Differential equations driven by Hölder continuous functions of order greater than 1/2

Yaozhong Hu * and David Nualart
Department of Mathematics, University of Kansas
405 Snow Hall, Lawrence, Kansas 66045-2142

Abstract
We derive estimates for the solutions to differential equations driven by a Hölder continuous function of order \( \beta > 1/2 \). As an application we deduce the existence of moments for the solutions to stochastic partial differential equations driven by a fractional Brownian motion with Hurst parameter \( H > 1/2 \).

1 Introduction
We are interested in the solutions of differential equations on \( \mathbb{R}^m \) of the form

\[
x_t = x_0 + \int_0^t f(x_r)dy_r,
\]

where the driving force \( y : [0, \infty) \to \mathbb{R}^m \) is a Hölder continuous function of order \( \beta > 1/2 \). If the function \( f : \mathbb{R}^d \to \mathbb{R}^{md} \) has bounded partial derivatives which are Hölder continuous of order \( \lambda > \frac{1}{\beta} - 1 \), then there is a unique solution \( x : \mathbb{R}^m \to \mathbb{R} \) which has bounded \( \frac{1}{\beta} \)-variation on any finite interval. These results have been proved by Lyons in [2] using the \( p \)-variation norm and the technique introduced by Young in [6]. The integral appearing in (1.1) is then a Riemann-Stieltjes integral.

In [7] Zähle has introduced a generalized Stieltjes integral using the techniques of fractional calculus. This integral is expressed in terms of fractional derivative operators and it coincides with the Riemann-Stieltjes integral \( \int_0^T f dg \), when the functions \( f \) and \( g \) are Hölder continuous of orders \( \lambda \) and \( \mu \), respectively and \( \lambda + \mu > 1 \) (see Proposition 2.1 below). Using this formula for the Riemann-Stieltjes integral, Nualart and Răşcanu have obtained in [3] the existence of a unique solution for a class of general differential equations that includes (1.1). Also they have proved that the solution of (1.1) is bounded on a finite interval.

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\[ [0, T] \text{ by } C_1 \exp(C_2 \|y\|_{0, T, \beta}^\frac{1}{\beta}), \text{ where } \kappa > \frac{1}{\beta} \text{ if } f \text{ is bounded and } \kappa > \frac{1}{1 - 2\beta} \text{ is } f \text{ has linear growth. Here } \|y\|_{0, T, \beta} \text{ denotes the } \beta\text{-Hölder norm of } y \text{ on the time interval } [0, T]. \text{ These estimates are based on a suitable application of Gronwall’s lemma. It turns out that the estimate in the linear growth case is unsatisfactory because } \kappa \text{ tends to infinity as } \beta \text{ tends to } 1/2. \]

The main purpose of this paper is to obtain better estimates for the solution \( x_t \) in the case where \( f \) is bounded or has linear growth using a direct approach based on formula (2.8). In the case where \( f \) is bounded we estimate \( \sup_{0 \leq t \leq T} |x_t| \) by

\[
C \left( 1 + \|y\|_{0, T, \beta}^\frac{1}{\beta} \right)
\]

and if \( f \) has linear growth we obtain the estimate

\[
C_1 \exp \left( C_2 \|y\|_{0, T, \beta}^\frac{1}{\beta} \right). \]

In Theorem 3.1 we provide explicit dependence on \( f \) and \( T \) for the constants \( C, C_1 \) and \( C_2 \).

Another novelty of this paper is that we establish the explicit dependence of the solution \( x_t \) to (1.1) on the initial condition \( x_0 \), the driving control \( y \) and the coefficient \( f \) (Theorem 3.2). Similar results are obtained for the case \( 1/3 < \beta < 1/2 \) in a forthcoming paper [1].

As an application we deduce the existence of moments for the solutions to stochastic partial differential equations driven by a fractional Brownian motion with Hurst parameter \( H > \frac{1}{2} \). We also discuss the regularity of the solution in the sense of Malliavin Calculus, improving the results of Nualart and Saussereau [4], and we apply the techniques of the Malliavin calculus to establish the existence of densities under suitable non-degeneracy conditions.

