NOTES ABOUT THE KP/BKP CORRESPONDENCE

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We present a set of remarks related to previous work. These are remarks on polynomials solutions, the application of the Wick theorem, examples of creation of polynomial solutions with the help of vertex operators, the eigenproblem for polynomials, and a remark on the conjecture by Alexandrov and Mironov, Morozov about the ratios of the projective Schur functions. New results on the bilinear relations between characters of the symmetric group and the Sergeev group and on bilinear relations between skew Schur and projective Schur functions and also between shifted Schur and projective Schur functions are added. Certain new matrix models are discussed.

Keywords: KP tau function, BKP tau function, Schur function, projective Schur function, shifted Schur function, character of symmetric group, character of Sergeev group, symmetric polynomial, vertex operator, eigenvalue problem

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1. Introduction

The goal of these notes is to present a number of remarks and observations concerning the KP [1]–[3], TL [4], BKP [5], [6], and Veselov–Novikov [7] (2DKP, see [8]) tau functions. It is well known that these hierarchies have a number of remarkable applications in mathematics and physics. This paper adds some details to works [9]–[11], which were in turn based on [12]–[17]. Sections 2 and 5 contains reviews and add details to some pieces discussed in [9]–[11]. Sections 3, 4, and 6 contain new results.

The KP equation was integrated in [18] and in the famous work of Zakharov and Shabat in 1974 [19]. A great number of applications in mathematics and mathematical physics have been developed done due to the work of the Kyoto school [1]–[3]. They introduced the concept of a tau function and used free fermions. Using the free fermion approach, they related the KP hierarchy to the $A_\infty$ root system and expanded the integrability to different root systems. An analogue of the KP hierarchy related to the root system $B_\infty$ is called the BKP hierarchy. It was also studied in [6].

We recall the basic notions very briefly. But before that, the following remark is in order.

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Remark 1.1. The notation in this paper is slightly different from the notation in [9]–[11]. In the KP case, instead of the higher times \( t = (t_1, t_2, t_3, \ldots) \) common in soliton theory, the power-sum variables are used: \( p = (p_1, p_2, p_3, \ldots) \), the relation being \( mt_m = p_m \). Moreover, in the BKP case, the power sums \( p^n = (p_1, p_3, p_5, \ldots) \) are related to the BKP higher times \( t^n = (t_1, t_3, t_5, \ldots) \) as \( mt_m = 2p_m \). Instead of \( \tau(t), s_\lambda(t), \gamma(t) \) and instead of \( \tau^n(t^n), Q_\nu(t^n/2) \), we respectively write \( \tau(p), s_\lambda(p), \gamma(p) \) and \( \tau^n(2p^n), Q_\nu(p^n), \gamma^n(2p^n) \).

**Charged and neutral fermions.** The fermion creation and annihilation operators satisfy the anti-commutation relations

\[
[\psi_j, \psi_k^\dagger]_+ = [\psi_j^\dagger, \psi_k^\dagger]_+ = 0, \quad [\psi_j, \psi_k^\dagger]_+ = \delta_{jk}.
\]

(1.1)

We recall that a nonincreasing set of nonnegative integers \( \lambda_1 \geq \ldots \geq \lambda_k \geq 0 \) is called a partition \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \), and the \( \lambda_i \) are called the parts of \( \lambda \). The sum of parts is called the weight \( |\lambda| \) of \( \lambda \). The number of nonzero parts of \( \lambda \) is called the length of \( \lambda \), it is denoted by \( \ell(\lambda) \) (see [20] for the details). Partitions are denoted by Greek letters: \( \lambda, \mu, \ldots \). The set of all partitions is denoted by \( P \). The set of all partitions with odd parts is denoted by \( DP \). Partitions with distinct parts are called strict partitions, we prefer letters \( \alpha, \beta \) to denote them. The set of all strict partitions is denoted by \( DP \). For partitions \( \alpha|\beta = \lambda \in P \), we let \( \alpha, \beta \) denote the Frobenius coordinated (we recall that the coordinates \( \alpha = (\alpha_1, \ldots, \alpha_k) \in DP \) determine the lengths of arms counted from the main diagonal of the Young diagram of \( \lambda \) while \( \beta = (\beta_1, \ldots, \beta_k) \in DP \) determine the lengths of legs counted from the main diagonal of the Young diagram of \( \lambda \), with \( k \) being the length of the main diagonal of \( \lambda \); see [20] for the details).

The vacuum element \( |n\rangle \) in each charge sector \( F_n \) is the basis element corresponding to the trivial partition \( \lambda = \varnothing \):

\[
|n\rangle := |\varnothing; n\rangle = e_{n-1} \wedge e_{n-2} \wedge \cdots.
\]

(1.2)

Elements of the dual space \( F^* \) are called the bra vectors and are denoted as \( \langle w | \), with the dual basis \( \{ |\lambda; n\rangle \} \) for \( F_n^* \) defined by the pairing \( \langle \lambda; n|\mu; m \rangle = \delta_{\lambda\mu} \delta_{nm} \). For the KP \( \tau \)-functions, we only need to consider the \( n = 0 \)-charge sector \( F_0 \), and generally drop the charge \( n \) symbol, denoting the basis elements simply as \( |\lambda\rangle := |\lambda; 0\rangle \).

For \( j > 0 \), \( \psi_{-j} \) and \( \psi_{-j-1} \) (respectively \( \psi_{-j}^\dagger \) and \( \psi_{j-1} \)) annihilate the right (respectively left) vacua: for any \( j > 0 \),

\[
\psi_{-j}|0\rangle = 0, \quad \psi_{-j-1}|0\rangle = 0, \quad \langle 0|\psi_{j-1} = 0, \quad \langle 0|\psi_{-j} = 0.
\]

(1.3)

Neutral fermions \( \phi^+_j \) and \( \phi^-_j \) are defined [5] by

\[
\phi^+_j := \frac{\psi_j + (-1)^j \psi_{-j}^\dagger}{\sqrt{2}}, \quad \phi^-_j := \frac{\psi_j - (-1)^j \psi_{-j}^\dagger}{\sqrt{2}}, \quad j \in \mathbb{Z}
\]

(1.4)

(where \( i = \sqrt{-1} \)), and satisfy the relations

\[
[\phi^+_j, \phi^-_k]_+ = 0, \quad [\phi^+_j, \phi^+_k]_+ = [\phi^-_j, \phi^-_k]_+ = (-1)^j \delta_{j+k,0}.
\]

(1.5)

In particular, \( (\phi^+_0)^2 = (\phi^-_0)^2 = 1/2 \). Acting on the vacua \(|0\rangle \) and \(|1\rangle \), we have

\[
\phi^+_j|0\rangle = \phi^-_{-j}|0\rangle = \phi^+_j|1\rangle = \phi^-_{-j}|1\rangle = 0, \quad \langle 0|\phi^-_j = \langle 0|\phi^+_j = \langle 1|\phi^-_j = \langle 1|\phi^+_j = 0, \quad j > 0,
\]

\[
\phi^+_0|0\rangle = -i\phi^-_0|0\rangle = \frac{1}{\sqrt{2}} \psi_0|0\rangle = \frac{1}{\sqrt{2}} |1\rangle, \quad \langle 0|\phi^+_0 = i\langle 0|\phi^-_0 = \frac{1}{\sqrt{2}} \langle 0|\psi_0^\dagger = \frac{1}{\sqrt{2}} |1\rangle.
\]

(1.6)
We set
\[
\Psi_{\alpha,\beta} := (-1)^{\sum_{j=1}^{\beta}(-1)^{r-1/2}\psi_{\alpha_1} \cdots \psi_{-\beta_1-1} \cdots \psi_{-\beta_r-1}},
\]  
\[
\Phi_{\alpha}^\pm := 2^{k/2}\phi_{\alpha_1}^\pm \cdots \phi_{\alpha_k}^\pm,
\]  
where \(\lambda = (\alpha|\beta)\). We note that \(|\lambda\rangle = \Psi_{\alpha,\beta}|0\rangle, \langle\lambda| = \langle 0|\Phi_{\alpha,\beta}^\dagger\).  
For \(\alpha = (\alpha_1, \ldots, \alpha_k)\in\text{DP}\), we introduce  
\[
\Phi_{\alpha} := (-1)^{\sum_{j=1}^{k\alpha}\alpha^j} 2^{k/2}\phi_{\alpha_k}^\pm \cdots \phi_{\alpha_1}^\pm.
\]  
We then obtain  
\[
\langle 0|\Phi_{\alpha}^\dagger \Phi_{\alpha'}^\dagger |0\rangle = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'},
\]  
whose bosonized version is the scalar product of the Schur functions  
\[
\langle s_{\alpha,\beta}, s_{\alpha',\beta'} \rangle = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'} \quad \text{(see \cite{20})},
\]  
and  
\[
\langle 0|\Phi_{\alpha}^\mp |0\rangle = 2^{\ell(\alpha)}\delta_{\alpha,\alpha'}
\]  
The bosonized version of this equation is the scalar product of the projective Schur functions:  
\[
\langle Q_\alpha, Q_{\alpha'} \rangle = 2^{\ell(\alpha)}\delta_{\alpha,\alpha'}.
\]  
In what follows, we sometimes write \(\phi_\alpha\) and \(\Phi_\alpha\) instead of \(\phi_\alpha^\pm\) and \(\Phi_\alpha^\pm\).

