Abstract. Reverse Mathematics (RM for short) is a program in the foundations of mathematics with the aim of finding the minimal axioms required for proving theorems about countable and separable objects. RM usually takes place in second-order arithmetic and due to this choice of framework, continuous real-valued functions have to be represented by so-called codes. Kohlenbach has shown that the RM-definition of continuity-involving-codes constitutes a slight constructive enrichment of the epsilon-delta definition, namely in the form of a modulus of continuity. In this paper, we show that the RM-definition of continuity also gives rise to a ‘nonstandard’ enrichment in the form of nonstandard continuity. This observation allows us to (i) establish that RM-theorems related to continuity are implicitly higher-order statements and (ii) prove equivalences between RM-theorems concerning continuity and their associated higher-order versions. In conclusion, higher-order statements are present in second-order RM due the RM-definition of continuity.

1. Introduction

In two words, the topic of this paper is the implicit presence of higher-order statements in second-order Friedman-Simpson Reverse Mathematics. In particular, we show that the definition of continuity-via-codes used in the latter, gives rise to higher-order statements. We first introduce the aforementioned italicised notions.

Reverse Mathematics (RM for short) is a program in the foundations of mathematics initiated by Friedman [9,10], and developed extensively by Simpson and others (See [20,21] for an overview and introduction). The aim of RM is to find the axioms necessary to prove a given theorem of ordinary, i.e. about countable and separable objects, mathematics, assuming the ‘base theory’ RCA0, a weak system of computable mathematics. RM usually takes place in second-order arithmetic, i.e. a system of first-order logic with two sorts: natural numbers and sets of the latter (equivalently: Only type 0 and 1 objects are available). By contrast, in Kohlenbach’s higher-order RM (See [13] for details), all finite types are available. Thus, objects of type ‘higher than 1’ shall be informally referred to as ‘higher-order’.

In RM, real numbers are represented by fast-converging Cauchy sequences as in [21 II.4.4]; This implies that real-valued functions are not ‘directly’ available in RM (as they have type 1 → 1). To this end, continuous functions are represented by (type 1) codes as in [21 II.6.1]. In [13 §4], Kohlenbach proves that this RM-definition of continuity involves a slight constructive enrichment of the usual epsilon-delta definition of continuity, namely in the form of a modulus of continuity.
In Section 3.1 we show that the RM-definition of continuity gives rise to a ‘nonstandard’ enrichment, namely that standard RM-continuous functions are nonstandard continuous, and vice versa. In Section 3.2 we explore how this observation gives rise to a higher-order statement, namely the existence of a modulus-of-continuity functional, implicit in the base theory RCA\(^0\). Furthermore, in Section 3.3 we show that second-order theorems of RM relating to continuity are implicitly higher-order statements. In particular, the following statement

**There is a functional which witnesses the uniform continuity of every continuous functional on Cantor space.**

is implicit in the RM-theorem that every continuous function on Cantor space is uniformly continuous (See [21, I.10.3.4]). Similar higher-order statements are implicit in other RM-theorems (not necessarily concerning continuity), as explored in Sections 3.3 and 3.4.

Now, some readers would perhaps be more easily convinced of the veracity of our claim (that higher-order statements are implicit in second-order RM) if no nonstandard methods were used. Hence, in Section 4, we present our ‘smoking gun’: Without the use of nonstandard methods, we show that the statement

**Every RM-continuous function on Cantor space is uniformly RM-continuous.**

is equivalent to the statement that

**There is a functional which witnesses the uniform RM-continuity of every RM-continuous functional on Cantor space.**

We also show that this equivalence only goes through because of the use of RM-continuity, as the latter has greatly reduced quantifier complexity compared to the usual definition of continuity. Similar equivalences hold for other RM-theorems related to continuity.

In conclusion, the results in this paper suggest that insisting on formalising mathematics in second-order arithmetic is self-defeating: The RM-definition of continuity brings in higher types ‘through the back door’. Note that we do not claim that such a formalisation is pointless: We merely point out that the reduction in ontological commitment (provided by to the use of second-order arithmetic in RM) should not be exaggerated.

Finally, in the next section, we introduce the higher-order base theory, a conservative extension of RCA\(^0\), in which we shall work.

2. About and around the base theory RCA\(^0\)

In this section, we introduce the base theory RCA\(^0\) in which we will work. We discuss some basic results and introduce some notation.

In two words, RCA\(^0\) is a conservative extension of Kohlenbach’s base theory RCA\(^0\) from [13] with certain axioms from Nelson’s Internal Set Theory ([15]) based on the approach from [7, 8]. This conservation result is proved in [8], while certain partial results are implicit in [7]. The system RCA\(^0\) is a conservative extension of RCA\(^0\) for the second-order language by [13 Prop. 3.1].

In Nelson’s syntactic approach to Nonstandard Analysis ([15]), as opposed to Robinson’s semantic one ([16]), a new predicate ‘st(\(x\))’, read as ‘\(x\) is standard’ is added to the language of ZFC. The notations (∀\(x\))(\(x\) standard) and (∃\(y\))(\(y\) standard) are short for (∀\(x\))(\(x\) standard) and (∃\(y\))(\(y\) standard). The three axioms Idealization, Standard Part, and Transfer govern the new predicate ‘st’ and give rise to a conservative extension of ZFC. Nelson’s approach has been studied in the context of higher-type arithmetic in e.g. [1, 7, 8]. We use the same notations, introduced in Notation 2.3.
Following Nelson’s approach in arithmetic, we define $\text{RCA}^\Omega_0$ as the system
\[
\text{E-PRA}^{\omega^+}_0 + \text{QF}-\text{AC}^{1,0} + I + \text{HAC}_{\text{int}} + \text{PF-TP}_{\forall}
\]
from $[5]$ [§3.2-3.3]. Nelson’s idealization axiom $I$ is available in $\text{RCA}^\Omega_0$, but to guarantee that the latter is a conservative extension of $\text{RCA}^{\omega^+}_0$, Nelson’s axiom Transfer has to be limited to universal formulas without parameters, as in $\text{PF-TP}_{\forall}$. We have the following theorem

2.1. **Theorem.** The system $\text{E-PRA}^{\omega^+}_0 + \text{HAC}_{\text{int}} + I + \text{PF-TP}_{\forall}$ is a conservative extension of $\text{E-PRA}^\omega$. The system $\text{RCA}^\Omega_0$ is a $\Pi^0_2$-conservative extension of $\text{PRA}$.

**Proof.** See $[8$, Cor. 9].

The conservation result for $\text{E-PRA}^{\omega^+}_0 + \text{QF}-\text{AC}^{1,0}$ is trivial. Furthermore, omitting $\text{PF-TP}_{\forall}$, the theorem is implicit in $[7$, Cor. 7.6] as the proof of the latter goes through as long as EFA is available.

We now discuss the Transfer principle included in $\text{RCA}^\Omega_0$, which is as follows.

2.2. **Principle** (PF-TP$_\forall$). For any internal formula $\varphi(x^r)$ with all parameters shown, we have $(\forall^s x^r)\varphi(x) \rightarrow (\forall x)\varphi(x)$.

