Variance Estimation For Dynamic Regression via Spectrum Thresholding

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Abstract
We consider the dynamic linear regression problem, where the predictor vector may vary with time. This problem can be modeled as a linear dynamical system, where the parameters that need to be learned are the variance of both the process noise and the observation noise. While variance estimation for dynamic regression is a natural problem, with a variety of applications, existing approaches to this problem either lack guarantees or only have asymptotic guarantees without explicit rates. In addition, all existing approaches rely strongly on Gaussianity of the noises. In this paper we study the global system operator: the operator that maps the noise vectors to the output. In particular, we obtain estimates on its spectrum, and as a result derive the first known variance estimators with finite sample complexity guarantees. Moreover, our results hold for arbitrary sub Gaussian distributions of noise terms. We evaluate the approach on synthetic and real-world benchmarks.

1 Introduction
A dynamic linear regression (West and Harrison [1997] Chapter 3), or non-stationary regression, is a situation where we are given a sequence of scalar observations \( \{Y_t\}_{t \leq T} \subset \mathbb{R} \), and observation vectors \( \{u_t\}_{t \leq T} \subset \mathbb{R}^n \) such that \( Y_t = \langle X_t, u_t \rangle + z_t \) where \( X_t \in \mathbb{R}^n \) is a regressor vector, and \( z_t \) a random noise term. In contrast to a standard linear regression, the vector \( X_t \) may change with time. One common objective for this problem is at time \( T \), to estimate the trajectory of \( X_t \) for \( t \leq T \), given the observation vectors and observations, \( \{u_t\}_{t \leq T}, \{Y_t\}_{t \leq T} \), and possibly to forecast \( Y_{T+1} \) if \( u_{T+1} \) is also known.

In this paper we model the problem as follows:

\[
X_{t+1} = X_t + h_t \quad (1)
\]

\[
Y_t = \langle X_t, u_t \rangle + z_t \quad (2)
\]
where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$, $z_t$, the observation noise, are zero-mean sub Gaussian random variables, with variance $\eta^2$, and the process noise variables $h_t$ take values in $\mathbb{R}^n$, such that coordinates of $h_t$ are sub Gaussian, independent, and have variance $\sigma^2$. All $h_t$ and $z_t$ variables are assumed to be mutually independent. The vectors $u_t$ are an arbitrary sequence in $\mathbb{R}^n$, and the observed, known, quantities at time $T$ are $\{Y_t\}_{t \leq T}$ and $\{u_t\}_{t \leq T}$.

The system (1)-(2) is a special case of a Linear Dynamical System (LDS). As is well known, when the parameters $\sigma, \eta$ are given, the mean-squared loss optimal forecast for $Y_{T+1}$ and estimate for $X_T$ are obtained by the Kalman Filter (Anderson and Moore, 1979; Hamilton, 1994; Chui and Chen, 2017). In this paper we are concerned with estimators for $\sigma, \eta$, and sample complexity guarantees for these estimators.

Let us first make a few remarks about the particular system (1)-(2). First, as a natural model of time varying regression, this system is useful in a considerable variety of applications. We refer to West and Harrison (1997), Chapter 3, for numerous examples. In addition, an application to electricity consumption time series as a function of the temperature is provided in the experiments section of this paper. Second, one may regard the problem of estimating $\sigma, \eta$ in (1)-(2) as a pure case of finding the optimal learning rate for $X_t$. Indeed, the Kalman filter equations for (1)-(2), are given by (3)-(4) below, where (3) describes the filtered covariance update and (4) the filtered state update. Here $\bar{x}_t$ is the estimated state, given the observations $Y_1, \ldots, Y_t$, see West and Harrison (1997).

\begin{align*}
C_{t+1} &= \frac{\eta^2}{(C_t + \sigma^2 I)u_{t+1}, u_{t+1}) + \eta^2 (C_t + \sigma^2 I)} \quad (3) \\
\bar{x}_{t+1} &= \bar{x}_t + \frac{C_{t+1}}{\eta^2} u_{t+1} \cdot (Y_{t+1} - \langle \bar{x}_t, u_{t+1} \rangle) \quad (4)
\end{align*}

In particular, following (4), the role of $\sigma$ and $\eta$ may be interpreted as regulating how much the estimate of $\bar{x}_{t+1}$ is influenced, via the operator $\frac{C_{t+1}}{\eta^2}$, by the most recent observation and input $Y_{t+1}, u_{t+1}$. Roughly speaking, higher values of $\sigma$ or lower values of $\eta$ would imply that the past observations are given less weight, and result in an overfit of the forecast to the most recent observation (an illustration is given in Figure 3, Appendix A). On the other hand, very low $\sigma$ or high $\eta$ would make the problem similar to the standard linear regression, where all observations are given equal weight, and result in a lag of the forecast. Finally, it is worth mentioning that the system (1)-(2) is closely related to the study of online gradient (OG) methods (Zinkevich, 2003; Hazan, 2016). In this field, assuming quadratic cost, one considers the update

\begin{equation}
\bar{x}_{t+1} = \bar{x}_t + \alpha \cdot u_{t+1} \cdot (Y_{t+1} - \langle \bar{x}_t, u_{t+1} \rangle), \quad (5)
\end{equation}

where $\alpha$ is the learning rate, and studies the performance guarantees of the forecaster $\langle \bar{x}_t, u_{t+1} \rangle$. Compared to (1), the update (5) is simpler, and uses a scalar rate $\alpha$ instead of the input-dependent operator rate $\frac{C_{t+1}}{\eta^2}$ of the Kalman filter. However, due to the similarity, every domain of applicability of the OG methods is also a natural candidate for the model (1)-(2) and vice-versa.
As an illustration, we compare the OG to Kalman filter based methods with learned $\sigma, \eta$ in the experiments section.

In this paper we introduce a new estimation algorithm for $\sigma, \eta$, termed STVE (Spectrum Thresholding Variance Estimator), and prove finite sample complexity bounds for it. In particular, our bounds are an explicit function of the parameters $T$ and $\{u_i\}_{i=1}^T$ for any finite $T$, and indicate that the estimation error decays roughly as $T^{-\frac{1}{2}}$, with high probability. To the best of our knowledge, these are the first bounds of this kind. As we discuss in detail in Section 2, most existing estimation methods for LDSs, such as subspace identification (van Overschee and de Moor, 1996), or improper learning (Anava et al., 2013; Hazan et al., 2017; Kozdoba et al., 2019), do not apply to the system (1)-(2), due to non-stationarity. On the other hand, the methods that do apply to (1)-(2) lack guarantees and rely strongly on Gaussianity of the noises.

Moreover, our approach differs significantly from the existing methods. We show that the structure of equations (1)-(2) is closely related to, and inherits several important properties from the classical discrete Laplacian operator on the line — a line of reasoning that was not explored in the literature so far. In particular, we use this connection to show that an appropriate inversions of the system produce estimators that are concentrated enough so that $\sigma$ and $\eta$ may be recovered.

The rest of the paper is organized as follows: The related work is discussed in Section 2. In Section 3 we describe in general lines the methods and main results of this paper. The main technical statements are given in the Section 4. In Section 5 we present experimental results on synthetic and real data, and conclusions and future work are discussed in Section 6. Due to space constraints, while we outline the main arguments in the text, the full proofs are deferred to the Appendix.

2 Literature

We refer to Chui and Chen (2017); Hamilton (1994); Anderson and Moore (1979); Shumway and Stoffer (2011) for a general background on LDSs, the Kalman Filter and maximum likelihood estimation.

Existing approaches to the variance estimation problem may be divided into three categories: (i) General methods for parameter identification in LDS, either via maximum likelihood estimation (MLE) (Hamilton, 1994), or via subspace identification (van Overschee and de Moor, 1996). (ii) Methods designed specifically to learn the noise parameters of the system, developed primarily in the control theory community, in particular via the innovation auto-correlation function, such as the classical Mehra (1970); Belanger (1974), or for instance more recent Wang et al. (2017); Dunik et al. (2018). (iii) Improper Learning methods, such as Anava et al. (2013); Hazan et al. (2017); Kozdoba et al. (2019). In these approaches, one does not learn the LDS directly, but instead learns a model from a certain auxiliary class and shows that this auxiliary model produces forecasts that are as good as the forecasts of an LDS with “optimal” parameters.
Despite the apparent simplicity of the system (1)-(2), most of the above methods do not apply to this system. This is due to the fact that most of the methods are designed for time invariant, asymptotically stationary systems, where the observation operator (\(u_t\) in our notation) is constant and the Kalman gain (or, equivalently \(C_t\) in eq. (3)) converges with \(t\). However, if the observation vector sequence \(u_t\) changes with time – a necessary property for the dynamic regression problem – the system will no longer be asymptotically stationary. In particular, due to this reason, neither the subspace identification methods, nor any of the improper learning approaches above apply to system (1)-(2).

Among the methods that do apply to (1)-(2) are the general MLE estimation, and some of the auto-correlation methods (Belanger, 1974; Dunik et al., 2018). On one hand, both types of approaches may be applicable to systems apriori more general than (1)-(2). On the other hand, the situation with consistency guarantees – the guarantee that one recovers true parameters given enough observations – for these methods is somewhat complicated. Due to the non-convexity of the likelihood function, the MLE method is not guaranteed to find the true maximum, and as a result the whole method has no guarantees. The results in Belanger (1974); Dunik et al. (2018) do have asymptotic consistency guarantees. However, these rely on some explicit and implicit assumptions about the system, the sequence \(u_t\) in our case, which can not be easily verified. In particular, Belanger (1974); Dunik et al. (2018) assume uniform observability of the system, which we do not assume, and in addition rely on certain assumption about invertibility and condition number of the matrices related to the sequence \(u_t\). Moreover, even if one assumes that the assumptions hold, the results are purely asymptotic, and for any finite \(T\), do not provide a bound of the expected estimation error as a function of \(T\) and \(\{u_t\}_{t=1}^T\).

