Association schemes, classical RCFT’s, and centres of monoidal functor categories

Brian Day
April 7, 2019

Abstract

Here we describe three straightforward examples of what was called a graphic Fourier transformation in [4]. At least two of these examples may be viewed simply as monoidal comonads on suitable monoidal closed functor categories, but the third example, which involves “centres” of monoidal closed functor categories, is generally not comonadic. For the first two examples (i.e., association schemes and RCFT’s), a more elaborate “probicategory” set-up was envisaged in an earlier version of this note, but many readers missed the main point so it is simplified (hopefully) below.

1 Introduction

A functor category such as \([\mathcal{A}, \text{Vect}]\) or \([\mathcal{A}, \text{Set}]\) behaves naïvely like a function algebra, both types of structure being familiar manifestations of “form” and “function”.

Upon setting \(\mathcal{V} = \text{Vect}\) (or \(\text{Set}\)) and taking \(\mathcal{A}\) and \(\mathcal{X}\) to be two (small) promonoidal \(\mathcal{V}\)-categories, we let \([\mathcal{A}, \mathcal{V}]\) and \([\mathcal{X}, \mathcal{V}]\) denote the corresponding convolution functor categories into \(\mathcal{V}\) (as in [2]). A \(\mathcal{V}\)-functor

\[ \hat{K} : [\mathcal{A}, \mathcal{V}] \to [\mathcal{X}, \mathcal{V}] \]

is then called a “graphic” Fourier transformation [4] if

1. \(\hat{K}\) is multiplicative (i.e., \(\hat{K}\) preserves the tensor product and tensor unit of \([\mathcal{A}, \mathcal{V}]\) up to natural isomorphism),
2. \(\hat{K}\) is conservative (i.e., \(\hat{K}\) reflects isomorphisms),
3. \(\hat{K}\) is cocontinuous (i.e., \(\hat{K}\) has a right \(\mathcal{V}\)-adjoint, denoted here by \(\check{K}\)).

Sometimes the convolutions \([\mathcal{A}, \mathcal{V}]\) and \([\mathcal{X}, \mathcal{V}]\) both have (contravariant) involutions on them and we ask in addition that \(\hat{K}\) should preserve this.

Below we mention two elementary examples in which (in the first instance) the category \(\mathcal{A}\) is discrete. We also describe a third type of example based on the “centres” of certain monoidal closed functor categories into \(\mathcal{V}\).

For the general theory, the “image” of \(\hat{K}\) – here called the Wiener category (cf. [4] §1.3) and denoted by Wien(\(\hat{K}\)) – was discussed briefly in [4]. Roughly
speaking, its objects are the functors of the form \( \hat{K}(f) \) for some \( f \in [A, V] \), and its maps are precisely those natural transformations \( \alpha : \hat{K}(f) \to \hat{K}(g) \) in \([X, V]\) for which
\[
\hat{K}(\alpha) \hat{K}(\eta) = \hat{K}(\eta) \alpha,
\]
where \( \eta \) denotes the (monomorphic) unit of the basic adjunction \( \hat{K} \dashv \check{K} \). These so-called “regular” morphisms \( \alpha \in [X, V] \) have been further characterized in many special examples in the literature.

Moreover, the property that \( \hat{K} \) should be multiplicative amounts, under the left Kan extension process, to the existence of two natural isomorphisms:
\[
\int_{y, z \in X} K(a, y) \otimes K(b, z) \otimes p(y, z, x) \cong \int_{c \in A} K(c, x) \otimes p(a, b, c)
\]
\[
j(x) \cong \int_{c \in A} K(c, x) \otimes j(c)
\]
in \( V \), where the functor
\[
\hat{K} : A^{op} \otimes X \to V
\]
denotes the tensor-hom transpose in \( V\text{-Cat} \) of the composite
\[
\mathcal{A}^{op} \xrightarrow{\text{Yoneda}} [A, V] \xrightarrow{\hat{K}} [X, V],
\]
and was called the (multiplicative) kernel of \( \hat{K} \) in [4].

