Abstract. We consider a $C^*$-algebra $E_{\mathcal{M}}$ of a smooth 4-dimensional manifold $\mathcal{M}$ introduced by Gábor Etesi. It is proved that the $E_{\mathcal{M}}$ is a stationary AF-algebra. We calculate the topological and smooth invariants of $\mathcal{M}$ in terms of the K-theory of the $C^*$-algebra $E_{\mathcal{M}}$. It is shown that the smooth structures on $\mathcal{M}$ form a torsion abelian group under the connected sum operation. The latter is isomorphic to the Brauer group of a number field generated by traces on the $K_0(E_{\mathcal{M}})$.

1. Introduction

Algebraic topology of the 4-dimensional manifolds $\mathcal{M}$ is a vast uncharted area of mathematics. Unlike dimensions 2 and 3, the smooth structures are detached from the topology of $\mathcal{M}$. Due to the works of Rokhlin, Freedman and Donaldson, it is known that $\mathcal{M}$ can be non-smooth and if there exists a smooth structure, it need not be unique. The classification of all smoothings of $\mathcal{M}$ is an open problem.

Let $\mathcal{M}$ be a smooth 4-dimensional manifold. Denote by $\text{Diff}(\mathcal{M})$ a group of the orientation-preserving diffeomorphisms of $\mathcal{M}$ and let $\text{Diff}_0(\mathcal{M})$ be a connected component of $\text{Diff}(\mathcal{M})$ containing the identity. The group $\text{Diff}(\mathcal{M})/\text{Diff}_0(\mathcal{M})$ is discrete and therefore locally compact.

Definition 1.1. By the Etesi $C^*$-algebra $E_{\mathcal{M}}$ one understands a group $C^*$-algebra of the locally compact group $\text{Diff}(\mathcal{M})/\text{Diff}_0(\mathcal{M})$.

Remark 1.2. The original definition uses a unitary representation of the group $\text{Diff}(\mathcal{M})$ alone, see [Etesi 2016] [3, Section 2]. Notice that the $\text{Diff}(\mathcal{M})$ is a Fréchet manifold and therefore the standard construction of group $C^*$-algebra of a locally compact group [Blackadar 1986] [1, Section 11.1] fails; hence 1.1. However, the two definitions are essentially equivalent.

The aim of our note is a classification of the smooth structures on $\mathcal{M}$ based on the K-theory of the Etesi $C^*$-algebra $E_{\mathcal{M}}$. To formalize our results, recall that a $C^*$-algebra is called Approximately Finite-dimensional (AF-) if it is an inductive limit of the multi-matrix $C^*$-algebras $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ [Blackadar 1986] [1, Section 7]. The AF-algebra is called stationary, if the inductive limit depends on a single positive integer matrix $A \in GL(n, \mathbb{Z})$ [Blackadar 1986] [1, Section 7.2] or [10, Section 3.5.2]. Our main result can be formulated as follows.

Theorem 1.3. The Etesi $C^*$-algebra $E_{\mathcal{M}}$ is a stationary AF-algebra.
Let $\lambda_A > 1$ be the Perron-Frobenius eigenvalue of the positive matrix $A$ defined by $E_A$. Consider a number field $K = \mathbb{Q}(\lambda_A)$. The eigenvector $(v_1, \ldots, v_n)$ corresponding to $\lambda_A$ can always be scaled so that $v_i \in K$. By $m := \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ we understand a $\mathbb{Z}$-module in the field $K$ and by $\Lambda$ the ring of endomorphisms of $m$. Let $[m]$ be an ideal class of $m$ in the ring $\Lambda$. The K-theory of stationary AF-algebras says that the triples $(\Lambda, [m], K)$ in a one-to-one correspondence with the Morita equivalence classes of $E_M$ [Handelman 1981] [6], see also [10, Theorem 3.5.4]. Finally, we let $Br(K)$ to be the Brauer group of the number field $K$, i.e. a torsion abelian group of the Morita equivalence classes of the central simple algebras over $K$. An application of theorem 1.3 is as follows.

**Corollary 1.4.** If $\mathcal{M}$ is a compact 4-manifold, then there exists a one-to-one correspondence between:

(i) topological type of $\mathcal{M}$ and the Handelman triples $(\Lambda, [m], K)$;

(ii) smoothings of $\mathcal{M}$ and elements of the Brauer group $Br(K)$.

In particular, all smoothings of $\mathcal{M}$ form a torsion abelian group $Br(K)$ under operation of the connected sum of manifolds.

**Remark 1.5.** Since $Br(K)$ classifies the division algebras over $K$, one gets a functor from the smooth 4-manifolds to the hyper-algebraic number fields, i.e. fields with non-commutative multiplication. Such a functor appears independently in the arithmetic topology of the 4-dimensional manifolds [11].

