The anomalous transport of axial charge: topological vs non-topological fluctuations

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ABSTRACT: Axial charge imbalance is an essential ingredient in novel effects associated with chiral anomaly such as chiral magnetic effects (CME). In a non-Abelian plasma with chiral fermions, local axial charge can be generated a) by topological fluctuations which would create domains with non-zero winding number b) by conventional non-topological thermal fluctuations. We provide a holographic evaluations of medium’s response to dynamically generated axial charge density in hydrodynamic limit and examine if medium’s response depends on the microscopic origins of axial charge imbalance. We show a local domain with non-zero winding number would induce a non-dissipative axial current due to chiral anomaly. We illustrate holographically that a local axial charge imbalance would be damped out with the damping rate related to Chern-Simon diffusive constant. By computing chiral magnetic current in the presence of dynamically generated axial charge density, we found that the ratio of CME current over the axial charge density is independent of the origin of axial charge imbalance in low frequency and momentum limit. Finally, a stochastic hydrodynamic equation of the axial charge is formulated by including both types of fluctuations.

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1. Introduction

The parity-odd response of a medium with chiral fermions and its deep relationship to topology and quantum anomalies have attracted significant interest. One such effect under extensive study is the chiral magnetic effect (CME) [1–4], which is the appearance of a vector current along the direction of an external magnetic field in the presence of axial charge imbalance (see Refs. [5,6] for a recent review). The CME has been demonstrated in various theoretical frameworks, such as in hydrodynamics [7–13], kinetic theories [14–19], perturbative theories [4,20,21], effective theories [22–26] and in the AdS/CFT correspondence [27–32]. A closely related effect is the chiral vortical effects (CVE), which is the appearance of a current along the direction of vorticity. CVE and its relation to mixed anomalies has been studied in [33–37]. Interesting properties of chiral media have been discussed in [38–40].

Those anomalous effects are not only theoretically well-motivated, but also phenomenologically important. In a heavy ion collision, a very strong magnetic field, on the order of $eB \sim m^2$, is created from the incoming nuclei that are positively charged and move at nearly the speed of light. Therefore, CME will convert axial charge fluctuations generated in heavy-ion collisions into (vector) charge-dependent correlation which could be potentially detected by experimental observables. Recently, there have been significant experimental efforts in searching for CME and other anomalous transport effects (see [41] for a review) in heavy ion collision experiments [42–45].

One essential ingredient in those anomaly-related effects is the presence of axial charge imbalance. For example, in terms of chiral charge imbalance parametrized by the axial chemical potential $\mu_A$, CME can be expressed as:

$$j_V^{\text{CME}} = C_A \mu_A eB, \quad C_A = \frac{N_c}{2\pi^2}.$$  \hspace{1cm} (1.1)

Previously, most studies were based on introducing axial charge asymmetry by hand, after which the response of the medium to a magnetic field is investigated (see Ref. [46,47] for exceptional cases). However, axial charge density $n_A(t, x)$ is a local and dynamical quantity depending on space and time. The medium’s response to time-dependent, in-homogeneous axial charge density and the connections of this response to anomaly have been rarely studied before. One motivation of this paper is to fill this gap.

A distinctive feature of local axial charge density (in contrast to vector charge density) is that there are microscopically two different mechanisms for local generation of axial charge imbalance. The first one is by topological fluctuations of gluonic fields which would create domains with non-zero winding number. The resulting topological charge will in turn convert into axial charge density via anomaly relation:

$$\partial_\mu j_A^\mu = -2q, \quad q = \frac{g^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} G_{\mu\nu} G_{\rho\sigma},$$  \hspace{1cm} (1.2)

where $q$ is topological charge density (Pontryagin density) and $G_{\mu\nu}$ denotes field strength of gluonic fields. Such topological fluctuations would in general create both global axial charge and local axial charge imbalance. The second mechanism is through thermal fluctuations.
In the absence of topological fluctuations, while the whole system is a grand canonical ensemble (of axial charge), each fluid cell of that system can be considered as a canonical ensemble (of axial charge). Therefore axial charge inside a fluid cell still fluctuates due to thermal fluctuations. For a non-Abelian plasma with chiral fermions, both mechanisms would contribute to local axial charge density fluctuations. Would response of the medium depend on the way that axial charge density is generated?

In this paper, we will consider a de-confined non-Abelian plasma with chiral fermions at finite temperature. We will begin by studying the medium’s response to the interplay between local axial charge density \( n_A(t, x) \) and \( j_A(t, x) \) and \( q(t, x) \) in long time and long wavelength limit (i.e. in hydrodynamic regime). In particular, we will examine if the relation among those one point functions depends on the microscopic origin of local axial charge density. As a basis for this study, we will work in a top-down holographic model, namely Sakai-Sugimoto model [48, 49]. The Sakai-Sugimoto model is considered to be close to the large \( N_c \) QCD with massless chiral quarks in quenched approximation. It has been widely applied to study anomaly-related effects (e.g. [27, 50, 51]). To model gluonic fluctuations and implement anomaly relation (1.2), we will consider the dynamics of \( C_7 \) Ramond-Ramond field and its Wess-Zumino coupling to flavor sector. As we are working in the long time, long wave length limit, the results presented in this paper are analytic.

We found axial current \( j_A(t, x) \) can be created by in-homogeneity of topological domains. It is well known that in-homogeneity of axial charge densities leads to diffusion: \( j_A(t, x) = -D \nabla n_A(t, x) \) where \( D \) is the conventional diffusive constant. However, if the axial charge density results from a local topological domain which will be represented by an effective field “\( \theta(t, x) \)” in this paper, such topological domain will also induce an axial charge density and non-dissipative axial current in addition to diffusive current:

\[
n_A = \frac{\Gamma_{CS}}{T} \theta(t, x), \quad j_A^{\text{new}} = \kappa_{CS} \nabla \theta.
\]

Here, \( \Gamma_{CS} \) and \( \kappa_{CS} \) is related to the behavior of retarded Green’s function \( G_{qq}^{R}(t, x) \sim \langle q(t, x), q(0, 0) \rangle \) in hydrodynamic regime:

\[
G_{qq}^{R}(\omega, k) = \frac{1}{2} \left[ -i \frac{\Gamma_{CS}}{T} \omega - \kappa_{CS} k^2 \right],
\]

and \( T \) denotes temperature. It is worthy noting that \( \kappa_{CS} \) term in (1.3) is opposite to the direction of diffusive current and non-dissipative. One way to understand current (1.3) is that a topological domain also carries kinetic energy which would be transfered to chiral fermions via anomaly relation (1.2). In Ref. [52] by us, we have derived (1.3) based on a generic setting and presented a brief verification of (1.3) in the Sakai-Sugimoto model. We provide more details on this calculation in Sec. 3.1 and elucidate how axial current (1.3) is generated from gravity side of the duality.

Our calculation in Sec. 3.2 confirms that a local chiral charge imbalance \( n_A(t, x) \) will induced a non-zero \( q(t, x) \) and they are related by

\[
q(t, x) = \frac{n_A(t, x)}{2 \tau_{sph}}.
\]
\(\tau_{sph}\) here can be interpreted as the axial charge damping time. Indeed, substituting (1.3) into (1.2), one has: \(\partial_\mu J^\mu_A = -n_A/\tau_{sph}\). Eqs. (1.5) hence implies that a non-zero axial charge density will eventually be damped out by inducing a non-zero \(q\). Furthermore, we verify by holographic computations that \(\tau_{sph}\) is related to Chern-Simon diffusive rate \(\Gamma_{CS}\) and susceptibility \(\chi\):

\[
\tau_{sph} = \frac{\chi T}{2\Gamma_{CS}}.
\]

(1.6)

Previously, relation (1.6) has been derived based on the standard fluctuations-dissipation argument [53] (see also Sec. 3). Very recently, \(\tau_{sph}\) has also been computed numerically in a bottom-up holographic model [54]. To best of our knowledge, current work is the first direct verification of relation (1.6) in strong coupling regime.

As we find that the axial current in response to axial charge density depend on how such axial charge imbalance is generated, it is natural to ask if chiral magnetic current (1.1) also depends on the origin of axial charge imbalance. To be quantitative, we consider the ratio between CME current and axial charge density in low frequency, small momentum limit in the presence of constant magnetic field:

\[
(\chi_{\text{dyn}})^{-1} \equiv \lim_{\omega,k \to 0} \left[ \frac{\mu_A(\omega,k)}{n_A(\omega,k)} \right], \quad \mu_A(\omega,k) \equiv \frac{j_V^{CME}(\omega,k)}{C_A e B}.
\]

(1.7)

For a system with constant axial charge density \(n_A\), the ratio \(j_V^{CME}/(CeB)\) equals to axial chemical potential \(\mu_A\) due to (1.3). However, if \(n_A\) is space-time dependent, the definition of axial chemical potential \(\mu_A\) is ambiguous. If one takes the ratio \(j_V^{CME}/(CeB)\) as the generalized definition of axial chemical potential, the ratio \(n_A(\omega,k)/\mu_A(\omega,k)\) can be interpreted as susceptibility. For this reason, we will call \(\chi_{\text{dyn}}\) the “dynamical axial susceptibility”.

We would like to emphasis that \(\chi_{\text{dyn}}\) (1.7) is conceptually different from chiral magnetic conductivity [55] which is the proportionality coefficient of CME current to the time-dependent magnetic field for a medium with homogeneous, time-independent axial chemical potential. In (1.7) however, magnetic field is constant while \(n_A\) is space-time dependent. It is worthy noting that in realistic situations such as quark-gluon plasma (QGP) created in heavy-ion collisions, \(\chi_{\text{dyn}}\) would be a relevant measure of CME as in those situations, the axial charge density is always generated dynamically. Previously, chiral magnetic conductivity has been calculated for plasma in equilibrium at both weak coupling limit [20,21,55] and strong coupling [50,56,57] and for plasma out-of-equilibrium [58]. However, we are not aware any existing literature discussing the “dynamic axial susceptibility” \(\chi_{\text{dyn}}\) and its universality.

