ON THE LIFTING OF HILBERT CUSP FORMS TO HILBERT-SIEGEL CUSP FORMS

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Abstract. Starting from a Hilbert cusp form of weight \(2\kappa\), we will construct a Hilbert-Siegel cusp form of weight \(\kappa + \frac{m}{2}\) and degree \(m\) and its transfer to inner forms of symplectic groups. Applications include a relation between Fourier coefficients of Hilbert cusp forms of weight \(\kappa + \frac{1}{2}\) and certain weighted sum of the representation numbers of a quadratic form of rank \(2n\) by a quadratic form of rank \(4n\).

Partant d’une forme modulaire parabolique de Hilbert de poids \(2\kappa\), nous construisons une forme modulaire parabolique de Hilbert-Siegel de poids \(\kappa + \frac{m}{2}\) et de degré \(m\) et son transfert aux formes intérieures des groupes symplectiques. Comme application, on obtient entre autres une relation entre les coefficients de Fourier de formes modulaires paraboliques de Hilbert de poids \(\kappa + \frac{1}{2}\) et une certaine somme pondérée des nombres de représentation d’une forme quadratique de rang \(2n\) par une forme quadratique de rang \(4n\).

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1. Introduction

The present investigation deals with the following problem: starting from simple automorphic data such as cusp forms on $\GL_2$, construct more complicated automorphic forms on groups of higher degree. Toward this problem, Ikeda [22] has constructed a lifting associating to an elliptic cusp form a Siegel cusp form of even genus. This paper generalizes it to Hilbert cusp forms with different methods. The resulting Hilbert-Siegel cusp forms are applied to the theory of quadratic forms.

To illustrate our results, let $F$ be a totally real number field of degree $d$ with adèlic ring $\mathbb{A}_F$. We write $\mathbb{A}_F$ and $\mathbb{A}_\infty$ for the finite part and the infinite part of the adèlic ring. We denote the set of $d$ real primes of $F$ by $\mathfrak{S}_\infty$ and the normalized absolute value by $\alpha = \prod_v \alpha_v : \mathbb{A}_\times \to \mathbb{R}_+^\times$.

Let $\Sym_m = \{ z \in M_m | \| z \| = z \}$ be the space of symmetric matrix of size $m$ and $W_m$ a symplectic vector space of dimension $2m$. We take matrix representation

$$ Sp_m = \left\{ g \in \GL_{2m} \mid g \begin{pmatrix} 0 & -1_m \\ 1_m & 0 \end{pmatrix} g^t = \begin{pmatrix} 0 & -1_m \\ 1_m & 0 \end{pmatrix} \right\} $$

of the associated symplectic group $Sp(W_m)$ by choosing a Witt basis of $W_m$. We define homomorphisms $m : \GL_m \to Sp_m$ and $n : \Sym_m \to Sp_m$ by

$$ m(a) = \begin{pmatrix} a & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad n(b) = \begin{pmatrix} 1_m & b \\ 0 & 1_m \end{pmatrix}. $$

Let $\Mp(W_m)_\mathbb{A} \to Sp_m(\mathbb{A})$ be the metaplectic double cover. Denote the inverse image of $Sp_m(\mathbb{A}_\infty)$ (resp. $Sp_m(\mathbb{A}_F)$) by $\Mp(W_m)_\infty$ (resp. $\Mp(W_m)_F$).

Define the character $e_\infty : \mathbb{C}^d \to \mathbb{C}_\times$ by $e_\infty(z) = \prod_v e^{2\pi \sqrt{-1} z_v}$. Let $\psi = \prod_v \psi_v$ be the additive character of $\mathbb{A}/F$ whose restriction to $\mathbb{A}_\infty$ is $e_\infty|_{\mathbb{R}^d}$. We let $p$ denote a finite prime of $F$ and do not use $p$ for an archimedean place. For $\xi \in \Sym_m(F)$ we define the character $\psi^\xi = \prod_p \psi^\xi_p : \Sym_m(\mathbb{A}_\ell) \to \mathbb{C}_\times$ by $\psi^\xi_p(z) = \prod_p \psi_p(\tr(\xi z_p))$. For $t \in F_\ell^\times$ there is an $8$th root of unity $\gamma(\psi^t)$ such that for all Schwartz functions $\phi$ on $F_v$

$$ \int_{F_v} \phi(x_v)\psi_v(tx_v^2)\,dx_v = \gamma(\psi^t_v)|2t|_v^{-1/2} \int_{F_v} F_\phi(x_v)\psi_v \left( -\frac{x_v^2}{4t} \right) \,dx_v, $$

where $dx_v$ is the self-dual Haar measure on $F_v$ with respect to the Fourier transform $F_\phi(y) = \int_{F_v} \phi(x_v)\psi_v(x_v y)\,dx_v$. Set $\gamma(\psi_v) = \gamma(\psi^t)/\gamma(\psi^t_v)$. We denote the set of totally positive elements of $F$ by $F_+^\times$, the set of totally positive definite symmetric matrices of rank $m$ over $F$ by $\Sym^+_m$, and the set of all complex symmetric matrices of size $m$ with positive definite imaginary part by $\mathcal{H}_m$. For $t \in F_+^\times$ we write $\hat{\chi}^t = \prod_p \hat{\chi}^t_p$ for the quadratic character of $\mathbb{A}_\times/F^\times$ associated to the extension $F(\sqrt{t})/F$ and denote its restriction to the finite idèle group $\mathbb{A}_\mathbb{F}$ by $\hat{\chi}^t$. For $\ell \in \mathbb{R}^d$ we will set $|t|_v = \prod_{v \in \mathfrak{S}_\infty} |t|^\ell_v$.

The real metaplectic group $\Mp(W_m)_v$ acts on $\mathcal{H}_m$ through $Sp_m(F_v)$ for $v \in \mathfrak{S}_\infty$. There is a unique factor of automorphy $j : \Mp(W_m)_v \times \mathcal{H}_m \to \mathbb{C}_\times$
satisfying \( j(\tilde{g}_v, Z_v)^2 = \det(C_v Z_v + D_v) \). We here write the projection of \( \tilde{g}_v \) to \( S_{p_m}(F_v) \) as \( \begin{pmatrix} * & * \\ C_v & D_v \end{pmatrix} \). Let \( \ell \) be a tuple of \( d \) positive half integers such that \( 2\ell_v \equiv 2\ell_{v'} \mod 2 \) for all \( v, v' \in \Sigma_\infty \). We set \( J_\ell(\tilde{g}, Z) = \prod_{v \in \Sigma_\infty} j(\tilde{g}_v, Z_v)^{2\ell_v} \) for \( \tilde{g} \in \text{Mp}(W_m)_\infty \) and \( Z \in \mathcal{H}_m^d \). If \( \ell \in \mathbb{Z}^d \), then \( J_\ell \) descends to the function on \( S_{p_m}(A_\infty) \times \mathcal{H}_m^d \). Even when \( \ell \notin \mathbb{Z}^d \), one can define it on some congruence subgroup \( \Gamma_m \) of \( S_{p_m}(F) \). A Hilbert-Siegel modular form \( F \) of weight \( \ell \) with respect to a congruence subgroup \( \Gamma \) of \( S_{p_m}(F) \) is a holomorphic function on \( \mathcal{H}_m^d \) which satisfies \( F(\gamma Z) = F(Z) J_\ell(\gamma, Z) \) for every \( \gamma \in \Gamma \) and also the additional condition at infinity if \( m = 1 \) and \( F = \mathbb{Q} \). Let \( M_\ell(\Gamma) \) denote the vector space of such Hilbert-Siegel modular forms. The vector space \( \mathcal{S}_\ell(\Gamma) \) of Hilbert-Siegel cusp forms consists of functions \( F \in M_\ell(\Gamma) \) such that \( F(\gamma Z) \sqrt{J_\ell(\gamma, Z)}^{-1} \) has a Fourier expansion of the form \( \sum_{\alpha \in \text{Sym}^m_m} A(\xi) e_{\infty}(\text{tr}(\xi Z)) \) for all \( \gamma \in S_{p_m}(F) \), where \( \sqrt{J_\ell(\gamma, Z)} \) means any branch of the square root of \( J_\ell(\gamma, Z) \). Let \( \mathcal{S}_\ell^{(m)} \) denote the union of \( \mathcal{S}_\ell(\Gamma) \) for congruence subgroups \( \Gamma \subset \Gamma_m^d \). The group \( \text{Mp}(W_m)_\mathfrak{f} \) acts on the space \( \mathcal{S}_\ell^{(m)} \) and it is important to know which representations appear in this space. We shall explicitly construct a rather small irreducible submodule, which is neither tempered nor generic at any finite prime.

We define the space \( \mathcal{C}_{2\kappa} \) of Hilbert cusp forms on \( \text{PGL}_2 \) of weight \( 2\kappa \) in Definition \( \ref{def:2} \). Let \( \pi_\mathfrak{f} \simeq \oplus_\mathfrak{p} \pi_\mathfrak{p} \) be an irreducible admissible unitary generic representation of \( \text{PGL}_2(A_\mathfrak{f}) \). For some reason (see Remark \( \ref{rem:3} \)) we suppose that none of \( \pi_\mathfrak{p} \) is supercuspidal, i.e., there is a collection of continuous characters \( \mu_\mathfrak{p} \) of the multiplicative groups of nonarchimedean local fields \( F_\mathfrak{p} \) such that \( \pi_\mathfrak{f} \) is equivalent to the unique irreducible submodule of the principal series representation \( \oplus_\mathfrak{p} I(\mu_\mathfrak{p}, \mu_\mathfrak{p}^{-1}) \), where \( \mu_\mathfrak{p} \) is unramified for almost all \( \mathfrak{p} \). Put \( \mu_\mathfrak{f} = \prod_\mathfrak{p} \mu_\mathfrak{p} \). We form the restricted tensor product \( I_{n, \mathfrak{f}}^{(\oplus)}(\mu_\mathfrak{f}) = \oplus_\mathfrak{p} I_{n, \mathfrak{p}, \mathfrak{f}}^{(\oplus)}(\mu_\mathfrak{p}) \), where \( I_{n, \mathfrak{p}, \mathfrak{f}}^{(\oplus)}(\mu_\mathfrak{p}) \) is the representation of the local metaplectic group \( \text{Mp}(W_{m})_{\mathfrak{p}} \) on the space of smooth functions \( h_\mathfrak{p} \) on \( \text{Mp}(W_{m})_{\mathfrak{p}} \) transforming on the left according to

\[
h_\mathfrak{p}((m(a)n(b), \zeta)\tilde{g}) = \zeta^{m_\mathfrak{f}} \gamma_\mathfrak{f}(\det a)^{m_\mathfrak{f}} \mu_\mathfrak{f}(\det a) | \det a|^{(m+1)/2} h_\mathfrak{p}(\tilde{g})
\]

for all \( \zeta \in \{ \pm 1 \} \), \( a \in \text{GL}_m(F_\mathfrak{p}) \), \( z \in \text{Sym}_m(F_\mathfrak{p}) \) and \( \tilde{g} \in \text{Mp}(W_{m})_{\mathfrak{p}} \). This representation has a unique irreducible submodule \( A_{m, \mathfrak{f}}(\mu_\mathfrak{f}) \), which is unitary.

**Theorem 1.1.** Notation being as above, \( A_{m, \mathfrak{f}}(\mu_\mathfrak{f}) \) appears in \( S_{(2\kappa+\kappa)/2}^{(m)} \) if and only if \( \pi_\mathfrak{f} \) appears in \( \mathcal{C}_{2\kappa} \) and \((-1)^{2\kappa} \prod_\mathfrak{p} \mu_\mathfrak{p}(-1) = 1 \).

We here denote the tuple \( (\kappa_\mathfrak{v} + m/2) \) \( \nu \in \Sigma_\infty \in \frac{1}{2}\mathbb{Z}^d \) simply by \( (2\kappa+m)/2 \). The representation \( \pi_\mathfrak{f} \) is the Shimura correspondence of \( A_{1, \mathfrak{f}}^{(\oplus)}(\mu_\mathfrak{f}) \) and \( A_{2, \mathfrak{f}}^{(\oplus)}(\mu_\mathfrak{f}) \) is the Saito-Kurokawa lifting of \( \pi_\mathfrak{f} \). Both are theta liftings. We discuss the connection of this result with Arthur’s endoscopic classification in \( \S 3 \).

Though the trace formula will ultimately lead to another proof, our proof,
which relies heavily on the theory of the Shimura correspondence but not on the Saito-Kurokawa lifting, is completely elementary. If $\kappa_v < m$ for every $v \in \mathcal{O}_\infty$, then we obtain an irreducible cuspidal automorphic representation which is nontempered at all the places.

More importantly, our proof gives more precise information. We can describe how the representation $A_m^{\psi_f}(\mu_f)$ is embedded in $S_{(2\kappa+m)/2}^{(m)}$ quite explicitly. Fix a Haar measure $db = \otimes_p db_p$ on $\text{Sym}_{m} (A_f)$. Then we can associate to each $\xi \in \text{Sym}_{m}$ a basis vector $w_{\xi}^{\mu_f}$ of the one-dimensional vector space $\text{Hom}_{\text{Sym}_{m}}(A_f)(I_m^{\psi_f}(\mu_f) \circ n, \psi_{\xi}^{*})$ by

$$w_{\xi}^{\mu_f}(\otimes_p h_p) = \prod_p w_{\xi}^{\mu_f}(h_p),$$

where $w_{\xi}^{\mu_f} \in \text{Hom}_{\text{Sym}_{m}}(I_m^{\psi_f}(\mu_f) \circ n, \psi_{\xi}^{*})$ is defined by

$$w_{\xi}^{\mu_f}(h_p) = \int_{\text{Sym}_{m}(F_p)} h_p \left( \left( \begin{array}{cc} 0 & 1_m \\ -1_m & 0 \end{array} \right) n(b_p), 1 \right) \psi_{\xi}^{*}(h_p) \, db_p$$

$$\times \frac{|\det \xi|^{(m+1)/4} |(m+1)/2|}{L(\frac{1}{2}, \mu_f \chi_{\xi}^{*})} \prod_{j=1}^{\text{det} \xi} L(2j-1, \mu_f^2) \times \begin{cases} 1 & \text{if } 2 \nmid m, \\ \left\{ \begin{array}{ll} \frac{m+1}{2} & \mu_p \chi_p^{*}(-1)^{m/2} \\
 & \text{if } 2|m. \end{array} \right. \end{cases}$$

The integral diverges but makes sense as it stabilizes. One can check that $w_{\xi}^{\mu_f}(h_p) = 1$ for almost all $p$.

**Theorem 1.2.** If $\pi_f$ appears in $\mathcal{C}_{2\kappa}$ and $(-1)^{\sum_{v \in \mathcal{O}_\infty} \kappa_v} \prod_p \mu_p(-1) = 1$, then $A_m^{\psi_f}(\mu_f)$ appears in the decomposition of $S_{(2\kappa+m)/2}^{(m)}$ with multiplicity one, and there is a set $\{c_t\}_{t \in F_+^*}$ of complex numbers such that the $\text{Mp}(W_m)f$-intertwining embeddings $i_{\eta}^{\mu_f}: A_m^{\psi_f}(\mu_f \chi_{f}^{*}) \hookrightarrow S_{(2\kappa+m)/2}^{(m)}$ are given for all $m$ and $\eta \in F_+^*$ by means of the Fourier expansion

$$i_{\eta}^{\mu_f}(h)(Z) = \sum_{\xi \in \text{Sym}_{m}} |\det \xi|^{(2\kappa+m)/4} c_\eta \det \xi e_{\kappa}(\text{tr}(\xi)Z)) w_{\xi}^{\mu_f \chi_{f}^{*}}(h).$$

The constant $c_t$ is a mysterious part of the $t$th Fourier coefficient of a Hilbert cusp form of weight $\kappa + \frac{1}{2}$. When $F = \mathbb{Q}$, Kohnen and Zagier [33] have given an exact relation between the square $c_t^2$ and the central value $L(\frac{1}{2}, \pi \otimes \chi_{f}^{*})$. We refer to Theorem 12.3 of [117] for its extension to Hilbert cusp forms. The formula of Fourier coefficients looks like the classical Maass relation.

Our proof is direct and simple. Section 6 is the technical heart of this paper. Lemmas 7.2 and 7.4 play the important role in the proof of Theorem 6.4. The series $i_{\eta}^{\mu_f}(h)$ is a cuspidal form if and only if so are all its Fourier-Jacobi coefficients of degree 1, thanks to Lemma 6.7, which can apply to arbitrary Hilbert-Siegel cusp forms. Taking into account the inductive structure described in Lemma 7.4, we will explicitly compute those Fourier-Jacobi series.
in \(\S\S\S.8\). From (8.3) we can choose the coefficients \(c_t\) so that they are the Shintani lifts of \(\pi_f\). We will prove Theorems 8.1 and 8.2 in \(\S\S\S.8\) except for the multiplicity of \(A_m^\text{tf}(\mu_f)\), which is determined in \(\S\S\S.8\). The proof in \(\S\S\S.8.4-8.5\) is most technically difficult in this paper and may be skipped by readers, though a characterization of the lifting is given at the end of \(\S\S\S.8\). Namely, if an irreducible cuspidal automorphic representation of \(\text{Mp}(W_m)\) has degenerate principal series as local components at nonarchimedean primes and has lowest weight representations of scalar \(K\)-type as archimedean components, then it is our lifting.

Theorem 8.4 constructs analogous liftings of \(\pi_f\) for inner forms of symplectic groups of even rank, which are given by similar Fourier series with the same coefficients \(c_t\). The series naturally extends to a cusp form on the similitude group for even \(m\) (see Remark 8.2.1). In the proof of this case we shall use transfers of the Saito-Kurokawa lifts instead of the Shimura correspondence due to the lack of Fourier-Jacobi coefficients of degree 1.

We will let \(m = 2n\) and construct Hilbert-Siegel cuspidal Hecke eigenforms of even degree by making Theorems 8.2 and 8.4 explicit with the test function \(h_p\) invariant under a maximal compact subgroup of \(\text{Sp}_{2n}(F_p)\). Let \(\mathcal{S}_{2n}(\mu_f) = I_{2n}^f(\mu_f \chi_f^{(-1)} n)\) be a representation of \(\text{Sp}_{2n}(F_p)\). We denote the integer rings of \(F\) and \(F_p\) by \(\mathfrak{o}\) and \(\mathfrak{p}\), respectively, the different, trace, norm of \(F/\mathfrak{o}\) by \(\mathfrak{d}\), \(\text{Tr}_{F/\mathfrak{o}}\), \(N_{F/\mathfrak{o}}\), and the cardinality of the residue field \(\mathfrak{o}/\mathfrak{p}\) by \(q_p\). Put
\[
\mathcal{R}_m = \{ \xi \in \text{Sym}_m(F) \mid \text{tr}(\xi z) \in \mathfrak{o} \text{ for every } z \in \text{Sym}_m(\mathfrak{o}) \}.
\]
Set \(\mathcal{R}_m^+ = \mathcal{R}_m \cap \text{Sym}_m^+\). With fractional ideals \(\mathfrak{b}, \mathfrak{c}\) of \(\mathfrak{o}\) such that \(\mathfrak{b} \mathfrak{c} \subset \mathfrak{o}\), we put
\[
(1.2) \quad \Gamma_m[\mathfrak{b}, \mathfrak{c}] = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{Sp}_m(F) \mid \alpha, \delta \in M_m(\mathfrak{o}), \beta \in M_m(\mathfrak{b}), \gamma \in M_m(\mathfrak{c}) \right\}.
\]

The norm and the order of a fractional ideal of \(\mathfrak{o}\) are defined by \(N(\mathfrak{p}^j) = q_p^j\) and \(\text{ord}_p \mathfrak{p}^j = j\). We denote the conductor of \(\chi^n\) by \(\mathfrak{o}_n\) and put
\[
f_p^n = \frac{1}{2}(\text{ord}_p \eta - \text{ord}_p \mathfrak{o}_n) \in \mathbb{Z}, \quad \eta = \sqrt{\frac{N_{F/Q}(\eta)}{N(\mathfrak{o}_n)}} \in \mathbb{Q}^\times.
\]
We put \(\delta_p(\eta) = 1\) if \(\sqrt{\eta} \in F_p\), \(\delta_p(\eta) = -1\) if \(F_p(\sqrt{\eta})\) is an unramified quadratic extension of \(F_p\), and \(\delta_p(\eta) = 0\) if \(F_p(\sqrt{\eta})\) is a ramified quadratic extension of \(F_p\). We define \(\Psi_p(\eta, X) \in \mathbb{C}[X + X^{-1}]\) by
\[
\Psi_p(\eta, X) = \begin{cases} \frac{X^{f_p^n} - X^{-f_p^n} - 1}{X - X^{-1}} + q_p^{-1/2} \delta_p(\eta) \frac{X^{f_p^n} - X^{-f_p^n}}{X - X^{-1}} & \text{if } f_p^n \geq 0, \\ 0 & \text{if } f_p^n < 0. \end{cases}
\]
For \(\xi \in \text{Sym}_2^+\), we set
\[
D(\xi) = (-1)^n \det(2\xi), \quad \delta_p(\xi) = \delta_p(D(\xi)), \quad f_p^\xi = f_p^{D(\xi)}, \quad f_\xi = f_{D(\xi)}.
\]
The Siegel series associated to $\xi \in \mathcal{H}_{2n}$ and $p$ is defined by
\[
b_p(\xi, s) = \sum_{z \in \text{Sym}_{2n}(F_p)/\text{Sym}_{2n}(o_p)} \psi'_p(-\text{tr}(\xi z))\nu[z]^{-s},
\]
where $\nu[z] = [z^{2n+1} + z^{2n}]$ and $\psi'_p$ is an arbitrarily fixed additive character on $F_p$ of order zero. We define the polynomial $\gamma_p(\xi, X) \in \mathbb{Z}[X]$ by
\[
\gamma_p(\xi, X) = \frac{1 - X}{1 - \delta_p(\xi)q_p^nX} \prod_{j=1}^{n} (1 - q_p^{2j}X^2).
\]

There exists a monic polynomial $F_p(\xi, X) \in \mathbb{Z}[X]$ such that
\[
b_p(\xi, s) = F_p(\xi, q_p^{-s})\gamma_p(\xi, q_p^{-s}), \quad F_p(\xi, X) = q_p^{2n+1}X^{2f'_p}F_p(\xi, q_p^{-2n+1}X^{-1})
\]
(see [24, 25]). We define $\tilde{F}_p(\xi, X) \in \mathbb{C}[X + X^{-1}]$ by
\[
\tilde{F}_p(\xi, X) = X^{-f'_p}F_p(\xi, q_p^{-(2n+1)/2}X).
\]

An explicit formula for $\tilde{F}_p(\xi, X)$ is given by Ikeda and Katsurada (cf. [26]).

We consider the parallel weight $(k, \ldots, k)$, which is denoted simply by $k$, for the rest of this section. When $d(k + n)$ is even, the Kohnen plus subspace $S^+_{(2k+1)/2}$ of the space of Hilbert cusp forms of weight $k + \frac{1}{2}$ with respect to $n$ and the congruence subgroup $\Gamma_1[\mathfrak{o}^{-1}, \mathfrak{o}]$ is defined in [33] if $F = \mathbb{Q}$ and in [10] in general. It should be remarked that our notation differs from that used in [10]. The superscript $^n$ indicates that one should take $\eta = (-1)^n$ in Definition 13.2 of [10]. Let $\pi_f \simeq \otimes'_p I(\alpha_p^{s_p}, \alpha_p^{-s_p})$ be an irreducible summand of $\mathcal{C}_{2k}$. There is a Hilbert cusp form in $S^+_{(2k+1)/2}$ which has a Fourier expansion of the form
\[
h_n(\mathcal{Z}) = \sum_{\eta \in F_{(2k-1)/2}} c(\eta) e_\infty(\eta \mathcal{Z}) f(\eta) \prod_{p} \Psi_p((-1)^n\eta, q_p^{s_p})
\]
and which generates $\otimes'_p \mathcal{F}_n(\alpha_p^{s_p}, \chi_p^{(-1)^n})$.

The following result is a special case of Corollary 14.1 and the generalization of the main result of [22].

**Corollary 1.3.** Notations and assumptions being as above, the series
\[
\text{Lift}_{2n}(\pi)(\mathcal{Z}) = \sum_{\xi \in \mathcal{O}_{2n}} c(\det(2\xi)) f^{(2k-1)/2} \prod_{p} \tilde{F}_p(\xi, q_p^{s_p}) e_\infty(\text{tr}(\xi \mathcal{Z}))
\]
is a Hilbert-Siegel cusp form in $S_{k+n}(\Gamma_{2n}[\mathfrak{o}^{-1}, \mathfrak{o}])$ and generates $\otimes'_p \mathcal{F}_{2n}(\alpha_p^{s_p})$.

All the results and the proofs in this paper are applicable, with some modifications, to holomorphic cusp forms on tube domains of other types, which generalizes the level one holomorphic cusp forms constructed in [24, 31, 31] to higher level. Actually, analogous Hilbert-Hermitian cusp forms are constructed for all the Hilbert cusp forms in a similar way. The assumption on $\pi_f$ should not be essential, but if supercuspidal representations had been
included, then the construction would not have been so neat (cf. Remark 10.3). Kim and Yamauchi [32] currently constructed higher level holomorphic cusp forms on the exceptional group of type \( E_7 \), generalizing their previous work [30]. Moreover, we shall construct Hilbert-Siegel cusp forms of Miyawaki type in Corollary 1.4, following the same technique as in [23]. Atobe [3] currently used our construction of Hilbert-Siegel cusp forms to establish a theory of Miyawaki liftings. Miyawaki liftings were constructed in a classical setting for quasisplit unitary groups in [22] and for \( \mathrm{GSpin}(2,10) \) in [31]. Furthermore, our lifting combined with the theta lifting for the dual pair \( M_p(W_m) \times O(2m+1) \), \( Sp_{2m} \times O(2m) \) or \( Sp(n,n) \times O^*(4n) \) produces even more Hilbert-Siegel cusp forms and CAP representations of orthogonal or quaternion unitary groups.

Let \( n \) be a positive integer such that \( dn \) is even. Then there exists a \( 4n \)-dimensional totally positive definite quadratic space \( (V_n, q_{V_n}) \) over \( F \) which is split over every nonarchimedean local field \( F_p \). We write \( \mathcal{M}(O(V_n)) \) for the space of locally constant functions on \( O(V_n, F) \bs O(V_n, \mathbb{A}_F) \). A function in \( \mathcal{M}(O(V_n)) \) is called an algebraic modular form (of trivial weight) for \( O(V_n) \) (cf. [14]). Since \( O(V_n, F_p) \) is the split orthogonal group, it has a parabolic subgroup \( \mathbb{P}_{2n}(F_p) \) whose Levi subgroup is isomorphic to \( \mathrm{GL}_{2n}(F_p) \). Let \( \mathbb{I}_{2n}((\mu_p)) = \operatorname{Ind}_{\mathbb{P}_{2n}(F_p)}^{O(V_n,F_p)} \mu_p \circ \det \) be the normalized induced representation. We here do not work in greatest possible generality, but rather consider a reasonable special case of irreducible principal series, in which the main ideas of the theory become clear.

**Corollary 1.4.** Let \( \pi = \otimes_v \pi_v \) be an irreducible cuspidal automorphic representation of \( \mathrm{PGL}_2(\mathbb{A}) \) such that \( \pi_v \) is a discrete series with minimal weight \( \pm 2n \) for \( v \in \mathcal{S}_\infty \) and such that \( \pi_p \simeq I(\mu_p, \mu_p^{-1}) \) for every prime \( p \). If \( dn \) is even, then \( \mathbb{I}_{2n}(\mu_p) \simeq \otimes_p \mathbb{I}_{2n}(\mu_p) \) occurs in \( \mathcal{M}(O(V_n)) \) with multiplicity one.

This \( O(V_n, \mathbb{A}_F) \)-intertwining embedding is here denoted by

\[
\mathbb{J}_n : \mathbb{I}_{2n}(\mu_F) \hookrightarrow \mathcal{M}(O(V_n)).
\]

Theorem 10.3 applied to \( \pi \otimes \chi^{(-1)^n} \) implies that \( \mathfrak{J}_{2n}(\mu_F) \) appears in \( \mathcal{S}_{2n}^{(2n)} \) if and only if \( \varepsilon(\frac{1}{2}, \pi) = 1 \). The Schrödinger model of the Weil representation yields a representation \( \omega_{V_n}^{\psi_F} \) of \( Sp_{2n}(\mathbb{A}_F) \times O(V_n, \mathbb{A}_F) \) on the space \( \mathcal{S}(V_n^{2n}(\mathbb{A}_F)) \) of Schwartz functions on \( V_n^{2n}(\mathbb{A}_F) \). The theta function associated to \( \phi \in \mathcal{S}(V_n^{2n}(\mathbb{A}_F)) \) is defined by

\[
\theta(Z, \phi) = \sum_{x \in V_n^{2n}(F)} \phi(x) e_\infty \left( \frac{1}{2} \operatorname{tr}(q_{V_n}(x) Z) \right),
\]

where \( q_{V_n}(x) = (q_{V_n}(x_i, x_j)) \in \operatorname{Sym}_{2n}(F) \) is the matrix of inner products of the components of \( x = (x_1, \ldots, x_{2n}) \in V_n^{2n}(F) \). We will explicitly construct a nonzero \( Sp_{2n}(\mathbb{A}_F) \)-intertwining, \( O(V_n, \mathbb{A}_F) \)-invariant map

\[
\mathbb{J}_{2n}^{\psi_F} : \omega_{V_n}^{\psi_F} \otimes \mathbb{I}_{2n}(\mu_F) \to \mathfrak{J}_{2n}(\mu_F)
\]
in §1.4. Such a map is unique up to scalar by the Howe principle.

Corollary 1.5. Notation being as in Corollary 1.3, if \( L(\frac{1}{2}, \pi) \neq 0 \), then there is a nonzero constant \( c \) such that

\[
(1.3) \quad i_{2n}(\varphi_n(\phi \otimes f))(\mathcal{Z}) = c \int_{O(V_n, F) \setminus O(V_n, \mathbb{A}_F)} j_n(f)(g) \theta(\mathcal{Z}, \omega_{V_n}(g)\phi) \, dg
\]

for all \( \phi \in \mathcal{S}(V_n^{2n}(\mathbb{A}_F)) \) and \( f \in \mathbb{I}_{2n}(\mu_{\mathcal{F}}) \).

This identity is analogous to the Siegel-Weil formula (cf. [51, 52, 63]). The \( \xi \)th Fourier coefficient of \( i_{2n}(h) \) is built out of the local quantity \( w_{\xi}^{(\mu_{\mathcal{F}}X^{-1})}(h_p) \) and the \( \det(2\xi) \)th Fourier coefficient of a Hilbert cusp form in \( I_{V_n}^{\mathcal{F}}(\mu_{\mathcal{F}}X^{-1}) \).

On the other hand, the theta function involves the representation numbers of quadratic forms, essentially diophantine quantities. Our identity thus contains a link between local and global information. We will prove Corollaries 1.2 and 1.3 in §1.4 and 1.5, respectively.

When \( \mathcal{L} \) is a lattice in \( V_n(F) \), it is one of the classical tasks of number theory to determine the number of representations of \( \xi \in \text{Sym}_n(F) \) by \( \mathcal{L} \)

\[
N(L, \xi) = \# \{(x_1, \ldots, x_m) \in \mathcal{L}^m \mid q_{V_n}(x_a, x_b) = 2\xi_{ab} \text{ for } 1 \leq a, b \leq m\}.
\]

Let \( \Xi_n \) be the finite set of isomorphism classes of maximal lattices in \( V_n(F) \). To solve this problem for \( \mathcal{L} \in \Xi_n \), it is sufficient to determine the sum

\[
R_n(\xi, f) = \sum_{L \in \Xi_n} f(L) \frac{N(L, \xi)}{\#O(L)}
\]

for all Hecke eigenfunctions \( f : \Xi_n \rightarrow \mathbb{C} \). It is important to note that \( R_n(\xi, f) \) appears in the \( \xi \)th Fourier coefficient of the theta lift of \( f \). When \( f : \Xi_n \rightarrow \mathbb{C} \) is associated to an irreducible everywhere unramified cuspidal automorphic representation of \( \text{PGL}_2(\mathbb{A}_F) \) with parallel minimal weights \( \pm 2n \) in the sense of Corollary 1.3, we will give a product formula for \( R_n(\xi, f) \) with \( \xi \in \mathcal{R}_{2n} \).

