Majorization and additivity for multimode bosonic Gaussian channels

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Abstract

Recently, the longstanding Gaussian optimizer conjecture was proven for bosonic Gaussian gauge-covariant or contravariant channels \cite{3}. In \cite{11} this result was strengthened for one-mode channels by establishing that the output for the vacuum or coherent input majorizes the output for any other input.

In the present paper we give the multimode extension of the result of \cite{11}, including sufficient conditions under which the coherent states are the only optimizers. We also discuss direct implications of this multimode majorization result to the positive solution of the additivity problem for the Gaussian channels. In particular, we demonstrate the additivity of the output Rényi entropies of arbitrary order \(p > 1\). Finally, we present an alternative derivation of a majorization property of Glauber’s coherent states by Lieb and Solovej \cite{10}, basing on the method of the work \cite{3}.

1 Introduction

Recently, the longstanding Gaussian optimizer conjecture in quantum information theory was proven for the class of bosonic Gaussian gauge-covariant or contravariant channels \cite{3}. The conjecture says that the minimum output entropy of a bosonic Gaussian channel is attained on the vacuum state
(as well as on any coherent state). In [11] this result was strengthened for one-mode channels by establishing that the output for the vacuum or coherent input majorizes the output for any other input, in that it minimizes a broad class of concave functionals of the output states. For a detailed discussion of motivation and applications of these advances to quantum optics and communications we refer to [3], [11].

In this present paper we obtain further results in this direction. In Sec. 2 we give the multimode extension of the result of [11], and in particular, a precise formulation of sufficient conditions under which the coherent states are the only minimizers. We also discuss direct implications of this multimode majorization result to the positive solution of another famous conjecture, namely the additivity problem for the Gaussian channels. In particular, we demonstrate the additivity of the output Rényi entropies of arbitrary order $p > 1$, which generalizes a result of Giovannetti and Lloyd [4] for integer $p$ and special channels.

In Sec. 4 we generalize a majorization result of Lieb and Solovej [10], basing on the method of the work [3]. Wehr [12] introduced the “classical entropy” of a quantum state $\rho$ by the formula

$$S_{cl}(\rho) = -\int_{C^s} \langle z|\rho|z\rangle \log \langle z|\rho|z\rangle \frac{d^{2s}z}{\pi^s},$$

where $\langle z|\rho|z\rangle$ is the Husimi function, $|z\rangle$ are Glauber’s coherent vectors, $s$ – number of the modes. Lieb [9] used exact constants in the Hausdorff-Young inequality (Fourier transform) and Young inequality (convolution) to prove Wehrl’s conjecture: $S_{cl}(\rho)$ is minimized by any coherent state $\rho = |\zeta\rangle\langle\zeta|$. Recently Lieb and Solovej [10] gave another derivation based on the limit version of a similar result for Bloch spin coherent states. Moreover, in this way they could establish the majorization property of Glauber’s coherent states. In Sec. 4 we suggest yet a different (and perhaps most natural) approach to the proof of this property and its generalization motivated by the recent solution of the Gaussian optimizers problem [3].

## 2 Majorization for gauge-covariant channels

We start with repeating some definitions and notations from [3], restricting to the case of channels with identical input and output spaces. Consider $s$– dimensional complex Hilbert space $\mathbf{Z}$ which can be considered as
2s—dimensional real space equipped with the symplectic form $z, z' \rightarrow \text{Im } z^* z'$. We will consider vectors in $\mathbb{Z}$ as $s$—dimensional complex column vectors, in which case (complex-linear) operators in $\mathbb{Z}$ are represented by complex $s \times s$—matrices, and $^*$ denotes Hermitian conjugation. The gauge group acts in $\mathbb{Z}$ as multiplication by $e^{i\phi}$, where $\phi$ is real number called phase. The Weyl quantization is described by the unitary displacement operators $D(z)$ acting irreducibly in the representation space $\mathcal{H}$ and satisfying the canonical commutation relation

$$D(z)D(z') = \exp(-i\text{Im } z^* z') D(z + z').$$

Introducing the annihilation-creation operators of the system $a_j, a_j^\dagger; j = 1, \ldots, s$, which satisfy the commutation relations $[a_j, a_k^\dagger] = \delta_{jk}I$, the operator $D(z)$ can be expressed as

