Estimates for the Lowest Eigenvalue of Magnetic Laplacians

T. Ekholm*, H. Kovařík†, F. Portmann‡

Abstract

We prove various estimates for the first eigenvalue of the magnetic Dirichlet Laplacian on a bounded domain in two dimensions. When the magnetic field is constant, we give lower and upper bounds in terms of geometric quantities of the domain. We furthermore prove a lower bound for the first magnetic Neumann eigenvalue in the case of constant field.

1 Introduction

Let \( \Omega \) be a bounded open domain in \( \mathbb{R}^2 \) and \( B \in L^\infty_{\text{loc}}(\mathbb{R}^2) \) a real-valued function, the magnetic field. To \( B \) we associate a vector potential \( A \in L^\infty(\Omega) \) such that \( B = \text{curl} A = \partial_1 A_2 - \partial_2 A_1 \) in \( \Omega \), see Section 2 for an explicit construction of \( A \). The magnetic Dirichlet Laplacian on \( \Omega \),

\[
H^D_{\Omega,B} := (-i \nabla + A)^2
\]

is then defined through the Friedrichs extension of the quadratic form

\[
h^D_{\Omega,A}[u] := \int_{\Omega} |(-i \nabla + A) u(x)|^2 \, dx,
\]
on \( C^\infty_c(\Omega) \). Altogether, there is a huge amount of literature dealing with spectral properties of the operator \( H^D_{\Omega,B} \) on bounded as well as unbounded domains in \( \mathbb{R}^2 \). We refer to the [AHS, CFKS] for an introduction on Schrödinger operators with magnetic fields. Various estimates for sums and Riesz means of eigenvalues of \( H^D_{\Omega,B} \) on bounded domains were established in [ELV, KW, LLR, LS]. Hardy-type inequalities for \( h^D_{\Omega,A} \) were studied in [BLS, LW, W]. For a version of the well-known Faber-Krahn inequality in the case of constant magnetic field we refer to [Er].

*tomase@kth.se, Royal Institute of Technology KTH, Sweden
†hynek.kovarik@unibs.it, Università degli studi di Brescia, Italy
‡Corresponding Author; fabian@math.ku.dk, University of Copenhagen, Denmark
The main object of interest in this note will be the quantity \( \lambda_1(\Omega, B) := \inf \text{spec } H^D_{\Omega, B} = \inf_{u \in C_\infty^c(\Omega)} \frac{h^D_{\Omega, A}[u]}{\|u\|^2} \geq 0. \)

Since \( \Omega \) is bounded, our conditions on \( B \) imply that the form domain of \( H^D_{\Omega, B} \) is \( H^1_0(\Omega) \) and \( \lambda_1(\Omega, B) \) is indeed the lowest eigenvalue of \( H^D_{\Omega, B} \). There exist two well-known lower bounds for \( \lambda_1(\Omega, B) \). By a commutator estimate, see e.g. \[AHS\], one obtains
\[
h^D_{\Omega, A}[u] \geq \pm \int_\Omega B(x) |u(x)|^2 \, dx. \tag{1.2}
\]

For a constant magnetic field \( B(x) = B_0 \), inequality (1.2) yields
\[
\lambda_1(\Omega, B_0) \geq \pm B_0. \tag{1.3}
\]

The pointwise diamagnetic inequality (see for example \[LL\, Theorem 7.21\])
\[
|(-i \nabla + A) u(x)| \geq |\nabla|u(x)|, \quad \text{for a.e. } x \in \Omega, \tag{1.4}
\]
on the other hand tells us that
\[
\inf_{u \in H^1_0(\Omega)} \frac{h^D_{\Omega, A}[u]}{\|u\|^2} \geq \inf_{u \in H^1_0(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\|u\|^2} = \inf_{v \geq 0} \frac{\int_\Omega |\nabla v|^2}{\|v\|^2}.
\]

This implies that
\[
\lambda_1(\Omega, B) \geq \lambda_1(\Omega, 0). \tag{1.5}
\]

Under very weak regularity conditions on \( B \) it was shown in \[He\] that inequality (1.5) is in fact strict; \( \lambda_1(\Omega, B) > \lambda_1(\Omega, 0). \)

Let us briefly discuss the Neumann case. The quadratic form corresponding to the magnetic Neumann Laplacian \( H^N_{\Omega, B} \) is given by
\[
h^N_{\Omega, A}[u] := \int_{\Omega} |(-i \nabla + A) u(x)|^2 \, dx,
\]
and the form domain is now \( H^1(\Omega) \). Again,
\[
\mu_1(\Omega, B) := \inf \text{spec } H^N_{\Omega, B}
\]
is the first eigenvalue of \( H^N_{\Omega, B} \), provided \( \Omega \) is sufficiently regular. The estimate (1.5) remains valid in the Neumann case (since (1.4) holds a.e.), and gives:
\[
\mu_1(\Omega, B) \geq \mu_1(\Omega, 0) = 0. \tag{1.6}
\]
The corresponding estimate (1.2) (resp. (1.3)) is a priori not available due to the different boundary conditions. A lot of attention has been paid to the asymptotic behavior of \( \mu_1(\Omega, B) \) for large values of the magnetic field, see e.g. \[Bo\, \[FH1\], \[LP\], \[Ra\], \[Si\].

1.1 Overview of the Main Results

A natural question which arises in this context is whether estimates (1.2), (1.3) and (1.5) can be improved by adding a positive term to their righthand sides.

It is clear that their combinations cannot be achieved by simple addition; already for the constant magnetic field any lower bound of the type

$$\lambda_1(\Omega, B_0) \geq \lambda_1(\Omega, 0) + cB_0,$$

with $c > 0$ independent of $B_0$, must fail. Indeed, since the eigenfunction of $H^{D}_{\Omega, 0} = -\Delta^{D}_{\Omega}$ relative to the eigenvalue $\lambda_1(\Omega, 0)$ may be chosen real-valued, analytic perturbation theory yields

$$\lambda_1(\Omega, B_0) = \lambda_1(\Omega, 0) + O(B_0^2), \quad B_0 \to 0. \quad (1.8)$$

This clearly contradicts (1.7) for $B_0$ small enough.

