Short time regularity of Navier-Stokes flows with locally $L^3$ initial data and applications

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Abstract

We prove short time regularity of suitable weak solutions of 3D incompressible Navier-Stokes equations near a point where the initial data is locally in $L^3$. The result is applied to the regularity problems of solutions with uniformly small local $L^3$ norms, and of forward discretely self-similar solutions.

Keywords: Navier-Stokes equations, regularity, Herz space, discretely self-similar.

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1 Introduction

The Navier-Stokes equations describe the evolution of a viscous incompressible fluid’s velocity field $v$ and its associated scalar pressure $\pi$. They are required to satisfy

$$\partial_t v - \Delta v + v \cdot \nabla v + \nabla \pi = 0, \quad \nabla \cdot v = 0,$$

in the sense of distributions. For our purposes, (NS) is applied on $\mathbb{R}^3 \times (0, \infty)$ and $v$ evolves from a prescribed, divergence free initial data $v_0 : \mathbb{R}^3 \to \mathbb{R}^3$. Solutions to (NS) satisfy a natural scaling: if $v$ satisfies (NS), then for any $\lambda > 0$

$$v^\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t),$$

is also a solution with pressure

$$\pi^\lambda(x, t) = \lambda^2 \pi(\lambda x, \lambda^2 t),$$

and initial data

$$v_0^\lambda(x) = \lambda v_0(\lambda x).$$

A solution is called self-similar (SS) if $v^\lambda(x, t) = v(x, t)$ for all $\lambda > 0$ and is discretely self-similar with factor $\lambda$ (i.e. $v$ is $\lambda$-DSS) if this scaling invariance holds for a given $\lambda > 1$. Similarly, $v_0$ is self-similar (a.k.a. $(-1)$-homogeneous) if $v_0(x) = \lambda v_0(\lambda x)$ for all $\lambda > 0$ or $\lambda$-DSS if this holds for a given $\lambda > 1$. These solutions can be either forward or backward if they are defined on $\mathbb{R}^3 \times (0, \infty)$ or $\mathbb{R}^3 \times (-\infty, 0)$ respectively. In this paper we work exclusively with forward solutions and omit the qualifier “forward”.

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Self-similar solutions are interesting in a variety of contexts as candidates for ill-posedness or finite time blow-up of solutions to the 3D Navier-Stokes equations (see [11, 15, 16, 24, 30, 31] and the discussion in [2]). Forward self-similar solutions are compelling candidates for non-uniqueness [16, 11]. Until recently, the existence of forward self-similar solutions was only known for small data (see the references in [2]). Such solutions are necessarily unique. In [15], Jia and Šverák constructed forward self-similar solutions for large data where the data is assumed to be Hölder continuous away from the origin. This result has been generalized in a number of directions by a variety of authors [2, 3, 4, 5, 7, 21, 23, 32]; see also the survey [17].

The motivating problem for this paper is the following question: It is shown in Tsai [32] that, if a $\lambda$-DSS initial data $v_0 \in C^0_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$, $0 < \alpha < 1$, with $M = \|v_0\|_{C^0(B_2 \setminus B_1)} < \infty$, and if $\lambda - 1 \leq c_1(M)$ for some sufficiently small positive constant $c_1$ depending on $M$, then there is a $\lambda$-DSS solution $v$ with initial data $v_0$ such that $v$ is regular, that is, $v \in L^\infty_{\text{loc}}(\mathbb{R}^4)$. The question is: What if we weaken the assumption of $v_0$ so that $v_0$ is in $L^{3,\infty}(\mathbb{R}^3)$ (weak $L^3$)? Note that for $v_0 \in L^{3,\infty}(\mathbb{R}^3)$ that is $\lambda$-DSS and divergence free, Bradshaw and Tsai [2] constructs at least one $\lambda$-DSS local Leray solution, based on a weak solution approach. Thus regularity cannot be obtained as a by-product of the existence proof, and one needs to prove the regularity of any such solutions if $\lambda - 1$ is sufficiently small.

Motivated by this problem, we need to study solutions whose initial data is locally in $L^3$, as it is also shown in [2] that, when $v_0$ is $\lambda$-DSS, then $v_0 \in L^{3,\infty}(\mathbb{R}^3)$ if and only if $v_0 \in L^3(B_\lambda \setminus B_1)$. The following local theorem is our first main result.

**Theorem 1.1.** There are positive constants $\epsilon_0, C_1$ such that the following holds. For any $M > 0$, there exist $T_1 = T_1(M) \in (0, 1)$ such that, if $(v, \pi)$ is any suitable weak solution of the Navier-Stokes equations (NS) in $B_1 \times (0, T_1)$ with initial data $v_0$,

$$\|v_0\|_{L^3(B_1)} \leq \epsilon_0,$$

and

$$\|v\|_{L^\infty_t L^3_x \cap L^4_t H^1_x(\mathbb{R}^3 \times (0, T_1))} + \|\pi\|_{L^3_t L^{3/2}_x(\mathbb{R}^3 \times (0, T_1))} \leq M,$$

then $v$ is regular in $B_{1/4} \times (0, T_1)$ with

$$|v(x, t)| \leq \frac{C_1}{\sqrt{t}}, \quad \text{in} \quad B_{1/4} \times (0, T_1),$$

and

$$\sup_{\frac{1}{r} \in (0, T_1)} \sup_{0 < r < \infty} \frac{1}{r^2} \int_{Q_{z_0, r} \cap B_1 \times (0, T_1)} |v|^3 \, dz \leq 1.$$

We can choose $T_1(M) = \epsilon(1 + M)^{-6}$ for some sufficiently small $\epsilon$ independent of $M$.

Above, we define the parabolic cylinder by $Q_{z_0, r} = B_r(x_0) \times (t_0 - r^2, t_0)$ for $z_0 = (x_0, t_0)$.

**Comments for Theorem 1.1:**

1. Our result holds for locally (in space) defined suitable weak solutions. In particular, no boundary condition is assumed on $\partial B_1 \times (0, T_1)$.

2. The quantities bounded by $M$ in (1.5) both have dimension 1 in the sense of [6]. This is convenient for the tracking of constants in Corollary 1.2.
3. It should be noted that the constant $C_1$ is independent of $M$. Intuitively, the nonlinear term has no effect before $T_1 = T_1(M)$, and hence the solution behaves like a linear solution, and its size is given by the initial data.

4. The boundedness of $\pi$ in $L^3_t L^{3/2}_x$ is natural for the Leray-Hopf weak solutions defined in $\mathbb{R}^3$, as $\pi$ is given by $\pi = R_i R_j (v_i v_j)$, where $R_j = (-\Delta)^{-1/2} \partial_j$ is the Riesz transform, and

$$\|\pi\|_{L^3_t L^{3/2}_x(\mathbb{R}^3 \times (0,T))} \leq C\|v\|^2_{L^4_t L^2_x(\mathbb{R}^3 \times (0,T))} \leq C\|v\|^2_{L^\infty L^2 \cap L^2 \dot{H}^1(\mathbb{R}^3 \times (0,T))}.$$ 

We will prove the local-in-space pressure bound for local energy solutions in Lemma 3.5.

5. The assumption $\|\pi\|_{L^3_t L^{3/2}_x(B_1 \times (0,T))} \leq M$ can be replaced by, e.g., $\|\pi\|_{L^q(\mathbb{R}^3)} \leq M$ for some $q \in (3/2, 5/3)$. It ensures that $\int_0^T \int_{B_1} |v|^3 + |p|^{3/2}$ is small for sufficiently small $T = T(M)$; thus $q = 3/2$ is not allowed. Our choice of exponents is to maximize the time exponent, so that $T_1(M) = \epsilon (1 + M)^{-m}$ has the smallest $m = 6$.

6. Theorem 1.1 is an extension of Jia-Sverak [15, Theorem 3.1], in which the initial data is assumed in $L^m(B_1)$, $m > 3$. This is similar to the extension of the mild solution theory for the scale subcritical data $v_0 \in L^m(\mathbb{R}^3)$, $m > 3$, of Fabes-Jones-Rivi`ere [8] to the critical data $v_0 \in L^3(\mathbb{R}^3)$ of Weissler [34], Giga-Miyakawa [10], Kato [19] and Giga [9].

Our first set of applications (Corollaries 1.2-1.4) of Theorem 1.1 is concerned with local energy solutions defined globally in $\mathbb{R}^3$, which are weak solutions of (NS) in $\mathbb{R}^3 \times (0, \infty)$ that are uniformly in time bounded in $L^2_{uloc}$, and satisfying the local energy inequality. See Section 3 for their definitions and properties. In order to state our result, we introduce the uniformly local $L^q$ spaces. For $q \in [1, \infty)$, we say $f \in L^q_{uloc}$ if $f \in L^q_{uloc}(\mathbb{R}^3)$ and

$$\|f\|_{L^q_{uloc}} = \sup_{x \in \mathbb{R}^3} \|f\|_{L^q(B_1(x))} < \infty. \quad (1.8)$$

We also denote for $\rho > 0$

$$\|f\|_{L^q_{uloc,\rho}} = \sup_{x \in \mathbb{R}^3} \|f\|_{L^q(B_\rho(x))}.$$  

Let $E^q$ be the closure of $C_c^\infty(\mathbb{R}^3)$ in $L^q_{uloc}$-norm. Equivalently, $E^q$ consists of those $f \in L^q_{uloc}$ with $\lim_{|x| \to \infty} \|f\|_{L^q(B_1(x))} = 0$, see [22].

In the following corollary we assume that the initial data belongs to $L^3(B_\delta) \cap E^2$.

**Corollary 1.2.** Let $\epsilon_0$ and $C_1$ be the constants from Theorem 1.1. Suppose $v$ is a local Leray solution of the Navier-Stokes equations (NS) with initial data $v_0 \in E^2$ and there is $\delta \in (0, 1]$ such that

$$\|v_0\|_{L^3(B_\delta)} \leq \epsilon_0. \quad (1.9)$$

Then, there exist $T_2 = T_2(\delta, \|v_0\|_{L^2_{uloc}}) > 0$, such that $v$ is regular in $B_{\delta/4} \times (0, T_2)$ with

$$|v(x, t)| \leq C_1 \sqrt{t}, \quad \text{in} \quad B_{\delta/4} \times (0, T_2),$$

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and
\[\sup_{z_0 \in B_{\delta/4} \times (0,T_2)} \sup_{0 < r < \infty} \frac{1}{r^2} \int_{Q_{z_0,r} \cap [B_{\delta/4} \times (0,T_2)]} |v|^3 \, dz \leq 1.\]

Furthermore, we can take \(T_2 = T_1(M)\delta^2\) with \(M = \frac{C}{\delta} \sup_{x_0} \int_{B_{\delta}(x_0)} |v_0|^2\).

In Corollary 1.3 we assume the initial data \(v_0 \in L^3_{uloc}(\mathbb{R}^3) \cap E^2\).

**Corollary 1.3.** Let \(\epsilon_0\) be the small constant from Theorem 1.1. Suppose \(v\) is a local Leray solution of the Navier-Stokes equations (NS) with initial data \(v_0 \in L^3_{uloc} \cap E^2\) and there is \(\delta \in (0, \infty)\) such that
\[\sup_{x_0 \in \mathbb{R}^3} \int_{B_{\delta}(x_0)} |v_0|^3 \leq \epsilon_0^3.\] (1.10)

Then, there is \(T > 0\) such that \(v\) is regular in \(\mathbb{R}^3 \times (0,T)\) with
\[|v(x,t)| \leq \frac{C}{\sqrt{t}}, \quad (0 < t < T).\] (1.11)

This result is similar to Maekawa-Terasawa [28, Theorem 1.1 (iii)]. Indeed, [28] constructs local in time mild solutions in the intersection of \(L^\infty(0,T;L^3_{uloc})\) and (1.11), for \(v_0 \in L^3_{uloc}\) that satisfies the smallness condition (1.10). They have \(T = C\delta^2 \|u\|^{-4}_{L^3_{uloc,\delta}}\), and do not assume spatial decay \(v_0 \in E^2\).

In contrast, Corollary 1.3 is a regularity theorem, assuming further the spatial decay of \(v_0\).

In Corollary 1.4 we consider general initial data \(v_0 \in E^2\). Let
\[\rho(x;v_0) = \sup \left\{ r > 0 : v_0 \in L^3(B_r(x)), \int_{B_r(x)} |v_0|^3 \leq \epsilon_0^3 \right\}.\]

We let \(\rho(x;v_0) = 0\) if such \(r\) does not exist.

**Corollary 1.4.** Suppose \(v_0 \in E^2\) and \(\text{div} \ v_0 = 0\). Let \(\bar{\rho}(x) = \min(\rho(x;v_0),1) \geq 0\), and \(N_r = \sup_{x_0 \in \mathbb{R}^3} \frac{1}{4} \int_{B_r(x_0)} |v_0|^2 \, dx\). Let
\[T(x) = \epsilon(1 + N_{\bar{\rho}(x)})^{-6} \bar{\rho}(x)^2 \geq 0,\]
where the constant \(\epsilon > 0\) is sufficiently small. Then, any local Leray solution \(v\) of the Navier-Stokes equations (NS) with initial data \(v_0\) is regular in the region
\[\Omega = \{(x,t) : x \in \mathbb{R}^3, \ 0 < t < T(x)\},\]
and
\[|v(x,t)| \leq \frac{C_1}{\sqrt{t}} \quad \text{in} \ \Omega.\]

Of course this corollary is interesting only near those \(x\) with \(\rho(x;v_0) > 0\). It is a consequence of Corollary 1.2.

Our second set of applications is for solutions with initial data in the Herz spaces. These spaces contain self-similar and DSS solutions, and are of particular interest to the study of DSS solutions since they are weighted spaces with a particular choice of centre. We now
recall the definitions and basic properties of Herz spaces \cite{13, 29, 33}. Let \( A_k = \{ x \in \mathbb{R}^n : 2^{k-1} \leq |x| < 2^k \} \). For \( n \in \mathbb{N}, s \in \mathbb{R} \) and \( p, q \in (0, \infty) \), the \textit{homogeneous Herz space} \( \dot{K}^s_{p,q}(\mathbb{R}^n) \) is the space of functions \( f \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) with finite norm

\[
\|f\|_{\dot{K}^s_{p,q}} = \begin{cases} 
\left( \sum_{k \in \mathbb{Z}} 2^{ksq} \|f\|_{L^p(A_k)}^q \right)^{1/q} & \text{if } q < \infty, \\
\sup_{k \in \mathbb{Z}} 2^{ks} \|f\|_{L^p(A_k)} & \text{if } q = \infty.
\end{cases}
\]

The \textit{weak Herz space} \( W\dot{K}^s_{p,q}(\mathbb{R}^n) \) are defined similarly, with \( L^p(A_k) \)-norm in the definition replaced by its weak version, \( L^{p,\infty}(A_k) \)-norm.