**Fermions and tau functions: KP and BKP cases.** According to \cite{2}, the KP tau functions can be represented in the form of the vacuum expectation value (VEV)  
\[
\tau_\alpha(p) = \langle 0|\gamma(p)g|0\rangle
\]  
where \(g\) is an exponential of an expression bilinear in \(\{\psi_i\}\) and \(\{\psi_i^\dagger\}\). Here,  
\[
\gamma(p) = \exp\left(\sum_{m>0} \frac{1}{m}p_m J_m\right), \quad J_m = \sum_{i\in\mathbb{Z}} \psi_i^\dagger \psi_{i+m}
\]  
and \(p\) is the set of parameters \((p_1, p_2, \ldots)\); the numbers \(p_m/m\) are called the KP higher times.

Similarly, the BKP tau function can be represented as  
\[
\tau_\alpha^n(2p^B) = \langle 0|\gamma^{n\pm}(2p^B)h^\pm|0\rangle
\]  
where \(h^\pm\) is an exponential of an expression quadratic in \(\{\phi_\alpha^\pm\}\). Here,  
\[
\gamma^{n\pm}(2p^B) = \exp\left(\sum_{m>0, m \text{ odd}} \frac{2}{m}p_m J_m^\pm\right), \quad J_m^\pm = \sum_{i\in\mathbb{Z}} (-1)^i\phi_{i+m}^\pm
\]  
with \(p^B = (p_1, p_3, \ldots)\). The set of \(t_m^B = 2p_m^B, m = 1, 3, 5, \ldots\), is called the set of the BKP higher times.

After article \cite{1}, symmetric functions have become part of the soliton theory; power sum variables, Schur functions and later (see \cite{21}, \cite{22}) the projective Schur functions were used there.

A wonderful result of Sato’s school \cite{1}, \cite{2} is the fermionic formula for the Schur polynomial  
\[
s_\lambda(t) = \langle 0|\gamma(t)\Psi_{\alpha,\beta}|0\rangle.
\]  
In the BKP case, a similar formula was found in \cite{21}:
\[
Q_\alpha(p^B) = \langle 0|\gamma^{n\pm}(2p^B - |)\Phi_\alpha^\pm|0\rangle.
\]
**Bosonization.** We recall that there exists the fermion–boson correspondence in 2D space. The first work was the preprint by Pogrebkov and Sushko, which preceded paper [23]. This correspondence turned out to be very important in the theory of solitons, as it was shown in a series of excellent works of the Kyoto school. We recall some facts.

The fermionic Fock space is in one-to-one correspondence with the vertex operators, where

\[ \psi(z) \leftrightarrow V^+(z), \quad \psi^\dagger(z) \leftrightarrow V^-(z) \quad (1.18) \]

where

\[ V^\pm(z) = \exp \left( \pm \sum_{j=1}^{\infty} z^j p_j - \frac{\partial}{\partial p_j} \right), \quad z \in S^1, \quad (1.19) \]

and where \( e^{\mp \partial_n z^\pm n} \) is the shift operator that acts on the variable \( n \) and \( \partial_n, [\partial_n, n] = 1, \) are bosonic mutually conjugate zero-mode operators) and

\[ \psi(z) = \sum_{i \in \mathbb{Z}} z^i \psi_i, \quad \psi^\dagger(z) = \sum_{i \in \mathbb{Z}} z^{-i} \psi_i^\dagger. \quad (1.20) \]

In the BKP case, we have a one-to-one correspondence between the Fock space of neutral fermions and the bosonic Fock space that can be viewed as the space of polynomials in a set \( \mathbf{p}^a = \frac{1}{2}(t_1, t_3, t_5, \ldots) \) and a Grassmann parameter \( \xi (\xi^2 = 0) \),

\[ \Phi^\pm_\alpha |0\rangle \leftrightarrow Q_\alpha (\mathbf{p}^a) \begin{cases} 1, & \ell(\alpha) \text{ even}, \\ \xi, & \ell(\alpha) \text{ odd} \end{cases} \quad (1.21) \]

where \( Q_\lambda (\mathbf{p}^a) \) is the projective Schur polynomial written in terms of the power-sum variables \( \mathbf{p}^a \) [20]. The Fermi operators are in one-to-one correspondence with the vertex operators, where

\[ V^{\pm a}(z) = \frac{1}{\sqrt{2}} \left( \xi + \frac{\partial}{\partial \xi} \right) \exp \left( \sum_{j=1, j \text{ odd}}^{\infty} 2 z^j - p_j^\pm \right) \exp \left( - \sum_{j=1, j \text{ odd}}^{\infty} z^{-j} \frac{\partial}{\partial p_j^\mp} \right), \quad z \in S^1, \]

and

\[ \phi^\pm(z) = \sum_{i \in \mathbb{Z}} z^i \phi_i^\pm. \quad (1.22) \]

We rewrite formulas (1.16) and (1.17) for the Schur functions as

\[
\begin{align*}
 s_{\lambda}(x) \Delta(x) &= \langle 0 | \psi^\dagger(x_1^{-1}) \ldots \psi^\dagger(x_N^{-1}) \Phi_{\alpha,\beta} | N \rangle = \langle N | \Phi_{\alpha,\beta}^\dagger \psi(x_1) \ldots \psi(x_N) | 0 \rangle, \\
 Q_{\alpha}(x) \Delta^*(x) &= 2^{-N/2} \langle 0 | \phi(-x_1^{-1}) \ldots \phi(-x_N^{-1}) \Phi_{\alpha} | 0 \rangle = 2^{-N/2} \langle 0 | \Phi_{-\alpha} \phi(x_1) \ldots \phi(x_N) | 0 \rangle.
\end{align*}
\quad (1.23)
\]

We here use the notation

\[ \Delta(x) := \prod_{i<j} (x_i - x_j), \quad \Delta^*(x) := \prod_{i<j} \frac{x_i - x_j}{x_i + x_j}, \quad (1.24) \]
**Fermi fields.** For the Fermi fields and for the Fourier components of the Fermi fields, we have quite similar relations

\[
\psi(-z^{-1}) = \frac{\phi^+(z^{-1}) - i\phi^-(z^{-1})}{\sqrt{2}}, \quad \psi^\dagger(z^{-1}) = \frac{\phi^{+\dagger}(z^{-1}) + i\phi^{-\dagger}(z^{-1})}{\sqrt{2}},
\]

\[
\psi^\dagger_j = \frac{\phi^+_j - i\phi^-_j}{\sqrt{2}}, \quad (-1)^j\psi^\dagger_{-j} = \frac{\phi^{+\dagger}_j + i\phi^{-\dagger}_j}{\sqrt{2}},
\]

which result in

\[
\psi^\dagger(z^{-1})\psi(-z^{-1}) = -i\phi^+(z^{-1})\phi^-(z^{-1}), \quad (-1)^j\psi^\dagger_j\psi_j = -i\phi^{+\dagger}_j\phi^{-\dagger}_j.
\]

Next, we introduce

\[
\Psi(x, y) := \psi(x_1)\ldots\psi(x_N)\psi^\dagger(-y_1)\ldots\psi^\dagger(-y_N),
\]

\[
\Psi^*(x, y) := \psi(-y_N^{-1})\ldots\psi(-y_1^{-1})\psi^\dagger(x_N^{-1})\ldots\psi^\dagger(x_1^{-1})
\]

and

\[
\Phi^{+\pm}(x) = 2^{-N/2}\phi^{\pm}(x_1^{-1})\ldots\phi^{\pm}(x_N^{-1}),
\]

\[
\Phi^{-\pm}(x) = 2^{-N/2}\phi^{-\pm}(x_1)\ldots\phi^{\pm}(x_N).
\]

Here, we consider \(N\) to be an even number.

