Santilli Autotopisms of Partial Groups

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Abstract: This paper deals with those partial groups that contain a given Santilli isotopism in their autotopism group. A classification of these autotopisms is explicitly determined for partial groups of order n ≤ 4.

Keywords: Partial Group, Isotopism, Classification

1. Introduction

In 1942, Albert [1] introduced the concept of isotopy of algebras: Two algebras \((A_1, \cdot)\) and \((A_2, \ast)\) over a field \(K\) are said to be isotopic if there exist three regular linear transformations \(f, g\) and \(h\) from \(A_1\) to \(A_2\) such that

\[
f(u) \ast g(v) = h(u \cdot v), \quad \text{for all } u, v \in A_1.
\]

1) The algebra \(A_2\) is then said to be isotopic to the algebra \(A_1\), or, equivalently, \(A_2\) is an isotope of \(A_1\). The triple \(\Theta = (f, g, h)\) is an isotopy or isotopism between both algebras \(A_1\) and \(A_2\). If \(f = g = h\), then this is indeed an isomorphism. If the elements of \(A_1\) and \(A_2\) coincide, then the isotopism \(\Theta\) is said to be principal if \(h\) is the trivial transformation \(Id\), that is, if \(h(u) = Id(u) = u\), for all \(u \in A_1\). In this case, the algebra \(A_2\) is said to be a principal isotope of \(A_1\). In his original paper, Albert proposed the question as to whether a principal isotope of a Lie algebra is Lie. In this regard, he proved that a principal isotope \(A_2\) of a Lie algebra \(A_1\) with respect to a principal isotopism \((f, g, Id)\) is a Lie algebra if and only if, for all \(u, v, w \in A_1\), it verified that

\[
f(f(u) \cdot g(v)) = - f(v) \cdot g(u).
\]

2) In 1944, Bruck [2] introduced the concept of isotopically simple algebra as a simple algebra such that all their isotopic algebras are simple. He proved in particular that the Lie algebra of order \(n \cdot (n-1)/2\), consisting of all skew-symmetric matrices over any subfield of the field of all reals, under the multiplication \([u, v] = u \cdot v - v \cdot u\), is an isotopically simple algebra over \(R\).

More recently, in 1978, Santilli [3] generalized the associative product \(u \cdot v\) between Hermitian generators of the universal enveloping associative algebra by considering the new product

\[
u \ast v = u \cdot T \cdot v
\]

where \(T\) is a positive-definite operator called the isotopic element, which can indeed depend on distinct factors

\[
T = T(x, x', x'', ..., \mu, \tau)
\]

The product

\[
[u, v] = u \ast v - v \ast u
\]

preserves the Lie axioms and is called the Lie-isotopic product. The application to Lie’s theory (enveloping algebras, Lie algebras and Lie groups) that emerges from this new product is the so-called Lie-Santilli isotheory (see [3, pp. 287-290 and 329-374] and also [4-9]).

In the development of the isotheory, Santilli extended the unit of the ground field to the generalized unit or isounit

\[
I = I(x, x', x'', ..., \mu, \tau) = T^{-1}
\]

He defined then the isonumbers

\[
u = u \ast I(x, x', x'', ..., \mu, \tau), \quad \text{for all } u \in A.
\]

and the isoproduct

\[
[u, v] = u \ast v - v \ast u
\]
This isoproduct constitutes the Lie product of an isomorphic Lie algebra of $A$ whenever the isomorphism $f$ is constant. In any other case, this determines a generalization of the classical notion (2) of isomorphism. In order to analyze this fact, the authors [10] reinterpreted in 2006 the dependence on distinct factors of the isomorphism as a family of classical Bruck’s isomorphisms. This reinterpretation became clearer shortly after [11] once the attention was focused not on isomorphisms of algebras, but on isomorphisms of partial quasi-groups.

