Pathwise unique solutions and stochastic averaging for mixed stochastic partial differential equations driven by fractional Brownian motion and Brownian motion

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Abstract

This paper is devoted to a system of stochastic partial differential equations (SPDEs) that have a slow component driven by fractional Brownian motion (fBm) with the Hurst parameter $H > 1/2$ and a fast component driven by fast-varying diffusion. It improves previous work in two aspects: Firstly, using a stopping time technique and an approximation of the fBm, we prove an existence and uniqueness theorem for a class of mixed SPDEs driven by both fBm and Brownian motion; Secondly, an averaging principle in the mean square sense for SPDEs driven by fBm subject to an additional fast-varying diffusion process is established. To carry out these improvements, we combine the pathwise approach based on the generalized Stieltjes integration theory with the Itô stochastic calculus. Then, we obtain a desired limit process of the slow component which strongly relies on an invariant measure of the fast-varying diffusion process.

Keywords. Pathwise unique solutions, stochastic averaging, fast-slow, fractional Brownian motion, mixed stochastic partial differential equations

Mathematics subject classification. 60G22, 60H10, 60H05, 34C29

1. Introduction

It is widely known that there are many phenomena the well-studied theory of semimartingales cannot describe. For example, telecommunication connections, climate, weather derivatives and other objects have long memory \cite{8,21,35}. Brownian motion (Bm) with independent increments which has no memory turns out to be insufficient to describe this effect. Another example could be that the concept of turbulence in hydrodynamics can be described with the help of stationary (dependent) increments \cite{25}. Thus, the long-range dependence properties of fractional Brownian motion (fBm) make this process a suitable candidate to describe this kind of phenomena.

For $H \in (0,1)$, a continuous centered Gaussian process $\beta^H = (\beta^H(t))_{t \geq 0}$ with the covariance function $\mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2} \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$, $t, s \geq 0$, is called one-dimensional fBm \cite{21} with the Hurst parameter $H$. An fBm differs significantly from an Bm and semimartingales. It is characterized by the stationarity of its (dependent) increments and long-memory property only for $H \in (\frac{1}{2},1)$. In the case $H \in (0,\frac{1}{2})$ it is a process with short memory. Note that if $H \neq \frac{1}{2}$, an fBm is not a semimartingale nor a Markov process.

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Now, we recall the definition of an infinite-dimensional fBm following [12, 22]. Let \((V, \langle \cdot, \cdot \rangle)\) be a separable Hilbert space. Its norm is denoted by \(| \cdot |\). For a sequence \(\{\lambda_i\}_{i \in \mathbb{N}}\) of positive real numbers with \(\sum_{i=1}^{\infty} \lambda_i < \infty\) and an orthonormal basis \(\{e_i\}_{i \in \mathbb{N}}\) of \(V\), a \(V\)-valued fBm \(B^H\) is defined by
\[
B^H_t = \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i \beta^H_i(t), \quad t \geq 0,
\]
where \(\{\beta^H_i\}_{i \in \mathbb{N}}\) is a sequence of independent one-dimensional fBm’s. It is known that the right hand side is convergent in \(L^2\) for every \(t\) and has a continuous modification in \(t\).

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a suitable probability space with a filtration satisfying the usual condition. We assume that a \(V\)-valued \(\{\mathcal{F}_t\}\)-Brownian motion \(W\) and \(\{\mathcal{F}_t\}\)-adapted one-dimensional fBm’s \(\beta^H_i, i \in \mathbb{N}\), are defined on this probability space. We further assume that \(W\) and \(\beta^H_i, i \in \mathbb{N}\), are all independent. (For the definition of a \(V\)-valued Brownian motion, see Proposition 4.3 where is called \(Q\)-Wiener process.)

From now on we assume \(H \in \left(\frac{1}{4}, 1\right)\) and work on the time interval \([0, T]\), where \(T > 0\) is arbitrary but fixed. Let \(A\) be the infinitesimal generator of an analytic semigroup \(S\) on \(V\) and assume that \(-A\) has discrete spectra \(0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots\) and \(\lim_{k \to \infty} \lambda_k = \infty\).

This paper firstly will prove an existence and uniqueness theorem for a class of mixed stochastic partial differential equations (SPDEs) driven by both fBm and Bm with a given initial value \(u_0\), which is given by
\[
\frac{du_t}{dt} = (Au_t + f(u_t))dt + \sigma(u_t)dW_t + g(u_t)dB^H_t, \quad 0 \leq t \leq T. \tag{1.1}
\]
Precise conditions on the nonlinear coefficients \(f, \sigma, g\) will be given in Section 3.

The idea of this part is based on a pathwise approach developed by Zähle [44], who defined the stochastic integral with respect to fBm based on a sort of generalization of integration by parts formula with respect to fractional derivatives. Nualart and Răşcanu [32] and Garrido-Atienza et al. [14] investigated stochastic differential equations (SDEs) in finite dimension. Infinite-dimensional equations were treated with the same success as finite-dimensional ones, e.g. Tindel et al. [36] and Garrido-Atienza et al. [12, 13]. Pathwise solutions of this kind of equation without Bm term (\(\sigma(\cdot) = 0\) in Eq. (1.1)) were studied in Maslowski and Nualart [22] and Garrido-Atienza, Lu, and Schmalfuss [13], and recently by Chen, Gao, Garrido-Atienza, and Schmalfuss [6] when the stochastic evolution equations are driven by a Hölder continuous function with Hölder exponent in \(\left(\frac{1}{4}, 1\right)\) and with nontrivial multiplicative noise. Guerra and Nualart [16] proved an existence and uniqueness theorem of solutions to multidimensional SDEs driven by fBm with Hurst parameter \(H > \frac{1}{4}\) and Bm. Using the theory of Wiener integral, Caraballo, Garrido-Atienza and Taniguchi [4] investigated the existence and exponential behavior of solutions to stochastic delay evolution equations with an additive fractional noise.

However, the method proposed in [16] fails for the infinite-dimensional case and the method in [4] fails for the multiplicative fractional noise case. The main difference (and, of course, difficulty) is that we cannot apply directly the existence and uniqueness results in [13, 16] and [4, 6]. Thus, to close this gap, as one of two main results of our paper, we obtain pathwise unique solutions to Eq. (1.1) relying on a pathwise approach, a stopping time technique and an approximation for the fractional noise (See Theorem 5.5).

Then, as the second main result, this paper will establish an averaging principle in the mean square sense for a class of SPDEs driven by fBm subject to an additional fast-varying diffusion process, which is given by
\[
\begin{aligned}
\frac{dX^\varepsilon_t}{dt} &= (AX^\varepsilon_t + b(X^\varepsilon_t, Y^\varepsilon_t))dt + g(X^\varepsilon_t)dB^H_t, \\
\frac{dY^\varepsilon_t}{dt} &= \frac{1}{\varepsilon}(AY^\varepsilon_t + F(X^\varepsilon_t, Y^\varepsilon_t))dt + \frac{1}{\sqrt{\varepsilon}}G(X^\varepsilon_t, Y^\varepsilon_t)dW_t,
\end{aligned} \tag{1.2}
\]
where \(b, g, F, G\) are nonlinear coefficients and \(X^\varepsilon_0 = X_0, Y^\varepsilon_0 = Y_0\) are initial values. The parameter \(0 < \varepsilon \ll 1\) represents the ratio between the natural time scale of the \(X^\varepsilon_t\) and \(Y^\varepsilon_t\) variables. For more precise setting and assumptions, see Section 4.
The theory of stochastic averaging principles has been studied extensively (see for instance the paper by Khasminskii \cite{18} and Freidlin and Wentzell \cite{6}, or the recent paper by Xu et al. \cite{38,39,41,42,43}, Liu \cite{20}, Liu, Röckner, Sun and Xie \cite{19} and Thompson, Kuske, and Monahan \cite{37}) and is used in many applications. Cerrai and Freidlin \cite{3} developed stochastic averaging principles for two-time-scale stochastic reaction diffusion equations whose additive noise is included in the fast motion. In this infinite-dimensional setting, there are also interesting papers such as Bréhier \cite{3}, Xu and Miao \cite{40}, Fu et al. \cite{10,11}, Bao, Yin, and Yuan \cite{1} and Sun and Zhai \cite{34}. However, the literature concerning fast-slow mixed SPDEs driven by multiplicative fractional noise is still, to some extent, in its infancy. Pei, Xu and Yin \cite{29} established an averaging principle for a system of SPDEs that have a slow component driven by an additive fractional noise and a fast component driven by fast-varying diffusion. Pei et al. \cite{30,31} examined averaging principles for SPDEs driven by an additive fractional noise with two-time-scale Markovian switching processes. But, till now, in multiplicative fractional noise case, only the averaging results for SDE cases were obtained. Hairer and Li \cite{17} considered slow-fast systems where the slow system is driven by fBm and proved the convergence to the averaged solution took place in probability. Very recently, Pei, Inahama and Xu \cite{27} answered affirmatively that an averaging principle still holds for fast-slow mixed SDEs if disturbances involve both fBm and long-range dependence modeled by fBm and \( \beta \)-dimensional fBm \( H \) for fast-slow mixed SDEs if disturbances involve both Bm and long-range dependence modeled recently, Pei, Inahama and Xu \cite{27} answered affirmatively that an averaging principle still holds for fast-slow mixed SDEs if disturbances involve both Bm and long-range dependence modeled by fBm and \( \beta \)-dimensional fBm \( H \).

To the best of our knowledge, the second part of our paper is the first attempt to study stochastic averaging for fast-slow mixed SPDEs driven by multiplicative fractional noise. The main goal of this part is to generalize the results in \cite{27,28,29} by using directly a pathwise approach to deal with multiplicative fractional noise term. In order to reach this objective, we shall borrow the construction of stochastic integral with respect to infinite-dimensional fBm given in \cite{22,23} and the stopping time technique to control the fBm term given in \cite{24} which will be recalled in Section 2 and Section 3, respectively (See Theorem 4.4). Although the results on fractional calculus and stochastic integrals with respect to the one-dimensional fBm \( \beta \)- and \( V \)-valued infinite-dimensional fBm \( B^H \) have already been done in the recent paper \cite{2,12,13,22,23,24}, we present them here for the sake of completeness.

For \( T > 0 \) and \( 0 < \alpha < \frac{1}{2} \), we denote \( W^{\alpha,1}(0,T;V) \), the space of measurable functions \( h : [0,T] \to V \) such that

\[
|h|_{\alpha,1} := \int_0^T \left( \frac{|h(s)|}{s^\alpha} + \int_0^s \frac{|h(s) - h(r)|}{(s-r)^{\alpha+1}} dr \right) ds < \infty.
\]

For \( a < t < c \) the Weyl derivatives are given by

\[
D^\alpha_{a+} h(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{h(t)}{(t-a)^\alpha} + \alpha \int_a^t (t-\zeta)^{\alpha-1} h(\zeta) d\zeta \right),
\]

\[
D^{-\alpha}_{c-} l_{c-}(t) = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left( \frac{l(t) - l(c)}{(c-t)^{1-\alpha}} + (1-\alpha) \int_t^c \frac{l(\zeta) - l(c)}{(\zeta-t)^{2-\alpha}} d\zeta \right),
\]

where, for \( 0 \leq a < c \leq T, l_{c-}(r) := l(r) - l(c) \), and \( \Gamma \) denotes the Gamma function. Then,
the pathwise integral Eq. (2.3) is well-defined on Ω where the convergence of the sums in (2.3) is understood as an operator valued map such that

\[ G = \sum_{n \in \mathbb{N}} G_n, \]

Remark 2.1. The following result can be found in [22, Proposition 2.1].

Let \( H \in (\frac{1}{2}, 1) \), take a parameter \( \alpha \in (1 - H, \frac{1}{2}) \) which will be fixed throughout this paper. For \( h \in W^{\alpha,1}(0, T; V) \) the integral

\[ \int_0^T h(s) dt \]

will be understood in the sense of definition (2.1) pathwise, which makes sense because \( \Lambda \) is finite a.s. (cf. [22]).

Let \( L(V) \) denote the space of linear bounded operators on \( V \) and let \( G : \Omega \times [0, T] \to L(V) \) be an operator valued map such that \( G(\omega, \cdot) e_i \in W^{\alpha,1}(0, T; V) \) for each \( i \in \mathbb{N} \) and almost \( \omega \in \Omega \). We define

\[ \int_0^T G(s) dB^H_s := \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^T G(s) e_i dB^H_i(s), \tag{2.3} \]

where the convergence of the sums in (2.3) is understood as \( \mathbb{P} \)-a.s. convergence in \( V \).

From now on, to make the pathwise integral (2.3) well-defined, we assume \( \sum_{i=1}^{\infty} \sqrt{\lambda_i} < \infty \). The following result can be found in [22, Proposition 2.1].

Remark 2.1. Assume that \( \sum_{i=1}^{\infty} \sqrt{\lambda_i} < \infty \). Then there exists \( \Omega_1 \subset \Omega \), \( \mathbb{P}(\Omega_1) = 1 \), such that the pathwise integral Eq. (2.3) is well-defined on \( \Omega_1 \) for each \( G : \Omega_1 \times [0, T] \to L(V) \) satisfying \( G(\omega, \cdot) e_i \in W^{\alpha,1}(0, T; V) \), for \( \omega \in \Omega_1 \), such that \( \sup_{i \in \mathbb{N}} |G(\omega, \cdot) e_i|_{\alpha,1} < \infty \). In addition

\[ \left| \int_0^T G(s) dB^H_s \right| \leq \Lambda_{0,T}^{\alpha,B_H} \sup_{i \in \mathbb{N}} |G(\cdot) e_i|_{\alpha,1}, \quad \omega \in \Omega_1, \]

where \( \Lambda_{0,T}^{\alpha,B_H} := \sum_{i=1}^{\infty} \sqrt{\lambda_i} \Lambda_i^{\alpha,T}(\beta_H) \). Note that \( \Lambda_{0,T}^{\alpha,B_H} \) is finite a.s.

We recall the following two auxiliary technical lemmas from [12].

Lemma 2.2. For any positive constants \( a, d \), if \( a + d - 1 > 0 \) and \( a < 1 \), one has

\[ \int_0^T (r-s)\alpha(t-s)^{-d} ds \leq (t-r)^{1-a-d} B(1-a,d+a-1), \]

\[ \int_r^T (s-r)\alpha(t-s)^{-d} ds \leq (t-r)^{1-a-d} B(1-a,d+a-1), \]

where \( r \in (0, t) \) and \( B \) is the Beta Function.
Lemma 2.3. For any non-negative $a$ and $d$ such that $a + d < 1$, and for any $\rho \geq 1$, there exists a positive constant $C$ such that
\[
\int_0^t e^{-\rho(t-r)}(t-r)^{-a}r^{-d} dr \leq C\rho^{a+d-1}.
\]
In addition, for $d \leq 0$ and $0 < a < 1$, and for any $\rho \geq 1$, we have
\[
\int_0^t e^{-\rho(t-r)}(t-r)^{-a}r^{-d} dr \leq \Gamma(1-a)t^{-d}\rho^{a-1}.
\]

Please note that $C$ and $C_*$ denote certain positive constants that may depend on the parameters $\alpha, \beta, T$ and the initial values and vary from line to line. $C_*$ is used to emphasize that the constant depends on the corresponding parameter $*$ which is one or more than one parameter.

3. Mixed SPDEs driven by fBm and Bm

This section will prove an existence and uniqueness theorem for the mixed SPDEs driven by both fBm and Bm. Let $V_\beta, \beta \geq 0$, denote the domain of the fractional power $(-A)^\beta$ equipped with the graph norm $|x|_\beta := |(-A)^\beta x|, x \in V_\beta$. For shortness, denote, $|x|_\beta := |x|_\beta$. We recall here some properties of the analytic semigroup, which will be used later in our analysis.