In addition, as mentioned earlier, MLE methods by definition must assume that the noises are Gaussian (or belong to some other predetermined parametric family) and autocorrelation based methods also strongly use the Gaussianity assumption. Our approach, on the other hand, requires only sub Gaussian noises with independent coordinates. It is an interesting question whether our approach may be extended further, by removing the independent coordinates assumption. The operator analysis part of this paper does not depend on the distribution of the noises. Therefore, to achieve such an extension, one would only need to correspondingly extend our main probabilistic tool, the Hanson-Wright inequality (Hanson and Wright, 1971; Rudelson et al., 2013, see also Section 3 and Appendix E). It is worth noting that such an extension was recently obtained in Hsu et al. (2012), but only for upper deviations, rather than for concentration around the mean. In this form, it is not sufficient for the purposes of this work.

3 Overview of the approach

We begin by rewriting (1)-(2) in a vector form. To this end, we first encode sequences of \(T\) vectors in \(\mathbb{R}^n\), \(\{a_t\}_{t \leq T} \subset \mathbb{R}^n\), as a vector \(a \in \mathbb{R}^{Tn}\), constructed
by concatenation of $a_t$’s. Next, we define the summation operator $S' : \mathbb{R}^T \to \mathbb{R}^T$ which acts on any vector $(h_1, h_2, \ldots, h_T) \in \mathbb{R}^T$ by

$$S'(h_1, h_2, \ldots, h_T) = (h_1, h_1 + h_2, \ldots, \sum_{i \leq T-1} h_i, \sum_{i \leq T} h_i).$$

(6)

Note that $S'$ is an invertible operator. Next, we similarly define the summation operator $S : \mathbb{R}^{Tn} \to \mathbb{R}^{Tn}$, an $n$-dimensional extension of $S'$, which sums $n$-dimensional vectors. Formally, for $(h_i)_{i=1}^n \in \mathbb{R}^{Tn}$, and for $1 \leq j \leq n, 1 \leq t \leq T$, $(Sh)_{(t-1)n+j} = \sum_{i \leq t} h_{(i-1)n+j}$. Observe that if the sequence of process noise terms $h_1, \ldots, h_T \in \mathbb{R}^n$ is viewed as a vector $h \in \mathbb{R}^{Tn}$, then by definition $Sh$ is the $\mathbb{R}^{Tn}$ encoding of the sequence $X_t$.

Next, given a sequence of observation vectors $u_1, \ldots, u_T \in \mathbb{R}^n$, we define the observation operator $O_u : \mathbb{R}^{Tn} \to \mathbb{R}^T$ by $(O_u x)_t = \langle u_t, \langle x_{(t-1)n+1}, \ldots, x_{(t-1)n+n} \rangle \rangle$. In words, coordinate $t$ of $O_u x$ is the inner product between $u_t$ and $t$-th part of the vector $x \in \mathbb{R}^{Tn}$. Define also $Y = (Y_1, \ldots, Y_T) \in \mathbb{R}^T$ to be the concatenation of $Y_1, \ldots, Y_T$. With this notation, one may equivalently rewrite the system (1)-(2) as follows:

$$Y = O_u Sh + z,$$

(7)

where $h$ and $z$ are independent zero-mean random vectors in $\mathbb{R}^{Tn}$ and $\mathbb{R}^T$ respectively, with independent sub Gaussian coordinates. The variance of each coordinate of $h$ is $\sigma^2$ and each coordinate of $z$ has variance $\eta^2$.

Up to now, we have reformulated our data model as a single vector equation. Note that in that equation, the observations $Y$ and both operators $O_u$ and $S$ are known to us. Our problem may now be reformulated as follows: Given $Y \in \mathbb{R}^T$, assuming $Y$ was generated by (7), provide estimates of $\sigma, \eta$.

As a motivation, we first consider taking the expectation of the norm squared of eq. (7). For any operator $A : \mathbb{R}^m \to \mathbb{R}^m$ and zero-mean vector $h$ with independent coordinates and coordinate variance $\sigma^2$, we have $E |Ah|^2 = \|A\|_{HS}^2 \sigma^2$, where $\|A\|_{HS}$ is the Hilbert-Schmidt (or Frobenius) norm of $A$. Taking the norm and expectation of (7), and dividing by $T^2$, we thus obtain

$$\frac{E |Y|^2}{T^2} = \frac{\|O_u S\|_{HS}^2 \sigma^2}{T^2} + \frac{T}{T^2} \eta^2.$$

(8)

Next, note that $\|O_u S\|_{HS}^2$ is known, and an elementary computation shows that $\frac{\|O_u S\|_{HS}^2}{T^2}$ is of constant order (as a function of $T$; see (25)), while the coefficient of $\eta^2$ is $\frac{1}{T}$. Thus, if the quantity $\frac{|Y|^2}{T^2}$ were close enough to its expectation with high probability, we could take this quantity as a (slightly biased) estimator of $\sigma^2$. However, as it will become apparent later, the deviations of $\frac{|Y|^2}{T^2}$ around the expectation are also of constant order, and thus $\frac{|Y|^2}{T^2}$ cannot be used as an estimator. The reason for these high deviations of $\frac{|Y|^2}{T^2}$ is that the spectrum of $O_u S$ is extremely peaked. The highest squared singular value of $O_u S$ is of order $T^2$, the same order as sum of all of them, $\|O_u S\|_{HS}^2$. Contrast this with the case...

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of identity operator, \( \text{Id} : \mathbb{R}^{Tn} \to \mathbb{R}^{Tn} \): We have \( \mathbb{E} |\text{Id}(h)|^2 = \mathbb{E} |h|^2 = Tn\sigma^2 \), and one can also easily show that, for instance, \( \text{Var} |\text{Id}(h)|^2 = Tn\sigma^2 \), and thus the deviations are of order \( \sqrt{Tn\sigma} \) – a smaller order than \( \mathbb{E} |\text{Id}(h)|^2 \). While for the identity operator the computation is elementary, for a general operator \( A \) the situation is significantly more involved, and the bounds on the deviations of \( |Y|^2 \) will be obtained from the Hanson-Wright inequality [Hanson and Wright, 1971; see also Rudelson et al. (2013)], combined with standard norm deviation bounds for isotropic sub Gaussian vectors.

With these observations in mind, we proceed to flatten the spectrum of \( O_u S \) by taking the pseudo-inverse. Let \( R : \mathbb{R}^T \to \mathbb{R}^{Tn} \) be the pseudo-inverse, or Moore-Penrose inverse of \( O_u S \). Specifically, let

\[
O_u S = U \circ \text{Diag}(\gamma_1, \ldots, \gamma_T) \circ W,
\]

be the singular value decomposition of \( O_u S \), where \( \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_T \) are the singular values.

For the rest of the paper, we will assume that all of the observation vectors \( u_t \) are non-zero. This assumption is made solely for notational simplicity and may easily be avoided, as discussed later in this section. Under this assumption, since \( S \) is invertible and \( O_u \) has rank \( T \), we have \( \lambda_t > 0 \) for all \( t \leq T \). For \( i \leq T \), denote \( \chi_i = \gamma_{T+1-i}^{-1} \). Then \( \chi_i \) are the singular values of \( R \), arranged in a non-increasing order, and we have by definition

\[
R = W^* \circ \text{Diag}(\chi_T, \chi_{T-1}, \ldots, \chi_2, \chi_1) \circ U^*,
\]

where \( W^*, U^* \) denote the transposed matrices of \( U, V \), defined in (9). Similarly to Eq. (8), we apply \( R \) to (7), and since \( \|RO_u S\|_{HS} = T \), by taking the expectation of the squared norm we obtain

\[
\frac{|RY|^2}{T} = \sigma^2 + \frac{\|R\|^2}{HS} \eta^2 + \left( \frac{|RY|^2}{T} - \mathbb{E} \frac{|RY|^2}{T} \right).
\]

In this equation, the deviation term \( \left( \frac{|RY|^2}{T} - \mathbb{E} \frac{|RY|^2}{T} \right) \) is of order \( O(\frac{1}{\sqrt{T}}) \) with high probability (Theorem 1). Moreover, the coefficient of \( \sigma^2 \) is 1, and the coefficient of \( \eta^2 \), which is \( \frac{\|R\|_{HS}^2}{T} \), is of order at least \( \Omega(\frac{\log^2 T}{T}) \) (Theorem 7) – much larger order than \( \frac{1}{\sqrt{T}} \). Since \( |RY|^2 \) and \( \|R\|_{HS}^2 \) are known, it follows that we have obtained one equation satisfied by \( \sigma^2 \) and \( \eta^2 \) up to an error of \( \frac{1}{\sqrt{T}} \), where both coefficients are of order larger than the error.

