A four node quadrilateral shell element for free vibration response of functionally graded spherical shell panels

R B Dahele¹, S D Kulkarni², V A Dagade¹

¹Research Scholar, ²Former Associate Professor
Department of Civil Engineering, College of Engineering Pune, Pune 411005, Maharashtra, India.
rajendra2299@gmail.com

Abstract. The present work involves the free vibration analysis of functionally graded spherical shells using a four-node quadrilateral shell element. This four-node, shell element has seven DOF (Degrees of Freedom) per node namely; three displacements, two rotations of mid-plane, and two transverse shear strain components. The element is developed by modifying the discrete Kirchhoff quadrilateral shell element based on Reddy’s third-order theory developed earlier by the authors for the cylindrical shell. By transforming all the degrees of freedom from local to global coordinates using a transformation matrix converts the plate bending element into a flat shell element, resulting in the present formulation being suitable to give results for thin shells as well as for thick shells. In the present work functionally graded spherical shell panels with simply supported as well as clamped boundary conditions, with various radius to span ratios and with different volume fraction indices are analyzed for free vibration response. The power law property variation through the thickness is considered in this study. To assess the performance of the developed element, the results, of non-dimensionals frequencies are compared with the results presented in the literature. Comparison of results shows that the developed element yields quite accurate results even with the coarser mesh, which indicates the computational efficiency. It is also seen that the percentage difference between the present results and the other results available in the literature is less than 2 in most of the cases.

Keywords- Functionally Graded Material, Spherical Shell, Free Vibration, Finite Element, Third-order.

1. Introduction

The Functionally graded materials are useful for structural elements that are subjected to high differential temperatures. Laminated composite can also be used for the above applications, but at the layer interface, there is discontinuity of mechanical properties and are susceptible to delamination. FG materials do not suffer from delamination as the material properties vary smoothly across the thickness and thus advantages over the laminated composite materials. FG material is microscopically heterogeneous and usually contains two material phases namely; Ceramic and metal. Ceramic acts as a thermal barrier and metal impart the required ductility. In the literature, various 2D and 3D solutions have been presented by various researchers. In the present study, a four-node quadrilateral shell element is developed based on Reddy’s third-order theory which was used earlier by for free vibration response of functionally graded cylindrical shells, since the finite element technique is useful for the analysis of spherical shells with various displacements-based boundary conditions.
Earlier, Dahale et.al. [1] have developed a four node quadrilateral element based on Reddy’s theory for the free vibration analysis of functionally graded cylindrical shell panels, where $C^1$ continuity problem posed by Reddy’s third order theory in finite element formulation is successfully circumvented by using improved discrete Kirchhoff interpolation functions. The Present finite element results are compared with the finite element results based on Sander’s theory and Donnell’s theory of Pandey et.al. [3] and Liu et.al. [4] The present results are also compared with the finite element results based on higher order shear deformation theory with eight-node and nine degrees of freedom per node of Pradyumna et.al. [5] It is seen from the literature that no researcher has so far used the four node quadrilateral plate element using the novel concept of obtaining the transformation matrix, which is used in present formulation for the analysis of functionally graded spherical shell panels.

The present element is observed to yield quite accurate results. The use of two separate coordinate systems for translational and rotational degrees of freedom and the use of the plate theory is the novelty of the present approach.

1.1. Displacement Approximation for the Reddy’s Third Field Order Theory

Consider a spherical shell panel as shown in ‘Figure 1’ with thickness ‘$h$’. The shell's middle surface is known to be a reference plane where $z$ is zero and the top surface is at $z = h/2$ and bottom surface is at $z = -h/2$.

![Figure 1. Geometry of FGM spherical shells.](image_url)

The displacement field approximation of Reddy’s third order theory assumes a third order variation for the in-plane displacement and a uniform transverse displacement across the thickness. The condition of zero shear stress at the top and bottom of the plate imposed and has same number of primary variables as the first order shear deformation theory. Reddy’s third order theory does not require the shear correction factor unlike first order shear deformation theory. This theory can yield quite accurate results for thick as well as thin plates.

