Positivity-preserving hybrid DG/FV method with subcell resolution 
for compressible Euler equations with stiff source terms

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Abstract In numerical investigations of stringent problems in science and engineering research, non-physical negative density or pressure may emerge and cause blow-ups of the computation. We build on the realization that the positivity of density and pressure can be preserved in an efficient way of exploiting the advantage of discontinuous Galerkin and finite volume (DG/FV) hybrid computation framework. We thus present an effective and simple method with great practical significance to maintain the positivity of density and pressure in solving the reactive Euler equations. The approach is able to maximize the multiscale capability of the DG method by achieving both positivity-preserving and oscillation-free solution in the subgrid level. In the designed scheme, a priori detection and computation with hyperbolic tangent function prevent the occurrence of negativity in the flux evaluation, while a posteriori detection and computation with the first-order Godunov scheme guarantee the positivity of subcell solution. The a priori computation achieves bounded reconstruction and less oscillatory solution so that the a posteriori computation can be active in few cells where the extremely complex computation condition appears. Furthermore, the valuable information from the reconstruction process is utilized to design the indication strategy that identifies the DG and FV cells, and the technique of adaptively choosing reconstruction candidates is adopted to overcome the excessive numerical dissipation in the shock-capturing scheme. Numerical tests, including demanding examples in stiff detonation simulation, demonstrate the positivity-preserving, non-oscillatory and subcell resolution property of the present method.

Keywords DG/FV method, Positivity-preserving, Subcell resolution, Detonation, A priori detection, A posteriori computation, BVD

1 Introduction

The hyperbolic system of conservation laws describes the physics phenomena in several aspects of science and engineering. One important feature in simulating the hyperbolic conservation laws is that the discontinuous solutions may appear despite smooth initial conditions. When the solution is discontinuous and the hyperbolic conservation laws contain a stiff source term, spurious numerical solutions may be produced owing to the different time scale of transport part and source term [1]. For instance, in the area of combustion and high speed chemical reacting flows, the source term represents chemical reactions that can be much faster than the gas flow. It leads to many difficulties in designing numerical schemes to accurately solve the hyperbolic system with stiff source term.

Continuous efforts have been made to improve numerical schemes in order to deal with such stringent

Research supported by the National Natural Science Foundation of China (11702015 and 11721202).

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computational problems. In practice, it is often to encounter the situation where the concerned physical quantity, like the density or pressure, becomes negative. A loss of density or pressure positivity renders the system of equations non-convex, thereby provoking further problems in its numerical solution [2]. There is a growing interest in designing high order methods with positivity-preserving property [3]. Successful high order schemes, like weighted essentially non-oscillatory (WENO) [4] and discontinuous Galerkin (DG) [5], have been constructed to preserve the positivity of density and pressure. The main difficulty for high order schemes to satisfy a strict positivity-preserving property is to develop such treatments that lead to positivity-preserving and meanwhile also maintain high order accuracy for smooth solutions [6]. In a series of work of [7-10], the authors have established a general framework, for arbitrary order of accuracy, to construct a positivity preserving limiter for the finite volume (FV) and the DG methods. More recently, the method has been extended to the convection-diffusion equations and the Navier-Stokes equations in [11, 12]. Alternative positivity-preserving strategies also include self-adjusting positivity-preserving methodology presented in [2] and the strategy of introducing the first-order positivity-preserving flux like in [13-15].

In this work we focus on rendering numerical schemes stable for solving the reacting Euler equations that usually describe gaseous detonation and combustion systems. The low order numerical schemes can be used in the simulation of detonation waves, but numerical results have some deviation from the experimental results [16]. Since the excessive numerical dissipation is the cause of wrong propagation speed of discontinuities, previous studies attempted to employ high order schemes to overcome this difficulty. The work of [17-19] utilized WENO scheme and adaptive mesh to improve the ability to resolve flow structures. Several studies [6, 20] extended the DG method to detonation simulations by locally increasing the degrees of freedom (DOFs) as new prognostic variables. Apart from high order methods, the second-order schemes were also developed for combustion and detonation problems by reducing numerical diffusion and enhancing discontinuity-resolving ability [21, 22], and promising results were obtained.

We have recently proposed a hybrid algorithm for DG and FV method to resolve both smooth and discontinuous solutions with high fidelity [23]. Different from previous DG/FV method, see for example [24-26], the proposed methodology of using different reconstruction candidates shows great advantages in resolving discontinuity and small-scale flow structures with high fidelity on even relatively coarse grids. Since the numerical viscosity in a shock-capturing scheme is critical to track the wave front while the hybrid DG/FV achieves high-fidelity discontinuity-resolving, it inspired us to develop a positivity-preserving hybrid DG/FV scheme to solve the reactive Euler equations. We believe this work has provided two new aspects for solving the hyperbolic systems with the stiff source term: firstly, most researches have concentrated on designing schemes with provable positivity-preserving property for DG or FV method within one discretization framework. The approaches in this work are radically different from previous ones by preserving the positivity for DG and FV method in a hybrid discretization framework. The resulting scheme is thus simple and effective to maintain the positivity of density and pressure for stringent problems in practice. Secondly, researches that explore the multiscale property of DG method are rarely seen for stiff discontinuity-capturing problems [27]. Compared to most positivity-preserving DG methods, the present approach could maximize the multiscale capability of DG method by achieving both positivity-preserving and oscillation-free solution in the subgrid level, which has allowed one to obtain better subcell resolution on rather coarse grids, as will be demonstrated in the numerical experimentation.

The remainder of the paper is organized as follows. Section 2 briefly introduces the governing equations. Section 3 describes the positivity-preserving hybrid DG/FV method in detail. Numerical results are presented in Section 4 and some concluding remarks are drawn in Section 5.
2 Governing equations

The governing equations for the reactive Euler equations are written in the following form

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \\
\frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho uu + p \delta) = 0 \\
\frac{\partial \rho e}{\partial t} + \nabla \cdot ((\rho e + p)u) = 0 \\
\frac{\partial \rho \alpha}{\partial t} + \nabla \cdot (\rho \alpha u) = -K(T)\rho \alpha
\]

where \( \rho \) is the density, \( t \) is the time, \( u \) is the velocity vector, \( p \) is the static pressure, \( e \) is the total energy per unit mass, \( \delta \) is the Kronecker tensor and \( \alpha \) is the reactant mass fraction. The temperature is given as

\[ T = \frac{p}{\rho} \]

while the pressure is calculated by

\[ p = (\gamma - 1) \rho (e - \frac{1}{2} u \cdot u - aq_0) \]

where \( q_0 \) denotes the chemical heat release, and \( \gamma \) is the ratio of the specific heats. This work models the reaction rate with two types of forms. The first form is Arrhenius kinetics expressed as

\[ K(T) = K_0 e^{-\frac{G_{\text{ign}}}{T}} \]

where \( K_0 \) is the reaction rate constant, and \( T_{\text{ign}} \) is the ignition temperature. The second form of the reaction rate is Heaviside kinetics with which the stiffness of source term generally becomes more severe. The form of Heaviside kinetics reads

\[ K(T) = -\frac{1}{\xi} H(T - T_{\text{ign}}) \]

with \( \xi \) representing the reaction time. In Eq. (5) we have \( H(x) = 1 \) for \( x \geq 0 \) and \( H(x) = 0 \) for \( x < 0 \). Moreover, in the above equations, if we omit the last equation of Eq. (1) and set the chemical heat release \( q_0 \) to be zero in Eq. (3), we then obtain the governing equations for the non-reactive Euler equations that are also studied in this work.

