Finite free point processes

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Abstract

We use techniques from finite free probability to analyze matrix processes related to eigenvalues, singular values, and generalized singular values of random matrices. The models we use are quite basic and the analysis consists entirely of expected characteristic polynomials. A number of our results match known results in random matrix theory, however our main result (regarding generalized singular values) seems to be more general than any of the standard random matrix processes (Hermite/Laguerre/Jacobi) in the field. To test this, we perform a series of simulations of this new process that, on the one hand, confirms that this process can exhibit behavior not seen in the standard random matrix processes, but on the other hand provides evidence that the true behavior is captured quite well by our techniques. This, coupled with the fact that we are able to compute the same statistics for this new model that we are for the standard models, suggests that further investigation could be both interesting and fruitful.

1 Introduction

For integers $m, n$, let $\mathcal{M}_{m,n}$ denote the collection of real $m \times n$ matrices and let $X_{m \times n} \in \mathcal{M}_{m,n}$ have entries consisting of independent, identically distributed symmetric\(^1\) random variables with variance 1 (and all other moments bounded). We consider the following questions:

1. **Eigenvalues**: If $A \in \mathcal{M}_{n,n}$, how do the eigenvalues of

   $$A + \sqrt{\theta}X_{n \times n}$$

   evolve for $\theta \geq 0$?

2. **Hermitian Eigenvalues**: If $B \in \mathcal{M}_{n,n}$, how do the eigenvalues of

   $$(B + \sqrt{\theta}X_{n \times n}) + (B + \sqrt{\theta}X_{n \times n})^\dagger$$

   evolve for $\theta \geq 0$?

3. **Singular Values**: If $C \in \mathcal{M}_{m,n}$, how do the (squares of the) singular values of

   $$C + \sqrt{\theta}X_{m,n}$$

   evolve for $\theta \geq 0$?

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\(^{1}\)\(\mathbb{P}[Y] = \mathbb{P}[-Y]\).
4. Generalized Singular Values: If $D_1 \in M_{n_1,k}$ and $D_2 \in M_{n_2,k}$, how do the (squares of the) generalized singular values of the pair

$$\left\{ D_1 + \sqrt{\theta} X_{n_1 \times k}, D_2 + \sqrt{\theta} X_{n_2 \times k} \right\}$$

evolve for $\theta \geq 0$?

In this paper we investigate topics 2., 3., 4. using techniques from finite free probability\(^2\). Our aim is to show that these simple models and the information we can get about them connect deeply to the standard models in random matrix theory.

We will first show that 2. is related to a problem that has already been written about by Tao [22], so (in fact) the real focus in this paper will be on 3. and 4. and the majority of the work will lie in developing a tool that is robust enough to handle both cases at once (Section 4). Our main tool will be finite free probability — or, in other words, expected characteristic polynomials. Despite only containing a fraction of the information about a distribution, we will find that expected characteristic polynomials capture essentially all of the “non-random” elements of random matrix processes. In particular, the point process derived from 2. will reproduce the Hermite (or Gaussian) process (Section 2) and the point process derived from 3. will reproduce the Laguerre (or Wishart) process (Section 3).

While those familiar with random matrix processes might naturally expect the point process derived from 4. to reproduce the Jacobi process, we find that is not the case (though the two are related — see Section 5.2). Most notably, the dynamics of 4. depend very heavily on the initial matrix, a feature not present in the Hermite, Laguerre, and Jacobi processes. As far as we can tell, the dynamics displayed by 4. are (in this sense) more complex than any of the processes typically studied in random matrix theory, and so may suggest a fruitful direction of further research.

In order to get a better understanding of the point process derived from 4., we provide the results of a simulation that produces two vastly different behaviors from the same starting points and then compare this to the information provided by the expected characteristic polynomials. This, along with some other experimental results, is presented in Section 6. Finally, we end with some concluding comments and acknowledgments in Section 7.

1.1 Conventions

As mentioned in the introduction, we will let $\mathcal{M}_{m,n}$ denote the collection of $m \times n$ real matrices and will always use $X_{m \times n} \in \mathcal{M}_{m,n}$ to denote a matrix with entries consisting of independent, identically distributed symmetric random variables with variance 1 (and all other moments bounded)\(^3\). The choice of reals is arbitrary — one could substitute the complex (and conjugate transpose for transpose) and get the same results (except for Section 6).

For an operator $L : \mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}[x_1, \ldots, x_n]$, we will write

$$L \{ p \}$$

to denote $L$ applied to the polynomial $p$.

\(^2\)Topic 1. has the added complication of having a decomposition whose matrices may or may not be unitary, which is something that must be addressed by alternative means (see [21] for developments in this direction).

\(^3\)It is likely this condition can be weakened substantially, but that is not something we will be concerned with in this paper.
We will use $\mathcal{SPM}_n \subseteq \mathcal{M}_{n,n}$ to denote the set of signed permutation matrices of size $n$. We will say that a random matrix $Y \in \mathcal{M}_{m,n}$ is $\mathcal{SPM}$–bi-invariant if

$$\mathbb{P}[Y] = \mathbb{P}[Q_m Y] = \mathbb{P}[Y Q_n]$$

for $Q_m \in \mathcal{SPM}_m$ and all $Q_n \in \mathcal{SPM}_n$ and we will say that a random matrix $Z \in \mathcal{M}_{n,n}$ is $\mathcal{SPM}$–conjugation-invariant if

$$\mathbb{P}[Y] = \mathbb{P}[Q_n Y Q_n^T]$$

for all $Q_n \in \mathcal{SPM}_n$. It should be clear that (by construction) $X_{m,n}$ is $\mathcal{SPM}$–bi-invariant for all $n,m$. It is then easy to check that $X_{n,n} + X_{n,n}^T$ is $\mathcal{SPM}$-conjugation–symmetric.

Lastly, we note that the convention in random matrix theory is to consider the squares of the (generalized) singular values, as this can then be viewed as an “eigenvalue” problem and we will do the same (in order to compare results effectively).

### 1.2 Some Random Matrix Theory

Here we list the relevant random matrix processes along with the known results concerning the evolution of their derived point processes. In each case, the behavior of the point process differs depending on whether the Brownian motion involved is real valued ($\beta = 1$) or complex valued ($\beta = 2$) valued$^4$.

#### 1.2.1 Hermite Process

The real (complex) matrix Hermite process was introduced and studied by Dyson [7]. It can be constructed as a Wiener process $(H_t)_{t \in [0,\infty)}$ with $H_0$ a fixed symmetric (Hermitian) matrix and with each increment $H_t - H_s$ equal to $\sqrt{t-s}$ times a Gaussian orthogonal (unitary) matrix. Assuming $H_0$ has distinct eigenvalues, one can show that the eigenvalues of $H_t$ form a point process $\lambda = \lambda(t)$ which is the unique strong solution to the stochastic differential equation (SDE)

$$d\lambda_i = dW_i + \frac{\beta}{2} \left[ \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right] dt \quad (1)$$

where the $W_i$ are (1-dimensional) independent Brownian motions.

#### 1.2.2 Laguerre Process

The real (complex) matrix Laguerre process — also known as the Wishart process — was introduced and studied in [3]. It can be constructed as a Wiener process $(L_t)_{t \in [0,\infty)}$ with $L_0$ a fixed $m \times n$ real (complex) matrix with $m \leq n$ and with each increment $L_t - L_s$ equal to an $m \times n$ matrix with entries independent real (complex) Gaussian random variables with variance $t-s$. Assuming $L_0$ has distinct singular values, the $m$ eigenvalues of $L_t L_t^\top$ (so the squares of the singular values of $L_t$) then form a point process $\lambda = \lambda(t) \in \mathbb{R}^m$ which is the unique strong solution to the SDE

$$d\lambda_i = 2 \sqrt{\lambda_i} dW_i + \beta \left[ n + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right] dt \quad (2)$$

where the $W_i$ are (1-dimensional) independent Brownian motions.

