Abstract

A distributed computing protocol consists of three components: (i) Data Localization: a network-wide dataset is decomposed into local datasets separately preserved at a network of nodes; (ii) Node Communication: the nodes hold individual dynamical states and communicate with the neighbors about these dynamical states; (iii) Local Computation: state recursions are computed at each individual node. Information about the local datasets enters the computation process through the node-to-node communication and the local computations, which may be leaked to dynamics eavesdroppers having access to global or local node states. In this paper, we systematically investigate this potential computational privacy risks in distributed computing protocols in the form of structured system identification, and then propose and thoroughly analyze a Privacy-Preserving-Summation-Consistent (PPSC) mechanism as a generic privacy encryption subroutine for consensus-based distributed computations. The central idea is that the consensus manifold is where we can both hide node privacy and achieve computational accuracy. In this first part of the paper, we demonstrate the computational privacy risks in distributed algorithms against dynamics eavesdroppers and particularly in distributed linear equation solvers, and then propose the PPSC mechanism and illustrate its usefulness.

1 Introduction

The development of distributed control and optimization has become one of the central streams in the study of complex network operations, due to the rapidly growing volume and dimension of data and information flow from a variety of applications such as social networking, smart grid, and intelligent transportation [3].
Celebrated results have been established for problems ranging from optimization [4,5] and learning [6] to formation [7] and localization [8], where distributed control and decision-making solutions provide resilience and scalability for large-scale problems through allocating information sensing and decision making over individual agents. The origin of this line of research can be traced back to the 1980s from the work of distributed optimization and decision making [9] and parallel computations [10]. Consensus algorithms serve as a basic tool for the information dissemination of distributed algorithms [11,12], where the goal is to drive the node states of a network to a common value related to the network initials by node interactions respecting the network structure, usually just the average.

The primary power of consensus algorithms lies in their distributed nature in the sense that throughout the recursions of the algorithms, individual nodes only share their state information with a few other nodes that are connected to or trusted which are called neighbors. As a result, building on consensus algorithms, many algorithms can be designed which utilize the consensus mechanism to achieve network-level control and computation objectives, e.g., the recent work on network linear equation solvers [13,14], and convex optimization of a sum of local objective functions [5]. This emerging progress on distributed computation is certainly an extension of the classical development of distributed and parallel computation, but the new emphasis is on how a distributed computing algorithm for a network-wide problem can be designed with robustness and resilience over a given communication structure [13,14], instead of breaking down a high-dimensional system to a number of computing units with all-to-all communications.

Under a distributed computing structure, the information about the network-wide computation problem is encoded in the individual node initial values or update rules, which are not shared directly among the nodes. Rather, nodes share certain dynamical states based on neighboring communications and local problem datasets, in order to find solutions of the global problem. The node dynamical states may contain sensitive private information directly or indirectly, but in distributed computations to achieve the network-wide computation goals nodes have to accept the fact that their privacy, or at least some of it, will be inevitably lost. It was pointed out that indeed nodes lose their privacy in terms of initial values if an attacker, a malicious user, or an eavesdropper knows part of the node trajectories and the network structure [17,18]. In fact, as long as observability holds, the whole network initial values become reconstructable with only a segment of node state updates at a few selected nodes. Several insightful privacy-preserving consensus algorithms have been presented in the literature [19–23] in the past few years, where the central idea is to inject random noise or offsets in the node communication and iterations, or employ local maps as node state masks. The classical notion on differential privacy in computer science has also been introduced to distributed optimization problems [34,35], following the earlier work on differentially private filtering [36].

In this paper, we systematically investigate the potential computational privacy risks in distributed computing protocols, and then propose and thoroughly analyze a Privacy-Preserving-Summation-Consistent (PPSC) mechanism as a generic privacy encryption step for consensus-based distributed computations.
In distributed computing protocols, a network-wide dataset is decomposed into local datasets which are separately located at a network of nodes; the nodes hold individual dynamical states, based on which communication packets are produced and shared according to the network structure; local computations are then carried out at each individual node. The information about the local datasets enters the computation process through the node-to-node communication or the local computations; while techniques from system identification may be able to recover such datasets by observations of the node states. In this first part of the paper, we reveal the identification-type computational risk in distributed algorithms and particularly in distributed linear equation solvers, and then define a so-called PPSC mechanism for privacy preservation. The main results of the paper are summarized as follows.

- We show that distributed computing is exposed to privacy risks in terms the local datasets from both global and local eavesdroppers having access to the entirety or part of the node states. Particularly, for network linear equations as a basic computation task, we show explicitly how the computational privacy has indeed been completely lost to global eavesdroppers for almost all initial values, and partially to local passive or active eavesdroppers who can monitor or alter selected node states.

- We propose a Privacy-Preserving-Summation-Consistent (PPSC) mechanism which is shown to be useful as a generic privacy encryption step for consensus-based distributed computations. The central idea is to make use of the consensus manifold to on one hand hide node privacy, and on the other hand ensure computation accuracy and efficiency.

We remark that the existence of lower-dimensional manifold for the network-wide computation goal is very much not unique only for consensus-based distributed computing, and therefore the idea of the PPSC mechanism may be extended to other distributed computing methods. In the second part of the paper, we demonstrate that PPSC mechanism can be realized by conventional gossip algorithms, and establish their detailed privacy-preserving capabilities and convergence efficiencies.

Some preliminary ideas and results of this work were presented at IEEE Conference on Decision and Control in Dec. 2018 [1, 2]. In the current manuscript, we have established a series of new results on privacy loss characterizations and privacy preservation quantifications, in addition to a few new illustration examples and technical proofs.

The remainder the paper is organized as follows. In Section 2 we introduce a general model for distributed computing protocols, based on which we define the resulting computational privacy. The connection between computational privacy and network structure and initial-value privacies is also discussed. In Section 3 we investigate the potential loss of computational privacies in distributed computing algorithms, and in particular in distributed linear equation solvers. We also show that conventional methods for differential privacy often lead to non-ideal computation result for distributed computings facing fundamental privacy vs. accuracy trade-off in differential privacy can be challenging for a recursive computing
process. In Section 4, we define the PPSC mechanism and illustrate how we can design privacy-preserving algorithms for average consensus, distributed linear equation solvers, and in general for distributed convex optimization based on PPSC mechanism. Section 5 extends the discussion on computational privacy loss to local eavesdroppers and establish some results using classical system identification methods. Finally Section 6 concludes the paper with a few remarks. The various statements established in this paper is collected in the appendix.

**Notation.** We let $H_{ij}, H_i$ denote the $ij$-th entry and the transpose of the $i$-th row of a matrix $H$, respectively. For a vector $v$, $[v]_i$ denotes its $i$-th component. Let $\text{diag}(a_1, \ldots, a_n)$ with $a_i \in \mathbb{R}$ denote a diagonal matrix with diagonal entries being $a_1, \ldots, a_n$ in order. The transpose of a matrix or a vector $H$ are denoted by $H^\top$. Let $\text{tr}(\cdot)$ denote the trace of a square matrix. The orthogonal complement of a vector space $S$ is denoted by $S^\perp$. Let $\otimes$ denote the Kronecker product of two matrices or the direct sum of two sets. Conventionally, $\| \cdot \|_p$ is the $p$-norm of a vector or a matrix, and $\| \cdot \|$ is its 2-norm. Let $\lambda_m(\cdot), \lambda_M(\cdot)$ be the smallest and the largest eigenvalue of a real symmetric matrix. Let $\sigma_M(\cdot), \sigma_m(\cdot)$ denote the maximum and minimum absolute value of eigenvalues of a real symmetric matrix. Let $\| \cdot \|_F$ denote the Frobenius norm of a matrix. The range of a matrix or a function is denoted as range($\cdot$). We use $\mathbb{Z}, \mathbb{Z}^\geq 0, \mathbb{Z}^+, \mathbb{R}, \mathbb{R}^+$ to represent the set of integers, the set of nonnegative integers, the set of positive integers, the set of real numbers, and the set of positive real numbers, respectively. We let $\text{Pr}(\cdot)$ be the probability of an event in some probability space. We let $\text{pdf}(\cdot), \mathbb{E}(\cdot)$ denote the probability density function (PDF) and the expected value of a random variable, respectively. Let $\text{Lap}(v)$ with $v > 0$ denote the zero-mean Laplace distribution with variance $2v^2$. Correspondingly, $\text{Lap}^n(v)$ represents an $n$-dimensional vector of identical and independent random variables distributed according to $\text{Lap}(v)$.

2 Privacy Notions in Distributed Computation

In this section, we first introduce a basic model for distributed computation protocols which clearly indicates the problem decomposition, node communication, and node state update along the recursions. Next, we discuss potential eavesdroppers for distributed computing protocols and the resulting categories of privacy losses at both network and individual node levels.

2.1 Distributed Computations

Consider a network of $n$ nodes indexed in $V = \{1, \ldots, n\}$ whose communication structure is described by a graph $G = (V, E)$. We assume $G$ is an undirected connected graph for the sake of having a simplified presentation. Let $\mathcal{D}$ be a dataset whose elements are embedded in different dimensional Euclidean spaces. We write $\mathcal{D} = \mathcal{D}'$ if $\mathcal{D}$ and $\mathcal{D}'$ have identical records (up to some equivalence relation). Suppose $\mathcal{D}$ is allocated over the network by partition $\mathcal{D} = \mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_n$, where node $i$ holds $\mathcal{D}_i$ and $\otimes$ represents
direct sum to indicate potential logical or computational structure of the data points $\mathcal{D}$. Each node holds a dynamical state $x_i(t) \in \mathbb{R}^m$. Let $N_i = \{j : \{i, j\} \in E\}$ be the neighbor set of node $i$. The following protocol describes a general procedure for distributed computing algorithms.

**DCP Distributed Computing Protocols**

1: Set $t \leftarrow 0$ and $x_i(0) \in \mathbb{R}^m$ for $i \in V$.

2: Each node $i$ computes $c_i(t) = l_{i,t}(x_i(t), D_i)$ and sends $c_i(t)$ to all its neighbors $j \in N_i$.