Next, we would like to obtain another linear relation between \( \sigma^2, \eta^2 \). To this end, choose some \( p = \alpha T \), where \( 0 < \alpha < 1 \) is of constant order. The possible choices of \( p \) are discussed later in this section. We define an operator \( R' : \mathbb{R}^T \to \mathbb{R}^{Tn} \) to be a version of \( R \) truncated to the first \( p \) singular values. If \( R' = W^* \circ \text{Diag}(0, \ldots, \chi_p, \chi_{p-1}, \ldots, \chi_1) \circ U^* \),
Algorithm 1: Spectrum Thresholding Variance Estimator (STVE)

1: **Input:** Observations $Y_t$, observation vectors $u_t$, with $t \leq T$, and $p = \alpha T$.
2: Compute the SVD of $O_u S$,

$$O_u S = U \circ \text{Diag}(\gamma_1, \ldots, \gamma_T) \circ W,$$

where $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_T > 0$. Denote $\chi_i = \gamma_{T+1-i}$ for $1 \leq i \leq T$.
3: Construct the operators

$$R = W^* \circ \text{Diag}(\chi_T, \ldots, \chi_1) \circ U^*$$

and

$$R' = W^* \circ \text{Diag}(0, 0, \ldots, 0, \chi_p, \ldots, \chi_1) \circ U^*$$

4: Produce the estimates:

$$\hat{\eta}^2 = \left( \frac{|R'Y|^2}{p} - \frac{|RY|^2}{T} \right) / \left( \frac{\|R'\|_{HS}^2}{p} - \frac{\|R\|_{HS}^2}{T} \right) \quad (13)$$

$$\hat{\sigma}^2 = \frac{|RY|^2}{T} - \frac{\|R\|_{HS}^2}{T} \hat{\eta}^2.$$

Similarly to the case for $R$, we have

$$\frac{|R'Y|^2}{p} = \sigma^2 + \frac{\|R'\|_{HS}^2}{p} \eta^2 + \left( \frac{|R'Y|^2}{p} - \frac{\mathbb{E}|R'Y|^2}{p} \right). \quad (12)$$

The deviations in (12) are also described by Theorem 1. Note also that since $\|R'\|_{HS}^2$ is the sum of $p$ largest squared singular values of $R$, by definition it follows that $\frac{\|R'\|_{HS}^2}{p} \geq \frac{\|R\|_{HS}^2}{T}$. Now, given two equations in two unknowns, we can solve the system to obtain the estimates $\hat{\sigma}^2$ and $\hat{\eta}^2$. The full procedure is summarized in Algorithm 1, and the bounds implied by Theorem 1 on the estimators $\hat{\sigma}^2$ and $\hat{\eta}^2$ are given in Corollary 2. We first state these results, and then discuss in detail the various parameters appearing in the bounds.

**Theorem 1.** Consider a random vector $Y \in \mathbb{R}^T$ of the form $Y = O_u S h + z$ where $h \sim ISG_{T \alpha}(\sigma, \kappa)$ and $z \sim ISG_T(\eta, \kappa)$. Set $|u_{min}| = \min_t |u_t|$. Then for

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1 Sub-Gaussian vectors with independent coordinates. See Section 4.
any $0 < \delta < 1$,

\[
P \left( \left| \frac{|RY|^2}{T} - \left( \sigma^2 + \frac{\|R\|_{HS}^2}{T} \eta^2 \right) \right| \geq c \frac{(1 + \kappa^2) \left( 1 + \left| u_{\min} \right|^{-2} \right) \log \frac{1}{\delta}}{\sqrt{T}} \right) \leq 4\delta.
\]  

(14)

\[
P \left( \left| \frac{|R'|^2}{p} - \left( \sigma^2 + \frac{\|R'\|_{HS}^2}{p} \eta^2 \right) \right| \geq c \frac{(1 + \kappa^2) \left( 1 + \left| u_{\min} \right|^{-2} \right) \log \frac{1}{\delta}}{\sqrt{p}} \right) \leq 4\delta.
\]  

(15)

The bounds on the estimators of Algorithm 1, stated in Corollary 2, are a Corollary to Theorem 1 and to Theorem 7, which is stated in Section 4.

Corollary 2. Let $\hat{\sigma}^2, \hat{\eta}^2$ be the estimators of $\sigma^2, \eta^2$ obtained from Algorithm 1 with $p \geq \frac{\bar{T}}{4}$. Set $|u_{\max}| = \max_t |u_t|$. Then for any $0 < \delta < 1$, with probability at least $1 - 8\delta$,

\[
\left| \hat{\sigma}^2 - \sigma^2 \right| \leq c \frac{(1 + \kappa^2) \left( 1 + \left| u_{\min} \right|^{-2} \right) \log \frac{1}{\delta}}{\sqrt{T}} \left( 1 - \frac{p \|R\|_{HS}^2}{T \|R'\|_{HS}^2} \right)^{-1},
\]  

(16)

\[
\left| \hat{\eta}^2 - \eta^2 \right| \leq c \frac{(1 + \kappa^2) \left( 1 + \left| u_{\min} \right|^{-2} \right) \log \frac{1}{\delta}}{\sqrt{T}} \left( 1 - \frac{p \|R\|_{HS}^2}{T \|R'\|_{HS}^2} \right)^{-1} n^2 |u_{\max}|^2 \log^2 T.
\]  

(17)

We first discuss the assumption $|u_{\min}| > 0$. This assumption is made solely for notational convenience, as detailed below. To begin, note that some form of lower bound on the norms of the observation vectors $u_t$ must appear in the bounds. This is simply because if one had $u_t = 0$ for all $T$, then clearly no estimate of $\sigma$ would have been possible. On the other hand, our use of the smallest value $|u_{\min}|$ may seem restrictive at first. We note however, that instead of considering the observation operator $O_u : \mathbb{R}^{Tn} \rightarrow \mathbb{R}^T$, one may consider the operator $O_{\bar{u}} : \mathbb{R}^{Tn} \rightarrow \mathbb{R}^{T}$ for any subsequence $\{\bar{u}_t\}_{t=1}^{\bar{T}}$. The observation vector $Y$ would be correspondingly restricted to the subsequence of indices. This allows us to treat missing values and to exclude any outlier $u_t$ with small norms. All the arguments in Theorems 1 and 7 hold for this modified $O_{\bar{u}}$ without change. The only price that will be paid is that $T$ will be replaced by $\bar{T}$ in the bounds. Moreover, we note that typically we have $|u_t| \geq 1$ by construction, see for instance Section 5.2. Additional discussion of missing values may be found in Appendix H.

Next, up to this point, we have obtained two equations, (11)-(12), in two unknowns, $\sigma^2, \eta^2$. Note that in order to be able to obtain $\eta^2$ from these equations, at least one of the coefficients of $\eta^2$, either $\frac{\|R\|_{HS}^2}{T}$ or $\frac{\|R'\|_{HS}^2}{p}$ must be of larger order than $\frac{1}{\sqrt{T}}$, the order of deviations. Providing lower bounds on
these quantities is one of the main technical contributions of this work. This analysis uses the connection between the operator $S$ and the Laplacian on the line, and resolves the issue of translating spectrum estimates for the Laplacian into the spectral estimates for $R$. We note that there are no standard tools to study the spectrum of $R$, and our approach proceeds indirectly via the analysis of the nuclear norm of $O_u S$. These results are stated in Theorem 7 below. In particular, we show that $\frac{\|R\|^2}{T}$ is $\Omega(\frac{1}{\log^2 T})$, which is the source of the log factor in (17).

Finally, in order to solve the equations (11)-(12), not only the equations must have large enough coefficients, but the equations must be different. This is reflected by the term $\left(1 - \frac{p\|R\|^2}{T\|R\|^2} \right)^{-1}$ in (16), (17). Equivalently, while $\frac{\|R\|^2}{p\|R\|^2 T} \geq 1$ by definition, we would like to have

$$\frac{\|R\|^2}{p\|R\|^2 T} \geq 1 + \text{const}$$

for the bounds (16), (17) to be stable. Note that since both $\|R\|^2$ and $\|R\|^2$ are computed in Algorithm 1, the condition (18) can simply be verified before the estimators $\hat{\sigma}^2$, $\hat{\eta}^2$ are returned.

It is worth emphasizing that for simple choices of $p$, say $p = \frac{1}{4} T$, the condition (18) does hold in practice. Note that, for any $p$, we can have $\frac{\|R\|^2}{p\|R\|^2 T} = 1$ only if the spectrum of $R$ is constant. Thus (18) amounts to stating that the spectrum of $R$ exhibits some decay. As we show in experiments below, the spectrum (squared) of $R$, for $u_t$ derived from daily temperature features, or for random Gaussian $u_t$, indeed decays. See Section 5, Figures 1a and 1b. In particular, in both cases (18) holds with const $> 1$.

While the quantity $\left(1 - \frac{p\|R\|^2}{T\|R\|^2} \right)^{-1}$ may simply be taken as a parameter of the problem, which measures the decay of $R$, it is nevertheless desirable to have some simpler bounds on this quantity. Here we provide a partial result in this direction, which bounds $\left(1 - \frac{p\|R\|^2}{T\|R\|^2} \right)^{-1}$ under an additional condition on $u_t$. 

