An Induction Accelerator of Cosmic Rays on the Axis of an Accretion Disk.

A. A. Shatskiy and N. S. Kardashev
Astro Space Center, Lebedev Physical Institute, Moscow, RussiaReceived January 3, 2002; revised February 1, 2002.

ABSTRACT
The structure and magnitude of the electric field created by a rotating accretion disk with a poloidal magnetic field is found for the case of a vacuum approximation along the axis. The accretion disk is modeled as a torus filled with plasma and the frozen-in magnetic field. The dimensions and location of the maximum electric field are found, as well as the energy of the accelerated particles. The gravitational field is assumed to be weak.

1 INTRODUCTION
Recently, there has been wide discussion of various mechanisms for accelerating particles around supermassive black holes (SMBHs) in the nuclei of galaxies and stellar-mass black holes in our Galaxy, in connection with studies of synchrotron radiation and inverse Compton scattering in the well collimated jets observed from radio to gamma-ray wavelengths. Extremely high-angular-resolution observations obtained via radio interferometry show that these jets become very narrow (comparable to the gravitational radius) with approach to the black hole. Explanations of particle acceleration near relativistic objects (black holes and neutron stars) are usually based on two types of mechanisms: acceleration by electric fields and magnetohydrodynamical acceleration (the Blandford-Znajek mechanism [1]). Acceleration by an electric field, and the very existence of the electric field, are inseparably linked with the low density of plasma in this volume. Conditions justifying a vacuum approximation are probably realized in the magnetospheres of pulsars and in some types of SMBHs [2, 3]. In this case, it is possible to accelerate particles to extremely high energies [4]. The limiting charge densities for which the vacuum approximation remains valid are determined by the formula [5]

\[ n_e < \frac{|\Omega H|}{2\pi ce} \lesssim \frac{1}{P} \cdot \frac{H}{10^4} \cdot 10^{-2} \cdot 3. \]  

Here, \( \Omega \) is the angular velocity of rotation, \( H \) the characteristic magnetic-field strength, \( P \) the rotational period, \( c \) the velocity of light, and \( e \) the electron charge. It is evident from this expression that we should have for typical quasars \( n_e < 10^{-2} cm^{-3} \). Note that, in intergalactic space, \( n_e \approx 10^{-6}cm^{-3} \), and in the Galaxy, \( n_e \approx 1cm^{-3} \). The presence of a black hole in the center of the Galaxy also leads to a decrease in \( n_e \) near the center. In addition, the magnetic fields near SMBHs can reach values of the order of \( 10^9G \) [4]. In any case, the question of the applicability of the vacuum approximation is rather complex, and must be solved taking into account the physics of black holes. In this paper, we will assume that the conditions for the
vacuum approximation are satisfied; this will enable us to investigate the structure of the electric field excited by a rotating accretion disk with a poloidal magnetic field. The formulation of this problem is analogous to that considered by Deutsch [6], who found the structure of the electric field created by a dipolar magnetic field frozen in a rotating star. If a conductor rotates together with a frozen-in magnetic field, then, in a rotating coordinate system in which the conductor is at rest, there must be no electric field inside the conductor. Therefore, in an inertial system, an electric field is induced inside the conductor due to the presence of the magnetic field, and this electric field gives rise to a surface charge (in the special case of a magnetic dipole with a quadrupolar distribution). This surface charge is the source of the external electric field. We will consider the analogous problem for an accretion disk.

2 Construction of the Model

Let us consider a stationary system consisting of a rotating accretion disk in the form of a regular torus filled with plasma and surrounded by a poloidal magnetic field. We neglect motion of the torus toward the center. The geometry of the torus is determined by the two radii \( a \) and \( R \) (Fig. 1). Due to the high conductivity of the plasma, the magnetic field is frozen inside the torus, and currents flow only in the toroidal direction; there is no matter outside the torus. In a coordinate system comoving with the plasma, the electric field vanishes due to the relation

\[
j'_\alpha = \sigma' F'_{\alpha 0},
\]

where \( j'_\alpha = 0 \) are the poloidal components of the current density, \( \sigma' \) is the plasma conductivity, and \( F'_{\alpha 0} \) are the covariant components of the electric field tensor in a system comoving with the plasma. We introduce the following coordinates (Fig. 1): \( x \) is the distance from an arbitrary point to the center inside the torus in the same meridional plane, \( \psi \) is the angle between the direction toward the center of the system and the direction to a given point from the center inside the torus in the same meridional plane, and \( \varphi \) is the position angle defining this plane. Thus, the differential coordinates \( dx, d\psi, \) and \( d\varphi \) form a righthanded orthogonal vector triad. We can write the square of a linear element in Minkowski space for the differentials of these