The article is organized as follows. Some preliminary facts can be found in Section 2. Theorem 1.3 and corollary 1.4 are proved in Section 3. We conclude by remarks in Section 4.

2. Preliminaries

In this section we briefly review the $C^*$-algebras, their K-theory and the 4-dimensional manifolds. We refer the reader to [Blackadar 1986] [1], [Dixmier 1977] [2] and [Gompf 1984] [5] for the details.

2.1. $C^*$-algebras. The $C^*$-algebra is an algebra $\mathcal{A}$ over $\mathbb{C}$ with a norm $a \mapsto ||a||$ and an involution $\{ a \mapsto a^* \mid a \in \mathcal{A} \}$ such that $\mathcal{A}$ is complete with respect to the norm, and such that $||ab|| \leq ||a|| ||b||$ and $||a^*a|| = ||a||^2$ for every $a, b \in \mathcal{A}$. Each commutative $C^*$-algebra is isomorphic to the algebra $C_0(X)$ of continuous complex-valued functions on some locally compact Hausdorff space $X$. Any other algebra $\mathcal{A}$ can be thought of as a noncommutative topological space.

An AF-algebra (Approximately Finite-dimensional $C^*$-algebra) is defined to be the norm closure of an ascending sequence of finite dimensional $C^*$-algebras $M_n$, where $M_n$ is the $C^*$-algebra of the $n \times n$ matrices with entries in $\mathbb{C}$. Here the index $n = (n_1, \ldots, n_k)$ represents the semi-simple matrix algebra $M_n = M_{n_1} \oplus \cdots \oplus M_{n_k}$. The ascending sequence mentioned above can be written as

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \ldots ,$$

(2.1)

where $M_i$ are the finite dimensional $C^*$-algebras and $\varphi_i$ the homomorphisms between such algebras. If $\varphi_i = Const$, then the AF-algebra $\mathcal{A}$ is called stationary.

The homomorphisms $\varphi_i$ can be arranged into a graph as follows. Let $M_i = M_1 \oplus \cdots \oplus M_{n_k}$ and $M_{i'} = M_1' \oplus \cdots \oplus M_{n_k}'$ be the semi-simple $C^*$-algebras and $\varphi_i : M_i \to M_{i'}$ the homomorphism. One has two sets of vertices $V_1, \ldots, V_{ik}$ and
$V_1, \ldots, V_k$ joined by $a_{rs}$ edges whenever the summand $M_{k_j}$ contains $a_{rs}$ copies of the summand $M_i$ under the embedding $\varphi_i$. As $i$ varies, one obtains an infinite graph called the Bratteli diagram of the AF-algebra. The matrix $A = (a_{rs})$ is known as a partial multiplicity matrix; an infinite sequence of $A_i$ defines a unique AF-algebra. If $\mathscr{A}$ is a stationary AF-algebra, then $A_i = \text{Const}$ for all $i \geq 1$.

