Plane symmetric thin-shell wormholes: solutions and stability

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Using the cut-and-paste procedure, we construct static and dynamic plane symmetric wormholes by surgically grafting together two spacetimes of plane symmetric vacuum solutions with a negative cosmological constant. These plane symmetric wormholes can be interpreted as domain walls connecting different universes, having planar topology, and upon compactification of one or two coordinates, cylindrical topology or toroidal topology, respectively. A stability analysis is carried out for the dynamic case by taking into account specific equations of state, and a linearized stability analysis around static solutions is also explored. It is found that thin shell wormholes made of a dark energy fluid or of a cosmological constant fluid are stable, while thin shell wormholes made of phantom energy are unstable.

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I. INTRODUCTION

There has been much interest in traversable wormholes since the important Morris-Thorne article [1]. It was found that these geometries, which act as tunnels from one region of spacetime to another, possess a peculiar property, namely exotic matter, involving a stress-energy tensor that violates the null energy condition [1,2,3], see also [4] for a review and references therein. In fact, traversable wormholes violate all of the pointwise energy conditions and averaged energy conditions (see, e.g., [5]). As the violation of the energy conditions is a particularly problematic issue [6], it is useful to minimize the usage of exotic matter. Recently, Visser et al [7,8], noting the fact that the energy conditions do not actually quantify the “total amount” of energy condition violating matter, developed a suitable measure for quantifying this notion by introducing a “volume integral quantifier”. Although the null energy and averaged null energy conditions are always violated for wormhole spacetimes, they considered specific examples of spacetime geometries containing wormholes that are supported by arbitrarily small quantities of averaged null energy condition violating matter.

Another elegant way of minimizing the usage of exotic matter is to construct a simple class of wormhole solutions using the cut and paste technique [9,10] (see also [3]), in which the exotic matter is concentrated at the wormhole throat, i.e., a thin shell wormhole solution.

The surface stresses of the exotic matter were determined by invoking the Darmois-Israel formalism [11]. These thin-shell wormholes are extremely useful as one may apply a stability analysis for the dynamical cases, either by choosing specific surface equations of state [12], or by considering a linearized stability analysis around a static solution [13,14,15,16,17,18], in which a parametrization of the stability of equilibrium is defined, so that one does not have to specify a surface equation of state. The dynamical analysis in [13,14,15] was generalized by considering solutions with electrical charge [16], solutions in the presence of a cosmological constant [17], and spherically symmetric dynamical solutions [13]. More recently, by using the cut and paste technique, several solutions were analyzed, such as, the dynamics of non rotating cylindrical thin-shell wormholes [19], charged thin-shell Lorentzian wormholes in dilaton gravity [20], five dimensional thin-shell wormholes in Einstein-Maxwell theory with a Gauss-Bonnet term [21], solutions in higher dimensional Einstein-Maxwell theory [22], thin-shell wormholes associated with global cosmic strings [23], solutions in heterotic string theory [24], spherical thin-shell wormholes supported by a Chaplygin gas [25], a new class of thin-shell wormhole by surgically grafting two black hole solutions localized on a three brane in five dimensional gravity in the Randall-Sundrum scenario [26], and spherically symmetric thin-shell wormholes in a string cloud background in (3+1)-dimensional spacetime were also analyzed [27]. Other wormhole solutions have been analyzed in [28,29,30,31].

One can go beyond thin shell solutions, and construct wormhole solutions by matching an interior wormhole to an exterior vacuum solution, at a junction surface. In

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particular, a thin shell around a traversable wormhole, with generic surface stresses was analyzed in [32], a particular case of this, a wormhole with a zero surface energy density and constant redshift function, having been studied in [4]. A similar analysis for the plane symmetric case, with a negative cosmological constant, was done in [33]. The plane symmetric traversable wormhole is a natural extension of topological black hole solutions with a negative cosmological constant [34, 35, 36, 37], upon addition of exotic matter. These plane symmetric wormholes can be interpreted as domain walls connecting different universes, having planar topology, and upon compactification of one or two coordinates, cylindrical topology or toroidal topology, respectively. The construction of these wormholes does not alter the topology of the background spacetime (i.e., spacetime is not multiply-connected), so that these solutions can instead be considered domain walls. Thus, in general, these wormhole solutions do not allow time travel.

