Oscillations of oblate drop between heterogeneous plates under uniform electric field

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Abstract. The forced oscillations of the incompressible fluid drop under the action of the uniform electric field are considered. In equilibrium, the drop has the form of a circular cylinder bounded axially by the parallel solid planes; the contact angle is right. An incompressible fluid of different density surrounds the drop. The external electric field acts as an external force that causes motion of the contact line. In order to describe this contact line motion, the modified Hocking boundary condition is applied: the velocity of the contact line is proportional to the deviation of the contact angle and the speed of the fast relaxation processes, whose frequency is proportional to twice the frequency of the electric field. The case of heterogeneous plates is investigated. We assume that the Hocking parameter depends on the polar angle in this case. The function describing the change in the coefficient of the interaction between the plate and the fluid (the contact line) is expanded in a series of the Laplace operator eigenfunctions.

1. Introduction

In most articles on electrowetting-on-dielectric (EWOD) (see fig. 1) [1-4], the fundamental equation in the EWOD study is the Young–Lippmann equation for the contact angle $\theta$ written as:

$$\cos \theta = \cos \theta_0 + E_w \cos \delta_0 = \frac{\sigma_{ip} - \sigma_{ip}}{\sigma_{ip}}, \quad E_w = 0.5CV^2\sigma_{ip}, \quad C = \varepsilon_0d^{-1},$$

where $\theta_0$ is the contact angle without applied voltage – the equilibrium contact angle, which is defined by the well-known Young’s equation, $\sigma$ is the interfacial tension between the drop of conducting fluid ($c$), the isolating (surrounding) fluid ($i$) and the dielectric plate ($p$), $E_w$ is the EW number (it represents the ratio of the electrostatic energy to the liquid – surrounding fluid interfacial energy), $C$ is capacitance per unit area, $V$ is the value of the applied DC voltage, $d$ is the thickness of the dielectric film, $\varepsilon_0$ and $\varepsilon$ are the vacuum and dielectric layer permittivity, respectively. For the alternating current (AC) instead of $V^2$ in equation (1) the square value of the voltage, or effective voltage, $U^2$ is used [1]. Although the Young–Lippmann equation (1) is now universally recognized as a basic equation of EWOD treatment, different approaches have been proposed to derive it [1,4]. Now EWOD has found wide application in various fields, such as electronic display technology [5,6], variable-focus liquid lenses [7,8], digital (droplet) microfluidic devices for bioanalysis (lab-on-a-chip) [9,10], etc.

However, the available experimental results proved to be very much different from the theoretical predictions of the Young–Lippmann equation (1). It is anticipated that a zero contact angle will be
obtained after some critical voltage angle (complete wetting and the contact angle remains unchanged), but in fact the value of the contact angle obtained in experiments is always finite [1,4]. The mechanism of the contact angle saturation is still not clearly understood and is the subject for discussion [1].

In [11], for a cylindrical drop we proposed another effective boundary condition on the basis of Hocking’s equation [12]:

\[
\frac{\partial \zeta^*}{\partial t'} = \pm \Lambda^* \left( \frac{\partial \zeta^*}{\partial z^*} + A^* \cos \left( 2\omega^* t' \right) \right),
\]

(2)

where \( \zeta^* \) is the deviation of the drop interface from the equilibrium position, \( z^* \) is the axial coordinate, \( \Lambda^* \) is a phenomenological constant (the so-called wetting parameter or Hocking parameter), having the dimension of velocity, \( A^* \) is the effective amplitude, \( \omega^* \) is the AC frequency. The second term in boundary condition (2) is the external action, which is written in the same fashion as in the Young–Lippmann equation (1): \( \cos \vartheta = E_{w} = \overline{E}_{w} V^2 \) as \( \cos \vartheta_0 = 0 \) in our problem, so that \( \zeta_k \sim \tan \vartheta = E_{w} \sqrt{1 - E_{w}^2} = \overline{E}_{w} V^2 \sqrt{1 - \overline{E}_{w}^2 V^2} \approx \overline{E}_{w} V^2 + O \left( V^4 \right) \approx \overline{E}_{w} V^2 \sim V^2 \). In article [11] we also investigated the effect of inhomogeneous electric field. Also a general discussion of the effective boundary conditions can be found in articles [13,14].

Another important problem is the non-uniform wetting of the surface, along which the contact line moves [15,16]. Heterogeneous substrates can cause different effects [17], for example, hysteresis of the contact angle [18,19]. However, most these articles consider the effect of mechanical (acoustical) vibrations [16].