![Figure 1: Accretion disk in the form of a torus (side view).](image-url)
coordinates \(^1\):

\[
\left\{ \begin{array}{c}
 ds^2 = dt^2 - dx^2 - x^2 \, d\psi^2 - (R - x \cos \psi)^2 \, d\varphi^2, \\
 \sqrt{-g} = x |R - x \cos \psi|.
\end{array} \right.
\]  

(3)

Let the plasma in the torus rotate with angular velocity \(\Omega\) with respect to a distant observer.

Then, the coordinate transformation is given by \(^2\)

\[ dx^i = dx'^k [\delta^i_k + \Omega \delta^i_\varphi \delta^0_k] \]  

(4)

We now introduce the covariant four-vector potential of the electromagnetic (EM) field \(A_i\). Due to the axial symmetry of the system, only \(A_0\), the potential of the electric field, and \(A_\varphi\), the potential of the magnetic field, differ from zero. Therefore, the EM-field tensor \(F_{ij}\) has only poloidal components:

\[ F_{\alpha0} = \partial_\alpha A_0, \quad F_{\alpha\varphi} = \partial_\alpha A_\varphi. \]  

(5)

These components transform in accordance with (4) \(^7, \S 83\):

\[ F'_{\alpha0} = F_{\alpha0} + \Omega F_{\alpha\varphi}, \quad F'_{\alpha\varphi} = F_{\alpha\varphi}, \quad A'_0 = A_0 + \Omega A_\varphi, \quad A'_\varphi = A_\varphi. \]  

(6)

Since \(F'_{\alpha0} = 0\) in the plasma [see (2)], we have for \(x < a\):

\[ F_{\alpha0} = -\Omega F_{\alpha\varphi}. \]  

(7)

It follows from (5) and (7) that, when \(x < a\), \(A_0 = \text{const} - \Omega A_\varphi\). It follows from the third equation of (6) that this constant is \(A'_0\) inside the torus.

The continuous boundary conditions for the tangential electric and normal magnetic components of the EM field act at the interface between the plasma and vacuum (at the surface of the torus). These components should vanish at the equator, due to the axial symmetry of the system and the mirror (anti-)symmetry of the components of the EM field. Hence, the normal component of the magnetic field can be expanded in a Fourier sine series:

\[ F_{\psi\varphi} = n R_n(x) \sin(\psi n). \]  

(8)

Here and below, the summation over \(n\) is assumed, where \(n\) runs through all numbers of the natural series. It follows from (7) and (8) that the boundary condition for the tangential electric field is

\[ F_{\psi 0(a, \psi)} = -\Omega n R_n(a) \sin(\psi n). \]  

(9)

### 3 Potential and Structure of the Electric Field Near the Torus

Near the torus [see formulas (30), (32), and (37) in the Appendix], the main contribution to the potential \(A_0\) is made by the first term of the first harmonic of the Fourier series. Far from the torus, the potential dies away. The kinetic energy of a charged particle accelerated by the system is determined primarily by this part of the potential. Accordingly, we obtain the main approximation for the difference in the potentials at the torus surface between the angles \(\psi\) and \(\psi = \tilde{\psi}\):

\[
\Delta A_0 = \Omega \cdot R \cdot H_0 \cdot a \cdot (\ln(4/b) - 1) \cdot [1 - \cos \tilde{\psi}] / \pi.
\]  

(10)

---

\(^1\)We take the velocity of light to be unity: \(c = 1\).