The displacement field approximation is as follows:

\begin{align}
  w(x, y, z, t) &= w_0(x, y, t) \\
  u(x, y, z, t) &= u_0(x, y, t) - z w_{0z} + R(z) \varphi_0(x, y, t)
\end{align}

Where,

\begin{align}
  u &= \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \\
  w_{0d} &= \begin{bmatrix} w_{0x} \\ w_{0y} \end{bmatrix}, \\
  u_0 &= \begin{bmatrix} u_{0x} \\ u_{0y} \end{bmatrix}, \\
  \varphi_0 &= \begin{bmatrix} \varphi_{0x} \\ \varphi_{0y} \end{bmatrix}
\end{align}

and

\begin{align}
  R(z) &= \left[ z - \frac{4z^2}{3h^2} \right]
\end{align}

1.2. Material Properties of FGM
Consider a spherical shell panel as shown in ‘Figure 1’. The properties changes gradually from ceramic phase to the metal phase in functionally graded material. Voigt’s rule of mixtures (ROM) is used for predicting the material property variation, through the thickness as follows.

$$P_z = P_b + \left( P_t - P_b \right) \left( 0.5 + \frac{z}{h} \right)^n$$

In above equation $n$ is the power law index which can vary from 0 to $\infty$. $P_b$ represents the properties of the material at bottom and that of the top of the shell is $P_t$. $P_z$ is the properties of the material at any $z$-coordinates. Poisson’s ratio is assumed to be constant.

1.3. Finite Element Formulation

The spherical shell shown in ‘Figure 1’ is discretized using the quadrilateral elements developed by the same authors and presented in Dahale et.al. [1], with necessary modifications, as shown in ‘Figure 2’. The actions and displacements are transformed from local to global coordinate system by using the concept of two co-ordinates systems as given by Clough et.al. [2]. The orientations of local, global, and surface coordinate axes are shown in ‘Figure 2’.

![Figure 2. Discretized spherical shells with coordinate systems.](image)

Cartesian coordinates ($x$, $y$, $z$) and surface coordinates ($\xi_1$, $\xi_2$, $\xi_3$) are shown in ‘Figure 2’. In surface coordinate $\xi_3$ is taken normal to the shell surface at every node of the element. It is assumed that rotation about the normal to the tangent plane would be negligible in the actual shell. Accordingly, rotational degrees of freedom referred to the surface coordinates ($\xi_1$, $\xi_2$) per nodal point are only included in the base system describing the assembled structure by Clough et.al. [2]”. Thus in this approach, the local translational degrees $u_{0x}$, $u_{0y}$, $w_0$ are transformed to base coordinate system with reference to global Cartesian coordinates ($x$, $y$, $z$) and the rotational degrees $\theta_{0x}$, $\theta_{0y}$, $\psi_{0x}$, $\psi_{0y}$ are transformed to surface coordinate system ($\xi_1$, $\xi_2$, $\xi_3$), neglecting the contribution to ($\xi_3$).

$U^{e'}$ is local and $U^e$ is the global displacement vector for $i^{th}$ node shown in the following expressions.

$$U^{e'} = \begin{bmatrix} u_{0x}^{i} & u_{0y}^{i} & w_0^{i} & \theta_{0x}^{i} & \theta_{0y}^{i} & \psi_{0x}^{i} & \psi_{0y}^{i} \end{bmatrix}^T$$

$$U^e = \begin{bmatrix} u_{0x}^{i} & u_{0y}^{i} & w_0^{i} & \theta_{0x}^{i} & \theta_{0y}^{i} & \psi_{0x}^{i} & \psi_{0y}^{i} & \psi_{0z}^{i} \end{bmatrix}^T$$

Transformation matrix $T^i$ for transforming the local translational DOF to global translational DOF for an $i^{th}$ node is given by the following expression.
where $\lambda_{11}$, $\lambda_{12}$ etc. are the direction cosines of the local (primed) axes with respect to global (unprimed) axes.

