Defining free damped oscillation in technological systems

D Y Ershov¹,², I N Lukyanenko¹, A O Smirnov¹ and E E Aman¹

¹ Department of Higher Mathematics and Mechanics, St. Petersburg State University of Aerospace Instrumentation, 190000, St. Petersburg, st. B. Morskaya, 67A
² Department of Machine building, St. Petersburg Mining University, 199106, St. Petersburg, Vasilyevsky Island, 21 lines d.2

E-mail: fetcat@mail.ru, irina.n.lukyanenko@gmail.com

Abstract. The paper reviews approaches to defining free damped oscillation in mechanical drive of the technological equipment based on identifying the intrinsic oscillation frequencies given the Rayleigh dissipation function under non-linear dynamic effect. It presents a mathematical model for damped oscillation accounting for dissipative forces and their effect on the system vibration activity.

1. Introduction

Generally, a physical drive assembly is a complex elastic inertial assembly of shafts with cogwheels, pulleys, clutches, etc. [1-3]. Shafts are joined to deformable bearing supports. All links in the assembly, as well as flexible and rigid (spline and keyway) joints, are deformed, to a certain extent, in operation [4-7]. Hence, a complete description of the assembly will require determining every movement of every point as derivatives of time and coordinates. Torsion causes both elasticity and inelastic resistance moments due to the shaft material internal friction [12]. Linear oscillation theory views the latter as proportional to strain rate [8-11].

2. Mathematical model

Hence the torque in transverse section can be presented as

\[ M_{j-1,j} = c_{j-1,j} (\varphi_{j-1,j} - \varphi_j) + b_{j-1,j} (\dot{\varphi}_{j-1,j} - \dot{\varphi}_j) \]  

(1)

\( M_{j-1,j} \) value determines the torque in the shaft section between \( j-1 \) and \( j \) disks. The first summand determines the elastic forces moment, while the second, that of inelastic resistance, with \( b_{j-1,j} \) proportionality constant.

Figure 1 illustrates free oscillations in technological system. The mechanical drive consists of the rotor 1, transmission gear 2 and actuator 3 given as disks in the Figure 1, \( c_1 \) and \( c_2 \) indicate the torsional stiffness in shaft sections, while \( b_1 \) and \( b_2 \) indicate proportionality constant for inelastic resistance moments.

Given the disks rotation angles \( \varphi_1 \), \( \varphi_2 \) and \( \varphi_3 \), the system kinetic (K) and potential (P) energies are
Inelastic resistance forces are derived from the Rayleigh dissipation function. The equation for the function is analogous to potential energy equation.

\[ F = \frac{1}{2} b_2 (\dot{\varphi}_2 - \dot{\varphi}_3)^2 + \frac{1}{2} b_3 (\dot{\varphi}_3 - \dot{\varphi}_3)^2 \]  

\[(3)\]

Free oscillations in the system in question are conditioned on two coordinates

\[ \varphi_1 = \psi_0 + \psi_1 + \psi_2 \]
\[ \varphi_2 = \psi_0 + \mu_1 \psi_1 + \mu_2 \psi_2 \]
\[ \varphi_3 = \psi_0 + \mu_3 \psi_1 + \mu_4 \psi_2 \]  

\[(4)\]

The system kinetic and potential energies for generalized coordinates are calculated as

\[ K = \frac{1}{2} J_1 \dot{\varphi}_1^2 + \frac{1}{2} J_2 \dot{\varphi}_2^2 + \frac{1}{2} J_3 \dot{\varphi}_3^2 \]
\[ P = \frac{1}{2} c_1 (\varphi_1 - \varphi_2)^2 + \frac{1}{2} c_2 (\varphi_2 - \varphi_3)^2 \]  

\[(5)\]

Quasi-elasticity ratios are

\[ c_i = c_i \left(1 - \mu_{i1}\right)^2 + c_2 \left(\mu_{21} - \mu_{31}\right)^2 \cdot \mu_{32} \]
\[ c_{ii} = c_i \left(1 - \mu_{i2}\right)^2 + c_4 \left(\mu_{22} - \mu_{32}\right)^2 \]  

\[(6)\]