2 Fractional integrals and derivatives

Let \( a, b \in \mathbb{R} \) with \( a < b \). Let \( f \in L^1(a, b) \) and \( \alpha > 0 \). The left-sided and right-sided fractional Riemann-Liouville integrals of \( f \) of order \( \alpha \) are defined for almost all \( x \in (a, b) \) by

\[
I_{a+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \, ds
\]

and

\[
I_{b-}^\alpha f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) \, ds,
\]

respectively, where \((-1)^{-\alpha} = e^{-i\pi\alpha}\) and \( \Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} \, dr \) is the Euler gamma function. Let \( I_{a+}^\alpha(L^p) \) (resp. \( I_{b-}^\alpha(L^p) \)) be the image of \( L^p(a, b) \) by the
operator $I_{a+}^\alpha$ (resp. $I_{b-}^\alpha$). If $f \in I_{a+}^\alpha (L^p)$ (resp. $f \in I_{b-}^\alpha (L^p)$) and $0 < \alpha < 1$ then the Weyl derivatives are defined as

$$D_{a+}^\alpha f (t) = \frac{1}{\Gamma (1 - \alpha)} \left( \frac{f (t)}{(t - a)^\alpha} + \alpha \int_a^t \frac{f (s)}{(s - t)^{\alpha+1}} ds \right)$$

and

$$D_{b-}^\alpha f (t) = \frac{(-1)^\alpha}{\Gamma (1 - \alpha)} \left( \frac{f (t)}{(b - t)^\alpha} + \alpha \int_t^b \frac{f (s)}{(s - t)^{\alpha+1}} ds \right)$$

where $a \leq t \leq b$ (the convergence of the integrals at the singularity $s = t$ holds point-wise for almost all $t \in (a, b)$ if $p = 1$ and moreover in $L^p$-sense if $1 < p < \infty$).

For any $\lambda \in (0, 1)$, we denote by $C^\lambda (a, b)$ the space of $\lambda$-Hölder continuous functions on the interval $[a, b]$. We will make use of the notation

$$\|x\|_{a,b,\beta} = \sup_{a \leq \theta < r \leq b} \frac{|x_r - x_\theta|}{|r - \theta|^{\beta}},$$

and

$$\|x\|_{a,b,\infty} = \sup_{a \leq r \leq b} |x_r|,$$

where $x : \mathbb{R}^d \to \mathbb{R}$ is a given continuous function.

Recall from [5] that we have:

- If $\alpha < \frac{1}{p}$ and $q = \frac{p}{1 - \alpha}$ then
  $$I_{a+}^\alpha (L^p) = I_{b-}^\alpha (L^p) \subset L^q (a, b).$$

- If $\alpha > \frac{1}{p}$ then
  $$I_{a+}^\alpha (L^p) \cup I_{b-}^\alpha (L^p) \subset C^{\alpha - \frac{1}{p}} (a, b).$$

The following inversion formulas hold:

$$I_{a+}^\alpha (D_{a+}^\alpha f) = f, \quad \forall f \in I_{a+}^\alpha (L^p)$$

$$I_{a-}^\alpha (D_{a-}^\alpha f) = f, \quad \forall f \in I_{a-}^\alpha (L^p)$$

and

$$D_{a+}^\alpha (I_{a+}^\alpha f) = f, \quad D_{a-}^\alpha (I_{a-}^\alpha f) = f, \quad \forall f \in L^1 (a, b).$$

On the other hand, for any $f, g \in L^1(a, b)$ we have

$$\int_a^b I_{a+}^\alpha f(t) g(t) dt = (-1)^\alpha \int_a^b f(t) I_{b-}^\alpha g(t) dt,$$

and for $f \in I_{a+}^\alpha (L^p)$ and $g \in I_{a-}^\alpha (L^p)$ we have

$$\int_a^b D_{a+}^\alpha f(t) g(t) dt = (-1)^{-\alpha} \int_a^b f(t) D_{b-}^\alpha g(t) dt.$$
Suppose that \( f \in C^\lambda(a, b) \) and \( g \in C^\mu(a, b) \) with \( \lambda + \mu > 1 \). Then, from the classical paper by Young [6], the Riemann-Stieltjes integral \( \int_a^b f \, dg \) exists. The following proposition can be regarded as a fractional integration by parts formula, and provides an explicit expression for the integral \( \int_a^b f \, dg \) in terms of fractional derivatives (see [7]).

