Almost Global Consensus on the $n$-Sphere

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Abstract—This paper establishes novel results regarding the global convergence properties of a large class of consensus protocols for multi-agent systems that evolve in continuous time on the $n$-dimensional unit sphere or $n$-sphere. For any connected, undirected graph and all $n \in \mathbb{N}\setminus\{1\}$, each protocol in said class is shown to yield almost global consensus. The feedback laws are negative gradients of Lyapunov functions and one instance generates the canonical intrinsic gradient descent protocol. This convergence result sheds new light on the general problem of consensus on Riemannian manifolds: the $n$-sphere for $n \in \mathbb{N}\setminus\{1\}$ differs from the circle and $SO(3)$ where the corresponding protocols fail to generate almost global consensus. Moreover, we derive a novel consensus protocol on $SO(3)$ by combining two almost globally convergent protocols on the $n$-sphere for $n \in \{1, 2\}$. Theoretical and simulation results suggest that the combined protocol yields almost global consensus on $SO(3)$.

Index Terms—Consensus, agents and autonomous systems, cooperative control, aerospace, nonlinear systems.

I. INTRODUCTION

Consider a network of $N$ agents whose states are points on an $n$-dimensional manifold $\mathcal{M}$. Each agent has a limited capability to sense certain information that pertains to some of the other agents. Distributed control protocols allow a multi-agent system to synchronize its agents, i.e., for all agents to reach a consensus as information propagates over time by means of local interactions [1]. There are a number of results concerning the case when the initial states of all agents belong to a geodesically convex subset of $\mathcal{M}$ [2]–[4], but the likelihood of encountering such a scenario by chance decreases exponentially with $N$. The problem of almost global consensus on Riemannian manifolds is largely unexplored and requires further study [5], [6]. This paper establishes almost global convergence for a large class of consensus protocols on all $n$-spheres except the circle, a rather unexpected finding.

Consensus problems on the circle and the sphere arise in a number of engineering applications, including cooperative reduced rigid-body attitude control [7], [8], planetary scale mobile sensing networks [9], and self-synchronizing chemical and biological oscillators described by the Kuramoto model [10], [11].

The reduced attitude provides a model for the orientation of objects that for various reasons, such as task redundancy, cylindrical symmetry, actuator failure, etc., lack one degree of rotational freedom in three-dimensional space. The orientation of such objects corresponds to a pointing direction with the rotation about the axis of pointing being of little to no importance [12]. The reduced attitude synchronization problem is equivalent to the consensus problem on the 2-sphere. The problem of cooperative control on the $n$-sphere in $\mathbb{R}^{n+1}$, denoted $\mathbb{S}^n$, has received some attention in the literature [13]–[19] but comparatively less than equivalent problems on $SO(3)$ for which there is a considerable literature [2], [6], [20]–[28].

The problem of almost global consensus has been studied on $S^1$ [14]–[16], on $SO(3)$ [29], [30], on $S^2$ in the special case of a complete graph [13], [17], and on other Riemannian manifolds [5]. Tron et al. [29] apply an optimization based method to characterize the stability of all equilibria on $SO(3)$ for a particular discrete-time consensus protocol. Their result is akin to almost global consensus over any connected graph topology. The algorithm makes use of a reshaping function which depends on a parameter that must exceed a bound whose value cannot be calculated from local information. Moreover, the overall convergence speed of the algorithm decreases with increasing values of the parameter. In contrast to [29], this paper shows that almost global convergence of a large class of consensus protocols on $\mathbb{S}^n$ for $n \in \mathbb{N}\setminus\{1\}$ can be established without the use of a reshaping function or any non-local knowledge of the graph. Furthermore, we show how this class can be extended to a class of protocols on $SO(3)$ that only depend on an upper bound of $N$ and display convergence properties that rival those of [29].

The 2-sphere is diffeomorphic to the quotient space $SO(3)/SO(2)$ and, as such, many results obtained for $SO(3)$ also apply to $S^2$. Special cases sometimes allow for stronger results. This paper shows that the conditions for achieving almost global consensus are more favorable on $\mathbb{S}^n$ for $n \in \mathbb{N}\setminus\{1\}$ than what is implied by previously known results concerning $S^1$ and $SO(3)$. A large class of intrinsic consensus protocols over connected, undirected graph topologies renders all equilibria but the consensus set unstable on $\mathbb{S}^n$. By contrast, analysis of the corresponding consensus protocols on $S^1 \approx SO(2)$ [15], [16] and simulations on $SO(3)$ [29] show that certain graph topologies yield equilibrium sets aside from consensus that are asymptotically stable on $SO(n)$ for $n \in \{2, 3\}$.

The literature on continuous-time cooperative control on the $n$-sphere has largely been focused on special cases. Previous work either concerns the case of a specific graph topology [8], [13], [16], [30], a specific sphere [5], [8], [13], [14], [16], [18], or a specific control law [8], [13], [29], [31]. Many of them also lack a rigorous proof of almost global convergence [5], [14], [29], [31]. They only show that all equilibrium sets except the consensus set are unstable, which is a weaker result in general [32]. We provide a rigorous proof of almost global convergence for a large class of analytic consensus protocols over any connected graph by, roughly speaking, showing that the region of attraction of any set of exponentially unstable equilibria have measure zero on $(\mathbb{S}^n)^n$.

In the literature survey [33], it is observed that almost
global convergence of consensus protocols on nonlinear spaces (in particular $S^1$) is graph dependent. The survey discusses three control design procedures to circumvent this problem: resocusing functions [14], [15], [29], gossip algorithms [15], and dynamic feedback [5]. The main contribution of this paper is to show that consensus on $S^n$ is not graph dependent for any $n \in \mathbb{N}\{1\}$, and that almost global consensus can be achieved without utilizing any of the three design procedures in [5]. This leads to the contra-intuitive but intriguing notion that almost global consensus is more difficult to achieve on $S^1$ than any other sphere. Preliminary results are found in [31], conjectures are made in [7], [13].

II. Problem Description

The following notation is used in this paper. The inner and outer product of $x, y \in \mathbb{R}^n$ are denoted by $\langle x, y \rangle$ and $x \otimes y$, respectively. The inner product of $A, B \in \mathbb{R}^{n \times n}$ is $\langle A, B \rangle = \text{tr} A^T B$. Let $\| \cdot \|$ denote the Euclidean norm of a vector and $\| \cdot \|_F$ the Frobenius norm of a matrix. The gradient in Euclidean space is denoted $\nabla : f(x) \mapsto \nabla f(x) \in \mathbb{R}^n$, the intrinsic gradient map at a point $x$ on a manifold $\mathcal{M}$ is denoted $\nabla f(x) \mapsto \nabla f(x) \in T_x \mathcal{M}$. We represent manifolds by their canonical embeddings in Euclidean space. The special orthogonal group is $SO(n) = \{ R \in \mathbb{R}^{n \times n} \mid R^T = R^{-1}, \det R = 1 \}$. The Lie algebra of $SO(n)$ is $so(n) = T_I SO(n) = \{ S \in \mathbb{R}^{n \times n} \mid S^T = -S \}$. The $n$-sphere is $S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$. The tangent space of $S^n$ is $T_x S^n = \{ y \in \mathbb{R}^{n+1} \mid \langle y, x \rangle = 0 \} \cong \mathbb{R}^n$. An undirected, simple graph is a pair $G = (V, E)$ where $V \subset \mathbb{N}$ is the node set and $E \subset \{ e \subset V \mid |e| = 2 \}$ is the edge set. A graph $G$ is said to be connected if it contains a tree subgraph with $|V| - 1$ edges.

A. Distributed Control on the n-Sphere

Consider a multi-agent system where each agent corresponds to an index $i \in V$ and has a state $x_i \in S^n$ expressed in a world coordinate frame $W$. Agent $i$ uses a body-fixed frame $B_i$ that relates to $W$ by a rotation matrix $R_i(t) \in SO(n)$ for all $t \in [0, \infty)$. At each $t$, $R_i(t)$ yields a map $R_i : [v]_{B_i} \mapsto [v]_W$, where the bracket $[\cdot]_F$ denotes that its content is expressed in a frame $F$. If the frame is omitted, then $W$ is presupposed. Chose the reduced attitude $x_a$ of agent $i$ to satisfy $[x_a]_{B_i} = e_1$. Thus $[x_i]_W = R_i[x_i]_{B_i} = R_i e_1$, i.e., $[x_i]_W$ is given by the first column of $R_i$. The agents are capable of limited local sensing. The topology of the communication network is described by an undirected connected graph $G = (V, E)$, where $V = \{ i \in \mathbb{N} \mid i \leq N \}$, and $\{ i, j \} \in E$ implies that there are two neighboring agents $i$ and $j$ can sense the so-called relative information $[I_{ij}]_{B_i}, [I_{ji}]_{B_j}$ regarding the displacement of their states $x_i$ and $x_j$. All relative information agent $i$ has access to is compounded into a set $I_i$, the precise nature of which may differ between applications.