Then the dissipative function conditioned on the generalized coordinates is

\[ F = \frac{1}{2} \beta_1 \dot{\varphi}_1^2 + \frac{1}{2} \beta_2 \dot{\varphi}_2^2 + \frac{1}{2} \beta_3 \dot{\varphi}_3^2 \cdot \beta_4 \]  

\[(7)\]

where

\[ \beta_i = b_1 \left(1 - \mu_{i1}\right)^2 + b_2 \left(\mu_{21} - \mu_{31}\right)^2 \]
\[ \beta_2 = b_1 \left(1 - \mu_{i2}\right)^2 + b_3 \left(\mu_{22} - \mu_{32}\right)^2 \]
\[ \beta_3 = b_1 \left(1 - \mu_{i3}\right) \left(1 - \mu_{i2}\right) + b_2 \left(\mu_{23} - \mu_{33}\right) \left(\mu_{22} - \mu_{32}\right) \]  

\[(8)\]

The Lagrange equation of the second kind with kinetic and potential energies and dissipative functions give the differential equations of motion:
\[ J_1 \ddot{\psi}_1 + \beta_1 \dot{\psi}_1 + c_{1t} \psi_1 + \beta_{1t} \psi_2 = 0 \]
\[ J_2 \ddot{\psi}_2 + \beta_2 \dot{\psi}_2 + c_{2t} \psi_2 + \beta_{2t} \psi_1 = 0 \]
\[ J_0 \ddot{\psi}_0 = 0 \]  

(9)

3. Solving equations

The resulting equations 1 and 2 (9) for the system describing the disks oscillations are linked by dissipation rate \( \beta_{12} \), while equation 3 describing the motion in technological system as a solid unit has no link to equations 1 and 2. The solution for equation 3 runs as

\[ \dot{\psi}_0 = \omega_0, \]
\[ \psi_0 = \omega_0 t + C \]  

(10)

Motion conditioned on \( \psi_0(t) \) function, the shaft deformation and inelastic resistance moments are non-existent.

The solution for equations 1 and 2 (9) given the function runs as

\[ \psi_1 = A_1 e^{\lambda t} \]
\[ \psi_2 = A_2 e^{\lambda t} \]  

(11)

where \( A_1, A_2 \) and \( \lambda \) are unknown values. By substituting (11) to (9) we arrive at the linear homogeneous system of algebraic equations with \( A_1 \) and \( A_2 \):

\[ (J_1 \lambda^2 + \beta_1 \lambda + c_{1t}) A_1 + \beta_{1t} \lambda A_2 = 0 \]
\[ \beta_{12} \lambda A_1 + (J_2 \lambda^2 + \beta_2 \lambda + c_{2t}) A_2 = 0 \]  

(12)

The solution (12) will be nonzero if the determinant equals zero:

\[ \Delta(\lambda) = \left| \begin{array}{cc} J_1 \lambda^2 + \beta_1 \lambda + c_{1t} & \beta_{1t} \lambda \\ \beta_{12} \lambda & J_2 \lambda^2 + \beta_2 \lambda + c_{2t} \end{array} \right| = 0 \]  

(13)

Expanding the determinant we arrive at equation that allows to calculate \( \lambda \):

\[ J_1 J_2 \lambda^4 + (J_1 \beta_2 + J_2 \beta_1) \lambda^3 + (J_1 c_{1t} + J_2 c_{2t} + \beta_0 \beta_2 - \beta_{12} \lambda) \lambda^2 + \]
\[ + (\beta_1 c_{1t} + \beta_2 c_{2t}) \lambda + c_{1t} c_{2t} = 0 \]  

(14)

The equation includes four roots allowing for the following options:

- All roots are real;
- Two roots are real and two roots are complex conjugate;
- Two pairs of complex conjugate roots.

