Quantum entanglement is a quite amazing physical phenomenon. It was first discussed in depth in 1935 by Einstein, Podolsky and Rosen (EPR) \cite{Einstein}, and by Schrödinger \cite{Schr}, and since then by many others \cite{Bell, Peres, Greenberger}. This phenomenon has attracted intensive interest in recent years due to its applications in quantum computation, teleportation and cryptography, as well as to its connections to quantum chaos. Special importance is attributed to the frontier which, in mixed states, separates the quantum entangled from the so-called separable states. Separability, precisely defined in what follows, is essentially a classical concept, and it is the one on which the possibility of a description in terms of local realism (the classical concept, and it is the one on which the possibility of a description in terms of local realism (e.g., for understanding the interface between the classical and quantum descriptions of the real world) and practically (e.g., for efficiently implementing the input and output of a device such as a quantum computer). Some specific \(N\)-body systems have already been discussed in the literature (see \cite{Peres, Greenberger} for some classes of systems with arbitrary \(N\), and \cite{Abe} for the Greenberger-Horne-Zeilinger state of \(N = 3\) spins 1/2; see \cite{Tsallis} for a review).

Let us consider a \(N\)-body system composed by spin-\(S\) operators acting on a Hilbert space \(H = H_1 \otimes \ldots \otimes H_n\) of dimension \(D^N = (2S + 1)^N\). An arbitrary density matrix acting on this space should have all its eigenvalues non-negative. Consider an arbitrary orthogonal basis \(|s_1, \ldots, s_N\rangle\rangle > (s_i = -S, \ldots, S)\) spanning the Hilbert space. The Peres criterion appropriate to this \(N\)-body systems asserts that from a given density matrix \(\rho\), with elements \(<s_1, \ldots, s_N|\rho|s'_1, \ldots, s'_N\rangle\rangle\), it is possible to define other density matrices \(\sigma^{i_1 \cdots i_n}\), where the spin coordinates \(i_1, \ldots, i_n\) \((i_1, \ldots, i_n = 1, \ldots, N)\), are transposed ("in \(\leftrightarrow\) out "), i.e., \(<s_1 \cdots s_i - 1 s'_1, s_{i+1} + 1 \cdots s'_1 s_{i+n} + 1 \cdots s_N|\sigma^{i_1 \cdots i_n}|s'_1 s_{i+1} \cdots s_n + 1 \ldots s_N\rangle\rangle = \langle s_1 \cdots s_i - 1 s'_1 s_{i+1} + 1 \cdots s'_1 s_{i+n} + 1 \cdots s_N|\sigma^{i_1 \cdots i_n}|s'_1 s_{i+1} \cdots s_n + 1 \ldots s_N\rangle\rangle\).
The non-negativity of these new density matrices give us the Peres criterion, namely all eigenvalues of \( \sigma^{i_1 \cdots i_n} \) should be non-negative.

In order to probe and compare the above mentioned methods we will consider several sets of density matrices of general type

\[
\rho(x) = (1 - x)\rho(0) + x\tilde{\rho}
\]  

where \( \rho(0) = \rho_1 \otimes \cdots \otimes \rho_N = I_{DN}/D^N \), \( I_{DN} \) being the \( D^N \times D^N \) unity matrix, is a fully separable state and \( \tilde{\rho} = |\tilde{\phi} > < \tilde{\phi}| \) is a density matrix corresponding to a known entangled state \( \rho = |\phi > < \phi| \) of general type where \( \rho \) is the matrix calculated from (2). If we denote \( \tilde{\rho} = |\tilde{\phi} > < \tilde{\phi}| \), we provide a condition for quantum information of the system, are given by

\[
S_q(A_1 + \cdots + A_{i_1-1} + A_{i_1+1} + \cdots + A_{i_n-1} + A_{i_n+1} + \cdots + A_N | A_{i_1} + A_{i_2} + \cdots + A_{i_n}) = \frac{S_q(A_1 + \cdots + A_N) - S_q(A_{i_1} + A_{i_2} + \cdots + A_{i_n})}{1 + (1 - q)\tilde{S}_q(A_{i_1} + A_{i_2} + \cdots + A_{i_n})}
\]

