SCHUR PARAMETERS, TOEPLITZ MATRICES, AND KREÎN SHORTED OPERATORS

YURY ARLINSKIĬ

ABSTRACT. We establish connections between Schur parameters of the Schur class operator-valued functions, the corresponding simple conservative realizations, lower triangular Toeplitz matrices, and Kreîn shorted operators. By means of Schur parameters or shorted operators for defect operators of Toeplitz matrices necessary and sufficient conditions for a simple conservative discrete-time system to be controllable/observable and for a completely non-unitary contraction to be completely non-isometric/completely non-co-isometric are obtained. For the Schur problem a characterization of central solution and uniqueness criteria to the solution are given in terms of shorted operators for defect operators of contractive Toeplitz matrices, corresponding to data.

1. Introduction

In this Section we briefly describe notations, the basic objects, and the main goal of this paper.

1.1. Notations. In what follows the class of all continuous linear operators defined on a complex Hilbert space $H_1$ and taking values in a complex Hilbert space $H_2$ is denoted by $L(H_1, H_2)$ and $L(H_1) := L(H_1, H_1)$. All infinite dimensional Hilbert spaces are supposed to be separable. We denote by $I$ the identity operator in a Hilbert space and by $P_L$ the orthogonal projection onto the subspace (the closed linear manifold) $L$. The notation $T ↾ L$ means the restriction of a linear operator $T$ on the set $L$. The range and the null-space of a linear operator $T$ are denoted by $\text{ran } T$ and $\ker T$, respectively. We use the usual symbols $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{N}_0$ for the sets of complex numbers, positive integers, and nonnegative integers, respectively. The Schur class $S(H_1, H_2)$ is the set of all function $\Theta(\lambda)$ analytic on the unit disk $D = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \}$ with values in $L(H_1, H_2)$ and such that $\| \Theta(\lambda) \| \leq 1$ for all $\lambda \in D$. An operator $T \in L(H_1, H_2)$ is said to be

- contractive if $\| T \| \leq 1$;
- isometric if $\| Tf \| = \| f \|$ for all $f \in H_1$ $\iff$ $T^* T = I$;
- co-isometric if $T^*$ is isometric $\iff$ $T T^* = I$;
- unitary if it is both isometric and co-isometric.

Given a contraction $T \in L(H_1, H_2)$, the operators $D_T := (I - T T^*)^{1/2}$ and $D_{T^*} := (I - TT^*)^{1/2}$ are called the defect operators of $T$, and the subspaces $\mathcal{D}_T = \overline{\text{ran } D_T}$, $\mathcal{D}_{T^*} = \overline{\text{ran } D_{T^*}}$ the defect subspaces of $T$. The defect operators satisfy the following relations $T D_T = D_T^* T$, $T^* D_{T^*} = D_T T^*$.

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1.2. The Schur algorithm. Given a scalar Schur class function \( f(\lambda) \), which is not a finite Blaschke product, define inductively

\[
f_0(\lambda) = f(\lambda), \quad f_{n+1}(\lambda) = \frac{f_n(\lambda) - f_n(0)}{\lambda(1 - f_n(0)f_n(\lambda))}, \quad n \in \mathbb{N}_0.
\]

It is clear that \( \{f_n\} \) is an infinite sequence of Schur functions called the associated functions and neither of its terms is a finite Blaschke product. The numbers \( \gamma_n := f_n(0) \) are called the Schur parameters. Note that

\[
f_n(\lambda) = \frac{\gamma_n + \lambda f_{n+1}(\lambda)}{1 + \gamma_n \lambda f_{n+1}(\lambda)} = \gamma_n + (1 - |\gamma_n|^2)\frac{\lambda f_{n+1}(\lambda)}{1 + \gamma_n \lambda f_{n+1}(\lambda)}, \quad n \in \mathbb{N}_0.
\]

The method of labeling \( f \in \mathcal{S} \) by its Schur parameters is known as the Schur algorithm and is due to I. Schur [13]. In the case when \( f \) is a finite Blaschke product of order \( N \), the Schur algorithm terminates at the \( N \)-th step, i.e., the sequence of Schur parameters \( \{\gamma_n\}_{n=0}^N \) is finite, \( |\gamma_n| < 1 \) for \( n = 0, 1, \ldots, N-1 \), and \( |\gamma_N| = 1 \).

The next theorem goes back to Shmul’yan [14, 15] and T. Constantinescu [27] (see also [8, 19, 23, 30, 31]) and plays a key role in the Schur algorithm for operator-valued functions.

**Theorem 1.1.** Let \( \mathcal{M} \) and \( \mathcal{N} \) be Hilbert spaces and let the function \( \Theta(\lambda) \) be from the Schur class \( \mathcal{S}(\mathcal{M}, \mathcal{N}) \). Then there exists a function \( Z(\lambda) \) from the Schur class \( \mathcal{S}(\mathcal{D}_{\Theta(0)}, \mathcal{D}_{\Theta^*(0)}) \) such that

\[
\Theta(\lambda) = \Theta(0) + D_{\Theta^*(0)}Z(\lambda)(I + \Theta^*(0)Z(\lambda))^{-1}D_{\Theta(0)}, \quad \lambda \in \mathbb{D}.
\]

The representation (1.1) of a function \( \Theta(\lambda) \) from the Schur class is called the Möbius representation of \( \Theta(\lambda) \) and the function \( Z(\lambda) \) is called the Möbius parameter of \( \Theta(\lambda) \). Clearly, \( Z(0) = 0 \) and from Schwartz’s lemma one obtains that

\[
\lambda^{-1} Z(\lambda) \in \mathcal{S}(\mathcal{D}_{\Theta(0)}, \mathcal{D}_{\Theta^*(0)}).
\]

The operator Schur’s algorithm [19]. For \( \Theta \in \mathcal{S}(\mathcal{M}, \mathcal{N}) \) put \( \Theta_0(\lambda) = \Theta(\lambda) \) and let \( Z_0(\lambda) \) be the Möbius parameter of \( \Theta \). Define

\[
\Gamma_0 = \Theta(0), \quad \Theta_1(\lambda) = \lambda^{-1} Z_0(\lambda) \in \mathcal{S}(\mathcal{D}_{\Gamma_0}, \mathcal{D}_{\Gamma_0^*}), \quad \Gamma_1 = \Theta_1(0) = Z_0'(0).
\]

If \( \Theta_0(\lambda), \ldots, \Theta_n(\lambda) \) and \( \Gamma_0, \ldots, \Gamma_n \) have been chosen, then let \( Z_{n+1} \in \mathcal{S}(\mathcal{D}_{\Gamma_n}, \mathcal{D}_{\Gamma_n^*}) \) be the Möbius parameter of \( \Theta_n \). Put

\[
\Theta_{n+1}(\lambda) = \lambda^{-1} Z_{n+1}(\lambda), \quad \Gamma_{n+1} = \Theta_{n+1}(0).
\]

The contractions \( \Gamma_0 \in \mathcal{L}(\mathcal{M}, \mathcal{N}), \quad \Gamma_n \in \mathcal{L}(\mathcal{D}_{\Gamma_{n-1}}, \mathcal{D}_{\Gamma_{n-1}^*}), \quad n = 1, 2, \ldots \) are called the Schur parameters of \( \Theta(\lambda) \) and the function \( \Theta_n \in \mathcal{S}(\mathcal{D}_{\Gamma_{n-1}}, \mathcal{D}_{\Gamma_{n-1}^*}) \) is called the \( n \)-th associated function. Thus,

\[
\Theta_n(\lambda) = \Gamma_n + \lambda D_{\Gamma_n} \Theta_n(\lambda)(I + \lambda^* \Theta_n(\lambda))^{-1}D_{\Gamma_n}, \quad \lambda \in \mathbb{D},
\]

and

\[
\Theta_{n+1}(\lambda) \upharpoonright \text{ran } D_{\Gamma_n} = \lambda^{-1} D_{\Gamma_n}(I - \Theta_n(\lambda)\Gamma_n^*)^{-1}(\Theta_n(\lambda) - \Gamma_n)D_{\Gamma_n}^{-1} \upharpoonright \text{ran } D_{\Gamma_n}.
\]

Clearly, the sequence of Schur parameters \( \{\Gamma_n\} \) is infinite if and only if the operators \( \Gamma_n \) are non-unitary. The sequence of Schur parameters consists of finite number of operators \( \Gamma_0, \Gamma_1, \ldots, \Gamma_N \) if and only if \( \Gamma_N \in \mathcal{L}(\mathcal{D}_{\Gamma_{N-1}}, \mathcal{D}_{\Gamma_{N-1}^*}) \) is unitary. If \( \Gamma_N \) is non-unitary but isometric (respect., co-isometric), then \( \Gamma_n = 0 \in \mathcal{L}(0, \mathcal{D}_{\Gamma_N}) \) (respect., \( \Gamma_n = 0 \in \mathcal{L}(\mathcal{D}_{\Gamma_N}, 0) \)) for all \( n > N \). The following theorem is the operator generalization of Schur’s result.
Theorem 1.2. \([19, 27]\). There is a one-to-one correspondence between the Schur class \(S(\mathcal{M}, \mathcal{N})\) and the set of all sequences of contractions \(\{\Gamma_n\}_{n \geq 0}\) such that
\[
\Gamma_0 \in \mathcal{L}(\mathcal{M}, \mathcal{N}), \quad \Gamma_n \in \mathcal{L}(\mathcal{D} \Gamma_{n-1}, \mathcal{D} \Gamma_{n-1}), \quad n \geq 1.
\]
Notice that a sequence of contractions of the form (1.2) is called the choice sequence \([26]\).

1.3. The lower triangular Toeplitz matrices. Let \(\Theta\) be holomorphic in \(\mathbb{D}\) operator valued function acting between Hilbert spaces \(\mathcal{M}\) and \(\mathcal{N}\) and let
\[
\Theta(\lambda) = \sum_{n=0}^{\infty} \lambda^n C_n, \quad \lambda \in \mathbb{D}, \quad C_n \in \mathcal{L}(\mathcal{M}, \mathcal{N}), \quad n \geq 0
\]
be the Taylor expansion of \(\Theta\). Consider the lower triangular (analytic) Toeplitz matrix
\[
T_\Theta := \begin{bmatrix}
C_0 & 0 & 0 & 0 & \cdots & \cdots \\
C_1 & C_0 & 0 & 0 & \cdots & \cdots \\
C_2 & C_1 & C_0 & 0 & \cdots & \cdots \\
C_3 & C_2 & C_1 & C_0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
C_n & C_{n-1} & C_{n-2} & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

As is well known \([19, 33]\)
\[
\Theta \in S(\mathcal{M}, \mathcal{N}) \iff T_\Theta \in \mathcal{L}(l_2(\mathcal{M}), l_2(\mathcal{N})) \quad \text{is a contraction.}
\]
Set for \(n = 0, 1, \ldots\)
\[
\mathcal{M}^{n+1} = \mathcal{M} \oplus \mathcal{M} \oplus \cdots \oplus \mathcal{M}, \quad \mathcal{N}^{n+1} = \mathcal{N} \oplus \mathcal{N} \oplus \cdots \oplus \mathcal{N}.
\]
Clearly, if \(T_\Theta\) is a contraction, then the operator \(T_{\Theta,n} \in \mathcal{L}(\mathcal{M}^{n+1}, \mathcal{N}^{n+1})\) given by the block operator matrix
\[
T_{\Theta,n} := \begin{bmatrix}
C_0 & 0 & 0 & \cdots & 0 \\
C_1 & C_0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
C_n & C_{n-1} & C_{n-2} & \cdots & C_0
\end{bmatrix}
\]
is a contraction for each \(n\). There are connections, established by T. Constantinescu \([27]\), between the Taylor coefficients \(\{C_n\}_{n \geq 0}\) and Schur parameters of \(\Theta \in S(\mathcal{M}, \mathcal{N})\). These connections are given by the relations
\[
C_0 = \Gamma_0,
\]
\[
C_n = \text{formula}_n(\Gamma_0, \Gamma_1, \ldots, \Gamma_{n-1}) + D_{\Gamma_0} D_{\Gamma_1} D_{\Gamma_{n-1}}, \quad n \geq 1.
\]
Here \(\text{formula}_n(\Gamma_0, \Gamma_1, \ldots, \Gamma_{n-1})\) is a some expression, depending on \(\Gamma_0, \Gamma_1, \ldots, \Gamma_{n-1}\).
Let now \( \{C_k\}_{k=0}^{\infty} \) be a sequence of operators from \( \mathbf{L} (\mathcal{M}, \mathcal{N}) \). Then ([19] Theorem 2.1) there is a one-to-one correspondence between the set of contractions

\[
T_{\infty} := \begin{bmatrix}
    C_0 & 0 & 0 & 0 & 0 & \ldots \\
    C_1 & C_0 & 0 & 0 & 0 & \ldots \\
    C_2 & C_1 & C_0 & 0 & 0 & \ldots \\
    C_3 & C_2 & C_1 & C_0 & 0 & \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} : l_2(\mathcal{M}) \rightarrow l_2(\mathcal{N})
\]

and the set of all choice sequences \( \Gamma_0 \in \mathbf{L}(\mathcal{M}, \mathcal{N}), \Gamma_k \in \mathbf{L}(\mathcal{D}_{\Gamma_{k-1}}, \mathcal{D}_{\Gamma_k}) \), \( k = 1, \ldots, \). The connections between \( \{C_k\} \) and \( \{\Gamma_k\} \) is also given by ([19]). The operators \( \{\Gamma_k\} \) can be by successively defined ([19] proof of Theorem 2.1], using parametrization of contractive block-operator matrices (see Section 2), from the matrices

\[
T_0 = C_0 = \Gamma_0, \ T_1 = \begin{bmatrix} C_0 & 0 \\ C_1 & C_0 \end{bmatrix}, \ T_2 = \begin{bmatrix} C_0 & 0 & 0 \\ C_1 & C_0 & 0 \\ C_2 & C_1 & C_0 \end{bmatrix}, \ldots.
\]

Moreover, \( T_{\infty} = T_{\Theta}, \Theta(\lambda) = \sum_{n=0}^{\infty} \lambda^n C_n, \lambda \in \mathbb{D} \), and \( \{\Gamma_k\}_{k \geq 0} \) are the Schur parameters of \( \Theta \) ([19] Proposition 2.2]. Put

\[
\tilde{\Theta}(\lambda) := \Theta^*(\lambda), \ |\lambda| < 1.
\]

Then \( \tilde{\Theta}(\lambda) = \sum_{n=0}^{\infty} \lambda^n C_n^* \). Clearly, if \( \{\Gamma_0, \Gamma_1, \ldots\} \) are the Schur parameters of \( \Theta \), then \( \{\Gamma_0^*, \Gamma_1^*, \ldots\} \) are the Schur parameters of \( \tilde{\Theta} \).

1.4. The Schur problem. The following problem is called the Schur problem:

Let \( \mathcal{M} \) and \( \mathcal{N} \) be Hilbert spaces. Given the operators \( C_k \in \mathbf{L}(\mathcal{M}, \mathcal{N}), k = 0, 1, \ldots, N \), it is required to (a) find conditions for the existence of \( \Theta \in \mathbf{S}(\mathcal{M}, \mathcal{N}) \) such that \( C_0, C_1, \ldots, C_N \) are the first \( N + 1 \) Taylor coefficients of \( \Theta \); (b) give an explicit description of all solutions \( \Theta \) (if there any) to problem (a).

