Iterative Algorithm for Discrete Structure Recovery

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Abstract

We propose a general modeling and algorithmic framework for discrete structure recovery that can be applied to a wide range of problems. Under this framework, we are able to study the recovery of clustering labels, ranks of players, and signs of regression coefficients from a unified perspective. A simple iterative algorithm is proposed for discrete structure recovery, which generalizes methods including Lloyd’s algorithm and the iterative feature matching algorithm. A linear convergence result for the proposed algorithm is established in this paper under appropriate abstract conditions on stochastic errors and initialization. We illustrate our general theory by applying it on three representative problems: clustering in Gaussian mixture model, approximate ranking, and sign recovery in compressed sensing, and show that minimax rate is achieved in each case.

Keywords. \(k\)-means clustering, approximate ranking, high-dimensional statistics, Hamming distance, variable selection

1 Introduction

Discrete structure is commonly seen in modern statistics and machine learning, and various problems can be formulated into tasks of recovering the underlying discrete structure. A leading example is clustering analysis [37], where the discrete structure of the data is parametrized by a vector of clustering labels. Theoretical and algorithmic understandings of clustering analysis have received much attention in the recent literature especially due to the interest in community detection of network data [34, 43, 52, 69]. Other important examples of discrete structure recovery include ranking [11, 48], variable selection [13, 38], crowdsourcing [19, 28], estimation of unknown permutation [15, 55], graph matching [16, 22], and recovery of hidden Hamiltonian cycle [6, 12].

Despite the the progress of understanding discrete structures in various specific problems, a general theoretical investigation has been lacking in the literature. This is partly due to the fact that theory of discrete structure recovery can be quite different from traditional statistical estimation of continuous parameters. In fact, it has been argued that the nature
of discrete structure recovery is closely related to hypothesis testing theory [29]. In addition, the existing literature on the statistical guarantees of discrete structure recovery mostly focuses on characterizing the condition of exact recovery [1, 6, 45, 49, 51, 63, 71]. Let 
\( z^* = (z^*_1, z^*_2, \ldots, z^*_p) \) represent a discrete structure of interest, where each \( z^*_j \) parametrizes a discrete status of either the \( j \)th sample or the \( j \)th variable of the data set. The exact recovery is achieved by some estimator \( \hat{z} \) if \( \hat{z}_j = z^*_j \) for all \( j \in [p] \). However, exact recovery of discrete structure usually requires a strong signal to noise ratio condition. A more interesting, more realistic, but harder problem is when only partial recovery of discrete structures. We first propose a general structured linear model which unifies various problems of discrete structure recovery into the same framework. A simple iterative algorithm is then proposed for recovering \( z^* \), which can be informally written in the following form

\[
\text{argmin}_z \sum_{j=1}^p \left| \left| T_j - \nu_j \left( \tilde{B}(z^{(t-1)}), z_j \right) \right| \right|^2 \quad \text{for all } t \geq 1.
\]  

(1)

Here, \( T_j \) is some local statistic whose distribution depends both on the \( j \)th label \( z^*_j \) and the global continuous parameter \( B^* \) of the model. Because of the separability of the objective function across \( j \in [p] \), each \( z_j^{(t)} \) takes the value of \( z_j \) such that \( \nu_j(\tilde{B}(z^{(t-1)}), z_j) \) is the closest to \( T_j \), and therefore computation of (1) is straightforward. The general iterative procedure (1) recovers some interesting algorithms, among which perhaps the most important one is Lloyd’s algorithm [44] for \( k \)-means clustering. In the clustering context, \( T_j \) is the \( j \)th data point, and \( \nu_j(\tilde{B}(z^{(t-1)}), z_j) \) is the \( z_j \)th estimated clustering center computed based on the clustering labels \( z^{(t-1)} \) from the previous step. In addition, (1) also leads to algorithms in approximate ranking and sign recovery that will be studied in details in this paper.

The main result of our paper characterizes conditions under which (1) converges with respect to some loss function \( \ell(\cdot, \cdot) \) to be defined later. An informal statement of the result is given below,

\[
\ell(z^{(t)}, z^*) \leq 2\xi_{\text{ideal}}(\delta) + \frac{1}{2} \ell(z^{(t-1)}, z^*) \quad \text{for all } t \geq 1,
\]  

(2)

2
with high probability. That is, the value of $\ell(z^{(t)}, z^*)$ converges at a linear rate to $4\xi_{\text{ideal}}(\delta)$. Here, we use $\xi_{\text{ideal}}(\delta)$ to characterize the error of an ideal procedure,

$$\hat{z}_{\text{ideal}} = \arg\min_z \sum_{j=1}^P \| T_j - \nu_j \left( \hat{B}(z^*), z_j \right) \|_2^2, \quad (3)$$

and the definition of $\xi_{\text{ideal}}(\delta)$ with a general $\delta > 0$ will be given in Section 3. The convergence result (2) is established with some $\delta > 0$ arbitrarily close to 0. We note that the ideal procedure (3) is not realizable because of its dependence on the true $z^*$, but (2) shows that the iterative algorithm (1) achieves almost the same statistical performance of (3). The general abstract result is then applied to three examples: clustering for Gaussian mixture model, approximate ranking, and sign recovery in compressed sensing, which represent three different types of discrete structure recovery problems. Moreover, in each of the examples, we can relate $\xi_{\text{ideal}}(\delta)$ to the minimax rate of the problem, and therefore claim that the simple algorithm (1) is both computationally efficient and minimax optimal.

Another popular method that is suitable for discrete structure recovery is the EM algorithm [20]. The global convergence of EM algorithm has been established under the setting of unimodal likelihood [64] and the setting of two-component Gaussian mixtures [18, 65, 66, 67]. Local convergence results for general settings are obtained by [7]. However, the most important difference between [7] and our work, besides the obvious difference of algorithms, is that our convergence guarantee (2) is established for the estimation error of the discrete structure $z^*$, while the convergence result in [7] for the EM algorithm is established for the estimation error of the continuous model parameter $B^*$. Results like (2) may be possibly established for the EM algorithm in the context of clustering using the techniques suggested by the paper [70] $^1$, but whether (2) can be proved for the EM algorithm in general settings is unknown.

The most related work to us in the literature is the analysis of Lloyd’s algorithm in Gaussian mixture models by [46]. Since Lloyd’s algorithm is a special case of (1), our convergence result (2) recovers the result in [46] as a special case with even a slightly weaker condition on the number of clusters. We also mention the recent paper [53] that studies a variant of Lloyd’s algorithm and improves the signal to noise ratio condition in [46] for the two-component Gaussian mixtures.

Organization. Our general modeling and algorithmic framework will be introduced in Section 2. In Section 3, we formulate abstract conditions under which we can establish the convergence of the algorithm. Applications to specific examples will be discussed afterwards, including clustering in Gaussian mixture model (Section 4), approximate ranking (Section 5), and sign recovery in compressed sensing (Section 6). Finally, all the technical proofs will be given in Section 7.

$^1$The paper [70] established the convergence of mean-field coordinate ascent and Gibbs sampling in the sense of (2) for community detection in stochastic block models. Due to the connection and similarity between the EM algorithm and variational Bayes, we believe the techniques used in (2) can also be applied to the analysis of EM algorithms for clustering problems.
Notation. For $d \in \mathbb{N}$, we write $[d] = \{1, \ldots, d\}$. Given $a, b \in \mathbb{R}$, we write $a \lor b = \max(a, b)$ and $a \land b = \min(a, b)$. For two positive sequences $a_n$ and $b_n$, we write $a_n \lesssim b_n$ to mean that there exists a constant $C > 0$ independent of $n$ such that $a_n \leq Cb_n$ for all $n$; moreover, $a_n \asymp b_n$ means $a_n \lesssim b_n$ and $b_n \lesssim a_n$. For a set $S$, we use $1_S$ and $|S|$ to denote its indicator function and cardinality respectively. For a vector $v = (v_1, \ldots, v_d)^T \in \mathbb{R}^d$, we define $\|v\|^2 = \sum_{\ell = 1}^d v_\ell^2$. The trace inner product between two matrices $A, B \in \mathbb{R}^{d_1 \times d_2}$ is defined as $\langle A, B \rangle = \sum_{\ell = 1}^{d_1} \sum_{\ell' = 1}^{d_2} A_{\ell \ell'} B_{\ell' \ell}$, while the Frobenius and operator norms of $A$ are given by $\|A\|_F = \sqrt{\langle A, A \rangle}$ and $\|A\| = s_{\max}(A)$ respectively, where $s_{\max}(\cdot)$ denotes the largest singular value. The notation $\mathbb{P}$ and $\mathbb{E}$ are generic probability and expectation operators whose distribution is determined from the context.

2 A General Framework of Models and Algorithms

We start with the introduction of structured linear model. Consider a pair of random vectors $Y \in \mathbb{R}^N$ and $X \in \mathbb{R}^D$. We impose the relation that

$$\mathbb{E}(Y|X) = \mathcal{S}_{z^*}(B^*).$$

(4)

On the right hand side of (4), $z^* = (z_1^*, \ldots, z_p^*)$ is a vector of discrete labels, and each $z_j^*$ is allowed to take its value from a label set of size $k$. For simplicity, we assume the label set to be $[k]$ without loss of generality. The vector $B^*$ is the model parameter that lives in a linear subspace indexed by $z^*$. We use the notation $B_{z^*}$ for this linear subspace. Finally, $\mathcal{S}_{z^*}$ is a linear operator jointly determined by $X$ and $z^*$. It maps from $B_{z^*}$ to $\mathbb{R}^N$.

The general structured linear model (4) can be viewed as a slight variation of the one introduced by [32]. It is particularly suitable for the research of label recovery and includes some important examples that will be studied in this paper.

To estimate the labels $z_1^*, \ldots, z_p^*$, one strategy is to first compute a local statistic $T_j = T_j(X, Y) \in \mathbb{R}^d$ and then infer $z_j^*$ from $T_j$ for each $j \in [p]$. We require that

$$\mathbb{E}(T_j|X) = \mu_j(B^*, z_j^*).$$

(5)

Then, suppose the model parameter $B^*$ was known, a natural procedure to estimate $z_j^*$ would find an $a \in [k]$ such that $\|T_j - \mu_j(B^*, a)\|^2$ is the smallest. However, for some applications, the form of $\mu_j(B^*, z_j^*)$ may not be available, and thus we need to associate each $\mu_j(B^*, z_j^*)$ with a surrogate $\nu_j(B^*, z_j^*)$. An oracle procedure that uses the knowledge of $B^*$ is given by

$$z_j^{\text{oracle}} = \arg\min_{a \in [k]} \|T_j - \nu_j(B^*, a)\|^2.$$  

(6)

On the other hand, since $B^*$ is unknown in practice, we need to replace the $B^*$ in (6) by an estimator. A natural procedure is the least-squares estimator $\hat{B}(z^*)$, where for a given $z$, $\hat{B}(z)$ is defined by

$$\hat{B}(z) = \arg\min_{B \in B_z} \|Y - \mathcal{S}_z(B)\|^2.$$  

(7)
This time we need to know $z$ in (7) to compute $\hat{B}(z)$. Therefore, we shall combine (6) and (7) and obtain the following iterative algorithm.

**Algorithm 1: Iterative discrete structure recovery**

**Input**: The data $Y$, $X$ and the number of iterations $t_{\text{max}}$.

**Output**: The estimator $\hat{z} = z^{(t_{\text{max}})}$.

1. Compute the initializer $z^{(0)}$.
2. For $t$ in $1 : t_{\text{max}}$, compute
   
   $$B^{(t)} = \arg\min_{B \in B^{(t-1)}} \|Y - \mathcal{X}_z(B)\|^2, \quad (8)$$
   
   and $z_j^{(t)} = \arg\min_{a \in [k]} \|T_j(X,Y) - \nu_j(B^{(t)},a)\|^2 \quad \forall j \in [p]. \quad (9)$

Let us now discuss a few important examples. Though we regard $X$ and $Y$ to be vectors in our general framework, in some specific examples, it is often more convenient to arrange the data into matrices instead of vectors. Of course, the two representations are equivalent and the relation can be precisely described with the operations of vectorization and Kronecker product.

**Clustering in Gaussian Mixture Model.** Consider $Y \in \mathbb{R}^{d \times p}$ with $Y_1, \ldots, Y_p$ standing for its columns. We assume that $Y_j \sim \mathcal{N}(\theta_j^*, I_d)$ independently for $j \in [p]$. Here, $z_1^*, \ldots, z_p^* \in [k]$ are $p$ clustering labels and $\theta_1^*, \ldots, \theta_k^* \in \mathbb{R}^d$ are $k$ clustering centers. In our general framework, we have $N = dp$, $B^*$ is the concatenation of the $k$ clustering centers, and $B_{z^*} = \mathbb{R}^{d \times k}$. The linear operator $\mathcal{X}_{z^*}$ maps the matrix $\{\theta_a^*\}_{a \in [k]} \in \mathbb{R}^{d \times k}$ to the matrix $\{\theta_j^*\}_{j \in [p]} \in \mathbb{R}^{d \times p}$. For the algorithm to recover the clustering labels, the obvious local statistic is $T_j = Y_j$ for $j \in [p]$. Moreover, we set $\nu_j(B^*,a) = \mu_j(B^*,a) = \theta_a^*$. Then, Algorithm 1 is specialized into the following iterative procedures:

$$\theta_a^{(t)} = \frac{\sum_{j=1}^p 1\{z_j^{(t-1)} = a\} Y_j}{\sum_{j=1}^p 1\{z_j^{(t-1)} = a\}}, \quad a \in [k],$$

$$z_j^{(t)} = \arg\min_{a \in [k]} \|Y_j - \theta_a^{(t)}\|^2, \quad j \in [p].$$

This is recognized as Lloyd’s algorithm [44], the most popular way to solve $k$-means clustering.

**Approximate Ranking.** In the task of ranking, we consider the observation of pairwise interaction data $Y_{ij}$ for $(i, j) \in [p]^2$ and $i \neq j$. The rank or the position of the $j$th player is specified by an integer $z_j^* \in [p]$. What is known as the pairwise comparison model assumes that $Y_{ij} \sim \mathcal{N}(\beta^*(z_i^* - z_j^*), 1)$ for some signal strength parameter $\beta^* \in \mathbb{R}$. Our goal is to estimate the discrete position $z_j^*$ for each player $j \in [p]$. This is known as the approximate ranking problem [27], which is different from exact ranking where $z^*$ corresponds to a permutation. It is easy to see that this approximate ranking model is a special case of our general
structured linear model. To be specific, we have \( N = p(p-1) \), \( B^* \) is identified with \( \beta^* \), and \( B_{z^*} = \mathbb{R} \). The linear operator \( \mathcal{X}_{z^*} \) maps \( \beta^* \) to \( \{\beta^*(z^*_i - z^*_j)\}_{1 \leq i \neq j \leq p} \). To recover \( z^*_j \), it is natural to define

\[
T_j = \frac{1}{\sqrt{2(p-1)}} \sum_{i \in [p] \setminus \{j\}} (Y_{ji} - Y_{ij}). \tag{10}
\]

Thus, we have

\[
\mu_j(B^*, a) = \frac{2p}{\sqrt{2(p-1)}} \beta^* \left( a - \frac{1}{p} \sum_{i=1}^p z_i^* \right). \tag{11}
\]

Because of the dependence of \( \mu_j(B^*, a) \) on the unknown \( \frac{1}{p} \sum_{i=1}^p z_i^* \), we also introduce \( \nu_j(B^*, a) \) that replaces \( \frac{1}{p} \sum_{i=1}^p z_i^* \) with a fixed value \( \frac{p+1}{2} \),

\[
\nu_j(B^*, a) = \frac{2p}{\sqrt{2(p-1)}} \beta^* \left( a - \frac{p+1}{2} \right).
\]

The choice of \( \frac{p+1}{2} \) is due to the parameter space of \( z^* \) that will be introduced in Section 5. This leads to the following iterative algorithm:

\[
\beta^{(t)} = \frac{\sum_{1 \leq i \neq j \leq p} (z_i^{(t-1)} - z_j^{(t-1)}) Y_{ij}}{\sum_{1 \leq i \neq j \leq p} (z_i^{(t-1)} - z_j^{(t-1)})^2},
\]

\[
z_j^{(t)} = \arg\min_{a \in [p]} \left| \sum_{i \in [p] \setminus \{j\}} (Y_{ji} - Y_{ij}) - 2p\beta^{(t)} \left( a - \frac{p+1}{2} \right) \right|^2, \quad j \in [p]. \tag{12}
\]

Since (12) is recognized as feature matching [15], this is the iterative feature matching algorithm suggested by [27] for approximate ranking.

**Sign Recovery in Compressed Sensing.** In a standard regression problem, we assume \( Y|X \sim \mathcal{N}(X\beta^*, I_n) \). Consider a random design setting, where \( X_{ij} \overset{iid}{\sim} \mathcal{N}(0, 1) \) for \( (i, j) \in [n] \times [p] \). We study the sign recovery problem, which is equivalent to estimating \( z_j^* \in \{-1, 0, 1\} \), where the three possible values of \( z_j^* \) standing for \( \beta_j^* \) being negative, zero, and positive. We also define the sparsity level \( s = \sum_{j=1}^p |z_j^*| \). In order that sign recovery is information-theoretically possible, we assume that either \( \beta_j^* = 0 \) or \( |\beta_j^*| \geq \lambda \). The same setting has been considered by [54]. The sparse linear regression model is clearly a special case of our general framework with the choices \( N = n \), \( B^* = \beta^* \), and \( B_{z^*} = \{\beta \in \mathbb{R}^p : \beta_j = \beta_j|z_j^*|\} \). The linear operator \( \mathcal{X}_{z^*} \) maps \( \beta^* \) to \( X\beta^* \). Following [54], we use the local statistic

\[
T_j = \|X_j\|^{-1}X_j^TY \tag{13}
\]

to recover \( z_j^* \). Here, \( X_j \in \mathbb{R}^n \) stands for the \( j \)th column of \( X \). Computing its conditional expectation, we obtain

\[
\mu_j(B^*, a) = a\|X_j\| \max\{|\beta_j^*|, \lambda\} + \|X_j\|^{-1} \sum_{t \in [p] \setminus \{j\}} \beta_t^* X_j^T X_t, \tag{14}
\]
for \( a \in \{-1, 0, 1\} \). Replacing \( \max\{|\beta_j^*|, \lambda\} \) in the above formula by some threshold level \( 2t(X_j) \), we get
\[
\nu_j(B^*, a) = 2a\|X_j\|t(X_j) + \|X_j\|^{-1} \sum_{l \in [p]\backslash\{j\}} \beta_j^* X_j^T X_l,
\] (15)
for \( a \in \{-1, 0, 1\} \). The threshold level is specified by
\[
t(X_j) = \frac{\lambda}{2} + \log \frac{p-s}{s} \|X_j\|^2,
\] (16)
which can be derived from a minimax analysis [13, 54]. Specializing Algorithm 1 to the current context gives
\[
\beta(t) = \arg\min_{\beta \in \mathbb{R}^p: \beta_j = \beta_j^{(t-1)}} \|y - X\beta\|^2, \quad \text{(17)}
\]
\[
z_j^{(t)} = \begin{cases} 
1 & \frac{X_j^T y - \sum_{l \in [p]\backslash\{j\}} \beta_l^{(t)} X_j^T X_l}{\|X_j\|^2} > t(X_j) \\
0 & \frac{X_j^T y - \sum_{l \in [p]\backslash\{j\}} \beta_l^{(t)} X_j^T X_l}{\|X_j\|^2} \leq t(X_j) \\
-1 & \frac{X_j^T y - \sum_{l \in [p]\backslash\{j\}} \beta_l^{(t)} X_j^T X_l}{\|X_j\|^2} < -t(X_j)
\end{cases}
\] (18)
We note that (18) is a slight modification of the variable selection procedure in [54]. The main difference is that [54] uses an estimator of \( \beta^* \) computed with an independent data set, while we compute a least-squares procedure (17) restricted on the support of \( z^{(t-1)} \) obtained from the previous step using the same data set.

3 Convergence Analysis

In this section, we formulate abstract conditions under which we can derive the statistical and computational guarantees of Algorithm 1.

A General Loss Function Our goal is to establish a bound for every \( t \geq 1 \) with respect to the loss \( \ell(z^{(t)}, z^*) \). The loss function is defined by
\[
\ell(z, z^*) = \sum_{j=1}^p \|\mu_j(B^*, z_j) - \mu_j(B^*, z_j^*)\|^2.
\] (19)
It has a close relation to the Hamming loss \( h(z, z^*) = \sum_{j=1}^p \mathbb{1}_{\{z_j \neq z_j^*\}} \). Define
\[
\Delta_{\text{min}}^2 = \min_{j \in [p]} \min_{1 \leq a \neq b \leq k} \|\mu_j(B^*, a) - \mu_j(B^*, b)\|^2,
\]
and then we immediately have
\[
\ell(z, z^*) \geq \Delta_{\text{min}}^2 h(z, z^*).
\] (20)
Error Decomposition  By (5), we can decompose each local statistic as

\[ T_j = \mu_j(B^*, z^*_j) + \epsilon_j. \]  \hfill (21)

We usually have \( \epsilon_j \sim \mathcal{N}(0, I_d) \), but this is not required, and we shall also note that the \( \epsilon_j \)'s may not even be independent across \( j \in [p] \). By (9), if we start the algorithm from any \( z \), then \( z^*_j \) will be incorrectly estimated after one iteration if \( z^*_j \neq \text{argmin}_{a \in [k]} \|T_j - \nu_j(\B(z), a)\|^2 \). Consequently, assume \( z^*_j = a \), and it is important to analyze the event

\[ \|T_j - \nu_j(\B(z), b)\|^2 \leq \|T_j - \nu_j(\B(z), a)\|^2, \]  \hfill (22)

for any \( b \in [k] \setminus \{a\} \). Recall the definition of \( \B(z) \) in (7). We plug (21) into (22), and then after some rearrangement, we can see that the event (22) is equivalent to

\[ \langle \epsilon_j, \nu_j(\B(z^*), a) - \nu_j(\B(z^*), b) \rangle \leq -\frac{1}{2} \Delta_j(a, b)^2 + F_j(a, b; z) + G_j(a, b; z) + H_j(a, b; z). \]  \hfill (23)

On the right hand side of (23), \( \Delta_j(a, b)^2 \) is the main term that characterizes the difference between the two labels \( a \) and \( b \). It is defined as

\[ \Delta_j(a, b)^2 = \|\mu_j(B^*, a) - \nu_j(B^*, b)\|^2 - \|\mu_j(B^*, a) - \nu_j(B^*, a)\|^2. \]

Note that with the notation \( \Delta_j(a, b)^2 \), we have implicitly assume that \( \Delta_j(a, b)^2 \geq 0 \) throughout the paper. This assumption is easily satisfied in all the examples considered in the paper. The other three terms in (23) are the error terms that we need to control. Their definitions are given by

\[
F_j(a, b; z) = \left\langle \epsilon_j, \left( \nu_j(\B(z^*), a) - \nu_j(\B(z), a) \right) \right\rangle - \left( \nu_j(\B(z^*), b) - \nu_j(\B(z), b) \right) \right\rangle,
\]

\[
G_j(a, b; z) = \frac{1}{2} \|\mu_j(B^*, a) - \nu_j(\B(z), a)\|^2 - \|\mu_j(B^*, a) - \nu_j(\B(z^*), a)\|^2 - \frac{1}{2} \|\mu_j(B^*, a) - \nu_j(\B(z), b)\|^2 - \|\mu_j(B^*, a) - \nu_j(\B(z^*), b)\|^2,
\]

\[
H_j(a, b; z) = \frac{1}{2} \|\mu_j(B^*, a) - \nu_j(\B(z^*), a)\|^2 - \|\mu_j(B^*, a) - \nu_j(\B(z^*), b)\|^2 - \|\mu_j(B^*, a) - \nu_j(B^*, a)\|^2 - \|\mu_j(B^*, a) - \nu_j(B^*, b)\|^2.
\]

With these quantities defined as above, we can check that (23) is indeed equivalent to (22). The reason to have such decomposition (23) is as follows.

- By ignoring the three error terms, the event \( \langle \epsilon_j, \nu_j(\B(z^*), a) - \nu_j(\B(z^*), b) \rangle \leq -\frac{1}{2} \Delta_j(a, b)^2 \) contributes to the ideal recovery error rate. That is, even if we were given the true \( z^* \), applying one iteration in Algorithm 1, i.e., (9) would still result in some error.

- The error terms \( F_j(a, b; z) \) and \( G_j(a, b; z) \) can be controlled by the difference between \( \B(z) \) and \( \B(z^*) \), which further depends on \( \ell(z, z^*) \). We will treat \( F_j(a, b; z) \) and \( G_j(a, b; z) \) differently because the former involves the additional randomness of \( \epsilon_j \).

- The error term \( H_j(a, b; z) \) can be controlled by the difference between \( \B(z^*) \) and \( B^* \). In fact, unlike \( F_j(a, b; z) \) or \( G_j(a, b; z) \), \( H_j(a, b; z) \) does not depend on \( z \), and thus its value remains unchanged throughout the iterations.
Conditions for Algorithmic Convergence  

Now we need to discuss how to analyze the error terms $F_j(a, b; z)$, $G_j(a, b; z)$ and $H_j(a, b; z)$. There are three types of conditions that we will impose.

**Condition A** ($\ell_2$-type error control). Assume that

$$
\max_{\{z: \ell(z, z^*) \leq \tau\}} \sum_{j=1}^{p} \max_{b \in [k] \setminus \{z_j^*\}} \frac{F_j(z_j^*, b; z)^2 \|\mu_j(B^*, b) - \mu_j(B^*, z_j^*)\|^2}{\Delta_j(z_j^*, b)^4 \ell(z, z^*)} \leq \frac{1}{256} \delta^2
$$

holds with probability at least $1 - \eta_1$, for some $\tau, \delta, \eta_1 > 0$.

**Condition B** (restricted $\ell_2$-type error control). Assume that

$$
\max_{\{z: \ell(z, z^*) \leq \tau\}} \frac{1}{\Delta_{\min}[T]} \sum_{t \in [T]} \max_{b \in [k] \setminus \{z_j^*\}} G_j(z_j^*, b; z)^2 \|\mu_j(B^*, b) - \mu_j(B^*, z_j^*)\|^2 \leq \frac{1}{256} \delta^2
$$

holds with probability at least $1 - \eta_2$, for some $\tau, \delta, \eta_2 > 0$.

**Condition C** ($\ell_\infty$-type error control). Assume that

$$
\max_{\{z: \ell(z, z^*) \leq \tau\}} \max_{j \in [p]} \max_{b \in [k] \setminus \{z_j^*\}} \frac{|H_j(z_j^*, b; z)|}{\Delta_j(z_j^*, b)^2} \leq \frac{1}{4} \delta
$$

holds with probability at least $1 - \eta_3$, for some $\tau, \delta, \eta_3 > 0$.

Conditions A, B and C are for the error terms $F_j(a, b; z)$, $G_j(a, b; z)$ and $H_j(a, b; z)$, respectively. Because of the difference of the three terms that we have mentioned earlier, they are controlled in different ways. Both Conditions A and B impose $\ell_2$-type controls and relate $F_j(a, b; z)$ and $G_j(a, b; z)$ to the loss function $\ell(z, z^*)$. On the other hand, $H_j(a, b; z)$ is controlled by an $\ell_\infty$-type bound in Condition C.

Next, we define a quantity referred to as the ideal error,

$$
\xi_{\text{ideal}}(\delta) = \sum_{j=1}^{p} \sum_{b \in [k] \setminus \{z_j^*\}} \|\mu_j(B^*, b) - \mu_j(B^*, z_j^*)\|^2 \mathbb{1}_{\{b \in [p]: \ell_j(B^*, z_j^*) - \ell_j(B(z, b)) \leq - \frac{\delta}{2} \Delta_j(z_j^*, b)^2\}},
$$

We note that $\xi_{\text{ideal}}(\delta)$ is a quantity that does not change with $t$. In fact, with some $\delta > 0$, $\xi_{\text{ideal}}(\delta)$ can be shown to be an error bound for the ideal procedure $\hat{z}_j^{\text{ideal}}$ defined in (3). We therefore choose $\xi_{\text{ideal}}(\delta)$ with a small $\delta > 0$ as the target error that $z(t)$ converges to. In specific examples studied later in Sections 4-6, we will show $\xi_{\text{ideal}}(\delta)$ can be bounded by the minimax rate of each problem.

