1. Introduction

In this paper we investigate how germs of real functions can change under deformation. In particular we look at deformations of germs of isolated singularities from $\mathbb{R}^n$ to $\mathbb{R}^k$ ($n \geq k$) and the relation with there natural stratification in some tame categorie (algebraic, analytic, semi-algebraic, subanalytic, $o$-minimal structure polynomially bounded). The word tame in this paper will refer to one of these categories.

We say that two germs are topologically equivalent $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^k, 0)$ and $g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^k, 0)$ are topologically equivalents if there exists a germ of homeomorphism $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ such that $g \circ h = f$ the topological type of a germ is its right equivalence class.

A family of germs is the germ at $\{0\} \times \mathbb{R}^p$ of some function $F : (\mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^p \times \{0\}) \rightarrow (\mathbb{R}^k, 0)$.

We shall usually denote a family of germs by $f_t : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^k, 0)$, $t \in \mathbb{R}^p$, where $f_t(x) = F(x, t)$.

A stratification of a set, e.g., an algebraic variety, is roughly speaking, a partition of it into smooth manifolds so that these manifolds fit together with respect to some regularity condition.

We are interested in regularity condition that insure topological triviality, which in consequence implies the constancy of the topological type.\textsuperscript{e}

Stratification theory was introduced by R. Thom and H. Whitney for algebraic and analytic sets.

(see [GM] and [PW] for some examples of applications of stratification theory).

We consider in the paper the classical Whitney’s conditions:

**Definition 1.1.** Let $X, Y$ be disjoint manifolds in $\mathbb{R}^m$, and let $y \in Y \cap \overline{X}$. A triple $(X, Y, y)$ is called $(a)$ (resp. $(b)$)-regular if

(a) when a sequence $\{x_n\} \subset X$ tends to $y$ and $T_{x_n}X$ tends in the Grassmanian bundle to a subspace $\tau_y$ of $\mathbb{R}^m$, then $T_yY \subset \tau_y$;

(b) when sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ each tends to $y$, the unit vector $\frac{(y_n - x_n)}{|y_n - x_n|}$ tends to a vector $v$, and $T_{x_n}X$ tends to $\tau_y$, then $v \in \tau_y$.\textsuperscript{1}
X is called (a) (resp. (b))-regular over Y if each triple (X, Y, y) is (a)(resp. (b))-regular.

**Definition 1.2.** A stratification of a subset V in \( \mathbb{R}^m \) is a disjoint decomposition \( V = \bigsqcup_{i \in I} V_i \), \( V_i \cap V_j = \emptyset \) for \( i \neq j \) into smooth submanifolds \( \{ V_i \}_{i \in I} \), called strata, is called an (a)(resp. (b))-regular stratification if

1. each point has a neighborhood intersecting only finitely many strata;
2. the frontier \( V_j \setminus V_j \) of each stratum \( V_j \) is a union of other strata \( \bigsqcup_{i \in J(i), i} V_i \);
3. any triple \( (V_j, V_i, x) \) such that \( x \in V_i \subset V_j \) is (a)(resp. (b))-regular.

**Theorem 1.3.** For any tame set \( V \) in \( \mathbb{R}^m \) there is an (a)(resp. b)-regular stratification.

For the proof of this theorem i.e. the existence of stratifications, in various tame categories, see [Wh], [Th], [Lo], [Hi], [BCR], [DM].

For a family of germs of isolated singularities \( F : (\mathbb{R}^n \times \mathbb{R}^p, \{0\} \times \mathbb{R}^p) \to (\mathbb{R}, 0) \), we associate the canonical stratification of \( \mathbb{R}^n \times \mathbb{R}^p \) given by the partition

\[
\{ \mathbb{R}^n \times \mathbb{R}^p - F^{-1}(0), F^{-1}(0) - \{0\} \times \mathbb{R}^p, \{0\} \times \mathbb{R}^p \}
\]

We shall denote by \( \pi \) the projection on the second factor, \( V = F^{-1}(0), Y = \{0\} \times \mathbb{R}^p, X = V - Y \) and \( X_t = \{x \in \mathbb{R}^n | F(x, t) = 0\} \).

Since \( X_t \) has an isolated singularity at \((0, t)\) i.e. the critical set of the restriction of \( \pi \) to \( V \) is \( Y \).

Then \( X \) is a smooth manifold , and for each point \((x, t) \in X \), we have

\[
T_{(x,t)}X = \{(u, v) \in \mathbb{C}^n \times \mathbb{C} | \sum_{i=1}^{n} u_i \frac{\partial F}{\partial x_i}(x, t) + \sum_{j=1}^{p} v_j \frac{\partial F}{\partial t_j}(x, t) = 0 \} = (\mathbb{R}dF)^\perp .
\]

we use the following notation \( dF = \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial t_1}, \ldots, \frac{\partial F}{\partial t_p} \right) \), \( d_x F = \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right) \) and \( \|d_x F\|^2 = \sum_{i=1}^{n} \|\frac{\partial F}{\partial x_i}\|^2 \).