**Proposition 3.** Given the sequence $\{u_t\}_{t=1}^T \subset \mathbb{R}^n$, define the scalar sequence $\tilde{u}_t = \sum_{i \leq n} u_{ti}$ and set

$$|\tilde{u}_{\min}| = \min_{t} |\tilde{u}_t| \text{ and } |\tilde{u}_{\max}| = \max_{t} |\tilde{u}_t|.$$

Then for $p = \frac{1}{4} T$,

$$\frac{\|R\|^2}{p} - \frac{\|R\|^2}{T} \geq c \left( \frac{|\tilde{u}_{\min}|}{n |\tilde{u}_{\max}|} \right) \frac{\|R\|^2}{T}.$$  

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The general idea behind the proof of Proposition 3 is to show that for \( n = 1 \), the ratio \( \left( 1 - \frac{\|R\|_2}{\|R\|_2} \right)^{-1} \) can be controlled. This is done in Lemmas 11 and 12 in Appendix G. In particular, Lemma 11 is a general statement about integrals of monotone real functions under certain order constraints, and Lemma 12 provides a relation of the spectrum of \( R \) to that of \( S^{-1} \). It is then shown that for arbitrary \( n \), \( O_\mu S \) contains a certain copy of an \( n = 1 \)-dimensional operator with parameters \( \tilde{u}_t \), which implies the bounds.

### 4 Properties of \( O_\mu S \) and \( R \)

In this section we introduce the necessary notation, and state the key properties of the operators \( O_\mu S \) and \( R \) defined in Section 3. We refer to Bhatia (1997) and Vershynin (2018) as general references on the notation introduced below, for operators and sub Gaussian variables, respectively.

Let \( A : \mathbb{R}^n \to \mathbb{R}^m \) be an operator with a singular value decomposition \( A = U \cdot \text{Diag}(\lambda_1, \ldots, \lambda_s) \cdot W \), where \( s \leq \min\{m, n\} \) and \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s > 0 \). Note that singular values are strictly positive by definition (that is, vectors corresponding to the kernel of \( A \) do not participate in the decomposition \( A = U \cdot \text{Diag}(\lambda_1, \ldots, \lambda_s) \cdot W \)). The Hilbert-Schmidt (Frobenius), nuclear and operator norms are defined, respectively, as

\[
\|A\|_{HS} = \sqrt{\sum_{i=1}^{s} \lambda_i^2}, \quad \|A\|_{nuc} = \sum_{i=1}^{s} \lambda_i, \quad \|A\|_{op} = \lambda_1.
\]  

(21)

A centered (\( \mathbb{E}X = 0 \)) random variable \( X \) is sub-Gaussian with constant \( \kappa \), denoted \( X \sim SG(\kappa) \), if for all \( t > 0 \) it satisfies

\[
\mathbb{P}(|X| > t) \leq 2 \exp \left( -\frac{t^2}{\kappa^2} \right).
\]  

(22)

A random vector \( X = (X_1, \ldots, X_m) \) is \( \kappa \) sub-Gaussian, denoted \( X \sim SG_m(\kappa) \), if for every \( v \in \mathbb{R}^m \) with \( |v| = 1 \) the random variable \( \langle v, X \rangle \) is \( \kappa \) sub-Gaussian. A random vector \( X \) is \( \sigma \)-isotropic if for every \( v \in \mathbb{R}^m \) with \( |v| = 1 \), \( \mathbb{E}\langle v, X \rangle = \sigma^2 \).

Finally, a random vector \( X = (X_1, \ldots, X_m) \) is \( \sigma \)-isotropically \( \kappa \) sub-Gaussian with independent components, denoted \( X \sim ISG_m(\sigma, \kappa) \) if \( X_i \) are independent, and for all \( i \leq m \), \( \mathbb{E}X_i = 0 \), \( \mathbb{E}X_i^2 = \sigma^2 \) and \( X_i \sim SG(\kappa) \). Clearly, if \( X \sim ISG_m(\sigma, \kappa) \) then \( X \) is \( \sigma \)-isotropic. Recall also that \( X \sim ISG_m(\sigma, \kappa) \) implies \( X \sim SG_m(\kappa) \) (Vershynin, 2018). The noise variables we discuss in this paper are \( ISG(\kappa, \sigma) \).

Finally, throughout the paper, absolute constants are denoted by \( c, c', c'' \), etc. Their values may change from line to line.

The main result of this Section is Theorem 7. To prove it we require several Lemmas. First, we obtain the following bounds on the spectrum of \( O_\mu S \).
Lemma 4. The singular values of $O_uS$ satisfy the following:

\begin{align}
\lambda_1(O_uS) &\leq |u_{\text{max}}| \cdot T \quad \text{and} \quad \lambda_T(O_uS) \geq \frac{1}{2} |u_{\text{min}}|, \\
\|O_uS\|_{\text{nu}} &= \sum_{t \leq T} \lambda_t(O_uS) \leq 4n |u_{\text{max}}| T \log T, \\
\frac{1}{4} |u_{\text{min}}|^2 T^2 &\leq \|O_uS\|_{\text{HS}}^2 = \sum_{t \leq T} t |u_t|^2 \leq \frac{1}{2} |u_{\text{max}}|^2 T^2. 
\end{align}

The proof of this lemma uses the following auxiliary result: Let $D_T : \mathbb{R}^T \to \mathbb{R}^{T-1}$ be the difference operator, $(D_T x)_t = x_{t+1} - x_t$ for $t \leq T - 1$. In the field of Finite Difference methods, the operator $D_T D_T^*$ is known as the Laplacian on the line, or as the discrete derivative with Dirichlet boundary conditions, and is well studied. The eigenvalues of $D_T D_T^*$ may be derived by a direct computation, and correspond to the roots of the Chebyshev polynomial of second kind of order $T$. In particular, the following holds:

Lemma 5. The operator $D_T$ has kernel of dimension 1 and singular values

\begin{align}
\lambda_l(D_T) &= 2 \sin \left( \frac{\pi(T-l)}{2T} \right) \quad \text{for} \quad l = 1, \ldots, T-1.
\end{align}

We refer to [Mitchell and Griffiths (1980)] for the proof of Lemma 5. Next, in Lemma 6, we show that the inverse of the operator $S'$, defined in (6), is a one dimensional perturbation of $D_T$, which implies bounds on singular values of $S'^{-1}$.

Lemma 6. The singular values of $S'^{-1}_T$ satisfy

\begin{align}
2 \sin \left( \frac{\pi(T-t)}{2(T+1)} \right) &\leq \lambda_t(S'^{-1}_T) \leq 2 \sin \left( \frac{\pi(T+1-t)}{2(T+1)} \right), \\
\frac{1}{T} &\leq \lambda_T(S'^{-1}_T) \leq 2 \sin \left( \frac{\pi}{2(T+1)} \right),
\end{align}

for $1 \leq t \leq T - 1$, and

With the estimates of Lemma 6, the bounds in Lemma 4 follow easily.

Finally, as discussed in Section 3, the following bounds play an essential role in the analysis of the variance estimators:

Theorem 7. Let $R : \mathbb{R}^T \to \mathbb{R}^{Tn}$ be the pseudoinverse of $O_uS$. Then

\begin{align}
\frac{1}{n |u_{\text{max}}| \log T} &\leq \|R\|_{\text{op}} \leq 2 |u_{\text{min}}|^{-1}, \\
\frac{1}{n^2 |u_{\text{max}}|^2 \log^2 T} &\leq \|R\|_{\text{HS}}^2 \leq 4 |u_{\text{min}}|^{-2} T, \\
\frac{1}{n^4 |u_{\text{max}}|^4 \log^4 T} &\leq \|R^* R\|_{\text{HS}}^2 \leq 16 |u_{\text{min}}|^{-4} T.
\end{align}

The main technical issues in Theorem 7 is obtaining lower bounds on $\|R\|_{\text{HS}}^2$. Our approach is rather indirect, and we obtain these bounds from the nuclear norm bound (24) via a Markov type inequality.
5 Experiments

In this section we empirically evaluate the STVE algorithm. In the Synthetic Data section we demonstrate the method on a simulated system with Gaussian random inputs $u_t$. We verify in particular that the estimation error decays roughly at the rate of $T^{-\frac{1}{2}}$ as $T$ grows, as predicted by the theory. In Temperatures and Electricity Consumption section we study the relation between daily temperatures and electricity consumption via the dynamic regression model. We compare the following methods: a stationary regression, an online gradient, and a Kalman filter for a dynamic regression, with parameters learned via MLE or STVE. We find that the Kalman filter methods provide the best performance, with no significant difference between STVE and MLE derived systems.

5.1 Synthetic Data

We generated synthetic data by the LDS (1)-(2) where $\sigma^2 = 0.5, \eta^2 = 2, n = 5.$ The values of $u_t$ were sampled from $N(0, I)$. We then run the STVE algorithm for different values of $T$, where for each $T$ we sampled the data 150 times. Figure 1c shows the average (over 150 runs) estimation error for both process and observation noise variances for various values of $T$. As expected from the bounds in Corollary 2, it may be observed in Figure 1c that the estimation errors decay roughly at the rate of $const/\sqrt{T}$. A typical spectrum of $R$ is shown in Figure 1b. For larger $T$, the spectra also exhibits similar decay.

5.2 Temperatures and Electricity Consumption

In this section we examine the relation between daily temperatures and electricity consumption in the data from Hong et al. (2014) (see also Hong (2016)). In this data, we have total daily electricity consumption (load) for a certain region, for the period Jan-2004 to Jun-2008. We also have average daily temperatures, $v_t$, for the region. We refer to Appendix[4] for additional details on the data, the preprocessing, and an extended discussion of the analysis.
An elementary inspection of the data reveals that the load may be reasonably approximated as a quadratic function of the temperature,

\[ y_t = x_{t,1} \cdot 1 + x_{t,2} \cdot v_t + x_{t,3} \cdot v_t^2, \]

(31)

where \( u_t = (1, v_t, v_t^2) \) is the observation vector (features), and \( x_t = (x_{t,1}, x_{t,2}, x_{t,3}) \) is the possibly time varying regression vector. This is shown in Figure 2a, where we fit a stationary (time invariant) regression of the form (31), using either only the first or only the second half of the data. We note that these regressions differ\(^2\) – the regression vector changes with time. We will therefore attempt to track it via online regression.

