CP violation for four generations of quarks

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We discuss the generalization of the Jarlskog condition of CP conservation for the case of the Standard Model with 4 quark generations. We express this condition in terms of the 3 Jarlskog invariants of the CKM matrix. Next we present the test for the existence of the 4-th quark generation in terms of the Jarlskog invariants involving only known particles.

I. INTRODUCTION

The Standard Model [1–9] (SM) is constructed in such a way that it reproduces all known phenomenological information about the spectrum and interactions of elementary particles. When one considers an extension of the SM, then certain properties of the construction are automatically generalized, but frequently generalizations are not simple and require a careful analysis.

Recently, a new evidence for a fourth, sterile neutrino, has appeared [10]. This has again opened a possibility for a fourth generation [11, 12] of quarks and leptons in the SM, though its nature, if it exists, might not be clear and it may also require an extension of the Higgs sector [13]. It should be also noted that the Standard Model with four generations provides sufficient CP violation to explain the Universe baryon asymmetry [14].

In this paper we will concentrate our attention on the conditions for the CP-conservation in the four generations SM (SM4). The general (necessary and sufficient) condition for the CP conservation has been formulated in [15, 16] and requires that there exists a unitary matrix \( U_L \) such that the quark mass matrices \( M_q \) fulfill the following conditions

\[
U_L^\dagger H_q U_L = H_q^*, \quad H_q = M_q M_q^\dagger, \quad q = u, d. \tag{1}
\]

Condition (1) is valid for any number of generations, but it is not expressed in terms of the observables. In Ref. [15, 16], there is a theorem that determines the equivalence of condition (1) with the vanishing of the imaginary part of the products of powers of the \( H_q \) matrices, which are observables (quark masses and elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix). For the SM with 3 generations (SM3) condition (1) is equivalent to vanishing of the following expression

\[
\text{Im}(\text{Tr}(H_u^2 H_d H_u H_d^2)) = (m_t^2 - m_c^2)(m_c^2 - m_u^2)(m_b^2 - m_s^2)
\times (m_b^2 - m_s^2)(m_b^2 - m_d^2)(m_s^2 - m_d^2) \text{Im}(V_{ud} V_{cd}^* V_{cs} V_{us}^*) = 0. \tag{2}
\]

Eq. (2) is the necessary and sufficient condition for the CP conservation in the SM with 3 generations. This condition states that there is no CP violation if the Jarlskog invariant \( \text{Im}(V_{ud} V_{cd}^* V_{cs} V_{us}^*) \) vanishes or if any two masses within up or down quark sectors are equal. The equality of masses within the up or down sectors implies new properties of the CKM matrix: if one pair of quark masses within a multiplet were equal (e.g., \( m_u = m_c \)) then the CKM matrix would depend on two angles only (no CP violation) and if all quark masses of the same type were equal (e.g., \( m_u = m_c = m_t \)), then the CKM matrix would be an identity matrix. Such is the mechanism of CP conservation for the case of equal masses, so the necessary and sufficient condition of CP conservation for 3 generations in the SM3 model is only the vanishing of the Jarlskog invariant. What should be stressed, is that for the case of two equal quark masses the CP conservation requires that the equality of masses must be exact and this would require some kind of fine tuning and would probably be a consequence of new conservation laws.

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On the other hand the conditions for CP conservation for the SM4, in terms of observables, are more involved. It turns out that the necessary and sufficient conditions for CP conservation in case of non-degenerate masses consist in the vanishing of the following expressions

\[ \text{Im}(\text{Tr}(H_u^2 H_d H_u H_d^2)) = 0, \quad \text{Im}(\text{Tr}(H_d^2 H_u H_d H_u^2)) = 0 \]

\[ \text{Im}(\text{Tr}(H_u^2 H_d H_u H_d^2)) = 0, \quad \text{Im}(\text{Tr}(H_u H_d H_u H_d^2 H_d^2 H_u^2)) = 0 \]  

(3)