For the canonical stratification associated to a family of germs of isolated singularities, to be (a) (resp. (b)) regular, can be made more practical by the following form:

**Definition 1.4.** We say that \( F \) is Whitney regular at 0 if its canonical stratification is Whitney regular and this is equivalent to the following conditions are satisfied:

**condition (a):**

\[
\lim_{(x,t) \to (0,0)} \left( \frac{\frac{\partial F}{\partial x_j}(x, t)}{\|d_x F(x, t)\|} \right) = 0 \quad \text{for each } 1 \leq j \leq p.
\]

**condition (b'):**
\[
\lim_{(x,t) \to 0} \left( \frac{\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i}(x,t)}{\|x\|\|d_x F(x,t)\|} \right) = 0.
\]

Remark 1.5. (1) It is known that condition \(a + b'\) is equivalent to condition \(b\). (see [Th, Ma])

(2) A Whitney regular family of function germs is topologically trivial, then the topological type is constant in such family. (see [Th, Ma])

2. Uniform radius and vanishing folds

A family of germs is said to have no coalescing of critical points (in the sense of H.King [K]) if there exists a neighbourhood \(U\) of \(\{0\} \times \mathbb{R}^p\) in \(\mathbb{R}^n \times \mathbb{R}^p\), such that the restriction of \(f_t\) to \(U \cap (\mathbb{R}^n \setminus \{0\} \times \{t\})\) has no critical point (i.e. submersion) for each \(t \in \mathbb{R}^p\).

For example the family \(f_t(x) = x^3 - 3tx^2\) has a coalesing, and \(f_0\) is topologically equivalent to the identity, but for \(t \neq 0\), \(f_t\) has a maximum or a minimum at 0 and then fails to be topologically equivalent to the identity.

For a germ of isolated singularity \(f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)\) we denote by \(\mu(f_t) = \dim \mathcal{O}_{n}(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)\) its the Milnor number.

We say that a family of isolated singularities is \(\mu\)-constant family if \(\mu(f_t) = \mu(f_0)\) for each \(t \in \mathbb{R}^p\).

Remark 2.1. For complex analytic germs no coalescing of critical points is equivalent to the family is \(\mu\)-constant.

Definition 2.2. Let \(f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)\) be a tame germ with isolated singularity. Let \(\rho : (\mathbb{R}^n, 0) \to \mathbb{R}^+\) be a germ of a tame submersion such that \(\rho^{-1}(0) = \{0\}\).

The \(\rho\)-Milnor radius of \(f, \rho(f)\), is the smallest critical value of the restriction of \(\rho\) to the smooth variety \(f^{-1}(0) - \{0\}\) i.e.
\[
\rho(f) = \inf \{\rho(x) | x \in f^{-1}(0) - \{0\} \text{ and } d_x f = \lambda d_x \rho, \text{ for some } \lambda \in \mathbb{R} \}.
\]

If there are no critical values then \(\rho(f) = \infty\).

To extend this notion to tame maps \(f : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)\) we use the following notation.

Definition 2.3. Let \(f : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)\) be a tame germ with isolated singularity. Let \(\rho : (\mathbb{R}^n, 0) \to \mathbb{R}^+\) be a germ of a tame submersion such that \(\rho^{-1}(0) = \{0\}\).

The \(\rho\)-Milnor radius of \(f, \rho(f)\), is the smallest critical value of the restriction of \(\rho\) to the smooth variety \(f^{-1}(0) - \{0\}\) i.e.
\[
\rho(f) = \inf \{\rho(x) | x \in f^{-1}(0) - \{0\} \text{ and } \sum_{1 \leq i_1 < \ldots < i_{m+1} \leq n} \left| \frac{D(f_1, \ldots, f_k, \rho)}{D(x_{i_1}, \ldots, x_{i_{m+1}})}(x) \right|^2 = 0 \}.
\]

If there are no critical values then \(\rho(f) = \infty\).
Definition 2.4 (uniform Milnor radius). Let \( \{f_t\} \), \( t \in \mathbb{R}^p \) be a family of germs at \( \{0\} \) of isolated singularity.

We say that \( \{f_t\} \) has uniform \( \rho \)-Milnor radius if there is an \( \epsilon > 0 \) such that \( \rho(f_t) > \epsilon \) for all \( t \in \mathbb{R}^p \).

We call such function \( \rho \) a control function.

We say that a point \( p \in f^{-1}(0) \) is a \( \rho \)-Kink (or simply a Kink) of \( f^{-1}(0) \) if \( p \) is non-singular point of \( f \) and if \( p \) is a critical point of \( \rho \) restricted to the manifold of smooth points of \( f^{-1}(0) \).