Corollary 1.6. Let \( \pi_{\mathcal{F}} \simeq \otimes_p I(\alpha_p, \alpha_p^{s_p}) \) be an irreducible summand of \( \mathfrak{C}_{2n} \).

Assume that \( dn \) is even. Let \( f : \Xi_n \rightarrow \mathbb{C} \) be a common eigenfunction of all Hecke operators whose standard \( L \)-function is \( \prod_{j=1}^{2n} L(s + n + \frac{1}{2} - j, \pi) \).

1. If \( L(\frac{1}{2}, \pi) = 0 \), then \( R_n(\xi, f) = 0 \) for every \( \xi \in \mathcal{R}_{2n} \).
2. If \( L(\frac{1}{2}, \pi) \neq 0 \), then \( R_n(\xi, f) = 0 \) unless \( \xi \in \mathcal{R}_{2n}^+ \), in which case

\[
R_n(\xi, f) = c(\det(2\xi))f(2n-1)/2 \prod_p \tilde{F}_p(\xi, q_\xi^{s_p}).
\]

The constant \( c(\eta) \) satisfies \( c(\eta a^2) = c(\eta) \prod_{\nu \in E_{\infty}} \text{sgn}_\nu(a)^n \) for every \( a \in F^{\times} \), and the series

\[
\sum_{\eta \in F_\infty} c(\eta) e_{\infty}(\eta Z)^{(2n-1)/2} \prod_p \Psi_p((-1)^n \eta, q_\xi^{s_p})
\]
belongs to $S_{(2n+1)/2}^+$ and generates $I_1^S(\mu_1 \chi_1^{(-1)^n})$.

One can view this result, which is a consequence of Corollary 14.3, as a generalization of the Siegel formula. The function $f$ exists uniquely up to scalar by Corollary 14.2. Actually, one can obtain $f$ as an eigenvector of the Kneser $p$-neighbor matrix with eigenvalue

$$q_p^{(2n-1)/2}q_p^{2n} - 1 \frac{q_p^{sp} + q_p^{-sp}}{q_p - 1}$$

thanks to Proposition 14.1 (cf. 14.3).

Since $\{c(\eta)\}$ is determined uniquely up to scalar, one gets relations between the ratios of the various $R_n(\xi, f)$ and those of $c(\det(2\xi))$. In particular, if $L\left(\frac{1}{2}, \pi \otimes \chi^{D(\xi)}\right) = 0$, then $R_n(\xi, f) = 0$. Whereas the generalized Kohnen-Zagier formula mentioned above gives an explicit formula for $|c(\eta)|^2$, our formula involves $c(\eta)$ itself.

When $F = \mathbb{Q}(\sqrt{2})$, the genus of 8-dimensional totally positive definite even unimodular lattices over $\mathbb{Z}[\sqrt{2}]$ has 6 classes $E_8, 2\Delta'_4, \Delta_8, 2D_4, 4\Delta_2, \emptyset$. As an application we shall determine everywhere unramified representations of $O(V_2, H_f)$ occurring in $\mathcal{M}(O(V_2))$ and their degrees and investigate the subspace of $M_4(Sp_4(\mathbb{Z}[\sqrt{2}]))$ spanned by the 6 associated theta series in Section 14. This subspace is 6-dimensional and intersects the space of cusp forms in a one dimensional subspace, which is spanned by the lift of a Hilbert cusp form $\phi_4$ of weight 4. It corresponds to a Hilbert cusp form $\phi_{5/2}(\mathcal{Z}) = \sum \eta b(\eta)e_\infty(\eta \mathcal{Z})$ in the Kohnen plus space of weight $\frac{5}{2}$ (see 14.2.3 for their constructions). We will give the following example of Corollary 14.6:

$$\frac{N(E_8, \xi)}{2^8 \cdot 3^2 \cdot 5 \cdot 7} + \frac{N(2\Delta'_4, \xi)}{2^9 \cdot 3^2} - \frac{N(\Delta_8, \xi)}{2^{8} \cdot 3 \cdot 7} + \frac{N(2D_4, \xi)}{2^{8} \cdot 3^2} - \frac{N(4\Delta_2, \xi)}{2^9 \cdot 3} + \frac{N(\emptyset, \xi)}{2^7 \cdot 3^2 \cdot 5} = \frac{b(w_\mathcal{E})}{\xi} + \prod_p F_p(\xi, \beta_p).$$

Here, $\xi \in \mathcal{M}^+_4$, $w_\mathcal{E}$ is a totally positive generator of $\mathcal{E}$, $\{\beta_p, \beta_p^{-1}\}$ is the $p$-Satake parameter of $\phi_4$, and $\phi_{5/2}$ has been normalized so that $b(1) = 1$.

Theorem 14.2 is a generalization of the lifting constructed by Ikeda [22], where he discussed the case in which $F = \mathbb{Q}$, $m$ is even and $\mu_{\mathcal{E}, \chi_p^{(-1)^m/2}}$ is an unramified unitary character of $\mathbb{Q}_p^\times$ for all rational primes $p$. Ikeda’s first proof in [22] is rather indirect and uses the Siegel Eisenstein series which also has a similar Fourier series. As it uses the algebraic independence of the $p^{-s}$, it works only over rationals. This method cannot apply to nonsplit inner forms of symplectic groups even when $F = \mathbb{Q}$. Subsequently, Ikeda invented a more general approach and proved in his unpublished preprint that if $\otimes_p^f(\mu_p, \mu_p^{-1})$ occurs in the space $\mathcal{E}_{2\kappa}$ and its root number is 1, then $I_1^S(\mu_f)$ occurs in the space $S_{(2\kappa+m)/2}^{(m)}$ with multiplicity one. Later, Yamana refined and generalized this new approach, giving the explicit Fourier expansions and combining it with theta correspondence. The numerical examples in
Section 1 were added by Ikeda. The present article was written finally by combining Yamana’s manuscript with Ikeda’s original preprint.

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2. Preliminaries

2.1. Notation. For an associative ring $\mathcal{O}$ with identity element we denote by $\mathcal{O}^\times$ the group of all its invertible elements and by $M^m_n(\mathcal{O})$ the $\mathcal{O}$-module of all $m \times n$ matrices with entries in $\mathcal{O}$. Put $\mathcal{O}^n = M^1_1(\mathcal{O})$, $M_n(\mathcal{O}) = M^n_n(\mathcal{O})$ and $GL_n(\mathcal{O}) = M_n(\mathcal{O})^\times$. The zero element of $M^m_n(\mathcal{O})$ is denoted by 0 and the identity element of the ring $M_n(\mathcal{O})$ is denoted by $1_n$. If $x_1, \ldots, x_k$ are square matrices, then $\text{diag}[x_1, \ldots, x_k]$ denotes the matrix with $x_1, \ldots, x_k$ in the diagonal blocks and 0 in all other blocks. Assume that $\mathcal{O}$ has an involution $a \mapsto a^\dagger$. For a matrix $x$ over $\mathcal{O}$, let $x^\dagger$ be the transpose of $x$ and $x^\ast = \ell x^\dagger$ the conjugate transpose of $x$. Given $\epsilon \in \{\pm 1\}$, we let $S^\epsilon_m = \{z \in M_m(\mathcal{O}) \mid z^\ast = \epsilon z\}$ be the space of $\epsilon$-Hermitian matrices of size $m$. Set $z[x] = x^\ast z x$ for matrices $z \in S^\epsilon_m$ and $x \in M^m_n(\mathcal{O})$. Given $\epsilon$-Hermitian matrices $B \in S^\epsilon_k$ and $\Xi \in S^\epsilon_j$, we sometimes write $B \oplus \Xi$ instead of $\text{diag}[B, \Xi] \in S^\epsilon_{j+k}$, particularly when we view them as $\epsilon$-Hermitian forms.

We say that $\Xi$ is represented by $B$ if there is a matrix $x \in M^k_1(\mathcal{O})$ such that $B[x] = \Xi$.

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}_+^\times$, $\mathbb{S}$ and $\mu_k$ the set of strictly positive rational integers, the ring of rational integers, the fields of rational, real, complex numbers, the groups of strictly positive real numbers, complex numbers of absolute value 1 and $k^{\text{th}}$ roots of unity. We define the sign character $\text{sgn} : \mathbb{R}^\times \to \mu_2$ by $\text{sgn}(x) = x/|x|$. When $X$ is a totally disconnected locally compact topological space or a smooth real manifold, we write $S(X)$ for the space of Schwartz-Bruhat functions on $X$.

2.2. Quaternionic unitary groups. Let $F$ for the moment be an arbitrary field and $D$ a quaternion algebra over $F$, by which we understand a central simple algebra over $F$ such that $[D : F] = 4$. We denote by $i$ the main involution of $D$, by $x^\ast = i x^\dagger$ the conjugate transpose of a matrix $x \in M_n(D)$, by $\nu : GL_n(D) \to \mathbb{G}_m$ the reduced norm and by $r : M_n(D) \to \mathbb{G}_a$ the reduced trace, where $\mathbb{G}_m = \text{GL}_1$ and $\mathbb{G}_a = M_1$ are the multiplicative and additive
groups in one variable over $F$. If $n = 1$, then $\nu(x) = xx^t$ and $\tau(x) = x + x^t$ for $x \in D$. Put

$$S_n = \{ B \in \text{M}_n(D) \mid B^* = -B \}, \quad S_n^{\text{nd}} = S_n \cap \text{GL}_n(D).$$

When $n = 1$, we simply write $D_1 = S_1$ and $D_n = S_n^{\text{nd}}$. We identify $S_n$ with the space of $D$-valued skew Hermitian forms on the right $D$-module $D^n$, by which we understand an $F$-linear map $B : D^n \times D^n \to D$ such that

$$B(x, y) = -B(y, x), \quad B(xa, yb) = a^t B(x, y)b \quad (a, b \in D; \ x, y \in D^n).$$

We frequently regard $D$ as an algebraic variety over $F$ and define the $F$-algebraic group $\mathcal{G}_n$ by

$$\mathcal{G}_n = \{ g \in \text{GL}_{2n}(D) \mid gJ_n g^* = \lambda_n(g) J_n \text{ with } \lambda_n(g) \in \mathbb{G}_m \},$$

where

$$J_n = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \in \text{GL}_{2n}(F) \subset \text{GL}_{2n}(D).$$

We call $\lambda_n : \mathcal{G}_n \to \mathbb{G}_m$ the similitude character. We are interested in its kernel $G_n = \{ g \in \mathcal{G}_n \mid \lambda_n(g) = 1 \}$. For $A \in \text{GL}_n(D)$, $z \in S_n$ and $t \in \mathbb{G}_m$ we define matrices in $\text{GL}_{2n}(D)$ by

$$m(A) = \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix}, \quad n(z) = \begin{pmatrix} 1_n & z \\ 0 & 1_n \end{pmatrix}, \quad d(t) = \begin{pmatrix} 1_n & 0 \\ 0 & t \cdot 1_n \end{pmatrix}.$$  

Let $P_n$ be the parabolic subgroup of $G_n$ which has a Levi factor $M_n = \{ m(A) \mid A \in \text{GL}_n(D) \}$ and the unipotent radical $N_n = \{ n(z) \mid z \in S_n \}$.

### 2.3. The split case

We include the case in which $D$ is the matrix algebra $M_2(F)$ of degree 2 over $F$. Let us now take this case. We often identify $\text{M}_m(D)$ with $M_{2m}(F)$ by viewing an element $(x_{ij})$ of $\text{M}_m(D)$ as a matrix of size $2m$ whose $(i, j)$-block of size 2 is $x_{ij}$. Put

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{GL}_2(F), \quad B_m = \text{diag}[J, \ldots, J] \in \text{GL}_{2m}(F).$$

Then we easily see that $X^* = B_m^{-1} X B_m$ for $X \in \text{M}_m(D)$, where $X^*$ denotes the transpose of $X$ as a matrix of size $2m$. We are led to

$$\sigma_n G_n \sigma_n^{-1} = \text{Sp}_{2n}, \quad B_n S_n = \text{Sym}_{2n},$$

where $\sigma_n = \text{diag}[1_{2n}, B_n]$. Since all automorphisms of $\text{Sp}_{2n}$ are inner, we conclude that $G_n$ is an inner form of $\text{Sp}_{2n}$ for every $D$.

### 2.4. The case $n = 1$

When $\mathcal{G}$ is an algebraic group over a field $F$ and $Z$ is its center, we write $PG$ for the adjoint group $\mathcal{G}/Z$. It is important to note that the group $PG_1$ is isomorphic to a certain orthogonal group. To see this relation, we recall some well-known facts on Clifford algebras. The basic reference is [11]. For the time being, we will take $V$ to be a finite dimensional vector space over a field $F$ of characteristic different from 2, and let $q_V : V \to F$ be a nondegenerate quadratic form. The associated symmetric bilinear form is defined by $q_V(x, y) = \frac{1}{2}(q_V(x+y) - q_V(x) - q_V(y))$.\]
A Clifford algebra of $(V, q_V)$ is an $F$-algebra $A$ with an $F$-linear map $p : V \to A$ satisfying the following properties:

- $A$ has an identity element, which we denote by $1_A$;
- $A$ as an $F$-algebra is generated by $p(V)$ and $1_A$;
- $p(x)^2 = q_V(x)1_A$ for every $x \in V$;
- $A$ has dimension $2^{\ell}$ over $F$, where $\ell = \dim V$.

It is known that such a pair $(A, p)$ is unique up to isomorphism. Moreover, $p$ is injective, and as such, $V$ can be viewed as a subspace of $A$ via $p$. We denote this algebra $A$ by $A(V)$. The basic equalities are $xy + yx = 2q_V(x, y)$ for $x, y \in V$.

There is an automorphism $\beta \mapsto \beta'$ of $A(V)$ such that $\beta' = -\beta'$ for every $\beta' \in \mathbb{V}$. Similarly, there is an anti-automorphism $\beta \mapsto \beta^\rho$ of $A(V)$ such that $\beta^\rho = -\beta^\rho$ for every $\beta^\rho \in \mathbb{V}$. The orthogonal group $O(V)$ consists of elements $g \in \text{GL}(V)$ which satisfy $q_V(gx) = q_V(x)$ for all $x \in V$. The Clifford group $G(V)$ consists of elements $\beta \in A(V)^{\times}$ such that $\beta V \beta^{-1} = V$. Let us put

$$A^+(V) = \{ \beta \in A(V) \mid \beta' = \beta \}, \quad A^-(V) = \{ \beta \in A(V) \mid \beta' = -\beta \},$$

$$G^+(V) = G(V) \cap A^+(V), \quad G^-(V) = G(V) \cap A^-(V), \quad G(V) = G^+(V) \cup G^-(V).$$

It is known that $G(V)$ is a subgroup of $G(V)$. Put $\mu_1(\beta) = \beta^\rho$ for $\beta \in A(V)$. The map $\mu_1$ gives a homomorphism of $G(V)$ to $F^{\times}$. For $\beta \in G(V)$ we can define $\vartheta(\beta) \in \text{GL}(V)$ by $\vartheta(\beta)v = \beta v\beta^{-1}$ ($v \in V$). In \S2 we will use the following result, which is proved in Theorem 3.6 of [51].

**Lemma 2.1.**

1. The map $\vartheta$ gives an isomorphism of $G^+(V)/F^{\times}$ onto the special orthogonal group $SO(V) = \text{SL}(V) \cap O(V)$.

2. If $\ell$ is even, then $G^+(V) = G(V)$ and $\vartheta$ is an isomorphism of $G(V)/F^{\times}$ onto $O(V)$.

By restricting the symmetric bilinear form on $D^2$ given by $(x, y) \mapsto \frac{1}{2} \tau(xy)$, we obtain a three dimensional quadratic space $V_D = (D_-, q_{D_-})$ of discriminant 1. In what follows we take $V = Fe \oplus V_D \oplus Fe'$ and define the quadratic form $q_V$ by

$$q_V(re + x + r'e') = rr' + q_{D_-}(x) = rr' - \nu(x) \quad (r, r' \in F; x \in D_-).$$

**Lemma 2.2.** Notation and assumption being as above, there is an $F$-linear ring homomorphism $\Psi : A(V) \to M_2(D)$ such that

$$\Psi(\beta^\rho) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi(\beta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for all $\beta \in A(V)$ and whose restriction gives isomorphisms

$$A^+(V) \simeq M_2(D), \quad G^+(V) \simeq G_1.$$
(\varphi \circ \Psi^{-1})(J_j)(re + x + r'e') = r'e - x + r'e'.

\textit{Proof.} This isomorphism is explained in §§A4.2 and A4.3 of [31], to which we refer for additional explanation. The last assertion can be verified by a simple calculation. \hfill \Box

3. Degenerate Whittaker functions

The ground field $F$ is a totally real number field or its completion. Excluding the case of the real field, we let $\mathfrak{o}$ be the maximal order of $F$ and fix a maximal order $O$ of $D$. We define the additive character of $\mathbb{C}$ by $\psi(z) = e^{2\pi \sqrt{-1}z}$ for $z \in \mathbb{C}$. In the real case we set $\psi = e_{|\mathbb{R}|}$. When $F$ is an extension of $\mathbb{Q}_p$, we define the character $\psi$ of $F$ by $\psi(x) = e(-y)$ with $y \in \mathbb{Z}[p^{-1}]$ such that $\text{Tr}_{F/\mathbb{Q}_p}(x) - y \in \mathbb{Z}_p$. In the global case we put $\psi_{\infty}(x) = \prod_{\nu \in \mathfrak{S}_\infty} e(\nu(x))$, $\psi_t(x) = \prod_{\nu \in \mathfrak{P}} \psi_{\nu}(x_\nu)$ and $\psi(x) = \psi_{\infty}(x)\psi_t(x)$ for $x \in \mathbb{A}$.

3.1. Degenerate principal series. In this and the next subsections $F$ is a completion at a nonarchimedean prime. We denote by $\omega$ the order of the residue field of the valuation ring $\mathfrak{o}$, by $\alpha(t) = |t|$ the normalized absolute value of $t \in F^\times$ and by $\chi^B$ the quadratic character of $F^\times$ associated to $F(\sqrt{t})/F$ via class field theory. For $B \in S_n^{\text{nd}}$ we set $\chi^B = \chi^{(-1)^{\nu(B)}}$. We write $\Omega(F^\times)$ for the group of all continuous homomorphisms from $F^\times$ to $\mathbb{C}^\times$. Continuous homomorphisms of the form $\alpha^s$ for some $s \in \mathbb{C}$ are called unramified. When $\mu \in \Omega(F^\times)$, we define $\Re\mu$ as the unique real number such that $\mu \alpha^{-\Re\mu}$ is unitary.

For $\mu \in \Omega(F^\times)$ the normalized induced representation $J_n(\mu)$ is realized on the space of smooth functions $f : \mathcal{G}_n \rightarrow \mathbb{C}$ satisfying

$$f(\mathbf{d}(t)\mathbf{m}(A)\mathbf{n}(z)g) = \mu(t^{-n}\nu(A))|t^{-n}\nu(A)|^{(2n+1)/2}f(g)$$

for all $t \in F^\times$, $A \in \text{GL}_n(D)$, $z \in S_n$ and $g \in \mathcal{G}_n$. We denote its restriction to $G_n$ by $I_n(\mu)$. Since $\mathcal{G}_n = \{\mathbf{d}(t) \mid t \in F^\times\} \ltimes G_n$, these representations can also be realized on the space of smooth functions $f : G_n \rightarrow \mathbb{C}$ satisfying

$$f(\mathbf{m}(A)\mathbf{n}(z)g) = \mu(\nu(A))|\nu(A)|^{(2n+1)/2}f(g)$$

for all $A \in \text{GL}_n(D)$, $z \in S_n$ and $g \in G_n$.

Let $B \in S_n$. We define the character $\psi^B : S_n \rightarrow \mathbb{S}$ by $\psi^B(z) = \psi(\tau(B)z)$ for $z \in S_n$. For any smooth representation $\Pi$ of $G_n$ we put

$$\text{Wh}_B(\Pi) = \text{Hom}_{S_n}(\Pi \circ \mathbf{n}, \psi^B).$$

\textbf{Proposition 3.1 ([28, 39, 61]).} (1) If $-\frac{1}{2} < \Re\mu < \frac{1}{2}$, then $I_n(\mu)$ is irreducible and unitary.

(2) $\dim \text{Wh}_B(I_n(\mu)) = 1$ for all $\mu \in \Omega(F^\times)$ and $B \in S_n^{\text{nd}}$.

(3) Assume that $\mu^2 = \alpha$. Then $I_n(\mu)$ has a unique irreducible subrepresentation $A_n(\mu)$, which is unitary. Moreover, $A_n(\mu)$ is the unique irreducible subrepresentation of $J_n(\mu)$. Furthermore, $\text{Wh}_B(A_n(\mu))$ is nonzero if and only if $\chi^B \neq \mu\alpha^{-1/2}$. 

Proof. The module structure of $I_n(\mu)$ is determined by Kudla-Rallis \cite{Kudla-Rallis} in the symplectic case and by Yamana \cite{Yamana} in the quaternion case. The unitarity follows from the general fact on irreducible subquotients of ends of complementary series explained in Section 3 of \cite{Ikeda-Yamana}. The second part is proved in \cite{Ikeda-Yamana}. We will prove (i). Since $J_n(d(t), \mu)A_n(\mu)$ is an irreducible submodule of $I_n(\mu)$, we know by its uniqueness that $J_n(d(t), \mu)A_n(\mu) = A_n(\mu)$ for all $t \in F^\times$. It is known that

$$I_n(\mu)/A_n(\mu) \cong \sum_{\chi B' = \mu \alpha^{-1/2}} R_{n}^{\psi}(B'),$$

where $B'$ extends over all equivalence classes of nondegenerate skew Hermitian matrices of size $n$ with character $\mu \alpha^{-1/2}$ and $R_{n}^{\psi}(B')$ is an irreducible representation arising via the Weil representation of $G_n \times U(B')$ associated to $B'$, where $U(B') = \{ g \in GL_n(D) \mid B'[g] = B' \}$. In the symplectic case $Wh_B(R_{n}^{\psi}(B'))$ is nonzero if and only if $B$ is equivalent to $B'$ by Lemma 3.5 of \cite{Ikeda-Yamana}. One can see that this result is valid in the quaternion case by a basic calculation based on Lemma on p. 73 of \cite{Ikeda-Yamana}. The claimed fact derives from the exactness of the Jacquet functor combined with these observations. \hfill \square

3.2. Jacquet integrals. For an ideal $\mathfrak{c}$ of $\mathfrak{o}$ we put

$$K_n^D[\mathfrak{c}] = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_n \mid \alpha, \delta \in M_n(\mathfrak{o}), \gamma \in \mathfrak{o}M_n(\mathfrak{o}) \right\}.$$

For $g \in G_n$ the quantity $\varepsilon_{\mathfrak{c}}(g)$ is defined by writing $g = pk$ with $p = m(A)n(z) \in P_n$, $k \in K_n^D[\mathfrak{c}]$, and setting $\varepsilon_{\mathfrak{c}}(g) = |\nu(A)|$. For $\mu \in \Omega(F^\times)$, $f \in I_n(\mu)$ and $s \in \mathbb{C}$ we define $f(s) \in I_n(\mu\alpha^s)$ by $f(s)(g) = f(g)\varepsilon_{\mathfrak{c}}(g)^s$. Put $R_n = S_n \cap M_n(\mathfrak{o})$, $R_{2n}^D = \{ B \in S_n \mid \tau(BR_n) \subset \mathfrak{o} \}$, $R_{2n}^D, s = R_{2n} \cap S_n$.

Define a Haar measure $dz$ on $S_n$ so that the measure of $R_n$ is 1.

When $\mu$ is unramified, we put $L(s, \mu) = (1 - \mu(A)q^{-s})^{-1}$, where $\omega$ is a generator of $\mathfrak{p}$. Set $L(s, \mu) = 1$ if $\mu$ is ramified. When $\mu$ is the trivial character, we write $\zeta(s) = L(s, \mu)$. For $B \in S_n^s$ the integral

$$w_B^{\mu, s}(f(s)) = \int_{S_n} f(s)(J_n(n(z)))\psi_B(z)dz$$

defines a formal Dirichlet series in the variable $s$, which is absolutely convergent for $\Re s > n + \frac{1}{2} - \Re \mu$. Actually, the integral stabilizes and consequently, it is a polynomial of $q^{-s}$, from which we can evaluate $w_B^{\mu, s}(f(s))$ at $s = 0$ so as to get a basis vector $w_B^{\mu} \in Wh_B(I_n(\mu))$. From now on we assume that $\Re \mu > -\frac{1}{2}$ and set

$$w_B^{\mu}(f) = |\nu(B)|(2n+1)!w_B^{\mu}(f)L(n+\frac{1}{2}, \mu) \prod_{j=1}^{n} L(2j-1, \mu^2).$$

We write $q$ (resp. $\psi$) for the right regular action of $G_n$ (resp. $G_n$) on the space of smooth functions on $G_n$ (resp. $G_n$).
Lemma 3.2. Let $B \in S_n^{\text{nd}}$ and $\mu \in \Omega(F^\times)$. Assume that $\Re \mu > -\frac{1}{2}$.

1. $0 \neq w_\mu^B \in \text{Wh}_B(I_n(\mu))$.
2. When $\mu^2 = \alpha$, the restriction of $w_\mu^B$ to $A_n(\mu)$ is nonzero if and only if $\chi^B \neq \mu \alpha^{-1/2}$.
3. If $t \in F^\times$ and $A \in \text{GL}_n(D)$, then

$$w_\mu^B \circ \varphi(d(t)m(A)) = \mu(t^{-n}\nu(A))^{-1}w_{t^{-1}B[A]}^\mu.$$ 

Proof. The first part is clear. The second part is evident from Proposition 3.4. The third part can be verified by obvious changes of variables. \qed

We refer the reader to Lemma 3.3 of [22] for the following bound.

Lemma 3.3. Let $f \in I_n(\mu)$. For any compact subset $C$ of $G_n$ there are a Schwartz function $\Phi$ on $S_n$ and a positive constant $M$ such that for all $\Delta \in C$ and $B \in S_n^{\text{nd}}$

$$|w_\mu^B(g(\Delta)f)| \leq |\nu(B)|^{-M}\Phi(B).$$

The Siegel series associated to $B \in \mathcal{R}_{2n}^D$ is defined by

$$b(B, s) = \sum_{z \in S_n/\mathcal{R}_n} \psi'(-\tau(Bz))\nu[z]^{-s},$$

where $\nu[z] = [z\mathcal{O}^n + \mathcal{O}^n : \mathcal{O}^n]^{1/2}$ and $\psi'$ is an arbitrarily fixed additive character on $F$ of order zero. We define the function $\gamma(B, s)$ by

$$\gamma(B, s) = \zeta(s)^{-1}L(s-n, \chi^B) \times \begin{cases} \prod_{j=1}^{[n]} \zeta(2s-2j)^{-1} & \text{if } D \simeq M_2(F), \\ \prod_{j=1}^{[n/2]} \zeta(2s-4j)^{-1} & \text{otherwise}. \end{cases}$$

Put $F(B, q^{-s}) = b(B, s)\gamma(B, s)^{-1}$. Then $F(B, X)$ is a polynomial of degree $f^B$ with constant term 1 by [23]. [24] (see Section 14 for the definition of $f^B$).

We denote the different of $F/\mathbb{Q}_p$ by $\mathfrak{d}$. 

Lemma 3.4. If $B \in \mathcal{R}_{2n}^{D, \text{nd}}$, then there is a nonzero constant $c(s)$ independent of $B$ such that $w_\mu^B(\xi^{s-(2n+1)/2}) = c(s)|\nu(B)|^{(2n+1)/4}F(B, q^{-s-(2n+1)/2})$.

If $\mathfrak{d} = \mathfrak{o}$ and $D \simeq M_2(F)$, then $c(s) = 1$ and there is a positive constant $M$, depending only on $n$, such that $|w_\mu^B(\xi^{s-(2n+1)/2})| \leq |\nu(B)|^{-M}$ for $B \in S_n^{\text{nd}}$ and $\Re s > -\frac{1}{2}$.

Proof. We refer the reader to Lemma 4.5 of [22] except for the last statement. In the proof of Lemma 4.1 of [22] Ikeda shows that when $\Re s > 0$,

$$|F(B, q^{-s})| \leq |\nu(B)|^{-(13n^2+13n+4)/2},$$

from which we obtain the desired estimate. \qed
3.3. Degenerate Whittaker functions: the archimedean case. We discuss the case in which \( F = \mathbb{R} \) and \( D = M_2(\mathbb{R}) \). Let \( \text{Sym}_m(\mathbb{R})^+ \) be the set of positive definite symmetric matrices of rank \( m \) over \( \mathbb{R} \). Put
\[
S_n^+ = \{ B \in S_n \mid BB_n \in \text{Sym}_{2n}(\mathbb{R})^+ \}, \quad G_n^+ = \{ g \in G_n \mid \lambda_n(g) > 0 \}.
\]
We can define the action of \( G_n^+ \) on the space
\[
\mathcal{H}_n = \{ Z \in M_{2n}(\mathbb{C}) \mid \Psi(ZB_n^{-1}) = ZB_n^{-1}, \, \Psi(ZB_n^{-1}) \in \text{Sym}_{2n}(\mathbb{R})^+ \}
\]
and the automorphy factor on \( G_n^+ \times \mathcal{H}_n \) by
\[
gZ = (\alpha Z + \beta)(\gamma Z + \delta)^{-1}, \quad j(g, Z) = \nu(g)^{-1/2}\nu(\gamma Z + \delta)
\]
for \( Z \in \mathcal{H}_n \) and \( g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_n^+ \) with matrices \( \alpha, \beta, \gamma, \delta \) of size \( n \) over \( M_2(\mathbb{R}) \). There is a biholomorphic isomorphism from \( \mathcal{H}_n \) onto the Siegel upper half space \( \mathcal{H}_{2n} \) (cf. \( H_{2n} \)). Define the origin of \( \mathcal{H}_n \) and the standard maximal compact subgroup of \( G_n \) by
\[
i = \sqrt{-1}B_n \in \mathcal{H}_n, \quad K_n = \{ g \in G_n \mid g(i) = i \}.
\]
For \( \ell \in \mathbb{N} \) and \( B \in S_n^+ \) we define a function \( W_B^{(\ell)} : G_n^+ \to \mathbb{C} \) by
\[
W_B^{(\ell)}(g) = \nu(B)^{\ell/2}\epsilon(\nu(Bg(i)))j(g, i)^{-\ell}.
\]
Clearly,
\[
(3.2) \quad W_B^{(\ell)}(n(z)d(t)m(A)gk) = \epsilon(\tau(Bz))\text{sgn}(\nu(A))^{\ell}W_{t^{-1}B[A]}^{(\ell)}(g)j(k, i)^{-\ell}
\]
for \( z \in S_n, \, A \in GL_{2n}(\mathbb{R}), \, t \in \mathbb{R}_+^\times, \, g \in G_n^+ \) and \( k \in K_n \).

3.4. Degenerate Whittaker functions: the global case. Until the end of the next section \( D \) is a totally indefinite quaternion algebra over a totally real number field \( F \). For each prime \( v \) of \( F \) and an algebraic group \( V \) defined over \( F \), let \( F_v \) be the \( v \)-completion of \( F \) and put \( V_v = V(F_v) \) to make our exposition simpler. The adèlle group, the finite part of the adèlle group, the infinite part of the adèlle group and its connected component of the identity are denoted by \( V(\mathbb{A}), \, V(\mathbb{A}_f), \, V(\mathbb{A}_\infty) \) and \( V(\mathbb{A}_\infty)^+ \), respectively. For an adèlle point \( x \in V(\mathbb{A}) \) we denote its projections to \( V(\mathbb{A}_f), \, V(\mathbb{A}_\infty) \) and \( V_v \) by \( x_f, \, x_\infty \) and \( x_v \), respectively.

