Geometric ergodicity for some space-time max-stable Markov chains

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Abstract

Max-stable processes are central models for spatial extremes. In this paper, we focus on some space-time max-stable models introduced in Embrechts et al. (2016). The processes considered induce discrete-time Markov chains taking values in the space of continuous functions from the unit sphere of $\mathbb{R}^3$ to $(0, \infty)$. We show that these Markov chains are geometrically ergodic. An interesting feature lies in the fact that the state space is not locally compact, making the classical methodology inapplicable. Instead, we use the fact that the state space is Polish and apply results presented in Hairer (2010).

Key words: Geometric ergodicity; Markov chains with non locally compact state space; Space-time max-stable processes on a sphere; Spectral separability.

1 Introduction

Max-stable processes constitute an extension of multivariate extreme-value theory to the level of stochastic processes (see, e.g., de Haan, 1984; de Haan and Ferreira, 2007) and turn out to be fundamental for the modelling of spatial extremes. In the related literature, measurements are often assumed to be independent in time and, thus, only the spatial structure is studied (see, e.g., Padoan et al., 2010). To the best of our knowledge, only Davis et al. (2013), Huser and Davison (2014), Buhl and Klüppelberg (2016) and Embrechts et al. (2016) propose space-time max-stable processes. The class of models introduced in Embrechts et al. (2016), i.e. the space-time max-stable models with spectral separability, allows to overcome some drawbacks inherent to the approach taken in the other mentioned papers; see Embrechts et al. (2016) for details.

In this study, we focus on a generalised version of the subclass of “models of type 2” defined in [Embrechts et al. (2016), Section 2.1.2. One remarkable feature of the associated models is to be space-time max-stable models on the unit sphere of $\mathbb{R}^3$. Although max-stable processes on a sphere have, to the best of our knowledge, only been considered in Embrechts et al. (2016), such processes can be relevant for applications due to the natural

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spherical shape of planets and stars. As will be seen, in the discrete-time case, some of
the models mentioned directly above induce Markov chains taking values in the space
of continuous functions from the unit sphere of $\mathbb{R}^3$ to $(0, \infty)$. For an excellent review
of Markov chains theory, we refer the reader to Meyn and Tweedie (2009). The main
result of the present paper is the geometric ergodicity of the just mentioned Markov
chains. Since the state space is not locally compact, geometric ergodicity cannot be
obtained using classical results, contained, e.g., in Meyn and Tweedie (2009). Instead,
we take advantage of the fact that the state space is Polish and apply results for Markov
chains with Polish state spaces to be found in Hairer (2010). Conditions for geometric
ergodicity of Markov chains with Polish state spaces have been barely considered in
the literature so far. Geometric ergodicity is a very powerful property: under some
specific moment condition, any transformation on a geometrically ergodic Markov chain
satisfies a central limit theorem; see, e.g., Ibragimov and Linnik (1971), Chapter 19,
Section 1, for uniformly geometrically ergodic (i.e. uniformly ergodic) Markov chains
and Häggström (2005), especially Theorem 1.2, for geometrically ergodic Markov chains.
Moreover, Miasojedow (2014) has shown that some functions of $\pi$-irreducible reversible
generically ergodic Markov chains with Polish state spaces satisfy an inequality of
Hoeffding type; see Remark 3.2 and Theorem 3.3. Hence, geometric ergodicity allows
to obtain results for statistics based on the Markov chain and to carry out statistical
inference.

The remainder of the paper is organised as follows. Section 2 describes the previously
mentioned class of “models of type 2” as well as its generalised version. Then, our Markov
chains are presented and their geometric ergodicity is shown in Section 3. Finally, Section
4 provides a short summary as well as some perspectives. Throughout the paper, “$\sqcup$”
denotes the supremum when the latter is taken over a countable set. Additionally, $\equiv$
stands for equality in distribution. In the case of stochastic processes, this must be
understood as equality of finite-dimensional distributions.