2.2. $K$-theory of AF-algebras. By $M_\infty(\mathscr{A})$ one understands the algebraic direct limit of the $C^*$-algebras $M_n(\mathscr{A})$ under the embeddings $a \mapsto \text{diag}(a, 0)$. The direct limit $M_\infty(\mathscr{A})$ can be thought of as the $C^*$-algebra of infinite-dimensional matrices whose entries are all zero except for a finite number of the non-zero entries taken from the $C^*$-algebra $\mathscr{A}$. Two projections $p, q \in M_\infty(\mathscr{A})$ are equivalent, if there exists an element $v \in M_\infty(\mathscr{A})$, such that $p = vv^* v$ and $q = vv^*$. The equivalence class of projection $p$ is denoted by $[p]$. We write $V(\mathscr{A})$ to denote all equivalence classes of projections in the $C^*$-algebra $M_\infty(\mathscr{A})$, i.e. $V(\mathscr{A}) := \{[p] : p = p^* = p^2 \in M_\infty(\mathscr{A})\}$. The set $V(\mathscr{A})$ has the natural structure of an abelian semi-group with the addition operation defined by the formula $[p] + [q] := \text{diag}(p, q) = [p' \oplus q']$, where $p' \sim p, q' \sim q$ and $p' \perp q'$. The identity of the semi-group $V(\mathscr{A})$ is given by $[0]$, where $0$ is the zero projection. By the $K_0$-group $K_0(\mathscr{A})$ of the unital $C^*$-algebra $\mathscr{A}$ one understands the Grothendieck group of the abelian semi-group $V(\mathscr{A})$, i.e. a completion of $V(\mathscr{A})$ by the formal elements $[p] - [q]$. The image of $V(\mathscr{A})$ in $K_0(\mathscr{A})$ is a positive cone $K_0^+(\mathscr{A})$ defining the order structure on the abelian group $K_0(\mathscr{A})$. The pair $(K_0(\mathscr{A}), K_0^+(\mathscr{A}))$ is known as a dimension group of the $C^*$-algebra $\mathscr{A}$. The scale $\Sigma(\mathscr{A})$ is the image in $K_0^+(\mathscr{A})$ of the equivalence classes of projections in the $C^*$-algebra $\mathscr{A}$. The $\Sigma(\mathscr{A})$ is a generating, hereditary and directed subset of $K_0^+(\mathscr{A})$, i.e. (i) for each $a \in K_0^+(\mathscr{A})$ there exist $a_1, \ldots, a_r \in \Sigma(\mathscr{A})$ such that $a = a_1 + \cdots + a_r$; (ii) if $0 \leq a \leq b \in \Sigma(\mathscr{A})$, then $a \in \Sigma(\mathscr{A})$ and (iii) given $a, b \in \Sigma(\mathscr{A})$ there exists $c \in \Sigma(\mathscr{A})$, such that $a, b \leq c$. Each scale can always be written as $\Sigma(\mathscr{A}) = \{a \in K_0^+(\mathscr{A}) \mid 0 \leq a \leq u\}$, where $u$ is an order unit of $K_0^+(\mathscr{A})$. The pair $(K_0(\mathscr{A}), K_0^+(\mathscr{A}))$ and the triple $(K_0(\mathscr{A}), K_0^+(\mathscr{A}), \Sigma(\mathscr{A}))$ are invariants of the Morita equivalence and isomorphism class of the $C^*$-algebra $\mathscr{A}$, respectively. If $\mathbb{A}$ is an AF-algebra, then its scaled dimension group (dimension group, resp.) is a complete invariant of the isomorphism (Morita equivalence, resp.) class of $\mathbb{A}$, see e.g. [10, Theorem 3.5.2].

Let $\tau$ be the canonical trace on the AF-algebra $\mathbb{A}$. Such a trace induces a homomorphism $\tau_\mathbb{A} : K_0(\mathbb{A}) \to \mathbb{R}$ and we let $m := \tau_\mathbb{A}(K_0(\mathbb{A})) \subset \mathbb{R}$. If $\mathbb{A}$ is the stationary AF-algebra given by a matrix $A \in GL(n, \mathbb{Z})$, then $m$ is a $\mathbb{Z}$-module in the number field $K = \mathbb{Q}(\lambda_A)$ generated by the Perron-Frobenius eigenvalue $\lambda_A$ of the matrix $A$. The endomorphism ring of $m$ is denoted by $\Lambda$ and the ideal class of $m$ is denoted by $[m]$. The triple $(\Lambda, [m], K)$ is an invariant of the Morita equivalence class of $\mathbb{A}$ [Handelman 1981] [6].

Remark 2.1. Each stationary AF-algebra defines a torsion abelian group. Indeed, let $?_n(x)$ be the $n$-dimensional Minkowski question-mark function, see [Minkowski 1904] [9, p.172] for $n = 2$ and [Panti 2008] [12, Theorem 3.5] for $n \geq 2$. The $?_n(x) : [0,1]^{n}\to [0,1]^{n}$ is a continuous function with the following properties: (i) $?_n(0) = 0$ and $?_n(1) = 1$; (ii) $?_n(Q^{n-1}) = (\mathbb{Z}[\frac{1}{2}])^{n-1}$ are dyadic rationals and (iii) $?_n(K^{n-1}) = (\mathbb{Q} - \mathbb{Z}[\frac{1}{2}])^{n-1}$, where $K$ are algebraic numbers of degree $n$ over $\mathbb{Q}$. It is not hard to see, that (iv) $?_n(\Delta) = \Delta$ is a monotone function, where $\Delta = [0, 1]$ is the normalized diagonal of the simplex $[0,1]^{n-1}$. Recall that $\tau_\mathbb{A}(K_0(\mathbb{A})) = m$.
and \( \tau_*(\Sigma(\mathbb{A})) = \mathfrak{m} \cap [0, 1] \), where \( \tau \) is the canonical trace on the AF-algebra \( \mathbb{A} \) and \( \mathfrak{m} \) is a \( \mathbb{Z} \)-module in the number field \( K \). We assume that \( \tau_*(K_0(\mathbb{A})) \subset \Delta \). By the properties (iii) and (iv) of the Minkowski question-mark function, one gets the following inclusion:

\[
\mathcal{Y} := \{ n(\tau_*(\Sigma(\mathbb{A}))) \} \subset \mathbb{Q}/\mathbb{Z}.
\]  

(2.2)

**Definition 2.2.** By the Minkowski group \( M_i(K) \) of stationary AF-algebra we understand a torsion abelian group generated by the elements of set \( \mathcal{Y} \).

