Measuring outcome correlation for Bell cat-state and geometric phase induced spin parity effect

Yan Gu, Haifeng Zhang, Zhigang Song
Institute of Theoretical Physics and Department of Physics,
State Key Laboratory of Quantum Optics and Quantum Optics Devices,
Shanxi University, Taiyuan, Shanxi 030006, China

J. -Q. Liang*
Institute of Theoretical Physics and Department of Physics,
State Key Laboratory of Quantum Optics and Quantum Optics Devices,
Shanxi University, Taiyuan, Shanxi 030006, China and
*jqliang@sxu.edu.cn

L. -F. Wei
State Key Laboratory of Optoelectronic Materials and Technologies,
School of Physics and Engineering, Sun Yat-Sen University, Guangzhou 510275, China and
Quantum Optoelectronics Laboratory, School of Physics and Technology,
Southwest Jiaotong University, Chengdu 610031, China

(Received textdate; Revised textdate; Accepted textdate; Published textdate)

In terms of quantum probability statistics the Bell inequality (BI) and its violation are extended to spin-s entangled Schrödinger cat-state (called the Bell cat-state) with both parallel and antiparallel spin-polarizations. Except the spin-1/2 the BI is never ever violated by the Bell cat-states with the measuring outcomes including entire Hilbert space. If, on the other hand, measuring outcomes are restricted in the subspace of spin coherent state (SCS), a universal Bell-type inequality (UBI), \( p_{lc} \leq 0 \), is formulated in terms of the local realistic model. We observe a spin parity effect that the UBI can be violated only by the Bell cat-states of half-integer but not the integer spins. The violation of UBI is seen to be a direct result of non-trivial Berry phase between the SCSs of south- and north-pole gauges for half-integer spin, while the geometric phase is trivial for the integer spins. A maximum violation bound of UBI is found as \( p_{lc}^{max} = 1 \), which is valid for arbitrary half-integer spin-s states.

PACS numbers: 03.65.Ud; 03.65.Vf; 03.67.Lx; 03.67.Mn
Keywords: Bell inequality; non-locality; Berry phase; spin coherent state.

1. Introduction

Non-locality[1–3] as one of the most peculiar characteristic of quantum mechanics does not coexist with the relativistic causality in our intuition of space and time. The two-particle entangled state is a typical example of non-locality, which as a matter of fact was originally considered by Einstein, Podolsky, and Rosen (EPR) to question the completeness of quantum mechanics.[4] Since it leads to apparently contradictory results with the locality and reality criterion in classical theory.[4] Nevertheless, the entanglement has become an essential ingredients in quantum information and computation.[5–7]

With the spin version of EPR argument Bell formulated a quantitative test of non-local correlations[8] known as Bell’s inequality (BI), which is derived in terms of classical statistics with assumptions of local hidden-variables[9] and measuring-outcome independence. The correlation properties of entangled states are fundamentally different from the classical world.[10–12] The overwhelming experimental evidence for the violation[13–20] of BI opens up a most intriguing aspect of non-locality in quantum mechanics, although the underlying physics is obscure.[21] Following the idea of Bell various extensions of the original BI were proposed such as Clauser-Horne-Shimony-Holt[22] (CHSH) and Wigner[23] inequalities. The experimental tests[17, 24, 25] for the violation of BI have been also reported using the spin entangled-states of a nitrogen-vacancy defect in diamond.[26]

In our previous work[27, 28] the original BI based on the two-spin singlet is extended to a unified form, which is valid for the general entangled states with both antiparallel and parallel polarizations. The Bell-CHSH-Wigner
inequalities and their violation are formulated in a unified way by the spin coherent-state (SCS) quantum probability statistics.\cite{29, 30} The density operator of entangled state can be separated into the local and nonlocal parts. All inequalities are derived in terms of the local part alone. While the nonlocal part leads to the violation, which is a result of the coherent interference between two components of the entangled state. The maximum violations are found for both BI and Wigner inequality, which are shown to be equally convenient for the experimental test.\cite{29, 30} It has been pointed out that not all entangled states can be used to demonstrate the Bell nonlocality.\cite{31, 32} A natural question is whether or not the BI is violated by all entangled states? By the explicit calculation it was shown that the BI is not violated by the spin-1 entangled state.\cite{27, 28} We in the present paper extend this investigation to entangled cat-state or Bell cat-state for arbitrary spin-$s$. The cat state originated from Schrödinger\cite{33} refers to any quantum superposition of macroscopically distinct state called the macroscopic quantum state, which gives rise to the minimum Heisenberg uncertainty relation. The cat states received quite a lot of attention over the last decade.\cite{34, 35} It is of fundamental importance both theoretically and experimentally to examine the Bell correlation in the entangled cat-state of arbitrary spin. The violation of BI and phase effect of quantum states (Aharonov-Bohm phase, Berry phase) are both considered as nonlocal phenomena in quantum mechanics. One goal of the present study is tried to establish a relation between them in terms of the Bell cat-states.