2 The subclass of space-time max-stable models of type
2 and its generalisation

First, we recall the definition of the subclass composed of the “models of type 2” specified
in Embrechts et al. (2016), Section 2.1.2. Before doing so, we need to introduce some
notations and concepts. Let $\mathcal{T}$ be the set of time indices. The mentioned models are
either continuous-time ($\mathcal{T} = \mathbb{R}$) or discrete-time ($\mathcal{T} = \mathbb{Z}$). Let $\lambda$ be the Lebesgue
measure on $\mathbb{R}$ (case $\mathcal{T} = \mathbb{R}$) or the counting measure $\sum_{z \in \mathbb{Z}} \delta_{\{z\}}$, where $\delta$
stands for the Dirac measure (case $\mathcal{T} = \mathbb{Z}$). Denoting by $\|\cdot\|$ the Euclidean norm, we introduce
$S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$, the unit sphere in $\mathbb{R}^3$. Moreover, let $\lambda_{S^2}$ be the Lebesgue
measure on $S^2$. Let $f$ be the von Mises-Fisher probability density function (see, e.g.,
Mardia and Jupp, 1999, Section 9.3.2) on $S^2$ with parameters $\mu \in S^2$ and $\kappa \geq 0$:

$$f(x; \mu, \kappa) = \frac{\kappa}{4\pi \sinh(\kappa)} \exp (\kappa \mu'x), \ x \in S^2,$$

where $\sinh$ designates the hyperbolic sine function and $'$ denotes transposition. The
parameters $\mu$ and $\kappa$ are termed the mean direction and concentration parameter, respec-
tively. The higher the value of $\kappa$, the greater the concentration of the distribution around
the mean direction $\mu$. The distribution is uniform on the sphere for $\kappa = 0$ and unimodal.
for $\kappa > 0$. In addition, for $\mathbf{u} = (u_x, u_y, u_z)' \in S^2$, let $R_{\theta, \mathbf{u}}$ be the rotation matrix of angle $\theta \in \mathbb{R}$ around an axis in the direction of $\mathbf{u}$. The latter is written

$$R_{\theta, \mathbf{u}} = \cos \theta I_3 + \sin \theta [\mathbf{u}]_\times + (1 - \cos \theta) \mathbf{u} \mathbf{u}' ,$$

where $I_3$ is the identity matrix of $\mathbb{R}^3$ and $[\mathbf{u}]_\times$ the cross product matrix of $\mathbf{u}$, defined by

$$[\mathbf{u}]_\times = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix} .$$

Furthermore, let $g$ be a probability density function (case $\mathcal{T} = \mathbb{R}$) or a probability mass function (case $\mathcal{T} = \mathbb{Z}$). Finally, we recall the way we define a Poisson point process on $Z$ in [Embrechts et al. (2016)]. Let $(N_k)_{k \in \mathbb{Z}}$ be independent and identically distributed Poisson$(1)$, where, for $\lambda_p > 0$, Poisson$(\lambda_p)$ stands for the Poisson distribution with parameter $\lambda_p$. The process $N$ defined by $N(A) = \sum_{k \in A} N_k$, $A \subset \mathbb{Z}$, is a Poisson point process on $\mathbb{Z}$ with intensity one. Indeed, the quantity $N(A)$ is Poisson distributed with parameter $\lambda(A)$ and, for any $l \geq 1$ and $A_1, \ldots, A_l$ disjoint sets in $\mathbb{Z}$, the $N(A_i)$, $i = 1, \ldots, l$, are independent random variables. The definition of the “models of type 2” introduced in [Embrechts et al. (2016)] is recalled immediately below.

**Definition 1.** The “models of type 2” in [Embrechts et al. (2016)] are defined by

$$(X(t, x))_{(t, x) \in \mathcal{T} \times S^2} = \left(\bigvee_{i=1}^{\infty} \{U_i g(t - B_i) f \left( R_{\theta_i} \mathbf{u}_i, x \right) \} \right)_{(t, x) \in \mathcal{T} \times S^2} ,$$

where $(U_i, B_i, \mathbf{u}_i)_{i \geq 1}$ are the points of a Poisson point process on $(0, \infty) \times \mathcal{T} \times S^2$ with intensity $u^{-2} du \times \lambda(db) \times \lambda_{S^2}(d\mathbf{u})$.

Second, the class of processes presented right above can be extended by allowing in (2) any probability density function $\tilde{f} : S^2 \rightarrow [0, \infty)$ involving one mean direction parameter in $S^2$, and not only the function $f$ defined in (1). The resulting models constitute the so called generalised subclass of models of type 2. For several examples of probability density functions on spheres, we refer the reader to [Mardia and Jupp (1999)], Section 9.3.

**Remark 1.** The generalised subclass of models of type 2 is included in the class of space-time max-stable models with spectral separability introduced in [Embrechts et al. (2016)], Definition 1. For an explanation about the interest of this class of space-time max-stable models compared to those previously introduced in the literature and an interpretation of its different components, we refer the reader to [Embrechts et al. (2016)].