The relationship between the local rotations and global rotations for an $i^{th}$ node is given by the following expression.

\[
\begin{bmatrix}
\theta_{0x}' \\
\theta_{0y}' \\
\theta_{0z}'
\end{bmatrix} =
\begin{bmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} \\
\lambda_{31} & \lambda_{32} & \lambda_{33}
\end{bmatrix}
\begin{bmatrix}
\theta_{0x} \\
\theta_{0y} \\
\theta_{0z}
\end{bmatrix}
\]

\[\text{i.e. } \theta_0' = T \theta_0\]  \hspace{1cm} (7)

The relationship between the surface rotations $\theta_0'^x$ and $\theta_0'^y$ for the $i^{th}$ node can be obtained as follows. For a spherical shell, the angle between the global x-axis and tangent $\xi_1$ can be obtained easily as this tangent is actually tangent to the circle $x^2 + z^2 = R^2$ in the x-z plane. Differentiating this expression for the circle yields, $\tan \alpha = -x/z$. $\alpha$ is the angle between the axes $\xi_1$ and x. In addition, as the axis $\xi_2$ is along the global y-axis, is the tangent to the circle $y^2 + z^2 = R^2$ in y-z plane, considering the angle between the global y axis and $\xi_2$ is $\beta$ i.e. $\tan \beta = -y/z$. Using this, the relationship between $\theta_0'^x$ and $\theta_0'^y$ for the $i^{th}$ node can be obtained. Transformation matrix $T_{\xi}^i$ for transforming the surface rotational DOF to global rotational DOF for $i^{th}$ node is given by the following expression.

\[
\begin{bmatrix}
\theta_{0x} \\
\theta_{0y} \\
\theta_{0z}
\end{bmatrix} =
\begin{bmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} \\
\lambda_{31} & \lambda_{32} & \lambda_{33}
\end{bmatrix}
\begin{bmatrix}
\theta_{0x}' \\
\theta_{0y}' \\
\theta_{0z}'
\end{bmatrix}
\]

\[\text{i.e. } \theta_0 = T_{\xi}^i \theta_0'\]  \hspace{1cm} (8)

Using eq. (8) and eq. (9), relationship between the $\theta_0'^x$ and $\theta_0'^y$ is

\[
\theta_0'^x = T_{\xi}^i T_{\xi} \theta_0^x
\]

\[\text{i.e. } \theta_0'^x = T_{\xi}^i \theta_0^x\]  \hspace{1cm} (9)

The transformation matrix $T_{\xi}^i$ of size $2 \times 2$, which relates the local rotations $\theta_0'^x$, $\theta_0'^y$ to tangential rotations $\theta_0'^x$, $\theta_0'^y$ are obtained by removing the third row and third column of transformation matrix $T_{\xi}^i$. $T_{\xi}^i$ is also used for transforming $\psi_0'^x$, $\psi_0'^y$, to tangential shell directions. Finally, the element transformation matrix of size 28×28 $T^e$ is given as,
\[ T^e = \begin{bmatrix} T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & T_1^\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_1^\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_2^\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & T_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & T_3^\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_3^\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_4^\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_4^\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{bmatrix} \] (11)

After deriving the expressions for transformation matrix \( T^e \), local stiffness matrix \( K^e \), local mass matrix \( M^e \) and local load vector \( P^e \), finally the element stiffness matrix \( K_e \) of size 28×28, element mass matrix \( M_e \) of size 28×28 and element load vector \( P_e \) of size 28×1 with reference to global axes are given as

\[ M^e = T^e^T M^e' T^e, \]
\[ K^e = T^e^T K^e' T^e, \]
\[ P^e = T^e^T P^e'. \] (12)

1.4. Assembly of element matrices

Summing up the contributions for all elements Hamilton’s principle yields

\[ M \ddot{U} + KU = \bar{P} \] (13)

where \( \bar{K}, \bar{M}, \) and \( \bar{P} \) are the assembled counterparts of matrices \( K^e, M^e, \) and \( P^e \). For synchronous vibration, the \( \bar{P} \) (load vector) is set to zero. The undamped natural frequencies \( \omega_n \) are obtained by solving eigenvalue problem.

