Renewal approximation for the absorption time of a decreasing Markov chain

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\textbf{Abstract} We consider a Markov chain \((M_n)_{n \geq 0}\) on the set \(\mathbb{N}_0\) of nonnegative integers which is eventually decreasing, i.e. \(\mathbb{P}\{M_{n+1} < M_n | M_n \geq a\} = 1\) for some \(a \in \mathbb{N}\) and all \(n \geq 0\). We are interested in the asymptotic behaviour of the law of the stopping time \(T = T(a) := \inf\{k \in \mathbb{N}_0 : M_k < a\}\) under \(\mathbb{P}_n := \mathbb{P}(\cdot | M_0 = n)\) as \(n \to \infty\). Assuming that the decrements of \((M_n)_{n \geq 0}\) given \(M_0 = n\) possess a kind of stationarity for large \(n\), we derive sufficient conditions for the convergence in minimal \(L^p\)-distance of \(\mathbb{P}_n((T - a_n)/b_n, \cdot)\) to some non-degenerate, proper law and give an explicit form of the constants \(a_n\) and \(b_n\).

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\section{1 Introduction}

The purpose of this paper is to study the asymptotic behavior of a class of integer-valued stochastic sequences \((T_n)_{n \geq 0}\) that satisfy a random recursive equation of the form \(2\) stated below. In many, though not all applications, such \(T_n\) arise as absorption times for certain decreasing Markov chains.

To be more specific, let \((M_n)_{n \geq 0}\) be a temporally homogeneous Markov chain on \(\mathbb{N}_0\) with absorbing state 0 and transition matrix \(P = (p_{i,j})_{i,j \geq 0}\) such that \(p_{i,j} = 0\) for \(1 \leq i < j\). The last condition means that the chain is

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decreasing in the sense that
\[ P(M_{n+1} < M_n | M_n \geq 1) = 1 \]
for all \( n \geq 0 \). Given this property, the time until absorption at 0, viz.
\[ T := \inf\{k \geq 0 : M_k = 0\}, \]
is clearly an a.s. finite stopping time under each \( P_n := P(\cdot | M_0 = n) \). Our purpose is to study the distribution of \( T \) under \( P_n \) as \( n \to \infty \). Obviously, for this goal the situation stated in the abstract, which is more general and also more commonly encountered in applications, can always be cast into the present framework by relabeling the states \( i \geq a \) and collapsing all states \( <a \) into one absorbing state.

Our analysis embarks on the simple observation that
\[
P_n(T \in \cdot) = \sum_{k=1}^{n} P_n(M_1 = n - k, T \in \cdot) = \sum_{k=1}^{n} p_{n,n-k} P_k(T \in \cdot) \tag{1}
\]
for all \( n \in \mathbb{N} \). Introducing random variables \( T_0, \hat{T}_0, T_1, \hat{T}_1, \ldots \) and \( I_1, I_2, \ldots \) on a common probability space \((\Omega, \mathcal{A}, \mathbb{P})\) such that
- \( P(T_n \in \cdot) = P(\hat{T}_n \in \cdot) = P_n(T \in \cdot) \) for each \( n \in \mathbb{N}_0 \),
- \( P(I_n \in \cdot) = P_n(n - M_1 \in \cdot) = (p_{n,n-k})_{1 \leq k \leq n} \) for each \( n \in \mathbb{N} \),
- \( (T_n)_{n \geq 0}, (\hat{T}_n)_{n \geq 0} \) and \( (I_n)_{n \geq 1} \) are independent,
we may restate (1) in the form of a random recursive equation, namely
\[
T_n = 1 + \hat{T}_{n-I_n} \quad \text{for all } n \geq 1, \tag{2}
\]
where \( T_0 := 0 \) and \( d \) means equality in distribution.

\section*{1.1 Bibliographic notes}

The random variable \( T \) arises in different areas of applied probability and also in varying disguises, for instance, the number of blocks in regenerative compositions \cite{7, 10, 12}, the number of collisions in exchangeable coalescents \cite{8, 15, 23, 26}, the number of positive jumps in a random walk with a barrier \cite{19, 22}, or the number of cuts needed to isolate the root of a random recursive tree \cite{18}.

Besides the asymptotic analysis of \( Q_n := P_n(T \in \cdot) \) for particular Markov chains like those appearing in the afore-mentioned (as well as many other) models, there is also work on the behavior of \( Q_n \) in a general context. Under the assumption that the large jumps of the chain are rare and occur at rates
that behave like a negative power of the current state, the scaling limit of
\((M_n)_{n \geq 0}\) was derived in [15]. As a byproduct of the main result, the authors
also obtained the convergence in distribution of \(T_n\) properly normalized to
the law of \(\int_0^\infty e^{-U_t} \, dt\), where \((U_t)_{t \geq 0}\) is an increasing Lévy process (subordi-
nator). These results where extended in the recent preprint [3] to the case of
arbitrary Markov chains (not necessarily decreasing) with negative drift. On
the other hand, there are situations where the distributions of the jumps are
almost identical for all states far enough from 0. In such cases it is natural
to expect that, given \(M_0 = n\) for large \(n\), the trajectory of \((M_k)_{k \geq 0}\) stays
close to the trajectory of \((n - S_k)_{k \geq 0}\) for a suitable zero-delayed random walk
\((S_k)_{k \geq 0}\) with positive increments, implying that \(Q_n\) is close to the law of
the first passage time \(\inf\{k \geq 0 : n - S_k < a\}\) and therefore to some stable
law after normalization. Using such a “renewal approximation”, the simplest
case of a random walk with increments having finite variance, was treated
in [27], where some sufficient conditions were derived for the convergence of
\(T_n\), properly centered and normalized, to the standard normal law. Finally,
we mention a short note [25], where a representation of \(T_n\) as a sum of inde-
pendent indicators was provided under the assumption that the increment
distribution can be decomposed as the product of a function of the current
state and a function of the jump size.

1.2 Renewal approximation

The main purpose of this paper is to provide some general results concerning
the distributional convergence of \(T_n\) in the context of the “renewal approx-
imation” mentioned above and to extend this approach to another class of
decreasing Markov chains with so-called “multiplicative” stationary de-
crements. More precisely, we will assume that one of the following conditions
holds true as \(n \to \infty\):

\(\text{(Add)} \quad I_n \xrightarrow{d} \xi\) for a random variable \(\xi\).

\(\text{(Mult)} \quad n^{-1}I_n \xrightarrow{d} 1 - \eta\) for a \([0, 1]\)-valued random variable \(\eta\).

Here the labels are chosen to serve as mnemonic acronyms for “additive”
and “multiplicative”. Let us stipulate that \(\xi\) and \(\eta\) are always assumed to be
independent of any other occurring random variables.

We continue with an outline of the main idea behind our approach. Assume
first that condition (Add) holds. Let \((\xi_n)_{n \geq 1}\) be a sequence of independent
copies of \(\xi\) and \((S_n)_{n \geq 0}\) the associated zero-delayed random walk, viz.

\[ S_0 := 0 \quad \text{and} \quad S_n := S_{n-1} + \xi_n \quad \text{for } n \geq 1. \quad (3) \]

Consider the renewal counting process
\[ N_n := \sum_{k \geq 0} 1_{\{S_k < n\}} = \inf\{k \geq 0 : S_k \geq n\}, \quad n \in \mathbb{N}_0, \quad (4) \]

thus \( N_0 := 0 \). Standard renewal arguments lead to the distributional identity

\[ N_n \overset{d}{=} 1 + \tilde{N}_{n-\xi \wedge n} \quad \text{for} \quad n \geq 1, \quad (5) \]

where \((\tilde{N}_n)_{n \geq 0}\) denotes a copy of \((N_n)_{n \geq 0}\) independent of \(\xi\). Comparing (2) with (5) under the hypothesis (Add), one may expect that the distribution of \(T_n\) can be approximated for large \(n\) by the distribution of \(N_n\) at least if some additional assumptions on the “closeness in distribution” of \(I_n\) and \(\xi \wedge n\) are imposed. In what follows we call the case, when assumption (Add) is in force, the “additive case”, highlighting that the approximating process \((N_n)_{n \geq 0}\) in constructed from the standard additive random walk.