Another set of conditions for the CP conservation for the 4 generations SM was given in Ref [19] and reads

\[ I_1 = \text{Im}(\text{Tr}(H_u^2 H_d H_u H_d^2)) = 0 \]
\[ I_2 = \text{Im}(\text{Tr}(H_u^3 H_d H_u H_d^2)) = 0 \]
\[ I_3 = \text{Im}(\text{Tr}(H_u^4 H_d H_u H_d^2 H_d^2 H_u^2)) = 0 \]
\[ I_4 = \text{Im}(\text{Tr}(H_u^5 H_d H_u H_d^2 H_u^2 H_d H_u H_d)) = 0 \]
\[ I_5 = \text{Im}(\text{Tr}(H_u^3 H_d H_u H_d^3)) = 0 \]
\[ I_6 = \text{Im}(\text{Tr}(H_u^4 H_d H_u H_d^3)) = 0 \]
\[ I_7 = \text{Im}(\text{Tr}(H_u^5 H_d H_u H_d^3 H_u^3 H_u^2 H_d^3)) = 0 \]
\[ I_8 = \text{Im}(\text{Tr}(H_u^2 H_d H_u H_d^5 H_d^2 H_u H_d^3 H_u H_d^4 H_u H_d^3 H_u H_d^2)) = 0. \]  

(4)

All invariants in (3) and \( I_k \) in (1) can be written as sums of expressions that contain the function \( G(i, j; k, l) \) (where \( m_u \) and \( m_d \) are the up and down quark masses, respectively)

\[ G(i, j; k, l) = -(m_u^2 - m_d^2)(m_u^2 - m_u^2)(m_u^2 - m_u^2)(m_u^2 - m_u^2)(m_u^2 - m_u^2)(m_u^2 - m_u^2)(m_u^2 - m_u^2) \text{Im}(V_{ik} V_{jk} V_{jl} V_{il}). \]

(5)

multiplied by polynomials of squares of quark masses. From Eq. (3) one can see that the Jarlskog type invariants also play an important role in the conditions for the CP conservation for the SM4.

The CKM matrix in the SM4 has three phases and the vanishing of these 3 phases is also a condition for CP conservation. The most remarkable fact is that there are 8 relations in Eq. (3), which put conditions on the 3 phases. It thus seems that 8 conditions in Eq. (1) contain a mixture of fine tuned conditions and also conditions for the CKM matrix only. The fine tuned conditions imply the vanishing of the phases, so eventually the conditions for the CKM matrix are important. In this paper we will find the necessary and sufficient conditions for the CP conservation in the SM4 in terms of the Jarlskog invariants only.

II. REPHASING TRANSFORMATIONS

In our study we will consider only the case of non-degenerate quark masses within the up and down quarks. In such a case the SM is invariant under the rephasing transformation of the quark fields (and not a bigger unitary group) and this implies that the CKM matrix is determined up to the rephasing transformation. From this it follows that the 3 generations CKM matrix has 1 independent phase and 3 angles, while the 4 generation CKM matrix depends on 3 independent phases and 6 angles. The vanishing of these phases is a sufficient and necessary condition for the CP conservation. In general the CP-conservation means that for \( n \)-generations there must exist two diagonal matrices \( D_L = \text{Diag}(e^{i\phi_1}, e^{i\phi_2}, \ldots, e^{i\phi_n}) \) and \( D_R = \text{Diag}(1, e^{i\phi_1}, \ldots, e^{i\phi_{n-1}}) \), such that the following equation is fulfilled

\[ \text{Im}(D_L V_{\text{CKM}} D_R) = 0. \]

(6)

One might consider Eq. (3) as an analogue of condition (1), but the mathematical requirements for Eqs. (1) and (3) are different since Eq. (3) is applied only to one matrix, \( V_{\text{CKM}} \), and the matrices \( D_L \) and \( D_R \), which are to be determined, are diagonal unitary matrices. Step by step we will find the conditions on the CKM4 matrix, stemming from the CP-conservation.