**Condition D** (ideal error). Assume that

$$
\xi_{\text{ideal}}(\delta) \leq \frac{1}{4} \tau,
$$

with probability at least $1 - \eta_4$, for some $\tau, \delta, \eta_4 > 0$.
Finally, we need a condition on $z^{(0)}$, the initialization of Algorithm 1.

**Condition E (initialization).** Assume that

$$\ell(z^{(0)}, z^*) \leq \tau,$$

with probability at least $1 - \eta_5$, for some $\tau, \eta_5 > 0$.

**Convergence Guarantee** With all the conditions specified, we establish the convergence guarantee for Algorithm 1.

**Theorem 3.1.** Assume Conditions A, B, C, D and E hold for some $\tau, \delta, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5 > 0$. We then have

$$\ell(z^{(t)}, z^*) \leq 2\xi_{\text{ideal}}(\delta) + \frac{1}{2}\ell(z^{(t-1)}, z^*)$$

for all $t \geq 1$, with probability at least $1 - \eta$, where $\eta = \sum_{i=1}^{5} \eta_i$.

The theorem shows that the error of $z^{(t)}$ converges to $4\xi_{\text{ideal}}(\delta)$ at a linear rate. Among all the conditions, Conditions A, B and C are the most important ones. The largest $\tau$ that makes Conditions A, B and C hold simultaneously will be the required error bound for the initialization in Condition E. With (20), Theorem 3.1 also implies that the iterative algorithm achieves an error of $4\xi_{\text{ideal}}(\delta)/\Delta^2_{\text{min}}$ in terms of Hamming distance.

In Sections 4-6, we will apply Theorem 3.1 to the three examples mentioned in Section 2, covering different categories of discrete structures: clustering label, rank, and variable sign. The clustering labels are discrete objects without order or any topological structure. This is in contrast to the ranks that are ordered objects in the space of natural numbers. The variable signs are similar to the clustering labels except two differences. The first difference is the prior knowledge that most variables are zero in the context of sparse linear regression. The second difference is that a nonzero sign only implies a range of a variable instead of its specific value. Despite all the differences between these discrete structures, we are able to analyze them in a unified framework with the same algorithm.

# 4 Clustering in Gaussian Mixture Model

We assume the data matrix $Y \in \mathbb{R}^{d \times p}$ is generated from a Gaussian mixture model. This means we have $Y_j = \theta_{z^*_j} + \epsilon_j \sim \mathcal{N}(\theta_{z^*_j}, I_d)$ independently for $j \in [p]$, where $z^* \in [k]^p$ is the vector of clustering labels that we aim to recover. Specializing Algorithm 1 to the clustering problem, we obtain the well-known Lloyd’s algorithm, which can be summarized as

$$z_j^{(t)} = \arg\min_{a \in [k]} \|Y_j - \tilde{\theta}_a(z^{(t-1)})\|^2,$$

where for each $z \in [k]^p$, we use the notation

$$\tilde{\theta}_a(z) = \frac{\sum_{j=1}^{p} 1\{z_j = a\} Y_j}{\sum_{j=1}^{p} 1\{z_j = a\}}, \quad a \in [k].$$
Even though general k-means clustering is known to be NP-hard [3, 17, 47], local convergence of the Lloyd’s iteration can be established under certain data-generating mechanism [5, 40]. In particular, the recent work [46] shows that under the Gaussian mixture model, the misclustering error of $z^{(t)}$ in the Lloyd’s iteration linearly converges to the minimax optimal rate. In this section, we show that our theoretical framework developed in Section 3 leads to a result that is comparable to the one in [46].

4.1 Conditions

To analyze the algorithmic convergence, we note that $\mu_j(B^*, a) = \nu_j(B^*, a) = \theta_a^*$, $\Delta_j(a, b)^2 = \|\theta_a^* - \theta_b^*\|^2$, $\ell(z, z^*) = \sum_{j=1}^p \|\theta_{z_j}^* - \theta_{z_j}^*\|^2$, and $\Delta_{\min} = \min_{1 \leq a < b \leq k} \|\theta_a^* - \theta_b^*\|$ in the current setting. The error terms that we need to control are

$$F_j(a, b; z) = \left\langle \epsilon_j, \hat{\theta}_a(z^*) - \hat{\theta}_a(z) - \hat{\theta}_b(z^*) + \hat{\theta}_b(z) \right\rangle,$$

$$G_j(a, b; z) = \frac{1}{2} \left( \|\theta_a^* - \hat{\theta}_a(z)\|^2 - \|\theta_a^* - \hat{\theta}_a(z^*)\|^2 - \|\theta_a^* - \hat{\theta}_b(z)\|^2 + \|\theta_a^* - \hat{\theta}_b(z^*)\|^2 \right),$$

$$H_j(a, b; z) = \frac{1}{2} \left( \|\theta_a^* - \hat{\theta}_a(z^*)\|^2 - \|\theta_a^* - \hat{\theta}_b(z)\|^2 + \|\theta_a^* - \theta_b^*\|^2 \right).$$

The following lemma controls the error terms $F_j(a, b; z)$, $G_j(a, b; z)$ and $H_j(a, b; z)$.

**Lemma 4.1.** Assume that $\min_{a \in [k]} \sum_{j=1}^p 1\{z_j^* = a\} \geq \frac{a p}{k}$ and $\tau \leq \frac{\Delta_{\min}^{\text{opt}}}{2k}$ for some constant $\alpha > 0$. Then, for any $C' > 0$, there exists a constant $C > 0$ only depending on $\alpha$ and $C'$ such that

$$\max_{\{z: \ell(z, z^*) \leq \tau\}} \max_{b \in [k] \setminus \{z_j^*\}} \frac{F_j(z_j^*, b; z)^2}{\Delta_j(z_j^*, b)^4 \ell(z, z^*)} \leq C \frac{k^2 (kd/p + 1)}{\Delta_{\min}^2} \left( 1 + \frac{k(d/p + 1)}{\Delta_{\min}^2} \right),$$

$$\max_{\{z: \ell(z, z^*) \leq \tau\}} \max_{T \subset [p]} \frac{G_j(z_j^*, b; z)^2}{\Delta_j(z_j^*, b)^4 \ell(z, z^*)} \leq C \left( \frac{k\tau}{p\Delta_{\min}^2} + \frac{k(d/p)^2}{p\Delta_{\min}^2} \right),$$

and

$$\max_{\{z: \ell(z, z^*) \leq \tau\}} \max_{j \in [p]} \max_{b \in [k] \setminus \{z_j^*\}} \frac{|H_j(z_j^*, b; z)|}{\Delta_j(z_j^*, b)^2} \leq C \left( \frac{k(d + \log p)}{p\Delta_{\min}^2} + \frac{k(d + \log p)}{p\Delta_{\min}^2} \right),$$

with probability at least $1 - p^{-C'}$.

From the bounds (26)-(28), we can see that a sufficient condition that Conditions A, B and C hold is $\frac{\tau}{\Delta_{\min}^2} \rightarrow 0$ and

$$\frac{\Delta_{\min}^2}{k^2 (kd/p + 1)} \rightarrow \infty.$$
In fact, under this sufficient condition, we can set $\delta = \delta_p$ to be some sequence $\delta_p$ converging to 0 in Conditions A, B and C.

Next, we need to control $\xi_{\text{ideal}}(\delta)$ in Condition D. This is given by the following lemma.

**Lemma 4.2.** Assume $\frac{\Delta^2_{\text{min}}}{\log k + kd/p} \to \infty$, $p/k \to \infty$, and $\min_a \sum_{j=1}^p 1\{z_j^* = a\} \geq \frac{\alpha p}{k}$ for some constant $\alpha > 0$. Then, for any sequence $\delta_p = o(1)$, we have

$$
\xi_{\text{ideal}}(\delta_p) \leq p \exp \left( -\frac{1}{2} \frac{\Delta^2_{\text{min}}}{\Delta_{\text{min}}} \right),
$$

with probability at least $1 - \exp(-\Delta_{\text{min}})$.

We note that the signal condition $\frac{\Delta^2_{\text{min}}}{\log k + kd/p} \to \infty$ required by Lemma 4.2 is implied by the stronger condition (29). Therefore, we need to require (29) for the Conditions A, B, C and D to hold simultaneously.

### 4.2 Convergence

With the help of Lemma 4.1 and Lemma 4.2, we can specialize Theorem 3.1 into the following result.

**Theorem 4.1.** Assume (29) holds, $p/k \to \infty$, and $\min_a \sum_{j=1}^p 1\{z_j^* = a\} \geq \frac{\alpha p}{k}$ for some constant $\alpha > 0$. Suppose $z^{(0)}$ satisfies

$$
\ell(z^{(0)}, z^*) = o\left( \frac{p^2}{k} \right),
$$

with probability at least $1 - \eta$. Then, we have

$$
\ell(z^{(t)}, z^*) \leq p \exp \left( -\frac{1}{2} \frac{\Delta^2_{\text{min}}}{\Delta_{\text{min}}} \right) + \frac{1}{2} \ell(z^{(t-1)}, z^*) \quad \text{for all } t \geq 1,
$$

with probability at least $1 - \eta - \exp(-\Delta_{\text{min}}) - p^{-1}$.

**Remark 4.1.** Our result is comparable to the main result of [46]. The main difference is that the convergence analysis in [46] is for the misclustering error, defined by

$$
\text{Misclust}(z, z^*) = \frac{1}{p} \sum_{j=1}^p 1\{z_j \neq z_j^*\},
$$

while Theorem 4.1 is established for an $\ell_2$ type loss function, which is more natural in our general framework. The main condition of Theorem 4.1 is the signal requirement (29). Interestingly, this is exactly the same condition used in [46]. On the other hand, we only require $k = o(p)$ for the number of clusters allowed, whereas [46] assumes a slightly stronger condition $k = o(p/(\log p)^{1/3})$. 

12
In the context of clustering, the loss function (31) may be more natural than $\ell(z, z^*)$. Given the relation that

$$\text{Misclust}(z, z^*) \leq \ell(z, z^*),$$

we immediately obtain the following corollary on the misclustering error.

**Corollary 4.1.** Assume (29) holds, $p/k \to \infty$, and $\min_{a \in [k]} \sum_{j=1}^{p} 1 \{z_j^* = a\} \geq 2p \forall k$ for some constant $\alpha > 0$. Suppose $z^{(0)}$ satisfies (30) with probability at least $1 - \eta$. Then, we have

$$\text{Misclust}(z^{(t)}, z^*) \leq \exp \left(- (1 + o(1)) \frac{\Delta_{\min}^2}{8} \right) + 2^{-t} \text{ for all } t \geq 1,$$

(32)

with probability at least $1 - \eta - \exp (- \Delta_{\min}) - p^{-1}$.

According to a lower bound result in [46], the quantity $\exp \left(- (1 + o(1)) \frac{\Delta_{\min}^2}{8} \right)$ is the minimax rate of recovering $z^*$ with respect to the loss function $\text{Misclust}(z, z^*)$ under the Gaussian mixture model. Since $\text{Misclust}(z, z^*)$ takes value in the set $\{j/p : j \in [p] \cup \{0\}\}$, the term $2^{-t}$ in (32) is negligible as long as $2^{-t} = o(p^{-1})$. We therefore can claim

$$\text{Misclust}(z^{(t)}, z^*) \leq \exp \left(- (1 + o(1)) \frac{\Delta_{\min}^2}{8} \right) \text{ for all } t \geq 3 \log p.$$

In other words, the minimax rate is achieved after at most $\lceil 3 \log p \rceil$ iterations.

### 4.3 Initialization

To close this section, we discuss how to initialize Lloyd’s algorithm. In the literature, this is usually done by spectral methods [5, 40, 46]. We consider the following variation that is particularly suitable for Gaussian mixture models. Our initialization procedure has two steps:

1. Perform a singular value decomposition on $Y$, and obtain $Y = \sum_{l=1}^{p \land n} \hat{d}_l \hat{u}_l \hat{v}_l^T$ with $\hat{d}_1 \geq \ldots \geq \hat{d}_{p \land n} \geq 0$, $\{\hat{u}_l\}_{l \in [p \land n]} \in \mathbb{R}^d$ and $\{\hat{v}_l\}_{l \in [p \land n]} \in \mathbb{R}^p$. With $\hat{U} = (\hat{u}_1, \ldots, \hat{u}_k) \in \mathbb{R}^{d \times k}$, we define

$$\hat{\mu} = \hat{U}^T Y \in \mathbb{R}^{k \times p}. \quad (33)$$

2. Find some $\beta_1^{(0)}, \ldots, \beta_k^{(0)} \in \mathbb{R}^k$ and $z^{(0)} \in [k]^p$ that satisfy

$$\sum_{j=1}^{p} \|\hat{\mu}_j - \beta_j^{(0)}\|^2 \leq M \min_{\beta_1, \ldots, \beta_k \in \mathbb{R}^k} \sum_{j=1}^{p} \|\beta_j - \beta_j^{(0)}\|^2,$$

(34)

where $\hat{\mu}_j$ is the $j$th column of $\hat{\mu}$.

The first step (33) serves as a dimensionality reduction procedure, which reduces the dimension of data from $d$ to $k$. Then, the columns of $\hat{\mu}$ are collected to compute the $M$-approximation of the $k$-means objective in (34). We note that approximation of the $k$-means
objective can be computed efficiently in polynomial time \([4, 39, 41]\). For example, the \(k\)-means++ algorithm \([4]\) can efficiently solve (34) with \(M = O(\log k)\). However, we shall treat \(M\) flexible here, and its value will be reflected in the error bound of \(z^{(0)}\). The second step (34) can also be replaced by a greedy clustering algorithm used in \([30]\). The theoretical guarantee of \(z^{(0)}\) is given in the following proposition.

**Proposition 4.1.** Assume \(\min_{a \in [k]} \sum_{j=1}^{p} 1_{\{z_j^{*} = a\}} \geq \frac{\alpha}{k}\) for some constant \(\alpha > 0\) and \(\Delta^2_{\min} / ((M + 1)k^2(1 + d/p)) \to \infty\). For any \(C' > 0\), there exists a constant \(C > 0\) only depending on \(\alpha\) and \(C'\) such that

\[
\min_{\pi \in \Pi_k} \ell(\pi \circ z^{(0)}, z^*) \leq C(M + 1)k(p + d),
\]

with probability at least \(1 - e^{-C'(p+d)}\), where \(\Pi_k\) denotes the set of permutations on \([k]\).

We remark that a signal to noise ratio condition that is sufficient for both the conclusions of Proposition 4.1 and Theorem 4.1 is given by

\[
\frac{\Delta^2_{\min}}{(M + 1)k^2(kd/p + 1)} \to \infty,
\]

which is almost identical to (29). Note that the clustering structure is only identifiable up to a label permutation, and this explains the necessity of the minimum over \(\Pi_k\) in (35). In other words, (35) implies that there exists some \(\pi \in \Pi_k\), such that \(\ell(z^{(0)}, \pi^{-1} \circ z^*) \leq C(M + 1)k^2(p + d)\). Then, under the condition (36), (30) is satisfied with \(z^*\) replaced by \(\pi^{-1} \circ z^*\). Therefore, Theorem 4.1 implies that \(\ell(z^{(0)}, \pi^{-1} \circ z^*)\) converges to the minimax error with a linear rate.

## 5 Approximate Ranking

In this section, we study the estimation of \(z^* \in [p]^p\) using the pairwise interaction data generated according to \(Y_{ij} \sim \mathcal{N}(\beta^*(z^*_i - z^*_j), 1)\) independently for all \(1 \leq i \neq j \leq p\). This model can be viewed as a special case of the more general pairwise comparison model \(Y_{ij} \sim \mathcal{N}(\theta_i^* - \theta_j^*, 1)\), where \(\theta_i^*\) parametrizes the ability of the \(i\)th player, and the choice \(\theta_i^* = \alpha^* + \beta_i^*\) leads to \(Y_{ij} \sim \mathcal{N}(\beta^*(z^*_i - z^*_j), 1)\) that will be studied in this section. Let \(\Pi_p\) be the set of all possible permutations of \([p]\). We assume the rank vector \(z^*\) belongs to the following class,

\[
\mathcal{R} = \left\{ z \in [p]^p : \min_{\bar{z} \in \Pi_p} \|z - \bar{z}\|^2 \leq c_p \right\},
\]

for some sequence \(1 \leq c_p = o(p)\). In other words, \(\mathcal{R}\) is a set of approximate permutations. A rank vector \(z^* \in \mathcal{R}\) is allowed to have ties and not necessarily to start from 1. To be more precise, a \(z^* \in \mathcal{R}\) should be interpreted as discrete positions of the \(p\) players in the latent space of their abilities. This is in contrast to the exact ranking problem, also known as “noisy sorting” in the literature, where \(z^*\) is assumed to be a permutation \([11, 48, 58]\).
For the loss function
\[ L_2(z, z^*) = \frac{1}{p} \sum_{j=1}^{p} (z_j - z_j^*)^2, \] (38)
the minimax rate of estimating \( z^* \) takes the following formula,
\[ \inf_{\hat{z}} \sup_{z^* \in \mathcal{R}} \mathbb{E} L_2(\hat{z}, z^*) \asymp \begin{cases} \exp \left( -(1 + o(1)) \frac{p(\beta^*)^2}{4} \right), & p(\beta^*)^2 > 1, \\ \frac{1}{p(\beta^*)^2} \wedge p^2, & p(\beta^*)^2 \leq 1. \end{cases} \] (39)
See Theorems 2.2 and 2.3 in [27]. Interestingly, the minimax rate either takes a polynomial form or an exponential form, depending on the signal strength parametrized by \( p(\beta^*)^2 \). In the paper [27], a combinatorial procedure is constructed to achieve the optimal rate (39), and whether (39) can be achieved by a polynomial-time algorithm is unknown. This is where our proposed iterative algorithm comes. We will particularly focus on the regime of \( p(\beta^*)^2 \to \infty \), where the minimax rate takes an exponential form.

Specializing Algorithm 1 to the approximate ranking problem, we can write the iterative feature matching algorithm as
\[ z_j^{(t)} = \arg\min_{a \in [p]} \left| \sum_{i \in [p] \setminus \{j\}} (Y_{ji} - Y_{ij}) - 2p\hat{\beta}(z^{(t-1)}) \left( a - \frac{p + 1}{2} \right) \right|^2, \quad j \in [p], \]
where for each \( z \in [p]^p \), we use the notation
\[ \hat{\beta}(z) = \frac{\sum_{1 \leq i \neq j \leq p} (z_i - z_j)Y_{ij}}{\sum_{1 \leq i \neq j \leq p} (z_i - z_j)^2}. \] (40)

5.1 Conditions

From (11), we have
\[ \Delta_j(a, b)^2 = \frac{2p^2(\beta^*)^2}{p - 1} \left[ (a - b)^2 - 2(a - b) \left( \frac{1}{p} \sum_{j=1}^{p} z_j^* - \frac{p + 1}{2} \right) \right], \] (41)
and
\[ l(z, z^*) = \frac{2p^2(\beta^*)^2}{p - 1} \sum_{j=1}^{p} (z_j - z_j^*)^2 \] (42)
in the current setting. It is easy to check that \( \Delta_j(a, b)^2 > 0 \) for all \( a \neq b \) as long as \( z^* \in \mathcal{R} \). From (10) and (11), we have \( T_j = \mu_j(B^*, z_j^*) + \epsilon_j \) where \( \epsilon_j \sim \mathcal{N}(0, 1) \) for all \( j \in [p] \). The
\footnote{The paper [27] considers a parameter space that is slightly different from \( \mathcal{R} \). However, the proof of [27] can be modified so that the same minimax rate also applies to \( \mathcal{R} \).}
error terms that we need to control are

\[ F_j(a, b; z) = \frac{2p}{\sqrt{2(p-1)}} (\hat{\beta}(z^*) - \hat{\beta}(z))(a-b), \]

\[ G_j(a, b; z) = \frac{p^2}{p-1} \left( \beta^* \left( a - \frac{1}{p} \sum_{j=1}^{p} z_j^* \right) - \hat{\beta}(z) \left( a - \frac{p+1}{2} \right) \right)^2 \]

\[ - \frac{p^2}{p-1} \left( \beta^* \left( a - \frac{1}{p} \sum_{j=1}^{p} z_j^* \right) - \hat{\beta}(z^*) \left( a - \frac{p+1}{2} \right) \right)^2 \]

\[ + \frac{p^2}{p-1} \left( \beta^* \left( a - \frac{1}{p} \sum_{j=1}^{p} z_j^* \right) - \hat{\beta}(z^*) \left( b - \frac{p+1}{2} \right) \right)^2, \]

\[ H_j(a, b; z) = \frac{p^2}{p-1} \left( \beta^* \left( a - \frac{1}{p} \sum_{j=1}^{p} z_j^* \right) - \hat{\beta}(z^*) \left( a - \frac{p+1}{2} \right) \right)^2 \]

\[ - \frac{p^2}{p-1} \left( \beta^* \left( a - \frac{1}{p} \sum_{j=1}^{p} z_j^* \right) - \beta^* \left( a - \frac{p+1}{2} \right) \right)^2 \]

\[ - \frac{p^2}{p-1} \left( \beta^* \left( a - \frac{1}{p} \sum_{j=1}^{p} z_j^* \right) - \hat{\beta}(z^*) \left( b - \frac{p+1}{2} \right) \right)^2 \]

\[ + \frac{p^2}{p-1} \left( \beta^* \left( a - \frac{1}{p} \sum_{j=1}^{p} z_j^* \right) - \beta^* \left( b - \frac{p+1}{2} \right) \right)^2. \]

Lemma 5.1. Assume \( z^* \in \mathcal{R}, \tau = o(p^2(\beta^*)^2), \) and \( p(\beta^*)^2 \geq 1. \) Then, for any \( C' > 0, \) there exists a constant \( C > 0 \) only depending on \( C' \) such that

\[ \max_{\{z \in \mathcal{R}\}} \sum_{j=1}^{p} \max_{b \in [p] \setminus \{z \}} \frac{F_j(z_j^*, b; z)}{\Delta_j(z_j^*, b)4\ell(z, z^*)} \leq C\tau^{-2}, \tag{43} \]

\[ \max_{\{z \in \mathcal{R}\}} \max_{T \subset [p]} \frac{\sum_{j \in T} \max_{b \in [p] \setminus \{z \}} G_j(z_j^*, b; z)}{4\Delta_{\min}^2 T + \tau} \leq C \left( \frac{\tau}{p|\beta^*|^2} + \frac{1}{p|\beta^*|^2} \right), \tag{44} \]

and

\[ \max_{\{z \in \mathcal{R}\}} \max_{\mathcal{J} \subset [p]} \max_{b \in [p] \setminus \{z \}} \frac{|H_j(z_j^*, b; z)|}{\Delta_j(z_j^*, b)^2} \leq C \frac{1}{\sqrt{p}|\beta^*|}, \tag{45} \]

with probability at least \( 1 - (C'p)^{-1} \) for a sufficiently large \( p. \)
Lemma 5.1 implies that Conditions A, B and C hold with some sequence $\delta = \delta_p = o(1)$ as long as $\tau = o(p^2(\beta^*)^2)$ and $p(\beta^*)^2 \to \infty$.

Next, we need to control $\xi_{\text{ideal}}(\delta)$ in Condition D. This is given by the following lemma.

**Lemma 5.2.** Assume $p(\beta^*)^2 \to \infty$. Then, for any sequence $\delta_p = o(1)$, we have

$$\xi_{\text{ideal}}(\delta_p) \leq p \exp \left( -(1 + o(1)) \frac{p(\beta^*)^2}{4} \right),$$

with probability at least $1 - \exp \left( -\sqrt{p(\beta^*)^2} \right) - p^{-1}$.

We note that the signal condition $p(\beta^*)^2 \to \infty$ implies that Conditions A, B, C and D to hold simultaneously.

### 5.2 Convergence

With the help of Lemma 5.1 and Lemma 5.2, we can specialize Theorem 3.1 into the following result.

**Theorem 5.1.** Assume $p(\beta^*)^2 \to \infty$ and $z^* \in \mathcal{R}$. Suppose $z^{(0)}$ satisfies $\ell(z^{(0)}, z^*) = o(p^2(\beta^*)^2)$ with probability at least $1 - \eta$. Then, we have

$$\ell(z^{(t)}, z^*) \leq p \exp \left( -(1 + o(1)) \frac{p(\beta^*)^2}{4} \right) + \frac{1}{2} \ell(z^{(t-1)}, z^*) \quad \text{for all } t \geq 1,$$

with probability at least $1 - \eta - \exp \left( -\sqrt{p(\beta^*)^2} \right) - 2p^{-1}$.

Using the relation from (38) and (42) that

$$L_2(z, z^*) = \frac{p - 1}{2p^3(\beta^*)^2} \ell(z, z^*),$$

we immediately obtain the following result on the loss $L_2(z, z^*)$.

**Corollary 5.1.** Assume $p(\beta^*)^2 \to \infty$ and $z^* \in \mathcal{R}$. Suppose $z^{(0)}$ satisfies $\ell(z^{(0)}, z^*) = o(p^2(\beta^*)^2)$ with probability at least $1 - \eta$. Then, we have

$$L_2(z^{(t)}, z^*) \leq \exp \left( -(1 + o(1)) \frac{p(\beta^*)^2}{4} \right) + 2^{-t} \quad \text{for all } t \geq 1,$$

with probability at least $1 - \eta - \exp \left( -\sqrt{p(\beta^*)^2} \right) - 2p^{-1}$.

We observe that $L_2(z, z^*)$ takes value in the set $\{j/p : j \in \mathbb{N} \cup \{0\}\}$, the term $2^{-t}$ in (47) is negligible as long as $2^{-t} = o(p^{-1})$. We therefore can claim

$$L_2(z^{(t)}, z^*) \leq \exp \left( -(1 + o(1)) \frac{p(\beta^*)^2}{4} \right) \quad \text{for all } t \geq 3 \log p.$$

Hence, by (39) the iterative feature matching algorithm achieves the minimax rate of approximate ranking in the regime of $p(\beta^*)^2 \to \infty$ after at most $\lceil 3 \log p \rceil$ iterations.
5.3 Initialization

To initialize the iterative feature matching algorithm, we consider a simple ranking procedure based on the statistics \(\{T_j\}_{j \in [p]}\). That is, letting \(T_{(1)} \leq \cdots \leq T_{(p)}\) be the order statistics of \(\{T_j\}_{j \in [p]}\), we define \(z^{(0)}\) to be a permutation vector that satisfies \(T_{z^{(0)}_j} = T_j\) for all \(j \in [p]\).

Proposition 5.1. Assume \(z^* \in \mathcal{R}\) and \(\beta^* > 0\). Then, we have

\[
L_2(z^{(0)}, z^*) \lesssim \begin{cases} o(1), & (\beta^*)^2 \to \infty, \\
\frac{1}{p(\beta^*)^2} \land p^2, & (\beta^*)^2 = O(1), 
\end{cases}
\]

with probability at least \(1 - p^{-1}\).

Note that the additional condition \(\beta^* > 0\) guarantees that \(z^{(0)}\) estimates \(z^*\) instead of its reverse order. In the regime of \(p(\beta^*)^2 \to \infty\), the initialization procedure achieves \(L_2(z^{(0)}, z^*) = o(1)\) with high probability. Given the relation (46), this implies that \(\ell(z^{(0)}, z^*) = o(p^2(\beta^*)^2)\), and thus the initialization condition of Theorem 5.1 is satisfied. In the regime of \(p(\beta^*)^2 = O(1)\), the initialization procedure achieves the rate \(\frac{1}{p(\beta^*)^2} \land p^2\), which is already minimax optimal according to (39), and there is no need for the improvement via the iterative algorithm.