We have computed \(\chi_{\text{dyn}}\) with \(n_A(\omega,k)\) generated by topological and thermal (non-topological) fluctuations in Sakai-Sugimoto model. We found such ratio \(\chi_{\text{dyn}}\) (1.7) is independent of the origin of axial charge imbalance and equals to static susceptibility \(\chi\). Moreover, we derive a simple analytic expression (4.15) relating the (integration of) gravity metric to \(\chi_{\text{dyn}}\) which applies to a large class of holographic model. From such expression, we obtain a condition on the universality of \(\chi_{\text{dyn}}\).
Having established the fact that axial charge generate by both topological fluctuations and thermal fluctuations would contribution to CME current, it is then important to incorporate both fluctuations in the framework of stochastic hydrodynamics. Recently, there are encouraging progress on applying anomalous hydrodynamics to simulate charge separation effects \cite{59, 60} and chiral magnetic wave effects \cite{61, 62} in heavy-ion collisions. In those studies, axial charge density enters as the initial conditions while the fluctuations of axial charge density during hydrodynamic evolution have been neglected. Motivated by findings in this paper, we formulate a stochastic hydrodynamic equation of axial charge density in Sec. 5. Such hydrodynamic equation includes stochastic noise from both topological fluctuations and thermal fluctuations. While it is a direct generalization of the general framework \cite{63–65}, to our knowledge, stochastic equation (5.8) is new in literature. We hope our theory would be applied to simulate phenomenology of anomalous transport in the future.

The paper is organized as follows. We will begin with a brief review of pertinent ingredients of Sakai-Sugimoto model and realization of anomaly relation (1.2) in Sec. 2. Sec. 3 is devoted to studying medium’s response to axial charge density. The computation of $\chi_{\text{dyn}}$ is presented in Sec. 4. The stochastic hydrodynamic equation for axial charge is formulated in Sec. 5. We conclude in Sec. 6.

2. Sakai-Sugimoto model and chiral anomaly

2.1 Set-up of the model and realization of anomaly

In this paper, we will work in Sakai-Sugimoto model \cite{48, 49}. In this model, the de-confined phase is given by $D4$ black-brane metric, which is a warped product of a 5d black hole and $S^1 \times S^4$ \cite{66, 67}. The $D4$ brane background is given by \cite{66}:

$$ds^2 = \left(\frac{U}{R}\right)^{3/2} (-f(U)dt^2 + dx^2 + dx_4^2) + \left(\frac{R}{U}\right)^{3/2} \left(U^2 d\Omega_4^2 + \frac{dU^2}{f(U)}\right),$$  \hspace{1cm} (2.1)

$$F^{RR(4)} = \frac{2\pi N_c \epsilon_4}{V_4}, \quad \epsilon_4 = g_s \left(\frac{U}{R}\right)^{3/4}, \quad R^3 = \pi g_s N_c f_s^3, \quad f(U) = 1 - \left(\frac{U_T}{U}\right)^3.$$  \hspace{1cm} (2.2)

Here $x_4$ is the coordinates of $S_1$ and $\epsilon_4$ is the volume form of the four sphere $S_4$. In addition, $V_4 = 8\pi^2/3$ is the volume of $S_4$ and $g_s, l_s$ are string coupling and string length respectively. The location of the horizon $U_T$ is related to the inverse temperature:

$$\frac{4\pi}{3} \frac{R^3/2}{U_T^{1/2}} = \frac{1}{T}.$$  \hspace{1cm} (2.3)

The periodicity of $x_4$ is given by

$$\delta \tau = 2\pi R_4 = \frac{2\pi}{M_{KK}}.$$  \hspace{1cm} (2.4)
The background (2.1) is stable for $T > 1/(2\pi R_4) = M_{KK}/2\pi$ [67]. Finally, 't Hooft coupling $\lambda$ is given by:

$$\lambda = N_c \frac{(2\pi)^2 g_s l_s}{2\pi R_4} = 2\pi N_c g_s l_s M_{KK}.$$  \hspace{1cm} (2.5)

To model gluonic fluctuations, we will consider the dynamics of $C_7$ Ramond-Ramond form. The kinetic energy of $C_7$ are given by

$$S_{RR} = - \frac{(2\pi l_s)^6}{4\pi} \int_{10} dC_7 \wedge (*dC_7).$$  \hspace{1cm} (2.6)

In Sakai-Sugimoto model, right and left handed quarks are introduced by $N_f D_8$ branes and $N_f \bar{D}_8$ branes [48, 49]. Right-handed (left-handed) $U(1)$ gauge field $A_R$ ($A_L$) lives on $D8$ ($\bar{D}8$) and is dual to right-handed (left-handed) current $J_R^\mu$ ($J_L^\mu$) on the boundary. The $D8/\bar{D}8$ branes are separated along the $x_4$ direction, with $D8$ branes located at $x_4 = 0$ and $\bar{D}8$ branes located at $x_4 = \pi R_4$. In this work will consider $N_f = 1$ though generalization to the case of multi-flavors is straightforward.

The action of bulk gauge field $A_{R,L}$ or its field strength $F_{R,L}$ is given by the summation of Dirac-Born-Infeld (DBI) term

$$S_{DBI}^{R,L} = - \frac{1}{(2\pi)^{8} l_s^{9}} \int d^{9}x e^{-\phi} \sqrt{-\det (g_{MN} + (2\pi \alpha') F_{R,L}^{MN})},$$  \hspace{1cm} (2.7)

and Wess-Zumino (WZ) term which couples $A_{R,L}$ to Ramond-Ramond form:

$$S_{WZ}^{R,L} = \pm \int_{\Sigma_9} \Sigma_q C_{q+1} \wedge \text{tr} e^{\frac{F_{R,L}}{2\pi^2}}.$$  \hspace{1cm} (2.8)

We normalize the RR forms $C$ as in [48] and use hermitian worldvolume gauge field $A$. The DBI action is identical for $A_R$ and $A_L$. In WZ term, plus/minus sign is corresponding to $A_R/A_L$ (i.e. $F_{R,N}^{MN}/F_{L,N}^{MN}$) respectively. Axial gauge field and vector gauge field are related to right-handed and left-handed gauge field by:

$$A = \frac{A^R - A^L}{2}, \quad V = \frac{A^R + A^L}{2}.$$  \hspace{1cm} (2.9)

The total action we will study then becomes:

$$S = S_{RR} + S_{DBI} + S_{WZ}.$$  \hspace{1cm} (2.10)

It is instructive to show how axial anomaly relation (1.2) is realized in the current holographic model. Following holographic dictionary, the axial current $j^\mu_A$ is given by the variation of holographic on-shell action $S_{\text{holo}}$ with respect the boundary value of $a_\mu \equiv A_\mu(U \to \infty)$ and $q$ is given by the variation of $S_{\text{holo}}$ with respect to $\theta$:

$$j^\mu_A = \frac{\delta S_{\text{holo}}}{\delta a_\mu}, \quad q = \frac{\delta S_{\text{holo}}}{\delta \theta}.$$  \hspace{1cm} (2.11)

Here $\theta$ is determined by the holonomy of $C_1$ on the compactified $x_4$ direction [68]:

$$\theta(t, x) = \lim_{U \to \infty} \int dx_4 (C_1^{(4)}).$$  \hspace{1cm} (2.12)
One may note that the normalization in (2.12) is consistent with the one found by considering action of multiple probe color branes.

$C_1$ is related to $C_7$ and $A_{R,L}$ field [69]:

$$dC_1 = (2\pi l_s)^6 * dC_7 - A^R \wedge (\delta(x_4 - \pi R_4)dx_4) + A^L \wedge (\delta(x_4)dx_4) .$$  \hspace{1cm} (2.13)

It is clear from (2.13) that $C_7$ is invariant under axial gauge transformation:

$$\delta_\Lambda A^R = d\Lambda, \quad \delta_\Lambda A^L = -d\Lambda, \quad \delta_\Lambda C_1 = -\Lambda \delta(x_4)dx_4 - \Lambda \delta(x_4 - \pi R_4)dx_4 .$$  \hspace{1cm} (2.14)

where $\Lambda$ is an arbitrary scalar function. At boundary and after integrating out $x_4$ dependence, the axial gauge transformation is reduced to

$$\delta_\Lambda a_\mu(t, x) = \partial_\mu \Lambda(t, x), \quad \delta_\Lambda \theta(t, x) = -2\Lambda(t, x) .$$  \hspace{1cm} (2.15)

Since the action (2.10) expressed in terms of $C_7$ and field strength $F_R/F_L$ are manifestly invariant under axial gauge transformation (2.14), $S_{\text{holo}}$ would also be invariant under (2.15). We therefore have that for an infinitesimal transformation $\delta\Lambda$:

$$\delta_\Lambda S_{\text{holo}} = \int d^4x \left[ \frac{\delta S_{\text{holo}}}{\partial a_\mu(t, x)} \partial_\mu (\delta \Lambda(t, x)) + \frac{\delta S_{\text{holo}}}{\partial \theta(t, x)} (-2\delta \Lambda(t, x)) \right]$$

$$= \int d^4x \left[ j_\mu^\Lambda(t, x) (\partial_\mu \Lambda(t, x)) - 2q(t, x)\delta \Lambda(t, x) \right] = -\int d^4x \left[ \partial_\mu j_\mu^\Lambda(t, x) + 2q(t, x) \right] \delta \Lambda(t, x) = 0 .$$  \hspace{1cm} (2.16)

The anomaly relation (1.2) then follows from the requirement that (2.16) holds for arbitrary $\delta\Lambda$. This is the holographic realization of axial anomaly in the current model. Realization of axial anomaly in general $Dp/Dq$ brane can be found in [70].

2.2 Fluctuations of bulk fields

We wish to study medium’s response to local axial charge imbalance. To model that process in holography, we need to introduce sources on the boundary. Those sources will excite bulk fields which in turn would generate one point functions such as axial current $j_\mu^\Lambda$ and $q$ on the boundary. As we discussed in the introduction, local axial charge imbalance can be generated by a gluonic configuration with non-zero winding number and by thermal fluctuations. Correspondingly, we will create axial charge imbalance by putting a non-zero $\theta(t, x)$ and by putting a non-zero axial gauge field $a_\mu(t, x)$ on the boundary. In this work, we will restrict ourselves to the longitudinal fluctuations, i.e., the source on the boundary are $\theta(t, x), a_t(t, x), a_x(t, x)$ where $x$ corresponds to the direction of non-vanishing current and sources will only depend on $t, x$. Consequently, non-zero bulk fields are $A_t, A_x, A_U, C_7$ and they would only depend on $t, x, U^1$. For later convenience, we introduce a dimensionless radial coordinate:

$$u \equiv U/U_T .$$  \hspace{1cm} (2.17)

Note that $C_7$ can also depend on $x_4$. This can happen when we consider backreaction of D8 brane. We will restrict ourselves to lowest mode on $S^1$ with no $x_4$ dependence in this work.
We will also rescale all other dimensionful quantities by
\[ \tilde{T} \equiv \sqrt{\frac{U_T}{R^3}} = \frac{4\pi}{3} T = KT, \quad K \equiv \frac{4\pi}{3}. \] (2.18)
where we have used (2.3).

