Parabose – Parafermi Supersymmetry

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Abstract

The \((p = 2)\) parabose – parafermi supersymmetry is studied in general terms. It is shown that the algebraic structure of the \((p = 2)\) parastatistical dynamical variables allows for (symmetry) transformations which mix the parabose and parafermi coordinate variables. The example of a simple parabose – parafermi oscillator is discussed and its symmetries investigated. It turns out that this oscillator possesses two parabose – parafermi supersymmetries. The combined set of generators of the symmetries forms the algebra of supersymmetric quantum mechanics supplemented with an additional central charge. In this sense there is no relation between the parabose – parafermi supersymmetry and the parasupersymmetric quantum mechanics. A precise definition of a quantum system involving this type of parabose – parafermi supersymmetry is offered, thus introducing \((p = 2)\) Supersymmetric Paraquantum Mechanics. The spectrum degeneracy structure of general \((p = 2)\) supersymmetric paraquantum mechanics is analyzed in detail. The energy eigenvalues and eigenvectors for the parabose – parafermi oscillator are then obtained explicitly. The

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latter confirms the validity of the results obtained for general super-symmetric paraquantum mechanics.
1 Introduction

In a preceding article [1], an attempt is made to simplify the study of the algebraic structure of dynamical systems involving \((p = 2)\) parabose and parafermi variables. The approach presented in [1] is aimed to facilitate the analysis of systems possessing parabose – parafermi supersymmetry, thus providing the necessary framework for investigating the relation between the conventional parastatistics of Green [2] and the more recent developments of parasupersymmetric quantum mechanics [3, 4, 5, 6]. More specifically the purpose of the present article is to answer the question:

*Is parabose – parafermi supersymmetry the same as parasupersymmetry?*

One should note that the so-called “parasupersymmetric oscillators” studied in the literature, e.g., [3, 4], are constructed using some specific matrix representation of the parafermi operators. The analysis presented in this paper does not restrict to matrix representations and treats the parafermi and parabose operators (variables) as fundamental mathematical objects.

The paper is organized as follows: In Sec. 2, the main results of [1] are quoted and the possibility of the existence of parabose – parafermi (supersymmetry) transformations is investigated. In Sec. 3, the analogy between the \((p = 2)\) parabose – parafermi supersymmetry and the ordinary bose – fermi supersymmetry is discussed. The example of the supersymmetric os-
cillator is then reviewed and the \((p = 2)\) parabose–parafermi oscillator is introduced by analogy. In Sec. 4, the parabose–parafermi supersymmetries of the oscillator are studied. In Sec. 5, the super Lie algebra of the symmetries of the oscillator is used to define the notion of \((p = 2)\) *Supersymmetric Paraquantum Mechanics*. This section also offers a detailed treatment of the degeneracy structure of general \((p = 2)\) supersymmetric paraquantum mechanics. Sec. 6 is devoted to an analysis of the energy eigenstates and the spectrum degeneracy of the parabose–parafermi oscillator. Here, the explicit form of a complete set of energy eigenstate vectors is obtained. Sec. 7 includes the conclusions.

For brevity we shall use the notation \(\pi b, \pi f, \pi SUSY\) for \((p = 2)\) parabose, parafermi, and parabose–parafermi supersymmetry, and abbreviations SQM, PSQM, SPQM for *supersymmetric quantum mechanics*, parasupersymmetric quantum mechanics, and *supersymmetric paraquantum mechanics*, respectively. We shall follow Einstein summation convention of summing over repeated indices throughout the paper, unless otherwise indicated.

## 2 Algebraic Structure of Classical \(\pi SUSY\)

In this section, first we recall the constructions developed in \([\square]\).

The algebra of the creation \(a^\mu_\dagger\) and annihilation operators \(a^\mu\) for the \((p = 2)\) \(\pi b\) \((\mu = 0)\) and \(\pi f\) \((\mu = 1)\) variables is given by:

\[
a^\mu = \sum_{\alpha=0}^1 \zeta^\alpha_\mu, \tag{1}
\]
\[ \theta_{k_1}^{\alpha \mu} := \sqrt{\frac{\hbar}{2}} (\zeta_k^{\alpha \mu} + \zeta_k^{\alpha \mu \dagger}) , \tag{2} \]

\[ \theta_{k_0}^{\alpha \mu} := -i \sqrt{\frac{\hbar}{2}} (\zeta_k^{\alpha \mu} - \zeta_k^{\alpha \mu \dagger}) , \tag{3} \]

\[ [[\theta_{im}^{\alpha \mu}, \theta_{jn}^{\beta \nu}]] := \hbar \delta_{ij} \delta^{\alpha \beta} [i(1 - \mu)(1 - \nu)\epsilon_{mn} + \mu \nu \delta_{mn}] , \tag{4} \]

where \( \zeta \)'s are the Green components of \( a \)'s \([2]\), \( \alpha, \beta, \mu, \nu = 0, 1, m, n = 1, 2 \), and \([ [ , ] ]\) is the parabracket:

\[ [[\theta_{im}^{\alpha \mu}, \theta_{jn}^{\beta \nu}]] := \theta_{im}^{\alpha \mu} \theta_{jn}^{\beta \nu} - (-1)^{\mu \nu + \alpha + \beta} \theta_{jn}^{\beta \nu} \theta_{im}^{\alpha \mu} , \tag{5} \]

introduced in \([4]\). Note that Eq. \((4)\) is the statement of the canonical quantization rule for the Green components \( \theta \) on the one hand, and the expression of the normal relative statistics \([3, 4]\) on the other. The classical analogs of the self-adjoint operators \( \theta \) are obtained by setting \( \hbar = 0 \) in Eq. \((4)\).

