GEODESIC RAYS IN THE DONALDSON–UHLENBECK–YAU THEOREM

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Abstract. We give new proofs of two implications in the Donaldson–Uhlenbeck–Yau theorem. Our proofs are based on geodesic rays of Hermitian metrics, inspired by recent work on the Yau–Tian–Donaldson conjecture.

1. Introduction

Let \((X^n, \omega)\) be a compact Kähler manifold, and \(E\) a holomorphic vector bundle of rank \(r\) over \(X\). A Hermite-Einstein metric on \(E\) is a Hermitian metric \(h\) which satisfies

\[
\Theta(h) \wedge \omega^{n-1} = \gamma \omega^n \text{Id}_E,
\]

where \(\Theta(h)\) is the curvature of \(h\), \(\text{Id}_E\) is the identity endomorphism, and \(\gamma\) is a cohomological constant.

The celebrated Donaldson–Uhlenbeck–Yau theorem states that \(E\) admits a Hermite-Einstein metric if and only if \(E\) is slope stable \([\text{Don87}; \text{UY86}]\). We will consider the following version:

**Theorem 1.1.** Suppose that \(E\) is a holomorphic vector bundle over a compact Kähler manifold \((X^n, \omega)\). Then the following conditions are equivalent:

1. \(E\) is slope stable;
2. The Donaldson functional \(\mathcal{M}\) is proper on the space of Hermitian metrics;
3. \(E\) admits a unique Hermite-Einstein metric.

Here \(E\) is slope stable if and only if \(\mu_F < \mu_E\) for any nontrivial holomorphic torsion free subsheaf \(E \subseteq F\), where \(\mu_F\) denotes the slope of a holomorphic sheaf \(F\) over \(X\) with respect to \(\omega\). The Donaldson functional \(\mathcal{M}\) is a functional on the space of Hermitian metrics whose minimizers are exactly the Hermite–Einstein metrics. See §2 for details.

The equivalence \(1 \Leftrightarrow 3\) in Theorem 1.1 was the first result to link solvability of a geometric PDE to a stability condition in the sense of GIT, and has proven to be deeply influential in shaping subsequent results and conjectures, e.g. the Yau–Tian–Donaldson conjecture, as discussed shortly.

We have chosen the above formulation of Theorem 1.1 in order to emphasize the similarities with the variational approach to the Yau–Tian–Donaldson (YTD) conjecture on the existence of (unique) cscK metrics on polarized complex manifolds \((X, L)\). Despite much recent progress, this conjecture is still open in general, but it is settled for Fano manifolds, when \(L = -K_X\), see \([\text{Ber16}; \text{CDS15a}; \text{CDS15b}; \text{CDS15c}; \text{Tia15}; \text{DS16}; \text{CSW18}; \text{BBJ21}]\), and even for (possibly singular) log Fano pairs \([\text{Li22}]\).

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In the general YTD conjecture, the Donaldson functional is replaced by the Mabuchi K-energy functional, and slope stability by a suitable version of K-stability, a condition on the space of test configurations for \((X, L)\) \cite{Tia97, Don02}.

The analogue of the equivalence \((2) \iff (3)\) is known in full generality \cite{BBEGZ19, DR17, CC21a, CC21b}, and versions of \((3) \implies (1)\) and \((1) \implies (2)\) have been shown in \cite{BHJ19} and \cite{Li21}, respectively (albeit with two different, conjecturally equivalent \cite{Li21, BJ22} definitions of stability). In each of these proofs, the notion of a geodesic ray in the space of singular semipositive metrics on \(L\) plays a crucial role.

Going back to the Hermite–Einstein problem, there is also a natural notion of geodesic rays in the space of Hermitian metrics on \(E\), and the goal of the present paper is to give new proofs of the implications \((3) \implies (1)\) and \((1) \implies (2)\) by utilizing geodesic rays, paralleling the recent work on the YTD conjecture.

Our first result constructs a geodesic ray from any filtration. As in the YTD conjecture, we will actually construct rays of singular metrics. Let \(H^1, p\) be the space of trace-free endomorphisms on \(E\) with coefficients in the Sobolev space \(W^{1, p}\), for \(1 \leq p \leq \infty\).

**Theorem A.** Let \(h_0\) be a hermitian metric on \(E\) and

\[
0 =: \mathcal{E}_{m+1} \subset \mathcal{E}_m \subset \ldots \subset \mathcal{E}_1 := E
\]

a filtration of \(E\) by holomorphic subsheaves. Let \(\mathcal{F}_i := \mathcal{E}_i / \mathcal{E}_{i+1}\), \(1 \leq i \leq m\). Then there exists \(w \in H^1, \infty\), such that the geodesic ray of singular hermitian metrics \(h_t := e^{tw}h_0\), \(t \geq 0\), satisfies:

\[
(1.1) \quad \lim_{t \to \infty} \frac{M_\omega(h_t, h_0)}{t} = \sum_{k=1}^m 2\pi (m - k + 1) \operatorname{rk}(\mathcal{F}_k) (\mu_\mathcal{F}_k - \mu_E),
\]

where \(\operatorname{rk}(\mathcal{F}_k)\) is the rank of \(\mathcal{F}_k\), and \(\mu_\mathcal{F}_k\) is its slope (with respect to \(\omega\)).

Theorem A, which can be viewed as an analogue of Theorem A in \cite{BHJ19}, easily gives the implication \((3) \implies (1)\) in Theorem 1.1 using the fact that Hermite–Einstein metric are exactly the minimizers of the Donaldson functional, which is furthermore convex along any geodesic ray. Indeed, when \(m = 2\), the sign of the right-hand side of \((1.1)\) is evidently related to the slope stability of \(E\), since for any \(\mathcal{E} \subset E\), \(\operatorname{rk}(\mathcal{E})(\mu_E - \mu_\mathcal{E}) = \operatorname{rk}(E/\mathcal{E})(\mu_{E/\mathcal{E}} - \mu_E)\).

We construct the desired ray in Theorem A by first passing to a smooth resolution \(\pi : \tilde{X} \to X\) of the filtration (see e.g. \cite{Jac14, Sib15}). We then explicitly describe a smooth endomorphism \(w\) on \(\pi^*E\) which acts on \(\pi^*h_0\) by scaling the induced metrics on the quotient bundles \(\pi^*\mathcal{F}_k\) by \(k\); pushing \(w\) forward produces the desired ray downstairs.

As we see from Theorem A, it is useful to consider geodesics of singular metrics of the form \(e^{tw}h_0\), for some \(w \in H^{1, p}\) — note that \(e^{tw}h_0\) will generally be much more singular than \(w\). With this setup, the largest natural space of metrics to consider is:

\[
S := \{w \in H^{1, 1} \mid M(e^{tw}h_0, h_0) < \infty\}.
\]

Proposition 4.1 (\cite{Don87}) shows that actually \(S \subset H^{1, p_{\max}}\), where \(p_{\max} := \frac{2n}{2n-1}\), and it seems natural to interpret \(H^{1, p}\) as an analogue of the space \(E^1\) of metrics (or potentials) of finite energy in the study of cscK metrics (although it is not obvious to us that the...
constant $p_{\text{max}}$ is optimal, c.f. Theorem A). Our next result can be viewed as an analogue of [BBJ21] Theorem 2.16).

**Theorem B.** If the Donaldson functional $M$ is non-proper on $S$ with respect to the $W^{1,p}$-norm, for any $1 < p < p_{\text{max}}$, then there exists a geodesic ray in $S$ along which $M$ is bounded above.

Note that Proposition 4.1 actually implies that $M$ will be proper with respect to the $W^{1,p_{\text{max}}}$-norm if $E$ admits an HE metric, so that Theorem B can likely be improved.

There are two main ingredients in the proof of Theorem B. The first is the lower semi-continuity of $M$ in the weak $W^{1,p}$-topology, a fact for which we give a new, elementary proof, see Proposition 4.2. The analogous fact in the cscK case is that the Mabuchi functional is lsc with respect to the strong topology on the space of metrics of finite energy, see [BBEGZ19]. The second is a compactness statement, which here boils down to the Banach–Alaoglu theorem. The analogue in the cscK case is that sets of bounded entropy are strongly compact, as proved in [BBEGZ19].

Finally we go from geodesic rays to filtrations:

**Theorem C.** Suppose that $E$ admits a geodesic ray in $S$ along which the Donaldson functional is bounded from above. Then there exists a nontrivial filtration of $E$ by holomorphic subsheaves $\{E_k\}_{k=1}^{m+1}$ such that $\mu_{E_k} \geq \mu_E$ for at least one $k$. In particular, $E$ is not slope stable.

Theorem C follows from a formula for $M(h_t, h_0)$ in terms of the eigenvalues of $\log(h_0 h_t^{-1})$, due to Donaldson [Don87]. Using this, we show that $M(h_t)$ can only be bounded from above under very restrictive circumstances: essentially, the geodesic ray $(h_t)$ must have come from a construction similar to Theorem A. The weakly holomorphic $W^{1,2}$-projection theorem of Uhlenbeck-Yau [UY86; UY89] can then be used to produce the desired filtration; applying (1.1) shows that at least one of the subsheafs in the filtration has slope larger than $\mu_E$.

The role of Theorem C in the cscK case $(X, L)$ is played by Theorem 6.4 in [BBJ21], which to any geodesic ray (of linear growth) of metrics of finite energy associates a psh metric of finite energy on the Berkovich analytification of the line bundle with respect to the (non-Archimedean) trivial absolute value on $\mathbb{C}$. As proved by C. Li in [Li21], the slope at infinity of the Mabuchi functional along the ray is bounded below by the Mabuchi functional evaluated at the non-Archimedean metric. In the setting of Theorem C, the limiting object is simpler, given by a filtration of $E$ by holomorphic subsheaves.

The combination of Theorems B and C evidently give us the implication $(1) \Rightarrow (2)$ in Theorem 1.1. As already mentioned, $(3) \Rightarrow (1)$ follows from Theorem A. The remaining implication $(2) \Rightarrow (3)$ can be shown by an easy application of the Hermitian-Yang-Mills flow [Don85]. In the Kähler-Einstein case, any minimizer of the Mabuchi functional is a Kähler–Einstein metric [BBGZ13], and the corresponding result in the cscK case holds as well [CC21a; CC21b]. It is reasonable to believe that a minimizer in $H^{1,p}$ of the Donaldson functional must in fact be a (smooth) Hermitian metric.

