THE DETERMINISTIC AND STOCHASTIC SHALLOW LAKE PROBLEM

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ABSTRACT. We study the welfare function of the deterministic and stochastic shallow lake problem. We show that the welfare function is the viscosity solution of the associated Bellman equation, we establish several properties including its asymptotic behaviour at infinity and we present a convergent monotone numerical scheme.

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1. INTRODUCTION

The scope of this work is the theoretical study of the welfare function describing the economics of shallow lakes. Pollution of shallow lakes is a quite often observed phenomenon because of heavy use of fertilizers on surrounding land and an increased inflow of waste water from human settlements and industries. The shallow lake system provides conflicting services as a resource, due to the provision of ecological services of a clear lake, and as a waste sink, due to agricultural activities.

The economic analysis of the problem requires the study of an optimal control problem or a differential game in case of common property resources by various communities; see, for example, [Carpenter et al. ‘99], [Brock et al. ’2003] and [Maler et al. ’03].

Typically the model is described in terms of the amount, $x(t)$, of phosphorus in algae which is assumed to evolve according to the stochastic differential equation (sde for short)

\begin{equation}
\begin{aligned}
&dx(t) = \left( u(t) - bx(t) + \frac{x^2(t)}{x^2(t) + 1} \right) dt + \sigma x(t) dW(t), \\
&x(0) = x.
\end{aligned}
\end{equation}

The first term, $u(t)$, in the drift part of the dynamics represents the exterior load of phosphorus imposed by the agricultural community, which is assumed to be positive. The second term is the rate of loss $bx(t)$, which consists of sedimentation, outflow and sequestration in other biomass. The third term is the rate of recycling of phosphorus due to sediment resuspension resulting, for example, from waves, or oxygen depletion. This rate is typically taken to be a sigmoid function; see [Carpenter et al. ‘99]. The model also assumes an uncertainty in the recycling rate driven by a linear multiplicative white noise with diffusion strength $\sigma$.

The lake has value as a waste sink for agriculture modeled by $\ln u$ and it provides ecological services that decrease with the amount of phosphorus $-cx^2$. The positive parameter $c$ reflects the relative weight of this welfare component; large $c$ gives more weight to the ecological services of the lake.

Assuming an infinite horizon at a discount rate $\rho > 0$, the total benefit is

\begin{equation}
J(x; u) = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left( \ln u(t) - cx^2(t) \right) dt \right],
\end{equation}

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where $x(\cdot)$ is the solution to (1.1), for a given $u(\cdot)$, and $x(0) = x$. Optimal management requires to maximize the total benefit, over all exterior loads that act as controls by the social planner. Thus the welfare function is
\begin{equation}
V(x) = \sup_{u \in \mathcal{U}_x} J(x; u),
\end{equation}
where $\mathcal{U}_x$ is the set of admissible controls $u : [0, \infty) \to \mathbb{R}^+$ which are specified in the next section.

Dynamic programming arguments lead, under the assumption that the welfare function is decreasing, to the Hamilton-Jacobi-Bellman equation
\begin{equation}
\rho V = \left( \frac{x^2}{x^2 + 1} - bx \right) V_x - \left( \ln(-V_x) + x^2 + 1 \right) + \frac{1}{2} \sigma^2 x^2 V_{xx},
\end{equation}
and of the aims of this paper is to provide a rigorous justification of this fact.

In the deterministic case $\sigma = 0$, the optimal dynamics of the problem were fully investigated by analysing the possible equilibria of the dynamics given by Pontryagin maximum principle; see, for example, [Mäler et al. ’03] and [Wagener ’03]. The possibility to steer the combined economic-ecological system towards the trajectory of optimal management via optimal state-dependent taxes was also considered; see [Kossioris et al. ’11].

On the other hand, there is not much in the literature about the stochastic problem. Formal asymptotics expansions of the solution for small $\sigma$ for Hamilton-Jacobi-Bellman equations like (OHJB) have been presented in [Grass et al. ’15]. In the same paper, the authors also give a formal phenomenological bifurcation analysis based on a geometric invariant quantity, along with some numerical computations of the stochastic bifurcations based on (formal) asymptotics for small $\sigma$.

The connection between stochastic control problems and Hamilton-Jacobi-Bellman equation, which is based on the dynamic programing principle, has been studied extensively. The correct mathematical framework is that of the Crandall-Lions viscosity solutions introduced in [Crandall et al. ’83]; see the review article [Crandall et al. ’92]). The deterministic case leads to the study of a first order Hamilton-Jacobi equation; see, for example, [Bardi & Dolcetta ’97], [Barles ’94] and references therein. For the general stochastic optimal control problem we refer to [Krylov ’80], [Lions ’83a] - [Lions ’83c], [Fleming & Soner ’93], [Yong & Zhou ’99] and references therein.

The stochastic shallow lake problem has some nonstandard features and, hence, it requires some special analysis. At first, the problem is formulated as a state constraint one on a semi-infinite domain. Viscosity solutions with state constraint boundary conditions were introduced for first order equations by [Soner ’86] and [Capuzzo Dolcetta & Lions ’90]. For second order equations one should consult [Katsoulakis ’94], [Lasry et al. ’89] and [Alvarez et al. ’97]. Apart from the left boundary condition, the correct asymptotic decay of the solution at infinity is necessary to establish a comparison result; see, for example, [Ishii & Lions ’90] and [Da Lio & Ley ’06].

The unboundedness of the controls along with the logarithmic term in the cost functional lead to a logarithm of the gradient variable in (OHJB). An a priori knowledge of the solution is required to guarantee that the Hamiltonian is well defined. Moreover, due the presence of the logarithmic term it is necessary to construct an appropriate test function to establish a comparison proof. The commonly used polynomial functions, see, for example, [Ishii & Lions ’90] and [Da Lio & Ley ’06], are not useful here since they do not yield a supersolution of the equation.

In the present work we study the stochastic shallow lake problem for a fixed $\sigma$. We first prove the necessary stochastic analysis estimates for the welfare function (1.3). We obtain directly various crucial properties for the welfare function, that is, boundary behaviour, local regularity, monotonicity and asymptotic estimates at infinity.

We prove, including the deterministic case, that (1.3) is the unique decreasing constrained viscosity solution to (OHJB) with quadratic growth at infinity. The comparison theorem is proved by considering a linearized equation and constructing a proper supersolution. Exploiting the well-posedness of the problem within the framework of constrained viscosity solutions we investigate the exact asymptotic behavior of the
solutions at infinity. The latter is used to construct a monotone convergent numerical scheme that along with the optimal dynamics equation can be used to reconstruct numerically the stochastic optimal dynamics.

2. The General Setting and the Main Results

We assume that there exists a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) satisfying the usual conditions, and a Brownian motion \(W(\cdot)\) defined on that space. An admissible control \(u(\cdot) \in \mathcal{U}_x\) is an \(\mathcal{F}_t\)-adapted, \(\mathbb{P}\)-a.s. locally integrable process with values in \(U = (0, \infty)\), satisfying

\[
E \left[ \int_0^\infty e^{-\rho t} \ln u(t) dt \right] < \infty, \tag{2.1}
\]
such that the problem (1.1) has a unique strong solution \(x(\cdot)\).

The shallow lake problem has an infinite horizon. Standard arguments based on the dynamic programming principle (see [Fleming & Soner ’93], Section III.7) suggest that the welfare function \(V\) given by (1.3) satisfies the HJB equation

\[
\rho V = \sup_{u \in U} G(x, u, V_x, V_{xx}), \tag{2.2}
\]

with \(G\) defined by

\[
G(x, u, p, P) = \frac{1}{2} \sigma^2 x^2 P + \left( u - bx + \frac{x^2}{x^2 + 1} \right) p + \ln u - x^2. \tag{2.3}
\]

One difficulty in the study of this problem is related to the fact that the control functions \(u\) take values in the unbounded set \(\mathbb{R}^+\) so that supremum in (2.2) might take infinite values. Indeed, when \(U = \mathbb{R}^+\), setting

\[
H(x, p, P) = \sup_{u \in \mathbb{R}^+} G(x, u, p, P),
\]
we find

\[
H(x, p, P) = \begin{cases} 
\left( \frac{x^2}{x^2 + 1} - bx \right) p - (\ln(-p) + x^2 + 1) + \frac{1}{2} \sigma^2 x^2 P & \text{if } p < 0, \\
+\infty & \text{if } p \geq 0.
\end{cases} \tag{2.4}
\]

It is natural to expect that since shallow lake looses its value with a higher concentration of phosphorus, the welfare function is decreasing as the initial state of phosphorus increases. Assuming that \(V_x < 0\), (2.2) becomes (OHJB).