This study is intended as an extension to work [11]. We consider the behavior of a droplet sandwiched between two heterogeneous flat plates under the applied uniform ac voltage. In order to describe the motion of the contact line the modified boundary condition (2) is used: the velocity of the contact line is proportional to the deviation of the contact angle and the speed of the fast relaxation processes, whose frequency is proportional to twice the electric field frequency. In the future, we plan to compare the obtained results with the experimental data [20].

2. Problem formulation

The problem formulation largely coincides with that presented in article [13]. We consider the dynamic behavior of the incompressible liquid drop of density \( \rho_i^* \) surrounded by another liquid of density \( \rho_e^* \) (here and in the following, the quantities with subscript \( i \) refer to the drop, and those with subscript \( e \) to the surrounding liquid). The system is bounded by two parallel solid surfaces (see figure 2) with the inter-plate spacing \( h^* \). The equilibrium contact angle \( \vartheta_0 \) between the lateral surface of the drop and the solid surface is equal to \( 0.5\pi \). The external uniform alternating electric field acts as an external force that causes the contact line motion [11].

Because of the problem symmetry it is convenient to introduce the cylindrical coordinates \( r^*, \alpha^*, z^* \). The azimuthal angle \( \alpha^* \) is reckoned from the \( x \)-axis. Let the surface of the droplet be described by
the equation \( r' = R_0' + \zeta t' (\alpha, \zeta, t') \). Assuming a potential liquid motion, we introduce the velocity potential \( \tilde{\nu} = \tilde{\nu} \phi \). Taking length \( R_0' \), height \( h' \), density \( \rho_0' + \rho_z' \), time \( \sigma^{-1/2} \sqrt{(\rho_0' + \rho_z')} R_0^{3/2} \), velocity potential \( A' \sqrt{\sigma} (\rho_0' + \rho_z')^{-1/2} \), pressure \( A' \sigma (\rho_0')^{-3/2} \), and deviation of the surface \( A' \) as characteristic quantities, we pass to dimensionless variables and obtain the following linear problem

\[
p_j = -\rho_j \phi_j, \quad \Delta \phi_j = 0, \quad j = i, e,
\]

\[
\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2} + b^2 \frac{\partial^2}{\partial z^2},
\]

\[
r = 1: \left[ \phi_j \right] = 0, \quad \zeta_i = \phi_i, \quad \left[ p \right] = \zeta + \zeta_{aa} + b^2 \zeta_{zz}, \quad z = \pm \frac{1}{2} \cdot \phi_z = 0,
\]

\[
r = 1, \quad z = \pm \frac{1}{2}: \quad \zeta_i = \mp \lambda (\alpha) (\zeta_z + a \cos(2\omega t)),
\]

where \( p \) is the fluid pressure, \( \lambda (\alpha) \) describes the heterogeneous condition of plates, \( a \) is forced oscillation amplitude, the square brackets denote a jump in the quantity at the interface between the external liquid and the drop. The boundary-value problem (3)–(6) involves six parameters: the aspect ratio, the dimensionless density, the wetting parameter, the AC frequency and amplitude

\[
b = R_0 h^{-1}, \quad \rho_i = \rho_i' (\rho_0' + \rho_z')^{-1}, \quad \rho_e = \rho_e' (\rho_0' + \rho_z')^{-1}, \quad \lambda = \Lambda \sigma^{-1/2} b \sqrt{(\rho_0' + \rho_z')} R_0',
\]

\[
\omega = \omega^* \sigma^{-1/2} (\rho_0' + \rho_z') R_0^{3/2}, \quad a = A' C \sigma^{-1/2} (\rho_0' + \rho_z') R_0^{3/2}/2.
\]

Figure 2. Problem geometry (1 – electrode, 2 – dielectric layer).