We then assume the following “time ordering” in the \(\Psi, \Psi^\dagger, \) and \(\Phi^{\pm}\):

\[
|\psi(x_i^{-1})| > \cdots > |\psi(x_N^{-1})|, \quad |\psi^\dagger(y_i^{-1})| > \cdots > |\psi^\dagger(y_N^{-1})|.
\]

This ordering is similar allows using partitions similarly to how this was done in (1.7) and (1.8). We have

\[
\Psi(x, y) = (-1)^{N(N-1)/2}(-i)^N\Phi^+(x)\Phi^-(x),
\]

\[
\Psi^*(x, y) = (-1)^{N(N-1)/2}(-i)^N\Phi^{++}(x)\Phi^{--}(x).
\]

Let

\[
p_m(x, y) := \sum_{i=1}^{N} (x^m - (-y)^m)
\]

where \(N\) is even. It then follows that

\[
s_\lambda(x, y)\Delta(x, y) = \langle 0|\Psi^*(x, y)\Psi_{\alpha, \beta}|0 \rangle \prod_{i=1}^{N} x_i^{-1} = \langle 0|\Psi^\dagger_{\alpha, \beta}\Psi(x, y)|0 \rangle \prod_{i=1}^{N} y_i^{-1}
\]

where

\[
\Delta(x, y) = (-1)^{N(N-1)/2} \prod_{i=1}^{N} \frac{\Delta(x)\Delta(y)}{\prod_{i, j=1}^{N} (x_i + y_j)},
\]

\[
Q_{\alpha}(x)\Delta^+(x) = \langle 0|\Phi(x)\Phi_{\alpha}|0 \rangle = \langle 0|\Phi_{-\alpha}\Phi^*(x)|0 \rangle.
\]

It is worth comparing the relation for \(\Delta(x, y)\) with the relation

\[
s_\lambda(p_1) = \frac{\Delta(\alpha)\Delta(\beta)}{\prod_{i, j=1}^{N} (\alpha_i + \beta_j + 1)} \prod_{i=1}^{N} \frac{1}{\alpha_i!\beta_i!} = \frac{\dim \lambda}{|\lambda|!}, \quad p_1 = (1, 0, 0, \ldots),
\]

where \(\lambda = (\alpha|\beta)\) and \(\dim \lambda\) is the number of ways to build the \(\lambda\) Young diagram by adding one node to a previous diagram, starting with the empty diagram, such that we have a Young diagram at each step.

**Remark 1.2.** In the literature (see, e.g., [20]) the notation \(s_\lambda(x/y)\) is sometimes used, which is \(s_\lambda(x, -y)\) in our present notation. Our notation is convenient for our purposes.
2. On the KP vs BKP correspondence in [9]–[11]

2.1. Preliminaries. Let the KP higher times be parameterized by

\[ t_m = t_m(\{x, y\}) = \sum_{i=1}^{N} (x_i^m - (-y_i)^m) \]  \hspace{1cm} (2.1)

and let the Frobenius parts of the partition \( \lambda = (\alpha|\beta) \) be denoted by \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \text{DP} \) and \( \beta = (\beta_1, \ldots, \beta_r) \in \text{DP} \). Our formulas are of the type

\[ s_{(\alpha|\beta)}(x, y) = \sum_{P_F, P_x} A(P_F) D(P_x) Q_{\nu^+}^{BKP}(z^+) Q_{\nu^-}^{BKP}(z^-), \]  \hspace{1cm} (2.2)

or more generally [24],

\[ \tau_{(\alpha|\beta)}^{KP}(x, y) = \sum_{P_F, P_x} A(P_F) D(P_x) \tau_{\nu^+}^{BKP}(z^+) \tau_{\nu^-}^{BKP}(z^-), \]  \hspace{1cm} (2.3)

where \( A(P_F) \) and \( D(P_x) \) are special combinatorial numbers [9], \( z^+ \) is a subset of the set \( z \) of coordinates \( x_1, \ldots, x_M, y_1, \ldots, y_M \) and \( z^- \) is the complementary subset \( z \setminus z^+ \). The partition \( \nu^+ \) is an ordered subset of the set \( F \) of numbers \( \alpha_1, \ldots, \alpha_r, \beta_1 + 1, \ldots, \beta_r + 1 \) and the partition \( \nu^- \) is the complementary subset \( F \setminus \nu^+ \) under the condition: in case a pair of equal numbers occur in the set \( F \) (say, \( \alpha_i = \beta_j + 1 \)), then these numbers belong to different subsets (these are either \( \nu^+ \) or \( \nu^- \)). The combinatorial coefficient \( D(P_x) \) is similar to \( A(P_F) \), with the partitions \( \alpha, \beta, \nu^+, \nu^- \) replaced by continuous variables \( x, y, z^+, z^- \), and with an extra Vandermonde-type factor that depends on these continuous variables. Symbols \( P_F \) and \( P_x \), which we call polarizations, denote the selection of the subsets.

Papers [9], [10] were respectively devoted to finding the weights \( A(P_F) \) in cases (2.2) and (2.3); the weights turned out to coincide. In [24] the coefficient \( D(P_x) \) was found ‘in both cases.

Here, we also deal with formulas that relate the polynomials naturally arising in the KP theory and polynomials naturally arising in the BKP theory.

2.2. Four key lemmas. There are four statements that follow from Eqs. (1.25) and (1.26) and from Eqs. (1.3) and (1.6). We need some notation.

For a given partition \( \lambda = (\alpha|\beta) \), we have an ordered set of numbers

\[ (\alpha, I(\beta)) = \alpha_1, \ldots, \alpha_r, \beta_1 + 1, \ldots, \beta_r + 1 \]

that consists of two strict partitions of equal length: the left \( \alpha = (\alpha_1, \ldots, \alpha_r) \) and the right \( I(\beta) = (\beta_1 + 1, \ldots, \beta_r + 1) \). In this set, we perform a permutation \( \sigma \) that satisfies the following conditions.

1. The resulting sequence of numbers also consists of two consecutive strict partitions \( (\nu^+, \nu^-) \): a left strict partition \( \nu^+ \) and a right strict partition \( \nu^- \):

\[ \nu^+ = (\nu^+_1, \ldots, \nu^+_p), \quad \nu^+_1 > \cdots > \nu^+_p \geq 0, \quad \nu^- = (\nu^-_1, \ldots, \nu^-_q), \quad \nu^-_1 > \cdots > \nu^-_q \geq 0. \]

We call \( m(\nu^+) = p_1 \) (or \( m(\nu^-) = p_2 \)) the cardinality of the partition \( \nu^+ (\nu^-) \).

2. In the original set \( (\alpha, \beta) \), there can be pairs of equal numbers (for example, \( \alpha_i = \beta_j + 1 \) for some \( i \) and \( j \)). We let \( s \) denote the number of such pairs. It is required that the element of each pair that was in the left partition (that is, in \( \alpha \)) remain in the left partition \( \nu^+ \) (accordingly, the right element \( \beta \) of the pair must end up in the right partition \( \nu^- \)).
We call the pairs \((\nu^+, \nu^-)\) the \textit{polarization} of \((\alpha|\beta)\). For a given \(\lambda = (\alpha|\beta)\), we let \(P(\alpha, \beta)\) denote the set of all possible pairs \((\nu^+, \nu^-)\) under the conditions above.

We let \(\text{sgn}(\nu^+, \nu^-)\) denote the sign of the permutation from the ordered set \((\alpha, I(\beta))\) to the ordered set \((\nu^+, \nu^-)\), and let \(\pi(\nu^\pm) := \#(\alpha \cap \nu^\pm)\) denote the cardinality of the intersection of \(\alpha\) with \(\nu^\pm\) (also see Appendix A for a more formal treatment).

We introduce

\[
A_{\alpha,\beta}^{\nu^+,\nu^-} := \frac{(-1)^{r(r+1)/2+s}}{2^{r-s}} \text{sgn}(\nu)(-1)^{\pi(\nu^-)}i^m(\nu^-),
\]

\(\text{Lemma 2.1.}\) \textbf{The following representations hold:}

\[
\Psi_{(\alpha|\beta)} = \sum_{(\nu^+, \nu^-) \in P(\alpha, \beta)} A_{\alpha,\beta}^{\nu^+,\nu^-} \Phi^+_{\nu^+} \Phi^-_{\nu^-},
\]

\[
\Psi^\dagger_{(\alpha|\beta)} = \sum_{(\nu^+, \nu^-) \in P(\alpha, \beta)} (-1)^{|\nu^+|+|\nu^-|} A_{\alpha,\beta}^{\nu^+,\nu^-} \Phi^+_{\nu^+} \Phi^-_{\nu^-}.
\]

\textbf{Proof.} We recall the sign factor \(\text{sgn}(\nu)\) corresponding to the order of the neutral fermion factors and to the powers of \(-1\) and \(i\); then, noting that there are \(2^n\) resulting identical terms, we arrive at (A.7). The sign factor in (2.6) can be checked using the equality \(\langle 0|\Psi^\dagger_{(\alpha|\beta)} \Psi_{(\alpha'|\beta')} |0 \rangle = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}\) (see (1.10) and (1.11)).

By similar relations (1.25) and (1.26), we obtain a direct analogue of Lemma 2.1.

