The serial harness interacting with a wall

Pablo A. Ferrari\textsuperscript{a} Luiz R. G. Fontes\textsuperscript{a} Beat M. Niederhauser\textsuperscript{a} Marina Vachkovskaia\textsuperscript{b}

\textsuperscript{a}Universidade de São Paulo
\textsuperscript{b}Universidade Estadual de Campinas

AMS 1991 subject classifications. 60K35 82B 82C

Abstract

The serial harnesses introduced by Hammersley describe the motion of a hypersurface of dimension \(d\) embedded in a space of dimension \(d+1\). The height assigned to each site \(i\) of \(\mathbb{Z}^d\) is updated by taking a weighted average of the heights of some of the neighbors of \(i\) plus a “noise” (a centered random variable). The surface interacts by exclusion with a “wall” located at level zero: the updated heights are not allowed to go below zero. We show that for any distribution of the noise variables and in all dimensions, the surface delocalizes. This phenomenon is related to the so called “entropic repulsion”. For some classes of noise distributions, characterized by their tail, we give explicit bounds on the speed of the repulsion.

Key words: harness, surface dynamics, entropic repulsion

1 Introduction and results

Hammersley (1965) introduced the serial harness, a discrete-time stochastic process that models the time evolution of a hypersurface of dimension \(d\) embedded in a \(d+1\) dimensional space. A quantity \(Y_n(i) \in \mathbb{R}\) stays for the height of the surface at site \(i \in \mathbb{Z}^d\) at (integer) time \(n \geq 0\). The initial configuration is the flat surface \(Y_0(i) = 0\) for all \(i\). Under the evolution, at each moment \(n \geq 0\) the height at each site is substituted by a weighted average of the heights at the previous moment plus a symmetric random variable.

Let \(P = \{p(i,j)\}_{i,j \in \mathbb{Z}^d}\) be a stochastic matrix, i.e. \(p(i,j) \geq 0\) and \(\sum_j p(i,j) = 1\), which satisfies \(p(i,j) = p(0,j-i) =: p(j-i)\) (homogeneity), \(\sum_j j p(j) = 0\), and \(p(j) = 0\) for all \(|j| > v\) for some \(v\) (finite range). Assume also that \(P\) is truly \(d\)-dimensional: \(\{j \in \mathbb{Z}^d : p(j) \neq 0\}\) generates \(\mathbb{Z}^d\).
Let $E = (\varepsilon, (\varepsilon_n(i), i \in \mathbb{Z}^d), n \in \mathbb{Z})$ be a family of i.i.d. integrable symmetric random variables. Let $\mathbb{P}$ and $\mathbb{E}$ denote the probability and expectation in the probability space generated by $E$. (We use preliminary $n \in \mathbb{N}$ in the definitions but later it will be useful to have $n \in \mathbb{Z}$.)

The serial harness $(Y_n, n \geq 0)$ is the discrete-time Markov process in $\mathbb{R}^{\mathbb{Z}^d}$ defined by

$$Y_n(i) = \begin{cases} 0, & \text{if } n = 0, \\ \sum_{j \in \mathbb{Z}^d} p(i, j)Y_{n-1}(j) + \varepsilon_n(i), & \text{if } n \geq 1. \end{cases} \quad (1.1)$$

Here $Y_n(i)$ denotes the height of the serial harness at site $i$ at time $n$. In other words, the evolution is given by

$$Y_n = \mathbb{P}Y_{n-1} + \varepsilon_n, \quad (1.2)$$

where $\varepsilon_n = (\varepsilon_n(i), i \in \mathbb{Z}^d)$. Since the “noise variable” $\varepsilon$ is symmetric and thus has zero mean, we have that $\mathbb{E}Y_n(i) = 0$ for all $i, n$. We can interpret $p(i, j)$ as transition probabilities of a random walk on $\mathbb{Z}^d$; let $p_m(i, j)$ be its $m$-step transition probabilities. By homogeneity, $p_m(i, j) = p_m(0, j-i) =: p_m(j-i)$.

Iterating (1.1),

$$Y_n(i) = \sum_{r=1}^{n} \sum_{j \in \mathbb{Z}^d} p_{n-r}(i, j)\varepsilon_r(j) \overset{d}{=} \sum_{r=0}^{n-1} \sum_{j \in \mathbb{Z}^d} p_r(j)\varepsilon_r(j), \quad (1.3)$$

for all $n \geq 1, i \in \mathbb{Z}^d$, where $\overset{d}{=}$ means equidistributed. Hammersley (1965) obtained that

$$\mathbb{E}(Y_n(i))^2 = \sigma^2 s(n) \quad (1.4)$$

where $\sigma^2$ is the variance of $\varepsilon$ and

$$s(n) := \sum_{r=0}^{n-1} \sum_{j \in \mathbb{Z}^d} p_r(j)^2. \quad (1.5)$$

is the expected number of encounters up to time $n$ of two independent copies of a random walk starting at 0 with transition probabilities $\mathbb{P}$. Equality (1.4) follows immediately from (1.3). Since $s(n) \sim \sqrt{n}$ for $d = 1$, $s(n) \sim \log n$ for $d = 2$ and $s(n)$ is uniformly bounded in $n$ for $d \geq 3$ (see, for example, Spitzer (1976)), the surface delocalizes in dimensions $d \leq 2$ and stays localized in dimensions $d \geq 3$. Toom (1997) studies localization of the surface and surface-differences in function of the decay of the distribution of $\varepsilon$.