(6)

where \( S_q(A_1 + \cdots + A_n) \) is given by (3) with

\[
\rho_{A_1 + \cdots + A_n} = \text{Tr}_{A_1 \cdots A_{i_1-1}, A_{i_1+1} \cdots A_{i_n-1}, A_{i_n+1}, \cdots A_N} [\rho_{A_1 + \cdots + A_N}].
\]

(7)

The condition for non-negative conditional probability imply that, for arbitrary combinations \( \{i_1, \ldots, i_n\} \), \( \tilde{\rho} \) should be non-negative, and consequently the bound for \( x_c \) is evaluated by imposing

\[
\text{Tr}_{A_1 + \cdots + A_n} \tilde{\rho}_{A_1 + \cdots + A_n}^q = \text{Tr}_{A_1 + \cdots + A_n} \tilde{\rho}_{A_1 + \cdots + A_n}^q = \frac{D^N - 1)(1 - x)D^N}{x} + \frac{(1 + (D^N - 1)x)}{x^q}.
\]

(8)

The left side of the above equation is simple to calculate for the special set of density matrices (3) by going to the basis where \( \tilde{\rho} \) is diagonal, and we obtain

\[
\text{Tr}_{A_1 + \cdots + A_n} \tilde{\rho}_{A_1 + \cdots + A_n}^q = \frac{D^N - 1)(1 - x)}{x} + \frac{(1 + (D^N - 1)x)}{x^q}.
\]

(9)

From (3) and (7) we obtain

\[
\rho_{A_1 + \cdots + A_n} = \frac{1 - x}{D^N}D^N - x\tilde{\rho}_{A_1 + \cdots + A_n},
\]

(10)

where

\[
\tilde{\rho}_{A_1 + \cdots + A_n} = \text{Tr}_{A_1, \ldots A_{i_1-1}, A_{i_1+1}, \ldots A_{i_n-1}, A_{i_n+1}, \ldots A_N} [\tilde{\rho}_{A_1 + \cdots + A_N}].
\]

(11)

is a \( D^N \times D^N \) matrix calculated from (3). If we denote by \( v_i \) \( (i = 1, \ldots, D^n) \) the eigenvalues of \( \tilde{\rho} \) the relations (6)-(10) give us

\[
(D^N - 1)(1 - x)D^N + \frac{(1 + (D^N - 1)x)}{x^q} = \sum_{i=1}^{D^n} \frac{1 - x}{D^N}D^N - x v_i)^q.
\]

(12)

This equation will give us \( x_c(q) \) for each value of \( q \). Due to the monotonic properties of the entropy (3), the lowest bound for \( x_c \) is obtained in the limiting case \( q \to \infty \). In this limit we can neglect the first term in the left hand side of (12) and keep in the sum only the term corresponding to the largest eigenvalue \( \bar{v}(N) \) of the matrix \( \tilde{\rho}_{A_1 + \cdots + A_n} \); which gives

\[
x_c^S = \frac{1}{1 + \frac{D^N}{D^N - 1} \bar{v}}.
\]

(13)
where the superscript refers to entropy. In principle we should consider the conditional probabilities of arbitrary subsystems. In our applications the largest eigenvalue, i.e., the one which produces the most restrictive bound, is obtained when we consider, in these conditional probabilities, the subsystem $A_1 + \ldots + A_{N-1}$.

