ASYMPTOTIC HEAT KERNELS IN
QUANTUM FIELD THEORY

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Abstract. Asymptotic expansions were first introduced by Henri Poincaré in 1886. This paper describes their application to the semi-classical evaluation of amplitudes in quantum field theory with boundaries. By using zeta-function regularization, the conformal anomaly for a massless spin-$\frac{1}{2}$ field in flat Euclidean backgrounds with boundary is obtained on imposing locally supersymmetric boundary conditions. The quantization program for gauge fields and gravitation in the presence of boundaries is then introduced by focusing on conformal anomalies for higher-spin fields. The conditions under which the covariant Schwinger-DeWitt and the non-covariant, mode-by-mode analysis of quantum amplitudes agree are described.

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1. Introduction

In 1886, Henri Poincaré published a paper on the irregular integrals of linear equations [1]. Section I of [1] is devoted to the asymptotic series, and Poincaré begins by discussing the peculiar properties of Stirling’s series:

$$\log \Gamma(x + 1) = \frac{1}{2} \log(2\pi) + \left( x + \frac{1}{2} \right) \log(x) - x + \frac{B_1}{1 \cdot 2} \frac{1}{x} - \frac{B_2}{3 \cdot 4} \frac{1}{x^2} + \frac{B_3}{5 \cdot 6} \frac{1}{x^3} - \ldots \quad \text{(1.1)}$$

Poincaré points out that this series is always diverging, but one can use it at large $x$. What happens is that, after decreasing very rapidly, the terms become unboundedly large. Nevertheless, if one takes the smallest term, the corresponding error in the evaluation of $\log \Gamma(x + 1)$ is very small. These properties lead to the following definitions, hereafter presented in the more general case of functions defined on a subset of complex numbers.

Let $f$ be a function defined in an unbounded domain $\Omega$. A power series $\sum_{n=0}^{\infty} a_n z^{-n}$, converging or diverging, is said to be an asymptotic expansion of $f$ if, $\forall$ fixed $N \geq 0$, one has

$$f(z) = \sum_{n=0}^{N} a_n z^{-n} + O\left(z^{-\left(N+1\right)}\right) \quad \text{as} \quad z \to \infty. \quad \text{(1.2)}$$

as $z \to \infty$. Hence one finds

$$\lim_{z \to \infty} z^N \left| f(z) - S_N(z) \right| = 0 \quad \text{as} \quad z \to \infty, \quad \text{(1.3)}$$

where $S_N$ is the sum of the first $N + 1$ terms of the series, and one writes

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^{-n} \quad \text{as} \quad z \to \infty. \quad \text{(1.4)}$$
The asymptotic expansions (hereafter denoted by A.E.) have some basic properties, which are standard (by now) but very useful. They are as follows.

(i) The A.E. of \( f \), if it exists, is unique.

(ii) A.E. may be summed, i.e. if \( f(z) \sim \sum_{n=0}^{\infty} a_n z^{-n} \) and \( g(z) \sim \sum_{n=0}^{\infty} b_n z^{-n} \), then

\[
\alpha f(z) + \beta g(z) \sim \sum_{n=0}^{\infty} \left( \alpha a_n + \beta b_n \right) z^{-n},
\]

as \( z \to \infty \) in \( \Omega \).

(iii) A.E. can be multiplied, i.e.

\[
f(z)g(z) \sim \sum_{n=0}^{\infty} c_n z^{-n},
\]

where \( c_n \equiv \sum_{s=0}^{n} a_s b_{n-s} \).

(iv) If \( f \) is continuous in the domain \( \Omega \) defined by \( |z| > a \), \( \arg(z) \in [\theta_0, \theta_1] \), and if (1.4) holds, then

\[
\int_{z}^{\infty} \left[ f(t) - a_0 - \frac{a_1}{t} \right] dt \sim \sum_{n=1}^{\infty} \frac{a_{n+1}}{n} z^{-n},
\]

as \( z \to \infty \) in \( \Omega \), where the integration is taken along a line \( z \to \infty \) with fixed argument.

This is what one means by term-by-term integration of A.E.