We consider the serial harness interacting by exclusion with a wall located at the origin. The wall process $(W_n, n \geq 0)$ is the Markov process in $(\mathbb{R}^+)^{\mathbb{Z}^d}$ defined by
\[ W_n(i) = \begin{cases} 
0, & \text{if } n = 0, \\
\left(\sum_{j \in \mathbb{Z}^d} p(i, j)W_{n-1}(j) + \varepsilon_n(i)\right)^+, & \text{if } n \geq 1, 
\end{cases} \tag{1.6} \]

for \( i \in \mathbb{Z}^d \), where for \( a \in \mathbb{R} \), \( a^+ = a \lor 0 = \max(a, 0) \); this can be reexpressed as

\[ W_n = (\mathcal{P}W_{n-1} + \varepsilon_n)^+. \tag{1.7} \]

We say that the law of a random surface \( Z \) is an *invariant measure* for the wall process if \( Z \overset{d}{=} (\varepsilon_0 + \mathcal{P}Z)^+ \), with \( \varepsilon_0 \) and \( Z \) independent. We show in Section 2 that

\[ W_n \leq W_{n+1} \text{ stochastically.} \tag{1.8} \]

This implies that \( W_n \) is stochastically non-decreasing and thus their laws converge to a limit (that could give positive weight to infinity). If the limit is nondegenerate, then it is an invariant measure for the wall process. Monotonicity (1.8) implies in particular

\[ \mu_n := \mathbb{E}W_n(0) \]

is nondecreasing and thus converges either to a finite limit or to \( \infty \). Our first result is general and rules out the former possibility, showing however that \( \mu_n \) goes to infinity slower than \( n \).

**Theorem 1.1** (a) There is no nondegenerate invariant measure for the wall process \((W_n)\); (b) \( W_n \to \infty \) in probability; (c) \( \mu_n \to \infty \) as \( n \to \infty \); (d) \( \mu_n/n \to 0 \) as \( n \to \infty \).

This theorem is proven in Section 2.

Let \( F \) be the law of \( \varepsilon \), \( \bar{F}(x) = \mathbb{P}(\varepsilon > x) \) and define

\[ \mathcal{L}_\alpha^- := \{F : \bar{F}(x) \leq ce^{-c'x^\alpha}, x > 0, \text{ for some positive } c, c'\} \tag{1.9} \]

\[ \mathcal{L}_\alpha^+ := \{F : \bar{F}(x) \geq ce^{-c'x^\alpha}, x > 0, \text{ for some positive } c, c'\} \tag{1.10} \]

and

\[ \mathcal{L}_\alpha := \mathcal{L}_\alpha^- \cap \mathcal{L}_\alpha^+ \tag{1.11} \]

We next state our main result. It consists of upper and lower bounds for \( \mu_n \) for different noise distributions.

**Theorem 1.2** There exist constants \( c \) and \( C \) that may depend on the dimension such that
(i) for \( d = 1 \) if \( F \in \mathcal{L}_1^- \),
\[
   cn^{1/4} \leq \mu_n \leq Cn^{1/4}\sqrt{\log n};
\]
(1.12)

(ii) for \( d = 2 \), if \( F \in \mathcal{L}_\alpha \), for some \( \alpha \geq 1 \),
\[
   c(\log n)^{\frac{1}{\alpha} + \frac{1}{2}} \leq \mu_n \leq C \log n;
\]
(1.13)

(iii) for \( d \geq 3 \), if \( F \in \mathcal{L}_\alpha \), for some \( 1 \leq \alpha < 1 + d/2 \),
\[
   c(\log n)^{\frac{1}{\alpha}} \leq \mu_n \leq C(\log n)^{\frac{1}{\alpha} + \frac{2}{d}};
\]
(1.14)

(iv) for \( d \geq 3 \) if \( F \in \mathcal{L}_{1+d/2} \),
\[
   c(\log n)^{\frac{1}{\alpha}} \leq \mu_n \leq C(\log n)^{\frac{2}{\alpha}}(\log \log n)^{\frac{d}{\alpha}}.
\]
(1.15)

Our upper bound in (1.15) can be slightly improved, see (6.4) and Remark 6.2 below. The lower bound in (i) can be shown to hold under weaker conditions; that is also the case for some cases of (ii); see (6.7) and Remark 6.7 below. If the noise distribution is in \( \mathcal{L}_\alpha \) for some \( \alpha \geq 1 \), then our lower and upper bounds to \( \mu_n \) are of the same order in the case that \( d \geq 3 \), \( 1 \leq \alpha < 1 + d/2 \) (which includes the Gaussian case \( \alpha = 2 \) for all such dimensions), and also in the case that \( d = 2 \), \( \alpha = 1 \).

Theorems 1.1 and 1.2 catch the effect of the “entropic repulsion” in a stochastically moving surface interacting with a wall by exclusion.

Many papers deal with the problem of entropic repulsion in Equilibrium Statistical Mechanics. The role of the entropic repulsion in the Gaussian free field was studied by Lebowitz and Maes (1987), Bolthausen, Deuschel and Zeitouni (1995), Deuschel (1996), Deuschel and Giacomin (1999) and Bolthausen, Deuschel and Giacomin (2001). In the Ising, SOS and related models the matter was discussed in Bricmont, El Mellouki and Fröhlich (1986), Bricmont (1990), Cesari and Martinelli (1996), Dinaburg and Mazel (1994), Holicky and Zahradnik (1993), and Ferrari and Martínez (1998).

The exponent \( 1/4 \) for dynamic entropic repulsion in \( d = 1 \) was predicted by Lipowsky (1985) using scaling arguments. This exponent was then found numerically by Mon, Binder, Landau (1987), Binder (1990), De Coninck, Dunlop and Menu (1993). Dunlop, Ferrari and Fontes (2001) proved bounds (slightly worse than) (1.12) for a one dimensional interface related to the phase separation line in the two dimensional Ising model at zero temperature. Funaki and Olla (2001) studied a one dimensional model in a finite box rescaled as the square of the time.

The strategy to show part of Theorem 1.2 is to compare the wall process with a “free process” — in our case the serial harness — as proposed by Dunlop,
Ferrari and Fontes (2001). The following lemmas are the basic ingredients in this approach. The first two concern moderate deviations of the serial harness $Y_n$; they are then extended to the wall process $W_n$ in the last one.

