QUANTIZATION OF PROBABILITY DISTRIBUTIONS ON R-TRIANGLES

DO˘GAN C ¸¨OMEZ AND MRINAL KANTI ROYCHOWDHURY

Abstract. In this paper, we have considered a Borel probability measure \( P \) on \( \mathbb{R}^2 \) which has support the R-triangle generated by a set of three contractive similarity mappings on \( \mathbb{R}^2 \). For this probability measure, the optimal sets of \( n \)-means and the \( n \)th quantization error are determined for all \( n \geq 2 \). In addition, it is shown that the quantization dimension of this measure exists, but the quantization coefficient does not exist.

1. Introduction

The theory of quantization studies the process of approximating probability measures, which are invariant for certain systems, with discrete probabilities having a finite number of points in their support. Of particular interest are the types of behaviors which may be encountered in the quantization process for various measures. For an extensive survey of the history of the subject one is referred to [10]. For mathematical foundation of quantization theory one is referred to [8, 9]. The same mathematical results are used in pattern recognition (optimal sets of prototypes), economics (optimal location of service centers), numerical integration (optimal location of knots) and the theory of convex sets (optimal approximation by polytopes). Let us consider a Borel probability measure \( P \) on \( \mathbb{R}^d \) and a natural number \( n \in \mathbb{N} \). Then, the \( n \)th quantization error for \( P \) is defined by:

\[
V_n := V_n(P) = \inf \left\{ \int \min_{a \in \alpha} \| x - a \|^2 dP(x) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \right\},
\]

where \( \| \cdot \| \) denotes the Euclidean norm on \( \mathbb{R}^d \). A set \( \alpha \) for which the infimum is achieved is called an optimal set of \( n \)-means for the probability measure \( P \) and the points in an optimal set are called optimal points. Of course, this makes sense only if the mean squared error or the expected squared Euclidean distance \( \int \| x \|^2 dP(x) \) is finite (see [1, 6, 7, 8]). It is known that for a continuous probability measure an optimal set of \( n \)-means always has exactly \( n \)-elements (see [8]). The numbers

\[
\underline{D}(P) := \liminf_{n \to \infty} \frac{2 \log n}{-\log V_n(P)}, \quad \text{and} \quad \overline{D}(P) := \limsup_{n \to \infty} \frac{2 \log n}{-\log V_n(P)},
\]

are respectively called the lower and upper quantization dimensions of the probability measure \( P \). If \( \underline{D}(P) = \overline{D}(P) \), the common value is called the quantization dimension of \( P \) and is denoted by \( D(P) \). Quantization dimension measures the speed at which the specified measure of the error tends to zero as \( n \) approaches to infinity. For any \( s \in (0, +\infty) \), the numbers

\[
\liminf_n n^{\frac{s}{2}} V_n(P) \quad \text{and} \quad \limsup_n n^{\frac{s}{2}} V_n(P)
\]

are respectively called the \( s \)-dimensional lower and upper quantization coefficients of \( P \). If the \( s \)-dimensional lower and upper quantization coefficients of \( P \) are finite and positive, then \( s \) coincides with the quantization dimension of \( P \). The quantization coefficients provide us with more accurate information about the asymptotics of

2010 Mathematics Subject Classification. 60Exx, 28A80, 94A34.

Key words and phrases. R-triangle, R-measure, optimal quantizers, quantization error, quantization dimension, quantization coefficient.

The research of the second author was supported by U.S. National Security Agency (NSA) Grant H98230-14-1-0320.
the quantization error than the quantization dimension. Compared to the calculation of quantization dimension, it is usually much more difficult to determine whether the lower and upper quantization coefficients are finite and positive. For more details in this direction one can see \[8\, \text{[12]}\]. Optimal quantization of probability distributions is also connected with centroidal Voronoi tessellations. Given a finite subset \(\alpha \subset \mathbb{R}^d\), the Voronoi region generated by \(a \in \alpha\) is defined by

\[ M(a|\alpha) = \{x \in \mathbb{R}^d : \|x - a\| = \min_{b \in \alpha} \|x - b\|\} \]

i.e., the Voronoi region generated by \(a \in \alpha\) is the set of all points \(x\) in \(\mathbb{R}^d\) such that \(a\) is a nearest point to \(x\) in \(\alpha\), and the set \(\{M(a|\alpha) : a \in \alpha\}\) is called the Voronoi diagram or Voronoi tessellation of \(\mathbb{R}^d\) with respect to \(\alpha\). A Voronoi tessellation is called a centroidal Voronoi tessellation (CVT), if the generators of the tessellation are also the centroids of their own Voronoi regions with respect to the probability measure \(P\). A Borel measurable partition \(\{A_a : a \in \alpha\}\), where \(\alpha\) is an index set, of \(\mathbb{R}^d\) is called a Voronoi partition of \(\mathbb{R}^d\) if \(A_a \subset M(a|\alpha)\) for every \(a \in \alpha\). Let us now state the following proposition (see \[5\, \text{[8]}\]):

**Proposition 1.1.** Let \(\alpha\) be an optimal set of \(n\)-means and \(a \in \alpha\). Then,

(i) \(P(M(a|\alpha)) > 0\), (ii) \(P(\partial M(a|\alpha)) = 0\), (iii) \(a = E(X : X \in M(a|\alpha))\), and (iv) \(P\)-almost surely the set \(\{M(a|\alpha) : a \in \alpha\}\) forms a Voronoi partition of \(\mathbb{R}^d\).

Let \(\alpha\) be an optimal set of \(n\)-means and \(a \in \alpha\), then by Proposition 1.1 we have

\[ a = \frac{1}{P(M(a|\alpha))} \int_{M(a|\alpha)} x dP = \frac{\int_{M(a|\alpha)} x dP}{\int_{M(a|\alpha)} dP}, \]

which implies that \(a\) is the centroid of the Voronoi region \(M(a|\alpha)\) associated with the probability measure \(P\) (see also \[4\, \text{[14]}\]).

Let \(P\) be a Borel probability measure on \(\mathbb{R}\) given by \(P = \frac{1}{4}P \circ S_1^{-1} + \frac{1}{4}P \circ S_2^{-1}\) where \(S_1(x) = \frac{1}{3}x\) and \(S_2(x) = \frac{1}{3}x + \frac{2}{3}\) for all \(x \in \mathbb{R}\). Then, \(P\) has support the classical Cantor set \(C\). For this probability measure Graf and Luschgy gave a closed formula to determine the optimal sets of \(n\)-means and the \(n\)th quantization error for all \(n \geq 2\); they also proved that the quantization dimension of this distribution exists and is equal to the Hausdorff dimension \(\beta := \log 2/\log 3\) of the Cantor set, but the \(\beta\)-dimensional quantization coefficient does not exist (see \[3\]). Later for \(n \geq 2\), L. Roychowdhury gave an induction formula to determine the optimal sets of \(n\)-means and the \(n\)th quantization error for a Borel probability measure \(P\) on \(\mathbb{R}\) given by \(P = \frac{1}{4}P \circ S_1^{-1} + \frac{1}{4}P \circ S_2^{-1}\) where \(S_1(x) = \frac{1}{3}x\) and \(S_2(x) = \frac{1}{3}x + \frac{2}{3}\) for all \(x \in \mathbb{R}\) (see \[13\]). Let \(P\) be a Borel probability measure on \(\mathbb{R}^2\) such that \(P = \frac{1}{3} \sum_{i,j=1}^{3} P \circ S_{i,j}^{-1}\), where \(S_{i,j}\) are similarity mappings on \(\mathbb{R}^2\) given by \(S_{i,j} := (U_i, U_j)\) where \(1 \leq i, j \leq 2\), \(U_1(x) = \frac{1}{3}x\) and \(U_2(x) = \frac{1}{3}x + \frac{2}{3}\) for all \(x \in \mathbb{R}\). For this probability measure, Çemeş and Roychowdhury determined the optimal sets of \(n\)-means and the \(n\)th quantization error (see \[3\]).

Let us now consider a set of three contractive similarity mappings \(S_1, S_2, S_3\) on \(\mathbb{R}^2\), such that \(S_1(x_1, x_2) = \frac{1}{3}(x_1, x_2), S_2(x_1, x_2) = \frac{1}{3}(x_1, x_2) + \frac{2}{3}(1, 0),\) and \(S_3(x_1, x_2) = \frac{1}{3}(x_1, x_2) + \frac{2}{3}(1, \frac{2\sqrt{3}}{3})\) for all \((x_1, x_2) \in \mathbb{R}^2\). The limit set of the iterated function system \(\{S_{i,j}\}_{i,j=1}^{3}\) is a version of the Sierpiński gasket, which is constructed as follows: (i) Start with an equilateral triangle; (ii) delete the open middle third from each side of the triangle and join the end points of the adjacent sides to construct three smaller congruent equilateral triangles; (iii) repeat step (ii) with each of the remaining smaller triangles. At each step the new triangles appear as radiated from the center of the triangle in the previous step towards the vertices. In order to distinguish it from the classical Sierpiński gasket we will call it the \(R\)-triangle generated by the contractive mappings \(S_1, S_2, S_3\). It is easy to see that the area and the circumference of a \(R\)-triangle are zero and it has Hausdorff dimension 1 (see also Section 4). Let \(P = \frac{1}{3} \sum_{j=1}^{3} P \circ S_j^{-1}\). Then, \(P\) is a unique Borel probability measure on \(\mathbb{R}^2\) with support the \(R\)-triangle generated
by $S_1, S_2, S_3$. We call it the $R$-measure, or more specifically the $R$-measure generated by $S_1, S_2$ and $S_3$. For this $R$-measure, in this paper, we determine the optimal sets of $n$-means and the $n$th quantization error. In addition, we show that the quantization dimension of the $R$-measure exists which is equal to one, and it coincides with the Hausdorff dimension of the $R$-triangle, the Hausdorff and packing dimensions of the $R$-measure, i.e., all these dimensions are equal to one. Moreover, we show that the $s$-dimensional quantization coefficient for $s = 1$ of the $R$-measure does not exist.