We will denote the group of totally positive elements of \( F \) by \( F_+^\times \). Put
\[
S_n^+(F) = \{ B \in S_n(F) \mid B \in S_n(F_v)^+ \text{ for all } v \in \mathcal{S}_\infty \}.
\]
When \( n = 1 \), we write \( D_1^+ = S_1^+ \). For \( B \in S_n(F) \) we define the characters
\[
\psi^B : S_n(\mathbb{A}) \to \mathbb{S} \text{ by } \psi^B(z) = \psi(\tau(Bz)) \text{ whose restriction to } S_n(\mathbb{A}_f) \text{ is denoted by } \psi_f^B.
\]
For any smooth representation \( \Pi \) of \( G_n(\mathbb{A}_f) \) we put
\[
\text{Wh}_B(\Pi) = \text{Hom}_{S_n(\mathbb{A}_f)}(\Pi \circ n, \psi_f^B).
\]
For $\mu_1, \mu_2 \in \Omega(F_p^\times)$ let $I(\mu_1, \mu_2)$ denote the representation of $GL_2(F_p)$ on the space of all smooth functions $f$ on $GL_2(F_p)$ satisfying

$$f \left( \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} g \right) = \mu_1(a_1)\mu_2(a_2)\alpha_p(a_1a_2^{-1})^{1/2}f(g)$$

for all $a_1, a_2 \in F_p^\times$; $b \in F_p$ and $g \in GL_2(F_p)$. This representation $I(\mu_1, \mu_2)$ is irreducible unless $\mu_1\mu_2^{-1} \in \{\alpha_p, \alpha_p^{-1}\}$. If $\mu_1\mu_2^{-1} = \alpha_p$, then $I(\mu_1, \mu_2)$ has a unique irreducible submodule, which we denote by $A(\mu_1, \mu_2)$.

In what follows we fix, once and for all, an irreducible admissible unitary generic representation $\pi_f$ of $PGL_2(\mathbb{A}_F)$ whose local components are not supercuspidal. Then $\pi_f$ is equivalent to the unique irreducible submodule $A(\mu_1, \mu_2^{-1})$ of $I(\mu_1, \mu_2^{-1}) = \otimes'_p I(\mu_p, \mu_p^{-1})$ for some character $\mu_f$ of the finite idèle group $\mathbb{A}_F^\times$ whose restriction to $F_p^\times$ is denoted by $\mu_p$ and which fulfills the following conditions:

- $-\frac{1}{2} < \Re \mu_p \leq \frac{1}{2}$ for all finite primes $p$;
- $\mu_p = \alpha_p$ whenever $\Re \mu_p = \frac{1}{2}$;
- $\mu_p$ is unramified and $\Re \mu_p < \frac{1}{2}$ for almost all finite primes $p$.

Let $\mathcal{S}_{\pi_f}$ be the set of nonarchimedean primes $p$ such that $\Re \mu_p = \frac{1}{2}$.

Let $I_n(\mu_f)$ and $J_n(\mu_f)$ be the degenerate principal series representations of $G_n(\mathbb{A}_F)$ and $G_n(\mathbb{A}_F)$ induced from the character $d(n)\mathfrak{m}(A) \mapsto \mu_f(t^{-n}v(A))$. They have factorizations $I_n(\mu_f) \simeq \otimes'_p I_n(\mu_p)$ and $J_n(\mu_f) \simeq \otimes'_p J_n(\mu_p)$. For $B \in S_n^+$ Lemma 4.3 defines a nonzero vector $w_{B \mu_f} \in \text{Wh}_B(I_n(\mu_f))$ by

$$w_{B \mu_f}(f) = \prod_p w_{B \mu_p}(f_p),$$

provided that $f = \otimes_p f_p$ is factorizable. Let us set

$$A_n(\mu_f) = (\otimes_{p \in \mathcal{S}_{\pi_f}} A_n(\mu_p)) \otimes (\otimes_{p \notin \mathcal{S}_{\pi_f}} I_n(\mu_p)).$$

We can view $A_n(\mu_f)$ as the unique irreducible subrepresentation of both $I_n(\mu_f)$ and $J_n(\mu_f)$. We also regard $A_n(\mu_f)$ as a subrepresentation of the space of smooth functions on $G_n(\mathbb{A}_F)$ or $G_n(\mathbb{A}_F)$ on which $G_n(\mathbb{A}_F)$ or $G_n(\mathbb{A}_F)$ acts by the right regular action $\varrho$ or $\varphi$. Put

$$S_{n, \pi_f}^\varrho = \{B \in S_n^+ \mid \chi_B \neq \mu_p\alpha_p^{-1/2} \text{ for all } p \in \mathcal{S}_{\pi_f}\}.$$

When $n = 1$, we will sometimes write $D_{\pi_f}^\varrho = S_{1, \pi_f}^\varrho$. Proposition 3.1 and Lemma 3.2 give the following result:

**Lemma 3.5.** Let $B \in S_n^+$. Then $\text{Wh}_B(A_n(\mu_f))$ is nonzero if and only if the restriction of $w_{B \mu_f}$ to $A_n(\mu_f)$ is nonzero if and only if $B \in S_{n, \pi_f}^\varrho$.

4. **Holomorphic Cusp Forms on Quaternion Unitary Groups**

When $\mathcal{F}$ is a smooth function on $N_n(F)\backslash G_n(\mathbb{A})$ and $B \in S_n(F)$, let

$$W_B(g, \mathcal{F}) = \int_{S_n(F)\backslash S_n(\mathbb{A})} \mathcal{F}(\mathfrak{n}(z)g)\psi_B(z)\,dz$$
be the $B$th Fourier coefficient of $\mathcal{F}$. For $\ell \in \mathbb{Z}^d$ and $B \in S_n^+\mathbb{R}$ we define a function $W^{(\ell)}_B : G_n(\mathbb{A}_\infty) \to \mathbb{C}$ by

$$W^{(\ell)}_B(g) = \prod_{v \in S_\infty} W^{(\ell_v)}_B(g_v).$$

**Definition 4.1.** The symbol $\mathfrak{G}^n_G$ (resp. $\mathfrak{C}^n_G$, $\mathfrak{C}^n_G$) denotes the space of all smooth functions $\mathcal{F}$ on $G_n(F) \backslash G_n(\mathbb{A})$ (resp. $P_n(F) \backslash G_n(\mathbb{A})$, $G_n(F) \backslash G_n(\mathbb{A})$) that admit Fourier expansions of the form

$$\mathcal{F}(g) = \sum_{B \in S_n^+\mathbb{R}} w_B(gr, \mathcal{F}) W^{(\ell)}_B(g_{\infty})$$

which is absolutely and uniformly convergent on any compact neighborhood of $g = g_{\infty}gr \in G_n(\mathbb{A}_\infty)^+G_n(\mathbb{A}_\infty)$ (resp. $g = g_{\infty}gr \in G_n(\mathbb{A})$).

The functions in $\mathfrak{G}^n_G$ (resp. $\mathfrak{C}^n_G$) are cuspidal automorphic forms on $G_n(\mathbb{A})$ (resp. $G_n(\mathbb{A})$) with scalar $K$-type $k \mapsto \prod_{v \in S_\infty} j(k_v, i)^{-\ell}$ in the sense of Langlands (cf. Proposition 4.3) and related to classical holomorphic cusp forms in the standard way (cf. Remark 3.2).

**Remark 4.2.** Put

$$\mathcal{P}_n = \{d(t)p \mid t \in \mathcal{G}_m, p \in P_n\}, \quad \mathcal{P}_n^+ = \{d(t)p \mid t \in F_+^\times, p \in P_n(F)\}.$$ Since $G_n(\mathbb{A}) = \mathcal{P}_n(F)G_n(\mathbb{A}_\infty)^+G_n(\mathbb{A}_f)$, smooth functions on $\mathcal{P}_n(F) \backslash G_n(\mathbb{A})$ can naturally be identified with smooth functions on $\mathcal{P}_n^+ \backslash G_n(\mathbb{A}_\infty)^+G_n(\mathbb{A}_f)$.

**Lemma 4.3.** If $i : A_n(\mu_f) \to \mathfrak{C}^n_G$ is a $G_n(\mathbb{A}_f)$-intertwining map, then there are complex numbers $C_B$ such that

$$i(g, f) = \sum_{B \in S_n^+\mathbb{R}} C_B W^{(\ell)}_B(g_{\infty}) w^\mu_B(p(gr)f)$$

for all $f \in A_n(\mu_f)$, $g = g_{\infty}gr$, $g_{\infty} \in G_n(\mathbb{A}_\infty)$ and $gr \in G_n(\mathbb{A}_f)$.

**Proof.** Notice that the coefficient $w_B(gr, \mathcal{F})$ is given by

$$w_B(gr, \mathcal{F}) = |\nu(B)|^{-\ell_2} e_{\infty}(-\tau(B(1, \ldots, 1))) W^{(\ell)}_B(gr, \mathcal{F}).$$

Recall that $|\nu(B)|^{\ell_2} = \prod_{v \in S_\infty} |\nu(B)|_{v}^{\ell_v/2}$. In particular, for $z \in S_n(\mathbb{A}_f)

$$w_B(n(z)gr, \mathcal{F}) = \psi^\beta(z)w_B(gr, \mathcal{F}).$$

Therefore the $\mathbb{C}$-linear functional $f \mapsto w_B(1, i(f))$ belongs to the space $Wh_B(A_n(\mu_f))$. There is a complex number $C_B$ such that $w_B(1, i(f)) = C_B w^\mu_B(f)$ for all $f \in A_n(\mu_f)$ in view of Lemma 3.3 \hfill $\square$

We associate to $f \in A_n(\mu_f)$, $\ell = (\ell_v)_{v \in S_\infty} \in \mathbb{Z}^d$ and complex numbers $C_B$ indexed by $B \in S_n^+\mathbb{R}$ the Fourier series

$$\mathcal{F}_\ell(g; f, \{C_B\}) = \sum_{B \in S_n^+\mathbb{R}} C_B W^{(\ell)}_B(g_{\infty}) w^\mu_B(p(gr)f), \quad g = g_{\infty}gr \in G_n(\mathbb{A}),$$

assuming that the series is absolutely convergent.
Lemma 4.4. Notation being as above, if for any lattice $L$ in $S_n(F)$ there are positive constants $C$ and $M$ such that $|C_B| \leq CN_{F/Q}(\nu(B))^M$ for all $B \in S_n^+ \cap L$, then the series $F_\ell(g; f, \{C_B\})$ is absolutely and uniformly convergent on any compact subset of $G_n(\mathbb{A})$ for every $f \in A_n(\mu_\ell)$.

Proof. The proof goes along the same lines of the arguments in Section 4 of [22]. It suffices to show that the series

$$\sum_{B \in S_n^\ell} C_B |\nu(B)|^{\ell/2} w_B^{\mu_\ell}(f) e_\infty(\tau(BZ))$$

is absolutely and uniformly convergent on any compact subset of $S_n^\ell$. Put $R_n = S_n(F) \cap \mathbb{M}_n(\mathbb{O})$. Take a natural number $N$ such that $w_B^{\mu_\ell}(f) = 0$ unless $B \in N^{-1}R_n$. Lemmas 3.3 and 3.4 say that $w_B^{\mu_\ell}(f) \leq C'N_{F/Q}(\nu(B))^M'$ for all $B \in S_n^\ell$ with constants $C'$ and $M'$ depending only on $f$. Note that $N_{F/Q}(\nu(B)) \leq \max(1, (nd)^{-1}(\text{Tr}_F/Q_B)^{1/2})^{2nd}$. The number of $B \in N^{-1}R_n \cap S_n^\ell$ such that $\text{Tr}_F/Q_B \leq T$ is $O(T^{dn(2n+1)})$. From these estimates the series converges absolutely and uniformly on

$$\{Z \in S_n^\ell \mid \Im(Z_vB_v^{-1}) < \epsilon \} \in \text{Sym}_{2n}(F_v) \quad \text{for all } v \in \mathfrak{S}_\infty$$

for any positive constant $\epsilon$. \hfill \Box

Definition 4.5. Let $T_\ell^n(\mu_\ell)$ be the vector space which consists of sets $\{C_B\}_{B \in S_n^\ell}$ of complex numbers such that the series $F_\ell(g; f, \{C_B\})$ converges absolutely and uniformly on any compact subset of $G_n(\mathbb{A})$ for all $f \in A_n(\mu_\ell)$ and such that for all $B \in S_n^\ell$ and $A \in \text{GL}_n(D(F))$

$$C_B[A] = C_B\mu_\ell(\nu(A))^{-1} \prod_{v \in \mathfrak{S}_\infty} \text{sgn}_v(\nu(A))^{\ell_v}.$$ 

Let $C_\ell^n(\mu_\ell)$ (resp. $G_\ell^n(\mu_\ell)$) stand for the space of coefficients $\{C_B\} \in T_\ell^n(\mu_\ell)$ such $F_\ell(f, \{C_B\}) \in \mathfrak{C}_\ell^n$ (resp. $\mathfrak{G}_\ell^n$) for all $f \in A_n(\mu_\ell)$.

Lemma 4.6. (1) $F_\ell(f, \{C_B\}) \in \mathfrak{C}_\ell^n$ for $\{C_B\} \in T_\ell^n(\mu_\ell)$ and $f \in A_n(\mu_\ell)$.

(2) Let $\{C_B\} \in \mathfrak{C}_\ell^n(\mu_\ell)$. Then $\{C_B\} \in \mathfrak{G}_\ell^n(\mu_\ell)$ if and only if $C_{1B} = C_B\mu_\ell(t)^{-\nu}$ for all $t \in F_+^\times$.

Proof. Fix $f \in A_n(\mu_\ell)$ and put $w_B^{(\ell)}(g) = W_B^{(\ell)}(g_\infty)w_B^{\mu_\ell}(\varphi(g_\ell)f)$. Then Lemma 3.2(3) and (3.2) say that

$$w_B^{(\ell)}(n(x)d(t)m(A)g) = \frac{\psi_B(z)}{\mu_\ell(t-n\nu(A))} \prod_{v \in \mathfrak{S}_\infty} \text{sgn}_v(\nu(A))^{\ell_v}$$

for $z \in S_n(\mathbb{A})$, $A \in \text{GL}_n(D(F))$, $t \in F_+^\times$ and $g \in G_n(\mathbb{A}_\infty)^+\mathfrak{G}_n(\mathbb{A}_\ell)$, from which $F_\ell(f, \{C_B\})$ is left invariant under $P_n(F)$. This proves (\textit{i}).

We can define the function $F_\ell(f; \{C_B\}) : \mathfrak{G}_n(\mathbb{A}_\infty)^+\mathfrak{G}_n(\mathbb{A}_\ell) \to \mathbb{C}$ by

$$F_\ell(f; \{C_B\}) = \sum_{B \in S_n^\ell} C_B W_B^{(\ell)}(g_\infty)w_B^{\mu_\ell}(\varphi(g_\ell)f), \quad f \in A_n(\mu_\ell).$$
where $W$ is a group homomorphism, by which we view such a way that

$$\text{Put } \text{Sym}_n W$$

the symplectic group of rank $m$ as in $[\text{[11]}]$. These matrices generate the parabolic subgroup $\mathcal{P}_m$ of $Sp(W_m)$ with unipotent radical $\mathcal{U}_m = \{u(b) | b \in \text{Sym}_m\}$. 

Put $\text{Sym}_m^{nd} = \text{Sym}_m \cap GL_m$.

Let $F$ be a local field of characteristic zero. We exclude the complex case. The metaplectic group $Mp(W_m)$ is the nontrivial central extension of $Sp(W_m)$ by $\mu_2$. Let $\mu_2$ inject into the center of $Mp(W_m)$. We call functions on $Mp(W_m)$ or representations of $Mp(W_m)$ genuine if $-1 \in \mu_2$ acts by multiplication by $-1$. We choose the section $s : Sp(W_m) \to Mp(W_m)$ in such a way that

$$\zeta s(g) \cdot \zeta' s(g') = \zeta \zeta' c(g, g') s(gg') \quad (\zeta, \zeta' \in \mu_2 \subseteq Mp(W_m); g, g' \in Sp(W_F))$$

where $c(g, g')$ is the Rao two cocycle on $Sp(W_F)$. The restriction of $s$ to $\mathcal{U}_m$ is a group homomorphism, by which we view $\mathcal{U}_m$ as a subgroup of $Mp(W_m)$.

We will write $\tilde{m} = s \circ m$ and $\tilde{n} = s \circ n$. Note that

$$\tilde{m}(a) \tilde{m}(a') = \chi^{\text{det } a} \tilde{m}(a) \tilde{m}(a')^{-1} = \tilde{n}(ab'a)$$

for $a, a' \in \text{GL}_m(F)$ and $b \in \text{Sym}_m(F)$. For a subgroup $H$ of $Sp_m(F)$ we denote the inverse image of $H$ in $Mp(W_m)$ by $\tilde{H}$. As in Section $4$ we define the Weil index $\gamma^\psi : F^\times \times \gamma^\psi : F^\times \to \mu_4$, which possesses the following properties:

$$\gamma^\psi(t t') = \gamma^\psi(t) \gamma^\psi(t') \gamma^\psi(t')$$

When $F$ is of odd residual characteristic, there is a unique splitting $Sp_m(\mathfrak{o}) \hookrightarrow Mp(W_m)$, by which we regard $Sp_m(\mathfrak{o})$ as a subgroup of $Mp(W_m)$. In other words there is a continuous map $\zeta : Sp_m(F) \to \mu_2$ such that $c(k, k') = \zeta(k) \zeta(k') \zeta(kk')$ for $k, k' \in Sp_m(\mathfrak{o})$. We shall set $\zeta(g) = 1$ in the real and dyadic cases to make our exposition uniform. We use a cocycle

$$c(g, g') = c(g, g') \zeta(g) \zeta(g') \zeta(g')$$

with global applications in view, i.e., we identify $Mp(W_m)$ with the product $Sp_m(F) \times \mu_2$ whose group law is given by $(g, \zeta)(g', \zeta') = (gg', \zeta \zeta' c(g, g'))$. It should be remarked that the section $s$ is now given by $s(g) = (g, \zeta(g))$.

The real metaplectic group acts on the Siegel upper half-space $\mathcal{H}_m$ through $Sp_m(\mathbb{R})$. There exists a unique factor of automorphy $j : Mp(W_m) \times \mathcal{H}_m \to \mathbb{C}^\times$ satisfying $j(\zeta s(g), z)^2 = \text{det}(Cz + D)$ for $\zeta \in \mu_2$ and $g = \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in$
Mp(W_m). For each \( \ell \in \frac{1}{2}\mathbb{Z} \) we put \( J_\ell(\tilde{g}, \mathcal{Z}) = j(\tilde{g}, \mathcal{Z})^{2\ell} \). For each positive definite \( \xi \in \text{Sym}_m(\mathbb{R}) \) we define a function on the real metaplectic group by
\[
W^{(\ell)}_\xi(\tilde{g}) = \langle \det \xi \rangle^{\ell/2} e(\text{tr}(\tilde{g}^2 \theta)) J_\ell(\tilde{g}, \sqrt{-1} \mathbf{1}_m)^{-1}.
\]

Put \( \mathcal{K}_m = \{ g \in \text{Sp}_m(\mathbb{R}) \mid g(\sqrt{-1} \mathbf{1}_m) = \sqrt{-1} \mathbf{1}_m \} \). Let \( \tilde{\mathcal{K}}_m \) be the preimage of \( \mathcal{K}_m \). If \( \zeta \in \mu_2 \), \( a \in \text{GL}_m(\mathbb{R}) \), \( \tilde{g} \in \text{Mp}(W_m) \) and \( \tilde{k} \in \tilde{\mathcal{K}}_m \), then
\[
W^{(\ell)}_\xi(\zeta \tilde{m}(a) \tilde{g} \tilde{k}) = \zeta^{2\ell} \gamma^\psi(\det a)^{2\ell} W^{(\ell)}_\xi(\tilde{g}) J_\ell(\tilde{k}, \sqrt{-1} \mathbf{1}_m)^{-1}.
\]

### 5.2. Representations of the metaplectic group.

Now we assume \( F \) to be nonarchimedean. For \( \xi \in \text{Sym}_m^{\text{ad}} \) and a smooth representation \( \Pi \) of \( \text{Mp}(W_m) \) we set \( W\xi(\Pi) = \text{Hom}_{\text{Sym}_m(F)}(\Pi \circ \tilde{\mathfrak{n}}, \psi^\xi) \). When \( \mu \in \Omega(F^c) \) and \( \Re \mu > -\frac{1}{2} \), we define the representation \( I^\xi_m(\mu) \) of \( \text{Mp}(W_m) \) and the nonzero functional \( w^\mu_\xi \in W\xi(I^\mu_m(\mu)) \) as in Section 3. We write \( g \) for the right regular action of \( \text{Mp}(W_m) \) on the space of smooth functions on \( \text{Mp}(W_m) \).

**Proposition 5.1** (\cite[Lemma 3]{57}, \cite{53}).

1. If \( -\frac{1}{2} < \Re \mu < \frac{1}{2} \), then \( I^\xi_m(\mu) \) is irreducible and unitary.
2. \( \dim W\xi(I^\mu_m(\mu)) = 1 \) for all \( \xi \in \text{Sym}_m^{\text{ad}} \) and \( \mu \in \Omega(F^c) \).
3. Assume that \( \mu^2 = \alpha \). Then \( I^\psi_\xi(\mu) \) has a unique irreducible subrepresentation \( A^{\psi}_\xi(\mu) \), which is unitary. Furthermore, \( W\xi(A^{\psi}_\xi(\mu)) \) is nonzero if and only if the restriction of \( w^\mu_\xi \) to \( A^{\psi}_\xi(\mu) \) is nonzero if and only if \( \chi^\text{det}\xi \neq \mu \alpha^{-1/2} \).
4. If \( F \) is not dyadic, \( \psi \) is of order 0, \( \mu \) is unramified, \( \xi \in \text{Sym}_m^{\text{ad}} \cap \text{GL}_m(\mathfrak{o}) \) and \( h(k) = 1 \) for \( k \in \text{Sp}_m(\mathfrak{o}) \), then \( w^\mu_\xi(h) = 1 \).
5. For all \( \xi \in \text{Sym}_m^{\text{ad}} \) and \( a \in \text{GL}_m(F) \)
\[
w^\mu_\xi \circ g(\zeta \tilde{m}(a)) = \zeta^m \gamma^\psi(\det a)^m \mu(\det a)^{-1} w^\mu_\xi[\tilde{a}]^m.
\]

**Proof.** When \( m \) is even, all the results are included in Proposition 5.1 and Lemma 5.2. The fourth part is Theorem 16.2 of \cite{11}. The other assertions are included in \cite{57} or can be derived analogously. \( \square \)

### 5.3. Holomorphic cusp forms on \( \text{Mp}(W_m) \).

Let \( F \) be a totally real number field. For each place \( v \) of \( F \) we adopt the notation by adding a subscript \( v \) for objects associated to \( F_v \). We define the adèlic cocycle
\[
c : \text{Sp}_m(\mathbb{A}) \times \text{Sp}_m(\mathbb{A}) \to \mu_2,
\]
for \( g, g' \in \text{Sp}_m(\mathbb{A}) \). Recall that \( c_v(g_v, g'_v) = 1 \) for almost all \( v \) by the definition of the local cocycle \( c_v \). We write \( \text{Mp}(W_m)_\mathbb{A} \) for the central extension of \( \text{Sp}_m(\mathbb{A}) \) associated to \( c \).

Let \( \prod_v \text{Mp}(W_m)_v \) denote the restricted direct product with respect to the subgroups \( \{ \text{Sp}_m(\mathfrak{o}_p) \}_{p \mid 2} \). There exist a canonical embedding \( \text{Mp}(W_m)_v \hookrightarrow \text{Mp}(W_m)_{\mathbb{A}} \) and a canonical surjection \( \prod_v \text{Mp}(W_m)_v \twoheadrightarrow \text{Mp}(W_m)_{\mathbb{A}} \). The image of \( (\tilde{g}_v) \in \prod_v \text{Mp}(W_m)_v \) is also denoted by \( (\tilde{g}) \). If we are given a
collection of genuine admissible representations $\sigma_v$ of $Mp(W_m)_v$ such that the space of $Sp_m(p)$-invariant vectors in $\sigma_p$ is one-dimensional for almost all finite primes $p$, then we can form a genuine admissible representation of $Mp(W_m)_\mathbb{A}$ by taking a restricted tensor product $\otimes_v \sigma_v$.

It is well-known that $Mp(W_m)_\mathbb{A}$ splits over the subgroup of rational points $Sp_m(F)$. An explicit splitting is given by

$$s : Sp_m(F) \to Mp(W_m)_\mathbb{A}, \quad s(\gamma) = (\gamma, \prod_v \zeta_v(\gamma)),$$

where if $\gamma \in Sp_m(F)$, then $\zeta_v(\gamma) = 1$ for almost all $v$. Though the expression $\prod_v \zeta_v(g_v)$ does not make sense for all $g \in Sp_m(\mathbb{A})$, we will denote the element $(g, \prod_v \zeta_v(g_v))$ by $s(g)$ whenever it makes sense. For example, $s(m(a)n(b))$ is defined for $a \in GL_m(\mathbb{A})$ and $b \in Sym_m(\mathbb{A})$. We will regard $Sp_m(F)$ and $\mathbb{U}_m(\mathbb{A})$ as subgroups of $Mp(W_m)_\mathbb{A}$ via $s$. For $\xi \in Sym_m(F)$ and a smooth function $F : \mathbb{U}_m(F) \setminus Mp(W_m)_\mathbb{A} \to \mathbb{C}$ let

$$W_\xi(\tilde{g}, F) = \int_{Sym_m(F) \setminus Sym_m(\mathbb{A})} F(s(n(b))\tilde{g})\psi^\xi(-b) \, db$$

denote the $\xi$th Fourier coefficient of $F$.

Let $\ell \in \frac{1}{2}\mathbb{Z}^d$ be such that $2(\ell_v - \ell_{v'})$ is even for every $v, v' \in \mathbb{S}_\infty$. For $\tilde{g} \in Mp(W_m)_\mathbb{A}$ and a function $F$ on $\mathcal{H}_m^d$ we define another function $F_{|\ell}\tilde{g}$ on $\mathcal{H}_m^d$ by $F_{|\ell}\tilde{g}(Z) = F(\tilde{g}Z)J_{\ell}(\tilde{g}, Z)^{-1}$. Define the origin of $\mathcal{H}_m^d$ and the standard maximal compact subgroup of $Mp(W_m)_\mathbb{A}$ by

$$i_m = (\sqrt{-1}i_1, \ldots, \sqrt{-1}i_d) \in \mathcal{H}_m^d, \quad \tilde{K}_m = \{ \tilde{g} \in Mp(W_m)_\mathbb{A} \mid \tilde{g}(i_m) = i_m \}.$$

**Definition 5.2.** The symbol $\mathcal{E}_\ell^{(m)}$ (resp. $\mathcal{E}_\ell^{(m)}$) denotes the space of all smooth functions $F$ on $Mp(W_m)_\mathbb{A}$ which is left invariant under $Sp_m(F)$ (resp. $\mathcal{P}_m(F)$) and transforms on the right by the character $\tilde{k} \mapsto J_{\ell}(\tilde{k}, i_m)^{-1}$ of $\tilde{K}_m$ and such that $F_{\Delta}$ is a holomorphic function on $\mathcal{H}_m^d$ having a Fourier expansion of the form

$$F_{\Delta}(Z) = \sum_{\xi \in \text{Sym}^{\ell}_m} |\det \xi|^{\ell/2}w_\xi(\tilde{\Delta}, F)e_{\infty}(\text{tr}(\xi Z))$$

for each $\tilde{\Delta} \in Mp(W_m)_\mathbb{F}$, where $w_\xi(F)$ is a function on $Mp(W_m)_\mathbb{F}$ and the function $F_{\Delta} : \mathcal{H}_m^d \to \mathbb{C}$ is defined by

$$F_{\Delta}|_{\ell}\tilde{g}(i_m) = F(\tilde{g}\tilde{\Delta}), \quad \tilde{g} \in Mp(W_m)_\mathbb{A}.$$

**Remark 5.3.** The group $Mp(W_m)_\mathbb{F}$ acts on these spaces by right translation. The linear map $F \mapsto F_{\tilde{e}_m}$ is a bijection of $\mathcal{E}_\ell^{(m)}$ onto the space $S_\ell^{(m)}$ of Hilbert-Siegel cusp forms of weight $\ell$ defined in Section 4 by strong approximation in $Sp_m$, where $\tilde{e}_m$ is the identity element of $Mp(W_m)_\mathbb{F}$. If $F \in \mathcal{E}_\ell^{(m)}$ is right invariant under the open compact subgroup $D$ of $Mp(W_m)_\mathbb{F}$, then $F_{\tilde{e}_m} \in S_\ell(\Gamma)$, where $\Gamma = Sp_m(F) \cap D$. 
For $\xi \in \text{Sym}_m^+$ we define a function $W^{(l)}_\xi$ on $\text{Mp}(W_m)_\infty$ by

$$W^{(l)}_\xi(\tilde{g}) = \prod_{v \in \mathcal{S}_\infty} W^{(l_v)}_\xi(\tilde{g}_v).$$

The group $\text{GL}_2(\mathbb{A}_\infty)^+ := \{ (g_v) \in \text{GL}_2(\mathbb{A}_\infty) \mid \det g_v > 0 \text{ for all } v \in \mathcal{S}_\infty \}$ acts componentwise on $\mathcal{H}^d_1$. Given $g \in \text{GL}_2(\mathbb{A}_\infty)^+$ and a function $\mathcal{F}$ on $\mathcal{H}^d_1$, we define a function $\mathcal{F}|_\kappa g$ on $\mathcal{H}^d_1$ by

$$\mathcal{F}|_\kappa g(\mathcal{Z}) = \mathcal{F}(g\mathcal{Z}) J_\kappa(g, \mathcal{Z})^{-1}, \quad J_\kappa(g, \mathcal{Z}) = |\det g|^{-\kappa/2} \prod_{v \in \mathcal{S}_\infty} (c_v \mathcal{Z}_v + d_v)^{\kappa_v},$$

where $g_v = \left( \begin{array}{cc} * & * \\ c_v & d_v \end{array} \right)$. Put $\mathcal{K}_m = \{ g \in \text{Sp}_m(\mathbb{A}_\infty) \mid g(\mathfrak{i}_m) = \mathfrak{i}_m \}$.

**Definition 5.4.** A Hilbert cusp form $\mathcal{F}$ on $\text{PGL}_2$ of weight $2\kappa$ is a smooth function on $\text{GL}_2(\mathbb{A})$ satisfying

$$\mathcal{F}(z \gamma g k) = \mathcal{F}(g) J_{2\kappa}(k, \mathfrak{i}_1)^{-1}, \quad (z \in \mathbb{A}^\times, \gamma \in \text{GL}_2(F), g \in \text{GL}_2(\mathbb{A}), k \in \mathcal{K}_1)$$

and having a Fourier expansion of the form

$$\mathcal{F}_\Delta(\mathcal{Z}) = \sum_{t \in \mathfrak{F}_+^\times} |t|^\kappa \mathcal{w}_t(\Delta, \mathcal{F}) \mathcal{e}_\infty(t \mathcal{Z})$$

for each $\Delta \in \text{GL}_2(\mathbb{A})$, where $\mathcal{w}_t(\mathcal{F})$ is a function on $\text{GL}_2(\mathbb{A})$ and the function $\mathcal{F}_\Delta : \mathcal{H}_1^d \to \mathbb{C}$ is defined by $\mathcal{F}_\Delta|_{2\kappa} g(\mathfrak{i}_1) = \mathcal{F}(g\Delta)$ for $g \in \text{GL}_2(\mathbb{A}_\infty)^+$.

We write $\mathcal{E}_{2\kappa}$ for the space of Hilbert cusp forms on $\text{PGL}_2$ of weight $2\kappa$.

Notation being as in [32], we form the restricted tensor product

$$A^\psi_m(\mu_\tau) = (\otimes_{p \in \mathcal{S}_{\pi_\tau}} A^\psi_{m_\tau}(\mu_\tau)) \otimes (\otimes_{p \in \mathcal{S}_{\pi_\tau} \cup \mathcal{S}_{\pi_f}} I^\psi_{m_\tau}(\mu_\tau)).$$

For $\xi \in \text{Sym}_m^+$ and a smooth representation $\Pi$ of $\text{Mp}(W_m)_\mathfrak{f}$ we put

$$\text{Wh}_\xi(\Pi) = \text{Hom}_{\text{Sym}_m(\mathbb{A})}(\Pi \circ \mathfrak{s} \circ \mathfrak{n}, \psi_\xi).$$

We can define $w^{\mu_\tau}_\xi \in \text{Wh}_\xi(A^\psi_{m_\tau}(\mu_\tau))$ by setting $w^{\mu_\tau}_\xi(h) = \prod_{p \in \mathcal{S}_{\pi_\tau}} \psi_{m_\tau}^\psi(h_p)$ for factorizable vectors $h = \otimes_p h_p \in A^\psi_{m_\tau}(\mu_\tau)$. Put

$$\text{Sym}_{m_\tau}^\pi = \{ \xi \in \text{Sym}_{m_\tau}^+ \mid \chi_p^\det \xi \neq \mu_\tau \alpha_p^{1/2} \text{ for all } p \in \mathcal{S}_{\pi_\tau} \}, \quad \mathfrak{F}_{\pi_\tau} = \text{Sym}_{m_\tau}^{\pi_\tau}.$$ 

Proposition 5.4 tells us that $\text{Wh}_\xi(A^\psi_{m_\tau}(\mu_\tau))$ is nonzero if and only if the restriction of $w^{\mu_\tau}_\xi$ to $A^\psi_{m_\tau}(\mu_\tau)$ is nonzero if and only if $\xi \in \text{Sym}_{m_\tau}^\pi$.

