HOLOGRAPHIC FORMULA FOR $Q$-CURVATURE

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Introduction

In this paper we give a formula for $Q$-curvature in even-dimensional conformal geometry. The $Q$-curvature was introduced by Tom Branson in [B] and has been the subject of much research. There are now a number of characterizations of $Q$-curvature; see for example [GZ], [FG1], [GP], [FH]. However, it has remained an open problem to find an expression for $Q$-curvature which, for example, makes explicit the relation to the Pfaffian in the conformally flat case.

Theorem 1. The $Q$-curvature of a metric $g$ in even dimension $n$ is given by

\begin{equation}
2nc_{n/2}Q = nv^{(n)} + \sum_{k=1}^{n/2-1} (n - 2k)p^*_{2k} v^{(n-2k)},
\end{equation}

where $c_{n/2} = (-1)^{n/2} [2n(n/2)!((n/2) - 1)!]^{-1}$.

Here the $v^{(2j)}$ are the coefficients appearing in the asymptotic expansion of the volume form of a Poincaré metric for $g$, the differential operators $p_{2k}$ are those which appear in the expansion of a harmonic function for a Poincaré metric, and $p^*_{2k}$ denotes the formal adjoint of $p_{2k}$. These constructions are recalled in §1 below. We refer to the papers cited above and the references therein for background about $Q$-curvature.

Each of the operators $p^*_{2k}$ for $1 \leq k \leq n/2 - 1$ can be factored as $p^*_{2k} = \delta q_k$, where $\delta$ denotes the divergence operator with respect to $g$ and $q_k$ is a natural operator from functions to 1-forms. So the second term on the right hand side is the divergence of a natural 1-form. In particular, integrating (0.1) over a compact manifold recovers the result of [GZ] that

\begin{equation}
2c_{n/2} \int_M Qdv_g = \int_M v^{(n)}dv_g.
\end{equation}

This quantity is a global conformal invariant; the right hand side occurs as the coefficient of the log term in the renormalized volume expansion of a Poincaré metric (see [G]).

The work of the first author was partially supported by NSF grant DMS-0505701. The work of the second author was supported by SFB 647 “Raum-Zeit-Materie” of DFG.
As we also discuss in [1] if \( g \) is conformally flat then
\[
v^{(n)} = (-2)^{-n/2}(n/2)!^{-1} \text{Pff},
\]
where \( \text{Pff} \) denotes the Pfaffian of \( g \). So in the conformally flat case, Theorem 1 gives a decomposition of the \( Q \)-curvature as a multiple of the Pfaffian and the divergence of a natural 1-form. A general result in invariant theory ([BGP]) establishes the existence of such a decomposition, but does not produce a specific realization.

We refer to (0.1) as a holographic formula because its ingredients come from the Poincaré metric, involving geometry in \( n + 1 \) dimensions. Our proof is via the characterization of \( Q \)-curvature presented in [FG1] in terms of Poincaré metrics; in some sense Theorem 1 is the result of making explicit the characterization in [FG1]. However, passing from the construction in [FG1] to (0.1) involves a non-obvious application of Green’s identity. The transformation law of \( Q \)-curvature under conformal change, probably its most fundamental property, is not transparent from (0.1), but it is from the characterization in [FG1]. In [2] we derive another identity involving the \( p^*_x v^{(n-2k)} \) which is used in [3] and we discuss relations to the paper [CQY]. In [3] we describe the relation between holographic formulae for \( Q \)-curvature and the theory of conformally covariant families of differential operators of [J], and in particular explain how this theory leads to the conjecture of a holographic formula for \( Q \).

We are grateful to the organizing committee of the 2007 Winter School ‘Geometry and Physics’ at Srní, particularly to Vladimir Soucek, for the invitation to this gathering, which made possible the interaction leading to this paper.

We dedicate this paper to the memory of Tom Branson. His insights have led to beautiful new mathematics and have greatly influenced our own respective work.