In Section 2 the Bell correlation is examined for the spin-3/2 entangled states, which do not lead to the violation of BI due to the cancellation of non-local interference in the quantum probability statistics. The non-violation of BI is approved for spin-$s$ entangled cat-states in Sec.3. Sec.4 is devoted to spin parity effect with measurement restricted in the subspace of SCSs only. A universal Bell-type inequality (UBI) is formulated for the general entangled cat-states. In Section 5 we demonstrate the maximum violation of UBI and the corresponding entangled states.

2. Non-Violation of BI for Spin-3/2 Entangled State

In the previous works\cite{27, 28} it was shown by the explicit evaluation that the BI is not violated by the spin-1 entangled state $|\psi\rangle = c_1|+1, -1\rangle + c_2|-1, +1\rangle$. We extend the investigation to arbitrary spin-$s$ state. As an example we first consider the spin-3/2 entangled state.

2.1. Anti-parallel spin polarization

The spin-3/2 entangled state with anti-parallel polarization is

$$|\psi\rangle = c_1|+\frac{3}{2}, -\frac{3}{2}\rangle + c_2|+\frac{3}{2}, +\frac{3}{2}\rangle - \frac{3}{2} + \frac{3}{2},$$  \hspace{1cm} (1)

in which the normalized superposition-coefficients are parameterized as $c_1 = e^{i\eta}\sin\xi$, $c_2 = e^{-i\eta}\cos\xi$ with $\xi$, $\eta$ being arbitrary real numbers. The measuring outcome-correlation-probability respectively along two random directions $a$ and $b$ is evaluated by the quantum probability statistics

$$P(a, b) = Tr[\hat{\Omega}(a, b)\hat{\rho}],$$  \hspace{1cm} (2)

in which

$$\hat{\Omega}(a, b) = (\hat{s} \cdot a) \otimes (\hat{s} \cdot b)$$

is the two-spin correlation operator. The density operator $\hat{\rho} = |\psi\rangle \langle \psi|$ of entangled state can be separated into the local and non-local parts

$$\hat{\rho} = \hat{\rho}_{lc} + \hat{\rho}_{nlc},$$

with

$$\hat{\rho}_{lc} = \sin^2\xi \left[\frac{3}{2}\frac{3}{2} - \frac{3}{2}\right] \left[\frac{3}{2}\frac{3}{2} + \cos^2\xi \left[-\frac{3}{2} + \frac{3}{2}\right] \left[-\frac{3}{2} + \frac{3}{2}\right]\right],$$

$$\hat{\rho}_{nlc} = \sin\xi\cos\xi \left(e^{2i\eta}\frac{3}{2} - \frac{3}{2}\right) \left[-\frac{3}{2} + \frac{3}{2}\right] + e^{-2i\eta}\left[-\frac{3}{2} + \frac{3}{2}\right] \left[-\frac{3}{2} + \frac{3}{2}\right].$$
The measuring outcome correlation thus can be separated as the local
\[ P_{lc}(a,b) = Tr[\hat{\Omega}(a,b)\hat{\rho}_{lc}], \]  
and nonlocal part
\[ P_{nlc}(a,b) = Tr[\hat{\Omega}(a,b)\hat{\rho}_{nlc}]. \]

