Poisson structures on the homology of the space of knots

KEICHI SAKAI

In this article we study the Poisson algebra structure on the homology of the totalization of a fibrant cosimplicial space associated with an operad with multiplication. This structure is given as the Browder operation induced by the action of little disks operad, which was found by McClure and Smith. We show that the Browder operation coincides with the Gerstenhaber bracket on the Hochschild homology, which appears as the $E^2$-term of the homology spectral sequence constructed by Bousfield. In particular we consider a variant of the space of long knots in higher dimensional Euclidean space, and show that Sinha’s homology spectral sequence computes the Poisson algebra structure of the homology of the space. The Browder operation produces a homology class which does not directly correspond to chord diagrams.

55P48; 55P35

1 Introduction

In [14] McClure and Smith proved that the totalization of a cosimplicial space $O^*$ associated with a non-symmetric topological operad $O$ with multiplication (a base point in the operadic sense) admits an action of an operad weakly equivalent to the little disks operad. As an immediate consequence, there exists a natural bracket on the (rational) homology of $\text{Tot} O^*$, called the Browder operation.

On the other hand, Bousfield [1] constructed a spectral sequence computing the homology of a totalization (under some conditions). When moreover the cosimplicial space arises from the operad $O$ with multiplication, then its $E^1$-term is the Hochschild complex of the homology operad $H_*(O)$. It is known (Gerstenhaber–Voronov [6], Turchin [17, 18]) that there exist a natural product and a bracket on such a complex which induce the Gerstenhaber algebra structure, the degree one Poisson algebra structure, on the homology. Note that the Gerstenhaber algebra structure also comes from the action of the chains of the little disks operad (Deligne’s conjecture; see McClure–Smith [13]).

The main result of this article states that the above two actions correspond with each other at least on the homology level, or the two brackets coincide with each other.
Namely, for any operad \( O \) with multiplication, Bousfield’s spectral sequence computes \( H_*(\text{Tot} O) \) as a Poisson algebra. In some cases this gives us a method to compute the (topological) Browder operation purely algebraically, as was done by Turchin [18].

In particular we concentrate on the space \( \mathcal{K}_n \) of long knots in \( \mathbb{R}^n, n > 3 \), or its variant \( \mathcal{K}_n' \) (for definition see Sinha [16] or Section 2 below). For \( \mathcal{K}_n' \), Sinha [16] constructed a model using the Kontsevich operad, essentially equivalent to the little balls operad, based on the embedding calculus due to Goodwillie [7, 8] and Goodwillie–Weiss [9, 19]. In this case Bousfield’s spectral sequence rationally degenerates at \( E^2 \) because of the formality of the operad (see Kontsevich [10], Lambrechts–Turchin–Volić [11] and Lambrechts–Volić [12]). Thus the homology of the space of long knots is isomorphic to the Hochschild homology of the Kontsevich operad. Our result shows that this is an isomorphism of Poisson algebras.

To prove the main result, we explicitly write down McClure–Smith action, which compares with Budney’s (possibly another) one [2] defined on certain embedding spaces. In the case of long knots, our explicit description suggests that McClure–Smith action might be equivalent to Budney’s one, but no rigorous proof of their consistency has been given. Budney’s action is quite geometric; “shrink one (framed) knot and make it go through another knot.” Budney proved that the space of (framed) long knots in \( \mathbb{R}^3 \) is a free 2-cubes object, hence the homology operations are highly non-trivial (as studied by Budney and Cohen [3]). In higher dimensional case we might be able to approach the similar freeness problem by means of the spectral sequence.

Note that Salvatore has announced our result [15, Proposition 22], but not written the proof. He proved that the space \( \mathcal{K}_n \) \( (n > 3) \) is a double loop space but the projection \( \mathcal{K}_n' \to \mathcal{K}_n \) is not a double loop map, by use of the main result of this paper. Here we emphasize that our result implies the non-triviality of the (topological) Browder operation, and this produces a homology class of \( \mathcal{K}_n \) which does not directly correspond to any chord diagram (see Section 3).

## 2 The spaces of knots and the result

A long knot is an embedding \( \mathbb{R}^1 \hookrightarrow \mathbb{R}^n, n \geq 3 \), which agrees with a fixed line outside a compact set. A long immersion is defined similarly.

Let \( \mathcal{K}_n \) and \( \text{Imm}_n \) be the spaces of long knots and long immersions in \( \mathbb{R}^n \) respectively. Consider the space

\[
\mathcal{K}'_n := \text{the homotopy fiber of } \mathcal{K}_n \hookrightarrow \text{Imm}_n.
\]
Sinha’s model for the space is as follows.

**Theorem 2.1** [16] There exists a topological operad $X_n = \{X_n(k)\}_{k \geq 0}$ with multiplication such that

1. there exist homotopy equivalences from $\text{Conf}(\mathbb{R}^n, k) = (\mathbb{R}^n)^k \setminus \Delta$ to $X_n(k)$ for all $k \geq 0$.
2. when $n > 3$, the homotopy invariant totalization $\widehat{\text{Tot}} X^*_n$ is weakly equivalent to $K'_n$ where $X^*_n$ is the associated cosimplicial space.

The operad $X_n$ is called the Kontsevich operad by Sinha [16]. Since it is multiplicative, we can construct an associated cosimplicial space (see Section 4.1.1), denoted by $X^n$ in the above.

Below we deal with general cases. For any cosimplicial space $Y^*$, its homotopy invariant totalization is defined as

$$\widehat{\text{Tot}} Y^* = \text{Map}(\hat{\Delta}^*, Y^*)$$

where $\hat{\Delta}^*$ is a cofibrant replacement of the standard cosimplicial space $\Delta^*$. The ordinary totalization $\text{Tot} Y^*$ is defined by replacing $\hat{\Delta}^*$ by $\Delta^*$ in the above. There is a map $\text{Tot} Y^* \to \widehat{\text{Tot}} Y^*$ induced by a canonical map $\hat{\Delta}^* \to \Delta^*$.

In the following let $O$ be an operad with multiplication, assuming $O(0) = \{\ast\}$. Denote by $O^*$ its associated cosimplicial space. One of the remarkable features of such a cosimplicial space $O^*$ is the existence of a little disks action on the totalization.