**Proposition 2.1** Suppose that \( f \in C^\lambda(a, b) \) and \( g \in C^\mu(a, b) \) with \( \lambda + \mu > 1 \). Let \( \lambda > \alpha \) and \( \mu > 1 - \alpha \). Then the Riemann Stieltjes integral \( \int_a^b f \, dg \) exists and it can be expressed as

\[
\int_a^b f \, dg = (-1)^{\alpha} \int_a^b D^\alpha_{a+} f(t) D^{1-\alpha}_{b-} g(t) \, dt,
\]

where \( g_{b-}(t) = g(t) - g(b) \).

### 3 Estimates for the solutions of differential equations

Suppose that \( y : [0, \infty) \to \mathbb{R}^m \) is a Hölder continuous function of order \( \beta > 1/2 \). Fix an initial condition \( x_0 \in \mathbb{R}^d \) and consider the following differential equation

\[
x_t = x_0 + \int_0^t f(x_r) \, dy_r,
\]

where \( f : \mathbb{R}^d \to \mathbb{R}^{md} \) is given function. Lyons has proved in [2] that Equation (3.1) has a unique solution if \( f \) is continuously differentiable and it has a derivative \( f' \) which is bounded and locally Hölder continuous of order \( \lambda > \frac{1}{\beta} - 1 \).

Our aim is to obtain estimates on \( x_t \) which are better than those given by Nualart and Răşcanu in [3].

**Theorem 3.1** Let \( f \) be a continuously differentiable such that \( f' \) is bounded and locally Hölder continuous of order \( \lambda > \frac{1}{\beta} - 1 \).

(i) **Assume that** \( f \) **is also bounded.** Then, there is a constant \( k \), which depends only on \( \beta \), such that for all \( T \),

\[
\sup_{0 \leq t \leq T} |x_t| \leq |x_0| + kT \| f \|_{\infty} \| f' \|_{\infty}^{1-\beta} \| y \|_{0, T, \beta}^{1/\beta},
\]

(ii) **Assume that** \( f \) **satisfies the linear growth condition**

\[
|f(x)| \leq a_0 + a_1 |x|,
\]

**where** \( a_0 \geq 0 \) **and** \( a_1 \geq 0 \). **Then there is a constant** \( k \) **depending only on** \( \beta \), **such that for all** \( T \),

\[
\sup_{0 \leq t \leq T} |x_t| \leq 2kT \| f \|_{\infty} \| y \|_{0, T, \beta}^{1/\beta}(|x_0| + 1).
\]
Proof. Without loss of generality we assume that \( d = m = 1 \). Assume first that \( f \) is bounded. Set \( \|y\|_\beta = \|y\|_{0,T,\beta} \). Let \( \alpha > 1/2 \) such that \( \alpha > 1 - \beta \). First we use the fractional integration by parts formula given in Proposition 2.1 to obtain for all \( s, t \in [0, T] \),

\[
| \int_s^t f(x_r) dy_r | \leq \int_s^t |D^{\alpha}_s f(x_r) D_{t-}^{1-\alpha} y_{t-}(r)| dr.
\]

From (2.2) and (2.1) it is easy to see

\[
|D^{1-\alpha}_t y_{t-}(r)| \leq k \|y\|_{r,t,\beta} |t-r|^\alpha \beta - 1 \leq k \|y\|_\beta |t-r|^\alpha \beta - 1
\]

(3.5)

and

\[
|D^{\alpha}_s f(x_r)| \leq k \left[ \|f\|_\infty (r - s)^{-\alpha} + \|f'|_\infty \|x\|_{s,t,\beta} (r - s)^{\beta - \alpha} \right].
\]

(3.6)