Remark 2.5. For \( k = 1 \), an easy computation shows that a nonsingular \( p \in f^{-1}(0) \) is a kink if and only if \( df(p) = \lambda dx \rho(p) \) for some \( \lambda \) in \( \mathbb{R} - \{0\} \).

We suppose that for every \( t \in \mathbb{R}^p \), \( f_t(0) = 0 \) and 0 is an isolated critical point of \( f_t \).

Let \( \gamma : [0, \epsilon] \to \mathbb{R}^n \times [0, 1] \) be a real analytic path \( \gamma(s) = (x(s), t(s)) \) such that:

1) \( \gamma(0) = (0, 0) \)
2) \( |x(s)| > 0 \) and \( |t(s)| > 0 \) for all \( 0 < s < \epsilon \), and
3) \( f(x(s), t(s)) = 0 \) for all \( 0 \leq s \leq \epsilon \).

Definition 2.6. The path \( \gamma \) will be called a \( \rho \)-vanishing fold of \( f \) (centered at 0) if \( x(s) \) is a \( \rho \)-kink of \( f_t^{-1}(0) \) for every \( s \in (0, \epsilon] \).

Proposition 2.7. \( \{f_t\} \) has a \( \rho \)-uniform radius if and only if it has no \( \rho \)-vanishing fold in \( U(\epsilon_0) = \{ x \in \mathbb{K}^n : \rho(x) < \epsilon_0 \} \) for some \( \epsilon_0 > 0 \).

Remark 2.8. In the analytic complex case we have the following theorem which relies the jump of Milnor number with the existence of vanishing folds.

We obtain a generalisation of [O]:

Theorem 2.9. \( K = \mathbb{C} \).

Let \( F : (\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0) \) a family of germs of isolated singularities and \( X_t = \{f_t^{-1}(0)\} \) the corresponding family of hypersurfaces. Let \( \mu_t \equiv \mu \) be the Milnor number of \( f_t \) at the origine and suppose that \( \mu_t = \mu \) is constant for \( 0 < t \leq 1 \) and \( \mu < \mu_0 \).

Then, the family \( \{f_t^{-1}(0)\} \) admits a vanishing fold centered at 0.

Remark 2.10. It is not difficult, to see that any family of isolated singularity of quasihomogeneous functions can not have a vanishing fold with \( \rho = \sum_{i=1}^{n} |x_i|^2 \). i.e. has \( \rho \)-Milnor uniforme radius.
In fact, if \( \{ f_t \} \) is quasihomogeneous family of type \((w_1, \ldots, w_n; D)\), the "Euler formula" gives:

\[
\sum_{i=1}^{n} w_i x_i \frac{\partial f_t}{\partial x_i} = D.f_t.
\]

And now if \( f \) has a kink at \( p = (x, t) \) then there exists \( \lambda \in \mathbb{R}^* \) such that
\[
d_x f(p) = \lambda d_x \rho(p) \quad i.e. \quad \frac{\partial f_t}{\partial x_i} = \lambda x_i
\]
Since \( p \in f^{-1}(0) \) the Euler formula gives
\[
\lambda \sum_{i=1}^{n} w_i |x_i|^2 = 0
\]
which implies \( p \in \{ 0 \} \times \mathbb{R}^p \) then it’s not in the smooth part.

We can also show that it has uniform \( \rho \)-milnor radius, for \( \rho \) a quasihomogeneous control function with respect to the system of weights, for example:
\[
\sum_{i=1}^{n} |x_i|^{2\rho}.
\]

**Remark 2.11.** Having a uniform \( \rho \)-Milnor radius for some "control function" \( \rho \) doesn’t for a family of germs is in general weaker than Whitney regular.

For example, take the Briançon-speder family \( F(x, y, z, t) = z^5 + ty^6 z + y^7 x + x^{15} \), its a family quasihomogeneous polynomials of type \((1, 2, 3; 15)\).

The Milnor number is given for an isolated singularity quasihomogeneous \( f \) of type \((w_1, \ldots, w_n; D)\) by the formula:
\[
\mu(f) = \frac{(D - w_1)(D - w_2) \ldots (D - w_n)}{w_1.w_2.\ldots.w_n}
\]
which gives in our case \( \mu(f_t) = 364 \).

The family \( f_t \) is \( \mu \)-constant but for a generic hyperplan \( H \) in \( \mathbb{R}^3 \), its equation can be written \( z = ax + by \) with \( a, b \in \mathbb{R} - \{ 0 \} \), and so the restriction family \( g_t = f_t|_H \) is the family of polynomials \( g_t(x, y) = x^5 + txy^6 + y^7(ax + by) + (ax + by)^{15} \).