2. Basic definitions and lemmas

In this section, we give the basic definitions and lemmas that will be instrumental in our analysis. Let $S_1, S_2$ and $S_3$ be the generating maps of the $R$-triangle as defined in the previous section. Write $I := \{1, 2, 3\}$. By a word $\omega$ of length $k$ over the alphabet $I$, we mean $\omega := \omega_1 \omega_2 \cdots \omega_k \in I^k$. A word of length zero is called the empty word and is denoted by $\emptyset$. By $I^*$, it is meant the set of all words over the alphabet $I$ including the empty word $\emptyset$. By the concatenation of two words $\omega := \omega_1 \omega_2 \cdots \omega_k$ and $\tau := \tau_1 \tau_2 \cdots \tau_l$, denoted by $\omega \tau$, it is meant $\omega \tau := \omega_1 \cdots \omega_k \tau_1 \cdots \tau_l$. For $\omega = \omega_1 \omega_2 \cdots \omega_k \in I^k$, set $S_\omega := S_{\omega_1} \circ \cdots \circ S_{\omega_k}$. Let $\Delta$ be the equilateral triangle with vertices $(0,0), (1,0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. The sets $\{\Delta_\omega : \omega \in I^k\}$ are just the $3^k$ triangles in the $k$th level in the construction of the $R$-triangle. The triangles $\Delta_{\omega_1}, \Delta_{\omega_2}$ and $\Delta_{\omega_3}$ into which $\Delta_\omega$ is split up at the $(k + 1)$th level are called the basic triangles of $\Delta_\omega$. The set $R = \bigcap_{k \in \mathbb{N}} \bigcup_{\omega \in I^k} \Delta_\omega$ is the $R$-triangle and equals the support of the probability measure $P$ given by $P = \frac{1}{3} \sum_{j=1}^{3} P \circ S_j^{-1}$.

Let us now give the following lemma.

**Lemma 2.1.** Let $f : \mathbb{R} \rightarrow \mathbb{R}^+$ be Borel measurable and $k \in \mathbb{N}$. Then,

$$\int f \, dP = \frac{1}{3^k} \sum_{\omega \in I^k} \int f \circ S_\omega \, dP.$$  

**Proof.** We know $P = \frac{1}{3} \sum_{j=1}^{3} P \circ S_j^{-1}$, and so by induction $P = \frac{1}{3^k} \sum_{\omega \in I^k} P \circ S_\omega^{-1}$, and thus the lemma is yielded. \hfill \Box

Let $S_{(1)}, S_{(2)}$ be the horizontal and vertical components of the transformations $S_i$ for $1 \leq i \leq 3$. Then, for any $(x_1, x_2) \in \mathbb{R}^2$ we have $S_{(1)}(x_1) = \frac{1}{3} x_1$, $S_{(2)}(x_2) = \frac{1}{3} x_2$, $S_{(21)}(x_1) = \frac{1}{3} x_1 + \frac{2}{3}$, $S_{(22)}(x_2) = \frac{1}{3} x_2$, $S_{(31)}(x_1) = \frac{1}{3} x_1 + \frac{1}{3}$, and $S_{(32)}(x_2) = \frac{1}{3} x_2 + \frac{\sqrt{3}}{3}$. Let $X = (X_1, X_2)$ be a bivariate random variable with distribution $P$. Let $P_1, P_2$ be the marginal distributions of $P$, i.e., $P_1(A) = P(A \times \mathbb{R}) = P \circ \pi_1^{-1}(A)$ for all $A \in \mathcal{B}$, and $P_2(B) = P(\mathbb{R} \times B) = P \circ \pi_2^{-1}(B)$ for all $B \in \mathcal{B}$, where $\pi_1, \pi_2$ are two projection mappings given by $\pi_1(x_1, x_2) = x_1$ and $\pi_2(x_1, x_2) = x_2$ for all $(x_1, x_2) \in \mathbb{R}^2$. Here $\mathcal{B}$ is the Borel $\sigma$-algebra on $\mathbb{R}$. Then $X_1$ has distribution $P_1$ and $X_2$ has distribution $P_2$.

The statement below provides the connection between $P$ and its marginal distributions via the components of the generating maps $S_i$. The proof is similar to Lemma 2.2 in [3].

**Lemma 2.2.** Let $P_1$ and $P_2$ be the marginal distributions of the probability measure $P$. Then,

$$P_1 = \frac{1}{3} P_1 \circ S_{(1)}^{-1} + \frac{1}{3} P_1 \circ S_{(21)}^{-1} + \frac{1}{3} P_1 \circ S_{(31)}^{-1} \quad \text{and}$$

$$P_2 = \frac{1}{3} P_2 \circ S_{(12)}^{-1} + \frac{1}{3} P_2 \circ S_{(22)}^{-1} + \frac{1}{3} P_2 \circ S_{(32)}^{-1}.$$
For words $\beta, \gamma, \cdots, \delta$ in $I^*$, by $a(\beta, \gamma, \cdots, \delta)$ we mean the conditional expectation of the random variable $X$ given $\Delta_\beta \cup \Delta_\gamma \cup \cdots \cup \Delta_\delta$, i.e.,

\begin{equation}
(1) \quad a(\beta, \gamma, \cdots, \delta) = E(X|X \in \Delta_\beta \cup \Delta_\gamma \cup \cdots \cup \Delta_\delta) = \frac{1}{P(\Delta_\beta \cup \cdots \cup \Delta_\delta)} \int_{\Delta_\beta \cup \cdots \cup \Delta_\delta} x dP.
\end{equation}

**Lemma 2.3.** Let $E(X)$ and $V(X)$ denote the the expectation and the variance of the random variable $X$. Then,

$$E(X) = (E(X_1), E(X_2)) = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$$

and the corresponding quantization error is the variance $V := V(X) = E\|X - \left(\frac{1}{2}, \frac{1}{2}\right)\|^2 = \frac{1}{6}$.

**Proof.** We have

$$E(X_1) = \int x_1 dP_1 = \frac{1}{3} \int x_1 dP_1 \circ S_{(11)}^{-1} + \frac{1}{3} \int x_1 dP_1 \circ S_{(21)}^{-1} + \frac{1}{3} \int x_1 dP_1 \circ S_{(31)}^{-1}$$

$$= \frac{1}{3} \int (\frac{1}{3} x_1)^2 dP_1 + \frac{1}{3} \int (\frac{2}{3} x_1)^2 dP_1 + \frac{1}{3} \int \left(\frac{1}{3} x_1 + \frac{1}{3}\right)^2 dP_1$$

$$= \frac{1}{3} \int (\frac{1}{9} x_1^2) dP_1 + \frac{1}{3} \int \left(\frac{4}{9} x_1^2 + \frac{4}{9} x_1 + \frac{4}{9}\right) dP_1 + \frac{1}{3} \int \left(\frac{1}{9} x_1^2 + \frac{2}{9} x_1 + \frac{1}{9}\right) dP_1$$

$$= \frac{3}{27} E(X_1^2) + \frac{6}{27} E(X_1) + \frac{5}{27} = \frac{1}{9} E(X_1^2) + \frac{8}{27},$$

which implies $E(X_1^2) = \frac{1}{3}$. Thus, we see that $V(X_1) = E(X_1^2) - (E(X_1))^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$. Similarly, one can show that $V(X_2) = \frac{1}{12}$. Hence,

$$E\|X - \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)\|^2 = \int \int_{\mathbb{R}^2} \left((x_1 - \frac{1}{2})^2 + (x_2 - \frac{\sqrt{3}}{6})^2\right) dP(x_1, x_2)$$

$$= \int (x_1 - \frac{1}{2})^2 dP_1(x_1) + \int (x_2 - \frac{\sqrt{3}}{6})^2 dP_2(x_2) = V(X_1) + V(X_2) = \frac{1}{6},$$

which completes the proof of the lemma. \hfill \Box

**Note 2.4.** From Lemma 2.3 it follows that the optimal set of one-mean is the expected value and the corresponding quantization error is the variance $V$ of the random variable $X$. For $\omega \in I^k$, $k \geq 1$, since $a(\omega) = E(X : X \in J_\omega)$, using Lemma 2.1 we have

$$a(\omega) = \frac{1}{P(\Delta_\omega)} \int_{\Delta_\omega} x dP(x) = \int_{\Delta_\omega} x dP \circ S_\omega^{-1}(x) = \int S_\omega(x) dP(x) = E(S_\omega(X)) = S_\omega(\frac{1}{2}, \frac{\sqrt{3}}{6}).$$

For any $(a, b) \in \mathbb{R}^2$, $E\|X - (a, b)\|^2 = V + \|(\frac{1}{2}, \frac{\sqrt{3}}{6}) - (a, b)\|^2$. In fact, for any $\omega \in I^k$, $k \geq 1$, we have $\int_{\Delta_\omega} \|x - (a, b)\|^2 dP = \frac{1}{3^k} \int \|x_1, x_2) - (a, b)\|^2 dP \circ S_\omega^{-1}$, which implies

\begin{equation}
(2) \quad \int_{\Delta_\omega} \|x - (a, b)\|^2 dP = \frac{1}{3^k} \left(\frac{1}{9^k} V + \|a(\omega) - (a, b)\|^2\right).
\end{equation}

In the next section, we determine the optimal sets of $n$-means for all $n \geq 2$. 
3. Optimal sets of \( n \)-means for all \( n \geq 2 \)

Recall that \( \alpha_n \) represents an optimal set of \( n \)-means for all \( n \geq 1 \), and for any \( \omega \in I^k \) by \( a(\omega) \) it is meant \( a(\omega) = S_\omega(E(X)) \). Also, recall the notation given by (\( \square \)). In this section let us first prove the following proposition.