The complete eigenstate-set of projection spin-operators \( \hat{s} \cdot a \) and \( \hat{s} \cdot b \) must be taken into account in the quantum probability statistics. The eigenstates of spin projection operator along direction, say \( r \), are obtained by solving the eigenstate equation, \( \hat{s} \cdot r |r_m\rangle = m |r_m\rangle \) (in the unit convention \( \hbar = 1 \)) with the magnetic quantum number \( m = 3/2, 1/2, -1/2, -3/2 \). With the unit vector parameterized by the polar and azimuthal angles \( \theta \), \( \phi \) such that \( r = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \), we obtain the four eigenstates
\[ |r_+\rangle = \cos^3 \frac{\theta}{2} + \sin \frac{\theta}{2} e^{i\phi} |\frac{1}{2}\rangle + \sin^3 \frac{\theta}{2} e^{3i\phi} |\frac{-3}{2}\rangle, \]
\[ |r_-\rangle = \sin^3 \frac{\theta}{2} + \sin \frac{\theta}{2} e^{2i\phi} |\frac{1}{2}\rangle - \cos^3 \frac{\theta}{2} e^{3i\phi} |\frac{-3}{2}\rangle, \]
\[ |r_\frac{3}{2}\rangle = \sqrt{3} \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} |\frac{1}{2}\rangle - \left( 1 - 3 \sin^2 \frac{\theta}{2} \right) \cos \frac{\theta}{2} e^{i\phi} |\frac{1}{2}\rangle + \frac{1}{2} \]
\[ + \left( 1 - 3 \sin^2 \frac{\theta}{2} \right) \sin \frac{\theta}{2} e^{2i\phi} |\frac{-1}{2}\rangle - \sqrt{3} \sin \frac{\theta}{2} \cos \frac{\theta}{2} e^{3i\phi} |\frac{-3}{2}\rangle, \]
\[ |r_\frac{-3}{2}\rangle = \sqrt{3} \sin^2 \frac{\theta}{2} \cos \frac{\theta}{2} |\frac{1}{2}\rangle + \left( 1 - 3 \cos^2 \frac{\theta}{2} \right) \sin \frac{\theta}{2} e^{i\phi} |\frac{1}{2}\rangle + \frac{1}{2} \]
\[ + \left( 1 - 3 \cos^2 \frac{\theta}{2} \right) \cos \frac{\theta}{2} e^{2i\phi} |\frac{-1}{2}\rangle + \sqrt{3} \sin \frac{\theta}{2} \cos \frac{\theta}{2} e^{3i\phi} |\frac{-3}{2}\rangle, \]  
in which only the two states \( |r_{\pm s}\rangle \) ( \( s = 3/2 \) in the present case) are known as the SCSSs
\[ \text{(macroscopic quantum states)} \]  
satisfying the minimum uncertainty relation. For the measurement along two random directions \( a \) and \( b \) the outcome correlation probability Eq. (2) is evaluated by the trace over 16 base vectors, which may be grouped as
\[ |+, +\rangle = \{ |a_+, b_+\rangle, |a_+, b_-\rangle, |a_-, b_+\rangle, |a_-, b_-\rangle \} \]
\[ |+, -\rangle = \{ |a_+, b_-\rangle, |a_-, b_+\rangle, |a_-, b_-\rangle, |a_+, b_+\rangle \} \]
\[ |-, +\rangle = \{ |a_-, b_+\rangle, |a_+, b_+\rangle, |a_+, b_-\rangle, |a_-, b_-\rangle \} \]
\[ |-, -\rangle = \{ |a_-, b_-\rangle, |a_+, b_-\rangle, |a_-, b_+\rangle, |a_+, b_+\rangle \}, \]
where the notation \( |a_m, b_{m'}\rangle \) denotes product eigenstates of spin projection operators respectively along directions \( a \) and \( b \) with eigenvalues \( m, m' \). Each group including four eigenstates corresponds to one measuring outcome state of the spin-1/2 case. The measuring outcome correlation-probability of the local part is derived from Eq. (3) by the straightforward but tedious algebra as
\[ P_{lc}(a,b) = -\frac{9}{4} \cos \theta_a \cos \theta_b. \]  
While the nonlocal part of correlation evaluated from Eq. (4) vanishes by the quantum probability statistics with the complete set of eigenstates
\[ P_{nlc}(a,b) = 0. \]
We may define a normalized correlation such that

\[ p(a, b) = \frac{P(a, b)}{s^2}. \tag{7} \]

Then the typical Bell correlation is recovered\cite{27, 28}

\[ p(a, b) = p_{lc}(a, b) = -\cos \theta_a \cos \theta_b. \]

The BI is not violated by the entangled cat-state of spin-3/2 with antiparallel polarization.

### 2.2. Parallel spin polarization

For the spin-3/2 entangled state with parallel polarization

\[ |\psi\rangle = c_1|+\frac{3}{2}, +\frac{3}{2}\rangle + c_2|+\frac{3}{2}, -\frac{3}{2}\rangle \]

the local and non-local parts of state density operator become

\[ \hat{\rho}_{lc} = \sin^2 \xi \left[ +\frac{3}{2} + \frac{3}{2} \right] \left[ +\frac{3}{2} + \frac{3}{2} \right] + \cos^2 \xi \left[ -\frac{3}{2} - \frac{3}{2} \right] \left[ -\frac{3}{2} - \frac{3}{2} \right], \]

\[ \hat{\rho}_{nlc} = \sin \xi \cos \xi \left( e^{2i\eta} \left[ +\frac{3}{2} + \frac{3}{2} \right] \left[ +\frac{3}{2} + \frac{3}{2} \right] + e^{-2i\eta} \left[ -\frac{3}{2} - \frac{3}{2} \right] \left[ -\frac{3}{2} - \frac{3}{2} \right] \right). \]

The normalized local correlation probability is

\[ p_{lc}(a, b) = \cos \theta_a \cos \theta_b, \]

which is again exactly the same as Bell correlation of spin-1/2 entangled state with parallel spin polarization.\cite{28, 30} The nonlocal part vanishes by the quantum probability statistics \( p_{nlc}(a, b) = 0 \). We conclude that the modified\cite{28} and extended\cite{30} BIs

\[ |p(a, b) - p(a, c)| - |p(b, c)| \leq 1 \tag{8} \]

are not violated at all by the entangled spin-3/2 cat state for both antiparallel and parallel spin polarizations.

