Overlapping Cover Local Regression Machines

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Abstract We present the Overlapping Domain Cover (ODC) notion for kernel machines, as a set of overlapping subsets of the data that covers the entire training set and optimized to be spatially cohesive as possible. We show how this notion benefit the speed of local kernel machines for regression in terms of both speed while achieving while minimizing the prediction error. We propose an efficient ODC framework, which is applicable to various regression models and in particular reduces the complexity of Twin Gaussian Processes (TGP) regression from cubic to quadratic. Our notion is also applicable to several kernel methods (e.g. Gaussian Process Regression(GPR) and IWTGP regression, as shown in our experiments). We also theoretically justified the idea behind our method to improve local prediction by the overlapping cover. We validated and analyzed our method on three benchmark human pose estimation datasets and interesting findings are discussed.

1 Introduction

Estimation of a continuous real-valued or a structured-output function from input features is one of the critical problems that appears in many machine learning applications. Examples include predicting the joint angles of the human body from images, head pose, object viewpoint, illumination direction, and a person’s age and gender. Typically, these problems are formulated by a regression model. Recent advances in structure regression encouraged researchers to adopt it for formulating various problems with high-dimensional output spaces, such as segmentation, detection, and image reconstruction, as regression problems. However, the computational complexity of the state-of-the-art regression algorithms limits their applicability for big data. In particular, kernel-based regression algorithms such as Ridge Regression [12], Gaussian Process Regression (GPR) [18], and the Twin Gaussian Processes (TGP) [2] require inversion of kernel matrices (\(O(N^3)\), where \(N\) is the number of the training points), which limits their applicability for big data. We refer to these non-scalable versions of GPR and TGP as full-GPR and full-TGP, respectively.

Khandekar et. al. [13] discussed properties and benefits of overlapping clusters for minimizing the conductance from spectral perspective. These properties of overlapping clusters also motivate studying scalable local prediction based on overlapping kernel machines. Figure 1 illustrates the notion by starting from a set of points, diving them into either disjoint and overlapping subsets, and finally learning a kernel prediction function on each (i.e., \(f_i(x^*)\) for subset \(i\), \(x^*\) is testing point). In summary, the main question, we address in this paper, is how local kernel machines with overlapping training data could help speedup the computations and gain accurate predictions. We achieved considerable speedup and good performance on GPR, TGP, and IWTGP (Importance Weighted TGP) applied to 3D pose estimation datasets. To the best of our knowledge, our framework is the first to achieve quadratic prediction complexity for TGP. The ODC concept is also novel in the context of kernel machines and is shown here to be successfully applicable to multiple kernel-machines. We studies in this work GPR and TGP and IWTGP (a third model) kernel machines. The remainder of this paper is organized as follows: Section 2 and 4 presents some motivating kernel machines and the related work. Section 5 presents our approach and a theoretical justification for our ODC concept. Section 6 and 7 presents our experimental validation and conclusion.
2 Background on Full GPR and TGP Models

In this section, we show example kernel machines that motivated us to propose the ODC framework to improve their performance and scalability. Specifically, we review GPR for single output regression, and TGP for structured output regression. We selected GPR and TGP kernel machines for their increasing interest and impact. However, our framework is not restricted to them.

GPR \cite{choi2019learning} assumes a linear model in the kernel space with Gaussian noise in a single-valued output, i.e., \( y = f(x) + N(0, \sigma_n^2) \), where \( x \in \mathbb{R}^d \) and \( y \in \mathbb{R} \). Given a training set \( \{x_i, y_i, i = 1 : N\} \), the posterior distribution of \( y \) given a test point \( x_* \) is:

\[
p(y|x_*) = N(\mu_y = k(x_*)^T(K + \sigma_n^2 I)^{-1}f, \sigma_y^2 = k(x_*, x_*) - k(x_*)^T(K + \sigma_n^2 I)^{-1}k(x_*))
\]

where \( k(x, x') \) is kernel defined in the input space, \( K \) is an \( N \times N \) matrix, such that \( K(l, m) = k(x_l, x_m) \), \( k(x_*) = [k(x_*, x_1), ..., k(x_*, x_N)]^T \), \( I \) is an identity matrix of size \( N \), \( \sigma_n \) is the variance of the measurement noise, \( f = [y_1, \cdots, y_N]^T \). GPR could predict structured output \( y \in \mathbb{R}^d \) by training a GPR model for each dimension. However, this indicates that GPR does not capture dependency between output dimensions which limit its performance.

TGP \cite{choi2019learning} encodes the relation between both inputs and outputs using GP priors. This was achieved by minimizing the Kullback-Leibler divergence between the marginal GP of outputs (e.g., poses) and observations (e.g., features). Hence, TGP prediction is given by:

\[
\hat{y}(x_*) = \arg\min_y [k_Y(y, y) - 2k_Y(y)^T(K_X + \lambda_X I)^{-1}k_X(x_*) - \eta \log(k_Y(y, y) - k_Y(y)^T(K_Y + \lambda_Y I)^{-1}k_Y(y))]
\]

where \( \eta = k_X(x_* x_*) - k_X(x_*)^T(K_X + \lambda_X I)^{-1}k_X(x_*) \), \( k_X(x, x') = \exp(-\frac{||x - x'||^2}{2\rho_x^2}) \) and \( k_Y(y, y') = \exp(-\frac{||y - y'||^2}{2\rho_y^2}) \) are Gaussian kernel functions for input feature \( x \) and output vector \( y \), \( \rho_x \) and \( \rho_y \) are the kernel bandwidths for the input and the output. \( k_Y(y) = [k_Y(y, y_1), ..., k_Y(y, y_N)]^T \), where \( N \) is the number of the training examples. \( k_X(x_*) = [k_X(x_*, x_1), ..., k_X(x_*, x_N)]^T \), and \( \lambda_X \) and \( \lambda_Y \) are regularization parameters to avoid overfitting. This optimization problem can be solved using a quasi-Newton optimizer with cubic polynomial line search \cite{choi2019learning}; we denote the number of steps to convergence as \( l_2 \).
3 Importance Weighted Twin Gaussian Processes
(IWTGP)

Yamada et al. [26] proposed the importance-weighted variant of twin Gaussian processes [2] called IWTGP. The weights are calculated using RuLSIF [27] (relative unconstrained least-squares importance fitting). The weights were modeled as 
\[ w_{\alpha}(x, \theta) = \sum_{l=1}^d \theta_l k(x, x_l) \] 
where \( k(x, x_l) = \exp(- \|x - x_l\|^2) \). The weights are calculated using RuLSIF [27] (relative unconstrained least-squares importance fitting).