In what follows we take \( q = \infty \), which is most suitable for our purpose. In this case, \( \dot{K}^s_{p,\infty} \)-norm is equivalent to

\[
\|f\|_{s,p} = \sup_{x \neq 0} \left\{ |x|^s \cdot \|f\|_{L^p(B_{|x|}(x))} \right\}.
\]

Also note \( \dot{K}^s_{p,\infty} \subseteq \dot{B}^{-s}_{p,\infty} \) if \( 1 < p < \infty \) and \( 0 < s < n(1 - 1/p) \), see \cite[Theorem 1.6 (ii)]{33}.

For \( n = 3 \), let

\[
K_p := \dot{K}^{1-3/p}_{p,\infty}, \quad p \geq 3.
\]

It is invariant under the scaling \( f(x) \to \lambda f(\lambda x) \), i.e., (1.3), the natural scaling of stationary \textit{(NS)} and the following relation holds

\[
K_p \subset \dot{B}^{3/p-1}_{p,\infty} \subset BMO^{-1} \quad (3 < p < \infty).
\]

The space \( K_p \) contains those DSS in \( L^p_{\text{loc}}(\mathbb{R}^3 \setminus \{0\}) \), and thus \( K_3 \) contains all initial data considered in \cite{2, 3}.

We are interested in the Herz spaces because they seem to be natural spaces for DSS solutions of \textit{(NS)}. The existence problem of mild solutions of \textit{(NS)} in the Herz spaces has been studied extensively by Tsutsumi \cite{33}. He proves short time existence for large data in subcritical weak Herz spaces \( W\dot{K}^s_{p,\infty}(\mathbb{R}^3) \), \( 0 \leq s < 1 - 3/p \), and global existence for small data in the critical weak Herz space \( W\dot{K}^0_{3,\infty}(\mathbb{R}^3) \).

**Theorem 1.5.** Let \( \epsilon_0 \) and \( C_1 \) be the constants from Theorem 1.1. Let \( v \) be a local Leray solution of the Navier-Stokes equations \textit{(NS)} with initial data \( v_0 \in K_3 \cap E^2 \). Assume further that there is \( \mu \in (0, 1) \) such that

\[
\sup_{0 \neq x \in \mathbb{R}^3} \int_{B_{|x|}(x)} |v_0|^3 \leq \epsilon_0^3. \tag{1.12}
\]

Then there exist \( \sigma_1 = \sigma_1(\|v_0\|_{K_3}) > 0 \), \( C_2 = C_2(\|v_0\|_{K_3}) \) and \( \sigma_2 = \sigma_2(\mu, \|v_0\|_{K_3}) \in (0, \sigma_1) \), such that, for any \( R > 0 \),

\[
\sup_x \frac{1}{R} \int_{B_R(x)} |v|^2 dx + \sup_x \frac{1}{R} \int_0^{\sigma_1 R^2} \int_{B_R(x)} |\nabla v|^2 dx dt \leq C_2, \tag{1.13}
\]

and

\[
|v(x, t)| \leq \frac{C_1}{\sqrt{t}}, \quad \text{for} \quad 0 < t < \sigma_2 |x|^2. \tag{1.14}
\]
Comments for Theorem 1.5:

1. Estimate (1.14) gives a regularity estimate for the solution below the paraboloid \( t = \sigma_2|x|^2 \), i.e., in the region bounded by \( t = \sigma_2|x|^2 \) and \( t = 0 \).

2. Note that \( K_3 \subset L^2_{uloc} \), and (1.13) is a property for all local Leray solutions, see Section 3. However, We still need to assume \( v_0 \in E^2 \), since \( K_3 \) is not a subset of \( E^2 \) as the following example shows

\[
v_0(x) = \sum_{k=1}^{\infty} \zeta(x - 2^k e_1)
\]

where \( \zeta \) is a smooth cut-off function supported in \( B_1 \) and \( e_1 = (1,0,0) \).

**Corollary 1.6.** Let \( v \) be a local Leray solution of the Navier-Stokes equations (NS) with initial data \( v_0 \in K_p, \ p > 3 \). Then, the same conclusion of Theorem 1.5 is true, with the constants depending on \( \| v_0 \|_{K_p} \).

This corollary is a direct consequence of Theorem 1.5 since \( K_p \subset K_3 \) and (1.12) follows for \( \mu = C(\varepsilon_0/\| v_0 \|_{K_p})^{p/(p-3)} \) since

\[
\| v_0 \|_{L^3(B_{\mu|x|}(x))} \leq (C\mu|x|)^{1-3/p} \| v_0 \|_{L^p(B_{\mu|x|}(x))} \leq C\delta^{1-3/p} \| v_0 \|_{K_p}.
\]

**Corollary 1.7.** Let \( \lambda > 1 \) and \( v \) be a \( \lambda \)-DSS local Leray solution of the Navier-Stokes equations (NS) with \( \lambda \)-DSS initial data \( v_0 \in L^{3,\infty}(\mathbb{R}^3) \). Then \( v_0 \in K_3 \), (1.12) holds for some \( \mu > 0 \), and the same conclusion of Theorem 1.5 is true.

Furthermore, there exists \( \lambda_* = \lambda_*(\mu) \in (1,2) \) such that if \( 1 < \lambda < \lambda_* \), then \( v \) is regular at any \( (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+ \), with

\[
|v(x,t)| \leq \frac{C}{\sqrt{t}} \text{ in } \mathbb{R}^3 \times \mathbb{R}^+.
\]

This corollary answers our motivating problem.

The rest of the paper is organized as follows. In Section 2 we recall auxiliary results, including the theorems of Caffarelli-Kohn-Nirenberg [6], Kato [19], and the localization of divergence free vector fields. In Section 3 we discuss various definitions and properties of local energy solutions including a priori estimates for the pressure. In Section 4 we prove the interior regularity result for the perturbed Stokes equation. Then we address the local analysis of the Navier-Stokes equations and the proof of theorem 1.1 in Section 5. In Section 6 we consider local energy solutions with local \( L^3 \) data, and prove Corollaries 1.2-1.4. In Section 7 we discuss solutions with data in Herz spaces, and prove Theorem 1.5 and Corollaries 1.6-1.7. In Section 8 Appendix 1, we prove properties of local Leray solutions stated in Section 3.

We thank Professors Barker and Prange who kindly sent us their preprint [1] while we are finishing this paper. The preprint [1] contains a result similar to our Corollary 1.2 for local energy weak solutions.
2 Preliminaries

We first recall the following rescaled version of Caffarelli-Kohn-Nirenberg [6, Proposition 1]. It is formulated in the present form in [30, 25], and is the basis for many regularity criteria, see e.g. in [12]. For a suitable weak solution \((v, \pi)\), let

\[
C(r) = \frac{1}{r^2} \int_{Q_r} |v|^3 \, dx \, dt , \quad D(r) = \frac{1}{r^2} \int_{Q_r} |\pi|^{3/2} \, dx \, dt .
\]

**Lemma 2.1.** There are absolute constants \(\varepsilon_{CKN}\) and \(C_{CKN} > 0\) with the following property. Suppose \((v, \pi)\) is a suitable weak solution of NS with zero force in \(Q_{r_1}\), \(r_1 > 0\), with

\[
C(r_1) + D(r_1) \leq \varepsilon_{CKN},
\]

then \(v \in L^\infty(Q_{r_1/2})\) and

\[
\|v\|_{L^\infty(Q_{r_1/2})} \leq \frac{C_{CKN}}{r_1} .
\]  

(2.1)

We next recall the results due to Kato [19] and Giga [9].

**Lemma 2.2.** There is \(\varepsilon_2 > 0\) such that if \(v_0 \in L^2_\sigma(\mathbb{R}^3)\) with \(\varepsilon = \|v_0\|_2 \leq \varepsilon_2\), then there is a unique mild solution \(v \in L^\infty(0, \infty; L^3(\mathbb{R}^3))\) of (NS) with zero force and initial data \(v_0\) that satisfies

\[
\|v\|_{L^\infty L^3(\mathbb{R}^3)} + \sup_{t > 0} t^{1/2} \|v(t)\|_{L^\infty(\mathbb{R}^3)} \leq C \varepsilon .
\]  

(2.2)

We will need the following localization lemma for divergence free vector fields.

**Lemma 2.3** (localization). Let \(1 < p < \infty\) and \(0 < r < R\). There is a linear map \(\Phi\) from \(V = \{v \in L^p(B_R; \mathbb{R}^3) : \operatorname{div} v = 0\}\) into itself, and a constant \(C = C(p, r/R) > 0\) such that for \(v \in V\) and \(a = \Phi v \in V\), we have \(\operatorname{supp} a \subset B_{\frac{3}{2}(r+R)}\), \(v = a\) in \(B_r\), and \(\|a\|_{L^p(B_R)} \leq C \|v\|_{L^p(B_R)}\).

**Proof.** We may assume \(R = 1\), since the general case follows by scaling \(v(x) \rightarrow \tilde{v}(y) = v(Ry), \ x \in B_R\) and \(y \in B_1\). Fix \(\chi \in C_c^\infty(\mathbb{R}^3)\) with \(\chi = 1\) in \(B_r\) and \(\chi(x) = 0\) if \(|x| \geq \frac{1}{2}(r+1)\). We will take

\[
a = \chi v - b,
\]

where the correction \(b\) satisfies

\[
\text{div} b = \nabla \chi \cdot v , \quad \operatorname{supp} b \subset \bar{A}, \quad A := B_{\frac{3}{2}(r+1)} \setminus B_r .
\]

It can be defined by \(b = \Pi(\nabla \chi \cdot v)\), where \(\Pi\) is a Bogovskii-map from \(L^p_0(A)\) to \(W^{1,p}_0(A)\), where

\[
L^p_0(A) = \{ f \in L^p(A) : \int_A f = 0 \} ,
\]

such that \(\text{div} \Pi f = f\) and \(\|\Pi f\|_{W^{1,p}_0(A)} \leq C \|f\|_{L^p_0(A)}\). Since \(\int_A \nabla \chi \cdot v = 0\), \(b\) is defined and we have \(\|b\|_p \leq C \|\nabla b\|_p \leq C \|v\|_p\). Thus \(\|a\|_p \leq C \|v\|_p\). \(\Box\)

We will also recall the following lemma, which is proved by Jia-Sverak [15, Lemma 2.1].
Lemma 2.4. Let \( f \) be a nonnegative nondecreasing bounded function defined on \([0,1]\) with the following property: for some constants \( 0 < \sigma < 1, 0 < \theta < 1, M > 0, \beta > 0 \), we have

\[
f(s) \leq \theta f(t) + \frac{M}{(t-s)^\beta}, \quad \sigma < s < t < 1.
\]

Then,

\[
\sup_{s \in [0,\sigma]} f(s) \leq C(\sigma, \theta, \beta)M,
\]

for some positive constant \( C \) depending only on \( \sigma, \theta, \beta \).

3 Local energy solutions and Leray solutions

In this section we discuss the various definitions and properties of local energy solutions, or local Leray solutions, of \((\text{NS})\). We will also show a slightly better time integrality of the pressure.

The class of local Leray solutions was introduced by Lemarié-Rieusset in [22] to provide a local analogue of Leray’s weak solutions [24]. He constructed global in time local Leray solutions if \( v_0 \) belongs to \( E^2 \). (Recall \( L^q_{uloc} \), \( L^q_{uloc,\rho} \), and \( E^q \) are defined in the paragraph after (1.8).) See Kikuchi-Seregin [20] for another construction which treats the pressure carefully. Note that [22], [20] and Jia-Sverak [14, 15] contain alternative definitions of local Leray solutions. As some key properties of the solutions are not explicitly included in the definition of [22], we will discuss only the relation of local energy solutions of [20], and local Leray solutions of [14, 15].

Definition 3.1 (Local energy solutions [20]). A vector field \( v \in L^2_{uloc}(\mathbb{R}^3 \times [0,\infty)) \) is a local energy solution to \((\text{NS})\) with divergence free initial data \( v_0 \in E^2 \) if:

1. for some \( \pi \in L^{3/2}_{uloc}(\mathbb{R}^3 \times [0,\infty)) \), the pair \((v, \pi)\) is a distributional solution to \((\text{NS})\),

2. for any \( R > 0 \),

\[
\text{esssup} \sup_{0 \leq t < R} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} |v(x,t)|^2 \, dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^R \int_{B_R(x_0)} |\nabla v(x,t)|^2 \, dx \, dt < \infty, \quad (3.1)
\]

3. for all compact subsets \( K \) of \( \mathbb{R}^3 \) we have \( v(t) \to v_0 \) in \( L^2(K) \) as \( t \to 0^+ \),

4. \( v \) is suitable in the sense of Caffarelli-Kohn-Nirenberg, i.e., for all cylinders \( Q \) compactly supported in \( \mathbb{R}^3 \times (0,\infty) \) and all non-negative \( \phi \in C^\infty_c(Q) \), we have

\[
\int |v|^2 \phi(t) \, dx + 2 \int_0^t \int |\nabla v|^2 \phi \, dx \, dt \\
\leq \int_0^t \int |v|^2 (\partial_t \phi + \Delta \phi) \, dx \, dt + \int_0^t \int (|v|^2 + 2\pi)(v \cdot \nabla \phi) \, dx \, dt. \quad (3.2)
\]

5. for every \( x_0 \in \mathbb{R}^3 \), there exists \( c_{x_0} \in L^{3/2}(0,T) \) such that

\[
\pi(x,t) - c_{x_0}(t) = \frac{1}{3} |v(x,t)|^2 + \int_{B_2(x_0)} K(x-y) : v(y,t) \otimes v(y,t) \, dy \\
+ \int_{\mathbb{R}^3 \setminus B_2(x_0)} (K(x-y) - K(x_0-y)) : v(y,t) \otimes v(y,t) \, dy \quad (3.3)
\]

in \( L^{3/2}(0,T; L^{3/2}(B_{3/2}(x_0)))) \), where \( K(x) = \text{p.v.} \nabla^2 (\frac{1}{4\pi|x|}) \).
6. for any compact supported \( w \in L^2(\mathbb{R}^3) \),

\[
\text{the function } t \mapsto \int_{\mathbb{R}^3} v(x, t) \cdot w(x) \, dx \text{ is continuous on } [0, \infty). \tag{3.4}
\]

Property 6 in Definition 3.1 is rather mild: Vector fields satisfying Properties 1-5 can be redefined at a subset of time of zero measure so that Property 6 is also satisfied, similar to Leray-Hopf weak solutions.