**Definition 5.5.** Assume that $2\ell - m$ is even for every $v \in \mathcal{S}_\infty$. Let $\hat{T}^{(m)}(\mu_\tau)$ (resp. $\hat{\mathcal{C}}^{(m)}(\mu_\tau)$) denote the vector space which consists of sets $\{ C_\xi \}$ of complex numbers indexed by $\xi \in \text{Sym}_{m_\tau}^\pi$ such that the series

$$\mathcal{F}_\ell(\tilde{g}; h, \{ C_\xi \}) = \sum_{\xi \in \text{Sym}_{m_\tau}^\pi} C_\xi W^{(l)}_\xi(\tilde{g}_\infty) w^{\mu_\tau}_\xi(g(\tilde{g}_\mathfrak{f}) h)$$

belongs to $\hat{\mathcal{C}}^{(m)}$ (resp. $\hat{\mathcal{C}}^{(m)}$) for every $h \in A^\psi_{m_\tau}(\mu_\tau)$. 
Lemma 5.6.  
(1) $\tilde{C}_{\ell}^{(m)}(\mu_f) \neq \{0\}$ if and only if there is a $\text{Mp}(W_{m})_f$ intertwining embedding $A_{m}^{\psi}(\mu_f) \rightarrow \tilde{C}_{\ell}^{(m)}$.
(2) Assume that $F_{\ell}(h, \{C_{\xi}\})$ converges for all $h \in A_{m}^{\psi}(\mu_f)$. Then $\{C_{\xi}\} \in \tilde{T}_{\ell}^{(m)}(\mu_f)$ if and only if for all $a \in \text{GL}_m(F)$ and $\xi \in \text{Sym}_m^+$,
$$C_{\xi[a]} = C_{\xi}\mu_f(\det a)^{-1} \prod_{v \in \mathcal{S}} \text{sgn}_v(\det a)^{(2\ell_v - m)/2}.$$  
(3) If $\tilde{T}_{\ell}^{(m)}(\mu_f) \neq \{0\}$, then $\mu_f(-1)(-1)^{\sum_{v \in \mathcal{S}}(2\ell_v - m)/2} = 1$.
(4) $\dim \tilde{C}_{\ell}^{(1)}(\mu_f) = 1$.
(5) If $\{c_t\} \in \tilde{C}_{\ell}^{(1)}(\mu_f)$, then $\{c_{\eta}t\} \in \tilde{C}_{\ell}^{(1)}(\mu_f \chi_{\ell}^{\eta})$ for all $\eta \in F_{m}^\times$.
(6) We choose $0 \neq \{c_t\} \in \tilde{C}_{\ell}^{(1)}(\mu_f)$, assuming that $\tilde{C}_{\ell}^{(1)}(\mu_f) \neq \{0\}$. Then $c_t \neq 0$ if and only if $t \in F_{m}^\times$ and $L\left(\frac{1}{2}, \pi \otimes \chi_{\ell}^{t}\right) \neq 0$.

Proof. The proof of (i) is similar to that of Lemma 5.3, Proposition 5.17 and (6.2) prove (ii). One sees that the third statement is its simple consequence by taking $a \in \text{GL}_m(F)$ so that $\xi[a] = \xi$ and $\det \xi = -1$. The assertion (ii) follows from the fact proved by Waldspurger [50] that every irreducible representation of $\text{Mp}(W_1)_H$ occurring in the decomposition of the space of cusp forms on $\text{Mp}(W_1)$ appears with multiplicity one.

Now we prove (6). Let $h \in A_{1}^{\psi}(\mu_f)$. Put $F = F_{\ell}(h, \{c_t\})$. Let us set $d(a) = \text{diag}[1, a]$ for $a \in \mathbb{A}^\times$. We can realize $\text{Mp}(W_1)_H$ as a normal subgroup of a double cover of $\text{GL}_2(\mathbb{A})$ constructed by using the Kubota two cocycle. The conjugation action of $\{d(a) \mid a \in F_{m}^\times\}$ on $\text{SL}_2(\mathbb{A})$ has a lift to $\text{Mp}(W_1)_H$, which preserves the subgroup $\text{SL}_2(F)$. We define a cusp form $F' : \text{Mp}(W_1)_H \rightarrow \mathbb{C}$ by $F'(\tilde{g}) = F(d(\eta)\tilde{g}(\eta)^{-1}\gamma^{-1})$, where $\gamma = (m(\sqrt{\eta}), 1) \in \text{Mp}(W_1)_\infty$. Then $F'$ equals $F_{\ell}(h', \{c_{\eta}t\})$ up to scalar, where $h'$ is defined by $h'(\tilde{g}) = h'(d(\eta)\tilde{g}(\eta)^{-1})$. One can see that $h' \in A_{1}^{\psi}(\mu_f \chi_{\ell}^{\eta})$. The last assertion is Theorem 4.1 of [17].

6. Main theorem

6.1. Liftings to inner forms of $\text{Sp}_{2n}$.

Theorem 6.1. Notations and assumptions being as in Theorem 5.2, the assignment
$$\nu = \nu(B) \Delta(Z) = \sum_{B \in \mathcal{S}^+_n} |\nu(B)|^{(n+1)/2} c_{\nu(B)} e_{\infty}(\tau(BZ)) u_B^{\mu_f \chi_{\ell}^{(-1)^n} \eta}$$
defines an embedding $\nu : A_n(\mu_f \chi_{\ell}^{(-1)^n} \eta) \hookrightarrow \mathfrak{e}_{\kappa+n}^n$ for every $n$ and $\eta \in F_{m}^\times$, where $\Delta \in G_n(\mathbb{A}_F)$ and $Z \in S^+_n$.

Remark 6.2. (1) In light of Lemma 6.6, this embedding naturally defines an embedding $A_n(\mu_f \chi_{\ell}^{(-1)^n} \eta) \hookrightarrow \mathfrak{e}_{\kappa+n}^n$. 

(2) The multiplicity of \( A_1(\mu_f \overline{\chi_f}) \) in \( \mathfrak{g}^1_{n+1} \) is one by Corollary 7.7 of [3]. However, we do not know if this result can imply the multiplicity of \( A_1(\mu_f \overline{\chi_f}) \) in \( \mathfrak{g}^1_{n+1} \).

6.2. Compatibility with Arthur’s endoscopic classification. We explain how Theorem 5[?] can be viewed in the framework of Arthur’s conjecture. For details the reader should consult [1, 2]. The conjecture specialized to our current case is discussed in [3], Section 14 of [22] and [63].

Let \( \mathcal{L} \) be the hypothetical Langlands group over \( F \). Hypothetically, there is a bijective correspondence between the set of all equivalence classes of \( \mathfrak{m} \)-dimensional irreducible automorphic representations of \( GL_2(\mathbb{A}) \), then \( \pi \) corresponds to a map \( \rho(\pi) : \mathcal{L} \to \text{SL}_2(\mathbb{C}) \). Let \( \text{sym}^{m-1} \) be the irreducible \( \mathfrak{m} \)-dimensional representation of \( \text{SL}_2(\mathbb{C}) \). We may assume that \( \text{sym}^{2n-1}(\text{SL}_2(\mathbb{C})) \subset Sp_n(\mathbb{C}) \). Thus \( \rho(\pi) \boxtimes \text{sym}^{2n-1} \) gives rise to a homomorphism \( \mathcal{L} \times \text{SL}_2(\mathbb{C}) \to \text{SO}_{4n}(\mathbb{C}) \). Embedding \( \text{SO}_{4n}(\mathbb{C}) \) into \( \text{SO}_{4n+1}(\mathbb{C}) = \overline{G}_n \), we get a homomorphism \( \mathcal{L} \times \text{SL}_2(\mathbb{C}) \to \overline{G}_n \).

One postulates that for each place \( v \) there is a natural conjugacy class of embeddings \( \mathcal{L}_v \hookrightarrow \mathcal{L} \), where \( \mathcal{L}_v \) is the Weil group of \( F_v \) if \( v \in \mathfrak{s}_\infty \), and the Weil-Deligne group of \( F_v \) if \( v \notin \mathfrak{s}_\infty \). We obtain a homomorphism \( \rho(\pi_v) \boxtimes \text{sym}^{2n-1} : \mathcal{L}_v \times \text{SL}_2(\mathbb{C}) \to \overline{G}_n \) for each \( v \).

The Arthur conjecture suggests that there exists a finite set \( \Pi_n(\pi_v) = \Pi_v^+(\pi_v) \sqcup \Pi_v^-(\pi_v) \) of equivalence classes of unitary admissible representations of \( G_{n,v} \) associated to \( \rho(\pi_v) \boxtimes \text{sym}^{2n-1} \). Moreover, it is required that if \( G_{n,v} \simeq Sp_{2n}(F_v) \), then \( \Pi_v^+(\pi_v) \) contains the Langlands quotients \( I_n^+(\pi_v) \) of

\[
\text{Ind}_{P_{2,\ldots,n}(F_v)}^{Sp_{2n}(F_v)} (\pi_v \otimes \alpha_v^{(2n-1)/2}) \boxtimes (\pi_v \otimes \alpha_v^{(2n-3)/2}) \boxtimes \cdots \boxtimes (\pi_v \otimes \alpha_v^{1/2}),
\]

where \( P_{2,\ldots,n} \) is the standard parabolic subgroup of \( Sp_{2n} \) with Levi subgroup \( GL_2 \times \cdots \times GL_2 \). Choose \( \Pi_v \in \Pi_v^+(\pi_v) \) for each \( v \). Then \( \mathcal{S}_v^\circ \Pi_v \) is an automorphic representation of \( G_n(\mathbb{A}) \) generated by square-integrable automorphic forms if and only if \( \prod_v \epsilon_v = \epsilon(\frac{1}{2}, \pi) \).

Let \( p \) be a finite prime such that \( D_p \simeq M_2(F_p) \). Let \( \mu \in \Omega(F_p^\times) \). If \( \Re \mu > -\frac{1}{2} \), then we obtain an intertwining map

\[
I_n(\mu) \to \text{Ind}_{P_{2,\ldots,n}(F_p)}^{Sp_{2n}(F_p)} (I(\mu, \mu^{-1}) \otimes \alpha_p^{(2n-1)/2}) \boxtimes \cdots \boxtimes (I(\mu, \mu^{-1}) \otimes \alpha_p^{-1/2})
\]

by applying Proposition 1 of [110] or Proposition 4.1 and Lemma 5.1 of [131] repeatedly. Therefore if \( -\frac{1}{2} < \Re \mu < \frac{1}{2} \), then this map is nonzero by Lemma 8 of [110], so that \( I_n(\mu) \simeq I_n^+(I(\mu, \mu^{-1})) \). If \( \mu^2 = \alpha_p \) and \( \mu \neq \alpha_p^{1/2} \), then \( A_n(\mu) \simeq I_n^+(A(\mu, \mu^{-1})) \) by Proposition 3.11(2) of [27]. On the other hand, \( A_n(\alpha_p^{1/2}) \) is the Langlands quotient of

\[
\text{Ind}_{Q_{2,\ldots,n}(F_p)}^{Sp_{2n}(F_p)} (\alpha_p^\circ \alpha_p^{n-1}) \boxtimes \cdots \boxtimes A(\alpha_p^2, \alpha_p) \boxtimes A_1(\alpha_p^{1/2})
\]

by Proposition 3.10(2) of [27], where \( Q_{2,\ldots,n} \) is the standard parabolic subgroup of \( Sp_{2n} \) with Levi subgroup \( GL_2 \times \cdots \times GL_2 \times Sp_2 \). We guess that
A_p(α_p^{1/2}) \in \Pi_n^-(A(α_p^{1/2}, α_p^{-1/2})). We presume that the reasoning above is correct even when \( D_p \) is division.

Let \( \pi_v \simeq (\otimes_{v \in \mathcal{S}_\infty} \pi_v) \otimes \pi_T \) be an irreducible cuspidal automorphic representation of \( \mathrm{PGL}_2(\mathbb{A}) \) on which we impose the following conditions:

(i) \( \pi_v \) is the discrete series with extremal weights \( \pm 2\kappa_v \) for \( v \in \mathcal{S}_\infty \);

(ii) \( \pi_T \simeq A(\mu_T, \mu_T^{-1}) \);

(iii) \( \mu_T(-1)(-1)^{\sum_{v \in \mathcal{S}_\infty} \kappa_v} = 1 \).

Let \( v \in \mathcal{S}_\infty \) and assume that \( \kappa_v > n \). Then \( W_B^{(\kappa_v + n)} \) generates a holomorphic discrete series representation of \( \mathrm{Sp}_{2n}(F_v) \). Fix \( \eta \in F^*_n \). Put

\[
\sigma = \pi \otimes \chi^{(-1)^{\kappa_v} \eta}, \quad \ell_\sigma = \# \{ \mathfrak{p} \notin \mathcal{S}_\infty \mid \sigma_{\mathfrak{p}} \simeq A(\alpha_p^{1/2}, \alpha_p^{-1/2}) \}.
\]

The holomorphic discrete series representation with lowest \( K \)-type \( (\det)^{\kappa_v + n} \) belongs to \( \Pi_n^{(-n)}(\pi_v) \). Since

\[
\epsilon(1/2, \sigma) = (-1)^{\ell_\sigma + \sum_{v \in \mathcal{S}_\infty} \kappa_v} \mu_T(-1)^{\chi^{(-1)^{\kappa_v} \eta}(-1) = (-1)^{\ell_\sigma + n d},
\]
the restriction (iii) is compatible with the Arthur conjecture.

**Remark 6.3.** If \( \pi_v \) is supercuspidal, then we no longer have a description of an element of \( \Pi_n(\pi_v) \) in terms of degenerate principle series, and we have no simple expression of its degenerate Whittaker functionals. This is the reason why we assume that \( \pi_v \) is not supercuspidal.

If \( m \) is odd, then we obtain a homomorphism \( \rho(\sigma) \boxtimes \mathrm{sym}^{m-1} : \mathcal{L} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{m}(\mathbb{C}) = \mathrm{L} \mathrm{SO}_{2m+1}, \) which should be the Arthur parameter of the lifting constructed in Theorems 2 and 3. Our result is compatible with an analogue of the Arthur conjecture for metaplectic groups formulated by Wee Teck Gan [33]. We refer to [33] for a description of the associated \( A \)-packet in the metaplectic case. The homomorphism \( \rho(\sigma) \boxtimes \mathrm{sym}^{2n-1} : \mathcal{L} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n}(\mathbb{C}) = \mathrm{L} \mathrm{SO}_{4n} \) should be the Arthur parameter of the lifting constructed in Corollary 4.

Let \( \rho(g) : \mathcal{L} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C}) = \mathrm{L} \mathrm{Sp}_{n} \) be the Arthur parameter for the cuspidal automorphic representation generated by a Hecke eigenform \( g \in S_{\kappa+n+r}(1_r(\delta^{-1}, \delta)) \). The homomorphism \( \rho(g) \times (\rho(\sigma) \boxtimes \mathrm{sym}^{2n-1}) : \mathcal{L} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n+4n+1}(\mathbb{C}) = \mathrm{L} \mathrm{Sp}_{2n+r} \) should be the Arthur parameter of the Miyawaki lifting constructed in Corollary 5.

### 7. Fourier-Jacobi modules

**7.1. Jacobi groups.** Fix \( 0 \leq i \leq n \). Put \( n' = n - i \). For \( z \in S_i \) and \( x, y, y \in M_{n'}(D) \) we use the notation

\[
v(x, y; z) = \begin{pmatrix}
1_i & x & z - yx^* & y \\
0 & 1_{n'} & -y^* & 0 \\
0 & 1_i & 0 & -x^* \\
0 & 1_{n'} & 1_i & 1_{n'}
\end{pmatrix}, \quad \eta_i = \begin{pmatrix}
n_{n'} & 1_i \\
1_i & 1_{n'}
\end{pmatrix} \in G_n.
\]
We define some subgroups of $G_n$ by
\[ X_i = \{ v(x, 0; 0) \mid x \in M^i_n(D) \}, \quad Y_i = \{ v(0, y; 0) \mid y \in M^i_n(D) \}, \]
\[ Z_i = \{ v(0, 0; z) \mid z \in S_i \}, \quad N_i = \{ v(x, y; z) \mid x, y \in M^i_n(D), \, z \in S_i \}. \]
We identify $X_i$ and $Y_i$ with the space $M^i_n(D)$. We view $G_i$ and $G_n'$ as subgroups of $G_n$ via the embeddings
\[ g_1 \mapsto \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} 1_n' \\ 1_n' \end{pmatrix}, \quad g_2 \mapsto \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} 1_i \\ 1_i \end{pmatrix}, \]
where we write a typical element $g_1 \in G_i$ in the form $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ with matrices $a_1, b_1, c_1, d_1$ of size $i$ over $D$, and similarly for $g_2 \in G_n'$. We also write
\[ m'(A) = \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^* \end{pmatrix}, \quad n'(z) = \begin{pmatrix} 1_{n'} & z \\ 0 & 1_{n'} \end{pmatrix} \]
for $A \in \text{GL}_{n'}(D)$ and $z \in S_{n'}$. We will frequently specialize to the case $i = n - 1$ in our application to the proof of main theorems.

7.2. Weil representations of Jacobi groups. Let $F$ be a local field. Fix $S \in S^\text{nd}$. We regard $S$ as a homomorphism $Z_i \to G_a$ by $z \mapsto \tau(Sz)$. Then $N_i/\text{Ker}S$ is a Heisenberg group with center $Z_i/\text{Ker}S$ and a natural symplectic structure $N_i/Z_i$. The Schrödinger representation $\omega^\psi_S$ of $N_i$ with central character $\psi^S$ is realized on the Schwartz space $\mathcal{S}(X_i)$ by
\[ (\omega^\psi_S(v(x, y; z))\phi)(u) = \phi(u + x)\psi^S(z)\psi(2\tau(u^*Sz)) \] (7.1)
for $\phi \in \mathcal{S}(X_i)$. By the Stone-von Neumann theorem, $\omega^\psi_S$ is the unique irreducible representation of $N_i$ on which $Z_i$ acts by $\psi^S$.

The embedding $G_n' \hookrightarrow G_n$ and the conjugating action give an embedding $G_n' \hookrightarrow \text{Sp}(N_i/Z_i)$ and Kudla [77] gave an explicit local splitting $G_n' \hookrightarrow \text{Mp}(N_i/Z_i)$, where $\text{Mp}(N_i/Z_i)$ is the metaplectic extension of $\text{Sp}(N_i/Z_i)$. The representation $\omega^\psi_S$ of $N_i$ extends to the Weil representation of $\text{Mp}(N_i/Z_i) \times N_i$ whose pullback to $G_n'$ is characterized by the following formulas:
\[ (\omega^\psi_S(m'(A))\phi)(u) = \hat{\chi}^S(\nu(A))\nu(A)|^i\phi(uA), \]
\[ (\omega^\psi_S(n'(z))\phi)(u) = \psi^S(uzu^*)\phi(u), \]
\[ (\omega^\psi_S(J_{n'})\phi)(u) = \gamma^S(S)(\mathcal{F}_S\phi)(u) \]
(7.2)
for $\phi \in \mathcal{S}(X_i)$, $u \in X_i$, $A \in \text{GL}_{n'}(D)$ and $z \in S_{n'}$, where $\gamma^S(S)$ is a certain 8th root of unity and $\mathcal{F}_S\phi$ is the Fourier transform defined by
\[ (\mathcal{F}_S\phi)(u) = \int_{X_i} \phi(x)\psi(2\tau(x^*Su)) \, dx. \]
7.3. The nonarchimedean case. Let $F$ be a finite extension of $\mathbb{Q}_p$.

**Lemma 7.1.** Let $f \in I_n(\mu)$ and $\phi \in \mathcal{S}(X_i)$. If $\Re \mu \gg 0$, then the integral

$$B^\psi_S(g'; f \otimes \bar{\phi}) = \int_{Y_i \cap N_i} f(\eta v g') (\omega^\psi_S(v g') \phi)(0) \, dv$$

is absolutely convergent. Moreover, it possesses analytic continuation to all $\mu \in \Omega(F^\times)$ and gives an $N_i$-invariant and $G_{n'}$-intertwining map

$$B^\psi_S : I_n(\mu) \otimes \omega^\psi_S \to I_n'(\mu \hat{\chi}^S).$$

**Proof.** The integral over $Z_i$ can be viewed as a Jacquet integral of the restriction of $f$ to $G_i$, which belongs to $I_i(\mu \alpha^{n'})$. It is entire on the whole of the complex manifold $\Omega(F^\times)$. Since $(\omega^\psi_S(xg') \phi)(0) = (\omega^\psi_S(g') \phi)(x)$ for $x \in X_i$, the integral over $X_i$ is convergent for all $\mu$. When $D \simeq M_2(F)$, Ikeda showed that $B^\psi_S(f \otimes \bar{\phi}) \in I_n'(\mu \hat{\chi}^S)$ in the proof of Theorem 3.2 of [21]. The computation applies equally well to the quaternion case. \hfill $\square$

We modify the integral above in the following way:

$$\beta^\psi_S(g'; f \otimes \bar{\phi}) = \frac{L(n + \frac{1}{2}, \mu)}{L(n' + \frac{1}{2}, \mu \hat{\chi}^S)} B^\psi_S(g'; f \otimes \bar{\phi}) \prod_{j=1}^i L(2n' + 2j - 1, \mu^2).$$

The reason for this normalization is Corollary (1).\hfill $\square$

**Lemma 7.2.** Let $S \in S^\text{nd}_i$. Put $n' = n - i$. There exists a nonzero constant $E_S$ such that for all $\Xi \in S^\text{nd}_i$, $f \in I_n(\mu)$ and $\phi \in \omega^\psi_S$

$$w^\mu_{\Xi} \beta_S^\psi (f \otimes \bar{\phi})) = E_S |\nu(\Xi)|^{-i}/2 \int_{X_i} \phi(x) w^\mu_{\Xi} (\bar{\psi}(x)f) \, dx.$$

**Proof.** Since $\eta_i = J_n \cdot J_{n'}$ and $J_{n'}v(0,y,z)J_{n'} = v(y,0,z)$, we observe that

$$\int_{X_i} f(J_n v J_{n'} g') (\omega^\psi_S(v J_{n'} g') \phi)(0) \, dv$$

$$= \int_{Z_i} \int_{Y_i} \int_{X_i} f(J_n x z y J_{n'} g') (\omega^\psi_S(z y J_{n'} g') \phi)(x) \, dxdydz$$

$$= \int_{S_1} \int_{M'_{n}(D)} \int_{X_i} f(J_n v(0,y,z)J_{n'} g') (\omega^\psi_S(J_{n'} v(y,0,z) g') \phi)(x) \, dxdydz$$

$$= \int_{S_1} \int_{M'_{n}(D)} f(\eta_i v(y,0,z)J_{n'} g') (\mathcal{F} \omega^\psi_S(J_{n'} v(y,0,z) g') \phi)(0) \, dydz$$

$$= \gamma(S)^{-1} \int_{Z_i} \int_{X_i} f(\eta_i v(y,0,z)g') (\omega^\psi_S(v(y,0,z) g') \phi)(0) \, dydz$$

by (1), (2) and the Fourier inversion. Since

$$|\nu(\Xi)|^{(2n'+1)/4} \frac{L(n' + \frac{1}{2}, \mu \hat{\chi}^S)}{L(\frac{1}{2}, \mu \hat{\chi}^S \Xi)} \prod_{j=1}^{n'} L(2j - 1, \mu^2) \frac{L(n + \frac{1}{2}, \mu)}{L(n' + \frac{1}{2}, \mu \hat{\chi}^S)} \prod_{j=1}^i L(2n' + 2j - 1, \mu^2)$$

\hfill $\square$
\[
|\nu(S)|^{-(2n+1)/4}|\nu(\Xi)|^{-i/2}|\nu(S \oplus \Xi)|^{(2n+1)/4} \frac{L(n + \frac{1}{2}, \mu)}{L(\frac{1}{2}, \mu) \chi^{S \oplus \Xi}} \prod_{j=1}^{n} L(2j - 1, \mu^2),
\]

it suffices to prove
\[
\int_{S_n} \int_{N_i} f(J_n v n'(u))(\omega_S^\psi(v n'(u)) \phi)(0) \psi^\Xi(-u) \, du \, dv
\]
\[
= \int_{X_i} \int_{S_n} f(J_n n(z)x) \phi(x) \psi^{S \oplus \Xi}(-z) \, dz \, dx
\]
for \( \Re \mu \gg 0 \). The left hand side is equal to
\[
\int_{S_n} \int_{N_i} f(J_n n'(u)v)(\omega_S^\psi(n'(u)v) \phi)(0) \psi^\Xi(-u) \, du \, dv
\]
\[
= \int_{S_n} \int_{N_i} f(J_n n'(u)v)(\omega_S^\psi(v) \phi)(0) \psi^\Xi(-u) \, du \, dv
\]
\[
= \int_{Y_i} \int_{Z_i} \int_{X_i} f(J_n n'(u)v x) \phi(x) \psi^S(z) \psi^\Xi(-u) \, dx \, dz \, dy \, du
\]
\[
= \int_{X_i} \int_{S_n} f(J_n n(z)x) \phi(x) \psi^{S \oplus \Xi}(-z) \, dx \, dz.
\]
Since this integral is absolutely convergent for \( \Re \mu \gg 0 \), we can exchange the order of integration.

**Corollary 7.3.**

1. If \( p \neq 2 \), \( D \cong M_2(F) \), \( \psi \) is of order 0, \( \mu \) is unramified, \( S \in S_1^{\text{sd}} \cap \text{GL}_i(\mathcal{O}) \), \( \phi \) is the characteristic function of \( M_1(\mathcal{O}) \) and \( f \in I_n(\mu) \) satisfies \( f(k) = 1 \) for all \( k \in K_1^D[\mathfrak{o}] \), then \( \beta^\psi_S(I_{2n'}; f \otimes \phi) = 1 \).
2. If \( -\frac{1}{2} < \Re \mu < \frac{1}{2} \), then \( \beta^\psi_S'(I_n(\mu) \otimes \omega_S^\psi) = I_{n'}(\mu \chi^S) \).
3. If \( \mu^2 = \alpha \), then \( \beta^\psi_S(A_n(\mu) \otimes \omega_S^\psi) = A_{n'}(\mu \chi^S) \).

**Proof.** Since \( E_S = 1 \) in the unramified case, we can derive (1) from Lemmas 6.1 and 6.3. If \( \chi^\Xi = \mu \chi^S \alpha^{-1/2} \), then \( \chi^{S \oplus \Xi} = \mu \alpha^{-1/2} \), and hence \( w_{\Xi}^{\mu \chi^S}(\beta_S^\psi(f \otimes \phi)) = 0 \) for all \( f \in A_n(\mu) \) and \( \phi \in \omega_S^\psi \) by Lemmas 6.2 and 6.6.

It follows from the proof of Proposition 6.3 that \( \beta_S^\psi(A_n(\mu) \otimes \omega_S^\psi) \subset A_{n'}(\mu \chi^S) \).

Since the target spaces are irreducible, it is sufficient to show the nonvanishing of the intertwining maps. Lemma 7.3 enables us to take \( \Xi \in S_n^{\text{sd}} \) and \( f \) so that \( w_{\Xi}^{\mu \chi^S}(\beta_S^\psi(f \otimes \phi)) \neq 0 \). Using Lemma 7.3 and choosing \( \phi \) to be supported in a small neighborhood, one can show that \( w_{\Xi}^{\mu \chi^S}(\beta_S^\psi(f \otimes \phi)) \neq 0 \). \( \square \)

7.4. **The metaplectic case.** Fix \( 0 \leq i \leq m \). Put \( m' = m-i \). For \( z \in \text{Sym}_i \), \( x, y \in M_{i'} \), \( a \in \text{GL}_{m'} \) and \( b \in \text{Sym}_{m'} \) we use the notation
\[
u(x, y; z) = \begin{pmatrix}
1_i & x & z - y^t x & y \\
0 & 1_{m'} & t_y & 0 \\
1_i & 0 & t_y^t & 1_{m'} \\
0 & -t_y^t & 1_{m'} & 0
\end{pmatrix},
\]
\[\eta_i = \begin{pmatrix}
1_{i'} & -1_i \\
1_i & 0 \\
0 & 1_{m'}
\end{pmatrix},\]
\[ m'(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad n'(b) = \begin{pmatrix} 1_{m'} & b \\ 0 & 1_{m'} \end{pmatrix}. \]

We define the subgroups of \( Sp_m \) by \( \mathcal{J}_i = Sp_{m'} \cdot \mathcal{N}_i \) and

\[ \mathcal{J}_i = \{ u(x, 0; 0) \mid x \in M_i \}, \quad \mathcal{Y}_i = \{ u(0, y; 0) \mid y \in M_{m'} \}, \]
\[ \mathcal{Z}_i = \{ u(0, 0; z) \mid z \in \text{Sym}_i \}, \quad \mathcal{N}_i = \{ u(x, y; z) \mid x, y \in M_i, \; z \in \text{Sym}_i \}. \]

For the time being let \( F \) be a local field. Fix \( R \in \text{Sym}^{nd}_i \). Recall the parabolic subgroup \( \mathcal{P}_{m'} \) of \( Sp_{m'} \) defined in \( \mathcal{O}_i \). The pullback to \( \mathcal{P}_{m'} \ltimes \mathcal{N}_i \) of the Weil representation \( \omega^\psi_R \) of \( \text{Mp}(W_{m'}) \ltimes \mathcal{N}_i \) is given by

\[
(\omega^\psi_R(u(x, y; z))\phi)(u) = \phi(u + x)\psi^R(z)\psi(2\text{tr}(uRy)),
\]
\[
(\omega^\psi_R(n'(b)m'(a))\phi)(u) = \psi^{R[a]}(b)\gamma^\psi(\text{det} a)\hat{\chi}^{\text{det} R}(\text{det} a)\text{det} a|^{1/2}\phi(ua)
\]
for \( \phi \in S(\mathcal{Z}_i) \), \( u \in \mathcal{Z}_i \), \( a \in \text{GL}_{m'}(F) \) and \( b \in \text{Sym}_m(F) \).

**Lemma 7.4.** Let \( F \) be a finite extension of \( \mathbb{Q}_p \). Assume that \( m' \) is odd. The following integral makes sense for all \( R \mu > -\frac{m'}{2} - 1 \) and gives an \( \mathcal{N}_i \)-invariant and \( \text{Mp}(W_{m'}) \)-intertwining map \( \beta^\psi_R : \mathcal{I}_{m'}(\mu) \otimes \omega^\psi_R \to \mathcal{I}_{m'}(\mu\hat{\chi}^{\text{det} R}) \):

\[
\beta^\psi_R(g'; h \otimes \phi) = \prod_{j=1}^{[i/2]} L(m' + 2j, \mu^2) \int_{\mathcal{J}_i \setminus \mathcal{N}_i} h(s(\eta; v)g') (\omega^\psi_R(vg')\phi)(0) \, dv
\]
\[
\times \begin{cases} 1 & \text{if } 2 \nmid m, \\ L(m' + 1, \mu\hat{\chi}(-1)^{m/2}) & \text{if } 2 \mid m. \end{cases}
\]

Moreover, for \( T \in \text{Sym}^{nd}_{m'} \), \( h \in \mathcal{I}_{m'}(\mu) \) and \( \phi \in \omega^\psi_R \)

\[
w^\mu^\chi^{\text{det} R}_T(\beta^\psi_R(h \otimes \phi)) = E_{R|\det T}^{-1/4} \int_{\mathcal{J}_i} \overline{\phi(x)}w^\mu_R \phi(xh) \, dx.
\]

We omit the proof as it is the same as those of Lemmas [1] and [2]. We can deduce the following corollary from Proposition [3] and Lemma [4] by the same reasoning as in the proof of Corollary [3].