From recent astronomical observations, it seems that we presently live in a world with a positive cosmological constant, $\Lambda > 0$. However, a spacetime with $\Lambda < 0$ is also of significant relevance, since it allows a consistent physical interpretation in the context of supergravity and superstring theories [36]. Indeed, one can enlarge general relativity into a gauged extended supergravity theory, in which the vacuum state has an energy density given by $\Lambda = -3g^2/(4\pi pG)$, where $g$ is the coupling constant of the theory, and $G$ is the gravitational constant. Thus, the vacuum of the theory is described by an anti-de Sitter spacetime. It is important to emphasize that if these theories are correct, they imply that the anti-de Sitter spacetime should be considered as a symmetric phase of the theory, although it must have been broken, since we do not presently live in a universe with $\Lambda < 0$. In addition, note that anti-de Sitter spacetimes have other interesting features: (i) they are one of the rare gravitational backgrounds yielding a consistent interaction with massless higher spins, (ii) they permit a consistent theory of strings in any dimension, (iii) they allow valid definitions for the mass, angular momentum and charge, and (iv) it has been conjectured that they have a direct correspondence with conformal field theory on the boundary of that space, the AdS-CFT conjecture, see, e.g., [38]. Moreover, even with its preference to negative cosmological constant scenarios, string theory can in a contrived way, produce a landscape of positive cosmological constant universes [39], indicating perhaps that one can transit between both signs of the cosmological constant. So, it is certainly of interest to study astrophysical structures that appear within a positive cosmological constant scenario, as well as to study micro structures that may appear within a negative cosmological constant scenario. Here we analyze wormhole structures within a negative cosmological context. Within this context other structures, such as black holes with spherical, toroidal, and hyperbolical topologies, have been analyzed (see [39] for a review).

The paper is organized as follows. By using the cut-and-paste procedure for a thin shell, we construct static and dynamic wormholes by surgically grafting together two spacetimes of plane symmetric vacuum solutions, with planar, cylindrical or toroidal topologies, with a negative cosmological constant. This analysis is displayed in Section III. We shall also consider the specific case of static wormhole solutions, and consider several equations of state. In Section IV we consider a dynamical stability analysis. In particular, a stability analysis is carried out for the dynamic case by taking into account specific surface equations of state, and a linearized stability analysis around static solutions is also further explored. Finally, in Section V we conclude.

II. BLACK MEMBRANE SURGERY AND STATIC WORMHOLES

A. Cut and paste technique

1. General considerations, general equation of state

The planar black hole or a black membrane metric is given by [34, 35, 36, 37]

$$ds^2 = -\left(\alpha^2 r^2 - \frac{M}{\alpha r}\right) dt^2 + \left(\alpha^2 r^2 - \frac{M}{\alpha r}\right)^{-1} dr^2 + \alpha^2 r^2 (dx^2 + dy^2),$$

(1)

where $\alpha$ is the inverse of the characteristic length of the system, which here we adopt as given by the negative cosmological constant, i.e., we put $\alpha^2 = -\Lambda/3$. The ranges of $t$ and $r$ are $-\infty < t < +\infty$ and $0 \leq r < +\infty$. The range of the coordinates $x$ and $y$ determine the topology of the plane symmetric metric. For the planar case, the topology of the two-dimensional space, $t = \text{constant}$ and $r = \text{constant}$, is $R^2$, with coordinate range $-\infty < x < +\infty$ and $-\infty < y < +\infty$. For the cylindrical case the topology is $R \times S^1$, with $-\infty < x < +\infty$ and $0 \leq \alpha y < 2\pi$. For the toroidal case the topology is $S^1 \times S^1$ (i.e., the torus $T^2$), with $0 \leq \alpha x < 2\pi$ and $0 \leq \alpha y < 2\pi$. A scalar polynomial singularity occurs at $r = 0$, which can be demonstrated through the Weyl scalar, given by $C^{\mu\nu\alpha\beta}C_{\mu\nu\alpha\beta} = 12M^2/(\alpha^2 r^6)$, where Greek indices are spacetime indices, $\mu = t, r, x, y$. It can also be found through the Kretschmann scalar, given by $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = 24\alpha^4 + 12M^2/(\alpha^2 r^4)$, but since the first term is innocuous, the Weyl scalar suffices. In particular, we shall consider the planar topology case, and the analysis considered in this work applies equally well for solutions with toroidal and cylindrical topologies. Note that considering a positive value for $M$, an event horizon occurs at $r_h = M^{1/3}/\alpha$, consequently resulting in a planar black hole solution, or black membrane. Considering a negative value for $M$, we verify that no event horizons occur, implying the existence of a singular naked membrane. A remark is in order here: cylindrically thin-shell
wormholes were analyzed in [19], which are different to the planar, cylindrical or toroidal topological solutions we analyze, as for our case infinity carries the same topology as the throat.

An interesting wormhole solution consists in applying the cut-and-paste construction. We consider two copies of the black membrane or planar black hole solution, Eq. [19], removing from each spacetime the four-dimensional regions described by

$$\Omega^\pm \equiv \{ \tau^\pm = 0 | a > M^{1/3}/a \}$$

where a is a constant. The condition $a > M^{1/3}/a$ is important to avoid the presence of an event horizon. The removal of these two regions in each manifold, geodesically incomplete, with boundaries given by the following timelike hypersurfaces

$$\partial \Omega^\pm \equiv \{ \tau^\pm = 0 | a > M^{1/3}/a \}$$

We now identify these two timelike hypersurfaces, $\partial \Omega^+ = \partial \Omega^-$, which results in a manifold, now geodesically complete, with two regions connected by a wormhole. The throat of the wormhole, i.e., the junction surface, is situated at $\partial \Omega$, and may be viewed as behaving like a domain wall connecting two universes. One can then write the intrinsic metric to $\partial \Omega$ as

$$ds^2_{\partial \Omega} = -d\tau^2 + a^2(\tau) (dx^2 + dy^2)$$

where $\tau$ is the proper time on $\partial \Omega$.