One also generalizes the definition of the parabracket to arbitrary polynomials in \( \theta \)'s, according to:

\[ [[M, N]] = MN - (-)^{\eta(M,N)}MN , \tag{6} \]

where \( M \) and \( N \) are monomials:

\[ M := \theta_{i_1 m_1}^{\alpha_1 \mu_1} \cdots \theta_{i_r m_r}^{\alpha_r \mu_r} , \]

\[ N := \theta_{j_1 n_1}^{\beta_1 \nu_1} \cdots \theta_{j_s n_s}^{\beta_s \nu_s} , \]

\[ \eta(M, N) := (\sum_{k=1}^{r} \mu_k)(\sum_{l=1}^{s} \nu_l) + r \sum_{l=1}^{s} \beta_l + s \sum_{k=1}^{r} \alpha_k , \tag{7} \]

and bilinearity of the parabracket. In the classical limit the parabracket of any two polynomials vanishes identically.
In Ref. [1], it is also argued that in the Lagrangian formulation of the para-classical mechanics, the Green components of the $\pi b$ coordinate variables are $\theta_{im=1}^{\alpha \mu = 0}$. Thus, one introduces a collective index $I = (i, m)$ which may take $(i = 1, \cdots, n_{\pi b}; m = 1)$ for $\mu = 0$ and $(i = 1, \cdots, n_{\pi f}; m = 1, 2)$ for $\mu = 1$, and denote the Green components of the coordinate variables by $\theta_I^{\alpha \mu}$. The physical quantities, such as a Lagrangian, is chosen from the algebra of polynomials in the coordinates

$$\psi_I^\mu := \sum_{\alpha=0}^1 \theta_I^{\alpha \mu}$$

and the velocities $\dot{\psi}_I^\alpha$. For computational convenience, they are then expressed in terms of the Green components $\theta_I^{\alpha \mu}$ and $\dot{\theta}_I^{\alpha \mu}$.

As a polynomial in (the classical) $\theta$’s and $\dot{\theta}$’s, a Lagrangian must satisfy (up to total time derivatives) the following conditions [1]:

1) It must be real.

2) It must be an even polynomial in both $\pi b$ ($\mu = 0$) and $\pi f$ ($\mu = 1$) variables.

To define the notion of reality in the algebra of polynomials in $\theta$’s and $\dot{\theta}$’s (alternatively in $\psi$’s and $\dot{\psi}$’s), one first introduces a *-operation satisfying:

$$(\xi_{x_1} \cdots \xi_{x_n})^* = \xi_{x_n} \cdots \xi_{x_1} ,$$

$$(\lambda_1 P_1 + \lambda_2 P_2)^* = \lambda_1^* P_1^* + \lambda_2^* P_2^* ,$$

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where $\xi_{x_i}$ are any of the generators: $\theta$’s and $\dot{\theta}$’s (resp. $\psi$’s and $\dot{\psi}$’s), $\lambda_a \in \mathbb{C}$ with $a = 1, 2$, $\lambda_a^*$ are their complex conjugates, and $P_a$ are polynomials in $\xi_{x_i}$. Then a polynomial $P$ is defined to be real if $P^* = P$.

The classical dynamics of the system is given by the least action principle, where the action functional has the form: $S = \int L \, dt$. This leads to the analogs of the Euler-Lagrange equations:

$$\frac{d}{dt}(\frac{\partial}{\partial \dot{\theta}}) - L \frac{\partial}{\partial \theta} = 0.$$ 

(9)

Here the indices on $\theta$’s are suppressed for simplicity and the left partial derivatives with respect to $\theta$’s and $\dot{\theta}$’s are defined in Refs. [8, 1].

Having reviewed the basic elements of the Lagrangian formulation of para-classical systems, we would like to address the question:

**Does the algebraic structure of ($p = 2$) parastatistical dynamical variables allow for a transformation of $\pi_b$ variables into $\pi_f$ variables and vice versa?**

Unlike, the case of ordinary ($p = 1$) fermi – bose systems, where the product of two fermi variables is a commutative algebraic object and thus behaves as a bose variable, the algebraic structure of the ($p = 2$) variables is too complicated to have such a simple grading. Nevertheless, in view of the formalism developed in [1], one can easily respond to the above mentioned question in the positive.
To see this, consider the algebra $B$ of the real Green components generated by $\xi_{k}^{\alpha\mu}$, and the algebra $A$ generated by:

$$\gamma_{k}^{\mu} = \sum_{\alpha=0}^{1} \xi_{k}^{\alpha\mu}.$$ 

The elements of $A$ (resp. $B$) will be used as non-dynamical parameters added to the algebra of polynomials in dynamical variables $\psi$’s and $\dot{\psi}$’s (resp. $\theta$’s and $\dot{\theta}$’s). Then in the enlarged algebra, it is not difficult to check that the multiplication of dynamical variables $\psi_{I}^{\mu}$ and $\dot{\psi}_{I}^{\mu}$ by the real parameters:

$$\gamma_{k} = \{\gamma_{k}^{0}, \gamma_{k}^{1}\} := \sum_{\alpha,\beta=0}^{1} \{\xi_{k}^{\alpha\mu=0}, \xi_{k}^{\beta\mu=1}\}, \quad (10)$$

changes their parity. Here there is no summation over the index $k$. This can be easily verified by defining $\psi_{I}^{\alpha\mu'} := \psi_{I}^{\alpha\mu} \gamma_{k}$ and examining their commutation properties by first decomposing them into their Green components. One can further show that $\gamma_{k}$ commute with all the parabose variables and anticommute with all the parafermi variables.

Presence of $\gamma_{k}$ allows for the existence of the $\pi$SUSY transformations. We shall examine examples of such symmetry transformations in the next section. We shall also introduce $\delta\gamma_{k}$ which are analogs of the (fermionic) parameters of the infinitesimal supersymmetry transformations.

3 SUSY and $\pi$SUSY Oscillators

A thorough discussion of the supersymmetric (SUSY) oscillator is offered in Ref. [4]. The Hamiltonian operator of one-dimensional SUSY oscillator is
the sum of the Hamiltonians of a fermi and a bose oscillators with identical frequencies, i.e.,
\[ \hat{H} = \hat{H}^0 + \hat{H}^1, \]  
(11)
\[ \hat{H}^0 := \frac{\omega}{2} \{ \hat{a}^\dagger, \hat{a} \}, \]  
(12)
\[ \hat{H}^1 := \frac{\omega}{2} [\hat{a}^\dagger, \hat{a}]. \]  
(13)

Here, \( \hat{a} \) and \( \hat{a}^\dagger \) (resp. \( \hat{\alpha} \) and \( \hat{\alpha}^\dagger \)) stand for the bosonic (resp. fermionic) annihilation and creation operators and the hats are placed to distinguish the quantum mechanical operators and the classical dynamical variables.

The combined system of two oscillators (11) serves as a simple example of a supersymmetric system. To reveal the supersymmetry of this system, we shall first switch to the self-adjoint operators:
\[ \hat{x} := \sqrt{\frac{\hbar}{2\omega}} (\hat{a} + \hat{a}^\dagger), \]
\[ \hat{p} := -i\sqrt{\frac{\omega\hbar}{2}} (\hat{a} - \hat{a}^\dagger), \]  
(14)
\[ \hat{\psi}_1 = \sqrt{\frac{\hbar}{2}} (\hat{\alpha} + \hat{\alpha}^\dagger), \]
\[ \hat{\psi}_2 = -i\sqrt{\frac{\hbar}{2}} (\hat{\alpha} - \hat{\alpha}^\dagger). \]

Then the Hamiltonian (11) takes the form:
\[ \hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{x}^2) + \frac{i\omega}{2} \epsilon_{mn} \hat{\psi}_m \hat{\psi}_n, \]  
(15)
where \( \epsilon_{mn} \) are the components of the Levi Civita symbol.
The classical counterpart of the SUSY oscillator is obtained by dropping the hats in the above relations and treating $x$ and $p$ as bosonic (commuting or even) and $\psi_m$ as fermionic (anticommuting or odd) supernumbers [9], respectively.

