A Composite Likelihood-based Approach for Change-point Detection in Spatio-temporal Process

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April 15, 2019

Abstract

This paper develops a unified, accurate and computationally efficient method for change-point inference in non-stationary spatio-temporal processes. By modeling a non-stationary spatio-temporal process as a piecewise stationary spatio-temporal process, we consider simultaneous estimation of the number and locations of change-points, and model parameters in each segment. A composite likelihood-based criterion is developed for change-point and parameters estimation. Asymptotic theories including consistency and distribution of the estimators are derived under mild conditions. In contrast to classical results in fixed dimensional time series that the asymptotic error of change-point estimator is $O_p(1)$, exact recovery of true change-points is guaranteed in the spatio-temporal setting. More surprisingly, the consistency of change-point estimation can be achieved without any penalty term in the criterion function. A computational efficient pruned dynamic programming algorithm is developed for the challenging criterion optimization problem. Simulation studies and an application to U.S. precipitation data are provided to demonstrate the effectiveness and practicality of the proposed method.

Keywords: Dynamic programming; increasing domain asymptotics; multiple change-points; pairwise likelihood; structural breaks.

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1 Introduction

In analyzing dependent data, many widely-used statistical models in practice require the assumption of stationarity to establish the validity of statistical methodologies. However, stationary models are often inadequate when the data are collected over a long period. A growing body of empirical evidence reveals that real data do exhibit non-stationarities, such as in climate statistics (Kelly and Ó Gráda, 2014; Nam et al., 2015), financial data (Fryzlewicz, 2014) and genetics (Fearnhead and Liu, 2007; Fearnhead and Rigaill, 2018). Instead of developing complicated models for describing the non-stationary behavior, it is often more intuitive and effective to incorporate a change-point model which segments the data into stationary pieces. With the advance in data collection technology, many long or high frequency data sets are available, and hence change-point analysis is becoming increasingly popular in recent decades.

Change-point estimation has been extensively studied for time series data, see, for example, Davis et al. (2006); Aue et al. (2009); Matteson and James (2014); Preuss et al. (2015); Cho and Fryzlewicz (2015); Yau and Zhao (2016). Recently, change-point analysis has gained popularity in high-dimensional statistics (Wang and Samworth, 2018) and functional data analysis (Berkes et al., 2009; Aston and Kirch, 2012; Aue et al., 2018). However, inference of change-points for spatio-temporal data remains largely unexplored. Only a few literature have considered change-point estimation for spatio-temporal processes. Moreover, the existing methods are subject to various limitations. For example, Bayesian methods such as Majumdar et al. (2005) and Altieri et al. (2015) usually require very specific model structures and are computationally intensive. Gromenko et al. (2017) focus on at most one change-point in the mean function of a spatio-temporal process. Furthermore, all existing literature require the separability of space-time covariance. The major difficulty of change-point analysis for spatio-temporal processes stems from the theoretical and computational challenges of spatio-temporal modeling under the increase in dimension of both space and time with the presence of unknown change-points.

In this paper, we propose a likelihood-based procedure for multiple change-point estimation in spatio-temporal process with spatial locations at a possibly irregular grid. The procedure adopts pairwise likelihood to alleviate the computational difficulty of the full likelihood for spatio-temporal data while maintaining statistical efficiency. The proposed method is a general approach that can handle a wide range of spatio-temporal models including both separable and non-separable space-time covariance. Furthermore, it works under model misspecification and can detect changes beyond first and second moments.

The contribution of this paper is two-folded. First, in terms of statistical theory, we show that, in contrast to the commonly used likelihood-based change-point estimation approach for time series
(e.g. Davis et al., 2006; Ma and Yau, 2016), the pairwise likelihood for spatio-temporal data induces an edge effect from the boundary of each change-point. This edge effect is non-ignorable under the spatio-temporal setting as the spatial dimension grows, and causes inconsistency of change-point estimation. To tackle this problem, we carefully modify the pairwise likelihood by introducing a marginal likelihood term to correct the edge effect. Interestingly, unlike traditional criterion functions such as Bayesian information criterion (BIC) and minimum description length (MDL) which involve a penalty term, the consistency of the change-point estimation can be achieved solely by the modified pairwise likelihood. Moreover, in contrast to classical results in fixed dimensional time series that the asymptotic error of change-point estimation is $O_p(1)$, we show that exact recovery of true change-points is guaranteed in the spatio-temporal setting under mild conditions. To further achieve consistent model selection for each segment and enhance finite sample performance, an MDL-based criterion function is developed. We prove that, even if the model is misspecified, the number and locations of change-points can be consistently estimated under mild assumptions. The asymptotic distributions of the estimated change-points and the spatio-temporal model parameters in each stationary segment are also derived.

Second, in terms of statistical computing, we develop a computationally efficient algorithm for the optimization of the criterion function. Computational feasibility is a major challenge in change-point estimation since it involves optimization over a large number of change-point location configurations. Popular optimization methods, such as binary segmentation (Zhang et al., 2010) and genetic algorithm (Davis et al., 2006), are fast but only provide approximate solutions. On the other hand, dynamic programming algorithms (Killick et al., 2012; Maidstone et al., 2017) provide exact solutions with a linear to quadratic computational cost. In particular, the pruned exact linear time (PELT) algorithm (Killick et al., 2012) reduces the computational cost substantially by pruning away irrelevant computational configurations. In this paper, PELT is adapted to the spatio-temporal setting to yield a computationally efficient and asymptotically exact solution to the optimization problem.

The rest of this paper is organized as follows. Section 2 provides the background and derivation of the composite likelihood based criterion for change-point inference. The main results including estimation consistency and asymptotic distribution of the estimators are presented in Section 3. Numerical experiments and an application to U.S. precipitation data are given in Section 4.
2 Background

2.1 Setting and notations

On a set of spatial locations $S$ with cardinality $S = \text{Card}(S)$, consider a spatio-temporal process $Y = \{y_{t,s} : t \in [1, T] \cap \mathbb{N}, s \in S\} = \{y_t : 1 \leq t \leq T, t \in \mathbb{N}\}$, where $y_t = \{y_s : s \in S\}$ denotes the observations of all locations at time $t$. There are in total $S \cdot T$ observations. We focus on $S \subset \mathbb{R}^2$, while the theory can be easily generalized to $\mathbb{R}^d$ with $d \geq 3$.

We assume that the spatio-temporal process can be partitioned into $m + 1$ strictly stationary segments along the time dimension. In other words, there are $m$ unknown distinct change-points $\tau_1, \ldots, \tau_m$ in the observations. Set $\tau_0 = 0, \tau_{m+1} = T$, and let $\lambda_j = \tau_j/T, j = 0, \ldots, m + 1$ be the normalized change-points. Assume that $0 = \lambda_0 < \cdots < \lambda_m < \lambda_{m+1} = 1$. The asymptotic results are based on increasing $S, T$ with the $\lambda_j$s being fixed.

For $j = 1, \ldots, m + 1$, the $j$th stationary segment is modeled by a stationary process $X_j = \{x^{(j)}_t\}_{t \in \mathbb{N}}$ with $x^{(j)}_t = \{x_{t,s} : s \in S\}$, such that
\[
x^{(j)}_{t-\tau_{j-1}} = y_t, \quad t = \tau_{j-1} + 1, \ldots, \tau_j.
\]
(1)

Let $T_j = \tau_j - \tau_{j-1}$ be the length of the $j$th segment. The observed spatio-temporal process $Y$ can be written as

\[
Y = (x^{(1)}_1, \ldots, x^{(1)}_{T_1}, x^{(2)}_1, \ldots, x^{(2)}_{T_2}, \ldots, x^{(m+1)}_1, \ldots, x^{(m+1)}_{T_{m+1}}).
\]

As in Davis et al. (2008) and Aue et al. (2009), we first assume for simplicity that the data across different segments $X_j (j = 1, \ldots, m + 1)$ are independent. This assumption is commonly found in spatio-temporal change-point literatures, such as Altieri et al. (2015). Also, Gromenko et al. (2017) assume independence between functional observations, which implies the independence between segments. See Remark 1 for discussions on its relaxations.

Given the locations of the change-points, each stationary segment is modeled by a member of a pre-specified finite class of models, $\mathcal{M}$. Each element in $\mathcal{M}$ is a model specified by an integer-valued parameter $\xi$ of dimension $c$ that represents the model order. Let $\xi_j$ be the $c_j$-dimensional integer-valued vector that specifies the model order for the $j$th segment $X_j$. Note that, by definition, the form of the parametric model and thus the form of the likelihood function for the $j$th segment $X_j$ is completely determined by $\xi_j$. Given $\xi_j$, the model for $X_j$ depends on a real-valued parameter $\theta_j = \theta(\xi_j)$ of dimension $d_j = d_j(\xi_j)$. That is, the distribution of $X_j$ is completely determined by $\theta_j$. Here, $\xi_j$ is regarded as a parameter for the model order and $\theta_j$ is the parameter that specifies
the exact model. This framework, for example, covers the general spatio-temporal process
\[ Y_{t,s} = \mu_{t,s} + \varepsilon_{t,s}, \]
where \( t = 1, \ldots, T, \ s \in S \) and \( \varepsilon_{t,s} \) is a zero-mean error process. The mean \( \mu_{t,s} \) of \( Y_{t,s} \) can take a constant \( \mu \) or a regression form such as \( z_t' \beta, \) where \( z_t \) denotes a \( p \)-dimensional covariates associated with \((t,s)\). The covariance structure \( \phi_{t,s} = \text{Cov}(Y_{t_1,s_1}, Y_{t_1+s_1}) \) can take \( K \) possible parametric forms of separable or non-separable spatio-temporal dependence. Here, a model class may be specified by \( \xi = (i_1, \ldots, i_p, k) \), where \( i_q = 1 \) or 0 indicates whether the \( q \)th covariate is selected in the model or not, and \( k \in \{1, \ldots, K\} \) indicates the parametric form for the covariance structure. Also, \( \theta(\xi) \) contains the coefficients of the selected covariates and the covariance parameters.

Define \( \psi_j = (\xi_j, \theta_j) \) as the complete model parameter set of the \( j \)th segment. Assume that \( \theta_j \) is in the interior of a compact parameter space \( \Theta_j = \Theta_j(\xi_j) \subset \mathbb{R}^{d_j} \). The change-points and the model parameter vector are denoted by \( \Lambda = \{\lambda_1, \ldots, \lambda_m\} \) and \( \Psi = \{\psi_1, \ldots, \psi_{m+1}\} \), respectively. To lighten the notations, we use \( x_{t,s} \) for \( x_{t,s}^{(j)} \) and suppress the \( j \) in \( T_j \) and \( \Theta_j \) when there is no possibility of confusion.

2.2 Composite likelihood and pairwise likelihood

Although likelihood based methods generally achieve high statistical efficiency, when the likelihood function involves high-dimensional inverse covariance matrices or integrals, computations become infeasible. To overcome this limitation, Lindsay (1988) considers the composite likelihood, which is a weighted product of likelihoods for some subsets of the data. By specifying the subsets, different classes of composite likelihood are obtained. One popular class is the pairwise likelihood, which is the product of the bivariate densities of all possible pairs of observations,
\[ L_P(\theta; X) = \prod_{i,j} L_i(\theta; x_i, x_j)^{w_{i,j}}, \]
where \( w_{i,j} \) are the weights. Composite likelihood often enjoys computational efficiency while statistical efficiency is retained; see Lindsay (1988); Varin et al. (2011). Owing to its flexibility and attractive asymptotic properties, composite likelihood has been widely used in genetics (Larribe and Fearnhead, 2011), longitudinal data (Bartolucci and Lupparelli, 2016), time series (Davis and Yau, 2011), spatial statistics (Guan et al., 2015) and spatio-temporal statistics (Bevilacqua et al., 2012; Huser and Davison, 2014).