3: Each node $i$ updates their states $x_i(t+1) \leftarrow f_{i,t}(x_i(t); c_j(t), j \in N_i; D_i)$.

4: Set $t \leftarrow t + 1$ and go to Step 2.

The $c_i(t)$ are the intermediate quantities for communications among the nodes, which often coincide with the nodes states $x_i(t)$. The mappings $l_{i,t}$ and $f_{i,t}$ can be deterministic or random, and they are distributed in the sense that they are mappings of the local data $D_i$ and information received from neighbors. The closed-loop of the above distributed computing protocol is described by

$$x_i(t+1) = f_{i,t}(x_i(t); l_{j,t}(x_j(t), D_j), j \in N_i; D_i). \quad (1)$$

In the following, we present a few examples in the literature showing how this DCP model covers average consensus and consensus-based distributed computing algorithms. To this end, let $W = [w_{ij}]$ be a doubly stochastic matrix in $\mathbb{R}^{n \times n}$ that complies with the graph $G$, i.e., $w_{ij} > 0$ if and only if $j \in N_i$ over the graph $G$ for $i \neq j$. In particular, we assume $w_{ii} > 0$ for all $i \in N_i$.

### 2.1.1 Average Consensus

Each node $i$ is assigned $D_i = \beta_i \in \mathbb{R}$. The initial state $x_i(0)$ is set as $\beta_i$. The mapping $l_{i,t}$ is identity, and therefore $c_i(t) = x_i(t)$. Then

$$x_i(t+1) = f_{i,t}(x_i(t); c_j(t), j \in N_i; D_i) = \sum_{j \in N_i \cup \{i\}} w_{ij} x_j(t) \quad (2)$$

represents a standard average consensus algorithm.

### 2.1.2 Distributed Linear Equations

Each node $i$ holds $D_i = (H_i, z_i)$ with $H_i \in \mathbb{R}^{m \times n}$ and $z_i \in \mathbb{R}$ defining a linear equation

$$\mathcal{E}_i : \quad H_i^\top y = z_i \quad (3)$$

with respect to an unknown $y \in \mathbb{R}^m$. The overall dataset $\mathcal{D}$ forms the following linear algebraic equation

$$\mathcal{E} : \quad Hy = z \quad (4)$$
with $H \in \mathbb{R}^{n \times m}$, $z \in \mathbb{R}^n$, where $H_i^\top$ is the $i$-th row of $H$ and $z_i$ denotes the $i$-th component of $z$. Let $P_i : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be the orthogonal projection onto the subspace $\{y \in \mathbb{R}^m : H_i^\top y = z_i\}$, which encodes the information of $D_i$. The following two algorithms are common distributed linear equation solvers.

- **[Consensus+Projection Algorithm (CPA)]** Let $c_i(t) = x_i(t)$. The following algorithm is an Euler approximation of the so–called “consensus + projection” flow [13–15]:
  \[
  x_i(t+1) = f_{i,t}(x_i(t); c_j(t), j \in N_i; D_i) = \sum_{j \in N \cup \{i\}} w_{ij} x_j(t) + \alpha \left( P_i(x_i(t)) - x_i(t) \right),
  \]
  where $\alpha > 0$ is a step-size parameter.

- **[Projection Consensus Algorithm (PCA)]** Let $c_i(t) = P_i(x_i(t))$. We can also have the following algorithm as a distributed linear equation solver [5,16]:
  \[
  x_i(t+1) = f_{i,t}(x_i(t); c_j(t), j \in N_i; D_i) = \sum_{j \in N \cup \{i\}} w_{ij} P_j(x_j(t)).
  \]

2.1.3 Distributed Convex Optimization

Each node holds $D_i$ being a differentiable convex function $f_i : \mathbb{R}^m \rightarrow \mathbb{R}$. The overall dataset $D = \sum_{i=1}^n f_i(\cdot)$. Again $c_i(t) = x_i(t)$, and the node update is given by [4]

\[
  x_i(t+1) = f_{i,t}(x_i(t); c_j(t), j \in N_i; D_i) = \sum_{j \in N \cup \{i\}} w_{ij} x_j(t) - \epsilon_i \nabla f_i(x_i(t)),
\]

which describes a distributed gradient descent algorithm.

2.2 Communication and Dynamics Eavesdroppers

In practice, there can be two typical types of malicious eavesdroppers for a distributed computing process. Communication eavesdroppers are adversaries who can intercept node-to-node communications, i.e., the $c_i(t)$. Dynamics eavesdroppers are adversaries who can monitor the node dynamical states $x_i(t)$. For the choice of the communication mechanism

\[
c_i(t) = \ell_{i,t}(x_i(t), D_i),
\]

we can see that eavesdroppers that have access to both the node states and node communications may easily identify $D_i$ to the extent that $c_i(t)$ depends on the data $D_i$. For the two categories of eavesdroppers, they can also have distinct capabilities.

(i) Eavesdroppers may have different knowledge about the network, e.g., knowledge of the number of nodes, network structure, and mechanisms of the communication and node updates.
Eavesdroppers can be local or global in the sense that they are able to monitor the communication or node states of the entire network or only at one or a few nodes.

Eavesdroppers can be active or passive by whether they are able to or willing to influence the observed data. E.g., a passive communication eavesdropper records $c_i(t)$ at a node $i$ only, while an active eavesdropper might aim to replace $c_i(t)$ with some injected data.

In this paper, we focus on dynamics eavesdroppers, motivated by applications of distributed computing where the node dynamical states $x_i(t)$ might represent physical quantities, e.g. the powers and voltages of generators in a smart grid, the velocities and accelerations of trucks in a platoon, and the workloads of computing servers in a cloud. Such physical quantities might be potentially measurable by malicious sensors. We limit our discussion on the effect of communication eavesdroppers because on the one hand, existing cryptography methods have the capacity to enable secure node-to-node communication; and on the other hand, communication eavesdroppers fall to the category of dynamics eavesdroppers when $c_i(t) = x_i(t)$, which holds true for many existing distributed computing algorithms.

### 2.3 Privacy Notions and Related Works

There can be various notions of privacies in a dynamical network process. First of all, the node dynamics reveal information about the network itself. As a result, dynamics eavesdroppers may infer statistics or even the topology of the network. This type of study has been investigated in the context network tomography and network identification problems [27–30]. Furthermore, for an individual node $i$, its dynamical state $x_i(t)$, and especially its initial value $x_i(0)$ may directly contain sensitive information. We present the following illustrative example.

**Example 1.** Let $i$ represent an individual in a company of in an organization. Let $\beta_i \in \mathbb{R}$ be the monthly salary of the individual $i$. For each month, the individuals can run the average consensus algorithm (2) with $x_i(0) = \beta_i$, and as $t$ tends to infinity, each node will be able to learn the value of $\sum_{j=1}^{n} \beta_j / n$. Along this computation process [2], it has been shown that with observability condition and knowledge about the weight matrix $W$, it is possible to recover $x(0)$ from observation of a finite trajectory at a single node $i$ [18]. On the other hand, the network structure $G$ may be re-constructible by a node knockout process established in [27]. Additionally, in this process, node $i$ reveal its states to its neighbors, which means that it loses its privacy with respect to $\beta_i$ to neighbors right at the first step.

In this paper, we are primarily interested in the privacy issues defined by the local datasets $D_i$ against dynamics eavesdroppers. These local datasets $D_i$ may be sensitive in terms of privacy in general dynamical networks, e.g., the local cost functions in distributed economical MPC directly reflects the economy of individual subsystems [24]. We aim to establish concrete results on the possibilities of identifying the
local datasets $D_i$ in distributed computing schemes. This problem is precisely a parameter identification problem in classical system identification literature \cite{25,26}, but with specific dynamics structure inherited from the network topology and problem setup. We term this type of privacy as computational privacy in a distributed computing protocol, and use distributed solvers for network linear equations as a basic example to illustrate the computational privacy loss.

3 Dynamical Privacy and PPSC Mechanism

In this section, we define and specify computational risks in a distributed computing protocol, subject to either global or local eavesdroppers of node dynamical states.

3.1 Dynamics Eavesdroppers

We introduce the following definition.

**Definition 1.** For a distributed computing protocol (1), a global dynamics eavesdropper is a party who has access to the node states $(x(t))_{t=0}^\infty$ with initial condition $x(0) = x_0$. For global dynamics eavesdroppers, the distributed computing protocol (1) is computationally private with respect to the dataset $D$ under initial condition $x_0 \in \mathbb{R}^{nm}$ if the dataset $D$ is not recoverable from $(x(t))_{t=0}^\infty$.

A distributed computing protocol (1) induces a mechanism $\mathcal{M}$ which maps from the space of the data $D$ to the space of the trajectories $(x(t))_{t=0}^\infty$, by

$$\mathcal{M}_{x_0}(D) = (x(t))_{t=0}^\infty.$$ 

The following statement provides a generic condition for computational privacy.

**Proposition 1.** For the distributed computing protocol (1), the following statements hold.

(i) With deterministic update, the distributed computing protocol (1) is globally computationally private with respect to the dataset $D$ under initial condition $x_0 \in \mathbb{R}^{nm}$ if and only if there exists $D' \neq D$ such that $\mathcal{M}_{x_0}(D) = \mathcal{M}_{x_0}(D')$.

(ii) With randomized update, the distributed computing protocol (1) is globally computationally private under initial condition $x_0 \in \mathbb{R}^{nm}$ with respect to the dataset $D$, if and only if there exists $D' \neq D$ such that pdf$(\mathcal{M}_{x_0}(D)) = \text{pdf}(\mathcal{M}_{x_0}(D'))$.

In the following, we show some more explicit characterizations of the computational privacy in network linear equation solvers. Let $\mathcal{E} = \{a^\top y = b : a \in \mathbb{R}^m, b \in \mathbb{R}\}$ be the space of linear equations over $\mathbb{R}^m$, and then let $\mathcal{E}_{nm}^* = \{Ay = b : A \in \mathbb{R}^{n \times m}, b \in \mathbb{R}^n, \text{ b \in range(A)}\}$ be the space of the solvable $n$-dimensional linear equations over $\mathbb{R}^m$.  