6 Sign Recovery in Compressed Sensing

We consider a regression model \(Y = X\beta^* + w \in \mathbb{R}^n\), where \(X \in \mathbb{R}^{n \times p}\) is a random design matrix with i.i.d. entries \(X_{ij} \sim \mathcal{N}(0, 1)\), and \(w\) is an independent noise vector with i.i.d. entries \(w_i \sim \mathcal{N}(0, 1)\). Our goal is to recover the signs of the regression coefficients \(\beta^*_j\)'s. Formally speaking, we assume

\[
z^* \in Z_s = \left\{ z \in \{-1, 0, 1\}^p : \sum_{j=1}^p |z_j| = s \right\},
\]

and \(\beta^* \in B_{z^*, \lambda}\), where for some \(z \in \{-1, 0, 1\}^p\) and some \(\lambda > 0\), the space \(B_{z, \lambda}\) is defined by

\[
B_{z, \lambda} = \left\{ \beta \in \mathbb{R}^p : \beta_j = z_j|\beta_j|, \min_{\{j \in [p] : z_j \neq 0\}} |\beta_j| \geq \lambda \right\}.
\]

The problem is to estimate the sign vector \(z^*\). A closely related problem is support recovery, which is equivalent to estimating the vector \(\{|z^*_j|\}_{j \in [p]}\). This problem has received much attention in the literature of compressed sensing, where one usually has control over the distribution of the design matrix. Necessary and sufficient conditions on \((n, p, s, \lambda)\) for exact support recovery have been derived in [2, 25, 56, 57, 61, 62] and references therein. Recently, the minimax rate of partial support recovery with respect to the Hamming loss has been derived in [54]. Their results can be easily modified to the estimation of the sign vector \(z^*\)
as well. We will state the lower bound result in [54] as our benchmark. To do that, we need to introduce the normalized Hamming loss

$$H_{(s)}(z, z^*) = \frac{1}{s} h(z, z^*) = \frac{1}{s} \sum_{j=1}^{p} 1\{z_j \neq z_j^*\}.$$ 

We also define the signal-to-noise ratio of the problem by

$$\text{SNR} = \frac{\lambda \sqrt{n} - \log p - s \lambda}{\lambda \sqrt{n}}.$$ (48)

**Theorem 6.1** (Ndaoud and Tsybakov [54]). Assume $\lim \sup \frac{s}{p} < \frac{1}{2}$ and $s \log p \leq n$. If $\text{SNR} \to \infty$, we have

$$\inf \sup \sup_{\tilde{z}, z^* \in \mathbb{Z}, \beta^* \in B_{s^*}, \lambda} E H_{(s)}(\tilde{z}, z^*) \geq \exp \left( -\frac{(1 + o(1)) \text{SNR}^2}{2} \right) - 4e^{-s/8}.$$ 

Otherwise if $\text{SNR} = O(1)$, we then have

$$\inf \sup \sup_{\tilde{z}, z^* \in \mathbb{Z}, \beta^* \in B_{s^*}, \lambda} E H_{(s)}(\tilde{z}, z^*) \geq c,$$

for some constant $c > 0$.

We remark that the lower bound result in [54] is stated in a more general non-asymptotic form. Here, we choose to work out its asymptotic formula (by Lemma 7.5) so that we can better compare the lower bound with the upper bound rate achieved by our algorithm. In [54], the minimax rate is achieved by a thresholding procedure that requires sample splitting. Though theoretically sound, the requirement of splitting the data into two halves may not be appealing in practice. This is where our general Algorithm 1 comes. We will show that Algorithm 1 can achieve the minimax rate without sample splitting.

Our analysis is focused in the regime where $\text{SNR} \to \infty$, which is necessary for consistency under the loss $H_{(s)}(\tilde{z}, z^*)$ according to Theorem 6.1. Specializing Algorithm 1 to the current setting, we obtain the following iterative procedure

$$z_j^{(t)} = \begin{cases} 
1 & \frac{x_j^T y - \sum_{i \in [p] \setminus \{j\}} \hat{\beta}_i(z^{(t-1)}) x_i}{\|x_j\|^2} > t(X_j) \\
0 & t(X_j) \leq \frac{x_j^T y - \sum_{i \in [p] \setminus \{j\}} \hat{\beta}_i(z^{(t-1)}) x_i}{\|x_j\|^2} \leq t(X_j), \quad j \in [p], \\
-1 & \frac{x_j^T y - \sum_{i \in [p] \setminus \{j\}} \hat{\beta}_i(z^{(t-1)}) x_i}{\|x_j\|^2} < -t(X_j)
\end{cases}$$ (49)

where $t(X_j)$ is defined by (16). Here, for some $z \in \{-1, 0, 1\}^p$, we use the notation

$$\hat{\beta}(z) = \arg\min_{\beta \in \mathbb{R}^p : \beta_j = \beta_j(z)} \|y - X \beta\|^2.$$ 

In other words, $\hat{\beta}(z)$ is the least-squares solution on the support of $z$. The formula (49) resembles the thresholding procedure proposed in [54]. In [54], $\hat{\beta}_i(z^{(t-1)})$ is replaced by some
estimator \( \hat{\beta}_t \) computed from an independent data set. In comparison, we use \( \hat{\beta}_t(z^{(t-1)}) \) and thus avoid sample splitting. The iteration (49) is also different from existing algorithms in the literature for support/sign recovery in compressed sensing. For example, the popular iterative hard thresholding algorithm [9] updates the regression coefficients with a gradient step instead of a full least-squares step. The hard thresholding pursuit algorithm [26] has a full least-squares steps, but updates the support by choosing the \( s \) variables with the largest absolute values.

6.1 Conditions

For any \( j \in [p] \), \( T_j \) is the local statistic defined in (13) and it can be decomposed as \( T_j = \mu_j(B^*, z_j^*) + \epsilon_j \), with \( \epsilon_j = \| X_j \|^{-1} X_j^T w \sim \mathcal{N}(0,1) \). To analyze the algorithmic convergence, we need to specialize the abstract objects \( \| \mu_j(B^*, z_j^*) - \mu_j(B^*, b) \|^2 \), \( \Delta_j(z_j^*, b)^2 \), and \( \ell(z, z^*) \) into the current setting. With the formulas (14) and (15), we have

\[
\| \mu_j(B^*, z_j^*) - \mu_j(B^*, b) \|^2 = \begin{cases} 
\lambda^2 \| X_j \|^2 & z_j^* = 0 \text{ and } b \neq 0 \\
|\beta_j^*|^2 \| X_j \|^2 & z_j^* \neq 0 \text{ and } b = 0 \\
4|\beta_j^*|^2 \| X_j \|^2 & z_j^*b = -1,
\end{cases}
\]

which leads to the formula of the loss function

\[
\ell(z, z^*) = \sum_{j=1}^{p} \left( \lambda^2 \| X_j \|^2 \mathbf{1}_{\{z_j^* = 0, z_j \neq 0\}} + |\beta_j^*|^2 \| X_j \|^2 \mathbf{1}_{\{z_j^* \neq 0, z_j = 0\}} + 4|\beta_j^*|^2 \| X_j \|^2 \mathbf{1}_{\{z_j^*z_j = -1\}} \right).
\]

By (20), we have the relation

\[
H(s)(z, z^*) \leq \frac{\ell(z, z^*)}{s \Delta_{\text{min}}^2},
\]

where \( \Delta_{\text{min}}^2 = \lambda^2 \min_{j \in [p]} \| X_j \|^2 \) in the current setting. Lastly, the formula of \( \Delta_j(z_j^*, b)^2 \) is given by

\[
\Delta_j(z_j^*, b)^2 = \begin{cases} 
4t(X_j)^2 \| X_j \|^2 & z_j^* = 0 \text{ and } b \neq 0 \\
4t(X_j)(|\beta_j^*| - t(X_j)) \| X_j \|^2 & z_j^* \neq 0 \text{ and } b = 0 \\
8t(X_j)|\beta_j^*| \| X_j \|^2 & z_j^*b = -1.
\end{cases}
\]

One may question whether we always have \( \Delta_j(z_j^*, b)^2 > 0 \) for all \( b \neq z_j^* \) and \( j \in [p] \). We note that this property is guaranteed by Lemma 7.6 with high probability.

Next, we analyze the error terms. In the current setting, they are

\[
\mathcal{F}_j(a, b; z) = 0,
\]

\[
\mathcal{G}_j(a, b; z) = 2(a - b)t(X_j) \sum_{l \in [p] \setminus \{j\}} \left( \hat{\beta}_l(z) - \hat{\beta}_l(z^*) \right) X_j^T X_l,
\]

\[
\mathcal{H}_j(a, b; z) = 2(a - b)t(X_j) \sum_{l \in [p] \setminus \{j\}} \left( \hat{\beta}_l(z^*) - \beta^* \right) X_j^T X_l.
\]
Lemma 6.1. Assume $s \log p \leq n$ and $\tau \leq C_0 sn\lambda^2$ for some constant $C_0 > 0$. Then, for any $C' > 0$, there exists a constant $C > 0$ only depending on $C_0$ and $C'$ such that

\[
\max_{\{z : \ell(z, z^*) \leq \tau \}} \max_{T \subset [p]} \frac{\tau}{4\Delta_m^2 |T|} + \max_{j \in [p]} \frac{\max_{b \in \{-1,0,1\}\setminus\{z_j^*\}} G_j(z_j^*, b; z) + \mu_j(B^*, b) - \mu_j(B^*, z_j^*)^2}{\Delta_j(z_j^*, b)^4 \ell(z, z^*)}
\]

\[
\leq C \frac{s \log p^2}{n} \left(1 + \frac{1}{n\lambda^2}\right) \max_{j \in [p]} \left[ \frac{\lambda^2}{\Delta_j(z_j^*, b)^2} \vee \frac{\lambda^2}{t(X_j)^2} \right],
\]

and

\[
\max_{\{z : \ell(z, z^*) \leq \tau \}} \max_{j \in [p]} \frac{|H_j(z_j^*, b; z)|}{\Delta_j(z_j^*, b)^2} \leq C \sqrt{\frac{s \log p^2}{n}} \frac{1}{\min_{j \in [p]} \sqrt{n} \lambda^2} |t(X_j)|,
\]

with probability at least $1 - p^{-C'}$.

The two error bounds (53) and (54) are complicated. However, by Lemma 7.6, if we additionally assume $\limsup s/p < \frac{1}{2}$, $\text{SNR} \to \infty$, and $s \log p^4 = o(n)$, the right hand sides of (53) and (54) can be shown to be of order $o((\log p)^{-1})$. Therefore, Conditions A, B and C hold with some $\delta = \delta_p = o((\log p)^{-1})$.

The following lemma controls $\xi_{\text{ideal}}(\delta)$ in Condition D.

Lemma 6.2. Assume $\limsup s/p < \frac{1}{2}$, $s \log p \leq n$, and $\text{SNR} \to \infty$. Then for any sequence $\delta_p = o((\log p)^{-1})$, we have

\[
\xi_{\text{ideal}}(\delta_p) \leq 2\lambda \exp \left( - \frac{(1 + o(1))\text{SNR}^2}{2} \right),
\]

with probability at least $1 - \exp(-\text{SNR}) - p^{-1}$.

6.2 Convergence

With Lemma 6.1 and Lemma 6.2, we then can specialize Theorem 3.1 into the following result.

Theorem 6.2. Assume $\limsup s/p < \frac{1}{2}$, $s \log p^4 = o(n)$, and $\text{SNR} \to \infty$. Suppose $\ell(z(0), z^*) \leq C_0 sn\lambda^2$ with probability at least $1 - \eta$ for some constant $C_0 > 0$. Then, we have

\[
\ell(z(t), z^*) \leq sn\lambda^2 \exp \left( - \frac{(1 + o(1))\text{SNR}^2}{2} \right) + \frac{1}{2} \ell(z(t-1), z^*) \quad \text{for all } t \geq 1,
\]

with probability at least $1 - \eta - \exp(-\text{SNR}) - 2p^{-1}$.

The relation (51) and a simple concentration result for $\min_{j \in [p]} \|X_j\|^2$ immediately implies a convergence result for the loss $H(s)(z, z^*)$. 

21
Corollary 6.1. Assume \( \limsup s/p < \frac{1}{2} \), \( s \log p = o(n) \), and \( \text{SNR} \to \infty \). Suppose \( \ell(z(0), z^*) \leq C_0 sn\lambda^2 \) with probability at least \( 1 - \eta \) for some constant \( C_0 > 0 \). Then, we have
\[
H(s)(z(t), z^*) \leq \exp\left(-\frac{(1 + o(1))SNR^2}{2}\right) + 2^{-t} \quad \forall t \geq 1,
\]
with probability at least \( 1 - \eta - \exp(-\text{SNR}) - 2p^{-1} \).

Since the loss function \( H(s)(z, z^*) \) takes value in the set \( \{j/s : j \in [p] \cap \{0\}\} \), the term \( 2^{-t} \) in (55) is negligible as long as \( 2^{-t} = o(s^{-1}) \). We therefore can claim
\[
H(s)(z(t), z^*) \leq \exp\left(-\frac{(1 + o(1))SNR^2}{2}\right) \quad \forall t \geq 3 \log s,
\]
when \( s \to \infty \). If instead we have \( s = O(1) \), then any \( t \to \infty \) will do. This implies after at most \([3 \log p]\) iterations, Algorithm 1 achieves the minimax rate.

Remark 6.1. The leading term of the non-asymptotic minimax lower bound in [54] with respect to the loss \( H(s)(z, z^*) \) takes the form of \( \psi(n, p, s, \lambda, 0)/s \), where
\[
\psi(n, p, s, \lambda, \delta) = sP(\epsilon > (1 - \delta)\|\beta\|_1 + (p - s)P(\epsilon > (1 - \delta)\|\beta\|_1))
\]
with \( \epsilon \sim \mathcal{N}(0, 1) \) and \( \zeta \sim \mathcal{N}(0, I_n) \) independent of each other. By scrutinizing the proof of Lemma 6.2, we can also write (55) as
\[
H(s)(z(t), z^*) \leq \psi(n, p, s, \lambda, \delta_p)/s + 2^{-t} \quad \forall t \geq 1,
\]
with high probability with some \( \delta_p = o((\log p)^{-1}) \).

6.3 Initialization

Our final task in this section is to provide an initialization procedure that satisfies the bound \( \ell(z(0), z^*) \leq C_0 sn\lambda^2 \) with high probability. We consider a simple procedure that thresholds the solution of the square-root SLOPE [8, 10, 21, 59]. It has the following two steps:

1. Compute
\[
\hat{\beta} = \arg\min_{\beta \in \mathbb{R}^p} (\|Y - X\beta\| + A\|\beta\|_{\text{SLOPE}}),
\]
where the penalty takes the form \( A\|\beta\|_{\text{SLOPE}} = \sum_{j=1}^p \sqrt{\log (2p/j)}|\beta|_1 \). Here \( |\beta|_1 \geq |\beta|_2 \geq \cdots \geq |\beta|_p \) is a non-increasing ordering of \( |\beta|_1, |\beta|_2, \cdots, |\beta|_p \).

2. For any \( j \in [p] \), compute \( z_j^{(0)} = \text{sign}(\hat{\beta}_j)1_{\{\hat{\beta}_j \geq \lambda/2\}} \).

The theoretical guarantee of \( z(0) \) is given by the following proposition.

Proposition 6.1. Assume \( \limsup s/p < \frac{1}{4} \), \( s \log p \leq n \), and \( \text{SNR} \to \infty \). For some sufficiently large constant \( A > 0 \) in (57) and any constant \( C' > 0 \), there exist some \( C_0 \) and \( C_1 \) only depending on \( A \) and \( C' \), such that
\[
\ell(z(0), z^*) \leq C_0 sn\lambda^2
\]
with probability at least \( 1 - e^{-C_1 s \log(ep/s)} - p^{-C'} \).
7 Proofs

7.1 Proof of Theorem 3.1

Suppose \( \ell(z^{(t-1)}, z^*) \leq \tau \), and we will show \( \ell(z^{(t)}, z^*) \leq 2\xi_{\text{ideal}}(\delta) + \frac{1}{2}\ell(z^{(t-1)}, z^*) \). By the definition of the loss (19), we have

\[
\ell(z^{(t)}, z^*) = \sum_{j=1}^{p} \|\mu_j(B^*, z_j^{(t)}) - \mu_j(B^*, z_j^*)\|^2
\]

\[
= \sum_{j=1}^{p} \sum_{b \in [k] \setminus \{z_j^*\}} \|\mu_j(B^*, b) - \mu_j(B^*, z_j^*)\|^2 \mathbf{1}_{\{z_j^{(t)} = b\}}. \tag{58}
\]

To bound (58), we have

\[
\mathbf{1}_{\{z_j^{(t)} = b\}} \leq \mathbf{1}_{\{\|T_j - \nu_j(\tilde{B}(z^{(t-1)}), b)\|^2 \leq \|T_j - \nu_j(\tilde{B}(z^{(t-1)}), z_j^*)\|^2\}} \tag{59}
\]

\[
= \mathbf{1}_{\{\langle \nu_j(\tilde{B}(z^*), z_j^*) - \nu_j(\tilde{B}(z^*), \xi) \rangle \leq -\frac{1}{2}\Delta_j(z_j^*, b) + F_j(z_j^*, b; z^{(t-1)}) + G_j(z_j^*, b; z^{(t-1)}) + H_j(z_j^*, b; z^{(t-1)})\}} \tag{60}
\]

\[
\leq \mathbf{1}_{\{\langle \nu_j(\tilde{B}(z^*), z_j^*) - \nu_j(\tilde{B}(z^*), \xi) \rangle \leq -\frac{1}{2}\Delta_j(z_j^*, b) \quad \text{and} \quad \frac{3}{4}\Delta_j(z_j^*, b)^2 \leq F_j(z_j^*, b; z^{(t-1)}) + G_j(z_j^*, b; z^{(t-1)}) + H_j(z_j^*, b; z^{(t-1)})\}} \tag{61}
\]

\[
\leq \mathbf{1}_{\{\langle \nu_j(\tilde{B}(z^*), z_j^*) - \nu_j(\tilde{B}(z^*), \xi) \rangle \leq -\frac{1}{2}\Delta_j(z_j^*, b) \quad \text{and} \quad \frac{3}{4}\Delta_j(z_j^*, b)^2 \leq F_j(z_j^*, b; z^{(t-1)}) + G_j(z_j^*, b; z^{(t-1)}) + H_j(z_j^*, b; z^{(t-1)})\}} \tag{62}
\]

\[
+ \frac{32F_j(z_j^*, b; z^{(t-1)})^2}{\delta^2\Delta_j(z_j^*, b)^4} + \frac{32G_j(z_j^*, b; z^{(t-1)})^2}{\delta^2\Delta_j(z_j^*, b)^4}. \tag{63}
\]

The inequality (59) is due to the definition that \( z_j^{(t)} = \arg\min_{a \in [k]} \|T_j - \nu_j(\tilde{B}(z^{(t-1)}), a)\|^2 \). Then, the equality (60) uses the equivalence between (22) and (23). The inequality (61) uses a union bound, and (62) applies Condition C. Finally, (63) follows Markov’s inequality.
Apply the bound (63) to (58), and then \( \ell(z^{(t)}, z^*) \) can be bounded by

\[
\sum_{j=1}^{p} \sum_{b \in [k] \setminus \{z_j^*\}} \|\mu_j(B^*, b) - \mu_j(B^*, z_j^*)\|^2 \mathbb{1}_{\{z_j \neq z_j^*\}} \leq \frac{1}{8} \Delta_{\text{min}} h(z^{(t)}, z^*) + \frac{1}{4} \ell(z^{(t)}, z^*)
\]



This leads to

\[
\ell(z^{(t)}, z^*) \leq 2 \xi_{\text{ideal}}(\delta) + \frac{1}{2} \ell(z^{(1)}, z^*) + \frac{1}{4} \ell(z^{(t-1)}, z^*)
\]

which can be rearranged into

\[
\ell(z^{(t)}, z^*) \leq 2 \xi_{\text{ideal}}(\delta) + \frac{1}{2} \ell(z^{(t-1)}, z^*).
\]

To prove the conclusion of Theorem 3.1, we use a mathematical induction argument. First, Condition D asserts that \( \ell(z^{(0)}, z^*) \leq \tau \). This leads to \( \ell(z^{(1)}, z^*) \leq 2 \xi_{\text{ideal}}(\delta) + \frac{1}{2} \ell(z^{(0)}, z^*) \leq \tau \), together with Condition C that \( \xi_{\text{ideal}}(\delta) \leq \frac{1}{2} \tau \). Suppose \( \ell(z^{(t-1)}, z^*) \leq \tau \), we then have \( \ell(z^{(t)}, z^*) \leq 2 \xi_{\text{ideal}}(\delta) + \frac{1}{2} \ell(z^{(t-1)}, z^*) \leq \tau \). Hence, \( \ell(z^{(t-1)}, z^*) \leq \tau \) for all \( t \geq 1 \), which implies that \( \ell(z^{(t)}, z^*) \leq 2 \xi_{\text{ideal}}(\delta) + \frac{1}{2} \ell(z^{(t-1)}, z^*) \) for all \( t \geq 1 \), and the proof is complete.

### 7.2 Proofs in Section 4

In this section, we present the proofs of Lemma 4.1, Lemma 4.2 and Proposition 4.1. The conclusions of Theorem 4.1 and Corollary 4.1 are direct consequences of Theorem 3.1, and thus their proofs are omitted. We first list some technical lemmas. The following \( \chi^2 \) tail probability is Lemma 1 of [42].
Lemma 7.1. For any $x > 0$, we have
\[
\Pr\left(\chi^2_d \geq d + 2\sqrt{dx} + 2x\right) \leq e^{-x},
\]
\[
\Pr\left(\chi^2_d \leq d - 2\sqrt{dx}\right) \leq e^{-x}.
\]

Lemma 7.2. Consider i.i.d. random vectors $\epsilon_1, \ldots, \epsilon_p \sim \mathcal{N}(0, I_d)$ and some $z^* \in [k]^p$ and $k \in [p]$. Then, for any constant $C' > 0$, there exists some constant $C > 0$ only depending on $C'$ such that
\[
\max_{a \in [k]} \left\| \frac{\sum_{j=1}^p 1_{\{z_j^* = a\}} \epsilon_j}{\sum_{j=1}^p 1_{\{z_j^* = a\}}} \right\| \leq C \sqrt{d + \log p},
\]
(68)
\[
\max_{T \subset [p]} \left\| \frac{1}{\sqrt{|T|}} \sum_{j \in T} \epsilon_j \right\| \leq C \sqrt{d + p},
\]
(69)
\[
\max_{a \in [k]} \frac{1}{d + \sum_{j=1}^p 1_{\{z_j^* = a\}}} \left\| \sum_{j=1}^p 1_{\{z_j^* = a\}} \epsilon_j^T \right\| \leq C,
\]
(70)

with probability at least $1 - p^{-C'}$. We have used the convention that $0/0 = 0$.

Proof. By Lemma 7.1, we have $\Pr(\chi^2_d \geq d + 2\sqrt{dx} + 2x) \leq e^{-x}$. Then, a union bound argument leads to (68). The inequalities (69) and (70) are Lemmas A.1 and A.2 in [46]. We need to slightly extend Lemma A.2 in [46], but this can be done by a standard union bound argument.

With the two lemmas above, we are ready to state the proofs of Lemma 4.1 and Lemma 4.2.

Proof of Lemma 4.1. We write $\epsilon_j = Y_j - \theta_{z_j^*}$ and consider the event that the three inequalities (68)-(70) hold. For any $z \in [k]^p$ such that $\ell(z, z^*) \leq \tau \leq \frac{\Delta^2_{\text{min}} \alpha p}{2k}$, we have
\[
\sum_{j=1}^p 1_{\{z_j = a\}} \geq \sum_{j=1}^p 1_{\{z_j^* = a\}} - \sum_{j=1}^p 1_{\{z_j \neq z_j^*\}} \\
\geq \sum_{j=1}^p 1_{\{z_j^* = a\}} - \frac{\ell(z, z^*)}{\Delta^2_{\text{min}}} \\
\geq \frac{\alpha p}{k} - \frac{\alpha p}{2k} \\
= \frac{\alpha p}{2k},
\]
which implies
\[
\min_{a \in [k]} \sum_{j=1}^p 1_{\{z_j = a\}} \geq \frac{\alpha p}{2k},
\]
(71)
We then introduce more notation. We write \( \hat{\theta}_a(z) = \mathbb{E}\tilde{\theta}_a(z) \) and
\[
\tilde{c}_a(z) = \frac{\sum_{j=1}^p 1_{\{z_j = a\}} \epsilon_j}{\sum_{j=1}^p 1_{\{z_j = a\}}}.
\]

We first derive bounds for \( \max_{a \in [k]} \|\hat{\theta}_a(z^*) - \theta_a^*\| \), \( \max_{a \in [k]} \|\theta_a(z) - \theta_a(z^*)\| \) and \( \max_{a \in [k]} \|\tilde{c}_a(z) - \tilde{c}_a(z^*)\| \). By (68) and (71), we have
\[
\max_{a \in [k]} \|\tilde{c}_a(z) - \tilde{c}_a(z^*)\| = \max_{a \in [k]} \left| \frac{\sum_{j=1}^p 1_{\{z_j = a\}} \epsilon_j}{\sum_{j=1}^p 1_{\{z_j = a\}}} \right|
\]
\[
\leq \sqrt{\frac{k}{\max_{a \in [k]}} \left( \sqrt{\frac{\sum_{j=1}^p 1_{\{z_j^* = a\}} \epsilon_j}{\sum_{j=1}^p 1_{\{z_j = a\}}} - \sqrt{\frac{\sum_{j=1}^p 1_{\{z_j = a\}} \epsilon_j}{\sum_{j=1}^p 1_{\{z_j = a\}}}} \right)}
\]
\[
\leq \sqrt{\frac{k(d + \log p)}{p}}. \quad \quad (72)
\]

By (71), we have
\[
\max_{a \in [k]} \|\theta_a(z) - \theta_a(z^*)\| = \left| \frac{1}{\sum_{j=1}^p 1_{\{z_j = a\}}} \sum_{j=1}^p \sum_{b \in [k] \setminus \{a\}} 1_{\{z_j = a, z_j^* = b\}} (\theta_b^* - \theta_a^*) \right|
\]
\[
\leq \frac{2k}{\alpha p} \sum_{j=1}^p \sum_{b \in [k] \setminus \{a\}} \|\theta_b^* - \theta_a^*\| 1_{\{z_j = a, z_j^* = b\}}
\]
\[
\leq \frac{2k}{\alpha p \Delta_{\text{min}}(z, z^*)}. \quad \quad (73)
\]

By (71), we have
\[
\max_{a \in [k]} \|\tilde{c}_a(z) - \tilde{c}_a(z^*)\|
\]
\[
= \max_{a \in [k]} \left| \frac{\sum_{j=1}^p 1_{\{z_j = a\}} \epsilon_j}{\sum_{j=1}^p 1_{\{z_j = a\}}} \right|
\]
\[
\leq \max_{a \in [k]} \left| \frac{\sum_{j=1}^p 1_{\{z_j = a\}} \epsilon_j}{\sum_{j=1}^p 1_{\{z_j = a\}}} \right|
\]
\[
\leq \frac{2k}{\alpha p} \max_{a \in [k]} \left| \sum_{j=1}^p (1_{\{z_j = a\}} - 1_{\{z_j^* = a\}}) \epsilon_j \right|
\]
\[
+ \frac{2k}{\alpha p} \left( \frac{\sum_{j=1}^p 1_{\{z_j = a\}} - \sum_{j=1}^p 1_{\{z_j^* = a\}}}{\sqrt{\sum_{j=1}^p 1_{\{z_j = a\}}}} \right) \left| \max_{a \in [k]} \left| \frac{\sum_{j=1}^p 1_{\{z_j = a\}} \epsilon_j}{\sum_{j=1}^p 1_{\{z_j = a\}}} \right| \right|, \quad (74)
\]
where the first term in the above bound can be bounded by

\[
\frac{2k}{\alpha p} \max_{a \in [k]} \left| \sum_{j=1}^{p} 1\{z_j = a, z_j^* \neq a\} \epsilon_j \right| + \frac{2k}{\alpha p} \max_{a \in [k]} \left| \sum_{j=1}^{p} 1\{z_j^* = a, z_j \neq a\} \epsilon_j \right|
\]

\[
\lesssim \frac{k \sqrt{d + p}}{d} \sqrt{\frac{\ell(z, z^*)}{\Delta_{\min}^2}},
\]

because of the facts that \( \max_{a \in [k]} \sum_{j=1}^{p} 1\{z_j = a, z_j^* \neq a\} \leq \frac{\ell(z, z^*)}{\Delta_{\min}^2}, \) \( \max_{a \in [k]} \sum_{j=1}^{p} 1\{z_j^* = a, z_j \neq a\} \leq \frac{\ell(z, z^*)}{\Delta_{\min}^2}, \) and the inequality (69), and the second term can be bounded by

\[
\frac{2k}{\alpha p} \sqrt{\frac{k}{\max_{a \in [k]} \Delta_{\min}^2}} \left| \sum_{j=1}^{p} 1\{z_j^* = a\} \epsilon_j \right| \left( \max_{a \in [k]} \sum_{j=1}^{p} 1\{z_j = a, z_j^* \neq a\} + \max_{a \in [k]} \sum_{j=1}^{p} 1\{z_j^* = a, z_j \neq a\} \right)
\]

\[
\lesssim \frac{4k}{\alpha p} \sqrt{\frac{k}{\max_{a \in [k]} \Delta_{\min}^2}} \left| \sum_{j=1}^{p} 1\{z_j^* = a\} \epsilon_j \right| \sqrt{\sum_{j=1}^{p} 1\{z_j^* = a\}}
\]

\[
\lesssim \frac{k \sqrt{\ell(z, z^*)} \sqrt{d + \log p}}{d \sqrt{\Delta_{\min}^2}}.
\]