We take following ansatz for \( C_7 \):
\[ C_7 = B_M(t, x; u) \, dx^M \wedge d\sigma_{12} \wedge \epsilon_4 = [B_t(t, x; u) \, dt + B_x(t, x; u) \, dx + B_u(t, x; u) \, du] \wedge d\sigma_{12} \wedge \epsilon_4. \] (2.19)

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where we have used (2.3).
2.3 The prescription for computing one point function

In this section, we will derive the explicit holographic prescription for computing one point function $n_A, j_A, q$ from definition (2.11). For this purpose, we need to first obtain holographic on-shell action and then perform the variation with respect to the sources, $\theta, a_t, a_x$. It is therefore more natural to rewrite the holographic action in terms of $A$ and $C_1$ field instead of $C_7$. The reason is that $C_1$ is directly related to the boundary source $\theta$ as seen in Eq.(2.12). From (2.13) and (2.19), we have explicitly:

$$\sqrt{-g} \epsilon_{LMN} G^{MN} = 2K [\partial_L M + 2A_L].$$

(2.26)

where $M$ is the $x_4$ component of $C_1$, integrated over the $x_4$ circle:

$$M \equiv \int dx_4 C_1^{(4)}.$$

(2.27)

The boundary value of $M$ is $\theta$. Note that $G^{MN}$ and $(\partial_N M + 2A_N)$ are invariant under both flavor and $C_7$ gauge transformations (2.14). Using (2.26), (2.25) can be written in terms of $M$ as

$$N_c \partial_M (\sqrt{-\gamma} F^{MN}) = 2K^2 \sqrt{-g'} (\partial^N M + 2A^N),$$

(2.28)

where we have defined:

$$\sqrt{-g'} \equiv \frac{(-g_{tt}g_{xx}g_{uu})}{\sqrt{-g}}.$$

(2.29)

Moreover, the Bianchi identity of $G_{MN}$ reads

$$\epsilon^{LMN} \partial_L G_{MN} = 0,$$

(2.30)

together with relation (2.26) gives

$$\partial_N \left[ \sqrt{-g'} (2A^N + \partial^N M) \right] = 0.$$

(2.31)

It is easy to see the action in terms of the $M$ and $A_M$ fields, which would lead to equation of motion (2.28) and (2.31) is

$$S = \int d^4x du \left[ -\frac{1}{4} N_c \sqrt{-\gamma} F^{MN} F_{MN} - \frac{K^2}{2} \sqrt{-g'} (\partial_Q M + 2A_Q) \left( \partial^Q M + 2A^Q \right) \right],$$

(2.32)

as it is analyzed in [69], [70]. The action has the same form as Eq. (9) in [71] from a bottom-up model. However our action differs in the $N_c$ dependence of different terms in (2.32), which is absent in [71].

In order to compute the one point functions of $q, n_A$ and $j_A$ we use the action (2.32) instead of (2.23), since it is expressed in terms of the bulk fields which are directly related to the sources of the boundary operators. To obtain the on-shell action, we do variation of (2.32):

$$\delta S = \int d^4x du \left[ \partial_M \left( N_c \sqrt{-\gamma} F^{MN} - 2K^2 \sqrt{-g'} (\partial^N M + 2A^N) \right) \delta A_N + K^2 \sqrt{-g'} \partial_N \left( \partial^N M + 2A^N \right) \delta M \right]$$

$$+ \int d^4x \left[ N_c \sqrt{-\gamma} F^{NU} \delta A_N - K^2 \sqrt{-g'} (\partial^u M + 2A^u) \delta M \right].$$

(2.33)
The bulk term vanishes by Eqs. (2.28), (2.31), and the boundary term gives the on-shell action. We therefore have from (2.11)

\[
q(t, x) = \lim_{u \to \infty} \left[ -K^2 \sqrt{-g} \left( \partial^u M(t, x; u) + 2A^u(t, x; u) \right) \right]_{\text{Ren}} = \lim_{u \to \infty} \left[ -KG_{tx}(t, x; u) \right]_{\text{Ren}},
\]

and

\[
n_A(t, x) = N_c \lim_{u \to \infty} \left[ \sqrt{-\gamma} F^{tu}(t, x; u) \right]_{\text{Ren}}, \quad j_A(t, x) = N_c \lim_{u \to \infty} \left[ \sqrt{-\gamma} F^{xu}(t, x; u) \right]_{\text{Ren}}.
\]

In (2.35), we have introduced a “bulk axial current” [72]:

\[
J^\mu_A(u) \equiv N_c \sqrt{-\gamma} F^{\mu u}(u).
\]

As in general bulk current $J^\mu_A(u)$ and bulk field $G_{tx}(u)$ might be divergent near the boundary $u \to \infty$, we use the subscript “Ren” in (2.35) and (2.34) to denote the subtraction of such divergences in (2.34). The correspondence: (2.35) and (2.34) has been used in Ref. [52]. It is also interesting to note that from the $u$-component of the first equation in (2.25a):

\[
N_c \partial_\mu \left[ \sqrt{-\gamma} F^{\mu u} \right] = 2KG_{tx},
\]

the anomaly relation (1.2) will be reproduced by taking $u \to \infty$ limit on both side of (2.37) using (2.35) and (2.34).

We would like to comment on the nature of current $j_A$. Naively $j_A$ obtained by a functional derivative is by definition a consistent current with respect to flavor gauge. In fact, it is also the covariant current. We can confirm this by noting that boundary source entering the bulk field strength only through boundary values of $E_A$ and $G$, which are axial gauge invariant, therefore $j_A$ is manifestly axial gauge invariant. The agreement of consistent and covariant currents may appear odd: this is because the QCD anomaly studied in this section is realized with an on-shell action that is manifestly axial gauge invariant. Therefore, the current obtained from functional derivative is also invariant, as if it were an ordinary current. In contrast, the QED anomaly is realized with an anomalous on-shell action under axial gauge transform. In this case, we can not have a current which is both conserved (consistent) and invariant (covariant), in the presence of external axial field. As an example, we will see that the covariant current is not conserved in section 4, where we expand our study to include QED anomaly. Furthermore, because holography has access to gauge invariant (with respect to $SU(N_c)$ gauge) quantities only, $j_A$ is also covariant current with respect to $SU(N_c)$ gauge transform. We stress that the action (2.32) is very different from what we would have obtained by a naive substitution of (2.26) into (2.22). In particular, the kinetic term of $M$ would have an opposite sign, which would lead to a wrong sign for $q$, [48, 69, 70].

### 2.4 Computing one point function

We now ready to compute one point function. We will work in Fourier space: $\partial_t \to -i\omega$, $\partial_x \to ik$. Then using (2.37) and Bianchi identity, one would express bulk current $J^\mu_A, J^\nu_A$.
Here and hereafter, we use prime to denote the derivative with respect to $N$. Accordingly, we will compute one point function such as $J_A(u; \omega, k)$. To the leading nontrivial order, the solutions are not affected by the back-reaction. At order $1/N$, while $G(u; \omega, k)$ satisfies in-homogeneous equation: 

$$G''(u; \omega, k) = \left( \frac{5}{2u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)} \right) E_A'(u; \omega, k) + \frac{(\omega^2 - k^2 f)}{u^2 f^2} E_A(u; \omega, k) = 0 ,$$

(2.38a)

and $J_A(u; \omega, k)$ to first non-trivial order in power series of $1/N_c$: 

$$J_A(u; \omega, k) \equiv F_{tx}(u; \omega, k), \quad G(u; \omega, k) \equiv -G_{tx}(u; \omega, k).$$

(2.39)

Here and hereafter, we use prime to denote the derivative with respect to $u$. From (2.38), we observe that we only need to solve equations for $G(u; \omega, k), E_A(u; \omega, k)$ to obtain one point function $n_A, j_A, q$. From (2.24), one finds:

$$G'' + \left( \frac{1}{u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)} \right) G'' + \frac{(\omega^2 - k^2 f)}{u^2 f^2} G = \left( \frac{1}{N_c} \right) \left[ \frac{4K^2}{\sqrt{-\gamma}} \right] \[ + \frac{2K\omega kf'}{\sqrt{-g f(\omega^2 - k^2 f)}} \right] E_A ,$$

(2.40a)

$$E_A'' + \left( \frac{5}{2u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)} \right) E_A' + \frac{(\omega^2 - k^2 f)}{u^3 f^2} E_A = \left( \frac{1}{N_c} \right) \left[ \frac{4K^2}{\sqrt{-g f(\omega^2 - k^2 f)}} \right] E_A + \left( \frac{1}{N_c} \right) \left[ \frac{2K\omega kf'}{\sqrt{-g f(\omega^2 - k^2 f)}} \right] G ,$$

(2.40b)

It is understood that the back-reaction of the flavor branes will induce $1/N_c$ correction to the black-brane metric. Analysis shows that the correction to the metric could induce terms $\sim G/N_c$ and $\sim E_A/N_c$ to (2.40a) and terms $\sim E_A/N_c$ and $\sim G/N_c^2$ to (2.40b). We will seek solutions to the leading nontrivial order in power series of $1/N_c$:

$$G = G^{(0)} + \frac{1}{N_c} G^{(1)} + \ldots , \quad E_A = E_A^{(0)} + \frac{1}{N_c} E_A^{(1)} + \ldots .$$

(2.41)

To the leading nontrivial order, the solutions are not affected by the back-reaction. Accordingly, we will compute one point function such as $q, n_A, j_A$ to first non-trivial order in $N_c$. From (2.24) and (2.41), we found that $E^{(0)}$ satisfies the homogeneous equation:

$$E^{(0)}'' + \left( \frac{5}{2u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)} \right) E^{(0)}' + \frac{(\omega^2 - k^2 f)}{u^2 f^2} E^{(0)} = 0 ,$$

(2.42a)

while $G^{(0)}$ satisfies inhomogeneous equation:

$$G^{(0)}'' + \left( \frac{1}{u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)} \right) G^{(0)}' + \frac{(\omega^2 - k^2 f)}{u^3 f^2} G^{(0)} = \frac{2K\omega u f'}{g f(\omega^2 - k^2 f)} E^{(0)} ,$$

(2.42b)

At order $1/N_c$, we further have:

$$E^{(1)}'' + \left( \frac{5}{2u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)} \right) E^{(1)}' + \frac{(\omega^2 - k^2 f)}{u^3 f^2} E^{(1)} = \frac{2K\omega u f'}{C_g u^{5/2} f(\omega^2 - k^2 f)} G^{(0)} ,$$

(2.42c)
and a similar equation for \( G^{(1)} \).