The classical SUSY oscillator may also be described using the Lagrangian:

$$L = \frac{1}{2}(x^2 - \omega^2 x^2) + i\frac{\delta_{mn}}{2}(\psi_m \dot{\psi}_n - \dot{\psi}_m \psi_n) - \frac{i\omega}{2} \epsilon_{mn} \psi_m \psi_n,$$

where $m, n = 1, 2$. Then it is an easy exercise to check that this Lagrangian is invariant (up to total time derivatives) under the transformation:

$$\delta x = i\psi_m \delta \zeta_m,$$

$$\delta \psi_m = (\delta_{mn} \dot{x} + \omega \epsilon_{mn} x) \delta \zeta_n,$$

where $\delta \zeta_n$ are “infinitesimal” fermionic supernumber parameters [9]. The corresponding Nöther charges of this symmetry – the supercharges – are given by:

$$Q_m = \lambda(\delta_{mn} \dot{x} - \omega \epsilon_{mn} x) \psi_n,$$

where $\lambda \in \mathbb{C}$ is an arbitrary non-zero coefficient. Upon quantization of this system one can easily show that the supercharges, that generate the supersymmetry transformations, and the Hamiltonian satisfy the defining algebra of SQM. In particular, taking $\lambda = 1/\sqrt{\hbar}$, one has:

$$\{\hat{Q}_m, \hat{Q}_n\} = 2\delta_{mn}\hat{H}.$$

Next, let us introduce the para-generalization of the SUSY oscillator. We shall denote this by $\pi$SUSY oscillator for simplicity.
In general, the Hamiltonian for the parabose and parafermi oscillators is given by Eqs. (12) and (13), with \( \hat{a} \) and \( \hat{\alpha} \), now, denoting the parabose and parafermi annihilation operators, respectively, \[8\]. Returning to our notation of Sec. 2, we set \( \hat{a} := \hat{a}^{\mu=0} \) and \( \hat{\alpha} := \hat{a}^{\mu=1} \). In terms of the self-adjoint operators: \( \hat{\psi}^{\mu} := \hat{\psi}^{\mu}_{i=1} \) of Eq. (8) and their Green components \( \theta^{\alpha,\mu} := \theta^{\alpha,\mu}_{i=1} \), we have:

\[
\hat{H}^{\mu} = \frac{\omega}{2} \left[(1 - \mu) \delta_{mn} \hat{\psi}^{\mu}_m \hat{\psi}^{\mu}_n + i \mu \epsilon_{mn} \hat{\psi}^{\mu}_m \hat{\psi}^{\mu}_n\right], \quad (21)
\]

\[
\hat{H}^{\mu} = \frac{\omega}{2} \left[(1 - \mu) \delta_{mn} \hat{\theta}^{\alpha,\mu}_m \hat{\theta}^{\alpha,\mu}_n + i \mu \epsilon_{mn} \hat{\theta}^{\alpha,\mu}_m \hat{\theta}^{\alpha,\mu}_n\right], \quad (22)
\]

where \( \mu = 0, 1 \) correspond to \( \pi_b \) and \( \pi_f \) oscillators, respectively.

The \((p = 2) - \pi\) SUSY oscillator is then defined by Eq. (11):

\[
\hat{H} = \hat{H}^0 + \hat{H}^1 = \sum_{\alpha=0}^1 \left\{ \frac{1}{2} (\hat{\chi}^{\alpha})^2 + \frac{\omega}{2} (\hat{\chi}^{\alpha})^2 + \frac{i \omega}{2} \epsilon_{IJ} \hat{\tau}^{\alpha}_I \hat{\tau}^{\alpha}_J \right\}, \quad (23)
\]

where \( \hat{\chi}^{\alpha} := \hat{\theta}^{\alpha,\mu=0}/\sqrt{\omega} \), \( \hat{\chi}^{\alpha} := \sqrt{\omega} \hat{\theta}^{\alpha,\mu=0} \) are the Green components of the \( \pi_b \) coordinate and momentum operators, and \( \hat{\tau}^{\alpha}_I := \hat{\theta}^{\alpha,\mu=1} \) are those of the \( \pi_f \) coordinate operators.

The Lagrangian associated with the \( \pi \) SUSY oscillator is given by:

\[
L = \frac{1}{2} (\dot{x}^2 - \omega^2 x^2) + \frac{i}{4} \delta_{IJ} (\psi_I \dot{\psi}_J - \dot{\psi}_I \psi_J) - \frac{i \omega}{2} \epsilon_{IJ} \psi_I \psi_J \quad (24)
\]

\[
L = \sum_{\alpha=0}^1 \left\{ \frac{1}{2} (\dot{\chi}^{\alpha})^2 - \omega^2 (\chi^{\alpha})^2 + \frac{i}{2} \delta_{IJ} \dot{\tau}^{\alpha}_I \dot{\tau}^{\alpha}_J - \frac{i \omega}{2} \epsilon_{IJ} \tau^{\alpha}_I \tau^{\alpha}_J \right\} \quad (25)
\]

where \( x = \sum_{\alpha=0}^1 \chi^{\alpha} \), \( \psi_I = \sum_{\alpha=0}^1 \tau^{\alpha}_I \) are the \((p = 2) - \pi_b \) and \( \pi_f \) dynamical variables, respectively.
The form of the $\pi b$ and $\pi f$ kinetic terms in (24) is obtained in Ref. [1] in an attempt to consistently generalize the Peierls bracket quantization scheme to the paraclasical systems.

The Peierls bracket quantization of this system leads to the following paracommutation relations:

\[
[\hat{\chi}^{\alpha}, \hat{\chi}^{\beta}] = 0, \\
[\hat{\chi}^{\alpha}, \hat{\dot{\chi}}^{\beta}] = i\hbar \delta^{\alpha\beta}, \\
[\hat{\dot{\chi}}^{\alpha}, \hat{\dot{\chi}}^{\beta}] = 0, \\
[\hat{\tau}_{I}^{\alpha}, \hat{\tau}_{J}^{\beta}] = \hbar \delta^{\alpha\beta} \delta_{IJ}, \\
[\hat{\chi}^{\alpha}, \hat{\tau}_{I}^{\beta}] = 0, \\
[\hat{\dot{\chi}}^{\alpha}, \hat{\tau}_{I}^{\beta}] = 0,
\]

which become identical with the canonical quantization relations (4) if one only considers the momenta $\pi^{\alpha}$ conjugate to $\chi^{\alpha}$ and identifies them with $\dot{\chi}^{\alpha}$.