(v) Term-by-term differentiation can also be performed, providing in the domain \( \Omega \) defined by \( |z| > R \), \( \arg(z) \in ]\theta_0, \theta_1[ \), the function \( f \) satisfying (1.4) has continuous derivative \( f' \),
and $f'$ has an A.E. as $z \to \infty$ in $\Omega$. Then

$$f'(z) \sim -\sum_{n=1}^{\infty} n a_n z^{-(n+1)} \quad \text{as} \quad z \to \infty \quad \text{in} \quad \Omega \quad . \quad (1.8)$$

2. Zeta-function and heat kernels

We are here interested in the approach to quantum field theory in terms of Feynman path integrals. Hence we study the amplitudes of going from data on a spacelike surface $\Sigma_1$ to data on a spacelike surface $\Sigma_2$. For example, in the case of real scalar fields $\phi$ in a curved background $M$, the data are the induced 3-metric $h$ and a linear combination of $\phi$ and its normal derivative: $a\phi + b\frac{\partial \phi}{\partial n}$. The latter reduces to homogeneous Dirichlet conditions if $b = 0$, and Neumann conditions if $a = 0$. Otherwise, it is a Robin boundary condition. The quantum amplitudes are functionals of these boundary data. On making a Wick rotation and using the background-field method, one expands both the 4-metric $g$ and the field $\phi$ around solutions of the classical field equations as $g = g_0 + \mathcal{G}$ and $\phi = \phi_0 + \phi$. If second-order cross-terms vanish in the Euclidean action $I_E$, the logarithm of the quantum amplitude $Z$ takes the asymptotic form [2]

$$\log(Z) \sim -I_E(g_0) + \log \int \mu_1[\phi]e^{-I_2[\phi]} + \log \int \mu_2[\mathcal{G}]e^{-I_2[\mathcal{G}]} \quad , \quad (2.1)$$

where $\mu_1$ and $\mu_2$ are suitable measures on the spaces of scalar-field and metric perturbations, respectively. The part $I_2[\phi]$ of the action which is quadratic in scalar-field perturbations involves a second-order elliptic operator $\mathcal{B}$. Assuming completeness of the set $\{\varphi_n\}$
of eigenfunctions of $B$, with eigenvalues $\lambda_n$, the corresponding contribution to one-loop quantum amplitudes involves an infinite product of Gaussian integrals, i.e.

\[
\prod_{n=n_0}^{\infty} \int \mu \, dy_n \, e^{-\frac{\lambda_n}{2} y_n^2} = \frac{1}{\sqrt{\det \left( \frac{1}{2} \pi^{-1} \mu^{-2} B \right)}} .
\]

To make sense of this infinite product of eigenvalues, one is thus led to use zeta-function regularization. This is a rigorous mathematical tool which relies on the spectral theorem, according to which for any elliptic, self-adjoint, positive-definite operator $A$, its complex powers $A^{-s}$ can be defined. Hence its zeta-function is defined as

\[
\zeta_A(s) \equiv \text{Tr} \left[ A^{-s} \right] = \sum_{\lambda > 0} \lambda^{-s} , \tag{2.2}
\]

where the eigenvalues in (2.2) are counted with their degeneracies. If $n$ is the dimension of our Riemannian manifold $M$, and $m$ is the order of $A$, its zeta-function has an analytic continuation to the whole complex-$s$ plane as a meromorphic function, given by [3-4]

\[
\zeta_A(s)'' = \sum_{k \neq 0, k = -n}^{N} \frac{a_k}{s + \frac{k}{m}} + H_N , \tag{2.3}
\]

where $H_N$ is holomorphic for $\text{Re}(s) > -\frac{N}{m}$. Thus, on using analytic continuations, $\zeta_A(0)$ is actually finite, and its value gives information about one-loop divergences of physical theories and scaling properties of quantum amplitudes. The relation $\det A = e^{-\zeta'(0)}$, first obtained by formal differentiation, is then used to define $\det A$ after performing the suitable analytic continuation.
Coming back to the operator $\mathcal{B}$ in the Euclidean action for real scalar fields, one now begins by studying the corresponding heat equation, whose Green’s function reads

$$
F(x, y, t) = \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} e^{-\lambda_n m t} \varphi_{n,m}(x) \otimes \varphi_{n,m}(y) .
$$

The corresponding integrated heat kernel, defined by

$$
G(t) \equiv \int_{M} \text{Tr} \ F \sqrt{\det g_0} \ d^4x ,
$$

is then related to the zeta-function by an inverse Mellin transform:

$$
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} G(t) \ dt .
$$

Since, as $t \to 0^+$, $G(t)$ has an A.E. in the form

$$
G(t) \sim b_0 t^{-2} + b_1 t^{-\frac{3}{2}} + b_2 t^{-1} + b_3 t^{-\frac{1}{2}} + b_4 + O(\sqrt{t}) ,
$$

whenever the boundary conditions ensure self-adjointness of $\mathcal{B}$, the $\zeta(0)$ value is equal to the constant coefficient $b_4$ appearing in (2.7).

