GAUSSIAN INTEGRAL MEANS OF ENTIRE FUNCTIONS: LOGARITHMIC CONVEXITY AND CONCAVITY

CHUNJIE WANG AND JIE XIAO

ABSTRACT. For \( 0 < p < \infty \) and \( \alpha \in (-\infty, \infty) \) we determine when the \( L^p \) integral mean on \( \{ z \in \mathbb{C} : |z| \leq r \} \) of an entire function with respect to the Gaussian area measure \( e^{-\alpha |z|^2} dA(z) \) is logarithmic convex or logarithmic concave.

1. INTRODUCTION

Let \( dA \) be the Euclidean area measure in the finite complex plane \( \mathbb{C} \). For any real number \( \alpha \) and \( 0 < p < \infty \), the Gaussian integral means of an entire function \( f : \mathbb{C} \to \mathbb{C} \) are defined by

\[
M_{p,\alpha}(f, r) = \frac{\int_{\{z \in \mathbb{C} : |z| \leq r\}} |f(z)|^p e^{-\alpha |z|^2} dA(z)}{\int_{\{z \in \mathbb{C} : |z| \leq r\}} e^{-\alpha |z|^2} dA(z)}, \quad 0 < r < \infty.
\]

This concept lies in the theory of Fock spaces; see [7].

The famous Hadamard’s three circles theorem for the above entire function \( f \) (cf. [1]) states that if

\[
\left\{ 0 < r_1 < r < r_2 < \infty ; \right. \\
\left. M(f, s) = \max\{|f(z)| ; |z| \leq s\} \text{ for } s \in (0, \infty), \right.
\]

then

\[
\ln M(f, r) \leq \left( \ln \frac{r_2}{r_1} \right) \ln M(f, r_1) + \left( \ln \frac{r}{r_1} \right) \ln M(f, r_2),
\]

i.e., \( \ln M(f, r) \) is convex in \( \ln r \). Continuing from [2] and its prior work [3, 4, 5, 6], this paper investigates such an analogous problem: When is the function \( r \mapsto \ln M_{p,\alpha}(f, r) \) convex or concave in \( \ln r \) ? In what follows, we will see that a resolution of this question depends on the parameter \( \alpha \) and its induced function

\[
\varphi(x) = \frac{1 - e^{-\alpha x}}{\alpha}, \quad 0 < x < \infty.
\]

**Theorem 1.** Let \( \alpha < 0 \). Suppose both \( x \mapsto M(x) \) and \( x \mapsto M'(x) \) are positive on \( (0, \infty) \). Then the function

\[
x \mapsto \ln \frac{\int_0^x M(t)e^{-\alpha t} \, dt}{\int_0^x e^{-\alpha t} \, dt}
\]

is convex in \( \ln x \) for \( x \) in an open interval \( I \subset (0, \infty) \) provided that the following conditions are satisfied:

(i) \( x \mapsto \ln M(x) \) is convex in \( \ln x \) for \( x \in I \);
Corollary 2. Let \((\alpha, p) \in (-\infty, 0) \times (0, \infty)\). If \(f : \mathbb{C} \to \mathbb{C}\) is an entire function, then \(r \mapsto \ln M_{p,\alpha}(f, r)\) is convex in \(\ln r\) for \(r \in \left(0, \sqrt{\frac{t_0}{\alpha}}\right)\), where \(t_0 = 1.79 \cdots\) is the unique root of \(u(t) = e^t - 1 - t - t^2\) on \((0, \infty)\).

During the process of extending Theorem 1 from \(I\) to \((0, \infty)\), we find the following assertion.