Behavior of \( G^{(0)}(u; \omega, k) \), \( E_A^{(0)}(u; \omega, k) \) and \( E_A^{(1)}(u; \omega, k) \) near boundary can be determined directly from (2.42):

\[
G^{(0)}(u; \omega, k) = a_2(\omega, k)u^2 + b_0(\omega, k)(1 + \cdots), \quad (2.43a)
\]

\[
E_A^{(0)}(u; \omega, k) = E_0(\omega, k)(1 + \cdots) + E_1(\omega, k)
\left( u^{-3/2} + \cdots \right), \quad (2.43b)
\]

\[
E_A^{(1)}(u; \omega, k) = E_1(\omega, k)
\left( u^{-3/2} + \cdots \right), \quad (2.43c)
\]

where \( \cdots \) denote terms higher order in \( 1/u \). In (2.43c) we have defined the solution to in-homogeneous function (2.42c) \( E_A^{(1)} \) in such a way that \( E_A^{(1)}(u \to \infty) = 0 \). \( a_2 \) here is related to \( a, a_x, \theta \) by

\[
a_2 = \frac{K}{2C_g} \left[ (\omega^2 - k^2)\theta + 2i\omega a_t + 2ika_x \right]. \quad (2.44)
\]

In deriving (2.44), we have used the relation:

\[
u^{-1}\partial_u G = -\frac{K}{2C_g} \left[ f^{-1}\partial_t (\partial_t M + 2A_t) - \partial_x (\partial_x M + 2A_x) \right], \quad (2.45)
\]

which can be derived from (2.31) and (2.31). It is useful to note that \( E_0, a_2 \) are invariant under transformation (2.15).

As usual, we impose the infalling wave condition at the black hole horizon for (2.42):

\[
\lim_{u \to u_H} G^{(0)}(u; \omega, k), E_A^{(0)}(u; \omega, k), E_A^{(1)}(u; \omega, k) \to (u - 1)^{-i\omega/3}. \quad (2.46)
\]

Here \( u_H = 1 \) denotes the location of horizon. Consequently, with given boundary value \( a_2, E_0, (2.42) \) can be solved and \( b_0(\omega, k), E_1(\omega, k), E_1(\omega, k) \) will be determined from the resulting solutions. They are related to one point function \( q, n_A, j_A \) via (2.38) and definition (2.35), (2.34). We therefore have

\[
q(\omega, k) = K b_0(\omega, k), \quad (2.47a)
\]

\[
n_A(\omega, k) = \frac{1}{\omega^2 - k^2} \left[ -\frac{3C_\gamma}{2} (ik) \left( N_c E_1^{(0)}(\omega, k) + E_1^{(1)}(\omega, k) \right) - 2K (i\omega) b_0(\omega, k) \right], \quad (2.47b)
\]

\[
j_A(\omega, k) = \frac{1}{\omega^2 - k^2} \left[ -\frac{3C_\gamma}{2} (i\omega) \left( N_c E_1^{(0)}(\omega, k) + E_1^{(1)}(\omega, k) \right) - 2K (ik) b_0(\omega, k) \right]. \quad (2.47c)
\]

3. Medium’s response to chiral charge imbalance

In section, we will solve (2.42) with two different boundary conditions and consider the relation between \( j_A, q \) and \( n_A \). Physically, we would like to use those two different boundary conditions to model two different mechanisms for the generation of axial charge imbalance. In particular, we consider:
Case 1 \textbf{axial charge imbalance is generated by a domain with non-zero winding number.} To model this situation, we set axial gauge field to be zero at boundary, i.e., $a_t, a_x = 0$ but turn on a non-zero $\theta(\omega, k)$. Consequently, boundary condition for (2.42) becomes:

$$a_2(\omega, k) = \frac{K}{2C_g} \left( \omega^2 - k^2 \right) \theta(\omega, k), \quad E_0(\omega, k) = 0. \quad (3.1)$$

Case 2 \textbf{axial charge imbalance is generated by non-topological fluctuations.} To model this situation, we instead set $a_2 = 0$ and consider a non-zero axial electric field on the boundary:

$$a_2(\omega, k) = 0, \quad E_0(\omega, k) \neq 0. \quad (3.2)$$

3.1 Medium’s response to axial charge imbalance generated by topological fluctuations

In this section, we will study medium’s response to axial charge imbalance generated by topological fluctuations. As we discussed previously, this amounts to solve (2.42) with boundary condition (3.1) and (2.43c). As in this case, there is no source term for (2.42a), $E^{(0)}(u; \omega, k) = 0$ trivially satisfies (2.42a) and consequently (2.42b) becomes an homogeneous equation:

$$G^{(0)''} + \left( -\frac{1}{u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)} \right) G^{(0)'} + \frac{(\omega^2 - k^2 f)}{u^3 f^2} G^{(0)} = 0. \quad (3.3)$$

We will seek the in-falling solution for (3.3) in hydrodynamic regime $\omega, k \ll 1$. In this regime, the solution can be obtained analytically by first solving (3.3) order by order in power of $\omega, k$ away from horizon and then determining integration constants by matching with in-falling wave boundary conditions near the horizon. Away from the horizon, we can drop the third term in (3.3) to obtain the following solution

$$g_h(u) = 1 - \left( \frac{i\omega}{3} \right) \left[ \int_{u_H}^u du' \left( \frac{3C_g(1 - s^2 f)}{\sqrt{-gf}} - \frac{1}{u' - 1} \right) + \log(u - 1) \right], \quad (3.4)$$

where we have defined

$$s \equiv \frac{k}{\omega}, \quad (3.5)$$

to save notations. It is easy to check that behavior of (3.4) near the horizon $u \to 1$ can be matched to the infalling wave behavior in small $\omega$ limit:

$$(u - 1)^{-i\phi} = 1 - \frac{i\omega}{3} \log(u - 1) + O(\omega^2). \quad (3.6)$$

It is also worthy mentioning that the integral over $u'$ in (3.4) is convergent as we have explicitly taken the log($u - 1$) outside the integral.

From boundary condition (3.1), we then fix the normalization of $G^{(0)}$:

$$G^{(0)}(u; \omega, k) = \left( \frac{iK\omega}{2C_g} \theta \right) g_h(u; \omega, k). \quad (3.7)$$
Expanding (3.7) near the boundary, we obtain
\[ q(\omega, k) = 2Kb_0(\omega, k) = \left( \frac{K^2}{C_g} \right) \{i\omega - k^2 \lim_{u \to \infty} \left[ \int_{u_H}^u du' \frac{C_g}{\sqrt{-g}} - \frac{u^2}{2} \right] + \mathcal{O}(\omega^2) \} \theta(\omega, k). \]
(3.8)

In (3.8), the subtraction is necessary to remove the divergence near the boundary. As the ratio \(-q(\omega, k)/\theta(\omega, k)\) should be matched to the behavior of retarded Green’s function (1.4), we identify \(\Gamma_{CS}, \kappa_{CS}\) in the present model:
\[ \Gamma_{CS} = \frac{2K^2}{C_g} = \frac{2K^2}{C_g} \hat{T}^3 = \frac{8\lambda^3 T^6}{729\pi M_{K\overline{K}}^2}, \]
(3.9)
\[ \kappa_{CS} = -2 \left( \frac{K^2}{C_g} \right) \lim_{u \to \infty} \left[ \int_{u_H}^u du' \frac{u'^2}{2} \right] = \frac{1}{2} \left( \frac{\Gamma_{CS}}{T} \right) \hat{T}^{-1} = \frac{\lambda^3 T^4}{243\pi^2 M_{K\overline{K}}}, \]
(3.10)
where in the last step, we recover the units and used (2.24) and definition (2.18). The \(\Gamma_{CS}\) in Sakai-Sugimoto model was computed previously in [73] (see Ref. [74] for \(\Gamma_{CS}\) in other holographic models).