Since the problem is set in \((0, \infty)\), it is necessary to introduce boundary conditions guaranteeing the well-posedness of the corresponding boundary value problem.

Given the possible degeneracies of Hamilton-Jacobi-Bellman equations at \(x = 0\), the right framework is that of continuous viscosity solutions in which boundary conditions are considered in the weak sense. Since at the boundary point \(x = 0\)

\[
\inf_{u \in U} \left\{ -u + bx - \frac{x^2}{x^2 + 1} \right\} < 0, \tag{2.5}
\]
that is, there always exists a control that can drive the system inside \((0, \infty)\), the problem should be considered as a state constraint one on the interval \([0, \infty)\), meaning that \(V\) is a subsolution in \([0, \infty)\) and a supersolution in \((0, \infty)\).

Next we present the main results of the paper. The proofs are given in Section 4. The first is about the relationship between the welfare function and (OHJB).

**Theorem 2.1.** If \(\sigma^2 < \rho + 2b\), the welfare function \(V\) is a continuous in \([0, \infty)\) constrained viscosity solution of the equation (OHJB) in \([0, \infty)\).

The second result is the following comparison principle for solutions of (OHJB).
Theorem 2.2. Assume that \( u \in C([0, \infty)) \) is a bounded from above strictly decreasing subsolution of (OHJB) in \([0, \infty)\) and \( v \in C([0, \infty)) \) is a bounded from above strictly decreasing supersolution of (OHJB) in \([0, \infty)\) such that \( v \geq -c(1 + x^2) \) and \( Du \leq -\frac{b}{\rho} \) in the viscosity sense, for \( c, c' \) positive constants. Then \( u \leq v \) in \([0, \infty)\).

The next theorem describes the exact asymptotic behavior of \( (1.3) \) at \( +\infty \). Let
\[
A = \frac{1}{\rho + 2b - \sigma^2} \quad \text{and} \quad K = \frac{1}{\rho} \left( \frac{2b + \sigma^2}{2\rho} - \frac{A(\rho + 2b)}{(b + \rho)^2} - 1 \right).
\]

Theorem 2.3. As \( x \to \infty \),
\[
(2.6) \quad V(x) = -A \left( x + \frac{1}{b + \rho} \right)^2 - \frac{1}{\rho} \ln \left( x + \frac{1}{b + \rho} \right) + K + o(1).
\]

An important ingredient of the analysis is the following proposition which collects some key properties of \( V \) that are used in the proofs of Theorem 2.1 and Theorem 2.3 and show that \( V \) satisfies the assumptions of Theorem 2.2. The proof is presented in Section 3.

Proposition 1. Suppose \( \sigma^2 < \rho + 2b \).

(i) There exist constants \( K_1, K_2 > 0 \), such that, for any \( x \geq 0 \), we have
\[
(2.7) \quad K_1 \leq V(x) + A \left( x + \frac{1}{b + \rho} \right)^2 + \frac{1}{\rho} \ln \left( x + \frac{1}{b + \rho} \right) \leq K_2.
\]

(ii) There exist \( C > 0 \) and \( \Phi : [0, +\infty) \to \mathbb{R} \) increasing such that, for any \( x_1, x_2 \in [0, +\infty) \) with \( x_1 < x_2 \),
\[
(2.8) \quad -\Phi(x_2) \leq \frac{V(x_2) - V(x_1)}{x_2 - x_1} \leq -C.
\]

(iii) \( V \) is differentiable at zero and
\[
(2.9) \quad \ln \left( -V'(0) \right) + \rho V(0) + 1 = 0.
\]

It is shown in the next section that the assumption \( \sigma^2 < \rho + 2b \) is necessary, otherwise \( V(x) = -\infty \).

Relation (2.6) is important for the numerical approximation of \( (1.3) \). Given that the computational domain is finite, the correct asymptotic behavior of \( (1.3) \) at \( +\infty \) is necessary for an accurate computation of \( V \). In this connection, in Section 5 we present a monotone numerical scheme approximating \( (1.3) \).

3. The proof of Proposition 1

Properties of the dynamics. The first result states that, if \( x \geq 0 \), the solution to \( (1.1) \) stays nonnegative.

Lemma 3.1. If \( x \geq 0, u \in U_x \), and \( x(\cdot) \) is the solution to \( (1.1) \), then \( P[x(t) \geq 0, \forall t \geq 0] = 1 \).

Proof: Elementary stochastic analysis calculations yield that
\[
(3.1) \quad x(t) = xZ_t + \int_0^t \frac{Z_s}{Z_t} \left( u(s) + \frac{x^2(s)}{1 + x^2(s)} \right) ds,
\]
where
\[
(3.2) \quad Z_t = e^{\sigma W_t - (b + \frac{\sigma^2}{2})t},
\]
and the claim is now obvious, since \( u \) takes positive values. \( \square \)

The next assertion is that the set of admissible controls is actually independent of the starting point \( x \).

Lemma 3.2. For all \( x, y \geq 0, U_x = U_y \).
\textbf{Proof:} Fix }u \in \mathcal{U}_x\text{ and }x \in [0, \infty)\text{ and let }x(\cdot)\text{ the unique strong solution to } (1.1)\text{ with }x(0) = x, \text{ and, for any } y \geq 0, \text{ consider the sde}
\begin{equation}
(3.3)\begin{cases}
dw(t) = \left\{-b(w(t) - \frac{(x(t)-w(t))^2}{1+(x(t)-w(t))^2}) + \frac{x(t)^2}{1+x(t)^2}\right\} dt + \sigma(t) dw(t), \\
w(0) = x - y,
\end{cases}
\end{equation}
\begin{align}
\text{and note that the coefficients are Lipschitz and grow at most linearly. It follows that (3.3) has a unique strong solution defined for all } t \geq 0. It is easy to see now that the process } y(t) = x(t) - w(t)\text{ satisfies (1.1) with } y(0) = y. \text{ Moreover, the uniqueness of } y(\cdot)\text{ follows from that of } x(\cdot), \text{ so } u \in \mathcal{U}_y. \quad \Box
\end{align}

In view of Lemma 3.2, we will denote the set of admissible controls by }\mathcal{U}\text{, regardless of the starting point }x\text{ in (1.1).

\textbf{Lemma 3.3.} Suppose } x(\cdot), y(\cdot)\text{ satisfy (1.1) with controls } u_1, u_2 \in \mathcal{U}, \text{ respectively, and } x(0) = x, y(0) = y. \text{ If } x \leq y \text{ and } \mathbb{P}[u_1(t) \leq u_2(t), \forall t \geq 0] = 1, \text{ then}
\begin{equation}
\mathbb{P}[y(t) - x(t) \geq (y - x)Z_t, \forall t \geq 0] = 1.
\end{equation}
\begin{align}
\text{Proof:} \text{ The proof is an immediate consequence of (3.1) and Gronwall's inequality, since } w(t) = y(t) - x(t) \text{ satisfies}
\begin{align}
w(t) &= (y - x)Z_t + \int_0^t \frac{Z_s}{Z_t} (u_2(s) - u_1(s) + w(s)) \frac{y(s) + x(s)}{(1 + y^2(s))(1 + x^2(s))} ds.
\end{align}
\end{align}

\textbf{Properties of the welfare function.} The results here follow from the properties of (1.1). In the rest of the paper, for the sake of convenience, we assume that } c = 1. \text{ Throughout this section, we refer to quantities depending only on } \rho, b \text{ and } \sigma^2 \text{ as constants.}

We remark that, if } \sigma^2 \geq \rho + 2b, \text{ then } V(x) \equiv -\infty. \text{ Indeed, when } u \in \mathcal{U} \text{ and } x(\cdot)\text{ satisfies (1.1), (3.4) implies that } \mathbb{P}[x(t) \geq M_t(u), \forall t \geq 0] = 1, \text{ with}
\begin{equation}
M_t(u) = \int_0^t \frac{Z_s}{Z_t} u(s) ds,
\end{equation}
where \{Z_t\}_{t \geq 0} \text{ is as in (3.2).

In view of this observation we will hereafter assume that}
\begin{equation}
\sigma^2 < \rho + 2b.
\end{equation}
\begin{align}
\text{We will first prove three lemmata before we proceed with the proof of Proposition 1.