The term quasi-group was introduced in 1937 by Haussmann and Ore [12] to denote a nonempty set $Q$ endowed with a product $·$, such that if any two of the three symbols $u, v$ and $w$ in the equation $u · v = w$ are given as elements of $Q$, then the third is uniquely determined as an element of $Q$. Its order is the cardinality of the underlying set, that is, the number of elements of the quasigroup $Q$. This is said to be a loop if it contains a unit element, that is, there exists an element $e ∈ Q$ such that $e · u = u · e = u$ for all $u ∈ Q$. Every associative loop is indeed a group. The multiplication table of a quasigroup of order $n$ is a Latin square of order $n$, that is, an $n × n$ array with elements chosen from a set of $n$ distinct symbols such that each symbol occurs precisely once in each row and each column (see Figure 1).

![Figure 1. Latin square of order 4.](image)

A partial Latin square of order $n$ is an $n × n$ array with elements chosen from a set of $n$ distinct symbols such that each symbol occurs at most once in each row and each column (see Figure 2). It constitutes the multiplication table of a finite partial quasigroup $(Q, ·)$ of order $n$. Let $u, v ∈ Q$. The product $u · v$ is then an element of $Q$ or it is undefined. This last case is denoted as $u · v = Ø$. By abuse of notation, it is also considered that $u · Ø = Ø · u = Ø$, for all $u ∈ Q$ and hence, the partial quasigroup is associative if $(u · v) · w = u · (v · w)$, for all $u, v, w ∈ Q$. It is a partial loop if there exists an element $e ∈ Q$ such that $e · u = u · e = u$ for all $u ∈ Q$ and there does not exist an element $e' ≠ e$ such that $e' · u = u$ or $e · e' = u$. Every associative partial loop constitutes a partial group.

![Figure 2. Partial Latin square of order 4.](image)

In 1943-44, Albert [13, 14] together with Bruck [15] extended the definition of isomorphism from algebras to quasi-groups. Particularly, two quasi-groups $(Q_1, ·)$ and $(Q_2, ·)$ of the same order are said to be isotopic if there exist three bijections $f, g$ and $h$ between their sets of elements such that

$$f(u) · g(v) = h(u · v), \text{ for all } u, v ∈ Q.$$  

(10)

The definition can be naturally extended to partial quasi-groups once it is considered $h(Ø) = Ø$. The triple $Θ = (f, g, h)$ is said to be an isotopism between $Q_1$ and $Q_2$ and it is denoted $Q_2 = Q_1Θ$. If $Q_2 = Q_1$, then the isotopism $Θ$ is said to be an autotopism of $Q_1$ and $f, g$ and $h$ are permutations of the elements of $Q_1$. The set of autotopisms of a (partial) quasigroup constitutes, therefore, a group with the component-wise composition of permutations.

In 2007, the authors [11] introduced the concept of Santilli isotopism between partial quasi-groups. Specifically, an isotopism $Θ = (f, g, h)$ between two partial quasi-groups $(Q_1, ·)$ and $(Q_2, ·)$ is said to be a Santilli isotopism if there exist three elements $i_1, i_2$ and $i_3$ in $Q_1$ such that

$$f(u) = u · i_1, \quad g(v) = v · i_2, \quad h(u) = v · i_3, \text{ for all } u ∈ P_1.$$  

(11)

The triple $(i_1, i_2, i_3)$ is denoted by $S(Θ, Q_1)$. If $Q_2 = Q_1$, then the isotopism $Θ$ is said to be a Santilli autotopism of $Q_1$.

In [11], there were exposed several properties of the set of partial quasi-groups having a Santilli autotopism that fixes at least one of the symbols. An exhaustive study of Santilli autotopisms is, however, currently required. This paper is established as a first contribution in this regard. In Section 2, some new general properties of the set of Santilli isotopisms of (associative) partial quasi-groups, partial loops and partial groups are analyzed. In Section 3, a classification of the Santilli autotopisms of groups of order $n ≤ 6$ is explicitly given. Remark that, even if the number of quasi-groups is known for order up to 11 [16, 17], that of partial quasi-groups is only known for order up to four [18, 19].

