PERMUTATIONS WITH ARITHMETIC CONSTRAINTS

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Abstract. Let $S_{\text{lcm}}(n)$ denote the set of permutations $\pi$ of $[n] = \{1, 2, \ldots, n\}$ such that $\text{lcm}(j, \pi(j)) \leq n$ for each $j \in [n]$. Further, let $S_{\text{div}}(n)$ denote the number of permutations $\pi$ of $[n]$ such that $j \mid \pi(j)$ or $\pi(j) \mid j$ for each $j \in [n]$. Clearly $S_{\text{div}}(n) \subset S_{\text{lcm}}(n)$. We get upper and lower bounds for the counts of these sets, showing they grow geometrically. We also prove a conjecture from a recent paper on the number of “anti-coprime” permutations of $[n]$, meaning that each $\gcd(j, \pi(j)) > 1$ except when $j = 1$.

In memory of Eduard Wirsing (1931–2022)

1. Introduction

Recently, in [7] some permutation enumeration problems with an arithmetic flavor were considered. In particular, one might count permutations $\pi$ of $[n] = \{1, 2, \ldots, n\}$ where each $\gcd(j, \pi(j)) = 1$ and also permutations $\pi$ where each $\gcd(j, \pi(j)) > 1$ except for $j = 1$. It was shown in [7] that the coprime count is between $n!/c_1^n$ and $n!/c_2^n$ for all large $n$, where $c_1 = 3.73$ and $c_2 = 2.5$. Shortly after, Sah and Sawhney [9] showed that there is an explicit constant $c_0 = 2.65044 \ldots$ with the count of the shape $n!/(c_0 + o(1))^n$ as $n \to \infty$. The “anti-coprime” count was shown in [7] to exceed $n!/(\log n)^{(\alpha+o(1))n}$ as $n \to \infty$, where $\alpha = e^{-\gamma}$, with $\gamma$ Euler’s constant. It was conjectured in [7] that this lower bound is sharp, which we will prove here.

There are several papers in the literature that have considered the divisibility graph on $[n]$ where $i \neq j$ are connected by an edge if $i$ divides $j$ or vice versa, and the closely related lcm graph, where edges correspond to $\text{lcm}(i, j) \leq n$. In particular, it was shown in [6] that the length of the longest simple path in such graphs is $o(n)$, and this has been improved to order-of-magnitude $n/\log n$, see Saias [10] for a recent paper on the topic. One might also consider permutations of $[n]$ compatible with these graphs. Let $S_{\text{div}}(n)$ denote the set of permutations $\pi$ of $n$ such that for each $j \in [n]$, either $j \mid \pi(j)$ or $\pi(j) \mid j$.

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Further, let $S_{\text{lcm}}(n)$ denote the set of permutations $\pi$ of $[n]$ such that for each $j \in [n]$, $\text{lcm}[j, \pi(j)] \leq n$. Clearly, $S_{\text{div}}(n) \subset S_{\text{lcm}}(n)$. There is a small literature on these topics. In particular, counts for $\#S_{\text{div}}(n)$ are on OEIS [5] (due to Heinz and Farrokhi), which we reproduce here, together with new counts for $\#S_{\text{lcm}}(n)$.

The table suggests that $\#S_{\text{lcm}}(n) > \#S_{\text{div}}(n) > 2^n$ for $n$ large, and that there may be a similar upper bound. In this note we will prove that $(\#S_{\text{div}}(n))^{1/n}$ is bounded above 1 and $(\#S_{\text{lcm}}(n))^{1/n}$ is bounded below.

Table 1. Counts for $S_{\text{div}}(n)$ and $S_{\text{lcm}}(n)$ and their $n$th roots.