Given a stationary segment \( X = \{x_1, \ldots, x_T\} \) with parameter \( \psi \), the pairwise likelihood is defined as the product of the bivariate densities \( f(\cdot, \cdot; \psi) \) of each distinct pair of observations. However, in many spatio-temporal processes, the dependence between observations diminishes as the time lag or spatial distance increases, so there is a loss of efficiency because most of the pairs
are not informative. Therefore, it suffices to consider pairs up to a small time lag \( k \) and a small spatial distance \( d \). Denote \( \mathcal{N}' \equiv \mathcal{N}(s) = \{ s' | s' \in \mathcal{S}, s' \neq s, \text{dist}(s, s') \leq d \} \) as a distance-based neighborhood of location \( s \), where the distance can be Euclidean metric. Note that \( \mathcal{N} \) depends on \( d \) implicitly. Given observations \( X \) and \((k, \mathcal{N})\), the pairwise log-likelihood is defined as

\[
\text{PL}_{ST}(\psi;X) = \sum_{(t,i,s_1,s_2) \in D_{k,N}} \log f(x_{t,s_1}, x_{t+i,s_2}; \psi) = \sum_{(t,i,s_1,s_2) \in D_{k,N}} l_{\text{pair}}(\psi; x_{t,s_1}, x_{t+i,s_2}), \quad (4)
\]

where \( D_{k,N} = \{ (t,i,s_1,s_2) : 1 \leq t, t+i \leq T, 0 \leq i \leq k, s_1 \in \mathcal{S}, s_2 \in \mathcal{S} \cup \mathcal{N}(s_1), \text{if } i = 0, s_1 \neq s_2 \} \) is the collection of pairs of observations that are at most \( k \) time units and \( d \) spatial distance apart.

For the choices of \( k \) and \( \mathcal{N} \), intuitively, larger neighborhood can be used if there exists strong spatial correlation across \( \mathcal{S} \), and smaller neighborhood should be favored if the spatial correlation is weak; see Varin and Vidoni (2005) and Bai et al. (2012). On the other hand, similar to Ma and Yau (2016), if the main focus is estimating change-points rather than model parameters, then it usually suffices to use the smallest \( k \) and \( d \) that ensure identifiability of the models in the candidate model set \( \mathcal{M} \). See more discussions below Assumption 3 in Section 5.

### 2.3 Edge effect and a remedial composite likelihood

By the definition of \( D_{k,N} \) in (4), each data point in a stationary segment does not appear in \( \text{PL}_{ST} \) for the same number of times. For example, \( x_1 \) can only be paired with \{ \( x_t : t = 1, \ldots, k+1 \) \}, and thus appears approximately half frequently than an ordinary observation \( x_\tilde{t} \), \( k+1 \leq \tilde{t} \leq T-k \), which can be paired with \{ \( x_t : \tilde{t} - k \leq t \leq \tilde{t} + k \) \}. This can be viewed as that different weights are implicitly assigned to the observations in \( X = \{x_1, \ldots, x_T\} \), and observations on the edge of a segment receive less weights. We call this phenomenon in pairwise likelihood the “edge effect”.

When we want to decide whether a change point \( \tau \) exists in a segment, say \{ \( x_t : t_1 \leq t < t_2 \) \}, we need to compare two quantities: the pairwise likelihood formed by \{ \( x_t : t_1 \leq t < t_2 \) \}, and the sum of pairwise likelihoods formed by \{ \( x_t : t_1 \leq t < \tau \) \} and \{ \( x_t : \tau + 1 \leq t < t_2 \) \}, respectively. For the latter quantity, all observations within \( k \) time units from \( \tau \) suffer from the edge effect and receive less weights. Essentially, the latter quantity has \( O(S) \) fewer terms than the former one. When \( S \to \infty \), the edge effect is non-ignorable and can cause inconsistency of the pairwise likelihood based method, as the following example shows.

**Example 1.** Consider the simple example of \( k = 1 \) and let the criterion function be in the form

\[
IC(m) = - \sum_{j=1}^{m+1} \text{PL}_{ST}(\hat{\psi}_j; X_j) + C(m + 1) \log(ST),
\]

where \( m \) is the number of change-points, \( \text{PL}_{ST}(\hat{\psi}_j; X_j) \) is the pairwise log-likelihood of the \( j \)th segment evaluated under a parameter estimate \( \hat{\psi}_j \), and the penalty \( C(m + 1) \log(ST) \) is of similar
magnitude as some common information criteria such as BIC. We compare between two scenarios:

- Scenario 1: No change point is assumed and the pairwise log-likelihood function gives an estimate \( \hat{\psi} \) over the entire observations \( Y \). We have
  \[
  IC(0) = -PL_{ST}(\hat{\psi}; Y) + C \log(ST),
  \]
  where
  \[
  PL_{ST}(\hat{\psi}; Y) = \sum_{(t, i, s_1, s_2) \in D_{k,N}} \log f(y_{t, s_1}, y_{t+i, s_2}; \hat{\psi}),
  \]
  \( D_{k,N} = \{(t, i, s_1, s_2) : 1 \leq t, t + i \leq T; 0 \leq i \leq 1; 1 \leq s_1 \leq S; s_2 \in s_1 \cup N(s_1); \text{ if } i = 0, s_1 \neq s_2\} \).

- Scenario 2: A change point is fixed at \( \frac{T}{2} \) and the pairwise log-likelihoods give the estimates \( \hat{\psi}_1, \hat{\psi}_2 \) in the two segments, respectively. Thus,
  \[
  IC(1) = -\sum_{j=1}^{2} PL_{ST}(\hat{\psi}_j; X_j) + 2C \log(ST),
  \]
  where
  \[
  PL_{ST}(\hat{\psi}_j; X_j) = \sum_{(t, i, s_1, s_2) \in D_{k,N}^j} \log f(y_{t, s_1}, y_{t+i, s_2}; \hat{\psi}_1) + \sum_{(t, i, s_1, s_2) \in D_{k,N}^2} \log f(y_{t, s_1}, y_{t+i, s_2}; \hat{\psi}_2),
  \]
  \( D_{k,N}^j = \{(t, i, s_1, s_2) : 1 \leq t, t + i \leq \frac{T}{2}; 0 \leq i \leq 1; 1 \leq s_1 \leq S; s_2 \in s_1 \cup N(s_1); \text{ if } i = 0, s_1 \neq s_2\}, \text{ and } D_{k,N}^2 = \{(t, i, s_1, s_2) : \frac{T}{2} + 1 \leq t, t + i \leq T; 0 \leq i \leq 1; 1 \leq s_1 \leq S; s_2 \in s_1 \cup N(s_1); \text{ if } i = 0, s_1 \neq s_2\} \).

Suppose that there is no change-point in the data, i.e., Scenario 1 is true, and the true parameter value is \( \psi_o \). Then, under some regularity conditions, the estimators \( \hat{\psi}, \hat{\psi}_1 \) and \( \hat{\psi}_2 \) converge in probability to \( \psi_o \) with order \( \sqrt{ST} \). Hence, by Taylor’s expansion, it can be seen that

\[
IC(0) - IC(1) = \sum_{j=1}^{2} PL_{ST}(\psi_o; X_j) - PL_{ST}(\psi_o; Y) - C \log(ST) + O_p(1) .
\]  

(5)

Compared to \( PL_{ST}(\psi_o; Y) \), due to the edge effect, \( \sum_{j=1}^{2} PL_{ST}(\psi_o; X_j) \) does not contain the pairwise log-likelihoods at \( D_{k,N}^c := \{(\frac{T}{2}, \frac{T}{2} + 1, s_1, s_2)|1 \leq s_1 \leq S, s_2 \in s_1 \cup N(s_1)\} \), which is of order \( O(S) \). In other words,

\[
\sum_{j=1}^{2} PL_{ST}(\psi_0; X_j) - PL_{ST}(\psi_0; Y) = - \sum_{(s_1, s_2) \in D_{k,N}^c} \log f(y_{\frac{T}{2}, s_1}, y_{\frac{T}{2}+1, s_2}; \psi_o) .
\]

(6)

Note that for a pairwise log-likelihood evaluated at \( y = (y_1, y_2) \), we have

\[
- \log f(y_1, y_2; \psi_o) = - \log \frac{1}{2\pi \sqrt{\Sigma}} \exp(-\frac{(y - \mu)^\Sigma^{-1}(y - \mu)/2}{2}) = \log 2\pi + 1/2 \log |\Sigma| + (y - \mu)^\Sigma^{-1}(y - \mu)/2 .
\]

If \( |\Sigma| > 1 \), then the term in (6) is strictly positive and is of order \( O_p(S) \), which dominates the penalty term of order \( O(\log ST) \). Combining with (5), \( IC(0) - IC(1) > 0 \) with probability approaching one, indicating that a change-point model is wrongly selected. \( \square \)

The non-ignorable edge effect in Example 1 is due to the special feature of \( S \rightarrow \infty \) in the spatio-
temporal setting, in contrast to the fixed cross-sectional dimension in multivariate time series. To fix the edge effect and hence achieve consistency, one should compensate the missing pairs in the edge so that each data point appears the same number of times in the likelihood function. This can be achieved by introducing additional marginal likelihoods for data points observed on the edge $t = (1, \ldots, k) \cup (T_j - k + 1, \ldots, T_j)$, resulting in a new composite likelihood (for the $j$th segment)

$$L_{ST}^{(j)}(\psi; X_j) = \text{PL}_{ST}(\psi; X_j) + \sum_{(i,s) \in E_{k,N}} \log f(x_{i,s}^{(j)}; \psi_j) + \sum_{(i,s) \in E_{k,N}} \log f(x_{T_j-i+1,s}^{(j)}; \psi_j)$$

$$= \sum_{(t,i,s_1,s_2) \in D_{k,N}^{(j)}} l_{pair}(\psi_j; x_{t,i,s_1}^{(j)}, x_{t+i,s_2}^{(j)}) + \sum_{(i,s) \in E_{k,N}} \left[ l_{\text{marg}}(\psi_j; x_{i,s}^{(j)}) + l_{\text{marg}}(\psi_j; x_{T_j-i+1,s}^{(j)}) \right],$$

where $l_{\text{marg}}(\psi; x) = \log f(x; \psi)$ and $l_{\text{pair}}(\psi; x_1, x_2) = \log f(x_1, x_2; \psi)$, $D_{k,N}^{(j)} = \{(t, i, s_1, s_2) : 1 \leq t, t+i \leq T_j, 0 \leq i \leq k, s_1 \in S, s_2 \in s_1 \cup N(s_1), \text{if } i = 0, s_1 \neq s_2\}$, and $E_{k,N} = \{(i,s) : 1 \leq i \leq k, s \in S\}$, repeat $(i, s)$ by $(k - i + 1)(1 + |N(s)|)$ times is the collection of sites where marginal likelihoods are included for compensation of the edge effect. Note that both $D_{k,N}^{(j)}$ and $E_{k,N}$ depend implicitly on $(S, T_j)$, which is suppressed for notational simplicity.