It holds that complex conjugate roots in the characteristic equation determine the oscillatory motion in the mechanical system, while real roots determine the aperiodic oscillation. The studies show the mechanical drives of technological systems characterized by small values of inelastic resistance forces compared to elastic forces moments. Hence, technological systems out of equilibrium reveal oscillation (vibration). This results in two pairs of complex conjugate roots, while the real part of the root indicative [12].

Given the ratios of characteristic equation
\[
a_0 = J_J N, \\
a_1 = J_J \beta_2 + J_N \beta_1, \\
a_2 = J_J c_N + J_N c_1 + \beta_1 \beta_2 - \beta_{12}, \\
a_3 = \beta_1 c_N + \beta_2 c_1, \\
a_4 = c_1 c_N \\
\]

are positive and satisfy
\[
a_4 (a_4 a_2 - a_0 a_0) - a_4 a_1^2 > 0, \\
\]

the complex conjugate roots have negative real parts. The roots themselves can be done as
\[
\lambda_1 = -n_1 + ik_1, \\
\lambda_2 = -n_1 - ik_1, \\
\lambda_3 = -n_2 + ik_2, \\
\lambda_4 = -n_2 - ik_2. \\
\]

where \( n_1 > 0 \) and \( n_2 > 0 \).

The system of equations (12) gives the ratio of amplitudes
\[
\mu_2 = \frac{A_2}{A_1} = - \frac{J_J \lambda_2^2 + \beta_1 \lambda_1 + c_1}{\beta_{12} \lambda_1} \\
\]

Substituting \( \lambda_1 = -n_1 + ik_1 \) to (18), we get
\[
\mu_{21} = \frac{A_{21}}{A_1} = - \frac{J_J \lambda_1^2 + \beta_1 \lambda_1 + c_1}{\beta_{12} \lambda_1} = - \frac{J_J (-n_1 + ik_1)^2 + \beta_1 (-n_1 + ik_1) + c_1}{\beta_{12} (-n_1 + ik_1)} = \mu_{21}^0 e^{\gamma_{21}} \\
\]

where
\[
\mu_{21}^0 = - \frac{\sqrt{b_1^2 + d_1^2}}{\sqrt{b_2^2 + d_2^2}}, \quad \gamma_{21} = \gamma_1 - \gamma_2, \\
b_1 = c_1 - J_J k_1^2 - n_2 \left( \beta_1 - J_J n_1 \right) d_1 = k_1 \left( \beta_1 - 2 J_J n_1 \right), b_2 = -\beta_2 n_1, d_2 = \beta_{12} k_1, \\
\gamma_1 = \text{arctg} \left( \frac{d_1}{b_1} \right), \quad \gamma_2 = \text{arctg} \left( \frac{d_2}{b_2} \right). \\
\]

Substituting \( \lambda_2 = -n_1 - ik_1 \) to (18), we get
\[
\bar{\mu}_{21} = \frac{A_{21}}{A_1} = - \frac{J_J \lambda_2^2 + \beta_1 \lambda_2 + c_1}{\beta_{12} \lambda_2} = - \frac{J_J (-n_1 - ik_1)^2 + \beta_1 (-n_1 - ik_1) + c_1}{\beta_{12} (-n_1 - ik_1)} = \bar{\mu}_{21}^0 e^{-\gamma_{21}} \\
\]

In the same way \( \lambda_2 \) and \( \lambda_4 \) root forms ratios can be done
\[
\mu_{22} = \frac{A_{22}}{A_{12}} = \mu_{22}^0 e^{\gamma_{22}} \\
\]

(21)
\[ \overline{p}_{22} = \frac{\overline{A}_2}{A_2} = \mu_2 e^{-\gamma_2 t} \]  

(22)

It follows from the above that forms ratios are complex values.