\textbf{Lemma 3.4.} The function } x \mapsto V(x) + Ax^2 \text{ is decreasing on } [0, +\infty).
\end{align}
\begin{align}
\text{Proof:} \text{ Fix } x_1, x_2 \geq 0 \text{ with } x_1 \leq x_2. \text{ It suffices to show that, for any control } u \in \mathcal{U},
J(x_2; u) + Ax_2^2 \leq J(x_1; u) + Ax_1^2.
\end{align}
\begin{align}
\text{Since this holds trivially if } J(x_2; u) = -\infty, \text{ we may assume that } J(x_2; u) > -\infty.
\text{ Consider now the solutions } x_1(\cdot), x_2(\cdot) \text{ to (1.1) with initial conditions } x_1, x_2 \text{ and a common control } u \in \mathcal{U}. \text{ Lemma 3.3 implies that, P-a.s. and for all } t \geq 0 \text{ and } i = 1, 2,
\begin{align}
x_1(t) \geq x_1Z_t, \quad \text{and} \quad x_2(t) - x_1(t) \geq (x_2 - x_1)Z_t.
\end{align}
\end{align}
\begin{align}
\text{Note that since } u \in \mathcal{U} \text{ and } J(x_2; u) > -\infty,
\int_0^\infty e^{-\rho t} x_2^2(t) dt < +\infty \Rightarrow \int_0^\infty e^{-\rho t} x_1^2(t) dt < +\infty \Rightarrow J(x_1; u) > -\infty.
\end{align}
In particular,
\[
J(x_2; u) - J(x_1; u) = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left( \ln u(t) - x_2^2(t) \right) dt - \int_0^\infty e^{-\rho t} \left( \ln u(t) - x_1^2(t) \right) dt \right] \\
= -\mathbb{E} \left[ \int_0^\infty e^{-\rho t} (x_2(t) - x_1(t))(x_2(t) + x_1(t)) dt \right] \\
\leq -(x_2^2 - x_1^2) \int_0^\infty e^{-\rho t} \mathbb{E} [Z_t^2] dt = -A(x_2^2 - x_1^2).
\]

Lemma 3.5. The welfare function at zero satisfies \( V(0) \leq \frac{1}{\rho} \ln \left( \frac{b + \rho}{\sqrt{2e}} \right) \).

Proof: Recall that, for any \( u \in \mathcal{U} \), \( x(t) \geq M_t(u) \). Using Jensen’s inequality and part (i) of Lemma A.1, we find
\[
\mathbb{E} \left[ \int_0^\infty e^{-\rho t} \ln u(t) dt \right] \leq \frac{1}{\rho} \ln \mathbb{E} \left[ \int_0^\infty \rho e^{-\rho t} u(t) dt \right] \\
= \frac{1}{\rho} \ln \mathbb{E} \left[ \int_0^\infty \rho(b + \rho)e^{-\rho t} M_t(u) dt \right] \\
\leq \frac{\ln(b + \rho)}{\rho} + \frac{1}{\rho} \ln \mathbb{E} \left[ \int_0^\infty \rho e^{-\rho t} x(t) dt \right] \\
\leq \frac{\ln(b + \rho)}{\rho} + \frac{1}{2\rho} \ln \mathbb{E} \left[ \int_0^\infty \rho e^{-\rho t} x^2(t) dt \right].
\]
In view of (2.1), we need only consider \( u \in \mathcal{U} \) such that \( D = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} x^2(t) dt \right] < \infty \). Then
\[
\mathbb{E} \left[ \int_0^\infty e^{-\rho t} [\ln u(t) - x^2(t)] dt \right] \leq \frac{\ln(b + \rho)}{\rho} + \frac{\ln(\rho D)}{2\rho} - D \leq \frac{1}{\rho} \ln \left( \frac{b + \rho}{\sqrt{2e}} \right),
\]
and the assertion holds.

It follows from Lemma 3.4 and Lemma 3.5 that \( V < +\infty \) in \([0, \infty)\). The next result is a special case of the dynamic programming principle.

Lemma 3.6. Fix \( x_1, x_2 \in [0, \infty) \) with \( x_1 < x_2 \), and, for \( u \in \mathcal{U} \), let \( x(\cdot) \) be the solution to (1.1) with control \( u \) and \( x(0) = x_1 \). If \( \tau_u \) is the hitting time of \( x(\cdot) \) on \([x_2, +\infty)\), that is,
\[
\tau_u = \inf \{ t \geq 0 : x(t) \geq x_2 \},
\]
then
\[
V(x_1) = \sup_{u \in \mathcal{U}} \mathbb{E} \left[ \int_0^{\tau_u} e^{-\rho t} \left( \ln u(t) - x^2(t) \right) dt + e^{-\rho \tau_u} V(x_2) \right].
\]

Proof: We have
\[
J(x_1; u) = \mathbb{E} \left[ \int_0^{\tau_u} e^{-\rho t} \left( \ln u(t) - x^2(t) \right) dt \right] + \mathbb{E} \left[ \int_{\tau_u}^{\infty} e^{-\rho t} \left( \ln u(t) - x^2(t) \right) dt; \tau_u < +\infty \right].
\]
Conditioning on the \( \sigma \)-field \( \mathcal{F}_{\tau_u} \), and applying the strong Markov property, the rightmost term becomes
\[
\mathbb{E} \left[ e^{-\rho \tau_u} \mathbb{E} \left[ \int_{\tau_u}^{\infty} e^{-\rho (t-\tau_u)} \left( \ln u(t) - x^2(t) \right) dt \mid \mathcal{F}_{\tau_u} \right]; \tau_u < +\infty \right] \leq \mathbb{E} \left[ e^{-\rho \tau_u} \right] V(x_2),
\]
since on the event \( \{ \tau_u < +\infty \} \), \( x(\tau_u + \cdot) \) satisfies (1.1) with initial condition \( x(\tau_u) = x_2 \) and control \( u(\tau_u + \cdot) \). Taking the supremum over \( u \in \mathcal{U} \) we see that the left hand side of (3.5) is less than or equal the right hand one.
For the reverse inequality, take any \( u \in \U \) and consider \((1.1)\) driven by the Brownian motion \( B(t) = W(\tau_u + t) - W(\tau_u) \), and, for \( \epsilon > 0 \), choose a control \( u_\epsilon \) such that
\[
V(x_2) < J(x_2; u_\epsilon) + \epsilon.
\]

Define now the new control \( u^*_t \in \U \) as
\[
u^*_t = \begin{cases} 
  u(t) & \text{for } t \leq \tau_u \\
  u_\epsilon(t - \tau_u) & \text{for } t > \tau_u.
\end{cases}
\]

Just as in the proof of the upper bound we get
\[
V(x_1) \geq J(x_1; u^*_t) = \mathbb{E} \left[ \int_0^{\tau_u} e^{-\rho t} \left( \ln u(t) - x^2(t) \right) dt + e^{-\rho \tau_u} J(x_2; u_\epsilon) \right] > \mathbb{E} \left[ \int_0^{\tau_u} e^{-\rho t} \left( \ln u(t) - x^2(t) \right) dt + e^{-\rho \tau_u} V(x_2) \right] - \epsilon,
\]
which concludes the proof. \( \Box \)

We next give the proof of Proposition \((1.1)\) which is subdivided in several parts.

**Proof of Proposition \((1.1)\):**

**Proof lower bound, part (i):** The claim will follow by choosing the control \( u(t) = \frac{1 + x(t)}{1 + x^2(t)} \), which is clearly admissible. Then, \((3.1)\) gives
\[
x(t) = xZ_t + \int_0^t \frac{Z_t}{Z_s} \left( 1 + \frac{x(s)}{1 + x^2(s)} \right) ds = xZ_t + M_t(1) + \int_0^t \frac{Z_t}{Z_s} \frac{x(s)}{1 + x^2(s)} ds
\]
and, hence,
\[
x^2(t) = x^2 Z_t^2 + 2xZ_t M_t(1) + M^2_t(1) + \left( \int_0^t \frac{Z_t}{Z_s} \frac{x(s)}{1 + x^2(s)} ds \right)^2 + M_t(1) \int_0^t \frac{Z_t}{Z_s} \frac{2x(s)}{1 + x^2(s)} ds + \int_0^t \frac{Z_t^2}{Z_s} \frac{2x(s)}{1 + x^2(s)} ds.
\]

To estimate the rightmost term from above, note that, in view of \((3.6)\), \( x \leq x(s)Z_s^{-1} \), while for the third and fourth terms of the sum we use that \( \frac{x(s)}{1 + x^2(s)} \leq \frac{1}{2} \). It follows that
\[
x^2(t) \leq x^2 Z_t^2 + 2xZ_t M_t(1) + \frac{9}{4} M^2_t(1) + 2 \int_0^t \frac{Z_t^2}{Z_s} ds,
\]

It is easy to see that
\[
\mathbb{E} \left[ \int_0^\infty e^{-\rho t} x^2 Z_t^2 dt \right] = x^2 \int_0^\infty e^{-(\rho + 2b - \sigma^2)t} dt = Ax^2,
\]