Theorem 3. Let \(\alpha < 0\). Suppose both \(x \mapsto M(x)\) and \(x \mapsto M'(x)\), are positive on \((0, \infty)\). Then the function

\[
x \mapsto \ln \frac{\int_{0}^{x} M(t)e^{-\alpha t}dt}{\int_{0}^{x} e^{-\alpha t}dt}
\]

is convex in \(\ln x\) for \(x \in (0, \infty)\) provided that

\[
\left(\frac{M'(x)}{M(x)}\right)' \geq \begin{cases} 0, & x \in (0, x_0); \\ \frac{[x(1-\alpha x) - \varphi(x)]^2}{4x[\varphi(x) - x]^2}, & x \in [x_0, \infty), \end{cases}
\]

where \(x_0 = -t_0/\alpha\) is the unique root of \(x(1-\alpha x) - \varphi(x)\) on \((0, \infty)\).

As a by-product of Theorem 3, the following corollary extends the logarithmic convexity of \(M_{p,\alpha}(f, \cdot)\) from \(I\) to \((0, \infty)\).

Corollary 4. Let \((\alpha, p) \in (-\infty, 0) \times (0, \infty)\). If \(f : \mathbb{C} \to \mathbb{C}\) is an entire function, then \(r \mapsto \ln M_{p,\alpha}(f, r)\) is convex in \(\ln r\) for \(r \in (0, \infty)\) if

\[
\left(\frac{M'(x)}{M(x)}\right)' \geq \frac{[x(1-\alpha x) - \varphi(x)]^2}{4x[\varphi(x) - x]^2}, \quad x \geq x_0,
\]

where \(x = r^2, M(x) = \int_{0}^{2\pi} |f(\sqrt{x}e^{i\theta})|^{p} \, d\theta\) and \(x_0\) is as the same as above.

However, whenever handling the logarithmic concavity we have only one situation as follows.

Theorem 5. Let \(\alpha \geq 0\). Suppose \(M(x)\) \& \(M'(x)\) are positive and \(M''(x)\) exists for \(x \in (0, \infty)\). If \(x \mapsto \ln M(x)\) is concave in \(\ln x\) for \(x \in (0, \infty)\), then the function

\[
x \mapsto \ln \frac{\int_{0}^{x} M(t)e^{-\alpha t}dt}{\int_{0}^{x} e^{-\alpha t}dt}
\]

is also concave in \(\ln x\) for \(x \in (0, \infty)\).

Note that for any nonnegative integer \(k\) the classical integral mean of \(z^k\) is both logarithmic convex and logarithmic concave. So, we obtain the following corollary, which is part (i) of [2, Theorem 7].

Corollary 6. Let \((\alpha, p) \in [0, \infty) \times (0, \infty)\). If \(k\) is a nonnegative integer, then the function \(r \mapsto \ln M_{p,\alpha}(z^k, r)\) is concave in \(\ln r\) for \(r \in (0, \infty)\).

Notation. In the forthcoming sections, we will employ the symbol \(\equiv\) when a new notation is introduced, but also use the notation \(U \sim V\) when \(U\) and \(V\) have the same sign.
2. FOUR LEMMAS

This section collects four lemmas which will be used in the proofs of Theorems 1, 3, 5.

The first two lemmas come from [2].

**Lemma 7.** Suppose $f$ is positive and twice differentiable on $(0, \infty)$. Then

(a) $f(x)$ is convex in $\ln x$ if and only if $f(x^2)$ is convex in $\ln x$ and $f(x)$ is concave in $\ln x$ if and only if $f(x^2)$ is concave in $\ln x$.

(b) Let

$$D(f(x)) = \frac{f'(x)}{f(x)} + x \frac{f''(x)}{f(x)} - x \left(\frac{f'(x)}{f(x)}\right)^2.$$  

Then $\ln f(x)$ is convex in $\ln x$ if and only if $D(f(x)) \geq 0$ and $\ln f(x)$ is concave in $\ln x$ if and only if $D(f(x)) \leq 0$ for all $x \in (0, \infty)$.