A similar construction can be given when (Mult) holds. Let \((\eta_n)_{n \geq 1}\) be a sequence of independent copies of \(\eta\), \((\Pi_n)_{n \geq 0}\) the associated multiplicative random walk, viz.

\[ \Pi_0 := 1 \quad \text{and} \quad \Pi_n := \Pi_{n-1}\eta_n \quad \text{for} \quad n \geq 1, \quad (6) \]

and \((A_t)_{t \geq 0}\) the renewal counting process corresponding to \((-\log \Pi_n)_{n \geq 0}\), thus

\[ A_t := \sum_{k \geq 0} 1\{-\log \Pi_k \leq t\} = \inf\{k \geq 0 : -\log \Pi_k > t\}, \quad t \in \mathbb{R}. \quad (7) \]

Upon setting \(L_t := A_{\log t}\) for \(t > 0\) and with \((\tilde{L}_t)_{t \geq 0}\) having the obvious meaning, we obtain the distributional identity

\[ L_t \overset{d}{=} 1 + \tilde{L}_{t\eta} = 1 + \tilde{L}_{t-(1-\eta)} \quad \text{for} \quad t \geq 1 \quad (8) \]

with \(L_t := 0\) for \(0 < t < 1\), which again looks similar to (2) for \(t = n\), because \(I_n \approx n(1-\eta)\) for large \(n\) if (Mult) holds. Since the approximating process \((L_t)_{t \geq 0}\) emerges here from a multiplicative random walk we call this situation the "multiplicative case".

### 1.3 Motivating examples

We proceed with a series of examples from applied probability, where “additive” or “multiplicative” renewal approximations come into play.
**Coalescents with multiple collisions**

A coalescent with multiple collisions (or \(\Lambda\)-coalescent) is a continuous-time Markov process \((\Sigma_t^{(n)})_{t \geq 0}\) on the space of partitions of \(\{1, 2, \ldots, n\}\) with one type of transitions: if at some time \(t \geq 0\) there are \(m\) blocks in \(\Sigma_t^{(n)}\), then each \(k\)-tuple of them merges into one block at rate

\[
\lambda_{m,k} = \int_0^1 x^{k-2} (1-x)^{m-k} \, \Lambda(dx), \quad 2 \leq k \leq m,
\]

where \(\Lambda\) is a finite measure on \([0, 1]\). Let \((\hat{\Sigma}_k^{(n)})_{k \geq 0}\) denote the embedded discrete-time Markov chain at jump epochs and \(|\hat{\Sigma}_k^{(n)}|\) the number of blocks in \(\hat{\Sigma}_k^{(n)}\). Since the transition rates depend on a given state only through its size (number of blocks), it is clear that \(|\hat{\Sigma}_k^{(n)}|_{k \geq 0}\) forms a Markov process as well. Moreover, it is decreasing with absorbing state \(a = 1\), and the total number of collisions, say \(C_n\), equals the number of transitions of the chain \(|\hat{\Sigma}_k^{(n)}|_{k \geq 0}\) before absorption, hence

\[
C_1 = 0 \quad \text{and} \quad C_n \overset{d}{=} 1 + \hat{C}_{n-I_n+1} \quad \text{for } n \geq 2,
\]

where

- \(\hat{C}_n\) is a copy of \(C_n\) for \(n \geq 1\);
- \(I_n \in \{2, \ldots, n\}\) denotes a copy of \(n - |\hat{\Sigma}_1^{(n)}| + 1\) and thus describes the number of blocks involved in the first collision event for \(n \geq 2\);
- the \(\hat{C}_n\) and \(I_n\) are independent for \(n \geq 2\).

For the case when \(\Lambda\) is a \(\beta(a, b)\)-distribution with parameter \((a, b) \in (0, 1) \times (0, \infty)\), it is known (see, for instance, formulae (9) and (10) in [9] or formula (11) in [11]) that

\[
I_n - 1 \overset{d}{\to} I_\infty \quad \text{as } n \to \infty,
\]

where

\[
\mathbb{P}(I_\infty = k) = \frac{(2-a) \Gamma(k+a-1)}{\Gamma(a)(k+1)!} \quad \text{for } k \in \mathbb{N},
\]

and \(C_n\) exhibits the same weak asymptotics as the renewal process \(N_n := \inf\{k \geq 0 : S_k^{\infty} \geq n\}\), where \((S_k^{\infty})_{k \geq 0}\) denotes a random walk with generic increment \(I_\infty\).

We refer to the seminal papers [23, 26] for the construction and basic properties of \(\Lambda\)-coalescents, and to the lecture notes [2] for a survey and further references. More information on the limiting behaviour of \(C_n\) as well as other functionals of the \(\Lambda\)-coalescent can be found in [8].

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\[1\] The stated weak convergence for \(I_n - 1\) holds also for \(a \in (1, 2)\) but the renewal approximation is no more valid in this range.
The Bernoulli sieve

In a classical occupancy scheme balls are allocated independently in an infinite array of boxes with probability \( p_k \) of hitting box \( k \in \mathbb{N} \), \( k = 1, 2, \ldots \) for a fixed probability sequence \((p_k)_{k \geq 1}\) with positive entries, also called frequencies. The Bernoulli sieve (see [7, 12, 13]) forms an extension of the occupancy scheme with random frequencies

\[
p_k := W_1 W_2 \cdots W_{k-1} (1 - W_k), \quad k \in \mathbb{N},
\]

where the \( W_k \) are independent copies of a random variable \( W \in (0, 1) \). If \( K_n \) denotes the number of occupied boxes after \( n \) placements, then (see equation (6) in [12])

\[
K_0 = 0 \quad \text{and} \quad K_n \overset{d}{=} 1 + \widehat{K}_{n^-} - J_n \quad \text{for } n \geq 1
\]

where \( K_n, \widehat{K}_n, J_n \) are independent, \( \widehat{K}_n \overset{d}{=} K_n \) and the law of \( J_n \) equals the conditional law of the number of balls in the first box given that this number is positive, viz.

\[
P(J_n = k) = \frac{1}{1 - EW^n} \binom{n}{k} E(1 - W)^k W^{n-k} \quad \text{for } k = 1, 2, \ldots, n.
\]

By the law of large numbers,

\[
n^{-1}J_n \overset{d}{\to} 1 - W \quad \text{as } n \to \infty,
\]

and, under additional assumptions on the rate of this convergence, \( K_n \) has the same weak asymptotics as \( L_n := \inf\{k \geq 0 : W_1 W_2 \cdots W_k < 1/n\} \). This has been proved in [13] if \( E|\log(1 - W)| < \infty \), while the case of infinite mean was treated in [7] by using different approach.

