INTERSECTION BODIES WITH CERTAIN SYMMETRIES

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Abstract. We generalize the class of intersection bodies in \( \mathbb{R}^n \) by imposing invariance under a certain subgroup of orthogonal transformations. We show that this class of bodies shares many properties with their real counterparts.

Introduction

Intersection bodies were introduced by E. Lutwak in 1988 in his celebrated paper [31] in connection with the Busemann-Petty problem. We recall that an origin-symmetric star body \( K \) in \( \mathbb{R}^n \) is an intersection body of an origin-symmetric star body \( L \) if the radius of \( K \) in every direction equals to the \((n-1)\)-dimensional volume of the central hyperplane section of \( L \) perpendicular to this direction. In other words, for every unit vector \( \xi \) in \( \mathbb{R}^n \),

\[
\|\cdot\|_K^{-1} = |L \cap \xi^\perp|,
\]

where \( \| \cdot \|_K \) is the Minkowski functional of \( K \), \( \xi^\perp \) is the hyperplane perpendicular to \( \xi \) and \( | \cdot | \) denotes the Euclidean volume. Using polar coordinates, equation (1) becomes

\[
\|\xi\|_K^{-1} = \frac{1}{n-1} \int_{S^{n-1}} \|\theta\|_{L}^{-n+1} d\theta = \frac{1}{n-1} \mathcal{R}_{n-1}(\|\cdot\|_{L}^{-n+1})(\xi),
\]

where \( \mathcal{R}_{n-1} \) is the spherical Radon transform. Hence, a star body \( K \) in \( \mathbb{R}^n \) is the intersection body of a star body if and only if \( \| \cdot \|_K^{-1} \) is the spherical Radon transform of a continuous positive function on \( S^{n-1} \).

A more general class of intersection bodies in \( \mathbb{R}^n \) was introduced by P. Goodey, E. Lutwak and W. Weil in 1996 in [13]. A star body \( K \) is an intersection body if there exists a finite non-negative Borel measure \( \mu \) on the sphere so that \( \| \cdot \|_K^{-1} = \mathcal{R}_{n-1}\mu \).

Intersection bodies in \( \mathbb{R}^n \) have been an object of extensive study for many years, see [8, 17, 22, 29] and the references therein. The analogous class of bodies in \( \mathbb{C}^n \) was studied by A. Koldobsky, G. Paouris and M. Zymonopoulou in [28]. Let \( K \) be a complex convex body in \( \mathbb{C}^n \), i.e. \( K \) is a convex body in \( \mathbb{R}^{2n} \) that is invariant under the block diagonal subgroup of \( \text{SO}(2n) \) of the form

\[
\{ \text{diag}(g, \ldots, g) : g \in \text{SO}(2) \},
\]
where \( \text{SO}(\cdot) \) stands for the special orthogonal group over the reals. Intersection bodies in \( \mathbb{C}^n \) were defined along the same lines as intersection bodies in \( \mathbb{R}^n \), taking into account the above invariance. They inherit many properties of their real counterparts.

The goal of this paper is to study intersection bodies in \( \mathbb{R}^\kappa n \) that are invariant under the block diagonal subgroup of \( \text{SO}(\kappa n) \) of the form

\[
\{ \text{diag}(g, \ldots, g) : g \in \text{SO}(\kappa) \},
\]

where \( \kappa \in \mathbb{N} \) is fixed. Subsets of \( \mathbb{R}^\kappa n \) that satisfy the above invariance will be called \( \kappa \)-balanced. In this paper we only concern ourselves with \( \kappa \)-balanced sets. By \( \mathbb{K}^n \) we denote the space \( \mathbb{R}^\kappa n \) with the property that all geometric objects in this space (such as star shaped bodies, linear subspaces, etc.) satisfy the above invariance, see Section 1. For \( \kappa = 1, 2, 4, \mathbb{K}^n \) can be thought of as the \( n \)-dimensional real, complex or quaternionic vector space, respectively; however our results hold in more generality for any \( \kappa \in \mathbb{N} \).

In our discussion we follow ideas from [28]. We generalize to \( \mathbb{K}^n \) many known results from the theory of intersection bodies in \( \mathbb{R}^n \) and \( \mathbb{C}^n \). We organized this paper as follows. In Section 1 we define intersection bodies of star bodies in \( \mathbb{K}^n \). In Section 2 we introduce the spherical Radon transform on \( \mathbb{K}^n \) and prove that it coincides with the Fourier transform of distributions on the class of \((\kappa n + \kappa)\)-homogeneous functions on \( \mathbb{R}^\kappa n \) that are \( \kappa \)-invariant, see Lemma 7. This allows to express the volume of sections of star bodies in \( \mathbb{K}^n \) in Fourier analytic terms, see Theorem 1. Intersection bodies in \( \mathbb{K}^n \) are introduced in Section 3; here we also prove their Fourier analytic characterization in Theorem 2. In Section 4 we use the above characterization to show that intersection bodies in \( \mathbb{K}^n \) coincide with two generalizations of real intersection bodies due to A. Koldobsky and G. Zhang: the \( \kappa \)-balanced \( \kappa \)-intersection bodies in \( \mathbb{R}^\kappa n \) and \( \kappa \)-balanced generalized \( \kappa \)-intersection bodies in \( \mathbb{R}^\kappa n \), see Corollary 2 and Proposition 1. In turn, this allows to extend to \( \mathbb{K}^n \) the result of P. Goodey and W. Weil that intersection bodies in \( \mathbb{R}^n \) can be obtained as the closure in the radial metric of radial sums of ellipsoids, see Theorem 3. In Section 5 the main results are Theorems 4-6. Theorem 5 deals with the Busemann-Petty problem in \( \mathbb{K}^n \) for arbitrary measures. From the stability consideration in this problem we derive the inequality for the volume of sections by \( \kappa \)-dimensional subspaces in Theorem 6. In Lemma 12 and its Corollaries we describe inequalities obtained from the stability consideration mentioned above; here we take advantage of the fact that we solve the stability question with different density functions for the volume of the body and the volume of sections. In Theorem 4 and Corollary 6 we consider the stability in the Busemann-Petty problem in \( \mathbb{K}^n \) and derive the related inequality for the Euclidean volume of sections by \( \kappa \)-dimensional subspaces. Finally, intersection bodies of convex bodies in \( \mathbb{K}^n \) are studied in Section 6; here, in Theorem 7 and Corollary 11 we
extend to $\mathbb{K}^n$ two classical results about intersection bodies of convex bodies in $\mathbb{R}^n$: Busemann’s and Hensley-Borell theorems. We introduce the notation and preliminaries throughout the article as needed.

1. Intersection Bodies of Star Bodies in $\mathbb{K}^n$

Let $\kappa \in \mathbb{N}$ and $x = (x_1, x_2, \ldots, x_{\kappa n}) \in \mathbb{R}^n$. We view $x$ as an ordered set of $n$ ordered $\kappa$-tuples. For every $\sigma \in \text{SO}(\kappa)$ define

$$R_\sigma(x) := (\sigma(x_1), \ldots, x_\kappa, \ldots, \sigma(x_{\kappa(n-1)+1}), \ldots, x_{\kappa n})$$

to be the vector obtained by rotating the ordered $\kappa$-tuples of $x$. A set $M$ in $\mathbb{R}^n$ is called $\kappa$-balanced if

$$\|x\|_M = \|\sigma(x_1, \ldots, x_\kappa, \ldots, x_{\kappa(n-1)+1}, \ldots, x_{\kappa n})\|_M$$

for every $x \in \mathbb{R}^n$ and for every $\sigma \in \text{SO}(\kappa)$. We work exclusively with geometric objects in $\mathbb{R}^n$ that are $\kappa$-balanced. For clarity and the easy of notation, we denote by $\mathbb{K}^n$ the space $\mathbb{R}^n$ with the additional property that all geometric objects in this space satisfy the above invariance.

We call a set in $\mathbb{K}^n$ a convex body if it is a compact $\kappa$-balanced convex set in $\mathbb{R}^n$ with non-empty interior. Recall that a compact subset $K$ of $\mathbb{R}^n$ containing the origin as an interior point is called a star body if every line through the origin crosses the boundary in exactly two points different from the origin. Its Minkowski functional is defined by

$$\|x\|_K := \min\{a \geq 0 : x \in aK\},$$

with $x \in \mathbb{R}^n$, and its radial function by

$$\rho_K(x) := \max\{a > 0 : ax \in K\}.$$  

For $x \in S^{n-1}$, $\rho_K(x) = \|x\|^{-1}_K$, is the Euclidean distance from the origin to the boundary of $K$ in the direction $x$. The set of $\kappa$-balanced star bodies in $\mathbb{R}^n$ forms the class of star bodies in $\mathbb{K}^n$.

Now we introduce the notion of a hyperplane in $\mathbb{K}^n$. For $\kappa \geq 2$, fix an orthogonal sequence $\{I = \sigma_0, \sigma_1, \ldots, \sigma_{\kappa-1}\}$ in $\text{SO}(\kappa)$, meaning that for every $x \in \mathbb{R}^n$ the vectors $x, \sigma_1(x), \ldots, \sigma_{\kappa-1}(x)$ are mutually orthogonal. Then the sequence of rotations $\{I = R_{\sigma_0}, R_{\sigma_1}, \ldots, R_{\sigma_{\kappa-1}}\}$ in $\text{SO}(\kappa n)$ is orthogonal as well. For an element $\xi \in S^{\kappa n-1}$, we denote by $H^\perp_\xi$ the $\kappa$-dimensional subspace of $\mathbb{R}^n$ spanned by the vectors $\{R_{\sigma_i}(\xi)\}_{i=0}^{\kappa-1}$, and by $H_\xi$ its orthogonal complement. $H_\xi$ is the $(\kappa n - \kappa)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to the vectors $\{R_{\sigma_i}(\xi)\}_{i=0}^{\kappa-1}$. We call $H_\xi$ the hyperplane in $\mathbb{K}^n$ determined by the vector $\xi$. Note that the $\text{SO}(\kappa)$-orbit of a vector $x \in \mathbb{R}^n$, in other words the set $\{R_\sigma x : \sigma \in \text{SO}(\kappa)\}$, is contained in the subspace $H^\perp_x$.  

Definition 1. Let $K$ and $L$ be star bodies in $\mathbb{R}^n$. We call $K$ an intersection body of $L$ in $\mathbb{R}^n$ and denote it by $K = I_K(L)$ if for every $\xi \in S^{n-1}$

$$|K \cap H_\xi| = |L \cap H_\xi|.$$  

(2)

Observe that for a $\kappa$-balanced star body, the set $K \cap H_\xi$ is a $\kappa$-dimensional ball of radius $\|\xi\|_K^{-1}$ and thus, by the polar formula for the volume, equation (2) becomes

$$\frac{\Omega_\kappa}{\kappa}\|\xi\|_{I_K(L)} = |L \cap H_\xi|,$$

(3)

where $\Omega_\kappa$ stands for the surface area of the unit ball in $\mathbb{R}^\kappa$.

2. The Radon and Fourier Transforms of $\kappa$-invariant Functions

We call a function $f$ on $\mathbb{R}^n$ $\kappa$-invariant if $f(x) = f(R_\sigma x)$ for every $x \in \mathbb{R}^n$ and $\sigma \in SO(\kappa)$, and denote continuous $\kappa$-invariant real-valued functions on the unit sphere by $C_\kappa(S^{n-1})$. The spherical Radon transform on $\mathbb{K}^n$, denote it by $\mathcal{R}^\kappa$, is an operator from $C_\kappa(S^{n-1})$ to itself, defined by

$$\mathcal{R}^\kappa f(\xi) = \int_{S^{n-1} \cap H_\xi} f(x)dx.$$  

The polar formula for the volume yields

$$|L \cap H_\xi| = \frac{1}{\kappa n - \kappa} \mathcal{R}^\kappa(\|\cdot\|_{L^{\kappa n+\kappa}})(\xi)$$

(4)

for any star body $L$ in $\mathbb{K}^n$ and $\xi \in S^{n-1}$. Moreover, condition (2) becomes

$$\|\xi\|_{I_K(L)} = \frac{1}{(n-1)\Omega_\kappa} \mathcal{R}^\kappa(\|\cdot\|_{L^{\kappa n+\kappa}})(\xi).$$

(5)

We conclude that a star body $K$ in $\mathbb{K}^n$ is an intersection body of a star body if and only if the function $\|\xi\|_{K^{-\kappa}}$ is a spherical Radon transform on $\mathbb{K}^n$ of a positive $\kappa$-invariant continuous function on $S^{n-1}$.

We will generalize several classical facts, connecting the Radon and Fourier transforms. We start by recalling the relevant concepts and facts in $\mathbb{R}^n$.

One of the main tools used in this paper is the Fourier transform of distributions, see [11] for details. Denote by $S(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, also referred to as test functions, and by $S'(\mathbb{R}^n)$ the space of distributions on $\mathbb{R}^n$, the continuous dual of $S(\mathbb{R}^n)$. The Fourier transform $\hat{f}$ of a distribution $f$ is defined by

$$\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle$$

for every test function $\varphi$. For an even test function $\varphi$, the Fourier transform is self-invertible up to a constant factor: $(\varphi^\vee)^\wedge = (2\pi)^n \varphi$.

