Insertion and Lie Bracket Concerning Finite Sets

Zhou Mai *

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Abstract

In this article we discuss the operations of partitions (sequence of disjoint finite subsets) which are quotient, insertion, composition and Lie bracket. Moreover, we discuss applications of those operations for Feynman diagrams and Kontsevich’s graphs.

Keywords: partition, quotient, insertion, Lie bracket, Feynman diagrams, admissible graphs.

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1 Introduction

In this article we discuss some operations of partitions, where the partition means the sequence of disjoint finite subsets. The operations what we focus on include quotient, insertion, composition and Lie bracket. All of operation is generalization of ones concerning Feynman diagrams (see A. Connes and D. Kreimer [1, 2], A. Connes and M. Marcolli [3], D. Kreimer [6]) and Kontsevich’s graphs (see L. M. Ionescu [4], M. Kontsevich [7]). The construction in this article is suitable for the cases of Feynman diagrams and Kontsevich’s graphs, actually, if we consider some structure maps additionally, the operations concerning Feynman diagrams and Kontsevich’s graphs can be reduced to our construction. Here the construction of quotient follows the ideas in Zhou Mai [9], but, some modification occurs such that it is more suitable for the cases of Feynman diagrams and Kontsevich’s graphs.

*address: Colleague of Mathematical Science, Nankai University, Weijin Road, Tianjin City, Republic China; email address: zhoumai@nankai.edu.cn
diagrams and Kontsevich’s graphs. Our construction is suitable for ordinary graphs in the sense of graphic theory as well. Somehow, a ordinary graph can be regarded as a Feynman diagram without external lines, but, the case of subgraph is different. It seems that three types of above graphs can be dealt with in an uniform way.

This paper is organized as follows. In section 2 we discuss the quotient and insertion of partitions in details. Based on the quotient we construct the coproduct which will result in a hopf algebra, but we do not discuss this issue more. In section 3 we construct the composition and Lie bracket. Here two types of composition are considered, both of them will result in well defined Lie bracket. Finally, in section 4 we discuss the cases of Feynman diagrams, Kontsevich’s graphs and ordinary graphs starting from our construction.

2 Quotient and insertion of the partitions

2.1 Notations concerning the partitions

Partitions: Firstly, we introduce some notations which will be useful for discussion below.

- For a finite set $A$, we denote the power set of $A$ by $\mathcal{P}(A)$. The reversion map $\mathcal{R} : \mathcal{P}(\mathcal{P}(A)) \rightarrow \mathcal{P}(A)$ is defined to be:
  $$\mathcal{R}(\{I_i\}) = \bigcup_i I_i, \{I_i\} \in \mathcal{P}(\mathcal{P}(A)).$$

Let $\mathcal{P}_{\text{dis}}(A)$ denote a subset of $\mathcal{P}(\mathcal{P}(A))$,

$$\mathcal{P}_{\text{dis}}(A) = \{\{I_i\} \in \mathcal{P}(\mathcal{P}(A))| \{I_i\} \in \text{Part}(\mathcal{R}(\{I_i\}))\}.$$  

We call the element $\{I_i\} \in \mathcal{P}_{\text{dis}}(A)$ the partition in $A$. If we ignore the order of $\{I_i\}$, then we identify $\{I_i\}$ with $\{I_{\sigma(i)}\}$, where $\sigma \in S_m$ and $m = |\{I_i\}|$ denotes the number of the elements in a finite set $B$.

A partition $\{I_i\}_{i=1}^m \in \mathcal{P}_{\text{dis}}(A)$ can be derived by a function $f$ from $I = \mathcal{R}(\{I_i\})$ to $\mathbb{N}$ with $|\text{Im}(f)| = m$. Precisely, let $\text{Im}(f) = \{i_1, \cdots, i_m\}$ ($0 < i_1 < \cdots < i_m$), then $\{f^{-1}(i_k)\}_{k=1}^m$ is a partition in $A$. If $f^{-1}(i_k) = I_k$, then the function $f$ defines the partition $\{I_i\}_{i=1}^m$. For a permutation $\sigma : \{i_1, \cdots, i_m\} \rightarrow \{i_{\sigma(1)}, \cdots, i_{\sigma(m)}\}$, it is obvious that $\sigma \circ f$ defines a same partition. On the other hand, let $\tau$ be a map from $\{i_1, \cdots, i_m\}$ to $m = \{1, \cdots, m\}$, $\tau(i_k) = k$ ($k = 1, \cdots, m$), then $\tau \circ f$ defines the same partition also. Without loss of generality, we can always assume $\text{Im}(f) = m$. We call the function $f : I \rightarrow m$ satisfying $f^{-1}(i) = I_i (i = 1, \cdots, m)$ the defining function of the partition $\{I_i\}_{i=1}^m$, denoted by $f(I_i)$.

- Let $\{I_i\}_{i=1}^m, \{J_j\}_{j=1}^n \in \mathcal{P}_{\text{dis}}(A)$ with defining functions $f(I_i), f(J_j), \mathcal{R}(\{I_i\}) \cap \mathcal{R}(\{J_j\}) = \emptyset$. Then $\{I_1, \cdots, I_m, J_1, \cdots, J_n\} \in \mathcal{P}_{\text{dis}}(A)$, the new partition as above is denoted by $\{I_i\} \cup \{J_j\}$. Let $f(I_i) \cup f(J_j)$ be the defining function of $\{I_i\} \cup \{J_j\}$, then $f(I_i) \cup f(J_j) = f(I_i) \cup f(J_j), f(I_i) \cup f(J_j) = f(I_i) \cup f(J_j)$, where $I = \mathcal{R}(\{I_i\}), J = \mathcal{R}(\{J_j\})$ and $i(j) = j + m (j = 1, \cdots, n)$. We denote $f(I_i) \cup f(J_j)$ by $f(I_i) \cup f(J_j)$ also. It is obvious that $f(I_i) \cup f(J_j)$ defines same partition, therefore, we will identify $f(I_i) \cup f(J_j)$ with $f(I_i) \cup f(J_j)$.

- Let $\{I_i\}, \{J_j\} \in \mathcal{P}_{\text{dis}}(A)$, we say $\{J_j\} \subset \{I_i\}$, if for any $j$, there is an $i$, such that $J_j \subset I_i$. An important situation is $\{J_j\} = \{I_i \cap B\}$, where $B \subset \mathcal{R}(\{I_i\})$. It is obvious that $\mathcal{R}(\{J_j\}) = B$. In this situation we call $\{J_j\}$ the restriction of $\{I_i\}$ on $B$, denoted by $\{J_j\} = \{I_i \cap B\}$. The defining function of $\{J_j\}$ can be determined as follows. Let $\{i_1, \cdots, i_n\} = \{i|1 \leq i \leq m, I_i \cap B \neq \emptyset\}$, where $i_1 < \cdots < i_n$ and $n = |\{J_j\}|$. If we take $J_j = I_i \cap B$, then $f(I_i) \cup f(J_j)$ should be such a function, $f(I_i) : B \rightarrow \mathbb{Z}$.  

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(f_{(J,j)})^{-1}(j) = (f_{(I,i)})^{-1}(i) \cap B = J_j.

Thus, \( f_{(I,i)}|_B = \iota \circ f_{(I,i)} \), where \( \iota : \{i_1, \cdots, i_n\} \rightarrow \{i_1, \cdots, i_n\}, i(j) = i_j \) \((j = 1, \cdots, n)\). If we do not distinguish \( f_{(J,j)} \) and \( \iota \circ f_{(I,i)} \), then we have \( f_{(I,i)}|_B = f_{(I,i)} \). In fact the functions to define a partition is an equivalent class under a natural equivalent relation, above formula is exactly valid in the sense of the equivalent relation.

If \( D \subset B \subset \mathcal{R}({I_i}), \{J_j\} = \{I_i\}_{(B)}, \) we have \( \{J_j\}_{(D)} = \{I_i\}_{(D)} \).

**Map-union of the partitions** Let \( \{I_i\} \in \mathcal{P}_d^2(A), B \subset A \) be a subset, \( B \cap \mathcal{R}({I_i}) = \emptyset, f : B \rightarrow \mathcal{R}({I_i}) \) ba a map, we define a new partition \( \{f^{-1}(I_i \cup I)\} \) which is called the map-union of \( \{I_i\} \) and \( B \) by \( f \), denoted by \( B \cup_f \{I_i\} \).

Now we discuss some properties of the map-union.

- Let \( \{I_i\}_{i=1}^m \in \mathcal{P}_d^2(A) \) with defining function \( f_{(I_i)} \), \( B \subset A \) be a subset, \( B \cap I = \emptyset \) \((I = \mathcal{R}({I_i}))\), for two maps \( f : B \rightarrow I, g : B \rightarrow I \), then

\[
B \cup_f \{I_i\} = B \cup_g \{I_i\} \iff f^{-1}(I_i) = g^{-1}(I_i) (1 \leq i \leq m) \iff f_{(I_i)} \circ f = f_{(I_i)} \circ g.
\]

Thus, the map-union \( B \cup_f \{I_i\} \) depends only on \( f_{(I_i)} \circ f \). With the help of defining function \( f_{(I_i)} \), there is a one-one correspondence between \( I_i \) and its index \( i \), the map \( f_{(I_i)} \circ f \) can be regarded as a map \( \iota : B \rightarrow \{I_i\} \), and the map-union \( B \cup_f \{I_i\} \) can be expressed as \( \{\iota^{-1}(I_i \cup I)\} \) denoted by \( B \cup_f \{I_i\} \).

- For two partitions \( \{I_i\}, \{J_j\} \in \mathcal{P}_d^2(A) \), and two subsets \( B, C \subset A \), which satisfy \( I \cap J = \emptyset, B \cap C = \emptyset, (B \cup C) \cap (I \cup J) = \emptyset \), where \( I = \mathcal{R}({\{I_i\}}), J = \mathcal{R}({\{J_j\}}) \), it is obvious that

\[
(I \cup J) \cup_{f \cup g} \{I_i\} \cup \{J_j\} = (I \cup J) \cup_{f \cup g} \{I_i\} \cup \{J_j\},
\]

where \( f : B \rightarrow I, g : C \rightarrow J, f \cup g : B \cup C \rightarrow I \cup J, (f \cup g)|_B = f, (f \cup g)|_C = g \).

- Let \( \{I_i\} \in \mathcal{P}_d^2(A), B, C \subset A \), which satisfy \( B \cap I = \emptyset, C \subset I \), where \( I = \mathcal{R}({\{I_i\}}) \), then

\[
B \cup f \{I_i\} = (B \cup f_{(I_i)} \{I_i\}) \cup (B_f \cup g \{I_i\}) = B \cup f_{(I_i)} \{I_i\} \cup (B_f \cup g \{I_i\}),
\]

where \( f : B \rightarrow I, B_1 = f^{-1}(I \cap C), B_2 = f^{-1}(I \setminus C) \).

- Let \( B, C \subset A \), \( \{I_i\} \in \mathcal{P}_d^2(A), B \cap C = \emptyset, I \cap (B \cup C) = \emptyset, I = \mathcal{R}({\{I_i\}}) \), then we have

\[
(B \cup C) \cup_{f \cup g} \{I_i\} = B \cup f_{(C \cup g \{I_i\})} = C \cup g \{B \cup f \{I_i\}\},
\]

where \( f : B \rightarrow I, g : C \rightarrow I, f \cup g : B \cup C \rightarrow I, (f \cup g)|_B = f, (f \cup g)|_C = g \).

**2.2 Quotient of the partitions**

Now we define the quotient between two partitions.

**Definition 2.1.** Let \( \{I_i\} \in \mathcal{P}_d^2(A), B \subset \mathcal{R}({\{I_i\}}) \), we define the quotient of \( \{I_i\} \) by \( \{I_i\}_{(B)} \), denoted by \( \{I_i\} / \{I_i\}_{(B)} \), to be

\[
\{I_i\} / \{I_i\}_{(B)} = \{I_i\}_{I \cap B = \emptyset} \cup \{\mathcal{R}({\{I_i\}_{I \cap B = \emptyset}}) \setminus B\}.
\]

We call \( \mathcal{R}({\{I_i\}_{I \cap B = \emptyset}}) \setminus B \) the ideal part of the quotient \( \{I_i\} / \{I_i\}_{(B)} \).
Lemma 2.1.

Otherwise, above decomposition is not valid. From definition 2.1 we can easily see that

\[ \mathcal{R}([I_i]) = \mathcal{R}([I_i]/[I_i](B)) \cup B, \] \[ \mathcal{R}([I_i]/[I_i](B)) \cap B = \emptyset. \]

Therefore

\[ \mathcal{R}([I_i]/[I_i](B)) = \mathcal{R}([I_i]) \setminus B. \]

Remark 2.1.

• It is convenience to denote \( \mathcal{R}([I_i]_{I_i \cap B \neq \emptyset}) \) by \( \mathcal{R}_{I_i,B} \), or \( \mathcal{R}_B \) for short sometime. From definition 2.1 we can easily see that

\[ \mathcal{R}([I_i]) = \mathcal{R}([I_i]/[I_i](B)) \cup B, \] \[ \mathcal{R}([I_i]/[I_i](B)) \cap B = \emptyset. \]

Particularly, when \( \mathcal{R}_B = B \), which means that if \( I_i \cap B \neq \emptyset \) we have \( I_i \subset B \), or, \( [I_i](B) = [I_i]_{I_i \cap B \neq \emptyset} \) which is a subset of \( [I_i] \), then \( [I_i]/[I_i](B) = ([I_i] \setminus [I_i](B)) \cup \{ \emptyset \} \). We will identify \( ([I_i] \setminus [I_i](B)) \cup \{ \emptyset \} \) with \( [I_i] \setminus [I_i](B) \). In this special situation the ideal part of quotient \( [I_i]/[I_i](B) \) is \( \{ \emptyset \} \), we call this special situation the trivial quotient. Furthermore, \( [I_i]/[I_i](I) = [I_i] \setminus [I_i] = \emptyset \).

• Noting that \( [I_i] = [I_i]_{I_i \cap B = \emptyset} \cup [I_i]_{I_i \cap B \neq \emptyset} \), and \( [I_i]_{I_i \cap B = \emptyset}/[I_i](B) = \{ \emptyset \} \cup \{ \mathcal{R}_B \setminus B \} \), thus we can rewrite the formula (2.1) in the following form:

\[ [I_i]/[I_i](B) = [I_i]_{I_i \cap B = \emptyset} \cup ([I_i]_{I_i \cap B \neq \emptyset}/[I_i](B)). \]

• Let \( J \subset A, J \cap I = \emptyset, B \subset I, I = \mathcal{R}([I_i]) \), then the quotient \( [I_i]/[I_i](B) \) induces the quotient of map-union \( J \cup f[I_i] \), where \( f : J \rightarrow [I_i] \) is a map. Actually, let \( p_B \) be the projection from \( \mathcal{R}([I_i]) \) to \( [I_i]/[I_i](B) \), \( p_B(I_i) = I_i \) for \( I_i \cap B = \emptyset \), \( p_B = \mathcal{R}_B \setminus B \) for \( I_i \cap B \neq \emptyset \), it is easy to check that

\[ (J \cup f[I_i])/[I_i](B) = J \cup p_B f ([I_i]/[I_i](B)). \]

We will take a look at the properties of the quotient. Here we focus on the case of

\[ ([I_i]/[I_i](B))/([I_i]/[I_i](B))(C), \]

where \( B, C \subset I = \mathcal{R}([I_i]) \), \( B \cap C = \emptyset \). The key point is that when \( \mathcal{R}_B \setminus \mathcal{R}_C = \emptyset \), we have the following decomposition

\[ [I_i]_{I_i \cap (B \cup C) \neq \emptyset} = [I_i]_{I_i \cap B \neq \emptyset} \cup [I_i]_{I_i \cap C \neq \emptyset}. \]

Otherwise, above decomposition is not valid.