Then, for \( t \neq 0 \), \( g_t \) is semiquasihomogeneous with leading term \( x^5 + txy^6 \) is of type \((3, 2; 15)\). Using the fact that \( \mu(g_t) = \mu(x^5 + txy^6) \) we obtains
\[
\mu(H \cap Y_t) = 26.
\]

But for \( t = 0 \), \( g_0(x, y) = x^5 + y^7(ax + by) + (ax + by)^{15} = x^5 + by^8 + (axy^7 + (ax + by)^{15}) \) is semiquasihomogeneous with leading term \( x^5 + by^8 \) is of type \((8, 5; 40)\) then:
\[
\mu(H \cap Y_0) = 28.
\]

Since the Milnor number jumps, this family has \( \rho \)-vanishing fold.

In fact the family of curves we obtain by the intersection with a generic hyperplan must have a vanishing folds.
In the complex, Briançon and Speder show that this family is not Whitney regular (using the fact that Whitney regular family must have constant Milnor numbers after intersection by generic hyperplan).

3. Vanishing folds and Whitney condition

Let $F$ an analytic function from $\mathbb{R}^n \times \mathbb{R}^p$ to $\mathbb{R}$, in a neighbourhood of $0$

$$F : \mathbb{R}^n \times \mathbb{R}^p, 0 \to \mathbb{R}, 0 \quad (x, t) \mapsto F(x, t)$$

$F(0, t) = 0$

We denote by $\pi$ the projection on the second factor, $V = F^{-1}(0)$, $Y = \{0\} \times \mathbb{R}$ and $X_t = \{x \in \mathbb{R}^n/F(x, t) = 0\}$.

We suppose $X_t$ has an isolated singularity at $(0, t)$ i.e. the critical set of the restriction of $\pi$ to $V$ is $Y$.

Then $X = V - Y$ is an analytic complex manifold of dimension $n$, and for each point $(x, t) \in X$ we have

$$T_{(x, t)}X = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^p / \sum_{i=1}^n u_i \frac{\partial F}{\partial x_i}(x, t) + \sum_{j=1}^p u_iv_j \frac{\partial F}{\partial t}(x, t) = 0\} = (\mathbb{R}dF)^\perp.$$  

where $dF = (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial t})$, $d_x F = (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n})$.

Let $\mathcal{G}$ be the set of analytic applications germs from $\mathbb{R}^n \times \mathbb{R}^p, 0$ to $\mathbb{R}^n \times \mathbb{R}^p, 0$ of the following type :

$\Phi(y, \tau) = (\Psi(y, \tau), \lambda(\tau)) = (x, t)$,

where $\Psi$ for small $\tau$ is a germ of automorphisms of $(\mathbb{R}^n, 0)$ (i.e. det $\left(\frac{\partial \Phi}{\partial y}\right) \neq 0$ and $\Psi(0, \tau) = 0$).

We suppose given $F : \mathbb{R}^n \times \mathbb{R}^p, 0 \to \mathbb{R}, 0$ is an analytic deformation of $f = f_0$ such that $F(0, t) = \frac{\partial F}{\partial x_1} = \ldots = \frac{\partial F}{\partial x_n} = 0$, $X = F^{-1}(0)$, $X_t = f_t^{-1}(0)$ and $Y = \{0\} \times \mathbb{R}$.

The following theorem says that Whitney regularity is equivalent to the stability of the uniform $\rho$-Milnor property with respect to families of linear change of variable in $x$.

**Theorem 3.1.** Let $F$ be a $\mu$-constant deformation. The following conditions are equivalent

(i) $F$ is Whitney regular
(ii) $\forall \Phi \in \mathcal{G}$, $F \circ \Phi$ has no vanishing fold.

**Proof:** (i) $\Rightarrow$ (ii)

We have seen that, if a deformation is Whitney regular then it’s has no vanishing folds. We have then only to show that if $F$ is Whitney then so is $F \circ \Phi$ for all $\Phi \in \mathcal{G}$.

By definition $F \circ \Phi(y, \tau) = F(\Psi(y, \tau), \lambda(\tau))$; this suggest to do it in the two following steps:
Firstly, for \( \lambda = Id_{\mathbb{R}^p}, \Phi_1(y, \tau) = (\Psi(y, \tau), \tau), \) is then an analytic diffeomorphism of \( \mathbb{R}^{n+p} \), since Whitney’s conditions are invariant by diffeomorphism (see [Ma]), if \( F \) is Whitney regular, so is \( F \circ \Phi_1 \), where \( \Phi_1(y, \tau) = (\Psi(y, \tau), \tau) \).

Secondly, if \( F \) is Whitney regular, then so is \( F \circ \Phi_2 \), where \( \Phi_2(y, \tau) = (y, \lambda(\tau)) \) and \( \lambda : \mathbb{R}^p, 0 \rightarrow \mathbb{R}^p, 0 \).