**Corollary 7.5.**

1. If \( p \neq 2 \), \( \psi \) is of order 0, \( \mu \) is unramified, \( R \in \text{Sym}^{nd}_i \cap \text{GL}_i(\mathfrak{o}) \), \( \phi \) is the characteristic function of \( M_{m'}(\mathfrak{o}) \) and \( h \in \mathcal{I}_{m'}(\mu) \) satisfies \( h(k) = 1 \) for all \( k \in Sp_m(\mathfrak{o}) \), then \( \beta^\psi_R(1_{2m'}; f \otimes \phi) = 1 \).

2. If \( -\frac{1}{2} < R \mu < \frac{1}{2} \), then \( \beta^\psi_R(I_{m'}(\mu) \otimes \omega^\psi_R) = \mathcal{I}_{m'}(\mu\hat{\chi}^{\text{det} R}) \).

3. If \( \mu^2 = \alpha \), then \( \beta^\psi_R(A_{m'}(\mu) \otimes \omega^\psi_R) = A_{m'}(\mu\hat{\chi}^{\text{det} R}) \).

**7.5. The Archimedean Case.** When \( m = i + m' \), \( R \in \text{Sym}_m^{nd} \), \( T \in \text{Sym}_{m'}^{nd} \) and \( x \in M_{m'}(F) \), we put

\[
\begin{pmatrix} R & 0 \\ 0 & T \end{pmatrix} \cdot \begin{pmatrix} 1_i & x \\ 0 & 1_{m'} \end{pmatrix} = \begin{pmatrix} R & Rx \\ t_{xR}T + R[x] \end{pmatrix}.
\]
Lemma 7.6. Suppose that $F \simeq \mathbb{R}$, $m = i + m'$ and $R \in \text{Sym}^+_1$. Define $\varphi_R \in S(\mathcal{F}_i)$ by $\varphi_R(x) = e^{-2\pi \text{tr}(R|x|)}$ for $x \in \mathcal{F}_i$. Then there is a nonzero constant $E_R$ such that for $T \in \text{Sym}^+_{m'}$ and $\tilde{g}' \in \text{Mp}(W_{m'})$

$$\int_{\mathcal{F}_i} W_R^{(T)}(s(x)\tilde{g}')\overline{\omega_R^{\psi}(\tilde{g}')}\varphi_R(x) \, dx = E_R^{-1}(\det T)^{i/4} W_T^{(i-2)}(\tilde{g}').$$

Proof. Since the Gaussian is an eigenfunction for the action of $\tilde{K}_{m'}$ with eigencharacter $k' \mapsto J_{i/2}(k',1)^{-1}$ by [6], one can readily verify that

$$(\omega_R^{\psi}(\tilde{g}'))(x) = J_{i/2}(\tilde{g}',i_{m'})^{-1} e(\text{tr}(R|x|\tilde{g}'(i_{m'})))$$

for $x \in \mathcal{F}_i$ and $\tilde{g}' \in \text{Mp}(W_{m'})$. Put $Y = \exists \tilde{g}'(i_{m'})$. Since

$$W_R^{(T)}(s(x)\tilde{g}') = W_R^{(T)}\left(\tilde{m}\left(\begin{array}{c} i \\ 0 \\ 1_{m'} \end{array}\right)\right) = W^{(T)}(\tilde{g}')$$

by (5.2), the left hand side is equal to

$$\frac{\det(R \oplus T)^{i/2}}{J_{i/2}(\tilde{g}',1_{m'})} \int_{\mathcal{F}_i} e\left(\text{tr}\left(R_{T,x}\left(\begin{array}{c} i \\ \tilde{g}'(i_{m'}) \end{array}\right)\right)\right) \left(\frac{e(\text{tr}(R[x]|\tilde{g}'(i_{m'})))}{J_{i/2}(\tilde{g}',i_{m'})}\right) \, dx$$

$$= W^{(T-2)}(\tilde{g}')(\det R)^{i/2} \frac{\det(R \oplus T)^{i/2}}{e^{-2\pi \text{tr}(R|Y)|}} \int_{\mathcal{F}_i} e^{-4\pi \text{tr}(R|x|Y)} \, dx.$$ 

The factor $(\det Y)^{i/2} \int_{\mathcal{F}_i} e^{-2\pi \text{tr}(R|x|Y)} \, dx$ is a constant independent of $Y$. □

7.6. The global case. For the rest of this section $D$ is a totally indefinite quaternion algebra over a totally real number field $F$. Fix $S \in S^+_1$. Take a maximal abelian subgroup $\mathcal{A}$ of $\mathcal{N}_i(\mathcal{A})$ to which the character $\psi_S$ has an extension $\psi_S$. The Schrödinger representation is equivalent to $\text{Ind}_{\mathcal{A}}^{\mathcal{N}_i(\mathcal{A})} \psi_S$. Having chosen $\mathcal{A} = Y_i(\mathcal{A}) \oplus Z_i(\mathcal{A})$, we obtain the Schrödinger model of the Weil representation $\omega_S^{\psi} \simeq \otimes \psi_S$ realized on $S(\mathcal{X}_i(\mathcal{A}))$. If we choose $\mathcal{A} = \mathcal{N}_i(F)Z_i(\mathcal{A})$, then the space $\text{Ind}_{\mathcal{A}}^{\mathcal{N}_i(\mathcal{A})} \psi_S = C_{\psi_S}^\infty (\mathcal{N}_i(F) \setminus \mathcal{N}_i(\mathcal{A}))$ consists of smooth functions on $\mathcal{N}_i(F) \setminus \mathcal{N}_i(\mathcal{A})$ on which the center $Z_i(\mathcal{A})$ acts by $\psi_S$. The equivariant isomorphism $S(\mathcal{X}_i(\mathcal{A})) \simeq C_{\psi_S}^\infty (\mathcal{N}_i(F) \setminus \mathcal{N}_i(\mathcal{A}))$ is given by

$$\Theta(\omega_S^{\psi}(v)\varphi) = \sum_{x \in X_i(F)} (\omega_S^{\psi}(v)\varphi)(x).$$

Denote by $\omega_S^{\psi} \simeq \otimes \psi_S^{\psi}$ the finite part of the global Weil representation. For $\phi \in S(\mathcal{X}_i(\mathcal{A}))$ we define a Schwartz function $\phi_S$ on $X_i(\mathcal{A})$ by

$$\phi_S(x) = \phi(x)\varphi_S(x_{\infty}), \quad \varphi_S(x_{\infty}) = e^{-2\pi \sum_{v \in S_{\infty}} r(S[x_v]B_n)}$$

for $x = (x_v) \in X_i(\mathcal{A})$. We write $\rho$ for the right regular action of $G_n(\mathcal{A}_f)$ on $\mathcal{X}_i^n$. For $F \in \mathcal{X}_i^n$, we define the $(S,\phi)^{th}$ Fourier-Jacobi coefficient of $F$ by

$$F_{\phi}^{(S)}(g') = \int_{\mathcal{N}_i(F) \setminus \mathcal{N}_i(\mathcal{A})} F(vg')\Theta(\omega_S^{\psi}(vg')\phi_S) \, dv.$$
Lemma 7.7. Let $S \in S_i^+$. Put $n' = n - i$. Let
\[ F(g) = \sum_{B \in S_i^+} w_B(g_f, F) W_B^{(\ell + n)}(g_\infty), \]
be the Fourier expansion of $F \in \mathcal{F}_n^+$. Then $F^S_\phi(g')$ is equal to
\[ \sum_{\Xi \in S_{n'}^+} N_{F/\mathbb{Q}}(\nu(\Xi))^{i/2} W^{(\ell + n')}_{\Xi}(g'_\infty) \int_{X_i(\mathbb{A})} w_{S \otimes \Xi}(xg'_\Xi, F)(\omega^{(\psi)}_S(g'_\Xi)\phi(x)) dx \]
up to a nonzero constant multiple. Moreover, $F \in \mathcal{C}^{n_+}_F$ if and only if $(\rho(\Delta)F)_\phi^S \in \mathcal{C}^{n'}_{F_n}$ for all $\Delta \in G_n(\mathbb{A}_F)$, $S \in S_i^+$ and $\phi \in S(X_i(\mathbb{A}_F))$. 

Remark 7.8. When $D \simeq M_2(F)$, one can prove an analogous result for $R \in \text{Sym}_i^+$ and $F \in \tilde{\mathfrak{F}}^{(m)}_{\ell+m}$ in the same way.

Proof. We abuse notation in writing $w_B^{(\ell + n)}(g, F) = w_B(g_f, F) W_B^{(\ell + n)}(g_\infty).$ The calculation in the proof of [13, Lemma 4.1] shows that
\[ F^S_\phi(g') = \sum_{\Xi \in S_{n'}^+} \int_{X_i(\mathbb{A})} w_{S \otimes \Xi}^{(\ell + n)}(xg'_\Xi, F)(\omega^{(\psi)}_S(g'_\Xi)\phi_S(x)) dx \]
for $g' \in G_{n'}(\mathbb{A})$. Employing Lemma 7.4, we arrive at the stated formula.

The “only if” part is clear. The proof of the other direction is similar to that of the Saito-Kurokawa lifting (cf. Proposition 1.3 of [21] and Section 9 of [22]). Recall that we regard $G_{n'}$ as a subgroup of $G_n$ as in §4.1. Note that
\[ F(g) = \sum_{S \in S_i^+} F^S(g), \quad F^S(g) = \sum_{\Xi \in S_{n'}^+} \sum_{u \in X_i(F)} w_{S \otimes \Xi}^{(\ell + n)}(ug, F). \]
Since the subgroups $P_n(F)$ and $G_{n'}(F)$ generate $G_n(F)$, if $F^S$ is left invariant under $G_{n'}(F)$ for all $S \in S_i^+$, then $F \in \mathcal{C}^{n_+}_F$. Fix $A \in \text{GL}_d(D(\mathbb{A}_\infty))$. Put $A' = m(\text{diag}[A, 1_{n'}])$ and $C = e^{-2\pi \sum_{v \mid \mathbb{A}_v} \tau(S[A_v]B_v) \prod_{v} \nu(S)^{(\ell + n'}/2)_{\psi}(A_v)_{\ell + n'}}$. Then for $v \in N(\mathbb{A}_\infty)$ and $g' \in G_{n'}(\mathbb{A}_\infty)$
\[ W_{S \otimes \Xi}^{(\ell + n')}(uvA'g') = CN_{F/\mathbb{Q}}(\nu(\Xi))^{i/2}[\omega^{(\psi)}_S(vg')\varphi_\infty](u)W_{\Xi}^{(\ell + n')}(g') \]
by (2.1) and (2.3). Let $\mathcal{C}$ be a compact subgroup of $G_{n'}(\mathbb{A}_F) \times N_i(\mathbb{A}_\mathfrak{f})$ under which $F^S$ is right invariant. Take an orthonormal basis $\{\phi^1, \ldots, \phi^k\}$ of the finite dimensional space $\{\phi \in \omega^{\psi}_{S} \mid \omega^{\psi}_{S}(c)\phi = \phi \text{ for } c \in \mathcal{C}\}$. Then for $A \in \text{GL}_d(D(\mathbb{A}_\infty))$, $v \in N(\mathbb{A}_\infty)$ and $g' \in G_{n'}(\mathbb{A}_\infty)$
\[ F^S(vA'g') = C \sum_{l=1}^{k} \Theta(\omega^{\psi}_{S}(vg')\phi^l_\mathfrak{f}) \mathcal{F}_{\phi^l}^S(g'). \]
Therefore, if $(\rho(\Delta)F)^S_\phi$ is left invariant under $G_{n'}(F)$ for all $\Delta \in G_n(\mathbb{A}_F)$ and $\phi \in S(X_i(\mathbb{A}_F))$, then so is $F^S$. □
Lemma 7.9. Let \( \{C_B\}_{B \in S^n_{n'}} \in T^n_{n'}(\mu_\ell) \). Then \( \{C_B\} \in C^n_{n'+}(\mu_\ell) \) if and only if \( \{C_S \in S^n_{n'} \}
exists \in C^1_{n+}(\mu_\ell) \) for all \( S \in S_{n-1}^n \).

Proof. Taking Corollary \( \circ \) into account, we define a surjection

\[ \beta_S^{\psi'} = \bigotimes \beta_S^{\psi'} : \beta_A(\mu_\ell) \otimes \omega_S^{\psi'} \rightarrow \beta_A(\mu_\ell \hat{C}_S) \].

Lemma \( \otimes \) says that \( F_{n+}(f, \{C_B\}) \in \mathfrak{T}_{n+}^n \) for all \( f \in A_n(\mu_\ell) \). In view of Lemma \( \circ \) and \( \otimes \), the \((S, \psi)\)'th Fourier-Jacobi coefficient of \( F_{n+}(f, \{C_B\}) \) equals \( F_{n+}(\beta_S^{\psi'}(f \otimes \omega), \{C_S \in S^n_{n'} \}) \) up to a nonzero constant multiple. The last statement of Lemma \( \triangle \) finally proves the equivalence.

\[ \square \]

8. Proofs of Theorems \( \square \) and \( \triangle \)

8.1. Theta lifts from \( \text{Mp}(W_m) \). We give a brief account of theta correspondence for the dual pair \( \text{Mp}(W_m) \times \text{O}(V) \). For a detailed treatment one can consult \cite{34, 35, 36, 37, 38, 39}. Let \((V, q_V)\) be a quadratic space of dimension \( l \). In the case of interest in this paper \( m = 1 \) and \( V = V_D \) or \( V = Fe \oplus V_D \oplus Fe' \). In the former case \( G^+(V) \simeq D^k \) and in the latter case \( G^+(V) \simeq S^k \). We define the symplectic vector space \((\mathbb{W}, \langle , \rangle, \gg)\) of dimension \( 2ml \) by \( \mathbb{W} = V \otimes W_m \) and \( \langle , \rangle, \gg \rangle = ( , ) \otimes ( , ) \).

We have natural homomorphisms

\[ \text{Sp}(W_m) \hookrightarrow \text{Sp}(\mathbb{W}), \quad G^+(V) \rightarrow \text{SO}^+(V) \rightarrow \text{O}(V) \hookrightarrow \text{Sp}(\mathbb{W}). \]

The groups \( \text{O}(V) \) and \( \text{Sp}(W_m) \) form a dual pair inside \( \text{Sp}(\mathbb{W}) \).

Fix \( \eta \in F^\times \). We obtain the representation \( \omega_V^{\psi'} = \bigotimes \omega_V^{\psi'} \) by pulling back to \( \text{O}(V, A) \times \text{Mp}(W_m) \) or \( \text{G}^+(V, A) \times \text{Mp}(W_m) \). The metaplectic double cover \( \text{Mp}(\mathbb{W}) \) of \( \text{Sp}(\mathbb{W}) \) associated to \( \psi' \). Note that \( \omega_V^{\psi'} \simeq \omega_V^{\psi'} \), where \( \eta \) is the space \( V \) equipped with the quadratic form \( \eta V \). When \( m = l = 1 \), the local Weil representation \( \omega_V^{\psi'} = \omega_1^{\psi'} \) is realized in \( \text{Sp}(F) \) and is the direct sum of two irreducible representations: \( \omega_1^{\psi'} = \omega_1^{\psi'} \oplus \omega_1^{\psi'} \), where \( \omega_1^{\psi'} \) (resp. \( \omega_1^{\psi'} \)) consists of the even (resp. odd) \( \omega_1^{\psi'} \) functions in \( \text{Sp}(F) \). Given an irreducible admissible genuine representation \( \sigma_v \) of \( \text{Mp}(W_m)_v \) the maximal quotient of \( \omega_V^{\psi'} \) on which \( \text{Mp}(W_m)_v \) acts as a multiple of \( \sigma_v \) is of the form \( \sigma_v \otimes \Theta_{V_v}^{\psi'}(\sigma_v) \), where \( \Theta_{V_v}^{\psi'}(\sigma_v) \) is a representation of \( \text{O}(V_v) \). We say that \( \Theta_{V_v}^{\psi'}(\sigma_v) \) is zero if \( \sigma_v \) does not occur as a quotient of \( \omega_V^{\psi'} \). Let \( \Theta_{V_v}^{\psi'}(\sigma_v) \) be the maximal semisimple quotient of \( \Theta_{V_v}^{\psi'}(\sigma_v) \). Then \( \Theta_{V_v}^{\psi'}(\sigma_v) \) is either zero or irreducible by the Howe conjecture, which was proved by Wee Teck Gan and Takeda \cite{39}.

It turns out that there is a natural \( \text{Sp}_{2ml}(F) \)-invariant map \( \Theta : \omega_V^{\psi'} \rightarrow \mathbb{C} \). Let \( \sigma \) be an irreducible genuine cuspidal automorphic representation of
Mp(W_m). For \( h \in \sigma \) and \( \phi \in \omega_V^\psi \) we set
\[
\theta_V^\psi(g; h, \phi) = \int_{Sp_m(F) \backslash Sp_m(\mathbb{A})} h(\tilde{x})\Theta(\omega_V^\psi(\tilde{x}, g) \phi) \, dx,
\]
where \( \tilde{x} \) denotes a preimage of \( x \) in \( Mp(W_m) \). Note that since the integrand is a product of two genuine functions on the double cover of \( Sp_m(\mathbb{A}) \), it is defined on \( Sp_m(\mathbb{A}) \). Then \( \theta_V^\psi(h, \phi) \) is an automorphic form on \( O(V) \). We write \( \theta_V^\psi(\sigma) \) for the subspace of the space of automorphic forms on \( O(V) \) spanned by \( \theta_V^\psi(h, \phi) \) for all \( h \in \sigma \) and \( \phi \in \omega_V^\psi \). It is a simple consequence of the Howe conjecture that if \( \theta_V^\psi(\sigma) \) is nonzero and contained in the space of square-integrable automorphic forms on \( O(V) \), then \( \theta_V^\psi(\sigma) \simeq \otimes'_n \theta_{V_n}^\psi(\sigma_n^\psi) \). The correspondence \( \Pi \mapsto \theta_{W_n}^\psi(\Pi) \) in the opposite direction can be defined similarly.

Let \( \mathcal{A}_{00} \) denote the space of genuine cuspidal forms on \( Mp(W_1) \) orthogonal to elementary theta series of the Weil representation \( \omega_\eta^\psi \) for any \( \eta \in F^\times \). The following result can be deduced from the Rallis inner product formula.

**Proposition 8.1** (Theorem 2.8 of [3]). Assume that \( m = 1 \) and \( l \geq 5 \). Let \( \sigma \) be an irreducible genuine cuspidal automorphic representation in \( \mathcal{A}_{00} \). Then \( \theta_V^\psi(\sigma) \) is nonzero if and only if \( \Theta_{V_n}^\psi(\sigma_v) \) is nonzero for all \( v \).

### 8.2. The work of Waldspurger

The space \( \mathcal{A}_{00} \) satisfies multiplicity one but does not satisfy strong multiplicity one: there are nonequivalent cuspidal automorphic representations \( \sigma \) and \( \sigma' \) whose local components are equivalent for almost all places. We say that such \( \sigma \) and \( \sigma' \) are nearly equivalent.

Waldspurger has described the near equivalence classes of representations in \( \mathcal{A}_{00} \). In his papers [64, 67] he defined a surjective map \( Wd_{\psi_v} \) from the set of irreducible admissible genuine unitary representations of \( Mp(W_1)_v \) which are not equivalent to \( \omega_\eta_{V_v}^\psi \) for any \( \eta \in F_v^\times \) to the set of irreducible infinite dimensional unitary representations of \( PGL_2(F_v) \). Let \( \sigma_v \) be such a representation of \( Mp(W_1)_v \). Let \( V_v^+ \) (resp. \( V_v^- \)) stand for a three dimensional split (resp. anisotropic) quadratic space over \( F_v \) of discriminant 1. Then precisely one of the representations \( \theta_{V_v^+}^\psi(\sigma_v) \) and \( \theta_{V_v^-}^\psi(\sigma_v) \) is nonzero. Set \( Wd_{\psi_v}(\sigma_v) = \theta_{V_v^+}^\psi(\sigma_v) \) if it is nonzero. Otherwise \( Wd_{\psi_v}(\sigma_v) \) corresponds to \( \theta_{V_v^-}^\psi(\sigma_v) \) via the Jacquet-Langlands correspondence.

Given an irreducible infinite dimensional unitary representation \( \pi_v \) of \( PGL_2(F_v) \), we put \( \Pi_{\psi_v}(\pi_v) = Wd_{\psi_v}^{-1}(\pi_v) \). If \( \pi_v \) is a discrete series, then \( \# \Pi_{\psi_v}(\pi_v) = 2 \). Otherwise \( \Pi_{\psi_v}(\pi_v) \) is a singleton. In the first case the set \( \Pi_{\psi_v}(\pi_v) \) has a distinguished element \( \sigma_{V_v^+}^\psi(\pi_v) \), which is characterized by the fact that \( \sigma_{V_v^+}^\psi(\pi_v) \otimes \pi_v \) is a quotient of the Weil representation \( \omega_{V_v^+}^\psi \). The other element of \( \Pi_{\psi_v}(\pi_v) \) is denoted by \( \sigma_{V_v^-}^\psi(\pi_v) \): it is characterized by the
fact that $\sigma^{\psi_v}_v(\pi_v) \otimes \pi_{v1}$ is a quotient of $\omega^{\psi_v}_{v1}$, where $\pi_{v1}$ is the Jacquet-Langlands correspondence of $\pi_v$. In the second case we shall let $\sigma^{\psi_v}_v(\pi_v)$ be the unique element in $\Pi^{\psi_v}_v(\pi_v)$, and set $\sigma^{\psi_v}_v(\pi_v) = 0$. This partition of representations of $\mathrm{Mp}(W)_v$ into packets and their parametrization in terms of representations of $\mathrm{PGL}_2(F_v)$ depend on the choice of $\psi_v$. But it is quite explicit. Denote the discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ with extremal weights $\pm 2\ell$ by $D_{2\ell}$ and the lowest weight representation of the real metaplectic group of rank $m$ with lowest $K$-type $k \mapsto J((2\ell+1)/2(k,\sqrt{-1}1_m))^{-1}$ by $\mathcal{D}^{(m)}_{(2\ell+1)/2}$.

**Proposition 8.2** (Propositions 4, 5, 9 of [17]). (1) If $\pi_p$ is an irreducible principal series representation $I(\mu_p, \mu_p^{-1})$, then $\sigma^{\psi_v}_v(\pi_p) \simeq I^{\psi_v}_1(\mu_p)$.

(2) If $\pi_p \simeq A(\alpha_p^{1/2}, \alpha_p^{-1/2})$, then $\sigma^{\psi_v}_v(\pi_p) \simeq A^{\psi_v}_1(\alpha_p^{1/2})$.

(3) If $\mu_p^2 = \alpha_p$, $\mu_p \neq \alpha_p^{1/2}$ and $\pi_p \simeq A(\mu_p, \mu_p^{-1})$, then $\sigma^{\psi_v}_v(\pi_p) \simeq A^{\psi_v}_1(\mu_p)$.

(4) If $v \in \mathcal{S}_\infty$, then $\sigma^{\psi_v}_v(D_{2\kappa_v}) \simeq \mathcal{D}^{(1)}_{(2\kappa_v+1)/2}$.

Bear in mind the assumption that $\psi_v = e|_{F_v}$ for $v \in \mathcal{S}_\infty$. Given an irreducible cuspidal automorphic representation $\pi = \otimes'_v \pi_v$ of $\mathrm{PGL}_2(\mathbb{A})$, we define a set of irreducible unitary representations of $\mathrm{Mp}(W)_\mathbb{A}$ by

$$\Pi^{\psi}_v(\pi) = \{ \otimes'_v \sigma^{\psi_v}_v(\pi_v) \mid \epsilon_v \in \{ \pm \} \},$$

for all most all $v$, $\epsilon_v = \pm$.

For given $\sigma = \otimes'_v \sigma^{\psi_v}_v(\pi_v) \in \Pi^{\psi}_v(\pi)$, we set $\epsilon(\sigma) = \prod_v \epsilon_v$. Corollaries 1 and 2 on p. 286 of [17] say that

$$\sigma^{A_{00}} \simeq \bigoplus_{\pi \in \Pi^{\psi}(\pi) : \epsilon(\sigma)=\epsilon(1/2,\pi)} \sigma,$$

where the sum ranges over all irreducible cuspidal automorphic representations $\pi$ of $\mathrm{PGL}_2(\mathbb{A})$ such that $L(\frac{1}{2}, \pi \otimes \chi_t) \neq 0$ for some $t \in F^\times$.

This result includes the special case of Theorem 1.3 in which $m = 1$.

**Lemma 8.3.** $A^{\psi_1}_1(\mu_f)$ appears in $\tilde{\mathcal{C}}^{(1)}_{(2\kappa_v+1)/2}$ if and only if $\otimes'_p A(\mu_p, \mu_p^{-1})$ appears in $\mathcal{C}_{2\kappa}$ and $\mu_f(-1)(-1)^{\sum_\epsilon \kappa_\epsilon} = 1$.

**Proof.** Put

$$\pi = (\otimes_{v \in \mathcal{S}_\infty} D_{2\kappa_v}) \otimes (\otimes'_p A(\mu_p, \mu_p^{-1})), \quad \sigma = (\otimes_{v \in \mathcal{S}_\infty} \mathcal{D}^{(1)}_{(2\kappa_v+1)/2}) \otimes A^{\psi_1}_1(\mu_f).$$

If $\sigma$ is an irreducible cuspidal automorphic representation, then so is $\pi = \otimes'_v \mathrm{Wd}_{\psi_v}(\sigma_v)$. Since $\pi_v$ is a discrete series for $v \in \mathcal{S}_\infty$, Theorem A.2 of [17] says that $L(\frac{1}{2}, \pi \otimes \chi_t) \neq 0$ for some $t$. Let $\mathcal{S}_\pi^-$ be the set of nonarchimedean primes $p$ of $F$ such that $\mu_p = \alpha_p^{1/2}$. Put $\ell^-_\pi = \# \mathcal{S}_\pi^-$. Then

$$\epsilon(1/2, \pi) = \mu_f(-1)(-1)^{\ell^-_\pi + \sum_\epsilon \kappa_\epsilon}.$$
Note that
\[ \sigma \simeq (\otimes_{p \in \mathbb{P}} \sigma_{\psi_p}^v(\pi_p)) \otimes (\otimes_{\nu \in \mathbb{N}} \sigma_{\psi_v}^v(\pi_v)) \in \Pi^v(\pi) \]
by Proposition 5.3. Thus \( \epsilon(\sigma) = \epsilon(1/2, \pi) \) is equivalent to the sign condition. We have the desired conclusion by \( \square \).

Remark 8.4. Lemma 5.3 is consistent with Lemma 5.6(\#).

8.3. Construction of an embedding \( A^\psi_m(\mu_R) \hookrightarrow \tilde{\mathcal{C}}^{(m)}_{(2m+n)/2} \). We now prove Theorem 8.2. We hereafter assume that \( m > 1 \). Fix \( R \in \text{Sym}^m_{m-1} \). Corollary 5.2 gives a \( \text{Mp}(W_1)_R \)-isointertwining surjective homomorphism
\[ \beta^\psi_R = \otimes_p \beta^\psi_p : A^\psi_m(\mu_R) \otimes \omega^\psi_R \to A^\psi_1(\mu_R^\circ \det R) \]
We associate to \( \phi \in \mathcal{S}(\mathcal{X}_{m-1}(\mathbb{A}_R)) \) the function \( \phi_R = \phi \otimes (\otimes_{\nu \in \mathbb{N}} \varphi_R) \in \mathcal{S}(\mathcal{X}_{m-1}(\mathbb{A})) \) and the theta function on \( \mathcal{X}_{m-1}(F) \setminus \mathcal{X}_{m-1}(\mathbb{A}) \) defined by
\[ \Theta(\omega_R(vg'))\phi_R = \sum_{\ell \in \mathcal{X}_{m-1}(F)} (\omega_R(vg')\phi_R)(l) \quad (v \in \mathcal{X}_{m-1}(\mathbb{A}), \ g' \in \text{Mp}(W_1)_\mathbb{A}). \]
The \((R, \phi)^{\text{th}}\) Fourier-Jacobi coefficient of \( F \in \tilde{\mathcal{C}}^{(m)}_{\ell} \) is defined by
\[ F^R(\ell') = \int_{\mathcal{X}_{m-1}(F) \setminus \mathcal{X}_{m-1}(\mathbb{A})} F(s(v)\ell') \Theta(\omega_R(vg')\phi_R) \, dv. \]

Recall the function \( F_{(2k+m)/2}(h, \{C_\xi\}) \) defined in Definition 5.3. Lemmas 6.4 and 6.5 give a nonzero constant \( E_R \) such that
\[ F_{(2k+m)/2}(h, \{C_\xi\})^R = E_R F_{(2k+1)/2}(\beta^\psi_R(h \otimes \bar{\phi}), \{C_R, \Gamma_{1/2}\}) \]
for all \( \{C_\xi\} \in \tilde{\mathcal{C}}^{(m)}_{(2k+1)/2}(\mu_R) \) and \( h \in A^\psi_m(\mu_R) \) in view of Remark 6.8.

Lemma 8.5. If \( \{c_\xi\} \in \tilde{\mathcal{C}}^{(1)}_{(2k+1)/2}(\mu_R) \), then \( \{c_{\eta, \det, \xi}\} \in \tilde{\mathcal{C}}^{(m)}_{(2k+m)/2}(\mu_R \chi_f^\eta) \) for all \( m \) and \( \eta \in F^+_k \).

Proof. The series \( i^m_{n}(h) = F_{(2k+m)/2}(h, \{c_{\eta, \det, \xi}\}) \) is convergent for all \( h \in A^\psi_m(\mu_R \chi_f^\eta) \) by Lemma 8.6 and the estimate of \( \{c_\xi\} \) given in Proposition 8.1 of [15], and so by Lemma 8.1(1) \( i^m_{n}(A^\psi_m(\mu_R \chi_f^\eta)) \subset \tilde{\mathcal{C}}^{(m)}_{(2k+m)/2} \). Lemma 8.6(\#), (8.3) and (8.4) show that
\[ i^m_{n}(h)^R = E_R i^m_{n}(h \otimes \bar{\phi}) \in \tilde{\mathcal{C}}^{(1)}_{(2k+1)/2} \]
for all \( R \in \text{Sym}^m_{m-1} \) and \( \phi \in \mathcal{S}(\mathcal{X}_{m-1}(\mathbb{A}_R)) \). Lemma 8.6 eventually concludes that \( i^m_{n}(A^\psi_m(\mu_R \chi_f^\eta)) \subset \tilde{\mathcal{C}}^{(m)}_{(2k+m)/2} \) (cf. Remark 8.8). \( \square \)

We have thus seen that the Fourier series given in Theorem 8.2 is a Hilbert-Siegel cusp form of weight \( \kappa + n/2 \). In Section 9 we will show that it reduces to the Duke-Imamoglu-Ikeda lift when \( F = \mathbb{Q}, \ m = 2n \) and \( h \) is right invariant under \( \prod_p Sp_{2n}(\mathbb{Z}_p) \).
8.4. Some lemma on quadratic forms. It remains to verify that the multiplicity of $A_{2k}^+(\mu_2)$ is at most one. The proof relies on certain technical results. When $\pi$ is an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ of the form $([\Xi, \mathbb{Z}])$, we put

$$\mathcal{S}_m = \{ \xi \in \mathrm{Sym}^+_m | L(1/2, \pi \otimes \chi^{\det \xi}) \neq 0, \chi_p^{\det \xi} \neq \mu_p \alpha_p^{-1/2} \text{ for } p \in \mathfrak{S}_\pi \}. $$

**Lemma 8.6.** If $\tilde{\mathcal{C}}^{(1)}_{(2k+1)/2}(\mu_2) \neq \{0\}$ and $\Xi \in \mathcal{S}^\pi_3$ represents $t_1, t_2 \in F^\times_+$, then there are a quadratic form $S \in \mathrm{Sym}^+_2$ representing $t_1$ and represented by $\Xi$ and a quadratic form $T \in \mathcal{S}_3^\pi$ representing both $S$ and $t_1 \oplus t_2$.