The optimal \( \hat{\theta} \) vector is computed in a closed form solution as follows:

\[
\hat{\theta} = (\mathbf{H} + \nu I)^{-1} \mathbf{h}
\]

where \( \mathbf{H}_{ij} = \frac{1}{n_{tr}} \sum_{l=1}^{n_{te}} k(x_i^t, x_j^r) + \alpha n_{tr} \sum_{j=1}^{n_{te}} k(x_j^r, x_j^r) \), \( \mathbf{h} \) is an \( n_{tr} \times 1 \) dimensional vector with the \( i \)-th element \( h_i = \frac{1}{n_{te}} \sum_{l=1}^{n_{te}} k(x_i^t, x_l^t) \). \( \mathbf{K} \) is an \( n_{te} \times n_{te} \times n_{te} \) dimensional identity matrix, where \( n_{te} \) and \( n_{tr} \) are the number of testing and training points respectively. Model selection of RuLSIF is based on cross-validation with respect to the squared-error criterion \( J \) in [27]. Having computed \( \hat{\theta} \), each input and output examples are simply re-weighted by \( w_{\alpha} \) [26]. Therefore, the output of the importance weighted TGP (IWTGP) is given by

\[
\hat{y} = \text{argmin}_y [K_Y(y, y) - 2 \eta_y log(K_Y(y, y)) - k_y(y) \mathbf{W}^T (\mathbf{W}^T K_Y \mathbf{W}^T + \lambda I)^{-1} \mathbf{W} k_y(y)]
\]

where \( \eta_y = \mathbf{W}^T (\mathbf{W}^T K_Y \mathbf{W}^T + \lambda I)^{-1} \mathbf{W} k_y(x) \), \( \eta_{\text{w}} = k_X(x, x) - k_x(x) \) for GPR, Similar to TGP, IWTGP can also be solved using a second order, BFGS quasi-Newton optimizer with cubic polynomial line search for optimal step size selection.

Table 1 shows the training an testing complexity of full GPR and TGP models, where \( d_Y \) is the dimensionality of the output. Table 1 also summarizes the computational complexity of the related approximation methods, discussed in the following section, and our method, N.
involve complicated integration, approximated by computationally expensive sampling or Monte Carlo simulation.

Park et al. [16] proposed a large-scale approach for GPR by domain decomposition on up to 2D grid on input, where a local regression function is inferred for each subdomain such that they are consistent on boundaries. This approach obviously lacks a solution to high-dimensional input data because the size of the grid increases exponentially with the dimensions, which limits its applicability. More recently, [5] proposed a Recursive Partitioning Scheme (RPC) to decompose the data into non-overlapping equal-size clusters, and they built a GPR on each cluster. They showed that this local scheme gives better performance than FIC [22] and other methods. However, this partitioning scheme obviously lacks consistency on the boundaries of the partitions and it was restricted to single-output GPR. Table 1 shows the complexity of this scheme denoted by local-RPC for GPR.

Beyond GPR, we found that local regression was adopted differently in structured regression models like Twin Gaussian Processes (TGP) [2], and also an data bias version of it, denoted by IWTGP [26]. TGP and IWTGP outperform not only GPR in this task, but also various regression models including Hilbert Schmidt Independence Criterion (HSIC) [10], Kernel Target Alignment (KTA) [6], and Weighted-KNN [18]. Both TGP and IWTGP have no closed-form expression for prediction. Hence, the prediction is made by gradient descent on a function that needs to compute the inverse of both the input and output kernel matrices, \( O(N^3) \) complexity. Practically, both approaches have been applied by finding the \( M \ll N \) Nearest-Neighbors (NN) of each test point in [2] and [26]. The prediction of a test point is \( O(M^3) \) due to the inversion of \( M \times M \) input and output kernel Matrices. However, NN scheme has three drawbacks: (1) A regression model is computed for each test point, which results in a scalability problems in prediction (i.e., Matrix inversions on \( M \times M \) complexity), (2) Number of neighbors might not be large enough to create an accurate prediction model since it is constrained by the first drawback, (3) It is inefficient compared with the other schemes used for GPR. Table 1 shows the complexity of this NN scheme.

5 ODC Framework

The problems of the existing approaches, presented above, motivated us to develop an approach that satisfies the properties listed in Table 1. The table also shows which of these properties are satisfied for the relevant methods. In order to satisfy all the properties, we present the Overlapping Domain Cover (ODC) notion. We define the ODC as a collection of overlapping subsets of the training points, denoted by subdomains, such that they are as spatially coherent as possible. During training, an ODC is computed such that each subdomain overlaps with the neighboring subdomains.

Then, a local prediction model (kernel machine) is created for each subdomain and the computations that does not depend on the test data are factored out and precomputed (e.g. inversion of matrices). The nature of the ODC generation makes these kernel machines consistent in the overlapped regions, which are the boundaries since we constraint the subdomains to be coherent. This is motivated by the notion that data lives on a manifold with local properties and consistent connections between its neighboring regions. On prediction, the output is calculated as a reduction function of the predictions on the closed subdomain(s). Table 1 (the last row) shows the complexity for our generalized ODC framework, detailed in Sec 5.1 and 5.2. In contrast to the prior work, our ODC framework is designed to cover structured regression setting, \( dy > 1 \) and to be applicable to GPR, TGP, and many other models.

Notations. Given a set of input data \( X = \{x_1, \cdots, x_N\} \), our prediction framework firstly generates a set of non-overlapping equal-size partitions, \( C = \{C_1, \cdots, C_K\} \), such that \( \cup_i C_i = X \), \( |C_i| = N/K \). Then, the ODC is defined based on them as \( D = \{D_1, \cdots, D_K\} \), such that \( |D_i| = M \forall i \), \( D_i = C_i \cup O_i \forall i \). \( O_i \) a the set of points that overlaps with the other partitions, i.e., \( O_i = \{ x : x \in \cup_{j\neq i} C_j \} \), such that \( |O_i| = p \cdot M, |C_i| = (1-p) \cdot M, 0 \leq p \leq 1 \) is the ratio of points in each overlapping subdomain, \( D_i \), that belongs to/overlaps with partitions, other than its own, \( C_i \).

