{q}(\xi^k + u_k \xi^q) + u_{kl} G^{kl}(u_q \xi^u - \eta_u) + u_k [ -\xi_q G_{kl} - \eta G_{kl}^{q} + u_p \xi^p G^{kl}_{u_q} + u_p u_q \xi^p G^{kl}_{u_u} - \eta_q G^{kl}_{u_u} - u_q \eta_u G^{kl}_{u_q} ] + u_t [ \eta(L^2) u - u_p(L^2) u_q \xi^p - u_p u_q (L^2) u_s \xi^p + (L^2) u_p \xi^p - L^2 u_p \xi^p - L^2 \tau_t ] + (L^2) u_q \eta_q + L^2 \eta_q + u_p (L^2) u_q \eta_u] + F_0(u_1, ..., u_n), \] (4.17)
where \( (L^2) u_q \) and \( G^{kl}_{u_q} \) is partial derivatives with respect to \( u, q \), and \( F_0(u_1, ..., u_n) \) is polynomial with respect to \( u_1, ..., u_n \).

Inserting the dependence relation \( \Psi^1 = 0 \) and collect again the second order derivatives of \( u \), we find that the coefficient of \( u_{kl} \) for each \( (k, l) \) is
\[-\xi^s_\sigma_{sk}^{kl} - \eta \sigma_{ru}^{kl} + \xi_s^{k} \sigma_{s}^{sk} + \xi_l^{s} \sigma_{sl}^{ks} - \tau_t \sigma_{kl}^{kl} = 0, \quad (4.18)\]

where \( \sigma_{sk}^{kl} \) and \( \sigma_{ru}^{kl} \) are partial derivatives of \( \sigma^{kl} \) with respect to \( x^s \) and \( u \) respectively. Since \( \sigma_{kl}^{kl} = -\sigma_{ij}^{\alpha} \sigma_{ik}^{\sigma} \sigma_{jl}^{\eta} \), \( (4.19) \)

\( (4.18) \) becomes

\[ \xi^s (\sigma_{kl})_s + \eta (\sigma_{kl})_u + \xi_l^s \sigma_{ks} + \xi_s^k \sigma_{sl} + \tau_t \sigma_{kl} = 0 \quad (4.20) \]

All terms of the form \( u^p u_{kl} \) are collected as

\[ (\eta^k + \xi^k_u) u_s u_{sk} \]

So

\[ \eta^k + \xi^k_u = 0. \quad (4.22) \]

Lowering the indices, we get for each \( k \)

\[ \eta_k + \xi^s_u \sigma_{sk} = 0. \quad (4.23) \]

All terms of the form \( u^i u^j u_{kl} \) can be divided into two parts: \( u^k u^l u_{kl} \) and \( u^i u^j u_{kl} \) for all \((i, j) \neq (k, l)\). The term of \( u^k u^l u_{kl} \) is

\[ (2\eta_u - \tau_t) u^k u^l u_{kl} \quad (4.24) \]

If we can check that \( u^k u^l u_{kl} \) and \( u^i u^j u_{kl} \) for all \((i, j) \neq (k, l)\) are independent, then we get

\[ 2\eta_u - \tau_t = 0. \quad (4.25) \]

We suppose otherwise there exist constants \( C_0 \) and \( C_{ij}^{pq} \), which are not all zero, such that

\[ C_0 u^k u^l u_{kl} + \sum_{(p, q) \neq (i, j)} C_{pq}^{ij} u^p u^q u_{ij} = 0 \quad (4.26) \]

actually we can further assume that \( C_{pq}^{ij} \) are symmetric both in \((i, j)\) and in \((p, q)\), since \( u^p u^q u_{ij} \) have the same symmetries. Differentiating (4.26) with respect to \( u_{ij} \) we get

\[ C_0 u^i u^j + \sum_{(p, q) \neq (i, j)} C_{pq}^{ij} u^p u^q = 0 \quad (4.27) \]

Differentiating it further with respect to \( u_i \) and by the symmetry of \( C_{pq}^{ij} \) we get for \( q \neq j \)

\[ C_0 u^i u^j + 2 \sum_{q \neq j} C_{iq}^{ij} u^q = 0 \quad (4.28) \]

Since \( u_j \) and \( u_q \) for \( q \neq j \) are independent, we have \( C_0 = 0 \) and \( C_{iq}^{ij} = 0 \), then (4.26) becomes for \( p \neq i, j \) and \( q \neq i, j \)

\[ \sum_{p \neq i, j, q \neq i, j} C_{pq}^{ij} u^p u^q u_{ij} = 0. \quad (4.29) \]

Similarly we can get \( C_{pq}^{ij} = 0 \), which is a contradiction.
In summary, we obtain the following system of equations from (4.20), (4.23) and (4.25)

\[
\begin{align*}
\xi^s(\sigma_{kl})_s + \eta(\sigma_{kl})_u + \xi^s_l \sigma_{ks} + \xi^s_k \sigma_{sl} &= \tau_t \sigma_{kl} \\
\eta_k + \xi^s_k \sigma_{sk} &= 0 \\
2\eta_u &= \tau_t.
\end{align*}
\] (4.30)

It can be checked that the left hand side of equations of (4.30) are exactly the components of $L_X g$, so they are tensor equations which is independent of the choice of coordinates of $M$, thus we can obtain that $\tau_t = 0$ by differentiating the first equation of (4.30) with respect to $t$ and choosing a normal coordinate system $(x, u)$ around a point of $M$. Therefore, for some constant $\lambda$ the system (4.30) can be written as

\[
\begin{align*}
\xi^s(\sigma_{kl})_s + \eta(\sigma_{kl})_u + \xi^s_l \sigma_{ks} + \xi^s_k \sigma_{sl} &= 2\lambda \sigma_{kl} \\
\eta_k + \xi^s_k \sigma_{sk} &= 0 \\
\eta_u &= \lambda.
\end{align*}
\] (4.31)

It can be checked that (4.31) is exactly $L_X g = 2\lambda g$ satisfied by the homothetically conformal vector field.