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Definition 2. (i) Let $E : a^\top y = b$ and $E' : a'^\top y = b'$ be two equations in $\mathcal{E}$. We term that $E$ and $E'$ are equivalent if there exists $\alpha \in \mathbb{R} \setminus \{0\}$ such that $a' = \alpha a$, $b' = \alpha b$.

(ii) Let $E : Ay = b$ and $E' : A'y = b'$ be two equations in $\mathcal{E}_{[nm]}^*$. We say that $E$ and $E'$ are equivalent if there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{R} \setminus \{0\}$ such that $A' = \text{diag}(\alpha_1, \ldots, \alpha_n)A$, $b' = \text{diag}(\alpha_1, \ldots, \alpha_n)b$.

It is worth mentioning that equipped with the equivalence relation in Definition 2, each equation $E \in \mathcal{E}_{[nm]}^*$ corresponds to a unique equivalence class: $\{E' \in \mathcal{E}_{[nm]}^* : E' \sim E\}$. This further specifies the quotient space $\mathcal{E}_{[nm]}^*/\sim := \{\{E' \in \mathcal{E}_{[nm]}^* : E' \sim E\} : E \in \mathcal{E}_{[nm]}^*\}$. It can be easily verified that the linear equations in the same equivalence class share a common solution set. Therefore, with Definition 2, we have effectively identified the information of a solvable linear equation with its solution set for equations in $\mathcal{E}_{[nm]}^*$. For the two distributed linear equation solvers in (5) and (6), we have the following result.

Theorem 1. Consider a linear equation $E \in \mathcal{E}_{[nm]}^*$. Assume that the weight matrix $W$ is public knowledge.

(i) Suppose $\alpha$ is also public knowledge. The CPA (5) is not computationally private in the quotient space $\mathcal{E}_{[nm]}^*/\sim$ against global dynamics eavesdroppers for $E$ under almost all initial conditions.

(ii) The PCA (6) is not computationally private in the quotient space $\mathcal{E}_{[nm]}^*/\sim$ against global dynamics eavesdroppers for $E$ under almost all initial conditions.

The following example illustrates how the information about the linear equation has been lost during the recursion of a distributed linear equation solver, which partially explains the intuition behind Theorem 1.

Example 2. Consider a 4–node star graph centered at node 1 and the following linear equation:

$$E : \begin{bmatrix} 3 & -1 \\ 1.5 & 0.8 \\ -2 & 1.5 \\ -1.2 & 4 \end{bmatrix} y = \begin{bmatrix} 5 \\ -0.1 \\ -5 \\ -9.2 \end{bmatrix}.$$  

Let the CPA (5) with $\alpha = 0.1$ and

$$W = \begin{bmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0.3 & 0.7 & 0 & 0 \\ 0.2 & 0 & 0.8 & 0 \\ 0.4 & 0 & 0 & 0.6 \end{bmatrix}$$

starting from some initial condition be implemented.

A geometric illustration of the way that a global dynamics eavesdropper reconstructs $E$ is shown in Figure 1. In Figure 1(a), the states $x_i(t), t = 0, 1, \ldots, 10$ of nodes $i = 1, 2, 3, 4$ are plotted by the dot points of different colors. Then the projections $P_i(x_i(t)), t = 0, 1, \ldots, 9$ for all $i$ are computed by the eavesdropper according to (5), as marked by the stars in Figure 1(b). By $P_i(x_i(t))$ and the perpendicularity of the
Figure 1: (a) shows the trajectories of the node states $x_i(t)$ with $t = 0, 1, \ldots, 10$ and $i = 1, 2, 3, 4$. (b) demonstrates the computed projection points $P_i(x_i(t)), t = 0, 1, \ldots, 9$ by eavesdroppers. (c) illustrates the disclosed solution affine space of the linear equations.
projections, the solution spaces of $E'_i$s are identified as shown in Figure 1 (c). As a result, an equivalent set $\{E': E' \sim E\}$ has been reconstructed by the eavesdropper.

### 3.2 Differentially Private Computation

From the above investigation of distributed linear equation solvers, we now have a concrete example where the local datasets for a distributed computation setup may indeed be lost during the computation process. We acknowledge that information nodes have to reveal certain amount of information about the local datasets $D_i$ in the overall closed-loop dynamical equation (1): If absolutely no information about $D_i$ enters (1), the eventual computation outcome will not depend on $D_i$ and therefore will never be a network-wide solution. It has been shown how we can borrow the idea of differential privacy in distributed optimization problems, e.g., [34,35], following the earlier work on differentially private filtering [36].

#### 3.2.1 Differentially Private Distributed Computation

Differential privacy is a concept that statistically quantifies the ability of an information protocol to protect the privacy of confidential databases, and meanwhile provide as accurate query output as possible. It was formally proposed in [33], with the intuition being that a query that comes from a database is nearly indistinguishable even if a record in the database is modified or removed. Recall that the dataset $D$ is allocated over the network by partition $D = D_1 \otimes \cdots \otimes D_n$. We say $D' = D'_1 \otimes \cdots \otimes D'_n$ is adjacent with $D$ if there exists $i \in \{1, \ldots, n\}$ such that

$$D_j = D'_j, \quad \forall j \neq i.$$ 

Recall that $\mathcal{M}_{x_0}(\cdot)$ is the mechanism along a randomized distributed computing protocol defined by $\mathcal{M}_{x_0}(D) = (x(t))_{t=0}^\infty$. We can then borrow conventional differential privacy notion to define that $\mathcal{M}_{x_0}$ is $\epsilon$-differentially private with some privacy budget $\epsilon > 0$ if for all adjacent $D, D'$, and for all $R \subset \text{range}(\mathcal{M}_{x_0})$,

$$\Pr(\mathcal{M}_{x_0}(D) \in R) \leq e^\epsilon \cdot \Pr(\mathcal{M}_{x_0}(D') \in R),$$

where the probability is taken over the randomness used by $\mathcal{M}_{x_0}$.

In the differential privacy literature [33], one popular approach is the so-called Laplace mechanism, where Laplace noise with appropriate randomness is added to the output returned from the input data. One can apply this approach to the distributed computing protocol (1) by simply viewing $\mathcal{M}_{x_0}$ as a data processing mechanism and resolving differential privacy challenges with added noise [34,35]. However, adding noise to (1) will in general make it impossible for the protocol to converge to the true value, and in many cases one can only establish error guarantee that is increasingly conservative as time moves forward under rather restrictive conditions [34,35]. In contrast, classical differential privacy results manage
Definition 4. Consider a distributed solver for linear equations in the space \( \mathcal{D}P–DLES \) (Differentially Private Distributed Linear Equation Solver) data [33]. The reason for this sharp comparison is that, the dimension of the outputs of \( \mathcal{M}_{x_0} \) (i.e., the \( x(t) \)) increases linearly with time steps, controlling the error the way how a standard differentially private query system works becomes challenging or even impossible. In the following, we illustrate this point for distributed LAE solvers.

3.2.2 Differentially Private Network LAE Solving

For network linear equations, we can define adjacency properties of different datasets over the quotient space.

**Definition 3.** Consider two linear equations \( \mathcal{E} : A y = b, \mathcal{E}' : A' y = b' \in \mathcal{E}^*_{[nm]} \). We call them to be \( (\delta_A, \delta_b)–adjacent \) if there exists \( i \in \{1, \ldots, n\} \) such that \( \mathcal{E}_j \sim \mathcal{E}'_j, \forall j \neq i \) and

\[
\left\| \frac{A_i A_i^\top}{A_i^\top A_i} - \frac{A_i' A_i'^\top}{A_i'^\top A_i'} \right\| \leq \delta_A, \quad \left\| \frac{b_i}{A_i^\top A_i} - \frac{b_i'}{A_i'^\top A_i'} \right\| \leq \delta_b.
\]

**Definition 4.** Consider a distributed solver for linear equations in the space \( \mathcal{E}^*_{[nm]} \). Let \( \mathcal{M}_{x_0}(\cdot) \) be the resulting mechanism with \( \mathcal{M}_{x_0}(\mathcal{E}) = (x(t))_{t=0}^{\infty} \). Then the solver is called \( \epsilon \)-differentially private under \( (\delta_A, \delta_b)–adjacency \) if

\[
\Pr(\mathcal{M}_{x_0}(\mathcal{E}) \in R) \leq e^{\epsilon \cdot \Pr(\mathcal{M}_{x_0}(\mathcal{E}') \in R)},
\]

for all \( (\delta_A, \delta_b)–adjacent \) linear equations \( \mathcal{E} : A y = b, \mathcal{E}' : A' y = b' \in \mathcal{E}^*_{[nm]} \), for any initial condition \( x(0) \in \mathbb{R}^m \), and for all network state trajectories in range(\( \mathcal{M}_{x_0} \)).

We assume the network of nodes have prior knowledge about a linear equation \( \mathcal{E} \) that one of its solutions falls in a compact and convex set \( \Omega \subset \mathbb{R}^m \). We define a projection \( \mathcal{P}_\Omega : \mathbb{R}^m \to \mathbb{R}^m \) with\( \mathcal{P}_\Omega(\mathcal{v}) = \inf_{\mathcal{v} \in \Omega} \| \mathcal{v} - \mathcal{u} \| \). By adapting the deterministic CPA [5], we now present the following algorithm as a Laplace–mechanism–based differentially private distributed LAE solver, where Laplace noise \( \text{Lap}^m(c \phi^t) \) with \( c > 0, 0 < \phi < 1 \) is injected prior to node state broadcasting at each time \( t \). Define \( \alpha(t) = \lambda \psi^t \) with \( \lambda > 0 \) and \( 0 < \psi < 1 \) as a time–varying step size. Following the idea of [34], we propose the following differentially private distributed linear equation solver based on the Laplace approach.