For $0 \leq \gamma \leq \varsigma \leq 1$ and $v \in [0, 1), \mu \in (0, 1 - v)$, there exists a constant $C > 0$, such that for $0 \leq s < t \leq T$, we have

\[
|S_t|_{L(V_\gamma, V_\mu)} \leq Ct^{-\varsigma}e^{-\lambda_1 t},
\]
\[
|S_{t-s} - \text{id}|_{L(V_\nu, V_\mu)} \leq C(t-s)^\nu.
\]

We also note that, for $q, \nu \in [0, 1]$ and $0 \leq \nu < \gamma + \varsigma$, there exists $C > 0$ such that for $0 \leq q \leq r \leq s \leq t$, we derive

\[
|S_{t-r} - S_{t-q}|_{L(V_\gamma, V_\mu)} \leq C(r - q)^\varsigma(t-r)^{-\gamma + \nu},
\]

and

\[
|S_{t-r} - S_{t-q} + S_{s-q}|_{L(V, V)} \leq C(t-s)^\varsigma(r-q)^\varsigma(s-r)^{-\gamma + \nu}.
\]

From now on, we use the symbol $\| \cdot \|$ to denote $| \cdot |_{L(V, V)}$ for shortness.

Let $f : V \to V$ and $\sigma : V \to L_2(V)$ be measurable and satisfy Lipschitz and linear growth conditions with constants $L_f$ and $L_\sigma$ respectively, where $L_2(V)$ is the family of Hilbert-Schmidt operators from $V$ to itself. We assume that $g : V \to L(V)$ is Fréchet $C^1$ and that $g$ and $g' : V \to L(V, L(V))$ are Lipschitz continuous with constants $L_g, M_g$ in the following senses:

\[
\sup_{i \in \mathbb{N}} |g(v_1) e_i - g(v_2) e_i| \leq L_g |v_1 - v_2|,
\]
\[
\sup_{i \in \mathbb{N}} |g'(v_1) e_i - g'(v_2) e_i|_{L(V)} \leq M_g |v_1 - v_2|,
\]

where $\{e_i\}_{i \in \mathbb{N}}$ is the complete orthonormal basis in $V$.

Remark 3.1. For $v_1, v_2, u_1, u_2 \in V$, there exist $c_1, c_2 > 0$, such that

\[
\sup_{i \in \mathbb{N}} |g(v_1) e_i| \leq \sup_{i \in \mathbb{N}} |g(v_1) e_i - g(0) e_i| + \sup_{i \in \mathbb{N}} |g(0) e_i| \leq c_1(1 + |v_1|),
\]

here $c_1 := \max\{L_g, \sup_{i \in \mathbb{N}} |g(0) e_i|\}$ and by [32, Lemma 7.1],

\[
\sup_{i \in \mathbb{N}} |g(v_1) e_i - g(v_2) e_i - g(u_1) e_i + g(u_2) e_i| \leq c_2 |v_1 - v_2 - u_1 + u_2| + c_2 |v_1 - v_2|||v_1 - u_1| + |v_2 - u_2|),
\]

holds.
Taking a parameter $1 - H < \alpha < \frac{1}{2}$, for the measurable functions $h : [0, T] \to V$, let
\[ \|h(t)\|_\alpha = |h(t)| + \int_0^t \frac{|h(t) - h(s)|}{(t - s)^{\alpha + 1}} \, ds. \] (3.9)

Denote by $\mathcal{B}^{\alpha,2}(0, T; V)$ the space of measurable functions $h : [0, T] \to V$ endowed with the norm $\| \cdot \|_{\alpha, T}$ defined by
\[ \|h\|_{\alpha, T}^2 := \sup_{t \in [0, T]} |h(t)|^2 + \int_0^T \left( \int_0^t \frac{|h(t) - h(s)|}{(t - s)^{\alpha + 1}} \, ds \right)^2 \, dt < \infty. \]

**Remark 3.2.** Note that $\mathcal{B}^{\alpha,2}(0, T; V)$ is continuously embedded in $W^{\alpha,1}(0, T : V)$.

**Definition 3.3.** For $\alpha \in (1 - H, \frac{1}{2})$, $V$-valued process $(u_t)_{t \in [0, T]}$ is a solution of Eq. (1.1) in the mild sense if the following two conditions are satisfied:
1. $u \in \mathcal{B}^{\alpha,2}(0, T; V)$ a.s.;
2. $(u_t)$ is adapted to $(\mathcal{F}_t)$ and satisfies the following integral equation:
\[ u_t = S_t u_0 + \int_0^t S_{t-s} f(u_s) \, ds + \int_0^t S_{t-s} \sigma(u_s) \, dW_s + \int_0^t S_{t-s} g(u_s) \, dB_s^H. \] (3.10)

The following Lemma 3.4, which will be proved in Appendix B provides the basic estimates needed to prove the pathwise unique solutions of Eq. (1.1).

**Lemma 3.4.** Taking $\alpha < \alpha' < 1 - \beta$ and $\alpha < \beta$, there exists a constant $C > 0$, such that for any $0 \leq s < t \leq T$, $u, v \in V$,
\[
K_1(s, t) := \left| \int_s^t S_{t-r} g(u_r) \, dB_r^H \right| \\
\leq CA_{\alpha, B}^{0,t} \int_s^t \left[ (r - s)^{-\alpha} + (t - r)^{-\alpha} \right] |u_r - u_0| (1 + |u_r|) + \int_s^t \frac{|u_r - u_q|}{(r - q)^{1+\alpha}} \, dq, \\
K_2(0, s) := \left| \int_0^s (S_{t-r} - S_{s-r}) g(u_r) \, dB_r^H \right| \\
\leq CA_{\alpha, B}^{0,s} \int_0^s \left[ (s - r)^{-\beta} + (s - r)^{-\beta - \beta} \right] (1 + |u_r|) \, dr + CA_{\alpha, B}^{0,s} \int_0^s (s - r)^{-\beta} \left( \int_0^r \frac{|u_r - u_q|}{(r - q)^{1+\alpha}} \, dq \right) \, dr, \\
K_3(s, t) := \left| \int_s^t S_{t-r} (g(u_r) - g(v_r)) \, dB_r^H \right| \\
\leq CA_{\alpha, B}^{0,t} \int_s^t \left[ (r - s)^{-\alpha} + (t - r)^{-\alpha} \right] |u_r - v_r| \, dr \\
+ CA_{\alpha, B}^{0,s} \int_0^s |u_r - v_r| \left( \int_0^r \frac{|u_r - u_q| + |v_r - v_q|}{(r - q)^{1+\alpha}} \, dq \right) \, dr \\
+ CA_{\alpha, B}^{0,s} \int_0^s \left( \int_0^r \frac{|u_r - v_r - u_q + v_q|}{(r - q)^{1+\alpha}} \, dq \right) \, dr, \\
K_4(0, s) := \left| \int_0^s (S_{t-r} - S_{s-r}) (g(u_r) - g(v_r)) \, dB_r^H \right| \\
\leq CA_{\alpha, B}^{0,s} \int_0^s \left[ (s - r)^{-\beta} + (s - r)^{-\beta - \beta} \right] |u_r - v_r| \, dr \\
+ CA_{\alpha, B}^{0,s} \int_0^s (s - r)^{-\beta} |u_r - v_r| \left( \int_0^r \frac{|u_r - u_q| + |v_r - v_q|}{(r - q)^{1+\alpha}} \, dq \right) \, dr \\
+ CA_{\alpha, B}^{0,s} \int_0^s (s - r)^{-\beta} \left( \int_0^r \frac{|u_r - v_r - u_q + v_q|}{(r - q)^{1+\alpha}} \, dq \right) \, dr.
\]
Theorem 3.5. Assume that \( f, \sigma \) satisfy Lipschitz and linear growth conditions, and \( g \) and \( g' \) satisfy (3.3) and (3.6), respectively. Then, for any initial value \( u_0 \in V_\beta \), \( \beta > \alpha \), there exists a unique mild pathwise solution to Eq. (1.1).

Note that the unique solution for given \( u_0 \) is independent of \( \alpha \). The proof of Theorem 3.5 will be divided into several logical steps.

Step1: Construction of approximations. We recall the following auxiliary technical lemma from [24]. The proof can be obtained by [24, Lemma 2.1, Proposition 2.1], thus, we omit it.

Lemma 3.6. Let \( \varpi \in (0, 1] \) and \( h : [0, T] \to \mathbb{R} \) be a \( \varpi \)-Hölder continuous function. Define for \( \epsilon > 0 \),

\[
h'(t) = \epsilon^{-1} \int_{0 \vee (t - \epsilon)}^{t} h(s)ds.
\]

Then, for \( \alpha \in (1 - \varpi, 1) \), there exists a constant \( C > 0 \) such that

\[
\|h - H\|_{\alpha, 0, T} \leq CK_{\varpi}(h)e^{\alpha + \alpha - 1},
\]

where \( K_{\varpi}(h) = \sup_{0 \leq s \leq t \leq T} |h(t) - h(s)|/(t - s)^{\varpi} \) is the \( \varpi \)-Hölder seminorm of \( h \).

Fix \( N \geq 1 \), we define the following stopping time \( \tau_N \),

\[
\tau_N := \inf\{t \geq 0 : \Lambda_{\alpha, B}^{0, t} \geq N \} \land T,
\]

(3.11)

where \( \Lambda_{\alpha, B}^{0, t} := \sum_{i=1}^{\infty} \sqrt{\lambda_i} \Lambda_{\alpha}^{0, t}(\beta_{i}^{H}) \).

Put \( \beta_{i}^{H,N}(t) = \beta_{i}^{H}(t \land \tau_N), t \in [0, T], i \in \mathbb{N} \) and taking \( \epsilon = \frac{1}{n} \) in Lemma 3.6, then, for each \( n, i \in \mathbb{N} \), define an approximation of \( \beta_{i}^{H,N,i} \) by

\[
\beta_{i}^{H,N,i}(t) = n \int_{(t - \epsilon)}^{t} \beta_{i}^{H,N}(s)ds.
\]

(3.12)

Similarly, denote \( \Lambda_{\alpha, B}^{0, t} := \sum_{i=1}^{\infty} \sqrt{\lambda_i} \Lambda_{\alpha}^{0, t}(\beta_{i}^{H,N,i}) \) which will be used in next step.

Lemma 3.7. For any \( i \in \mathbb{N} \) and \( N \), we have

\[
\|\beta_{i}^{H,N,i}\|_{\alpha, 0, T} \leq C\|\beta_{i}^{H}\|_{\alpha, 0, T},
\]

almost surely, where \( C \) is a constant which is independent of \( N \) and \( i \).

Note that the above lemma will be proven in Appendix B.

To proceed, by Lemma 3.6 and Lemma 3.7, for any \( i \in \mathbb{N} \) and \( \varpi \in (1 - \alpha, H) \), we have

\[
\|\beta_{i}^{H,N,i} - \beta_{i}^{H,N}\|_{\alpha, 0, T} \leq CK_{\varpi}(\beta_{i}^{H,N})(\frac{1}{n})^{\alpha + \alpha - 1} \leq C\|\beta_{i}^{H,N}\|_{1 - \varpi, 0, T}(\frac{1}{n})^{\alpha + \alpha - 1} \leq C\|\beta_{i}^{H}\|_{1 - \varpi, 0, T}(\frac{1}{n})^{\alpha + \alpha - 1},
\]

(3.13)

almost surely. Moreover, since \( \varpi \in (1 - \alpha, H) \), by Lemma 3.7 and (3.13), we have

\[
\Lambda_{\alpha, B}^{0, t} \leq \sum_{i=1}^{\infty} \sqrt{\lambda_i}(\|\beta_{i}^{H,N,i} - \beta_{i}^{H,N}\|_{\alpha, 0, T} + \|\beta_{i}^{H,N}\|_{\alpha, 0, T}) \leq \frac{C}{1 - \alpha} \sum_{i=1}^{\infty} \sqrt{\lambda_i}(\|\beta_{i}^{H}\|_{1 - \varpi, 0, T} + \|\beta_{i}^{H}\|_{1 - \varpi, 0, T})
\]

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\begin{align}
\leq CN. \quad (3.14)
\end{align}

Now, let us consider
\begin{align}
u_t^{N,n} &= S_t u_0 + \int_0^t S_{t-s} f(u_s^{N,n}) ds + \int_0^t S_{t-s} \sigma(u_s^{N,n}) dW_s \\
&\quad + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t S_{t-s} g(u_s^{N,n}) e_i d\beta_i^{H,N,n}(s), \quad (3.15)
\end{align}
or equivalently
\begin{align}
u_t^{N,n} &= S_t u_0 + \int_0^t S_{t-s} f(u_s^{N,n})(u_s^{N,n}) ds + \int_0^t S_{t-s} \sigma(u_s^{N,n}) dW_s, \quad (3.16)
\end{align}
where \( f^{N,n}(u) := f(u) + g(u) \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i \beta_i^{H,N,n}(s) \) is a random drift.

Such equations were studied in [7, Section 7]. To proceed, it is easy to obtain
\begin{align}
\left[ \sum_{i=1}^{\infty} \sqrt{\lambda_i} \frac{d}{ds} \beta_i^{H,N,n}(s) \right] &\leq \sum_{i=1}^{\infty} \sqrt{\lambda_i} \frac{d}{ds} \beta_i^{H,N,n}(s) \\
&\leq n \sum_{i=1}^{\infty} \sqrt{\lambda_i} |\beta_i^{H,N}(s) - \beta_i^{H,N}((s - \frac{1}{n}) V 0)| \\
&\leq n \sum_{i=1}^{\infty} \sqrt{\lambda_i} \Lambda_n^{0,s}(\beta_i^{H,N}) \\
&\leq C_n N, \quad (3.17)
\end{align}
where \( C_n N \) is a constant dependent on \( n \) and \( N \). Thus, by (3.17), we obtain the function \( f^{N,n} \) satisfies Lipschitz and growth conditions. Then, by [7, Theorem 7.4], there exists a unique mild solution \( u^{N,n} \) to Eq. (3.14).

**Step 2: Convergence of approximations.** To obtain the convergence of approximations, we give the following two key lemmas which will be proved in Appendix B.

**Lemma 3.8.** Under the assumptions of Theorem 3.5, there exists a constant \( C_N \), such that
\begin{align}
E[\|u^{N,n}\|_{\alpha,T}^2] \leq C_N.
\end{align}

**Lemma 3.9.** Under the assumptions of Theorem 3.5. Then, there exists a constant \( C_{N,R} \), such that
\begin{align}
E\left[\|u^{N,n} - u^{N,m}\|_{\alpha,T}^2 1_{D_T^{N,m}}\right] \leq C_{N,R} E\left[\left( \sum_{i=1}^{\infty} \sqrt{\lambda_i} \|\beta_i^{H,N,n} - \beta_i^{H,N,m}\|_{\alpha,T} \right)^2\right]. \quad (3.18)
\end{align}

where \( D_T^{N,R} := \{\|u^{N,n}\|_{\alpha,T} \leq R, \|u^{N,m}\|_{\alpha,T} \leq R\} \).

For fixed \( N \geq 1 \), we will show that the sequence \( \{u^{N,n}, n \geq 1\} \) is Cauchy sequence in the norm \( \|\cdot\|_{\alpha,T} \).