**Proposition 3.1.** The set \( \{ (\frac{1}{2}, \frac{\sqrt{3}}{6\sqrt{3}}), (\frac{1}{2}, \frac{\sqrt{3}}{6\sqrt{3}}) \} \) is an optimal set of two-means with quantization error \( V_2 = \frac{5}{54} = 0.0925926 \).

**Proof.** Note that with respect to any of its medians, the R-triangle has the maximum symmetry, i.e., with respect to any of its medians the R-triangle is geometrically symmetric as well as symmetric with respect to the probability distribution \( P \). By the symmetric with respect to the probability distribution \( P \), it is meant that if the two basic triangles of similar geometrical shape lie in the opposite sides of a median, and are equidistant from the median, then they have the same probability. Due to this, among all the pairs of two points which have the boundaries of the Voronoi regions oblique lines passing through the centroid \( (\frac{1}{2}, \frac{\sqrt{3}}{6}) \), the two points which have the boundary of the Voronoi regions the line perpendicular to a median will give the smallest distortion error. Without any loss of generality, to get an optimal set of two-means we consider the median passing through the vertex \( (\frac{1}{2}, \frac{\sqrt{3}}{6}) \). Let \( \alpha := \{ (p, b_1), (p, b_2) \} \) be an optimal set of two-means with \( b_1 \leq b_2 \). Since the optimal points are the centroids of their own Voronoi regions, by the properties of centroids, we have

\[
(p, b_1)P(M((p, b_1)|\alpha)) + (p, b_2)P(M((p, b_2)|\alpha)) = \left( \frac{1}{2}, \frac{\sqrt{3}}{6} \right),
\]

which implies \( p = \frac{1}{2} \) and \( b_1P(M((p, b_1)|\alpha)) + b_2P(M((p, b_2)|\alpha)) = \frac{\sqrt{3}}{6} \).

Thus, one can see that the two optimal points are \( (\frac{1}{2}, b_1) \) and \( (\frac{1}{2}, b_2) \), and they lie in the opposite sides of the point \( (\frac{1}{2}, \frac{\sqrt{3}}{6}) \). This yields the fact that \( \Delta_1 \cup \Delta_2 \subset M((\frac{1}{2}, b_1)|\alpha) \) and \( \Delta_3 \subset M((\frac{1}{2}, b_2)|\alpha) \). Again, the optimal points are the centroids of their own Voronoi regions, and so by equation (\( \square \)), we have

\[
\left( \frac{1}{2}, b_1 \right) = E(X : X \in \Delta_1 \cup \Delta_2) = \frac{1}{3} + \frac{1}{3} \left( \frac{1}{3}a(1) + \frac{1}{3}a(2) \right) = \left( \frac{1}{2}, \frac{1}{6\sqrt{3}} \right),
\]

\[
\left( \frac{1}{2}, b_2 \right) = E(X : X \in \Delta_3) = a(3) = \left( \frac{1}{2}, \frac{7}{6\sqrt{3}} \right),
\]

and then the quantization error is

\[
V_2 = \int_{\Delta_1 \cup \Delta_2} \min_{a \in \alpha} \| x - a \|^2 dP + \int_{\Delta_3} \min_{a \in \alpha} \| x - a \|^2 dP
\]

\[
= \int_{\Delta_1} \| x - \left( \frac{1}{2}, \frac{1}{6\sqrt{3}} \right) \|^2 dP + \int_{\Delta_2} \| x - \left( \frac{1}{2}, \frac{1}{6\sqrt{3}} \right) \|^2 dP + \int_{\Delta_3} \| x - \left( \frac{1}{2}, \frac{7}{6\sqrt{3}} \right) \|^2 dP
\]

\[
= \frac{5}{54} = 0.0925926.
\]

Hence, the proof of the proposition is complete. \( \square \)

**Remark 3.2.** Due to symmetry, the sets \( \{ (\frac{1}{2}, \frac{1}{6\sqrt{3}}), (\frac{2}{3}, \frac{2}{3\sqrt{3}}) \} \) and \( \{ (\frac{5}{6}, \frac{1}{6\sqrt{3}}), (\frac{1}{3}, \frac{2}{3\sqrt{3}}) \} \) also form optimal sets of two-means with quantization error \( V_2 = \frac{5}{54} \) (see Figure (\( \square \))).

**Lemma 3.3.** Let \( \alpha \) be an optimal set of \( n \)-means with \( n \geq 3 \). Then \( \alpha \cap \Delta_i \neq \emptyset \) for all \( 1 \leq i \leq 3 \).
Proof. Let us consider the three-point set $\beta$ given by $\beta = \{a(1), a(2), a(3)\}$. Then, the distortion error is
\[
\int \min_{a \in \alpha} \|x - a\|^2 dP = \sum_{i=1}^{3} \int_{\Delta_i} \|x - a(i)\|^2 dP = 3 \cdot \frac{1}{162} = \frac{1}{54} = 0.0185185.
\]
Since, $V_n$ is the quantization error for $n \geq 3$, we have $0.0185185 \geq V_3 \geq V_n$. Let $\alpha$ be an optimal set of $n$-means for $n \geq 3$. As the optimal points are the centroids of their own Voronoi regions we have $\alpha \subset \Delta$. To prove the lemma, let us proceed as follows:

Suppose that $\alpha$ does not contain any point from $\bigcup_{i=1}^{3} \Delta_i$. If all the points of $\alpha$ are below the line $x_2 = \frac{2\sqrt{3}}{9}$, for any $(x_1, x_2) \in \Delta_3$ we have $\min_{(a,b)\in \alpha} \|(x_1, x_2) - (a,b)\|^2 \geq (\frac{\sqrt{3}}{3} - \frac{2\sqrt{3}}{9})^2 = \frac{1}{27}$, and for any $(x_1, x_2) \in \Delta_{11} \cup \Delta_{22}$ we have $\min_{(a,b)\in \alpha} \|(x_1, x_2) - (a,b)\|^2 \geq \frac{1}{27}$, and then the distortion error is obtained as
\[
\int \min_{a \in \alpha} \|x - a\|^2 dP > \int_{\Delta_3} \min_{a \in \alpha} \|x - a\|^2 dP + \int_{\Delta_{11} \cup \Delta_{22}} \min_{a \in \alpha} \|x - a\|^2 dP
\]
\[
\geq \frac{1}{3} \cdot \frac{1}{27} + 2 \cdot \frac{1}{9} \cdot \frac{1}{27} = \frac{5}{243} = 0.0205761 > V_3,
\]
which is a contradiction. If $\alpha$ does not contain any point below the line $x_2 = \frac{2\sqrt{3}}{9}$, for any $(x_1, x_2) \in \Delta_{11} \cup \Delta_{12} \cup \Delta_{21} \cup \Delta_{22}$ we have $\min_{a\in \alpha} \|(x_1, x_2) - a\|^2 \geq \frac{1}{12}$, and then the distortion
error is obtained as
\[
\int \min_{a \in \alpha} \|x - a\|^2 dP > \int \min_{a \in \alpha} \|x - a\|^2 dP \geq 4 \cdot \frac{1}{9} \cdot \frac{1}{12} = \frac{1}{27} = 0.037037 > V, 
\]
which is a contradiction. Thus, we conclude that \( \alpha \) contains points both above and below the line \( x_2 = \frac{2\sqrt{3}}{9} \). If \( \alpha \) contains two or more points below the line \( x_2 = \frac{2\sqrt{3}}{9} \), then the quantization error can be strictly reduced by moving points below the line \( x_2 = \frac{2\sqrt{3}}{9} \) to \( \Delta_1 \) and \( \Delta_2 \), and by moving the points above the line \( x_2 = \frac{2\sqrt{3}}{9} \) to \( \Delta_3 \), and so, we assume that \( \alpha \) contains only one point below the line \( x_2 = \frac{2\sqrt{3}}{9} \). Due to symmetry we can assume that this point lies on the line \( x_1 = \frac{1}{2} \). Then, notice that \( a(12, 21) = (\frac{1}{2}, \frac{1}{18\sqrt{3}}) \) and it is the midpoint of the line segment joining the centroids of \( \Delta_{12} \) and \( \Delta_{21} \); the point of intersection of the lines \( x_2 = \sqrt{3}x_1 \) and \( x_2 = \frac{2\sqrt{3}}{9} \) is \( (\frac{2}{9}, \frac{2\sqrt{3}}{9}) \), and the base of the perpendicular passing through \( (0, 0) \) of the triangle \( \Delta_1 \) is \( (\frac{1}{2}, \frac{1}{4\sqrt{3}}) \). Hence, we obtain
\[
(3) \int \min_{a \in \alpha} \|x - a\|^2 dP \\
\geq \int_{\Delta_{12} \cup \Delta_{21}} \min_{a \in \alpha} \|x - a\|^2 dP + \int_{\Delta_{13} \cup \Delta_{23}} \min_{a \in \alpha} \|x - a\|^2 dP + \int_{\Delta_{11} \cup \Delta_{22}} \min_{a \in \alpha} \|x - a\|^2 dP \\
\geq 2 \int_{\Delta_{12}} \|x - (\frac{1}{2}, \frac{1}{18\sqrt{3}})^2 dP + 2 \int_{\Delta_{13}} \|x - (\frac{2}{9}, \frac{2\sqrt{3}}{9})\|^2 dP + 2 \int_{\Delta_{11}} \|x - (\frac{1}{4}, \frac{1}{4\sqrt{3}})\|^2 dP \\
= \frac{25}{2187} + \frac{5}{729} + \frac{17}{1458} = \frac{131}{6372} = 0.0299497 > V, 
\]
which is a contradiction. Thus, we arrive at a contradiction under the assumption that \( \alpha \) does not contain any point from \( \Delta_1 \cup \Delta_2 \cup \Delta_3 \). Hence, \( \alpha \) contains at least one point from \( \bigcup_{i=1}^{3} \Delta_i \).

