New Modular Fixed-Point Theorem in the Variable Exponent Spaces $\ell_p(.)$

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Abstract: In this work, we prove a fixed-point theorem in the variable exponent spaces $\ell_p(.)$ when $p^− = 1$ without further conditions. This result is new and adds more information regarding the modular structure of these spaces. To be more precise, our result concerns $\rho$-nonexpansive mappings defined on convex subsets of $\ell_p(.)$ that satisfy a specific condition which we call “condition of uniform decrease”.

Keywords: electrorheological fluid; fixed point; modular vector space; Nakano; strictly convex; uniformly convex

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1. Introduction

Variable exponent spaces first appeared in a work of Orlicz in 1931 [1] (see also [2]), where he defined the following space:

$$X = \{\{x_n\} \in \mathbb{R}^N, \sum_{n=0}^{\infty} |x_n|^{p(n)} < \infty, \text{ for some } \lambda > 0 \}.$$  

They became very important because of their use in the mathematical modeling of non-Newtonian fluids [3,4]. The typical example of such fluids are electrorheological fluids, the viscosity of which exhibits dramatic and sudden changes when exposed to an electric or magnetic field. The necessity of a clear understanding of the spaces with variable integrability is reinforced by their potential applications.

The properties of this vector space have been extensively studied in [5–7]. The norm that was commonly used to investigate the geometrical properties of $X$ is the Minkowski functional associated to the modular unit ball and it is known as the Luxembourg norm. Whereas in the case of classical $\ell_p$ spaces, the natural norm is suitable for making calculations, the Luxembourg norm on $X$ is very difficult to manipulate.

In 1950, Nakano [8] introduced for the first time the notion of modular vector space (see also [9,10]). This abstract point of view has been crucial to the development of the research on geometrical and topological properties of the variable exponent spaces $\ell_p(.)$.

In this work, we will introduce a class of subsets of $\ell_p(.)$ that have some interesting geometrical properties. This will allow us to prove a new fixed-point theorem concerning $\ell_p(.)$ spaces. For the study of metric fixed-point theory, we recommend the book [9].
2. Basic Notations and Terminology

For a function \( p : \mathbb{N} \rightarrow [1, +\infty) \), define the vector space

\[
\ell_{p(.)} = \left\{ \{x_n\} \in \mathbb{R}^\mathbb{N}, \sum_{n=0}^{\infty} \frac{1}{p(n)} |\lambda x_n|^p < \infty, \text{ for some } \lambda > 0 \right\}.
\]

Nakano [8,11] introduced the concept of modular vector space.

**Proposition 1** ([6,9]). Consider the function \( \rho : \ell_{p(.)} \rightarrow [0, +\infty] \) defined by

\[
\rho(x) = \rho(\{x_n\}) = \sum_{n=0}^{\infty} \frac{1}{p(n)} |x_n|^p(n)
\]

then \( \rho \) satisfies the following properties

1. \( \rho(x) = 0 \) if and only if \( x = 0 \),
2. \( \rho(ax) = \rho(x), \text{ if } |a| = 1 \),
3. \( \rho(ax + (1-a)y) \leq a\rho(x) + (1-a)\rho(y), \forall a \in [0,1] \).

for any \( x, y \in X \). The function \( \rho \) is called a convex modular.

For any subset \( I \) of \( \mathbb{N} \), we consider the functional

\[
\rho_I(x) = \sum_{n \in I} |x_n|^p(n).
\]

If \( I = \emptyset \), we set \( \rho_I(x) = 0 \). We define on modular spaces a modular topology which is similar to the topology induced by a metric.

**Definition 1.** Consider the vector space \( \ell_{p(.)} \).

(a) We say that a sequence \( \{x_n\} \subset \ell_{p(.)} \) is \( \rho \)-convergent to \( x \in \ell_{p(.)} \) if and only if \( \rho(x_n - x) \rightarrow 0 \). The \( \rho \)-limit is unique if it exists.

(b) A sequence \( \{x_n\} \subset \ell_{p(.)} \) is called \( \rho \)-Cauchy if \( \rho(x_n - x_m) \rightarrow 0 \) as \( n, m \rightarrow +\infty \).

(c) A nonempty subset \( C \subset \ell_{p(.)} \) is called \( \rho \)-closed if for any sequence \( \{x_n\} \subset C \) which \( \rho \)-converges to \( x \) implies that \( x \in C \).

(d) A nonempty subset \( C \subset \ell_{p(.)} \) is called \( \rho \)-bounded if and only if

\[
\delta_\rho(C) = \sup \{ \rho(x - y), \ x, y \in C \} < \infty.
\]

Note that \( \rho \) satisfies the Fatou property, i.e.,

\[
\rho(x - y) \leq \liminf_{n \to +\infty} \rho(x - y_n),
\]

holds whenever \( \{y_n\} \) \( \rho \)-converges to \( y \), for any \( x, y, y_n \in \ell_{p(.)} \). Throughout, we will use the notation \( B_\rho(x, r) \) to denote the \( \rho \)-ball with radius \( r \geq 0 \) centered at \( x \in \ell_{p(.)} \) and defined as

\[
B_\rho(x, r) = \left\{ y \in \ell_{p(.)}, \ \rho(x - y) \leq r \right\}.
\]

Note that Fatou property holds if and only if the \( \rho \)-balls are \( \rho \)-closed. That is, all \( \rho \)-balls are \( \rho \)-closed in \( \ell_{p(.)} \).