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$^4$Technically, quaternion valued ($\beta = 4$) is possible, but that presents complications we wish to avoid here.
1.2.3 Jacobi Process

The real (complex) matrix Jacobi process was introduced and studied in [6]. The method of construction given in [6] is to first construct a Brownian motion \( Q_n \) on the group of \( n \times n \) orthogonal (unitary) using a heat kernel as in [1]. Then for fixed parameters \( m, p, q \leq n \) with \( q = n - p \), one forms the \( m \times m \) matrix

\[
J_m := P_m Q_n P_m^T P_q Q_n P_q^T P_m
\]

where \( P_k \) denotes the \( k \times n \) projection matrix

\[
P_k = \begin{bmatrix} I_k & 0_{k \times (n-k)} \end{bmatrix}.
\]

It was shown in [6] that when \( J_m \) is real (complex) and \( \max\{p, q\} \geq m - 1 + 2/\beta \) for \( \beta = 1 \) (\( \beta = 2 \)), the eigenvalue process \( \lambda = \lambda(t) \) is the unique strong solution of the stochastic differential equation

\[
d\lambda_i = 2\sqrt{\lambda_i(1 - \lambda_i)} \, dW_i + \beta \left[ p(1 - \lambda_i) - q\lambda_i + \sum_{j \neq i} \frac{\lambda_i(1 - \lambda_j) + \lambda_j(1 - \lambda_i)}{\lambda_i - \lambda_j} \right] \, dt
\]

(3)

where the \( W_i \) are (1-dimensional) independent Brownian motions.

Remark 1.1. It should be noted that the construction of the Jacobi process is quite different from the previous two. We suspect that this is because the natural way to generalize the constructions of the Hermite and Laguerre processes to generalized singular values does not reduce to a simple SDE (this is discussed in Section 5.1).

1.3 Some remarks

As we claimed earlier, the goal is to show (and attempt to exploit) a seemingly strange connection between the point processes discussed in the previous section and ones derived from expected characteristic polynomials. A common theme in random matrix theory is the fact that certain distributions depend on the number field over which the entries are drawn. This dependence on the number field is typically denoted \( \beta \) and the general goal is to express solutions as a function of \( \beta \) in such a way that \( \beta = 1, 2, 4 \) corresponds to the real, complex, and quaternion number systems. Once one has an object (say, for example, a distribution) expressed in terms of \( \beta \) in this way, one can consider the collection of objects one would get by plugging in any \( \beta \geq 0 \) as a parameter. While these may not necessarily correspond to random matrix, understanding the dependence of the objects on the parameter \( \beta \) (apart from being of interest on its own) can often give insights into the original random matrix problems corresponding to the specific cases \( \beta = 1, 2, 4 \). The idea that random matrices represent three distinct points on a continuum of probability models was called the “Threefold Way” by Dyson in his original paper. The author shares a belief first put forth by Edelman\(^5\) that there is at least one other “Way” — that is, a fourth distinct point on the continuum of probability models that can be modeled using random matrices. This distinct “point” is the limit \( \beta \to \infty \), with the corresponding model being the algebra of expected characteristic polynomials known as “finite free probability”.

It is worth noting that a common interpretation of the \( \beta \) parameter is as an inverse temperature (stemming from Dyson’s original paper [7] where he noted that a number of the distributions he

\(^5\)At least as far as the author knows.
derived would be similar to what one would get from an abstract “log-gas”). In that respect, one would expect that a model for $\beta \to \infty$ would consist of the “non-random” contributions to a given random process. While we will not attempt to define “non-random” rigorously (in general), it is clear what this would mean in the specific context of the processes listed in Section 1.2. Each of these processes is a drift–diffusion process, with the parts that integrate against $dt$ being referred to as the “drift” and the parts that integrate against Brownian motions being referred to as the “diffusion.” Hence in this respect, we should expect a $\beta \to \infty$ model to exhibit similar drift behavior to the $\beta = 1, 2$ models, but without any diffusion term.

2 Hermitian eigenvalues

We start by discussing the case of Hermitian eigenvalues. Our main tool will be a theorem that first appeared (in spirit) in [19] but was proved in the generality of $\mathcal{SPM}$–conjugation-invariance in [18].

**Theorem 2.1.** Let $A, B \in \mathcal{M}_{n,n}$ be independent random matrices with $B$ being $\mathcal{SPM}$–conjugation-invariant. Furthermore, let $P, Q$ be power series for which the operators $P(\partial)$ and $Q(\partial)$ satisfy

$$E_A \{ \det [xI + A] \} = P(\partial) \{ x^n \} \quad \text{and} \quad E_B \{ \det [xI + B] \} = Q(\partial) \{ x^n \}.$$  

Then

$$E_{A,B} \{ \det [xI + A + B] \} = (PQ)(\partial) \{ x^n \}$$

where $PQ$ denotes the multiplication of the two power series.

As mentioned in Section 1.1, $X_{n,n} + X_{n,n}^\top$ is $\mathcal{SPM}$-conjugation–invariant, and so Theorem 2.1 gives us (in theory) a way to compute

$$E \left\{ \det \left[ xI - (A + \sqrt{\theta}X_{n,n}) - (A + \sqrt{\theta}X_{n,n})^\top \right] \right\}$$

for any $A \in \mathcal{M}_{n,n}$ simply by knowing

$$E_A \{ \det [xI - (A + A^\top)] \} \quad \text{and} \quad E \left\{ \det \left[ xI - \sqrt{\theta}(X_{n,n} + X_{n,n}^\top) \right] \right\}.$$  

To do so, however, we will need to find a power series $Q$ which satisfies

$$E \{ \det [xI - \theta(X_{n,n} + X_{n,n}^\top)] \} = Q(\partial) \{ x^n \}$$

This can be derived from a result of Edelman which showed that the expected characteristic polynomial of the Gaussian Unitary Ensemble is a Hermite polynomial [8]. We give an alternative derivation here:

**Lemma 2.2.** For all integers $n > 0$, we have

$$E \left\{ \det \left[ xI - \sqrt{\theta}(X_{n\times n} + X_{n\times n}^\top) \right] \right\} = e^{-\theta \beta^2} \{ x^n \}$$

where we are writing $e^{-\theta \beta^2}$ as shorthand for the operator

$$e^{-\theta \beta^2} = \sum_{i=0}^{\infty} \frac{(-\theta)^i}{i!} \partial^{2i}.$$  

6This also has the benefit of revealing the underlying semigroup.
Proof. Let $Y = \frac{X_{n \times n} + X_{n \times n}^\top}{\sqrt{2}}$. This is a symmetric matrix with random variable entries, each of which has expectation 0 and variance 1. Furthermore, all of the entries of $Y$ are independent except for the imposed symmetry $Y_{i,j} = Y_{j,i}$. We now consider the expansion of the polynomial

$$E \{ \det [xI + Y] \}$$

using the formula

$$E \{ \det [A] \} = \sum_{\sigma \in S_n} (-1)^{\text{sgn} (\sigma)} E \{ A_{1,\sigma(1)} \ldots A_{n,\sigma(n)} \} .$$

Of particular note is the fact that the diagonal entries of $Y$ do not contribute in any way to (4) — the expansion is affine in terms of each diagonal entry and so when the expectation is taken, these terms will all become 0. Hence we can replace the diagonal entries of $Y$ with (deterministic) 0 entries and not change (4) — we call the remaining matrix $Z$.

The utility of this transformation is that $Z$ can now be viewed as the weighted adjacency matrix of the complete graph with independent random variables on the edges, each of which has mean 0 and variance 1. It is now a somewhat well-known result of Godsil and Gutmann that the expected characteristic polynomial of such an adjacency matrix is the matching polynomial of the graph [12]. It is also well known (see [11]) that the matching polynomial of the complete graph on $n$ vertices is the Hermite polynomial, which has the formula [24]

$$H_n(x) = e^{- \frac{x^2}{2}} \{ x^n \} .$$

From this, it is easy to check that

$$E \{ \det [xI - tZ] \} = t^n H_n(x/t) = e^{-t^2 \frac{x^2}{2}} \{ x^n \} .$$

The theorem follows by setting $t = \sqrt{2 \theta}$.