**Lemma 8.** Suppose $f = f_1/f_2$ is a quotient of two positive and twice differentiable functions on $(0, \infty)$. Then

$$D(f(x)) = D(f_1(x)) - D(f_2(x))$$  

for $x \in (0, \infty)$. Consequently, $\ln f(x)$ is convex in $\ln x$ if and only if

$$D(f_1(x)) - D(f_2(x)) \geq 0$$  

on $(0, \infty)$ and $\ln f(x)$ is concave in $\ln x$ if and only if

$$D(f_1(x)) - D(f_2(x)) \leq 0$$  

on $(0, \infty)$.

We next establish several estimates for the function $\varphi$.

**Lemma 9.** Suppose

$$\begin{cases}
\alpha \in \mathbb{R}; \\
x \in [0, \infty); \\
\varphi = \varphi(x) \equiv \int_0^x e^{-\alpha t^2} 2t^2 dt = \int_0^x e^{-\alpha t} dt = \frac{1}{\alpha}(e^{-\alpha x} - 1).
\end{cases}$$

Then

(a) $1 - \alpha \varphi(x) = \varphi'(x)$.

(b) $\varphi(x) - x \geq 0$ when $\alpha \leq 0$ and $\varphi(x) - x \leq 0$ when $\alpha \geq 0$.

(c) $g_1(x) \equiv x(1 - \alpha x) - \varphi(x) \leq 0$ when $\alpha \geq 0$.

(d) $g_2(x) \equiv \alpha \varphi^2(x) - 2(1 + \alpha x) \varphi(x) + 2x \leq 0$.

(e) $g_3(x) \equiv x - (1 + \alpha x) \varphi(x)$ is nonnegative when $\alpha \leq 0$ and not positive when $\alpha \geq 0$.

**Proof.** Part (a) follows from the fact that

$$\varphi(x) = \frac{1}{-\alpha}(e^{-\alpha x} - 1), \quad \varphi'(x) = e^{-\alpha x}.$$  

Part (b) follows from the fact that $e^{-\alpha x} \geq 1$ for $\alpha \leq 0$ and $x \in [0, \infty)$. A direct computation shows that

$$g_1'(x) = 1 - 2\alpha x - \varphi'(x)$$  

and

$$g_1''(x) = -2\alpha - \varphi''(x) = \alpha(\varphi'(x) - 2).$$  

It follows that $g_1''(x) \leq 0$ when $\alpha \geq 0$. So we have $g_1'(x) \leq g_1'(0) = 0$ and $g_1(x) \leq g_1(0) = 0$ for all $x \in [0, \infty)$. This proves (c).
Another computation gives
\[ g'_2(x) = 2\alpha \varphi \varphi' - 2\alpha \varphi - 2(1 + \alpha x)\varphi' + 2 \]
\[ = 2\alpha \varphi \varphi' - 2(1 + \alpha x)\varphi' + 2\varphi' \]
\[ = 2\alpha (\varphi - x)\varphi'. \]
By part (b), we have \( g'_2(x) \leq 0 \) for all \( x \in [0, \infty) \). Therefore, \( g_2(x) \leq g_2(0) = 0 \) for all \( x \in [0, \infty) \). This proves (d).

A similar computation produces
\[ g'_3(x) = 1 - \alpha \varphi(x) - (1 + \alpha x)\varphi'(x) \]
which yields \( g_3(x) \geq g_3(0) = 0 \) for all \( x \in [0, \infty) \) when \( \alpha \leq 0 \) and \( g_3(x) \leq g_3(0) = 0 \) for all \( x \in [0, \infty) \) when \( \alpha \geq 0 \). This proves (e) and completes the proof of the lemma. \( \square \)

Finally, Lemma 9 is applied to derive the following fundamental property.