### 3. Non-Violation of BI for Spin-s Entangled State by the Quantum Probability Statistics

We now turn to the entangled cat-state\cite{27} of arbitrary spin-s with anti-parallel polarization

\[ |\psi\rangle = c_1|+s, -s\rangle + c_2|-s, +s\rangle. \tag{9} \]

It is impossible to obtain all \((2s + 1)\) analytic eigenstates of the projection spin-operator \( \hat{s} \cdot \mathbf{r} \). We evaluate the measuring outcome correlation by the trace over eigenstates of spin operator \( \hat{s}_z |m\rangle = m|m\rangle \) instead

\[ P(a, b) = \sum_{m_1, m_2} \langle m_1, m_2 | \hat{\Omega}(a, b) | m_1, m_2 \rangle \]

\[ = P_{lc}(a, b) + P_{nlc}(a, b), \]

which can be worked out by straightforward calculation. The local part is the same form

\[ P_{lc}(a, b) = -s^2 \cos \theta_a \cos \theta_b. \]
While the nonlocal part again vanishes by the quantum average over \((2s + 1)^2\)-dimension base vectors. For the entangled state with parallel spin polarization

\[ |\psi\rangle = c_1|+s, +s\rangle + c_2|-s, -s\rangle, \]

the local part of correlation is

\[ P_{lc}(a, b) = s^2 \cos \theta_a \cos \theta_b. \]

The nonlocal part vanishes by the quantum average. We have the normalized correlation probability

\[ p(a, b) = p_{lc}(a, b), \]

for both cases. The extended BI is not violated by the Bell cat states except \(s = 1/2\).

### 4. Berry Phase Induced Spin Parity Effect and UBI

We have demonstrated by explicit calculation that the BI is not violated by the entangled cat-states of spin-\(s\) \((s \neq 1/2)\) in the quantum probability statistics, in which the nonlocal part of measuring outcome correlation vanishes by the average over complete set of eigenstates. It is an interesting problem if the measurements are restricted in the subspace of SCSs, namely only the maximum spin values \(\pm s\) are measured.

#### 4.1. Measuring outcome correlation in the subspace of SCSs

The SCSs for projection spin-operator \(\hat{s} \cdot \mathbf{r}\) in the direction of unit vector \(\mathbf{r} = \mathbf{a}, \mathbf{b}\) can be derived from the eigenstate equations

\[ \hat{s} \cdot \mathbf{r} |\pm \mathbf{r}\rangle = \pm s |\pm \mathbf{r}\rangle. \]

The explicit forms of SCSs in the Dicke-state representation are given by

\[
|+\mathbf{r}\rangle = \sum_{m=-s}^{s} \left( \begin{array}{c} 2s \\ s + m \end{array} \right) \frac{1}{2^{2s}} K_r^{s+m} \Gamma_r^{s-m} \exp \left[ i (s - m) \phi_r \right] |m\rangle \]

\[
|-\mathbf{r}\rangle = \sum_{m=-s}^{s} \left( \begin{array}{c} 2s \\ s + m \end{array} \right) \frac{1}{2^{2s}} K_r^{s-m} \Gamma_r^{s+m} \exp \left[ i (s - m) (\phi_r + \pi) \right] |m\rangle
\]

in which

\[ K_r^{s\pm m} = \left( \cos \frac{\theta_r}{2} \right)^{s\pm m} \]

and

\[ \Gamma_r^{s\pm m} = \left( \sin \frac{\theta_r}{2} \right)^{s\pm m}. \]

The two orthogonal states \(|\pm \mathbf{r}\rangle\) are known as SCSs of north- and south- pole gauges, in which a phase factor \(\exp[i(s - m)\pi]\) difference between two gauges plays a key role in the spin parity effect. The eigenstates of projection spin-operators \(\hat{s} \cdot \mathbf{a}\) and \(\hat{s} \cdot \mathbf{b}\) form a measuring-outcome independent base-vectors, if the measurements are restricted in the maximum spin-values, \(\pm s\). The four base-vectors are labeled as

\[
|1\rangle = |+\mathbf{a}, +\mathbf{b}\rangle, |2\rangle = |+\mathbf{a}, -\mathbf{b}\rangle, |3\rangle = |-\mathbf{a}, +\mathbf{b}\rangle, |4\rangle = |-\mathbf{a}, -\mathbf{b}\rangle,
\]

for the sake of simplicity. The measuring outcome correlation\(^\text{(27-30)}\) evaluated by the trace over the subspace of SCSs can be also divided into local and non-local parts

\[ P(a, b) = \text{Tr}[\hat{\Omega}(a, b)\hat{\rho}] = P_{lc}(a, b) + P_{nlc}(a, b), \]
with the normalized correlation probabilities Eq.(7) given by

\[ p_{lc}(a,b) = \rho_{11}^l - \rho_{22}^l - \rho_{33}^l + \rho_{44}^l, \]

and

\[ p_{nlc}(a,b) = \rho_{11}^{nlc} - \rho_{22}^{nlc} - \rho_{33}^{nlc} + \rho_{44}^{nlc}. \]

Where

\[ \rho_{ii} = \langle i | \hat{\rho} | i \rangle = \rho_{11}^{lc} + \rho_{11}^{nlc} \]

\( (i = 1, 2, 3, 4) \) denotes matrix elements of the density operator.