\(^2\)Where not indicated otherwise, \(x_i = t, x, \psi, \varphi\), and \(x_\alpha = x, \psi\).
Figure 2: Appearance of the electromagnetic field lines in the system. The solid curves show the magnetic field lines and the dashed curves the electric field lines. \( z \) and \( \rho \) are expressed dashed curves the electric field lines. \( z \) and \( \rho \) are expressed in fractions of \( R \).

This same expression can also be obtained in another way [8, §63]. The corresponding invariant result has the form

\[
\varepsilon \equiv U^i \Delta A_i = \int_C [U^i F_{ji}] \, d\alpha^j = \int_0^{\psi} U^{i\psi} F'_{\psi \phi} \, d\psi. \tag{11}
\]

Here, \( U^i \) denotes the four-velocity of the observer \(^3\) at the measurement point, and the contour for the integral is chosen for convenience to be on the inside surface of the torus in a reference system comoving with the torus. After substituting into this expression \( U^{i\psi} \approx \Omega \) and the expression for \( F'_{\psi \phi} = F_{\psi \phi} \) we obtain (10).

The electric field inside the torus is zero, since the torus is conducting. In an inertial reference system, this is accomplished by the compensation of two fields: that induced by the rotation of the magnetic field and the field of the surface charge on the torus. The surface density of the electric charge on the torus is \( \rho_e = -F_{x0}/(4\pi) \). \( F_{x0} \) is the normal component of the electric field on the torus surface [see (7)]. At large distances from the torus surface \((x >> a)\), these charges represent a superposition of dipoles (Fig. 2). However, at distances \( x >> R \), the field from all these dipoles has a quadrupole character. It is not difficult to obtain an expression for the dipole moment per unit angle \( \varphi \):

\[
d = \frac{\Omega R^2 a^2 H_0}{2\pi} \left[ 2 - \ln(4/b) \right]. \tag{12}
\]

The electric field and its potential \( \phi \) can be obtained by integrating all the dipoles over the angle \( \varphi \) [7, ˘40]. As a result, we obtain for the potential and components of the electric field in cylindrical coordinates the quadrature expressions

\[
\phi(\rho, z) = \frac{2d}{R^2} \int_0^\pi \left[ \frac{1 - \tilde{\rho} \cos \varphi}{(1 + \tilde{\rho}^2 + \tilde{z}^2 - 2\tilde{\rho} \tilde{z} \cos \varphi)^{3/2}} \right] \, d\varphi. \tag{13}
\]

\[
E_\rho(\rho, z) = \frac{2d}{R^3} \int_0^\pi \left[ \frac{\cos \varphi(\tilde{z}^2 - 2\tilde{\rho}^2 - 2) + 3\tilde{\rho} + \tilde{\rho} \cos^2 \varphi}{(1 + \tilde{\rho}^2 + \tilde{z}^2 - 2\tilde{\rho} \tilde{z} \cos \varphi)^{5/2}} \right] \, d\varphi. \tag{14}
\]

\(^3\)Who is at rest relative to the distant stars.
Figure 3: Magnitude of the poloidal component of the electric field $E_{||}(\rho/R, z/R)$ in units of its value at the saddle point. $Z$ and $\rho$ are expressed in fractions of $R$.

$$E_z(\rho, z) = \frac{2d}{R^3} \int_0^\pi \frac{3\tilde{z}(1 - \tilde{\rho} \cos \varphi)}{(1 + \tilde{\rho}^2 + \tilde{z}^2 - 2\tilde{\rho}\tilde{z} \cos \varphi)^{5/2}} d\varphi.$$  \hspace{1cm} (15)

Here, $\tilde{\rho} = \rho/R$ and $\tilde{z} = z/R$ are dimensionless cylindrical coordinates. The last three formulas are expressed in terms of derivatives of the full elliptical integrals. The result of this integration is shown in Fig. 2.