A distribution $f$ on $\mathbb{R}^n$ is even homogeneous of degree $p \in \mathbb{R}$, if

$$\langle f(x), \varphi \left( \frac{x}{\alpha} \right) \rangle = |\alpha|^{n+p} \langle f, \varphi \rangle$$
for every test function $\varphi$ and every $\alpha \in \mathbb{R}, \alpha \neq 0$. The Fourier transform of an even homogeneous distribution of degree $p$ is an even homogeneous distribution of degree $-n - p$. We call a distribution $f$ positive definite if its Fourier transform is a positive distribution, i.e. $\langle \hat{f}, \varphi \rangle \geq 0$ for every non-negative test function $\varphi$. A measure $\mu$ is tempered if for some $\beta > 0$
\[ \int_{\mathbb{R}^n} (1 + |x|^2)^{-\beta} d\mu(x) < \infty. \]
A distribution is positive definite if and only if it is the Fourier transform of a tempered measure on $\mathbb{R}^n$, see [10], p.152. Let $K$ be an origin-symmetric star body in $\mathbb{R}^n$. For $0 < p < n$, the function $\| \cdot \|_K^{-p}$ is locally integrable on $\mathbb{R}^n$, and represents an even homogeneous distribution of degree $-n - p$, see [22], Lemma 2.1. In case $\| \cdot \|_K^{-p}$ is also positive definite, then its Fourier transform is a tempered measure and a homogeneous distribution of degree $-n + p$; we have
\[ \int_{\mathbb{R}^n} \|x\|_K^{-p} \varphi(x) dx = \int_{S^{n-1}} \left( \int_0^\infty t^{n-1} \hat{\varphi}(t\xi) \right) d\mu(\xi), \]
for every test function $\varphi$, see [22], Corollary 2.26 (i).

Let $f$ be an even continuous function on $S^{n-1}$ and let $p$ be a non-zero real number. We extend $f$ to an even homogeneous function on $\mathbb{R}^n$ of degree $p$ in the usual way as follows. Let $x \in \mathbb{R}^n$, then $x = r\theta$ with $r = |x|_2$ and $\theta = x/|x|_2$. We put
\[ f \cdot r^p(x) = f(\theta)r^p. \]
It was shown in [22], Lemma 3.16, that for an infinitely-smooth function $f$ on $S^{n-1}$ and $-n < p < 0$, the Fourier transform of $f \cdot r^{-p}$ is an infinitely-smooth function on $\mathbb{R}^n \setminus \{0\}$, homogeneous of degree $-n + p$.

We shall often use Parseval’s formula on the sphere:

**Lemma 1.** ([22], Lemma 3.22) Let $f$ and $g$ be even infinitely-smooth functions on $S^{n-1}$ and let $0 < p < n$. Then
\[ \int_{S^{n-1}} (f \cdot r^{-p})^\wedge(\theta)(g \cdot r^{-n+p})^\wedge(\theta)d\theta = (2\pi)^n \int_{S^{n-1}} f(\theta)g(\theta)d\theta. \]

Another basic fact from Fourier analysis is the following.

**Lemma 2.** ([22], Lemma 3.24) Let $0 < k < n$, and let $\varphi \in S(\mathbb{R}^n)$ be an even test function. Then for any $(n-k)$-dimensional subspace $H$ of $\mathbb{R}^n$
\[ (2\pi)^k \int_H \varphi(x) dx = \int_{H^\perp} \hat{\varphi}(x) dx. \]

The spherical version of the above lemma allows to express the volume of lower-dimensional sections of an origin-symmetric star body $K$ in $\mathbb{R}^n$ in Fourier analytic terms.
Lemma 3. ([22], Lemma 3.25) Let $0 < k < n$, and let $\varphi$ be an even infinitely-smooth function on $S^{n-1}$. Then for any $(n-k)$-dimensional subspace $H$ of $\mathbb{R}^n$

$$(2\pi)^k \int_{S^{n-1} \cap H} \varphi(x) dx = \int_{S^{n-1} \cap H^\perp} (\varphi \cdot r^{-n+k})^\wedge(x) dx.$$ 

The $\kappa$-invariance of a function translates into a certain invariance of its Fourier transform. The following lemma is a functional analog of the Lemma 2 in [42].

Lemma 4. Suppose that $f$ is an even infinitely-smooth $\kappa$-invariant function on $S^{\kappa n-1}$. Then for any $0 < p < \kappa n$ and any $\xi \in S^{\kappa n-1}$ the Fourier transform of the distribution $f \cdot r^{-p}$ is a constant function on $S^{\kappa n-1} \cap H^\perp_\xi$.

Proof: The Fourier transform of $f \cdot r^{-p}$ is a continuous function outside of the origin in $\mathbb{R}^{\kappa n}$ by Lemma 3.16 in [22]. Since the function $f$ is $\kappa$-invariant, by the connection between the Fourier transform of distributions and linear transformations, the Fourier transform of $f \cdot r^{-p}$ is also $\kappa$-invariant. Recall that the $\kappa$-dimensional space $H^\perp_\xi$ is spanned by the vectors $\{R_{\sigma_i}(\xi)\}_{i=0}^{\kappa-1}$, where $\{R_{\sigma_i}\}_{i=0}^{\kappa-1}$ is an orthogonal sequence of $\kappa$-tuple-wise rotations in $SO(\kappa n)$, see Section 1. Consequently, every vector in $S^{\kappa n-1} \cap H^\perp_\xi$ is the image of $\xi$ under one of the $\kappa$-tuple-wise rotations in $SO(\kappa n)$, so the Fourier transform of $f \cdot r^{-p}$ is a constant function on $S^{\kappa n-1} \cap H^\perp_\xi$.

Lemma 5. Let $\varphi$ be an even infinitely-smooth $\kappa$-invariant function on $S^{\kappa n-1}$, then for $\xi \in S^{\kappa n-1}$

$$\mathcal{R}^\kappa \varphi(\xi) = \frac{\Omega_\kappa}{(2\pi)^\kappa} (\varphi \cdot r^{-\kappa n+k})^\wedge(\xi).$$

Proof: By Lemma 3, we have

$$\mathcal{R}^\kappa \varphi(\xi) = \int_{S^{\kappa n-1} \cap H^\perp_\xi} \varphi(x) dx = \frac{1}{(2\pi)^\kappa} \int_{S^{\kappa n-1} \cap H^\perp_\xi} (\varphi \cdot r^{-\kappa n+k})^\wedge(x) dx.$$ 

Since $\varphi$ is $\kappa$-invariant, by Lemma 4 the integrand on the right-hand side is a constant function on $S^{\kappa n-1} \cap H^\perp_\xi$, which itself is a $\kappa$-dimensional Euclidean unit sphere. Hence

$$\int_{S^{\kappa n-1} \cap H^\perp_\xi} (\varphi \cdot r^{-\kappa n+k})^\wedge(x) dx = \Omega_\kappa (\varphi \cdot r^{-\kappa n+k})^\wedge(\xi).$$

The smoothness assumption in the above lemma can be removed. It is an analog of Lemma 3.7 in [22], see also Lemma 4 in [28]. Beforehand we need the following fact.
Lemma 6. The spherical Radon transform on $\mathbb{R}^n$ is self-dual, i.e. for any even continuous $\kappa$-invariant functions $f, g$ on $S^{n-1}$

$$
\int_{S^{n-1}} \mathcal{R}^\kappa f(\xi)g(\xi)d\xi = \int_{S^{n-1}} f(\theta)\mathcal{R}^\kappa g(\theta)d\theta.
$$

Proof: We can assume that functions $f, g$ are infinitely-smooth. The Fourier transform of the homogeneous extension of $g$ of degree $-\kappa n + \kappa$ is an infinitely-smooth $\kappa$-invariant homogeneous function of degree $-\kappa$ on $\mathbb{R}^n \setminus \{0\}$, so for some infinitely-smooth $\kappa$-invariant function $h$ on $S^{n-1}$

$$(g \cdot r^{-\kappa n + \kappa})^\wedge = (2\pi)^\kappa h \cdot r^{-\kappa}.$$ 

Using Lemma 5 and spherical Parseval’s formula, we now compute

$$
\int_{S^{n-1}} \mathcal{R}^\kappa f(\xi)g(\xi)d\xi = \frac{\Omega_\kappa}{(2\pi)^\kappa} \int_{S^{n-1}} (f \cdot r^{-\kappa n + \kappa})^\wedge(\xi)(g \cdot r^{-\kappa n + \kappa})(\xi)d\xi
$$

$$
= \frac{\Omega_\kappa}{(2\pi)^\kappa} \int_{S^{n-1}} (f \cdot r^{-\kappa n + \kappa})^\wedge(\xi)(h \cdot r^{-\kappa})^\wedge(\xi)d\xi
$$

$$
= \frac{\Omega_\kappa}{(2\pi)^\kappa} \int_{S^{n-1}} f(\theta)(h \cdot r^{-\kappa})(\theta)d\theta
$$

$$
= \frac{\Omega_\kappa}{(2\pi)^\kappa} \int_{S^{n-1}} f(\theta)(g \cdot r^{-\kappa n + \kappa})^\wedge(\theta)d\theta
$$

$$
= \int_{S^{n-1}} f(\theta)\mathcal{R}^\kappa g(\theta)d\theta.
$$

We say that a distribution $f$ on $\mathbb{R}^n$ is $\kappa$-invariant if $\langle f, \varphi(R_\sigma \cdot) \rangle = \langle f, \varphi \rangle$ for every test function $\varphi$ and for every $\sigma \in \text{SO}(\kappa)$. Note that if two $\kappa$-invariant distributions coincide on the set of $\kappa$-invariant test functions, then they are equal.

Lemma 7. Let $f$ be an even continuous $\kappa$-invariant function on $S^{n-1}$, then for $\xi \in S^{n-1}$

$$
\mathcal{R}^\kappa f(\xi) = \frac{\Omega_\kappa}{(2\pi)^\kappa} (f \cdot r^{-\kappa n + \kappa})^\wedge(\xi),
$$

where $\Omega_\kappa$ stands for the surface area of the unit ball in $\mathbb{R}^\kappa$.

Proof: Let $\varphi$ be any $\kappa$-invariant test function, then

$$
\int_{H^\perp_\xi} \hat{\varphi}(x)dx = \int_{S^{n-1} \cap H^\perp_\xi} \int_0^\infty \hat{\varphi}(t\theta)t^{\kappa-1}drd\theta = \Omega_\kappa \int_0^\infty \hat{\varphi}(t\xi)t^{\kappa-1}dr.
$$
Using this observation we compute
\[
\langle (f \cdot r^{-\kappa n+\kappa})^\wedge, \varphi \rangle = \int_{S^{n-1}} f(\theta) \int_0^\infty \hat{\varphi}(t\theta)r^{\kappa-1}drd\theta
\]
by Lemma 2
\[
= \frac{(2\pi)^\kappa}{\Omega_\kappa} \int_{S^{n-1}} f(\theta) \int_{H_\theta} \hat{\varphi}(x)dx d\theta
\]
by Lemma 6
\[
= \frac{(2\pi)^\kappa}{\Omega_\kappa} \int_{S^{n-1}} f(\theta) \mathcal{R}^\kappa \left( \int_0^\infty \varphi(r\cdot)r^{\kappa n-\kappa-1}dr \right)(\theta)d\theta
\]
This shows that \(\kappa\)-invariant distributions \((f \cdot r^{-\kappa n+\kappa})^\wedge\) and \(\frac{(2\pi)^\kappa}{\Omega_\kappa}|x|_2^{-\kappa} \mathcal{R}^\kappa f\left(\frac{x}{|x|_2}\right)\) coincide on the set of \(\kappa\)-invariant test functions and are therefore equal.

The above lemma allows to express the volume of sections of star bodies as a Fourier transform of a certain function. The real version of this fact was proved in [19], the complex version was proved in [26] and [28], see also Theorem 1 in [42] for a different proof of this result for infinitely-smooth bodies.

**Theorem 1.** For any origin-symmetric star body \(K\) in \(\mathbb{R}^n\) and for any unit vector \(\xi \in \mathbb{R}^n\) we have
\[
|K \cap H_\xi| = \frac{\Omega_\kappa}{(2\pi)^\kappa(\kappa n - \kappa)}(\|\cdot\|_K^{-\kappa n+\kappa})^\wedge(\xi),
\]
where \(H_\xi\) is the hyperplane in \(\mathbb{R}^n\) determined by \(\xi\), see Section 1 for the definition, and \(\Omega_\kappa\) is the surface area of the unit ball in \(\mathbb{R}^n\).

**Proof:** By (4) and Lemma 7 we compute
\[
|K \cap H_\xi| = \frac{1}{\kappa n - \kappa} \mathcal{R}^\kappa(\|\cdot\|_K^{-\kappa n+\kappa})(\xi) = \frac{\Omega_\kappa}{(2\pi)^\kappa(\kappa n - \kappa)}(\|\cdot\|_K^{-\kappa n+\kappa})^\wedge(\xi).
\]
\(\square\)
As in \( \mathbb{R}^n \), Theorem 1 provides a simple proof of the Minkowski’s uniqueness theorem saying that an origin-symmetric star body is uniquely determined by the volume of its central hyperplane sections.

**Corollary 1.** Let \( K, L \) be origin-symmetric star bodies in \( \mathbb{K}^n \). If for every direction \( \xi \in S^{\kappa n-1} \)
\[
|K \cap H_\xi| = |L \cap H_\xi|,
\]
then \( K = L \).