**Definition 2.** Let \( C \subset \ell_{p(.)} \) be a nonempty subset. A mapping \( T : C \rightarrow C \) is called \( \rho \)-Lipschitzian if there exists a constant \( K \geq 0 \) such that

\[
\rho(T(x) - T(y)) \leq K \rho(x - y), \ \forall x, y \in C.
\]
If $K = 1$, $T$ is called $\rho$-nonexpansive. A point $x \in C$ is called a fixed point of $T$ if $T(x) = x$.

The concept of modular uniform convexity was first introduced by Nakano [11], but a weaker definition of modular uniform convexity called (UUC2) was introduced in [9] and seems to be more suitable to hold in $\ell_p$ when weaker assumptions on the exponent function $p(\cdot)$ hold. The following definition is given in terms of subsets because of the subsequent results discovered in this work.

**Definition 3** ([9]). Consider the vector space $\ell_p$. Let $C$ be a nonempty subset of $\ell_p$.
1. Let $r > 0$ and $\varepsilon > 0$. Define
   
   \[
   D_2(r, \varepsilon) = \{(x, y) \in \ell_p \times \ell_p, \rho(x) \leq r, \rho(y) \leq r, \rho\left(\frac{x - y}{2}\right) \geq \varepsilon r\}.
   \]
   If $D_2(r, \varepsilon) \cap (C \times C) \neq \emptyset$, let
   
   \[
   \delta_{2,C}(r, \varepsilon) = \inf \left\{1 - \frac{1}{r} \rho\left(\frac{x + y}{2}\right), (x, y) \in D_2(r, \varepsilon) \cap (C \times C)\right\}.
   \]
   If $D_2(r, \varepsilon) \cap (C \times C) = \emptyset$, we set $\delta_2(r, \varepsilon) = 1$. We say that $\rho$ satisfies (UUC2) on $C$ if for every $r > 0$ and $\varepsilon > 0$, we have $\delta_{2,C}(r, \varepsilon) > 0$. When $C = \ell_p$, we remark that for every $r > 0$, $D_2(r, \varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough. In this case, we will use the notation $\delta_{2,\ell_p}(r, \varepsilon) = \delta_2$.
2. We say that $\rho$ satisfies (UUC2) on $C$ if for every $s \geq 0$ and $\varepsilon > 0$, there exists $\eta_2(s, \varepsilon) > 0$ depending on $s$ and $\varepsilon$ such that
   
   \[
   \delta_{2,C}(r, \varepsilon) \geq \eta_2(s, \varepsilon) > 0 \quad \text{for} \quad r > s.
   \]
3. We say that $\rho$ is strictly convex on $C$ (in short (SC)), if for every $x, y \in C$ such that
   \[
   \rho(x) = \rho(y) \quad \text{and} \quad \rho\left(\frac{x + y}{2}\right) = \frac{\rho(x) + \rho(y)}{2}\quad \text{imply}\quad x = y.
   \]

In the study of the properties of $\ell_p$ (see [12]), the following values are very important:

\[
p^+ = \sup_{n \in \mathbb{N}} p(n) \quad \text{and} \quad p^- = \inf_{n \in \mathbb{N}} p(n).
\]

In [5], the authors proved that for $\ell_p$ with $p^- > 1$, the modular is (UUC2). This modular geometrical property allows to prove the following fixed-point result:

**Theorem 1.** Consider the vector space $\ell_p$. Assume $p^- > 1$. Let $C$ be a nonempty $\rho$-closed convex $\rho$-bounded subset of $\ell_p$. Let $T : C \rightarrow C$ be a $\rho$-nonexpansive mapping. Then $T$ has a fixed point.

In [13], the authors proved a similar fixed-point theorem in the case where $\{n \in \mathbb{N}, p(n) = 1\}$ has at most one element which is an improvement from $p^- > 1$.

Before we close this section, we recall the following lemma, of a rather technical nature, which plays a crucial role when dealing with $\ell_p$ spaces.

**Lemma 1.** The following inequalities hold:
(i) [14]. If $p \geq 2$, then
   \[
   \left|\frac{a + b}{2}\right|^p + \left|\frac{a - b}{2}\right|^p \leq \frac{1}{2} \left(|a|^p + |b|^p\right),
   \]
   for any $a, b \in \mathbb{R}$. 

(ii) [15]. If \(1 < p \leq 2\), then
\[
\left| a + b \right|^{p} + \frac{p(p - 1)}{2} \left| \frac{a - b}{|a| + |b|} \right|^{2} \left| a - b \right|^{p} \leq \frac{1}{2} \left( |a|^{p} + |b|^{p} \right),
\]
for any \(a, b \in \mathbb{R}\) such that \(|a| + |b| \neq 0\).

In this work, using a different approach, we obtain some fixed-point results when \(p^{-} = 1\) without the known conditions on the function \(p(\cdot)\).

3. Uniform Decrease Condition

First, we introduce an interesting class of subsets of \(\ell_{p(\cdot)}\), which will play an important part in our work. In particular, they enjoy similar modular geometric properties as \(\ell_{p(\cdot)}\) when \(p^{-} > 1\). Before, let us introduce the following notations:

\[
I_{a} = \{ n \in \mathbb{N}; \; p(n) \geq a \} \quad \text{and} \quad J_{a} = \mathbb{N} \setminus I_{a} = \{ n \in \mathbb{N}; \; p(n) < a \},
\]
where \(a \in [1, +\infty)\).