Acceleration by the electric field is associated only with the component parallel to the magnetic field. This acceleration is especially efficient on the $\Omega$ axis, where the electric and magnetic lines of force are parallel. We can readily see from (15) that the maximum electric field on the $\Omega$ axis is reached at the point $z = R/2$. However, this point is not a local maximum of the modulus of the longitudinal electric-field component $E_{||} \equiv (\mathbf{E} \mathbf{H})/H$. Figure 3 shows the surface of the $E_{||}$ force field, where we can see that the point $z = R/2$ on the axis is a saddle point of the distribution of $E_{||}$. With approach to the torus, the conductivity of the medium should increase and the field should become force-free: $E_{||} \to 0$.

According to (13), the potential at the saddle point is approximately 28% lower than the potential at the center of the system, and is roughly an order of magnitude lower than the maximum potential on the torus surface. The extent of the region of acceleration along the $z$ axis (at the level of 0.5 of the value of $E_{||}$ at the saddle point) is determined by the points $z_1 \approx 0.25$, $z_2 \approx 1.25$ in fractions of $R$. Assuming that a mass $M$ with its corresponding gravitational radius $r_g$ is at the center of the system, we can use (13) to estimate the energy to which particles with the elementary charge initially at the saddle point on the axis can be accelerated:

$$E_k \approx \left(\frac{6.5 \cdot a}{R}\right) \cdot \left(\frac{\Omega R}{c}\right) \cdot \left(\frac{H_0}{10^4}\right) \cdot \left(\frac{a}{r_g}\right) \cdot \left(\frac{M}{10^9 M_\odot}\right) \cdot \left[\ln(4/b) - 2\right] \cdot 10^{20} eV.$$ \hspace{1cm} (16)

We can see that the factors in parantheses can be of the order of unity in quasars, so that the kinetic energy of particles accelerated by such a system can reach $10^{20}$ eV.

\footnote{Note that, in a Blandford-Znajek model, the field is forcefree everywhere by definition, so that efficient acceleration is not possible.}
4 CONCLUSIONS

We can draw the following conclusions from our results.

1) The magnetic field at large distances approaches a dipolar field, while the electric field corresponds to a quadrupolar distribution for the charge induced on the torus surface. It follows from Fig. 2 that, in the model considered here, in contrast to a Blanford-Znajek model, we find a tendency for the electric field lines to become more concentrated near the \( \Omega \) axis, which can explain the observed focusing (collimation) of the accelerated relativistic particles.

2) The dimensions and locations of the regions of cosmic-ray acceleration we have found can be used to compare our results with observational data.

3) Thanks to its covariance, our method for the computation of the electromagnetic field in a system with toroidal symmetry can be generalized to the case of a gravitationally curved space-time.

4) The mechanism considered here yields accelerated-particle energies with the same order of magnitude as the Blandford-Znajek mechanism (see, for example, [1, 9, 10]).

5 ACKNOWLEDGEMENTS

This work was supported by the Russian Foundation for Basic Research (project codes 01-02-16812, 00-15-96698, and 01-02-17829).

6 APPENDICES

Finding the External Solution

Let us write Maxwell’s equations for arbitrary curvilinear coordinates in the axially symmetric and stationary case [7, §90]:

\[
\begin{align*}
\epsilon^{\alpha\beta\varphi} \partial_\beta F_{\alpha\varphi} & = 0; \quad \partial_\beta \left( \sqrt{-g} g^{\alpha\beta} g^{\varphi\varphi} F_{\alpha\varphi} \right) = 4\pi \sqrt{-g} j^\varphi; \\
\epsilon^{\alpha\beta\varphi} \partial_\beta F_{\alpha0} & = 0; \quad \partial_\beta \left( \sqrt{-g} g^{\alpha\beta} g^{00} F_{\alpha0} \right) = 4\pi \sqrt{-g} j^0.
\end{align*}
\]

(17)

Here, \( \epsilon^{\alpha\beta\varphi} \) is a Levi-Civita symbol, \( \sqrt{-g} \) and \( g^{ij} \) are defined by expression (3), and \( j^i \) is the current 4-vector, which is identically equal to zero when \( x > a \). We obtain for the magnetic field from (8) and the first of equations (17):

\[
F_{x\varphi} = -\partial_x R_n(x) \cos(\psi n), \quad A_\varphi = -R_n(x) \cos(\psi n).
\]

(18)