**Proof:** By Theorem 1 the hypothesis of the corollary implies that homogeneous of degree \(-\kappa\) continuous functions on \( \mathbb{R}^{\kappa n} \setminus \{0\} \), \((\| \cdot \|^{-\kappa+\kappa})^\wedge\) and \((\| \cdot \|_L^{-\kappa+\kappa})^\wedge\) coincide on the sphere \( S^{\kappa n-1} \). Thus they coincide as distributions on the whole \( \mathbb{R}^{\kappa n} \). The result follows by the uniqueness theorem for the Fourier transform of distributions.

\[\square\]

3. Intersection Bodies in \( \mathbb{K}^n \)

Intersection bodies of star bodies in \( \mathbb{K}^n \) were introduced in Section 1. Now we define a more general class of intersection bodies by extending the equality (5) to measures, as it was done in [13] for the real case and in [28] for the complex case. A finite Borel measure \( \mu \) on the sphere \( S^{\kappa n-1} \) is called \( \kappa \)-invariant if for any continuous function \( f \) on the sphere \( S^{\kappa n-1} \) and for any \( \sigma \in SO(\kappa) \)
\[
\int_{S^{\kappa n-1}} f(x) d\mu(x) = \int_{S^{\kappa n-1}} f(R_\sigma x) d\mu(x).
\]
The spherical Radon transform on \( \mathbb{K}^n \) of an \( \kappa \)-invariant measure \( \mu \) on the sphere \( S^{\kappa n-1} \) is defined as a functional \( \mathcal{R}^{\kappa} \mu \) on the space of \( C_\kappa(S^{\kappa n-1}) \) by
\[
(\mathcal{R}^{\kappa} \mu, f) = \int_{S^{\kappa n-1}} \mathcal{R}^{\kappa} f(x) d\mu(x).
\]
Surely, the spherical Radon transform on \( \mathbb{K}^n \) of a finite \( \kappa \)-invariant Borel measure \( \mu \) on \( S^{\kappa n-1} \), \( \mathcal{R}^{\kappa} \mu \), is again a finite \( \kappa \)-invariant Borel measure on \( S^{\kappa n-1} \). From the self-duality of the spherical Radon transform on \( \mathbb{K}^n \), Lemma 6, it follows that if the measure \( \mu \) has density \( f \), then the measure \( \mathcal{R}^{\kappa} \mu \) has density \( \mathcal{R}^{\kappa} f \).

**Definition 2.** An origin-symmetric star body \( K \) in \( \mathbb{K}^n \) is called an intersection body in \( \mathbb{K}^n \) if there exists a finite \( \kappa \)-invariant Borel measure \( \mu \) on the sphere \( S^{\kappa n-1} \) so that \( \| \cdot \|_K^{-\kappa} \) and \( \mathcal{R}^{\kappa} \mu \) are equal as functionals on \( C_\kappa(S^{\kappa n-1}) \); that is, if for any \( f \in C_\kappa(S^{\kappa n-1}) \)
\[
\int_{S^{\kappa n-1}} \| x \|_K^{-\kappa} f(x) dx = \int_{S^{\kappa n-1}} \mathcal{R}^{\kappa} f(x) d\mu(x).
\]
It follows from the self-duality of the spherical Radon transform on $\mathbb{K}^n$ and equation (5), that every intersection body of a star body in $\mathbb{K}^n$ is an intersection body in $\mathbb{K}^n$ in the sense of Definition 2.

It was shown in [20] that real intersection bodies admit the following Fourier analytic characterization: an origin-symmetric star body $K$ in $\mathbb{R}^n$ is an intersection body if and only if the function $\| \cdot \|^{-1}_K$ represents a positive definite distribution. Intersection bodies in $\mathbb{K}^n$ allow for a similar characterization. It is easy to see this for intersection bodies of star bodies in $\mathbb{K}^n$. By Theorem 1 we have:

$$\| \xi \|^{-\kappa}_{I_K(L)} = \frac{\kappa}{\Omega_\kappa} |L \cap H_\xi| = \frac{1}{(2\pi)^{\kappa}(n-1)}(\| \cdot \|^\kappa_{L^\kappa})^\wedge(\xi).$$

Both sides are even homogeneous functions of degree $-\kappa$ and agree on $S^{\kappa n-1}$, so they are equal as distributions on $\mathbb{R}^\kappa$. Since the Fourier transform of even distributions is self-invertible up to a constant factor, we get

$$\| \cdot \|^\kappa_{I_K(L)}^{-1} = (2\pi)^\kappa (2\pi)^{\kappa(n-1)} \| \cdot \|^{-\kappa+\kappa}_L > 0.$$  \hspace{1cm} (6)

Thus $\| \cdot \|^\kappa_{I_K(L)}^{-1}$ is positive definite. Furthermore, if the Fourier transform of $\| \cdot \|^\kappa_K$ is an even strictly positive $\kappa$-invariant function on the sphere, then using equation (6) we can construct a star body $L$ in $\mathbb{K}^n$ so that $K = I_K(L)$.

Next we show that this Fourier analytic characterization holds for arbitrary intersection bodies in $\mathbb{K}^n$.

**Theorem 2.** An origin-symmetric star body $K$ in $\mathbb{K}^n$ is an intersection body in $\mathbb{K}^n$ if and only if $\| \cdot \|^\kappa_K$ represents a positive definite distribution on $\mathbb{R}^{\kappa n}$.

**Proof:** Suppose that $K$ is an intersection body in $\mathbb{K}^n$ with the corresponding measure $\mu$. It is enough to show $\langle (\| \cdot \|^\kappa_K)^\wedge, \varphi \rangle \geq 0$ for every even $\kappa$-invariant non-negative test function $\varphi$. We compute

$$\langle (\| \cdot \|^\kappa_K)^\wedge, \varphi \rangle = \int_{\mathbb{K}^{\kappa n}} \| x \|^\kappa_K \hat{\varphi}(x) dx$$

$$= \int_{S^{\kappa n-1}} \| \theta \|^\kappa_K \left( \int_0^\infty \hat{\varphi}(r\theta) r^{\kappa n-\kappa-1} dr \right) d\theta$$

by Definition 2

$$= \int_{S^{\kappa n-1}} \mathcal{R}^\kappa \left( \int_0^\infty \hat{\varphi}(r) r^{\kappa n-\kappa-1} dr \right) (\theta) d\mu(\theta)$$

$$= \int_{S^{\kappa n-1}} \int_{S^{\kappa n-1} \cap H_\theta} \int_0^\infty \hat{\varphi}(r x) r^{\kappa n-\kappa-1} dr dx d\mu(\theta)$$

$$= \int_{S^{\kappa n-1}} \left( \int_{H_\theta} \hat{\varphi}(x) dx \right) d\mu(\theta)$$
by Lemma 2
\[ = (2\pi)^{\kappa n - \kappa} \int_{S^{n-1}} \left( \int_{H^\perp_\theta} \varphi(x) dx \right) d\mu(\theta) \geq 0. \]

Now suppose that \( \| \cdot \|_{K}^{-\kappa} \) is a positive definite distribution, then there exists a finite Borel measure \( \mu \) on \( S^{n-1} \) so that for every test function \( \varphi \)
\[ \int_{\mathbb{R}^n} \| x \|_{K}^{-\kappa} \varphi(x) dx = \int_{S^{n-1}} \left( \int_{0}^{\infty} t^{\kappa - 1} \hat{\varphi}(t\xi) \right) d\mu(\xi), \]
see Section 2. Since the body \( K \) is \( \kappa \)-balanced, we can assume that the measure \( \mu \) is \( \kappa \)-invariant as well. Recall from the proof of Lemma 7, that for a \( \kappa \)-invariant test functions
\[ \int_{H^\perp_\xi} \hat{\varphi}(x) dx = \Omega_\kappa \int_{0}^{\infty} \hat{\varphi}(t\xi) r^{\kappa - 1} dr. \]

Thus for even \( \kappa \)-invariant test functions, the right-hand side of equation (7) can be written as
\[ \frac{1}{\Omega_\kappa} \int_{S^{n-1}} \left( \int_{H^\perp_\xi} \hat{\varphi}(x) dx \right) d\mu(\xi) = \frac{(2\pi)^{\kappa}}{\Omega_\kappa} \int_{S^{n-1}} \left( \int_{H^\perp_\xi} \varphi(x) dx \right) d\mu(\xi), \]
where we used Lemma 2, and now writing the interior integral in polar coordinates gives, we obtain
\[ = \frac{(2\pi)^{\kappa}}{\Omega_\kappa} \int_{S^{n-1}} \mathcal{R}^\kappa \left( \int_{0}^{\infty} \varphi(r\cdot) r^{\kappa n - \kappa - 1} dr \right)(\xi) d\mu(\xi). \]

Writing the left-hand side in equation (7) in polar coordinates, we obtain that for any even \( \kappa \)-invariant test function \( \varphi \)
\[ \int_{S^{n-1}} \| \theta \|_{K}^{-\kappa} \left( \int_{0}^{\infty} \varphi(r\theta) r^{\kappa n - \kappa - 1} dr \right) d\theta = \frac{(2\pi)^{\kappa}}{\Omega_\kappa} \int_{S^{n-1}} \mathcal{R}^\kappa \left( \int_{0}^{\infty} \varphi(r\cdot) r^{\kappa n - \kappa - 1} dr \right)(\xi) d\mu(\xi). \]

Let \( u \) be some non-negative test function on \( \mathbb{R} \) and let \( v \) be an arbitrary infinitely-smooth even \( \kappa \)-invariant function on \( S^{n-1} \). For \( x \in \mathbb{R}^n \), set \( \varphi(x) = u(r) v(\theta) \), where \( x = r\theta \) with \( r \in [0, \infty) \) and \( \theta \in S^{n-1} \). Evaluating equation (8) for such test functions \( \varphi \), yields
\[ \int_{S^{n-1}} \| \theta \|_{K}^{-\kappa} v(\theta) d\theta = \frac{(2\pi)^{\kappa}}{\Omega_\kappa} \int_{S^{n-1}} \mathcal{R}^\kappa v(\xi) d\mu(\xi). \]
Since infinitely-smooth functions on the sphere are dense in the space of continuous functions on the sphere, the latter equation holds for $v \in C_\kappa(S^{kn-1})$, which implies that $K$ is an intersection body in $\mathbb{K}^n$.

4. Characterization of intersection bodies in $\mathbb{K}^n$

Intersection bodies in $\mathbb{K}^n$ are related to two generalizations of real intersection bodies. Consequently they inherit many of their properties.

One generalization, $k$-intersection bodies, was introduced by A. Koldobsky in [20, 21] as follows. Let $M, L$ be star bodies in $\mathbb{R}^n$ and let $k$ be an integer, $0 < k < n$. We say that $M$ is a $k$-intersection body of $L$ if for every $(n-k)$-dimensional subspace $H$ of $\mathbb{R}^n$,

$$|M \cap H^\perp| = |L \cap H|.$$ 

A more general class of $k$-intersection bodies was defined in [21] as follows.

**Definition 3.** Let $0 < k < n$. We say that an origin-symmetric star body $M$ in $\mathbb{R}^n$ is a $k$-intersection body if there exists a measure $\mu$ on $S^{n-1}$ such that for every test function $\varphi$ in $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} \|x\|^{-k}_M \varphi(x) dx = \int_{S^{n-1}} \left( \int_0^\infty t^{k-1} \hat{\varphi}(t\xi) \right) d\mu(\xi).$$

Equivalently, $k$-intersection bodies can be viewed as limits in the radial metric of $k$-intersection bodies of star bodies, see [32, 35]. They are related to a certain generalization of the Busemann-Petty problem in the same way as intersection bodies are related to the original problem, see Section 5.2 in [22].

An origin-symmetric star body $K$ in $\mathbb{R}^n$ is a $k$-intersection body if and only if $\|\cdot\|_K^{-k}$ represents a positive definite distribution, see [21]. Thus Theorem 2 implies,

**Corollary 2.** An origin-symmetric star body in $\mathbb{K}^n$ is an intersection body in $\mathbb{K}^n$ if and only if it is a $\kappa$-balanced $\kappa$-intersection body in $\mathbb{R}^{\kappa n}$.

It was shown in [42] that every origin-symmetric $\kappa$-balanced convex body in $\mathbb{R}^{\kappa n}$ is a $k$-intersection body provided that $k > 0$ and satisfies $\kappa n - \kappa - 2 \leq k < \kappa n$. Hence, it follows that

**Corollary 3.** Every origin-symmetric convex body in $\mathbb{K}^2$ is an intersection body in $\mathbb{K}^2$.

This is no longer true for $\mathbb{K}^n$ with $n \geq 3$. For $q > 2$, the unit ball

$$B_q^{\kappa n} = \{ x \in \mathbb{R}^{\kappa n} : (x_1^2 + \cdots + x_\kappa^2)^{\frac{q}{2}} + \cdots + (x_{\kappa (n-1)+1}^2 + \cdots + x_{\kappa n}^2)^{\frac{q}{2}} \leq 1 \}$$

is not a $\kappa$-intersection body for $\kappa > \frac{2}{n-2}$, see [42].
Corollary 4. An origin-symmetric convex body in $\mathbb{K}^n$ is an intersection body in $\mathbb{K}^n$ only in the following cases: (i) $n = 2, \kappa \in \mathbb{N}$, (ii) $n = 3, \kappa \leq 2$ and (iii) $n = 4, \kappa = 1$.

Proof: For $n = 2, \kappa \in [\kappa - 2, 2\kappa)$ for any $\kappa \in \mathbb{N}$. For $n = 3, \kappa \in [2\kappa - 2, 3\kappa)$ only for $\kappa \leq 2$, and for $\kappa > 2$, $B_{q}^{3k}$ with $q > 2$ is not an intersection body in $\mathbb{K}^3$. For $n = 4, \kappa \in [3\kappa - 2, 4\kappa)$ only for $\kappa = 1$, and for $\kappa > 1$, $B_{q}^{4k}$ with $q > 2$ is not an intersection body in $\mathbb{K}^4$. For $n \geq 5$, $B_{q}^{\kappa n}$ with $q > 2$ is not an intersection body in $\mathbb{K}^n$ for $\kappa \in \mathbb{N}$.