In fact, the condition \( b' \) is trivially satisfied since it does not make use of the partial derivative relatively to the parameter.

To check, the \((a)\) condition, we compute
\[
\frac{\partial F \circ \Phi_2(y, \lambda(\tau))}{\partial t_j} = \sum_{m=1}^{p} \frac{\partial \lambda_m}{\partial t_j}(\tau) \frac{\partial F}{\partial \lambda_m}(y, \lambda(\tau)),
\]

since \( F \) satisfy \((a)\) condition, we have
\[
\lim_{(y, \tau) \to 0 \atop (x, \eta) \in X \times Y} \left( \frac{\partial F \circ \Phi_2(y, \tau)}{\partial t_j} \right) = 0.
\]

Now (1) and (2) implies that for any \( \Phi \in \mathcal{G}, F \circ \Phi \) is Whitney regular, then it has no vanishing folds.

(ii) \( \Rightarrow \) (i)

Firstly, since \( F \) is a \( \mu \)-constant deformation in a neighborhood of 0, its satisfy the \((a)\) regularity condition (in fact we have more, \( \mu \)-constant implie “good stratification” in the sens of Thom, see [LS], [BS], [T]).

Let us suppose that \( b \) fails, which in turn implies \( b' \) fails, since \( a \) holds.

Let \( \Delta(z, \tau) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(z, \tau) \frac{\partial \Phi_j}{\partial \lambda}(\tau) \frac{\partial F}{\partial \lambda_m}(y, \tau) \), where \((z, \tau) \in X - Y\).

Then there exists a real analytic curve \( \gamma : [0, \varepsilon] \rightarrow X, \gamma(s) = (x(s), t(s)) \) and \( \delta_0 > 0 \) such that:

1) \( \gamma(0) = (0, 0) \)
2) \( f(z(s), \tau(s)) = 0 \) for all \( 0 \leq s \leq \varepsilon \), and
3) \( \lim_{s \to 0} \Delta(z(s), \tau(s)) = 0 \neq 0 \)

Let us dnote par \( v \) the valuation of \( O_{\mathbb{R}, 0} \) associated to \( \gamma \).

We will use the following notations:
\[
v(z) = \inf_{1 \leq i \leq n} z_i \text{ for } z \in \mathbb{R}^n, v(\frac{\partial f}{\partial x}) = \inf_{1 \leq i \leq n} v(\frac{\partial f}{\partial x_i}).
\]

In this conditions, if we denote \( v(z) = p \) and \( v(\frac{\partial f}{\partial x}) = q \), we can suppose (change the order of variables if needed) that \( v(z_1) = p \).

Since, \( \Delta \circ \gamma(s) \) has a non zero limit when \( s \) tends to 0, we may conclude that
\[
v(z_1, \gamma(s)) = v(z) + v(\frac{\partial f}{\partial x}(z, \tau)) = p + q
\]

Let us now denotes \( \gamma(s) = (p_1(s), \ldots, p_n(s), \lambda(s)), \)
\[
\frac{\partial F}{\partial x} \circ \gamma(s) = (g_1(s), \ldots, g_n(s) \text{ and } v(z, \frac{\partial F}{\partial x} > \gamma(s)) = u(s).\]

Let us now define
\[
\Phi : \mathbb{R}^n \times \mathbb{R}, 0 \rightarrow \mathbb{R}^n \times \mathbb{R}, 0 \text{ by: } \Phi(y_1, \ldots, y_n, \tau) = (\Psi(y, \tau), \lambda(\tau))
\]
with $\Psi(y, \tau) = (y_1 - \frac{p_1}{u} h, y_2 + \frac{p_2}{p_1} y_1 - \frac{p_2}{u} h, \ldots, y_n + \frac{p_n}{p_1} y_1 - \frac{p_n}{u} h)$
where $h = q_2 y_2 + q_3 y_3 + \ldots + q_n y_n$.
We may first check that, $\Phi \in \mathcal{G}$:
1) $\Phi$ is analytic.
We use for this the valuation along $\gamma$.
If $j \neq 1$, then for $y_j + \frac{p_j}{p_1} y_1 - \frac{p_j}{u} h$, we have by hypothesis $v(\frac{p_j}{p_1} y_1) \geq v(p_1) + v(y_1) - v(p_1) \geq 0$ and $v(\frac{p_j}{u} h) = v(p_j) + v(h) - v(u) \leq (p + q) - (p + q) = 0$.
If $j = 1$ for $y_1 - \frac{p_1}{u} h$, we have $v(\frac{p_1}{u} h) = v(p_1) + v(h) - v(u) = (p + q) - (p + q) = 0$. 
2) $\Psi(0, \tau) = 0$ 
3) The jacobian of $\Psi$ is invertible in a neighborhood of 0.
For this we compute the determinant of this Jacobian and show it equals 1.
Let $\Phi_1(y, \tau) = y_1 - \frac{p_1}{u} h$ and $\Phi_j(y, \tau) = y_j + \frac{p_j}{p_1} y_1 - \frac{p_j}{u} h$ for $2 \leq j \leq n$.
Then $\frac{\partial \Phi_1}{\partial y_j} = 1$ and $\forall j \geq 2$, \[\frac{\partial \Phi_j}{\partial y_j} = -\frac{p_j}{u} q_j,\]
$\forall i, j \geq 2, i \neq j$, $\frac{\partial \Phi_i}{\partial y_j} = -\frac{p_j}{u} q_i$
$\forall i \geq 2$, $\frac{\partial \Phi_i}{\partial y_i} = 1 - \frac{p_i}{u} q_i$ and $\frac{\partial \Phi_i}{\partial y_j} = \frac{p_i}{p_1}$.