The Schur problem is often called the Carathéodory or the Carathéodory-Fejér problem. This problem was studied in many papers, see monographs [19], [31], [33], and references therein. It is well known that the Schur problem has a solution if and only if the Toeplitz operator from \( \mathbf{L}(\mathcal{M}^{N+1}, \mathcal{N}^{N+1}) \)

\[
(1.6) \quad T_N = T_N(C_0, C_1, \ldots, C_N) := \begin{bmatrix} C_0 & 0 & 0 & \ldots & 0 \\ C_1 & C_0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_N & C_{N-1} & C_{N-2} & \ldots & C_0 \end{bmatrix}
\]

is a contraction. By means of relations (1.5) contractions \( T_0, T_1, \ldots, T_N \) determine choice parameters

\[
\Gamma_0 := C_0, \ \Gamma_1 \in \mathbf{L}(\mathcal{D}_{\Gamma_0}, \mathcal{D}_{\Gamma_0}), \ldots, \Gamma_N \in \mathbf{L}(\mathcal{D}_{\Gamma_{N-1}}, \mathcal{D}_{\Gamma_{N-1}})
\]

If \( T_N \) is a contraction, then operators \( \{C_k\}_{k=0}^{N} \) are said to be the Schur sequence [31]. Let us formulate known conditions for a uniqueness solution to the Schur problem.

**Theorem 1.3.** ([19] Proposition 2.3). Let the complex numbers \( \{C_k\}_{k=0}^{N} \) be the Schur sequence. Then the following assertions are equivalent:

(i) the Schur problem with data \( \{C_k\}_{k=0}^N \) has a unique solution;
(ii) there exists a number \( r, \ 0 \leq r \leq N \) such that \( |\Gamma_r| = 1 \);
(iii) \( \det D_{T_r}^2 = 0 \) for some \( 0 \leq r \leq N \), but \( \det D_{T_p}^2 \neq 0 \) for \( 0 \leq p < r \);
(iv) \( \det D_{T_N}^2 = 0 \).

**Theorem 1.4.** [19, Theorem 2.6]. Consider a solvable Schur problem with the data
\[ C_0, \ldots, C_N \in L(\mathcal{H}, \mathcal{M}). \]
Then the solution is unique if and only if the corresponding choice parameters \( \{\Gamma_n\}_{n=0}^N \), determined by the operator \( T_N \), satisfy the condition: one of \( \Gamma_n, \ 0 \leq n \leq N \) is an isometry or a co-isometry.

1.5. **Simple conservative discrete time-invariant systems and their transfer functions.** Here we recall some results from the theory of conservative discrete time-invariant systems cf. [3, 4, 12, 13, 25, 34, 20, 48].

A collection
\[ \tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H} \right\} \]
is called the linear discrete time-invariant systems with the state space \( \mathcal{H} \) and the input and output spaces \( \mathcal{M} \) and \( \mathcal{N} \), respectively. A system \( \tau \) is called conservative if the linear operator
\[ T_\tau = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \mathcal{M} \oplus \mathcal{H} \rightarrow \mathcal{N} \oplus \mathcal{H} \]
is unitary. The transfer function
\[ \Theta_\tau(\lambda) := D + \lambda C (I - \lambda A)^{-1} B, \quad \lambda \in \mathbb{D}, \]
of a conservative system \( \tau \) belongs to the Schur class \( S(\mathcal{M}, \mathcal{N}) \). Conservative systems are also called unitary colligations and their transfer functions are called the characteristic functions [25]. The subspaces
\[ \mathcal{H}_c^\tau := \text{span} \{A^n BM : n = 0, 1, \ldots\} \]
and
\[ \mathcal{H}_o^\tau := \text{span} \{A^n C^n N : n = 0, 1, \ldots\} \]
are said to be the controllable and observable subspaces of the system \( \tau \), respectively. The system \( \tau \) is said to be controllable (respect., observable) if \( \mathcal{H}_c^\tau = \mathcal{H} \) (respect., \( \mathcal{H}_o^\tau = \mathcal{H} \)), and it is called minimal if \( \tau \) is both controllable and observable. The system \( \tau \) is said to be simple if \( \mathcal{H} = \text{clos} \{\mathcal{H}_c^\tau + \mathcal{H}_o^\tau\} \) (the closure of the span). Two discrete time-invariant systems
\[ \tau_1 = \left\{ \begin{bmatrix} D & C_1 \\ B_1 & A_1 \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H}_1 \right\} \quad \text{and} \quad \tau_2 = \left\{ \begin{bmatrix} D & C_2 \\ B_2 & A_2 \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H}_2 \right\} \]
are said to be unitarily similar if there exists a unitary operator \( U \) from \( \mathcal{H}_1 \) onto \( \mathcal{H}_2 \) such that
\[ A_1 = U^{-1} A_2 U, \quad B_1 = U^{-1} B_2, \quad C_1 = C_2 U. \]
As is well known, two simple conservative systems with the same transfer function are unitarily similar. It is important that any function \( \Theta \in S(\mathcal{M}, \mathcal{N}) \) can be realized as the transfer function of a linear conservative and simple discrete-time system.
1.6. M. Kreĭn’s shorted operators. For every nonnegative bounded operator $S$ in the Hilbert space $\mathcal{H}$ and every subspace $K \subset \mathcal{H}$ M.G. Kreĭn [37] defined the operator $S_K$ by the relation

$$S_K = \max \{ Z \in L(\mathcal{H}) : 0 \leq Z \leq S, \text{ran} \ Z \subseteq K \}.$$ 

The equivalent definition

$$(S_K f, f) = \inf_{\varphi \in K^\perp} \{(S(f + \varphi), f + \varphi)\}, \quad f \in \mathcal{H}. \tag{1.7}$$

Here $K^\perp := H \oplus K$. The properties of $S_K$, were studied by M. Kreĭn and by other authors (see [8] and references therein). $S_K$ is called the shorted operator (see [5, 6]). Let the subspace $\Omega$ be defined as follows

$$\Omega = \{ f \in \overline{\text{ran}} S : S^{1/2} f \in K \} = \overline{\text{ran}} S \ominus S^{1/2} K^\perp.$$ 

It is proved in [37] that $S_K$ takes the form

$$S_K = S^{1/2} P_\Omega S^{1/2}.$$ 

Hence, $\ker S_K \supseteq K^\perp$. Moreover [37],

$$\text{ran} S_K^{1/2} = \text{ran} S^{1/2} \cap K. \tag{1.8}$$

It follows that

$$S_K = 0 \iff \text{ran} S^{1/2} \cap K = \{0\}. \tag{1.9}$$

1.7. The goal of this paper. In this paper we establish connections between the Schur parameters of $\Theta \in S(\mathcal{M}, \mathcal{N})$, a simple conservative realization of $\Theta$, the operators $T_\Theta$ and $T_{\Theta, n}$, and the Kreĭn shorted operators. These connections allows to

1. give criterions of controllability and observability for the corresponding to $\Theta$ simple conservative system in terms of Schur parameters/ Kreĭn shorted operators $(D^2_{T_\Theta})_{2\mathbb{R}}$ and $(D^2_{T_{\Theta, n}})_{2\mathbb{R}}$,

2. to obtain necessary and sufficient conditions for a completely non-unitary contraction $A$ to be completely non-isometric or completely non-co-isometric [20] in terms of Schur parameters / Kreĭn shorted operators $(D^2_{T_{\Psi}})_{\Delta_A}$, $(D^2_{T_{\Psi}})_{\Delta_A^*}$ of Sz-Nagy–Foias characteristic function $\Psi$ of $A$ [19],

3. give a characterization of the central (maximal entropy) solution to the Schur problem,

4. give a uniqueness criterion to the solution of the operator Schur problem in terms of the Kreĭn shorted operators for the defect operators of the Toeplitz matrices, constructed from problem’s data.

The paper is organized as follows. Sections 2–4 deal with additional background material concerning parametrization of $2 \times 2$ contractive and unitary block operator matrices, the theory of completely non-unitary contractions, defect functions of the Schur class functions, and conservative realization of the Schur algorithm. New results about the Kreĭn shorted operators are given in Section 5. Main results of the paper are presented in Section 6. Relying on the results of Section 5, we prove that the Kreĭn shorted operators $\{(D^2_{T_k})_{2\mathbb{R}} \mid \mathfrak{M}\}$ forms a non-increasing sequence, where $T_k = T_k(C_0, C_1, \ldots, C_k)$ are the Toeplitz operators constructed from the Schur sequence. We study in more detail the central solution to the
Schur problem and obtain a uniqueness solution criteria. The latter is closed to results of V.M. Adamyan, D.Z. Arov, and M.G. Krein obtained in [1] and [2] concerning to scalar and operator Nehari problem [40]. These authors did not use the Krein shorted operators in explicit form, their approach is essentially rely on the extension theory of isometric operators. Different approaches to the descriptions of all solutions to the Schur problem can be found in [31] for finite dimensional \( \mathcal{M} \) and \( \mathcal{N} \), in [19, 33] for general case. The Schur problem can be reduced to the above mentioned Nehari problem [15]. All solutions to this problem are obtained in [1, 2, 15, 35] (see also [41]).

2. Parametrization of contractive block-operator matrices

Let \( \mathcal{H}, \mathcal{K}, \mathcal{M} \) and \( \mathcal{N} \) be Hilbert spaces. The following theorem goes back to [18, 29, 46]; other proofs of the theorem can be found in [7, 11, 36, 39, 41].

**Theorem 2.1.** Let \( A \in \mathbf{L}(\mathcal{H}, \mathcal{K}), B \in \mathbf{L}(\mathcal{M}, \mathcal{K}), C \in \mathbf{L}(\mathcal{H}, \mathcal{N}), \) and \( D \in \mathbf{L}(\mathcal{M}, \mathcal{N}) \). The following conditions are equivalent:

(i) the operator \( T = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \mathcal{H} \oplus \mathcal{M} \rightarrow \mathcal{K} \oplus \mathcal{N} \) is a contraction;

(ii) the operator \( A \in \mathbf{L}(\mathcal{H}, \mathcal{K}) \) is a contraction and

\[
B = D_A\cdot M, \quad C = K D_A, \quad D = -K A^* M + D_K \cdot X D_M,
\]

where \( M \in \mathbf{L}(\mathcal{M}, \mathcal{D}_A^*), K \in \mathbf{L}(\mathcal{D}_A, \mathcal{M}), \) and \( X \in \mathbf{L}(\mathcal{D}_M, \mathcal{D}_K^*) \) are contractions;

(iii) the operator \( D \in \mathbf{L}(\mathcal{M}, \mathcal{N}) \) is a contraction and

\[
B = F D_D, \quad C = D_D \cdot G, \quad A = -F D^* G + D_F \cdot L D_G.
\]

where the operators \( F \in \mathbf{L}(\mathcal{D}_D, \mathcal{K}), G \in \mathbf{L}(\mathcal{H}, \mathcal{D}_D^*) \) and \( L \in \mathbf{L}(\mathcal{D}_G, \mathcal{D}_F^*) \) are contractions.

Moreover, if \( T \) is a contraction, then the operators \( K, M, \) and \( X \) in (2.1) and operators \( F, G, \) and \( L \) in (2.2) are uniquely determined.

**Corollary 2.2.** [8, 9]. Let

\[
T = \begin{bmatrix} -K A^* M + D_K \cdot X D_M & K D_A \\ D_A \cdot M & A \end{bmatrix} = \begin{bmatrix} D & D_D \cdot G \\ F D_D & -F D^* G + D_F \cdot L D_G \end{bmatrix} : \mathcal{H} \oplus \mathcal{M} \rightarrow \mathcal{K} \oplus \mathcal{N}
\]

be a contraction. Then

1. \( (D_T^2)_H = D_M D_X^2 D_M P_H, (D_T^2)_K = D_K \cdot D_X^2 D_K \cdot P_K, \)
2. \( (D_T^2)_H = D_G D_L^2 D_G P_H, (D_T^2)_K = D_F \cdot D_L^2 D_F \cdot P_K, \)
3. \( T \) is isometric if and only if

\[
D_K D_A = 0, \quad D_X D_M = 0, \quad D_F D_D = 0, \quad D_L D_G = 0,
\]

4. \( T \) is co-isometric if and only if

\[
D_M \cdot D_A^* = 0, \quad D_X \cdot D_K^* = 0, \quad D_G \cdot D_D^* = 0, \quad D_L \cdot D_F^* = 0.
\]

If \( T \) is unitary, then \( D_K^* = 0 \iff D_M = 0 \) and \( D_F^* = 0 \iff D_G = 0. \)

Let us give connections between the parametrization of a unitary block-operator matrix given by (2.1) and (2.2).
Proposition 2.3. [9, Proposition 4.7]. Let
\[ U = \begin{bmatrix} -KA^*M + D_{K^*}XD_M & KD_A \\ D_{A^*}M & A \end{bmatrix} \]
\[ = \begin{bmatrix} D & D_{D^*}G \\ FD_D & -FD^*G + D_{F^*}LD_G \end{bmatrix} : \mathcal{M} \oplus \mathcal{N} \to \mathcal{H} \oplus \mathcal{H} \]
be a unitary operator matrix. Then
\[ (2.3) \quad D_D = M^*D_{A^*}M, \quad \mathfrak{D}_D = \text{ran } M^*, \quad D_{D^*} = KD_AK^*, \quad \mathfrak{D}_{D^*} = \text{ran } K, \]
\[ (2.4) \quad F^* = M^*P_{\mathfrak{D}_A^*}, \quad F = M|_{\mathfrak{D}_D}, \quad G = KP_{\mathfrak{D}_A}, \quad G^* = K^*|_{\mathfrak{D}_{D^*}}, \]
\[ (2.5) \quad GFf = KP_{\mathfrak{D}_A}Mf, \quad f \in \mathfrak{D}_D. \]

