Factorization in 2D String Theory

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Abstract

We show the factorization of correlation functions of tachyon operators in 2D string theory using the discretized approach. Our results can be understood in terms of the operator product expansion of tachyon operators. We also give a systematic way of computing correlation functions of tachyon operators.

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Developments in the last few years in two-dimensional quantum gravity coupled to \( c \leq 1 \) conformal field theories [1]-[5] have cast new light on theories of gravity in higher dimensions. The most interesting is the \( c = 1 \) Liouville theory which can be regarded as a critical string theory in a two-dimensional target space. Tachyon correlation functions have been studied by many people from the point of view of continuum approach [6]-[8]. Sakai and Tanii demonstrated the factorization of the \( N \)-point functions in the \( c = 1 \) Liouville theory [9]. They understood the factorization as a result of the short-distance singularity arising from the operator product expansion of two tachyon operators. Because of the limitations of the path-integral technique, their argument was restricted to a particular kinematical situation.

On the other hand, Dijkgraaf, Moore, and Plesser showed that the tachyon amplitudes can be reformulated in terms of a scattering process for free fermions [10]. They derived a compact expression for the generating functional of the correlation functions of tachyon operators. Their generating functional can naturally be considered a tau function of the KP hierarchy. It is very natural to apply their powerful method to the problem of factorization.

The purpose of the present article is to provide a more general demonstration of the factorization than in the paper of Sakai and Tanii. We find that the factorization is independent of the topology of the two-dimensional manifold. We can understand this from the local nature of the operator product expansion of tachyon operators. We calculate correlation functions of tachyon operators explicitly for some simple cases and confirm that these correlation functions indeed satisfy our factorization rule.

In this article we follow the notation of refs.[8], [9]. The \( c = 1 \) Liouville theory is described by the action

\[
S[\phi, X; \mu] = S_{\text{Liouville}} + S_{\text{matter}},
\]

\[
S_{\text{Liouville}} = \frac{1}{8\pi} \int d^2 z \sqrt{\hat{g}} \left( \hat{g}^{ab} \partial_a \phi \partial_b \phi - 2\sqrt{2} \hat{R} \phi + 8\mu e^{-\sqrt{2}\phi} \right),
\]

\[
S_{\text{matter}} = \frac{1}{8\pi} \int d^2 z \sqrt{\hat{g}} \hat{g}^{ab} \partial_a X \partial_b X,
\]

where \( \phi \) is the Liouville field and \( X \) a massless scalar field with central charge \( c = 1 \). The physical states in the \( c = 1 \) Liouville theory are well-known from the study of BRST cohomology [11]. We have two kinds of states: tachyon states and boundary states. Now we concentrate on the tachyon states. We shall use the Euclidean signature spacetime. In the \( c = 1 \) Liouville theory, the tachyon operators are given
in the form \[ \mathcal{O}_p = \int d^2 z \sqrt{g} \mathcal{T}_p(z) = \int d^2 z \sqrt{g} e^{ipX} e^{\beta(p)\phi}, \] (2)

where \( p \) is the momentum parameter of the tachyon. From BRST invariance we have the on-shell condition

\[
\frac{1}{2} p^2 - \frac{1}{2} \beta (\beta + 2\sqrt{2}) = 1. \tag{3}
\]

In spite of the name *tachyon*, this state corresponds to a massless ground state in the language of string theory in a two-dimensional target space \[\textcolor{red}{[7]}\]. We can easily understand this statement as follows. The relation between the tachyon operator \( \mathcal{T}_p \) and the wave function of the corresponding state is given by

\[ \mathcal{T}_p = g_{\text{st}} \Psi(\phi, X), \]

where \( g_{\text{st}} \) is the string coupling constant. If we recall that \( g_{\text{st}} \propto \exp(-\sqrt{2}\phi) \), the wave function has the following form

\[ \Psi(\phi, X) = \exp\left[ipX + (\beta + \sqrt{2})\phi \right]. \tag{4} \]