### 4 Symmetries of the $\pi$SUSY Oscillator

Setting $\tau_{I}^{0} = \tau_{I}^{1}$ and $\chi^{0} = \chi^{1}$ in (25), one recovers the Lagrangian for the SUSY oscillator (16). This may be used as a hint to seek similar symmetries for the $\pi$SUSY oscillator.

Following this hint, consider the $\pi$-SUSY transformation:

\[
\delta \chi^{\alpha} = i \tau_{J}^{\alpha} \delta \gamma_{J}, \\
\delta \tau_{I}^{\alpha} = (\delta_{IJ} \dot{\chi}^{\alpha} + \omega \epsilon_{IJ} \chi^{\alpha}) \delta \gamma_{J},
\]
where $\delta \gamma_J$ are the “infinitesimal” analogs of $\gamma_J$ of Eq. (10). It is not difficult
to check that the action functional and therefore the dynamical equations
remain invariant under this transformation. Indeed, one finds:

$$
\delta L \propto \frac{d}{dt}(\dot{\chi}^\alpha \tau_I^\alpha - \omega \epsilon_{IJ} \tau_I^\alpha \chi^\alpha) \delta \gamma_J .
$$

(29)

Thus the corresponding conserved charges have the form:

$$
Q^1_J = \lambda (\chi^\alpha \tau_I^\alpha - \omega \epsilon_{IJ} \tau_I^\alpha \chi^\alpha) .
$$

(30)

Here the superscript “1” is placed for later use and $\lambda$ is a non-zero numerical
coefficient.

In the remainder of this paper, we shall set $\hbar = 1$ for simplicity.

The quantum analog of $Q^1_J$ with an appropriate normalization is given
by:

$$
\hat{Q}^1_J := \hat{\chi}^\alpha \hat{\tau}_I^\alpha - \omega \epsilon_{IJ} \hat{\chi}^\alpha \hat{\tau}_I^\alpha .
$$

(31)

In view of the paracommutation relations (26), it is not difficult to check
that $Q^1_J$ generate the transformations (27) and (28), i.e.,

$$
[[\hat{\chi}^\alpha, \hat{Q}^1_J \delta \gamma_J]] = i \hat{\tau}_I^\alpha \delta \gamma_J = \delta \hat{\chi}^\alpha ,
$$

(32)

$$
[[\hat{\tau}_I^\alpha, \hat{Q}^1_J \delta \gamma_J]] = (\delta_{IJ} \hat{\chi}^\alpha + \omega \epsilon_{IJ} \hat{\chi}^\alpha) \delta \gamma_J = \delta \hat{\tau}_I^\alpha ,
$$

(33)

and that they satisfy the defining algebra of SQM, namely:

$$
\{\hat{Q}^1_I, \hat{Q}^1_J\} = 2 \delta_{IJ} \hat{H} .
$$

(34)

Note also that $\hat{Q}^1_I$ are self-adjoint operators by construction (31).
Another important point in handling \((p = 2)\) para-dynamical systems is that the Green components are not the physical dynamical variables. In other words, one must be able to express all physical quantities in terms of the variables, \(x, \dot{x}, \psi_I,\) and \(\dot{\psi}_I\). This also applies to the \(Q^1_I\). In fact, one can show that:

\[
\hat{Q}^1_J = \frac{1}{2}\{\delta_{JK}\dot{x} - \omega\epsilon_{JK}\dot{x}, \dot{\psi}_K\}.
\]

(35)

Here use is made of the identities:

\[
\hat{x}^\alpha \hat{\tau}^\alpha_J = \frac{1}{2}\{\dot{x}, \dot{\psi}\}, \quad \hat{\chi}^\alpha \hat{\tau}^\alpha_J = \frac{1}{2}\{\dot{x}, \dot{\psi}_J\}.
\]

(36)

The \(\pi\)SUSY transformations (27) and (28) mix the Green components \(\chi^\alpha\) and \(\tau^\alpha_J\) with the same Green index \(\alpha\). Since the Green components are not physical quantities, there must be no difference between say \(\tau^0_J\) and \(\tau^1_J\). This suggests the possibility of symmetry transformations which mix \(\chi^\alpha\) with \(\tau^{\alpha+1}\). Here the values of the Green indices is taken in \(\mathbb{Z}_2\), i.e., they are calculated modulo 2. The following is such a symmetry transformation:

\[
\delta \chi^\alpha = -i\tau^{\alpha+1}_J \delta \gamma_J,
\]

\[
\delta \tau^\alpha_J = (\delta_{IJ}\chi^{\alpha+1} + \omega\epsilon_{IJ}\chi^{\alpha+1}) \delta \gamma_J.
\]

(37)

(38)

The associated conserved charges to this symmetry are given by

\[
Q^2_J = \lambda' (\tau^\alpha_J \hat{\chi}^{\alpha+1} - \omega\epsilon_{JK}\hat{\tau}^\alpha_K \hat{\chi}^{\alpha+1}),
\]

(39)

where the summation over \(\alpha\) is understood. Quantizing the system and taking:

\[
\hat{Q}^2_J := i(\hat{\tau}^\alpha_J \hat{\chi}^{\alpha+1} - \omega\epsilon_{JK}\hat{\tau}^\alpha_K \hat{\chi}^{\alpha+1}),
\]

(40)

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one obtains another set of self-adjoint $\pi$SUSY charges. They generate the transformations (37) and (38) and are expressed in terms of the physical variables $x$ and $\psi$ according to:

$$\hat{Q}_J^2 = \frac{-i}{2} [\delta_{JK} \hat{x} - \omega \epsilon_{JK} \hat{x}, \hat{\psi}_K] .$$  \hfill (41)

Here use is made of the identities:

$$\tau^\alpha_I \hat{x}^{\alpha+1} = \frac{1}{2} [\hat{\psi}_I, \hat{x}] , \quad \tau^\alpha_I \hat{\psi}^{\alpha+1} = \frac{1}{2} [\hat{\psi}_I, \hat{x}] .$$  \hfill (42)

Furthermore, the superalgebra relation:

$$\{ \hat{Q}_I^2, \hat{Q}_J^2 \} = 2 \delta_{IJ} \hat{H} ,$$  \hfill (43)

also holds.