### 2.4 Derivation of the criterion

With the modified composite likelihood, in this section we derive a criterion function for estimating the change-points and model parameters in each segment. The criterion is based on the minimum description length (MDL) principle, which aims to select the best-fitting model that requires the minimum amount of code length to store the data (Rissanen 2012), and has been shown to have promising performance for change-point detection; see Davis et al. (2006, 2008), Aue and Lee (2011). One classical way to construct the MDL is the two-stage approach (Hansen and Yu 2001, Lee 2001), which splits the code length, $\text{CL}(Y)$, into two components:

$$\text{CL}(Y) = \text{CL}(\hat{M}) + \text{CL}(Y \mid \hat{M})$$

where $\text{CL}(\hat{M})$ is the code length for the fitted model $\hat{M}$ and $\text{CL}(Y \mid \hat{M})$ is the information in $Y$ unexplained by $\hat{M}$. Recall that all model parameters in the $j$th segment are specified by $\psi_j = (\xi_j, \theta_j)$, where $\xi_j$ is the model order and $\theta_j$ is the model parameter. Given $\xi_j$, the composite likelihood estimator of $\theta_j$ is obtained by $\hat{\theta}_j = \arg \max_{\theta_j} L_{ST}^{(j)}\{(\xi_j; \theta_j); X_j\}$, where $L_{ST}^{(j)}$ is defined in [7]. Since the fitted model $\hat{M}$ can be completely described by $m$, $\lambda$s and $\xi$s, we have

$$\text{CL}(\hat{M}) = \text{CL}(m) + \text{CL}(\lambda_1) + \cdots + \text{CL}(\lambda_m) + \text{CL}(\xi_1) + \cdots + \text{CL}(\xi_{m+1})$$

$$+ \text{CL}(\hat{\theta}_1) + \cdots + \text{CL}(\hat{\theta}_{m+1}).$$

Since $(\lambda_1, \ldots, \lambda_m)$ contains information equivalent to the integer-valued vector $(T_1, \ldots, T_{m+1})$, and the code lengths for an integer $I$ and an estimate of a real-valued parameter from $N$ observations
are \( \log_2 I \) and \( (\log_2 N)/2 \), respectively, we have
\[
CL(\hat{\mathcal{M}}) = \log_2 m + \sum_{j=1}^{m+1} \log_2 T_j + \sum_{j=1}^{m+1} \sum_{i=1}^{c_j} \log_2 \xi_{i,j} + \frac{d_j}{2}(\log_2 T_j + \log_2 S),
\]
where \( \xi_j = (\xi_{1,j}, \ldots, \xi_{c_j,j}) \); see Hansen and Yu (2001) and Lee (2001).

As demonstrated by Rissanen (2012), \( CL(Y \mid \hat{\mathcal{M}}) = -\log_2 L \), where \( L \) is the maximized full likelihood. By regarding the composite likelihood as an approximation to the full likelihood, and using logarithm base \( e \) rather than base 2, the sum of the negative of (7) can be used to define \( CL(Y \mid \hat{\mathcal{M}}) \). However, by construction, each data point appears in several parts of the composite likelihood function. In particular, the number of times that an observation is used in the composite log-likelihood \( L_{ST}(\psi, X) \) equals to
\[
C_{k,N} = \frac{2\text{Card}(D_{k,N}) + 2\text{Card}(E_{k,N})}{ST},
\]
where \( \text{Card}(\cdot) \) denotes the cardinality of a set. Indeed, if the data are identically and independently distributed, then the composite likelihood is a product of marginal densities, and \( L_{ST}(\psi, X) \) is exactly \( C_{k,N} \) times the full likelihood. Thus, we compensate the code length by magnifying \( CL(\hat{\mathcal{M}}) \) by a factor \( C_{k,N} \), and define the composite likelihood-MDL (CLMDL) criterion as
\[
\text{CLMDL}(m, \Lambda, \Psi) = C_{k,N}\left\{ \log m + \sum_{j=1}^{m+1} \sum_{i=1}^{c_j} \log \xi_{i,j} + \frac{m+1}{2}(\log S + \frac{d_j}{2} \log S) \right\}
\]
\[
- \sum_{j=1}^{m+1} L_{ST}^{(j)}(\hat{\psi}_j; \hat{X}_j).
\]

To ensure identifiability of the change-points, assume that there exists an \( \epsilon_\lambda > 0 \) such that \( \min_{1 \leq j \leq m+1} |\lambda_j - \lambda_{j-1}| \geq \epsilon_\lambda \). In other words, we require \( \Lambda \in A_{\lambda}^m \), where
\[
A_{\lambda}^m = \{ \Lambda \in (0,1)^m : 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_m < \lambda_{m+1} = 1, \lambda_j - \lambda_{j-1} \geq \epsilon_\lambda, j = 1, \ldots, m+1 \}.
\]
Under this constraint, the number of change-points is bound by \( M_\lambda = [1/\epsilon_\lambda] + 1 \). The estimates of the number of change-points, the locations of the change-points and the parameters in each of the segments are given by the vector \((\hat{m}, \hat{\Lambda}_{ST}, \hat{\Psi}_{ST})\), where
\[
(\hat{m}, \hat{\Lambda}_{ST}, \hat{\Psi}_{ST}) = \arg\min_{m \leq M_\lambda, \psi_j \in \Theta, \Lambda \in A_{\lambda}^m} \text{CLMDL}(m, \Lambda, \Psi).
\]
Denote \( \hat{\Lambda}_{ST} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_\hat{m}) \) and \( \hat{\Psi}_{ST} = (\hat{\psi}_1, \ldots, \hat{\psi}_{\hat{m}+1}) \), where \( \hat{\psi}_j = (\hat{\xi}_j, \hat{\theta}_{ST}^{(j)}) \). Note that
\[
\hat{\theta}_{ST}^{(j)} = \arg\max_{\theta_j \in \Theta_j} L_{ST}^{(j)}(\hat{\xi}_j; \hat{X}_j)
\]
is the composite likelihood estimator of the model parameters of the \( j \)th estimated segment of the spatio-temporal process, \( \hat{X}_j = \{ y_t : [T_{\hat{\lambda}_{j-1}} + 1 \leq t \leq T_{\hat{\lambda}_j}] \} \).
3 Main Results

In this section, we impose some mild regularity conditions on the composite log-likelihood, and on the strong-mixing coefficients of the piecewise stationary spatio-temporal process, and then we present the main results.

For the asymptotic theory, we assume that the spatio-temporal process is generated by a random field in $\mathcal{D} = \mathbb{N} \times \mathcal{S} \subset \mathbb{N} \times \mathbb{R}^2$. Also, the data $Y$ is observed on $\mathcal{D}_n = \mathcal{T}_n \times \mathcal{S}_n$ with $\mathcal{T}_n = [1, T_n] \cap \mathbb{N}$ and $|\mathcal{S}_n| = S_n$, in other words,

$$Y = \{y_{t,s} : t \in [1, T_n] \cap \mathbb{N}, s \in \mathcal{S}_n, \mathcal{S}_n \subset \mathbb{R}^2\}.$$ 

The asymptotic theory is based on $n \to \infty$, where the number of observations $|\mathcal{D}_n| = S_n T_n > |\mathcal{D}_{n'}| = S_{n'} T_{n'}$ whenever $n > n'$. For notational simplicity, in the following we use $(S, T)$ instead of $(S_n, T_n)$ when there is no possibility of confusion.

We define a metric $\rho$ on $\mathcal{D}$ by $\rho(d_1, d_2) = \max(|t_2 - t_1|, |s_2^1 - s_1^1|, |s_2^2 - s_1^2|)$, where $d_i = (t_i, s_i)$, $s_i = (s_i^1, s_i^2)$, $i = 1, 2$, are indexes in $\mathcal{D}$. Also, the distance between any subsets $U, V \in \mathcal{D}$ is defined as

$$\rho(U, V) = \inf\{\rho(d_1, d_2) : d_1 \in U \text{ and } d_2 \in V\}.$$

The asymptotic results follow an increasing domain framework as in [Jenish and Prucha (2009)] and [Bai et al. (2012)], as is made explicit by Assumption [1].

**Assumption 1.** The (possibly unevenly spaced) lattice $\mathcal{D} \subset \mathbb{N} \times \mathbb{R}^2$ is countably infinite. All elements in $\mathcal{D}$ are located at distances of at least $\rho_0 > 0$ from each other, i.e., for all $d_1, d_2 \in \mathcal{D}$, we have $\rho(d_1, d_2) \geq \rho_0$.

Assumption [1] allows for unevenly spaced locations and general forms of sample regions, which is often encountered in real data. By Assumption [1], we can assume the maximum cardinality of the neighborhood set $\mathcal{N}(s)$ is bounded by a constant $B_N$. Throughout this section, we assume the time lag used in CLMDL is $k$ and the maximum cardinality of $\mathcal{N}(s)$ in CLMDL is $B_N$.

**Assumption 2 (r).** There exists an $\epsilon > 0$ such that for any fixed model order $\xi$, we have

(i) for $a = 0$ and $j = 1, \ldots, m_o + 1$,

$$\sup_{S, T} \sup_{(t, s_1, s_2) \in D_{k, N}^{(j)}} \mathbb{E} \left[ \sup_{\theta \in \Theta(\xi)} |\tilde{l}_{\text{pair}}^{[a]}(\xi, \theta, x_{t, s_1}, x_{t+i, s_2})|^{r+\epsilon} \right] < \infty,$$

(ii) for $a = 1, 2$, the above moment conditions hold with $r = 2$,

where $l_{\text{marg}}$ and $l_{\text{pair}}$ are defined in [7] and $a = 0, 1, 2$ in $[a]$ stands for the $a$th order derivative.
derivatives

For each stationary segment that if the data \( Y \) w.r.t. \( \theta \) the supremum w.r.t. \( \psi \) have

\[
\sup_{\theta} \{ \{ \xi_j, \theta_j \} \} = \arg \max_{\xi_j, \theta_j} \tilde{L}_{ST}^{(j)}((\xi_j, \theta_j)) .
\]

The model \( \xi^*_j \) is uniquely identifiable in the sense that if there exists another model \( (\xi^*_j, \theta^*_j) \neq (\xi^*_j, \theta^*_j) \)

with \( \theta^*_j \in \mathbb{R}^{d_k} \) and \( \tilde{L}_{ST}^{(j)}((\xi^*_j, \theta^*_j)) = L_{ST}^{(j)}((\xi^*_j, \theta^*_j)) \), then \( d_k > d_j \). Moreover, for every \( \delta > 0 \), we have

\[
\sup_{S,T} \frac{1}{s} \left( \sup_{|\theta - \theta_j^*| > d_j} \tilde{L}_{ST}^{(j)}((\xi_j, \theta)) - L_{ST}^{(j)}((\xi_j, \theta_j^*)) \right) < 0 ;
\]

\[
\sup_{S,T} \frac{1}{s} \left( L_{ST}^{(j)}(\psi^*_{j-1}) - L_{ST}^{(j)}(\psi^*_{j}) \right) < 0 \text{ and } \sup_{S,T} \frac{1}{s} \left( L_{ST}^{(j-1)}(\psi^*_{j}) - L_{ST}^{(j-1)}(\psi^*_{j-1}) \right) < 0 ,
\]

where \( \psi^*_{j-1} \) and \( \psi^*_{j} \) are defined in (i).