**Proof.** Choose vectors $x, y \in F^3$ such that $\Xi[x] = t_1$ and $\Xi[y] = t_2$. If $\Xi(x, y) = \langle x, y \rangle = 0$, then we can take $S = t_1 \oplus t_2$ and $T = \Xi$. Suppose that $\Xi(x, y) \neq 0$. Define functions $S : F^3 \rightarrow \mathrm{Sym}^2_2(F)$ and $T : F^3 \rightarrow \mathrm{Sym}^3_3(F)$ by

$$S(z) = \begin{pmatrix} \Xi(x, x) & \Xi(x, z) \\ \Xi(x, z) & \Xi(z, z) \end{pmatrix}, \quad T(z) = \begin{pmatrix} \Xi(x, x) & \Xi(x, z) & 0 \\ \Xi(x, z) & \Xi(z, z) & \Xi(y, z) \\ 0 & \Xi(y, z) & \Xi(y, y) \end{pmatrix}. $$

Clearly, $S(z)$ and $T(z)$ fulfill all the requirements besides the condition that $T(z) \in \mathcal{S}_3^\pi$. We define a quadratic form $Q$ of three variables by

$$Q[z] = \det T(z) - \Xi(x, x)\Xi(y, y) \Xi(z, z) - \Xi(x, y)\Xi(x, z)^2 - \Xi(x, z)\Xi(y, z)^2. $$

By a direct calculation $\det Q = -\Xi \cdot \Xi(x, y)^2 \Xi(x, y)^2 \Xi(x, y)^2 \in -F^\times_+$. Let $\mathfrak{S}_Q$ be the set of places of $F$ at which $Q$ is anisotropic. If $z \neq 0$ and $\Xi(x, z) = \Xi(y, z) = 0$, then $Q[z] = \Xi(x, x)\Xi(y, y)\Xi(z, z) \in F^\times_+$. Thus $\mathfrak{S}_Q$ consists of finite primes. Since $T(z) \simeq t_1 \oplus t_2 \oplus (t_1 t_2)^{-1}Q[z]$, if $Q[z] \in F^\times_+$, then $T(z)$ is totally positive definite. Then $Q$ represents any element $t \in F^\times_+$ such that $t \notin (\det \Xi)F^\times_+\mathfrak{S}_Q$ for all $p \in \mathfrak{S}_Q$.

Take $\eta \in F^\times_+\mathfrak{S}_Q$ such that $\eta \notin (\det \Xi)F^\times_+\mathfrak{S}_Q$ for all $p \in \mathfrak{S}_Q$. Since $\varepsilon(\frac{1}{2}, \pi \otimes \chi^{\det \xi}) = 1$ by Lemma 3.12 (cf. (3.3)), for any $\varepsilon > 0$ and a finite set $\mathfrak{S}$ of primes of $F$, Theorem 4 of [37] gives an element $t \in F^\times_+$ which satisfies $|t - \eta|_v < \varepsilon$ for $v \in \mathfrak{S}$ and such that $L\left(\frac{1}{2}, \pi \otimes \chi^{\det \xi}\right) \neq 0$. □

**Lemma 8.7.** Suppose that $\tilde{\mathcal{C}}^{(1)}_{(2k+1)/2}(\mu_2) \neq \{0\}$. For $\xi_0, \xi_3 \in \mathcal{S}_m^\pi$ there are $\xi_1, \xi_2 \in \mathcal{S}_m^\pi$ and $R_1, R_2, R_3 \in \mathrm{Sym}^+_m-1$ such that $R_i$ is represented by both $\xi_{i-1}$ and $\xi_i$ for all $i = 1, 2, 3$.

**Proof.** If $m \geq 3$, then $\xi_0 \oplus (-\xi_3)$ must have a totally isotropic subspace of dimension $m - 2$ and hence there are $\xi \in \mathrm{Sym}^+_m$ and $\xi' \in \mathcal{S}^\pi_2$ such that $\xi \simeq \xi_0 \oplus \xi''$ and $\xi' \simeq \xi_3 \oplus \xi''$. We may therefore assume that $m = 2$.

Set $\Xi = 1 \oplus \xi_0$ and $\Xi' = 1 \oplus \xi_3$. Choose $R_2 \in F^\times_+$ represented by $\Xi$ and $\Xi'$. Applying Lemma 3.10 to $\Xi, 1$ and $R_2$, we find a quadratic form $S \in \mathrm{Sym}^+_2$ representing $1$ and represented by $\Xi$ and find a quadratic form $T \in \mathcal{S}_3^\pi$ representing both $S$ and $1 \oplus R_2$. Put $R_1 = \det S$. Then $S \simeq 1 \oplus R_1$. There is $\xi_1 \in \mathcal{S}_2^\pi$ such that $T \simeq 1 \oplus \xi_1$. Then $\xi_1$ represents $R_2$. Since both $\Xi = 1 \oplus \xi_0$ and $\Xi' \simeq 1 \oplus \xi_1$ represent $S \simeq 1 \oplus R_1$, both $\xi_0$ and $\xi_1$ represent
5.6. Similarly, we can find a quadratic form $\xi_2 \in \mathcal{S}_2^\pi$ representing $R_2$ and find a totally positive element $R_3$ represented by both $\xi_2$ and $\xi_3$.

8.5. **Multiplicity of $A_m^{\psi_f}(\mu_f)$.** In light of Lemma 5.6.11, giving a $\text{Mp}(W_m)_{\mathfrak{f}}$-intertwining map from $A_m^{\psi_f}(\mu_f)$ into $\tilde{C}_{(2k+m)/2}^{(m)}(\mu_f)$ is equivalent to giving complex numbers $\{C_\xi\}_{\xi \in \text{Sym}_{\mathfrak{g}}^{\pi_f}} \in \tilde{C}_{(2k+m)/2}^{(m)}$. We now prove a stronger result.

**Lemma 8.8.** Suppose that there is a $\text{Mp}(W_m)_{\mathfrak{f}}$-intertwining embedding $i : A_m^{\psi_f}(\mu_f) \hookrightarrow \tilde{C}_{(2k+m)/2}^{(m)}$. Then $\mu_f(-1)\sum_{v \in \mathcal{E}_{\infty}} \kappa_v = 1$, $\otimes'_p A(\mu_p, \mu_p^{-1})$ occurs in $\mathcal{C}_{2k}$ and there is $\{C_\xi\} \in \tilde{C}_{(1)}^{(1)}(\mu_f)$ such that $i(h) = \mathcal{F}_{(2k+m)/2}(h, \{C_{\det \xi}\})$ for all $h \in A_m^{\psi_f}(\mu_f)$. In particular, $\tilde{C}_{(2k+m)/2}^{(m)}(\mu_f) = 1$.

**Proof.** As mentioned above, Lemma 5.6.11 gives $0 \neq \{C_\xi\} \subset \tilde{C}_{(2k+m)/2}^{(m)}(\mu_f)$ such that $i(h) = \mathcal{F}_{(2k+m)/2}(h, \{C_{\xi}\})$. Hence $\{C_R\} \subset \tilde{C}_{(2k+m)/2}^{(m)}(\mu_f, \chi^{\det R})$ for all $R \in \text{Sym}_{m-1}^\pi$ by (8.2) and (8.3). This together with Lemma 8.7 proves one implication of Theorem 8.11

Fix a basis vector $\{e_1\}$ of the one dimensional vector space $\tilde{C}_{(2k+1)/2}^{(1)}(\mu_f)$. For each $R \in \text{Sym}_{m-1}^\pi$ Lemma 8.5.3 gives a complex number $\delta_R$ such that $C_{R \otimes t} = \delta_R c_{t \det R}$ for all $t \in F \cdot \chi^{\det R}$. Let $\xi \in \text{Sym}_{\mathfrak{g}}^{\pi_f}$. If there exists $a \in \text{GL}_m(F)$ such that $\xi = (R \oplus t)[a]$, then

$$C_\xi = \delta_R c_{t \det R} \xi = \delta_R c_{t \det R} \xi = \delta_R c_{t \det R} \xi = \delta_R c_{t \det R} \xi = \delta_R c_{t \det R} \xi$$

by Lemma 8.5.2. Lemma 8.5.3 tells us that $C_\xi \neq 0$ only if $\xi \in \mathcal{S}_m^\pi$. Lemma 8.7 now says that

$$\frac{C_{\xi_0}}{c_{\det \xi_0}} = \delta_{R_1} = \frac{C_{\xi_1}}{c_{\det \xi_1}} = \delta_{R_2} = \frac{C_{\xi_2}}{c_{\det \xi_2}} = \delta_{R_3} = \frac{C_{\xi_3}}{c_{\det \xi_3}}$$

for all $\xi_0, \xi_3 \in \mathcal{S}_m^\pi$, which completes our proof.

Let $\mathcal{S}_{\text{cusp}}(\text{Mp}(W_m))$ be the space of cusp forms on $\text{Sp}_m(F) \backslash \text{Mp}(W_m)_{\mathfrak{f}}$ and $\mathfrak{g}$ the complexified Lie algebra of $\text{Mp}(W_m)_{\mathfrak{f}}$.

**Proposition 8.9.**

1. $\tilde{C}_\ell^{(m)} \subset \mathcal{S}_{\text{cusp}}(\text{Mp}(W_m))$.

2. If $e$ is a lowest weight vector of $\bigotimes_{v \in \mathcal{E}_{\infty}} \mathcal{D}_{\ell_v}^{(m)}$ and $\iota : \bigotimes_{v \in \mathcal{E}_{\infty}} \mathcal{D}_{\ell_v}^{(m)} \rightarrow \mathcal{S}_{\text{cusp}}(\text{Mp}(W_m))$ is a $(\mathfrak{g}, \hat{\mathcal{C}}_m)$ intertwining map, then $\iota(e) \in \tilde{C}_\ell^{(m)}$.

**Proof.** Lemma 5 of [8] says that Hilbert-Siegel cusp forms are cusp forms in the sense of Langlands, which just amounts to (1). By Lemma 7 of [8] $\iota(e) \hat{\Delta}$ is a holomorphic function on $\mathcal{H}_m^d$. We may assume that $m > 1$ as $\iota(e) \in \tilde{C}_\ell^{(m)}$ is a part of the definition if $m = 1$. Then $\iota(e)$ is a Hilbert-Siegel modular...
form by the Koecher principle. Since \( \vartheta(e) \in \mathcal{S}_{\text{cusp}}(\text{Mp}(W_m)) \), Proposition A4.5(2) of [18] says that \( \vartheta(e) \in \tilde{\mathcal{C}}^{(m)}_f \) as expected. 

We conclude this section by giving the following characterization.

**Corollary 8.10.** Let \( \Pi = \otimes_v \pi_v \) be an irreducible cuspidal automorphic representation of \( \text{Mp}(W_m) \). Assume that there is a character \( \mu_\ell = \prod_p \mu_p \in \Omega(\mathbb{A}_F^\times) \) with \( -\frac{1}{2} < \Re \mu_p \leq \frac{1}{2} \) satisfying the following conditions:

- \( \Pi_v \simeq D^{(m)}_{(2\kappa_v+m)/2} \) with \( \kappa_v \in \mathbb{N} \) for every \( v \in \mathcal{S}_\infty \);
- \( \Pi_p \) is equivalent to a subrepresentation of \( I_{\infty}^p(\mu_p) \) for every \( p \).

Then the unique irreducible subrepresentation of \( \otimes'_p I(\mu_p, \mu_p^{-1}) \) is a summand of \( \mathcal{C}_2, -1^{\sum_{v \in E_{\infty}} \kappa_v} \mu_\ell(-1) = 1 \) and \( \Pi \) is generated by \( i_m^!(A^{\text{ef}}_m(\mu_\ell)) \).

This can be derived as a corollary from Lemma 8.3 and Proposition 8.4.

9. TRANSFER TO INNER FORMS

We retain the notation of [27]. Thus \( V = Fe \oplus V_D \oplus F' \) and \( G^+(V) \simeq G_1 \). In the first half of this section we switch to a local setting. Thus \( F = F_v \) is a local field of characteristic zero.

9.1. **The Schrödinger model vs. the mixed model.** The Weil representation \( \omega^\psi_V \) can be realized on \( \mathcal{S}(V) \) and has the following formulas:

\[
(9.1) \quad (\omega^\psi_V(\zeta \tilde{m}(t))\Phi)(v) = \zeta \gamma^\psi(t)t^{3/2}\Phi(tv),
\]

\[
(9.2) \quad (\omega^\psi_V(\tilde{n}(b))\Phi)(v) = \psi(bq_V(v))\Phi(v),
\]

\[
(9.3) \quad (\omega^\psi_V(\beta)\Phi)(v) = \Phi(\vartheta(\beta)^{-1}v)
\]

for \( \zeta \in \mu_2, t \in F^\times, b \in F, \beta \in G_1 \) and \( v \in V \). Since the map

\[
(\mathcal{F}\Phi)(x;u,u') = \int_F \Phi(re + x + ue')\psi(ru') \, dr
\]

is a \( \mathbb{C} \)-linear isomorphism from \( \mathcal{S}(V) \) onto \( \mathcal{S}(V_D \oplus F^2) \), we can define the action \( \Omega^\psi_V \) of \( \text{Mp}(W_1) \times G_1 \) on \( \mathcal{S}(V_D \oplus F^2) \) so that

\[
\Omega^\psi_V(\tilde{x},g)\mathcal{F}\Phi = \mathcal{F}(\omega^\psi_V(\tilde{x},g)\Phi), \quad (\tilde{x},g) \in \text{Mp}(W_1) \times G_1.
\]

The following formulas are derived easily or read of from Lemma 46 of [27]:

\[
(9.4) \quad (\Omega^\psi_V(\text{d}(t)\text{m}(A))\varphi)(x;u,u') = \left[ \frac{\nu(A)}{t} \right] \varphi\left( A^{-1}xA; \frac{\nu(A)u}{t}, \frac{\nu(A)u'}{t} \right),
\]

\[
(9.5) \quad (\Omega^\psi_V(\text{n}(z))\varphi)(x;u,u') = \psi(-\tau(xz + uz)v(x - uz;u,u'),
\]

\[
(9.6) \quad (\Omega^\psi_V(\tilde{m}(t))\varphi)(x;u,u') = \gamma^\psi(t)t^{3/2}\varphi(tx;tu,t^{-1}u'),
\]

\[
(9.7) \quad (\Omega^\psi_V(\tilde{n}(b))\varphi)(x;u,u') = \psi(-bv(x))\varphi(x;u,u' + ub),
\]

\[
(9.8) \quad (\Omega^\psi_V(\text{s}(J))\varphi)(x;u,u') = \gamma^\psi_\Delta \int_{D^+} \varphi(y;-u',u)\psi(\tau(xy)) \, dy
\]
for $A \in D^\times$; $t \in F^\times$; $z, x \in D_-$ and $b, u, u' \in F$, where $\gamma^\psi_D$ is a certain 8th root of the unitary and $dy$ is the self-dual Haar measure on $D_-$ with respect to the Fourier transform defined by

$$(\mathcal{F}_D \phi)(x) = \int_{D_-} \phi(y) \psi(\tau(xy)) dy, \quad \phi \in \mathcal{S}(D_-) .$$

Remark 9.1. In the notation of [22] $n_2(z) = n(-z)$ (cf. Lemma 9.2).

9.2. Compatibility of Jacquet integrals. We first discuss the $p$-adic case. Recall that $\hat{\chi}^{-1}$ corresponds to $F(\sqrt{-1})/F$ via class field theory.

Lemma 9.2. Let $h \in I^\psi_1(\mu)$ and $\Phi \in \mathcal{S}(V)$. If $\Re \mu > -\frac{3}{2}$, then the integral

$$\Gamma^\psi(g; h \otimes \Phi) = L(3/2, \mu \hat{\chi}^{-1})^{-1} \int_{\mathfrak{g}_1 \setminus \mathbf{SL}_2(F)} h(\tilde{x})(\omega^\psi_V(\tilde{x}, g) \Phi)(e) dx$$

is absolutely convergent. It gives a $\text{Mp}(W_1)$-invariant and $\mathcal{G}_1$-intertwining map $\Gamma^\psi_{1} : I^\psi_1(\mu) \otimes \omega^\psi_V \rightarrow J_1(\mu \hat{\chi}^{-1})$. If $F$ is not dyadic, $D \simeq M_2(F)$, $\psi$ is of order 0, $\mu$ is unramified, $\Phi$ is the characteristic function of $\phi \otimes \mathcal{R}_1 + \phi'$ and $h(k) = 1$ for all $k \in \mathbf{SL}_2(o)$, then $\Gamma^\psi_1(1_2; h \otimes \Phi) = 1$.

Proof. The integral defining $\Gamma^\psi(g; h \otimes \Phi)$ makes sense by (9.2) and equals

$$\int_{F^\times} \int_{\mathbf{SL}_2(o)} h(\tilde{a})(\omega^\psi_V(\tilde{a}, g) \Phi)(e)|a|^{-2} da$$

by (9.1) and (9.3). It is absolutely convergent for $\Re \mu > -\frac{3}{2}$. It follows from (9.1), (9.3) and Lemma 9.2 that for $A \in D^\times$, $t \in F^\times$ and $z \in D_-$

$$L(3/2, \mu \hat{\chi}^{-1}) \Gamma^\psi(d(t) \mathbf{m}(A) n(z); h \otimes \Phi) = \int_{\mathfrak{g}_1 \setminus \mathbf{SL}_2(F)} h(\tilde{x})(\omega^\psi_V(\tilde{x}, g) \Phi)(t \nu(A)^{-1} e) dx$$

by $$(\hat{\chi}^{-1} - \mu \alpha^{3/2})(t^{-1} \nu(A)) \int_{\mathfrak{g}_1 \setminus \mathbf{SL}_2(F)} h(\tilde{x})(\omega^\psi_V(\tilde{x}, g) \Phi)(t \nu(A)^{-1} e) dx$$

Therefore $\Gamma^\psi(h \otimes \Phi) \in J_1(\mu \hat{\chi}^{-1})$. \hfill \Box

Recall that $J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{GL}_2(D)$ in the quaternion case.

Lemma 9.3. Let $h \in I^\psi_1(\mu)$, $\Phi \in \mathcal{S}(V)$ and $\Xi \in D^{\text{reg}}$. Put $\varphi = \mathcal{F} \Phi \in \mathcal{S}(V_D \oplus F^2)$. If $\Re \mu > -\frac{3}{2}$, then

$$\Gamma^\psi(J_1; h \otimes \Phi) = L(3/2, \mu \hat{\chi}^{-1})^{-1} \int_{\mathbf{SL}_2(F)} h(\tilde{x})(\Omega^\psi_V(\tilde{x}) \varphi)(0; 1, 0) dx.$$
Proof. The product $L(3/2, \mu \overline{\chi}_2^{-1}) \Gamma^V(J_1; h \otimes \Phi)$ equals
\[
\int_{\mathcal{F}_1 \setminus SL_2(F)} h(\tilde{x})(\omega_V^2(\tilde{x}, J_1) \Phi)(e') \, dx = \int_{\mathcal{F}_1 \setminus SL_2(F)} h(\tilde{x})(\omega_V^2(\tilde{x}) \Phi)(e') \, dx
\]
by (1.3) and Lemma 2.2. The Fourier inversion says that
\[
(\mathcal{F}^{-1} \varphi)(re + x + r'e') = \int_F \varphi(x; r') \psi(-ru) \, du.
\]
We use this formula and (1.3) to see that the left hand side equals
\[
\int_{\mathcal{F}_1 \setminus SL_2(F)} h(\tilde{g}) \mathcal{F}^{-1}(\Omega_V^\psi(\tilde{g}) \varphi)(e') \, d\tilde{g} = \int_{\mathcal{F}_1 \setminus SL_2(F)} h(\tilde{g}) \int_F \langle \Omega_V^\psi(\tilde{g}) \varphi \rangle(0; 1, u) \, dud\tilde{g}
\]
\[
= \int_{\mathcal{F}_1 \setminus SL_2(F)} h(\tilde{g}) \int_F \langle \Omega_V^\psi(\tilde{n}(u) \tilde{g}) \varphi \rangle(0; 1, 0) \, dud\tilde{g}.
\]
We combine the integrals over $\mathcal{F}_1$ and $\mathcal{F}_1 \setminus SL_2(F)$ into an integral over $SL_2(F)$ to obtain the stated formula. The integral thus obtained equals
\[
\int_{SL_2(F)} \int_F |a| |\mu(a)| (h(k)|a|^{3/2} \hat{\chi}^{-1}(a) (\Omega_V^\psi(k) \varphi)(0; a, ua^{-1})|a|^{-2} \, dudak
\]
and converges absolutely for $\Re \mu > -\frac{3}{2}$, which justifies all the manipulations.

Lemma 9.4. Notation being as in Lemma 9.3, there is a constant $C$ which is independent of $\Xi$ and such that $w_{\Xi}^{\widehat{\chi}_2^{-1}}(\Gamma^V(h \otimes \Phi))$ is equal to
\[
C|\nu(\Xi)|^{1/4} \int_{\mathcal{F}_1 \setminus SL_2(F)} w_{\nu(\Xi)}^{\mu}(h(\tilde{n}) \varphi(\Omega_V^\psi(\tilde{n}) \varphi)(-\Xi; 0, 1) \, d\tilde{g}.
\]
Proof. By Lemma 9.3 we can write $w_{\Xi}^{\mu \hat{\chi}^{-1} - \alpha}(\Gamma^V(h(s) \otimes \Phi))$ as the product of
\[
|\nu(\Xi)|^{3/4} \frac{L(2s + 1, \mu^2)}{L(s + \frac{1}{2}, \mu \hat{\chi}^{-1} \overline{\chi} \Xi)} = |\nu(\Xi)|^{1/4} \cdot \frac{|\nu(\Xi)|^{1/2} L(2s + 1, \mu^2)}{L(s + \frac{1}{2}, \mu \hat{\chi} \nu(\Xi))}
\]
and the integral
\[
\int_{D} \int_{SL_2(F)} h(s)(J \tilde{g})(\Omega_V^\psi(J \tilde{g}, n(z)) \varphi)(0; 1, 0) \overline{\psi}(z) \, d\tilde{g} \, dz
\]
\[
= \int_{D} \int_{SL_2(F)} h(s)(J \tilde{g})(\Omega_V^\psi(J \tilde{g}) \varphi)(-z; 1, 0) \psi(-\tau(\Xi z)) \, d\tilde{g} \, dz
\]
by (1.3). Since the double integral is absolutely convergent for $\Re s \gg 0$, we may interchange the order of integration. Using (1.8) and (1.7), we get
\[
\frac{1}{\gamma_D^{\psi}} \int_{SL_2(F)} h(s)(J \tilde{g})(\Omega_V^\psi(J^2 \tilde{g}) \varphi)(\Xi; 0, -1) \, d\tilde{g}
\]
\[
= \frac{\gamma_D^{\psi} (-1)}{\gamma_D^{\psi}} \int_{SL_2(F)} h(s)(J \tilde{g})(\Omega_V^\psi(\tilde{g}) \varphi)(-\Xi; 0, 1) \, d\tilde{g}
\]
By (4.10) the outer integral converges absolutely for all $s$. The proof is complete by evaluating the equality at $s = 0$. □

Corollary 9.5. (1) If $-\frac{1}{2} < \Re \mu < \frac{3}{2}$, then $\Gamma^\psi(\ell^\psi_1(\mu) \otimes \omega^\psi_1) = J_1(\mu \hat{x}^{-1})$.

(2) If $\mu^2 = \alpha$, then $\Gamma^\psi(A^\psi_1(\mu) \otimes \omega^\psi_1) = A_1(\mu \hat{x}^{-1})$.

Proof. If $\hat{x} = \mu \hat{x}^{-1} \alpha^{-1/2}$, then $\chi^\nu(\Xi) = \mu \alpha^{-1/2}$, and hence

$$w^\mu_{\Xi^{-\hat{x}}}(\Gamma^\psi(h \otimes \Phi)) = 0$$

for all $h \in A^\psi_1(\mu)$ and $\Phi \in \omega^\psi_1$ by Proposition 9.6 and Lemma 9.7. We can therefore infer from Proposition 9.4 that $\Gamma^\psi(A^\psi_1(\mu) \otimes \omega^\psi_1) \subseteq A_1(\mu \hat{x}^{-1})$. Employing Proposition 9.5 again, we can take $\Xi \in D^{\eta \nu}$ and a test vector $h$ for which $w^\mu_{\nu}(h) \neq 0$. If $\varphi = \varphi' \otimes \varphi''$ with $\varphi' \in \mathcal{S}(D)$ and $\varphi'' \in \mathcal{S}(F^2)$, then we obtain

$$w^\mu_{\Xi^{-\hat{x}}}(\Gamma^\psi(h \otimes \Phi)) = C|\nu(\Xi)|^{1/4} \int_F \int_{F \setminus \{0\}} \varphi''(c, a)$$

$$\times w^\mu_{\nu}(\Xi) \left( \theta \left( \left( \begin{array}{cc} a^{-1} & 0 \\ c & a \end{array} \right) \right) h \right) \left( \omega^\psi_{\nu D} \left( \left( \begin{array}{cc} a^{-1} & 0 \\ c & a \end{array} \right) \right) \varphi' \right) (-\Xi) \, \, \, d \, \, \, dc$$

by rewriting the formula in Lemma 9.4, where $da$ and $dc$ are Haar measures on $F$. This integral can be made nonzero by choosing $\varphi''$ to be supported in a small neighborhood of $(0, 1)$. Thus $\Gamma^\psi$ is nonzero, which verifies the claimed results as the target spaces are irreducible. □

Remark 9.6. The theta lift $\theta^\psi_1(\sigma)$ of an irreducible admissible representation $\sigma$ of $\text{Mp}(W_1)$ is defined to be the unique irreducible quotient of $\sigma^\psi \otimes \omega^\psi_1$ (see [8,4]). Since the map $\Gamma^\psi_1 : \ell^\psi_1(\mu) \otimes \omega^\psi_1 \to J_1(\mu \hat{x}^{-1})$ is $\text{Mp}(W_1)$-invariant and $G_1$-equivariant by Lemma 9.5, we see by Corollary 9.7 that

$$\theta^\psi_1(\ell^\psi_1(\mu)^\vee) \simeq J_1(\mu \hat{x}^{-1}), \quad \theta^\psi_1(A^\psi_1(\mu)^\vee) \simeq A_1(\mu \hat{x}^{-1}).$$

This result is stated in Propositions 5.2 and 6.3 of [3].

In Lemma 7.6 of [20] Ichino explicitly constructed a Schwartz function $\Lambda \in \mathcal{S}(D \otimes \mathbb{R}^2)$ with the following property (cf. Remark 9.1).

Lemma 9.7. Suppose that $F = \mathbb{R}$ and $D \simeq M_2(\mathbb{R})$. There is $\Lambda \in \mathcal{S}(V_D \oplus \mathbb{R}^2)$ such that for all $\Xi \in D_+^+$

$$W^{(f)}_{\Xi}(g) = 2^f \nu(\Xi)^{1/4} \int_{\mathcal{E}_1 \setminus \text{SL}_2(\mathbb{R})} W^{(2f-1)/2}_{\nu}(\xi)(\Omega^\psi_1(\hat{x}, g)\Lambda)(-\Xi; 0, 1) \, dx$$

and for all $\Xi \in -D_+^+$

$$\int_{\mathcal{E}_1 \setminus \text{SL}_2(\mathbb{R})} W^{(2f-1)/2}_{\nu}(\xi)(\Omega^\psi_1(\hat{x}, g)\Lambda)(-\Xi; 0, 1) \, dx = 0.$$
9.3. Fourier coefficients for Saito-Kurokawa liftings. Let \( F \) be a totally real number field and \( D \) a totally indefinite quaternion algebra over \( F \). We denote by \( \omega^{\psi}_{V_v} \simeq \otimes_v' \omega^{\psi}_{V_v} \) the Schrödinger model of the global Weil representation and by \( \Omega^{\psi}_{V} \) its mixed model. These models are related by the intertwining map \( \mathcal{F} : \mathcal{S}(V(\mathcal{A})) \to \mathcal{S}(V_D(\mathcal{A}) \oplus \mathbb{A}^2) \). We write \( \omega^{\psi}_{Vf} \) and \( \Omega^{\psi}_{\chi_f} \) for their finite parts. For \( \Phi \in \mathcal{S}(V(\mathcal{A})) \) we define a Schwartz function \( \Phi_\Lambda \) on \( V(\mathcal{A}) \) by

\[
\Phi_\Lambda(x) = \Phi(x_f) \prod_{v \leq 6, \infty} (\mathcal{F}_v^{-1})(x_v), \quad x = (x_v) \in V(\mathcal{A}).
\]

Taking Lemma 4.2 and Corollary 4.3 into account, we define a surjective homomorphism

\[
\Gamma^{\psi} = \otimes_p \Gamma^{\psi}_f : A_1^1(\mu_f) \otimes \omega^{\psi}_V \to A_1^1(\mu_f \chi_f^{-1}).
\]

The following result is nothing but Lemma 47 of \([77]\) (cf. Remark 4.1).

**Lemma 9.8.** If \( \mathcal{F} \in \mathcal{A}_0 \), \( \varphi \in \mathcal{S}(V_D(\mathcal{A}) \oplus \mathbb{A}^2) \) and \( 0 \neq \Xi \in D_{-}(\mathcal{F}) \), then

\[
W_\Xi(\theta^\psi_V(\mathcal{F}, \varphi)) = \int_{\mathcal{A}(\mathcal{A}) \backslash SL_2(\mathcal{A})} W_\nu(\Xi)(\varphi; \Omega^{\psi}_V)(\varphi)(-\Xi; 0, 1) \, d\nu.
\]

**Lemma 9.9.** If \( 0 \neq \{c_t\}_{t \in \mathcal{F}_\mathcal{F}} \in \tilde{\mathcal{C}}_{(2k+1)/2}(\mu_f) \), then

\[
0 \neq \{c_t(\xi)\}_{\xi \in D_{-}^{\mathcal{F} \otimes \chi_f^{-1}}} \in C^1_{k+1}(\mu_f \chi_f^{-1}).
\]

**Proof.** By assumption \( \mathcal{F}_{(2k+1)/2}(h, \{c_t\}) \in \tilde{\mathcal{C}}_{(2k+1)/2}(\mu_f) \) for all \( h \in A^1_1(\mu_f) \). We consider its Saito-Kurokawa lift

\[
\theta^\psi_V(g; \mathcal{F}_{(2k+1)/2}(h, \{c_t\}), \Phi_\Lambda) = \sum_{\Xi \in D_{-}(\mathcal{F})} W_\Xi(g, \theta^\psi_V(\mathcal{F}_{(2k+1)/2}(h, \{c_t\}), \Phi_\Lambda)).
\]

Anti-holomorphic discrete series representations of \( \text{Mp}(W_1)_v \) do not occur in the quotient of the Weil representation \( \omega^{\psi}_{V_D} \) for \( v \in \mathcal{G}_\infty \). Consequently, \( \theta^\psi_V(\tilde{\mathcal{C}}_{(2k+1)/2}) = \{0\} \), and so by the tower property, the space \( \theta^\psi_V(\tilde{\mathcal{C}}_{(2k+1)/2}) \) consists of cuspidal automorphic forms on \( \mathcal{G}_1 \). It follows that

\[
W_0(\theta^\psi_V(\mathcal{F}_{(2k+1)/2}(h, \{c_t\}), \Phi_\Lambda)) = 0.
\]

Lemmas 4.3 and 4.8 show that

\[
W_\Xi(\theta^\psi_V(\mathcal{F}_{(2k+1)/2}(h, \{c_t\}), \Phi_\Lambda)) = 0
\]

unless \( \Xi \in D_{-}^{\mathcal{F} \otimes \chi_f^{-1}} \), in which case Lemma 4.3 gives a nonzero constant \( C \) which is independent of \( \Xi \) and such that

\[
W_\Xi(g_\infty, \theta^\psi_V(\mathcal{F}_{(2k+1)/2}(h, \{c_t\}), \Phi_\Lambda)) = C_{c_\nu(\xi)} W_\Xi^{(k+1)}(g_\infty) W_\Xi^\mu \chi_f^{-1} (\Gamma^{\psi}(h, \Phi)).
\]

We conclude that

\[
\theta^\psi_V(g; \mathcal{F}_{(2k+1)/2}(h, \{c_t\}), \Phi_\Lambda) = C_{\mathcal{F}_{k+1}}(g_\infty; \Gamma^{\psi}(h, \omega^{\psi}_{V}(g_f)\Phi), \{c_\nu(\xi)\})
\]
9.4. End of the proof of Theorem 5.11. Let \( \{c_t\}_{t \in \mathcal{F}_{\kappa+1}} \in \hat{\mathcal{C}}_{\kappa+1}(\mathfrak{m}_\mathfrak{f}) \).