Under the condition that \( \ell(z, z^*) \leq \tau \leq \frac{\alpha p}{2k}, \) we have

\[
\max_{a \in [k]} \left\| \bar{e}_a(z) - \bar{e}_a(z^*) \right\|
\]

\[
\lesssim \frac{k \sqrt{d + p}}{p} \sqrt{\frac{\ell(z, z^*)}{\Delta_{\min}^2}} + \frac{k \sqrt{\ell(z, z^*)} \sqrt{d + \log p}}{p \sqrt{\Delta_{\min}^2}}
\]

\[
\lesssim \frac{k \sqrt{d + p}}{p} \sqrt{\frac{\ell(z, z^*)}{\Delta_{\min}^2}}. \tag{74}
\]

Combining the two bounds (73) and (74) and using triangle inequality, we also have

\[
\max_{a \in [k]} \left\| \tilde{\theta}_a(z) - \tilde{\theta}_a(z^*) \right\|
\]

\[
\leq \max_{a \in [k]} \left\| \theta_a(z) - \theta_a(z^*) \right\| + \max_{a \in [k]} \left\| \bar{e}_a(z) - \bar{e}_a(z^*) \right\|
\]

\[
\lesssim \frac{k}{p \Delta_{\min}} \ell(z, z^*) + \frac{k \sqrt{d + p}}{p \Delta_{\min}} \sqrt{\ell(z, z^*)}. \tag{75}
\]
Now we proceed to prove (26)-(28). For (26), we have

\[
\sum_{j=1}^{p} \max_{b \in [k] \setminus \{z_j^*\}} \frac{F_j(z_j^*, b; z)^2 \| \mu_j(B^*, b) - \mu_j(B^*, z_j^*) \|^2}{\Delta_j(z_j^*, b)^4 \ell(z, z^*)} \leq \sum_{j=1}^{p} \sum_{b=1}^{k} \frac{\left| \langle \epsilon_j, \hat{\theta}_{z_j^*}^*(z^*) - \hat{\theta}_{z_j^*}(z) - \hat{\theta}_b(z) + \hat{\theta}_b(z) \rangle \right|^2}{\| \theta_{z_j^*}^* - \theta_b^* \|^2 \ell(z, z^*)}
\]

\[
\leq \sum_{b=1}^{k} \sum_{a \in [k] \setminus \{b\}} \sum_{j=1}^{p} \mathbf{1}_{\{z_j^* = a\}} \frac{\left| \langle \epsilon_j, \hat{\theta}_a(z^*) - \hat{\theta}_a(z) - \hat{\theta}_b(z^*) + \hat{\theta}_b(z) \rangle \right|^2}{\| \theta_{a}^* - \theta_b^* \|^2 \ell(z, z^*)} \leq \frac{k^2 (kd/p + 1)}{\Delta_{\min}^2} \left( 1 + \frac{k(d/p + 1)}{\Delta_{\min}^2} \right)
\]

where we have used (70), (75) and the condition that \( \ell(z, z^*) \leq \tau = \frac{\Delta_{\min}^2 \exp}{2k} \). Next, for (27), we have

\[
|G_j(a, b; z)| \leq \frac{1}{2} \| \hat{\theta}_a(z) - \hat{\theta}_a(z^*) \|^2 + \frac{1}{2} \| \hat{\theta}_b(z) - \hat{\theta}_b(z^*) \|^2 + \max_{a \in [k]} \| \hat{\theta}_a(z) - \hat{\theta}_a(z^*) \| + \max_{a \in [k]} \| \hat{\theta}_a(z^*) - \theta_a^* \| + \max_{a \in [k]} \| \hat{\theta}_b(z) - \hat{\theta}_b(z^*) \| + \max_{a \in [k]} \| \hat{\theta}_b(z^*) - \theta_b^* \|
\]

\[
\leq \max_{a \in [k]} \| \hat{\theta}_a(z) - \hat{\theta}_a(z^*) \|^2 + \left( \max_{a \in [k]} \| \hat{\theta}_a(z^*) - \theta_a^* \| \right) \left( \max_{a \in [k]} \| \hat{\theta}_a(z) - \hat{\theta}_a(z^*) \| \right) + \max_{a \in [k]} \| \theta_a^* - \theta_b^* \| \left( \max_{a \in [k]} \| \hat{\theta}_a(z) - \hat{\theta}_a(z^*) \| \right).
\]
This implies for any subset $T \subset [p]$, we have

$$
\begin{align*}
\frac{\tau}{4\Delta_{\min}^2 |T|} & + \frac{\tau}{4\Delta_{\min}^2 |T|} \sum_{j \in T, b \in [k] \setminus \{z\}} \max G_j(z_j^*, b; z)^2 \frac{\mu_j(B^*, b) - \mu_j(B^*, z_j^*)}{\Delta_j(z_j^*, b)^4 \ell(z, z^*)} \\
& \leq \frac{\tau}{4\Delta_{\min}^2 |T|} \sum_{j \in T} 3 \max_{a \in [k]} \left| \hat{\theta}_a(z) - \hat{\theta}_a(z^*) \right|^4 \\
& + \frac{\tau}{4\Delta_{\min}^2 |T|} \sum_{j \in T} 4 \max_{a \in [k]} \left| \hat{\theta}_a(z) - \hat{\theta}_a(z^*) \right|^2 \\
& + \frac{\tau}{4\Delta_{\min}^2 |T|} \sum_{j \in T} \max_{a \in [k]} \left| \hat{\theta}_a(z) - \hat{\theta}_a(z^*) \right|^2 \\
& = \frac{3\tau \max_{a \in [k]} \left| \hat{\theta}_a(z) - \hat{\theta}_a(z^*) \right|^4}{4\Delta_{\min}^4 \ell(z, z^*)} + \frac{3\tau \max_{a \in [k]} \left| \hat{\theta}_a(z) - \hat{\theta}_a(z^*) \right|^2}{4\Delta_{\min}^2 \ell(z, z^*)} \\
& + \frac{\tau}{4\Delta_{\min}^2 |T|} \sum_{j \in T} 12 \max_{a \in [k]} \left| \hat{\theta}_a(z^*) - \theta_a^* \right|^2 \left( \max_{a \in [k]} \left| \hat{\theta}_a(z) - \hat{\theta}_a(z^*) \right|^2 \right) \\
& \leq \frac{k\tau}{p\Delta_{\min}^2} + \frac{k(d + p)}{p\Delta_{\min}^2} + \frac{k^2(d + p)^2}{p^2\Delta_{\min}^4},
\end{align*}
$$

where we have used (72), (75), and the condition that $\ell(z, z^*) \leq \tau \leq \Delta_{\min}^{\alpha p}$. Finally, for (28), the bound (72) leads to

$$
\frac{|H_j(a, b; z)|}{\Delta_j(a, b)^2} \leq \frac{1}{2} \left| \hat{\theta}_a(z^*) - \theta_a^* \right|^2 + \frac{1}{2} \left| \hat{\theta}_b(z^*) - \theta_b^* \right|^2 + \left| \theta_a^* - \theta_b^* \right| \left| \hat{\theta}_a(z^*) - \theta_a^* \right| \\
\leq \frac{k(d + \log p)}{p\Delta_{\min}^2} + \sqrt{\frac{k(d + \log p)}{p\Delta_{\min}^2}}.
$$

By taking maximum, we have obtained (26)-(28). The proof is complete. \hfill \square

**Proof of Lemma 4.2.** Note that

$$
\begin{align*}
\mathbb{P} \left( \langle \epsilon_j, \hat{\theta}_a(z^*) - \hat{\theta}_b(z^*) \rangle \leq -\frac{1 - \frac{\delta}{2}}{2} \left| \theta_a^* - \theta_b^* \right|^2 \right) \\
\leq \mathbb{P} \left( \langle \epsilon_j, \theta_a^* - \theta_b^* \rangle \leq -\frac{1 - \frac{\delta}{2}}{2} \left| \theta_a^* - \theta_b^* \right|^2 \right) \\
+ \mathbb{P} \left( \langle \epsilon_j, \hat{\theta}_a(z^*) - \theta_a^* \rangle \leq -\frac{\delta}{4} \left| \theta_a^* - \theta_b^* \right|^2 \right) \\
+ \mathbb{P} \left( -\langle \epsilon_j, \hat{\theta}_b(z^*) - \theta_b^* \rangle \leq -\frac{\delta}{4} \left| \theta_a^* - \theta_b^* \right|^2 \right),
\end{align*}
$$

where $\bar{\delta} = \frac{\delta}{2}$ is some sequence to be chosen later, and we need to bound the three terms on the right hand side of the above inequality respectively. For the first term, a standard
Gaussian tail bound gives
\[
\mathbb{P}\left( \epsilon_j, \theta_a^* - \theta_b^* \right) \leq \exp\left(-\frac{(1 - \delta - \bar{\delta})^2}{8} \|\theta_a^* - \theta_b^*\|^2 \right).
\]

To bound the second term, we note that
\[
\left\langle \epsilon_j, \hat{\theta}_a(z^*) - \theta_a^* \right\rangle = \frac{1}{p} \frac{\|\epsilon_j\|^2}{\sum_{l=1}^p 1 \{ z_l^* = a \}} + \frac{\sum_{l \in [p] \setminus \{ j \}} 1 \{ z_l^* = a \} \epsilon_j^T \epsilon_l}{\sum_{l=1}^p 1 \{ z_l^* = a \}} \geq \frac{\sum_{l \in [p] \setminus \{ j \}} 1 \{ z_l^* = a \} \epsilon_j^T \epsilon_l}{\sum_{l=1}^p 1 \{ z_l^* = a \}}.
\]

This implies
\[
\mathbb{P}\left( \epsilon_j, \hat{\theta}_a(z^*) - \theta_a^* \right) \leq -\frac{\bar{\delta}}{4} \|\theta_a^* - \theta_b^*\|^2
\]
\[
\leq \mathbb{P}\left( \sum_{l \in [p] \setminus \{ j \}} 1 \{ z_l^* = a \} \frac{\epsilon_j^T \epsilon_l}{\sum_{l=1}^p 1 \{ z_l^* = a \}} \leq -\frac{\bar{\delta}}{4} \|\theta_a^* - \theta_b^*\|^2
\]
\[
\leq \mathbb{P}\left( \sum_{l \in [p] \setminus \{ j \}} 1 \{ z_l^* = a \} \frac{\epsilon_j^T \epsilon_l}{\sum_{l=1}^p 1 \{ z_l^* = a \}} \leq -\frac{\bar{\delta}}{4} \|\theta_a^* - \theta_b^*\|^2 \right) + \mathbb{P}\left( \|\epsilon_l\|^2 > d + 2\sqrt{xd} + 2x \right)
\]
\[
\leq \mathbb{E}\left( \exp\left(-\frac{\bar{\delta}^2 \|\theta_a^* - \theta_b^*\|^2 \sum_{l=1}^p 1 \{ z_l^* = a \}}{32 \|\epsilon_l\|^2} \right) \right) + \mathbb{P}\left( \|\epsilon_l\|^2 > d + 2\sqrt{xd} + 2x \right)
\]
\[
\leq \exp\left(-\frac{\bar{\delta}^2 \|\theta_a^* - \theta_b^*\|^2 \alpha p}{32k (d + 2\sqrt{xd} + 2x)} \right) + \exp(-x).
\]

Choosing \( x = \bar{\delta} \|\theta_a^* - \theta_b^*\|^2 \sqrt{\alpha p/k} \), we have
\[
\mathbb{P}\left( \epsilon_j, \hat{\theta}_a(z^*) - \theta_a^* \right) \leq -\frac{\bar{\delta}}{4} \|\theta_a^* - \theta_b^*\|^2
\]
\[
\leq \exp\left(-C \frac{\bar{\delta}^2 \|\theta_a^* - \theta_b^*\|^2 \alpha p}{kd} \right) + \exp\left(-C \frac{\bar{\delta} \|\theta_a^* - \theta_b^*\|^2 \sqrt{p}}{k} \right).
\]

To bound the third term, we note that
\[
-\left\langle \epsilon_j, \hat{\theta}_b(z^*) - \theta_b^* \right\rangle = \frac{1}{p} \frac{\|\epsilon_j\|^2}{\sum_{l=1}^p 1 \{ z_l^* = b \}} - \frac{\sum_{l \in [p] \setminus \{ j \}} 1 \{ z_l^* = b \} \epsilon_j^T \epsilon_l}{\sum_{l=1}^p 1 \{ z_l^* = b \}}.
\]

30
and we then have
\[
\mathbb{P}\left( \left\langle \epsilon_j, \hat{\theta}_b(z^*) - \theta_b^* \right\rangle \leq -\frac{\delta}{4} \|\theta_a^* - \theta_b^*\|^2 \right)
\leq \mathbb{P}\left( -\frac{1}{2} \frac{\|\epsilon_j\|^2}{\sum_{l=1}^p 1_{\{z_l^* = b\}}} \leq -\frac{\delta}{8} \|\theta_a^* - \theta_b^*\|^2 \right)
\leq \mathbb{P}\left( \|\epsilon_j\|^2 \geq \frac{\delta}{8} \|\theta_a^* - \theta_b^*\|^2 2 \alpha p \right)
\leq \exp\left( -C \frac{\delta}{8} \|\theta_a^* - \theta_b^*\|^2 \frac{2}{k} \frac{p}{p} \right),
\]
under the condition \( \frac{\Delta_{\text{min}}^2}{\log k + kd/p} \to \infty \). Combining the bounds above, we have
\[
\mathbb{P}\left( \left\langle \epsilon_j, \hat{\theta}_a(z^*) - \hat{\theta}_b(z^*) \right\rangle \leq -\frac{1 - \delta - \bar{\delta}}{2} \|\theta_a^* - \theta_b^*\|^2 \right)
\leq \exp\left( -\frac{(1 - \delta - \bar{\delta})^2}{8} \|\theta_a^* - \theta_b^*\|^2 \right) + \exp\left( -C \frac{\delta}{8} \|\theta_a^* - \theta_b^*\|^2 \frac{2}{k} \right)
+ 2 \exp\left( -C \frac{\delta}{8} \|\theta_a^* - \theta_b^*\|^4 \frac{p}{k} \frac{p}{p} \right) + 2 \exp\left( -C \frac{\delta}{8} \|\theta_a^* - \theta_b^*\|^2 \frac{2}{k} \right)
\leq 6 \exp\left( -\frac{(1 - \delta - \bar{\delta})^2}{8} \|\theta_a^* - \theta_b^*\|^2 \right),
\]
where the last inequality above is obtained under the condition that \( \frac{\Delta_{\text{min}}^2}{\log k + kd/p} \to \infty \) and \( p/k \to \infty \), so that we can choose some \( \bar{\delta} = \delta_p = o(1) \) that is slowly diverging to zero.

Now we are ready to bound \( \xi_{\text{ideal}}(\delta) \). We first bound its expectation. We have
\[
\mathbb{E} \xi_{\text{ideal}}(\delta) = \sum_{j=1}^p \sum_{b \in [k] \setminus \{z_j^*\}} \|\theta_b^* - \theta_{z_j^*}^*\|^2 \mathbb{P}\left( \left\langle \epsilon_j, \hat{\theta}_{z_j^*}^*(z^*) - \hat{\theta}_b(z^*) \right\rangle \leq -\frac{1 - \delta - \bar{\delta}}{2} \|\theta_{z_j^*}^* - \theta_b^*\|^2 \right)
\leq 6 \sum_{j=1}^p \sum_{b \in [k] \setminus \{z_j^*\}} \|\theta_b^* - \theta_{z_j^*}^*\|^2 \exp\left( -\frac{(1 - \delta - \bar{\delta})^2}{8} \|\theta_{z_j^*}^* - \theta_b^*\|^2 \right).
\]
With \( \delta = \delta_p = o(1) \), we then have
\[
\mathbb{E} \xi_{\text{ideal}}(\delta_p) \leq \sum_{j=1}^p \sum_{b \in [k] \setminus \{z_j^*\}} \exp\left( -(1 + o(1)) \frac{\|\theta_{z_j^*}^* - \theta_b^*\|^2}{8} \right) \leq p \exp\left( -(1 + o(1)) \frac{\Delta_{\text{min}}^2}{8} \right),
\]
31
under the condition that \( \frac{\Delta_{\text{min}}^2}{\log k + ka/p} \to \infty \). Finally, by Markov’s inequality, we have

\[
\mathbb{P}(\xi_{\text{ideal}}(\delta_p) > \mathbb{E}\xi_{\text{ideal}}(\delta_p) \exp(\Delta_{\text{min}})) \leq \exp(-\Delta_{\text{min}}).
\]

In other words, with probability at least \( 1 - \exp(-\Delta_{\text{min}}) \), we have

\[
\xi_{\text{ideal}}(\delta_p) \leq \mathbb{E}\xi_{\text{ideal}}(\delta_p) \exp(\Delta_{\text{min}}).
\]

By the fact that \( \Delta_{\text{min}} \to \infty \), we have

\[
\mathbb{E}\xi_{\text{ideal}}(\delta_p) \exp(\Delta_{\text{min}}) \leq p \exp\left(- (1 + o(1)) \frac{\Delta_{\text{min}}^2}{8}\right),
\]

and thus the proof is complete.

Finally, we prove Proposition 4.1.

**Proof of Proposition 4.1.** We divide the proof into three steps.

**Step 1.** Define \( \hat{P} = \hat{U} \hat{U}^T Y \in \mathbb{R}^{d \times p} \) with \( \hat{P}_j \) being the jth column of \( \hat{P} \). Since \( \hat{P}_j = \hat{U} \mu_j \) for all \( j \in [p] \), we have \( \|\hat{P}_j - \hat{P}_{j'}\| = \|\mu_j - \mu_{j'}\| \) for all \( j, j' \in [p] \). This implies

\[
\min_{\theta_1, \ldots, \theta_k \in \mathbb{R}^k} \sum_{j=1}^p \|\hat{P}_j - \theta_j\|^2 = \min_{\beta_1, \ldots, \beta_k \in \mathbb{R}^k} \sum_{j=1}^p \|\hat{\mu_j} - \beta_j\|^2.
\]

Similarly, define \( \theta_a^{(0)} = \hat{U} \beta_a^{(0)} \) for all \( a \in [k] \), we have

\[
\sum_{j=1}^p \|\hat{P}_j - \theta_a^{(0)}\|_z^2 = \sum_{j=1}^p \|\hat{U} \beta_a^{(0)}\|_z^2 = \sum_{j=1}^p \|\hat{\mu}_j - \beta_a^{(0)}\|_z^2.
\]

Thus, (34) leads to

\[
\sum_{j=1}^p \|\hat{P}_j - \theta_a^{(0)}\|_z^2 \leq M \min_{\beta_1, \ldots, \beta_k \in \mathbb{R}^k} \sum_{j=1}^p \|\hat{P}_j - \theta_j\|^2.
\] (77)

That is, any \( z^{(0)} \in [k]^p \) that satisfies (34) with some \( \beta_1^{(0)}, \ldots, \beta_k^{(0)} \) also satisfies (77) with some \( \theta_1^{(0)}, \ldots, \theta_k^{(0)} \).

**Step 2.** It is sufficient to study any \( \theta_1^{(0)}, \ldots, \theta_k^{(0)} \in \mathbb{R}^d \) and \( z^{(0)} \in [k]^p \) that satisfies (77). Let us define \( P^* = \mathbb{E}Y \), and we have \( P^*_j = \theta_{z_j}^* \) according to the model assumption. We first give an error bound for \( \|\hat{P} - P^*\|_F^2 \). Since \( \hat{P} \) is the rank-k approximation of \( Y \), we have \( \|Y - \hat{P}\|_F^2 \leq \|Y - P^*\|_F^2 \), which implies that \( \|\hat{P} - P^*\|_F^2 \leq 4 \max_{\{A \in \mathbb{R}^{d \times p}, \|A\| \leq 1, \text{rank}(A) \leq 2k\}} \|A, Y - P^*\|_2^2 \). Use a standard random matrix theory result [60], we have \( \|Y - P^*\|_F^2 \leq p + d \) with probability
at least $1 - e^{-C'(d + p)}$. For any $A$ such that $\|A\|_F \leq 1$ and $\text{rank}(A) \leq 2k$, its singular value decomposition can be written as $A = \sum_{l=1}^{2k} d_l u_l v_l^T$, where $\sum_{l=1}^{2k} d_l^2 \leq 1$. Thus, we have

$$|\langle A, Y - P^* \rangle|^2 = \sum_{l=1}^{2k} |d_l u_l^T (Y - P^*) v_l|^2 \leq \sum_{l=1}^{2k} |u_l^T (Y - P^*) v_l|^2 \leq 2k \|Y - P^*\|^2 \lesssim k(p + d).$$

Taking maximum over $A$, we have $\|\hat{P} - P^*\|_F^2 \lesssim k(p + d)$ with probability at least $1 - e^{-C'(d + p)}$.

By (77), we have

$$\sum_{j=1}^p \|\hat{P}_j - \theta_z^{(0)}\|^2 \leq M \|\hat{P} - P^*\|_F^2 \lesssim Mk(p + d),$$

and as a consequence,

$$\sum_{j=1}^p \|\theta_z^{*} - \theta_z^{(0)}\|^2 \leq 2 \sum_{j=1}^p \left(\|\hat{P}_j - \theta_z^{(0)}\|^2 + \|\theta_z^{*} - \hat{P}_j\|^2\right) \lesssim (M + 1)k(p + d). \tag{78}$$

Define

$$S = \left\{ j \in [p] : \|\theta_z^{*} - \theta_z^{(0)}\| \geq \frac{\Delta_{\min}}{2} \right\},$$

and we have

$$|S| \leq \frac{\sum_{j=1}^p \|\theta_z^{*} - \theta_z^{(0)}\|^2}{\left(\frac{\Delta_{\min}}{2}\right)^2} \lesssim \frac{(M + 1)k(p + d)}{\Delta_{\min}^2}.$$

We are now going to show that all the data points in $S^c$ are all correctly clustered. We define

$$C_a = \{ j \in [p] : z_j^* = a, j \in S^c \},$$

for all $a \in [k]$. Under the assumption $\Delta_{\min}^2/((M + 1)k^2(1 + d/p)) \rightarrow \infty$, we have

$$|S| = o(p/k). \tag{79}$$

We have the following arguments:

- For each $a \in [k]$, $C_a$ cannot be empty, as

$$|C_a| \geq |\{ j \in [p] : z_j^* = a \}| - |S| \geq \frac{|\{ j \in [p] : z_j^* = a \}|}{2} \geq \frac{\alpha p}{2k}. \tag{80}$$

- For each pair $a, b \in [k], a \neq b$, there cannot exist some $j \in C_a, j' \in C_b$ such that $z_j^{(0)} = z_{j'}^{(0)}$. Otherwise $\theta_z^{(0)} = \theta_{z_j^{(0)}}$ would imply

$$\|\theta_z^* - \theta_b^*\| = \|\theta_z^{*} - \theta_{z_{j'}^{*}}\| \leq \|\theta_z^{*} - \theta_{z_{j'}^{(0)}}\| + \|\theta_{z_{j'}^{(0)}} - \theta_z^{(0)}\| + \|\theta_z^{(0)} - \theta_{z_{j'}^{(0)}}\| \leq \Delta_{\min},$$

contradicting the definition of $\Delta_{\min}$. 

33
Since $z_j^{(0)}$ can only take values in $[k]$, we conclude that \{z_j^{(0)} : j \in C_a\} contains only one and different element for all $a \in [k]$. That is, there exists a permutation $\pi_0 \in \Pi_k$, such that

$$z_j^{(0)} = \pi_0(z_j^*), \quad (81)$$

for all $j \in S^c$.

Step 3. The last step is to establish an upper bound for $\ell(\pi_0^{-1} \circ z^{(0)}, z^*)$. By (78), (80) and (81), we have

$$\|\theta^*_a - \theta^{(0)}_{\pi_0(a)}\|^2 = \sum_{j \in C_a} \frac{\|\theta^*_j - \theta^{(0)}_{\pi_0^{-1}(z_j^{(0)})}\|^2}{|C_a|} \leq \frac{\sum_{j=1}^p \|\theta^*_j - \theta^{(0)}_{\pi_0^{-1}(z_j^{(0)})}\|^2}{|C_a|} \lesssim (M + 1)k^2 \left(1 + \frac{d}{p}\right),$$

for all $a \in [k]$. As a result, together with (78), (79) and (81), we have

$$\ell(\pi_0^{-1} \circ z^{(0)}, z^*) = \sum_{j \in [p]} \|\theta^*_j - \theta^*_{\pi_0^{-1}(z_j^{(0)})}\|^2 = \sum_{j \in [p]} \|\theta^*_j - \theta^*_{\pi_0^{-1}(z_j^{(0)})}\|^2 \mathbb{1}_{\{z_j^* \neq \pi_0^{-1}(z_j^{(0)})\}}$$

$$\leq 2 \sum_{j \in [p]} \|\theta^*_j - \theta^{(0)}_{\pi_0^{-1}(z_j^{(0)})}\|^2 + \mathbb{1}_{\{z_j^* \neq \pi_0^{-1}(z_j^{(0)})\}}$$

$$\leq 2 \sum_{j \in [p]} \|\theta^*_j - \theta^{(0)}_{\pi_0^{-1}(z_j^{(0)})}\|^2 + \max_{a \in [k]} \|\theta^*_a - \theta^*_{\pi_0^{-1}(a)}\|^2 \sum_{j=1}^p \mathbb{1}_{\{z_j^* \neq \pi_0^{-1}(z_j^{(0)})\}}$$

$$\leq 2 \sum_{j \in [p]} \|\theta^*_j - \theta^{(0)}_{\pi_0^{-1}(z_j^{(0)})}\|^2 + |S| \max_{a \in [k]} \|\theta^*_a - \theta^*_{\pi_0(a)}\|^2 \lesssim (M + 1)k (p + d).$$

The proof is complete. \qed

7.3 Proofs in Section 5

This section collects the proofs of Lemma 5.1, Lemma 5.2, and Proposition 5.1. The conclusions of Theorem 5.1 and Corollary 5.1 are direct consequences of Theorem 3.1, and thus we omit their proofs. We first need the following technical lemma.

Lemma 7.3. Consider i.i.d. random variables $w_{ij} \sim \mathcal{N}(0,1)$ for $1 \leq i \neq j \leq p$. Then, for
any constant $C' > 0$, there exists some constant $C > 0$ only depending on $C'$ such that

$$
\max_{a \in \mathbb{R}^p} \left| \frac{\sum_{1 \leq i \neq j \leq p} (a_i - a_j)w_{ij}}{\sqrt{\sum_{1 \leq i \neq j \leq p} (a_i - a_j)^2}} \right| \leq C\sqrt{p}, \quad (82)
$$

$$
\sum_{j=1}^{p} \left( \frac{1}{\sqrt{2(p-1)}} \sum_{i \in [p] \setminus \{j\}} (w_{ji} - w_{ij}) \right)^2 \leq Cp, \quad (83)
$$

$$
\max_{j \in [p]} \left| \frac{1}{\sqrt{2(p-1)}} \sum_{i \in [p] \setminus \{j\}} (w_{ji} - w_{ij}) \right| \leq C\sqrt{\log p}, \quad (84)
$$

with probability at least $1 - (C'p)^{-1}$. We have used the convention that $0/0 = 0$.