System 1. The system is given by $N$ agents, an undirected and connected graph $G = (V, E)$, agent states $x_i \in S^n$, where $n \in \mathbb{N}$, and dynamics

$$\dot{x}_i = u_i - \langle u_i, x_i \rangle x_i = (I - X_i) u_i = P_i u_i,$$

where $u_i : I_i \rightarrow \mathbb{R}^{n+1}$ is the input signal of agent $i$, $X_i = x_i \otimes x_i$, and $P_i = I - X_i$ for all $i \in V$.

Control is based on relative information. The information that agent $i$ has access to regarding its neighbor agent $j$ could be defined to include

$$[\text{pos} \{ x_j - x_i \}]_{B_i} \subseteq [I_{ij}]_{B_i} \subseteq \mathbb{R}^{n+1},$$

which is the relative information customary to the ambient space $\mathbb{R}^{n+1}$. The set of neighbors of agent $i$ is $N_i = \{ j \in V \mid \{ i, j \} \in E \}$. The set of relative information known to agent $i \in V$ is $[I_{ij}]_{B_i} = \text{pos} \cup_{j \in N_i} [I_{ij}]_{B_i}$. The dynamics (1) of agent $i$ projects the input of agent $j$ on the tangent space $T_x S^n$, i.e., on a hyperplane orthogonal to $x_i$.

Remark 2. It can be argued that

$$\text{pos} \{ P_i (x_j - x_i) \} \subseteq I_{ij},$$

where $P_i : \mathbb{R}^{n+1} \rightarrow T_x S^n$ is an orthogonal projection matrix, is preferable to (2) since it confines $I_{ij}$ to an intrinsic rather than an ambient space. However, we believe that the constraints on $I_{ij}$ tend to come from limited sensing capabilities rather than rigid-body dynamics, and that most applications on $S^2$ involve sensors that measure features of ambient rather than intrinsic space. Note that the dynamics (1) remain the same in both cases since $P_i^2 = P_i$.

While agent $i$ may not be able to calculate some $[u_i]_{B_i} \in [I_{ij}]_{B_i}$ based on the information (2) obtained from all its neighbors, that agent may still be able to calculate an input $[u_i]_{B_i} \in [I_{ij}]_{B_i}$ whose projection on $T_x S^n$ by the dynamics of $x_i$ is identical to that of $u_i$. This holds for inputs that belongs to $\cup_{j \in N_i} x_j$, and in particular for elements of the positive cone $\text{pos} \cup_{j \in N_i} x_j$. Intuitively speaking, it is reasonable to assume that agent $i$ should be able to sense the bearing and distance to any of its neighbors, and we therefore set $[I_{ij}]_{B_i} = [\text{pos} \cup_{j \in N_i} [x_j]_{B_i}]_{B_i}$.

The results and proofs in this paper are carried out in the world frame $W$. To implement the control law in a distributed fashion, $u_i$ must be transferred to $B_i$ for all $i \in V$. Let a control law in $W$ be given by $[u_i]_W = \sum_{j \in N_i} f_{ij} [x_j]_W$. Hence $[u_i]_{B_i} = \sum_{j \in N_i} f_{ij} R_i^T [x_j]_W = \sum_{j \in N_i} f_{ij} [x]_B_i$. Moreover, $[x_i]_B_i = R_i^T [x_i]_W = R_i^T [u_i]_W - \langle [u_i]_W, [x_i]_W \rangle R_i^T [x_i]_W = [u_i]_{B_i} - \langle [u_i]_{B_i}, [x_i]_{B_i} \rangle [x_i]_{B_i}$, (3)

since inner products are invariant under orthogonal changes of coordinates. It is clear from (3) that (1) can be implemented in a distributed fashion.

The problem of multi-agent consensus on $S^n$ concerns the design of distributed control protocols $(u_i)_{i=1}^N$ based on relative information that stabilize the consensus set

$$C = \{ (y_i)_{i=1}^N \in (S^n)^N \mid y_i = y_j, \forall i, j \in V \} = \{ (y_i)_{i=1}^N \in (S^n)^N \mid y_i = y_j, \forall \{ i, j \} \in E \}$$

of System (1) where the second equality hinges on the assumption that $G$ is connected. If the states of all agents assume the same value on the $n$-sphere, then they are said to
reach consensus. Terms such as consensus, synchronization, rendezvous, and state-aggregation are used interchangeably in this paper, but note that some authors, see e.g., [5], [17], assign the definitions of these concepts subtle nuances.

B. Problem Statement

This paper concerns some aspects of control design but the main focus is stability analysis. Algorithm 3 is arguably the most basic conceivable feedback for consensus on $S^n$ by virtue of its correspondence with the linear consensus protocol on $\mathbb{R}^{n+1}$ for single integrator dynamics given by $x_i = u_i$ for all $i \in V$. Algorithm 3 is the negative gradient of the Lyapunov function $V = \frac{1}{2} \sum_{(i,j) \in E} \|x_i - x_j\|^2$ and generates what may be referred to as the canonical intrinsic gradient descent consensus protocol. As such, it is of interest to determine the limits of Algorithm 3’s performance, i.e., the global level stability of the consensus set $C$ as an equilibrium set of System 1.

It is important to establish that the region of attraction of the undesired equilibria is of negligible size, e.g., meager in the sense of Baire and of measure zero [32].

Algorithm 3. The feedback is given by $u_i = \sum_{j \in N_i} f_{ij} x_j$, where the constants $f_{ij} \in (0, \infty)$ satisfy $f_{ij} = f_{ji}$ for all $\{i, j\} \in E$.

Definition 4 (Measure zero). A set $N \subset (S^n)^N$ has measure zero if for every chart $\phi : D \rightarrow \mathbb{R}^{N(n+1)}$ in some atlas of $(S^n)^N$, it holds that $\phi(D \cap N)$ has Lebesgue measure zero.

Definition 5 (Almost global attractiveness). Consider a system that evolves on $(S^n)^N$. A set of equilibria $D \subset (S^n)^N$ is said to be almost globally attractive if for all initial conditions $(x_i(0))_{i=1}^N \in (S^n)^N \setminus N$, where $N$ is some set of zero measure, it holds that $\lim_{t \rightarrow \infty} (x_i(t))_{i=1}^N \in D$.

Problem 6. Show that there is a large class of consensus protocols for System 1 including Algorithm 3 such that the consensus set $C$ is stable and almost globally attractive.

Problem 6 concerns the global behavior of System 1. Under certain assumptions regarding the connectivity of $G$, local consensus on $S^n$ can be established with the region of attraction being the largest geodesically convex sets on $S^n$, i.e., open hemispheres [19]. See also [25] in the case of an undirected graph and [27] in the case of a directed and time-varying graph. A global stability result for discrete-time consensus on $SO(3)$ is provided in [29]. Almost global asymptotical stability of the consensus set on the $n$-sphere is known to hold when the graph is a tree [25] or is complete in the case of first- and second-order models [13], [17]. The author of [13] conjectures that global stability also holds for a larger class of topologies whereas [15], [16] provides counterexamples of basic consensus protocols that fail to generate consensus on $S^n$.

Remark 7. Global consensus on $S^n$ cannot be achieved by means of a continuous feedback due to topological constraints [17]. It is however possible to achieve almost global asymptotical stability, as has been demonstrated on the circle [15], [16]. To prove almost global convergence to the consensus set is challenging since basic tools such as the Hartman-Grothman theorem or stable-unstable manifold theorems are unavailable due to the equilibria being nonhyperbolic [35]. Feasible approaches include dual Lyapunov stability theory [36] and a technique based on stability in the first approximation [32] that applies to convergent systems. We take the latter approach.

III. STABILITY OF THE CONSENSUS MANIFOLD

This section and the next concern System 1 governed by Algorithm 3 which is an extension of Algorithm 4. Algorithm 8 provides a large class of smooth continuous-time consensus protocols on the $n$-sphere. The stability properties of all equilibria are fully determined, as is those of the overall system.

A. Control Design

Consider a class of consensus protocols that formalizes the idea of increasing system cohesion by moving an agent into the convex hull of its state and those of its neighbors.