3 Positivity-preserving hybrid DG/FV method

3.1 Hybrid DG/FV methodology with subcell resolution

Now we rewrite Eq. (1) as

\[
\frac{\partial Q}{\partial t} + \nabla \cdot f_s(Q) = s
\]

where \( Q \) is the conservative state vector, \( f_s(Q) \) is the inviscid flux tensor and \( s \) is the source term. In this paper, the governing equation is solved at the aid of so-called hybrid DG/FV methodology. The methodology carries out the DG discretization on the main element and the FV discretization on the subcell element that is usually embedded in the main element. The hybrid DG/FV method simultaneously achieves high order of accuracy
provided by the DG method and shock-capturing capability provided by the FV method.

Here we introduce the methodology by firstly formulating the DG discretization on the main element. The finite element solution $\mathbf{Q}_h$ is introduced as

$$
\mathbf{Q}_h = \sum_{i=1}^{N(p)} v_{h,i} \mathbf{\hat{Q}}_i(t)
$$

where $\mathbf{Q}_i$ denotes the degree of freedom (DOF), $v_{h,i}$ is piecewise polynomial test function and $N(p)$ is the number of modes. For the test function $v_{h,i}$ the Legendre basis is employed in the current work. By inserting $\mathbf{Q}_h$ in the governing equation and performing an integration by parts, the following weak formulation of Eq. (6) can be obtained as

$$
\int_{\Omega} v_h \frac{\partial \mathbf{Q}_h}{\partial t} d\Omega + \int_{\partial \Omega} v_h \mathbf{f}_c \cdot \mathbf{n} d\sigma - \int_{\Omega} \nabla v_h \cdot \mathbf{f}_c (\mathbf{Q}_h) d\Omega = \int_{\Omega} v_h s d\Omega
$$

where $\mathbf{n}$ is the outward unit normal vector to the boundary. We utilize Gauss quadrature points for the domain and boundary integrals in Eq. (8). And by assembling all the cell contributions together, we show the semi-discrete form of Eq. (8) as

$$
M \frac{d\mathbf{Q}}{dt} + \mathbf{R} (\mathbf{Q}) = 0
$$

where we have the mass matrix $M$ and the residual vector $\mathbf{R} (\mathbf{Q})$ for the DG method.

From the expression in Eq. (7), multiple DOFs are existed in the DG main element. It has been realized that the discontinuities can be captured within one single element by exploiting the multiscale information in the DG method [28], and the embedded subcell is one effective way to exploit this information. In Fig. 1 we show a representative set of subcells constructed in the main element for the 4th order DG/FV method.

On the embedded subcell the FV method is then formulated. By conducting integration over the subcell, for instance subcell $i,j$ shown in Fig. 1, the integration formulation of Eq. (6) can be obtained as

$$
\int_{\Omega} \mathbf{\hat{v}} \frac{\partial \mathbf{\hat{q}}}{\partial t} d\Omega + \int_{\partial \Omega} \mathbf{\hat{v}} \cdot \mathbf{\hat{f}}_c (\mathbf{\hat{q}}) d\Omega = \int_{\Omega} s d\Omega
$$

Here we use $\mathbf{q}$ to indicate the conservative state vector defined at the subcell level. Based on the structured index in Fig. 1, we have

$$
\frac{d \mathbf{\bar{q}}_j(t)}{dt} + \mathbf{\hat{F}}_{c,ij+1/2} - \mathbf{\hat{F}}_{c,ij-1/2} = \Omega_j s (\mathbf{\bar{q}}_j)
$$

In Eq. (11) $\mathbf{\hat{F}}_{c,ij+1/2}$ is the convective flux at the subcell interface and $\mathbf{\bar{q}}_j$ is the subcell average defined by

$$
\mathbf{\bar{q}}_j(t) = \frac{1}{\Omega_j} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{q}(x,y,t) dy dx
$$

Also $ij \pm 1/2$ indicates both $x$ and $y$ components that include $i \pm 1/2, j$ and $i,j \pm 1/2$, and $\Omega_j$ denotes the subcell volume. It should be notified that since the discontinuous solutions are permitted in both DG and FV methods, the interface flux, e.g. $\mathbf{f}_c \cdot \mathbf{n}$ in Eq. (8) and $\mathbf{\hat{F}}_{c,ij+1/2}$ in Eq. (11), has to be evaluated with a monotone numerical Riemann flux function. In this article the local Lax–Friedrichs flux function [5] is employed to do the job.

After the spatial discretization, the system of Eq. (9) of the DG method and the system of Eq. (11) of the FV method are both discretized in time by a third order version of TVD Runge–Kutta (RK) method [29]. For instance, the RK method to discretize the system of Eq. (9) in time is formulated as
\[
Q' = Q^n - \Delta t M^{-1} R(Q)
\]
\[
Q'' = \frac{3}{4} Q^n + \frac{1}{4} Q' - \frac{1}{4} \Delta t M^{-1} R(Q')
\]
\[
Q^{n+1} = \frac{1}{3} Q^n + \frac{2}{3} Q'' - \frac{2}{3} \Delta t M^{-1} R(Q'')
\]

where \( Q^n \) and \( Q^{n+1} \) are the solutions at the present and next time steps, meanwhile \( Q' \) and \( Q'' \) denote the intermediate solutions at substeps. The time step \( \Delta t \) is set according to
\[
\Delta t = \frac{CFL}{\lambda_{\text{max}} (1 / \Delta x + 1 / \Delta y) + K_0}
\]

where \( \lambda_{\text{max}} = \max \{\|u\|, \|v\| + \|c\|\} \) with velocity components \( u, v \) and sound speed \( c \) that is computed by
\[
c = \sqrt{\frac{\gamma p}{\rho}}
\]

Fig. 1. FV subcells (dashed line) and FV solution (●) within the main element (solid line) for the 4th-order DG/FV method.