We thus interpret the Liouville field as Euclidean time and regard the energy of this state as \( E = \beta + \sqrt{2} \). We can rewrite (3) in the following form

\[ E^2 = p^2 + m^2, \quad m^2 = 0. \tag{5} \]

Now let us briefly discuss the factorization of the tachyon amplitudes in the continuum approach. The \( N \)-point correlation function of tachyon operators is given by a path integral \[\textcolor{red}{[8]}\]

\[
\langle \mathcal{O}_{p_1} \cdots \mathcal{O}_{p_N} \rangle = \frac{\mathcal{D}\phi \mathcal{D}X}{V_{\text{CKV}}} \frac{e^{-S[\phi,X;\mu]}}{V_{\text{CKV}}} \mathcal{O}_{p_1} \cdots \mathcal{O}_{p_N}
\]

\[ = 2\pi\delta\left(\sum_{k=1}^{N} p_k\right) \frac{\Gamma(-s)}{\sqrt{2}} \tilde{A}(p_1, \ldots, p_n), \tag{6} \]

\[
\tilde{A} = \int \frac{[d\tau]}{V_{\text{CKV}}} \prod_{k=1}^{N} \int d^2 z_k \sqrt{\hat{g}(z_k)} \left\langle \prod_{k=1}^{N} e^{ip_k X(z_k)} \right\rangle \tilde{X} \\
\times \left\langle \prod_{k=1}^{N} e^{\beta(p_k)\phi(z_k)} \left( \frac{\mu}{\pi} \int d^2 w \sqrt{\hat{g}(w)} e^{-\sqrt{2}\phi(w)} \right)^s \right\rangle \tilde{\phi}, \tag{7} \]

\[3\]
\[ s = 2 - 2h - N + \frac{1}{\sqrt{2}} \sum_{k=1}^{N} |p_k| , \]

where \( s \) is the number of the cosmological term insertion and the volume \( V_{\text{CKV}} \) is that generated by conformal Killing vectors.

In the \( c = 1 \) Liouville theory, the notion of \textit{chirality} appears [8]. As a solution of (3) we choose \( \beta(p) = -\sqrt{2} + |p| \) following the argument of refs. [4], [5]. We define the chirality of the tachyon operator (2) to be positive if \( p > 0 \) and to be negative if \( p < 0 \). Any zero-momentum tachyon operators decouple in (6) and make the path integral vanish. In the sphere topology, the integral for non-zero modes can be explicitly performed after fixing the \( SL(2, \mathbb{C}) \) gauge. In particular, if \( p_1 < 0, p_2, \ldots, p_N > 0 \), the amplitude is given in a form that allows analytic continuation with respect to

\[
\left\langle O_{p_1}(-) O_{p_2}^{(+)} \cdots O_{p_N}^{(+)} \right\rangle_{h=0} \equiv A(p_1^{(-)}, p_2^{(+)}, \ldots, p_N^{(+)}) = \pi^{N-3} \mu^s \frac{\Gamma(-s)}{\Gamma(N + s - 2)} \prod_{k=2}^{N} \Delta(1 - \sqrt{2}|p_k|),
\]

where \( s = 2 - N + \sqrt{2}|p_1| \) and \( \Delta(x) \equiv \Gamma(x)/\Gamma(1 - x) \). In (8), we redefined the \( N \)-point function as follows

\[
\langle O_{p_1} \cdots O_{p_N} \rangle_{h=0} \equiv \Gamma(-s) \tilde{A}(p_1, \ldots, p_N). \quad (9)
\]