**Comparison with previous works:** In terms of history, Lübke [Liub83] and Kobayashi [Kob87] first proved the implication $(3) \Rightarrow (1)$ in Theorem 1.1 by using vanishing theorems for $E$. The more difficult implication $(1) \Rightarrow (3)$ was proved by Donaldson [Don83].
for projective surfaces and by Uhlenbeck and Yau [UY86] in general. Donaldson gave another proof in [Don87] for projective manifolds, using induction on dimension and a theorem of Mehta–Ramanathan [MR84], and the implications (1) ⇒ (2) and (2) ⇒ (3) can be extracted from results in that paper.

Subsequent work of Simpson simultaneously unified and generalized the approaches in [UY86] and [Don87], establishing a version of Theorem 1.1 for Higgs bundles over certain non-compact Kähler manifolds. His usage of a blow-up argument along a non-proper ray to extract a limiting endomorphism and subsequent filtration is similar in spirit to our proof of Theorem B, but with several differences; firstly, his notion of properness is less general than ours, only holding for a special subclass of metrics with $L^1$-curvature. Several simplifications follow from this – for instance, it is easy to show the lower semi-continuity of $\mathcal{M}$ on this smaller space, and he has no need to work with regularizations of subsheaves.

Quite recently, Hashimoto and Keller [HK19, HK21] have given a new proof of the implication (3) ⇒ (1), and a conditional new proof of (3) ⇒ (1), both in the polarized case when $\omega \in c_1(L)$, for an ample line bundle $L$. Like ours, their approach is variational in nature, but uses geodesics in the space of Hermitian norms on global sections of $E \otimes L^k$ for $k \gg 0$.

There has also been a great deal of work on singular versions of Theorem 1.1. In [BS94], Bando and Siu introduced the notion of an admissible Hermite-Einstein metric on a torsion-free sheaf, and showed that a reflexive sheaf on a Kähler manifold admits an admissible Hermite-Einstein metric if and only if it is polystable. Subsequent work has focused on similar results on singular varieties, see e.g. Chen and Wentworth [CW21].

[BS94] also introduced a regularization procedure for holomorphic subsheaves of $E$, which was elaborated upon by Jacobs [Jac14] and Sibley [Sib15] (see also [Buc99]). This procedure is an important tool in our proofs (c.f. Theorem 3.7). It was used by Jacobs [Jac14] to generalize Theorem 1.1 to semi-stable bundles, motivated by work of Kobayashi [Kob87].

Other generalizations of Theorem 1.1 include generalizations to compact Hermitian manifolds ([Buc99, LY87]), and very recent work of Feng-Liu-Wan [FLW18], which expanded Theorem 1.1 to include the existence of Finsler-Einstein metrics.

**Organization:** In Section 2 we set some definitions and collect several background results. In Section 3 we prove Theorem A (c.f. Theorem 3.7). In Section 4 we prove Proposition 4.1 which can be seen as a reverse Sobolev inequality for $w \in \mathcal{S}$, and show the lower semicontinuity of $\mathcal{M}$ on $\mathcal{H}^{1,p}$. We then show Theorems B and C in Section 5.

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2. Background Material

2.1. Slope stability and Sobolev Endomorphisms. For any holomorphic, torsion-free sheaf $\mathcal{E}$ on $X$, write:

$$\mu_{\mathcal{E}} := \frac{\int_X c_1(\mathcal{E}) \wedge \omega^{n-1}}{\text{rk}(\mathcal{E})},$$
for the \textit{slope} of $E$, with respect to $\omega$. We say $E$ is \textit{slope stable} if:

$$\mu_E < \mu_E$$

for all proper, saturated torsion-free subsheaves $E \subset E$. $E$ is said to be \textit{slope semi-stable} if $\mu_E \leq \mu_E$ for all such $E$, and \textit{slope unstable} otherwise. A subsheaf satisfying $\mu_E > \mu_E$ is said to be \textit{destabilizing}.

Fix a Hermitian metric $h_0$ on $E$. We can identify $\text{Herm}(E)$, the space of all smooth Hermitian metrics on $E$, with $\tilde{\mathcal{H}}$, the space of $h_0$-self adjoint endomorphisms of $E$ by:

$$h \in \text{Herm}(E) \mapsto \log(hh^{-1});$$

note that geodesics in $\text{Herm}(E)$ map to straight line segments in $\tilde{\mathcal{H}}$, and visa versa.

We have $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathbb{R}$, where $\mathcal{H}$ is the (geodesically complete) subspace of trace-free endomorphims.

Write $\mathcal{H}^{1,p} := \mathcal{H} \otimes_{C^\infty} W^{1,p}$, for any $p \geq 1$. We refer to straight line segments in $\mathcal{H}^{1,p}$ as \textit{weak} geodesics, sometimes without the adjective if the lack of regularity is clear from context. For any $w \in \mathcal{H}^{1,p}$, we define $\|w\|_{L^p}$ to be the $L^p$-norm of the operator norm of $w$ i.e.

$$\|w\|_{L^p}^p = \int_X \text{tr}(ww^*)^{p/2} \omega^n.$$

The duality pairing between $\mathcal{H}^{1,p}$ and $\mathcal{H}^{1,q}$, $q := \frac{p}{p-1}$, is defined to be:

$$\langle w, u \rangle := \int_X \text{tr}(w\overline{u}) \omega^n + \sqrt{-1} \int_X \text{tr}(D'w \wedge \overline{u}) \wedge \omega^{n-1},$$

where $D = D' + \overline{\partial}$ is the Chern connection of $h_0$; it follows that $\mathcal{H}^{1,q}$, is linearly dual to $\mathcal{H}^{1,p}$. Standard functional analysis implies $\mathcal{H}^{1,p}$ is reflexive for $p > 1$, so by the Banach–Alaoglu Theorem, any bounded subset of $\mathcal{H}^{1,p}$ is weakly compact, i.e. if $\|\overline{\partial}w_i\|_{L^p} \leq C$, then there exists a subsequence of the $i$ such that, after relabeling:

$$w_i \rightharpoonup w \in \mathcal{H}^{1,p},$$

in the sense that:

$$\langle w_i, u \rangle \to \langle w, u \rangle,$$

for every $u \in \mathcal{H}^{1,q}$.

For the purposes of this paper, it will be convenient to fix $1 < p \leq p_{\text{max}} := \frac{2n}{2n-1}$. Then $p < 2$, and the Sobolev conjugate of $p$ is $p' := \frac{2np}{n-p}$ and:

$$p' > \frac{p}{2-p} =: p^*$$

Note that $p_{\text{max}}' = p_{\text{max}}^*$. The Gagliardo-Nirenberg-Sobolev inequality can be given as:

\begin{equation}
\|w\|_{L^{p'}} \leq C_{\text{Sob}} \|w\|_{W^{1,p}} \text{ for any } w \in \mathcal{H}^{1,p},
\end{equation}

and by the Sobolev embedding theorem, the inclusion $W^{1,p} \hookrightarrow L^{p^*}$ is compact – this will be the only reason we need to restrict to $p < p_{\text{max}}$ in the proof of Theorem B.

Given $w \in \mathcal{H}$, write $\lambda_1 \geq \ldots \geq \lambda_r$ for the eigenvalues of $w$; these will be Lipschitz functions on $X$ such that $\sum_{i=1}^r \lambda_i = 0$. When $w \in \mathcal{H}^{1,p}$, the $\lambda_i$ may only be in $L^{p'}$. 
For any $w \in \mathcal{H}^{1,p}$, we define $e^w$ via the power series formula. The resulting self-adjoint endomorphism will be measurable and a.e. finite, but may not be integrable, and it follows that the same applies to $e^w h_0$.

2.2. Donaldson Functional and Properness. Recall now the Donaldson functional; if $w \in \mathcal{H}$, the functional was originally defined by Donaldson [Don87] as:

$$\mathcal{M}(w) := \mathcal{M}(e^w h_0, h_0) = \int_0^1 \int_X \text{tr}(v_s \Theta_s) \wedge \omega^{n-1} \wedge ds,$$

where $h_s := e^{sw} h_0$, $v_s := (d_s h_s) h_s^{-1}$, and $\Theta_s := \sqrt{-1} \bar{\partial}(\partial h_s) h_s^{-1}$ is the curvature of $h_s$ (note the $\sqrt{-1}$ factor). Recall that $\mathcal{M}(tw)$ is convex in $t$, and $\mathcal{M}(0) = 0$. By construction, $\mathcal{M}$ is a Lagrangian for the Hermite-Einstein equation, and it is standard to check that if $w \in \mathcal{H}$ is a minimizer of $\mathcal{M}$ and $\Theta$ is the curvature of $h = e^w h_0$, then:

$$\Theta \wedge \omega^{n-1} = \gamma I_E \cdot \omega^n,$$

with $\gamma = \frac{2\pi n \omega}{\int_X e^w}$, i.e. $h$ is Hermite-Einstein.

There is another formulation of the Donaldson functional in terms of the eigenvalues of $w$. For a point $z \in X$, let $\{e_i\}$ be a unitary frame for $E_z$ which diagonalizes $w$ at $z$, with $w(z) = \text{Diag}(\lambda_1 \text{Id}_{r_1}, \ldots, \lambda_m \text{Id}_{r_m})$ such that $\lambda_1 > \cdots > \lambda_m$ and $\sum_i r_i = r$. Write $\eta_j^i \in A^{0,1}(X)$ for the $r_i \times r_j$ block matrix of $(\bar{\partial}w)(z)$ with respect to the $\{e_i\}$-basis, and define:

$$|\eta_j^i|^2(z) := n \frac{\sqrt{-1} \text{tr}(\eta_j^i \wedge (\eta_j^i)^*) \wedge \omega^{n-1}}{\omega^n}(z),$$

where the $(\eta_j^i)_l^k$ are the components of the block matrix $\eta_j^i$. We then define the function:

$$(2.2) \quad f_w(z) := \sum_{i,j=1}^m |\eta_j^i|^2 \frac{e^{\lambda_i - \lambda_j} - (\lambda_i - \lambda_j) - 1}{(\lambda_i - \lambda_j)^2},$$

where we interpret $\frac{x - x - 1}{x}$ as $\frac{1}{2}$ when $x = 0$.