For all \( \varepsilon > 0, R \geq 1 \), we have
\begin{align}
P(\|u^{N,n} - u^{N,m}\|_{\alpha,T} > \varepsilon) &\leq P(\|u^{N,n}\|_{\alpha,T} > R \text{ or } \|u^{N,m}\|_{\alpha,T} > R) \\
&\quad + P(\|u^{N,n} - u^{N,m}\|_{\alpha,T} > \varepsilon, \|u^{N,n}\|_{\alpha,T} \leq R, \|u^{N,m}\|_{\alpha,T} \leq R).
\end{align}

Since \( \varpi \in (1 - \alpha, H) \), by (3.13), we have
\begin{align}
\sup_{i \in \mathbb{N}} E[\|\beta_i^{H,N,n} - \beta_i^{H,N}\|^2_{\alpha,T}] \leq C(\frac{1}{n})^{2(\varpi + \alpha - 1)} \sup_{i \in \mathbb{N}} E[\|\beta_i^H\|^2_{1 - \varpi,0,T}]
\end{align}
where $C$ is a constant which is independent of $i$ and $N$.

Then, by Cauchy-Schwarz’s inequality, we have

$$E\left[\left(\sum_{i=1}^{\infty} \lambda_i \beta_{i}^{H,N,n} - \beta_{i}^{H,N}\right)^2\right] = E\left[\left(\sum_{i=1}^{\infty} \sqrt{\lambda_i} \beta_{i}^{H,N,n} - \beta_{i}^{H,N}\right)^2\right] \leq \sum_{i=1}^{\infty} \lambda_i \left(\sum_{i=1}^{\infty} \lambda_i E[\|\beta_{i}^{H,N,n} - \beta_{i}^{H,N}\|_{\alpha,0,T}^2]\right).$$

Thus, the condition $\sum_{i=1}^{\infty} \sqrt{\lambda_i} < \infty$ and (3.19) yield that

$$E\left[\left(\sum_{i=1}^{\infty} \sqrt{\lambda_i} \beta_{i}^{H,N,n} - \beta_{i}^{H,N}\right)^2\right] \leq C\left(\frac{1}{n}\right)^{2(\pi + \alpha - 1)} \left(\sum_{i=1}^{\infty} \lambda_i\right)^2 \to 0 \text{ as } n \to \infty.$$  (3.20)

Due to Lemma 3.9 and Markov’s inequality, we see that for all $\varepsilon > 0$, $R \geq 1$,

$$P\left(\|u_{N,n} - u_{N,m}\|_{\alpha,T} > \varepsilon, \|u_{N,n}\|_{\alpha,T} \leq R, \|u_{N,m}\|_{\alpha,T} \leq R\right) \to 0 \quad \text{as } n \to \infty,$$

and

$$\lim_{n,m \to \infty} P\left(\|u_{N,n}\|_{\alpha,T} > R \text{ or } \|u_{N,m}\|_{\alpha,T} > R\right) \leq 2 \sup_{u \in \mathbb{N}} P\left(\|u_{N,n}\|_{\alpha,T} > R\right).$$

Next, Lemma 3.8 and Markov’s inequality imply that

$$\sup_{u \in \mathbb{N}} P\left(\|u_{N,n}\|_{\alpha,T} > R\right) \to 0 \quad \text{as } R \to \infty.$$

Thus,

$$\|u_{N,n} - u_{N,m}\|_{\alpha,T} \to 0 \quad \text{as } n,m \to \infty,$$

in probability. Then there exists a random process $u^N$ such that

$$\|u_{N,n} - u^N\|_{\alpha,T} \to 0 \quad \text{as } n \to \infty,$$

in probability. Denoting an almost surely convergent subsequence by the same symbol, we have

$$\|u_{N,n} - u^N\|_{\alpha,T} \to 0 \quad \text{as } n \to \infty,$$

almost surely.

**Step 3: The limit provides a solution.** Since $\|u_{N,n} - u^N\|_{\alpha,T} \to 0$, as $n \to \infty$, a.s., we easily obtain

$$\int_0^t S_{t-s} f(u^N_s) ds \to \int_0^t S_{t-s} f(u^N_s) ds \quad \text{as } n \to \infty,$$

almost surely.

Similar to the cases of the proof in Lemma 3.9, denoting $1_{t} := 1\{\|u_{N,n}\|_{\alpha,t} \leq R, \|u_{N,b}\|_{\alpha,t} \leq R\}$, we have

$$\left|\sum_{i=1}^{\infty} \lambda_i \int_0^t S_{t-s} (g(u^N_s) - g(u^N_s)) e_i d\beta_{i}^{H,N,n}(s) 1_{t}\right|^2 \leq C\left(\sum_{i=1}^{\infty} \lambda_i\right)\|\beta_{i}^{H,N,n}\|_{\alpha,0,T} \int_0^t |r^{-\alpha} + (t-r)^{-\alpha}| |u^N_r - u^N_t| dr 1_{t}^2.$$
\[ + C \left( \sum_{i=1}^{\infty} \sqrt{\lambda_i} \| \beta_{i,H,N} \|_{\alpha,0,T} \right) \int_0^t |u_r^{N,n} - u_r^{N} (| (\int_0^r |u_r^{N,n} - u_r^{N}| \frac{1}{(r-q)^{1+\alpha}} dq) dr) 1_t \right)^2 \\
\leq C_{N,R} \int_0^t |u^{N,n} - u^N|_2^2 1_t + C_N |u^{N,n} - u^N|_{\alpha,t}^2 1_t. \]

Thus, it is easy to obtain
\[
\sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t S_{t-s} g(u_s^{N,n}) e_i d\beta_i^{H,N,n}(s) \rightarrow \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t S_{t-s} g(u_s^{N,n}) e_i d\beta_i^{H,N,n}(s) \quad \text{as } n \rightarrow \infty,
\]
almost surely. Finally, we will obtain
\[
\left| \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t S_{t-s} g(u_s^{N,n}) e_i d(\beta_i^{H,N,n}(s) - \beta_i^{H,N}(s)) \right| 1_t \leq C \left( \sum_{i=1}^{\infty} \sqrt{\lambda_i} \| \beta_i^{H,N,n} - \beta_i^{H,N} \|_{\alpha,0,T} \right) \int_0^t (r-\alpha + (t-r)^{-\alpha}) (1 + \|u^N\|_{\alpha,r}) dr 1_t \\
+ C \left( \sum_{i=1}^{\infty} \sqrt{\lambda_i} \| \beta_i^{H,N,n} - \beta_i^{H,N} \|_{\alpha,0,T} \right) \int_0^t \left( \int_0^r \frac{|u_r^{N,n} - u_r^{N}|}{(r-q)^{1+\alpha}} dq \right) dr 1_t \\
\leq C_R \sum_{i=1}^{\infty} \sqrt{\lambda_i} \| \beta_i^{H,N,n} - \beta_i^{H,N} \|_{\alpha,0,T}.
\]

Since \( \| \beta_i^{H,N,n} - \beta_i^{H,N} \|_{\alpha,0,T} \rightarrow 0, n \rightarrow \infty, \) a.s., then, we have
\[
\left| \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t S_{t-s} g(u_s^{N,n}) e_i d(\beta_i^{H,N,n}(s) - \beta_i^{H,N}(s)) \right| 1_t \rightarrow 0 \quad \text{as } n \rightarrow \infty,
\]
almost surely.

Next, we have
\[
\mathbb{E} \left[ \left| \int_0^t S_{t-s} (\sigma(u_s^{N,n}) - \sigma(u_s^{N})) dW_s \right|^2 1_t \right] \leq \int_0^t \mathbb{E} \left[ |\sigma(u_s^{N,n}) - \sigma(u_s^{N})|^2 \right] d\mathbb{L}_2(V) 1_s ds \\
\leq \int_0^t \mathbb{E} \left[ |u^{N,n} - u^N|_2^2 1_s \right] ds.
\]

Thus, by Lemma 3.9, it is easy to obtain
\[
\mathbb{E} \left[ \left| \int_0^t S_{t-s} (\sigma(u_s^{N,n}) - \sigma(u_s^{N})) dW_s \right|^2 1_t \right] \rightarrow 0 \quad \text{as } n \rightarrow \infty,
\]
and, consequently
\[
\int_0^t S_{t-s} \sigma(u_s^{N,n}) dW_s - \int_0^t S_{t-s} \sigma(u_s^{N}) dW_s \rightarrow 0 \quad \text{as } n \rightarrow \infty,
\]
in probability. Since \( |u^{N,n} - u^N|_{\alpha,T} \rightarrow 0, n \rightarrow \infty, \) a.s., we have the convergence of the integrals in probability on \( \{ |u^N|_{\alpha,T} \leq R \} \), where \( R \geq 1 \) is arbitrary, therefore the convergence holds on \( \Omega \). This means that \( u^N \) is a solution to
\[
u^N_t = S_{t}u_0 + \int_0^t S_{t-s}f(u_s^{N}) ds + \int_0^t S_{t-s}\sigma(u_s^{N}) dW_s + \int_0^t S_{t-s}g(u_s^{N}) dB_s^{H,N}. \quad (3.21)
\]
Step 4: Letting $N \to \infty$ and uniqueness. It follows from Lemma 3.9 that the processes $u^N$ and $u^M$ with $M \geq N$ coincide almost surely on the set $A_{N,T} = \{ A_{N,T} \leq N \}$. Hence, there exists a process $u$ such that $u^N = u$, a.s. on $A_{N,T}$ for each $N \geq 1$, hence, almost surely. Finally, the pathwise uniqueness follows in similar way. Thus, the proof is finished. □

4. Fast-slow SPDEs Driven by fBm and Bm

Throughout this section, we assume that the following conditions are fulfilled. We assume that

(A1) The coefficients $b(x,y) : V \times V \to V, F(x,y) : V \times V \to V, G(x,y) : V \times V \to L_2(V)$ of Eq. (1.2) are globally Lipschitz continuous in $x, y$, i.e., there exist two positive constants $C_1, C_2$, such that

$$|b(x_1,y_1) - b(x_2,y_2)|^2 \leq C_1(|x_1 - x_2|^2 + |y_1 - y_2|^2),$$

$$|F(x_1,y_1) - F(x_2,y_2)|^2 + |G(x_1,y_1) - G(x_2,y_2)|^2_{L_2(V)} \leq C_2(|x_1 - x_2|^2 + |y_1 - y_2|^2),$$

for all $x_1, x_2, y_1, y_2 \in V$.

(A2) The coefficients $b(x,y), F(x,y), G(x,y)$ of Eq. (1.2) satisfy linear growth conditions, i.e., there exist two positive constants $C_3, C_4$ such that

$$|F(x,y)|^2 + |G(x,y)|^2_{L_2(V)} \leq C_3(1 + |x|^2 + |y|^2),$$

$$|b(x,y)|^2 \leq C_4(1 + |x|^2 + |y|^2),$$

for all $x, y \in V$.

(A3) $g : V \to L(V)$ and $g' : V \to L(V, L(V))$ are Lipschitz continuous in the senses of Eq. (3.5) and Eq. (3.6).

(A4) There exist constants $\beta_1, C_5 > 0$ and $\beta_2, \beta_3 \in \mathbb{R}$ which are independent of $(x, y_1, y_2)$, such that

$$\langle y_1, F(x,y_1) \rangle \leq -\beta_1|y|^2 + \beta_2,$$

$$\langle y_1 - y_2, F(x_1,y_1) - F(x_2,y_2) \rangle \leq \beta_3|y_1 - y_2|^2 + C_5|x_1 - x_2|^2,$$

for all $x, y_1, y_2 \in V$.

(A5) $\eta := 2\lambda_1 - 2\beta_3 - C_2 > 0, \kappa := 2\lambda_1 + 2\beta_1 - C_3 > 0, \theta := \lambda_1^2 - C_3 > 0$, where $\lambda_1$ is the first eigenvalue of $-A, C_i, i = 2, 3$ and $\beta_i, i = 1, 2, 3$ were given in (A1), (A2) and (A4).

Remark 4.1. Assumptions (A4) and (A5) are known as the strong dissipative conditions that imply the existence of a unique invariant measure and moreover, it has exponentially mixing property for the Markov semigroup associated to the fast variable.

Through Theorem 3.3 and a similar argument as in the proof of Theorem 2.2, it is easy to prove that Eq. (1.2) has a unique mild pathwise solution. Here, we omit the proof.

Lemma 4.2. Suppose that conditions (A1)-(A3) are satisfied. Then, for any initial values $X_0, Y_0 \in V_0, \beta > \alpha$, Eq. (1.2) has a unique mild pathwise solution $(X^\varepsilon_t, Y^\varepsilon_t)$, i.e.,

$$X^\varepsilon_t = S_tX_0^\varepsilon + \int_0^t S_{t-s}b(X^\varepsilon_s, Y^\varepsilon_s)ds + \int_0^t S_{t-s}g(X^\varepsilon_s)dB^H_s, \quad X_0^\varepsilon = X_0,$$  

$$Y^\varepsilon_t = S_tY_0^\varepsilon + \frac{1}{\varepsilon} \int_0^t S_{t-s}F(X^\varepsilon_s, Y^\varepsilon_s)ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t S_{t-s}G(X^\varepsilon_s, Y^\varepsilon_s)dW_s, \quad Y_0^\varepsilon = Y_0.$$  

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Denote by $\tilde{X}_t, t \in [0, T]$, the solution of the following SPDEs driven by fBm,  
\[
d\tilde{X}_t = (A\tilde{X}_t + b(\tilde{X}_t))dt + g(\tilde{X}_t)dB^H_t, \quad \tilde{X}_0 = X_0, \tag{4.3}
\]  
where  
\[
b(x) = \int_{V} b(x, z)\mu^x(dz), \quad x \in V,
\]  
and $\mu^x$ is the unique invariant measure on $V$ of the transition semigroups for the following frozen equation:  
\[
dY_t^x = (AY_t^x + F(x, Y_t^x))dt + G(x, Y_t^x)dW_t, \quad Y_0^x = y \in V. \tag{4.5}
\]  
According to the definition of $\bar{b}$ and conditions (A1)-(A5), it is easy to prove $\bar{b}$ also satisfies the Lipschitz and growth conditions. Then, we have the following lemma.

**Lemma 4.3.** Suppose that conditions (A1)-(A5) are satisfied. Then, for any initial value $X_0 \in V$, $\beta > \alpha$, Eq. (4.3) has a unique mild pathwise solution.

**Proof:** For any $x_1, x_2, x \in V$ and any initial value $y \in V$, by (A.2) and (A.3) in Appendix A, we have  
\[
|\bar{b}(x_1) - \bar{b}(x_2)| \leq \left| \int_{V} b(x_1, z)\mu^x(dz) - \mathbb{E}[b(x_1, Y_s^{x,y})] \right|^2 
+ \left| \int_{V} b(x_2, z)\mu^x(dz) - \mathbb{E}[b(x_2, Y_s^{x,y})] \right|^2 
+ \left| \mathbb{E}[b(x_1, Y_s^{x,y}) - b(x_2, Y_s^{x,y})] \right|^2 
\leq C e^{-\gamma y}(1 + |x_1|^2 + |x_2|^2 + |y|^2) + C|x_1 - x_2|^2,
\]  
Let $s \to \infty$, then we obtain that $\bar{b}_1$ is Lipschitz continuous in $x$, and  
\[
|\bar{b}(x)|^2 \leq \left( \int_{V} |b(x, z)|\mu^x(dz) \right)^2 \leq C(1 + |x|^2).
\]  
So, $\bar{b}_1$ satisfies the growth condition. Thus, according to Theorem 3.5, (4.3) has a unique strong solution.

From now on, we assume $\beta \in (\frac{1}{2}, 1 - \alpha)$ and to present our main averaging result, we need to impose another condition.

(B1) $\sup_{x, y \in V} (|b(x, y)| + |G(x, y)|) < \infty$.

**Theorem 4.4.** Suppose that conditions (A1)-(A5) and (B1) hold. Then, we have  
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \|X^\varepsilon - \bar{X}\|_{2, T}^2 \right] = 0.
\]

**Remark 4.5.** To obtain the strong convergence, it is known that the diffusion coefficient $g$ in Eq. (4.2) should not depend on the fast variable $Y^\varepsilon$ (see e.g. [13]).