It remains to check the non-degeneracy condition.

**Step 3.** We first check the local solvability. By the same argument, we can assume without loss of generality that the initial data for the local solvability is

\[
(x_0, 0, u_0, (u_t)_0, (u_i)_0, (u_{ij})_0, (u_{it})_0, (u_{tt})_0),
\] (4.32)

for all $1 \leq i, j \leq n$, where $x_0 = (x_0^1, ..., x_0^n)$, subject to the condition:

\[
\begin{align*}
\Psi^1(x_0, t_0, u_0^{(2)}) &= 0 \\
\Psi^{kl}(x_0, t_0, u_0^{(2)}) &= 0.
\end{align*}
\] (4.33)

for all $k$ and $l$.

We are to construct a solution to the mean curvature flow $u = f(x, t)$ such that $u_0^{(2)} = pr^{(2)} f(x_0, 0)$. We choose local normal coordinate system $\{x^1, ..., x^n\}$ centered at the point $x_0$ of $M$, and define the initial hypersurface by

\[
f_0(x) := f(x, 0) = u_0 + (u_i)_0 (x^i - x_0^i) + \frac{1}{2} (u_{ij})_0 (x^i - x_0^i)(x^j - x_0^j) + \frac{1}{4!} C (x^1 - x_0^1)^4,
\] (4.34)

where $C$ is a constant to be determined. And $u = f(x, t)$ satisfies

\[
\begin{align*}
\Psi^1(x, t, f^{(2)}) &= 0 \\
\Psi^{kl}(x, t, f^{(2)}) &= 0 \\
f(x, 0) &= f_0(x).
\end{align*}
\] (4.35)
Using $\Psi^1(x_0, t_0, u_0^{(2)}) = 0$, and noting that $\Gamma^\alpha_{\beta\gamma}$ and $\sigma_{ij}$ depend only on $(x, u)$, we can obtain for all $i$ and $j$,
\[ f(x_0, 0) = u_0, \quad f_i(x_0, 0) = (u_i)_0, \quad f_{ij}(x_0, 0) = (u_{ij})_0, \quad f_t(x_0, 0) = (u_t)_0. \quad (4.36) \]
By $\Psi^2(x_0, u_0^{(2)}) = 0$ and $\Psi^{kl}(x, f^{(2)}) = 0$, we see that for all $k$
\[ f_{kt}(x_0, 0) = (u_{kt})_0 \quad (4.37) \]

Since
\[ u_{tt} = \frac{\partial L}{\partial t} H + L \frac{\partial H}{\partial t}, \quad (4.38) \]
and
\[ \frac{\partial L^2}{\partial t} = 2u_{kt}u_k, \quad (4.39) \]
according to Lemma 10.7 of [17], we have
\[ \frac{\partial H}{\partial t} = \Delta H + H(|A|^2 + Ric_N(\nu, \nu)) \quad (4.40) \]
Thus there exists a function $\alpha(x, u^{(3)})$, such that at the point $(x_0, u_0)$,
\[ \frac{\partial H}{\partial t}(x_0, u_0) = \frac{1}{L_0} g^{ij}_0 (f_0)_{ijkk} + \alpha(x_0, f_0^{(3)}) \]
\[ = \frac{C}{L_0} + \alpha(x_0, f_0^{(3)}) \quad (4.41) \]
where $L_0 = L(x_0, u_0)$ and $g^{ij}_0 = g^{ij}(x_0, u_0)$. Inserting (4.41) and (4.39) into (4.38) and evaluating at the point $(x_0, u_0)$, since $L_0$ is positive, we can always choose proper $C$ such that $f_{tt}(x_0, u_0) = (u_{tt})_0$.

**Step 4.** Now we check the condition of maximal rank. If there is a point $(x_0, t_0, u_0^{(2)}) \in S := \{(x, t, u^{(2)}) : \Psi^1(x, t, u^{(2)}) = 0, \Psi^{kl}(x, t, u^{(2)}) = 0, 1 \leq k, l \leq n\}$ such that $J_\Psi$ is of maximal rank, then there is a neighbourhood of $(x_0, t_0, u_0^{(2)})$ such that $J_\Psi$ also have maximal rank due to smoothness of $\Psi$, so the condition is fulfilled in that neighbourhood where we can apply Theorem 3. Thus, we only have to consider the case when there is no such a point, in another word, the determinants of all $(n^2 + 1)$ order minor matrices of $J_\Psi$ are zero for every point of $S$.