To understand the applications presented in section 3, it is also necessary to define the zeta-function at large $x$, i.e.

$$
\zeta(s, x^2) \equiv \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} \left( \lambda_{n,m} + x^2 \right)^{-s} .
$$

The A.E. (2.7) is here re-written as

$$
G(t) \sim \sum_{n=0}^{\infty} B_n t^{\frac{n}{2}-2} \quad t \to 0^+ .
$$
For problems with boundaries, the eigenfunctions are usually expressed in terms of Bessel functions. By virtue of the boundary conditions, a linear (or non-linear) combination of Bessel functions is set to zero. Denoting by $F_p$ the function occurring in this eigenvalue condition, one has the identity [2,5]

$$\Gamma(3)\zeta(3, x^2) = \sum_{p=0}^{\infty} N_p \left( \frac{1}{2x} \frac{d}{dx} \right)^3 \log \left[ (ix)^{-p} F_p(ix) \right], \quad (2.10)$$

where $N_p$ is the corresponding degeneracy. On the other hand, by virtue of (2.9) one finds

$$\Gamma(3)\zeta(3, x^2) = \int_{0}^{\infty} t^2 e^{-x^2 t} G(t) \, dt \sim \sum_{n=0}^{\infty} B_n \Gamma \left( 1 + \frac{n}{2} \right) x^{-n-2}. \quad (2.11)$$

Thus, by comparison, one finds that $\zeta(0) = B_4$ is half the coefficient of $x^{-6}$ in the uniform A.E. of the right-hand side of (2.10).

3. Conformal anomalies for massless spin-$\frac{1}{2}$ fields

The analysis of boundary conditions in quantum field theory has motivated the introduction of locally supersymmetric boundary conditions for bosonic and fermionic fields. We here focus on a massless fermionic field at one-loop about a flat Euclidean background bounded by a 3-sphere, following [2,6]. Using 2-component spinor notation, such a field is expressed by a pair of independent spinor fields $\psi^A$ and $\tilde{\psi}^A$. Their expansion on a
family of 3-spheres centred on the origin can be written as ($\tau$ being the Euclidean-time coordinate)

$$
\psi^A = \frac{\tau^{-\frac{3}{2}}}{2\pi} \sum_{n=0}^{\infty} \sum_{p,q=1}^{(n+1)(n+2)} \alpha_n^{pq} \left[ m_{np}(\tau) \rho^{nqA} + \tilde{r}_{np}(\tau) \sigma^{nqA} \right],
$$

(3.1)

$$
\tilde{\psi}'^A = \frac{\tau^{-\frac{3}{2}}}{2\pi} \sum_{n=0}^{\infty} \sum_{p,q=1}^{(n+1)(n+2)} \alpha_n^{pq} \left[ \tilde{m}_{np}(\tau) \rho^{nqA'} + r_{np}(\tau) \sigma^{nqA'} \right].
$$

(3.2)

With our notation, the $\alpha_n^{pq}$ are block-diagonal matrices with blocks $\left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right)$, and the $\rho$- and $\sigma$-harmonics obey the identities described in [2,6]. Our boundary conditions are

$$
\sqrt{2} e_n A' \psi^A = \epsilon \tilde{\psi}'^A \text{ on } S^3,
$$

(3.3)

where $\epsilon \equiv \pm 1$, and $e_n A'$ is the Euclidean normal to $S^3$ [2,6]. As shown in [2,6], the corresponding eigenvalue condition is found to be

$$
F(E) \equiv \left[ J_{n+1}(E) \right]^2 - \left[ J_{n+2}(E) \right]^2 = 0 \quad \forall n \geq 0.
$$

(3.4)

Remarkably, the function $F$ occurring in (3.4) admits a canonical-product representation in terms of its eigenvalues $\mu_i$ as ($\gamma$ being a constant)

$$
F(z) = \gamma z^{2(n+1)} \prod_{i=1}^{\infty} \left( 1 - \frac{z^2}{\mu_i^2} \right).
$$

(3.5)