### 3 Markovian models and geometric ergodicity

In this section, if $\mathcal{T} = \mathbb{R}$, let $g$ be the density of a standard exponential random variable whereas if $\mathcal{T} = \mathbb{Z}$, let $g$ correspond to the probability mass function of a geometric random variable:

$$g(t) = \begin{cases} \nu \exp(-\nu t) \mathbf{1}_{\{t \geq 0\}} & \text{if } \mathcal{T} = \mathbb{R}, \\ (1 - \phi) \phi^t \mathbf{1}_{\{t \geq 0\}} & \text{if } \mathcal{T} = \mathbb{Z}, \end{cases} \quad (3)$$
where $\nu > 0$ and $\phi \in (0, 1)$. Let us denote by $a$ the constant $\exp(-\nu)$ if $T = \mathbb{R}$ and the constant $\phi$ if $T = \mathbb{Z}$. Combining (2) and (3), the models of type 2 described in Section 2 become, for $t \in T$ and $x \in \mathbb{S}^2$,

\[
X(t, x) = \begin{cases} 
\sqrt{\frac{\nu}{2\pi}} \exp(-\nu t) \int_{\mathbb{R}^2} f(R_{t,s}(t, B_t) u x; \mu_t, \kappa) \, du \, d\mu_t & \text{if } T = \mathbb{R}, \\
\sqrt{\frac{\nu}{2\pi}} \int_{\mathbb{R}^2} f(R_{t,s}(t, B_t) u x; \mu_t, \kappa) \, du \, d\mu_t & \text{if } T = \mathbb{Z},
\end{cases}
\]

(4)

where $(U_i, B_t, \mu_t)_{i \geq 1}$ are the points of a Poisson point process on $(0, \infty) \times \mathcal{T} \times \mathbb{S}^2$ with intensity $u^{-2} du \times \lambda(du) \times \lambda_m(u) \mu(t, \kappa)$ and $f$ is given by (1). Similarly as in the case of Markovian models of types 1 and 4 defined in [Embrechts et al. (2016)], Section 3.1, we have the following result.

**Theorem 1.** The process $X((t, x))_{(t, x) \in T \times \mathbb{S}^2}$ defined in (1) satisfies, for all $t, s \in T$ such that $s > 0$ and $x \in \mathbb{S}^2$,

\[
X(t, x) = \max \left\{ a^s X(t - s, R_{\theta s}(t, x)), (1 - a^s) Z(t, x) \right\},
\]

(5)

where the process $(Z(t, x))_{x \in \mathbb{S}^2}$ is independent of $(X(t - s, x))_{x \in \mathbb{S}^2}$ and

\[
(Z(t, x))_{x \in \mathbb{S}^2} \overset{d}{=} \left( \int_{\mathbb{S}^2} f(U_i; \mu_t, \kappa) \right)_{x \in \mathbb{S}^2},
\]

(6)

where $(U_i, \mu_t)_{i \geq 1}$ are the points of a Poisson point process on $(0, \infty) \times \mathbb{S}^2$ with intensity $u^{-2} du \times \lambda_m(du) \mu(t, \kappa)$.

**Proof.** The proof is similar as that of Theorem 3, Bullet (i), in [Embrechts et al. (2016)]. We highlight now the main differences. We consider, for $t, b \in T$ and $x \in \mathbb{S}^2$, $R_{t,b}(t, x) = R_{\theta(t-b), \mu(t,b)}(t, \kappa)$. Furthermore, in order to establish (5), we use the fact that, for all $M \in \mathbb{N}\setminus\{0\}$, $x_1, \ldots, x_M \in \mathbb{R}^2$, $t, b \in T$, $\theta \in \mathbb{R}$, $u \in \mathbb{S}^2$ and $\kappa \geq 0$,

\[
\int_{\mathbb{S}^2} \sum_{m=1}^{M} f(R_{\theta(t-b), \mu(t,b)} x_1; \mu_t, \kappa) \lambda_m(du) = \int_{\mathbb{S}^2} \sum_{m=1}^{M} f(x_1; \mu_t, \kappa) \lambda_m(du).
\]

(7)

The latter inequality has been shown in the proof of Theorem 1 in [Embrechts et al. (2016)].