**DP–DLES Differentially Private Distributed Linear Equation Solver**

1. Set \( t \leftarrow 0 \) and \( x(0) \).
2. Each node \( i \in V \) computes \( x_i^0(t) = \mathcal{P}_\Omega(x_i(t)) \).
3. Each node \( i \in V \) draws \( \omega_i(t) \) from the distribution \( \text{Lap}^m(c \phi^t) \).
4. Each node \( i \in V \) computes \( x_i^0(t) \leftarrow x_i^0(t) + \omega_i(t) \) and propagates \( x_i^0(t) \) to all its neighbors \( j \in N_i \).
5. Each node \( i \in V \) computes \( x_i(t + 1) \leftarrow \sum_{j \in N_i} W_{ij} x_j^0(t) + \alpha(t) \left( \mathcal{P}_i(x_i^0(t)) - x_i^0(t) \right) \).
6. Set \( t \leftarrow t + 1 \) and go to Step 2.
We present the following results.

**Theorem 2.** Suppose $W \in \mathbb{R}^{n \times n}$ has full rank. Let $B = \sup_{v \in \Omega} \|v\|$. Then the DP–DLES preserves $\epsilon$–differential privacy under $(\delta_A, \delta_b)$–adjacency if $\psi < \phi$ with

$$\frac{\phi}{\phi - \psi} \cdot \frac{\lambda}{c} \cdot \frac{\sqrt{nm(B\delta_H + \delta_z)}}{\sigma_m(W)} \leq \epsilon.$$  \hspace{1cm} (8)

Note that $\epsilon$ is the privacy budget which is expected to be a small number. As a result, for a fixed amount of noise injected to the computation process represented by $c$ and $\phi$ in $\text{Lap}^m(c\phi^t)$, $\lambda$ and $\psi$ to be small enough for (8) to hold. The values of $\lambda$ and $\psi$, however, determine how much local data information $D_i$ is used in the computation step through the $P_i$. Therefore, small $\lambda$ and $\psi$ means greater computation error will take place. This reveals the privacy vs computation dilemma, which can be seen from the following example.

**Example 3.** Consider the same linear equation, network structure and weight matrix $W$ as Example 1. We let $\Omega$ be a ball centered at $y^*$ with radius one. We tune the parameters of DP–DLES such that DP–DLES preserves $(\epsilon, 1, 1)$–differential privacy with $\epsilon = 2, 4, 6, 8$. Corresponding to the parameter setup for each $\epsilon$, we independently execute DP–DLES from starting from initial states sampled from $[-1, 1]$ at random for $10^3$ times, and plot the averaged

$$\| \sum_{i=1}^4 x_i(t) - y^* \|$$

in Figure 2. It is clear from the numerical result that when better differential privacy is achieved, worse computational accuracy is recorded.

![Figure 2: The plot of the averaged $\| \sum_{i=1}^4 x_i(t) - y^* \|$ along DP–DLES with $\epsilon = 2, 4, 6, 8$.](image)
4 PPSC Mechanism

In view of the above discussions, we now see that on one hand, dynamical privacy indeed exists in distributed computing processes; and on the other hand, adding noise to the computational recursions improves privacy but in the meantime jeopardizes the computation accuracy. Noticing the fact that in average consensus and in consensus-based distributed computing algorithms, the consensus manifold is where the network-wide solution lies in, for which the node states can stay by local interactions even in the presence of additional random noise. In this section, we show that we can effectively use this observation to hide node privacy, under a so-called PPSC mechanism.

4.1 Mechanism Definition

**Definition 5.** Let \( \beta_i \in \mathbb{R}^m \) be held by node \( i \) over the network \( G = (V, E) \). An algorithm running over the network is called a distributed Privacy-Preserving-Summation-Consistent (PPSC) mechanism, which produces output \( \beta^\# = (\beta^\#_1^\top \ldots \beta^\#_n^\top)^\top \) from the network input \( \beta = (\beta_1^\top \ldots \beta_n^\top)^\top \), if the following conditions hold:

(i) (Graph Compliance) Each node \( i \) communicates only with its neighbors in the set \( N_i \);

(ii) (Local Privacy Preservation) Each node \( i \) never reveals its initial value \( \beta_i \) to any other agents or a third party;

(iii) (Global Privacy Preservation) \( \beta \) is non-identifiable given \( \beta^\# \) with even an infinite number of independent samples;

(iv) (Summation Consistency) \( \sum_{i=1}^{n} \beta_i^\# = \sum_{i=1}^{n} \beta_i \).

The condition (i) requires that the information flow of the algorithm must comply with the underlying graph; the condition (ii) says that during running of the algorithm no node will have to directly reveal its initial value \( \beta_i \); the condition (iii) further suggests that even if final node states are known, it should not be possible to use that knowledge to recover \( \beta_i \); the condition (iv) asks for that sum of the output of the algorithm should be consistent with that of the input. The overall mechanism of PPSC is denoted as the following mapping:

\[
\beta^\# = \mathcal{P}(\beta). \tag{9}
\]

4.2 PPSC for Distributed Computation

Now we show that the PPSC mechanism can be used as a privacy preservation step for distributed computing algorithms.
**Average Consensus** For the average consensus algorithm \(^{(2)}\), we can run a distributed PPSC algorithm over \((\beta_1, \ldots, \beta_n)\) and generate \((\beta_1^{\#}, \ldots, \beta_n^{\#})\). Letting \(x_i(0) = \beta_i^{\#}, \ i \in V\) in instead, there still holds
\[
\lim_{t \to \infty} x_i(t) = \frac{\sum_{j=1}^{n} \beta_j}{n}, \text{ but } \beta_i \text{ has been kept as private information for node } i.
\]

**Network Linear Equations** For the network linear equation \(^{(4)}\), the PPSC mechanism can provide distributed solver with privacy guarantees.

**Privacy Preserving Linear Equation Solver**

1: Set \(t \leftarrow 0\) and \(y_1(t), \ldots, y_n(t) \in \mathbb{R}^r\);
2: Run a PPSC algorithm with input \(y_1(t), \ldots, y_n(t)\) and produce output \(y_1^{\#}(t), \ldots, y_n^{\#}(t)\);
3: Run the averaging consensus algorithm with input \(y_1^{\#}(t), \ldots, y_n^{\#}(t)\) and output the agreement \(\bar{y}(t)\) at each node \(i\);
4: Each node \(i\) computes \(y_i(t+1) \leftarrow \mathcal{P}_i(\bar{y}(t))\). Set \(t \leftarrow t + 1\) and go to Step 2.

The above privacy preserving linear equation solver produces the following recursion:
\[
y_i(t + 1) = \mathcal{P}_i \left( \frac{1}{n} \sum_{j=1}^{n} y_j(t) \right), \quad t \in \mathbb{Z}^{\geq 0}, \ i \in V,
\]
which falls to the category of the projected consensus algorithms in \(^{(5)}\). According to Proposition 2 in \(^{(5)}\), each node state \(y_i(t)\) converges to a solution of the linear equation \(^{(4)}\) if it admits exact solutions, i.e., \(z \in \text{span}\{H\}\).

**Distributed Gradient Descent** With the PPSC mechanism, the distributed gradient descent algorithm \(^{(7)}\) can also be made privacy-preserving, by the following procedure.

**Privacy Preserving Distributed Optimizer**

1: Set \(t \leftarrow 0\) and \(y_1(t), \ldots, y_n(t) \in \mathbb{R}^r\);
2: Each node \(i\) computes \(y_i^{\#}(t) \leftarrow y_i(t) - \frac{1}{\sqrt{t+1}} \nabla f_i(y_i(t))\);
3: Run a PPSC algorithm with input \(y_1^{\#}(t), \ldots, y_n^{\#}(t)\) and produce output \(y_1^{\#}(t), \ldots, y_n^{\#}(t)\);
4: Run the averaging consensus algorithm with input \(y_1^{\#}(t), \ldots, y_n^{\#}(t)\) and output the agreement \(\bar{y}(t)\) at each node \(i\);
5: Each node \(i\) sets \(y_i(t+1) \leftarrow \bar{y}(t)\).
6: Set \(t \leftarrow t + 1\) and go to Step 2.

The underlying dynamics of the privacy-preserving distributed optimizer is described by
\[
y_i(t + 1) = \frac{1}{n} \sum_{j=1}^{n} y_j(t) - \frac{1}{\sqrt{t+1}} \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(y_j(t)),
\]
Eq. (11)
4.3 Privacy Preservation by PPSC

From the above three examples of integrating a PPSC step in each round of recursion of distributed computing algorithms, we can see that the PPSC mechanism effectively achieves the following advantages in a distributed computing protocol:

(i) Nodes keep their real states \( x_i(t) \) strictly from themselves, and receive only virtual states \( \hat{x}_i(t) \) produced by the PPSC mechanism \( \mathcal{P} \). The impossibility of identifying \( x(t) \) from \( \hat{x}(t) = \mathcal{P}(x(t)) \) (with even an arbitrary number of realizations of \( \hat{x}(t) \) for each \( t \)) provides the nodes with plausible deniability about their real states \( x(t) \) for all \( t \), which immediately implies the same deniability about the \( D_i \).

(ii) The price paid in the PPSC step is that averaging the \( \hat{x}(t) = \mathcal{P}(x(t)) \) may take more time than averaging the \( x(t) \) over the same graph \( G \). Since \( \hat{x}_i(t) \) has proven privacy guarantee, there is no need to share it with only neighbors for privacy concerns. Each node \( i \) can broadcast \( \hat{x}_i(t) \) to the entire network whenever possible, and therefore may compensate the efficiency lost in the averaging step.

(iii) The sum consistency property from PPSC mechanism guarantees that the computation produces the same result with or without the PPSC privacy preservation steps in the above consensus-based linear equation solving and gradient descent algorithms.

In Part II of the paper, we will show that conventional gossip algorithms can be used to realize PPSC mechanism in distributed manners over a network by only a finite number of steps, and beyond non-identifiability of the PPSC mechanism \( \mathcal{P} \), we can even easily ensure \( \mathcal{P} \) to be differentially private.

We also acknowledge that a weakness of the above PPSC-based distributed computation is that it involves running PPSC and average consensus to completion for every gradient step, which is likely to be very slow over the network \( G \) and lead to truncation errors when we use only a finite number of average consensus. However, since \( \beta^\hat{} \) can be proven privacy preserving, each node \( i \) can broadcast \( \beta_i \) to the entire network for the average consensus, instead of communicating with neighbors in \( G \) only. As a result, averaging \( \beta^\hat{} \) can be much faster compared to averaging \( \beta \) over \( G \), producing the same consensus value. This would not resolve the effect of finite-step truncation errors, which is left to future work.