**Lemma 10.** Given a nonconstant entire function \( f : \mathbb{C} \to \mathbb{C} \), suppose

\[
\begin{align*}
\alpha & \in \mathbb{R}; \\
x & \in [0, \infty); \\
p & \in (0, \infty); \\
M(x) & \equiv M_p(f, \sqrt{x}) = \int_0^{2\pi} |f(\sqrt{x}e^{i\theta})|^p \, d\theta; \\
h & = h(x) \equiv \int_0^x M_p(f, t)e^{-\alpha t^2} \, 2tdt = \int_0^x M(t)e^{-\alpha t} \, dt.
\end{align*}
\]

Let

\[
\begin{align*}
A & = A(x) \equiv \frac{\varphi(x) - x}{\varphi^2(x)}; \\
B & = B(x) \equiv (1 - \alpha x) + x \frac{M'(x)}{M(x)}; \\
C & = C(x) \equiv x\varphi'(x); \\
\Delta(x) & \equiv D(h(x)) - D(\varphi(x)).
\end{align*}
\]

Then

(a) \( S = S(x) = \sqrt{B^2 - 4AC} > 0 \) \( \forall x \in (0, \infty) \).

(b) \( \Delta(x) \sim -A \frac{h^2}{M^2} + B \frac{h}{M} - C \) \( \forall x \in (0, \infty) \).

**Proof.** (a) Noticing that \( M' > 0 \) and \( M > 0 \), together with \( x - (1 + \alpha x)\varphi \geq 0 \) by Lemma 9, we have

\[
B^2 - 4AC = \left[ (1 - \alpha x) + x \frac{M'(x)}{M} \right]^2 - 4x \varphi' \frac{\varphi - x}{\varphi^2}
\]
\[
> (1 - \alpha x)^2 - 4x \varphi' \frac{\varphi - x}{\varphi^2}
\]
\[
= \frac{(2x - (1 + \alpha x)\varphi)^2}{\varphi^2}
\]
\[
> 0.
\]

(b) Since
\[
\varphi' = e^{-\alpha x}, \quad \varphi'' = -\alpha e^{-\alpha x},
\]
we have

\[
D(\varphi(x)) = \frac{(1 - \alpha x)\varphi'}{\varphi} - x \frac{(\varphi')^2}{\varphi^2} = \frac{(\varphi - x)\varphi'}{\varphi^2}.
\]

On the other hand,
\[
h' = h'(x) = M(x)\varphi',
\]
and
\[ h'' = h''(x) = [M'(x) - \alpha M(x)] \varphi'. \]

It follows from simple calculations that
\[ D(h) = \frac{hh' + xhh'' - x(h')^2}{h^2} = \frac{(1 - \alpha x)Mh + xM'h - xM^2\varphi'}{h^2}. \]

Therefore,
\[ \Delta(x) = \frac{\varphi'M}{h^2}(hB - CM) - \varphi'A \]
\[ \sim -A\frac{h^2}{M^2} + Bh - C. \]

\[ \square \]

3. PROOFS OF THEOREMS 1 & 3

To verify Theorems 1 & 3, we use (ii) of Lemma 7 to show the logarithmic convexity of \( h(x)/\varphi(x) \) on \((0, \infty)\). According to Lemma 8, this will be accomplished if we can prove \( \Delta(x) \geq 0 \).

Suppose that \( \alpha < 0 \). From Lemma 9 it follows that \( A(x), B(x) \) and \( C(x) \) are all positive on \((0, \infty)\) as \( \alpha \leq 0 \) and \( M'/M > 0 \). By some direct computations, we have

\[ xA'(x) = \frac{x}{\varphi^3} \left[ \alpha \varphi^2 - 2(1 + \alpha x)\varphi + 2x \right], \]
\[ B' = -\alpha + \left( \frac{M'(x)}{M(x)} \right) ', \]
\[ xC' = x(1 - \alpha x)\varphi' = (1 - \alpha x)C. \]

Thus, an application of Lemma 10 yields that \( \Delta(x) \geq 0 \) is equivalent to

\[ \frac{-\sqrt{B^2 - 4AC}}{2A} \leq \frac{h}{M} - \frac{B}{2A} \leq \frac{\sqrt{B^2 - 4AC}}{2A}. \]