2. Santilli Autotopisms

From now on, every partial quasi-group of order $n$ is considered to be formed by the set of elements $\{1, ..., n\}$. The set of isotopisms of partial quasi-groups of order $n$ is then denoted as $I_n = S_n × S_n × S_n$, where $S_n$ is the symmetric group on $\{1, ..., n\}$. The set of fixed symbols in a permutation $π ∈ S_n$ is denoted as

$$Fix(π) = \{u ∈ \{1, ..., n\} \text{ such that } π(u) = u\}.$$  

(12)

Let $Θ ∈ I_n$ and let $SQ(Θ), SL(Θ), SAQ(Θ)$ and $SG(Θ)$ be, respectively, the sets of partial quasi-groups, partial loops, associative partial quasi-groups and partial groups that have $Θ$ as a Santilli autotopism. The next results are satisfied.

Lemma 2.1. Let $Θ = (f, g, h) ∈ I_n$ and $(Q, ·) ∈ SQ(Θ)$ be such that $S(Θ, Q) = (i_1, i_2, i_3)$. Then, $i_0 = g(i_3)$. As a consequence,

$$(i · i_j) · (f · i_j) = (i · j) · (i_1 · i_j), \text{ for all } i, j ∈ Q.$$  

(13)

Proof. Given $v ∈ Q$, let $u ∈ Q$ be such that $f(u) = v$. Then, $v · i_0 = h(v) = h(f(u)) = h(u · i_j) = f(u) · g(i_3) = v · g(i_3)$. The result holds from the fact that $Q$ is a partial quasigroup and $h(v) ∈ Q$.

Proposition 2.2. Let $Θ = (f, g, h) ∈ I_n$ and $(Q, ·) ∈ SQ(Θ)$ be such that $S(Θ, Q) = (i_1, i_2, i_3)$. If $h = f$, then $i_1 ∈ Fix(g)$.

Proof. The result follows straightforward from Lemma 2.1 and the fact of being $h = f$. 

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Lemma 2.3. Let $\Theta = (f, g, h) \in I_\nu$. If there exist two permutations $\alpha, \beta \in \{f, g, h\}$ such that $\alpha(u_\beta) = \beta(u_\alpha)$ for some $u_\beta \in Q$, then $\alpha = \beta$.

Proof. Let $(Q, \cdot)$ be a partial quasigroup in $SQ(\Theta)$ and let $i_\nu \in Q$ be such that $\alpha(u) = u \cdot i_\nu$ and $\beta(u) = u \cdot i_\nu$ for all $u \in Q$. Particularly, $u_\alpha \cdot i_\nu = \alpha(u_\beta) = \beta(u_\alpha) = u_\alpha \cdot i_\nu$. This product is not undefined because $\alpha(u_\beta) \in Q$. Since $Q$ is a partial quasigroup, it must be then $u_\alpha = i_\mu$ and hence, $\alpha = \beta$.

Proposition 2.4. Let $\Theta = (f, g, h) \in I_\nu$ be such that $Fix(g) = \emptyset$. Then, $\nu(u) \neq h(u)$ for all $u \in Q$.

Proof. Let $u \in Q$ be such that $\nu(u) = h(u)$. From Lemma 2.1, it must be $f = h$. Thus, from Lemma 2.1, it is $i_\mu = g(i_\nu)$ and hence, $i_\mu \in Fix(g)$, which is a contradiction.

Lemma 2.5. Let $\Theta = (f, g, h) \in I_\nu$ and $(Q, \cdot) \in SQ(\Theta)$ be such that $S(\Theta(Q) = (i_\nu, i_\mu, i_\delta)$. If there exists $u_0 \in Q$ such that $h^m(g(u_0)) = g(f^m(u_0))$ for some positive integer $m$, then $i_\mu \in Fix(g^m)$. As a consequence, if $Fix(g^m) = \emptyset$ for some positive integer $m$, then $h^m(g(u_0)) \neq g^m(f^m(u_0))$, for all $u \in Q$.