**Lemma 1.3** If the distribution of $\varepsilon$ is in $L_1^{-}$, then in $d \leq 2$ there exist constants $k, c, c' > 0$ such that for all $K > 0$ and $0 \leq l \leq n$,

$$
P[Y_l(0) \geq K \sqrt{s(n) \log n}] \leq kn^{1/e'} K. \tag{1.16}
$$

**Lemma 1.4** If the distribution of $\varepsilon$ is in $L_\alpha^{-}$ for some $\alpha \geq 1$, then in $d \geq 3$ there exist constants $k, c, c' > 0$ such that, for all $K > 0$ and $0 \leq l \leq n$,

(i) if $\alpha \neq 1 + d/2$, then

$$
P[Y_l(0) \geq K (\log n)^{\frac{1}{\alpha'} + \frac{d}{2}}] \leq kn^{e' - c' K}; \tag{1.17}
$$

(ii) and if $\alpha = 1 + d/2$, then

$$
P[Y_l(0) \geq KL_n(1 + 2/d)] \leq kn^{e' - c' K}, \tag{1.18}
$$

where $L_n(\cdot)$ is defined in (6.1) below.

**Lemma 1.5** The bounds of Lemmas 1.4 and 1.3 hold for $l = n$ if we replace $Y_n$ with $W_n$, possibly with worse constants $k, c$.

We conclude this introduction with a remark concerning the form (1.6) of the interaction with the wall. Two other choices are also natural. First, if the noise would push the process below zero, simply do nothing. Or, in the same case, only take the convex combination without a noise. Formally, these two cases are, respectively

$$
W'_0(i) = W''_0(i) \equiv 0,
$$

and for $n \geq 1$

$$
W'_n(i) = \begin{cases} 
\sum_{j \in Z^d} p(i, j)W'_{n-1}(j) + \varepsilon_n(i), & \text{if this is positive,} \\
W'_{n-1}(i), & \text{otherwise;}
\end{cases} \tag{1.19}
$$

and

$$
W''_n(i) = \begin{cases} 
\sum_{j \in Z^d} p(i, j)W''_{n-1}(j) + \varepsilon_n(i), & \text{if this is positive,} \\
\sum_{j \in Z^d} p(i, j)W''_{n-1}(j), & \text{otherwise.}
\end{cases} \tag{1.20}
$$

Coupling $W, W', W''$ by the same realization of the noise variables, one sees that, stochastically, both $W' \geq W$ and $W'' \geq W$. This implies immediately
that any lower bound for $\mu_n$ (in particular the ones in this paper) hold for $\mu'_n := EW'_n(0)$ and $\mu''_n := EW''_n(0)$ as well. These dominations also imply immediately the validity of the results of Theorem 1.1 (a-c) for $W'$ and $W''$. For the analogue of Theorem 1.1 (d), domination does not help (it goes in the wrong direction). An argument along the same lines as the one for $W$ can be made for $W''$ straightforwardly; see paragraphs containing (2.14) and (3.2). Under the assumption that $\mathbb{P}(0,0) > 0$, one can also make a similar argument for $W'$; otherwise, the matter is more delicate, and we do not have an argument.

As for upper bounds for $\mu'_n, \mu''_n$, the ones we get for $\mu_n$ also hold for both of them, since the proof only relies on the free process started at some height $r$ dominating stochastically the wall process started at the same height, and this holds for all three choices.

2 Delocalization

In this section we show Theorem 1.1. The wall process is \textit{attractive}, that is,

$$\text{if } W \leq W' \text{ then } (PW + \varepsilon_0)^+ \leq (PW' + \varepsilon_0)^+ \text{ a.s.} \quad (2.1)$$

coordinatewise, which implies

$$\text{if } W_n \leq W'_n \text{ stochastically, then } W_{n+1} \leq W'_{n+1} \text{ stochastically.} \quad (2.2)$$

Since for the process with initial flat surface $0 \equiv W_0 \leq W_1$ a.s. this implies (1.8).

Theorem 1.1 is a consequence of the following three lemmas.

\textbf{Lemma 2.1} There is no invariant measure for $(W_n)$ with finite mean.

\textbf{Proof.} Suppose there exists an invariant measure $\nu_o$ with finite mean $m_o$. Let $I = [-c, c]$ be the support of the distribution of $\varepsilon$. Then there exists $0 < c' < c$ such that $\mathbb{P}[\varepsilon < -c'] > 0$ and, by Markov’s inequality, for any $n$, $\mathbb{P}[\sum_j p_n(0, j)W(j) < 2m_o] > \frac{1}{2}$, where $p_n$ are the $n$-step transition probabilities.

The preceding implies that the process started from the invariant measure $\nu_o$ reaches the wall at the origin in $n' = 2m_o/c'$ steps with strictly positive probability. This yields a positive drift, contradicting the assumption.

\textbf{Lemma 2.2} Every invariant measure for $(W_n)$ dominates stochastically

$$\lim_n \mathbb{P}(W_n \in \cdot).$$
Proof. Attractiveness (2.2) implies that the law of \( W_n \) is stochastically non-decreasing and hence converges to a limit. Since the initial flat configuration is dominated by any other, any invariant measure dominates stochastically that limit. □

Consider the family of processes \( ((W^k_n, n \geq k), k \in \mathbb{Z}) \) defined by

\[
  W^k_n = \begin{cases} 
    0, & \text{if } n = k, \\
    (\mathcal{P}W^k_{n-1} + \varepsilon_n)^+, & \text{if } n \geq k + 1,
  \end{cases}
\]

(2.3)

\( (W^k_n, n \geq k) \) is the wall process evolving from time \( k \) on, having flat configuration at initial time \( k \). It is clear that for \( k \geq 0 \),

\[
  W_0^{-k} \overset{d}{=} W_k^0 (= W_k).
\]

(2.4)

Since \( 0 = W_k^k \leq W_k^{k-1} \), by attractiveness (2.1), \( W_n^k \leq W_n^{k-1} \) for all \( n \geq k \), and in particular:

\[
  W_0^k \leq W_0^{k-1}
\]

(2.5)

so that \( W_0^{-\infty} = \lim_{k \to \infty} W_0^{-k} \) is well defined (but could be infinity).

Lemma 2.3 \( W_0^{-\infty} \) (and hence \( W_n^{-\infty} \) for all \( n \)) is almost surely identically infinity.