Since the function \( M \) is positive and increasing, we have
\[ B(x) \geq 1 - \alpha x \geq 0, \quad h(x) \leq \int_0^x M(t)\varphi'(t)dt = M(x)\varphi(x). \]

It follows from this, the proof of Lemma 10 part (b) of Lemma 9 and the triangle inequality that
\[ \frac{B + \sqrt{B^2 - 4AC}}{2A} \geq \frac{(1 - \alpha x) + \left| 1 + \alpha x - \frac{2x}{\varphi} \right|}{2A} \]
\[ \geq \frac{1 - \alpha x + 1 + \alpha x - \frac{2x}{\varphi}}{2A} \]
\[ = \frac{2(\varphi - x)}{2A\varphi} = \varphi \geq \frac{h}{M}. \]

This proves the right half of (1).

To prove the left half of (1), we write
\[ \delta = \delta(x) = h - M\frac{B - \sqrt{B^2 - 4AC}}{2A} \]
for \( x \in (0, \infty) \) and proceed to show that \( \delta(x) \) is nonnegative. It follows from the elementary identity
\[ \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{2C}{B + \sqrt{B^2 - 4AC}} \]
that \( \delta(x) \to 0 \) as \( x \to 0^+ \). If we can show that \( \delta'(x) \geq 0 \) for all \( x \in (0, \infty) \), then we will obtain

\[
\delta(x) \geq \lim_{t \to 0^+} \delta(t) = 0, \quad x \in (0, \infty).
\]

The rest of the proof is thus devoted to proving the inequality \( \delta'(x) \geq 0 \) for \( x \in (0, \infty) \).

By a direct computation, we have

\[
\delta'(x) = M \varphi' - \frac{M'A - MA'}{2A^2} \left( B - \sqrt{B^2 - 4AC} \right)
- \frac{M}{2A} \left( B' - \frac{BB' - 2(A'C + AC')}{\sqrt{B^2 - 4AC}} \right)
= \frac{M}{x} \left[ C - \left( \frac{xM'}{M} - \frac{xA'}{A} \right) \frac{B - S}{2A}
+ \frac{xB'}{2AS} (B - S) - \frac{xA'C + (1 - \alpha x)AC'}{AS} \right] .
\]

Noticing that \( B + S \geq 0 \) for any \( \alpha \in (-\infty, \infty) \). Multiplying \( \frac{xS(B+S)}{MC} \) on both sides of the above expressions of \( \delta'(x) \), and then using (1) and (2), we obtain that

\[
\delta' \sim (B+S)S - 2S \left( \frac{xM'}{M} - \frac{xA'}{A} \right)
+ 2xB' - \left( \frac{xA'}{A} + 1 - \alpha x \right) (B + S)
= \left( \frac{xM'}{M} - \frac{xA'}{A} \right) (B - S) + 2xB' - 4AC
= \left( \frac{xM'}{M} - \frac{xA'}{A} \right) (B - S) + 2xA' \varphi + 2xD(M(x)) \equiv d_1 .
\]

We will determine the sign of \( d_1 \). To this end, we let

\[
y = \frac{xM'}{M}
\]

and

\[
d_2 = \left( y - \frac{xA'}{A} \right) (B - S) + 2xA' \varphi.
\]

Note that \( A, C, A' \) are independent of \( y \) and \( B = 1 - \alpha x + y \). A simple computation shows that \( S'(y) = B/S \). Multiplying

\[
\frac{B + S}{-2xA' \varphi}
\]

on the both sides of the above expressions of \( d_2 \), we obtain that

\[
d_2 \sim \left( y - \frac{xA'}{A} \right) \cdot \frac{4AC}{-2xA' \varphi} - (B + S)
= \frac{\alpha}{\varphi A'} y + \left( \frac{2x}{\varphi} - 1 - \alpha x \right) - S \equiv d_3 .
\]

**Proof of Theorem 1, Continued.** Since \( \alpha < 0 \), using the assumption \( D(M(x)) \geq 0 \) we have \( d_1 \geq d_2 \).