3.2 Analyses of positivity constraints for hybrid DG/FV methodology

In practice, the negative value leads to an ill-posed problem, often seen as blow-ups of the numerical simulation, when the negative value appears under the radical sign. In the analysis of detailed computational procedure of DG and FV discretization of the reactive Euler equations, we find two types of constraints with respect to the positivity of density and pressure: the first one lies in the evaluation of numerical flux, where the density and pressure at the left and right side of the cell interface are required to be positive; the second one lies in the evaluation of time step, where the density and pressure of volume-integrated average are required to be positive. According to this analysis, we design the corresponding strategy to avoid the negative value appearing under the radical sign, therefore to obtain the positivity-preserving property of the computation method.

In the discretization framework of hybrid DG/FV method, preserving the positivity could have become more complex because there are two types of discretization methods to deal with. Nevertheless, the scheme presented in this work can make things easy by taking advantage of the hybrid computation. The basic idea is to preserve the positive values for the FV subcell solution based on a posteriori detection/computation paradigm.
while preventing the negative values being used during the evaluation of numerical flux based on a priori detection/computation paradigm.

Based on this idea, we design current positivity-preserving strategy. A sketch that illustrates the basic strategy along with the procedure of hybrid DG/FV computation is shown in Fig. 2. The positivity-preserving strategy, indicated with dashed line in Fig. 2, is explicitly applied to the FV computation and implicitly applied to the DG computation through the update of FV solution. Also one can see that the present method reverts to a general hybrid DG/FV method, if the positivity-preserving strategy is completely omitted. The concept and the detail of the strategy shown in Fig. 2 will be given in the following subsections.

![Fig. 2. Sketch of basic positivity-preserving strategy for hybrid DG/FV methodology: * indicates candidate solution.](image)

### 3.3 Preserving the positivity with a priori and a posteriori computation

Now we catalogue the details of the positivity-preserving strategy shown in Fig. 2. To keep it simple, we consider each explicit sub-step discretization of TVD Runge-Kutta method that can correspond to the convex combination of several explicit steps.

Firstly, we define the subcell data representation used in this study, which is also denoted as the projection and recovery process in Fig. 2. In this work the projection process obtains the FV solution through the $L_2$ projection of DG main cell solution on to the subcell, which means the FV solution is the cell average value of the DG solution on the subcell. For instance in Fig. 1, the FV solution of subcell $i, j$ is obtained by

$$
\bar{q}_{ij} = \frac{1}{\Omega_{ij}} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} Q_h(x, y, t)dydx
$$

We see that the projection process of DG solution is consistent with the definition of FV solution in Eq. (12). The Gauss quadrature rules are used in the numerical integration of Eq. (16), and the same number of Gauss quadrature points is utilized in the subcell integration just as in the main cell integration performed in the DG method.

On the other hand, the recovery process is to obtain the DG solution through the reverse process of the projection, which means we use the corresponding FV subcell solutions defined by Eq. (16) to obtain the DOFs of DG solution on the main element. For instance, for the 4th order DG/FV method we introduce Eq. (7) into Eq. (16) and list all FV subcell solutions shown in Fig. 1 as
In this work, the number of subcells is equal to the number of DOFs of DG solution. As a result, we can compute and store the matrices of $M_{16 \times 16}$ and the reverse matrices of $M_{16 \times 16}$ for the projection and recovery process described above.

The cell average definition of Eq. (16) is useful in preserving the conservation between the FV solution on the subcell and the DG solution on the main element. From Eq. (16), one can see that the cell-averaged DG solution on the main element is just the summation of all FV subcell solutions. Hence as long as we preserve positive FV solution on all subcells, the positivity of cell-averaged DG solution is straightforward guaranteed.

According to the analysis in Section 3.2, there are two types of positivity constraints. Herein we begin to design the strategy to satisfy the first type of positivity constraint, which is related to the density and pressure at the left and right side of the cell interface. Given the positive FV solution $\bar{q}$ on each subcell, we can apply numerical scheme that preserves the boundedness of the reconstruction in order to ensure the positivity for $q_i$ and $q_r$ that denote the variables at the left and right side of the interface. Currently we choose the tangent of hyperbola for interface capturing (THINC) method to do the job. The THINC method [30, 31] was originally proposed in multiphase simulations to preserve the boundedness of the volume of fluid (VOF) function, which requires the solution bounded by 0 and 1. Later the method was extended as shock-capturing scheme for solving hyperbolic conservation laws with stiff source term [21, 32]. Here we use the THINC scheme to obtain the interface values that can be bounded by the neighboring solutions.

Using the hyperbolic tangent function, the THINC scheme constructs the reconstruction function

$$q_i(x) = \bar{q}_{\min} + \frac{\bar{q}_{\max}}{2} \left[ 1 + \theta \tanh \left( \beta \left( \frac{x - x_i - x_{i/2}}{x_{i+1/2} - x_{i-1/2}} - \tilde{x}_i \right) \right) \right]$$  \hspace{1cm} (17)

with $\bar{q}_{\min} = \min(\bar{q}_{i,1}, \bar{q}_{i+1})$, $\bar{q}_{\max} = \max(\bar{q}_{i,1}, \bar{q}_{i+1}) - \bar{q}_{\min}$ and $\theta = sg(\bar{q}_{i,1} - \bar{q}_{i+1})$. Here we illustrate the THINC method in one dimension and hence drop the subscript $j$ for the variables. In Eq. (17) the parameter $\beta$ controls the jump thickness of the THINC reconstruction and $\tilde{x}_i$ represents the location of the jump center and is computed from constraint condition

$$\bar{q}_i = \frac{1}{\Delta x} \int_{x_{i+1/2}}^{x_{i-1/2}} q_i(x) dx$$  \hspace{1cm} (18)

As a sigmoid function, the hyperbolic tangent function fits well a step-like discontinuity and also preserves the boundedness of the reconstruction. Given the positive solution of $\bar{q}$ on each subcell, the interface values are bounded through the THINC reconstruction, and therefore are guaranteed to be positive. Since the THINC reconstruction is conducted before the candidate solution $\bar{q}_{\ast,n+1}$ in Fig. 2 is obtained, the procedure belongs to a priori computation that is taken during the process of FV computation on the subcell, as shown in Fig. 2.

Although the THINC reconstruction is adopted to satisfy the first type of positivity constraint, the method is still not positivity-preserving, because the candidate solution $\bar{q}_{\ast,n+1}$ in Fig. 2 can still become negative. This leads to the positivity-preserving strategy with respect to the second type of positivity constraint, which is also shown in Fig. 2 as a posteriori detection/computation procedure. The detailed procedure is summarized in pointwise form as follows:
(1) Obtain \( \bar{q}^{\text{r},n+1} \) either by directly solving Eq. (11) with FV computation on the subcell or through the projection process, as displayed in Fig. 2.

(2) Check \( \bar{q}^{\text{r},n+1} \) for negative density or pressure. Use \( \bar{q}_{\text{negative}}^{\text{r},n+1} \) to flag the subcell with the negative value and use \( \bar{q}_{\text{negative}}^{\text{r},n+1} \) to flag the neighboring subcell that shares a common face with subcell \( \bar{q}^{\text{r},n+1} \).