Owing to the results in Theorem 1 (especially (5)), we focus in the remainder of the paper on the Markov chain $((X(t, x))_{x \in \mathbb{S}^2})_{t \in \mathbb{Z}}$ satisfying, for all $t \in \mathbb{Z}$ and $x \in \mathbb{S}^2$,

\[
X(t, x) = \max \{ a^X(t - 1, R_{\theta, u}(x)), (1 - a^X) Z(t, x) \},
\]

(8)

where the $(Z(t, x))_{x \in \mathbb{S}^2}$, $t \in \mathbb{Z}$, are independent replications of the process $(Z(x))_{x \in \mathbb{S}^2}$ defined by

\[
(Z(x))_{x \in \mathbb{S}^2} \overset{d}{=} \left( \int_{\mathbb{S}^2} f(U_i; \mu_t, \kappa) \right)_{x \in \mathbb{S}^2},
\]

(9)

where $(U_i, \mu_t)_{i \geq 1}$ are the points of a Poisson point process on $(0, \infty) \times \mathbb{S}^2$ with intensity $u^{-2} du \times \lambda_m(du)$. Now, let $C_{\mathbb{S}^2} = C(S^2, (0, \infty))$ be the space of continuous functions from
$S^2$ to $(0, \infty)$ with the topology induced by the uniform metric, i.e. $d(h_1, h_2) = \|h_1 - h_2\|_\infty$, where, for $h \in C_{S^2}$, $\|h\|_\infty = \sup_{x \in S^2} \{|h(x)|\}$. Since $Z$ has standard Fréchet margins, it is almost surely (a.s.) positive. Moreover, as $f$ is defined on a compact set and is continuous, it is bounded. Hence, using similar arguments as for Theorem 4 in [Schlather (2002)], there exists an a.s. finite integer $I$ such that

$$(Z(x))_{x \in S^2} = \left(\bigvee_{i=1}^{I} \{U_i, f(x; \mu_i, \kappa)\}\right)_{x \in S^2}. \tag{10}$$

Accordingly, since $f$ is continuous, we directly obtain that $Z$ is sample-continuous. Hence, it follows from (8) that the Markov chain $(X(t, x))_{x \in S^2}$ takes values in $C_{S^2}$. In addition, $X$ is time-homogeneous since the distribution of the innovation processes $(Z(t, x))_{x \in S^2}, t \in \mathbb{Z}$, does not depend on $t$. We have the following result.

**Theorem 2.** Let $(Z(t, x))_{x \in S^2}, t \in \mathbb{Z}$, be independent replications of the process $(Z(x))_{x \in S^2}$ defined by (9). The Markov chain $X$ has a unique invariant probability measure on $C_{S^2}$ which is entirely characterised by the finite-dimensional distributions of the process

$$(\bigvee_{j=0}^{\infty} \{a^j(1 - a)Z(t - j, R_{\theta_j, u}x)\})_{x \in S^2}, \tag{10}$$

for any $t \in \mathbb{Z}$.

**Proof.** Using similar arguments as in the proof of Proposition 2.2 in [Davis and Resnick (1989)], it is easily shown that (9) has a unique time-stationary solution. This yields that the Markov chain $X$ has a unique invariant probability measure. Moreover, using the fact that, for all $\theta \in \mathbb{R}$, $u \in S^2$ and $j = 0, 1, \ldots$, $R_{\theta_j, u}R_{\theta, u} = R_{\theta(j+1), u}$, it is readily shown that the process in (10) is a time-stationary solution of (9). From the uniqueness of the solution, we deduce that the Markov chain $X$ has at each date the same finite-dimensional distributions as (10). Finally, the distribution (in the sense of the induced probability measure on $C_{S^2}$) of $X$ is entirely characterised by its finite-dimensional distributions. This concludes the proof. \hfill $\Box$

It is clear that $C_{S^2}$ equipped with the previously defined uniform norm is an infinite-dimensional normed vector space. Hence, Riesz Theorem (see, e.g., [Aldrovandi and Pereira, 2017, Section 1.3.14]) immediately gives that this space is not locally compact. Thus, the most classical results about geometric ergodicity of Markov chains, to be found e.g. in [Meyn and Tweedie (2009)], cannot be applied. However, since $S^2$ is a compact and metrisable space and $(0, \infty)$ is a Polish space, Theorem 4.19 in [Kechris (1995)] gives that $C_{S^2}$ is a Polish space. Thus, instead, we use results for Markov chains with Polish state spaces. Such results are uncommon in the literature and here we use those by [Hairer (2010)].