1. Derivation

Let \( g \) be a metric of signature \( (p,q) \) on a manifold \( M \) of even dimension \( n \). In this paper, by a Poincaré metric for \((M, g)\) we will mean a metric \( g^+ \) on \( M \times (0, a) \) for some \( a > 0 \) of the form
\[
g^+ = x^{-2}(dx^2 + g_x),
\]
where \( g_x \) is a smooth 1-parameter family of metrics on \( M \) satisfying \( g_0 = g \), such that \( g^+ \) is asymptotically Einstein in the sense that \( \text{Ric}(g^+) + ng^+ = O(x^{n-2}) \) and \( \text{tr}_{g^+}(\text{Ric}(g^+) + ng^+) = O(x^{n+2}) \). Such a Poincaré metric always exists and \( g_x \) is unique up addition of a term of the form \( x^n h_x \), where \( h_x \) is a smooth 1-parameter family of symmetric 2-tensors on \( M \) satisfying \( \text{tr}_g(h_0) = 0 \) on \( M \). The Taylor expansion of \( g_x \) is even through order \( n \) and the derivatives \( (\partial_x)^k g_x|_{x=0} \) for \( 1 \leq k \leq n/2 - 1 \) and the trace \( \text{tr}_g((\partial_x)^n g_x|_{x=0}) \) are determined inductively from the Einstein condition and are given by polynomial formulae in terms of \( g \), its inverse, and its curvature tensor and covariant derivatives thereof. See [GH] for details.
The first ingredient in our formula for $Q$-curvature consists of the coefficients in the expansion of the volume form

$$dv_{g^+} = x^{-n-1}dv_{g_x}dx.$$  

Because the expansion of $g_x$ has only even terms through order $n$, it follows that

$$dv_{g_x} = \left( \frac{\det g_x}{\det g} \right)^{1/2} dv_g = (1 + v^{(2)} x^2 + \cdots + v^{(n)} x^n + \cdots) dv_g,$$

where each of the $v^{(2k)}$ for $1 \leq k \leq n/2$ is a smooth function on $M$ expressible in terms of the curvature tensor of $g$ and its covariant derivatives. Set $v^{(0)} = 1$.

The second ingredient in our formula is the family of differential operators which appears in the expansion of a harmonic function for the metric $g^+$. Given $f \in C^\infty(M)$, one can solve formally the equation $\Delta g^+ u = O(x^n)$ for a smooth function $u$ such that $u|_{x=0} = f$, and such a $u$ is uniquely determined modulo $O(x^n)$. The Taylor expansion of $u$ is even through order $n - 2$ and these Taylor coefficients are given by natural differential operators in the metric $g$ applied to $f$ which are obtained inductively by solving the equation $\Delta g^+ u = O(x^n)$ order by order. See [GZ] for details. We write the expansion of $u$ in the form

$$u = f + p_2 f x^2 + \cdots + p_n f x^{n-2} + O(x^n);$$

then $p_{2k}$ has order $2k$ and its principal part is $(-1)^k \frac{\Gamma(n/2 - k)}{2^{2k} k! \Gamma(n/2)} \Delta^k$. (Our convention is $\Delta = -\nabla^i \nabla_i$.) Set $p_0 f = f$.

We remark that the volume coefficients $v^{(2k)}$ and the differential operators $p_{2k}$ also arise in the context of an ambient metric associated to $(M, [g])$. If an ambient metric is written in normal form relative to $g$, then the same $v^{(2k)}$ are coefficients in the expansion of its volume form, and the same operators $p_{2k}$ appear in the expansion of a harmonic function homogeneous of degree 0 with respect to the ambient metric.

Let $g^+$ be a Poincaré metric for $(M, g)$. In [FG1] it is shown that there is a unique solution $U \mod O(x^n)$ to

$$\Delta g^+ U = n + O(x^{n+1} \log x)$$

of the form

$$U = \log x + A + B x^{n} \log x + O(x^n),$$

with

$$A, B \in C^\infty(M \times [0, a)), \quad A|_{x=0} = 0.$$  

Also, $A \mod O(x^n)$ is even in $x$ and is formally determined by $g$, and

$$B|_{x=0} = -2c_n Q.$$
The proof of (1.7) presented in [FG1] used results from [GZ] about the scattering matrix, so is restricted to positive definite signature. However, a purely formal proof was also indicated in [FG1]. Thus (1.7) holds in general signature.

Proof of Theorem 1. Let \( g_+ \) be a Poincaré metric for \( g \) and let \( U \) be a solution of (1.5) as described above. Let \( f \in C^\infty(M) \) have compact support. Let \( u \) be a solution of \( \Delta g_+ u = O(x^n) \) with \( u|_{x=0} = f \); for definiteness we take \( u \) to be given by (1.4) with the \( O(x^n) \) term set equal to 0. Let \( 0 < \varepsilon < x_0 \) with \( \varepsilon, x_0 \) small.

Consider Green’s identity

\[
\int_{\varepsilon < x < x_0} (U \Delta g_+ u - u \Delta g_+ U) \, dv_{g_+} = \left( \int_{x=x_0} + \int_{x=x} \right) (U \partial_\nu u - u \partial_\nu U) \, d\sigma,
\]

where \( \nu \) denotes the inward normal and \( d\sigma \) the induced volume element on the boundary, relative to \( g_+ \). Both sides have asymptotic expansions as \( \varepsilon \to 0 \); we calculate the coefficient of \( \log \varepsilon \) in these expansions.