For any domain \( \Omega \subset \mathbb{R}^3 \), we say \((v, \pi)\) is a suitable weak solution in \( \Omega \times (0, T) \) if it satisfies \((\text{NS})\) in the sense of distributions in \( \Omega \times (0, T) \),

\[
v \in L^\infty L^2(Q) \cap L^2 \dot{H}^1(Q), \quad \pi \in L^{3/2}(Q),
\]

and local energy inequality (3.2) for all cylinders \( Q \) compactly supported in \( \Omega \times (0, T) \) and all non-negative \( \phi \in C^\infty_c(Q) \).

**Definition 3.2** (local Leray solutions of [14, 15]). A vector field \( v \in L^2_{\text{loc}}(\mathbb{R}^3 \times [0, \infty)) \) is a local Leray solution to \((\text{NS})\) with divergence free initial data \( v_0 \in E^2 \) if properties 1-4 of Definition 3.1 are satisfied, while properties 5-6 are replaced by

7. for any \( R > 0 \),

\[
\lim_{|x_0| \to \infty} \int_0^{|x_0|} \int_{B_R(x_0)} |v(x, t)|^2 \, dx \, dt = 0, \tag{3.5}
\]

On one hand, Definition 3.1 requires the pressure decomposition formula (3.3) in \( B_1(x_0) \) for every \( x_0 \). On the other hand, in Definition 3.2, the formula (3.3) is replaced by the decay condition (3.5) at spatial infinity. Jia and Šverák claim in [14, 15] that, if \( v \) exhibits this decay, then the pressure decomposition formula (3.3) is valid. Since the decay property is easier to verify for a given solution, this justifies using it in place of the explicit pressure formula (3.3). Since [14, 15] do not provide a proof and we need a better estimate for the pressure, we will prove the equivalence of the two definitions, using ideas contained in a recent preprint of Maekawa, Prange and the second author [27] on the construction of local energy solutions in the half space.

The following lemma from [20] shows that a local energy solution is also a local Leray solution.

**Lemma 3.3** ([20], Lemma 2.2). Let \( \chi_R(x) = \chi(\frac{x}{R}) \) and \( \chi(x) \) be a smooth cut-off function in \( \mathbb{R}^3 \) so that \( \chi(x) = 0 \) for \(|x| < 1 \) and \( \chi(x) = 1 \) for \(|x| > 2 \). A local energy solution \((v, \pi)\) with divergence free initial data \( v_0 \in E^2 \) in the sense of Definition 3.1 has the decay estimate

\[
\begin{aligned}
\text{ess sup}_{0 < t < T} \alpha_R(t) + \beta_R(T) + \gamma_R^2(T) + \delta_R^4(T) &\leq C(T, A) \left\{ \left\| \chi_R v_0 \right\|_{L^2_{\text{loc}}}^2 + R^{-2/3} \right\}, \tag{3.6}
\end{aligned}
\]

for any \( T \in (0, \infty) \) and \( R \in (1, \infty) \), where \( A = \text{ess sup}_{0 < t < T} \alpha_0(t) + \beta_0(T) + \gamma_0^2(T) \) and

\[
\begin{aligned}
\alpha_R(t) &= \left\| \chi_R v(\cdot, t) \right\|_{L^2_{\text{loc}}}^2, \quad \beta_R(t) = \sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{B_1(x_0)} |\chi_R \nabla v|^2 \, dx \, dt, \\
\gamma_R(t) &= \sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{B_1(x_0)} |\chi_R v|^2 \, dx \, dt, \quad \delta_R(t) = \sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{B_1(x_0)} |\chi_R (p - c_{x_0})|^{3/2} \, dx \, dt.
\end{aligned}
\]

In particular, a local energy solution \((v, \pi)\) to \((\text{NS})\) satisfies (3.5) and is a local Leray solution to \((\text{NS})\) in the sense of Definition 3.2.
It is standard to see that $A$ is finite by using properties 2–5 and the Sobolev embedding.

The following lemma shows that a local Leray solution is also a local energy solution. It is stated in [14, 15] without a proof. We will give a proof in Appendix 1 (§8) for the sake of completeness.

**Lemma 3.4** (pressure decomposition). Suppose $(v, \pi)$ is a local Leray solution to (NS) with divergence free initial data $v_0 \in E^2$ in the sense of Definition 3.2. For any $x_0 \in \mathbb{R}^3$, $r > 0$, and $T > 0$, we have for $(x, t) \in Q := B_r(x_0) \times (0, T)$,

$$
\pi(x, t) = \pi_{loc}(x, t) + \pi_{far}(x, t) + c_{x_0,r}(t)
$$

for some function $c_{x_0,r}(t) \in L^{3/2}(0,T)$. In particular, $(v, \pi)$ is a local energy solution to (NS) with initial data $v_0$ in the sense of Definition 3.1.

The decomposition (3.7) is stronger than (3.3) since the radius $r$ is arbitrary.

We do not have a bound of $c_{x_0,r}$, which is not needed anyway, since the quantity in the equation (NS) is $\nabla \pi$. Formally $c_{x_0,r}(t) = \int_{\mathbb{R}^3 \backslash B_{2r}(x)} K(x-y)(v \otimes v)(y,t) \, dy$, but the integral does not converge.

With both Lemmas 3.3 and 3.4, we can treat local energy solutions and local Leray solutions as the same.

The following lemma is the a priori bounds for the local Leray solutions. In particular the first estimate (3.8) is proved in [14, Lemma 2.2]. We will give a proof in Appendix 1 (§8).

**Lemma 3.5** (a priori bounds). Suppose $(v, \pi)$ is a local Leray solution to (NS) with divergence free initial data $v_0 \in E^2$ and $\pi$ is decomposed as in Lemma 3.4 in every $B_r(x_0)$. For any $s, q > 1$ with $\frac{2}{s} + \frac{2}{q} = 3$, there exists a positive constant $C(s, q)$ such that

$$
\text{ess sup}_{0 \leq t \leq \sigma^2} \sup_{x_0 \in \mathbb{R}^3} \frac{1}{r} \int_{B_r(x_0)} |v|^2 \, dx + \sup_{x_0 \in \mathbb{R}^3} \frac{1}{r} \int_0^{\sigma^2} \int_{B_r(x_0)} |\nabla v|^2 \, dx \, dt < C_0 N_r, \tag{3.8}
$$

$$
\sup_{x_0 \in \mathbb{R}^3} \frac{1}{r} \|\pi - c_{x_0,r}(t)\|_{L^s(0,\sigma^2;L^q(B_r(x_0)))} \leq C(s, q) N_r \tag{3.9}
$$

where

$$
N_r = \sup_{x_0 \in \mathbb{R}^3} \frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \, dx, \quad \sigma = \sigma(r) = c_0 \min \left\{ (N_r)^{-2}, 1 \right\},
$$

for universal constants $C_0$ and $c_0 > 0$.

Note that both estimates are stated as the dimension free form in the sense of [6]. Instead of (3.9), the following is given in [15, (3.6)],

$$
\sup_{x_0 \in \mathbb{R}^3} \frac{1}{r^2} \int_0^{\sigma^2} \int_{B_r(x_0)} |\pi - c_{x_0,r}(t)|^{3/2} \, dx \, dt < C N_r^{3/2}, \tag{3.10}
$$
which is a consequence of (3.9) by Hölder inequality. In [20] and [14, 15], the estimate of $\|\pi - c_{x_0,r}(t)\|$ is only in $L^{3/2}(B_r(x_0) \times (0,T))$. This is however not sufficient for our purpose: We need the exponent for time integration to be larger than $3/2$ for the application to Theorem 1.1. In fact, that $p \in L^{5/3}_{t,x,loc}$ seems to be implicitly used in the proof of [15, Theorem 3.1], as explained below [32, (3.3)]. In the regularity theory for (NS), one can often improve the spatial regularity but not the temporal regularity. Hence it is advantageous to start with a higher exponent for the time integrability.

We end this section with summarizing other fundamental properties proved in [20] for the sake of completeness.

Lemma 3.6 ([20], Theorem 1.4). Suppose $(v, \pi)$ is a local energy solution to (NS) with divergence free initial data $v_0 \in E^2$. Then $v(t) \in E^2$ for all $t$, $v(t) \in E^3$ for a.e. $t$, and

$$\lim_{t \to 0^+} \|v(t) - v_0\|_{L^2_{uloc}} = 0.$$ (3.11)

4 Perturbed Stokes system

The following interior result for the perturbed Stokes system is similar to [15, Lemma 2.2]. Instead of Hölder continuity, we claim $L^q$-integrability for any finite $q$ under a weaker assumption for the perturbed term. Recall $Q_r = B_r \times (-r^2, 0)$.

Proposition 4.1. For any $q \in [5, \infty)$, there is $\delta_0 = \delta_0(q) > 0$ such that the following hold. For any $M > 0$, if $G \in L^5(Q_1; \mathbb{R}^{3 \times 3})$ with $\|G\|_{L^5(Q_1)} \leq M$, $a \in L^5(Q_1)$ with $\text{div} a = 0$, and $\|a\|_{L^5(Q_1)} \leq \delta_0$, $\xi \in \mathbb{R}^3$, $|\xi| \leq 1$, $u \in L^\infty L^2 \cap L^2 H^1(Q_1)$, $p \in L^{3/2}(Q_1)$,

$$\|u\|_{L^3(Q_1)} + \|p\|_{L^{3/2}(Q_1)} \leq M$$

solve the a-perturbed Stokes equations

$$u_t - \Delta u + (a + \xi) \cdot \nabla u + u \cdot \nabla a + \text{div} G + \nabla p = 0, \quad \text{div} u = 0,$$ (4.1)

in $Q_1$, then we have

$$u \in L^q(Q_{1/2}), \quad \|u\|_{L^q(Q_{1/2})} \leq C(q) M.$$

Proof of Proposition 4.1. Step 1. Initial bounds and localization.

Bounds. Since this equation is linear, we may assume $M = 1$. Because $a \in L^5$ and $u$ is in energy space, we can prove local energy inequality for the $a$-perturbed Stokes equations (4.1). Fix $1 - 10^{-10} < \sigma < 1$. By the local energy inequality for (4.1), a calculation similar to that in [15, page 242] shows that, for $\sigma < r_1 < r_2 < 1$,

$$E(r_1) \leq \frac{C}{(r_2 - r_1)^2} + (C \|a\|_{L^5(Q_1)} + \frac{1}{2}) E(r_2),$$

where

$$E(r) = \text{ess sup} \int_{-r^2}^{0} \int_{B_r} \frac{|u|^2}{2} dx + \int_{-r^2}^{0} \int_{B_r} |\nabla u|^2 dxdt.$$ 

By Lemma 2.4, if $\|a\|_{L^5(Q_1)}$ is sufficiently small, we have $E(\sigma) < C$, i.e.,

$$\|u\|_{L^\infty L^2 \cap L^2 H^1(Q_0)} \leq C.$$
Taking the divergence of (4.1), we get \( p \)-equation
\[-\Delta p = \partial_i \partial_j \tilde{G}_{ij}, \quad \tilde{G}_{ij} = (a + \xi)_i u_j + u_i a_j + G_{ij}.\]

Note that
\[\|\tilde{G}\|_{L^{3/2}L^{18/7}(Q_1)} \lesssim (\|a\|_{L^5} + |\xi|)\|u\|_{L^{15/7}L^{90/17}} + \|G\|_{L^5} \leq C.\]

Hence by the elliptic estimate we have
\[\|p\|_{L^{3/2}((-\sigma^2,0);L^{18/7}(R^3,2))} \leq \|\tilde{G}_{ij}\|_{L^{3/2}L^{18/7}(Q_4)} + \|p\|_{L^{3/2}(Q_1)} \leq C. \tag{4.2}\]

Localization of \( a \). By the Bogovski map, we can solve \( \tilde{a} : \mathbb{R}^3 \times (-1,0) \to \mathbb{R}^3 \) such that
\[\text{div} \tilde{a} = 0, \quad \tilde{a}(x,t) = a(x,t) \quad \text{if } |x| < \sigma, \quad a(x,t) = 0 \quad \text{if } |x| > 1,
\]
\[\|\tilde{a}\|_{L^5(\mathbb{R}^3 \times (-1,0))} \leq C\|a\|_{L^5(Q_1)},\]
for a constant \( C \).

Localization of \( u \). Choose \( \chi_0 \in C_c^\infty(\mathbb{R}^3) \), radial, \( \chi_0 \geq 0 \), \( \chi_0 = 1 \) on \( B_\sigma \), \( \chi_0 = 0 \) on \( B_\sigma^c \). Let \( \chi_k(x) = \chi_0(\sigma^{-k}x) \). Let \( \chi = \chi_2 \). Let
\[w = u\chi - \nabla \eta, \quad \pi = p\chi + \partial_t \eta,\]
where \( \eta \) is the function which satisfies \( \Delta \eta = u \cdot \nabla \chi \) and it is given by
\[\eta(x,t) = \int \frac{1}{4\pi|x-y|} (u \cdot \nabla \chi)(y, t) \, dy.\]

Since \( u \cdot \nabla \chi \) is supported in \( Q_1 \), the Calderon-Zygmund estimate shows
\[\|\nabla^2 \eta\|_{L^{10}_t((-\sigma^2,0);(L^{30/13}_x \cap L^{6/5}_x)(\mathbb{R}^3))} \leq C\|u\|_{L^{10}_t(L^{30/13}_x \cap L^{6/5}_x)(Q_4)} \leq C.\]

By the Sobolev embedding and Riesz potential estimates, we have
\[\|\nabla \eta\|_{L^{10}_t((-\sigma^2,0);(L^{6/5}_x \cap L^2_x)(\mathbb{R}^3))} \leq C, \quad \|\nabla \eta\|_{L^{10/3}_t((-\sigma^2,0);(L^6 \cap L^6)(\mathbb{R}^3))} \leq \|u\|_{L^{10/3}_t(L^6 \cap L^6)(Q_4)} \leq C. \tag{4.3}\]

We also have
\[\|\nabla^2 \eta\|_{L^{10/3}_t((-\sigma^2,0) \times \mathbb{R}^3)} \leq \|u\|_{L^{10/3}_t(Q_4)} \leq C
\]
\[\|\nabla \eta\|_{L^{10/3}_t((-\sigma^2,0);L^{\infty}(\mathbb{R}^3))} \leq \|\nabla^2 \eta\|_{L^{10/3}_t((-\sigma^2,0);L^6(\mathbb{R}^3))} + \|\nabla \eta\|_{L^{10/3}_t} \leq C. \tag{4.4}\]

Therefore from (4.3), we obtain
\[\|w\|_{L^{\infty}_tL^2 \cap L^2 \dot{H}^1((-\sigma^2,0) \times \mathbb{R}^3)} \leq C. \tag{4.5}\]

Moreover, \( w \) satisfies
\[w_t - \Delta w + (\tilde{a} + \xi) \cdot \nabla w + w \cdot \nabla \tilde{a} + \nabla \pi = f_0 + \nabla F, \quad \text{div } w = 0 \quad \text{in } \mathbb{R}^3 \times (-1,0), \tag{4.6}\]
where
\[f_0 = u \Delta \chi + [(\tilde{a} + \xi) \cdot \nabla \chi]u + [u \cdot \nabla \chi] \tilde{a} + p \nabla \chi + (\nabla \chi) \cdot G, \]
\[F = F_0 - \xi \otimes \nabla \eta, \tag{4.7}\]
\[F_0 = -2 \nabla \chi \otimes u - \nabla \eta \otimes \tilde{a} - \tilde{a} \otimes \nabla \eta - \chi G.\]
Note that both \( f_0 \) and \( F_0 \) are localized. In terms of size,

\[
f_0 \approx p + \tilde{a}u + u + G, \quad F \approx u + a\nabla \eta + \nabla \eta + G.
\]

Recall that \( p \) and \( \tilde{a}u \) are estimated by (4.2), and \( \nabla \eta \) by (4.3). Thus

\[
\begin{align*}
\|f_0\|_{L^{3/2}(-\sigma^2, 0; (L^{18/7} \cap L^2)(\mathbb{R}^3))} & \leq C, \\
\|F\|_{L^{10/3}(-\sigma^2, 0; (L^{10/3} \cap L^2)(\mathbb{R}^3))} & \leq C.
\end{align*}
\]

(4.8)

Note that \( p \) appears in \( f_0 \), but not in \( F_0 \). This is very helpful because we cannot improve its time integrability. Also note that the correction term \( \nabla \eta \) in the definition of \( w \) is defined by the nonlocal Newtonian potential, which enables us to hide its time derivative \( \partial_t \eta \) in \( \pi \), so that we don’t need to estimate \( \partial_t \eta \). This technique has been used, for example, in [18, 26].

