CANONICAL INVOLUTION ON DOUBLE JET BUNDLES

Hülya Kadioğlu
Yildiz Technical University, Esenler, Istanbul TURKEY,
Department of Mathematics, Idaho State University, Pocatello, ID 83209-8085, USA,
email: hkadio@yildiz.edu.tr

Abstract

In this study, we generalize double tangent bundles to double jet bundles. We present a secondary vector bundle structure on a 1-jet of a vector bundle. We show that 1-jet of a vector bundle carries two vector bundle structures, namely primary and secondary structures. We also show that the manifold charts induced by primary and secondary structures belong to the same atlas. We prove that double jet bundles can be considered as a quotient of second order jet bundle. We show that there exists a natural involution that interchanges between primary and secondary vector bundle structures on double jet bundles.

1 Introduction

In general, there are two ways to define $k-$ jets. The first definition is based on using the sections of a fibered manifold. In this definition, a $k-$jet is an equivalence class determined by an equivalence relation $\sim_k$. Two sections of a fibered manifold are called $k-$related by the relation $\sim_k$, if they have the same Taylor polynomial expansion at the point $x$ truncated at order $k$. This definition usually leads to a geometric approach which is applied to study of systems of differential equations (we refer to [1], [2], [4], [8], [11], and [10] for more details).

The second and the more general definition of jet bundles is based on using the functions from $N$ to $M$, where $N$ and $M$ are smooth manifolds. In this definition, a $k-$jet of a function $f$ at $x$ is an equivalence class defined by an equivalence relation $\sim_k$. The equivalence of two functions is defined by the same way as the equivalence of sections. The collection of all $k-$jets is called a $k-$jet bundle. One particular case is when $N = \mathbb{R}$. In this case, the jet bundle is called tangent bundle of higher order. This jet bundle possesses a certain kind of geometric structure, which is called almost tangent structure of higher order. More generally, if $N = \mathbb{R}^p$,  

2010 Mathematics Subject Classification: Primary 58A20 ; Secondary 55R25, 58A05.

Key words and Phrases: Double Jet Bundle, Double Vector Bundle, Second Order Jets, Canonical Involution, Tangent Bundle of Higher Order.

Correspond to: Hulya Kadioğlu, Yildiz Technical University, Istanbul-TURKEY, email: hkadio@yildiz.edu.tr
then the jet bundle is called the tangent bundle of \( p^k \) velocities. This concept was introduced by Ehresmann to develop classical field theory in autonomous sense [9].

Throughout this paper, we will be considering the second definition: 1-jets with source at the origin of \( \mathbb{R}^p \), and targeted in \( M \). In section 2, we give some preliminary theorems that we need for the paper. In section 3, we study 1-jet of an arbitrary vector bundle and present a secondary vector bundle structure on a 1-jet of the vector bundle. Due to two vector bundle structures, 1-jet of a jet bundle is considered as a double vector bundle (DVB). In terms of the induced manifold structures on the total space of DVB, we prove that both belongs to the same atlas. We also prove that 1-jet of a jet bundle (double jet bundle) can be considered as a quotient of a second order jet bundle. Moreover, we show that two vector bundle structures are isomorphic on a double jet bundle by defining a canonical involution that interchanges between such structures. In Appendix section (Section 4), we prove some identities and statements that we used throughout the paper.

2 Preliminaries

In this section we summarize some necessary preliminary materials that we need for a self contained presentation of our paper.

2.1 First Order Jets: \( J^1_p M \)

Let \( C^\infty(\mathbb{R}^p) \) be the algebra of \( C^\infty \)-functions on the Euclidean space \( \mathbb{R}^p \) with natural coordinates \((u_1, u_2, ..., u_p)\). Let \( f, g \in C^\infty(\mathbb{R}^p) \); \( f \) is equivalent to \( g \) if \( f(0) = g(0) \) and \( \partial/\partial u_i(f) = \partial/\partial u_i(g) \) at \( u = 0 \in \mathbb{R}^p \) for every \( i = 1, 2, ..., p \); clearly, this is an equivalence relation.

Now, let \( M \) be an \( m \) dimensional manifold. Consider the set \( C^\infty(\mathbb{R}^p; M) \) of all smooth maps \( \phi : \mathbb{R}^p \to M \), and take elements \( \phi, \xi \in C^\infty(\mathbb{R}^p; M) \). Then \( \phi \) is equivalent to \( \xi \) if \( f \circ \phi \) is equivalent to \( f \circ \xi \) for every \( f \in C^\infty(M) \). This is again an equivalence relation; and denoted by \( j^1(\phi) \) the equivalence class of \( \phi \in C^\infty(\mathbb{R}^p; M) \), and called a 1-jet in \( M \) at \( \phi(0) \). We denote \( J^1_p M \) the set of all equivalence classes in \( C^\infty(\mathbb{R}^p; M) \). If \((U, x_1, ...x_m)\) is a local chart in \( M \), then \( (j^1_p U, x_1, ..., x_m, x^1_{\alpha}, ..., x^m_{\alpha}) \) is the local chart for \( J^1_p M \), with \( \alpha = 1, 2, ..., p \) by

\[
\begin{align*}
x_i(j^1(\phi)) &= x_i(\phi(0)) \\
x^i_{\alpha}(j^1(\phi)) &= \frac{\partial(x_i \circ \phi)}{\partial u_{\alpha}}|_0.
\end{align*}
\]

Let \( j^1(\phi) \) be an arbitrary point, and let \( \phi_\alpha : \mathbb{R} \to M \) be the differentiable curve given by \( \phi_\alpha(u) = \phi(0, ..., u, ..., 0) \), with \( u \) at the \( \alpha^{th} \) place; then, associated to \( j^1(\phi) \) there is a unique \((p + 1)\)-tuple \([x; X_1, ..., X_p]\) given by

\[
x = \phi(0) \quad \quad X_\alpha = \phi_\alpha\left(\frac{d}{du}\right)|_0,
\]
where $d/du$ is the canonical vector field tangent to $\mathbb{R}$. From now on, we shall write $[x; X_1, ..., X_p]$ simply as $[x; X_\alpha]$ and shall identify $j^1 \phi \equiv [x; X_\alpha]$ if there is no confusion.

**Remark 1.** In later sections, we sometimes use the notation $X_\alpha$ for $\frac{\partial \phi}{\partial u_\alpha}|_0$.

Now we focus on some functorial properties of jet bundles.

**Theorem 2.** [5]

(i) If $h : M \to N$ is a differentiable function, then $h$ induces a canonical differentiable map $h^1 : J^1_p M \to J^1_p N$ given by

$$h^1(j^1 \phi) = j^1(h \circ \phi), \quad \forall j^1 \phi \in J^1_p M$$

and in terms of previous identification, we have, $h^1([x; X_\alpha]) = [h(x); h_* X_\alpha]$.

(ii) If $h$ is a diffeomorphism, then the induced map is also a diffeomorphism and, moreover $(h^1)^{-1} = (h^{-1})^1$. Also for the manifolds $M$ and $N$, $J^1_p(M \times N)$ is diffeomorphic to $J^1_p M \times J^1_p N$.

(iii) If $M$ is a real vector space of dimension $m$, then $J^1_p M$ inherits a vector space structure: for any $j^1 f, j^1 g \in J^1_p M$ and $\lambda \in \mathbb{R}$, operations are given by

$$j^1 f + j^1 g = j^1(f + g), \quad \lambda j^1 f = j^1(\lambda f),$$

where $f + g$ and $\lambda f$ are defined in the usual way. Vector space operations of $J^1_p M$ as $[x; X_\alpha] + \lambda[y; Y_\alpha] = [x + \lambda y; X_\alpha + \lambda Y_\alpha]$.