Random walks with a barrier

Let \( \zeta_1, \zeta_2, \ldots \) be independent copies of a random variable \( \zeta \) taking values in \( \mathbb{N} \). The associated random walk \((R_k^{(n)})_{k \geq 0}\) with barrier \( n \in \mathbb{N} \) is then defined as follows:

\[
R_0^{(n)} := 0 \quad \text{and} \quad R_k^{(n)} := R_{k-1}^{(n)} + \zeta_k 1_{\{R_{k-1}^{(n)} + \zeta_k < n\}} \quad \text{for } k \geq 1.
\]

Obviously, \((n - R_k^{(n)})_{k \geq 0}\) forms a nonincreasing Markov chain on \( \mathbb{N} \) with absorbing state 1 which is eventually attained under the additional condition that \( P(\zeta = 1) > 0 \). Defining the number of jumps of \((R_k^{(n)})_{k \geq 0}\), viz.
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\[ Z_n := \sum_{k \geq 1} 1_{\{R_{k-1}^{(n)} \neq R_k^{(n)}\}} = \sum_{k \geq 1} 1_{\{R_{k-1}^{(n)} + \zeta_k \geq n\}}, \]

one can easily see that

\[ Z_1 = 0, \quad \text{and} \quad Z_n = 1 + \hat{Z}_{n-\vartheta_n} \quad \text{for} \quad n \geq 2, \]

where as usual \( Z_n, \hat{Z}_n \) and \( \vartheta_n \) are independent, \( \hat{Z}_n \overset{d}{=} Z_n \) and the law of \( \vartheta_n \) equals the conditional law of \( \zeta \) given \( \zeta < n \), thus

\[ \mathbb{P}(\vartheta_n = k) = \mathbb{P}(\zeta = k|\zeta < n) \quad \text{for} \quad k = 1, 2, \ldots, n - 1. \]

Using the obvious fact that

\[ \vartheta_n \overset{d}{\to} \zeta \quad \text{as} \quad n \to \infty \]

the corresponding renewal approximation has been obtained in [19] under the assumption that \( \zeta \) lies in the domain of attraction of a stable law of index \( \alpha \in [1, 2) \).

The number of zero jumps of \((R_k^{(n)})_{k \geq 0}\) (those dismissed in the unrestricted walk because they would lead to value greater or equal to \( n \)), i.e.

\[ V_n := \sum_{k=1}^{T_n} 1_{\{R_{k-1}^{(n)} + \zeta_k \geq n\}} \]

with \( T_n := \inf\{k \in \mathbb{N}_0 : R_k^{(n)} = n - 1\} \) being the absorption time, provides another functional of interest and a further example involving a multiplicative renewal approximation. In [22], it was shown that the sequence \( V'_n := V_n + 1_{\{n > 1\}} \) satisfies

\[ V'_1 = 0, \quad \text{and} \quad V'_n \overset{d}{=} 1 + \hat{V}'_n \quad \text{for} \quad n \geq 2, \]

where \( Y_n \) denotes the undershoot at \( n \) of a standard random walk with generic increment \( \zeta \) and the usual independence assumptions are made. Further assuming that \( \mathbb{P}(\zeta > n) = cn^{-\alpha} + O(n^{-(\alpha+\varepsilon)}) \) for some \( c > 0, \alpha \in (0, 1) \) and \( \varepsilon > 0 \) and using the classical observation due to Dynkin that

\[ n^{-1}Y_n \overset{d}{\to} \eta_\alpha \quad \text{as} \quad n \to \infty, \]

where \( \eta_\alpha \) has a \( \beta(1-\alpha, \alpha) \)-distribution, it was proved in [22] that a renewal approximation can be used with a multiplicative random walk having generic step size \( \eta_\alpha \).
A simple decreasing Markov chain

Our last example shows that the convergence condition \((Add)\) alone does not suffice for the renewal approximation to work. In fact, the distributions of \(X_n\) and \(N_n\) may exhibit a completely different asymptotic behaviour as the following example from [21] demonstrates. Consider a decreasing Markov chain \((M_k)_{k \geq 0}\) with absorbing state 0, transition probabilities

\[ p_{i,0} = 1 - p_{i,i-1} = \frac{1}{i}, \quad i \in \mathbb{N}, \]

absorption time \(T\) and random variables \(T_n, \hat{T}_n, I_n\) as defined at the beginning of this section, thus satisfying (2). Then one can easily verify that

\[ \mathbb{P}(I_n = n) = 1 - \mathbb{P}(I_n = 1) = \frac{1}{n} \]

for all \(n \geq 1\) and particularly

\[ I_n \xrightarrow{p} \xi = 1 \quad \text{as} \quad n \to \infty \]

Consequently, the corresponding renewal process with generic step size \(\xi\) is degenerate. On the other hand, it can be checked using generating functions that \(T_n\) has a uniform law on \(\{1, 2, \ldots, n\}\) for all \(n \in \mathbb{N}\) whence

\[ \frac{T_n}{n} \xrightarrow{d} \text{Unif}(0,1), \quad n \to \infty. \]

1.4 Minimal \(L^p\)-distance

As illustrated by the last example, assumption \((Add)\) alone does not suffice for our renewal approximation to work, and the same is true for \((Mult)\). Indeed, some extra conditions on the rate of convergence of \(I_n\) to \(\xi\) in the additive case, and of \(n^{-1}I_n\) to \(1 - \eta\) in the multiplicative case are necessary.

For the approach of this paper, which has already been used in [9] and [22], the rate of convergence in \((Add)\) and \((Mult)\) will be measured in terms of the minimal \(L^p\)-distance, for the following reasons a natural choice:

- the convergence in the chosen distance implies convergence in distribution.
- the distance is invariant, in a certain sense, under affine transformations of the laws, thus making calculations with centered and/or normalized random variables easy.

Let us briefly recall the definition and basic properties of the minimal \(L^p\)-distance, defined on the set \(\mathcal{D}^p\) of probability distributions on \(\mathbb{R}\) with finite absolute \(p\)-th moment. A pair of random variables \((X, Y)\), defined on
a common probability space, is called a \((F,G)\)-coupling for \(F,G \in \mathcal{D}_p\), if \(\mathcal{L}(X) = F\) and \(\mathcal{L}(Y) = G\), where \(\mathcal{L}(X)\) denotes the law of \(X\). We then write \((X,Y) \sim (F,G)\). The minimal \(L^p\)-distance between \(F\) and \(G\) is now defined by

\[
d_p(F,G) := \inf_{(X,Y) \sim (F,G)} (\mathbb{E}|X - Y|^p)^{1/p} = \inf_{(X,Y) \sim (F,G)} \|X - Y\|_p. \tag{10}
\]

In what follows, we also write, in slight abuse of language, \(d_p(X,Y)\) and \(d_p(X,G)\) for \(d_p(\mathcal{L}(X), \mathcal{L}(Y))\) and \(d_p(\mathcal{L}(X), G)\), respectively.

The following properties of \(d_p\) for \(p \geq 1\), summarized for our convenience, are well known (see, for instance [1, 6, 20, 24, 28]).

**Proposition 1.** Let \(p \geq 1\) and \(X,Y\) be random variables with laws \(F,G \in \mathcal{D}_p\), respectively. Further, let \(F^\rightarrow(x) := \inf\{y : F(y) \geq x\}\) denote the pseudoinverse of \(F\) and \(U\) a \(\text{Unif}(0,1)\) random variable. The function \(d_p(\cdot, \cdot)\) has the following properties:

\begin{enumerate}[(P1)]
  \item The infimum in Equation (10) is attained for the \((F,G)\)-coupling \((F^\rightarrow(U), G^\rightarrow(U))\), thus
    \[
d_p(X,Y) = \left( \int_0^1 |F^\rightarrow(x) - G^\rightarrow(x)|^p \, dx \right)^{1/p}.
    \]
    In particular,
    \[
d_1(X,Y) = \int_0^1 |F^\rightarrow(x) - G^\rightarrow(x)| \, dx = \int_\mathbb{R} |F(x) - G(x)| \, dx.
    \]
  \item If \(p = 1\), **Kantorovich-Rubinstein representation**, holds, viz.
    \[
d_1(X,Y) = \sup_{f \in \mathcal{F}} |\mathbb{E}f(X) - \mathbb{E}f(Y)|,
    \]
    where \(\mathcal{F}\) denotes the class of all Lipschitz functions \(f : \mathbb{R} \to \mathbb{R}\) with Lipschitz constant one, that is \(|f(x) - f(y)| \leq |x-y|\) for all \(x, y \in \mathbb{R}\).
  \item \(d_p(X+Z,Y+Z) \leq d_p(X,Y)\) for any further random variable \(Z \in \mathcal{D}_p\) independent of \((X,Y)\).
  \item \(d_p(aX + b, aY + b) = |a|d_p(X,Y)\) for any \(a, b \in \mathbb{R}\).
  \item If \((X_n)_{n \geq 1}\) denotes a sequence of random variables with laws in \(\mathcal{D}_p\), then \(d_p(X_n,X) \to 0\) holds iff \(X_n \overset{d}{\to} X\) and \(\mathbb{E}|X_n|^p \to \mathbb{E}|X|^p\), as \(n \to \infty\).
\end{enumerate}

The rest of the paper is organized as follows. Section 2 contains our main results, which are stated as Theorem 1 and Theorem 2. The proofs can be

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2 A representation of this form does not exist for \(p > 1\), see [24, Lemma 4.3.2], but it holds for \(p \in (0,1)\) with \(|x-y|\) replaced by \(|x-y|^p\) in the definition of the set \(\mathcal{F}\).
found in Section 3 and some necessary auxiliary results, including the ones on the convergence of the renewal processes \((N_n)_{n \geq 0}\) and \((A_t)_{t \geq 0}\) in minimal \(L^p\)-distance, are collected in an appendix.