Proof. Let $m$ be such that $h^m(g(u_0)) = g^m(f^m(u_0))$ for some $u_0 \in Q$. It is then $f^m(u_0) \cdot g(i_\nu) = h^m(u_0 \cdot i_\mu) = h^m(g(u_0)) = g(f^m(u_0)) = f^m(u_0 \cdot i_\mu)$. This product is not undefined because $h^m(g(u_0)) \in Q$. Since $Q$ is a partial quasigroup, it must be then $i_\mu \in Fix(g^m)$. The consequence is immediate.

Lemma 2.6. Let $\Theta = (f, g, h) \in I_\nu$ be such that $|Fix(f)| \cdot |Fix(g)| \cdot |Fix(h)| > 0$. Let $(Q, \cdot) \in SQ(\Theta)$ be such that $S(\Theta(Q) = (i_\nu, i_\mu, i_\delta)$. If there exist $u_0 \in Q$, $w_0 \in Fix(h)$ and $\alpha \in \{f, g, h\}$ such that $\alpha(u_\beta) = w_0$, then $i_\mu \in Fix(g)$. Further, if $i_\mu \in Fix(g)$, then $g(u) \in Fix(h)$ for all $u \in Q$.

Proof. It is satisfied that $u_0 \cdot i_\mu = \alpha(u_\beta) = w_0 = h(w_0) = h(0) = f(u_0 \cdot i_\mu) = g(i_\nu) = u_0 \cdot g(i_\nu)$. Since $w_0 \in Q$ and $Q$ is a quasigroup, it must be $i_\mu \in Fix(g)$. Let us suppose now that $i_\mu \notin Fix(g)$ and let us consider an element $u \in Fix(f)$. Then $g(u) = u \cdot i_\mu = f(u) \cdot g(i_\mu) = h(u \cdot i_\mu) = h(g(u))$ and hence, $u \in Fix(h)$.

The next three results deal with the set of partial loops $SL(\Theta)$ having a Santilli isotopism $\Theta$ in their autotopism group.

Proposition 2.7. Let $\Theta = (f, g, h) \in I_\nu$ and $(Q, \cdot) \in SL(\Theta)$ be a partial loop with unit element $e$. Then, $S(\Theta(Q) = (f(e), g(e), g(f(e)))$.

Proof. Let $S(\Theta(Q) = (i_\mu, i_\nu, i_\delta)$. The result follows straightforward from Lemma 2.1 and the fact that $e \in Q$. Hence, $\mu(e) = e \cdot i_\mu = i_\mu \cdot i_\nu = e \cdot i_\nu = e \cdot i_\nu$ = $\mu(e) = e \cdot i_\nu$ = $\mu(e)$. Hence, $\mu(e) = e \cdot i_\nu$ = $\mu(e)$. Hence, $\mu(e) = e \cdot i_\nu$ = $\mu(e)$.

Lemma 2.8. Let $\Theta = (f, g, h) \in I_\nu$. If there exists a permutation $\pi \in \{f, g, h\}$ such that $Fix(\pi) \neq \emptyset$, then $\pi = Id$.

Proof. Let $(Q, \cdot) \in SL(\Theta)$ and $S(\Theta(Q) = (i_\mu, i_\nu, i_\delta)$. Let $\pi \in \{f, g, h\}$ and $u_0 \in Q$ be such that $\pi(u_\beta) = u_0$. Since $u_0 = u_0 \cdot i_\delta$, the element $i_\delta$ is the unit element of the partial loop. Let $u \in Q$. Since $\pi(u) \in Q$, it is $\pi(u) = u \cdot i_\mu = u$ and hence, $\pi = Id$.

Lemma 2.9. Let $\Theta = (f, g, h) \in I_\nu$ and $(Q, \cdot) \in SL(\Theta)$ be a partial loop with unit element $e$. If $e \in Fix(f^m)$ for some positive integer $m$, then $h^m = f^m$.