2.2 Second Order Jets: $J^2_{2p} M$

Now we consider the 2- jets by taking classes having equivalence up to all derivatives of second order. The natural atlas of this bundle can be obtained as follows:

Let $\phi, \phi' \in C^\infty(\mathbb{R}^{2p}, M)$. We say $\phi$ is equivalent to $\phi'$ if

$$\phi(0, 0) = \phi'(0, 0)$$

$$\left. \frac{\partial \phi}{\partial u_\alpha} \right|_{(0, 0)} = \left. \frac{\partial \phi'}{\partial u_\alpha} \right|_{(0, 0)}$$

$$\left. \frac{\partial^2 \phi}{\partial u_\alpha \partial u_\beta} \right|_{(0, 0)} = \left. \frac{\partial^2 \phi'}{\partial u_\alpha \partial u_\beta} \right|_{(0, 0)}, \quad 1 \leq \bar{\alpha}, \bar{\beta} \leq 2p$$

Let $E$ be a finite dimensional real vector space, then $J^2_{2p} E$ is regarded as a finite dimensional real vector space. Let $L(\mathbb{R}^{2p}, E)$ denote the vector space of linear functions $A : \mathbb{R}^{2p} \to E$ and
let $S_2(\mathbb{R}^p; E)$ denote the vector space of all symmetric bi-linear functions $B : \mathbb{R}^2 \times \mathbb{R}^2 \to E$. The function $j^2 \phi \to (\phi(0,0), \frac{\partial \phi}{\partial u_\alpha}|_{(0,0)}, \frac{\partial^2 \phi}{\partial u_\alpha \partial u_\beta}|_{(0,0)})$, $1 \leq \bar{\alpha}, \bar{\beta} \leq 2p$

is a canonical isomorphism, where

$$\bar{u}_\alpha = \begin{cases} u_\alpha & \bar{\alpha} = 1, 2, ..., p, \\ w_\alpha & \bar{\alpha} = p + 1, ..., 2p. \end{cases}$$

We note that we identify $\mathbb{R}^{2p}$ with $\mathbb{R}^p \times \mathbb{R}^p$. Let $\varphi : U \to E$ be a local chart of $M$ that, without loss of generality, we assume maps onto a vector space $E$. Let $(x, A, B) \in J_2^2(\mathbb{R}^p; E)$; the corresponding 2-jet is the one represented by

$$\phi(\bar{u}) = x + A.\bar{u} + \frac{1}{2} \bar{u}^T.B.\bar{u}.$$ 

In terms of curve notations discussed for the first order jets, we may identify $j^2 \phi \in J_2^2(\mathbb{R}^p; E)$ with the triple

$$j^2 \phi = [x; A_\bar{\alpha}; B_{\bar{\alpha}\bar{\beta}}], \quad 1 \leq \bar{\alpha}, \bar{\beta} \leq 2p$$

where

$$A_{\bar{\alpha}} = \frac{\partial \phi}{\partial u_{\bar{\alpha}}}|_{(0,0)}$$

and

$$B_{\bar{\alpha}\bar{\beta}} = \frac{\partial^2 \phi}{\partial u_{\bar{\alpha}} \partial u_{\bar{\beta}}}|_{(0,0)}.$$ 

3 Jet Bundle to a Vector Bundle

Let $\pi_E : E \to M$ be a vector bundle with the local bundle trivialization

$$\psi : \pi_E^{-1}(U) \to U \times E$$

where $U$ is an open subset of $M$. Then, $J_1^1E$ can be considered as a jet bundle on $E$ (the total space of the VB $\pi_E$). The bundle trivialization on $J_1^1E$ is

$$\tilde{\psi} : J_1^1E \to E \times L(\mathbb{R}^p, \mathbb{R}^{m+k})$$

by

$$\tilde{\psi}(j^1\Phi) = (\Phi(0), \frac{\partial \Phi}{\partial u_\alpha}|_0)$$

where $\Phi \in C^\infty(\mathbb{R}^p, E)$. Then, $\Phi(0) = (\psi)^{-1}(x, y)$, and
\[
\frac{\partial \Phi}{\partial u_\alpha} \bigg|_0 = \left( \frac{\partial(x_i \circ \Phi)}{\partial u_\alpha} \bigg|_0, \frac{\partial(y_j \circ \Phi)}{\partial u_\alpha} \bigg|_0 \right)
\]

where, \(x_i = x_i \circ \pi_E\), \(y_j = y_j \circ pr_2 \circ \psi\), and \(x_i, y_j\) are the local coordinate functions of \(M\) and \(E\) respectively.

Using curve notation gives:

\[ j^1 \Phi \equiv [x, y; X_\alpha, Y_\alpha] \]

where

\[ X_\alpha = \frac{\partial(x_i \circ \Phi)}{\partial u_\alpha} \bigg|_0 \]

and

\[ Y_\alpha = \frac{\partial(y_j \circ \Phi)}{\partial u_\alpha} \bigg|_0. \]

Moreover, for each smooth manifold \(M\), \(J^1_p M\) carries a vector bundle structure (see Lemma [12]). Since \(E\) is a smooth manifold, then \(J^1_p E\) carries a VB structure, where the intrinsic operations are given by the following:

For a local bundle chart \(\tilde{\psi} : (\pi_E \circ \tilde{\pi})^{-1} \rightarrow \pi_E^{-1}(U) \times L(\mathbb{R}^p, \mathbb{R}^{m+k})\), the mapping \(+_1\) and \(\bullet_1\) on \((\tilde{\pi})^{-1}\{(x, y)\}\) are defined locally as

\[ j^1 \Phi +_1 j^1 \Phi' = (\tilde{\psi})^{-1}((\psi)^{-1}(x, y), (pr_2 \circ \tilde{\psi})(j^1 \Phi) + (pr_2 \circ \tilde{\psi})(j^1 \Phi')) \]

and

\[ \lambda \bullet_1 j^1 \Phi = (\tilde{\psi})^{-1}((\psi)^{-1}(x, y), \lambda.(pr_2 \circ \tilde{\psi})(j^1 \Phi)). \]

Using curve notation, one can see that

\[ j^1 \Phi +_1 j^1 \Phi' = [x, y; X_\alpha, Y_\alpha] +_1 [x, y; X'_\alpha, Y'_\alpha] = [x, y; X_\alpha + X'_\alpha, Y_\alpha + Y'_\alpha] \]

and

\[ \lambda \bullet_1 [x, y; X_\alpha, Y_\alpha] = [x, y; \lambda X_\alpha, \lambda Y_\alpha] \]

where \(U\) is an open subset of \(M\).

**Remark 3.** Hereafter, we will refer to above vector bundle structure as the primary structure.
3.1 Secondary VB Structure on $J^1_pE$

We recall that there exists an induced a canonical smooth function $\pi^1_E : J^1_pE \to J^1_pM$, where $\pi_E : E \to M$ a smooth bundle projection of $E$. One can easily prove that $(\pi_E)^1$ is a surjective map. Moreover, it can be seen from Theorem 2 that $J^1_pE$ is a vector space (isomorphic to $L(\mathbb{R}^p, E)$). If we let the local trivialization of $E$ be the map $\psi$, then $\psi^1 : J^1_p(\pi^{-1}_E(U)) \to J^1_p(U \times E)$ is a diffeomorphism.