As a first step we will find conditions for a general \( n \times n \) matrix to be equivalent to a real matrix as a result of a rephasing transformation. These conditions are given by the following

**Theorem 1.** A \( n \times n \) complex matrix \( M = m_{ij} \neq 0 \) is equivalent to a real \( n \times n \) matrix \( \tilde{M} = |m_{ij}| \) by the following rephasing transformation, \( M = D_L \tilde{M} D_R \), iff

\[ \text{Im}(m_{11} m_{kl} m_{*1} m_{*k}^*) = 0, \quad k, l = 2, \ldots, n. \]  

(7)
Here

\[ D_L = \text{Diag}(e^{i\phi_1}, e^{i\phi_2}, \ldots, e^{i\phi_n}), \quad D_R = \text{Diag}(1, e^{i\psi_1}, \ldots, e^{i\psi_{n-1}}) \]

with \( \phi_i \) and \( \psi_i \) being real phases.

The proof of this theorem is given in the Appendix. From Theorem 1 we see that for a \( n \times n \) arbitrary matrix there are \((n-1)^2\) conditions for the rephasing equivalence to a real matrix, i.e., for a \( 3 \times 3 \) matrix there are 4 conditions and for a \( 4 \times 4 \) matrix there are 9 conditions. Also note that the conditions are expressed in terms of the rephasing invariants of Jarlskog type, \( \text{Im}(m_{ij}m_{lk}^*m_{ik}m_{lj}^*) \), \( i < k, j < l \), and do not require calculation of the phases \( \phi_i \) and \( \psi_i \). One should also see that there is an additional freedom in condition (7): instead of \( m_{11} \) one can choose an arbitrary fixed element \( m_{i_0j_0} \) and then \( k \in 1, 2, \ldots, n \) and \( k \neq i_0 \) and \( l \neq j_0 \). For a general \( n \times n \) matrix there exist \( \left( \frac{n(n-1)}{2} \right)^2 \) Jarlskog invariants (see Appendix), and Theorem 1 determines that only \((n-1)^2\) conditions are necessary for the remaining Jarlskog invariants to vanish. This implies the existence of relations between Jarlskog invariants for \( n > 2 \).

Condition (6) for the CP-conservation is imposed on the CKM matrix, which is unitary while Theorem 1 gives its equivalence to a real matrix for an arbitrary matrix. The unitarity of a matrix reduces the number of conditions for the CP-conservation. This happens, because the unitarity of a matrix (in particular CKM) imposes relations between the Jarlskog invariants.

The next step in our analysis is to consider the relations between the Jarlskog invariants for a unitary matrix, which are given in the following

**Theorem 2.** The unitarity of the \( n \times n \) matrix \( V \) implies the following set of linear relations between the Jarlskog invariants

\[
\text{Im} \left( \sum_{l \neq i} V_{ij}^* V_{ik} V_{lk}^* V_{kj}^* \right) = 0, \quad j < k, \quad i, j = 1, \ldots, n, \quad k = 2, \ldots, n \tag{8a}
\]

\[
\text{Im} \left( \sum_{l \neq i} V_{ji}^* V_{jl} V_{lk}^* V_{kl}^* \right) = 0, \quad j < k, \quad i, j = 1, \ldots, n, \quad k = 2, \ldots, n. \tag{8b}
\]

The proof of the theorem is given in the Appendix. Relations (8a) follow from the orthogonality of the rows of the matrix \( V \) and relations (8b) follow from the orthogonality of the columns. There are altogether \( n^2(n-1) \) relations of the type (8) between \( \left( \frac{n(n-1)}{2} \right)^2 \) Jarlskog invariants. Not all relations (8) are linearly independent, in the case \( n = 3 \) remains 1 independent invariant and for \( n = 4 \) remain 4 independent invariants.

### III. CONDITIONS FOR THE CP CONSERVATION

**General case**

Let us start from the case of the \( n \times n \) CKM matrix \( V \). From Theorem 1 the conditions for the CKM matrix to be real after the rephasing are

\[
\text{Im}(V_{11}V_{ij}^*V_{1j}^*) = 0, \quad i, j = 2, \ldots, n. \tag{9}
\]

Eq. (9) requires the vanishing of the \((n-1)^2\) Jarlskog invariants for the CP conservation.