2. Numerical Results

A computer program is developed in FORTRAN for the free vibration analysis based on the formulation in section 2. A computer program and finite element formulation are assessed by comparing the result of non-dimensionalised natural frequencies obtained using developed element with the results in the literature. In all the problems considered in the study, the full shell panel is discretized with \( N \times N \) mesh of equal size elements. The shell type and material properties considered are specified in ‘Table 1’.

| Table 1. Material Properties. |
|-----------------------------|
| Shell type                  | Material                | Modulus of Elasticity GPa | Density kg/m$^3$ | Poisson’s ratio |
| FG M1 (top ceramic)         | Ceramic - Silicon carbide | $E_c = 427$                | $\rho_c = 3100$  | $\nu_c = 0.3$   |
| FG M2 (top ceramic)         | Ceramic - Alumina       | $E_m = 70$                 | $\rho_m = 2707$  | $\nu_m = 0.3$   |
| FG M1 (top ceramic)         | Metal - Alumina         | $E_c = 380$                | $\rho_c = 3000$  | $\nu_c = 0.3$   |
| FG M2 (top ceramic)         | Metal - Alumina         | $E_m = 70$                 | $\rho_m = 2707$  | $\nu_m = 0.3$   |
The obtained natural frequencies are non-dimensionalised as:

- \( \overline{\omega} = \alpha \frac{a^2}{h} \sqrt{\frac{\rho}{E}} \) \( \rho m a h E \omega = (14) \)

- \( \overline{\omega} = \alpha a^2 \sqrt{\frac{\rho h}{D_m}} \), where \( D_m = \frac{E h^3}{12(1-\nu^2)} \) \( \) \( \) \( \) (15)

\[2.1. \text{All-round simply supported spherical shell panel with square plan-form} \]

The results of the natural frequencies of fundamental mode are presented for a spherical shell panel for ‘a/h’ = 5, 10, and 20 for two different ‘R/a’ ratios. The present results are compared with the results of Pandey et.al. [3] obtained using eight-node element with nine degrees of freedom per node shell element based on Sander’s theory and Donnell’s shell theories. In the analysis of the shell with mesh size 8 × 8 gives total 225 nodes and 2025 DOF. The results of Liu et.al. [4] obtained using four-node with nine degrees of freedom per node shell element based on the same theories are also given in the table for comparison. In the analysis of the shell with mesh size 8 × 8 gives total 81 nodes and 729 DOF. It is observed that for thick to thin shell panels the present element results are quite comparable with those of Pandey et.al. [3] and Liu et.al. [4]. It is noted that the present element is not based on shell theory. It is also observed that even though the element is based on plate theory it gives sufficiently accurate results not only for the shallow shell but for deep shells too. The 24 × 24 mesh size shell having 625 total nodes and 4375 total degrees of freedom. It is also to be noted that as the volume fraction increases, the values of the non-dimensionalised frequencies decrease because of the increase in metal part and thereby reducing the stiffness.