We use the first half of the data (train set) to learn the parameters \( \sigma, \eta \) of the online regression (1)-(2) via MLE optimization and using STVE. We also use the train set to find the optimal learning rate \( \alpha \) for the OG forecaster described by the update equation (5). This learning rate is chosen as the rate that yields smallest least squares forecast error on the train set, and is found via the L-BFGS-B optimization algorithm. In addition, we learn a time independent, stationary regression on the first half of the data.

We then employ the learned parameters to make predictions of the load given the temperature, by all four methods. The predictions for the system (1)-(2) are made with a Kalman filter (at time \( t \), we use the filtered state estimate \( \hat{x}_t \), which depends only on \( y_1, \ldots, y_t \) and \( u_1, \ldots, u_t \), and make the prediction \( \hat{y}_{t+1} = \langle \hat{x}_t, u_{t+1} \rangle \).

Daily squared prediction errors (that is, \( (y_t - \hat{y}_t)^2 \)) are shown in Figure 2b (smoothed with a moving average of 50 days). We see that the adaptive models (MLE, STVE, OG) outperform the stationary regression already on the train set (first half of the data), and that the difference in performance becomes dramatic on the second half (test). It is also interesting to note that the performance of the Kalman filter based methods (MLE, STVE) is practically identical, but\(^2\) Possible reasons for this are discussed in Appendix I.
both are somewhat better than the simpler OG approach (see also Figure 5 in Appendix I).

Finally, we note that by construction, we have \( |u_{\text{min}}| \geq 1 \) in this experiment, due to the constant 1 coordinate, and also \( |u_{\text{max}}| \leq 5 \), due to the normalization. Since these operations are typical for any regression problem, we conclude that the direct influence of \( |u_{\text{min}}| \) and \( |u_{\text{max}}| \) on the bounds in Theorem 1 and Corollary 2 will not usually be significant.

6 Conclusion and Future Work

In this work we introduced the STVE algorithm for estimating the variance parameters of LDSs of type (1)-(2), and obtained sample complexity guarantees for the estimators.

We conclude by mentioning a few extensions and open questions. There are a few extensions that may be obtained by our methods virtually without change. For instance, an extension to vector rather than scalar valued observations is straightforward and can be done along the lines of the current arguments. An extension to control settings, where the equation of the state (1) is \( X_{t+1} = X_t + f_t + h_t \), with known inputs \( f_t \), is also immediate. Next, it is easy to see that the error rate of the estimators in Corollary 2 \( T^{-\frac{1}{2}} \), is optimal in \( T \). Indeed, even for a single sub Gaussian variable \( h \), the error rate of the standard variance estimator \( \frac{1}{T} \sum_{t=1}^{T} h_t^2 \) is lower-bounded by \( T^{-\frac{1}{2}} \), by the Central Limit Theorem. However, it is not clear whether the dependence on \( u \) and the dimension \( n \) in (16)-(17) is also tight. Finally, a question of independent interest is the question of stability: Given error bounds for \( \tilde{\sigma}, \tilde{\eta} \), what can be said about the prediction error of the Kalman filter with parameters \( \tilde{\sigma}, \tilde{\eta} \) compared to the error of the filter with true parameters \( \sigma, \eta \)? To the best of our knowledge, this fundamental question is still open.

References

Anava, O., Hazan, E., Mannor, S., and Shamir, O. (2013). Online learning for time series prediction. In COLT 2013 - The 26th Annual Conference on Learning Theory, June 12-14, 2013, Princeton University, NJ, USA.

Anderson, B. and Moore, J. (1979). Optimal Filtering. Prentice Hall.

Belanger, P. R. (1974). Estimation of noise covariance matrices for a linear time-varying stochastic process. Automatica, 10(3).

Bhatia, R. (1997). Matrix Analysis. Graduate Texts in Mathematics. Springer New York.

Chui, C. and Chen, G. (2017). Kalman Filtering: with Real-Time Applications. Springer International Publishing.
Dunik, J., Kost, O., and Straka, O. (2018). Design of measurement difference autocovariance method for estimation of process and measurement noise covariances. *Automatica*, 90.

Gohberg, I. and Krein, M. (1969). *Introduction to the Theory of Linear Non-selfadjoint Operators*. Translations of mathematical monographs. American Mathematical Society.

Hamilton, J. (1994). *Time Series Analysis*. Princeton University Press.

Hanson, D. L. and Wright, E. T. (1971). A bound on tail probabilities for quadratic forms in independent random variables. *Ann. Math. Statist.*, 42.

Hazan, E. (2016). Introduction to online convex optimization. *Found. Trends Optim.*

Hazan, E., Singh, K., and Zhang, C. (2017). Online learning of linear dynamical systems. In *Advances in Neural Information Processing Systems*, pages 6686–6696.

Hong, T. (2016). Tao hongs’s blog. [http://blog.drhongtao.com/2016/07/gefcom2012-load-forecasting-data.html](http://blog.drhongtao.com/2016/07/gefcom2012-load-forecasting-data.html) Accessed: 1/8/2019.

Hong, T., Pinson, P., and Fan, S. (2014). Global energy forecasting competition 2012. *International Journal of Forecasting*, 30:357–363.

Hsu, D., Kakade, S., Zhang, T., et al. (2012). A tail inequality for quadratic forms of subgaussian random vectors. *Electronic Communications in Probability*, 17.

Kozdoba, M., Marecek, J., Tchrakian, T. T., and Mannor, S. (2019). Online learning of linear dynamical systems: Exponential forgetting in Kalman filters. *AAAI*.

Mehra, R. (1970). On the identification of variances and adaptive kalman filtering. *IEEE Transactions on Automatic Control*, 15(2).

Mitchell, A. and Griffiths, D. (1980). *The finite difference method in partial differential equations*. Wiley-Interscience publication. Wiley.

Petris, G. (2010). An R package for dynamic linear models. *Journal of Statistical Software*.

Rudelson, M., Vershynin, R., et al. (2013). Hanson-wright inequality and subgaussian concentration. *Electronic Communications in Probability*, 18.

Shumway, R. and Stoffer, D. (2011). *Time Series Analysis and Its Applications (3rd ed.)*.

van Overschee, P. and de Moor, L. (1996). *Subspace identification for linear systems: theory, implementation, applications*. Kluwer Academic Publishers.
Vershynin, R. (2018). *High-Dimensional Probability: An Introduction with Applications in Data Science*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press.

Wang, H., Deng, Z., Feng, B., Ma, H., and Xia, Y. (2017). An adaptive kalman filter estimating process noise covariance. *Neurocomputing*, 223.

West, M. and Harrison, J. (1997). *Bayesian Forecasting and Dynamic Models (2nd ed.)*. Springer-Verlag.

Zinkevich, M. (2003). Online convex programming and generalized infinitesimal gradient ascent. *ICML.*
A Outline

This Appendix is organized as follows: Theorems 1, 7, Corollary 2, and all intermediate results are proved in Sections B to F. In Section G we prove Proposition 3. Section H contains the discussion of missing values in STVE. Additional details on the electricity consumption experiment are given in Section I.

Finally, Figure 3 illustrates the behavior of the Kalman filter on a fixed data for various values of $\sigma, \eta$. The data (observations, red) was generated from the system (1)-(2) with one dimensional state ($n = 1$), and we set $u_t = 1$. The states produced by the Kalman filter with ground truth values of $\sigma, \eta$ are shown in green, while states obtained from other choices of parameters are shown in black and blue. Clearly, with wrong choices of $\sigma, \eta$, the filter can either severely lag after the signal, or overfit to the latest observation.

B Proof of Lemma 6

Proof. Recall that the operator $S_{T-1}^{T-1}$ is given by $S_{T-1}^{T-1}x = (x_1, x_2 - x_1, \ldots, x_T - x_{T-1})$ and the operator $D = D_T : \mathbb{R}^T \to \mathbb{R}^{T-1}$ is given by $Dx = (x_2 - x_1, \ldots, x_T - x_{T-1})$. Let $V = \text{span}\{e_2, \ldots, e_T\}$ be the subspace spanned by all but the first coordinate. Let $P_V : \mathbb{R}^T \to \mathbb{R}^T$ be the projection onto $V$, i.e., a restriction to second to $T$’th coordinate. Observe that the action of $DP_V$ is equivalent to that of $S_{T-1}^{T-1}$. Therefore the singular values of $S_{T-1}^{T-1}$ are identical to those of $DP_V$. Thus, by the Cauchy’s Interlacing Theorem (Bhatia, 1997, Corollary III.1.5),

$$\lambda_t(D^*D) \geq \lambda_t(P_V D^*DP_V) \geq \lambda_{t+1}(D^*D)$$ (32)

for all $t \leq T - 1$. In conjunction with Lemma 3 this provides us with the estimates for all but the smallest singular value of $S_{T-1}^{T-1}$ (since $\lambda_T(D^*D) = 0$).