**Definition 4.** Consider the vector space \(\ell_{p(\cdot)}\). A nonempty subset \(C\) of \(\ell_{p(\cdot)}\) is said to satisfy the uniform decrease condition (in short \((UD)\)) if for any \(\alpha > 0\), there exists \(r > 1\) such that
\[
\sup_{x \in C} \rho_{J_{a}}(x) \leq \alpha.
\]

Obviously the condition \((UD)\) passes from a set to its subsets. Moreover, if \(p(\cdot)\) is identically equal to 1, then the only \((UD)\) subset is \(C = \{0\}\). Since this case is not interesting, we will assume throughout that \(p(\cdot)\) is not identically equal to 1. Moreover, if \(p^{-} > 1\), then any nonempty subset of \(\ell_{p(\cdot)}\) satisfies the condition \((UD)\). Indeed, let \(C\) be a nonempty subset of \(\ell_{p(\cdot)}\) and \(a > 0\). Let \(a \in (1, p^{-})\). Then \(J_{a} = \emptyset\) which implies
\[
\sup_{x \in C} \rho_{J_{a}}(x) = 0 \leq \alpha.
\]
Therefore, the condition \((UD)\) is interesting to study only when \(p^{-} = 1\) and \(p(\cdot)\) is not identically equal to 1, which will be the case throughout.

**Example 1.** Consider the function \(p(\cdot)\) defined by
\[
p(n) = 1 + \frac{1}{n + 1}, \quad n \in \mathbb{N}.
\]

Consider the subset
\[
C = \left\{ x \in \ell_{p(\cdot)}; \; |x_{n}| \leq \frac{1}{(n + 1)^{2}}, \; n \in \mathbb{N} \right\}.
\]

\(C\) is nonempty, convex and \(p\)-closed. Let us show that it satisfies the condition \((UD)\). Indeed, fix \(\alpha > 0\). Let \(N \geq 1\) be such that \(\sum_{k \geq N} \frac{1}{(k + 1)^{2}} \leq \alpha\). Set \(a = 1 + \frac{1}{N}\). We have
\[ \rho_{\ell_1}(x) = \sum_{n \in J_n} \frac{|x_n|^p(n)}{p(n)} \leq \sum_{n \geq N} \frac{|x_n|^p(n)}{p(n)} \leq \sum_{n \geq N} \frac{1}{p(n)(n+1)^2} \leq \sum_{n \geq N} \frac{1}{a(n+1)^2} \leq \alpha, \]

for all \( x \in C \), which proves our claim that \( C \) is \((UD)\).

Before we give a characterization of subsets which satisfy the condition \((UD)\), we need to introduce a new class of subsets of \(\ell_p(\cdot)\).

**Definition 5.** Consider the vector space \(\ell_p(\cdot)\) such that \(p^- = 1\) and \(p(\cdot)\) not identically equal to 1. Let \( f : (0, +\infty) \to (1, 2) \) be a nondecreasing function. Define the set \( C_f \) to be

\[ C_f = \left\{ x \in \ell_p(\cdot); \; \rho_{\ell_1}(x) \leq \alpha, \; \text{for all } \alpha > 0 \right\}. \]

Note that \( C_f \) is never empty since \( 0 \in C_f \). Some of the basic properties of \( C_f \) are given in the following lemma.

**Lemma 2.** Consider the vector space \(\ell_p(\cdot)\) such that \(p^- = 1\) and \(p(\cdot)\) not identically equal to 1. Let \( f : (0, +\infty) \to (1, 2) \) be a non-decreasing function. Then the following properties hold:

1. \( C_f \) is convex.
2. \( C_f \) is symmetrical, i.e., \(-z \in C_f \) whenever \( z \in C_f \).
3. The Fatou property implies easily that \( C_f \) is \(p\)-closed as a subset of \(\ell_p(\cdot)\) which in turn implies that \( C_f \) is \(p\)-complete.

**Proposition 2.** Consider the vector space \(\ell_p(\cdot)\) such that \(p^- = 1\) and \(p(\cdot)\) not identically equal to 1. A subset \( C \) of \(\ell_p(\cdot)\) satisfies the condition \((UD)\) if and only if there exists \( f : (0, +\infty) \to (1, 2) \) non-decreasing such that \( C \subset C_f \).

**Proof.** First, we prove that \( C_f \) satisfies the condition \((UD)\). Fix \( \alpha > 0 \). If we take \( a = f(\alpha) \), we obtain

\[ \sup_{x \in C_f} \rho_{\ell_1}(x) \leq \alpha, \]

which proves our claim. Clearly, any subset \( C \) of \( C_f \) will also satisfy the condition \((UD)\). Conversely, let \( C \) be a nonempty subset of \(\ell_p(\cdot)\) which satisfies the condition \((UD)\). For any \( \alpha > 0 \), there exists \( a > 1 \) such that \( \sup_{x \in C} \rho_{\ell_1}(x) \leq \alpha \). Set

\[ [\alpha] = \left\{ a > 1; \; \sup_{x \in C} \rho_{\ell_1}(x) \leq \alpha \right\}. \]

Define

\[ f(\alpha) = \begin{cases} 2 \sup_{[\alpha]} \left( [\alpha] \cap (1, 2) \right) & \text{if } [\alpha] \subset [2, +\infty), \vspace{1mm} \\
\infty & \text{if } [\alpha] \cap (1, 2) \neq \emptyset. \end{cases} \]