A combination of Theorem 2.1 and Lemma 2.2 imply that for a given matrix $B \in \mathcal{M}_{n,n}$ with

$$p_B(x) = \det [xI - B - B^\top]$$

we have the formula

$$E \{ \det [xI - (B + X_{n,n}) - (B + X_{n,n})^\top] \} = e^{-\theta x^2} \{ p_B(x) \} .$$

At this point we could more or less refer to the reader to [22], where an exploration of this process was conducted for other reasons (and contains a number of interesting observations that are beyond the scope of this paper). However it will be instructive to outline relevant parts of the analysis briefly since we take a slightly different approach which will be somewhat instructive for the results in Section 5. To analyze (5), we start by finding a differential equation that it satisfies. Let $p(x)$ be an arbitrary polynomial and let

$$q(x, t) = e^{-\theta x^2} \{ p(x) \} .$$

Now define the root path $r(\theta)$ so that $q(r(\theta), \theta) = 0$. Taking a derivative in $\theta$, we have

$$0 = \frac{\partial}{\partial \theta} q(r(\theta), \theta) = q_1(r(\theta), \theta)r'(\theta) + q_2(r(\theta), \theta)$$

...
so (setting $\theta = 0$)

$$0 = p'(r)r' - p''(r)$$

or

$$r'(0) = \frac{p''(r)}{p'(r)}.$$ 

Now if we set

$$p(x) = \prod_{i=0}^{d} (x - \lambda_i)$$

then at $x = \lambda_i$ we have

$$\frac{p''(\lambda_i)}{p'(\lambda_i)} = \sum_{j \neq i} \frac{2}{\lambda_i - \lambda_j}. \quad (6)$$

Hence evolution of the roots of (5) satisfy the differential equations

$$\frac{\partial \lambda_i}{\partial \theta} = \sum_{j \neq i} \frac{2}{\lambda_i - \lambda_j}.$$ 

At this point in his presentation, Tao [22] notes “Curiously, this system resembles that of Dyson Brownian motion (except with the Brownian motion part removed, and time reversed$^7$).” Specifically, he is noting that this evolution matches (up to a constant) the drift term in the Hermite process (1).

The “curiosity” comes from the fact that one would not expect this to be the case — in some sense, this correspondence can be interpreted as asserting that a random variable $Y$ (in our case, the characteristic polynomial of a certain random matrix) satisfies

$$f(\mathbb{E} \{Y\}) = \mathbb{E} \{f(Y)\}$$

for some nonlinear function $f$ (in our case, the “$k$th root” function applied to a polynomial). This is of course not true in general, but (as we shall see in Section 3) seems to occur more generally in this context.

### 3 Singular Values

For the sake of observing the parallels with the previous section, we will quickly discuss the case of singular values. Our main tool will be a theorem that first appeared (in spirit) in [15] but was proved in the generality of $S\mathcal{P}\mathcal{M}$–bi-invariance in [17].

**Theorem 3.1.** Let $A$ and $B$ be $m \times n$ independent random matrices with $B$ being $S\mathcal{P}\mathcal{M}$–bi-invariant. Furthermore, let $P, Q$ be power series for which the operators $P(\partial_x \partial_y)$ and $Q(\partial_x \partial_y)$ satisfy

$$y^{n-m} \det [xyI - AA^T] = P(\partial_x \partial_y) \{x^m y^n\} \quad \text{and} \quad y^{n-m} \det [xyI - BB^T] = Q(\partial_x \partial_y) \{x^m y^n\}.$$ 

Then

$$\mathbb{E}_{Q,R} \left\{ y^{n-m} \det [xyI - (A + QBR)(A + QBR)^T] \right\} = (PQ)(\partial_x \partial_y) \{x^m y^n\} \quad (7)$$

where $PQ$ denotes the multiplication of the two power series.

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$^7$His comment about time reversal stems from the fact that he was using the operator $e^{t\partial^2}$ instead of $e^{-t\partial^2}$. 

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As mentioned in Section 1.1, $X_{m,n}$ is $\mathcal{SM}$-bi-invariant, and so Theorem 3.1 gives us (in theory) a way to compute
\[
E \left\{ \det \left[ xI - (A + \sqrt{\theta}X_{m,n})(A + \sqrt{\theta}X_{m,n})^\top \right] \right\}
\]
for any $A \in M_{m,n}$ simply by knowing
\[
E \left\{ \det [xI - AA^\top] \right\} \quad \text{and} \quad E \left\{ \det [xI - \theta X_{m,n}X_{m,n}^\top] \right\}.
\]
This can again be derived from a result of Edelman which showed that the expected characteristic polynomial of the Wishart Ensemble is a Laguerre polynomial [8]. In Section 4.2 we will give an alternative derivation of the following fact (see Lemma 4.2):

**Lemma 3.2.** For all integers $m, n > 0$, we have
\[
E \left\{ y^{n-m} \det [xyI - \theta X_{m,n}X_{m,n}^\top] \right\} = e^{-\theta \partial_x \partial_y} \{ x^m y^n \}
\]
where we are writing $e^{-\theta \partial_x \partial_y}$ as shorthand for the operator
\[
e^{-\theta \partial_x \partial_y} = \sum_{i=0}^{\infty} \frac{(-\theta)^i}{i!} \partial_x^i \partial_y^i.
\]
A combination of Theorem 3.1 and Lemma 3.2 imply that for a given matrix $B \in M_{m,n}$ with
\[
p_{BB^\top}(x) = \det [xI - BB^\top]
\]
we have the formula
\[
y^{n-m}E \left\{ \det [xyI - (B + X_{m,n})(B + X_{m,n})^\top] \right\} = e^{-\theta \partial_x \partial_y} \{ y^{n-m}p_{BB^\top}(xy) \}.
\]

We now proceed similarly to Section 2. For the sake of simplicity, we will assume $n \geq m$ and then find a differential equation that (8) satisfies. Let $p(x, y)$ be an arbitrary polynomial and let
\[
q(x, y, t) = e^{-\theta \partial_x \partial_y} \{ p(x, y) \}.
\]
Now define the root path $r(\theta)$ so that $q(r(\theta), 1, \theta) = 0$. Taking a derivative in $\theta$, we have
\[
0 = \frac{\partial}{\partial \theta} q(r(\theta), 1, \theta) = q_1(r(\theta), 1, \theta)r'(\theta) + q_3(r(\theta), 1, \theta)
\]
so (setting $\theta = 0$)
\[
0 = p_1(r, 1)r' - p_{12}(r, 1)
\]
or
\[
r'(0) = \frac{p_{12}(r, 1)}{p_1(r, 1)}.
\]
We can now use the fact that $x$ and $y$ play similar roles in (8), so that for $p(x, y) = y^{n-m}p_{BB^\top}(xy)$ we have
\[
p_1 = y^{n-m+1}p'_{BB^\top}(xy) \quad \text{and} \quad p_2 = (n-m)y^{n-m-1}p_{BB^\top}(xy) + xy^{n-m}p'_{BB^\top}(xy).
\]
Hence we can write
\[ p_2 = (n - m) \frac{p}{y} + \frac{x}{y}p_1 \quad \text{and} \quad p_{12} = (n - m + 1) \frac{p_1}{y} + \frac{x}{y}p_{11} \]

Now if we set \( y = 1 \), we see that (9) becomes
\[ r'(0) = (n - m + 1) + \frac{r p_1(r, 1)}{p_1(r, 1)}, \]
and so for \( x = \lambda_i \), (6) implies the evolution of the roots of \( p_{BB^T} \) satisfy the differential equations
\[ \frac{\partial \lambda_i}{\partial \theta} = (n - m + 1) + \lambda_i \sum_{j \neq i} \frac{2}{\lambda_i - \lambda_j} = n + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \]

So, in particular, we see that the evolution again exhibits behavior similar to the drift term in the Brownian motion system (2). Furthermore, one can consider this correspondence to be “curious” for the same reasons as the Hermite case.

4 A tool for generalized singular values

We start by briefly reviewing the generalized singular value decomposition and and its associated “characteristic polynomial” as motivation.