**Theorem 2.2** [14] Let $O$ be a topological operad with multiplication, and $O^*$ its associated cosimplicial space. Then $\widehat{\text{Tot}} O^*$ admits an action of the operad $\mathcal{D}$ which is weakly equivalent to the little disks operad. That is, there exist maps

$$\gamma_k : \mathcal{D}(k) \times (\widehat{\text{Tot}} O^*)^k \to \widehat{\text{Tot}} O^*, \quad k \geq 1,$$

satisfying the associativity conditions. Similar action exists on $\text{Tot} O^*$.

We denote the induced Browder operation by $\lambda$:

$$\lambda : H_p(\text{Tot} O^*, \mathbb{Q}) \otimes H_q(\text{Tot} O^*, \mathbb{Q}) \to H_{p+q+1}(\text{Tot} O^*, \mathbb{Q})$$

We regard $\lambda$ as a Poisson algebra structure on the homology (see Cohen [5]).

On the other hand, $H_*(\text{Tot} O^*)$ can be computed by means of Bousfield spectral sequence [1]. To do this, we choose a fibrant replacement $R : O^* \xrightarrow{\sim} RO^*$ of the cosimplicial
space $O^\ast$. Tot $RO^\ast$ may not be acted on by little disks since $RO^\ast$ need not come from an operad. But there exists a sequence of weak equivalences

$$\tilde{\text{Tot}} O^\ast \xrightarrow{\sim} \text{Tot} RO^\ast \leftarrow \text{Tot} RO^\ast,$$

so there exists a Poisson algebra structure on $H_s(\text{Tot} RO^\ast)$ (we also call the bracket in this case the Browder operation), induced from $H_s(\tilde{\text{Tot}} O^\ast)$.

If $RO^\ast$ satisfies the convergence conditions [1, Theorem 3.2], then the filtration (defined below) on a double complex $S_s(RO^\ast)$, the modules of singular chains of $RO^\ast$, yields a second quadratic spectral sequence $E^r$ converging to $H_s(\text{Tot} RO^\ast)$. Indeed Bousfield proved that (under the same conditions) the total complex $TS(RO^\ast)$ is an algebraic model for $\text{Tot} RO^\ast$, that is, there exists a quasi-isomorphism $\varphi: S_s(\text{Tot} RO^\ast) \longrightarrow TS(RO^\ast)$ (for its definition see [1, section 2] or Section 4 below).

Denote its associated quotient by $GH_s(TS(RO^\ast))$:

$$GH_q(TS(RO^\ast)) = \bigoplus_p F_p H_q(TS(RO^\ast))/F_{p+1} H_q(TS(RO^\ast)).$$

$H_s(\tilde{\text{Tot}} O^\ast)$ is also filtered via isomorphism $\varphi^{-1}$. We will see (Theorem 4.6) that the Browder operation $\lambda$ preserves this filtration, hence via $\varphi$ a Poisson structure passes on $GH_s(TS(RO^\ast))$.

Thus we have two Poisson algebra structures on $GH_s(TS(RO^\ast))$: One induced via $\varphi$ and the other arising from the Hochschild homology. Our main result states that they are same.
Theorem 2.3  Let $O^*$ be the cosimplicial space associated with a multiplicative operad $O$ satisfying the convergence condition [1, Theorem 3.2]. Choose a fibrant replacement $RO^*$. Then the Browder operation induced on $GH_*(TS(RO^*))$ via $\varphi$ coincides up to sign with the Gerstenhaber structure defined on the Hochschild homology.

In the case of $O = X_n$ we have the rational degeneracy of the spectral sequence.

Theorem 2.4  [11, 12] When $n > 3$, the above homology spectral sequence degenerates at $E^2$-term rationally. Thus $H_*(K'_n, \mathbb{Q})$ is the Hochschild homology of the Poisson algebras operad.

This theorem holds since the formality of the operad $X_n$ allows us to replace the vertical differentials of the double complex by zero maps. Hence we can compute the Browder operation on $H_*(K'_n, \mathbb{Q})$ by calculating $E^2$-term, without referring Proposition 4.8.

The proof of Theorem 2.3 will be given later. First we compute some homology groups of $K'_n$.

3 Some computations

Below we assume $n > 3$. The subgroup $\bigoplus_{k \geq 0} H_{n-3+k}(K_n)$ is known to be non-trivial (Cattaneo–Cotta-Ramusino–Longoni [4], Sinha [16], Turchin [18]), since it contains the subalgebra isomorphic to the algebra $A$ of chord diagrams modulo 4-term and 1-term relations, which we now explain. An example of a chord diagram and the 4-term relator can be seen in Figures 1 and 2 respectively. Since we consider the long knots, the chords are on the line, not on a circle.

Figure 1: an example of a chord diagram

By 1-term relation we regard a diagram $\Gamma$ as zero if $\Gamma$ has an isolated chord. Here a chord $c$ with endpoints $a, b \in \mathbb{R}^1$ ($a < b$) is said to be isolated if there is no other chord one of whose endpoints is in $(a, b)$ and another outside $[a, b]$.

The degree of a chord diagram is $(n - 3)k$ if it has $k$ chords. The space of chord diagrams forms a graded algebra, whose product is defined as the concatenation of
Given a chord diagram $\Gamma$ with $k$ chords, we have a long immersion $f_\Gamma$ with $k$ transversal self-intersections determined by the chords of $\Gamma$ (see Figure 3). At each self-intersection we have resolutions of the intersection parametrized by $S^{n-3}$. Considering all the resolutions of $k$ self-intersections, we have a map

$$s(\Gamma) : (S^{n-3})^k \to K_n.$$

More explicitly, the knot $s(\Gamma)(z_1, \ldots, z_k)$ is defined as follows (see Cattaneo–Cotta-Ramusino–Longoni [4]); if the $i$-th doublepoint of $f_\Gamma$ is $f_\Gamma(s_i) = f_\Gamma(u_i)$ ($s_i < u_i$), then

$$s(\Gamma)(z_1, \ldots, z_k)(t) = f_\Gamma(t) + \delta z_i \exp \left[ \frac{1}{(t - s_i)^2 - \varepsilon^2} \right]$$

for $t \in (s_i - \varepsilon, s_i + \varepsilon)$, where $\delta$ and $\varepsilon$ are small positive numbers, and otherwise $s(\Gamma)(z)(t) = f_\Gamma(t)$.