We define
\[ \Lambda^{(k,l)}_{(i,j)} := \frac{1}{2L^2} \frac{\partial \Psi^{kl}}{\partial u_{ij}} \quad (4.42) \]
and
\[ E^{(k,l)}_i := \frac{1}{2L^2} \frac{\partial \Psi^{kl}}{\partial u_{it}} \quad (4.43) \]
The index $(k, l)$ and $(i, j)$ denote respectively the row and the column of the matrix $\Lambda^{(k,l)}_{(i,j)}$ with the lexicographical order. Then $\Lambda^{(k,l)}_{(i,j)}$ is a $n^2 \times n^2$ matrix and $E^{(k,l)}_i$ is $n^2 \times n$ matrix.
We can calculate the elements of \( \Lambda_{(i,j)}^{(k,l)} \) and \( E_i^{(k,l)} \) as follows.

\[
\Lambda_{(i,j)}^{(k,l)} = \begin{cases} 
H + g^{ij}h_{ij} & (k, l) = (i, j) \\
g^{ij}h_{kl} & (k, l) \neq (i, j),
\end{cases}
\tag{4.44}
\]

where there is no summation in \( g^{ij}h_{ij} \).

\[
E_i^{(k,l)} = \begin{cases} 
u_i & k = l = i \\
\frac{\nu_k}{2} & k \neq l, k = i \\
\frac{\nu_k}{2} & k \neq l, l = i \\
0 & \text{else.}
\end{cases}
\tag{4.45}
\]

For any \( t \in I \), we can choose a local normal coordinates \( \{x_1, ..., x_n\} \) of some open subset \( O \subset M_t \) around any given point \( p \), such that the metric \( g(t) \) and the second fundamental form \( h(t) \) of \( M_t \) can be simultaneously diagonalized at \( p \). Now we replace the column of \( \Lambda_{(i,j)}^{(k,l)} \) with the corresponding the column of \( E_i^{(k,l)} \) for all \( 1 \leq i \leq n \), and denote the new minor by \( \tilde{\Lambda}_{(i,j)}^{(k,l)} \).

**Lemma 4.1.** Each diagonal element of \( \tilde{\Lambda}_{(i,j)}^{(k,l)} \) is the only non-zero element of the corresponding row or column.

Proof. By (4.44), (4.45) and the definition of \( \tilde{\Lambda}_{(i,j)}^{(k,l)} \), we have

\[
\tilde{\Lambda}_{(i,j)}^{(k,k)} = \begin{cases} 
E_i^{(k,k)} = 0, & i = j, k \neq i \\
u_i, & i = j, k = i \\
\Lambda_{(i,j)}^{(k,k)} = 0, & i \neq j
\end{cases}
\tag{4.46}
\]

and for \( i \neq j \),

\[
\tilde{\Lambda}_{(i,j)}^{(k,l)} = \Lambda_{(i,j)}^{(k,l)} \begin{cases} 
H, & (k, l) = (i, j) \\
0, & (k, l) \neq (i, j)
\end{cases}
\tag{4.47}
\]

We note that the diagonal elements of \( \tilde{\Lambda} \) are \( \tilde{\Lambda}_{(i,j)}^{(k,l)} \) with \( (k, l) = (i, j) \). By (4.46), the only non-zero element of \( (k, k) \) row is the \( \{(k, k), (k, k)\} \)-th element which is \( u_i \). By (4.47), the only non-zero element of \( (i, j) \) column for all \( i \neq j \) is the \( \{(k, l), (i, j)\} \)-th element with \( (k, l) = (i, j) \) which is \( H \). Thus we complete the proof.

By Lemma 4.1 and Laplace expansion, the determinant of \( \tilde{\Lambda}_{(i,j)}^{(k,l)} \) is the product of the diagonal elements

\[
\det\{\tilde{\Lambda}_{(i,j)}^{(k,l)}\} = H^{n^2-n} \prod_{i=1}^{n} u_i.
\tag{4.48}
\]
We note that
\[ \frac{\partial \Psi^1}{\partial u_t} \neq 0 \] (4.49)
and for all \( k \) and \( l \).
\[ \frac{\partial \Psi^{kl}}{\partial u_t} = 0, \] (4.50)

By the assumption that each minor of order \( n^2 + 1 \) is degenerate on \( \mathcal{O} \), we obtain
\[ \det \{ \tilde{\Lambda}^{(k,l)} \} = 0. \] (4.51)

which implies \( H = 0 \) or \( u_i = 0 \) on \( \mathcal{O} \) for some \( i \) at each time \( t \in I \). Since the minimal hypersurface \( H = 0 \) is a stationary solution to the mean curvature flow which admits only diffeomorphisms, it is a trivial case for our purpose. Thus we assume \( H \neq 0 \) and \( u_1 = 0 \) without loss of generality.