3. Completely non-unitary contractions

A contraction \( A \) acting in a Hilbert space \( \mathfrak{H} \) is called completely non-unitary \([49]\) if there is no nontrivial reducing subspace of \( A \), on which \( A \) generates a unitary operator. Given a contraction \( A \) in \( \mathfrak{H} \), then there is a canonical orthogonal decomposition \([49, Theorem I.3.2]\)
\[ \mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1, \quad A = A_0 \oplus A_1, \quad A_j = A|_{\mathfrak{H}_j}, \quad j = 0, 1, \]
where \( \mathfrak{H}_0 \) and \( \mathfrak{H}_1 \) reduce \( A \), the operator \( A_0 \) is a completely non-unitary contraction, and \( A_1 \) is a unitary operator. Moreover,
\[ \mathfrak{H}_1 = \left( \bigcap_{n \geq 1} \ker D_{A^n} \right) \cap \left( \bigcap_{n \geq 1} \ker D_{A^n} \right). \]
Since
\[ (3.1) \quad \bigcap_{k=0}^{n-1} \ker (D_A A^k) = \ker D_{A^n}, \quad \bigcap_{k=0}^{n-1} \ker (D_{A^*} A^k) = \ker D_{A^n}, \]
we get
\[ (3.2) \quad \bigcap_{n \geq 1} \ker D_{A^n} = \mathfrak{H} \ominus \overline{\text{span}} \{ A^n D_A \mathfrak{H}, \quad n \in \mathbb{N}_0 \}, \]
\[ \bigcap_{n \geq 1} \ker D_{A^n} = \mathfrak{H} \ominus \overline{\text{span}} \{ A^n D_{A^*} \mathfrak{H}, \quad n \in \mathbb{N}_0 \}. \]
It follows that
\[ (3.3) \quad A \text{ is completely non-unitary} \iff \left( \bigcap_{n \geq 1} \ker D_{A^n} \right) \cap \left( \bigcap_{n \geq 1} \ker D_{A^n} \right) = \{0\} \]
\[ \iff \overline{\text{span}} \{ A^n D_A, \quad A^m D_{A^*}, \quad n, m \in \mathbb{N}_0 \} = \mathfrak{H}. \]
Note that
\[ \ker D_A \supset \ker D_{A^2} \supset \cdots \supset \ker D_{A^n} \supset \cdots, \]
\[ A \ker D_{A^n} \subset \ker D_{A^{n+1}}, \quad n = 2, 3, \ldots. \]
From (3.2) we get that the subspaces \( \bigcap_{n \geq 1} \ker D_{A^n} \) and \( \bigcap_{n \geq 1} \ker D_{A^n} \) are invariant with respect to \( A \) and \( A^* \), respectively, and the operators \( A|_{\bigcap_{n \geq 1} \ker D_{A^n}} \) and \( A^*|_{\bigcap_{n \geq 1} \ker D_{A^n}} \) are unilateral shifts, moreover, these operators are the maximal unilateral shifts contained in \( A \) and
A*, respectively [32, Theorem 1.1, Corollary 1]. By definition [32] the operator A contains a co-shift V if the operator A* contains the unilateral shift V*. In accordance with the terminology of [20], a contraction A in \( H \) is called completely non-isometric (c.n.i.) if there is no nonzero invariant subspace for A on which A is isometric. This equivalent to (see [20])

\[
\bigcap_{n \geq 1} \ker D_{A^n} = \{0\}.
\]

A contraction A is called completely non-co-isometric (c.n.c.-i.) if \( A^* \) is completely non-isometric. Thus, for a completely non-unitary contraction A we have

\[
\bigcap_{n \geq 1} \ker D_{A^n} = \{0\} \iff A \text{ is c.n.i.} \iff A \text{ does not contain a unilateral shift},
\]

\[
\bigcap_{n \geq 1} \ker D_{A^*n} = \{0\} \iff A^* \text{ does not contain a unilateral shift}.
\]

If \( \tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H} \right\} \) is a conservative system, then \( \tau \) is simple if and only if the state space operator A is a completely non-unitary contraction [25, 20]. Moreover,

\[
\mathcal{H}_\tau^0 = \overline{\text{span}} \{ A^n D A^*, \; n \in \mathbb{N}_0 \}, \quad \mathcal{H}_\tau^0 = \overline{\text{span}} \{ A^*n D_A, \; n \in \mathbb{N}_0 \}.
\]

Let A be a contraction in a separable Hilbert space \( \mathcal{H} \). Suppose \( \ker D_A \neq \{0\} \). Define the subspaces [9]

\[
\begin{cases}
\mathcal{H}_{0,0} := \mathcal{H} \\
\mathcal{H}_{n,0} := \ker D_{A^n}, \quad \mathcal{H}_{0,m} := \ker D_{A^*m}, \\
\mathcal{H}_{n,m} := \ker D_{A^n} \cap \ker D_{A^*m}, \quad m, n \in \mathbb{N}.
\end{cases}
\]

Let \( P_{n,m} \) be the orthogonal projection in \( \mathcal{H} \) onto \( \mathcal{H}_{n,m} \). Define the contractions [9]:

\[
A_{n,m} := P_{n,m} A|_{\mathcal{H}_{n,m}} \in \mathcal{L}(\mathcal{H}_{n,m}).
\]

Observe that (see [9]) the following relations are valid:

\[
\ker D_{A^n_{n,m}} = \mathcal{H}_{n+k,m}, \quad \ker D_{A^*n_{n,m}} = \mathcal{H}_{n,m+k}, \quad k = 1, 2, \ldots,
\]

\[
(A_{n,m})_{k,l} = A_{n+k,m+l},
\]

the operators \( A|_{\mathcal{H}_{n,m}} \in \mathcal{L}(\mathcal{H}_{n,m}, \mathcal{H}_{n-1,m+1}) \) are unitary, \( A_{n-1,m}|_{\mathcal{H}_{n,m}} = A|_{\mathcal{H}_{n,m}} \), and

\[
A_{n-1,m+1} A f = AA_{n,m} f, \quad f \in \mathcal{H}_{n,m}, \quad n \geq 1,
\]

i.e., the operators

\[
A_{n,0}, A_{n-1,1}, \ldots, A_{n-k,k}, \ldots, A_{0,n}
\]

are unitarily equivalent. The relation (3.8) yields the following picture for the creation of the operators \( A_{n,m} \):
The process terminates at the $N$-th step if and only if
\[
\ker D_{A^N} = \{0\} \iff \ker D_{A^{N-1} \cap \ker D_{A^*}} = \{0\} \iff \ldots \iff \ker D_{A^{N-k} \cap \ker D_{A^k}} = \{0\} \iff \ldots \ker D_{A^N} = \{0\}.
\]

The following result [49, Proposition V.4.2] is needed in the sequel.

**Theorem 3.1.** Let $\mathcal{M}$ be a separable Hilbert space and let $N(\xi), \xi \in \mathbb{T}$, be an $L(\mathcal{M})$-valued measurable function such that $0 \leq N(\xi) \leq I$. Then there exist a Hilbert space $\mathcal{K}$ and an outer function $\varphi(\lambda) \in S(\mathcal{M}, \mathcal{K})$ satisfying the following conditions:

1. $\varphi^*(\xi)\varphi(\xi) \leq N^2(\xi)$ a.e. on $\mathbb{T}$;
2. if $\mathcal{K}$ is a Hilbert space and $\mathcal{F}(\lambda) \in S(\mathcal{M}, \mathcal{K})$ is such that $\mathcal{F}^*(\xi)\mathcal{F}(\xi) \leq N^2(\xi)$ a.e. on $\mathbb{T}$, then $\mathcal{F}^*(\xi)\mathcal{F}(\xi) \leq \varphi^*(\xi)\varphi(\xi)$ a.e. on $\mathbb{T}$.

Moreover, the function $\varphi(\lambda)$ is uniquely defined up to a left constant unitary factor.

Assume that $\Theta \in S(\mathcal{M}, \mathcal{M})$ and denote by $\varphi_\Theta(\xi)$ and $\psi_\Theta(\xi)$, $\xi \in \mathbb{T}$ the outer functions which are solutions of the factorization problem described in Theorem 3.1 for $N^2(\xi) = I - \Theta^*(\xi)\Theta(\xi)$ and $N^2(\xi) = I - \Theta(\xi)\Theta^*(\xi)$, respectively. Clearly, if $\Theta(\lambda)$ is inner or co-inner, then $\varphi_\Theta = 0$ or $\psi_\Theta = 0$, respectively. The functions $\varphi_\Theta(\lambda)$ and $\psi_\Theta(\lambda)$ are called the right and left defect functions (or the spectral factors), respectively, associated with $\Theta(\lambda)$; cf. [19, 21, 22, 23, 32]. The following result has been established in [32, Theorem 1.1, Corollary 1] (see also [22, Theorem 3], [23, Theorem 1.5]).

**Theorem 3.2.** Let $\Theta \in S(\mathcal{M}, \mathcal{M})$ and let $\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \mathcal{M}, \mathcal{N}, \mathcal{H} \right\}$ be a simple conservative system with transfer function $\Theta$. Then

1. the functions $\varphi_\Theta(\lambda)$ and $\psi_\Theta(\lambda)$ take the form
   \[
   \varphi_\Theta(\lambda) = P_\Omega(I_\mathcal{B} - \lambda A)^{-1}B, \\
   \psi_\Theta(\lambda) = C(I_\mathcal{B} - \lambda A)^{-1}A^*\Omega_*,
   \]
   where
   \[
   \Omega = (\mathcal{H}_r^\perp \ominus A(\mathcal{H}_r^\perp) \ominus (\mathcal{H}_r^\perp) \ominus A^*(\mathcal{H}_r^\perp) \ominus \Omega_*, \]
2. $\varphi_\Theta(\lambda) = 0$ ($\psi_\Theta(\lambda) = 0$) if and only if the system $\tau$ is observable (controllable).
The defect functions play an essential role in the problems of the system theory, in particular, in the problem of similarity and unitary similarity of the minimal passive systems with equal transfer functions \[16, 17\] and in the problem of optimal and (*) optimal realizations of the Schur function \[13, 14\].

4. **Conservative realization of the Schur algorithm**

**Theorem 4.1.** \[9\]. 1) Let the system

\[
\tau = \left\{ \begin{bmatrix} D & D_D \cdot G \\ FD_D & -FD^*G + D_F \cdot LD_G \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H} \right\}
\]

be conservative and simple and let \( \Theta \) be its transfer function. Suppose that the first associated function \( \Theta_1 \) is non-unitary constant. Then the systems

\[
\zeta_1 = \left\{ \begin{bmatrix} GF & G \\ LD_GF & LD_G \end{bmatrix} ; \mathcal{D}_D, \mathcal{D}_D^*, \mathcal{D}_F^* \right\},
\]

\[
\zeta_2 = \left\{ \begin{bmatrix} GF & GL \\ D_GF & D_GL \end{bmatrix} ; \mathcal{D}_D, \mathcal{D}_D^*, \mathcal{D}_G \right\}
\]

are conservative and simple and their transfer functions are equal to \( \Theta_1 \).

2) Let \( \Theta \in \mathcal{S}(\mathcal{M}, \mathcal{N}) \), \( \Gamma_0 = \Theta(0) \) and let \( \Theta_1 \) be the first associated function. Suppose

\[
\tau = \left\{ \begin{bmatrix} \Gamma_0 & C \\ B & A \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H} \right\}
\]

is a simple conservative system with transfer function \( \Theta \). Then the simple conservative systems

\[
\zeta_1 = \left\{ \begin{bmatrix} D_{\Gamma_0}^{-1}C(D_{\Gamma_0}^{-1}B^*) & D_{\Gamma_0}^{-1}C \mid \ker D_A^* \\ AP_{\ker D_A}D_{A^*}B & P_{\ker D_A}D_{A^*}A \mid \ker D_A^* \end{bmatrix} ; \mathcal{D}_{\Gamma_0}, \mathcal{D}_{\Gamma_0}^*, \ker D_A^* \right\},
\]

\[
\zeta_2 = \left\{ \begin{bmatrix} D_{\Gamma_0}^{-1}C(D_{\Gamma_0}^{-1}B^*) & D_{\Gamma_0}^{-1}CA \mid \ker D_A \\ P_{\ker D_A}D_{A^*}B & P_{\ker D_A}D_{A^*}A \mid \ker D_A \end{bmatrix} ; \mathcal{D}_{\Gamma_0}, \mathcal{D}_{\Gamma_0}^*, \ker D_A \right\}
\]

have transfer functions \( \Theta_1 \). Here the operators \( D_{\Gamma_0}^{-1}, D_{\Gamma_0}^{-1}, \) and \( D_{A^*}^{-1} \) are the Moore–Penrose pseudo-inverses.

**Theorem 4.2.** \[9\]. Let \( \Theta \in \mathcal{S}(\mathcal{M}, \mathcal{N}) \) and let \( \tau_0 = \left\{ \begin{bmatrix} \Gamma_0 & C \\ B & A \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H} \right\} \) be a simple conservative realization of \( \Theta \). Then for each \( n \geq 1 \) the unitarily equivalent simple conservative systems

\[
\tau_n^{(k)} = \left\{ \begin{bmatrix} \Gamma_n & D_{\Gamma_n}^{-1} \cdots D_{\Gamma_0}^{-1} (CA^{n-k}) \\ A^k \left( D_{\Gamma_n}^{-1} \cdots D_{\Gamma_0}^{-1} (B^* \mid \mathcal{H}_{n,0}) \right)^* A_{n-k,k} \end{bmatrix} ; \mathcal{D}_{\Gamma_n}, \mathcal{D}_{\Gamma_n}^*, \mathcal{H}_{n-k,k} \right\},
\]

\( k = 0, 1, \ldots, n \)

are realizations of the \( n \)-th associated function \( \Theta_n \) of the function \( \Theta \). Here the operator

\[
B_{n} = \left( D_{\Gamma_n}^{-1} \cdots D_{\Gamma_0}^{-1} (B^* \mid \mathcal{H}_{n,0}) \right)^* \in \mathcal{L}(\mathcal{D}_{\Gamma_n}, \mathcal{H}_{n,0})
\]

is the adjoint to the operator

\[
D_{\Gamma_n}^{-1} \cdots D_{\Gamma_0}^{-1} (B^* \mid \mathcal{H}_{n,0}) \in \mathcal{L}(\mathcal{H}_{n,0}, \mathcal{D}_{\Gamma_{n-1}}).
\]
Notice that the systems $\tau_n^{(0)}, \tau_n^{(1)}, \ldots, \tau_n^{(n)}$ are unitarily similar. In addition
\[
(\tau_n^{(k)})_m = \tau_{n+k}^{(k+l)}, \quad k = 0, 1, \ldots, n, \quad l = 0, \ldots m.
\]
This property can be illustrated by the following picture

5. SOME NEW PROPERTIES OF THE KREĬN SHORTED OPERATORS

The next statement is well known.