The main results of our paper are the following. In Section 2 we give quantitative lower bounds on the quadratic form

$$h^{D}_{\Omega, A}[u] = \int_{\Omega} B(x) |u(x)|^2 \, dx,$$

denoted by estimates of the first type. Estimates for the difference

$$\lambda_1(\Omega, B) - \lambda_1(\Omega, 0),$$

referred to as estimates of the second type, are studied in Section 3. In both cases, particular attention will be devoted to the case of constant magnetic field. Last but not least, we will also establish a lower bound of the second type for the lowest eigenvalue of the magnetic Neumann Laplacian in the case of constant magnetic field in Section 3.2.

Notation:

Given $x \in \Omega$ and $r > 0$, we denote by $B(x, r)$ the open disc of radius $r$ centered in $x$. We also introduce the distance function

$$\delta(x) := \text{dist}(x, \partial\Omega),$$

and the in-radius of $\Omega$,

$$R_{in} := \sup_{x \in \Omega} \delta(x).$$

Finally, given a positive real number $x$ we denote by $[x]$ its integer part.
2 Estimates of the First Type

In this section we will derive lower bounds on the forms

\[ h^{B\Omega,A}_{\Omega,A}[u] = \int_{\Omega} B(x) |u(x)|^2 \, dx. \]

Instead of introducing a vector potential \( A \) associated to \( B \), we decide link both quantities through a so called super potential, an approach that is well-known in the study of the Pauli operator, see e.g. [EV]. For our magnetic fields however, this approach is equivalent with the standard definition given in the introduction.

Let \( r > 0 \) be such that \( \Omega \subset \mathcal{B}(0,r) \). For any \( B \in L^\infty_{\text{loc}}(\mathbb{R}^2) \) let

\[ \mathcal{F}(B) := \{ \Psi : \mathbb{R}^2 \to \mathbb{R} : \Delta \Psi = B \text{ in } \mathcal{B}(0,r) \} \]

be the family of super potentials associated to \( B \). Note that \( \mathcal{F}(B) \) is not empty. Indeed, the function

\[ \Psi_0(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus \mathcal{B}(0,r)} \log |x-y| B(y) \, dy, \quad x \in \mathbb{R}^2, \]

which is well defined in view of the regularity of \( B \), solves

\[ \Delta \Psi_0(x) = \begin{cases} B(x) & x \in \mathcal{B}(0,r), \\ 0 & \text{elsewhere}, \end{cases} \quad (2.1) \]

in the distributional sense. Since \( B \in L^\infty_{\text{loc}}(\mathbb{R}^2) \), standard regularity theory implies that \( \Psi_0 \in W^{2,p}(\mathcal{B}(0,r)) \) for every \( 1 \leq p < \infty \), see [GT, Thm. 9.9]. Moreover, for any \( \Psi \in \mathcal{F}(B) \) the difference \( \Psi - \Psi_0 \) is a harmonic function in \( \mathcal{B}(0,r) \). Hence for any \( \Psi \in \mathcal{F}(B) \) and any \( p \in [1, \infty) \) we have \( \Psi \in W^{2,p}(\mathcal{B}(0,r)) \). By Sobolev’s embedding theorem it then follows that \( \Psi \) is continuous on \( \mathcal{B}(0,r) \), so we may define the oscillation of \( \Psi \) over \( \Omega \);

\[ \text{osc}(\Omega, \Psi) = \sup_{x \in \Omega} \Psi(x) - \inf_{x \in \Omega} \Psi(x). \]

Accordingly we set

\[ \mathcal{D}(\Omega, B) := \inf_{\Psi \in \mathcal{F}(B)} \text{osc}(\Omega, \Psi). \]

Note also that a vector field \( A : \mathcal{B}(0,r) \to \mathbb{R}^2 \) defined by

\[ A := (-\partial_2 \Psi, \partial_1 \Psi), \quad \Psi \in \mathcal{F}(B), \]

belongs to \( W^{1,p}(\mathcal{B}(0,r)) \) for every \( 1 \leq p < \infty \), in view of the regularity of \( \Psi \), and satisfies

\[ \text{curl} A(x) = B(x) \text{ in } \mathcal{B}(0,r) \]

in the distributional sense. Hence by the Sobolev embedding theorem we have \( A \in L^\infty(\mathcal{B}(0,r)) \) and furthermore \( \text{div} A = 0 \) almost everywhere on \( \mathcal{B}(0,r) \).
Theorem 2.1. Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain and suppose that $B \in L^\infty_{\text{loc}}(\mathbb{R}^2)$. Then
\[
h_{\Omega,A}^B[u] \geq \pm \int_{\Omega} B(x) |u(x)|^2 \, dx + e^{-2\mathcal{P}(\Omega,B)} \lambda_1(\Omega,0) \int_{\Omega} |u(x)|^2 \, dx \tag{2.2}
\]
holds true for all $u \in C^\infty_c(\Omega)$.