2 Main results

2.1 Weak convergence in the additive case

**Theorem 1.** Suppose that (Add) holds and the law of \(\xi\) is 1-arithmetic and nondegenerate with finite mean \(\mu\).

(a) If \(\sigma^2 := \text{Var}\,\xi < \infty\) and
\[
d_2(I_n, \xi \land n) = o(n^{-1/2}) \quad \text{as } n \to \infty,
\]
then
\[
d_2\left(\frac{T_n - \mu^{-1}n}{\sigma \mu^{-3/2}n^{1/2}}, N(0,1)\right) \overset{n \to \infty}{\to} 0,
\]
where \(N(0,1)\) denotes the standard normal law.

(b) If \(\sigma^2 = \infty\),
\[
\ell(n) := \mathbb{E}[\xi^2 \mathbb{1}_{\{\xi \leq n\}}]
\]
is slowly varying at infinity and
\[
d_1(I_n, \xi \land n) = o(n^{-1}c(n)), \quad \text{as } n \to \infty
\]
for a positive function \(c(t)\) such that
\[
\lim_{n \to \infty} \frac{n\ell(c(n))}{c(n)^2} = 1,
\]
then
\[
d_1\left(\frac{T_n - \mu^{-1}n}{\mu^{-3/2}c(n)}, N(0,1)\right) \overset{n \to \infty}{\to} 0.
\]

(c) If \(\ell(n) := n^\alpha \mathbb{P}(\xi \geq n)\) is slowly varying at infinity for some \(\alpha \in (1,2)\) and Condition [12] holds for a positive function \(c(t)\) satisfying
\[
\lim_{n \to \infty} \frac{n\ell(c(n))}{c(n)^\alpha} = 1,
\]
then
\[
d_1\left(\frac{T_n - \mu^{-1}n}{\mu^{-(\alpha+1)/\alpha}c(n)}, S_\alpha\right) \overset{n \to \infty}{\to} 0,
\]
where \(S_\alpha\) denotes the \(\alpha\)-stable law with characteristic function...
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\[ t \to \exp \left[ -|t|^\alpha \Gamma(1 - \alpha)(\cos(\pi \alpha/2) + i \sin(\pi \alpha/2) \text{sign}(t)) \right]. \quad (15) \]

Remark 1. Regarding the existence and further properties of the function \( c \) appearing in parts (b) and (c) of Theorem 1 as well as Theorem 2 we refer to Remark 2 in the Appendix.

2.2 Weak convergence in the multiplicative case

The case of multiplicative renewal approximation is treated by our second theorem.

Theorem 2. Suppose that (Mult) holds and the distribution of \(|\log \eta|\) is nonarithmetic with finite mean \( \mu_0 \).

(a) If \( \sigma_0^2 := \text{Var} (|\log \eta|) < \infty \) and

\[
d_1 \left( \log \left( 1 - \frac{I_n}{n} \right), \log \eta \right) = o \left( \frac{1}{\log^{1/2} n} \right) \quad \text{as } n \to \infty, \quad (16)
\]

then

\[
d_1 \left( \frac{T_n - \mu_0^{-1} \log n}{\sigma_0 \mu_0^{-3/2} \log^{1/2} n}, \mathcal{N}(0, 1) \right) \xrightarrow{n \to \infty} 0.
\]

(b) If \( \sigma_0^2 = \infty \),

\[
\ell(t) := \mathbb{E}[\log^2 \eta 1_{|\log \eta| \leq t}]
\]

is slowly varying at infinity and

\[
d_1 \left( \log \left( 1 - \frac{I_n}{n} \right), \log \eta \right) = o \left( \frac{c(\log n)}{\log n} \right) \quad \text{as } n \to \infty \quad (18)
\]

for a positive function \( c(t) \) satisfying

\[
\lim_{t \to \infty} \frac{t \ell(c(t))}{c(t)^2} = 1,
\]

then

\[
d_1 \left( \frac{T_n - \mu_0^{-1} \log n}{\mu_0^{-3/2} c(\log n)}, \mathcal{N}(0, 1) \right) \xrightarrow{n \to \infty} 0.
\]

(c) If \( \ell(t) := t^\alpha \mathbb{P}(|\log \eta| > t) \) is slowly varying at infinity for some \( \alpha \in (1, 2) \) and Condition 18 holds for some positive function \( c(t) \) satisfying

\[
\lim_{t \to \infty} \frac{t \ell(c(t))}{c(t)^\alpha} = 1,
\]

then
3 Proofs

3.1 Proof of Theorem 1

Recall that \((N_n)_{n \geq 0}\) denotes the renewal counting process of a random walk \((S_n)_{n \geq 0}\) with generic step size \(\xi\). We start by pointing out that

\[
\lim_{n \to \infty} E I_n = E \xi = \mu. \tag{21}
\]

Indeed with \(p = 2\) in (a) and \(p = 1\) in (b), (c), we have

\[
d_p(I_n, \xi) \leq d_p(I_n, \xi \wedge n) + d_p(\xi \wedge n, \xi),
\]

where the first summand tends to zero by (11) and (12), respectively, while the second one does so by the dominated convergence theorem. Hence (21) follows by (P5) in Proposition 1, and it ensures that \(\psi_n = n\) satisfies Condition (C1) of Lemma 1 in the Appendix with \(p_{n,k} := P(I_n = n - k)\), a fact to be used further below (see before (24)).

In what follows, we let \(c(t)\) be given by \(\sigma t^{1/2}\) in part (a), by (13) in part (b), and by (14) in part (c). We also put \(\alpha = 2\) in (a) and (b), and we write \(d_{1(2)}\) to denote \(d_p\) with \(p = 2\) in (a) and \(p = 1\) in (b) and (c). Finally, let \(W\) be the standard normal law \(N(0,1)\) in (a) and (b) and the stable law \(S_\alpha\) with characteristic function (15) in (c). Our task is then to show that

\[
d_{1(2)} \left( \frac{T_n - \mu^{-1} n}{\mu^{-(\alpha+1)/\alpha} c(n)}, W \right)
\]

converges to 0 as \(n \to \infty\). Using the triangle inequality, we can bound this distance by

\[
d_{1(2)} \left( \frac{T_n - \mu^{-1} n}{\mu^{-(\alpha+1)/\alpha} c(n)}, N_n \right) \leq d_{1(2)} \left( \frac{N_n - \mu^{-1} n}{\mu^{-(\alpha+1)/\alpha} c(n)}, W \right) + d_{1(2)} \left( \frac{N_n}{\mu^{-(\alpha+1)/\alpha} c(n)}, W \right),
\]

and Proposition 2 (1-arithmetic case) ensures that the second term converges to zero. As for the first one, it is enough to prove that

\[
e_n := d_{1(2)}(T_n, N_n) = o(c(n)) \quad \text{as} \quad n \to \infty \tag{22}
\]

by property (P4) in Proposition 1.
Using the recursions (2.5) and again property (P4), we have for $n \geq 1$

$$
e_n = d_{1(2)}(\hat{T}_{n-I_n}, \hat{N}_{n-\xi \land n})
\leq d_{1(2)}(\hat{N}_{n-I_n}, \hat{N}_{n-\xi \land n}) + d_{1(2)}(\hat{T}_{n-I_n}, \hat{N}_{n-I_n})
\leq d_{1(2)}(\hat{N}_{n-I_n}, \hat{N}_{n-\xi \land n}) + \sum_{k=0}^{n-1} \mathbb{P}(I_n = n-k) d_{1(2)}(T_k, N_k)