4.2. Spin parity effect for entangled state with antiparallel spin-polarization

For the antiparallel spin-polarizations Eq.(9), the density-matrix elements of local part are obtained as

\[ \rho_{11}^l = \sin^2 \xi K_a^4 \Gamma_b^4 + \cos^2 \xi \Gamma_a^4 K_b^4, \]

\[ \rho_{22}^l = \sin^2 \xi K_a^4 K_b^4 + \cos^2 \xi \Gamma_a^4 \Gamma_b^4, \]

\[ \rho_{33}^l = \sin^2 \xi \Gamma_a^4 K_b^4 + \cos^2 \xi K_a^4 K_b^4, \]

\[ \rho_{44}^l = \sin^2 \xi \Gamma_a^4 \Gamma_b^4 + \cos^2 \xi K_a^4 \Gamma_b^4. \]

The non-local parts are

\[ \rho_{11}^{nlc} = \rho_{44}^{nlc} = \sin 2 \xi K_a^2 \Gamma_b^2 + \cos 2 \xi \Gamma_a^2 K_b^2 \cos [2s (\phi_a - \phi_b) + 2\eta] \]

\( (12) \)

and

\[ \rho_{22}^{nlc} = \rho_{33}^{nlc} = (-1)^{2s} \rho_{11}^{nlc}. \]

(13)

It may be worthwhile to remark that the density matrix elements of nonlocal part for the same-direction measurement of two spins differ with that of opposite directions by a phase factor \((-1)^{2s} = \exp(i2s\pi)\), which resulted from the geometric phase between SCSs of the north- and south-pole gauges. The normalized local correlation probability is found as

\[ p_{lc}(a,b) = - \left( K_a^{4s} - \Gamma_a^{4s} \right) \left( K_b^{4s} - \Gamma_b^{4s} \right). \]

(14)

While the nonlocal part of correlation is simply

\[ p_{nlc}(a,b) = 2 \left[ 1 - (-1)^{2s} \right] \rho_{11}^{lc}, \]

(15)

which vanishes for integer spin-\( s \) but not for the half-integer spin. This spin parity effect is a result of geometric phase.

4.3. Parallel polarizations

For the state with parallel polarization Eq.(10), the density-matrix elements of local part are modified as

\[ \rho_{11}^l = \sin^2 \xi K_a^{4s} K_b^{4s} + \cos^2 \xi \Gamma_a^{4s} \Gamma_b^{4s} \]

\( (10) \)
\[
\rho_{22}^{l} = \sin^2 \xi K_a^{4s} \Gamma_b^{4s} + \cos^2 \xi \Gamma_a^{4s} \Gamma_b^{4s}
\]
\[
\rho_{33}^{l} = \sin^2 \xi \Gamma_a^{4s} \Gamma_b^{4s} + \cos^2 \xi K_a^{4s} \Gamma_b^{4s}
\]
\[
\rho_{44}^{l} = \sin^2 \xi \Gamma_a^{4s} \Gamma_b^{4s} + \cos^2 \xi K_a^{4s} \Gamma_b^{4s}.
\]

(16)

The four matrix elements of nonlocal part have the same relation with the antiparallel case that
\[
\rho_{11}^{nlc} = \rho_{44}^{nlc} = \sin 2\xi K_a^{2s} \Gamma_b^{2s} K_b^{2s} \Gamma_a^{2s} \cos \left[2s (\phi_a + \phi_b) + 2\eta\right],
\]
\[
\rho_{22}^{nlc} = \rho_{33}^{nlc} = (-1)^{2s} \rho_{11}^{nlc}.
\]

(17)

and \(\rho_{22}^{nlc} = \rho_{33}^{nlc} = (-1)^{2s} \rho_{11}^{nlc} \). The local correlation probability
\[
p_{lc}(a, b) = \left( K_a^{4s} - \Gamma_a^{4s} \right) \left( K_b^{4s} - \Gamma_b^{4s} \right),
\]

(18)

possesses a positive sign different from the antiparallel case Eq.\([13]\). The nonlocal correlation probability is also the same as Eq.\([15]\) \(p_{nlc}(a, b) = 2 \left[ 1 - (-1)^{2s} \right] \rho_{11}^{nlc}\), here the matrix element \(\rho_{11}^{nlc}\) Eq.\([17]\) is slightly different from antiparallel polarizations in Eq.\([12]\). The spin parity effect holds for the parallel spin-polarizations. Particularly for \(s = 1/2\) the measuring outcome correlations reduce to the well known results\([27, 28, 30]\).

4.4. UBI for entangled state of half-integer spin

The original BI, which requires the total measuring-probability to be one, is not suitable for the incomplete measurement in the SCS subspace, where the measured probability is less than one. Following Bell we also consider the three-direction (a, b, c) measurements. The UBI is then formulated as
\[
|p_{lc}(b, c)| \geq p_{lc}(a, b) p_{lc}(a, c),
\]

(19)

which is suitable for both antiparallel and parallel polarizations. Notice the forms of local correlation \(p_{lc}\) in Eqs.\([14, 18]\), the validity of the UBI is obvious for the local realistic model. The UBI is also derived from classical statistics with the hidden variable in Appendix. To find the maximum violation of UBI Eq.\([19]\) we may define a quantity of probability difference that
\[
p_{lc}^{f} = p_{lc}(a, b) p_{lc}(a, c) - |p_{lc}(b, c)| \leq 0.
\]

(20)

Any positive \(p_{lc}\) indicates the violation of UBI.