Proof. We first prove inequality (2.2) with the plus sign on the right hand side. To do so, we pick any $\Psi \in \mathcal{F}(B)$ and perform the ground state substitution $u(x) =: v(x) e^{-\Psi(x)}$ and obtain, after a relatively lengthy (but straightforward) calculation,
\[
h_{\Omega,A}^B[u] - \int_{\Omega} B(x) |u(x)|^2 \, dx = \int_{\Omega} e^{-2\Psi} |(-i\partial_1 - \partial_2)v|^2 \, dx. \tag{2.3}
\]
Next, we have
\[
\int_{\Omega} e^{-2\Psi} |(-i\partial_1 - \partial_2)v|^2 \, dx \geq e^{-2\sup_{x \in \Omega} \Psi(x)} \int_{\Omega} |(-i\partial_1 - \partial_2)v|^2 \, dx
\]
\[
= e^{-2\sup_{x \in \Omega} \Psi(x)} \int_{\Omega} |\nabla v|^2 \, dx,
\]
where in the last step we used that
\[
\int_{\Omega} ((\partial_1 v)\partial_2 \bar{v} - (\partial_1 \bar{v})\partial_2 v) \, dx = 0.
\]
It then follows that
\[
h_{\Omega,A}^B[u] - \int_{\Omega} B(x) |u(x)|^2 \, dx \geq e^{-2\sup_{x \in \Omega} \Psi(x)} \int_{\Omega} |\nabla v|^2 \, dx
\]
\[
\geq e^{-2\sup_{x \in \Omega} \Psi(x)} \lambda_1(\Omega,0) \int_{\Omega} |v|^2 \, dx
\]
\[
\geq e^{-2\text{osc}(\Omega,\Psi)} \lambda_1(\Omega,0) \int_{\Omega} |u|^2 \, dx.
\]
To prove the corresponding lower bound with the minus sign in front of $B$ on the right hand side, we note that the substitution $u(x) =: w(x) e^{\Psi(x)}$ gives
\[
h_{\Omega,A}^B[u] + \int_{\Omega} B(x) |u(x)|^2 \, dx = \int_{\Omega} e^{2\Psi} |(-i\partial_1 + \partial_2)w|^2 \, dx.
\]
Moreover, since $\text{osc}(\Omega,-\Psi) = \text{osc}(\Omega,\Psi)$, the same procedure as above gives an identical lower bound. To complete the proof of (2.2) it now suffices to optimize the right hand side with respect to $\Psi \in \mathcal{F}(B)$, keeping in mind that the spectral properties of $h_{\Omega,A}^B$ only depend on $B$. 

\[\square\]
2.1 Estimates for the Constant Magnetic Field

In this section we consider the case of a constant magnetic field:

\[ B(x) = B_0 > 0. \]

Clearly, all \( \Psi \in F(B_0) \) are smooth, and optimizing estimate (2.2) amounts to minimizing the oscillation of \( B_0 \tilde{\Psi} \), where \( \tilde{\Psi} \) satisfies

\[ \Delta \tilde{\Psi} = 1 \quad \text{in} \quad \Omega. \]

The optimal \( \tilde{\Psi} \) depends very much on the geometry of \( \Omega \), and we start with a rather general result.

We pick any point \( x_0 \in \Omega \) and a rotation \( R(x_0, \theta) \in SO(2) \), parametrized by an angle \( \theta \in [0, 2\pi) \) and center of rotation \( x_0 \). Set

\[
\ell(\Omega, x_0, \theta) := \sup_{x \in R(x_0, \theta)\Omega} x_2 - \inf_{x \in R(x_0, \theta)\Omega} x_2,
\]

the maximal \( x_2 \)-distance of the rotated set \( R(x_0, \theta)\Omega \). The quantity \( \ell(\Omega) \) is then defined as follows:

\[
\ell(\Omega) := \inf_{\theta \in [0, 2\pi)} \ell(\Omega, x_0, \theta). \tag{2.5}
\]

It is easily seen that \( \ell(\Omega) \) is independent of the choice of \( x_0 \in \Omega \) and finite, since \( \Omega \) is bounded.

**Theorem 2.2.** Let \( \Omega \subset \mathbb{R}^2 \) a bounded open domain. Then

\[
\lambda_1(\Omega, B_0) \geq B_0 + e^{-\frac{B_0}{4} \ell(\Omega)^2} \lambda_1(\Omega, 0). \tag{2.6}
\]

**Proof.** In view of estimate (2.2) we have

\[
\lambda_1(\Omega, B_0) \geq B_0 + e^{-2 \sec(\Omega, \Psi)} \lambda_1(\Omega, 0), \quad \forall \Psi \in F(B_0). \tag{2.7}
\]

The rotational symmetry of the problem allows us to assume that \( \Omega \) has been rotated such that

\[ \ell(\Omega) = \sup_{x \in \Omega} x_2 - \inf_{x \in \Omega} x_2. \]

Let \( \alpha := \inf_{x \in \Omega} x_2 \) and \( \beta := \sup_{x \in \Omega} x_2 \). We then chose the super potential

\[
\Psi(x_1, x_2) = \frac{B_0}{2} (x_2 - a)^2, \tag{2.8}
\]

where \( a \) is a free parameter. Observe that we may assume that the entire line between \( \alpha \) and \( \beta \) is contained in \( \Omega \), because the oscillation of a function over
a domain can only increase as the domain is increased (and $\Psi$ is globally well-defined). Next, we calculate $\text{osc}(\Omega, \Psi)$ and minimize the result with respect to $a$. A direct calculation shows that the best choice is $a = (\alpha + \beta)/2$, which gives

$$\text{osc}(\Omega, \Psi) \leq \frac{B_0}{8} (\beta - \alpha)^2 = \frac{B_0}{8} \ell(\Omega)^2.$$ 

This in combination with (2.7) implies (2.6).

**Remark 2.3.** It was shown in [Er, Er2] that for any $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that

$$\lambda_1(\mathcal{B}(0, R), B_0) \geq B_0 + \frac{C(\varepsilon)}{R^2} e^{-B_0(\frac{1}{2} + \varepsilon)R^2},$$

(2.9)

Together with the Faber-Krahn inequality [Er] this yields

$$\lambda_1(\Omega, B_0) \geq \lambda_1(\mathcal{B}(0, R), B_0)$$

$$\geq B_0 + \frac{C(\varepsilon)}{R^2} e^{-B_0(\frac{1}{2} + \varepsilon)R^2}, \quad \varepsilon > 0,$$

(2.10)

where $R$ is such that $|\mathcal{B}(0, R)| = |\Omega|$. It is clear that (2.6) is an improvement of the estimate (2.10) for domains that are geometrically very far from the disc, as for example very wide rectangles or thin ellipses.

**Proposition 2.4.** Let $\Omega \subset \mathbb{R}^2$ be any bounded convex open domain, then

$$2R_{\text{in}} \leq \ell(\Omega) \leq 3R_{\text{in}}.$$ 

(2.11)

**Proof.** Let $\mathcal{B} \subset \Omega$ be a disc of radius $R_{\text{in}}$ contained in $\Omega$. Independently of $\Omega$ being convex or not we have

$$\ell(\Omega, x_0, \theta) \geq 2R_{\text{in}}, \quad \forall \theta \in [0, 2\pi), x_0 \in \Omega.$$ 

(2.12)

This follows directly from (2.4); for any $\theta \in [0, 2\pi)$ and $x_0 \in \Omega$ we have that $\ell(\Omega, x_0, \theta)$ is larger or equal to the length of the intersection of $\Omega$ with the vertical line passing through the center of $\mathcal{B}$. The latter is obviously larger or equal to $2R_{\text{in}}$, hence equation (2.12).