$$D(z) = \exp \sum_{j=1}^s \left( z_j a_j^\dagger - \bar{z}_j a_j \right),$$

The gauge group has the unitary representation $\phi \rightarrow U_\phi = e^{i\phi N}$ in $\mathcal{H}$ where $N = \sum_{j=1}^s a_j^\dagger a_j$ is the total number operator. The representation of the gauge group in $\mathcal{H}$ acts according to the relation $U_\phi^* D(z) U_\phi = D(e^{i\phi} z)$, $\phi \in [0, 2\pi]$. A state $\rho$ is then said to be gauge-invariant if it commutes with all $U_\phi$ or, equivalently, if its characteristic function $\phi(z) = \text{Tr} \rho D(z)$ is invariant under the action of the gauge group. In particular Gaussian gauge-invariant states are given by the characteristic function of the form

$$\phi(z) = \exp (-z^* \alpha z),$$

where $\alpha$ is a complex correlation matrix satisfying $\alpha \geq I/2$, $I$ – the unit $s \times s$—matrix. The vacuum state $|0\rangle\langle 0|$ corresponds to $\alpha = I/2$.

A channel $\Phi$ in $\mathcal{H}$ is completely positive trace preserving map of the Banach space of trace-class operators in $\mathcal{H}$, see, e.g. [6] for detail. The channel is called gauge-covariant if

$$\Phi[U_\phi \rho U_\phi^*] = U_\phi \Phi[\rho] U_\phi^*.$$  

In the Heisenberg picture, a bosonic Gaussian gauge-covariant channel $\Phi$ [3] is described by the action of its adjoint $\Phi^*$ onto the displacement operators as follows:

$$\Phi^*[D(z)] = D(K^* z) \exp (-z^* \mu z),$$

$3$
where $K$ is a complex matrix and $\mu$ is Hermitian matrix satisfying the inequality

$$\mu \geq \pm \frac{1}{2} (I - KK^*).$$

(6)

The gauge-covariant channel is *quantum-limited* if $\mu$ is a minimal solution of the inequality (6). Special cases of the maps (5) are provided by the attenuator and amplifier channels, characterized by matrix $K$ fulfilling the inequalities, $KK^* \leq I$ and $KK^* \geq I$, respectively. We are particularly interested in *quantum-limited attenuator* which corresponds to

$$KK^* \leq I, \quad \mu = \frac{1}{2} (I - KK^*),$$

(7)

and *quantum-limited amplifier*

$$KK^* \geq I, \quad \mu = \frac{1}{2} (KK^* - I).$$

(8)

These channels are diagonalizable: by using singular value decomposition $K = V_B K_d V_A$, where $V_A, V_B$ are unitaries and $K_d$ is a diagonal matrix with nonnegative values on the diagonal, we have $KK^* = V_B K_d K_d^* V_B^*$, and

$$\Phi[\rho] = U_B \Phi_d[U_A \rho U_A^*] U_B^*,$$

(9)

where $\Phi_d$ is a tensor product

$$\Phi_d = \otimes_{j=1}^s \Phi_j$$

(10)

of one-mode quantum-limited channels defined by the matrix $K_d$ and where $U_A, U_B$ are the canonical unitary transformations acting on $\mathcal{H}$, such that

$$U_B^* D(z) U_B = D(V_B^* z), \quad U_A^* D(z) U_A = D(V_A^* z),$$

(notice that $U_A |0\rangle = |0\rangle, U_B |0\rangle = |0\rangle$).

**Theorem 1** (i) Let $\Phi$ be a Gaussian gauge-covariant channel, and let $f$ be a concave function on $[0, 1]$, such that $f(0) = 0$, then

$$\operatorname{Tr} f(\Phi[\rho]) \geq \operatorname{Tr} f(\Phi[|\zeta\rangle\langle\zeta|]) = \operatorname{Tr} f(\Phi[|0\rangle\langle 0|]).$$

(11)

for all states $\rho$ and any coherent state $|\zeta\rangle\langle\zeta|$ (the value on the right is the same for all coherent states by the unitary covariance property of a Gaussian channel (6)).
(ii) If $f$ is strictly concave, and the channel $\Phi$ satisfies one of the two conditions:

a) $K$ is invertible and

$$\mu > \frac{1}{2} (KK^*-I);$$  \hspace{1cm} (12)

b) $KK^* > I$ and $\mu = \frac{1}{2} (KK^*-I)$ (hence $\Phi$ is a quantum-limited amplifier),

then the equality in (11) is attained only when $\rho$ is a coherent state.