This map determines an $(n-3)k$-cycle $\alpha(\Gamma)$ of $K_n$. The correspondence $\alpha$ preserves the 4-term relations.
**Proposition 3.1** [4, 18] The correspondence $\Gamma \mapsto \alpha(\Gamma)$ determines an injective homomorphism of algebras

\[ \alpha : \mathcal{A} \longrightarrow \bigoplus_{k \geq 0} H_{(n-3)k}(\mathcal{K}_n). \]

Thus we have a subalgebra of $H_*(\mathcal{K}_n)$ which is isomorphic to $\mathcal{A}$ and concentrated in the degrees $(n - 3)k$, $k \geq 0$. But few homology classes except those in $\alpha(\mathcal{A})$ have been identified.

**Theorem 2.3** implies that the spectral sequence computes the Poisson algebra structure on $H_*(\mathcal{K}_n', \mathbb{Q})$. By computing the $E^2$-term explicitly (see Turchin [18]), we find that the Browder operation produces new homology classes other than those in $\alpha(\mathcal{A})$.

To state the result, we need the following.

**Lemma 3.2** [16] For any $n \geq 3$, the inclusion $\mathcal{K}_n \hookrightarrow \text{Imm}_n$ is null-homotopic. Consequently there exists a homotopy equivalence $\mathcal{K}_n' \simeq \mathcal{K}_n \times \Omega^2 S^{n-1}$.

The topology of $\Omega^2 S^{n-1}$ is well known:

\[ H_*(\Omega^2 S^{n-1}, \mathbb{Q}) \cong \begin{cases} \Lambda^*[\lambda_{\Omega^2 S^{n-1}}(\iota, \iota)] \otimes P[\iota] & n \text{ is odd,} \\ \Lambda^*[\iota] & n \text{ is even,} \end{cases} \]

where $\iota$ is the image of $1 \in \pi_{n-3}(\Omega^2 S^{n-1})$ via the Hurewicz isomorphism, and $\lambda_{\Omega^2 S^{n-1}}$ is the Browder operation induced by the little disks action on $\Omega^2 S^{n-1}$. Using this, we can determine the generators of some low-degree homology groups of $\mathcal{K}_n'$ and $\mathcal{K}_n$.

**Theorem 3.3** Suppose $n > 4$. Then

\[ H_{3n-8}(\mathcal{K}_n', \mathbb{Q}) \cong \begin{cases} \mathbb{Q}^2 & n \text{ is odd,} \\ \mathbb{Q} & n \text{ is even.} \end{cases} \]

When $n$ is odd, the above group is generated by $\iota\lambda(\iota, \iota)$ and $\lambda(\iota, v_2)$, where $v_2 \in H_{2(n-3)}(\mathcal{K}_n)$ is the class made from a chord diagram with two interleaving chords (see Figure 4). By the Künneth formula

\[ H_{3n-8}(\mathcal{K}_n', \mathbb{Q}) \cong \mathbb{Q} \]

and this group is generated by the homology class corresponding to $\lambda(\iota, v_2)$, which is the first example of a homology class of $\mathcal{K}_n$ which does not directly correspond to chord diagrams.
In other words, we can regard $H_*(\Omega^2 S^{n-1}; \mathbb{Q})$ as acting on $H_*(K_n, \mathbb{Q})$ via the Browder operation, and this action is non-trivial when $n$ is odd. The homology class $\lambda(\iota, v_2)$ (when $n$ is odd) is not in $\alpha(A)$, since it is not in degrees $(n-3)k$.

When $n$ is even, $\iota$ belongs to the center (Turchin [18]). This is because the map of spectral sequences $E^2(K_n') \rightarrow E^2(K_n)$ is a map of Gerstenhaber algebras, and under the map $\iota$ must be sent to 0 for the dimensional reason. But $\text{rank } H_{3n-8}(K_n') = 1$ still holds, and the generator is unknown yet.

Salvatore [15] pointed out that $K_n$ is also a double loop space, but the existence of $\lambda(\iota, v_2)$ shows that the projection $K_n' \rightarrow K_n$ does not preserve the Browder operation (since $\iota$ is mapped to zero by the dimensional reason), hence this projection is not the map of double loop spaces.

The above computation suggests that we might be able to obtain more homology classes by applying $\lambda(\iota, \cdot)$ to the elements of $\alpha(A)$. It is possible in principle to compute them, but it would become more exhausting as the degrees increase (see Cattaneo–Cotta-Ramusino–Longoni [4], Turchin [18]).

4 Proof of the main theorem

4.1 Disks action on $\text{Tot } O^*$

First we let $O$ be a topological operad with multiplication and suppose the associated cosimplicial space $O^*$ is fibrant.

To obtain the explicit formula for the Browder operation on $H_*(\text{Tot } O^*)$, it suffices to study in detail the map

$$\gamma: S^1 \times (\text{Tot } O^*)^2 \longrightarrow \text{Tot } O^*$$

($S^1 \sim \mathcal{D}(2)$) defined by McClure and Smith [14]. We think $g^* \in \text{Tot } O^*$ as a sequence of maps $g^l: \Delta^l \rightarrow O^l \ (l \geq 0)$, compatible with the cosimplicial structure maps. Defining the above map $\gamma$ is equivalent to defining the maps

$$\gamma(\tau; g_1^*, g_2^l): \Delta^l \longrightarrow O^l, \quad l \geq 0$$
compatible with cosimplicial structure maps, for any \( \tau \in S^1 \), \( g^* \in \text{Tot} \mathcal{O}^* \). After the fashion of McClure and Smith’s work [13], we first look at the easiest example.

### 4.1.1 Conventions

We now proceed to operadic computations. For such notions see McClure–Smith [13, 14]. We use the notation \( \circ_i \) for operadic structure:

\[
\circ_i : \mathcal{O}(p) \times \mathcal{O}(q) \longrightarrow \mathcal{O}(p + q - 1), \quad a \circ_i b = a(\text{id}, \ldots, \text{id}, b, \text{id}, \ldots, \text{id})
\]

for any operad \( \mathcal{O} \). The symbol \( \circ_i \) represents the ‘inserting’ operation.