3. Forced oscillations

The function \( \lambda (\alpha) \) is expanded into the Fourier series in eigen functions of the Laplace operator. Let us consider a particular case of the heterogeneous plates: \( \lambda (\alpha) = \lambda \left[ \sin \left( k_i \cos (\alpha) \right) \right], \) where \( k_i \) is the wavenumbers. Note, that we used the above-mentioned function in [11] for non-uniform electric filed. The solutions for the velocity potential and the surface deviation are written as

\[
\phi_i (r, z, t) = \text{Re} \left( i 2 \omega \sum_{m=0} a_{mk} I_m \left( (2k + 1) \pi br \right) \sin \left( (2k + 1) \pi z \right) \cos (2m \alpha) e^{2i \omega t} \right),
\]

\[
\phi_e (r, z, t) = \text{Re} \left( i 2 \omega \sum_{m=0} b_{mk} K_m \left( (2k + 1) \pi br \right) \sin \left( (2k + 1) \pi z \right) \cos (2m \alpha) e^{2i \omega t} \right),
\]
$$\zeta(z,t) = \text{Re} \left\{ \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} c_{mk} \sin\left((2k+1)\pi z\right) \cos(2m\alpha) + d_0 \sin\left(\frac{z}{b}\right) + \sum_{m=1}^{\infty} d_m \sinh\left(\frac{\sqrt{4m^2 - 1}}{b} z\right) \cos(2m\alpha) e^{z^2\alpha} \right\},$$

where $I_m$ and $K_m$ are the $m$-th order modified Bessel functions. Substituting solutions (10)–(12) into (3)–(6), we obtain the linear system of non-homogeneous equations for the unknown amplitudes $d_0$ and $d_m$. Other coefficients $d_{mk}$, $b_{mk}$, $c_{mk}$ are calculated through these amplitudes.

**Figure 3.** Deviation of contact line $|\zeta_s|$ (a-c), drop surface $|\zeta_q|$ (d-f) and contact angle $\theta$ at upper plate (g-i) vs frequency $\omega$

(a) $\zeta_s$ (b) $|\zeta_s|$ (c) $|\zeta_q|$ (d) $|\zeta_q|$ (e) $|\zeta_q|$ (f) $|\zeta_q|$ (g) $\theta$ (h) $\theta$ (i) $\theta$

$(\alpha = 5, \ b = 1, \ \rho_i = 0.7), \ \lambda_0 = 0.1$ – solid line, $\lambda_0 = 1$ – dotted, $\lambda_0 = 10$ – dashed.
The dependence of the amplitude of the contact line and the surface oscillations at \( z = 0.25 \) and the contact angle at the upper plate on the frequency of the driving force is given in figure 3 for different values of the Hocking parameter and wavenumber. As it is seen from the plots in the fig.3a-c, there are no “antiresonances”, i.e. the amplitude of the contact line goes to zero. These resonances were observed for a homogeneous substrate both in the case of mechanical vibrations [21,22] and in the case of electric field [11]: at certain values of \( \omega \) the drop motion is independent of the wetting parameter and the contact line remains motionless at any \( \lambda \). Thus, the eigenfrequencies of different oscillation modes change because of the heterogeneous surface and this effect of “antiresonances” drops out.

The wavenumber \( k_i \) plays the role of the effective wetting coefficient \( \lambda_0 \). The smaller the Hocking parameter \( \lambda \), the greater the interaction between the contact line and the substrate and the smaller the amplitude of oscillations (see fig. 3a,c). Consequently, the long-wave heterogeneity of the substrate leads to a decrease in the oscillation amplitude. The deviation of the contact angle is less than 0.5\( \pi \) (see fig. 3g,i), i.e. complete wetting does not occur.

In figure 4, the deviation of the contact angle is plotted against the square root of the amplitude \( \alpha \) (i.e. proportional to AC potential \( V \)) for different values of the Hocking parameter \( \lambda \) and AC frequency \( \omega \). The responses obtained qualitatively agree with the experimental data. However, the
maximum deviation of the contact angle tends to $\pi/2$, i.e. $\theta \to 0$ or $\theta \to \pi$, whereas the contact angle found in the experiments is finite.

4. Conclusions
The behavior of the cylindrical drop between two solid plates has been considered taking into account the dynamics of the contact angle under the action of uniform electric field. The solid plates have heterogeneous surfaces described by the function $\lambda(\alpha) = \lambda_0 \sin(k \cos(\alpha))$. The main purpose of this paper is to develop a method for studying the forced oscillations of the drop on heterogeneous substrates and determining the contact angle. The investigation of forced oscillations has shown that the wavenumber $k_\lambda$ plays the role of the effective wetting parameter.

For small values of the parameter $\lambda_0$, at which the energy of interaction between the contact line and plate is strong, the oscillations amplitude is small. In the opposite case, the amplitude of the surface forced oscillations is large and tends to infinity in the limit $\lambda_0 \to \infty$. There are no "antiresonant" frequencies, i.e. such external frequencies, at which the contact line does not move and the contact angle does not change.

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5. References
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