**Proposition 2.1.** $f_w$ is a well-defined function on $X$.

**Proof.** Fix $z \in X$ and suppose we have another unitary frame $\{f_i\}$ which diagonalizes $w(z)$. Since the eigenvalues of $w(z)$ are independent of the choice of a basis, the matrix of $w(z)$ with respect to the $\{f_i\}$-basis will be still be $\text{Diag}(\lambda_1 \text{Id}_{r_1}, \ldots, \lambda_m \text{Id}_{r_m})$, and we can relate the $\{e_i\}$ and $\{f_i\}$ bases by a block diagonal unitary transformation $\text{Diag}(A_1, \ldots, A_m)$, where each $A_i$ is an $r_i \times r_i$ unitary matrix.

It follows that $(\bar{\partial}w)(z)$ with respect to the $\{f_i\}$-basis is given by $A_i^* \eta_j^i A_j^*$, and so it suffices to check that $\text{tr}(\eta_j^i \wedge (\eta_j^i)^*) = \text{tr}(A_i^* \eta_j^i A_j^* \wedge (A_i^* \eta_j^i A_j^*)^*)$. This can be seen by...
computing:

\[
\begin{aligned}
&\text{tr}(A_i \eta^j A_j^* \wedge A_i \eta^j A_j^*) = \text{tr}(A_i \eta^j A_j^* \wedge A_j^* \eta^j A_j^*) \\
&= \text{tr}(A_i \eta^j \wedge (\eta^j)^* A_j^*) \\
&= \text{tr}(A_i^* A_i \eta^j \wedge (\eta^j)^*) \\
&= \text{tr}(\eta^j \wedge (\eta^j)^*),
\end{aligned}
\]

since each \(A_i\) and \(A_j\) are unitary. \(\square\)

**Remark 2.2.** Note that in the above definition the ranks of the block matrices, \(r_i\) could vary with \(z \in X\). However, if the eigenvalues of \(w\) are constant over \(X\), then the ranks \(r_i\) do not depend on the points \(z \in X\). If \(U \subset X\) is such that \(E|_U\) is a trivial vector bundle, we can also think of \(\eta\) as matrix-valued functions on \(U\).

Write \(\Theta_0\) for the curvature of \(h_0\).

**Proposition 2.3.** For any \(w \in \mathcal{H}\), we have:

\[
\mathcal{M}(w) = \int_X f_w \omega^n + \int_X \text{tr}(\Theta_0 w) \wedge \omega^{n-1}.
\]

**Proof.** The Donaldson functional satisfies the following [Kob87, Equations 6.3.27 and 6.3.33]:

\[
\begin{aligned}
\mathcal{M}(0) &= 0 \\
\frac{d\mathcal{M}}{dt} \bigg|_{t=0} &= \int_X \text{tr}(w \Theta_0) \wedge \omega^{n-1} \\
\frac{d^2\mathcal{M}}{dt^2} &= \int_X |\partial w|^2 h_1 \omega^{n-1} \\
&= \sum_{1 \leq i,j \leq m} \int_X e^{t(\lambda_i - \lambda_j)} |\eta^j_i|^2 \omega^n.
\end{aligned}
\]

For \(z \in X\), consider the following system of ODEs, for some \(\phi_z : \mathbb{R}_{\geq 0} \to \mathbb{R}\).

\[
\begin{aligned}
\phi_z(0) &= 0 \\
\phi_z'(0) &= \frac{\text{tr}(w(z) \cdot \Theta_0(z)) \wedge \omega^{n-1}}{\omega^n} \\
\phi_z''(t) &= \sum_{i,j} e^{t(\lambda_i - \lambda_j)} |\eta^j_i|^2.
\end{aligned}
\]

It is easy to see that the solution to this system is given by

\[
\phi_z(t) = f_tw(z) + \frac{\text{tr}(w(z) \cdot \Theta_0(z)) \wedge \omega^{n-1}}{\omega^n} t.
\]

Furthermore, we have that \(\mathcal{M}(tw) = \int_X \phi_z(t) \omega^n\). Setting \(t = 1\), we get the required result. \(\square\)

The function \(f_w\) makes sense for any \(w \in \mathcal{H}^{1,1} \supset \mathcal{H}^{1,p}\), and is always \(\geq 0\). This lets us extend the definition of \(\mathcal{M}\) as:
Definition 2.4. Given $w \in H^{1,1}$, we define:
\[
M(w) := \int_X f_w \omega^n + \int_X \text{tr}(\Theta_0 w) \wedge \omega^{n-1},
\]
if $f_w \in L^1(X)$; otherwise, set $M(w) := +\infty$. We define:
\[
S := \{ w \in H^{1,1} \mid M(w) < \infty \}.
\]

As already remarked in the introduction, it will follow from Proposition 4.1 that $S \subset H^{1,p_{\text{max}}}$.

Proposition 2.5. Suppose that $w \in H^{1,1}$. Then $M$ is convex on the weak geodesic ray $(tw)_{t \geq 0}$.

Proof. Note that
\[
f_{tw}(z) := \sum_{i,j=1}^m |\eta^i_j|^2 \frac{e^{t(\lambda_i-\lambda_j)} - t(\lambda_i - \lambda_j) - 1}{(\lambda_i - \lambda_j)^2}.
\]
The convexity of $e^x - 1 - x$ implies that for $t, s \in \mathbb{R}_{\geq 0}$ and $\alpha \in (0, 1)$ we have
\[
f_{(\alpha s + (1 - \alpha)t)w} \leq \alpha f_{sw} + (1 - \alpha) f_{tw}.
\]
Thus, we get that
\[
M((\alpha s + (1 - \alpha)t)w) = \int_X f_{(\alpha s + (1 - \alpha)t)w} \omega^n + (\alpha s + (1 - \alpha)t) \int_X \text{tr}(\Theta_0 w) \wedge \omega^{n-1}
\]
\[
\leq \alpha M(sw) + (1 - \alpha) M(tw).
\]
□

Definition 2.6. We say that $M$ is proper on $S$ with respect to the $W^{1,p}$-norm if there exists a constant $C > 0$ such that:
\[
M(w) \geq C \|w\|_{W^{1,p}} - C
\]
for all $w \in S$.

If $M$ is proper with respect to $W^{1,p}$, then clearly it is proper with respect to $W^{1,q}$ for any $1 \leq q < p$.

By Proposition 2.5, $M$ cannot be bounded below if there exists a weak geodesic ray $(tw)_{t \geq 0}$ along which $M$ has negative asymptotic slope, i.e.
\[
\lim_{t \to \infty} \left. \frac{d}{ds} \right|_{s=t} M(sw) < 0.
\]
Similarly, $M$ cannot be proper if there exists a weak geodesic ray along which $M$ has non-positive asymptotic slope.

2.3. Weakly Holomorphic Projections. Finally, we recall the weak holomorphicity theorem of Uhlenbeck-Yau (see also [Pop05]):

Theorem 2.7. (Uhlenbeck-Yau) Suppose that $\pi \in \tilde{H}^{1,2}$ is a weakly holomorphic projection, i.e.:
\[
\pi^2 = \pi \quad \text{and} \quad (I - \pi) \bar{\partial} \pi = 0.
\]
Then $\pi$ is actually the projection onto a holomorphic subsheaf of $E$. 
3. Geodesic Rays Associated to a Filtration

In this section, we show how to construct a natural geodesic ray from a filtration of $E$ and a strictly decreasing set of real numbers $\lambda_1 > \ldots > \lambda_m$; the asymptotic slope of $\mathcal{M}$ along the resulting ray will be computed by the slopes of the filtration, weighted by the $\lambda_i$.

3.1. The case of a subbundle filtration. Fix a Hermitian metric $h_0$ on $E$. Suppose first that we have a filtration of $E$ by holomorphic subbundles $E = E_1 \supset \ldots \supset E_m \supset E_{m+1} = \{0\}$. Let $F_i := E_i/E_{i+1}$; we have a smooth, orthogonal splitting $E = \bigoplus_{i=1}^m G_i$, such that $G_i \cong F_i$ for each $i = 1, \ldots, m - 1$, and $G_m = E_m$.

Let $r_i := \text{rk}(F_i)$ and $s_k = \sum_{i=k}^m r_k = \text{rk}(E_k)$. Consider $\{e_i\}_{i=1}^r$, a local holomorphic frame of $E$, such that $\{e_i\}_{i=r-s_k+1}^r$ frames $E_k$ for each $k = 1, \ldots, m$. Write $\{f_i\}_{j=1}^r$ for the orthogonal projections of the $\{e_i\}_{i=1}^r$ onto the decomposition $\bigoplus_{i=1}^m G_i$; for each $k = 1, \ldots, m$, $\{f_i\}_{i=r-s_k+1}^{r-s_k}$ is a (smooth) frame for $G_k$ (note $e_i = f_i$ for $r - s_m - 1 < i \leq r$).

Let $\lambda_1 > \ldots > \lambda_m$ and set $\delta := \min_{i<j}\{\lambda_i - \lambda_j\} > 0$. Define $w \in \mathcal{H}$ by $w(v) = e^{\lambda_i}v$ for all $v \in G_i$, and $h_t := e^{tw}h_0$, $t \geq 0$; by definition, this is a geodesic ray in $\mathcal{H}$.

**Lemma 3.1.** Let $D_0$ denote the Chern connection of $h_0$. The connection form $\alpha_0$ for $D_0$ can be written with respect to the $\{f_i\}$-basis as

$$\alpha_0 = \begin{pmatrix} \beta_{11} & \ldots & \beta_{1m} \\ \vdots & \ddots & \vdots \\ \beta_{m1} & \ldots & \beta_{mm} \end{pmatrix}.$$  

Using the identification $G_i \cong F_i$, the $\beta_{ij}$ have the following description.