Particular solutions for the roots in characteristic equation run as follows

\[ \psi_{11} = A_1 e^{i\kappa_1 t} = A_1 e^{(-n_1 + ik_1)t} \]
\[ \overline{\psi}_{11} = \overline{A}_1 e^{\overline{\gamma}_1 t} = \overline{A}_1 e^{(-n_1 + ik_1)t} \]
\[ \psi_{12} = A_2 e^{i\kappa_2 t} = A_2 e^{(-n_2 + ik_2)t} \]
\[ \overline{\psi}_{12} = \overline{A}_2 e^{\overline{\gamma}_2 t} = \overline{A}_2 e^{(-n_2 + ik_2)t} \]

\[ \psi_{21} = A_{11} e^{i\kappa_{12} t} = A_{11} e^{(-n_1 + ik_1)t} \]
\[ \overline{\psi}_{21} = \overline{A}_{11} e^{\overline{\gamma}_{12} t} = \overline{A}_{11} e^{(-n_1 + ik_1)t} \]
\[ \psi_{22} = A_{22} e^{i\kappa_{12} t} = A_{22} e^{(-n_2 + ik_2)t} \]
\[ \overline{\psi}_{22} = \overline{A}_{22} e^{\overline{\gamma}_{12} t} = \overline{A}_{22} e^{(-n_2 + ik_2)t} \]

The general solution for \( \psi_0 \) coordinate is

\[ \psi_1 = \psi_{11} + \overline{\psi}_{11} + \psi_{12} + \overline{\psi}_{12} = e^{-n_1 t} \left( A_1 e^{i\kappa_1 t} + \overline{A}_1 e^{-i\kappa_1 t} \right) + e^{-n_2 t} \left( A_2 e^{i\kappa_2 t} + \overline{A}_2 e^{-i\kappa_2 t} \right) \]  

(23)

given

\[ A e^{i\alpha t} + \overline{A} e^{-i\alpha t} = a \sin(kt + \alpha) \]  

(24)

hence

\[ A e^{i\alpha t} + \overline{A} e^{-i\alpha t} = A \left( \cos kt + i \sin kt \right) + \overline{A} \left( \cos kt - i \sin kt \right) = \left( A + \overline{A} \right) \cos kt + i \left( A - \overline{A} \right) \sin kt \]  

(25)

\( A \) and \( \overline{A} \) constants of integration are complex conjugate values \( A = B + iC \), \( \overline{A} = B - iC \), hence \( A + \overline{A} = 2B \) and \( A - \overline{A} = -2iC \).

This gives

\[ A e^{i\alpha t} + \overline{A} e^{-i\alpha t} = 2B \cos kt + 2C \sin kt = a \sin(kt + \alpha) \]  

(26)

where

\[ a = \sqrt{4B^2 + 4C^2}, \]
\[ \alpha = \arctg \left( \frac{B}{C} \right) \]

On return to \( \psi_1 \) we get

\[ \psi_1 = a_1 e^{-n_1 t} \sin(k_1 t + \alpha_1) + a_2 e^{-n_2 t} \sin(k_2 t + \alpha_2) \]  

(27)

In (27) \( a_{11}, a_{12}, a_1 \) and \( a_2 \) values are constants of integration, while \( n_1, n_2, k_1 \) and \( k_2 \) are constants of the system physical properties.
The general solution for \( \psi_2 \) coordinate is

\[
\psi_2 = \psi_{21} + \overline{\psi}_{21} + \psi_{22} + \overline{\psi}_{22} = \mu_2^0 e^{-\eta^0 \omega f} \left[ A_1 e^{i(k_f + \gamma_{21})} + \overline{A}_1 e^{-i(k_f + \gamma_{21})} \right] + \\
+ \mu_2^{12} a_{12} e^{-\eta^1 \omega f} \left[ A_1 e^{i(k_f + \gamma_{22})} + \overline{A}_1 e^{-i(k_f + \gamma_{22})} \right] = \\
= \mu_2^0 a_{11} e^{-\eta^0 \omega f} \sin(k_f t + \gamma_{12} + \alpha_1) + \mu_2^{12} a_{12} e^{-\eta^1 \omega f} \sin(k_f t + \gamma_{22} + \alpha_2)
\]