**Proposition 5.1.** [6] Let $\mathcal{K}$ be a subspace in $\mathcal{H}$. Then

1. if $S_1$ and $S_2$ are nonnegative selfadjoint operators then
   \[
   (S_1 + S_2)_{\mathcal{K}} \geq (S_1)_{\mathcal{K}} + (S_2)_{\mathcal{K}};
   \]

2. $S_1 \geq S_2 \geq 0 \Rightarrow (S_1)_{\mathcal{K}} \geq (S_2)_{\mathcal{K}}$;

3. if $\{S_n\}$ is a nonincreasing sequence of nonnegative bounded selfadjoint operators and $S = s - \lim_{n \to \infty} S_n$ then
   \[
   s - \lim_{n \to \infty} (S_n)_{\mathcal{K}} = S_{\mathcal{K}}.
   \]

Let $\mathcal{K}^\perp = \mathcal{H} \ominus \mathcal{K}$. Then a bounded selfadjoint operator $S$ has the block-matrix form
\[
S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{pmatrix} : \begin{array}{c} \mathcal{K} \\ \mathcal{K}^\perp \end{array} \rightarrow \begin{array}{c} \mathcal{K} \\ \mathcal{K}^\perp \end{array}.
\]

It is well known (see [38]) that

the operator $S$ is nonnegative if and only if

\[
S_{22} \geq 0, \quad \text{ran} S_{12}^* \subset \text{ran} S_{22}^{1/2}, \quad S_{11} \geq \left(S_{22}^{-1/2} S_{12}^*\right)^* \left(S_{22}^{-1/2} S_{12}^*\right)
\]

and the operator $S_\mathcal{K}$ is given by the block matrix
\[
S_\mathcal{K} = \begin{pmatrix} S_{11} - \left(S_{22}^{-1/2} S_{12}^*\right)^* & 0 \\ 0 & 0 \end{pmatrix}.
\]
If $S_{22}^{-1} \in \mathbb{L}(\mathcal{K}^\perp)$ then the right hand side of (5.2) is of the form

$$
\begin{pmatrix}
S_{11} - S_{12}S_{22}^{-1}S_{12}^* & 0 \\
0 & 0
\end{pmatrix}
$$

and is called the *Schur complement* of the matrix $S$. From (5.2) it follows that

$$
S_{K} = 0 \iff \text{ran } S_{12}^* \subset \text{ran } S_{22}^{1/2} \quad \text{and} \quad S_{11} = \left( S_{22}^{-1/2}S_{12}^* \right)^* \left( S_{22}^{-1/2}S_{12} \right).
$$

**Proposition 5.2.** Let a bounded nonnegative self-adjoint operator $S$ be given by

$$
S = \begin{bmatrix}
S & 0 \\
0 & I
\end{bmatrix} : \mathcal{L} \oplus \mathcal{M} \rightarrow \mathcal{L} \oplus \mathcal{M}
$$

and let $\mathcal{K}$ be a subspace of $\mathcal{L}$. Then

$$
S_{\mathcal{K}} = S_{\mathcal{K}}P_{\mathcal{L}}.
$$

**Proof.** The inclusion $\mathcal{K} \subset \mathcal{L}$ yields

$$
\Omega := \{ f \in \text{ran } S : S^{1/2}f \in \mathcal{K} \} = \{ f \in \text{ran } S : S^{1/2}f \in \mathcal{K} \} \subset \mathcal{L}.
$$

It follows that

$$
S_{\mathcal{K}} = S^{1/2}P_{\Omega}S^{1/2} = S^{1/2}P_{\Omega}S^{1/2}P_{\mathcal{L}} = S_{\mathcal{K}}P_{\mathcal{L}}.
$$

**Proposition 5.3.** Let $S$ be a bounded nonnegative selfadjoint operator in the Hilbert space $\mathcal{H}$, $P$ be an orthogonal projection in $\mathcal{H}$, and let $\mathcal{K}$ be a subspace in $\mathcal{H}$ such that $\mathcal{K} \subseteq \text{ran } (P)$. Then

$$
(PSP)_{\mathcal{K}} \geq S_{\mathcal{K}}.
$$

**Proof.** Let $f \in \mathcal{K}$. Then by (1.7) and taking into account that $PK^\perp \subset K^\perp$ we get

$$
(PSP)_{\mathcal{K}}f, f = \inf_{\varphi \in K^\perp} \left\{ \|S^{1/2}P(f + \varphi)\|^2 \right\}
$$

$$
= \inf_{\varphi \in K^\perp} \left\{ \|S^{1/2}(f + P\varphi)\|^2 \right\} = \inf_{\psi \in \text{ran } (P) \cap K^\perp} \left\{ \|S^{1/2}(f + \psi)\|^2 \right\}
$$

$$
\geq \inf_{\varphi \in K^\perp} \left\{ \|S^{1/2}(f + \varphi)\|^2 \right\} = (S_{\mathcal{K}}f, f).
$$

Now the equalities

$$
(PSP)_{\mathcal{K}}| K^\perp = (S)_{\mathcal{K}}| K^\perp = 0,
$$

yield that $(PSP)_{\mathcal{K}} \geq S_{\mathcal{K}}$. \hfill \Box

**Remark 5.4.** Let $S \geq 0$ be given by a block-operator matrix

$$
S = \begin{pmatrix}
S_{11} & S_{12} \\
S_{12}^* & S_{22}
\end{pmatrix} : \mathcal{K} \oplus \mathcal{K}^\perp \rightarrow \mathcal{K} \oplus \mathcal{K}^\perp,
$$

and let $P = P_{\mathcal{K}}$. Then $(P_{\mathcal{K}}SP_{\mathcal{K}})_{\mathcal{K}} = P_{\mathcal{K}}SP_{\mathcal{K}} = \begin{pmatrix} S_{11} & 0 \\ 0 & 0 \end{pmatrix}$. If $S_{12} \neq 0$, then from (5.1) and (5.2) it follows that $S_{\mathcal{K}}| K \neq S_{11}$. Therefore, in general $(PSP)_{\mathcal{K}} \neq S_{\mathcal{K}}$.

**Theorem 5.5.** Let $X$ be a nonnegative contraction in the Hilbert space $\mathcal{H}$. Assume
(1) there is a sequence \( \{X_n\} \) of nonnegative contractions strongly converging to \( X \),

(2) there is a subspace \( \mathcal{K} \) in \( \mathcal{H} \) such that the sequence of operators \( \{(I - X_n)\mathcal{K}\} \) is non-increasing.

Then
\[
(5.3) \quad s - \lim_{n \to \infty} (I - X_n)\mathcal{K} \leq (I - X)\mathcal{K}.
\]

Proof. We will use the equality (see [8, Theorem 2.2])
\[
(5.4) \quad (I - X)\mathcal{K} = P_\mathcal{K} - ((I - X^{1/2}P_\mathcal{K}X^{1/2})^{-1/2}X^{1/2}P_\mathcal{K})^*(I - X^{1/2}P_\mathcal{K}X^{1/2})^{-1/2}X^{1/2}P_\mathcal{K}.
\]

for a nonnegative selfadjoint contraction \( X \) in \( \mathcal{H} \).

As is well-known if \( B \) is an arbitrary nonnegative selfadjoint operator, then
\[
(5.5) \quad \sup_{g \in \text{dom} B^* \{0\}} \frac{|(h, g)|^2}{(Bg, g)} = \begin{cases} ||B^{-1/2}h||^2, & h \in \text{ran} B^{1/2} \\
+\infty, & h \notin \text{ran} B^{1/2} \end{cases},
\]

where \( B^{-1/2} \) is the Moore-Penrose pseudo-inverse. Hence, equality (5.4) for all \( X_n \) and each \( f, g \in \mathcal{H} \) yields
\[
\frac{|(X_n^{1/2}P_\mathcal{K}f, g)|^2}{||g||^2 - ||P_\mathcal{K}X_n^{1/2}g||^2} \leq ||P_\mathcal{K}f||^2 - ((I - X_n)\mathcal{K}f, f).
\]

Since the sequence of operators \( \{(I - X_n)\mathcal{K}\} \) is non-increasing, there exists
\[
W := s - \lim_{n \to \infty} (I - X_n)\mathcal{K}.
\]

Therefore
\[
\frac{|(X_n^{1/2}P_\mathcal{K}f, g)|^2}{||g||^2 - ||P_\mathcal{K}X_n^{1/2}g||^2} \leq ||P_\mathcal{K}f||^2 - (Wf, f).
\]

One can prove that
\[
X = s - \lim_{n \to \infty} X_n \Rightarrow X^{1/2} = s - \lim_{n \to \infty} X_n^{1/2}
\]

Therefore
\[
\frac{|(X^{1/2}P_\mathcal{K}f, g)|^2}{||g||^2 - ||P_{\mathcal{K}+}X^{1/2}g||^2} \leq ||P_\mathcal{K}f||^2 - (Wf, f).
\]

By virtue (5.5) for \( B = I_\mathcal{H} - X^{1/2}P_\mathcal{K}X^{1/2} \), we obtain
\[
||(I_\mathcal{H} - X^{1/2}P_\mathcal{K}X^{1/2})^{-1/2}X^{1/2}P_\mathcal{K}f||^2 \leq ||P_\mathcal{K}f||^2 - (Wf, f).
\]

Now (5.4) yields (5.3). \( \square \)

6. Main results

6.1. Shorted operators for defect operators of Toeplitz matrices. Let \( \Theta \in S(\mathfrak{M}, \mathfrak{N}) \) and let \( \Theta(\lambda) = \sum_{n=0}^{\infty} \lambda^n C_n \). Recall that by definition \( \tilde{\Theta}(\lambda) := \Theta^*(\lambda) \), \( |\lambda| < 1 \). We identify \( \mathfrak{M} \) \((\mathfrak{N}, \text{respectively})\) with the subspace
\[
\mathfrak{M} \oplus \{0\} \oplus \{0\} \oplus \cdots \oplus \{0\} \quad \text{and} \quad \mathfrak{N} \oplus \{0\} \oplus \{0\} \oplus \cdots \oplus \{0\}.
\]
in \( M^{n+1} (\mathfrak{N}^{n+1}) \), and with \( M \oplus \bigoplus_{k=1}^{\infty} \{0\} \) \( (M \oplus \bigoplus_{k=1}^{\infty} \{0\}) \) in \( l_2(M) \) \((l_2(M))\).

**Theorem 6.1.** Let \( \Theta \in S(M, N) \) and let \( \{\Gamma_0, \Gamma_1, \cdots\} \) be the Schur parameters of \( \Theta \). Then for each \( n \) the relations

\[
\begin{align*}
(D_{T_{\Theta,n}}^2)_{\mathfrak{N}} &= D_{\Gamma_0}D_{\Gamma_1} \cdots D_{\Gamma_{n-1}}D_{\Gamma_n}^2D_{\Gamma_{n-1}} \cdots D_{\Gamma_1}D_{\Gamma_0}P_{\mathfrak{N}}, \\
(D_{T_{\Theta,n}}^2)_{\mathfrak{R}} &= D_{\Gamma_0}D_{\Gamma_1} \cdots D_{\Gamma_{n-1}}D_{\Gamma_n}^2D_{\Gamma_{n-1}} \cdots D_{\Gamma_1}D_{\Gamma_0}P_{\mathfrak{R}},
\end{align*}
\]

hold.

**Proof.** Let

\[
\mathfrak{M}_n := \{0\} \oplus \{0\} \oplus \cdots \oplus \{0\} \oplus M, \quad \mathfrak{N}_n := \{0\} \oplus \{0\} \oplus \cdots \oplus \{0\} \oplus N.
\]

Clearly, the operator

\[
S_{\Theta,n} := \begin{bmatrix}
0 & 0 & \cdots & 0 & C_0 \\
0 & 0 & \cdots & C_0 & C_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
C_0 & C_1 & \cdots & C_{n-1}
\end{bmatrix} \in \mathcal{M}(\mathfrak{M}^{n+1}, \mathfrak{N}^{n+1})
\]

is a contraction. The matrix \( S_{\Theta,n} \) we represent in the block matrix form

\[
S_{\Theta,n} = \begin{bmatrix}
Q_{n-1} & B_{n-1} \\
B_{n-1}^T & C_{n-1}
\end{bmatrix} : \mathfrak{M}_n \oplus \mathfrak{N}_n \rightarrow \mathfrak{M}_n \oplus \mathfrak{N}_n,
\]

where

\[
Q_{n-1} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & C_0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & C_0 & C_1 & \cdots & C_{n-1}
\end{bmatrix}, \quad B_{n-1} = \begin{bmatrix}
C_0 \\
C_1 \\
\vdots \\
C_{n-1}
\end{bmatrix}, \quad B_{n-1}^T = \begin{bmatrix}
C_0 & C_1 & \cdots & C_{n-1}
\end{bmatrix}.
\]

Since \( S_{\Theta,n} \) is a contraction, by Theorem 2.1 (see (2.2)) we have

\[
B_{n-1} = D_{Q_{n-1}}G_{n-1}, \quad B_{n-1}^T = F_{n-1}D_{Q_{n-1}}^*, \\
C_{n} = -F_{n-1}Q_{n-1}^*G_{n-1} + D_{F_{n-1}}^*L_{n-1}D_{G_{n-1}}.
\]

In [19] it is proved that

\[
||D_{F_{n-1}}^*f||^2 = ||D_{F_{n-1}} \cdots D_{\Gamma_0}^*f||^2, \quad f \in \mathfrak{N}, \\
||D_{G_{n-1}}^*h||^2 = ||D_{\Gamma_{n-1}} \cdots D_{\Gamma_0}h||^2, \quad h \in \mathfrak{M}.
\]

Therefore,

\[
D_{F_{n-1}}^*f = Y_{n-1}D_{\Gamma_{n-1}}^* \cdots D_{\Gamma_0}^*f, \quad f \in \mathfrak{N}, \\
D_{G_{n-1}}^*h = Z_{n-1}D_{\Gamma_{n-1}} \cdots D_{\Gamma_0}h, \quad h \in \mathfrak{M}.
\]
where \( Y_{n-1} \in \mathcal{L}(\mathcal{D}_{n-1}, \mathcal{D}_{n-1}) \) and \( Z_{n-1} \in \mathcal{L}(\mathcal{D}_{n-1}, \mathcal{D}_{n-1}) \) are unitary operators. It follows that

\[
\Gamma_n = Y_{n-1} Z_{n-1}, \quad D_{\Gamma_n}^2 = Z_{n-1} D_{\Gamma_n}^2 Z_{n-1}, \quad D_{\Gamma_n}^2 = Y_{n-1} D_{\Gamma_n}^2 Y_{n-1}.
\]

Hence

\[
\begin{align*}
D_{\Gamma_n}^2 &= D_{\Gamma_{n-1}}^2 D_{\Gamma_{n-1}}^2 \quad \forall n \geq 1, \\
D_{\Gamma_n}^2 &= D_{\Gamma_{n-1}}^2 D_{\Gamma_{n-1}}^2 \quad \forall n \geq 1.
\end{align*}
\]

Now from Corollary 2.2 and (6.4) it follows that

\[
\begin{align*}
D_{\Gamma_n}^2 &= D_{\Gamma_{n-1}}^2 \quad \forall n \geq 1, \\
D_{\Gamma_n}^2 &= D_{\Gamma_{n-1}}^2 \quad \forall n \geq 1.
\end{align*}
\]

Let the operator \( J_n \in \mathcal{L}(\mathfrak{M}^{n+1}, \mathfrak{M}^{n+1}) \) be given by

\[
J_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & I_{\mathfrak{M}} \\ 0 & 0 & \cdots & I_{\mathfrak{M}} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_{\mathfrak{M}} & 0 & \cdots & 0 & 0 \end{bmatrix}.
\]

The operator \( J_n \) is selfadjoint and unitary, \( J_n \mathfrak{M}_n = \mathfrak{M}_n \), and, clearly,

\[
S_{\Theta, n} = T_{\Theta, n} J_n, \quad D_{S_{\Theta, n}}^2 = J_n D_{T_{\Theta, n}}^2 J_n.
\]

It follows that

\[
\left( D_{S_{\Theta, n}}^2 \right)_{\mathfrak{M}_n} = J_n \left( D_{T_{\Theta, n}}^2 \right)_{\mathfrak{M}_n} J_n.
\]

This relation and (6.5) lead to (6.1). Replacing \( \Theta \) by \( \tilde{\Theta} \) we get (6.2). \( \square \)

Notice that the relation \( S_{\Theta, n}^* = J_n T_{\Theta, n}^* \) yields

\[
\left( D_{T_{\Theta, n}}^2 \right)_{\mathfrak{M}_n} = \left( D_{S_{\Theta, n}}^2 \right)_{\mathfrak{M}_n} = D_{\Gamma_0} D_{\Gamma_1} \cdots D_{\Gamma_{n-1}} D_{\Gamma_n}^2 D_{\Gamma_{n-1}} \cdots D_{\Gamma_1} D_{\Gamma_0} P_{\mathfrak{M}_n}.
\]

The next statement is an immediate consequence of equalities (6.1), (6.2), and (1.8).