To prove Theorem 4.4, firstly, following the discretization techniques inspired by Khasminskii in [13], we introduce an auxiliary process $(\tilde{X}_t^\varepsilon, \tilde{Y}_t^\varepsilon)$ and divide $[0, T]$ into intervals of size $\delta$, where $\delta \in (0, 1)$ is a fixed number depending on $\varepsilon$ and $\delta > \varepsilon$, which will be chosen later. Then, we construct auxiliary processes $\tilde{Y}_t^\varepsilon$ and $\tilde{X}_t^\varepsilon$, by  
\[
\tilde{Y}_t^\varepsilon = S_{t-s}Y_s + \frac{1}{\varepsilon} \int_{0}^{t} S_{t-s}F(X_{s(\delta)}^\varepsilon, \tilde{Y}_s^\varepsilon)ds + \frac{1}{\sqrt{\varepsilon}} \int_{0}^{t} S_{t-s}G(X_{s(\delta)}^\varepsilon, \tilde{Y}_s^\varepsilon)dW_s, \tag{4.6}
\]
\[
\tilde{X}_t^\varepsilon = S_{t-s}X_s + \int_{0}^{t} S_{t-s}b(X_{s(\delta)}^\varepsilon, \tilde{Y}_s^\varepsilon)ds + \int_{0}^{t} S_{t-s}g(X_{s(\delta)}^\varepsilon)dB^H_s. \tag{4.7}
\]
where \( s(\delta) = \lfloor s/\delta \rfloor \delta \) is the nearest breakpoint preceding \( s \). For \( t \in [k\delta, \min\{(k + 1)\delta, T]\} \), we assume the fast component \( Y_{k\delta}^\varepsilon \) is reset to equal \( Y_{k\delta}^\varepsilon \) at each breakpoint \( k\delta \). To proceed, we can derive uniform bounds \( \|X^\varepsilon - \bar{X}\|_{\alpha,T}^2 \). Next, based on the ergodic property of the frozen equation, we obtain appropriate control of \( \|X^\varepsilon - \bar{X}\|_{\alpha,T}^2 \). Finally, we can estimate \( \|X^\varepsilon - \bar{X}\|_{\alpha,T}^2 \).

4.1. A Priori Estimate

Estimates of the auxiliary process \((X_t^\varepsilon, Y_t^\varepsilon)\) will be given in this subsection.

**Lemma 4.6.** Suppose that conditions (A1)-(A3) and (B1) are satisfied. Then, for any \( p \geq 2 \), there exists a constant \( C_p \), which is independent of \( \varepsilon \) such that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|X_t^\varepsilon\|_{\alpha}^p \right] \leq C_p,
\]

where \( \| \cdot \|_{\alpha} \) was defined in \([k,\beta]\).

**Proof:** For shortness, denote, \( \Lambda := \Lambda_{\alpha,B}^{\beta,T} \vee 1 \), and for \( \rho \geq 1 \), let

\[
\|f\|_{\rho,T} := \sup_{t \in [0,T]} e^{-\rho t} |f(t)|,
\]

\[
\|f\|_{1,\rho,T} := \sup_{t \in [0,T]} e^{-\rho t} \int_0^t |f(t) - f(r)| dr.
\]

Using techniques similar to those used in \([33, \text{Lemma 4.1}]\), we start by estimating \( \|X^\varepsilon\|_{\rho,T} \). By the similar step as for the terms \( M_{21}, M_{42} \) in Appendix B and using Lemma \( 34 \) (A1)-(A3) and (B1), we have

\[
\|X^\varepsilon\|_{\rho,T} = \sup_{t \in [0,T]} e^{-\rho t} \left| S_t X_0 + \int_0^t S_{t-r} b(X_r^\varepsilon, Y_r^\varepsilon) dr + \int_0^t S_{t-r} g(X_r^\varepsilon) dB_r^H \right| \leq C \left( 1 + |X_0|_{\beta} + \Lambda_{\alpha,B}^{\beta,T} \sup_{t \in [0,T]} \int_0^t e^{-\rho(t-r)} [(r^{-\alpha} + (t-r)^{-\alpha}) \|X^\varepsilon\|_{\rho,T} + \|X^\varepsilon\|_{1,\rho,T}] dr \right) \leq KA \left( 1 + \rho^{-\alpha} \|X^\varepsilon\|_{\rho,T} + \rho^{-1} \|X^\varepsilon\|_{1,\rho,T} \right),
\]

with some constant \( K \) (which is dependent on \( |X_0|_{\beta} \) and can be assumed to be greater than 1 without loss of generality).

For \( \|X^\varepsilon\|_{1,\rho,T} \), by the similar step as for the terms \( M_{41} \) in Appendix B and using Fubini's theorem, Lemma \( 23 \) (A1)-(A3) and (B1), we have

\[
\|X^\varepsilon\|_{1,\rho,T} \leq C \sup_{t \in [0,T]} \int_0^t e^{-\rho t} \left| \left( S_t - S_s \right) X_0 \right| \frac{ds}{(t-s)^{1+\alpha}} + C \sup_{t \in [0,T]} \int_0^t e^{-\rho t} \left| \int_s^t S_{t-r} b(X_r^\varepsilon, Y_r^\varepsilon) dr \right| \frac{ds}{(t-s)^{1+\alpha}} + C \sup_{t \in [0,T]} \int_0^t e^{-\rho t} \left| \int_0^{t-r} \left( S_{t-r} - S_{s-r} \right) b(X_r^\varepsilon, Y_r^\varepsilon) dr \right| ds + C \sup_{t \in [0,T]} \int_0^t e^{-\rho t} \left| \int_0^{t-r} S_{t-r} g(X_r^\varepsilon) dB_r^H \right| ds + C \sup_{t \in [0,T]} \int_0^t e^{-\rho t} \left| \int_0^{t-r} \left( S_{t-r} - S_{s-r} \right) g(X_r^\varepsilon) dB_r^H \right| ds \leq C(1 + |X_0|_{\beta})
\]

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Thus, we have

\[ + CA_{\alpha,B_0}^0 \sup_{t \in [0,T]} \int_0^t e^{-\rho t} \int_s^t (r-s)^{-\alpha} + (t-r)^{-\alpha} \left( 1 + |X_r^\varepsilon| \right) dr ds \]

\[ + CA_{\alpha,B_0}^0 \sup_{t \in [0,T]} \int_0^t e^{-\rho t} \int_s^t (t-s)^{-1-\alpha} \left( \int_0^r \frac{|X_s^\varepsilon - X_q^\varepsilon|}{(r-q)^{1+\alpha}} dq \right) dr ds \]

\[ + CA_{\alpha,B_0}^0 \sup_{t \in [0,T]} \int_0^t e^{-\rho t} \int_s^t (t-s)^{-\beta} r^{-\alpha} + (s-r)^{-\alpha-\beta} \left( 1 + |X_r^\varepsilon| \right) dr ds \]

\[ + CA_{\alpha,B_0}^0 \sup_{t \in [0,T]} \int_0^t e^{-\rho t} \int_s^t (t-s)^{-1-\alpha+\beta} (s-r)^{-\beta} \left( \int_0^r \frac{|X_s^\varepsilon - X_q^\varepsilon|}{(r-q)^{1+\alpha}} dq \right) dr ds \]

\[ \leq C(1 + |X_0|) \]

Putting \( \rho = (4K\Lambda)^{\frac{1}{\alpha}} \), we get from the inequality (4.8) that

\[ \|X_r^\varepsilon\|_{\rho,T} \leq \frac{4}{3} K\Lambda \left( 1 + \rho^{-1}\|X_t^\varepsilon\|_{1,\rho,T} \right). \]  \( (4.10) \)

Plugging this into the inequality (4.8) and making simple transformations, we arrive at

\[ \|X_r^\varepsilon\|_{1,\rho,T} \leq \frac{3}{2} K\Lambda + 2(K\Lambda)^{\frac{1}{\alpha}} \leq CA^{\frac{1}{\alpha}}. \]

Substituting this into (4.10), we get

\[ \|X_r^\varepsilon\|_{\rho,T} \leq CA^{\frac{1}{\alpha}}. \]

Thus, we have

\[ \sup_{t \in [0,T]} \|X_t^\varepsilon\|_{\alpha} \leq e^{\rho T} \left( \|X_r^\varepsilon\|_{\rho,T} + \|X_r^\varepsilon\|_{1,\rho,T} \right) \]

\[ \leq C \exp\left( CA^{\frac{1}{\alpha}} \right) \Lambda^{\frac{1}{\alpha}} \]

\[ \leq C \exp\left( \left( A_{\alpha,B_0}^{0,T} \right)^{\frac{1}{\alpha}} \right). \]

Since \( 0 < \frac{1}{\alpha} < 2 \), by the classical Fernique’s theorem, we have

\[ \mathbb{E} \left[ \exp \left( \left( A_{\alpha,B_0}^{0,T} \right)^{\frac{1}{\alpha}} \right) \right] < \infty. \]

Then, the statement follows.

Using similar techniques as for the Lemma (4.8) we have the following remark.

**Remark 4.7.** Suppose that conditions (A1)-(A5) and (B1) are satisfied. Then, for any \( p \geq 2 \), we have

\[ \sup_{t \in [0,T]} \left( \|\hat{X}_t^\varepsilon\|_{\alpha} + \|\hat{X}_t\|_{\alpha} \right) \leq C \exp\left( \left( A_{\alpha,B_0}^{0,T} \right)^{\frac{1}{\alpha}} \right), \]

\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \left( \|\hat{X}_t^\varepsilon\|^p_{\alpha} + \|\hat{X}_t\|^p_{\alpha} \right) \right] \leq C_p. \]
Lemma 4.8. Suppose that conditions (A1)-(A3) and (B1) are satisfied. Then, we have

$$\mathbb{E}[|X_t^\varepsilon - X_s^\varepsilon|^2] \leq C|t - s|^{2\beta},$$

where $C$ is independent of $\varepsilon, t, s$.

Proof: From Lemma 4.8, we have

$$|X_t^\varepsilon - X_s^\varepsilon| \leq |(S_t - S_s)X_0| + \int_s^t |S_{t-r}b(X_r^\varepsilon, Y_r^\varepsilon)|dr + \int_s^t |(S_{t-r} - S_{s-r})b(X_r^\varepsilon, Y_r^\varepsilon)|dr + \int_s^t |S_{t-r}g(X_r^\varepsilon)dB_r^H| + \int_s^t |(S_{t-r} - S_{s-r})g(X_r^\varepsilon)dB_r^H| =: \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 + \mathcal{V}_4 + \mathcal{V}_5.$$

Since $X_0 \in V_3$, by Lemma 4.9 and (B1), we have

$$\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 \leq |(S_{t-s} - \id)S_sX_0| + \int_s^t |S_{t-r}b(X_r^\varepsilon, Y_r^\varepsilon)|dr + \int_s^t |(S_{t-r} - S_{s-r})b(X_r^\varepsilon, Y_r^\varepsilon)|dr \leq C|X_0|_{\beta}(t-s)^\beta + C(t-s) + C(t-s)^\beta.$$

Next, for $\mathcal{V}_4$ and $\mathcal{V}_5$, by Remark 2.1 and Lemma 3.4 we have

$$\mathcal{V}_4 + \mathcal{V}_5 \leq C\Lambda_{\alpha,B^H}^{0,T} \int_s^t |(r-s)^{-\alpha} + (t-r)^{-\alpha}|(1 + \|X_r^\varepsilon\|_\alpha)dr + C(t-s)^\beta \Lambda_{\alpha,B^H}^{0,T} \int_s^t |(r-s)^{-\beta} + (s-r)^{-\alpha}|(1 + \|X_r^\varepsilon\|_\alpha)dr \leq C[(t-s)^{1-\alpha} + (t-s)^{2\beta}]\Lambda_{\alpha,B^H}^{0,T} \sup_{\tau \in [0,T]} \|X_{\tau}^\varepsilon\|_\alpha.$$

Finally, by Lemma 4.6 we have

$$\mathbb{E}[|X_t^\varepsilon - X_s^\varepsilon|^2] \leq C|t - s|^{2\beta}.$$

Then, the statement follows. \qed

Lemma 4.9. Suppose that conditions (A1)-(A5) and (B1) are satisfied. Then, we have

$$\sup_{\tau \in [0,T]} \mathbb{E}[|Y_\tau^\varepsilon|^2] \leq C,$$

where $C$ is a positive constant which is independent of $\varepsilon$.

Proof: Note that

$$\sup_{\tau \in [0,T]} \mathbb{E}[|Y_\tau^\varepsilon|^2] \leq \sup_{\tau \in [0,T]} \mathbb{E}[|S_{\tau}Y_0|^2] + \sup_{\tau \in [0,T]} \mathbb{E}\left[\frac{1}{\varepsilon} \int_0^\tau S_{s-\varepsilon}F(X_s^\varepsilon, Y_s^\varepsilon)ds\right]^2 + \sup_{\tau \in [0,T]} \mathbb{E}\left[\frac{1}{\sqrt{\varepsilon}} \int_0^\tau S_{s-\varepsilon}G(X_s^\varepsilon, Y_s^\varepsilon)dW_s\right]^2.$$
The resetting of the auxiliary process at the breakpoints $C$

Suppose that conditions Lemma 4.10.

Here, $\square$

Then, the statement follows.

Therefore, due to (A5), $C_3 < 2\lambda_1^2$, we have

$$\sup_{t \in [0, T]} E[|Y_t^\varepsilon|^2] \leq C.$$ 

Then, the statement follows.

Using similar techniques, under conditions (A1)-(A5) and (B1), we can prove

$$\sup_{t \in [0, T]} E[|Y_t^\varepsilon|^2] \leq C.$$

Here, $C$ is also a positive constant which is independent of $\varepsilon$.

Now, using the definitions of $Y_t^\varepsilon$ (1.12) and $\hat{Y}_t^\varepsilon$ (1.14), we proceed to estimate $E[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2]$.

Lemma 4.10. Suppose that conditions (A1)-(A5) are satisfied. Then, we have

$$\int_{k \delta}^{\min\{(k+1) \delta, T\}} E[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2] dt \leq C \varepsilon \delta^{2\beta} e^{\frac{C}{\varepsilon}},$$

(4.11)

where $C$ is a constant which is independent of $\varepsilon, \delta, k$.