Lemma 2.1. Let \( [I_i] \in P^2_{\text{dis}}(A), B, C \subset \mathcal{R}([I_i]), B \cap C = \emptyset, \) then following formulas are valid.

• \( ([I_i]/[I_i](B))(C) = [I_i]_{I_i \cap (B \cup C)} / [I_i](B). \) (2.2)

• \( ([I_i]/[I_i](B))/(I_i)/[I_i](B))(C) \)

\( = ([I_i]/[I_i](B))/([I_i](B))/(I_i)(B). \) (2.3)

• If \( \mathcal{R}_B \setminus \mathcal{R}_C = \emptyset, \) then

\[ [I_i](B \cup C)/[I_i](B) = [I_i](C), \] \[ ([I_i]/[I_i](B))(C) = [I_i](C). \] (2.4)

and

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Proof. At first, in order to prove the formula (2.2), we calculate the left side of the formula. From definition 2.1 we have

\[
\frac{\{I_i\}}{\{I_i\}_{(B)}} = \left\{ \frac{I_i \cap B = \emptyset \cup \{R_B \setminus B\}}{C} \right\} = \{I_i \cap C\}I_i \cap B = \emptyset \cup \{R_B \setminus C\}.
\]

On the other hand, for the right side of the formula (2.2) we have

\[
\frac{\{I_i\}}{\{I_i\}_{(B)}} = \left\{ \frac{I_i \cap (B \cup C)}{I_i \cap B = \emptyset \cup \{R_B \cap (B \cup C) \setminus B\}} \right\} = \{I_i \cap C\}I_i \cap B = \emptyset \cup \{R_B \cap C\}.
\]

Therefore, the formula (2.2) is valid. The formula (2.3) is the corollary of the formula (2.2).

We turn to prove the formula (2.4). Because \(R_B \cap R_C = \emptyset\), we know that

\[
I_i \cap B \neq \emptyset \iff I_i \cap C = \emptyset \text{ or } I_i \cap C \neq \emptyset \iff I_i \cap B = \emptyset, \text{ if } I_i \cap (B \cup C) \neq \emptyset.
\]

Thus

\[
\{I_i\}_{(B \cup C)} = \{I_i\}_{(B)} \cup \{I_i\}_{(C)}.
\]

then, in this situation the quotient is trivial, that is

\[
\{I_i\}_{(B \cup C)}/\{I_i\}_{(B)} = \{I_i\}_{(B \cup C)}/\{I_i\}_{(B)} = \{I_i\}_{(B \cup C)} \setminus \{I_i\}_{(B)} = \{I_i\}_{(C)}.
\]

Combining the formulas (2.2), (2.4) we can get the last formula in lemma 2.1.

\[ \square \]

**Proposition 2.1.** Let \( \{I_i\} \in P_\text{dis}^2(A), B, C \subset R(\{I_i\}), B \cap C = \emptyset \), then following formulas are valid.

- When \( R_B \cap R_C \neq \emptyset \),

\[
\frac{\{I_i\}}{\{I_i\}_{(B)}}/\{I_i\}_{(B \cup C)}/\{I_i\}_{(B)} = \{I_i\}_{(B \cup C)}.
\]

- When \( R_B \cap R_C = \emptyset \), we have

\[
\frac{\{I_i\}}{\{I_i\}_{(B)}}/\{I_i\}_{(B \cup C)}/\{I_i\}_{(B)} = \{I_i\}_{(B \cup C)}/\{I_i\}_{(C)}.
\]

**Proof.** From definition of quotient, we know that

\[
\{I_i\}_{(B)} = \{I_i\}_{I_i \cap B = \emptyset} \cup \{R_B \setminus B\},
\]

and

\[
\{I_i\}_{(B \cup C)} = \{I_i\}_{I_i \cap (B \cup C) = \emptyset} \cup \{R_B \setminus (B \cup C)\}.
\]

The discussions will be divided into two situations.

**Case of** \( R_B \cap R_C \neq \emptyset \):

By the definition of the quotient we have
Now we reach the formula (2.6).

Above formula is (2.5) exactly.

\[ \{(I_i)_{I_i(B)}\} / \{(I_i)_{I_i(B)}\}_{(C)} \]
\[ = \{I_i\}_{I_i(B)C} \cup \{(R(I_i)_{I_i(B)C} \cup R_B)\} \setminus C. \]

Noting \( R(I_i)_{I_i(B)C} \) \( \cup \) \( \{\} \) \( B \) \( \emptyset \), thus
\[ R(I_i)_{I_i(B)C} \cup \{\} \cup R_B = R_{I_i(B)C} B. \]

In summary, we get
\[ \{(I_i)_{I_i(B)}\} / \{(I_i)_{I_i(B)}\}_{(C)} = \{I_i\}_{I_i(B)C} \cup \{\} \cup R_B \cup \{\} \cup R_C \setminus C. \]

Above formula is (2.5) exactly.

**Case of** \( R_B \cap R_C = \emptyset : \)

In this situation, we have \((R_B \setminus B) \cap B = \emptyset \), and \((R\{I_i\}_{I_i(B)C} \cup \{\} \cup R_B) = R\{I_i\}_{I_i(B)C} \cup \{\} \cup R_B \setminus B \cup \{\} \cup R_C \setminus C. \]

Now we reach the formula (2.6).

\[ \square \]

**Corollary 2.1.**

\[ \{(I_i)_{I_i(B)}\} / \{(I_i)_{I_i(B)}\}_{(C)} = \{(I_i)_{I_i(C)}\} / \{(I_i)_{I_i(C)}\}_{(B)}. \]

(2.7)

\[ \{(I_i)_{I_i(B)}\} / \{(I_i)_{I_i(C)}\} = \{(I_i)_{I_i(C)}\} / \{(I_i)_{I_i(B)}\}. \]

(2.8)

Now we consider more general situation.

**Definition 2.2.** Let \( \{I_i\}, \{J_j\} \in \mathcal{P}_{d_A}(A), \ R(\{J_j\}) \subset R(\{I_i\}) \), we define the following quotient inductively,

\[ \{I_i\} / (J_1) \equiv \{I_i\} / (J_1, J_2) \equiv \{I_i\} / (J_1, \ldots, J_{k+1}) \equiv \{I_i\} / (J_1, \ldots, J_{k+1}). \]

**Remark 2.2.** By the formula (2.7) in corollary 2.1, it is easy to check that the quotient in definition 2.2 does not depend on the order of \( J_j \). Precisely, let \( n = |\{J_j\}|, \tau \in S_n \) be a permutation on \( \tau \_i \), then,

\[ \{I_i\} / (J_\tau) = \{I_i\} / (J_{\tau(i)}). \]

**Definition 2.3.** Let \( \{I_i\}, \{J_j\} \in \mathcal{P}_{d_A}(A), \ R(\{J_j\}) \subset R(\{I_i\}) \), and
\[ R_{\{I_i\}, J_j} \cap R_{\{I_i\}, J_{j'}} = \emptyset, j \neq j', \]

(2.9)

we call \( \{J_j\} \) admits to \( \{I_i\} \), denoted by \( \{J_j\} \subset \{I_i\} \).

**Remark 2.3.** The condition (2.9) is equivalent to the following conditions
\[ R_{\{I_i\}, J_j} \cap J_{j'} = \emptyset \text{ or } R_{\{I_i\}, J_j} \cap J_j = \emptyset, j \neq j'. \]

\[ R_{\{I_i\}, J_j} \] will be shortly denoted by \( R_{\{I_i\} J_j} \) later.
Actually, the fact \( \{J_j\}_{j=1}^{p} \subset \{I_i\} \) means that \( \{I_i\} \) adapts to a decomposition as follows,
\[
\{I_i\} = \{I_i\}_{i \cap J = \emptyset} \cup \bigcup_j \{I_i\}_{i \cap J \neq \emptyset},
\]
where \( J = \mathcal{R}(\{J_j\}) \). Therefore, we have
\[
\{I_i\}/(J_j) = \{I_i\}_{i \cap J = \emptyset} \cup \bigcup_j \{I_i\}_{i \cap J \neq \emptyset}/(I_i)(J_j)
= \{I_i\}_{i \cap J = \emptyset} \cup \bigcup_j (\mathcal{R}_{J_j} \setminus J_j).
\]

Generally, \( \{J_j\} \) may do not admit to \( \{I_i\} \), even though \( \mathcal{R}(\{J_j\}) \subset \mathcal{R}(\{I_i\}) \). But if we consider the quotient, the situation can always be induced to the simple case.

**Proposition 2.2.** Let \( \{J_j\}, \{I_i\} \in \mathcal{P}_{\text{dis}}^2(A) \), \( \mathcal{R}(\{J_j\}) \subset \mathcal{R}(\{I_i\}) \). Then, there is a partition \( \{L_l\} \in \mathcal{P}_{\text{dis}}^2(A) \) satisfying

- \( \{L_l\} \subset \{I_i\} \).
- \( \{J_j\} \subset \{L_l\} \), and \( \mathcal{R}(\{J_j\}) = \mathcal{R}(\{L_l\}) \).
- If \( \{K_k\} \in \mathcal{P}_{\text{dis}}^2(A) \) satisfies \( \{J_j\} \subset \{K_k\} \subset \{I_i\} \), then we have \( \{L_l\} \subset \{K_k\} \).

If we ignore the order in \( \{L_l\} \), \( \{L_l\} \) is unique.

**Proof.** In \( \{J_j\} \) we define an equivalent relation as follows. Let \( n = |\{J_j\}| \). For any \( j, j' \in \mathbb{N} \), we say \( J_j \sim J_{j'} \), if there is a subset \( \{j_0, j_1, \cdots, j_m\} \) of \( \mathbb{N} \) such that \( j_0 = j, j_m = j' \), and \( \mathcal{R}_{J_{j_k}} \cap \mathcal{R}_{J_{j_{k+1}}} \neq \emptyset \) (\( k = 0, \cdots, m - 1 \)). It is obvious that \( \sim \) is an equivalent relation.

Under the equivalent relation defined above \( \{J_j\} \) can be divided into the set of equivalent class, i.e. we have
\[
\{J_j\} = \bigcup_{j \in E_l} \{J_j\},
\]
where \( \{E_l\} \in \text{Part}(\mathcal{G}) \), each \( \{J_j\}_{j \in E_l} \) is an equivalent class under \( \sim \). We take \( L_l \) to be \( L_l = \bigcup_{j \in E_l} J_j \). It is easy to check that \( \mathcal{R}(\{L_l\}) = \mathcal{R}(\{J_j\}) \) and \( \mathcal{R}_{L_l} \cap \mathcal{R}_{L_{l'}} = \emptyset \) for \( l \neq l' \). From proposition 2.1 we know that
\[
\{I_i\}/(J_j)_{j \in E_l} = \{I_i\}/(\bigcup_{j \in E_l} J_j) = \{I_i\}/(\bigcup_{j \in E_l} J_j).
\]

By the previous discussion, we can reach the formula
\[
\{I_i\}/(J_j) = \{I_i\}/(L_l).
\]

Let \( \{K_k\} \in \mathcal{P}_{\text{dis}}^2(A) \) satisfy \( \{J_j\} \subset \{K_k\} \subset \{I_i\} \), then we can prove that for each \( l \), there is \( k \) such that \( \mathcal{R}_{(L_l), L_k} \subset \mathcal{R}_{(L_l), K_k} \). On the other hand, \( \{J_j\} \subset \{K_k\} \), thus we have \( \{L_l\} \subset \{K_k\} \).

Let \( \{L_{l'}\} \in \mathcal{P}_{\text{dis}}^2(A) \) satisfy the conditions same as ones of \( \{L_l\} \), then both of \( \{L_l\} \subset \{L_{l'}\} \) and \( \{L_{l'}\} \subset \{L_l\} \) are valid, which implies \( \{L_l\} = \{L_{l'}\} \).

\[\square\]

We denote \( \{L_l\} \) by \( \{J_j\}_{\text{ad}, \{I_i\}} \). From proposition 2.2, when we discuss the quotient \( \{I_i\}/(J_j) \), we can always assume \( \{J_j\} \subset \{I_i\} \).
Corollary 2.2. Let \( \{I_i\}, \{J_j\}, \{K_k\} \in \mathcal{P}^2_{dis}(A) \) satisfying \( J \cap K = \emptyset, J, K \subset I \), where \( I = \mathcal{R}({\{I_i\}}, J = \mathcal{R}({\{J_j\}}), K = \mathcal{R}({\{K_k\}}) \). Then we have
\[
\{(I_i) \cup \{K_k\}\}_{ad, \{I_i\}}
= \{(J_j) \cup \{K_k\}_{ad, \{J_j\}}\}_{ad, \{I_i\}}
= \{(I_i) \cup \{K_k\}_{ad, \{I_i\}} \cap R\}_{ad, \{I_i\}}
= \{(K_k) \cup \{J_j\}_{ad, \{J_j\}} \cap R\}_{ad, \{I_i\}}.
\]

2.3 Insertion of the partitions

Now we turn to the discussion of insertion. Let \( \{I_i\}, \{J_j\} \in \mathcal{P}^2_{dis}(A), \mathcal{R}({\{I_i\}}) \cap \mathcal{R}({\{J_j\}}) = \emptyset \). We hope to define the insertion of \( \{J_j\} \) into \( \{I_i\} \) at \( I_a \).

Definition 2.4. Let \( \{I_i\}_{1 \leq i \leq m}, \{J_j\}_{1 \leq j \leq n} \in \mathcal{P}^2_{dis}(A), \mathcal{R}({\{I_i\}}) \cap \mathcal{R}({\{J_j\}}) = \emptyset \), and \( \iota : I_a \rightarrow \{J_j\}_{1 \leq j \leq n} \) be a map (\( 1 \leq \alpha \leq m \)). The insertion of \( \{J_j\}_{j=1}^n \) into \( \{I_i\}_{i=1}^m \) at \( I_a \) by \( \iota \) is a partition \( \{I_i\} \circ \iota^{\iota}_{I_a} \{J_j\} \) \( \in \mathcal{P}^2_{dis}(A) \), where
\[
\{I_i\} \circ \iota^{\iota}_{I_a} \{J_j\} = \{I_1, \ldots, I_{a-1}, I_a \cap \iota^{-1}(J_1), \ldots, J_n \cup \iota^{-1}(J_n), I_{a+1}, \ldots, I_m\}. \tag{2.10}
\]

\( I_a \) is called the position of insertion, and \( \iota \) is called insertion map.