Proof. The event \( \{ W_0^{-\infty} = \infty \} \) belongs to the tail \( \sigma \)-algebra of \( \{ \varepsilon_k : k \leq 0 \} \), and is thus trivial. Write

\[
  W_0^{-\infty} = (\varepsilon_0 + \mathcal{P}W_{-1}^{-\infty})^+ = \ldots
  = (\varepsilon_0 + \mathcal{P}(\varepsilon_{-1} + \ldots \mathcal{P}(\varepsilon_{-k+1} + \mathcal{P}W_{-k}^{-\infty})^+ \ldots)^+)^+
  \geq U_k + \mathcal{P}^k W_{-k}^{-\infty},
\]

for \( k > 0 \), where \( U_k = \sum_{i=0}^{k-1} \mathcal{P}^i \varepsilon_{-i} \). Notice that \( U_k \) is symmetric and that \( U_k \) and \( W_{-k}^{-\infty} \) are independent: \( U_k \) is a function of \( (\varepsilon_i : -k + 1 \leq i \leq 0) \) while \( W_{-k}^{-\infty} \) is function of \( (\varepsilon_i : i \leq -k) \). Since \( W_{-k}^{-\infty} \overset{d}{=} W_0^{-\infty} \), for all \( k \geq 0 \)

\[
  W_0^{-\infty} \geq V_k + \mathcal{P}^k W_0^{-\infty}, \text{ stochastically}
\]

(2.9)

with \( V_k \overset{d}{=} U_k, V_k \) and \( W_0^{-\infty} \) independent.

A key observation is that \( W_0^{-\infty} \) is ergodic for spatial shifts. This follows from the fact that \( W_0^{-\infty} \) is a function of \( \varepsilon_n(i)'s \) for a cone of indices \( (n, i) \in -\mathbb{N} \times \mathbb{Z}^d \) with vertex in \( (0, x) \). Now, \( \mathbb{E}(W_0^{-\infty}) = \infty \), the Ergodic Theorem implies that
\[ P^k W_0^{-\infty} \to \infty \] almost surely as \( k \to \infty \). Indeed,
\[ P^k W_0^{-\infty}(0) = \sum_{i \in \mathbb{Z}^d} p_k(i) W_0^{-\infty}(i) \geq \frac{c}{k^{d/2}} \sum_{|i| \leq \sqrt{k}} W_0^{-\infty}(i) \to \infty, \] (2.10)
as \( k \to \infty \), by the Ergodic Theorem. We have used the positivity of \( W_0^{-\infty} \) and the well known Local Central Limit Theorem estimate to the effect that \( \inf_{|i| \leq \sqrt{k}} p_k(i) \geq c/k^{d/2} \) for some \( c > 0 \). For this estimate, aperiodicity is required; we leave the necessary and straightforward adaptations for the periodic case to the reader.

Now, (2.9), (2.10) and the symmetry of \( V_k \) imply that for arbitrary \( M > 0 \)
\[ \mathbb{P}(W_0^{-\infty} > M) \geq \liminf_{k \to \infty} \mathbb{P}(V_k + P^k W_0^{-\infty} > M) \] (2.11)
\[ \geq \liminf_{k \to \infty} \mathbb{P}(V_k \geq 0) \mathbb{P}(P^k W_0^{-\infty} > M) \] (2.12)
\[ \geq \frac{1}{2} \liminf_{k \to \infty} \mathbb{P}(P^k W_0^{-\infty} > M) = \frac{1}{2}. \] (2.13)
Thus \( \mathbb{P}(W_0^{-\infty} = \infty) \geq 1/2 \) and triviality implies \( \mathbb{P}(W_0^{-\infty} = \infty) = 1. \)

**Proof of Theorem 1.1.** (a) is immediate consequence of Lemmas 2.2 and 2.3: any invariant surface dominates stochastically \( W_0^{-\infty} \) and \( W_0^{-\infty} \) is almost surely identically infinity. (b) follows from Lemma 2.3 and (2.4). (c) follows from the identity \( \mu_n = \mathbb{E}W_n(0) = \mathbb{E}W_0^{-n} \) and the monotone convergence theorem. Finally, in (3.2) below it is shown that
\[ \mu_n - \mu_{n-1} = \mathbb{E} \int_{P^k W_{n-1}}^\infty \mathbb{P}(\varepsilon > x) \, dx \] (2.14)
Since \( \varepsilon \) is integrable and \( P^k W_{n-1} \) increases to infinity in probability, (2.14) converges to zero, and we get (d).

**3 A generic lower bound**

From (1.7),
\[ W_n(i) = (P W_{n-1}(i) + \varepsilon_n(i))^+ = P W_{n-1}(i) + \varepsilon_n(i) + (-P W_{n-1}(i) - \varepsilon_n(i))^+ \] (3.1)
Taking expectations, since \( \varepsilon \) is symmetric,
\( \mu_n = \mu_{n-1} + \mathbb{E} \int_{PW_{n-1}}^{\infty} P(\varepsilon > x) \, dx. \) \hfill (3.2)

As \( \int_y^{\infty} P(\varepsilon > x) \, dx \) is a convex function of \( y \),

\[ \mu_n \geq \mu_{n-1} + \int_{E(PW_{n-1})}^{\infty} P(\varepsilon > x) \, dx = \mu_{n-1} + \mathbb{E}(\varepsilon - \mu_{n-1})^+. \] \hfill (3.3)

For \( s \geq 0 \), let \( G(s) = \mathbb{E}(\varepsilon - s)^+ \), \( H(s) = s + G(s) \), and \( \nu(t) \) be such that \( \int_0^{\nu(t)} [G(s)]^{-1} \, ds = t \).