In general terms, the position of the junction surface is given by $x^\mu(\tau, x, y)$, and the respective 4-velocity is

$$u^\mu = \frac{dx^\mu}{d\tau}$$

where $\tau$ is the proper time on the surface. We shall use the Darmois-Israel formalism to determine the surface stresses at the junction boundary [11]. The intrinsic surface stress-energy tensor, $S_{ij}$, is given by the Lanczos equations in the form

$$S_{ij} = \frac{1}{2a} \left( \kappa^i_j + \delta^i_j \kappa^k_k \right)$$

where Latin indices are intrinsic surface indices which run through $i = \tau, x, y$. For notational convenience, the discontinuity in the second fundamental form or extrinsic curvatures is given by $\kappa^i_j = K^+_i j - K^-_i j$. The second fundamental form is defined as

$$K^+_i j = \frac{\partial x^j}{\partial \xi^i} \frac{\partial x^j}{\partial \xi^i} n_{\alpha} n_{\beta}$$

$$= -n_\gamma \left( \frac{\partial^2 x^j}{\partial \xi^i \partial \xi^j} + \Gamma^\alpha_{\beta \gamma} \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \right)$$

where $n_\gamma$ is the unit 4-normal to $\partial \Omega$, and the superscripts $\pm$ correspond to the exterior and interior spacetimes, respectively. The parametric equation for $\partial \Omega$ is given by $f(x^\mu(\xi^i)) = 0$. The unit 4-normal to $\partial \Omega$ is given by

$$n^\mu = \pm \left| g^{\alpha \beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial f}{\partial x^\beta} \right|^{1/2} \frac{\partial f}{\partial x^\alpha}$$

with $n^\mu n_\mu = +1$.

Applying to our particular case, the position of the junction surface is given by $x^\mu(\tau, x, y) = (t(\tau), a(\tau), x, y)$, and the respective 4-velocity is

$$u^\mu = \left( \frac{dt}{d\tau}, \frac{da}{d\tau}, 0, 0 \right) = \left( \frac{\sqrt{\alpha^2 a^2 - \frac{M}{\alpha a} + \hat{a}^2}}{\alpha^2 a^2 - \frac{M}{\alpha a}}, \hat{a}, 0, 0 \right)$$

where the overdot denotes a derivative with respect to $\tau$, which is defined as the proper time on $\partial \Omega$. The unit normal to the junction surface may be determined by Eq. [7] or by the contractions, $u^\mu n_\mu = 0$ and $n^\mu n_\mu = +1$, and is given by

$$n_{\mu} = \left( -\hat{a}, \frac{\sqrt{\alpha^2 a^2 - \frac{M}{\alpha a} + \hat{a}^2}}{\alpha^2 a^2 - \frac{M}{\alpha a}}, 0, 0 \right)$$

Considerable simplifications occur due to plane symmetry, namely $\kappa^\alpha_j = \text{diag}(\kappa^\alpha_\tau, \kappa^\alpha_x, \kappa^\alpha_x)$. Thus, the surface stress-energy tensor may be written in terms of the surface energy density, $\sigma$, and the surface pressure, $P$, as $S^\mu_j = \text{diag}(\sigma, P, P)$, which taking into account the Lanczos equations, reduce to

$$\sigma = \frac{1}{4\pi} \kappa^\alpha_x$$

$$\mathcal{P} = \frac{1}{8\pi} (\kappa^\alpha_\tau + \kappa^\alpha_x)$$

This simplifies the determination of the surface stress-energy tensor to that of the calculation of the non-trivial components of the extrinsic curvature, or the second fundamental form. Therefore, taking into account the metric of Eq. [11] and the definition of the second fundamental form, Eq. [10], we have

$$K^\alpha_\tau = \pm \frac{\alpha^2 a + \frac{M}{\alpha a} + \hat{a}}{\sqrt{\alpha^2 a^2 - \frac{M}{\alpha a} + \hat{a}^2}}$$

$$K^\alpha_x = \pm \frac{1}{\alpha} \sqrt{\alpha^2 a^2 - \frac{M}{\alpha a} + \hat{a}^2}$$