Table 2. Non-dimensionalised frequencies of all-round simply supported FGM1 spherical shell.

| R/a | a/h | n  | Pandey et. al. [3] 8 × 8 | Donnell’s [3] 8 × 8 | Sander’s [4] 8 × 8 | Donnell’s [4] 8 × 8 | DKTOTSS-FG (Present) |
|-----|-----|----|-------------------------|-------------------|------------------|------------------|---------------------|
| 0.5 | 5   | 1  | 10.9134                 | 10.2602           | 11.0092          | 11.0076          | 10.9861             |
| 0.5 | 5   | 2  | 8.3062                  | 7.8821            | 8.6135           | 8.6083           | 8.5911              |
| 0.5 | 5   | 5  | 7.3351                  | 7.2739            | 7.6106           | 7.6022           | 7.5903              |
| 1   | 5   | 1  | 9.5173                  | 9.2754            | 9.7803           | 9.7794           | 9.7584              |
| 1   | 5   | 2  | 8.0760                  | 8.1172            | 8.6135           | 8.6083           | 8.5911              |
| 1   | 5   | 5  | 7.1920                  | 7.4478            | 7.6106           | 7.6022           | 7.5903              |
| 2   | 5   | 1  | 12.6384                 | 12.4847           | 13.2238          | 13.1906          | 13.1887             |
| 10  | 5   | 1  | 20.0728                 | 19.9198           | 20.8099          | 20.7455          | 20.6781             |
| 20  | 5   | 1  | 15.0152                 | 14.9544           | 15.6747          | 15.6249          | 15.6420             |
| 0.5 | 5   | 1  | 10.0214                 | 9.6165            | 10.0214          | 10.0214          | 10.0214             |
| 0.5 | 5   | 2  | 8.8273                  | 8.7506            | 8.8295           | 8.8484           | 8.8804              |
| 0.5 | 5   | 5  | 7.7408                  | 7.6733            | 7.7825           | 7.7712           | 7.7643              |
| 1   | 5   | 1  | 9.5173                  | 9.2754            | 9.7803           | 9.7794           | 9.7584              |
| 1   | 5   | 2  | 8.0760                  | 8.1172            | 8.6135           | 8.6083           | 8.5911              |
| 1   | 5   | 5  | 7.1920                  | 7.4478            | 7.6106           | 7.6022           | 7.5903              |
| 2   | 5   | 1  | 12.6384                 | 12.4847           | 13.2238          | 13.1906          | 13.1887             |
| 10  | 5   | 1  | 20.0728                 | 19.9198           | 20.8099          | 20.7455          | 20.6781             |
| 20  | 5   | 1  | 15.0152                 | 14.9544           | 15.6747          | 15.6249          | 15.6420             |

The non-dimensionalised fundamental frequencies of FGM2 spherical shells are analyzed in ‘Table 3’ for ‘R/a’ = 1, 5, and 10 with ‘a/h’ ratio is equal to 10. The shells of the power law indices
‘n’ varying from 0, 0.2, 0.5 and 1 are presented in the table. The present results obtained using 12 × 12, 16 ×16, and 24 × 24 mesh are compared with those presented by Pradyumna et. al [5] obtained using eight-node element having nine degrees of freedom per node. The formulation is based on the higher-order shear deformation theory. It is observed that for deep to shallow shell panels the present element results are quite comparable with those of Pradyumna et. al [5].

Table 3. Non-dimensionalised frequencies of all-round simply supported FGM2 spherical shell.

| R/a | n    | Mesh size | 0    | 0.2  | 0.5  | 1    |
|-----|------|-----------|------|------|------|------|
|     | Pradyumna [5] | 8×8 | 78.2306 | 72.6343 | 66.5025 | 59.8521 |
|     | DKTOTSS-FG    | 12×12 | 82.6259 | 76.7050 | 70.2166 | 63.2263 |
| 1   |      | 16×16 | 81.8414 | 75.9609 | 69.5103 | 62.5522 |
|     |      | 24×24 | 80.9123 | 75.1218 | 68.7532 | 61.8604 |
|     | Pradyumna [5] | 8×8 | 44.0073 | 41.7782 | 38.7731 | 34.6004 |
|     | DKTOTSS-FG    | 12×12 | 43.8301 | 42.0462 | 37.8716 | 33.6194 |
| 5   |      | 16×16 | 43.8747 | 42.0790 | 37.8763 | 33.6161 |
|     |      | 24×24 | 43.9458 | 42.1399 | 37.9145 | 33.5979 |
|     | Pradyumna [5] | 8×8 | 42.3579 | 40.2608 | 37.3785 | 33.3080 |
|     | DKTOTSS-FG    | 12×12 | 42.1685 | 40.4163 | 36.3157 | 32.1643 |
| 10  |      | 16×16 | 42.1953 | 40.4358 | 36.156 | 32.1430 |
|     |      | 24×24 | 42.2171 | 40.4522 | 36.3176 | 32.1299 |