**Theorem 3.1** \( \mu_n \geq \nu(n) \) for all \( n \geq 0 \).

**Remark 3.2** This general lower bound does not depend on the dimension.

**Corollary 3.3** If the distribution of \( \varepsilon \) belongs to \( \mathcal{L}_+^\alpha \) for some \( \alpha > 0 \), then there exists \( c_2 = c_2(\alpha) > 0 \) such that

\[ \mu_n \geq c_2 (\log n)^{\frac{1}{\alpha}}. \] \hfill (3.4)

**Corollary 3.4** Suppose that the distribution of \( \varepsilon \) decays at most polynomially, i.e. \( P(\varepsilon > x) \geq c_0 x^{-\alpha} \) for all \( x > 1 \) and some positive constants \( c_0 \) and \( \alpha > 1 \). Then there exists \( c_1 = c_1(\alpha) > 0 \) such that

\[ \mu_n \geq c_1 n^{\frac{1}{\alpha}}. \] \hfill (3.5)

**Proof of Theorem 3.1.** Notice first that \( \nu(t) \) is a solution of

\[ \nu(t) = \int_0^t G(\nu(s)) \, ds \]

and thus satisfies

\[ \nu(n) = \nu(n-1) + \int_{n-1}^{n} G(\nu(s)) \, ds. \]

Notice also that \( G(x) \) is decreasing and \( H(x) \) is increasing. We prove the lemma by induction. First, \( \mu_0 = \nu(0) = 0 \). Suppose that \( \mu_{n-1} \geq \nu(n-1) \). Then,

\[ \nu(n) = \nu(n-1) + \int_{n-1}^{n} G(\nu(s)) \, ds \leq \nu(n-1) + G(\nu(n-1)) \]

\[ = H(\nu(n-1)) \leq H(\mu_{n-1}) \leq \mu_n, \]

where the last inequality is (3.3).
Proof of Corollary 3.4. Note that

\[ G(x) = \mathbb{E}(\varepsilon - x)^+ = - \int_{\mathbb{R}} (y - x) d\mathbb{P}[\varepsilon \geq y] = \int_{\mathbb{R}} \mathbb{P}[\varepsilon \geq y] dy \]

and thus

\[ g(t) := \int_0^t \frac{ds}{G(s)} = \int_0^t \frac{ds}{\int_0^{+\infty} \mathbb{P}[\varepsilon \geq y] dy} \tag{3.6} \]

Thus, from the assumption in the statement of Corollary 3.4,

\[ g(t) \leq \frac{1}{c_0} \int_0^t \frac{ds}{\frac{1}{\alpha - 1} s^{1-\alpha}} = \frac{\alpha - 1}{c_0 \alpha} t^\alpha \tag{3.7} \]

and

\[ \nu(t) \geq c_1 t^{\frac{1}{\alpha}} \]

follows immediately. \[ \blacksquare \]

Proof of Corollary 3.3. As above, we have

\[ g(t) \leq \frac{1}{c} \int_0^t \frac{ds}{\int_0^{+\infty} e^{-c'y^\alpha} dy} \leq c_1 \int_0^t e^{c_2 s^\alpha} ds \leq c_3 e^{c_4 t^\alpha} \tag{3.8} \]

and the result follows. \[ \blacksquare \]

4 Moderate deviations for the serial harness

The proofs of Lemmas 1.3 and 1.4 are based on the behavior of \( \mathbb{E}(e^{lY_{l}(0)}) \) for small and large \( l \), established in Lemmas 4.1 and 4.3 below.

Lemma 4.1 Let \( l_n \) be a sequence of positive numbers such that

\[ \bar{l}_n := l_n/\sqrt{s(n)} \leq 1. \tag{4.1} \]

Then there exists a constant \( c \) such that for all \( 0 \leq l \leq n \)

\[ \mathbb{E}[e^{l_n Y_l(0)}] \leq e^{cl^2}. \tag{4.2} \]

Proof. For all \( 0 \leq l \leq n \)
\begin{equation}
\mathbb{E}\left[e^{l_n Y_n(0)}\right] = \prod_{r=0}^{l-1} \prod_{j \in \mathbb{Z}^d} \mathbb{E}\left[e^{l_n p_r(j)\varepsilon}\right] \leq \prod_{r=0}^{l-1} \prod_{j \in \mathbb{Z}^d} e^{c l_n^2 p_r(j)^2}
= \exp\{c l_n^2 s(n)^{-1} s(l)\} \leq e^{c l_n^2},
\end{equation}

where \(c = \mathbb{E}(\varepsilon^2)\) and we have used that for a symmetric random variable \(W\), if \(|\lambda| \leq 1\), then

\begin{equation}
\mathbb{E}(e^{\lambda W}) \leq 1 + \mathbb{E}(e^W)\lambda^2 \leq e^{\mathbb{E}(e^W)\lambda^2}
\end{equation}

and the fact that \(s(\cdot)\) is nondecreasing.

**Proof of Lemma 1.3.**

\[
\mathbb{P}[Y_l(0) \geq K \sqrt{s(n)} \log n] = \mathbb{P}[l_n Y_n(0) \geq \log n c' K] \leq n^{-c' K} \mathbb{E}[e^{l_n Y_n(0)}],
\]

where \(l_n = c'' \sqrt{\log n}\), for an appropriate constant \(c''\), and Lemma 4.1 yields the result.

For the proof of Lemma 1.4, we will use that in \(d \geq 3\)

\[
s := \lim_{n \to \infty} s(n) < \infty.
\]

We will also need the following converse of (4.3).

**Lemma 4.2** If the distribution of \(W\) is in \(L^-_\alpha\) for some \(\alpha > 1\), then there exists a constant \(c\) such that

\begin{equation}
\mathbb{E}(e^{\lambda W}) \leq e^{c \lambda^\beta},
\end{equation}

for all \(l \geq 1\), where \(\beta = \alpha/(\alpha - 1)\).

**Proof.** We have that

\begin{equation}
\mathbb{E}e^{\lambda W} \leq 1 + c \int_0^\infty e^{tx} e^{-c x^\alpha} dx = 1 + c_1 \int_0^\infty e^{\tilde{\lambda} x} e^{-x^\alpha} dx,
\end{equation}

where \(\tilde{\lambda} = l/c^{1/\alpha}\). Now, we write the integral in (4.6) as

\begin{equation}
\int_0^{(2\tilde{\lambda})^{\beta-1}} e^{\tilde{\lambda} x} dx + \int_0^{\infty} e^{\tilde{\lambda} x-x^\alpha} dx.
\end{equation}

The former integral is bounded above by \(e^{c'n \lambda^\beta}\). The latter one is bounded above by a uniform constant.
Lemma 4.3 In $d \geq 3$, if the distribution of $\varepsilon$ is in $\mathcal{L}_\alpha$ for some $\alpha > 1$, then there exist a constant $c$ such that for all large $q$

$$\mathbb{E}(e^{q_0 Y_n(0)}) \leq \begin{cases} e^{c q \beta (1+2/d)}, & \text{if } \alpha \neq 1 + d/2; \\ e^{c q_0^2 / \log q}, & \text{if } \alpha = 1 + d/2, \end{cases} \quad (4.7)$$

where $\beta = \alpha / (\alpha - 1)$ as before.