- $\beta_{ii} = \alpha_{F_i,0}$, where $\alpha_{F_i,0}$ is the connection form for $D_{F_i,0}$, the Chern connection of the Hermitian metric induced by $h_0$ on $F_i$.
- For $i > j$, $\beta_{ij}$ is a Hom$(F_i,F_j)$-valued $(1,0)$-form (i.e. an $r_i \times r_j$ matrix of $(1,0)$-forms). For $i < j$, $\beta_{ij}$ is a Hom$(F_j,F_i)$-valued $(0,1)$-form.
- $\beta_{ji}$ is the $h_0$-adjoint of $-\beta_{ij}$ for $i \neq j$, i.e.

$$h_0(\beta_{ij}v,w) + h_0(v,\beta_{ji}w) = 0 \text{ for all } v \in F_i, w \in F_j.$$  

**Proof.** The first and second part of the claim can by easily proved by induction on $m$. In the case of $m = 2$, the proof can be found in [Kob87, Propositions 1.6.4-1.6.6].

To see that last part of the claim, let $v \in F_i$ and $w \in F_j$. By abuse of notation, let $v, w$ also denote their respective images in $E$ under the identifications $F_i \cong G_i$ and $F_j \cong G_j$. Then, $h_0(v,w) = 0$ and we get

$$dh_0(v,w) = h_0(D_0v,w) + h_0(v,D_0w)$$

$$0 = \sum_k h_0(\beta_{ik}v,w) + \sum_l h_0(v,\beta_{lj}w)$$

$$0 = h_0(\beta_{ij}v,w) + h_0(v,\beta_{ji}w).$$  

$\square$
Lemma 3.2. Let $\alpha_t$ be the connection form of $D_t$, the Chern connection of $h_t$. Then, the matrix of $\alpha_t$ can be written as $\alpha_t = (\beta_{t,ij})$ with respect to the $\{f_i\}$-basis, where,
\[
\beta_{t,ij} = \begin{cases} 
  e^{t(\lambda_j - \lambda_i)} \beta_{ij} & \text{if } i > j \\
  \beta_{ij} & \text{otherwise},
\end{cases}
\]
and $\alpha_0 = (\beta_{ij})$, as in Lemma 3.1.

Proof. It is enough to check that $D_t := d + \alpha_t$ satisfies the properties of the Chern connection. First we check that $D_t'' = \overline{\partial}_E$. It is enough to check that if $v \in G_i$, then $D_t''v = \overline{\partial}_Ev$. We know that $D_t''v$ is the $(0, 1)$ part of $D_tv = dv + \sum_j \beta_{t,ij}v$. Using Lemma 3.1, we get that
\[
D_t''v = D_{F_i}''v + \sum_{j > i} \beta_{t,ij}v = D_{F_i}''v + \sum_{j > i} \beta_{ij}v.
\]
Since $D_0$ is the Chern connection on $E$ with respect to $h_0$, we have that $D_t''v = D_0''v = \overline{\partial}_Ev$.

To show that $D_t$ is compatible with $h_t$, we need to show that
\[
dh_t(v, w) = h_t(D_tv, w) + h_t(v, D_tw),
\]
for $v, w \in E$. It is enough show this in the case when $v \in G_i$ and $w \in G_j$. If $i = j$, then
\[
dh_t(v, w) = e^{\lambda_i}dh_0(v, w)
= e^{\lambda_i}(h_0(D_0v, w) + h_0(v, D_0w))
= e^{\lambda_i}(h_0(D_{F_i}v, w) + h_0(v, D_{F_i}w))
= h_t(D_tv, w) + h_t(v, D_tw).
\]
Now consider the case of $i \neq j$. Without loss of generality, assume that $i > j$. We have that $dh_t(v, w) = 0$ and the right hand side is given by
\[
h_t(D_tv, w) + h_t(v, D_tw) = h_t(\beta_{t,ij}v, w) + h_t(v, \beta_{ji}w)
= e^{\lambda_j}h_0(e^{-t(\lambda_j - \lambda_i)} \beta_{ij}v, w) + e^{\lambda_i}h_0(v, \beta_{ji}w)
= h_t(\beta_{ij}v, w) + h_0(v, \beta_{ji}w)
= 0.
\]
where the last equality follows from Lemma 3.1. □

Theorem 3.3. Along the geodesic ray $h_t$, the Donaldson functional is given by
\[
\mathcal{M}(tw) = 2\pi \sum_{i=1}^m \lambda_i \text{rk}(F_i)(\mu_{F_i} - \mu_E)t - \sum_{1 \leq i < j \leq m} B_{ji}(1 - e^{-t(\lambda_i - \lambda_j)})
\]
where $B_{ji} = \int_X |\beta_{ji}|^2 h_0 \omega^n$ is a non-negative integer.
Proof. Write \( \Theta_t \) for the curvature of \( h_t \). In the \( \{f_i\} \)-basis, we can write \( \Theta_t = (\theta_{t,ij}) \), for some \( \text{Hom}(F_i, F_j) \)-valued \((1, 1)\)-forms \( \theta_{t,ij} \).

Recall that the Donaldson functional along a geodesic is given by

\[
\mathcal{M}(tw) = \int_0^t \int_X \text{tr}(w\Theta_s) \wedge \omega^{n-1} ds - c \int_X \text{tr}(w) \omega^n.
\]

Since the matrix for \( w \) in the \( f \)-basis is \( \text{Diag}(\lambda_1 \text{Id}_{r_1}, \ldots, \lambda_m \text{Id}_{r_m}) \), we can locally write:

\[
\int_0^t \text{tr}(w\Theta_s) ds = \sum_i \lambda_i \int_0^t \text{tr}(\theta_{s,ii}).
\]

Using Lemma 3.2, we can compute \( \theta_{s,ii} \) by using the local expression for the curvature \( \Theta_s = d\alpha_s + \alpha_s \wedge \alpha_s \), which gives:

\[
\theta_{s,ii} = \Theta_{s} + \sqrt{-1} \sum_{i \neq j} e^{-s|\lambda_i - \lambda_j|} \beta_{ji} \wedge \beta_{ij},
\]

writing \( \Theta_t \) for the curvature of the metric induced on \( F_i \) by \( h_0 \). We can now integrate (3.1) in \( s \) to get:

\[
\int_0^t \text{tr}(w\Theta_s) ds = \sum_i t \lambda_i \text{tr}(\Theta_t) + \sqrt{-1} \sum_{i \neq j} \frac{\lambda_i (1 - e^{-t|\lambda_i - \lambda_j|})}{|\lambda_i - \lambda_j|} \text{tr}(\beta_{ji} \wedge \beta_{ij}).
\]

This can be simplified by using the relations \( \beta_{ij} = -\beta_{ji}^* \) and \( \lambda_j < \lambda_i \) for \( i < j \) to give:

\[
\int_0^t \text{tr}(w\Theta_s) ds = t \sum_i \lambda_i \text{tr}(\Theta_t) - \sqrt{-1} \sum_{i < j} (1 - e^{-t(\lambda_i - \lambda_j)}) \text{tr}(\beta_{ji} \wedge \beta_{ji}^*).
\]

Now the second term in \( \mathcal{M}(tw) \) is given by

\[
\gamma \cdot t \int_X \text{tr}(w) \omega^n = \gamma \cdot t \sum_i \lambda_i r_i \int_X \omega^n
\]

so by integrating (3.2) over \( X \) and combining the two expressions, we get the result. \( \Box \)

**Corollary 3.4.** Consider a two-step filtration given by a subbundle \( E_2 \subset E_1 = E \). Set \( \lambda_1 = 1, \lambda_2 = 0 \), and \( F = E_1/E_2 \), and let \( h_t \) be the geodesic ray constructed in Theorem 3.3 from this data. Then:

\[
\mathcal{M}(h_t) = 2\pi \text{rk}(F)(\mu_F - \mu_E)t - B(1 - e^{-t}),
\]

for some non-negative constant \( B \) depending only on \( h_0 \). It follows that:

\[
\lim_{t \to \infty} \left. \frac{d}{ds} \right|_{s = t} \mathcal{M}(h_s) = 2\pi \text{rk}(F)(\mu_F - \mu_E).
\]

We conclude that if \( E_2 \) is a destabilizing subbundle of \( E \), then \( \mathcal{M} \) has negative slope along \( h_t \); if \( \mu_{E_2} = \mu_E \), then \( \mathcal{M} \) is monotonically decreasing along \( h_t \), with asymptotic slope 0.

Moreover, we see that \( \mathcal{M} \) is zero along \( h_t \) if and only if there exists a holomorphic splitting \( E = E_2 \oplus E_2^{h_0} \) with \( \mu_{E_2} = \mu_E \).
Proof. The expression for $\mathcal{M}(h_t)$ and the asymptotic slope follow from Theorem 3.3. If $E_2$ is destabilizing, then $\mu_{E_2} > \mu_E$ and $\mu_F < \mu_E$, and the asymptotic slope is negative. If $\mu_{E_2} = \mu_E$, then $\mu_E = \mu_F$ and $\mathcal{M}(h_t) = -B(1 - e^{-t})$ is decreasing in $t$ as $B \geq 0$.

Finally, since $B = \|\beta\|^2_{h_0}$, where $\beta$ is the second fundamental form of $E_2$ in $(E, h_0)$, we have $B = 0$ if and only if there is a holomorphic splitting $E = E_2 \oplus E_2^{\perp h_0}$ (see Proposition 1.6.14). If $E = E_2 \oplus E_2^{\perp h_0}$ is a holomorphic splitting and $\mu_{E_2} = \mu_E$, then $\mu_E = \mu_F$ and $B = 0$ and $\mathcal{M}$ is zero along $h_t$. Conversely, if $\mathcal{M}$ is zero along $h_t$, then $\mu_E = \mu_F$ and $B = 0$, which implies that $E = E_2 \oplus E_2^{\perp h_0}$ is a holomorphic splitting with $\mu_E = \mu_{E_2}$.