It is important to note that, the ODC could be specified by two parameters, \( M \) and \( p \), which are the number of points in each subdomain and the ratio of overlap respectively; this is since \( K = N/(1-p)M \). This parameterization of ODC generation is reasonable for the following reasons. First, \( M \) defines the number of points that are used to train each local kernel machine, which controls the performance of the local prediction. Second, given \( M \) and that \( K = N/(1-p)M \), \( p \) defines how coarse/fine the distribution of kernel machines are. It is not hard to see that as \( p \) goes to 0, the generated ODC reduces to the set of non-overlapping clusters. Similarly, as \( p \) approaches \( 1 \), the ODC reduces to generating a cluster at each point with maximum overlap with other clusters, i.e., \( K = N, |C_i| = 1 \), and \( |O_i| = M-1 \). Our main claim is two fold. First, precomputing local kernel machines (e.g. GPR, TGP, IWTGP) during training on the ODC significantly increase the speedup on prediction time. Second, given a fixed \( M \) and \( N \), as \( p \) increases, local prediction per-
formance increases, theoretically supported by Lemma [57]

**Lemma 51.** Under ODC notion, as the overlap $p$ increases, the closer the nearest model to an arbitrary test point and the more likely that model get trained on a big neighborhood of the test point.

**Proof.** We start by outlining the main idea behind the proof, which is directly connected to the fact that $K = \frac{N}{(1-p)M}$, which indicates that the number of local models increases as $p$ increases given fixed $N$ and $M$. Under the assumption that the local models are spatially cohesive, $p \to 1$ theoretically indicates that there is a local model centered at each point in the space (i.e., $K = \infty$). Hence, as $p$ increases, the distribution of the kernel machines is the finest and the more likely a test point to find the closest kernel machines trained on a big neighborhood of it leading to more accurate prediction. Meanwhile, as $p$ goes to 0, the distribution is the coarsest and the less likely a test point finds, the closest kernel machines, trained on a big neighborhood.

Let’s assume that each kernel machine is defined on $M$ points that are spatially cohesive, covering the space of $N$ points with $\frac{N}{(1-p)M}$. Let’s assume that center of the $M$ points in kernel machine $i$ is $\mu_i$, the the Co-variance matrix of these points is $\Sigma_i$. Hence

$$p(x|D_i) = N(\mu_i, \Sigma_i) = (2\pi)^{-\frac{d}{2}}|\Sigma_i|^{-\frac{1}{2}}e^{-\frac{1}{2}(x-\mu_i)^T\Sigma_i^{-1}(x-\mu_i)}$$

where $N(\mu_i, \Sigma_i)$ is a normal distribution of mean $\mu_i$ and Co-variance matrix $\Sigma_i$.

Let’s assume that there are two ODCs, $ODC_1$ and $ODC_2$, defined on the same $N$ points, the first one has overlap $p_1$ and the second one is with overlap $p_2$, such that, $p_2 > p_1$. Let’s assume that the number of kernel machines in $ODC_1$ and $ODC_2$ are $K_1$ and $K_2$, respectively. Hence,

$$K_1 = \frac{N}{(1-p_1)M}, \quad K_2 = \frac{N}{(1-p_2)M}$$

Since $p_2 > p_1$, $0 \leq p_1 < 1$ and $0 \leq p_2 < 1$, then $K_2 > K_1$, which indicates that the number of kernel machines in $ODC_2$ with higher overlap is bigger than the number of kernel machines in $ODC_2$. Let’s assume that there is an test point $x^*$ and define that the probability that $x^*$ is captured by the ODC to be proportional to the maximum probability of $x^*$ among the domains.

$$p(x^*) = \sum_{i=1}^{K} p(x^*, D_i)$$

$$= \sum_{i=1}^{K} (p(x^*|D_i)\delta(p(x^*|D_i) - \max_{j=1}^{K}(p(x^*|D_i)))$$

$$= \max_{i=1}^{K} p(x^*|D_i)$$

$$= (2\pi)^{-\frac{d}{2}}\max_{i=1}^{K}|\Sigma_i|^{-\frac{1}{2}}e^{-\frac{1}{2}(x^*-\mu_i)^T\Sigma_i^{-1}(x^*-\mu_i)}$$

where $\delta(0) = 1, 0$ otherwise. The reason behind this definition of $p(x^*)$ is that our method select the domain of prediction based on $\arg\max_{i=1}^{K} p(x^*|D_i)$. Hence $p_{ODC_1}(x^*) = \max_{i=1}^{K_1} p_{ODC_1}(x^*|D_i)$ and $p_{ODC_2}(x^*) = \max_{i=1}^{K_2} p_{ODC_2}(x^*|D_i)$.

We start by the case where the points are uniformly distributed in the space. Under this condition and assuming that spatially cohesive domain cover, this leads to that $p(x^*|D_i) \approx N(\mu_i, \Sigma)|\Sigma_i\rangle$, where $\Sigma_1 = \Sigma_2 \cdots = \Sigma_K = \Sigma$. Hence

$$p(x^*|D_i) \propto e^{-\frac{1}{2}(x^*-\mu_i)^T \Sigma^{-1}(x^*-\mu_i)}$$

$$ln(p(x^*|D_i)) \propto -(x^*-\mu_i)^T \Sigma^{-1}(x^*-\mu_i)$$

Then

$$p(x^*) = \max_{i=1}^{K} p(x^*|D_i)$$

$$= (2\pi)^{-\frac{d}{2}}\max_{i=1}^{K}|\Sigma|^{-\frac{1}{2}}e^{-\frac{1}{2}(x-\mu_i)^T \Sigma^{-1}(x-\mu_i)}$$

$$\propto \max_{i=1}^{K} e^{-\frac{1}{2}(x-\mu_i)^T \Sigma^{-1}(x-\mu_i)}$$

$$ln(p(x^*)) \propto \max_{i=1}^{K} x^T \Sigma^{-1} x - 2\mu_i^T \Sigma^{-1} x + \mu_i^T \Sigma \mu_i$$