Taking \( k = l = 1 \) in the evolution equation of the metric, we get at the point \( p \)
\[ 0 = 2u_k u_{1t} = -2Hh_{11} = 2H\mu_1, \] (4.52)
where \( \mu_1 \) is a principal curvature in the direction of \( e_1 \). We have \( \mu_1 = 0 \) due to \( H \neq 0 \), and it is independent on our choice of the coordinates. Thus the \( \{(1,1),(1,1)\} \)-th element of the matrix \( \tilde{\Lambda}^{(k,l)} \) is \( H \), now we replace the column of \( \tilde{\Lambda}^{(k,l)} \) with the column of \( E_1^{(k,l)} \) for \( 2 \leq i \leq n \), and the determinant of the resulting matrix, which is also denoted by \( \tilde{\Lambda}^{(k,l)} \), should also be zero by the degenerate assumption, that is
\[ \det \{ \tilde{\Lambda}^{(k,l)} \} = H^{n^2-n+1} \prod_{i=2}^{n} u_i. \] (4.53)

By the same arguments, we obtain \( u_2 = 0 \). Repeating the process, we obtain that \( u_i = 0 \) for \( 1 \leq i \leq n \), and the determinant of the original matrix \( \Lambda^{(k,l)} \) becomes
\[ \det \{ \Lambda^{(k,l)} \} = H^{n^2}. \] (4.54)

Therefore, we obtain that when the solution is not a stationary minimal hypersurface, the system
\[ \begin{align*}
\Psi^1(x, t, u^{(2)}) &= 0 \\
\Psi^{kl}(x, t, u^{(2)}) &= 0
\end{align*} \] (4.55)
is of maximal rank.
In this section, we derive the characterizing equation of the mean curvature flow solitons and give examples of the homothetic solitons in non-Euclidean surfaces.

We first give the definition of the mean curvature flow solitons

**Definition 5.1.** A smooth solution $F : M^n \times I \rightarrow (N^{n+1}, \bar{g})$ to the mean curvature flow is called a soliton if there exists a one-parameter subgroup $\omega_t, t \in I$ of the symmetry group of the mean curvature flow, such that $F(x,t) = \omega_t \cdot F_0(x)$, where $x \in M$ and $F_0(x) = F(x,0)$.

Let $F(x,t)$ be the mean curvature flow soliton, such that

\[
\begin{cases}
\left( \frac{\partial F}{\partial t} \right)^\perp = H \nu \\
F(x,0) = F_0(x)
\end{cases}
\]

and we assume that $\omega_t$ is a local one-parameter homothetic transformations of $N$ satisfying $\omega_t^* (\bar{g}) = c^2(t) \bar{g}$ for some positive function $c(t)$, and

\[
\begin{cases}
d\omega_t(x) = X(\omega_t(x)) \\
\omega_0(x) = x.
\end{cases}
\]

We call $X$ the homothetic vector field corresponding to the one-parameter homothetic transformations, and we have the following relations

\[
L_X \omega_t^* (\bar{g}) = \lim_{\varepsilon \to 0} \frac{\omega_{t+\varepsilon}^* (\bar{g}) - \omega_t^* (\bar{g})}{\varepsilon} = 2c'(t)c(t)\bar{g}.
\]

By the definition of solitons, we have

\[
\left( \frac{\partial F}{\partial t} \right)^\perp (\omega(x,t), t) = X^\perp(F(x,t)) = H(F(x,t))\nu(F(x,t))
\]

Applying tangential mappings $(\omega_t^{-1})_* = (\omega_{-t})_*$ to the above equation, we obtain

\[
(\omega_{-t})_* (X^\perp(F(x,t))) = H(F(x,t))(\omega_{-t})_* (\nu(F(x,t))).
\]

Since $(\omega_t)_*$ is an isomorphism between $T_{F_0(x)} M$ and $T_{F(x,t)} M$, in addition, the conformal mapping preserves the orthogonality, we have

\[
(\omega_{-t})_* (X^\perp(F(x,t))) = \left[(\omega_{-t})_* (X(F(x,t))) \right]^\perp
\]

and we can also fix an orientation of the hypersurface such that

\[
(\omega_{-t})_* (\nu(F(x,t))) = \frac{1}{c(t)} \nu(x,0).
\]

The corresponding mean curvatures can be related by

\[
H(F(x,t)) = \frac{1}{c(t)} H(F_0(x)).
\]
Therefore the characterizing equation of the mean curvature flow solitons is

\[
\left[ (\omega_t)^* (X(F(x,t))) \right]^\perp = \frac{1}{c^2(t)} H(x,0) \nu(x,0).
\] (5.9)

Particularly, when \( t = 0 \), we have

\[
X^\perp = H \nu
\] (5.10)

As an example we recover the classical homothetic solutions in Euclidean spaces.

**Example 5.1.** The homothetic transformation of \( \mathbb{R}^{n+1} \) is defined by

\[
\omega_t(x) = c(t)x,
\] (5.11)

for \( x = (x^1, ..., x^{n+1}) \in \mathbb{R}^{n+1} \), and a homothetic soliton is \( F(x,t) = c(t)F_0(x) \), for some given initial hypersurface \( F_0 : M \to \mathbb{R}^{n+1} \). The corresponding vector field \( X \) of \( \omega_t(x) \) is then

\[
X(F(x,t)) = c'(t)x(F(x,t))
\] (5.12)

Since \( \omega_t^{-1} = \frac{1}{c(t)}x \), we have

\[
(\omega_t^{-1})^* = \frac{1}{c(t)} Id
\] (5.13)

then

\[
(\omega_t^{-1})^* (X(F(x,t))) = \frac{c'(t)}{c(t)} x(\omega_t(F(x,t))) = \frac{c'(t)}{c(t)} F_0(x).
\] (5.14)

Thus by (5.9), we get

\[
\left[ (\omega_t^{-1})^* (X(F(x,t))) \right]^\perp = \frac{1}{c^2(t)} H(x,0) \nu(x,0) = \frac{c'(t)}{c(t)} F_0^\perp(x),
\] (5.15)

that is,

\[
H(x,0) \nu(x,0) = c(t)c'(t) F_0^\perp(x).
\] (5.16)

This is exactly the characterizing equation of the homothetic solutions in Euclidean spaces see [5] for further accounts.