Let $L$ be a function from $C_{S^2}$ to $[0, \infty)$ and let us introduce a weighted supremum norm on the space of functions from $C_{S^2}$ to $\mathbb{R}$ in the following way. For any function $\varphi : C_{S^2} \mapsto \mathbb{R}$, we define

$$\|\varphi\|_L = \sup_{h \in C_{S^2}} \left\{ \frac{|\varphi(h)|}{1 + L(h)} \right\}.$$

\(^1\)In this paper, stationarity refers to strict stationarity.
Furthermore, for a probability measure $\eta$ on $C_{S^2}$ and $\varphi : C_{S^2} \mapsto \mathbb{R}$, let us denote $\eta(\varphi) = \int_{C_{S^2}} \varphi(y) \eta(dy)$. We denote by $\mathcal{B}(C_{S^2})$ the Borel $\sigma$-field of $C_{S^2}$. For $h \in C_{S^2}$, $B \in \mathcal{B}(C_{S^2})$ and $n \in \mathbb{N}\{0\}$, let $\mathcal{P}(h, B)$ and $\mathcal{P}^n(h, B)$ be respectively the transition probability and the $n$-step transition probability from $h$ to $B$ associated with the Markov chain $((X(t, x))_{x \in S^2})_{t \in \mathbb{Z}}$. Finally, we denote by $\mathcal{P}$ its transition kernel, defined by $\mathcal{P} = \{\mathcal{P}(h, B), h \in C_{S^2}, B \in \mathcal{B}(C_{S^2})\}$. Likewise, let $\mathcal{P}^n$ be its $n$-step transition kernel, written as $\mathcal{P}^n = \{\mathcal{P}^n(h, B), h \in C_{S^2}, B \in \mathcal{B}(C_{S^2})\}$. As in [Hairer (2010)], we also use the notations $\mathcal{P}$ and $\mathcal{P}^n$ for the operators defined, for any measurable function $V : C_{S^2} \mapsto \mathbb{R}$ and any $h \in C_{S^2}$, by

$$(\mathcal{P}V)(h) = \int_{C_{S^2}} V(y) \mathcal{P}(h, dy) \quad \text{and} \quad (\mathcal{P}^nV)(h) = \int_{C_{S^2}} V(y) \mathcal{P}^n(h, dy). \quad (11)$$

Our main result states the geometric ergodicity of the Markov chain $((X(t, x))_{x \in S^2})_{t \in \mathbb{Z}}$.

**Theorem 3** (Geometric ergodicity). Let $\pi_*$ be the unique invariant probability measure of the Markov chain $((X(t, x))_{x \in S^2})_{t \in \mathbb{Z}}$. Furthermore, let, for $h \in C_{S^2}$, $L(h) = \|h^\gamma\|_{\infty}$, where $\gamma \in (0, 1)$. Then there exist constants $C > 0$ and $\rho \in (0, 1)$ such that

$$\|\mathcal{P}^n\varphi - \pi_*(\varphi)\|_L \leq C \rho^n \|\varphi - \pi_*(\varphi)\|_L$$

holds for every measurable function $\varphi : C_{S^2} \mapsto \mathbb{R}$ such that $\|\varphi\|_L < \infty$.

**Proof.** We show that the two assumptions required in Theorem 3.6 in [Hairer (2010)] are satisfied.

**Assumption 1** (Assumption 3.1 in [Hairer, 2010]). There exists a function $L : C_{S^2} \rightarrow [0, \infty)$ and constants $K \geq 0$ and $\beta \in (0, 1)$ such that

$$(\mathcal{P}L)(h) \leq \beta L(h) + K,$$

for all $h \in C_{S^2}$.