Remark 2.4.
- We can explain the insertion in terms of map-union. If we ignore the order of the partitions, the insertion \( \{I_i\} \circ \iota_{I_a} \{J_j\} \) can be expressed as \( \{I_i\}_{\neq a} \cup (I_a \sqcup \{J_j\}) \). In fact, the map-union is a special situation of insertion, \( B \sqcup \{J_j\} = \{B\} \circ \iota^B_{B} \{J_j\} \) \( \cap R({\{J_j\}}) = \emptyset \). For simplicity, we can denote \( \{I_i\} \circ \iota_{I_a} \{J_j\} \) in the following intuitive way:
\[
\{I_1, \ldots, I_{a-1}, I_a \sqcup \{J_j\}, \ldots, I_m\} \uparrow \{I_1, \ldots, I_{a-1}, I_a \sqcup \{J_j\}, \ldots, I_m\}.
\]

- Particularly, we can always identify \( \{I_i\} \) with \( \{I_i\} \cup \{0\} \), then the insertion of \( \{J_j\} \) into \( \{I_i\} \) at \( 0 \) is defined as \( \{I_i\} \cup \{J_j\} \). We call \( \{I_i\} \circ \iota_{0} \{J_j\} \) the trivial insertion. Let \( \{K_k\} \in \mathcal{P}^2_{dis}(A) \) satisfying \( I \cup J \cap \mathcal{R}({\{K_k\}}) = \emptyset \), then, it is obvious that \( \{I_i\} \cup \{J_j\} \circ \iota_{I_a}^k \{K_k\} = \{I_i\} \cup \{J_j\}_\circ \iota_{I_a}^k \{K_k\} \), or \( \{I_i\} \circ \iota_{0} \{J_j\} \circ \iota_{I_a}^k \{K_k\} = \{I_i\} \circ \iota_{0} \{J_j\} \circ \iota_{I_a}^k \{K_k\} \).

In the case of non-trivial insertion we have:

Proposition 2.3. Let \( \{I_i\}, \{J_j\}, \{K_k\} \in \mathcal{P}^2_{dis}(A) \) satisfying \( I \cap J = \emptyset, I \cup J = K \) \( \cap \mathcal{R}({\{I_i\}}, J = \mathcal{R}({\{J_j\}}), K = \mathcal{R}({\{K_k\}}), \{J_j\} = \{K_k\}_{(j), \mathcal{R}({\{K_k\}}) \cap J \neq \emptyset} \). Then
\[
\{K_k\} \sqcup \{J_j\} = \{I_i\},
\]
if and only if, there is an insertion map \( \iota : I_a \rightarrow \{J_j\} \) for some \( I_a \) \( (1 \leq a \leq \#\{I_i\}) \) such that
\[
\{K_k\} = \{I_i\} \circ \iota_{I_a}^i \{J_j\}, \text{ (or } \{K_k\} = \{(I_i) / \{J_j\} \circ \iota_{I_a}^i \{J_j\}\} \).
Proof. Let \( \{ K_k \} = \{ I_i \} \circ_{I_a} \{ J_j \} \). By proposition 2.2 and remark 2.2 we have

\[
\{ K_k \} = \{ I_i \}_{i \neq a} \cup (I_a \cup \{ J_j \}).
\]

Thus \( \{ K_k \}_{K_k \cap J = \emptyset} = \{ I_i \}_{i \neq a} \), \( \{ K_k \}_{K_k \cap J \neq \emptyset} = I_a \cup \{ J_j \} \) and \( \{ K_k \}_{(J)} = \{ J_j \} \). Then we have

\[
\{ K_k \}_{K_k \cap J \neq \emptyset} / \{ J_j \} = (I_a \cup \{ J_j \}) / \{ J_j \} = \{ I_a \}.
\]

Finally, we get

\[
\{ K_k \} / \{ J_j \} = \{ K_k \}_{K_k \cap J = \emptyset} \cup (\{ K_k \}_{K_k \cap J \neq \emptyset} / \{ J_j \}) = \{ I_i \}_{i \neq a} \cup \{ I_a \} = \{ I_i \},
\]

i.e.

\[
(\{ I_i \} \circ_{I_a} \{ J_j \}) / \{ J_j \} = \{ I_i \}.
\]

Conversely, we assume \( \{ K_k \} \) satisfies \( \{ K_k \} / \{ J_j \} = \{ I_i \} \). Let \( K = \mathcal{R}(\{ K_k \}) \), \( \mathcal{R}_{J} = \mathcal{R}(\{ K_k \}_{K_k \cap J = \emptyset}) \), then we have \( K = I \cup J \), \( \{ J_j \} = \{ K_k \}_{(J)} = \{ K_k \cap J \}_{K_k \cap J \neq \emptyset} \), and

\[
\{ K_k \}_{K_k \cap J = \emptyset} \cup \{ \mathcal{R}_{J} \setminus J \} = \{ I_i \}.
\]

In order to recover \( \{ K_k \} \) by insertion, the position of the insertion should be taken to be \( I_a = \mathcal{R}_{J} \setminus J \) for some \( a \) (1 \( \leq a \leq \#(I_i)) \). Thus \( \{ K_k \}_{K_k \cap J = \emptyset} = \{ I_i \}_{i \neq a} \). Let \( \{ K_k \}_{K_k \cap J = \emptyset} = \{ K_k \} \), and \( K_{k_j} = J_j \cup L_j \), where \( J_j \cap L_j = \emptyset \), we get a decomposition of \( I_a \) that is \( I_a = \bigcup J_j \).

Above decomposition define a map \( \iota : I_a \to \{ J_j \} \), such that \( \iota^{-1}(J_j) = L_j \). It is obvious that \( \{ K_k \} = I_a \cup \{ J_j \} \). Up to now, we have proved

\[
\{ K_k \} = \{ I_i \} \circ_{I_a} \{ J_j \}.
\]

\[\square\]

Remark 2.5. In the first situation of proposition 2.3, the position of insertion is the ideal part of quotient, and the insertion map is taken in a canonical way, thus we denote this insertion by symbol \( \circ_{\text{ideal}_\alpha} \). Let \( \{ I_i \} \in \mathcal{P}_{\text{dis}}^2(A) \), \( B \subset \mathcal{R}(\{ I_i \}) \).

- When \( \mathcal{R}(\{ I_i \}_{I_i \cap B \neq \emptyset}) \setminus B \neq \emptyset \), we have

\[
(\{ I_i \} / (I_i)_{(B)}) \circ_{\text{ideal}_\alpha} (I_i)_{(B)} = \{ I_i \},
\]

where the position of insertion is at \( \mathcal{R}(\{ I_i \}_{I_i \cap B \neq \emptyset}) \setminus B \), and the insertion map is taken to be \( \iota^{-1}(I_i \cap B) = I_i \setminus B \) if \( I_i \setminus B \neq \emptyset \).

- When \( \mathcal{R}(\{ I_i \}_{I_i \cap B \neq \emptyset}) = B \), we have

\[
\{ I_i \} = (\{ I_i \} / (I_i)_{(B)}) \circ_{\emptyset} (I_i)_{(B)},
\]

where the insertion is trivial one.

There is a conclusion about insertion and map-union as follows.

Proposition 2.4. Let \( \{ I_i \}, \{ J_j \} \in \mathcal{P}_{\text{dis}}^2(A) \), \( B \subset A \), \( I \cap J = \emptyset \), \( B \cap (I \cup J) = \emptyset \), \( I = \mathcal{R}(\{ I_i \}) \), \( J = \mathcal{R}(\{ J_j \}) \). For a pair \((f, \iota)\) there is an unique pair \((f', \iota')\) such that

\[
(B \sqcup_f \{ I_i \}) \circ_{I_a} \{ J_j \} = B \sqcup_{f'} \{ (I_i) \circ_{I_a} \{ J_j \} \},
\]

where \( \circ_{I_a} \) denotes the insertion map.
where \( 1 \leq a \leq |\{I_i\}|, f: B \to \{I_i\}, f': B \to \{I_i\} \circ^\kappa_{a'} \{J_j\}, \iota: f^{-1}(I_a) \cup I_a \to \{J_j\}, \) and vice-versa, \( \iota': I_a \to \{J_j\} \). On the other hand, we also have

\[
(B \cup f (\{I_i\} \circ^\kappa_{I_a} \{J_j\}) \cap \{J_j\}) = B \cup f_{\|_{I_a} \circ^\kappa_{I_a} \{J_j\}) \cap \{J_j\}) = B \cup f \circ \|_{I_a} \circ^\kappa_{I_a} \{J_j\},
\]

where \( p: \{I_i\} \circ^\kappa_{I_a} \{J_j\} \to \{I_i\} \) is a projection satisfying

\[
p(I_i) = I_i (i \neq a) p((\iota')^{-1}(J_j) \cup J_j) = I_a (\forall j), f' \circ p = f.
\]

**Proof.** Let \((f, \iota)\) be a given pair, we want to construct the pair \((f', \iota')\) based on \((f, \iota)\). By the definition of insertion and map-union we have

\[
(B \cup_f \{I_i\} \circ^\kappa_{f^{-1}(I_a) \cup I_a} \{J_j\}) = \{f^{-1}(I_i) \cup I_i \}_{i \neq a} \cup \{f^{-1}(I_a) \cup I_a \} \cup I_i \{J_j\},
\]

and

\[
B \cup_f ((\{I_i\} \circ^\kappa_{f^{-1}(I_a) \cup I_a} \{J_j\}) = \{f^{-1}(I_i) \cup I_i \}_{i \neq a} \cup (B_1 \cup_{f \|_{I_a}} (I_a \cup \iota \{J_j\})),
\]

where \( B_1 = f^{-1}(I_a \cup J) \). The previous formulas imply \( f^{-1}(I_i) = f^{-1}(I_i) (i \neq a) \), and \( f^{-1}(I_a) = f^{-1}(I_a \cup J) \). Thus we have \( B_1 = f^{-1}(I_a) \) and \( f|_{B_1} = f|_{B_1} \). If we regard \( \iota, \iota', f|_{B_1} \) as maps from some subsets to \( \{J_j\} \), then the first formula in proposition 2.4 means that the formula \( \iota = f'|_{B_1} \cup \iota' \) should be valid, i.e. we have

\[
f'|_{B_1} = \iota|_{B_1}, \iota' = \iota|_{I_a}.
\]

Conversely, starting from \((f', \iota')\) we can determine \((f, \iota)\) in similar way.

The second formula in proposition 2.4 is obviously valid.

\[\square\]

**Remark 2.6.** Observing the proof of proposition 2.4, in the procedure of from \((f', \iota')\) to \((f, \iota)\), \( f' \) determines \( f \) uniquely, where \( f' \) and \( f \) depend on “a” only. If we fix \( f' \), there is an one-one corresponding between \( \iota \) and \( \iota' \).

There is another version of proposition 2.1 in terms of the insertion.

**Proposition 2.5.** Let \( \{I_i\}, \{J_j\}, \{K_k\} \in \mathcal{P}^2(A), \emptyset \in \emptyset, K \cap (I \cup J) = \emptyset, \emptyset = \mathcal{R}(\emptyset) \), \( J = \mathcal{R}(\emptyset) \), \( K = \mathcal{R}(\emptyset) \). Let \( \{I_i\} = (\{K_k\} \circ^\kappa_{K_k} \{I_i\}) \circ^\kappa_{\tau} \{J_j\}, \) then

- \( \mathcal{R}_{\{L_i\}, I \cap \mathcal{R}_{\{L_i\}, I} = \emptyset \iff \{I_i\} = (\{K_k\} \circ^\kappa_{K_k} \{I_i\}) \circ^\kappa_{\tau} \{J_j\}, a \neq b. \)
- \( \mathcal{R}_{\{L_i\}, I \cap \mathcal{R}_{\{L_i\}, I} \neq \emptyset \iff \{I_i\} = (\{K_k\} \circ^\kappa_{K_k} \{I_i\}) \circ^\kappa_{\tau^{-1}(I_a) \cup I_a} \{J_j\}. \)

In this situation, we have

\[
\{I_i\} = \{K_k\} \circ^\kappa_{K_k} ((\{I_i\} \circ^\kappa_{I_a} \{J_j\})
\]

and

\[
\{I_i\|_{I \cup J} = \{I_i\} \circ^\kappa_{I_a} \{J_j\},
\]

for some \( \iota' \) and \( \kappa \).
Proof. Case of $\mathcal{R}(L_1) \cap \mathcal{R}(L_1) \cap J = \emptyset$:

Let $\mathcal{R}(L_1) \cap \mathcal{R}(L_1) \cap J = \emptyset$, then

$$\{L_1\} = \{L_1 \cap (L_1 \cap J) \cap \emptyset \cup \{L_1 \cap J \neq \emptyset \cup \{L_1 \cap J \neq \emptyset \}.$$ 

Noting $\{L_1\}_J = \{J\}$, by proposition 2.1 and proposition 2.2 we have

$$\{L_1\} \cap \{J\} = \{L_1 \cap (L_1 \cap J) \cap \emptyset \cup \{L_1 \cap J \neq \emptyset \cup \mathcal{R}(L_1) \cap \{J\} = \{K_k \} \cap \{J\}.$$ 

Furthermore, we now that $\{L_1\}_J = \{I_i\}$. Thus

$$\{L_1\} \cap \{J\} = \{L_1 \cap (L_1 \cap J) \cap \emptyset \cup \mathcal{R}(L_1) \cap \{J\} \cup \mathcal{R}(L_1) \cap \{J\} = \{K_k \}.$$ 

Above formula means that

$$\{\mathcal{R}(L_1) \cap \{I\} \cap \{J\} = \{L_1 \cap (L_1 \cap J) \cap \emptyset \cup \mathcal{R}(L_1) \cap \{I\} \cap \{J\} = \{K_k \}.$$ 

and

$$\{\mathcal{R}(L_1) \cap \{J\} = \{L_1 \cap (L_1 \cap J) \cap \emptyset \cap \mathcal{R}(L_1) \cap \{J\} = \{K_k \}.$$ 

for some $a$ and $b$ ($a \neq b$).

Conversely, let

$$\{L_1\} = \{\{K_k \} \cap \{I_i\} \cap \{J_j\}, a \neq b \}.$$ 

Then, we have

$$\{L_1\} = \{\{K_k \} \cap \{I_i\} \cap \{J_j\}, a \neq b \}.$$ 

and

$$\{L_1\} = \{\{K_k \} \cap \{I_i\} \cap \{J_j\}, a \neq b \}.$$ 

Above facts imply $\mathcal{R}(L_1) \cap \mathcal{R}(L_1) \cap J = \emptyset$.