Assumption 3(i) only assumes the existence of a pseudo-true model \( \psi^*_{j} \) in \( \mathcal{M} \), which is of the simplest form. That is, the model cannot be expressed by another model \( \psi^*_{j} = (\xi^*_j, \theta^*_j) \) in \( \mathcal{M} \), where \( \theta^*_j \) is of a smaller dimension. The last statement in Assumption 3(i) asserts that for any model order \( \xi^*_j \), the point \( \theta^*_j \) is the unique parameter value that maximizes the expected composite likelihood.

Assumption 3(ii) rules out the degenerate case that the \((j-1)\)th and \(j\)th stationary segments are indistinguishable by the composite likelihood. In general, Assumption 3(ii) may fail if the two adjacent segments follow different models but have the same expected composite likelihood. In theory, this situation can always be avoided by using a larger lag \( k \) and neighborhood \( N \) when defining \( \mathcal{M} \). As noted by Ma and Yau (2016), this situation seldom occurs in practice and \( k = 1 \) or \( 2 \) is usually sufficient to distinguish between stationary segments.

The next two assumptions regulate the dependence structure of the spatio-temporal process by imposing strong mixing conditions on the underlying random field. For \( U, V \subset \mathcal{D} \), let \( \sigma(U) = \sigma(x_{t,s}; (t,s) \in U) \) be the sigma field generated by the random variables in \( U \). Define

\[
\alpha(U, V) = \sup \{ |P(A \cap B) - P(A)P(B)| ; A \in \sigma(U), B \in \sigma(V) \} .
\]
The $\alpha$-mixing coefficient for the $j$th stationary random field $X_j$ is defined as

$$\alpha_{X_j}(m; u, v) = \sup \{\alpha(U,V); |U| \leq u, |V| \leq v, \rho(U,V) \geq m \}.$$ 

Also, denote $M = (1 + k)(1 + B_N)$, where $2M$ upper bounds the number of composite likelihood components $l_{pair}$ that any point $(t,s)$ can have in $D_{k,N}$.

**Assumption 4 (r).** For each stationary segment $X_j$ of the random field, where $j = 1, \ldots, m_o + 1$, there exist some $\epsilon > 0$ and $c \in 2\mathbb{N}$ where $c > r$, such that for all $u, v \in \mathbb{N}^+$, $u + v \leq c$, $u, v \geq 2$, we have

$$\sum_{m=1}^{\infty} (m + 1)^{3(c-u+1)-1}[\alpha_{X_j}(m; Mu, Mv)]^{\epsilon/(c+\epsilon)} < \infty.$$ 

Assumption 4 requires a polynomial decaying rate for the mixing coefficient of the random field, which is mild and is used for invoking the moment inequality in Doukhan (1994) that controls the asymptotic size of the composite likelihood. Note that the mixing rate in Assumption 4 depends on $M = (1 + k)(1 + B_N)$. This is intuitive since a longer time lag $k$ and a larger neighborhood size $B_N$ induce a slightly higher dependence among the composite likelihood and thus require stronger conditions on the mixing coefficients. Define also

$$\alpha^*_X(m) = \sup \{\alpha((-\infty, t_1] \times S, [t_2, \infty] \times S); t_2 - t_1 = m \},$$

which characterizes the dependence of the random field along the time dimension. It can be regarded as an analogue to the classic strong mixing coefficient in the time series setting.

**Assumption 5.** For each stationary segment $X_j$ of the random field, where $j = 1, \ldots, m_o + 1$, there exist $r > 2$, $\delta > 0$ and $\tau > r(r + \delta)/(2\delta)$, such that for any fixed model order $\xi$,

1. $\sup_{S,T} \sup_{(t_1, s_1, 2) \in D_{k,N}} E \left[ \sup_{\theta \in \Theta(\xi)} |l_{pair}^{[1]}((\xi, \theta); x_{t_1, s_1}, x_{t_1+i, s_2})|^{r+\delta} \right] < \infty$,

2. $\alpha^*_X(m) \leq Cm^{-\tau}$ for some $C > 0$.

Assumption 5(i) is a moment condition on the first order derivative of the composite likelihood, which is similar to the one in Assumption 2. Assumption 5(ii) is a mild mixing condition along the time dimension and is used for invoking an maximal moment inequality in Yang (2007) that controls the asymptotic size of the first order derivative of the composite likelihood at the true parameter value. Note that for a fixed moment condition $c = r + \delta$, the slowest polynomial decaying rate for the mixing condition is $\tau > \frac{c}{c+2}$, which is achieved when $r \to 2$. Thus, a higher order moment condition on the first order derivative requires a weaker mixing condition.

Next, we impose Assumptions 6 and 7 which are standard in establishing the asymptotic distribution of the parameter estimator $\hat{\theta}_{ST}^{(j)}$ in a stationary segment; see Assumption 3 in Jenish and Prucha (2009) and Assumptions 6-8 in Bai et al. (2012).
Assumption 6. For each stationary segment $X_j$ of the random field, where $j = 1, \ldots, m_o + 1$, there exists some $\delta > 0$ such that

\begin{align*}
(i) \quad & \sum_{m=1}^{\infty} m^{3(2+\delta)/\delta - 1} < \infty, \\
(ii) \quad & \sum_{m=1}^{\infty} m^2 \alpha(m; M, M) < \infty \quad \text{for } u + v \leq 4, \\
(iii) \quad & \alpha(m; M, \infty) = O(m^{-3-\epsilon}) \quad \text{for some } \epsilon > 0.
\end{align*}

Assumption 7. For each stationary segment $X_j$ of the random field, where $j = 1, \ldots, m_o + 1$, we have

\begin{align*}
(i) \quad & \frac{1}{ST} \text{Var}(L^j_{ST}(\psi_j; X_j)) \rightarrow \Sigma_1^{(j)}, \\
(ii) \quad & -\frac{1}{ST} E(L''^j_{ST}(\psi_j; X_j)) \rightarrow \Sigma_2^{(j)},
\end{align*}

where $\Sigma_1^{(j)}$ and $\Sigma_2^{(j)}$ are positive definite matrices.

3.1 Consistency of CLMDL

We now present the main results of this paper. We begin with a somewhat surprising results on the consistency of change-point estimation using the composite likelihood function alone without any penalty for model complexity. It requires the existence of the $(2 + \epsilon)$th moments of the composite likelihood in (7) and its derivatives, and a mild divergence rate requirement between $T$ and $S$.

Proposition 1. Let $Y$ be a piecewise stationary spatio-temporal process specified by the vector $(m_o, \Lambda^o, \Psi^o)$, and Assumptions 2(1), 3 and 4(1) hold with some $r > 2$. Analogous to (12), define the estimator

\begin{equation}
(\hat{m}, \hat{\Lambda}_{ST}, \hat{\Psi}_{ST}) = \arg \min_{m \leq M_\lambda, \psi_j \in M_j, \Lambda \in \Lambda^o} -\sum_{j=1}^{m+1} L^j_{ST}(\psi_j; X_j),
\end{equation}

where $\hat{\Lambda}_{ST} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_{\hat{m}}), \hat{\Psi}_{ST} = (\hat{\psi}_1, \ldots, \hat{\psi}_{\hat{m}+1}), \hat{\psi}_j = (\hat{\xi}_j, \hat{\theta}^j_{ST}), \hat{\theta}^j_{ST} = \arg \max_{\theta_j \in \Theta_j(\hat{\xi}_j)} L^j_{ST}\{(\hat{\xi}_j, \theta_j); \tilde{X}_j\}$, and $\tilde{X}_j = \{y_t: [T\hat{\lambda}_{j-1}]^i \leq t \leq [T\hat{\lambda}_j]\}$. Then, for $j = 1, \ldots, m_o$, there exists a $\hat{\lambda}_{ij} \in \hat{\Lambda}_{ST}$, where $1 \leq i_j \leq \hat{m}$, such that

\[
P\left(\left|\left|T\lambda^o_j\right| \right| - \left|T\hat{\lambda}_{ij}\right|\right| = 0 \right) \rightarrow 1,
\]

provided that $S, T \rightarrow \infty$. If, in addition, $T \cdot S^{-r/2} \rightarrow 0$ and Assumption 5 holds, then we have

\[
\hat{m} \rightarrow m_o, \quad [T\hat{\Lambda}_{ST}] = [T\Lambda^o],
\]

and $\hat{\xi}_j$ has a dimension greater than $d_j$ for $j = 1, \ldots, m_o + 1$, in probability.

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The consistency of change-point estimation in Proposition 1 can be viewed as the opposite phenomenon to the inconsistency result in Example 1. The intuition behind is as follows. By assigning one false change-point to a stationary segment, the difference introduced on the composite likelihood criterion function in (13) is the difference between the dropped pairwise likelihood and the corresponding compensated marginal likelihood. For a stationary segment, by Assumption 3 and information inequality, the expected value of the dropped pairwise likelihood minus the compensated marginal likelihood is positive and is of order $O(S)$. That is, a model with an over-estimated change-point has a significantly larger criterion function value than the true model, and thus will not be selected. This phenomenon is due to the use of the proposed composite likelihood and that $S \rightarrow \infty$ (which is non-ignorable). In contrast, the classical time series setting lacks this feature and the consistency needs to be achieved by an extra penalty term in the criterion function to suppress over-estimation.

Proposition 1 only guarantees consistency in change-point estimation but not model selection. Indeed, introducing penalty term ensures consistency of model selection in each segment and improves the finite sample performance in terms of reducing false positive when there is no change-point. Theorem 1 states the consistency of our procedure based on the CLMDL criterion.

Theorem 1. Let $Y$ be a piecewise stationary spatio-temporal process specified by the vector $(m_o, \Lambda^o, \Psi^o)$, and Assumptions 1, 2($r$), 3 and 4($r$) hold with some $r > 2$. Then, for the estimator $(\hat{m}, \hat{\Lambda}_{ST}, \hat{\Psi}_{ST})$ defined in (12), we have that for $j = 1, \ldots, m_o$, there exists a $\hat{\lambda}_{ij} \in \hat{\Lambda}_{ST}, 1 \leq i_j \leq \hat{m}$, such that

$$P \left( \left| \left[ T \hat{\lambda}_{ij} \right] - \left| T \hat{\lambda}_{ij} \right| \right| = 0 \right) \rightarrow 1,$$

provided that $S, T \rightarrow \infty$. If, in addition, $T \cdot S^{-r/2} \rightarrow 0$ and Assumption 5 holds, then we have

$$\hat{m} \rightarrow m_o, \quad [T \hat{\Lambda}_{ST}] = [T \Lambda^o], \quad \hat{\Psi}_{ST} \rightarrow \Psi^o,$$

in probability.

In Proposition 1 and Theorem 1, we have $P([T \hat{\Lambda}_{ST}] = [T \Lambda^o]) \rightarrow 1$, which indicates that asymptotically we can recover the exact position of the change point without error. This result is different from the one under the multivariate time series setting, where an $O_p(1)$ error is observed asymptotically. This difference is due to the expansion of $S$ under the spatio-temporal setting, which intuitively provides more information for the detection of change points.

To guarantee the consistency of the estimated number of change points, Proposition 1 and Theorem 1 require an additional polynomial rate condition on the divergence rate between $S$ and $T$, i.e. $T \cdot S^{-r/2} \rightarrow 0$. This technical condition is needed for controlling the asymptotic size of the edge effect via a union bound. Compared to the classical multivariate time series setting, e.g. Ma and Yau (2016), the condition here is more complicated due to the expansion of $S$. Note
that a higher moment assumption \( r \) on the composite likelihood function implies a less restrictive divergence rate requirement. In the case where \( Y \) is Gaussian, all moments of the composite likelihood exist, and the rate requirement becomes minimal.