= e'_n + \sum_{k=0}^{n-1} \mathbb{P}(I_n = n-k) e_k \quad \text{with} \quad e'_n := d_{1(2)}(\hat{N}_{n-I_n}, \hat{N}_{n-\xi \land n}).$$

Assuming we have already proved

$$e'_n = o(n^{-1}c(n)) \quad \text{as} \quad n \to \infty,$$

we can apply Lemma 2 with $\psi_n = n$ and $r_n = \varepsilon c(n)/n$ for arbitrarily small $\varepsilon > 0$ to infer that, as $n \to \infty$,

$$e_n = \varepsilon O\left(\sum_{k=1}^{n} \sup_{j \geq k} j^{-1} c(j)\right) = \varepsilon O\left(\sum_{k=1}^{n} k^{-1} c(k)\right) = \varepsilon O(c(n)),$$

where we have utilized Theorem 1.5.3 and Proposition 1.5.8 in [4] and the fact that $c(x)$ is a regularly varying function of index $1/\alpha$, see Remark 2 in Appendix.

Left with the proof of (23), let $(I'_n, \xi')$ be a $(\mathcal{L}(I_n), \mathcal{L}(\xi))$-coupling such that $d_{1(2)}(I_n, \xi \land n) = d_{1(2)}(I'_n, \xi' \land n) = \|I'_n - \xi' \land n\|_{1(2)}$ and $(S'_n)_{n \geq 0}$ be a copy of $(S_n)_{n \geq 0}$ which is independent of $(I'_n, \xi')$. Denote by $(N'_n)_{n \geq 0}$ the corresponding renewal counting process, clearly a copy of $(N_n)_{n \geq 0}$. Then

$$
e'_n = d_{1(2)}(\hat{N}_{n-I_n}, \hat{N}_{n-\xi \land n})
\leq \|N'_n - I'_n - N''_n - \xi' \land n\|_{1(2)}
\leq \|I'_n - \xi' \land n\|_{1(2)},$$

where the last inequality follows from the fact that, in view of $\mathbb{P}(\xi_1 \geq 1) = 1$, the number of points $S'_n$ falling in some interval $[a, b], a, b \in \mathbb{N}$, cannot exceed the length of this interval. Consequently,

$$e'_n \leq d_{1(2)}(I'_n, \xi' \land n) = d_{1(2)}(I_n, \xi \land n),$$

and thus (23) by (11) in part (a) and by (12) in parts (b) and (c). This completes the proof.
3.2 Proof of Theorem 2

The proof of Theorem 2 uses similar ideas as the previous one in the additive case, but for technical reasons it is more convenient to work with the stationary version of the renewal counting process \((\Lambda_t)_{t \geq 0}\) associated with \((- \log \Pi_n)_{n \geq 0}\) (see (6) and (7) for the definition of \(\Pi_n\) and \(\Lambda_t\)).

Let \(\eta_0^* \in (0,1)\) be a random variable independent of \((\Pi_k)_{k \geq 0}\) such that

\[
r(t) := \mathbb{P}(|\log \eta_0^*| \leq t) = \int_0^t \mathbb{P}(|\log \eta| > s) \, ds, \quad t \geq 0.
\]  

(25)

Define the delayed multiplicative random walk \((\Pi_k^*)_{k \geq 0}\) by

\[
\Pi_0^* = \eta_0^* \quad \text{and} \quad \Pi_k^* := \eta_0^* \eta_1 \cdots \eta_k \quad \text{for} \quad k \in \mathbb{N},
\]

the stationary renewal counting process associated with \((- \log \Pi_k^*)_{k \geq 0}\) by

\[
\Lambda_t^* := \sum_{k \geq 0} 1(- \log \Pi_k^* \leq t), \quad t \in \mathbb{R},
\]

and finally \(L_t^* := \Lambda_{\log t}^*\) for \(t > 0\). Then it is well-known that

\[
\mathbb{E} \Lambda_t^* = \frac{t^+}{\mu_0} \quad \text{for all} \quad t \in \mathbb{R},
\]  

(26)

a fact frequently be used hereafter. Moreover, as \(A_t = \inf\{k : - \log \Pi_k > t\}\), Wald’s identity ensures (see e.g. [14, Theorem 2.5.1])

\[
\mathbb{E} A_t = \frac{t}{\mu_0} + o(t) \quad \text{as} \quad t \to \infty,
\]  

(27)

and \(o(t)\) may be replaced with \(o(c(t))\) under the assumptions of part (b) and (c) of Theorem 2 (see p. 5 in [16]). The counterpart of (8) for the process \((L_t^*)_{t \geq 0}\) is given by

\[
L_t^* \overset{d}{=} 1_{\{\eta_0^* > 1/t\}} + \tilde{L}_{t \eta_0}, \quad t \geq 0.
\]  

(28)

where \((\tilde{L}_t^*)_{t > 0}\) is independent of \(\eta\) and obtained as a copy of \((L_t^*)_{t \geq 0}\) by replacing the \(\eta_1, \eta_2, \ldots\) by i.i.d. copies in the definition of the underlying multiplicative random walk \((\Pi_k^*)_{k \geq 0}\) while keeping the delay \(\eta_0^*\) fixed.

Returning to the proof of Theorem 2 we consider again all three parts (a)–(c) simultaneously and use analogous notation as in Section 3.1. This means that \(c(t)\) equals \(\sigma_0 t^{1/2}\) in part (a), is given by \(\nu_0\) in part (b) and given by \(\nu_1\) in part (c), \(\alpha = 2\) in parts (a) and (b). Also, \(\mathcal{W}\) stands for the limiting distribution, thus for \(\mathcal{N}(0,1)\) in (a) and (b), and for \(\mathcal{S}_\alpha\) in (c).

Using the triangle inequality, we obtain
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\[ d_1 \left( \frac{T_n - \mu_0^{-1} \log n}{\mu_0^{-1/\alpha} c(\log n)}, \mathcal{W} \right) \leq d_1 \left( \frac{T_n - \mu_0^{-1} \log n}{\mu_0^{-1/\alpha} c(\log n)}, \frac{L_n^* - \mu_0^{-1} \log n}{\mu_0^{-1/\alpha} c(\log n)} \right) + d_1 \left( \frac{L_n^* - \mu_0^{-1} \log n}{\mu_0^{-1/\alpha} c(\log n)}, \frac{L_n - \mu_0^{-1} \log n}{\mu_0^{-1/\alpha} c(\log n)} \right) + d_1 \left( \frac{L_n - \mu_0^{-1} \log n}{\mu_0^{-1/\alpha} c(\log n)}, \mathcal{W} \right). \]

The third summand converges to zero by Proposition 2 so that it is enough to prove (use (P4) of Proposition 1)

\[ d_1(T_n, L_n^*) = o(c(\log n)) \quad \text{as} \quad n \to \infty \quad (29) \]

and

\[ d_1(L_n, L_n^*) = o(c(\log n)) \quad \text{as} \quad n \to \infty. \quad (30) \]

We first consider (30) and recall that the renewal counting process \((\Lambda_t)_{t \in \mathbb{R}}\) is subadditive in distribution, viz.

\[ A_{u+v} - A_u \overset{d}{\leq} A_v \quad \text{for all} \quad u, v \in \mathbb{R}, \quad (31) \]

where \(\overset{d}{\leq}\) denotes stochastic ordering: \(X \overset{d}{\leq} Y \iff \mathbb{P}(X > x) \leq \mathbb{P}(Y > x)\) for all \(x \in \mathbb{R}\).