\textbf{Lemma 2.2.}\textbf{ The following representation holds:}

\[
\Psi(x, y) = \sum_{(z^+, z^-) \in P(x, y)} A_{x,y}^{z^+,z^-} \Phi^+(z^+)\Phi^-(z^-).
\]

We used these assertions in [24].

We now define

\[
J_\Delta := \prod_{i=1}^{\ell(\Delta)} J_{\Delta_i}, \quad J^-_\Delta := \prod_{i=1}^{\ell(\Delta)} J^-_{\Delta_i}, \quad \Delta \in P,
\]

\[
J^{\pm}_\Delta := \prod_{i=1}^{\ell(\Delta)} J^\pm_{\Delta_i}, \quad J^\pm_{-\Delta} := \prod_{i=1}^{\ell(\Delta)} J^\pm_{-\Delta_i}, \quad \Delta \in \text{OP}.
\]

For a given \(\Delta \in \text{OP}\), we can split its parts into two ordered odd partitions:

\[
\Delta = \Delta^+ \cup \Delta^-, \quad \Delta^+ \cup \Delta^- \in \text{OP}, \quad \ell(\Delta^+) + \ell(\Delta^-) = \ell(\Delta).
\]

We let \(\text{OP}(\Delta)\) denote the set of all such \((\Delta^+, \Delta^-)\).

Because \(J_n = J^+_n + J^-_n\) for \(n\) odd (see [2]), we obtain the following lemma.

\textbf{Lemma 2.3.}\textbf{ The following equalities hold:}

\[
J_\Delta = \sum_{(\Delta^+, \Delta^-) \in \text{OP}} J^{\pm}_{\Delta^+} J^\pm_{\Delta^-}, \quad J^-_\Delta = \sum_{\Delta^+ \in \text{OP}, \Delta^+ \cup \Delta^+ = \Delta} J^+_\Delta^- J^-_{\Delta^-},
\]

\textbf{Proof.}
**Lemma 2.4** (factorization). If $U^\pm$ are either even- or odd-degree elements of the subalgebra generated by the respective operators $\{\phi^\pm_i\}_{i \in \mathbb{Z}}$, the VEV of their product can be factored as

$$
\langle 0|U^+U^-|0\rangle = \begin{cases} 
(0|U^+|0\rangle)(0|U^-|0\rangle), & \text{if } U^+ \text{ and } U^- \text{ are both of even degree,} \\
0, & \text{if } U^+ \text{ and } U^- \text{ have different parity,} \\
2i(0|U^+\phi^+_0|0\rangle)(0|U^-\phi^-_0|0\rangle), & \text{if } U^+ \text{ and } U^- \text{ are both of odd degree.}
\end{cases}
$$

In [9]–[11], we used Lemmas 2.1 and 2.4 to evaluate VEVs. For this, the following notion is useful.

**Definition.** If $\nu$ is a strict partition of cardinality $m(\nu)$ (with 0 allowed as a part), we define the associated supplemented partition $\hat{\nu}$ to be

$$
\hat{\nu} := \begin{cases} 
\nu, & \text{if } m(\nu) \text{ is even,} \\
(\nu, 0), & \text{if } m(\nu) \text{ is odd.}
\end{cases}
$$

We let $m(\hat{\nu})$ denote the cardinality of $\hat{\nu}$.

For instance, as a result of the application of Lemmas 2.1 and 2.4 to (1.16) and (1.17), we obtain [9]

$$
s_{(\alpha|\beta)}(p') = \sum_{(\nu^+, \nu^-) \in P_{(\alpha, \beta)}} a_{\nu^+, \nu^-}^{\nu^+, \nu^-} Q_{\nu^+}(p^n)Q_{\nu^-}(p^n),
$$

where the power-sum variables are $p' = (t_1, 0, t_3, 0, t_5, 0, \ldots)$, $p^n = \frac{1}{2}(t_1, t_3, t_5, \ldots)$, and where

$$
a_{\alpha, \beta}^{\nu^+, \nu^-} := \frac{(-1)^{r(r+1)/2+s}}{2^{r-s}} \text{sgn}(\nu)(-1)^{m(\nu^-)+m(\hat{\nu}^-)/2}
$$

Equation (2.12) follows from (2.4) by replacing the factor $i^{m(\nu^-)}$ with $(-1)^{m(\hat{\nu}^-)/2}$ (we note the hat above $\nu^-$, which is the result of the application of Lemma 2.4).

3. **Relation between characters of the symmetric group and characters of the Sergeev group**

It is known [20] that the power sums labeled by partitions

$$
p_{\Delta} = p_{\Delta_1}p_{\Delta_2} \cdots, \quad \Delta \in P
$$

are uniquely expressed in terms of the Schur polynomials

$$
p_{\Delta} = \sum_{\lambda \in P} \chi_{\lambda}(\Delta)s_{\lambda}(p), \quad p = (p_1, p_2, p_3, \ldots),
$$

while the odd power-sum variables (power sums labeled by odd numbers)

$$
p_{\Delta} = p_{\Delta_1}p_{\Delta_2} \cdots, \quad \Delta \in \text{OP},
$$

also denoted $p^n_{\Delta}$, are uniquely expressed in terms of projective Schur polynomials:

$$
p^n_{\Delta} = \sum_{\alpha \in DP} \chi^{n}_{\alpha}(\Delta)Q_{\alpha}(p^n), \quad p^n = (p_1, p_3, p_5, \ldots).
$$
We recall that the coefficients $\chi_\lambda(\Delta)$ in (3.2) have the meaning of the irreducible characters of the symmetric group $S_d$, $d = |\lambda|$, evaluated on the cycle class $C_\Delta = (\Delta_1, \ldots, \Delta_k)$, $|\lambda| = |\Delta| = d$ (see [20]), and we can write it as

$$\chi_\lambda(\Delta) = \langle 0 | J_\Delta \Psi_{\alpha, \beta} | 0 \rangle$$  \hspace{1cm} (3.5)$$

where $J_\Delta$ and $\Psi_{\alpha, \beta}$ are respectively given by (2.8) and (1.7) (see, e.g., [17] for the details). The characters of symmetric groups have a very wide use, in particular, in mathematical physics; we refer to two works [25], [26] as an example.

The notion of the Sergeev group and of the central character of the Sergeev group was introduced in [27] (see Appendix B). The coefficient $\chi_B^\mu$ is an irreducible character of this group [27], [28]. As was shown in [27], the so-called spin Hurwitz numbers (introduced in that paper) can be expressed in terms of these characters. As was pointed out in [29], [30], the generating function for the spin Hurwitz numbers can be related to the BKP hierarchy similarly to how the generating function for the usual Hurwitz numbers is related to the KP (and also to the TL) hierarchies (see the pioneering works [31]–[33] and also [34]–[41]).

We can write these characters in terms of the BKP currents $J_B^m$ ($m$ odd) [17]:

$$\chi_B^\mu(\Delta) = 2^{-\ell(\alpha)} \langle 0 | J_B^\mu \Phi^\alpha \rangle.$$  \hspace{1cm} (3.6)

From Lemma 2.1 and (2.3) we deduce the following theorem.

**Theorem 3.1.** The character $\chi_\lambda$, $\lambda = (\alpha|\beta)$ evaluated on an odd cycle $\Delta \in \text{OP}$ is a bilinear function of the Sergeev characters:

$$\chi_\lambda(\Delta) = \sum_{(\nu^+, \nu^-) \in \mathcal{P}(\alpha, \beta)} \sum_{(\Delta^+, \Delta^-) \in \text{OP}(\Delta)} 2^{\ell(\nu^+)+\ell(\nu^-)} a_{\alpha, \beta}^{\nu^+, \nu^-} \chi_{\nu^+}(\Delta^+) \chi_{\nu^-}(\Delta^-),$$  \hspace{1cm} (3.7)

where $a_{\alpha, \beta}^{\nu^+, \nu^-}$ are given by (2.13).

**4. Relation between generalized skew Schur polynomials and generalized skew projective Schur polynomials**

This section can be regarded as a remark to [9].

We find the relation between the following quantities:

$$s_{\lambda/\mu}(p') := \langle 0 | \Psi_\mu^\lambda \gamma(p') \Psi_{\alpha, \beta} | 0 \rangle,$$  \hspace{1cm} (4.1)

$$Q_{\nu/\theta}(p^B) := \langle 0 | \Phi_{-\theta}^\gamma \Phi^\theta(2p^B) | 0 \rangle,$$  \hspace{1cm} (4.2)

where the power-sum variables are

$$p' = (p_1, 0, p_3, 0, p_5, 0, \ldots), \quad p^B = \frac{1}{2}(p_1, p_3, p_5, \ldots),$$

and $\lambda = (\alpha|\beta)$, $\mu = (\gamma|\delta)$ with $\alpha, \beta, \gamma, \delta, \theta, \nu \in \text{DP}$.