Therefore

\[
| \int_s^t f(x_r) dy_r | \leq k \|y\|_\beta \int_s^t \left[ \|f\|_\infty (r - s)^{-\alpha} (t - r)^{\alpha + \beta - 1} \\
+ \|f'|_\infty \|x\|_{s,t,\beta} (r - s)^{\beta - \alpha} (t - r)^{\alpha + \beta - 1} \right] dr
\]

\[
\leq k \|y\|_\beta \left[ \|f\|_\infty (t - s)^{\beta} + \|f'|_\infty \|x\|_{s,t,\beta} (t - s)^{2 \beta} \right].
\]

Consequently, we have

\[
\|x\|_{s,t,\beta} \leq k \|y\|_\beta \left[ \|f\|_\infty + \|f'|_\infty \|x\|_{s,t,\beta} (t - s)^{\beta} \right].
\]

Hence,

\[
\|x\|_{s,t,\beta} \leq k \|y\|_\beta \|f\|_\infty (1 - k \|f'|_\infty \|y\|_\beta (t - s)^{\beta})^{-1}.
\]

Therefore,

\[
\|x\|_{s,t,\infty} \leq |x_s| + \|x\|_{s,t,\beta} (t - s)^{\beta}
\]

\[
\leq |x_s| + k \|y\|_\beta \|f\|_\infty (1 - k \|f'|_\infty \|y\|_\beta (t - s)^{\beta})^{-1} (t - s)^{\beta}.
\]

Let \( A := k \|f'|_\infty \|y\|_\beta \). Divide the interval \([0, T]\) into \( n = T/\Delta \) subintervals and apply the above inequality on the interval \([0, \Delta], [\Delta, 2\Delta]\) and so on recursively to obtain

\[
\sup_{0 \leq t \leq T} |x_t| \leq |x_0| + kT \|f\|_\infty \|y\|_\beta (1 - A \Delta^{\beta})^{-1} \Delta^{\beta - 1}.
\]

With the choice \( \Delta = \left( \frac{1-\beta}{A} \right)^{\frac{1}{\beta}} \) we get

\[
\sup_{0 \leq t \leq T} |x_t| \leq |x_0| + kT \|f\|_\infty \|y\|_\beta \left( \frac{1}{\beta (1 - \beta)} \right)^{\frac{1}{\beta}} (k \|f'|_\infty \|y\|_\beta)^{\frac{1-\beta}{\beta}}
\]

\[
= |x_0| + kT \|f'\|_\infty \|y\|_\beta \frac{1-\beta}{\beta}.
\]

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This proves the inequality (3.2).

Assume now that $f$ satisfies (3.3). In this case, instead of (3.6) we have
\[
|D^\alpha_s f(x_r)| \leq k \left[(a_0 + a_1|x_r|)(r-s)^{-\alpha} + \|f'||_\infty \|x\|_{s,t,\beta}(r-s)^{\beta-\alpha}\right].
\]
As a consequence,
\[
\|x\|_{s,t,\beta} \leq k \|y\|_{\beta} \left[a_0 + a_1 \|x\|_{s,t,\infty} + \|f'||_\infty \|x\|_{s,t,\beta}(t-s)^{\beta}\right].
\]
Or
\[
\|x\|_{s,t,\beta} \leq k \|y\|_{\beta} \left(a_0 + a_1 \|x\|_{s,t,\infty}\right) (1 - k \|f'||_\infty \|y\|_{\beta}(t-s)^{\beta})^{-1}.
\]
Therefore,
\[
|x_t| \leq |x_s| + k \|y\|_{\beta} \left(1 - k \|f'||_\infty \|y\|_{\beta}(t-s)^{\beta}\right)^{-1}
\times \left(a_0 + a_1 \|x\|_{s,t,\infty}\right) (t-s)^{\beta}.
\]
As before, divide the interval $[0,T]$ into $n = T/\Delta$ subintervals and set $\Delta = t-s$.