In order to prove (30), notice that

\[ A_t^* = A_{t - |\log \eta_0^*|} \quad \text{for all} \quad t \in \mathbb{R} \quad (32) \]

with \(\eta_0^*\) being independent of \((A_t)_{t \geq 0}\). Using this, we infer

\[ d_1(L_n, L_n^*) = d_1(A_{\log n}, A_{\log n}^*) \leq \mathbb{E}|A_{\log n} - A_{\log n}^*| \leq \mathbb{E}A_{\log n} - \mathbb{E}A_{\log n}^* = o(c(\log n)), \]

where \(A_t^* \leq A_t\) for all \(t \in \mathbb{R}\) by \(32\) has been utilized for the penultimate equality, and \(26, 27\) plus subsequent remark for the final estimate.

Left with the proof of (29), use the recursions (28) and (2) to find that

\[ e_n := d_1(T_n, L_n^*) \]

\[ = d_1(1 + \hat{T}_{n-I_n}, 1_{\{\eta_0^* > 1/n\}} + \hat{L}_n^*) \]

\[ = d_1(\hat{T}_{n-I_n}, \hat{L}_{n-I_n} - 1_{\{\eta_0^* \leq 1/n\}}) \]

\[ \leq d_1(\hat{L}_{n-I_n}, 1_{\{\eta_0^* \leq 1/n\}} + d_1(\hat{T}_{n-I_n}, \hat{L}_{n-I_n}) \]

\[ = e_n^* + \sum_{k=0}^{n-1} \mathbb{P}(I_n = n-k) e_k, \]
where \( e'_n := d_1(\hat{L}_{n-I_n}^*, \hat{L}_{n\eta'}^* - 1_{\{n\theta \leq 1/n\}}) \). Let us assume for a moment that

\[
e'_n = o \left( \frac{c\log n}{\log n} \right) \text{ as } n \to \infty \tag{33}
\]

is already known and further note that (Mult) yields

\[
\mathbb{E}(n - I_n) \simeq n \mathbb{E} |\log \eta| = \mu_0 n \text{ as } n \to \infty. \tag{34}
\]

The latter implies that the sequence \( \psi_n = 1 \) satisfies condition (C1) of Lemma 1 when putting \( p_{n,k} := P(I_n = n - k) \). An appeal to Lemma 2 together with (33) provides us with

\[
e_n = O \left( \sum_{k=1}^{n} \sup_{j \geq k} \frac{c(j)}{j \log j} \right) = o \left( \sum_{k=3}^{n} \frac{c(k \log k)}{k \log k} \right),
\]

where we have used [4, Theorem 1.5.3] and the fact that \( c(t)/t \log t \) is regularly varying with index \(-1\) (see Remark 2 in the Appendix). Note that the sum \( \sum_{k=3}^{n} \frac{c(k \log k)}{k \log k} \) diverges as \( n \) tends to \( \infty \), because \( c(t) \) varies regularly with index \( 1/\alpha > 0 \). Consequently, as \( n \to \infty \),

\[
\sum_{k=3}^{n} \frac{c(k \log k)}{k \log k} \simeq \int_{c}^{n} \frac{c(y \log y)}{y \log y} dy = \int_{1}^{\log n} \frac{c(u)}{u} du \simeq \text{const} \cdot c(\log n)
\]

by [4, Theorem 1.6.1] and (33) is proved.

It remains to show that (33) holds. Let \((I'_n, \eta')\) be a \((\mathcal{L}(I_n), \mathcal{L}(\eta))\)-coupling such that

\[
d_1 \left( \log \left( 1 - \frac{I_n}{n} \right), \log \eta \right) = \left\| \log \left( 1 - \frac{I_n}{n} \right) - \log \eta' \right\|_1
\]

and \((\log \hat{\Pi}_n^*)_{n \geq 0}\) be a copy of \((\log \Pi^*_n)_{n \geq 0}\) with delay variable \( \hat{\eta}_0^* \) and independent of \((I'_n, \eta')\). Further defining

\[
\hat{\Lambda}_t^* := \sum_{k \geq 0} 1_{(-\log \hat{\Pi}_t^* \leq t)} \text{ for } t \in \mathbb{R}
\]

and \( \hat{\Pi}_t^* := \hat{\Lambda}_{t \log t} \) for \( t > 0 \), we have

\[
e'_n = d_1(\hat{L}_{n-I_n}^*, \hat{L}_{n\eta'}^* - 1_{\{n\theta \leq 1/n\}}) \leq \left\| \hat{L}_{n-I_n}^* - \hat{L}_{n\eta'}^* - 1_{\{n\theta_0 \leq 1/n\}} \right\|_1 \leq \left\| \hat{L}_{n-I_n}^* - \hat{L}_{n\eta'}^* \right\|_1 + \mathbb{P}(\hat{\eta}_0^* \leq 1/n),
\]
where \( \eta_0^* = \hat{H}_0^* \overset{d}{=} \eta_0^* \). Recalling (25), we see that
\[
\mathbb{P}(\eta_0^* \leq 1/n) = \mathbb{P}(\log n \geq \log n)
= \frac{1}{\mu_0} \int_{\log n}^{\infty} \mathbb{P}(|\log \eta| > s) \, ds = 1 - r(\log n).
\]
(35)

Since in all parts (a)-(c)
\[
\lim_{t \to \infty} \frac{t(1 - r(t))}{c(t)} = 0,
\]
we find that
\[
\mathbb{P}(\eta_0^* \leq 1/n) = o(c(\log n)/\log n) \quad \text{as } n \to \infty.
\]

In order to bound \( ||\hat{L}_{n-I_n^*} - \hat{L}_{n\eta'}^*||_1 \), we estimate
\[
||\hat{L}_{n-I_n^*} - \hat{L}_{n\eta'}^*||_1 = ||\hat{A}_{\log(n-I_n^*)} - \hat{A}_{\log(n\eta')}||_1
= \left\| \sum_{k=0}^{\infty} \mathbb{1}_{\{\log(n-I_n^*) < \log \lambda_k \leq \log(n-I_n^*) \wedge \log(n\eta')\}} \right\|_1
\leq \sum_{k=0}^{\infty} \mathbb{P}(\log(n-I_n^*) < \log \lambda_k \leq \log(n-I_n^*) \wedge \log(n\eta'))
\leq \mathbb{E} \hat{A}_{\log(n-I_n^*) \wedge \log(n\eta')} - \mathbb{E} \hat{A}_{\log(n-I_n^*) \wedge \log(n\eta')}^c.
\]

Now use \( \mathbb{E} \hat{A}_t^* = \mu_0^{-1} t^* \) to see that the last line can be further estimated by
\[
= \frac{1}{\mu_0} \left( (\mathbb{E} \log(n-I_n^*) \wedge \log(n\eta')) - \mathbb{E} \log(n-I_n^*) \wedge \log(n\eta')) \right)
= \frac{1}{\mu_0} \mathbb{E} \log(n-I_n^*) - \log(n\eta')
\leq \frac{1}{\mu_0} \mathbb{E} \log \left( \frac{1-I_n^*}{n} \right) - \log n + \frac{1}{\mu_0} \mathbb{E} \log(n\eta') - \log(n\eta')
= \frac{1}{\mu_0} d_1 \left( \log \left( \frac{1-I_n^*}{n} \right), \log \eta \right) + \frac{1}{\mu_0} \mathbb{E} \left( (-\log(\eta)) \mathbb{1}_{\{\eta \leq 1\}} \right).
\]

