An alternative to hypercovers

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August 28, 2020

Abstract

I introduce a class of diagrams in a Grothendieck site called atlases which can be used to study hyperdescent, and show that hypersheaves take atlases to limits using an indexed ‘nerve’ construction that produces hypercovers from atlases. Atlases have the flexibility to be at the same time more explicit and more universal than hypercovers.

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1 Introduction

A presheaf $F$ on a topological space $X$ is said to satisfy descent along an open cover $\{U_i \subseteq V\}_{i:I}$ of an open set $V \subseteq X$ if its sections over $V$ can be computed
by taking the limit over the Čech nerve

\[ \cdots \Longrightarrow (\coprod_i U_i) \times_X (\coprod_i U_i) \Longrightarrow \coprod_i U_i \longrightarrow X \]  

(1.0.1)

(for sheaves of sets, the first two terms, which appear here explicitly, are enough). It is called a sheaf if it satisfies descent along all open covers. This perspective, popularised in [SGA4], is the starting point for topos theory and the modern approach to numerous flavours of cohomology theory.

For computing global sections — i.e. cohomology — it is useful to be able to restrict attention to covers belonging to a certain site for \(X\). For example, if \(X\) is a manifold, then a convenient site is the poset of open sets diffeomorphic to \(\mathbb{R}^n\). Since these are exactly the contractible open sets, the classifying space of this poset has the same homotopy type as \(X\). Hence, this site is useful for computing homotopy invariants of \(X\), such as the theory of local systems on \(X\).

Because contractible subsets of \(X\) are not closed under intersection, the Čech nerve construction is not available for every cover in the site \(\mathcal{U}^o(X)\). The preceding definition of descent along open covers does not, therefore, translate easily into the new context. What we need is a generalisation in which a binary intersection of open subsets can be resolved with its own open cover, and again for the intersections of those open sets, and so on \textit{ad infinitum}. This is the intuition that the hypercovers of [SGA4, Exp. V, 7.3] attempt to capture.

Today, the theory of hypercovers is woven into the fabric of nearly all flavours of homotopical sheaf theory (with the bulk of [HTT, Chap. 6] being a notable exception). They admit an elegant characterisation in terms of the model structure on simplicial presheaves [DHI04].

In practice, we must often construct hypercovers by conversion from more basic ‘naturally occurring’ data. This process destroys finiteness and, perhaps, intuitiveness. Why not try instead to capture these ‘naturally occurring’ data directly? D. Quillen is supposed to have said that only when he “freed [himself] from the shackles of the simplicial way of thinking” was he able to discover his Q-construction in algebraic \(K\)-theory [Gra13]. The notion of \textit{atlas} I now define arose in the search for a similar liberty in the context of descent theory:

1.1 Definition (Atlas). Let \(T\) be an \(\infty\)-category equipped with a Grothendieck topology in the sense of [HTT, Def. 6.2.2.1], \(X : T\) an object. An \(\infty\)-functor
$U : I \rightarrow T/X$ is said to be an atlas for $X$ if for any finite diagram $\alpha : K \rightarrow I$, the set
\[
\{ U \widetilde{\alpha} \rightarrow \lim_{k:K} U_{\alpha(k)} \}_{\alpha: I} \)
\]
is a covering in $\text{PSh}(T)$, where $T/\alpha$ is the overcategory of [HTT, §1.2.9] (also known as the category of left cones), and $U \widetilde{\alpha}$ stands for the value of $U \circ \widetilde{\alpha} : K^{op} \rightarrow T/X$ on the cone point.

**1.2 Definition** (Descent along atlases). An $\infty$-presheaf $F : \text{Top}^{op} \rightarrow \text{Spc}$ is said to satisfy descent along atlases if each open $V \subseteq X$ and atlas $U \mid I$ of $V$ induces an equivalence of spaces
\[
F(V) \cong \lim_{i:I} F(U_i).
\]
The relevance of this definition is captured by the following statement:

**1.3 Conjecture.** The following conditions on an $\infty$-presheaf $F : \text{Top}^{op} \rightarrow \text{Spc}$ are equivalent:

1. $F$ satisfies descent along atlases.

2. $F$ satisfies hyperdescent.

In particular, the full subcategory of $\text{PSh}_\infty(X)$ spanned by the functors that satisfy descent along atlases is a hypercomplete topos.

The main application I have for this theory of atlases is a formulation of the universal property of (derived) geometry, which appears in a separate work [Mac17]. In the present paper, I prove a truncated version of Conjecture 1.3 that applies to topological spaces — more precisely, to locales — and is sufficient for the application of op. cit..

**1.4 Theorem.** Let $X$ be a locale with lattice of open sets $\mathcal{U}(X)$, and let $F : \mathcal{U}(X)^{op} \rightarrow \text{Spc}$ be a hypercomplete $\infty$-sheaf. Then $F$ satisfies descent along atlases indexed by posets.