A special case of the previous can be found in Avigad’s system NPRA$^\omega$ from $[1]$. The omission of parameters in $\text{PF-TP}_{\forall}$ is essential, as is clear from Theorem 2.3, relating to the following principles:

\[
(\forall^s f^1)[(\forall^s n)f(n) = 0 \rightarrow (\forall n)f(n) = 0], \quad (\Pi^s_1\text{-TRANS})
\]
\[
(\exists x^2)[(\exists x^0)g(x) = 0 \leftrightarrow \varphi(g) = 0]. \quad (\exists^2)
\]

Note that standard parameters are allowed in $f^1$, and that $(\exists^2)$ is the functional version of $\text{ACA}_0$ ([21, III]), i.e. arithmetical comprehension.

2.3. **Theorem.** The system $\text{RCA}^\Omega_0$ proves $\Pi^0_2\text{-TRANS} \leftrightarrow (\exists^2)$.

**Proof.** By $[8$, Cor. 12].

The absence of parameters notwithstanding, $\text{PF-TP}_{\forall}$ is extremely useful, as we shall observe in the next remark. By way of context for the latter, recall that extending the language of a logical system with symbols representing certain functionals is common practice in mathematical logic: Indeed, examples may be found in e.g. $[1$ p. 935, §4.5], $[21$ [§2.5] and $[5,6]$.

2.4. **Remark** (Standard functionals). We discuss some important advantages of the axiom $\text{PF-TP}_{\forall}$. First of all, we show that ‘functionals defined by an internal formula’ are standard. To this end, consider the the fan functional ([13 p. 293]):

\[
(\exists \Theta^3)(\forall \varphi^2)(\forall f^1, g^1 \leq 1)[\Theta(\Omega(\varphi)) = 0 \Rightarrow \Theta(\varphi) = 0 \varphi(f) = 0 \varphi(g)], \quad (\text{MUC})
\]

We immediately obtain, via the contraposition of $\text{PF-TP}_{\forall}$, that

\[
(\exists \Theta^3)(\forall \varphi^2)(\forall f^1, g^1 \leq 1)[\Theta(\varphi) = 0 \Rightarrow \varphi(f) = 0 \varphi(g)]. \quad (2.1)
\]

In other words, we may assume that the fan functional is standard. The same holds for any functional of which the internal definition does not involve parameters.

Secondly, again for the fan functional, we may assume $\Omega(\varphi)$ is the least number as in (MUC), which implies that $\Theta(\varphi)$ from (2.1) can also be assumed to have this property. However, then $\Theta(\varphi) = 0 \Omega(\varphi)$ for any $\varphi^2$, implying $\Theta = 0 \Omega$, i.e. if it exists, the fan functional is unique and standard. The same holds for any uniquely-defined functional of which the definition does not involve additional parameters.
The two above observations prompted the addition to RCA₀ the of axioms reflecting the uniqueness and standardness of certain functionals (See [8 §3.3]). In particular, the language of RCA₀ contains a new symbol Ω₀ and the system itself contains:

\[ \text{st}(Ω₀) \land (\forall^st ξ^3)[M^st(ξ) \to (\forall^st ϕ^2)(Ω₀(ϕ) =_0 ξ(ϕ))], \tag{2.2} \]

where \( M(Ω) \) is the formula in square brackets in \( (\text{MUC}) \), with the addition that \( Ω(ϕ) \) is the least number with this property. As noted in [8 §3.3], such axioms still result in a conservative \( [\text{MUC}] \) extension of RCA₀.

Clearly, the axiom (2.2) expresses that, if it exists, the fan functional is standard and unique, reflecting the standardness and uniqueness properties we have proved in the previous two paragraphs, assuming \( (\text{MUC}) \). Furthermore, as noted in [8 §3.3], RCA₀ contains axioms like (2.2) for uniquely defined (via an internal formula) functionals. An advantage of (2.2) and its kin is that RCA₀ proves \( (\text{MUC})^st \to (\text{MUC}) \) by applying PF-TP to \( M^st(Ω₀) \), as discussed for the \( μ \)-operator in [8 §3.3]. In this way, \( (\text{MUC}) \leftrightarrow (\text{MUC})^st \) follows easily.

Finally, for the purpose of this paper, we note that RCA₀ includes (2.2) with all quantifiers \((∀ϕ^2)\) replaced by \((∀ϕ^2 \in C(2^N))\), i.e. corresponding to the classically valid restriction of the fan functional to type 2 functionals continuous on Cantor space. This axiom will play an important role as \( (\text{MUC})^3 \) below.

We stress that axioms such as (2.2) merely reflect basic properties of functionals, such as uniqueness and standardness, as axioms of a base theory are wont to do. The equivalence one obtains due to the addition of (2.2) is almost an unintended result. We finish this section with two remarks on notation. First of all, we shall use Nelson’s notations, as sketched now.

2.5. Remark (Notations). We write \((∀^st x^r) Φ(x^r)\) and \((∃^st x^r) Ψ(x^r)\) as short for \((∀x^r)[\text{st}(x^r) \to Φ(x^r)]\) and \((∃x^r)[\text{st}(x^r) \land Ψ(x^r)]\). We also write \((∀x^0 ∈ Ω) Φ(x^0)\) and \((∃x^0 ∈ Ω) Ψ(x^0)\) as short for \((∀x^0)[¬\text{st}(x^0) \to Φ(x^0)]\) and \((∃x^0)[¬\text{st}(x^0) \land Ψ(x^0)]\). Furthermore, if \( ¬\text{st}(x^0) \) (resp. \( \text{st}(x^0) \)), we also say that \( x^0 \) is ‘infinite’ (resp. finite) and write ‘\( x^0 \in Ω \)’. Finally, a formula \( A \) is ‘internal’ if it does not involve \( \text{st} \), and \( A^st \) is defined from \( A \) by appending ‘st’ to all quantifiers (except bounded number quantifiers).

Secondly, we shall use the usual notations for rational and real numbers and functions as introduced in [13] p. 288-289] (and [21] I.8.1 and II.4.4] for the foremen).

2.6. Remark (Real number). A (standard) real number \( x \) is a (standard) fast-converging Cauchy sequence \( q^1_n \), i.e. \((∀n)(|q_n - q_{n+1}| < 0 \frac{1}{2^n})\). We freely use of Kohlenbach’s ‘hat function’ from [13 p. 289] to guarantee that every sequence \( f^1 \) can be viewed as a real. Two reals \( x, y \) represented by \( q^1_n \) and \( r^1_n \) are equal, denoted \( x = y \), if \((∀n)(|q_n - r_n| ≤ \frac{1}{2^n})\). Inequality \( < \) is defined similarly. We also write \( x \approx y \) if \((∀^n)(|q_n - r_n| ≤ \frac{1}{2^n})\) and \( x ≫ y \) if \( x > y \land x \not= y \). Real-valued functions \( F : R \to R \) are represented by functionals \( Φ^1 \) such that \((∀x,y)(x = y \to Φ(x) = Φ(y))\), i.e. equal reals are mapped to equal reals.

\(^1\)To see this, note that every model \( M \) of a ‘reasonable’ logical system \( T \) can easily be extended to validate (2.2): If \( M \) includes a functional \( Ξ^3 \) which (in \( M \)) is standard and satisfies \( M^st(Ξ) \), then interpret \( Ω₀ \) as \( Ξ \in M \); Otherwise, interpret \( Ω₀ \) as \( θ^3 \).
3. Higher-order statements implicit in second-order Reverse Mathematics

In this section, we show that higher-order statements are implicit in second-order RM. We start by establishing that the RM-definition of continuity actually constitutes nonstandard continuity (and vice versa).