**Theorem 4.1.** We have

$$s_{\lambda/\mu}(p') = \sum_{(\nu^+, \nu^-) \in \mathcal{P}(\alpha, \beta)} \sum_{(\theta^+, \theta^-) \in \mathcal{P}(\gamma, \delta)} (-1)^{|\theta^+|+|\theta^-|} a_{\alpha, \beta}^{\nu^+, \nu^-} a_{\gamma, \delta}^{\theta^+, \theta^-} Q_{\nu^+/\theta^+}(p^B) Q_{\nu^-/\theta^-}(p^B).$$  \hspace{1cm} (4.3)
By the Wick theorem, we obtain
\[ g = g(C) = \exp \sum C_{ij} \psi_i \psi_j^\dagger, \quad h^\pm = h^\pm(A) = \exp \sum A_{ij} \phi_i^\pm \phi_j^\pm, \] (4.4)
where the entries \( C_{ij} \) form a matrix \( C \) and the entries \( A_{ij} = -A_{ji} \) form a skew-symmetric matrix \( A \). We suppose that
\[ g|0\rangle = c_1, \quad h^\pm|0\rangle = |0\rangle c_2, \quad c_1, c_2 \in \mathbb{C}, \quad c_1, c_2 \neq 0. \] (4.5)
We then define the generalized Schur polynomials and the generalized projective Schur polynomials by
\[ s_{\lambda/\mu}(p|g) := \langle \mu| \gamma(p) |g \lambda \rangle, \] (4.6)
\[ Q_{\nu/\theta}(p^{|h^\pm}) := \langle (0) | \Phi_{-\theta} B_{\nu/\theta} (2pq^\dagger) h^\pm \Phi_{\nu} |0\rangle. \] (4.7)

**Remark 4.1.** In [10], \( s_{\lambda/\mu}(p|g) \) and \( Q_{\nu/\theta}(p^{|h^\pm}) \) are defined as \( s_{\lambda/\mu}(p|C) \) and \( Q_{\nu/\theta}(p^{|A}) \).

The polynomiality of (4.6) in \( p_1, \ldots, p_{|\lambda|-|\mu|} \) and of (4.7) in \( p_1, \ldots, p_{|\nu|-|\theta|} \) follows from (4.5). We can treat a given \( p \) as a constant and study \( s_{\lambda/\mu}(p|A) \) as the function of discrete variables \( \lambda - i \) and \( \mu - i \).

**Theorem 4.2.** We suppose that \( g = h^+ h^- \) and (4.4) is true. Then
\[ s_{\lambda/\mu}(p^{|g}) = \sum_{(\nu^+, \nu^-) \in \mathcal{P}(\alpha, \beta), (\theta^+, \theta^-) \in \mathcal{P}(\gamma, \delta)} (-1)^{\theta^+ + \theta^-} a_{\alpha, \beta}^{\nu^+, \nu^-} a_{\gamma, \delta}^{\theta^+, \theta^-} Q_{\nu^+/\theta^+}(p|g) Q_{\nu^-/\theta^-}(p^{|g}). \] (4.10)

The proof is based on the same reasoning as the proof of a theorem in [10]. We omit it.

We introduce the functions
\[ s_{\lambda/\mu}(p|r) := \langle 0| \Phi_{\mu} \gamma_r(p) \Phi_{\alpha, \beta} |0\rangle, \] (4.8)
\[ Q_{\nu/\theta}(p^{|r}) := \langle 0| \Phi_{-\theta} B_{\nu/\theta} (2pq^\dagger) \Phi_{\nu} |0\rangle, \] (4.9)
where
\[ \gamma_r(p) := \exp \left( \sum_{j=1}^{\infty} \frac{1}{j} \sum_{k \in \mathbb{Z}} \psi_k \psi_k^\dagger r(k+1) \ldots r(k+j) \right), \] (4.10)
\[ \gamma^B_r(2p^\dagger) := \exp \left( \sum_{j=1}^{\infty} \frac{1}{j} \sum_{k \in \mathbb{Z}} (-1)^j \phi_k^\pm \phi_k^\dagger r(k+1) \ldots r(k+j) \right). \] (4.11)
By the Wick theorem, we obtain
\[ s_{\lambda/\mu}(p|r) = \det \left( r(\lambda_i - i + 1) \ldots r(\lambda_j - j) s_{\lambda_i - \mu_j - i+j}(p) \right)_{i,j}. \] (4.12)

A similar relation for \( Q_{\nu/\theta}(p^{|r}) \) is more bulky cumbersome, the Wick theorem yields a Nimmo-type Pfaffian formula, and we omit it. We have
\[ s_{\lambda/\mu}(p^{|r}) = \sum_{(\nu^+, \nu^-) \in \mathcal{P}(\alpha, \beta), (\theta^+, \theta^-) \in \mathcal{P}(\gamma, \delta)} (-1)^{\theta^+ + \theta^-} a_{\alpha, \beta}^{\nu^+, \nu^-} a_{\gamma, \delta}^{\theta^+, \theta^-} Q_{\nu^+/\theta^+}(r) Q_{\nu^-/\theta^-}(p^{|r}). \]
Relation between shifted Schur and shifted projective Schur functions. We recall the notion of the shifted Schur function introduced by Okounkov and Olshanski [42]. It can be defined as

\[ s^*_\mu(\lambda) = \frac{\dim \lambda/\mu}{\dim \lambda} n(n-1) \cdots (n-k+1) = \frac{s_{\lambda/\mu}(p_1)}{s_\lambda(p_1)} \] (4.13)

where \( n = |\lambda|, k = |\mu|, p_1 = (1, 0, 0, \ldots) \) and

\[ \dim \lambda/\mu = s_{\lambda/\mu}(p_1)(n-k)!, \quad \dim \lambda = s_\lambda(p_1)n! \]

are the numbers of the standard tableaux of the respective shape \( \lambda/\mu \) and \( \lambda \) (see [20]). The function \( s^*_\mu(\lambda) \) viewed as a function of the Frobenius coordinates is also known as the Frobenius–Schur function \( FS(\alpha, \beta) \).

On the other hand, Ivanov [43] introduced a projective analogue of the shifted \( Q \)-functions:

\[ Q^*_\theta(\nu) = \frac{Q_{\nu/\theta}(p_1)}{Q_{\nu}(p_1)}. \] (4.14)

Therefore,

\[ s^*_\mu(\lambda)s_\lambda(p_1) = \sum_{(\nu^+, \nu^-) \in P(\alpha, \beta), \theta^+ \in \Delta(\gamma, \delta)} (\nu^+ - \nu^-) a^{\theta^+}_{\alpha, \beta} a_{\gamma, \delta}^{-\theta^+} Q^*_\theta(\nu^+)Q^*_{\theta^-}(\nu^-)Q_{\nu^+}(p_1)Q_{\nu^-}(p_1). \]

We also note that both \( s^* \) and \( Q^* \) were used in [30], [34] to describe the generalized cut-and-join structure in the context of Hurwitz and spin Hurwitz numbers.

5. Minor remarks to [10], [11]

5.1. A note on polynomial solutions. A simple and, in fact, trivial remark that should nonetheless be made is that when we speak of polynomial solutions, we should always clarify what variables we are actually meant.

We consider any KP tau function written in the Sato form

\[ \tau(p) = \sum_{\lambda \in P} \pi_\lambda s_\lambda(p) \] (5.1)

where \( P \) is the set of all partitions, \( s_\lambda \) are Schur functions [20], and \( \pi_\lambda \) are Plücker coordinates of a point in the Sato Grassmannian [1]. For tau function (5.1) to be a polynomial in the variables \( p_1, p_2, \ldots \), it is obviously necessary that only a finite number of \( \pi_\lambda \) be nonzero. But more often we mean symmetric polynomials in the \( x_1, x_2, \ldots \) variables that arise from the substitutions

\[ p_m = \pm \sum_{i=1}^N x_i^m. \]

We suppose that \( \pi_\lambda = 0 \) if the length \( \ell(\lambda) > M \). Then it suffices to choose

\[ p_m = p_m(\mathbf{x}) = - \sum_{i=1}^N x_i^m, \] (5.2)
for $\tau(p(x))$ to be a symmetric polynomial of the weight at most $MN$ in the $x = (x_1, \ldots, x_N)$. An example is given by the hypergeometric family of tau functions. Such polynomials can have a determinant representation in the an form

$$\tau(p(x)) = \frac{1}{\Delta(x)} \det \left( \langle 0 | \psi_1^j \psi(x_j^{-1} g | 0) \rangle \right)_{i,j},$$

(5.3)

which can be deduced from the bosonization formulas and Wick’s theorem. However, whether the last equality is true depends on the choice of $g$. In the next section, we discuss in what cases we can apply Wick’s $g$ theorem written as in (5.3).