**Proof.** It is well-known that hypercompleteness means that $F$ takes hypercovers to limits. See [HAG-I, §3] (below Definition 3.4.8) for a proof in the context of simplicially enriched categories; to translate into the language of quasi-categories, use [HTT, Prop. 4.2.4.4] and [HTT, Rmk. 6.5.2.15].
The data of a hypercover can be equivalently formulated as a certain kind of diagram indexed by the total space of a left fibration over $\Delta^{op}$, called its index diagram (4.6), which satisfies certain local filling conditions (Prop. 6.1).

The result now follows from a nerve construction which associates to each diagram $U : I \rightarrow \mathcal{U}(V)$ of open sets of $V$, indexed by a poset $I$, a diagram

$$
\begin{array}{c}
\int \tilde{N}(I) \\
\downarrow \text{l-fib} \\
\Delta^{op}
\end{array}
\xrightarrow{\varepsilon} I
$$

with the property that $U \circ \varepsilon : \int \tilde{N}(I) / \Delta^{op} \rightarrow \mathcal{U}(V)$ is the index diagram of a hypercover if and only if $U$ is an atlas of $V$ (Theorem 7.7). Moreover, $\varepsilon$ is cofinal (Proposition 7.3), whence a colimit over $I$ can be computed on its restriction to $\int \tilde{N}(I)$. Thus if $F$ is hypercomplete and $U$ is an atlas, then $U$ — and therefore also $U$ — takes $\int \tilde{N}(U)$ to a limit. \hfill \Box

2 Atlases for locales

If we restrict attention from arbitrary $\infty$-categories to posets, unsurprisingly we find some substantial simplifications to the general theory.

2.1 (Atlas for a locale). Let $X$ be a locale with frame of open sets $\mathcal{U}(X)$ [Joh82]. A diagram of open subsets of $X$ is a monotone map of posets $U : I \rightarrow \mathcal{U}(X)$. We write $U_i \subseteq X$ for the open set associated to $i : I$ by $U$.

2.2 Proposition. Let $U : I \rightarrow \mathcal{U}(X)$ be a diagram of open subsets of $X$ indexed by a poset $I$. The following conditions are equivalent:

1. $X = \bigcup_{i : I} U_i$, and for any $i, j : I$, $U_i \cap U_j = \bigcup_{k \leq i, j} U_k$.

2. For any finite $J \subseteq I$,

$$
\bigsqcup_{i \in J \forall j \in J} U_i \rightarrow \bigcap_{j : J} U_j
$$

is a covering.

3. $U$ is an atlas.

\text{\textsuperscript{1}}If the reader prefers, he may instead let $X$ be a topological space without affecting the arguments. Descent theory is in any case mediated through the associated locale.
Proof. 1 ⇔ 2 by induction on the cardinality of $J$.

The implication $3 \Rightarrow 2$ is clear. Conversely, suppose that $U : I \to \mathcal{U}(X)$ satisfies 2, and let $K \to I$ be a finite diagram with $K$ some $\infty$-category. Then

$$\mathcal{U}(X)_{\alpha} = \mathcal{U}(X)_{\tau_0 \alpha}$$

because $\mathcal{U}(X)$ is 0-truncated (i.e. a poset) and the formation $K \to K^\triangleright$ of the left cone commutes with truncation. So, replacing $K$ with its truncation, we may assume that it is a finite poset. But then also

$$\mathcal{U}(X)_{\alpha} = \mathcal{U}(X)_{\alpha | K_0}$$

where $K_0$ is $K$ regarded as a poset with the trivial ordering. The atlas condition is now handled by 2. \qed

The language of atlases is more flexible than that of hypercovers: many hypercovers of interest are obtained by conversion from a naturally arising diagram which may itself already be an atlas.

2.3 Example. Let $X = U \cup V$ be a topological space expressed as a union of two open sets. The diagram

$$U \cap V \to V \to U$$

is an atlas for $X$.

Suppose now we have a further decomposition $U \cap V = A \cup B$ with $A \cap B = \emptyset$. Then the diagram

$$\begin{array}{ccc}
A & \to & U \\
\downarrow & & \downarrow \\
B & \to & U \\
\end{array}$$

is an atlas for $X$. Notice that the index poset for this atlas has the weak homotopy type of $S^1$. The reader can no doubt imagine a way to realise this diagram as an atlas of contractible open sets in the case $X = S^1$. 

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2.4 Example (Basis). A basis $I \subseteq \mathcal{U}(X)$ for the topology of $X$ is an atlas for $X$: condition 2 of Proposition 2.2 follows easily from the definition of a basis.

2.5 Example (Schemes). A locally ringed space $X$ is a scheme if and only if the inclusion $\mathcal{U}[\text{aff}](X) \rightarrow \mathcal{U}(X)$ of the poset of open immersions from affine schemes is an atlas for $X$. Note that this poset is closed under finite intersections (resp. binary intersections) if and only if $X$ is an affine scheme (resp. has affine diagonal).