Due to symmetry without any loss of generality we can assume that \( \alpha \) contains at least one point from \( \Delta_3 \) and does not contain any point from \( \Delta_1 \cup \Delta_2 \). Then, notice that Voronoi region of any point of \( \alpha \) which are below the line \( x_2 = \frac{2\sqrt{3}}{9} \) does not contain any point from \( \Delta_3 \); if it does then the quantization error can be strictly reduced by relocating the points, and it will contradict the fact that \( \alpha \) is an optimal set. Hence, if \( \alpha \) contains two or more points below the line \( x_2 = \frac{2\sqrt{3}}{9} \), quantization error can be strictly reduced by moving points to \( \Delta_1 \) and to \( \Delta_2 \). So, we assume that \( \alpha \) contains only one point below the line \( x_2 = \frac{2\sqrt{3}}{9} \). Then as shown in (3), we have the distortion error as
\[
\int \min_{a \in \alpha} \|x - a\|^2 dP \geq \frac{131}{4374} = 0.0299497 > V, 
\]
which is a contradiction. Thus, we conclude that \( \alpha \) does not contain any point from \( \Delta \) below the line \( x_2 = \frac{2\sqrt{3}}{9} \). But, then,
\[
\int \min_{a \in \alpha} \|x - a\|^2 dP \geq \int_{\Delta_1 \cup \Delta_2} \min_{a \in \alpha} \|x - a\|^2 dP \\
\geq 2 \left( \int_{\Delta_{11} \cup \Delta_{13}} \|x - (\frac{2}{9}, \frac{2\sqrt{3}}{9})\|^2 dP + \int_{\Delta_{12}} \|x - (\frac{5}{18}, \frac{2\sqrt{3}}{9})\|^2 dP \right) = \frac{101}{1458} = 0.069273, 
\]
which is larger than \( V \), and so another contradiction arises. All these contradictions arise due to our assumption that \( \alpha \) contains at least one point from \( \Delta_3 \), and does not contain any point
from $\Delta_1 \cup \Delta_2$. We now assume that $\alpha$ contains points from any two of the basic triangles $\Delta_1$, $\Delta_2$ and $\Delta_3$. Due to symmetry, without any loss of generality, we can now assume that $\alpha$ contains points from $\Delta_1$ and $\Delta_2$, but does not contain any point from $\Delta_3$. In this situation, suppose that $\alpha$ does not contain any point above the line $x_2 = \frac{2\sqrt{3}}{9}$. Then, for any $(x_1, x_2) \in \Delta_1 \cup \Delta_2$, we have $\min_{(a,b) \in \alpha} \| (x_1, x_2) - (a, b) \|^2 \geq \left( \frac{\sqrt{3}}{3} - 2\sqrt{3} \right)^2 = \frac{1}{27}$; and for any $(x_1, x_2) \in \Delta_3$, we have $\min_{(a,b) \in \alpha} \| (x_1, x_2) - (a, b) \|^2 \geq \left( \frac{4\sqrt{3}}{9} - 2\sqrt{3} \right)^2 = \frac{1}{27}$. Thus, the distortion error is obtained as

$$\int_{a \in \alpha} \min \| x - a \|^2 dP > \int_{\Delta_1 \cup \Delta_2} \min \| x - a \|^2 dP + \int_{\Delta_3} \min \| x - a \|^2 dP \geq \frac{2}{9} \cdot \frac{1}{27} + \frac{1}{9} \cdot \frac{4}{27} = \frac{2}{81} = 0.0246914 > V_3$$

which is a contradiction. So, we can assume that $\alpha$ contains at least one point above the line $x_2 = \frac{2\sqrt{3}}{9}$. Moreover, $\alpha$ contains points from both $\Delta_1$ and $\Delta_2$. Now, if $\alpha$ contains only one point above the line $x_2 = \frac{2\sqrt{3}}{9}$, then the quantization error can be strictly reduced by moving the point to $\Delta_3$. If $\alpha$ contains two or more points above the line $x_2 = \frac{2\sqrt{3}}{9}$, then the quantization error can be strictly reduced by moving at least one point which are above the line $x_2 = \frac{2\sqrt{3}}{9}$ to $\Delta_3$. This contradicts the fact that $\alpha$ is an optimal set of $n$-means with $n \geq 3$. Hence, $\alpha$ contains points from $\Delta_i$ for all $1 \leq i \leq 3$, i.e., $\alpha \cap \Delta_i \neq \emptyset$ for all $1 \leq i \leq 3$. □

**Lemma 3.4.** Let $\alpha$ be an optimal set of $n$-means with $n \geq 3$. Then $\alpha \subset \bigcup_{i=1}^{3} \Delta_i$, and $|n_i - n_j| = 0$, or $1$ for $1 \leq i \neq j \leq 3$ where $n_k = \text{card}(\alpha \cap \Delta_k)$, $1 \leq k \leq 3$.

**Proof.** Consider the following cases:

*Case 1: $n = 3k$ for some positive integer $k \geq 1$.*

Then, due to symmetry we can assume that $\alpha$ contains $k$ points from each of $\Delta_i$, otherwise, quantization error can be strictly reduced by redistributing the points in $\alpha$ equally among $\Delta_i$ for $1 \leq i \leq 3$. So, in this case $\alpha$ does not contain any point from $\Delta \setminus \bigcup_{i=1}^{3} \Delta_i$ and $|n_i - n_j| = 0$ for $1 \leq i \neq j \leq 3$.

*Case 2: $n = 3k + 1$ for some positive integer $k \geq 1$.*

In this case, due to symmetry, we can assume that $\alpha$ contains $k$ points from each of $\Delta_i$, and the remaining one point is $(a, b)$. If possible, let $(a, b) \not\in \bigcup_{i=1}^{3} \Delta_i$. Due to symmetry we assume that $(a, b)$ lies on the line $x_1 = \frac{1}{2}$. Then, if $(a, b)$ lies on or above the line $x_2 = \frac{\sqrt{3}}{6}$, then $M((a, b)|\alpha)$ does not contain any point from $\Delta_1 \cup \Delta_2$. So, quantization error can be strictly reduced by moving the point $(a, b)$ to $\Delta_3$, which is a contradiction. We now assume that $(a, b)$ is below the line $x_2 = \frac{\sqrt{3}}{6}$, then $M((a, b)|\alpha)$ does not contain any point from $\Delta_1$. Let us first assume that $k = 1$, i.e., $\alpha$ contains only one point from each of $\Delta_1$, $\Delta_2$ and $\Delta_3$. Let $(a_1, b_1)$ be the points that $\alpha$ contains from $\Delta_i$ for $1 \leq i \leq 3$. For any position of $(a, b)$ on the line $x_1 = \frac{1}{2}$, always $\Delta_{11} \subset M((a_1, b_1)|\alpha)$. If $M((a_1, b_1)|\alpha)$ does not contain any point from $\Delta_{13} \cup \Delta_{12}$, then we have $(a_1, b_1) = a(11) = (\frac{1}{18}, \frac{1}{18}\sqrt{3})$. But, then $M((a, b)|\alpha)$ does not contain any point from $\Delta_{11} \cup \Delta_{13} \cup \Delta_{121}$, and so $M((a_1, b_1)|\alpha)$ must contain $\Delta_{11} \cup \Delta_{13} \cup \Delta_{121}$. If $M((a_1, b_1)|\alpha)$ does not contain any point from $\Delta_{122} \cup \Delta_{123}$, then, $(a_1, b_1) = a(11, 12, 121) = (\frac{7}{54}, \frac{73}{378}\sqrt{3})$. But, then if we draw the boundary of the Voronoi regions of $(a_1, b_1)$ and $(a, b)$, we see that $M((a, b)|\alpha)$ does not contain any point from $\Delta_{11} \cup \Delta_{13} \cup \Delta_{121} \cup \Delta_{123}$ and it covers largest area from $\Delta_1$.
if \((a, b) = \left(\frac{1}{2}, 0\right)\). Thus, we can take
\[
(a_1, b_1) = a(11, 13, 121, 123) = \left(\frac{4}{27}, \frac{5}{27\sqrt{3}}\right) \text{ and } (a, b) = \left(\frac{1}{2}, 0\right).
\]

Write \(A := \Delta_{11} \cup \Delta_{13} \cup \Delta_{121} \cup \Delta_{123}\). If \(A \subset M((a_1, b_1)|\alpha)\) and \(\Delta_{122} \subset M((a, b)|\alpha)\), then the distortion error is obtained as
\[
\int \min_{c \in \alpha} \|x-c\|^2dP = 2\left(\int \|x-(a_1, b_1)\|^2 + dP + \int \|x-(a, b)\|^2dP\right) + \int \|x-(a_3, b_3)\|^2dP
\]
\[
= 2\left(\int \|x-\left(\frac{4}{27}, \frac{5}{27\sqrt{3}}\right)\|^2 + dP + \int \|x-\left(\frac{1}{2}, 0\right)\|^2dP\right) + \int \|x-a(3)\|^2dP
\]
\[
= \frac{1100}{59049} = 0.0186286,
\]
which is larger than 0.015775, where \(\frac{22}{138} = 0.015775\) is the distortion error due to the four-point set \(\beta\) given by \(\beta := \{a(13), a(11, 12), a(2), a(3)\}\) which contradicts the optimality of \(\alpha\). Note that in the above calculation we assumed \(\Delta_{11} \cup \Delta_{13} \cup \Delta_{121} \cup \Delta_{123} \subset M((a_1, b_1)|\alpha)\) and \(\Delta_{122} \subset M((a, b)|\alpha)\). If not, then \(M((a_1, b_1)|\alpha)\) will contain points from \(\Delta_{122}\), and then the boundary of the Voronoi regions of the points \(a_1, b_1\) and \((a, b)\) will move further right from the current position, and proceeding similarly we can show that a contradiction arises. Similarly, we can show that if \(k \geq 2\), contradiction arises. Thus, the point \((a, b)\) must belong to either \(\Delta_1, \Delta_2\), or \(\Delta_3\), i.e., \(\alpha\) must contain \((k+1)\) points from one of \(\Delta_i\) for \(1 \leq i \leq 3\), and \(k\) points from each of the remaining two triangles.