4.1 The generalized singular value decomposition

The generalized singular value decomposition (GSVD) is often described as a decomposition on a pair of matrices, however we prefer the approach of [23] of using a single block matrix. For simplicity, we will restrict to the case when this block matrix has full column rank, however we will find a way to eliminate this requirement shortly.

For a matrix \( M \in M_{n_1+n_2,k} \) with rank \( k \) and block form
\[ M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \]

the GSVD provides a simultaneous decomposition of \( M_1 \) and \( M_2 \) into
\[ M_1 = U_1 Ch \quad \text{and} \quad M_2 = U_2 Sh \]

where

- \( U_1 \in M_{n_1,n_1} \) satisfies \( U_1^T U_1 = U_1 U_1^T = I_{n_1} \)
- \( U_2 \in M_{n_2,n_2} \) satisfies \( U_2^T U_2 = U_2 U_2^T = I_{n_2} \)
- \( C \in M_{n_1,k} \) and \( S \in M_{n_2,k} \) are pseudo-diagonal with \( C^T C + S^T S = I_k \), and
- \( H \) is an \( k \times k \) invertible matrix.
The diagonal entries of \( C \) and \( S \) satisfy \( c_i^2 + s_i^2 = 1 \), and because of this, these matrices are often referred to as \textit{cosine} and \textit{sine} matrices.

The standard singular value decomposition can then be recovered by letting \( M_2 = I \) and noting that this implies that \( M_1 \) has the decomposition

\[
M_1 = U_1 CS^{-1} U_2.
\]

When \( M \) has rank \( k \), there is an easy way to calculate the generalized singular values without needing to form the entire decomposition. Letting \( W_1 = M_1^\top M_1 \) and \( W_2 = M_2^\top M_2 \), it is easy to check that

\[
W = (W_1 + W_2)^{-1/2} W_1 (W_1 + W_2)^{-1/2}
\]

is a positive semidefinite Hermitian matrix that has the same eigenvalues\(^8\) as \( C^\top C \). Letting

\[
\det \left[ xI - (W_1 + W_2)^{-1/2} W_1 (W_1 + W_2)^{-1/2} \right] = \det [W_1 + W_2]^{-1} \det [x(W_1 + W_2) - W_1].
\]

and so we can use the characteristic polynomial

\[
\det [x(W_1 + W_2) - W_1] = \det [(x - 1)W_1 + xW_2]
\]

which has the added benefit that it does not require \( W_1 + W_2 \) to be invertible.

\subsection{A generalized characteristic polynomial}

For matrices \( A \in \mathcal{M}_{n_1,k} \) and \( B \in \mathcal{M}_{n_2,k} \) we will consider the polynomial

\[
p_{A,B}(x,y,z) = \det \left[xI + yA^\top A + zB^\top B \right].
\]

(13)

Note that \( p_{A,B}(x,0,-1) \) reduces to the polynomial \( p_{BB^\top}(x) \) from Section 3 and \( p_{A,B}(0,z - 1, z) \) reduces to (12). Hence understanding the evolution of the polynomial

\[
P_{A + \sqrt{\theta}X, B + \sqrt{\theta}X}
\]

will allow us to derive the information we want regarding both decompositions.

The main property of \( X_{m,n} \) that we will exploit in our analysis is the fact that it is \( S\mathcal{P}\mathcal{M} - \text{bi-invariant} \). In particular, it was shown in [17] that for any \( S\mathcal{P}\mathcal{M} - \text{bi-invariant} \) matrices \( C, D \), the coefficients of

\[
\mathbb{E}\left\{ p_{A+C,B+D}(x,y,z) \right\}
\]

are each multilinear functions of the coefficients of

\[
\mathbb{E}\left\{ p_{A,B}(x,y,z) \right\} \quad \text{and} \quad \mathbb{E}\left\{ p_{C,D}(x,y,z) \right\}.
\]

To state the correspondence, it will be beneficial (for the moment) to instead work with a slight transformation of the polynomials \( p_{A,B} \):

\[
q_{A,B}(x,y,z) = y^{n_1} z^{n_2} p(x, 1/y, 1/z).
\]

The results of [17] then imply the following:

---

\(^{8}\)What one actually calls the “generalized singular values” (\( c_i?, c_i/s_i? \)) is subject to interpretation, but all interpretations are functions of the eigenvalues of \( C^\top C \), which will always be our subject of interest in this paper.
**Theorem 4.1.** Let \( A, C \in \mathcal{M}_{n_1,k} \) and \( B, D \in \mathcal{M}_{n_2,k} \) be random matrices with \( C \) and \( D \) SPM-bi-invariate. Now let \( F \) and \( G \) be bivariate polynomials for which
\[
\mathbb{E} \{ q_{A,B}(x,y,z) \} = F(\partial_x \partial_y, \partial_x \partial_z) \left\{ x^k y^{n_1} z^{n_2} \right\} \quad \text{and} \quad \mathbb{E} \{ q_{C,D}(x,y,z) \} = G(\partial_x \partial_y, \partial_x \partial_z) \left\{ x^k y^{n_1} z^{n_2} \right\}.
\]

Then
\[
\mathbb{E} \{ q_{A+C,B+D}(x,y,z) \} = F(\partial_x \partial_y, \partial_x \partial_z) G(\partial_x \partial_y, \partial_x \partial_z) \left\{ x^k y^{n_1} z^{n_2} \right\}.
\]

Given Theorem 4.1, an obvious next step to computing the polynomial in (14) would be to find the value of the polynomial \( \mathbb{E} \left\{ q_{aX_{n_1,k},bX_{n_2,k}}(x,y,z) \right\} \) for general \( a, b \in \mathbb{R} \). This can be done combinatorially (as in Theorem 2.2), but we will find it easier (and more instructive) to use a central limit theorem argument.

**Lemma 4.2.** Let \( \{A_i\}_{i=1}^\infty \subseteq \mathcal{M}_{n_1,k} \) and \( \{B_i\}_{i=1}^\infty \subseteq \mathcal{M}_{n_2,k} \) be sequences of independent random SPM-bi-invariate matrices for which
\[
\mathbb{E} \{ \text{Tr} [A_i A_i^\top] \} = \theta_1 n_1 k \quad \text{and} \quad \mathbb{E} \{ \text{Tr} [B_i B_i^\top] \} = \theta_2 n_2 k
\]
for all \( i \). Define the random matrices
\[
C_m = \frac{\sum_{i=1}^n A_i}{\sqrt{m}} \quad \text{and} \quad D_m = \frac{\sum_{i=1}^n B_i}{\sqrt{m}}.
\]

Then
\[
\lim_{m \to \infty} \mathbb{E} \{ q_{C_m,D_m}(x,y,z) \} = e^{\theta_1 \partial_x \partial_y + \theta_2 \partial_x \partial_z} \left\{ x^k y^{n_1} z^{n_2} \right\}
\]

**Proof.** Due to the boundedness of the underlying random variables, we can write
\[
\mathbb{E} \{ q_{A_i / \sqrt{m}, B_i / \sqrt{m}}(x,y,z) \} = x^k y^{n_1} z^{n_2} + \frac{\theta_1 n_1 k}{m} x^{k-1} y^{n_1-1} z^{n_2} + \frac{\theta_2 n_2 k}{m} x^{k-1} y^{n_1} z^{n_2-1} + O \left( \frac{1}{m^2} \right)
\]
\[
= 1 + \frac{\theta_1}{m} \partial_x \partial_y + \frac{\theta_2}{m} \partial_x \partial_z + O \left( \frac{1}{m^2} \right) \left\{ x^k y^{n_1} z^{n_2} \right\}
\]
as \( m \to \infty \). So by Theorem 4.1, we have
\[
\mathbb{E} \{ q_{C_m,D_m}(x,y,z) \} = \left( 1 + \frac{\theta_1}{m} \partial_x \partial_y + \frac{\theta_2}{m} \partial_x \partial_z + O \left( \frac{1}{m^2} \right) \right)^m \left\{ x^k y^{n_1} z^{n_2} \right\}
\]
which converges to the claimed polynomial as \( m \to \infty \). \( \square \)

We now show that (15) is (in fact) the same polynomial as \( \mathbb{E} \left\{ q_{aX_{n_1,k},bX_{n_2,k}}(x,y,z) \right\} \).