As was discussed in [8], all the singularities in \( p_k, (k = 2, \ldots, N) \) come from the short-distance singularities between \( T_{p_1}(z) \) and \( T_{p_k}(0) \),

\[
: e^{ip_1 X} e^{\beta(p_1) \phi}(z) : e^{ip_k X} e^{\beta(p_k) \phi}(0) :
\]

\[
\sim \sum_{n=0}^{\infty} \left( \frac{1}{n!} \right)^2 |z|^{2\vec{p}_1 \cdot \vec{p}_k + 2n} : e^{ip_1 X} e^{\beta(p_1) \phi}(0) \partial^n \bar{\partial}^n e^{ip_k X} e^{\beta(p_k) \phi}(0) :, \quad (10)
\]

where \( \vec{p}_i = (p_i, -i\beta(p_i)) \). The operator that appears in the \( n = 0 \) term of the right-hand side of (10) has the same form as a tachyon operator and its corresponding state has the following mass

\[
M^2 = (\beta(p_1) + \beta(p_k) + \sqrt{2})^2 - (p_1 + p_k)^2 = -2(2\vec{p}_1 \cdot \vec{p}_k + 2) = 2(1 + \sqrt{2}p_1)(1 - \sqrt{2}p_k). \quad (11)
\]

Therefore the pole at \( p_k = 1/\sqrt{2} \) \( (n = 0) \) is due to a tachyon intermediate state. In the case of (8), the residue of the pole at \( p_k = 1/\sqrt{2} \) is indeed given by a \((N-1)\)-point
tachyon amplitude

\[ A(p_1, p_2, \ldots, p_N) \approx \frac{\pi}{(1 + \sqrt{2}p_1)(1 - \sqrt{2}p_k)} A(p, p_2, \ldots, p_{k-1}, p_{k+1}, \ldots, p_N), \]  

(12)

where \( p = p_1 + 1/\sqrt{2} \) is the momentum of the intermediate tachyon operator whose chirality must be negative.

From the above mentioned argument, all we have to do to study the factorization of the tachyon amplitudes (6) is take the limit of \( p_k \rightarrow 1/\sqrt{2} \) (\( p_k \rightarrow -1/\sqrt{2} \)) for a tachyon operator \( T_{p_k} \) with positive (negative) chirality and confirm that the residue of the pole is indeed that given by the \((N-1)\)-point tachyon amplitude. To do this the discrete approach of Dijkgraaf, Moore, and Plesser is more convenient than the path integral approach of Sakai and Tanii.

Dijkgraaf, Moore, and Plesser expressed the tachyon amplitudes (3) as vacuum expectation values in a conformal field theory [10]. We adjust their result to fit the results of the continuum theory (\( X \) field is compactified in radius \( \beta \))

\[ A(O^{(-)}_{p_1}, \ldots, O^{(-)}_{p_m}, O^{(+)}_{p_{m+1}}, \ldots, O^{(+)}_{p_N}) = \lim_{\beta \rightarrow \infty} (-i)^N \pi^{N-3} \frac{1}{\beta} \prod_{j=1}^{N} \frac{\Gamma(-n_j/\beta)}{\Gamma(n_j/\beta)}  \mu^{\sum_{j=1}^{N} n_j} \times \langle 0| \alpha_{nN} \cdots \alpha_{nm+1} S \alpha_{-nm} \cdots \alpha_{-n_1} |0 \rangle, \]  

(13)

where \( \alpha \) denotes the creation or annihilation operator of a free boson and \( |0 \rangle \) is the standard \( SL(2, \mathbb{C}) \) invariant vacuum. The correspondence between the momentum parameters of the tachyons and the subscripts of the \( \alpha \) modes is given by

\[ \sqrt{2}p_k = -n_k/\beta, \quad \text{for} \quad k = 1, \ldots, m, \]
\[ \sqrt{2}p_l = n_l/\beta, \quad \text{for} \quad l = m+1, \ldots, N. \]  

(14)