Case of $\mathcal{R}(L_1) \cap \mathcal{R}(L_1) \cap J \neq \emptyset$:

The conclusion in this situation is the corollary of the one in the case of $\mathcal{R}(L_1) \cap \mathcal{R}(L_1) \cap J = \emptyset$. Let

$$\{L_1\} = \{\{K_k \} \cap \{I_i\} \cap \{J_j\} \cap \mathcal{R}(L_1) \cap \{J\}.$$ 

By proposition 2.4 we have

$$\{K_k \} \cap \{I_i\} \cap \{J_j\} = K_k \cap \{I_i\} \cap \{J_j\}.$$ 

for some $\kappa$ and $\kappa$. It is natural that we have

$$\{L_1\} = \{I_i\} \cap \{J_j\}.$$ 

□

Proposition 2.5 can be generalized to more general situation. Let $\{I_i\}, \{J_{j_1}\}, \cdots, \{J_{j_n}\} \in \mathcal{P}_{\text{dis}}(A)$, $J_{(i)} \cap J_{(i')} = \emptyset$ ($i \neq i'$), $I \cap (\bigcup_{i=1}^{n} J_{(i)}) = \emptyset$, $I = \mathcal{R}(\{I_i\})$, $J_{(i)} = \mathcal{R}(\{J_{j_i}\})$ ($1 \leq i \leq n$),

$$\{K_{(1)} \} = \{I_i\} \cap \{J_{(1)}\}, \{K_{(2)} \} = \{K_{(1)}\} \cap \{J_{(2)}\}, \cdots,$$

$$\{K_{(n-1)} \} = \{K_{(n-1)}\} \cap \{J_{(n-1)}\}, \{K_{(n)} \} = \{K_{(n)}\} \cap \{J_{(n)}\}.$$
Corollary 2.3. $\mathcal{R}(I_k, J^{(l)}) \cap \mathcal{R}(K_k, J^{(l')}) = \emptyset$ if and only if there are $a_1, \ldots, a_n$ such that

$$\{K_k\} = \{I_i\}_{i \neq a_1, \ldots, a_n} \cup \left( \bigcup_{i=1}^n I_{a_i} \cup \{J^{(l)}_{j_i}\} \right).$$

Remark 2.7. The conclusion of corollary 2.2 can be described by a different way, which is

$$\{J^{(l)}\}_{i=1}^n \subset \{K_k\} \Leftrightarrow \{K_k\} = \{I_i\}_{i \neq a_1, \ldots, a_n} \cup \left( \bigcup_{i=1}^n I_{a_i} \cup \{J^{(l)}_{j_i}\} \right),$$

for some $a_1, \ldots, a_n$.

2.4 Coproduct

Now we consider the coproduct as an application of quotient. At first we introduce

$$\mathcal{P}_{\text{dis}, k}^2(A) \subset \mathcal{P}_{\text{dis}}^2(A) \times \cdots \times \mathcal{P}_{\text{dis}}^2(A),$$

$$\{(I_{i_1}^{(1)}), \ldots, (I_{i_k}^{(k)})\} \in \mathcal{P}_{\text{dis}, k}^2(A) \Leftrightarrow \{I_{i_1}^{(1)}\} \in \mathcal{P}_{\text{dis}}^2(A), \lambda = 1, \ldots, k;$$

$$\mathcal{R}(\{I_{i_1}^{(1)}\}) \cap \mathcal{R}(\{I_{i_k}^{(k)}\}) = \emptyset, 1 \leq \lambda < \mu \leq k.$$

Similar to definition 2.3 we have

Definition 2.5. Let $\{J_j\} \in \mathcal{P}_{\text{dis}}^2(A), \{(I_{i_1}^{(1)}), \ldots, (I_{i_k}^{(k)})\} \in \mathcal{P}_{\text{dis}, k}^2(A)$, we say $\{J_j\} \subset \{(I_{i_1}^{(1)}), \ldots, (I_{i_k}^{(k)})\}$, if $\mathcal{R}(\{J_j\}) \subset \bigcup_{\lambda = 1}^k \mathcal{R}(\{I_{i_{\lambda}}^{(\lambda)}\})$, and $\{J_j\} \cap \mathcal{R}(I_{i_{\lambda}}^{(\lambda)}) \neq \emptyset \subset \{I_{i_{\lambda}}^{(\lambda)}\}, (\lambda = 1, \ldots, k)$. In this situation, we define

$$\{(I_{i_1}^{(1)}), \ldots, (I_{i_k}^{(k)})\}/(J_j) = ((I_{i_1}^{(1)})/(J_j), \ldots, (I_{i_k}^{(k)})/(J_j)),$$

where

$$\{I_{i_{\lambda}}^{(\lambda)}\}/(J_j) = (I_{i_{\lambda}}^{(\lambda)})/(J_j)_{J_j \subset \mathcal{R}(I_{i_{\lambda}}^{(\lambda)})}, \lambda = 1, \ldots, k.$$

Similarly, we can discuss the insertion for the case of $\{(I_{i_1}^{(1)}), \ldots, (I_{i_k}^{(k)})\}$ in a obvious way.

Let $\mathbb{K}$ be a field of characteristic zero, $V_{\mathbb{K}, A} = \bigoplus_{1 \leq k \leq |A|} \text{Span}_A(\mathcal{P}_{\text{dis}, k}^2(A))$. Based on the above discussions we now define coproduct on $V_{\mathbb{K}, A}, \triangle : V_{\mathbb{K}, A} \rightarrow V_{\mathbb{K}, A} \otimes V_{\mathbb{K}, A}$.

Definition 2.6. Let $\{I_i\} \in \mathcal{P}_{\text{dis}}^2(A), \{(I_{i_1}^{(1)}), \ldots, (I_{i_k}^{(k)})\} \in \mathcal{P}_{\text{dis}, k}^2(A)$, we define

$$\bigtriangleup\{I_i\} = \emptyset \otimes \{I_i\} + \{I_i\} \otimes \emptyset +$$

$$\sum_{\{J_j\} \subset \{I_i\}} (\{I_i\}_{(J_j)} \cdots, \{I_i\}_{(J_j')} \otimes \{I_i\}/(J_j), \ldots, \{I_i\}/(J_j')) \quad (2.11)$$

$$\bigtriangleup((I_{i_{1}}^{(1)}), \ldots, (I_{i_{k}}^{(k)})) = \emptyset \otimes ((I_{i_{1}}^{(1)}), \ldots, (I_{i_{k}}^{(k)})) + ((I_{i_{1}}^{(1)}), \ldots, (I_{i_{k}}^{(k)})) \otimes \emptyset +$$

$$\sum_{\{J_j\} \subset ((I_{i_{1}}^{(1)}), \ldots, (I_{i_{k}}^{(k)}))} ((I_{i_{1}}^{(1)}), \ldots, (I_{i_{k}}^{(k)})/\{J_j\}) \cdots, \{I_i\}/(J_j), \ldots, \{I_i\}/(J_j')) \quad (2.12)$$

where $\{I_i\} = \bigcup_\lambda \{I_{i_{\lambda}}^{(\lambda)}\}$.
For associativity of coproduct defined in definition 2.6 we need the following conclusion.

**Theorem 2.1.** Let \( \{I_i, \{J_j, \{K_k \in \mathcal{P}^2_{\text{dis}}(A), \{J_j \sqsubset \{I_i, \{K_k \sqsubset \{I_i, (J_j). If we take \( \{M_\mu \} = ((J_j) \cup \{K_k\})_{\text{ad}, (I_i), then \)

\[
\{M_\mu \} = \{J_j\}_{(J_j) \cap K = \emptyset} \cup \{N_k\},
\]

where \( N_k = K_k \cup \bigcup_{(J_j) \cap K = \emptyset \neq J_j} \), \( K = \mathcal{R}(\{K_k\}), k = 1, \ldots, |\{K_k\}|. \{M_\mu \} satisfies the following conditions:

- \( \{K_k\} = \{M_\mu\}/(J_j) \).
- \( \{M_\mu\} \sqsubset \{I_i\} \).
- \( \{J_j\} \sqsubset (\cdots, \{I_i\}_{(M_\mu)}, \cdots) \).
- \( (\cdots, (\{I_i\}/(J_j))_{(K_k)}, \cdots) = (\cdots, (\{I_i\}_{(M_\mu)}, \cdots)/(J_j). \)
- \( (\{I_i\}/(J_j))/(K_k) = \{I_i\}/(M_\mu). \)

**Proof.** At first, we prove \( \{M_\mu\} = \{J_j\}_{(J_j) \cap K = \emptyset} \cup \{N_k\}. \)

Let \( \{L_\lambda\} = \{I_i\}/(J_j) \), then, from definition of quotient we have

\[
\{L_\lambda\} = \{I_i\}_{I_i \cap J = \emptyset} \cup \{\mathcal{R}_{J_j \setminus J_j^p}\}_{j = 1},
\]

where \( p = |\{J_j\}|, J = \mathcal{R}(\{J_j\}_{j = 1}^p), \) and \( \mathcal{R}_{J_j} = \mathcal{R}_{(I_i), J_j}. \) Recalling definition 2.3, \( \{K_k\} \sqsubset \{L_\lambda\} \) means that

\[
\mathcal{R}_{(L_\lambda)} \cap \mathcal{R}_{(L_\lambda), K_k'} = \emptyset, k \neq k',
\]

On the other hand, according to \( \{K_k\} \) there is the decomposition of \( \{L_\lambda\} \) as follows,

\[
\{L_\lambda\} = \{L_\lambda\}_{L_\lambda \cap K = \emptyset} \cup \{L_\lambda\}_{L_\lambda \cap K \neq \emptyset} \cup \cdots \cup \{L_\lambda\}_{L_\lambda \cap K_q \neq \emptyset},
\]

where \( K = \mathcal{R}(\{K_k\}), \) and \( q = |\{K_k\}|. \) It is obvious that

\[
\{L_\lambda\}_{L_\lambda \cap K = \emptyset} = \{I_i\}_{I_i \cap (J_j \cup K) = \emptyset} \cup \{\mathcal{R}_{J_j \setminus J_j^p}\}_{J_j \cap K = \emptyset},
\]

and

\[
\{L_\lambda\}_{L_\lambda \cap K \neq \emptyset} = \{I_i\}_{I_i \cap J \neq \emptyset} \cup \{\mathcal{R}_{J_j \setminus J_j^p}\}_{J_j \cap K_k \neq \emptyset},
\]

where \( k = 1, \ldots, q. \) Notong \( J \cap K = \emptyset, \) thus

\[
(\mathcal{R}_{J_j \setminus J_j^p} \cap K_k \neq \emptyset) \Leftrightarrow \mathcal{R}_{J_j \cap K_k \neq \emptyset}.
\]

In summary, we know that for each \( \mathcal{R}_{J_j} \) there two possibilities:

- \( \mathcal{R}_{J_j} \cap K = \emptyset, \)

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or, there is an unique $k$ such that $\mathcal{R}_{J_j} \cap K_k \neq \emptyset$.

Up to now we have proved

$$\{M_\mu\} = \{J_j\}_{\mathcal{R}_{J_j} \cap K = \emptyset} \cup \{N_k\}.$$  

We now begin to prove that $\{M_\mu\}$ satisfies the conditions in theorem 2.1 step by step.

Let $\{G_i\} = \{J_j\}_{\mu \in \{M_\mu\}}$, then we have

$$\{G_i\} = \{J_j\}_{\mathcal{R}_{J_j} \cap K = \emptyset} \cup \left( \bigcup_{k} \{J_j\}_{\mathcal{R}_{J_j} \cap K_k \neq \emptyset} \right),$$

and $\{M_\mu\} \cap \{J_j\} = \{M_\mu\} \cap \{G_i\}$. It is easy to check that $\{K_k\} = \{M_\mu\} \cap \{G_i\}$.

Noting

$$\{I_i\}_{I_i \cap N_k \neq \emptyset} = \{I_i\}_{I_i \cap J = \emptyset, I_i \cap K \neq \emptyset} \cup \left( \bigcup_{j} \{I_i\}_{I_i \cap J = \emptyset, \mathcal{R}_{J_j} \cap K \neq \emptyset} \right),$$

thus

$$\mathcal{R}_{\{I_i\}_{I_i \cap N_k \neq \emptyset}} = \mathcal{R}_{\{L_\lambda\}_{K_k}} \cup \left( \bigcup_{\mathcal{R}_{J_j} \cap K \neq \emptyset} J_j \right), k = 1, \cdots, q.$$  

Because $\{K_k\} \subset \{L_\lambda\}$, which means

$$\mathcal{R}_{\{L_\lambda\}_{K_k} \cap \mathcal{R}_{\{L_\lambda\}_{K_k'}} = \emptyset, k \neq k',$$

hence, $\{N_k\} \subset \{I_i\}$.

$$\{I_i\}_{\{J_j\}_{I_i \cap N_k} \subset \{L_\lambda\}_{K_k}} = \{I_i\}_{\{J_j\}_{I_i \cap N_k} \subset \{L_\lambda\}_{K_k}}, k = 0, 1, \cdots, q.$$  

By the definition of $\{N_k\}$ we know that for each $J_j$, $J_j \cap K \neq \emptyset$, there is a $N_k$ such that $J_j \subset N_k$. Thus we have

$$\{I_i\}_{\{N_k\}} = \{I_i \cap K\}_{I_i \cap J = \emptyset, I_i \cap K \neq \emptyset} \cup \left( \bigcup_{\mathcal{R}_{J_j} \cap K \neq \emptyset} \{I_i\}_{\mathcal{R}_{J_j} \cap (J \cup K)} \right).$$

Noting $\mathcal{R}_{\{I_i\}_{\{N_k\}}} = N_k$, ($1 \leq k \leq q$), thus

$$J_j \subset N_k \Leftrightarrow \mathcal{R}_{J_j} \cap K_k \neq \emptyset.$$  

It is obvious that we have $\{J_j\}_{J_j \subset N_k} \subset \{I_i\}_{\{N_k\}}$.

$$\cdots, (\{I_1\} \cap \{J_j\}) (K_k), \cdots) = \cdots, (\{I_1\} (M_\mu), \cdots) \cap \{J_j\} :$$

Noting

$$\{I_i\}_{\{M_\mu\}} = \left\{ \begin{array}{ll} \{I_i\}_{\{J_j\}}, & \mathcal{R}_{J_j} \cap K = \emptyset, \\
\{I_i\}_{\{N_k\}}, & \text{other}, \end{array} \right.$$  

thus, in the situation of $\mathcal{R}_{J_j} \cap K = \emptyset$, $\{I_i\}_{\{J_j\}} \cap \{I_i\}_{\{J_j\}} = \emptyset$, and in the part of $\{I_i\}_{\{N_k\}}$, by the expressions of $\{I_i\}_{\{N_k\}}$ and $\{L_\lambda\}_{L_j \subset K_k \neq \emptyset}$, we have

$$\{L_\lambda\}_{\{K_k\}} = \{I_i\}_{\{N_k\}} \cap \{J_j\}_{J_j \subset N_k},$$

$$((\{I_i\} \cap \{J_j\}) \cap \{K_k\}) = \{I_i\} \cap \{M_\mu\}.$$
We note that

\[ \{I_i\}/(M_p) = \{I_i\}_{I_i \cap M = \emptyset} \cup \left( \bigcup_{j, R_j \cap K = \emptyset} \{I_i\}_{I_i \cap J_j \neq \emptyset} \right) \cup \left( \bigcup_{k=0}^{q} \{I_i\}_{I_i \cap M_k \neq \emptyset} \right) \]

and

\[ \{L_\lambda\}/(K_k) = \{L_\lambda\}_{L_\lambda \cap K = \emptyset} \cup \left( \bigcup_{k=1}^{q} \{L_\lambda\}_{L_\lambda \cap K_k \neq \emptyset} \right) \]

It is obvious that

\[ \{L_\lambda\}_{L_\lambda \cap K = \emptyset} = \{I_i\}_{I_i \cap M = \emptyset} \cup \left( \bigcup_{j, R_j \cap K = \emptyset} \{I_i\}_{I_i \cap J_j \neq \emptyset} \right) \]

Noting \(M_k = K_k \cup (\bigcup_{j, R_j \cap K_k \neq \emptyset} J_j)\), hence, we have

\[ R_{(L_\lambda), K} \setminus K_k = R_{(I_i), M_k} \setminus M_k. \]

Above formula implies

\[ \{L_\lambda\}_{L_\lambda \cap K_k \neq \emptyset} \cap \{L_\lambda\}_{(K_k)} = \{I_i\}_{I_i \cap M_k \neq \emptyset} \cap \{I_i\}_{(M_k)}, \]

where \(1 \leq k \leq q\).