Another generalization of intersection bodies was introduced by G. Zhang in [40] as follows. For $1 \leq k \leq n - 1$, let $G(n, n - k)$ be the Grassmanian of $(n - k)$-dimensional subspaces of $\mathbb{R}^n$. Recall that the $(n - k)$-dimensional spherical Radon transform is an operator $R_{n-k} : C(S^{n-1}) \to C(G(n, n - k))$ defined by

$$R_{n-k}f(H) = \int_{S^{n-1} \cap H} f(x)dx,$$

for $H \in G(n, n - k)$. Denote the image of the operator $R_{n-k}$ by $X$:

$$R_{n-k}(C(S^{n-1})) = X \subset C(G(n, n - k)).$$

Let $M^+(X)$ be the space of positive linear functionals on $X$, that is, for every $\nu \in M^+(X)$ and for every non-negative function $f \in X$, we have $\nu(f) \geq 0$.

Definition 4. An origin-symmetric star body $K$ in $\mathbb{R}^n$ is called a generalized $k$-intersection body if there exists a functional $\nu \in M^+(X)$ so that for every $f \in C(S^{n-1})$

$$\int_{S^{n-1}} \|x\|^{-k} K f(x)dx = \nu(R_{n-k}f).$$

The generalized $k$-intersection bodies are related to the lower-dimensional Busemann-Petty problem, see [40].

Proposition 1. An origin-symmetric star body $K$ in $\mathbb{K}^n$ is an intersection body in $\mathbb{K}^n$ if and only if it is a $\kappa$-balanced generalized $k$-intersection body in $\mathbb{R}^{\kappa n}$.

Proof: Let $K$ be an intersection body in $\mathbb{K}^n$, then there exists a $\kappa$-invariant Borel measure $\mu$ on $S^{\kappa n-1}$ so that for every $f \in C(S^{\kappa n-1})$

$$\int_{S^{\kappa n-1}} \|x\|^{-k} K f(x)dx = \int_{S^{\kappa n-1}} R^\kappa f(\xi)d\mu(\xi).$$

Consider the mapping from $S^{\kappa n-1} \to G(\kappa n, \kappa n - \kappa)$ given by $\xi \mapsto H_\xi$. The image of the measure $\mu$ under this mapping is a measure on $G(\kappa n, \kappa n - \kappa)$, denote it by $\nu$, then

$$\int_{S^{\kappa n-1}} \|x\|^{-k} K f(x)dx = \int_{G(\kappa n, \kappa n - \kappa)} R_{\kappa n-k}f(H)d\nu(H).$$
We can view the measure $\nu$ as a positive linear continuous functional on $X$ acting by

$$\nu(\mathcal{R}_{\kappa n} - \kappa f) = \int_{G(\kappa n, \kappa n - \kappa)} \mathcal{R}_{\kappa n} - \kappa f(H)d\nu(H).$$

Hence $K$ is a generalized $\kappa$-intersection body in $\mathbb{R}^{\kappa n}$.

It was shown in [21] that every generalized $k$-intersection body is a $k$-intersection body. The result now follows by Corollary 2.

Together Proposition 1 and Corollary 2 imply that for $\kappa$-balanced origin-symmetric star bodies in $\mathbb{R}^{\kappa n}$ the class of $\kappa$-intersection bodies and the class of generalized $\kappa$-intersection bodies coincide. This is not true in general as was shown by E. Milman in [33].

P. Goodey and W. Weil proved in [14] that all intersection bodies in $\mathbb{R}^n$ can be obtained as the closure in the radial metric of radial sums of ellipsoids. This result was extended by E. Grinberg and G. Zhang to generalized $k$-intersection bodies with the radial sum replaced by the $k$-radial sum. E. Milman gave a different proof of the latter result in [32]. The complex version of this result was proved in [28]. We now prove this result in $\mathbb{K}^n$ by adjusting the proof from [29] to our setting.

Define the radial sum of two star bodies $K, L$ in $\mathbb{K}^n$, $K + \kappa L$, as a star body in $\mathbb{K}^n$ whose radial function satisfies

$$\rho^\kappa_{K + \kappa L} = \rho^\kappa_K + \rho^\kappa_L,$$

or equivalently as

$$\| \cdot \|_{K + \kappa L} = \| \cdot \|_K + \| \cdot \|_L.$$  

We will prove the following theorem in several steps.

**Theorem 3.** Let $K$ be an origin-symmetric star body in $\mathbb{K}^n$. Then $K$ is an intersection body in $\mathbb{K}^n$ if and only if $\| \cdot \|_K$ is the limit, in the space $C_K(S^{\kappa n-1})$, of finite sums of the form

$$\| \cdot \|_{E_1} + \cdots + \| \cdot \|_{E_m},$$

where $E_1, \ldots, E_m$ are ellipsoids in $\mathbb{K}^n$.

For a vector $\xi$ on the sphere and $a > 0, b > 0$, let $E_{a,b}(\xi)$ be an ellipsoid in $\mathbb{R}^{\kappa n}$ with the norm

$$\| x \|_{E_{a,b}(\xi)} = \left(\frac{\sum_{i=0}^{\kappa-1} (x, R_{\sigma_i} \xi)^2}{a^2} + \frac{|x|_2^2 - \sum_{i=0}^{\kappa-1} (x, R_{\sigma_i} \xi)^2}{b^2}\right)^{\frac{1}{2}}.$$
with $x \in \mathbb{R}^{kn}$. Note that $(\sum_{i=0}^{\kappa n-1} (x, R_{\sigma_i} \xi)^2)^{1/2}$ is the length of the projection of the vector $x$ onto the subspace $H^\perp_\xi$. Since for $\sigma \in SO(\kappa)$, the projection of $R_{\sigma}x$ onto $H^\perp_\xi$ has the same length as the projection of $x$ itself, $E_{a,b}(\xi)$ is a $\kappa$-balanced ellipsoid or an ellipsoid in $\mathbb{K}^n$.

Recall the formula for the Fourier transform of powers of the Euclidean norm in $\mathbb{R}^n$:

$$\left(\| \cdot \|_{E_{a,b}(\xi)}^{-\kappa}\right)^\wedge(\theta) = \frac{\pi^{\kappa n/2} \Gamma\left(\frac{\kappa n+\kappa}{2}\right)}{\Gamma\left(\frac{\kappa n-\kappa}{2}\right)} |\theta|^{-\kappa - n - p},$$

and the formula connecting the Fourier transform and linear transformations

$$(f(T \cdot))^\wedge(y) = |\det T|^{-1} \hat{f}((T^*)^{-1} y),$$

where $T$ is a linear transformation and $T^*$ denotes the adjoint of $T$.

**Lemma 8.** For $\theta \in S^{kn-1}$

$$\left(\| \cdot \|_{E_{a,b}(\xi)}^{-\kappa}\right)^\wedge(\theta) = \frac{C(n, \kappa)}{a^{\kappa(n-2)}} \|\theta\|^{-\kappa + \kappa} \left(\| \cdot \|_{E_{b,a}(\xi)}^{-\kappa}\right)^\wedge(\theta),$$

with $C(n, \kappa) = \frac{\pi^{\kappa n/2} \Gamma\left(\frac{\kappa n-\kappa}{2}\right)}{\Gamma\left(\frac{\kappa n+\kappa}{2}\right)}$.

**Proof:** Let $T$ be a linear operator so that $T B_2^{kn} = E_{a,b}(\xi)$, then $T$ is a composition of a diagonal operator and a rotation.

$$\left(\| \cdot \|_{E_{a,b}(\xi)}^{-\kappa}\right)^\wedge(\theta) = \left(\| T^{-1} \cdot \|_{2}^{-\kappa}\right)^\wedge(\theta)$$

$$= |\det T|^{-1} \left(\| \cdot \|_{2}^{-\kappa}\right)^\wedge(T^\ast \theta)$$

$$= |\det T| C(n, \kappa) |T^\ast \theta|^{-\kappa + \kappa}$$

$$= |\det T| C(n, \kappa) \|\theta\|^{-\kappa + \kappa} \left(\| \cdot \|_{E_{b,a}(\xi)}^{-\kappa}\right)^\wedge(\theta)$$

$$= |\det T| C(n, \kappa) \|\theta\|^{-\kappa + \kappa}$$

$$= \frac{C(n, \kappa)}{a^{\kappa(n-2)}} \|\theta\|^{-\kappa + \kappa} E_{b,a}(\xi).$$

**Lemma 9.** Let $K$ be an origin-symmetric star body in $\mathbb{K}^n$, then $\| \cdot \|^{-\kappa}_K$ can be approximated in the space of $C_{\kappa}(S^{kn-1})$ by functions of the form

$$f_{a,b}(\xi) = \frac{1}{a^{\kappa(n-2)}} \int_{S^{kn-1}} \|\theta\|_{K}^{-\kappa + \kappa} \|\theta\|^{-\kappa + \kappa} \left(\| \cdot \|_{E_{b,a}(\xi)}^{-\kappa}\right)^\wedge(\theta) \, d\theta$$

for an appropriate choice of $b$ and $a \to 0$. 
Proof: Using the formula for the Fourier transform of powers of the Euclidean norm, Parseval's formula on the sphere and previous lemma, we get

\[
\int_{S^{n-1}} \|\theta\|^{-\kappa n + \kappa} d\theta = \frac{C(n, \kappa)}{(2\pi)^{\kappa n}} \int_{S^{n-1}} \|\theta\|^{-\kappa n + \kappa} (|\cdot|^2)^{-\kappa n + \kappa} (\theta) d\theta
\]

\[
= \frac{C(n, \kappa)}{(2\pi)^{\kappa n}} \int_{S^{n-1}} (|\cdot|_{E_{b,a}(\xi)})^{-\kappa n + \kappa} (\theta) d\theta
\]

\[
= a^{\kappa(n-2)} \int_{S^{n-1}} \|\theta\|^{-\kappa} d\theta .
\]

Thus

\[
\frac{1}{a^{\kappa(n-2)}} \int_{S^{n-1}} \|\theta\|^{-\kappa n + \kappa} d\theta = \int_{S^{n-1}} \left( \sum_{i=0}^{\kappa-1} (\theta, R_{\sigma_i} \xi)^2 \frac{1}{a^2} + \frac{1 - \sum_{i=0}^{\kappa-1} (\theta, R_{\sigma_i} \xi)^2}{b^2} \right) d\theta .
\]

Note that for a fixed \(a\) this integral approaches infinity as \(b \to \infty\) and it goes to zero as \(b \to 0\). Hence for every \(a\) there exists \(b = b(a)\) such that

\[
\frac{1}{a^{\kappa(n-2)}} \int_{S^{n-1}} \|\theta\|^{-\kappa n + \kappa} d\theta = 1 .
\]

Since the measure in the above integral is rotation invariant, \(b(a)\) does not depend on \(\xi\). Hence for every \(\xi\) on the sphere and for any \(\delta \in (0, 1)\), we have

\[
\left| \|\xi\|^{-\kappa}_{K} - \frac{1}{a^{\kappa(n-2)}} \int_{S^{n-1}} \|\theta\|^{-\kappa}_{K} \|\theta\|^{-\kappa n + \kappa}_{E_{b(a),a}(\xi)} d\theta \right|
\]

\[
\leq \frac{1}{a^{\kappa(n-2)}} \int_{S^{n-1}} \left( \|\xi\|^{-\kappa}_{K} - \|\theta\|^{-\kappa}_{K} \|\theta\|^{-\kappa n + \kappa}_{E_{b(a),a}(\xi)} d\theta \right)
\]

\[
= \frac{1}{a^{\kappa(n-2)}} \left( \sum_{i=0}^{\kappa-1} (\theta, R_{\sigma_i} \xi)^2 \geq \delta \right) + \left( \sum_{i=0}^{\kappa-1} (\theta, R_{\sigma_i} \xi)^2 < \delta \right) \left( \|\xi\|^{-\kappa}_{K} - \|\theta\|^{-\kappa}_{K} \|\theta\|^{-\kappa n + \kappa}_{E_{b(a),a}(\xi)} d\theta \right)
\]

\[
= I_1 + I_2 .
\]

By the uniform continuity of the function \(\cdot\|_K\) on the sphere, for any \(\epsilon > 0\), there is \(\delta \in (0, 1)\), \(\delta\) close to one, so that \(\|\xi\|^{-\kappa}_{K} - \|\theta\|^{-\kappa}_{K} < \frac{\epsilon}{2}\) for \((\theta, \xi) \geq \delta^\frac{1}{2}\). For \(\theta\) on the sphere with \(\sum_{i=0}^{\kappa-1} (\theta, R_{\sigma_i} \xi)^2 \geq \delta\), let \(\sigma \in SO(\kappa)\) be so that \((R_{\sigma_0} \theta, R_{\sigma_1} \xi) = 0\) for \(i \neq 0\), which means that \((R_{\sigma_0} \theta, \xi) \geq \delta^\frac{1}{2}\). Since \(K\) is \(\kappa\)-balanced, we get \(\|\xi\|^{-\kappa}_{K} - \|\theta\|^{-\kappa}_{K} < \frac{\epsilon}{2}\). Thus with this choice of \(\delta\) we can
Lemma 10. Let estimate the first integral as follows:

\[
I_1 = \frac{1}{a^{n-2}} \sum_{i=0}^{n-1} (\theta, R_{\sigma_i} \xi) ^{2} \geq \delta \left\| \| \theta \|_{K}^{-\kappa} - \| \theta \|_{K}^{-\kappa} \right\| \left\| \theta \right\|_{E_{b(a), \theta}}^{\kappa-n+\kappa} d \theta
\]