Case 3: \(n = 3k + 2\) for some positive integer \(k \geq 1\). In this case, due to symmetry, we can assume that \(\alpha\) contains \(k\) points from each of \(\Delta_i\), and the other two points are symmetrically distributed over the triangle \(\Delta\) with respect to one of the medians, say the median passing through the vertex \(\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\). Then, due to symmetry \(\alpha\) must contain \((k+1)\) points from \(\Delta_1\) and \((k+1)\) points from \(\Delta_2\), otherwise quantization error can be strictly reduced by moving one point to \(\Delta_1\) and one point to \(\Delta_2\).

Hence, by Case 1, Case 2 and Case 3, we see that if \(\alpha\) is an optimal set of \(n\)-means with \(n \geq 3\), then \(\alpha \subset \bigcup_{i=1}^{3} \Delta_i\), and \(|n_i - n_j| = 0, \text{ or } 1 \leq i \neq j \leq 3\) where \(n_k = \text{card}(\alpha \cap \Delta_k)\), \(1 \leq k \leq 3\). Thus, the proof of the lemma is complete. \(\square\)

As an immediate consequence of Lemma 3.4 we obtain the statement below.

**Corollary 3.5.** The set \(\{a(1), a(2), a(3)\}\) is a unique optimal set of three-means for the R-measure \(P\) with quantization error \(V_3 = \frac{1}{54} = 0.0185185\) (see Figure 7).

The following lemma plays an important role in the paper.

**Lemma 3.6.** Let \(n \geq 3\) and let \(\alpha\) be an optimal set of \(n\)-means. For \(1 \leq i \leq 3\), set \(\beta_i := \alpha \cap \Delta_i\) and \(n_i := \text{card}(\beta_i)\). Then, \(S_i^{-1}(\beta_i)\) is an optimal set of \(n_i\)-means, and \(V_n = \sum_{i=1}^{3} \frac{1}{27} V_{n_i}\).

**Proof.** For \(n \geq 3\), by Lemma 3.3 and Lemma 3.4 we have \(\alpha = \bigcup_{i=1}^{3} \beta_i\), \(n = n_1 + n_2 + n_3\), and so
\[
V_n = \sum_{i=1}^{3} \int \min_{a \in \beta_i} \|x-a\|^2dP.
\]
If \(S_i^{-1}(\beta_i)\) is not an optimal set of \(n_i\)-means for \(P\), then there exists a set \(\gamma_1 \subset \mathbb{R}^2\) with \(\text{card}(\gamma_1) = n_1\) such that \(\int \min_{a \in \gamma_1} \|x-a\|^2dP < \int \min_{a \in S_i^{-1}(\beta_i)} \|x-a\|^2dP\).
But then, \( \delta := S_1(\gamma_1) \cup \beta_2 \cup \beta_3 \) is a set of cardinality \( n \), and since

\[
\int_{\Delta_1} \min_{a \in S_1(\gamma_1)} \|x-a\|^2dP = \int_{\Delta_1} \min_{a \in \gamma_1} \|x-S_1(a)\|^2dP = \frac{1}{3} \int_{\Delta_1} \min_{a \in \gamma_1} \|x-S_1(a)\|^2d(P \circ S_1^{-1})
\]

\[
= \frac{1}{27} \int_{\Delta_1} \min_{a \in \gamma_1} \|x-a\|^2dP < \frac{1}{27} \int_{\Delta_1} \min_{a \in S_1^{-1}(\beta_1)} \|x-a\|^2dP = \frac{1}{27} \int_{\Delta_1} \min_{a \in \beta_1} \|x-S_1^{-1}(a)\|^2dP
\]

\[
= \frac{1}{3} \int_{\Delta_1} \min_{a \in \beta_1} \|x-a\|^2d(P \circ S_1^{-1}) = \int_{\Delta_1} \min_{a \in \beta_1} \|x-a\|^2dP,
\]

we have

\[
\int_{\Delta_1} \min_{a \in \delta} \|x-a\|^2dP = \int_{\Delta_1} \min_{a \in S_1(\gamma_1)} \|x-a\|^2dP + \sum_{i=2}^{3} \int_{\Delta_i} \min_{a \in \beta_i} \|x-a\|^2dP < \int_{\Delta_1} \min_{a \in \alpha} \|x-a\|^2dP,
\]

which contradicts the fact that \( \alpha \) is an optimal set of \( n \)-means for \( P \). Similarly, it can be proved that \( S_2^{-1}(\beta_2) \) and \( S_3^{-1}(\beta_3) \) are optimal sets of \( n_2 \)- and \( n_3 \)-means respectively. Thus,

\[
V_n = \frac{1}{3} \int_{\Delta_1} \min_{a \in \beta_1} \|x-a\|^2d(P \circ S_1^{-1}) = \frac{1}{3} \int_{\Delta_1} \min_{a \in S_1^{-1}(\beta_1)} \|x-a\|^2dP = \frac{1}{3} \int_{\Delta_1} \min_{a \in \alpha} \|x-a\|^2dP = \frac{1}{27} V_n,
\]

which is the lemma.

**Lemma 3.7.** Let \( P = \sum_{\omega \in P} \frac{1}{3} P \circ S_\omega^{-1} \) for some \( k \geq 1 \). Let \( \alpha \) be an optimal set of \( n \)-means for the \( R \)-measure \( P \). Then, \( \{S_\omega(a) : a \in \alpha\} \) is an optimal set of \( n \)-means for the image measure \( P \circ S_\omega^{-1} \). The converse is also true: If \( \beta \) is an optimal set of \( n \)-means for the image measure \( P \circ S_\omega^{-1} \), then \( \{S_\omega^{-1}(a) : a \in \beta\} \) is an optimal set of \( n \)-means for \( P \).

**Proof.** If \( \{S_\omega(a) : a \in \alpha\} \) is not an optimal set of \( n \)-means for the image measure \( P \circ S_\omega^{-1} \), then we can find a set \( \gamma \subset \mathbb{R}^2 \) with \( \text{card}(\gamma) = n \) such that

\[
\int_{\gamma} \min_{a \in \gamma} \|x-a\|^2d(P \circ S_\omega^{-1}) < \int_{\alpha} \min_{a \in \alpha} \|x-S_\omega(a)\|^2d(P \circ S_\omega^{-1}),
\]

which implies \( \int_{\gamma} \min_{a \in \gamma} \|S_\omega(x) - a\|^2dP < \int_{\alpha} \min_{a \in \alpha} \|S_\omega(x) - S_\omega(a)\|^2dP \), i.e.,

\[
\int_{\gamma} \min_{a \in S_\omega^{-1}(\gamma)} \|x-a\|^2dP < \int_{\alpha} \min_{a \in \alpha} \|x-a\|^2dP.
\]

Note that \( S_\omega^{-1}(\gamma) \) has cardinality \( n \), and so the last inequality contradicts the fact that \( \alpha \) is an optimal set of \( n \)-means for \( P \). Hence, \( \{S_\omega(a) : a \in \alpha\} \) is an optimal set of \( n \)-means for the image measure \( P \circ S_\omega^{-1} \). To prove the converse, let \( \beta \) be an optimal set of \( n \)-means for the image measure \( P \circ S_\omega^{-1} \). If \( S_\omega^{-1}(\beta) \) is not an optimal set of \( n \)-means for \( P \), then there exists a set \( \delta \subset \mathbb{R}^2 \) with \( \text{card}(\delta) = n \) such that \( \int_{\delta} \min_{a \in \delta} \|x-a\|^2dP < \int_{\beta} \min_{a \in S_\omega^{-1}(\beta)} \|x-a\|^2dP \), which implies

\[
\int_{\delta} \min_{a \in \delta} \|S_\omega(x) - S_\omega(a)\|^2dP < \int_{\beta} \min_{a \in S_\omega^{-1}(\beta)} \|S_\omega(x) - S_\omega(a)\|^2dP
\]

i.e.,

\[
\int_{\delta} \min_{a \in S_\omega(\delta)} \|x-a\|^2d(P \circ S_\omega^{-1}) < \int_{\beta} \min_{a \in \beta} \|x-a\|^2d(P \circ S_\omega^{-1}).
\]

Note that \( S_\omega(\delta) \) has cardinality \( n \), and so the last inequality contradicts the fact that \( \beta \) is an optimal set of \( n \)-means for \( P \circ S_\omega^{-1} \). Thus, we can say that \( \{S_\omega^{-1}(a) : a \in \beta\} \) is an optimal set of \( n \)-means for \( P \) if \( \beta \) is an optimal set of \( n \)-means for the image measure \( P \circ S_\omega^{-1} \). Hence, the proof of the lemma follows. \( \square \)
Remark 3.8. If \( \beta \) is an optimal set of \( n \)-means for the image measure \( P \circ S_\omega^{-1} \), and \( \gamma \) is an optimal set of \( \ell \)-means for the image measure \( P \circ S_\tau^{-1} \), then \( S_\omega^{-1}(\beta) \cup S_\tau^{-1}(\gamma) \) is not necessarily an optimal set of \((n + \ell)\)-means for \( P \).