Proof: The resetting of the auxiliary process at the breakpoints $k \delta$ implies that $Y_{k \delta}^\varepsilon = \hat{Y}_{k \delta}^\varepsilon$, for all $k$. Then, for $t \in [k \delta, \min\{(k+1) \delta, T\}]$, we start with

$$e^{2\lambda_1 \frac{t}{\varepsilon}} E[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2] \leq e^{2\lambda_1 \frac{t}{\varepsilon}} E \left[ \left( \frac{1}{\varepsilon} \int_{k \delta}^{t} S_{\frac{s}{\varepsilon}} (F(X_s^\varepsilon, Y_s^\varepsilon) - F(X_{k \delta}^\varepsilon, \hat{Y}_{k \delta}^\varepsilon)) ds \right)^2 \right]$$

$$+ e^{2\lambda_1 \frac{t}{\varepsilon}} E \left[ \left( \frac{1}{\varepsilon} \int_{k \delta}^{t} S_{\frac{s}{\varepsilon}} (G(X_s^\varepsilon, Y_s^\varepsilon) - G(X_{k \delta}^\varepsilon, \hat{Y}_{k \delta}^\varepsilon)) dW(s) \right)^2 \right]$$

$$\leq C \left( \delta + \frac{1}{\varepsilon} \right) \int_{k \delta}^{t} e^{2\lambda_1 \frac{s}{\varepsilon}} E[|X_s^\varepsilon - X_{k \delta}^\varepsilon|^2] ds$$

$$\leq C \left( \delta + \frac{1}{\varepsilon} \right)^2 \int_{k \delta}^{t} e^{2\lambda_1 \frac{s}{\varepsilon}} ds + \int_{k \delta}^{t} e^{2\lambda_1 \frac{s}{\varepsilon}} E[|Y_s^\varepsilon - \hat{Y}_s^\varepsilon|^2] ds. \quad (4.12)$$

By Grönwall’s inequality, we have

$$e^{2\lambda_1 \frac{t}{\varepsilon}} E[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2] \leq C \varepsilon \delta^{2\beta} \left( \int_{k \delta}^{t} e^{2\lambda_1 \frac{s}{\varepsilon}} ds \right) e^{C \left( \frac{\delta}{\varepsilon} + \frac{1}{\varepsilon} \right) (t - k \delta)}. \quad (4.13)$$
It is clear that
\[
\mathbb{E}[|Y_r^\varepsilon - \hat{Y}_r^\varepsilon|^2] \leq C\delta^{2\beta} + \varepsilon e^{C(\frac{\alpha}{\beta} + \frac{1}{2})}(t-k\delta). \tag{4.14}
\]

Integrate (4.14) from \(k\delta\) to \(\min((k+1)\delta, T)\), we have
\[
\int_{k\delta}^{\min((k+1)\delta, T)} \mathbb{E}[|Y_r^\varepsilon - \hat{Y}_r^\varepsilon|^2] dr \leq C\delta^{2\beta} + \varepsilon \int_{k\delta}^{\min((k+1)\delta, T)} e^{C(\frac{\alpha}{\beta} + \frac{1}{2})(t-k\delta)} dt \\
\leq C\delta^{2\beta} + \varepsilon \left(\frac{1}{\frac{\alpha}{\beta} + \frac{1}{2}}\right) (e^{C(\frac{\alpha}{\beta} + \frac{1}{2})\delta} - 1) \\
\leq C\varepsilon \delta^{2\beta} e^{C\frac{\alpha}{\beta}}.
\]

This completes the proof of Lemma \ref{lemma4.10}. \qed

4.2. The Proof of Theorem \ref{lemma4.11}

We divide the proof into three steps.

**Step 1:** This step will estimate \(||X^\varepsilon - \hat{X}^\varepsilon||_{\alpha, T}\). By \ref{lemma4.11} and \ref{lemma4.17}, we have
\[
\mathbb{E}[||X^\varepsilon - \hat{X}^\varepsilon||_{\alpha, T}^2] \leq C\mathbb{E}\left[\left|| \int_0^T S_{t-s}(b(X_s^\varepsilon, \hat{Y}_s^\varepsilon) - b(X_s^\varepsilon, \hat{Y}_s^\varepsilon)) ds\right|^2_{\alpha, T}\right] + C\mathbb{E}\left[\left|| \int_0^T S_{t-s}(b(X_s^\varepsilon, Y_s^\varepsilon) - b(X_s^\varepsilon, \hat{Y}_s^\varepsilon)) ds\right|^2_{\alpha, T}\right] \\
=: I_1 + I_2.
\]

Following the similar steps as for the terms \(M_{21}\) and \(M_{22}\) in Appendix B, we obtain
\[
I_1 \leq C\int_0^T \mathbb{E}[|X_s^\varepsilon - X_{r(s)}^\varepsilon|^2] ds \\
+ C\sup_{t \in [0, T]} \mathbb{E}\left[\left(\int_0^t \int_0^t (t-s)^{-1-\alpha} ds |b(X_r^\varepsilon, \hat{Y}_r^\varepsilon) - b(X_r^\varepsilon, Y_r^\varepsilon)| ds\right)^2\right] \\
+ C\sup_{t \in [0, T]} \mathbb{E}\left[\left(\int_0^t \int_0^t (t-s)^{-1-\alpha} (s-r)^{-\beta} ds |b(X_r^\varepsilon, \hat{Y}_r^\varepsilon) - b(X_r^\varepsilon, Y_r^\varepsilon)| ds\right)^2\right] \\
\leq C\int_0^T \mathbb{E}[|X_s^\varepsilon - X_{r(s)}^\varepsilon|^2] ds \\
\leq C\delta^{2\beta}.
\]

Next, for \(I_2\), we have
\[
I_2 \leq C\mathbb{E}\left[\sup_{t \in [0, T]} \left| \int_0^t S_{t-r}(b(X_r^\varepsilon, Y_r^\varepsilon) - b(X_r^\varepsilon, \hat{Y}_r^\varepsilon)) dr\right|^2\right] \\
+ C\sup_{t \in [0, T]} \mathbb{E}\left[\left(\int_0^t (t-s)^{-1-\alpha} \left| \int_0^t S_{t-r}(b(X_r^\varepsilon, Y_r^\varepsilon) - b(X_r^\varepsilon, \hat{Y}_r^\varepsilon)) dr\right| ds\right)^2\right] \\
+ C\sup_{t \in [0, T]} \mathbb{E}\left[\left(\int_0^t (t-s)^{-1-\alpha} \left| \int_0^s (S_{t-r} - S_{s-r})(b(X_r^\varepsilon, Y_r^\varepsilon) - b(X_r^\varepsilon, \hat{Y}_r^\varepsilon)) dr\right| ds\right)^2\right] \\
=: I_{21} + I_{22} + I_{23}.
\]

If we set \(t := \{t \geq \left(\frac{1}{\beta} + 2\delta\right)\}, \ell := \{t < (\frac{1}{\beta} + 2\delta)\}, \beta := \{\left|\frac{1}{\beta}\right| \leq 1\} \) and \(\beta := \{\left|\frac{1}{\beta}\right| > 1\}\) and by Hölder’s inequality and the fact that \((\frac{1}{\beta} - \left|\frac{1}{\beta}\right| - 1) \leq \frac{\delta}{\delta}\), for any \(0 \leq s \leq t\), we obtain
\[
I_{21} + I_{22}
\]
\[ \begin{align*}
&\leq C \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_{t_{(k)}}^{t} S_{t-r}(b(X_r^\varepsilon, Y_r^\varepsilon) - b(X_r^\varepsilon, \hat{Y}_r^\varepsilon))dr \right|^2 \right] \\
&\quad + C \delta^{-1} \sum_{k=0}^{(1/\nu)-1} \mathbb{E} \left[ \int_{k\delta}^{(k+1)\delta} \left| S_{t-r}(b(X_r^\varepsilon, Y_r^\varepsilon) - b(X_r^\varepsilon, \hat{Y}_r^\varepsilon))dr \right|^2 \right] \\
&\quad + C \sup_{t \in [0, T]} \mathbb{E} \left[ \int_{0}^{t} (t-s)^{-\frac{\alpha}{2}} \int_{s}^{t} S_{t-r}(b(X_r^\varepsilon, Y_r^\varepsilon) - b(X_r^\varepsilon, \hat{Y}_r^\varepsilon))dr ds \right] \\
&\leq C \delta^2 + C \delta^{-2} \max_{0 \leq k \leq \left( \frac{1}{2} \right)-1} \mathbb{E} \left[ \left| \int_{k\delta}^{(k+1)\delta} S_{t-r}(b(X_r^\varepsilon, Y_r^\varepsilon) - b(X_r^\varepsilon, \hat{Y}_r^\varepsilon))dr \right|^2 \right] \\
&\quad + C \delta^{-\alpha} + C \sup_{t \in [0, T]} \left\{ \int_{0}^{t} (t-s)^{-\frac{\alpha}{2}} \mathbb{E} \left[ \left| \int_{k\delta}^{(k+1)\delta} S_{t-r}(b(X_r^\varepsilon, Y_r^\varepsilon) - b(X_r^\varepsilon, \hat{Y}_r^\varepsilon))dr \right|^2 \right] \right\} \\
&\quad + C \delta^{-1} \sup_{t \in [0, T]} \int_{0}^{t} \left[ \int_{k\delta}^{(k+1)\delta} S_{t-r}(b(X_r^\varepsilon, Y_r^\varepsilon) - b(X_r^\varepsilon, \hat{Y}_r^\varepsilon))dr \right] \right] \\
&\leq C \delta + C \delta^{-2} \max_{0 \leq k \leq \left( \frac{1}{2} \right)-1} \mathbb{E} \left[ \left| \int_{k\delta}^{(k+1)\delta} S_{t-r}(b(X_r^\varepsilon, Y_r^\varepsilon) - b(X_r^\varepsilon, \hat{Y}_r^\varepsilon))dr \right|^2 \right].
\end{align*} \]

Once again, by Hölder’s inequality and inequality (1.22) and taking \( \beta' \in (\frac{1+2\alpha}{\alpha}, \frac{1}{2}) \), we have
and it tends to 0.

The following inequality holds:

\[\sum_{k=0}^{n} f_{k\delta}^{(k+1)\delta} (s - k\delta) b(X_{k\delta}, Y_{k\delta}) \leq C \delta^{2} + C \delta^{2-2\beta}\]

+ \sup_{t \in [0, T]} \left\{ \int_{0}^{t} E \left[ \sum_{k=0}^{n} f_{k\delta}^{(k+1)\delta} (s - k\delta) b(X_{k\delta}, Y_{k\delta}) - b(X_{k\delta}, \hat{Y}_{k\delta}) \right] ds \right\}

\leq C \delta^{2}\sup_{t \in [0, T]} \left\{ \int_{0}^{t} E \left[ \sum_{k=0}^{n} f_{k\delta}^{(k+1)\delta} (s - k\delta) b(X_{k\delta}, Y_{k\delta}) - b(X_{k\delta}, \hat{Y}_{k\delta}) \right] ds \right\}

\leq C \delta^{2} + C \delta^{2-2\beta}.

As a consequence, we have

\[I_{\delta} \leq C \delta^{-1} \sup_{t \in [0, T]} \left\{ \int_{0}^{t} E \left[ \sum_{k=0}^{n} f_{k\delta}^{(k+1)\delta} (s - k\delta) b(X_{k\delta}, Y_{k\delta}) - b(X_{k\delta}, \hat{Y}_{k\delta}) \right] ds \right\}

\leq C \delta^{2}\sup_{t \in [0, T]} \left\{ \int_{0}^{t} E \left[ \sum_{k=0}^{n} f_{k\delta}^{(k+1)\delta} (s - k\delta) b(X_{k\delta}, Y_{k\delta}) - b(X_{k\delta}, \hat{Y}_{k\delta}) \right] ds \right\}

\leq C \delta^{-2} + C \delta^{2-2\beta}.

Note that for 2\beta' < 1,

\[\sum_{k=0}^{n} (s - k\delta)^{-2\beta'} \leq \delta^{-2\beta'} \sum_{k=1}^{n} k^{-2\beta'} \leq \delta^{-2\beta'} \int_{1}^{\frac{n}{\delta}} u^{-2\beta'} du \leq \delta^{-2\beta'} \frac{1}{1 - 2\beta' \delta} \frac{1}{\delta^{1-2\beta'}} \leq C \delta^{-1}, \]

holds, then by Lemma 4.10 and applying Hölder’s inequality again

\[I_{\delta} \leq C \delta^{2} + C \delta^{-1} \sup_{t \in [0, T]} \left\{ \int_{0}^{t} E \left[ \sum_{k=0}^{n} f_{k\delta}^{(k+1)\delta} (s - k\delta) b(X_{k\delta}, Y_{k\delta}) - b(X_{k\delta}, \hat{Y}_{k\delta}) \right] ds \right\}

\leq C \delta^{2} + C \delta^{2} e^{C \delta^{2} \epsilon C \delta^{2}} \leq C \delta^{2} + C \epsilon^{-1}\delta^{2} e^{C \delta^{2} \epsilon C \delta^{2}}.

Therefore, we have

\[E[\|X_{\epsilon} - \hat{X}_{\epsilon}\|_{2,\alpha}^{2}] \leq C (\epsilon \delta^{-1} \delta^{2\beta} e^{C \delta^{2} \epsilon} + \delta^{2\beta}). \tag{4.16}\]

This completes the proof of Step 1. \(\square\)

**Step 2:** This step will estimate \(\|\hat{X}_{\epsilon} - \hat{X}\|_{\alpha, T}\).

**Lemma 4.11.** The following inequality hold:

\[P(\tau_{N} < T) \leq N^{-1} E[\Lambda_{\alpha, B\alpha}^{0, T}],\]

and it tends to 0 when \(N \to \infty\).
Proof: By Chebyshev’s inequality, we have

$$
P(\tau_N < T) \leq \mathbb{P} \left( \sum_{i=1}^{\infty} \sqrt{\lambda_i} \Lambda^{0,T}(\beta_i^H) \geq N \right) \leq N^{-1} \sum_{i=1}^{\infty} \sqrt{\lambda_i} \mathbb{E} \left[ \Lambda^{0,T}(\beta_i^H) \right].$$

Because $\Lambda^{0,T}(\beta^H)$ has moments of all order, see [32, Lemma 7.5], thus we have

$$\lim_{N \to \infty} N^{-1} \mathbb{E} \left[ \Lambda^{0,T}_{\alpha,B^H} \right] = 0.$$

This completes the proof of Lemma 4.11. □

Then, by (4.1) and (4.3), we get

$$E[\|X^\varepsilon - \hat{X}\|_{\alpha,T}^2] \leq E[\|X^\varepsilon - \hat{X}\|_{\alpha,T}^2 1_{\{\tau_N < T\}}] + E[\|X^\varepsilon - \hat{X}\|_{\alpha,T}^2 1_{\{\tau_N \geq T\}}].$$

For the first term on the right-hand side of the above inequality, by Chebyshev’s inequality, we have

$$E[\|X^\varepsilon - \hat{X}\|_{\alpha,T}^2 1_{\{\tau_N < T\}}] \leq \sqrt{E[\|X^\varepsilon - \hat{X}\|_{\alpha,T}^4]} \cdot \sqrt{P(\tau_N < T)}.$$

It follows from Lemma 4.11 that $P(\tau_N < T) \leq N^{-1} \mathbb{E} \left[ \Lambda^{0,T}_{\alpha,B^H} \right]$. Then, by Lemma 4.6 summing up all bounds we obtain

$$E[\|X^\varepsilon - \hat{X}\|_{\alpha,T}^2 1_{\{\tau_N < T\}}] \leq C \sqrt{N^{-1} \mathbb{E} \left[ \Lambda^{0,T}_{\alpha,B^H} \right]}.$$  (4.17)

For the second term, set $A_{N,T} = \{\Lambda^{0,T}_{\alpha,B^H} \leq N\}$, we have

$$E[\|X^\varepsilon - \hat{X}\|_{\alpha,T}^2 1_{A_{N,T}}] \leq C E \left[ \int_0^T S_{-s}(b(X^\varepsilon_{s(\delta)}, \hat{Y}^\varepsilon_s) - \hat{b}(X^\varepsilon_s)) ds \right]_{1_{A_{N,T}}}^2 + C E \left[ \int_0^T S_{-s}(b(X^\varepsilon_{s(\delta)}) - \hat{b}(X^\varepsilon_s)) ds \right]_{1_{A_{N,T}}}^2 + C E \left[ \int_0^T S_{-s}(\hat{b}(X^\varepsilon_s) - \hat{b}(X^\varepsilon_s)) ds \right]_{1_{A_{N,T}}}^2 + C E \left[ \int_0^T S_{-s}(g(X^\varepsilon_s) - g(X^\varepsilon_s)) db^{H_s} \right]_{1_{A_{N,T}}}^2 + C E \left[ \int_0^T S_{-s}(\hat{g}(X^\varepsilon_s) - g(X^\varepsilon_s)) db^{H_s} \right]_{1_{A_{N,T}}}^2.$$

For $J_1$, we only need to replace the estimate for $(b(X^\varepsilon_s, Y^\varepsilon_s) - b(X^\varepsilon_s, \hat{Y}^\varepsilon_s))$ which appeared in the estimation of $I_2$ in (4.15) by the corresponding estimate of $(b(X^\varepsilon_{s(\delta)}, \hat{Y}^\varepsilon_s) - b(X^\varepsilon_{s(\delta)}))$.