Fix \( \eta \in \mathcal{F}_n \). Put \( C_B = c_{\mu(B)} \) for \( B \in S_n^{\mathfrak{m}_\mathfrak{f} \otimes \chi_{\mathfrak{f}}^{(-1)^n \eta}} \). Thanks to the estimate of Fourier coefficients given in Proposition A.6.4 of \([99]\), we can invoke Lemma 4.3 to guarantee convergence of the series \( \mathcal{F}_{\kappa+n}(f, \{C_B\}) \) for \( f \in A_n(\mu_\mathfrak{f} \chi_{\mathfrak{f}}^{(-1)^n \eta}) \). Moreover, \( \{C_B\} \in T_n^{\mathfrak{m}_\mathfrak{f} \chi_{\mathfrak{f}}^{(-1)^n \eta}} \) by Lemma 5.11(3), (5). Note that \( \{c_{\mu(S)t}\}_{t \in \hat{\mathcal{C}}_{\kappa+1}(\mathfrak{m}_\mathfrak{f} \chi_{\mathfrak{f}}^{Q(S)})} = C_{\kappa+1}(\mu_\mathfrak{f} \chi_{\mathfrak{f}}^{(-1)^n \eta} \mathcal{S}_\mathfrak{f}) \)

for all \( S \in S_{n-1} \) by Lemma 5.11, our proof is complete by Lemma 7.10.

10. Translation to classical language

10.1. Hilbert-Siegel cuspidal Hecke eigenforms with respect to \( \Gamma_n^D[3] \).

We shall translate obtained results from adèle language into classical terminology. If \( h \) is right invariant under an open compact subgroup \( \mathcal{D} \) of \( \text{Mp}(W_m)_\mathfrak{f} \), then \( \mu_{\mathfrak{f}}(h) \in S_{2m+1/2}(\Gamma) \) by Remark 5.2, where \( \Gamma = \text{Sp}_m(F) \cap \mathcal{D} \). When \( \pi_\mathfrak{f} \) is ramified, it is a laborious task to find a suitable \( \mathcal{D} \), choose a good test vector \( h \) and calculate \( u^\mu_{\mathfrak{f}} \hat{\chi}_\mathfrak{f}(h) \). Here we let \( m = 2n \) and explicate \( \mu_{\mathfrak{f}}(f) \) in Theorem 6.1 when \( \pi_\mathfrak{f} \) is everywhere unramified and \( f \) is fixed by the maximal compact subgroup \( \prod_p K_n^D[\mathfrak{d}_p] \) of \( G_n(\mathfrak{a}_\mathfrak{f}) \).

The norm and the order of a fractional ideal of \( \mathfrak{d} \) are defined by \( \mathfrak{n}(\mathfrak{p}^k) = q_\mathfrak{p}^k \) and \( \text{ord}_p \mathfrak{p}^k = k \). We denote the different of \( F/\mathbb{Q} \) by \( \mathfrak{d} \), the product of all the prime ideals of \( \mathfrak{d} \) ramified in \( D \) by \( \mathfrak{c}^D \) and the Dedekind zeta function of \( F \) by \( \zeta_F(s) = \prod_p (1 - q_\mathfrak{p}^{-s})^{-1} \). Put

\[
\Gamma_n^D[\mathfrak{c}] = \left\{ \left( \begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \end{array} \right) \in G_n(F) \bigg| \begin{array}{c} \alpha, \delta \in M_n(\mathcal{O}) \\ \beta \in \mathfrak{c}^{-1}M_n(\mathcal{O}), \ \gamma \in \text{cM}_n(\mathcal{O}) \end{array} \right\},
\]

\[
\mathcal{H}_n^{2m+1} = \{ B \in S_n^{2m+1} | \tau(Bz) \in \mathfrak{c} \text{ for all } z \in S_n(F) \cap M_n(\mathcal{O}) \}
\]

for a fractional ideal \( \mathfrak{c} \) of \( \mathfrak{d} \).

We consider only the parallel weight case merely for expediency, so the weight \( (k, \ldots, k) \) is simply denoted by \( k \). A Hilbert-Siegel cusp form \( \mathcal{F} \) of weight \( k \) with respect to an arithmetic subgroup \( \Gamma \) of \( G_n(F) \) is a holomorphic function on \( \mathcal{H}_n^{2m+1} \) which satisfies \( \mathcal{F}|_{k\mathfrak{c}} = \mathcal{F} \) for every \( \gamma \in \Gamma \) and such that \( \mathcal{F}|_{k\mathfrak{c}} \) has a Fourier expansion of the form \( \sum_{B \in S_n^{2m+1}} c(B) e_{\mathfrak{c}}(\tau(Bz)) \) for all \( \gamma \in G_n(F) \).
For \( t \in F^\times \) and \( B \in S^+_1 \) we denote the conductors of \( \chi^t \) and \( \chi^B \) by \( t^t \) and \( t^B \), respectively, and define rational numbers \( f_B \), \( D_B \) and \( f_B \) by

\[
\begin{align*}
\hat{t} &= \sqrt{\frac{|N_{F/Q}(t)|}{\mathfrak{N}(t^t)}},
D_B &= \mathfrak{N}(t^D)^{2((n+1)/2)}N_{F/Q}(\nu(2B)),
\hat{f}_B &= \sqrt{\frac{D_B}{\mathfrak{N}(t^B)}}.
\end{align*}
\]

We write \( t \equiv \square \pmod{4} \) if there is \( y \in \mathfrak{o} \) such that \( t \equiv y^2 \pmod{4} \). For a finite prime \( p \) we define \( f^t_p \in \mathbb{Z} \) by \( f^t_p = \frac{1}{2}(\ord_p t - \ord_p t^t) \). Recall the polynomials \( \Psi_p(t, X) \in \mathbb{C}[X + X^{-1}] \) and \( F_p(B, X) \in \mathbb{Z}[X] \) defined in Section \( \|$ and \& Section \( 3.2 \). Set

\[
\tilde{F}_p(B, X) = X^{-f^B_p} F_p(B, q_p^{-2(n+1)/2} X),
\]

where

\[
f^B_p = f^t_p(-1)^{\nu(2B)} + \begin{cases} 
0 & \text{if } p \nmid \epsilon^D, \\
\frac{n+1}{2} & \text{otherwise}.
\end{cases}
\]

Then \( \tilde{F}_p(B, X) = \tilde{F}_p(B, X)^{-1} \) by \( \{ 21, 23, 32 \} \).

Let \( k \) be a natural number such that \( d(k+n) \) is even. Let \( \pi_f \simeq \otimes_p \mathcal{I}(\alpha_p^{\epsilon_p}, \alpha_p^{-\epsilon_p}) \) be an irreducible summand of \( \mathcal{E}_{2k} \). Then \( \pi_f \) is isomorphic as a Hecke module to a certain subspace of the space \( \tilde{\mathcal{E}}_{2(k+1)/2}^{(1)} \) by the works of Shimura and Waldspurger among others. Theorems 9.4, 10.1, 13.5 and Remark 9.1 of \( [10] \) give a Hecke eigenform \( h_n \) in the Kohnen plus subspace of weight \( k + \frac{1}{2} \) with respect to \( \Gamma_1[0^{-1}, 40] \) and \( \eta = (-1)^n \) whose Fourier expansion is given by

\[
h_n(Z) = \sum_{t \in \mathfrak{o} \cap F^\times, (-1)^nt \equiv \square \pmod{4}} c(t) \epsilon_\infty(tZ) t^{k-1/2} \prod_p \Psi_p((-1)^nt, q_p^{\epsilon_p}).
\]

This result is a generalization of the work of Kohnen \( [33] \). We extend \( c \) to a function on \( F^\times / F^\times 2 \) when \( D \neq M_2(F) \). One can obtain the following explicit result from Theorem \( \{ 17 \} \) and Lemma \( \{ 61 \} \) which is a strengthening of Theorems 3.2 and 3.3 of \( [22] \).

**Corollary 10.1.** Notations and assumptions being as above, we define a function \( \text{Lift}_2^D(\pi) : \mathcal{E}_{2n} \to \mathbb{C} \) by

\[
\text{Lift}_2^D(\pi)(Z) = \sum_{B \in \mathfrak{A}_2^+} c(\nu(2B)) \epsilon_\infty(\tau(BZ)) \tilde{f}_B^{(2k-1)/2} \prod_p \tilde{F}_p(B, q_p^{\epsilon_p}).
\]

Then \( \text{Lift}_2^D(\pi) \) defines a cuspidal Hecke eigenform of weight \( k + n \) with respect to \( \Gamma_1[0] \) whose standard (partial) \( L \)-function is equal to

\[
\zeta_{\pi}^{D}(s) \prod_{i=1}^{2n} L^{D,i} \left( s + n - i + \frac{1}{2}, \pi \right),
\]

where \( \zeta_{\pi}^{D} \) and \( L^{D,i} \) are defined with Euler factors for primes in \( \epsilon^D \) removed.
Proof. Put $\beta_f = \prod_p \alpha_p^{s_p}$. We will apply Theorem 11.1 with $\eta = 1$ and $\mu_f = \beta_f \lambda_f^{(-1)^n}$. Proposition 4.5 of [13] gives an element $f^+_K \in I^\psi_1(\mu_f)$ such that

$$|t|^{(2k+1)/4} u^\mu_t(f^+_K) = |t|^{(2k-1)/4} \prod_p \frac{\Psi_p(((-1)^n t, q_p^{s_p})}{q_p^{s_p} f_p^{(-1)^n t}}$$

for every $t \in F^+_+$ (cf. Section 6 of [13]). Put

$$c_t = \mathfrak{N}(\mathfrak{d}(-1)^n t)^{-1} \Psi_t \prod_p q_p^{s_p} t^{(-1)^n t}$$. 

Since $|t|^{(2k-1)/4} = \mathfrak{N}(\mathfrak{d}(-1)^n t)^{(2k-1)/4} \prod_p q_p^{s_p} t^{(-1)^n t}$, we see that $\{c_t\} \subset C(1)$ in view of the Fourier expansion of $h_n$. Put $c^D_n = \prod_p q_p^{(n+1)/2 s_p}$. Observe that for $B \in \mathcal{B}_{2n}^D$,

$$c(\nu(2B)) = c^D_n \mathfrak{N}(\mathfrak{d}^B)^{(2k-1)/4} c_n \prod_p q_p^{s_p} F_p^B$$

Let $f_p = \epsilon_{p(2n+1)/2} \in I_n(\alpha_p^{s_p})$ be the normalized $K^D_n[\mathfrak{d}_p]$-invariant element. Put $f = \otimes_p f_p \in I_n(\beta_f)$. Lemma 12.5 shows that for $B \in \mathcal{B}_{2n}^D$,

$$|\nu(B)|^{(k+n)/2} w_B^\beta_f(f) = c_n \mathfrak{N}(\mathfrak{d}^B)^{(2k-1)/4} \prod_p \frac{F_p(B, q_p^{s_p})}{F_p^{s_p} F_p^B}$$

where $c_n$ is a constant independent of $B$. Thus $\text{Lift}_{2n}^D(\pi) = \frac{c^D_n}{c_n} \text{Lift}_{n}^D(f)_{1_{2n}}$. \hfill \Box

Remark 10.2. (1) When $n = 1$ and $D \simeq M_2(F)$, a weaker version of this Fourier expansion was proved in [13].

(2) When $n = 1$, $F = \mathbb{Q}$ and $D$ is division, this lifting was explicitly computed by Oda [15] and Sugano [33].

(3) One can prove Conjectures 10.1 and 10.2 of [33] in a parallel way.

10.2. Miyawaki liftings. For simplicity, we let $D \simeq M_2(F)$. Let $\kappa$ be a tuple of $d$ natural numbers such that $\sum_{v \in \mathfrak{S}_{\infty}^+} \kappa_v \equiv d(n + r) \pmod{2}$. Let $\pi_f \simeq \otimes_p I^f(\alpha_p^{s_p}, \alpha_p^{-s_p})$ be an irreducible summand of $\mathfrak{G}_2n$. Theorem 12.2 gives a Hecke eigenform $F \in S_{\kappa+n+r}(\Gamma_2(n+1)[\mathfrak{d}^{-1}, \mathfrak{d}])$ which generates $\otimes_p \mathfrak{G}_2(n+r)(\alpha_p^{s_p})$ (cf. Corollary 12.1). Let $g \in S_{\kappa+n+r}(\Gamma_r[\mathfrak{d}^{-1}, \mathfrak{d}])$ be a Hecke eigenform. We consider the Miyawaki type integral

$$\mathcal{F}_{\pi, \mathfrak{d}, \mathcal{G}}^{(n)}(Z) = \int_{\Gamma_r[\mathfrak{d}^{-1}, \mathfrak{d}] \backslash \mathcal{H}_r^d} F \left( \begin{pmatrix} \mathcal{W}_{\pi} & 0 \\ 0 & Z \end{pmatrix} \right) g(-\mathcal{W}) \prod_v (\det \mathcal{S} \mathcal{W}_\pi)^{\kappa_v+n-1} d\mathcal{W}.$$

Then $\mathcal{F}_{\pi, \mathfrak{d}, \mathcal{G}}^{(n)} \in S_{\kappa+n+r}(\Gamma_2(n+1)[\mathfrak{d}^{-1}, \mathfrak{d}])$. One can prove the following result by the same type of reasoning as in [23].
Corollary 10.3. If \( \mathcal{F}_{\pi, \mathcal{G}}^{(n)} \) is not identically zero, then \( \mathcal{F}_{\pi, \mathcal{G}}^{(n)} \) is a Hecke eigenform whose standard \( L \)-function is equal to

\[
L(s, \mathcal{F}_{\pi, \mathcal{G}}^{(n)}, st) = L(s, g, st) \prod_{i=1}^{2n} L \left( s + n - i + \frac{1}{2}, \pi \right).
\]

11. The Duke-Imamoglu-Ikeda lifts and theta functions

11.1. Theta correspondence between degenerate principal series.

We define the split quadratic form on the vector space \( \mathbb{H}_m \) of 2\( m \)-dimensional column vectors by \( q_{\mathbb{H}_m}(x) = 2^t x H_m x \), where \( H_m = \frac{1}{2} \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix} \). Let \( F \) be a local field and \( \mu \in \Omega(F^\times) \) in this subsection. We write \( \mathbb{P}_m \) for the parabolic subgroup of the orthogonal group \( O(\mathbb{H}_m) \) stabilizing the maximal isotropic subspace \( \{ (x, 0, \ldots, 0) \in F^{2m} \mid x \in F^m \} \). Let \( I_m(\mu) \) denote the space of smooth functions \( f \) on \( O(\mathbb{H}_m) \) which satisfy

\[
f \left( \begin{pmatrix} a & b \{ t \} a^{-1} \\ 0 & \{ t \} a^{-1} \end{pmatrix} g \right) = \mu(\det a) | \det a |^{(m-1)/2} f(g)
\]

for all \( a \in \text{GL}_m(F) \), \( g \in O(\mathbb{H}_m) \) and skew symmetric matrices \( b \) of size \( m \).

In the \( p \)-adic case we put \( \mathbb{K}_m = O(\mathbb{H}_m) \cap \text{GL}_{2m}(\mathfrak{o}) \), define the open compact subgroup \( K_m[\mathfrak{b}, \mathfrak{c}] \) of \( \text{Sp}_m(F) \) and the right \( K_m[\mathfrak{o}^{-1}, \mathfrak{o}] \)-invariant function \( \varepsilon_\phi : \text{Sp}_m(F) \to \mathbb{R}_+^\times \) as in \( \S \) and \( \S \) respectively, and let \( f_m^{(s)}(k) = 1 \) for every \( k \in \mathbb{K}_m \). The normalized induced representation \( \mathcal{F}_m(\mu) \) is realized on the space of functions

\[
h(m(a)n(b)g) = \mu(\det a) | \det a |^{(m+1)/2} h(g)
\]

for all \( a \in \text{GL}_m(F) \), \( g \in \text{Sp}_m \) and \( b \in \text{Sym}_m \).

The Weil representation \( \omega_{H_{2n}}^\psi \) is realized on the Schwartz space \( \mathcal{S}(\mathbb{M}_m^{2m}(F)) \) as in \( \S \). Let \( L^{GJ}(s, \mu^{-1} \circ \det_{\text{GL}_m}) = \prod_{j=1}^{m} L(s + \frac{m+1}{2} - j, \mu^{-1}) \) denote the Godement-Jacquet \( L \)-factor of the one-dimensional representation \( \mu \circ \det \) of \( \text{GL}_m(F) \). Following \( \S \), we consider the following integral

\[
Z(\phi, s, \mu) = \int_{\text{GL}_{2n}(F)} \mu(\det a)^{-1} \phi \left( \begin{pmatrix} a \\ 0 \end{pmatrix} \right) | \det a |^{s+(2n-1)/2} da
\]

for \( \phi \in \mathcal{S}(\mathbb{M}_m^{2m}(F)) \). This integral converges absolutely for \( \Re s > n + \Re \mu - \frac{1}{2} \) and is meromorphically continued to the whole \( \mathbb{C} \). We can take the limit

\[
Z(\phi, \mu) = \lim_{s \to 0} L^{GJ}(s, \mu^{-1} \circ \det_{\text{GL}_{2n}})^{-1} Z(\phi, s, \mu).
\]

A simple calculation gives

\[
Z \left( \langle \omega_{H_{2n}}^\psi \left( \begin{pmatrix} a_1 & b_1 \\ 0 & t a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & t a_2^{-1} \end{pmatrix} \right) \phi, \mu \right) = \mu(\det a_1) | \det a_1 |^{(2n+1)/2} \mu(\det a_2)^{-1} \mu(\det a_2)^{-1} Z(\phi, \mu).
\]
We obtain a $Sp_{2n}$-intertwining, $O(\mathbb{H}_{2n})$-invariant map 
\[ \vartheta_n^\psi : \omega_{H_{2n}}^\psi \otimes I_{2n}(\mu) \to \mathcal{I}_{2n}(\mu) \]
by setting 
\[ \vartheta_n^\psi (h, \phi \otimes f) = \int_{\mathbb{P}_{2n} \backslash O(\mathbb{H}_{2n})} Z(\omega_{H_{2n}}^\psi (h, g) \phi, \mu) f(g) \, dg. \]

**Proposition 11.1.** If $-\frac{1}{2} < \Re \mu < \frac{1}{2}$, then the following statements hold.

1. $I_{2n}(\mu)$ is irreducible.
2. $\vartheta_n^\psi (\mathcal{I}_{2n}(\mu)) \simeq I_{2n}(\mu)$.

**Proof.** The first part is included in Theorem 8.1 of [11]. Since the map $Z(\mu : \omega_{H_{2n}}^\psi \to \mathcal{I}_{2n}(\mu) \otimes I_{2n}(\mu^{-1})$ is nonzero on account of the definition of the Godement-Jacquet $L$-factor, the map $\vartheta_n^\psi$ is nonzero, which implies that $\vartheta_n^\psi (\mathcal{I}_{2n}(\mu)) \simeq I_{2n}(\mu)^V \simeq I_{2n}(\mu^{-1}) \simeq I_{2n}(\mu)$. \qed

When $\psi$ is trivial on $\mathfrak{c}^{-1}$ but nontrivial on $\mathfrak{p}^{-1} \mathfrak{c}^{-1}$, we put $\text{ord } \psi = \mathfrak{c}$.

**Lemma 11.2.**

1. If $F$ is $p$-adic, $\phi_0 \in \mathcal{S}(M_{2n}^4(\mathfrak{o}))$ is the characteristic function of $M_{2n}^4(\mathfrak{o})$ and $\mathfrak{d} = \text{ord } \psi$, then $\vartheta_n^\psi (\phi_0 \otimes f_{2n}^{(s)})$ equals $\varepsilon_\mathfrak{d}^{s+(2n+1)/2}$ up to a nonzero scalar multiple.
2. If $F = \mathbb{R}$ and $V$ is a positive definite quadratic space of dimension $m$, then the $O(V)$-invariant subspace $\mathcal{S}(V^!)^{O(V)}$ of $\omega_V^\psi$ is equivalent to $\mathcal{D}^{(l)}_{m/2}$.

**Proof.** It is the well-known fact that $Z(\phi_0, \alpha^s)$ is a nonzero constant. Observe that $\phi_0$ is invariant under the action of the product $K_{2n}[\mathfrak{o}^{-1}, \mathfrak{d}] \times \mathbb{K}_{2n}$ of maximal compact subgroups. Thus $\vartheta_n^\psi (\phi_0 \otimes f_{2n}^{(s)})$ is nonzero, right $K_{2n}[\mathfrak{o}^{-1}, \mathfrak{d}]$-invariant, and necessarily equal to a multiple of $\varepsilon_\mathfrak{d}^{s+(2n+1)/2}$. It is well-known that $\vartheta_n^\psi (1_{O(V)}^!) \simeq \mathcal{D}^{(l)}_{m/2}$, where $1_{O(V)}^!$ denotes the trivial representation of $O(V)$. Since $\vartheta_n^\psi (1_{O(V)}^!) \simeq \mathcal{S}(V^!)^{O(V)}$ is irreducible by Section 3 of [12], the second part follows. \qed

**11.2. The embedding $I_{2n}(\mu_{\mathfrak{f}}) \hookrightarrow \mathfrak{M}(O(V_n))$.** From non on $F$ is a totally real number field of degree $d$ and $n$ is a positive integer such that $dn$ is even. Let $V_n$ be a $4n$-dimensional totally positive definite quadratic space over $F$ such that $V_n(F_{\mathfrak{p}}) \simeq \mathbb{H}_{2n}(F_{\mathfrak{p}})$ for every prime $\mathfrak{p}$. Recall that the space $\mathfrak{M}(O(V_n))$ of algebraic modular forms for $O(V_n)$ consists of locally constant functions on $O(V_n, A_{\mathfrak{f}})$ which are left invariant under $O(V_n, F)$. We now prove Corollary 1.4.

**Proof.** We can choose $\eta \in F^\times$ in such a way that $L\left(\frac{1}{2}, \pi \otimes \chi_\eta^s\right) \neq 0$ on account of Theorem A.2 of [17]. Then $\Pi \simeq \mathcal{D}_{2n}^{(2n|\mathfrak{f}|d)} \otimes \mathcal{I}_{2n}(\mu_{\mathfrak{f}}\eta)$ occurs in $\mathcal{M}_{\text{cusp}}(Sp_{2n})$ with multiplicity one by Theorem 13.1 and Proposition 13.4.
The standard $L$-function of $\Pi$ is $\zeta_F(s) \prod_{j=1}^{2n} L(s + n + \frac{1}{2} - j, \pi \otimes \chi^v)$. Since the archimedean $L$-factor of $\pi$ is holomorphic for $\Re s > 1/2 - n$, the partial function $L(s, \pi \otimes \chi^v)$ has no zero for $1/2 - n < \Re s \leq -1/2$. Therefore the standard $L$-function has a pole at $s = 1$. Since $\mathfrak{D}_n^{(2n)}$ is a limit of discrete series, its standard $L$-factor is holomorphic for $\Re s > 0$ by Lemma 7.2 of [51]. Thus the complete standard $L$-function has a pole at $s = 1$ and is holomorphic for $\Re s > 1$, and so by the functional equation, it has a pole at $s = 0$ and is holomorphic for $\Re s < 0$.

We can now apply [51, Theorem 10.1] or [11, Theorem 11.6] to see that $\theta_{V_n}^\psi(\Pi)$ is nonzero. The compact group $O(V_n, A_\infty)$ acts trivially on $\theta_{V_n}^\psi(\Pi)$ in view of Lemma 11.2. Thus $\theta_{V_n}^\psi(\Pi)$ occurs in $\mathfrak{m}(O(V_n))$. It is equivalent to $\mathbb{I}_{2n}(\mu_f \chi_f^\psi)$ by Proposition 11.4. Using the notation in §2.4, we define the spinor norm $\text{spin}_v : O(V_n, F_v) \to F_v^\times / F_v^{\times 2}$ as follows. Given $g_v \in O(V_n, F_v)$, take an element $\beta_v \in G(V_n(F_v))$ so that $\vartheta(\beta_v) = g_v$. We denote by $\text{spin}_v(g_v)$ the coset represented by $\mu_1(\beta_v)$ in $F_v^\times / F_v^{\times 2}$. Define a homomorphism $\text{spin} : O(V_n, A) \to \mathbb{A}^\times / \mathbb{A}_F^{\times 2}$ by $\text{spin}(g) = (\text{spin}_v(g_v))$ for $g = (g_v) \in O(V_n, A)$. The composition $\chi^\psi \circ \text{spin}$ is an automorphic character of $O(V_n, A)$. Let $T$ be a finite set of places of $F$. When the cardinality of $T$ is even, we can define an automorphic character $\text{sgn}^T : O(V_n, A) \to \mu_2$ by $\text{sgn}^T(g) = \prod_{v \in T} \text{det } g_v$.

We can choose $T$ so that $\text{sgn}^T(g) = \chi^\psi(\text{spin}(g))$ for $g \in O(V_n, A_\infty)$. Then

$$\mathbb{I}_{O(V_n, A_\infty)} \otimes \mathbb{I}_{2n}(\mu_f) \simeq \theta_{V_n}^\psi(\Pi) \otimes \text{sgn}^T(\chi^\psi \circ \text{spin}).$$

by Lemma 4.9 of [51]. Therefore $\mathbb{I}_{2n}(\mu_f)$ appears in $\mathfrak{m}(O(V_n))$.

Let $\sigma$ be an irreducible summand of $\mathfrak{m}(O(V_n))$ which is equivalent to $\mathbb{I}_{2n}(\mu_f \chi_f^\psi)$. Since its standard $L$-function is entire and has no zero at $s = 1$, Theorem 2 and Lemma 10.2 of [11] say that $\theta_{W_{2n}}^\psi(\sigma)$ is nonzero and cuspidal. Since $\theta_{W_{2n}}^\psi(\sigma)$ is equivalent to $\Pi$ as an abstract representation, it is equal to $\Pi$ by Corollary 11.1. It follows that $\sigma = \theta_{V_n}^\psi(\Pi)$ by Lemma 11.2. As such, the multiplicity of $\mathbb{I}_{2n}(\mu_f \chi_f^\psi)$, which is equal to that of $\mathbb{I}_{2n}(\mu_f)$, is one. □

11.3. An identity of Hilbert-Siegel cusp forms. We have defined a nonzero intertwining map $\vartheta_{V_n}^\psi = \otimes_v \vartheta_{V_n}^\psi : \omega_{V_n} \otimes \mathbb{I}_{2n}(\mu_f) \to \mathcal{S}_{2n}(\mu_f)$ in §11.1.

The space $\mathcal{S}(V_n(A_\infty))_{2n}(O(V_n))$ consists of Schwartz functions $\varphi$ on $V_n(A_\infty)$ which satisfy $\omega_{V_n}(k, g) \varphi = J_{2n}(k, i)\varphi$ for every $k \in K_i$ and $g \in O(V_n, A_\infty)$. Since the lowest $K$-type occurs in $\mathfrak{D}_{2n}$ with multiplicity one, this space is one-dimensional by Lemma 11.2.

We are now ready to prove Corollary 11.2.

**Proof.** As we have seen in the previous subsection, the standard $L$-function of $\mathbb{I}_{2n}(\mu_f)$ is entire and has no zero at $s = 1$ by assumption, and hence its theta lift to $Sp_{2n}$ is nonzero, cuspidal and equivalent to $\mathfrak{D}_{2n} \otimes \mathcal{S}_{2n}(\mu_f)$. 


Fix a Gaussian $0 \neq \varphi \in S(V_{n}^{2n}(\Lambda_\infty))_{2n}^{O(V_n)}$. Then
\[
\Theta(\omega^V_{V_n}(h,g)(\varphi \otimes \phi)) = \theta(h(i_{2n}),\omega^V_{V_n}(g)\phi)/J_{2n}(h,i_{2n})
\]
is nonzero for some $h \in Sp_{2n}(\Lambda_\infty)$, $g \in O(V_n,\Lambda_F)$ and $\phi \in S(V_{n}^{2n}(\Lambda_F))$. The right hand side of (12.4) is therefore equal to $\theta^V_{W_{2n}}(j_n(f),\varphi \otimes \phi)$. We can define a nonzero $O(V_n,\Lambda_F)$-invariant, $Sp_{2n}(\Lambda_F)$-intertwining map $\omega^V_{V_n} \otimes I_{2n}(\mu_F) \to S_{2n}^{(2n)}$ by $\phi \otimes f \mapsto \theta^V_{W_{2n}}(j_n(f),\varphi \otimes \phi)$. The Howe principle forces it to factor through the quotient map $\vartheta^V_{W_{2n}}$. The resulting map is proportional to $i_{2n}^{1}$ by uniqueness.

We shall translate this result into a more classical language. Let $L$ be a lattice in $V_n(F)$. For $g \in O(V_n,\Lambda)$, we write $gL$ for the lattice defined by $(gL)_p = g_pL_p$, where we denote the closure of $L$ in $V_n(F_p)$ by $L_p$. We mean by the genus (resp. class) of $L$ the set of all lattices of the form $gL$ with $g \in O(V_n,\Lambda)$ (resp. $g \in O(V_n,F)$). We denote the genus of $L$ by $\Upsilon(L)$ and the class of $L$ by $[L]$. Put
\[
\mathcal{K}_L = \{g \in O(V_n,\Lambda) \mid gL = L\}, \quad O(L) = O(V_n,F) \cap \mathcal{K}_L, \quad E(L) = \sharp O(L).
\]
Then we can identify the set of classes in the genus of $L$ with double cosets for $O(V_n,F) \setminus O(V_n,\Lambda)/\mathcal{K}_L$ via the map $g \mapsto [gL]$. The $\mathcal{K}_L$-invariant subspace of $\mathfrak{M}(O(V_n))$ can be considered to be the $\mathbb{C}$-vector space $V(\Upsilon(L))$ with basis $\{[L] \mid L \in \Upsilon(L)\}$ via the map $f \mapsto \sum [L] f(L)[L]$.

Define the dual lattice of $L$ by $L^* = \{x \in V_n(F) \mid q_{V_n}(x,L) \subset 6\}$. We call $L$ integral if $L \subset L^*$, even if $q_{V_n}(L) \subset 2\delta$, and unimodular if $L = L^*$. The set of even unimodular lattices in $V_n$ forms a genus $\Upsilon_n$. The theta series associated to $L \in \Upsilon_n$ is a Hilbert-Siegel modular form of degree $m$ and weight $2n$ defined by
\[
\theta^{(m)}_{L}(Z) = \sum_{x \in \Lambda^m} e_{\infty} \left( \frac{1}{2} \text{tr}(q_{V_n}(x)Z) \right).
\]
We define the theta lift of $f \in V(\Upsilon_n)$ by
\[
(11.1) \quad \Theta^{(m)}(Z,f) = \sum_{[L]} \frac{f(L)}{E(L)} \theta^{(m)}_{L}(Z).
\]

**Corollary 11.3.** Let $\pi_L \simeq \otimes_p f_\pi^{(m)}(\alpha_p,\alpha_p^{-m})$ be an irreducible summand of $\mathcal{C}_{2n}$. Assume that $dn$ is even. Let $f \in V(\Upsilon_n)$ be a common eigenfunction of all Hecke operators whose standard $L$-function is $\prod_{j=1}^{2n} L(s + n + \frac{1}{2} - j, \pi)$. If $L(\frac{1}{2},\pi) \neq 0$, then $\Theta^{(2n)}(f)$ is a nonzero constant multiple of Lift$_{2n}^{(2n)}(\pi)$. If $L(\frac{1}{2},\pi) = 0$, then $\Theta^{(2n)}(f)$ is zero.

**Proof.** Let $\phi^{(2n)}_{L_{p}}$ be the characteristic function of $L_{p}^{2n}$ and $f_{2n}$ a nonzero $\mathcal{K}_{L}$-invariant element of $\otimes_p^{\sharp} I_{2n}(\alpha_p^{*})$. Then $f$ equals $j_n(f_{2n})$ up to scaling by
\[ \mathbb{C}^x. \] Put \( \phi_{\mathcal{L}}^{(2n)} = \otimes_p \phi_{\mathcal{L}_p}^{(2n)} \). The theta lift
\[
\int_{O(V_n,F) \backslash O(V_n,K)} j_n(f_{2n})(g) \theta(\mathcal{Z}, \omega_{V_n}^{(2n)}(g) \phi_{\mathcal{L}}^{(2n)}) \, dg
\]
\[
= \sum_{g \in O(V_n,F) \backslash O(V_n, A)/\mathcal{K}_\mathcal{L}} j_n(f_{2n})(g) \theta(\mathcal{Z}, \phi_{g,\mathcal{L}}^{(2n)}) \int_{O(g \mathcal{L}) \backslash O(g \mathcal{L})} \, dk
\]
is equal to \( \text{vol}(\mathcal{K}_{\mathcal{L}}) \Theta^{(2n)}(j_n(f_{2n})) \). If \( L(\frac{1}{2}, \pi) = 0 \), then the standard \( L \)-function of \( f \) has a zero at \( s = 1 \), and so by Theorem 2 of [41], the theta lift is zero. Suppose that \( L(\frac{1}{2}, \pi) \neq 0 \). Then \( \Theta^{(2n)}(f) \) equals \( i_{2n}^1(\phi^{(2n)}_{\mathcal{L}} \otimes f_{2n}) \) by Corollary 1.2. Since \( \phi^{(2n)}_{\mathcal{L}}(\phi^{(2n)}_{\mathcal{L}} \otimes f_{2n}) \) is a nonzero \( \prod_p K_{2n} \mathfrak{D}^{p-1}_{2n}, \mathfrak{d}_p \)-invariant element of \( \otimes_p \mathfrak{J}_{2n}(\alpha^{(2n)}_{\mathcal{L}}) \) by Lemma 1.2.11, its image under \( i_{2n}^1 \) coincides with \( \text{Lift}_{2n}(\pi) \) by Corollary 1.2.11.