2.3. 4-dimensional manifolds. We denote by \( \mathcal{M} \) a smooth 4-dimensional manifold and always assume \( \mathcal{M} \) to be compact. Let \( S^4 \) be the 4-dimensional sphere and \( X_g \) be a closed 2-dimensional orientable surface of genus \( g \geq 0 \). By the knotted surface \( \mathcal{X} := X_{g_1} \cup \ldots \cup X_{g_n} \) in \( \mathcal{M}^4 \) one understands a transverse immersion of a collection of \( n \geq 1 \) surfaces \( X_{g_i} \) into \( \mathcal{M} \). We refer to \( \mathcal{X} \) a surface knot if \( n = 1 \) and a surface link if \( n \geq 2 \).

**Theorem 2.3.** ([Piergallini 1995] [13]) Each smooth 4-dimensional manifold \( \mathcal{M} \) is the 4-fold PL cover of the sphere \( S^4 \) branched at the points of a knotted surface \( \mathcal{X} \subset S^4 \).

Let \( S^2 \) be the 2-dimensional sphere. By \( S(k) \) we understand a smooth 4-dimensional manifold corresponding to a connected sum

\[
S(k) := (S^2 \times S^2) \# \ldots \# (S^2 \times S^2).
\]

(2.3)

**Theorem 2.4.** ([Gompf 1984] [5]) Let \( \mathcal{M} \) and \( \mathcal{M}' \) be two different smoothings of a topological manifold \( \mathcal{M}_{top} \). Then for sufficiently large \( k \) there exists an orientation-preserving diffeomorphism:

\[
\mathcal{M} \# S(k) \rightarrow \mathcal{M}' \# S(k).
\]

(2.4)

**Remark 2.5.** The smooth structures on \( \mathcal{M}_{top} \) can be classified using 2.4. Indeed, one can see that \( \pi_1(\mathcal{M} \# S(k)) \cong \pi_1(\mathcal{M}) \). Assuming the Borel Conjecture for the manifolds in question, we conclude that \( \mathcal{M} \# S(k) \) and \( \mathcal{M} \) are homeomorphic 4-manifolds. However, the manifolds \( \mathcal{M} \# S(k) \) and \( \mathcal{M} \) are not diffeomorphic, since otherwise by (2.4) the manifolds \( \mathcal{M} \) and \( \mathcal{M}' \) would be diffeomorphic. Such a conclusion contradicts the assumption of Gompf’s Theorem. Moreover, each pair of distinct smoothing of \( \mathcal{M}_{top} \) can be obtained using formula (2.4).

3. Proofs

3.1. **Proof of theorem 1.3.** We shall split the proof in two lemmas.

**Lemma 3.1.** The \( C^* \)-algebra \( E_{\mathcal{M}} \) is an AF-algebra.

**Proof.** The lemma follows from an observation that the \( E_{\mathcal{M}} \) is a group \( C^* \)-algebra of a locally compact group, see definition 1.1. Then the \( C^* \)-algebra \( E_{\mathcal{M}} \) is an AF-algebra, see e.g. [Blackadar 1986] [1, Corollary 11.1.2]. This fact was proved independently by Gábor Etesi in terms of a von Neumann algebra related to \( E_{\mathcal{M}} \). Namely, such an algebra was shown to be hyperfinite, see [Etesi 2017] [4, Lemma 2.3]. For the sake of clarity, we adapt the proof to the case of the \( C^* \)-algebra \( E_{\mathcal{M}} \).
Let $\mathcal{G} := \text{Diff}(\mathcal{M})/\text{Diff}_0(\mathcal{M})$. Consider a profinite completion of the discrete group $\mathcal{G}$, i.e.

$$\widehat{\mathcal{G}} := \lim_{\leftarrow} \mathcal{G}/N,$$

(3.1)

where $N$ ranges through the open normal finite index subgroups of $\mathcal{G}$. Recall that if $G$ is a finite group, then the group algebra $C[G]$ has the form

$$C[G] \cong M_{n_1}(C) \oplus \cdots \oplus M_{n_k}(C),$$

(3.2)

where $n_i$ are degrees of the irreducible representations of $G$ and $h$ is the total number of such representations [Serre 1967] [14, Proposition 10]. In view of (3.1), we have

$$\widehat{\mathcal{G}} \cong \lim_{\leftarrow} G_i,$$

(3.3)

where $G_i$ is a finite group. Consider a group algebra

$$C[G_i] \cong M_{n_i}^{(i)}(C) \oplus \cdots \oplus M_{n_h}^{(i)}(C),$$

(3.4)

corresponding to $G_i$. Notice that the $C[G_i]$ is a finite-dimensional $C^*$-algebra. The inverse limit (3.3) defines an ascending sequence of the finite-dimensional $C^*$-algebras of the form

$$\lim\limits_{\leftarrow} M_{n_i}^{(i)}(C) \oplus \cdots \oplus M_{n_h}^{(i)}(C).$$