From Eqs. [10]-[11], we deduce the surface stress-energy components given by

$$\sigma = -\frac{1}{2\alpha a} \sqrt{\alpha^2 a^2 - \frac{M}{\alpha a} + \hat{a}^2}$$

$$\mathcal{P} = \frac{1}{4\alpha a} \frac{2\alpha^2 a^2 - \frac{M}{\alpha a} + \hat{a} + \alpha a}{\sqrt{\alpha^2 a^2 - \frac{M}{\alpha a} + \hat{a}^2}}$$

respectively. Obviously $\sigma$ and $\mathcal{P}$ obey the conservation equation

$$(\sigma a^2)' + \mathcal{P} (a^2)' = 0$$

In order to be able to solve the two nontrivial equations [12]-[15] for $\sigma(\tau)$ and $a(\tau)$, one needs to specify
an equation of state. Here we impose a cold equation of state

\[ P = \mathcal{P} (\sigma), \]  

(17)

which follows if the temperature is zero, \( T = 0 \), in a general equation of state of the form \( P = \mathcal{P}(\Sigma, T) \), \( \sigma = \sigma(\Sigma, T) \), where \( \Sigma \) is the baryonic mass density. In our study we will make the analysis initially with a generic cold equation of state of the form (17), i.e., without invoking any specific equation of state, see [3], and then particularizing to a specific cold equation of state, see [3].

2. Specific equation of state: dark energy

Besides doing the analysis for a general equation of state of the form \( P = \mathcal{P}(\sigma) \), as in Eq. (17), we want to study specific cases by giving an equation of state. We resort in the following to a particularly interesting equation of state, namely, an equation of state analogous to the dark energy equation of state considered in cosmology, i.e.,

\[ P = \omega \sigma, \text{ with } \omega < 0. \]  

(18)

Such a dark energy fluid can be divided into three cases: a normal dark energy fluid when \(-1 < \omega < 0\), a cosmological constant fluid when \( \omega = -1 \), and a phantom energy fluid when \( \omega < -1 \). All of these fluids are possible candidates responsible for the accelerated expansion of the Universe. But in addition to its cosmological interest, the equation of state (18) can also be used for domain walls and wormholes, connecting opposite spacetime regions.

B. Static wormholes

1. General considerations, general equation of state

The case of a static wormhole is particularly simple, yet it provides interestingly enough results. Equations (14)-(15) reduce to

\[ \sigma = - \frac{1}{2\pi a} \sqrt{\alpha^2 a^2 - \frac{M}{\alpha a}}, \]  

(19)

\[ \mathcal{P} = \frac{1}{2\pi a} \frac{\alpha^2 a^2 - \frac{M}{\alpha a}}{\sqrt{\alpha^2 a^2 - \frac{M}{\alpha a}}}. \]  

(20)

Since there is a generic equation of state of the type given in Eq. (17), the constants \( M, a, \) and \( \alpha \) are inter-related.

The surface energy density in (19) is negative, implying the violation of the weak and dominant energy conditions. The surface pressure is always positive. The null energy condition is verified if \( \sigma + \mathcal{P} > 0 \) is satisfied. Thus, taking into account the relationship

\[ \sigma + \mathcal{P} = \frac{1}{4\pi a} \frac{3M}{\alpha a} \frac{1}{\sqrt{\alpha^2 a^2 - \frac{M}{\alpha a}}}, \]  

(21)

we see that the null energy condition is verified only for \( M > 0 \), and violated for \( M < 0 \). The strong energy condition is satisfied if \( \sigma + \mathcal{P} > 0 \) and \( \sigma + 2\mathcal{P} > 0 \), and by continuity implies the null energy condition. Using the condition

\[ \sigma + 2\mathcal{P} = \frac{1}{2\pi a} \frac{\alpha^2 a^2 + \frac{M}{\alpha a}}{\sqrt{\alpha^2 a^2 - \frac{M}{\alpha a}}}, \]  

(22)

we verify that the strong energy condition is readily satisfied for \( M > 0 \).

It is also of interest to analyze the attractive or repulsive character [28] of this plane symmetric traversable wormhole. The four-velocity of a static observer is \( u^\mu = dx^\mu / d\tau = (u^t, 0, 0, 0) = (1/\sqrt{\alpha^2 r^2 - M/\alpha r}, 0, 0, 0) \). The observer’s four-acceleration is \( a^\mu = u^\mu, \nu u^\nu \), so that taking into account Eq. (1), the only non-zero component is given by

\[ a^r = \Gamma^r_\mu \left( \frac{dt}{d\tau} \right)^2 = \alpha^2 r + \frac{M}{2\alpha r^2}. \]  

(23)

From the geodesic equation, a radially moving test particle which starts from rest initially has the equation of motion

\[ \frac{d^2 r}{d\tau^2} = -\Gamma^r_\mu \left( \frac{dt}{d\tau} \right)^2 = -a^r. \]  

(24)

Therefore, \( a^r \) is the radial component of proper acceleration that an observer must maintain in order to remain at rest at constant \((r, x, y)\). It is interesting to note that a wormhole is “attractive” if \( a^r > 0 \), i.e., observers must maintain an outward-directed radial acceleration to keep from being pulled into the wormhole; and “repulsive” if \( a^r < 0 \), i.e., observers must maintain an inward-directed radial acceleration to avoid being pushed away from the wormhole. Note that for \( M > 0 \), the wormhole is attractive. For \( M < 0 \), the wormhole is repulsive for the values \( r < \sqrt{|M|/2/\alpha} \), and attractive for \( r > \sqrt{|M|/2/\alpha} \). In particular, for \( r = \sqrt{|M|/2/\alpha} \) static observers are geodesic.