Algorithm 8. The input is given by

$$u_i = \sum_{j \in N_i} f_{ij}(s_{ij})x_j,$$

where $s_{ij} = 1 - \langle x_i, x_j \rangle$ and the feedback gains $f_{ij} : \mathbb{R} \rightarrow \mathbb{R}$ are real analytic functions that satisfy

1. $f_{ij} > 0$,
2. $f_{ij} = f_{ji}$,
3. $(n - 2 + s_{ij})s_{ij}f_{ij} - (2 - s_{ij})s_{ij}^2f_{ij} > 0$, for all $s_{ij} \in (0, 2)$ and all $\{i, j\} \in E$.

Note that $f_{ij}$ depends on $s_{ij} : S^n \times S^n \rightarrow [0, 2]$ given by

$$s_{ij} = \frac{1}{2} \|x_i - x_j\|^2 = 1 - \langle x_i, x_j \rangle,$$

which is invariant under orthogonal changes of coordinates. Algorithm 8 therefore complies with the requirements of Section II-A regarding distributed feedback laws over the $n$-sphere. Various forms of the closed loop dynamics of System 1 under Algorithm 8 is stated on the readers behalf and for the sake of completeness.

$$\dot{x}_i = u_i - \langle u_i, x_i \rangle x_i = \sum_{j \in N_i} f_{ij}(s_{ij})x_j - \sum_{j \in N_i} f_{ij}(s_{ij})\langle x_j, x_i \rangle x_i = \sum_{j \in N_i} f_{ij}(s_{ij})(x_j - (1 - s_{ij})x_i) = (1 - X_i) \sum_{j \in N_i} f_{ij}(s_{ij})x_j = P_i \sum_{j \in N_i} f_{ij}(s_{ij})x_j.$$

Remark 9. Algorithm 8 comprises a class of algorithms which includes those of Algorithm 3 for all $n \in \mathbb{N} \setminus \{1\}$. If $f_{ij} = k \in (0, \infty)$ for all $\{i, j\} \in E$, then (iii) evaluates to $k(n - 2 + s_{ij})s_{ij} \geq k^2 s_{ij}^2 > 0$ for all $s_{ij} \in (0, 2]$ when $n \geq 2$ but for $n = 1$ we obtain

$$(1 + s_{ij})s_{ij} \cdot k + (2 - s_{ij})s_{ij}^2 \cdot 0 = -k(1 - s_{ij})s_{ij} \leq 0$$
for all \( s_{ij} \in [0, 1] \). Note that the class grows with \( n \). For example, if \( f_{ij} = s_i^k \) for some \( k \in \mathbb{N} \) then (iii) evaluates to

\[
(n - (k + 1)(2 - s_{ij}))s_{ij}^{k+1},
\]

which is positive on \((0, 2)\) when \( k < \frac{n}{2} - 1 \). To see that the class is empty for \( n = 1 \), note that (iii) can be rewritten as

\[
\frac{f_{ij}}{s_{ij}} < \frac{-1 + s_{ij}}{(2 - s_{ij})s_{ij}} \quad (6)
\]

for all \( \{i, j\} \in \mathcal{E} \) and all \( s_{ij} \in (0, 2) \). This implies \( \lim_{s_{ij} \to 0} f_{ij}^{\prime}\left(s_{ij}\right) = -\infty \). Since \( f_{ij}^{\prime} \) is continuous, it is bounded on \([0, 2]\) whereby \( f_{ij}(0) = 0 \) and \( f_{ij}(0) \leq 0 \). Even if \( f_{ij}^{\prime}(0) = 0 \), the inequality (6) still implies that \( f_{ij}(s_{ij}) < 0 \) for all \( s \in (0, \delta) \) for some \( \delta \in (0, \infty) \). By continuity there exists an \( \varepsilon \in (0, \infty) \) such that \( f_{ij}(s_{ij}) < 0 \) for all \( s_{ij} \in \{0, \varepsilon\} \), which contradicts requirement (i) of Algorithm \[\text{Algorithm 8}\].

**Remark 10.** For some feedback gains \( f_{ij} \) there is a ball in the space \( \mathcal{C}^\omega \) of real analytic functions consisting entirely of feedback gains of other elements of Algorithm \[\text{Algorithm 8}\]. For instance, Algorithm \[\text{Algorithm 8}\] still converges if instead of a constant \( f_{ij} \) agent \( i \) and \( j \) use \( f_{ij} + g_{ij} \), where \( g_{ij} \in \mathcal{C}^\omega \) is of sufficiently small norm. This could be interpreted as a form of robustness against analytic radial errors, e.g., constant measurement errors due to biased sensors.

Algorithm \[\text{Algorithm 8}\] can be derived by taking the gradient of the candidate Lyapunov function \( V \) of the system \((n) \) defined on \( \mathcal{E} \) by

\[
V((s_{ij})_{i=1}^n) = \sum_{\{i, j\} \in \mathcal{E}} f_{ij}(r)dr,
\]

where \( s_{ij} = 1 - f_{ij}(r) \). Let \( U : (\mathbb{R}^{n+1})^n \to \mathbb{R} \) be the extension of \( V \) obtained by just changing the domain, i.e.,

\[
U((s_{ij})_{i=1}^n) = \sum_{\{i, j\} \in \mathcal{E}} f_{ij}(r)dr.
\]

The functions \( f_{ij} \) being analytic on \( \mathbb{R} \) by assumption implies that \( U \) is smooth since integrals of analytic functions are analytic \[\text{[37]}\]. Denote \( \nabla U = (\nabla U)^{\prime}_{i=1} \), where \( \nabla_i = \nabla x_i \).

Then

\[
\nabla_i U = \sum_{j \in \mathcal{N}_i} \frac{dU}{ds_{ij}} \nabla s_{ij} = -\sum_{j \in \mathcal{N}_i} f_{ij}(s_{ij})x_j \quad (8)
\]

It follows that \( u_i = -\nabla_i U \) and \( \dot{x}_i = -P_i \nabla_i U \) for all \( i \in \mathcal{V} \).

**Proposition 11.** System \[\text{System 1}\] under Algorithm \[\text{Algorithm 8}\] converges to an equilibrium set in \((n) \). At any equilibrium point, each input is parallel to the state of its agent.

**Proof.** Consider the potential function (7). It holds that

\[
\dot{V} = \sum_{\{i, j\} \in \mathcal{E}} f_{ij}(s_{ij}) - \sum_{\{i, j\} \in \mathcal{E}} f_{ij}(\dot{x}_i, \dot{x}_j) + \left( \dot{x}_i, \dot{x}_j \right) + \left( x_i, x_j \right) - \sum_{i \in \mathcal{V}} \left( \dot{x}_i, \sum_{j \in \mathcal{N}_i} f_{ij}x_j \right) - \sum_{j \in \mathcal{V}} \left( \sum_{i \in \mathcal{N}_j} f_{ij}x_i, \dot{x}_j \right)
\]

\[
= -2 \sum_{i \in \mathcal{V}} \left( \dot{x}_i - \{u_i, u_i\}x_i, \dot{u}_i \right) = -2 \sum_{i \in \mathcal{V}} \|u_i\|^2 - \langle u_i, x_i \rangle^2.
\]

System \[\text{System 1}\] converges to the set \{(\cdot)^N_{i=1} | V = 0\} by LaSalle’s theorem. The Cauchy-Schwarz inequality applied to (9) shows that the input and state of each agent align up to sign asymptotically. This implies \( x_i = 0 \) for all \( i \in \mathcal{V} \), i.e., that the system is at an equilibrium by inspection of \[\text{Algorithm 8}\].