In the case of one mode such a result was obtained in [11]. Our goal here is to generalize it to the case of many modes, in particular by making precise the conditions in the statement (ii).

**Proof.** (i) By concavity it is sufficient to prove (11) for pure states $\rho = |\psi\rangle\langle\psi|$. As shown in [3] (see also Appendix 1, Proposition 5), any gauge-covariant channel can be represented as concatenation $\Phi = \Phi_2 \circ \Phi_1$ of quantum-limited attenuator $\Phi_1$ with operator $K_1$ and quantum-limited amplifier $\Phi_2$ with operator $K_2$. Then an argument similar to [11] shows that it is sufficient to prove (11) only for the amplifier $\Phi_2$. Indeed, assume that for any state vector $|\psi\rangle$

$$\text{Tr} f(\Phi_2 |\psi\rangle\langle\psi|) \geq \text{Tr} f(\Phi_2 |0\rangle\langle0|).$$   \hspace{1cm} (13)

Consider the spectral decomposition $\Phi_1 |\psi\rangle\langle\psi| = \sum_j p_j |\phi_j\rangle\langle\phi_j|$, where $p_j > 0$, then

$$\text{Tr} f(\Phi |\psi\rangle\langle\psi|) = \text{Tr} f(\Phi_2 [\Phi_1 |\psi\rangle\langle\psi|])$$  \hspace{1cm} (14)

$$\geq \sum_j p_j \text{Tr} f(\Phi_2 |\phi_j\rangle\langle\phi_j|)$$  \hspace{1cm} (15)

$$\geq \text{Tr} f(\Phi_2 |0\rangle\langle0|)$$  \hspace{1cm} (16)

$$= \text{Tr} f(\Phi_2 [\Phi_1 |0\rangle\langle0|]) = \text{Tr} f(\Phi [0\rangle\langle0|]),$$  \hspace{1cm} (17)

because vacuum is an invariant state of a quantum-limited attenuator.

Let us now prove (13). Since

$$\min_{\rho} \text{Tr} f(\Phi_2[\rho]) = \min_{\rho} \text{Tr} f(U_B \Phi_1 U_A^* \rho U_A U_B^*) = \min_{\rho} \text{Tr} f(\Phi_1[\rho]),$$

1 For Hermitian matrices $M, N$, the strict inequality $M > N$ means that $M - N$ is positive definite.
it is sufficient to consider the diagonal amplifier. The proof for one-mode quantum-limited amplifier is based on the fact that the complementary channel has the representation (also based on Proposition 5 [3])

\[ \tilde{\Phi}_2 = T \circ \Phi_2 \circ \Phi'_1, \]  

(18)

where T is transposition defined by the relation \( T[D(z)] = D(-\bar{z}) \) (\( \bar{z} \) is the complex conjugate vector), and \( \Phi'_1 \) is another quantum-limited attenuator defined by the operator \( \tilde{K}_2 = \sqrt{I - K_2^{-2}} \). But for a diagonal multimode amplifier the expression for the complementary channel and also representation (18) (with diagonal \( \Phi'_1 \)) follows from the results for each mode.

The representation (18) implies that nonzero spectra of the density operators \( \Phi_2[\rho] \) and \( \Phi_2 \circ \Phi'_1[\rho] \) coincide for pure inputs \( \rho = |\psi\rangle\langle \psi| \) [3]. Then similarly to (14)-(15)

\[
\text{Tr}_f(\Phi_2[|\psi\rangle\langle \psi|]) = \text{Tr}_f(\Phi_2[\Phi'_1[|\psi\rangle\langle \psi|]]) \\
\geq \sum_j p'_j \text{Tr}_f(\Phi_2[|\phi'_j\rangle\langle \phi'_j|]),
\]

(19)

where

\[
\Phi'_1[|\psi\rangle\langle \psi|] = \sum_j p'_j |\phi'_j\rangle\langle \phi'_j|, \quad p'_j > 0,
\]