Clearly, \( f \) is well defined and \( f(\alpha) \in (1, 2] \), for all \( \alpha > 0 \). Let \( \alpha < \beta \) be such that \( 0 < \alpha \leq \beta \). We claim that \( f(\alpha) \leq f(\beta) \). Indeed, it is easy to see that \( [\alpha] \subset [\beta] \). If \( [\alpha] \cap (1, 2] \neq \emptyset \), then we have \( [\beta] \cap (1, 2] \neq \emptyset \) which easily implies \( f(\alpha) \leq f(\beta) \). Otherwise, assume \( [\alpha] \subset [2, +\infty) \).
Let \( a \in [\alpha] \). We have \( a \geq 2 \) and \( a \in [\beta] \). By definition of the sets \( I \), we have \( J_2 \subset I_\alpha \). Since \( \rho_{J_2}(x) \leq \rho_{I_\alpha}(x) \), for all \( x \in \ell_{p(\cdot)} \), we obtain
\[
\sup_{x \in C} \rho_{J_2}(x) \leq \sup_{x \in C} \rho_{I_\alpha}(x) \leq \beta,
\]
i.e., \( 2 \in [\beta] \). This fact, will force \( f(\beta) = 2 \). In all cases, we have \( f(\alpha) \leq f(\beta) \). In other words, the function \( f : (0, +\infty) \rightarrow (1, 2] \) is non-decreasing. Finally, let us show that \( C \subset \ell_g \), where \( g(x) = (1 + f(x))/2 \), for all \( a > 0 \). Since \( 1 < f(\alpha) \), then we have \( 1 < g(a) < f(\alpha) \), for all \( a > 0 \). If \( [\alpha] \subset [2, +\infty) \), pick \( \alpha \in [\alpha] \). Then \( g(\alpha) = 3/2 < a \) which implies \( I_g(\alpha) \subset I_\alpha \). Hence
\[
\rho_{I_g(\alpha)}(x) \leq \rho_{I_\alpha}(x), \quad \text{for all } x \in C,
\]
which implies \( \sup_{x \in C} \rho_{I_g(\alpha)}(x) \leq \sup_{x \in C} \rho_{I_\alpha}(x) \leq \alpha \). Otherwise, assume \( [\alpha] \cap (1, 2] \neq \emptyset \), then \( f(\alpha) = \sup \{ [\alpha] \cap (1, 2] \} \). Since \( g(a) < f(\alpha) \), there exists \( \alpha \in [\alpha] \) such that \( g(\alpha) < a \leq f(\alpha) \). Similar argument will show that
\[
\sup_{x \in C} \rho_{I_g(\alpha)}(x) \leq \sup_{x \in C} \rho_{I_\alpha}(x) \leq \alpha.
\]
In both cases, we showed that \( \sup_{x \in C} \rho_{I_g(\alpha)}(x) \leq \alpha \), for all \( a > 0 \), i.e., \( C \subset C_g \) as claimed. \( \square \)

Proposition 2 allows us to focus on the subsets \( C_f \) instead of subsets which satisfy the condition \( (ULD) \). The next result is amazing and surprising since it tells us that the subsets \( C_f \) enjoy nice modular geometric properties despite the fact that \( p^- = 1 \).

**Theorem 2.** Consider the vector space \( \ell_{p(\cdot)} \) such that \( p^- = 1 \) and \( p(\cdot) \) not identically equal to 1. Let \( f : (0, +\infty) \rightarrow (1, 2] \) be a non-decreasing function. Then, \( \rho \) is \( (ULIC2) \) on \( C_f \).