Using the form of the expansion of \( U \) and the fact that \( \Delta g_+ u = O(x^n) \), one sees that the expansion of \( U \Delta g_+ u \) has no \( x^{-1} \) term, so \( \int_{\varepsilon < x < x_0} U \Delta g_+ u \, dv_{g_+} \) has no \( \log \varepsilon \) term. Using (1.2), (1.3), (1.4), and (1.5), one finds that the \( \log \varepsilon \) coefficient of

\[
\int_{\varepsilon < x < x_0} u \Delta g_+ U \, dv_{g_+}
\]

is

\[
(1.9) \quad \sum_{k=0}^{n/2-1} \int_M u^{(n-2k)} p_{2k} f \, dv_g.
\]

On the right hand side of (1.8), \( \int_{x=x_0} \) is independent of \( \varepsilon \), and

\[
\int_{x=x} (U \partial_\nu u - u \partial_\nu U) \, d\sigma = \varepsilon^{1-n} \int_{x=x} (U \partial_\nu u - u \partial_\nu U) \, dv_{g_+}.
\]

A \( \log \varepsilon \) term in the expansion of this quantity can arise only from the \( \log x \) or \( x^n \log x \) terms in the expansion of \( U \). Substituting the expansions, one finds without difficulty that the \( \log \varepsilon \) coefficient is

\[
\int_M \left( \sum_{k=1}^{n/2-1} 2k u^{(n-2k)} p_{2k} f - n B f \right) \, dv_g.
\]

Equating this to (1.9), using (1.7), and moving all derivatives off \( f \) gives the desired identity.

Since \( \Delta g_+ 1 = 0 \), it follows that \( p_{2k} 1 = 0 \) for \( 1 \leq k \leq n/2 - 1 \). Thus these \( p_{2k} \) have no constant term, so \( p_{2k}^* = \delta q_k \) for some natural operator \( q_k \) from functions to 1-forms, where \( \delta \) denotes the divergence with respect to the metric \( g \). So in (0.1), the second term on the right hand side is the divergence of a natural 1-form. As mentioned in the introduction, integration gives (0.2). The proof of Theorem 1 presented above in the special case \( u = 1 \) is precisely the proof of (0.2) presented in [FG1].
Theorem 1 provides an efficient way to calculate the $Q$ curvature. Solving for the beginning coefficients in the expansion of the Poincaré metric and then expanding its volume form shows that the first few of the $v^{(2k)}$ are given by:

$$v^{(2)} = -\frac{1}{2} J$$

$$v^{(4)} = \frac{1}{8} (J^2 - |P|^2)$$

$$v^{(6)} = \frac{1}{48} \left(-\frac{2}{n-4} P^{ij} B_{ij} + 3 J |P|^2 - J^3 - 2 P^{ij} P^k P_{kj}\right)$$

where

$$P_{ij} = \frac{1}{n-2} \left(R_{ij} - \frac{R}{2(n-1)} g_{ij}\right)$$

$$J = \frac{R}{2(n-1)} = P^i_i$$

$$B_{ij} = P_{ij,k} - P_{ik,j} + P^{kl} W_{kijl}$$

and $W_{ijkl}$ denotes the Weyl tensor. Similarly, one finds that the operators $p_2$ and $p_4$ are given by:

$$-2(n-2)p_2 = \Delta$$

$$8(n-2)(n-4)p_4 = \Delta^2 + 2 J \Delta + 2(n-2) P^{ij} \nabla_i \nabla_j + (n-2) J, \nabla_i.$$

For $n = 2$, Theorem 1 states $Q = -2v^{(2)} = \frac{1}{2} R$. For $n = 4$, substituting the above into Theorem 1 gives:

$$Q = 2(J^2 - |P|^2) + \Delta J,$$

and for $n = 6$:

$$Q = 8 P^{ij} B_{ij} + 16 P^{ij} P^k P_{kj} - 24 J |P|^2 + 8 J^3$$

$$+ \Delta^2 J + 4 \Delta(J^2) + 8 P^{ij} J,_{ij} - 4 \Delta(|P|^2).$$

In the formula for $n = 6$, the first line is $(12c_3^{-1}) v^{(6)}$ and the second line is $(12c_3)^{-1} \left(4p_2^* v^{(4)} + 2p_4^* v^{(2)}\right)$. Details of these calculations will appear in [J].