The CKM matrix is unitary and from Theorem 2 we know that the Jarlskog invariants are linearly dependent and those from Eq. (9) fulfill the relations

\[
\text{Im} \left( \sum_{l=2}^{n} V_{1l}^* V_{lk}^* V_{1k} \right) = 0, \quad k = 2, \ldots, n, \tag{10a}
\]

\[
\text{Im} \left( \sum_{l=2}^{n} V_{1l}^* V_{kl}^* V_{1k}^* \right) = 0, \quad k = 2, \ldots, n. \tag{10b}
\]

Eqs. (10) contain \( 2(n-1) \) relations, which are not linearly independent and there exist one relation between them: the sum of Eqs. (10a) is equal to the sum of Eqs. (10b). This means that Eqs. (10) reduce the number of conditions by...
2(n - 1) - 1. Thus after including the unitarity of the CKM matrix the number of conditions for the CP conservation is equal

\[(n - 1)^2 - 2(n - 1) = (n - 2)^2.\] \hspace{1cm} (11)

The number of conditions in Eq. (11) should be compared to the number of phases in the CKM matrix, what is done in Table I. From Table I one confirms the well known fact that for 3 generations the number of conditions is equal to the number of phases. For more generations the number of conditions is larger than the number of the phases. By the parameter counting these two numbers should be equal. One thus may ask the question: Is there a mechanism which makes that these two numbers should differ or there is a way to prove that these two numbers are equal? The answer is in the next subsection.

**CP violation for 4 quark generations**

To find conditions for the CP conservation for 4 generations we will use an explicit, slightly modified parameterization of the CKM4 matrix from [20, 21]. As we know the CKM4 matrix is parameterized by 6 angles \(\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}\) and 3 phases \(\beta_{13}, \beta_{14}, \beta_{24}\). We will not write here the full expressions for the CKM4 matrix, but we will only include the following three Jarlskog invariants (\(c_{ij} = \cos \alpha_{ij}\) and \(s_{ij} = \sin \alpha_{ij}\))

\[
\begin{align*}
\text{Im}(V_{11}V_{22}V_{12}^{*}V_{21}^{*}) &= -\sin \beta_{14}c_{13}c_{14}c_{24}s_{13}s_{14}s_{23}s_{24} \\
\text{Im}(V_{11}V_{23}V_{13}^{*}V_{21}^{*}) &= \cos \beta_{14} \sin \beta_{13}c_{12}c_{13}c_{14}c_{23}c_{24}s_{12}s_{14}s_{23}s_{24} \\
&\quad + \cos \beta_{13} \sin \beta_{14}c_{12}c_{13}c_{14}c_{23}c_{24}s_{12}s_{14}s_{23}s_{24} \\
&\quad + \sin \beta_{14}c_{12}c_{13}c_{14}c_{23}c_{24}s_{12}s_{14}s_{23}s_{24} \\
\text{Im}(V_{11}V_{32}V_{12}^{*}V_{31}^{*}) &= -\sin \beta_{24}c_{13}c_{14}c_{23}c_{24}s_{13}s_{14}s_{34} + \sin \beta_{14}c_{13}c_{14}c_{23}c_{24}s_{13}s_{14}s_{34}s_{34}. \hspace{1cm} (12a)
\end{align*}
\]

From Eqs. (12a) we obtain:

From Eq. (12a): \(\text{Im}(V_{11}V_{22}V_{12}^{*}V_{21}^{*}) = 0 \iff \beta_{14} = 0 \text{ or } \pi\),

From Eqs. (12a) and (12b): \(\text{Im}(V_{11}V_{22}V_{12}^{*}V_{21}^{*}) = 0 \text{ and } \text{Im}(V_{11}V_{23}V_{13}^{*}V_{21}^{*}) = 0 \iff \beta_{14} = 0 \text{ or } \pi \text{ and } \beta_{13} = 0 \text{ or } \pi\),

From Eqs. (12a) and (12c): \(\text{Im}(V_{11}V_{22}V_{12}^{*}V_{21}^{*}) = 0 \text{ and } \text{Im}(V_{11}V_{32}V_{12}^{*}V_{31}^{*}) = 0 \iff \beta_{14} = 0 \text{ or } \pi \text{ and } \beta_{24} = 0 \text{ or } \pi\).