2.6 Example (Manifolds). A paracompact Hausdorff space $X$ is a topological manifold if and only if the preorder of open immersions $\mathbb{R}^n \hookrightarrow X$ is an atlas for $X$. This preorder is never closed under binary intersections (unless $X$ is a point). This example, or its $C^\infty$ analogue, was what first motivated me to formulate the notion of atlas.

Atlases give us an easy way to describe certain counterexamples to hypercompleteness:

2.7 Example (Hilbert cube). Consider the Hilbert cube $Q = [0,1]^\mathbb{N}$ as in [HTT, Ex. 6.5.4.8]. The set of open subsets homeomorphic to $Q \times [0,1)$ forms a base for the topology; in particular, it is an atlas. Borel-Moore homology defines a sheaf (by excision) whose restriction to this atlas is zero, but whose global sections are nonzero; hence, by Theorem 1.4 it cannot be hypercomplete.

3 Simplices

Begin with some preliminary remarks on simplicial sets.

3.1 (Simplex categories). As usual, $\Delta$ is the geometric simplex category of inhabited totally ordered sets, while $\Delta_+$ is the subcategory consisting of injective maps. By [HTT, Lemma 6.5.3.7], the inclusion $\Delta_+ \rightarrow \Delta$ is $\infty$-coinitial, i.e. for each $n$, $\Delta_+ \downarrow_{\Delta} \Delta^n$ is weakly contractible.

3.2 (Simplices of a simplicial set). A simplicial set $F$ can be realised as a left fibration in sets by integrating over $\Delta^{\text{op}}$. We denote this fibration by $\int F / \Delta^{\text{op}}$, its total space (source) by $\int F$. It is called the category of simplices of $F$. We use this construction to make sense of the category of maps from a simplicial set $F$ to a category $I$.

$$I(F) := \text{Fun}(\int F, I)$$

(3.2.1)
Similarly, a semisimplicial set $G$ can be realised as a left fibration $f_+ G / \Delta^\text{op}_+$. It is called the category of nondegenerate simplices of $G$. We use $f_+$ to make sense of the category
\[
I(G) = \text{Fun}(f_+ G, I).
\]

of maps from $G$ to a category $I$. Let $\text{Lan}_{\Delta+/\Delta} G$ be the simplicial envelope of $G$ — that is, simplicial set obtained from $G$ by left Kan extension along $\Delta_+ \subset \Delta$. Then precomposition with the natural functor $f_+ G \to f \text{Lan}_{\Delta+/\Delta} G$ gives us a map
\[
I(\text{Lan}_{\Delta+/\Delta} G) \to I(G), \quad s \mapsto s_+
\]
which restricts a face diagram to its nondegenerate part.

3.3 Example (Simplex). We write $\Delta^+_n$ for the semi-simplicial set represented by an ordered set with $n + 1$ elements. The category $f_+ \Delta^+_n$ of nondegenerate simplices is the opposite of the poset of inhabited subsets of $[n + 1]$ — in particular, it is finite.

The simplicial envelope of $\Delta^+_n$ — the simplicial set represented by the same ordered set as an object of $\Delta$ — is written $\Delta^n$. Its category of simplices $f\Delta^n$ is infinite, but taking the image of a map $\Delta^k \to \Delta^n$ defines a coreflector onto the finite subset $f_+ \Delta^+_n$ of nondegenerate simplices. In particular, it is left finite in that limits over $f\Delta^n$ are equivalent to limits over a finite category — cf. (5.1).

3.4 Example (Simplex boundary). Write $\partial\Delta^+_n$ for the semi-simplicial set representing the boundary of the $n$-simplex, and $\partial\Delta^n$ for the associated simplicial set. The category
\[
f_+ \partial\Delta^+_n = f_+ \Delta^n \setminus \{[n + 1]\}
\]
of nondegenerate simplices of $\partial\Delta^n$ is the poset of inhabited proper subsets of $[k + 1]$. In particular,
\[
f_+ \Delta^+_n = (f_+ \partial\Delta^+_n)^\text{op}
\]
is a categorical left cone over the category of simplices of $\partial\Delta^+_n$.

Similarly, $\partial\Delta^n = \Delta^n \times_{\Delta^+_n} \partial\Delta^+_n$ is the category of simplices equipped with a non-surjective map to $\Delta^n$. It is a sieve in $\Delta^n$. It contains $\partial\Delta^+_n$ as a coreflective subcategory and is therefore left finite.
### 3.5 Example (Mixed category of simplices of a simplex)

The square

\[
\begin{array}{ccc}
\int_+ \partial \Delta^n_+ & \rightarrow & \int_+ \Delta^n_+ \\
\downarrow & & \downarrow \\
\int \partial \Delta^n & \rightarrow & \int \Delta^n
\end{array}
\]

of fully faithful functors isn’t quite a pushout in `Cat`: rather, the pushout is the full subcategory \( \int \partial \Delta^n \cup \int_+ \Delta^n_+ \) of \( \int \Delta^n \) spanned by simplices which are either nondegenerate or factor through the boundary. This subcategory is coreflective. The value of the coreflector on \( \sigma : \Delta^k \rightarrow \Delta^n \) is either \( \sigma \) itself, if the image of \( \sigma \) is contained in the boundary, or \( \text{Im}(\sigma) \) otherwise.