Recall that an operad \( \mathcal{O} \) is said to be multiplicative if we can choose basepoints \( \mu_k \in \mathcal{O}(k), k \geq 0 \), in the operadic sense;

\[
\mu_k(\mu_{j_1}, \ldots, \mu_{j_k}) = \mu_{j_1 + \cdots + j_k}.
\]

In this case \( \mathcal{O} \) can be seen as a cosimplicial space by letting \( \mathcal{O}_k = \mathcal{O}(k) \) and defining

\[
d^i : \mathcal{O}^k \longrightarrow \mathcal{O}^{k+1}, \quad 0 \leq i \leq k + 1, \\
s^i : \mathcal{O}^k \longrightarrow \mathcal{O}^{k-1}, \quad 0 \leq i \leq k - 1,
\]

by

\[
d^i(x) = \begin{cases} 
\mu_2 \circ_2 x & i = 0, \\
x \circ_i \mu_2 & 1 \leq i \leq k, \\
\mu_2 \circ_1 x & i = k + 1,
\end{cases}
\]

\[
s^i(x) = x \circ_{i+1} e
\]

where \( e \in \mathcal{O}(0) \). They indeed satisfy the cosimplicial identities.

Below by convention

\[
\Delta^k := \begin{cases} 
\{*\} & k = 0, \\
\{(t_1, \ldots, t_k) \in [-1, 1]^k | t_1 \leq \cdots \leq t_k\} & k \geq 1,
\end{cases}
\]

\[
S^1 := [-1, 1]/\sim.
\]

### 4.1.2 The first case

As the easiest case we construct

\[
\gamma(\tau; g_1^*, g_2^*) : \Delta^1 \longrightarrow \mathcal{O}^1
\]

(assuming \( \mathcal{O}^0 = \{e\} \), the case \( l = 0 \) is obvious). Imitating a work of McClure and Smith [13], we roughly illustrate the definition of this map (see Figure 5).
Remark 4.1 Broadly speaking, as $\tau$ increases from $-1$ to 0, the knot $g_2$ ‘goes through’ the knot $g_1$ ($\circ_i$ will represent the ‘insertion’). By $\tau = 0$, $g_2$ will get away from $g_1$ and ‘juxtapose’ to $g_1$ ($\mu_2 \in O(2)$ will indicate ‘concatenation’). When $0 < \tau < 1$, now $g_1$ passes through $g_2$, and juxtaposes to $g_2$ when $\tau = 1$. See also the pictures in Budney’s paper [2]. The reason why we may regard $\circ_i$ and $\mu_2$ as respectively ‘insertion’ and ‘concatenation’ can be found in the definition of Poisson algebras operad $H_*(X_n)$ (see Turchin [17, 18]).

More precise definition is as follows. When $-1 < \tau < 0$, define non-negative integers $i_0, i_1$ by

$$ (i_0, i_1) = \begin{cases} 
(1, 1) & t < \tau, \\
(0, 1) & \tau \leq t < 1 + \tau, \\
(0, 0) & 1 + \tau \leq t, 
\end{cases} $$

and define $(u_1, u_2) \in \Delta^{2+i_0-i_1} \times \Delta^{i_1-i_0}$ by

$$ (u_1, u_2) = \begin{cases} 
((2t + 1, 1 + 2\tau); \ast) \in \Delta^2 \times \Delta^0 & t < \tau, \\
(2\tau + 1; 2(t - \tau) - 1) \in \Delta^1 \times \Delta^1 & \tau \leq t < 1 + \tau, \\
((1 + 2\tau, 2t - 1); \ast) \in \Delta^2 \times \Delta^0 & 1 + \tau \leq t.
\end{cases} $$

Then

$$ \gamma(\tau; g_1^\ast, g_2^\ast)(t) := g_1^{2+i_0-i_1} \circ_i 1 + 1 g_2^{i_1-i_0}(u_2) \in O^1. $$

Figure 5: the ‘definition’ of $\gamma(\tau; g_1^\ast, g_2^\ast)(t)$
When $0 < \tau < 1$, define

$$(j_0, j_1) = \begin{cases} 
(1, 1) & t < -\tau, \\
(0, 1) & -\tau \leq t < 1 - \tau, \\
(0, 0) & 1 - \tau \leq t,
\end{cases}$$

and define $(v_1, v_2) \in \Delta^{l - j_0} \times \Delta^{2 + j_0 + j_1}$ by

$$(v_1, v_2) = \begin{cases} 
(*; (2t + 1, 1 - 2\tau)) \in \Delta^0 \times \Delta^2 & t < -\tau, \\
(2(t - \tau) - 1; 1 - 2\tau) \in \Delta^1 \times \Delta^1 & -\tau \leq t < 1 - \tau, \\
(*; (1 - 2\tau, 2t - 1)) \in \Delta^0 \times \Delta^2 & 1 - \tau \leq t.
\end{cases}$$

Then

$$\gamma(\tau; g^1_1, g^2_2)^1(t) := g^{j_i - j_0}_2(v_2) \circ_{j_0 + 1} g^{2 + j_0 - j_i}_1(v_1).$$

The cases $\tau = 0, \pm 1$ may be contained in the above definitions; even in those cases $\gamma(\tau; g^1_1, g^2_2)^1(t)$ is well-defined. But it would be better to give the definitions for $\tau = 0, \pm 1$ separately to make the meaning of $\gamma$ clearer. We define

$$\gamma(0; g^1_1, g^2_2)^1(t) := \mu_2(g^0_1(w_1), g^{1 - j_0}_2(w_2)),$$

$$\gamma(\pm 1; g^1_1, g^2_2)^1(t) := \mu_2(g^{j_i}_2(w_2), g^{1 - j_i}_1(w_1)),$$

where $i_0$ (for $\tau = 0$) and $j_1$ (for $\tau = +1$) are as above, and

$$(w_1, w_2) = \begin{cases} 
(2t + 1; *) \in \Delta^1 \times \Delta^0 & t < 0, \\
(*; 2t - 1) \in \Delta^0 \times \Delta^1 & t \geq 0.
\end{cases}$$