Hence, $p(x^*)$ gets maximized as it get closer to one of the centers of the domains $\mu_i$, defined by the ODC. It is not hard to see that that chances of $x^*$ to be closer to one of the centers covered by $ODC_2$ is higher than $ODC_2$, especially when $p_2 \gg p_1$. This is since $K_1 = \frac{N}{(1-p_1)M}$, $K_2 = \frac{2p}{2p-1}$ and $K_2 = \frac{2p}{2p-1}$.
\[ \frac{N}{(1-p_2)M} \] Hence \( K_2 \gg K_1 \) when \( p_2 \gg p_1 \). For instance, when \( p_1 = 0 \) and \( p_2 = 0.9 \), this leads to that ODC will generate \( K_1 = \frac{N}{2} \) domains, while ODC will generate \( K_2 = 10.9N = 10K_1 \), which is ten times more domains and centers. The fact that there are much more domains if \( K_2 \gg K_1 \) together with the fact that there domains are spatially cohesive leads to \( \max_{i=1}^{K_1} - (x^i - \mu_1^i)^T \Sigma^{-1} (x^i - \mu_1^i) \gg \max_{i=1}^{K_2} - (x^i - \mu_2^i)^T \Sigma^{-2} (x^i - \mu_2^i) \). The proof of this statement derives from the fact that \( \max_{i=1}^{K} (x^i - \mu_i)^T \Sigma^{-1} (x^i - \mu_i) \) is could maximized by (1) if \( x^i \) gets very close to one of \( \mu_i \), \( i = 1 : K \) and (2) smaller variance \( |\Sigma| \), which is minimized by the nature by which ODC is created, since each domain \( i \) is created by neighboring points to its center (i.e., \( |\Sigma_1| \gg |\Sigma_2| \)). This directly leads to that if \( K_2 \gg K_1 \) then \( \max_{i=1}^{K_1} - (x^i - \mu_1^i)^T \Sigma^{-1} (x^i - \mu_1^i) \gg \max_{i=1}^{K_2} - (x^i - \mu_2^i)^T \Sigma^{-2} (x^i - \mu_2^i) \). Hence, \( p_{ODC_2}(x^i) \gg p_{ODC_1}(x^i) \).

Even if the points are not uniformly distributed, it is still more likely that an ODC with higher overlap would have higher \( p(x^i) \), since \( x^i \) is close under expectation to one of the centers if more spatially cohesive domains are generated which increases with higher overlap. Our experiments also proves that the ODC concept generalizes on three real dataset where the training points are not distributed uniformly.

5.1 Training

There are several overlapping clustering methods that include (e.g. [17] and [3]), which looks relevant for our framework. However these methods do not fit our purpose both equal-size constraints for the local kernel machines. We also found them very slow in practice because their complexity varies from cubic to quadratic (with a big constant factor) on the training-set. These problems motivated us to propose a practical method that builds overlapping local kernel machines with spatial and equal-size constraints. These constraints are critical for our purpose since the number of points in each kernel-machine determine its local performance. Hence, our training phase is two steps: (1) the training data is split into \( K = N/(1-p)M \) equal-sized clusters of \( (1-p)M \) points. (2) an ODC with \( K \) overlapping subdomains is generated by augmenting each cluster with \( p \cdot M \) points from the neighboring clusters.

5.1.1 Equal-size Clustering

There are efficient algorithms that deal with size constraints in clustering. For example, [29] formulated the problem of clustering with size constraints as a linear programming problem. However such algorithms are not computationally efficient, especially for large scale datasets (e.g., Human3.6M).

We study two efficient ways to generate equal size clusters; see Table 1 (last row) for their ODC-complexity.

Recursive Projection Clustering (RPC) [5]. In this method, the training data is partitioned to perform GPR prediction. Initially all data points are put in one cluster. Then, two points are chosen randomly and orthogonal projection of all the data onto the line connecting them is computed. Depending on the median value of the projections, the data is then split into two equal size subsets. The same process is then applied to each cluster to generate \( 2^l \) clusters after \( l \) repetitions. The iterations stops once \( 2^l > K \). As indicated, the number of clusters in this method has to be a power of two and it might produce long thin clusters.

Equal-Size K-means (EKmeans). We propose a variant of k-means clustering [11] to generate equal-size clusters. The goal is to obtain disjoint partitioning of \( X \) into clusters \( C = \{C_1, \ldots, C_K\} \), similar to the k-means objective, minimizing the within-cluster sum of squared Euclidean distances, \( C = \arg \min_{C} J(C) = \min \sum_{i=1}^{K} \sum_{x_i \in C_i} d(x_i, \mu_i) \), where \( \mu_i \) is the mean of cluster \( C_i \), and \( d(\cdot, \cdot) \) is the square distance. Optimizing this objective is NP-hard and k-means iterates between the assignment and update steps as a heuristic to achieve a solution; \( l_i \) denotes number of iterations of k-means. We add equal-size constraints \( \forall (1 \leq i \leq K), |C_i| = N/K \).

In order to achieve this partitioning, we propose an efficient heuristic algorithm, denoted by Assign and Balance (AB) EKmeans. It mainly modifies the assignment step of the k-means to bound the size of the resulting clusters. We first assign the points to their closest see center as typically done in the assignment step of k-means. We use \( \tilde{C}(x_p) \) to denote the cluster assignment of a given point \( x_p \). This results in three types of clusters: balanced, overfull, and underfull clusters. Then some of the points in the overfull clusters are redistributed to the underfull clusters by assigning each of these points to the closest underfull cluster. This is achieved by initializing a pool of overfull points defined as \( \tilde{X} = \{x_{p_0} : x_{p_0} \in C_i, |C_i| > N/K \} \); see Figure 3.

Let us denote the set of underfull clusters by \( \tilde{C} = \{C_p : |C_p| < N/K \} \). We compute the distances \( d(x_i, \mu_j), \forall x_i \in \tilde{X} \) and \( \mu_j \in \tilde{C} \). Iteratively, we pick the minimum distance pair \((x_{p_0}, \mu_j)\) and assign \( x_{p_0} \) to cluster \( \tilde{C} \) instead of cluster \( \tilde{C}(x_{p_0}) \). The point is then removed from the overfull pool. Once an underfull cluster becomes full it is removed from the underfull pool, once an overfull cluster is balanced, the remaining points of that cluster are removed from overfull pool. The intuition behind this algorithms is that, the cost associated with the initial optimal assignment (given the computed means) is minimally increased by each swap since we pick the minimum distance pair in each iteration. Hence the cost is kept as low as possible while balancing the clusters. We denote the the name of this Algorithm as Assign and Balance EKmeans. Algorithm 1 illustrates the overall assignment step and Fig. 4 visualizes the balancing step.
5.1.2 Overlapping Domain Cover (ODC) Model