We next construct examples of non-Euclidean homothetic solitons generated by a special one-parameter homothetic transformations \( \omega_t \) defined by (5.2), where we further assume that \( L_{X^\perp} g = 2\lambda g \) for some non-zero constant \( \lambda \). From the following lemmas, we can see that there is an obstruction to the existence of such homothetic transformations.

**Lemma 5.1.** (Kobayashi [13], pp. 242, Lemma 2) If \( M \) is a complete Riemannian manifold which is not locally Euclidean, then any homothetic transformation of \( M \) is an isometry.

**Lemma 5.2.** If a complete Riemannian manifold \( M \) admits one-parameter homothetic transformations defined above, then \( M \) is isometric to a Euclidean space.
Proof. By Lemma 5.1, $M$ must be locally Euclidean and thus a flat manifold. Since $M$ is a complete flat manifold, its universal Riemannian covering space is a Euclidean space (cf. [13]). Let $\tilde{V}$ be the homothetic vector field induced by the family of transformations and $V$ be its horizontal lift. Since Riemannian covering is local isometry, the vector field $V$ is a homothetic vector field on $(\mathbb{R}^n, g_0)$, where $g_0 = \sum_{i=1}^{n} (dx^i)^2$. Suppose $V = \sum_{i=1}^{n} V^i \frac{\partial}{\partial x^i}$, such that $L_V g_0 = 2\lambda g_0$, where $\lambda$ is non-zero constant, then we have

$$g_0(D \frac{\partial}{\partial x^j} V, \frac{\partial}{\partial x^i}) + g_0(D \frac{\partial}{\partial x^i} V, \frac{\partial}{\partial x^j}) = 2\lambda \delta_{ij}. \tag{5.17}$$

which can be further reduced to

$$V^j_i + V^i_j = 2\lambda \delta_{ij}, \tag{5.18}$$

where $V^j_i = \frac{\partial V^j}{\partial x^i}$. We see that $V^j_i$ satisfies

$$\begin{cases} V^i_i = \lambda, & 1 \leq i \leq n \\ V^j_i = -V^i_j, & 1 \leq i \neq j \leq n. \end{cases} \tag{5.19}$$

Thus for each $i$, there exists a function $\alpha^i$ satisfying

$$\begin{cases} \frac{\partial \alpha^i}{\partial x^i} = 0, & 1 \leq i \leq n \\ \frac{\partial \alpha^i}{\partial x^j} = -\frac{\partial \alpha^j}{\partial x^i}, & 1 \leq i \neq j \leq n. \end{cases} \tag{5.20}$$

such that

$$V^i(x) = \lambda x^i + \alpha^i(x^1, ..., \hat{x}^i, ..., x^n) \tag{5.21}$$

where $\hat{x}^i$ represents omitting the variable.

If we define

$$W := \sum_{i=1}^{n} \alpha^i(x^1, ..., \hat{x}^i, ..., x^n) \frac{\partial}{\partial x^i}, \tag{5.22}$$

then it is straightforward to check that $L_W g_0 = 0$, so $W$ is a Killing vector field. Since a Killing vector field on $\mathbb{R}^n$ can be expressed by the linear combination of vector fields corresponding to translations and rotations, we have

$$W = \sum_{i=1}^{n} T^i \frac{\partial}{\partial x^i} + \sum_{p \neq q}^{n} R_{pq} \{-x^p \frac{\partial}{\partial x^q} + x^q \frac{\partial}{\partial x^p}\}, \tag{5.23}$$

and we can choose $R_{pq}$ to be an skew symmetric matrix without loss of generality.

We assume the covering map is

$$\pi : \mathbb{R}^n \to \mathbb{R}^n/\Gamma \tag{5.24}$$

where $\Gamma$ is a lattice, namely

$$\Gamma = \text{Span}_\mathbb{Z} \{\omega_1, ..., \omega_s\} \tag{5.25}$$
where $\omega_1, ..., \omega_s$, $1 \leq s \leq n$ are independent vectors of $\mathbb{R}^n$. Since $\tilde{V} = (\pi)_*(V)$ is global vector field, $V$ must be $\Gamma-$periodic, that is

$$V(x) = V(x + k\omega_i), \quad (5.26)$$

for arbitrary $k \in \mathbb{Z}$ and $1 \leq i \leq s$. For any non-zero vector $\omega \in \Gamma$, supposing its coordinates are $(a^1, ..., a^n)$, we get by (5.26),

$$\sum_{i=1}^{n} \lambda x^i \frac{\partial}{\partial x^i} + W = \sum_{i=1}^{n} \lambda(x^i + ka^i) \frac{\partial}{\partial x^i} + \sum_{i=1}^{n} T^i \frac{\partial}{\partial x^i} + \sum_{p \neq q}^{n} R_{pq} \{ - (x^p + ka^p) \frac{\partial}{\partial x^q} + (x^q + ka^q) \frac{\partial}{\partial x^p} \}. \quad (5.27)$$

By the skew symmetry of $R := R_{pq}$, we obtain

$$(\lambda - 2R)\omega = 0. \quad (5.28)$$

since $\omega \neq 0$, $\lambda$ is an eigenvalue of $2R$. Since the only real eigenvalue of an skew symmetric matrix is zero, it contradicts with the assumption.