Using (9), the external solution for the electric field can be expanded in a Fourier series in the variable \( \psi \). Then, in accordance with the third equation of (17), we obtain the solution for the electric field (for \( x > a \))

\[
F_{\psi0} = nZ_n(x) \sin(\psi n), \quad F_{x0} = -\partial_x Z_n(x) \cos(\psi n), \quad A_0 = -Z_n(x) \cos(\psi n).
\]

(19)

It follows from (8) and (19) that the external electric field is generated by the normal component of the magnetic field at the torus boundary. Everywhere where the expression \( (R - x \cos \psi) \) is positive, such as in the range \( a < x < \bar{R} \), we can remove the modulus signs in the second expression of (3). With this in mind, introducing the dimensionless variable \( y = x/(2R) \) and denoting a derivative with respect to this variable with a prime, we substitute (19) into the fourth Maxwell equation (17) and obtain

\[
y [y^2 Z_n'' + 2y Z_n' - n(n - 1)Z_n] \cos\{(n - 1)\psi\} - \\
- [y^2 Z_n'' + y Z_n' - n^2 Z_n] \cos\{\psi n\} + \\
y [y^2 Z_n'' + 2y Z_n' - n(n + 1)Z_n] \cos\{(n + 1)\psi\} = 0.
\]

(20)
The boundary conditions for $Z_n(y)$ when $y = b \equiv a/(2R)$ and $n > 0$ follow from (9):

$$Z_n(b) = -\Omega R_n(b), \quad (n > 0).$$

(21)

The necessary condition for the total electric charge in the torus to be zero can be written

$$Q = \oint F_{x0} g^{00} g^{xx} \sqrt{-g} 2\pi d\psi = 0 \quad (x : a < x < R).$$

(22)

This expression flows from integration of the fourth equation of (17). We thus obtain

$$Z'_0 = yZ'_1$$

(23)

Setting the expressions with the same harmonics in (20) equal to zero, we obtain a recurrent system of equations for $Z_n(y)$. Equation (23) is contained in the Maxwell equation (20) with harmonic $n = 0$. The condition that the total charge of the system be equal to zero can also be written for $x > R$. For this, we must take a contour passing around the system, so that $(R - x \cos \psi)$ does not change sign; i.e., the contour should be located to the right of the halfplane from the $\Omega$ axis. We choose this contour as shown in Fig. 4. Thus, we pass by the torus along a semicircle with radius $x > a$ on the right-hand side, and we allow the contour to approach infinity at the angles $\psi = \pi/2$ and $\psi = -\pi/2$ along half-lines parallel and antiparallel to the $\Omega$ axis. The two (upper and lower) "caps" of the finite area $\pi R^2$ remain unclosed at infinity, but since the field there asymptotically approaches zero, the integrals on these "caps" can be neglected. We can write the surface integral over this contour in accordance with the fourth of equations (17) taking into account the mirror symmetry for the electric field in the equatorial plane:

$$Q = \int_{\pi/2}^{\pi} F_{x0} g^{00} g^{xx} \sqrt{-g} 4\pi d\psi + \int_{x}^{\infty} F_{\psi0} g^{00} g^{\psi\psi} \sqrt{-g} 4\pi dx \bigg|_{\psi=\pi/2} = 0.$$  

(24)

After straightforward but cumbersome manipulations, we obtain

$$yZ'_n \left[ \pi(y\delta_n^1 - \delta_n^0) + 2y \frac{\cos(\frac{\pi n}{2})}{n^2 - 1} + \frac{\sin(\frac{\pi n}{2})}{n} \right] + \sin \left( \frac{\pi n}{2} \right) \int_{y}^{\infty} \frac{Z_n}{y} dy = 0.$$  

(25)
Let us solve this problem with accuracy to within the first three terms in the Fourier expansion \((Z_0, Z_1, \text{and} \ Z_2)\). In this case, taking into account (23), the last expression becomes

\[
(1 - 2y^2)yZ_1' + \int_{y}^{\infty} \frac{Z_1}{y} dy \approx \frac{2}{3} y^2 Z_2'.
\]  

(26)