5 Local Dynamical Privacy

Finally, we turn our attention to local dynamics eavesdroppers, where two types of behaviors, passive or active, are distinguished for the role of a local eavesdropper in the network dynamics.
5.1 Local Dynamics Eavesdroppers

We introduce the following definition.

Definition 6. (i) A passive local dynamics eavesdropper is a party adherent to one node \( i \in V \) and therefore has access to the observations of \( \{x_j(t), j \in N_i\}_{t=0}^\infty \).

(ii) An active local dynamics eavesdropper is a party adherent to one node \( i \in V \), has access to the observations of \( \{x_j(t), j \in N_i\}_{t=0}^\infty \), and can arbitrarily alter the dynamics of \( x_i(t) \).

In either of the two cases in Definition 6 we assume without loss of generality that such a local dynamics eavesdropper is precisely a node \( i \in V \). From the perspectives of local eavesdroppers, recovering information about the \( D \) from \( \{x_j(t), j \in N_i\}_{t=0}^\infty \) defines a structured system identification problem. Let \( N_i = \{j_1, \ldots, j_{|N_i|}\} \) with \( j_1 < \cdots < j_{|N_i|} \). Define \( E_i \in \mathbb{R}^{|N_i|\times n} \) as the matrix whose entries in the \( k \)-th row are all zeros except for the \( kj_k \)-th entry being one. Denote \( F = W \otimes I_m - \alpha \text{diag} \left( \frac{H_iH_i^T}{H_i^TH_i}, \cdots, \frac{H_iH_n^T}{H_n^TH_n} \right) \).

Introduce

\[
\mathcal{J} = \{ T \in \mathbb{R}^{nm \times nm} : T \text{ is invertible}, (E_i \otimes I_m)T = [I_{|N_i|} \ 0] \}.
\]

For the CPA against passive local dynamics eavesdroppers, we can establish the following understanding based on the result on obtaining the minimal realization of LTI systems from input–output data in time domain established in \([31]\).

Theorem 3. Consider the distributed linear equation solver CPA for \( \mathcal{E} \subset \mathcal{E}_{[nm]} \). Assume that the weight matrix \( W \) and the step size \( \alpha \) are public knowledge. Suppose the following two conditions hold: (i) \((F, E_i \otimes I_m)\) is a completely observable pair; (ii) There exist \( t_1 < \cdots < t_{nm} \) such that the vectors \( x(t_k), k = 1, \ldots, nm \) are linearly independent. Then for the passive local dynamics eavesdropper \( i \), it can be determined from \( \{x_j(t), j \in N_i\}_{t=0}^\infty \) a subset \( \mathcal{E}^\dagger \subset \mathcal{E}_{[nm]} \) such that \( \mathcal{E} \in \mathcal{E}^\dagger \), where

\[
\mathcal{E}^\dagger = \left\{ Ay = b : \text{diag} \left( \frac{A_1A_1^T}{A_1^TA_1}, \cdots, \frac{A_nA_n^T}{A_n^TA_n} \right) = (W \otimes I - T^{-1}F_sT)/\alpha; b = A \lim_{t \to \infty} x_i(t); T \in \mathcal{J} \right\}.
\]

Here \( F_s \in \mathbb{R}^{nm \times nm} \) depends on \( \{x_j(t), j \in N_i\}_{t=0}^\infty \) only.

For the CPA against an active local eavesdropper \( i \), we provide the following result based on the time–domain–based system identification method proposed in \([32]\).

Theorem 4. Consider the distributed linear equation solver CPA for \( \mathcal{E} \subset \mathcal{E}_{[nm]} \). Assume that \( \mathcal{E} \) admits a unique solution \( y^* \in \mathbb{R}^m \). Assume further that the weight matrix \( W \), the step size \( \alpha \) and the solution \( y^* \) are public knowledge. Then there is a simple strategy for an active local dynamics eavesdropper \( i \) by adding a periodic signal \( r : \mathbb{Z}^{\geq 0} \to \mathbb{R}^m \) to \( x_i(t) \) at each time \( t \) under which it can identify \( \mathcal{E}^\dagger \subset \mathcal{E}_{[nm]} \) with
$\mathcal{E} \in \mathcal{E}^\dagger$, given by

$$
\mathcal{E}^\dagger = \left\{ Ay = b : \text{diag} \left( \frac{A_1 A_1^\top}{A_1}, \ldots, \frac{A_n A_n^\top}{A_n} \right) = (W \otimes I - T^{-1} F_\ast T)/\alpha; \ b = Ay^*; \ T \in \mathcal{T} \right\},
$$

$$
\mathcal{T} = \left\{ T \in \mathbb{R}^{nm \times nm} : T \text{ is invertible, } (E_i \otimes I_m) T = C_\ast \right\}.
$$

Here $F_\ast \in \mathbb{R}^{nm \times nm}$ and $C_\ast \in \mathbb{R}^{\lvert N_i \rvert \times nm}$ depend on $r(t)$ and $\{ x_j(t), j \in N_i \}_{t=0}^\infty$ only.

From Theorem 3 and Theorem 4, a passive or active eavesdropper eventually finds $A_\ast \in \mathbb{R}^{nm \times nm}$ and $C_\ast \in \mathbb{R}^{\lvert N_i \rvert \times nm}$ such that $Z_H = \text{diag} \left( H_1 H_1^\top, \ldots, H_n H_n^\top \right)$ satisfies

$$(W \otimes I_m - \alpha Z_H) T = TA_\ast,
$$

$$(E_i \otimes I_m) T = C_\ast.
$$

Introduce a new matrix variable $Q = Z_H T$. Then solving $Q$ and $T$ from the above equations will lead to the recovering of $Z_H$, which further yields $E$ in the quotient space in view of the knowledge of $y^\ast$. Then we can establish by vectorization that

$$
\begin{bmatrix}
S_\ast & -\alpha I_{nm^2} \\
I_{nm} \otimes (E_i \otimes I_m) & 0
\end{bmatrix}
\begin{bmatrix}
\text{vec}(T) \\
\text{vec}(Q)
\end{bmatrix}
= \begin{bmatrix} 0 \\ \text{vec}(C_\ast) \end{bmatrix}
$$

(12)

where $S_\ast = I_{nm} \otimes (W \otimes I_m) - A_\ast \otimes I_{nm}$. This linear equation about

$$(\text{vec}(T)^\top, \text{vec}(Q)^\top)^\top$$

is underdetermined, and therefore exploring the linear structure of the obtained equations is not enough for the eavesdroppers to recover $E$. However, the $Q$ and $T$ have additional constraints: $T$ must be nonsingular, and $QT^{-1}$ has rank-one diagonal blocks. These highly nonlinear but strong constraint conditions might eventually locate a unique solution from the solution set of (12).

5.2 Numerical Example

**Example 4.** Let the CPA (5) be implemented with the same setup in Example 1 on the graph in Figure 1 for solving a linear equation $E : Hy = z$ with

$$
H = \begin{bmatrix}
71.5 & -65.5 \\
-95 & 47.1 \\
-35.5 & 100 \\
86.5 & -69
\end{bmatrix}, \ z = \begin{bmatrix}
-202.5 \\
189.2 \\
235.5 \\
-224.5
\end{bmatrix}.
$$
Suppose a local eavesdropper is at node 2 and system identification methods in [31,32] have led to

\[
A^* = \begin{bmatrix}
0.86 & 1.09 & -0.87 & -0.73 & 0.47 & 0.05 & 0.61 & -0.87 \\
0.61 & 0.59 & -0.47 & -0.16 & -0.36 & -0.70 & -0.61 & 0.20 \\
0.78 & 1.10 & -0.94 & -1.06 & 0.06 & -0.65 & 0.15 & -0.75 \\
1.03 & 0.64 & -1.12 & 0.27 & -0.28 & -0.85 & -0.68 & 0.09 \\
-0.73 & -1.37 & 1.72 & 1.18 & 0.30 & 0.38 & -0.53 & 1.19 \\
-1.22 & -0.78 & 1.40 & 0.84 & 0.47 & 2.10 & 0.92 & -0.01 \\
2.03 & 1.60 & -2.97 & -1.82 & -0.33 & -1.94 & 0.03 & -0.96 \\
0.36 & 0.19 & -0.35 & -0.21 & -0.12 & -0.39 & -0.30 & 0.80
\end{bmatrix},
\]

\[
C^* = \begin{bmatrix}
-60.68 & 83.44 & 6.16 & -67.56 & 37.84 & 7.67 & 63.46 & -63.63 \\
-49.78 & -42.83 & 55.83 & 58.86 & 49.63 & 99.23 & 73.74 & -47.24 \\
23.21 & 51.44 & 86.80 & -37.76 & -9.89 & -84.36 & -83.11 & -70.89 \\
-5.34 & 50.75 & -74.02 & 5.71 & -83.24 & -11.46 & -20.04 & -72.79
\end{bmatrix}.
\]

Let the eavesdropper attempt to approximate \(Z_H\) by the following optimization problem:

\[
\min_{H \in \mathbb{R}^{n \times m}, T \in \mathbb{R}^{nm \times nm}} \| (W \otimes I_m - \alpha Z_H)T - TA^* \|_F^2
\]

\[
\text{s.t. } (E_i \otimes I_m)T = C_i
\]

(13)

It turns out that the interior-points method leads to iterations converging to local minima in general. However, when the initial estimate of the interior–point method is sufficiently close to the true \(H\), the interior–point method can find the value of \(H\).