### 4 Index diagrams

Hypercoverings of \( X \) are defined in [SGA4, Exp. V, §7.3], [DHI04, Def. 4.2], [HAG-I, Def. 3.4.8] as certain simplicial objects in the category of presheaves on \( \mathcal{U}(X) \). Some manipulation is required to convert these into diagrams in \( \mathcal{U}(X) \) itself.

#### 4.1 Definition (Semi-representable presheaves)

Let \( C \) be a poset. Recall from [SGA4, Exp. V,7.3] that a presheaf of sets on \( C \) is said to be **semi-representable** if it can be represented as a coproduct of representables. The full subcategory of the 1-category \( \text{PSh}_1(C) \) of presheaves of sets on \( C \) spanned by the semi-representable objects is denoted \( \text{SR}(C) \).

#### 4.2 (Set indexed objects)

An \( S \)-indexed element of \( C \), where \( S : \text{Set} \), is a map \( S \rightarrow C \). We write

\[ C^S = \text{Fun}(S, C) \]

for the poset of \( S \)-indexed elements of \( C \). Integrating, the category of all set-indexed elements of \( C \) is defined

\[ \int_S C^S = \int_{S : \text{Set}} C^S. \]

It is Cartesian-fibred over \( \text{Set} \). Integrating the projection induces a co-Cartesian fibration

\[ \text{idx} \rightarrow \int_S C^S, \]
the universal index set of set-indexed objects of $C$.

4.3. The functor $\int_{S: \text{Set}} \text{Fun}(S, C) \to \text{SR}(C)$ is constructed as follows:

- An object $X : S \to C$ goes to the presheaf $\coprod_{s : S} X_s : \text{PSh}_1(C)$;
- A morphism
  \[
  \begin{array}{ccc}
  S_0 & \xrightarrow{f} & S_1 \\
  \downarrow & \Rightarrow & \downarrow \\
  X_0 & \to & X_1
  \end{array}
  \]
  gets sent to the map
  \[
  \coprod_{s : S_0} X_{0,s} \cong \coprod_{s : S_1} (\text{Lan}_f X_0)_s \to \coprod_{s : S_1} X_{1,s}
  \]
  obtained from the universal property of the left Kan extension $\text{Lan}_f(X_0)$ of $X_0$ along $f$.

Compatibility with composition follows from uniqueness of the map from the left Kan extension.

4.4 Proposition. The groupoid of representations of a semi-representable presheaf as a coproduct of representables is contractible. There is a natural equivalence

\[
\text{SR}(C) \cong \int_{S: \text{Set}} \text{Fun}(S, C)
\]

between the 1-category of semi-representable presheaves on $C$ and the 1-category of set-indexed objects of $C$.

**Proof.** Let $F = \coprod_{i : I} X_i$ be semi-representable. We will show that the groupoid whose objects are maps $Y : S \to C$ together with an isomorphism $\coprod_{s : S} Y_s \cong F$, is trivial.

The right fibration $C \downarrow F / C$ decomposes as a disjoint union $\coprod_{i : I}(C \downarrow X_i / C)$ of fibrations whose total space is connected. This is the decomposition of a category into its connected components, hence it is unique. Finally, each $X_i$ is determined uniquely by the fibration $C \downarrow X_i / C$.

It remains to show that $\Pi_0 : \text{SR}(C) \to \text{Set}$ is a Cartesian fibration. Let $F : \text{SR}(C)$ and let $\phi : S \to \Pi_0(F)$ be a map. Then

\[
\begin{array}{ccc}
\text{Map}(X, \coprod_{s : S} F_{\phi s}) & \longrightarrow & \text{Map}(X, F) \\
\downarrow & & \downarrow \\
S & \xrightarrow{\phi} & \Pi_0(F)
\end{array}
\]
for any $X : C$, because $\text{Map}(X, -)$ commutes with coproducts and coproducts of sets are universal.

4.5 Proposition (Families of set indexed objects). The 1-category of set-indexed objects of $C$ classifies diagrams

$$
\begin{array}{c}
I \longrightarrow C \\
\downarrow^{\text{l-fib}} \\
K
\end{array}
$$

where $I \to K$ is a left fibration in sets.