The next natural step in the study of the symmetries of the $\pi$SUSY oscillator is to investigate the algebraic properties of both types of $\pi$SUSY’s. Proceeding in this direction, one finds:

$$[[\hat{Q}_J^a, \hat{Q}_K^b]] = \{ \hat{Q}_J^a, \hat{Q}_K^b \} = 2 \delta_{JK} \delta^{ab} \hat{H} - 2 \epsilon^{ab} \epsilon_{JK} \hat{Q} ,$$  \hfill (44)

where

$$\hat{Q} := i \omega \hat{x}^{\alpha+1} + \frac{\omega}{2} \tau^\alpha_I \tau^\alpha_I$$  \hfill (45)

is another (self-adjoint) conserved charge.

Repeating this procedure, i.e., including $\hat{Q}$ in the set of the generators of symmetries and investigating the parabrack of $\hat{Q}$ and other generators, one obtaines

$$[[\hat{Q}_J^a, \hat{Q}]] = [\hat{Q}_J^a, \hat{Q}] = 0 .$$  \hfill (46)
Thus the superalgebra consisting of the generators of $\pi$SUSY of the $\pi$SUSY oscillator closes. Summarizing the superalgebra relations, one has:

$$[[\hat{Q}^a_J, \hat{H}]] = [[\hat{Q}^a_J, \hat{H}]] = 0$$

$$[[\hat{Q}^a_J, \hat{Q}^b_K]] = \{\hat{Q}^a_J, \hat{Q}^b_K\} = 2\delta_{JK}\delta^{ab}\hat{H} - 2\epsilon^{ab}\epsilon_{JK}\hat{Q},$$

$$[[\hat{Q}, \hat{H}]] = [[\hat{Q}, \hat{H}]] = 0 \quad (47)$$

$$[[\hat{Q}^a_J, \hat{Q}]] = [[\hat{Q}^a_J, \hat{Q}]] = 0.$$

The generators $Q^a_J$ behave as the “odd” elements of the super Lie algebra and $H$ and $Q$ as the “even” (central) elements.

The (central) charge $Q$ is also expressed in terms of the physical variables. One has:

$$Q = \frac{i\omega}{2}(\hat{x}\hat{x} - \hat{x}\hat{x}) + \frac{\omega}{2}\delta_{IJ}\hat{\psi}_I\hat{\psi}_J.$$

Here, one uses the following identities:

$$\delta_{IJ}\hat{\psi}_I\hat{\psi}_J = \hat{\tau}^a_\alpha\hat{\tau}^{\alpha+1}_a + 2, \quad \hat{x}\hat{x} - \hat{x}\hat{x} = 2\hat{\chi}^\alpha\hat{\chi}^{\alpha+1} + 2i.$$

One can also examine the symmetry transformations generated by $Q$. These are obtained by computing:

$$[[\hat{\chi}^\alpha, Q\delta\epsilon]] = \{\hat{\chi}^\alpha, Q\}\delta\epsilon = \omega\hat{\chi}^{\alpha+1}\delta\epsilon,$$

$$[[\hat{\tau}^\alpha_I, Q\delta\epsilon]] = \{\hat{\tau}^\alpha_I, Q\}\delta\epsilon = \omega\hat{\tau}^{\alpha+1}_I\delta\epsilon.$$

Thus:

$$\delta_Q\hat{\chi}^\alpha = \omega\hat{\chi}^{\alpha+1}\delta\epsilon, \quad \delta_Q\hat{\tau}^\alpha_I = \omega\hat{\tau}^{\alpha+1}_I\delta\epsilon.$$
Here $\delta \epsilon$ is an infinitesimal commuting parameter. In terms of the physical dynamical variables, one has:

\[ \delta Q_x = \omega x \delta \epsilon, \quad \delta Q^I = \omega \psi_I \delta \epsilon. \]

We would like to conclude this section by emphasizing the enormous advantage of using parabrackt \( \{ \} \) in performing the tedious computations necessary for establishing the superalgebra relations Eqs. (47). The details of these computations have been omitted due to the space limitations.

## 5 Degeneracy Structure of General SPQM

Let us first define SPQM:

**Definition:** Let \( \mathcal{H} \) be a \( \mathbb{Z}_2 \)-graded Hilbert space with grading involution \( \hat{\tau} \). Then a quantum mechanical system with \( \mathcal{H} \) as the Hilbert space and self-adjoint symmetry generators \( \hat{Q}_{J_n}, \hat{Q}_n, n = 1, \cdots N, I_n, a_n = 1, 2 \), and the Hamiltonian operator \( \hat{H} \) satisfying the super Lie algebra relations:

\[ [\hat{Q}_{J_n}, \hat{H}] = [\hat{Q}_n, \hat{H}] = [\hat{Q}_{J_n}, \hat{Q}_n] = 0 \]  
\[ \{\hat{Q}_{J_n}, \hat{Q}_{K_m}\} = \delta_{nm}(2\delta_{J_nK_n}\delta^{ab}\hat{H} - 2\epsilon_{ab\epsilon_JK_n}\hat{Q}_n), \]

and parity properties:

\[ \{\hat{\tau}, \hat{Q}_{J_n}\} = 0, \quad [\hat{\tau}, \hat{Q}_n] = [\hat{\tau}, \hat{H}] = 0, \]

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for all $I_n, a_n$ and $n = 1, \ldots, N$, is called a $(p = 2)$ - supersymmetric paraquantum mechanical (SPQM) system of type $N$.

In this section, we shall present a detailed analysis of the spectrum degeneracy structure of general $(p = 1)$-SPQM systems of type $N = 1$.

For $N = 1$ we suppress the index $n = 1$ and recover the super Lie algebra of the $\pi$SUSY oscillator, i.e., Eqs. (47). For simplicity we shall drop the hats and introduce the notation:

$Q_1 \equiv Q_1^1, \quad Q_2 \equiv Q_2^1, \quad Q_3 \equiv Q_1^2, \quad Q_4 \equiv Q_2^2$.

Then Eqs. (47) are written as:

\begin{align*}
Q_i^2 &= H, \quad (51) \\
\{Q_1, Q_2\} &= 0, \quad (52) \\
\{Q_1, Q_3\} &= 0, \quad (53) \\
\{Q_1, Q_4\} &= -2 Q, \quad (54) \\
\{Q_2, Q_3\} &= 2 Q, \quad (55) \\
\{Q_2, Q_4\} &= 0, \quad (56) \\
\{Q_3, Q_4\} &= 0, \quad (57) \\
\left[Q_i, Q\right] &= 0, \quad (58)
\end{align*}

where $i = 1, 2, 3, 4$.