Equations (27) and (28) allow to calculate the rotation angle of the first disk in the mathematical model (figure 1):

\[
\phi_1 = \psi_0 + \psi_1 + \psi_2 = \omega_0 t + C + a_{11} e^{-\eta^0 \omega f} \sin(k_f t + \alpha_1) + a_{12} e^{-\eta^1 \omega f} \sin(k_f t + \alpha_2) + \\
+ \mu_2^0 a_{11} e^{-\eta^0 \omega f} \sin(k_f t + \gamma_{12} + \alpha_1) + \mu_2^{12} a_{12} e^{-\eta^1 \omega f} \sin(k_f t + \gamma_{22} + \alpha_2) = \\
= \omega_0 t + C + a_{11} e^{-\eta^0 \omega f} \sin(k_f t + \alpha_1) + \mu_2^0 a_{11} e^{-\eta^0 \omega f} \sin(k_f t + \gamma_{12} + \alpha_1) + \\
+ \mu_2^{12} a_{12} e^{-\eta^1 \omega f} \sin(k_f t + \gamma_{22} + \alpha_2)
\]

(29)

Given the most generalized conditions the changes in \( \phi_1 \) rotation angle can be viewed as the sum of three motions: rotation at \( \omega_0 \) constant angle velocity, oscillation with \( k_1 \) and \( k_2 \) frequencies. Since \( n_1 > 0 \) and \( n_2 > 0 \) the disc vibration conditioned on its rotation will dump resulting in rotation at \( \omega_0 \) angle velocity. Such rotation at \( \omega_0 \) angle velocity is possible unless the shaft bearing friction is taken. Changes in disks 2-3 rotation angles will bothersome.

4. Conclusion

Hence, once out of equilibrium the system will reveal free damped oscillation at \( k_1 \) and \( k_2 \) frequencies conditioned on the rotation of the system as of a solid unit at constant angle velocity rate. Oscillation damped, the system will rotate.

Hence inelastic resistance moments under the shaft deformation result in damped oscillation, i.e. lower mechanical energy in the system compared to the initial one. This means inelastic resistance moments result in dissipation of energy. Such forces are called dissipative. The further study of the effect of those forces on the dynamics of technological systems is required.

References

[1] Bessekersky V, Popov E 1972 Theory of automatic regulation systems. (Moscow, Russia)
[2] Ershov D, Zlotnikov E and Koboyankwe L 2017 IOP Conf. Ser.: Earth Environ. Sci. 87(8) 82016
[3] Eliseev S, Khomenko A 2014 Kinematic disturbances in dynamics of mechanical oscillatory systems. Modern technologies. System analysis. Simulation. V 4(44) 8-18
[4] Eliseev S, Bolshakov R, Kinash N and Nguyen D 2015 Structural mathematical modeling, mechanical reactions in mechanical oscillatory systems Modern trends in science and technologies 3 63-69
[5] Eliseev S, Artyushkin A 2016 Applied oscillation theory in dynamics of linear mechanical systems 459
[6] Kuleshov M, Syromyatnikov V 2017 Optimization of conveyor drive parameters at random load changes Proceedings of Higher Educational Institutions. Machine Building 10(691) 69-76
[7] Nekrasov R, Putilova U, Starikov A, Solovyov I and Nekrasov Yu 2015 Modeling processes of diagnostics and processing control for CNC machinery Systems. Methods. Technologies 2(26) 54-9
[8] Panovko Ya 2015 Fundamentals of applied oscillation and shock theory (Moscow, Russia)
[9] Pirogov D, Shlyapugin R and Selezniov S 2017 Dissipative forces-based study of mechanical parameters of cam-weaving metal loom Reliability and life of machinery. 7th Conf. proc. 330-3
[10] Rasskazova N 2016 Improved damping efficiency in Duffing-type systems with discreet communication of elastic parts Dynamics of systems, mechanisms and machinery 1 80-8
[11] Wolfson I 2017 On corrections in evaluation of motor effect on drive vibration activity VNTR Vestnik 2 (114) 11-23
[12] Yershov D 2018 Determining natural oscillation frequencies in 3-mass dynamic systems IPDME-2018 1 25