Thus, setting $m \equiv n + 2$, one finds [2,6]

$$
J_{m-1}^2(x) - J_m^2(x) = J'_m^2 + \left( \frac{m^2}{x^2} - 1 \right)J_m^2 + 2 \frac{m}{x} J_m J'_m.
$$

(3.6)
Thus, on making the analytic continuation $x \to ix$ and then defining $\alpha_m \equiv \sqrt{m^2 + x^2}$, one obtains [2,6]

$$\log\left((ix)^{-2(m-1)}(J_{m-1}^2 - J_m^2)(ix)\right) \sim -\log(2\pi) + \log(\alpha_m) + 2\alpha_m$$

$$- 2m \log(m + \alpha_m) + \log(\tilde{\Sigma}). \quad (3.7)$$

In the A.E. (3.7), $\log(\tilde{\Sigma})$ admits an asymptotic series in the form

$$\log(\tilde{\Sigma}) \sim \left[\log(c_0) + \frac{A_1}{\alpha_m} + \frac{A_2}{\alpha_m^2} + \frac{A_3}{\alpha_m^3} + \ldots\right], \quad (3.8)$$

where, on using the Debye polynomials for uniform A.E. of Bessel functions [2], one finds (hereafter $t \equiv \frac{m}{\alpha_m}$)

$$c_0 = 2(1 + t), \quad (3.9)$$

$$A_1 = \sum_{r=0}^{2} k_{1r} t^r, \quad A_2 = \sum_{r=0}^{4} k_{2r} t^r, \quad A_3 = \sum_{r=0}^{6} k_{3r} t^r, \quad (3.10)$$

where [2,6]

$$k_{10} = -\frac{1}{4}, \quad k_{11} = 0, \quad k_{12} = \frac{1}{12}, \quad (3.11)$$

$$k_{20} = 0, \quad k_{21} = -\frac{1}{8}, \quad k_{22} = k_{23} = \frac{1}{8}, \quad k_{24} = -\frac{1}{8}, \quad (3.12)$$

$$k_{30} = \frac{5}{192}, \quad k_{31} = -\frac{1}{8}, \quad k_{32} = \frac{9}{320}, \quad k_{33} = \frac{1}{2}, \quad (3.13)$$

$$k_{34} = -\frac{23}{64}, \quad k_{35} = -\frac{3}{8}, \quad k_{36} = \frac{179}{576}. \quad (3.14)$$
The corresponding zeta-function at large (cf. section 2) has a uniform A.E. given by

\[ \Gamma(3)\zeta(3, x^2) \sim W_\infty + \sum_{n=5}^{\infty} \hat{q}_n x^{-2-n}, \quad (3.15) \]

where, defining

\[ S_1(m, \alpha_m(x)) \equiv -\log(\pi) + 2\alpha_m, \quad (3.16) \]

\[ S_2(m, \alpha_m(x)) \equiv -(2m - 1) \log(m + \alpha_m), \quad (3.17) \]

\[ S_3(m, \alpha_m(x)) \equiv \sum_{r=0}^{2} k_{1r} m^r \alpha_m^{r-1}, \quad (3.18) \]

\[ S_4(m, \alpha_m(x)) \equiv \sum_{r=0}^{4} k_{2r} m^r \alpha_m^{r-2}, \quad (3.19) \]

\[ S_5(m, \alpha_m(x)) \equiv \sum_{r=0}^{6} k_{3r} m^r \alpha_m^{r-3}, \quad (3.20) \]

\( W_\infty \) can be obtained as [2,6]

\[ W_\infty = \sum_{m=0}^{\infty} \left( m^2 - m \right) \left( \frac{1}{2x} \frac{d}{dx} \right)^3 \left[ \sum_{i=1}^{5} S_i(m, \alpha_m(x)) \right]. \quad (3.21) \]

The resulting \( \zeta(0) \) value receives contributions from \( S_2, S_4 \) and \( S_5 \) only, and is given by [2,6]

\[ \zeta(0) = -\frac{1}{120} + \frac{1}{24} + \frac{1}{2} \sum_{r=0}^{4} k_{2r} - \frac{1}{2} \sum_{r=0}^{6} k_{3r} = \frac{11}{360}. \quad (3.22) \]
Of course, for a massless Dirac field, the full $\zeta(0)$ is twice the value in (3.22):

$$\zeta_{\text{Dirac}}(0) = \frac{11}{180} .$$  \hfill (3.23)