Proof. Beginning from the alias

$$
\int^S \mathfrak{C} \mathfrak{S} = \text{Set} \downarrow \mathsf{Cat}_0 \{C\}
$$

we find

$$
\begin{array}{c}
\text{Fun}(K, \int^S C^S) \\ \cong \left\{ \begin{array}{c}
\text{Set} \\
K \longrightarrow C \longrightarrow \mathsf{Cat}_0
\end{array} \right. \\
\cong \left\{ \begin{array}{c}
I \longrightarrow K \times C \\
\text{l-fib} \\
K
\end{array} \right. \text{ via } \int^K \\
\cong \left\{ \begin{array}{c}
I \longrightarrow C \\
\text{l-fib} \\
K
\end{array} \right.
\end{array}
$$

where arrows marked l-fib are constrained to the category of left fibrations in sets.

4.6 (Simplicial semi-representable presheaves). Combining Propositions 4.2 and 4.5, a simplicial object $U_{\bullet}$ of $\text{SR}(C)$ consists of the data of a diagram

$$
\begin{array}{c}
\text{idx}_U \longrightarrow \text{idx} \longrightarrow C \\
\downarrow^J \\
\Delta^{op} \longrightarrow \int^S C^S
\end{array}
$$
where \( \text{idx}_U \to \Delta^{\text{op}} \) is a left fibration. The category \( \text{idx}_U = \int_{\Delta^{\text{op}}} (\text{idx} \circ U) \) is called the index category or category of indices of the object \( U_* \), and the data \( \tilde{U} : \text{idx}_U \to C \) is called the index diagram of \( U_* \).

4.7 Example (Tensor-representable objects). Let \( K_* \) be a simplicial set and \( V : C \). Then \( K_* \otimes V \) is a tensor-representable simplicial presheaf on \( C \) with index diagram

\[
\int K \xrightarrow{V} C
\]

\[\Delta^{\text{op}}\]

If \( U : \int K \to C \) is another diagram, then morphisms \( K_* \otimes V \to U \) are the same as cones over \( U \) with vertex \( V \). In particular, if \( U \) admits a limit in \( C \), then it admits a final object in the category of maps from tensor-representable objects.

4.8 Example (Local isomorphisms). Via the inverse of the index category construction, a commuting triangle

\[
\begin{array}{ccc}
S_0 & \xrightarrow{f} & S_1 \\
\downarrow & & \downarrow \\
C & & C \\
X_0 & \leftarrow & X_1
\end{array}
\]

induces a morphism of simplicial semi-representable presheaves. A morphism in \( s\text{SR}(C) \) induced in this way is called a local isomorphism. In other words, local isomorphisms are the morphisms which are Cartesian for the forgetful functor \( \text{idx} : s\text{SR}(C) \to s\text{Set} \); we also write \( X_0 = f^* X_1 \).

The intuition behind this terminology is as follows: a morphism \( \phi : K \to L \) in \( \text{SR}(C) \) is called a local isomorphism if for each connected component \( K' \subseteq K \), the restriction of \( \phi \) to \( K' \) exhibits it as a connected component of \( L \). This concept extends term-wise to \( s\text{SR}(C) \).

5 Local filling conditions

Throughout this section and for the rest of the paper, \( X \) will denote a locale with lattice of open subsets \( \mathcal{O}(X) \).
5.1 (Finite intersections). Let us say that a 1-category $K$ is left 0-finite if any functor from $K$ into a finitely complete poset admits a limit cone. For example, this is the case for any $K$ admitting a 0-coinitial functor from a finite category; in particular, when $K$ admits an initial object.

Let $U \mid I$ be a diagram in $\mathcal{U}(X)$. If $K$ is a left 0-finite category and $\alpha : K \to I$ is a functor, write

$$U_\alpha := \lim_{k:K} U_{ak} \in \mathcal{U}(X)$$

for the limit of $U$ in $\mathcal{U}(X)$ over the diagram $\alpha$. Since $\mathcal{U}(X)$ is a poset, this limit can be computed as an intersection

$$U_\alpha = \bigcap_{k:K_0} U_{ak}$$

where $K_0 \subseteq K$ is any 0-coinitial subset, that is, such that any element of $K$ is bounded below by an element of $K_0$. If in particular $K$ admits an initial object $e$, then of course $U_\alpha = U_{\alpha(e)}$.

5.2 Example (Simplex boundary). The category of elements of the simplex boundary $\int \partial \Delta^n$ has a 0-coinitial subset

$$\{\Delta^{n-1} \xrightarrow{\sigma_j} \Delta^n\}_{j=0}$$

comprising the facets of $\Delta^n$. In particular, $\int \partial \Delta^n$ is left 0-finite.

For any diagram $U \mid I$ in $\mathcal{U}(X)$ and map $\tau : \partial \Delta^n \to I$, we calculate

$$U_\tau = \bigcap_{j=0}^n U_{\tau \sigma_j}.$$
• $K \to L$ is a fully faithful functor; especially, but not always, equivalent to $K \subset K^\triangleleft$.