(20)

is the spectral decomposition of the output of the quantum-limited attenuator \( \Phi'_1 \). Assume for a moment that \( f \) is strictly concave, then one arrives to the conclusion that for any pure minimizer \( \rho = |\psi\rangle\langle \psi| \) of \( \text{Tr}_f(\Phi_2[|\psi\rangle\langle \psi|]) \) the sum (20) necessarily contains only one term, i.e.

\[
\Phi'_1[|\psi\rangle\langle \psi|] = |\phi'\rangle\langle \phi'|.
\]

(21)

Indeed, otherwise by the strict concavity the inequality in (19) is strict, contradicting the assumption that \( |\psi\rangle\langle \psi| \) is a minimizer of \( \text{Tr}_f(\Phi_2[|\psi\rangle\langle \psi|]) \) (strict concavity of \( f \) also excludes non-pure minimizers). Next, we first consider the amplifier with \( K_2 > I \), then the associated attenuator \( \Phi'_1 \) is defined by the operator \( \tilde{K}_2 = \sqrt{I - K_2^{-2}} \), such that \( 0 < \tilde{K}_2 < I \). We then apply the following

**Lemma 2** Let \( \Phi'_1 \) be a diagonal quantum-limited attenuator defined by the operator \( \tilde{K}_2 \), such that \( 0 < \tilde{K}_2 < I \). Then (21) implies that \( |\psi\rangle\langle \psi| \) is a coherent state.
For one mode, this is Lemma 2 from \[11\] which implies that any pure input \(\rho\), such that \(\Phi'_{1}[\rho]\) is also a pure state, is a coherent state. The proof is based on the explicit expression for the complementary channel \(\tilde{\Phi}'_{1}\). By using this expression for each mode, one can generalize the proof to the case of the diagonal multimode channel \(\Phi'_{1}\).

This proves (13) for strictly concave \(f\) and for the amplifiers \(\Phi_{2}\) with \(K_{2} > I\). An arbitrary concave \(f\) can then be monotonically approximated by strictly concave functions by setting \(f_{\varepsilon}(x) = f(x) - \varepsilon x^{2}\), and passing to the limit \(\varepsilon \downarrow 0\) in (13).

In the case of the diagonal amplifier \(\Phi_{2}\) with \(K_{2} \geq I\), we take any sequence of diagonal operators \(K_{(n)} \uparrow I\), \(K_{(n)} \rightarrow K_{2}\), and consider the corresponding diagonal amplifiers \(\Phi_{2}^{(n)}\). Then \(\|\Phi_{2}^{(n)}[\rho] - \Phi_{2}[\rho]\|_{1} \rightarrow 0\) and \(\text{Tr}f(\Phi_{2}^{(n)}[\rho]) \rightarrow \text{Tr}f(\Phi_{2}[\rho])\) for any concave polygonal function \(f\) on \([0, 1]\), such that \(f(0) = 0\). This follows from the fact that any such function is Lipschitz, \(|f(x) - f(y)| \leq \kappa|x - y|\), hence \(\|\text{Tr}f(\Phi_{2}^{(n)}[\rho]) - \text{Tr}f(\Phi_{2}[\rho])\|_{1} \leq \kappa\|\Phi_{2}^{(n)}[\rho] - \Phi_{2}[\rho]\|_{1}\). This implies that (13) holds for polygonal concave functions \(f\) and all quantum-limited amplifiers, hence by (16) the inequality (11) with such \(f\) holds for all Gaussian gauge-covariant channels. For arbitrary concave \(f\) on \([0, 1]\) there is a monotonously nondecreasing sequence of concave polygonal functions \(f_{m}\) converging to \(f\) pointwise. Passing to the limit \(m \rightarrow \infty\) gives the first statement.