Now we consider the trivialization domains:

**Lemma 4.** Let $\pi^{-1}_E(U)$ be a local trivialization domain of the bundle $E$. Then

$$J^1_p(\pi^{-1}_E(U)) = (\pi^1_E)^{-1}(J^1_pU) = (\pi_E \circ \tilde{\pi})^{-1}(U).$$  \hspace{1cm} (2)

**Proof.** To prove equation (2), we will prove followings:

(i) $J^1_p(\pi^{-1}_E(U)) \subset (\pi^1_E)^{-1}(J^1_pU),$

(ii) $(\pi^1_E)^{-1}(J^1_pU) \subset (\pi_E \circ \tilde{\pi})^{-1}(U)$, and

(iii) $(\pi_E \circ \tilde{\pi})^{-1}(U) \subset J^1_p(\pi^{-1}_E(U)).$

Suppose that

$$j^1\phi \in J^1_p(\pi^{-1}_E(U)).$$

By its own definition, $\phi(0) \in \pi^{-1}_E(U)$, then

$$\pi_E(\phi(0)) \in U \implies j^1(\pi_E \circ \Phi) \in J^1_pU$$

$$\implies \pi_E(j^1\Phi) \in J^1_pU$$

$$\implies j^1\phi \in (\pi^1_E)^{-1}(J^1_pU)$$

Then $J^1_p((\pi^1_E)^{-1}(U)) \subset (\pi^1_E)^{-1}(J^1_pU)$. The first statement is proven.

Suppose that $j^1\Phi \in (\pi^1_E)^{-1}(J^1_pU)$. Then
\[\pi^1_E(j^1\Phi) \in J^1_p U \Rightarrow j^1(\pi_E \circ \Phi) \in J^1_p U\]
\[\Rightarrow (\pi_E \circ \Phi)(0) \in U\]
\[\Rightarrow \Phi(0) \in (\pi_E)^{-1}(U)\]
\[\Rightarrow \tilde{\pi}^{-1}(\Phi(0)) \in \tilde{\pi}^{-1}(\pi^{-1}_E(U))\]
\[\Rightarrow j^1\Phi \in (\pi_E \circ \tilde{\pi})^{-1}(U).\]

Then \((\pi_E)^{-1}(J^1_p U) \subset (\pi_E \circ \tilde{\pi})^{-1}(U)\). The second statement is proven.

On the other hand, suppose that \(j^1\Phi \in (\pi_E \circ \tilde{\pi})^{-1}(U)\). Then
\[\pi_E(\tilde{\pi}(j^1\Phi)) \in U \Rightarrow \pi_E(\Phi(0)) \in U\]
\[\Rightarrow \Phi(0) \in \pi^{-1}_E(U)\]
\[\Rightarrow j^1\Phi \in J^1_p(\pi^{-1}_E(U))\]

which shows that \((\pi_E \circ \tilde{\pi})^{-1}(U) \subset J^1_p(\pi^{-1}_E(U))\). This completes the proof.

**Proposition 5.** Let \(\pi_E : E \to M\) be a vector bundle. Then \(\pi^{-1}_E : J^1_p E \to J^1_p M\) is a vector bundle so that the manifold \(J^1_p E\) has two vector bundle structures, namely, its primary vector bundle structure as the first jet bundle of manifold \(E\), and a secondary structure with \(J^1_p M\) as the base manifold. Moreover the induced charts on \(J^1_p E\) from primary and secondary structures belong to the same atlas.

**Proof.** First, we start the proof by showing that \(\psi^1\) is the bundle trivialization of the secondary jet bundle \(J^1_p E\):

Since \(Pr_1 \circ \psi^1 = \pi^1_E\), then \(\pi^1_E : J^1_p E \to J^1_p M\) is a smooth fiber bundle. (Here \(Pr_1 : J^1_p U \times J^1_p E \to J^1_p U\) represents the first projection. We note that \(J^1_p(U \times E)\) and \(J^1_p U \times J^1_p E\) are diffeomorphic by setting \(j^1 f \cong (j^1(f_1), j^1(f_2))\), where \(f = (f_1, f_2) : \mathbb{R}^p \to U \times E\).)

Now we identify \(j^1\Phi \in J^1_p E\) as quadruple \([x, X_\alpha; y, Y_\alpha]\) such that
\[X_\alpha = \frac{\partial(\pi_E \circ \Phi)}{\partial u_\alpha}|_0\] (3)

and
\[Y_\alpha = \frac{\partial(pr_2 \circ \psi \circ \Phi)}{\partial u_\alpha}|_0.\] (4)

Showing that the fiber map being linear, proves that \(\pi^1_E\) is a vector bundle over \(J^1_p M\).
Let \( j^1 \Phi, j^1 \Phi' \in (\pi^1_E)^{-1}\{j^1 \theta\} \), for each \( j^1 \theta = [x; X_\alpha] \in J^1_p M \). Then \( j^1 \Phi = [x, X_\alpha; y, Y_\alpha] \) for \( \alpha = 1, 2, \ldots, p \), and \( j^1 \Phi' = [x, X_\alpha; y', Y_\alpha'] \). The secondary vector bundle operations on this fiber are defined by:

\[
j^1 \Phi +_2 j^1 \Phi' = (\psi^1)^{-1}(j^1 \theta, (pr_2 \circ \psi^1)(j^1 \Phi) + (j^1 \theta, pr_2 \circ \psi^1)(j^1 \Phi'))
\]
\[
= [x, X_\alpha; y + y', Y_\alpha + Y_\alpha']
\]

and

\[
\lambda \cdot_2 j^1 \Phi = (\psi^1)^{-1}(j^1 \theta, \lambda (pr_2 \circ \psi^1)(j^1 \Phi))
\]
\[
= [x, X_\alpha; \lambda y, \lambda Y_\alpha]
\]

where \( j^1 \Phi, j^1 \Phi' \in (\pi^1_E)^{-1}\{j^1 \theta\} \), and \( \lambda \in \mathbb{R} \). Thus,

\[
\psi^1_{[x, X_\alpha]}([x, X_\alpha; y, Y_\alpha] +_2 \lambda \cdot_2 [x, X_\alpha; y', Y_\alpha']) = \psi^1([x, X_\alpha; y + \lambda y', Y_\alpha + Y_\alpha'])
\]
\[
= [y + \lambda y', Y_\alpha + Y_\alpha']
\]
\[
= [y, Y_\alpha] + \varepsilon \lambda \cdot [y', Y_\alpha']
\]

which equals to

\[
\psi^1_{[x, X_\alpha]}([x, X_\alpha; y, Y_\alpha]) +_2 \lambda \cdot \varepsilon \psi^1_{[x, X_\alpha]}([x, X_\alpha; y', Y_\alpha'])
\]

This shows that \( \psi^1_{[x, X_\alpha]} \) is a linear function. Therefore, \( J^1_p E \) is a vector bundle with its secondary structure.

By now, we have shown that \( J^1_p E \) carries two vector bundle structures (namely primary and secondary). One can easily see that these two structures define coordinate charts on the total space \( J^1_p E \). Now we will show that these two charts belong to the same atlas. To do this we will show that identity map of \( J^1_p E \) is a diffeomorphism between two VB structures based on \( J^1_p M \) and \( E \).

Let \( E \) has the local coordinate maps \( \bar{x}_i, \bar{y}_j \) with \( 1 \leq i \leq m, 1 \leq j \leq k \). Considering the primary structure on \( J^1_p E \): For all \( j^1 \phi \in J^1_p E \), there exists triples \( (x, y, \rho) \) with \( \Phi(0) = \psi^{-1}(x, y) \) and \( \rho : \mathbb{R}^p \to \mathbb{R}^{m+k} \) which is a linear function that corresponds to the matrix \( A = \begin{bmatrix} \frac{\partial \Phi_a}{\partial a} \end{bmatrix}_{a=0} \). Here the term \( \Phi_a \) is defined by

\[
\Phi_a = \left\{ \begin{array}{ll}
(\bar{x}_i \circ \Phi) & a = 1, 2, \ldots, m, \\
(\bar{y}_j \circ \Phi) & a = m + 1, \ldots, m + k.
\end{array} \right.
\]

Therefore the matrix \( A \) consists of two sub matrices: they are
On the other hand, considering secondary structure on \( J^1_pE \), given any \( j^1 \Phi \in J^1_pE \), there exists quadruple \((x, f; y, g)\), where \( \Phi(0) = (x, y) \), \( f = \left[ \frac{\partial (\bar{y}_i \circ \Phi)}{\partial u^\alpha} \big|_0 \right] \), and \( g = \left[ \frac{\partial (\bar{y}_j \circ \Phi)}{\partial u^\alpha} \big|_0 \right] \).