![Approximate Reconstruction of H](image)

Figure 3: The log–log plot of \(\|H_k - H\|_F^2\) along the interior–point method starting from different initial estimates \(H_0\). Here \(H_k\) is the \(k\)–th iteration of the interior–point method implemented over the problem (13).
6 Conclusions

We have shown that distributed computing inevitably led to privacy risks in terms the local datasets by both global and local eavesdroppers having access to the entirety of part of the node states. Such threats deserves attention as much of the dynamical states in distributed computing protocols represents physical states, which is sensitive in terms of both privacy to individuals and vulnerability to eavesdroppers. After presenting a general frame work, we showed that for network linear equations, the computational privacy has indeed been completely lost to global eavesdroppers for almost all initial values following existing distributed linear equation solvers, and partially to local passive or active eavesdroppers who can monitor or alter one selected node. We proposed a Privacy-Preserving-Summation-Consistent (PPSC) mechanism, and showed that it can be used as a generic privacy encryption step for consensus-based distributed computations. The central idea of using the consensus manifold as a place to hide node privacy while achieving computation accuracy may be extended to non-consensus based distributed computing tasks in future work.

Appendices

A. Proof of Proposition 1

(i) Suppose there exist datasets \( \mathcal{D}' \neq \mathcal{D} \) that yield the same node state trajectory \( \mathcal{M}_{x_0}(\mathcal{D}) = \mathcal{M}_{x_0}(\mathcal{D}') \) under \( x_0 \). Due to the existence of \( \mathcal{D}' \), \( \mathcal{D} \) cannot be uniquely determined from \( \mathcal{M}_{x_0}(\mathcal{D}) \), implying that \( \mathcal{M}_{x_0}(\mathcal{D}) \) is globally computationally private with respect to \( \mathcal{D} \) under \( x_0 \). On the other hand, we suppose that \( \mathcal{M}_{x_0}(\mathcal{D}) = \mathcal{M}_{x_0}(\mathcal{D}') \) implies \( \mathcal{D} = \mathcal{D}' \). As a result, \( \mathcal{M}_{x_0} \) is injective and \( \mathcal{M}_{x_0}^{-1} \) exists. Then from \( \mathcal{M}_{x_0}(\mathcal{D}) \), a unique dataset \( \mathcal{D} \) always be uniquely determined by \( \mathcal{M}_{x_0}^{-1} \). Therefore, \( \mathcal{M}_{x_0}(\mathcal{D}) \) is not globally computationally private with respect to \( \mathcal{D} \) under \( x_0 \).

(ii) Along (1) with randomized update, the observation sequence \( \mathcal{M}_{x_0}(\mathcal{D}) \) becomes an array of random variables, with its statistical properties fully characterized by its PDF. Define a mapping \( q \) from the space of \( \mathcal{D} \) to the space of pdf(\( \mathcal{M}_{x_0}(\mathcal{D}) \)). Then \( q(\mathcal{D}) = q(\mathcal{D}') \) if and only if \( \mathcal{D} = \mathcal{D}' \). Thus the proof of (ii) is analogous to (i) and omitted here.

B. Proof of Theorem 1

We first provide the following lemma that assists with the proof.

**Lemma 1.** Consider a linear equation \( E \in \mathcal{E}_{[nm]} \) and an initial condition \( x_0 \in \mathbb{R}^{nm} \).
(i) Along the CPA (5), $E$ is recoverable in the quotient space $E_i^*([m])/ \sim$ by a global dynamics eavesdropper under $x_0$ if and only if for each $i \in V$, either of the following conditions holds: (a) There exists $s_i \in \mathbb{Z}^{\geq 0}$ such that $P_i(x_i(s)) \neq x_i(s_i)$; (b) There exist $r^i_1, \ldots, r^i_m \in \mathbb{Z}^{\geq 0}$ such that $x_i(r^i_1), \ldots, x_i(r^i_m)$ are affinely independent.

(ii) Along the the PCA (4), $E$ is recoverable in the quotient space $E_i^*([n])/ \sim$ by a global dynamics eavesdropper under $x_0$ if and only if for each $i \in V$, either of the following conditions holds: (a) There exists $s_i \in \mathbb{Z}^{\geq 0}$ such that $\sum_{j \in N_i} w_{ij} P_i(x_j(s)) \neq \sum_{j \in N_i} w_{ij} x_j(s)$; (b) There exist $r^i_1, \ldots, r^i_m \in \mathbb{Z}^{\geq 0}$ such that $\sum_{j \in N_i} w_{ij} x_j(r^i_j), \ldots, \sum_{j \in N_i} w_{ij} x_j(r^i_m)$ are affinely independent.

Proof. (Necessity.) Suppose there exists $i^* \in V$ such that neither (a) nor (b) holds. Then $P_{i^*}(x_{i^*}(t)) = x_{i^*}(t)$ for all $t \in \mathbb{Z}^{\geq 0}$. As a result, the dynamics (5) becomes
\[
\begin{aligned}
x_{i^*}(t + 1) &= \sum_{j \in N_{i^*}} w_{ij} x_j(t); \\
x_i(t + 1) &= \sum_{j \in N_i} w_{ij} x_j(t) + \alpha \left( P_i(x_i(t)) - x_i(t) \right), \quad i \neq i^*.
\end{aligned}
\]

This means that the trajectory of $x(t)$ contains no information about $E_{i^*}$, except for the knowledge $P_{i^*}(x_{i^*}(t)) = x_{i^*}(t)$. By the definition of $P_i$, the eavesdropper can only attempt to recover $E_{i^*}$ from
\[
H_{i^*}^T x_{i^*}(t) = z_{i^*}, \quad t \in \mathbb{Z}^{\geq 0}.
\]

Since the eavesdropper does not know $z_{i^*}$, from the perspective of the eavesdropper who does not know $z_{i^*}$, the infinite equation set (14) is equivalent to
\[
H_{i^*}^T x_{i^*}(0) = H_{i^*}^T x_{i^*}(1) = \cdots.
\]

It is clear that (15) is equivalent to the constraint that $H_{i^*}$ is perpendicular to $x_{i^*}(t) - x_{i^*}(s)$ for all $t, s \in \mathbb{Z}^{\geq 0}$. When (b) does not hold, the vectors $x_{i^*}(t)$ for all $t \in \mathbb{Z}^{\geq 0}$ are affinely dependent, which yields $\dim \{ x_{i^*}(t) - x_{i^*}(s) : \forall t, s \in \mathbb{Z}^{\geq 0} \} < m - 1$. This implies the solution space of $H_{i^*}$ satisfying (15) has dimension larger than one. Therefore, $E_{i^*}$, and there by $E$, is not recoverable. This implies that the CPA (5) is computationally private.

(Sufficiency of Condition (a)) Suppose the condition (a) holds for any $i \in V$. Then according to (5), one has
\[
x_i(s_i + 1) \neq \sum_{j \in N_i} w_{ij} x_j(s_i).
\]

Direct calculation shows $P_i(x) = \left( I_m - \frac{H_i H_i^T}{H_i^T H_i} \right) x + \frac{z_i H_i}{H_i^T H_i}. \quad \text{Based on the dynamics (5), we have}$
\[
\frac{\alpha H_i}{H_i^T H_i} (z_i - H_i^T x_i(t)) = x_i(t + 1) - \sum_{j \in N_i} w_{ij} x_j(t).
\]
Clearly, (16) guarantees that both sides of (17) are nonzero when \( t = s_i \), which further implies that \( \mathbf{H}_i \) and \( \mathbf{x}_i(t + 1) - \sum_{j \in \mathbb{N}_i} w_{ij} \mathbf{x}_j(t) \) are parallel nonzero vectors. Defining

\[
\hat{\mathbf{H}}_i = \frac{\mathbf{x}_i(s_i + 1) - \sum_{j \in \mathbb{N}_i} w_{ij} \mathbf{x}_j(s_i)}{\left\| \mathbf{x}_i(s_i + 1) - \sum_{j \in \mathbb{N}_i} w_{ij} \mathbf{x}_j(s_i) \right\|}, \tag{18}
\]

there exists \( \lambda \neq 0 \) such that

\[
\mathbf{H}_i = \lambda \hat{\mathbf{H}}_i. \tag{19}
\]

By replacing \( \mathbf{x}_i(t + 1) - \sum_{j \in \mathbb{N}_i} w_{ij} \mathbf{x}_j(t) \) and \( \mathbf{H}_i \) in (17) based on the relations given in (18) and (19), one has

\[
\alpha \hat{\mathbf{H}}_i \left( \frac{\mathbf{z}_i}{\lambda} - \hat{\mathbf{H}}_i^\top \mathbf{x}_i(s_i) \right) = \hat{\mathbf{H}}_i \cdot \left\| \mathbf{x}_i(s_i + 1) - \sum_{j \in \mathbb{N}_i} w_{ij} \mathbf{x}_j(s_i) \right\|. \tag{20}
\]

It can be observed that both sides of (20) are a product of a scalar and the nonzero vector \( \hat{\mathbf{H}}_i \). Thus one can obtain the following equation by letting the scalars at the left-hand and right-hand side be equal.

\[
\alpha \left( \frac{\mathbf{z}_i}{\lambda} - \hat{\mathbf{H}}_i^\top \mathbf{x}_i(s_i) \right) = \left\| \mathbf{x}_i(s_i + 1) - \sum_{j \in \mathbb{N}_i} w_{ij} \mathbf{x}_j(s_i) \right\|. \tag{21}
\]

By letting \( \lambda = 1 \) in (21), one can immediately recover the following linear equation.

\[
\hat{\mathbf{e}}_i : \left( \mathbf{x}_i(s_i + 1) - \sum_{j \in \mathbb{N}_i} w_{ij} \mathbf{x}_j(s_i) \right)^\top \mathbf{y} = \left( \mathbf{x}_i(s_i + 1) - \sum_{j \in \mathbb{N}_i} w_{ij} \mathbf{x}_j(s_i) \right)^\top \mathbf{x}_i(s_i)
+ \frac{1}{\alpha} \left\| \mathbf{x}_i(s_i + 1) - \sum_{j \in \mathbb{N}_i} w_{ij} \mathbf{x}_j(s_i) \right\|^2. \tag{22}
\]

It can be easily verified \( \hat{\mathbf{e}}_i \sim \mathcal{E}_i \).