Differentiating, we obtain

\[
(1 - 2y^2)y^2 Z_1'' + (1 - 6y^2)y Z_1' - Z_1 \approx \frac{2}{3} y(y^2 Z_2'' + 2y Z_2').
\]  

(27)

We now write the Maxwell equation (20) corresponding to the harmonic \(n = 1\):

\[
(1 - 2y^2)y^2 Z_1'' + (1 - 6y^2)y Z_1' - Z_1 - y(y^2 Z_2'' + 2y Z_2' - 2Z_2) = 0.
\]  

(28)

An equation for \(Z_2\) follows from these last two expressions:

\[
y^2 Z_2'' + 2y Z_2' - 6Z_2 = 0.
\]  

(29)

The solution vanishing at infinity has the form

\[Z_2(y) = C_2/y^3.\]  

(30)

The constant \(C_2\) is found from the boundary conditions (21). We can find \(Z_1(y)\) using any of equations (27) or (28):

\[
(1 - 2y^2)y^2 Z_1'' + (1 - 6y^2)y Z_1' - Z_1 = 4C_2/y^2.
\]  

(31)

A partial solution of the inhomogeneous equation (31) has the form \(Z_1^1 = 4C_2/(3y^2)\). However, it does not satisfy (26), of which (27) is a consequence, so that we must search for a solution of \(Z_1(y)\) from (26). The substitution \(\int_{y}^{\infty} (Z_0''/y) dy = f_{\infty} - f(y)\) brings the homogeneous equation (26) into the form \((1 - 2y^2)y(yf')' - f + f_{\infty} = 0\). Hence, in the limit \(y \to 0\), we obtain \(Z_0^0 \to C_1 y^{\pm 1}\).

In the limit \(y \to \infty\), the substitution \(y = \exp(\xi)\) reduces (26) to

\[
f''(\xi) + e^{-2\xi} f/2 = 0, \quad f''(\xi) + e^{-2\xi} f/2 = e^{-2\xi} f_{\infty}/2, \quad f = f + f_{\infty}.
\]

In the main approximation in \(1/y\), the solution has the form \(f(y) = \exp[-1/(8y^2)] + 1/(8y^2)\). Hence, the asymptotic for \(y \to \infty\) has the form \(Z_0^0 \to C_1/(32y^4)\).

\(Z_0(y)\) can be found using (23) after \(Z_1(y)\) is found. We present the asymptotics of these solutions for small and large \(y\):

\[
Z_0 \xrightarrow{y \to 0} -C_1 \ln(y) + 8C_2/(3y), \quad Z_1 \xrightarrow{y \to 0} C_1/y + 4C_2/(3y^2),
\]

\[
Z_0 \xrightarrow{y \to \infty} -(C_1 + 8C_2)/(24y^3), \quad Z_1 \xrightarrow{y \to \infty} -(C_1 + 8C_2)/(32y^4).
\]  

(32)

Here, \(C_1\) is the coefficient for the solution of the homogeneous equation (26); like \(C_2\), it is determined by the specific configuration of the magnetic field from the boundary conditions.

\textit{Calculation of Coefficients for the Special Case of a Current Ring in the Torus}

8
Let us now find the magnetic field in a simple case. Let the current in the torus be in the form of a ring along the torus axis with a delta-function distribution [7, §43]:

$$ j^\varphi = \lim_{x_0 \to 0} J^\varphi \delta(x - x_0) \delta(\psi - \psi_0)/(2\pi \sqrt{-g}), \quad j^\alpha = 0. $$

To find the potential $A_\varphi$ corresponding to this current, we introduce the physical components of vectors in accordance with the definition

$$ H_{\text{phys}} \equiv \hat{H} = \{H^\alpha \sqrt{|g_{\alpha\beta}|}\}. $$

This is necessary because the Biot-Savart law in its usual form [7, §43] is written in physical coordinates:

$$ \hat{A}^\alpha(x,\psi) = \hat{e}^\alpha \oint \frac{\hat{e}^\gamma}{|\mathbf{r} - \mathbf{R}|} \sqrt{-g} \, d\tilde{x} \, d\psi \, d\tilde{\varphi}. $$

(34)

Here, $\hat{e}^\alpha$ is a unit vector in the direction of the angle $\varphi$, $(\hat{\mathbf{j}}^\gamma \cdot \hat{\mathbf{e}}_\gamma) = \hat{j}^\gamma \sqrt{|g_{\varphi\varphi}|} \cos \tilde{\varphi}$, and $|\mathbf{r} - \mathbf{R}|^2 = 2R(R - x \cos \tilde{\varphi}) + x^2$ is the square of the distance from the segment of current with coordinate $\tilde{\varphi}$ to the observation point.