Denote
\[
A = k \|f'||_\infty \|y\|_{\beta}, \quad B = k a_0 \|y\|_{\beta}, \quad C = k a_1 \|y\|_{\beta}, \quad D = (1 - (1 - A \Delta^\beta)^{-1} C \Delta^\beta)^{-1}, \quad F = D B (1 - A \Delta^\beta)^{-1} \Delta^\beta.
\]
We have
\[
\|x\|_{s,t,\infty} \left[1 - k \|y\|_{\beta}(1 - A \Delta^\beta)^{-1} a_1 \Delta^\beta\right] \leq |x_s| + k a_0 \|y\|_{\beta}(1 - A \Delta^\beta)^{-1} \Delta^\beta.
\]
This implies
\[
\sup_{0 \leq r \leq t} |x_r| \leq (1 - (1 - A \Delta^\beta)^{-1} C \Delta^\beta)^{-1} \left[\sup_{0 \leq r \leq s} |x_r| + B(1 - A \Delta^\beta)^{-1} \Delta^\beta\right].
\]
Or
\[
\sup_{0 \leq r \leq t} |x_r| \leq D \sup_{0 \leq r \leq s} |x_r| + F.
\]
Denote
\[
Z_n = \sup_{0 \leq r \leq n \Delta} |x_r|,
\]
where $n = \frac{T}{\Delta}$. Then
\[
Z_n \leq D Z_{n-1} + F \leq \cdots \leq D^n Z_0 + \sum_{k=0}^{n-1} D^k F.
\]
This yields
\[
\sup_{0 \leq t \leq T} |x_t| \leq (1 - (1 - A\Delta^\beta)^{-1} C\Delta^\beta)^{-T/\Delta} |x_0|
\]
\[+ \sum_{k=0}^{n-1} (1 - (1 - A\Delta^\beta)^{-1} C\Delta^\beta)^{-k-1} B(1 - A\Delta^\beta)^{-1} \Delta^\beta.\]

Then we let \(\Delta\) satisfy
\[
A\Delta^\beta \leq 1/3, C\Delta^\beta \leq 1/3, B\Delta^\beta \leq 1/3
\]
Namely, we take
\[
\Delta = \left(\frac{1}{3(A \lor B \lor C)}\right)^{1/\beta}.
\]
Then
\[
\sup_{0 \leq t \leq T} |x_t| \leq 2^{T/\Delta}(|x_0| + 1)
\]
\[\leq 2^k T\left[\|f\|_\infty \lor a_0 \lor a_1\right]^{1/\beta} \left[\|y\|_{0,T,\beta}^{1/\beta}\right] (|x_0| + 1).
\]
The proof of the theorem is now complete. ■

Suppose now that we have two differential equations of the form
\[
x_t = x_0 + \int_0^t f(x_s) dy_s,
\]
and
\[
\tilde{x}_t = \tilde{x}_0 + \int_0^t \tilde{f}(\tilde{x}_s) \tilde{y}_s,
\]
where \(y\) and \(\tilde{y}\) are Hölder continuous functions of order \(\beta > 1/2\), and \(f\) and \(\tilde{f}\) are two functions which are continuously differentiable with Hölder continuous derivatives of order \(\lambda > 1/\beta - 1\). Then, we have the following estimate.

**Theorem 3.2** Suppose in addition that \(f\) is twice continuously differentiable and \(f''\) is bounded. Then there is a constant \(k\) such that
\[
\sup_{0 \leq r \leq T} |x_r - \tilde{x}_r| \leq 2^k D^{1/\beta} \|y\|_{0,T,\beta}^{1/\beta} T
\]
\[\times \left\{ |x_0 - \tilde{x}_0| + \|y\|_{0,T,\beta} \left[\|f - \tilde{f}\|_\infty + \|x\|_{0,T,\beta} \|f' - \tilde{f}'\|_\infty \right] \right. \]
\[+ \left[\|f\|_\infty + \|\tilde{f}\|_\infty \right] \|x\|_{0,T,\beta} \|y - \tilde{y}\|_{0,T,\beta} \right\}
\]
where
\[
D = \|f'\|_\infty \lor \left(\|f''\|_\infty \|y\|_{0,T,\beta} + \|f''\|_\infty (\|x\|_{0,T,\beta} + \|\tilde{x}\|_{0,T,\beta}) T^\beta \right).
\]
Proof. Fix $s, t \in [0, T]$. Set

$$x_t - \tilde{x}_t - (x_s - \tilde{x}_s) = I_1 + I_2 + I_3$$

where

$$I_1 = \int_s^t [f(x_r) - f(\tilde{x}_r)] \, dy_r$$

$$I_2 = \int_s^t [f(\tilde{x}_r) - \tilde{f}(\tilde{x}_r)] \, dy_r$$

$$I_3 = \int_s^t \tilde{f}(\tilde{x}_r) \, dy_r$$

The terms $I_2$ and $I_3$ can be estimated easily.