**Proof.** To bound the first inequality, we define

$$
\mathcal{A} = \left\{ A = \{a_{ij}\}_{(i,j) \in [p]^2} : a_{ij} = a_i - a_j \text{ for some } a \in \mathbb{R}^p, \|A\|_F \leq 1 \right\},
$$

and

$$
\mathcal{B} = \left\{ B = \{b_{ij}\}_{(i,j) \in [p]^2} : \text{rank}(B) \leq 2, \|B\|_F \leq 1 \right\}.
$$

Then, we have $\mathcal{A} \subset \mathcal{B}$, and

$$
\max_{a \in \mathbb{R}^p} \left| \frac{\sum_{1 \leq i \neq j \leq p} (a_i - a_j)w_{ij}}{\sqrt{\sum_{1 \leq i \neq j \leq p} (a_i - a_j)^2}} \right| = \max_{A \in \mathcal{A}} | \langle A, W \rangle |.
$$

By Lemma 3.1 of [14], the covering number of the low-rank set $\mathcal{B}$ is bounded by $e^{O(p)}$, which further implies the same covering number bound for $\mathcal{A}$ by the fact that $\mathcal{A} \subset \mathcal{B}$. In other words, there exists $A_1, \ldots, A_m \in \mathcal{A}$, such that $m \leq e^{C_1p}$, and for any $A \in \mathcal{A}$, $\min_{1 \leq l \leq m} \|A_l - A\|_F \leq 1/2$. Let us choose any $A \in \mathcal{A}$, and then let $A_l$ be the matrix in the covering set that satisfies $\|A_l - A\|_F \leq 1/2$. We then have

$$
| \langle A, W \rangle | \leq \|A - A_l\|_F \left\langle \frac{A - A_l}{\|A - A_l\|_F}, W \right\rangle + | \langle A_l, W \rangle | \leq \frac{1}{2} \max_{A \in \mathcal{A}} | \langle A, W \rangle | + | \langle A_l, W \rangle |,
$$

which implies

$$
\max_{A \in \mathcal{A}} | \langle A, W \rangle | \leq \frac{1}{2} \max_{A \in \mathcal{A}} | \langle A, W \rangle | + \max_{1 \leq l \leq m} | \langle A_l, W \rangle |.
$$

After rearrangement, we get $\max_{A \in \mathcal{A}} | \langle A, W \rangle | \leq 2\max_{1 \leq l \leq m} | \langle A_l, W \rangle |$. Then, the conclusion follows by a standard union bound argument.

For the second inequality, we use the notation $r_j = \frac{1}{\sqrt{2(p-1)}} \sum_{i \in [p] \setminus \{j\}} (w_{ji} - w_{ij})$. It is clear that $r_j \sim \mathcal{N}(0, 1)$ for all $j \in [p]$, and thus we have $E \left( \sum_{j=1}^{p} r_j^2 \right) = p$. We then calculate the variance. We have

$$
\text{Var} \left( \sum_{j=1}^{p} r_j^2 \right) = \sum_{j=1}^{p} \sum_{i=1}^{p} E(r_j^2 - 1)(r_i^2 - 1).
$$
For $j = l$, we get $\mathbb{E}(r_j^2 - 1)^2 = 2$. For $j \neq l$, we have $\mathbb{E}(r_j^2 - 1)(r_l^2 - 1) = \mathbb{E}r_j^2r_l^2 - 1$, and
\[
\mathbb{E}r_j^2r_l^2 = \frac{1}{4(p - 1)^2}\mathbb{E}\left(\sum_{i \in [p] \setminus \{l\}} (w_{ji} - w_{ij} + (w_{jl} - w_{lj}))^2 \sum_{i \in [p] \setminus \{j\}} (w_{li} - w_{il}) + (w_{lj} - w_{lj})\right).
\]
Since the three terms $\sum_{i \in [p] \setminus \{l\}} (w_{ji} - w_{ij})$, $\sum_{i \in [p] \setminus \{j\}} (w_{li} - w_{il})$ and $(w_{lj} - w_{lj})$ are independent, we can expand the above display and calculate the expectation of each term in the expansion, and we get
\[
\mathbb{E}r_j^2r_l^2 = \frac{4(p - 2)^2 + 4 + 8(p - 2)}{4(p - 1)^2} = 1.
\]
Therefore, $\text{Var}\left(\sum_{j=1}^p r_j^2\right) = 2p$, and the desired conclusion is obtained by Chebyshev's inequality. Finally, the last inequality is a direct consequence of a union bound argument.

Now we are ready to state the proofs of Lemma 5.1 and Lemma 5.2.

**Proof of Lemma 5.1.** For any $z \in [p]^p$ such that $\ell(z, z^*) \leq \tau = o(p^2(\beta^*)^2)$, we have
\[
\sum_{1 \leq i \neq j \leq p} (z_i - z_i^* - z_j + z_j^*)^2 \leq 4(p - 1) \sum_{j=1}^p (z_j - z_j^*)^2 \leq \frac{2}{(\beta^*)^2}\ell(z, z^*) = o(p^2).
\]
For any $z^* \in \mathcal{R}$, we have $\sum_{1 \leq i \neq j \leq p} (z_i^* - z_j^*)^2 \leq p^4$. Moreover, by the definition of $\mathcal{R}$, there exists a $\tilde{z} \in \Pi_p$ such that $\|z^* - \tilde{z}\|^2 \leq c_p$. This implies
\[
\left(\frac{1}{p} \sum_{j=1}^p z_j^* - \frac{p - 1}{2}\right)^2 = \left(\frac{1}{p} \sum_{j=1}^p z_j^* - \sum_{j=1}^p \tilde{z}_j\right)^2 \leq \frac{1}{p} \|z^* - \tilde{z}\|^2 \leq \frac{c_p}{p} = o(1),
\]
and
\[
\left|\sum_{j=1}^p (z_j^*)^2 - \sum_{j=1}^p z_j^*\right| = \|z^*\|^2 - \|\tilde{z}\|^2 \leq \|z^* - \tilde{z}\| (\|z^*\| + \|\tilde{z}\|) \lesssim c_p^{1/2}p^{1.5} = o(p^2).
\]
Thus,
\[
\sum_{1 \leq i \neq j \leq p} (z_i^* - z_j^*)^2 = 2p \sum_{j=1}^p (z_j^*)^2 - 2 \left(\sum_{j=1}^p z_j^*\right)^2 \geq 2p \left(\sum_{j=1}^p j^2 - o(p^2)\right) - (1 + o(1))2 \left(\sum_{j=1}^p j\right)^2 \geq \frac{p^4}{12}.
\]
Therefore,
\[
\sum_{1 \leq i \neq j \leq p} (z_i - z_j)^2 \geq \frac{1}{2} \sum_{1 \leq i \neq j \leq p} (z_i^* - z_j^*)^2 - \sum_{1 \leq i \neq j \leq p} (z_i - z_j^* - z_i^* + z_j^*)^2 \\
\geq \frac{p^4}{24} - o(p^2) \\
\geq \frac{p^4}{25},
\]
where the last inequality assumes \( p \) is sufficiently large. We then introduce more notations. We define
\[
\beta(z) = \beta^* \frac{\sum_{1 \leq i \neq j \leq p} (z_i - z_j)(z_i^* - z_j^*)}{\sum_{1 \leq i \neq j \leq p} (z_i - z_j)^2},
\]
and
\[
\bar{w}(z) = \frac{\sum_{1 \leq i \neq j \leq p} (z_i - z_j)w_{ij}}{\sum_{1 \leq i \neq j \leq p} (z_i - z_j)^2}.
\]
We write \( w_{ij} = Y_{ij} - \beta^*(z_i^* - z_j^*) \) so that \( \epsilon_j = \frac{1}{\sqrt{2(p-1)}} \sum_{i \in [p] \setminus \{j\}} (w_{ji} - w_{ij}) \). We consider the event that the three inequalities (82)-(84) hold. We first derive bounds for \(|\hat{\beta}(z^*) - \beta^*|, |\beta(z) - \beta^*|\) and \(|\bar{w}(z) - \bar{w}(z^*)|\). By (82), we have
\[
|\hat{\beta}(z^*) - \beta^*| = \frac{\sum_{1 \leq i \neq j \leq p} (z_i^* - z_j^*) w_{ij}}{\sum_{1 \leq i \neq j \leq p} (z_i^* - z_j^*)} \leq p^{-1.5}. \tag{89}
\]
By (85) and (88), we have
\[
|\beta(z) - \beta^*| = \left| \beta^* \frac{\sum_{1 \leq i \neq j \leq p} (z_i - z_j)(z_i^* - z_j^* - z_i + z_j)}{\sum_{1 \leq i \neq j \leq p} (z_i - z_j)^2} \right| \\
\leq \frac{25|\beta^*|}{p^4} \sqrt{\sum_{1 \leq i \neq j \leq p} (z_i - z_j)^2} \sqrt{\sum_{1 \leq i \neq j \leq p} (z_i^* - z_j^* - z_i + z_j)^2} \\
\leq \frac{25|\beta^*|}{p^2} \sqrt{\frac{2}{(\beta^*)^2} \ell(z, z^*)} \\
\leq \frac{\sqrt{\ell(z, z^*)}}{p^2}. \tag{90}
\]
Next, we bound \(|\bar{w}(z) - \bar{w}(z^*)|\). We have
\[
|\bar{w}(z) - \bar{w}(z^*)| \leq \frac{\sum_{1 \leq i \neq j \leq p} (z_i - z_j - z_i^* + z_j^*) w_{ij}}{\sum_{1 \leq i \neq j \leq p} (z_i - z_j)^2} \\
+ \frac{\sum_{1 \leq i \neq j \leq p} (z_i - z_j)^2 - \sum_{1 \leq i \neq j \leq p} (z_i^* - z_j^*)^2}{\sum_{1 \leq i \neq j \leq p} (z_i - z_j)^2} \sqrt{\sum_{1 \leq i \neq j \leq p} (z_i^* - z_j^*)^2} \frac{\sum_{1 \leq i \neq j \leq p} (z_i^* - z_j^*) w_{ij}}{\sqrt{\sum_{1 \leq i \neq j \leq p} (z_i^* - z_j^*)^2}}.
\]
We bound the two terms on the right hand side of the above inequality separately. The first term can be bounded by
\[
\frac{\sum_{1 \leq i \neq j \leq p} (z_i - z_j)^2}{\sqrt{\sum_{1 \leq i \neq j \leq p} (z_i - z_j)^2}} \leq \frac{25}{p^4} \sqrt{\frac{2}{(\beta^*)^2} \ell(z, z^*)} \frac{\sum_{1 \leq i \neq j \leq p} (z_i - z_j - z_i^* + z_j^*) w_{ij}}{\sqrt{\sum_{1 \leq i \neq j \leq p} (z_i - z_i^* - z_j + z_j^*)^2}} \lesssim \frac{\sqrt{p \ell(z, z^*)}}{|\beta^*| p^4},
\]
where we have used the inequalities (82), (85), and (88). By (82) and (88), the second term can be bounded by
\[
C_1 p^{-5.5} \left| \sum_{1 \leq i \neq j \leq p} (z_i - z_j)^2 - \sum_{1 \leq i \neq j \leq p} (z_i^* - z_j^*)^2 \right| \leq \frac{1}{p^{(p-1)}} \sum_{i \in [p]} |w_{ji} - w_{ij}| \lesssim \frac{\sqrt{p \ell(z, z^*)}}{|\beta^*| p^4},
\]
where we have used (85) in the last inequality. Combining the two bounds, we obtain
\[
|\tilde{w}(z) - \tilde{w}(z^*)| \lesssim \frac{\sqrt{p \ell(z, z^*)}}{|\beta^*| p^4}. \tag{91}
\]
From (89), (90) and (91), we can further derive
\[
|\tilde{\beta}(z) - \tilde{\beta}(z^*)| \leq |\beta(z) - \beta^*| + |\tilde{w}(z) - \tilde{w}(z^*)| \lesssim \frac{\sqrt{\ell(z, z^*)}}{p^2}, \tag{92}
\]
under the condition $p(\beta^*)^2 \geq 1$.

We are ready to prove (43)-(45). Recall that $\epsilon_j = \frac{1}{\sqrt{2(p-1)}} \sum_{i \in [p] \setminus \{j\}} (w_{ji} - w_{ij})$, and we have
\[
\sum_{j=1}^{p} \epsilon_j^2 \leq \sum_{j=1}^{p} \left( \frac{1}{\sqrt{2(p-1)}} \sum_{i \in [p] \setminus \{j\}} (w_{ji} - w_{ij}) \right)^2 \lesssim p, \tag{93}
\]
by (83). Moreover, from (41), since $z^* \in \mathcal{R}$, we have

$$\Delta_j(a, b)^2 = (1 + o(1)) \frac{2p^2(\beta^*)^2}{p - 1} (a - b)^2. \tag{94}$$

Thus,

$$\sum_{j=1}^{p} \max_{b \in [k] \setminus \{z_j^*\}} \frac{F_j(z_j^*, b; z)^2 \| \mu_j(B^*, b) - \mu_j(B^*, z_j^*) \|^2}{\Delta_j(z_j^*, b)^4 \ell(z, z^*)}$$

$$= \frac{|\tilde{\beta}(z) - \tilde{\beta}(z^*)|^2}{|\beta^*|^2 \ell(z, z^*)} \sum_{j=1}^{p} \epsilon_j^2$$

$$\lesssim \frac{1}{p^2},$$

where we have used (92), (93), (94) and the condition $p(\beta^*)^2 \geq 1$ in the last inequality.

Taking maximum, we obtain (43). For (44), we note that

$$|G_j(a, b; z)| \leq \frac{p^2}{p - 1} \left| \left( a - \frac{p + 1}{2} \right)^2 - \left( b - \frac{p + 1}{2} \right)^2 \right| |\tilde{\beta}(z) - \tilde{\beta}(z^*)|^2$$

$$+ \frac{2p^2}{p - 1} \left| a - \frac{p + 1}{2} \right|^2 |\tilde{\beta}(z) - \tilde{\beta}(z^*)| |\tilde{\beta}(z^*) - \beta^*|$$

$$+ \frac{2p^2}{p - 1} \left| b - \frac{p + 1}{2} \right| |\tilde{\beta}(z) - \tilde{\beta}(z^*)| \left| b - \frac{p + 1}{2} \right) \tilde{\beta}(z^*) - \left( a - \frac{p + 1}{2} \right) \beta^*|$$

$$+ \frac{2p^3}{p - 1} |\beta^*| |\tilde{\beta}(z) - \tilde{\beta}(z^*)| \left| \frac{1}{p} \sum_{j=1}^{p} z_j^* - \frac{p + 1}{2} \right|$$

$$\leq \frac{p^4}{p - 1} |\tilde{\beta}(z) - \tilde{\beta}(z^*)|^2 + \frac{4p^4}{p - 1} |\tilde{\beta}(z) - \tilde{\beta}(z^*)| |\tilde{\beta}(z^*) - \beta^*|$$

$$+ \frac{4p^3}{p - 1} |a - b| |\beta^*| |\tilde{\beta}(z) - \tilde{\beta}(z^*)|.$$
obtain (44). Finally, for (45), we have

\[
\frac{|H_j(a, b; z)|}{\Delta_j(a, b)^2} \leq \frac{1}{2|\beta^*|^2} (\tilde{\beta}(z^*) - \beta^*)^2 \left( a - \frac{p + 1}{2} \right)^2 + \frac{1}{2|\beta^*|^2} (\tilde{\beta}(z^*) - \beta^*)^2 \left( b - \frac{p + 1}{2} \right)^2 \\
+ 2p \frac{|\tilde{\beta}(z^*) - \beta^*|}{|\beta^*|} \\
\lesssim \frac{1}{\sqrt{p}|\beta^*|}.
\]

where we have used (89). We thus obtain (45) by taking maximum. The proof is complete. \(\square\)

**Proof of Lemma 5.2.** By (94), there exists some \(\delta' = \delta'_p = o(1)\), such that

\[
1 \{ \{ \varepsilon_j \nu_j(\tilde{B}(z^*), z_j^*) - \nu_j(\tilde{B}(z^*), b) \leq -\frac{1}{2} \Delta_j(z_j^*, h) \} \}
\leq 1 \{ \{ \varepsilon_j \sqrt{2p(p-1)} \tilde{\beta}(z^*) (z_j^* - b) \leq -\frac{1}{2} \delta' \sqrt{2p(p-1)} (z_j^* - b) \} \}
\leq 1 \{ \{ \varepsilon_j \sqrt{2p(p-1)} \beta^*(z_j^* - b) \leq -\frac{1}{2} \delta' \sqrt{2p(p-1)} (z_j^* - b) \} \}
+ 1 \{ \{ \varepsilon_j \sqrt{2p(p-1)} (\tilde{\beta}(z^*) - \beta^*) (z_j^* - b) \leq -\frac{1}{2} \delta' \sqrt{2p(p-1)} (z_j^* - b) \} \}.
\]

By (84), (89), and \(p(\beta^*)^2 \to \infty\), we have

\[
\max_{j \in [p]} \left| \varepsilon_j \sqrt{2p(p-1)} (\tilde{\beta}(z^*) - \beta^*) (z_j^* - b) \right| = o \left( \frac{\sqrt{\log p}}{p} \right),
\]

with probability at least \(1 - p^{-1}\). Therefore, we can set \(\delta = \delta_p\) for some sequence \(\delta_p \to 0\) and \(\delta_p \geq \sqrt{\log p} \) \(\frac{1}{p} \), and then

\[
1 \{ \{ \varepsilon_j \sqrt{2p(p-1)} (\tilde{\beta}(z^*) - \beta^*) (z_j^* - b) \leq -\frac{1}{2} \delta' \sqrt{2p(p-1)} (z_j^* - b) \} \} = 0,
\]

for all \(j \in [p]\) with probability at least \(1 - p^{-1}\). This immediately implies that \(\xi_{\text{ideal}}(\delta_p) \leq \tilde{\xi}_{\text{ideal}}(\delta_p + \delta'_p + \delta_p)\) with high probability, where

\[
\tilde{\xi}_{\text{ideal}}(\delta_p + \delta'_p + \delta_p) = \frac{2p^2(\beta^*)^2}{p-1} \sum_{j=1}^{p} \sum_{b \in [p] \setminus \{z_j^*\}} (z_j^* - b)^2 \left\{ \varepsilon_j \sqrt{2p(p-1)} \beta^*(z_j^* - b) \leq -\frac{1}{2} \delta - \delta'_p - \delta_p \sqrt{2p(p-1)} (z_j^* - b) \right\}.
\]

A standard Gaussian tail bound implies

\[
\mathbb{E}[\xi_{\text{ideal}}(\delta_p + \delta'_p + \delta_p)]
\leq \frac{2p^2(\beta^*)^2}{p-1} \sum_{j=1}^{p} \sum_{b \in [p] \setminus \{z_j^*\}} (z_j^* - b)^2 \mathbb{P} \left( \mathcal{N}(0, 1) \leq -\frac{1}{2} \delta - \delta'_p - \delta_p \sqrt{2p(p-1)} (z_j^* - b) \right)
\leq \sum_{j=1}^{p} \sum_{l=1}^{\infty} \frac{4p^2(\beta^*)^2}{p-1} \exp \left( - \left( \frac{1 - \delta - \delta'_p - \delta_p}{2} \sqrt{2p(p-1)} (z_j^* - b) \right) ^2 \right)
\leq p \exp \left( -(1 + o(1)) \frac{p(\beta^*)^2}{4} \right),
\]

40
where we have used the conditions $p(\beta^*)^2 \to \infty$ and $\delta_p + \delta_p' + \tilde{\delta}_p = o(1)$ in the last inequality. Finally, by Markov’s inequality, with probability at least $1 - \exp\left(\frac{-p(\beta^*)^2}{4}\right)$, we have

$$\tilde{\xi}_{\text{ideal}}(\delta_p + \delta_p' + \tilde{\delta}_p) \leq \mathbb{E}\tilde{\xi}_{\text{ideal}}(\delta_p + \delta_p' + \tilde{\delta}_p) \exp\left(\sqrt{p(\beta^*)^2}\right)$$

$$\leq p \exp\left(\frac{-1 + o(1)}{4}\right),$$

as $p(\beta^*)^2 \to \infty$. Since $\xi_{\text{ideal}}(\delta_p) \leq \tilde{\xi}_{\text{ideal}}(\delta_p + \delta_p' + \tilde{\delta}_p)$, the proof is complete. \qed

Finally, we state the proof of Proposition 5.1.

**Proof of Proposition 5.1.** Note that we have the following fact. Consider any $x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{R}^m$. Let $y = \arg\min_{z \in \Pi_p} \sum_{i=1}^p (x_i - z_i)^2$. Then for any pair $(i, j)$ such that $x_i < x_j$, we must have $y_i < y_j$. Otherwise if $y_i > y_j$, since $((x_i - y_i)^2 + (x_j - y_j)^2) - ((x_i - y_j)^2 + (x_j - y_j)^2) = -2(x_i - x_j)(y_i - y_j) > 0$, we can always swap $y_i$ and $y_j$ to make $\sum_{i=1}^p (x_i - y_i)^2$ strictly smaller. This indicates that $y$ preserves the order of $x$.

As a result, since a linear transformation does not change the rank, we can write $z^{(0)}$ as

$$z^{(0)} = \arg\min_{z \in \Pi_p} \sum_{j=1}^p \left(\frac{\sqrt{2(p-1)}}{2p\beta^*} T_j + \frac{p+1}{2} - z_j^{(0)}\right)^2. \tag{95}$$

Since $z^* \in \mathcal{R}$, there exists some $\bar{z} \in \Pi_p$ such that $L_2(\bar{z}, z^*) = o(1)$. By (95),

$$\sum_{j=1}^p \left(\frac{\sqrt{2(p-1)}}{2p\beta^*} T_j + \frac{p+1}{2} - z_j^{(0)}\right)^2 \leq \sum_{j=1}^p \left(\frac{\sqrt{2(p-1)}}{2p\beta^*} T_j + \frac{p+1}{2} - \bar{z}_j\right)^2.$$

We then have

$$L_2(z^{(0)}, \bar{z}) = p^{-1} \sum_{j=1}^p (z_j^{(0)} - \bar{z}_j)^2$$

$$\leq 2p^{-1} \sum_{j=1}^p \left(\frac{\sqrt{2(p-1)}}{2p\beta^*} T_j + \frac{p+1}{2} - z_j^{(0)}\right)^2 + 2p^{-1} \sum_{j=1}^p \left(\frac{\sqrt{2(p-1)}}{2p\beta^*} T_j + \frac{p+1}{2} - \bar{z}_j\right)^2$$

$$\leq 4p^{-1} \sum_{j=1}^p \left(\frac{\sqrt{2(p-1)}}{2p\beta^*} T_j + \frac{p+1}{2} - \bar{z}_j\right)^2$$

$$\leq 12p^{-1} \sum_{j=1}^p \left(\frac{\sqrt{2(p-1)}}{2p\beta^*} T_j + \frac{1}{p} \sum_{j=1}^p z_j^* - z_j^*\right)^2 + 12p^{-1} \sum_{j=1}^p (\bar{z}_j - z_j^*)^2 + 12 \left(\frac{1}{p} \sum_{j=1}^p z_j^* - \frac{p+1}{2}\right)^2$$

$$= \frac{6(p-1)}{p^3\beta^*} \sum_{j=1}^p z_j^2 + 12L_2(\bar{z}, z^*) + 12 \left(\frac{1}{p} \sum_{j=1}^p z_j^* - \frac{p+1}{2}\right)^2,$$
where the last equation is due to the fact that \( T_j = \mu_j(B^*, z_j^*) + \epsilon_j \). By (93), (86), and \( L_2(\zeta, z^*) = o(1) \), we have

\[
L_2(z^{(0)}, \zeta) \lesssim \frac{1}{p(\beta^*)^2} + o(1),
\]

with probability at least \( 1 - p^{-1} \). By \( L(z^{(0)}, z^*) \leq 2L(z^{(0)}, \zeta) + 2L_2(\zeta, z^*) \), we have

\[
L_2(z^{(0)}, z^*) \lesssim \frac{1}{p(\beta^*)^2} + o(1),
\]

with high probability. When \( p(\beta^*)^2 \to \infty \), we clearly have \( L_2(z^{(0)}, z^*) = o(1) \). When \( p(\beta^*)^2 = O(1) \), we have \( L_2(z^{(0)}, z^*) \lesssim \min\left( p^2, \frac{1}{p(\beta^*)^2} \right) \), where \( L_2(z^{(0)}, z^*) \lesssim p^2 \) is by the definition of the loss. \( \square \)

### 7.4 Proofs in Section 6

In this section, we will prove results presented in Section 6. Most efforts will be devoted to the proofs of Lemma 6.1, Lemma 6.2, and Proposition 6.1. With these results established, the conclusions of Theorem 6.1, Theorem 6.2, and Corollary 6.1 easily follow.

Let us introduce some more notation to facilitate the proofs. Given some vector \( v \in \mathbb{R}^d \), some matrix \( A \in \mathbb{R}^{d \times d} \) and some set \( S \subseteq [d] \), we use \( v_S \in \mathbb{R}^{|S|} \) for the sub-vector \( (v_j : j \in S) \) and \( A_S \in \mathbb{R}^{d \times |S|} \) for the sub-matrix \( (A_{ij} : i \in [d'], j \in S) \). We denote \( \text{span}(A) \) to be the space spanned by the columns of \( A \). For any \( j \in [d] \), we denote \( [v]_j = v_j \) to be the \( j \)th coordinate of \( v \). We also write \( \phi_S : [d] \to [|S|] \) for the map that satisfies \( v_j = [v_S]_{\phi_S(j)} \). The domain of the map \( \phi_S \) can also be extended to sets so that for any \( S' \subseteq S \), we can write \( v_{S'} = [v_S]_{\phi_S(S')} \). For any \( j \in S \), we write \( z_{-j} = S \setminus \{j\} \). We use \( \mathbf{I}_d \) for the \( d \times d \) identity matrix, and sometimes just write \( I \) for simplicity if the dimension is clear from the context.

Given any square matrix \( A \in \mathbb{R}^{d \times d} \), we use \( \text{diag}(A) \) for the diagonal matrix whose diagonal entries are identical to those of \( A \). For two random elements \( X \) and \( Y \), we write \( X \overset{d}{=} Y \) if their distributions are identical, and \( X \perp Y \) if they are independent of each other.

We first state and prove three technical lemmas.

**Lemma 7.4.** Assume \( s \log p \leq n \). Consider a random matrix \( X \in \mathbb{R}^{n \times p} \) with i.i.d. entries \( X_{ij} \sim \mathcal{N}(0, 1) \), an independent \( w \sim \mathcal{N}(0, I_n) \), some \( S^* \subseteq [p] \) satisfying \( |S^*| = s \), and some \( \beta^* \in \mathbb{R}^p \). For any \( S \subseteq [p] \), denote \( P_S = X_S (X_S^T X_S)^{-1} X_S^T \) to be the projection matrix onto the subspace \( \text{span}(X_S) \). We also use the notation \( P_j = X_j X_j^T / \|X_j\|^2 \), where \( X_j \) represents the \( j \)th column of \( X \). Then, for any constants \( C_0, C' > 0 \), there exists some constant \( C > 0 \)
only depending on $C_0,C'$ such that

$$
\max_{j \in [p]} \left\| X_j \right\|^2 - n \leq C \sqrt{n \log p} \leq n/2, \quad \tag{96}
$$

$$
\max_{S \subseteq [p]: |S| \leq 2C_0s} \left\| (X_S^T X_S)^{-1} \right\| \leq \frac{C}{n}, \quad \tag{97}
$$

$$
\max_{S,T \subseteq [p]: S \cap T = \emptyset, |S|, |T| \leq 2C_0s} \left\| (X_S^T X_S)^{-1} X_S^T X_T \right\|^2 \leq \frac{C s \log p}{n}, \quad \tag{98}
$$

$$
\max_{S,T \subseteq [p]: S \cap T = \emptyset, |S|, |T| \leq 2C_0s} \frac{1}{|T|} \left\| (X_T^T (I - P_S) X_T)^{-1} X_T^T (I - P_S) w \right\|^2 \leq \frac{C \log p}{n}, \quad \tag{99}
$$

$$
\max_{S,T \subseteq [p]: S \cap T = \emptyset, |S|, |T| \leq 2C_0s} \left| X_j^T P_S X_j \right| \leq C s \log p, \quad \tag{100}
$$

$$
\max_{S \subseteq [p]: |S| \leq 2C_0s} \frac{1}{|S^* \cap S^c| + |S^c \cap S|} \sum_{j \in S^* \cap S^c} \left( X_j^T P_S w \right)^2 \leq C s \log^2 p, \quad \tag{101}
$$

$$
\max_{j \in S^*} \left( X_j^T P_S w \right)^2 \leq C s \log p, \quad \tag{102}
$$

$$
\max_{S \subseteq [p]: |S| \leq 2C_0s} \left\| X_S^T w \right\|^2 \leq C n \log p, \quad \tag{103}
$$

$$
\max_{S \subseteq [p]: |S| \leq 2C_0s} \left\| \beta_{S^c \cap S^\star} \right\|^2 |T| \| T \| \leq C n \log p, \quad \tag{104}
$$

$$
\max_{S \subseteq [p]: |S| \leq 2C_0s} \frac{1}{|S^* \cap S^c| + |S^c \cap S|} \frac{1}{|T|} \left\| X_T^T (P_S - \bar{P}_S) w \right\|^2 \leq C \log^2 p, \quad \tag{105}
$$

$$
\max_{j \in S^*} \left\| X_j \right\|^{-1} X_j^T X_{S^*} (X_{S^*} (I - P_j) X_{S^*})^{-1} X_{S^*}^T (I - P_j) w \right\| \leq \sqrt{\frac{C s \log^2 p}{n}}, \quad \tag{106}
$$

with probability at least $1 - \exp(-C' \log p)$. We have used the convention that $0/0 = 0$.