Up to now we have proved the proposition.

As the corollary of theorem 2.1 we have:

**Proposition 2.6.** The coproduct in definition 2.6 satisfies

\[ (\triangle \otimes \text{id}) \triangle = (\text{id} \otimes \triangle) \triangle. \quad (2.13) \]

**Proof.** For simplicity, we will replace the coproduct \(\triangle\) in (2.13) by the reduce coproduct \(\triangle'\), where

\[ \triangle'(\cdot) = \triangle(\cdot) - \triangle(\cdot) \otimes \emptyset - \emptyset \otimes (\cdot). \]

Here we only give the proof in the situation of

\[ (\triangle' \otimes \text{id}) \triangle' (\{I_i\}) = (\text{id} \otimes \triangle') \triangle' (\{I_i\}), \quad (*) \]

where \(\{I_i\} \in \mathcal{P}_{d_{\mu}}^2(A)\). The general situation is similar.

Recalling the formula (2.11) in definition 2.6, we have

\[ \triangle'(\{I_i\}) = \sum_{\{J_1, \ldots, J_l\} \subset \{I_i\}} (\{I_i\}_{(J_1)} \otimes \cdots \otimes \{I_i\}_{(J_l)}) / (J_j). \]

The right side of (*) should be of the following form,

\[ = \sum_{\{J_1, \ldots, J_l\} \subset \{I_i\}} (\{I_i\}_{(J_1)} \otimes \cdots \otimes \{I_i\}_{(J_l)}) / (J_j) \]

\[ = \sum_{\{J_1, \ldots, J_l\} \subset \{I_i\}, \{K_k\} \subset \{L_\lambda\}} (\{I_i\}_{(J_1)} \otimes \cdots \otimes \{I_i\}_{(J_l)}) / (J_j) \]

\[ \otimes (\{L_\lambda\}_{(K_k)}) / (K_k), \]

\[ = (\text{id} \otimes \triangle') \triangle' (\{I_i\}) / (J_j). \]

\[ \square \]
where \( \{L_\lambda\} = \{I_i\}/(J_j) \). Recalling proposition 2.8, we know that there is \( \{M_\mu\} \in P^2_{\text{dis}}(A) \) satisfying the following conditions:

\[
\{M_\mu\} \subseteq \{I_i\}, \{J_j\} \subseteq (\cdots, \{I_i\}(M_\mu), \cdots),
\]

\[
(\cdots, ((I_i)/(J_j))(K_k), \cdots) = (\cdots, \{I_i\}(M_\mu), \cdots)/(J_j),
\]

\[
((I_i)/(J_j))/(K_k) = \{I_i\}/(M_\mu),
\]

then, we have

\[
\sum_{\{M_\mu\} \subseteq \{I_i\}, \{J_j\} \subseteq (\{I_i\}(M_\mu))} (id \otimes \Delta')(\{I_i\}) = \sum_{\{M_\mu\} \subseteq \{I_i\}, \{J_j\} \subseteq (\{I_i\}(M_\mu))} (\{I_i\}(M_\mu))/\{I_i\}/(M_\mu).
\]

Because \( R(\{J_j\}) \subseteq R(\{M_\mu\}) \), we have \( \{I_i\}(M_\mu \cap J_j) = \{I_i\}(J_j) \). Finally, we have

\[
= (id \otimes \Delta')(\{I_i\}) = (\{I_i\}(J_j)) \otimes (\{I_i\}/(M_\mu)) = (\Delta' \otimes id)(\{I_i\}).
\]

\[ \square \]

**Remark 2.8.** Proposition 2.9 means that \( V_{\mathbb{K},A} \) is a coalgebra under the coproduct in definition 2.6. It is easy to check that for each \( \{I_i\} \in P^2_{\text{dis}}(A) \) (or \( \{I_i^{(1)}\}, \cdots, \{I_i^{(k)}\} \in P^2_{\text{dis}, k}(A) \) \( (\Delta')^m(\{I_i\}) = 0 \) (or \( (\Delta')^m(\{I_i^{(1)}\}, \cdots, \{I_i^{(k)}\}) = 0 \) for some positive integer \( m \), thus, from \( V_{\mathbb{K},A} \), we can construct a Hopf algebra in a standard way (see D. E. Radford [3]).

### 3 Composition and Lie bracket

Let \( \mathbb{K} \) be a field of characteristic zero, \( \mathbb{K}(P^2_{\text{dis}}(A)) \) denote the vector space over \( \mathbb{K} \) spanned by \( P^2_{\text{dis}}(A) \). We now go to the discussion of the composition between \( \{I_i\} \) and \( \{J_j\} \), where \( \{I_i\}, \{J_j\} \in P^2_{\text{dis}}(A), \mathcal{R}(\{I_i\}) \cap \mathcal{R}(\{J_j\}) = \emptyset \). When we focus on the composition, we restrict us to consider the non-travail insertion only. Similar to the cases of Feynman diagrams and Kontsevich’s graphs, we have:

**Definition 3.1.** We define

\[
\{I_i\} \circ_a \{J_j\} = \sum_i \{I_i\} \circ_i a \{J_j\}, \tag{3.1}
\]

and

\[
\{I_i\} \circ a \{J_j\} = \sum_a \{I_i\} \circ_a \{J_j\}. \tag{3.2}
\]

The composition \( \circ \) defined by (3.2) is not associative generally. Let \( \{I_i\}, \{J_j\}, \{K_k\} \in P^2_{\text{dis}}(A) \) satisfying \( I \cap J = \emptyset, K \cap (I \cup J) = \emptyset \), where \( I = \mathcal{R}(\{I_i\}), J = \mathcal{R}(\{J_j\}) \) and \( K = \mathcal{R}(\{K_k\}) \). We are interested in the difference between \( (\{I_i\} \circ \{J_j\}) \circ \{K_k\} \) and \( \{I_i\} \circ (\{J_j\} \circ \{K_k\}) \). In general, we have

\[
(\{I_i\} \circ \{J_j\}) \circ \{K_k\} \neq \{I_i\} \circ (\{J_j\} \circ \{K_k\}).
\]

Observing \( (\{I_i\} \circ \{J_j\}) \circ \{K_k\} \), by (3.1) and (3.2) we know that

\[
(\{I_i\} \circ \{J_j\}) \circ \{K_k\} = \sum_{a,i} \{I_i\} \circ_i a \{J_j\} \circ \{K_k\}.
\]
and

\[
\{\{I_i\} \circ \{J_j\} \circ \{K_k\} = \sum_{b, \tau, a \neq b} (\{I_i\} \circ \{J_j\}) \circ \{K_k\} + \sum_{c, \kappa} (\{I_i\} \circ \{J_j\}) \circ \{K_k\}. \\
\]

Recalling proposition 2.5 we know that

\[
\{I_i\} \circ \{J_j\} \circ \{K_k\} = \{I_i\} \circ \{J_j\} \circ \{K_k\}
\]

for some \(\iota'\) and \(\kappa'\). By proposition 2.4 we know that, for fixed \(I_a\) and \(J_c\), there is an one-one corresponding between \((\iota, \kappa)\) and \((\iota', \kappa')\). Summarizing previous discussions, we reach the following formula.

**Proposition 3.1.**

\[
\{\{I_i\} \circ \{J_j\} \circ \{K_k\} = \sum_{a, b, \alpha \neq b}(\{I_i\} \circ \{J_j\}) \circ \{K_k\} + \{I_i\} \circ \{J_j\} \circ \{K_k\}. \quad (3.3)
\]

We introduce a compact notation

\[
<\{I_i\}, \{J_j\}, \{K_k\}> = \sum_{a, b, \alpha \neq b}(\{I_i\} \circ \{J_j\}) \circ \{K_k\},
\]

then the formula (3.3) can be rewritten as

\[
(\{I_i\} \circ \{J_j\}) \circ \{K_k\} = \{I_i\} \circ \{J_j\} \circ \{K_k\} + <\{I_i\}, \{J_j\}, \{K_k\}>
\]

Now we have

**Corollary 3.1.**

\[
<\{I_i\}, \{J_j\}, \{K_k\}> = <\{I_i\}, \{K_k\}, \{J_j\}>.
\]

**Proof.** Noting

\[
\{\{I_i\} \circ \{J_j\} \circ \{K_k\} = \{I_i\}_a \cup \{I_a \cup \{J_j\}_b \cup \{K_k\}_{\iota, \kappa} \},
\]

the conclusion will be implied immediately. \(\square\)

**Corollary 3.2.**

\[
\{I_i\} \circ (\{J_j\} \circ \{K_k\}) = (\{I_i\} \circ \{J_j\}) \circ \{K_k\} = \{I_i\} \circ \{K_k\} \circ \{J_j\} = (\{I_i\} \circ \{K_k\}) \circ \{J_j\}.
\]

Based on the discussion as above we are able to define the Lie bracket for partitions.

**Definition 3.2.** Let \(\{I_i\}, \{J_j\} \in \mathcal{P}^2_{dis}(A)\) with \(\mathcal{R}(\{I_i\}) \cap \mathcal{R}(\{J_j\}) = \emptyset\), their Lie bracket is defined to be

\[
[\{I_i\}, \{J_j\}] = \{I_i\} \circ \{J_j\} - \{J_j\} \circ \{I_i\}. \quad (3.4)
\]

In order to prove the bracket (3.4) is Lie bracket, it is enough for us to check that Jacobi identity is valid.

**Theorem 3.1.** The bracket (3.4) satisfies Jacobi identity.
Proof. Let \( \{I_i\}, \{J_j\}, \{K_k\} \in \mathcal{P}_\text{dis}^2(A) \) satisfying \( \mathcal{R}(\{I_i\}) \cap \mathcal{R}(\{J_j\}) \neq \emptyset, \mathcal{R}(\{I_i\}) \cap \mathcal{R}(\{K_k\}) \neq \emptyset, \mathcal{R}(\{J_j\}) \cap \mathcal{R}(\{K_k\}) \neq \emptyset \), we want to prove Jacobi identity

\[
[\{I_i\}, [\{J_j\}, \{K_k\}]] + \text{cocycle} = 0.
\]

By a straightforward calculation we get

\[
[\{I_i\}, [\{J_j\}, \{K_k\}]] = \{I_i\} \circ ([J_j] \circ [K_k]) - [J_j] \circ ([I_i] \circ [K_k]) - ([I_i] \circ [K_k]) \circ [J_j] + ([I_i] \circ [J_j]) \circ [K_k],
\]

With the help of corollary 3.2 we can get Jacobi identity. \(\Box\)

The previous discussions can be generalized to more general situations. We assume that every partition \( \{I_i\} \in \mathcal{P}_\text{dis}^2(A) \) assigns to a function \( f : \{I_i\} \to \mathbb{N} \). Now we consider the Lie bracket concerning the pair \( (\{I_i\}, f) \). Let \( (\{I_i\}, f), (\{J_j\}, g) \) be two pairs, where \( \{I_i\}, \{J_j\} \in \mathcal{P}_\text{dis}^2(A) \), \( \mathcal{R}(\{I_i\}) \cap \mathcal{R}(\{J_j\}) = \emptyset \). Then the insertion of two pairs is defined to be

\[
(\{I_i\}, f) \circ_{\text{pair}} ([J_j], g) = \{\{I_i\}, f\} \circ_{\text{i_process}} ([J_j], g),
\]

where \( f \circ_a g \) is defined as follows:

\[
f \circ_a g(I_i) = f(I_i) (i \neq a), \quad f \circ_a g(i^{-1}(J_j) \cup J_j) = g(J_j).
\]

Above formula shows that \( f \circ_a g \) is independent of \( i \). Now we define the composition of the pairs to be

\[
(\{I_i\}, f) \circ_a (\{J_j\}, g) = \sum_i (\{I_i\}, f) \circ_{\text{i_process}} ([J_j], g),
\]

and

\[
(\{I_i\}, f) \circ_{\text{pair}} (\{J_j\}, g) = \sum_a (-1)^{f(I_i)}(\{I_i\}, f) \circ_a (\{J_j\}, g).
\]

We hope that the Lie bracket arising from the composition \( \circ_{\text{pair}} \) will be well defined. Hence, we need to prove a conclusion similar to corollary 3.2 is valid. Let \( (\{I_i\}, f), (\{J_j\}, g), (\{K_k\}, h) \) be three pairs, it is enough for us to consider the composition \( (f \circ_a g) \circ_b h \). Similar to the situation of proposition 3.3, it is necessary for us to discuss the following two possibilities:

- **Case of** \( (\{I_i\} \circ_{\text{i_process}} \{J_j\}) \circ_{\text{i_process}} \{K_k\} (a \neq b) \): In this situation we have

\[
\begin{align*}
(f \circ_a g) \circ_b h(I_i) &= (f \circ_a g)(I_i) = f(I_i), \quad i \neq a, b, \\
(f \circ_a g) \circ_b h(i^{-1}(J_j) \cup J_j) &= (f \circ_a g)(i^{-1}(J_j) \cup J_j) = g(J_j), \\
(f \circ_a g) \circ_b h(i^{-1}(K_k) \cup K_k) &= h(K_k).
\end{align*}
\]

In this situation we have

\[
(f \circ_a g) \circ_b h = (f \circ_b h) \circ_a g.
\]
Case of $\{I_i\} o_{I_a}^* \{J_j\} o_{J_c}^{-1}(J_d) \cup J_e \{K_k\}$ (a \neq b) : In this situation we have

$$((f o_a g) o_b h)(I_i) = (f o_a g)(I_i) = f(I_i), i \neq a,$$

$$((f o_a g) o_b h)((J_j) \cup J_j) = (f o_a g)((J_j) \cup J_j) = g(J_j), j \neq c,$$

$$((f o_a g) o_b h)(K_k) = h(K_k).$$

In this situation, it is easy to check that

$$(f o_a g) o_b h = f o_a (g o_c h).$$

Similar to previous discussions, we introduce the following notation

$$<((I_i), f), ((J_j), g), (\{K_k\}, h)>_{\text{pair}} = \sum_{i, j, \tau} (-1)^{f(I_i) + f(J_j)} ((I_i) o_{I_a}^* (J_j) o_{J_c}^* (K_k), (f o_a g) o_b h).$$

Now we reach the following conclusion:

Proposition 3.2.

$$<((I_i), f), ((J_j), g), (\{K_k\}, h)>_{\text{pair}} = <((I_i), f), ((J_j), g), (\{K_k\}, h)>_{\text{pair}} + (\{I_i\}, f) o_{\text{pair}} ((\{J_j\}, g) o_{\text{pair}} (\{K_k\}, h)).$$

(3.8)

$$<((I_i), f), ((J_j), g), (\{K_k\}, h)>_{\text{pair}} = <((I_i), f), ((J_j), g), (\{K_k\}, h)>_{\text{pair}}.$$

(3.9)

Up to now, we can easily see the conclusion similar to corollary 3.2 is valid.