The quantum correlation probability of measuring outcomes along a, b reads
\[
p(a, b) = \mp \left( K_a^{4s} - \Gamma_a^{4s} \right) \left( K_b^{4s} - \Gamma_b^{4s} \right) + 4 \sin 2\xi K_a^{2s} \Gamma_a^{2s} K_b^{2s} \Gamma_b^{2s} \cos \left[2s (\phi_a \mp \phi_b) + 2\eta\right]
\]

(21)

respectively for antiparallel and parallel spin-polarizations. It reduces to the well known two-state result\([27, 28, 30]\) when \(s = 1/2\). The UBI can be violated by the quantum correlation Eq.\([21]\) for half-integer spin-\(s\) entangled states.

5. Maximum Violation of UBI

5.1. Spin-1/2 entangled state

First of all we examine the maximum violation of UBI Eq.\([20]\) by the spin-1/2 entangled states. For the antiparallel spin-polarization, the quantum correlation probability Eq.\([21]\) becomes the known result\([27, 29]\)
\[
p(a, b) = -\cos \theta_a \cos \theta_b + \sin 2\xi \sin \theta_a \sin \theta_b \cos (\phi_a - \phi_b + 2\eta).
\]

(22)
Since the polar angle is restricted by $0 \leq \theta \leq \pi$, it is easy to verify that the UBI can be violated by the quantum correlation probability Eq. (22). The quantum correlation-probability difference is bounded by a maximum violation value $P_{s}^{\text{max}}$ that

$$
p_{\pm} = p(a, b)p(a, c) - |p(b, c)| \leq \cos (\theta_{a} + \theta_{b}) \cos (\theta_{a} + \theta_{c}) \leq 1 = P_{s}^{\text{max}}.
$$

As a matter of fact when $\theta_{a} = \theta_{b} = \theta_{c} = \pi/2$, the probability difference is

$$
p_{\pm} = (\sin 2\xi \cos (\phi_{a} - \phi_{b} + 2\eta)) (\sin 2\xi \cos (\phi_{a} - \phi_{c} + 2\eta)) - |\sin 2\xi \cos (\phi_{b} - \phi_{c} + 2\eta)|.
$$

Choosing the entangled state angles $\xi = \eta = (\pi/4) \mod 2\pi$, we have

$$
p_{\pm} = \sin (\phi_{a} - \phi_{b}) \sin (\phi_{a} - \phi_{c}) - |\sin (\phi_{b} - \phi_{c})|.
$$

The maximum violation $P_{s}^{\text{max}} = 1$ is then realized with measuring direction angles $\phi_{a} = \pi/2, \phi_{b} = \phi_{c} = 0$.

The entangled state to realize the maximum violation is

$$
|\psi\rangle = \frac{1}{\sqrt{2}} \left( e^{i\frac{\pi}{4}}| + \frac{1}{2}, - \frac{1}{2} \rangle + e^{-i\frac{\pi}{4}}| - \frac{1}{2}, \frac{1}{2} \rangle \right).
$$

The three-direction measurements should be set up respectively with the polar and azimuthal angles $\theta_{a} = \theta_{b} = \theta_{c} = \pi/2, \phi_{a} = \pi/2, \phi_{b} = \phi_{c} = 0$, namely $a, b, c$ are perpendicular to the original spin polarization with $a$ along $y$-direction, $b, c$ along $x$-direction.

With the same analyses the state with parallel spin-polarization to realize the maximum violation of UBI is seen to be

$$
|\psi\rangle = \frac{1}{\sqrt{2}} \left( e^{i\frac{\pi}{4}}| + \frac{1}{2}, + \frac{1}{2} \rangle + e^{-i\frac{\pi}{4}}| - \frac{1}{2}, - \frac{1}{2} \rangle \right),
$$

for which the three-direction measuring angles are the same as that of antiparallel polarizations.

### 5.2. Arbitrary half-integer spin-$s$

Including the nonlocal part the quantum correlation probability difference defined in Eq. (21) is

$$
p_{s} = p(a, b)p(a, c) - |p(b, c)|,
$$

any positive value of which indicates the violation of UBI. We now find the maximum violation value $p_{s}^{\text{max}}$, the corresponding state parameters $\xi, \eta$, and the three measuring directions. From the spin-1/2 case we know that the three measuring directions $a, b, c$ are perpendicular to the original spin polarization, namely $\theta_{a} = \theta_{b} = \theta_{c} = \pi/2$, which leads to

$$
K_{r}^{4s} - \Gamma_{r}^{4s} = 0,
$$

with $r = a, b, c$. The quantum correlation probabilities thus become

$$
p(a, b) = 2^{-2(2s-1)} \sin 2\xi \cos [2s (\phi_{a} \mp \phi_{b}) + 2\eta],
$$

$$
p(a, c) = 2^{-2(2s-1)} \sin 2\xi \cos [2s (\phi_{a} \mp \phi_{c}) + 2\eta],
$$