Then we obtain the monoidal equivalence of categories
\[
\hat{K} : [A, V] \to \text{Wien}(K).
\]

## 2 Terminology

A Set-promonoidal category \((A, n, i)\) where \( A \) is a groupoid with antipode \( S : A^{op} \to A \) \((S^2 \cong 1)\) is called \((S-)\)precompact if both
\[
n(a, b, c) \cong n(Sb, Sa, Sc) \text{ and }
\]
\[
i(c) \cong i(Sc)
\]
dinaturally in \( a, b, c \in A \). Then, if \( n \) and \( i \) are finite, we have both
\[
k[n](a, b, c)^* \cong k[n](Sb, Sa, Sc) \text{ and }
\]
\[
k[i](c)^* \cong k[i](Sc)
\]
naturally in \( a, b, c \in A \), where \( k \) is any field. Thus, we call any finite Vect\(_k\)-promonoidal category \((A, p, j)\), with a \( k\)-linear antipode \( S \) \((S^2 \cong 1)\), precompact if both
\[
p(a, b, c)^* \cong p(Sb, Sa, Sc) \text{ and }
\]
\[
j(c)^* \cong j(Sc)
\]
naturally in \( a, b, c \in A \).
Note that, when the above data is available we usually have the “structure” maps

\[ f^* \otimes g^* \to (g \otimes f)^* \]

\[ I \to I^* , \]

where \( f^* \) is defined by \( f^*(a) = f(Sa)^* \), giving \((-)^*\) the structure of a contravariant monoidal functor on \([A, \text{Vect}_{k}]\). This is so, for example, when \( A \) is “Frobenius” (see [5] — there is a family of maps

\[ \delta : A(a, b) \to A(c, b) \otimes A(a, c) \]

in \( V \) which is natural in \( a, b \in A \). In this case there is a “comparison of integrals”

\[ \int_a h(a, a) \to \int_b h(b, b) \]

for each \( k\)-linear functor \( h : A^{op} \otimes A \to \text{Vect}_k \) which is derived directly from the composite natural transformation

\[ h(a, a) \xrightarrow{\delta} A(a, b) \otimes h(b, b) \xrightarrow{\mu} h(b, b). \]

Thus one has the composite:

\[ (f^* \otimes g^*)(c) := \int_a f(Sa)^* \otimes g(Sb)^* \otimes p(a, b, c) \quad \text{(by definition of } f^* \otimes g^*) \]

\[ \cong \int_a f(a)^* \otimes g(b)^* \otimes p(Sa, Sb, c) \quad \text{(} S^2 \cong 1 \text{)} \]

\[ \to \int_a g(b)^* \otimes f(a)^* \otimes p(Sa, Sb, c) \quad \text{(by “comparison of integrals”)} \]

\[ \cong \left( \int_a g(b) \otimes f(a) \otimes p(b, a, Sc) \right)^* \quad \text{(when } A \text{ is precompact)} \]

\[ = (g \otimes f)(Sc)^* \quad \text{(by definition of } g \otimes f) \]

\[ = (g \otimes f)^*(c). \]

## 3 Discrete association schemes

Let \( X \) be a set and let \( S \subset P(X \times X) \) be a fixed partition of \( X \times X \) such that \( \Delta \in S \) and \( s^* \in S \) when \( s \in S \), where \( \Delta \subset X \times X \) is the diagonal set, and \( s^* \) is the reverse relation to \( s \). The relevant kernel

\[ K : S \times X \times X \to \text{Set} \]

is given here by

\[ K(s, x, y) = s(x, y) = \begin{cases} 1 & \text{if } (x, y) \in s \\ 0 & \text{otherwise}. \end{cases} \]

The resulting transformation

\[ \tilde{K} : [S, \text{Set}] \to [X \times X, \text{Set}] \]

is
then maps $f$ to

$$
\hat{K}(f)(x, y) = \sum_s K(s, x, y) \times f(s)
$$

$$
= \sum_{(x, y) \in s} f(s).
$$

A (discrete) association scheme on $X$ (after [10], for example) consists of the given partition $S$ and a map

$$
N : S \times S \times S \to \text{Set}
$$

together with a bijection

$$
\sum_r N(s, t, r) \times r \cong s \circ t
$$

for each $(s, t) \in S \times S$, where $s \circ t$ denotes the $\text{Set}$-matrix composite