We also have:

Corollary 3.5. Suppose that $\{E_i\}_{i=1}^m$ is a filtration of $E$, and $h_t$ is a geodesic of the form constructed in Theorem 3.3, for some $\lambda_1 > \ldots > \lambda_m > 0$. If $\mathcal{M}(h_t)$ has non-positive asymptotic slope, then at least one of the $E_i$ is such that $\mu_{E_i} \geq \mu_E$.

Proof. Using Lemma 3.3, we have that
\[ \sum_i \lambda_i r_i (\mu_{F_i} - \mu_E) \leq 0. \]

We need to show that $\mu_{E_i} \geq \mu_E$ for some $2 \leq i \leq m$. Note that $\deg(E_i) = \sum_{j=1}^m \deg(F_j) = \sum_{j=1}^m r_j \mu_{F_j}$, i.e. we need to show that
\[ \frac{r_i \mu_{F_i} + r_{i+1} \mu_{F_{i+1}} + \cdots + r_m \mu_{F_m}}{r_i + r_{i+1} + \cdots + r_m} \geq \mu_E. \]

Rearranging, we see that it is enough to show that
\[ \sum_{j=1}^m r_j (\mu_{F_j} - \mu_E) \geq 0. \]

for some $2 \leq i \leq m$. Applying Lemma 3.6 with $a_i = r_i (\mu_{F_i} - \mu_E)$, we get the required result.

Lemma 3.6. Let $\lambda_1, \ldots, \lambda_m, a_1, \ldots, a_m \in \mathbb{R}$ such that $\lambda_1 > \ldots > \lambda_m$, $\sum_i a_i = 0$, and $\sum_i \lambda_i a_i < 0$ (respectively $< 0$). Then, there exist $2 \leq i \leq m$ such that $a_i + \cdots + a_m > 0$ (respectively $> 0$).

Proof. We prove the result by induction on $m$ for the non-strict inequality. The case of the strict inequality follows similarly. First consider the base case when $m = 2$. Since replacing $(\lambda_1, \lambda_2)$ by $(\lambda_1 + c, \lambda_2 + c)$ for $c \in \mathbb{R}$ leaves the sum $\lambda_1 a_1 + \lambda_2 a_2$ unchanged, we may assume that $\lambda_1 + \lambda_2 = 0$ i.e. $\lambda_2 = -\lambda_1$ and $\lambda_1$ is positive. If $a_i = -a_2$, then $-2\lambda_1 a_2 = \lambda_1 a_1 + \lambda_2 a_2 \leq 0$, giving $a_2 \geq 0$.

Now consider the case when $m > 2$. If $a_m \geq 0$, we pick $i = m$ and we are done. So assume that $a_m < 0$. Since $\lambda_{m-1} > \lambda_m$, we have that $\lambda_{m-1} a_m < \lambda_m a_m$. Then,
\[ 0 \geq \sum_{i=1}^m \lambda_i a_i \geq \lambda_1 a_1 + \ldots + \lambda_{m-2} a_{m-2} + \lambda_{m-1} (a_{m-1} + a_m). \]
Applying induction hypothesis to \((\lambda_1, \ldots, \lambda_{m-1}), (a_1, \ldots, a_{m-2}, a_{m-1} + a_m)\), we get the claim. \(\square\)

3.2. General Filtrations. Consider a general filtration of saturated holomorphic sub-sheaves \(E = \mathcal{E}_1 \supset \mathcal{E}_2 \supset \cdots \supset \mathcal{E}_m \supset \{0\}\). Again let \(\lambda_1 > \ldots > \lambda_m\) and \(F_i = \mathcal{E}_i/\mathcal{E}_{i+1}\).

By regularizing the \(\mathcal{E}_i\), as in \cite[Proposition 4.3]{Sib15}, we can construct a geodesic ray as in Theorem 3.3, we can find a smooth endomorphism \(\tilde{\mu}\) such that the saturation of \(f\) as in Theorem 3.3 on a proper modification of \(X\) from fixed data.

**Theorem 3.7.** There exist \(w \in \mathcal{H}^{1, p}\) such that

\[
\mathcal{M}(tw) = 2\pi \sum_{i=1}^{m} \lambda_i \text{rk}(F_i)(\mu_{F_i} - \mu_E)t - \sum_{1 \leq i < j \leq m} B_{ji}(1 - e^{-t(\lambda_i - \lambda_j)}),
\]

where \(B_{ji}\) is a non-negative integer.

Note that Theorem A is a direct corollary of this result.

**Proof.** Using \cite[Proposition 4.3]{Sib15}, we can find a proper modification \(f : \tilde{X} \to X\) such that the saturation of \(f^*\mathcal{E}_i\) in \(f^*E\) is a sub-bundle for all \(i\). Let \(E_i\) denote the saturation of \(f^*\mathcal{E}_i\) in \(f^*E\) and \(F_i\) denote the quotient bundle \(f^*E/E_i\). Applying the construction of Theorem 3.3, we can find a smooth endomorphism \(\tilde{w} \in \mathcal{H}_{\tilde{X}}\) such that

\[
\mathcal{M}_{\tilde{X}}(t\tilde{w}) = 2\pi \sum_{i=1}^{m} \lambda_i \text{rk}(F_i)(\mu_{F_i} - \mu_E)t - \sum_{1 \leq i < j \leq m} B_{ji}(1 - e^{-t(\lambda_i - \lambda_j)}),
\]

where \(\mathcal{M}_{\tilde{X}}\) is the Donaldson functional on \(\tilde{X}\) with respect to the reference metric \(f^*h_0\) and the form \(f^*\omega\). Note that the form \(f^*\omega\) is no longer a Kähler form on \(\tilde{X}\); however we do not require this in the construction done in Theorem 3.3.

Since \(f\) is smooth and the pushforward on currents commutes with \(d\), we see that \(w := f_*\tilde{w} \in \mathcal{H}^{1, \infty}\). To finish the proof, it is enough to show that \(\mathcal{M}_{\tilde{X}}(t\tilde{w}) = \mathcal{M}(tw)\) and that \(\mu_{F_i} = \mu_{\mathcal{E}_i}\). The first part follows from applying the change of variables formula to the definition of \(\mathcal{M}_{\tilde{X}}\) and \(\mathcal{M}\). To show that \(\mu_{F_i} = \mu_{\mathcal{E}_i}\), it is enough to check that \(\mu_{E_i} = \mu_{\mathcal{E}_i}\). Since \(\mathcal{E}_i\) is torsion-free, \(f^*\mathcal{E}_i\) and its saturation can only differ along a submanifold of codimension at least 2. Thus \(f^*\det(\mathcal{E}_i)\) and \(\det(E_i)\) are line bundles that differ along a submanifold of codimension at least 2 and therefore must be isomorphic. We have

\[
\mu_{\mathcal{E}_i} = \frac{\deg_\omega(\det(\mathcal{E}_i))}{\text{rk}(\mathcal{E}_i)} = \frac{\deg_{f^*\omega}(\det(E_i))}{\text{rk}(E_i)} = \mu_{E_i}.
\]

The following result follows from Corollary 3.5

**Proposition 3.8.** Let \(w\) be as constructed in Theorem 3.7. If we further have that \(\mathcal{M}(tw) \leq 0\) for all \(t \geq 0\), then there exists \(2 \leq i \leq m\) such that \(\mu_{\mathcal{E}_i} \geq \mu_E\). \(\square\)

4. Further Properties of the Donaldson Functional

Throughout this section and the next, we use \(C\) to represent a positive constant, whose exact value may change from line to line, but which can always be computed from fixed data.
4.1. **Reverse Sobolev Inequality.** The following estimate is essentially due to Donaldson; it immediately gives a reverse Sobolev inequality for \( w \in \mathcal{S} \), showing that \( \|w\|_{W^{1,p}} \) is equivalent to \( \|w\|_{L^{p^*}} \) when both the norms are large.

**Proposition 4.1.** Suppose that \( w \in H^{1,p} \). Then there exists a constant \( C \geq 0 \), depending only on \( (X,\omega) \) and \( (E,h_0) \), such that:

\[
\|w\|_{W^{1,p}} \leq CM(w) + C(\|w\|_{L^{p^*}} + 1).
\]

**Proof.** Recall that \( M(w) = \int_X f_w \omega^n + \int_X \Theta_0 w \wedge \omega^{n-1} \). Therefore,

\[
\int_X f_w \omega^n \leq M(w) + \left| \int_X \Theta_0 w \wedge \omega^{n-1} \right|.
\]

Now we find a lower bound for \( \int_X f_w \omega^n \). Since

\[
e^x - 1 - x \geq \frac{1}{2(|x| + 1)},
\]

we have that

\[
\int_X f_w \omega^n \geq \frac{1}{2} \int_X \frac{\sum_{i,j} |h_{ij}|^2}{|w| + 1} \omega^n = \frac{1}{2} \int_X \frac{|\bar{\omega} \partial w|^2}{|w| + 1} \omega^n,
\]

where \( |w| \) and \( |\bar{\omega} \partial w| \) denote the pointwise operator norms of \( w \) and \( \bar{\omega} \partial w \). Writing \( |\bar{\omega} \partial w|^p = \frac{|\bar{\omega} \partial w|^p}{(|w| + 1)^{p/2}} \cdot (|w| + 1)^{p/2} \) and applying the H"older inequality with conjugate exponents \( 2/p \) and \( 2/(2 - p) \) gives:

\[
\|\bar{\omega} \partial w\|_{L^p} \leq \left\| \frac{|\bar{\omega} \partial w|}{\sqrt{|w| + 1}} \right\|_{L^2} \cdot \left( |w| + 1 \right)^{1/2},
\]

since \( p^* = \frac{p}{2-p} \). It follows that

\[
\int_X f_w \omega^n \geq \frac{1}{2} \left\| \frac{|\bar{\omega} \partial w|^2}{|w| + 1} \right\|_{L^p} \geq \frac{1}{2} \left\| \bar{\omega} \partial w \right\|_{L^p}^2 + C.
\]

On the other hand, since \( h_0 \) is a smooth Hermitian metric, we also have that

\[
\left| \int_X \Theta_0 w \wedge \omega^{n-1} \right| \leq C\|w\|_{L^{p^*}}.
\]

Combining the two inequalities above, we get that

\[
\left\| \bar{\omega} \partial w \right\|_{L^p}^2 \leq 2(M(w) + C\|w\|_{L^{p^*}}).
\]

Simplifying, we get that

\[
\left\| \bar{\omega} \partial w \right\|_{L^p} \leq 2(M(w) + C\|w\|_{L^{p^*}})\left(\|w\|_{L^{p^*}} + C\right)
\]

\[
\leq \frac{M(w) + C\|w\|_{L^{p^*}} + C}{\sqrt{2}},
\]

by the AM-GM inequality. Since \( p \leq p^* \), we conclude. \( \square \)
4.2. Lower-semi-continuity.