$$
p(b, c) = 2^{-2(2s-1)} \sin 2\xi \cos [2s (\phi_{b} \mp \phi_{c}) + 2\eta],
$$

respectively for the antiparallel and parallel spin-polarizations. The front number-factor $2^{-2(2s-1)}$ would lead to the correlation probability negligibly small with large spin $s$. This is easy to understand that we measure only in the four SCSs, while entire dimension of Hilbert space is $(2s + 1)^2$. We may consider the relative or scaled correlation probability

$$
p_{rl}(a, b) = \frac{p(a, b)}{N},
$$

(27)
where

\[ N = \sum_{i=1}^{4} |\langle i | \psi \rangle|^2 = \sum_{i=1}^{4} \rho_{ii}, \]

is the total probability of entangled state \(|\psi\rangle\) in the four measuring base-vectors of SCS given by Eq. (11).

Using the matrix elements in Eqs. (12,16,17) with \(\theta_a = \theta_b = \theta_c = \pi/2\), the total probability gives rise to just the same number factor

\[ N = 2^{-2(2s-1)}, \tag{28} \]

which is 1 for \(s = 1/2\). In the following the scaled probability of Eq. (27) is adopted without the subscript "rl" for the sake of simplicity. The scaled quantity of correlation probability difference Eq. (25) becomes

\[ p_a = \sin [2s (\phi_a \mp \phi_b)] \sin [2s (\phi_a \mp \phi_c)] - |\sin [2s (\phi_b \mp \phi_c)]|, \tag{29} \]

with the polar angles of three measuring directions \(\theta_a = \theta_b = \theta_c = \pi/2\) and entangled-state parameters \(\xi = \eta = (\pi/4) \mod 2\pi\). From the Eq. (29) we can determine the maximum violation value

\[ p_a^{\text{max}} = 1 \tag{30} \]

in the case of azimuthal measuring-angles \(\phi_b = \phi_c = 0\), and \(\phi_a = \pi/2\) or \(3\pi/2\). Namely a is perpendicular to the two colinear directions b, c. The entangled states to generate the maximum violation of UBI are respectively

\[ |\psi\rangle = \frac{1}{\sqrt{2}} \left( e^{i\frac{\pi}{4}} | +s, -s \rangle + e^{-i\frac{\pi}{4}} | -s, +s \rangle \right) \]

and

\[ |\psi\rangle = \frac{1}{\sqrt{2}} \left( e^{i\frac{\pi}{4}} | +s, +s \rangle + e^{-i\frac{\pi}{4}} | -s, -s \rangle \right), \]

for antiparallel and parallel spin-polarizations in consistence with the spin-1/2 states Eqs. (23,24).

**6. Conclusion**

We reformulate the BI and its violation in a unified formalism by means of the quantum probability statistics, in which the measuring outcome correlation-probability is divided into the local and non-local parts. The BI is derived from the local correlation, while the nonlocal one gives rise to its violation. Based on the two-spin model we conclude that the violation of BI depends not only on the specific entangled states but also on the measurements. For the entangled cat-states the BI is not violated at all except the spin-1/2 case, if the measurements are performed over entire Hilbert space. The non-local correlation between two components of entangled state is canceled out completely by the quantum statistical average. On other hand, when the measurements are restricted in the subspace of SCSs, namely only the maximum spin values \(\pm s\) are taken into account, the UBI proposed in the present paper is only violated by the entangled states of half-integer spin but not the integer spin. This spin parity phenomenon is seen to be a direct result of Berry phase between the SCSs of north- and south- pole gauges. It is a common belief that there exist two types of non-locality. One is the phase effect of quantum states such as Aharonov-Bohm and Berry phases. The other is the violation of BI by entangled states. It is a long standing problem to find the relation between them. The present paper provides an example to relate the violation of BI with the geometric phase. The UBI, \(p_{lc}^{\text{max}} \leq 0\), derived from local-realistic model is suitable to detect the non-locality of arbitrary entangled cat-states of spin-s for both antiparallel and parallel polarizations. A maximum violation bound is found as \(p_a^{\text{max}} = 1\) for the half-integer spin (including the spin-1/2) entangled-states.

**Acknowledgements**

This work was supported in part by National Natural Science Foundation of China, under Grants No. 11275118, U1330201.
Appendix

Following Bell the proof of UBI is trivial from the classical statistics.