(ii) a) Notice that the conditions on the channel \(\Phi\) imply that in the decomposition \(\Phi = \Phi_{2} \circ \Phi_{1}\) the attenuator \(\Phi_{1}\) is defined by the operator \(K_{1}\) such that \(0 < K_{1}^{*}K_{1} < I\) (see Appendix 1, Remark 6). Applying the argument involving the relations (19) with strictly concave \(f\) to the relations (14)-(17), we obtain that for any pure minimizer \(\rho = |\psi\rangle\langle\psi|\) of \(\text{Tr}f(\Phi[|\psi\rangle\langle\psi|])\) the output of the quantum-limited attenuator \(\Phi_{1}[|\psi\rangle\langle\psi|]\) is necessarily a pure state. Applying Lemma 2 to the attenuator \(\Phi_{1}\) we conclude that \(|\psi\rangle\langle\psi|\) is necessarily a coherent state.

b) In this case we just apply the argument involving the relations (19) with strictly concave \(f\) to the quantum-limited attenuator \(\Phi = \Phi_{2}\). ■

Theorem 1 can be extended to Gaussian gauge-contravariant channel satisfying \(\Phi[U_{\phi}\rho U_{\phi}^{*}] = U_{\phi}^{*}\Phi[\rho]U_{\phi}\) instead of (1). The proof follows from the fact that the complementary \(\tilde{\Phi}_{2}\) of the diagonal quantum-limited amplifier \(\Phi_{2}\) is just the diagonal quantum-limited gauge-contravariant channel (see 3 for detail).
3 Implications for the additivity

For any $p > 1$ the output purity of a channel $\Phi$ is defined as
\[ \nu_p(\Phi) = \sup_{\rho \in \mathcal{S}(\mathcal{H})} \text{Tr}[\rho]^p. \]

**Corollary 3** For any Gaussian gauge-covariant channel $\Phi$ the output purity is equal to $\nu_p(\Phi) = \text{Tr}[|0\rangle\langle 0|^p]$. The multiplicativity property
\[ \nu_p(\Phi \otimes \Psi) = \nu_p(\Phi)\nu_p(\Psi) \] (22)
holds for any two Gaussian gauge-covariant channels $\Phi$ and $\Psi$.

**Proof.** The first statement follows from Proposition [1] by taking $f(x) = -x^p$, so that $\nu_p(\Phi) = -\min_\rho \text{Tr}(\Phi[\rho])$. The second statement then follows from the fact that the channel $\Phi \otimes \Psi$ is also gauge-covariant and from multiplicativity of the vacuum state. □

The output purity for the channel (5) can be explicitly computed as
\[ \nu_p(\Phi) = \det \left( \left[ (\mu + KK^*/2 + I/2)^p - (\mu + KK^*/2 - I/2)^p \right] \right). \]
The formula follows from the fact that the state $\Phi[|0\rangle\langle 0|]$ is Gaussian with the covariance matrix $\mu + KK^*/2$ and from the expression for the spectrum of a Gaussian density operator [7].

The minimal output Rényi entropy of a channel $\Phi$ is expressed via its output purity as follows
\[ \tilde{R}_p(\Phi) = \frac{1}{1-p} \log \nu_p(\Phi) \]
and multiplicativity property (22) can be rewritten as the additivity of the minimal output Rényi entropy
\[ \tilde{R}_p(\Phi \otimes \Psi) = \tilde{R}_p(\Phi) + \tilde{R}_p(\Psi). \] (23)

In the limit $p \downarrow 1$ (or taking $f(x) = -x \log x$) we recover the additivity of the minimal output von Neumann entropy established in [3].

\[ \min_{\rho_{12}} H((\Phi \otimes \Psi)[\rho_{12}]) = \min_{\rho_1} H(\Phi[\rho_1]) + \min_{\rho_2} H(\Phi[\rho_2]). \]

The additivity result in [3] is more general in that it allows the case where one of the channels is gauge-covariant, while the other is contravariant. On the other hand, the proof in [3] is restricted to states with finite second moments, while the present one does not require this.
4 Majorization for quantum-classical Gaussian channel

It is helpful to consider the map $\rho \rightarrow \langle z | \rho | z \rangle$ as a “quantum-classical Gaussian channel” which transforms Gaussian density operators into Gaussian probability densities. We will consider a more general transformation:

$$\rho \rightarrow p_\rho(z) = \text{Tr}\rho D(z)\rho_0 D(z)^*,$$

where $D(z)$ are the displacement operators, $\rho_0$ is the Gaussian gauge-invariant state with the quantum characteristic function $\phi_0(z) = \exp(-z^*\alpha_0 z)$, where $\alpha_0 \geq \frac{I}{2}$. Notice that $p_\rho(z) = \langle z | \rho | z \rangle$ if $\rho_0$ is the vacuum state corresponding to $\alpha_0 = \frac{I}{2}$.