Then $\gamma(\tau; g^1_1, g^2_2)^1(t)$ is indeed well-defined. For example, let us see the continuity at $t = 1 + \tau$ when $-1 < \tau < 0$. By definition

$$\gamma(\tau; g^1_1, g^2_2)^1(t) = \begin{cases} 
g^1_1(2\tau + 1) \circ_1 g^2_2(2(t - \tau) - 1) & t < 1 + \tau, \\
g^2_2(2\tau + 1, 2\tau + 1) \circ_1 g^0_1(*) & t = 1 + \tau.
\end{cases}$$

Then we can show

$$\lim_{t \to 1 + \tau} \gamma(\tau; g^1_1, g^2_2)^1(t) = g^1_1(2\tau + 1) \circ_1 g^2_2(1) = \gamma(\tau; g^1_1, g^2_2)^1(1 + \tau)$$

as follows;

$$g^1_1(2\tau + 1) \circ_1 g^2_2(1) = g^1_1(2\tau + 1) \circ_1 g^2_2(d^1(*) = g^1_1(2\tau + 1) \circ_1 d^1 g^0_2(*) = g^1_1(2\tau + 1) \circ_1 \mu_2(g^0_2(*), \text{id}) = g^1_1(2\tau + 1) \circ_1 (\mu_2 g^0_2(*), \text{id}) = g^1_1(2\tau + 1)(\mu_2 g^0_2(*), \text{id})) = (d^1 g^1_1(2\tau + 1) g^0_2(*), \text{id}) = g^1_1 d^1(2\tau + 1) g^0_2(*) \circ_1 \text{id} = g^1_1(2\tau + 1, 2\tau + 1) \circ_1 g^0_2(*) = \gamma(\tau; g^1_1, g^2_2)^1(1 + \tau).$$
We see one more point; $\gamma$ is continuous at $\tau = 0, \pm 1$. For example we show
$$\lim_{\tau \to -1} \gamma(\tau; g_1^*, g_2^*)^1(t) = \gamma(-1; g_1^*, g_2^*)^1(t)$$
when $-1 < t < 0$. In this case the above limit is equal to
$$g_1^1(-1) \circ_1 g_2^1(2t + 1) = g_1^1(d\mu_0^*(g_2^1(2t + 1))) = (d\mu_0^*(g_1^*(g_2^1(2t + 1)))) = \mu_2(id, g_2^0(\ast))(g_2^1(2t + 1)) = \mu_2(g_2^1(2t + 1), g_1^0(\ast)) = \gamma(-1; g_1^*, g_2^*)^1(t).$$
The proofs for other cases go in similar ways.

### 4.1.3 General cases

Here we describe the complete definition of the map
$$\gamma: S^1 \times (\text{Tot } O^*)^2 \longrightarrow \text{Tot } O^*,$$
a special case of the construction by McClure and Smith [14], and the induced map on homology.

First we define the integers $i_\varepsilon, j_\varepsilon$ ($\varepsilon = 0, 1$), which determine where $g_{\varepsilon(2)}$ is ‘inserted’ in $g_{\varepsilon(1)}$, for any $t = (t_1, \ldots, t_l) \in \Delta^l$ and $\tau \in S^1$. When $-1 < \tau < 0$, define
$$i_0 := \min\{i; t_{i+1} \geq \tau\}, \quad i_1 := \min\{i; t_{i+1} \geq 1 + \tau\},$$
and when $0 < \tau < 1$, define
$$j_0 := \min\{j; t_{j+1} \geq -\tau\}, \quad j_1 := \min\{j; t_{j+1} \geq 1 - \tau\}.$$

Next, define
$$(u_1(\tau, t), u_2(\tau, t)) \in \Delta^{l_0-i_0+1} \times \Delta^{l_1-i_1}$$
(when $-1 < \tau < 0$) and
$$(v_1(\tau, t), v_2(\tau, t)) \in \Delta^{j_0} \times \Delta^{l_0-j_0+1}$$
(when $0 < \tau < 1$) by
$$u_1(\tau, t) = (2t_1 + 1, \ldots, 2t_{i_0} + 1, 1 + 2\tau, 2t_{i_1+1} - 1, \ldots, 2t_l - 1),$$
$$u_2(\tau, t) = (2t_{i_0+1} - 2\tau - 1, \ldots, 2t_{i_1} - 2\tau - 1),$$
$$v_1(\tau, t) = (2t_{j_0+1} - 2\tau - 1, \ldots, 2t_{j_1} - 2\tau - 1),$$
$$v_2(\tau, t) = (2t_1 + 1, \ldots, 2t_{j_0} + 1, 1 - 2\tau, 2t_{j_1+1} - 1, \ldots, 2t_l - 1).$$

Here we note a consequence of a straightforward computation.
Lemma 4.2 For any given \( l \) and \( 0 \leq i_0 \leq i_1 \leq l \), define
\[
\Delta(i_0, i_1) := \{(\tau, t) \in (-1, 0) \times \Delta^{l} \mid i_{\epsilon}(\tau, t) = i_{\epsilon} \ (\epsilon = 0, 1)\}.
\]
Then the correspondence
\[
u = (v_1, v_2) : \Delta(i_0, i_1) \longrightarrow \Delta^{l} \times \Delta^{l-i_0},
\]
defined in the above remark, is a homeomorphism on their interior. Similarly, if we define
\[
\Delta'(j_0, j_1) := \{(\tau, t) \in (0, 1) \times \Delta^{l} \mid j_{\epsilon}(\tau, t) = j_{\epsilon} \ (\epsilon = 0, 1)\},
\]
for any given \( l \) and \( 0 \leq j_0 \leq j_1 \leq l \), then
\[\gamma(\tau; g_1, g_2)(t) = \begin{cases} \quad g_1^{l+i_0-i_1+1}(u_1(\tau, t)) \circ_{i_0+1} g_2^{i_1-i_0}(u_2(\tau, t)) & -1 < \tau < 0, \\ \quad g_2^{j_j-j_0}(v_2(\tau, t)) \circ_{j_0+1} g_1^{l+1-j_1-j_0}(v_1(\tau, t)) & 0 < \tau < 1. \end{cases}\]