Corollary 3.3.

$$((I_i), f) o_{\text{pair}} ((J_j), g) o_{\text{pair}} (\{K_k\}, h) = ((I_i), f) o_{\text{pair}} (\{K_k\}, h) o_{\text{pair}} ((J_j), g) = (\{I_i\}, f) o_{\text{pair}} (\{K_k\}, h) o_{\text{pair}} ((J_j), g).$$

(3.10)

Summarizing the previous discussions, we know that the following Lie bracket will be well defined.

Definition 3.3.

$$[[((I_i), f), ((J_j), g)]_{\text{pair}} = ((I_i), f) o_{\text{pair}} ((J_j), g) - ((J_j), g) o_{\text{pair}} ((I_i), f).$$

(3.11)

4 Applications to graphs

The construction discussed in section 2 and section 3 is suitable for graphs, for example, Feynman diagrams, Kontesvich’s graphs and the ordinary graphs in the sense of graphic theory. For Feynman diagrams, we follow the notations in Jean-Louis Loday and N. M. Nikolov [5], actually, which is one of our motivation for the construction in this article. For simplicity, we do not consider the coloured Feynman diagrams. As a preparation we introduce some notations. Let $A$ be a finite set, $\sigma : A \to A$ be an involution, $\sigma^2 = \sigma$. A subset $J \subset A$ is called a $\sigma$ - invariant subset, if $J = \sigma(J)$. Actually, for a subset $J \subset A$, it is easy to check that $J \cap \sigma(J)$ and $J \cup \sigma(J)$ are $\sigma -$ invariant. Roughly speaking, a graph can be viewed as a finite set endowed with an involution and a decomposition.
4.1 The graphs in the sense of graphic theory

Definition 4.1.  • A ordinary graph is a pair \((\sigma, \{I_i\})\), where \(\{I_i\} \in \mathcal{P}^2_{dis}(A)\), \(\sigma\) is a map from \(I\) to itself without fixed points, \(\sigma^2 = \sigma\), where \(I = \mathcal{R}(\{I_i\})\). A graph \((\sigma, \{I_i\})\) is also denoted by \(\Gamma_{\sigma, \{I_i\}}\), or, \(\Gamma_{\{I_i\}}\) for short.

• If for any \(I_i\) and \(I_{i'}\), there are some positive integers \(i_0, i_1, \ldots, i_m\) (\(i_0 = i, i_m = i'\)), such that \(\sigma(I_k) \cap I_{k+1} \neq \emptyset (0 \leq k < m)\), we say \(\Gamma_{\{I_i\}}\) is a connected graph. Otherwise, we say \(\Gamma_{\{I_i\}}\) is disconnected. We call \(I_{i_0}, I_{i_1}, \ldots, I_{i_m}\) the chain in \(\Gamma_{\{I_i\}}\) connecting \(I_i\) and \(I_{i'}\).

• Let \((\sigma, \{I_i\})\) be an ordinary graph, \(J\) be a \(\sigma\)–invariant subset of \(I\). A subgraph of \(\Gamma_{\{I_i\}}\) related to \(J\) is a pair \((\sigma|_J, \{I_i\}_{J})\), denoted by \(\Gamma_J \subset \Gamma_{\{I_i\}}\) also. When \(J = \mathcal{R}(\{I_i\}_{J})\cap \sigma(\mathcal{R}(\{I_i\}_{J}))\), we call \(\Gamma_J\) a induced subgraph.

Remark 4.1.  • We call \(\{I_i\}\) the set of vertices of \(\Gamma_{\{I_i\}}\), denoted by \(\text{Vert}(\Gamma_{\{I_i\}})\). \(I\) is called the total set of \(\Gamma_{\{I_i\}}\). For a subgraph \(\Gamma_J\), we identify \(\text{Vert}(\Gamma_J) = \{I_i\}_{I \cap J \neq \emptyset}\).

• By the assumption of the involution \(\sigma\), we know that there is no \(e \in I\) such that \(\sigma(e) = e\). Now we define an equivalent relation in \(I\) in the following way. We say \(e_1 \sim e_2 (e_1, e_2 \in I)\), if \(e_1 = \sigma(e_2)\). It is easy to check that \(\sim\) is an equivalent relation, and each equivalent class consists of two pairs \((e, \sigma(e))\) and \((\sigma(e), e)\) \((e \in I)\). Each equivalent class assigns an edge of \(\Gamma_{\{I_i\}}\), then \(I/\sim\) is the set of all edges of \(\Gamma_{\{I_i\}}\).

• Let \(\tau : m \rightarrow m \) \((m = |\{I_i\}|)\) be a permutation, we identify \((\sigma, \{I_i\})\) with \((\sigma, \{I_{\tau(i)}\})\).

Lemma 4.1. Let \(\Gamma_{\sigma, \{I_i\}}\) be a graph, \(\Gamma_J\) and \(\Gamma_{J'}\) are two connected subgraphs of \(\Gamma_{\sigma, \{I_i\}}\). If \(\mathcal{R}(\{I_i\}) \cap \mathcal{R}(\{I_i\}_{J'}) \neq \emptyset\), then \(\Gamma_{J \cup J'}\) is a connected subgraph of \(\Gamma_{\sigma, \{I_i\}}\).

Proof. We need to prove a fact i.e. for any \(I_i\) and \(I_{i'}\) with \(I_i \cap J \neq \emptyset\) and \(I_i \cap J' \neq \emptyset\) (or \(I_i \subset \mathcal{R}(\{I_i\})\) and \(I_i' \subset \mathcal{R}(\{I_i\})_{J'}\)) there is a chain in \(\Gamma_J \cup \Gamma_{J'}\) connecting \(I_i \cap (J \cup J')\) and \(I_i' \cap (J \cup J')\). Because of \(\mathcal{R}(\{I_i\}) \cap \mathcal{R}(\{I_i\}_{J'}) \neq \emptyset\), there is \(I_n\) such that \(I_n \subset \mathcal{R}(\{I_i\})_{J'}\) and \(I_n \subset \mathcal{R}(\{I_i\})_{J'}\), which means \(I_n \cap J \neq \emptyset\) and \(I_n \cap J' \neq \emptyset\). Noting both \(\Gamma_J\) and \(\Gamma_{J'}\) are connected, thus there is a chain in \(\Gamma_J\) connecting \(I_i \cap J\) with \(I_n \cap J\), and there is a chain in \(\Gamma_{J'}\) connecting \(I_i' \cap J'\) with \(I_n \cap J'\). Above two chains will result in a chain in \(\Gamma_J \cup \Gamma_{J'}\) connecting \(I_i \cap (J \cup J')\) with \(I_i' \cap (J \cup J')\). Therefore, we have proved the conclusion.

\[\square\]

Proposition 4.1. A graph \(\Gamma_{\{I_i\}}\) is disconnected if and only if \(\Gamma_{\{I_i\}}\) adapts the following decomposition
\[\Gamma_{\{I_i\}} = \bigcup_j \Gamma_{J_j},\] (4.1)
where \(\{J_j\} \in \mathcal{P}^2_{dis}(A)\) (\(2 \leq |\{J_j\}|\)) satisfying the following conditions:

• \(\mathcal{R}(\{J_j\}) = \mathcal{R}(\{I_i\}), \) and \(\{J_j\} \subset \{I_i\}\).

• \(\sigma(J_j) = J_j, \) \(\Gamma_{J_j}\) is connected subgraph for each \(j\).

• \(\text{Vert}(\Gamma_{J_j}) \cap \text{Vert}(\Gamma_{J_{j'}}) = \emptyset, \) \(j \neq j'\).

Each \(\Gamma_{J_j}\) is a connected component of \(\Gamma_{\{I_i\}}\). If we ignore the order in the decomposition (4.1), the decomposition (4.1) is unique.
Proof. For \( I_i \) and \( \Gamma_{\nu} \), we say \( I_i \sim \Gamma_{\nu} \), if there is a chain connecting \( I_i \) and \( \Gamma_{\nu} \). is an equivalent relation obviously. Thus there is a partition \( \{ K_k \} \in \text{Part}(m) \) such that

\[
\{ I_i \}_{i=1}^m = \bigcup_{k} \{ I_i \}_{i \in K_k},
\]

where each \( \{ I_i \}_{i \in K_k} \) is an equivalent class. We take \( J_j = \mathcal{R}(\{ I_i \}_{i \in K_j}) \). We need to prove \( \{ J_j \} \) constructed in such a way satisfies the conditions in proposition. it is obvious that \( \mathcal{R}(\{ J_j \}) = \mathcal{R}(\{ I_i \}) \), and \( \{ J_j \} \subset \{ I_i \} \). We now prove \( \sigma(J_j) = J_j \) for each \( j \). It is for us to prove for any \( j \) and \( j' \) (\( j \neq j' \)), we have \( \sigma(J_j) \cap J_{j'} = \emptyset \). Otherwise, there are \( j \) and \( j' \) such that \( \sigma(J_j) \cap J_{j'} \neq \emptyset \). Then, there are \( i \in K_j \) and \( i' \in K_{j'} \), \( \sigma(I_i) \cap I_i' \neq \emptyset \), which means \( \{ I_i \}_{i \in K_j} \cup \{ I_i \}_{i \in K_{j'}} \) should be included in some equivalent class. That is a contradiction. The procedure to construct \( \Gamma_j \) implies each \( \Gamma_j \) is a connected subgraph of \( \Gamma_{\{I_i\}} \). It is easy to check that for each connected subgraph \( \Gamma_j \) there is a \( j \) such that \( J \subset J_j \). Thus each \( \Gamma_j \) is a connected component of \( \Gamma_{\{I_i\}} \). Furthermore, we know that \( \Gamma_{\{I_i\}} \) is disconnected if and only if \(|\{ J_j \}| \geq 2 \). The uniqueness of the decomposition (4.1) is obvious.

\[\square\]

Remark 4.2. Let \( \Gamma_{\sigma(I_i)} \) be a graph, \( \{ J_j \} \in \mathcal{P}_{\text{dis}}^2(A) \). If \( \{ J_j \} \subset \{ I_i \} \) and \( \sigma(J_j) = J_j \) for any \( j \), we say \( \{ J_j \} \sim \sigma \subset \{ I_i \} \) denoted by \( \{ J_j \} \subset \sigma \{ I_i \} \).

Proposition 4.2. Let \( \Gamma_{\sigma(I_i)} \) be a connected graph, \( \{ J_j \} \in \mathcal{P}_{\text{dis}}^2(A) \) satisfy \( J \subset I \) and \( \sigma(J_j) = J_j \) for any \( j \), where \( I = \mathcal{R}(\{ I_i \}) \), \( J = \mathcal{R}(\{ J_j \}) \). Then, the subgraph \( \Gamma_j \) adapts the following decomposition

\[
\Gamma_j = \bigcup_{l} \Gamma_{L_l}, \quad \text{Vert}(\Gamma_{L_l}) \cap \text{Vert}(\Gamma_{L_{l'}}) = \emptyset, \quad l \neq l',
\]

where \( \{ L_l \} = \{ J_j \}_{\text{ad}(I_i); j \subset \{ I_i \}} \) and each \( \Gamma_{L_l} \) is a connected component of \( \Gamma_j \).

Proof. According to proposition 4.1, for subgraph \( \Gamma_j \) there is the unique decomposition based on its connected components. Here we need to prove this decomposition is exactly given by \( \{ L_l \} = \{ J_j \}_{\text{ad}(I_i); j \subset \{ I_i \}} \).

It is obvious that \( \sigma(L_l) = L_l \) for each \( l \), thus each \( \Gamma_{L_l} \) is a subgraph of \( \Gamma_j \). By proposition 2.6 we know that \( \{ L_l \} \subset \{ I_i \}_{j \subset \{ I_i \}} \). In addition, noting \( \mathcal{R}(\{ L_l \}) = J \), it is easy to prove \( \mathcal{R}(\{ L_l \}_{j \subset \{ I_i \}}) = L_l \) for each \( l \). Recalling the procedure to construct \( \{ L_l \} \) in the proof of proposition 2.6, combining with lemma 4.1, we know that \( \Gamma_{L_l} \) is a connected subgraph of \( \Gamma_j \) for each \( l \). It is obvious that as subgraphs of \( \Gamma_j \), \( \text{Vert}(\Gamma_{L_l}) \cap \text{Vert}(\Gamma_{L_{l'}}) = \emptyset \) \( (l \neq l') \). Therefore, \( \{ L_l \} \) satisfies all conditions in proposition 4.2, which means \( \Gamma_{L_l} \) is a connected component of \( \Gamma_j \) for each \( l \).

\[\square\]

We now turn to the quotient of the ordinary graphs. The discussions below will follow the idea of Connes-Kriemer theory, but a different description will be provided based on the setting in this article.

Definition 4.2. Let \( \Gamma_{\sigma(I_i)} \) be a connected ordinary graph, \( \Gamma_j \) be a connected subgraph of \( \Gamma_{\sigma(I_i)} \) determined by a \( \sigma \)-invariant subset \( J \subset I \). We define the quotient of \( \Gamma_{\sigma(I_i)} \) by \( \Gamma_j \) to be a pair \( (\sigma_{\Gamma_j}, \{ I_i \}/(\{ J_j \})) \), denoted by \( \Gamma_{\sigma(I_i)}/\Gamma_j \) also, where \( I = \mathcal{R}(\{ I_i \}), \{ J_j \} = \{ I_i \}_{j \subset \{ I_i \}} \).

Remark 4.3. Let \( \{ J_j \} \in \mathcal{P}_{\text{dis}}^2(A) \) satisfy \( \mathcal{R}(\{ J_j \}) \subset \mathcal{R}(\{ I_i \}) \) and \( \sigma(J_j) = J_j \) for any \( j \). Then, it is natural for us to define the quotient of \( \Gamma_{\sigma(I_i)} \) by the sequence of the subgraphs \( \{ \Gamma_j \} \) to be \( (\sigma_{\Gamma_j}, \{ I_i \}/(\{ J_j \})) \), where \( J = \mathcal{R}(\{ J_j \}) \). Above quotient can also be denoted by \( \Gamma_{\sigma(I_i)}/(\Gamma_j) \). Combining proposition 2.7 and proposition 4.1, we have \( \Gamma_{\sigma(I_i)}/(\Gamma_j) = \Gamma_{\sigma(I_i)}/(\Gamma_{L_l}) \), where
\(\{I_i\} = \{J_j\}_{i \cap J_j \neq \emptyset}\) and each \(\Gamma_{i_j}\) is an connected component of \(\Gamma_J\). Thus, when we discuss the quotient of the graphs, we can always assume \(\{J_j\} \sqsubset \{I_i\}\).