The scattering matrix \( S \) of the scattering process discussed in [12] is obtained using matrix quantum mechanics. It is described in terms of fermion modes

\[ \alpha_{n} = \sum_{m} \psi_{-(m+\frac{1}{2})} \overline{\psi}_{n+m+\frac{1}{2}}, \quad \{ \psi_{-(m+\frac{1}{2})}, \overline{\psi}_{n+m+\frac{1}{2}} \} = \delta_{n+m,0}, \]  

(15)

as follows

\[ S =: \exp \left[ \sum_{m} \left( \log R_{pm} \right) \psi_{-(m+\frac{1}{2})} \overline{\psi}_{m+\frac{1}{2}} \right]. \]  

(16)
where $\sqrt{2}p_m = \frac{1}{\beta}(m + \frac{1}{2})$ is the momentum of the loop fermion \[12\] which is discretized as a result of the compactification. In (16), $R_p$ is the reflection coefficient of the scattering process

$$R_p = e^{i\mu \log \mu - \sqrt{2}p}$$ \quad \text{(17)}$$

Note that (17) is valid only for sufficiently large $\mu$. As the formula (13) has its origin in matrix quantum mechanics, it has the form of a sum over all topologies. One can expand it in terms of the genus as follows \[3\], \[10\]

$$A(p_1^{(-)}, \ldots, p_m^{(-)}, p_{m+1}^{(+)}, \ldots, p_N^{(+)}) = \sum_{h=0}^{N} \mu^{-2h} A(p_1^{(-)}, \ldots, p_m^{(-)}, p_{m+1}^{(+)}, \ldots, p_N^{(+)})_h \quad \text{(18)}$$

This asymptotic expansion is only true for large cosmological constant $\mu$.

Now let us expand (13) in terms of the genus and perform an explicit computation of the tachyon amplitudes for some cases. First we rewrite it as follows

$$A(p_1^{(-)}, \ldots, p_m^{(-)}, p_{m+1}^{(+)}, \ldots, p_N^{(+)}) = \lim_{\beta \to \infty} (i)^N (\pi)^{N-3} \frac{1}{\beta} \prod_{j=1}^{N} \frac{\Gamma(-n_j/\beta)}{\Gamma(n_j/\beta)} \mu^{\frac{1}{2\beta} \sum_{j=1}^{N} n_j} \times \langle 0 | \alpha_{nN} \cdots \alpha_{n_{m+1}} \prod_{k=0}^{m-1} (S \alpha_{n_{m-k}} S^{-1}) | 0 \rangle$$

$$= \lim_{\beta \to \infty} (i)^N (\pi)^{N-3} \frac{1}{\beta} \prod_{j=1}^{N} \frac{\Gamma(-n_j/\beta)}{\Gamma(n_j/\beta)} \mu^{\frac{1}{2\beta} \sum_{j=1}^{N} n_j} \times \langle 0 | \alpha_{nN} \cdots \alpha_{n_{m+1}} \prod_{k=0}^{m-1} \sum_{n} R_{pn} R_{pn+1}^{*} \bar{\psi}_{n_{m-k}+\sqrt{2}p_n} \bar{\psi}_{-(n+1/2)} \bar{\psi}_{-(n_{m-k}+1/2)} | 0 \rangle \quad \text{(19)}$$

We can expand $RR^*$ in (13) using the following asymptotic expansion derived from the Stirling formula,

$$z^{b-a} \frac{\Gamma(z+a)}{\Gamma(z+b)} \approx \sum_{n=0}^{\infty} \left( \frac{B}{n} \right)^{[n/2]} \sum_{m=0}^{[n/2]} (-1)^m I_m(B) \frac{d^{2m}}{dA^{2m}} \left( \frac{A}{z} \right)^n \quad \text{as } z \to \infty,$$

$$A = (a+b-1)/2, \quad B = a - b,$$

$$\text{(20)}$$
where the $I_m(x)$ is a polynomial of $x$ of $m$-th order and the first few polynomials are given by

\[
I_0 = 1, \quad I_1 = \frac{1}{4^4} (2x + 1), \quad I_2 = \frac{1}{8 \cdot 6^4} (2x + 1)(10x + 7),
\]

\[
\ldots
\]