Lemma 3.9. The set \( \{a(1), a(2), a(33), a(31, 32)\} \) is an optimal set of four-means with quantization error \( V_4 = \frac{1}{27}(2V_1 + V_2) \).

Proof. Let \( \alpha \) be an optimal set of four-means. Let \( \beta_i = \alpha \cap \Delta_i \) for \( 1 \leq i \leq 3 \). By Lemma 3.3 and Lemma 3.4, we can assume that \( \text{card}(\beta_1) = \text{card}(\beta_2) = 1 \) and \( \text{card}(\beta_3) = 2 \), and \( \alpha = \bigcup_{i=1}^{3} \beta_i \).

By Lemma 3.6, both \( S_\tau^{-1}(\beta_1) \) and \( S_\tau^{-1}(\beta_2) \) are optimal sets of one-mean, and \( S_\tau^{-1}(\beta_3) \) is an optimal set of two-means. Thus, we can take \( S_\tau^{-1}(\beta_1) = S_\tau^{-1}(\beta_2) = (\frac{1}{3}, \sqrt[3]{3}) \), and \( S_\tau^{-1}(\beta_3) = \{a(3), a(1, 2)\} \) yielding \( \beta_1 = \{a(1)\} \), \( \beta_2 = \{a(2)\} \), and \( \beta_3 = \{a(33), a(31, 32)\} \). By Lemma 3.6 we have the quantization error as \( V_4 = \frac{1}{27}(2V_1 + V_2) \), which completes the proof of the lemma.

Remark 3.10. Due to symmetry, there are nine optimal sets of four-means with quantization error \( V_4 = \frac{23}{1458} \) (see Figure 1).

Lemma 3.11. Let \( n = 3^\ell(n) + 1 \) for some positive integer \( \ell(n) \). Then, \( \{a(\omega) : \omega \in I^{\ell(n)} \setminus \{\tau\} \} \cup S_\tau(\alpha_2) \) is an optimal set of \( n \)-means for any \( \tau \in I^{\ell(n)} \).

Proof. Let us prove it by induction. If \( n = 4 \) then it is true by Lemma 3.9. Let us assume that it is true if \( n = 3^k + 1 \) for some positive integer \( k \). Let \( \alpha \) be an optimal set of \( n \)-means for \( n = 3^{k+1} + 1 \). Let \( \beta_i = \alpha \cap \Delta_i \) for \( 1 \leq i \leq 3 \). By Lemma 3.3 and Lemma 3.4, we can assume that \( \text{card}(\beta_1) = \text{card}(\beta_2) = 3^k \) and \( \text{card}(\beta_3) = 3^k + 1 \), and \( \alpha = \bigcup_{i=1}^{3} \beta_i \). Then, by Lemma 3.6 both \( S_\tau^{-1}(\beta_1) \) and \( S_\tau^{-1}(\beta_2) \) are optimal sets of \( 3^k \)-means, and \( S_\tau^{-1}(\beta_3) \) is an optimal set of \((3^k + 1)\)-means. Thus, we can write \( \beta_1 = \{a(1\omega) : \omega \in I^k\} \), \( \beta_2 = \{a(2\omega) : \omega \in I^k\} \) and \( \beta_3 = \{a(3\omega) : \omega \in I^k \setminus \{\tau\}\} \cup S_\tau(\alpha_2) \) for some \( \tau \in I^k \). Hence, \( \alpha = \{a(\omega) : \omega \in I^{k+1} \setminus \{\tau\}\} \cup S_\tau(\alpha_2) \) for some \( \tau \in I^{k+1} \) is an optimal set of \( n \)-means for \( n = 3^{k+1} + 1 \). Thus, by the Principle of Mathematical Induction, the proof of the lemma is complete.

Now we prove the following propositions which provide further information on the optimal sets of \( n \)-means.

Proposition 3.12. Let \( n \in \mathbb{N} \) be such that \( n = 3^\ell(n) \) for some positive integer \( \ell(n) \). Then, the set \( \alpha_{3^\ell(n)} := \{a(\omega) : \omega \in I^{\ell(n)}\} \) is a unique optimal set of \( n \)-means for \( P \) with quantization error \( V_n = \frac{1}{6} \cdot \frac{1}{9^{\ell(n)}} \).

Proof. Let us prove it by induction. By Corollary 3.5 it is true if \( \ell(n) = 1 \). Let us assume that it is true for \( n = 3^k \) for some positive integer \( k \). We now show that it is also true if \( n = 3^{k+1} \). Let \( \beta \) be an optimal set of \( 3^{k+1} \)-means. Set \( \beta_i := \beta \cap \Delta_i \) for \( 1 \leq i \leq 3 \). Note that \( \text{card}(\beta_i) = 3^k \). Then, by Lemma 3.3 and Lemma 3.4 and Lemma 3.6, \( S_\tau^{-1}(\beta_i) \) is an optimal set of \( 3^k \)-means, and so \( S_\tau^{-1}(\beta_i) = \{a(\omega) : \omega \in I^k\} \) which implies \( \beta_i = \{a(\omega) : \omega \in I^k\} \). Thus, \( \beta = \beta_1 \cup \beta_2 \cup \beta_3 = \{a(\omega) : \omega \in I^{k+1}\} \) is an optimal set of \( 3^{k+1} \)-means. Since \( a(\omega) \) is the centroid of \( \Delta_\omega \) for each \( \omega \in I^{k+1} \), the set \( \beta \) is unique. Now, by Lemma 3.6 we have the quantization error as

\[
V_{3^{k+1}} = \sum_{i=1}^{3} \frac{1}{27} V_{3^k} = \frac{1}{9} \cdot \frac{1}{6} \cdot \frac{1}{9^k} = \frac{1}{6} \cdot \frac{1}{9^{k+1}}.
\]

Thus, by the Principle of Mathematical Induction, the proof of the proposition is complete.

Proposition 3.13. Let \( 3^\ell(n) < n \leq 2 \cdot 3^\ell(n) \) for some positive integer \( \ell(n) \). Choose \( J \subset I^{\ell(n)} \) with \( \text{card}(J) = n - 3^\ell(n) \), and then the set

\[
\alpha_n(J) := \{a(\omega) : \omega \in I^{\ell(n)} \setminus J\} \cup \bigcup_{\omega \in J} S_\omega(\alpha_2)
\]

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is an optimal set of $n$-means for the $R$-measure $P$.

**Proof.** Let us prove it by induction. If $\ell(n) = 1$, i.e., when $3 < n \leq 2 \cdot 3$, the proposition is true, and it can be proved proceeding as in Lemma 3.1. Let the proposition be true if $\ell(n) = m$ for some positive integer $m$. We now show that it is also true if $\ell(n) = m + 1$. Let $\beta$ be an optimal set of $n$-means where $n = 3^{m+1} + k$ and $1 \leq k \leq 3^{m+1}$. Let $J \subset I^{m+1}$ be such that $\text{card}(J) = k$ for $1 \leq k \leq 3^{m+1}$. Set $\beta := \beta \cap \Delta_i$ for $1 \leq i \leq 3$. Then, $\beta = \bigcup_{i=1}^{3} \beta_i$ and card($\beta_i$) = $3^m + k_i$, where $k_i := \text{card}\{\omega \in J : a(\omega) \in \Delta_i \cap \beta_i\}$ for $1 \leq i \leq 3$. Notice that $0 \leq k_i \leq 3^m$ and $k = k_1 + k_2 + k_3$. By Lemma 3.6, $S_i^{-1}(\beta_i)$ is an optimal set of $(3^m + k_i)$-means, and so we can write

$$S_i^{-1}(\beta_i) = \{a(\omega) : \omega \in I^m \setminus J_i\} \cup \bigcup_{\omega \in J_i} S_\omega(\alpha_2)$$

where $J_i \subset I^m$ with card($J_i$) = $k_i$. Note that if card($J_i$) = 0 then the set $\bigcup_{\omega \in J_i} S_\omega(\alpha_2)$ is an empty set. Thus, we have

$$\beta_i = \{a(i\omega) : \omega \in I^m \setminus J_i\} \cup \bigcup_{\omega \in J_i} S_\omega(\alpha_2).$$

Hence, $\beta = \beta_1 \cup \beta_2 \cup \beta_3 = \{a(\omega) : \omega \in I^{m+1} \setminus J\} \cup \bigcup_{\omega \in J} S_\omega(\alpha_2)$ is an optimal set of $n$-means for $n = 3^{m+1} + k$. Therefore, by the Principle of Mathematical Induction, the proposition is true.

**Proposition 3.14.** Let $n \in \mathbb{N}$ be such that $2 \cdot 3^{\ell(n)} < n < 3^{\ell(n) + 1}$. Choose $J \subset I^{\ell(n)}$ with card($J$) = $n - 2 \cdot 3^{\ell(n)}$, and then the set

$$\alpha_n(J) := \bigcup_{\omega \in J} S_\omega(\alpha_3) \cup \bigcup_{\omega \in I^{\ell(n)} \setminus J} S_\omega(\alpha_2)$$

is an optimal set of $n$-means for the $R$-measure $P$.