**Step 2.** A bootstrap lemma.

In this step we prove a bootstrap lemma to improve integrability. We will use the following potential estimates.

**Lemma 4.2.** Let \( Q = \mathbb{R}^3 \times I, I = (0, T), 0 < T < \infty \). Let

\[
\Phi_0 f_0(t) = \int_0^t e^{(t-s)\Delta} P f_0(s) ds, \quad \Phi_1 F(t) = \int_0^t e^{(t-s)\Delta} P \nabla \cdot F(s) ds,
\]

for \( f_0 \in L^{3/2}(I; L^r(\mathbb{R}^3; \mathbb{R}^3)) \) and \( F \in L^m(\mathbb{R}^3 \times I; \mathbb{R}^{3 \times 3}) \). Here \( P \) denotes the Helmholtz projection on \( \mathbb{R}^3 \). We have

\[
\begin{align*}
\|\Phi_0 f_0\|_{L^q(Q)} & \lesssim T^{\frac{2}{3} \left( 1 - \frac{3}{q} + \frac{3}{2r} \right)} \|f_0\|_{L^{3/2}(I; L^r)}, \quad \frac{1}{q} \geq \frac{3}{5r} - \frac{2}{15}, \quad 1 < r \leq q < \infty; \\
\|\nabla \Phi_0 f_0\|_{L^q(Q)} & \lesssim T^{\frac{2}{3} \left( 1 - \frac{3}{q} + \frac{3}{2r} \right)} \|f_0\|_{L^{3/2}(I; L^r)}, \quad \frac{1}{q} \geq \frac{3}{5r} + \frac{1}{15}, \quad 1 < r \leq q < \infty; \\
\|\Phi_1 F\|_{L^q(Q)} & \lesssim T^{\frac{2}{3} \left( 1 - \frac{m}{q} + \frac{3}{2} \right)} \|F\|_{L^m(Q)}, \quad \frac{1}{q} \geq \frac{1}{m} - \frac{1}{5}, \quad 1 < m \leq q < \infty.
\end{align*}
\]

(4.9) (4.10) (4.11)

**Proof.** By the decay estimates of \( e^{(t-s)\Delta} P \nabla \cdot \), we have

\[
\|\Phi_1 F(t)\|_{L^q_\varepsilon} \lesssim \int_0^t |t-s|^{-\alpha} \|F(s)\|_{L^q_\varepsilon} ds,
\]

where \( \alpha = \frac{2}{2} \left( \frac{1}{m} - \frac{1}{q} \right) + \frac{1}{2} \in \left[ \frac{1}{2}, 1 \right) \), hence \( 0 \leq \frac{1}{m} - \frac{1}{q} < \frac{1}{2} \). By the Hardy-Littlewood-Sobolev inequality and \( T < \infty \), we get \( \|\Phi_1 F\|_{L^q(Q)} \lesssim T^{\frac{2}{3} \left( 1 - \frac{m}{q} + \frac{3}{2} \right)} \|F\|_{L^m(Q)} \) if

\[
\frac{1}{q} + 1 \geq \alpha + \frac{1}{m}, \quad \text{i.e.,} \quad \frac{1}{m} - \frac{1}{q} \leq \frac{1}{5}.
\]

This shows (4.11). Similarly,

\[
\|\Phi_0 f_0(t)\|_{L^q_\varepsilon} \lesssim \int_0^t |t-s|^{-\alpha} \|f_0(s)\|_{L^q_\varepsilon} ds,
\]

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where \( \alpha = \frac{3}{2} \left( \frac{1}{q} - \frac{1}{r_1} \right) \in [0, 1) \), hence \( 0 \leq \frac{1}{q} < \frac{2}{3} \). By the Hardy-Littlewood-Sobolev inequality and \( T < \infty \), we get \( \| \Phi_0 f_0 \|_{L^q(Q)} \lesssim T^{\frac{1}{2} \left( \frac{1}{q} - \frac{1}{r} + \frac{1}{r'} \right)} \| f_0 \|_{L^{3/2}(I; L^r)} \) if

\[
\frac{1}{q} + 1 \geq \alpha + \frac{2}{3}, \quad \text{i.e.,} \quad \frac{1}{q} \geq \frac{3}{5r} - \frac{2}{15},
\]

which implies \( \frac{1}{q} > \frac{1}{r} - \frac{2}{r_1} \). This shows (4.9). Since the estimate (4.10) is similar to estimates above, we skip its details. \( \square \)

Next we show the bootstrap lemma.

**Lemma 4.3.** Let \( 3 \leq r < \infty, 2 \leq r_0 \leq 9/2 \) and \( 2 \leq r_1 \leq 5 \). There are small \( \epsilon = \epsilon(r, r_0, r_1) > 0 \) and \( \tau_0 = \tau_0(r, r_0, r_1) \in (0, 1) \) such that the following hold. Let \( w \in L^\infty L^2 \cap L^2 H^1(Q) \) be a weak solution of the perturbed Stokes system (4.6) in \( Q = \mathbb{R}^3 \times I, I = (t_0, t_1), t_0 < t_1 \leq t_0 + \tau_0 \). Assume that \( w(t_0) \in L^r \cap L^2(\mathbb{R}^3), f_0 \in L^{3/2}(I; L^{r_0} \cap L^2(\mathbb{R}^3)), F \in L^{r_1}(I; L^{r_1} \cap L^2), \left| \xi \right| \leq 1, \text{ div} \tilde{a} = 0, \text{ and } \| \tilde{a} \|_{L^5(Q)} \leq \epsilon \). Let

\[
N := \| w(t_0) \|_{L^r \cap L^2} + \| f_0 \|_{L^{3/2}(I; L^{r_0} \cap L^2)} + \| F \|_{L^{r_1}(I; L^{r_1} \cap L^2)},
\]

Then \( w \in X \) with

\[
\| w \|_X \leq C(r, r_0, r_1) N, \quad X = L^\infty L^2 \cap L^2 \tilde{H}^1 \cap L_q^{t, x}(Q).
\]

Here \( q = \min \left( \frac{3}{r}, q_0, q_1 \right), q_0 = (\frac{3}{2r_0} - \frac{2}{3r})^{-1}, q_1 = (\frac{1}{r_1} - \frac{1}{5})^{-1}, \max (r_0, r_1) \leq q < \infty \).

**Proof.** Without loss of generality, we may assume \( t_0 = 0 \). We may also assume \( N = 1 \) since it is a linear equation. We define a sequence of approximation solutions of (4.6) by

\[
w_1(t) = e^{t \Delta} w(0) + \Phi_1 F(t) + \Phi_0 f_0(t),
\]

\[
w_{k+1}(t) = w_1(t) - \Phi_1 ((\tilde{a} + \xi) \otimes w_k + w_k \otimes \tilde{a})(t) \quad k = 1, 2, \ldots
\]

Due to the energy inequality for the Stokes system and (4.8), we have

\[
\| e^{t \Delta} w(0) + \Phi_1 F \|_{L^q_t L^2_x \cap L^q_t \tilde{H}^1_x} \leq C \| w(0) \|_{L^2} + \| F \|_{L^q_t L^2_x} \leq C
\]

We observe by (4.10) of Lemma 4.2 that

\[
\| \Phi_0 f_0 \|_{L^q_t L^2_x \cap L^q_t \tilde{H}^1_x} \leq C t_1^{\frac{1}{5}} \| f_0 \|_{L^q_t L^2_x}^{\frac{3}{5}} L^2_x.
\]

Therefore,

\[
\| w_1 \|_{L^q_t L^2_x \cap L^q_t \tilde{H}^1_x} \leq C \| w(0) \|_{L^2} + C \| F \|_{L^q_t L^2_x} + C t_1^{\frac{1}{5}} \| f_0 \|_{L^q_t L^2_x}^{\frac{3}{5}} L^2_x \leq C.
\]

On the other hand, using the \( L^q_t \) estimate of \( e^{t \Delta} w(0) \) (see Giga [9]) and Lemma 4.2, we also have \( \| w_1 \|_{L^q_t L^2_x} \leq C, \) and hence \( \| w_1 \|_{X} \leq K \) for some \( K > 0 \).

We show that \( w_k \in X \) with \( \| w_k \|_X \leq 2K \) provided that \( t_k \) is sufficiently small by induction. Suppose that \( w_k \in X \) with \( \| w_k \|_X \leq 2K \) for some \( k = 0, 1, 2, \ldots \). The energy inequality shows that

\[
\| \Phi_1 ((\tilde{a} + \xi) \otimes w_k + w_k \otimes \tilde{a}) \|_{L^\infty L^2 \cap L^2 \tilde{H}^1} \lesssim \| (\tilde{a} + \xi) \otimes w_k + w_k \otimes \tilde{a} \|_{L^\infty L^2 x} 
\]

\[
\lesssim \| \tilde{a} \|_{L^\infty L^2 x} \| w_k \|_{L^{10/3}} + t_1^{\frac{1}{5}} \| w_k \|_{L^\infty L^2}.
\]
Next using Lemma 4.2, we can see that

\[ \| \Phi_1((\tilde{a} + \xi) \otimes w_k + w_k \otimes \tilde{a}) \|_{L^2} \leq C(\| \tilde{a} \|_{L^3} + t_1^{1/2}) \| w_k \|_{L^2}. \]

Thus if \( t_0 \) and \( \| \tilde{a} \|_{L^2_{t,x}} \) are sufficiently small, we have

\[ \| \Phi_1((\tilde{a} + \xi) \otimes w_k + w_k \otimes \tilde{a}) \|_{X} \leq \frac{1}{2} K, \]

which shows the uniform bound of \( w_k \) in \( X \) by \( 2K \).

Denoting \( \delta_k w := w_{k+1} - w_k \), we get

\[ \delta_k w = -\Phi_1((\tilde{a} + \xi) \otimes \delta_{k-1} w + \delta_{k-1} w \otimes \tilde{a}), \quad k = 1, 2, \ldots . \]

Following the estimates above, we see that

\[ \| \delta_k w(t) \|_{X} \leq C \left( \| \tilde{a} \|_{L^2_{t,x}} + t_1^{1/2} \right) \| \delta_{k-1} w(s) \|_{X}. \]

Therefore, if \( \| \tilde{a} \|_{L^2_{t,x}} \) is sufficiently small, and if \( t_1 > 0 \) is so small such that

\[ C t_1^{1/2} \leq 1/4, \quad (4.12) \]

then \( w_k \) is a Cauchy sequence in \( X \), and converges to a mild solution \( \tilde{w} \) of (4.1) in \( X \) with \( \tilde{w}(0) = w(0) \in L^2 \cap L^r(R^3) \). By the uniqueness of the weak solution of the perturbed Stokes system (4.6) in the energy class (which can be proved by energy estimate and Gronwall inequality, using \( \tilde{a} \in L^5_{t,x} \)), we see that \( w = \tilde{w} \). This completes the proof. \( \square \)

**Step 3.** Intermediate bounds.

Let \( r = 6, \ r_0 = 18/7, \) and \( r_1 = 10/3 \). We have \( f \in L^{3/2}(-7/8, 0; L^{r_0} \cap L^2) \) and \( F \in L^{r_1}(-7/8, 0; L^{r_1} \cap L^2) \) by (4.8). Choose \( \tau = \min(\frac{1}{8}, \tau_0) \), where \( \tau_0 = \tau_0(r = 6, r_0 = 18/7, r_1 = 10/3) \) is decided by Lemma 4.3. Note \( \frac{5}{3}r - q_0 = q_1 = 10 \). Thus we take \( q = 10 \).