The equilibria that are characterized by Proposition \[\text{Proposition 11}\] can be divided into three categories:

\[
(x_i, u_i) \in \left\{-\left(\frac{u_i}{\|u_i\|}\right), \left(\frac{u_i}{\|u_i\|}\right), (x_i, 0)\right\}, \quad (10)
\]

where \( u_i = \sum_{j \in \mathcal{N}_i} f_{ij}x_j \) for all \( i \in \mathcal{V} \). The case of \( u_i = 0 \) for all \( i \in \mathcal{V} \) is illustrated by Figure 1. The agent states in Figure 1 correspond to the six corners of an octahedron, which is one of the five platonic solids. Likewise, the tetrahedral graph (i.e., the complete graph over four nodes) has the tetrahedron as an equilibrium with \( x_i = -u_i/\|u_i\| \) for all \( i \in \mathcal{V} \); whereas the cube, icosahedral, and dodecahedral graphs have respectively the cube, icosahedron, and dodecahedron as equilibria with \( x_i = u_i/\|u_i\| \) for all \( i \in \mathcal{V} \).

\[\text{Fig. 1. An equilibrium of a system on } \mathbb{S}^2 \text{ (left) with an octahedral graph (right). The sum of neighbor states is zero, as is the projection of said sum on the tangent plane } T_x \mathbb{S}^2.\]

The following result, Proposition \[\text{Proposition 12}\], concerns consensus over the largest geodesically convex sets on \( \mathbb{S}^n \), i.e., open hemispheres. Analogues to Proposition \[\text{Proposition 12}\] and various generalizations thereof are known to the control community. For example, \[\text{[4]}\] uses invariant convex hulls in a manner that was preceded in \[\text{[2]}, \text{[38]}\] to prove local convergence of time switched consensus protocols on \( \mathbb{S}_i \). To solve Problem \[\text{Problem 8}\] this paper provides a companion to Proposition \[\text{Proposition 12}\] Theorem \[\text{Theorem 13}\] which characterizes all equilibrium sets of System \[\text{System 1}\] under Algorithm \[\text{Algorithm 8}\] in terms of attractiveness and stability. Although Proposition \[\text{Proposition 12}\] is used in the proof of Theorem \[\text{Theorem 13}\] its full power is not needed. Rather, it is included as a contrast to highlight the greater generality achieved by our analysis.

**Proposition 12.** Consider System \[\text{System 1}\] under Algorithm \[\text{Algorithm 8}\]. The consensus set \( \mathcal{C} \) is asymptotically stable. Moreover, the system reaches consensus asymptotically if there is some finite time such that all agents belong to an open hemisphere.

**Proof.** Let \( \mathcal{H} \) denote the open hemisphere. Since \( f_{ij} \in [0, \infty) \) for all \( j \in \mathcal{N}_i \), \( \dot{x}_i = P_i \sum_{j \in \mathcal{N}_i} f_{ij}x_j \) points towards the geodesically convex hull of \( \{x_j | j \in \mathcal{N}_i\} \) on \( \mathbb{S}^n \) along the tangent space \( T_x \mathbb{S}^n \) superimposed on \( \mathbb{S}^n \) at \( x_i \). This shows \( \mathcal{H} \) to be invariant and \( \mathcal{C} \) to be stable. It remains to show attractiveness. Proposition \[\text{Proposition 11}\] establishes that System \[\text{System 1}\] under Algorithm \[\text{Algorithm 8}\] converges to an equilibrium set. Since \( \mathcal{H} \) is
invariant the desired result follows if the only equilibrium configuration on \(H\) is a consensus.

There must be at least one agent \(k\) that minimizes the distance to the boundary of \(H\). At any equilibrium, it holds that \(x_i = \parallel u_i\parallel\) for all \(i \in V\) by Proposition 1.

Since all agents belong to an open hemisphere it follows that \(0, -x_i \notin \text{pos}(x_i)\). By (10), only \(x_i = (\text{pos}(x_i))\) remains. Agent \(k\) belongs to an extreme ray of the convex cone \(\text{pos}(x_i)\). But then \(x_k = \text{pos}(x_i)\) if and only if \(x_j = \text{pos}(x_i)\) for all \(j \in N_k\). An induction argument can be applied to show that the system is at a consensus due to \(G\) being a connected graph. □

B. Main Result

In light of the previous section, we state our main result.

Theorem 13. Consider System 1 under Algorithm 3 in the case of \(n \in \mathbb{N}\setminus\{1\}\). The consensus set \(C\) given by (3) is almost globally asymptotically stable. The rate of convergence is locally exponential if the feedback gains \(f_{ij}\) are nonzero over \(C\) for all \(i, j \in E\). Moreover, each trajectory of the system converges to some point. The region of attraction of the set of all unstable equilibria is meager.

The proof of Theorem 13 is given in Section IV-D. Let us briefly sketch the main ideas. That the consensus set is asymptotically stable follows from Proposition 12. To prove the exponential instability of the undesired equilibria we use the indirect method of Lyapunov. The system is linearized around an equilibrium on the \(n\)-sphere. Perturbing all agents in one direction, i.e., towards the consensus set, increases cohesion in one half of the sphere while depleting it in the other half. One such perturbation corresponds to a direction of instability for the linearized system. Finally, a known result establishes conditions under which any set of exponentially unstable equilibria have a region of attraction that is of measure zero and meager.

IV. INSTABILITY OF UNDESIRABLE EQUILIBRIUM SETS

The global behavior of the system is determined by the stability and attractiveness of all its equilibria, which often can be characterized locally by means of linearization. To establish almost global convergence we must show that the set of all unstable equilibria has a region of attraction with measure zero. It is possible for a set of exponentially unstable equilibria to have a region of attraction with non-zero measure, but only if the system fails to be convergent as shown in Proposition 20.

A. Linearization on the \(N\)-Fold \(n\)-Sphere

Let us study the signs of the real part of the eigenvalues of the linearization of System 1 under Algorithm 3. This matrix is also the negative Riemannian Hessian, \(H = \mathbb{R}^V\), of the potential function \(V\) given by (7). The Riemannian Hessian of a function can be expressed in terms of its Euclidean gradient, Euclidean Hessian, and the Weingarten map [39].

Proposal 14 (P.-A. Absil, R. Mahony & J. Trumpf [39]). Let \(f : M \to \mathbb{R}\) be a function defined on a Riemannian submanifold \(M\) of \(\mathbb{R}^n\). The intrinsic Hessian map \(\mathbb{R}^2 f : M \times T_xM \to T_xM : (x, w) \mapsto \mathbb{R}^2 f(x)w\) is given by

\[
\mathbb{R}^2 f(x)w = P \nabla^2 g(x)w + W(w, (I - P)\nabla g(w)),
\]

where \(P : M \times \mathbb{R}^n \to T_xM\) is an orthogonal projection, \(g : \mathbb{R}^n \to \mathbb{R}\) is a smooth extension of \(f\) to \(\mathbb{R}^n\), and \(W : T_xM \times (T_xM)^\perp \to T_xM\) is the Weingarten map.

Proposal 15. The blocks of the linearization matrix \(H = (H_{ij}) \in \mathbb{R}^{(n+1) \times (n+1)}\) of System 2 under Algorithm 8 are given by

\[
H_{ii} = -\langle u_i, x_i \rangle P_i - \sum_{j \in N_i} f_{ij}' P_j x_j P_i
\]

for all \(i \in V\),

\[
H_{ij} = P_i \left( f_{ij} I - f_{ij}' x_j \otimes x_i \right) P_j
\]

for all \(i, j \in E\), and \(H_{ij} = 0\) otherwise.

Proof. We use the technique of Proposition 14 The Euclidean gradient is \(\nabla U = [\nabla_i U]\) where \(\nabla_i U = -u_i\) as seen by (6). The Euclidean Hessian is \(\mathbb{R}^2 U\), where \(\mathbb{R}^2 U \in \mathbb{R}^{(n+1) \times (n+1)}\) is given by

\[
\mathbb{R}^2 U = \begin{cases} f_{ij}' x_j & \text{if } j = i, \\ -f_{ij} I + f_{ij}' x_j \otimes x_i & \text{if } j \in N_i, \\ 0 & \text{otherwise}, \end{cases}
\]

as can be seen by calculation. The projection \(P\) is a block-diagonal matrix whose \(i\)th block is given by \(P_i = I - X_i\). The \(i\)th block of the matrix \(I - P\) is hence \(X_i\). The Weingarten map \(W\) at \((x_i)_{i=1}^N \in \{S^n\}^N\) is given by

\[
W([t_j], [n_i]) = [M(t_j, u_i)],
\]

where \(M\) is the Weingarten map on \(S^n\). The Weingarten map at a point \(x \in S^n\) is derived in [39] as

\[
M(t, u) = -(u, x)t,
\]

where \(t \in T_xS^n\) and \(u \in (T_xS^n)^\perp\).