$$J_1 \leq C \delta^{-1} \sup_{t \in [0,T]} \int_0^t \sum_{k=0}^{\lfloor \frac{t}{\delta} \rfloor - 1} \left( (s-k\delta)^{-2} \mathbb{E} \left[ \left| \int_{k\delta}^{(k+1)\delta} (b(X^\varepsilon_{s(\delta)}, \hat{Y}^\varepsilon_s) - b(X^\varepsilon_{s(\delta)})) dr \right|^2 \right] \right) ds + C \delta^{2\gamma^2} + C \delta^{-2} \max_{0 \leq k \leq \lfloor \frac{t}{\delta} \rfloor - 1} \mathbb{E} \left[ \left| \int_{s}^{(s+1)\delta} S_{t-s}(b(X^\varepsilon_{s(\delta)}, \hat{Y}^\varepsilon_s) - b(X^\varepsilon_{s(\delta)})) dr \right|^2 \right].$$

(4.19)

By a time shift transformation, we note that it follows from the definition of $\hat{Y}^\varepsilon_s$ that for $s \in [0, \delta]$, we have

$$\hat{Y}^\varepsilon_{s+k\delta} = S_{s+k\delta} \hat{Y}^\varepsilon_{k\delta} + \frac{1}{\delta} \int_0^\delta S_{s+k\delta} F(X^\varepsilon_{k\delta}, Y^\varepsilon_{s+k\delta}) dr + \frac{1}{\sqrt{\delta}} \int_0^\delta S_{s+k\delta} G(X^\varepsilon_{k\delta}, Y^\varepsilon_{s+k\delta}) dW^*_r.$$  (4.20)
where $W^*_r := W_{r+k\delta} - W_{k\delta}$ is the shift of $W_r$, both of which have the same distribution. Let $W$ be a $V$-valued Brownian motion defined on the same stochastic basis and independent of $(B^H, W)$. Construct a process $Y_x^{X_z, Y_{k\delta}}$ by means of

$$
Y_x^{X_z, Y_{k\delta}} = S_x \hat{Y}^x_{k\delta} + \int_0^\tau S_x \hat{Y}^x_{k\delta} F(X^x_{k\delta}, Y^x_{k\delta}, \tilde{Y}^x_{k\delta}) \, dt + \int_0^\tau S_x \hat{Y}^x_{k\delta} G(X^x_{k\delta}, Y^x_{k\delta}, \tilde{Y}^x_{k\delta}) \, d\tilde{W}_r
$$

$$
= S_x \hat{Y}^x_{k\delta} + \frac{1}{\varepsilon} \int_0^\tau S_x \hat{Y}^x_{k\delta} F(X^x_{k\delta}, Y^x_{k\delta}) \, dt + \frac{1}{\sqrt{\varepsilon}} \int_0^\tau S_x \hat{Y}^x_{k\delta} G(X^x_{k\delta}, Y^x_{k\delta}) \, d\tilde{W}_r^*,
$$

(4.21)

where $\tilde{W}_r^* := \sqrt{\varepsilon} \tilde{W}_r$ is the shift of $\tilde{W}_r$ with the same distribution. Because both $W^*$ and $\tilde{W}^*$ are independent of $(X^x_{k\delta}, \hat{Y}^x_{k\delta})$, comparison of (120) and (121) yields

$$
(X^x_{k\delta}, \{\hat{Y}^x_{r+k\delta}, Y^x_{k\delta}\}, \tau \in [0,\delta]) \sim (X^x_{k\delta}, \{Y^x_{r+k\delta}, Y^x_{k\delta}\}, \tau \in [0,\delta]),
$$

where $\sim$ denotes a coincidence in distribution sense.

To proceed, for $t \in [0, T]$, we have

$$
E \left[ \int_{k\delta}^{(k+1)\delta} S_{t-r}(b(X^x_{k\delta}, \hat{Y}^x_{r}) - \hat{b}(X^x_{k\delta})) \, d\tilde{W}_r \right]^2 \leq C\varepsilon^2 \int_0^\tau \int_0^{\frac{\tau}{\varepsilon}} J_k(s, \tau) \, ds \, d\tau,
$$

where

$$
J_k(s, \tau) = E \left[ (S_{t-r}(b(X^x_{k\delta}, \hat{Y}^x_{r}) - \hat{b}(X^x_{k\delta})), S_{t-\delta-s-r}(b(X^x_{k\delta}, \hat{Y}^x_{r+\delta}) - \hat{b}(X^x_{k\delta}))) \right]
$$

$$
= E \left[ (S_{t-k\delta-s-r}(b(X^x_{k\delta}, Y^x_{k\delta}, \tilde{Y}^x_{k\delta}) - b(X^x_{k\delta})), S_{t-k\delta-r-s}(b(X^x_{k\delta}, Y^x_{k\delta}, \tilde{Y}^x_{k\delta}) - b(X^x_{k\delta}))) \right].
$$

We now present a key lemma for an estimate of $J_k(s, \tau)$ which will be proved in Appendix B.

**Lemma 4.12.** For any $k$, we have

$$
J_k(s, \tau) \leq C e^{-\frac{\eta}{\delta}(s-\tau)} E[1 + |X^x_{k\delta}|^2 + |\hat{Y}^x_{k\delta}|^2],
$$

where $\eta$ is defined in condition (A5) and $C > 0$ is a constant independent of $\varepsilon, \delta, k, s, \tau$.

According to Lemma 4.12 and by choosing $\delta = \delta(\varepsilon)$ such that $\delta$ is sufficiently large, we have

$$
E \left[ \int_{k\delta}^{(k+1)\delta} S_{t-r}(b(X^x_{k\delta}, \hat{Y}^x_{r}) - \hat{b}(X^x_{k\delta})) \, d\tilde{W}_r \right]^2 \leq C\varepsilon^2 \int_0^\tau \int_0^{\frac{\tau}{\varepsilon}} J_k(s, \tau) \, ds \, d\tau
$$

$$
\leq C\varepsilon^2 \int_0^\tau \int_0^{\frac{\tau}{\varepsilon}} e^{-\frac{\eta}{\delta}(s-\tau)} \, ds \, d\tau
$$

$$
\leq C\varepsilon^2 \left( \frac{2}{\eta \delta} \frac{\delta^2}{\varepsilon} \right).
$$

Hence, we get

$$
J_1 \leq C\delta^{2\beta'} + C\delta^{-2\beta'} \left( \frac{2}{\eta \varepsilon} - \frac{4}{\eta^2} + e^{-\frac{\delta}{4}} \right) \leq C(\varepsilon \delta^{-1} + \delta^{2\beta'}).
$$

(4.22)

By similar calculations as for the terms $M_{21}, M_{22}$ in Appendix B, we have

$$
J_2 + J_3 + J_4 \leq C \int_0^T E[|X_r^x - X^x_{r(\delta)}|^2 1_{A_{N,r}}] \, dr
$$
Using similar calculations as for the terms $N_{33}, N_{34}$ and $N_{36}$ in Appendix B, we have

\[ J_5 \leq C_N \left[ \sup_{t \in [0,T]} \left| \int_0^t S_{t-r}(g(X_r^\varepsilon) - g(\hat{X}_r^\varepsilon))dB_r^H \right|^2 \right] 1_{A_{N,T}} \]
\[ + C_N \mathbb{E} \left[ \int_0^T \left( \int_0^t (t-s)^{-\alpha} \left| \int_s^t S_{t-r}(g(X_r^\varepsilon) - g(\hat{X}_r^\varepsilon))dB_r^H \right| ds \right)^2 dt 1_{A_{N,T}} \right] \]
\[ + C_N \mathbb{E} \left[ \int_0^T \left( \int_0^t (t-s)^{-\alpha} \left| \int_s^t (S_{t-r} - S_{s-r})(g(X_r^\varepsilon) - g(\hat{X}_r^\varepsilon))dB_r^H \right| ds \right)^2 dt 1_{A_{N,T}} \right] \]
\[ =: J_{51} + J_{52} + J_{53}, \]

where

\[ J_{51} \leq C_N \int_0^T \mathbb{E} \left[ \|X_r^\varepsilon - \hat{X}_r^\varepsilon\|_{a,r}^2 1_{A_{N,T}} \right] dr \]
\[ + C_N \mathbb{E} \left[ \int_0^T \left( \int_0^t \frac{|X_r^\varepsilon - \hat{X}_r^\varepsilon - X_s^\varepsilon + \hat{X}_s^\varepsilon|}{(t-s)^{1+\alpha}} ds \right)^2 dt 1_{A_{N,T}} \right] \]
\[ \leq C_N \int_0^T \mathbb{E} \left[ \|X_r^\varepsilon - \hat{X}_r^\varepsilon\|_{a,r}^2 1_{A_{N,T}} \right] dr + C_N \mathbb{E} \left[ \|X_r^\varepsilon - \hat{X}_r^\varepsilon\|_{a,T}^2 1_{A_{N,T}} \right], \]

and

\[ J_{52} + J_{53} \leq C_N \int_0^T \mathbb{E} \left[ \|X_r^\varepsilon - \hat{X}_r^\varepsilon\|_{a,r}^2 1_{A_{N,T}} \right] dr. \]

This yields that

\[ J_5 \leq C_N (\varepsilon^{1-2\beta} C_N^{2\beta} + \delta^{2\beta}). \]

For $J_6$, using the same step as in the proof of $J_5$, we have

\[ J_6 \leq C_N \left[ \sup_{t \in [0,T]} \left| \int_0^t S_{t-r}(g(X_r^\varepsilon) - g(\hat{X}_r^\varepsilon))dB_r^H \right|^2 \right] 1_{A_{N,T}} \]
\[ + C_N \mathbb{E} \left[ \int_0^T \left( \int_0^t (t-s)^{-\alpha} \left| \int_s^t S_{t-r}(g(X_r^\varepsilon) - g(\hat{X}_r^\varepsilon))dB_r^H \right| ds \right)^2 dt 1_{A_{N,T}} \right] \]
\[ + C_N \mathbb{E} \left[ \int_0^T \left( \int_0^t (t-s)^{-\alpha} \left| \int_s^t (S_{t-r} - S_{s-r})(g(X_r^\varepsilon) - g(\hat{X}_r^\varepsilon))dB_r^H \right| ds \right)^2 dt 1_{A_{N,T}} \right] \]
\[ =: J_{61} + J_{62} + J_{63}, \]

where

\[ J_{61} \leq C_N \int_0^T \mathbb{E} \left[ \|X_r^\varepsilon - \hat{X}_r^\varepsilon\|_{a,r}^2 1_{A_{N,T}} \right] dr + C_N \mathbb{E} \left[ \int_0^T \left( \int_0^t \frac{|X_r^\varepsilon - \hat{X}_r^\varepsilon - X_s^\varepsilon + \hat{X}_s^\varepsilon|}{(t-s)^{1+\alpha}} ds \right)^2 dt 1_{A_{N,T}} \right], \]

and

\[ J_{62} + J_{63} \leq C_N \int_0^T \mathbb{E} \left[ \|X_r^\varepsilon - \hat{X}_r^\varepsilon\|_{a,r}^2 1_{A_{N,T}} \right] dr. \]
For $J_{61}$, from Eq. (4.13) and Eq. (4.7), using the fact that

$$
X^\varepsilon_t - X^\varepsilon_t = \int_0^t S_{t-r}(b(X^\varepsilon_{r(\varepsilon)}), \hat{Y}^\varepsilon_r) - b(\hat{X}_r)dr + \int_0^t S_{t-r}(g(X^\varepsilon_r) - \hat{g}(\hat{X}_r))dB^H_r,
$$

we have

$$
J_{61} \leq CN \int_0^T \mathbb{E}[\|\hat{X}^\varepsilon - \bar{X}\|_{\alpha,r}^2 1_{A_{N,r}}]dr
+ CN \mathbb{E} \left[ \int_0^T \left( \int_0^t (t-s)^{-1-\alpha} \left| \int_0^s S_{t-r}(b(X^\varepsilon_{r(\varepsilon)}), \hat{Y}^\varepsilon_r) - b(\hat{X}_r)dr \right|^2 ds \right)^{1/2} d\mathbf{1}_{A_{N,r}} \right]
+ CN \mathbb{E} \left[ \int_0^T \left( \int_0^t (t-s)^{-1-\alpha} \left| \int_0^s S_{t-r}(g(X^\varepsilon_r) - \hat{g}(\hat{X}_r))dr \right|^2 ds \right)^{1/2} d\mathbf{1}_{A_{N,r}} \right]
+ CN \mathbb{E} \left[ \int_0^T \left( \int_0^t (t-s)^{-1-\alpha} \left| \int_0^s S_{t-r}(g(X^\varepsilon_r) - \hat{g}(\hat{X}_r))dr \right|^2 ds \right)^{1/2} d\mathbf{1}_{A_{N,r}} \right]
\leq CN \int_0^T \mathbb{E}[\|\hat{X}^\varepsilon - \bar{X}\|_{\alpha,r}^2 1_{A_{N,r}}]dr + \sum_{i=1}^{4} J_i + J_{52} + J_{53} + J_{62} + J_{63}
\leq CN \int_0^T \mathbb{E}[\|\hat{X}^\varepsilon - \bar{X}\|_{\alpha,r}^2 1_{A_{N,r}}]dr + CN \int_0^T \mathbb{E}[\|\hat{X}^\varepsilon - \bar{X}\|_{\alpha,r}^2 1_{A_{N,r}}]dr
+ CN \varepsilon^{-1} \delta^{2\beta} e^{C_{2\beta}^2} + CN (\delta^{2\beta'} + \delta^{2\beta}).
$$

This yields that

$$
J_6 \leq CN \int_0^T \mathbb{E}[\|\hat{X}^\varepsilon - \bar{X}\|_{\alpha,r}^2 1_{A_{N,r}}]dr + CN (\varepsilon \delta^{-1} \delta^{2\beta} e^{C_{2\beta}^2} + \delta^{2\beta'}).\n$$

Putting above results together, by Gronwall’s lemma, we have

$$
\mathbb{E}[\|\hat{X}^\varepsilon - \bar{X}\|_{\alpha,r}^2 1_{A_{N,r}}] \leq CN (\delta^{-1} \delta^{2\beta} e^{C_{2\beta}^2} + \delta^{2\beta'}).\n$$

Finally, we have

$$
\mathbb{E}[\|\hat{X}^\varepsilon - \bar{X}\|_{\alpha,r}^2] \leq CN (\varepsilon \delta^{-1} \delta^{2\beta} e^{C_{2\beta}^2} + \delta^{2\beta'}) + C N^{-1} \mathbb{E}[A_{0,T}].\n$$

(4.23)

**Step 3**: Putting (4.16) and (4.23) together, then, taking $\delta = \varepsilon \sqrt{-\ln \varepsilon}$, we have

$$
\lim_{\varepsilon \to 0} \mathbb{E}[\|X^\varepsilon - \bar{X}\|_{\alpha,T}^2] = 0.
$$

This completes the proof of Theorem 4.4. □

**Appendix A.**

To make our paper self-contained, we recall the ergodicity for fast motion which was introduced by Fu and Liu [10]. Consider the problem associated to the fast motion with frozen show component Eq. (4.15). In this section, we replace the initial value $y$ defined in Eq. (4.5) by $z$. 