**Proof.** Let \( r > 0 \) and \( \epsilon > 0 \). Let \( x, y \in C_f \) such that \( \rho(x) \leq r, \rho(y) \leq r \) and \( \rho(\frac{x-y}{2}) \geq \epsilon r \). Since \( \rho \) is convex, we have
\[
\epsilon r \leq \rho(\frac{x-y}{2}) \leq \frac{\rho(x) + \rho(y)}{2} \leq r,
\]
which implies \( \epsilon \leq 1 \). Set \( \alpha = \frac{\epsilon r}{2} \). The properties of \( C_f \) imply \( \frac{x-y}{2} \in C_f \). So
\[
\rho_{I_f(\alpha)}(\frac{x-y}{2}) \leq \alpha,
\]
which implies
\[
\rho_{I_f(\alpha)}(\frac{x-y}{2}) = \rho(\frac{x-y}{2}) - \rho_{I_f(\alpha)}(\frac{x-y}{2}) \geq \epsilon r - \alpha = \frac{\epsilon r}{2}.
\]
Next, set
\[
K = I_f(\alpha) \cap \{ n, p(n) \geq 2 \} \quad \text{and} \quad L = I_f(\alpha) \cap \{ n, p(n) < 2 \}.
\]
Since \( I_f(\alpha) = K \cup L \), we obtain \( \rho_{I_f(\alpha)}(z) = \rho_K(z) + \rho_L(z) \), for all \( z \in C_f \). From our assumptions, we have
\[
\rho_K(\frac{x-y}{2}) \geq \frac{\epsilon r}{4} \quad \text{or} \quad \rho_L(\frac{x-y}{2}) \geq \frac{\epsilon r}{4}.
\]
Assume first that
\[
\rho_K(\frac{x-y}{2}) \geq \frac{\epsilon r}{4}.
\]
Using Lemma 1, we obtain
\[ \rho_K \left( \frac{x+y}{2} \right) + \rho_K \left( \frac{x-y}{2} \right) \leq \frac{\rho_K(x) + \rho_K(y)}{2}, \]
which implies
\[ \rho_K \left( \frac{x+y}{2} \right) \leq \frac{\rho_K(x) + \rho_K(y)}{2} - \varepsilon r. \]
Using the convexity of the modular, we have
\[ \rho_{L \cup J_{f(a)}} \left( \frac{x+y}{2} \right) \leq \frac{\rho_{L \cup J_{f(a)}}(x) + \rho_{L \cup J_{f(a)}}(y)}{2}, \]
which implies
\[ \rho \left( \frac{x+y}{2} \right) \leq \frac{\rho(x) + \rho(y)}{2} - \varepsilon r \leq r \left( 1 - \frac{\varepsilon}{4} \right). \]
For the second case, assume
\[ \rho_L \left( \frac{x-y}{2} \right) \geq \frac{\varepsilon r}{4}. \]
Set
\[ c = \frac{\varepsilon r}{8}, \quad L_1 = \left\{ n, \ |x_n - y_n| \leq c \left( |x_n| + |y_n| \right) \right\} \text{ and } L_2 = L \setminus L_1. \]
Since \( c < 1 \), we obtain
\[ \rho_{L_1} \left( \frac{x-y}{2} \right) \leq \sum_{n \in L_1} c^{p(n)} \left( \frac{|x_n| + |y_n|}{2} \right)^{p(n)} \leq \frac{c}{2} \sum_{n \in L_1} \frac{|x_n|^{p(n)} + |y_n|^{p(n)}}{p(n)}. \]
Hence
\[ \rho_{L_1} \left( \frac{x-y}{2} \right) \leq \frac{c}{2} \left( \rho_{L_1}(x) + \rho_{L_1}(y) \right) \leq \frac{c}{2} \left( \rho(x) + \rho(y) \right) \leq \frac{c}{2} r. \]
Our assumption on \( \rho_L \left( \frac{x-y}{2} \right) \) implies
\[ \rho_{L_2} \left( \frac{x-y}{2} \right) = \rho_L \left( \frac{x-y}{2} \right) - \rho_{L_1} \left( \frac{x-y}{2} \right) \geq r \frac{\varepsilon r}{4} \geq \frac{\varepsilon}{8}. \]
For any \( n \in L_2 \), we have
\[ f \left( \frac{\varepsilon r}{2} \right) - 1 = f(\alpha) - 1 \leq p(n) - 1 \leq p(n) (p(n) - 1) \]
\[ c \leq c^{2-p(n)} \leq \left( \frac{|x_n - y_n|}{|x_n| + |y_n|} \right)^{2-p(n)}. \]
Using Lemma 1, we obtain
\[ \left| \frac{x_n + y_n}{2} \right|^{p(n)} + \left( \frac{f(\alpha) - 1}{2} \right) c \left| \frac{x_n - y_n}{2} \right|^{p(n)} \leq \frac{1}{2} \left( |x_n|^{p(n)} + |y_n|^{p(n)} \right), \]
for any \( n \in L_2 \). Hence
\[ \rho_{L_2} \left( \frac{x+y}{2} \right) \leq \frac{\rho_{L_2}(x) + \rho_{L_2}(y)}{2} \leq \frac{r (f(\alpha) - 1) \varepsilon^2}{128}, \]
which implies
\[ \rho \left( \frac{x+y}{2} \right) \leq r \left( 1 - \frac{(f(\alpha) - 1) \varepsilon^2}{128} \right). \]
Both cases imply that \( \rho \) is \((UIC2)\) on \( C_f \) with

\[
\delta_{2,C_f}(r,\varepsilon) \geq \min \left( \frac{\varepsilon}{4} \left( \frac{f\left(\frac{\varepsilon}{2}\right)}{128} - 1 \right)^2 \right) > 0,
\]

since \( f(a) > 1 \), for any \( a > 0 \). Since \( f(\cdot) \) is nondecreasing, we may set

\[
\eta_2(r,\varepsilon) = \min \left( \frac{\varepsilon}{4} \left( \frac{f\left(\frac{\varepsilon}{2}\right)}{128} - 1 \right)^2 \right)
\]

to see that in fact \( \rho \) is \((UUC2)\) on \( C_f \) which completes the proof of Theorem 2. \( \Box \)

The following lemma will be useful:

**Lemma 3.** Consider the vector space \( \ell_{p(\cdot)} \) such that \( p^{-} = 1 \) and \( p(\cdot) \) not identically equal to 1. Let \( f : (0, +\infty) \to (1, 2] \) be a non-decreasing function. Set \( g(a) = f\left(\frac{a}{4}\right) \), for \( a > 0 \). We have

\[
C_f + C_f = \{ x + y; \ x, y \in C_f \} \subset C_g.
\]

**Proof.** Let \( x, y \in C_f \). For any \( n \in I_{g(a)} = \{ n; \ p(n) \leq f\left(\frac{a}{4}\right) \} \), we have

\[
\left| \frac{x_n + y_n}{2} \right|^{p(n)} \leq \frac{1}{2} \left( |x_n|^{p(n)} + |y_n|^{p(n)} \right),
\]

which implies

\[
\frac{1}{p(n)} |x_n + y_n|^{p(n)} \leq \frac{2^{p(n)-1}}{p(n)} \left( |x_n|^{p(n)} + |y_n|^{p(n)} \right).
\]

Hence

\[
\rho_{\frac{a}{4} + \frac{a}{4}}(x + y) \leq 2^{f\left(\frac{a}{4}\right)-1} \left( \rho_{\frac{a}{4}}(x) + \rho_{\frac{a}{4}}(y) \right)
\]

\[
\leq 2\left(\frac{a}{4} + \frac{a}{4}\right) = a.
\]

Therefore \( \rho_{\frac{a}{4}}(x + y) \leq a \), that is \( x + y \in C_g \), which completes the proof of Lemma 3. \( \Box \)

In the next section, we will prove a fixed-point theorem for modular nonexpansive mappings.