(3.5)

The group $C^*(\widehat{\mathcal{G}})$ of the profinite group $\widehat{\mathcal{G}}$ is the norm closure of the group algebra $C[\widehat{\mathcal{G}}]$ [Dixmier 1977] [2, Section 13.9]. One concludes from (3.5) that the $C^*(\widehat{\mathcal{G}})$ is an AF-algebra.

On the other hand, the canonical homomorphism $\mathcal{G} \to \widehat{\mathcal{G}}$ gives rise to an extension of the $C^*$-algebras:

$$C^*(\mathcal{G}) \to C^*(\widehat{\mathcal{G}}) \to \mathcal{B}.$$ 

(3.6)

Since $C^*(\widehat{\mathcal{G}})$ is an AF-algebra, both $C^*(\mathcal{G})$ and $\mathcal{B}$ must be AF-algebras [Handelman 1982] [7, Lemma I.5(a)]. It remains to recall that $C^*(\mathcal{G}) := \mathbb{E}_\mathcal{M}$, see definition 1.1. Thus $\mathbb{E}_\mathcal{M}$ is an AF-algebra. Lemma 3.1 is proved. □

**Lemma 3.2.** The AF-algebra $\mathbb{E}_\mathcal{M}$ is stationary.

**Proof.** (i) Let $S^4$ be the 4-dimensional sphere. By the Piergallini Theorem 2.3, there exists a 4-fold covering map $\mathcal{M} \to S^4$ branched at the points of a knotted surface $\mathcal{X}$ defined by an embedding:

$$\mathcal{X} \hookrightarrow S^4.$$ 

(3.7)

In view of the inclusion $\text{Diff}(S^4) \subset \text{Diff}(S^4 - \mathcal{X})$, one gets an injective homomorphism of the $C^*$-algebras:

$$\mathbb{E}_{S^4} \hookrightarrow \mathbb{E}_\mathcal{M}.$$ 

(3.8)

(ii) Let us show that $\mathbb{E}_{S^4} \cong C$. Indeed, since $\text{Diff}(S^4) \cong \text{Diff}_0(S^4)$, the group $\text{Diff}(S^4)/\text{Diff}_0(S^4)$ is trivial. In particular, the group $C^*$-algebra $\mathbb{E}_{S^4}$ is commutative. The Gelfand Theorem says that $\mathbb{E}_{S^4} \cong C_0(X)$, where $C_0(X)$ is the $C^*$-algebra of continuous complex-valued functions on a locally compact Hausdorff space $X$, see Section 2.1. But $X \cong \text{pt}$ is a singleton and therefore $C_0(\text{pt}) \cong C$. Thus one gets $\mathbb{E}_{S^4} \cong C$.

(iii) The AF-algebra $\mathbb{E}_{S^4}$ is given by an ascending sequence (2.1) of the form:

$$C \overset{1}{\to} C \overset{1}{\to} \cdots,$$

(3.9)
where \( \mathbf{1} \) is the identity homomorphism. It follows from (3.9) that the \( E_{S^4} \) is a stationary AF-algebra.

(iv) Since \( K_0(C) \cong \mathbb{Z} \), we conclude that the dimension group of the AF-algebra \( E_{S^4} \) is isomorphic to \((\mathbb{Z}, \mathbb{Z}^+)\), where \( \mathbb{Z}^+ \) is the semi-group of positive integers. It is easy to see, that the Handelman triple corresponding to the stationary AF-algebra \( E_{S^4} \) has the form \((\mathbb{Z}, [\mathbb{Z}], \mathbb{Q})\).

(v) On the other hand, the map (3.8) induces an inclusion of the abelian groups:

\[
K_0(E_{S^4}) \subset K_0(E_{\mathbb{M}}).
\]  

Moreover, if \( \tau \) is the canonical trace on the AF-algebra \( E_{\mathbb{M}} \), one gets from (3.10) the following inclusion of the additive groups of the real line:

\[
\mathbb{Z} \subset \tau_*(K_0(E_{\mathbb{M}})).
\]  

(vi) Since \( \mathbb{Z} \) is a ring, the group inclusion (3.11) can be extended to such of the rings. But the only finite degree extension of the ring \( \mathbb{Z} \) coincides (up to a scaling constant) with an order \( \Lambda \) in the number field \( K = \Lambda \otimes \mathbb{Q} \). We conclude that

\[
\tau_*(K_0(E_{\mathbb{M}})) \subset K,
\]  

where \( K \) is a real number field.