We now give examples of homothetic solitons on surface patches.

**Example 5.2.** We consider a Riemannian surface patch $(\Sigma, g)$, where $\Sigma$ is a connected domain with isothermal coordinates $\{u, v\}$, such that

$$g = e^{2\rho(u,v)}(du^2 + dv^2). \quad (5.29)$$

Suppose also that

$$X = \phi(u, v) \frac{\partial}{\partial u} + \psi(u, v) \frac{\partial}{\partial v} \quad (5.30)$$

is a conformal vector field on $\Sigma$ such that $L_X g = 2\lambda g$ for some constant $\lambda$, and we are particularly interested in the case when $\lambda \neq 0$. By straightforward calculation, the condition $L_X g = 2\lambda g$ implies that $\phi$ and $\psi$ satisfies

$$\begin{align*}
\phi_u + \phi \rho_u + \psi \rho_v &= \lambda \\
\phi_u - \psi_v &= 0 \\
\phi_v + \psi_u &= 0.
\end{align*} \quad (5.31)$$

Suppose $r : I \rightarrow \Sigma$ is a smooth curve, where $I$ is an open interval, and we choose $s$ to be the arc-length parameter. Then the tangent field of $r(s) = (u(s), v(s))$ is

$$r_s(\frac{\partial}{\partial s}) = u' \frac{\partial}{\partial u} + v' \frac{\partial}{\partial v} \quad (5.32)$$

where $u'$ and $v'$ are partial derivatives with respect to $s$, satisfying

$$|r_s(\frac{\partial}{\partial s})| = 1, \quad (5.33)$$
which implies
\[(u')^2 + (v')^2 = e^{-2\rho}. \tag{5.34}\]

The left-ward pointing unit normal \(\nu\) is then
\[\nu = -v' \frac{\partial}{\partial u} + u' \frac{\partial}{\partial v}. \tag{5.35}\]

By Gauss formula we obtain the geodesic curvature \(k_g\) of \(r\)
\[k_g = e^{2\rho}(-v'f_1 + u'f_2), \tag{5.36}\]
where
\[
\begin{align*}
  f_1 &:= u'' + (u')^2 \rho_u - (v')^2 \rho_u + 2u'v' \rho_u \\
  f_2 &:= v'' - (u')^2 \rho_v + (v')^2 \rho_v + 2u'v' \rho_u
\end{align*} \tag{5.37}\]

By the characterizing equation of the mean curvature flow soliton (5.10), we obtain
\[(\phi - f_1) v' + (f_2 - \psi) u' = 0. \tag{5.38}\]

Therefore in order to determine the mean curvature flow solitons on the surface \((\Sigma, g)\), we have to solve the following system of equations
\[
\begin{cases}
  (\phi - f_1) v' + (f_2 - \psi) u' = 0 \tag{5.39a} \\
  \phi_u + \phi \rho_u + \psi \rho_v = \lambda \tag{5.39b} \\
  \phi_u - \psi_v = 0 \tag{5.39c} \\
  \phi_v + \psi_u = 0 \tag{5.39d}
\end{cases}
\]

Some special solutions to this system are easily obtained.

(I) Firstly, we observe that a special type of solutions to (5.39c) and (5.39d) are
\[
\begin{align*}
  \phi &= au + bv + c_1 \\
  \psi &= -bu + av + c_2.
\end{align*} \tag{5.40}\]

where \(a, b, c_1\) and \(c_2\) are constants. Secondly, we observe that (5.39b) is a first order linear partial differential equation with respect to \(\rho\), thus we can solve it in the following cases.

(i) \(a = b = 0\). The characteristic ODE system of (5.39b) is
\[
\begin{align*}
  \frac{du}{dt} &= c_1 \\
  \frac{dv}{dt} &= c_2 \\
  \frac{d\rho}{dt} &= \lambda. \tag{5.41}\end{align*}
\]

and its first integrals are
\[
I_1 := c_2u - c_1v \quad \text{and} \quad I_2 := \lambda u - c_1 \rho \tag{5.42}
\]
or
\[
I_1 := c_2u - c_1v \quad \text{and} \quad I_2 := \lambda v - c_2 \rho \tag{5.43}
\]
the corresponding solutions of $\rho$ are

$$\rho = \frac{\lambda}{c_1} u + Q(c_2 u - c_1 v)$$

(5.44)

or

$$\rho = \frac{\lambda}{c_2} v + Q(c_2 u - c_1 v)$$

(5.45)

where $Q$ is any smooth function.

Now the Gauss curvature $K$ of $\Sigma$ can be computed by

$$K = -e^{-2\rho}(\rho_{uu} + \rho_{vv}).$$

(5.46)

which is generally not zero, we thus obtain non-Euclidean homothetic solitons when $\lambda \neq 0$.