(3) For subcell \( \bar{q}^{\text{r},n+1} \), use the last step solution \( \bar{q}^{n} \) and perform the 1st-order Godunov scheme to recompute the spatial residual. For subcell \( \bar{q}_{\text{neighbor}}^{\text{r},n+1} \), reassemble the residual because the common interface flux has been recomputed by the 1st-order Godunov scheme.

(4) Solve Eq. (11) to obtain the new solution \( \bar{q}_{\text{negative}}^{\text{new},n+1} \) and \( \bar{q}_{\text{neighbor}}^{\text{new},n+1} \) with recomputed and reassembled residual for subcell \( \bar{q}_{\text{negative}}^{\text{r},n+1} \) and subcell \( \bar{q}_{\text{neighbor}}^{\text{r},n+1} \), respectively.

(5) Return to step (2) and this time only check \( \bar{q}_{\text{neighbor}}^{\text{new},n+1} \) for negative density or pressure until no negative density or pressure is detected for \( \bar{q}^{\text{r},n+1} \).

(6) Set \( \bar{q}^{n+1} = \bar{q}^{\text{r},n+1} \), perform the recovery process to obtain DG solution \( \bar{Q}^{n+1} \) and then continue the computation as displayed in Fig. 2.

The above procedure detects the candidate solution \( \bar{q}^{\text{r},n+1} \) and obtains \( \bar{q}^{n+1} \) by solving the governing equation using \( \bar{q}^{n} \) from last time step, which is therefore denoted as a posteriori detection/computation in Fig. 2. We mention the idea here is very similar to a newly proposed multi-dimensional optimal order detection (MOOD) paradigm [33, 34]. This paradigm has recently been applied as a posteriori detection strategy for designing shock-capturing scheme in the field of DG, FV and finite difference methods, see for example [26, 35-37]. Although being able to achieve positivity-preserving characteristics, the MOOD method has rarely been carried out in the literature to design the straightforwardly positivity-preserving scheme. Comparing to the original method in [33, 34] where the order reduction loops as well as numerical detection criteria are involved, the approach presented here can be considered as a simplified version of MOOD because it only applies one loop using the 1st-order scheme for those cells with non-physical negative values. More importantly, the reason why the MOOD method can be implemented in such simplified way is mainly that the a priori computation with THINC reconstruction has already achieved bounded reconstruction and less oscillatory solution. Thus a posteriori computation is in fact only activated in few cells where the extremely complex computation condition appears, which will also be demonstrated in the following numerical experimentation.

### 3.4 Indication strategy for positivity-preserving hybrid DG/FV method

In the hybrid DG/FV method, the so-called troubled-cell indicator is required to identify the discontinuity, the process of which is denoted as determining the FV and DG cell in Fig. 2. It should be mentioned that the indication of troubled cells is still not easy task for either DG or FV method. To better preserve the positivity, the indication strategy should use detection criteria that contain physics-based admissibility properties, such as the positive density and pressure. More importantly, the indication strategy in this work also utilizes valuable information from a priori reconstruction process. The detailed procedure is formulated as follows:

Step 1: on each subcell, we perform both high order and low order reconstructions to the subcell interface. Two sets of interface values are obtained by high order and low order reconstructions, and thus are indicated by subscript \(<\text{high-order}>\) and \(<\text{low-order}>\) in following Eqs. (19)-(22). Also \( q_{ij}^{LR} \) denotes the variable at the left and right side of interface in \( x \) direction for subcell \( i,j \). Then, we calculate the total boundary variation (TBV) values in \( x \) direction for high order reconstruction

\[
TBV_{<\text{high-order}>} = \left| q_{i-1/2}^{<\text{high-order}>} - q_{i+1/2}^{<\text{high-order}>} \right| + \left| q_{i+1/2}^{<\text{high-order}>} - q_{i+1/2}^{<\text{high-order}>} \right|
\]  

(19)
and for low order reconstruction
\[
TBV_i^{\text{low-order}} = |q_{i-1/2}^L - q_{i+1/2}^R| + |q_{i+1/2}^L - q_{i-1/2}^R|
\]
(20)
The same can be done to obtain \( TBV_i^{\text{high-order}} \) and \( TBV_j^{\text{low-order}} \) in y direction.

Step 2: We give a so-called relaxed boundary variation diminishing (BVD) condition [38, 23] in the following Eq. (21) and Eq. (22) to assess the TBV values in x and y direction, respectively. Then for subcell \( ij \), we use \( N_{rec_{ij}} \) to count the total number of reconstructions that satisfy either Eq. (21) or Eq. (22). Having obtained \( N_{rec} \) for all subcells in main cell \( K \), we then use \( N_{sub_K} \) to count the total number of subcells that have \( N_{rec_{im}}>0 \), \( l=i-1,\ldots,i+2 \) and \( m=j-1,\ldots,j+2 \) as shown in Fig. 1.

\[
TBV_i^{\text{high-order}} - TBV_i^{\text{low-order}} > \max(e_1,e_2(TBV_i^{\text{high-order}} + TBV_i^{\text{low-order}}))
\]
(21)

\[
TBV_j^{\text{high-order}} - TBV_j^{\text{low-order}} > \max(e_1,e_2(TBV_j^{\text{high-order}} + TBV_j^{\text{low-order}}))
\]
(22)

Step 3: Main cell \( K \) is determined as FV cell if any \( N_{rec_{im}}>1 \) for \( l=i-1,\ldots,i+2 \) and \( m=j-1,\ldots,j+2 \) or \( N_{sub_K}>N(p)/2 \) where \( N(p) \) is the total number of subcells in main cell \( K \).

Step 4: Otherwise, for main cell \( K \) we calculate the variables at all Gauss quadrature points by using the DG polynomial and check the positivity of density and pressure. If the negativity is detected, main cell \( K \) is determined as FV cell.

Step 5: Main cell \( K \) is determined as DG cell if it is not determined as FV cell by both step 3 and step 4.

The above indication strategy only requires a priori reconstruction and does not involved solving the PDE, therefore in Fig. 2 the strategy is denoted as a priori detection applied to determine the FV and DG cell. There have been researches in using subcell data in designing troubled-cell indicator for DG method, see for instance [39]. The novelty of the present method is taking advantage of valuable information from a priori process of subcell data reconstruction. In above step 1 and step 2, a priori high and low order reconstructions are conducted simultaneously for all subcell data. According to [38], high order reconstruction is inclined to produce oscillatory solutions around the discontinuity, thus gaining larger TBV values, while low order reconstruction is able to achieve stabilized solutions around the discontinuity, thus gaining smaller TBV values. Therefore by using the condition of Eqs. (21)-(22), we can obtain the information in terms of the smoothness of subcell data. Based on the information from all subcells, we then determine the main cell to be DG or FV cell in step 3. Although the procedure in step 3 is rather empirical, the applicability of the method will be demonstrated in practice.