**Sufficiency of Condition (b)** Suppose the condition (b) holds for any \( i \in \mathcal{V} \). We have shown above the linear equation \( \mathcal{E} \) is recoverable if the condition (a) holds, and thus in this case we suppose the condition (a) does not hold, namely \( \mathcal{P}_i(\mathbf{x}_i(t)) = \mathbf{x}_i(t) \) for all \( t \). By the definition of \( \mathcal{P}_i \), we know

\[
\mathbf{H}_i^\top \mathbf{x}_i(r_k^i) = \mathbf{z}_i, \quad k = 1, \ldots, m. \tag{23}
\]

Further, it follows from (23)

\[
\mathbf{H}_i^\top \left( \mathbf{x}_i(r_{k+1}^i) - \mathbf{x}_i(r_k^i) \right) = 0, \quad k = 1, \ldots, m - 1,
\]

which can be written into a compact form \( \mathbf{U} \mathbf{H}_i = 0 \), where \( \mathbf{U} \) is an \( m - 1 \)-by-\( m \) matrix with each row being \( \mathbf{U}_k = \mathbf{x}_i(r_{k+1}^i) - \mathbf{x}_i(r_k^i), \quad k = 1, \ldots, m - 1 \). According to (38), \( \text{rank}(\mathbf{U}) = m - 1 \) and thereby the dimension of the kernel of \( \mathbf{U} \) is one. We can pick any \( \mathbf{H}_i^j \neq 0 \) in the kernel of \( \mathbf{U} \) and compute \( \mathbf{z}_i^j = \mathbf{H}_i^j \mathbf{x}_i(r_k^i) \) with any \( k \). Finally, \( \mathcal{E}_i^j : \mathbf{H}_i^j \mathbf{x}_i = \mathbf{z}_i^j \) that is equivalent to \( \mathcal{E}_i \) can be recovered.

We have shown above that if either the condition (a) or the condition (b) holds for any \( i \), then \( \mathcal{E}_i^j \sim \mathcal{E}_i \) is recoverable.
(ii) For all \( i \in V \), there holds according to \([6] \)

\[
\frac{H_i}{H_i H_i^\top} (z_i - H_i^\top \sum_{j \in N_i} w_{ij} x_j(t_i)) = x_i(t_i + 1) - \sum_{j \in N_i} w_{ij} x_j(t_i).
\] (24)

It is worth noting that \((24)\) along the PCA \([6] \) is similar to \((17)\) along the CPA \([5] \). By the same arguments as the proof for the CPA \([5] \), we can establish the desired statement as well.

(i) Based on Lemma \([1] \) we know if \( E \) is recoverable under \( x(0) \) along the CPA \([5] \), then there exists \( i^* \in V \) such that neither the condition (a) nor the condition (b) holds. Correspondingly, we define sets

\[
S_t = \{ x(0) \in \mathbb{R}^{nm} : \mathcal{P}_i(x_i(t)) = x_i(t), \text{ where } x_i(t) \text{ evolves according to } \mathcal{P}_i \}, \quad t \in \mathbb{Z}_{\geq 0}
\]

\[
T = \{ x(0) \in \mathbb{R}^{nm} : x_i(t), t \in \mathbb{Z}_{\geq 0} \text{ are affinely dependent, where } x_i(t) \text{ evolves according to } \mathcal{P}_i \}.
\]

According to the claim above, \( x(0) \in \bigcap_{t=0}^{\infty} S_t \cap T \). Clearly, \( x(0) \in S_0 \) says \( E(x_i(0)) = 0 \). Recall that \( E_i \) for all \( i \) are assumed to be nontrivial. Then the subspace \( S_0 \) has dimension \( nm - 1 \), and thus has measure zero in \( \mathbb{R}^{nm} \). Since \( \bigcap_{t=0}^{\infty} S_t \cap T \) is a subset of \( S_0 \), it also has measure zero. This finishes proving that the CPA \([5] \) is not computationally private under almost all initial conditions.

(ii) Analogously, if \( E \) is recoverable under \( x(0) \) along the PCA \([6] \), then based on Lemma \([1] \) there necessarily exists \( i^* \in V \) such that

\[
\frac{H_i^\top}{H_i^\top H_i^\top} \sum_{j \in N_i} w_{i^*j} x_j(0) = z_{i^*},
\]

which restricts the dimension of the subspace of \( x(0) \) to \( nm - 1 \). This leads to the same conclusion as the CPA \([5] \).

C. Proof of Theorem \([2] \)

The DP-DLES can be written as

\[
x_i^O(t) = \mathcal{P}_\Omega(x_i(t))
\]

\[
x_i^O(t) = x_i^* + \omega_i(t)
\]

\[
x_i(t) = \sum_{j \in N_i} W_{ij} x_j^O(t) + \alpha(t) (\mathcal{P}_i(x_i^O(t)) - x_i^O(t)).
\] (25)

Define \( \mathcal{P}_\Omega(x(t)) = [\mathcal{P}_\Omega(x_1(t))^\top \ldots \mathcal{P}_\Omega(x_n(t))^\top]^\top \) and \( \omega(t) = [\omega_1(t)^\top \ldots \omega_n(t)^\top]^\top \). By removing the intermediate states \( x_i^O(t) \) and \( x_i^O(t) \), we can rewrite \((25)\) compactly as

\[
x(t + 1) = (W \otimes I_m)(x(t) + \omega(t)) + \alpha(t) (z_{i^*} - \tilde{Z}_H \mathcal{P}_\Omega(x(t))),(26)
\]

where \( \tilde{Z}_H = \diag(H_i H_i^\top, \ldots, H_n H_n^\top) \). We now associate each iteration of DP-DLES at time \( t \in \mathbb{Z}_{\geq 0} \) with a mechanism \( \mathcal{M} : \mathcal{E}_{\text{nm}}^* \times \mathbb{R}^{nm} \rightarrow \mathbb{R}^{nm} \) satisfying \( \mathcal{M}(\mathcal{E}, x(t)) = x(t + 1) \).
Consider two adjacent linear equations \( \mathcal{E} : \mathbf{H} \mathbf{y} = \mathbf{z}, \mathcal{E}' = \mathbf{H}' \mathbf{y} = \mathbf{z}' \in \mathcal{E}_{[\mathcal{M}]} \). We define \( \mathcal{Z}_{H, H}', \mathbf{z}_H \) for \( \mathcal{E}' \) similarly as \( \mathcal{Z}_H, \mathbf{z}_H \) for \( \mathcal{E} \). Then for \( \mathcal{E}, \mathcal{E}' \), there holds

\[
\frac{\Pr(\mathcal{M}(\mathcal{E}, \mathbf{x}(t)) = \mathbf{x}(t+1))}{\Pr(\mathcal{M}(\mathcal{E}', \mathbf{x}(t)) = \mathbf{x}(t+1))} \geq \frac{\text{pdf}(\mathbf{W} \otimes \mathbf{I}_m)^{-1}(\mathbf{x}(t+1) - \alpha(t)(\mathbf{z}_H - \mathcal{Z}_H^\top \mathbf{P}_H^\top(\mathbf{x}(t)))) - \mathbf{x}(t))}{\text{pdf}(\mathbf{W} \otimes \mathbf{I}_m)^{-1}(\mathbf{x}(t+1) - \alpha(t)(\mathbf{z}'_H - \mathcal{Z}_H^\top \mathbf{P}_H^\top(\mathbf{x}(t)))) - \mathbf{x}(t))}
\leq \exp\left(\frac{\alpha(t)}{c_0^\delta}(\mathbf{W} \otimes \mathbf{I}_m)^{-1}(\mathcal{Z}_H - \mathcal{Z}_H') \mathbf{P}_H^\top(\mathbf{x}(t)) - (\mathbf{z}_H - \mathbf{z}_H')\right)_1
\leq \exp\left(\frac{\alpha(t)}{c_0^\delta}(\mathbf{W}^{-1} \otimes \mathbf{I}_m)_1\left\| \mathcal{Z}_H - \mathcal{Z}_H' \mathbf{P}_H^\top(\mathbf{x}(t)) - (\mathbf{z}_H - \mathbf{z}_H') \right\|ight)
\leq \exp\left(\frac{\alpha(t)\sqrt{nm}}{c_0^\delta} \mathbf{W}^{-1}\left\| \mathbf{H}_i \mathbf{H}_i^\top - \mathbf{H}_i' \mathbf{H}_i'^\top \mathbf{P}_H^\top(\mathbf{x}(t)) \right\| + \left\| \mathbf{z}_i \mathbf{H}_i - \mathbf{z}_i' \mathbf{H}_i' \right\|ight)
\leq \exp\left(\frac{\lambda t^{\delta} \sqrt{nm}}{c_0^\delta \sigma_m(\mathbf{W})}(B_{\delta H} + \delta_x)\right),
\]

where a) uses \(26\) and the PDF of Laplace random variables, b) is obtained from norm inequalities, and c) comes from the linear–equation adjacenc. By omitting the notation \( \mathcal{E} \), the following composition relation holds \( \mathcal{M} = (\mathcal{M}_0, \mathcal{M}_1 \circ \mathcal{M}_0, \ldots, \cdots \circ \mathcal{M}_1 \circ \mathcal{M}_0) \). To let \( \mathcal{M} \) be \((\epsilon, \delta_H, \delta_x)\)-differentially private, based on \(27\), the following must hold from the composition property of differential privacy (see, e.g., Theorem 1 of \(37\)):

\[
\sum_{t=0}^{\infty} \frac{\lambda t^{\delta} \sqrt{nm}}{c_0^\delta \sigma_m(\mathbf{W})}(B_{\delta H} + \delta_x) \leq \epsilon,
\]

which further implies the desired result.

D. Proof of Theorem 3

We divide the proof into three steps.