The expansion of the Poincaré metric $g_+$ was identified explicitly in the case that $g$ is conformally flat in [SS]. (Since we are only interested in local considerations, by conformally flat we mean locally conformally flat.) The two dimensional case is somewhat anomalous in this regard, but the identification of $Q$ curvature is trivial when $n = 2$, so we assume $n > 2$ for this discussion. The conclusion of [SS] is that if $g$ is conformally flat and $n > 2$ (even or odd), then the expansion of the Poincaré metric terminates at second order and

$$\langle g_+ \rangle_{ij} = g_{ij} - P_{ij} x^2 + \frac{1}{4} P_{ik} P^k_{,j} x^4.$$
Proposition 1. If \( g \) is conformally flat and \( n > 2 \), then
\[
v^{(2k)} = \begin{cases} 
(-2)^{-k} \sigma_k(P) & 0 \leq k \leq n \\
0 & n < k
\end{cases}
\]
where \( \sigma_k(P) \) denotes the \( k \)-th elementary symmetric function of the eigenvalues of the endomorphism \( P^j_i \).

Proof. Write \( g^{-1}P \) for \( P^j_i \). Then the \( \sigma_k(P) \) are given by
\[
\det(I + g^{-1}P t) = \sum_{k=0}^{n} \sigma_k(P) t^k.
\]
Equation (1.11) can be rewritten as \( g^{-1}g_x = (I - \frac{1}{2}g^{-1}P_x)^2 \). Taking the determinant and comparing with (1.3) gives the result. \( \square \)

We remark that for \( g \) conformally flat, \( g_x \) given by (1.11) is uniquely determined to all orders by the requirement that \( g_+ \) be hyperbolic. So in this case the \( v^{(2k)} \) are invariantly determined and given by Proposition 1 for all \( k \geq 0 \) in all dimensions \( n > 2 \).

Returning to the even-dimensional case, we define the Pfaffian of the metric \( g \) by
\[
2^n(n/2)! \Pf = (-1)^{\mu_{i_1...i_n}} \mu_{j_1...j_n} R_{i_1j_2i_3j_3} \cdots R_{i_{n-1}j_ni_nj_{n-1}j_1},
\]
where \( \mu_{i_1...i_n} = \sqrt{|\det(g)|} \epsilon_{i_1...i_n} \) is the volume form and \( \epsilon_{i_1...i_n} \) denotes the sign of the permutation. For a conformally flat metric, one has \( R_{iijkl} = 2(P_{i[k}g_{lj]} - P_{j[k}g_{li}]). \) Using this in (1.12) and simplifying gives
\[
\Pf = (n/2)! \sigma_{n/2}(P)
\]
(see Proposition 8 of [V] for details). Combining with Proposition 1 we obtain for conformally flat \( g \):
\[
v^{(n)} = (-2)^{-n/2}(n/2)!^{-1} \Pf.
\]
Hence in the conformally flat case, (0.1) specializes to
\[
2Q = 2^{n/2}(n/2 - 1)! \Pf + (nc_{n/2})^{-1} \sum_{k=1}^{n/2-1} (n - 2k)p^{2k}_{2k} v^{(n-2k)},
\]
and again the second term on the right hand side is a formal divergence.
2. A Related Identity

In this section we derive another identity involving the $p_{2k}^* v^{(n-2k)}$. It is in general impossible to choose the $O(x^n)$ term in (1.4) to make $\Delta_{g_+} u = O(x^n)$; in fact $x^{-n}\Delta_{g_+} u|_{x=0}$ is independent of the $O(x^n)$ term in (1.4) and is a conformally invariant operator of order $n$ applied to $f$, namely a multiple of the critical GJMS operator $P_n$. Following [GZ], we consider the limiting behavior of the corresponding term in the expansion of an eigenfunction for $\Delta_{g_+}$ as the eigenvalue tends to 0.

Let $g_+$ be a Poincaré metric as above. If $0 \neq \lambda \in \mathbb{C}$ is near 0, then for $f \in C^\infty(M)$, one can solve formally the equation $(\Delta_{g_+} - \lambda(n-\lambda))u_\lambda = O(x^{n+\lambda+1})$ for $u_\lambda$ of the form

$$u_\lambda = x^\lambda \left( f + p_{2,\lambda} f x^2 + \cdots + p_{n,\lambda} f x^n + O(x^{n+1}) \right),$$