**Corollary 4.3.**
\[
\mathbb{E} \left\{ q_{aX_{n_1,k},bX_{n_2,k}}(x,y,z) \right\} = e^{a \partial_x \partial_y + b \partial_x \partial_z} \left\{ x^k y^{n_1} z^{n_2} \right\}
\]

**Proof.** Let \( G_1 \in \mathcal{M}_{n_1,k} \) and \( G_2 \in \mathcal{M}_{n_2,k} \) be matrices of independent standard Gaussians and let \( a \) and \( b \) be real numbers. We will apply Lemma 4.2 in the case where the \( A_i \) are independent copies of \( aG_1 \) and the \( B_i \) are independent copies of \( bG_2 \).
On the one hand, it is easy to see using the properties of standard Gaussians that
\[ C_m = \sum_{i=1}^{n} A_i \sqrt{\frac{m}{m}} = aG_1 \quad \text{and} \quad D_m = \sum_{i=1}^{n} B_i \sqrt{\frac{m}{m}} = bG_2 \]
for all \( m \). Hence we have
\[
\lim_{m \to \infty} \mathbb{E} \{ q_{C_m, D_m}(x, y, z) \} = \mathbb{E} \{ q_{aG_1, bG_2}(x, y, z) \}.
\]

On the other hand, it is easy to calculate that
\[
\mathbb{E} \{ \text{Tr} [a^2 G_1^2] \} = a^2 n_1 k \quad \text{and} \quad \mathbb{E} \{ \text{Tr} [b^2 G_2^2] \} = b^2 n_2 k
\]
so Lemma 4.2 implies that
\[
\lim_{m \to \infty} \mathbb{E} \{ q_{C_m, D_m}(x, y, z) \} = e^{\theta_1 \partial x \partial y + \theta_2 \partial x \partial z} \left\{ x^k y^{n_1} z^{n_2} \right\}
\]
Equating the two gives us
\[
\mathbb{E} \{ q_{aG_1, bG_2}(x, y, z) \} = e^{a^2 \partial x \partial y + b^2 \partial x \partial z} \left\{ x^k y^{n_1} z^{n_2} \right\}
\]
The remainder of the proof follows from the fact that \( q_{A, B} \) is at most quadratic in the entries of the matrices \( A \) and \( B \). So when those entries are random variables, the only contributions to the polynomial come from the first two moments. Hence having a matrix of independent standard Gaussians will give the same result as any set of independent mean 0, variance 1 random variables, and so we have
\[
\mathbb{E} \{ q_{aX_{n_1, k}, bX_{n_2, k}}(x, y, z) \} = \mathbb{E} \{ q_{aG_1, bG_2}(x, y, z) \} = e^{a^2 \partial x \partial y + b^2 \partial x \partial z} \left\{ x^k y^{n_1} z^{n_2} \right\}
\]
as claimed.

Theorem 4.1 combined with Corollary 4.3 gives us the following formula for any matrices \( A \in \mathcal{M}_{n_1, k} \) and \( B \in \mathcal{M}_{n_2, k} \):
\[
\mathbb{E} \left\{ q_{A+aX_{n_1, k}, B+bX_{n_2, k}}(x, y, z) \right\} = e^{a^2 \partial x \partial y + b^2 \partial x \partial z} \left\{ q_{A, B}(x, y, z) \right\}.
\] (16)
We end this subsection by showing how one can translate this back to the polynomials we are truly interested in. For the sake of notation, for a polynomial \( f = f(y) \), let us define the operator \( U_k^y \) as
\[
U_k^y \{ p(y) \} = y^k p \left( \frac{1}{y} \right).
\] (17)

**Lemma 4.4.** Let \( i, j \) be integers with \( j \leq s \). Then
\[
U_y^s \circ e^{\theta \partial_x \partial_y} \circ U_y^s \left\{ x^i y^j \right\} = (1 + \theta y \partial_x)^{s-j} \left\{ x^i y^j \right\}
\]
Proof. We compute:

\[ U^s_y \circ e^{\theta \partial_y} \circ U^s_y \{ x^i y^j \} = U^s_y \circ e^{\theta \partial_y} \{ x^i y^{s-j} \} \]

\[ = U^s_y \left\{ \sum_k \frac{\theta^k}{k!} \partial_x^k \partial_y^k \{ x^i y^{s-j} \} \right\} \]

\[ = U^s_y \left\{ \sum_k \theta^k \partial_x^k \left( s-j \atop k \right) x^i y^{s-j-k} \right\} \]

\[ = (1 + \theta y \partial_x)^{s-j} \left\{ x^i y^j \right\}. \]

\[ \qed \]

4.3 Returning to the decompositions

We now return the respective decompositions to give a proof of Lemma 3.2 and the corresponding result for generalized singular values. Consider formula (16) in the case when \( A \) and \( B \) are \( 0 \)-matrices and \( a = b = 1 \). Then \( q_{A,B}(x, y, z) = x^k \) and so Lemma 4.4 implies

\[ \mathbb{E}\left\{ p_{X_{n_1,k},X_{n_2,k}}(x, y, z) \right\} = (1 + y \partial_x)^{n_1} (1 + z \partial_x)^{n_2} \left\{ x^k \right\}. \]

As noted at the beginning of Section 4.2, we can get the singular value polynomial by setting \( y = -1 \) and \( z = 0 \):

\[ \mathbb{E}\left\{ \det \left[ xI - X_{n_1,k}^\top X_{n_1,k} \right] \right\} = (1 - \partial_x)^{n_1} \left\{ x^k \right\} = (-1)^k k! L_k^{(n_1-k)}(x) \]

where \( L_k^{(\alpha)}(x) \) is a Laguerre polynomial [24]. Setting \( x = 0 \) and \( y = z = -1 \), on the other hand, gives us the generalized singular value polynomial:

\[ \mathbb{E}\left\{ \det \left[ (z-1)X_{n_1,k}^\top X_{n_1,k} + zX_{n_2,k}^\top X_{n_2,k} \right] \right\} = (1 + (z-1) \partial_x)^{n_1} (1 + z \partial_x)^{n_2} \left\{ x^k \right\} = P_k^{(n_2-k,n_1-k)}(2x-1) \]

where \( P_k^{(\alpha,\beta)}(x) \) is a Jacobi polynomial [24]. For those familiar with random matrix theory, these two results are perhaps not surprising as the random matrix ensembles one would get by using i.i.d. standard normals random variables in the random matrices \( X_{n,k}, X_{n_1,k} \) and \( X_{n_2,k} \) are often called the Hermite, Laguerre, and Jacobi ensembles (for this reason) [10].

5 Brownian Motion

Using the results in Section 4.2, we have reduced the problem of interest to understanding the effects of a certain differential operator on a generalized characteristic polynomial. The operator of interest is \( Q_{s,t}^\theta \) defined by

\[ Q_{s,t}^\theta \{ x^i y^j z^k \} \mapsto (1 + \theta y \partial_x)^{s-j} (1 + \theta z \partial_x)^{t-k} \left\{ x^i y^j z^k \right\} \]

and then extended to general polynomials linearity.
To understand the evolution of these polynomials, we first derive a differential equation that this operator satisfies. For a fixed polynomial \( p(x, y, z) \), we define the polynomial

\[
\tilde{p}(x, y, z, \theta) = Q_{s,t}^\theta \{ p(x, y, z) \}
\]

and then consider variables \( \tilde{x} = \tilde{x}(\theta), \tilde{y} = \tilde{y}(\theta), \tilde{z} = \tilde{z}(\theta) \) for which

\[
\tilde{p}(\tilde{x}(\theta), \tilde{y}(\theta), \tilde{z}(\theta), \theta) = 0.
\]

In particular, we can take a derivative in \( \theta \) to get

\[
\tilde{p}_1 \tilde{x}' + \tilde{p}_2 \tilde{y}' + \tilde{p}_3 \tilde{z}' + \tilde{p}_4 = 0.
\]