For \( \tau = 0, \pm 1 \) (at which \( g_1 \) and \( g_2 \) 'switch'), we separately give the definition; when \( \tau = 0 \), define \( i_0 \) as above and
\[
\tilde{u}_1(\tau, t) := (2t_1 + 1, \ldots, 2t_{i_0} + 1) \in \Delta^{i_0},
\]
\[
\tilde{u}_2(\tau, t) := (2t_{i_0+1} - 1, \ldots, 2t_l - 1) \in \Delta^{l-i_0}.
\]
Then we define
\[ \gamma(0; g_1^*, g_2^*; t) := \mu_2(g_1^0(\bar{u}_1), g_2^{l_0}(\bar{u}_2)). \]

When \( \tau = \pm 1 \), define \( j_1 \) as above (for \( \tau = +1 \)) and
\[
\bar{v}_1(\tau, t) := (2t_{i_1} + 1, \ldots, 2t_l - 1) \in \Delta^{l-j_i},
\]
\[
\bar{v}_2(\tau, t) := (2t_l + 1, \ldots, 2t_{j_1} + 1) \in \Delta^j.
\]

Then we define
\[ \gamma(\pm 1; g_1^*, g_2^*; t) := \mu_2(g_2^j(\bar{v}_2), g_1^{l-j}(\bar{v}_1)). \]

Then the map \( \gamma(\tau; g_1^*, g_2^*; t) \) is well-defined, which is proven similarly as when \( l = 1 \).
Moreover \( \gamma \) indeed defines a map to a totalization.

**Lemma 4.3** The sequence of maps \( \gamma(\tau; g_1^*, g_2^*; t) \) is compatible with the cosimplicial structure maps. \( \square \)

For example, if \( (\tau, t) \in \Delta^i(i_0, i_1) \), \( 0 < i_0 < i_1 \) and \( 0 < i \leq i_0 \), then \( d^i(t) = (\ldots, i, \ldots, i_{i_0}, \ldots) \) and \( (\tau, d^i(t)) \in \Delta^{i+1}(i_0 + 1, i_1 + 1) \).

\[
\gamma(\tau; g_1^*, g_2^*; d^{i+1}(\tau, t))
\]
\[
= g_1(\ldots, 2t_l + 1, 2t_l + 1, \ldots) \circ_{i_0+2} g_2(2t_{i_0} + 2 - 2\tau - 1, \ldots, 2t_i - 2\tau - 1)
\]
\[
= g_1(d^i(\ldots, 2t_l + 1, \ldots)) \circ_{i_0+2} g_2(u_2(\tau, t))
\]
\[
= (d^i g_1(u_1(\tau, t))) \circ_{i_0+2} g_2(u_2(\tau, t)) = (g_1(u_1) \circ \mu_2) \circ_{i_0+2} g_2(u_2)
\]
\[
= (g_1(u_1) \circ_{i_0+2} g_2(u_2)) \circ \mu_2 = d^i(\gamma(\tau; g_1^*, g_2^*; t))
\]
(\text{the fifth equality uses} \( i < i_0 \)). Other cases can be proven similarly.

**Remark 4.4** In fact the definitions for \( \tau = 0, \pm 1 \) can be obtained as the limits of those for \(-1 < \tau < 0 \) and \( 0 < \tau < 1 \). But we give them separately to clarify the meaning of the definition. \( \square \)

For the induced map on homology, we only need to pre-compose the Eilenberg-MacLane map (see Section 4.2). In the following we use the same symbols as above.

**Theorem 4.5** Let \( g_1^* \in S_q(\text{Tot} \mathcal{O}^*) \) and \( g_2^* \in S_0(\text{Tot} \mathcal{O}^*) \) be cycles. Define the map
\[
(g_1^*, g_2^*; l) : \Delta^q \times \Delta^l \times \Delta^l \rightarrow \mathcal{O}^l, \quad l \geq 0
\]
Poisson structures on the homology of the space of knots

by

\[
\langle g_1, g_2 \rangle^l(x, y, \tau, t) := \begin{cases} 
\mu_2 \left( g_2^i(y)(\bar{\nu}), g_1^{l-j_1}(x)(\bar{\nu}_1) \right) & \tau = \pm 1, \\
\mu_2 \left( g_1^i(x)(\bar{u}_1), g_2^{l-j_0}(y)(\bar{u}_2) \right) & \tau = 0, \\
g_2^{l+j_0-j_1+1}(y)(\bar{u}_2) \circ_{j_0+1} g_1^{j_1-j_0}(x)(\bar{u}_1) & 0 < \tau < 1. 
\end{cases}
\]

Then the map

\[
\lambda : H_q(Tot O^*) \otimes H_s(Tot O^*) \to H_{q+s+1}(Tot O^*)
\]

given by

\[
\lambda(g_1^*, g_2^*) := (-1)^{q+1}\langle g_1^*, g_2^* \rangle \circ EM \in H_{q+s+1}(Tot O^*)
\]
is the Browder operation, where EM is the Eilenberg-MacLane map.

\[\square\]

4.2 Algebraic model for Tot \(O^*\)

Let \(TS(O^*)\) be the total complex of the double complex from §2: its degree \(k\) part is

\[
TS(O^*)_k = \prod_{l \geq 0} S_{k+l}(O^l),
\]

and the differential is

\[
\partial_T = \partial + (-1)^p \delta,
\]

where \(\partial = \sum (-1)^l d^l\) (\(d^l\) was defined in Section 4.1.1), a signed sum of the coface maps, and \(\delta\) is the usual boundary map of singular chain complex. Bousfield [1] constructed a quasi-isomorphism

\[
\varphi : S_*(Tot O^*) \to TS(O^*)
\]

when \(O^*\) satisfies some conditions. This is defined as follows. We regard a chain \(f^* = \sum a_i f_i^* \in S_q(Tot O^*)\) as the sum of maps \(f_i^* = \{f_i^l\}_{l \geq 0}\),