3.1. The nonstandard enrichment of continuity. In this section, we show that the RM-definition of continuity as in [21, II.6.1] constitutes a ‘nonstandard’ enrichment of the usual epsilon-delta-definition of continuity. In particular, we show that standard functions which are continuous in the sense of RM, i.e. given by codes, are also nonstandard continuous. Conversely, we show that a nonstandard continuous type 2 functional has a code in the standard world.

Our development takes place inside RCA_0^Ω. For simplicity, we shall work in Baire space (See [21, II.5.5]), where the definition of continuity (Φ ∈ C for short) is:

\[(∀α_{1})(∃N_{0})(∀β_{1})(αN = 0 βN → Φ(α) = 0 Φ(β))\]  

(3.1)

We say that the functional Φ^2 is standard continuous if it satisfies (3.1)st, and that the functional Φ^2 is nonstandard continuous if

\[(∀α^2)(∀β^1)(α ≈_1 β → Φ(α) = 0 Φ(β))\]  

(3.2)

where α ≈_1 β if (∀st n_0)[α(n) = β(n)]. If (3.1) holds limited to binary sequences, we say that Φ is continuous on Cantor space, and write ‘Φ ∈ C(2^N)’ for short.

We now show that standard functions continuous in the sense of RM are also nonstandard continuous as in (3.2). In Theorem 3.2 below, we also show the ‘converse’, namely that every type 2 functional which is nonstandard continuous as in (3.2), has a RM-code (relative to ‘st’). By [14, Prop. 4.6], nonstandard continuity thus constitutes a constructive enrichment.

First of all, with regard to known results, Kohlenbach shows in [14, §4] that the RM-definition of continuity includes a constructive enrichment in the form of a modulus of (pointwise) continuity, in contrast to Simpson’s claim (See [21, I.8.9 and IV.2.8]) that Reverse Mathematics analyses theorems ‘as they stand’, i.e. without constructive enrichments. Notwithstanding this negative result, Kohlenbach also shows in [14, §4] that the enrichment present in [21, II.6.1] is in general harmless. In particular, there is no change to the RM-equivalences of weak König’s lemma.

In more detail, Friedman-Simpson style Reverse Mathematics takes place in second-order arithmetic, i.e. only type 0 and 1 (numbers and sets of the latter) objects are available. Simpson motivates this choice as follows:

[the second-order language is the weakest one that is rich enough to express and develop the bulk of core mathematics. (21 Preface)]

As a result of this choice of framework, one cannot define real-valued functions ‘directly’ in RM, as the latter objects have type 1 → 1. For this reason, a real-valued continuous function is represented in Reverse Mathematics by a (type 1) code as in [21 II.6.1]. Kohlenbach shows in [14 Prop. 4.4] that the existence of a code for a continuous functional Φ^2, is equivalent to the existence of an associate for Φ as in [14 Def. 4.3], and equivalent to the existence of a modulus of continuity for Φ. Since associates are more amenable to our framework, we shall therefore work with the former, instead of RM-codes. The definition is as follows.

3.1. Definition. The function α^1 is an associate of a continuous functional Φ^2 if:

(i) (∀β^1)(∃k^0)α(βk) > 0,

(ii) (∀β^1, k^0)(α(βk) > 0 → Φ(β) + 1 = 0 α(βk)).
Note that we assume that every associate is a neighbourhood function as in [13].

The range of $\beta$ in the previous definition may be restricted if $\Phi^2$ is only continuous on a subspace. Finally, if the two items from Definition 3.1 only hold relative to ‘st’, then we say that $\alpha$ is an associate for $\Phi^2$ relative to ‘st’.

Secondly, since the Reverse Mathematics definition of ‘continuity-via-codes’ implicitly involves a modulus of continuity (again, by [14] Prop. 4.4), we shall make the latter explicit. Hence, we represent a continuous function $\phi$ on Baire space via a pair of codes $(\alpha^1, \beta^1)$, where $\alpha$ codes $\phi$ and $\beta$ codes its continuous modulus of pointwise continuity $\omega_\phi$. In more technical detail, $\alpha$ and $\beta$ satisfy

$$(\forall \gamma^1)(\exists N^0)(\exists^N K)\{K \geq \alpha(\gamma N) > 0\} \land (\forall \gamma^1)(\exists N^0)(\beta(\gamma N) > 0),$$

and the values of $\omega_\phi$ and $\phi$ at $\gamma^1 \leq 1$, denoted $\omega_\phi(\gamma)$ and $\phi(\gamma)$, are $\beta(\gamma k) - 1$ and $\alpha(\gamma k) - 1$ for any $k^0$ such that $\beta(\gamma k) > 0$ and $\alpha(\gamma k) > 0$. With the previous definitions in place, the following formula makes sense and expresses that $\omega_\phi$ is the modulus of continuity of $\phi$:

$$(\forall \zeta^1, \gamma^1)(\zeta \omega_\phi(\zeta) = \tau \omega_\phi(\zeta) \rightarrow \phi(\zeta) = \phi(\gamma)).$$

(3.3)

In short, the representation of a functional $\phi$ on Baire space via the RM-definition of continuity is equivalent to our representation (3.3).

Thirdly, a basic property of any standard functional is that it maps standard inputs to standard outputs. This ‘standardness’ property is a basic axiom2 of all the systems in [18] and a cornerstone of Nonstandard Analysis. Thus, to represent a standard continuous function $\phi$ on Cantor space, we should require that $\phi(\gamma)$ and $\omega_\phi(\gamma)$ are standard for standard $\gamma$. To accomplish this, we require that $\alpha$ and $\beta$ are standard and that these codes additionally satisfy:

$$(\forall^\ast \gamma^1)(\exists N^0)(\exists^N K)[K \geq \alpha(\gamma N) > 0] \land (\forall^\ast \gamma^1)(\exists N^0)(\exists^N K^0)[K \geq \beta(\gamma N) > 0].$$

(3.4)

Obviously, there are other ways of guaranteeing that $\phi$ and $\omega_\phi$ map standard sequences to standard numbers. Whichever way we guarantee that $\omega_\phi$ and $\phi$ are standard for standard input, (3.3) yields that

$$(\forall^\ast \zeta^1)(\exists^N K^0)(\exists^N \gamma^1)(\tau N = \gamma N \rightarrow \phi(\zeta) = \phi(\gamma)),$$

(3.5)

since $\omega_\phi(\zeta)$ is assumed to be standard for standard $\zeta^1$. Furthermore, we may assume the number $N^0$ as in (3.3) is minimal (though this number depends on the choice of the code for $\phi$). Clearly, (3.5) implies that $\phi$ is also nonstandard pointwise continuous, i.e.

$$(\forall^\ast \zeta^1)(\forall^\ast \gamma^1)(\zeta \approx_1 \gamma \rightarrow \phi(\zeta) = \phi(\gamma)),$$

which is the ‘nonstandard enrichment’ we mentioned previously. Thus, a standard and continuous $\phi$ on Baire space represented by an associate, is automatically nonstandard continuous. We now prove the ‘converse’ in the following theorem.

3.2. Theorem. In RCA$_0$, a functional $\Phi^2$ which is nonstandard continuous on Baire space, has an associate relative to ‘st’ there.