Next estimate the second integral as follows:

\[
I_2 = \frac{1}{a^{n-2}} \sum_{i=0}^{n-1} (\theta, R_{\sigma_i} \xi) ^{2} < \delta \left\| \| \theta \|_{K}^{-\kappa} - \| \theta \|_{K}^{-\kappa} \right\| \left\| \theta \right\|_{E_{b(a), \theta}}^{\kappa-n+\kappa} d \theta
\]

\[
\leq \frac{2 \max \| x \|_{K}}{a^{n-2}} \sum_{i=0}^{n-1} (\theta, R_{\sigma_i} \xi) ^{2} \left\| \theta \right\|_{E_{b(a), \theta}}^{\kappa-n+\kappa} d \theta
\]

\[
\leq \frac{2 \max \| x \|_{K}}{a^{n-2}} \sum_{i=0}^{n-1} (\theta, R_{\sigma_i} \xi) ^{2} \left( 1 - \sum_{i=0}^{n-1} (\theta, R_{\sigma_i} \xi) ^{2} \right) \left\| \theta \right\|_{E_{b(a), \theta}}^{\kappa-n+\kappa} d \theta
\]

\[
= 2a^{\kappa} \max \| x \|_{K} \int_{S^{n-1}} \left( 1 - \sum_{i=0}^{n-1} (\theta, R_{\sigma_i} \xi) ^{2} \right) d \theta
\]

\[
\leq 2a^{\kappa} \max \| x \|_{K} \int_{S^{n-1}} \left( 1 - \sum_{i=0}^{n-1} (\theta, R_{\sigma_i} \xi) ^{2} \right) \left\| \theta \right\|_{E_{b(a), \theta}}^{\kappa-n+\kappa} d \theta
\]

Now we can choose \( a \) so small that \( I_2 \leq \frac{\epsilon}{2} \).

\[
\square
\]

Lemma 10. Let \( \mu \) be a finite measure on \( S^{n-1} \) and \( a > 0, b > 0 \). The function

\[
f(\xi) = \int_{S^{n-1}} \left\| \theta \right\|_{E_{a,b}(\theta)}^{-\kappa} d \mu(\theta)
\]

is the limit, in the space \( C_{\kappa}(S^{n-1}) \), of sums of the form

\[
\sum_{i=1}^{m} \| \xi \|_{E_{i}^{-\kappa}}
\]

where \( E_1, \ldots, E_m \) are \( \kappa \)-balanced ellipsoids.

Proof: For \( \epsilon > 0 \), choose a finite covering of the sphere by spherical \( \epsilon \)-balls:

\( B_{\epsilon}(\xi_i) = \{ \theta \in S^{n-1} : |\theta - \xi_i| < \epsilon \}, \xi_i \in S^{n-1}, i = 1, \ldots, m = m(\epsilon) \). Define

\[
\tilde{B}_{\epsilon}(\xi_1) = B_{\epsilon}(\xi_1) \quad \text{and} \quad \tilde{B}_{i}(\xi_i) = B_{\epsilon}(\xi_i) \setminus \bigcup_{j=1}^{i-1} B_{\epsilon}(\xi_j), \quad \text{for } i=2, \ldots, m.
\]
Set \( p_i = \mu(\tilde{B}_r(\xi_i)) \), then \( p_1 + \cdots + p_m = \mu(S_i^{n-1}) \).

Denote as \( \rho(E_{a,b}(\xi), x) \) the value of the radial function of the ellipsoid \( E_{a,b}(\xi) \) at the point \( x \). Note that \( \rho(E_{a,b}(\xi), x) = \rho(E_{a,b}(x), \xi) \), since \( \rho(E_{a,b}(\xi), x) \) depends only on \( \sum_{i=1}^{\kappa-1} (x, R_a, \xi)^2 \) and \( (x, R_a, \xi) = (R_a^*, x, \xi) \). Hence

\[
|\rho^\kappa(E_{a,b}(\xi), x) - \rho^\kappa(E_{a,b}(\theta), x)| \leq C_{a,b}|\xi - \theta|,
\]

for some constant \( C_{a,b} \) depending only on \( a \) and \( b \). We are now ready to estimate

\[
\left| \int_{S_i^{n-1}} \rho^\kappa(E_{a,b}(\xi), x) d\mu(\xi) - \sum_{i=1}^{m} p_i \rho^\kappa(E_{a,b}(\xi_i), x) \right|
\]

\[
= \sum_{i=1}^{m} \left( \int_{\tilde{B}_r(\xi_i)} \rho^\kappa(E_{a,b}(\xi), x) d\mu(\xi) - \int_{\tilde{B}_r(\xi_i)} \rho^\kappa(E_{a,b}(\xi_i), x) d\mu(\xi) \right)
\]

\[
\leq \sum_{i=1}^{m} \int_{\tilde{B}_r(\xi_i)} |\rho^\kappa(E_{a,b}(\xi), x) - \rho^\kappa(E_{a,b}(\xi_i), x)| d\mu(\xi)
\]

\[
\leq \sum_{i=1}^{m} \int_{\tilde{B}_r(\xi_i)} C_{a,b} |\xi - \xi_i| d\mu(\xi)
\]

\[
\leq \epsilon C_{a,b} \mu(S_i^{n-1}).
\]

The result follows by letting \( \epsilon \to 0 \) and defining \( \|\xi\|_{E_i}^{-\kappa} = p_i \rho^\kappa(E_{a,b}(\xi_i), x) \).

\[\square\]

**Proof of Theorem 3.** The 'if' part follows immediately from Theorem 2 and Lemma 8.

To prove the converse, suppose \( K \) is an intersection body in \( \mathbb{K}^n \). By Lemma 9, \( \|\xi\|_{K}^{-\kappa} \) as a uniform limit of functions of the form

\[
\frac{1}{\alpha^{(n-2)}} \int_{S_i^{n-1}} \|\theta\|_{E_{a,b}(\xi)}^{-\kappa} \|\theta\|_{E_{a,b}(\xi)}^{-\kappa n + \kappa} d\theta,
\]

as \( a \to 0 \). By Parseval’s formula on the sphere, this equals to

\[
\frac{1}{C(n, \kappa)} \int_{S_i^{n-1}} \|\theta\|_{E_{a,b}(\xi)}^{-\kappa} \|\theta\|_{E_{a,b}(\xi)}^{-\kappa n + \kappa} d\theta = \int_{S_i^{n-1}} \|\theta\|_{E_{a,b}(\xi)}^{-\kappa} \|\theta\|^\wedge_{E_{a,b}(\xi)} d\mu(\theta),
\]

where \( d\mu = 1/C(n, \kappa)\|\cdot\|_K^{-\kappa} \|\cdot\|_K^{-\kappa} d\theta \). By Lemma 10, the above is the uniform limit of sums of the form

\[
\|\xi\|_{E_1}^{-\kappa} + \cdots + \|\xi\|_{E_m}^{-\kappa}.
\]

\[\square\]
5. Stability in the Busemann-Petty problem and related inequalities

Intersection bodies played an important role in the solution of the Busemann-Petty problem, which can be posed as follows. Given two origin-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ such that for every $\xi \in S^{n-1}$

$$|K \cap \xi^\perp| \leq |L \cap \xi^\perp|,$$

does it follow that

$$|K| \leq |L|?$$

The answer is affirmative for $n \leq 4$ and negative for $n \geq 5$. This problem, posed in 1956 in [5], was solved in the late 90’s as a result of a sequence of papers [1, 3, 6, 7, 9, 12, 19, 20, 30, 31, 34, 38, 39], see [22], p. 3-5, for the history of the solution. One of the main steps in the solution was the connection established by E. Lutwak in [31] between this problem and intersection bodies: For an intersection body $K$ and any star body $L$ the Busemann-Petty problem has a positive answer. For an origin-symmetric convex body $L$ that is not an intersection body, one can construct a counterexample. The complex version of this problem was considered in [26, 28].

The Busemann-Petty problem in $\mathbb{K}^n$ can be formulated as follows: Given two origin-symmetric $\kappa$-balanced convex bodies $K$ and $L$ in $\mathbb{R}^{\kappa n}$ such that $|K \cap H_\xi| \leq |L \cap H_\xi|$, for every $\xi \in S^{\kappa n-1}$. Does it follow that $|K| \leq |L|$? It was proved in [42] (see also [36]) that the answer is affirmative in the following cases: (i) $n = 2$, $\kappa \in \mathbb{N}$, (ii) $n = 3$, $\kappa \leq 2$, (iii) $n = 4$, $\kappa = 1$, and negative for any other values of $n$ and $\kappa$. The solution uses a connection to intersection bodies in $\mathbb{K}^n$, analogous to Lutwak’s connection: If $K$ is an intersection body in $\mathbb{K}^n$ and $L$ is any star body in $\mathbb{K}^n$, then the Busemann-Petty problem in $\mathbb{K}^n$ has an affirmative answer. If there exists an origin-symmetric convex body $L$ in $\mathbb{K}^n$ that is not an intersection body in $\mathbb{K}^n$, then one can construct another origin-symmetric convex body $K$ in $\mathbb{K}^n$, so that the pair of bodies $K, L$ provides a counterexample. This connection was formulated in [42] in terms of positive-definite distributions. Via Theorem 2 it transforms into an assertion in terms of intersection bodies in $\mathbb{K}^n$.

A. Zvavitch generalized the Busemann-Petty problem in $\mathbb{R}^n$ to arbitrary measures in place of volume and proved that the answer is affirmative for $n \leq 4$ and negative for $n \geq 5$, see [41]. M. Zymonopoulou proved a complex version of this result in [43]. In this section we extend Zvavitch’s result to $\mathbb{K}^n$ and consider the associated stability question along with the stability in the Busemann-Petty problem in $\mathbb{K}^n$. Stability in the original Busemann-Petty problem was established in [23], for the complex version in [24] and for arbitrary measures in [25], other stability results include [18, 27].
Theorem 4. Let \( K, L \) be origin-symmetric star bodies in \( \mathbb{R}^n \) and let \( \epsilon > 0 \). Suppose \( K \) is an intersection body in \( \mathbb{R}^n \) and for every \( \xi \in S^{n-1} \)

\[ |K \cap H_\xi| \leq |L \cap H_\xi| + \epsilon, \]

then

\[ |K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} + \epsilon \frac{|B_2^n|^{\frac{n-1}{n}}}{|B^{n-\kappa}_2|}. \]

Proof: By (4) the inequality for sections can be written as

\[ R^\kappa(\| \cdot \|_K^{\kappa n + \kappa})(\xi) \leq R^\kappa(\| \cdot \|_L^{\kappa n + \kappa})^\wedge(\xi) + \kappa(n-1). \]

Let \( \nu \) be the measure which corresponds to the body \( K \) by Definition 2. Integrating the above inequality over the sphere with respect to \( \nu \) and applying the equality condition of Definition 2, yields

\[ \int_{S^{n-1}} \| x \|_K^{-\kappa} dx \leq \int_{S^{n-1}} \| x \|_L^{-\kappa} \| x \|_L^{-\kappa n + \kappa} dx + \kappa(n-1) \int_{S^{n-1}} d\nu(\xi). \]

Applying Hölder’s inequality and using polar formula for the volume, gives

\[ \kappa n |K| \leq \kappa n |K|^\frac{1}{n} |L|^\frac{1}{n} + \kappa(n-1) \int_{S^{n-1}} d\nu(\xi). \]

The spherical Radon transform on \( \mathbb{R}^n \) of the constant function one, is the constant function with value \( \Omega_{\kappa n - \kappa} \). Using the equality of Definition 2 and Hölder’s inequality, we obtain

\[ \int_{S^{n-1}} d\nu(\xi) = \frac{1}{\Omega_{\kappa n - \kappa}} \int_{S^{n-1}} R^\kappa 1(\xi) d\nu(\xi) \]

\[ = \frac{1}{\Omega_{\kappa n - \kappa}} \int_{S^{n-1}} \| x \|_K^{-\kappa} dx \]

\[ = \frac{1}{\Omega_{\kappa n - \kappa}} \kappa n |K|^\frac{1}{n} |B_2^n|^\frac{n-1}{n}. \]

Altogether, we have

\[ |K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} + \epsilon \frac{|B_2^n|^{\frac{n-1}{n}}}{|B^{n-\kappa}_2|}. \]

Interchanging the roles of \( K \) and \( L \) in the above theorem and letting \( \epsilon = \max_{\xi \in S^{n-1}} ||K \cap H_\xi| - |L \cap H_\xi|| \), we obtain the corresponding volume difference inequality.
**Corollary 5.** Suppose $K, L$ are intersection bodies in $\mathbb{K}^n$, then
\[
\left|K^{\frac{n-1}{n}} - |L|^{\frac{n-1}{n}}\right| \leq \frac{|B_2^{kn}|^{\frac{n-1}{n}}}{|B_2^{kn-\kappa}|} \max_{\xi \in S^{kn-1}} ||K \cap H_\xi| - |L \cap H_\xi||.
\]
Setting $L = \delta B_2^{kn}$ and letting $\delta$ go to zero, we obtain:

**Corollary 6.** Suppose $K$ is an intersection body in $\mathbb{K}^n$, then
\[
|K|^{\frac{n-1}{n}} \leq \frac{|B_2^{kn}|^{\frac{n-1}{n}}}{|B_2^{kn-\kappa}|} \max_{\xi \in S^{kn-1}} |K \cap H_\xi|.
\]

For $\kappa = 1, 2$, Corollary 6 reduces to the previously known hyperplane inequalities corresponding to the stability problem in the original [23] and in the complex version [24] of Busemann-Petty problem.

Corollary 6 is related to the famous Hyperplane Conjecture, which can be formulated as follows. Does there exist an absolute constant $C$ so that for any origin-symmetric convex body $K$ in $\mathbb{R}^n$
\[
|K|^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|,
\]
where $\xi^\perp$ stands for the central hyperplane perpendicular to $\xi$? This problem is still open. The best known estimate $C \sim n^{1/4}$ is due to B. Klartag [16], who slightly improved the previous estimate of J. Bourgain [4].