Proof. We first present a fact that will be used repeatedly in the proof. For two independent $\xi_1, \xi_2 \sim \mathcal{N}(0, I_d)$, we have $\xi_1^T \xi_2 = \left\| \xi_1 \right\|^2 \left( \left\| \xi_1 \right\|^{-1} \xi_1 \right)^T \xi_2 \overset{d}{=} \left\| \xi_1 \right\| \xi_2$, where $\xi \sim \mathcal{N}(0, 1)$ and $\xi \perp \xi_1$. Throughout the proof, we will use $c, c', c_1, c_2, \ldots$ as generic constants whose values may change from place to place. We refer to Lemma 7.1 for the $\chi^2$ tail probability bound.

Equation (96): We have $\left\| X_j \right\|^2 \sim \chi^2_n$. Then the $\chi^2$ tail bound and a union bound argument over $j \in [p]$ lead to the desired bound.

Equation (97): It is sufficient to study the smallest eigenvalue of $X_S^T X_S$. For a fixed $S$ and $\theta \in \mathbb{R}^{|S|}$ such that $\left\| \theta \right\| = 1$, we have $\theta^T X_S^T X_S \theta \sim \chi^2_n$. Thus $\mathbb{P} \left( \left| \theta^T X_S^T X_S \theta - n \right| \leq \frac{n}{2} \right) \leq \frac{n}{2}$.
2 \exp(-n/16). By a standard \(\epsilon\)-net argument \([60]\), we can obtain
\[
\mathbb{P}\left( \min_{\theta \in \mathbb{R}^{|S|}: \|\theta\|=1} \theta^T X_S^T X_S \theta \leq c_1 n \right) \leq c_2^{|S|} \exp(-n/16).
\]
We then take a union bound over \(S\) to obtain
\[
\mathbb{P}\left( \min_{S \subseteq [p]: |S| \leq 2C_0 s} \min_{\theta \in \mathbb{R}^{|S|}: \|\theta\|=1} \theta^T X_S^T X_S \theta \leq c_1 n \right) \leq 2 \left( \frac{p}{2C_0 s} \right)^{|S|} \exp(-n/16).
\]
Thus
\[
\mathbb{P}\left( \max_{S \subseteq [p]: |S| \leq 2C_0 s} \|X_S^T X_S\|^{-1} \geq \frac{1}{c_1 n} \right) \leq \exp(-c_3 n),
\]
for some constant \(c_3 > 0\).

Equation (98): Conditioning on \(X_S\), we have \((X_S^T X_S)^{-1} X_S^T X_T \overset{d}{=} (X_S^T X_S)^{-\frac{3}{2}} \zeta\), where \(\zeta \sim \mathcal{N}(0, I_{|S|})\). Thus \(\|X_S^T X_S\|^{-1} X_S^T X_T \|2^d = \zeta^T (X_S^T X_S)^{-1} \zeta \leq \|X_S^T X_S\|^{-1} \|\zeta\|^2\). We have
\[
\mathbb{P}\left( \|X_S^T X_S\|^{-1} X_S^T X_T \|2^d \geq 4c \|X_S^T X_S\|^{-1} s \log p \left| \begin{array} {c} X_S \end{array} \right. \right) \leq \exp(-cs \log p).
\]
A union bound over \(T\) gives
\[
\mathbb{P}\left( \max_{T \subseteq [p]: |T| \leq 2C_0 s} \|X_S^T X_S\|^{-1} X_S^T X_T \|2^d \geq 4c \|X_S^T X_S\|^{-1} s \log p \left| X_S \right. \right) \leq \left( \frac{p}{2C_0 s} \right) \exp(-cs \log p).
\]
Consequently, for a fixed \(S\),
\[
\mathbb{P}\left( \max_{T \subseteq [p]: |T| \leq 2C_0 s} \|X_S^T X_S\|^{-1} X_S^T X_T \|2^d \geq \frac{4cs \log p}{c_1 n} \right)
\]
\[
\leq \left( \frac{p}{2C_0 s} \right) \exp(-cs \log p) + \mathbb{P}\left( \|X_S^T X_S\|^{-1} \geq \frac{1}{c_1 n} \right).
\]
Using the result established above when proving (97), together with a union bound over \(S\), we have
\[
\mathbb{P}\left( \max_{S,T \subseteq [p]: S \subseteq T = \emptyset, |S|, |T| \leq 2C_0 s} \|X_S^T X_S\|^{-1} X_S^T X_T \|2^d \geq \frac{4cs \log p}{c_1 n} \right)
\]
\[
\leq \left( \frac{p}{2C_0 s} \right) \left( \frac{p}{2C_0 s} \right) \exp(-cs \log p) + \mathbb{P}\left( \max_{S \subseteq [p]: |S| \leq 2C_0 s} \|X_S^T X_S\|^{-1} \geq \frac{1}{c_1 n} \right)
\]
\[
\leq \left( \frac{p}{2C_0 s} \right) \left( \frac{p}{2C_0 s} \right) \exp(-cs \log p) + \exp(-c_2 n)
\]
\[
\leq \exp(-c_3 s \log p).
\]

Equation (99): The proof of (99) is very similar to that of (98). Conditioning on \(X_S, X_T\), we have \((X_T^T (I - P_S) X_T)^{-1} X_T^T (I - P_S) w \overset{d}{=} (X_T^T (I - P_S) X_T)^{-1} \zeta\) where \(\zeta \sim \mathcal{N}(0, I_{|S|})\).
\( \mathcal{N}(0, I_{|T|}) \). Consequently, \( \left\| (X_T^T (I - P_S) X_T)^{-1} X_T^T (I - P_S) w \right\|^2 \) is stochastically dominated by \( \left\| (X_T^T (I - P_S) X_T)^{-1} \right\| \| w \|^2 \). Similar to the proof of (98), we have

\[
\mathbb{P} \left( \max_{S,T \subset [p]: S \cap T = \emptyset, |S|, |T| \leq 2C_0s} \left\| \frac{1}{T} (X_T^T (I - P_S) X_T)^{-1} X_T^T (I - P_S) w \right\|^2 \geq \frac{4c \log p}{c_1n} \right). 
\]

\[
\leq \sum_{S \subset [p]: |S| \leq 2C_0s} \sum_{m=0}^{2C_0s} \sum_{T \subset [p]: |T| = m} \mathbb{P} \left( \chi_m^2 \geq 4cm \log p \right) 
\]

\[
+ \mathbb{P} \left( \max_{S,T \subset [p]: S \cap T = \emptyset, |S|, |T| \leq 2C_0s} \left\| (X_T^T (I - P_S) X_T)^{-1} \right\| \geq \frac{1}{c_1n} \right)
\]

\[
\leq \left( \frac{p}{m} \right) \sum_{m=0}^{2C_0s} \left( \frac{p}{2C_0s} \right) \exp (-cm \log p) + \exp (-cn)
\]

\[
\leq \exp \left( -c' \log p \right),
\]

where in the second to the last inequality, we use (106), which will be proved later.

**Equation (100):** First we fix some \( S \). Then \( X_j^T P_S X_j \) is stochastically dominated by a \( \chi_2^2 \). We have \( \mathbb{P} \left( X_j^T P_S X_j \geq 4cs \log p \right) \leq \exp (-cs \log p) \). A union bound over \( S \) and \( j \) leads to the desired result.

**Equation (101):** For any pair (\( S, T \)), we have

\[
\left\| X_T^T (I - P_S) X_T - \text{diag} \left\{ X_T^T (I - P_S) X_T \right\} \right\|
\leq \left\| X_T^T (I - P_S) X_T - (n - |S|) I \right\| + \left\| (n - |S|) I - \text{diag} \left\{ X_T^T (I - P_S) X_T \right\} \right\|.
\]

The first term can be controlled by (105), to be proved later. For the second term, we have

\[
\left\| (n - |S|) I - \text{diag} \left\{ X_T^T (I - P_S) X_T \right\} \right\| = \max_{j \in T} \left| X_j^T (I - P_S) X_j - (n - |S|) \right|
\]

\[
\leq \max_{j \in T} \left| X_j \right|^2 - n + \max_{j \in T} \left| X_j^T P_S X_j - |S| \right|,
\]

which can be bounded by (96) and (100). Combining the two terms together gives the desired result.

**Equation (102):** For a fixed \( S \) and any \( j \in S^* \cap S^c \), using the fact we give at the beginning of the proof, we have \( X_j^T P_s w \overset{d}{=} \left\| P_S w \right\| \xi_j \) where \( \xi_j \sim \mathcal{N}(0, 1) \) and \( \xi_j \perp \left\| P_S w \right\| \). Since \( \xi_j \) only depends on \( X_j^T (\left\| P_S w \right\|^{-1} P_S w) \), we have the independence among \( \{\xi_j\}_{j \in S^* \cap S^c} \). As a result, we have \( \sum_{j \in S^* \cap S^c} \left( X_j^T P_s w \right)^2 \overset{d}{=} \zeta \), where \( \zeta \sim \chi_2^2 \), \( \xi \sim \chi_2^2 \) and \( \zeta \perp \xi \). Similar arguments will also be used later to prove (103)-(104) and (107)-(110) and will be omitted there. Then

\[
\mathbb{P} \left( \sum_{j \in S^* \cap S^c} \left( X_j^T P_s w \right)^2 \geq 16c^2 s \log^2 p \left( |S^* \cap S^c| + |S^* \cap S| \right) \right)
\]

\[
\leq \mathbb{P} \left( \zeta \geq 4cs \log p \right) + \mathbb{P} \left( \xi \geq 4c \left( |S^* \cap S^c| + |S^* \cap S| \right) \log p \right)
\]

\[
\leq \exp (-cs \log p) + \exp (-c \left( |S^* \cap S^c| + |S^* \cap S| \right) \log p).
\]
After applying union bound, we get
\[
\mathbb{P}\left(\max_{S \subseteq [p]:|S| \leq 2C_0s} \frac{1}{|S^c \cap S| + |S^c \cap S|} \sum_{j \in S^c \cap S} (X_j^T P_S w)^2 \geq 16c^2 s \log^2 p \right) \\
\leq \sum_{m=0}^{2C_0s} \binom{p}{m} (\exp(-cs \log p) + \exp(-cm \log p)) \\
\leq \exp(-c' \log p).
\]

*Equation (103):* For each \( j \notin S^* \), we have \((X_j^T P_S w)^2\) stochastically dominated by \(\xi\) where \(\xi \sim \chi^2\), \(\zeta \sim \chi^2\), and \(\xi \perp \zeta\). We get the desired result by the \(\chi^2\) tail bound and a union bound over \( j \notin S^* \).

*Equation (104):* By (106) to be proved later, it is sufficient to establish
\[
\mathbb{P}\left(\max_{S,T \subseteq [p]:|S|,|T| \leq 2C_0s} \frac{1}{|T|} \left\|X_T^T P_S w\right\|^2 \geq cs \log p \right) \leq \exp(-c' \log p).
\]
Note that for any fixed \( S, T \), we have \(\|X_T^T P_S w\|^2\) stochastically dominated by \(\xi\) where \(\xi \sim \chi^2_{2C_0s}\), \(\zeta \sim \chi^2_{|T|}\), and \(\xi \perp \zeta\). Then we have
\[
\mathbb{P}\left(\|X_T^T P_S w\|^2 \geq c^2 s \log p \right) \leq \mathbb{P} (\zeta \geq c |T|) + \mathbb{P} (\xi \geq cs \log p)
\]
which can be controlled by the \(\chi^2\) tail bound. A union bound is then sufficient to complete the proof.

*Equation (105):* For any fixed \( S, T \), and any \( \theta \in \mathbb{R}^{|T|} \) such that \(\|\theta\| = 1\), we have
\[
\theta^T (X_T(I - P_S)X_T)^{-1} \theta \sim \chi^2_{n - |S|},
\]
and \(\theta^T (n - |S|)I_T \theta = (n - |S|)\). By a standard \(\epsilon\)-net argument [60], the \(\chi^2\) tail bound, and a union bound over \( S, T \), we conclude its proof.

*Equation (106):* Its proof is similar to that of (97). We can show
\[
\mathbb{P}\left(\max_{S,T \subseteq [p]:|S|,|T| = 0,|S|,|T| \leq 2C_0s} \left\|(X_T(I - P_S)X_T)^{-1}\right\| \geq \frac{1}{cn} \right) \leq \exp(-c'n)
\]
for some \(c, c'\). Its proof is omitted here.

*Equation (107):* We have \(\|X_T^S w\|^2 = \xi \zeta\) where \(\xi \sim \chi^2_n\), \(\zeta \sim \chi^2_{|S|}\) and \(\xi \perp \zeta\). Thus,
\[
\mathbb{P}\left(\|X_T^S w\|^2 \geq c^2 n |S| \log p \right) \leq \mathbb{P} (\xi \geq cn) + \mathbb{P} (\zeta \geq c |S| \log p).
\]
A union bound over integers \(0 \leq m \leq 2C_0s\) and over all sets \(\{S \subseteq [p]:|S| = m\}\) leads to the desired result.

*Equation (108):* For a fixed pair \( S, T \), we have \(\|\beta_{S \cap S^*}^{-1} X_{S \cap S^*} \beta_{S \cap S^*}\| \sim \mathcal{N}(0, I_n)\), and consequently \(\|\beta_{S \cap S^*}^{-1} X_{S \cap S^*} \beta_{S \cap S^*}\|^{-2} \|X_T(I - P_S)X_{S \cap S^*} \beta_{S \cap S^*}\|\) is stochastically dominated by \(\xi \zeta\) where \(\xi \sim \chi^2_n\), \(\zeta \sim \chi^2_{|T|}\) and \(\xi \perp \zeta\). Note that \(\xi\) only depends on \(S^c \cap S^*\) and \(\zeta\) only depends
on $T$. For a fixed $S$, in order to take a union bound over $T$, we add a subscript to $\zeta$ as in $\zeta_T$ to make the dependence explicit. We have
\[
\mathbb{P}\left( \max_{T \subset [p]} \frac{1}{|T| \cup s} \|\beta_{S \cap \tilde{S}^*}^T(I - P_S)X_{S \cap \tilde{S}^*} \| \leq 4c \log p \right) \\
\leq \mathbb{P}\left( \xi \geq 4c n \right) + \mathbb{P}\left( \max_{T \subset [p]} \frac{1}{|T| \cup s} \zeta_T \geq 4c \log p \right) \\
\leq \exp(-cn) + \sum_{m=0}^p \sum_{|T| = m} \exp(-c(m \land s) \log p) \\
\leq \exp(-c's \log p).
\]

The proof is completed by an additional union bound argument over $S$.

**Equation (109):** Consider a fixed pair $S, T$. For any $x \in \mathbb{R}^n$, we have $(P_{S^*} - P_S)x = P_{S,1}x - P_{S,2}x$, where $P_{S,1}$ is the projection matrix onto the space span$(X_{S^*}) \setminus (\text{span}(X_{S^*}) \cap \text{span}(X_S))$, and $P_{S,2}$ is the projection matrix onto the space span$(X_S) \setminus (\text{span}(X_{S^*}) \cap \text{span}(X_S))$. Then we have
\[
\|X_T^T(P_{S^*} - P_S)w\|^2 = \|X_T^T P_{S,1}w - X_T^T P_{S,2}w\|^2 \leq 2 \left( \|X_T^T P_{S,1}w\|^2 + \|X_T^T P_{S,2}w\|^2 \right).
\]

Note that span$(X_{S \cap \tilde{S}^*}) \subset \text{span}(X_{S^*}) \cap \text{span}(X_S)$, and thus the rank of $P_{S,1}$ is bounded by $|S^* \cap S|$. Hence, $\|X_T^T P_{S,1}w\|^2$ is stochastically dominated by $\xi \zeta$ where $\xi \sim \chi^2_{|S^* \cap S| + |S^\circ \cap S|}$, $\zeta \sim \chi^2_{|T|}$ and $\xi \perp \zeta$. Note that $\xi$ only depends on $S$ and $\zeta$ only depends on $T$. For a fixed $S$, in order to take a union bound over $T$, we add a subscript to $\zeta$ as in $\zeta_T$ to make the dependence explicit. We have
\[
\mathbb{P}\left( \max_{T \subset [p]: T \cap (S \cup S^*) = \emptyset} \frac{1}{|T| \cup s} \|X_T^T P_{S,1} w\|^2 \geq 16c^2 (|S^* \cap S^c| + |S^\circ \cap S|) \log^2 p \right) \\
\leq \mathbb{P}\left( \xi \geq 4c (|S^* \cap S^c| + |S^\circ \cap S|) \log p \right) + \mathbb{P}\left( \frac{1}{|T| \cup s} \zeta_T \geq cs \log p \right) \\
\leq \exp(-c (|S^* \cap S^c| + |S^\circ \cap S|) \log p) + \sum_{m=0}^p \sum_{|T| = m} \exp(-c(m \land s) \log p) \\
\leq \exp(-c (|S^* \cap S^c| + |S^\circ \cap S|) \log p) + \exp(-c's \log p).
\]

Then we take a union bound of $S$.
\[
\mathbb{P}\left( \max_{S \subset [p]: |S| \leq 2C_0 s} \max_{T \subset [p]: T \cap (S \cup S^*) = \emptyset} \frac{1}{|T| \cup s} \|X_T^T P_{S,1} w\|^2 \geq 16c^2 \log^2 p \right) \\
\leq \sum_{m'=0}^{2C_0 s} \sum_{|S| \leq 2C_0 s} \exp(-cm' \log p) + \exp(-c's \log p) \\
\leq \exp(-c'' \log p).
\]

A similar result holds for the term related to $P_{S,2}$. Putting them together, we complete the proof.
Equation (110): Define $B_j = \|X_j\|^{-1} X_j^T X_{S_j^c} \left( X_{S_j^c}^T (I - P_j) X_{S_j^c} \right)^{-1} X_{S_j^c}^T X_j \|X_j\|^{-1}$ for all $j \in S^*$. Note that $\|X_j\|^{-1} X_j^T X_{S_j^c} \left( X_{S_j^c}^T (I - P_j) X_{S_j^c} \right)^{-1} X_{S_j^c}^T (I - P_j) w$ is identically distributed by $\sqrt{\psi_j} \xi_j$ with $\xi_j \sim N(0, 1)$ and $\xi_j \perp B_j$. Here we have the subscript for both $\xi_j$ and $B_j$ to make their dependence on $j$ explicit. Then,
\[
\mathbb{P} \left( \|X_j\|^{-1} X_j^T X_{S_j^c} \left( X_{S_j^c}^T (I - P_j) X_{S_j^c} \right)^{-1} X_{S_j^c}^T (I - P_j) w \geq \sqrt{\frac{c^2 s \log^2 p}{n}} \right)
\leq \mathbb{P} \left( \xi_j \geq \sqrt{c \log p} \right) + \mathbb{P} \left( B_j \geq \frac{c s \log p}{n} \right),
\]
and thus
\[
\mathbb{P} \left( \max_{j \in S^*} \|X_j\|^{-1} X_j^T X_{S_j^c} \left( X_{S_j^c}^T (I - P_j) X_{S_j^c} \right)^{-1} X_{S_j^c}^T (I - P_j) w \geq \sqrt{\frac{c^2 s \log^2 p}{n}} \right)
\leq \mathbb{P} \left( \max_{j \in S^*} \xi_j \geq \sqrt{c \log p} \right) + \mathbb{P} \left( \max_{j \in S^*} B_j \geq \frac{c s \log p}{n} \right).
\]
The first term can be easily bounded by $s \exp \left( -2^{-1} c \log p \right) \leq \exp \left( -c' \log p \right)$. For the second term, we have
\[
B_j \leq \left\| \left( X_{S_j^c}^T (I - P_j) X_{S_j^c} \right)^{-1} \right\| \left\| X_{S_j^c}^T X_j \|X_j\|^{-1} \right\|^2,
\]
for all $j \in S^*$. By a similar analysis as in (106), we can show $\max_{j \in S^*} \left\| \left( X_{S_j^c}^T (I - P_j) X_{S_j^c} \right)^{-1} \right\| \leq c_1/n$ with probability at least $1 - \exp \left( -c_2 n \right)$. Note that $\left\| X_{S_j^c}^T X_j \|X_j\|^{-1} \right\|^2 \sim \chi^2_{s-1}$. Easily we can show $\max_{j \in S^*} \left\| X_{S_j^c}^T X_j \|X_j\|^{-1} \right\|^2 \leq 4 c_3 s \log p$ with probability at least $1 - \exp \left( -c_4 s \log p \right)$. As a result,
\[
\mathbb{P} \left( \max_{j \in S^*} B_j \geq \frac{c s \log p}{n} \right) \leq \exp \left( -c_2 n \right) + \exp \left( -c_4 s \log p \right),
\]
which completes the proof. \hfill \Box

Lemma 7.5. Define
\[
\tilde{\psi}(n, p, s, \lambda, \delta, C) = s \mathbb{P} \left( \epsilon > (1 - \delta) \|\zeta\| (\lambda - t(\zeta)) \& \left| \|\zeta\|^2 - n \right| \leq C \sqrt{n \log p} \right)
+ (p - s) \mathbb{P} \left( \epsilon > (1 - \delta) \|\zeta\| t(\zeta) \& \left| \|\zeta\|^2 - n \right| \leq C \sqrt{n \log p} \right), \tag{111}
\]
where $\epsilon \sim N(0, 1)$, $\zeta \sim N(0, I_n)$, and they are independent of each other. Assume $s \log p \leq n$, $\limsup s/p < 1/2$, and $\text{SNR} \to \infty$. For any $\delta \leq 1/ \log p$ and any constant $C > 0$, we have
\[
\tilde{\psi}(n, p, s, \lambda, \delta, C) = s \exp \left( \frac{-(1 + o(1)) \text{SNR}^2}{2} \right).
\]
48
Proof. Throughout the proof, we use \( g(x) \) and \( G(x) \) for the density and survival functions of \( \mathcal{N}(0, 1) \). A standard Gaussian tail analysis gives
\[
\frac{1}{2x} g(x) \leq G(x) \leq \frac{1}{x} g(x), \tag{112}
\]
for all \( x \geq 2 \). With a slight abuse of notation, we also use the notation
\[
t(u) = \frac{\lambda}{2} + \frac{\log \frac{p-s}{s}}{u^2}, \tag{113}
\]
for all \( u > 0 \). We first focus on deriving an upper bound for \( \tilde{\psi}(n, p, s, \lambda, \delta, C) \). For any \( u > 0 \), we define
\[
m(u) = u \left( \lambda - \frac{\log \frac{p-s}{s}}{\lambda u^2} \right) = \lambda u - \frac{\log \frac{p-s}{s}}{\lambda u}.
\]
Recall the definition of \( t(u) \) in (113). Define \( u_{\min} = \sqrt{n - C \sqrt{n \log p}} \) and \( u_{\max} = \sqrt{n + C \sqrt{n \log p}} \), and \( U = [u_{\min}, u_{\max}] \). Since \( u (\lambda - t(u)) \) is an increasing function of \( u > 0 \). We have
\[
m(u_{\min}) \leq u (\lambda - t(u)) \leq m(u_{\max}),
\]
for all \( u \in U \). This gives
\[
s \mathbb{P} \left( \frac{\epsilon}{1 - \delta} \geq \| \xi \| (\lambda - t(\xi)) \& \| \| \xi \| - n \| \leq C \sqrt{n \log p} \right)
\leq s \mathbb{P} \left( \frac{\epsilon}{1 - \delta} \geq m(u_{\min}) \right)
\leq \frac{s}{\sqrt{2\pi} (1 - \delta) m(u_{\min})} \exp \left( -\frac{1}{2} (1 - \delta)^2 m^2(u_{\min}) \right),
\]
where the last inequality is by (112). In addition, we have the identity \( (ut(u))^2 = 2 \log \frac{p-s}{s} + m^2(u) \). This leads to
\[
2 \log \frac{p-s}{s} + m^2(u_{\min}) \leq (ut(u))^2 \leq 2 \log \frac{p-s}{s} + m^2(u_{\max}),
\]
for all \( u \in U \). As a result, using (112), we have
\[
(p-s) \mathbb{P} \left( \frac{\epsilon}{1 - \delta} \geq \| \xi \| (\lambda - t(\xi)) \& \| \| \xi \| - n \| \leq C \sqrt{n \log p} \right)
= (p-s) \mathbb{E}_{u^2 \sim \chi^2_n} \left[ \mathbb{P} \left( \frac{\epsilon}{1 - \delta} \geq |u|t(u) \right) \mathbb{1}_{\{|u^2-n| \leq C \sqrt{n \log p}\}} \right]
\leq \frac{p-s}{\sqrt{2\pi}} \mathbb{E}_{u^2 \sim \chi^2_n} \left[ \frac{1}{ut(u)} \exp \left( -\frac{1}{2} (1 - \delta)^2 (ut(u))^2 \right) \mathbb{1}_{\{|u^2-n| \leq C \sqrt{n \log p}\}} \right]
\leq \frac{p-s}{\sqrt{2\pi}} \mathbb{E}_{u^2 \sim \chi^2_n} \left[ \frac{1}{\min_{u \in U} ut(u)} \exp \left( -\frac{1}{2} (1 - \delta)^2 \left( 2 \log \frac{p-s}{s} + m^2(u_{\min}) \right) \right) \mathbb{1}_{\{|u^2-n| \leq C \sqrt{n \log p}\}} \right]
\leq \frac{p-s}{\sqrt{2\pi} \min_{u \in U} ut(u)} \exp \left( -(1 - \delta)^2 \log \frac{p-s}{s} - \frac{1}{2} (1 - \delta)^2 m^2(u_{\min}) \right)
= \frac{s}{\sqrt{2\pi} \min_{u \in U} ut(u)} \exp \left( (2\delta - \delta^2) \log \frac{p-s}{s} - \frac{1}{2} (1 - \delta)^2 m^2(u_{\min}) \right).
\]
Combining the above results together, we have

\[ \tilde{\psi}(n,p,s,\lambda,\delta,C) \leq \frac{s}{\sqrt{2\pi (1-\delta) m(u_{\min})}} \exp\left( -\frac{1}{2} (1-\delta)^2 m^2(u_{\min}) \right) \]
\[ + \frac{s}{\sqrt{2\pi \min_{u \in U} ut(u)}} \exp\left( (2\delta - \delta^2) \log \frac{p-s}{s} - \frac{1}{2} (1-\delta)^2 m^2(u_{\max}) \right). \]

Now we derive a lower bound for \( \tilde{\psi}(n,p,s,\lambda,\delta,C) \). Note that \( \mathbb{P}\left( ||\mathbf{\zeta}|| - n \leq C \sqrt{n \log p} \right) \leq 1/2 \). We therefore have

\[ s \mathbb{P}\left( \epsilon \geq ||\mathbf{\zeta}|| (\lambda - t(\mathbf{\zeta})) \& \ ||\mathbf{\zeta}|| - n \leq C \sqrt{n \log p} \right) \]
\[ \geq \frac{s}{2} \mathbb{P}\left( \epsilon \geq m(u_{\max}) \right) \]
\[ \geq \frac{s}{4\sqrt{2\pi} m(u_{\max})} \exp\left( -\frac{1}{2} m^2(u_{\max}) \right), \]

and

\[ (p-s) \mathbb{P}\left( \epsilon \geq ||\mathbf{\zeta}|| t(\mathbf{\zeta}) \& \ ||\mathbf{\zeta}|| - n \leq C \sqrt{n \log p} \right) \]
\[ \geq \frac{s}{4\sqrt{2\pi} \max_{u \in U} ut(u)} \exp\left( (2\delta - \delta^2) \log \frac{p-s}{s} - \frac{1}{2} (1-\delta)^2 m^2(u_{\max}) \right). \]

Consequently,

\[ \tilde{\psi}(n,p,s,\lambda,\delta,C) \geq \frac{s}{4\sqrt{2\pi} m(u_{\max})} \exp\left( -\frac{1}{2} m^2(u_{\max}) \right) \]
\[ + \frac{s}{4\sqrt{2\pi} \max_{u \in U} ut(u)} \exp\left( (2\delta - \delta^2) \log \frac{p-s}{s} - \frac{1}{2} (1-\delta)^2 m^2(u_{\max}) \right). \]

Since \( \delta \leq 1/\log p \) and \( \text{SNR} \to \infty \), with the same arguments used in the proof of Lemma 7.6, we can show for all \( u \in U \), we have \( \lambda_{u} - \frac{\log \frac{p-s}{s}}{\Lambda_{u}} = (1 + o(1)) \text{SNR} \) and \( \lambda_{u} + \frac{\log \frac{p-s}{s}}{\Lambda_{u}} \to \infty \). This leads to \( \tilde{\psi}(n,p,s,\lambda,\delta,C) = s \exp\left( -\frac{(1+o(1))\text{SNR}^2}{2} \right) \) as desired, which completes the proof.