\(^3\) In (a) the numerator is bounded, since \( 1 - r(t) \) is integrable, while in (b), (c) the numerator varies regularly with index \( 2 - \alpha \) and the denominator varies regularly with index \( 1/\alpha > 2 - \alpha \) (with \( \alpha = 2 \) in parts (a) and (b)).
The first summand is \( o(c(\log n) / (\log n)) \), \( n \to \infty \), by Condition (16) in part (a) and by Condition (18) in (b) and (c). As for the second summand, an integration by parts yields

\[
\frac{1}{\mu_0} \mathbb{E}(\frac{1}{n} \sum_{n \leq \xi_n} 1_{\{\xi_n \leq 1\}}) = \frac{1}{\mu_0} \int_{\log n}^{\infty} (s - \log n) \mathbb{P}(\log \eta < s) \, ds
\]

\[
= \frac{1}{\mu_0} \int_{\log n}^{\infty} \mathbb{P}(\log \eta < s) \, ds
\]

\[
= 1 - r(\log n),
\]

which is of the order \( o(c(\log n) / (\log n)) \) as \( n \to \infty \) by (35) and subsequent remarks. This completes the proof. \( \square \)

4 Appendix

4.1 Convergence of renewal quantities in minimal \( L^p \)-distance

**Proposition 2.** Let \( \xi, \xi_1, \xi_2, ... \) be iid positive and nonarithmetic random variables with finite mean \( \mu \) and associated zero-delayed random walk \( (S_n)_{n \geq 0} \). For \( t \geq 0 \), let

\[
N_t := \sum_{n \geq 0} 1_{\{S_n \leq t\}}
\]

denote the number of renewals in \([0, t]\).

(R1) If \( \sigma^2 := \text{Var} \xi < \infty \), then

\[
d_2 \left( \frac{N_t - \mu^{-1} t}{\sigma \mu^{-3/2} t^{1/2}}, \mathcal{N}(0, 1) \right) \xrightarrow{t \to \infty} 0.
\]

(R2) If \( \sigma^2 = \infty \),

\[
\ell(t) := \mathbb{E} \left[ \xi^2 1_{\{\xi \leq t\}} \right]
\]

is slowly varying at infinity and \( c(t) \) a positive continuous function such that

\[
\lim_{t \to \infty} \frac{t \ell(c(t))}{c^2(t)} = 1,
\]

then

\[
d_1 \left( \frac{N_t - \mu^{-1} t}{\mu^{-3/2} c(t)}, \mathcal{N}(0, 1) \right) \xrightarrow{t \to \infty} 0.
\]

(R3) If, for some \( \alpha \in (1, 2) \),
\[
\ell(t) := t^\alpha \mathbb{P}(\xi > t)
\]

is slowly varying at infinity and \(c(t)\) a positive function satisfying

\[
\lim_{t \to \infty} \frac{t\ell(c(t))}{c^\alpha(t)} = 1,
\]

then

\[
d_1 \left( \frac{N_t - \mu^{-1} t}{\mu^{-(1+\alpha)/\alpha} c(t)}, \mathcal{S}_\alpha \right) \xrightarrow{t \to \infty} 0,
\]

where \(\mathcal{S}_\alpha\) denotes the \(\alpha\)-stable law with characteristic function \(\text{[15]}\).

All assertions remain valid with \(t = nd\) if \(\xi\) is \(d\)-arithmetical for some \(d > 0\).

**Proof.** Replacing the convergence in \(d_p\) for \(p = 1\) or \(2\) with weak convergence, the proposition is well-known (see, for instance, \[14, \text{Chapter III, Section 5}\]). Since \(\sigma^2 < \infty\) further implies

\[
\lim_{t \to \infty} \mathbb{E} \left( \frac{N_t - \mu^{-1} t}{\sqrt{\sigma^2 \mu^{-3} t}} \right)^2 = \mathbb{E} Z^2 = 1, \quad Z \stackrel{d}{=} \mathcal{N}(0,1)
\]

(see e.g. \[14, \text{Theorem 8.4 in Chapter III}\]), the assertion in (R1) follows by an appeal to (P5) of Proposition \[1\]. But (R2) and (R3) follow in the same manner when invoking Lemma A.1 in \[17\] which states the convergence of the first absolute moment of \((N_t - \mu^{-1} t)/c(t)\) as \(t \to \infty\) to the first absolute moment of the respective limiting distribution. \(\square\)

**Remark 2.** The function \(c(t)\) appearing in the cases (R2) and (R3) always exists provided that \(\mathbb{P}(\xi > t)\) varies regularly at infinity, i.e. \(\mathbb{P}(\xi > t) = t^{-\alpha} \ell(t)\) for some slowly varying function \(\ell\) and \(\alpha \in (1,2]\). This function can be equivalently defined, see e.g. \[3, \text{Theorem 7}\], by the asymptotic relation \(\mathbb{P}(\xi > c(t)) \simeq 1/t\), as \(t \to \infty\), in particular one can choose

\[
c(t) := \left(1/\mathbb{P}(\xi > t)\right)^\downarrow.
\]

Moreover, by Proposition 1.5.15 in \[4\]

\[
c(t) \simeq t^{1/\alpha}(\ell^\#(t))^{1/\alpha} \quad \text{as} \quad t \to \infty,
\]

where \(\ell^\#\) denotes the de Bruijn conjugate of the slowly varying function \(L(t) := 1/\ell(t^{1/\alpha})\). Since \(\ell^\#\) is slowly varying as well, \(c\) is regularly varying of index \(1/\alpha\), a fact which has repeatedly been used in our proofs.

---

\[4\] In the case (R2) slow variation of \(\ell(t)\) is equivalent to regular variation of \(\mathbb{P}(\xi > t)\) with index \(-2\).
4.2 A linear recursion

Fixing \( a \in \mathbb{N}_0 \) and a sequence \((r_n)_{n \geq 1}\) of positive reals, let \((p_{n,k})_{0 \leq k < n}\) be an arbitrary probability distribution on \(\{0, \ldots, n - 1\}\) for each \( n > a \) and then \((s_n)_{n \geq 0}\) the unique solution to the linear recursion

\[
s_n = r_n + \sum_{k=0}^{n-1} p_{n,k} s_k, \quad n > a,
\]

(36)

with given initial values \(s_0, s_1, \ldots, s_a\). The following result forms a slight extension of Lemma 6.1 from [9].

**Lemma 1.** Suppose there exists a sequence \((\psi_n)_{n \geq 1}\) such that

(C1) \( \liminf_{n \to \infty} n^{-1} \psi_n \sum_{k=0}^{n-1} (n - 1 - k)p_{n,k} > 0 \),

(C2) the sequence \((r_k\psi_k/k)_{k \geq 1}\) is nonincreasing.

Then \((s_n)_{n \geq 0}\), defined by (36), satisfies

\[
s_n = O \left( \sum_{k=1}^{n} \frac{r_k\psi_k}{k} \right) \quad \text{as} \quad n \to \infty.
\]

(37)

**Proof.** By assumption (C1), there exists \( n_0 > a \) such that

\[
c^{-1} := \inf_{n > n_0} \frac{\psi_n}{n} \sum_{k=0}^{n-1} p_{n,k}(n - 1 - k) > 0,
\]

thus \( c < \infty \). We show by induction that

\[
s_n \leq \sum_{i=0}^{n_0} s_i + c \sum_{i=1}^{n} \frac{r_i\psi_i}{i} \quad \text{for all} \quad n \in \mathbb{N}_0.
\]