We are going to consider initially density matrices written in vector basis invariant under charge conjugation, or spin-reversal symmetry. In the $S^2$-basis the basis vectors are given by the non-null combinations $|\phi_k^i\rangle = (|s_1, \ldots, s_N \rangle + c| - s_1, \ldots, -s_N \rangle)$, $(i = 1, \ldots, D^N, c = \pm 1)$. For $S = \frac{1}{2}$ and $N = 2$ this basis recovers the standard Bell basis. We now consider the density matrix $\rho(x)$ given by $\rho = |\phi_k^i\rangle \langle \phi_k^i|$ is formed by an arbitrary basis vector with charge conjugation eigenvalue $c = -1$, i.e., $|\phi_k^i\rangle = (|1\rangle - |2\rangle)/\sqrt{2}$, with $|1\rangle = |s_1, \ldots, s_N \rangle$ and $|2\rangle = | - s_1, \ldots, -s_N \rangle$ ($\{ s_k \}$ arbitrary). It is simple to convince ourselves that in the standard basis the only non-zero elements of $\rho$ in $|1\rangle$ are given by $< 1|\rho|1 >= -2|\rho|2 >= - < 2|\rho|1 >= 1/2$. In order to apply the Peres criterion we consider the density matrix $\sigma$ defined in $|1\rangle$ by the transposition of the last spin. The only non-zero terms of the related matrix $\sigma$ are given by $< 1|\sigma|1 >= 2|\sigma|2 >= - < 2|\sigma|1 >= -1/2$, where $|3\rangle = |s_1, \ldots, s_N-1, -s_N \rangle$ and $|4\rangle = | - s_1, \ldots, -s_N-1, s_N \rangle$. This last matrix has the lowest eigenvalue $-1/2$; consequently, from $|1\rangle$, we get the bound

$$x_c = x^{P} = \frac{1}{1 + D^N/2},$$

(14)

for all values of $D$ and $N$.

In the application of the entropic bound, the most restrictive condition happens when we consider the conditional probability of the subsystem composed by $(N - 1)$ spins and it is not difficult to see that $\tilde{\rho}$ in $|1\rangle$ has only two nonzero elements ($= 1/2$) in the diagonal, that gives the eigenvalue $\tilde{\nu} = 1/2$ and from $|2\rangle$, the bound

$$x_c^S = \frac{1}{1 + \frac{D^N - 1}{2^{N-1}}}.$$  

(15)

Comparing $|4\rangle$ and $|2\rangle$ we see that for $S = 1/2$ both criteria give us the same bound $x^{P} = x^{S} = 1/2$, but for general values of the spin $S \neq 1/2$ the bound coming from Peres criterion is more restrictive than that of the entropic criterion since $x^{P} < x^{S}$. It is interesting to remark that in the case $S = 1/2$, where the bounds coincide the value is known to be the exact value for $x_c$, for $N = 2$ $|2\rangle$ or $N > 2$ $|3\rangle$. The bounds $|4\rangle$ give us an interesting result. Suppose we need, as would occur in a quantum computer, to couple the special entangled state $|\phi_k\rangle >$ with a white spectrum environment. The mixed system will be described by the density matrix given by $|4\rangle$, where the parameter $x$ controls the coupling with the environment. The bound $|4\rangle$ tell us that we keep the whole system entangled as long as the ratio between the component of our special vector $|\phi_k\rangle >$ and any other vector in the environment, forming the density matrix, is larger than $(1 + (D^N - 1)x^{P})/(1 - x^{P})$ that as $N \to \infty$ tend towards 3, independently of the value of $S$.

A second class of $N$ spins $S$ density matrices we consider is a generalization of the Werner density matrices $|3\rangle$, given by $|3\rangle$ with

$$|\phi > = \frac{1}{N} \sum_{k=1}^{s} a_k|k, k, \ldots, k >, \quad N = \sum_{k=1}^{s} |a_k|^2.$$  

(16)