$$|I_2| \leq k\|y\|_\beta \left[ \|f - \tilde{f}\|_\infty (t-s)^\beta + \|f' - \tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}(t-s)^{2\beta} \right]$$

and

$$|I_3| \leq k\|y - \tilde{y}\|_\beta \left[ \|\tilde{f}\|_\infty (t-s)^\beta + \|\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}(t-s)^{2\beta} \right],$$

where $\|y\|_\beta = \|y\|_{0,T,\beta}$ and $\|y - \tilde{y}\|_\beta = \|y - \tilde{y}\|_{0,T,\beta}$. The term $I_1$ is a little more complicated.

$$|I_1| \leq \int_s^t |D^\alpha_{s+} [f(x_r) - f(\tilde{x}_r)] - D^{1-\alpha}_{t-} y_-(r)| \, dr$$

$$\leq k \int_s^t \|y\|_{s,t,\beta}(t-r)^{\alpha+\beta-1} \left[ \|f(x_r) - f(\tilde{x}_r)(r-s)^{-\alpha} \right.$$\n
$$+\|f'\|_\infty \|x - \tilde{x}\|_{s,r,\beta} (r-s)^{2-\alpha} \right.$$\n
$$+\|f''\|_\infty \|x - \tilde{x}\|_{s,r,\infty} \left[ \|x\|_{s,r,\beta} + \|\tilde{x}\|_{s,r,\beta} \right] (r-s)^{2-\alpha} \left. \right] \, dr$$

$$\leq k\|y\|_\beta \left\{ \|f'\|_\infty \|x - \tilde{x}\|_{s,t,\infty} (t-s)^{\beta} + \|f''\|_\infty \|x - \tilde{x}\|_{s,t,\beta} (t-s)^{2\beta} \right.$$\n
$$+\|f''\|_\infty \|x - \tilde{x}\|_{s,t,\infty} \left[ \|x\|_{s,t,\beta} + \|\tilde{x}\|_{s,t,\beta} \right] (t-s)^{2\beta} \right\}.$$

Therefore

$$\|x - \tilde{x}\|_{s,t,\beta} \leq k\|y\|_\beta \left\{ \|f'\|_\infty \|x - \tilde{x}\|_{s,t,\infty} + \|f''\|_\infty \|x - \tilde{x}\|_{s,t,\beta} (t-s)^{\beta} \right.$$\n
$$+\|f''\|_\infty \|x - \tilde{x}\|_{s,t,\infty} \left[ \|x\|_{s,t,\beta} + \|\tilde{x}\|_{s,t,\beta} \right] (t-s)^{\beta} \right.$$\n
$$+\|f - \tilde{f}\|_\infty + \|f' - \tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta} (t-s)^{\beta} \right\}$$

$$+ k\|y - \tilde{y}\|_\beta \left[ \|\tilde{f}\|_\infty + \|\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta} (t-s)^{\beta} \right].$$

Rearrange it to obtain

$$\|x - \tilde{x}\|_{s,t,\beta} \leq k(1 - k\|f''\|_\infty \|y\|_\beta (t-s)^{\beta})^{-1} \left\{ \|y\|_\beta \left[ \|f''\|_\infty \|x - \tilde{x}\|_{s,t,\infty} \right. \right.$$
\[ + \| f'' \|_\infty \| x - \tilde{x} \|_{s,t,\infty} \left[ \| x \|_{s,t,\beta} + \| \tilde{x} \|_{s,t,\beta} (t - s)^\beta \right] \\
+ \| f - \tilde{f} \|_\infty + \| f' - \tilde{f}' \|_\infty \| \tilde{x} \|_{s,t,\beta} (t - s)^\beta \]

\[ + k \| y - \tilde{y} \|_\beta \left[ \| \tilde{f} \|_\infty + \| \tilde{f}' \|_\infty \| \tilde{x} \|_{s,t,\beta} (t - s)^\beta \right] \}.