By using Lemma 9(e) we can easily see that

\[
\frac{\alpha}{\varphi A'} \geq 1 .
\]
It follows from elementary calculus that we consider
\[ d_2 \sim \left( \frac{\alpha}{\varphi A'} y + \left(\frac{2x}{\varphi} - 1 - \alpha x\right) \right)^2 - S^2 \]
\[ = y \left( \frac{\alpha^2}{(\varphi A')^2} - 1 \right) y + 2 \frac{\alpha}{\varphi A'} \left( \frac{2x}{\varphi} - 1 - \alpha x \right) - 2(1 - \alpha x) \]
\[ \sim y - y_0, \]
where
\[ y_0 = \frac{x(1 - \alpha x) - \varphi}[\alpha \varphi^2 - 2(1 + \alpha x) \varphi + 2x]{(\varphi - x)(x - (1 + \alpha x) \varphi)} \]
Since
\[ -\alpha [\varphi - x(1 - \alpha x)] = e^{-\alpha x} - 1 + \alpha x - \alpha^2 x^2, \]
we consider
\[ u(t) = e^t - 1 - t - t^2, \quad t > 0. \]
It follows from elementary calculus that \( u(t) \) has a unique root \( t_0 = 1.79 \cdots \) on \((0, \infty)\) and \( u(t) < 0 \) on \((0, t_0)\) and \( u(t) > 0 \) on \((t_0, \infty)\). Hence \( \varphi - x(1 - \alpha x) \) has a unique root \( t_0 = t_0/(-\alpha) \) on \((0, \infty)\) and \( \varphi - x(1 - \alpha x) < 0 \) on \((0, x_0)\) and \( \varphi - x(1 - \alpha x) > 0 \) on \((x_0, \infty)\).

Note that \( y_0 \sim \varphi - x(1 - \alpha x) \). When \( x \leq x_0, y_0 \leq 0 \), so we have \( d_2 \geq 0 \) and hence \( d_1 \geq 0 \) on \((0, x_0)\). In particular, the function
\[ x \mapsto \ln \frac{\int_0^x M(t)e^{-\alpha t} dt}{\int_0^x e^{-\alpha t} dt} \]
is convex in \( \ln x \) for \( x \in (0, x_0) \). As for \( x \in I \cap (x_0, \infty) \), we have \( y_0 \geq 0 \), the assumption \( y \geq y_0 \) when \( x \in I \) implies \( d_2 \geq 0 \) and hence \( d_1 \geq 0 \) on \( I \cap (x_0, \infty) \). This shows that \( d_1 \) is always nonnegative and completes the proof of Theorem. \( \Box \)

**Proof of Theorem** Continued. Since \( \alpha < 0 \), it follows from Lemma that
\[ B - S \sim B^2 - S^2 = 4AC \geq 0 \quad \& \quad \frac{x A'}{A} \leq 0. \]
Hence
\[ d'_2(y) = B - S + \left( y - \frac{x A'}{A} \right)(1 - \frac{B}{S}) \]
\[ \sim S - \left( y - \frac{x A'}{A} \right)^2 \]
\[ \sim S^2 - \left( y - \frac{x A'}{A} \right)^2 \]
\[ = \frac{2(\varphi^2 - x(3 + \alpha x) \varphi + 2x^2)}{\varphi(\varphi - x)} y \]
\[ \sim \frac{\varphi(\varphi - x)}{\varphi(\varphi - x)^2} \frac{(\varphi - x)(-1 + 2\alpha x) \varphi^2 + x(5 + 3\alpha x) \varphi - 4x^2)}{\varphi(\varphi - x)^2} \]
It follows from Lemma that
\[ \varphi - x \geq 0 \quad \& \quad \varphi^2 - x(3 + \alpha x) \varphi + 2x^2 = (\varphi - x)^2 + x(x - (1 + \alpha x) \varphi) \geq 0, \]
and
\[-(1 + 2\alpha x)\varphi^2 + x(5 + 3\alpha x)\varphi - 4x^2
\]
\[= x(\varphi - x) + (x - (1 + \alpha x)\varphi)(\varphi - x) - x[\alpha\varphi^2 - 2(1 + \alpha x)\varphi + 2x]\]
\[\geq 0.\]