It is clear that the the matrix representations of \( f \) and \( g \) are defined the same as in Equation 5. Let \( \Omega \) be the identity map of \( J^1_pE \), \( \Psi_E \), \( \Psi_M \) and \( \Psi_E \) be the local trivializations of jet bundles \( J^1_pE, J^1_pM \) and \( J^1_pE \) respectively, and \( \varphi : U \subset M \to \mathbb{R}^m \) be a coordinate chart. Then the local form of \( \Omega \) is given by the following commutative diagram:

\[
\begin{array}{ccc}
\pi_1^E \circ j^1_pU & \overset{\Omega}{\longrightarrow} & (\pi \circ \tilde{\pi})^{-1}(U) \\
\downarrow \varphi_2 & & \downarrow \varphi_1 \\
\varphi(U) \times L(\mathbb{R}^p, \mathbb{R}^m) \times E \times L(\mathbb{R}^p, \mathbb{R}^k) & \overset{\hat{\Omega}}{\longrightarrow} & \varphi(U) \times E \times L(\mathbb{R}^p, \mathbb{R}^m) \times L(\mathbb{R}^p, \mathbb{R}^k)
\end{array}
\]

where

\[
\varphi_1 = (\varphi \times id_{E \times L(\mathbb{R}^p, \mathbb{R}^m) \times L(\mathbb{R}^p, \mathbb{R}^k)}) \circ (\psi \times \xi) \circ \tilde{\psi},
\]

and

\[
\varphi_2 = (\varphi \times id_{L(\mathbb{R}^p, \mathbb{R}^m) \times E \times L(\mathbb{R}^p, \mathbb{R}^k)}) \circ (\Psi_M \times \Psi_E) \circ \psi^1
\]

and

\[
\xi : L(\mathbb{R}^p, \mathbb{R}^{m+k}) \to L(\mathbb{R}^p, \mathbb{R}^m) \times L(\mathbb{R}^p, \mathbb{R}^k)
\]

\[
f_g \to \xi(f_g) = (f, g)
\]

where \( f_g(u) = (f(u), g(u)) \).

From Lemma 16, the local form \( \hat{\Omega} \) is \((x, f; y, g) \to (x, y, f, g)\). It is clear that the local form is a surjective map. Since \( \hat{\Omega} = pr_1 \times pr_3 \times pr_2 \times pr_4 \), then the local form is differentiable with its own inverse. Therefore \( \hat{\Omega} \) is a diffeomorphism which implies that two structure charts belong to the same atlas.

**Remark 6.** The secondary structure defined on tangent bundles can be found in [7].
3.2 \( J^1_p(J^1_pM) \) as a Quotient Manifold

Let us consider the case \( E = J^1_pM \) for the smooth manifold \( M \). By Lemma 12, \( J^1_pM \) is a vector bundle. By Proposition 5, \( J^1_p(J^1_pM) \) has two vector bundle structures both based on \( J^1_pM \). To define the double jet manifold, we begin with the idea of smooth functions on \( J^1_pM \).

Let \( \phi : \mathbb{R}^p \times \mathbb{R}^p \to M \) be a smooth function. We define

\[
\Phi : \mathbb{R}^p \to J^1_pM \\
\quad u \to \Phi(u) = J^1_p(\phi_u)
\]

where \( \phi_u \) is defined by

\[
\phi_u : \mathbb{R}^p \to M \\
\quad v \to \phi_u(v) = \phi(u, v)
\]

It can easily be seen that \( \phi_u \) is a smooth function.

Here

\[
(\psi \circ \Phi)(u) = (\phi_u(0), \frac{\partial \phi_u}{\partial w_{\alpha}}|_{w=0}), \quad u \in \mathbb{R}^p.
\]

\[
= (\phi(u, 0), \frac{\partial \phi_u}{\partial w_{\alpha}}|_{w=0})
\]

is a smooth function. Therefore, \( \Phi \in C^\infty(\mathbb{R}^p, J^1_pM) \).

**Proposition 7.** Let \( \Phi \) and \( \phi \) be the functions defined by Equation 6. Then \( J^1_p(J^1_pM) \) is a quotient of \( J^2_pM \), with the quotient map \( \Lambda \) defined by the following:

\[
\Lambda : \ J^2_p(M) \to \ J^1_p(J^1_pM) \\
J^2\phi \to J^1\Phi.
\]
Proof. Let \( \psi \) be a local trivialization of the vector bundle \( J_1^1 M \), then considering local charts for \( J_2^2 M \), and \( J_1^1 (J_1^1 M) \), we have the commutative diagram:

\[
\begin{array}{ccc}
J_2^2 M & \xrightarrow{\Lambda} & J_1^1 (J_1^1 M) \\
\downarrow{\psi^2} & & \downarrow{\bar{\psi} \circ \psi^1 = \tilde{\psi}} \\
M \times L(\mathbb{R}^{2p}, \mathbb{R}^m) \times S_2(\mathbb{R}^{2p}, \mathbb{R}^m) & \xrightarrow{\hat{\Lambda}} & M \times L(\mathbb{R}^p, \mathbb{R}^m) \times L(\mathbb{R}^p, \mathbb{R}^m) \times L(\mathbb{R}^p, \mathbb{R}^{mp})
\end{array}
\]

(7)

where \( \tilde{\psi} \) denotes the local trivialization of \( J_1^1 (M \times L(\mathbb{R}^p, \mathbb{R}^m)) \), and \( \psi^2 \) stands for the local trivialization of \( J_2^2 M \). We note that due to Proposition 5 it is also possible to use the induced chart of primary structure.

For any \( A \in L(\mathbb{R}^{2p}, \mathbb{R}^m) \), we can identify \( A \) in terms of the sub matrices of form \( m \times p \) with \( A = [A_\alpha A_\dot{\alpha}] \) where \( A_\alpha, A_\dot{\alpha} \in L(\mathbb{R}^p, \mathbb{R}^m) \). Similarly, for any \( B \in S_2(\mathbb{R}^{2p}, \mathbb{R}^m) \), we can identify \( B \) as the sub matrices

\[
B_{\dot{\alpha}\beta} = \begin{pmatrix} B_{\alpha\beta} & B_{\dot{\alpha}\beta} \\ B_{\dot{\alpha}\beta} & B_{\dot{\alpha}\dot{\beta}} \end{pmatrix}.
\]

Suppose that \( (x, A, B) \in M \times L(\mathbb{R}^{2p}, \mathbb{R}^m) \). Let \( (\psi^2)^{-1}(x, A, B) = j^2 \phi \) with the following properties:

\[
x = \phi(0, 0), \quad A = \frac{\partial \phi}{\partial u_\alpha}|_{(0,0)}, \quad [B_{\alpha\beta}] = \frac{\partial^2 \phi}{\partial u_\alpha u_\beta}|_{(0,0)}.\]

Therefore the sub matrices \( A_\alpha, A_\dot{\alpha}, B_{\alpha\beta}, B_{\dot{\alpha}\beta}, B_{\alpha\dot{\beta}} \) and \( B_{\dot{\alpha}\dot{\beta}} \) are

\[
A_\alpha = \frac{\partial \phi}{\partial u_\alpha}|_{(0,0)} \quad A_\dot{\alpha} = \frac{\partial \phi}{\partial w_\alpha}|_{(0,0)}
\]

and

\[
[B_{\alpha\beta}] = \frac{\partial^2 \phi}{\partial u_\alpha u_\beta}, \quad [B_{\dot{\alpha}\beta}] = \frac{\partial^2 \phi}{\partial w_\alpha u_\beta}, \quad [B_{\alpha\dot{\beta}}] = \frac{\partial^2 \phi}{\partial u_\alpha w_\beta}, \quad [B_{\dot{\alpha}\dot{\beta}}] = \frac{\partial^2 \phi}{\partial w_\alpha w_\beta}.
\]