Let \( a_n = -n\tau/2, \) and \( I_n = [a_n, a_{n-1}], \) \( n \in \mathbb{Z} \). Choose smallest integer \( N \) so that \( I_N \subset (-\frac{4}{8}, -\frac{4}{3}) \). Since \( w \) is in the energy class, we have

\[ \int_{I_n} \left( \int_{R^3} \left| \nabla w \right|^2 + \left| w \right|^2 dx \right) dt \leq C, \quad 1 \leq n \leq N. \]

Thus, there is \( t_n \in I_n \) such that

\[ \int_{R^3} \left| \nabla w(t_n) \right|^2 + \left| w(t_n) \right|^2 dx \leq \frac{C}{\tau/2} = C, \quad 1 \leq n \leq N. \]

By Sobolev imbedding, \( \int |w(t_n)|^6 dx \leq C \). By Lemma 4.3, we have

\[ w \in L^{10}(J_n \times R^3), \quad J_n = (t_n, t_n + \tau) \cap (-1, 0), \quad 1 \leq n \leq N. \]

Since

\[ t_n + \tau \geq a_n + \tau = a_{n-2} \geq t_{n-1}, \quad 2 \leq n \leq N, \]

we have

\[ (-3/4, 0) \subset J := \bigcup_{n=1}^{N} J_n = (t_N, 0), \quad t_N \leq -3/4. \]
Hence

\[ \|w\|_{L^{10}(J \times \mathbb{R}^3)} \leq C. \quad (4.13) \]

We now show \( w \in L^{\infty}(J; L^{10/3}) \). Indeed, for \( t \in J \),

\[ \|w(t)\|_{L_x^{10/3}} \lesssim \|w(t_N)\|_{L_x^6 \cap L^2} + \|\Phi_0 f_0(t)\|_{L_x^{10/3}} + \|\Phi_1 F(t)\|_{L_x^{10/3}} + \|\Phi_1 F_1(t)\|_{L_x^{10/3}} \]

where \( F_1 = (\tilde{a} + \xi) \otimes w + w \otimes \tilde{a} \), and \( \Phi_0 \) and \( \Phi_1 \) are redefined with initial time \( t_N \). Note

\[ \|\Phi_1 F_1(t)\|_{L_x^{10/3}} \lesssim \int_{t_N}^t |t - s|^{-1/2} \|F_1(s)\|_{L_x^{10/3}} \, ds \]
\[ \lesssim \|F_1\|_{L_t^{10/3}(J \times \mathbb{R}^3)} \]
\[ \lesssim (1 + \|a\|_{L_x^6}) \|w\|_{L^{10} \cap L^{10/3}(J \times \mathbb{R}^3)}. \]

Similarly,

\[ \|\Phi_0 f_0(t)\|_{L_x^{10/3}} \lesssim \|f_0\|_{L_x^{3/2}L_x^{18/7}} \]
\[ \|\Phi_1 F(t)\|_{L_x^{10/3}} \lesssim \|F\|_{L_t^{10/3}}. \]

Thus

\[ \|w\|_{L^{\infty}(J; L^{10/3}(\mathbb{R}^3))} \leq C. \]

We also claim \( \nabla w \in L^{10/3}(J \times \mathbb{R}^3) \) and \( w \in L^{10/3}_t(\mathbb{R}^3) \). Indeed,

\[ \|\nabla w\|_{L_t^{10/3}(J \times \mathbb{R}^3)} \leq \|\nabla e^{(t-t_N)} w(t_N)\|_{L_t^{10/3}(J \times \mathbb{R}^3)} + \|\nabla \Phi_0 f_0\|_{L_t^{10/3}(J \times \mathbb{R}^3)} + \|\nabla \Phi_1 (F + F_1)\|_{L_t^{10/3}(J \times \mathbb{R}^3)}. \]

By energy estimate,

\[ \|\nabla e^{(t-t_N)} w(t_N)\|_{L_t^{10/3}(J \times \mathbb{R}^3)} \leq C \|\nabla w(t_N)\|_{L^2(\mathbb{R}^3)} \leq C. \]

By (4.10) of Lemma 4.2,

\[ \|\nabla \Phi_0 f_0(t)\|_{L_x^{10/3}} \lesssim \|f_0\|_{L_x^{3/2}L_x^{18/7}}. \]

By maximal regularity in \( L_{t,x}^{10/3} \),

\[ \|\nabla \Phi_1 (F + F_1)\|_{L_t^{10/3}(J \times \mathbb{R}^3)} \leq C \|F + F_1\|_{L_t^{10/3}(J \times \mathbb{R}^3)} \leq C. \]

Thus \( \|\nabla w\|_{L^{10/3}(J \times \mathbb{R}^3)} \leq C \), and by Sobolev inequality,

\[ \|w\|_{L_t^{10/3}(J \times \mathbb{R}^3)} \lesssim \|\nabla w\|_{L_t^{10/3}(J \times \mathbb{R}^3)} + \|w\|_{L_t^{10/3}(J \times \mathbb{R}^3)} \leq C. \]

By \( w = u\chi - \nabla \eta \), \( \chi = 1 \) on \( B_{a^3} \), and \( \nabla \eta \) estimates in (4.3)–(4.4), we have

\[ \|u\|_{L_t^{10/3} \cap L_x^6 \cap L_x^{10/3} \cap L_x^{10/3} \cap L_x^6 (-\frac{3}{4}, 0) \times B_{a^3}} \lesssim \|\nabla u\|_{L_t^{10/3} (-\frac{3}{4}, 0) \times B_{a^3}} \leq C. \quad (4.14) \]

Note

\[ \|au\|_{L^{3/2}((-\frac{3}{4}, 0); L^6(B_{a^3}))} \lesssim \|a\|_{L^5((-\frac{3}{4}, 0); L^5(B_{a^3}))} \|u\|_{L^{15/7}((-\frac{3}{4}, 0); L^\infty(B_{a^3}))} \leq C. \]
From elliptic estimate \((4.2)\) again with spatial exponent 5, we get
\[
\|p\|_{L^{3/2}(\frac{-3}{4}, 0; L^5(B_{\sigma^3}))} \leq \|au\|_{L^{3/2}L^5} + \|u\|_{L^{3/2}L^5} + \|G\|_{L^{3/2}L^5} + \|p\|_{L^{3/2}L^{3/2}} \leq C. \tag{4.15}
\]
Above, the last 4 norms are taken over \((-\frac{3}{4}, 0) \times B_{\sigma^3}.

**Step 4.** Refined bounds.

We repeat the localization and define again \(w = u\chi - \nabla \eta\), but with \(\chi\) replaced by \(\chi_4(x) = \chi_0(\sigma^{-3}x)\). We now have better estimates than those in Step 1 because of \((4.14)\) and \((4.15)\). We first have
\[
\|\nabla^2 \eta\|_{L^{3/2}_{t,\tau}((-\frac{3}{4}, 0) \cap L^6_{x}) \cap L^5_{x}((-\frac{3}{4}, 0) \times \mathbb{R}^3)} \leq C,
\]
and thus
\[
\|w\|_{L^{3/2}_{t,\tau} \cap L^\infty_{x}(L^2 \cap L^5_{x}) \cap L^5_{x}((-\frac{3}{4}, 0) \times \mathbb{R}^3)} + \|\nabla w\|_{L^{3/2}_{t,\tau} \cap L^\infty_{x}(L^2 \cap L^5_{x})((-\frac{3}{4}, 0) \times \mathbb{R}^3)} \leq C.
\]
Thanks to \(\nabla \eta \in L^\infty_{t,x}\), we have
\[
\|F\|_{L^5_{t,x}((-\frac{3}{4}, 0) \times \mathbb{R}^3)} \leq C\|\tilde{a}\| + \|u\| + \|G\|_{L^5_{t,x}((-\frac{3}{4}, 0) \times B_{\sigma^3})} + \|\nabla \eta\|_{L^5_{t,x}((-\frac{3}{4}, 0) \times \mathbb{R}^3)} \leq C,
\]
and
\[
\|f_0\|_{L^{3/2}_{t,\tau}((-\frac{3}{4}, 0) \cap L^5_{x}(B_{\sigma^3}))} \leq C\|p\| + \|u\| + \|au\| + \|G\|_{L^{3/2}_{t,\tau}((-\frac{3}{4}, 0) \cap L^5_{x}(B_{\sigma^3}))} \leq C.
\]

For any \(q \in (10, \infty)\), let \(r = 3q/5\), \(r_0 = 9/2\), and \(r_1 = 5\). Note \(q_0 = q_1 = \infty\), and \(w \in L^{10/3}((-\frac{3}{4}, 0) \cap L^r(\mathbb{R}^3))\). Choose \(\tau = \min(\frac{1}{\sigma}, \tau_0)\), where \(\tau_0 = \tau_0(r, r_0 = 9/2, r_1 = 5)\) is decided by Lemma 4.3.

By the same argument as in Step 3, we may use time interval partition to get \(u(t_\epsilon) \in L^2 \cap L^r(\mathbb{R}^3)\), and Lemma 4.3 to get \(w \in L^q_{t,x}((-1/2, 0) \times \mathbb{R}^3)\).

Since \(w = u\chi_4 - \nabla \eta\) and \(\nabla \eta \in L^\infty_{t}(L^2 \cap L^\infty_{x})\), we have, if \(1/2 \leq \sigma^5\),
\[
u \in L^q_{t,x}((-1/2, 0) \times B_{1/2}(0)).
\]
This finishes the proof of Proposition 4.1. \(\Box\)

**Remark.** We can repeat the second part of Step 3 and prove, e.g., \(w \in L^\infty L^q\) for any finite \(q\) if \(\|\tilde{a}\|_{L^q_{t,x}} \leq \delta(q)\) is sufficiently small.

## 5 Local analysis for the Navier-Stokes equations

In this section we prove Theorem 1.1. The proof is split into 3 subsections.

### 5.1 Decay estimates for Navier-Stokes

Let \((u, p)\) be a suitable weak solution of the following \(a\)-perturbed Navier-Stokes equations in \(Q = B_1 \times (0, T)\), with \(a \in L^5(Q)\), \(\text{div} \: a = 0\),
\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + (a + u) \cdot \nabla u + u \cdot \nabla a + \nabla p &= 0, \\
\text{div} \: u &= 0.
\end{align*}
\tag{5.1}
\]
That is, \( u \in L^\infty L^2(Q) \cap L^2 \dot{H}^1(Q) \), \( p \in L^{3/2}(Q) \), the pair solves (5.1) in the distributional sense, and satisfies the \textit{perturbed local energy inequality}: For all non-negative \( \phi \in C^\infty_c(Q) \), we have

\[
\int |u|^2 \phi(t) \, dx + 2 \int_0^t \int |\nabla u|^2 \phi \, dx \, dt \\
\leq \int_0^t \int |u|^2 (\partial_t \phi + \Delta \phi) \, dx \, dt + \int_0^t \int \left( |u|^2 (u + a) + 2pu \right) \cdot \nabla \phi \, dx \, dt \\
+ \int_0^t \int u_j a_i \partial_j (u_i \phi) \, dx \, dt.
\]

This is equivalent to (3.2) for \( v = u + a \) if \( v \) is a weak solution of (NS) in \( Q \) and \( a \) is a strong solution of (NS); see the argument after (5.19) for details.

Let \( z = (x, t) \) and \( Q_r(z) = B_r(x) \times (t - r^2, t) \). We denote

\[
\varphi(u, p, r, z) := \left( \frac{1}{r^2} \int_{Q_r(z)} |u - (u)_{Q_r(z)}|^3 \right)^{\frac{1}{3}} + \left( \frac{1}{r^2} \int_{Q_r(z)} |p - (p)_{B_r(x)}(t)|^{3/2} \right)^{\frac{2}{3}}
\]

where

\[
(u)_{Q_r(z)} = \frac{1}{|Q_r(z)|} \int_{Q_r(z)} u, \quad (p)_{B_r(x)}(t) = \frac{1}{|B_r(x)|} \int_{B_r(x)} p(y, t) \, dy.
\]

Note that \( \varphi \) is dimension-free in the sense of [6], and its form is invariant under scaling.

**Lemma 5.1** (Decay estimate). For any \( \alpha \in (0, 1) \), there is a small \( \delta_0 > 0 \) such that the following hold. Let \( (u, p) \) be a suitable weak solution to the perturbed Navier-Stokes equations (5.1) in \( Q_r(z) \), with \( a \in L^5(Q_r(z)) \), \( \text{div} \, a = 0 \), \( \|a\|_{L^5(Q_r(z))} = \delta \leq \delta_0 \). Denote \( (u)_r = (u)_{Q_r(z)} \). Then, for any \( \theta \in (0, 1/3) \) there exist \( \epsilon = \epsilon(\theta, \alpha) > 0 \) and \( C = C(\alpha) > 0 \) independent of \( \theta \) such that if

\[
r |(u)_r| \leq 1, \quad \varphi(u, p, r, z) + r |(u)_r| \delta < \epsilon,
\]

then

\[
\theta r |(u)_{1\theta}| \leq 1, \quad \varphi(u, p, \theta r, z) \leq C \theta^\alpha \left[ \varphi(u, p, r, z) + r |(u)_r| \delta \right].
\]

**Proof.** Choose \( q \in (5, \infty) \) such that \( \alpha < 1 - \frac{5}{q} \). Choose \( \delta_0 = \delta_0(q(\alpha)) \) according to Proposition 4.1. Since \( \varphi \) and \( r(u)_r \) are dimension-free, we may assume \( r = 1 \). We may assume \( z = 0 \) and skip the \( z \)-dependence in \( \varphi \) without loss of generality. We first show (5.5). Indeed,

\[
\theta |(u)_\theta| \leq \theta |(u - (u)_1)_\theta| + \theta |(u)_1| \\
\leq \theta |Q_\theta|^{-\frac{3}{2}} \|u - (u)_1\|_{L^3(Q_\theta)} + \theta \\
\leq C_3 \theta^{-\frac{2}{3}} \varphi(1) + \theta,
\]

with \( C_3 = |Q_1|^{-\frac{1}{3}} \). By (5.4), \( \varphi(1) \leq \epsilon \), hence \( \theta |(u)_\theta| < 1 \) if

\[
\epsilon \leq \theta^{2/3} / 2C_3.
\]

Next we show the decay estimate (5.6). Here we use a contradiction argument, following a similar argument as given in e.g. [25, Lemma 3.2] and [15, Lemma 2.3]. Since some
modification is required, we give the details for completeness. Suppose that this is not the case. Then there exist solutions \((u_i, p_i)\) of (5.1), \(a_i\) and \(\epsilon_i\) with \(\lim_{i \to \infty} \epsilon_i = 0\) such that
\[
\xi_i = (u_i)_{Q_1}, \quad |\xi_i| \leq 1, \quad \|a_i\|_{L^5(Q_1)} \leq \delta_0, \quad \text{div} \ a_i = 0,
\]
\[
\varphi(u_i, p_i, 1) + |\xi_i|\|a_i\|_{L^5(Q_1)} = \epsilon_i,
\]
\[
\varphi(u_i, p_i, \theta) \geq C_2 \theta^\alpha \epsilon_i.
\]
Here \(C_2 > 0\) is a large constant to be chosen later. Setting \(v_i = (u_i - (u_i)_1)/\epsilon_i\) and \(q_i = (p_i - (p_i)_1(t))/\epsilon_i\), it follows that
\[
\|v_i\|_{L^3(Q_1)} + \|q_i\|_{L^\infty(Q_1)}^2 + \frac{|\xi_i|}{\epsilon_i} \|a_i\|_{L^5(Q_1)} = 1,
\]
\[
\left(\frac{1}{\theta^2} \int_{Q_\theta} |v_i - (v_i)_{Q_\theta}|^3 \right)^{\frac{1}{3}} + \left(\frac{1}{\theta^2} \int_{Q_\theta} |q_i - (q_i)_{B_r(t)}|^3 \right)^{\frac{1}{3}} \geq C_2 \theta^\alpha
\]  
(5.9)
and \((v_i, q_i)\) satisfies
\[
\partial_t v_i - \Delta v_i + (\epsilon_i v_i + a_i + \xi_i) \cdot \nabla v_i + \left(v_i + \frac{\xi_i}{\epsilon_i}\right) \cdot \nabla a_i + \nabla q_i = 0, \quad \text{div} \ v_i = 0.
\]
Denote
\[
E_i(r) = \text{ess sup}_{-r^2 < t < 0} \int_{B_r} \frac{|v_i|^2}{2} \, dx + \int_0^{-r^2} \int_{B_r} |\nabla v_i|^2 \, dx \, dt.
\]
By the local energy inequality for (5.1), the calculation in [15, page 242] shows that, for \(3/4 < r_1 < r_2 < 1\),
\[
E_i(r_1) \leq \frac{C}{(r_2 - r_1)^2} + (C\|a_i\|_{L^5(Q_1)} + \frac{1}{2}) E_i(r_2),
\]
By Lemma 2.4, if \(\|a_i\|_{L^5(Q_1)} \leq \delta_0\) is sufficiently small, we have \(E_i(3/4) < C\) for all \(i\).