It remains to prove the second inequality of (2.11). Let $\mathcal{B} \subset \Omega$ be a disc of radius $R_{\text{in}}$. Assume first that $\partial \mathcal{B} \cap \partial \Omega$ contains at least two distinct points $P_1$ and $P_2$ and that the vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are linearly dependent. By convexity, $\Omega$ is contained in an infinite rectangle of height $2R_{\text{in}}$, and $\ell(\Omega) \leq 3R_{\text{in}}$.
Assume now that $\partial B \cap \partial \Omega$ is distributed in such a way that there is a closed, connected set $\Gamma \subset \partial B$ of length $\pi R_{\text{in}}$ with the property that the distance
\[
\rho(x) := \inf_{y \in \partial \Omega} |x - y|, \quad x \in \Gamma,
\]
is positive.

Since $\rho$ is continuous and $\Gamma$ is closed there is an $\varepsilon$ such that $\rho(x) \geq \varepsilon > 0$, for all $x \in \Gamma$. Hence we can move the disc $B$ a distance $\varepsilon/2$ much in the direction of $u$, such that $B$ becomes a proper subset of $\Omega$. This contradicts that the inner radius of $\Omega$ is $R_{\text{in}}$.

Assume that $\partial B \cap \partial \Omega$ contains at least three points $P_1$, $P_2$ and $P_3$. They must be distributed in such a way that there is no such $\Gamma$ as above. Since $\Omega$ is convex, it is contained in a triangle given by the tangent lines of the intersection points.

For a triangle, $\ell(T)$ is given by the smallest height, which is maximized for the equilateral triangle. Hence $\ell(\Omega) \leq 3R_{\text{in}}$. □

Note that the second term on the righthand side of (2.6) decays exponentially fast to zero as $B_0$ tends to infinity. This was in fact already observed in [FH2, Remark 1.4.3], where the authors observed that
\[
\frac{\lambda_1(\Omega, B_0)}{B_0} = 1 + O(\exp(-\alpha B_0)), \quad B_0 \to \infty,
\]
and $\alpha$ is a positive constant. The optimal value of $\alpha$ is in general unknown, however it was conjectured in [FH2, Remark 1.4.3] that $\alpha$ is proportional to $R_{\text{in}}^2$. This is in agreement with Proposition 2.4 and the following result.

**Proposition 2.5.** Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain and suppose that $B_0 R_{\text{in}}^2 \geq 4$. Then
\[
\lambda_1(\Omega, B_0) \leq B_0 + e B_0^2 R_{\text{in}}^2 e^{-\frac{B_0}{2R_{\text{in}}}}. \hspace{1cm} (2.14)
\]
Proof. In view of (2.3), any $v \in H^1_0(\Omega)$ satisfies

$$\lambda_1(\Omega, B_0) \leq B_0 + \frac{\int_{\Omega} e^{-2\Psi} |(-i\partial_1 - \partial_2)v|^2 \, dx}{\int_{\Omega} e^{-2\Psi} |v|^2 \, dx}. \tag{2.15}$$

Without loss of generality we may assume that the largest disc contained in $\Omega$ is centered in the origin. Hence $\mathscr{B}(0, R_{in}) \subset \Omega$. We choose the superpotential in the form

$$\Psi(x) = \frac{B_0}{4} |x|^2,$$

and apply inequality (2.15) with $v(x) = \begin{cases} 1 & |x| \leq R_{in} - \frac{1}{B_0}, \\ R_{in}B_0(R_{in} - |x|) & R_{in} - \frac{1}{B_0} < |x| < R_{in}, \\ 0 & \text{elsewhere}. \end{cases}$

Obviously $v \in H^1_0(\Omega)$. Note also that since $v$ is real valued, we have $|(-i\partial_1 - \partial_2)v|^2 = |\nabla v|^2$. Performing both integrations in (2.15) in polar coordinates and taking into account the condition $B_0R_{in}^2 \geq 4$, we find that

$$\int_{\Omega} e^{-2\Psi} |(-i\partial_1 - \partial_2)v|^2 \, dx = 2\pi B_0^2 R_{in} \int_{R_{in} - \frac{1}{B_0}}^{R_{in}} e^{-\frac{B_0}{2} r^2} \, r \, dr$$

$$= 2\pi B_0 R_{in}^2 e^{-\frac{B_0}{2} R_{in}^2} \left( \exp \left( 1 - \frac{1}{2B_0^2 R_{in}^2} \right) - 1 \right)$$

$$\leq 2\pi (e - 1) B_0 R_{in}^2 e^{-\frac{B_0}{4} R_{in}^2}$$

and

$$\int_{\Omega} e^{-2\Psi} |v|^2 \, dx \geq 2\pi \int_0^{R_{in} - \frac{1}{B_0}} e^{-\frac{B_0}{2} r^2} \, r \, dr \geq \frac{2\pi}{B_0} (1 - e^{-1})$$

which proves (2.14). \qed

In the special case when $\Omega$ is a disc we have

**Proposition 2.6.** Let $\Omega = \mathscr{B}(0, R)$ and let $B_0R^2 \geq 4$. Then

$$1 + \frac{j_{0,1}^2}{B_0R^2} e^{-\frac{1}{2}B_0R^2} \leq \frac{\lambda_1(\mathscr{B}(0, R), B_0)}{B_0} \leq 1 + e B_0R^2 e^{-\frac{1}{2}B_0R^2}, \tag{2.16}$$

where $j_{0,1} \simeq 2.405$ is the first zero of the Bessel function $J_0$. 