where $p_{2k,\lambda}$ is a natural differential operator in the metric $g$ of order $2k$ with principal part $(-1)^k \frac{\Gamma(n/2 - k - \lambda)}{2 k! \Gamma(n/2 - \lambda)} \Delta^k$ such that $\frac{\Gamma(n/2 - \lambda)}{\Gamma(n/2 - k - \lambda)} p_{2k,\lambda}$ is polynomial in $\lambda$. Set $p_{0,\lambda} f = f$. The operators $p_{2k,\lambda}$ for $k < n/2$ extend analytically across $\lambda = 0$ and $p_{2k,0} = p_{2k}$ for such $k$, where $p_{2k}$ are the operators appearing in (1.4). But $p_{n,\lambda}$ has a simple pole at $\lambda = 0$ with residue a multiple of the critical GJMS operator $P_n$. Now $P_n$ is self-adjoint, so it follows that $p_{n,\lambda} - p_{n,\lambda}^*$ is regular at $\lambda = 0$. We denote its value at $\lambda = 0$ by $p_{n,\lambda} - p_{n,\lambda}^*$; a natural operator of order at most $n - 2$. Our identity below involves the constant term $(p_n - p_n^*) 1$. Note that since $P_n 1 = 0$, both $p_{n,\lambda} - p_{n,\lambda}^*$ and $p_{n,\lambda}^* 1$ are regular at $\lambda = 0$. We denote their values at $\lambda = 0$ by $p_n 1$ and $p_n^* 1$; then $(p_n - p_n^*) 1 = p_n 1 - p_n^* 1$. Moreover, (4.7), (4.13), (4.14) of [GZ] show that

$$p_n 1 = -c_{n/2} Q.$$

It is evident that $\int_M p_n 1 \ dv_g = \int_M p_n^* 1 \ dv_g$. The next proposition expresses the difference $p_n 1 - p_n^* 1$ as a divergence.

**Proposition 2.**

$$n (p_n - p_n^*) 1 = \sum_{k=1}^{n/2-1} 2 k p_{2k}^* v^{(n-2k)}$$

**Proof.** Take $f \in C^\infty(M)$ to have compact support, let $0 \neq \lambda$ be near 0, and define $u_\lambda$ as in (2.1) with the $O(x^{n+1})$ term taken to be 0. Define $w_\lambda$ by the corresponding expansion with $f = 1$:

$$w_\lambda = x^\lambda \left( 1 + p_{2,\lambda} 1 x^2 + \cdots + p_{n,\lambda} 1 x^n \right).$$

As in the proof of Theorem 1 consider Green’s identity

$$\int_{0 < r < x_0} (u_\lambda \Delta_{g_+} w_\lambda - w_\lambda \Delta_{g_+} u_\lambda) \ dv_{g_+} = e^{1-n} \int_{r=x} (u_\lambda \partial_x w_\lambda - w_\lambda \partial_x u_\lambda) \ dv_{g_+} + c_{x_0},$$

where
where $c_{x_0}$ is the constant (in $\epsilon$) arising from the boundary integral over $x = x_0$. Consider the coefficient of $\epsilon^{2\lambda}$ in the asymptotic expansion of both sides. The left hand side equals

$$\int_{\epsilon<x<x_0} \left[ u_\lambda (\Delta g_+ - \lambda(n - \lambda)) w_\lambda - w_\lambda (\Delta g_+ - \lambda(n - \lambda)) u_\lambda \right] dv_{g_+}.$$ 

Now $u_\lambda (\Delta g_+ - \lambda(n - \lambda)) w_\lambda \ dv_{g_+}$ and $w_\lambda (\Delta g_+ - \lambda(n - \lambda)) u_\lambda \ dv_{g_+}$ are of the form $x^{2\lambda} \psi \ dx dv_g$ where $\psi$ is smooth up to $x = 0$. It follows that the asymptotic expansion of the left hand side of (2.4) has no $\epsilon^{2\lambda}$ term. Consequently the coefficient of $\epsilon^{n+2\lambda}$ must vanish in the asymptotic expansion of

$$\int_{\chi=\epsilon} \left( u_\lambda x \partial_x w_\lambda - w_\lambda x \partial_x u_\lambda \right) dv_{g_+}.$$ 

This is the same as the coefficient of $\epsilon^n$ in the expansion of

$$\int_M \left[ \left( \sum_{k=0}^{n/2} p_{2k,\lambda} \epsilon^{2k} \right) \left( \sum_{k=0}^{n/2} (2k + \lambda)p_{2k,\lambda} \epsilon^{2k} \right) - \left( \sum_{k=0}^{n/2} p_{2k,\lambda} \epsilon^{2k} \right) \left( \sum_{k=0}^{n/2} (2k + \lambda)p_{2k,\lambda} \epsilon^{2k} \right) \right] \left( \sum_{k=0}^{n/2} v^{(2k)} \epsilon^{2k} \right) dv_g.$$ 

Evaluation of the $\epsilon^n$ coefficient gives

$$\int_M \sum_{0 \leq k, l, m \leq n/2 \atop k + l + m = n/2} (2l - 2k)(p_{2l,\lambda} f)(p_{2l,\lambda} v^{(2m)}) dv_g = 0,$$

and then moving the derivatives off $f$ results in the pointwise identity

$$\sum_{0 \leq k, l, m \leq n/2 \atop k + l + m = n/2} (2l - 2k) p_{2k,\lambda}^* \left( (p_{2l,\lambda} v^{(2m)}) \right) = 0.$$ 