**Lemma 5.1.**

\[
\tilde{p}_4(x, y, z, 0) = (n_1 y + n_2 z)\tilde{p}_1(x, y, z, 0) - y^2 \tilde{p}_{12}(x, y, z, 0) - z^2 \tilde{p}_{13}(x, y, z, 0)
\]

**Proof.** Write

\[
p(x, y, z) = \sum_{i,j,k} c_{i,j,k} x^i y^j z^k
\]

where the degree of \( y \) is at most \( s \) and the degree of \( z \) is at most \( t \). Then

\[
Q_{s,t}^\theta \{ p(x, y, z) \} = \sum_{i,j,k} c_{i,j,k} (1 + \theta y \partial_x)^{n_1 - j} (1 + \theta z \partial_x)^{n_2 - k} \left\{ x^i y^j z^k \right\}
\]

\[
= \sum_{i,j,k} c_{i,j,k} \left( 1 + \theta ((n_1 - j)y \partial_x + (n_2 - k)z \partial_x) + O(\theta^2) \right) \left\{ x^i y^j z^k \right\}
\]

\[
= \sum_{i,j,k} c_{i,j,k} \left( x^i y^j z^k + i(n_1 - j)\theta x^{i-1} y^j z^k + i(n_2 - k)\theta x^i y^{j-1} z^{k+1} \right) + O(\theta^2)
\]

and so

\[
\frac{\partial}{\partial \theta} Q_{n_1,n_2}^\theta \{ p(x, y, z) \} \bigg|_{\theta=0} = \sum_{i,j,k} i c_{i,j,k} \left( (n_1 - j)x^{i-1} y^j z^k + (n_2 - k)x^i y^{j-1} z^{k+1} \right).
\]

Now we rewrite

\[
i(n_1 - j)x^{i-1} y^j z^k = in_1 x^{i-1} y^j z^k - ijx^{i-1} y^j z^k = n_1 y \partial_x \left\{ x^i y^j z^k \right\} - y^2 \partial_x \partial_y \left\{ x^i y^j z^k \right\}
\]

and similarly

\[
i(n_2 - k)x^i y^{j-1} z^{k+1} = in_2 x^{i-1} y^j z^{k+1} - ikx^{i-1} y^j z^{k+1} = n_2 z \partial_x \left\{ x^i y^j z^k \right\} - z^2 \partial_x \partial_z \left\{ x^i y^j z^k \right\}.
\]

Hence

\[
\tilde{p}_4(x, y, z, 0) = \frac{\partial}{\partial \theta} Q_{n_1,n_2}^\theta \{ p(x, y, z) \} \bigg|_{\theta=0} = (n_1 y + n_2 z)p_1(x, y, z) - y^2 p_{12}(x, y, z) - z^2 p_{13}(x, y, z)
\]

and the result follows from the fact that \( \tilde{p}(x, y, z, 0) = p(x, y, z) \).
It is worth mentioning that the equation we have derived:
\[ \tilde{p}_1 \tilde{x}' + \tilde{p}_2 \tilde{y}' + \tilde{p}_3 \tilde{z}' = -(n_1 y + n_2 z) \tilde{p}_1 + y^2 \tilde{p}_{12} + z^2 \tilde{p}_{13} \]
still has a bit of freedom to be manipulated due to the homogeneity if \( p \). How we do that manipulation will depend on how we intend to use the polynomial. For example, if our goal is going to be to plug in \( \tilde{y} = \tilde{w} - 1 \) and \( \tilde{z} = \tilde{w} \) at some point, then the terms
\[ y^2 p_{12} \quad \text{and} \quad z^2 p_{13} \]
will not reduce to functions of \( \tilde{w} \) easily. We can fix this as follows:

**Lemma 5.2.** Let \( p(x, y, z) \) be a \( k \)-homogeneous polynomial. Then
\[ y^2 p_{12} + z^2 p_{13} = (k - 1)(y + z)p_1 - x(y + z)p_{11} - yz(p_{12} + p_{13}). \]

**Proof.** Since \( p \) is \( k \)-homogeneous, \( p_1 \) is \( k - 1 \) homogeneous, and so
\[ xf_{11} + yf_{12} + zf_{13} = (k - 1)f_1 \]
Solving for \( f_{13} \) (and separately for \( f_{12} \)) then gives
\[ z^2 f_{13} = -xzf_{11} - yzf_{12} + z(d - 1)f_1 \quad \text{and} \quad y^2 f_{12} = -xyf_{11} - yzf_{13} + y(d - 1)f_1. \]
and the lemma follows by adding them together. \( \square \)

Hence we have derived the following equation for \( p \) at the point \( (\tilde{x}, \tilde{y}, \tilde{z}) \) for \( \theta = 0 \):
\[ p_1 \tilde{x}' + \tilde{p}_2 \tilde{y}' + \tilde{p}_3 \tilde{z}' = -(n_1 - k + 1)\tilde{y}p_1 - (n_2 - k + 1)\tilde{z}p_1 - \tilde{x}(\tilde{y} + \tilde{z})p_{11} - \tilde{y}\tilde{z}(p_{12} + p_{13}). \quad (19) \]

### 5.1 Jacobi Process

We now make the substitution described in (12), letting
\[ h(x, u) = p(x, u - 1, u) \]
and setting \( \tilde{y} = \tilde{w} - 1 \) and \( \tilde{z} = \tilde{w} \) in (19) we get
\[ \tilde{x}' h_1(\tilde{x}, \tilde{w}) + \tilde{w}' h_2(\tilde{x}, \tilde{w}) = -(n_1 - k + 1)(\tilde{w} - 1)h_1(\tilde{x}, \tilde{w}) - (n_2 - k + 1)\tilde{w}h_1(\tilde{x}, \tilde{w}) \]
\[ - \tilde{x}(2\tilde{w} - 1)h_{11}(\tilde{x}, \tilde{w}) - \tilde{w}(\tilde{w} - 1)h_{12}(\tilde{x}, \tilde{w}) \]
which for \( \tilde{x} = 0 \) simplifies to
\[ \frac{\tilde{w} h_2}{h_1} = -(n_1 - k + 1)(\tilde{w} - 1) - (n_2 - k + 1)\tilde{w} - \tilde{w}(\tilde{w} - 1) \frac{h_{12}}{h_1}. \quad (20) \]

We would like to compare this to the evolution of the random matrix version of the Jacobi process given in (3) with parameters \( p = n_1 \) and \( q = n_2 \). At first glance, these are not the same, but we claim that this is because (20) is actually significantly more general than (3).
Before addressing this, however, we wish to point out that not only are (20) and (3) different, they are fundamentally different. In particular, (3) is an eigenvalue evolution that only depends on the original eigenvalues. That is, if we let
\[ A_\theta = A + \sqrt{\theta}X_{n_1,k} \quad \text{and} \quad B_\theta = B + \sqrt{\theta}X_{n_2,k} \]
then the evolution of (3) would imply that one can compute the eigenvalues of
\[ (A_\theta A_\theta^T + B_\theta B_\theta^T)^{-1/2} A_\theta A_\theta^T (A_\theta A_\theta^T + B_\theta B_\theta^T)^{-1/2} \]  
(21)
as a function of the eigenvalues of
\[ (A_0 A_0^T + B_0 B_0^T)^{-1/2} A_0 A_0^T (A_0 A_0^T + B_0 B_0^T)^{-1/2}. \]
This is not the case for (20), and for good reason. As we will see in Section 6, the eigenvalues of (21) will depend on (among other things) the relationship between the eigenvectors of \(A_0 A_0^T\) and \(B_0 B_0^T\), and so to some extent it is actually surprising that we can get all of the necessary information even from the complete polynomial \(h(\tilde{x}, \tilde{w})\).