2.2. All-round clamped spherical shell panel with square plan-form

The fundamental non-dimensionalised frequencies of all-round clamped square planform FGM1 spherical shells for ‘a/h’ = 5, 10, and 20 for two different ‘R/a’ ratios are presented in ‘Table 4’. The results are compared with Liu et.al. [4]. The percentage difference between Liu et.al. [4] and present element using 24 × 24 mesh size are given in the table. It is noted from ‘Table 4’, that the results for this case also are in good agreement with those given by Liu et.al. [4].

Table 4. Non-dimensionalised frequencies of all-round clamped FGM1 spherical shell.

| R/a | a/h | n   | Liu et. al. [4] | DKTOTSS-FG | % diff. |
|-----|-----|-----|----------------|------------|---------|
|     |     | 8 × 8 | 12 × 12 | 16 × 16 | 24 × 24 |
| 2   | 5   | 0.5  | 14.1023 | 13.7174 | 13.4032 | 0.45 |
|     |     | 1    | 29.6243 | 30.3452 | 30.3793 | 30.4040 | 3.89 |
|     |     | 5    | 16.9965 | 15.9336 | 15.9174 | 15.9019 | 1.87 |
|     | 10  | 0.5  | 15.9159 | 15.9972 | 15.9294 | 15.8687 | 0.30 |
|     |     | 1    | 14.1023 | 14.2154 | 14.0937 | 14.0382 | 0.45 |
|     |     | 2    | 13.3124 | 12.7446 | 12.2826 | 12.2301 | 0.67 |
|     |     | 5    | 10.6242 | 10.6858 | 10.6247 | 10.3877 | 2.23 |
| 5   | 10  | 0.5  | 16.9965 | 17.2719 | 17.2433 | 17.2161 | 1.29 |
|     |     | 1    | 19.6830 | 20.2246 | 20.2254 | 20.2244 | 2.75 |
|     |     | 5    | 15.6730 | 15.9864 | 15.9772 | 15.6595 | 0.09 |
The results for the first six mode shapes for FGM1 shallow shell panel with ‘R/a’ = 5 and ‘a/h’ = 5 for volume fraction index = 1 are plotted in Fig 4 and Fig 5 for the all-round clamped and for one edge clamped and other three free (cantilever) boundary conditions respectively. The mesh size of $24 \times 24$ is used for obtaining mode shapes. These results will serve as a benchmark for the other researchers.

\begin{align*}
\omega_1 &= 14.0382 \\
\omega_2 &= 24.5212 \\
\omega_3 &= 24.5289 \\
\omega_4 &= 33.2571 \\
\omega_5 &= 33.9456 \\
\omega_6 &= 33.9553
\end{align*}

**Fig 4.** First six mode shapes of all-round clamped FGM1 shells for ‘R/a’=5, ‘a/h’=5 ‘n’ = 1.

\begin{align*}
\tilde{\omega}_1 &= 1.6844 \\
\tilde{\omega}_2 &= 3.6752 \\
\tilde{\omega}_3 &= 6.0828 \\
\tilde{\omega}_4 &= 8.7855 \\
\tilde{\omega}_5 &= 11.2717 \\
\tilde{\omega}_6 &= 12.1602
\end{align*}