Step 1. The CPA (5) can be compactly rewritten as

\[
\mathbf{x}(t+1) = \mathbf{F} \mathbf{x}(t) + \alpha \mathbf{z}_H,
\]

where \( \mathbf{F} = \mathbf{W} \otimes \mathbf{I}_m - \alpha \text{ diag} \left( \frac{\mathbf{H}_1 \mathbf{H}_1^\top}{\mathbf{H}_1' \mathbf{H}_1'^\top}, \ldots, \frac{\mathbf{H}_m \mathbf{H}_m^\top}{\mathbf{H}_m' \mathbf{H}_m'^\top} \right) \) and \( \mathbf{z}_H = \left[ \frac{\mathbf{z}_1 \mathbf{H}_1^\top}{\mathbf{H}_1' \mathbf{H}_1'^\top}, \ldots, \frac{\mathbf{z}_m \mathbf{H}_m^\top}{\mathbf{H}_m' \mathbf{H}_m'^\top} \right] \). Any passive local dynamics eavesdropper \( i \in \mathcal{V} \) knows a solution \( \mathbf{y}^* \) to the linear equation \( \mathcal{E} \) as a result of implementing the CPA (5). By introducing \( \mathbf{\gamma}(t) = \mathbf{x}(t) - 1 \otimes \mathbf{y}^* \), the eavesdropper can rewrite (28) into

\[
\mathbf{\gamma}(t+1) = \mathbf{F} \mathbf{\gamma}(t).
\]

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Since the eavesdropper $i$ has access to only the state observations of the nodes in $N_i$, its observation can be described by

$$v(t) = (E_i \otimes I_m)\gamma(t) + 1 \otimes y^*.$$  

Alternatively, knowing $y^*$, this passive eavesdropper has access to

$$y(t) = v(t) - 1 \otimes y^* = (E_i \otimes I_m)\gamma(t).$$  

(30)

Step 2. We apply the system identification approach from [31] to the system described by (29)–(30) with state $\gamma(t)$ and output $y(t)$. According to [31], if $(F, E \otimes I_m)$ is completely observable, i.e., the matrix

$$O = \begin{bmatrix} E_i \otimes I_m \\ (E_i \otimes I_m)F \\ \vdots \\ (E_i \otimes I_m)F^{nm-|N_i|m} \end{bmatrix}$$

has full column rank, and there exist $t_1 < \cdots < t_{nm}$ such that the vectors $x(t_k)$, $k = 1, \ldots, nm$ are linearly independent, then the passive local dynamics eavesdropper $i$ can identify $(F^*, C^*) \in \mathbb{R}^{nm \times nm} \times \mathbb{R}^{|N_i|m \times nm}$ from $(y(t))_{t=0}^\infty$ by the following procedure.

(i) Collect a number of consecutive outputs of the system (29)–(30) $(y(t))_{t=k}^{t_k+nm-|N_i|m}$ for $k = 1, \ldots, nm$ and construct

$$Y = \begin{bmatrix} y(t_1) & \cdots & y(t_{nm}) \\ \vdots & \ddots & \vdots \\ y(t_1 + nm - |N_i|m - 1) & \cdots & y(t_{nm} + nm - |N_i|m - 1) \end{bmatrix},$$

$$\bar{Y} = \begin{bmatrix} y(t_1 + 1) & \cdots & y(t_{nm} + 1) \\ \vdots & \ddots & \vdots \\ y(t_1 + nm - |N_i|m) & \cdots & y(t_{nm} + nm - |N_i|m) \end{bmatrix}.$$  

Since $Y = O[x(t_1) \cdots x(t_{nm})]$ and $O$ has full column rank, $\text{rank}(Y) = \text{rank}[x(t_1) \cdots x(t_{nm})] = nm$.

(ii) Write the state–output relation for the collected output data as

$$\bar{Y} = O(W \otimes I_m - \alpha Z_H) \begin{bmatrix} x(t_1) & \cdots & x(t_{nm}) \end{bmatrix},$$

$$\bar{Y} = O \begin{bmatrix} x(t_1) & \cdots & x(t_{nm}) \end{bmatrix}.$$  

(31)

(iii) Choose a row–selecting matrix $S : \mathbb{R}^{nm \times |N_i|(nm-|N_i|m)}$ such that $S\bar{Y}$ is invertible, and further, based on [31], obtains $F_* := S'O(W \otimes I_m - \alpha Z_H)(SO)^{-1} = S\bar{Y}(S\bar{Y})^{-1}$ and then $C_* = [I_m|N_i| 0]$.

The obtained pair $(A_*, C_*)$ is different from $(F, E_i \otimes I_m)$ subject to a coordinate change based on Proposition 2.3 of [39]:

$$T^{-1}FT = F^*, \ (E_i \otimes I_m)T = C_*$$  

(32)
Step 3. We now come back to the point of view of the eavesdropper. Since the local dynamics eavesdropper
$i$ knows $E_i$, it can determine the set $\mathcal{T}$ based on (32). By (32), there exists some $T \in \mathcal{T}$ such that
$\text{diag} \left( H_1 H_1^\top, \ldots, H_n H_n^\top \right) = (W \otimes I - T^{-1} F_s T)/\alpha$. It is worth noting that the eavesdropper can compute $z = H \lim_{t \to \infty} x_i(t)$. This completes the proof.

E. Proof of Theorem 4

We first establish a lemma that assists with the presentation of the privacy of CPA (5) against the active
local eavesdropper $i$.

Lemma 2. The eigenvalues of $W \otimes I_m - \alpha Z_H$ for $\alpha \in (0, \lambda_m(W) + 1)$ are all strictly less than one in
absolute value if $E \in \mathcal{E}_{[nm]}^*$ has a unique solution.

Proof. By the randomness of $W$ and Lemma 8.1.21 [40], for any $v = [v_1^\top \ldots v_n^\top]^\top \neq 0$ with $v_i \in \mathbb{R}^m$, there holds $\lambda_m(W) \|v\|^2 \leq v^\top (W \otimes I_m) v \leq \|v\|^2$, where the right–hand equality holds if and only if $v_1 = \cdots = v_n$. In addition, $0 \leq v^\top Z_H v = \sum_{i=1}^n \|H_i^\top v_i\|^2 \leq \|v\|^2$ for all $v$, where the left–hand equality holds if and only if each $v_i$ is perpendicular to $H_i$. Then it can be concluded that $(\lambda_m(W) - \alpha) \|v\|^2 \leq v^\top (W \otimes I_m - \alpha Z_H) v \leq \|v\|^2$ for all $v$, where the right–hand equality holds if and only if $v_1 = \cdots = v_n \in \ker(H)$. Since $\alpha < \lambda_m(W) + 1$ and $\ker(H) = \emptyset$, $-\|v\|^2 < v^\top (W \otimes I_m - \alpha Z_H) v < \|v\|^2$, implying that all its
eigenvalues fall in $(-1, 1)$.

Again we divide the proof into three steps.

Step 1. Let a $T$–periodic signal $r : \mathbb{Z}^\geq 0 \to \mathbb{R}^m$ be added to $x_i(t)$ at each time $t$ by the eavesdropper $i$ with $T \geq 2nm + 1$ and

$$
R = \begin{bmatrix}
  r(0) & r(1) & \cdots & r(T-1) \\
  r(T-1) & r(0) & \cdots & r(T-2) \\
  \vdots & \vdots & \ddots & \vdots \\
  r(1) & r(2) & \cdots & r(0)
\end{bmatrix}
$$

having full rank. Now the system (29) becomes

$$
\gamma(t+1) = F \gamma(t) + B r(t), \quad t > t^*,
$$

(33)

where $B \in \mathbb{R}^{nm}$ is the Kronecker product of an all–zeros vector except for the $i$–the component being one and the $m$–by–$m$ identity matrix. From Lemma 2 $F$ is a stable matrix if $E$ has a unique solution.

Step 2. We apply the system identification approach from [32] to the system (30)–(33) with state $\gamma(t)$ and output $y(t)$ by the following procedure:
(i) Collect $(y(t))_{t=kT}^{kT+T-1}$ for some large $k$, namely consecutive outputs of the system (30)–(33) within one period $T$.

(ii) Compute

$$\sum_{j=0}^{\infty} [G_{jT} \quad \ldots \quad G_{jT+T-1}] = [y(kT) \quad \ldots \quad y(kT + T - 1)] \cdot R^{-1},$$

where $G_j = (E_i \otimes I_m)(W \otimes I_m - \alpha Z_H)^{j-1}b$ and $G_0 = 0$ are the impulse responses of the system (30)–(33). The result is finite because the system (33) is stable by Lemma 2.

(iii) Reorganize $\sum_{j=0}^{\infty} G_{jT+l}$ with $l = 0, \ldots, T - 1$ in a Hankel matrix $M \in \mathbb{R}^{p \times q}$

$$M = \begin{bmatrix}
\sum_{j=0}^{\infty} G_{jT+1} & \cdots & \sum_{j=0}^{\infty} G_{jT+q} \\
\vdots & \ddots & \vdots \\
\sum_{j=0}^{\infty} G_{jT+p} & \cdots & \sum_{j=0}^{\infty} G_{jT+p+q-1}
\end{bmatrix}$$

with $p > nm, p + q = T$ and computes its singular value decomposition as

$$M = \begin{bmatrix} U_s & U_o \end{bmatrix} \begin{bmatrix} \Sigma_s & 0 \\ 0 & \Sigma_o \end{bmatrix} \begin{bmatrix} V_s^\top \\ V_o^\top \end{bmatrix}, \quad \Sigma_s \in \mathbb{R}^{nm \times nm}.$$

Then $A_* = ([I_{[N_i|m(p-1)]}]U_s)^+ = [I_{[N_i|m(p-1)]}]U_s$ and $C_* = ([I_{[N_i|m]}]0)U_s$, where $T^{-1}(W \otimes I_m - \alpha Z_H)T = A_*$, $(E_i \otimes I_m)T = C_*$ and $T \in \mathbb{R}^{nm \times nm}$ is invertible.

According to Theorem 1 in [32], with $F$ being stable, the obtained pair $(F_*, C_*)$ and $(F, C)$ also satisfy the relation in (32).

Step 3. From (32), the eavesdropper can determine the set $\mathcal{T}$. By (32), there exists some $T \in \mathcal{T}$ such that

$$\text{diag} \left( \frac{H_1H_1^\top}{H_1}, \ldots, \frac{H_nH_n^\top}{H_n} \right) = (W \otimes I - T^{-1}F_*T)/\alpha.$$ 

Further, the eavesdropper can compute $z = Hy^*$. This completes the proof.

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