By the uniform bound \(E_i(3/4) < C\) for all \(i\), there exist \((v, q) \in (L^3 \times L^{3/2})(Q_{3/4})\), \(\xi \in \mathbb{R}^3\) and \(a, G \in L^3(Q_{3/4})\) such that (if necessary, subsequence can be taken)
\[
v_i \to v \quad \text{strongly in} \quad L^3(Q_{3/4}), \quad \xi_i \to \xi,
\]
\[
q_i \to q \quad \text{weakly in} \quad L^\frac{3}{2}(Q_{3/4}), \quad a_i \to a \quad \text{weakly in} \quad L^5(Q_{3/4}),
\]
\[
\frac{(u_i)_1}{\epsilon_i} \otimes a_i \to G \quad \text{weakly in} \quad L^5(Q_{3/4}),
\]
as \(i \to \infty\). Furthermore, \((v, q)\) solves the linear perturbed Stokes system in \(Q_{3/4}\)
\[
\partial_t v - \Delta v + \xi \cdot \nabla v + a \cdot \nabla v + v \cdot \nabla a + \text{div} \, G + \nabla q = 0, \quad \text{div} \ v = 0.
\]
Due to Proposition 4.1, it follows that \(v \in L^q(Q_{1/2}), q > 5\), for the exponent \(q\) chosen at the beginning of the proof. Thus, by the strong convergence of \(v_i\) to \(v\) in \(L^3(Q_{3/4})\), we have for sufficiently large \(i\)
\[
\left(\frac{1}{\theta^2} \int_{Q_\theta} |v_i - (v_i)_a|^3 \, dz \right)^{\frac{1}{3}} \leq C \theta^{1 - \frac{5}{q}}.
\]  
(5.10)
On the other hand, by the pressure equation, we decompose $q_i = q_i^R + q_i^H$ such that

$$q_i^R = \frac{1}{(-\Delta)^{-1}} \text{div} \text{div} \left( [v \otimes v_i + v_i \otimes a_i + a_i \otimes v_i] \chi_{B^{3/4}_3} \right),$$

Here $\chi_{B^{3/4}_3}$ is the characteristic function of $B^{3/4}_3$. We then see that $q_i^R$ converges strongly to $q_i^R$ in $L^3_t(Q_{3/4})$, where $q_i^R$ is

$$q_i^R = (-\Delta)^{-1} \text{div} \text{div} \left( [v \otimes a + a \otimes v] \chi_{B^{3/4}_3} \right).$$

We note that $q_i^R \in L^{l}((Q_{1/2}))$, where $l = \frac{1}{q} + \frac{1}{5}$. Therefore,

$$\left( \frac{1}{\theta^2} \int_{Q_\theta} |q_i^R|^\frac{2}{3} \, dz \right)^{\frac{3}{2}} \leq C \theta^{2-\frac{5}{2}} = C \theta^{1-\frac{5}{q}}.$$

Thus, for large $i$, we also have

$$\left( \frac{1}{\theta^2} \int_{Q_\theta} |q_i^R|^\frac{2}{3} \, dz \right)^{\frac{3}{2}} \leq C \theta^{1-\frac{5}{q}}.$$

Since $q_i^H$ is harmonic (in $x$) in $Q_{3/4}$, we see that

$$\left( \frac{1}{\theta^2} \int_{Q_\theta} |q_i^H - (q_i^H)_{B_\theta(t)}|^\frac{3}{2} \, dz \right)^{\frac{3}{2}} \leq C \theta^{\frac{5}{2}}.$$

Adding up the above estimates,

$$\left( \frac{1}{\theta^2} \int_{Q_\theta} |q_i - (q_i)_{B_\theta(t)}|^\frac{3}{2} \, dz \right)^{\frac{3}{2}} \leq C \theta^{1-\frac{5}{q}}. \tag{5.11}$$

The sum of (5.10) and (5.11) contradicts (5.9) if we take $C_2$ sufficiently large. This completes the proof. \hfill \Box

### 5.2 Regularity criterion for perturbed Navier-Stokes

In this subsection we prove the following regularity criterion for perturbed Navier-Stokes equations (5.1). It is an extension of the result [15, Theorem 2.2] for the perturbed term $a \in L^m(Q_1)$ with $m > 5$.

**Lemma 5.2** (Regularity criterion). For any fixed $\beta \in (0,1)$, there exist small $\epsilon_1(\beta), \delta(\beta) > 0$ with the following properties: Let $(u, p)$ be a suitable weak solution to the perturbed Navier-Stokes equations (5.1) in $Q_{3/4}$, with $a \in L^5(Q_{3/4})$, $\text{div} \, a = 0$, $\|a\|_{L^5(Q_{3/4})} \leq \delta$, and

$$\int_{Q_{3/4}} |u|^3 + |p|^\frac{3}{2} \leq \epsilon_1. \tag{5.12}$$

Then we have

$$\sup_{z_0 = (x_0, t_0) \in Q_{3/4}} \sup_{r < \frac{1}{4}} \frac{1}{r^{2+3\beta}} \int_{Q_r(z_0)} |u|^3 + |p - (p)_{B_r(z_0)}(t)|^{3/2} \, dz < C(\beta). \tag{5.13}$$
Remark. Unlike [15, Theorem 2.2] and [6, Proposition 1], estimate (5.13) does not imply Hölder continuity, but Morrey type regularity.

Proof. For fixed $\beta \in (0, 1)$, choose $\alpha = (1 + \beta)/2$ so that $\alpha \in (\beta, 1)$, and choose $\theta \in (0, 1/3)$ so that the factor $C\theta^{\alpha}$ in (5.6) is bounded by $\frac{1}{2}\theta^{\beta}$, and $\theta^{1-\beta} < \frac{1}{2}$.

In the following we omit the dependence on $z_0 \in Q_{1/4}$ to simplify the notation.

Let $B(r) = r \mid (u)_{Q_+}$ and $\varphi(r)$ be defined by (5.3). It is proved in (5.7) for $r = 1$ that

$$B(\theta r) \leq C_3 \theta^{-\frac{4}{1+\beta}} \varphi(r) + \theta B(r),$$

(5.14)

where $C_3 = |Q_1|^{-1/3}$. The proof for general $r$ is the same. Let

$$\Psi(r) = \varphi(r) + (2C_3)^{-1} \theta^{\frac{2}{1+\beta}} B(r),$$

where $C_3$ is the constant in (5.14). We want to show by induction that

condition (5.4) is valid, and $\Psi(\theta r) \leq \theta^\beta \Psi(r)$, (5.15)

for $r \in I_k = [\frac{\theta^{k+1}}{4}, \frac{\theta^k}{4}]$, for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let

$$\Psi_k = \sup_{z_0 \in Q_{1/4}, \ r \in I_k} \Psi(r; z_0), \ k \in \mathbb{N}_0.$$

By (5.12),

$$\Psi_0 \leq C(\beta)\epsilon_1^{1/3} \leq \epsilon$$

if $\epsilon_1 = \epsilon_1(\beta)$ is sufficiently small. In particular, the condition (5.4) is uniformly satisfied for every $z_0 = (x_0, t_0) \in Q_{1/4}$ and $r \in I_0$.

Suppose that (5.15) has been proved for $r \in \cup_{j<k} I_j$ and condition (5.4) is satisfied for $r \in I_k$ for some $k \in \mathbb{N}_0$. By (5.6) of Lemma 5.1 and (5.14),

$$\Psi(\theta r) = \varphi(\theta r) + (2C_3)^{-1} \theta^{\frac{2}{1+\beta}} B(\theta r)$$

$$\leq \frac{\theta^\beta}{2} \varphi(r) + \frac{\theta^\beta}{2} \delta B(r) + \frac{\theta^\beta}{2} \varphi(r) + (2C_3)^{-1} \theta^{\frac{2}{1+\beta}} B(r)$$

$$= \theta^\beta \varphi(r) + \theta^\beta \left( C_3 \delta \theta^{-\frac{2}{1+\beta}} + \theta^{1-\beta} \right) (2C_3)^{-1} \theta^{\frac{2}{1+\beta}} B(r),$$

which is bounded by $\theta^\beta \Psi(r)$ if $\delta \leq \min\{\delta_0(\alpha), (2C_3)^{-1} \theta^{\frac{2}{1+\beta}}\}$. This shows (5.15) for $r \in I_k$.

As a result, $\Psi_{k+1} \leq \theta^\beta \Psi_k \leq \ldots \leq \theta^{(k+1)\beta} \Psi_0 \leq \theta^{(k+1)\beta} \epsilon$. Hence

$r \mid (u)_r = B(r) \leq 2C_3 \theta^{-\frac{4}{1+\beta}} \Psi_{k+1} \leq 2C_3 \theta^{-\frac{4}{1+\beta}} \theta^\beta \epsilon \leq 1$

by (5.8),

$r \mid (u)_r \mid \leq 1 \cdot \delta \leq \epsilon/2,$

and

$\varphi(u, p, r, z_0) \leq \Psi_{k+1} \leq \theta^\beta \epsilon \leq \epsilon/2$

for $r \in I_{k+1}$. That is, condition (5.4) is valid for $r \in I_{k+1}$.

By induction, we have shown (5.15) for all $r \leq 1/4$ and all $z_0 \in Q_{1/4}$. In particular, if $r \in I_k$,

$$\Psi(r, z_0) \leq \Psi_k \leq \theta^{k\beta} \epsilon \leq C \epsilon r^\beta,$$

which implies (5.13).
5.3 Proof of Theorem 1.1

We now prove Theorem 1.1. Choose $\alpha = 1/2$, $\beta = 1/4$ and choose $\theta > 0$ so small that $\theta^{\alpha-\beta}$, $\theta^{1-\beta}$ and $\theta^{3/2}$ are sufficiently small in the proof of Lemma 5.2.

By Lemma 2.3, there is $a_0 \in L^3(\mathbb{R}^3)$ with

$$a_0 = v_0 \text{ in } B_{3/4}, \quad a_0 = 0 \text{ in } B_1^c, \quad \text{div} \, a_0 = 0, \quad \|a_0\|_{L^3} \leq C(3, \frac{3}{4})\|v_0\|_{L^3} \leq \varepsilon_2,$$

where $\varepsilon_2$ is the constant in Lemma 2.2. By Lemma 2.2, there is a unique mild solution $a$ of (NS) with zero force and initial data $a(0) = a_0$ that satisfies (2.2). In particular,

$$\|a\|_{L^5_tL^3_x(\mathbb{R}^4)} \leq C\varepsilon_2. \quad (5.16)$$

Let $\pi_a$ be its corresponding pressure. We have $\pi_a = R_iR_ja_i a_j$, and

$$\|\pi_a\|_{L^{5/2}_tL^4_x(\mathbb{R}^4)} \leq C\|a\|_{L^5_tL^3_x(\mathbb{R}^4)} \leq C\varepsilon_2. \quad (5.17)$$

By the maximal regularity for the inhomogeneous Stokes system, we have

$$\nabla a \in L^{5/2}(\mathbb{R}^4), \quad \nabla \pi_a \in L^{5/3}(\mathbb{R}^4). \quad (5.18)$$

Let $b_0 = v_0 - a_0$, $b = v - a$, and $\pi_b = \pi - \pi_a$. Denote $T = T_1$. Observe that $(b, \pi_b)$ is a weak solution of the $\varepsilon$-perturbed Navier-Stokes equations (5.1) in $Q = B_1 \times (0, T)$, with $b(x, 0) = b_0(x)$, and $b_0(x) = 0$ in $B_{3/4}$. We claim that $(b, \pi_b)$ satisfies the perturbed local energy inequality (5.2). This is also stated in [15] without detailed explanation. Indeed, (5.2) and (3.2) for $v = a + b$ are equivalent because they differ by an equality which is the sum of the weak form of $a$-equation with $2v\phi$ as the test function and the weak form of $b$-equation with $2a\phi$ as the test function. This equality can be proved because $a$ is a strong solution satisfying (5.16)–(5.18).

By the interpolation, $\|v\|_{L^4_tL^3_x(\mathbb{R}^3)} \leq C\|v\|_{L^{\infty}_tL^2_x \cap L^3_xH^1_x(\mathbb{R}^3)}$. Hence the assumption (1.5) shows

$$\|v\|_{L^3(\mathbb{R}^3)} \leq C\|v\|_{L^4_tL^3_x(\mathbb{R}^3)} T^{\frac{1}{12}} \leq C\sqrt{MT}^{\frac{1}{12}},$$

and

$$\|\pi\|_{L^{3/2}_tL^2_x(\mathbb{R}^3)} \leq C\|\pi\|_{L^{5/2}_tL^4_x(\mathbb{R}^3)} T^{\frac{1}{6}} \leq CMT^{\frac{1}{6}}.$$

Thus, if $T \leq \varepsilon_1^2 M^{-6}$ with $\varepsilon$ sufficiently small, we get

$$\int_0^T \int_{B_1} |b|^3 + |\pi_b|^3 \leq C\varepsilon + C\varepsilon^2 \leq \varepsilon_1, \quad (5.19)$$

where $\varepsilon_1$ is the small constant in (5.12) of Lemma 5.2.

Extend $a$, $b$, and $\pi_b$ by zero for $t < 0$ and denote $Q^T_r := B_r \times (T - r^2, T)$. Using that $b_0(x) = 0$ in $B_{3/4}$ and $\|a\|_{L^5}$ is sufficiently small, the standard energy estimate shows that $(b, \pi_b)$ is a suitable weak solution of (5.1) in $Q^T_{3/4} r$ satisfying the perturbed local energy inequality (5.2), and $\frac{3}{4}(b)_{Q^T_{3/4}} \leq 1$. In particular, $(b, \pi_b)$ satisfies (5.1) across $t = 0$ in the sense of distributions.