**Fig 5.** First six mode shapes of cantilever FGM1 spherical shell for ‘R/a’=5, ‘a/h’=5 ‘n’ = 1.

The non-dimensionalised frequencies of the all-round clamped FGM2 spherical shells are analyzed in ‘Table 5’ for ‘R/a’ = 1, 5, and 10 and ‘a/h’ ratio of 10. The present results obtained are compared with those presented by Pradyumna et.al. [5]. It is observed that for deep to shallow shell panels the present element results are quite comparable with those of Pradyumna et.al. [5].
Table 5. Non-dimensionalised frequencies of all-round clamped FGM2 spherical shell.

| R/a | Mesh size | 0    | 0.2  | 0.5  | 1    |
|-----|-----------|------|------|------|------|
|     | Pradyumna [5] |      |      |      |      |
| 1   | 8 × 8      | 120.9210 | 112.2017 | 102.5983 | 92.2147 |
|     | 12 × 12    | 122.1768 | 113.9150 | 104.6969 | 94.6296 |
|     | 16 × 16    | 122.2951 | 114.1103 | 104.8073 | 94.7293 |
|     | 24 × 24    | 122.3743 | 114.1103 | 104.8850 | 94.7997 |
|     | % diff.    | 1.20  | 1.70  | 2.23  | 2.80  |
| 5   | Pradyumna [5] |      |      |      |      |
|     | 8 × 8      | 73.5550 | 69.6597 | 64.6114 | 57.8619 |
|     | 12 × 12    | 74.2272 | 71.2583 | 64.2484 | 57.0335 |
|     | 16 × 16    | 74.1496 | 71.1737 | 64.1603 | 56.9321 |
|     | 24 × 24    | 74.0600 | 71.0880 | 64.0772 | 56.8439 |
|     | % diff.    | 0.69  | 2.05  | 0.83  | 1.76  |
| 10  | Pradyumna [5] |      |      |      |      |
|     | 8 × 8      | 71.4659 | 67.7257 | 62.8299 | 56.2222 |
|     | 12 × 12    | 72.3846 | 69.4558 | 62.5596 | 55.4823 |
|     | 16 × 16    | 72.2942 | 69.3684 | 62.4696 | 55.3791 |
|     | 24 × 24    | 72.1926 | 69.2713 | 62.3759 | 55.2806 |
|     | % diff.    | 1.02  | 2.28  | 0.72  | 1.67  |

3. Conclusions

The present element gives sufficiently accurate results for natural frequencies for all cases of functionally graded shells including the shallow and deep shells considered in this study. The percentage difference between the present results and those reported in literature is even less than 0.5 in some cases and mostly below 2. It is also observed from the tables that the results obtained are converging as the mesh becomes fine. It is also observed that, the present formulation is suitable for capturing the modes, which the 2D analytical theory fails to capture. The element developed for free vibration response of functionally graded spherical shells are suitable for the general purpose finite element programming, and do not suffer from any locking problem, hence it is useful to thick as well as thin shell. This proves the suitability of the developed element for the functionally graded spherical shells. The use of the two coordinate system in the transformation of translational and rotational degrees of freedom from local to global system, is an important contribution of the present study.

References

1. Dahale R B, Kulkarni S D and Dagade V A 2021 Free Vibration Response of Functionally Graded Cylindrical Shells Using a Four Node Flat Shell Element. Advances in construction technology and management-2021, International conference 2021
2. Clough R W and Johnson CP 1968 A finite element approximation for the analysis of thin shells. International Journal of Solids and Structures 4 (1) 43-60
3. Pandey S and Pradyumna S 2015 A layerwise finite element formulation for free vibration analysis of functionally graded sandwich shells. Composite Structures vol 133, pp 438–450
4. Liu B, Ferreira A J M, Xing Y F and Neves A M A 2016 Analysis of functionally graded sandwich and laminated shells using a layerwise theory and a differential quadrature finite element method. Composite Structures vol 136, pp 546–553
5. Pradyumna S and Bandyopadhyay J N 2008 Free vibration analysis of functionally graded curved panels using a higher-order finite element formulation. Journal of Sound and Vibration vol 318, pp 176–192