Proof.

$$\mathbb{E}(e^{q_0 Y_n(0)}) \leq \prod_{k=0}^{\infty} \prod_{x \in \mathbb{Z}^d} \mathbb{E}(e^{q p_k(x) \varepsilon}) \leq \prod_{k, x : q p_k(x) > 1} e^{c(q p_k(x))^\beta} \prod_{k, x : q p_k(x) \leq 1} e^{c(q p_k(x))^2} = \exp \left\{ c \left[ \sum_{k, x : q p_k(x) > 1} (q p_k(x))^\beta + \sum_{k, x : q p_k(x) \leq 1} (q p_k(x))^2 \right] \right\}. \quad (4.8)$$

We now estimate the expression within square brackets in (4.8). If $\beta \geq 2$ or, equivalently, $1 < \alpha \leq 2$, then that expression is bounded above by

$$q^\beta \sum_{k, x} p_k^2(x) = q^\beta s. \quad (4.9)$$

For the case $1 < \beta < 2$ (equivalently, $\alpha > 2$), we use the well known estimate on $p_k := \sup_{x \in \mathbb{Z}^d} p_k(x)$: there exists a constant $C$ such that for all $k \geq 1$

$$p_k \leq C k^{-d/2} \quad (4.10)$$

(see e.g. Spitzer (1976)) to conclude that the expression within square brackets in (4.8) is bounded above by

$$q^\beta \sum_{k=0}^{(Cq)^{2/d}} p_k^{\beta-1} + q^2 \sum_{k=(Cq)^{2/d}} p_k \leq C' q^\beta \sum_{k=1}^{(Cq)^{2/d}} k^{-d(\beta-1)/2} + C'' q^{1+2/d} \quad (4.11)$$

for some constants $C', C''$. The result follows. 

Proof of Lemma 1.4. Let $Q_n$ be a sequence of positive numbers such that $Q_n = o(\log n)$ and $q_n = (\log n)/Q_n$. Then

$$\mathbb{P}[Y_l(0) \geq K Q_n] \leq \mathbb{P}[q_n Y_l(0) \geq K (\log n)] \leq n^{-K} \mathbb{E}(e^{q_n Y_l(0)}). \quad (4.12)$$

We can thus use Lemma 4.3 for $q_n$. Therefore, if $1 < \alpha \neq 1 + d/2$, making $Q_n = (\log n)^{1/\alpha \vee \frac{2}{d+2}}$, we have $q_n = (\log n)^{1 - (1/\alpha \vee \frac{2}{d+2})} = (\log n)^{1/\alpha \vee \frac{2}{d+2}}$ and thus, from (4.7)

$$\mathbb{P}[Y_l(0) \geq K (\log n)^{1/\alpha \vee \frac{2}{d+2}}] \leq n^{-c-K}. \quad (4.13)$$
If $\alpha = 1 + d/2$, we make $Q_n = L_n(1 + 2/d)$, and thus $q_n = (\log n)/L_n(1 + 2/d) = \ell_n(1 + 2/d)$. From (4.7) and the definition of $\ell_n(1 + 2/d)$ (above (6.1) below)

$$
\mathbb{P}[Y_t(0) \geq KL_n(1 + 2/d)] \leq n^{c-K}.
$$

(4.14)

For $\alpha = 1$, we have

$$
\mathbb{E} e^{Y_n(0)} = \prod_{k,x} \mathbb{E} e^{p_k(x)\varepsilon} \leq e^{c\sum_{k,x} p_k(x)} = e^{cs},
$$

(4.15)

where we have used (4.3). Thus, we obtain that

$$
\mathbb{P}[Y_n(0) > K \log n] \leq C n^{-K}.
$$

(5.1)

Then,

$$
\mathbb{P}[W_n(0) \geq a_n] \leq \mathbb{P}[W_n^{0,r}(0) \geq a_n]
$$

(5.2)

To get a bound for the probability in (5.2) of the form (1.16-1.18), we take $r = a_n/2$ and use (1.16-1.18).

5 Moderate deviations for the wall process

In this section we show Lemma 1.5. Introduce new processes $W_n^{0,r}$ and $Y_n^{0,r}$, which have the same evolution as $W_n$, respectively $Y_n$, but are started at time zero at height $r \in \mathbb{N}$. That is, $W_0^{0,r}(i) = Y_0^{0,r}(i) = r$, for all $i \in \mathbb{Z}^d$.

Let

$$
a_n = \begin{cases} 
2K(\log n)^{\frac{1}{\alpha} + \frac{2+d}{2+2d}}, & \text{for the extension of (1.17);} \\
2KL_n(1 + 2/d), & \text{for the extension of (1.18);} \\
2K\sqrt{s(n)\log n}, & \text{for the extension of (1.16).}
\end{cases}
$$

(5.1)

Then,

$$
\mathbb{P}[W_n(0) \geq a_n] \leq \mathbb{P}[W_n^{0,r}(0) \geq a_n]
$$

(5.2)

To get a bound for the probability in (5.2) of the form (1.16-1.18), we take $r = a_n/2$ and use (1.16-1.18).
The probability in (5.3) is treated as follows. Note that

\[ W_n^0, r(0) \] 

differ if a discrepancy occurs in the cone \( v \) is the maximal speed of a discrepancy.

\[ \{(l, j) \in \mathbb{N}_0 \times \mathbb{Z}^d : l \leq n, |j| \leq v(n - l)\}, \quad (5.4) \]

that is,

\[ \{Y_n^0, r(0) \neq W_n^0, r(0)\} = \{Y_l^0, r(j) < 0 \text{ for some } (l, j) \text{ with } l \leq n, |j| \leq v(n - l)\}. \]

Since \( Y_n^0, r(0) \) has the same law as \( Y_n(0) + r \) and by symmetry, we have

\[ \mathbb{P}[Y_l^0, r(j) < 0] = \mathbb{P}[Y_l(j) < -r] = \mathbb{P}[Y_l(j) > r]. \quad (5.5) \]