**Corollary 6.2.** The following conditions are equivalent:

(i) \( \mathfrak{M} \subset \text{ran} D_{T_{\Theta, n}} \),

(ii) \( \mathfrak{N} \subset \text{ran} D_{T_{\Theta, n}} \),

(iii) operators \( \Gamma_0, \ldots, \Gamma_n \) have norms less than 1.

**Theorem 6.3.** The equalities

\[
\begin{align*}
\left( D_{T_{\Theta}}^2 \right)_{\mathfrak{M}} &= s - \lim_{n \to \infty} \left( D_{\Gamma_0} D_{\Gamma_1} \cdots D_{\Gamma_{n-1}} D_{\Gamma_n}^2 D_{\Gamma_{n-1}} \cdots D_{\Gamma_1} D_{\Gamma_0} \right) P_{\mathfrak{M}}, \\
\left( D_{T_{\Theta}}^2 \right)_{\mathfrak{N}} &= s - \lim_{n \to \infty} \left( D_{\Gamma_0} D_{\Gamma_1} \cdots D_{\Gamma_{n-1}} D_{\Gamma_n}^2 D_{\Gamma_{n-1}} \cdots D_{\Gamma_1} D_{\Gamma_0} \right) P_{\mathfrak{N}}.
\end{align*}
\]

hold.
Proof. Let $P_n$ be the orthogonal projection onto $\mathcal{M}^{n+1}$ in $L_2(\mathcal{M})$ and let $\widehat{T}_{\Theta,n} := P_n T_\Theta P_n$. Then $T_n$ takes the block operator matrix form

$$\widehat{T}_{\Theta,n} = \begin{bmatrix} T_{\Theta,n} & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{M}^{n+1} \oplus (\mathcal{M}^{n+1})^\perp \to \mathcal{M}^{n+1} \oplus (\mathcal{M}^{n+1})^\perp,$$

where

$$(\mathcal{M}^{n+1})^\perp = L_2(\mathcal{M}) \otimes \mathcal{M}^{n+1}, \quad (\mathcal{M}^{n+1})^{\perp} = L_2(\mathcal{M}) \otimes \mathcal{M}^{n+1}.$$  

Hence

$$D^2_{T_{\Theta,n}} = \begin{bmatrix} D^2_{T_{\Theta,n}} & 0 \\ 0 & I \end{bmatrix}.$$ 

Since $\mathcal{M} \subset \mathcal{M}^{n+1}$, from Proposition 5.2 it follows that

$$(6.9) \quad \left( D^2_{T_{\Theta,n}} \right) \mathcal{M} = \left( D^2_{T_{\Theta,n}} \right) \mathcal{M}^{\perp} + \mathcal{M}.$$ 

In addition

$$\|D_{T_{\Theta,n}}^2 f\|^2 = \|D_{T_{\Theta}} P_n f\|^2 + \| (I - P_n) f\|^2 + \| (I - P_n) T_{\Theta} P_n f\|^2, \quad f \in L_2(\mathcal{M}).$$

It follows that

$$D^2_{T_{\Theta,n}} \geq P_n D^2_{T_{\Theta}} P_n.$$ 

Using Propositions 5.1 and 5.3 we get

$$(6.10) \quad \left( D^2_{T_{\Theta,n}} \right)_{\mathcal{M}} \geq (P_n D^2_{T_{\Theta}} P_n)_{\mathcal{M}} \geq (D^2_{T_{\Theta}})_{\mathcal{M}}.$$ 

Let $X = T_{\Theta} T_{\Theta}$ and $X_n = \widehat{T}_{\Theta,n} \widehat{T}_{\Theta,n} = P_n T_{\Theta} P_n T_{\Theta} P_n$, $n = 1, 2, \ldots$. Then $X$ and $X_n$ are nonnegative selfadjoint contractions and

$$s - \lim_{n \to \infty} X_n = X, \quad D^2_{T_{\Theta,n}} = I - X_n, \quad D^2_{T_{\Theta}} = I - X.$$ 

From (6.10) it follows that the sequence $\{D^2_{T_{\Theta,n}}\}_{n=1}^\infty$ is non-increasing. Therefore, by Theorem 5.5 we get that

$$s - \lim_{n \to \infty} \left( D^2_{T_{\Theta,n}} \right)_{\mathcal{M}} \leq (D^2_{T_{\Theta}})_{\mathcal{M}}.$$ 

On the other hand (6.10) implies

$$s - \lim_{n \to \infty} \left( D^2_{T_{\Theta,n}} \right)_{\mathcal{M}} \geq (D^2_{T_{\Theta}})_{\mathcal{M}}.$$ 

Hence

$$s - \lim_{n \to \infty} \left( D^2_{T_{\Theta,n}} \right)_{\mathcal{M}} = (D^2_{T_{\Theta}})_{\mathcal{M}}.$$  

Now from (6.10) and (6.9) we obtain (6.7) and similarly (6.8). \[\square\]

Notice that it is proved the equalities

$$(6.11) \quad s - \lim_{n \to \infty} \left( D^2_{T_{\Theta,n}} \right)_{\mathcal{M}} = (D^2_{T_{\Theta}})_{\mathcal{M}} \mathcal{M},$$

$$s - \lim_{n \to \infty} \left( D^2_{T_{\Theta,n}} \right)_{\mathcal{M}} = (D^2_{T_{\Theta}})_{\mathcal{M}} \mathcal{M}.$$

**Corollary 6.4.** The following conditions are equivalent:

(i) $\mathcal{M} \subset \text{ran} \ D_{T_{\Theta}}$,
(ii) $\mathcal{M} \subset \text{ran } D_{T\Theta}$,
(iii) all Schur parameters $\{\Gamma_k\}_{k=0}^{\infty}$ of $\Theta$ have norms less than 1.

Proof. Since $(D^2_{T\Theta}) M \leq (D^2_{T\Theta}) M$ for each $n$, the condition $\mathcal{M} \subset \text{ran } D_{T\Theta}$ implies $\mathcal{M} \subset \text{ran } D_{T\Theta}$ for each $n$. Then equivalence of (i), (ii), (iii) follows from Corollary 6.2. □

Let $H^2(\mathcal{M})$, $H^2(\mathcal{N})$ be the Hardy spaces [49]. Denote by $\mathcal{P}(\mathcal{M})$ ($\mathcal{P}(\mathcal{N})$) the linear manifolds of all polynomial from $H^2(\mathcal{M})$ ($H^2(\mathcal{N})$) and by $\mathcal{P}_n(\mathcal{M})$ ($\mathcal{P}_n(\mathcal{N})$) the linear space of all polynomials of degree at most $n$. By $P^2_n$ ($P^3_n$) we denote the orthogonal projection in $H^2(\mathcal{M})$ ($H^2(\mathcal{N})$) onto $\mathcal{P}_n(\mathcal{M})$ ($\mathcal{P}_n(\mathcal{N})$).

**Theorem 6.5.** Let $\Theta \in S(\mathcal{M}, \mathcal{N})$ and let $\{\Gamma_0, \Gamma_1, \cdots\}$ be the Schur parameters of $\Theta$. Then

$$
\inf_{p \in \mathcal{P}_n(\mathcal{M}), p(0) = 0} \left\{ \frac{2\pi}{2} \int_0^{2\pi} \left( \|f - p(e^{it})\|_{\mathcal{M}}^2 - \|P^2_n\Theta(e^{it})(f - p(e^{it}))\|_{\mathcal{M}}^2 \right) dt \right\}
= \|D_{\Gamma_n} D_{\Gamma_{n-1}} \cdots D_{\Gamma_1} D_{\Gamma_0} f\|_2^2, \quad f \in \mathcal{M},
$$

$$
\inf_{p \in \mathcal{P}_n(\mathcal{M}), p(0) = 0} \left\{ \frac{2\pi}{2} \int_0^{2\pi} \left( \|f - p(e^{it})\|_{\mathcal{M}}^2 - \|\Theta(e^{it})(f - p(e^{it}))\|_{\mathcal{M}}^2 \right) dt \right\}
= \lim_{n \to \infty} \|D_{\Gamma_n} D_{\Gamma_{n-1}} \cdots D_{\Gamma_1} D_{\Gamma_0} f\|_2^2, \quad f \in \mathcal{M},
$$

$$
\inf_{p \in \mathcal{P}_n(\mathcal{N}), p(0) = 0} \left\{ \frac{2\pi}{2} \int_0^{2\pi} \left( \|h - p(e^{it})\|_{\mathcal{N}}^2 - \|P^2_n\Theta(e^{it})(h - p(e^{it}))\|_{\mathcal{N}}^2 \right) dt \right\}
= \|D_{\Gamma_n} D_{\Gamma_{n-1}} \cdots D_{\Gamma_1} D_{\Gamma_0} h\|_2^2, \quad h \in \mathcal{N},
$$

$$
\inf_{p \in \mathcal{P}_n(\mathcal{N}), p(0) = 0} \left\{ \frac{2\pi}{2} \int_0^{2\pi} \left( \|h - p(e^{it})\|_{\mathcal{N}}^2 - \|\Theta(e^{it})(h - p(e^{it}))\|_{\mathcal{N}}^2 \right) dt \right\}
= \lim_{n \to \infty} \|D_{\Gamma_n} D_{\Gamma_{n-1}} \cdots D_{\Gamma_1} D_{\Gamma_0} h\|_2^2, \quad h \in \mathcal{N}.
$$

Proof. One can easily see that

$$
\|T_{\Theta, \bar{a}}\|_{l_2(\mathcal{M})}^2 = \|\Theta a\|_{H^2(\mathcal{M})}^2 = \frac{1}{2\pi} \int_0^{2\pi} \|\Theta(e^{it}) a(e^{it})\|_{\mathcal{M}}^2 dt,
$$

where $\bar{a} = (a_0, a_1, \ldots) \in l_2(\mathcal{M})$, $a(z) = \sum_{k=0}^{\infty} a_k z^k \in H^2(\mathcal{M})$. If $p(z) = p_0 + p_1 z + \cdots p_n z^n$ and $\bar{p} = (p_0, p_1, \ldots, p_n) \in \mathcal{M}^{n+1}$, then

$$
\|T_{\Theta, \bar{p}}\|_{l_2(\mathcal{M})}^2 = \|P^2_n\Theta \bar{p}\|_{H^2(\mathcal{M})}^2.
$$

To complete the proof of the theorem we use definition (1.7) of the shorted operator and equalities (6.1), (6.7), (6.1), and (6.8). □
6.2. Schur parameters, controllability, and observability.

**Theorem 6.6.** Let \( \tau = \bigg\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \mathcal{M}, \mathcal{N}, \mathcal{H} \bigg\} \) be a simple conservative system with transfer function \( \Theta \). Let \( \{ \Gamma_0, \Gamma_1, \ldots \} \) be the Schur parameters of \( \Theta \). Then for each \( n \) the relations

\[
||P_{n,0}Bh||^2 = ||D_{\Gamma_n}D_{\Gamma_{n-1}} \cdots D_{\Gamma_0}h||^2, \quad h \in \mathcal{M},
\]

\[
||P_{0,n}C^*f||^2 = ||D_{\Gamma_n}D_{\Gamma_{n-1}} \cdots D_{\Gamma_0}f||^2, \quad f \in \mathcal{N}
\]

hold.

**Proof.** Clearly \( D = \Gamma_0 \). The unitary operator

\[
U = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \mathcal{M} \oplus \mathcal{N} \rightarrow \mathcal{H} \oplus \mathcal{H}
\]

admits the representations (see Theorem 2.1)

\[
U = \begin{bmatrix} -KA^*M + DK^*XD_M & KD_A \\ D_A^*M & A \end{bmatrix} = \begin{bmatrix} \Gamma_0 & D_{\Gamma_0}^*G \\ FD_{\Gamma_0} & -FD^*G + D_{F^*L}D_G \end{bmatrix}.
\]

From Theorem 4.1 it follows \( \Gamma_1 = GF \). Equality (2.5) yields

\[
\Gamma_1 = KP_{\mathcal{D}_A}M.
\]

Now taking into account that \( F \in \mathcal{L}(\mathcal{D}_{\Gamma_0}, \mathcal{M}) \) is isometry and relation (2.4), for \( f \in \mathcal{D}_{\Gamma_0} \) we get

\[
||D_{\Gamma_1}f||^2 = ||f||^2 - ||KP_{\mathcal{D}_A}Mf||^2
= ||Mf||^2 - ||P_{\mathcal{D}_A}Mf||^2 = ||P_{1,0}Mf||^2.
\]

Hence

\[
||D_{\Gamma_1}D_{\Gamma_0}f||^2 = ||P_{1,0}MD_{\Gamma_0}f||^2, \quad f \in \mathcal{M}.
\]

Because \( M^* \in \mathcal{L}(\mathcal{D}_{\Gamma_0}, \mathcal{M}) \) is an isometry, from (2.3) we have \( MD_{\Gamma_0} = D_{A^*}M = B \). Thus

\[
||D_{\Gamma_1}D_{\Gamma_0}f||^2 = ||P_{1,0}Bf||^2, \quad f \in \mathcal{M}.
\]

By Theorem 4.2 the simple conservative system

\[
\tau^{(0)}_1 = \bigg\{ \begin{bmatrix} \Gamma_1 \\ (D_{\Gamma_0}^{-1}(B^* \mid \mathcal{H}_{1,0}))^* \\ D_{\Gamma_0}^{-1}(CA) \end{bmatrix} : \mathcal{D}_{\Gamma_0}, \mathcal{D}_{\Gamma_0}, \mathcal{H}_{1,0} \bigg\}
\]

has transfer function \( \Theta_1 \). Let

\[
B_1 = (D_{\Gamma_0}^{-1}(B^* \mid \mathcal{H}_{1,0}))^* \in \mathcal{L}(\mathcal{D}_{\Gamma_0}, \mathcal{H}_{1,0}).
\]

Since the Schur parameters of \( \Theta_1 \) are \( \{ \Gamma_1, \Gamma_2, \ldots \} \), starting from the system \( \tau^{(0)}_1 \) and using the equality \( (\mathcal{H}_{1,0})_{1,0} = \mathcal{H}_{2,0} \) (see (3.7)), we obtain similarly to (6.14) the relation

\[
||D_{\Gamma_2}D_{\Gamma_1}f||^2 = ||P_{2,0}B_1f||^2, \quad f \in \mathcal{D}_{\Gamma_0}.
\]

Let us show that

\[
(6.15) \quad B_1D_{\Gamma_0} = P_{1,0}B.
\]
Actually for \( \varphi \in \mathcal{M} \) and \( \psi \in \mathcal{H}_{1,0} \) one has
\[
(B \Gamma_0 \varphi, \psi) = ((D_{\Gamma_0}^{-1} (B^\ast \mathcal{H})_0)) \ast D_{\Gamma_0} \varphi, \psi) = (D_{\Gamma_0} \varphi, D_{\Gamma_0}^{-1} (B^\ast \mathcal{H})_0 \psi)
\]
= \((\varphi, B^\ast \psi) = (B \varphi, \psi) = (P_{1,0} B \varphi, \psi)\).