Similarly, any BKP tau function can be written in the form

$$\tau^B(2p^B) = \sum_{\alpha \in DP} A_\alpha Q_\alpha (p^B)$$

(5.4)

where $Q_\alpha$ are the projective Schur functions [20] and $A_\alpha$ are Cartan coordinates of a point in the isotropic Grassmannian [44]; DP denotes the set of all strict partitions (partitions with distinct parts).

We suppose that $A_\lambda = 0$ if the length $\ell(\lambda) > M$. Then it suffices to choose

$$p_m = p_m (x) = -2 \sum_{a=1}^N x_a^m.$$  

(5.5)

As examples, we mention orthogonal polynomials in Appendix 6 of [45], these are: $q$-Askey–Wilson polynomials, continuous $q$-Jacobi polynomials, $q$-Gegenbauer polynomials, Clebsh–Gordan coefficients $C_q$, $q$-Hahn polynomials, and $q$-Racah polynomials.

### 5.2. How to use Wick’s theorem: a remark to [11]

Another simple remark can be added about the use of Wick’s theorem, which is the main tool in deriving various Pfaffian and determinant expressions for multivariate polynomials that naturally appear in the framework of the KP and BKP hierarchies.

For an even number of fermionic operators $(w_1, \ldots, w_{2L})$ that anticommute, $[w_j, w_k]_+ = 0$, $1 \leq j, k \leq 2L$, the matrix with the elements $\langle 0 | w_j w_k | 0 \rangle$ is skew symmetric, and Wick’s theorem implies that $\langle 0 | w_1 \ldots w_{2L} | 0 \rangle$ is given by its Pfaffian:

$$\langle 0 | w_1 \ldots w_{2L} | 0 \rangle = \text{Pf}(\langle 0 | w_j w_k | 0 \rangle).$$

(5.6)

On the other hand, if the odd elements $w_1, w_3, \ldots$ are linear combinations of creation operators $\{\psi_j\}_{j \in \mathbb{Z}}$ and the even elements $w_2, w_4, \ldots$ are linear combinations of annihilation operators $\{\psi_j^\dagger\}_{j \in \mathbb{Z}}$, Wick’s theorem implies

$$\langle 0 | w_1 \ldots w_{2L} | 0 \rangle = \text{det}(\langle 0 | w_j w_k | 0 \rangle)_{j=1,3,\ldots, k=2,4,\ldots}.$$  

(5.7)

The next problem is to represent the Fock vector $\gamma(p) | g | 0 \rangle$ in factored form. This is possible if $g$ factors as

$$g = g^+ g^0 g^-, \quad g^0 | 0 \rangle = | 0 \rangle, \quad g^0 | 0 \rangle = e | 0 \rangle, \quad \langle 0 | g^+ = (0 |.$$  

(5.8)

But this may be impossible, for example, if $g = O_\lambda$, where $O_\lambda$ is the operator that creates the state $O_\lambda | 0 \rangle = | \lambda \rangle$, where $| \lambda \rangle$ is the basis Fock vector labeled by a partition $\lambda$. In this context, paper [46] devoted to the study of Bruhat cells of the $A_\infty$ group can be useful.

### 6. Polynomials and vertex operators, a remark to [10]

Polynomial BKP tau functions were studied in [47], [48] and in [10], [11].

In Example 4.4 in [10], we considered polynomials denoted by $s_\lambda (t | \hat{A}^r(p))$ and $Q_\mu (\frac{1}{2} t | \hat{A}^r(p^n))$ there. Below, these polynomials are respectively denoted as $s_\lambda (t | p, r)$ and $Q_\mu (t^n | p^B, r)$ (see Remark 1.1).

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The KP case. The following formula is known:

\[ s_\lambda(t + p) = \langle 0 | \gamma(t) \gamma(p) | \lambda \rangle = \sum_{\rho \in P} \langle 0 | \gamma(t) | \rho \rangle \langle \rho | \gamma(p) | \lambda \rangle = \sum_{\rho \subseteq \lambda} s_{\lambda/\rho}(p) s_\rho(t). \]

In [10], we studied examples of polynomial \( \tau \)-functions and introduced the generalized Schur function parameterized in terms of two sets of infinite collections of parameters \( r := \{ r(j) \}_{j \in \mathbb{Z}} \) and \( p = (p_1, p_2, \ldots) \):

\[ s_\lambda(t|p,r) = \langle 0 | \gamma(t) \gamma_r(p) | \lambda \rangle = \sum_{\rho \subseteq \lambda} r_{\lambda/\rho} s_{\lambda/\rho}(p) s_\rho(t), \]  

(6.1)

where \( s_{\lambda/\rho}(p) \) is the skew Schur function corresponding to the skew partition \( \lambda/\rho \),

\[ r_{\lambda/\rho} := \prod_{(i,j) \in \lambda/\rho} r(j - i) \]

is the content product over the diagram \( \lambda \) (for instance, \( r_{(2,2)/(11)} = r(1)r(-1)r(0) \)), and \( \gamma_r(p) \) is defined in (4.10) such that \( \gamma(p) = \gamma_r(p) \) for \( r \equiv 1 \). An example of such polynomials is given by the Laguerre symmetric function \( \mathcal{L}_\lambda \) introduced in [49] (see Definition 4.3 there). In the specific case under consideration, we choose

\[ p = t_0 := (1, 0, 0, \ldots), \quad t_j = t_j(x) = \sum_{k=1}^N x_k^j, \quad r(j) = r(j; z, z') := -(z + j)(z' + j), \]

and then Eq. (6.1) takes the form

\[ \mathcal{L}_\lambda(x) = \sum_{\rho \subseteq \lambda} (-1)^{|\rho| - |\lambda|} \frac{\dim \mathcal{S}_{1/\rho}}{(|\lambda| - |\rho|)!} s_\rho(t(x)) \prod_{(i,j) \in \lambda/\rho} (z + j - i)(z' + j - i). \]  

(6.2)

We consider the vertex operator introduced in the context of classical integrable systems in a series of articles by the Kyoto school (see, e.g., [2])

\[ V^\pm(z) = \exp \left( \pm \sum_{j=1}^\infty \frac{1}{j} z^j p_j \right) z^{\pm N} e^{|\partial N} \exp \left( \mp \sum_{j=1}^\infty z^{-j} \frac{\partial}{\partial p_j} \right), \quad z \in S^1. \]  

(6.3)

We set

\[ A_m(r) = \text{res}_z :\left( \left( \frac{1}{z} r(D) \right)^m V^+(z) V^-(z) \right) \frac{dz}{z}, \quad m \neq 0, \]  

(6.4)

where \( r(D) \) acts on functions on the circle as \( r(D) \cdot z^n = r(n)z^n \) (\( D = z \partial/\partial z \) is the Euler operator on the circle); we let \( :X: \) denote the bosonic normal ordering, which means that all derivatives \( \partial/\partial p_j \) are moved to the right. For example,

\[ \left( \frac{1}{z} r(D) \right)^2 = \frac{1}{z^2} r(D - 1)r(D) \quad \text{and} \quad : \frac{\partial^2}{\partial p_k^2} \frac{\partial}{\partial p_j} : = p_k \frac{\partial^3}{\partial p_k^2 \partial p_j}. \]

In the case \( r \equiv 1 \),

\[ A_m(r) = \begin{cases} p_m, & m > 0, \\ \frac{\partial}{\partial p_m}, & m < 0. \end{cases} \]

We have a realization of the Heisenberg algebra \( [A_m(r), A_n(r)] = n \delta_{m,n} \), and therefore

\[ \exp \sum_{m < 0} \frac{p_m}{m} A_m(r) \cdot \exp \sum_{m > 0} \frac{p_m}{m} A_m(r) = \exp \sum_{m > 0} \frac{p_m p_{-m}}{m} \cdot \exp \sum_{m > 0} \frac{p_m}{m} A_m(r) \cdot \exp \sum_{m < 0} \frac{p_m}{m} A_m(r). \]  

(6.5)

\(^{1}\)The vertex operator was first introduced in [23] in a different context.
Proposition 6.1. We have the representation

\[ s_\lambda(t|p,r) = \exp\left( \sum_{m<0} \frac{p_m}{m} A_m(r) \right) s_\lambda(t). \]  

(6.6)

In particular,

\[ L_\lambda(x) = (e^{A_{-1}(r)} s_\lambda(t)) \big|_{x=x(t)}. \]  

(6.7)

The BKP case. This case is similar to the preceding one. Here, we have

\[ Q_\alpha(t^n+p^B) = \langle 0 | \gamma^n(2t^B) \gamma^n(2p^B) \Phi_\alpha | 0 \rangle = \sum_{\theta \in DP} 2^{-\ell(\theta)} \langle 0 | \gamma^n(2t^B) \Phi_\theta | 0 \rangle \langle 0 | \Phi_\theta \gamma^n(2p^B) \Phi_\alpha | 0 \rangle = \sum_{\theta \subseteq \alpha} 2^{-\ell(\theta)} Q_{\alpha/\theta}(p^B) Q_\theta(t^B), \]

where \( Q_{\alpha/\theta} \) is the skew projective Schur function [20].