The function $L$ is here $L(h) = \|h^\gamma\|_{\infty}, h \in C_{S^2}$, where $\gamma \in (0, 1)$. Using the first part of (11) and the fact that $((X(t, x))_{x \in S^2})_{t \in \mathbb{Z}}$ is a time-homogeneous Markov chain, we see that, for all $h \in C_{S^2}$ and $t \in \mathbb{Z}$,

$$(\mathcal{P}L)(h) = \mathbb{E}[L(X(t, \cdot))|X(t-1, \cdot) = h(\cdot)]. \quad (12)$$

Now, observe that, if we denote $(\Gamma_i)_{i \geq 1} = (U_i^{-1})_{i \geq 1}$, where the $(U_i)_{i \geq 1}$ are as in (9), then the $(\Gamma_i)_{i \geq 1}$ are the points of an homogeneous Poisson point process on $(0, \infty)$ with constant intensity equal to one. Hence, the highest $U_i$ corresponds to the smallest $\Gamma_i$, which follows the standard exponential distribution. Thus, its inverse follows the standard Fréchet distribution. Moreover, for $f$ defined in (11), $\|f\|_{\infty}$ is reached for $x = \mu$ and is finite. Therefore, the process $Z$ defined in (9) satisfies

$$\|Z^\gamma\|_{\infty} \overset{d}{=} Y^\gamma \|f^\gamma\|_{\infty}, \quad (13)$$

where $Y$ is a random variable with standard Fréchet distribution. Moreover, for all $t \in \mathbb{Z}$,

$$\|\max\{ah(R_{\theta, u^r}, (1 - a)Z(t, \cdot))^\gamma\|_{\infty} = \|\max\{a\gamma h^\gamma(R_{\theta, u^r}, (1 - a)^\gamma Z^\gamma(t, \cdot))\|_{\infty} = \max\{a\gamma \|h^\gamma\|_{\infty}, (1 - a)^\gamma \|Z^\gamma\|_{\infty}\}. \quad (14)$$
Using (8), (12), (13), (14) and denoting by $\Gamma$ the gamma function, we obtain, for all $h \in \mathcal{C}_{\mathbb{S}^2}$ and $t \in \mathbb{Z}$,

$$
(\mathcal{P}L)(h) = \mathbb{E}[\max\{a^\gamma \|h^\gamma\|_\infty, (1-a)\|f^\gamma\|_\infty\}] = a^\gamma \|h^\gamma\|_\infty \mathbb{P}( Y^\gamma \geq \frac{a^\gamma \|h^\gamma\|_\infty}{(1-a)^\gamma \|f^\gamma\|_\infty} ) + (1-a)^\gamma \|f^\gamma\|_\infty \mathbb{E} \left[ Y^\gamma 1_{\{Y^\gamma \geq \frac{a^\gamma \|h^\gamma\|_\infty}{(1-a)^\gamma \|f^\gamma\|_\infty}\}} \right] \\
\leq a^\gamma \|h^\gamma\|_\infty + (1-a)^\gamma \|f^\gamma\|_\infty \Gamma(1-\gamma) = \beta L(h) + K,
$$

where $\beta = a^\gamma \in (0,1)$ (since $a \in (0,1)$) and $K = (1-a)^\gamma \|f^\gamma\|_\infty \Gamma(1-\gamma) \geq 0$ (since $\Gamma(1-\gamma) > 0$). Hence, Assumption 1 is satisfied.

**Assumption 2** (Assumption 3.4 in [Hairer (2010)]). We denote by $\|\cdot\|_{TV}$ the total variation distance between two probability measures. For every $R > 0$, there exists a constant $\alpha > 0$ such that

$$
sup_{h_1, h_2 \in D_R} \{\|\mathcal{P}(h_1, \cdot) - \mathcal{P}(h_2, \cdot)\|_{TV}\} \leq 2(1-\alpha),
$$

where $D_R = \{h_1, h_2 : L(h_1) + L(h_2) \leq R\}$.

Remark 3.5 in [Hairer (2010)] gives that Condition (15) is equivalent to the fact that

$$
\|\mathcal{P}(\varphi)(h_1) - \mathcal{P}(\varphi)(h_2)\| \leq 2(1-\alpha)
$$

holds uniformly on $\mathcal{G} = \{\varphi : \mathcal{C}_{\mathbb{S}^2} \to \mathbb{R} : \varphi$ measurable and $\|\varphi\|_\infty \leq 1\}$. Consequently, taking advantage of (12), we see that we need to prove that, for all $t \in \mathbb{Z}$,

$$
\sup_{h_1, h_2 \in D_{R_1}, \varphi \in \mathcal{G}} \{\|\mathbb{E}[\varphi(X(t, \cdot))X(t-1, \cdot) - \mathbb{E}[\varphi(X(t, \cdot))]X(t-1, \cdot) = h_2(\cdot)]\| \leq 2(1-\alpha).
$$