This proves (6.15). Since \( \mathcal{H}_{2,0} \subseteq \mathcal{H}_{1,0} \), we have the equalities
\[
P_{2,0} B = P_{2,0} P_{1,0} B = P_{2,0} B_{1} D_{\Gamma_0},
\]
which lead to
\[(6.16) \quad ||D_{\Gamma_2} D_{\Gamma_1} D_{\Gamma_0} f||^2 = ||P_{2,0} B f||^2, \ f \in \mathcal{M}.
\]
By induction, using the equality \((A_{n,0})_{1,0} = A_{n+1,0}\) (see (3.8)), we obtain (6.12) and similarly (6.13).

Using (5.1), Theorem 2.1 and Corollary 2.2 we may interpret equalities (6.12) and (6.13) as follows
\[
\inf_{\{\varphi_k\}_{k=0}^{n-1} \subset \mathcal{M}} \left\{ \left\| B h - \sum_{k=0}^{n-1} A^k C^* \varphi_k \right\|^2 \right\} = \left\| D_{\Gamma_{n}} D_{\Gamma_{n-1}} \cdots D_{\Gamma_{0}} h \right\|^2, \ h \in \mathcal{M}, \ n \geq 1,
\]
\[
\inf_{\{\psi_k\}_{k=0}^{n-1} \subset \mathcal{M}} \left\{ \left\| C^* f - \sum_{k=0}^{n-1} A^k B \psi_k \right\|^2 \right\} = \left\| D_{\Gamma_{n}} D_{\Gamma_{n-1}} \cdots D_{\Gamma_{0}} f \right\|^2, \ f \in \mathcal{M}, \ n \geq 1.
\]

**Corollary 6.7.** Let \( \Theta \in S(\mathcal{M}, \mathcal{M}) \) and let \( \varphi_\Theta \) and \( \psi_\Theta \) be the right and the left defect functions of \( \Theta \), respectively. Then
\[(6.17) \quad \varphi_\Theta(0) \varphi_\Theta(0) = s - \lim_{n \to \infty} \left( D_{\Gamma_{0}} D_{\Gamma_{1}} \cdots D_{\Gamma_{n-1}} D_{\Gamma_{n}}^2 D_{\Gamma_{n-1}} \cdots D_{\Gamma_{1}} D_{\Gamma_{0}} \right),
\]
and
\[(6.18) \quad \psi_\Theta(0) \psi_\Theta(0) = s - \lim_{n \to \infty} \left( D_{\Gamma_{0}} D_{\Gamma_{1}} \cdots D_{\Gamma_{n-1}} D_{\Gamma_{n}}^2 D_{\Gamma_{n-1}} \cdots D_{\Gamma_{1}} D_{\Gamma_{0}}^2 \right),
\]
where \( \{\Gamma_0, \Gamma_1, \ldots\} \) are the Schur parameters of \( \Theta \).

**Proof.** Let \( \tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix} \right\} : \mathcal{M}, \mathcal{M}, \mathcal{H} \right\} \) be a simple conservative system with transfer function \( \Theta \). Since
\[
\mathcal{H}_{1,0} \supseteq \mathcal{H}_{2,0} \supseteq \cdots \mathcal{H}_{n,0} \supseteq \cdots,
\]
the sequence of orthogonal projections \( \{P_{n,0}\} \) strongly converges to the orthogonal projection \( P_{\mathcal{H}_{0}} \), where
\[
\mathcal{H}_0 := \bigcap_{n \geq 1} \mathcal{H}_{n,0} = (\mathcal{H}_n^\perp)^\perp.
\]
Therefore
\[
P_{\mathcal{H}_{0}} B h = \lim_{n \to \infty} P_{n,0} B h, \ h \in \mathcal{M}.
\]
From (6.12) it follows
\[(6.19) \quad ||P_{\mathcal{H}_{0}} B h||^2 = \lim_{n \to \infty} \left\| D_{\Gamma_{n}} D_{\Gamma_{n-1}} \cdots D_{\Gamma_{0}} h \right\|^2, \ h \in \mathcal{M}.
\]
The operator \( A \upharpoonright \mathcal{H}_0 \) is a unilateral shift, therefore \( D_A x = 0 \) for all \( x \in \mathcal{H}_0 \). Since the operator

\[
U = \begin{bmatrix} D & C \\ B & A \end{bmatrix}
\]

is unitary, the operator \( B \) is of the form \( B = D_A^* M \), where \( M^* \in L(\mathcal{O}_A, \mathfrak{M}) \) is isometry. Hence for \( h \in \mathfrak{M} \) and \( x \in \mathcal{H}_0 \) one obtains

\[
(P_{\Omega} Bh, Ax) = (D_A^* Mh, Ax) = (Mh, D_A^* Ax) = (Mh, AD_A x) = 0.
\]

Thus,

\[
P_{\Omega} Bh = P_{\Omega} B h, \quad h \in \mathfrak{M},
\]

where \( \Omega = \mathcal{H}_0 \ominus A \mathcal{H}_0 \). Theorem 3.2 yields that \( P_{\Omega} B h = \varphi_\Theta(0) h \) and since the sequence of operators

\[
\{ D_{\Gamma_0} D_{\Gamma_1} \cdots D_{\Gamma_n}, D_{\Gamma_{n-1}} \cdots D_{\Gamma_1} D_{\Gamma_0} \}_{n=0}^\infty
\]

in non-increasing, we obtain (6.17), and similarly (6.18). \( \square \)

Using equalities (6.7), (6.8), (6.17), and (6.18) we arrive at the next two corollaries.

**Corollary 6.8.** 1) The following conditions are equivalent

(i) the system \( \tau \) is observable,

(ii)

\[
\lim_{n \to \infty} \left( D_{\Gamma_0} D_{\Gamma_1} \cdots D_{\Gamma_n}, D_{\Gamma_{n-1}} \cdots D_{\Gamma_1} D_{\Gamma_0} \right) = 0,
\]

(iii) \( \left( D_{\Gamma_0}^2 \right)_{\mathfrak{M}} = 0 \).

2) The following conditions are equivalent

(i) the system \( \tau \) is controllable,

(ii)

\[
\lim_{n \to \infty} \left( D_{\Gamma_0} D_{\Gamma_1} \cdots D_{\Gamma_n}^*, D_{\Gamma_{n-1}} \cdots D_{\Gamma_1}^* D_{\Gamma_0} \right) = 0.
\]

(iii) \( \left( D_{\Gamma_0}^2 \right)_{D_A} = 0 \).

**Corollary 6.9.** Let \( A \) be a completely non-unitary contraction in the Hilbert space \( \mathcal{H} \), let

\[
\Psi_A(\lambda) = (-A + \lambda D_A^* (I - \lambda A^*)^{-1} D_A) \upharpoonright \mathcal{D}_A, \quad \lambda \in \mathbb{D}
\]

be the Sz.-Nagy–Foias characteristic function of \( A \) \( [49] \), and let \( \{ \gamma_n \}_{n \geq 0} \) be the Schur parameters of \( \Psi_A \).

1) The following conditions are equivalent

(i) \( A \) is completely non-isometric,

(ii)

\[
\lim_{n \to \infty} \left( D_{\gamma_0} D_{\gamma_1} \cdots D_{\gamma_n} D_{\gamma_{n-1}} \cdots D_{\gamma_1} D_{\gamma_0} \right) = 0,
\]

(iii) \( \left( D_{\psi_A^2} \right)_{\mathcal{D}_A} = 0 \).

2) The following conditions are equivalent

(i) \( A \) is completely non-co-isometric,

(ii)

\[
\lim_{n \to \infty} \left( D_{\gamma_0} D_{\gamma_1} \cdots D_{\gamma_n}^2 D_{\gamma_{n-1}} \cdots D_{\gamma_1}^2 D_{\gamma_0} \right) = 0,
\]

(iii) \( \left( D_{\psi_A^2} \right)_{\mathcal{D}_A^*} = 0 \).
Proof. The function $\Psi_A$ is the transfer function of the simple conservative system

$$\Sigma = \left\{ \begin{bmatrix} -A & D_A \\ D_A & A^* \end{bmatrix} ; \mathcal{D}_A, \mathcal{D}_{A^*}, \mathcal{H} \right\}.$$ 

Now statements follow from Corollary 6.8. □

Let us make a few remarks. If $\mu$ is a nontrivial scalar probability measure on the unit circle $\mathbb{T} = \{ \xi \in \mathbb{C} : |\xi| = 1 \}$ ( $\mu$ is not supported on a finite set), then with $\mu$ are associated the monic polynomials $\Phi_n(z, \mu)$ (or $\Phi_n$ if $\mu$ is understood) orthogonal in the Hilbert space $L^2(\mathbb{T}, d\mu)$, connected by the Szegő recurrence relations

$$(6.22) \quad \Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n(\mu)\Phi_n^*(z)$$

with some complex numbers $\alpha_n(\mu)$, called the Verblunsky coefficients [47]. By definition

$$\Phi(z) = \sum_{j=0}^{n} \rho_j z^j \Rightarrow \Phi^*(z) = \sum_{j=0}^{n} \bar{\rho}_{n-j} z^j.$$ 

The norm of the polynomials $\Phi_n$ in $L^2(\mathbb{T}, d\mu)$ can be computed by:

$$||\Phi_n||^2 = \prod_{j=0}^{n-1} (1 - |\alpha_j(\mu)|^2), \quad n = 1, 2, \ldots.$$ 

A result of Szegő – Kolmogorov – Krein reads that

$$\prod_{j=0}^{\infty} (1 - |\alpha_j(\mu)|^2) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \ln \mu'(t) \, dt \right),$$

where $\mu'$ is the Radon – Nikodym derivative of $\mu$ with respect to Lebesgue measure $d\mu$.

Define the Carathéodory function by

$$F(z) = F(z, \mu) := \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta).$$

$F$ is an analytic function in $\mathbb{D}$ which obeys $\Re F > 0$, $F(0) = 1$. The Schur function is then defined by

$$f(z) = f(z, \mu) := \frac{1}{z} \frac{F(z) - 1}{F(z) + 1}, \quad F(z) = \frac{1 + zf(z)}{1 - zf(z)},$$

so it is an analytic function in $\mathbb{D}$ with $\sup_{\mathbb{D}} |f(z)| \leq 1$. A one-to-one correspondence can be easily set up between the three classes (probability measures, Carathéodory and Schur functions). Under this correspondence $\mu$ is trivial, that is, supported on a finite set, if and only if the associate Schur function is a finite Blaschke product. Let $\{\gamma_n(f)\}$ be the Schur parameters of $f$. According to Geronimous theorem the equalities $\gamma_n(f) = \alpha_n(\mu)$ hold for all $n \geq 0$.

If a Schur function $f$ is not a finite Blaschke product, the connection between the non-tangential limit values $f(\zeta)$ and its Schur parameters $\{\gamma_n\}$ is given by the formula (see [23])

$$\prod_{n=0}^{\infty} (1 - |\gamma_n|^2) = \exp \left\{ \int_{\mathbb{T}} \ln(1 - |f(\zeta)|^2) \, d\mu \right\}.$$
Thus, (cf. [47, Theorem 1.5.7]): for any nontrivial probability measure $\mu$ on the unit circle, the following are equivalent:

(i) $\lim_{n \to \infty} ||\Phi_n|| = 0$;
(ii) $\prod_{j=0}^{\infty} (1 - |\alpha_j(\mu)|^2) = 0$;
(iii) the system $\{\phi_n = \Phi_n/||\Phi_n||\}_{n=0}^{\infty}$ is the orthonormal basis in $L^2(\mathbb{T}, d\mu)$,
(iv) $\ln \mu' \not\in L^1(\mathbb{T})$,
(v) $\ln(1 - |f(\xi)|^2) \not\in L^1(\mathbb{T})$,
(vi) a simple conservative system with transfer function $f$ is controllable and observable.

In the case, when $f \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ and the norms of all Schur parameters $\{\Gamma_n\}_{n \geq 0}$ of $f$ are less than 1, in [19, Corollary 4.8] is mentioned that

$$s - \lim_{n \to \infty} \left(D \Gamma_0 D \Gamma_1 \cdots D \Gamma_{n-1} D \Gamma_n \right) = G_{\mu}^*(0) G_{\mu}(0),$$

where $G_{\mu}$ is the spectral factor of the operator-valued measure $\mu$ from the integral representation

$$F(z) = \int \frac{\zeta + z}{\zeta - z} d\mu(\zeta)$$

of the function $F(z) = (1 + z f(z))(1 - z f(z))^{-1}$.

Analogs of formulas (6.2) and (6.8) have been established by G. Popescu in [42] for a positive definite multi-Toeplitz kernel and corresponding generalized Schur parameters.

6.3. Central solution to the Schur problem. Now we return to the Schur problem (see Subsection 1.4) with data $\{C_k\}_{k=0}^{N} \subset \mathbf{L}(\mathfrak{M}, \mathfrak{N})$. The necessary and sufficient condition of solvability is the contractiveness of the operator

$$T_N = \begin{bmatrix} C_0 & 0 & 0 & \ldots & 0 \\ C_1 & C_0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_N & C_{N-1} & C_{N-2} & \ldots & C_0 \end{bmatrix}.$$ 

Suppose that $T_N$ is a contraction and let

$$\Gamma_0 = C_0, \quad \Gamma_k \in \mathbf{L}(\mathfrak{D}_{\Gamma_{k-1}}, \mathfrak{D}_{\Gamma_{k-1}}), \quad k = 1, \ldots, N$$

be the choice sequence determined by the contractive operator $T_N$. Notice that equalities (6.1) and (6.2) remain true. Moreover, it follows from (6.1), (6.2) that the sequence of operators $\{(D^2_{T_k})_{\mathfrak{N}} \vert \mathfrak{M}\}_{k=0}^{N}$ and $\{(D^2_{\tilde{T}_k})_{\mathfrak{N}} \vert \mathfrak{M}\}_{k=0}^{N}$ are non-increasing. Below this fact we establish directly.

**Proposition 6.10.** Let $C_0, C_1, \ldots, C_N \in \mathbf{L}(\mathfrak{M}, \mathfrak{N})$ be the Schur sequence and let $T_N$ be given by (1.6). Then for $T_k = T_k(C_0, C_1, \ldots, C_k)$, $\tilde{T}_k = T_k(C^*_0, C^*_1, \ldots, C^*_k)$, $k = 0, 1, \ldots, N$ the inequalities

$$(D^2_{T_k})_{\mathfrak{N}} \vert \mathfrak{M} \geq (D^2_{T_{k+1}})_{\mathfrak{N}} \vert \mathfrak{M}, \quad k = 0, \ldots, N - 1,$$
$$\left(D^2_{\tilde{T}_k}\right)_{\mathfrak{N}} \vert \mathfrak{N} \geq \left(D^2_{\tilde{T}_{k+1}}\right)_{\mathfrak{N}} \vert \mathfrak{N}, \quad k = 0, \ldots, N - 1,$$
hold.