Set \( \Delta = t - s \), and \( A = k \| f' \|_\infty \| y \|_\beta \). Then

\[ \| x - \tilde{x} \|_{s,t,\infty} \leq |x_s - \tilde{x}_s| + \| x - \tilde{x} \|_{s,t,\beta} (t - s)^\beta \]

\[ \leq |x_s - \tilde{x}_s| + k(1 - A\Delta^\beta)^{-1} \Delta^\beta \left\{ \| y \|_\beta \left[ \| f' \|_\infty \| x - \tilde{x} \|_{s,t,\infty} \\
+ \| f'' \|_\infty \| x - \tilde{x} \|_{s,t,\infty} \left[ \| x \|_{s,t,\beta} + \| \tilde{x} \|_{s,t,\beta} \right] \Delta^\beta \\
+ \| f - \tilde{f} \|_\infty + \| f' - \tilde{f}' \|_\infty \| \tilde{x} \|_{s,t,\beta} \Delta^\beta \right] \\
+ k \| y - \tilde{y} \|_\beta \left[ \| \tilde{f} \|_\infty + \| \tilde{f}' \|_\infty \| \tilde{x} \|_{s,t,\beta} \Delta^\beta \right] \}.
\]

Denote

\[ B = k \| y \|_\beta \left( \| f' \|_\infty + \| f'' \|_\infty (\| x \|_{0,T,\beta} + \| \tilde{x} \|_{0,T,\beta}) T^{\beta} \right) \].

Then

\[ \| x - \tilde{x} \|_{s,t,\infty} \leq \left( 1 - (1 - A\Delta^\beta)^{-1} \Delta^\beta B \right)^{-1} \]

\[ \times \left\{ |x_s - \tilde{x}_s| + k(1 - A\Delta^\beta)^{-1} \Delta^\beta \right. \]

\[ \times \left[ \| y \|_\beta \left[ \| f - \tilde{f} \|_\infty + \| f' - \tilde{f}' \|_\infty \| \tilde{x} \|_{s,t,\beta} \Delta^\beta \right] \\
+ \| y - \tilde{y} \|_\beta \left[ \| \tilde{f} \|_\infty + \| \tilde{f}' \|_\infty \| \tilde{x} \|_{s,t,\beta} \Delta^\beta \right] \}.
\]

Let \( \Delta \) satisfy

\[ A\Delta \leq 1/3, \quad B\Delta \leq 1/3 \]

Namely, we take

\[ \Delta = \left( \frac{1}{3(A \lor B)} \right)^{1/\beta} \].

Then

\[ \| x - \tilde{x} \|_{s,t,\infty} \leq 2 \left[ |x_s - \tilde{x}_s| + C\Delta^\beta \right], \]
where
\[
C = \frac{3}{2} k \left[ \|y\|_{\beta} \left[ \|f - \tilde{f}\|_{\infty} + \|f' - \tilde{f}'\|_{\infty} \|\tilde{x}\|_{s,t,\beta} \Delta^{\beta} \right] \\
+ \|y - \tilde{y}\|_{\beta} \left[ \|\tilde{f}\|_{\infty} + \|\tilde{f}'\|_{\infty} \|\tilde{x}\|_{s,t,\beta} \Delta^{\beta} \right] \right].
\]

Applying the above estimate recursively we obtain
\[
\sup_{0 \leq r \leq T} |x_r - \tilde{x}_r| \leq 2^n \left[ |x_0 - \tilde{x}_0| + C \Delta^{\beta} \right],
\]
where \( T = n \Delta \). Or we have
\[
\sup_{0 \leq r \leq T} |x_r - \tilde{x}_r| \leq 2^k \left( \|f\|_{\infty} + \|f'\|_{\infty} \|x\|_{s,t,\beta} \|\tilde{x}\|_{s,t,\beta} \Delta^{\beta} \right) T^{1/\beta} \left| y_{0,T,\beta} \right|^ {1/\beta} T \times \left( |x_0 - \tilde{x}_0| + \|y\|_{0,T,\beta} \left[ \|f - \tilde{f}\|_{\infty} + \|f\|_{0,T,\beta} \right] + \left[ \|f\|_{\infty} + \|f\|_{0,T,\beta} \right] \|y - \tilde{y}\|_{0,T,\beta} \right).
\]