Hence,

\[ \mathbb{P}[Y_n^0, r(0) \leq W_n^0, r(0)] = \mathbb{P}[\exists (l, j) \text{ with } l \leq n, |j| \leq v(n - l) : Y_l(j) > r] \]

\[ \leq \sum_{l=0}^{n} \sum_{|j| \leq v(n-l)} \mathbb{P}[Y_l(j) > r]. \]

Taking \( r = a_n/2 \) as before and using (1.16-1.18), we obtain

\[ \mathbb{P}[Y_n^0, r(0) \neq W_n^0, r(0)] \leq kn^{c - c'K} \sum_{l=0}^{n} \sum_{|j| \leq v(n-l)} 1 \leq k'n^{c'' - c'K}, \quad (5.6) \]

for some \( k', c'' \).

\[ \square \]

6 Bounds for the wall process

For \( \gamma > 1 \), define \( \ell_n(\gamma) \) as the solution of \( x^\gamma \log x = \log n \), and let

\[ L_n(\gamma) = (\log n)/\ell_n(\gamma). \quad (6.1) \]

Note that

\[ (\log n)^{1 - \frac{1}{\gamma}} \leq L_n(\gamma) \leq (\log n)^{1 - \frac{1}{\gamma}}(\log \log n)^{\frac{1}{\gamma}} \text{ for all } n. \quad (6.2) \]

**Theorem 6.1** Suppose that the distribution of \( \varepsilon \) belongs to \( \mathcal{L}_{\alpha}^- \) for some \( \alpha \geq 1 \). If \( d \geq 3 \), then there exists \( c_3 = c_3(\alpha, d) > 0 \) such that

(i) if \( 1 \leq \alpha \neq 1 + \frac{d}{2} \), then

\[ \mu_n \leq c_3(\log n)^{\frac{1}{\gamma} + \frac{d}{2d + 2}}; \quad (6.3) \]

(ii) if \( \alpha = 1 + \frac{d}{2} \), then for all \( \delta > 0 \) we have

\[ \mu_n \leq c_3 L_n(1 + 2/d); \quad (6.4) \]
If \( d = 2 \), then there exists \( c_3 \) such that

\[
\mu_n \leq c_3 \log n. \tag{6.5}
\]

**Remark 6.2** From (6.4) and (6.2), a slightly weaker alternative to (6.4) is

\[
\mu_n \leq c_3 (\log n)^{\frac{d}{2+\alpha}} (\log \log n)^{\frac{d}{2}}. \tag{6.6}
\]

We now restrict attention to the class of exponentially decaying noise distributions. When the noise distribution is in \( \mathcal{L}_\alpha, \alpha \geq 1 \), the results in Corollary 3.3 and Theorem 6.1 are our best explicit bounds (to leading order) for \( d \geq 3 \) and \( d = 2, 1 \leq \alpha \leq 2 \). For \( d = 1, \alpha \geq 1 \) and \( d = 2, \alpha > 2 \), we have better bounds, which we discuss now.

**Theorem 6.3** If the distribution of \( \varepsilon \) is in \( \mathcal{L}_1^- \), then for \( d \leq 2 \), there exist constants \( c, C > 0 \) such that

\[
c\sqrt{s(n)} \leq \mu_n \leq C \sqrt{s(n) \log n}, \tag{6.7}
\]

where \( s(n) \) is defined in (1.5). In particular

(i) for \( d = 1 \),

\[
 cn^{1/4} \leq \mu_n \leq Cn^{1/4} \sqrt{\log n}; \tag{6.8}
\]

(ii) and for \( d = 2 \),

\[
c\sqrt{\log n} \leq \mu_n \leq C \log n \tag{6.9}
\]

**Remark 6.4** The lower bound in (6.7) actually holds under the weaker assumption that \( \mathbb{E}(\varepsilon^2) < \infty \). See Remark 6.7 below.

We prove first the lower bound (6.7). The first step is to calculate the variance of the serial harness, which will give us the proper scaling. From (1.3) we get (this is already contained in Hammersley (1965)) \( \mathbb{E} Y_n(0) = 0 \) and \( \mathbb{E} Y_n(0)^2 = \sigma^2 s(n) \).

The correct scaling for the serial harness is therefore \( s(n)^{1/2} \), and we define accordingly

\[
\tilde{Y}_n(0) \equiv s(n)^{-\frac{1}{2}} Y_n(0). \tag{6.10}
\]

Analogously we define \( \tilde{W}_n(0) \) for the wall process. We now show that \( \tilde{Y}_n(0) \) is uniformly integrable (with respect to \( n \)).

**Lemma 6.5** The process \( (\tilde{Y}_n(0))_n \) satisfies \( \sup_n \mathbb{E}(e^{\mid \tilde{Y}_n(0) \mid}) < \infty \).

**Proof.** By symmetry of the \( \varepsilon \), \( \mathbb{E}(e^{\mid \tilde{Y}_n(0) \mid}) \leq 2 \mathbb{E}(e^{\tilde{Y}_n(0)}) \leq 2e^c \), where the last inequality follows from Lemma 4.1 with \( l_n \equiv 1 \).
From Lemma (6.5) it follows immediately that $s(n)^{-1}Y_n(0)^2$ is uniformly integrable.

**Lemma 6.6** *There exists a constant $c > 0$ such that for all $n$*

$$\mathbb{E}[\tilde{Y}_n(0)] > c.$$  \hfill (6.11)

**Proof.** Clearly, for any positive $M$,

$$\mathbb{E}[\tilde{Y}_n(0)^2] = \mathbb{E}\left[\tilde{Y}_n(0)^2 \mathbf{1}\{|\tilde{Y}_n(0)| > M\}\right] + \mathbb{E}\left[\tilde{Y}_n(0)^2 \mathbf{1}\{|\tilde{Y}_n(0)| \leq M\}\right]$$

$$\leq \mathbb{E}\left[\tilde{Y}_n(0)^2 \mathbf{1}\{|\tilde{Y}_n(0)| > M\}\right] + M\mathbb{E}[|\tilde{Y}_n(0)|].$$  \hfill (6.12)

Since $\tilde{Y}_n(0)^2$ is uniformly integrable, for each $\delta > 0$ we can choose $M > 0$ such that

$$\mathbb{E}\left[\tilde{Y}_n(0)^2 \mathbf{1}\{|\tilde{Y}_n(0)| > M\}\right] < \delta,$$  \hfill (6.13)

uniformly in $n$. Thus

$$\mathbb{E}[|\tilde{Y}_n(0)|] \geq \frac{\mathbb{E}[\tilde{Y}_n(0)^2] - \delta}{M} = \frac{\sigma^2 - \delta}{M} > c > 0,$$  \hfill (6.14)

for some $\delta > 0$.