Theoretically, any high order and low order reconstruction can be used in the above procedure. Previous researches have investigated including WENO and THINC in [38], or MUSCL and THINC in [32, 40] as high and low order reconstruction method. Following more recent studies in [41, 42, 23], we choose upwind linear reconstruction and THINC reconstruction as high order and low order reconstruction method in this work. Also based on our numerical experience, the 5th and 7th order linear upwind reconstructions both work well as high order reconstruction method, and in this article we adopt the 7th order one.

Up to now, the positivity-preserving strategy for hybrid DG/FV method has been completed. In accordance with two types of positivity constraints analyzed in Section 3.2, we summarize the corresponding approaches here: the positivity constraint in numerical flux evaluation is guaranteed by a priori detection described in this section and the THINC method described in Section 3.3; the positivity constraint for volume-integrated average is guaranteed by the subcell data definition and a posteriori detection/computation described in Section 3.3.
Furthermore, since our method is able to preserve the positivity of subcell solutions, the subcell resolution of DG method can be fully exploited, which will be demonstrated in the numerical results.

3.5 Extension to the reactive Euler equations with adaptive reconstruction

The positivity-preserving DG/FV method presented above is essentially a shock-capturing scheme that still produces a few transition points in the shock. For the stiff case, like detonation problem considered in this work, these transition points are often responsible for incorrect propagation speed of the discontinuities and nonphysical spurious waves. Furthermore, the numerical dissipations of shock-capturing scheme, although in the subgrid level, still cause certain accuracy loss for the hybrid DG/FV method.

Here we provide the technique to reduce the numerical dissipation for the present positivity-preserving scheme. The basic idea is to hybridize other reconstruction candidates in the reconstruction process therefore to achieve desirable characteristics. According to [38, 40], the BVD algorithm with the THINC reconstruction works best within a cell where a discontinuity exists. Therefore we choose the THINC reconstruction that has more compressive or anti-diffusion effect in order to achieve high fidelity resolution for strong discontinuities. In practice, the THINC reconstruction is conducted by adaptively choosing the sharpness parameter $\beta$ in Eq. (17). To be specific, firstly, the interface values are obtained by THINC reconstruction with small $\beta_i$ and large $\beta_l$ using Eq. (17), which is therefore denoted as $d_{i+1/2}^{L,R}\beta_i\beta_l$ and $q_{i+1/2}^{L,R}\beta_i\beta_l$. Then using similar equations as Eqs. (19)-(20), the TBV values are calculated and denoted as $TBV_i^{\beta_i\beta_l}$ and $TBV_i^{\beta_i\beta_l}$ for $\beta_i$ and $\beta_l$, respectively. Lastly, the final reconstruction is determined by

$$q_{i_{Final}}^{\beta_i\beta_l}(x) = \begin{cases} q_{i_{Final}}^{\beta_i\beta_l}(x) & \text{if } TBV_i^{\beta_i\beta_l} > TBV_i^{\beta_i\beta_l} \\ q_{i_{Final}}^{\beta_i\beta_l}(x) & \text{otherwise} \end{cases}$$

(23)

To make the implementation more convenient, we perform the THINC method with $\beta_i$ on all subcells as it is required by either positivity-preserving reconstruction in Section 3.3 or indication strategy in Section 3.4, while we perform the THINC method with $\beta_l$ only for the identified FV cell in the adaptive reconstruction to further reduce numerical dissipations. A figure that illustrates the connection between DG/FV computation and the adaptive reconstruction is shown in Fig. 3, where one dimensional reconstruction is displayed for simplicity. Let’s assume the discontinuity exists in subcell $i$, firstly the 7th-order linear reconstruction and THINC reconstruction with $\beta_i$ is performed for subcell $i-5,..,i+6$, to identify the DG and FV cell shown in Fig. 3. Then the reconstruction with $\beta_l$ is performed for subcell $i-1,..,i+2$, only to modify the reconstruction in subcell $i$ into $q_{i_{Final}}^{\beta_i\beta_l}(x)$ while retaining $q_{i_{Final}}^{\beta_i\beta_l}(x)$ for the rest subcells of $i-1,i+1$ and $i+2$.

The adaptive reconstruction presented above can already give favorable results for the cases considered in this work. Nevertheless, one can still add other reconstruction function to the reconstruction process to achieve even less diffusive method. For instance, if possible, we can make high order reconstruction selected on subcell $i+2$ in Fig. 3 by adding another BVD selection process between THINC function with $\beta_i$ and the desired high order reconstruction function. The reference [41, 42] is referred to for constructing high order polynomial of $n$-degree and THINC function of $m$-level using the BVD paradigm. Here we do not list the details for this method because the positivity-preserving characteristic is still the main concern of this study. Still we have examined the applicability of introducing high order reconstruction for one test case in Section 4.1.5.
Fig. 3. Illustration of adaptive reconstruction for hybrid DG/FV methodology.

At last we give some implementation details before we begin the numerical experimentation. Firstly, in solving the reactive Euler equations, the reactant mass fraction \( \alpha \) tends to become negative in some cases. If \( \alpha < 0 \), current method sets \( \alpha \) to be zero especially when it is used in the computation of the source term. Secondly, the time step is determined on the subgrid level in this work, and the CFL number is chosen as 0.3 for all numerical test cases. Thirdly, the primitive, conservative or characteristic variable can be used in the reconstruction process. Current work adopted the characteristic variable in solving the non-reactive Euler equations and the primitive variable in solving the reactive Euler equations. Lastly, the present positivity-preserving strategy is applicable to arbitrary order DG/FV scheme as well as to the unstructured meshes, although in this work we mainly test the 4th-order scheme on the rectangular mesh, as we described the method’s details in above sections.

4 Numerical tests

In this section, we test the proposed method for the non-reactive and reactive Euler equations. In solving the reactive Euler equations, we compare the present scheme with the fifth-order WENO scheme by using the same DOFs. Not only stringent test suite but also test cases containing both shocks and smooth flow structures are carried out in order to better examine the effectiveness of positivity-preserving strategy.