• Either $J = \Delta^\text{op}$ or $\Delta^1$, in which case the maps from $K$, $L$, and $I$ are left fibrations, or $J = \text{pt}$.

The diagram $U : I \to \mathcal{U}(X)$ is said to admit local fillers or satisfy the local filling condition for the above data if $U_\sigma$ is covered by sets of the form $U_\tau$, where $\tau$ ranges over fillings of the square.

If $U$ satisfies the local filling condition for fixed $K \subseteq L$ and any $\sigma : K \to I$ over $J$, then we say that $U$ satisfies the local filling conditions $K / L / J$. We are particularly interested in the case that $L$ has an initial object, for example when $L = K^\triangleleft$ is a left cone over $K$.

5.4 Proposition. The set of local filling conditions satisfied by a diagram $U : I \to \mathcal{U}(X)$ is stable under composition and pushout.

Proof. The argument is the same as for any class of arrows defined by a right lifting property.

5.5 Example. The condition (2.2), 2, for a diagram $U | I$ to be an atlas is the local filling condition for inclusions $K / K^\triangleleft$ where $K$ is a finite set. In particular, condition 1 is the local filling condition for $\int_+ \partial \Delta^1 / \int_+ \Delta^1$.

5.6 Lemma (Local filling conditions reduce to a coinital subset). Let $K$ be a finite poset, $K_0 \subseteq K$ a 0-coinital subset. Let $U : I \to \mathcal{U}(X)$ be a poset-indexed diagram. Then $U$ satisfies the local filling conditions for $K \subset K^\triangleleft$ if and only if it satisfies those for $K_0 \subset K_0^\triangleleft$.

Proof. By Lemma 5.7.

5.7 Lemma. Let $K$ be a poset, $K_0 \subseteq K$ a 0-coinital subset. Then

$$
\begin{array}{ccc}
K_0 & \longrightarrow & K \\
\downarrow & & \downarrow \\
K_0^\triangleleft & \longrightarrow & K^\triangleleft
\end{array}
$$

is a pushout in the category of posets.
Proof. The statement for underlying sets is obvious, so we are just checking that all the order relations on $K = K \sqcup \{e\}$ factorise as strings of relations that lift to $K^<_0$ and $K$. The only relations $i \leq j$ in $K$ that do not lift to $K$ are those for which $i = e$ is the cone point. In this case, by the hypothesis on $K_0$ there is some $i' \leq j$ with $i' \in K_0$, and the relation factorises as $i = e \leq i' \leq j$ with $e \leq i'$ coming from $K^<_0$. \hfill \square

5.8 Lemma. The local filling conditions for $\int_+ \partial \Delta^n / \int \partial \Delta^n$ and for $\int \partial \Delta^n / \int \Delta^n$ are equivalent.

Proof. Any extension problem for $\int_+ \partial \Delta^n / \int \partial \Delta^n$ extends to a diagram

\[
\begin{array}{ccc}
\int_+ \partial \Delta^n & \rightarrow & \int \partial \Delta^n \\
\downarrow & & \downarrow \\
\int_+ \Delta^n & \rightarrow & \int \Delta^n
\end{array}
\]

By Example 3.5, $\int \Delta^n$ retracts onto the pushout $\int_+ \Delta^n \cup \int \partial \Delta^n$.

Conversely, every extension problem for $\int \partial \Delta^n / \int \Delta^n$ can be reduced to an extension problem for $\int \partial \Delta^n / \int \Delta^n$ because the former is a retract of the latter. \hfill \square

6 Hypercovers

Hypercovers are defined in terms of local lifting conditions in the category of simplicial presheaves [DHI04, §3]. Via the construction of 4.6, these translate nicely into local filling conditions in the sense of 5.3. The key difference is that the lifting conditions of op. cit. restrict attention to tensor-representable objects (4.7), while our filling conditions are restricted to local isomorphisms (4.8).

Denote by $\mathrm{SR}(X)$ the category of semi-representable presheaves on $\mathcal{U}(X)$, and by $\mathrm{sSR}(X)$ its category of simplicial objects.

6.1 Proposition. The local filling condition for a diagram

\[
\begin{array}{ccc}
\int \partial \Delta^n & \rightarrow & \int K \\
\downarrow & & \downarrow \\
\int \Delta^n & \rightarrow & \Delta^{op}
\end{array}
\]
is equivalent to the square

\[
\begin{array}{ccc}
\partial \Delta^n \otimes U_\sigma & \xrightarrow{\sigma} & U_* \\
\downarrow & & \downarrow \\
\Delta^n \otimes U_\sigma & \rightarrow & pt
\end{array}
\]  

(admitting local liftings in the sense of [DHI04, §3].)

In particular, an object \(U_* : SR(X)\) is a hypercover if and only if its index diagram admits local fillings for all \(f \partial \Delta^n / \Delta^n / \Delta^{op}\).