Next, we use the simultaneous eigenstate vectors $|E, q_1, q\rangle$, with $E, q_1, q \in \mathbb{R}$, of $H, Q_1$ and $Q$ to span the Hilbert space. We shall assume that these
state vectors form an orthonormal basis and attempt to represent all the relevant operators in this basis. These properties are summarized by the following set of relations:

\[ H |E, q_1, q\rangle = E |E, q_1, q\rangle, \quad Q_1 |E, q_1, q\rangle = q_1 |E, q_1, q\rangle, \quad (59) \]
\[ Q |E, q_1, q\rangle = q |E, q_1, q\rangle, \quad \langle E', q'_1, q'|E, q_1, q\rangle = \delta_{E'E} \delta_{q_1', q_1} \delta_{q'q}. \quad (60) \]

A simple consequence of Eq. (51) with \( i = 1 \), is that the energy spectrum is non-negative. Furthermore, for any energy level \( E \), one has:

\[ q_1 = \pm \sqrt{E}, \quad (61) \]
\[ |q_1, q\rangle = 0 \Leftrightarrow |-q_1, q\rangle = 0, \quad (62) \]
\[ Q_2 |q_1, q\rangle = C_2(q_1, q) |q_1, q\rangle, \quad C_2(q_1, q) \in \mathbb{C} - \{0\}, \quad (63) \]
\[ Q_3 |q_1, q\rangle = C_3(q_1, q) |q_1, q\rangle, \quad C_3(q_1, q) \in \mathbb{C} - \{0\}, \quad (64) \]

where use is made of Eqs. (51) – (53) and abbreviation \(|q_1, q\rangle\) is used for \(|E, q_1, q\rangle\). Enforcing Eq. (55), one finds:

\[ C_2(q_1, q) C_3(-q_1, q) + C_3(q_1, q) C_2(-q_1, q) = 2q. \quad (65) \]

Then by acting both sides of Eqs. (51), with \( i = 2, 3 \), on \(|q_1, q\rangle\), one has:

\[ C_2(q_1, q) C_2(-q_1, q) = E, \quad C_3(q_1, q) C_3(-q_1, q) = E. \quad (66) \]

Next, we calculate:

\[ E = (\langle q_1, q | Q_2 (Q_2 | q_1, q\rangle) = C_2(q_1, q)^* C_2(q_1, q). \]
A similar relation holds for $C_3$. These relations together with Eqs. (66) imply:

$$C_2(\pm q_1, q) = \sqrt{E} e^{\pm i\alpha_2(q)} , \quad C_3(\pm q_1, q) = \sqrt{E} e^{\pm i\alpha_3(q)} . \quad (67)$$

Combining the latter equations with Eq. (65), one is led to:

$$\frac{C_2(q_1, q)}{C_3(q_1, q)} + \frac{C_3(q_1, q)}{C_2(q_1, q)} = \frac{2q}{E} . \quad (68)$$

Eqs. (67) and (68), in turn, yield:

$$\cos[\alpha_2(q) - \alpha_3(q)] = \frac{q}{E} . \quad (69)$$

Next, we act both sides of Eqs. (54), (56), and (57) on $|q_1, q\rangle$ on the left. This gives rise to:

$$Q_1 Q_4 |q_1, q\rangle = -q_1 Q_4 |q_1, q\rangle - 2q |q_1, q\rangle , \quad (70)$$

$$Q_2 Q_4 |q_1, q\rangle = -\sqrt{E} e^{i\alpha_2} Q_4 |q_1, q\rangle , \quad (71)$$

$$Q_3 Q_4 |q_1, q\rangle = -\sqrt{E} e^{i\alpha_3} Q_4 |q_1, q\rangle . \quad (72)$$

To pursue our analysis further, we express the action of $Q_4$ on the basic kets $|q_1, q\rangle$ as the following linear combination:

$$Q_4 |q_1, q\rangle =: a(q_1, q)|\sqrt{E}, q\rangle + b(q_1, q)|-\sqrt{E}, q\rangle . \quad (73)$$

where $a$ and $b$ are complex numbers $a \text{ priori}$ depending on $q_1, q$ and of course $E$. Substituting this expression in Eq. (70), one finds:

$$a(q_1 = \sqrt{E}, q) = -\frac{q}{\sqrt{E}} \quad a(q_1 = -\sqrt{E}, q) = 0 . \quad (74)$$
Repeating the same procedure for Eqs. (71) and (72), and performing the simple algebra, one finally obtains:

\[ b(q_1 = \sqrt{E}, q) = 0, \quad b(q_1 = -\sqrt{E}, q) = -a(q_1 = \sqrt{E}, q) = \frac{q}{\sqrt{E}} \] (75)

\[ e^{i\alpha_2(q)} = \pm 1 = e^{i\alpha_3(q)}. \] (76)

The last pair of equations together with Eq. (53) imply:

\[ q = E\eta, \quad \eta = \pm 1. \] (77)

Having obtained all the unknowns of our construction and appealing to the gauge freedom of the phases of the initial basic eigenstate vectors – which allows us to set, say, \( \alpha_2 = 0 \) so that \( e^{i\alpha_3} = \eta \) – we are in a position to present matrix representations of all the charges. However, before presenting these representations, we would like to remark that although \( |\sqrt{E}, q\rangle \neq 0 \iff | - \sqrt{E}, q\rangle \neq 0 \), this relation does not imply that \( |\pm \sqrt{E}, q\rangle \neq 0 \) for some \( q \), i.e., in general it may be the case that for some values of \( E \) the state vectors corresponding to either \( q = +E \) or \( q = -E \) vanish. In this case \( E \) will be doubly degenerate. Otherwise it will be quadruply degenerate. For the latter case the symmetry generators are represented by:

\[ Q_1|_{H_E} = \sqrt{E} \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{E}} & 0 \\ 0 & -1 & \frac{1}{\sqrt{E}} & 0 \\ 1 & 0 & \frac{1}{\sqrt{E}} & 0 \\ 0 & -1 & \frac{1}{\sqrt{E}} & 0 \end{pmatrix} \]

\[ Q_2|_{H_E} = \sqrt{E} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \]
Here we have identified:

\[ | \sqrt{E}, E \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad | -\sqrt{E}, E \rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \]

\[ | \sqrt{E}, -E \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad | -\sqrt{E}, -E \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \]

the empty blocks consist of vanishing entries, and \( \mathcal{H}_E \) denotes the degeneracy Hilbert space associated with the energy \( E > 0 \).