By Proposition 14, the intrinsic Hessian on \(\{S^n\}^N\) can be expressed as a block matrix \([\mathbb{R}^2_i V]\), where \(\mathbb{R}^2_i\) denote \(\mathbb{R}^2_{x_i}\), which satisfies

\[
\sum_{j \in V} \mathbb{R}^2_{ij} V_{w_j} = P_i \nabla_i^2 U w_i + \sum_{j \in N_i} P_j \nabla_j^2 U w_j + M(w_i, (I - P_i)\nabla_i U).
\]

Since \(w_i \in T_xS^n\), it holds that \(w_i = P_i v_i\) for some \(v_i \in \mathbb{R}^{n+1}\), whereby

\[
\sum_{j \in V} \mathbb{R}^2_{ij} V_{w_j} = f_{ij}' P_i x_j P_j v_i +
\]

\[
P_j \sum_{j \in N_i} (-f_{ij} I + f_{ij}' x_j \otimes x_i) P_j v_j -
\]

\[
-\langle X_i u_i, x_i \rangle P_i v_i = ((u_i, x_i) P_i + f_{ij}' P_i x_j P_i) v_i -
\]

\[
\langle u_i, x_i \rangle P_i f_{ij}' x_j P_i v_i,
\]

\[
\langle u_i, x_i \rangle P_i f_{ij}' x_j P_i v_i.
\]
This equation gives the Riemannian Hessian, $\Box^2 V$, by inspection; negating it gives the linearization matrix. The blocks on the diagonal are symmetric, and the off-diagonal blocks satisfy $\Box_{ji}^2 V = \Box_{ij}^2 V^T$ as we would expect from a Hessian. □

B. Instability of Undesired Equilibria

Consider an equilibrium such that all agents belong to the intersection of $S^n$ and a hyperplane in $\mathbb{R}^{n+1}$. Perturb all agents into an open hemisphere by an arbitrarily small movement along a direction orthogonal to the hyperplane. By Proposition 12, the perturbed system converges to a consensus. The spectral properties of a linearized system determine how it reacts to perturbations. This is the basic idea behind Proposition 16, perturb all agents in the same direction, e.g., towards the north pole. This increases cohesion in the north hemisphere while depleting it in the south. We show that such perturbation corresponds to a direction of exponential instability.

**Proposition 16.** Any equilibrium $(x_i)_{i=1}^N \notin \mathcal{C}$ of System 1 under Algorithm 8 is exponentially unstable.

**Proof.** The proof makes use of the linearization provided by Proposition 15. The Courant-Fischer-Weyl min-max principle bounds the range of the Rayleigh quotient of a symmetric matrix by its minimal and maximal eigenvalues [40]. If the Rayleigh quotient is positive for some argument, then the maximal eigenvalue is positive. Recall that if $H$ has a positive eigenvalue at an equilibrium, then that equilibrium is unstable by the indirect method of Lyapunov [41]. Let $v = [y^\top \ldots y^\top]^\top \in T_e(S^n)^N$, i.e., $y \in \mathbb{R}^{n+1}$ since $\cup_{x \in S^n} T_x S^n \simeq \mathbb{R}^{n+1}$, and consider

$$
(v, Hv) = \sum_{i \in V} \langle y, H_{ii} y \rangle + \sum_{j \in N_i} \langle y, H_{ij} y \rangle
$$

$$
= \left( y, \left( \sum_{i \in V} H_{ii} + \sum_{j \in N_i} H_{ij} \right) y \right).
$$

Denote $G = \sum_{i \in V} H_{ii} + \sum_{j \in N_i} H_{ij}$. The matrix $G$ is symmetric since

$$
G = \sum_{i \in V} H_{ii} + \sum_{(i,j) \in \mathcal{E}} H_{ij} + \sum_{(j,i) \in \mathcal{E}} H_{ji} = \sum_{i \in V} H_{ii} + \sum_{(i,j) \in \mathcal{E}} H_{ij} + H_{ij}^T
$$

wherefore $\sigma(G) \subset \mathbb{R}$ by the spectral theorem. If $G$ has a strictly positive eigenvalue, then for the corresponding eigenvector $z \in \mathbb{R}^{n+1}$ it holds that $\langle z, Gz \rangle > 0$ whereby setting $y = z$ yields $\langle v, Hv \rangle > 0$. The min-max principle then implies that $H$ has a strictly positive eigenvalue, i.e., the equilibrium is exponentially unstable.

Let us prove that $G$ has a positive eigenvalue. Consider

$$
\text{tr } G = \sum_{i \in V} -n\langle u_i, x_i \rangle + \sum_{j \in N_i} \left( -f_{ij}'(1 - (x_i, x_j)^2) + f_{ij}(n - 1 + (x_i, x_j)^2 - f_{ij}'(x_i, x_j)(x_i, x_j)^2 - 1) \right)
$$

$$
= \sum_{i \in V} -n\langle u_i, x_i \rangle + \sum_{j \in N_i} \left( f_{ij}(n - 1 + (x_i, x_j)^2) - f_{ij}'(x_i, x_j)(x_i, x_j)^2 - 1 \right)
$$

$$
= \sum_{i \in V} -n\langle u_i, x_i \rangle + \sum_{j \in N_i} \left( f_{ij}(n - 1 + (x_i, x_j)^2) - f_{ij}'(x_i, x_j)(x_i, x_j)^2 - 1 \right)
$$

$$
= \sum_{i \in V} (n - 2 + s_{ij})s_{ij} - f_{ij}'(2 - s_{ij})s_{ij}^2.
$$

where we used that $\text{tr } X_i = \|x_i\|^2 = 1$ and $s_{ij} = 1 - \langle x_i, x_j \rangle$.

Recall that

$$
(n - 2 + s_{ij})s_{ij} - (2 - s_{ij})s_{ij}^2 > 0
$$

for all $s_{ij} \in (0, 2]$ and all $(i, j) \in \mathcal{E}$ by condition (ii) of Algorithm 8. Since $\text{tr } G \geq 0$ with strict inequality unless $s_{ij} = 0$ for all $(i, j) \in \mathcal{E}$, i.e., unless $(x_i)_{i=1}^N$, it follows that $G$ has a strictly positive eigenvalue. □

**Remark 17.** Requirement (iii) in Algorithm 8 arises from the lower bound on the largest eigenvalue of $H$ implied by the sign of $\text{tr } G$. This lower bound is likely to be conservative with respect to the requirements on $f_{ij}$ for all $(i, j) \in \mathcal{E}$ that results in $H$ having a positive eigenvalue. The class of control signals that yield almost global consensus on $S^n$ should hence be larger than Algorithm 8.

Proposition 18 is used to prove Theorem 13. The version presented here is particularized for our purposes, a more general result and its proof may e.g., be found in [32].

**Proposition 18 (R.A. Freeman [32]).** Consider a system $x = f(x)$ that evolves on a state-space $X$, where $f \in C^1$. Let $\mathcal{S} \subset X$ be a set consisting entirely of exponentially unstable equilibria. If each trajectory of the system converges to some equilibrium, then the region of attraction of $\mathcal{S}$ is of zero measure and meager in $X$.

C. Point-Wise Convergence

The instability requirements of Proposition 18 are satisfied by Proposition 16. However, to show that every trajectory of the system converges to a point, i.e., that the system is so-called pointwise convergent [42], requires some additional analysis. Point-wise convergence is of importance since Proposition 17 only establishes convergence to equilibrium sets, all of which have $n$ degrees of rotational invariance. In theory, it would be possible for each agent to traverse its sphere indefinitely: each agent would move along a path of rotational invariance of the full agent configuration, while the system as a
whole approaches an equilibrium set. The use of Proposition 19 a corollary of the Łojasiewicz gradient inequality [43], may be not necessary but suffices to establish point-wise convergence. This is the reason that we assume the feedback gains $f_{ij} \in \mathbb{C}^+$ for all $(i,j) \in \mathbb{E}$ rather than $f_{ij} \in \mathbb{C}^1$.

**Proposition 19 (S. Łojasiewicz [42, 43]).** Let $\mathcal{M}$ be a real analytic Riemannian manifold and $f : \mathcal{M} \to \mathbb{R}$ be a real analytic function. For the Riemannian gradient flow $\dot{x} = -\nabla f$ it either holds that $\lim_{t \to \infty} x(t) = y$ for some $y \in \mathcal{M}$ or the set of $\omega$-limit points is empty.

**Proposition 20.** Each trajectory of System \[ under Algorithm 8] converges to an equilibrium.

**Proof.** The $n$-sphere is a real analytic manifold, and so is $(S^n)^N$. Sums, composite functions, integrals, and derivatives of multivariate analytic functions are analytic [37]. By analyticity of the feedback gains in Algorithm 8 it follows that the candidate Lyapunov function $V$ given by (7) is analytic.

Equation (8) only provides the extrinsic gradient $\nabla U : (R^{n+1})^N \to (R^{n+1})^N$ of (7) without regard to the fact that $(x_i)^{N}_{i=1} \in (S^n)^N$. The intrinsic gradient $\Box V : (S^n)^N \to \Sigma (S^n)^N$ is given by

$$\Box V = (\Box_i V)^{N}_{i=1} = (P_i \nabla_i U)^{N}_{i=1},$$

where $\Box_i = \Box x_i$.