**Proof.** Given a filler \(\tau\) for (6.1.1) with cone point \(V \subseteq U_\sigma\), we can construct a diagram

\[
\begin{array}{ccc}
\partial \Delta^n \otimes V & \xrightarrow{\sigma} & \sigma^* U \\
\downarrow & & \downarrow \\
\Delta^n \otimes V & \rightarrow & pt
\end{array}
\]

and hence a filler for (6.1.2) on the restriction of \(U_*\) to \(V\).

For the converse, given a lifting problem (6.1.1) we can at least formulate the problem (6.1.2). We now pause to record a lemma:

**6.2 Lemma.** With notation as it stands, let \(V \ast U : \int \Delta^n \to \mathcal{U}(X)\) be the map defined by the formula

\[
V \ast U(z) := \begin{cases} 
U_z & z \in \int \partial \Delta^n \\
V & z \in \int \Delta^n \setminus \int \partial \Delta^n .
\end{cases}
\]

Then the square

\[
\begin{array}{ccc}
\partial \Delta^n \otimes V & \xrightarrow{\sigma} & \sigma^* U \\
\downarrow & & \downarrow \\
\Delta^n \otimes V & \rightarrow & V \ast \sigma^* U
\end{array}
\]

is a pushout in \(sSR(C)\).

**Proof.** It is enough to compute the pushout in the category of functors \(f \Delta^n \to \text{PSh}(|\mathcal{U}(X)|)\).

\[
\begin{array}{ccc}
\text{Lan}_{f \partial \Delta^n / \Delta^n} V_{\partial \Delta^n} & \rightarrow & \text{Lan}_{f \partial \Delta^n / \Delta^n} \tilde{U} \circ \sigma \\
\downarrow & & \downarrow \\
V_{\Delta^n} & \rightarrow & V_{\Delta^n}
\end{array}
\]
This left Kan extension is easy to compute: the slice $\int \partial \Delta^n \downarrow \int \Delta^n z$ is empty for any $z \in \int \Delta^n \setminus \int \Delta^n$, so it takes the value $\varnothing$.

Calculating this pushout pointwise, we find a square of the form:

$$
\begin{array}{c}
V \\
\downarrow \\
\emptyset
\end{array} \quad \quad
\begin{array}{c}
U_z \\
\downarrow \\
\emptyset
\end{array}
$$

for $z \in \int \partial \Delta^n$; otherwise.

Thus the pushout is the functor $\int \Delta^n \rightarrow \mathcal{U}(X)$ whose restriction to $\int \partial \Delta^n$ is $\tilde{U}$ and which sends all dominant simplices to $V$.

By Lemma 6.2, a filler of (6.1.2) over $V \subseteq U_\sigma$ yields an extension $\tau : V \star \sigma^* U \rightarrow U_*$. By the Cartesian property of local isomorphisms (4.8), this factorises as

$$
V \star \sigma^* U \longrightarrow \text{id}_{\tau}^* U \longrightarrow_{\text{local-iso}} U
$$

where $\text{id}_{\tau} : \int \Delta \rightarrow I$ is the map of index diagrams induced by $\tau$. The right-hand arrow is a solution to the filling problem with vertex $U_{\text{id}_{\tau}(e)}$ where $e$ is the initial object of $\int \Delta$.

Now, since $U_\sigma$ is covered by $V$ over which (6.1.2) admits a lift, and $V \subseteq U_{\text{id}_{\tau}(e)}$, the local lifting condition is satisfied.

7 Nerves of atlases

To generate a hypercover — a kind of simplicial diagram — from an atlas — which may be indexed by an arbitrary category — we need a kind of ‘nerve’ construction. More precisely, we would like a construction that takes in an arbitrary diagram of open sets and outputs a diagram indexed by (the category of elements of) a simplicial set, and which transforms the atlas condition into the hypercover condition.

7.1 Remark (Why not just use the standard nerve?). It is easy enough to see that the standard nerve construction [HTT, p. 9] won’t work. Let $I$ be a category, and let $f : j(\partial^1) = 2 \rightarrow I$ be a functor. This functor admits an
extension to \([0 \rightarrow 1]\) if and only if \(f(0) \rightarrow f(1)\). Even if \(I\) indexes an atlas, it is of course not realistic to expect this for arbitrary \(f\).

We need to associate a more flexible family of categories to the standard simplices \(\Delta^k\). As we will see, the tautological test functor \(\tilde{N}\) provided by Grothendieck’s theory of test categories [Mal05] is good enough.