Now we turn to the Busemann-Petty problem in $\mathbb{K}^n$ for arbitrary measures. Let $f$ be a non-negative locally-integrable even function on $\mathbb{R}^{kn}$ and $\mu$ be the measure on $\mathbb{R}^{kn}$ with density $f$. Let $g$ be a non-negative locally-integrable even function on $\mathbb{R}^{kn}$, which is locally-integrable on every $H_\xi$, $\xi \in S^{kn-1}$, and define a measure $\gamma$ on $H_\xi$, for any $\xi \in S^{kn-1}$, by
\[
\gamma(B) = \int_B g(x)dx,
\]
for any bounded Borel set $B \subset H_\xi$. The Busemann-Petty problem in $\mathbb{K}^n$ for arbitrary measures can be formulated as follows:

Given $K, L$ two origin-symmetric $\kappa$-balanced convex bodies in $\mathbb{R}^{kn}$ satisfying
\[
\gamma(K \cap H_\xi) \leq \gamma(L \cap H_\xi),
\]
for every $\xi \in S^{kn-1}$, does it follow that
\[
\mu(K) \leq \mu(L) ?
\]

Since we work with $\kappa$-balanced sets, we can assume that the measures $\mu, \gamma$ are $\kappa$-invariant, consequently the functions $f, g$ are $\kappa$-invariant as well. We need a polar formula for the measure of star bodies in $\mathbb{K}^n$ as well as for the measure of their sections.

\[
\mu(K) = \int_K f(x)dx = \int_{S^{kn-1}} \int_0^{||x||_K^2} f(rx)r^{kn-1}drdx.
\]
Lemma 11. For an even continuous \( \kappa \)-invariant function \( g \) on \( \mathbb{R}^{kn} \setminus \{0\} \) and \( \xi \in S^{kn-1} \), using Lemma 7, we obtain
\[
\gamma(K \cap H_\xi) = \int_{S^{kn-1} \cap H_\xi} \left( \left\| x \right\|_K^{-1} \right) g(rx) r^{\kappa n - \kappa - 1} dr dx
\]
\[
= \int_{S^{kn-1} \cap H_\xi} \left( \left\| x \right\|_2^{-\kappa + \kappa} \right) \left( \int_0^\infty g \left( \frac{rx}{|x|_2} \right) r^{\kappa n - \kappa - 1} dr \right) dx
\]
\[
= \mathcal{R} \left( \left\| x \right\|_2^{-\kappa + \kappa} \right) \left( \int_0^\infty g \left( \frac{rx}{|x|_2} \right) r^{\kappa n - \kappa - 1} dr \right) (\xi)
\]
\[
= \frac{\Omega_\kappa}{(2\pi)^n} \left( \left\| x \right\|_2^{-\kappa + \kappa} \right) \left( \int_0^\infty g \left( \frac{rx}{|x|_2} \right) r^{\kappa n - \kappa - 1} dr \right) ^\wedge (\xi).
\] 

The following elementary lemma is an analog of a lemma used by A. Zvavitch in [41].

**Lemma 11.** Let \( \kappa, n \in \mathbb{N} \), \( \kappa \geq 1, n \geq 2 \) and let \( a, b \geq 0 \). For non-negative integrable functions \( \alpha, \beta \) on \([0, \max\{a, b\}]\) so that \( t^{\frac{\alpha(t)}{\beta(t)}} \) is non-decreasing, we have
\[
a^\kappa \frac{\alpha(a)}{\beta(a)} \int_a^b t^{\kappa n - \kappa - 1} \beta(t) dt \leq \int_a^b t^{\kappa n - \kappa} \alpha(t) dt.
\]

**Proof:** Compute
\[
a^\kappa \frac{\alpha(a)}{\beta(a)} \int_a^b t^{\kappa n - \kappa - 1} \beta(t) dt = \int_a^b t^{\kappa n - \kappa} \alpha(t) \left( a^\kappa \frac{\alpha(a)}{\beta(a)} \right) \left( t^{\kappa \frac{\alpha(t)}{\beta(t)}} \right)^{-1} dt
\]
\[
\leq \int_a^b t^{\kappa n - \kappa} \alpha(t) dt.
\]

**Proposition 2.** Let \( \epsilon > 0 \) and let \( f, g \) be even non-negative \( \kappa \)-invariant continuous functions on \( \mathbb{R}^{kn} \setminus \{0\} \) so that \( t^{\kappa \frac{f(tx)}{g(tx)}} \) is a non-decreasing function in \( t \) for any fixed \( x \in S^{kn-1} \). Suppose that an origin-symmetric star body \( K \) in \( \mathbb{K}^n \) has the property that \( \|x\|_K^{-\kappa} f(x\|x\|_K^{-1}) \) is a positive-definite distribution on \( \mathbb{K}^{kn} \). Then for any origin-symmetric star body \( L \) in \( \mathbb{K}^n \) satisfying
\[
\gamma(K \cap H_\xi) \leq \gamma(L \cap H_\xi) + \epsilon,
\]
for every \( \xi \in S^{kn-1} \), it follows that
\[
\mu(K) \leq \mu(L) + \epsilon \frac{1}{\Omega_{\kappa n - \kappa}} \int_{S^{kn-1}} \left\| x \right\|_K^{-\kappa} f(x\|x\|_K^{-1}) g(x\|x\|_K^{-1}) dx.
\]
Proof: Using equation (10), the inequality for sections can be written as

\[
R^\kappa \left( |x|_2^{-\kappa+\kappa} \int_0^{|x|_K} g \left( \frac{r x}{|x|_2} \right) r^{\kappa n-\kappa-1} dr \right) (\xi) \\
\leq R^\kappa \left( |x|_2^{-\kappa+\kappa} \int_0^{|x|_L} g \left( \frac{r x}{|x|_2} \right) r^{\kappa n-\kappa-1} dr \right) (\xi) + \epsilon .
\]

Define an auxiliary star body \( D \) by

\[
\|x\|_D^{-\kappa} = \|x\|_K^{-\kappa} \frac{f(x\|x\|_K^{-1})}{g(x\|x\|_K^{-1})}.
\]

Note that \( D \) is an even \( \kappa \)-balanced star body and \( \| \cdot \|_D^{-\kappa} \) is positive-definite, thus \( D \) is an intersection body in \( \mathbb{K}^n \). By Definition 2 there is a measure \( \nu \) on \( S^{n-1}_\kappa \) corresponding to the body \( D \). Integrating the above inequality over the sphere with respect to the measure \( \nu \) and applying the equality of Definition 2, yields

\[
\int_{S^{n-1}_\kappa} \|x\|_D^{-\kappa} \int_0^{\|x\|_K^{-1}} g(r x) r^{\kappa n-\kappa-1} dr dx \\
\leq \int_{S^{n-1}_\kappa} \|x\|_D^{-\kappa} \int_0^{\|x\|_L^{-1}} g(r x) r^{\kappa n-\kappa-1} dr dx + \epsilon \int_{S^{n-1}_\kappa} d\nu(\xi) .
\]

By Lemma 11, with \( a = \|x\|_K^{-1}, b = \|x\|_L^{-1}, \alpha(r) = f(r x), \beta(r) = g(r x) \), we also have

\[
\int_0^{\|x\|_K^{-1}} f(r x) r^{\kappa n-1} dr - \|x\|_D^{-\kappa} \int_0^{\|x\|_K^{-1}} g(r x) r^{\kappa n-\kappa-1} dr \\
\leq \int_0^{\|x\|_L^{-1}} f(r x) r^{\kappa n-1} dr - \|x\|_D^{-\kappa} \int_0^{\|x\|_L^{-1}} g(r x) r^{\kappa n-\kappa-1} dr .
\]

Integrating equation (12) over the sphere and adding the resulting equation to equation (11), we obtain

\[
\int_{S^{n-1}_\kappa} \|x\|_K^{-1} \int_0^{\|x\|_K^{-1}} f(r x) r^{\kappa n-1} dr dx \\
\leq \int_{S^{n-1}_\kappa} \|x\|_L^{-1} \int_0^{\|x\|_L^{-1}} f(r x) r^{\kappa n-1} dr dx + \epsilon \int_{S^{n-1}_\kappa} d\nu(\xi) ,
\]
which reads as
\[ \mu(K) \leq \mu(L) + \epsilon \int_{S^{n-1}} d\nu(\xi). \]

Finally, since the spherical Radon transform on \( \mathbb{K}^n \) of the constant function one, is the constant function with value \( \Omega_{\kappa n} \), using the equality of Definition 2, we obtain
\[
\int_{S^{n-1}} d\nu(\xi) = \frac{1}{\Omega_{\kappa n}} \int_{S^{n-1}} \mathcal{R}^\kappa 1(\xi) d\nu(\xi) \\
= \frac{1}{\Omega_{\kappa n}} \int_{S^{n-1}} \|x\|^{-\kappa} dx \\
= \frac{1}{\Omega_{\kappa n}} \int_{S^{n-1}} \|x\|^{-\kappa} f(x\|x\|^{-1}) \frac{1}{g(x\|x\|^{-1})} dx.
\]

\( \blacksquare \)

**Proposition 3.** Let \( f, g \) be even strictly positive \( \kappa \)-invariant continuous functions on \( \mathbb{R}^{kn} \setminus \{0\} \) and so that \( t^\kappa \frac{f(tx)}{g(tx)} \) is a non-decreasing function in \( t \) for any fixed \( x \in S^{n-1} \). Let \( l = \max\{2, \kappa - 2\} \) and assume also that \( g \in C^l(\mathbb{R}^{en} \setminus \{0\}) \).

Suppose \( L \) is an infinitely-smooth origin-symmetric convex body in \( \mathbb{K}^n \) with strictly positive curvature so that
\[
\|x\|^{-\kappa} f(x\|x\|^{-1}) \frac{1}{g(x\|x\|^{-1})} 
\]
is in \( C^{kn-\kappa}(\mathbb{R}^{kn} \setminus \{0\}) \) and does not represent a positive-definite distribution on \( \mathbb{R}^{kn} \). Then there is an origin-symmetric convex body \( K \) in \( \mathbb{K}^n \) satisfying
\[
\gamma(K \cap H_\xi) \leq \gamma(L \cap H_\xi),
\]
for every \( \xi \in S^{n-1} \), but
\[
\mu(K) > \mu(L).
\]

**Proof:** Since the function (13) is in \( C^{kn-\kappa-1}(\mathbb{R}^{kn} \setminus \{0\}) \), it follows by Corollary 3.17 (i) in [22], that its Fourier transform is a continuous function on the sphere. Hence, by continuity, its Fourier transform must be negative on some open subset \( \Omega \) of the sphere. From the \( \kappa \)-invariance of the function (13), it follows that the set \( \Omega \) is \( \kappa \)-balanced. Let \( h \) be an infinitely-smooth positive \( \kappa \)-invariant function on the sphere with support contained in the set \( \Omega \). Extend \( h \) to a homogeneous function of degree \(-\kappa\), then the Fourier transform of this extension is a homogeneous function of degree \(-\kappa n + \kappa\), i.e. there is an infinitely smooth function \( v \) on the sphere so that \( (h \cdot r^{-\kappa})^\wedge = v \cdot r^{-kn+\kappa} \).
Let $\epsilon > 0$, define another body $K$ by
\[
|x|_2^{-\kappa n+\kappa} \int_0^{\frac{|x|_2}{|x|_L}} r^{-\kappa n+\kappa-1} g \left( \frac{r x}{|x|_2} \right) dr
= |x|_2^{\kappa n+\kappa} \int_0^{\frac{|x|_2}{|x|_L}} r^{-\kappa n+\kappa-1} g \left( \frac{r x}{|x|_2} \right) dr - \epsilon |x|_2^{-\kappa n+\kappa} \gamma \left( \frac{x}{|x|_2} \right).
\]

As $g \in C^2(\mathbb{R}^{kn} \setminus \{0\})$, by Lemma 5.16 in [22], $K$ is convex for $\epsilon$ small enough. Since the function $h$ is positive, using equation (10), it follows
\[
\gamma(K \cap H_\xi) = \frac{\Omega_\kappa}{(2\pi)^\kappa} \left( |x|_2^{-\kappa n+\kappa} \int_0^{\frac{|x|_2}{|x|_L}} g \left( \frac{r x}{|x|_2} \right) r^{\kappa n-\kappa-1} dr \right) \wedge \Omega_\kappa (2\pi)^{\kappa n} h(\xi)
= \frac{\Omega_\kappa}{(2\pi)^\kappa} \left( |x|_2^{-\kappa n+\kappa} \int_0^{\frac{|x|_2}{|x|_L}} g \left( \frac{r x}{|x|_2} \right) r^{\kappa n-\kappa-1} dr \right) \wedge \Omega_\kappa (2\pi)^{\kappa n} h(\xi)
\leq \gamma(L \cap H_\xi).
\]

On the other hand, the function $h$ is supported on the set where the Fourier transform of the function (13) is negative, hence
\[
\left( \|x\|_L^{-\kappa} \frac{f(x\|x\|_L^{-1})}{g(x\|x\|_L^{-1})} \right) \wedge \left( |x|_2^{-\kappa n+\kappa} \int_0^{\frac{|x|_2}{|x|_L}} g \left( \frac{r x}{|x|_2} \right) r^{\kappa n-\kappa-1} dr \right) \wedge \left( \|x\|_L^{-\kappa} \frac{f(x\|x\|_L^{-1})}{g(x\|x\|_L^{-1})} \right) \wedge \left( |x|_2^{-\kappa n+\kappa} \int_0^{\frac{|x|_2}{|x|_L}} g \left( \frac{r x}{|x|_2} \right) r^{\kappa n-\kappa-1} dr \right)
= \epsilon (2\pi)^{\kappa n} \left( \|x\|_L^{-\kappa} \frac{f(x\|x\|_L^{-1})}{g(x\|x\|_L^{-1})} \right) \wedge \left( |x|_2^{-\kappa n+\kappa} \int_0^{\frac{|x|_2}{|x|_L}} g \left( \frac{r x}{|x|_2} \right) r^{\kappa n-\kappa-1} dr \right)
\leq \epsilon (2\pi)^{\kappa n} \left( \|x\|_L^{-\kappa} \frac{f(x\|x\|_L^{-1})}{g(x\|x\|_L^{-1})} \right) \wedge \left( |x|_2^{-\kappa n+\kappa} \int_0^{\frac{|x|_2}{|x|_L}} g \left( \frac{r x}{|x|_2} \right) r^{\kappa n-\kappa-1} dr \right)
\]

Note that the above inequality is strict on $\Omega$.