4.1 Numerical results for the Euler equations

4.1.1 Accuracy test

We test the accuracy of the method for solving a two-dimensional low density problem. The exact solution of the problem is \( \rho = 1 + 0.99 \sin \theta \), \( u = 1 \), \( v = 1 \), \( p = 1 \), \( \theta = x + y - 2 \tau \). The minimum density of the exact solution is 0.01. The convergence test is conducted on the domain of \([0, 2\pi] \times [0, 2\pi]\) from the time \( t=0 \) to \( t=0.1 \). The positivity-preserving hybrid DG/FV method is performed and the errors of the cell averages of density are listed in Table 1. By monitoring the computation process, we observe some FV cells activated (or then deactivated) by the present indication strategy. From Table 1 one can see that the designed order of accuracy can be generally obtained for the current method.

| Mesh size | DG/FV (P1) | DG/FV (P2) | DG/FV (P3) |
|-----------|------------|------------|------------|
|           | \( L^2 \) error | Order | \( L^2 \) error | Order | \( L^2 \) error | Order |
| 5×5       | 3.76e-3    | —       | 8.90e-4    | —       | 4.95e-5    | —       |
| 10×10     | 1.10e-3    | 1.77     | 1.32e-4    | 2.75     | 2.90e-6    | 4.09     |
| 20×20     | 3.08e-4    | 1.84     | 1.82e-5    | 2.86     | 1.99e-7    | 3.87     |
| 40×40     | 7.72e-5    | 2.00     | 2.40e-6    | 2.92     | 1.34e-8    | 3.89     |
4.1.2 Sedov blast wave

Now we consider a 2D Sedov blast wave, see also [6, 8]. The computational domain is a square given by [0, 1.1] × [0, 1.1]. The initial density is 1, velocity is 0, total energy is $10^{-12}$ everywhere except that in the lower left corner the energy is the constant $E/({\Delta x}{\Delta y})$ with $E=0.244816$. The boundary conditions are that of a reflecting surface on the left and bottom edges. The numerical experiment is performed on the mesh with element size of 1.1/100, which is considered to be relatively coarse for the test problem. The computation lasts until time $t=1$ and the solutions on the main cell and the solutions on the subcell are both illustrated in Fig. 4. One can see the subcell solutions are positive as well as non-oscillatory. On the subcell scale the method is able to further exploit the DG solution on the main element, and as a result the peak value matches the exact solution better and sharper solutions are produced around the shocks. Also a posteriori computation process has not been activated in this test case, which is consistent to the analysis in Section 3.3.

![Fig. 4. 2D Sedov blast, plot of density. Left: solution on the main cell; right: solution on the subcell.](image)

4.1.3 Mach 5.09 shock diffracting over a block step

The shock diffracting problem is a common example that is usually utilized to test the positivity-preserving ability of the method [3, 8, 10]. The computational domain contains the union of [0,1]×[6,11] and [1,13]×[0,11]. The initial condition is a shock of Mach=5.09, located at $x=0.5$, $6 \leq y \leq 11$, moving into undisturbed air with the density of 1.4 and the pressure of 1. The boundary conditions are inflow at $x=0$, $6 \leq y \leq 11$, outflow at $x=13$, $0 \leq y \leq 11$, reflective at $0 \leq x \leq 1$, $y=6$ and at $x=1$, $0 \leq y \leq 6$, and Neumann at $1 \leq x \leq 13$,
y = 0 and at 0 ≤ x ≤ 13, y = 11. The computational grid has the cell edge length of 1/32. In Fig. 5 the density solutions on the main cell and subcell at t=2.3 are presented. The fact that the solutions are positive as well as non-oscillatory on the subcell scale is emphasized. The identified FV cells are shown in Fig. 6 along with the local view of density counters superimposed on the computational grid. One can see the shock is captured by less than one single cell length. Again the computation has been completed without triggering a posteriori computation for any FV subcell, indicating the robustness of the current method.

\textbf{Fig. 5.} Mach 5.09 shock diffracting problem. Density: 20 equally spaced contour lines from 0.066227 to 7.0668. Left: solution on the main cell; right: solution on the subcell.

\textbf{Fig. 6.} Mach 5.09 shock diffracting problem. Left: identified FV cells marked with red color; right: local view of Fig. 5(right) with computational grid.

4.1.4 2D Mach 800 dense adiabatic jet

This challenging case from [2] is patterned after galactic astrophysical jets, for which the Mach number of the gas flow is extremely high. Similar cases are also considered in [8, 43, 44]. According to [8], the big challenge for solving the high Mach number jet is that, negative pressure could appear since the internal energy is very small compared to the huge kinetic energy. And even with an unmodified second order TVD scheme, this kind of problems cannot be properly simulated using simple density and pressure floors [2]. Hence it provides suitable test cases to validate positivity-preserving ability for the designed method. The computational setup is the same as [2]. The domain of [-0.5, 0.5]×[0, 1.5] is initially full of the ambient gas that has density of
0.14 and pressure of 1. A jet with a width of 0.1 enters from the bottom of the computation zone with a velocity of 800. The jet has density of 1.4 and the same pressure as the ambient gas. The computation stops at \( t = 0.002 \).

Fig. 7 shows the results of density, pressure and magnitude of the velocity that are obtained on \( 400 \times 600 \) cells. Overall the results are comparable to the ones reported in [2]. The identified FV cells are given in Fig. 8 and show relatively symmetrical distribution, which also indicates the favorable performance of the present method. For this challenging test case, a posteriori computation process has to be activated in order to obtain the positive density and pressure for the subcell solution.

![Fig. 7. 2D Mach 800 dense adiabatic jet: a) Logarithmic density (base-10), b) Logarithmic pressure (base-10), c) Magnitude of the velocity](image)

![Fig. 8. 2D Mach 800 dense adiabatic jet: identified FV cells marked with red color.](image)

4.1.5 Mach 10 Shock reflection and diffraction

The test is also known as Schardin’s problem [45]. A Mach 10 shock passes an equilateral triangle, which is a representative test case for positivity-preserving high order scheme. The size of the computational domain, the initial conditions, and the boundary conditions are the same as those in [12], and the rectangular meshes used in the computation have cell edge length around 1/80. An illustration of the domain and the mesh is in Fig.
9. The results at time $t=0.245$ obtained by the positivity-preserving DG/FV method are shown in Fig. 10. We see the shocked, complex structures in Fig. 10 a)-b) and the subcell resolution on the discontinuity-capturing in Fig. 10 c). The solutions on the subcell scale are again positive, non-oscillatory, and giving more details of the flow.

Furthermore, we examine an even less diffusive method, as suggested in Section 3.5 by adding another BVD selection with the fifth-order linear reconstruction upon the present algorithm. The general procedure for constructing polynomial of $n$-degree and THINC function of $m$-level using the BVD paradigm can be found in [41, 42]. We select this test case to investigate the resulting algorithm because according to [12, 46], this test case has not only low density and pressure but also complicated structures due to the Kelvin Helmholtz instability. As expect, the scheme with the high order reconstruction introduced has captured more delicate structures in Fig. 11, although the method using only adaptive THINC reconstruction has produced favorable flow details in Fig. 10. Also there are more FV cells activated when the high order reconstruction is introduced, see the comparison between Fig. 10 d) and Fig. 11 b), however it does not affect the algorithm to capture more fine details such as the roll-ups of contact lines in Fig. 11 a). In addition, compared with results for instance in Fig. 2 or Fig. 15 in [12], the present method, with or without high order reconstruction introduced, can achieve favorable balance in terms of reducing oscillations and capturing delicate flow structures.