Having generated the disjoint equal size clusters, we generate the ODC subdomains based on the overlapping ratio $p$, such that $p \cdot M$ points are selected from the neighboring clusters. Let’s assume that we select only the closest $r$ clusters to each cluster, $C_i$ is closer to $C_j$ than $C_k$ if $\|\mu_i - \mu_j\| < \|\mu_i - \mu_k\|$. It is important to note that $r$ must be greater than $p/(1-p)$ in order to supply the required $p \cdot M$ points; this is since number of points in each cluster is $(1-p)M$. Hence, the minimum value for $r$ is $\lceil (p \cdot M)/(1-p) \cdot M \rceil = \lceil p/(1-p) \rceil$ clusters. Hence, we parametrize $r$ as $r = \lceil t \cdot p/(1-p) \rceil$, $t \geq 1$. We study the effect of $t$ in the experimental results section. Having computed $r$ from $p$ and $t$, each subdomain $D_i$ is then created by merging the points in the cluster $C_i$ with $p \cdot M$ points, retrieved from the $r$ neighboring clusters. Specifically, the points are selected by sorting the points in each of $r$ clusters by the distance to $\mu_i$. The number of points retrieved for each of the $r$ neighboring clusters is inversely proportional to the distance of its center to $\mu_i$. If a subset of the $r$ clusters are requested to retrieve more than its capacity (i.e., $(1-p)M$), the set of the extra points are requested from the remaining clusters giving priority to the closer clusters (i.e., starting from the nearest neighboring cluster to the cluster on which the subdomain is created). As $t = 1$ and $p$ increases, all points that belong to the $r$ clusters tends to be merged with $C_i$. In our framework, we used FLANN [15] for fast NN-retrieval; see pseudo-code of ODC generation in Appendix C.

After the ODC is generated, we compute the the sample normal distribution using the points that belong to each subdomain. Then, a local kernel machine is trained for each of the overlapping subdomains. We denote the point set normal distribution of the subdomains as $p(x|D_i) = \mathcal{N}(\mu'_i \in \mathbb{R}^{d \times d}, \Sigma'_i \in \mathbb{R}^{d \times d \times d})$, $\Sigma'_i^{-1}$ is precomputed during the training for later use during the prediction. Finally, we factor out all the computations that does not depend on the test point (for GPR, TGP, IWTGP) and store them with each subdomain as its local kernel machine. We denote the training model for subdomain $i$ as $\mathcal{M}_i$, which is computed as follows for GPR and TGP respectively.

**GPR.** Firstly, we precompute $(K_j + \sigma^2_{n_j} I)^{-1}$, where $K_j$ is an $M \times M$ kernel matrix, defined on the input points in $D_i$. Each dimension $j$ in the output could have its own hyper-parameters, which results in a different kernel matrix for each dimension $K_j$. We also precompute $(K_j + \sigma^2_{n_j} I)^{-1} y_j$ for each dimension. Hence $\mathcal{M}_{GPR} = \{(K_j + \sigma^2_{n_j} I)^{-1}, (K_j + \sigma^2_{n_j} I)^{-1} y_j, j = 1 : d_y \}$.

**TGP.** The local kernel machine for each subdomain in TGP case is defined as $\mathcal{M}_{TGP} = \{(K_X + \lambda_X I)^{-1}, (K_Y + \lambda_Y I)^{-1}\}$, where $K_X$ and $K_Y$ are $M \times M$ kernel matrices de-
fined on the input points and the corresponding output points respectively, which belong to domain $i$.

**IWTGP.** It is not obvious how to factor out computations that does not depend on the test data in the case of IWTGP, since the computational extensive factor (i.e., $(W^T)^2 K_X W^T + \lambda I)^{-1}, (W^T)^2 K_X W^T + \lambda I)^{-1}$) does depend on the test set since $W^T$ is computed on test time. To help factor out the computation, we used linear algebra to show that

\[(D A D + \lambda I)^{-1} = D^{-1} A^{-1} D^{-1} - \frac{\lambda D^{-2} A^{-2}}{1 + \lambda \cdot tr(D^{-1} A^{-1} D^{-1})}\]  

(10)

where $D$ is a diagonal matrix, $I$ is the identity matrix, and $tr(B)$ is the trace of matrix $B$.

**Proof.** Kenneth Miller [14] proposed the following Lemma on Matrix Inverse.

\[(G + H)^{-1} = G^{-1} - \frac{1}{1 + tr(GH^{-1})} G^{-1} H G^{-1}\]  

(11)

Applying Miller’s lemma, where $G = DAD$ and $H = \lambda I$, leads directly to Eq. (10).

Mapping $D$ to $W^T K_X W^T + \lambda I$ to either of $K_X$ or $K_Y$, we can compute $M^i$ = \{K_X^{-1}, K_Y^{-1}\}. Having computed $W^i$ on test time, $(W^T)^2 K_X W^T + \lambda I)^{-1}, (W^T)^2 K_Y W^T + \lambda I)^{-1}$ could be computed in quadratic time given $M^i$ following equation (10) since the inverse and the power of $W^T$ has linear computational complexity since it is diagonal.

5.2 Prediction

ODC-Prediction is performed in three steps.

1. Finding the closest subdomains. The closest $K' \ll K$ subdomains are determined based on the covariance norm of the displacement of the test input from the means of the subdomain distribution (i.e. $\|x - \mu_i\|_{\Sigma^{-1}}$, $i = 1 : K$), where $\|x - \mu_i\|_{\Sigma^{-1}} = (x - \mu_i)^T \Sigma^{-1}(x - \mu_i)$. The reason behind using the covariance norm is that it captures details of the density of the distribution in all dimensions. Hence, it better models $p(x|D_i)$, indicating better prediction of $x$ on $D_i$.

2. Closest subdomains Prediction. Having determined the closest subdomains, predictions are made for each of the closest clusters. We denote these predictions as $Y_{K'} x_i \{\mu_j\}_{j=1}^{K'}$. Each of these predictions are computed according to the selected kernel machine. For GPR, predictive mean and variance are $O(M \cdot d_X)$ and $O(M^2 \cdot d_Y)$ respectively, for each output dimension. For TGP, the prediction is $O(l \cdot M^2 \cdot d_Y)$; see Eq. (2).

3. Subdomains weighting and Final prediction. The final predictions are formulated as $Y(x) = \sum_{i=1}^{K'} a_i Y_{x_i}$, $a_i > 0, \sum_{i=1}^{K'} a_i = 1$. $a_i$ are computed as follows. Let the distribution of domain $\{D^i_{x_1} = ||x - \mu_i||_{\Sigma^{-1}}\}_{i=1}^{K'}$ denotes to the distances to the closest subdomains, $L^i_{x_1} = 1/D^i_{x_1}, a_i = L^i_{x_1}/\sum_{i=1}^{K'} L^i_{x_1}$.