$$
s \circ t(x, y) = \sum_{z \in X} s(x, z) \times t(z, y).
$$

It follows readily from the associativity of this composition that there is a proassociativity bijection

$$
\alpha : \sum_u N(s, t, u) \times N(u, r, v) \cong \sum_u N(s, u, v) \times N(t, r, u)
$$

for all $s, t, r, v \in S$ since $\sum_v M(v) \times v \cong \sum_v P(v) \times v$ implies $M(v) \cong P(v)$ for all $v \in S$ by the definition of “partition”. Together with the diagonal “object” $\Delta$, this data represents a discrete promonoidal structure called $(S, N, J)$ on the “category” $S$ (where the prounit $J : S \to \text{Set}$ is given by

$$
J(s) = \begin{cases} 
1 & \text{if } s = \Delta \\
0 & \text{otherwise}. 
\end{cases}
$$

Moreover, we may suppose (see [10]) that

$$
N(s, t, r) = N(t^*, s^*, r^*)
$$

so that $(S, N, J)$ is what we have called precompact in §2. Additionally, if also $X$ is finite and

$$
N(s, t, r^*) = N(t, r, s^*),
$$

then the monoidal closed convolution structure on $[S, \text{Vect}_{fd}]$ is in fact compact, as shown in the arXiv note [3].

Now it is easily seen that the transformation

$$
\hat{K} : [S, \text{Set}] \to [X \times X, \text{Set}]
$$
is multiplicative (where \([S, \text{Set}]\) has the convolution structure from \((S, N, J)\)),
conservative, and cocontinuous. Also, if we set \(f^*(s) = f(s^*)\) then we get
\[
\hat{K}(f^*)(x, y) = \sum_{(x, y) \in s} f^*(s)
= \sum_{(x, y) \in s} f(s^*)
= \sum_{(y, x) \in s} f(s^*)
= \sum_{(y, x) \in \xi} f(t)
= \hat{K}(f)^*(x, y).
\]
Thus, \(\hat{K}\) becomes a graphic Fourier transformation on \([S, \text{Set}]\), in the sense of [4].

In the case where the set \(X\) is finite, we can further view each element \(s \in S\) as a “\(k\)-bimodule” \(M_s\), for the given field \(k\), and write
\[
M_s \circ M_t \cong \bigoplus_r N(s, t, r) \cdot M_r
\]
where “\(\circ\)” denotes matrix composition, and “\(\cdot\)” denotes copowers in \([X \times X, \text{Vect}_{fd}]\). The functor
\[
\hat{K} : [S, \text{Vect}_{fd}] \to [X \times X, \text{Vect}_{fd}]
\]
given by
\[
\hat{K}(f)(x, y) = \bigoplus_{(x, y) \in s} f(s)
\]
then becomes a \(k\)-linear graphic Fourier transformation on \([S, \text{Vect}_{fd}]\). Here we can actually set
\[
f^*(s) := f(s^*)^*
\]
and obtain
\[
\hat{K}(f^*)(x, y) \cong \hat{K}(f)^*(x, y)
\]
for all \((x, y) \in X \times X\); i.e.,
\[
\hat{K}(f^*) \cong \hat{K}(f)^*.
\]

4 Rational conformal field theories

A discrete RCFT consists of a finite (discrete) set \(S\), containing a distinguished element 0, and a set of finite sets \(N(x, y, z)\) indexed by elements \(x, y, z\) of \(S\). We suppose also that there are proassociativity and prounit maps (bijections)
\[
\alpha : \sum_u N(x, y, u) \times N(u, z, v) \xrightarrow{\cong} \sum_u N(x, u, v) \times N(y, z, u)
\]
\[
\lambda : N(0, y, z) \xrightarrow{\cong} S(y, z)
\]
\[
\rho : N(x, 0, z) \xrightarrow{\cong} S(x, z),
\]

which satisfy some standard coherence conditions. There is usually assumed to be an involution \((-)^*\) on \(S\) satisfying the cyclic relation

\[ N(x, y, z^*) \cong N(y, z, x^*), \]

together with a “braiding”

\[ N(x, y, z) \cong N(y, x, z). \]

(See Moore and Sieberg [8] for details.)