The limit as $\lambda \to 0$ exists of all $p_{2l,\lambda}$ with $0 \leq l \leq n/2$ and all $p_{2k,\lambda}^*$ with $0 \leq k \leq n/2 - 1$. Since $k = n/2$ forces $l = m = 0$, the operator $p_{n,\lambda}^*$ occurs only applied to 1. Thus we may let $\lambda \to 0$ in (2.5). Using $p_{2l} = 0$ for $1 \leq l \leq n/2 - 1$ results in

$$np_n - \sum_{0 \leq k, m \leq n/2 \atop k + m = n/2} 2k p_{2k} v^{(2m)} = 0.$$ 

Separating the $k = n/2$ term in the sum gives (2.3).

Proposition 2 may be combined with (0.1) and (2.2) to give other expressions for $Q$-curvature. However, (0.1) seems the preferred form, as the other expressions all involve some nontrivial linear combination of $p_n$ and $p_n^*$.  

\qed
We remark that the generalization of (2.5) obtained by replacing \( p_{2l,\lambda}^1 \) by \( p_{2l,\lambda}^1 f \) remains true for arbitrary \( f \in C^\infty(M) \). This follows by the same argument, taking \( w_\lambda \) to be given by the asymptotic expansion of the same form but with arbitrary leading coefficient.

We conclude this section with some observations concerning relations to the paper [CQY]:

(1) Recall that Theorem 1 was proven by consideration of the \( \log \epsilon \) term in (1.8), generalizing the proof of (0.2) in [FG1] where \( u = 1 \). In [CQY], it was shown that for a global conformally compact Einstein metric \( g^+ \), consideration of the constant term in

\[
\int_{x > \epsilon} \Delta_{g^+} U \, dv_{g^+} = \int_{x = \epsilon} \partial_\nu U \, d\sigma
\]

for \( U \) a global solution of \( \Delta_{g^+} U = n \) gives a formula for the renormalized volume \( V(g^+, g) \) of \( g^+ \) relative to a metric \( g \) in the conformal infinity of \( g^+ \). In our notation this formula reads

\[
(2.6) \quad V(g^+, g) = - \int_M \left. \frac{d}{ds} \right|_{s=n} (S(s)1) \, dv_g + \frac{1}{n} \int_M \sum_{k=1}^{n/2} 2k \hat{p}^*_{2k} v^{(n-2k)} \, dv_g,
\]

where \( \hat{p}_{2k} = \frac{d}{d\lambda}|_{\lambda=0} p_{2k,\lambda} \) (which exists for \( k = n/2 \) when applied to 1) and \( S(s) \) denotes the scattering operator relative to \( g \). The operators \( \hat{p}_{2k} \) arise in this context because the coefficient of \( x^{2k} \) in the expansion of \( U \) is \( \hat{p}_{2k}^* 1 \) for \( 1 \leq k \leq n/2 - 1 \), and the coefficient of \( x^n \) involves \( \hat{p}_n 1 \). Likewise, consideration of the constant term in

\[
\int_{x > \epsilon} u \Delta_{g^+} U \, dv_{g^+} = \int_{x = \epsilon} (u \partial_\nu U - U \partial_\nu u) \, d\sigma
\]

for harmonic \( u \) gives an analogous formula for the finite part of \( \int_{x > \epsilon} u \, dv_{g^+} \) in terms of boundary data.

(2) There is an analogue of Proposition 2 involving the \( \hat{p}^*_{2k} v^{(n-2k)} \). Differentiating (2.5) with respect to \( \lambda \) at \( \lambda = 0 \) and rearranging gives the identity

\[
\sum_{k=1}^{n/2} 2k \left( \hat{p}^*_{2k} v^{(n-2k)} - (\hat{p}_{2k}^* 1) v^{(n-2k)} \right) = \sum_{k=2}^{n/2} \sum_{l=1}^{k-1} (4l - 2k) \hat{p}^*_{2k-2l} \left( (\hat{p}_{2l}^* 1) v^{(n-2k)} \right)
\]

which expresses the left hand side as a divergence.