Proof. Clearly, nonstandard continuity (3.2) implies by definition that:

$$(\forall^\ast \alpha^1)(\forall^\beta^1)(\exists^N N^0)(\exists^N N =_0 N \rightarrow \Phi(\alpha) =_0 \Phi(\beta)).$$

Applying the idealization axiom I for fixed standard $\alpha^1$, we obtain

$$(\forall^\ast \alpha^1)(\exists^N K^0)(\forall^\beta^1)(\exists N^0 \leq K)(\exists^N N =_0 N \rightarrow \Phi(\alpha) =_0 \Phi(\beta)).$$

2 In particular, the axiom $(\forall^\ast x^\ast, y^\ast \rightarrow \tau)(st(y(x)))$ is part of $T^\ast$ by [3 §2] and [7 §2].
We may remove the bounded quantifier as follows:

\[(\forall^\ast \alpha^1)(\exists^\ast K^0)(\forall \beta^1)(\alpha K = 0 \rightarrow \Phi(\alpha) = 0 \Phi(\beta)),\]  

(3.6)

and apply HAC\textsubscript{int} to (3.6) obtain a standard functional Ξ\textsubscript{1}→0 such that

\[(\forall^\ast \alpha^1)(\exists \beta^1)(\forall \beta (\alpha K = 0 \rightarrow \Phi(\alpha) = 0 \Phi(\beta))).\]  

(3.7)

Now define Ψ(α) as the maximum of all Ξ(α)(i) for i < |Ξ(α)|. Then Ψ\textsuperscript{2} is a (standard) modulus of pointwise continuity for Φ, as follows:

\[(\forall^\ast \alpha^1)(\forall \beta^1)(\alpha \Psi(\alpha) = 0 \rightarrow \Phi(\alpha) = 0 \Phi(\beta)).\]  

(3.8)

3.3. Corollary. For standard Φ\textsuperscript{2} as in the theorem, its associate is also standard.

Proof. Immediate by Kohlenbach’s effective definition of an associate in terms of a modulus in the proof of [14, Prop. 4.4].

3.4. Remark. The correspondence between ‘continuity-via-an-associate’ and nonstandard continuity established above, can be explained as follows: Intuitively speaking, both definitions of continuity remove the innermost universal quantifier (involving β\textsuperscript{1}) in (3.1); Indeed, this reduction in quantifier complexity is literally part of the definition of associate (See item i) in Definition 3.1), while nonstandard continuity gives rise to (3.6), in which the innermost \textit{internal} universal quantifier (involving β\textsuperscript{1}) ‘does not count’ from the point of view of HAC\textsubscript{int}, as the latter applies to \textit{all} internal formulas. In both cases, the (literal or not) removal of this innermost universal quantifier allows us to obtain a modulus of continuity.

Finally, Theorem 3.2 presents a ‘pointwise’ result in that we look at a particular Φ\textsuperscript{2} which is assumed to be nonstandard continuous. For this reason, we cannot apply PF-TP\textsubscript{τ} anywhere in the proof, as e.g. (3.7) contains Φ as a parameter. However, \textit{all} continuous functions in RM are represented by codes, and hence nonstandard continuous by the above. In other words, we should study the statement ‘All continuous and standard functions on Baire space are nonstandard continuous’. We explore this further in Section 3.2.

3.2. Theorems of the base theory. In the previous section, we showed that the representation of continuous functions by RM-codes gives rise to nonstandard continuity and vice versa. Thus, the following statement is a valid consequence of the RM-definition of continuity in second-order RM:

All continuous and standard functions on Baire space are nonstandard cont. (3.8)

In this section, we show that (3.8) formulated in the higher-type framework, is equivalent to the existence of a \textit{modulus-of-continuity functional}. Since RCA\textsubscript{ω} cannot prove the existence of such a functional (See [14] Prop. 4.4 and 4.6] or [23, \S 6, Theorem 2.6.7]), (3.8) gives rise to a \textit{strict higher-order enrichment} of the usual definition of continuity (3.1). In other words, due to the RM-definition of continuity, higher-order statements are implicit in second-order RM.

To establish the previous claims, consider the following statements:

\[(\forall^\ast \alpha^1)(\forall^\ast \beta^1)(\alpha \approx \beta \rightarrow \Phi(\alpha) = 0 \Phi(\beta)).\]  

[NC]

\[(\exists \Psi^3)(\forall \phi^2 \in C, \alpha^1, \beta^1)(\forall \Psi(\Phi(\alpha, \alpha) = 0 \Psi(\Phi(\alpha, \beta) \rightarrow \Phi(\alpha) = 0 \Phi(\beta))).\]  

[MC]

Clearly, [NC] is (3.8) in the higher-type framework and [MC] states the existence of a modulus-of-continuity functional.

3.5. Theorem. In RCA\textsubscript{0}, we have [MC] \rightarrow [NC] \rightarrow [MC]\textsuperscript{st}.
Proof. To obtain the first implication, apply PF-TP to \( (MC) \) to obtain a standard \( \Psi \) as in the latter. As standard objects are standard for standard input, we obtain
\[
(\forall^st \Phi^2 \in C, \alpha^1)(\exists^st K^0)(\forall^st \beta^1)(\exists^st N^0 \in C^0 \rightarrow \Omega(\Phi) = 0 \Phi(\beta)).
\]
which immediately yields \( (NC) \). For the final implication, assume the latter principle and obtain, as in the proof of Theorem 3.2, that \( (3.9) \) holds for all standard and continuous \( \Phi^2 \), i.e. we have:
\[
(\forall^st \Phi^2 \in C, \alpha^1)(\exists^st K^0)(\forall^st \beta^1)(\exists^st N^0 \in C^0 \rightarrow \Omega(\Phi) = 0 \Phi(\beta)).
\]
Now apply HAC\textsubscript{int} to obtain a standard functional \( \Xi^{(2 \times 1) \rightarrow 0^*} \) such that
\[
(\forall^st \Phi^2 \in C, \alpha^1)(\exists^st K^0 \in \Xi(\Phi, \alpha))(\exists^st \beta^1)(\exists^st N^0 \rightarrow \Omega(\Phi) = 0 \Phi(\beta)).
\]
Next, define \( \Psi(\Phi, \alpha) \) as the maximum of all \( \Xi(\Phi, \alpha)(i) \) for \( i < |\Xi(\Phi, \alpha)| \). Then \( \Psi^3 \) is a standard modulus-of-continuity functional as in
\[
(\forall^st \Phi^2 \in C, \alpha^1, \beta^1)(\exists^st N^0 \in C^0 \rightarrow \Omega(\Phi, \alpha) = 0 \Phi(\beta)),
\]
and the previous formula is exactly \( (MC)^{st} \).

While the previous theorem provides a higher-order statement implicit in \( (3.10) \), we would nonetheless like to obtain an equivalence in the previous theorem. We now present two ways of obtaining such.

3.6. Remark (Towards equivalence). First of all, as a crude solution, we could add e.g. the following axiom to RCA\textsuperscript{0} \( 0^* \): \( (\forall^st \Xi^3)(N(\Xi) \rightarrow N(\Psi_0)) \). Here, \( N(\Psi) \) is \( (3.10) \), and \( \Psi_0 \) is a new symbol in the language of RCA\textsuperscript{0} \( 0^* \). The resulting system is still a conservative extension of RCA\textsuperscript{0} \( 0^* \) (See Footnote 11 to Remark 2.4) and proves \( (MC)^{st} \rightarrow (MC) \), and hence \( (MC) \leftrightarrow (NC) \) in this extended version of RCA\textsuperscript{0} \( 0^* \). A more acceptable addition is discussed in [18].