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Proof. The upper bound is given by Proposition 2.5. For the lower bound we use Theorem 2.1 with the super potential \( \Psi(x) = \frac{B_0}{4} |x|^2 \), which is in this case better suited to the geometry of the domain than the one used in Proposition 2.4. Hence
\[
\text{osc}(\mathcal{B}(0, R), \Psi) = \frac{B_0 R^2}{4},
\]
and since \( \lambda(\mathcal{B}(0, R), 0) = \frac{\beta_0}{R^2} \), the lower bound in (2.16) follows from (2.2). \( \square \)

Remark 2.7. In [HM, Prop. 4.4] it was stated that
\[
\frac{\lambda_1(\mathcal{B}(0, R), B_0)}{B_0} \sim 1 + \frac{3^{3/2}}{\sqrt{\pi}} \sqrt{B_0} \Re e^{-\frac{1}{2} B_0 R^2}, \quad B_0 \to \infty. \quad (2.17)
\]
B. Helffer however pointed out to us that the argument used to establish the above was slightly flawed – the above is only true if taken as a lower bound. However, from Proposition 2.6 we easily see that
\[
\lim_{B_0 \to \infty} \frac{\log \left[ \lambda_1(\mathcal{B}(0, R), B_0) - B_0 \right]}{B_0 R^2} = -\frac{1}{2},
\]
which confirms, up to a pre-factor, the asymptotic (2.17) stated in [HM Prop. 4.4]. Note also that the lower bound in (2.16) improves qualitatively the lower bound (2.9), since it allows us to pass to the limit \( \varepsilon \to 0 \).

3 Estimates of the Second Type

3.1 Dirichlet boundary conditions

In this section we are going to establish a lower bound on the difference \( \lambda_1(\Omega, B) - \lambda_1(\Omega, 0) \). Given a point \( x \in \Omega \), we introduce the function
\[
\Phi(r, x) := \frac{1}{2\pi} \int_{\mathcal{B}(x, r)} B(x) \, dx, \quad 0 \leq r \leq \delta(x), \quad (3.1)
\]
the flux through \( \mathcal{B}(x, r) \). The next result shows that as soon as the magnetic field is not identically zero in \( \Omega \), the difference \( \lambda_1(\Omega, B) - \lambda_1(\Omega, 0) \) is strictly positive.

Theorem 3.1. Let \( \Omega \) be a bounded open domain in \( \mathbb{R}^2 \) and \( B \in L^\infty_{\text{loc}}(\mathbb{R}^2) \). If \( B \) is not identically zero in \( \Omega \), then there exists \( y \in \Omega \) such that
\[
\lambda_1(\Omega, B) - \lambda_1(\Omega, 0) \geq D(y, B), \quad (3.2)
\]
where \( D(y, B) > 0 \) is given by (3.9).
For the proof of Theorem 3.1 we are going to need the following elementary result.

**Lemma 3.2.** Let \( z_1, z_2 \in \mathbb{C} \) and let \( \beta, \gamma \) be positive constants. Then

\[
\beta |z_1|^2 + \gamma |z_1 + z_2|^2 \geq \frac{\beta \gamma}{\beta + \gamma} |z_2|^2.
\]

**Proof.** Since

\[
|\bar{z}_1 z_2| + |z_1 \bar{z}_2| \leq \varepsilon |z_1|^2 + \varepsilon^{-1} |z_2|^2, \quad \forall \varepsilon > 0,
\]

we have

\[
\beta |z_1|^2 + \gamma |z_1 + z_2|^2 \geq (\beta + \gamma(1 - \varepsilon))|z_1|^2 + \gamma(1 - \varepsilon^{-1})|z_2|^2.
\]

The claim now follows upon setting \( \varepsilon = \frac{\beta + \gamma}{\gamma} \).

**Proof of Theorem 3.1.** Let \( \varphi_1 \) be the positive normalized ground state of the Dirichlet Laplacian \( H_{\Omega,0}^D \),

\[
H_{\Omega,0}^D \varphi_1 = \lambda_1(\Omega,0) \varphi_1, \quad \int_{\Omega} \varphi_1^2 dx = 1.
\]

To simplify the notation, we abbreviate \( \lambda_1(\Omega,0) = \lambda_1 \) and accordingly for higher eigenvalues of \( H_{\Omega,0}^D \).

For \( u \in C_c^\infty(\Omega) \), we perform a groundstate substitution

\[
u(x) =: v(x) \varphi_1(x), \quad v \in C_c^\infty(\Omega),
\]

so that

\[
h_{\Omega,A}[u] - \lambda_1 \int_{\Omega} |v|^2 dx = \int_{\Omega} |(-i \nabla + A)v|^2 \varphi_1^2 dx
\]

by an explicit computation. Moreover, from the assumptions of the theorem it follows that there exists \( y \in \Omega \) and \( \rho \in (0, \delta(y)) \) such that \( \Phi(\cdot, y) \) is not identically zero in \( (0, \rho) \). By Lemma 3.1, we know that for any \( R \in (0, \delta(y)) \), there exists \( F_1 = F_1(y, B, R) \geq 0 \) such that

\[
\int_{\partial(y,R)} |(-i \nabla + A)v|^2 dx \geq F_1 \int_{\partial(y,R)} |v|^2 dx, \quad \forall v \in H^1(\Omega),
\]

(3.3)

and that the function \( F_1(y, B, \cdot) \) is not identically zero on \( (0, \delta(y)) \). We then write

\[
\int_{\Omega} |(-i \nabla + A)v|^2 \varphi_1^2 dx \geq \frac{1}{2} \int_{\Omega} |(-i \nabla + A)v|^2 \varphi_1^2 dx + \frac{1}{2} \int_{\partial(y,R)} |(-i \nabla + A)v|^2 \varphi_1^2 dx.
\]

(3.4)
The last term is estimated as follows:

$$\frac{1}{2} \int_{\mathcal{B}(y,R)} |(-i\nabla + A)v|^2 \varphi_1^2 \, dx \geq \frac{1}{2} \left( \inf_{x \in \mathcal{B}(y,R)} \varphi_1^2(x) \right) F_1 \int_{\mathcal{B}(y,R)} |v|^2 \, dx$$

$$\geq F_2 \int_{\mathcal{B}(y,R)} \varphi_1^2 |v|^2 \, dx = F_2 \int_{\mathcal{B}(y,R)} |u|^2 \, dx,$$

where

$$F_2(y, B, R) := \frac{1}{2} F_1(y, B, R) \frac{\inf_{x \in \mathcal{B}(y,R)} \varphi_1^2(x)}{\sup_{x \in \mathcal{B}(y,R)} \varphi_1^2(x)} \quad (3.5)$$