The standard spin-$S$ Werner density matrix, corresponds to $a_{-S} = \ldots = a_S$ and it was shown $|28\rangle$ that for this case the exact value where quantum entanglement takes place is $x_c = 1/(1 + D^N)$, for this particular case it was also shown recently $|24\rangle$ that the entropic bound $|3\rangle$ also coincides with the above exact value $x^{S}_{c} = x_c$. The application of the Peres criterion by transposing the last spin gives us the eigenvalues $\pm |a_l^* a_k|/N$ ($l \neq k$) or $|a_k|^2/N$ for the matrix $\tilde{\sigma}$ in $|1\rangle$ and consequently the bound $|1\rangle$ is given by

$$x^{P}_c = (1 + D^N \text{Max}\{|a_l^* a_k|\}) \sum_{k=-s}^{s} |a_k|^2, $$

(17)

where we denote by $\text{Max}\{|a_l^* a_k|\}$ the maximum value of the product $|a_l^* a_k|$ ($l \neq k$) of the coefficients forming the general density matrix $|3\rangle$ and $|1\rangle$. On the other hand the largest eigenvalue $\tilde{\nu} = \text{Max}\{|a_k|^2/N$ for the matrix $\tilde{\sigma}$. 

$$x^{S}_c = (1 + D^N \text{Max}\{|a_l^* a_k|\}) \alpha^{-1},$$

$$\alpha = \frac{\sum_{k=-s}^{s} |a_k|^2 - \text{Max}\{|a_k|^2\}}{(D - 1)\text{Max}\{|a_k|^2\}}.$$  

(18)

For arbitrary values of $\{ a_k \}$ we have $\alpha \leq 1$ and from (17) and (18) we see that the Peres criterion is in general more restrictive than the entropic one, i.e., $x^{P}_c < x^{S}_c$. For the standard Werner density matrix ($a_{-S} = \ldots = a_S = 1$) we have $\alpha = 1$ and the result $x^{P}_c = x^{S}_c = 1/(1 + D^N)$, that in this case coincides with the exact value of $x_c$ $|28\rangle$. Like in our first application $|4\rangle$ and $|2\rangle$, whenever $x^{P}_c = x^{S}_c$, these values turn out to be the exact one. Equation (17) and (18) tell us that, whenever at least one of the components $a_k = 0$ or at least a pair of coefficients exists such that $a_k \neq a_k$, in $|4\rangle$, $\alpha > 1$ and these bounds are distinct. The coincidence of bounds $x^{P}_c = x^{S}_c$ happens only for the special case $a_{-S} = \ldots = a_S$.

It is important to notice that interpreting $|3\rangle$ with $|1\rangle$ as the coupling of an entangled state $|\phi >$ with the white spectrum environment, in order that the system stays entangled, any component of the environment should not exceed a fraction $r_c = (1 - x_c)/(1 + (D^N - 1)x_c)$ of the component of the engineered entangled state $|\phi >$. 
Using the more restrictive bound $x_c^P$ [17], we obtain $r_c = \left(\sum_k |a_k|^2 - \text{Max}\{|a_i^* a_k|\}\text{Max}\{|a_i^* a_k|\}\right)$, that implies that $r_c$ decreases with the number of components where $a_k \neq 0$ in (16) and it is more difficult to keep the system entangled.

We have also studied other density matrices where the special entangled state $|\phi> > \text{are invariant under spatial translation on the spin ordering as well charge conjugation. In this case the results have to be carried numerically}$ [30] and our results indicate coincidences of the bounds $x_c^P$ and $x_c^S$ as the number of subsystems $N \to \infty$.

In conclusion we have shown that, although the Peres transposition method gives us more restrictive bounds than those coming from the nonextensive entropy, the application of both methods, that in general, at least numerically, is not a difficult problem, has the advantage of possibly providing, when the bounds $x_c^P$ and $x_c^S$ coincide, exact results, that are always of difficult derivation. Also we should remark that our analysis provides interesting results that should be useful for quantum computing, teleportation or cryptography. If the pure engineered entangled state $|\phi>$ is the desired information we are processing ($x = 1$ in (18)), while time runs, the coupling with the environment (represented by $x$ in (18)) grows ($x$ decreases) and the decoherence effect increases. As long as the system stays entangled, it might be possible to recover the desired information with appropriate correcting, distillation-like, procedures. Then, once we know the time dependence of $x(t)$, the bounds $x_c^P$ and $x_c^S$ will give us the time scale where the information should be processed.

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