Proof. Let us suppose that $e \in Fix(f^m)$ for some positive integer $m$. Let $u \in Q$. It is $g(u) = e \cdot g(u) = f(e) \cdot g(u) = h^m(e \cdot u)$. Since $g(u) \in Q$, it must be $e \cdot u = u$ and hence, $g^m(u) = h^m(u)$. The last assertion follows analogously.

We focus now on the set $SAQ(\Theta)$ of associative partial quasigroups having a Santilli autotopism in their autotopism group.

Proposition 2.10. Let $\Theta = (f, g, h) \in I_\nu$. If $SAQ(\Theta) \neq \emptyset$, then $h = g^m$.

Proof. Let $(Q, \cdot) \in SAQ(\Theta)$ and $S(\Theta(Q) = (i_\mu, i_\nu, i_\delta)$. From Lemma 2.1, we know that $i_\mu = g(i_\nu)$. Hence, for all $u \in Q$, it is verified that $h(u) = u \cdot i_\mu = u \cdot g(i_\nu) = (u \cdot i_\nu) \cdot i_\mu = (u \cdot i_\nu) \cdot i_\mu = g(\nu(u)))$.

Lemma 2.11. Let $\Theta = (f, g, h) \in I_\nu$ be such that $SAQ(\Theta) \neq \emptyset$ and let $m \leq n$ be a positive integer. Then

a) $SAQ(\Theta) \subseteq SAQ(\Theta^m)$. 

b) $SAQ(\Theta) = SAQ((f, g \cdot f^m, h \cdot f^m))$. 

Proof. Let $(Q, \cdot) \in SAQ(\Theta)$ be such that $S(\Theta(Q) = (i_\mu, i_\nu, i_\delta)$ and let $m \leq n$ be a positive integer. Then

1. The isotopism $\Theta^m$ is an autotopism of $(Q, \cdot)$ because $h^m(u \cdot v) = h^m(f(u) \cdot g(v)) = f(u) \cdot g^m(v)$, for all $u, v \in Q$. Since the quasigroup $(Q, \cdot)$ is associative, this is indeed a Santilli autotopism for which $S(\Theta^m(Q) = (i_\mu, i_\nu, i_\delta)$.

2. The isotopism $(f, g \cdot f^m, h \cdot f^m)$ is an autotopism of $(Q, \cdot)$ because $h^m(f(u) \cdot v) = h^m(f(u) \cdot v) = h^m(f(u) \cdot v) = f(u) \cdot g^m(v) = f(u) \cdot g^m(v) = f(u) \cdot g^m(v) = f(u) \cdot g^m(v) = f(u) \cdot v \in Q$. Since the quasigroup $(Q, \cdot)$ is associative, this is indeed a Santilli autotopism for which $S(\Theta^m(Q) = (i_\mu, i_\nu, i_\delta)$, hence, $SAQ(\Theta) \subseteq SAQ((f, g \cdot f^m, h \cdot f^m))$.

Let us consider now an associative partial quasigroup $(Q, \cdot) \in SAQ((f, g \cdot f^m, h \cdot f^m))$ such that $S(\Theta((f, g \cdot f^m, h \cdot f^m), Q) = (i_\mu, i_\nu, i_\delta)$.
Theorem 2.12. Let \( \Theta = (f, g, h) \in I_n \). If \( SG(\Theta) \neq \emptyset \) and \( Fix(f) \neq \emptyset \), then \( g = h \) and \( f = Id \).

Proof. The result follows straightforward from Lemma 2.8 and Proposition 2.10.

3. Santilli Autotopisms of Partial Groups of Order \( n \leq 4 \)

The results that have been exposed in Section 2 can be taken into account in order to determine explicitly the set of Santilli isotopisms that are autotopisms of partial groups of a given order. To this end, we say that two isotopisms \( \Theta_1 = (f_1, g_1, h_1) \) and \( \Theta_2 = (f_2, g_2, h_2) \) in \( I_n \) are equivalent if \( f_2 = f_1 \) and there exists a positive integer \( m \leq n \) such that \( g_2 = g_1 \circ f_1^m \) and \( h_2 = h_1 \circ f_1^m \).