Lemma \((A.1)\) (ii) gives
\[
\mathbb{E} \left[ \int_0^\infty e^{-\rho t} 2xZ_t M_t(1) dt \right] = 2Ax \int_0^\infty e^{-\rho t} \mathbb{E}[Z_t] dt = \frac{2Ax}{\rho + b}
\]
while Lemma \((A.1)\) (i) and Lemma \((A.1)\) (iii) yield
\[
\int_0^\infty e^{-\rho t} \mathbb{E}[M^2_t(1)] dt = 2A \int_0^\infty e^{-\rho t} \mathbb{E}[M_t(1)] dt = \frac{2A}{\rho(\rho + b)}.
\]

We also have
\[
\int_0^\infty e^{-\rho t} \int_t^\infty \mathbb{E} \left[ \frac{Z_s^2}{Z_t^2} \right] ds dt = \int_0^\infty e^{-\rho t} \int_0^t e^{(\sigma^2 - 2b)(t-s)} ds dt = \frac{A}{\rho}.
\]

Using the last four observations in \((3.7)\), we find for some constant \( B \),
\[
\int_0^\infty e^{-\rho t} \mathbb{E}[x^2(t)] dt \leq A \left[ (x + \frac{1}{\rho + b})^2 + B \right].
\]
On the other hand, using that, for all \( x \geq 0 \), \((1 + x)^2 \geq (1 + x^2)\), and Jensen’s inequality, we find
\[
\int_0^\infty e^{-\rho t}\mathbb{E}[\ln u(t)] dt \geq -\int_0^\infty e^{-\rho t}\mathbb{E}[\ln (1 + x(t))] dt \\
\geq -\frac{1}{\rho} \ln \left( \int_0^\infty e^{-\rho t}(1 + \mathbb{E}[x(t)]) dt \right) \\
= -\frac{1}{\rho} \ln \left( 1 + \rho \int_0^\infty e^{-\rho t}\mathbb{E}[x(t)] dt \right).
\]
By (3.6) it follows that \( \mathbb{E}[x(t)] \leq x\mathbb{E}[Z_t] + \frac{3}{2}\mathbb{E}[M_t(1)] = x e^{-\beta t} + \frac{3}{2} e^2 [M_t(1)]. \)
Hence, using Lemma A.1 (i), we obtain
\[
\int_0^\infty e^{-\rho t}\mathbb{E}[\ln u(t)] dt \geq -\frac{1}{\rho} \ln \left( 1 + \rho \frac{x}{\rho + b} + \frac{3}{2(\rho + b)} \right).
\]
The preceding estimate and (3.12) together imply that, for some suitable constant \( K_1 \),
\[
V(x) \geq J(x; u) = \mathbb{E}\left[ \int_0^\infty e^{-\rho t}\left( \ln u(t) - x^2(t) \right) dt \right] \\
\geq -A \left( x + \frac{1}{b + \rho} \right)^2 - \frac{1}{\rho} \ln \left( x + \frac{1}{b + \rho} \right) + K_1.
\]

**Proof of the upper bound, part (i):**
In view of Lemma 3.4 and Lemma 3.5 it suffices to find \( K_2 > 0 \), such that the asserted inequality holds for \( x \geq 1 \).
Fix \( u \in \mathcal{U} \). Then
\[
x^2(t) \geq x^2 Z_t^2 + 2x Z_t \int_0^t \frac{1}{Z_s} \left( u(s) + \frac{x^2(s)}{1 + x^2(s)} \right) ds \\
= x^2 Z_t^2 + 2x Z_t M_t(1 + u) - \int_0^t \frac{Z_s^2}{Z_s} \frac{2x Z_s}{1 + x^2(s)} ds.
\]
Since
\[
1 + x^2(t) \geq 1 + x^2 Z_t^2 \geq 2x Z_t,
\]
so we can further estimate \( x^2(t) \) from below by
\[
(3.13) \quad x^2(t) \geq x^2 Z_t^2 + 2x Z_t M_t(1 + u) - \int_0^t \frac{Z_s^2}{Z_s} ds.
\]
Using the elementary inequality that \( \ln a \leq ab - \ln b - 1 \), which holds for all \( a, b > 0 \), and Lemma A.1 (ii), we obtain, for some \( B \),
\[
\int_0^\infty e^{-\rho t}\mathbb{E}[\ln u(t)] dt \leq \mathbb{E}\left[ \int_0^\infty e^{-\rho t}\left\{ 2Ax Z_t u(t) - \ln \left( 2Ax Z_t \right) \right\} dt \right] \\
= \mathbb{E}\left[ \int_0^\infty e^{-\rho t} 2x Z_t M_t(u) dt \right] - \frac{\ln(2Ax)}{\rho} + \frac{2b + \sigma^2}{2\rho^2} \\
\leq \mathbb{E}\left[ \int_0^\infty e^{-\rho t}\left( x^2(t) - x^2 Z_t^2 - 2x Z_t M_t(1) + \int_0^t \frac{Z_s^2}{Z_s} ds \right) dt \right] - \frac{\ln x + B}{\rho},
\]
where in the final step we have used (3.13).

In view of (3.8), (3.9) and (3.11), for every \( u \in \mathcal{U} \) there exists \( K_2 > 0 \) such that
\[
J(x; u) \leq -A \left( x + \frac{1}{b + \rho} \right)^2 - \frac{1}{\rho} \ln \left( x + \frac{1}{b + \rho} \right) + K_2.
\]
The assertion now follows by taking the supremum over \( u \in \mathcal{U} \).
Proof of the lower bound, part (ii): Fix $x_1, x_2$ as in the statement. It follows from Lemma 3.4 that for any $\epsilon > 0$ there exists a control $u_\epsilon \in \mathcal{U}$ such that
\begin{equation}
V(x_1) \leq \mathbb{E} \left[ \int_0^{\tau_\epsilon} e^{-\rho t} \ln u_\epsilon(t) \, dt \right] + \mathbb{E} \left[ e^{-\rho \tau_\epsilon} V(x_2) + \epsilon(x_2^2 - x_1^2) \right],
\end{equation}
where $\tau_\epsilon$ is the hitting time on $[x_2, +\infty)$ of the solution $x_\epsilon(t)$ to (1.1) with $x(0) = x_1$ and control $u_\epsilon$.

Using the elementary inequality
\[ \ln u_\epsilon(t) \leq \ln c + \frac{u_\epsilon(t)}{c} - 1, \quad \text{with } c = e^{\rho V(x_2)+1}, \]
we find
\begin{equation}
V(x_1) - V(x_2) \leq \frac{1}{c} \mathbb{E} \left[ \int_0^{\tau_\epsilon} e^{-\rho t} u_\epsilon(t) \, dt \right] + \epsilon(x_2^2 - x_1^2).
\end{equation}

To conclude it suffices to show that
\begin{equation}
\frac{1}{c} \mathbb{E} \left[ \int_0^{\tau_\epsilon} e^{-\rho t} u_\epsilon(t) \, dt \right] \leq \Phi(x_2)(x_2 - x_1).
\end{equation}

To this end, we apply Itô’s rule to the semimartingale $Y_t = e^{-\rho t + \gamma x_\epsilon(t)}$, where $\gamma > 0$ is a constant to be determined, and find
\begin{align*}
Y_t - e^{\gamma x_1} &= \int_0^t Y_s \left( -\rho \, ds + \gamma \, dx_\epsilon(s) + \frac{\gamma^2}{2} \, d\langle x_\epsilon \rangle_s \right) \\
&= \int_0^t Y_s \left( -\rho + \gamma(u_\epsilon(s) - bx_\epsilon(s)) + \frac{x_\epsilon^2(s)}{1 + x_\epsilon^2(s)} + \frac{\gamma^2\sigma^2 x_\epsilon^2(s)}{2} \right) ds + M_t,
\end{align*}
where $M_t$ stands for the martingale $\gamma \sigma \int_0^t x_\epsilon(s) \, dW(s)$.