Secondly, a slightly more sophisticated approach is as follows: If \( \phi \) is a function on Cantor space represented by an associate \( \alpha^1 \), with a modulus of continuity \( \omega_0 \) as in \( (3.3) \) and \( (3.4) \), we may assume that the modulus outputs the least point of continuity for standard inputs. This becomes clear by considering \( (\forall^st \alpha)(\exists^st N^0 \alpha(\exists^st N^0) > 0 \) (a consequence of \( (3.4) \)) rather than \( (3.3) \). Indeed, the latter equation allows us to compute the least such \( N \), which is -prima facie- not the case for \( (3.5) \) due to the extra \( (\forall^st \beta^1) \)-quantifier.

In other words, the RM-definition of continuity not just constitutes the existence of a modulus of continuity, this modulus also outputs the minimal point of continuity (of course dependent on the choice of the associate representing \( \phi \)). Hence, to reflect the previous observation concerning second-order RM, we may assume a principle \( P \) which (relative to ‘st’) states that a modulus of continuity gives rise to a modulus outputting the minimal point of continuity.

In this way, working over RCA\textsuperscript{0} \( 0^* \) + \( P \), the functional \( \Psi \) from \( (3.10) \) may be assumed to output moduli which yield the minimal point of continuity. Such a functional \( \Psi \) is unique and as in Remark 2.4 and the proof of Theorem 3.8 below, we could obtain \( (MC)^{st} \rightarrow (MC) \), and hence \( (MC) \leftrightarrow (NC) \), assuming \( P \).

3.3. Theorems concerning continuity. In Section 3.1 we showed that the representation of continuous functions by RM-codes gives rise to nonstandard continuity and vice versa. In this section, we similarly show that the following is a valid consequence of [21, IV.2.3] due to the RM-definition of continuity:

All continuous and standard functions on Cantor space

are uniformly nonstandard continuous.
Furthermore, similar to Section 3.2, we show that in the higher-type framework, \(3.11\) is equivalent to the existence of a modulus-of-uniform-continuity functional. In other words, due to the RM-definition of continuity, higher-order statements are implicit in the (second-order) RM of weak König’s lemma. Note that by [21 IV.1.4], we could work over any compact subspace of Baire space, but we have chosen Cantor space for simplicity.

First of all, we consider the following theorem.

3.7. Theorem. In \(\text{RCA}_0^\Omega + \text{QF-AC}^{2,0}\), we have \(\text{WKL} \rightarrow \text{WKL}^\omega\).

Proof. We refer to the end of the proof of Theorem 4.1.

By the previous theorem, if weak König’s lemma is given, we also have access to this principle relative to ‘\(\exists^1\)’. Hence, working in \(\text{RCA}_0^\Omega + \text{QF-AC}^{2,0} + \text{WKL}\), the equation \(3.15\) yields that \(\forall^1 \leq 1)(\exists N^0)\alpha(\bar{N}) > 0\), which implies \(\forall^1 \leq 1)(\exists N^0 \leq k)\alpha(\bar{N}) > 0\) for some standard \(k\) by Theorem 3.7. Hence, \(3.5\) becomes

\[\forall^1 \leq 11(\exists N^0)(\forall^1, \gamma^1 \leq 1)(\bar{N} = \bar{N} \rightarrow \phi(\gamma) = \phi(\gamma)),\]

and we immediately obtain uniform nonstandard continuity as follows:

\[\forall^1, \gamma^1 \leq 11(\exists N^0)(\forall^1, \gamma^1 \leq 1)(\bar{N} = \bar{N} \rightarrow \phi(\gamma) = \phi(\gamma)),\]

(3.12)

Hence, we have established \(3.11\) assuming \(\text{WKL}\). Next, consider the following:

\[\forall^1 \leq 1(\forall^1, \gamma^1 \leq 1)(\exists N^0 \leq k)\alpha(\bar{N}) > 0\] \(\Rightarrow\) \(\forall^1 \leq 1)(\exists N^0 \leq k)\alpha(\bar{N}) > 0\)

Clearly, \(\forall^1 \leq 1\) in the higher-type framework and \(\text{MCU}\) states the existence of a modulus-of-uniform-continuity functional. Furthermore, we may assume \(\Psi(\varphi)\) is the least number as in \(\text{MCU}\) for \(\varphi^2 \in C\).

3.8. Theorem. In \(\text{RCA}_0^\Omega\), we have \(\text{UNC} \leftrightarrow \text{MCU}\).

Proof. To obtain the reverse direction, apply PF-TP\(\varphi\) to \(\text{MCU}\) to obtain a standard \(\Psi\) as in the latter. Hence, \(\Psi(\varphi)\) is standard for standard \(\varphi\), implying:

\[\forall^1 \leq 1)(\exists N^0 \leq k)\alpha(\bar{N}) > 0\]

which immediately yields \(\text{UNC}\). Now assume the latter and obtain

\[\forall^1 \leq 1)(\exists N^0 \leq k)\alpha(\bar{N}) > 0\]

in the same way as we derived \(3.9\) from \(\text{UNC}\). Now apply HAC to \(3.14\) to obtain standard \(\Xi\) such that

\[\forall^1 \leq 1)(\exists N^0 \leq k)\alpha(\bar{N}) > 0\]

Hence, again taking the maximum of the components of \(\Xi\), we observe that there is standard \(\Psi\) such that:

\[\forall^1 \leq 1)(\forall^1, \gamma^1 \leq 1)(\exists N^0 \leq k)\alpha(\bar{N}) > 0\]

Hence, we have obtained \(\text{MCU}\). To derive \(\text{MCU}\) from the latter, note that as stated in Remark 3.4 \(\text{RCA}_0^\Omega\) includes a version of \(3.2\) for continuous \(\varphi^2\) on Cantor space, i.e. with \(\varphi^2 \in C(\bar{N})\) everywhere, as follows:

\[\text{st}(\Gamma_0) \land (\forall^1 \Xi)(\forall^1 \varphi^2 \in C(\bar{N}))(\Gamma_0(\varphi) = 0) = \Xi(\varphi)\]

(3.16)

where \(\Gamma_0\) is a new symbol and \(N(\Xi)\) is \(3.10\) with the extra addition that \(\Psi(\Phi)\) is minimal. Clearly, \(3.10\) expresses that \(\Psi\) from \(3.15\), if it exists, is standard and unique (assuming the minimality mentioned above). Furthermore, \(3.15\) implies \(N(\Gamma_0)\) and the latter has no parameters, i.e. we apply PF-TP\(\varphi\) to yield \(\text{MCU}\).
Note that we can obtain \(\text{MCLU} \rightarrow \text{UNC} \rightarrow \text{MCLU}^s\) without \((3.10)\), i.e. in the absence of the latter, higher-order statements are still implicit in \((3.11)\).

In conclusion, we have established that higher-order statements are implicit in second-order theorems due to the RM-definition of continuity. We could establish similar results for RM-continuity on \([0, 1]\) and the associated theorems, e.g. about boundedness and Riemann integration (See [21, IV.2]). In this way, the following statements can be derived from the associated RM-theorems, which are [21, IV.2.3.3] and [21, IV.2.7.2], as follows:

All continuous on \([0, 1]\) and standard functions are bounded on all of \([0, 1]\).