**4. Application**

As an application to Theorem 2, we will prove a fixed-point result for modular nonexpansive mappings. The classical ingredients will be needed. First, we prove the proximinality of \( \rho \)-closed convex subsets which satisfies the condition \((UD)\).

**Proposition 3.** Consider the vector space \( \ell_{p(\cdot)} \) such that \( p^{-} = 1 \) and \( p(\cdot) \) not identically equal to 1. Let \( f : (0, +\infty) \to (1, 2] \) non-decreasing. Any nonempty \( \rho \)-closed convex subset \( C \) of \( C_f \) is proximinal, i.e., for any \( x \in C_f \) such that

\[
d_\rho(x, C) = \inf \left\{ \rho(x - y); \ y \in C \right\} < \infty,
\]

there exists a unique \( c \in C \) such that \( d_\rho(x, C) = \rho(x - c) \).
We claim that 
\[
\rho \left( \frac{y_{\phi(n)} - y_{\phi(m)}}{2} \right) \geq \varepsilon_0,
\]
for any \( n > m \geq 1 \). According to Lemma 3, \( \{ x - y_{\phi(n)} \} \) is in \( C_y \), where \( g(\alpha) = f(\alpha/4) \), for any \( \alpha > 0 \). Fix \( n > m \geq 1 \). We have

\[
\max \left\{ \rho \left( x - y_{\phi(n)} \right), \rho \left( x - y_{\phi(m)} \right) \right\} \leq R \left( 1 + \frac{\varepsilon_0}{\phi(m)} \right),
\]

Since
\[
\varepsilon_0 = R \left( 1 + \frac{1}{\phi(m)} \right) \frac{\varepsilon_0}{R \left( 1 + \frac{1}{\phi(m)} \right)} \geq R \left( 1 + \frac{1}{\phi(m)} \right) \varepsilon_1,
\]

with \( \varepsilon_1 = \frac{\varepsilon_0}{2R} \), and using Theorem 2, we obtain
\[
\rho \left( x - \frac{y_{\phi(n)} + y_{\phi(m)}}{2} \right) \leq R \left( 1 + 1/\phi(m) \right) \left( 1 - \delta \phi_n \left( 1 + 1/\phi(m) \right), \varepsilon_1 \right)
\]
\[
\leq R \left( 1 + 1/\phi(m) \right) \left( 1 - \eta_2(R, \varepsilon_1) \right),
\]

where
\[
\eta_2(R, \varepsilon_1) = \min \left( \frac{\varepsilon_1}{4}, \left( \frac{R \varepsilon_1}{2} - 1 \right) \frac{\varepsilon_1}{128} \right).
\]

Since \( y_{\phi(n)} \) and \( y_{\phi(m)} \) are in \( C \) and \( C \) is convex, we obtain
\[
R = d_\rho(x, C) \leq \rho \left( x - \frac{y_{\phi(n)} + y_{\phi(m)}}{2} \right) \leq R \left( 1 + 1/\phi(m) \right) \left( 1 - \eta_2(R, \varepsilon_1) \right).
\]

If we let \( m \to +\infty \), we obtain
\[
R \leq R \left( 1 - \eta_2(R, \varepsilon_1) \right) < R.
\]

This contradiction implies that \( \{ y_n/2 \} \) is \( \rho \)-Cauchy. Since \( \ell_{\rho(\cdot)} \) is \( \rho \)-complete, there exists \( y \in \ell_{\rho(\cdot)} \) such that \( \{ y_n/2 \} \) \( \rho \)-converges to \( y \). Since \( C \) is convex and \( \rho \)-closed, we conclude that \( 2y \in C \). Using the Fatou property, we have
\[
R = d_\rho(x, C) \leq \rho \left( x - 2y \right) \leq \liminf_{m \to +\infty} \rho \left( x - \left( y + \frac{y_m}{2} \right) \right) \leq \liminf_{m \to +\infty} \liminf_{n \to +\infty} \rho \left( x - \frac{y_n + y_m}{2} \right) \leq \liminf_{m \to +\infty} \liminf_{n \to +\infty} \rho \left( x - y_n \right) + \rho \left( x - y_m \right) = R = d_\rho(x, C).
\]

If we set \( c = 2y \), we obtain \( d(x, C) = \rho(x - c) \). The uniqueness of the point \( c \) comes from the fact that \( \rho \) is strictly convex on \( C_y \) since it is \((\text{UUC2})\). \( \square \)

The next result discusses an intersection property known as the property \((R)\) [9]. Recall that a nonempty \( \rho \)-closed convex subset \( C \) of \( \ell_{\rho(\cdot)} \) is said to satisfy the property \((R)\)
Proposition 4. Consider the vector space $\ell_{p(\cdot)}$ such that $p^- = 1$ and $p(\cdot)$ not identically equal to 1. Let $f : (0, +\infty) \to (1, 2]$ be a non-decreasing function. Then $C_f$ satisfies the property (R).