Indeed, if the spatio-temporal process \( Y \) is observed on a regular lattice, then under an additional assumption on the mixing coefficients, a finer result in terms of the divergence rate between \( S \) and \( T \) can be obtained.

**Theorem 2.** Let \( Y \) be a piecewise stationary random field specified by the vector \((m_0, \Lambda^o, \Psi^o)\), and Assumptions \( I \), \( II(r) \), \( \overline{III} \), \( \overline{IV}(r) \) and \( \overline{V} \) hold with \( r > 2 \). Moreover, assume that

\[
\alpha_{X_j}(n) := \sup_{u \geq 1} \sup_{v \geq 1} \alpha_{X_j}(n; u, v) \leq C n^{-\theta},
\]

for some \( C > 0 \), \( \theta > 2(1 + \eta)(q + \delta)/(q - 2 + \delta) \) with some \( q > 2 \), \( \delta > 0 \) and \( 0 < \eta < 1/2 \). Then, for the estimator \((\hat{m}, \hat{\Lambda}_{ST}, \hat{\Psi}_{ST})\) defined in \((12)\), we have

\[
\hat{m} \longrightarrow m_0, \quad [T \hat{\Lambda}_{ST}] = [T \Lambda^o], \quad \hat{\Psi}_{ST} \longrightarrow \Psi^o,
\]

in probability provided that \( S, T \longrightarrow \infty \) with \( \log T/S \longrightarrow 0 \).

The strong mixing coefficient \( \alpha_{X_j}(m) \) in \((16)\) is widely used in the literature, see Berkes and Morrow \(1981\). By definition, we have that \( \alpha_{X_j}(m) \geq \alpha_{X_j}^*(m) \). Thus, if the random field is strongly mixing with \( \alpha_{X_j}(m) \) decaying at a sufficiently fast polynomial rate, then Assumptions \( I \), \( II \), \( III \) and \( IV \) hold. By Section 2.1 of Doukhan \(1994\), a wide range of stationary Gaussian random fields have a polynomially decaying mixing coefficient \( \alpha_{X_j}(m) \), provided that their covariance functions decay at a sufficiently fast polynomial rate.

**Remark 1.** The consistency results given in Proposition \( I \) Theorems \( I \) and \( II \) are based on the independence assumption across segments, which we denote as IS. Based on IS, the likelihood function in the proposed CLMDL takes the form \( \sum_{j=1}^{m+1} L_{ST}^{(j)}(\psi_j; X_j) \) as a sum of likelihoods across the independent segments, which we denote as IL. As we show later, similar to the classical time series setting, CLMDL still gives consistency result under relaxation of IS when there is weak dependence across segments. On the other hand, relaxing IL is more difficult and requires specific model assumption about the dependence across segments, which is hard to define, especially for non-separable spatio-temporal covariance. As IL already enjoys consistency and relaxing IL may only have marginal improvement in efficiency but risks losing robustness, relaxing IL is not suggested.

Before stating the consistency result under relaxation of IS, we quantify the relaxation with Assumption \( \overline{VI} \) which is a spatio-temporal extension of Assumption 3.1 in Aue et al. \(2009\).

**Assumption 8.** The observed spatio-temporal process \( Y \) can be written as \( Y_{t,s} = f(Y_{t,s}^*, Z_{t,s}) \) with some measurable function \( f \), where \( Y^* \) is a piecewise stationary spatio-temporal process with IS.
and \( Z \) is another spatio-temporal process that introduces dependence across segments of \( Y \). There further exists a constant \( C_I \) such that

\[
\sup_{S,T} \sup_{\psi \in \mathcal{M}, 0 \leq t_1 < t_2 \leq T} E \left\{ \frac{1}{S} \left| L^{[a]}_{ST}(\psi; Y_{(t_1+1):t_2}) - L^{[a]}_{ST}(\psi; Y^*_S) \right| \right\} \leq C_I,
\]

for \( a = 0, 1, 2 \).

Assumption 8 is expected to hold under mild conditions. For example, let \( Y_{t,s} = Y^*_{t,s} + Z_{t,s} \) with \( Y^*_{t,s} \) being a Gaussian random field, then by Taylor expansion and Cauchy-Schwarz inequality, a sufficient condition for Assumption 8 is \( \sup_{S,T} 1/S \sum_{s \in S} \sum_{i=1}^{T} \sqrt{E(Z_{t,s}^2)} < \infty \), which requires \( \sqrt{E(Z_{t,s}^2)} \) to be absolutely summable along the time dimension. For a more concrete example, let \( Y^*(j)_{t,s} = \alpha(j)Y^*_{t-1,s} + \varepsilon_{t,s}^{(j)} \) for \( T\lambda_{j-1} + 1 \leq t \leq T\lambda_j, j = 1, \ldots, m_o + 1 \) be the piecewise stationary spatio-temporal process with IS, where the superscript \( j \) indicates the stationary segment that \( Y^*_{t,s} \) is from. To introduce dependence across segments, let \( Y_{1,s} = Y^*(1)_{t,s} \) and \( Y_{t,s} = \alpha(j)Y^*_{t-1,s} + \varepsilon_{t,s}^{(j)} \) for \( t > 1 \). Note that the initial observations of segment \( j \) depends on the last observations of segment \( j - 1 \). Simple algebra yields \( Z_{t,s} = (\alpha^{(j)})^{t-T\lambda_{j-1}}(Y_{T\lambda_{j-1} - 1} - Y^*_{T\lambda_{j-1}}) \) for \( T\lambda_{j-1} + 1 \leq t \leq T\lambda_j \), which clearly satisfies the absolutely summable condition.

The consistency result of CLMDL under the relaxed IS is presented in Corollary 1.

**Corollary 1.** Let Assumption 8 hold for the observed process \( Y \), and \( Y^* \) is a piecewise stationary spatio-temporal process specified by the vector \((m_o, \Lambda^0, \Psi^o)\) and satisfies conditions in Theorem 1 or Theorem 2, we have that

\[
[T\hat{\Lambda}_{ST}] - [T\Lambda^0] = O_p(1), \quad \text{and} \quad \hat{m} \rightarrow m_o, \quad \hat{\Psi}_{ST} \rightarrow \Psi^o,
\]

in probability provided that \( S, T \rightarrow \infty \) and \( S = O(T) \), with \( T \cdot S^{-r/2} \rightarrow 0 \) (Theorem 1) or with \( \log T/S \rightarrow 0 \) (Theorem 2).

By Corollary 1, the exact recovery property of the change-point estimator \( T\hat{\Lambda}_{ST} \) in Theorems 1 and 2 no longer holds. However, the relative change-point estimation remains consistent, i.e, \( \hat{\Lambda}_{ST} - \Lambda^0 = O_p(T^{-1}) \), which is common in classical time series setting. Intuitively, the presence of \( Z_{t,s} \) induces an extra \( O_p(S) \) term which blurs the observed \( Y_{t,s} \) process around the true change-point \( T\Lambda^0 \). Note that Corollary 1 additionally requires \( S = O(T) \) to limit the impact of the extra \( O_p(S) \) term, indicating the difficulty of detecting change-points with relaxation of IS.

### 3.2 Asymptotic distribution

We now investigate the asymptotic distribution of the change-point estimator. We focus on the case where IS holds for \( Y \) since derivation of the asymptotic distribution under relaxation of IS is difficult unless specific parametric assumption is made. See Remark 2 below for more discussion.
Theorems 1 and 2 indicate that under IS, the integer-valued change-points can be recovered exactly. Thus, there is no non-degenerate asymptotic distribution for the estimated change-points $\hat{\lambda}_{ST}$. Nevertheless, the following Theorem 3 gives some insights on the finite-sample behavior of $\hat{\lambda}_{ST}$. For simplicity, we only present the result for $k = 1$, the result for $k \geq 1$ is similar but notionally more complicated. Define $E_1 = \{(s_1, s_2) : s_1 \in S, s_2 \in s_1 \cup \mathcal{N}(s_1)\}$ and $E_2 = \{s \in S : \text{repeat } s \text{ by } 1 + \lfloor \mathcal{N}(s) \rfloor \text{ times}\}$.

For $q > 0$, define

$$A_1(q, \psi_j^0, \psi_{j+1}^0) = \sum_{s \in E_2} \left[ m_{\text{arg}}(\psi_j^0; x_{q, s}^{(j+1)}) + m_{\text{arg}}(\psi_{j+1}^0; x_{q+1, s}^{(j+1)}) \right] - \sum_{(s_1, s_2) \in E_1} l_{\text{pair}}(\psi_j^0; x_{q, s_1}^{(j+1)}, x_{q+1, s_2}^{(j+1)}),$$

$$A_2(\psi_j^0, \psi_{j+1}^0) = \sum_{(s_1, s_2) \in E_1} l_{\text{pair}}(\psi_j^0; x_{T_j, s_1}^{(j+1)}, x_{1, s_2}^{(j+1)}) - \sum_{s \in E_2} \left[ m_{\text{arg}}(\psi_j^0; x_{T_j, s}^{(j)}) + m_{\text{arg}}(\psi_{j+1}^0; x_{1, s}^{(j+1)}) \right],$$

$$A_3(q, \psi_j^0, \psi_{j+1}^0) = \sum_{(t, i, s_1, s_2) \in D_1(q)} l_{\text{pair}}(\psi_j^0; x_{T_j, s_1}^{(j+1)}, x_{1, s_2}^{(j+1)}) - \sum_{(t, i, s_1, s_2) \in D_1(q)} l_{\text{pair}}(\psi_{j+1}^0; x_{T_j, s_1}^{(j+1)}, x_{1, s_2}^{(j+1)}),$$

where $D_1(q) = \{(t, i, s_1, s_2) : 1 \leq t, t + i \leq q, 0 \leq i \leq 1, s_1 \in S, s_2 \in s_1 \cup \mathcal{N}(s_1), \text{ if } i = 0, s_1 \neq s_2\}$.

For $q < 0$, define

$$B_1(q, \psi_j^0, \psi_{j+1}^0) = \sum_{s \in E_2} \left[ m_{\text{arg}}(\psi_j^0; x_{T_j+q, s}^{(j)}) + m_{\text{arg}}(\psi_{j+1}^0; x_{T_j+q+1, s}^{(j+1)}) \right] - \sum_{(s_1, s_2) \in E_1} l_{\text{pair}}(\psi_j^0; x_{T_j+q, s}^{(j)}, x_{T_j+q+1, s}^{(j+1)}),$$

$$B_2(\psi_j^0, \psi_{j+1}^0) = \sum_{(s_1, s_2) \in E_1} l_{\text{pair}}(\psi_j^0; x_{T_j, s_1}^{(j)}, x_{1, s_2}^{(j+1)}) - \sum_{s \in E_2} \left[ m_{\text{arg}}(\psi_j^0; x_{T_j, s}^{(j+1)}) + m_{\text{arg}}(\psi_{j+1}^0; x_{1, s}^{(j+1)}) \right],$$

$$B_3(q, \psi_j^0, \psi_{j+1}^0) = \sum_{(t, i, s_1, s_2) \in D_2(q)} l_{\text{pair}}(\psi_j^0; x_{T_j, s_1}^{(j+1)}, x_{1, s_2}^{(j+1)}) - \sum_{(t, i, s_1, s_2) \in D_2(q)} l_{\text{pair}}(\psi_{j+1}^0; x_{T_j, s_1}^{(j+1)}, x_{1, s_2}^{(j+1)}),$$

where $D_2(q) = \{(t, i, s_1, s_2) : T_j + q \leq t, t + i \leq T_j, 0 \leq i \leq 1, s_1 \in S, s_2 \in s_1 \cup \mathcal{N}(s_1), \text{ if } i = 0, s_1 \neq s_2\}$. Note that $A_i$’s and $B_i$’s quantify the effects of the estimation error $|T \lambda_j - T \lambda_j^0| = q$ on the CLMDL. For $j = 1, \ldots, m_0$, we define a double-sided random walk for the $j$th change point,

$$W_{ST}^{(j)}(q; \psi_j^0, \psi_{j+1}^0) = \begin{cases} 
A_1(q, \psi_j^0, \psi_{j+1}^0) + A_2(\psi_j^0, \psi_{j+1}^0) + A_3(q, \psi_j^0, \psi_{j+1}^0), & q > 0, \\
0, & q = 0, \\
B_1(q, \psi_j^0, \psi_{j+1}^0) + B_2(\psi_j^0, \psi_{j+1}^0) + B_3(q, \psi_j^0, \psi_{j+1}^0), & q < 0.
\end{cases}$$

Theorem 3 gives an approximation of the finite-sample behavior of $\hat{\lambda}_{ST}$ and the asymptotic distribution of the estimated parameters $\hat{\theta}_{ST}^{(j)}$.