The two-spin measuring outcomes (normalized) along the direction $r$ are denoted respectively by $A(r, \lambda) = \pm 1$ and $B(r, \lambda) = \pm 1$ with $r = a, b, c$. The (normalized) measuring outcome correlation for two spins respectively in directions $a$ and $b$ is evaluated with the classical probability statistics under the local realistic model

$$p_{lc}(a, b) = \int \rho(\lambda) A(a, \lambda) B(b, \lambda) d\lambda,$$

in which $\rho(\lambda)$ is the probability density distribution of parameter $\lambda$. The product of two correlations is

$$p_{lc}(a, b) p_{lc}(a, c)$$

$$= \int \int \rho(\lambda) \rho(\lambda') A(a, \lambda) B(b, \lambda) A(a, \lambda') B(c, \lambda') d\lambda d\lambda'$$

$$\leq \int \rho(\lambda) A^2(a, \lambda) B(b, \lambda) B(c, \lambda) d\lambda$$

$$= \int \rho(\lambda) B(b, \lambda) B(c, \lambda) d\lambda$$

Using the hidden variable assumption that $B(r, \lambda) = \mp A(r, \lambda)$ for the entangled states respectively with antiparallel and parallel spin-polarizations, we have

$$\int \rho(\lambda) B(b, \lambda) B(c, \lambda) d\lambda = \mp \int \rho(\lambda) A(b, \lambda) B(c, \lambda) d\lambda$$

$$\leq |p_{lc}(b, c)|$$

The UBI

$$p_{lc}(a, b) p_{lc}(a, c) \leq |p_{lc}(b, c)|$$

is suitable to both parallel ($B(b, \lambda) = A(b, \lambda)$) and antiparallel ($B(b, \lambda) = -A(b, \lambda)$) polarizations with the total measurement probability $\int \rho(\lambda) d\lambda \leq 1,$ in which the equal sign is valid only for the spin-1/2 states.

References

[1] H. Y. Su, Y. C. Wu, J. L. Chen, C. Wu, L. C. Kwek, Phys. Rev. A 88 (2013) 022124.
[2] P. G. Kwiat et al., Nature 409 (2001) 1014.
[3] S. Popescu, Nat. Phys. 6 (2010) 151.
[4] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47 (1935) 777.
[5] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2011).
[6] C. H. Bennett and D. P. DiVincenzo, Nature 404 (2000) 247.
[7] D. Loss and D. P. DiVincenzo, Phys. Rev. A 57 (1998) 120.
[8] J. S. Bell, Physics 1 (1964) 195.
[9] D. Bohm, Phys. Rev. 85 (1952) 166.
[10] T. Paterek, A. Fedrizzi, S. Groblacher, T. Jennewein, M. Zukowski, M. Aspelmeyer, A. Zeilinger, Phys. Rev. Lett. 99 (2007) 210406.
[11] R. Rabelo, M. Ho, D. Cavalcanti, N. Brunner, V. Scarani, Phys. Rev. Lett. 107 (2011) 050502.
[12] C. Branciard, A. Ling, N. Gisin, C. Kurtsiefer, A. Lamas-Linares, V. Scarani, Phys. Rev. Lett. 99 (2007) 210407.
[13] A. Aspect, Nature 398 (1999) 189.
[14] L. F. Wei, Y. X. Liu and F. Nori, Phys. Rev. B 72 (2005) 104516; M. Ansmann et al., Nature 461 (2009) 504.
[15] H. Sakai, T. Saito, T. Ikeda, K. Itoh, T. Kawabata, H. Kuboki, Y. Maeda, N. Matsui, C. Rangacharyulu et al., Phys. Rev. Lett. 97 (2006) 150405.
[16] A. Cabello and F. Sciarrino, Phys. Rev. X 2 (2012) 021010.
[17] G. Waldherr, P. Neumann, S. F. Huelga, F. Jelezko, J. Wrachtrup, Phys. Rev. Lett. 107 (2011) 090401.
[18] M. A. Rowe et al., Nature 409 (2001) 791.
[19] N. Gisin and A. Peres, Phys. Lett. A 162 (1992) 15.
[20] K. F. Pal and T. Vertesi, Phys. Rev. A 96 (2017) 022123.
[21] M. Pawlowski et al., Nature 461 (2009) 1101.
[22] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. 23 (1969) 880.
[23] E. P. Wigner, Am. J. Phys. 38 (1970) 1005.
[24] B. Hensen et al., Sci. Rep. 6 (2016) 30289.
[25] B. Hensen et al., Nature 526 (2015) 682.
[26] N. Zhao et al., Nat. Nanotechnol. 6 (2011) 242.
[27] Z. G. Song, J. Q. Liang and L. F. Wei, Mod. Phys. Lett. B 28 (2014) 1450004.
[28] H. F. Zhang, J. H. Wang, Z. G. Song, J. Q. Liang and L. F. Wei, Mod. Phys. Lett. B 31 (2017) 1750032.
[29] Y. Gu, H. F. Zhang, Z. G. Song, J. Q. Liang and L. F. Wei, Int. J. Quantum Inf. 16 (2018) 1850041.
[30] Y. Gu, H. F. Zhang, Z. G. Song, J. Q. Liang and L. F. Wei, Chin. Phys. B 27 (2018) 100303.
[31] R. F. Werner, Phys. Rev. A 40 (1989) 4277.
[32] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. Lett. 80 (1998) 5239.
[33] E. Schröinger, Naturwissenschaften 23 (1935) 807.
[34] F. Fröwis and W. Dur, New J. Phys. 14 (2012) 093039.
[35] F. Fröwis, N. Sangouard and N. Gisin, Opt. Commun. 337 (2015) 2.
[36] J. Q. Liang and L. F. Wei, New Advances in Quantum Physics (Science Press, Beijing, 2011).