![Fig. 9. Illustration of the domain and the rectangular mesh with mesh size around 1/20.](image)

![Fig. 10. Density solution on the main cell and subcell.](image)
4.2 Numerical results for the reactive Euler equations

In the following numerical experimentation, we show test results for positivity-preserving DG/FV method solving the reactive Euler equations. As mentioned earlier, we verify the method’s performance by comparing the present scheme with the WENO scheme using the same DOFs.

4.2.1 One-dimensional detonation problems

Three one-dimensional detonation problems are considered in this subsection. The first example is the Chapman-Jouguet (C-J) detonation that has the chemical reaction modeled with an Arrhenius source term, see also [1, 47, 48]. A computation domain of [0, 30] is full of burned and unburned gas, and initially, the burned gas with state \((\rho_{CJ}, u_{CJ}, p_{CJ}, 0.0)\) is set on the left side of computation domain, while the unburned gas with state \((\rho_0, u_0, p_0, 1.0)\) is set on the right. The initial discontinuity is located at \(x=10\). Given any initial state of \(\rho_0, u_0\) and \(p_0\), the C-J initial state of \(\rho_{CJ}, u_{CJ}\), and \(p_{CJ}\) can be determined according to [49]. Here we choose \(\rho_0=1.0\), \(u_0=0.0\), and \(p_0=1.0\). The parameters are set as \(\gamma=1.4\), \(q_0=25\), \(T_{ign}=25\), and \(K_0=16,418\). We use the uniform mesh with cell number \(N=100\) for the DG/FV scheme and therefore \(N=400\) for WENO scheme. The computation evolves until final time \(t=1.8\) and the results of density, temperature and mass fraction are given in Fig. 12. The reference solution is computed by the WENO scheme with \(N=10,000\) cells. For the DG/FV method the identified FV computation cells are also marked in the first figure of Fig. 12, showing that the FV computation is only
activated around the detonation wave. Comparing to the WENO scheme, the DG/FV method, although using much coarser main elements, is able to capture the correct position of the wave.

The second example is the C-J detonation with the Heaviside model. The computation domain is [0, 0.05] and initially the discontinuity is set at \( x=0.005 \). We have the burned gas located at region of \( x<0.005 \) and the unburned gas located at region of \( x>0.005 \). The initial state for the unburned gas on the right is \( \rho=1.201 \times 10^{-3}, u=0.0, p=8.321 \times 10^{5}, \) and \( \alpha=1.0 \). Then the C-J initial state for the burned gas on the left is again determined by [49]. The parameters are set as \( \gamma=1.4, q_0=0.5196 \times 10^{10}, T_{\text{ign}}=0.1155 \times 10^{10}, \) and \( 1/\zeta=0.5825 \times 10^{10} \). Similar to the fist example, we use the uniform mesh with cell number \( N=100 \) for the DG/FV scheme and therefore \( N=400 \) for the WENO scheme. The computation evolves until final time \( t=3 \times 10^{-7} \) and the results of density, temperature and mass fraction are similarly given in Fig. 13. Again the identified FV cells are only around the detonation wave, and the DG/FV method has captured the correct position of the wave.

The third example taken from [50] involves a collision between a detonation wave with an oscillatory profile. The computation domain is \([0, 2\pi]\) and the initial conditions are

\[
(\rho, u, p, \alpha) = \begin{cases} 
(1.79463, 3.0151, 21.53134, 0), & x < \pi/2, \\
(1+0.5\sin 2x, 0, 1, 1), & \text{otherwise,}
\end{cases}
\]

The parameters are \( \gamma=1.2, q_0=50, T_{\text{ign}}=3, \) and \( 1/\zeta=1000 \) in this case. We use the uniform mesh with cell number \( N=80 \) for the hybrid DG/FV scheme and therefore \( N=320 \) for the WENO scheme. The computation is run for \( t=\pi/5 \) and the results of density, temperature and mass fraction are similarly plotted in Fig. 14. In this case the interaction of the detonation wave and oscillatory profile produces complicated flowfield. Nevertheless, the FV cells are still identified only around the discontinuity. As a result, the smooth structures are well resolved by the DG method on the coarse mesh while at the meantime the location of the detonation front is correctly computed by the embedded FV method.

![Fig. 12. C-J detonation wave with the Arrehenius source. Results obtained by the DG/FV (upper) and WENO (lower) scheme. Bulley asterisks in the first figure denote the identified FV cells for the DG/FV scheme.](image-url)
Fig. 13. C-J detonation wave with the Heaviside source. Results obtained by the DG/FV (upper) and WENO (lower) scheme. Blue asterisks in the first figure denote the identified FV cells for the DG/FV scheme.

Fig. 14. Collision of a detonation wave with an oscillatory profile. Results obtained by the DG/FV (upper) and WENO (lower) scheme. Blue asterisks in the first figure denote the identified FV cells for the DG/FV scheme.

4.2.2 A 2D detonation wave

The 2D detonation wave problem, also investigated in [1, 32, 48, 51], is examined here. The computational domain is \((x, y)=[0, 0.025] \times [0, 0.005]\). The initial conditions are
\[ (\rho, u, v, p, \alpha) = \begin{cases} (\rho_1, u_1, 0, p_1, 0), & \text{if } x \leq \psi(y), \\ (\rho_2, u_2, 0, p_2, 1), & \text{otherwise}, \end{cases} \] (25)

where

\[ \psi(y) = \begin{cases} 0.004 & \text{if } |y - 0.0025| \geq 0.001 \\ 0.005 - |y - 0.0025| & \text{if } |y - 0.0025| < 0.001 \end{cases} \] (26)

The right states are \( \rho_r = 1.201 \times 10^3, u_r = 0, p_r = 8.321 \times 10^5 \) which are the same as the second example in Section 4.2.1. The same states can be determined by the C-J detonation model for \( \rho_l = \rho_{CJ}, p_l = p_{CJ} \), whereas \( u_l = 8.162 \times 10^4 > u_{CJ} \). Also the Heaviside form with same parameters \( \gamma, q_0, T_{ign}, \) and \( 1/\xi \) as the second example in Section 4.2.1 is used to model the chemical reaction. One important feature of this case is the appearance of triple points that travel in the transverse direction and reflect back and forth from upper and lower boundaries [1]. We carry out the numerical experiment using \( 200 \times 40 \) grid cells for the DG/FV scheme, therefore \( 800 \times 160 \) grid cells for the WENO scheme. A solution from [32] calculated by WENO scheme with \( 2000 \times 400 \) grid cells is adopted here as the reference solution. The density contours with the subcell data are compared between two methods at the evolutionary time \( t = 0.3 \times 10^{-7}, t = 0.92 \times 10^{-7} \) and \( t = 1.7 \times 10^{-7} \) in Figs. 15-17. We also give the density distribution along \( y = 0.0025 \) at \( t = 1.7 \times 10^{-7} \) in Fig. 18. It can be seen that the DG/FV method is able to achieve comparable and even better solutions compared to the WENO scheme using the same DOFs. For example, the present method produces less spurious waves in front of the detonation in Fig. 16 and obtains better agreement with reference solution in Fig. 18. The solutions of hybrid DG/FV method are again without spurious oscillation in the subgrid level for stiff detonation wave simulation.