**Theorem 3.** Suppose that the conditions in Theorems 1 or 2 are satisfied, and Assumptions 6 and 7 hold. Given that the number of change-points and the model order are correctly identified, i.e., $m = m_0$ and $\hat{\xi}_j = \xi_j^0$ for $j = 1, \ldots, m_0 + 1$, then

$$\sqrt{ST}(\hat{\theta}_j - \theta_j^0) \longrightarrow N[0, \{\Sigma_2^{(j)}\}^{-1}\Sigma_1 \{\Sigma_2^{(j)}\}^{-1}] \text{ in distribution},$$

(18)
as \( S, T \rightarrow \infty \). If, additionally, \( S = o(T) \), we have for \( j = 1, \ldots, m_o + 1 \),

\[
T(\lambda_j - \lambda_o^j) = \arg \max_{q \in \mathbb{Z}} W_{ST}^{(j)}(q; \psi_j^o, \psi_{j+1}^o) + o_p(1).
\]

Moreover, \( \{\hat{\lambda}_1, \ldots, \hat{\lambda}_{m_o}, \hat{\theta}_1, \ldots, \hat{\theta}_{m_o+1}\} \) are asymptotically independent.

From the proof of Theorems 1 and 2, it can be shown that

\[
P\left( \arg \max_{q \in \mathbb{Z}} W_{ST}^{(j)}(q; \psi_j^o, \psi_{j+1}^o) = 0 \right) \rightarrow 1,
\]

which again indicates that the true change-points can be recovered without errors. Although \( T(\lambda_j - \lambda_o^j) \) eventually converges to a degenerate distribution, as is shown by the numerical experiments in Section 4, \( \arg \max_{q \in \mathbb{Z}} W_{ST}^{(j)}(q; \psi_j^o, \psi_{j+1}^o) \) can still give a reasonably accurate approximation to the finite-sample behavior of \( \hat{\Lambda}_{ST} \). The finite-sample approximation of \( T(\lambda_j - \lambda_o^j) \) in Theorem 3 requires \( S = o(T) \). For the case where \( S \) is greater than \( o(T) \), the approximation in (19) becomes inaccurate. The intuitive reason is that the distribution of \( T(\lambda_j - \lambda_o^j) \) converges too fast towards its degenerate limit when more information from the spatial dimension is available.

Since a closed-form expression for the distribution function of \( W_{ST}^{(j)}(\cdot, \psi_j^o, \psi_{j+1}^o) \) is unavailable, we need to simulate replicates of \( W_{ST}^{(j)}(\cdot, \hat{\psi}_j, \hat{\psi}_{j+1}) \) to conduct inference. However, the double-sided random walk \( W_{ST}^{(j)}(\cdot, \psi_j^o, \psi_{j+1}^o) \) depends not only on the pseudo-true parameters \( \psi_j^o \) and \( \psi_{j+1}^o \), but also on the true distributions of the \( j \)th and \((j + 1)\)th segments. Hence, this simulation procedure is valid only if the true models of the \( j \)th and \((j + 1)\)th segments are known. In other words, if the true model is not included in \( \mathcal{M} \), \( \hat{\Lambda}_{ST} \) is consistent, but inference cannot be made via Theorem 3.

**Remark 2.** Under Assumption 8 and the conditions of Theorem 3, the asymptotic distribution of \( \hat{\theta} \) in (18) still holds. However, the asymptotic distribution of \( \hat{\lambda} \) in (19) becomes \( T(\lambda_j - \lambda_o^j) = \arg \max_{q \in \mathbb{Z}} [W_{ST}^{(j)}(q; \psi_j^o, \psi_{j+1}^o) + O_p(S)] + o_p(1) \), where the \( O_p(S) \) term relates to \( q \) and the difference between \( L_{ST}(\psi_o; Y) \) and \( L_{ST}(\psi_o; Y^*) \) and cannot be further quantified unless specific parametric assumption is made.

### 4 Numerical Experiments

We begin this section by discussing the numerical optimization algorithm for solving the minimization in (12), and then present simulation studies and applications to real data.

#### 4.1 Optimization Using PELT

The optimization problem in (12) involves huge amount of combinatorial search for the best change-points configuration and thus is computationally intensive. Various searching algorithms have been
proposed in the literature for this optimization problem. In general, the searching algorithms can be classified as stochastic or deterministic. Stochastic methods such as genetic algorithm (Davis et al., 2006) require large scale simulation and give simulation-dependent solutions. Among the deterministic methods, the optimal partitioning (Jackson et al., 2005), which uses dynamic programming to simplify the exhaustive search on all possible choices, can still be computationally intensive for large datasets. Recently, pruning-based dynamic programming algorithms (Killick et al., 2012; Maidstone et al., 2017) are proposed to obtain exact optimization solution by discarding some unnecessary change-point configurations in dynamic programming.

In particular, the pruned exact linear time (PELT) algorithm proposed by Killick et al. (2012) is a modified dynamic programming algorithm that solves the minimization problem of the form

\[ F(T) = \sum_{j=1}^{m+1} C(Y_{(\tau_j-1+1):\tau_j}) + \beta f(m), \]  

where \( Y_{[a:b]} = \{y_t; t = a, \ldots, b\} \), \( m \) is the number of change-points, \( C \) is the cost function for a segment and \( \beta f(m) \) is a penalty function for the number of changes. If \( f(m) = m + 1 \), PELT solves (21) by recursively computing

\[ F(s) = \min_t \{ F(t) + C(Y_{[(t+1):s]}) + \beta \}, \]

where \( t < s < T \). The minimization is taken over all \( t < T \) excluding those satisfying the pruning condition

\[ F(t) + C(Y_{[(t+1):t']} + K \geq F(t'), \]

for some \( t' > t \) with \( t' - t \geq T\epsilon \lambda \). The constant \( K \) depends on \( C \) and is required to satisfy

\[ C(Y_{[(t+1):s]}) + C(Y_{[(s+1):T]}) + K \leq C(Y_{[(t+1):T]}), \]  

for all \( t < s < T \).

In practice, a considerable number of time points can be pruned by (22) and hence the efficiency of the dynamic programming algorithm can be greatly improved.

To implement PELT in the current spatio-temporal setting, we express the CLMDL in (10) in the format of (21) as

\[ C(Y_{(\tau_j-1+1):\tau_j}) = C_{k,N} \left\{ \sum_{i=1}^{\hat{d}_j} \log \hat{\xi}_{i,j} + \left( \frac{\hat{d}_j}{2} + 1 \right) \log(\tau_j - \tau_{j-1}) + \frac{\hat{d}_j}{2} \log S \right\} \]

− \( L_{ST}^{(j)}(\hat{\psi}_j; Y_{(\tau_j-1+1):\tau_j}) \),

where \( \beta = C_{k,N} \) and \( f(m) = \log m \). Although \( f(m) \) is not equal to \( m + 1 \), the minimization of (10) can still be performed by applying PELT iteratively with an adaptive choice of the penalty function \( f(m) \), until convergence; see Section 3.2 of Killick et al. (2012) for more detail.

Note that the additional requirement \( t' - t \geq T\epsilon \lambda \) is imposed by the identification condition in (11).
Compared to the classical multivariate time series setting which PELT is designed for, under the spatio-temporal setting, the constant $K$ in (23) is much harder to derive due to the non-ignorable edge effect when spatial dimension $S \rightarrow \infty$. If the condition (23) for $K$ is not satisfied for all $t < s < T$, the exact minimizer of CLMDL cannot be guaranteed to be found by PELT. The following lemma provides a potential choice of $K$, which can guarantee the asymptotic validity of applying PELT for the minimization of CLMDL.

**Lemma 1.** Consider all candidate model parameter sets $(\xi_j, \theta_j)$ in $M$, $\xi_j = (\xi_{1,j}, \ldots, \xi_{c,j})$ and $\theta_j \in \mathbb{R}^{d_j}$. Let $d_{\text{min}} = \min_j d_j$, $d_{\text{max}} = \max_j d_j$, $\xi_j^* = \sum_{i=1}^{c_j} \log \xi_{i,j}$, $\xi_{\text{min}}^* = \min_j \xi_j^*$ and $\xi_{\text{max}}^* = \max_j \xi_j^*$. Define

$$K = C_{k,N} \left\{ \left( \frac{d_{\text{min}}}{2} - d_{\text{max}} \right) \log ST + (2 + d_{\text{max}}) \log 2 + \xi_{\text{min}} - 2 \xi_{\text{max}} - \log T \right\}.$$ 

The true change points $T \Lambda^o$ will not be pruned asymptotically as $S, T \rightarrow \infty$ under the conditions of Theorem 1 and $T^2 \cdot S^{-r/2} \rightarrow 0$, or the conditions of Theorem 2 and $T \cdot S^{-r/2} \rightarrow 0$.

Since the true change-points $T \Lambda^o$ are not pruned asymptotically, by Theorem 1 or 2, PELT can find the minimizer of CLMDL as $S, T \rightarrow \infty$. Although there is no guarantee that PELT can obtain the exact solution of CLMDL when $S$ and $T$ are small, in practice, we find PELT works well as demonstrated by the numerical experiments in Section 4.2. Note that for small $S$ and $T$, an alternative is to directly implement optimal partitioning (Jackson et al. (2005)), which guarantees the exact minimization of CLMDL and requires a computational cost of $O(ST^2)$.

### 4.2 Simulation studies

For all the simulation studies, we set $k = 1$ and $d = 2$ in defining the composite likelihood, $\epsilon_\lambda = 0.1$ in the optimization, and set the number of replications to be 1000. Throughout the numerical experiments, we mainly consider a four-parameter autoregressive spatial model,

$$y_t - \mu = \phi(y_{t-1} - \mu) + \epsilon_t,$$  

(24)

where $y_t = \{y_{t,s} : s \in S\}$ is defined on a regular two-dimensional grid $S = \{(s_1, s_2) : s_1, s_2 = 1, \ldots, s\}$, i.e. $S = s^2$, and $\epsilon_t = \{\epsilon_{t,s} : s \in S\}$ is a Gaussian noise with exponential covariance function $\text{Cov}(\epsilon_{t,u}, \epsilon_{t,v}) = \sigma^2 \exp\left\{ - \|u - v\|^2 / \rho \right\}$ and $\text{Cov}(\epsilon_{t,u}, \epsilon_{t',v}) = 0$, when $t \neq t'$. The model is specified by $\theta = (\mu, \phi, \rho, \sigma^2)^T$, where $\phi \in (-1, 1)$, $\rho > 0$ and $\sigma^2 > 0$. Spatial and temporal dependence are determined by $\rho$ and $\phi$ respectively. Meanwhile, $\mu$ and $\sigma^2$ control the overall mean and variance.