5.2 The matrix Jacobi process, revisited

To see why the two solutions differ, let us first review how the standard matrix Jacobi process is defined. As we saw in Section 4.1, the method for obtaining the generalized singular values of
\[ M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \]
was via the formula (10) where \(W_1 = M_1^T M_1\) and \(W_2 = M_2^T M_2\). Note that \(W_1 + W_2 = M^T M\), and that\[ M_1 = \begin{bmatrix} I_{n_1} & 0 \end{bmatrix} M \]
and so another way to write (10) would be as
\[ W = (M^T M)^{-1/2} M^T \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \end{bmatrix} M (M^T M)^{-1/2} M^T. \]
One can simplify this using the (normal) singular value decomposition: letting \(M = U \Sigma V^T\), and simplifying, we get that
\[ W = \begin{bmatrix} I_k & 0 \end{bmatrix} U^T \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \end{bmatrix} U \begin{bmatrix} I_k \\ 0 \end{bmatrix} = C^T C \]
where \(C\) is the upper \(n_1 \times k\) corner of the unitary matrix \(U \in \mathcal{M}_{n_1+n_2,n_1+n_2}\).

When \(M\) is a matrix of Gaussians, the derived distribution on \(U\) is the Haar-distribution. This shows the equivalence of two methods for creating a Jacobi ensemble (a fact that has been used numerous times in the literature [4, 9]):

1. as the generalized singular values of a random Gaussian matrix
2. as the singular values of the upper corner of a Haar-distributed unitary matrix

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The issue occurs when one tries to turn this into a process. The standard matrix Jacobi process uses the “upper corner” model, replacing the Haar distribution with a unitary Brownian motion (starting, possibly at a fixed unitary matrix) \cite{6}. And this is precisely where the two processes diverge — for general matrices \( A_\theta \), approximating the left eigenvectors of \( A_\theta \) with a unitary Brownian motion centered at the left eigenvectors of \( A_0 \) is only that (an approximation). The actual distribution will depend on the singular values of \( A \), which (in turn) will depend on the relationship between the two blocks.

On the other hand, if one was able to artificially force the singular values of \( A_\theta \) to remain the same throughout the process, then this extra effect (while not necessarily zero) will not change, and so one can then hope to derive an evolution equation which holds for all \( A_\theta \). And this is (essentially) what is being done in the “upper corner process” — by using a unitary Brownian motion, one is forcing all of the singular values of the \( A_\theta \) to be 1 (for all \( \theta \)) \footnote{This is often how Brownian motion is defined — as the process on the space of unitary matrices which has instantaneous change equal to that of the additive process \cite{20}.}.

Lemma 5.3. Let \( A \in \mathcal{M}(n_1+n_2,k) \) be such that \( A^\top A = I_k \). Then \( h_1(0,u) = h_2(0,u) \) and

\[
\frac{h_{12}(0,\lambda_i)}{h_1(0,\lambda_i)} = \sum_{j \neq i} \frac{2}{\lambda_i - \lambda_j}
\]

where \( \{\lambda_i\}_{i=1}^k \) are the roots of \( h(0,u) \).

Proof. Letting

\[
A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}
\]

we have that \( A^\top A = I_k \) is equivalent to \( A_1^\top A_1 + A_2^\top A_2 = I \), so plugging these into the generalized characteristic polynomial (13), you get

\[
h(x,u) = p(x,u-1,u) = \det [xI + (u-1)A_1^\top A_1 + u(I - A_1^\top A_1)] = \det [(x+u)I - A_1^\top A_1].
\]

Hence derivatives in \( x \) and derivatives in \( u \) are the same, so (in particular) \( h_1(0,u) = h_2(0,u) \). For the same reason, we have

\[
\frac{h_{12}(0,u)}{h_1(0,u)} = \frac{h_{22}(0,u)}{h_2(0,u)}
\]

where if we let \( g(u) = h(0,u) \), then we can again use (6) to get

\[
\frac{h_{22}(0,\lambda_i)}{h_2(0,\lambda_i)} = \frac{g''(\lambda_i)}{g'(\lambda_i)} = \sum_{j \neq i} \frac{2}{\lambda_i - \lambda_j}
\]

as claimed. \( \square \)

Hence whenever \( A_\theta^\top A_\theta = I_k \), (20) reduces to

\[
\frac{\partial \lambda_i}{\partial \theta} = -(n_1 - k + 1)(\lambda_i - 1) - (n_2 - k + 1)\lambda_i - \sum_{j \neq i} \frac{2\lambda_i(\lambda_i - 1)}{\lambda_i - \lambda_j}
\]

\[
= -n_1(\lambda_i - 1) - n_2\lambda_i + \sum_{j \neq i} \frac{\lambda_i(1 - \lambda_j) + \lambda_j(1 - \lambda_i)}{\lambda_i - \lambda_j}
\]
matching the drift term in (3) exactly. To end the section, we find it worth mentioning that the GSVD of a unitary matrix is a decomposition that is well-studied in its own right (known as the CS-decomposition) [13]. This fact was noted in [9] as well, however only in the context of the Jacobi ensemble itself (not the process). Hence this might be the more appropriate “decomposition” to associate directly to the matrix Jacobi process and, if this is the case, then it suggests that a random matrix process related to the true GSVD has yet to be considered in full generality.

6 Simulation

The purpose of this section is 3-fold. Firstly, we hope to convince the reader of the assertion made in Section 5.1 that the point process described by 4. is fundamentally different than the point processes defined by 2. and 3. in the respect that future point configurations are not determined by the current point configuration alone.

Secondly, we hope to explore the degree to which the \( k \)th root of the expected characteristic polynomial correspond to the expected value of the \( k \)th root. The use of free probability as a tool for estimating random matrix statistics has a long history, hence there is hope that finite free probability has similar potential. The obvious upside of finite free probability is that it holds in fixed dimensions, and so one might hope that in situations where a fixed dimensions ensemble has features that do not remain in large dimensions that finite free probability could give better results. The obvious downside is that, unlike in the case of free probability, it is unclear how well the statistics that one is able to compute (the expected characteristic polynomials) relate to the statistics one might want to compute (moments or probabilities). Thus we hope to convey some indication as to what the potential is in this respect.

Thirdly, we wish to explore the recurring pattern in this paper of the point process defined by the polynomial convolution matching the “non-random” or “as \( \beta \to \infty \)” part of some random matrix process. While it is only a limited set, we wish to examine the \( \beta = 1 \) cases and \( \beta = 2 \) cases to see whether we can find support for such a claim experimentally and also to see whether there is some further intuition we can gain in this regard (like the appearance of monotonicity, for example).

6.1 The experiment

We first describe the experiment. We set \( k = 4, n_1 = 5, n_2 = 10 \), and use the following \( k \times k \) diagonal matrices:\(^{10}\)

\[
D_1 = \begin{bmatrix} 2 & 2 \\ 2 & 5 \\ 5 & 5 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 \\ 1 & 500 \end{bmatrix}, \quad K = \begin{bmatrix} 500 & 1 \\ 1 & 1 \end{bmatrix}
\]

\(^{10}\)One would not be wrong to consider the choice of matrices \( H \) and \( K \) to be somewhat extreme and not representative of a typical application. One can show that even small differences between \( H \) and \( K \) result in different behaviors, but “how different” is unclear. The choice of these particular matrices was made with the hopes of making various features visually recognizable.
We add \((n_1 - k)\) rows of 0 to \(D_1\) to form \(F_1 \in \mathcal{M}_{n_1,k}\) (and similarly to form \(F_2 \in \mathcal{M}_{n_2,k}\) from \(D_2\)). We then pick constants \(\delta, \epsilon\) and form
\[
A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \delta \begin{bmatrix} F_1 H \\ F_2 H \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \epsilon \begin{bmatrix} F_1 K \\ F_2 K \end{bmatrix}.
\]
with \(\delta, \epsilon\) chosen so that \(\text{Tr}[A^\top A] = \text{Tr}[B^\top B] = 1\).