**Proposition 4.2.** The Donaldson functional is lower-semi-continuous on $\mathcal{H}^{1,p}$ with respect to the weak $W^{1,p}$-topology, i.e. if $w_i \to w$ then:

$$\mathcal{M}(w) \leq \liminf_{k \to \infty} \mathcal{M}(w_k).$$

**Proof.** The proof will rely on several lemmas, whose proofs we provide at the end – the idea is to establish lower semi-continuity of the $w$ and $\partial w$ terms in $\mathcal{M}(w)$ separately, and then combine them.

We start with some setup; we can find a subsequence $\{w_{k_\ell}\}$ such that:

$$\lim_{\ell \to \infty} \mathcal{M}(w_{k_\ell}) = \liminf_{k \to \infty} \mathcal{M}(w_k).$$

By the Sobolev embedding theorem, $w_{k_\ell}$ converges to $w$ in $L^p$, so we can extract a further subsequence which converges pointwise a.e.. Hence, by relabeling, we may assume without loss of generality that $w_k \to w$ pointwise a.e, in $L^p$, and weakly in $W^{1,p}$.

Since $\Theta_0$ is smooth, $\int_X \text{tr}(\Theta_0 w_k) \wedge \omega^{n-1}$ is continuous in $k$, so we only need to show that:

$$\liminf_{k \to \infty} \int_X f_{w_k} \omega^n \geq \int_X f_w \omega^n.$$

The problem is local, so fix a trivializing open $U$ for $E$. Let $\{e_i\}$ be a unitary basis of $E|_U$ that such that, $D$, the matrix of $w$ is diagonal with respect to $\{e_i\}$ and has decreasing entries. We also consider unitary changes of bases $A_k$ which diagonalize $w_k$, so that, in the fixed $\{e_i\}$ basis, $w_k = A_k D_k A_k^*$ for a diagonal matrix $D_k$, whose entries are organized in decreasing order.

We seek to work with pointwise values for $w$, which requires us to deal with eigenvalues with multiplicity. To this end, consider:

$$\mathcal{R} = \left\{ (r_1, \ldots, r_m) \bigg| r_1, \ldots, r_m \in \mathbb{Z}_{>0} \text{ for some } m > 0, \text{ and } \sum_{i=1}^{m} r_i = r \right\},$$

the set of all partitions of $r$. For each $\underline{r} \in \mathcal{R}$, define $U_{\underline{r}}$ to be the set of all $z \in X$ such that $w$ has eigenvalues $\lambda_r > \ldots > \lambda_m$, with multiplicities given by $\underline{r}$ at $z$. Then $U_{\underline{r}}$ is a measurable set and:

$$U := \bigsqcup_{\underline{r} \in \mathcal{R}} U_{\underline{r}}.$$

Fix $\underline{r} \in \mathcal{R}$. Then, $D = \text{diag}(\lambda_1 \text{Id}_{r_1}, \ldots, \lambda_m \text{Id}_{r_m})$, on $U_{\underline{r}}$ where $\lambda_1 > \ldots > \lambda_m$, and $\lambda_i : U_{\underline{r}} \to \mathbb{R}_{\geq0}$ can vary with $z \in U_{\underline{r}}$.

Pointwise convergence implies that $A_k D_k A_k^* \to D$. If we write:

$$A_k = \begin{pmatrix} (A_k)^1_1 \cdots (A_k)^1_m \\ \vdots \\ (A_k)^m_1 \cdots (A_k)^m_m \end{pmatrix}$$

for $r_i \times r_j$-block matrices $(A_k)^i_j$, then we can apply Lemma 4.4 below to get that $D_k \to D$ and that each of the $(A_k)^i_j$ is almost unitary, in the sense that:

- $(A_k)^i_j ((A_k)^i_j)^* \to \text{Id}_{r_i}$ and $((A_k)^i_j)^* (A_k)^i_j \to \text{Id}_{r_i}$
\( (A_k)^i_j \to 0 \) for \( i \neq j \).

Define \( \phi(x) := \frac{e^x - 1 - x}{x^2} \), \( x \in \mathbb{R} \), with \( \phi(0) := \frac{1}{2} \). Write \( \lambda_{k,1} \geq \ldots \geq \lambda_{k,\rho} \) for the eigenvalues of \( w_k \), and define \( \Lambda_{k,i,a} := \lambda_{1,i+a}, \ldots, \lambda_{r_i,i+a} \), for any \( 1 \leq i \leq m \) and \( 1 \leq a \leq r_i \). We have that \( \Lambda_{k,i,a} \to \lambda_i \) from the pointwise convergence, and hence \( \phi(\Lambda_{k,i,a} - \Lambda_{k,j,b}) \to \phi(\lambda_i - \lambda_j) \) as \( k \to \infty \).

Write \( \overline{\partial}w_k = \eta_k \) in the \( A_k \)-basis; in the fixed \( \{e_i\} \)-basis, this becomes:
\[
\overline{\eta}_k = A_k \eta_k A_k^*,
\]
which by Lemma 4.3 weakly converges to \( \eta = \overline{\partial}w \). We now have that:
\[
\int_{U_r} f_{w_k} = \sum_{1 \leq i,j \leq m} \sum_{1 \leq a \leq r_i} \sum_{1 \leq b \leq r_j} \phi(\Lambda_{k,i,a} - \Lambda_{k,j,b}) |((\eta_k)^i_j)^b_a| \omega^n
\]
on all of \( U_r \); thus, we may conclude the proof by applying Lemma 4.5 with \( Y = U_r \),
\[
A = \{(a,b)| 1 \leq a \leq r_i, 1 \leq b \leq r_j\}, f_{k,\alpha} = \chi_U \cdot \phi(\Lambda_{k,i,a} - \Lambda_{k,j,b}), \ f = \chi_U \cdot \phi(\lambda_i - \lambda_j), \mu_{k,\alpha} = |((\eta_k)^i_j)^b_a|^2 \text{ and } \mu = |\eta_k|^2,
\]
provided that we first establish:
\[
\lim_{k \to \infty} \int_S |(\eta_k)^i_j|^2 \omega^n \geq \int_S |\eta_j|^2 \omega^n
\]
for every \( 1 \leq i, j \leq m \) and every measurable \( S \subset U_r \).

We show this by observing the the \( L^2 \) norm is lower semicontinuous with respect to weak \( L^1 \) convergence, essentially. Start by recalling that:
\[
\int_S |\eta_j|^2 \omega^n = \sup_{\|g\|_{L^2(S)} = 1} \left| \int_S \text{tr}(\eta_j^* g^*) \omega^n \right|
\]
where \( g \) is a \( r_i \times r_j \) matrix valued function. Fix such a \( g \), and note that:
\[
\int_S \text{tr}((\overline{\eta}_k)^i_j g^*) \omega^n \to \int_S \text{tr}(\eta_j^* g^*) \omega^n,
\]
by the weak convergence of the \( \overline{\eta}_k \).

Let \( \varepsilon > 0 \). Since \( (A_k)^i_j^*(A_k)^i_j) \to \text{Id}_n \) and \( (A_k)^i_j \to 0 \) pointwise, by Egorov’s theorem, there exists a set \( S_1 \subset S \) on which these convergences are uniform and the measure of \( S \setminus S_1 \) is as small as desired. By enlarging \( S_1 \) further, we may also assume that \( \|g\|_{L^q(S \setminus S_1)} \leq \varepsilon \), where \( q := \frac{p}{p-1} \).

Since \( \overline{\eta}_k = A_k \eta_k A_k^* \), we have
\[
(\overline{\eta}_k)^i_j = \sum_{a,b} (A_k)^i_a (\eta_k)^a_b ((A_k)^j_b)^*,
\]
we break up the integral \( \int_S \text{tr}((\overline{\eta}_k)^i_j g^*) \omega^n \) in to a sum of terms of two different types – the first type has either \( i \neq a \) or \( j \neq b \), and the second has \( i = a \) and \( j = b \).

Consider a term of the first type, and assume without loss of generality that \( i \neq a \). Then \( (A_k)^i_a \to 0 \) uniformly on \( S_1 \), so that \( |(A_k)^i_a| < \frac{\varepsilon}{\|g\|_{L^q(S)}} \) on \( S_1 \) for all \( k \gg 0 \). Thus,
we have
\[
\left| \int_S \text{tr}((A_k)^{i_j}(\eta_k)^{i_j}((A_k)^{i_j})^* g^*)^n \right|
\]
\[=
\left| \int_{S_1} \text{tr}((A_k)^{i_j}(\eta_k)^{i_j}((A_k)^{i_j})^* g^*)^n \right| + \left| \int_{S \setminus S_1} \text{tr}((A_k)^{i_j}(\eta_k)^{i_j}((A_k)^{i_j})^* g^*)^n \right|
\]
\[\leq \frac{\epsilon}{\|g\|_{L^2(S)}} \int_{S_1} |\eta_k||g|^n + \int_{S \setminus S_1} |\eta_k||g|^n
\]
\[\leq \frac{\epsilon}{\|g\|_{L^2(S)}} \|\partial w_k\|_{L^p} \|g\|_{L^q(S)} + \|\partial w_k\|_{L^p} \|g\|_{L^q(S \setminus S_1)}
\]
\[\leq \epsilon \|\partial w_k\|_{L^p} + \epsilon \|\partial w_k\|_{L^p}
\]
\[\leq C \epsilon,
\]
where the last line follows from the fact that $\partial w_k \rightarrow \partial w$ in $L^p$ and thus $\lim \sup_k \|\partial w_k\|_{L^p} < \infty$, independent of $k$.