The closed-loop dynamics of System 4 under Algorithm 8 can be written

$$\dot{x}_i = -\Box_i V$$

for all $i \in V$, i.e., it is an intrinsic gradient descent flow on $(S^n)^N$.

The conditions of Proposition 19 are satisfied by $(S^n)^N$ and (11). Since the canonical embedding of $(S^n)^N$ in $(R^{n+1})^N$ is compact, every sequence has a convergent subsequence by the Bolzano-Weierstrass theorem. The set of limit points is hence nonempty. It follows that $(x_i)^{N}_{i=1}$ converges to a single point, and by Proposition 19 that point is an equilibrium. □

**D. Proof of Main Theorem**

Recall that it remains to prove Theorem 13 Proposition 12, 16, 18 and 20 provide the sufficient tools to do so.

**Proof of Theorem 13.** The requirements of Proposition 18 are satisfied by Proposition 20 and Proposition 16. Since all system trajectories converge to equilibria by Proposition 20 and the set of initial conditions resulting in trajectories that converge to any equilibrium that does not belong to the consensus set is of zero measure and meager by Proposition 18 it follows that the set of trajectories converging to the consensus set is almost all of $(S^n)^N$. This establishes almost global attractiveness. Stability follows from Proposition 12.

It remains to show local exponential stability. The linearized system dynamics expressed in the variables $(y_i)^N_{i=1}$ when $x_i = c \in S^n$, $P_i = I - c \otimes c$ for all $i \in V$ are hence

$$\dot{y}_i = (I - c \otimes c) \sum_{j \in \mathcal{N}_i} f_{ij}(0)(y_j - y_i).$$ (12)

Each vector $y_i$ of the linearized system evolves along a hyperplane of codimension 1 given by $H = Tx_iS^n = \text{Im}(I - c \otimes c)$ for all $i \in V$. Since the graph is connected, and $f_{ij}(0)$ is strictly positive for all $(i,j) \in \mathbb{E}$, it follows that (12) reaches consensus exponentially if $y_i(0) \in H$ for all $i \in V$ [1]. □

**V. Perspectives**

Let us compare what is known with regard to consensus on $S^1$ and SO(3) in relation to Theorem 13.

**A. The Circle and the Sphere**

Algorithm 8 does not satisfy property (iii) of Algorithm 8 in the case of $n = 1$. This requirement is however only sufficient for almost global consensus. A counter-example that rules out almost global convergence is provided by [15]. The equilibrium set over cycle graphs where agents are spread out equidistantly over $S^1$ such that the geodesic distance $d_{ij} : S^1 \times S^1 \to [0, \pi]$ satisfies $d_{ij}(x_i, x_j) = \frac{\pi}{N}$ for all $(i,j) \in \mathbb{E}$ is asymptotically stable. This section explores the difference between $S^1$ and $S^2$ with regard to the preconditions for achieving almost global consensus.

**Example 21.** Consider six agents on $S^2$ and a cycle graph $G = (\{i \in \mathbb{N} \mid i \leq 6\}, \{1, 6\} \cup \{(i,j) \in V \times V \mid i - j = 1\})$ where we use the weights $f_{ij} = 1$ for all $(i,j) \in \mathbb{E}$. One equilibrium consists of the agents being equidistantly spread out over a great circle at a geodesic distance $\pi/3$ from one another, see Figure 2. The linearization matrix is

$$H = P(C \otimes I)P,$$

$$P = \begin{bmatrix} P_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & P_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & P_6 \end{bmatrix},$$

$$C = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

where $\otimes$ denotes the Kronecker product. The block diagonal elements of $P$ satisfy $P_i = I - R^{i-1}e_1 \otimes e_1 (R^T)^{i-1}$ for some $R \in \text{SO}(3)$ and $e \in S^2$. Note that $C$ is a circulant matrix for which all eigenpairs can be calculated explicitly [15].

There is no loss of generality in setting $e = e_1 = [1 \ 0 \ 0]^T$ and positioning all agents on the equator to decouple $P_i$ into a block diagonal matrix,

$$P_i = \begin{bmatrix} Q_i & 0 \\ 0 & 1 \end{bmatrix} = I - R^{i-1}e_1 \otimes e_1 (R^T)^{i-1},$$
The linearized dynamics are thereby decoupled into two independent subsystems, which we represent using the variables \( y_i \in \mathbb{R}^2 \) and \( z_i \in \mathbb{R} \) for all \( i \in V \). For perturbations \( (y_i)_i \) that belong to the equatorial plane, it follows that

\[
\dot{y}_i = Q_i (Q_{i-1} y_{i-1} + Q_{i+1} y_{i+1} - y_i),
\]

(13)

for all \( i \in V \), where the indices are added modulo 6. For perturbations that are normal to said plane, it holds that

\[
\dot{z} = Cz
\]

where \( z = [z_1, z_2, \ldots, z_6]^\top \).

The dynamics (13) can be written on the form

\[
\dot{y} = Q (C \otimes I) Q y,
\]

where \( Q \) is a block diagonal matrix with \( Q_i \) for \( i \in V \) as blocks, \( \otimes \) denotes the Kronecker product, \( I \) is the identity matrix of dimension 2, and \( y = [y_1, \ldots, y_6]^\top \in \mathbb{R}^{6N} \). Unlike \( \sigma (P (C \otimes I) P) \), the spectrum of \( Q (C \otimes I) Q \) belongs to the closed left-half complex plane. An eigenpair can be interpreted as a perturbation direction of the system resulting in an instantaneous response that is either aligned or negatively aligned with the perturbation. For example,

\[
(0, [v^1, (Tv)^\top, (T^2v)^\top, (T^3v)^\top, (T^4v)^\top, (T^5v)^\top])^\top,
\]

where \( T \in SO(2) \) is the leading principal submatrix of \( R \), is an eigenpair of \( Q (C \otimes I) Q \) for all \( v \in \mathbb{R}^2 \). It corresponds to the perturbation of moving each agent a fixed distance along its tangent space, thereby rotating the entire cyclic formation.

The dynamics of \( z \) are unstable since \( (1, [1111111]^\top) \) is an eigenpair of \( C \). This eigenpair can be interpreted as a perturbation that takes all agents into the north hemisphere, from where they reach consensus at the north pole. Another eigenpair is \( (-3, [1-1-1 1-1-1]^\top) \). The corresponding perturbation lifts and drops agents above and below the equator, thereby distancing any agent from the convex hull of itself and its neighbors. The response is hence a recoil towards the equator, as demonstrated by the negative eigenvalue. The effects of both these perturbations on the original nonlinear system are illustrated in Figure 3.

The unstable directions of perturbations are all orthogonal to the equator. The stability of a cycle equilibrium of System 1 under Algorithm 3 on \( S^1 \) is therefore not inherited by the embedding of \( S^1 \) in higher dimensional spheres. Aside from the instability, it is important to note such a perturbation bring all agents into a hemisphere from where they reach consensus by Proposition 12. This implies that the equator is unattractive. The circle can also be embedded on an infinite cylinder, but that case is not covered by this analysis.

The following corollary of Theorem 13 lack the generality of its precursor, but is nevertheless a result that we find to be interesting in its own right. It provides an exhaustive characterization of the stability properties of a particular dynamical system, both forwards and backwards in time. Recall that \( s_{ij} \) defined by (5) measures the extrinsic distance between two points on \( S^n \). Theorem 22 states that, under certain conditions, Algorithm 3 solves both the minimax and maximin problems of \( s_{ij} \) over all \( \{i,j\} \in E \) almost globally.

**Theorem 22.** Consider System 1 on \( S^2 \) under Algorithm 3 with \( f_{ij} = 1 \) for all \( \{i,j\} \in E \), where \( G = (V, E) \) is a cycle graph. The \( \alpha \)-limits of the flow from almost all initial conditions belong to

\[
\{(x_i)_{i=1}^N \in (S^n)^N | s_{ij} = \max_{s_{kl} | (k,l) \in E} \forall \{i,j\} \in E \}
\]

whereas the \( \omega \)-limits belong to the consensus set, i.e.,

\[
\{(x_i)_{i=1}^N \in (S^n)^N | s_{ij} = \min_{s_{kl} | (k,l) \in E} \forall \{i,j\} \in E \}.
\]

**Proof.** This is a direct consequence of Theorem 13 and the characterization of equilibria obtained by closing System 1 with the negation of Algorithm 3 provided in [7], [8].