Following [10] we set

\[ Q_\alpha(t^n|p^B, r) := \langle 0 | \gamma^n(2t^B) \gamma^r(2p^B) \Phi_\alpha | 0 \rangle = \sum_{\theta \subseteq \alpha} r_{\lambda/\rho}^\alpha Q_{\alpha/\theta}(p^B) Q_\theta(t^B) \]  

(6.8)

where \( t^n = (t_1, t_3, t_5, \ldots) \) and \( p^B = (p_1, p_3, p_5, \ldots) \) are sets of parameters and \( r = (r(1), r(2), r(3), \ldots) \) is another set of parameters. Then

\[ \gamma^r(2p^B) := \exp \left( \sum_{j=1}^{\infty} \frac{2}{j} \sum_{k \in \mathbb{Z}} (-1)^k \phi^+_k \phi^-_{k-j} r(k+1) \ldots r(k+j) \right) \]  

(6.9)

(such that \( \gamma^r(2p^B) = \gamma^r(2p^B) \) at \( r \equiv 1 \)) and

\[ r_{\alpha/\theta}^\alpha := \prod_{(i,j) \in \alpha/\theta} r(j) \]  

(6.10)

where the product goes over all nodes \((i,j)\) of the (skew) Young diagram, but each node \((i,j)\) is assigned a number \( r(j) \) that depends only on the coordinate \( j \). That is the “B-type” content product defined on the skew diagram \( \alpha/\theta \) (the product over all nodes with coordinates \((i,j)\) of the diagram, for example, \( r_{(5,2)/(1)}^B = r(2)r(3)r(4)r(5)r(1)r(2) \)).

Proposition 6.2. We have the representation

\[ Q_\alpha(t^n|p^B, r) = \exp \left( 2 \sum_{m < 0 \atop m \text{ odd}} \frac{p_m}{m} A_m(r) \right) Q_\alpha(t^n), \]  

(6.11)

where

\[ A_m^n(r) = \frac{1}{2} \text{res}_z \left( \left( \frac{1}{z} r(D) \right)^m V^B(z) \right) V^B(-z); \frac{dz}{z}, \quad m \text{ odd}. \]  

(6.12)
Remark 6.1. In the case \( r(j) = r(1 - j) \), we obtain
\[
\exp \left( \sum_{m < 0} \frac{p_m}{m} A_m(r) \right) = \exp \left( 2 \sum_{m < 0} \frac{p_m}{m} A_m^n(r) \right).
\]
If we take (6.6) and (6.11) as definitions of the polynomials \( s_{\lambda}(t|p, r) \) and \( Q_{\alpha}(t^\beta|p^\beta, r) \), then (2.12) implies the following assertion.

Corollary 6.1. We have the equality
\[
s_{(\alpha|\beta)}(t'|p', r) = \sum_{(\nu^+, \nu^-) \in P(\alpha, \beta)} a_{\alpha, \beta}^{\nu^+, \nu^-} Q_{\nu^+}(t^\beta|p^\beta, r)Q_{\nu^-}(t^\beta|p^\beta, r), \tag{6.13}
\]
where
\[
t' = (t_1, 0, t_3, 0, \ldots), \quad p' = (p_1, 0, p_3, 0, \ldots),
\]
\[
t^n = \frac{1}{2} (t_1, t_3, t_5, \ldots), \quad p^n = \frac{1}{2} (p_1, p_3, p_5, \ldots).
\]
In [10], this equality was derived for fermion operators.

Eigenproblem for polynomials: the KP case. Let
\[
T := \exp \left( - \sum_i : \psi_i \psi_i^\dagger : T_i \right) \equiv \exp \left( \text{res}_{z} (T(D) \cdot \psi(z)) \psi^\dagger(z) : \frac{dz}{z} \right), \tag{6.14}
\]
where \( T_i, i \in \mathbb{Z} \), is a set of numbers, \( D = z d/dz \), and we assume that \( T(D) \cdot z^k = T(k)z^k \). The colon in \( : A : \) denotes fermionic ordering \( A - \langle 0 | A | 0 \rangle \) of an expression quadratic in the fermions. Such diagonal operators were used in [50]–[52] and also in [12], [53]–[55] in quite different contexts. We have
\[
\langle 0 | \gamma(p) T | \lambda \rangle = s_{\lambda}(p) \exp \left( - \sum_{i=1}^{\ell(\lambda)} T_{h_i} \right).
\]
Let \( T_i = i^2 \). The bosonized version in this case is as follows:
\[
T^{\text{Bos}} = \exp \sum_{a + b = c} \left( p_a p_b \frac{\partial}{\partial p_c} + abp_c \frac{\partial^2}{\partial a \partial b} \right), \tag{6.15}
\]
(this operator is also known as the cut-and-join operator introduced in [56] in a different context). We set
\[
T(A) := \exp \left( \sum_{m < 0} \frac{1}{m} p_m A^\text{Fer}_m \right) T \exp \left( - \sum_{m < 0} \frac{1}{m} p_m A^\text{Fer}_m \right), \tag{6.16}
\]
where \( A^\text{Fer}_m \) are the fermionic counterparts of \( A_m, m = \pm 1, \pm 2, \ldots \), introduced by (6.4), i.e.,
\[
A^\text{Fer}_m = \text{res}_{z} \left( \left( \frac{1}{z} r(D) \right)^m \psi(z) \psi^\dagger(z) : \frac{dz}{z} \right), \tag{6.17}
\]
(such operators were introduced in [12]).
From fermionic expression for $s_{\lambda}(t)$, using the bosonization rule, we now obtain
\[
T^{\text{Bos}} \cdot s_{\lambda}(t) = s_{\lambda}(t) \exp \left(- \sum_{i=1}^{\ell(\lambda)} T_{h_i} \right) \tag{6.18}
\]
where $\ell(\lambda)$ is the length of $\lambda$ (the number of nonvanishing parts of $\lambda$), $h_i = \lambda_i - i + n$, and
\[
e^{\mathcal{H}_{\text{Bos}}} := \exp \left( \sum_{m<0} \frac{p_m}{m} A_m(r) \right) T^{\text{Bos}} \exp \left(- \sum_{m<0} \frac{p_m}{m} A_m(r) \right).
\]
It follows from Proposition 6.1 that
\[
e^{t\mathcal{H}_{\text{Bos}}} s_{\lambda}(t, p|r) = e^{tE(h_1, \ldots, h_N)} s_{\lambda}(t, p|r) \tag{6.19}
\]
where
\[
E(h_1, \ldots, h_N) = - \sum_{i=1}^{\ell(\lambda)} T_{h_i}.
\]

**Eigenproblem for polynomials: the BKP case.** Let
\[
T := \exp \left(- \sum :\phi_j \phi_j^\dagger : T_j \right) \equiv \exp \left( \text{res}_z (T(D) \cdot \phi(z)) \phi^\dagger(-z) : \frac{dz}{z} \right), \tag{6.20}
\]
where $T_i, i \in \mathbb{N}$, is a set of numbers. Such diagonal operators were used in [13]. We have
\[
\langle 0 | T^{\text{B}} (2p^\text{B}) T \Phi_\alpha | 0 \rangle = Q_\alpha (p^\text{B}) \exp \left(- \sum_{i=1}^{\ell(\alpha)} T_{\alpha_i} \right).
\]

Let $T_i = i^3$. The bosonization of the operator
\[
\frac{1}{2} \sum_{j \in \mathbb{Z}} j^3 (-1)^j :\phi_j \phi_{-j} : = \text{res}_z \left( \left( \frac{\partial}{\partial z} \right)^3 \phi(z) \right) \phi(-z) : \frac{dz}{z} \tag{6.21}
\]
gives [17]
\[
\frac{1}{2} \sum_{n>0} n^3 p_n \partial_n + \frac{1}{2} \sum_{n>0} np_n \partial_n + 4 \sum_{n_1, n_2, n_3 \text{ odd}} p_{n_1} p_{n_2} p_{n_3} (n_1 + n_2 + n_3) \partial_{n_1+n_2+n_3} + 3 \sum_{n_1+n_2=n_3+n_4 \text{ odd}} p_{n_1} p_{n_2} n_3 n_4 \partial_{n_3} \partial_{n_4} + \sum_{n_1,n_2,n_3 \text{ odd}} p_{n_1+n_2+n_3} \partial_{n_1} \partial_{n_2} \partial_{n_3}.
\]