**Proof.** Let $n = 2 \cdot 3^{\ell(n)} + k$ where $1 \leq k < 3^{\ell(n)}$. Let $\beta$ be an optimal set of $n$-means. Write $\beta_i := \beta \cap \Delta_i$ for $1 \leq i \leq 3$. First take $\ell(n) = 1$, then if $k = 1$, by Lemma 3.3, Lemma 3.4 and Lemma 3.6, we can assume that both $S_1^{-1}(\beta_1)$ and $S_2^{-1}(\beta_2)$ are optimal sets of two-means, and $S_3^{-1}(\beta_3)$ is an optimal set of three-means, which yields $\beta = \beta_1 \cup \beta_2 \cup \beta_3 = S_1(\alpha_2) \cup S_2(\alpha_2) \cup S_3(\alpha_3)$, i.e., $\alpha_7(\{3\}) = S_1(\alpha_2) \cup S_2(\alpha_2) \cup S_3(\alpha_3)$. Thus, the proposition is true if $\ell(n) = 1$ and $k = 1$. Similarly, we can prove that the proposition is true if $\ell(n) = 1$ and $1 \leq k < 3^{\ell(n)}$. Let us now assume that the proposition is true if $\ell(n) = m$ for some positive integer $m$, where $1 \leq k < 3^m$. Now proceeding as in the proof of Proposition 3.13, it can be shown that the proposition is also true for $\ell(n) = m + 1$. Therefore, by the Principle of Mathematical Induction, the proposition follows.

Let us now state and prove the following theorem which gives all the optimal sets of $n$-means and their numbers, and the corresponding quantization error for all $n \geq 3$.

**Theorem 3.15.** For $n \in \mathbb{N}$ with $n \geq 3$, let $\ell(n)$ be the unique natural number with $3^{\ell(n)} \leq n < 3^{\ell(n) + 1}$, and $\alpha_n$ be an optimal set of $n$-means. If $n = 3^{\ell(n)}$, then the set $\alpha_n := \{a(\omega) : \omega \in I^{\ell(n)}\}$ is a unique optimal set of $n$-means for $P$. If $3^{\ell(n)} < n \leq 2 \cdot 3^{\ell(n)}$, then the set $\alpha_n(J) = \{a(\omega) : \omega \in I^{\ell(n)} \setminus J\} \cup \bigcup_{\omega \in J} S_\omega(\alpha_2)$, where $J \subset I^{\ell(n)}$ with card($J$) = $n - 3^{\ell(n)}$, is an optimal set of $n$-means, and the number of such sets is $3^{\ell(n)} C_{n-3^{\ell(n)}} 3^{n-3^{\ell(n)}}$. On the other hand, if $2 \cdot 3^{\ell(n)} < n < 3^{\ell(n) + 1}$, then the set $\alpha_n(J) = \bigcup_{\omega \in I^{\ell(n)} \setminus J} S_\omega(\alpha_3) \cup \bigcup_{\omega \in I^{\ell(n)} \setminus J} S_\omega(\alpha_2)$, where $J \subset I^{\ell(n)}$ with card($J$) = $n - 2 \cdot 3^{\ell(n)}$, is an optimal set of $n$-means, and the number of such sets is $3^{\ell(n)} C_{n-2 \cdot 3^{\ell(n)}} 3^{3^{\ell(n) + 1} - n}$. The quantization error is given by $V_n = \frac{1}{2} \cdot \frac{1}{2^{\ell(n)+1}} (13 \cdot 3^{\ell(n)} - 4n)$. 
Proof. Let us first assume that $n = 3^{\ell(n)}$. Then, by Proposition 3.12, $\alpha_3^{\ell(n)} := \{a(\omega) : \omega \in I^{\ell(n)}\}$ is a unique optimal set of $n$-means for $P$ with quantization error

$$V_n = \sum_{\omega \in I^{\ell(n)}} \frac{1}{3^{\ell(n)}} \int \|x - a(\omega)\|^2 d(P \circ S_\omega^{-1}) = \sum_{\omega \in I^{\ell(n)}} \frac{1}{3^{\ell(n)}} \int \|S_\omega(x) - a(\omega)\|^2 dP$$

$$= \sum_{\omega \in I^{\ell(n)}} \frac{1}{3^{\ell(n)}} \frac{1}{9^{\ell(n)}} V = \frac{1}{6} \cdot \frac{1}{27^{\ell(n)+1}} \left(13 \cdot 3^{\ell(n)} - 4n\right).$$

Let us now assume that $3^{\ell(n)} < n \leq 2 \cdot 3^{\ell(n)}$. Then, by Proposition 3.13, $\alpha_n(J) := \{a(\omega) : \omega \in I^{\ell(n)} \setminus J\} \cup \bigcup_{\omega \in I^{\ell(n)} \setminus J} S_\omega(\alpha_2)$, where $J \subset I^{\ell(n)}$ with $\text{card}(J) = n - 3^{\ell(n)}$, is an optimal set of $n$-means.

Since the set $J$ from $I^{\ell(n)}$ can be chosen in $3^{\ell(n)} C_{3^{\ell(n)}}$ ways and for each $\omega \in J$ the set $S_\omega(\alpha_2)$ can be chosen in three different ways, the number of optimal sets of $n$-means in this case is given by $3^{\ell(n)} C_{3^{\ell(n)}} 3^{\ell(n)-3^{\ell(n)}}$. The quantization error is

$$V_n = \int \min_{a \in \alpha_n(J)} \|x - a\|^2 dP = \sum_{\omega \in I^{\ell(n)} \setminus J} \int \min_{a \in \alpha_n(J)} \|x - a\|^2 dP + \sum_{\omega \in J} \int \min_{a \in \alpha_n(J)} \|x - a\|^2 dP$$

$$= \sum_{\omega \in I^{\ell(n)} \setminus J} \int \|x - a(\omega)\|^2 dP + \sum_{\omega \in J} \int \min_{a \in S_\omega(\alpha_2)} \|x - a\|^2 dP$$

$$= \sum_{\omega \in I^{\ell(n)} \setminus J} \frac{1}{3^{\ell(n)}} \int \|x - a(\omega)\|^2 dP \circ S_\omega^{-1} + \sum_{\omega \in J} \frac{1}{3^{\ell(n)}} \int \min_{a \in S_\omega(\alpha_2)} \|x - a\|^2 dP \circ S_\omega^{-1}$$

$$= \sum_{\omega \in I^{\ell(n)} \setminus J} \frac{1}{3^{\ell(n)}} \frac{1}{9^{\ell(n)}} V + \sum_{\omega \in J} \frac{1}{3^{\ell(n)}} \frac{1}{9^{\ell(n)}} V_2 = \sum_{\omega \in I^{\ell(n)} \setminus J} \frac{1}{3^{\ell(n)}} \frac{1}{9^{\ell(n)}} V + \sum_{\omega \in J} \frac{1}{3^{\ell(n)}} \frac{1}{9^{\ell(n)}} V$$

$$= \frac{1}{6} \cdot \frac{1}{27^{\ell(n)+1}} \left(\text{card}(J) + 5 \text{card}(I^{\ell(n)} \setminus J)\right) = \frac{1}{6} \cdot \frac{1}{27^{\ell(n)+1}} \left(n - 2 \cdot 3^{\ell(n)} + 5(3^{\ell(n)+1} - n)\right)$$

$$= \frac{1}{2} \cdot \frac{1}{27^{\ell(n)+1}} \left(13 \cdot 3^{\ell(n)} - 4n\right).$$

Thus, the proof of the theorem is complete. \qed
We now give an example of an optimal set of eleven-means.

**Example 3.16.** $n = 11 = 3^2 + 2$. Take $J = \{11, 12\}$, where $J \subset I^2$ with card($J$) = 2. Take $\alpha_2 = \{a(1, 2), a(3)\}$. Then, by Theorem 3.15,

\[
\alpha_{11}(J) = \{a(\omega) : \omega \in I^2 \setminus J\} \cup \bigcup_{\omega \in J} S_\omega(\alpha_2)
\]

\[=
\{a(1, 3), a(2, 1), a(2, 2), a(2, 3), a(3, 1), a(3, 2), a(3, 3)\}
\]

\[\cup \{S_{11}(a(1, 2)), S_{11}(a(3)), S_{12}(a(1, 2)), S_{12}(a(3))\}\]

\[=
\{a(1, 3), a(2, 1), a(2, 2), a(2, 3), a(3, 1), a(3, 2), a(3, 3),
\]

\[a(111, 112), a(113), a(121, 122), a(123)\}.

Using equation \[\text{(2)}\], we obtain the distortion error as

\[
\int \min_{a \in \alpha_{11}(J)} \|x - a\|^2 dP
\]

\[= 7 \int_{\Delta_{13}} (x - a(13))^2 dP + 2 \int_{\Delta_{113}} (x - a(113))^2 dP + 2 \int_{\Delta_{111} \cup \Delta_{112}} (x - a(111, 112))^2 dP = \frac{73}{39366}.
\]

Now substituting $\ell(n) = 2$ and $n = 11$ in the formula given by Theorem 3.15, we also obtain

\[V_{11} = \frac{1}{2} \cdot \frac{1}{273} (13 \cdot 3^2 - 4 \cdot 11) = \frac{73}{39366}.
\]

The following example gives an optimal set of nineteen-means.

**Example 3.17.** $n = 19 = 2 \cdot 3^2 + 1$. Take $J = \{11\}$, where $J \subset I^2$ with card($J$) = 1. Take $\alpha_2 = \{a(1, 2), a(3)\}$. Notice that $\alpha_3 = \{a(1), a(2), a(3)\}$ which is unique. Then, by Theorem 3.15 we have

\[\alpha_{19}(J) = \bigcup_{\omega \in I^2 \setminus J} S_\omega(\alpha_2) \cup \bigcup_{\omega \in J} S_\omega(\alpha_3)
\]

\[=
\{a(121, 122), a(123), a(131, 132), a(133), a(211, 212), a(213), a(221, 222), a(223), a(231, 232),
\]

\[a(233), a(311, 312), a(313), a(321, 322), a(323), a(331, 332), a(333), a(111), a(112), a(113)\}.

Now substituting $\ell(n) = 2$ and $n = 19$ in the formula given by Theorem 3.15, we obtain

\[V_{19} = \frac{1}{2} \cdot \frac{1}{273} (13 \cdot 3^2 - 4 \cdot 19) = \frac{41}{39366},
\]

which can also be obtained by using equation \[\text{(2)}\].