2. Specific equation of state: dark energy

To justify the equation of state (15) let us first try a well known case, the surface stress-energy tensor without trace. As we will show now, this is no good. The traceless surface stress-energy tensor, \( S^i_\mu = 0 \), i.e., \(-\sigma + 2P = 0\), could be a case of particular interest, since the Casimir effect with a massless field gives a stress-energy tensor of this type. This effect is theoretically invoked to provide exotic matter to the system considered at hand. From \(-\sigma + 2P = 0\) we deduce \( \alpha^2 a^2 = M/(2aa) \), which is inside the event horizon of the planar black hole, or the black membrane, and so is no good. As in the spherical symmetric case [3], there is no solution of the Einstein
field equations because $\sigma$ and $P$ are imaginary, as one may readily verify from Eqs. (19), (20). So, one has to resort to the equation of state given in (18), i.e., $P = \omega \sigma$, with $\omega < 0$. That $\omega$ has to be less than zero can be seen directly from equations (19)- (20). Indeed, as the surface energy density is always negative, $\sigma < 0$, and the tangential surface pressure is always positive, $P > 0$, then the equation of state (18) imposes the condition $\omega < 0$, i.e., a dark energy equation of state. As mentioned above, one can subdivide the dark energy equation of state. As mentioned above, one can subdivide the dark energy fluid ($-\sigma - P$) are given by finding that this case is only valid when $\omega = -1$. From Eq. (21), one finds that this case is only valid when $M = 0$, thus from Eqs. (19)-(20) the surface energy density and surface pressure are given by $\sigma = -\Pi = -\frac{\omega}{4\pi}$. Another interesting example, is when $\omega < -1$, where it has been shown that spherically symmetric traversable wormholes can be theoretically supported by phantom energy [29, 30].

Note that if $\omega < -1$ reduces to $\sigma + P = 0$, which corresponds to the classical membrane, this being described by the three-dimensional generalization of the Nambu-Goto action, and being the simplest domain wall one may construct [3]. From Eq. (21), one finds that this case is only valid when $M = 0$, thus from Eqs. (19)-(20) the surface energy density and surface pressure are given by $\sigma = -\Pi = -\frac{\omega}{4\pi}$. Another interesting example, is when $\omega < -1$, where it has been shown that spherically symmetric traversable wormholes can be theoretically supported by phantom energy [29, 30].

Now, using Eqs. (19)-(20), we obtain the following relationship

$$M = \Gamma(\omega) \alpha^3 a^3,$$  

(25)

where

$$\Gamma(\omega) = \frac{1 + \omega}{1/4 + \omega}.$$  

(26)

Note that if $M > 0$ then $\omega$ has the range $-\infty < \omega < -1$ and $-1/4 < \omega < 0$. One should impose the additional no horizon condition, $M < \alpha^3 a^3$, since if the condition is not satisfied there are no static solutions, and moreover the wormholes would be inside their own gravitational radius, a situation which is of no interest here. Thus, imposing the additional no horizon condition one gets from Eq. (25) that $\Gamma(\omega) < 1$, which means the $M > 0$ case is restricted to $-\infty < \omega < -1$. If $M = 0$, i.e., the fluid is the spacetime background fluid with a negative cosmological constant, it naturally gives the corresponding equation of state, i.e., $\omega = -1$. If $M < 0$ then $\omega$ is restricted to the range $-1 < \omega < -1/4$. Thus the full range of $\omega$ is $-\infty < \omega < -1/4$, see Fig. 1.

III. DYNAMICAL STABILITY ANALYSIS

A. Stability analysis through matter fields

1. General considerations with a general equation of state

To analyze the dynamics of the wormhole, we consider that the throat is a function of the proper time, as measured by an observer comoving with the throat [3, 13, 14]. Assume that the position of the throat is given by $x^\mu(\tau, x, y) = (t(\tau), a(\tau), x, y)$. As mentioned, Eqs. (14)-(15) imply the energy conservation equation (10), which in turn can be put in the form

$$\dot{\sigma} = -2 (\sigma + P) \frac{\dot{a}}{a}.$$  

(27)

Equation (27) may also be rewritten as

$$\frac{da}{a} = -\frac{1}{2} \frac{d\sigma}{\sigma + P}.$$  

(28)

Taking into account an equation of state of the form given in Eq. (17), i.e., $P = P(\sigma)$, Eq. (28) can be integrated to yield

$$\ln(a) = -\frac{1}{2} \int \frac{d\sigma}{\sigma + P(\sigma)}.$$  

(29)