Applying it twice in $RR^*$, we get the following asymptotic expansion,

\[
R_{p_m} R_{n/\sqrt{2} \beta - p_m} \approx \sum_{k=0}^{\infty} \left( \frac{n/\beta}{k} \right) \sum_{l=0}^{[k/2]} (-1)^l (2l)! \left( \frac{k}{2l} \right) I_l(n/\beta) \left( m + \frac{1}{2} - \frac{n}{2} \right)^{k-2l} \left( \frac{i}{\beta \mu} \right)^k.
\]

From eqs. (19), (22) we can expand the amplitude (13) in terms of $\mu^{-1}$. We can compute each term of (18), since it is only a vacuum expectation value in a conformal field theory. We succeed in summarizing the results of the computation in compact form for the following three cases.

Case 1. $h = 0$, $(-, +, \ldots, +)$, $(s = \sqrt{2}|p_1| - N + 2)$,

\[
A(p_1^-, p_2^+, \ldots, p_N^+)_{h=0} = \lim_{\beta \to \infty} \left( -\pi \right)^{N-3} \frac{1}{\beta} \prod_{j=1}^{N} \frac{\Gamma(-n_j/\beta)}{\Gamma(n_j/\beta)} \mu^{\frac{1}{2} \sum_{j=1}^{N} n_j}
\]

\[
\times (0| \alpha_{n_1} \cdots \alpha_{n_2} \sum_{m} \left( \frac{i}{\beta \mu} \right)^{N-2} \left( \frac{n_1/\beta}{N-2} \right) m^{N-2} \psi_{-\frac{1}{2} + \frac{1}{2}} \psi_{-n_1+m+1} |0) = \lim_{\beta \to \infty} \left( -\pi \right)^{N-3} \prod_{j=1}^{N} \frac{\Gamma(-n_j/\beta)}{\Gamma(n_j/\beta)} \mu^{n_1/\beta + 2-N} \frac{\Gamma(n_1/\beta + 1)}{\Gamma(n_1/\beta - N + 3)} \frac{n_2 n_3 \cdots n_N}{\beta^{n_N}}
\]

\[
= \mu^s \frac{\Gamma(-s)}{\Gamma(N + s - 2)} \prod_{k=2}^{N} \Delta(1 - \sqrt{2}p_k).
\]

Case 2. $h = 1$, $(-, +, \ldots, +)$, $(s = \sqrt{2}|p_1| - N)$,

\[
A(p_1^-, p_2^+, \ldots, p_N^+)_{h=1} = \frac{\pi^{N-3}}{24 \mu^s} \frac{\Gamma(-s)}{\Gamma(s + N)} \prod_{k=2}^{N} \Delta(1 - \sqrt{2}p_k) \left( 2 \sum_{k=2}^{N} p_k^2 - \sqrt{2} \sum_{k=2}^{N} p_k - 1 \right).
\]

Case 3. $h = 0$, $(-, -\ldots, +)$,

\[
A(p_1^-, p_2^-, p_3^+, \ldots, p_N^+)_{h=0}
\]
\[ \frac{1}{2} (-1)^{N+1} \mu \sqrt{2} (|p_1| + |p_2|)^{2-N} \prod_{k=1}^{N} \Delta (1 - \sqrt{2}|p_k|) \]

\[ \times \left[ \prod_{i=1}^{N-3} (\sqrt{2}|p_i| - i) + \frac{1}{\sqrt{2}} \sum_{k=1}^{N-3-k-1} \prod_{i=1}^{N-3-k} (\sqrt{2}|p_i| - i) \prod_{j=1}^{N-3-k} (\sqrt{2}|p_j| - j) \right] \]

\[ \times \sum_{3 \leq n_1 < \ldots < n_k \leq N} \left[ \sum_{i=1}^{k} p_{n_i} + p_1 \right] + (p_1 \leftrightarrow p_2) \right]. \quad (25) \]