Using (8), we have, for all $\varphi \in \mathcal{G}$, $h_1, h_2 \in D_{R_1}$ and $t \in \mathbb{Z}$, that

$$
\mathbb{E}[\varphi(X(t, \cdot))X(t-1, \cdot) = h_1(\cdot)] - \mathbb{E}[\varphi(X(t, \cdot))]X(t-1, \cdot) = h_2(\cdot)] = \mathbb{E}[\varphi((\max\{ah_1(R_\theta, u), (1-a)Z(t, x)\})_{x \in \mathbb{S}^2}) - \varphi((\max\{ah_2(R_\theta, u), (1-a)Z(t, x)\})_{x \in \mathbb{S}^2})] \\
\leq \mathbb{E}[\varphi((\max\{ah_1(R_\theta, u), (1-a)Z(t, x)\})_{x \in \mathbb{S}^2}) - \varphi((\max\{ah_2(R_\theta, u), (1-a)Z(t, x)\})_{x \in \mathbb{S}^2})].
$$

Moreover, for $p, q, z \in \mathcal{C}_{\mathbb{S}^2}$, it is clear that if, for all $x \in \mathbb{S}^2$, max$p(x), q(x) \leq z(x)$, then, for all $\varphi \in \mathcal{G}$,

$$
|\varphi((\max\{p(x), z(x)\})_{x \in \mathbb{S}^2}) - \varphi((\max\{q(x), z(x)\})_{x \in \mathbb{S}^2})| = 0.
$$

Therefore, for all $\varphi \in \mathcal{G}$, using the fact that $\|\varphi\|_\infty \leq 1$, we have that

$$
\mathbb{E}[\varphi((\max\{ah_1(R_\theta, u), (1-a)Z(t, x)\})_{x \in \mathbb{S}^2}) - \varphi((\max\{ah_2(R_\theta, u), (1-a)Z(t, x)\})_{x \in \mathbb{S}^2})] \\
\leq 2 \left[ 1 - \mathbb{P} \left( \bigcap_{x \in \mathbb{S}^2} \left\{ Z(t, x) \geq \frac{a}{1-a} \max\{h_1(R_\theta, u), h_2(R_\theta, u)\} \right\} \right) \right] \\
\leq 2 \left[ 1 - \mathbb{P} \left( \inf_{x \in \mathbb{S}^2} \{ Z(t, x) \geq \frac{a}{1-a} \max\{\|h_1\|_\infty, \|h_2\|_\infty\} \} \right) \right].
$$

The quantity inf$_{x \in \mathbb{S}^2}\{f(x; \mu_1, \kappa)\}$ is reached for $x = -\mu_1$. Hence, the process $Z$ defined in (f) satisfies, for all $x \in \mathbb{S}^2$,

$$
Z(x) \geq \Gamma^{-1}_1 f(x; \mu_1, \kappa) \geq \Gamma^{-1}_1 \inf_{x \in \mathbb{S}^2} \{f(x; \mu_1, \kappa)\} = \Gamma^{-1}_1 \frac{\kappa}{4\pi \sinh(\kappa)} \exp(-\kappa),
$$
with $\Gamma_1 = U_1^{-1}$, where $U_1$ appears in the definition of $Z$. Since the $(Z(t, x))_{x \in S^2}$, $t \in \mathbb{Z}$, are independent replications of the process $(Z(x))_{x \in S^2}$, it follows, noting that $\max\{\|h_1\|_{\infty}, \|h_2\|_{\infty}\} \leq R$, that, for all $t \in \mathbb{Z}$,

$$
P \left( \inf_{x \in S^2\{Z(t, x)\}} \geq \frac{a}{1-a} \max\{\|h_1\|_{\infty}, \|h_2\|_{\infty}\} \right) \\
\geq P \left( \Gamma_1^{-1} \frac{\kappa}{4\pi \sinh(\kappa)} \exp(-\kappa) \geq \frac{a}{1-a} \max\{\|h_1\|_{\infty}, \|h_2\|_{\infty}\} \right) \\
\geq P \left( \Gamma_1^{-1} \geq \frac{4\pi \sinh(\kappa) a}{\kappa(1-a)} \exp(\kappa) R \right).
$$

(18)

Therefore, combining [16], [17] and [18], we obtain, for all $t \in \mathbb{Z}$ that,

$$
\sup_{h_1, h_2 \in D_{\mathcal{G}, \mathcal{F}}} |E \left[ \varphi(X(t, \cdot)) | X(t-1, \cdot) = h_1(\cdot) \right] - E \left[ \varphi(X(t, \cdot)) | X(t-1, \cdot) = h_2(\cdot) \right]| \\
\leq 2 \left( 1 - P \left( \Gamma_1^{-1} \geq \frac{4\pi \sinh(\kappa) a}{\kappa(1-a)} \exp(\kappa) R \right) \right) = 2(1-a),
$$

denoting

$$
\alpha = P \left( \Gamma_1^{-1} \geq \frac{4\pi \sinh(\kappa) a}{\kappa(1-a)} \exp(\kappa) R \right) > 0.
$$

Hence, Assumption 2 holds.