4. Quantization dimension and quantization coefficient of the R-measure

In this section, we study the quantization dimension and the quantization coefficient of the R-measure. Note that if $\beta$ is the Hausdorff dimension of the R-triangle, then $3\left(\frac{1}{2}\right)^\beta = 1$ which yields $\beta = 1$, i.e., the Hausdorff dimension of the R-triangle is one. Moreover, using the formula given by \[\text{(1)}\] Theorem A], we see that the Hausdorff dimension and the packing dimension of the R-measure are obtained as one. In the following theorem we show that the quantization dimension of the R-measure is also one, and it shows that all these dimensions coincide.

**Theorem 4.1.** Let $P$ be the R-measure as defined in this paper. Then, $\lim_{n \to \infty} \frac{2 \log n}{-\log V_n} = 1$, i.e., the quantization dimension of $P$ exists and equals one.

**Proof.** Let $n \geq 3$, be such that $3^{\ell(n)} \leq n < 3^{\ell(n)+1}$ for some $\ell(n) \in \mathbb{N}$. Then, by Theorem 3.15 we have

\[n^2 V_n \geq 9^{\ell(n)} V_{3^{\ell(n)+1}} = 9^{\ell(n)} \cdot \frac{1}{6} \cdot \frac{1}{9^{\ell(n)+1}} = \frac{1}{54}, \text{ and } n^2 V_n \leq 9^{\ell(n)+1} V_{3^{\ell(n)}} = 9^{\ell(n)+1} \cdot \frac{1}{6} \cdot \frac{1}{9^{\ell(n)}} = \frac{9}{6},
\]
Then, there exists a subsequence \((y_i)\) such that \(y_i \to y\), i.e., \(\lim_{\ell \to \infty} y_i = y\). To show that the set of accumulation points of the subsequence \((y_i)\) is equal to \([1,2]\), we see that \(\bigcup_{n=1}^{\infty} \left[\frac{n}{6}, \frac{n+1}{6}\right] = [1,2]\), and hence, the theorem follows.

**Lemma 4.2.** Define the function \(f : [1,2] \to \mathbb{R}\) by \(f(x) = \frac{1}{54} x^2 (13 - 4x)\). Then, \(f([1,2]) = \left[\frac{1}{6}, \frac{10}{27}\right]\).

**Proof.** We see that \(f'(x) = \frac{1}{27} x (13 - 6x)\), and so the function \(f\) is strictly increasing on the interval \([1,2]\), and \(f(1) = \frac{1}{6}\) and \(f(2) = \frac{10}{27}\). Hence, \(f([1,2]) = \left[\frac{1}{6}, \frac{10}{27}\right]\), which completes the proof of the lemma.

**Theorem 4.3.** \(s\)-dimensional quantization coefficient for \(s = 1\) of the R-measure does not exist.

**Proof.** We need to show that \(\lim_{n \to \infty} n^2 V_n\) does not exist. Let \((n_k)_{k \in \mathbb{N}}\) be a subsequence of the set of natural numbers such that \(3^{\ell(n_k)} \leq n_k < 3^{\ell(n_k)+1}\). To prove the theorem it is enough to show that the set of accumulation points of the subsequence \((n_k^2 V_{n_k})_{k \geq 1}\) equals \(\left[\frac{1}{6}, \frac{10}{27}\right]\). Let \(y \in \left[\frac{1}{6}, \frac{10}{27}\right]\). We now show that \(y\) is a subsequential limit of the sequence \((n_k^2 V_{n_k})_{k \geq 1}\). Since \(y \in \left[\frac{1}{6}, \frac{10}{27}\right]\), \(y = f(x)\) for some \(x \in [1,2]\). Set \(n_{k_\ell} = \lfloor x 3^\ell \rfloor\), where \(\lfloor x 3^\ell \rfloor\) denotes the greatest integer less than or equal to \(x 3^\ell\). Then, \(n_{k_\ell} < n_{k_{\ell+1}}\) and \(\ell(n_{k_\ell}) = \ell\), and there exists \(x_{k_\ell} \in [1,2]\) such that \(n_{k_\ell} = x_{k_\ell} 3^\ell\). Notice that by \(\ell(n_{k_\ell}) = \ell\) it is meant that \(3^\ell \leq n_{k_\ell} < 3^{\ell+1}\). Thus, putting the values of \(V_{n_{k_\ell}}\) from Theorem 3.13 we obtain

\[
n_{k_\ell}^2 V_{n_{k_\ell}} = n_{k_\ell}^2 \frac{1}{2} \cdot \frac{1}{27^\ell + 1} (13 \cdot 3^\ell - 4 n_{k_\ell}) = x_{k_\ell}^2 \frac{1}{2} \cdot \frac{1}{27^\ell + 1} (13 \cdot 3^\ell - 4 x_{k_\ell} 3^\ell),
\]

which yields

\[
\lim_{\ell \to \infty} n_{k_\ell}^2 V_{n_{k_\ell}} = f(x_{k_\ell}).
\]

Again, \(x_{k_\ell} 3^\ell \leq x 3^\ell < x_{k_\ell} 3^\ell + 1\), which implies \(x - \frac{1}{3^\ell} < x_{k_\ell} \leq x\), and so, \(\lim_{\ell \to \infty} x_{k_\ell} = x\). Since, \(f\) is continuous, we have

\[
\lim_{\ell \to \infty} n_{k_\ell}^2 V_{n_{k_\ell}} = f(x) = y,
\]

which yields the fact that \(y\) is an accumulation point of the subsequence \((n_k^2 V_{n_k})_{k \geq 1}\) whenever \(y \in \left[\frac{1}{6}, \frac{10}{27}\right]\). To prove the converse, let \(y\) be an accumulation point of the subsequence \((n_k^2 V_{n_k})_{k \geq 1}\). Then, there exists a subsequence \((n_k^2 V_{n_k})_{k \geq 1}\) of \((n_k^2 V_{n_k})_{k \geq 1}\) such that \(\lim_{i \to \infty} n_i^2 V_{n_i} = y\). Set \(\ell_i = \ell(n_{k_i})\) and \(x_{k_i} = \frac{n_{k_i}}{3^{\ell_i}}\). Then, \(x_{k_i} \in [1,2]\), and as shown in (4), we have

\[
n_{k_i}^2 V_{n_{k_i}} = f(x_{k_i}).
\]

Let \((x_{k_{ij}})_{j \geq 1}\) be a convergent subsequence of \((x_{k_i})_{i \geq 1}\), and then we obtain

\[
y = \lim_{i \to \infty} n_{k_i}^2 V_{n_{k_i}} = \lim_{j \to \infty} n_{k_{ij}}^2 V_{n_{k_{ij}}} = \lim_{j \to \infty} f(x_{k_{ij}}) \in \left[\frac{1}{6}, \frac{10}{27}\right].
\]

Thus, we have proved that the set of accumulation points of the subsequence \((n_k^2 V_{n_k})_{k \geq 1}\) equals \(\left[\frac{1}{6}, \frac{10}{27}\right]\), and hence, the proof of the theorem is complete.
5. Further remarks

In [2] some properties of “fat” Sierpiński triangles were studied. These are the attractors of iterated function systems defined by \( \{S_i\}_{i=1}^3 \), where

\[
S_i(x_1, x_2) = r(x_1, x_2) + (1-r)p_i, \quad r \in (\frac{1}{2}, 1),
\]

and \( p_i \) are three non-collinear points in \( \mathbb{R}^2 \). Their focus is on the calculation of the Hausdorff dimension of these fractals and, since such fractals do not satisfy the open set condition (OSC), the calculation of the Hausdorff dimension is highly non-trivial. They also mention, in passing, the attractors of the iterated function systems when \( r \in (0, 1/2] \) and observe that the resulting fractals satisfy the open set condition, essentially disjoint and have fractal dimension \( \frac{\log 3}{-\log r} \).

Of course, when \( 0 < r < \frac{1}{2} \), the fractals are totally disconnected. The R-triangle we studied above is actually the case \( r = \frac{1}{3} \).

**Remark 5.1.** Let \( 0 < r_1, r_2, r_3 < \frac{1}{2} \). Then, a general R-triangle can be constructed by the contractive mappings \( S_1, S_2, S_3 \) on \( \mathbb{R}^2 \), such that \( S_1(x_1, x_2) = r_1(x_1, x_2), \ S_2(x_1, x_2) = r_2(x_1, x_2) + (1-r_2)(1, 0), \) and \( S_3(x_1, x_2) = r_3(x_1, x_2) + (1-r_3)(\frac{1}{2}, \sqrt{3}/2) \) for all \( (x_1, x_2) \in \mathbb{R}^2 \); or, by the contractive mappings given by \( T_1(x_1, x_2) = r_1(x_1, x_2), \ T_2(x_1, x_2) = r_2(x_1, x_2) + (1-r_2)(1, 0), \) and \( T_3(x_1, x_2) = r_3(x_1, x_2) + (1-r_3)(0, 1) \) for all \( (x_1, x_2) \in \mathbb{R}^2 \). A general singular continuous probability measure \( P \) on a R-triangle can be defined by \( P = p_1 P \circ S_1^{-1} + p_2 P \circ S_2^{-1} + p_3 P \circ S_3^{-1} \) where \( (p_1, p_2, p_3) \) is a probability vector with \( p_i > 0 \) for all \( 1 \leq i \leq 3 \). If \( r_1 = r_2 = r_3 = r \), then a general R-triangle reduces to the triangle considered in this paper. For a general probability distribution on a general R-triangle the optimal sets of \( n \)-means and the \( n \)th quantization error are not known yet for all \( n \geq 2 \).

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