This result may be formally inverted to provide $\sigma = \sigma(a)$. Equation (14) may be recast into the following form

$$\dot{a}^2 - [(2\pi \sigma(a))^2 - \alpha^2] a^2 - \frac{M}{\alpha a} = 0,$$  

(30)

which may also be written as $\dot{a}^2 = -V(a)$, where the potential is defined as

$$V(a) = -[(2\pi \sigma)^2 - \alpha^2] a^2 - \frac{M}{\alpha a}.$$  

(31)

We sketch an analysis of this potential by dividing it into three cases. (i) When $(2\pi \sigma(a))^2 > \alpha^2$, from Eq. (30),
we can divide into two cases $M \geq 0$ and $M < 0$. For $M \geq 0$ we verify that the wormhole is dynamically unstable. Depending on the initial conditions, this wormhole solution will either collapse to $a = 0$ or explode to $a \to \infty$. For $M < 0$ the wormhole never collapses, and depending on the form of $\sigma(a)$ it may or may not explode. Thus, for $M < 0$ the solution may be stable. (ii) When $(2\pi\sigma(a))^2 = a^2$, then Eq. (33) reduces to $\dot{a}^2 - M/\alpha a = 0$. Thus, we can divide also into two cases $M \geq 0$ and $M < 0$ For $M \geq 0$ the differential equation has the following solutions $a(\tau) = \frac{1}{2\pi M} (\pm 12M^2 a^{-2}\tau^2 \mp 12M^2 a^{-2} C)^{2/3}$, where $C$ is a constant related to the initial radius of the wormhole, $C = \frac{2}{3} \sqrt{\alpha a_0^3}/M$. We verify that if $\tau \to \infty$, the wormhole explodes, $a \to \infty$. The wormhole collapses to $a = 0$ in the proper time $\tau = C$. In addition to $(2\pi\sigma)^2 = a^2$, considering $M = 0$, we have a stable and static wormhole solution, i.e., $\dot{a}(\tau) = 0$, as we verified whilst analyzing the classical membrane in static wormholes. For $M < 0$ there is no solution. (iii) When $(2\pi\sigma(a))^2 < a^2$, for $M \geq 0$ one has that the wormhole will not explode, but will collapse. For $M < 0$ there is no solution.

Before entering the analysis for the specific dark energy equation of state [18], we give a simple example of a dynamically stable solution with a specific choice for the matter field surface energy density. Note that to be dynamically stable, the potential has to be decreasing, i.e., $\dot{\sigma} = 0$, with $\sigma$ equal to Eq. (31), provides the following potential $V(a) = k_1(a - k_2)^2 - k_3$, which can easily be depicted qualitatively. The zeroes of $V(a)$ are situated at $a_{1,2} = k_2 \pm \sqrt{k_3/k_1}$, where the lower root $a_1 = k_2 - \sqrt{k_3/k_1}$ is required to obey $a_1 > M^{1/3}/\alpha$, in order to avoid a black hole solution.

2. Specific equation of state: dark energy

Considering the equation of state given in Eq. (18), i.e., $\rho = \omega \sigma$, from Eq. (29), we deduce

$$\sigma(a) = \sigma_0 \left(\frac{a}{a_0}\right)^{2(1+\omega)}, \quad (32)$$

where $a_0$ is the initial position of the throat and $\sigma_0 = \sigma(a_0)$. The qualitative behavior is transparent from Fig. 2. As $\omega \to 0$ and for high values of $\alpha = a/a_0$, then $\sigma \to 0$. On the other hand for decreasing values of the parameter $\omega$ and for high $\alpha = a/a_0$, then $\sigma \to -\infty$. For $\omega \to 0$ and $\alpha = a/a_0$, then $\sigma \to -\infty$. Note that considering the equation of state of the domain wall [3 15], $\sigma + \rho = 0$, from Eq. (27) we have $\dot{\sigma} = 0$, i.e., $\sigma = \text{const} < 0$.

Relative to the stability analysis, by substituting Eq. (32) into the potential, Eq. (31), yields the following relationship

$$U(a) = -\left[\left(\frac{a}{a_0}\right)^{-4(1+\omega)} - \alpha^2\right] \left(\frac{a}{a_0}\right)^2 - M \left(\frac{a}{a_0}\right), \quad (33)$$

where the following definitions were considered for notational simplicity

$$U(a) = \frac{V(a)}{(a_0 \sigma_0)^2}, \quad \bar{\sigma} = \frac{\sigma}{\sigma_0}, \quad \bar{M} = \frac{M}{a_0 \sigma_0}, \quad (34)$$

with $\sigma_0 = 2\pi\sigma_0$. A stability analysis is depicted in Fig. 3 for specific numerical values, which may be considered as representative for the present stability analysis. For positive values of $\bar{M}$, depicted by the dashed curve, we have considered the specific values of $\bar{M} = 0.15$, $\omega = -3/2$ and $\bar{\alpha} = 0.15$. One verifies that for this case, the wormhole is unstable. For $\bar{M} < 0$, depicted by the solid curve, we have considered the specific numerical values of $\bar{M} = -0.15$, $\omega = -1/2$ and $\bar{\alpha} = 0.15$, and one verifies the stability of the wormhole.