Next we apply the optional stopping theorem for the bounded stopping time $\tau_N = \min\{\tau_\epsilon, N\}$, with $N \in \mathbb{N}$, to find
\begin{equation}
\mathbb{E} [Y_{\tau_N}] - e^{\gamma x_1} = \mathbb{E} \left[ \int_0^{\tau_N} Y_s \left( -\rho + \gamma(u_\epsilon(s) - bx_\epsilon(s)) + \frac{x_\epsilon^2(s)}{1 + x_\epsilon^2(s)} + \frac{\gamma^2\sigma^2 x_\epsilon^2(s)}{2} \right) ds \right].
\end{equation}

Since $0 \leq x_\epsilon(s) \leq x_2$ in $[0, \tau_\epsilon]$,
\begin{equation}
e^{\gamma x_2} \mathbb{E} [e^{-\rho \tau_N}] - e^{\gamma x_1} \geq \mathbb{E} \left[ \int_0^{\tau_N} Y_s \left( -\rho + \gamma(u_\epsilon(t) - bx_\epsilon(t)) + \frac{\gamma^2\sigma^2 x_\epsilon^2(t)}{2} \right) dt \right],
\end{equation}
and
\begin{equation}
e^{\gamma x_2} - e^{\gamma x_1} \geq \gamma \mathbb{E} \left[ \int_0^{\tau_N} e^{-\rho t} u_\epsilon(t) \, dt \right] + \mathbb{E} \left[ \int_0^{\tau_N} Y_s \left( -b\gamma x_\epsilon(t) + \frac{\sigma^2\gamma^2 x_\epsilon^2(t)}{2} \right) dt \right].
\end{equation}

Note that the term in the parenthesis above is nonnegative if $\gamma x_\epsilon(t) \geq 2b/\sigma^2$, and greater than or equal to $-b^2/2\sigma^2$ in any case. Hence,
\begin{equation}
e^{\gamma x_2} - e^{\gamma x_1} \geq \gamma \mathbb{E} \left[ \int_0^{\tau_N} e^{-\rho t} u_\epsilon(t) \, dt \right] - \frac{b^2 e^{\frac{2b}{\sigma^2}}}{2\sigma^2} \mathbb{E} \left[ \int_0^{\tau_N} e^{-\rho t} dt \right].
\end{equation}

Letting $N \to \infty$ we get
\begin{equation}
e^{\gamma x_2} - e^{\gamma x_1} \geq \gamma \mathbb{E} \left[ \int_0^{\tau} e^{-\rho t} u_\epsilon(t) \, dt \right] - \frac{b^2 e^{\frac{2b}{\sigma^2}}}{2\sigma^2} \mathbb{E} \left[ \int_0^{\tau} e^{-\rho t} dt \right].
\end{equation}

With $\gamma$ still at our disposal, to show (3.16) it suffices to control the term $\mathbb{E} \left[ \int_0^{\tau} e^{-\rho t} dt \right]$ by $\mathbb{E} \left[ \int_0^{\tau} e^{-\rho t} u_\epsilon(t) \, dt \right]$.

Since without loss of generality we may assume that $\epsilon < A$, Lemmas 3.4 and (3.14) give
\begin{equation}
0 \leq V(x_1) - V(x_2) - \epsilon(x_2^2 - x_1^2) \leq \mathbb{E} \left[ \int_0^{\tau} e^{-\rho t} \ln u_\epsilon(t) \, dt \right] - \rho \mathbb{E} \left[ \int_0^{\tau} e^{-\rho t} dt \right].
\end{equation}
Jensen’s inequality then implies that
\[ E \left[ \int_0^\tau e^{-\rho t} u_c(t) \, dt \right] \geq e^{\rho \nu(x_2)} \ E \left[ \int_0^\tau e^{-\rho t} \, dt \right], \]
and \((3.17)\) gives, with \(C(x_2) = \frac{1^2}{2 \pi^2} e^{-\frac{\rho V(x_2)}{2}}\),
\begin{equation}
\gamma e^{\gamma \tau x_2}(x_2 - x_1) \geq e^{\gamma \tau x_2} - e^{\gamma x_1} \geq (\gamma - C(x_2)) \ E \left[ \int_0^\tau e^{-\rho t} u_c(t) \, dt \right].
\end{equation}

Even though \((3.16)\), and hence the assertion of the Theorem, follows now with a suitable choice of \(\gamma\), we will optimize the preceding inequality for later use. Choosing \(\gamma = q(x_2)/x_2\) in \((3.18)\), where
\begin{equation}
q(x_2) = \frac{x_2C(x_2)}{2} + \sqrt{\left(\frac{x_2C(x_2)}{2}\right)^2 + x_2C(x_2)},
\end{equation}
we obtain
\[ E \left[ \int_0^\tau e^{-\rho t} u_c(t) \, dt \right] \leq (x_2 - x_1)(1 + q(x_2)) e^{q(x_2)}. \]
Letting \(\varepsilon \to 0\), \((3.15)\) yields
\begin{equation}
\frac{V(x_2) - V(x_1)}{x_2 - x_1} \geq -(1 + q(x_2)) e^{q(x_2) - 1 - \rho V(x_2)},
\end{equation}
the claim now follows. \(\square\)

**Proof of the upper bound, part (ii):** In view of Lemma \ref{lem:3.4} it suffices to assume that \(x_2 \leq b\), since otherwise we have
\[ V(x_2) - V(x_1) \leq -A(x_2^2 - x_1^2) \leq -Ab(x_2 - x_1). \]

For a positive constant \(c\), choose a \(u_c \in \mathbb{U}\) that is constant an equal to \(c\) up to time \(\tau_c = \tau u_c\). Then, Lemma \ref{lem:3.6} yields
\[ V(x_1) \geq \frac{\ln c - x_2^2}{\rho}(1 - E[e^{-\rho \tau_c}]) + E[e^{-\rho \tau_c} V(x_2)], \]
or equivalently,
\begin{equation}
(V(x_2) - V(x_1)) E[e^{-\rho \tau_c}] \leq -(\ln c - \rho V(x_1) - x_2^2) E \left[ \int_0^\tau e^{-\rho t} \, dt \right].
\end{equation}

Consider now the solution \(x_c(\cdot)\) to \((1.1)\) with \(x(0) = x_1\) and control \(u_c\). Applying Itô’s formula to \(e^{-\rho t} x_c(t)\), followed by the optional stopping theorem for the bounded stopping time \(\tau_N = \tau_c \wedge N\), we get
\begin{equation}
E[e^{-\rho \tau_N} x_c(\tau_N)] - x_1 = E \left[ \int_0^{\tau_N} e^{-\rho t} \left( c - (b + \rho) x_c(t) + \frac{x_2^2(t)}{1 + x_2^2(t)} \right) \, dt \right].
\end{equation}
The leftmost term of \((3.22)\) is equal to \(x_2E[e^{-\rho \tau_c}; \tau_c \leq N] + e^{-\rho N} E[x_c(\tau_N); \tau_c > N]\).

On the other hand, since we have assumed that \(x_2 \leq b\), we have \(x_c(t) \leq b\) up to time \(\tau_c\). Thus, the right hand side of \((3.22)\) is bounded by \(E \left[ \int_0^{\tau_N} e^{-\rho t} \, dt \right]\).

Letting \(N \to \infty\) in \((3.22)\), by the monotone convergence theorem, we have
\[ x_2E[e^{-\rho \tau_c}] - x_1 \leq c E \left[ \int_0^\tau e^{-\rho t} \, dt \right] \implies (x_2 - x_1) E[e^{-\rho \tau_c}] \leq (c + \rho x_1) E \left[ \int_0^\tau e^{-\rho t} \, dt \right].
\]
Substituting this in \((3.21)\) and choosing \(\ln c = \rho V(x_1) + 1 + x_2^2\), we find
\begin{equation}
V(x_2) - V(x_1) \leq -(x_2 - x_1) \left( e^{\rho V(x_1)+1+x_2^2} + \rho x_1 \right)^{-1}.
\end{equation}
The assertion now follows letting \(C = Ab \wedge \left( e^{\rho V(0)+1+b^2} + \rho b \right)^{-1} > 0\). \(\square\)
Proof of part (iii): It follows from (3.23) that, for any \( \varepsilon \in (0, b] \),

\[
V(\varepsilon) - V(0) \leq -e^{-\rho V(0) - 1 - \varepsilon^2}.
\]

Letting \( \varepsilon \to 0 \) we get

\[
\limsup_{\varepsilon \to 0} \frac{V(\varepsilon) - V(0)}{\varepsilon} \leq -e^{-\rho V(0) - 1},
\]

while (3.20) gives

\[
\frac{V(\varepsilon) - V(0)}{\varepsilon} \geq -(1 + q(\varepsilon)) e^{q(\varepsilon) - 1 - \rho V(\varepsilon)}.
\]

Letting \( \varepsilon \to 0 \) and noting that \( q(\varepsilon) \to 0 \), we have

\[
\liminf_{\varepsilon \to 0} \frac{V(\varepsilon) - V(0)}{\varepsilon} \leq -e^{-\rho V(0) - 1},
\]

which proves the claim. \( \square \)

We conclude observing that since \( V \in C([0, \infty)) \), the general dynamic programming principle is also satisfied. For a proof we refer to [Touzi ‘13].

4. Viscosity solutions and the Hamilton–Jacobi–Bellman equation

Since the Hamiltonian (2.4) can take infinite values we have a singular stochastic control problem and the welfare function (1.3) should satisfy the proper variational inequality; see [Fleming & Soner ‘93], Section VIII and [Pham ‘09], Section 4. The proof of the next Lemma, except for the treatment of the boundary conditions, follows the lines of Proposition 4.3.2 of [Pham ‘09].