In view of Eqs. (50), we can also write down the matrix representation of the involution (chirality) operator in this basis. The result is given by

\[ \tau | \mathcal{H}_E \rangle = \begin{pmatrix} 0 & -i\epsilon_1 \\ i\epsilon_1 & 0 \\ 0 & -i\epsilon_2 \\ i\epsilon_2 & 0 \end{pmatrix}, \]
where $\epsilon_1, \epsilon_2 = \pm 1$.

It is an easy exercise to diagonalize the chirality involution(s) and to find out that in the diagonal form it has the form:

$$\tau|_{\mathcal{H}_E} = \text{diag}(1, -1, 1, -1).$$ \hfill (78)

This implies that the quadruply degenerate (positive) energy levels involve two odd (parafermionic) and two even (parabosonic) state vectors.

The representations of the symmetry generators and the involution operator for the doubly degenerate energy levels are given by either of the upper-left or lower-right blocks in the above list of matrix representations, according to whether $|\pm \sqrt{E}, -E\rangle = 0$ or $|\pm \sqrt{E}, +E\rangle = 0$, respectively. The situation is analogous to the ordinary supersymmetric case, \cite{5}.

The following lemma summarizes our results concerning $(p = 2)$-SPQM:

**Lemma 1:** The energy spectrum of any $(p = 2)$ supersymmetric paraquantum system is non-negative. The zero-energy eigenvalue, if exists, is non-degenerate\footnote{This is true provided that other quantum numbers are not present.}. The positive energy levels are either doubly or quadruply degenerate. They consist of pairs of odd and even parity eigenstates.

Moreover, one can define the Witten index according to

$$\text{index}_{\text{Witten}} := \text{trace}(\tau),$$
and use Eq. (78) to conclude that it counts the difference of the number of even and odd zero-energy states, and that it is a topological invariant.

### 6 Hilbert Space Structure of the $\pi$SUSY Oscillator

Ref. [10] offers an analysis of the energy eigenstates of the one-dimensional parabose oscillator of arbitrary order $p$. In the following, we shall use the results of [10], with $p = 1$, to construct a complete set of eigenstate vectors for the $\pi$SUSY oscillator.

The Hilbert space of the one-dimensional ($p = 1$) $\pi b$ oscillator is constructed using the following set of orthonormal energy eigenstate vectors:

$$|n\rangle := \frac{1}{\sqrt{2^n \left[ \frac{n}{2} \right]! \left[ \frac{n+1}{2} \right]!}} a^{|n\rangle},$$

(79)

where $a^\dagger$ and $|0\rangle$ are the $\pi b$ creation operator and the vacuum (ground) state, respectively, and $[k]$ stands for the largest integer smaller than or equal to $k \in \mathbb{R}$. One also has:

$$H^0|n\rangle = (n + 1)\omega,$$

(80)

$$a|n\rangle = \sqrt{2^n \left[ \frac{n}{2} \right] + 1}|n - 1\rangle,$$

(81)

$$a^\dagger|n\rangle = \sqrt{2^n \left[ \frac{n}{2} \right] + 2}|n + 1\rangle.$$  

(82)

For the $\pi$SUSY oscillator, one has also the $\pi f$ creation and annihilation operators. These have the property that $a^3 = 0$. So there is an apparent
triple grading intrinsic to the \((p = 2)\) πf operators. This has been used quite often in the context of parasupersymmetry. In the following we shall demonstrate that this is not the case for the πSUSY oscillator as one might expect in view of the treatment of Sec. 6.

It turns out that the following energy eigenstates form an orthonormal basis for the Hilbert space:

\[
|n, 1\rangle := \frac{1}{\sqrt{2^n n!}} a^\dagger n |0\rangle, \quad (n \geq 0)
\]

\[
|n, 2\rangle := \frac{1}{\sqrt{2^{n-1} n! (n-1)!}} \alpha^2 a^\dagger (n-1) |0\rangle, \quad (n \geq 1)
\]

\[
|n, 3\rangle := \frac{1}{\sqrt{2^{n-2} n! (n-2)!}} \alpha a^\dagger (n-2) |0\rangle, \quad (n \geq 2)
\]

\[
|n, 4\rangle := \frac{1}{\sqrt{2^{n-2} n! (n-2)!}} \alpha a^\dagger \alpha a^\dagger (n-2) |0\rangle, \quad (n \geq 2)
\]

(83)

Note that the state vector \(|n, 1\rangle\) is the same as \(|n\rangle\) of Eq (79). To establish the orthonormality of \(\{|n, a\rangle : a = 1, 2, 3, 4\}\), one needs to use the following set of paracommutation relations:

\[
\alpha \alpha^\dagger a^\dagger = -a^\dagger \alpha^\dagger \alpha + 2a^\dagger, \quad (84)
\]

\[
\alpha \alpha^\dagger 2 = -\alpha^\dagger 2 \alpha + 2\alpha^\dagger, \quad (85)
\]

\[
a a^\dagger \alpha = \alpha^\dagger a a + 2\alpha^\dagger, \quad (86)
\]

\[
\alpha a a^\dagger = a^\dagger a \alpha + 2\alpha, \quad (87)
\]

\[
\alpha a \alpha^\dagger = -a^\dagger a \alpha, \quad (88)
\]

and the identity:

\[
\alpha a^{2n} |0\rangle = 0. \quad (89)
\]

Relations (84)–(89) are most easily proved in the Green representation.
Furthermore, it is not difficult to check that indeed $|n, a\rangle$ are energy eigenvectors, i.e.,

$$H |n, a\rangle = E_n |n, a\rangle , \quad \text{with} \quad E_n := n\omega . \quad (90)$$

Finally, it is possible to show that $|n, a\rangle$ form a complete set of state vectors. This involves some lengthy algebraic manipulations. The completeness of $\{|n, a\rangle\}$ results from the following set of relations:

$$
\begin{align*}
a |n, 1\rangle &= \sqrt{2}\left[\frac{n}{2}\right] + 1 |n - 1, 1\rangle , \\
a^\dagger |n, 1\rangle &= \sqrt{2}\left[\frac{n}{2}\right] + 2 |n + 1, 1\rangle , \\
\alpha |n, 1\rangle &= 0 \\
\alpha^\dagger |n, 1\rangle &= \sqrt{2} |n + 1, 2\rangle , \\
a |n, 2\rangle &= \sqrt{\frac{(n - 1)/2(2[n/2] - 1)}{[n/2]}} |n - 1, 5\rangle , \\
a^\dagger |n, 2\rangle &= \sqrt{2}\left[\frac{n}{2}\right] + 2 |n + 1, 1\rangle , \\
\alpha |n, 2\rangle &= \sqrt{2} |n - 1, 1\rangle , \\
\alpha^\dagger |n, 2\rangle &= \sqrt{2} |n + 1, 3\rangle , \\
a |n, 3\rangle &= -\sqrt{2}[n/2] |n - 1, 3\rangle , \\
a^\dagger |n, 3\rangle &= -\sqrt{2}[n/2] |n + 1, 3\rangle , \\
\alpha |n, 3\rangle &= \sqrt{2} |n - 1, 2\rangle , \\
\alpha^\dagger |n, 3\rangle &= 0 , \\
a |n, 4\rangle &= \sqrt{\frac{(n - 1)/2(2[n/2] - 1) + 2}{[n/2]}} |n - 1, 2\rangle ,
\end{align*}
$$