We can now apply Lemma 5.2 to conclude that

$$\sup_{z_0 = (x_0, t_0) \in Q^T_r} \sup_{r < \frac{t_0}{2}} \frac{1}{r^{2+3\delta}} \int_{Q_r(z_0)} |b|^3 + |\pi_b - (\pi_b)_{B_r(x_0)}(t)|^{3/2} \, dz < C.$$
Choose largest \( r_1 \leq 1/4 \) such that \( Cr_1^{3\beta} \leq \frac{1}{2}\varepsilon_{\text{CKN}} \). Hence

\[
\sup_{z_0 = (x_0, t_0) \in Q^2_{\frac{3}{4}}} \sup_{r \leq r_1} r^2 \int_{Q_r(z_0)} |b|^3 + |\pi_b - (\pi_b)_{B_r(x_0)}(t)|^{3/2} dz < \frac{1}{2}\varepsilon_{\text{CKN}}.
\]

We may take \( T \leq 4r_1^2 \) as \( r_1 \) is an absolute constant. For \( r \geq r_1 \) we have

\[
\sup_{z_0 \in B_{\frac{1}{4}} \times (0, T)} \sup_{r \geq r_1} r^2 \int_{Q_r(z_0) \cap Q} |b|^3 dz < \frac{1}{r_1^2} C\varepsilon < \frac{1}{2}.
\]

Since \( v = a + b \) and \( a \in L^5(\mathbb{R}^4) \) small, we have

\[
\sup_{z_0 \in B_{\frac{1}{4}} \times (0, T)} \sup_{0 < r < \infty} \frac{1}{r^2} \int_{Q_r(z_0) \cap Q} |v|^3 dz < 1. \tag{5.20}
\]

Now for any \( z_0 = (x_0, t_0) \in B_{1/4} \times (0, T) \), take \( r = \frac{1}{2}\sqrt{t_0} \). We have \( r \leq r_1 \) and

\[
r^2 < t < 4r^2 \quad \text{if} \, (x, t) \in Q_r(z_0).
\]

For this \( r \), let

\[
\tilde{\pi} = \pi_a + \pi_b - (\pi_b)_{B_r(x_0)}(t).
\]

We have

\[
\frac{1}{r^2} \int_{Q_r(z_0)} |v|^3 + |\tilde{\pi}|^{3/2} dz < \varepsilon_{\text{CKN}}
\]

if \( \|a\|_{L^5} + \|\pi_a\|_{L^{5/2}} \leq C\varepsilon_0 \) is sufficiently small. Since \( v, \tilde{\pi} \) is a suitable weak solution of (NS) in \( Q_r(z_0) \), by Lemma 2.1, we get

\[
|v(z_0)| \leq \|v\|_{L^\infty(Q_{r/2}(z_0))} \leq \frac{C_{\text{CKN}}}{r/2} = \frac{4C_{\text{CKN}}}{\sqrt{t_0}}. \tag{5.21}
\]

This completes the proof of Theorem 1.1. \( \square \)

6 Local Leray solutions with local \( L^3 \) data

In this section, we present the proofs of Corollaries 1.2, 1.3 and 1.4.

Proof of Corollary 1.2. Let

\[
u_0(x) = \delta v_0(\delta x), \quad u(x, t) = \delta v(\delta x, \delta^2 t), \quad p(x, t) = \delta^2 \pi(\delta x, \delta^2 t). \tag{6.1}
\]

Then \((u, p)\) is a local Leray solution of (NS) with initial data \( u_0 \in E^2 \) and \( \|u_0\|_{L^3(B_1)} \leq \epsilon_0. \) By Lemma 3.5 with \((s, q) = (2, 3/2), \)

\[
\text{ess sup} \sup_{0 \leq t \leq \sigma} \int_{B_1(x_0)} |u|^2 dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^\sigma \int_{B_1(x_0)} |\nabla u|^2 dx dt < C_0N_1,
\]

\[
\sup_{x_0} \|p - c_{x_0, 1}(t)\|_{L^2(0, \sigma; L^{3/2}(B_1(x_0)))} \leq C(2, 3/2)N_1,
\]

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where
\[ N_1 = \sup_{x_0 \in \mathbb{R}^3} \int_{B_1(x_0)} |u_0|^2 \, dx = \sup_{x_0 \in \mathbb{R}^3} \frac{1}{\delta} \int_{B_\delta(x_0)} |v_0|^2 \, dx. \]

and \( \sigma(1) = c_0 \min \{(N_1)^{-2}, 1\} \). Note that we have used \( \delta < 1 \) in the last inequality.

We may replace \( p \) by \( p - c_{0,1}(t) \). Then \( u_0, u \) and \( p \) satisfy the assumptions in Theorem 1.1 with \( M = (C_0 + C(2,3/2))N_1 \). By Theorem 1.1, there exists \( T_1 = \epsilon(1 + M)^{-6} \in (0, \sigma] \) such that \( u \) is regular in \( B_{1/4} \times (0, T_1) \) with
\[ |u(x,t)| \leq \frac{C_1}{\sqrt{t}}, \quad \text{in} \quad B_{1/4} \times (0, T_1), \]

and
\[ \sup_{z_0 \in B_{1/4} \times (0,T_1)} \sup_{0 < r < \infty} \frac{1}{r^2} \int_{Q_r(z_0) \cap |B_1 \times (0, T_1)|} \frac{|u|^3}{dz} \leq 1. \]

The condition \( T_1 \leq \sigma \) is clearly satisfied if we had chosen \( \epsilon \leq c_0 \). Back to \( v \), we have
\[ |v(x,t)| \leq \frac{C_1}{\sqrt{t}}, \quad \text{in} \quad B_{\delta/4} \times (0, T_1 \delta^2), \]

and
\[ \sup_{z_0 \in B_{\delta/4} \times (0,T_1 \delta^2)} \sup_{0 < r < \infty} \frac{1}{r^2} \int_{Q_r(z_0) \cap |B_{\delta/4} \times (0, T_1 \delta^2)|} \frac{|v|^3}{dz} \leq 1. \]

This completes the proof of Corollary 1.2.

Proof of Corollary 1.3. If \( \delta \in (0,1] \), the Corollary is a direct consequence of Corollary 1.2, since smallness of \( L^3 \)-norm in \( B_{\delta}(x_0) \) is assumed to be uniform in \( x_0 \). If \( \delta > 1 \), then (1.10) is also valid for \( \delta = 1 \) and the Corollary follows from the case \( \delta = 1 \).

Proof of Corollary 1.4. Fix \( x_0 \in \mathbb{R}^3 \) such that \( \rho(x_0) = \rho(x_0; v_0) > 0 \). We may assume \( \rho(x_0) \leq 1 \). By Corollary 1.2, we obtain
\[ |v(x,t)| \leq \frac{C_1}{\sqrt{t}}, \quad \text{in} \quad B_{\rho(x_0)/4} \times (0, T(x_0)), \]

with \( T(x_0) = T_1(M)\rho^2(x_0) \) and \( M = CN_{\rho(x_0)} \). The claim then follows by taking \( T_1(M) = \epsilon(1 + M)^{-6} \).

7 Solutions with data in Herz spaces

In this section we prove Theorem 1.5 and Corollary 1.7 for initial data in the Herz space \( K_3 \).

Lemma 7.1. The inclusion \( K_3 \subset L^2_{uloc} \) holds. Moreover there exists a positive constant \( C \) such that
\[ N_R := \sup_{x \in \mathbb{R}^3} \frac{1}{R} \int_{B_R(x)} |v_0(x)|^2 \leq CR||v_0||^2_{K_3}, \]
holds for any \( R > 0 \).
Proof. Fix $R > 0$. If $|x| > 2R$, then
\[
\int_{B_R(x)} |v_0|^2 \leq \left( \int_{B_R(x)} |v_0|^3 \right)^{2/3} \|1\|_{L^3(B_R(x))} \leq \left( \int_{B_{1/2}(x)} |v_0|^3 \right)^{2/3} CR
\]
\[
\leq C\|v_0\|^2_{K_3 R}.
\]
On the other hand, if $|x| \leq 2R$, then
\[
\int_{B_R(x)} |v_0|^2 \leq \int_{B_{3R}(0)} |v_0|^2 \leq \sum_{k=0}^{\infty} \int_{x \sim 2^{-k}3R} |v_0|^2 \leq \sum_{k=0}^{\infty} C\|v_0\|^2_{K_3 2^{-k}R} = C\|v_0\|^2_{K_3 R}.
\]
These estimates show the desired bound. \hfill \square

Note that $K_3 \not\subset E^2$, as shown by the example (1.15).

Proof of Theorem 1.5. By Lemma 3.5 with $(s, q) = (2, 3/2)$, for any $R > 0$, we have
\[
\sup_{0 < \tau < \sigma R} \sup_{x \in \mathbb{R}^3} \frac{1}{R} \int_{B_R(x)} |v|^2 + \sup_{x \in \mathbb{R}^3} \frac{1}{R} \int_0^{\sigma R} \int_{B_R(x)} |\nabla v|^2 \leq CN_R \leq C\|v_0\|^2_{K_3},
\]
(7.1)
\[
\sup_{x \in \mathbb{R}^3} \frac{1}{R} \left( \int_0^{\sigma R} \left( \int_{B_R(x)} |\pi - c_{x, R}(t)|^{3/2} \right)^{4/3} \right)^{1/2} \leq C\|v_0\|^2_{K_3},
\]
(7.2)
for $\sigma = \sigma(\|v_0\|_{K_3})$ independent of $R$.

Let $\mu > 0$ be the small constant in (1.12). For $x_0 \in \mathbb{R}^3$ with $x_0 \neq 0$, let $\delta = \mu|x_0|$. By (1.12), we have
\[
\int_{B_{\delta}(x_0)} |v_0(x)|^3 dx \leq \epsilon_0^3.
\]
(7.3)
Here $\epsilon_0$ is the constant from Theorem 1.1.

By the same proof of Corollary 1.2, we have
\[
|v(x, t)| \leq \frac{C_1}{\sqrt{t}}, \text{ in } B_{\delta/4} \times (0, T \delta^2).
\]
and
\[
\sup_{z_0 \in B_{\delta/4} \times (0, T \delta^2)} \sup_{0 < \tau < \infty} \frac{1}{\tau^2} \int_{Q_t(z_0) \cap [B_{\delta/4} \times (0, T \delta^2)]} |v|^3 dz \leq 1.
\]
We do not need the assumption $\delta \leq 1$ since our a priori bounds (7.1) and (7.2) are valid for all $R \in (0, \infty)$.

This completes the proof of Theorem 1.5. \hfill \square

Proof of Corollary 1.7. By [2, Lemma 3.1], $v_0 \in K_3$. We will show $v_0$ satisfies (1.12) for some $\mu \in (0, 1)$. Because of the discrete self-similarity, it suffice to consider the region $A := \{x \in \mathbb{R}^3; \frac{1}{2} \leq |x| < \frac{3}{2} \lambda \}$. Since $v_0$ is locally $L^3$, there exists $L > 0$ such that $\int_A |v_0|^3 \leq L$. We now define $r_i$ ($i = 1, 2, \cdots$) iteratively as
\[
r_0 = \frac{1}{2}, \quad r_{i+1} = \sup \left\{ r > 0; \int_{r \leq |x| \leq r} |v_0|^3 \leq \frac{\varepsilon}{2} \right\},
\]
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unless \( r_{i+1} \geq \frac{3}{2} \lambda \). This iteration stops at finite steps, namely, there exists \( j \) such that \( r_j \geq \frac{3}{2} \lambda \). We then set
\[
\mu := \min \left\{ \frac{1}{2} \frac{r_i}{\lambda} : i = 1, 2, \ldots, j \right\}.
\]
If \( 1 \leq |x| \leq \lambda \), we can find some \( i = 1, 2, \ldots, j \) such that
\[
B_{\mu|x|}(x) \subset S_i \cup S_{i+1}
\]
where \( S_i := \{ x \in \mathbb{R}^3 : r_i \leq |x| \leq r_{i+1} \} \). Hence by the definition of \( r_i \), we see \( \int_{B_{\mu|x|}(x)} |v_0|^2 \leq \varepsilon \). Now the first part is a direct consequence of Theorem 1.5. The second part can be also shown by the arguments in [32, Lemma 3.3]. Since its verification is similar to that in [32, Lemma 3.3], we omit the details.

\[ \square \]

8 Appendix 1: Properties of local Leray solutions

In this appendix we prove Lemmas 3.4 and 3.5. We will prove Lemma 3.4 following the approach of Maekawa-Miura-Prange [27, §3] (our case in \( \mathbb{R}^3 \) is of course simpler), and using the estimates in Maekawa-Terasawa [28].