**Proposition 4.3.** Let \(\Gamma_{\sigma,\{I_i\}}\) be a connected ordinary graph, \(\Gamma_J\) be a connected subgraph of \(\Gamma_{\sigma,\{I_i\}}\) determined by a \(\sigma\)-invariant subset \(J \subseteq I\). Then \(\Gamma_{\sigma,\{I_i\}}/\Gamma_J\) is a connected graph.

**Proof.** If \(R_J \setminus J = \emptyset\), the conclusion is obviously valid. Now we assume \(R_J \setminus J \neq \emptyset\). Noting \(\text{Vert}(\Gamma_{\sigma,\{I_i\}}/\Gamma_J) = \{I_i\}_{i \cap J \neq \emptyset} \cup \{R_J \setminus J\}\), because \(\sigma_{\{I_i\}}\) is a connected graph, we know that \(\sigma(R(\{I_i\}_{i \cap J \neq \emptyset})) \cap R_J \neq \emptyset\). We need to prove \(\sigma(R(\{I_i\}_{i \cap J \neq \emptyset})) \cap (R_J \setminus J) \neq \emptyset\). Noting \(R(\{I_i\}_{i \cap J \neq \emptyset}) \cap J = \emptyset\), thus \(\sigma(R(\{I_i\}_{i \cap J \neq \emptyset})) \cap J = \emptyset\), which means \(\sigma(R(\{I_i\}_{i \cap J \neq \emptyset})) \cap R_J \subseteq R_J \setminus J\).

Combining proposition 4.2 and proposition 4.3 we have the following corollary.

**Corollary 4.1.** Let \(\Gamma_{\sigma,\{I_i\}}\) be a connected graph, \(\{J_j\} \in \mathcal{P}_{\text{dis}}^2(A)\) satisfying \(R(\{J_j\}) \subset R(\{I_i\})\) and \(\sigma(J_j) = J_j\) for any \(j\). Then \(\Gamma_{\sigma,\{I_i\}}/\Gamma_{J_j}\) is a connected graph.

In the situation of the ordinary graphs there is an analogue of theorem 2.1.

**Proposition 4.4.** Let \(\Gamma_{\sigma,\{I_i\}}\) be a connected graph, \(\{J_j\}, \{K_k\} \in \mathcal{P}_{\text{dis}}^2(A)\). \(\{I_i\}, \{J_j\}, \{K_k\}\) satisfy the following conditions:

- \(\{J_j\} \sqsubset \{I_i\}, \{K_k\} \sqsubset \{I_i\}/(J_j)\).
- \(J_j\) is \(\sigma\)-invariant and \(\Gamma_{J_j}\) is a connected subgraph in \(\Gamma_{\sigma,\{I_i\}}\) for each \(j\).
- \(K_k\) is \(\sigma_{\{I_i\}}\)-invariant and \(\Gamma_{K_k}\) is a connected subgraph of \(\Gamma_{\sigma,\{I_i\}}/\Gamma_{J_j}\) for each \(k\), where \(I = R(\{I_i\}), J = R(\{J_j\})\).

Then there is a partition \(\{M_\mu\} \in \mathcal{P}_{\text{dis}}^2(A)\) satisfying the following conditions:

- \(\sigma(M_\mu) = M_\mu\) and \(\Gamma_{M_\mu}\) is connected for each \(\mu\).
- \(\{M_\mu\} \sqsubset \{I_i\}, \{J_j\} \sqsubset (\cdots, \{I_i\}_{M_\mu} , \cdots)\).
- \(\cdots, (\{I_i\}/(J_j))_{(K_k)} , \cdots) = (\cdots, \{I_i\}_{(M_\mu)} , \cdots)/(J_j)\).
- \((\Gamma_{I_i}/\Gamma_{J_j})/(\Gamma_{K_k}) = \Gamma_{I_i} / \Gamma_{M_\mu}\).

**Proof.** The proof of proposition 4.4 is almost same as the proof of proposition 2.7. Here we only need to prove each \(M_\mu\) is \(\sigma\)-invariant. Recalling proposition 2.7, we know that

\[
\{M_\mu\} = \{J_j\}_{R_J \cap K = \emptyset} \cup \{N_k\},
\]

\[
N_k = K_k \cup \bigcup_{R_J \cap K \neq \emptyset} J_j, \quad k = 1, \ldots, |K_k|,
\]

where \(K = R(\{K_k\})\). It is easy to check that \(\sigma(M_\mu) = M_\mu\) for each \(\mu\).

We now consider the insertion of the ordinary graphs.

**Definition 4.3.** Let \((\sigma, \{I_i\}_{i=1}^m), (\lambda, \{J_j\}_{j=1}^n)\) be two connected ordinary graphs, with structure maps \(\sigma\) and \(\lambda\) respectively, \(I = R(\{I_i\}), J = R(\{J_j\}), I \cap J = \emptyset\). Then we define the insertion of \(\Gamma_{\{I_i\}}\) into \(\Gamma_{\{J_j\}}\) at \(I_a\) by \(i\) to be a pair \((\delta, \{I_i\}_{\sigma, \{J_j\}})\) denoted by \(\Gamma_{\{I_i\}} \circ_{\sigma} \Gamma_{\{J_j\}}\), or, \(\Gamma_{\{I_i\}} \circ_{\sigma} \{J_j\}\) also, where \(1 \leq a \leq m, i : I_a \rightarrow \{J_j\}, \delta_{|I} = \sigma, \delta_J = \lambda\).
It is obvious that we have:

**Proposition 4.5.** Let $I_{\sigma,\{I_i\}}, I_{\delta,\{J_i\}}$ be two connected graphs. Then, $I_{\{I_i\}} \circ_a I_{\{J_i\}}$ is a connected graph also.

Based on the discussions in section 3, we can prove there is a well-defined Lie bracket structure on the vector space $\text{Span}_{\mathbb{K}}(\mathcal{G}_c)$, where $\mathcal{G}_c$ denotes the set of all connected graphs, $\mathbb{K}$ is a field of characteristic zero. Here we only consider the composition $\circ$ in definition 3.1 for simplicity. The composition will be defined in a natural way. Let $I_{\sigma,\{I_i\}}, I_{\delta,\{J_i\}}$ be two connected graphs. We have:

$$I_{\sigma,\{I_i\}} \circ I_{\delta,\{J_i\}} = \sum_{a,e} I_{\{I_i\}} \circ_a I_{\{J_i\}}.$$ \hspace{1cm} (4.3)

The Lie bracket should be defined to be:

$$[I_{\sigma,\{I_i\}}, I_{\delta,\{J_i\}}] = I_{\sigma,\{I_i\}} \circ I_{\delta,\{J_i\}} - I_{\delta,\{J_i\}} \circ I_{\sigma,\{I_i\}}.$$ \hspace{1cm} (4.4)

### 4.2 Feynman diagrams

**Definition 4.4.**
- A Feynman diagram is a pair $(\sigma, \{I_i\})$, where $\{I_i\} \in \mathcal{P}_d^2(A)$, $\sigma$ is a map from $I$ to itself, $\sigma^2 = \sigma$, where $I = R(\{I_i\})$. A Feynman diagram $(\sigma, \{I_i\})$ is also denoted by $I_{\sigma,\{I_i\}}$, or, $I_{\sigma,\{I_i\}}$ for short. $I = R(\{I_i\})$ is called the total set of edges.
- We call $I_{\text{ext}} = \{e \in I| \sigma(e) = e\}$ the set of external lines and $I_{\text{int}} = (I \setminus I_{\text{ext}})/ \sim$ the set of internal lines. We say a Feynman diagram $(\sigma, \{I_i\})$ is connected if $(\sigma|\setminus I_{\text{ext}}, \{I_i\}|\setminus I_{\text{ext}})$ is connected.

**Remark 4.4.**
- It is obvious that $I \setminus I_{\text{ext}}$ is $\sigma$–invariant, and $\sigma|\setminus I_{\text{ext}}$ has no fixed points, thus $(\sigma|\setminus I_{\text{ext}}, \{I_i\}|\setminus I_{\text{ext}})$ is an ordinary graph denoted by $I_{\{I_i\}}{\text{int}}$. $I_{\{I_i\}}{\text{int}}$ is a Feynman diagram without external lines. A general Feynman diagram can be regarded as an extension of an ordinary graph. In fact, let $(\sigma, \{I_i\})$ be an ordinary graph, i.e. $I_{\text{ext}} = \emptyset$, now we want to add the set of external lines $I_{\text{ext}}$ to $I$ by a map $f : I_{\text{ext}} \rightarrow I_{\{I_i\}}$. The new Feynman diagram is naturally chosen to be the pair $(\omega, I_{\text{ext}} \cup f \{I_i\})$, the new structure map $\omega$ is defined to be $\omega|_I = \sigma$, $\omega|_{I_{\text{ext}}} = id$. A Feynman diagram $(\sigma, \{I_i\})$ with $I_{\text{ext}} \neq \emptyset$ can be rewritten as $(\sigma|\setminus I_{\text{ext}}, \{I_i\}|\setminus I_{\text{ext}})\sim f^{-1}(I_{\text{ext}} \setminus I_{\text{int}}) = I_{\{I_i\}}{\text{int}}$.
- We call $\{I_i\}$ the set of vertices of $I_{\{I_i\}}$, denoted by $\text{vert}(I_{\{I_i\}})$. We identify the vertices of $I_{\{I_i\}}$ with ones of $I_{\{I_i\}}{\text{int}}$, i.e. we have $\text{vert}(I_{\{I_i\}}) = \text{Vert}(I_{\{I_i\}}{\text{int}})$. Actually, for a graph $I_{\{I_i\}}$, there are two structure maps, $\sigma$ and projection $p : R(\{I_i\}) \rightarrow \{I_i\}$, $p(e) = I_i \iff e \in I_i$.

**Definition 4.5.** Let $(\sigma, \{I_i\})$ be a connected Feynman diagram, $J \subset I$ be a subset, where $I = R(\{I_i\})$. We call the pair $(\sigma_J, \{I_i\}|_{I \cap J \neq \emptyset})$ a sub-diagram of $(\sigma, \{I_i\})$, if $J$ satisfies the following conditions:

- $\sigma(J) \cap R_J = J \cap \sigma(R_J)$,
- If $I_i \cap J \neq \emptyset$, then $I_i \cap (\{J \cap \sigma(R_J)\} \setminus I_{\text{ext}}) \neq \emptyset$.
where $\mathcal{R}_J = \mathcal{R}(\{I_i\}_{i \in J \neq \emptyset})$ is the total set of $(\sigma_J, \{I_i\}_{i \in J \neq \emptyset})$. The structure map $\sigma_J : \mathcal{R}_J \rightarrow \mathcal{R}_J$ is defined to be:

$$\sigma_J(e) = \begin{cases} \sigma(e), & e \in J \cap \sigma(\mathcal{R}_J), \\ e, & e \in \mathcal{R}_J \setminus (J \cap \sigma(\mathcal{R}_J)). \end{cases}$$

The sub-diagram is also denoted by $\Gamma_J \subset \Gamma_{\{I_i\}}$. When $J \cap \sigma(\mathcal{R}_J) = \mathcal{R}_J \cap \sigma(\mathcal{R}_J)$ we call $\Gamma_J$ a subgraph.

**Remark 4.5.**

- The first condition in definition 4.5 means that $J \cap \sigma(\mathcal{R}_J)$ is $\sigma$–invariant.
- The second condition means that each vertex of $\Gamma_J$ is the endpoint of at least one internal line in $\Gamma_J$.
- For subgraph $\Gamma_J$, $\mathcal{R}_J \setminus (J \cap \sigma(\mathcal{R}_J))$ plays the role of external lines of $\Gamma_J$. Here, the subset $[(\mathcal{R}_J \cap \sigma(\mathcal{R}_J)) \setminus (J \cap \sigma(\mathcal{R}_J))] \setminus \text{I}_{\text{ext}}$ is regarded as a subset of external lines of $\Gamma_J$. Actually, the external momenta corresponding to this subset will be canceled each other.
- The set of external lines in $\Gamma_J$ should be

$$\left(\mathcal{R}_J \setminus (J \cap \sigma(\mathcal{R}_J))\right) \cup (\mathcal{R}_J \cap \text{I}_{\text{ext}}).$$

The set of internal lines in $\Gamma_J$ is

$$\left((J \cap \sigma(\mathcal{R}_J)) \setminus \text{I}_{\text{ext}}\right)/\sim.$$

Let $(\sigma, \{I_i\})$ be a Feynman diagram, we call a $\sigma$–invariant subset $J$ is internal, if $J \cap \text{I}_{\text{ext}} = \emptyset$. We can prove that each sub-diagram of $(\sigma, \{I_i\})$ can be uniquely determined by an internal $\sigma$–invariant subset.

**Proposition 4.6.** There is an one-one corresponding between the sub-diagrams and internal $\sigma$–invariant subsets.

**Proof.** Let $(\sigma, \{I_i\})$ be a Feynman diagram, $J \subset I = \mathcal{R}(\{I_i\})$ satisfy the conditions in definition 4.5. Then, $J' = (J \cap \sigma(\mathcal{R}_J)) \setminus \text{I}_{\text{ext}}$ is an internal $\sigma$–invariant subset. Due to the first condition in definition 4.5, we know that $\mathcal{R}_J = \mathcal{R}_{J'}$. Thus $\Gamma_J = \Gamma_{J'}$. Conversely, let $J \subset I$ be an $\sigma$–invariant subset, then, $J$ satisfies all conditions in definition 4.5. Thus $J$ determines a sub-diagram of $(\sigma, \{I_i\})$.

\[ \square \]

**Remark 4.6.**

- Let $(\sigma, \{I_i\})$ be a Feynman diagram, and $J \subset I = \mathcal{R}(\{I_i\})$ be an internal $\sigma$–invariant subset. Then, the set of external lines in $\Gamma_J$ just be $\mathcal{R}_J \setminus J$, and the set of internal lines in $\Gamma_J$ is $J/\sim$.
- By the definition 4.1 and 4.5, we know that there is much difference between the sub-diagrams of Feynman diagrams and subgraphs of the ordinary graphs. Noting $(\sigma|_{J \setminus \text{I}_{\text{ext}}}, \{I_i\}_{(J \setminus \text{I}_{\text{ext}})})$ can be regarded as a ordinary graph, the conclusion of proposition 4.6 means that there is an one-one correspondence between the sub-diagrams of Feynman diagram $(\sigma, \{I_i\})$ and the subgraphs of $(\sigma|_{J \setminus \text{I}_{\text{ext}}}, \{I_i\}_{(J \setminus \text{I}_{\text{ext}})})$ in the sense of the ordinary graphs. Thus, when we discuss the sub-diagrams of Feynman diagrams, the discussion can be reduced to the situations of the ordinary graphs.