\[
(3.1) \quad \det \left( \frac{\partial \Psi}{\partial y}(y, \tau) \right) = \begin{vmatrix}
1 & -\frac{p_1}{u} q_2 & \cdots & -\frac{p_1}{u} q_i & \cdots & -\frac{p_1}{u} q_n \\
\frac{p_2}{p_1} & 1 & -\frac{p_2}{u} q_2 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \cdots & 1 & -\frac{p_i}{u} q_i & \cdots \\
\vdots & \vdots & \cdots & \vdots & \ddots & \ddots \\
\frac{p_n}{p_1} & -\frac{p_n}{u} q_2 & \cdots & -\frac{p_n}{u} q_i & \cdots & 1 - \frac{p_n}{u} q_n
\end{vmatrix}
\]

If we denotes by $C_j$ the jth row, and applied the transformation $C_j = C_j + \frac{p_j}{p_1} C_1$ for $j = 1 \cdots m$
we see that

\[
(3.2) \quad \det \left( \frac{\partial \Psi}{\partial y}(y, \tau) \right) = \begin{vmatrix}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
\frac{p_2}{p_1} & 1 & \vdots & \vdots & \vdots & \vdots \\
\vdots & 0 & \ddots & 0 & \vdots & \vdots \\
\vdots & \vdots & \ddots & 1 & \vdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \ddots & 0 \\
\frac{p_n}{p_1} & 0 & \cdots & \cdots & 0 & 1
\end{vmatrix} = 1
\]
We can now conclude that $\Phi \in \mathcal{G}$. Moreover, by construction we have $\Phi(p_1(\tau), 0, \ldots, 0, \tau) = \gamma(\tau)$.

The computation gives us

$$\frac{\partial F \circ \Phi}{\partial y_i}(p_1(\tau), 0, \ldots, 0, \tau) = \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(\gamma(\tau)) \frac{\partial \Phi_j}{\partial y_i}(p_1(\tau), 0, \ldots, 0, \tau)$$

$$= \sum_{j=1}^{n} q_j(\tau) \frac{\partial \Phi_j}{\partial y_i}(p_1(\tau), 0, \ldots, 0, \tau)$$

$$= -\frac{q_1(\tau)q_i(\tau)p_1(\tau)}{u} + q_i(\tau) \left(1 - \frac{p_i(\tau)q_i(\tau)}{u}\right) + \sum_{j \neq 1, j \neq i} q_j(\tau)q_i(\tau)p_j(\tau)$$

$$= -\frac{q_1(\tau)}{u} \left(p_1(\tau)q_1(\tau) + p_i(\tau)q_i(\tau) + \sum_{j \neq 1, j \neq i} p_j(\tau)q_j(\tau)\right) + q_i(\tau)$$

$$= -\frac{q_1(\tau)}{u}. u + q_i(\tau) = 0.$$

If $i = 1$, $I = q_1(\tau) + \sum_{j=2}^{n} q_j(\tau) \frac{p_i(\tau)}{p_1(\tau)} = \frac{u}{p_1}$

Then, we obtain that $\frac{\partial F \circ \Phi}{\partial y_i}(p_1(\tau), 0, \ldots, 0, \tau) = \lambda(p_1(\tau), 0, \ldots, 0)$ with $\lambda = \frac{u}{|p_1|^2}$

This means that $F \circ \Phi$ has a vanishing fold.

**Remark 3.2.** In this proof we can replace $\mathcal{G}$ by the set

$$\mathcal{G}_i = \{ \Phi = (\Psi, \lambda) : \mathbb{R}^n \times \mathbb{R}, 0 \to \mathbb{R}^n \times \mathbb{R}, 0 \text{ such that } \Psi(\cdot, \tau) \in GL(\mathbb{R}^n) \}$$

A consequence of this theorem is, an example of $\mu$-constant deformation with a vanishing fold gives an example of non Whitney regular $\mu$-constant deformation.