**Lemma 7.6.** Consider some \( \beta^* \in \mathbb{R}^p \) that satisfies either \( |\beta_j^*| \geq \lambda \) or \( \beta_j^* = 0 \) for all \( j \in [p] \). Assume \( \limsup s/p < \frac{1}{2} \) and \( \text{SNR} \to \infty \). Then, for i.i.d. \( X_1, \ldots, X_p \sim \mathcal{N}(0, I_n) \), we have

\[ \min_{j \in [p]} \sqrt{n} |\beta_j^*| - t(X_j) > 1, \]
\[ \max_{j \in [p]} \frac{|\beta_j^*|}{||\beta_j^*| - t(X_j)|} \leq \sqrt{\log p}, \]

with probability at least \( 1 - e^{-p} \).
Proof. We first show that under the assumption that \( \lim \sup s/p < 1/2 \), the condition \( \text{SNR} \to \infty \) is equivalent to

\[
\frac{n\lambda^2 - 2 \log \frac{p-s}{s}}{\sqrt{\log \frac{p-s}{s}}} \to \infty.
\] (114)

A direct calculation gives

\[
\left( \frac{\lambda \sqrt{n}}{2} - \frac{\log \frac{p-s}{s}}{\lambda \sqrt{n}} \right)^2 = \left( \frac{n\lambda^2 - 2 \log \frac{p-s}{s}}{4n\lambda^2} \right)^2 = \left( \frac{n\lambda^2 - 2 \log \frac{p-s}{s}}{\sqrt{\log \frac{p-s}{s}}} \right)^2 \log \frac{p-s}{s} / 4n\lambda^2.
\]

If \( \text{SNR} \to \infty \) holds, we have \( n\lambda^2 \geq 2 \log \frac{p-s}{s} \), which leads to (114). For the other direction, if (114) holds, there exists some \( A \to \infty \) such that

\[
\frac{n\lambda^2 - 2 \log \frac{p-s}{s}}{\sqrt{\log \frac{p-s}{s}}} = 2 \log \frac{p-s}{s} + A \sqrt{\log \frac{p-s}{s}}.
\]

By the above identity, we have

\[
\left( \frac{\lambda \sqrt{n}}{2} - \frac{\log \frac{p-s}{s}}{\lambda \sqrt{n}} \right)^2 = \frac{\log \frac{p-s}{s}}{2 \left(2 \log \frac{p-s}{s} + A \sqrt{\log \frac{p-s}{s}}\right)} \to \infty.
\]

Thus we have shown that \( \text{SNR} \to \infty \) and (114) are equivalent.

Now we are going to prove the proposition under the high-probability event (96). Note that for any \( j \in [p] \) such that \( \beta_j^* = 0 \), we have \( \sqrt{n} |\beta_j^*| - t(X_j) = \sqrt{n}t(X_j) \geq \sqrt{n}\lambda/2 \to \infty \) by (114) and \( |\beta_j^*|/|\beta_j^*| - t(X_j)| = 0 \). Thus, we only need to consider the remaining \( j \in [p] \) such that \( \beta_j^* \neq 0 \). It is sufficient to prove \( \min_{j \in [p]:z_j^* \neq 0} \sqrt{n} (\lambda - t(X_j)) > 1 \) and \( \max_{j \in [p]:z_j^* \neq 0} \lambda \lambda - t(X_j) \leq \sqrt{\log p} \). Consider any \( j \in [p] \) such that \( z_j^* \neq 0 \). We have

\[
\sqrt{n} (\lambda - t(X_j)) = \frac{\lambda}{2\lambda} \left( \lambda^2 - 2 \frac{\log \frac{p-s}{s}}{\|X_j\|^2} \right) \geq \frac{\lambda}{2\lambda} \left( \lambda^2 - 2 \frac{\log \frac{p-s}{s}}{n - C\sqrt{n \log p}} \right),
\]

where the last inequality is by (96). By (114), there exists an \( A \to \infty \), such that

\[
n\lambda^2 = 2 \log \frac{p-s}{s} + A \sqrt{\log \frac{p-s}{s}}.
\]

Then, we have

\[
\sqrt{n} (\lambda - t(X_j)) \geq \frac{1}{2\sqrt{n}\lambda} \left( 2 \log \frac{p-s}{s} + A \sqrt{\log \frac{p-s}{s}} - 2 \frac{\log \frac{p-s}{s}}{n - C\sqrt{n \log p}} \right) \geq \frac{1}{2\sqrt{n}\lambda} \left( A \sqrt{\log \frac{p-s}{s}} - C' \left( \log \frac{p}{n} \log \frac{p-s}{s} \right) \right) \geq \frac{C'' A \sqrt{\log \frac{p-s}{s}}}{\sqrt{n}\lambda},
\]
for some constants \( C', C'' > 0 \). Starting from here, first we have

\[
\sqrt{n} (\lambda - t(X_j)) \geq C'' \sqrt{\frac{A^2 \log \frac{p-s}{s}}{n\lambda^2}} = C'' \sqrt{\frac{A^2 \log \frac{p-s}{s}}{2 \log \frac{p-s}{s} + A \sqrt{\log \frac{p-s}{s}}}} \to \infty,
\]

as \( A \to \infty \). Second, we have

\[
\frac{\lambda}{\lambda - t(X_j)} = \frac{\sqrt{n\lambda}}{\sqrt{n} (\lambda - t(X_j))} \leq \frac{\sqrt{n\lambda}}{C'' A \sqrt{\log \frac{p-s}{s}}} = \frac{n\lambda^2}{C'' A \sqrt{\log \frac{p-s}{s}}} = \frac{2 \log \frac{p-s}{s} + A \sqrt{\log \frac{p-s}{s}}}{C'' A \sqrt{\log \frac{p-s}{s}}} \leq o \left( \sqrt{\log \frac{p-s}{s}} \right).
\]

Hence, the proof is complete. \( \square \)

Now we are ready to prove Theorem 6.1, Theorem 6.2, and Corollary 6.1.

**Proof of Theorem 6.1.** By [54], we have

\[
\inf \sup_{\tilde{z}} \sup_{z^*} \mathbb{E} H_s(\tilde{z}, z^*) \geq \frac{1}{2s} \psi(n, p, s, \lambda, 0) - 4e^{-s/8},
\]

where \( \psi(n, p, s, \lambda, 0) \) is defined in (56). By Lemma 7.5,

\[
\frac{1}{2s} \psi(n, p, s, \lambda, 0) \geq \frac{1}{2s} \tilde{\psi}(n, p, s, \lambda, 0, C) = \exp \left( -\frac{(1 + o(1))SNR^2}{2} \right),
\]

and we obtain the desired conclusion. \( \square \)

**Proof of Theorem 6.2.** The condition of Theorem 6.2 allows us to apply Lemma 7.6 to the conclusion of Lemma 6.1. This implies that the right hand sides of (53) and (54) can be bounded by \( o((\log p)^{-1}) \), which then implies Conditions A-C hold with some \( \delta = o((\log p)^{-1}) \). Then, the desired conclusion is a special case of Theorem 3.1. \( \square \)

**Proof of Corollary 6.1.** By (51) and (96), we have

\[
H(s)(z, z^*) \leq \frac{\ell(z, z^*)}{s\lambda^2 \min_j \|X_j\|^2} \leq \frac{2\ell(z, z^*)}{sn\lambda^2},
\]

with high probability. Then, the conclusion is a direct consequence of Theorem 6.2. \( \square \)

Finally, we present the proofs of Lemma 6.1, Lemma 6.2, and Proposition 6.1.

**Proof of Lemma 6.1.** The proof will be established under the high-probability events (96)-(110). First we present a few important quantities closely related to \( \ell(z, z^*) \). By \( h(z, z^*) \leq \frac{\ell(z, z^*)}{\lambda^2 \min_j \|X_j\|^2} \) and (96), we have

\[
\frac{h(z, z^*)}{n\lambda^2} \leq \frac{2\ell(z, z^*)}{n\lambda^2}.
\]
By the definition of \( \ell(z, z^*) \), it is obvious

\[
\sum_{j=1}^p \beta_j^2 \mathbb{1}_{\{\|z_j\| \neq |z^*_j|\}} \leq \frac{\ell(z, z^*)}{\min_j \|X_j\|^2} \leq \frac{2\ell(z, z^*)}{n},
\]

where we have used (96) again. For any \( z \in \{0, 1, -1\}^p \) such that \( \ell(z, z^*) \leq \tau \leq C_0 s n \lambda^2 \), (115) implies

\[
h(z, z^*) \leq 2C_0 s.
\]

We will first prove the easier conclusion (54) and then prove (53).

**Proof of (54).** According to the definition, we have

\[
\left| \frac{1}{\|X_j\|} \sum_{t \in [p] \setminus \{j\}} \left( \widehat{\beta}_l(z^*) - \beta^* \right) X_j^T X_t \right| = \left| \frac{1}{\|X_j\|} X_j^T X \widehat{\beta}(z^*) - \frac{1}{\|X_j\|} X_j^T X \beta^* - \|X_j\| \left( \widehat{\beta}_l(z^*) - \beta^*_l \right) \right|
\]

\[
= \left| \frac{1}{\|X_j\|} X_j^T \left( X \widehat{\beta}(z^*) - X \beta^* \right) - \|X_j\| \left( \widehat{\beta}_l(z^*) - \beta^*_l \right) \right|.
\]

By the fact that \( \widehat{\beta}_{S^c}(z^*) = \beta_{S^c} + (X_{S^c}^T X_{S^c})^{-1} X_{S^c}^T w \) and \( \widehat{\beta}_{S^c}(z^*) = 0 \), we have

\[
\left| \frac{1}{\|X_j\|} \sum_{t \in [p] \setminus \{j\}} \left( \widehat{\beta}_l(z^*) - \beta^* \right) X_j^T X_t \right| = \begin{cases} \frac{1}{\|X_j\|} X_j^T w - \|X_j\| \left( (X_{S^c}^T X_{S^c})^{-1} X_{S^c}^T \right)_{\phi_{S^c}(j)} & j \in S^* \\ \frac{1}{\|X_j\|} X_j^T P_{S^c} w & j \notin S^* \end{cases}.
\]

- We first consider \( j \notin S^* \). By (96), (103) and (118), we have

\[
\max_{j \notin S^*} \left| \frac{1}{\|X_j\|} \sum_{t \in [p] \setminus \{j\}} \left( \widehat{\beta}_l(z^*) - \beta^* \right) X_j^T X_t \right| \leq \sqrt{\frac{2C_0 s \log p}{n}}.
\]

- Next, we consider \( j \in S^* \). Writing \( X_{S^c} \) into a block matrix form \( X_{S^c} = (X_j, X_{S_{-j}}) \), we have a block matrix inverse formula

\[
(X_{S^c}^T X_{S^c})^{-1} = \begin{pmatrix} \|X_j\|^2 & X_j^T X_{S_{-j}} \\ X_{S_{-j}}^T X_j & X_{S_{-j}}^T X_{S_{-j}} \end{pmatrix}^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},
\]

with

\[
B_{11} = \|X_j\|^2 + \|X_j\|^2 X_j^T X_{S_{-j}} (X_{S_{-j}}^T (I - P_j) X_{S_{-j}})^{-1} X_{S_{-j}}^T X_j \|X_j\|^2,
\]

\[
B_{12} = -\|X_j\|^2 X_j^T X_{S_{-j}} (X_{S_{-j}}^T (I - P_j) X_{S_{-j}})^{-1},
\]

\[
B_{21} = 0,
\]

\[
B_{22} = \|X_{S_{-j}}\|^2 + \|X_{S_{-j}}\|^2 X_{S_{-j}}^T X_{S_{-j}} (X_{S_{-j}}^T (I - P_j) X_{S_{-j}})^{-1} X_{S_{-j}}^T X_{S_{-j}} \|X_{S_{-j}}\|^2.
\]
and the explicit expressions of \( B_{21}, B_{22} \) not displayed since they are irrelevant to our proof. We have \( (X_{S^*}^T X_{S^*})^{-1} X_{S^*}^T w|_{\phi_{S^*}(j)} = B_{11} X_j^T w + B_{12} X_{S^*}^T w \). From (118), some algebra leads to

\[
\max_{j \in S^*} \left| \frac{1}{\|X_j\|} \sum_{t \in [p] \setminus \{j\}} \left( \tilde{\beta}_t(z^*) - \beta^* \right) X_j^T X_t \right| \\
= \max_{j \in S^*} \left| \frac{1}{\|X_j\|} X_j^T w - \|X_j\| \left( (X_{S^*}^T X_{S^*})^{-1} X_{S^*}^T w \right)_{\phi_{S^*}(j)} \right| \\
= \max_{j \in S^*} \left| \|X_j\|^{-1} X_j^T X_{S^*} \left( X_{S^*}^T (I - P_j) X_{S^*} \right)^{-1} X_{S^*}^T (I - P_j) w \right| \\
\leq \sqrt{\frac{Cs \log^2 p}{n}},
\]

where the last inequality is by (110).

Combining the two cases, we have

\[
\max_{j \in [p]} \left| \frac{1}{\|X_j\|} \sum_{t \in [p] \setminus \{j\}} \left( \tilde{\beta}_t(z^*) - \beta^* \right) X_j^T X_t \right| \leq \sqrt{\frac{2Cs \log^2 p}{n}}.
\]

Using (52), we get \( \left| \Delta_j(z_j^*, b)^2 \right| \geq 4t(X_j) \left| \beta_j^* - t(X_j) \right| \|X_j\|^2 \) for all \( j \in [p] \). Then,

\[
\max_{j \in [p]} \left| \frac{H_j(z_j^*, b; z)}{\Delta_j(z_j^*, b)^2} \right| \leq \max_{j \in [p]} \frac{4 \|X_j\| t(X_j) \left\| \right. \|X_j\|^{-1} \sum_{t \in [p] \setminus \{j\}} \left( \tilde{\beta}_t(z^*) - \beta^* \right) X_j^T X_t}{4t(X_j) \left| \beta_j^* - t(X_j) \right| \|X_j\|^2} \\
\leq \max_{j \in [p]} \left| \left. \|X_j\|^{-1} \sum_{t \in [p] \setminus \{j\}} \left( \tilde{\beta}_t(z^*) - \beta^* \right) X_j^T X_t \right| \left| \beta_j^* - t(X_j) \right| \|X_j\| \\
\leq \sqrt{\frac{2Cs \log^2 p}{n}} \frac{1}{\min_{j \in [p]} \left| \beta_j^* - t(X_j) \right| \|X_j\|} \\
\leq \sqrt{\frac{4Cs \log^2 p}{n}} \frac{1}{\min_{j \in [p]} \sqrt{n} \left| \beta_j^* - t(X_j) \right|},
\]

where in the last inequality we use (96).
Proof of (53). By (50) and (52), we have

\[
\begin{align*}
G_j(z_j^*, b; z)^2 & \frac{\|\mu_j(B^*, b) - \mu_j(B^*, z_j^*)\|^2}{\Delta_j(z_j^*, b)^4 \ell(z, z^*)} \\
& \leq \left( 4\|X_j\| t(X_j) \left( \|X_j\|^{-1} \sum_{t \in [p] \setminus \{j\}} (\hat{\beta}_t(z) - \hat{\beta}_t(z^*)) X_j^T X_t \right) \right)^2 \left( 4 \|\beta_j^*\|^2 \|X_j\|^2 1_{\{z_j^* = \pm 1\}} + \lambda^2 \|X_j\|^2 1_{\{z_j^* = 0\}} \right) \\
& = \frac{\left( \|X_j\|^{-1} \sum_{t \in [p] \setminus \{j\}} (\hat{\beta}_t(z) - \hat{\beta}_t(z^*)) X_j^T X_t \right)^2}{\ell(z, z^*)} \left( 4 \left( \left| \frac{\beta_j^*}{\beta_j^*} - t(X_j) \right| \right)^2 1_{\{z_j^* = \pm 1\}} + \left( \frac{\lambda}{t(X_j)} \right)^2 1_{\{z_j^* = 0\}} \right) \\
& \leq \frac{\left( \|X_j\|^{-1} \sum_{t \in [p] \setminus \{j\}} (\hat{\beta}_t(z) - \hat{\beta}_t(z^*)) X_j^T X_t \right)^2}{\ell(z, z^*)} \max \left\{ 4 \left( \left| \frac{\beta_j^*}{\beta_j^*} - t(X_j) \right| \right)^2, \left( \frac{\lambda}{t(X_j)} \right)^2 \right\}.
\end{align*}
\]

Define

\[
\hat{\beta}_j(z) = \|X_j\|^{-2} X_j^T \left( Y - \sum_{t \in [p] \setminus \{j\}} X_t \hat{\beta}_t(z) \right).
\]

We then have

\[
\begin{align*}
\|X_j\|^{-1} \sum_{t \in [p] \setminus \{j\}} (\hat{\beta}_t(z) - \hat{\beta}_t(z^*)) X_j^T X_t \\
& = -\|X_j\| \left( \|X_j\|^{-2} X_j^T \left( Y - \sum_{t \in [p] \setminus \{j\}} X_t \hat{\beta}_t(z) \right) - \|X_j\|^{-2} X_j^T \left( Y - \sum_{t \in [p] \setminus \{j\}} X_t \hat{\beta}_t(z^*) \right) \right) \\
& = -\|X_j\| \left( \hat{\beta}_j(z) - \hat{\beta}_j(z^*) \right).
\end{align*}
\]

Hence, we have

\[
\begin{align*}
\max_{b \in \{-1, 0, 1\} \setminus \{z_j^*\}} & \frac{\|X_j\| t(X_j) \left( \|X_j\|^{-1} \sum_{t \in [p] \setminus \{j\}} (\hat{\beta}_t(z) - \hat{\beta}_t(z^*)) X_j^T X_t \right) \right)^2}{\Delta_j(z_j^*, b)^4 \ell(z, z^*)} \\
& \leq \max \left\{ 4 \left( \left| \frac{\beta_j^*}{\beta_j^*} - t(X_j) \right| \right)^2, \left( \frac{\lambda}{t(X_j)} \right)^2 \right\} \|X_j\|^2 \left( \hat{\beta}_j(z) - \hat{\beta}_j(z^*) \right)^2 \ell(z, z^*) \\
& \leq \max \left\{ 4 \left( \left| \frac{\beta_j^*}{\beta_j^*} - t(X_j) \right| \right)^2, \left( \frac{\lambda}{t(X_j)} \right)^2 \right\} 2n \left( \hat{\beta}_j(z) - \hat{\beta}_j(z^*) \right)^2 \ell(z, z^*),
\end{align*}
\]
Thus, the explicit expression of $\tilde{\beta}(z)$ is given by

$$\tilde{\beta}_j(z) = \begin{cases} \beta_j^* + \left[ (X_S^T X_S)^{-1} X_S^T w \right]_{\phi_{S}(j)} & j \in S \\ \frac{1}{\|X_j\| X_j^T (I - P_S) w} & j \notin S \end{cases}$$

Similarly, we also have

$$\tilde{\beta}_j(z^*) = \begin{cases} \beta_j^* + \left[ (X_S^T X_{S^*})^{-1} X_S^T w \right]_{\phi_{S^*}(j)} & j \in S^* \\ \frac{1}{\|X_j\| X_j^T (I - P_{S^*}) w} & j \notin S^* \end{cases}$$

The analysis of $\tilde{\beta}_j(z^*) - \tilde{\beta}_j(z)$ will be studied in four different regimes. We divide $[p]$ into four disjoint sets,

$$S_1 = S^* \cap S^c, \quad S_2 = S^* \cap S, \quad S_3 = S^{*c} \cap S, \quad \text{and} \quad S_4 = S^{*c} \cap S^c.$$ 

Note that by (117), we have

$$|S_1| + |S_3| = h(z, z^*) \leq 2C_0 s. \quad (122)$$

We denote $X_l = X_{S_l}, l = 1, 2, 3, 4$ for simplicity. We also denote $P_l = X_l (X_l^T X_l)^{-1} X_l^T$ to be the projection matrix onto the subspace $\text{span}(X_l)$, for $l = 1, 2, 3, 4$.

**Regime 1**: $j \in S_1$. In this case, we have $\tilde{\beta}_j(z^*) = \beta_j^* + \left[ (X_S^T X_{S^*})^{-1} X_S^T w \right]_{\phi_{S^*}(j)}$. We can...
also write
\[
\bar{\beta}_j(z) = \frac{1}{\|X_j\|^2} X_j^T [(I - P_S)X_S\bar{\beta}_S^* + (I - P_S)w]
\]
\[
= \frac{1}{\|X_j\|^2} X_j^T (I - P_S)X_j\beta_j^* + \sum_{l \in S_1, l \neq j} \frac{1}{\|X_j\|^2} X_j^T (I - P_S)X_l\beta_l^* + \frac{1}{\|X_j\|^2} X_j^T (I - P_S)w
\]
\[
= \beta_j^* - \frac{1}{\|X_j\|^2} X_j^T P_S X_j\beta_j^* + \sum_{l \in S_1, l \neq j} \frac{1}{\|X_j\|^2} X_j^T (I - P_S)X_l\beta_l^* + \frac{1}{\|X_j\|^2} X_j^T (I - P_S)w.
\]
This leads to the decomposition
\[
\bar{\beta}_j(z) - \bar{\beta}_j(z^*) = -\frac{1}{\|X_j\|^2} X_j^T P_S X_j\beta_j^* + \sum_{l \in S_1, l \neq j} \frac{1}{\|X_j\|^2} X_j^T (I - P_S)X_l\beta_l^* - \frac{1}{\|X_j\|^2} X_j^T P_S w
\]
\[
+ \left( \frac{1}{\|X_j\|^2} X_j^T w - \left[ (X_S^T, X_S^*)^{-1} X_S^T, w \right]_{\phi_S^*(j)} \right).
\]
We will bound each term on the right hand side of the above equation.

(1.1) First, we have
\[
\left| -\frac{1}{\|X_j\|^2} X_j^T P_S X_j\beta_j^* \right| \leq \frac{2}{\min_j \|X_j\|^2} \max_{j \in S_1} \|X_j^T P_S X_j\| \|\beta_j^*\|
\]
\[
\leq \frac{2}{n} C_s \|\beta_j^*\| \log p,
\]
where the last inequality is by (96) and (100). Then
\[
\sum_{j \in S_1} \left( -\frac{1}{\|X_j\|^2} X_j^T P_S X_j\beta_j^* \right)^2 \leq \frac{4C_s^2 \log^2 p}{n^2} \|\beta_S^*\|^2.
\]

(1.2) For the second term, we have a matrix representation,
\[
\sum_{j \in S_1} \left( \sum_{l \in S_1, l \neq j} \frac{1}{\|X_j\|^2} X_j^T (I - P_S)X_l\beta_l^* \right)^2
\]
\[
\leq \frac{1}{\min_j \|X_j\|^2} \left\| \left( X_S^T (I - P_S)X_S - \text{diag} \{ X_S^T (I - P_S)X_S \} \right) \beta_S^* \right\|^2
\]
\[
\leq \frac{1}{\min_j \|X_j\|^2} \left\| X_S^T (I - P_S)X_S - \text{diag} \{ X_S^T (I - P_S)X_S \} \right\|^2 \|\beta_S^*\|^2
\]
\[
\leq \frac{2}{n^2} C_s \log p \|\beta_S^*\|^2.
\]
where the last inequality is by (96) and (101).
For the third term, we have
\[
\sum_{j \in S_1} \left( -\frac{1}{\|X_j\|^2} X_j^T P S w \right)^2 \leq \frac{1}{\min_j \|X_j\|^4} \sum_{j \in S_1} (X_j^T P S w)^2 \\
\leq \frac{2}{n^2} C s \log^2 p (|S_1| + |S_3|) \\
= \frac{2}{n^2} C sh(z, z^*) \log^2 p,
\]
where the second to the last inequality is by (96) and (102).

For the last term, using (96) and (120), we have
\[
\max_{j \in S^*} \left| \frac{1}{\|X_j\|^2} X_j^T w - \left[ (X_{S^*}^T X_{S^*})^{-1} X_{S^*}^T w \right]_{\phi_{S^*}(j)} \right| \\
\leq \max_{j \in S} \frac{1}{\|X_j\| \max_{j \in S^*} \|X_j\|} \left| \frac{1}{\|X_j\|^2} X_j^T w - \|X_j\| \left[ (X_{S^*}^T X_{S^*})^{-1} X_{S^*}^T w \right]_{\phi_{S^*}(j)} \right| \\
\leq \sqrt{\frac{2}{n} \frac{C s \log^2 p}{n^2}}.
\]

Hence,
\[
\sum_{j \in S_1} \left( -\frac{1}{\|X_j\|^2} X_j^T P S w \right)^2 \leq |S_1| \max_{j \in S^*} \left( \frac{1}{\|X_j\|^2} X_j^T w - \left[ (X_{S^*}^T X_{S^*})^{-1} X_{S^*}^T w \right]_{\phi_{S^*}(j)} \right)^2 \\
\leq \frac{2Cs |S_1| \log^2 p}{n^2} \\
\leq \frac{2Csh(z, z^*) \log^2 p}{n^2},
\]
where the last inequality is due to (122).

Combining the above results, we have
\[
\sum_{j \in S_2} \left( \tilde{\beta}_j(z) - \tilde{\beta}_j(z^*) \right)^2 \\
\leq 4 \left( \frac{4C^2 s^2 \log^2 p}{n^2} \|\beta_{S_1}^*\|^2 + \frac{2}{n^2} C s \log p \|\beta_{S_1}^*\|^2 + \frac{2}{n^2} C sh(z, z^*) \log^2 p + \frac{2Csh(z, z^*) \log^2 p}{n^2} \right) \\
\leq 4 \left( \frac{16C^2 s^2 \log^2 p}{n^2} + \frac{16s \log p}{n^2} \right) \frac{\ell(z, z^*)}{n}
\]
where the last inequality is by (115) and (116).

(2) Regime $j \in S_2$. In this case, $\tilde{\beta}_j(z) - \tilde{\beta}_j(z^*)$ can be written as
\[
\left[ (X_{S}^T X_{S})^{-1} X_{S}^T X_{S_1} \beta_{S_1}^* \right]_{\phi_{S}(j)} + \left[ (X_{S}^T X_{S})^{-1} X_{S}^T w \right]_{\phi_{S}(j)} - \left[ (X_{S}^T X_{S})^{-1} X_{S}^T w \right]_{\phi_{S^*}(j)}.
\]
We will bound the first term, and then the second and the third term will be analyzed together.

(2.1) For the first term, we have

\[
\sum_{j \in S_2} \left[ (X_S^T X_S)^{-1} X_S^T X_{S_1} \beta_{S_1}^* \right]_{\phi_S(j)}^2 \leq \sum_{j \in S} \left[ (X_S^T X_S)^{-1} X_S^T X_{S_1} \beta_{S_1}^* \right]_{\phi_S(j)}^2 = \left\| (X_S^T X_S)^{-1} X_S^T X_{S_1} \beta_{S_1}^* \right\|^2 \leq \left\| (X_S^T X_S)^{-1} X_S^T X_{S_1} \right\|^2 \beta_{S_1}^* \|_2^2 \leq C \frac{s \log p}{n} \| \beta_{S_1}^* \|_2^2,
\]

where the last inequality is due to (98) in Lemma 7.4.