To compute \(n_A, j_A\), one needs to solve in-homogenous equation (2.42c):
\[ E_A^{(1)\mu} + \left( \frac{5}{2u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)} \right) E_A^{(1)\nu} + \frac{(\omega^2 - k^2 f) R^3}{u^3 f^2} E_A^{(1)} = \frac{2Kk\omega f'}{C_g \sqrt{u^2 f^2 (\omega^2 - k^2 f)}} G^{(0)}, \]
(3.11)
with \(G^{(0)}(u; \omega, k)\) given by (3.7). At leading order in \(\omega, k\), the solution reads
\[ E_A^{(1)}(u; \omega, k) = - \left( \frac{2ikK^2}{3C_g} \right) \theta(\omega, k) u^{-3/2} (1 + \mathcal{O}(\omega, k)) \]
(3.12)
which can be easily verified by substituting it into (2.42c) and comparing results at leading order in \(\omega, k\). Now substituting (3.7) and (3.12) into (2.47), we have obtained the axial charge density generated by \(\theta\),
\[ n_A = \left( \frac{2K^2}{C_g} \right) \theta(\omega, k) [1 + \mathcal{O}(\omega, k)] = \left( \frac{\Gamma_{CS}}{T} \right) \theta(\omega, k), \]
(3.13)
and \(j_A\) vanishes at this order. To obtain \(j_A\), we consider \(t, x\) components of (2.25b) in the presence of \(G^{(0)}\) given by (3.7). We then obtain flow equations of bulk current \(J_A^t(u; \omega, k), J_A^x(u; \omega, k)\) along radial direction \(u\):
\[ \partial_u J_A^t = -2K G^{(0)}_{ux} + (ik\sqrt{-\gamma} g^{tt} g^{xx}) E_A^{(1)}, \quad \partial_u J_A^x = 2K G^{(0)}_{ut} + (i\omega \sqrt{-\gamma} g^{tt} g^{xx}) E_A^{(1)}. \]
(3.14)
Now using the relation between \(G^{(0)}_{tx}\) and \(G^{(0)}_{ux}, G^{(0)}_{ut}\) which can be obtained from (2.25a) in the absence of \(F\),
\[ G^{(0)}_{ut} = -\frac{ikf\partial_u G^{(0)}_{tx}}{\omega^2 - k^2 f}, \quad G^{(0)}_{ux} = \frac{i\omega \partial_u G^{(0)}_{tx}}{\omega^2 - k^2 f}, \]
(3.15)
\(^\ddagger\)our results [3.9] has a different normalization from [73]
we have from (3.7):
\[
\partial_u J^A_t = -\frac{2i\omega K^2\theta}{\sqrt{-gf}} + (ik\sqrt{-g} g^{tt} g^{xx}) E^{(1)}_A = \frac{-i\omega K^2\theta}{\sqrt{-gf}} [1 + O(\omega, k)] . \tag{3.16a}
\]
\[
\partial_u J^A_x = -\frac{2ikK^2\theta}{\sqrt{-g}} + (i\omega\sqrt{-g} g^{tt} g^{xx}) E^{(1)}_A = -\frac{2ikK^2\theta}{\sqrt{-g}} [1 + O(\omega, k)] , \tag{3.16b}
\]
By integrating over \(u\), we therefore have:
\[
n_A(\omega, k) = n_H^A(\omega, k) + \Delta n_A(\omega, k) , \quad j_A(\omega, k) = j_H^A(\omega, k) + \Delta j_A(\omega, k) . \tag{3.17}
\]
Here \(n_A^H, j_A^H\) are values of bulk current \(J^A_t(u; \omega, k), J^A_x(u; \omega, k)\) at horizon \(u = u_H\). We already know \(n_A \sim O(1)\) from (3.13) and
\[
\Delta n_A = (-2i\omega K^2\theta) \left[\int_{u_H}^\infty du' \frac{1}{\sqrt{-gf}}\right] \sim O(\omega, k) . \tag{3.18}
\]
Therefore we must have
\[
n^H_A = \left(\frac{2K^2}{C_g}\right) \theta(\omega, k) [1 + O(\omega, k)] . \tag{3.19}
\]
On the other hand,
\[
\Delta j_A = (-2ikK^2\theta) \left[\int_{u_H}^\infty du' \frac{1}{\sqrt{-g}}\right]_{\text{Ren}} = \left(-\frac{2ikK^2\theta}{C_g}\right) \lim_{u \to \infty} \left[\int_{u_H}^\infty du' u' - u^2/2\right] . \tag{3.20}
\]
By comparing (3.20) with (1.11), we have
\[
\Delta j_A = \kappa_{CS}(ik)\theta(\omega, k) . \tag{3.21}
\]
The current on the horizon \(j^H_A\) can be determined by substituting (3.12) into (2.38) and taking \(u \to u_H\) limit:
\[
j^H_A = \lim_{u \to u_H} [J^A_t(u; \omega, k)] = \lim_{u \to u_H} \left[\frac{\sqrt{-\gamma} (i\omega) f \partial_u E^{(1)}_A(u; \omega, k) - 2K(ik)f G^{(0)}(u; \omega, k)}{\omega^2 - k^2 f}\right]
\]
\[
= \lim_{u \to u_H} \left[\sqrt{-\gamma} E^{(1)}_A(u; \omega, k)\right] = \frac{2K^2}{3C_g} (ik\theta) . \tag{3.22}
\]
Here we have used a property of any function satisfying in-falling wave boundary condition, say \(Z_{in}(u)\) that
\[
\lim_{u \to u_H} (f \partial_u Z_{in}) = -i\omega \lim_{u \to u_H} Z_{in} . \tag{3.23}
\]
Comparing (3.13) and (3.22), we found that on the horizon, \(j^H_A\) and \(n^H_A\) are related by Fick’s law:
\[
j^H_A = -D\nabla n^H_A . \tag{3.24}
\]
To establish (3.24), we also used the value of diffusive constant \(D\) (3.30) in current model.
To sum up, in this subsection, we have studied axial current in response to axial charge imbalance created by topological fluctuations. To represent a domain with non-zero winding number, we first turn on a non-zero \( \theta(\omega, k) \) and found that it will induce a non-zero \( q(3.8) \) and consequently a non-zero axial charge density \( n_A(\omega, k) \). The axial charge density \( n_A(\omega, k) \) and \( \theta(\omega, k) \) are related by (3.13). Furthermore, the induced axial current can be divided into two parts. The first part is due to the diffusion of \( n_A(\omega, k) \) while the second part is in the opposite direction to the diffusive current and is proportional to \( \kappa_{CS} \), which quantifies the kinetic energy carried by a topological domain. We verified relation (1.3) as first proposed by us in Ref. [52]. It is interesting to note that holographically, the diffusive (dissipative) current coincides with the current on the horizon (3.24) while the non-dissipative current (1.3) is given by the integration from horizon to the boundary (c.f. (3.20) and (3.21)).

3.2 Medium’s response to axial charge fluctuations generated by non-topological fluctuations

We now consider medium’s response to axial charge imbalance generated by non-topological fluctuations. Following our discussion in Sec. 2.4 we will solve (2.42) with boundary condition (3.1) and (2.43c). We first need to solve the homogeneous solution (2.42a). Similarly to (3.7), the infalling wave solution to (2.42a) reads:

\[
e_h(u; \omega, k) = 1 - \left( \frac{i\omega}{3} \right) \left[ \int_{u_H}^{u} du' \left( \frac{3\gamma(1-s^2)f}{\sqrt{-\gamma f}} - \frac{1}{u'-1} \right) + \log(u - 1) \right].
\]

Consequently from boundary condition (3.2), we have:

\[
E_A^{(0)}(u; \omega, k) = \left[ \frac{E_0(\omega, k)}{c_E(\omega, k)} \right] e_h(u; \omega, k),
\]

where \( c_E(\omega, k) \) is defined by the value of \( e_h(u; \omega, k) \) at boundary:

\[
c_E(\omega, k) \equiv e_h(u \to \infty; \omega, k) = 1 + i\omega \left[ s^2 \int_{u_H}^{\infty} du' \frac{C_{\gamma}}{\sqrt{-\gamma}} + \mathcal{O}(1) \right] = 1 + \frac{2i\omega}{3}(s^2 + \mathcal{O}(1)).
\]

Plug (3.26) into (2.47), we obtain:

\[
\begin{align*}
    j_A(\omega, k) &= N_c C_{\gamma} \frac{E_0(\omega, k)}{c_E(\omega, k)}, \\
    n_A(\omega, k) &= N_c s C_{\gamma} \frac{E_0(\omega, k)}{c_E(\omega, k)}.
\end{align*}
\]

The conductivity \( \sigma \), diffusive constant \( D \) and susceptibility \( \chi \) can be extracted from (3.28) as follows. First of all, in the homogeneous limit \( s \to 0 \) of (3.28), we will reproduce Ohm’s law \( j_A = \sigma E_A \). Therefore:

\[
\sigma = N_c C_{\gamma} = \frac{2N_c \lambda T^2}{27\pi M_{KK}}.
\]

On the other hand, Eqs. (3.28) must have a hydrodynamic pole corresponding to diffusive mode at \( \omega = -i D k^2 \). This implies that \( c_E(\omega = -i D k^2, k) = 0 \) hence:

\[
D = \int_{u_H}^{\infty} du' \frac{C_{\gamma}}{\sqrt{-\gamma}} = \int_{u_H}^{\infty} du' (u')^{-5/2} = \frac{2}{3} \tilde{T}^{-1} = \frac{1}{2\pi T}.
\]
In (3.29) and (3.30), we have used expression (3.28) and have recovered the units at the last step. Finally, using the Einstein relation \( \sigma = \chi D \), we obtain the expression for \( \chi \) from (3.29) and (3.30):

\[
\chi^{-1} = \frac{D}{\sigma} = \frac{1}{N_c} \int_{u_H}^{\infty} \frac{du'}{\sqrt{u'}}.
\]

(3.31)

This relation between \( \chi \) and the bulk integration over \( \sqrt{-\gamma} \) is in agreement with general expression in Ref. [72].

Eq. (3.28) implies that turning on an external axial electric field \( E_0 \) would generate a local axial density and axial current. It would also create a non-zero \( q \), which can be determined by solving (2.42b):

\[
G^{(0)''} + \left( -\frac{1}{u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)} \right) G^{(0)'r} + \frac{(\omega^2 - k^2 f)}{u^2 f^2} G^{(0)} = \frac{2Kk\omegauf'}{C_g f(\omega^2 - k^2 f)} E_A^{(0)},
\]

(3.32)

with \( E_A^{(0)} \) given by (3.26). At leading order in \( \omega, k \), the in-homogeneous solution reads:

\[
G^{(0)}(u; \omega, k) = \left( \frac{sK}{C_g} \right) \left[ u^2 (1 + O(\omega, k)) + \frac{2g_h(u; \omega, k)}{i\omega(1 - s^2)} \right] \frac{E_0(\omega, k)}{c_E(\omega, k)}.
\]

(3.33)

As one can check, the first term, i.e., \( sKu^2/C_g \) term is a special solution to in-homogeneous equation (3.32) at leading order in \( \omega, k \). \( g_h(u) \) (3.4), the solution to homogeneous equation, is introduced to guarantee boundary condition (3.2). As a result, we have:

\[
q(\omega, k) = \left( \frac{2K^2}{C_g} \right) \left[ \begin{array}{c} -is \\ (1 - s^2)\omega \end{array} \right] \frac{E_0(\omega, k)}{c_E(\omega, k)} = \left( \frac{2K^2}{C_g} \right) \left[ \begin{array}{c} -is \\ -s^2\omega \end{array} \right] \frac{E_0(\omega, k)}{c_E(\omega, k)}.
\]

(3.34)

In the last step, we dropped the 1 in the bracket. This is justified in the diffusive regime where \( k^2 \sim \omega \). To find the response of \( q \) to \( n_A \), we first note that \( q \) in (3.34) contains responses to both \( n_A \) and axial electric field \( E_0 \). In fact, we can exclude the response to the latter by considering \( n_A \) induced by a normalizable mode. This occurs when \( i\omega s^2 = -3/2 \). According to (3.27) and (3.28), it implies that \( n_A \) remains finite while both \( c_E \) and \( E_0 \) vanish. Physically \( n_A \) in this case is induced by a diffusion wave. The response of \( q \) to \( n_A \) is then given by

\[
\frac{q}{n_A} = \frac{2K^2}{N_cC_gC_\gamma} \frac{1}{-i\omega s^2} = \frac{4K^2}{3N_cC_gC_\gamma},
\]

(3.35)

where in the last step we used \( i\omega s^2 = -3/2 \) and dropped higher order terms in \( \omega \). We note that (3.35) is precisely (1.4) as we advocated in the introduction.