3.1. **Example.** The Briançon and Speder example has vanishing folds (see [BS] [TH]).

From the theorem, Whitney faults is detected by vanishing folds. So to find a vanishing fold, it suffices to find an arc along which the Whitney regularity fails.

Let $F(x, y, z, t) = z^5 + ty^6z + y^7x + x^{15}$, then $F$ is quasihomogenous $\mu$-constant family of type $(3, 2, 1; 15)$. Then as we saw, the canonical stratification is $(a)$ regular.

$$\begin{cases}
\frac{\partial F}{\partial y} = y^7 + 15x^{14} \\
\frac{\partial F}{\partial y} = 6ty^5 + 7y^6 \\
\frac{\partial F}{\partial y} = 5z^4 + ty^6
\end{cases}$$

We shall construct an explicit analytic path $\gamma(s) = (x(s), y(s), z(s), t(s))$ containing in $V$ along which the $b'$ condition fails, that is $\Delta(x, y, z, t) = \left(\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i}(x, y, z, t)\right) / \|\text{grad}_xF(x, y, z, t)\|$ do not tends to 0 when $(x, y, z, t)$ tends to 0 along $\gamma(s)$. 

Let’s take it of the following form

\[
\begin{align*}
  x(s) &= \lambda s^5 \\
  y(s) &= \alpha s^5 \\
  z(s) &= s^8 \\
  t(s) &= -\frac{5}{\alpha} s^2
\end{align*}
\]

with \( \alpha \neq 0 \).

We must have \( F(\gamma(s)) = (1 - \frac{5}{\alpha^2} \alpha^6 + \lambda \alpha^7 + \lambda^1 s^{35}) s^{40} \equiv 0 \) i.e.

\[ G(\lambda, s) = -4 + \lambda \alpha^7 + \lambda^1 s^{35} \]

Since \( \frac{\partial G}{\partial \lambda}(\lambda, 0) = \alpha^7 \neq 0 \), by the implicit function theorem \( \lambda \) is a smooth function of \( s \).

Then we have along \( \gamma(s) \) nears \( s = 0 \):

\[
\begin{aligned}
  \frac{\partial F}{\partial x} &= y^7 + 15x^{14} = \alpha^7 s^{35} + 15\lambda s^{70} \sim \alpha^7 s^{35} \\
  \frac{\partial F}{\partial y} &= 6ty^5 + 7y^6 = \left( -\frac{30}{\alpha} + 7\alpha^6 \right) s^{35} \\
  \frac{\partial F}{\partial z} &= 5z^4 + ty^6 = 5s^{32} - \frac{5}{\alpha^5} \alpha^6 s^{32} \equiv 0
\end{aligned}
\]

The limit of orthogonal secant vectors \( \frac{\partial F}{\partial \|\| (x, y, z)} \) is \((1 : \alpha : 0)\) and the limit of normal vectors \( \frac{\partial F}{\partial \|\| (x, y, z)} \) is \((\alpha^7 : -\frac{30}{\alpha} + 7\alpha^6 : 0)\).

Then \( \Delta(\gamma(s)) \) tends to 0 if and only if \( 8\alpha^7 - 30 = 0 \). We can clearly choose \( \alpha \neq 0 \) such that \( 8\alpha^7 - 30 \neq 0 \), this means that Whitney condition fails along this curve.

Now the construction of the family analytique automorphisms in the proof gives a “control function” \( \rho \) such that the family \( \{ f_t(x, y, z) = z^5 + ty^6 + z + y^7 + x^{15} \} \) has a \( \rho \)-vanishing fold i.e. its \( \rho \)-Milnor radius is not uniform.

The control function is obtained by composing the standard disynthetic function \( |x|^2 + |y|^2 + |z|^2 \) by the family of automorphisms obtained this way.

It may be interesting to establish an analogue of theorem 3.1 for the condition \( C \) (see [B]) and the relation with uniform radii.

### 4. Tame Mappings

In this section we establish a version of Theorem 3.1 for family of germs of tame mappings (i.e. in the \( o \)-minimal category ). We will assume the reader familiar with the basic facts about \( o \)-minimal structure. The standard references are L. Van den Driess [D], L. Van den Driess and C. Miller [DM] and M. Coste [C].

Let us first recall the definition of an \( o \)-minimal structure extending the field \((\mathbb{R}, +, .)\).

**Definition 4.1.** Let \( \mathcal{S} = \bigcup_{n \in \mathbb{N}} \mathcal{S}_n \), where for each \( n \in \mathbb{N} \), \( \mathcal{S}_n \) is a family of subsets of \( \mathbb{R}^n \).