Then,
\[ \hat{\Lambda}(x, A, B) = ((\tilde{\psi} \circ \psi^1) \circ \Lambda \circ (\psi^2)^{-1})(x, A, B) \]
\[ = ((\tilde{\psi} \circ \psi^1) \circ \Lambda)(j^2(\phi)) \]
\[ = (\tilde{\psi} \circ \psi^1)(j^1(\Phi)) \]
\[ = \tilde{\psi}(j^1(\psi \circ \Phi)) \]
\[ = ((\psi \circ \Phi)(0), \frac{\partial(\psi \circ \Phi)}{\partial u^\alpha}|_{0}). \] (8)

From the last line of Equation 8, one can see that
\[ (\psi \circ \Phi)(0) = \psi(j^1(\phi_0)) = (\phi(0, 0), \frac{\partial \phi(0, w)}{\partial w^\alpha}|_{w=0}) \in M \times L(\mathbb{R}^p, \mathbb{R}^m), \]
and
\[ \frac{\partial(\psi \circ \Phi)}{\partial u^\alpha}|_{0} = (\frac{\partial \phi(u, 0)}{\partial u^\alpha}|_{u=0}, \frac{\partial^2 \phi}{\partial u^\alpha \partial w^\beta}|_{(0, 0)}) \in L(\mathbb{R}^p, \mathbb{R}^m) \times L(\mathbb{R}^p, \mathbb{R}^{mp}). \]
Continuing to Equation 8, we have
\[ \hat{\Lambda}(x, A, B) = (\phi(0, 0), \frac{\partial \phi(0, w)}{\partial w^\alpha}|_{w=0}, \frac{\partial^2 \phi}{\partial u^\alpha \partial w^\beta}|_{u=w=0}) \]
\[ = (\phi(0, 0), \frac{\partial \phi(u, w)}{\partial u^\alpha}|_{u=(0, 0)}, \frac{\partial}{\partial u^\alpha}|_{u=0}(\phi(u, 0), \frac{\partial \phi}{\partial w^\beta}|_{w=0}) \]
\[ = (x, A_\alpha, A_\alpha, B_\alpha). \]

The map \((A, B) \rightarrow (x, A_\alpha, A_\alpha, B_\alpha)\) is a surjective linear map. It follows immediately that \(\hat{\Lambda}\) is a submersion thus \(\Lambda\) is.

\[ \square \]

**Remark 8.** Using curve notation on each \(j^2\phi\) and \(j^1\Phi\) in \(J^2_{2p}M\) and \(J^1_{p}(J^1_{p}M)\) respectively are as follows:
\[ j^2\phi = [x; [A_\alpha A_\alpha]; B_{\alpha\beta}], \]
considering the primary structure on \(J^1_{p}(J^1_{p}M)\) we have
\[ j^1\Phi = [(x, X_\alpha); (Y_\alpha, C_{\alpha\beta})], \]
where \((x, X_\alpha) = \Phi(0),\) and \((Y_\alpha, C_{\alpha\beta}) = \frac{\partial \Phi}{\partial u_\alpha}|_{0}.\)

Considering secondary structure on \(J^1_{p}(J^1_{p}M),\) we have
\[ j^1\Phi = [(x, X'_\alpha); (Y'_\alpha, C'_{\alpha\beta})] \]
It follows from equations 3 and 4 that \(X'_\alpha, Y'_\alpha,\) and \(C'_{\alpha\beta}\) are defined by the following:
\[
X'_\alpha = \frac{\partial (\pi \circ \Phi)}{\partial u_\alpha}, \quad Y'_\alpha = (pr_2 \circ \psi \circ \Phi)(0) = \frac{\partial \phi_0}{\partial u_\alpha}|_0.
\]

and

\[
C'_{\alpha\beta} = \frac{\partial (pr_2 \circ \psi \circ \Phi)}{\partial u_\alpha}|_0 = \frac{\partial^2 \Phi}{\partial u_\alpha \partial w_\beta}|_0.
\]

The local form of \( \Lambda \) follows immediately that the curve notation (discussed in preliminaries section) is as following:

\[
\Lambda([x; [A_\alpha A_\dot{\alpha}]; B_{\alpha\beta}]) = [x, A_\dot{\alpha}; A_\alpha, B_{\alpha\beta}]
\]

Now we shall define an involution on second order jet bundles that will help us to define the involution of \( J^1_p(J^1_p M) \).

**Theorem 9.** There exist a canonical involution \( \hat{\ell} : J^2_2 p M \to J^2_2 p M \) that descends via \( \Lambda \) to \( J^1_p(J^1_p M) \) giving the involution \( \ell : J^1_p(J^1_p M) \to J^1_p(J^1_p M) \).

To prove this theorem, we need to show that \( \hat{\ell} \) is a diffeomorphism, and the induced map \( \ell \) is well defined.

**Proof.** Let \( f : \mathbb{R}^{2p} \to \mathbb{R}^{2p} \) denote the flip map \((u, w) \to (w, u)\) (the map that flips first and second \( p \)-tuples). Define

\[
\tilde{\ell} : J^2_2 p M \to J^2_2 p M
\]

by

\[
\tilde{\ell}(j^2 \phi) = j^2 (\phi \circ f)
\]

The local form of \( \tilde{\ell} \) is the involution \((x, A, B) \to (x, \bar{A}, \bar{B})\) such that

\[
(x, \bar{A}, \bar{B}) = (x, [A_\dot{\alpha} A_\alpha], \begin{pmatrix} B_{\alpha\beta} & B_{\alpha\beta} \\ B_{\alpha\beta} & B_{\alpha\beta} \end{pmatrix}).
\]  

(For detailed proof of Equation (9), see Appendix section, Corollary 15)

The linear function corresponding to matrix \( \bar{A} \) is

\[
\bar{A}(U, W) = A_\dot{\alpha} U + A_\alpha W = [A_\alpha A_\dot{\alpha}].(W, U) = A(f(U, W)).
\]

Therefore, \( \bar{A} = A \circ f \). The symmetric bi-linear form corresponding to matrix \( \bar{B} \) is
\[
\bar{B}((U, W), (\bar{U}, \bar{W})) = (\bar{U}, \bar{W})^T \tilde{B}(U, W) \\
= B_{\alpha\beta} U \bar{U} + B_{\alpha\beta} W \bar{W} + B_{\alpha\beta} U \bar{W} + B_{\alpha\beta} W \bar{U} \\
= f(U, W)^T \cdot B.f(U, W) \\
= B(f(U, W), f(\bar{U}, \bar{W})).
\]

Thus, \( \bar{B} = B \circ (f, f) \). Then, the local form of \( \tilde{\ell} \) is the function

\[(x, A, B) \rightarrow (x, A \circ f, B \circ (f, f))\]

which is a diffeomorphism.

To finish the proof, we need to show that above defined \( \ell \) is well defined.

Let \( j^2 \phi, j^2 \phi' \in J^2_{2p} M \).