We now show that the intuitive argument above could be established more rigourously in real space. For this purpose, it is convenient to perform the Fourier transform over \( \omega \) and directly consider \( n_A(t, k), j_A(t, k) \) and \( q(t, k) \). By definition and (3.28), we have:

\[
\frac{j_A(t, k)}{N_c} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} j_A(\omega, k) = -\gamma C \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left[ \frac{\omega E_0(\omega, k)}{\omega + iDk^2} \right]
\]

\[
= -\gamma C \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left[ \frac{\omega}{\omega + iDk^2} \right] \int dt' e^{i\omega t'} E_0(t', k).
\]

(3.36)
The above integral is nonvanishing for \( t > t' \) when we can pick up the diffusive pole in the lower half plane. We then obtain
\[
\frac{j_A(t, k)}{N_c} = -C_\gamma Dk^2 I_0(t, k).
\]
\( I_0(k) \) here is defined by:
\[
I_0(k, t) \equiv e^{-iDk^2t}E_0(\omega = -iDk^2, k) = \int_{-\infty}^{t} dt' e^{-Dk^2(t-t')}E(t', k).
\]

A similar computation gives:
\[
\frac{n_A(t, k)}{N_c} = (-iC_\gamma) kI_0(t, k),
\]
\( q(t, k) = -(2K^2/\zeta_g) (-iDk) I_0(t, k). \)

In (3.40), we have neglected a contribution higher order in \( k \).

We assume the external field \( E_0 \) only exists in a finite time window. At sufficiently late time \( t \), we can regard \( j_A \) and \( q \) as responses to \( n_A \). Comparing (3.36) and (3.39) and return to real space, we obtain Fick’s law:
\[
j_A(t, x) = -D \nabla n_A(t, x).
\]

Moreover, (3.39), (3.40) lead to the relation:
\[
q(t, x) = \left( \frac{2K^2}{N_c\zeta_g} \right) \left( \frac{D}{\zeta} \right) n_A(t, x) = \left( \frac{\Gamma_{CS}}{T} \right) n_A(t, x) = \left( \frac{\Gamma_{CS}}{\chi T} \right) n_A(t, x) = \frac{n_A(t, x)}{2\tau_{sph}},
\]
where we have used (3.9), (1.6).

The results of this section can be summarized by the following response matrix, characterizing the response of \( q, n_A \) and \( j_A \) to \( \theta \) and \( E_A \):
\[
\begin{pmatrix}
q \\
n_A \\
j_A
\end{pmatrix} = \begin{pmatrix}
\frac{i\omega \Gamma_{CS}}{2T} & \frac{\Gamma_{CS} i\omega}{\kappa_g} & \frac{\Gamma_{CS} i\omega}{\omega + iDk^2} \\
\frac{\Gamma_{CS}}{T} & \frac{\Gamma_{CS} i\omega}{\kappa_g} & \frac{\Gamma_{CS} i\omega}{\omega + iDk^2} \\
\frac{i}{k} (\kappa_{CS} - D\Gamma_{CS}) & \frac{\Gamma_{CS} i\omega}{\kappa_g} & \frac{\Gamma_{CS} i\omega}{\omega + iDk^2}
\end{pmatrix} \begin{pmatrix}
\theta \\
E_0
\end{pmatrix}.
\]

We keep only terms lowest order in small \( \omega \) and \( k \) limit. We further restrict ourselves to the regime \( \omega \sim k \) for the case with source \( \theta \), and to diffusive regime \( k^2 \sim \omega \) for the case with source \( E_0 \). The transport coefficients we presented are their first nonvanishing order in \( N_c \): the responses of \( n_A \) and \( j_A \) to \( E_A \) are at order \( O(N_c) \), while the rest responses are at order \( O(1) \).

Before closing this section, we would like to comment on the \( O(1) \) correction to the responses of \( n_A \) and \( j_A \) to \( E_0 \) above. This requires us to go beyond leading order in \( 1/N_c \) and compute \( E_A^{(1)} \) from (2.40b) with \( E_A^{(0)} \) given by (3.20). Similar analysis shows that near the boundary, \( E_A^{(1)} \sim u^{1/2} + \cdots \) as \( u \to \infty \), which would give divergent contributions to
$n_A$ and $j_A$. This is because the mixing of the bulk fields changes the dimension of the operator. We note that the change of operator dimension occurs immediately with mixing in bottom-up model in [71], while in our case it occurs from the subleading order in $1/N_c$. As we explained earlier that the solution this order in $1/N_c$ is incomplete without including back-reaction of the flavor branes. It is curious to see if including such backreaction would remove the potential divergence. Although these higher order corrections do not affect the results of our paper, we hope that we could revisit the puzzle in future.

4. Chiral Magnetic Effect and universality

In the previous section, we have considered two different situations where axial charge imbalance is generated. we now want to study whether the CME current would depend on the microscopic origin of axial charge density. In particular, we will compute the ratio between CME current $j_V^{\text{CME}}$ and axial charge density $n_A$ in low frequency and momentum limit as defined in (1.7). If axial charge density is static and homogeneous and CME current is universally given by (1.1), one would have:

$$\chi_{\text{dyn}} = \chi^\prime,$$

(4.1)
due to linearized equation of state $\delta n_A = \chi \delta \mu_A$. However, it is not obvious if (4.1) would still hold if $n_A$ is generated dynamically as considered in this paper. Of particular interest is the case considered in Sec. 3, that axial charge imbalance is created by topological fluctuations.

To compute (4.1), we turn on a small background magnetic field $F_{yz}^V = eB$, i.e. magnetic field is longitudinal to the direction of in-homogeneity as considered in the previous section. Then in the presence of bulk axial field $F_{tu}^A$, $F_{tx}^A$ and $F_{xz}^A$, $F_{tu}^V$, $F_{tx}^V$ and $F_{xz}^V$ components of vector field strength will be excited due to Wess-Zumino term:

$$S_{\text{WZ}} = \int C_3 \wedge \text{tr} F_R^{V/2\pi} - \int C_3 \wedge \text{tr} F_L^{V/2\pi},$$

(4.2)
with $F_4 = dC_3$ given in (2.1). The action for vector field consists of DBI term, which has the same form as that of axial gauge field $A$ and WZ term from (4.2):

$$S_V = -N_c \int d^4xdU \left[ \frac{1}{4} \sqrt{-g} F_V^{MN} F_V^{MN} + K_B \epsilon^{Q MN} A_Q F_V^{MN} \right],$$

(4.3)
where

$$K_B \equiv \left( -\frac{C_A eB}{2N_c} \right).$$

(4.4)
Here, we use subscript/superscript $V$ for vector gauge field strength $F_V^{MN}$ (below we will also use subscript/superscript $A$ for axial gauge field).

The variation of $S_V$ gives the equation of motion:

$$\partial_M \left( \sqrt{-g} F_V^{MN} \right) = K_B \epsilon^{N MQ} F_M^A.$$  

(4.5)
$F_{MQ}^A$ will be taken from solutions obtained in previous solutions. To compute (1.7), it is sufficient to work at linear order in $eB$. We therefore could neglect back-reaction due to $eB$ to the holographic background and solution $E_A$ obtained in the previous section.

As before, we define the bulk vector current

$$J^V_{\mu}(t, x; u) = N_c \sqrt{-\gamma} F^\mu_{\nu V}(t, x; u), \quad (4.6)$$

One point functions $n_V(t, x), j_V(t, x)$ are similarly given by the boundary values of bulk current:

$$n_V(t, x) \equiv \lim_{u \to \infty} J^V_I(t, x; u), \quad j_V(t, x) \equiv \lim_{u \to \infty} J^\nu_{VI}(t, x; u). \quad (4.7)$$

As before, the vector current defined here is a covariant current. Similar to (2.36), it would be convenient to express vector current in terms of $E_V$ and $E_A$ as

$$J^V_I(u) = N_c \frac{(ik) \sqrt{-\gamma} f E'_V(u) - (2i\omega K_B) E_A(u)}{\omega^2 - k^2 f(u)}, \quad (4.8a)$$

$$J^V_\nu(u) = N_c \frac{(-i\omega) \sqrt{-\gamma} f E'_V(u) - (2ikK_B) f E_A(u)}{\omega^2 - k^2 f(u)}, \quad (4.8b)$$

where we have introduced the short-handed notation for “bulk electric field”:

$$E_V(u; \omega, k) \equiv -F^\nu_{\mu V}(u; \omega, k). \quad (4.9)$$

We can easily verify using (4.8) that the covariant current is not conserved in the presence of external axial field $E_A$. The equation for $E_V(u; \omega, k)$ reads

$$E''_V + \left( \frac{5}{2u} + \frac{\omega^2 f'}{\omega^2 - k^2 f} \right) E'_V + \frac{(\omega^2 - k^2 f)}{u^3 f^2} E_V = \frac{2K_B}{C \gamma} \left[ \frac{\omega k f'}{u^{5/2} f(\omega^2 - k^2 f)} \right] E_A + \frac{1}{N_c} \frac{2K \omega k f'}{C \gamma u^5 f(\omega^2 - k^2 f)} G] + O((eB)^2). \quad (4.10)$$

We will solve (4.10) with $E_A, G$ determined in the previous section. To concentrate on vector current induced by axial charge imbalance, we will not turn on any source for vector field, i.e., imposing $E_V(u \to \infty) = 0$ on the boundary and use the standard in-falling wave boundary condition on the horizon.