(vii) To finish the proof of lemma 3.2, it remains to apply the result of [Handelman 1981] [6, Theorem II (iii)] saying that condition (3.12) is equivalent to the AF-algebra \( E_{\mathbb{M}} \) to be of a stationary type.

Lemma 3.2 is proved.

\[\square\]

Remark 3.3. The Etesi C*-algebra \( E_{\mathbb{M}} \) is simple, i.e. has only trivial two-sided ideals. This fact follows from lemma 3.2 and strict positivity of the matrix \( A \) corresponding to the stationary AF-algebra.

Returning to the proof of theorem 1.3, we apply lemmas 3.1 and 3.2. Theorem 1.3 follows.

3.2. Proof of corollary 1.4. We split the proof in a series of lemmas.

Lemma 3.4. The Etesi C*-algebras satisfy an isomorphism:

\[
E_{\mathbb{M}_1 \# \mathbb{M}_2} \cong E_{\mathbb{M}_1} \otimes E_{\mathbb{M}_2},
\]  

where \( \# \) is the connected sum of manifolds and \( \otimes \) is the tensor product of C*-algebras.
Proof. We let $\mathcal{G} := \text{Diff}(\mathcal{M}_1 \# \mathcal{M}_2)/\text{Diff}_0(\mathcal{M}_1 \# \mathcal{M}_2)$. It is not hard to see, that

$$\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2,$$

(3.14)

where $\mathcal{G}_1 = \text{Diff}(\mathcal{M}_1)/\text{Diff}_0(\mathcal{M}_1)$ and $\mathcal{G}_2 = \text{Diff}(\mathcal{M}_2)/\text{Diff}_0(\mathcal{M}_2)$. It is well known that the group ring $\mathbb{C}[\mathcal{G}]$ of the product (3.14) is given by the formula:

$$\mathbb{C}[\mathcal{G}] \cong \mathbb{C}[\mathcal{G}_1] \otimes \mathbb{C}[\mathcal{G}_2].$$

(3.15)

Since the $\mathbb{E}_{\mathcal{M}}$ is a nuclear $C^*$-algebra, the norm closure of a self-adjoint representation of (3.15) defines an isomorphism $\mathbb{E}_{\mathcal{M}_1 \# \mathcal{M}_2} \cong \mathbb{E}_{\mathcal{M}_1} \otimes \mathbb{E}_{\mathcal{M}_2}$. Lemma 3.4 is proved.

Lemma 3.5. The Etesi $C^*$-algebra of the 4-manifold $S^2 \times S^2$ is given by the formula:

$$\mathbb{E}_{S^2 \times S^2} \cong M_2(\mathbb{C}).$$

(3.16)

Proof. (i) It is known, that the 4-manifold $S^2 \times S^2$ is a 2-fold cover of the 4-sphere $S^4$ given by an involution $H := \{x, y, (x, y) \mapsto (y, x), (x, y) \mapsto (-x, -y), (x, y) \mapsto (y, -x) \mid (x, y) \in S^2 \times S^2\}$, such that

$$(S^2 \times S^2)/H \cong S^4,$$

(3.17)

see e.g. [Massey 1973] [8, p. 372]. The covering map (3.17) induces a homomorphism of the $C^*$-algebras $\mathbb{E}_{S^2 \times S^2} \rightarrow \mathbb{E}_{S^4}$ and a homomorphism of the corresponding abelian groups:

$$K_0(\mathbb{E}_{S^2 \times S^2}) \rightarrow K_0(\mathbb{E}_{S^4}).$$

(3.18)

From (3.9) one gets $K_0(\mathbb{E}_{S^4}) \cong \mathbb{Z}$. In view of the involution $H$, the kernel of the map (3.18) is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Thus one gets an isomorphism:

$$K_0(\mathbb{E}_{S^2 \times S^2}) \cong \mathbb{Z}.$$

(3.19)

(ii) On the other hand, it is known that the $\mathbb{E}_{S^2 \times S^2}$ is a stationary AF-algebra, see theorem 1.3. If $A$ is the corresponding matrix, then by (3.19) the eigenvalues of $A$ must be rational and equal to each other. In other words, $A = I_n$ is the identity matrix. Since the involution $N$ is a 2-fold cover of the AF-algebra (3.9), we conclude that $n = 2$. Altogether, the AF-algebra $\mathbb{E}_{S^2 \times S^2}$ is given by an ascending sequence (2.1) of the form:

$$M_2(\mathbb{C}) \left( \begin{array}{c} 1 \ 0 \\ 0 \ 1 \end{array} \right) M_2(\mathbb{C}) \left( \begin{array}{c} 1 \ 0 \\ 0 \ 1 \end{array} \right) \cdots$$

(3.20)

The sequence (3.20) converges to the $C^*$-algebra $M_2(\mathbb{C})$. We conclude that $\mathbb{E}_{S^2 \times S^2} \cong M_2(\mathbb{C})$. Lemma 3.5 is proved.