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Then, for any fixed $x \in V$ and initial value $z \in V$, Eq. (4.5) has a unique strong solution (also a mild solution) which will be denoted by $(Y_{t}^{x,z})_{t\geq 0}$. By energy equality (10) and conditions (A4)-(A5) and Poincaré inequality, one gets

$$
\mathbb{E}[|Y_{t}^{x,z}|^{2}] \leq |z|^{2} - (2\lambda_{1} + 2\beta_{1} - C_{3})\mathbb{E}\left[\int_{0}^{t}|Y_{s}^{x,z}|^{2}ds\right] + C_{2}(1 + |x|^{2})t.
$$

By Gronwall’s inequality, we have

$$
\mathbb{E}[|Y_{t}^{x,z}|^{2}] \leq C(1 + |z|^{2}e^{-\lambda_{1} + 2\beta_{1} - C_{3}t}),
$$

where $2\lambda_{1} + 2\beta_{1} - C_{3} > 0$ owing to condition (A5) and $C > 0$ is a constant.

Next, let $(Y_{t}^{x,z})_{t\geq 0}$ be a solution of Eq. (4.5) with the initial value $Y_{0}^{x} = z'$, by the Poincaré inequality, conditions (A4) and (A5), we obtain

$$
\mathbb{E}[|Y_{t}^{x,z} - Y_{t}^{x',z'}|^{2}] = |z - z'|^{2} + 2\mathbb{E}\left[\int_{0}^{t}\langle A(Y_{s}^{x,z} - Y_{s}^{x',z'}), Y_{s}^{x,z} - Y_{s}^{x',z'}\rangle ds\right]
$$

$$
+ 2\mathbb{E}\left[\int_{0}^{t}(F(x, Y_{s}^{x,z}) - F(x, Y_{s}^{x',z'}), Y_{s}^{x,z} - Y_{s}^{x',z'}) ds\right]
$$

$$
+ \int_{0}^{t}\mathbb{E}[[G(x, Y_{s}^{x,z}) - G(x, Y_{s}^{x',z'}))]^{2}_{2}(V)ds
$$

$$
\leq |z - z'|^{2} - (2\lambda_{1} - 2\beta_{3} - C_{2})\int_{0}^{t}\mathbb{E}[|Y_{s}^{x,z} - Y_{s}^{x',z'}|^{2}]ds.
$$

Therefore, by Gronwall’s inequality [13, pp. 584], we have

$$
\mathbb{E}[|Y_{t}^{x,z} - Y_{t}^{x',z'}|^{2}] \leq |z - z'|^{2}e^{-\eta t}, \quad (A.1)
$$

where $\eta = 2\lambda_{1} - 2\beta_{3} - C_{2} > 0$.

For any $x \in V$, denote by $(P_{t}^{x})_{t\geq 0}$ the Markov semigroup associated to Eq. (4.5) defined by

$$
P_{t}^{x}\Psi(z) = \mathbb{E}[\Psi(Y_{t}^{x,z})], \quad t \geq 0, \quad z \in V,
$$

for any $\Psi \in \mathcal{B}_{b}(V)$, the space of bounded functions on $V$. We recall that a probability $\mu^{x}$ on $V$ is called an invariant measure for $(P_{t}^{x})_{t\geq 0}$ if

$$
\int_{V}P_{t}^{x}\Psi d\mu^{x} = \int_{V}\Psi d\mu^{x}, \quad t \geq 0,
$$

for any bounded function $\Psi \in \mathcal{B}_{b}(V)$. As in [3], it is possible to show the existence of the unique invariant measure $\mu^{x}$ for the semigroup $(P_{t}^{x})_{t\geq 0}$ which satisfies $\int_{V}(|z|\mu^{x}(dz)) \leq (1 + |x|)$.

Furthermore, according to Lipschitz assumption on $b$ and Eq. (A.1), we have

$$
|\mathbb{E}[b(x, Y_{t}^{x,z})] - \int_{V}b(x, z)\mu^{x}(dz) | = \left| \int_{V}\mathbb{E}[b(x, Y_{t}^{x,z}) - b(x, Y_{t}^{x,z})]\mu^{x}(dz) \right|
$$

$$
\leq C\int_{V}\mathbb{E}[|Y_{t}^{x,z} - Y_{t}^{x',z'}|]\mu^{x}(dz)
$$

$$
\leq Ce^{-\frac{1}{2}\eta t}\int_{V}|y - z|\mu^{x}(dz)
$$

$$
\leq Ce^{-\frac{1}{2}\eta t}(1 + |x| + |y|), \quad (A.2)
$$

where $C > 0$ is a constant.

Suppose that (A1)-(A5) hold. For any given value $x_{1}, x_{2} \in V, z \in V$, we have

$$
\mathbb{E}[|Y_{t}^{x_{1},z} - Y_{t}^{x_{2},z}|^{2}] = 2\mathbb{E}\left[\int_{0}^{t}\langle A(Y_{s}^{x_{1},z} - Y_{s}^{x_{2},z}), Y_{s}^{x_{1},z} - Y_{s}^{x_{2},z}\rangle ds\right]
$$

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\[
+ 2\mathbb{E} \left[ \int_0^t (F(x_1, Y_{s_1}^{x_1}) - F(x_2, Y_{s_2}^{x_2}), Y_{s_1}^{x_1} - Y_{s_2}^{x_2}) ds \right] \\
+ \int_0^t \mathbb{E}||G(x_1, Y_{s_1}^{x_1}) - G(x_2, Y_{s_2}^{x_2})^2||_{L_2(V)} ds \\
\leq -(2\lambda_1 - 2\beta_3 - C_2) \int_0^t \mathbb{E}||Y_{s_1}^{x_1} - Y_{s_2}^{x_2}||^2 ds + C|x_1 - x_2|^2.
\]

Therefore, by Gronwall’s inequality \[15\), pp. 584], we have
\[
\mathbb{E}||Y_t^{x_1} - Y_t^{x_2}||^2 \leq C|x_1 - x_2|^2 e^{-\eta t}.
\]

**Appendix B.**

In this section, we present the proofs of Lemma 3.4, Lemma 3.7, Lemma 3.8, Lemma 3.9 and Lemma 4.2 that have been deferred from Section 3 and Section 4.

**The Proof of Lemma 3.4** Note that $0 \leq s < t \leq T$, by Remark 2.1 and Lemma 2.2 and Fubini’s theorem, it follows

\[
\mathcal{K}_1(s, t) \leq \Lambda^{0,t}_{\alpha, B} \sup_i \int_0^t \left[ \frac{||S_{s-r}||}{(r-s)^\alpha} \left( \int_0^r \frac{||S_{t-r} - S_{t-q}||}{(r-q)^{1+\alpha}} dq \right) dr \\
+ \int_0^r \frac{||S_{t-r} - S_{t-q}||}{(r-q)^{1+\alpha}} dq \right] dr \\
\leq C \Lambda^{0,t}_{\alpha, B} \int_0^t \left( \frac{1}{(r-s)^\alpha} \int_0^r \frac{||S_{t-r} - S_{t-q}||}{(r-q)^{1+\alpha}} dq \right) dr \\
\leq C \Lambda^{0,t}_{\alpha, B} \int_0^t \left( \frac{1}{(r-s)^\alpha} \int_0^r \frac{||S_{t-r} - S_{t-q}||}{(r-q)^{1+\alpha}} dq \right) dr.
\]

For $\mathcal{K}_2(0, s)$, taking $\alpha < \alpha' < 1 - \beta$, we obtain

\[
\mathcal{K}_2(0, s) \leq \Lambda^{0,t}_{\alpha, B} \sup_i \int_0^s \left[ \frac{||S_{s-r} - S_{s-r}||}{(r-s)^\alpha} \left( \int_0^r \frac{||S_{t-r} - S_{t-q}||}{(r-q)^{1+\alpha}} dq \right) dr \\
+ \int_0^r \frac{||S_{t-r} - S_{t-q}||}{(r-q)^{1+\alpha}} dq \right] dr \\
\leq C \Lambda^{0,t}_{\alpha, B} (t-s)^\beta \int_0^s \left( \frac{1}{(s-r)^{3+\alpha}} \int_0^r \frac{||S_{t-r} - S_{t-q}||}{(r-q)^{1+\alpha}} dq \right) dr \\
\leq C \Lambda^{0,t}_{\alpha, B} (t-s)^\beta \int_0^s \left( \frac{1}{(s-r)^{3+\alpha}} \int_0^r \frac{||S_{t-r} - S_{t-q}||}{(r-q)^{1+\alpha}} dq \right) dr.
\]

For $\mathcal{K}_3(s, t)$ and $\mathcal{K}_4(0, s)$, we only need to replace the estimate for $g(u_r)$ which appeared in the above proofs of $\mathcal{K}_1(s, t)$ and $\mathcal{K}_2(0, s)$ by the corresponding estimate of $\left(g(u_r) - g(v_r)\right)$. By (3.3), we have

\[
\mathcal{K}_3(s, t) \leq \Lambda^{0,t}_{\alpha, B} \sup_i \int_0^t \left( \frac{||S_{s-r}||}{(r-s)^\alpha} \left( \int_0^r \frac{||S_{t-r} - S_{t-q}||}{(r-q)^{1+\alpha}} dq \right) dr \\
\leq C \Lambda^{0,t}_{\alpha, B} \int_0^t \left( \frac{1}{(r-s)^\alpha} \int_0^r \frac{||S_{t-r} - S_{t-q}||}{(r-q)^{1+\alpha}} dq \right) dr.
\]
This completes the proof of Lemma 3.4.

The Proof of Lemma 3.4: Denote

\[ A_{s,t}(\beta^{t,H}_N) := \left( \frac{\|\beta^{H,N}_1(t) - \beta^{H,N}_1(s)\|}{(t-s)^{1-\alpha}} \right) + \int_s^t \frac{\|\beta^{H,N}_1(r) - \beta^{H,N}_1(s)\|}{(r-s)^{2-\alpha}} dr. \]

Then, we have

\[ A_{s,t}(\beta^{H,N}_1) \leq \left( \frac{\|\beta^{H,N}_1(t) - \beta^{H,N}_1(s)\|}{(t-s)^{1-\alpha}} \right) + \int_s^t \frac{\|\beta^{H,N}_1(r) - \beta^{H,N}_1(s)\|}{(r-s)^{2-\alpha}} dr 1_{\{s \leq t \leq \tau_N\}} \]

\[ + \left( \frac{\|\beta^{H,N}_1(t) - \beta^{H,N}_1(s)\|}{(t-s)^{1-\alpha}} \right) + \int_s^t \frac{\|\beta^{H,N}_1(r) - \beta^{H,N}_1(s)\|}{(r-s)^{2-\alpha}} dr 1_{\{t \leq s \leq t\}} \]

\[ + \left( \frac{\|\beta^{H,N}_1(t) - \beta^{H,N}_1(s)\|}{(t-s)^{1-\alpha}} \right) + \int_s^t \frac{\|\beta^{H,N}_1(r) - \beta^{H,N}_1(s)\|}{(r-s)^{2-\alpha}} dr 1_{\{s \leq t \leq \tau_N\}} \]

\[ \leq C \left( \frac{\|\beta^{H,N}_1(t) - \beta^{H,N}_1(s)\|}{(t-s)^{1-\alpha}} \right) + \int_s^t \frac{\|\beta^{H,N}_1(r) - \beta^{H,N}_1(s)\|}{(r-s)^{2-\alpha}} dr 1_{\{s \leq t \leq \tau_N\}} \]

\[ + \left( \frac{\|\beta^{H,N}_1(t) - \beta^{H,N}_1(s)\|}{(t-s)^{1-\alpha}} \right) + \int_s^t \frac{\|\beta^{H,N}_1(r) - \beta^{H,N}_1(s)\|}{(r-s)^{2-\alpha}} dr 1_{\{t \leq s \leq t\}} \]

\[ + \left( \frac{\|\beta^{H,N}_1(t) - \beta^{H,N}_1(s)\|}{(t-s)^{1-\alpha}} \right) + \int_s^t \frac{\|\beta^{H,N}_1(r) - \beta^{H,N}_1(s)\|}{(r-s)^{2-\alpha}} dr 1_{\{s \leq t \leq \tau_N\}} \]

\[ \leq C \left( \frac{\|\beta^{H,N}_1(t) - \beta^{H,N}_1(s)\|}{(t-s)^{1-\alpha}} \right) + \int_s^t \frac{\|\beta^{H,N}_1(r) - \beta^{H,N}_1(s)\|}{(r-s)^{2-\alpha}} dr 1_{\{s \leq t \leq \tau_N\}} \]

\[ + C \left( \frac{\|\beta^{H,N}_1(t) - \beta^{H,N}_1(s)\|}{(t-s)^{1-\alpha}} \right) + \int_s^t \frac{\|\beta^{H,N}_1(r) - \beta^{H,N}_1(s)\|}{(r-s)^{2-\alpha}} dr 1_{\{s \leq t \leq \tau_N\}}, \quad (B.1) \]
The Proof of Lemma 3.8

We start with

$$\|\beta^{H,N}_t\|_{\alpha,0,T} = \sup_{0 \leq s \leq t \leq T} A_{s,t}(\beta^{H,N}_t) \leq C\|\beta^{H}_t\|_{\alpha,0,T},$$

almost surely. This completes the proof of Lemma 3.8.

**The Proof of Lemma 3.8** We start with

$$E[\|u^{N,n}\|_{\alpha,T}^2] \leq C E[\|S u_0\|_{\alpha,T}^2] + CE\left[\left\| \int_0^T \frac{|S_t - S_s| u_0|}{(t-s)^{1+\alpha}} ds \right\|_{\alpha,T}^2 \right] + CE\left[\left\| \int_0^T S_{-\nu} \sigma(u^{N,n}_s) dW_s \right\|_{\alpha,T}^2 \right] + C E\left[\left\| \int_0^T S_{-s} g(u^{N,n}_s) dB^{H,N,n}_s \right\|_{\alpha,T}^2 \right] =: M_1 + M_2 + M_3 + M_4,$$

where $B^{H,n} := \sum_{i=1}^{\infty} \sqrt{\lambda_i} \partial_i^H\beta^{H,n}.$

Since $\alpha < \beta$, $u_0 \in V_\beta$, it is easy to obtain

$$M_1 \leq C E\left[\sup_{t \in [0,T]} |S_t u_0|^2 \right] + CE\left[\int_0^T \left( \int_0^t \frac{|S_t - S_s| u_0|}{(t-s)^{1+\alpha}} ds \right)^2 dt \right] \leq C + CE\left[\int_0^T \left( \int_0^t |u_0| \beta(t-s)^{\beta-1-\alpha} ds \right)^2 dt \right] \leq C.$$

For $M_2$, we have

$$M_2 \leq C E\left[\sup_{t \in [0,T]} \left\| \int_0^t S_{t-s} f(u^{N,n}_s) ds \right\|^2 \right] + CE\left[\int_0^T \left( \int_0^t \left| \int_0^t S_{t-s} f(u^{N,n}_s) ds \right| (t-s)^{1+\alpha} ds \right)^2 dt \right] =: M_{21} + M_{22}.$$

By Hölder’s inequality and the growth condition, we get

$$M_{21} \leq C E\left[\int_0^T \left\| S_{t-s} \right\|^2 |f(u^{N,n}_s)|^2 ds \right] \leq C \int_0^T (1 + E[|u^{N,n}_s|^2]) ds.$$

Next, for $M_{22}$, by Fubini’s theorem, Lemma 2.2 and Hölder’s inequality, we get

$$M_{22} \leq C E\left[\int_0^T \left( \int_0^t \left| \int_0^t S_{t-s} f(u^{N,n}_s) ds \right| \right)_{(t-s)^{\frac{1}{1+\alpha}}}^2 dt \right] + CE\left[\int_0^T \left( \int_0^t \left| \int_0^t (S_{t-s} - S_{s-t}) f(u^{N,n}_s) ds \right| (t-s)^{1+\alpha} ds \right)^2 dt \right] \leq C E\left[\int_0^T \left( \int_0^t (t-s)^{-1-\alpha} ds \right)^2 \right] + CE\left[\int_0^T \left( \int_r^t (t-s)^{-1-\alpha} ds \right)^2 dt \right] \leq C E\left[\int_0^T (t-r)^{-\alpha} |f(u^{N,n}_r)|^2 dr \right].$$
Applying again Hölder’s inequality, Burkholder-Davis-Gundy inequality, Fubini’s theorem, Lemma and then, by Burkholder-Davis-Gundy inequality,

\[ M \leq 4 \]