For the first term, we substitute back and use the diamagnetic inequality, so that

$$\frac{1}{2} \int_{\Omega} |(-i\nabla + A)v|^2 \varphi_1^2 \, dx = \frac{1}{2} \left( h_{\Omega,A}[u] - \lambda_1 \int_{\Omega} |u|^2 \, dx \right)$$

$$\geq \frac{1}{2} \int_{\Omega} |\nabla |u||^2 \, dx - \frac{\lambda_1}{2} \int_{\Omega} |u|^2 \, dx.$$

Putting the above estimates together, we obtain

$$h_{\Omega,A}[u] - \lambda_1 \int_{\Omega} |u|^2 \, dx \geq \frac{1}{2} \left( \int_{\Omega} |\nabla |u||^2 \, dx - \lambda_1 \int_{\Omega} |u|^2 \, dx \right)$$

$$+ F_2 \int_{\mathcal{B}(y,R)} |u|^2 \, dx.$$

Since we will be taking the infimum over all \( u \in H_0^1(\Omega) \), we have by the inclusion of sets

$$\inf_{u \in H_0^1(\Omega)} \left( h_{\Omega,A}[u] - \lambda_1 \int_{\Omega} |u|^2 \, dx \right)$$

$$\geq \inf_{u \in H_0^1(\Omega)} \left( \frac{1}{2} \int_{\Omega} |\nabla |u||^2 \, dx - \lambda_1 \int_{\Omega} |u|^2 \, dx + F_2 \int_{\mathcal{B}(y,R)} |u|^2 \, dx \right)$$

$$\geq \inf_{w \in H_0^1(\Omega)} \left( \frac{1}{2} \int_{\Omega} |\nabla w|^2 \, dx - \lambda_1 \int_{\Omega} |w|^2 \, dx + F_2 \int_{\mathcal{B}(y,R)} |w|^2 \, dx \right).$$

Next, we observe that any \( w \in H_0^1(\Omega) \) can be written as

$$w = \alpha \varphi_1 + f, \quad \alpha = (w, \varphi_1)_{L^2(\Omega)}$$

so that \( (\varphi_1, f) = 0 \). Hence,

$$\frac{1}{2} \left( \int_{\Omega} |\nabla w|^2 \, dx - \lambda_1 \int_{\Omega} |w|^2 \, dx \right) = \frac{1}{2} \left( \int_{\Omega} |\nabla f|^2 \, dx - \lambda_1 \int_{\Omega} |f|^2 \, dx \right)$$

$$\geq \frac{\Delta \lambda}{2} \int_{\Omega} |f|^2 \, dx,$$
where \( \Delta \lambda := \lambda_2 - \lambda_1 > 0 \). From this we conclude that
\[
h_{\Omega, A}^D[u] - \lambda_1 \int_{\Omega} |u|^2 \, dx \geq \Delta \frac{\lambda}{2} \int_{\Omega} |f|^2 \, dx + F_2 \int_{\mathcal{B}(y, R)} |\alpha \varphi_1 + f|^2 \, dx
\]
\[
\geq \Delta \frac{\lambda}{4} \int_{\Omega} |f|^2 \, dx + \int_{\mathcal{B}(y, R)} \left( \frac{\Delta \lambda}{4} |f|^2 + F_2 |\alpha \varphi_1 + f|^2 \right) \, dx.
\]
We then set \( \beta = \frac{\Delta \lambda}{4} \) and apply to the last term Lemma 3.2 with \( z_1 = f, z_2 = \alpha \varphi_1 \) and \( \gamma = F_2 \). This gives
\[
h_{\Omega, A}^D[u] - \lambda_1 \int_{\Omega} |u|^2 \, dx \geq \Delta \frac{\lambda}{4} \int_{\Omega} |f|^2 \, dx + |\alpha|^2 F_3 \int_{\mathcal{B}(y, R)} \varphi_1^2 \, dx,
\]
with
\[
F_3 = F_3(y, B, R) = \frac{F_3(y, B, R)}{F_2(y, B, R)} + \beta. \quad (3.6)
\]
To sum up, we obtain the lower bound
\[
\lambda_1(\Omega, B) - \lambda_1(\Omega, 0) \geq \inf_{f \in H^1_0(\Omega)} \inf_{\alpha \in C} \frac{\Delta \lambda}{4} \frac{||f||_2^2 + |\alpha|^2 F_3 \int_{\mathcal{B}(y, R)} \varphi_1^2}{|\alpha|^2 + ||f||_2^2}. \quad (3.7)
\]
The variational problem on the right hand side has an explicit solution:
\[
\inf_{f \in H^1_0(\Omega)} \inf_{\alpha \in C} \frac{\Delta \lambda}{4} \frac{||f||_2^2 + |\alpha|^2 F_3 \int_{\mathcal{B}(y, R)} \varphi_1^2}{|\alpha|^2 + ||f||_2^2} = \inf_{t \geq 0} \frac{\Delta \lambda}{4} + t F_3 \int_{\mathcal{B}(y, R)} \varphi_1^2 = \min \left\{ \frac{\Delta \lambda}{4}, F_3 \int_{\mathcal{B}(y, R)} \varphi_1^2 \right\} = F_3 \int_{\mathcal{B}(y, R)} \varphi_1^2, \quad (3.8)
\]
where the last equality follows from the definition of \( F_3 \) and the normalization of \( \varphi_1 \). Since (3.7) holds for any \( R \leq \delta(y) \), it follows that lower bound (3.2) holds with
\[
D(y, B) := \sup_{0 < R < \delta(y)} F_3(y, B, R) \int_{\mathcal{B}(y, R)} \varphi_1^2. \quad (3.9)
\]

**Remark 3.3.** The value of \( D(y, B) \) decreases when the support of the magnetic field gets closer to the boundary of \( \Omega \), which is natural. The precise value of \( D(y, B) \) might in general depend in a complicated way on \( B \) and on the geometry of \( \Omega \).