Using (10), one can easily check that multiplying on the right by an invertible matrix does not change the eigenvalues of \(C^\top C\) and so the squares of the generalized singular values of \(A\) and \(B\) will be the same:
\[
\begin{pmatrix} 4 & 4 & 25 & 25 \\ 13 & 13 & 34 & 34 \end{pmatrix} \approx (0.308, 0.308, 0.735, 0.735).
\]

For each trial, we consider two independent point processes: one starting at \(A_0 = A\) and the other at \(B_0 = B\). At time step \(i\), we generate a random matrix \(Z_i \in \mathcal{M}_{(n_1 + n_2),k}\) with independent normal entries \((\mu = 0, \sigma^2 = 10^{-8})\) to add to both \(A\) and \(B\):
\[
A^i \leftarrow A^{i-1} + Z_i \quad \text{and} \quad B^i \leftarrow B^{i-1} + Z_i.
\]
We calculate the squares of the generalized singular values \(c^2_{A^i}, c^2_{B^i} \in \mathbb{R}^k\) (both in increasing order). We then average over 500 trials, obtaining the (average) squared generalized singular values \(\bar{c}^2_{A^i}, \bar{c}^2_{B^i} \in \mathbb{R}^k\). These will be compared to the roots of the expected generalized singular value polynomials
\[
\tilde{p}_{A^i}(x, y, z) = Q_{n_1, n_2}^{(i\sigma^2)} \{ p_A(x, y, z) \} \quad \text{and} \quad \tilde{p}_{B^i}(x, y, z) = Q_{n_1, n_2}^{(i\sigma^2)} \{ p_B(x, y, z) \}.
\]

To simplify the wording slightly, rather than writing (for example) “notice that \(\tilde{p}_{A^i}\) (respectively \(\tilde{p}_{B^i}\)) are”, we will write simply “notice that the \(\tilde{p}_{M_i}\) are” (essentially anywhere there is an \(M\), it should be taken as being a statement for both \(A\) and \(B\), separately).

### 6.2 Results

We first hope to convey the fact that the evolution of \(A\) and \(B\) — despite starting at the same generalized singular values and using the same “random” matrices — are fundamentally different. For this purpose we can examine the evolution of the point process over various time scales (Figure 1). There are, of course, similarities, it is easy to see that both \(A\) and \(B\) exhibit repulsion between the paths early on in addition to converging to the roots of a Jacobi polynomial in the long term (both of which should be expected). However \(B\) seems to converge very quickly to the asymptotic limit whereas \(A\) takes a (very) long time. Of particular interest is the path of the largest value (in red); the path of \(A\) actually moves away from the eventual asymptotic limit for some time during the middle range. This is in direct contradiction to the other paths of \(A\) and all of the paths of \(B\) which seem to move in the direction of the asymptotic limit.

For the second goal, we will focus on the part of the process where the most action happens. Figure 2 shows the first 100 steps of the process for each matrix process (in colors) as well as the paths taken by the roots of the \(\tilde{p}_{M_i}\), in black). The third plot shows the two processes together.

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11The reason for adding the same matrix is to ensure that any differences that will appear are not a result of random effects. However numerous tests were done using independent matrices and each gave similar results.
Figure 1: Evolution of roots of $\tilde{p}_{A^i}$ (+) vs $\tilde{p}_{B^i}$ (-) vs asymptotes (dashed lines).

Figure 2 suggests the $\tilde{p}_{M^i}$ are capturing the general behavior of each $\bar{c}_{M^i}$ quite well. Furthermore, the distance between the two seems to be rather consistent. This is corroborated by Figure 3, which shows the actual distributions of $\bar{c}_{A_i}^2$, normalized so that the root of $\tilde{p}_{A_i}$ is at the center.

Figure 2: Experimental values (colors) versus polynomial predictions (black) for the first 100 steps.

Figure 3: The distributions of $\bar{c}_{A_i}^2$ shifted so that the centers are the roots of $\tilde{p}_{A_i}$ (for $i = 20, 100$).

To address the third goal we will find it more useful to examine the moments induced by the $\bar{c}_{M^i}$ instead of the points themselves. Figure 4 and Figure 5 show the relationship between the moments generated by the $\bar{c}_{M^i}$ (in yellow) and the moments generated by the roots of the $\tilde{p}_{M^i}$ (in blue). An additional data point (in green) consists of the moments of the $\bar{c}_{M^i}$ when we change the simulation slightly to use complex Gaussians when forming the random $Z_i$ (all transposes become conjugate transposes as well). Note that these are the moments of $\bar{c}_{M^i}$ and not $\bar{c}_{M^i}^2$, so the first image is (in some sense) the 1/2 moment of $\bar{c}_{M^i}^2$. We found this view to be more compelling as it shows the concave functions monotone increasing in $\beta$ and the convex ones monotone decreasing (signs of possible majorization). We will discuss conjectures related to this relationship in Section 7.

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12 One might also notice in Figure 2 that the errors in the largest value (red) are consistently positive, whereas the errors in the smallest value (blue) are negative (also signs of possible majorization).

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20
7 Conclusion

In many ways, it seems that the work in this paper suggests more interesting problems than it solves. A number of these questions were discussed at the beginning of Section 6, however we would like to comment a bit more on the theme of the polynomial convolution matching the “non-random” part of some random matrix process. One intriguing part of the polynomial convolutions is that (unlike many other things in random matrix theory) they do not depend on $\beta$, a fact that has been discovered in various contexts a number of times [2, 14]. It is also quite common for $\beta$-ensembles on $n \times n$ matrices to converge as $\beta \to \infty$ to a uniform distribution on a set of $n$ points. Any such distribution (finite, on $n$ points) is completely determined by the values of its expected elementary symmetric polynomials, and so one can hope to find a polynomial convolution which captures this behavior completely. The typical way to prove this, however, would be to express the behavior of interest as a function of $\beta$ and then take the limit. In situations where it is computationally infeasible to find such a function, are there other methods one could use to prove such a correspondence? If so, Section 6 suggests that there may be a measurable relationship between distributions as $\beta$ increases, and so such a result could lead to improved estimates for classical ensembles.

7.1 Random matrix theory

The most obvious question in this direction is whether a Gaussian point process that comes from 4. can be solved. That is, given matrices $A, G \in \mathcal{M}_{m,n}$ where $A$ is fixed and the entries of $G$ are i.i.d. Gaussians, can we find the exact distribution of the (squares of the) generalized singular values of $A + \theta G$? It is certainly understandable if any previous attempt seemed overly daunting — as we have mentioned, instantaneous behavior of the Hermite, Laguerre, and Jacobi matrix processes depend only on the current point configuration and a small number of parameters, whereas it should be clear from Section 6 that the point process derived from 4. depends on a much larger number of parameters. However the results of Section 5.1 suggest that these parameters can be captured by natural relationships between the two matrices, in which case an explicit formula could be possible.
We would think that such a formula would certainly be an interesting development in field.

7.2 Finite free probability

Finding a polynomial convolution for a matrix operation (when it exists) tends to be a fairly reasonable task; the results of [17], for example, reduce a number of possible convolution combinations to a straightforward calculation. The opposite direction — given an operation on polynomials, trying to find matrices (and matrix operations) that they correspond to — seems much harder. Even in the case where we know the operation, it tends to be hard to prove that there exist concrete (non-random) matrices that behave in this way. The major tool in this respect is a result of Helton and Vinnikov concerning real stable polynomials [16], however their result is quite quickly reaching the end of its utility (it is not true when the homogeneous polynomials in question have more than 3 variables). That said, it would not be surprising if polynomial convolutions that are based on matrix operations did not also maintain determinantal representation, and it would be interesting to find new ways to prove such statements.

The condition of the random matrix $X_{m,n}$ being symmetric turns out to be far more than we need — as we have seen, the expected characteristic polynomial (13) is (at most) quadratic in the entries of the individual matrices. The utility of having the symmetry condition is that it makes the resulting random matrix distribution $\mathcal{SPM}$–bi-invariant. This suggests that a weaker condition than $\mathcal{SPM}$–bi-invariance might be sufficient for gaining the required amount of symmetry to be able to use Theorem 4.1. One natural candidate to replace the signed permutation matrices is the collection of matrices in the standard representation of $S_{n+1}$ (a set of size $(n + 1)!$ instead of $2^n n!$). Furthermore, the validity of this replacement would follow easily from a conjecture in [17] that Theorem 4.1 holds in a slightly more general context (we refer the interested reader to [17] for more details).

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