As this estimate obviously holds for \( n \leq n_0 \), fix \( n > n_0 \) and suppose that it be true for all \( k < n \). Defining \( M^* := \sum_{i=0}^{n_0} s_i \), we find

\[
s_n = r_n + \sum_{k=0}^{n-1} p_{n,k} s_k \\
\leq r_n + \sum_{k=0}^{n-1} p_{n,k} \left( M^* + c \sum_{i=1}^{k} \frac{r_i\psi_i}{i} \right) \\
= M^* + r_n + c \sum_{k=1}^{n-1} p_{n,k} \sum_{i=1}^{k} \frac{r_i\psi_i}{i}.
\]
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\[ M^* + r_n + c \sum_{i=1}^{n-1} \frac{r_i \psi_i}{i} \sum_{k=1}^{i-1} p_{n,k} \]

\[ = M^* + r_n + c \sum_{i=1}^{n-1} \frac{r_i \psi_i}{i} \left( 1 - \sum_{k=0}^{i-1} p_{n,k} \right) \]

\[ \leq M^* + r_n + c \sum_{i=1}^{n-1} \frac{r_i \psi_i}{i} - c \frac{r_n \psi_n}{n} \sum_{i=1}^{n-1} \sum_{k=0}^{i-1} p_{n,k} \]

\[ \leq M^* + r_n + c \sum_{i=1}^{n-1} \frac{r_i \psi_i}{i} - c \frac{r_n \psi_n}{n} \sum_{k=0}^{n-1} p_{n,k} (n - 1 - k) \]

\[ \leq M^* + c \sum_{i=1}^{n-1} \frac{r_i \psi_i}{i}, \]

where the last inequality follows from the definition of \( c \). \( \square \)

We actually need the following generalization of the previous result.

**Lemma 2.** Suppose that \((s_n)_{n \geq 0}\) satisfies

\[ s_n \leq r_n + \sum_{k=0}^{n-1} p_{n,k} s_k \quad \text{for } n > a \quad (38) \]

for a nonnegative sequence \((r_n)_{n \geq 1}\) and initial values \(s_0, \ldots, s_a\). Suppose further the existence of a sequence \((\psi_n)_{n \geq 1}\) such that (C1) holds and

(C3) the sequence \((r_k \psi_k/k)_{k \geq 1}\) is bounded.

Then \((s_n)_{n \geq 0}\) satisfies

\[ s_n = O \left( \sum_{k=1}^{n} r_k^* \frac{\psi_k}{k} \right) \quad \text{as } n \to \infty, \quad (39) \]

where \(r_k^* = \frac{k}{\psi_k} \sup_{j \geq k} \frac{r_j \psi_j}{j} \) for \(k \geq 1\).

**Proof.** First note that \((\psi_n)_{n \geq 1}\) satisfies (C2) of the previous lemma with \((r_n^*)_{n \geq 1}\) instead of \((r_n)_{n \geq 1}\). Let \((s_n^*)_{n \geq 0}\) be defined by \(s_0^* := s_k \) for \(k = 0, \ldots, a\) and \(s_n^* = r_n^* + \sum_{k=0}^{n-1} p_{n,k} s_k^* \) for \(n > a\). Since \(r_n^* \geq r_n\) for each \(n\) and

\[ s_n^* - s_n \geq r_n^* - r_n + \sum_{k=0}^{n-1} p_{n,k} (s_k^* - s_k) \quad \text{for } n > a, \]

a simple induction shows that \(s_n \leq s_n^*\) for all \(n \geq 0\). Now use Lemma to infer
\[ s_n^* = O \left( \sum_{k=1}^{n} \frac{r_k^* \psi_k}{k} \right) \quad \text{as } n \to \infty, \]

which in combination with the previous statement proves (39). \(\square\)

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**References**

1. G. Alsmeyer. The smoothing transform: a review of contraction results. In *Random matrices and iterated random functions*, volume 53 of *Springer Proc. Math. Stat.*, pages 189–228. Springer, Heidelberg, 2013.
2. N. Berestycki. *Recent progress in coalescent theory*, volume 16 of *Ensaios Matemáticos [Mathematical Surveys]*. Sociedade Brasileira de Matemática, Rio de Janeiro, 2009.
3. J. Bertoin and I. Kortchemski. Self-similar scaling limits of markov chains on the positive integers, 2014. [www.arxiv.org:1412.1068](http://www.arxiv.org:1412.1068).
4. N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1989.
5. W. Feller. Fluctuation theory of recurrent events. *Trans. Amer. Math. Soc.*, 67:98–119, 1949.
6. C. R. Givens and R. M. Shortt. A class of Wasserstein metrics for probability distributions. *Michigan Math. J.*, 31(2):231–240, 1984.
7. A. Gnedin, A. Iksanov, and A. Marynych. Limit theorems for the number of occupied boxes in the Bernoulli sieve. *Theory Stoch. Process.*, 16(2):44–57, 2010.
8. A. Gnedin, A. Iksanov, and A. Marynych. \(\Lambda\)-coalescents: a survey. *J. Appl. Probab.*, 51A(Celebrating 50 Years of The Applied Probability Trust):23–40, 2014.
9. A. Gnedin, A. Iksanov, A. Marynych, and M. Möhle. On asymptotics of the beta coalescents. *Adv. in Appl. Probab.*, 46(2):496–515, 2014.
10. A. Gnedin and J. Pitman. Regenerative composition structures. *Ann. Probab.*, 33(2):445–479, 2005.
11. A. Gnedin and Y. Yakubovich. On the number of collisions in \(\Lambda\)-coalescents. *Electron. J. Probab.*, 12:no. 56, 1547–1567, 2007.
12. A. V. Gnedin. The Bernoulli sieve. *Bernoulli*, 10(1):79–96, 2004.
13. A. V. Gnedin, A. M. Iksanov, P. Negadajlov, and U. Rösler. The Bernoulli sieve revisited. *Ann. Appl. Probab.*, 19(4):1634–1655, 2009.
14. A. Gut. *Stopped random walks. Limit theorems and applications*. Springer Series in Operations Research and Financial Engineering. Springer, New York, 2nd edition, 2009.
15. B. Haas and G. Miermont. Self-similar scaling limits of non-increasing Markov chains. *Bernoulli*, 17(4):1217–1247, 2011.
16. A. Iksanov, A. Marynych, and M. Meiners. Moment convergence in renewal theory, 2012. [www.arxiv.org:1208.3964](http://www.arxiv.org:1208.3964).
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17. A. Iksanov, A. Marynych, and M. Meiners. Limit theorems for renewal shot noise processes with eventually decreasing response functions. *Stochastic Process. Appl.*, 124(6):2132–2170, 2014.

18. A. Iksanov and M. Möhle. A probabilistic proof of a weak limit law for the number of cuts needed to isolate the root of a random recursive tree. *Electron. Comm. Probab.*, 12:28–35, 2007.

19. A. Iksanov and M. Möhle. On the number of jumps of random walks with a barrier. *Adv. in Appl. Probab.*, 40(1):206–228, 2008.

20. O. Johnson and R. Samworth. Central limit theorem and convergence to stable laws in Mallows distance. *Bernoulli*, 11(5):829–845, 2005.

21. A. Marynych. Asymptotic behaviour of absorption time of decreasing Markov chains (in Ukrainian). *Bulletin of Kiev University, Ser. Phys.-Math. Sciences*, 1:118–121, 2010.

22. A. Marynych and G. Verovkin. Weak convergence of the number of zero increments in the random walk with barrier. *Electron. Commun. Probab.*, 19:no. 74, 11, 2014.

23. J. Pitman. Coalescents with multiple collisions. *Ann. Probab.*, 27(4):1870–1902, 1999.

24. S. T. Rachev. *Probability metrics and the stability of stochastic models*. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. John Wiley & Sons, Ltd., Chichester, 1991.

25. S. M. Ross. A simple heuristic approach to simplex efficiency. *European J. Oper. Res.*, 9(4):344–346, 1982.

26. S. Sagitov. The general coalescent with asynchronous mergers of ancestral lines. *J. Appl. Probab.*, 36(4):1116–1125, 1999.

27. B. Van Cutsem and B. Ycart. Renewal-type behavior of absorption times in Markov chains. *Adv. in Appl. Probab.*, 26(4):988–1005, 1994.

28. V. M. Zolotarev. *Modern theory of summation of random variables*. Modern Probability and Statistics. VSP, Utrecht, 1997.