Since $g \in C^{\kappa-2}(\mathbb{R}^{kn} \setminus \{0\})$, by Corollary 3.17 (i) in [22], functions $\xi \mapsto \gamma(K \cap H_\xi)$ and $\xi \mapsto \gamma(L \cap H_\xi)$ are continuous positive functions on the sphere. Integrating the latter inequality over the sphere and applying the spherical Parseval’s formula in the form of Corollary 3.23 in [22] with $k = \kappa n - \kappa$, which is justified by above observations and the fact that the function (13) is in
\(C^{\kappa n-\kappa}(\mathbb{R}^{\kappa n} \setminus \{0\})\), we obtain
\[
\int_{S^{\kappa n-1}} \|x\|_{L}^{-\kappa} \frac{f(x\|x\|)\|x\|_{L}^{-1}}{g(x\|x\|)\|x\|_{L}^{-1}} \int_{0}^{\|x\|_{K}^{-1}} g(rx) r^{\kappa n-\kappa-1} drdx
\]
\[
> \int_{S^{\kappa n-1}} \|x\|_{L}^{-\kappa} \frac{f(x\|x\|)\|x\|_{L}^{-1}}{g(x\|x\|)\|x\|_{L}^{-1}} \int_{0}^{\|x\|_{K}^{-1}} g(rx) r^{\kappa n-\kappa-1} drdx.
\]
This is equivalent to
\[
0 < \int_{S^{\kappa n-1}} \|x\|_{L}^{-\kappa} \frac{f(x\|x\|)\|x\|_{L}^{-1}}{g(x\|x\|)\|x\|_{L}^{-1}} \int_{0}^{\|x\|_{K}^{-1}} g(rx) r^{\kappa n-\kappa-1} drdx. \tag{14}
\]
By Lemma 11, with \(a = \|x\|_{K}^{-1}, b = \|x\|_{K}^{-1}, \alpha(r) = f(rx), \beta(r) = g(rx)\), we also have
\[
\|x\|_{L}^{-\kappa} \frac{f(x\|x\|)\|x\|_{L}^{-1}}{g(x\|x\|)\|x\|_{L}^{-1}} \int_{0}^{\|x\|_{K}^{-1}} g(rx) r^{\kappa n-\kappa-1} dr \leq \int_{0}^{\|x\|_{K}^{-1}} f(rx) r^{\kappa n-1} dr. \tag{15}
\]
Integrating equation (15) over the sphere and combining the resulting equation with inequality (14), yields
\[
\int_{S^{\kappa n-1}} f(rx) r^{\kappa n-1} drdx < \int_{S^{\kappa n-1}} f(rx) r^{\kappa n-1} drdx,
\]
which is equivalent to
\[
\mu(L) < \mu(K).
\]
\[
\square
\]
**Theorem 5.** Let \(f = g\) be equal even non-negative \(\kappa\)-invariant continuous functions on \(\mathbb{R}^{\kappa n} \setminus \{0\}\). Then the answer to the Busemann-Petty problem in \(\mathbb{K}^{n}\) for arbitrary measures is positive in the following cases: (i) \(n = 2, \kappa \in \mathbb{N}\), (ii) \(n = 3, \kappa \leq 2\) and (iii) \(n = 4, \kappa = 1\). In the remaining cases the answer to the Busemann-Petty problem in \(\mathbb{K}^{n}\) for arbitrary measures is negative for an even strictly positive \(\kappa\)-invariant function \(f \in C'(\mathbb{R}^{\kappa n} \setminus \{0\})\) with \(l = \max\{2, \kappa - 2\}\).

**Proof:** Since \(t^{\kappa} \frac{f(tx)}{g(tx)} = t^{\kappa}\) is a non-decreasing function, Propositions 2 and 3 apply. Suppose \(K\) is an intersection body in \(\mathbb{K}^{n}\), then \(\|x\|_{K}^{-\kappa} \frac{f(x\|x\|)\|x\|_{K}^{-1}}{g(x\|x\|)\|x\|_{K}^{-1}} = \|x\|_{K}^{-\kappa}\) is a positive-definite distribution on \(\mathbb{R}^{\kappa n}\). The affirmative part now follows from Corollary 4 and Proposition 2 with \(\epsilon = 0\).

For the negative part, note that in this case there is an origin-symmetric convex body \(L\) in \(\mathbb{K}^{n}\) that is not an intersection body in \(\mathbb{K}^{n}\), e.g. \(B_{q}^{\kappa n}\) with \(q > 2\), see Section 4. \(L\) can be approximated in the radial metric by a sequence of infinitely-smooth origin-symmetric convex bodies \(L_{m}\) in \(\mathbb{K}^{n}\) with strictly positive curvature so that each body \(L_{m}\) is not an intersection body in \(\mathbb{K}^{n}\). This follows from Lemma 4.10 in [22] and the connection between the convolution
and linear transformations. Thus we can assume that \( \|x\|_L^{-\kappa} f(x\|x\|_L^{-1}) = \|x\|_L^{-\kappa} \) is in \( C^\infty(\mathbb{R}^n \setminus \{0\}) \) and does not represent a positive definite distribution. The negative part now follows from Proposition 3.

\[
\Box
\]

Making different choices for functions \( f, g \) one can derive a series of interesting inequalities along the lines of \([41]\). We will refrain from doing this, as it will go beyond the scope of this paper, and only consider the stability question in the Busemann-Petty problem in \( \mathbb{K}^n \) for arbitrary measures.

The volume difference inequality is obtained by interchanging the roles of \( K \) and \( L \) in Proposition 2.

**Corollary 7.** Under the assumptions of Proposition 2, we have

\[
|\mu(K) - \mu(L)| \leq \frac{1}{\Omega_{\kappa n - \kappa}} \max_{\xi \in \mathbb{S}^{n-1}} |\gamma(K \cap H_{\xi}) - \gamma(L \cap H_{\xi})| \times \\
\times \max \left\{ \int_{\mathbb{S}^{n-1}} \|x\|_K^{-\kappa} f(x\|x\|_K^{-1}) \, dx, \int_{\mathbb{S}^{n-1}} \|x\|_L^{-\kappa} f(x\|x\|_L^{-1}) \, dx \right\}.
\]

**Theorem 6.** Let \( f = g \) be equal even non-negative \( \kappa \)-invariant continuous functions on \( \mathbb{R}^n \setminus \{0\} \). Let \( K \) be an intersection body in \( \mathbb{K}^n \), then

\[
\mu(K) \leq \frac{n}{n - 1} \frac{|B_{2\kappa}^{2\kappa-1}|}{|B_{2}^{2\kappa-\kappa}|} \max_{\xi \in \mathbb{S}^{n-1}} \gamma(K \cap H_{\xi}) |K|^{\frac{1}{n}}.
\]

**Proof:** Let \( f = g \) in the inequality of Corollary 7. Further, set \( L = \delta B_{2\kappa}^{\kappa} \), let \( \delta \) go to zero, and observe that by Hölder’s inequality

\[
\frac{1}{\Omega_{\kappa n - \kappa}} \int_{\mathbb{S}^{n-1}} \|x\|_K^{-\kappa} \, dx \leq \frac{1}{\Omega_{\kappa n - \kappa}} \left( \int_{\mathbb{S}^{n-1}} \|x\|_K^{-\kappa n} \, dx \right)^{\frac{n}{\kappa n}} \left( \Omega_{\kappa n} \right)^{\frac{n}{\kappa n}} = \frac{n}{n - 1} \frac{|B_{2\kappa}^{2\kappa-1}|}{|B_{2}^{2\kappa-\kappa}|} |K|^{\frac{1}{n}}.
\]

The constant is the best possible, this follows by a similar example as in \([25]\), Theorem 1.

\[
\Box
\]

For \( \kappa = 1, 2 \) the inequality of Theorem 6 reduces to the previously known hyperplane inequalities for arbitrary measures, see \([25]\) and \([28]\).

**Lemma 12.** Let \( M \) be an intersection body in \( \mathbb{K}^n \) and let \( K \) be any star body in \( \mathbb{K}^n \), then

\[
\int_{K} \|x\|_{M}^{-\kappa} \, dx \leq \frac{n}{n - 1} \frac{|B_{2\kappa}^{2\kappa-1}|}{|B_{2}^{2\kappa-\kappa}|} |M|^{\frac{1}{n}} \max_{\xi \in \mathbb{S}^{n-1}} \int_{K \cap H_{\xi}} \|x\|_{M}^{-\kappa} \, dx,
\]

for \( l < \kappa n - \kappa \).
Proof: Let $f(x) = \|x\|^{-1}_M$ and $g(x) = \|x\|^{-1}_M$, then $t^\kappa \frac{f(tx)}{g(tx)} = \|x\|^{-\kappa}_M$ is a non-decreasing function, $\|x\|^{-\kappa}_M f(tx) = \|x\|^{-\kappa}_M$ is a positive-definite distribution and hence Corollary 7 applies. The result follows by setting $L = \delta B_2^{\kappa n}$ and letting $\delta$ go to zero.

\[\square\]

Setting $l = -\kappa$ and $M = B_2^{\kappa n}$ in Lemma 12, yields Corollary 8. For any star body $K$ in $\mathbb{K}^n$, we have

\[|K| \leq \frac{n}{n-1} \frac{|B_2^{\kappa n}|}{|B_2^{\kappa n-\kappa}|} \max_{\xi \in S^{n-1}} \int_{K \cap H_\xi} |x|^n \, dx.\]

And setting $l = 0$ and $M = B_2^{\kappa n}$ in Lemma 12, we obtain Corollary 9. For any star body $K$ in $\mathbb{K}^n$, we have

\[\int_K |x|^{\kappa} \, dx \leq \frac{n}{n-1} \frac{|B_2^{\kappa n}|}{|B_2^{\kappa n-\kappa}|} \max_{\xi \in S^{n-1}} |K \cap H_\xi|.\]

6. Intersection bodies of convex bodies in $\mathbb{K}^n$

In this section we extend to $\mathbb{K}^n$ Busemann’s theorem, which says that the intersection body of an origin-symmetric convex body in $\mathbb{R}^n$ is convex.

The first part of the proof goes along the lines of the proof of Busemann’s theorem in $\mathbb{R}^n$, up to the inequality (17). The Busemann’s theorem in $\mathbb{R}^n$ is then obtained by a clever use of the arithmetic-geometric mean inequality. This step has to be replaced by the use of a result of K. Ball, as it was done in the complex case, see [28]. We will use the following form of Ball’s result as stated in [28].

Proposition 4. ([28], Corollary 5) Let $r_1, r_2 > 0$ and let $\alpha > 0$. Define $\lambda, r_3$ as follows:

\[\lambda = \frac{r_1}{r_1 + r_2}, \quad r_3 = \frac{\alpha}{r_1^{-1} + r_2^{-1}}.\]

Assume that $f_1, f_2, f_3 : [0, \infty) \to [0, \infty)$ such that $f_3(r_3) \geq f_1(r_1)^{(1-\lambda)} f_2(r_2)^{\lambda}$ for any $r_1, r_2 > 0$. Let $p \geq 1$ and denote

\[A^p = \int_0^\infty f_1(r)r^{p-1} \, dr, \quad B^p = \int_0^\infty f_2(r)r^{p-1} \, dr, \quad C^p = \int_0^\infty f_3(r)r^{p-1} \, dr.\]

Then

\[C \geq \frac{\alpha}{\lambda + \frac{\alpha}{B}}.\]
Theorem 7. (Busemann’s theorem in $\mathbb{R}^n$) Let $S$ be a $\kappa(n-2)$-dimensional $\kappa$-balanced subspace of $\mathbb{R}^n$ and $u \in S^{n-1} \cap S^\perp$. Denote by $S_u = \text{span}\{S, \{R_\sigma(u)\}_{i=0}^{\kappa-1}\}$. Define a function $r : S^{n-1} \cap S^\perp \to (0, \infty)$ by

$$r(u) = |K \cap S_u|^{1/\kappa}.$$  

Then the curve $r$ is the boundary of a $\kappa$-balanced convex body in $S^\perp$.

Proof: The curve $r$ is the boundary of a convex body in $S^\perp$ if and only if $r^{-1}$ satisfies the triangle inequality: For two linearly independent unit vectors $u_1, u_2$ in $S^\perp$

$$\frac{1}{r(u_1 + u_2)} \leq \frac{1}{r(u_1)} + \frac{1}{r(u_2)}. \quad (16)$$

Let $u_3 = \frac{u_1 + u_2}{|u_1 + u_2|}$, then (16) is equivalent to

$$\frac{|u_1 + u_2|}{r(u_3)} \leq \frac{1}{r(u_1)} + \frac{1}{r(u_2)}.$$  

We may assume that $H_{u_1}^\perp \cap H_{u_2}^\perp = \{0\}$, otherwise $H_{u_1}^\perp = H_{u_2}^\perp$ and (16) is trivially satisfied since $K \cap H_{u_1}^\perp \subset S^\perp$ is a ball.