We therefore estimate $\lambda_{T-1}(S_{T-1}^{T-1}) = \lambda_T(DP_V)$ directly, by bounding the norm of $S_{T-1}$. Indeed, for any $T$, by the Cauchy-Schwartz inequality,

$$|S_T^T x|^2 = \sum_{t \leq T} \left( \sum_{i \leq t} x_i \right)^2 \leq \sum_{t \leq T} \left( \sum_{i \leq t} x_i^2 \right) \left( \sum_{i \leq t} 1 \right)$$ (33)

$$= \sum_{t \leq T} t \left( \sum_{i \leq t} x_i^2 \right) \leq \sum_{t \leq T} t \left| x \right|^2 \leq \left| x \right|^2 T^2.$$ (34)

Thus we have $\|S_T^T\|_{op} \leq T$, which concludes the proof of the Lemma. $\square$
Figure 3: Lag and Overfitting to the most recent observation, for various variance values.

C Proof of Lemma 4

Proof. By Lemma 6,\
\|S\|_{op} \leq T \text{ and } |S_T' x| \geq \frac{1}{2} |x| \text{ for all } x \in \mathbb{R}^T. \quad (35)

Note also that $S$ is by definition a collection of $n$ independent copies of $S_T'$, and therefore the spectrum of $S$ is that of $S_T'$, but each singular value is taken with multiplicity $n$. In particular it follows that (35) holds also for $S$ itself. Since clearly $\|O_u\|_{op} \leq |u_{max}|$, the upper bound on $\|O_u S\|_{op}$ in (23) follows from (35).

For the lower bound, denote by $V'$ the orthogonal complement to the kernel of $O_u$, $V' = (\text{Ker}(O_u))^\perp$. Denote by $P_{V'} : \mathbb{R}^T \to V'$ the orthogonal projection onto $V'$. We have in particular that $O_u S = O_u P_{V'} S$. Next, the operator $S$ maps the unit ball $B_{Tn}$ of $\mathbb{R}^T$ into an ellipsoid $E_T$, and by (35), we have $\frac{1}{2} B_{Tn} \subset E_T$. It therefore follows that
\[ \frac{1}{2} B_{V_1} \subset (P_{V'} S)(B_{Tn}), \quad (36) \]
where $B_{V_1}$ is the unit ball of $V_1$. It remains to observe that for every $x \in V_1$, we have
\[ |O_u x| \geq |u_{min}| |x|. \quad (37) \]
Combining (36) and (37), we obtain the lower bound in (23).

To derive (24), recall that the nuclear norm is sub-multiplicative with respect to the operator norm:
\[ \|O_u S\|_{nuc} \leq \|O_u\|_{op} \cdot \|S\|_{nuc} \leq |u_{max}| \cdot \|S\|_{nuc}. \quad (38) \]
This follows for instance from the characterization of the nuclear norm as trace dual of the operator norm \cite{Bhatia1997} Propositions IV.2.11, IV.2.12). Next,
since the spectrum of $S$ is the spectrum of $S'$ taken with multiplicity $n$, we have
\[ \|S\|_{nuc} = n \|S'\|_{nuc}, \tag{39} \]
and it remains to bound the nuclear norm of $S'$.

Using the inequality
\[ \sin \frac{\pi}{2} \alpha \geq \alpha \text{ for all } \alpha \in [0, 1], \tag{40} \]
and Lemma 6, we have
\[ \|S'\|_{nuc} = T + 2 \sum_{t \leq T-1} \sin^{-1} \left( \frac{\pi T - t}{2(T+1)} \right) \leq 2T \sum_{t \leq T} \frac{1}{t+1} \leq 4T \log T. \tag{41} \]
Combining (38), (39) and (41), the inequality (24) follows.

It remains to estimate the Hilbert-Schmidt norm of $O_u S$, which can be done by a direct computation. Recall that for any operator $A : \mathbb{R}^m \to \mathbb{R}^m$ the Hilbert-Schmidt norm satisfies
\[ \|A\|^2_{HS} = \text{tr} A^* A = \sum_{i \leq m} \langle A^* A \phi_i, \phi_i \rangle = \sum_{i \leq m} \langle A \phi_i, A \phi_i \rangle = \sum_{i \leq m} |A \phi_i|^2, \tag{42} \]
for any orthonormal basis $\phi_i$ in $\mathbb{R}^m$. Let $e_{ti}$ be the standard basis vector in $\mathbb{R}^{Tn}$ corresponding to coordinate $i \leq n$ at time $t \leq T$. Let $e_{t', t''} \leq T$ denote the standard basis in $\mathbb{R}^T$. Then
\[ O_u S e_{ti} = \sum_{t' \leq t} u_{t' i} e_{t'}, \tag{43} \]
where $u_{t' i}$ is the $i$-th coordinate of $u_{t'}$. It follows that
\[ |O_u S e_{ti}|^2 = \sum_{t' \leq t} u_{t' i}^2 \tag{44} \]
and hence
\[ \|O_u S\|^2_{HS} = \sum_{t \leq T, t' \leq n} |O_u S e_{ti}|^2 = \sum_{t \leq T} \sum_{t' = t}^T |u_{t'}|^2 = \sum_{t \leq T} t |u_{t}|^2. \tag{45} \]
The bounds (25) follow directly from (45).

D Proof of Theorem 7

Proof. The bound on $\|R\|_{op}$ follows directly from the lower bound on the singular values of $O_u S$ in (23). Since $R$ is of rank $T$, the upper bounds on $\|R\|_{HS}$ follow directly from the $\|R\|_{op}$ bound.
The lower bounds on $\|R\|_{HS}^2$ and $\|R^* R\|_{HS}^2$ follow from the upper bounds on the nuclear norm in Lemma 4. Note that this argument would not have worked if we only had upper bounds on the Hilbert-Schmidt norm, rather than the nuclear norm in Lemma 4. Denote $\gamma_i = \lambda_i(O_u S)$. From (24) in Lemma 4, the number of $\gamma_i$ that are larger than $4n |u_{\text{max}}| \log T$ satisfies
\[
\# \left\{ \gamma_i \mid \gamma_i \geq 4n |u_{\text{max}}| \log T \right\} \leq \frac{T}{2}.
\] (46)
Since there are total of $T$ singular values $\gamma_i$ overall, we can equivalently rewrite (46) as
\[
\# \left\{ \gamma_i \mid \gamma_i^{-1} \geq \frac{1}{4n |u_{\text{max}}| \log T} \right\} \geq \frac{T}{2}.
\] (47)
This immediately implies the lower bounds on $\|R\|_{HS}^2$ and $\|R^* R\|_{HS}^2$.

\section*{E Proof of Theorem 1}

The two main probabilistic tools that we use are the Hanson-Wright inequality (Hanson and Wright [1971]), and a classical norm deviation inequality for sub-Gaussian vectors, as follows:

\textbf{Theorem 8 (Hanson-Wright Inequality).} Let $X = (X_1, \ldots, X_m) \in \mathbb{R}^m$ be a random vector such that the components $X_i$ are independent and $X_i \sim \text{SG}(\kappa)$ for all $i \leq m$. Let $A$ be an $m \times m$ matrix. Then, for every $t \geq 0$,
\[
P \left( |\langle AX, X \rangle - \mathbb{E} \langle AX, X \rangle| > t \right) \leq 2 \exp \left\{ -c \min \left( \frac{t^2}{\kappa^2 \|A\|_{HS}^2}, \frac{t^2}{\kappa^2 \|A\|_{op}} \right) \right\}.
\] (48)

In particular, recall that we are interested in concentration of $|RY|^2$. We may write:
\[
|RY|^2 = |RO_u Sh + Rz|^2 = |RO_u Sh|^2 + |Rz|^2 + \langle h, (RO_u S)^* Rz \rangle.
\] (49)

The deviations of the first two terms may be bounded via Theorem 8. For the third term, we use the following:

\textbf{Lemma 9.} For any $X \sim \text{SG}_m(\kappa)$ we have:
\[
P \left( |X| > 4\kappa \sqrt{m} + t \right) \leq \exp \left( -\frac{ct^2}{\kappa^2} \right).
\] (50)

Note that in Lemma 9 we do not require the coordinates of $X$ to be independent. This will be important in what follows. Lemma 9 is standard and can be proved via covering number estimates of the Euclidean ball, see for instance Vershynin [2018], Section 4.4.

In addition, the following observation is used throughout the text:
**Lemma 10.** Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be an operator, and let $h = (h_1, \ldots, h_m)$ have independent coordinates with $\mathbb{E} h_i^2 = \sigma^2$. Denote by $\{\lambda_i\}_{i=1}^k$, $k \leq \min(m, n)$, the singular values of $A$. Then

$$\mathbb{E} |Ah|^2 = \sigma^2 \|A\|^2_{HS} = \sigma^2 \sum_{i=1}^k \lambda_i^2.$$  

(51)

The elementary proof is omitted.

We now prove Theorem 1.

**Proof.** Let $0 < \delta < 1$ be given. We first bound the deviations from the expectation for the first term in (49), $|RO_uSh|^2$. We apply the Hanson-Wright inequality with $X = h$ and $A = (RO_uS)^*RO_uS$. By definition, $A$ has a single eigenvalue 1 with multiplicity $T$. Thus clearly $\|A\|^2_{HS} = T$ and $\|A\|_{op} = 1$.