Thus, in summary, for $\bar{M} > 0$, one sees that the wormholes are unstable, as depicted by the dashed curve in Fig. 3. For $\bar{M} = 0$, not depicted, the wormhole is static and neutrally stable. For $\bar{M} < 0$, which is depicted by the solid curve in Fig. 3 the wormhole is stable against expansion and collapse, it will oscillate between a maximum value and a minimum value of $a$.

B. Linearized stability analysis

1. General considerations with a general equation of state

An alternative to the partial stability analysis of Section III A is to consider a linear perturbation around a static solution with radius $a_0$. The respective static values of the surface energy density and the surface pressure
are given by Eqs. (19)-(20). To know whether the equilibrium solution is stable or not, one must analyze the shell’s equation of motion near the equilibrium solution. This means that in order to solve this equation one has to know the equation of state of the shell’s matter.

In this subsection, rather than choosing a specific equation of state, we shall follow the Poisson-Visser reasoning [14], and put quite generally $\mathcal{P} = \mathcal{P}(\sigma)$, see (17). Thus, the potential given by Eq. (31), may be rewritten as

$$V(a) = \alpha^2 a^2 - \frac{M}{a} - [2\pi a \sigma(a)]^2. \tag{35}$$

Linearizing around the stable solution at $a_0$, a second order expansion of $V(a)$ around $a_0$ provides

$$V(a) = V(a_0) + V'(a_0)(a-a_0) + \frac{1}{2} V''(a_0) (a-a_0)^2 + \mathcal{O}[(a-a_0)^3], \tag{36}$$

where the prime denotes a derivative with respect to $a$. To determine $V'(a)$ and $V''(a)$, it is useful to rewrite the conservation of the surface stress-energy tensor, Eq. (24), as $\sigma' a = -2(\sigma + \mathcal{P})$, taking into account $\sigma' = \dot{\sigma}/\dot{a}$. Defining the parameter $\eta(\sigma)$ by

$$\eta(\sigma) = d\mathcal{P}/d\sigma, \tag{37}$$

and since $\mathcal{P}(\sigma)' = (d\mathcal{P}/d\sigma)\sigma' = \eta \sigma'$, we have $\sigma' + 2\mathcal{P}' = \eta \sigma'(1+2\eta)$. Thus, $V'(a)$ and $V''(a)$, are given by

$$V'(a) = \frac{M}{a^2} + 2\alpha^2 a + 8\pi^2 \sigma a (\sigma + 2\mathcal{P}), \tag{38}$$

$$V''(a) = -\frac{2M}{a^{3/2}} + 2\alpha^2 - 8\pi^2 \left[(\sigma + 2\mathcal{P})^2 + 2\sigma (\sigma + \mathcal{P})(1+2\eta)\right], \tag{39}$$

respectively. Evaluated at the static solution, using Eqs. (19)-(20), and taking into account Eqs. (21)-(22), we find $V(a_0) = 0$ and $V'(a_0) = 0$, with $V''(a_0)$ given by

$$V''(a_0) = -\alpha^2 \frac{3M}{(a_0)^{3}} \left( \frac{\alpha^2 a_0^2 + M}{\sigma a_0} - 2\eta_0 \right), \tag{40}$$

where $\eta_0 = \eta(\sigma_0)$. For this, note that Eq. (40) may be expressed as

$$V''(a_0) = \alpha^2 \frac{3M}{(a_0)^{3}} \left( \frac{2P_0}{\sigma_0} + 1 + 2\eta_0 \right), \tag{41}$$

by taking into account the surface stresses evaluated at the static solution $a_0$, given by Eqs. (19)-(20). The potential $V(a)$, Eq. (35), is reduced to

$$V(a) = \frac{1}{2} V''(a_0)(a-a_0)^2 + \mathcal{O}[(a-a_0)^3], \tag{42}$$

so that the equation of motion for the wormhole throat presents the following form

$$\dot{a}^2 = -\frac{1}{2} V''(a_0)(a-a_0)^2 + \mathcal{O}[(a-a_0)^3], \tag{43}$$

to the order of the approximation considered. The solution is stable if and only if $V''(a_0) > 0$. Three cases may be distinguished, namely, $0 < M/(a_0)^{3} < 1$ (which essentially amounts to $M > 0$), $M/(a_0)^{3} = 0$ (which essentially amounts to $M = 0$), and $M/(a_0)^{3} < 0$ (which essentially amounts to $M < 0$). For $0 < M/(a_0)^{3} < 1$, the solution is stable if

$$\eta_0 > \frac{1}{2} \left[ \frac{(a_0)^2 + \frac{M}{(a_0)^3}}{(a_0)^2 - \frac{M}{(a_0)^3}} \right], \quad 0 < \frac{M}{(a_0)^{3}} < 1. \tag{44}$$