4. The BKKM function

So far, the most powerful algorithm for direct $\zeta(0)$ calculations is the one described and applied in [7-9], since it does not rely on the knowledge of the many coefficients appearing in the Debye polynomials for uniform A.E. of Bessel functions. With the notation in [7-9], one writes $f_n(M^2)$ for the function occurring in the equation obeyed by the eigenvalues by virtue of boundary conditions, and $d(n)$ for the degeneracy of the eigenvalues. One then defines the BKKM function [7-9]

$$I(M^2, s) \equiv \sum_{n=n_0}^{\infty} d(n) \, n^{-2s} \log \left[ f_n(M^2) \right] . \hfill (4.1)$$  

Such a function has an analytic continuation to the whole complex-$s$ plane as a meromorphic function, i.e.

$$\text{“}I(M^2, s)\text{”} = \frac{I_{\text{pole}}(M^2)}{s} + I^R(M^2) + O(s) .$$  \hfill (4.2)

The $\zeta(0)$ value is then obtained as

$$\zeta(0) = I_{\text{log}} + I_{\text{pole}}(\infty) - I_{\text{pole}}(0) .$$  \hfill (4.3)
where $I_{\log} = I_{\log}^R$ is the coefficient of $\log(M)$ from $I(M^2, s)$ as $M \to \infty$, and $I_{\text{pole}}(M^2)$ is the residue at $s = 0$. Remarkably, $I_{\log}$ and $I_{\text{pole}}(\infty)$ are obtained from the uniform A.E. of modified Bessel functions as their order tends to $\infty$ and $M \to \infty$, while $I_{\text{pole}}(0)$ is obtained from the limiting behaviour of such Bessel functions as $M \to 0$.

5. Recent results

The calculation outlined in section 3 is just an example of the many difficult calculations of conformal anomalies in the presence of boundaries appearing in the recent literature. The same $\zeta(0)$ value has been obtained by using the even more powerful technique elaborated in [7], as shown in [8-9]. The motivations for this analysis come from the quantization of closed cosmologies, from perturbative supergravity, and from the need to get a better understanding of different quantization techniques in field theory (i.e. reduction to physical degrees of freedom before quantization, or Faddeev-Popov technique, or Batalin-Fradkin-Vilkovisky method). Here we summarize the recent results for bosonic fields on using relativistic gauges within the Faddeev-Popov formalism [10-12].

(5.1) In the Lorentz gauge, the mode-by-mode analysis of one-loop amplitudes for vacuum Maxwell theory agrees with the results of the Schwinger-DeWitt technique, both in the one-boundary case (the disk) and in the two-boundary case (the ring).

(5.2) In the presence of boundaries, the effects of gauge modes and ghost modes do not cancel each other.
(5.3) When combined with the contribution of physical degrees of freedom, this lack of cancellation is exactly what one needs to achieve agreement with the results of the Schwinger-DeWitt technique. Thus, physical degrees of freedom are, by themselves, insufficient to recover the full information about one-loop amplitudes.

(5.4) Even on taking into account physical, non-physical and ghost modes, the analysis of relativistic gauges different from the Lorentz gauge yields gauge-invariant amplitudes only in the two-boundary case.

(5.5) The conditions under which one can decouple gauge modes in the presence of boundaries have been characterized.

(5.6) Changing the gauge leads to a continuous, multi-paramater variation of a matrix of elliptic self-adjoint operators. Out of the eigenvalues of such operators one can obtain a meromorphic function whose residue at the origin is invariant under homotopy (i.e. under the smooth variation mentioned above). Hence gauge invariance in the presence of boundaries may be proved by combining this result with a hard WKB analysis of coupled eigenvalue equations. Remarkably, one would then obtain yet another application of the Atiyah-Patodi-Singer theory of Riemannian 4-geometries with boundary [4].

(5.7) A mode-by-mode analysis of linearized gravity in the presence of boundaries in the de Donder gauge has just been completed, including gauge modes and ghost modes. Again, on taking a flat Euclidean background bounded by two concentric 3-spheres, the mode-by-mode analysis of Faddeev-Popov quantum amplitudes agrees with the result of covariant Schwinger-DeWitt formalism [12].
The developments presented in sections 3 and 4 may lead to a deeper understanding of conformal anomalies and gauge invariance in quantum field theory. Hence we think that Henri Poincaré would be pleased, if not surprised, to see how many applications of asymptotic analysis are relevant for modern quantum field theory.

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