It is not hard to see that when $K' = 1$, the prediction step reduces to regression using the closest subdomain to the test point. However it is reasonable in most of the prior work to make prediction using the closest model, we generalized it to $K'$ closest kernel machines and combining their predictions, so as to study how consistency of the combined prediction behaves as the overlap increases (i.e., $p$); see the experiments.

6 Experimental Results

**Equal-Size Kmeans Step Experiment:** We also tried another variant for Ekmeans that we call Iterative Minimum-Distance Assignments Ekmeans (IMDA- Ekmeans). Note that the algorithm presented earlier in the paper is denoted as Assign and Balance Kmeans (AB-Kmeans). The IMDA-Ekmeans algorithm works as follows. We initialize a pool of unassigned points $X = X$ and initialize all clusters as empty. Given the means computed from the previous update steps, we compute the distances $d(x_i, \mu_j)$ for all points/center pairs. We iteratively pick the minimum distance pair $(x_p, \mu_l) : d(x_p, \mu_l) \leq d(x_i, \mu_j) \forall x_i \in X$ and $|C_l| < N/K$ and assign point $x_p$ to cluster $l$. The point is then removed from the pool of unassigned points. if $|C_l| = N/K$, then it is marked as balanced and no longer considered. The process is repeated until the pool is empty; see Algorithm (2).

Table 3 presents the average cost over 10 runs of IMDA-Ekmens and AB-Ekmens algorithms. We initialize both the AB-Ekmens and IMDA-Ekmens algorithms by the cluster centers computed by running the standard k-means.

**Algorithm 2:** Iterative Minimum-Distance Assignments (IMDA) k-means: Assignment Step

1. Create a matrix $D \in R^{N \times K}$, where $D[i, j]$ is the distance between the $i^{th}$ point to the $j^{th}$ cluster center.
2. Get the coordinate $(x_*, j_*)$ that maps the smallest distance in $D$.
3. Remove the $i^{th}$ row from matrix $D$ and mark it as assigned to the $j^{th}$ cluster.
4. If the size of the cluster $j$ achieves the ideal size (i.e. $n/K$), then remove the $j^{th}$ column from matrix D.
5. Go to step 2 if there is still unassigned points.
As illustrated in Table 3, the AB-Ekmeans outperforms IMDA-Ekmeans in these experiments, which motivated us to utilize AB-Ekmeans, which is presented in the paper, against IMDA-Ekmeans under our ODC prediction framework. Our interpretation for these results is because AB-Ekmeans initializes the assignment with an assignment that minimizes the cost $J(C) = \min \sum_{j=1}^{K} \sum_{x_i \in C_j} d(x_i; \mu_j)$ given the cluster centers and then balance the clusters. In all the following experiments, we use AB-Ekmeans due to its clear superior performance to IMDA-Ekmeans.

**Table 3:** $J(C)$ of AB-kmeans and IMDA-kmeans on a dataset of 10,000 random 2D points, averaged over 10 runs

|        | K=5  | K=10 | K=50 |
|--------|------|------|------|
| AB-kmeans | 1077.3 | 540.241 | 105.305 |
| IMDA-kmeans | 1290.6 | 657.446 | 122.006 |
| Error Reduction | 16.53% | 17.83% | 13.52% |

Datasets and Setup. We evaluated our framework on three human pose estimation datasets, Poser, HumanEva, and Human3.6M; see Fig. 5 for summary of setup and representation for each. Poser dataset \(^1\) consists of 1927 training and 418 test images. The image features, corresponding to bag-of-words representation with silhouette-based shape-context features. The error is measured by the root mean-square error (in degrees), averaged over all joints angles, and is given by: $Error(\hat{y}, y^*) = \frac{1}{54} \sum_{m=1}^{54} ||\hat{y}^m - y^* mod 360||$, where $\hat{y} \in R^{54}$ is an estimated pose vector, and $y^* \in R^{54}$ is a true pose vector. HumanEva dataset \(^2\) contains synchronized multi-view video and Mocap data of 3 subjects performing multiple activities. We use HOG features \(^3\) (in $R^{270}$) proposed in \(^4\). We use training and validations subsets of HumanEva-I and only utilize data from 3 color cameras with a total of 9630 image-pose frames for each camera. This is consistent with experiments in \(^4\) and \(^5\). We use half of the data for training and half for testing. Human3.6M \(^6\) is a dataset of millions of Human poses. We managed to evaluate our proposed ODC-framework on six Subjects (S1, S2, S6, S7, S8, S9) from it, which is $\approx$ 0.5 million poses. We split them into 67% training 33% is testing. HOG features are extracted for 4 image-views for each pose and concatenated into 3060-dim vector. Error for each pose, in both HEeva (in mm) and Human 3.6 (in cm), is measured as $Error(\hat{y}, y^*) = \frac{1}{54} \sum_{m=1}^{54} ||\hat{y}^m - y^* mod 360||$.

There are four control parameters in our ODC framework: $M$, $p$, $t$, and $K'$. Figure 6 shows our parameter analysis with different values of $p$, $t$ and $K'$ on HumanEva dataset for GPR and TGP as local regression machines, where $M = 800$. Each sub-figure consists of six plots in two rows. The first row indicates the results using AB-Ekmeans clustering scheme, while the second row shows the results for RPC clustering scheme. Each row has three plots, one for $K' = 1$, 2, and 3 respectively. Each plot shows the error of different $t$ against $p$ from 0 to 0.95; i.e., it shows how the overlap affects the performance for different values of $t$. Each plot shows, on its top caption, the minimum and the maximum overlap regression errors where $t \rightarrow 1$. Looking at these plots, there are a number of observations:

1. As $t \rightarrow 1$ (the solid red line), the behavior of the error tends to reduce as $p$ increases, i.e., the overlap.

2. Comparing different $K'$, the behavior of the error indicates that combining multiple predictions (i.e., $K' = 2$ and $K' = 3$), gives poor performance, compared with $K' = 1$, when the overlap is small. However, all of them, $K' = 1$, 2, and 3, performs well as $p \rightarrow 1$; see column 2 and 3 in Fig 6 and Fig 8. This indicates consistent prediction of neighboring subdomains as $p$ increases; see also Fig. 7 for side by side comparison of different $K'$. The main reason behind this behavior is that as $p$ increases, the local models of the neighboring subdomains normally share more training points on their boundaries, which is reflected as shared constraints during the training of these models making them more consistent on prediction.