However, in the case of constant magnetic field it is possible to give a more explicit lower bound. We will state the result separately for small and large values of the magnetic field.
Corollary 3.4. Let $B = B_0 > 0$ be constant and assume that $B_0 \leq R_{in}^{-2}$, then

$$
\lambda_1(\Omega, B_0) - \lambda_1(\Omega, 0) \geq \frac{B_0^2}{8} \sup_{y \in \Omega} \sup_{R \leq \delta(y)} \frac{Q(y, R) \Delta \lambda}{B_0^2 Q(y, R)} \int_{\mathcal{B}(y, R)} \varphi_1^2,
$$

where

$$
Q(y, R) := R^2 \inf_{x \in \mathcal{B}(y, R)} \varphi_1^2(x), \quad \Delta \lambda := \lambda_2(\Omega, 0) - \lambda_1(\Omega, 0).
$$

Proof. Let $y \in \Omega$. Assume first that $B_0 \leq R_{in}^{-2}$ and let $R \leq \delta(y) \leq R_{in}$. A detailed inspection of Lemma A.1 (see in particular equations (A.2) and (A.3)) shows that in this case the quantity $F_1$ introduced in the proof of Theorem 3.1 satisfies

$$
F_1 \geq \frac{B_0^2 R^2}{12}, \quad \text{if } B_0 \leq R_{in}^{-2}.
$$

This in combination with (3.5), (3.6) and (3.9) implies that

$$
D(y, B_0) \geq \frac{B_0^2 \Delta \lambda}{8} \frac{Q(y, R) \int_{\mathcal{B}(y, R)} \varphi_1^2}{B_0^2 Q(y, R) + 3 \Delta \lambda}.
$$

Optimizing the right hand side first in $R$ and then in $y$ gives lower bound (3.10).

In order to state the result for larger values of $B_0$ we need some additional notation. Let $x_0 \in \Omega$ be the center of a disc of radius $R_{in}$ contained in $\Omega$. It is easily seen that the disc $\mathcal{B}(x_0, R_{in}/2)$ contains

$$
N(B_0) = \lfloor R_{in}^2 B_0 \rfloor
$$

disjoint squares of size $(2B_0)^{-1/2}$. Let $y_j$, $j = 1, \ldots, N(B_0)$, be the centers of these squares. It follows that $\mathcal{B}(x_0, R_{in}/2)$, and therefore $\Omega$, contains $N(B_0)$ disjoint disc of radius

$$
\rho = \frac{1}{2} B_0^{-1/2}
$$

centered in $y_j$. Let

$$
Q_j := \frac{\inf_{x \in \mathcal{B}(y_j, \rho)} \varphi_1^2(x)}{\sup_{x \in \mathcal{B}(y_j, \rho)} \varphi_1^2(x)}.
$$
Corollary 3.5. Let $B = B_0 > 0$ be constant. Assume that $B_0 > R_{\text{in}}^{-2}$. Then

$$\lambda_1(\Omega, B_0) - \lambda_1(\Omega, 0) \geq \frac{B_0}{8} \sum_{j=1}^{N(B_0)} \frac{Q_j \Delta \lambda}{B_0 Q_j + 12 \Delta \lambda} \int_{\mathcal{B}(y_j, \rho)} \varphi_1^2 (3.14)$$

where $N(B_0), y_j$, and $\rho$ are as above.

Proof. We will follow the proof of Theorem 3.1 and replace the disc $\mathcal{B}(y, R)$ by the family of discs $\mathcal{B}(y_j, \rho)$ with $j = 1, 2, \ldots, N(B_0)$. Since the latter are disjoint by construction, we obtain a modified version of inequality (3.4):

$$\int_{\Omega} |(-i \nabla + A)v|^2 \varphi_1^2 \, dx \geq \frac{1}{2} \int_{\Omega} |(-i \nabla + A)v|^2 \varphi_1^2 \, dx \quad (3.15)$$

$$+ \frac{1}{2} \sum_{j=1}^{N(B_0)} \int_{\mathcal{B}(y_j, \rho)} |(-i \nabla + A)v|^2 \varphi_1^2 \, dx.$$

Moreover, from Lemma A.1 we deduce that for any $v \in H^1(\Omega)$ it holds that

$$\int_{\mathcal{B}(y_j, \rho)} |(-i \nabla + A)v|^2 \, dx \geq \frac{B_0}{48} \int_{\mathcal{B}(y_j, \rho)} |v|^2 \, dx, \quad j = 1, \ldots, N(B_0). \quad (3.16)$$

Hence following the line of arguments of the proof of Theorem 3.1, see equations (3.5)–(3.8), we arrive at

$$\lambda_1(\Omega, B) - \lambda_1(\Omega, 0) \geq \sum_{j=1}^{N(B_0)} F_{3,j} \int_{\mathcal{B}(y_j, \rho)} \varphi_1^2,$$

where

$$F_{3,j} \geq \frac{B_0}{8} \frac{Q_j \Delta \lambda}{B_0 Q_j + 12 \Delta \lambda}.$$

The claim now follows. □

Remark 3.6. Note that by the Lebesgue property we have

$$\lim_{B_0 \to \infty} B_0 \int_{\mathcal{B}(y_j, \rho)} \varphi_1^2(x) \, dx = \frac{\pi}{4} \varphi_1^2(y_j).$$

Hence in view of (3.12) the right hand side of (3.14) remains bounded and strictly positive as $B_0$ tends to infinity.

Remark 3.7. Note that when $\Omega$ is convex, the righthand side of Corollary 3.4 and Corollary 3.5 can be further simplified by using the lower bound for the first spectral gap of the non-magnetic Laplacian. From [AC] we know that

$$\Delta \lambda = \lambda_2(\Omega, 0) - \lambda_1(\Omega, 0) \geq \frac{3\pi^2}{\text{diam}(\Omega)^2}.$$
3.2 Neumann boundary conditions

In the Neumann case, a simple perturbation argument with respect to the non-magnetic Laplacian shows that the corresponding estimate of (1.3) in the Neumann case must fail. By taking the (normalized) constant function on $\Omega$ as a trial state we obtain

$$\mu_1(\Omega, B_0) = O(B_0^2) \quad B_0 \to 0.$$  \hspace{1cm} (3.17)

This of course contradicts (1.3) with $\lambda_1(\Omega, B_0)$ replaced by $\mu_1(\Omega, B_0)$.