(3) In [CQY] it was also shown that under an infinitesimal conformal change, the scattering term

\[
S(g^+, g) \equiv \int_M \left. \frac{d}{ds} \right|_{s=n} (S(s)1) \, dv_g
\]
satisfies
\[
\frac{d}{d\alpha}\Bigg|_{\alpha=0} S(g_+, e^{2\alpha\Upsilon} g) = -2c_{n/2} \int_M \Upsilon Q \, dv_g.
\]
Comparing with
\[
\frac{d}{d\alpha}\Bigg|_{\alpha=0} V(g_+, e^{2\alpha\Upsilon} g) = \int_M \Upsilon v^{(n)} \, dv_g
\]
(see [G]) and using (2.6) and Theorem 1, one deduces the curious conclusion that the infinitesimal conformal variation of
\[
\int_M \frac{n}{2} \sum_{k=1}^{n/2} 2k \hat{p}_{2k}^* v^{(n-2k)} \, dv_g
\]
is
\[
-\int_M \Upsilon \sum_{k=1}^{n/2-1} (n-2k) \hat{p}_{2k}^* v^{(n-2k)} \, dv_g.
\]
This statement involves the conformal variation only of local expressions. For \(n = 2\) this is the statement of conformal invariance of \(\int_M R \, dv_g\), while for \(n = 4\) it is the assertion that the infinitesimal conformal variation of \(\int_M J^2 \, dv_g\) is \(2 \int_M \Upsilon \Delta J \, dv_g\).

3. \(Q\)-curvature and families of conformally covariant differential operators

In [J] one of the authors initiated a theory of one-parameter families of natural conformally covariant local operators
\[
D_N(X, M; h; \lambda) : C^\infty(X) \rightarrow C^\infty(M), \quad N \geq 0
\]
of order \(N\) associated to a Riemannian manifold \((X, h)\) and a hypersurface \(i : M \rightarrow X\), depending rationally on the parameter \(\lambda \in \mathbb{C}\). For such a family the conformal weights which describe the covariance of the family are coupled to the family parameter in the sense that
\[
e^{-\lambda - N} \omega D_N(X, M; \hat{h}; \lambda) e^{\lambda \omega} = D_N(X, M; h; \lambda), \quad \hat{h} = e^{2\omega} h
\]
for all \(\omega \in C^\infty(X)\) (near \(M\)).

Two families are defined in [J]: one via a residue construction which has its origin in an extension problem for automorphic functions of Kleinian groups through their limit set ([J2], chapter 8), and the other via a tractor construction. Whereas the tractor family depends on the choice of a metric \(h\) on \(X\), the residue family depends on the choice of an asymptotically hyperbolic metric \(h_+\) and a defining function \(x\), to which is associated the metric \(h = x^2 h_+\).
Fix an asymptotically hyperbolic metric $h_+$ on one side $X_+$ of $X$ in $M$ and choose a defining function $x$ for $M$ with $x > 0$ in $X_+$. Set $h = x^2 h_+$. To an eigenfunction $u$ on $X_+$ satisfying

$$\Delta_{h_+} u = \mu(n - \mu) u, \quad \text{Re} \mu = n/2, \quad \mu \neq n/2$$

is associated the family

$$\langle T_u(\zeta, x), \varphi \rangle \equiv \int_{X_+} x^\zeta u \varphi \, dv_h, \quad \varphi \in C^\infty_c(X)$$

of distributions on $X$. The integral converges for $\text{Re} \zeta > -n/2 - 1$ and the existence of a formal asymptotic expansion

$$u \sim \sum_{j \geq 0} x^{\mu + j} a_j(\mu) + \sum_{j \geq 0} x^{n - \mu + j} b_j(\mu), \quad x \to 0$$

with $a_j(\mu), b_j(\mu) \in C^\infty(M)$ implies the existence of a meromorphic continuation of $T_u(\zeta, x)$ to $\mathbb{C}$ with simple poles in the ladders

$$-\mu - 1 - \mathbb{N}_0, \quad -(n - \mu) - 1 - \mathbb{N}_0.$$

For $N \in \mathbb{N}_0$, its residue at $\zeta = -\mu - 1 - N$ has the form

$$\varphi \mapsto \int_M a_0 \delta_N(h; \mu + N - n)(\varphi) dv_{i^* h},$$

where

$$\delta_N(h; \lambda) : C^\infty(X) \to C^\infty(M)$$

is a family of differential operators of order $N$ depending rationally on $\lambda \in \mathbb{C}$. If $\hat{x} = e^{\omega} x$ with $\omega \in C^\infty(X)$, then $\hat{h} = e^{2\omega} h$ and it is easily checked that $\delta_N(h; \lambda)$ satisfies (3.2). (The family $\delta_N(h; \lambda)$ should more correctly be regarded as determined by $x$ and $h_+$, but we use this notation nonetheless.)