\[
f_i^l : \Delta^q \times \Delta^l \to O^l,
\]

which is compatible with the cosimplicial structure maps of the \(\Delta^l\)-factor. We choose the Eilenberg-MacLane map \(EM \in S_{q+l}(\Delta^q \times \Delta^l)\), \(l, q \geq 0\), which gives a chain equivalence

\[
S_*(M) \otimes S_*(N) \cong S_*(M \times N).
\]
for any spaces $M$ and $N$. Then the quasi-isomorphism $\varphi$ is defined by

$$\varphi(f^*) = \prod_{l \geq 0} S_{q+l}(O^l),$$

$$\varphi(f^*)_h := \sum a_i (f^*_i \circ EM) \in S_{q+l}(O^l),$$

where we write $h = (h_t)_{t \geq 0}$ with $h_t \in S_{q+l}(O^l)$ for any $h \in TS(O^*)_q$. Our main theorem states that the induced isomorphism on homology preserves the Poisson algebra structures.

The spectral sequence associated with the filtration defined in Section 2 on the double complex converges strongly when $E^1_{p,q} = 0$ for $p > q$ and, for any $m \geq 0$, there are only finitely many $(p, q)$ such that $q - p = m$ and $E^1_{p,q} \neq 0$. In the case of Kontsevich operad $X_n$, it turns out that the 'normalized' $E^1$-term

$$E^1 \cap \bigcap \ker s_*^i$$

satisfies those conditions (for definition of $s^i$ see Section 4.1.1). The proof uses the explicit form of the homology of $X_n(k) \simeq \text{Conf} (\mathbb{R}^n, k)$ (see Sinha [16]).

### 4.3 Browder operation in terms of $TS(O^*)$

Via the quasi-isomorphism $\varphi$, the Browder operation will be interpreted as follows.

**Theorem 4.6** For any $g^*_1 \in H_q(\text{Tot } O^*)$ and $g^*_2 \in H_s(\text{Tot } O^*)$,

$$\varphi(\lambda(g^*_1, g^*_2))_t = \sum_{p+r=t+1} (-1)^{(p+1)(r+1)+q+s} \left[ \sum_{i=1}^p (-1)^{\epsilon_i} \varphi(g^*_1)_p \circ_j \varphi(g^*_2)_r - (-1)^{(q+1)(s+1)} \sum_{j=1}^r (-1)^{\epsilon'_j} \varphi(g^*_2)_r \circ_j \varphi(g^*_1)_p \right]$$

where

$$\epsilon_i = (q - 1)(p - i) + (p - 1)(r + s), \quad \epsilon'_j = (p - 1)(r - j) + (p + q)(r - 1).$$

**Proof** The map

$$\Delta^q \times \Delta^s \times \Delta^1 \times \Delta^l \xrightarrow{(g_1, g_2)} X^l$$

pre-composed by $EM$ represents $\varphi(\lambda(g_1, g_2))_t$. By definition $\Delta^1 \times \Delta^l$ is decomposed by $\Delta(i_0, i_1)$'s and $\Delta^l(j_0, j_1)$'s ($0 \leq i_0 \leq i_1 \leq l$, $0 \leq j_0 \leq j_1 \leq l$), see Figure 6. By Theorem 4.5, $(g_1, g_2)$ is $g_1^p \circ_{i_0+1} g_2^r$ when restricted on $\Delta^q \times \Delta^s \times \Delta(i_0, i_1)$, where

$$p = l + i_0 - i_1 + 1, \quad r = i_1 - i_0.$$
and, when restricted on $\Delta^q \times \Delta^s \times \Delta'(j_0, j_1)$, $(g_1, g_2)^{\pm}$ is $g_2^{\pm j_0 \circ j_1} g_1^p$, where $p$ and $r$ are determined similarly.

Thus $\varphi(\lambda(g_1^1, g_2^2))$ should be a linear sum of $\varphi(g_1^1)^p \circ_i \varphi(g_2^2)^r$, $1 \leq i \leq p$, and $\varphi(g_2^2)^r \circ_j \varphi(g_1^1)^p$, $1 \leq j \leq r$. The coefficients are $\pm 1$ because of Lemma 4.2. The signs $\pm$ are those of the Jacobians of the maps

$$
\Delta^q \times \Delta^s \times \Delta(i_0, i_1) \xrightarrow{\sim} (\Delta^q \times \Delta^p) \times (\Delta^s \times \Delta'),
$$

and

$$
\Delta^q \times \Delta^s \times \Delta'(j_0, j_1) \xrightarrow{\sim} (\Delta^q \times \Delta^p) \times (\Delta^s \times \Delta'),
$$

given by $u, v$ as in Theorem 4.5. Explicit computations show that the signs are

$$
(-1)^{1+\delta(i_1-i_0)(l-i_1)+pr} = (-1)^{p+1}(r+1)+s+1(-1)^{r-1}(p-i_0-1)+(p-1)r+s,
$$

and

$$
(-1)^{1+j_0+(j_1-j_0)(l-j_1)+q(r+s)} = (-1)^{p+1}(r+1)+q+q+1 \times
$$

$$
(-1)^{p-1}(r-j_0-1)+(p+q)(r-1)
$$

respectively. They give the desired formula.

The isomorphism $\varphi$ introduces a filtration on $H_*(\Tot \mathcal{O})$;

$$
F_p H_*(\Tot \mathcal{O}) := \varphi^{-1}F_p H_*(\Tot \mathcal{O}).
$$

The definition of $F_*$ together with Theorem 4.6 says that the Browder operation $\lambda$ preserves this filtration in the sense

$$
x \in F_p, y \in F_r \implies \lambda(x, y) \in F_{p+r-1}.
$$

### 4.4 Poisson bracket on $E^2$

Here let $\mathcal{O}'$ be any operad of graded modules with multiplication (in our case $\mathcal{O}'$ will be $H_*(\mathcal{O})$). By unraveling the descriptions of the Hochschild complex $(\mathcal{O}', \partial)$ (see Gerstenhaber–Voronov [6], Turchin [17, 18]), we can see the following.