Suppose that \( \Lambda(j^2 \phi) = \Lambda(j^2 \phi') \). Then,

\[\tilde{\psi}(\Lambda(j^2 \phi)) = \tilde{\psi}(\Lambda(j^2 \phi'))\]

\[\Rightarrow (\tilde{\psi} \circ \Lambda)(j^2 \phi) = (\tilde{\psi} \circ \Lambda)(j^2 \phi')\]

By Diagram (7), we have

\[(\hat{\Lambda} \circ \psi^2)(j^2 \phi) = (\hat{\Lambda} \circ \psi^2)(j^2 \phi')\]

\[\Rightarrow (\phi(0,0), \frac{\partial \phi}{\partial w_\alpha}, \frac{\partial \phi}{\partial u_\alpha}, \frac{\partial^2 \phi}{\partial u_\alpha \partial w_\beta}) = (\phi'(0,0), \frac{\partial \phi'}{\partial w_\alpha}, \frac{\partial \phi'}{\partial u_\alpha}, \frac{\partial^2 \phi'}{\partial u_\alpha \partial w_\beta})\]

\[\Rightarrow (\phi(0,0), A_\alpha, A_\alpha, B_{\alpha\beta}) = (\phi'(0,0), A'_\alpha, A'_\alpha, B'_{\alpha\beta})\]

On the last equation, we can conclude that \( B_{\alpha\beta} = B'_{\alpha\beta} \). Since the matrices \( B \) and \( B' \) are symmetric, then

\[B_{\alpha\beta} = [B_{\alpha\beta}]^T\]

which implies

\[B_{\alpha\beta} = B'_{\alpha\beta}.\]

Now, we focus on the function \( \tilde{\ell} \). Using commutative Diagram (7), \( \tilde{\psi} \circ \Lambda = \hat{\Lambda} \circ \psi^2 \). Then
\[(\tilde{\psi} \circ \Lambda \circ \tilde{\ell})(j^2 \phi) = (\tilde{\psi} \circ \Lambda)(j^2(\phi \circ f)) = (\Lambda \circ \psi^2)(j^2(\phi \circ f)) = \Lambda(\phi(0,0), \frac{\partial(\phi \circ f)}{\partial \bar{u}_\alpha}, \frac{\partial^2(\phi \circ f)}{\partial \bar{u}_\alpha \partial \bar{u}_\beta}) = \Lambda(x, [A_{\tilde{\alpha}}, A_\alpha], \begin{pmatrix} B_{\tilde{\alpha}\beta} & B_{\alpha\beta} \\ B_{\alpha\beta} & B_{\alpha\beta} \end{pmatrix}) = (x, A_\alpha, A_{\tilde{\alpha}}, B_{\alpha\beta}) = (x', A'_\alpha, A'_{\tilde{\alpha}}, B'_{\alpha\beta}) = (\tilde{\psi} \circ \Lambda \circ \tilde{\ell})(j^2 \phi').\]

On the other hand, because \(\tilde{\psi}\) is injective, it follows immediately that

\[(\Lambda \circ \tilde{\ell})(j^2 \phi) = (\Lambda \circ \tilde{\ell})(j^2 \phi'). \tag{10}\]

Equation (10) implies that \(\ell\) is a function on \(J^1_p M\). Since \(\Lambda\) is a surjective submersion, and \(\tilde{\ell}\) is a diffeomorphism, then \(\ell\) is a differentiable function (see [3] and [6], 16.7.7,ii). Since \(\tilde{\ell}\) is an involution i.e. \(\tilde{\ell}\) has its own inverse, then \(\ell\) is also an involution, and the function \(\ell\) is a diffeomorphism.

In the following proposition, we define aforementioned involution \(\ell\).

**Proposition 10.** The involution \(\ell : J^1_p(J^1_p M) \to J^1_p(J^1_p M)\) is defined by the following:

\[\ell([y, X_\alpha; X_\alpha, C_{\alpha\beta}]) = [y, X_\alpha; Y_\alpha, [C_{\alpha\beta}]^T]\]

**Proof.** Suppose that \(v = [y, Y_\alpha; X_\alpha, C_{\alpha\beta}] \in J^1_p(J^1_p M)\). Due to \(\Lambda\) being a surjective function, there exists \(j^2 \phi = [x, A_{\tilde{\alpha}}, B_{\alpha\beta}] \in J^2_{2p} M\) such that

\[A_{\tilde{\alpha}} = [A_\alpha A_{\tilde{\alpha}}],\]

\[B_{\alpha\beta} = \begin{pmatrix} B_{\alpha\beta} & B_{\tilde{\alpha}\beta} \\ B_{\alpha\beta} & B_{\tilde{\alpha}\beta} \end{pmatrix},\]

and

\[[y, Y_\alpha; X_\alpha, C_{\alpha\beta}] = \Lambda(j^2 \phi) = \Lambda([x, A_{\tilde{\alpha}}, B_{\tilde{\alpha}\beta}]) = [x, A_\alpha, A_{\tilde{\alpha}}, B_{\alpha\beta}]\]

Then,

\[x = y, \quad Y_\alpha = A_{\tilde{\alpha}}, \quad X_\alpha = A_\alpha, \quad C_{\alpha\beta} = B_{\alpha\beta}\]
On the other hand, since the matrix $B_{\alpha\beta}$ is symmetric, then $[B_{\alpha\beta}]^T = B_{\alpha\beta}$. Then

$$\ell([y, Y_\alpha; X_\alpha, C_{\alpha\beta}]) = (\ell \circ \Lambda)(j^2 \phi) = (\Lambda \circ \tilde{\ell})(j^2 \phi) = [x, A_\alpha; A_\alpha, B_{\alpha\beta}],$$

$$= [y, X_\alpha; Y_\alpha, [B_{\alpha\beta}]^T]$$

which finishes the proof.

In the following proposition, we shall show that the primary and secondary vector bundle structures on $J_p^1(J_p^1 M)$ are isomorphic.

**Proposition 11.** For any smooth manifold $M$, the function

$$\ell : J_p^1(J_p^1 M) \rightarrow J_p^1(J_p^1 M)$$

which is defined in Theorem 9 is a $J^1 M$- isomorphism of vector bundles in the following way:

$$
\begin{array}{ccc}
J_p^1(J_p^1 M) & \xrightarrow{\ell} & J_p^1(J_p^1 M) \\
\downarrow \pi & & \downarrow \pi^1_M \\
J_p^1 M & = & J_p^1 M
\end{array}
$$

**Proof.** To prove the theorem, we only need to show that $\ell$ preserves fibers.

Suppose that $j^1 \phi \in \tilde{\pi}^{-1}([x, X_\alpha])$. Then from Remark 8, $j^1 \phi = [x, X_\alpha; Y_\alpha, C_{\alpha\beta}]$. Using Proposition 10 we have

$$\pi^1 \circ \ell(j^1 \phi) = \pi^1([x, X_\alpha; Y_\alpha, C_{\alpha\beta}]) = \pi([x, Y_\alpha; X_\alpha, [C_{\alpha\beta}]^T]) = [x, X_\alpha] = \tilde{\pi}(j^1 \phi)$$

which proves that $\ell$ preserves fibers. Thus $\ell$ is a bundle isomorphism.
4 Appendix

Below, we present some identities and statements that are used in this paper.

**Lemma 12.** $(J^1_pM, \pi, M, L(\mathbb{R}^p, \mathbb{R}^m))$ is a vector bundle with the bundle projection $\pi_M : J^1_pM \to M$ which is defined by $\pi_M(j^1 \phi) = \phi(0)$ for all $j^1 \phi \in J^1_pM$, the the above manifold structure on $J^1_pM$, and the intrinsic operations on $\pi^{-1}_M \{x\}$ defined by

$$[x; X_\alpha] + [x; X'_\alpha] = [x; X_\alpha + X'_\alpha] \quad \lambda \cdot [x; X_\alpha] = [x; \lambda X_\alpha]$$

(11)

for all $x \in M$.

**Proof.** In [5], it is proven that $J^1_pM$ is a bundle with the local trivialization

$$\Psi_M : J^1_pU \to U \times L(\mathbb{R}^p, \mathbb{R}^m)$$

where $J^1_pU = \{j^1 \phi : \phi(0) \in U\}$. Showing that $(\Psi_M)_x : \pi^{-1}_M \{x\} \to L(\mathbb{R}^p, \mathbb{R}^m))$ being an isomorphism, implies that $(J^1_pM, \pi, M, L(\mathbb{R}^p, \mathbb{R}^m))$ is a vector bundle.