Proof. It is sufficient to prove that \( (D^2_{T_k})_{2\mathbb{R}} | \mathcal{M} \geq (D^2_{T_{k+1}})_{2\mathbb{R}} | \mathcal{M} \), for \( k = 0, \ldots, N - 1 \). The operator \( T_{k+1} \) can be represented as follows

\[
T_{k+1} = \begin{bmatrix} T_k & 0 \\ B_k & C_0 \end{bmatrix},
\]

where \( B_k = [C_{k+1} \ C_{k+2} \ \cdots \ C_1] \). It follows that

\[
D^2_{T_{k+1}} = \begin{bmatrix} D^2_{T_k} - B_k^* B_k & -B_k^* C_0 \\ -C_0^* B_k & D^2_{C_0} \end{bmatrix}.
\]

Hence,

\[
D^2_{T_k} \geq D^2_{T_k} - B_k^* B_k = P_{2\mathbb{R}^{k+1}} D^2_{T_{k+1}} | \mathcal{M}^{k+1},
\]

and from Propositions[5.1][5.3]

\[
(D^2_{T_k})_{2\mathbb{R}} \geq (D^2_{T_k} - B_k^* B_k)_{2\mathbb{R}} = (P_{2\mathbb{R}^{k+1}} D^2_{T_{k+1}} | \mathcal{M}^{k+1})_{2\mathbb{R}} \geq (D^2_{T_{k+1}})_{2\mathbb{R}} | \mathcal{M}^{k+1}.
\]

Hence,

\[
(D^2_{T_k})_{2\mathbb{R}} | \mathcal{M} \geq (D^2_{T_{k+1}})_{2\mathbb{R}} | \mathcal{M}.
\]

\[\square\]

Corollary 6.11. Under conditions of Proposition[6.10] the equality

\[
(D^2_{T_k})_{2\mathbb{R}} = 0
\]

for some \( k \leq N - 1 \) implies

\[
(D^2_{T_{k+1}})_{2\mathbb{R}} = 0, \ (D^2_{T_{k+2}})_{2\mathbb{R}} = 0, \ldots, (D^2_{T_N})_{2\mathbb{R}} = 0.
\]

Corollary 6.12. Additionally to conditions of Proposition[6.10] suppose that \( \mathcal{M} \) and \( \mathcal{N} \) are one-dimensional. Then the following conditions are equivalent:

(i) \( \det D^2_{T_N} = 0 \),
(ii) \( (D^2_{T_N})_{2\mathbb{R}} = 0 \).

Proof. (ii)⇒(i). The equality \( (D^2_{T_N})_{2\mathbb{R}} = 0 \) is equivalent to \( \mathcal{M} \cap \text{ran} D^2_{T_N} = \{0\} \). It follows that \( \text{ran} (D^2_{T_N}) \neq \mathcal{M}^{N+1} \). Hence \( \det D^2_{T_N} = 0 \).

Let us prove (i)⇒(ii). Let \( m \leq N - 1 \) is such that \( \det D^2_{T_m} \neq 0 \) and \( \det D^2_{T_{m+1}} = 0 \). The matrix \( T_{m+1} \) takes the form

\[
T_{m+1} = \begin{bmatrix} C_0 & 0 \\ X_m & T_m \end{bmatrix},
\]

where

\[
X_m = \begin{bmatrix} C_1 \\ \vdots \\ C_{m+1} \end{bmatrix}.
\]

Then

\[
D^2_{T_{m+1}} = \begin{bmatrix} 1 - C_0^* C_0 - X_m^* X_m & -X_m^* T_m \\ -T_m^* X_m & D^2_{T_m} \end{bmatrix}.
\]
As is well known
\[ \det D_{m+1}^2 = \det D_{m}^2 (1 - C_0^* C_0 - X^* X_m - X^*_m T_m D_{m}^{-2} T^*_m X_m) \]
Since \( \det D_{m}^2 \neq 0 \) and \( \det D_{m+1}^2 = 0 \), we get
\[ 1 - C_0^* C_0 - X^* X_m - X^*_m T_m D_{m}^{-2} T^*_m X_m = 0. \]
But
\[ 1 - C_0^* C_0 - X^* X_m - X^*_m T_m D_{m}^{-2} T^*_m X_m = (D_{m+1}^2)_{2\mathbb{R}} \]
Thus \( (D_{m+1}^2)_{2\mathbb{R}} = 0 \). From Corollary 6.11 we obtain \( (D_{m}^2)_{2\mathbb{R}} = 0. \)

For a contraction \( S \in \mathcal{L}(\mathfrak{H}_1, \mathfrak{H}_2) \) define the Möbius transformation as follows
\[ \mathcal{M}_S(X) := S + D_S X (I + S^* X)^{-1} D_S, \ X \in \mathcal{L}(\mathcal{D}_S, \mathcal{D}_S^*), \ -1 \in \rho(S^* X). \]
Suppose that both subspaces \( \mathcal{D}_{\Gamma_N} \) and \( \mathcal{D}_{\Gamma_N^*} \) are non-trivial. Then all solutions to the Schur problem can be described as follows. Let \( W \) be an arbitrary function from \( \mathcal{S}(\mathcal{D}_{\Gamma_N}, \mathcal{D}_{\Gamma_N^*}) \).
Then define for \( \lambda \in \mathbb{D} \)
\[ W_1(\lambda) = \mathcal{M}_{\Gamma_N}(\lambda W(\lambda)), \ W_2(\lambda) = \mathcal{M}_{\Gamma_{N-1}}(\lambda W_1(\lambda)), \ldots, \]
\[ W_{N+1}(\lambda) = \mathcal{M}_{\Gamma_0}(\lambda W_N(\lambda)). \]
Due to the Schur algorithm, the function \( \Theta(\lambda) = W_{N+1}(\lambda) \) is a solution to the Schur problem.
We can write \( \Theta \) as
\[ \Theta(\lambda) = \mathcal{M}_{\Gamma_0} \circ \mathcal{M}_{\Gamma_1} \circ \cdots \circ \mathcal{M}_{\Gamma_N}(\lambda W). \]
If \( G_0 = W(0) \in \mathcal{L}(\mathcal{D}_{\Gamma_N}, \mathcal{D}_{\Gamma_N^*}), \ G_1 \in \mathcal{L}(\mathcal{D}_{G_0}, \mathcal{D}_{G_0^*}), \ldots \) are the Schur parameters of \( W \), then
\[ \Gamma_0, \Gamma_1, \ldots, \Gamma_N, G_0, G_1, \ldots \]
are the Schur parameters of \( \Theta \). This procedure, using the Redheffer product, leads to the representation of all solutions by means of fractional-linear transformation of \( W \) [19, 33].
We note also that all solutions to the Schur problem can be represented as transfer functions of simple conservative systems having block-operator CMV matrices [10] constructed by means of the choice sequence \( \Gamma_0, \Gamma_1, \ldots, \Gamma_N, G_0, G_1, \ldots \).

Apart from \( T_N \) we will consider the operator
\[ \tilde{T}_N = \begin{bmatrix} C_0^* & 0 & 0 & \cdots & 0 \\ C_1^* & C_0^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_N^* & C_{N-1}^* & C_{N-2}^* & \cdots & C_0^* \end{bmatrix} \]
Now we describe one step lifting of the Toeplitz matrix by means of Krein shorted operators \( (D_{m}^2)_{2\mathbb{R}} \) and \( (D_{m+1}^2)_{2\mathbb{R}} \).

**Proposition 6.13.** The Krein shorted operators \( (D_{m}^2)_{2\mathbb{R}} \) and \( (D_{m+1}^2)_{2\mathbb{R}} \) are of the forms
\[ (D_{m}^2)_{2\mathbb{R}} \upharpoonright \mathfrak{M} = I - C_0^* C_0 - \left( D_{m+1}^{-1} \begin{bmatrix} C_1 \\ \vdots \\ C_N \end{bmatrix} \right)^* \]
\[ \left( D_{m+1}^2 \right)_{2\mathbb{R}} \]
Due to (5.2) one has
\[ \lim_{x \to 1} (xI - T_{N-1}T_N)^{-1}T_N f, T_N f = \| (D_{T_N}^{-1} T_N f, T_N f) \|, \quad f \in \mathcal{M}, \]

and
\[ \lim_{x \to 1} (xI - T_{N-1}T_N)^{-1}T_N f, T_N f = -\| B_N f \| + x \| (xI - T_{N-1}T_N)^{-1/2} B_N f \|, \]

we obtain (6.24).

The operator $T_N^*$ can be represented as follows
\[ T_N^* = \begin{bmatrix} T_{N-1}^* & \hat{B}_N \\ 0 & C_0^* \end{bmatrix} : \mathcal{M}_N \to \mathcal{M}, \]

where $\hat{B}_N = \begin{bmatrix} C_0^* \\ \vdots \end{bmatrix}$. Recall that
\[ \mathcal{M}_N := \{0\} \oplus \{0\} \oplus \cdots \oplus \{0\} \oplus \mathcal{M}, \quad \mathcal{M} := \{0\} \oplus \{0\} \oplus \cdots \oplus \{0\} \oplus \mathcal{N}. \]

Then
\[ D_{T_N}^2 = \begin{bmatrix} D_{T_N}^2_{T_{N-1}} & -T_{N-1} \hat{B}_N \\ -\hat{B}_N^* T_{N-1} & I - \sum_{k=0}^N C_k C_k^* \end{bmatrix}. \]
As above we obtain
\[
\left( D_{T_{N+1}}^2 \right)_{\mathfrak{M}_N} \mid \mathfrak{M}_N = I - C_0 C_0^* - \left( D_{T_{N-1}}^{-1} \hat{B}_N \right)^* D_{T_{N-1}}^{-1} \hat{B}_N.
\]
Therefore (6.25) follows from (6.2) and (6.6).

**Theorem 6.14.** Let the data \( C_0, C_1, \ldots, C_N \in L(\mathfrak{M}, \mathfrak{M}) \) be the Schur sequence. Then the formula
\[
C_{N+1} = \hat{C}_{N+1} + \left( \left( D_{T_{N+1}}^2 \right)_{\mathfrak{M}} \mid \mathfrak{M} \right)^{1/2} Y \left( \left( D_{T_{N+1}}^2 \right)_{\mathfrak{M}} \mid \mathfrak{M} \right)^{1/2},
\]
where
\[
\hat{C}_{N+1} = - \left( D_{T_{N-1}}^{-1} \left[ \begin{array}{ccc} C_N^* & C_{N-1}^* & \vdots \\ C_{N-1}^* & \ddots & \vdots \\ \vdots & \ddots & \ddots \\ C_1^* & \cdots & C_0 \end{array} \right] \right)^* T_{N-1}^* D_{T_{N-1}}^{-1} \left[ \begin{array}{c} C_1 \\ C_2 \\ \vdots \\ C_N \end{array} \right]
\]
and \( Y \) is an arbitrary contraction from \( L \left( \text{ran} \left( (D_{T_{N+1}}^2)_{\mathfrak{M}} \right), \text{ran} \left( (D_{T_{N+1}}^2)_{\mathfrak{M}} \right) \right) \), describes all Schur sequences \( \{C_0, \ldots, C_N, C_{N+1}\} \).

**Proof.** Represent the matrix
\[
T_{N+1} = \left[ \begin{array}{cccccc} C_0 & 0 & 0 & \ldots & 0 & 0 \\ C_1 & C_0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_N & C_{N-1} & C_{N-2} & \ldots & C_0 & 0 \\ C_{N+1} & C_N & C_{N-1} & \ldots & C_1 & C_0 \end{array} \right] \in L(\mathfrak{M}^{N+2}, \mathfrak{M}^{N+2})
\]
in the form
\[
T_{N+1} = \left[ \begin{array}{cc} B & D \\ A & C \end{array} \right] : \mathfrak{M}^{N+2} \oplus \mathfrak{M}^{N+1} \rightarrow \mathfrak{M}^{N+1} \oplus \mathfrak{M}
\]
with
\[
B = \left[ \begin{array}{c} C_0 \\ C_1 \\ \vdots \\ C_N \end{array} \right], \quad D = \left[ \begin{array}{cccc} 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{N-1} & C_{N-2} & C_{N-3} & \ldots & C_0 & 0 \end{array} \right], \\
A = \left[ \begin{array}{c} C_{N+1} \end{array} \right], \quad C = \left[ \begin{array}{ccc} C_N & C_{N-1} & \ldots & C_0 \end{array} \right].
\]
On the other hand
\[
D = \left[ \begin{array}{cc} 0 & 0 \\ T_{N-1} & 0 \end{array} \right] : \mathfrak{M}^N \oplus \mathfrak{M} \rightarrow \mathfrak{M} \oplus \mathfrak{M}^N.
\]
The operator \( D \) is a contraction and
\[
D_D = \left[ \begin{array}{cc} D_{T_{N-1}}^{-1} & 0 \\ 0 & I \end{array} \right], \quad D_D^* = \left[ \begin{array}{cc} I & 0 \\ 0 & D_{T_{N-1}} \end{array} \right].
\]
From Theorem [2.1] it follows that \( T_{N+1} \) is a contraction if and only if \( A \) is of the form (see Section [2])
\[
A = -VD^*U + D_V Y D_U,
\]
where \( C = V D, \ B = D D^* U, \ Y \in L(\mathfrak{D}_U, \mathfrak{D}_V^*), \ ||Y|| \leq 1. \) Thus,
\[
A = -(D_D^{-1} C^*)^* D^* D_D^{-1} B + D_V^* Y D_U.
\]

We have
\[
V^* = D_D^1 C^* = \begin{bmatrix}
D_{T_{N-1}}^{-1} & C_N^* \\
& \vdots \\
& C_{N-1}^* \\
& C_0^*
\end{bmatrix}, \quad U = D_D^{-1} B = \begin{bmatrix}
C_0 \\
C_1 \\
& \vdots \\
& C_N
\end{bmatrix}.
\]

Then
\[
D^2_U = I - C^* C_0 - \left( D_{T_{N-1}}^{-1} \begin{bmatrix} C_1 \\
C_2 \\
& \vdots \\
C_N
\end{bmatrix} \right)^* D_{T_{N-1}}^{-1} \begin{bmatrix} C_1 \\
C_2 \\
& \vdots \\
C_N
\end{bmatrix},
\]
(6.28)

\[
D^2_{V^*} = I - C_0 C_0^* - \left( D_{T_{N-1}}^{-1} \begin{bmatrix} C_N^* \\
& \vdots \\
C_{N-1}^* \\
& C_1^*
\end{bmatrix} \right)^* D_{T_{N-1}}^{-1} \begin{bmatrix} C_N^* \\
& \vdots \\
C_{N-1}^* \\
& C_1^*
\end{bmatrix},
\]
(6.29)

and
\[
-VD^* U = - \left( D_{T_{N-1}}^{-1} \begin{bmatrix} C_N^* \\
& \vdots \\
& C_{N-1}^* \\
& C_1^*
\end{bmatrix} \right)^* D^* D_{T_{N-1}}^{-1} \begin{bmatrix} C_0 \\
C_1 \\
& \vdots \\
C_N
\end{bmatrix}
\]
\[
= - \left( D_{T_{N-1}}^{-1} \begin{bmatrix} C_N^* \\
& \vdots \\
C_{N-1}^* \\
C_1^*
\end{bmatrix} \right)^* T_{N-1}^* D_{T_{N-1}}^{-1} \begin{bmatrix} C_0 \\
C_1 \\
& \vdots \\
C_N
\end{bmatrix}.
\]

From (6.24) and (6.25) we get (6.27). \( \square \)

Remark 6.15. For finite dimensional \( \mathfrak{M} \) and \( \mathfrak{N} \) formulas (6.24), (6.28), and (6.29) can be found in [31].