From assertion (b) in Lemma 2.11, it is verified that \( SAQ(\Theta_1) = SAQ(\Theta_2) \). To be equivalent is then an equivalence relation in the set \( I_n \). Let \( \{ \Theta \} \) denote the equivalence class of \( \Theta \in I_n \). We expose in Table 1 these equivalence classes for Santilli autotopisms of partial groups of order \( n \leq 4 \). We focus on the case of non-trivial isotopisms, that is, those that do not coincide with \( (Id, Id, Id) \).

| \( n \) | \( \{ \Theta \} \) | \( SG(\Theta) \) |
|-------|----------------|----------------|
| 2     | \{ (12), (12), Id \} | \( A_2 \) |
|       | \{ (Id, (12), (12)) \} |               |
| 3     | \{ (123), (123), (123) \} | \( A_3 \) |
|       | \{ (Id, (123), (123)) \} |               |
| 4     | \{ (1234), (1234), (1234) \} | \( A_4 \) |
|       | \{ (Id, (1234), (1234)) \} |               |
|       | \{ (1234), (1234), (1234) \} |               |
|       | \{ (1234), (1234), (1234) \} |               |

We indicate for each class \( \{ \Theta \} \) in Table 1 the set \( SG(\Theta) \) of partial groups that have all the isotopisms of the class in their corresponding autotopism group. The multiplication tables of the elements of these sets are described in Figures 4–12.

Figure 4. Partial Latin squares related to \( A_2 \).

| \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) |
|-------|-------|-------|-------|
| 1     | 2     | 3     | 4     |
| 2     | 3     | 4     | 1     |
| 3     | 4     | 1     | 2     |
| 4     | 1     | 2     | 3     |

Figure 5. Partial Latin squares related to \( A_3 \).

| \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) |
|-------|-------|-------|-------|
| 1     | 2     | 3     | 4     |
| 2     | 3     | 4     | 1     |
| 3     | 4     | 1     | 2     |
| 4     | 1     | 2     | 3     |

Figure 6. Partial Latin squares related to \( A_4 \).

| \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) |
|-------|-------|-------|-------|
| 1     | 2     | 3     | 4     |
| 2     | 3     | 4     | 1     |
| 3     | 4     | 1     | 2     |
| 4     | 1     | 2     | 3     |

Figure 7. Partial Latin squares related to \( B_4 \).

| \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) |
|-------|-------|-------|-------|
| 1     | 2     | 3     | 4     |
| 2     | 3     | 4     | 1     |
| 3     | 4     | 1     | 2     |
| 4     | 1     | 2     | 3     |

Figure 8. Partial Latin squares related to \( C_4 \).

| \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) |
|-------|-------|-------|-------|
| 1     | 2     | 3     | 4     |
| 2     | 3     | 4     | 1     |
| 3     | 4     | 1     | 2     |
| 4     | 1     | 2     | 3     |

Figure 9. Partial Latin squares related to \( D_4 \).

| \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) |
|-------|-------|-------|-------|
| 1     | 2     | 3     | 4     |
| 2     | 3     | 4     | 1     |
| 3     | 4     | 1     | 2     |
| 4     | 1     | 2     | 3     |

Figure 10. Partial Latin squares related to \( E_4 \).

| \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) |
|-------|-------|-------|-------|
| 1     | 2     | 3     | 4     |
| 2     | 3     | 4     | 1     |
| 3     | 4     | 1     | 2     |
| 4     | 1     | 2     | 3     |

Figure 11. Partial Latin squares related to \( F_4 \).

| \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) |
|-------|-------|-------|-------|
| 1     | 2     | 3     | 4     |
| 2     | 3     | 4     | 1     |
| 3     | 4     | 1     | 2     |
| 4     | 1     | 2     | 3     |

Figure 12. Partial Latin squares related to \( G_4 \).

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