Introducing the notation $\tilde{\varphi} = \pi + 2\phi$, $\kappa^2 = (1 - 2y \cos \psi)/(1 - 2y \cos \psi + y^2)$, we can transform (34) to the form

$$ A_\varphi = (J^\varphi R/\pi) \sqrt{1 - 2y \cos \psi} \cdot \kappa \cdot \int_0^{\pi/2} \frac{2 \sin^2 \phi - 1}{\sqrt{1 - \kappa^2 \sin^2 \phi}} \, d\phi $$

or

$$ A_\varphi = 2(J^\varphi R/\pi) \cdot \sqrt{1 - 2y \cos \psi} \cdot \left\{K(\kappa)(1 - \kappa^2/2) - E(\kappa)\right\}/\kappa. $$

(35)

Here, $K(\kappa)$ and $E(\kappa)$ are full elliptical integrals.

We can find the asymptotic of (35) as $y \to 0$, or equivalently as $\kappa \to 1$. Further, using (18), we can obtain for the Fourier coefficients of the magnetic field:

$$ \lim_{y \to 0} : R_0 \to J^\varphi R/\pi \left[2 - \ln(4/y)\right], \quad R_1 \to -J^\varphi R/\pi \left[y \left(1 - \ln(4/y)\right)\right], \quad R_2 \to -J^\varphi R/\pi \left[y^2 \left(1 - \ln(4/y)/4\right)\right]. $$

(36)

It is clear from the boundary conditions (21) that, according to (36), $|Z_2(b)/Z_1(b)| \sim b \to 0$ as $b \to 0$, so that we can neglect the remaining terms of the Fourier series in the case of this magnetic-field configuration. We introduce the notation $H_0 \equiv |\partial_x A^\varphi|_{x=R, \psi=0} = J^\varphi/(2R)$ for the magnetic-field strength at the center of the system. We find the coefficients in this approximation from (21), (32), and (36):

$$ C_1 \approx 2\Omega H_o R^2 b^2 \left[1 - \ln(4/b)\right]/\pi, \quad C_2 \approx 2\Omega H_o R^2 b^5 \left[1 - \ln(4/b)/4\right]/\pi. $$

(37)

REFERENCES

[1] R. D. Blandford and R. L. Znajek, Mon. Not. R. Astron. Soc. 179, 433 (1977).
[2] R. D. Blandford, astro-ph/0110396.
[3] G. Barbiellini and F. Longo, astro-ph/0105464.
[4] N. S. Kardashev, Mon. Not. R. Astron. Soc. 276, 515 (1995).
[5] P. Goldreich and W. H. Julian, Astrophys. J. 157, 869 (1969).
[6] J. Deutsch, Ann. Astrophys. 1 (1), 1 (1955).
[7] L. D. Landau and E. M. Lifshitz, Course of Theoretical Physics, Vol. 2: The Classical Theory of Fields (Nauka, Moscow, 1988; Pergamon, Oxford, 1975).
[8] L. D. Landau and E.M. Lifshitz, Course of Theoretical Physics, Vol. 8: Electrodynamics of Continuous Media (Nauka, Moscow, 1992; Pergamon, New York, 1984).
[9] BlackHoles: the Membrane Paradigm, Ed. by K. S. Thorne, R. H. Price, and D. A. Macdonald (Yale Univ. Press, New Haven, 1986, Mir, Moscow, 1988).
[10] A. A. Shatskiy, Zh. Eksp. Teor. Fiz. 11, (2001).

Translated by D. Gabuzda
ASTRONOMY REPORTS Vol. 46 No. 8 2002