**Simulation 1.** (Change-point detection rate at different sample sizes and signal levels)
In this simulation, we demonstrate the estimation accuracy of CLMDL under different sample sizes and signal levels. We assume that the underlying process in each stationary segment follows (24) with $\mu = 0$ known, hence the process is specified by $\theta = (\phi, \rho, \sigma^2)^T$.

Let $\theta_1 = (-0.5, 0.6, 1)^T$ and $\theta_2 = (-0.5 + \delta\phi, 0.6 + \delta\rho, 1)^T$ be the underlying parameter vectors for the segments before and after the change-point, respectively. When there is no change-point (i.e. $\delta\phi = \delta\rho = 0$), the entire process is simulated from $\theta_1$, otherwise there is a change-point at $\lambda_0 = 0.5$.

We consider four scenarios corresponding to no change-point, change in temporal dependence ($\delta\phi$), change in spatial dependence ($\delta\rho$) and change in both spatial and temporal dependence.

Table 1 reports the estimated number of change-points under various settings. Under the no-change scenario, there is no false positive even when the sample size is small. Some over-fitting is observed for small sample when there is change in spatial dependence, which is probably due to the larger variation in estimating $\rho$. The over-fitting rate decreases when sample size increases. The detection power improves when either $S$, $T$ or signal level ($\delta\phi$, $\delta\rho$) increases. For example, for $(\delta\phi = 0.2, S = 6^2, T = 100)$, the detection power increases from 37% to 81% or 98% respectively, when $S$ increases to $10^2$ or $T$ increases to 200. To be expected, the proposed procedure is most powerful when there is change in both $\delta\phi$ and $\delta\rho$.

**Simulation 2.** (Distribution of change-point estimator)

This simulation compares the empirical distribution of the change-point estimator and its asymptotic distribution stated in Theorem 3. The underlying process follows (24) with $\mu = 0$ known and $\theta_1 = (-0.5, 0.6, 1)^T$ and $\theta_2 = (-0.3, 0.6, 1)^T$. In this simulation, $S = 8^2, 10^2$ and $T_1 = T_2 = 100$ are considered, each with 1000 replicates. When the number of change-points is correctly estimated, 100 replicates of $W_{ST}(\cdot)$ are simulated using $\hat{\psi}_1$ and $\hat{\psi}_2$ to compute $\arg\max_{q \in \mathbb{Z}} W_{ST}(q; \hat{\psi}_1, \hat{\psi}_2)$.

The top panel of Figure 1 gives the QQ plots of the empirical quantile of $\hat{\lambda}$ against its theoretical quantile based on $\arg\max_{q \in \mathbb{Z}} W_{ST}(q; \hat{\psi}_1, \hat{\psi}_2)$ in Theorem 3. Note that the plot closely aligns with the 45 degree line. The bottom panel of Figure 1 depicts the histograms of $\hat{\lambda}$. The distribution of $\hat{\lambda}$ is non-standard and becomes degenerate as the sample size increases, which aligns with the asymptotic results.

Table 2 further reports the performance of 90% confidence interval (CI) obtained from the theoretical quantiles of $\arg\max_{q \in \mathbb{Z}} W_{ST}(q; \hat{\psi}_1, \hat{\psi}_2)$. The width of CI decreases as sample size increases, and the empirical coverage probabilities are close to the nominal level.

**Simulation 3.** (Consistency without any penalty term)

This simulation demonstrates the consistency of change-point estimation by the composite likelihood when there is no penalty term, as stated in Proposition 1. The underlying process follows (24) with $\mu = 0$ known. Let $\theta_1 = (-0.5, 0.6, 1)^T$ and $\theta_2 = (-0.5 + \delta\phi, 0.6, 1)^T$ be the
Table 1: Percentage of estimated change-points $\hat{m}$ among 1000 replications under various spatial size $S$, temporal size $T$, and signal levels ($\delta_{\phi}, \delta_{\rho}$).

| $S$  | $T$ | $\delta_{\phi} \times 10$ | $\delta_{\rho} \times 10$ | % of $\hat{m}$ | $T$ | % of $\hat{m}$ |
|------|-----|-----------------|-----------------|---------------|-----|---------------|
|      |     |                 |                 | 0     | 1   | $\geq 2$     | 0     | 1   | $\geq 2$     |
| 6^2  | 100 | 0               | 0               | 100   | 0   | 0            | 100   | 0   | 0            |
|      |     | 1               | 0               | 94    | 6   | 0            | 88    | 12  | 0            |
|      |     | 2               | 0               | 63    | 37  | 0            | 20    | 81  | 0            |
|      |     | 3               | 0               | 22    | 79  | 0            | 0     | 100 | 0            |
|      |     | 0               | 6               | 81    | 18  | 2            | 38    | 61  | 1            |
|      |     | 0               | 8               | 51    | 46  | 4            | 6     | 92  | 2            |
|      |     | 0               | 10              | 16    | 81  | 3            | 3     | 96  | 2            |
|      |     | 2               | 2               | 54    | 47  | 0            | 10    | 91  | 0            |
|      |     | 3               | 3               | 11    | 89  | 1            | 4     | 95  | 0            |
| 8^2  | 100 | 0               | 0               | 100   | 0   | 0            | 100   | 0   | 0            |
|      |     | 1               | 0               | 90    | 10  | 0            | 70    | 31  | 0            |
|      |     | 2               | 0               | 29    | 71  | 1            | 1     | 99  | 0            |
|      |     | 3               | 0               | 1     | 99  | 1            | 0     | 100 | 0            |
|      |     | 0               | 6               | 39    | 60  | 2            | 13    | 82  | 5            |
|      |     | 0               | 8               | 7     | 89  | 4            | 3     | 95  | 2            |
|      |     | 0               | 10              | 1     | 95  | 4            | 1     | 98  | 2            |
|      |     | 2               | 2               | 14    | 86  | 1            | 0     | 100 | 0            |
|      |     | 3               | 3               | 1     | 99  | 1            | 0     | 100 | 0            |
| 10^2 | 100 | 0               | 0               | 100   | 0   | 0            | 100   | 0   | 0            |
|      |     | 1               | 0               | 76    | 24  | 0            | 37    | 63  | 0            |
|      |     | 2               | 0               | 2     | 98  | 0            | 0     | 100 | 0            |
|      |     | 3               | 0               | 0     | 100 | 0            | 0     | 100 | 0            |
|      |     | 0               | 6               | 7     | 91  | 2            | 0     | 98  | 2            |
|      |     | 0               | 8               | 2     | 94  | 4            | 0     | 99  | 1            |
|      |     | 0               | 10              | 1     | 96  | 3            | 0     | 100 | 0            |
|      |     | 2               | 2               | 0     | 100 | 0            | 0     | 100 | 0            |
|      |     | 3               | 3               | 0     | 100 | 0            | 0     | 100 | 0            |
Figure 1: Top: Q-Q plots of sample quantiles of $\arg\max_{q \in \mathbb{Z}} W_{ST}(q; \hat{\psi}_1, \hat{\psi}_2)$ against the sample quantiles of empirical $\hat{\lambda}$; bottom: Histogram of empirical $\hat{\lambda}$ when $\delta_{\phi} = 0.2$ and $T = 200$.

Table 2: Proportion of $\hat{m} = 1$, mean, empirical standard deviation (esd), mean of 90% confidence interval (CI) of $\hat{\lambda}$, and empirical coverage probability (CP) of the 90% CI when $\delta_{\phi} = 0.2, \delta_{\rho} = 0, T = 200$.

| S   | % of $\hat{m} = 1$ | $\hat{\lambda}$ | 90% CI          | CP  |
|-----|--------------------|------------------|------------------|-----|
|     |                    | mean  | esd    | [lower, upper]  |     |
| $8^2$| 99                 | 0.4964| 0.0310 | [0.4538, 0.5279]| 87.6|
| $10^2$| 100              | 0.4979| 0.0171 | [0.4726, 0.5189]| 91.6|

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underlying parameter vectors for the segments before and after the change-point, respectively. When there is no change-point (i.e. \( \delta_\phi = 0 \)), the entire process is simulated from \( \theta_1 \) and \( T = 100 \), otherwise there is a change-point at \( \lambda_1 = 0.5 \). Table 3 reports the estimated number of change-points using the proposed CLMDL and the composite likelihood (CL). Since there is no penalty term, we observe some over-fitting by CL when \( S = 30^2 \) and \( \delta_\phi = 0.2 \). On the other hand, CLMDL does not have false positive due to the penalty term. As the sample size increases, both criteria shows consistency, which agrees with the theoretical result. This example shows the penalty term in CLMDL can improve the estimation accuracy in finite sample and CL without penalty terms is consistent asymptotically.

| \( S \) | \( \delta_\phi \times 10 \) | CL | CLMDL |
|-------|-------------|-----|-------|
|       | % of \( \hat{m} \) | % of \( \bar{m} \) | % of \( \hat{m} \) | % of \( \bar{m} \) |
| 30^2  | 0           | 96  | 3     | 1     | 100  | 0       |
|       | 1           | 0   | 96  | 4     | 0    | 100  |
| 60^2  | 0           | 100 | 0    | 100   | 0    | 0     |

**Simulation 4.** (Robustness under misspecification)

In this simulation, we demonstrate the robustness of CLMDL against model misspecification. In particular, we consider the non-separable spatio-temporal covariance function in Cressie and Huang (1999),

\[
C(h, u|\theta) = \begin{cases} 
\frac{\sigma^2(2c^{d/2})}{(a^2u^2+1)^{(d/2)}} \left\{ b \left( \frac{a^2u^2+1}{a^2u^2+c} \right)^{1/2} h \right\}^{\nu} K_\nu \left( b \left( \frac{a^2u^2+1}{a^2u^2+c} \right)^{1/2} h \right), & \text{if } h > 0, \\
\frac{\sigma^2(2c^{d/2})}{(a^2u^2+1)^{(d/2)}} \frac{1}{(a^2u^2+c)^{d/2}}, & \text{if } h = 0, 
\end{cases}
\]

where \( h \) and \( u \) are respectively the space and time distance, \( d = 2 \) is the spatial dimension, \( \nu > 0 \) is the smoothness parameter which characterizes the behavior of the correlation function near the origin, \( K_\nu \) is the modified Bessel function of the second kind of order \( \nu \), \( a \geq 0 \) is the scaling parameter of time, \( b \geq 0 \) is the scaling parameter of space, \( c > 0 \) is the space-time interaction parameter, and \( \sigma^2 = C(0, 0|\theta) > 0 \) is the variance at the origin. The parameter vector \( \theta = (a, b, c, \nu, \sigma^2) \) completely specifies the covariance function in (25).

Note that \( C(h, u|\theta) \) in (25) generalizes various popular covariance functions. For example, if
\(\nu = 0\), then \(C(h, 0|\theta)\) becomes the Matérn spatial covariance function. When \(\nu = 0.5\), it is further reduced to an exponential covariance function. When \(\nu \to \infty\), the Gaussian covariance function is obtained. Moreover, a separable spatio-temporal covariance function is obtained when \(c = 1\). See Cressie and Huang (1999) for details.