Corollary 3.6. The Etesi $C^*$-algebra $\mathbb{E}_{\# S^2} \# S(k)$ is Morita equivalent to $\mathbb{E}_{\mathcal{M}}$.

Proof. Recall that the 4-manifold $S(k) = (S^2 \times S^2) \# \cdots \# (S^2 \times S^2)$ is a connected sum of the $k$ copies of $S^2 \times S^2$, see formula (2.3). From lemma 3.5 and formula (3.13) we get an isomorphism:

$$\mathbb{E}_{S(k)} \cong M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C}) \cong M_{2^k}(\mathbb{C}).$$

(3.21)

If $\mathcal{M}$ is a smooth 4-manifold, then by lemma 3.4 and formula (3.21) one obtains an isomorphism:

$$\mathbb{E}_{\mathcal{M} \# S(k)} \cong \mathbb{E}_{\mathcal{M}} \otimes M_{2^k}(\mathbb{C}).$$

(3.22)
Clearly, formula (3.22) implies the Morita equivalence of the $C^*$-algebras $E_{\mathcal{M}}$ and $E_{\mathcal{M}'\#S(k)}$. Corollary 3.6 is proved. □

**Lemma 3.7.** The topological type of manifold $\mathcal{M}$ is determined by the Handelman triple $(\Lambda, [m], K)$.

**Proof.** Roughly speaking, lemma 3.7 follows from Gompf’s Theorem 2.4, see also remark 2.5. The rest of the proof follows from corollary 3.6. We proceed in two steps.

(i) Theorem 2.4 says that for smoothings $\mathcal{M}$ and $\mathcal{M}'$ of a topological manifold $\mathcal{M}_{\text{top}}$, one finds $k \geq 0$ such that $\mathcal{M}\#S(k)$ and $\mathcal{M}'\#S(k)$ are diffeomorphic. Thus one gets an isomorphism:

$$E_{\mathcal{M}\#S(k)} \cong E_{\mathcal{M}'\#S(k)}.$$  (3.23)

In view of corollary 3.6 and formula (3.23), the $C^*$-algebra $E_{\mathcal{M}}$ is Morita equivalent to $E_{\mathcal{M}\#S(k)}$ and the $C^*$-algebra $E_{\mathcal{M}'\#S(k)}$ is Morita equivalent to $E_{\mathcal{M}'}$. Therefore the $C^*$-algebras $E_{\mathcal{M}}$ and $E_{\mathcal{M}'}$ are Morita equivalent by the transitivity property. By remark 2.5, the Morita equivalence class of the Etesi $C^*$-algebra $E_{\mathcal{M}}$ comprises all 4-dimensional manifolds which are homeomorphic but not diffeomorphic to each other.

(ii) Recall that the $E_{\mathcal{M}}$ is a stationary AF-algebra, see lemma 3.2. The Morita equivalence classes of such are described by the Handelman triples $(\Lambda, [m], K)$, see [Handelman 1981] [6], [10, Theorem 3.5.4] or Section 2.2. Comparing this fact with the result of item (i), we conclude that the $(\Lambda, [m], K)$ is an invariant of the topological type of the manifold $\mathcal{M}$.

Lemma 3.7 is proved. □

**Remark 3.8.** Lemma 3.7 says that the topological type of manifold $\mathcal{M}$ is defined by the Morita equivalence class of the Etesi $C^*$-algebra $E_{\mathcal{M}}$. Likewise, remark 2.5 implies that different smooth structures on $\mathcal{M}$ are distinguished by the isomorphism classes of the algebra $E_{\mathcal{M}}$.

**Lemma 3.9.** Let $K$ be a number field and let $Mi (K)$ be the Minkowski group, see definition 2.2. Then:

(i) the map $K \rightarrow Mi (K)$ is a covariant functor which maps isomorphic number fields $K$ to the isomorphic torsion abelian groups $Mi (K)$;

(ii) $Mi (K) \cong Br(K)$.

**Proof.** (i) Let $K$ be a number field corresponding to the Handelman triple $(\Lambda, [m], K)$. Denote by $[\mathcal{A}]$ the Morita equivalence class of stationary AF-algebras defined by the triple $(\Lambda, [m], K)$. Since $\tau$ is the canonical trace (i.e. $\tau(u) = 1$ for the order unit $u \in K_0^+ (\mathcal{A})$), we conclude that $\tau(\Sigma(\mathcal{A})) \subset [0, 1]$ does not depend on $\mathcal{A} \in [\mathcal{A}]$. Thus from (2.2) one gets a correctly defined map $K \rightarrow \mathcal{Y} := Mi (K)$. It can be verified directly that if $K'$ is a real embedding of $K$, then $Mi (K') \cong Mi (K)$. Item (i) is proved.