**Theorem 4.7** [6, 17, 18] The Poisson structure on $H_*(\mathcal{O}')$ is induced by the (degree-preserving) map

$$
\Psi : \mathcal{O}'(p) \times \mathcal{O}'(r) \to \mathcal{O}'(p + r - 1),
$$

$$
\Psi(x, y) := \sum_{1 \leq i \leq p} (-1)^{\epsilon_i} x \circ_i y - (-1)^{q+1}(s+1) \sum_{1 \leq j \leq r} (-1)^{\epsilon_j} y \circ_j x
$$

where $q = \deg x - p$, $s = \deg y - r$, and $\epsilon_i, \epsilon_j$ are as in Theorem 4.6.

---

*Geometry & Topology Monographs 13 (2008)*
In our case $\Psi$ is defined on $E^1$ and makes $E^2$ a Poisson algebra. Indeed $E^r$ is a spectral sequence of a Poisson algebra because of the following.

**Proposition 4.8** We have $d^r\Psi(x, y) = \Psi(d^r x, y) + (-1)^{|x|} r \Psi(x, d^r y)$ on $E^r$, $r \geq 2$. □

The proof uses the definition of the boundary operation of the Hochschild complex (see Section 4.2).

Thus $E^\infty$ inherits the induced Poisson bracket, and via the isomorphism

$$
\psi : E^\infty_{-p,q} \longrightarrow (F_p/F_{p+1})H_q(TS(O^*))
$$

$G H_*(TS(O^*))$ also becomes a Poisson algebra, where $G$ denotes the associated quotient. Comparing Theorem 4.6 with Theorem 4.7, we can see that this bracket coincides on $G H_*(TS(O^*))$ with the Browder operation induced via $\varphi$.

### 4.5 The case of the space of knots

Finally consider the case that $O^*$ is not fibrant (in particular the case of the space of knots, $O = X_n$). Though we must use the fibrant replacement $RO^*$, this does not change the formula from Theorem 4.5 except that we have to post-composing $R : O^* \rightarrow RO^*$. On the other hand the $E^1$-term of Bousfield spectral sequence $E^r(RO^*)$ for $\text{Tot} RO^*$ is equipped with the induced Gerstenhaber structure, whose formula is the same one from Theorem 4.7 with $R$ post-composed. Thus again via $\varphi$ the Browder operation on $H_*(\text{Tot} RO^*)$ and the Gerstenhaber bracket on $E^\infty(RO^*)$ coincide with each other. □

### Acknowledgements

The author expresses his great appreciation to Toshitake Kohno for his encouragement and advices. The author is also grateful to Victor Turchin and James McClure for reading the draft of the previous version of the paper and giving him many suggestions, to Dev Sinha for answering his questions, to Fred Cohen and Ryan Budney for teaching him about the little disks actions and so on, and to Paolo Salvatore for kindly giving the author his preprint.

This research is partially supported by the 21st century COE program at Graduate School of Mathematical Sciences, the University of Tokyo.
Poisson structures on the homology of the space of knots

References

[1] A K Bousfield, *On the homology spectral sequence of a cosimplicial space*, Amer. J. Math. 109 (1987) 361–394 MR882428

[2] R Budney, *Little cubes and long knots*, Topology 46 (2007) 1–27 MR2288724

[3] R Budney, F Cohen, *On the homology of the space of knots* arXiv:math.GT/0504206

[4] A S Cattaneo, P Cotta-Ramusino, R Longoni, *Configuration spaces and Vassiliev classes in any dimension*, Algebr. Geom. Topol. 2 (2002) 949–1000 MR1936977

[5] F R Cohen, *The homology of C_{n+1}–spaces, n > 0*, from: “The homology of iterated loop spaces”, Lecture Notes in Math. 533, Springer (1976) 207–352 MR0436146

[6] M Gerstenhaber, A A Voronov, *Homotopy G-algebras and moduli space operad*, Internat. Math. Res. Notices (1995) 141–153 MR1321701

[7] T G Goodwillie, *Calculus. II. Analytic functors*, K-Theory 5 (1991/92) 295–332 MR1162445

[8] T G Goodwillie, *Calculus. III. Taylor series*, Geom. Topol. 7 (2003) 645–711 MR2026544

[9] T G Goodwillie, M Weiss, *Embeddings from the point of view of immersion theory. II*, Geom. Topol. 3 (1999) 103–118 MR1694808

[10] M Kontsevich, *Operads and motives in deformation quantization*, Lett. Math. Phys. 48 (1999) 35–72 MR1718044

[11] P Lambrechts, V Turchin, I Volić, *The rational homology of the space of long knots in codimension > 2* arXiv:math.AT/0703649

[12] P Lambrechts, I Volić, *Formality of the little d-discs operad*, in preparation Available at http://palmer.wellesley.edu/~ivolic/home.html

[13] J E McClure, J H Smith, *A solution of Deligne’s Hochschild cohomology conjecture*, from: “Recent progress in homotopy theory (Baltimore, MD, 2000)”, Contemp. Math. 293, Amer. Math. Soc. (2002) 153–193 MR1890736

[14] J E McClure, J H Smith, *Cosimplicial objects and little n-cubes. I*, Amer. J. Math. 126 (2004) 1109–1153 MR2089084

[15] P Salvatore, *Knots, operads, and double loop spaces*, Int. Math. Res. Not. (2006) Art. ID 13628, 22 MR2276349

[16] D P Sinha, *Operads and knot spaces*, J. Amer. Math. Soc. 19 (2006) 461–486 MR2188133

[17] V Tourtchine, *On the homology of the spaces of long knots*, from: “Advances in topological quantum field theory”, NATO Sci. Ser. II Math. Phys. Chem. 179, Kluwer Acad. Publ., Dordrecht (2004) 23–52 MR2147415

Geometry & Topology Monographs 13 (2008)
[18] V Tourchine, *On the other side of the bialgebra of chord diagrams*, J. Knot Theory Ramifications 16 (2007) 575–629 MR2333307

[19] M Weiss, *Embeddings from the point of view of immersion theory. I*, Geom. Topol. 3 (1999) 67–101 MR1694812

Graduate School of Mathematical Science, University of Tokyo
3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan
ksakai@ms.u-tokyo.ac.jp

Received: 19 September 2006 Revised: 16 July 2007