In this simulation experiment, a model with change-point \((\delta > 0)\) consists of two segments with parameters \(\theta_1 = (1, 1, 3, 0.2, 1)^T\) and \(\theta_2 = (1 + \delta, 1 + \delta, 3, 0.2, 1)^T\) respectively, and \(T_1 = T_2 = 50\). A model without change-point consists of one segment with parameter \(\theta_1\) and \(T = 100\). We consider \(S = 6^2, 7^2, 8^2\) and \(\delta = 0, 0.5, 1.0, 0.15, 2.0\).

We conduct the change-point estimation using both the separable model (24) and the true model (25) to illustrate the robustness of CLMDL under misspecification. Table 4 summarizes the numerical result. It shows that CLMDL works well under model misspecification, with a low false positive rate when there is no change-point and high detection power when change-point exists. Compared to the true model, the loss in power due to model misspecification is small. It further shows that the empirical means of the change-point estimators based on both the misspecified model and the true model converge to the true value with the empirical standard deviations decrease gradually as sample size increases.

**Simulation 5.** (Model selection and multiple change-point detection)

In this example, we study the performance of CLMDL for model selection with the presence of multiple change-points and model misspecification. Specifically, we simulate a spatio-temporal process with four stationary segments. The underlying process for the first, second and fourth segments follow the four-parameter autoregressive spatial model (24) with \(\theta_1 = (0.0, -0.2, 0.6, 1.0)^T\), \(\theta_2 = (0.0, -0.5, 0.6, 1.0)^T\), \(\theta_4 = (0.3, -0.2, 0.9, 1.0)^T\) and \(T_1 = T_2 = T_4 = 50\). For the third segment, instead of exponential covariance function, Matérn spatial covariance function is used, i.e.

\[
\text{Cov}(\varepsilon_{t,u}, \varepsilon_{t',v}) = \frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)} \left( \sqrt{2\nu\frac{\|u - v\|_2^2}{\rho}} \right)^\nu K_\nu \left( \sqrt{2\nu\frac{\|u - v\|_2^2}{\rho}} \right),
\]

where \(K_\nu(\cdot)\) is the modified Bessel function of the second kind and \(\text{Cov}(\varepsilon_{t,u}, \varepsilon_{t',v}) = 0\), when \(t \neq t'\).

We set \(\phi = -0.5, \sigma^2 = 0.9, \mu = 0.3\) in (24) with \(\nu = 2\) and \(\rho = 0.9\) for the Matérn spatial covariance function and \(T_3 = 50\).

The three change-points correspond to change in \(\phi\), change in \(\mu\) and spatial covariance function, and change in \(\phi\) and spatial covariance function respectively.

The candidate model class \(\mathcal{M}\) consists of two models, \(M_1\) and \(M_2\), both in the form of (24) with exponential spatial covariance function, while \(M_1\) with \(\mu = 0\) fixed and \(M_2\) with \(\mu\) as a free parameter. The number of parameters of \(M_1\) and \(M_2\) are 3 and 4 respectively. Note that \(M_1\) is nested within \(M_2\) by setting \(\mu = 0\). For the third segment, the true model is not included in \(\mathcal{M}\). Since \(M_2\) allows non-zero \(\mu\), it acts as the pseudo-true model. In summary, the (pseudo-) true
Table 4: Percentage of estimated change-points $\hat{m}$, mean, empirical standard deviation (esd) of $\hat{\lambda}$ when $\hat{m} = 1$ using misspecified model (24) and true model (25).

| S   | $\delta \times 10$ | Misspecified model | True model |
|-----|---------------------|-------------------|------------|
|     |                     | $\%$ of $\hat{m}$ | $\hat{\lambda}$ | $\%$ of $\hat{m}$ | $\hat{\lambda}$ |
|     | 0                   | 1                | $\geq 2$ | mean | esd | 0 | 1 | $\geq 2$ | mean | esd |
| $6^2$ | 0                   | 97 | 2 | 0 | - | - | 97 | 2 | 0 | - | - |
|      | 5                   | 94 | 6 | 0 | 0.4244 | 0.1135 | 93 | 7 | 0 | 0.4237 | 0.1136 |
|      | 10                  | 33 | 67 | 1 | 0.5109 | 0.0407 | 30 | 70 | 1 | 0.5097 | 0.0375 |
|      | 15                  | 1 | 98 | 1 | 0.5039 | 0.0214 | 1 | 98 | 1 | 0.5037 | 0.0199 |
|      | 20                  | 0 | 99 | 1 | 0.5025 | 0.0136 | 0 | 99 | 1 | 0.4976 | 0.0135 |
| $7^2$ | 0                   | 98 | 2 | 0 | - | - | 98 | 2 | 0 | - | - |
|      | 5                   | 84 | 16 | 0 | 0.5280 | 0.0757 | 83 | 17 | 0 | 0.4744 | 0.0733 |
|      | 10                  | 3 | 97 | 1 | 0.5052 | 0.0253 | 3 | 97 | 1 | 0.4953 | 0.0229 |
|      | 15                  | 0 | 99 | 1 | 0.5022 | 0.0136 | 0 | 99 | 1 | 0.5021 | 0.0129 |
|      | 20                  | 0 | 99 | 1 | 0.4986 | 0.0089 | 0 | 99 | 1 | 0.5011 | 0.0073 |
| $8^2$ | 0                   | 99 | 1 | 0 | - | - | 99 | 1 | 0 | - | - |
|      | 5                   | 69 | 31 | 0 | 0.4886 | 0.0417 | 66 | 33 | 0 | 0.5104 | 0.0383 |
|      | 10                  | 0 | 99 | 1 | 0.5019 | 0.0111 | 0 | 99 | 1 | 0.4983 | 0.0109 |
|      | 15                  | 0 | 99 | 1 | 0.5008 | 0.0053 | 0 | 99 | 1 | 0.5007 | 0.0052 |
|      | 20                  | 0 | 99 | 1 | 0.4995 | 0.0039 | 0 | 100 | 0 | 0.4995 | 0.0037 |
models for the four segments are $M_1$, $M_1$, $M_2$ and $M_2$ respectively.

Table 5 reports the results of the multiple change-point estimation. It can be seen that the proportion of correctly estimated number of change-point increases, and both over-fitting and under-fitting diminish as $S$ increases. Moreover, the empirical standard deviations of the estimated location of change-point decrease gradually. Table 6 reports the model selection results, which further shows CLMDL performs well for model selection and maintains its efficiency when the model is misspecified.

### Table 5: Distribution of $\hat{m}$, mean and empirical standard deviation (esd) of $\hat{\lambda}_j$, $j = 1, 2, 3$, when $\hat{m} = m_o = 3$.

| $S$ | $\%$ of $\hat{m}$ | $\hat{m} = 3$ | $\hat{m} = 3$ | $\hat{m} = 3$ |
|-----|-------------------|----------------|----------------|----------------|
|     | 0 | 1 | 2 | 3 | $\geq$ 4 | $\hat{\lambda}_1$ | mean | esd | $\hat{\lambda}_2$ | mean | esd | $\hat{\lambda}_3$ | mean | esd |
| $6^2$ | 0 | 4.7 | 39.6 | 55.2 | 0.5 | 0.2579 | 0.0304 | 0.4996 | 0.0133 | 0.7505 | 0.0314 |
| $8^2$ | 0 | 0 | 2.3 | 96.9 | 0.8 | 0.2527 | 0.0158 | 0.4997 | 0.0040 | 0.7502 | 0.0143 |
| $10^2$ | 0 | 0 | 0 | 99.6 | 0.4 | 0.2513 | 0.0065 | 0.5001 | 0.0023 | 0.7498 | 0.0083 |

### 4.3 Application to U.S. precipitation data

Change-point detection in the amount of precipitation has been recognized as an important problem in climatology and environmental science (Gallagher et al., 2012; Gromenko et al., 2017). Gallagher et al. (2012) provides a review on some common approaches. Existing methods such as Gallagher et al. (2012) and Gromenko et al. (2017) only consider the at-most one change-point case, and require space-time separability of the covariance function, thus are not directly comparable with the proposed CLMDL.

We consider data from the Global historical climatological network database (GHCN), which is a main database for global climate monitoring. In particular, some key climate variables such as

### Table 6: Relative frequencies of the model selected given $\hat{m} = m_o = 3$.

| $S$ | Segment | 1 | 2 | 3 | 4 |
|-----|---------|---|---|---|---|
|     | $M_1$ | $M_2$ | $M_1$ | $M_2$ | $M_1$ | $M_2$ | $M_1$ | $M_2$ |
| $6^2$ | 95.7 | 4.3 | 99.1 | 0.9 | 0.2 | 99.8 | 0.2 | 99.8 |
| $8^2$ | 97.7 | 2.3 | 99.9 | 0.1 | 0.0 | 100.0 | 0.0 | 100.0 |
| $10^2$ | 98.2 | 1.8 | 99.7 | 0.3 | 0.0 | 100.0 | 0.0 | 100.0 |
the amount of precipitation are collected from stations located all over the world. The documentation and datasets are available from Menne et al. (2012) and GHCN official website. Similar to Gromenko et al. (2017), selected precipitation data from the Midwest region of U.S., including Illinois, Indiana, Iowa, Kansas, Michigan, Minnesota, Missouri, Nebraska, North Dakota, Ohio, South Dakota and Wisconsin are considered.

In the GHCN database, daily precipitation from each station is recorded. However, there are a lot of missing data in many stations. Therefore, we focus on the land surface stations that provide at least 5 daily records of in each month over the entire period, and compute the monthly average precipitation data for the analysis. In summary, we have the monthly average precipitation \( \{y_{t,s}^*\} \) in tenths of a millimeter (mm) from January 1941 to December 2010 for 76 stations (i.e. \( S = 76 \) and \( T = 720 \)). To alleviate the heavy tail behavior of the data, we transform the monthly average precipitation record by \( y_{t,s} = \log(y_{t,s}^* + 1) \).

We select all pairs with time lag within 2 month \( (k = 2) \) and spatial distance within 200 kilometers in the specification of composite likelihood. The geodesic distance (Karney, 2013) is used as the spatial distance, which corresponds to the shortest path between two points on the WGS84 ellipsoid. To handle the yearly seasonality, we apply an analysis-of-variance model to remove the month effects, treating each month as class variable as in Auchincloss et al. (2008) and Bai et al. (2012). The change-point detection is then conducted by CLMDL with the non-separable spatio-temporal covariance function (25) of Cressie and Huang (1999) on the resulting residuals.

![Location of 76 land surface stations selected.](image)

Figure 2: Location of 76 land surface stations selected.
The CLMDL detects two change-points at August 1952 and September 1971. Similar estimation results are observed under different choices of time lag and spatial distance for the neighborhood, indicating the robustness of the result. The first change-point is within the great drought and prolonged heatwave which had great impact on the Midwest region of U.S. (Mishra and Singh, 2010; Westcott, 2011). The second change-point is inline with the change in climate in North America proposed by Bartomeus et al. (2011) and Westcott (2011). Moreover, the second change-point is close to the detected change-point (1968) by Gromenko et al. (2017), which analyzes a similar dataset but allows at most one change-point. Based on Theorem 3, the 90% CIs for the change-point are (November 1951, March 1953) and (April 1971, February 1972) respectively.

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