**Lemma 8.1** (Linear \( L^p_{uloc} \) estimate in \( \mathbb{R}^d \) [28] Corollary 3.1). Let \( 1 \leq q \leq p \leq \infty \). For \( f \in L^q_{uloc}(\mathbb{R}^d) \) and \( m = 0, 1 \), we have
\[
\left\| \nabla^m e^{t \Delta} f \right\|_{L^p_{uloc}} \leq t^{-m/2} (1 + t^{-\frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right)}) \| f \|_{L^q_{uloc}}. \tag{8.1}
\]
\[
\left\| e^{t \Delta} \mathbb{P} \nabla \cdot F \right\|_{L^p_{uloc}} \leq t^{-1/2} (1 + t^{-\frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right)}) \| f \|_{L^q_{uloc}}. \tag{8.2}
\]

**Proof of Lemma 3.4.** By the definition of a local energy solution, there is \( A \in (0, \infty) \) such that
\[
\text{ess sup}_{0 < t < T} \| v(t) \|_{L^2_{uloc}} + \sup_{x \in \mathbb{R}^3} \int_0^T \int_{B_1(x)} |\nabla v|^2 \leq A.
\]
Since \( v_0 \in E^2 \), there are \( v_0^\epsilon \in C_c^\infty \), \( v_0^\epsilon \to v_0 \) in \( L^2_{uloc} \) as \( \epsilon \to 0 \). Fix a radial smooth nonnegative function \( \phi \) such that
\[
\phi = 1 \quad \text{in} \quad B_1, \quad \text{spt} \phi \subset B_{3/2}.
\]
For \( \phi_{\epsilon}(x) = \phi(\epsilon x) \), let
\[
v^\epsilon(t) = e^{t \Delta} v_0^\epsilon + \int_0^t e^{(t-s) \Delta} \mathbb{P} \nabla \cdot [\phi_{\epsilon} v \otimes v](s) ds,
\]
and
\[
\bar{v}(t) = e^{t \Delta} v_0 + \int_0^t e^{(t-s) \Delta} \mathbb{P} \nabla \cdot [v \otimes v](s) ds.
\]
By Lemma 8.1 and \( v \otimes v \in L^\infty(0, T; L^1_{uloc}) \), for \( 0 < t < T < \infty \) and \( 1 \leq q < 3/2 \)
\[
\left\| v^\epsilon(t), \bar{v}(t) \right\|_{L^q_{uloc}} \leq \| v_0 \|_{L^q_{uloc}} + \int_0^t s^{-1/2} (1 + s^{-\frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right)}) \| v \otimes v(s) \|_{L^1_{uloc}} ds
\]
\[
\lesssim \| v_0 \|_{L^2_{uloc}} + (T^{1/2} + T^{-\frac{d}{2} - 1}) \| v \otimes v \|_{L^\infty(0, T; L^1_{uloc})},
\]

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Thus for any $1 \leq q < 3/2$,
\[
\|v^\varepsilon, \bar{v}\|_{L^\infty(0,T;L^q_{uloc})} \leq C\|v^0\|_{L^2_{uloc}} + C(T^{1/2} + T^{3/2})A. \tag{8.3}
\]
We now show that
\[
\lim_{\varepsilon \to 0^+} \|v^\varepsilon - \bar{v}\|_{L^\infty(0,T;L^q_{uloc})} = 0, \quad \text{for all } q < 3/2. \tag{8.4}
\]
Indeed,\(^1\) for $f^\varepsilon = (1 - \phi_\varepsilon)v \otimes v$ and $0 < \delta < \min(t,1)$, we decompose
\[
\int_0^t e^{-(t-s)A}F^\varepsilon \cdot \nabla f^\varepsilon \, ds = \left( \int_0^{t-\delta} + \int_0^{t-\delta} \right) = I_1 + I_2.
\]
We have
\[
\|I_1\|_{L^q_{uloc}} \leq C \int_{|t-s|<\delta} |t-s|^{-2+\frac{3}{2q}} \|v\|_{L^\infty L^2}^2 \, ds \leq CA \delta^{\frac{3}{2q}-1}.
\]
Rewriting
\[
I_2(x,t) = \int_0^{t-\delta} \int_\mathbb{R}^3 \nabla_x S(x-y,t-s)f^\varepsilon(y,s) \, ds,
\]
where $S(x,y)$ is the Oseen tensor of the Stokes system in $\mathbb{R}^3$ and using the well-known estimate
\[
|\nabla_x S(x,t)| \leq C_m(|x| + \sqrt{t})^{-4}, \quad m \in \mathbb{N},
\]
we have
\[
|I_2(x,t)| \leq \sum_{k \in \mathbb{Z}^3} \int_0^{t-\delta} \int_{B_1(x+k)} |\nabla_x S(x-y,t-s)||f^\varepsilon(y,s)| \, dy \, ds
\]
\[
\leq \sum_{k \in \mathbb{Z}^3} C((|k| - 1) + \sqrt{\delta})^{-4} \int_0^{t-\delta} \int_{B_1(x+k)} |f^\varepsilon(y,s)| \, dy \, ds
\]
\[
\leq C \delta^{-2} F_\varepsilon,
\]
where $F_\varepsilon = \sup_{y \in \mathbb{R}^3} \int_0^T \int_{B_1(y)} |f^\varepsilon(y,s)| \, dy \, ds$. By assumption (3.5) with $R = \max(1, \sqrt{T})$, we know that $\lim_{\varepsilon \to 0} F_\varepsilon = 0$. For any $\tau > 0$, we can first choose $\delta > 0$ such that $CA \delta^{\frac{3}{2q}-1} < \frac{1}{2} \tau$, and then choose $\epsilon$ such that $C \delta^{-2} F_\varepsilon < \frac{1}{2} \tau |B_1|^{-1/q}$. Then
\[
\|I_1 + I_2\|_{L^q_{uloc}}(t) \leq \tau.
\]
Because the choices of $\delta$ and $\epsilon$ are uniform in $t$, we have shown (8.4).

Note $v^\varepsilon$ is a weak solution of the inhomogeneous Stokes system
\[
\partial_t v^\varepsilon - \Delta v^\varepsilon + \nabla p^\varepsilon = -G_\varepsilon, \quad \text{div } v^\varepsilon = 0, \quad v^\varepsilon|_{t=0} = v^0, \tag{8.6}
\]
\(^1\)This would be quite easy if we further assume
\[
\|(1 - \phi_\varepsilon)v \otimes v : L^\infty(0,T;L^1_{uloc})\| \to 0 \quad \text{as } \varepsilon \to 0. \tag{8.5}
\]
Note that our assumption (3.5) is slightly weaker than such an assumption.
Thus, by the maximal regularity estimate,
\[ \| \partial_t v^\varepsilon, \nabla^2 v^\varepsilon, \nabla p^\varepsilon \|_{L^2((0,T) \times \mathbb{R}^3)} \leq C(\epsilon, A, T), \]
and
\[ p^\varepsilon(x,t) = \int \frac{1}{4\pi|x-y|} \nabla \cdot G_\varepsilon(y,t)dy \]
\[ = -\frac{1}{3} \phi_x |v|^2(x,t) + \int K(x-y): (\psi \phi_x \otimes v)(y,t)dy, \] (8.7)
where \( K(y) = \text{p.v.} \nabla^2(4\pi|y|)^{-1} \).

We now decompose \( p^\varepsilon \). For fixed \( x_0 \in \mathbb{R}^3 \) and \( R > 0 \), we let \( \psi(x) = \phi(\frac{x-x_0}{2R}) \) and decompose (8.7) for \( x \in B_{\frac{3}{2}R}(x_0) \) as
\[ p^\varepsilon(x,t) = p^\varepsilon_{\text{loc}}(x,t) + p^\varepsilon_{\text{far}}(x,t) + c^\varepsilon(t) \]
\[ p^\varepsilon_{\text{loc}}(x,t) = -\frac{1}{3} \phi_x |v|^2(x,t) + \int K(x-y): (\psi \phi_x \otimes v)(y,t)dy \]
\[ p^\varepsilon_{\text{far}}(x,t) = \int [K(x-y) - K(x_0-y)] : ((1-\psi) \phi_x \otimes v)(y,t)dy \]
\[ c^\varepsilon(t) = \int K(x_0-y) : ((1-\psi) \phi_x \otimes v)(y,t)dy \]

By the Calderon-Zygmund estimate, for any \( q > 1 \) we have
\[ \int_{B_{\frac{3}{2}R}(x_0)} |p^\varepsilon_{\text{loc}}(x,t)|^q dx \leq c_q \int_{B_{3R}(x_0)} |v(x,t)|^{2q} dx. \]

Thus
\[ \| p^\varepsilon_{\text{loc}} \|_{L^q(0,T;L^q(B_{\frac{3}{2}R}(x_0)))} \leq c \| v \|_{L^{2q}(0,T;L^{2q}(B_{3R}(x_0)))}^2, \]
which is a priori bounded by \( A \) if \( \frac{2}{s} + \frac{3}{q} \geq \frac{3}{2} \), i.e., \( \frac{2}{s} + \frac{3}{q} \geq 3 \) and if \( 1 < q \leq 3 \).

For \( p^\varepsilon_{\text{far}} \) we have a pointwise bound,
\[ |p^\varepsilon_{\text{far}}(x,t)| \leq \int_{2R<|y-x_0|} \frac{cR}{|y-x_0|^4} |v(y,t)|^2 dy \]
\[ \leq \sum_{0 \neq k \in \mathbb{Z}^3} \int_{B_R(x_0+Rk)} \frac{cR}{|Rk|^4} |v(y,t)|^2 dy \]
\[ \leq cR^{-3} \| v(t) \|_{L^2_{\text{uloc},R}}^2. \]

Thus, for \( \frac{2}{s} + \frac{3}{q} \geq 3 \),
\[ \| p^\varepsilon_{\text{loc}} \|_{L^s(0,T;L^s(B_{\frac{3}{2}R}(x_0)))} + \| p^\varepsilon_{\text{far}} \|_{L^s(0,T;L^s(B_{\frac{3}{2}R}(x_0)))} \]
\[ \leq cA + cT^{1/s} R^{3/q-3} \text{ess sup}_{0<t<T} \| v(t) \|_{L^2_{\text{uloc},R}}^2 \]
\[ \leq c(T, R, s, q) A. \] (8.8)
By regarding $\nabla p^\epsilon$ as a given forcing term in (8.6) with the uniform estimate (8.8), we can invoke the local regularity estimate of the inhomogeneous heat equation, which results in, for any $\delta \in (0, T)$ and $q = r^*$, i.e., $1/r = 1/q - 1/3$,

$$
\|\partial_t v^\epsilon, \nabla^2 v^\epsilon\|_{L^q(\delta; T; L^q(B_R(x_0)))} \\
\leq c\|v^\epsilon\|_{L^1(0, T; L^1(B_{2R}(x_0)))} + c\|\phi_\epsilon v \otimes v\| + |p^\epsilon_{\text{loc}} + p^\epsilon_{\text{far}}|_{L^q(0, T; L^q(B_{\frac{2}{3}R}(x_0)))} \\
\leq C(\delta, T, A, R, s, q). \tag{8.9}
$$

These uniform estimates (8.8) and (8.9) enable us to pass limits. Let $\bar{p}_{x_0, R} = \bar{p}_{\text{loc}} + \bar{p}_{\text{far}}$ in $B_{\frac{2}{3}R}(x_0) \times (0, T)$ with

$$
\bar{p}_{\text{loc}}(x, t) = -\frac{1}{3}|v|^2(x, t) + \int K(x - y) : (\psi v \otimes v)(y, t) dy,
$$

$$
\bar{p}_{\text{far}}(x, t) = \int [K(x - y) - K(x_0 - y)] : ((1 - \psi)v \otimes v)(y, t) dy.
$$

We have the same bound as (8.8),

$$
\|\bar{p}_{x_0, R}\|_{L^q(0, T; L^q(B_{\frac{2}{3}R}(x_0)))} \leq c(T, R, s, q)A, \tag{8.10}
$$

and

$$
p^\epsilon_{\text{loc}} + p^\epsilon_{\text{far}} \rightharpoonup \bar{p}_{x_0, R} \quad \text{weakly in} \quad L^s(0, T; L^q(B_R(x_0)))).
$$

The limit of the weak form of (8.6) shows that $(\bar{v}, \bar{p}_{x_0, R})$ is a distributional solution of

$$
\partial_t \bar{v} - \Delta \bar{v} + \nabla \bar{p} = -\nabla \cdot (v \otimes v), \quad \text{div} \bar{v} = 0, \tag{8.11}
$$

and

$$
\lim_{t \to 0^+} (\bar{v}(t), \zeta) = (v_0, \zeta), \quad \forall \zeta \in C_c^\infty(\mathbb{R}^3).
$$

Note $\nabla \bar{p}_{x_0, R} = \nabla \bar{p}_{y_0, r}$ on $B_R(x_0) \cap B_r(y_0)$. In particular, we may specify $\bar{p}$ by setting $\bar{p} = \bar{p}_{0,1}$ on $B_1(x_0)$, and $\bar{p} = \bar{p}_{0,k} + c_k(t)$ on $B_k(x_0)$, $k \in \mathbb{N}$, where $c_k(t)$ is the unique function of $t$ such that $\bar{p}_{0,1} = \bar{p}_{0,k} + c_k(t)$ on $B_1(x_0)$. Thus $\bar{p}$ is defined in $\mathbb{R}^3 \times (0, T)$. Since $\nabla \bar{p} = \nabla \bar{p}_{x_0, R}$ on $B_R(x_0)$, we have

$$
\bar{p} = \bar{p}_{x_0, R} + c_{x_0, R}(t) \quad \text{on} \quad B_R(x_0)
$$

for some $c_{x_0, R}(t)$. Since $\bar{p}, \bar{p}_{x_0, R} \in L^s(0, T; L^q(B_R(x_0)))$, we get $c_{x_0, R} \in L^s(0, T)$. This establishes the pressure decomposition formula provided we have $\bar{v} = v$. To this end, we now show the spatial decay of $\bar{v}$. Fix any $q < 3/2$. For any $0 < \delta \ll 1$, by (8.4), there is $\epsilon > 0$ such that $\|\bar{v} - v^\epsilon\|_{L^q(0, T; L^q_{\text{uloc}})} \leq \delta$. Since $v^\epsilon \in L^2(\mathbb{R}^3 \times (0, T))$, there is $R > 0$ such that

$$
\sup_{|x| > R} \int_0^T \int_{B_1(x)} |v^\epsilon|^q < \delta.
$$

Thus

$$
\sup_{|x| > R} \int_0^T \int_{B_1(x)} |\bar{v}|^q \leq C\delta. \tag{8.12}
$$
It remains to show $\bar{v} = v$. Let $u = v - \bar{v}$. It satisfies by (3.5) and (8.12), for any $q < 3/2$,

$$u \in L^{\infty}(0, T; L^q_{uloc}), \quad \lim_{|x| \to \infty} \int_0^T \int_{B_1(x)} |u|^q = 0,$$

(8.13)

and

$$\partial_t u - \Delta u + \nabla \pi = 0, \quad \text{div } u = 0, \quad u(t = 0) = 0.$$ (8.14)

Let $\eta_\epsilon(x) = \epsilon^{-3} \phi(x/\epsilon)$ be a mollification kernel, and $\omega_\epsilon = \eta_\epsilon * \text{curl } u$. The vector field $\omega_\epsilon$ is a bounded solution of the heat equation with $\omega_\epsilon(t = 0) = 0$. Thus $\omega_\epsilon = 0$ for all $\epsilon > 0$. For any fixed $t$, $W_{\epsilon, t}(x) = \int_0^t \eta_\epsilon * u(s) \, ds$ is a bounded harmonic vector field which vanishes at spatial infinity. Thus $W_{\epsilon, t} \equiv 0$. Thus $u \equiv 0$.

**Proof of Lemma 3.5.** Let

$$A_R(t) = \sup_{x \in \mathbb{R}^3} \left\{ \sup_{0 < s < t} \frac{1}{R} \int_{B_R(x)} |v(y, s)|^2 dy + \frac{1}{R} \int_0^R \int_{B_R(x)} |\nabla v(y, s)|^2 dy ds \right\}.$$ (8.15)

The standard energy estimate as in [20, Lemma 3.2] based on the local energy inequality and the pressure estimate (8.10) gives

$$A_R(t) \leq N_R + CR^{-2} \int_0^t (A_R(s) + A_R(s)^3) \, ds.$$ (8.16)

Thus $A(t) \leq 2A_0$ for $t < \lambda R^2$ provided

$$\lambda \leq \frac{c}{1 + A_0^2},$$

which yields (3.8). Estimate (3.9) then follows from (8.10). We have shown Lemma 3.5.

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