From Equation (1) we have

$$(\Psi_M)_x(j^1 \phi) = \frac{\partial(x_i \circ \phi)}{\partial u_\alpha}|_0$$

If we denote $j^1 \phi = [x; X_\alpha]$ where $X_\alpha = \phi^*_\alpha(d/du)$, then

$$(\Psi_M)_x(j^1 \phi) = \left(X_\alpha[x_i]\right).$$

Therefore, we have

$$(\Psi_M)_x([x; X_\alpha] + [x; X'_\alpha]) = (X_\alpha + X'_\alpha)[x_i] = (X_\alpha[x_i] + X'_\alpha[x_i]) = (\Psi_M)_x([x; X_\alpha]) + (\Psi_M)_x([x; X'_\alpha]).$$

On the other hand, we have

$$(\Psi_M)_x(\lambda \cdot [x; X_\alpha]) = (\lambda X_\alpha)[x_i] = \lambda(X_\alpha[x_i]) = \lambda((\Psi_M)_x([x; X_\alpha]))$$

which shows that $(\Psi_M)_x$ is a linear function.

\[ \square \]

**Lemma 13.** The function $\Lambda$ defined in Lemma 9 is well defined.
**Proof.** Suppose that \( j^2 \phi = j^2 \phi' \), where \( j^2 \phi, j^2 \phi' \in J^2_{2p}(M) \). To prove the lemma, we should show that \( j^1 \Phi = j^1 \Phi' \).

The equality of 2-jets implies followings:

\[
\phi(0, 0) = \phi'(0, 0),
\]

\[
\frac{\partial \phi}{\partial \bar{u}_\alpha}|_{(0, 0)} = \frac{\partial \phi'}{\partial \bar{u}_\alpha}|_{(0, 0)},
\]

\[
\frac{\partial^2 \phi}{\partial \bar{u}_\alpha \bar{u}_\beta}|_{(0, 0)} = \frac{\partial^2 \phi'}{\partial \bar{u}_\alpha \bar{u}_\beta}|_{(0, 0)}.
\]

Equation (12) shows that

\[
\phi_0(0) = \phi'_0(0).
\]

The Equation (13) leads

\[
\frac{\partial \phi}{\partial \bar{u}_\alpha}|_{(0, 0)} = \frac{\partial \phi'}{\partial \bar{u}_\alpha}|_{(0, 0)} \quad \text{and} \quad \frac{\partial \phi}{\partial \bar{w}_\alpha}|_{(0, 0)} = \frac{\partial \phi'}{\partial \bar{w}_\alpha}|_{(0, 0)}.
\]

which implies

\[
\Rightarrow \frac{\partial \phi(u, 0)}{\partial \bar{u}_\alpha}\bigg|_{u=0} = \frac{\partial \phi'(u, 0)}{\partial \bar{u}_\alpha}\bigg|_{u=0}
\]

and

\[
\Rightarrow \frac{\partial \phi(0, w)}{\partial \bar{w}_\alpha}\bigg|_{w=0} = \frac{\partial \phi'(0, w)}{\partial \bar{w}_\alpha}\bigg|_{w=0}.
\]

Equation (17) leads

\[
\frac{\partial \phi_0}{\partial \bar{w}_\alpha} = \frac{\partial \phi'_0}{\partial \bar{w}_\alpha}.
\]

Combining Equations (15) and (18), we have

\[
\Phi(0) = j^1(\phi_0) = j^1(\phi'_0) = \Phi'(0).
\]

On the other hand, by setting \( \bar{u}_\alpha = u_\alpha \) and \( \bar{u}_\beta = w_\beta \) in Equation (14) we have

\[
\frac{\partial^2 \phi}{\partial \bar{u}_\alpha \bar{w}_\beta}|_{(0, 0)} = \frac{\partial^2 \phi'}{\partial \bar{u}_\alpha \bar{w}_\beta}|_{(0, 0)}.
\]

For all \( u \in \mathbb{R}^p \), \( j^1(\phi_u) \) can be associated to the ordered pair \( [\phi_u(0), \frac{\partial \phi_u}{\partial w_\beta}] \). Then, combining Equations (16) and (19) we have
\[
\frac{\partial \Phi}{\partial u_\alpha} \bigg|_{u=0} = \left( \frac{\partial \phi_u(0)}{\partial u_\alpha}, \frac{\partial^2 \phi}{\partial u_\alpha \partial w_\beta} \bigg|_{(0,0)} \right) \\
= \left( \frac{\partial \phi(u,0)}{\partial u_\alpha}, \frac{\partial^2 \phi}{\partial u_\alpha \partial w_\beta} \bigg|_{(0,0)} \right) \\
= \left( \frac{\partial \phi'_u(0)}{\partial u_\alpha}, \frac{\partial^2 \phi'}{\partial u_\alpha \partial w_\beta} \bigg|_{(0,0)} \right) \\
= \frac{\partial \Phi'}{\partial u_\alpha} \bigg|_{u=0}
\]

(20)

which shows that \( j^* \Phi = j^* \Phi' \). Thus \( \Lambda \) is well defined.

\[\□\]

**Lemma 14.** Let \( \phi \in C^\infty(\mathbb{R}^{2p}, M) \), and \( u^\alpha, w^\alpha \) denote the first \( p \) coordinate functions and the second \( p \) coordinate functions respectively on \( \mathbb{R}^{2p} \). Let \( f \) be the flip map defined in Lemma 9. Then there exists following equations:

\[
\frac{\partial(\phi \circ f)}{\partial u_\alpha} \bigg|_0 = \frac{\partial \phi}{\partial w_\alpha} \bigg|_0 \\
= \frac{\partial(\phi \circ f)}{\partial w_\alpha} \bigg|_0 = \frac{\partial \phi}{\partial u_\alpha} \bigg|_0 \label{eq:21}
\]

\[
\frac{\partial^2 (\phi \circ f)}{\partial u_\beta \partial u_\alpha} \bigg|_0 = \frac{\partial^2 \phi}{\partial w_\beta \partial w_\alpha} \bigg|_0 \label{eq:22}
\]

\[
\frac{\partial^2 (\phi \circ f)}{\partial w_\beta \partial u_\alpha} \bigg|_0 = \frac{\partial^2 \phi}{\partial u_\beta \partial w_\alpha} \bigg|_0 \label{eq:23}
\]

\[
\frac{\partial^2 (\phi \circ f)}{\partial w_\beta w_\alpha} \bigg|_0 = \frac{\partial \phi}{\partial u_\beta u_\alpha} \bigg|_0 \label{eq:24}
\]

**Proof.** Since \( f(u, w) = (w, u) \), then

\[
u_\alpha \circ f = w_\alpha, \quad w_\alpha \circ f = u_\alpha.
\]

Due to the Chern Rule, we have

\[
\frac{\partial(\phi \circ f)}{\partial u_\alpha} = \left( \frac{\partial \phi}{\partial u_\beta} \circ f \right) \cdot \frac{\partial(u_\beta \circ f)}{\partial u_\alpha}.
\]

(26)

Combining (25) and (26) at the point 0, we obtain Equation (21).