It is however possible to prove an analog of the estimate of the second type.

**Theorem 3.8.** Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with Lipschitz boundary and let $B = B_0 > 0$. Then

$$\mu_1(\Omega, B_0) \geq \frac{\pi}{4|\Omega|} \frac{B_0^2 R_{\text{in}}^4 \mu_2(\Omega, 0)}{B_0^2 R_{\text{in}}^2 + 6 \mu_2(\Omega, 0)} \quad \text{if} \quad B_0 \leq R_{\text{in}}^{-2},$$  \hspace{1cm} (3.18)

and

$$\mu_1(\Omega, B_0) \geq \frac{\pi}{32|\Omega|} \frac{N(B_0) \mu_2(\Omega, 0)}{B_0 + 12 \mu_2(\Omega, 0)} \quad \text{if} \quad B_0 > R_{\text{in}}^{-2},$$  \hspace{1cm} (3.19)

where $N(B_0)$ is given by (3.12).

**Proof.** Let $y \in \Omega$ be such that $\delta(y) = R_{\text{in}}$ and let $R \in (0, R_{\text{in}})$. Let $\psi_1$ be the normalized eigenfunction of the Neumann Laplacian associated to eigenvalue $\mu_1(\Omega, 0) = 0$:

$$\psi_1(x) = |\Omega|^{-1/2}.$$

We again use inequality (3.3) which, together with the diamagnetic inequality, leads to the lower bound

$$\mu_1(\Omega, B_0) \geq \frac{1}{2} \inf_{v \in H^1(\Omega)} \frac{\int_{\Omega} |\nabla v|^2 + F_1 \int_{\mathcal{A}(y; R)} |v|^2}{\int_{\Omega} |v|^2},$$  \hspace{1cm} (3.20)

cp. (3.5). Next we use the decomposition

$$v = \alpha \psi_1 + f, \quad \alpha = (v, \psi_1)_{L^2(\Omega)}.$$  \hspace{1cm} (3.21)

Following the proof of Theorem 3.1 and using the lower bound (3.11) we then find that

$$\mu_1(\Omega, B_0) \geq \frac{\mu_2 F_1}{4 F_1 + 2 \mu_2} \int_{\mathcal{A}(y; R)} \psi_1^2 dx = \frac{\pi}{|\Omega|} \frac{\mu_2 R^2 F_1}{4 F_1 + 2 \mu_2} \frac{\psi_1^2}{\int_{\mathcal{A}(y; R)} |v|^2}$$

$$\geq \frac{\pi}{4 |\Omega|} \frac{B_0^2 R_{\text{in}}^4 \mu_2}{B_0^2 R_{\text{in}}^2 + 6 \mu_2},$$

where we used the abbreviation $\mu_2 := \mu_2(\Omega, 0)$. Optimizing in $R$ then gives (3.18). As for inequality (3.19), this follows by mimicking the proof of Corollary 3.5. Indeed, if we replace $\varphi_1$ by $\psi_1$ and $\Delta \lambda$ by $\mu_2(\Omega, 0)$ we end up with (3.19). 

\hspace{1cm} $\square$
Remark 3.9. Notice that both lower bounds (3.10) and (3.18) are for small values of $B_0$ proportional to $B_0^2$. This is in agreement with the asymptotic expansions (1.8) and (3.17).

A Appendix

Let $y \in \Omega$ and let $R \in (0, \delta(y))$. Let

$$\mu(r) = \min_{k \in \mathbb{Z}} |k - \Phi(r, y)|,$$

where $\Phi(r, y)$ is given by (3.1). Define the function $\chi : [0, R] \to [0, 1]$ by

$$\chi(r) := \frac{\mu_0^2 \mu^2(r)}{r^2}, \quad \mu_0 := \left(\max_{[0, R]} \frac{\mu(r)}{r}\right)^{-1},$$

and let

$$\nu_0 := \max_{[0, R]} |r^{-2} (r \mu'(r) - \mu(r))|.$$

From the definition of $\chi$ it follows that there exists at least one $r_0 \in [0, R]$ such that

$$\chi(r_0) = 1. \quad (A.1)$$

To any such $r_0$ we define the constants

$$c_0 := 4 \max \left\{ j_{0,1}^2 r_0^2, (6r_0)^{-1}(2R^3 - 3R^2r_0 + r_0^3) \right\}, \quad (A.2)$$

$$c_1 := \max \left\{ 2\mu_0^2 + 4\mu_0^2 \nu_0^2, c_0 \right\}. \quad (A.3)$$

With this notation we can state [EK] Lemma 3.1:

**Lemma A.1** (Ekholm-Kovařík). Let $B \in L^\infty_{\text{loc}}(\mathbb{R}^2)$ and let $r_0$ satisfy (A.1). Then any $v \in H^1(\Omega)$ satisfies

$$c_1 \int_{B(y, R)} |(-i\nabla + A)v|^2 \, dx \geq \int_{B(y, R)} |v|^2 \, dx. \quad (A.4)$$

Aknowledgements

The authors would like to thank S. Fournais, B. Helffer, N. Raymond and J. P. Solovej for valuable discussions. Part of this work has been carried out during a visit at the Mathematisches Institut Oberwolfach during the Program “Eigenvalue Problems in Surface Superconductivity”. T. E. was supported by the Swedish Research Council grant Nr. 2009-6073. H. K. was supported by the Gruppo Nazionale per Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The support of MIUR-PRIN2010-11 grant for the
project “Calcolo delle variazioni” (H. K.) is also gratefully acknowledged.
F. P. acknowledges support from the Swedish Research Council grant Nr.
2012-3864 and ERC grant Nr. 321029 “The mathematics of the structure of
matter”.

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