4 Stochastic differential equations driven by a fBm

Let \( B = \{B_t, t \geq 0\} \) be an \( m \)-dimensional fractional Brownian motion (fBm) with Hurst parameter \( H > 1/2 \). That is, \( B \) is a Gaussian centered process with the covariance function \( E(B_i t B_j s) = R_H(t,s) \delta_{ij} \), where
\[
R_H(t,s) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).
\]

Consider the stochastic differential equation
\[
X_t = X_0 + \int_0^t \sigma(X_s) dB_s. \tag{4.1}
\]

This equation has a unique solution (see \[2\] and \[3\]) provided \( \sigma \) is continuously differentiable, and \( \sigma' \) is bounded and Hölder continuous of order \( \lambda > \frac{1}{H} - 1 \). The stochastic integral is interpreted as a path-wise Riemann-Stieltjes integral.

Then, using the estimate (3.4) in Theorem 3.1 we obtain the following estimate for the solution of Equation (4.1), if we choose \( \beta \in \left( \frac{1}{2}, H \right) \). Notice that \( \frac{1}{\beta} < 2 \).
\[
\sup_{0 \leq t \leq T} |X_t| \leq 2^k T \left( \|\sigma'\|_{\infty} \left| \sigma(0) \right| \right) \frac{T^{1/\beta}}{L_{0,T,\beta}} (|X_0| + 1). \tag{4.2}
\]
If \( \sigma \) is bounded we can make use of the estimate (3.2) and we obtain

\[
\sup_{0 \leq t \leq T} |X_t| \leq |X_0| + kT\|\sigma\|_\infty \|\sigma'\|_\infty \|B\|_{0, r, \beta},
\]

(4.3)

These estimates improve those obtained by Nualart and Răşcanu in [3] based on a suitable version of Gronwall’s lemma. The estimates (4.2) and (4.3) allow us to establish the following integrability properties for the solution of Equation (4.1).

**Theorem 4.1** Consider the stochastic differential equation (4.1). If \( \sigma' \) is bounded and Hölder continuous of order \( \lambda > \frac{1}{H} - 1 \), then

\[
E\left(\sup_{0 \leq t \leq T} |X_t|^p\right) < \infty
\]

(4.4)

for all \( p \geq 2 \). If furthermore \( \sigma \) is bounded, then

\[
E\left(\exp \lambda \left(\sup_{0 \leq t \leq T} |X_t|^\gamma\right)\right) < \infty
\]

(4.5)

for any \( \lambda > 0 \) and \( \gamma < 2\beta \).

If we apply these results to the linear equation satisfied by the derivative in the sense of Malliavin calculus of \( X_t \) then we get that \( X_t \) belongs to the Sobolev space \( D_{1,p} \) for all \( p \geq 2 \). This implies that if the coefficient \( \sigma \) is infinitely differentiable with bounded derivatives of all orders, then, \( X_t \) belongs to \( D^\infty \). This allows us to deduce the regularity of the density of the random vector \( X_t \) at a fixed time \( t > 0 \) assuming the following nondegeneracy condition:

(H) The vector space spanned by \( \left\{ (\sigma^{ij}(X_0))_{1 \leq i \leq d}, 1 \leq j \leq m \right\} \) is \( \mathbb{R}^m \).

That is, we have the following result.

**Theorem 4.2** Consider the stochastic differential equation (4.1). Suppose that \( \sigma \) is infinitely differentiable with bounded derivatives of all orders, and the assumption (H) holds. Then, for any \( t > 0 \) the probability law of \( X_t \) has a \( C^\infty \) density.

In [4] Nualart and Saussereau have proved that the random variable \( X_t \) belongs locally to the space \( D^\infty \), and, as a consequence, they have derived the absolute continuity of the law of \( X_t \) under the assumption (H).

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