Lemma 4.1. If \( \sigma^2 < \rho + 2b \), the welfare function \( V \) defined by (1.3) is a continuous constrained viscosity solution of

\[
\min \left[ \frac{\rho V - \sup_{u \in \mathbb{R}^+} \left( \frac{1}{2} \sigma^2 x^2 V_{xx} + (u - bx + \frac{x^2}{x^2 + 1}) V_x + \ln u - x^2 \right)}{V}, -V \right] = 0, \quad \text{in } [0, \infty).
\]

Proof: That \( V \) is a viscosity solution in \((0, \infty)\) follows as in [Pham ‘09], so we omit the details.

Here we briefly discuss the subsolution property at \( x = 0 \). Let \( \phi \) be a test function such that \( V - \phi \) has a maximum at \( x = 0 \) with \( V(0) - \phi(0) = 0 \), and, proceeding by contradiction, we assume that

\[
\rho \phi(0) - \sup_{u \in \mathbb{R}^+} G(0, u, \phi'(0), \phi''(0)) > 0 \quad \text{and} \quad -\phi'(0) > 0.
\]

Since \(-\phi'(0) > 0\), the supremum in the above inequality is finite, the Hamiltonian takes the standard form, and the first inequality in (4.2) becomes

\[
\rho \phi(0) + 1 + \ln \left( -\phi'(0) \right) > 0.
\]

On the other hand, in view of Proposition 1 (iii), we have \( \rho V(0) + 1 + \ln \left( -V'(0) \right) = 0 \), hence \( V'(0) > \phi'(0) \), contradicting that \( V - \phi \) has a maximum at \( x = 0 \).

We have now obtained all the necessary material for the proof of Theorem 2.1

Proof: The fact that (1.3) is a continuous constrained viscosity solution of the equation (OHJB) is a direct consequence of the above Lemma and the fact that inequality (2.8) implies that \( p \leq -C \) for any \( p \in D^2 V(x) \), with \( x \in (0, \infty) \). The regularity of \( V \) in \((0, \infty)\) follows from the classical results for uniformly elliptic operators. \( \square \)
Due to the extra regularity of the welfare function, the following verification equation holds in \((0, \infty)\), for any optimal pair \((\pi(t), \pi'(t))\),

\[
\rho V(\pi(t)) = \sup_{u \in U} G(\pi(t), u, V_x(\pi(t)), V_{xx}(\pi(t)))
\]

\[
= \left(\frac{\pi^2(t)}{\pi(t)} - \frac{\pi(t)}{\pi(t)} + 1\right) V_x(\pi(t)) - \frac{\ln(-V_x(\pi(t))) + \pi^2(t) + 1}{2} + \frac{1}{\sigma^2\pi^2(t)} V_{xx}(\pi(t)), \quad t \in [0, \infty) - a.e., \; P - a.e.;
\]

see [Fleming & Soner '93] and [Yong & Zhou '99].

Next we prove the proper comparison principle for (OHJB). The proof is along the lines of the strategy in [Lipton '97], where given a subsolution \(u\) and a supersolution \(v\) of (OHJB), \(u - v\) is a subsolution of the corresponding linearized equation. Then, one concludes by comparing \(u - v\) with the appropriate supersolution of the linearized equation; see also [Da Lio & Ley '06] and [Zariphopoulou '94]. The difference with the existing results is that, due to the presence of the logarithmic term, the commonly used functions of simple polynomials do not yield a supersolution of the equation.

Having in mind that we are looking for a viscosity solution that is strictly decreasing and satisfies (2.8), we prove the following lemma.

**Lemma 4.2.** Suppose \(u, v\) satisfy the assumptions of Theorem 2.2. Then \(\psi = u - v\) is a subsolution of

\[
\rho \psi + bx D\psi - (1 + c^0)|D\psi| - \frac{1}{2} \sigma^2 x^2 D^2 \psi = 0 \quad \text{in} \quad [0, \infty).
\]

**Proof:** Let \(\overline{x} \geq 0\) a maximum point of \(\psi - \phi\) for some smooth function \(\phi\) and set, following [Soner '86],

\[
\theta(x, y) = \phi(x) + \frac{(x - y + \varepsilon L)^2}{\varepsilon} + \delta(x - \overline{x})^4,
\]

where \(L, \delta\) are positive constants.

The assumptions on \(u, v\) imply that the function \((x, y) \mapsto u(x) - v(y) - \theta(x, y)\) is bounded from above and achieves its maximum at, say, \((x_e, y_e)\). It follows that \(x \mapsto u(x) - v(y) - \theta(x, y_e)\) has a local maximum at \(x_e\) and \(y \mapsto v(y) - u(x_e) + \theta(x_e, y)\) has a local minimum at \(y_e\). Moreover, (see Proposition 3.7 in [Crandall et al. '92]), as \(\varepsilon \to 0\),

\[
\frac{|x_e - y_e|^2}{\varepsilon} \to 0, \quad x_e \to \overline{x}, \quad \text{and} \quad u(x_e) - v(y_e) \to \psi(\overline{x}).
\]

The inequalities

\[
u(x_e) - v(y_e) - \theta(x_e, y_e) \leq \psi(\overline{x}) - \phi(\overline{x}) + v(x_e) - v(y_e) - \frac{|x_e - y_e + \varepsilon L|^2}{\varepsilon} - \delta(x_e - \overline{x})^4
\]

and

\[
u(x_e) - v(y_e) - \theta(x_e, y_e) \geq u(\overline{x}) - v(\overline{x} + \varepsilon L) - \theta(\overline{x}, \overline{x} + \varepsilon L) \geq \psi(\overline{x}) - \phi(\overline{x})
\]

together imply that

\[
\frac{|x_e - y_e + \varepsilon L|^2}{\varepsilon} + \delta(x_e - \overline{x})^4 \leq v(x_e) - v(y_e).
\]

Since \(v\) is decreasing we must have \(y_e \geq x_e\). In particular, \(y_e \in (0, \infty)\).

Therefore, setting \(p_e = \frac{2(x_e - y_e + \varepsilon L)}{\varepsilon}\) and \(q_e = \phi(\overline{x}) + 4\delta(x_e - \overline{x})^3\), Theorem 3.2 in [Crandall et al. '92] implies that, for every \(\alpha > 0\), there exist \(X, Y \in \mathbb{R}\) such that

\[
u(x_e) - H(x_e, p_e + q_e, X) \leq 0 \quad \text{and} \quad \nu(y_e) - H(y_e, p_e, Y) \geq 0
\]

and

\[
-(\frac{1}{\alpha} + \|M\|)I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq M + \alpha M^2
\]

with \(M = D^2\theta(x_e, y_e)\).
By subtracting the two inequalities in (4.5) we obtain
\begin{equation}
\rho u(x_\varepsilon) - \rho v(y_\varepsilon) + b(x_\varepsilon - y_\varepsilon)p_\varepsilon + \left(\frac{y_\varepsilon^2}{y_\varepsilon^2 + 1} - \frac{x_\varepsilon^2}{x_\varepsilon^2 + 1}\right)p_\varepsilon - \left(\frac{x_\varepsilon^2}{x_\varepsilon^2 + 1} - bx_\varepsilon\right)q_\varepsilon + \ln\left(-p_\varepsilon - q_\varepsilon\right) - \ln\left(-p_\varepsilon\right) - x_\varepsilon^2 - \frac{1}{2}\sigma^2 x_\varepsilon^2 X + \frac{1}{2}\sigma^2 y_\varepsilon^2 Y \leq 0.
\end{equation}

Our assumption on \( \sigma \) implies that \( p_\varepsilon + q_\varepsilon \leq -1/c^* \). Thus, the difference of the logarithmic terms in the above inequality can be estimated from below as
\[
\ln\left(\frac{p_\varepsilon + q_\varepsilon}{p_\varepsilon + q_\varepsilon}\right) \geq \frac{q_\varepsilon}{p_\varepsilon + q_\varepsilon} \geq -c^*|q_\varepsilon|,
\]
and inequality (4.7) gives
\begin{equation}
\frac{1}{2}\sigma^2 x_\varepsilon^2 X + \frac{1}{2}\sigma^2 y_\varepsilon^2 Y \geq \rho u(x_\varepsilon) - \rho v(y_\varepsilon) + bx_\varepsilon q_\varepsilon - (1 + c^*)|q_\varepsilon| + p_\varepsilon x_\varepsilon - y_\varepsilon \left(1 + \frac{x_\varepsilon}{1 + x_\varepsilon^2 + y_\varepsilon^2}\right) + x_\varepsilon^2 - y_\varepsilon^2
\end{equation}

On the other hand, the right-hand-side in (4.6) yields
\begin{equation}
\frac{1}{2}\sigma^2 x_\varepsilon^2 X - \frac{1}{2}\sigma^2 y_\varepsilon^2 Y \leq \frac{1}{2}\sigma^2 x_\varepsilon^2 \left(\phi_{xx}(x_\varepsilon) + 12\delta(x_\varepsilon - \bar{x})^2\right) + \frac{\sigma^2}{\varepsilon}(x_\varepsilon - y_\varepsilon)^2 + m(\frac{\alpha}{\varepsilon})\varepsilon, \tag{4.9}
\end{equation}
with \( m \) a modulus of continuity independent of \( \alpha, \varepsilon \).