So we have \(d'_2(y) \sim y - y^*\), where
\[y^* = \frac{(\varphi - x(1 - \alpha x))((-1 + 2\alpha x)\varphi^2 + x(5 + 3\alpha x)\varphi - 4x^2)}{2(\varphi - x)(\varphi^2 - x(3 + \alpha x)\varphi + 2x^2)}.\]

Note that \(y^* \sim \varphi - x(1 - \alpha x)\).

When \(x \leq x_0\), \(y^* \leq 0\), \(d'_2(y)\) and hence \(d_2(y)\) is nonnegative. As for \(x > x_0\), \(y^* \geq 0\), \(d_2(y)\) attains its minimum value at \(y^* \in (0, \infty)\). A direct computation shows that
\[d_2(y^*) = -\frac{1}{2} \left(1 + \frac{\alpha x^2}{\varphi - x}\right)^2.\]

So we have
\[d_1 \geq 2x D(M(x)) + d_2(y^*)\]
\[\sim D(M(x)) - \frac{1}{4x} \left(1 + \frac{\alpha x^2}{\varphi - x}\right)^2.\]

This shows that \(d_1\) is always nonnegative and completes the proof of Theorem 3.

□

4. PROOF OF THEOREM 5

To demonstrate Theorem 5, we indicate how to adapt the proof of Theorem 1 or Theorem 3 above to show that \(\Delta(x) \leq 0\).

Suppose \(\alpha > 0\). Then \(A < 0\) by Lemma 9 and so \(\Delta(x) \leq 0\) is equivalent to
\[-\frac{\sqrt{B^2 - 4AC}}{2A} \leq \frac{h}{M} - \frac{B}{2A}.\]

So we need only to prove that \(\delta = \delta(x)\) defined in (2) is not positive for all \(x \in (0, \infty)\). It is enough for us to prove that \(\delta'(x) \leq 0\) since \(\delta(0) = 0\). We have proved that \(\delta'(x) \sim d_1\). Since \(M(x)\) is logarithmic concave, that is, \(D(M(x)) \leq 0\), we obtain \(d_1 \leq d_2\). But \(d_2 \sim d_3\). Recall that
\[d_3 = \frac{\alpha}{\varphi A'}y + \left(\frac{2x}{\varphi} - 1 - \alpha x\right) - S.\]

Noticing that
\[\frac{\alpha}{\varphi A'} \leq 0\]
by Lemma 9 we have
\[d_3 \leq \left(\frac{2x}{\varphi} - 1 - \alpha x\right) - S.\]

By the proof of Lemma 10 we have
\[S \geq \left|\frac{2x}{\varphi} - 1 - \alpha x\right|.\]

Thus we get \(d_3 \leq 0\) and hence \(d_2 \leq 0\). This shows that \(d_1 \leq 0\) and completes the proof of Theorem 5.
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CHUNJIE WANG, DEPARTMENT OF MATHEMATICS, HEBEI UNIVERSITY OF TECHNOLOGY, TIANJIN 300401, CHINA
E-mail address: wcj@hebut.edu.cn

JIE XIAO, DEPARTMENT OF MATHEMATICS AND STATISTICS, MEMORIAL UNIVERSITY, ST. JOHN’S, NL A1C 5S7, CANADA
E-mail address: jxiao@mun.ca