Differentiating (5.34) and combining (5.39a), we obtain the ODE system satisfied by the initial curve $r$

$$\frac{d}{ds} \begin{bmatrix} u \\ v \\ w \\ z. \end{bmatrix} = \begin{bmatrix} w \\ z \\ h_1(u,v,w,z) \\ h_2(u,v,w,z) \end{bmatrix}$$

(5.47)

where

$$h_1(u,v,w,z) := -\frac{1}{w}\{zw' + e^{-2\rho}(wp_u + z\rho_v)\}$$

$$h_2(u,v,w,z) := \frac{z}{w}(w' + w^2 \rho_u - z^2 \rho_v + 2wz\rho_v - c_1)$$

$$+ w^2 \rho_v - z^2 \rho_v - 2wz\rho_u + c_2.$$

(ii) $b = 0$ and $a \neq 0$. Now (5.39b) becomes

$$(au + c_1)\rho_u + (av + c_2)\rho_v = \lambda - a.$$  

(5.49)

Its characteristic ODE system is

$$\begin{cases}
\frac{da}{dt} = au + c_1 \\
\frac{dv}{dt} = av + c_2 \\
\frac{dp}{dt} = \lambda - a.
\end{cases}$$

(5.50)

By

$$\frac{dp}{\lambda - a} = \frac{du}{au + c_1},$$

(5.51)

we obtain a special solution

$$\rho = \frac{\lambda - a}{a} \ln |au + c_1|.$$  

(5.52)

Multiplying

$$\frac{du}{au + c_1} = \frac{dv}{av + c_2} = 0$$

(5.53)
with an integrating factor

\[ \mu = \frac{a^2|au + c_1|^a}{|av + c_2|^a}, \tag{5.54} \]

we can get

\[ d(|au + c_1|^a|av + c_2|^{-a}) = 0, \tag{5.55} \]

and it implies that \( I_1 = |au + c_1|^a|av + c_2|^{-a} \) is a first integral. Therefore the solution can be written as

\[ \rho = \frac{\lambda - a}{a} \ln |au + c_1| + Q(|au + c_1|^a|av + c_2|^{-a}), \tag{5.56} \]

for \( u \neq -\frac{c_2}{a}, v \neq -\frac{c_1}{a} \) and an arbitrary smooth function \( Q \).

On a connected sub-domain

\[ D \subset \{(u, v) \in \Sigma : u \neq -\frac{c_2}{a}, v \neq -\frac{c_1}{a}\}, \tag{5.57} \]

a soliton can be solved by (5.47) defined on \( D \).

(iii) \( a = 0 \) and \( b \neq 0 \). Now (5.39b) is

\[ (bv + c_1)\rho_u + (bu - c_2)\rho_v = \lambda. \tag{5.58} \]

and its characteristic ODE system is

\[
\begin{align*}
\frac{du}{dt} &= bv + c_1 \\
\frac{dv}{dt} &= -bu + c_2 \\
\frac{d\rho}{dt} &= \lambda.
\end{align*}
\tag{5.59}
\]

By

\[ \frac{du}{bv + c_1} = \frac{d\rho}{\lambda}, \tag{5.60} \]

we obtain a special solution

\[ -\frac{\lambda}{b} \arctan \left( \frac{bv + c_1}{bu - c_2} \right), \tag{5.61} \]

where we assume that \( u \neq \frac{c_2}{b}, v \neq -\frac{c_1}{b} \). By

\[ \frac{du}{bv + c_1} = \frac{dv}{-bu + c_2} \tag{5.62} \]

we obtain a first integral

\[ I_1 := (bv + c_1)^2 + (bu - c_2)^2. \tag{5.63} \]

Thus the general solution can be written as

\[ \rho = -\frac{\lambda}{b} \arctan \left( \frac{bv + c_1}{bu - c_2} \right) + Q[(bv + c_1)^2 + (bu - c_2)^2], \tag{5.64} \]

for \( u \neq \frac{c_2}{b}, v \neq -\frac{c_1}{b} \) and an arbitrary smooth function \( Q \).

Similar to the example in case (II), a soliton solution can be constructed on some connected sub-domain of \( \Sigma \).
A homogeneous polynomial solution to (5.39c) and (5.39d) is
\[
\begin{align*}
\phi &= u^2 - v^2 \\
\psi &= 2uv
\end{align*}
\]  
(5.65)

and the corresponding characteristic ODE system is
\[
\begin{align*}
\frac{du}{dt} &= u^2 - v^2 \\
\frac{dv}{dt} &= 2uv \\
\frac{d\rho}{dt} &= \lambda - 2u.
\end{align*}
\]  
(5.66)

Since
\[
2uvdu + (v^2 - u^2)dv = 0,
\]  
(5.67)
we introduce the following coordinates transformation
\[
\begin{align*}
u &= r \sinh \theta \\
v &= r \cosh \theta,
\end{align*}
\]  
(5.68)
and obtain
\[
\frac{dr}{r} = -\frac{z^3 + 3z^2 - 3z - 1}{2z(z + 1)(z^2 + 1)}dz,
\]  
(5.69)
where
\[
z := e^{2\theta} = \frac{u + v}{v - u},
\]  
(5.70)
From this equation \(r\) can be expressed by \(z\) as follows
\[
r = C \sqrt{z(z + 1)},
\]  
(5.71)
for any positive constant \(C\). Inserting \(r\) into
\[
\frac{d\rho}{dz} = \frac{\lambda - 2u}{r^2} \frac{du}{dz},
\]  
(5.72)
we can solve \(\rho(z)\)
\[
\rho(z) = \frac{4\lambda}{c(z + 1)} + 4 \ln \frac{(z + 1)^2}{z^2 + 1} + D,
\]  
(5.73)
where \(D\) is an arbitrary constant.