We finally prove the result about the wall process by coupling it with the serial harness using the same disorder variables $\mathcal{E}$. By symmetry,

$$\mathbb{E}[|\tilde{Y}_n(0)|] = \mathbb{E}[(\tilde{Y}_n(0))^+] + \mathbb{E}[(-\tilde{Y}_n(0))^+] = 2\mathbb{E}[(\tilde{Y}_n(0))^+].$$  \hfill (6.15)

On the other hand, by construction, $\tilde{W}_n(0) \geq (\tilde{Y}_n(0))^+$, and therefore,

$$\mathbb{E}[\tilde{W}_n(0)] \geq \mathbb{E}[(\tilde{Y}_n(0))^+] \geq \frac{1}{2}\mathbb{E}[|\tilde{Y}_n(0)|] \geq c' > 0.$$  \hfill (6.16)

This proves the lower bound (6.7).

The upper bounds (6.3-6.5) and (6.9) follow from Lemma 1.5 in the same, following way. Let $a_n$ be as in (5.1) and $b_n = a_n/(2K)$. Then

$$\mu_n/b_n = \mathbb{E}[W_n(0)/b_n] = \int_0^\infty \mathbb{P}(W_n(0) > K b_n) \, dK$$

$$\leq c/c' + k \int_{c/c'}^\infty n^{c-c'K} \, dK \leq C,$$

for some constant $C$. \hfill ■
Remark 6.7 The lower bound in (6.7) actually holds under the weaker assumption that $\mathbb{E}(\varepsilon^2) < \infty$, since this is enough to have $\tilde{Y}_n(0)^2$ uniformly integrable.

Acknowledgments.

We thank Servet Martínez for many discussions on harnesses, in particular for pointing out the minimality of $W_{-\infty}^0$. We thank François Dunlop for references to the physical literature.

This paper is supported by Fundação de Apoio à Pesquisa do Estado de São Paulo (FAPESP), Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Programa Núcleos de Excelência (PRONEX).

References

[1] K. Binder (1990) Growth kinetics of wetting layers at surfaces. pp 31–44 in Kinetics of Ordering and Growth at Surfaces. Edited by M.G. Lagally. Plenum Press, New-York.

[2] E. Bolthausen, J.-D. Deuschel, G. Giacomin (2001) Entropic repulsion and the maximum of the two-dimensional harmonic crystal. Ann. Probab. 29 (4) 1670–1692.

[3] E. Bolthausen, J.-D. Deuschel, O. Zeitouni (1995) Entropic repulsion of the lattice free field. Comm. Math. Phys. 170 (2), 417–443.

[4] J. Bricmont (1990) Random surfaces in statistical mechanics in Wetting phenomena. Proceedings of the Second Workshop held at the University of Mons, Mons, October 17–19, 1988. Edited by J. De Coninck and F. Dunlop. Lecture Notes in Physics 354. Springer-Verlag, Berlin.

[5] J. Bricmont, A. El Mellouki, J. Fröhlich (1986) Random surfaces in Statistical Mechanics - roughening, rounding, wetting J. Stat. Phys. 42 (5/6), 743–798

[6] F. Cesi, F. Martinelli (1996) On the layering transition of an SOS surface interacting with a wall. I. Equilibrium results. J. Stat. Phys. 82 (3/4), 823–913.

[7] J. De Coninck, F. Dunlop, F. Menu (1993) Spreading of a Solid-On-Solid drop. Phys. Rev. E 47(3), 1820–1823.

[8] J.-D. Deuschel (1996) Entropic repulsion of the lattice free field. II. The 0-boundary case. Comm. Math. Phys. 181 (3), 647–665.

[9] F.M. Dunlop, P.A. Ferrari, L.R.G. Fontes (2002) A dynamic one-dimensional interface interacting with a wall. J. Statist. Phys. 107 (3-4), 705–727.
[10] J.-D. Deuschel, G. Giacomin (1999) Entropic repulsion for the free field: pathwise characterization in $d \geq 3$. *Comm. Math. Phys.* **206** (2), 447–462.

[11] E. Dinaburg, A.E. Mazel (1994) Layering transition in SOS model with external magnetic-field. *J. Stat. Phys.* **74** (3/4), 533–563.

[12] P.A. Ferrari, S. Martínez (1998) Hamiltonians on random walk trajectories. *Stochastic Process. Appl.* **78** (1), 47–68.

[13] T. Funaki, S. Olla (2001) Fluctuations for $\nabla \phi$ interface model on a wall. *Stochastic Process. Appl.* **94**, 1–27.

[14] J.M. Hammersley (1965/66) Harnesses. *Proc. Fifth Berkeley Sympos. Mathematical Statistics and Probability*, Vol. III, 89–117.

[15] P. Holický, M. Zahradník (1993) On entropic repulsion in low temperature Ising models. *Cellular automata and cooperative systems* (Les Houches, 1992), 275–287, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 396, Kluwer Acad. Publ., Dordrecht.

[16] J.L. Lebowitz, C. Maes (1987) The effect of an external field on an interface, entropic repulsion. *J. Stat. Phys.* **46** (1/2), 39–49.

[17] R. Lipowsky (1985) Nonlinear growth of wetting layers. *J. Phys. A* **18**, L585–L590.

[18] K.K. Mon, K. Binder, and D.P. Landau (1987) Monte Carlo simulation of the growth of wetting layers *Phys. Rev. B* **35** (7), 3683–3685.

[19] F. Spitzer (1976) *Principles of random walk*. Springer-Verlag, New York.

[20] A. Toom (1997) Tails in harnesses. *J. Statist. Phys.* **88** (1-2) 347–364.