Proof. Let $\{C_n\}$ be a decreasing sequence of nonempty $\rho$-closed convex subsets of $C_f$. Let $x \in C_1$. We have

$$d_\rho(x, C_n) = \inf \{\rho(x, x_n); x_n \in C_n\} \leq \sup \{\rho(x, y), \; x, y \in C_1\} = \delta_\rho(C_1) < \infty.$$ 

Since $\{C_n\}$ is decreasing, the sequence $\{d_\rho(x, C_n)\}$ is increasing bounded above by $\delta_\rho(C_1)$. Set $R = \lim_{n \to +\infty} d_\rho(x, C_n) = \sup d_\rho(x, C_n)$. If $R = 0$, then $x \in C_n$ for any $n \geq 1$, which will imply $\bigcap_{n \geq 1} C_n \neq \emptyset$. Otherwise, assume $R > 0$. Using Proposition 3, there exists $c_n \in C_n$ such that $d_\rho(x, C_n) = \rho(x - c_n)$, for any $n \geq 1$. Similar argument as the one used in the proof of Proposition 3 will show that $\{c_n/2\}$ is $\rho$-Cauchy and converges to $c \in \ell_{p(\cdot)}$. Since $\{C_n\}$ is a decreasing sequence of $\rho$-closed subsets, we conclude that $2c \in \bigcap_{n \geq 1} C_n$. Again this will show that $\bigcap_{n \geq 1} C_n \neq \emptyset$ which completes the proof of Proposition 4. Moreover, using Fatou property, we note that

$$\rho(x - 2c) \leq \liminf_{n \to +\infty} \liminf_{m \to +\infty} \rho\left(x - \frac{c_n + c_m}{2}\right),$$

which will imply

$$d_\rho\left(x, \bigcap_{n \geq 1} C_n\right) = \lim_{n \to +\infty} d_\rho(x, C_n).$$

\qed

Remark 1. Let us note that under the assumptions of Proposition 4, the conclusion still holds when we consider any family $\{C_\alpha\}_{\alpha \in \Gamma}$ of nonempty, convex, $\rho$-closed subsets of $C$, where $(\Gamma, \prec)$ is upward directed, such that there exists $x \in C$ which satisfies $\sup_{\alpha \in \Gamma} d_\rho(x, C_\alpha) < \infty$. Indeed, set $d = \sup_{\alpha \in \Gamma} d_\rho(x, C_\alpha)$. Without loss of generality, we may assume $d > 0$. For any $n \geq 1$, there exists $\alpha_n \in \Gamma$ such that

$$d\left(1 - \frac{1}{n}\right) < d_\rho(x, C_{\alpha_n}) \leq d.$$

Since $(\Gamma, \prec)$ is upward directed, we may assume $\alpha_n \prec \alpha_{n+1}$ which implies $C_{\alpha_{n+1}} \subset C_{\alpha_n}$. Proposition 4 implies $C_0 = \bigcap_{n \geq 1} C_{\alpha_n} \neq \emptyset$. Clearly $C_0$ is $\rho$-closed and using the last noted point in the proof of Proposition 4, we obtain

$$d_\rho(x, C_0) = \lim_{n \to +\infty} d_\rho(x, C_{\alpha_n}) = \sup_{n \geq 1} d_\rho(x, C_{\alpha_n}) = d.$$ 

Let $c_0 \in C_0$ such that $d_\rho(x, C_0) = \rho(x - c_0)$. We claim that $c_0 \in C_{\alpha_n}$ for any $\alpha \in \Gamma$. Indeed, fix $\alpha \in \Gamma$. If for some $n \geq 1$ we have $\alpha \prec \alpha_n$, then obviously we have $c_0 \in C_{\alpha_n} \subset C_{\alpha_n}$. Therefore let us assume that $\alpha \not\prec \alpha_n$, for any $n \geq 1$. Since $\Gamma$ is upward directed, there exists $\beta_n \in \Gamma$ such that $\alpha_n \prec \beta_n$ and $\alpha \prec \beta_n$, for any $n \geq 1$. We can also assume that $\beta_n \prec \beta_{n+1}$, for any $n \geq 1$. Again we have $C_1 = \bigcap_{n \geq 1} C_{\beta_n} \neq \emptyset$. Since $C_{\beta_n} \subset C_{\alpha_n}$, for any $n \geq 1$, we obtain $C_1 \subset C_0$. Moreover we have

$$d = d_\rho(x, C_0) \leq d_\rho(x, C_1) = \sup_{n \geq 1} d_\rho(x, C_{\beta_n}) \leq d.$$
Hence, $d_\rho(x, C_1) = d$ which implies the existence of a unique point $c_1 \in C_1$ such that $d_\rho(x, C_1) = \rho(x - c_1) = d$. Since $\rho$ is (SC) on $C_f$, we obtain $c_0 = c_1$. In particular, we have $c_0 \in C_{\beta_n}$, for any $n \geq 1$. Since $\alpha < \beta_n$, we conclude that $C_{\beta_n} \subset C_\alpha$, for any $n \geq 1$, which implies $c_0 \in C_\alpha$. Since $\alpha$ was taking arbitrary in $\Gamma$, we obtain $c_0 \in \bigcap_{\alpha \in \Gamma} C_\alpha$, which implies $\bigcap_{\alpha \in \Gamma} C_\alpha \neq \emptyset$ as claimed.

The next result is necessary to obtain the fixed-point theorem sought for $\rho$-nonexpansive mappings.