For the specific case of $M/(a_0)^{3} = 0$ we have a static and neutrally stable wormhole solution. This can be readily verified from Eq. (41), which implies $V''(a_0) = 0$, and consequently $\dot{a} = 0$, from Eq. (13). Thus any $\eta_0$ is possible, i.e.,

$$-\infty < \eta_0 < \infty, \quad \frac{M}{(a_0)^{3}} = 0. \tag{45}$$

Note that this is consistent with the analysis outlined in Sections III A 2 and III B 1. For $M/(a_0)^{3} < 0$, the stability region is dictated by the following inequality

$$\eta_0 < \frac{1}{2} \left[ \frac{(a_0)^2 - \frac{M}{(a_0)^3}}{(a_0)^2 + \frac{M}{(a_0)^3}} \right], \quad \frac{M}{(a_0)^{3}} < 0. \tag{46}$$
The stability regions are plotted in Figure 4. One can put Eqs. (44)-(46) in terms of the matter fields. These yield,

$$\eta_0 > - \left( \frac{\mathcal{P}_0}{\sigma_0} + \frac{1}{2} \right), \quad 0 < \frac{M}{(\alpha a_0)^3} < 1,$$  

$$- \infty < \eta_0 < \infty, \quad \frac{M}{(\alpha a_0)^3} = 0,$$  

$$\eta_0 < - \left( \frac{\mathcal{P}_0}{\sigma_0} + \frac{1}{2} \right), \quad \frac{M}{(\alpha a_0)^3} < 0,$$

respectively.

2. Specific equation of state: dark energy

It is also interesting to consider the stability regions, using the linearization approach, for the specific case of the dark energy equation of state, $\mathcal{P} = \omega \sigma$, (see Eq. (18)). By considering $\mathcal{P} = \omega \sigma$, we note that for this linear equation of state, we have $\eta_0 = \omega$. Now, using the stability condition $V''(a_0) > 0$, from Eqs. (49)-(47) we have the following stability conditions. For $M > 0$, one has to take into account the relation between $\omega$ and the mass $M$ given in Eq. (25) (see also Fig. 1), i.e., if $M > 0$ then $\omega$ is in the range $\omega < -1$, whereas the range $\omega > -\frac{1}{4}$ is physically unacceptable since it violates the no horizon condition. So, for $M > 0$, $\omega$ is in the range $\omega < -1$, and the solutions are unstable. For $M = 0$ one has $\omega = -1$, and the solution is stable. For $M < 0$, $\omega$ is in the range $-1 < \omega < -1/4$, and the solutions are stable. Thus, we finally have

$$- \infty < \omega < -1, \text{ i.e., } 0 < \frac{M}{(\alpha a_0)^3} < 1, \text{ unstable},$$  

$$\omega = -1, \text{ i.e., } \frac{M}{(\alpha a_0)^3} = 0, \text{ stable},$$  

$$-1 < \omega < -\frac{1}{4}, \text{ i.e., } \frac{M}{(\alpha a_0)^3} < 0, \text{ stable},$$

respectively. So, thin shell wormholes made of phantom energy ($\omega < -1, M > 0$) are unstable, while thin shell wormholes made of dark energy fluid ($-1 < \omega < -1/4, M < 0$) or of cosmological constant fluid ($\omega = -1, M = 0$) are stable.

C. Comparison between both stability analyses

Of course both stability analyses have to agree. That this is so one can deduce by comparing the specific examples considered in section III A 2 and the results given in section III B 2. We have found in section III A 2 that for $\omega = -3/2$ and $M > 0$ the solution is unstable, which is readily confirmed above. On the other hand we have found that for $\omega = -1/2$ and $M < 0$ the solution is stable which is also verified.

IV. CONCLUSION

Using the cut-and-paste technique, we have constructed an elegant and simple class of static and dynamical plane symmetric wormholes by surgically grafting together two spacetimes of plane symmetric vacuum solutions with a negative cosmological constant. We have further considered a dynamical stability analysis, first, by considering specific equations of state, and paid special attention to the dark energy equation of state. Second, a linearized stability analysis was also explored. Considering radial perturbations around a static solution, the respective stability regions were presented, and the specific case of a dark energy equation of state was also analyzed. It was found that thin shell wormholes made of a dark energy fluid or of a cosmological constant fluid are stable, while thin shell wormholes made of phantom energy are unstable. These plane symmetric wormholes may be viewed as domain walls connecting different universes. The construction of these wormholes does not alter the topology of the background spacetime, i.e., spacetime is not multiply-connected, so that, in general, these wormhole solutions do not allow time travel.

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