Thus we have a \(k\)-linear functor

\[ p : \mathcal{A}^{\text{op}} \otimes \mathcal{A}^{\text{op}} \otimes \mathcal{A} \to \text{Vect}_{\text{fd}}, \]

where \(\mathcal{A}\) denotes the discrete \(\text{Vect}\)-category on \(S\), by considering the free \(k\)-vector spaces on the relevant finite sets, that is

\[ p(x, y, z) = k[N](x, y, z). \]

Together with the corresponding induced proassociativity and prounit isomorphisms, we obtain a braided (finite) *-autonomous promonoidal \(\text{Vect}\)-category \((\mathcal{A}, p, j)\). We could clearly generalize this situation considerably (in theory at least). For example, the arXiv note [3] contains further details of compact convolution categories \([\mathcal{A}, \text{Vect}_{\text{fd}}]\) in the case when the domain category \(\mathcal{A}\) is more than just discrete or finite.

Always the representation (or “Cayley”) functor

\[ \hat{K} : [\mathcal{A}, \text{Vect}_{\text{fd}}] \to [\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \text{Vect}_{\text{fd}}], \]

which maps \(f\) to

\[ \hat{K}(f)(y, z) = \int^x f(x) \otimes p(x, y, z), \]

can be seen to be multiplicative, conservative, and cocontinuous (as is generally the case), thus providing examples of \(k\)-linear graphic Fourier transforms on certain compact convolutions of the form \([\mathcal{A}, \text{Vect}_{\text{fd}}]\). Indeed, if we suppose further that \((\mathcal{A}, p, j)\) is any closed category equipped with a suitable involution \((-)^* : \mathcal{A}^{\text{op}} \to \mathcal{A}\)

(that is, \(p(x, y, z) \cong \mathcal{A}(x, [y, z])\) with \([x, y]^* \cong [y, x]\)), then we also have

\[ \hat{K}(f^*)(x, y) = \int^z f^*(z) \otimes \mathcal{A}(z, [x, y]) \]

\[ \cong f^*[x, y] \]

\[ \cong f^*[y, x] \]

\[ \cong \hat{K}(f(y, x))^* \]

\[ = \hat{K}(f)(x, y), \]

thus completing the list of properties for a graphic transformation listed in the introduction.
5 Centres of some monoidal functor categories

If \((\mathcal{A}, p, j)\) is any small promonoidal \(\mathcal{V}\)-category, where \(\mathcal{V}\) is either \(\text{Vect}\) or \(\text{Set}\), then the centre \(\mathcal{Z}[\mathcal{A}, \mathcal{V}]\) of the monoidal convolution \([\mathcal{A}, \mathcal{V}]\) has the property that the underlying-object functor

\[ U : \mathcal{Z}[\mathcal{A}, \mathcal{V}] \to [\mathcal{A}, \mathcal{V}] \]

is multiplicative, conservative, and cocontinuous. Consequently, if we can show that a particular centre \(\mathcal{Z}[\mathcal{A}, \mathcal{V}]\) is monoidally equivalent to a complete convolution functor category of the form \([\mathcal{B}, \mathcal{V}]\), then the composite functor

\[ [\mathcal{B}, \mathcal{V}] \xrightarrow{\mathcal{K}} [\mathcal{A}, \mathcal{V}] \xrightarrow{U} \mathcal{Z}[\mathcal{A}, \mathcal{V}] \]

becomes a graphic Fourier transformation.

We remark that some efforts were made in [6] and [7] to find suitable representing categories \(\mathcal{B}\) for \(\mathcal{Z}[\mathcal{A}, \mathcal{V}]\) and, in particular, the functor

\[ \tilde{\Psi} : \mathcal{Z}[\mathcal{A}, \mathcal{V}] \to [\mathcal{Z}(\mathcal{A}), \mathcal{V}] \]

defined in [6] §2 is an equivalence of categories if and only if both

1. the canonical full embedding denoted by

\[ \Psi : \mathcal{Z}(\mathcal{A})^{\text{op}} \subset \mathcal{Z}[\mathcal{A}, \mathcal{V}] \]

in [6] is dense, and

2. existential quantification

\[ \exists U : [\mathcal{Z}(\mathcal{A}), \mathcal{V}] \to [\mathcal{A}, \mathcal{V}] \]

is conservative.