For an appropriate constant $c' > 0$, set $t = c' \kappa \sqrt{T \log \frac{1}{\delta}}$. Then,

$$P \left( |RO_uSh|^2 - T \sigma^2 \geq c'' \kappa \sqrt{T \log \frac{1}{\delta}} \right) \leq \delta.$$  

(52)

The deviation of the second term in (49) is similarly bounded using $A = R^*R$. Recall that by Theorem 7 we have $\|R\|_{op} \leq 2 |u_{min}|^{-1}$, and note that $\|R^*R\|^2_{HS} \leq T \|R\|^4_{op} \leq cT |u_{min}|^{-4}$. Set $t = c' \kappa |u_{min}|^{-2} \sqrt{T \log \frac{1}{\delta}}$. With this choice it follows that both terms in the minimum in (48) are larger than $c \log \frac{1}{\delta}$ and we have

$$P \left( |Rz|^2 - \eta^2 \|R\|^2_{HS} \geq c' \kappa^2 |u_{min}|^{-2} \sqrt{T \log \frac{1}{\delta}} \right) \leq \delta.$$  

(53)

Finally, we bound the third term in (49). Denote by $D$ the event

$$D = \left\{ |(RO_uS)^*Rz| > c |u_{min}|^{-1} \left( \kappa \sqrt{T} + \kappa \sqrt{\log \frac{1}{\delta}} \right) \right\},$$  

(54)

and by $E$ the event

$$E = \left\{ |\langle h, (RO_uS)^*Rz \rangle| > c' |u_{min}|^{-1} \left( \kappa \sqrt{T} + \kappa \sqrt{\log \frac{1}{\delta}} \right) \kappa \sqrt{\log \frac{1}{\delta}} \right\}. $$  

(55)

By Lemma 9 applied to $z$, and using the fact that $\|(RO_uS)^*R\|_{op} \leq 2 |u_{min}|^{-1}$,

$$P(D) \leq P \left( |z| > c \kappa \sqrt{T} + c \kappa \sqrt{\log \frac{1}{\delta}} \right) \leq \delta.$$  

(56)

Next, using independence of $h, z$ and $h \sim SG_{Tn}(\kappa)$,

$$P(E \mid D^c) \leq \delta.$$  

(57)
where $D^c$ is the complement of $D$. Therefore, combining (56) and (57),

$$P(E) = P(D)P(E|D) + P(D^c)P(E|D^c) \leq 2\delta.$$  

(58)

Combining (52), (53) and (58), we obtain via the union bound:

$$P\left(\left\|\frac{R}{T}Y\right\|^2 - \left(\sigma^2 + \frac{\|R\|^2_{HS}}{T} \eta^2\right) \geq c \left(1 + \kappa^2\right) \left(1 + |u_{min}|^{-2}\right) \log \frac{1}{\delta}\right) \leq 4\delta.$$  

(59)

Similarly, we obtain a bound for the equations involving $R'$:

$$P\left(\left\|\frac{R'Y}{p}\right\|^2 - \left(\sigma^2 + \frac{\|R'\|^2_{HS}}{p} \eta^2\right) \geq c \left(1 + \kappa^2\right) \left(1 + |u_{min}|^{-2}\right) \log \frac{1}{\delta}\right) \leq 4\delta.$$  

(60)

The only difference in the derivation of (60) compared to (59) is in the application of Lemma 9. In the later case, to replace $\sqrt{T}$ with $\sqrt{p}$ in (56), we apply Lemma 9 with $X = PVz$ rather than with $X = z$, where $PV$ is the projection onto the range of $R'$ – a $p$-dimensional space. Note that $PVz$ does not necessarily have a structure of $p$ independent coordinates, but is sub Gaussian and isotropic. Therefore Lemma 9 still applies.

\[\square\]

**F  **Proof of Corollary 2

We now turn to prove Corollary 2.

*Proof.* Denote

$$E(a) = \frac{(1 + \kappa^2) \left(1 + |u_{min}|^{-2}\right) \log \frac{1}{\delta}}{a}. $$  

(61)

Using (60) and (59) we may write

$$\frac{|RY|^2}{T} + e_1 = \sigma^2 + \frac{\|R\|^2_{HS}}{T} \eta^2,$$

(62)

$$\frac{|R'Y|^2}{p} + e_2 = \sigma^2 + \frac{\|R'\|^2_{HS}}{p} \eta^2,$$

(63)

where $e_1, e_2$ are error terms such that $|e_1| \leq E(\sqrt{T})$ and $|e_2| \leq E(\sqrt{p})$ holds with probability at least $1 - 8\delta$.

It follows that

$$\eta^2 = \left(\frac{|R'Y|^2}{p} - \frac{|RY|^2}{T}\right) \left(\frac{\|R'\|^2_{HS}}{p} - \frac{\|R\|^2_{HS}}{T}\right)^{-1} + (e_2 - e_1) \left(\frac{\|R'\|^2_{HS}}{p} - \frac{\|R\|^2_{HS}}{T}\right)^{-1}$$

(64)
\[
\frac{p\|R\|_{HS}^2 |RY|^2}{T\|R'\|_{HS}^2} + p\frac{\|R\|_{HS}^2}{T\|R'\|_{HS}^2} \sigma_2 = \frac{p\|R\|_{HS}^2}{T\|R'\|_{HS}^2} \sigma_2 + \frac{\|R\|_{HS}^2}{T} \eta^2.
\]  
\text{(65)}

Thus
\[
\sigma_2 = \left(1 - \frac{p\|R\|_{HS}^2}{T\|R'\|_{HS}^2}\right)^{-1} \left(\frac{|RY|^2}{T} - \frac{|R'Y|^2}{T} \frac{\|R\|_{HS}^2}{\|R'\|_{HS}^2}\right) 
+ \left(1 - \frac{p\|R\|_{HS}^2}{T\|R'\|_{HS}^2}\right)^{-1} \left(\epsilon_1 - \frac{p\|R\|_{HS}^2}{T\|R'\|_{HS}^2} \epsilon_2\right).
\text{(66)}

It remains to observe that
\[
\left(\frac{\|R'\|_{HS}^2}{p} - \frac{\|R\|_{HS}^2}{T}\right)^{-1} = \left(1 - \frac{p\|R\|_{HS}^2}{T\|R'\|_{HS}^2}\right)^{-1} \frac{p}{\|R'\|_{HS}^2} 
\leq c \left(1 - \frac{p\|R\|_{HS}^2}{T\|R'\|_{HS}^2}\right)^{-1} n^2 |u_{\max}|^2 \log^2 T,
\text{(67)}
\]

where the inequality follows from eq. (47) in the proof of Theorem 7.

\[ \square \]

**G Thresholding Gap Analysis, Proof of Proposition 3**

As discussed in the main text, we first prove Proposition 3 for the case \( n = 1 \), stated in Lemma 12. The argument consists of showing that the spectrum of \( R \) is upper and lower bounded by appropriately decaying functions, and therefore cannot be “too constant”. We first obtain general estimates for the integrals of such upper and lower bounded functions in the following Lemma:

**Lemma 11.** Let \( f : [0, 1] \rightarrow \mathbb{R} \) be a monotone non-increasing function such that for all \( x \in [0, 1] \),
\[
(1 - x)^2 \leq f(x) \leq M (1 - x)^2,
\text{(69)}
\]
for some \( M \geq 1 \). Set \( t_0 = \frac{1}{4} \). Then
\[
r(f) := \frac{1}{t_0} \int_0^{t_0} f(x)dx \geq 1 + \frac{c}{\sqrt{M}}.
\text{(70)}
\]

**Proof.** Denote
\[
I(a, b, f) = \int_a^b f(x)dx.
\text{(71)}
\]
Write
\[ r(f) = \frac{t_0^{-1}I(0,t_0,f)}{I(0,t_0,f) + I(t_0,1,f)} = \frac{t_0^{-1}}{1 + t_0 \frac{I(t_0,1,f)}{I(0,t_0,f)}} \] (72)
and set \( v := f(t_0) \). Then, among all \( f \) that satisfy (69) and \( f(t_0) = v \), \( r(f) \) is minimized on \( f \) with maximal \( I(t_0,1,f) \) and minimal \( I(0,t_0,f) \). Due to the form of the constraint (69) and monotonicity, this minimizer is given by
\[ \tilde{f}_v(x) = \begin{cases} (1-x)^2 & x \in [0,t_v^-] \\ v & x \in [t_v^-, t_v^+] \\ M(1-x)^2 & x \in [t_v^+, 1], \end{cases} \] (73)
where \( t_v^- := \max \{0,1 - \sqrt{\frac{v}{M}}\} \) and \( t_v^+ := 1 - \sqrt{\frac{v}{M}}. \) Our problem is therefore now reduced from a minimization of \( r(f) \) over the function space to a problem of minimizing \( r(v) := r(\tilde{f}_v) \) over a single scalar variable \( v \).

To this end, first note that we can assume w.l.o.g that \( M \geq (1 - t_0)^{-2}. \) Indeed, for larger \( M \), the lower bound on \( r(f) \) can only become smaller, since \( r(f) \) would be minimized over a larger set. By construction, the value \( v \) must satisfy \( (1 - t_0)^2 \leq v \leq M(1 - t_0)^2 \), and for \( M \geq (1 - t_0)^{-2} \), the value \( v = 1 \) satisfies these inequalities.

Next, by direct computation (taking the derivative in \( v \)) one verifies that for any \( M \geq (1 - t_0)^{-2}, \) \( r(v) \) as a function of \( v \) is minimized at \( v = 1. \) It remains to observe that \( r(1) = 1 - \frac{1}{3\sqrt{M}} \), which yields the statement of the Lemma. \( \square \)

We can now treat the \( n = 1 \) case.