To compute the vector current, it is also convenient to write down “flow equation” for bulk vector current by taking $t, x$ components of (1.3) and using definition (4.6):

$$\partial_u J^V_I(u) = \frac{K_B}{\sqrt{-\gamma} f} J^x_A(u) - (ik \sqrt{-\gamma} g^{t} g^{xx}) E_V(u) = \frac{K_B}{\sqrt{-\gamma} f} J^x_A(u) [1 + O(\omega, k)]. \quad (4.11a)$$

$$\partial_u J^V_\nu(u) = \frac{2K_B}{\sqrt{-\gamma} f} J^t_A(u) - (i \omega \sqrt{-\gamma} g^{t} g^{xx}) E_V(u) = \frac{2K_B}{\sqrt{-\gamma} f} J^t_A(u) [1 + O(\omega, k)]. \quad (4.11b)$$

On the R.H.S of (4.11), we have used the fact that $J^t_A$ term is always dominated over $E_V$ term in small $\omega, k$ limit due to additional gradients in front of $E_V$. This is because from (4.11) we observe that $E_V$ is the same order as $E_A$ and from (2.36), $J^t_A$ is at least the same order as $E_A$. 


Now integrating (4.11a) over \( u \) and using definition (4.7), we have:

\[
n_V(\omega, k) = \Delta n_V(\omega, k) + n_V^H(\omega, k), \quad j_V(\omega, k) = \Delta j_V(\omega, k) + j_V^H(\omega, k).
\]

(4.12)

Here \( n_V^H \) and \( j_V^H \) are values of bulk current \( J_V^t(u; \omega, k) \) and \( J_A^x(u; \omega, k) \) at horizon \( u = u_H \)

\[
\Delta n_V(\omega, k) \equiv K_B \left[ \int_{u_H}^{\infty} du' \frac{J_A^t(u'; \omega, k)}{f \sqrt{-\gamma}} \right], \quad \Delta j_V(\omega, k) \equiv 2K_B \left[ \int_{u_H}^{\infty} du' \frac{J_A^x(u'; \omega, k)}{\sqrt{-\gamma}} \right].
\]

(4.13)

We now claim that CME current should be identified with \( \Delta j_V \), i.e.,

\[
j_{V}^{\text{CME}} \equiv 2K_B \left[ \int_{u_H}^{\infty} du' \frac{J_A^x(u'; \omega, k)}{\sqrt{-\gamma}} \right].
\]

(4.14)

The physical motivation behind identification (4.14) is that generically in a holographic set-up, the current on the horizon is dissipative (see also example below). On the other hand, the CME current is non-dissipative. Therefore one should exclude the horizon current from the total current when identifying CME current holographically.

We now consider the implication of (4.14). With (4.14) and (3.31), \( \chi_{\text{dyn}} \) becomes:

\[
\chi_{\text{dyn}} = \lim_{\omega, k \to 0} \left\{ n_A(\omega, k) / \left[ \int_{u_H}^{\infty} du' \frac{J_A^t(u'; \omega, k)}{\sqrt{-\gamma}} \right] \right\}. \quad (4.15)
\]

It is clear that if in small \( \omega, k \) limit, bulk axial current is constant, i.e.,

\[
J_A^t(u; \omega, k) = n_A(\omega, k) \left[ 1 + O(\omega, k) \right], \quad (4.16)
\]

it follows one can replace \( J_A^t(u; \omega, k) \) with \( n_A \) in (4.15). Consequently, one will arrive at (4.1) by noting (3.31). Therefore (4.16) can be interpreted as a condition for the validity of (4.1).

In both cases considered in Sec. 3, the condition (4.16) is indeed satisfied, we therefore have (4.1) for those cases.

For completeness, we will calculate total \( j_V \) for both cases. For the first case (c.f. Sec. 3.1), it is straightforward to check that \( j_V^H \sim O(\omega, k) \theta \), which is sub-leading compared with \( j_V^{\text{CME}} \), we therefore have:

\[
j_V(\omega, k) = j_V^{\text{CME}}(\omega, k) = \left( \frac{K_B}{\chi} \right) \left( \frac{T \theta}{\Gamma_{\text{CS}}} \right) = C_A \left( \frac{\theta}{2\tau_{\text{sph}}} \right) eB, \quad (4.17)
\]

where we have used (1.3). By comparing (4.17) with (4.1), it is tempting to make the identification:

\[
\mu_A = \frac{\theta}{2\tau_{\text{sph}}}. \quad (4.18)
\]

In Ref [4], \( \mu_A \) is identified with \( \partial_t \theta \). (4.18) corresponds to replace \( \partial_t \theta \) with the inverse of characteristic time scale of sphaleron transition \( 1/\tau_{\text{sph}} \).
The situation is different for the second case (c.f. Sec. 3.2). In this case, $j_V^{\text{CME}}$ is the same order as $j_V^{\text{CME}}$ in small $\omega, k$ limit. From (3.28) and (4.15), we have:

$$j_V^{\text{CME}}(\omega, k) = s \left( \frac{2N_cK_B}{\chi} \right) \left( \frac{E_0(\omega, k)}{c_E(\omega, k)} \right) = \frac{4N_cK_B}{3} \left( \frac{kE_0(\omega, k)}{\omega c_E(\omega, k)} \right).$$  \hspace{1cm} (4.19)

On the other hand, to obtain $n_V, j_V$, we need to solve (4.10), at leading order in $N_c$, that

$$E''_V + \left( \frac{5}{2u} + \frac{\omega^2 f'}{\omega^2 - k^2 f} \right) E'_V + \frac{(\omega^2 - k^2 f)}{u^3 f^2} E_V = \frac{2K_B\omega kf'}{C_\gamma u^{5/2} f (\omega^2 - k^2 f)} E_A,$$ \hspace{1cm} (4.20)

where $E_A$ is given by (3.26). The leading order solution to (4.20) reads:

$$E_V = \frac{-4K_BE_0(\omega, k)}{3c_E(\omega, k)C_\gamma} \left[ u^{-3/2} + O(\omega, k) \right] s.$$ \hspace{1cm} (4.21)

Again, (4.21) can be easily verified by direct substitution. Now substituting (4.21) into (4.8), we have:

$$n_V = -2iN_cK_B \frac{E_0(\omega, k)}{\omega c_E(\omega, k)},$$ \hspace{1cm} (4.22)

and $j_V$ vanishes at this order. Since $\Delta n_V \sim O(1)$, we thus have

$$n_V^H(\omega, k) = n_V(\omega, k) = -2iN_cK_B \frac{E_0(\omega, k)}{\omega c_E(\omega, k)}.$$ \hspace{1cm} (4.23)

$j_V^H$ is obtained in the same way as $j_A^H$ in the previous section:

$$j_V^H(\omega, k) = \lim_{u \to u_H} [J^H_\mu(u; \omega, k)] = \lim_{u \to u_H} N_c \left[ \frac{(-i\omega)\sqrt{-\gamma}f \partial_\mu E_V(u; \omega, k) - 2(ikK_B)f E_A(u; \omega, k)}{\omega^2 - k^2 f} \right]$$

$$= \lim_{u \to u_H} -N_c \left[ \sqrt{-\gamma}E_V(u; \omega, k) \right] = -\frac{4N_cK_B kE_0(\omega, k)}{3} \frac{3}{\omega c_E(\omega, k)} = -iDkn_V^H(\omega, k).$$ \hspace{1cm} (4.24)

Again we see that $j_V^H$ can be interpreted as a diffusive current. As in this example, both diffusive current $j_A^H$ (4.24) and CME current (4.19) would contribute to the total vector current in small $\omega, k$ limit, to compute CME coefficient hence $\chi_{\text{dyn}}$ properly, it is crucial to identify CME contribution, i.e., (4.14).

To close this section, we would like to comment that while in this paper, we are working in a specific holographic model, the relation (4.12) still holds for holographic action for bulk vector field of the form (4.3). Consequently, assuming the identification of CME current (4.14), the condition (4.16) would warrant that $\chi_{\text{dyn}} = \chi$. Moreover, the violation of condition (4.16) would also break the relation (4.1).

5. Stochastic hydrodynamic equations for axial charge density

We now formulate a hydrodynamic theory for axial charge density by including stochastic noise from both topological fluctuations and thermal fluctuations. We will focus on the dynamics of axial charge thus setting temperature and fluid velocity $u^\mu$ to be homogeneous.
and time-independent. We could then work in the frame that the fluid is at rest: $u^\mu = (1, 0, 0, 0)$. To close anomaly:

$$\partial_t n_A(t, x) + \nabla \cdot \mathbf{j}_A(t, x) = -2q(t, x). \quad (5.1)$$

we want to express $q$ and $\mathbf{j}_A$ in terms of noise and $n_A$ (or its gradients). The constitute relation, which relates axial current $\mathbf{j}_A$ to $n_A$, is of the conventional form:

$$\mathbf{j}_A(t, x) = -D \nabla n_A(t, x) + \xi(t, x), \quad (5.2)$$

where $\xi(t, x)$ encodes axial charge generated by thermal fluctuations:

$$\langle \xi(t, x) \rangle = 0, \quad \langle \xi_i(t, x) \xi_j(t, x') \rangle = 2\sigma T \delta_{ij} \delta(t - t') \delta^3(x - x'). \quad (5.3)$$

Here $\langle \ldots \rangle$ denotes the average over noise and $i, j = 1, 2, 3$ run over spatial coordinates. The magnitude of $\xi(t, x)$ is given by the standard fluctuation-dissipation relation . Furthermore, $q(t, x)$ can be related to $n_A(t, x)$ using (1.5):

$$q(t, x) = \frac{n_A(t, x)}{2\tau_{sph}} + \xi_q(t, x). \quad (5.4)$$

$\xi_q(t, x)$ is the noise due to topological fluctuations:

$$\langle \xi_q(t, x) \rangle = 0, \quad \langle \xi_q(t, x) \xi_q(t, x') \rangle = \Gamma_{CS} \delta(t - t') \delta^3(x - x'), \quad (5.5)$$

and we will assume that there is no cross correlation between two different types of fluctuations:

$$\langle \xi_q(t, x) \xi_i(t, x') \rangle = 0. \quad (5.6)$$

This completely specifies our stochastic anomalous hydrodynamic equations. The noise due to topological fluctuations (5.5) have been considered previously in Refs. [75]. On the other hand, for a conserved current (such as vector $\mathbf{j}_V$), the noise of the form (5.3) is standard. Incorporating both fluctuations in (5.8) is new to the extent of our knowledge.