From proposition 4.2 and proposition 4.6 we have,
**Proposition 4.7.** Let \( \Gamma_{\{I\}} \) be a Feynman diagram, \( \Gamma_J \) be a sub-diagram, where \( J \) is an internal \( \sigma \)-invariant subset. If \( \Gamma_J \) is a disconnected sub-diagram, then \( \Gamma_J \) admits the decomposition as follows:

\[
\Gamma_J = \bigcup_J \Gamma_{J_j}
\]

where \( \{J_j\} \) satisfies the following conditions:

- \( \{J_j\} \in \text{Part}(J), |\{J_j\}| \geq 2, \)
- each \( J_j \) is an internal \( \sigma \)-invariant subset,
- \( \{J_j\} \subset \{I_i\}, \)
- each \( \Gamma_{J_j} \) is a connected component of \( \Gamma_J \).

The decomposition of \( \Gamma_J \) is unique.

We now consider the quotient of Feynman diagrams.

**Definition 4.6.** Let \( \Gamma_{\{I\}} \) be a connected Feynman diagram, \( \Gamma_J \) be a connected proper sub-diagram of \( \Gamma_{\{I\}} \) determined by a internal \( \sigma \)-invariant subset \( J \subset I \). We define the quotient of \( \Gamma_{\{I\}} \) by \( \Gamma_J \) to be a pair \( (\sigma|_{\Gamma \setminus J}, \{I_i\}/\{J_j\}) \), denoted by \( \Gamma_{\{I\}}/\Gamma_J \) also, where \( \{J_j\} = \{I_i\}_{\{J\}}. \)

**Remark 4.7.**

- Comparing with definition 4.2, in the situation of \( I_{\text{ext}} = \emptyset \), the quotient of Feynman diagrams is same as the quotient of the ordinary graphs exactly. Recalling the contents about \( \Gamma \)-union in section 2 and discussions in remark 4.1, the quotient of general Feynman diagrams can be reduced to the situation of the ordinary graphs. Actually, a Feynman diagram \( (\sigma, \{I_i\}) \) can be expressed as \( (\sigma, I_{\text{ext}} \uplus \{I_i\}_{\{J\}} \setminus I_{\text{ext}}) \), where \( f : I_{\text{ext}} \to \{I_i\}_{\{J\}} \setminus I_{\text{ext}} \) satisfies \( f^{-1}(I_{\text{ext}} \cap (I \setminus I_{\text{ext}})) = I_i \cap I_{\text{ext}} \). Therefore, for an internal \( \sigma \)-invariant subset \( J \), from the viewpoint of partition, the quotient \( \Gamma_{\{I\}}/\Gamma_J \) results in a projection \( p_J : \{I_i\}_{\{J\}} \to \{I_i \cap (I \setminus I_{\text{ext}})\} I_{\text{ext}} = \emptyset \cup \{(R_j \cap (I \setminus I_{\text{ext}})) \setminus J\} \), such that

\[
p_J(I_i \cap (I \setminus I_{\text{ext}})) = \begin{cases} (R_j \cap (I \setminus I_{\text{ext}})) \setminus J, & I_i \cap J \neq \emptyset, \\ I_i \cap (I \setminus I_{\text{ext}}), & I_i \cap J = \emptyset. \end{cases}
\]

Then

\[
\Gamma_{\text{ext} \cup f \{I_i\}_{\{J\}}/\Gamma_J} = \Gamma_{\text{ext} \cup p_J f \{I_i\}_{\{J\}}/\{I_i\}_{\{J\}}^\cdot}.
\]

- If a sub-diagram \( \Gamma_J \) determined by an internal \( \sigma \)-invariant subset \( J \) is not connected, then, according to proposition 4.1 or proposition 4.7, \( J \) admits the decomposition \( J = \bigcup_{j=1}^p J_j \), such that each \( \Gamma_{J_j} \) is a connected component of \( \Gamma_J \). The quotient \( \Gamma_{\{I\}} \) by \( \Gamma_J \) should be

\[
\Gamma_{\{I\}}/\Gamma_J = (\sigma|_{\Gamma \setminus J}, \{I_i\}/\{J_j\})
\]

\[
= ((\Gamma_{\{I\}}/\Gamma_{J_1})/\Gamma_{J_2}) \cdots)/\Gamma_{J_p}.
\]

Above quotient is also denoted by

\[
\Gamma_{\{I\}}/(\Gamma_{J}) = \Gamma_{\{I\}}/\{J_j\}.
\]
The previous discussions show that the quotient of Feynman diagrams can be reduced to the situation of the ordinary graphs. Then we have,

**Proposition 4.8.** Let \( I_\sigma \{I_i\} \) be a connected Feynman diagram, \( J \subset I = \mathcal{R}\{\{I_i\}\} \) be an internal \( \sigma \)-invariant subset. Then, \( I_\sigma \{I_i\}/J \) is a connected Feynman diagram.

Furthermore, a conclusion similar to proposition 4.4 or theorem 2.1 is valid.

**Definition 4.7.** Let \( (\lambda, \{J_{J,j}\}_{j=1}^m) \) be two connected Feynman diagrams satisfying \( I \cap J = \emptyset \), where \( I = \mathcal{R}\{\{I_i\}\}, J = \mathcal{R}\{\{J_j\}\}. \) Then we define the insertion of \( \Gamma(J,J) \) into \( \Gamma(I,I) \) at \( I_a \) by \( i \) to be a pair \( (\delta, \{I_i\}^{\circ}_{I_a} \{J_j\}) \) denoted by \( \Gamma(I_a) \circ_{I_a} \Gamma(J) \), or, \( \Gamma(I_a) \circ_{I_a} \Gamma(J_J,J_e) \) also, where \( 1 \leq a \leq m, i : I_a \to \{J_j\}, \delta \in I_a \cup J \) is a map from \( I_a \cup J \) to itself without fixed points.

**Remark 4.8.** In the case of Feynman diagram, by definition 4.4 it is easy to check that \( \{I_i\}^{\circ}_{I_a} \{J_j\} \subset \{J_j\} \subset \{J_j\} \) and \( \delta \) is a map from \( I_a \cup J \) to itself without fixed points.

By proposition 4.3 we can define the composition of Feynman diagrams as follows:

\[
\Gamma(I_i) \circ \Gamma(J) = \sum_{a,i} \Gamma(I_a) \circ_{I_a} \Gamma(J).
\]

The Lie bracket of Feynman diagrams can be defined to be

\[
[\Gamma(I_i), \Gamma(J)] = \Gamma(I_i) \circ \Gamma(J) - \Gamma(J) \circ \Gamma(I_i).
\]

**4.3 Kontsevich graphs**

**Definition 4.8.** An admissible graph is a triple \( (\lambda, \{I_i\}, \{J_j\}) \), where \( \{I_i\}, \{J_j\} \in \mathcal{P}_{\text{ad}}(A), I \cap J = \emptyset, \sigma \) is a map from \( I \cup J \) to itself without fixed points, \( \sigma^2 = \sigma \), where \( I = \mathcal{R}\{\{I_i\}\}, J = \mathcal{R}\{\{J_j\}\} \), and \( \{I_i\} \) and \( \{J_j\} \) satisfy

\[
\sigma(I_i) \cap I_i = \emptyset, \sigma(J_j) \cap J_j = \emptyset, |\sigma(I_i) \cap I_i| \leq 1, |\sigma(J_j) \cap J_j| \leq 1, \forall i, i', j, j', i \neq i'.
\]

We denote the admissible graph \( (\lambda, \{I_i\}, \{J_j\}) \) by \( \Gamma(\lambda, \{I_i\}, \{J_j\}) \), or, \( \Gamma(I_i), \{J_j\} \) for short. \( I \cup J \) is called the total set of \( \Gamma(I_i), \{J_j\} \). Vert(\( \Gamma(I_i), \{J_j\} \)) = \{I_i\} \cup \{J_j\}. The vertices from \( \{I_i\} \) are called the vertices of the first type, the vertices from \( \{J_j\} \) are called the vertices of the second type.

**Remark 4.9.** The conditions in (4.5) means that:
– There no loop in $\Gamma_{\{I\},\{J\}}$.
– For any two vertices there is at most one edge to connect them.
– There is not edge to connect any two vertices of the second type.

• In definition 4.8, we ignore the orientation of the edges. Now we discuss the issues of the orientation in details. Recalling the previous discussions, the edges of a graph are regarded as the equivalent classes under the equivalent relation defined above, each equivalent class consists of two elements $(e, \sigma(e))$ and $(\sigma(e), e)$ ($e \in I \cup J$). To indicate the orientation of an edge, we can only choose one element in each equivalent class, for example, $(e, \sigma(e))$, to represent an edge. If the first element $e \in I$, the second element $\sigma(e) \in I'$ (or $\sigma(e) \in J$), then $I$ will be the starting point of the edge $(e, \sigma(e))$, $I'$ (or $J$) will be the endpoint of this edge. Furthermore, each $J_j$ can not be the starting point of any edge.

• The connectivity of an admissible graph can be defined in the same way as definition 4.1.

Now we pay attention to subgraphs.

**Lemma 4.2.** Let $\Gamma_{\sigma,\{I\},\{J\}}$ be an admissible graph. Then $\sigma(J) \cup I$ and $I \setminus \sigma(J)$ are two $\sigma$–invariant subsets. Additionally, for a $\sigma$–invariant subset $K \subset I \cup J$, we have

$$K = (\sigma(K') \cup K') \cup K'',$$

where $K' = K \cap J$, $K'' = K \cap (I \setminus \sigma(J))$.

**Definition 4.9.** Let $(\sigma,\{I\},\{J\})$ be an admissible graph, and $K \subset I \cup J$ be a $\sigma$–invariant subset such that $K \cap J \neq \emptyset$, where $J = R(\{J\})$. A subgraph of $\Gamma_{\sigma,\{I\},\{J\}}$ determined by $K$ is a triple $(\sigma|_{\{I\}\setminus K},\{I\}_{\{I\}\setminus K},\{J\}_{\{K\}})$, also denoted by $\Gamma_{\sigma,\{I\},\{J\}}|_{\{K\}}$, or $\Gamma_K$ for short.

Now we discuss the quotient and insertion of the asmissible graphs.

**Definition 4.10.** Let $\Gamma_{\sigma,\{I\},\{J\}}$ be a connected admissible graph, $K \subset I \cup J$ be a $\sigma$–invariant subset satisfying:

- $|R_{\{J\}}(K \cap J) \setminus K | \leq 1$,
- $|R_{\{J\}}(K \cap J) \cap i | \leq 1$, $i \cap K = \emptyset$,
- $|R_{\{I\}}(K \cap J) \cap J_j | \leq 1$, $J_j \cap K = \emptyset$,
- $|R_{\{I\}}(K \cap J) \cap I_l | \leq 1$, $I_l \cap K = \emptyset$.

Then, we define the quotient of $\Gamma_{\sigma,\{I\},\{J\}}$ by $\Gamma_K$ to be a triple

$$(\sigma|_{\{I\}\setminus K},\{I\}_{\{I\}\setminus K},\{J\}_{\{K\}}),$$

denoted by $\Gamma_{\sigma,\{I\},\{J\}}/\Gamma_K$ also.

**Definition 4.11.** Let $\Gamma_{\{I\},\{J\}}$ and $\Gamma_{\{I\},\{K\}}$ be two connected admissible graphs with structure maps $\lambda$ and $\lambda'$ $(I \cup J) \cap (L \cup K) = \emptyset)$. Let $1 \leq a \leq \{\{I\}\}$, $1 \leq b \leq \{\{J\}\}$, $\iota : I_a \rightarrow \{L_i\}$, $\kappa : J_b \rightarrow \{K_k\}$, we define the insertion of $\Gamma_{\{I\},\{K\}}$ into $\Gamma_{\{I\},\{J\}}$ to be the triple

$$(\delta,\{I\} \circ_{I_a} \{L_i\},\{J\} \circ_{J_b} \{K_k\}),$$

where $\delta$ satisfies $\delta|_{\{I\}\setminus K} = \lambda$, $\iota = R(\{I\})$, $\delta|_{\{J\}\setminus K} = \lambda$, $J = R(\{J\})$, $L = R(\{L_i\})$, $K = R(\{K_k\})$.

The graph $\delta \circ_{I_a} \{L_i\},\{J\} \circ_{J_b} \{K_k\}$ is also denoted by $\Gamma_{\sigma,\{I\},\{J\}} \circ_{\iota,J_k} \Gamma_{\lambda,\{I\},\{K\}}$. 27
Similar to the situations in subsection 4.1 and 4.2, it is obvious that the following conclusions are valid.

**Proposition 4.9.**

- Let admissible graph \( \Gamma_{\sigma,\{I_i\},\{J_j\}} \) and its subgraph \( \Gamma_K \) are connected, where \( K \subset (I \cup J) \) is a \( \sigma \)-invariant subset. Then the quotient \( \Gamma_{\sigma,\{I_i\},\{J_j\}}/\Gamma_K \) is connected also.

- Let \( \Gamma_{\{I_i\},\{J_j\}} \) and \( \Gamma_{\{L_l\},\{K_k\}} \) be two connected admissible graphs. Then

\[
(\delta, \{I_i\} \circ_{I_a} \{L_l\}, \{J_j\} \circ_{J_b} \{K_k\})
\]

is a connected admissible graph.

Let \( (\sigma, \{I_i\}, \{J_j\}) \) and \( (\lambda, \{L_l\}, \{K_k\}) \) be two connected admissible graphs. Then, it is natural for us to define the composition of the admissible graphs to be

\[
(\sigma, \{I_i\}, \{J_j\}) \circ (\lambda, \{L_l\}, \{K_k\}) = \sum_{a,b,\iota,\kappa} (\delta, \{I_i\} \circ_{I_a} \{L_l\}, \{J_j\} \circ_{J_b} \{K_k\}).
\]  

(4.6)

Now we define the Lie bracket to be

\[
[[\sigma, \{I_i\}, \{J_j\}], (\lambda, \{L_l\}, \{K_k\})]
\]

\[
= (\sigma, \{I_i\}, \{J_j\}) \circ (\lambda, \{L_l\}, \{K_k\}) - (\lambda, \{L_l\}, \{K_k\}) \circ (\sigma, \{I_i\}, \{J_j\}).
\]  

(4.7)

Observing the definition 4.11 and the formula (4.6), the structure map \( \delta \) on \( (\delta, \{I_i\} \circ_{I_a} \{L_l\}, \{J_j\} \circ_{J_b} \{K_k\}) \) is independent of the choices of \( a, b, \iota, \kappa \), even the order of the insertion, thus, all admissible graphs concerning (4.7) adapt same structure map. Above fact implies that the Lie bracket (4.7) is similar to the cartesian product of Lie algebras, which means that the Lie bracket (4.7) will satisfy the Jacobi identity.

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