By combining (4.8) with (4.9), we conclude the proof taking first \( \alpha \to 0 \), then \( \varepsilon \to 0 \) and using (4.4). \( \square \)

We continue with the

**Proof of Theorem 2.2** The main step is the construction of a solution of the linearized equation. For this, we consider the ode
\begin{equation}
\rho w + \left(\frac{b x - (1 + c^*)}{2}\right)w' - \frac{1}{2}\sigma^2 x^2 w'' = 0, \tag{4.10}
\end{equation}
which has a solution of the form
\begin{equation}
w(x) = x^{-k}J\left(\frac{2 + 2c^*}{\sigma^2 x}\right), \tag{4.11}
\end{equation}
where \( k \) is a root of
\begin{equation}
k^2 + \left(1 + \frac{2b}{\sigma^2}\right)k - \frac{20}{\sigma^2} = 0, \tag{4.12}
\end{equation}
and \( J \) a solution of the degenerate hypergeometric equation
\begin{equation}
x y'' + (\bar{b} - x)y' - \bar{a}y = 0 \tag{4.13}
\end{equation}
with \( \bar{a} = k \) and \( \bar{b} = 2(k + 1 + b/\sigma^2) \).

Since we are looking for a solution of (4.10) with superquadratic growth at \( +\infty \), we choose \( k \) to be the negative root of (4.12). The assumption \( \sigma^2 < \rho + 2b \) implies \( -k > 2 \).

We further choose \( J \) to be the Tricomi solution of (4.13) which satisfies
\[
J(0) > 0 \quad \text{and} \quad J(x) = x^{-k}\left(1 + \frac{2\rho}{\sigma^2 x} + o(x^{-1})\right) \quad \text{as} \quad x \to \infty.
\]

With this choice, the function \( w \) defined in (4.11) for \( x > 0 \) and by continuity at \( x = 0 \), satisfies \( w(0) = w'(0) > 0 \) and \( w(x) \sim J(0)x^{-k} \), as \( x \to \infty \).

Note that \( w \) is increasing in \([0, \infty)\) since it would otherwise have a positive local maximum and this is impossible by (4.10). In particular, \( w \) satisfies (4.3).

Set now \( \psi = u - v \) and consider \( \epsilon > 0 \). Since \( \psi - \epsilon w < 0 \) in a neighborhood of infinity, there exists \( x^\varepsilon \in [0, \infty) \) such that
\[
\max_{x \geq 0} \left(\psi(x) - \epsilon w(x)\right) = \psi(x^\varepsilon) - \epsilon w(x^\varepsilon).
\]
By Lemma 4.2, \(\psi\) is a subsolution of (4.3). We now use \(\epsilon w\) as a test function to find that

\[
0 \geq \rho \psi(x^*) + \epsilon bx^* w(x^*) - \epsilon (1 + c^*) |w'(x^*)| - \frac{1}{2} \sigma^2 (x^*)^2 w''(x^*) = \psi(x^*) - \epsilon w(x^*).
\]

Hence, \(\psi(x) \leq \epsilon w(x)\) for all \(x \in [0, \infty)\). Since \(\epsilon\) is arbitrary, this proves the claim.

The stability property of viscosity solutions yields the following theorem.

**Theorem 4.1.** As \(\sigma \to 0\), the welfare function \(V\) defined by (4.3) converges locally uniformly to the constrained viscosity solution \(V(d)\) of the deterministic shallow lake equation in \([0, \infty)\).

\[
(OHJB_d) \quad \rho v(d) = \left(\frac{x^2}{x^2 + 1} - bx\right) v'(x) - \left(\ln(-v'(x)) + x^2 + 1\right).
\]

We next prove Theorem 2.3 that describes the asymptotic behaviour of \(V\) as \(x \to \infty\). The proof is based on a scaling argument and the stability properties of the viscosity solutions.

**Proof of Theorem 2.3.** We write \(V\) as

\[
V(x) = -A \left(x + \frac{1}{b + \rho}\right)^2 - \frac{1}{\rho} \ln \left(2A(x + \frac{1}{b + \rho})\right) + K + v(x).
\]

Straightforward calculations yield that \(v\) is a viscosity solution in \((0, \infty)\) of the equation

\[
(4.14) \quad \rho v' + \left(bx - \frac{x^2}{x^2 + 1}\right) v'' + \ln \left(1 + \frac{1 - \rho(x + \frac{1}{b + \rho})v'}{2\rho \left(x + \frac{1}{b + \rho}\right)^2}\right) - \frac{1}{2} \sigma^2 x^2 v'' + f = 0,
\]

where

\[
f(x) = \frac{b + \frac{x^2}{x^2 + 1} (b + \rho) \frac{2x}{x^2 + 1} + \frac{\sigma^2 x (b + \rho)}{2 \rho (1 + x (b + \rho))^2}}{2A \left(1 + x (b + \rho)\right)}.
\]

Note \(f\) is smooth on \([0, \infty)\) and vanishes as \(x \to \infty\).

Let \(v_\lambda(y) = v\left(\frac{y}{\lambda}\right)\) and observe that, if \(v_\lambda(1) \to 0\) as \(\lambda \to 0\), then \(v(x) \to 0\) as \(x \to \infty\). It turns out that \(v_\lambda\) solves

\[
\rho v_\lambda' + \left(bx - \frac{\lambda x^2}{x^2 + \lambda^2}\right) v_\lambda'' + \ln \left(1 + \frac{\lambda^2 (1 - \rho x + \frac{1}{b + \rho}) v_\lambda'}{2\rho \left(x + \frac{1}{b + \rho}\right)^2}\right) - \frac{1}{2} \sigma^2 x^2 v_\lambda'' + f(\frac{x}{\lambda}) = 0.
\]

Since, by (2.7) \(v_\lambda\) is uniformly bounded, we consider the half-relaxed limits \(v_+^* = \limsup_{x \to y, \lambda \to 0} v_\lambda(x)\) and \(v_*^*(y) = \liminf_{x \to y, \lambda \to 0} v_\lambda(x)\) in \((0, \infty)\), which are (see Barles & Perthame ’87) respectively sub- and super-solutions of

\[
(4.15) \quad \rho w + bw' - \frac{1}{2} \sigma^2 y^2 w'' = 0.
\]

It is easy to check that for any \(y > 0\) we have \(v_+^*(y) = \limsup_{x \to \infty} v(x)\) and \(v_*^*(y) = \liminf_{x \to \infty} v(x)\).

The subsolution property of \(v_+^*\) and the supersolution property of \(v_*^*\) give

\[
\limsup_{x \to \infty} v(x) \leq 0 \leq \liminf_{x \to \infty} v(x) \leq \limsup_{x \to \infty} v(x).
\]
5. A NUMERICAL SCHEME AND OPTIMAL DYNAMICS

A general argument to prove the convergence of monotone schemes for viscosity solutions of fully nonlinear second-order elliptic or parabolic, possibly degenerate, partial differential equations has been introduced in Barles & Souganidis ’91. Their methodology has been extensively used to approximate solutions to first-order equations, see for example Rouy & Tourin ’92, Sethian ’99, Kossioris et al. ’99, Qian ’06.

On the other hand, it is not always possible to construct monotone schemes for second-order equations in their full generality. However, various types of nonlinear second-order equations have been approximated via monotone schemes based on Barles & Souganidis ’91; see, for example, Osher & Fedkiw ’03, Barles & Jakobsen ’02, Bonnans & Zidani ’04, Froese & Oberman ’11.

Next, following Kossioris & Zohios ’12 which considered the deterministic problem, we construct a monotone finite difference scheme to approximate numerically the welfare function and recover numerically the stochastic optimal dynamics.

Let \( \Delta x \) denote the step size of a uniform partition \( 0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = l \) of \([0, l]\) for \( l > 0 \) sufficiently large. Having in mind (2.8), if \( V_i \) is the approximation of \( V \) at \( x_i \), we employ a backward finite difference discretization to approximate the first derivative in the linear term of the (OHJB), a forward finite difference scheme for the derivative in the logarithmic term and a central finite difference scheme to approximate the second derivative.