As an application, we will consider equal time axial charge correlation function:

$$C_{nn}(t, x) = \langle [n_A(t, x) - n_A(0, x)] [n_A(t, x) - n_A(0, x)] \rangle. \quad (5.7)$$

We start with equation for $n_A$ readily followed from the stochastic hydrodynamic equations (5.1):

$$\left[ \partial_t - D \nabla^2 + \tau_{sph}^{-1} \right] n_A(t, x) = -\nabla \xi(t, x) + 2\xi_q(t, x). \quad (5.8)$$

Under the initial condition $n_A(0, x) = 0$, $C_{nn}(t, x)$ characterize the magnitude of axial charge fluctuations at time $t$ and location $x$ due to (both) fluctuations.

To compute (5.7), we first Fourier transform (5.8) into $\mathbf{k}$ space (but keep $t$-dependence):

$$\left[ \partial_t + Dk^2 + \tau_{sph}^{-1} \right] n_A(t, k) = [-i\mathbf{k} \cdot \xi(t, k)] + 2\xi_q(t, k). \quad (5.9)$$
The solution of (5.4) under initial condition \( n_A(0, x) = 0 \) reads:

\[
n_A(t, k) = \int_0^t dt' e^{-(Dk^2+\tau_{sph}^{-1})(t-t')} \left[ (-i k) \cdot \xi(t', k) + 2\xi_q(t, k) \right]. \tag{5.10}
\]

We therefore have:

\[
\langle n_A(t, k)n_A(t, k') \rangle = \int_0^t dt_1 \int_0^t dt_2 e^{-(Dk^2+\tau_{sph}^{-1})(t-t_1)-(Dk^2+\tau_{sph}^{-1})(t-t_2)} \times \left[ k_i k'_j \langle \xi_i(t_1, k)\xi_j(t_2, k') \rangle + 4\langle \xi_q(t_1, k)\xi_q(t_2, k') \rangle \right]. \tag{5.11}
\]

Using (5.5) and (5.3) in Fourier space, we have:

\[
\langle \xi(t, k), \xi(t, k') \rangle = 0, \quad \langle \xi_i(t, k)\xi_j(t, k') \rangle = 2\sigma T \delta_{ij} \delta(t - t')\delta^3(k + k'), \tag{5.12a}
\]

\[
\langle \xi_q(t, k), \xi_q(t, k') \rangle = 0, \quad \langle \xi_q(t, k)\xi_q(t, k') \rangle = \Gamma_{CS} \delta(t - t')\delta^3(k - k'), \tag{5.12b}
\]

and performing the average over noise, we have:

\[
\langle n_A(t, k)n_A(t, k') \rangle = 2 \left( \sigma T k^2 + 2\Gamma_{CS} \right) \int_0^t dt_1 e^{-2(Dk^2+\tau_{sph}^{-1})(t-t_1)} \delta^3(k + k')
\]

\[
eq \frac{\sigma T k^2 + 2\Gamma_{CS}}{Dk^2 + \tau_{sph}^{-1}} \left[ 1 - e^{-2(Dk^2+\tau_{sph}^{-1})t} \right] \delta^3(k + k') = \chi T \left[ 1 - e^{-2(Dk^2+\tau_{sph}^{-1})t} \right] \delta^3(k + k'). \tag{5.13}
\]

In the last step, we have used Einstein relation \( \sigma = \chi D \) and (1.6). Now returning to real space, we have:

\[
C_{nn}(t, x) = \chi T \int \frac{d^3k}{(2\pi)^2} e^{ix \cdot k} \left[ 1 - e^{-2(Dk^2+\tau_{sph}^{-1})t} \right] = (\chi T) \left[ \delta^3(x) - \frac{1}{(8\pi Dt)^{3/2}} \frac{2\chi T}{\tau_{sph}} e^{-\frac{2\chi T}{\tau_{sph}} \frac{|x|^2}{8\pi Dt}} \right]. \tag{5.14}
\]

It is worthy noting that as we are in hydrodynamic regime, \( n_A(x) \) here should be understood as the coarse-grained axial charge density inside a fluid cell and \( x \) is the spatial coordinates labeling the corresponding fluid cell.

We now discuss the implication of (5.14). At very early time that \( Dt \ll L_{cell}^2 \) (therefore \( t \ll \tau_{sph} \)) where \( L_{cell} \) is the size of a fluid cell, the Gaussian appearing in (5.14) essentially becomes a delta function and we then have:

\[
C_{nn}(t, x) \approx \chi T \left[ 1 - e^{-\frac{2\chi T}{\tau_{sph}}} \right] \delta^3(x) \approx \frac{2\chi T}{\tau_{sph}} t \delta(x) = 4\Gamma_{CS} t \delta^3(x), \tag{5.15}
\]

where in the last step we have used (1.6). At this stage, there is no correlation among axial charge in each fluid cell (c.f. the delta function in (5.15)). Integrating (5.13) over volume \( \int d^3x \), we further recover relation between the fluctuation of axial charge and Chern-Simon diffusive constant:

\[
\langle Q^2 \rangle = 4\Gamma_{CS} V t, \tag{5.16}
\]

where \( V \) is the volume of the system. While it has been widely used in literature to estimate the fluctuation of axial charge, (5.16) is no longer valid at the stage that \( Dt \sim L_{cell}^2 \). In
this stage, the spatial dependence of axial charge fluctuations in (5.14) become important. The diffusion generates additional spatial correlation among axial charge density. Finally, in the long time limit $t \gg \tau_{sph}$, the second term in (5.14) are suppressed exponentially and axial charge fluctuations are given by:

$$C_{nn}(t \to \infty, x) \to (\chi T) \delta^3(x).$$

(5.17)

and

$$\lim_{t \to \infty} \langle Q^2 \rangle \to \chi T V,$$

(5.18)

This is of course expected as in long time limit, the $\langle Q^2 \rangle$ should approach its thermal equilibrium values (5.18).

Finally, we remark that to obtain (5.18), we have used the relation between Chern-Simon diffusive constant $\Gamma_{CS}$ and sphaleron damping rate $\tau_{sph}$ (1.6). Therefore like Einstein relation $\sigma = \chi D$ connecting conductivity $\sigma$ and diffusive constant $D$, the relation between $\Gamma_{CS}$ and $\tau_{sph}$ is also fixed by the requirement based on thermodynamics (5.18). It is reassuring that the relation (1.6) is also realized, as we discussed in Sec. 3.2 in the holographic model studied in this work.

6. Summary and Outlook

We have analyzed the anomalous transport of a non-Abelian plasma in a de-confined phase with dynamically generated axial charge using a top-down holographic model. In particular, we consider two separate cases in which the axial charge is generated due to a) topological b) non-topological thermal fluctuations. When the axial charge is generated by topological gluonic fluctuations, we show a non-dissipative current (1.3) is induced due to chiral anomaly in Sec. 3.1. We also illustrate holographically the damping of axial charge due to the interplay between flavor sector and gluonic sector. Furthermore, we consider the ratio of the CME current to the axial charge density at small $\omega$ and $k$ (c.f (1.7)). We interpret such ratio as (the inverse of) “dynamical axial susceptibility” $\chi_{dyn}$ (c.f. the discussion below (1.7)). We found in the context of current holographic model, dynamical susceptibility $\chi_{dyn}$ is independent of the microscopic origin of the axial charge and coincide with the static susceptibility $\chi$. One phenomenological implications of our work, in particular, Sec. 4 is that axial charge generated by topological fluctuations and non-topological thermal fluctuations would contribute to CME signature in heavy-ion collisions. For this reason, we propose a stochastic hydrodynamic equation of the axial charge where we incorporate noise both from both fluctuations in Sec. 5. We found that the magnitude of axial charge fluctuations depend on the time scale that such fluctuations are measured (c.f (5.16), (5.18)) and as (5.14) indicates, the diffusive mode would induce spatial correlations among axial charges.

There are several issues that can be further studied based on the above analysis. The axial charge response to gluonic fluctuations can be studied when the flavor degrees of freedom back-react to the glue part of the theory. In this case, the coupling of the axial current and the gluonic topological operator, $q$, is not suppressed.
In this work, we found in hydrodynamic limit, the “dynamical axial susceptibility” $\chi_{\text{dyn}}$ (1.7) is universal. It is both theoretically interesting and phenomenologically important to extend the definition of $\chi_{\text{dyn}}$ and study its independence on finite $\omega, k$. In this case, it is possible that the resulting $\chi_{\text{dyn}}$ would depend on the origin of axial charge imbalance.

In computing axial charge density correlation function $C_{nn}$ (5.7) from stochastic hydrodynamic equation formulated in Sec. 3, we consider a system in the absence of magnetic field. Once there is magnetic field, axial charge would also be transported by chiral magnetic wave [51]. Furthermore, a new diffusive model would emerge due to the interplay between chiral magnetic wave and sphaleron damping [71, 76]. It is interesting to see how those new modes would contribute to correlation among axial charge densities within the framework of stochastic hydrodynamics.

Acknowledgements

The authors would like to thank P. Arnold, U. Gursoy, C. Hoyos, A. Karch, D. Kharzeev, E. Kiritsis, K. Landsteiner, H. Liu, L. McLerran, G. Moore, R. Pisarski, E. Shuryak, H.-U. Yee and I. Zahed for useful discussions and the Simons Center for Geometry and Physics for hospitality where part of this work has been done. This work is supported in part by the DOE grant No. DE-FG-88ER40388 (I.I.) and in part by DOE grant No. DE-DE-SC0012704 (Y.Y.). S.L. is supported by RIKEN Foreign Postdoctoral Researcher Program.

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