The function $p_\rho(z)$ is bounded by 1 and is a continuous probability density, the normalization follows from the resolution of the identity

$$\int_{C^*} D(z)\rho_0 D(z) \frac{d^2 z}{\pi^s} = I_H.$$ 

Proposition 4 Let $f$ be a concave function on $[0, 1]$, such that $f(0) = 0$, then for arbitrary state $\rho$

$$\int_{C^*} f(p_\rho(z)) \frac{d^2 z}{\pi^s} \geq \int_{C^*} f(p_\rho(|\zeta\rangle\langle\zeta|)) \frac{d^2 z}{\pi^s}. \quad (24)$$

Proof. For any $c > 0$ consider “measure-reprepare” channel $\Phi_c$ defined by the relation

$$\Phi_c[\rho] = \int \frac{d^2 z}{\pi^s c^2} \text{Tr}[\rho D(c^{-1}z)\rho_0 D^*(c^{-1}z)] D(z)\rho'_0 D^*(z), \quad (25)$$

where $c > 0$ is a positive constant while $\rho'_0$ is another gauge-invariant Gaussian state with the characteristic function $\phi'_0(z) = \exp(-z^*\alpha'_0 z)$. The map (25) is a gauge-covariant bosonic Gaussian channel which in the Heisenberg representation acts on $D(z)$ as

$$\Phi_c^*[D(z)] = D(cz) \exp[-z^*(\alpha'_0 + c^2\alpha_0)z],$$

cf. [3]. Therefore by Theorem 1

$$\text{Tr}f(\Phi_c[\rho]) \geq \text{Tr}f(\Phi_c[|\zeta\rangle\langle\zeta|]) \quad (26)$$
for all states $\rho$ and any coherent state $|\zeta\rangle\langle\zeta|$. We will prove the Proposition by taking the limit $c \to \infty$.

In the proof we also use a simple generalization of the Berezin-Lieb inequalities \cite{1}:

$$\int_{C^s} f(p(z)) \frac{d^{2s}z}{\pi^s} \leq \operatorname{Tr} f(\sigma) \leq \int_{C^s} f(\bar{p}(z)) \frac{d^{2s}z}{\pi^s}, \quad (27)$$

valid for any quantum state admitting the representation

$$\sigma = \int_{C^s} p(z) D(z) \rho^0_D (z)^* \frac{d^{2s}z}{\pi^s}$$

with a probability density $p(z)$. In the right side of (27)

$$\bar{p}(z) = \operatorname{Tr} \sigma D(z) \rho^0_D (z).$$

The original inequalities refer to the case where $\rho^0$ is a pure state, but the proof applies to the more general case (see Appendix 2). In the inequalities (27) one has to assume that $f$ is defined on $[0, \infty)$ (in fact, $p(z)$ can be unbounded). We shall assume this for a while.

Taking $\sigma = \Phi_c[\rho]$, from (25) we have

$$p(z) = \frac{1}{c^{2s}} \operatorname{Tr} \rho D(c^{-1}z) \rho^0_D (c^{-1}z) = \frac{1}{c^{2s}} p_p(c^{-1}z).$$

while

$$\bar{p}(z) = \operatorname{Tr} \Phi_c[\rho] D(z) \rho^0_D (z)^* = \int_{C^s} p(w) \operatorname{Tr} \rho^0_D (z-w) \rho^0_D (z-w)^* \frac{d^{2s}w}{\pi^s}. \quad (28)$$

By using the quantum Parceval formula \cite{5}, we obtain

$$\pi^{-s} \operatorname{Tr} \rho^0_D (z) \rho^0_D (z)^* = \int_{C^s} \phi^0_0 (w)^2 \exp(2i \operatorname{Im} z^* w) \frac{d^{2s}w}{\pi^{2s}}$$

$$= \pi^{-s} \operatorname{det}(2\alpha^0_0)^{-1} \exp \left( -\frac{1}{2} z^* [\alpha^0_0]^{-1} z \right) \equiv q_{\alpha^0_0}(z)$$

– the probability density of a normal distribution. Substituting this into (28), we have

$$\bar{p}(z) = \int d^{2s}w \frac{p(w)}{c^{2s}} q_{\alpha^0_0}(z-w)$$

$$= \int d^{2s}w \frac{p(w')}{c^{2s}} q_{\alpha^0_0}(z-cw')$$

$$= \frac{1}{c^{2s}} \bar{p}_p \ast q_{\alpha^0_0/c^2}(c^{-1}z). \quad (29)$$
Here \( q_{0/c^2}(z) = c^{2s} q_{0/c^2}(cz) \) is the probability density of a normal distribution tending to \( \delta \)-function when \( c \to \infty \).