![2D detonation wave. Density counters at t=0.3×10^{-7} for WENO (left), hybrid DG/FV (middle) and reference solution (right).](image1)

![2D detonation wave. Density counters at t=0.92×10^{-7} for WENO (left), hybrid DG/FV (middle) and reference solution (right).](image2)
4.2.3 Detonation diffraction problem with 90° corner

The detonation wave passing through an obstacle with an angle of ninety degrees is conducted to valid the effectiveness of the method, see also [6, 52]. The computational domain is \((x, y)=[0, 5] \times [0, 5]\), and the initial conditions are

\[
(\rho, u, v, p, \alpha) = \begin{cases} 
(11, 6.18, 0, 970, 1), & x < 0.5, \\
(1, 0, 0, 55, 0), & \text{otherwise},
\end{cases}
\]  

(27)

The parameters are given as \(\gamma=1.2\), \(q_0=50\), \(T_{\text{ign}}=50\), and \(K_0=2566.4\). The reflective boundary conditions are applied at every boundary except that at \(x=0\), where we have \((\rho, u, v, E, \alpha) = (11, 6.18, 0, 970, 1)\). The numerical experiment is carried out on a uniform mesh with \(\Delta x=\Delta y=1/48\). The computation lasts until time \(t=0.6\) and the contours of density and pressure are displayed in Fig. 19 where we show the solutions on the main cell as well as the solutions on the subcell. The solutions on the main cell are overall comparable to those in [6], while the solutions on the subcell are kept almost similarly non-oscillatory but with better resolution on flow details. The identified FV cells are on the whole around discontinuities, demonstrating a successful implementation of the indication strategy. In this case, the WENO scheme is also carried out with positivity-preserving strategy in order to handle the sudden drop of the pressure or density close to zero. The strategy to preserve the positivity for WENO scheme is to use similar a posteriori computation process described in Section 3.3: once the negative values are detected in the reconstructed interface variable or the computed candidate solution, the 1st-order
Godunov scheme is performed until no negative values are detected for the candidate solution. Fig. 20 provides results of WENO scheme on the mesh with $\Delta x=\Delta y=1/192$. We see small oscillations ahead of the second shock, which are likely caused by current implementation without the characteristic decomposition. A very close look actually reveals slightly sharper shock transition by the DG/FV scheme in the subgrid level. Nevertheless in general, the two methods gain comparable results in this test case.

Fig. 19. Detonation diffraction problem with 90° corner obtained by hybrid DG/FV method.
4.2 Multiple obstacles

The last example is the detonation wave passing multiple rectangular obstacles, see also [6, 52], which is considered challenging and requires the positivity-preserving method to stabilize the computation. The computational domain is \((x, y)\in [0, 8.3] \times [0, 10]\), and the initial conditions are

\[
\begin{align*}
(p, u, v, E, \alpha) &= \begin{cases} 
(7, 0, 0, 200, 0), & x^2 + y^2 \leq 0.36, \\
(1, 0, 0, 55, 1), & \text{otherwise},
\end{cases}
\end{align*}
\]

The first obstacle is located at \([1.3, 3.3] \times [0, 2.6]\) and the second one is located at \([5.1, 8.3] \times [0, 4.3]\). The parameters are given as \(\gamma=1.2, q_0=50, T_{\text{ign}}=20,\) and \(K_0=2410.2\). The reflective boundary conditions are applied everywhere. The mesh is uniformly rectangular with \(\Delta x=\Delta y=1/20\) that is again coarser than most meshes used for this test case, see for example [6]. The contours of density and pressure for the subcell solution at final time \(t=1.4\) are displayed in Fig. 21 where the multiple shock waves are captured without obvious oscillations in the subgrid level. Some interesting features of the density solution are observed at the left side of the first obstacle, which is overall comparable to those in [6]. The identified FV cells are also on the whole around various discontinuities in Fig. 21. Fig. 22 displays the results of the WENO scheme on the mesh with \(\Delta x=\Delta y=1/80\). The same positivity-preserving strategy described in Section 4.2.3 is also applied to stabilize the computation. The counter lines by the WENO scheme are overall smoother, although the complex features at the left side of the first obstacle are smeared. There are certain deviations for shock structures at the bottom gap between two obstacles predicted by the WENO and DG/FV scheme, for which the DG/FV scheme seems to obtain results closer to those published in [6, 52].
Den den density counters on the subcell

Press pressure counters on the subcell

Identified FV cells marked with red color

Fig. 21. Multiple obstacles problem obtained by hybrid DG/FV method.
In this paper, a positivity-preserving DG/FV method is developed and implemented to the reactive Euler equations. Different from most existing positivity-preserving methods, the present approach seeks to preserve the positivity within the framework of hybrid DG/FV discretization. Through the analysis of positivity constraint on density and pressure during the hybrid computation, we design corresponding strategy consisting of both a priori and a posteriori detection and computation process. The a priori computation utilizes the THINC function with essentially monotone and bounded properties to preserve the boundedness of the reconstruction, while the a posteriori computation implements the first-order Godunov scheme to obtain the positive cell-averaged solution. For a priori detection we develop the indication strategy that explores physics-based admissibility properties as well as the valuable information from subcell data reconstruction. Furthermore, the technique of adaptive reconstruction is suggested in order to overcome the excessive numerical dissipation in the shock-capturing scheme. The resulting method is able to achieve both positivity-preserving and oscillation-free solution in the subgrid level so that the subcell resolution capability of the DG method can be maximized.

We have tested the fourth-order DG/FV scheme on a stringent test suite in which the density or pressure may become negative easily. Numerical tests show that the proposed scheme can capture flow details on the coarse mesh by achieving positive and non-oscillatory subcell solutions. For stiff detonation waves, the current method resolves the correct position of the detonation front and gains comparable and even better resolution in comparison with the WENO scheme under the condition of using the same DOFs.

In summary, the present scheme is an effective and simple method with great practical significance to maintain positivity of density and pressure while fully exploiting multiscale resolution of DG method. Further work will be carried out including the application to gaseous detonation with more complicated geometry as well as the extension to other complex flows, such as turbulence and acoustics with strong shock interactions.

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