If $g$ is a metric on $M$, then we can take $h_+ = g_+$ to be a Poincaré metric for $g$ on $X_+ = M \times (0, a)$ and $x$ to be the coordinate in the second factor, so that $h = dx^2 + g_x$. Then (assuming $N \leq n$ if $n$ is even), the family $\delta_N(h; \lambda)$ depends only on the initial metric $g$. The residue can be evaluated explicitly and for even orders $N = 2L$ one obtains

$$\delta_{2L}(h; \mu + 2L - n) = \sum_{k=0}^L \frac{1}{(2L - 2k)!} \left( \sum_{l=0}^k p_{2l, \mu}^* \circ v^{(2k - 2l)} \right) \circ i^* \partial_x^{2L - 2k},$$

where the $p_{2l, \mu}$ are the operators appearing in (2.1) and the coefficients $v^{(2j)}$ are used as multiplication operators. The corresponding residue family is defined by

$$D_{2L}^{es}(g; \lambda) = 2^{2L} L! \frac{\Gamma(-n/2 - \lambda + 2L)}{\Gamma(-n/2 - \lambda + L)} \delta_{2L}(h; \lambda);$$

where

$$\delta_{2L}(h; \lambda) : C^\infty(X) \to C^\infty(M)$$
the normalizing factor makes $D_{2L}^{\text{res}}(g; \lambda)$ polynomial in $\lambda$. We are interested in the critical case $2L = n$ for $n$ even. Using

$$\text{Res}_0(p_{n,\lambda}) = -c_{n/2} P_n$$

from [GZ], we see that

$$(3.5) \quad D_n^{\text{res}}(g; 0) = (-1)^{n/2} P_n(g) i^*.$$  

Direct evaluation from (3.3), (3.4) gives

$$\dot{D}_n^{\text{res}}(g; 0) = -(-1)^{n/2} c_{n/2}^{-1} \left( p_n^* 1 + \sum_{k=0}^{n/2-1} p_{2k}^* v^{(n-2k)} \right),$$

where the dot refers to the derivative in $\lambda$.

Suppose now that $g$ is transformed conformally: $\hat{g} = e^{2\Upsilon} g$ with $\Upsilon \in C^\infty(M)$. By the construction of the normal form in §5 of [GL], the Poincaré metrics $g_+$ and $\hat{g}_+$ are related by $\Phi^* \hat{g}_+ = g_+$ for a diffeomorphism $\Phi$ which restricts to the identity on $M$ and for which the function $\Phi^*(x)/x$ restricts to $e^\Upsilon$. Using this the residue construction easily implies

$$(3.6) \quad e^{-(\lambda-n)\Upsilon} D_n^{\text{res}}(\hat{g}; \lambda) = D_n^{\text{res}}(g; \lambda) (\Phi^*(x)/x)^{-\lambda} \Phi^*.$$  

Applying (3.6) to the function 1, differentiating at $\lambda = 0$, and using (3.5) and $P_n 1 = 0$ gives

$$e^{n\Upsilon} \dot{D}_n^{\text{res}}(\hat{g}; 0) 1 = \dot{D}_n^{\text{res}}(g; 0) 1 - (-1)^{n/2} P_{n/2} \Upsilon.$$  

This proves that the curvature quantity

$$-(-1)^{n/2} \dot{D}_n^{\text{res}}(g; 0) 1 = c_{n/2}^{-1} \left( p_n^* 1 + \sum_{k=0}^{n/2-1} p_{2k}^* v^{(n-2k)} \right)$$

satisfies the same transformation law as the $Q$-curvature. It is natural to conjecture that it equals the $Q$-curvature. Indeed, this follows from (0.1), (2.2), and (2.3):

$$p_n^* 1 + \sum_{k=0}^{n/2-1} p_{2k}^* v^{(n-2k)} = p_n 1 + \left( p_n^* 1 - p_n 1 + \frac{1}{n} \sum_{k=1}^{n/2-1} 2kp_k^* v^{(n-2k)} \right)$$

$$+ \left( \sum_{k=0}^{n/2-1} p_{2k}^* v^{(n-2k)} - \frac{1}{n} \sum_{k=1}^{n/2-1} 2kp_k^* v^{(n-2k)} \right).$$

The first term is $-c_{n/2} Q$ by (2.2), the second term is 0 by (2.3), and the last term is $2c_{n/2} Q$ by (0.1).

The relation $\dot{D}_n^{\text{res}}(g; 0) 1 = (-1)^{n/2+1} Q$ and (3.5) show that both the critical GJMS operator $P_n$ and the $Q$-curvature are contained in the one object $D_n^{\text{res}}(g; \lambda)$. In that
respect, $D_n^{\text{res}}(g; \lambda)$ resembles the scattering operator in [GZ]. However, the family $D_n^{\text{res}}(g; \lambda)$ is local and all operators in the family have order $n$.

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