Here \(U : \mathcal{Z}(\mathcal{A}) \to \mathcal{A}\) denotes the forgetful functor from the “promagmal” centre \(\mathcal{Z}(\mathcal{A})\) (see [6] §2) of the given promonoidal structure \((\mathcal{A}, p, j)\); thus \(\mathcal{Z}(\mathcal{A})\) reproduces the usual braided monoidal centre of \(\mathcal{A}\) when the promonoidal structure on \(\mathcal{A}\) is in fact monoidal.

In this regard, we point out that, in the case of \(\mathcal{V} = \text{Vect}\), the full embedding

\[ \Psi : \mathcal{Z}(\mathcal{A})^{\text{op}} \subset \mathcal{Z}[\mathcal{A}, \text{Vect}] \]

is dense whenever all epimorphisms split in \([\mathcal{A}, \text{Vect}]\) and each of the functors

\[ p(a, b, -) : \mathcal{A} \to \text{Vect}, \quad a, b \in \mathcal{A}, \]

is finitely presentable in \([\mathcal{A}, \text{Vect}]\); for example, if \(\mathcal{A}\) is monoidal then each functor

\[ p(a, b, -) \cong \mathcal{A}(a \otimes b, -) : \mathcal{A} \to \text{Vect} \]

is certainly finitely presentable in \([\mathcal{A}, \text{Vect}]\)
Example 5.1. (cf. [9]) All epimorphisms split in the functor category \([\text{Span}(C), \text{Vect}]\) when \(C = \text{finite } G\)-sets for a finite group \(G\), so that

\[
\Psi : Z(k, \text{Span}(C))^{\text{op}} \subset Z[k, \text{Span}(C), \text{Vect}]
\]

is dense in this case, because \(k, \text{Span}(C)\) (which denotes the \(k\)-linearization of \(\text{Span}(C)\)) is monoidal. Moreover,

\[
\tilde{\Psi} : Z[k, \text{Span}(C), \text{Vect}] \simeq Z(k, \text{Span}(C), \text{Vect})
\]

is an equivalence, which is not as refined as the main Theorem of [9] p. 4019, but is much more easily obtained.

Example 5.2. (cf. [1]) All epimorphisms split in the functor category \([A, \text{Vect}]\) for any compact monoidal fusion category \(A\) over a field \(k\) of suitable characteristic, so here also \(\Psi\) is dense, and

\[
\tilde{\Psi} : Z[A, \text{Vect}] \simeq Z(A, \text{Vect})
\]

is an equivalence of monoidal categories.

The duality and precompactness conditions mentioned in §2 can also be considered in this context.

Remark. We have considered just the centres of monoidal functor categories here. In fact, the results used above (from [6] and [7]) were originally established in terms of “lax” centres, but there is no significant difference in dealing with the centres.

References

[1] Alain Bruguières and Alexis Virelizier. Categorical centres and Reshetikhin-Turaev invariants, arXiv:0812.2426v1 [math.QA] Dec 2008.

[2] Brian Day. On closed categories of functors, Lecture Notes in Mathematics (Springer) 137 (1970) 1–38.

[3] Brian Day. Compact convolution, arXiv:math/0605463v1 [math.CT], May 2006.

[4] Brian Day. Monoidal functor categories and graphic Fourier transforms, arXiv:math/0612496v1 [math.QA], Dec. 2006.

[5] Brian Day. When is existential quantification conservative?, arXiv:0906.4594v1 [math.CT] Jan. 2009.

[6] B. Day, E. Pachchadcharam, and R. Street. Lax braidings and the lax centre, Contemporary Mathematics 431 (2007) 187–202.

[7] B. Day and R. Street. Centres of monoidal categories of functors, Contemporary Mathematics 441 (2007) 1–17.

[8] G. Moore and N. Sieberg. Classical and quantum conformal field theory, Communications in Mathematical Physics 123 (1989) 177–254.
[9] D. Tambara. The Drinfeld center of the category of Mackey functors, Journal of Algebra 319 no. 10 (2008) 4018–4101.

[10] P.-H. Zieschang. An algebraic approach to association schemes, Lecture Notes in Mathematics (Springer) 1628, 1996.

Department of Mathematics
Macquarie University
NSW, 2109, Australia