6. Proof of Theorem 2

As an application of Theorem 1, we consider the affine solutions to the mean curvature flow in the Euclidean space \(\mathbb{R}^n\).

Definition 6.1. A solution \(F(x,t)\) to the mean curvature flow is called an affine solution, if there exists a one-parameter family of affine transformations \(A(t)\) of \(\mathbb{R}^n\), such that \(M_t = A(t)(M_0)\).
Proof of Theorem 2. In a Cartesian coordinate system \( \{y^1, \ldots, y^n\} \) of \( \mathbb{R}^n \), the affine transformations can be represented as follows

\[
F(x, t) = R(t)F_0(x) + T(t),
\]

(6.1)

where \( T(t), F(x, t) \) and \( F_0(x) \) are regarded as column vectors and \( R(t) \) is an \( n \times n \) matrix. Here we assume that the determinant of \( R(t) \) is positive for any \( t \in I \). By the initial condition \( A(0) = Id \), we see that \( R(0) \) is a unit matrix \( I_n \), and \( T(0) = 0 \).

According to QR decomposition, a matrix can always be decomposed into the product of an orthogonal matrix and an upper triangular matrix. Thus we assume there exist an orthogonal matrix \( \tilde{U}(t) \) and an upper triangular matrix \( \tilde{V}(t) \), such that

\[
R(t) = \tilde{U}(t)\tilde{V}(t).
\]

(6.2)

Since \( \det\{R(t)\} > 0 \), the diagonal elements of \( \tilde{V}(t) \) are non-zero. Then we can further have the following decomposition

\[
R(t) = \begin{bmatrix}
  u_{11}(t) & \cdots & u_{1n}(t) \\
  \vdots & \ddots & \vdots \\
  u_{n1}(t) & \cdots & u_{nn}(t)
\end{bmatrix}
\begin{bmatrix}
  s(t) & 0 \\
  \vdots & \ddots \\
  0 & s(t)
\end{bmatrix}
\begin{bmatrix}
  1 & \cdots & v_{1n}(t) \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & v_{nn}(t)
\end{bmatrix},
\]

(6.3)

and we denote it by

\[
R(t) = U(t)S(t)V(t),
\]

(6.4)

where \( U(t) \) is a special orthogonal matrix, \( S(t) \) is a scalar matrix and \( V(t) \) is an upper triangular matrix, with \( U(0) = S(0) = V(0) = I_n \).

Now we define a one-parameter family of transformations acting on \( F(x, t) \) with \( t \in [t_0, t_2] \subset I \), such that they are still solutions to the mean curvature flow after the action. Let \( \varepsilon \in [-\delta, \delta] \), where \( \delta \) is small enough such that \( t + \varepsilon \in I \), the action is defined by

\[
A(\varepsilon)(F(x, t)) = F(x, t + \varepsilon),
\]

(6.5)

where \( t \in [t_0, t_1] \). In another word, \( A(\varepsilon) \) is a one-parameter subgroup of the symmetry group. Since \( A(t) \) is a one-parameter family of transformations, it satisfy \( A(t + \varepsilon) = A(\varepsilon)A(t) \), thus

\[
F(x, t + \varepsilon) = A(t + \varepsilon)(F_0(x)) = A(\varepsilon)A(t)(F_0(x)) = R(\varepsilon)[R(t)F_0(x) + T(t)] + T(\varepsilon).
\]

(6.6)

Differentiating \( F(x, t + \varepsilon) \) with respect to \( \varepsilon \) yields

\[
\left. \frac{\partial F(x, t + \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} = R'(0)F(x, t) + T'(0),
\]

(6.7)

and noting that

\[
R'(0) = U'(0) + S'(0) + V'(0),
\]

(6.8)
we have
\[ X := \left. \frac{\partial F(x, t)}{\partial t} \right|_{\varepsilon=0} = \frac{\partial F(x, t + \varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon=0} = U'(0)(F(x, t)) + S'(0)(F(x, t)) + V'(0)(F(x, t)) + T'(0). \] (6.9)

Recall that \( so(n) = \{ A \in gl(n) : A^t + A = 0 \} \), we see that \( V'(t) \) is not in \( so(n) \) unless \( V(t) \) is a unit matrix. Also \( V(t) \) is not scalar matrix unless \( V(t) \) is a unit matrix. So \( V(t) \) is generally neither a rotation nor a scaling and obviously not a translation. Since \( X \) is an infinitesimal symmetry, by Theorem 1, \( V'(t) \) must be tangent vector field on \( M \), so \( V(t) \) is a family of diffeomorphisms of \( M \). If we consider the mean curvature flow with only normal motion, then it is only possible that \( U(t) = I \). If we consider the general mean curvature flows, then by \( X^\perp = H_\nu \), they are exactly the self-similar solutions combining translation, rotation and scaling. And this proves Theorem 2.

\[ \square \]

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