We thus see that we have obtained

**Theorem 3.**

\[
\begin{pmatrix}
\text{Im}(V_{11}V_{22}V_{12}^{*}V_{21}^{*}) = 0 \\
\text{Im}(V_{11}V_{23}V_{13}^{*}V_{21}^{*}) = 0 \\
\text{Im}(V_{11}V_{32}V_{12}^{*}V_{31}^{*}) = 0
\end{pmatrix} \iff (\beta_{14} = 0, \beta_{13} = 0, \beta_{24} = 0).
\]

The vanishing of the phases \(\beta_{ij}\) of the CKM4 means that CP is conserved.

We have thus obtained the condition for the CP conservation in SM4. From Theorem 3 we obtain the conditions for the CP violation in SM4

**Theorem 4.** (symbol \(\lor\) denotes the logical or)

\((\text{Im}(V_{11}V_{22}V_{12}^{*}V_{21}^{*}) \neq 0 \lor \text{Im}(V_{11}V_{23}V_{13}^{*}V_{21}^{*}) \neq 0 \lor \text{Im}(V_{11}V_{32}V_{12}^{*}V_{31}^{*}) \neq 0) \iff (\beta_{13} \neq 0 \lor \beta_{14} \neq 0 \lor \beta_{24} \neq 0).
\]

It means that the non vanishing of any of the Jarlskog invariant implies the presence of the CP violation in SM4.
IV. DISCUSSION AND CONCLUSIONS

Theorems 3 and 4 constitute the main results of the paper and are the generalization of the Jarlskog’s conditions for CP conservation is the SM with 3 generations. Let us comment on these results.

1. Theorem 4 states that the non vanishing of any Jarlskog invariant implies the CP violation. Since CP is broken in SM3, then it means that this is violated in SM4 also.

2. Theorems 3 and 4 put conditions on the three following Jarlskog invariants: \( \text{Im}(V_{11}^*V_{22}) \), \( \text{Im}(V_{11}^*V_{32}) \), \( \text{Im}(V_{11}^*V_{32}) \). These three invariants are built from the \( 3 \times 3 \) CKM matrix alone. Had the CP been conserved in the SM4, then it would be possible to verify it from the CKM matrix of the SM3.

3. In Theorems 3 and 4 we can use any three Jarlskog invariants \( \text{Im}(V_{ij}V_{kl}V_{ij}^*V_{kl}^*) \), but the indices \((i,j,k)\) or \((i,j,l)\) of all of these three invariants cannot be equal.

4. The unitarity conditions 5 of the CKM matrix written for the CKM4 matrix have the form

\[
\text{Im}(V_{11}^*V_{22}^*V_{32}^*V_{21}) + \text{Im}(V_{11}^*V_{32}^*V_{31}^*V_{21}) + \text{Im}(V_{11}^*V_{42}^*V_{12}^*V_{41}) = 0 \tag{13a}
\]

and

\[
\text{Im}(V_{11}^*V_{22}^*V_{12}^*V_{21}) + \text{Im}(V_{11}^*V_{23}^*V_{13}^*V_{21}) + \text{Im}(V_{11}^*V_{24}^*V_{14}^*V_{21}) = 0. \tag{13b}
\]

The measurement of the first two Jarlskog invariants in each of equations (13) does not involve particles from the 4-th generation. The following conditions on these invariants

\[
\text{Im}(V_{11}^*V_{22}^*V_{32}^*V_{21}) + \text{Im}(V_{11}^*V_{32}^*V_{31}^*V_{21}) \neq 0 \tag{14a}
\]

or

\[
\text{Im}(V_{11}^*V_{22}^*V_{12}^*V_{21}) + \text{Im}(V_{11}^*V_{23}^*V_{13}^*V_{21}) \neq 0 \tag{14b}
\]

would imply the existence of the 4-th quark generation or non unitarity of the CKM3 matrix.