(2.2) Note that

\[
\sum_{j \in S_2} \left[ (X_S^T X_S)^{-1} X_S^T w \right]_{\phi_S(j)}^2 - \left[ (X_S^T X_{S_1})^{-1} X_S^T w \right]_{\phi_{S_1}(j)}^2 \leq \left\| (X_S^T X_S)^{-1} X_S^T w \right\|_{\phi_S(S_2)}^2 \leq \left\| (X_S^T X_{S_1})^{-1} X_S^T w \right\|_{\phi_{S_1}(S_2)}^2.
\]

Since $S$ is close to $S^*$, the two length-$|S_2|$ vectors on the right hand side of the above equation should also be close to each other. Applying block matrix inverse formula, we have

\[
(X_S X_S^T)^{-1} = \begin{pmatrix} X_S^T X_2 & X_S^T X_3 \\ X_2^T X_2 & X_3^T X_3 \end{pmatrix}^{-1} \triangleq \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\]

where

\[
A_{11} = (X_S^T X_2)^{-1} + (X_S^T X_2)^{-1} X_S^T X_3 (X_3^T (I - P_2) X_3)^{-1} X_3^T X_2 (X_S^T X_2)^{-1},
A_{12} = - (X_S^T X_2)^{-1} X_S^T X_3 (X_3^T (I - P_2) X_3)^{-1},
A_{21} = - (X_3^T (I - P_2) X_3)^{-1} X_3^T X_2 (X_S^T X_2)^{-1},
A_{22} = (X_3^T (I - P_2) X_3)^{-1}.
\]

With these notation, we have

\[
\left[ (X_S^T X_S)^{-1} X_S^T w \right]_{\phi_S(S_2)} = A_{11} X_2^T w + A_{12} X_3^T w = (X_S^T X_2)^{-1} X_2^T w - (X_S^T X_2)^{-1} X_S^T X_3 (X_3^T (I - P_2) X_3)^{-1} X_3^T (I - P_2) w,
\]

and

\[
\left[ (X_S^T X_{S_1})^{-1} X_S^T w \right]_{\phi_{S_1}(S_2)} = (X_S^T X_2)^{-1} X_2^T w - (X_S^T X_2)^{-1} X_S^T X_1 (X_1^T (I - P_2) X_1)^{-1} X_1^T (I - P_2) w.
\]
Thus
\[ \sum_{j \in S_2} \left( [X_S^T X_S]^{-1} X_S^T w]_{\phi S(j)} - [X_S^{*T} X_S^{*}]^{-1} X_S^{*T} w]_{\phi S^{*}(j)} \right)^2 \]
\[ = \left\| (X_S^T X_S)^{-1} X_S^T X_1 (X_1^T (I - P_2) X_1)^{-1} X_1^T (I - P_2) w - (X_S^T X_S)^{-1} X_S^T X_3 (X_3^T (I - P_2) X_3)^{-1} X_3^T (I - P_2) w \right\|^2 \]
\[ \leq \left\| (X_S^T X_S)^{-1} X_S^T X_1 \right\|^2 \left\| (X_1^T (I - P_2) X_1)^{-1} X_1^T (I - P_2) w \right\|^2 \]
\[ + \left\| (X_S^T X_S)^{-1} X_S^T X_3 \right\|^2 \left\| (X_3^T (I - P_2) X_3)^{-1} X_3^T (I - P_2) w \right\|^2 \]
\[ \leq \left( C_s \log p \right) \left( C |S_1| \log p + C |S_3| \log p \right) \]
\[ = C_2 \frac{s h(z, z^*) \log^2 p}{n^2}, \]
where the second to the last inequality is due to (98) and (99).

(2.3). Combining the above results, we have
\[ \sum_{j \in S_2} \left( \beta_j (z) - \beta_j (z^*) \right)^2 \leq 2 \left( \frac{C_s \log p}{n} \left\| \beta_S^* \right\|^2 + \frac{C^2 s h(z, z^*) \log^2 p}{n^2} \right) \]
\[ \leq \left( \frac{4C_s \log p}{n} + \frac{4C^2 s \log^2 p}{\lambda^2 n^2} \right) \ell (z, z^*), \]
where the last inequality is by (115) and (116).

(3) Regime $j \in S_3$. Since
\[ \beta_j (z) = \beta_j^* + \left( X_S^T X_S \right)^{-1} X_S^T X_S \beta_j^* \] 
\[ \beta_j (z^*) = \frac{1}{\left\| X_j \right\|^2} X_j^T (I - P_S^*) w, \]
and $\beta_j^* = 0$, we can write $\beta_j (z) - \beta_j (z^*)$ as
\[ \left( X_S^T X_S \right)^{-1} X_S^T X_S \beta_j^* \] 
\[ = \left( X_S^T X_S \right)^{-1} X_S^T X_S \beta_j^* \] 
\[ = \left( X_S^T X_S \right)^{-1} X_S^T X_S \beta_j^* \] 
\[ \leq C \frac{s \log p}{n} \left\| \beta_S^* \right\|^2 . \]
(3.2). For the second term, we have

\[
\sum_{j \in S_3} \left( \frac{1}{\|X_j\|^2} X_j^T P_{S_j} w \right)^2 \leq \frac{1}{\min_j \|X_j\|^4} \|X_j^T P_{S_j} w\|^2 \leq \frac{4}{n^2} C_3 S \log p \leq \frac{4}{n^2} C_3 h(z, z^*) \log p,
\]

where the second to last inequality is due to (96) and (103).

(3.3). For the last two terms, we can again apply block matrix inverse formula to simplify them. Using (123), we have

\[
\left( (X_S^T X_S)^{-1} X_S^T w \right)_{\phi_S(j)} = \left[ \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right] \left( X_S^T w \right)_{\phi_S(j)}
= [A_{21} X_2^T w + A_{22} X_3^T w]_{\phi_S(j)}
= - \left( (X_3^T (I - P_2) X_3)^{-1} X_3^T P_2 w \right)_{\phi_S(j)} + \left( (X_3^T (I - P_2) X_3)^{-1} X_3^T w \right)_{\phi_S(j)}.
\]

Then

\[
\left( (X_S^T X_S)^{-1} X_S^T w \right)_{\phi_S(j)} - \frac{1}{\|X_j\|^2} X_j^T w = - \left( (X_3^T (I - P_2) X_3)^{-1} X_3^T P_2 w \right)_{\phi_S(j)}
+ \left( (X_3^T (I - P_2) X_3)^{-1} X_3^T w \right)_{\phi_S(j)} - \frac{1}{\|X_j\|^2} X_j^T w.
\]

Consequently,

\[
\sum_{j \in S_3} \left( \left( (X_S^T X_S)^{-1} X_S^T w \right)_{\phi_S(j)} - \frac{1}{\|X_j\|^2} X_j^T w \right)^2
= \left\| (X_3^T (I - P_2) X_3)^{-1} X_3^T P_2 w + (X_3^T (I - P_2) X_3)^{-1} X_3^T w - D^{-1} X_3^T w \right\|^2
\leq 2 \left\| (X_3^T (I - P_2) X_3)^{-1} X_3^T P_2 w \right\|^2 + 2 \left\| (X_3^T (I - P_2) X_3)^{-1} X_3^T w - D^{-1} X_3^T w \right\|^2
\leq 2 \left\| (X_3^T (I - P_2) X_3)^{-1} X_3^T P_2 w \right\|^2 + 2 \left\| (X_3^T (I - P_2) X_3)^{-1} - D^{-1} \right\|^2 \left\| X_3^T w \right\|^2
\leq 2 \left\| (X_3^T (I - P_2) X_3)^{-1} X_3^T P_2 w \right\|^2 + 4 \left\| (X_3^T (I - P_2) X_3)^{-1} - (n - |S_2|)^{-1} I_{|S_3|} \right\|^2 \left\| X_3^T w \right\|^2
+ 4 \left\| (n - |S_2|)^{-1} I_{|S_3|} - D^{-1} \right\|^2 \left\| X_3^T w \right\|^2
\]

where \( D \in \mathbb{R}^{|S_3| \times |S_3|} \) is a diagonal matrix with diagonal entries \( \{1/\|X_j\|^2\}_{j \in S_3} \) and off-diagonal entries being 0. By (104), we have

\[
\left\| (X_3^T (I - P_2) X_3)^{-1} X_3^T P_2 w \right\|^2 \leq C n^{-2} S \log p.
\]

By (105), we have

\[
\left\| X_3^T (I - P_2) X_3 - (n - |S_2|) I_{|S_3|} \right\|^2 \leq C n s \log p.
\]

61
Together with (106), we have
\[
\left\| (X_3^T (I - P_2) X_3)^{-1} - (n - |S_2|)^{-1} I_{|S_3|} \right\|^2 \\
\leq \left\| (X_3^T (I - P_2) X_3)^{-1} \right\|^2 \left\| I_{|S_3|} - \frac{X_3^T (I - P_2) X_3}{n - |S_2|} \right\|^2 \\
\leq \frac{2C^3 s \log p}{n^3},
\]
where we have used the assumption $|S_3| \leq 2C_0 s$. By (96), we have
\[
\left\| (n - |S_2|)^{-1} I_{|S_3|} - D^{-1} \right\|^2 \leq \max_{j \in \mathbb{P}} \left\{ \left| \frac{1}{n - |S_2|} - \frac{1}{\| X_j \|^2} \right|^2, \left| \frac{1}{n - 2C_0 s} - \frac{1}{n} \right|^2 \right\} \\
\leq 2C \log p \frac{p}{n^3} + \frac{8s^2}{n^4}.
\]
By (107), we have
\[
\| X_3^T w \|^2 \leq C n |S_3| \log p.
\]
As a consequence, we have
\[
\sum_{j \in S_3} \left( \left( X_S^T X_S \right)^{-1} X_S^T w \left( I_{\phi_S(j)} - \frac{1}{\| X_j \|^2} X_j^T w \right) \right)^2 \\
\leq 2C n^{-2} |S_3| \log p + \left( \frac{4C^3 s \log p}{n^3} + \frac{8s^2}{n^4} \right) C n |S_3| \log p \\
\leq 8C^3 s \log p \frac{n^2}{n^2} |S_3| \log p \\
\leq 8C^3 s \log p \frac{n^2}{n^2} h(z, z^*) \log p.
\]
(3.4). Combining the above results, we have
\[
\sum_{j \in S_3} \left( \beta_j(z) - \tilde{\beta}_j(z^*) \right)^2 \\
\leq 3 \left( C \frac{s \log p}{n} \| \beta_{S_1}^* \|^2 + \frac{4}{n^2} C s h(z, z^*) \log p + 8C^3 s \log p \frac{p}{n^2} h(z, z^*) \log p \right) \\
\leq 3 \left( \frac{2Cs \log p}{n} + \frac{32C^3 s \log^2 p}{\lambda^2 n^2} \right) \frac{\ell(z, z^*)}{n},
\]
where the last inequality is by (115) and (116).

(4) Regime $j \in S_4$. In this case, we have
\[
\tilde{\beta}_j(z) - \tilde{\beta}_j(z^*) = \frac{1}{\| X_j \|^2} X_j^T (I - P_S) X_S \beta_{S_1}^* + \frac{1}{\| X_j \|^2} X_j^T (P_{S^*} - P_S) w.
\]
Then,
\[
\max_{T \subseteq S_4} \frac{1}{|T|} \sum_{j \in T} \left( \tilde{\beta}_j(z) - \tilde{\beta}_j(z^*) \right)^2 
\leq \frac{1}{\min_j \|X_j\|^2} \left( \max_{T \subseteq S_4} \frac{1}{|T|} \sum_{j \in T} (X_j^T (I - P_S) X_{S_1} \beta_{S_1}^*)^2 + \max_{T \subseteq S_4} \frac{1}{|T|} \sum_{j \in T} (X_j^T (P_{S^*} - P_S) w)^2 \right) 
\leq \frac{1}{\min_j \|X_j\|^2} \left( \max_{T \subseteq S_4} \frac{1}{|T|} \sum_{j \in T} X_j^T (I - P_S) X_{S_1} \beta_{S_1}^* \right)^2 + \max_{T \subseteq S_4} \frac{1}{|T|} \sum_{j \in T} \|X_j^T (P_{S^*} - P_S) w\|^2 
\leq \frac{4}{n^2} \left( Cn \log p \|\beta_{S_1}^*\|^2 + C (|S_1| + |S_3|) \log^2 p \right) 
= \frac{4}{n^2} \left( Cn \log p \|\beta_{S_1}^*\|^2 + Ch(z, z^*) \log^2 p \right).
\]
where the last inequality is by (96), (108) and (109). Then by (115) and (116), we have
\[
\max_{T \subseteq S_4} \frac{1}{|T|} \sum_{j \in T} \left( \tilde{\beta}_j(z) - \tilde{\beta}_j(z^*) \right)^2 \leq \left( \frac{8 C \log p}{n} + \frac{8 C \log^2 p}{\lambda^2 n^2} \right) \ell(z, z^*) \frac{n}{\lambda^2}.
\]
(5) Combining the bounds. Now we are ready to combine the bounds obtained in the four regimes. Let $T \subseteq [p]$ be any set. We have
\[
\sum_{j \in T} \left( \tilde{\beta}_j(z) - \tilde{\beta}_j(z^*) \right)^2 
\leq \sum_{j \in S_2} \left( \tilde{\beta}_j(z) - \tilde{\beta}_j(z^*) \right)^2 + \sum_{j \in S_1} \left( \tilde{\beta}_j(z) - \tilde{\beta}_j(z^*) \right)^2 + \sum_{j \in S_1} \left( \tilde{\beta}_j(z) - \tilde{\beta}_j(z^*) \right)^2 + \sum_{j \in S_1 \cap T} \left( \tilde{\beta}_j(z) - \tilde{\beta}_j(z^*) \right)^2 
\leq \left( \left( \frac{4 C \log p}{n} + \frac{4 C^2 s \log^2 p}{\lambda^2 n^2} \right) + 4 \left( \frac{16 C^2 s \log^2 p}{n^2} + \frac{16 s \log p}{\lambda^2 n^2} \right) + 3 \left( \frac{2 C \log p}{n} + \frac{32 C^3 s \log^2 p}{\lambda^2 n^2} \right) \right) \frac{\ell(z, z^*)}{n} 
+ \left( \frac{8 C \log p}{n} + \frac{8 C \log^2 p}{\lambda^2 n^2} \right) (|T| \vee s) \frac{\ell(z, z^*)}{n} 
\leq \left( \frac{128 C^2 \log p}{n} + \frac{256 C^3 \log^2 p}{\lambda^2 n^2} \right) (s + |T|) \frac{\ell(z, z^*)}{n}.
\]
Thus
\[
\max_{T \subseteq [p]} \frac{1}{s + |T|} \sum_{j \in T} \left( \tilde{\beta}_j(z) - \tilde{\beta}_j(z^*) \right)^2 \leq \left( \frac{128 C^2 \log p}{n} + \frac{256 C^3 \log^2 p}{\lambda^2 n^2} \right) \frac{\ell(z, z^*)}{n}.
\]
Together with (121), we have
\[
\max_{T \subseteq [p]} \frac{1}{s + |T|} \sum_{j \in T} \max_{b \in \{-1, 1, 0\} \setminus \{z_j^*\}} \frac{G_j(z_j^*, b; z) \|\mu_j(B^*, b) - \mu_j(B^*, z_j^*)\|^2}{\Delta_j(z_j^*, b) \ell(z, z^*)} 
\leq 2 \left( \frac{128 C^2 \log p}{n} + \frac{256 C^3 \log^2 p}{\lambda^2 n^2} \right) \max_{j \in [p]} \left\{ 4 \left( \frac{\|\beta_{z_j^*}^*\|}{\|\beta_{z_j^*}^*\| - t(X_j)} \right)^2, \left( \frac{\lambda}{t(X_j)} \right)^2 \right\}.
\]
Recall that $\Delta_{\min}^2 = \lambda^2 \min_j \|X_j\|^2 \geq n\lambda^2/2$. For any $T \subset [p]$, we have $\tau/|T| \leq \tau/(\tau + 2n\lambda^2 |T|) \leq C_0/(C_0 s + |T|)$, since $\tau \leq C_0 sn\lambda^2$. This gives us

$$
\max_T \frac{\tau}{\tau + 4\Delta_{\min}^2 |T|} \sum_{j \in [p]} \max_{b \in (-1,1) \setminus \{z_j^*\}} \frac{G_j(z_j^*, b; z)^2 \|\mu_j(B^*, b) - \mu_j(B^*, z_j^*)\|^2}{\Delta_j(z_j^*, b)^4 \ell(z, z^*)} \leq C^* s \left( \frac{\log^2 p}{n\lambda^2} + \frac{\log^2 p}{n\lambda^2} \right) \max_{j \in [p]} \left\{ \begin{array}{l}
4 \left( \frac{\|b_j^*\|}{\|t(X_j)\|} \right)^2, \\
\left( \frac{\lambda}{\|t(X_j)\|} \right)^2 \end{array} \right.,
$$

for some constant $C'$. The proof is complete. \(\square\)

**Proof of Lemma 6.2**. Recall for any $j \in [p]$, $T_j$ the local test to recover $z_j^*$ is defined in (13). We have the decomposition $T_j = \mu_j(B^*, z_j^*) + \epsilon_j$, where $\epsilon_j = \|X_j\|^{-1}X_j^T w \sim N(0, 1)$. Since $\nu_j(\hat{B}(z^*), z_j^*) = \nu_j(\hat{B}(z^*), B, b) = 2(z_j^* - b) \|X_j\| t(X_j)$, by (52), for any $0 < \delta < 1$, we have

$$
1 \left\{ \|\mu_j(B^*, b) - \mu_j(B^*, z_j^*)\|^2 \leq -\frac{1-\delta}{2} \Delta_j(z_j^*, b)^2 \right\} 
\leq \left\{ \begin{array}{l}
1 \left\{ z_j^* \epsilon_j \leq (1-\delta)\|X_j\| (|\beta_j^*| - t(X_j)) \right\}, \\
1 \left\{ -b_j \leq (1-\delta)\|X_j\| t(X_j) \right\} \end{array} \right. z_j^* \neq 0 \text{ and } b \neq z_j^*).
$$

Together with (50), for $b \neq z_j^*$, we have

$$
\|\mu_j(B^*, b) - \mu_j(B^*, z_j^*)\|^2 
\leq \left\{ \begin{array}{l}
\left( \frac{\beta_j^*}{\|X_j\|} \right)^2, \\
4 \lambda^2 \|X_j\|^2 \left( |X_j|^2 \right)
\end{array} \right. 1 \left\{ |\beta_j^*| - t(X_j) \right\} z_j^* = 0.
$$

As a consequence

$$
\xi_{\text{ideal}}(\delta) \leq 8 \sum_{j \in S^*} \|\beta_j^*\|^2 \|X_j\|^2 \left( |X_j|^2 \right) 1 \left\{ |\beta_j^*| - t(X_j) \right\} + 4 \sum_{j \in S^*} \lambda^2 \|X_j\|^2 1 \left\{ |\beta_j| \geq (1-\delta)\|X_j\| t(X_j) \right\}.
$$

Define $\mathcal{F}$ to be the event that (96) holds. Then by Lemma 7.4, we know that $P(\mathcal{F}) \geq 1 - p^{-C'}$. Under the event $\mathcal{F}$ and the condition that $\text{SNR} \to \infty$, we have

$$
\xi_{\text{ideal}}(\delta) 1_{\{\mathcal{F}\}} \leq 8 \sum_{j \in S^*} \|\beta_j^*\|^2 \|X_j\|^2 1 \left\{ |X_j|^2 \leq -C \sqrt{n \log p} \right\} 1 \left\{ |X_j|^2 \leq -n \leq C \sqrt{n \log p} \right\} + 4 \sum_{j \in S^*} \lambda^2 \|X_j\|^2 1 \left\{ |X_j|^2 \leq -n \leq C \sqrt{n \log p} \right\},
$$

which implies

$$
\mathbb{E}\xi_{\text{ideal}}(\delta) 1_{\{\mathcal{F}\}} \leq 16n \sum_{j \in S^*} \|\beta_j^*\|^2 \mathbb{P} \left( \frac{\epsilon_j}{1-\delta} \geq |X_j| (|\beta_j^*| - t(X_j)) \& |X_j|^2 \leq -n \leq C \sqrt{n \log p} \right) + 16n \sum_{j \in S^*} \lambda^2 \mathbb{P} \left( \frac{\epsilon_j}{1-\delta} \geq |X_j| t(X_j) \& |X_j|^2 \leq -n \leq C \sqrt{n \log p} \right).
$$
We are going to upper bound the above quantity by \( \tilde{\psi}(n, p, s, \lambda, \delta, C) \), defined in (111). To this end, we will first show the function \( f(y) = y^2 \mathbb{P} \left( \frac{\epsilon}{1 - \delta} \geq \| \zeta \| (y - t(\zeta)) \right) \leq C \sqrt{n \log p} \) is a decreasing function of \( y \) when \( y \geq \lambda \) and \( \lambda > 0 \). Since the function \( t(\zeta) \) only depends on \( \| \zeta \| \), we can also write \( t(\zeta) = t(\| \zeta \|) \) with a slight abuse of notation in (113). Define \( u_{\min} = \sqrt{n - C \sqrt{n \log p}} \) and \( u_{\max} = \sqrt{n + C \sqrt{n \log p}} \). Then, we have

\[
f(y) = y^2 \mathbb{P} \left( \frac{\epsilon}{1 - \delta} \geq \| \zeta \| (y - t(\zeta)) \right) \leq C \sqrt{n \log p}
= y^2 \int_{u_{\min}}^{u_{\max}} p(u) G \left( (1 - \delta) u (y - t(u)) \right) du,
\]
where \( p(\cdot) \) is the density of \( \| \zeta \| \). According to the same argument used in the proof of Lemma 7.6, it can be shown that \( \min_{u \in [u_{\min}, u_{\max}]} u(\lambda - t(u)) \to \infty \). Thus, \( u(y - t(u)) \geq u(\lambda - t(u)) > 0 \) for \( y \geq \lambda \) and \( u \in [u_{\min}, u_{\max}] \). Moreover, we also have \((1 - \delta)^2 u^2 y(y - t(u)) \geq (1 - \delta)^2 u^2 \lambda (\lambda - t(u)) > 2 \) for \( y \geq \lambda \) and \( u \in [u_{\min}, u_{\max}] \). Therefore,

\[
\frac{2}{(1 - \delta) u (y - t(u))} - y (1 - \delta) u = \frac{2 - (1 - \delta)^2 u^2 y (y - t(u))}{(1 - \delta) u (y - t(u))} \leq 2 - \frac{(1 - \delta)^2 u^2 \lambda (\lambda - t(u))}{(1 - \delta) u (y - t(u))} < 0.
\]

This gives

\[
f'(y) = \int_{u_{\min}}^{u_{\max}} p(u) \left( 2yG' \left( (1 - \delta) u (y - t(u)) \right) - y^2 (1 - \delta) u g \left( (1 - \delta) u (y - t(u)) \right) \right) du \leq \int_{u_{\min}}^{u_{\max}} p(u) g \left( \frac{2}{(1 - \delta) u (y - t(u))} - y (1 - \delta) u \right) g \left( (1 - \delta) u (y - t(u)) \right) du \leq 0,
\]

where we have used (112). As a result, \( f(y) \) is a decreasing function for all \( y \geq \lambda \), which implies

\[
\mathbb{E}\xi_{\text{ideal}}(\delta) \mathbf{1}_{\mathcal{F}} \leq 16n \sum_{j \in S^*} \lambda^2 \mathbb{P} \left( \frac{\epsilon_j}{1 - \delta} \geq \| X_j \| \left( \lambda - t(X_j) \right) \& \| X_j \|^2 - n \leq C \sqrt{n \log p} \right) + 16n \sum_{j \notin S^*} \lambda^2 \mathbb{P} \left( \frac{\epsilon_j}{1 - \delta} \geq \| X_j \| t(X_j) \& \| X_j \|^2 - n \leq C \sqrt{n \log p} \right) = 16n \lambda^2 \tilde{\psi}(n, p, s, \lambda, \delta, C).
\]

By applying Markov inequality, we have with probability at least \( 1 - w^{-1} \),

\[
\xi_{\text{ideal}}(\delta) \mathbf{1}_{\mathcal{F}} \leq 16wn \lambda^2 \tilde{\psi}(n, p, s, \lambda, \delta, C),
\]

where \( w \) is any sequence that goes to infinity. A union bound implies

\[
\xi_{\text{ideal}}(\delta) \leq 16wn \lambda^2 \tilde{\psi}(n, p, s, \lambda, \delta, C)
\]

(124)
holds with probability at least $1 - w^{-1} p^{-C'}$. Taking $\delta = \delta_p = o((\log p)^{-1})$ and $w = \exp(\text{SNR})$, the desired conclusion follows an application of Lemma 7.5. Thus, the proof is complete.

Proof of Proposition 6.1. By Proposition 5.1 of [54], we have

$$\|\tilde{\beta} - \beta^*\|^2 \leq \frac{C_1 s \log \frac{ep}{s}}{n},$$

with probability at least $1 - 2^{-C_2 s}$ for some constants $C_1, C_2 > 0$, as long as $A$ is chosen to be sufficiently large. In the rest part of the proof, we assume (96) holds. We divide the calculation of $\ell(\tilde{z}, z^*)$ into three parts. First we have

$$\sum_{j=1}^p \lambda^2 \|X_j\|^2 \mathbb{I}\{z_j \neq 0, z_j^* = 0\} \leq \sum_{j=1}^p \lambda^2 \|X_j\|^2 \mathbb{I}\{|\tilde{\beta}_j| > \frac{\lambda}{2}, \beta_j^* = 0\}$$

\[
\leq 4 \sum_{j=1}^p \|X_j\|^2 (\tilde{\beta}_j - \beta_j^*)^2 \mathbb{I}\{|\tilde{\beta}_j|^2 > \lambda, \beta_j^* = 0\} \]
\[
\leq 8n \sum_{j=1}^p (\tilde{\beta}_j - \beta_j^*)^2 \mathbb{I}\{|\tilde{\beta}_j| > \frac{\lambda}{2}, \beta_j^* = 0\}.
\]

Similarly, we have

$$\sum_{j=1}^p |\beta_j|^2 \|X_j\|^2 \mathbb{I}\{z_j^* \neq 0, \tilde{z}_j = 0\} \leq \sum_{j=1}^p |\beta_j|^2 \|X_j\|^2 \mathbb{I}\{|\beta_j^*|^2 \geq \lambda, |\tilde{\beta}_j| \leq \frac{\lambda}{2}\}$$

\[
\leq 8n \sum_{j=1}^p (\tilde{\beta}_j - \beta_j^*)^2 \mathbb{I}\{|\beta_j^*|^2 \geq \lambda, |\tilde{\beta}_j| \leq \frac{\lambda}{2}\},
\]

where in the last inequality, since $|\beta_j^*| \geq \lambda$ and $|\tilde{\beta}_j| \leq \frac{\lambda}{2}$, we have

$$|\beta_j^* - \tilde{\beta}_j| \geq |\beta_j^*| - |\tilde{\beta}_j| \geq \frac{|\beta_j|}{2} + \frac{\lambda}{2} - \frac{\lambda}{2} = \frac{|\beta_j|}{2}.$$
because when $\beta^*_j \leq -\lambda$ and $\tilde{\beta}_j > \frac{1}{2}$, we have

$$|\beta^*_j - \tilde{\beta}_j| = -\beta^*_j + \tilde{\beta}_j \geq |\beta^*_j|,$$

and when $\beta^*_j \geq \lambda$ and $\tilde{\beta}_j < -\frac{1}{2}$, we have

$$|\beta^*_j - \tilde{\beta}_j| = \beta^*_j - \tilde{\beta}_j \geq |\beta^*_j|.$$

Combining all of the above results together, we have

$$\ell(\tilde{z}, z^*) \leq 8n \left\| \beta - \beta^* \right\|^2 \leq 8C_1 s \log \frac{ep}{s}.$$

Under the assumption $\limsup s/p < 1/2$ and $\text{SNR} \to \infty$, we have $n\lambda^2 \geq 2 \log \frac{p-s}{s} > C_3 \log \frac{2p}{s}$. Thus, there exists some constant $C_0 > 0$ such that $\ell(\tilde{z}, z^*) \leq C_0 s n \lambda^2$. A union bound with the probability that the event (96) holds leads to the desired result.

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