To analyze terms of the second type, consider
\[
\left| \int_S \text{tr}((A_k)^{i_j}(\eta_k)^{i_j}((A_k)^{i_j})^* g^*)^n \right|
\]
\[\leq \|g\|_{L^2(S)} \left( \int_{S_1} \text{tr}((A_k)^{i_j}(\eta_k)^{i_j}((A_k)^{i_j})^* g^* (A_k)^{j_i}((\eta_k)^{j_i}((A_k)^{i_j})^* (A_k)^{i_j}) \right)^{\frac{1}{2}} + \int_{S \setminus S_1} \|\eta_k||g|^n.
\]
The second term is again under control by Hölder’s inequality:
\[
\int_{S \setminus S_1} |\eta_k||g|^n \leq \|\partial w_k\|_{L^p} \|g\|_{L^q(S \setminus S_1)} \leq C \epsilon.
\]
Since $\|g\|_{L^2(S_1)} \leq 1$, the first term can then be estimated as:
\[
\int_{S_1} \text{tr}((\eta_k)^{i_j}((A_k)^{j_i}((A_k)^{i_j})^* g^*)((\eta_k)^{j_i}((A_k)^{i_j})^* (A_k)^{i_j})
\]
\[\leq (1 + \epsilon)^2 \int_{S_1} |(\eta_k)^{i_j}|^2 \leq (1 + \epsilon)^2 \|\eta_k\|_{L^2(S)}^2.
\]

since $|(A_k)^{i_j}((A_k)^{i_j})^* - \text{Id}_r| < \epsilon$ on $S_1$ for all $i$ and for all $k \gg 0$.

We now have control over both types of terms, so taking $k \rightarrow \infty$ gives:
\[
\left| \int_S \text{tr}(\eta_j^i g^*)^n \right| \leq (1 + \epsilon) \left( \lim \inf_k \|\eta_k\|_{L^2(S)} \right) + C \epsilon.
\]

Using $\|\eta_j^i\|_{L^2(S)} = \sup_{g \in L^2(S) \setminus L^\infty(S), \|g\|_{L^2(S)} = 1} \int_S \text{tr}(\eta_j^i g^*)^n |g|$, we get that
\[
\|\eta_j^i\|_{L^2(S)} \leq (1 + \epsilon) \left( \lim \inf_k \|\eta_k\|_{L^2(S)} \right) + C \epsilon.
\]

Taking $\epsilon \rightarrow 0$, we get (4.1), as desired.
Lemma 4.3. Let $E|_U$ be a trivializable vector bundle. Consider an $L^\infty$ unitary basis $\{e_i\}$ of $E|_U$ i.e. a unitary basis of $E$ obtained by applying an $L^\infty$ unitary matrix to a smooth unitary basis of $E$. Then, $T_k \to T$ in $L^p$ for $T_k, T \in \operatorname{End}(E|_U)$ if and only if the same holds for their matrix representatives with respect to the basis $\{e_i\}$.

Proof. Note that $T_k \to T$ if and only if the same holds true for their matrix representatives with respect to a smooth unitary frame of $E$. By abuse of notation, denote these matrices as $T_k, T$ and let $A$ denote the change of basis from this smooth unitary frame to $\{e_i\}$. Then, we need to show that $T_k \to T$ if and only if $AT_kA^* \to ATA^*$. But this is true as $A \in L^\infty$.

Lemma 4.4. Let $r = \sum_{i=1}^m r_i$ and write $B^j_i$ for the (ordered) $r_i \times r_j$-blocks of any $B \in M_{r \times r}(\mathbb{C})$, as usual. Assume that $D_k, D \in M_{r \times r}(\mathbb{C})$ and $A_k \in U(r)$ are such that $D_k, D$ are diagonal, with their elements arranged in monotonically decreasing order, and $D = \operatorname{diag}(\lambda_1 \operatorname{Id}_{r_1}, \ldots, \lambda_m \operatorname{Id}_{r_m})$ for some $\lambda_1 > \cdots > \lambda_m$. Assume further that $A_k D_k A_k^* \to D$ as $k \to \infty$. Then,

- $D_k \to D$,
- $(A_k)_i^j(A_k^*)_i^j \to \operatorname{Id}_{r_j}$ and $(A_k^*)_i^j(A_k)_i^j \to \operatorname{Id}_{r_i}$, and
- $(A_k)_i^i \to 0$ for $i \neq j$.

as $k \to \infty$.

Proof. The first part follows from the fact that eigenvalues are a continuous function of the entries of a matrix and that $A_k D_k A_k$ and $D_k$ have the same eigenvalues.

Since $(D_k - D) \to 0$ and the entries of $A_k$ and $A_k^*$ are uniformly bounded, we also get that $A_k(D_k - D)A_k^* \to 0$. Thus, we see that $A_k D A_k^* \to D$, i.e.

$$\sum_{\ell} \lambda_i(A_k)_i^\ell(A_k^*)_i^\ell \to \begin{cases} 0 & \text{if } i \neq j \\ \lambda_i \operatorname{Id}_{r_i} & \text{if } i = j \end{cases}.$$  

In particular, picking $i = j = 1$, we get that

$$\sum_{\ell} \lambda_\ell(A_k)_1^\ell(A_k^*)_1^\ell \to \lambda_1 \operatorname{Id}_{r_1}$$

and

$$\sum_{\ell} (\lambda_1 - \lambda_\ell)(A_k)_1^\ell(A_k^*)_1^\ell \to 0.$$  

Since $A_k$ is unitary, we have that $\sum_{\ell}(A_k)_i^\ell(A_k^*)_i^\ell = 0$ if $i \neq j$ and $\sum_{\ell}(A_k)_i^\ell(A_k^*)_i^\ell = \operatorname{Id}_{r_i}$. Since $(\lambda_1 - \lambda_\ell) > 0$ for $\ell \geq 2$ and the diagonal entries of $(A_k)_1^\ell(A_k^*)_1^\ell$ are non-negative, we get that the diagonal entries of $(A_k)_1^\ell(A_k^*)_1^\ell$ converge to 0 for all $\ell \geq 2$ as $k \to \infty$, which implies that $(A_k)_1^\ell \to 0$ for $\ell \geq 2$. Now using $\sum_{\ell}(A_k)_i^\ell(A_k^*)_i^\ell = \operatorname{Id}_{r_i}$, we also get that $(A_k)_1^\ell(A_k^*)_1^\ell \to \operatorname{Id}_{r_1}$. Applying induction to

$$A = \begin{pmatrix} A_1^1 & \cdots & A_1^m \\ \vdots & \ddots & \vdots \\ A_m^1 & \cdots & A_m^m \end{pmatrix}$$

and $D = \operatorname{diag}(\lambda_2 \operatorname{Id}_{r_2}, \ldots, \lambda_m \operatorname{Id}_{r_m})$, we get the required result.
Lemma 4.5. Let $A$ be a finite set. For $k \in \mathbb{N}$ and for $\alpha \in A$, let $\mu_{k,\alpha}, \mu$ be positive Radon measures on a measure set $Y$ such that
\[
\liminf_{k \to \infty} \left( \sum_{\alpha \in A} \mu_{k,\alpha}(S) \right) \geq \mu(S)
\]
for all measurable $S \subset Y$. Let $f_{k,\alpha}, f$ be non-negative measurable functions on $U$ such that $f_{k,\alpha} \to f$ pointwise a.e. on $Y$ as $k \to \infty$ for all $\alpha \in A$. Then,
\[
\liminf_{k \to \infty} \left( \int_Y \sum_{\alpha} f_{k,\alpha} \mu_{k,\alpha} \right) \geq \int_Y f \mu.
\]

Proof. Let $Y_i$ be measurable sets and $\phi = \sum_{i=0}^{m} a_i I_{Y_i}$ for $0 < a_0 < \cdots < a_m < \infty$ be a simple function such that $\phi \leq f$ and $\sum_i \mu(Y_i) < \infty$. It is enough to prove that for all such $\phi$, we have
\[
\liminf_{k \to \infty} \int_X \sum_{\alpha} f_{k,\alpha} \mu_{k,\alpha} \geq \int_X \phi \mu.
\]
Fix $\epsilon > 0$. Let $S_k = \{x \in X : f_{k,\alpha}(x) \geq (1 - \epsilon) \phi \} \subset Y$. Thus, there exits an $k_0$ such that
\[
\mu(Y_i \cap S_k) \geq \mu(Y_i) - \frac{1}{2} \epsilon
\]
for all $\alpha \in A$. By hypothesis, there exists a $k_0$ such that
\[
\sum_{\alpha} \mu_{\alpha}(Y_i \cap S_k) \geq \mu(Y_i) - \frac{1}{2} \epsilon
\]
for all $k \geq k_0$ and all $i$. Since $\sum_{\alpha} \mu_{\alpha}(Y_i \cap S_k) \geq \mu(Y_i) - \epsilon$ for all $i$, we get
\[
\sum_{\alpha} \mu_{\alpha}(Y_i \cap S_k) \geq \mu(Y_i) - \epsilon
\]
for all $k \geq k_0$. Thus, for $k \geq k_0$, we get,
\[
\int_Y \sum_{\alpha} f_{k,\alpha} \mu_{k,\alpha} \geq \int_{S_k} \sum_{\alpha} f_{k,\alpha} \mu_{k,\alpha}
\]
\[
\geq (1 - \epsilon) \int_{S_k} \phi \sum_{\alpha} \mu_{k,\alpha}
\]
\[
= (1 - \epsilon) \sum_{i=0}^{m} a_i \left( \sum_{\alpha} \mu_{k,\alpha}(Y_i \cap S_k) \right)
\]
\[
\geq (1 - \epsilon) \sum_{i=0}^{m} a_i (\mu(Y_i) - \epsilon)
\]
\[
= (1 - \epsilon) \left( \int_S \phi \mu \right) - \epsilon(1 - \epsilon) \sum_i a_i.
\]

Letting $\epsilon \to 0$, we get the required result. \hfill \Box

5. Proofs of Theorems B and C

We are now ready to prove the remaining results in the Introduction.
5.1. Proof of Theorem B. We must show that if $\mathcal{M}$ fails to be proper on $\mathcal{H}^{1,p}$, then there exists some $w \in \mathcal{H}^{1,p}$ such that $\mathcal{M}(tw) \leq 0$ for all $t \geq 0$.