Appendix

The Jarlskog invariant of the CKM matrix is defined by the following expression

\[
\text{Im}(V_{ij}V_{kl}V_{ij}^*V_{kl}^*), \quad i \neq k \text{ and } j \neq l \tag{A.1}
\]

and it is invariant under the rephasing transformation. It fulfills two simple identities

\[
\text{Im}(V_{ij}V_{kl}V_{ij}^*V_{kl}^*) = \text{Im}(V_{kl}V_{ij}V_{ij}^*V_{kl}^*) = - \text{Im}(V_{kl}V_{ij}V_{ij}^*V_{kl}^*). \tag{A.2}
\]

It means that it is always possible to rearrange the order of indices in such a way that \( i < k \text{ and } j < l \). Taking this into account it is simple to demonstrate that for the \( n \times n \) matrix there are \( \left( \frac{n(n-1)}{2} \right)^2 \) independent Jarlskog invariants.

Proof of Theorem 4

\[
\Rightarrow \text{ From } M = D_L \tilde{M} D_R, \tilde{M} = |m_{ij}|, \text{ we have }
\]

\[
m_{11} = e^{i\phi_1} |m_{11}|, \quad m_{1l} = e^{i\phi_1} e^{i\psi_{l-1}} |m_{1l}|, \quad m_{kl} = e^{i\phi_k} |m_{kl}|, \quad k, l > 1. \tag{A.3}
\]

From Eq. (A.3) we have

\[
\text{Im}(m_{11}m_{kl}m_{1l}^*m_{kl}^*) = \text{Im}(e^{i\phi_1} |m_{11}| e^{i\phi_k} e^{i\psi_{l-1}} |m_{kl}| e^{-i\phi_1} e^{-i\psi_{l-1}} |m_{1l}| e^{-i\phi_k} |m_{kl}|) = \text{Im}(|m_{11}| |m_{kl}| |m_{1l}| |m_{kl}|) = 0. \tag{A.4}
\]
Now, when $m_{ij} = |m_{ij}|e^{i\omega_{ij}}$. The condition $\text{Im}(m_{11}m_{kl}m_{ik}^*m_{k1}^*) = 0$ implies
\[
\omega_{11} + \omega_{kl} - \omega_{1l} - \omega_{k1} = 0 \text{ or } \pi. \tag{A.5}
\]
Let us define
\[
\phi_k = \omega_{k1}, \quad \psi_{l-1} = \omega_{1l} - \omega_{11}. \tag{A.6}
\]
From Eq. (A.5) we find
\[
\phi_k + \psi_{l-1} = \omega_{k1} + \omega_{1l} - \omega_{11} = \omega_{kl}, \tag{A.7}
\]
thus $m_{kl} = e^{i\omega_{kl}}|m_{kl}| = e^{i\phi_k}e^{i\psi_{l-1}}|m_{kl}|$. Denoting $D_L = \text{Diag}(e^{i\phi_1}, e^{i\phi_2}, \ldots, e^{i\phi_n})$ and $D_R = \text{Diag}(1, e^{i\psi_1}, \ldots, e^{i\psi_{n-1}})$ we obtain
\[
M = D_L\tilde{M}D_R \tag{A.8}
\]
and this complete the proof.

**Proof of Theorem 2**

The CKM matrix is unitary, which imposes unitary relations between its matrix elements. Such relations also imply additional relations between the Jarlskog invariants. The standard method of the derivation of such relations is to multiply the $i$-th column (or row) by the complex conjugate of the $j$-th column (or row) of the $n \times n$ unitary matrix $V$
\[
\sum_{l=1}^{n} V_{lk}V_{ij}^* = 0, \text{ for } j \neq k. \tag{A.9}
\]
Now, if we multiply Eq. (A.9) by $V_{ij}V_{ik}^*$ we obtain
\[
\sum_{l \neq i} \text{Im}(V_{ij}V_{lk}V_{ik}^*) = 0, \quad j < k, \quad i, j, k = 1, \ldots, n. \tag{A.10a}
\]
Multiplying rows we obtain in the same way
\[
\sum_{l \neq i} \text{Im}(V_{ji}V_{kl}V_{jl}^*) = 0, \quad j < k, \quad i, j, k = 1, \ldots, n \tag{A.10b}
\]
and this completes the proof.

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