7.2 (Tautological nerve). Grothendieck integration \(f: \mathbf{sSet} \rightarrow \mathbf{Cat}\) admits a right adjoint

\[
\tilde{N}: K \rightarrow \text{Hom}(f(-), K),
\]

which is (opposite to) the tautological test functor attached to the Grothendieck test category \(\Delta\) [Mal05, Def. 1.3.7] and [Mal05, Prop. 1.5.13]. Using this adjunction we get an alternative expression of the category of maps from a simplicial set \(S\) to a 1-category \(J\) (3.2.1), to wit

\[
J(S) = \text{Hom}_{\mathbf{sSet}}(S, \tilde{N}J).
\]

The counit of the adjunction is the map

\[
\epsilon_\Delta: \int \tilde{N}(J) = \int_{\Delta^n: \Delta^{op}} \text{Fun}(f \Delta^n, J) \rightarrow J
\]

that evaluates a map \(\int \Delta^n \rightarrow J\) at the initial object.

7.3 Proposition. The counit \(\epsilon_\Delta\) is cofinal.

Proof. Let \(i: I\). We must show that \(i \downarrow I \int \tilde{N}(I)\) is weakly contractible. Using Lemma 7.4 we identify this with \(\int \tilde{N}(i \downarrow I)\). The latter is weakly contractible because \(i \downarrow I\) is weakly contractible and \(\tilde{N} - \int f\) induce inverse equivalences of homotopy categories (as \(\Delta\) is a test category [Mal05, Prop. 1.6.14]).
Proof. We may check this is a pullback on fibres of the projection to $\Delta^{op}$. The fibre over $\Delta^n$ is

$$\text{Fun}(f \Delta^n, i \downarrow I) \longrightarrow \text{Fun}(f \Delta^n, I)$$

$$\downarrow \quad \quad \downarrow$$

$$i \downarrow I \quad \longrightarrow \quad I$$

where, recall, the two vertical arrows are given by evaluation on the initial object. These are both Cartesian fibrations, and so it is enough to check that this is a pullback on the fibre over $j : I$. Moreover, the horizontal arrows are co-Cartesian fibrations; the fibre over $\phi : f \Delta^n \to I$ is the set of cones over $\phi$ with vertex $i$. The claim thus follows from the observation that the data of such a cone is the same as the data of a map from $i$ to $\phi(e)$, where $e : \Delta^n \to \Delta^n$ is the initial object of $f \Delta^n$. \qed

7.5 (Nerve of a diagram). Let $U : I \to \mathcal{U}(X)$ be a diagram. We associate to $U$ a simplicial semi-representable presheaf with index category $\bar{N}(I)$, whose realisation is the functor

$$\int \bar{N}(I) \xrightarrow{\epsilon_i^U} \mathcal{U}(X)$$

$$\downarrow$$

$$\Delta^{op}$$

obtained by precomposing $U_I$ with the counit $\epsilon_\Delta$. Denote this simplicial semi-representable presheaf $\bar{N}(U)_* : \Delta^{op} \to \text{SR}(\mathcal{U}(X))$.

7.6 Lemma. Let $K \to L$ be a morphism in $s\text{SR}(X)$. A diagram $U : I \to \mathcal{U}(X)$ of open sets satisfies the local lifting conditions for $\int K / \int L / \Delta^{op}$ if and only if $\int \bar{N}(U)_* \text{ satisfies the local lifting conditions for } \int K / \int L$.

Proof. The adjunction $\int \dashv \int \bar{N}$ (where we substitute for $s\text{Set}$ the category of left fibrations over $\Delta^{op}$) identifies the spaces

$$\left\{ \begin{array}{c} \int K \longrightarrow \int \bar{N}(I)_* \\ \downarrow \text{ dashed} \downarrow \end{array} \right\} \cong \left\{ \begin{array}{c} \int K \longrightarrow I \\ \downarrow \text{ dashed} \downarrow \end{array} \right\}$$

$$\left\{ \begin{array}{c} \int L \longrightarrow \Delta^{op} \\ \downarrow \end{array} \right\} \cong \left\{ \begin{array}{c} \int L \longrightarrow \text{pt} \\ \downarrow \end{array} \right\}$$

whence the lifting conditions are equivalent. \qed

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7.7 Theorem. Let $U : I \to \mathcal{U}(X)$ be a diagram. The following are equivalent:

1. $U$ is an atlas.

2. $\tilde{N}(U)_\ast$ is a hypercover.

In particular, every atlas admits a cofinal functor from the category of simplices of a hypercover.

Proof. By the chain of logical equivalences:

$U$ is an atlas

$\iff$ $U$ satisfies the local filling conditions for $[n]/[n]^\subset$ (Example 5.5).

$\iff$ $U$ satisfies the local filling conditions for $\int \partial \Delta^n / \int \Delta^n$ (Lemma 5.6 and Example 5.2).

$\iff$ $U$ satisfies the local filling conditions for $\int \Delta^n / \int \Delta^n$ (Lemma 5.8).

$\iff$ $\int \tilde{N}(U)_\ast$ satisfies the local filling conditions for $\int \Delta^n / \Delta^n / \Delta^{\mathsf{op}}$ (Lemma 7.6).

$\iff$ $\tilde{N}(U)_\ast$ is a hypercover (Proposition 6.1). 

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