We say that the collection \( \mathcal{S} \) is an \( o \)-minimal structure on \((\mathbb{R}, +, .)\) if:
1) each $S_n$ is a boolean algebra
2) if $A \in S_n$ and $B \in S_m$, then $A \times B \in S_{n+m}$
3) let $A \in S_{n+m}$ and $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be the projection on the first $n$ coordinates, then $\pi(A) \in S_n$.
4) all algebraic subsets of $\mathbb{R}^n$ are in $S_n$
5) the elements of $S_1$ are the finite unions of points and intervals.

A subset $A$ of $\mathbb{R}^n$ which belongs to $S_n$ is called a definable set in $S$. A map $f : A \to \mathbb{R}^m$ is definable in $S$ if its graph is a definable subset of $\mathbb{R}^n \times \mathbb{R}^m$ in $S$, if in addition, it is $C^k$ for some $k \in \mathbb{N}$, we call it a $C^k$ definable map in $S$.

We call a tame map map definable in some $o$-minimal structure.

Let $S$ be an $o$-minimal structure on $(\mathbb{R}, +, \cdot)$. We recall from [DM] the following notation:

**Notation:** Let $p$ be a natural number. Let $\Phi^p_S$ denote the set of all odd, strictly increasing bijections $\phi : \mathbb{R} \to \mathbb{R}^\mathbb{C}^p$ definable in $S$ and $p$-flat at $0$ (that is $\phi^{(l)}(0) = 0$ for $l = 0, \ldots, p$).

We quote also from this paper the following lemma (Lemma C.7. page 523):

**Lemma 4.2.** Let $f : A \times \mathbb{R}^* \to \mathbb{R}$ be a definable function in $S$, $A \subset \mathbb{R}^n$. Then, for any $p \in \mathbb{N}$, there exists $\phi \in \Phi^p_S$ such that $\lim_{t \to 0} \phi(t)f(x, t) = 0$ for each $x \in A$.

A $C^k$ version of this lemma is given in the following:

**Lemma 4.3.** Let $f : U \times \mathbb{R}^* \to \mathbb{R}$ be a $C^k$ definable function in $S$, and let $U$ be an open subset of $\mathbb{R}^n$. Then, for any $p \in \mathbb{N}$, there exists $\phi \in \Phi^p_S$ such that the function

$$g(x, t) = \begin{cases} 
\phi(t)f(x, t) & \text{if } t \neq 0, \\
0 & \text{if } t = 0.
\end{cases}$$

is a $C^k$ definable function.

**Proof.** Let $h : U \times \mathbb{R}^* \to \mathbb{R}$ denote any function from the collection of partial derivatives:

$$\{D^{\alpha}_{(x,t)}f ; \ \alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \in \mathbb{N}^{n+1} \text{ and } |\alpha| \leq k\}$$

By Lemma 4.2, there exists $\theta \in \Phi^p_S$ such that $\lim_{t \to 0} \theta(t)h(x, t) = 0$ for each $x \in U$. Then $\phi := \theta^{2k+1}$ satisfies the needs. \[
\]

**Definition 4.4.** A structure $S$ on the field $(\mathbb{R}, +, \cdot)$ is polynomially bounded if for any function $f : \mathbb{R} \to \mathbb{R}$ definable in $S$, there exists $N \in \mathbb{N}$ such that

$$|f(t)| \leq t^N$$

for all sufficiently large $t$. 


Lemma 4.5. If $S$ is polynomially bounded, then for any $\phi \in \Phi_S^p$, there exists $d \in \mathbb{N}$ and real number $\epsilon > 0$ such that:

$$|\phi(t)| \geq t^d$$

for any $t \in (-\epsilon, \epsilon)$.

Proof. We take $\theta(s) := \frac{1}{\phi(s)}$ where $s = \frac{1}{t}$. Since $\theta$ is definable in a polynomially bounded structure, there exists $d \in \mathbb{N}$ and $M \in \mathbb{N}$ such that $|\theta(s)| \leq s^d$ for $|s| > M$. Therefore $|\phi(t)| \geq t^d$ for any $t \in (-\epsilon, \epsilon)$ with $\epsilon = \frac{1}{M}$. \qed

Definition 4.6. A tame map is a map definable in some polynomially bounded o-minimal structure.

Using the lemmas above, we can show the following theorem which is a tame version of the main result.

Theorem 4.7. Given a tame family of isolated singularities germs $F : (\mathbb{R}^p \times \mathbb{R}^n, \mathbb{R}^p \times \{0\}) \to (\mathbb{R}^k, 0)$. We suppose the family $F$ is a $\mu$-constant deformation of $f_0 = f$. Then the following conditions are equivalent

(i) $F$ is Whitney regular.
(ii) $\forall \Phi \in \mathcal{G}$, $F \circ \Phi$ has no vanishing fold.

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