**Proposition 5.** Consider the vector space $\ell_{p(\cdot)}$ such that $p^- = 1$ and $p(\cdot)$ are not identically equal to 1. Let $f : (0, +\infty) \to [1, 2]$ be a nondecreasing function. Then $C_f$ has the $\rho$-normal structure property, i.e., for any nonempty $\rho$-closed convex $\rho$-bounded subset $C$ of $C_f$ not reduced to one point, there exists $x \in C$ such that

$$\sup_{y \in C} \rho(x - y) < \delta_\rho(C).$$

**Proof.** Let $C$ be a $\rho$-closed convex $\rho$-bounded subset $C$ of $C_f$ not reduced to one point. Since $C$ is not reduced to one point, we have $\delta_\rho(C) > 0$. Let $x, y \in C$ such that $x \neq y$. Set

$$\varepsilon_0 = \frac{1}{\delta_\rho(C)} \rho\left(\frac{x - y}{2}\right) > 0.$$

Fix $c \in C$. Using Lemma 3, we have $x - c$ and $y - c$ are in $C_f - C_f \subset C_g$, where $g(\alpha) = f(\alpha/4)$, for any $\alpha > 0$. So far we have

$$\max\{\rho(x - c), \rho(y - c)\} \leq \delta_\rho(C) \quad \text{and} \quad \rho\left(\frac{x - y}{2}\right) \geq \delta_\rho(C) \varepsilon_0.$$

Theorem 2 implies

$$\rho\left(c - \frac{x + y}{2}\right) \leq \delta_\rho(C) \left(1 - \delta_{2,C_g}(R, \varepsilon_0)\right).$$

Since $c$ was taken arbitrary in $C$, we conclude that

$$\sup_{c \in C} \rho\left(c - \frac{x + y}{2}\right) \leq \delta_\rho(C) \left(1 - \delta_{2,C_g}(\delta_\rho(C), \varepsilon_0)\right) < \delta_\rho(C) > 0.$$

Therefore the proof of Proposition 5 is complete. \(\square\)

Putting all this together, we are ready to prove the main fixed-point result of our work.

**Theorem 3.** Consider the vector space $\ell_{p(\cdot)}$ such that $p^- = 1$ and $p(\cdot)$ are not identically equal to 1. Let $C$ be a nonempty $\rho$-closed convex $\rho$-bounded subset of $\ell_{p(\cdot)}$, which satisfies the condition (UD). Any $\rho$-nonexpansive mapping $T : C \to C$ has a fixed point.

**Proof.** Since $C$ satisfies the condition (UD), Proposition 2 secures the existence of a non-decreasing function $f : (0, +\infty) \to [1, 2]$ such that $C$ is a subset of $C_f$. The conclusion is trivial if $C$ is reduced to one point. Therefore, we will assume that $C$ is not reduced to one point, i.e., $\delta_\rho(C) > 0$. Consider the family

$$\mathcal{F} = \{K \subset C, \ K \neq \emptyset, \ \rho \text{-closed convex and } T(K) \subset K\}$$

The family $\mathcal{F}$ is not empty since $C \in \mathcal{F}$. Since $C$ is bounded, we use Remark 1 to be able to use Zorn’s lemma and conclude that $\mathcal{F}$ contains a minimal element $K_0$. Let us show that $K_0$ is reduced to one point. Assume not, i.e., $K_0$ contains more than one point. Set $\text{co}(T(K_0))$ to be the intersection of all $\rho$-closed convex subset of $C$ containing $T(K_0)$. Hence $\text{co}(T(K_0)) \subset K_0$ since $K_0 \in \mathcal{F}$. Moreover, we have
which implies that \( \text{co}(T(K_0)) \subset T(K_0) \subset \text{co}(T(K_0)) \),

and we deduce the existence of \( x_0 \in K_0 \) such that

\[
r_0 = \sup_{y \in K_0} \rho(x_0 - y) < \delta_\rho(K_0).
\]

Define the subset

\[
K = \left\{ x \in K_0, \sup_{y \in K_0} \rho(x - y) \leq r_0 \right\}.
\]

Next, we prove that \( T(K) \subset K \). Indeed, let \( x \in K \). Since \( T \) is \( \rho \)-nonexpansive, we have

\[
\rho(T(x) - T(y)) \leq \rho(x - y) \leq r_0,
\]

for all \( y \in K_0 \). So we have \( T(y) \in B_\rho(T(x), r_0) \cap K_0 \), which implies \( T(K_0) \subset B_\rho(T(x), r_0) \).

Since \( K_0 = \text{co}(T(K_0)) \), we conclude that \( K_0 \subset B_\rho(T(x), r_0) \), which implies

\[
\rho(T(x) - y) \leq r_0,
\]

for all \( y \in K_0 \). Hence \( T(x) \in K \). Since \( x \) was taken as arbitrary in \( K \), we obtain \( T(K) \subset K \).

The minimality of \( K_0 \) will force \( K = K_0 \). Hence

\[
r_0 < \delta_\rho(K_0) = \delta_\rho(K) \leq r_0.
\]

This is a contradiction. Therefore, \( K_0 \) is reduced to one point and it is a fixed point of \( T \) because \( T(K_0) \subset K_0 \).

**Remark 2.** In Theorem 3, the condition (UD) can be replaced by the following condition which is slightly more general:

there exists \( x_0 \in \ell_{p(.)} \) such that \( x_0 + C \) satisfies the condition (UD).

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**Abbreviations**

The following abbreviations are used in this manuscript:

- MDPI Multidisciplinary Digital Publishing Institute
- DOAJ Directory of open access journals
- TLA Three letter acronym
- LD linear dichroism
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