Define consequentially the operators \( \hat{C}_{N+1}, \hat{C}_{N+2}, \ldots \) by means of (6.27) using \( \hat{T}_N, \hat{T}_{N+1}, \ldots \). The solution
\[
\Theta_0(\lambda) = \sum_{k=0}^{N} \lambda^k C_k + \sum_{n=1}^{\infty} \lambda^{N+n} \hat{C}_{N+n}
\]
of the Schur problem with data \( \{ C_k \}_{k=0}^{N} \) is called the central solution [31].

Theorem 6.16. Let the data \( C_0, C_1, \ldots, C_N \in L(\mathfrak{M}, \mathfrak{N}) \) be the Schur sequence. Then the following statements are equivalent:

(i) \( \Theta \) is the central solution to the Schur problem,

(ii) \( (D^2_{T^*})_{\mathfrak{M}} \upharpoonright \mathfrak{M} = (D^2_{T_N})_{\mathfrak{M}} \upharpoonright \mathfrak{M}, \)
(iii) \( \left( D^2_{T_{\Theta_0}} \right)_\mathfrak{N} \upharpoonright \mathfrak{M} = \left( D^2_{T_N} \right)_\mathfrak{N} \upharpoonright \mathfrak{N} \).

**Proof.** Let \( \Theta \) be a solution to the Schur problem, \( \Theta(\lambda) = \sum_{k=0}^{\infty} \lambda^k C_k \). Then \( C_{N+1} \) is given by (6.26) with some contraction \( Y \). For corresponding Toeplitz operators \( T_{N+1}, \tilde{T}_{N+1} \) from Proposition 6.10 we obtain

\[
\left( D^2_{T_{N+1}} \right)_\mathfrak{N} = \left( (D^2_{T_N})_{\mathfrak{N}} \right)^{1/2} D^2_{Y} \left( (D^2_{T_N})_{\mathfrak{N}} \right)^{1/2} P_{\mathfrak{N}},
\]

\[
\left( D^2_{\tilde{T}_{N+1}} \right)_\mathfrak{N} = \left( (D^2_{T_N})_{\mathfrak{N}} \right)^{1/2} D_Y^2 \left( (D^2_{T_N})_{\mathfrak{N}} \right)^{1/2} P_{\mathfrak{N}}.
\]

Since \( Y = 0 \) corresponds to \( \dot{C}_{N+1} \), we get

\[
\left( D^2_{T_{N+1}} \right)_\mathfrak{N} \upharpoonright \mathfrak{M} = \left( D^2_{T_N} \right)_\mathfrak{N} \upharpoonright \mathfrak{M}.
\]

Similarly

\[
\left( D^2_{\tilde{T}_{N+1}} \right)_\mathfrak{N} \upharpoonright \mathfrak{M} = \left( D^2_{T_N} \right)_\mathfrak{N} \upharpoonright \mathfrak{M}.
\]

By induction

\[
\text{(6.30)} \quad \left( D^2_{T_{N+n}} \right)_\mathfrak{N} \upharpoonright \mathfrak{M} = \left( D^2_{T_N} \right)_\mathfrak{N} \upharpoonright \mathfrak{M}, \quad \left( D^2_{\tilde{T}_{N+n}} \right)_\mathfrak{N} \upharpoonright \mathfrak{M} = \left( D^2_{T_N} \right)_\mathfrak{N} \upharpoonright \mathfrak{M}
\]

for each \( n \geq 1 \). Hence, if \( \Theta = \Theta_0 \) is the central solution, then (6.30) and (6.11) imply

\[
\left( D^2_{T_{\Theta_0}} \right)_\mathfrak{N} \upharpoonright \mathfrak{M} = \left( D^2_{T_{N+1}} \right)_\mathfrak{N} \upharpoonright \mathfrak{M}, \quad \left( D^2_{\tilde{T}_{\Theta_0}} \right)_\mathfrak{N} \upharpoonright \mathfrak{M} = \left( D^2_{T_N} \right)_\mathfrak{N} \upharpoonright \mathfrak{M}.
\]

Similarly (iii) \( \Rightarrow \) (i) and (ii) \( \Rightarrow \) (i). \( \square \)

Thus, for the lower triangular Toeplitz matrix \( T_{\Theta_0} \), corresponding to \( \Theta_0 \), we obtain the following statement.

**Theorem 6.17.** Let the data \( C_0, C_1, \ldots, C_N \in L(\mathfrak{M}, \mathfrak{N}) \) be the Schur sequence. Then the central solution \( \Theta_0 \in S(\mathfrak{M}, \mathfrak{N}) \) is a unique among other solutions \( \Theta \), satisfying

\[
\left( D^2_{T_{\Theta_0}} \right)_\mathfrak{N} \upharpoonright \mathfrak{M} = \max_{\Theta} \left\{ \left( D^2_{T_{\Theta}} \right)_\mathfrak{N} \upharpoonright \mathfrak{M} \right\}
\]

\[
\iff \left( D^2_{T_{\Theta_0}} \right)_\mathfrak{N} \upharpoonright \mathfrak{M} = \max_{\Theta} \left\{ \left( D^2_{T_{\Theta}} \right)_\mathfrak{N} \upharpoonright \mathfrak{M} \right\}.
\]

Note that the solution \( \Theta_0 \) is often called **maximal entropy solution** \([33]\). If the choice sequence of \( T_N \) are \( \Gamma_0 = C_0, \Gamma_1, \ldots, \Gamma_N \), then from (6.11) and Theorem 6.16

\[
\left( D^2_{T_{\Theta_0}} \right)_\mathfrak{N} = \left( D^2_{T_N} \right)_\mathfrak{N} = D_{\Gamma_0} D_{\Gamma_1} \cdots D_{\Gamma_{N-1}} D^2_{\Gamma_N} P_{\mathfrak{N}},
\]

\[
\left( D^2_{\tilde{T}_{\Theta_0}} \right)_\mathfrak{N} = \left( D^2_{T_N} \right)_\mathfrak{N} = D_{\Gamma_0} D_{\Gamma_1} \cdots D_{\Gamma_{N-1}} D^2_{\Gamma_N} P_{\mathfrak{N}}.
\]

From (6.30) and (6.11) it follows that the Schur parameters of \( \Theta_0 \) are operators

\[
\Gamma_0, \Gamma_1, \ldots, \Gamma_N, 0 \in L(\mathfrak{N}, \mathfrak{N}), 0 \in L(\mathfrak{N}, \mathfrak{N}), \ldots
\]
The function $\Theta_0$ is also given by (6.23) with $W(\lambda) = 0$, $\lambda \in \mathbb{D}$. Let $\{\Theta_n \in \mathcal{S}(\mathfrak{D}_{\Gamma_{n-1}}, \mathfrak{D}_{\Gamma_{n-1}})\}_{n \geq 0}$ be functions associated with $\Theta_0$ in accordance with Schur algorithm. Then $\Theta_{N+1} = \Theta_{N+2} = \cdots = 0 \in \mathcal{S}(\mathfrak{D}_{\Gamma_N}, \mathfrak{D}_{\Gamma_N})$. Let

$$\tau_0 = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix} ; \mathfrak{M}, \mathfrak{N}, \mathfrak{S} \right\}$$

be a simple conservative realization of the central solution $\Theta_0$. Clearly, $D = C = \Gamma_0$. Then by Theorem 4.2 the simple conservative systems

$$\tau_{N+1}^{(k)} = \left\{ \begin{bmatrix} A^k (D_{\Gamma_0}^{-1} \cdots D_{\Gamma_0}^{-1} (B^* | \mathfrak{S}_{N+1,0}))^* & D_{\Gamma_0}^{-1} \cdots D_{\Gamma_0}^{-1} (CA^{N+1-k}) \\ \vdots & A_{N+1-k,k} \end{bmatrix} ; \mathfrak{D}_{\Gamma_N}, \mathfrak{D}_{\Gamma_N}, \mathfrak{S}_{N+1-k,k} \right\},$$

$k = 0, 1, \ldots, N + 1$

realize the function $\Theta_{N+1} = 0$. Hence, the unitarily equivalent contractions $\{A_{N+1-k,k}\}_{k=0}^{N+1}$ are orthogonal sums of unilateral shifts and co-shifts of multiplicities $\dim \mathfrak{D}_{\Gamma_N}$ and $\dim \mathfrak{D}_{\Gamma_N}$, correspondingly [9].

### 6.4. Uniqueness solution to the Schur problem

Here we are interested in the case of uniqueness of the solution to the Schur problem. The following statement takes place.

**Theorem 6.18.** Let the data $C_0, C_1, \ldots, C_N \in \mathbf{L}(\mathfrak{M}, \mathfrak{N})$ be the Schur sequence. Then the following statements are equivalent

1. the Schur problem has a unique solution;
2. either $D_{\Gamma_N}^2|_{\mathfrak{M}} = 0$ or $D_{\Gamma_N}^2|_{\mathfrak{M}} = 0$;
3. either $\mathfrak{M} \cap \text{ran} D_{\Gamma_N} = \{0\}$ or $\mathfrak{N} \cap \text{ran} D_{\Gamma_N} = \{0\}$.

**Proof.** We give two proofs of the theorem.

The first proof. The equivalence of (ii) and (iii) follows from [1,9]. Let the Schur problem has a unique solution $\widehat{\Theta}(\lambda) = \sum_{k=0}^{N} \lambda^k C_k + \sum_{n=1}^{\infty} \lambda^{N+n} \widehat{C}_{N+n}$. Because $\{C_0, \ldots, C_N, \widehat{C}_{N+1}\}$ is the Schur sequence, from (6.26) it follows that $\widehat{C}_{N+1} = \widehat{C}_N$ and either $D_{\Gamma_N}^2|_{\mathfrak{M}} = 0$ or $D_{\Gamma_N}^2|_{\mathfrak{M}} = 0$. So, (i) implies (ii). In particular, we get that $\Theta = \Theta_0$.

If (ii) holds true, then again from (6.26) we get that $\Theta_0$ is a unique solution of the Schur problem.

The second proof. The matrix $T_N$ defines a sequence of contractions (the choice sequence)

$$\Gamma_0 = C_0, \Gamma_1 \in \mathbf{L}(\mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma_0}), \ldots, \Gamma_N \in \mathbf{L}(\mathfrak{D}_{\Gamma_{N-1}}, \mathfrak{D}_{\Gamma_{N-1}}).$$

Suppose that the Schur problem has a unique solution. Then by Theorem 1.4 one of $\Gamma_i$s is an isometry or co-isometry. Assume $\Gamma_p$ is isometry, where $p \leq N$. From Theorem 6.11 it follows that $D_{\Gamma_p}^2|_{\mathfrak{M}} = 0$. Corollary 6.11 yields the equality $D_{\Gamma_p}^2|_{\mathfrak{M}} = 0$. If we assume that $\Gamma_p$ is isometry, then similarly we get $D_{\Gamma_p}^2|_{\mathfrak{M}} = 0$.

Now suppose $D_{\Gamma_p}^2|_{\mathfrak{M}} = 0$. Let $p \leq N$ be such that $D_{\Gamma_p}^2|_{\mathfrak{M}} = 0$, but $D_{\Gamma_{p+1}}^2|_{\mathfrak{M}} \neq 0$. Note that in this case $\mathfrak{D}_{\Gamma_{p+1}} \neq \{0\}$, $\Gamma_p$ is isometry,

$$\mathfrak{D}_{\Gamma_p} = \mathfrak{D}_{\Gamma_{p+1}} = \cdots = \mathfrak{D}_{\Gamma_N} = \{0\}, \Gamma_{p+1} = \cdots = \Gamma_N = 0.$$

It follows that the solution to the Schur problem is unique and is of the form

$$\Theta(\lambda) = \mathcal{M}_{\Gamma_0} \circ \mathcal{M}_{\Gamma_1} \circ \cdots \circ \mathcal{M}_{\Gamma_{p-1}}(\lambda \Gamma_p), \lambda \in \mathbb{D}.$$
Similarly, the equality \((D_{2N}^2 \tilde{\mathcal{T}})_{\mathcal{N}} = 0\) implies the uniqueness. Thus \((i) \iff (ii)\). \(\square\)

Observe that \((D_{2N}^2 \tilde{\mathcal{T}})_{\mathcal{N}} = 0 \iff (D_{2N}^2 \mathcal{T}^*)_{\mathcal{N}} = 0\) (see (6.6)).

Remark 6.19. V.M. Adamyan, D.Z. Arov, and M.G. Kreın in [1] considered the following generalized Nehari–Carathéodory–Fejér problem: given a sequence of complex numbers \(\{\gamma_k\}_1^{\infty}\), find a function \(f \in L^\infty(\mathbb{T})\) with principal part \(\sum_{k=1}^{\infty} \gamma_k \zeta^{-k}\) and with minimal \(L^\infty\)-norm. By Nehari’s theorem [40] this problem has a solution if and only if the Hankel matrix \(\Gamma = \|\gamma_{j+k-1}\|\) is bounded in \(L^2\). A criteria of the uniqueness solution is established in the form [1, Theorem 2.1]

\[
\lim_{\rho \to \|\Gamma\|} ((\rho^2 I - \Gamma^* \Gamma)^{-1} \vec{e}, \vec{e}) = \infty,
\]

for the vector \(\vec{e} = (1, 0, 0, \ldots) \in L_2\). Because

\[
\lim_{x \to 0} \left( (B - xI)^{-1} g, g \right) = \begin{cases} \|B^{-1/2}g\|^2, & g \in \text{ran } B^{1/2}, \\ +\infty, & g \notin \text{ran } B^{1/2} \end{cases}
\]

for an arbitrary nonnegative selfadjoint operator \(B\) (\(B^{-1/2}\) is the Moore-Penrose pseudo-inverse), equality (6.31) means that

\(\vec{e} \notin \text{ran } (s^2 I - \Gamma^* \Gamma)\),

where \(s = \|\Gamma\|\). Then by (1.9) one has that (6.31) is equivalent to the equality

\[
(s^2 I - \Gamma^* \Gamma)_E = 0,
\]

where \(E = \{\lambda \vec{e}, \lambda \in \mathbb{C}\}\). The results of [1] have been extended to the case of operator-valued functions in the paper [2] (see also [11]). The corresponding uniqueness criteria [2, Theorem 1.3] also takes the limit form similar to the scalar case. As has been mentioned in Introduction the Schur problem can be reduced to the above problem and the matrix \(s^2 I - \Gamma^* \Gamma\) can be reduced to the square of the defect operator for a lower triangular Toeplitz matrix.

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Department of Mathematical Analysis, East Ukrainian National University, Kvartal Molodyozhny 20-A, Lugansk 91034, Ukraine

E-mail address: yury_arlinskii@yahoo.com