By assumption, there exists a sequence $w_k \in \mathcal{H}^{1,p}$ such that

$$\mathcal{M}(w_k) < k^{-1}\|w_k\|_{W^{1,p}} - k.$$  

We claim that $\|w_k\|_{W^{1,p}} \to \infty$ as $k \to \infty$. Recall that

$$\mathcal{M}(w_k) \geq -C\|w_k\|_{L^1} \geq -C\|w_k\|_{W^{1,p}},$$  

for some constant $C > 0$ depending only on $h_0, \omega$. Combining the two bounds for $\mathcal{M}(w_k)$, we get

$$-C\|w_k\|_{W^{1,p}} \leq k^{-1}\|w_k\|_{W^{1,p}} - k,$$

and simplifying, we see that

$$\|w_k\|_{W^{1,p}} \geq \frac{k}{C + k^{-1}}.$$  

In particular, we get that $\|w_k\|_{W^{1,p}} \to \infty$.

Let $\tilde{w}_k = \frac{w_k}{\|w_k\|_{W^{1,p}}}$. The $\tilde{w}_k$ are clearly bounded in $W^{1,p}$, and hence we can extract a weakly convergent subsequence, which we still denote $\tilde{w}_k$. Let $w \in \mathcal{H}^{1,p}$ be the weak limit.

We claim that $w \neq 0$. To see this, use the reverse Sobolev inequality, Proposition 4.1, to see that:

$$\|w_k\|_{W^{1,p}} \leq C(\|w_k\|_{L^{p^*}} + 1)$$

Taking $k$ sufficiently large and then applying the usual Sobolev inequality gives:

$$\frac{1}{2C} \|w_k\|_{W^{1,p}} \leq \|w_k\|_{L^{p^*}} \leq C \|w_k\|_{W^{1,p}},$$  

and so we conclude $\frac{1}{2C} \leq \|w\|_{L^{p^*}} \leq C$ for all $k \geq 0$.

By the Rellich–Kondrachov theorem, the embedding of $W^{1,p} \to L^{p^*}$ is compact (since $p^* < p'$) and we further have that $\tilde{w}_k \to w$ in $L^{p^*}$. Thus $\|w\|_{L^{p^*}} = \lim_{k \to \infty} \|\tilde{w}_k\|_{L^{p^*}} > 0$ so that $w \neq 0$.

We now examine the (non-trivial) geodesic ray $\{tw\}_{t \geq 0}$; fix some $t > 0$. By convexity of $\mathcal{M}$, we have that

$$\mathcal{M}(tw_k) \leq \frac{t\mathcal{M}(w_k)}{\|w_k\|_{W^{1,p}}} \leq \frac{tk}{k} - \frac{tk}{\|w_k\|_{W^{1,p}}} \leq \frac{tk}{k},$$

and hence $\liminf_k \mathcal{M}(tw_k) \leq 0$. Now by the semicontinuity of $\mathcal{M}$ (Proposition 4.2), we get that $\mathcal{M}(tw) \leq 0$ for all $t \geq 0$.

5.2. Proof of Theorem C. Now suppose we are given $w \in \mathcal{H}^{1,p}$, such that $\mathcal{M}(tw) \leq 0$ for all $t \geq 0$. We must prove that $w$ defines a filtration of $E$ by holomorphic subsheaves, and, after resolving the filtration, $e^{tw}h_0$ pulls back to a geodesic ray of the type constructed in Theorem 3.7 Moreover, one of these subsheaves is such that $\mu_{E_i} \geq \mu_E$.

Recall that $\mathcal{M}(tw) = \int_X f_{tw}^n + t\cdot\int_X \text{tr}(\omega_0w) \wedge \omega^{n-1}$. Since the first term is positive, and the second term grows linearly in $t$, we have that

$$\int_X f_{tw}^n \leq C_1 t$$
for all $t$ and for some positive constant $C_1$. Consider a unitary basis $\{e\}$ with respect to which $w$ is a diagonal matrix. Let the matrix of $w$ and $\partial w$ with respect to $\{e\}$ be $\text{diag}(\zeta_1, \zeta_2, \ldots, \zeta_r)$ and $\{\eta^a_b\}_{1 \leq a, b \leq r}$ respectively where we order the eigenvalues as $\zeta_1 \geq \zeta_2 \geq \ldots \geq \zeta_r$. Recall that

$$f_{tw} = \sum_{1 \leq a, b \leq r} |\eta^a_b|^2 \frac{e^{t(\zeta_i - \zeta_j)} - 1 - t(\zeta_i - \zeta_j)}{(\zeta_i - \zeta_j)^2}$$

Since $\frac{e^{x} - 1 - x}{x^2} \geq \frac{1}{2}$ when $x \geq 0$, we get that

$$f_{tw} \geq \frac{t^2}{2} \sum_{a \leq b} |\eta^a_b|^2.$$ 

Since $\int_X f_{tw} \omega^n < Ct$ for all $t$, we get that $\eta^a_b = 0$ almost everywhere for all $a \leq b$.

Now we claim that the eigenvalues of $w$ are constant on $X$. To see this, consider a change of basis $A$ from a holomorphic frame of $E$ to $\{e\}$. Then,

$$\eta = A^{-1} \partial (A \cdot D \cdot A^{-1}) A$$

$$\eta = A^{-1} (\partial A) D + \partial D - DA^{-1} \partial A,$$

where $D$ denotes the diagonal matrix $\text{diag}(\zeta_1, \ldots, \zeta_r)$. Computing the $(a, b)$-th entry, we get that

$$\eta^a_b = \begin{cases} (A^{-1} \partial A)^a_b (\zeta_b - \zeta_a) & \text{if } a \neq b \\ \partial \zeta_a & \text{if } a = b \end{cases}.$$ 

Using $\partial \zeta_a = (\eta^a_a = 0$, we get that the eigenvalues of $w$ are constant on $X$. For the remainder of the proof, let us relabel the eigenvalues of $w$ as $\lambda_1, \ldots, \lambda_m$ such that the matrix of $w$ in the basis $\{e\}$ is given by $\text{diag}(\lambda_1 \text{Id}_{r_1}, \ldots, \lambda_m \text{Id}_{r_m})$, and $\lambda_1 > \cdots > \lambda_m$. We also change our notation so that $\eta^i_j$ and $(A^{-1} \partial A)^i_j$ now denote the $r_i \times r_j$ block matrices of $\eta$ and $A^{-1} \partial A$ for $1 \leq i, j \leq m$.

Now using $\eta^i_j = 0$ for $i < j$, we get that $(A^{-1} \partial A)^i_j = 0$ for $i < j$. We now use this and Theorem 2.7 to show that projection to the sum of eigenspaces of $\lambda_1, \ldots, \lambda_m$ gives rise to a holomorphic filtration of $E$. Let $\pi_s$ denote the orthogonal projection to the sum of eigenspaces of $\lambda_s, \ldots, \lambda_m$ i.e. with respect to the basis $\{e\}$, the matrix of $\pi_s$ is

$$\Pi_s = \text{diag}(0 \cdot \text{Id}_{r_1 + \cdots + r_{s-1}}, \text{Id}_{r_s + \cdots + r_m}).$$

Let us also denote $M := A^{-1} \partial A$ and write

$$M = \begin{pmatrix} M_1 & 0 \\ M_2 & M_3 \end{pmatrix}$$

where $M_1$ is an $(r_1 + \cdots + r_{s-1}) \times (r_1 + \cdots + r_{s-1})$ block matrix and $M_3$ is an $(r_s + \cdots + r_m) \times (r_s + \cdots + r_m)$ block matrix.

Firstly, we claim that $\pi_s \in \mathcal{H}^{1, 2}$. It is clear that $\pi \in L^2$ as the operator norm of $\pi_s$ is 1. To see that $\partial \pi_s \in L^2$, the matrix of $\partial \pi_s$ with respect to the frame $\{e\}$ is given by
\[
A^{-1} \overline{\partial} (A \Pi_s A^{-1}) A = (A^{-1} \overline{\partial} A) \Pi_s - \Pi_s (A^{-1} \overline{\partial} A)
\]

\[
= \begin{pmatrix}
0 & 0 \\
M_2 & 0
\end{pmatrix}
\]

Since \( f_w \in L^1(X) \) and since the eigenvalues of \( w \) are constant, we get that \( \eta^i_j \) and thus \( (A^{-1} \overline{\partial} A)^i_j \) are in \( L^2 \). Since the entries of \( \overline{\partial} \pi_s \) are either \( (M_2)^i_j \) or 0, \( \overline{\partial} \pi_s \in L^2 \).

We also have that \((I - \pi_s) \overline{\partial} \pi_s \) in the frame \( \{e\} \) is given by

\[
\begin{pmatrix}
\text{Id}_{r_s} & \cdots & \text{Id}_{r_1} & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\
M_2 & 0
\end{pmatrix} = 0
\]

Using Theorem 2.7, we get that the image of \( \pi_s \) is a subsheaf \( E_s \subset E \) of rank \( r_s + \cdots + r_m \), and we get a filtration \( E = E_1 \supset \cdots \supset E_m \ni 0 \). Note that \( w \) acts by \( \lambda_i \) on \( \mathcal{F}_i = E_i/E_{i+1} \) and thus is of the form constructed in Theorem 3.7. Since \( \mathcal{M}(tw) \leq 0 \) for all \( t \geq 0 \), it follows that

\[
2\pi \sum_{i=1}^m \lambda_i \text{rk}(\mathcal{F}_i)(\mu_{\mathcal{F}_i} - \mu_E) \leq 0.
\]

The inequality \( \mu_{E_i} \geq \mu_E \) now follows from Corollary 3.5.

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