The form of the amplitude (13) is suitable for studying the factorization of tachyon amplitudes. Let \( p_l \) be the momentum parameter of a tachyon operator with positive chirality. As was discussed previously, we show the factorization of the \( N \)-point tachyon amplitude by taking the limit of \( p_l \rightarrow 1/\sqrt{2} \). To do this, we use the relation

\[ [\alpha, S_{\alpha-n} S^{-1}] = \frac{i n}{\mu} S_{\alpha-n+\beta} S^{-1} \], \quad (26)\]

which is derived from the formula

\[ S_{\alpha-n} S^{-1} = \sum_{m} R_{p_m} R_{n/\sqrt{2} \beta} \psi_{-(m+1/2)} \bar{\psi}_{-n+m+1/2}. \quad (27) \]

In (26), we assumed without loss of generality that \( \beta \) is integral-valued. Applying (26) repeatedly to (13), we evaluate the residue of the pole at \( p_l = 1/\sqrt{2} \)

\[ A(O_{p_1}^{(-)}, \ldots, O_{p_m}^{(-)}, O_{p_{m+1}}^{(+)}, \ldots, O_{p_n}^{(+)}) \]

\[ \lim_{p_l \rightarrow 1/\sqrt{2}} \lim_{\beta \rightarrow \infty} (-1)^{N} \pi^{-3} \frac{1}{\beta} \prod_{j=1}^{N} \Gamma(-n_j/\beta) \frac{1}{\Gamma(n_j/\beta)} \frac{1}{\mu \beta \Sigma_{j=1}^{N} n_j} \frac{1}{1 - n_l/\beta} \]

\[ \times \sum_{k=1}^{m} \alpha_{n,k} \ldots \alpha_{n_{k-1}} \alpha_{n_{k+1}} \ldots \alpha_{n_{m+1}} S_{\alpha-n_m} \ldots \alpha_{n_{k+1}} S^{-1} \]

\[ \times [\alpha, S_{\alpha-n_k} S^{-1}] S_{\alpha-n_{k+1}} \ldots \alpha_{n_1} |0 \rangle \]

\[ = \frac{-\pi}{1 - \sqrt{2} p_k} \sum_{k=1}^{m} \theta(1 - \sqrt{2} p_k) \times A(O_{p_1}^{(-)}, \ldots, O_{p_{k+1}}^{(-)}, \ldots, O_{p_m}^{(-)}, O_{p_{m+1}}^{(+)}, \ldots, O_{p_{l-1}}^{(+)}). \quad (28) \]

We now have a sum of \( m \) tachyon amplitudes, since we observe in (28) the short-distance singularity between \( O_{p_l}^{(+)} \) and each tachyon operator with negative chirality. The reason for the appearance of step functions is that an annihilation mode comes
to the right-hand side of the scattering matrix $S$. Eq. (28) is the result of a sum over all the topologies. By expanding both sides of (28) with respect to $\mu^{-1}$, we have a factorization like that in (28) for each value of the genus. This factorization is indeed consistent with the results of Sakai and Tanii. We can also show a similar factorization by taking a limit of $p_k \rightarrow -1/\sqrt{2}$ for a tachyon operator $T_{p_k}$ with negative chirality.

As a consequence, we obtain the following rules for the factorization of tachyon amplitudes.

1. There are no short-distance singularities in an operator product expansion between tachyon operators with the same chirality.

2. The pole of a physical tachyon state that is produced by an operator product expansion of two tachyon operators with opposite chiralities is of the first order. The chirality of the intermediate tachyon is opposite to that of the tachyon operator whose momentum approaches $\pm 1/\sqrt{2}$ in the limit.

3. The form of the factorization is genus-independent.

We can understand these rules from the point of view of the operator product expansion of two tachyon operators as in (10).

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