(ii) Let $Br(K)$ be the Brauer group of a number field $K$. It is well known, that the map $K \rightarrow Br(K)$ is a covariant functor which maps isomorphic number fields $K$ to the isomorphic torsion abelian groups $Br(K)$. Comparing with item (i) we
conclude that there exists a natural transformation between these two functors, see e.g. [10, Section 2.3]. In particular, such a transformation implies an isomorphism of the abelian groups $Br(K)$ and $Mi(K)$. Item (ii) and lemma 3.9 are proved. □

Remark 3.10. Lemma 3.9 can be viewed as part of a correspondence between the algebraic K-theory (e.g. the Milnor-Voevodsky K-theory) and the Galois cohomology. This subject is outside the scope of present note.

Corollary 3.11. Distinct smoothings of $M$ are classified by the elements of the Brauer group $Br(K)$. In particular, the set of all smoothings of $M$ form a torsion abelian group.

Proof. (i) Recall that from (2.2) we have:

$$\mathcal{Y} = ?_n(\tau_*(\Sigma(E_M))).$$

Since the Minkowski function $?_n(x)$ is monotone, formula (3.24) defines a bijective correspondence between generators of the torsion abelian groups $Mi(K) \cong Br(K)$ and the scale $\Sigma(E_M)$ of the Etesi $C^*$-algebra $E_M$.

(ii) Recall that the scale $\Sigma(E_M)$ is a generating subset of $K_0^+(E_M)$, i.e. for each $a \in K_0^+(E_M)$ there exist $a_1, \ldots, a_r \in \Sigma(E_M)$ such that $a = a_1 + \cdots + a_r$. We extend the correspondence of item (i) to a bijective map between the elements of the Brauer group $Br(K)$ and the elements of positive cone $K_0^+(E_M)$.

(iii) It is known that the pair $(K_0(E_M), K_0^+(E_M))$ is invariant of the Morita equivalence class of the AF-algebra $E_M$, while the triple $(K_0(E_M), K_0^+(E_M), \Sigma(E_M))$ is invariant of the isomorphism class of $E_M$, see Section 2.2. Moreover, the scale can be written as $\Sigma(E_M) = \{a \in K_0^+(E_M) \mid 0 \leq a \leq u\}$, where $u \in K_0^+(E_M)$ is fixed. Thus running through all $u \in K_0^+(E_M)$ one gets all possible scales $\Sigma(E_M)$ and vice versa. In other words, the elements $u \in K_0^+(E_M)$ parametrize isomorphism classes of $E_M$ within its Morita equivalence class.

(iv) To finish the proof of corollary 3.11, it remains to recall remark 3.8. Indeed, combing 3.8 with item (iii) we conclude that different smooth structures on $M$ are in bijection with the elements of $K_0^+(E_M)$. Moreover, the $K_0^+(E_M)$ has structure of a torsion abelian group isomorphic to the Brauer group $Br(K)$, see item (ii). Corollary 3.11 is proved. □

Corollary 1.4 follows from lemma 3.7 and corollary 3.11.

4. Remarks

Remark 4.1. If $M$ is a simply connected 4-manifold, then the number field $K$ is a totally real abelian extension of $Q$. This fact follows from [11, Corollary 1.2] saying that the Galois group of a division algebra over $K$ corresponding to a simply connected 4-manifold must be abelian. Indeed, since $Gal(K/Q)$ is a subgroup of the abelian group, one concludes that $Gal(K/Q)$ is an abelian group. Notice that an abelian extension of $Q$ are either a totally real or a CM-field. The latter case is precluded, since $A$ is a positive matrix with the Perron-Frobenius eigenvalue $\lambda_A > 1$ and therefore the number field $K = Q(\lambda_A)$ has a real embedding.
Remark 4.2. It would be interesting to understand the smooth Poincaré Conjecture in terms of 1.4. We know that $K_0(ES^4) \cong \mathbb{Z}$ and $K \cong \mathbb{Q}$. However, since $S^4$ is not aspherical, one cannot assume the Borel Conjecture (see remark 2.5) and apply the classification 1.4 directly. Thus the question is this: Let $\mathcal{S} \subseteq Br(Q)$ be a subgroup of the Brauer group formed by the smooth 4-manifolds homeomorphic to $S^4$. Is the group $\mathcal{S}$ trivial?

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