Let $r_j > 0$, $j = 1, 2$, and let $r_3u_3 = (1 - \lambda)r_1u_1 + \lambda r_2u_2$ be the intersection point of the line in the direction $u_3$ with the line segment with endpoints $r_1u_1, r_2u_2$, then

$$\lambda = \frac{r_1}{r_1 + r_2}, \quad \frac{r_3}{|u_1 + u_2|} = \frac{1}{r_1^{-1} + r_2^{-1}}.$$  

For $t > 0$, let $f_{u_j}(t) = |K \cap (S + tu_j)|$, $1 \leq j \leq 3$. Observe that $f_{u_j}(t) = f_{R_\sigma(u_j)}(t)$ for any $\sigma \in SO(\kappa)$. Indeed, since $K$ is $\kappa$-balanced

$$f_{u_j}(t) = \int_{S + tu_j} \chi(\|x\|^{-1}_K)dx = \int_{S + tR_\sigma(u_j)} \chi(\|x\|^{-1}_K)dx = f_{R_\sigma(u_j)}(t).$$

This, in turn, implies that

$$r(u_j) = \left(\Omega_\kappa \int_0^\infty f_{u_j}(t)t^{\kappa-1}dt\right)^{1/\kappa}, \quad 1 \leq j \leq 3,$$
since
\[ r^\kappa(u_j) = \int_{H_{\omega_j}} |K \cap (S + x)| \, dx \]
\[ = \int_0^\infty \int_{S^{n-1} \cap H_{\omega_j}} |K \cap (S + t\theta)| \, d\theta \, t^{\kappa-1} \, dt \]
\[ = \int_0^\infty \int_{S^{n-1} \cap H_{\omega_j}} f_\theta(t) \, d\theta \, t^{\kappa-1} \, dt \]
\[ = \Omega^\kappa \int_0^\infty f_{u_j}(t) \, d\theta \, t^{\kappa-1} \, dt . \]

Note that \( r \) is \( \kappa \)-invariant.

By construction the sets \( K \cap (S + r_j u_j), 1 \leq j \leq 3 \) lie in an affine subspace of \( \mathbb{R}^{\kappa n} \). Hence, by convexity of \( K \), for \( \lambda \) as defined above
\[
(1 - \lambda)(K \cap (S + r_1 u_1)) + \lambda(K \cap (S + r_2 u_2)) \subset K \cap (S + r_3 u_3) .
\]
Applying the Brunn-Minkowski inequality, we obtain
\[
f_{u_3}(r_3)^{1/(\kappa(n-2))} \geq (1 - \lambda)f_{u_1}(r_1)^{1/(\kappa(n-2))} + \lambda f_{u_2}(r_2)^{1/(\kappa(n-2))} ,
\]
and the arithmetic-geometric mean inequality yields
\[
f_{u_3}(r_3) \geq f_{u_1}(r_1)^{(1 - \lambda)} f_{u_2}(r_2)^{\lambda} . \tag{17}
\]
Now we apply Proposition 4 with \( p = \kappa \) and \( \alpha = |u_1 + u_2| \), this gives what we need
\[
\frac{|u_1 + u_2|}{r(u_3)} \leq \frac{1}{r(u_1)} + \frac{1}{r(u_2)} .
\]

\[\square\]

**Corollary 10.** Let \( K \) be an origin-symmetric convex body in \( \mathbb{K}^n \), then \( I_{\kappa}(K) \) is also an origin-symmetric convex body in \( \mathbb{K}^n \).

**Proof:** In case \( n = 2 \), \( H_\xi \) is \( \kappa \)-dimensional and hence \( K \cap H_\xi \) is a ball. This implies that \( I_{\kappa}(K) \) is a rotation of \( K \). Indeed, let \( \xi \in S^{\kappa n-1} \), then
\[
\frac{\Omega^\kappa}{\kappa} \|\xi\|_{I_{\kappa}(K)}^{-\kappa} = |K \cap H_\xi| = \frac{\Omega^\kappa}{\kappa} \|x\|_{K}^{-\kappa} ,
\]
for any \( x \in K \cap H_\xi \).

Now assume \( n \geq 3 \). A subset \( L \) of \( \mathbb{R}^{\kappa n} \) is convex if and only if all its two-dimensional sections are convex. In other words, for any linearly independent vectors \( x, y \), the section \( L \cap \text{span}\{x, y\} \) is convex. The condition that \( L \cap \text{span}\{H^\perp_x, H^\perp_y\} \) is convex, is stronger and hence implies that \( L \) is convex.
Let $S$ be a $\kappa(n - 2)$-dimensional $\kappa$-balanced subspace of $\mathbb{R}^{\kappa n}$ and $u, v \in S^{\kappa n - 1} \cap S^{\perp}$ so that $v \perp \{R_{\sigma_i}(u)\}_{i=0}^{\kappa-1}$. Observe that

$$|K \cap H_v| = |I_{K}(K) \cap H_v^\perp| = \frac{\Omega \kappa}{\kappa} \|v\|_{I_{K}(K)}^{-\kappa} = \frac{\Omega \kappa}{\kappa} \rho_{I_{K}(K)}(v).$$

Hence, in the notation of the Busemann’s theorem

$$\rho_{I_{K}(K)}(v) = \left(\frac{\kappa}{\Omega} |K \cap H_v|\right)^{1/\kappa} = \left(\frac{\kappa}{\Omega} |K \cap S_u|\right)^{1/\kappa} = \left(\frac{\kappa}{\Omega}\right)^{1/\kappa} r(u)^{1/\kappa}.$$

This shows that $I_{K}(K) \cap S^\perp$ is convex, and hence $I_{K}(K)$ is convex.

Together Corollaries 2 and 10 show that $\kappa$-intersection bodies of $\kappa$-balanced convex bodies in $\mathbb{R}^{\kappa n}$ exist and are convex, which is not true in general as was shown by V. Yaskin [37].

The result of D. Hensley [15] and C. Borell [2], that the intersection body of a convex body is isomorphic to an ellipsoid, extends to $K^n$ via a result from [17]. Recall that the Banach-Mazur distance of two origin-symmetric convex bodies $K, L$ in $\mathbb{R}^n$ is defined as

$$d_{BM}(K, L) = \inf\{a > 0 : K \subset TL \subset aK \text{ with } T \in GL_n\}.$$

**Proposition 5.** ([17], Theorem 1.2) Let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$ and assume that the $k$-intersection body of $K$, $I_k(K)$, exists and is convex, then

$$d_{BM}(I_k(K), B_2^n) \leq c(k),$$

where $c(k)$ only depends on $k$.

Combining the above proposition with Corollaries 10 and 2 yields

**Corollary 11.** Let $K$ be an origin-symmetric convex body in $K^n$, then

$$d_{BM}(I_K(K), B_2^{\kappa n}) \leq c(\kappa),$$

where $c(\kappa)$ only depends on $\kappa$.

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**References**

1. Keith Ball, *Some remarks on the geometry of convex sets*, Geometric aspects of functional analysis (1986/87), Lecture Notes in Math., vol. 1317, Springer, Berlin, 1988, pp. 224–231.
2. Christer Borell, *Complements of Lyapunov’s inequality*, Math. Ann. 205 (1973), 323–331.
3. Jean Bourgain, *On the Busemann-Petty problem for perturbations of the ball*, Geom. Funct. Anal. 1 (1991), no. 1, 1–13.
4. ______, *On the distribution of polynomials on high-dimensional convex sets*, Geometric aspects of functional analysis (1989–90), Lecture Notes in Math., vol. 1469, Springer, Berlin, 1991, pp. 127–137.
5. Herbert Busemann and Clinton M. Petty, *Problems on convex bodies*, Math. Scand. 4 (1956), 88–94.
6. Richard J. Gardner, *Intersection bodies and the Busemann-Petty problem*, Trans. Amer. Math. Soc. 342 (1994), no. 1, 435–445.
7. ______, *A positive answer to the Busemann-Petty problem in three dimensions*, Ann. of Math. (2) 140 (1994), no. 2, 435–447.
8. ______, *Geometric tomography*, second ed., Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, Cambridge, 2006.
9. Richard J. Gardner, Alexander Koldobsky, and Thomas Schlumprecht, *An analytic solution to the Busemann-Petty problem on sections of convex bodies*, Ann. of Math. (2) 149 (1999), no. 2, 691–703.
10. I. M. Gel’fand and N. Ya. Vilenkin, *Generalized functions. Vol. 4*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1964 [1977], Applications of harmonic analysis, Translated from the Russian by Amiel Feinstein.
11. Israel M. Gel’fand and Georgi E. Shilov, *Generalized functions. Vol. 1*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1964 [1977], Properties and operations, Translated from the Russian by Eugene Saletan.
12. Apostolos A. Giannopoulos, *A note on a problem of H. Busemann and C. M. Petty concerning sections of symmetric convex bodies*, Mathematika 37 (1990), no. 2, 239–244.
13. Paul Goodey, Erwin Lutwak, and Wolfgang Weil, *Functional analytic characterizations of classes of convex bodies*, Math. Z. 222 (1996), no. 3, 363–381.
14. Paul Goodey and Wolfgang Weil, *Intersection bodies and ellipsoids*, Mathematika 42 (1995), no. 2, 295–304.
15. Douglas Hensley, *Slicing convex bodies—bounds for slice area in terms of the body’s covariance*, Proc. Amer. Math. Soc. 79 (1980), no. 4, 619–625.
16. Boas Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Anal. 16 (2006), no. 6, 1274–1290.
17. A. Koldobsky, G. Paouris, and M. Zymonopoulou, *Isomorphic properties of intersection bodies*, J. Funct. Anal. 261 (2011), no. 9, 2697–2716.
18. Alexander Koldobsky, *Stability and separation in volume comparison problems*, preprint.
19. ______, *An application of the Fourier transform to sections of star bodies*, Israel J. Math. 106 (1998), 157–164.
20. ______, *Intersection bodies, positive definite distributions, and the Busemann-Petty problem*, Amer. J. Math. 120 (1998), no. 4, 827–840.
21. ______, *A functional analytic approach to intersection bodies*, Geom. Funct. Anal. 10 (2000), no. 6, 1507–1526.
22. ______, *Fourier analysis in convex geometry*, Mathematical Surveys and Monographs, vol. 116, American Mathematical Society, Providence, RI, 2005.
23. ______, *Stability in the Busemann-Petty and Shephard problems*, Adv. Math. 228 (2011), no. 4, 2145–2161.
24. ______, *Stability of volume comparison for complex convex bodies*, Arch. Math. (Basel) 97 (2011), no. 1, 91–98.
25. ______, A hyperplane inequality for measures of convex bodies in $\mathbb{R}^n$, $n \leq 4$, Discrete Comput. Geom. 47 (2012), no. 3, 538–547.
26. Alexander Koldobsky, Hermann König, and Marisa Zymonopoulou, The complex Busemann-Petty problem on sections of convex bodies, Adv. Math. 218 (2008), no. 2, 352–367.
27. Alexander Koldobsky and Dan Ma, Stability and slicing inequalities for intersection bodies, Geom. Dedicata 162 (2013), 325–335.
28. Alexander Koldobsky, Grigoris Paouris, and Marisa Zymonopoulou, Complex intersection bodies, preprint.
29. Alexander Koldobsky and Vladyslav Yaskin, The interface between convex geometry and harmonic analysis, CBMS Regional Conference Series in Mathematics, vol. 108, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2008.
30. David G. Larman and C. Ambrose Rogers, Rogers, The existence of a centrally symmetric convex body with central sections that are unexpectedly small, Mathematika 22 (1975), no. 2, 164–175.
31. Erwin Lutwak, Intersection bodies and dual mixed volumes, Adv. in Math. 71 (1988), no. 2, 232–261.
32. Emanuel Milman, Generalized intersection bodies, J. Funct. Anal. 240 (2006), no. 2, 530–567.
33. ______, Generalized intersection bodies are not equivalent, Adv. Math. 217 (2008), no. 6, 2822–2840.
34. Michael Papadimitrakis, On the Busemann-Petty problem about convex, centrally symmetric bodies in $\mathbb{R}^n$, Mathematika 39 (1992), no. 2, 258–266.
35. Boris Rubin, Intersection bodies and generalized cosine transforms, Adv. Math. 218 (2008), no. 3, 696–727.
36. ______, Comparison of volumes of convex bodies in real, complex, and quaternionic spaces, Adv. Math. 225 (2010), no. 3, 1461–1498.
37. Vlad Yaskin, Countereamples to convexity of k-intersection bodies, to appear in Proc. AMS.
38. Gao Yong Zhang, Intersection bodies and the Busemann-Petty inequalities in $\mathbb{R}^4$, Ann. of Math. (2) 140 (1994), no. 2, 331–346.
39. ______, A positive solution to the Busemann-Petty problem in $\mathbb{R}^4$, Ann. of Math. (2) 149 (1999), no. 2, 535–543.
40. Gaoyong Zhang, Sections of convex bodies, Amer. J. Math. 118 (1996), no. 2, 319–340.
41. Artem Zvavitch, The Busemann-Petty problem for arbitrary measures, Math. Ann. 331 (2005), no. 4, 867–887.
42. Marisa Zymonopoulou, A note on the Busemann-Petty problem for bodies of certain invariance, preprint, arXiv:0811.1593.
43. Marisa Zymonopoulou, The complex Busemann-Petty problem for arbitrary measures, Arch. Math. (Basel) 91 (2008), no. 5, 436–449.

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