**B. Simulations**

This section compares the global performance of two consensus protocols on \( S^n \) for \( n \in \{1, 2\} \) and on \( SO(3) \) respectively in simulation. To that end, consider the following multi-agent system on the special orthogonal group \( SO(n) \).

**System 23.** The system is given by \( N \) agents, an undirected graph \( G = (V, E) \), agent states \( R_i \in SO(n) \), and dynamics \( \dot{R}_i = \Omega_i R_i \) where \( \Omega_i \in so(n) \) for all \( i \in V \). It is assumed that \( G \) is connected and that the system can be actuated on a kinematic level, i.e., \( \Omega_i \) is the input signal of agent \( i \).

Recall that Algorithm 3 can be derived by taking the Riemannian gradient of the potential function (7). A related consensus protocol on \( SO(n) \) can be derived by taking the Riemannian gradient of the potential function \( V : (SO(n))^N \to [0, \infty) \) given by

\[
V((R_i)_{i=1}^N) = \frac{1}{2} \sum_{(i,j) \in E} \int_0^{s_{ij}} f_{ij}(r) dr,
\]
where \( f_{ij} : [0, 2\pi] \rightarrow [0, \infty) \) and \( s_{ij} = n - \langle R_i, R_j \rangle \) for all \( \{i, j\} \in E \). As such, Algorithm 23 is similar to the following algorithm on System (3).

**Algorithm 24.** The feedback is given by

\[
\Omega_i = \sum_{j \in N_i} f_{ij}(s_{ij})(R_i^T R_j - R_j^T R_i),
\]

where \( f_{ij} = f_{ji} \) for all \( \{i, j\} \in E \).

Table I displays the outcome of running \( 10^4 \) trials of Algorithm 3 on System 1 and \( 10^4 \) trials of Algorithm 24 on System 23 for three different graphs (we set \( f_{ij} = 5 \) for all \( \{i, j\} \in E \) for both algorithms). The initial conditions are drawn uniformly from the space using the fact that \( x \in N(0, I) \) implies that \( x/\|x\| \in U(S^n) \). This method is also used to draw from \( U(SO(3)) \) by first generating a uniform distribution on the unit sphere in quaternion space, \( i.e., \) drawing from \( U(S^3) \), and then mapping the sample to \( SO(3) \). By inspection of Table I note that Algorithm 3 fails to yield almost global consensus on \( S^1 \). Likewise, almost global consensus does not hold for Algorithm 24 on System 23 over \( SO(3) \). These results agree with those of [15], [29]. As predicted by Theorem 13, there were no failures to reach consensus on \( S^2 \) despite the high number of trials.

**TABLE I.** Number of failures to reach consensus on the space \( \mathcal{X} \in \{S^1, S^2, SO(3)\} \) over \( 10^4 \) random trials using Algorithm 3 and 24 with constant feedback gains.

| \( \mathcal{X} \) | \( S^1 \) | \( S^2 \) | \( SO(3) \) |
|-----------------|-------|-------|--------|
| \( 1504 \) | \( 2173 \) | \( 2126 \) | \( 711 \) |
| \( 0 \) | \( 0 \) | \( 0 \) | \( 66 \) |
| \( 86 \) | | | |

**C. Extension to the Special Orthogonal Group**

In Section V-A we learn that a certain undesired equilibrium set of System 1 under Algorithm 3 on \( S^1 \) is stable. Section V-B shows that the problem of multi-agent consensus on \( SO(3) \) poses similar challenges. In fact, if the reduced attitudes of all agents agree, then the remaining degree of rotational freedom of each agent is confined to a set that is diffeomorphic to \( S^1 \). On the \( n \)-sphere, a perturbation that is orthogonal to the equator will allow a system in such a configuration to reach consensus. On \( SO(3) \), the destabilizing effect of such a perturbation is counter-acted by the reduced attitude which, figuratively speaking, serves as a ballast that stabilizes the two other axes of all agents to a single great circle.

Let us utilize what we have learned about consensus on \( S^1 \) and \( S^2 \) to attempt to design a control law on \( SO(3) \) that stabilizes the consensus set almost globally. To that end, rewrite the variables \( R_i \) of (23) as \( R_i = [x_i, y_i, z_i] \), \( i.e., \) \( x_i = R_i e_1, y_i = R_i e_2, \) and \( z_i = R_i e_3 \) whereby

\[
\bar{R}_i = [\bar{x}_i, \bar{y}_i, \bar{z}_i] = \Omega_i[x_i, y_i, z_i]
\]

for all \( i \in V \). Let \( S : \mathbb{R}^3 \rightarrow so(3) \) be the bijective linear map defined by \( S(x) y \mapsto x \times y \) for all \( x, y \in \mathbb{R}^3 \). Denote \( \omega_i = S^{-1}(\Omega_i) \) for all \( i \in V \). The following algorithm decouples the evolution of \( x_i \) from any dependence on \( y_i \) and \( z_i \) by utilizing a decomposition of \( \omega_i \) into a part that is orthogonal to \( x_i \) and a part that is parallel to \( x_i \).

**Algorithm 25.** The feedback is given by

\[
\Omega_i = S \left( x_i \times u_i + \sum_{j \in N_i} g_{ij} x_i \right),
\]

where \( u_i \) is the input signal of Algorithm 8 and the locally Lipschitz function \( g_{ij} : I_i \rightarrow \mathbb{R} \) is related to the feedback gain of an almost globally convergent consensus protocol on \( S^1 \) for all \( \{i, j\} \in E \). More specifically, we require that the feedback gains \( g_{ij} \) are such that the system

\[
\dot{y}_i = \sum_{j \in N_i} g_{ij} y_i, \quad \dot{z}_i = -\sum_{j \in N_i} g_{ij} z_i,
\]

reach consensus for almost all initial conditions such that \( x_i(0) = x_j(0) \) for all \( \{i, j\} \in E \) (these dynamics evolve over a single great circle on \( S^2 \) since \( z_i = x_i \times y_i \) and \( x_i \) is constant, \( i.e., \) \( z_i \) can be expressed in terms of \( y_i \) for all \( i \in V \)).

**Remark 26.** To see that Algorithm 25 can be implemented by only using local and relative information, note that

\[
\begin{align*}
[\Omega_i]_{R_i} = R_i^T [\Omega_i]_{R_i} R_i &= S \left( R_i^T (x_i \times u_i) + \sum_{j \in N_i} g_{ij} R_i^T x_i \right) \\
&= S \left( R_i^T x_i \times (R_i^T u_i) + \sum_{j \in N_i} g_{ij} R_i^T x_i \right) \\
&= S \left( e_1 \times \sum_{j \in N_i} R_i^T R_j e_1 + \sum_{j \in N_i} g_{ij} e_1 \right).
\end{align*}
\]

The feedback hence only depends on the relative information \( (R_i^T R_j)_{j \in N_i} \) on \( SO(3) \).

The closed loop dynamics of System 1 under Algorithm 25 are given by

\[
\begin{align*}
\dot{x}_i &= (x_i \times u_i) \times x_i = u_i - (u_i, x_i)x_i = P_i u_i, \\
\dot{y}_i &= (x_i \times u_i) \times y_i + \sum_{j \in N_i} g_{ij} z_i, \\
&= -(u_i, y_i)x_i + \sum_{j \in N_i} g_{ij} z_i, \\
\dot{z}_i &= (x_i \times u_i) \times z_i - \sum_{j \in N_i} g_{ij} y_i, \\
&= -(u_i, z_i)x_i - \sum_{j \in N_i} g_{ij} y_i,
\end{align*}
\]

for all \( i \in V \).

Note that any implementation of Algorithm 25 involves the use of an almost globally convergent consensus protocol on \( S^1 \), \( e.g., \) that of [15], [16]. The protocol of [15], [16] requires an upper bound on the total number of agents, which is a weaker form of graph dependence than that of the protocol in [29]. The following result establishes that it is possible to
fuse Algorithm 8 with the protocol of \cite{15,16} in a manner which retains Lipschitz continuity.