All continuous on \([0, 1]\) and standard functions are nonstandard Riemann integrable.

Nonstandard Riemann integration involves the usual ‘infinitely fine’ partitions as in [22, Def. 5.3.1]. Similar to Theorem 3.8 the previous centered statements give rise to a functional respectively outputting an upper bound and the Riemann integral of a continuous function.

3.4. Theorems not related to continuity. In light of the results in the previous sections, one might wonder if the RM-definition of continuity is the ‘root of all higher-order evil’. We discuss this question in this section and try to provide a nuanced answer. We shall work informally so as to promote intuitive understanding.

3.4.1. The Heine-Borel lemma. In this section, we discuss how the Heine-Borel lemma (HB for short; See [21, IV.1]) gives rise to a uniform version of itself.

By [21, IV.1.2] and Theorem 5.7, HB implies HB\(^ s\). For simplicity, we only work with open covers \(I_n = (c_n, d_n)\) consisting of non-trivial intervals with rational end-points \(c_n, d_n\). Hence, HB\(^ s\) implies for all such standard covers \(I_n^{0 \rightarrow (0 \times 0)}\) that:

\[(\forall x \in [0, 1])((\exists x^0)(x \in (c_n, d_n))) \rightarrow ((\exists k^0)(\forall x \in [0, 1])(\exists n \leq k)(x \in (c_n, d_n))).\]

Again for simplicity, we assume that the existential quantifiers in \(x \in (c_n, d_n)\) and \(x \in [0, 1]\) are included in \((\exists x^0)\) and \((\exists n^0 \leq k)\). Now, it is easy to prove that if a finite cover (as in the consequent of the previous formula) covers all standard numbers in \([0, 1]\), this same cover also covers all numbers in \([0, 1]\), i.e. we have

\[(\forall x \in [0, 1])((\exists x^0)(x \in (c_n, d_n))) \rightarrow ((\exists k^0)(\forall x \in [0, 1])(\exists n \leq k)(x \in (c_n, d_n))).\]

Hence, for all standard \(I_n^{0 \rightarrow (0 \times 0)}\) and \(g^2\), we have

\[(\forall x \in [0, 1])(x \in (c_{g(x)}, d_{g(x)})) \rightarrow ((\exists k^0)(\forall x \in [0, 1])(\exists n \leq k)(x \in (c_n, d_n))).\]

Bringing all quantifiers outside, we have

\[(\forall x \in [0, 1])(x \in (c_{g(x)}, d_{g(x)})) \rightarrow ((\exists k^0)(\forall x \in [0, 1])(\exists n \leq k)(x \in (c_n, d_n))).\]

Applying \(\text{HAC}_{\text{int}}\) to the previous formula, we obtain a standard functional \(\Xi\) such that \((\exists k^0, x^1 \in \Xi(g, I_n))\) in the previous formula. Now define \(\Psi(g, I_n)\) as the maximum of \(\Xi(g, I_n)(1)(i)\) for \(i < \Xi(g, I_n)(1)\). It is now easy to prove that:

\[(\forall x \in [0, 1])(x \in (c_{g(x)}, d_{g(x)})) \rightarrow ((\exists k^0)(\forall x \in [0, 1])(\exists n \leq \Psi(g, I_n)(x \in (c_n, d_n)))).\]

Hence, we have derived, from the usual Heine-Borel lemma as in [21, IV.1.1], its standard uniform version \((3.17)\). In other words, there is a higher-order statement implicit in the Heine-Borel lemma, and the latter does not refer to continuity at all. Furthermore, we may assume \(\Psi\) from \((3.17)\) outputs the least such number as in the latter. Hence, we can obtain the internal version of \((3.17)\) as in Remark 2.4

\(^3\)Note that \(x \in (a, b)^*\) in the centered formula means \(x \ll y \ll b\) in line with the notations from Notation 2.6.
Unsurprisingly, it seems we can only obtain the higher-order version (3.17) because HB is about countable coverings. Indeed, HAC\textit{nat} only provides a finite sequence of witnesses, which only leads to a witness in special cases, such as for countable coverings and other ‘monotone objects’.

Finally, the absence of continuity in the Heine-Borel lemma can be argued against: Indeed, the definition of RM-continuity as in [21, II.6.1] is clearly based on open covers; This is particularly clear from the proof of [21, IV.2.2]. We hence discuss open sets in more details in the next section.

3.4.2. Theorems related to open sets. In this section, we discuss some examples of how the RM-definition of open set gives rise to higher-order theorems in RM.

Recall that a set $X$ (in a metric space) is open if for each $x \in X$, there is $r > 0$ such that the open ball around $x$ with radius $r$ is in $X$. The RM-definition of open set can be found in [21, §II.4 and II.5.6]. Essentially, open sets are represented in RM by countable collections of open balls. Since it can be proved that open sets in separable spaces are actually such collections, the RM-definition of open set seems faithful to the original. Furthermore, elementhood for an open set is a $\Sigma^0_1$-predicate (and vice versa by [21, II.5.7]) in RM, while the usual definition of open set of real numbers would necessarily be more complex.

We now provide two examples of RM-theorems which give rise to higher-order versions of themselves, directly due to the RM-definition of open set.

3.9. Example. We discuss [21, IV.1.7] which states that WKL$_0$ proves that for a compact metric space $X$ and a (code for a) closed set $C \subseteq X$, the assertion that $C \neq \emptyset$ is expressible by a $\Pi^0_1$-formula. The proof of this theorem involves proving that $C = \emptyset$ is actually a $\Sigma^0_1$-formula using HB. Indeed, since $X \setminus C$ is open by definition, $C = \emptyset$ implies that $X$ is (covered by) a countable cover of open intervals (defined only and explicitly in terms of the code of $C$). Applying HB now yields a suitable $\Sigma^0_1$-formula, and the converse is apparently easy.

Now in Section 3.4.1 we derived from HB a uniform version of itself as in (3.17). Combined with the proof of [21, IV.1.7], WKL$_0$ immediately yields a uniform version of the latter theorem where a functional $\Phi$ takes (the code of ) $C$ as input and outputs a function $f_C$ such that $C \neq \emptyset \iff (\forall n) f_C(n) = 0$. Note that with the usual definition of open set, we would require a functional converting an open set in a countable collection of open balls to guarantee a uniform version of [21, IV.1.7].

3.10. Example. We discuss [11, Theorem 5.8.(2)], which states that every closed set in a compact complete separable metric space $X$ is weakly located. A set $A$ is weakly located if $d(x, A) > r$ is a $\Sigma^0_1$-formula. Perhaps unsurprisingly, the implication WKL$_0 \rightarrow$ [11, Theorem 5.8.(2)] makes use of [21, IV.1.7] discussed previously. The proof of [11, Theorem 5.8.(2)] immediately implies that the latter theorem gives rise to a uniform version of itself where a functional provides the (bounded part of the) $\Sigma^0_1$-predicate needed for the weakly-locatedness.

The two previous examples show how the RM-definition of open set gives rise to higher-order statements, and how the latter ‘trickle down’ through RM.