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\[ a^\dagger |n, 4\rangle = \sqrt{2[(n+1)/2]} |n+1, 2\rangle , \]
\[ \alpha |n, 4\rangle = 0 , \]
\[ \alpha^\dagger |n, 4\rangle = 0 , \]

where in addition to Eqs. (81)–(88), the paracommutation relations:

\[ a \alpha \alpha^\dagger = a^\dagger \alpha^\dagger a , \]
\[ a \alpha^2 = -\alpha^2 a , \]
\[ \alpha a^\dagger \alpha^\dagger = -\alpha^\dagger a^\dagger \alpha , \]
\[ \alpha^\dagger a^\dagger \alpha^\dagger = 0 , \]

are also used.

To demonstrate the method of proof of such relations using the Green representation, a proof of the last equation is offered in the following. First note that

\[ a = \sum_{\alpha=0}^{1} \zeta^{\alpha 0} , \quad \alpha = \sum_{\beta=0}^{1} \zeta^{\beta 1} , \]
\[ \llbracket \zeta^{\alpha \mu} , \zeta^{\beta \nu} \rrbracket = 0 , \quad \llbracket \zeta^{\alpha \mu} , \zeta^{\beta \nu^\dagger} \rrbracket = \delta^{\alpha \beta} \delta^{\mu \nu} . \quad (91) \]

Here use is made of Eqs. (81)–(88). Next, one has:

\[ a^\dagger a^\dagger \alpha^\dagger = \sum_{\alpha,\beta,\gamma} \zeta^{\alpha 1^\dagger} \zeta^{\beta 0^\dagger} \zeta^{\gamma 1^\dagger} \]
\[ = \sum_{\alpha,\beta,\gamma} (-1)^{1+\beta+\gamma} \zeta^{\beta 0^\dagger} \zeta^{\gamma 1^\dagger} \zeta^{\alpha 1^\dagger} \]
\[ = -\sum_{\alpha,\beta,\gamma} \zeta^{\gamma 1^\dagger} \zeta^{\beta 0^\dagger} \zeta^{\alpha 1^\dagger} \]
\[ = -\alpha^\dagger a^\dagger \alpha^\dagger = 0 , \quad (92) \]
where in the second and third equalities use is made of Eqs. (5) and (9).

This concludes our investigation of the energy eigenstates of the πSUSY oscillator. We summarize the results of this section in the form of the following lemma:

**Lemma 2:** The energy spectrum of the πSUSY oscillator consists of a zero energy non-degenerate ground state (represented by \(|n = 0, 1\rangle\)), a doubly degenerate first excited state (level) of energy \(E_1 = \omega\) (with state vectors \(|n = 1, 1\rangle\) and \(|n = 1, 2\rangle\)), and higher excited states of \(E_n = n\omega \ (n \geq 2)\) which are quadruply degenerate (with state vectors \(|n, a\rangle, a = 1, 2, 3, 4\).)

This confirms our general results of Sec. 6.

### 7 Conclusion

There are dynamical systems involving \((p = 2)\) parastatistical degrees of freedom and symmetry transformations which mix the parabose and parafermi dynamical variables. The mixing which signifies a parabose – parafermi supersymmetry is shown to be present because of the non-trivial algebraic properties of such variables.

Having established the meaningfulness of the parabose – parafermi supersymmetry (πSUSY), one can investigate its relation with the ordinary (bose – fermi) supersymmetry and the parasupersymmetry. The simple example of an oscillator consisting of a parabosonic and a parafermionic sector is used
to demonstrate the nature of $\pi$SUSY. This oscillator possesses two ordinary supersymmetries. The study of the combined set of generators of these supersymmetries leads to the introduction of a central charge. Thus, it seems that there is no direct relation between parabose – parafermi supersymmetry and parasupersymmetry.

The oscillator considered in this article also serves as a useful example to demonstrate the practical importance of the parabrackt introduced in \cite{1}. Moreover, it is remarkable to check that indeed all the conserved charges depend on the physical dynamical variables and not on their Green components. This is quite non-trivial, for all the calculations are performed using the Green components. In view of these observations, one may conclude that there is no anomalous phenomena stemming from the unusual parastatistical nature of the $(p = 2)$ dynamical variables. In fact, it is shown that for example the $\pi$SUSY oscillator has a larger symmetry than the ordinary SUSY oscillator.

Another interesting observation regarding the symmetries of the $\pi$SUSY oscillator is that \textit{a priori} there is no parity associated with the quantities (polynomials) constructed out of the parastatistical variables, nevertheless the conserved charges and the Hamiltonian do possess parities, and they do form a super Lie algebra. This may be seen as the primary reason why one does not need trilinear algebraic relations between the symmetry generators. The latter has been shown \cite{3} to be unavoidable for an oscillator consisting
of ordinary bosons and \( (p = 2) \) parafermions.

The super Lie algebra associated with the \( \pi \text{SUSY} \) oscillator may be considered in a more general context. This line of reasoning leads to the introduction of supersymmetric paraquantum mechanics. The defining superalgebra of SPQM determines the degeneracy structure of the energy spectrum. The matrix representation of the conserved charges reveals the differences and the similarities between SPQM and SQM. The Witten index can also be defined for SPQM. It possesses the topological invariance property and signifies the breaking of \( \pi \text{SUSY} \), similarly to the ordinary SQM case.

The Hilbert space structure of the \( \pi \text{SUSY} \) oscillator is also analyzed in detail. A remarkable observation is that the presence of \( (p = 2) \) parafermi operators does not lead to a triple grading of the spectrum degeneracy. In fact, the general results obtained in the context of supersymmetric paraquantum mechanics are shown to be valid for the oscillator case. This serves as an independent check on the results obtained in Sec. 6.

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