3. Comparing the first row to the second row in each sub-figure, it is not hard to see that our AB-Ekmeans partitioning scheme consistently outperforms RPC \(^7\). e.g. the error in cases of GPR (M=800) is 47.48mm for AB-Ekmeans and 50.66mm for RPC, TGP (M=800) is 38.8mm for AB-Ekmeans and 39.8mm for RPC. This problem is even more severe when using smaller $M$, e.g. the error in case of TGP (M=400) is 39.5mm for AB-Ekmeans and 47.5mm for RPC; see a detailed plot for $M=400$ in Fig 9. We noticed significant drop in the performance as $M$ decreases. For instance when $M = 200$, the error for TGP best performance increased to 43.88mm instead of 38mm for $M = 800$.

4. TGP gives better prediction than GPR (i.e., 38mm using TGP compared with 47mm using GPR).

5. As $M$ increases, the prediction error decreases. For instance, when $M = 200$, The error for TGP best performance increased to 43.88mm instead of 38.9mm for $M = 800$. 

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\(^1\) http://www.pose-data.net/

\(^2\) http://humaneva-stats.csail.mit.edu/

\(^3\) http://www.cs.toronto.edu/~garriga-jurado/people/meanvod/meanvod.html

\(^4\) http://human3.6m.net/

\(^5\) http://www.columbia.edu/~majax/papers/overlapping-cover-regression/.html

\(^6\) http://www.robots.ox.ac.uk/~vgg/research/pose/poses.html

\(^7\) http://www.columbia.edu/~majax/papers/overlapping-cover-regression/.html
We found these observation to be also consistent on Poser dataset.

This analysis helped us conclude recommending choosing $t$ close to 1, big overlap ($p$ closer to 1), and $K' = 1$ is sufficient for accurate prediction.

Having accomplished the performance analysis which comprehensively interprets our parameters, we used the recommended setting to compare the performance with other methods and show the benefits of this framework. Figure 6 shows the speedup gained by retrieving the matrix inverses on test time, compared with computing them at test time by NN scheme. The figure shows significant speedup from precomputing local kernel machines.

Table 4 shows error, training time and prediction time of NN, FIC, and different variations of ODC on Poser and Human-Eva datasets. Training time is formatted as $(t_c + t_p)$, where $t_c$ is the clustering time and $t_p$ is the remaining training time excluding clustering. As indicated in the top part of table 4, TGP under our ODC-framework can significantly speedup the prediction compared with NN-scheme.

![Fig. 6: ODC framework Parameter Analysis of GPR and TGP on Human Eva Dataset](image)

**Table 4: Error & Time on Poser and Human Eva datasets (Intel core-i7 2.6GHZ), M = 800**

| Method | Error (deg) | Training Time | Prediction Time | Human-Eva | Error (mm) | Training Time | Prediction Time |
|--------|-------------|----------------|-----------------|-----------|-------------|----------------|-----------------|
| GPR    | 6.77        | 24 sec         |                 | 100       | 40          | 24 sec         |                 |
| TGP    |             |                |                 |           |             |                |                 |
| NN     |             |                |                 |           |             |                |                 |
| ODC($p = 0.9, t = 1, K' = 1$)-Ekmeans | 5.43 | 188.99 sec |  |        | 38.1 | 6364 sec |       |
| ODC($p = 0, t = 1, K' = 1$)-Ekmeans | 5.40 | (3.7 +25.1) sec | 16.5 sec |  | 38.9 | (2001 + 45.4) sec | 298 sec |
| ODC($p = 0.9, t = 1, K' = 1$)-RPC | 5.60 | (0.23 +41.6) sec | 15.8 sec |  | 41.87 | (240 + 4.9) sec | 257 sec |
| ODC($p = 0, t = 1, K' = 1$)-RPC | 7.70 | (0.15 + 1.7) sec | 13.89 sec |  | 39.9 | (0.45 + 49.1) sec | 277 sec |
| GPR    | 6.27        | (3.7 +11.1) sec | 0.56 sec | 100 | 49.3 | (2001 + 42.85) sec | 79 sec |
| TGP    |             |                |                 |           |             |                |                 |
| NN     |             |                |                 |           |             |                |                 |
| ODC($p = 0.9, t = 1, K' = 1$)-Ekmeans | 7.54 | (3.9 + 1.38 sec) | 0.35 sec |  | 49.6 | (240 + 6.4) sec | 48 sec |
| ODC($p = 0.9, t = 1, K' = 1$)-RPC | 6.45 | (0.23 + 17.3) sec | 0.52 sec |  | 52.8 | (0.49 + 46.06) sec | 64 sec |
| ODC($p = 0, t = 1, K' = 1$)-RPC | 7.46 | (0.15 + 1.5) sec | 0.27 sec |  | 54.6 | (0.26 + 4.4) sec | 44 sec |
| FIC    |             |                |                 |           |             |                |                 |
| TGP    |             |                |                 |           |             |                |                 |
| NN     |             |                |                 |           |             |                |                 |
| ODC($p = 0.9, t = 1, K' = 1$)-Ekmeans | 7.63 | (. + 20.63) sec | 0.3106 | 100 | 68.36 | 102 sec |       |
Fig. 7: HumanEva TGP different $K'$ as overlap increase, $M=800$

Fig. 8: Increasing $K'$ significantly heart the performance for small overlap (Human Eva TGP, $M=800$)

Fig. 10: Speedup of ODC framework prediction on either TGP or GPR while retrieving precomputed matrix inverses as $M$ increases, compared with computing them on test time by KNN scheme (log-log scale)

in [2], while achieving competitive performance; better in case Poser Dataset. As illustrated in our analysis in Figure 6, higher overlap ($p$) gives better performance. From time analysis perspective, higher $p$ costs more training time due that more subdomains are created and trained. While, Figure 6 and Table 4 indicates that AB-Ekmeans gives better performance than RPC under both GPR and TGP. AB-Ekmeans takes more time for clustering. Yet, it is feasible to compute in all the datasets, we used in our experiments. Our experiments also indicate that as $p \rightarrow 1$ in TGP and GPR, $K'=2$ and $K'=3$ takes double and triple the prediction time respectively, compared with $K'=1$, with almost no error reduction. We also compared our model to FIC in case of GPR, and our model achieved smaller error and smaller prediction time; see bottom part in Table 4. However, TGP consistently gives better results on both Poser and HumanEva datasets. We also tried full TGP and GPR on Poser and Human Eva Datasets. Full TGP error is 5.35 for Poser and 40.3 for Human Eva. Full GPR error is 6.10 for Poser and 59.62 for Human Eva. The results indicate that ODC achieves either better or competitive to the full models. Meanwhile, the speedup is significant for TGP prediction (21X for Human Eva and 11X for Poser Datasets); see Fig. 11. For GPR prediction, we achieved the best performance and the lowest prediction time compared to existing GPR prediction methods; see Fig. 12.