With the change of the integration variable \( c^{-1} z \to z \), the inequalities (27) become

\[
\int_{C_s} f(c^{-2s} p(z)) \frac{d^{2s} z}{\pi^s} \leq c^{-2s} \text{Tr} f(\Phi_c[\rho]) \leq \int_{C_s} f(c^{-2s} p * q_{0/c^2}(z)) \frac{d^{2s} z}{\pi^s},
\]

Substituting \( \rho = |\zeta\rangle \langle \zeta| \), we have

\[
\int_{C_s} f(c^{-2s} p_{|\zeta\rangle \langle \zeta|}(z)) \frac{d^{2s} z}{\pi^s} \leq c^{-2s} \text{Tr} f(\Phi_c[|\zeta\rangle \langle \zeta|]) \leq \int_{C_s} f(c^{-2s} p_{|\zeta\rangle \langle \zeta|} * q_{0/c^2}(z)) \frac{d^{2s} z}{\pi^s}.
\]

Combining the last two displayed formulas with (26) we obtain

\[
\int_{C_s} g(p(z)) \frac{d^{2s} z}{\pi^s} - \int_{C_s} g(p_{|\zeta\rangle \langle \zeta|}(z)) \frac{d^{2s} z}{\pi^s} \geq \int_{C_s} g(p(z)) \frac{d^{2s} z}{\pi^s} - \int_{C_s} g(p * q_{0/c^2}(z)) \frac{d^{2s} z}{\pi^s},
\]

where we denoted \( g(x) = f(c^{-2s} x) \), which is again a concave function. Moreover, arbitrary concave polygonal function \( g \) on \([0, 1]\), satisfying \( g(0) = 0 \), can be obtained in this way by defining

\[
f(x) = \begin{cases} 
  g(c^{2s} x), & x \in [0, c^{-2s}] \\
  g(1) + g'(1)(x - c^{-2s}), & x \in [c^{-2s}, \infty)
\end{cases}
\]

hence (30) holds for any such function. Then the right hand side of the inequality (30) tends to zero as \( c \to \infty \). Indeed, for polygonal function \( |g(x) - g(y)| \leq \kappa |x - y| \), and the asserted convergence follows from the convergence \( p * q_{0/c^2} \to p \) in \( L_1 \) : if \( p(z) \) is a bounded continuous probability density, then

\[
\lim_{c \to \infty} \int_{C_s} |p * q_{0/c^2}(z) - p(z)| d^{2s} z = 0.
\]

Thus we obtain (24) for the concave polygonal functions \( f \). But for arbitrary continuous concave \( f \) on \([0, 1]\) there is a monotonously nondecreasing sequence of concave polygonal functions \( f_n \) converging to \( f \). Applying Beppo-Levy’s theorem, we obtain the statement.
5 Appendix

1. The concatenation $\Phi = \Phi_2 \circ \Phi_1$ of two channels $\Phi_1$ and $\Phi_2$ obeys the rule

$$K = K_2 K_1,$$
$$\mu = K_2 \mu_1 K_2^* + \mu_2.$$

Proposition 5 [3] Any bosonic Gaussian gauge-covariant channel $\Phi$ is a concatenation of quantum-limited attenuator $\Phi_1$ and quantum-limited amplifier $\Phi_2$.