**Lemma 12.** Consider the case \( n = 1 \), and \( p = \frac{1}{4} T. \) Then
\[ \frac{\|R\|_{\text{HS}}^2}{p} - \frac{\|R\|_{\text{HS}}^2}{T} \geq c \left( \frac{\|u_{\min}\|}{\|u_{\max}\|} \right) \frac{\|R\|_{\text{HS}}^2}{T}. \] (74)

**Proof.** For any two operators \( A,B \) we have
\[ \lambda_i(AB) \leq \lambda_i(A) \|B\|_{op} \quad \text{and} \quad \lambda_i(BA) \leq \lambda_i(A) \|B\|_{op}, \] (75)
see Bhatia (1997); Gohberg and Krein (1969). Note that in the case \( n = 1 \) and \( |u_{\min}| > 0, \) \( O_u S' \) is invertible. Thus the singular values of \( O_u S' \) satisfy
\[ |u_{\min}| \lambda_i(S') \leq \lambda_i(O_u S') \leq |u_{\min}| \lambda_i(S'), \] (76)
where the first inequality follows by applying (75) to \( (O_u S')^{-1} , \) and the second by considering \( O_u S' \) itself. Equivalently, for \( i = 1, \ldots , T \)
\[ |u_{\max}|^{-1} \lambda_{T+1-i}(S')^{-1} \leq \lambda_{T+1-i}(R) \leq |u_{\min}|^{-1} \lambda_{T+1-i}(S')^{-1}, \] (77)
and using Lemma 4
\[ c \frac{|u_{\max}|^{-1} i}{T + 1} \leq \lambda_{T+1-i}(R) \leq |u_{\min}|^{-1} \pi \frac{|u_{\min}|^{-1} i}{T + 1}. \] (78)
By changing the index and taking squares, we have
\[ c|u_{\text{max}}|^2 \left(1 - \frac{i}{T+1}\right)^2 \leq \lambda_i^2(R) \leq c'|u_{\text{min}}|^2 \left(1 - \frac{i}{T+1}\right)^2. \] (79)

Now, using an elementary discretization argument, for \( p = \frac{1}{4}T \) by Lemma 11 it follows that
\[ \frac{\|R\|_{HS}^p}{\|R\|_{HS}^T} \geq 1 + c\frac{|u_{\text{max}}|}{|u_{\text{min}}|}, \] (80)
which implies (74).

Finally, we prove Proposition 3.

**Proof.** As discussed earlier, the key to a statement such as (20) is to show that the spectrum of \( R \) is non-constant, and in particular has enough small singular values. This is equivalent to providing appropriate lower bounds on the spectrum of \( O_u S \). Here we derive such bounds by comparison with an \( n = 1 \) case as follows: Let \( V \subset \mathbb{R}^{Tn} \) be a \( T \) dimensional subspace spanned by vectors for which for every time \( t \), all coordinates at time \( t \) are identical. Formally, for \( x \in V \), for every \( t \) we require that \( x(t-1)n+i = x(t-1)n+j \) for all \( i,j \leq n \).

Observe that the restriction of \( O_u S \) to \( V \), the operator \( O_u S_{PV} \), acts equivalently to the \( n = 1 \) case operator defined by \( O_u S' \). In particular, similarly to the argument in Lemma 12 it follows that
\[ |\tilde{u}_{\text{min}}| \lambda_i(S') \leq \lambda_i(O_u S'). \] (81)
Moreover, note that \((O_u S_{PV})^* O_u S_{PV} = P_{PV} S^* O_u S_{PV} \) and \((O_u S)^* O_u S \) are non-negative operators and \((O_u S)^* O_u S \geq (O_u S_{PV})^* O_u S_{PV} \) (that is, \((O_u S)^* O_u S - (O_u S_{PV})^* O_u S_{PV} \) is non-negative). It follows that for all \( i \leq T \),
\[ \lambda_i(O_u S) \geq \lambda_i(O_u S_{PV}). \] (82)
The upper bounds on the spectrum \( \lambda_i(O_u S) \) may again be obtained via (75). Indeed,
\[ \lambda_i(O_u S) \leq |u_{\text{max}}| \lambda_i(S) \leq |u_{\text{max}}| \lambda_i^\frac{1}{p} (S'), \] (83)
where the first inequality follows from (75) while the second is due to the multiplicity \( n \) of each singular value of \( S' \) in \( S \). The rest of the argument proceeds as in Lemma 12 □

### H Missing Values in STVE

We now discuss the treatment of missing values in STVE. Recall that our starting point is the vector form of the system, (7), which we rewrite here:
\[ Y = O_u Sh + z, \] (84)
where \( h \in \mathbb{R}^{T_n} \) and \( z \in \mathbb{R}^{T} \) are sub Gaussian vectors and \( Y = (y_1, \ldots, y_T) \in \mathbb{R}^{T} \) is the observation vector. Suppose that \( M \) out of \( T \) observation values are missing, at times \( t_1, \ldots, t_M \). Set \( T' = T - M \) and define a projection operator \( P_A : \mathbb{R}^T \to \mathbb{R}^{T'} \) as the operator that omits the coordinates \( t_1, \ldots, t_M \). Formally, let \( e(t), t \leq T \), be the standard basis vector in \( \mathbb{R}^{T} \), with 1 at coordinate \( t \) and zeros elsewhere. Then

\[
P_A(e(t)) = \begin{cases} 
0 & \text{if } t \in \{t_1, \ldots, t_M\}, \\
e(t) & \text{otherwise}.
\end{cases} \tag{85}
\]

We can then rewrite (84) as

\[
P_A Y = P_A O_u S h + P_A z. \tag{86}
\]

Note that the vector \( P_A Y \) contains only available, non-missing values of \( y_t \). Similarly to the case with no missing values, we define \( R \) as the Moore-Penrose inverse of \( P_A O_u S \). Then we have

\[
R P_A Y = R P_A O_u S h + R P_A z. \tag{87}
\]

Note that since a Moore-Penrose inverse of \( R \) only acts on the image of \( R \), we have \( R P_A = R \) and therefore

\[
R Y = R O_u S h + R z \tag{88}
\]

similarly to the case with no missing values. The only difference here will be that \( R \) will now have \( T' \) rather than \( T \) non-zero singular values. \( R' \) will be defined similarly. The analysis of the spectrum of \( R \) concerns only non-zero singular values of \( R \) and holds with no change other than that \( T \) should be replaced by \( T' \). Consequently, the whole approach works identically with \( T \) replaced by \( T' \) everywhere, and in the bounds of Theorems 1 and 7 in particular.

### I Electricity Consumption and Temperatures Data

The raw data in [Hong et al., 2014] contains electricity consumption levels for 20 nearby regions, also referred to as zones. Each zone had slightly different consumption characteristics. All figures in Section 5.2 refer to Zone 1. However, the results are similar for all zones. In particular, while Figure 2b shows the (smoothed) prediction errors of each method over time, the total error on the test set, normalized by the number of days, for each method and each zone, is shown in Figure 5. As with zone 1, adaptive methods perform better for all zones. STVE based estimator is better than MLE in half the cases, but the differences between STVE and MLE performance are negligible compared to errors in the stationary regression or OG. Note that Zone 9 is known to be an industrial zone [Hong, 2016] where the load does not significantly depend on the temperature, hence the higher error rates for all the methods.
All the data in the dataset is for the period Jan-2004 to Jun-2008. We note that there are missing values of the load in the data, for 9 non-consecutive weeks (out of about 234 weeks total). All methods discussed here, including STVE, can naturally incorporate missing values, as discussed in Appendix [1].

We have observed, as shown in Figure 2a, that regressions fit to the first and the second halves of the data differ. One immediate reason for the dependence of the load on temperature to change with time is that the load simply grows with time (assuming the temperatures do not exhibit a trend of the same magnitude). Indeed, it is easy to verify that there is an upward trend in the overall load (see Figure [1]). However, the upward trend is non-uniform in temperature, since the parabolas in Figure 2a are not a shift of each other by a constant. In any case, these observations show that the temperature/load dependence does change with time, and it is of interest to track it via online regression.

In the rest of this section we discuss the preprocessing of the data. The raw data contains hourly consumption levels for the 20 zones, and hourly temperatures from 11 weather stations with unspecified locations in the region. Here we are only interested in the total daily consumption – the total consumption over all hours of the day – for each zone. For each station we also consider the average daily temperature for the day. Moreover, since the average daily temperatures at different stations are strongly correlated ($\rho \geq 0.97$ for all pairs of stations), we only consider a single daily number, $v_t$ – an average temperature across all hours of the day and all stations.

From the total of about 234 weeks in the data, 9 non-consecutive weeks have missing loads (consumption) data, 4 in the first half of the period (train set), and 5 in the second half. For the purposes of training the stationary regression, we excluded the data points with missing values from the train set. The STVE, MLE and OG methods were trained with missing values. The prediction results of all methods were evaluated only at the points where actual values were known.

The temperatures and all the loads are normalized to have zero-mean and standard deviation 1 on the first half of the data (train set).

The MLE estimates of the parameters $\sigma^2, \eta^2$ were obtained using the DLM package of the R environment (Petris [2010]). The package uses an L-BGFS-B algorithm with numerically approximated derivatives as the underlying optimization procedure.
Figure 4: Load for Zone 1, with 6 month (orange) and year (green) moving average smoothing.

Figure 5: Prediction error for each zone.