On the other hand, taking partial derivatives of two sides of Equation (26) at the point 0 and using Chern rule gives Equations (22), (23) and (24).

\[\□\]

**Corollary 15.** Entries of the matrices \( \bar{A} \) and \( \bar{B} \) in Lemma 4 can be given as follows:

\[
\bar{A}_\alpha = A_\bar{\alpha} \quad \bar{A}_{\bar{\alpha}} = A_\alpha
\]
$$B_{\alpha\beta} = \bar{B}_{\alpha\beta}, \quad \bar{B}_{\alpha\beta} = B_{\alpha\beta}, \quad \bar{B}_{\alpha\beta} = B_{\alpha\beta}, \quad \bar{B}_{\alpha\beta} = B_{\alpha\beta}$$

**Proof.** Using the definition of the matrix $\bar{B}$ and Equations (22), (23), and (21) we have

$$[\bar{B}_{\alpha\beta}] = \frac{\partial^2 (\phi \circ \bar{f})}{\partial u_\alpha \partial u_\beta}|_0 = \frac{\partial^2 \phi}{\partial w_\alpha \partial w_\alpha}|_0 = B_{\alpha\beta}$$

and

$$[\bar{B}_{\alpha\beta}] = \frac{\partial^2 (\phi \circ \bar{f})}{\partial w_\alpha \partial u_\beta}|_0 = B_{\alpha\beta}$$

$$[\bar{B}_{\alpha\beta}] = \frac{\partial^2 (\phi \circ \bar{f})}{\partial w_\alpha \partial w_\beta}|_0 = \frac{\partial^2 \phi}{\partial u_\alpha \partial w_\alpha}|_0 = B_{\alpha\beta}.$$

\[\square\]

**Lemma 16.** The local form of the function $\Omega$ is

$$\hat{\Omega}(u, f, y, g) = (u, y, f, g).$$

**Proof.** For all $(u, f, y, g) \in \varphi_2((\pi_\mathcal{E})^{-1}(J_p^1U))$, then there exists a unique $j^1\Phi \in (\pi_\mathcal{E})^{-1}(J_p^1U)$ such that $\varphi_2(j^1\Phi) = (u, f, y, g)$. Thus,

$$(u, f, y, g)$$

$$= ((\varphi \times \text{id}_{L(\mathbb{R}^p, \mathbb{R}^m)} \times \text{pr}_{\mathcal{E}} \times \mathbb{R}^k) \circ (\Psi_M \times \Psi_\mathcal{E}) \circ \psi^1)(j^1\Phi)$$

$$= ((\varphi \times \text{id}_{L(\mathbb{R}^p, \mathbb{R}^m)} \times \text{pr}_{\mathcal{E}} \times \mathbb{R}^k) \circ (\Psi_M \times \Psi_\mathcal{E}))(j^1(\psi \circ \Phi))$$

$$= ((\varphi \times \text{id}_{L(\mathbb{R}^p, \mathbb{R}^m)} \times \text{pr}_{\mathcal{E}} \times \mathbb{R}^k) \circ (\Psi_M \times \Psi_\mathcal{E}))(j^1(\pi_\mathcal{E} \circ \Phi), j^1(pr_2 \circ \psi \circ \Phi))$$

$$= ((\varphi \circ \pi_\mathcal{E} \circ \Phi)(0), \frac{\partial(\pi_\mathcal{E} \circ \Phi)}{\partial u_\alpha}|_0, (pr_2 \circ \psi \circ \Phi)(0), \frac{\partial(pr_2 \circ \psi \circ \Phi)}{\partial u_\alpha}|_0).$$

Therefore, there exist the following equations:

$$u = (\varphi \circ \pi_\mathcal{E} \circ \Phi)(0), \quad f = \frac{\partial(\pi_\mathcal{E} \circ \Phi)}{\partial u_\alpha}|_0, \quad y = (pr_2 \circ \psi \circ \Phi)(0), \quad g = \frac{\partial(pr_2 \circ \psi \circ \Phi)}{\partial u_\alpha}|_0.$$

On the other hand, $\frac{\partial(\psi \circ \Phi)}{\partial u_\alpha}|_0 : \mathbb{R}^p \to \mathbb{R}^m$. The derivative can be written of form

$$\frac{\partial(\psi \circ \Phi)}{\partial u_\alpha}|_0 = [\frac{\partial(pr_1 \circ \psi \circ \Phi)}{\partial u_\alpha}|_0, \frac{\partial(pr_2 \circ \psi \circ \Phi)}{\partial u_\alpha}|_0].$$

Thus,
\[ \xi \left( \frac{\partial (\psi \circ \Phi)}{\partial u_\alpha} \right)_{\left| 0 \right.} = \left( \frac{\partial (pr_1 \circ \psi \circ \Phi)}{\partial u_\alpha} \right)_{\left| 0 \right.}, \frac{\partial (pr_2 \circ \psi \circ \Phi)}{\partial u_\alpha} \right)_{\left| 0 \right.}. \]

Using this equation, we have

\[ ((\varphi \times id) \circ (\psi \times \xi) \circ \tilde{\psi})(j^1 \Phi) = ((\varphi \times id) \circ (\psi \times \xi))(\Phi(0), \frac{\partial (\psi \circ \Phi)}{\partial u_\alpha}) \]
\[ = (\varphi \times id)((pr_1 \circ \psi \circ \Phi)(0), (pr_2 \circ \psi \circ \Phi)(0), \xi(\frac{\partial (\psi \circ \Phi)}{\partial u_\alpha})) \]
\[ = ((\varphi \circ \pi_E \circ \Phi)(0), (pr_2 \circ \psi \circ \Phi)(0), \left( \frac{\partial (pr_1 \circ \psi \circ \Phi)}{\partial u_\alpha} \right)_{\left| 0 \right.}, \frac{\partial (pr_2 \circ \psi \circ \Phi)}{\partial u_\alpha} \right)_{\left| 0 \right.}) \]
\[ = (u, y, f, g). \]

This proves that \( \hat{\Omega}(u, f, y, g) = (u, y, f, g). \)

\[ \square \]

Acknowledgements. The Author would like to thank to Idaho State University Department of Mathematics in particular Prof. Dr. Robert Fisher, Jr. for being such nice hosts during her visit in Fall 2015. She also acknowledges library support from ISU.

References

[1] Baker Richard, Doran Chris, Jet Bundles and the Formal Theory of Partial Differential Equations, In Dorst Leo, Doran Chris, Lasenby Joan, Applications of Geometric Algebra in Computer Science and Engineering, Chapter 12, Springer Science and Business Media, (2012), pp: 133–143.

[2] Barco Michael A. Solutions of Partial Differential Equations Using Symmetry and Symbolic Computation, PhD. Thesis, School of Mathematical and Statistical Sciences Faculty of Science, Technology and Engineering, La Trobe University.(2000)

[3] Brickell F. and Clark R. S. Differentiable Manifolds An Introduction, Van Nostrand Reinhold Company, London,(1970).

[4] Bocharov A.V., Chetverikov V.N, Duzhin S.V., Khorkova N.G ., Krasilshchik I.S ., Samokhin, A.V., Torkhov Yu.N., Verbovetsky A.M. and Vinogradov A.M. Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, Translations of Mathematical Monographs, 182, American Mathematical Society, Providence, Rhode Island, (1999).

[5] Cordero, L. A. and Dodson, C.T.J., and Leon, M. de Differential Geometry of Frame Bundles, Kluwer Academic Press, (1989).

[6] Dieudonne’ J., Treatise on Anaysis, 3, Academic Press,Inc., Newyork and London, (1972).
[7] Fisher Robert and Laquer H. Turner Second order tangent vectors in Riemannian geometry. J. Korean Math. Soc., 36 (5), (1999), 959–1008,

[8] Krasilshchik Joseph and Verbovetsky Alexander Geometry of Jet Spaces and Integrable Systems, Journal of Geometry and Physics, 61 (9), (2011), 1633–1674,

[9] Leon Manuel de and Rodrigues Paulo R. Generalized Classical Mechanics and Field Theory, NORTH-HOLLAND, (1985).

[10] Saunders D.J. The Geometry of Jet Bundles, Cambridge University Press, Cambridge-New York, (1989).

[11] Wu Wenyuan Geometric Symbolic-Numeric Methods for Differential and Algebraic Systems, PhD. Thesis, Faculty of Graduate Studies, The University of Western Ontario, (2007).