Proof. By inserting

$$\mu_1 = \frac{1}{2} (I - K_1 K_1^*) = \frac{1}{2} (I - |K_1^*|^2), \quad \mu_2 = \frac{1}{2} (K_2 K_2^* - I) = \frac{1}{2} (|K_2^*|^2 - I)$$

into (32) and using (31) we obtain

$$|K_2^*|^2 = K_2 K_2^* = \mu + \frac{1}{2} (KK^* + I) \geq \begin{cases} I \\ KK^* \end{cases}$$

from the inequality (33). By using operator monotonicity of the square root, we have

$$|K_2^*| \geq I, \quad |K_2^*| \geq |K^*|.$$  

The first inequality (33) implies that choosing

$$K_2 = |K_2^*| = \sqrt{\mu + \frac{1}{2} (KK^* + I)}$$

and the corresponding $\mu_2 = \frac{1}{2} (|K_2^*|^2 - I)$, we obtain diagonalizable quantum-limited amplifier, since $K_2$ and $\mu_2$ are commuting Hermitian operators.

Then with

$$K_1 = |K_2^*|^{-1} K$$

we obtain, taking into account the second inequality in (33)

$$K_1^* K_1 = K^* |K_2^*|^{-2} K = K^* \left[ \mu + \frac{1}{2} (KK^* + I) \right]^{-1} K \leq K^* (KK^*)^{-1} K \leq I,$$

where $-$ means generalized inverse, which implies $K_1^* K_1 \leq I$, hence $K_1$ with the corresponding $\mu_1 = \frac{1}{2} (I - K_1 K_1^*)$ give the quantum-limited attenuator.  

\[ \blacksquare \]
Remark 6 The inequality (12) via (36) implies $K_1^*K_1 < I$. Invertibility of $K$ implies $K_1^*K_1 > 0$.

2. For completeness we sketch the proof of the required generalization of Berezin-Lieb inequalities. Let $\mathcal{X}$ be a measurable space with $\sigma$–finite measure $\mu$, and $P(x)$ a weakly measurable function on $\mathcal{X}$ whose values are density operators in a separable Hilbert space $\mathcal{H}$ such that
\[ \int_{\mathcal{X}} P(x) \mu(dx) = I_{\mathcal{H}}, \]
where the integral converges in the sense of weak operator topology. Let $\rho$ be a density operator in $\mathcal{H}$ admitting representation $\rho = \int_{\mathcal{X}} p(x)P(x)\mu(dx)$, where $p(x)$ is a bounded probability density. Denote $\bar{p}(x) = \text{Tr}\rho P(x)$, which is a probability density uniformly bounded by 1. Then for a concave function $f$ defined on $[0, \infty)$ and satisfying $f(0) = 0$
\[ \int_{\mathcal{X}} f(p(x)) \mu(dx) \leq \text{Tr} f(\rho) \leq \int_{\mathcal{X}} f(\bar{p}(x)) \mu(dx). \quad (37) \]

Put $k = \max \{1, \sup_x p(x)\}$ and consider restriction of $f$ to $[0, k]$. Then there is a monotonously nondecreasing sequence of concave polygonal functions $f_n$ converging to $f$ pointwise on $[0, k]$ and satisfying $f_n(0) = 0$. Since $|f_n(x)| \leq k_n |x|$, the integrals and the trace in (37) with $f$ replaced by $f_n$ are finite for all $n$. Let us prove (37) for concave polygonal functions $f_n$ and then take the limit $n \to \infty$. This will also show that the integrals and trace in (37) are well defined although may take the value $+\infty$.

The second inequality follows from $\text{Tr} f(\rho) P(x) \leq f(\text{Tr} \rho P(x))$ which is a consequence of Jensen inequality applied along with the spectral decomposition of $\rho$. To prove the first inequality consider the positive operator-valued measure $M(B) = \int_B P(x) \mu(dx)$, $B \subseteq \mathcal{X}$, and its Naimark dilation to a projection-valued measure $\{E(B)\}$ in a larger Hilbert space $\tilde{\mathcal{H}} \supseteq \mathcal{H}$. Consider the bounded operator $R = \int_{\mathcal{X}} p(x)E(dx)$ in $\tilde{\mathcal{H}}$, then $f(R) = \int_{\mathcal{X}} f(p(x))E(dx)$ and
\[ \rho = PRP, \quad Pf(R)P = \int_{\mathcal{X}} f(p(x)) \mu(dx), \]
where $P$ is projection from $\tilde{\mathcal{H}}$ onto $\mathcal{H}$. The required inequality then follows from the more general fact $\text{Tr} Pf(R)P \leq \text{Tr} f(PP)$ [2].

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