Finally, one could argue that the RM-definition of open set is the underlying cause of why HB implies (3.17). Indeed, the RM-definition guarantees that elementhood of an open set has low complexity (namely $\Sigma^0_1$), which yields that HB has the same syntactic structure as the contraposition of WKL. With the usual definition of open set, this would definitely not be the case, and we could only obtain a uniform version of the Heine-Borel lemma if a functional would be available to convert open sets into countable collections of open balls.
4. Higher-order statements ‘explicit’ in second-order Reverse Mathematics

In the previous section, we discussed how nonstandard continuity was implicit in the RM-definition of continuity, and showed that this ‘nonstandard enrichment’ guarantees that higher-order statements are implicit in second-order theorems concerning continuity. In this section, we take a more direct approach and show that the following second-order statement:

Every RM-continuous function on Cantor space is uniformly continuous. (4.1)

is equivalent to the higher-order statement \( \text{URC} \) below (without the use of nonstandard methods). This equivalence is only possible because of the use of RM-continuity in (4.1), which greatly reduces the quantifier-complexity (just like nonstandard methods). This equivalence is only possible because of the use of RM-

The following continuity statement is (4.1), again noting that continuity via an associate (as opposed to (3.1)) is used, greatly reducing overall quantifier-complexity. Indeed, this reduction is essential for obtaining (4.2) (resp. (4.5)), to which QF-AC\( ^{\omega,0} \) (resp. HAC\( _{\text{int}} \)) can be applied. We now prove Theorem 4.1.

**Proof.** The equivalence \( \text{URC} \leftrightarrow \text{WKL} \) is straightforward; the same proof goes through for the equivalence relative to ‘st’. Furthermore, since QF-AC\( ^{1,0} \) is part of RCA\( _{0}^\omega \), \( \text{URC} \rightarrow \text{RC} \) is immediate. The same proof goes through for the implication relative to ‘st’, as HAC\( _{\text{int}} \) implies QF-AC\( ^{1,0} \) relative to ‘st’. We now prove the remaining implication in the first line of the proof.

Hence, assume \( \text{RC}^{\text{st}} \) and note that we have:

\[
(\forall^{\text{st}} \alpha^1, g^2)((\forall^{\text{st}} \beta \leq_1 1)\alpha(\overline{\beta}g(\beta)) > 0 \rightarrow (\exists^{\text{st}} k^0)(\forall^{\text{st}} \beta \leq_1 1)\exists N \leq k)\alpha(\overline{\beta}N) > 0 \],
\]

as \( g(\beta) \) is standard for standard \( \beta^{1} \). Trivially, we also have

\[
(\forall^{\text{st}} \alpha^1, g^2)((\forall^{\text{st}} \beta \leq_1 1)\alpha(\overline{\beta}g(\beta)) > 0 \rightarrow (\exists^{\text{st}} k^0)(\forall\gamma \leq_1 1)\exists N \leq k)\alpha(\overline{\gamma}N) > 0 \],
\]

as \( \overline{\gamma}N \) is standard for standard \( N^0 \). Bringing quantifiers to the front, we obtain

\[
(\forall^{\text{st}} \alpha^1, g^2)(\exists^{\text{st}} k^0, \beta^{1} \leq_1 1)[\alpha(\overline{\beta}g(\beta)) > 0 \rightarrow (\forall\gamma \leq_1 1)\exists N \leq k)\alpha(\overline{\gamma}N) > 0]. \quad (4.2)
\]
Since the formula in square brackets in (4.2) is internal, we may apply HAC\textsubscript{int}. Hence, there is standard $\Xi^{(1\times2)\rightarrow((0^*\times1)^*)}$ such that:

$$(\forall^a \alpha, g^2)(\exists k^0, \beta^1 \in \Xi(\alpha, g))[\beta \leq_1 1 \land \alpha(\overline{g}(\beta)) > 0] \rightarrow (\forall \gamma \leq_1 1)(\exists N \leq k)\alpha(\overline{\gamma}N) > 0].$$

(4.3)

Now define $\Psi(\alpha, g)$ as the maximum of $\Xi(\alpha, g)(1)(i)$ for $i < |\Xi(\alpha, g)(1)|$. Note that $\Psi$ completely ignores the second component of $\Xi$ (which contains a witness for $\beta^1$). Hence, since $\Xi(\alpha, g)$ is standard, $\Psi(\alpha, g)$ is standard as noted above, giving rise to (4.5). Again applying QF-AC\textsuperscript{2.0}, we obtain

$$(\forall^a \alpha, g^2)(\exists^a \beta^1)[\beta \leq_1 1 \land \alpha(\overline{g}(\beta)) > 0] \rightarrow (\forall \gamma \leq_1 1)(\exists N \leq \Psi(\alpha, g))\alpha(\overline{\gamma}N) > 0].$$

The previous formula is exactly (URC)\textsuperscript{st}, and the first line of the theorem is done.

Finally, we prove the remaining implication in the second line of the theorem. We proceed in roughly the same way as in the first paragraph of this proof, but with extra tricks to remove quantifiers prohibiting the use of QF-AC\textsuperscript{2.0} in the internal version of (4.2). Thus, consider (RC) and obtain the internal version of (4.2), i.e.

$$(\forall^a \alpha, g^2)(\exists k^0, \beta^1 \leq_1 1)[\alpha(\overline{g}(\beta))] > 0 \rightarrow (\forall \gamma \leq_1 1)(\exists N \leq k)\alpha(\overline{\gamma}N) > 0].$$

(4.4)

where $\overline{g}(\alpha)$ is the least $n \leq g(\alpha)$ such that $\overline{\alpha n} \notin T$, if such exists and zero otherwise. We now (trivially) weaken the antecedent of (4.4) as follows:

$$(\forall^a \alpha, g^2)(\exists k^0, \beta^1 \leq_1 1)[\alpha(\overline{g}(\beta))] > 0 \rightarrow (\forall \gamma^0 \leq_0 1)(\exists N \leq k)[|\gamma| = k \rightarrow \alpha(\overline{\gamma}N) > 0].$$

Now if $\beta \leq_1 1$ is such that $\alpha(\overline{g}(\beta)) > 0$, then clearly there is $\sigma^0 \leq_0 1$ such that $\alpha(\sigma * 00 \ldots \overline{g}(\sigma * 00 \ldots)) > 0$, namely take $\sigma = \overline{\beta g}(\beta)$. Hence, we obtain

$$(\forall^a \alpha, g^2)(\exists k^0, \sigma^0 \leq_0 1)[\alpha(\sigma * 00 \ldots \overline{g}(\sigma * 00 \ldots)) > 0] \rightarrow (\forall \gamma^0 \leq_0 1)(\exists N \leq k)[|\gamma| = k \rightarrow \alpha(\overline{\gamma}N) > 0].$$

Clearly, the formula in square brackets in (4.5) is quantifier-free. Hence, applying QF-AC\textsuperscript{2.0} to (4.5), we obtain $\Xi^{(1\times2)\rightarrow((0^*\times0)^*)}$ witnessing $(k, \sigma)$ in (4.5). Again ignoring the second component in $\Xi$ (involving $\sigma$), we obtain (URC).