For \( M_3 \), we get

\[
M_3 \leq C \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t S_{t-s} \sigma(u_s^{N,n}) dW_s \right|^2 \right] + \mathbb{E} \left[ \int_0^T \left( \int_0^t \left| \int_0^s S_{t-r} \sigma(u_r^{N,n}) dW_r - \int_0^s S_{t-r} \sigma(u_r^{N,n}) dW_r \left( t-s \right)^{1+\alpha} \right|^2 ds \right) dt \right] =: M_{31} + M_{32},
\]

and then, by Burkholder-Davis-Gundy inequality,

\[
M_{31} \leq C \mathbb{E} \left[ \int_0^T \|S_{t-s}\|^2 \|\sigma(u_s^{N,n})\|^2_{L^2(V)} ds \right] \leq C \int_0^T (1 + \mathbb{E}[|u_s^{N,n}|^2]) ds.
\]

Applying again Hölder’s inequality, Burkholder-Davis-Gundy inequality, Fubini’s theorem, Lemma and Lemma we have

\[
M_{32} \leq C \mathbb{E} \left[ \int_0^T \left( \int_0^t \left| \int_0^s S_{t-r} \sigma(u_r^{N,n}) dW_r \right| \left( t-s \right)^{1+\alpha} ds \right) dt \right] + \mathbb{E} \left[ \int_0^T \left( \int_0^t \left| \int_0^s (S_{t-r} - S_{s-r}) \sigma(u_r^{N,n}) dW_r \right| \left( t-s \right)^{1+\alpha} ds \right) dt \right] \leq C \int_0^T \int_0^t \left( t-s \right)^{-\frac{1}{2}} \left| \int_0^s S_{t-r} \sigma(u_r^{N,n}) dW_r \right| ddsdt \]

\[
+ \mathbb{E} \left[ \int_0^T \left( \int_0^t \left| \int_0^s (S_{t-r} - S_{s-r}) \sigma(u_r^{N,n}) dW_r \right| \left( t-s \right)^{1+\alpha} ds \right) dt \right] \leq C \int_0^T \int_0^t \left( t-s \right)^{-\frac{1}{2}} \left| \int_0^s S_{t-r} \sigma(u_r^{N,n}) \right| ddsdt \]

\[
+ \mathbb{E} \left[ \int_0^T \left( \int_0^t \left| \int_0^s (S_{t-r} - S_{s-r}) \sigma(u_r^{N,n}) dW_r \right| \left( t-s \right)^{1+\alpha} ds \right) dt \right] \leq C \int_0^T \int_0^t \left( t-s \right)^{-\frac{1}{2}} \left| \int_0^s S_{t-r} \sigma(u_r^{N,n}) \right| ddsdt
\]

To proceed, for \( M_4 \), we obtain

\[
M_4 \leq C \mathbb{E} \left[ \int_0^T \left( \int_0^t \left| \int_0^s S_{t-r} g(u_r^{N,n}) dB^{H,N,n}_r \right|^2 ds \right) dt \right] + \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t S_{t-r} g(u_r^{N,n}) dB^{H,N,n}_r \right|^2 \right] =: M_{41} + M_{42}.
\]

For \( M_{41} \), by Lemma and Hölder’s inequality, taking \( \alpha < \alpha' < 1 - \beta \), we have

\[
M_{41} \leq C \mathbb{E} \left[ \int_0^T \left( \int_0^t \left( t-s \right)^{-1-\alpha} \left| \int_0^s S_{t-r} g(u_r^{N,n}) dB^{H,N,n}_r \right| ds \right)^2 dt \right]
\]

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This completes the proof of Lemma 3.8.

Thus, we have

Finally, by Gronwall’s lemma, we have

\[ \mathbb{E}[\|u^{N,n}\|^2_{\alpha,T}] \leq C_N. \]

This completes the proof of Lemma 5.8. □
The Proof of Lemma 3.9: From (3.15), we have
\[
\begin{align*}
\mathbb{E}\left[\|u^{N,n} - u^{N,m}\|_{\alpha,T}^21_{D_{T,D}^N}\right] \\
\leq C\mathbb{E}\left[\left\|\int_0^T S_{-t}(f(u^{N,n}) - f(u^{N,m}))dt\right\|_{\alpha,T}^21_{D_{T,D}^N}\right] \\
+ C\mathbb{E}\left[\left\|\int_0^T S_{-t}(\sigma(u^{N,n}) - \sigma(u^{N,m}))dW_t\right\|_{\alpha,T}^21_{D_{T,D}^N}\right] \\
+ C\mathbb{E}\left[\left\|\int_0^T S_{-t}g(u^{N,n})dB_t^{H,N,n} - \int_0^T S_{-t}g(u^{N,m})dB_t^{H,N,m}\right\|_{\alpha,T}^21_{D_{T,D}^N}\right] \\
=: N_1 + N_2 + N_3.
\end{align*}
\]

The terms \(N_1\) and \(N_2\) can be estimated in the same way as the terms \(M_2\) and \(M_3\) in the proof of Lemma 3.8 using Lipschitz condition instead of the growth condition. This leads to the bounds
\[
N_1 + N_2 \leq C\int_0^T \mathbb{E}\left[\|u^{N,n} - u^{N,m}\|_{\alpha,T}^21_{D_{T,D}^N}\right]dt.
\]

For \(N_3\), we have
\[
\begin{align*}
N_3 &\leq C\mathbb{E}\left[\int_0^T \left(\int_0^t \int_0^s S_{t-s}g(u^{N,n})dB_t^{H,N,n} - \int_0^t S_{t-s}g(u^{N,m})dB_t^{H,N,m}\right)ds\right]^{1+\alpha}dt \\
&+ C\mathbb{E}\left[\int_0^T \left(\int_0^t \int_0^s S_{t-s}g(u^{N,n})dB_t^{H,N,n} - \int_0^t S_{t-s}g(u^{N,m})dB_t^{H,N,m}\right)ds\right]^{1+\alpha}dt \\
&+ C\mathbb{E}\left[\int_0^T \left(\int_0^t \int_0^s (S_{t-s}g(u^{N,n}) - g(u^{N,m}))dB_t^{H,N,n}\right)ds\right]^{1+\alpha}dt \\
&+ C\mathbb{E}\left[\int_0^T \left(\int_0^t \int_0^s (S_{t-s}g(u^{N,n}) - g(u^{N,m}))dB_t^{H,N,n}\right)ds\right]^{1+\alpha}dt \\
&+ C\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t S_{t-s}g(u^{N,n})dB_t^{H,N,n} - \int_0^t S_{t-s}g(u^{N,m})dB_t^{H,N,m}\right|\right]^{1+\alpha}dt \\
&+ C\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t S_{t-s}g(u^{N,n})dB_t^{H,N,n} - \int_0^t S_{t-s}g(u^{N,m})dB_t^{H,N,m}\right|\right]^{1+\alpha}dt \\
=: \sum_{i=1}^6 N_{3i}.
\end{align*}
\]

In the same way as for the term \(M_4\) in the proof of Lemma 3.8, we have
\[
\begin{align*}
N_{31} + N_{32} &\leq C\mathbb{E}\int_0^T \left(\int_0^t \left(1 + \|u^{N,n}\|_{\alpha,T}^2\right)dt\right)^2dt \\
&+ C\mathbb{E}\int_0^T \left(\int_0^t \frac{(t-r)^\alpha}{(t-r)^\alpha}(1 + \|u^{N,n}\|_{\alpha,T})dr\right)^2dt \\
&\leq C\mathbb{E}\int_0^T \left(\int_0^t \left(1 + \|u^{N,n}\|_{\alpha,T}^2\right)dt\right)^2dt.
\end{align*}
\]
\[ \leq C_{N,R,T,\alpha,B,N,n,m} \mathbb{E}\left[ \left( A_{\alpha,B,H,n,m}^{0,T} \right)^2 \right], \]

where \( A_{\alpha,B,H,n,m}^{0,T} := \sum_{k=1}^{\infty} \sqrt{\mathbb{X}_k} \beta_{H,n,m}^k - \beta_{H,n,m}^k \|_{\alpha,0,T}. \)

Next, for \( N_{33} \) and \( N_{34} \), by Lemma 3.4, we have

\[ N_{33} + N_{34} \leq C N_{R} \mathbb{E}\left[ \left\| u_{r}^{N,n} - u_{r}^{N,m} \right\|^2_{\alpha,B,D_{T,n}} \right]. \]

Next, for \( N_{35} \), we have

\[ N_{35} \leq C N_{E} \mathbb{E}\left[ \sup_{t \in [0,T]} \left( A_{\alpha,B,H,n,m}^{0,T} \right) \right] \left( \int_{0}^{t} (r^{-\alpha} + (t-r)^{-\alpha})(1 + |u_{r}^{N,n}|) dr \right)^2 \]

\[ \leq C N_{E} \mathbb{E}\left[ \left( A_{\alpha,B,H,n,m}^{0,T} \right)^2 \right]. \]

In the same way as for the term \( M_{42} \) in the proof of Lemma 3.8, we have

\[ N_{36} \leq C N_{E} \mathbb{E}\left[ \sup_{t \in [0,T]} \left( \int_{0}^{t} (r^{-\alpha} + (t-r)^{-\alpha})|u_{r}^{N,n} - u_{r}^{N,m}| dr \right)^2 \right] \]

\[ \leq C N_{E} \mathbb{E}\left[ \left( A_{\alpha,B,H,n,m}^{0,T} \right)^2 \right]. \]
Finally, by Gronwall’s lemma, we have

\[ E\left[ \|U^{N,n} - u^{N,m}\|_{\alpha,T}^2 1_{D^{N,R}_T} \right] \leq C_{N,R} E\left[ \left( \sum_{i=1}^{\infty} \sqrt{\lambda_i} \|\hat{\beta}_i H^{N,n} - \beta_i H^{N,m}\|_{\alpha,0,T} \right)^2 \right]. \]

This completes the proof of Lemma 3.9. □

**The Proof of Lemma 4.12** Let \( X^\\xi_{k,\delta} \), \( Y^\\xi_{k,\delta} \), \( X^\\epsilon_{k,\delta} \), \( Y^\\epsilon_{k,\delta} \) and \( \hat{W} \) be as Eq. (4.21) and let \( \mathcal{Q}^y \) denote the probability law of the diffusion process \( (Y^x_t)_{t \geq 0} \) which is governed by the SPDE

\[
dY^x_t = (AY^x_t + F(x, Y^x_t))dt + G(x, Y^x_t)d\hat{W}_t,
\]

with initial value \( Y^x_0 = y \) and we denote that solution by \( (Y^x_t)_{t \geq 0} \) (Without ambiguity, one can still use the notation \( Y \) to denote the solution of Eq. (B.2)). The expectation with respect to \( \mathcal{Q}^y \) is denoted by \( E^y \). Hence, we have

\[ E^y[\Psi(Y^x_t)] = E[\Psi(Y^x_t)], \quad t \geq 0, y \in V, \]

for all bounded function \( \Psi \). For more details on \( \mathcal{Q}^y \), the reader is referred to [26]. Let \( \mathcal{F}^x_t \) be the \( \sigma \)-field generated by \( \{Y^x_t, r \leq t\} \) and set

\[ \mathcal{J}_k(s, \tau, x, y) = E[\langle S_{(t-k\delta-\tau\varepsilon)}(b(x, Y^x_{s+\tau\varepsilon}) - \bar{b}(x)), S_{(t-k\delta-\tau\varepsilon)}(b(x, Y^x_{s+\tau\varepsilon}) - \bar{b}(x)) \rangle]. \]

Then, we have

\[
\begin{align*}
\mathcal{J}_k(s, \tau, x, y) & = E^y[\langle S_{(t-k\delta-\tau\varepsilon)}(b(x, Y^x_{s+\tau\varepsilon}) - \bar{b}(x)), S_{(t-k\delta-\tau\varepsilon)}(b(x, Y^x_{s+\tau\varepsilon}) - \bar{b}(x)) \rangle] \\
& = E^y[\langle S_{(t-k\delta-\tau\varepsilon)}(b(x, Y^x_{s+\tau\varepsilon}) - \bar{b}(x)), S_{(t-k\delta-\tau\varepsilon)} E^y[b(x, Y^x_{s+\tau\varepsilon}) - \bar{b}(x)] \rangle].
\end{align*}
\]

To proceed, by invoking the Markov property of \( Y^x \), we have

\[ \mathcal{J}_k(s, \tau, x, y) = E^y[\langle S_{(t-k\delta-\tau\varepsilon)}(b(x, Y^x_{s+\tau\varepsilon}) - \bar{b}(x)), S_{(t-k\delta-\tau\varepsilon)} E^y[b(x, Y^x_{s+\tau\varepsilon}) - \bar{b}(x)] \rangle], \]

where \( E^y[b(x, Y^x_{s+\tau\varepsilon}) - \bar{b}(x)] \) means the function \( E^y[b(x, Y^x_{s+\tau\varepsilon}) - \bar{b}(x)] \) evaluated at \( y = Y^x_{s+\tau\varepsilon} \).

Using first H"older’s inequality, then the contraction property of \( S_t \) and the boundedness of the function \( \bar{b} \), we obtain

\[ \mathcal{J}_k(s, \tau, x, y) \leq C(E^y[b(x, Y^x_{s+\tau\varepsilon}) - \bar{b}(x)]^2)^{\frac{1}{2}} (E^y[(E^y[b(x, Y^x_{s+\tau\varepsilon}) - \bar{b}(x)]^2)]^{\frac{1}{2}}, \]

where \( C > 0 \) is a constant independent of \( k, s, \tau \). Then, in view of Eq. (A.1) and Eq. (A.2), we have

\[ \mathcal{J}_k(s, \tau, x, y) \leq C(1 + |x|^2 + |y|^2)e^{-\frac{a}{2}(s-\tau)}. \]
\[
E\left[\left|\left(S_{t-k\delta-s\epsilon}(b(X_{\epsilon^{k\delta}},Y_{\epsilon^{k\delta}},\hat{Y}_{\epsilon^{k\delta}}) - \bar{b}(X_{\epsilon^{k\delta}})), S_{t-k\delta-s\epsilon}(b(X_{\epsilon^{k\delta}},Y_{\epsilon^{k\delta}},\hat{Y}_{\epsilon^{k\delta}}) - \bar{b}(X_{\epsilon^{k\delta}}))\right|, S_{t-k\delta-s\epsilon}(b(X_{\epsilon^{k\delta}},Y_{\epsilon^{k\delta}},\hat{Y}_{\epsilon^{k\delta}}) - \bar{b}(X_{\epsilon^{k\delta}}))\right|, S_{t-k\delta-s\epsilon}(b(X_{\epsilon^{k\delta}},Y_{\epsilon^{k\delta}},\hat{Y}_{\epsilon^{k\delta}}) - \bar{b}(X_{\epsilon^{k\delta}}))\right]\right] = E\left[J_k(s, \tau, x, y)\right]_{(x,y)=(X_{\epsilon^{k\delta}},\hat{Y}_{\epsilon^{k\delta}})}.
\]

which, with the aid of Eq. (B.3), yields
\[
J_k(s, \tau) \leq C E\left(1 + |X_{\epsilon^{k\delta}}|^2 + |\hat{Y}_{\epsilon^{k\delta}}|^2\right)e^{-\frac{\eta^2}{2}(s-\tau)}.
\]

This completes the proof of Lemma 5.12. \(\Box\)

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