Remark 11.4. The linear subspace of \( M_{12}(\mathbf{S}p_{12}(\mathbb{Z})) \) spanned by the theta series associated to the 24 different Niemeier lattices has been extensively studied in [1, 14, 22, 23, 7]. It intersects the space of cusp forms in a one-dimensional subspace whose basis vector is explicitly given in Theorem 4 of [6]. In Section 15 of [22] Ikeda verified that this sum of theta functions equals \( -\frac{1}{120} \text{Lift}_{12}(\Delta) \), where \( \Delta \in S_{12}(\mathbf{S}l_2(\mathbb{Z})) \) is the Ramanujan delta function and the corresponding cusp form \( h_6 \in S_{12/2}(\mathbf{S}l_0(4)) \) is normalized so that \( c(1) = 1 \). This gives an explicit example of Corollary 1.2.10 for the Niemeier lattices.

12. Hilbert-Siegel modular forms of degree 4 and weight 4 over \( \mathbb{Q}(\sqrt{2}) \)

12.1. Kneser neighbors and Hecke operators. For a prime \( p \) we denote the residue field \( \mathfrak{p}/p \) by \( \mathbb{F}_p \). Let \( L \in \mathcal{Y}_n \). The \( \mathbb{F}_p \)-vector space \( L/pL \) comes equipped with a nondegenerate quadratic form \( x \mapsto \frac{1}{2} q_{V_n}(x) \pmod{p} \). We say that \( K \in \mathcal{Y}_n \) is a \( p \)-neighbor of \( L \) if \( L/(L \cap K) \simeq K/(L \cap K) \simeq \mathbb{F}_p^2 \). The number of \( p \)-neighbors of \( L \) in the class of \( K \) is denoted by \( N(L,K,p^j) \). It is important to note that the matrix \( N(L,K,p^j) \) represents the action of a Hecke operator on algebraic automorphic forms for \( O(V_n) \).

Fix \( \mathcal{L} \in \mathcal{Y}_n \). Put \( \mathcal{K}_{\mathcal{L}_p} = \{ g \in O(V_n,F_p) \mid g \mathcal{L}_p = \mathcal{L}_p \} \). The space \( \mathfrak{M}(O(V_n)) \) comes equipped with the action of Hecke operators. Given \( f \in V(\mathcal{Y}_n) \) and \( y \in O(V_n,F_p) \), decompose the double coset \( \mathcal{K}_{\mathcal{L}_p} y \mathcal{K}_{\mathcal{L}_p} \) into a disjoint union of right cosets \( \bigsqcup_j y_j \mathcal{K}_{\mathcal{L}_p} \) and define \( f|_{\mathcal{K}_{\mathcal{L}_p} y \mathcal{K}_{\mathcal{L}_p}} \in V(\mathcal{Y}_n) \) by
\[
[f|_{\mathcal{K}_{\mathcal{L}_p} y \mathcal{K}_{\mathcal{L}_p}}(g) = \sum_j f(g y_j).
\]

We can take an \( \mathfrak{p} \)-basis \( \{ e_1, \ldots, e_{2n}, f_1, \ldots, f_{2n} \} \) of \( \mathcal{L}_p \) such that
\[
q_{V_n}(e_i) = q_{V_n}(f_j) = 0, \quad q_{V_n}(e_i, f_j) = \delta_{ij} \quad (1 \leq i, j \leq 2n),
\]
where $\delta_{ij}$ denotes Kronecker’s delta. In this basis we define a homomorphism $m : \text{GL}_{2n}(F_p) \to O(V_n, F_p)$ by (111). Theorem 5.11 of [13] says that

$$[L]|K_{\mathcal{L}_p}m(\text{diag}[\varpi_p, \varpi_p, \ldots, \varpi_p, 1, 1, \ldots, 1])K_{\mathcal{L}_p} = \sum_{[K]} N(K, L, p^j)[K].$$

We consider the $p$-neighbor operator on $V(\Upsilon_n)$

$$K_n(p) : [L] \mapsto \sum_{[K]} N(K, L, p)[K].$$

**Proposition 12.1.** If $f \in V(\Upsilon_n)$ is an eigenvector of all Hecke operators with $p$-Satake parameter $\{\beta_{p,1}^{\pm 1}, \ldots, \beta_{p,2n}^{\pm 1}\}$, then

$$K_n(p)f = q_p^{2n-1}f \sum_{i=1}^{2n} (\beta_{p,i} + \beta_{p,i}^{-1}).$$

**Proof.** We have only to show that

$$f|K_{\mathcal{L}_p}m(\text{diag}[\varpi_p, 1, 1, \ldots, 1])K_{\mathcal{L}_p} = q_p^{2n-1}f \sum_{i=1}^{2n} (\beta_{p,i} + \beta_{p,i}^{-1}).$$

Let $T_n$ be the maximal torus of $O(V_n, F_p)$ consisting of diagonal matrices. The representation of $O(V_n, F_p)$ generated by $f$ is equivalent to the unique class one component of the unramified principal series representation induced from the character of $T_n$ defined by $m(\text{diag}[t_1, \ldots, t_{2n}]) \mapsto \prod_{i=1}^{2n} \beta_{p,i}^{\text{ord}_p t_i}$. It is seen in Appendix A of [59] that

$$\sum_{[\gamma]} \nu[\gamma]^{-s}f|K_{\mathcal{L}_p}\gamma K_{\mathcal{L}_p} = \frac{f \prod_{j=1}^{2n} (1 - q_p^{1-2s-2j+4n-1})}{\prod_{j=1}^{2n} (1 - \beta_{p,j} q_p^{s+2n-1})(1 - \beta_{p,j}^{-1} q_p^{-s+2n-1})},$$

where $\gamma$ runs over all the representatives for $K_{\mathcal{L}_p}/O(V_n, F_p)/K_{\mathcal{L}_p}$. Here, we put $\nu[\gamma] = [\gamma \lambda_1^{4n} + \lambda_2^{4n} : \lambda_4^{4n}]$. We obtain the declared result by looking at the coefficient of $q_p^{-s}$.

**12.2. Unramified theta correspondence.**

**Lemma 12.2** (Moeglin [12]). Let $\sigma$ be an irreducible automorphic representation of $O(V_n, \mathbb{A})$ such that $\theta_{W_m}^\psi(\sigma)$ is nonzero and cuspidal. Then $\theta_{V_n}^{-1}(\theta_{W_m}^\psi(\sigma)) = \sigma$. If $l > m$, then $\theta_{W_l}^\psi(\sigma)$ is orthogonal to any cusp forms on $Sp_l(\mathbb{A})$.

We define the $\mathbb{C}$-linear map $\Theta^{(m)} : V(\Upsilon_n) \to M_{2n}(\Gamma_m[\delta^{-1}, \delta])$ as in (111). When $f$ is a Hecke eigenfunction, we define its degree by

$$\deg f = \min\{m \mid \Theta^{(m)}(f) \neq 0\}.$$

**Proposition 12.3.** Let $f \in V(\Upsilon_n)$ be a nonzero Hecke eigenfunction. Put $m = \deg f$.

1. $\Theta^{(m)}(f)$ is a nonzero Hecke eigenform in $S_{2n}(\Gamma_m[\delta^{-1}, \delta])$. 
(2) If \( l > m \), then \( \Theta^{(l)}(f) \) is nonzero and orthogonal to any cusp forms in \( S_{2n}(\Gamma_l[\mathfrak{o}^{-1}, \mathfrak{d}]) \).

**Proof.** Let \( \varphi \) be a nonzero element of \( S(V^m_n(\mathbb{A}_\infty))^{O(V_n)} \) and \( \phi^{(m)}_{\mathfrak{p}} \) the characteristic function of \( \mathcal{Z}^{m}_{\mathfrak{p}} \). Then \( \Theta^{(m)}(f) \) is a constant multiple of the theta lift \( \theta^\psi_{W_n}(f, \varphi \otimes (\otimes_{\mathfrak{p}} \phi^{(m)}_{\mathfrak{p}})) \), as we have seen in the proof of Corollary 2.6. This proves (ii) as \( \Theta^{(m)}(f) \) is invariant under \( K_m[\mathfrak{d}^{-1}, \mathfrak{d}] \) for every \( \mathfrak{p} \). Observe that the \( \left(\begin{array}{cc} 0 & 0 \\ 0 & \xi \end{array}\right) \) th Fourier coefficient of \( \Theta^{(l)}(f) \) is the \( \xi \) th Fourier coefficient of \( \Theta^{(l-1)}(f) \) for \( \xi \in \mathcal{R}_{l-1} \). Thus \( \Theta^{(m)}(f) \) is cuspidal in view of Proposition A4.5(4) of [13]. Moreover, \( \Theta^{(l)}(f) \) is nonzero and has no cuspidal component by Lemma 12.2. \( \square \)

**Lemma 12.4.** Let \( \mathcal{F} \in S_{2n}(\Gamma_m[\mathfrak{o}^{-1}, \mathfrak{d}] \) be a Hecke eigenform whose \( \mathfrak{p} \)-Satake parameter is denoted by \( \{ \beta_{1,\mathfrak{p}}, \ldots, \beta_{m,\mathfrak{p}} \} \). If \( \mathcal{F} \in \Theta^{(m)}(V(\mathcal{Y}_n)) \) and \( 2n \geq m \), then there exists a Hecke eigenform \( f \in V(\mathcal{Y}_n) \) whose \( \mathfrak{p} \)-Satake parameter is given by

\[
\{ \beta_{1,\mathfrak{p}}, \ldots, \beta_{m,\mathfrak{p}} \} \cup \{ q_{\mathfrak{p}}^0, 0 \leq j \leq 2n - m - 1 \}
\]

and such that \( \mathcal{F} = \Theta^{(m)}(f) \).

**Proof.** By Proposition 12.2(iii) there is a Hecke eigenform \( f \in V(\mathcal{Y}_n) \) such that \( \mathcal{F} = \Theta^{(m)}(f) \). We write \( \Pi \) for the automorphic representation generated by \( \mathcal{F} \). Then \( f \in \theta^\psi_{V_n}(\Pi) \) by Lemma 12.2. Therefore \( \theta^\psi_{V_n}(\Pi) \) is everywhere unramified and so by Corollary 2.6 of [36] its \( \mathfrak{p} \)-Satake parameter is given as above. \( \square \)

12.3. **Hilbert modular forms over** \( \mathbb{Q}(\sqrt{2}) \). Let \( F = \mathbb{Q}(\sqrt{2}) \). The narrow class number of \( F \) is one. Denote a totally positive generator of an ideal \( \mathfrak{a} \) by \( \varpi_\alpha \). For example, the different \( \mathfrak{d} \) is generated by \( \varpi_\mathfrak{d} = 4 - 2\sqrt{2} \).

For an integral ideal \( \mathfrak{b} \) of \( \mathfrak{d} \) we put \( \sigma_\mathfrak{b}(\mathfrak{a}) = \sum_{\mathfrak{d}|\mathfrak{b}} \mathcal{N}(\mathfrak{a})^k \). We denote by \( \mu \) the Möbius function for ideals and by \( \chi_\mathfrak{a} \) the ideal character associated to \( \varpi_\mathfrak{a} \). Put \( \widehat{\varpi}_\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{\nu_{\mathfrak{p}}} \). We consider the Eisenstein series of weight 2n

\[
G_{2n}(\mathcal{Z}) = \frac{\zeta_F(1-2n)}{2\mathfrak{d}} + \sum_{\xi \in \mathfrak{c}^{\times} F_+^{\times}} \sigma_{2n-1}((\xi)) e_\infty(\xi \mathcal{Z})
\]

and the Cohen-Su Eisenstein series (cf. [52]) of weight \( n + \frac{1}{2} \)

\[
G_{(2n+1)/2}(\mathcal{Z}) = \zeta_F(1-2n) + \sum_{(-1)^n \equiv 0 \pmod{4}} \sum_{\mathfrak{a}^{(-1)^n}} L(1-n, \chi^{(-1)^n} z) \mathcal{E}_n((-1)^n \eta e_\infty(z \mathcal{Z}),
\]

where \( \mathcal{E}_n(\eta) = \sum_{\mathfrak{b}|\mathfrak{d}} \mu(\mathfrak{b}) \chi^{\nu_{\mathfrak{b}}} (\mathfrak{b}) \mathcal{N}(\mathfrak{a})^{n-1} \sigma_{2n-1}(\widehat{\varpi}_\mathfrak{a}^{-1}). \)

The ring \( \bigoplus_{k \geq 0} M_{2k}(\text{SL}_2(\mathbb{Z}[\sqrt{2}])) \) of Hilbert modular forms of even parallel weight is a polynomial ring generated by Eisenstein series of weight 2, 4 and 6 (see [13]). This ring is isomorphic to \( \bigoplus_{k \geq 0} M_{2k}(\Gamma_1[\mathfrak{d}^{-1}, \mathfrak{d}]) \).
In particular, the ring \( \bigoplus_{k \geq 0} M_{2k}(\Gamma_1[\mathfrak{d}^{-1}, \mathfrak{d}]) \) is generated by \( G_2, G_4 \) and \( G_6 \). We have \( \dim S_4(\Gamma_1[\mathfrak{d}^{-1}, \mathfrak{d}]) = 1 \) and \( \dim S_6(\Gamma_1[\mathfrak{d}^{-1}, \mathfrak{d}]) = 2 \). The space \( S_3(\Gamma_1[\mathfrak{d}^{-1}, \mathfrak{d}]) \) is spanned by the normalized Hecke eigenform \( \phi_4(Z) = \sum_{\eta \in \mathcal{O} \cap F_\mathfrak{d}} a(\eta) e_\infty(\eta Z) \) given by

\[
\phi_4(Z) = 44G_2(Z)^2 - \frac{5}{6} G_4(Z) = e_\infty(Z) - 2e_\infty((2 - \sqrt{2})Z) - 4e_\infty(2Z) + \cdots.
\]

The Kohnen plus space of weight \( \frac{5}{2} \) with respect to \( \Gamma_1[\mathfrak{d}^{-1}, \mathfrak{d}] \) is spanned by the cusp form \( \phi_{5/2}(Z) = \sum_{\eta \equiv \square \pmod{4}} b(\eta) e_\infty(\eta Z) \) defined by

\[
\phi_{5/2}(Z) = 44G_2(4Z) \vartheta(Z) - 10G_{5/2}(Z) = e_\infty(Z) - 4e_\infty(2Z) + \cdots,
\]

where \( \vartheta(Z) = \sum_{\xi \in \mathfrak{d}} e_\infty(\xi^2Z) \). Corollary 13 gives the Hilbert-Siegel cusp form

\[
\text{Lift}_4(\phi_4)(Z) = \sum_{\xi \in \mathfrak{d}^+} b(\varpi_\xi^*) e_\infty(\text{tr}(\xi Z))^{3/2} \prod_p \bar{F}_p(\xi, \beta_p) \in S_4(\Gamma_4[\mathfrak{d}^{-1}, \mathfrak{d}]),
\]

where \( \beta_p \) satisfies \( \beta_p + \beta_p^{-1} = q_p^{-3/2}a(\varpi_p) \). The \( L \)-function of \( \phi_4 \) is given by

\[
L(s, \phi_4) = \prod_p (1 - a(\varpi_p)q_p^{-s} + q_p^{3-2s})^{-1}.
\]

It has a functional equation

\[
\Lambda(4-s, \phi_4) = \Lambda(s, \phi_4),
\]

where \( \Lambda(s, \phi_4) = 8^{-s}\Gamma_C(s)^2 L(s, \phi_4) \). By the generalized Kohnen-Zagier formula

\[
L(2, \phi_4) \neq 0
\]

(see Theorem 12.3 of [13]). Let \( q = (\sqrt{2}) \) be the prime ideal above 2. Since \( a(\varpi_q) = -2 \), the \( q \)-Satake parameter \( \beta_q \) is given by

\[
2^{3/2}(\beta_q + \beta_q^{-1}) = -2.
\]

On the other hand, \( S_6(\Gamma_1[\mathfrak{d}^{-1}, \mathfrak{d}]) \) is spanned by \( \{ \phi_6^+, \phi_6^- \} \), where

\[
\phi_6^+(Z) = \frac{-48240G_2G_4 + 2824320G_3^2 - 7G_6 \pm \sqrt{73}(14160G_2G_4 - 470400G_3^2 - 7G_6)}{1560}
\]

\[
= e_\infty(Z) + (-1 \pm \sqrt{73})e_\infty((2 - \sqrt{2})Z) + \cdots.
\]

The \( q \)-Satake parameter \( \gamma_q^+ \) of \( \phi_6^+ \) is given by

\[
2^{5/2}(\gamma_q^+ + (\gamma_q^+)^{-1}) = -1 \pm \sqrt{73}.
\]
12.4. 8-dimensional even unimodular lattices over \( \mathbb{Z} \sqrt{2} \). Totally positive definite even unimodular lattices of rank 8 over \( \mathbb{Z} \sqrt{2} \) form a genus, which we have denoted by \( \Upsilon_2 \). It is known that there are exactly 6 classes in this genus. They are labeled \( E_8 \), \( 2\Delta'_4 \), \( \Delta_8 \), \( 2D_4 \), \( 4\Delta_2 \) and \( \emptyset \). The orders of the automorphism groups are as in the following table (see [18, p. 371]).

**TABLE I.**

| L   | \( E_8 \) | \( 2\Delta'_4 \) | \( \Delta_8 \) | \( 2D_4 \) | \( 4\Delta_2 \) | \( \emptyset \) |
|-----|-----------|-----------------|--------------|-----------|----------------|---------|
| \( E(L) \) | \( 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 \) | \( 2^{17} \cdot 3^4 \) | \( 2^{15} \cdot 3^2 \cdot 5 \cdot 7 \) | \( 2^{14} \cdot 3^3 \) | \( 2^{18} \cdot 3 \) | \( 2^{14} \cdot 3^2 \cdot 5 \cdot 7 \) |

The numbers \( N(L, K, q) \) are given in the following table (see [18, p. 374]).

**TABLE II.**

| \( L \setminus K \) | \( E_8 \) | \( 2\Delta'_4 \) | \( \Delta_8 \) | \( 2D_4 \) | \( 4\Delta_2 \) | \( \emptyset \) |
|-------------------|-----------|-----------------|--------------|-----------|----------------|---------|
| \( E_8 \)          | 0         | 0               | 135          | 0         | 0              | 0       |
| \( 2\Delta'_4 \)   | 0         | 18              | 36           | 0         | 81             | 0       |
| \( \Delta_8 \)     | 2         | 35              | 28           | 70        | 0              | 0       |
| \( 2D_4 \)         | 0         | 0               | 3            | 96        | 36             | 0       |
| \( 4\Delta_2 \)    | 0         | 6               | 0            | 64        | 49             | 16      |
| \( \emptyset \)    | 0         | 0               | 0            | 0         | 105            | 30      |

We denote by \( f_1, \ldots, f_6 \) the eigenvectors of \( K_2(q) \). The coefficients of \( f_i \) (\( i = 1, \ldots, 6 \)) are as follows:

**TABLE III.**

| \( i \) | \( f_1 \) | \( f_2 \) | \( f_3 \) | \( f_4 \) | \( f_5 \) | \( f_6 \) |
|---------|--------|--------|--------|--------|--------|--------|
| \( E_8 \) | 1      | 1      | 1      | 1      | 1      | 1      |
| \( 2\Delta'_4 \) | 135    | -36    | -30    | 3      | -8     | 14     |
| \( \Delta_8 \) | -14175 | -216   | 840    | 81     | -304   | 840    |
| \( 2D_4 \) | -135   | -36    | -58    | -3     | 8      | 30     |
| \( 4\Delta_2 \) | 5775 - 525\sqrt{73} | -88 + 104\sqrt{73} | 560 | -8 + 13\sqrt{73} | 16 + 16\sqrt{73} | 560 |
| \( \emptyset \) | 5775 + 525\sqrt{73} | -8 - 104\sqrt{73} | 560 | -8 - 13\sqrt{73} | 16 - 16\sqrt{73} | 560 |

The eigenvalues \( \mu_i \) of \( K_2(q) \) corresponding to \( f_i \) are given by

**TABLE IV.**

| \( i \) | \( f_1 \) | \( f_2 \) | \( f_3 \) | \( f_4 \) | \( f_5 \) | \( f_6 \) |
|---------|--------|--------|--------|--------|--------|--------|
| \( f_1 \) | 135    | -30    | -8     | 58     | 33 + 3\sqrt{73} | 33 - 3\sqrt{73} |

It is important to note that the eigenvalues are mutually distinct.

Given an even integral lattice \( \mathcal{L} \) in \( V_n \) and a half-integral symmetric matrix \( \xi \) of size \( m \), we write \( N(\mathcal{L}, \xi) \) for the number of elements \( (x_1, \ldots, x_m) \in \mathcal{L}^m \) such that \( q_{V_n}(x_a, x_b) = 2\xi_{ab} \) for \( 1 \leq a, b \leq m \).

**Lemma 12.5.** If \( \xi \in \mathcal{B}_4^+ \) satisfies \( \det(2\xi) \in \sigma^\times \), then
\[
N(E_8, \xi) = N(\Delta_8, \xi) = N(2D_4, \xi) = N(4\Delta_2, \xi) = N(\emptyset, \xi) = 0, \quad N(2\Delta'_4, \xi) = 2^9 \cdot 3^2.
\]
Proof. It is known that \( \mathcal{T}_1 \) has only one class which we denote by \( \Delta_4' \). Therefore if \( L \) is a totally positive definite even unimodular lattice of rank 8 such that \( N(L, \xi) \neq 0 \), then \( L \simeq 2\Delta_4' \).

Let \( \xi' \) denote the Gram matrix of \( \Delta_4' \). We may assume that \( 2\xi = \xi' \).

Note that \( N(2\Delta_4', \xi) \) is equal to the number of pairs of matrices \( A', A'' \in M_4(\mathbb{Z}[\sqrt{2}]) \) such that \( \xi'[A'] + \xi'[A''] = \xi' \). Let \( (A', A'') \) be such a pair. Put

\[
e_1 = \iota(1, 0, 0, 0), \; e_2 = \iota(0, 1, 0, 0), \; e_3 = \iota(0, 0, 1, 0), \; e_4 = \iota(0, 0, 0, 1),
\]

\[
a'_i = A'e_i, \quad a''_i = A''e_i, \quad T' = \{ i \mid a'_i \neq 0 \}, \quad T'' = \{ i \mid a''_i \neq 0 \}.
\]

Since \( 2 = \xi_4'[e_i] = \xi_4'[a'_i] + \xi_4'[a''_i] \), we have \( T' \cap T'' = \emptyset \). Therefore if \( i \in T' \) and \( j \in T'' \), then since \( a'_i = 0 \) and \( a''_j = 0 \), we get

\[
\iota(e_i)\xi_4'\iota(e_j) = \iota(a'_i)\xi_4'a'_j + \iota(a''_i)\xi_4'a''_j = 0,
\]

which is a contradiction. Thus \( T' = \emptyset \) or \( T'' = \emptyset \) and hence \( A' = 0 \) or \( A'' = 0 \).

Since the mass of \( \Delta_4' \) is \( \frac{1}{8\cdot 32} \) (see p. 370 of [18]), we have \( N(2\Delta_4', \xi) = 2E(\Delta_4') = 2^9 \cdot 3^2 \).

Remark 12.6. (1) One can give an analytic proof by employing the Siegel-Eisenstein formula (cf. [31]).

(2) The proof gives \( E(2\Delta_4') = 2E(\Delta_4')^2 = 2^{17} \cdot 3^4 \).

12.5. The degree of \( f_i \).

Proposition 12.7. We have

\[
\deg f_1 = 0, \quad \deg f_2 = 4, \quad \deg f_4 = 1, \quad \deg f_5 = \deg f_6 = 2.
\]

Proof. Since \( \Theta^{(m)}(f_1) \) equals the Siegel Eisenstein series for every \( m \) by the Siegel-Weil formula [38], we understand that \( \deg f_1 = 0 \) (cf. Proposition 12.3). Indeed, the \( q \)-Satake parameter of the trivial representation is \( \{ 2^1, 2^2, 2, 1, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2} \} \) and \( 2^3(1 + \sum_{i=3}^3 2^i) = 135 \) is the eigenvalue of \( f_1 \) (see Table IV and Proposition 12.4).

We can appeal Corollary 11.10 to see that \( \text{Lift}_4(\phi_4) \in \Theta^{(4)}(V(\mathcal{Y}_2)) \). Since

\[
2^3(2^{3/2} + 2^{1/2} + 2^{-1/2} + 2^{-3/2})(\beta_q + \beta_q^{-1}) = -30
\]

agrees with the eigenvalue of \( f_2 \), Proposition 12.4 and Lemma 12.5 imply that \( \text{Lift}_4(\phi_4) \) and \( \Theta^{(4)}(f_2) \) are equal up to scalar. Hence \( \deg f_2 = 4 \).

Theorem 1 of [35] shows that \( \phi_4 \in \Theta^{(1)}(V(\mathcal{Y}_2)) \). Since

\[
2^3(\beta_q^2 + \beta_q^{-2} + 2^2 + 2 + 1 + 2^{-1} + 2^{-2}) = 58
\]

agrees with the eigenvalue of \( f_4 \), Proposition 12.4 and Lemma 12.5 conclude that \( \phi_4 \) equals \( \Theta^{(1)}(f_4) \) up to scalar. Hence \( \deg f_4 = 1 \).

We consider the Saito-Kurokawa lifting \( \text{Lift}_2(\phi_6^\pm) \in S_4(\Gamma_2[0^{-1}, \sigma]) \). We invoke Theorem 1 of [35] to see that \( \text{Lift}_2(\phi_6^\pm) \in \Theta^{(2)}(V(\mathcal{Y}_2)) \). Since

\[
2^3[(2^{-1/2} + 2^{1/2})(\gamma_q^\pm + (\gamma_q^\pm)^{-1}) + 2 + 1 + 2^{-1}] = 33 \pm 3\sqrt{73}
\]
agree with the eigenvalues of $f_5$ and $f_6$. Proposition 12.4 and Lemma 12.3 imply that $\text{Lift}_2(\phi^+_6)$ (resp. $\text{Lift}_2(\phi^-_6)$) is a multiple of $\Theta(2)(f_5)$ (resp. $\Theta(2)(f_6)$). Hence $\deg f_5 = \deg f_6 = 2$.

To determine $\deg f_3$, we follow the method of Ikeda [21]. We define a multiplication $x \circ y$ and an inner product $(\cdot, \cdot)$ on $V(Y_2)$ by

$$[L] \circ [K] = [L] \delta_{[L],[K]}, \quad ([L], [K]) = E(L)^{-1} \delta_{[L],[K]}.$$  

**Remark 12.8.** The normalization is different from that of Ikeda [21]. Our $[L]$ corresponds to $[L]/E(L)$ in [21]. In terms of algebraic automorphic forms, these are just the product and inner product of algebraic automorphic forms.

**Proposition 12.9.** Put

$$n_i = \deg f_i, \quad n_j = \deg f_j, \quad F_i = \Theta^{(n_i)}(f_i), \quad F_j = \Theta^{(n_j)}(f_j).$$

Then

$$\langle \Theta^{(n_i+n_j)}(f_k)|_{\mathcal{H}_2^2 \times \mathcal{H}_2^3}, F_i \times F_j \rangle = \frac{\langle F_i, F_i \rangle \langle F_j, F_j \rangle}{\langle f_i, f_i \rangle \langle f_j, f_j \rangle} (f_k, f_i \circ f_j).$$

In particular, if $(f_k, f_i \circ f_j) \neq 0$, then $\deg f_k \leq \deg f_i + \deg f_j$.

**Proof.** The proof is similar to Lemma 7.1 of [21] and omitted.

**Proposition 12.10.** $\deg f_3 = 3$.

**Proof.** One can easily verify $(f_3, f_4 \circ f_5) \neq 0$ and $(f_2, f_3 \circ f_4) \neq 0$. We combine Propositions 12.4 and 12.4 to have $\deg f_3 \leq 3$ and $\deg f_3 \geq 3$. (See also Nebe and Venkov [11, Proposition 2.3]).

**Corollary 12.11.** The Miyawaki lift $\mathcal{F}^{(1)}_{\phi_4,\phi_4}$ is nonzero.

**Proof.** We have seen that

$$0 \neq \langle \Theta^{(2)}(f_2)|_{\mathcal{H}_2^2 \times \mathcal{H}_2^3}, \phi_4 \times \Theta^{(3)}(f_3) \rangle = \langle \mathcal{F}^{(1)}_{\phi_4,\phi_4}, \Theta^{(3)}(f_3) \rangle$$

in the proof of Proposition 12.10.

**Remark 12.12.** One can show that $\mathcal{F}^{(1)}_{\phi_4,\phi_4}$ is a multiple of $\Theta^{(3)}(f_3)$ by employing Theorem 11.6 of [11] and arguing as in the proof of Proposition 12.4.

**Corollary 12.13.** The 6 theta series $\theta^{(4)}_{E_8}, \theta^{(4)}_{2\Delta_4}, \theta^{(4)}_{\Delta_8}, \theta^{(4)}_{2\Delta_4}, \theta^{(4)}_{4\Delta_2}$ and $\theta^{(4)}_{\Phi}$ are linearly independent. Every degree 4 cusp form spanned by the 6 theta series is a constant multiple of

$$\text{Lift}_4(\phi_4) = \frac{\theta^{(4)}_{E_8}}{2^8 \cdot 3^2 \cdot 5 \cdot 7} + \frac{\theta^{(4)}_{2\Delta_4}}{2^9 \cdot 3^2} - \frac{\theta^{(4)}_{\Delta_8}}{2^8 \cdot 3 \cdot 7} + \frac{\theta^{(4)}_{2\Delta_4}}{2^8 \cdot 3^2} - \frac{\theta^{(4)}_{4\Delta_2}}{2^7 \cdot 3^2 \cdot 5}.$$
Proof. Since $\Theta^{(i)}(f_i)$ (1 ≤ i ≤ 6) are nonzero and have different eigenvalues by Propositions 12.3 and 12.7, they are linearly independent. By Propositions 12.3, 12.7, and 12.10 $\Theta^{(4)}(f_2)$ is a cusp form but $\Theta^{(i)}(f_i)$ (i ≠ 2) are orthogonal to cusp forms. We combine (11.1) with Tables I and III to obtain

$$
\Theta^{(4)}(f_2) = \frac{\theta^{(4)}_{E_8}}{2^{14} \cdot 3^2 \cdot 5 \cdot 7} + \frac{\theta^{(4)}_{2\Delta'_4}}{2^{15} \cdot 3^2} - \frac{\theta^{(4)}_{2\Delta_4}}{2^{14} \cdot 3 \cdot 7} + \frac{\theta^{(4)}_{2\Delta'_2}}{2^{14} \cdot 3^2} - \frac{\theta^{(4)}_{4\Delta_2}}{2^{15} \cdot 3} + \frac{\theta^{(4)}_{\emptyset}}{2^{13} \cdot 3^2 \cdot 5}.
$$

Let $\xi'_4$ be the Gram matrix of $\Delta'_4$. The $\frac{1}{2}\xi'_4$th Fourier coefficient of $\Theta^{(4)}(f_2)$ is

$$
\frac{N(2\Delta'_4,\Delta'_4)}{2^{15} \cdot 3^2} = 2^6
$$

by Lemma 12.3. Since Lift$_4(\phi_4)$ is proportional to $\Theta^{(4)}(f_2)$ by Corollary 11.3 and its $\frac{1}{2}\xi'_4$th Fourier coefficient is 1, we conclude that Lift$_4(\phi_4) = 2^6\Theta^{(4)}(f_2)$.

□

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