Based on our comprehensive experiments on HumanEva and Poser datasets, we conducted an experiment on Human3.6M dataset with TGP kernel machine, where $M=1390$, $t=1$, $p=0.6$, $K'=1$, Ekmeans for clustering. We achieved a speedup of 41.7X on prediction time using our ODC framework compared with NN-scheme, i.e., 7 days if NN-scheme is used versus 4.03 hours in our case with our MATLAB implementation. The error is 13.5 (cm) for NN and 13.8 (cm) for ODC; see Fig. 13.

7 Conclusion

We proposed an efficient ODC framework for kernel machines and validated the framework on structured regression machines on three human pose estimation datasets. The key idea is to equally partition the data and create cohesive overlapping subdomains, where local kernel machines are computed for each of them. The framework is general and could be applied to various kernel machine beyond GPR, TGP, IWTGP validated in this work. Similar to TGP and IWTGP,
our framework could be easily applied to the recently proposed Generalized TGP [8] which is based on Sharma Mittal divergence, a relative entropy measure brought from Physics community. We also theoretically justified our framework’s notion.

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Appendices

A IWTGP-ODC Experiments

Tables 5 and 6 details the results of IWTGP-ODC experiments on Poser and HumanEva datasets in terms of error and speedup in prediction time.

|                  | IWTGP          | IWTGP-ODC      |
|------------------|----------------|----------------|
| (M = 800, M_{est} = 418) | (M = 800, M_{est} = 418) |
| error (deg)      | 6.1            | 5.32           |
| err reduction (deg) | -              | 0.783         |
| err reduction %  | -              | 12.836%        |
| Prediction Time (sec) | 360.0          | 26.61          |
| speedUp          | -              | 13.5           |

**Table 5**: POSER dataset IWTGP-NN vs IWTGP-ODC

|                  | IWTGP          | IWTGP-ODC      |
|------------------|----------------|----------------|
| (M = M_{est} = 800) | (M = M_{est} = 800) |
| error (mm)       | 39.1           | 39.3           |
| err reduction (mm) | -              | -0.2          |
| err reduction %  | -              | -0.512%        |
| Prediction Time (sec) | 7938.15        | 569.66         |
| speedUp          | -              | 13.92          |

**Table 6**: Human Eva dataset: IWTGPKN vs IWTGP-ODC

B More figures on AB Ekmeans

Figure 14 shows the clustering performance on 300000 random 2D point (K=5). Figure 15 shows the clustering output of our algorithm visualized on using the first three principal components of Human Eva training hog features. The figures shows that the cluster are spatially cohesive but not necessarily circular. This makes the elliptic distribution of the data captured by Mode 3 gives more accuracy membership measure me to the subdomains.

C Overlapping Domain Cover(ODC)
Generation-Algorithm

Algorithm 3 shows how the overlapping sub-domains are generated form the the equal size clusters from the closest r clusters.

D Local Kernel Machines hyper-parameters on each dataset

The hyper parameters were learnt using cross validation on the training set for GPR, TGP and IWTGP that we are interested in. The following subsection present the learnt hyper-parameters and the error measures on each dataset in case of TGP.

D.1 Poser Dataset

The parameters $2\rho^2_x$, $2\rho^2_y$, $\lambda_X$, and $\lambda_Y$ were assigned to 5, 5000, $10^{-4}$, and $10^{-4}$, respectively.

D.2 HumanEva Dataset

The parameters $2\rho^2_x$, $2\rho^2_y$, $\lambda_X$, and $\lambda_Y$ were assigned to 5, 500000, $10^{-3}$, and $10^{-3}$, respectively.
Input: Clusters \( \{C_k\}_{k=1}^K \)  
Output: Overlapping subdomains \( \{D_k\}_{k=1}^K \)  

foreach Cluster \( C_k \) do 
Compute the closest \( r \) clusters \( \{C'_i\}_{i=1}^r \) based on 
\( D_{K_i} = \|\mu_k - \mu_i\|, \quad i \neq k \) 
Let \( L_{K_i} = 1/D_{K_i} \), \( W_{K_i} = \frac{L_{K_i}}{\sum_{i=1}^{r} L_{K_i}} \) \( i = 1 : r \) 
Let \( NPK_i = \text{floor}(W_{K_i} \times \Omega_C), \quad i = 1 : r \) 
Let \( ExKPts = (1 - p)M - \sum_{i=1}^{r} NPK_i \) 
Let \( NPK_i = NPK_i + 1, i = 1 : ExKPts \) 
Let \( D_k = C_k \) 
Let \( overflow = 0 \) 
▷ The following for loop goes over the \( r \) clusters on an increasing order of \( D_{K_i} \) 
for \( i=1 : r \) do 
if \( NPK_i \geq |C_i| \) then 
\( overflow = overflow + NPK_i - |C_i| \) 
\( NPK_i = |C_i| \) 
endif 
if \( NPK_i < |C_i| \) then 
\( G_i = \min(overflow, |C_i| - NPK_i) \) 
\( NPK_i = NPK_i + G_i \) 
\( overflow = overflow - G_i \) 
\( Ps_i = \text{KNN}(OVC_{K_j}, NPK_i) \) 
\( D_k = D_k \cup Ps_i \) 
endif 
for \( i=1 : r \) do 
\( Ps_i = \text{KNN}(OVC_{K_j}, NPK_i) \) 
\( D_k = D_k \cup Ps_i \) 
▷ where KNN is the K-nearest neighbors algorithms. For high performance calculation of KNN, we use FLANN [15] to calculate KNN. 
\end{algorithm} 

Algorithm 3: Subdomains Generation (Note: All \( \{D_k\}_{k=1}^K \) are stored as indices to \( X \)). 

D.3 Human 3.6 Dataset 

The parameters \( 2\rho_x^2, 2\rho_y^2, \lambda_X, \) and \( \lambda_Y \) were assigned to 5, 500000, \( 10^{-3} \), and \( 10^{-2} \), respectively.
Fig. 9: Overlapping Domain Cover Parameter Analysis of GPR and TGP on Human Eva Dataset (best seen in color) (M=400)
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