**Proposition 27.** The class of feedback laws described by Algorithm 25 is nonempty.

**Proof.** We need to show that there exists at least one consensus protocol $g_{ij}$ with the required properties. Let

$$g_{ij} = g\left(\text{acos}\left(\frac{\langle y_i, y_j \rangle}{(y_i,y_j)^2 + (z_i,z_j)^2} \right) \text{sgn}(z_i, y_j)\right),$$

where $g$ is the almost globally convergent consensus protocol for the dynamics on $S^1$ in \cite{15,16}, i.e.,

$$g(\vartheta) = \begin{cases} -\frac{1}{N-1}(\pi + \vartheta) & \text{if } \vartheta \in [\pi, \pi], \\ \vartheta & \text{if } \vartheta \in [-\frac{\pi}{N}, \frac{\pi}{N}], \\ \frac{1}{N-1}(\pi - \vartheta) & \text{if } \vartheta \in (\frac{\pi}{N}, \pi], \end{cases}$$

To see that $g_{ij}$ is Lipschitz, note that the discontinuity of the sign function appears when $(y_i, z_i) = 0$ in which case the argument of $g$ is $\text{acos} \text{sgn}(y_i, y_j) \in (0, \pi)$ and $g(-\pi) = g(\pi)$.

Suppose that $x_i = x_j$ for all $(i,j) \in \mathcal{E}$. Let $(v_1, v_2)$ be an orthonormal basis of the plane $\mathcal{P}$ such that $y_i, z_i \in \mathcal{P}$ for all $i \in \mathcal{V}$. If $(R_i(0))_i = 1 \in S_2$, then

$$y_i = \cos \vartheta_i v_1 + \sin \vartheta_i v_2, \\
z_i = \cos(\vartheta_i + \frac{\pi}{2}) v_1 + \sin(\vartheta_i + \frac{\pi}{2}) v_2$$

for some $\vartheta_i \in (-\pi, \pi)$ for all $i \in \mathcal{V}$. Moreover,

$$\dot{y}_i = -\dot{\vartheta}_i \sin \vartheta_i v_1 + \dot{\vartheta}_i \cos \vartheta_i v_2 = \dot{\vartheta}_i z_i,$$

where

$$\dot{g}(\vartheta) = \begin{cases} -\frac{1}{N-1}(\pi + \vartheta) & \text{if } \vartheta \in [\pi, \pi], \\ \vartheta & \text{if } \vartheta \in [-\frac{\pi}{N}, \frac{\pi}{N}], \\ \frac{1}{N-1}(\pi - \vartheta) & \text{if } \vartheta \in (\frac{\pi}{N}, \pi], \end{cases}$$

Note that $y_j = (\langle y_i, y_j \rangle) y_j + (z_i, y_j) z_i$ on $\mathcal{P}$. The argument of $g$ is hence $\text{acos}(\langle y_i, y_j \rangle) \text{sgn}(z_i, y_j) = \vartheta_j - \vartheta_i$, which can be interpreted as a signed relative arc length on $S^1$. As such, the dynamics on of $\vartheta_i$ reduces to

$$\dot{\vartheta}_i = \sum_{j \in N_i} g(\vartheta_j - \vartheta_i),$$

i.e., to the form of the almost globally convergent consensus protocol \cite{15,16} on $S^1$. \hfill \Box

Note that the dynamics of $(x_i)_{i = 1}^N$ given by \cite{15} are precisely those of System 1 under Algorithm 8. The consensus set for the reduced attitudes $(x_i)_{i = 1}^N$ is hence almost globally asymptotically stable by Theorem 13. We will utilize the triangular structure of the system given by \cite{15} to establish a local convergence result. To this end, consider Proposition 28 from [47] which have been adapted to our setting.

**Proposition 28 (M.I. El-Hawwary & M. Maggiore [47]).** Consider a system $x = f(x)$, where $f$ is locally Lipschitz, that evolves on a compact state-space $X$. Let $S_1$ and $S_2$, where $S_1 \subset S_2 \subset X$, be two closed, positively invariant sets. Then, $S_1$ is asymptotically stable if the following conditions hold:

(i) $S_1$ is asymptotically stable relative to $S_2$.
(ii) $S_2$ is asymptotically stable.

**Remark 29.** There exists a global version of Proposition 28 from [47]. The case when convergence from $S_2$ to $S_1$ is global but convergence from $X$ to $S_2$ is almost global can be addressed by redefining $X$ to be the region of attraction of $S_2$, see [48]. However, it cannot be applied in our situation. The problem is that $S_1$ is only almost globally stable relative to $S_2$. We cannot guarantee that the convergence from $X$ to $S_2$ would not bring the system to a state at which convergence from $S_2$ to $S_1$ fails.

**Proposition 30.** The consensus set on $SO(3)$,

$$C = \{(R_i)_{i = 1}^N \in (SO(3))^N \mid R_i = R_j, \forall \{i, j\} \in \mathcal{E}\},$$

is an asymptotically stable equilibrium set of System 23 under Algorithm 25.

**Proof.** In terms of Proposition 28 let

$$S_1 = \{(R_i)_{i = 1}^N \in (SO(3))^N \mid R_i = R_j, \forall \{i, j\} \in \mathcal{E}\},$$

$$S_2 = \{(R_i)_{i = 1}^N \in (SO(3))^N \mid R_i e_1 = R_j e_1, \forall \{i, j\} \in \mathcal{E}\},$$

denote the consensus set and reduced attitude consensus set. Clearly $S_1$, $S_2$ are closed, positively invariant, nested sets. Property (ii) follows by application of Theorem 13 to the dynamics (15). To establish property (i), consider the case of $(R_i(0))_i = 1 \in S_2$. Then $y_i, z_i \in \mathcal{P}$ for all $i \in \mathcal{V}$, where $\mathcal{P}$ is the plane that has $x_i$ as normal for any $i \in \mathcal{V}$. The system (15)–(17) is hence on the form (14). The consensus set of the system (14) is almost globally asymptotically stable by our assumptions on $g_{ij}$ for all $\{i, j\} \in \mathcal{E}$, which implies (i). \hfill \Box

Let us return to the simulation problem of Section V-B. Generating uniformly distributed initial conditions on $SO(3)$ and simulating Algorithm 25 where the algorithm of \cite{15,16} is used to generate consensus on $S^1$ for the three graph topologies of Table 1 we find no failures to reach consensus. Algorithm 25 hence outperforms Algorithm 24 and rivals the practical performance of the algorithm in \cite{29}. Moreover, the version of Algorithm 25 based on \cite{15,16} only requires each agent to know an upper bound on $N$. Algorithm 25 also rivals the theoretical performance of \cite{29}, as shown in Theorem 31 of Theorem 13. Note that we cannot conclude that the consensus manifold is almost globally stable from the result of Theorem 31 since System 23 under Algorithm 25 is not a gradient descent flow.

**Theorem 31.** Suppose all feedback gains $g_{ij}$ for $\{i, j\} \in \mathcal{E}$ in Algorithm 25 are chosen such that all equilibria of system (14) are exponentially unstable except for those in $C$. Then all equilibria of System 23 under Algorithm 25 except those in $C$ are exponentially unstable. Moreover, $C$ is asymptotically stable.

**Proof.** Note that the linearization decouples like the dynamics (15)–(17), Theorem 13 establishes that the all equilibria except those belonging to the consensus set are unstable for the sub-system (15). Any candidate for a stable equilibrium must hence satisfy $x_i = x_j$ for all $\{i, j\} \in \mathcal{E}$. This requirement reduces
the dynamics \([15,17]\) to \([14]\) for which all equilibria apart from those in \(\mathcal{C}\) are exponentially unstable by assumption. That \(\mathcal{C}\) is asymptotically stable follows from Proposition \([30]\).

VI. Conclusions

This paper establishes almost global consensus on the \(n\)-sphere for general \(n \in \mathbb{N}\setminus\{1\}\), a class of intrinsic gradient descent consensus protocols, and all connected, undirected graph topologies. These results show that the conditions for achieving almost global consensus are more favorable on the \(n\)-sphere than known results regarding other Riemannian manifolds would suggest. In particular, almost global consensus on \(S^1\) \([15]\) and \(SO(3)\) \([29,30]\) requires protocols that are tailored for this specific purpose. The case of \(S^1\) differs from that of the general \(n\)-sphere due to its low dimension. There are asymptotically stable equilibrium sets on \(S^1\) that are disjunct from the consensus set. If these sets are embedded on the \(n\)-sphere for \(n \in \mathbb{N}\setminus\{1\}\) in the form of great circles then any normal to the corresponding equatorial plane is a direction of instability. The circle can also be embedded on \(SO(3)\), but there it gives rise to asymptotically stable undesired equilibria. By combing our understanding of almost global consensus on \(S^1\) and \(S^2\) we design a novel class of consensus protocol on \(SO(3)\) which renders undesired equilibria unstable and is shown to avoid them in simulation.

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