Finally, the application of Theorem 3.6 in [Hairer, 2011] yields the result.

The geometric ergodicity result of Theorem 3 has two strong implications. First, let $x_0 \in C_{S^2}$ be a realisation of the Markov chain $((X(t, x))_{x \in S^2})_{t \in \mathbb{Z}}$ at a date $t_0 \in \mathbb{Z}$. Geometric ergodicity implies that, for any $B \in \mathcal{B}(C_{S^2})$, $\lim_{t \to \infty} P((X(t_0 + t, x))_{x \in S^2} \in B | (X(t_0, x))_{x \in S^2} = x_0) = \pi_\star(B)$; note that the rate of this convergence is geometric. Hence, geometric ergodicity can be viewed as a loss of memory property. Second, assume that $X$, the Markov chain specified by [3], is now defined with $t \in \mathbb{N}$ and any initial (at time $t = 0$) distribution (probability measure on $C_{S^2}$) which is different from the invariant probability measure $\pi_\star$. Geometric ergodicity entails that the distribution of the chain at a given date tends at a geometric rate to $\pi_\star$.

It is worth highlighting the fact that the results of this section remain valid for all models belonging to the generalised subclass of models of type 2 as soon as $g$ is as in [3] and $\tilde{f}$ is positive, continuous and satisfies the adapted version of [7].

We conclude this section with the following remark, which shows that geometric ergodicity (in the sense of Theorem 3) can be shown for Markov chains defined in a similar way as in [3] but with a process $Z$ which is not necessarily max-stable.

**Remark 2.** For $a \in (0, 1)$, we now consider, provided it exists, the Markov chain $((X(t, x))_{x \in S^2})_{t \in \mathbb{Z}}$ satisfying, for all $t \in \mathbb{Z}$ and $x \in S^2$,

$$
X(t, x) = \max\{aX(t-1, R_{\theta, u}x), (1-a)Z(t, x)\},
$$

(19)

where the $(Z(t, x))_{x \in S^2}$, $t \in \mathbb{Z}$, are independent replications of a bounded, a.s. positive and sample-continuous stochastic process $(Z(x))_{x \in S^2}$. The same result as in Theorem 3 can
be obtained for the Markov chain \( X \) with similar arguments. Denoting by \( \mathcal{K} \) some compact set, the same holds true for, provided it exists, the Markov chain \( ((X(t, x))_{x \in \mathcal{K}})_{t \in \mathbb{Z}} \) satisfying, for all \( t \in \mathbb{Z} \) and \( x \in \mathcal{K} \):

\[
X(t, x) = \max\{aX(t-1, x), (1-a)Z(t, x)\},
\]

where the \( (Z(t, x))_{x \in \mathcal{K}}, \ t \in \mathbb{Z} \), are as in \( [19] \) apart from the replacement of \( S^2 \) with \( \mathcal{K} \). We thank the referee for pointing this out.

4 Conclusion

The main result of this paper concerns geometric ergodicity of some Markov chains induced by processes belonging to the class of space-time max-stable models with spectral separability introduced in Embrechts et al. (2016). Since the associated space state is not locally compact, we could not use the classical approach described, e.g., in Meyn and Tweedie (2009) and had to apply results for Markov chains with Polish state spaces to be found in Hairer (2010). Some future challenging work might consist in investigating whether or not the Markov chains described in Embrechts et al. (2016), Section 3.1, are geometrically ergodic. They take values in the space of continuous functions from \( \mathbb{R}^2 \) to \((0, \infty)\). The latter being non locally compact and not Polish (since non separable), neither the results in Meyn and Tweedie (2009) nor those from Hairer (2010) can be used in that case.

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\(^2\)The set \( \mathcal{K} \) is not necessarily a subset of \( \mathbb{R}^d \), whence the notation \( x \) instead of \( \mathbf{x} \).
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