These considerations yield, for \( i = 1, \ldots, N-1 \), the approximate equation

\[
(5.1) \quad V_i - \frac{1}{\rho} \left( \frac{x_i^2}{x_i^2 + 1} - b x_i \right) \frac{V_i - V_{i-1}}{\Delta x} + \frac{1}{\rho} \left[ x_i^2 + 1 + \ln \left( \frac{V_{i+1} - V_i}{\Delta x} \right) \right] \\
- \frac{\sigma^2}{2\rho} \frac{V_{i+1} + V_{i-1} - 2V_i}{(\Delta x)^2} = 0.
\]

Setting

\[
(5.2) \quad g(x, w, c, d) = \left[ (\Delta x)^2 - \frac{1}{\rho} \left( \frac{x^2}{x^2 + 1} - bx \right) \Delta x + \frac{\sigma^2}{\rho} \right] w + \\
\frac{1}{\rho}(x^2 + 1)(\Delta x)^2 + \frac{1}{\rho}(\Delta x)^2 \ln \left( \frac{c - w}{\Delta x} \right) + \frac{1}{\rho} \Delta x \left( \frac{x^2}{x^2 + 1} - bx \right) d - \frac{\sigma^2}{2\rho}(c + d),
\]

the numerical approximation of \( V \) satisfies

\[
(5.3) \quad g(x_i, V_i, V_{i+1}, V_{i-1}) = 0, \quad \text{for} \quad i = 1, \ldots, N-1,
\]

and the consistency is immediate.

For the monotonicity we observe that for two different approximation grid vectors \( (U_0, \ldots, U_N) \) and \( (V_0, \ldots, V_N) \) with \( U_i \geq V_i \) and \( U_i = V_i = w \), we have

\[
(5.4) \quad g(x_i, w, U_{i+1}, U_{i-1}) \leq g(x_i, w, V_{i+1}, V_{i-1}),
\]

provided \( \Delta x \left( \frac{x_i^2}{x_i^2 + 1} - bx \right) \leq \sigma^2/2 \). This condition is satisfied if \( b \geq 0.5 \) or if we take \( \Delta x \leq \sigma^2/2 \).

Since the welfare function solves a state constraint problem the equation is satisfied on the left boundary point.

It follows that the numerical scheme is monotone in the sense of Barles & Souganidis ’91 and converges to the unique constrained viscosity solution.

Since the computational domain of the problem is finite, a boundary condition has to be imposed at \( x = l \), for \( l \) sufficiently large, by exploiting the asymptotic behaviour of the welfare function \( V \) as \( x \to +\infty \). The boundary condition at the right endpoint \( x_N \) is provided by the asymptotic estimate (2.3).

The scheme above suggests a numerical algorithm for the computation of optimal dynamics governing the shallow lake problem. In this direction, the nondegeneracy of the shallow lake equation in \((0, \infty)\)
induces extra regularity for the function $V$ in $(0, \infty)$. Hence, the optimal dynamics for the shallow lake problem are described by

\begin{equation}
(5.5) \quad \begin{cases}
    d\bar{x}(t) = \left(-\frac{1}{V'(\bar{x}(t))} - b\bar{x}(t) + \frac{\bar{x}^2(t)}{\bar{x}^2(t) + 1}\right) dt + \sigma \bar{x}(t) dW(t),
    \\
    \bar{x}(0) = x
\end{cases}
\end{equation}

Using the numerical representation of $V$ via (5.3) and properly discretizing the SDE (5.5), we can reconstruct numerically the optimal dynamics. This is a direct approach to investigate numerically the stochastic properties of the optimal dynamics of the shallow lake problem for the various parameters $\rho, b, c, \sigma$ of the problem.

The exact numerical algorithm for the computation of the constrained viscosity solution along with the numerical study of the optimal dynamics and their stochastic properties for various $\sigma$’s will be presented elsewhere.

**APPENDIX A.**

**Lemma A.1.** Assume that $f$ is a positive $\mathbb{P}$-a.s. locally integrable $\mathcal{F}_t$ and let $M_t(f)$ be defined as in (3.4). Then,

(i) $\mathbb{E} \left[ \int_0^\infty e^{-\rho t} M_t(f) \, dt \right] = \frac{1}{\rho + b} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} f(t) \, dt \right].$

(ii) $\mathbb{E} \left[ \int_0^\infty e^{-\rho t} Z_t M_t(f) \, dt \right] = \begin{cases}
    A \mathbb{E} \left[ \int_0^\infty e^{-\rho t} Z_t f(t) \, dt \right] & \text{if } \sigma^2 < \rho + 2b, \\
    \mathbb{E} \left[ \int_0^\infty e^{-\rho t} f(t) M_t(f) \, dt \right] & \text{if } \sigma^2 \geq \rho + 2b.
\end{cases}$

(iii) $\mathbb{E} \left[ \int_0^\infty e^{-\rho t} M_t^2(f) \, dt \right] = \begin{cases}
    2A \mathbb{E} \left[ \int_0^\infty e^{-\rho t} f(t) M_t(f) \, dt \right] & \text{if } \sigma^2 < \rho + 2b, \\
    \mathbb{E} \left[ \int_0^\infty e^{-\rho t} f(t)(M_t(f))^2 \, dt \right] & \text{if } \sigma^2 \geq \rho + 2b.
\end{cases}$

**Proof:** (i) Since $f$ is $\mathcal{F}_t$-adapted we have

$$
\mathbb{E}[M_t(f)] = \mathbb{E} \left[ \int_0^t \mathbb{E}[Z_s | \mathcal{F}_s] \frac{f(s)}{Z_s} \, ds \right] = \int_0^t e^{-(t-s)} \mathbb{E}[f(s)] \, ds.
$$

Therefore,

$$
\mathbb{E} \left[ \int_0^\infty e^{-\rho t} M_t(f) \, dt \right] = \int_0^\infty e^{bs} \mathbb{E}[f(s)] \int_s^\infty e^{-(\rho + b)t} \, dt \, ds = \frac{1}{\rho + b} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} f(t) \, dt \right].
$$

(ii) Conditioning first on $\mathcal{F}_s$ we have

$$
\mathbb{E}[Z_t M_t(f)] = \mathbb{E} \left[ \int_0^t \mathbb{E}[Z_s^2 | \mathcal{F}_s] \frac{f(s)}{Z_s} \, ds \right] = \int_0^t e^{(\sigma^2 - 2b)(t-s)} \mathbb{E}[Z_s f(s)] \, ds,
$$

and, hence,

$$
\mathbb{E} \left[ \int_0^\infty e^{-\rho t} Z_t M_t(f) \, dt \right] = \int_0^\infty e^{(\sigma^2 - 2b)s} \mathbb{E}[Z_s f(s)] \int_s^\infty e^{-(\rho + 2b - \sigma^2)t} \, dt \, ds = \begin{cases}
    A \mathbb{E} \left[ \int_0^\infty e^{-\rho t} Z_t f(t) \, dt \right] & \text{if } \sigma^2 < \rho + 2b, \\
    \mathbb{E} \left[ \int_0^\infty e^{-\rho t} Z_t f(t) \, dt \right] & \text{if } \sigma^2 \geq \rho + 2b.
\end{cases}
$$
(iii) By Fubini’s theorem we have
\[
E[M_t^2(f)] = 2E \left[ \int_0^t \int_s^t \frac{Z_s^2}{Z_s Z_q} f(s) f(q) \, dq \, ds \right]
\]
\[
= 2E \left[ \int_0^t \int_s^t \frac{1}{Z_s Z_q} E[Z_s^2 | F_q] \frac{1}{Z_s Z_q} f(s) f(q) \, dq \, ds \right]
\]
\[
= 2E \int_0^t \int_s^t e^{(\sigma^2 - 2b)(t-q)} E \left[ \frac{Z_s}{Z_q} f(s) f(q) \right] \, dq \, ds
\]
\[
= 2E \int_0^t e^{(\sigma^2 - 2b)(t-q)} E \left[ f(q) M_q(f) \right] \, dq,
\]
and, therefore,
\[
E \left[ \int_0^\infty e^{-\rho t} M_t^2(f) \, dt \right] = \int_0^\infty e^{(2b - \sigma^2)q} E \left[ f(q) M_q(f) \right] \int_q^\infty e^{-(\rho + 2b - \sigma^2)q} \, dq \, dt
\]
\[
= \begin{cases} 
2A \int_0^\infty e^{-\rho t} f(t) M_t(f) \, dt & \text{if } \sigma^2 < \rho + 2b, \\
\infty & \text{if } \sigma^2 \geq \rho + 2b. \quad \Box
\end{cases}
\]

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