Finally, we present the proof of Theorem 3.7. Assuming WKL, (RC) follows as noted above, giving rise to (4.5). Again applying QF-AC\textsuperscript{2.0} to (4.5), we obtain $\Xi^{(1\times2)\rightarrow((0^*\times0)^*)}$ witnessing $(k, \sigma)$ in (4.5), i.e. we have

$$(\exists \Xi)[(\forall^a \alpha, g^2)[\alpha(\Xi(\alpha, g)(2) * 00 \ldots \overline{g}(\Xi(\alpha, g)(2) * 00 \ldots)) > 0] \rightarrow (\forall \gamma^0 \leq_0 1)(\exists N \leq \Xi(\alpha, g)(1))[|\gamma| = \Xi(\alpha, g)(1) \rightarrow \alpha(\overline{\gamma}N) > 0]].$$

The formula in big square brackets in (4.6) does not contain parameters other than $\Xi$, and applying PP-TP\textsubscript{α} to (4.6) implies that we may assume $\Xi$ is standard. Hence, for standard $\alpha^1, g^2$, the binary sequence $\Xi(\alpha, g)(2)$ and the number $\Xi(\alpha, g)(1)$ are standard. By (4.6), (URC)\textsuperscript{st} now immediately follows. As noted above, the latter is equivalent to WKL\textsuperscript{st} and we are done.

The restriction to Cantor space in (RC) and (URC) is only for convenience: In light of the equivalence between weak K\"{o}nig’s lemma and bounded K\"{o}nig’s lemma (See [21 IV.1.4]), one establishes Corollary 4.2 in exactly the same way as the theorem. Furthermore, the latter corollary contains similar results for [21 I.10.3.3] in which a continuous function is bounded on a compact subspace of Baire space.
4.2. Corollary. In $\text{RCA}_0 + \text{QF-AC}^{2,0}$, the following are equivalent to WKL:

\[
(\forall \alpha, \gamma^1)[(\forall \beta \leq 1 \gamma)(\exists N^0)\alpha(\beta N) > 0 \quad (\text{RC2})]
\]

\[
\Rightarrow (\exists \beta^0)(\forall \beta \leq 1 \gamma)(\exists N \leq k)\alpha(\beta N) > 0]
\]

\[
(\exists \Psi^3)(\forall \alpha, \gamma^1, g^2)[(\forall \beta \leq 1 \gamma)\alpha(\beta g(\beta)) > 0 \quad (\text{URC2})]
\]

\[
\Rightarrow (\forall \beta \leq 1 \gamma)(\exists N \leq \Psi(g, \alpha, \gamma))\alpha(\beta N) > 0]
\]

\[
(\forall \alpha, \gamma^1)[(\forall \beta \leq 1 \gamma)(\exists N^0)\alpha(\beta N) > 0 \quad (\text{RB})]
\]

\[
\Rightarrow (\forall \beta \leq 1 \gamma, N^0)(\alpha(\beta N) > 0 \Rightarrow \alpha(\beta N) \leq k]
\]

\[
(\exists \Psi^3)(\forall \alpha, \gamma^1, g^2)[(\forall \beta \leq 1 \gamma)\alpha(\beta g(\beta)) > 0 \quad (\text{URB})]
\]

\[
\Rightarrow (\forall \beta \leq 1 \gamma, N^0)(\alpha(\beta N) > 0 \Rightarrow \alpha(\beta N) \leq \Psi(\alpha, g)]
\]

In the same vein, we also have the following corollary, where $\text{FMU}$ is as follows:

\[
(\exists \Psi^3)(\forall \Phi^2 \in C, \gamma^1)(\forall \alpha, \beta \leq 1 \gamma)(\exists \Psi(\Phi, \gamma) = \overline{\Psi}(\Phi, \gamma) \Rightarrow \Phi(\alpha) = \Phi(\beta)), \quad (\text{FMU})
\]

and $\text{MC}_0$ is $\text{MC}$ with a similar extra quantifier $(\forall \gamma^1)$ guaranteeing a compact domain.

4.3. Corollary. In $\text{RCA}_0^-$, we have $\text{FMU} \leftrightarrow \text{WKL} + \text{MC}_0$.

Proof. The forward direction is immediate. For the reverse direction, again by the proof of [14] Prop. 4.4, $\text{MC}_0$ provides a functional $\Xi^2 \rightarrow^1$ such that $\Xi(\Phi, \gamma)$ is an associate for $\Phi^2 \in C$ on $\{\alpha^i : \alpha \leq 1 \gamma\}$. Hence, we obtain $\text{WKL} \Rightarrow (\text{URC2}) \Rightarrow (\text{MC})$, assuming $\text{MC}_0$.

As noted above, the use of associates in $\text{RC}$ is essential for obtaining the equivalences in Theorem 4.4 and Corollary 4.2. The proof of the former fails if we try to apply it to the following ‘higher-order’ version of $\text{RC}$:

\[
(\forall \Phi^2 \in C(2^N))(\exists N^0)(\forall \alpha^1, \beta^1 \leq 1)(\overline{\Psi} N =_0 \overline{\beta} N \Rightarrow \Phi(\alpha) =_0 \Phi(\beta)). \quad (4.7)
\]

Indeed, the antecedent of (4.7) involves (5.1) restricted to Cantor space, which results in a too high quantifier-complexity to apply QF-AC. Furthermore, we cannot weaken the consequent of (4.7) as in the proof of the theorem without access to an associate of $\Phi$ (uniformly via a functional).

In conclusion, we emphasise that on one hand, the choice of ‘continuity via an associate’ in $\text{RC}$, $\text{RC}_2$, and $\text{RB}$, yields that the latter are automatically equivalent to their respective uniform versions $\text{URC}$, $\text{URC}_2$, and $\text{URB}$. On the other hand, for the ‘non-associate’ version $\text{RM}$, an equivalence with $\text{FMU}$ is out of the question by Corollary 4.3 assuming $\text{WKL} \not\rightarrow \text{MC}_0$ over $\text{RCA}_0^-$. In other words, the choice of the RM-definition of continuity guarantees that:

"Every continuous function on Cantor space is uniformly continuous, is equivalent to the higher-order statement:

A functional witnesses the uniform continuity of every continuous function on Cantor space.

Note that by [14] Cor. 4.11, WKL guarantees that each $\Phi^2 \in C(2^N)$ has an associate on Cantor space, but the corresponding proof is highly non-uniform, i.e. a functional providing this associate seems unlikely (without the use of $(\exists^2)$). Furthermore, the proof of [3] Lemma, p. 65 seems to relativize to oracles, suggesting that $\text{WKL} \not\rightarrow \text{MC}_0$.\"
and such an equivalence does not follow for (MC) and (MCU), modulo the non-derivability of \( \text{MC}_0 \) from WKL.

Finally, with regard to further results and Section 3.4 we note that HB has the same syntactic structure as (RC), giving rise to the following theorem. Here, UHB is the obvious uniform version of HB as in Section 3.4.

4.4. Corollary. In RCA\(^0\), we have UHB\(^\text{st} \) ↔ HB\(^\text{st} \) ↔ WKL\(^\text{st} \).

In RCA\(^\omega \) + QF-AC\(^{\omega,0} \), we have UHB ↔ HB ↔ WKL.

**Proof.** Similar to the proof of Theorem 4.1, in light of Section 3.4. □

The author shows in [19] that the uniform version of ATR\(_0\) is equivalent to ATR\(_0\) itself. This result confirms the similarity between WKL\(_0\) (in the form of the fan theorem) and ATR\(_0\) as pointed out by Simpson in [21] I.11.7.

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