We set
\[
T(A) := \exp \left( \sum_{m<0} \frac{1}{m} p_m A^{\text{Fer,B}}_m \right) T \exp \left(- \sum_{m<0} \frac{1}{m} p_m A^{\text{Fer,B}}_m \right) \tag{6.22}
\]
where $A^{\text{Fer,B}}_m$ are the fermionic counterparts of $A^{\text{B}}_m, m = \pm 1, \pm 2, \ldots$, introduced by (6.4):
\[
A^{\text{Fer,B}}_m = \frac{1}{2} \text{res}_z \left( \left( \frac{1}{z} r(D) \right)^m \phi(z) \right) \phi(-z) : \frac{dz}{z} \tag{6.23}
\]
(such operators were introduced in [13]). From the fermionic expression for $Q_\alpha(p^n)$, using the bosonization rule, we then obtain

$$T^{\text{Bos}} \cdot Q_\alpha(t^n) = Q_\alpha(t^n) \exp\left(-\sum_{i=1}^{\ell(\alpha)} T_{\alpha_i}\right).$$

(6.24)

Then

$$e^{\mathcal{H}^{\text{Bos,B}}} := \exp\left(\sum_{m<0} \frac{p_m A^B_m(r)}{m}\right) T^{\text{Bos}} \exp\left(-\sum_{m<0} \frac{p_m A^B_m(r)}{m}\right).$$

(6.25)

It follows from Proposition 6.2 that

$$e^{\mathcal{H}^B} Q_\alpha(t^n, p^n | r) = e^{E(\alpha_1, \ldots, \alpha_N)} Q_\alpha(t^n, p^n | r),$$

where

$$E(\alpha_1, \ldots, \alpha_N) = -\sum_{i=1}^{\ell(\alpha)} T_{\alpha_i}.$$  

Appendix A: The key lemma [9]

**Definition.** A polarization of $\alpha|\beta = (\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r)$, is a pair $\nu^+, \nu^-$ of strict partitions with cardinalities (or lengths)

$$m(\nu^+) := \#(\nu^+), \quad m(\nu^-) := \#(\nu^-)$$

(A.1)

(including possibly a zero part $\nu^+_{m(\nu^+)} = 0$ or $\nu^-_{m(\nu^-)} = 0$), satisfying

$$\nu^+ \cap \nu^- = \alpha \cap I(\beta), \quad \nu^+ \cup \nu^- = \alpha \cup I(\beta),$$

(A.2)

where $I(\beta) := (I_1(\beta), \ldots, I_r(\beta))$ is the strict partition [20] with parts $I_j(\beta) = \beta_j + 1$, $j = 1, \ldots, r$. The set of all polarizations of $\alpha|\beta$ is denoted by $P(\alpha, \beta)$.

We let $S := \alpha \cap I(\beta)$ denote the strict partition obtained as the intersection of $\alpha$ and $I(\beta)$, and $s := \#(S)$ be its cardinality. Because both $\alpha$ and $I(\beta)$ have cardinality $r$, it follows that

$$m(\nu^+) + m(\nu^-) = 2r,$$

(A.3)

and hence $m(\nu^\pm)$ must have the same parity. It is easy to verify [9] that the cardinality of $P(\alpha, \beta)$ is $2^{2r-2s}$.

The following assertion was proved in [9].

**Lemma A.1.** For every polarization $\nu := (\nu^+, \nu^-)$ of $\lambda = (\alpha|\beta)$ there is a unique binary sequence $\epsilon(\nu) = (\epsilon_1(\nu), \ldots, \epsilon_{2r}(\nu))$ of length $2r$ with $\epsilon_j(\nu) = \pm, j = 1, \ldots, 2r$, such that

$$(\alpha_1, \epsilon_1(\nu)), \ldots, (\alpha_r, \epsilon_r(\nu)), (\beta_1 + 1, \epsilon_{r+1}(\nu)), \ldots, (\beta_r + 1, \epsilon_{2r}(\nu))$$

(A.4)

is a permutation of the sequence

$$(\nu^+_1, +), \ldots, (\nu^+_{m(\nu)+1}, +), (\nu^-_1, -), \ldots, (\nu^-_{m(\nu)}, -)$$

(A.5)

and

$$\epsilon_j(\nu) = + \quad \text{if} \quad \alpha_j \in S, \quad \epsilon_{r+j}(\nu) = - \quad \text{if} \quad \beta_j + 1 \in S, \quad \beta_j + 1 \in S, \quad j = 1, \ldots, r.$$

(A.6)
Definition. The sign of the polarization $(\nu^+, \nu^-)$, denoted by $\text{sgn}(\nu)$, is defined as the sign of the permutation that takes sequence (A.4) into sequence (A.5).

We let $\pi(\nu^\pm) := \#(\alpha \cup \nu^\pm)$ denote the cardinality of the intersection of $\alpha$ and $\nu^\pm$. It follows that

$$\pi(\nu^+) + \pi(\nu^-) = r + s.$$ Then we have the following assertion.

Lemma A.2. We have the decomposition

$$|\lambda| = \frac{(-1)^{r(r+1)/2 + s}}{2^{r+s}} \sum_{\nu \in \mathcal{P}(\alpha, \beta)} \text{sgn}(\nu)(-1)^{\pi(\nu^-)} m(\nu^-) \prod_{j=1}^{m(\nu^+)} \phi_{\nu^j}^+ \prod_{k=1}^{m(\nu^-)} \phi_{\nu_k}^- |0\rangle.$$ (A.7)

Proof. For the proof, we reorder the product over the factors $\psi_{\alpha_j} \psi_{-\beta_j-1}^\dagger$ such that the $\psi_{\alpha_j}$ terms precede the $\psi_{-\beta_j-1}$ ones, giving an overall sign factor $(-1)^{(r-1)/2}$. We then substitute

$$\psi_{\alpha_j} = \frac{1}{\sqrt{2}}(\phi_{\alpha_j}^+ - i\phi_{\alpha_j}^-), \quad \psi_{-\beta_j-1}^\dagger = \frac{(-1)^j}{\sqrt{2}}(\phi_{\beta_j+1}^+ + i\phi_{\beta_j+1}^-), \quad j \in \mathbb{Z},$$

in each factor and expand the product as a sum over monomial terms of the form

$$\prod_{j=1}^{m(\nu^+)} \phi_{\nu_j}^+ \prod_{k=1}^{m(\nu^-)} \phi_{\nu_k}^- |0\rangle.$$

Recalling the sign factor $\text{sgn}(\nu)$ that corresponds to the order of the neutral fermion factors and to the powers of $-1$ and $i$, and noting that there are $2^s$ resulting identical terms, we arrive at (A.7).

Appendix B: Sergeev group [27], [28]

As introduced in [27], the Sergeev group $C(d)$ is the semidirect product of the permutation group $S_d$ and the Clifford group $\text{Cliff}_d$ generated by the involutions $\xi_i, i = 1, \ldots, d$, and the central involution $\epsilon$, which are subject to the relations $\xi_i \xi_j = \epsilon \xi_j \xi_i$. The group $S_d$ acts on $\text{Cliff}_d$ by permuting the $\xi_i$. The irreducible representations of $C(d)$ are labeled by strict partitions. The normalized characters

$$f_\alpha(\Delta) := 2^{-\ell(\Delta)} |0\rangle |\Phi^B_\alpha\rangle |0\rangle \frac{1}{z_\Delta} Q_\alpha \{\delta_{k,1}\} = \frac{2^{\ell(\alpha)+\ell(\Delta)}}{z_\Delta Q_\alpha \{\delta_{k,1}\}} k^B_\alpha(\Delta)$$

(where $z_\Delta = \prod m_i! m_i! 2^m$ and $\Delta = (1^{m_1} 2^{m_2} \ldots) \in \text{OP}$, see [20]) are related to the $Q$ functions as

$$Q_\alpha \{p_k\} = Q_\alpha \{\delta_{k,1}\} \sum_{\Delta \in \text{OP}} f_\alpha(\Delta) p_\Delta$$ (B.1)

(this relation is dual to (3.4)). The normalized characters enter the formula for the spin Hurwitz numbers:

$$H^\pm(\Delta^1, \ldots, \Delta^k) = \sum_{\substack{\alpha \in \text{DP}, \\ |\alpha| = d, \ell(\alpha) \text{ even}}} (Q_\alpha \{\delta_{k,1}\})^2 f_\alpha(\Delta^1) \cdots f_\alpha(\Delta^k) - \sum_{\substack{\alpha \in \text{DP}, \\ |\alpha| = d, \ell(\alpha) \text{ even}}} (Q_\alpha \{\delta_{k,1}\})^2 f_\alpha(\Delta^1) \cdots f_\alpha(\Delta^k)$$

(see [17], [27], [29], [30]). The problem of evaluating the spin Hurwitz numbers can be related to the analysis of the BKP tau functions similarly to how evaluating the usual Hurwitz numbers is related to the KP and TL tau functions (see [31], [32], [34], [56]).
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