| $n$ | $\#S_{\text{div}}(n)$ | $(\#S_{\text{div}}(n))^{1/n}$ | $\#S_{\text{lcm}}(n)$ | $(\#S_{\text{lcm}}(n))^{1/n}$ |
|-----|----------------------|-----------------------------|----------------------|-----------------------------|
| 1   | 1                    | 1.0000                      | 1                    | 1.0000                      |
| 2   | 2                    | 1.4142                      | 2                    | 1.4142                      |
| 3   | 3                    | 1.4422                      | 3                    | 1.4422                      |
| 4   | 8                    | 1.6818                      | 8                    | 1.6818                      |
| 5   | 10                   | 1.5849                      | 10                   | 1.5849                      |
| 6   | 36                   | 2.8272                      | 56                   | 1.9560                      |
| 7   | 41                   | 1.6998                      | 64                   | 1.8114                      |
| 8   | 132                  | 1.8411                      | 192                  | 1.9294                      |
| 9   | 250                  | 1.8469                      | 332                  | 1.9060                      |
| 10  | 700                  | 1.9254                      | 1,184                | 2.0292                      |
| 11  | 750                  | 1.8254                      | 1,264                | 1.9142                      |
| 12  | 4,010                | 1.9965                      | 12,192               | 2.1556                      |
| 13  | 4,237                | 1.9011                      | 23,568               | 2.0708                      |
| 14  | 10,680               | 1.9398                      | 37,568               | 2.0721                      |
| 15  | 24,679               | 1.9626                      | 100,836              | 2.1556                      |
| 16  | 87,328               | 2.0362                      | 311,760              | 2.1903                      |
| 17  | 90,478               | 1.9569                      | 322,320              | 2.1087                      |
| 18  | 435,812              | 2.0573                      | 2,338,368            | 2.2585                      |
| 19  | 449,586              | 1.9839                      | 2,408,848            | 2.1671                      |
| 20  | 1,939,884            | 2.0623                      | 14,433,408           | 2.2802                      |
| 21  | 3,853,278            | 2.0588                      | 32,058,912           | 2.2773                      |
| 22  | 8,650,900            | 2.0669                      | 76,931,008           | 2.2828                      |
| 23  | 8,840,110            | 2.0046                      | 78,528,704           | 2.2043                      |
| 24  | 60,035,322           | 2.1091                      | 919,469,408          | 2.3631                      |
| 25  | 80,605,209           | 2.0714                      | 1,158,792,224        | 2.3044                      |
| 26  | 177,211,024          | 2.0761                      | 2,689,828,672        | 2.3051                      |
| 27  | 368,759,752          | 2.0757                      | 4,675,217,824        | 2.2811                      |
| 28  | 1,380,348,224        | 2.1205                      | 21,679,173,184       | 2.3396                      |
| 29  | 1,401,414,640        | 2.0673                      | 21,984,820,864       | 2.2731                      |
| 30  | 8,892,787,136        | 2.1460                      | 381,078,324,992      | 2.4324                      |
| 31  | 9,014,369,784        | 2.0947                      | 386,159,441,600      | 2.3646                      |
| 32  | 33,923,638,848       | 2.1334                      | 1,202,247,415,040    | 2.3851                      |
| 33  | 59,455,553,072       | 2.1208                      | 2,129,469,872        | 2.3728                      |
| 34  | 126,536,289,568      | 2.1210                      | 4,366,141,004        | 2.3157                      |
| 35  | 207,587,882,368      | 2.1055                      | 9,732,282,008        | 2.2802                      |
infinity. We conjecture they tend to limits, but we lack the numerical evidence or heuristics to suggest values for these limits.\footnote{This conjecture was very recently proved by McNew, see arXiv:2207.09652 [math.NT].} We will also show that $\#S_{\text{lcm}}(n)/\#S_{\text{div}}(n)$ tends to infinity geometrically.

One might also ask for the length of the longest cycle among permutations in $S_{\text{div}}(n)$ or in $S_{\text{lcm}}(n)$. This seems to be only slightly less (if at all) than the length of the longest simple chain in the divisor graph or lcm graph on $[n]$ mentioned above. Other papers have looked at tilings of $[n]$ with divisor chains, for example see [4]. This could correspond to asking about the cycle decomposition for permutations in $S_{\text{div}}(n)$ or in $S_{\text{lcm}}(n)$.

We mention the paper [2] of Erdős, Freud, and Hegyvári where some other arithmetic problems connected with integer permutations are discussed. Finally, we note the recent paper [1] which also has a similar flavor.

2. An upper bound for $\#S_{\text{lcm}}(n)$  

Theorem 1. We have $\#S_{\text{lcm}}(n) \leq e^{2.61n}$ for all large $n$.

Proof. Let $n$ be large. For $j \in [n]$, let $N(j)$ denote the number of $j' \in [n]$ with lcm$[j, j'] \leq n$. This condition can be broken down as follows: lcm$[j, j'] \leq n$ if and only if there are integers $a, b, c$ with

\begin{equation}
    j = ab, \quad j' = bc, \quad \gcd(a, c) = 1, \quad abc \leq n.
\end{equation}

That is, $N(j)$ is the number of triples $a, b, c$ with $j = ab$ satisfying (1). Since $a \mid j$, $b = j/a$, and $c \leq n/j$, we have $N(j) \leq \tau(j) n/j$, where $\tau$ is the divisor function (which counts the number of positive divisors of its argument). For $\pi \in S_{\text{lcm}}(n)$, the number of possible values for $\pi(j)$ is at most $N(j)$, so we have

\begin{equation}
    \#S_{\text{lcm}}(n) \leq \prod_{j \in [n]} N(j) \leq \prod_{j \in [n]} \tau(j) n/j.
\end{equation}

This quickly leads to an estimate for $\#S_{\text{lcm}}(n)$ that is of the form $n!^{o(1)}$ as $n \to \infty$, but to do better we will need to work harder. In particular we use a seemingly trivial property of permutations: they are one-to-one. In particular, there are not many values of $j$ with $\pi(j)$ small, since there are not many small numbers. This thought leads to versions of (2) where $\tau$ is replaced with a restricted divisor function that counts only small divisors.

Let $k = 30$. We partition the interval $(0, n]$ into subintervals as follows. Let $J_0 = (n/k, n]$. Let $i_0$ be the largest $i$ such that $L :=$
$k^{2^i} \leq \log n$, so that $(\log n)^{1/2} < L \leq \log n$. For $i = 1, \ldots, i_0$, let $J_i = (n/k^{2^i}, n/k^{2i-1}]$, and let $J_{i_0+1} = (0, n/L]$.

For $\pi \in S_{\text{lcm}(n)}$ we have sets $X_i, Y_i$ as follows:

$$X_i := \{j \in J_i : \pi(j) > n/k^{2^i}\}, \quad 0 \leq i \leq i_0,$$

$$Y_i := \{j > n/k^{2i-1} : \pi(j) \in J_i\}, \quad 1 \leq i \leq i_0 + 1.$$

These sets depend on the choice of $\pi$, but the number of choices for the sets $Y_i$ is not so large. We begin by counting the number of possibilities for the sequence of sets $Y_1, \ldots, Y_{i_0+1}$.

Since $\pi$ is a permutation it follows that $y_i := \#Y_i$ is at most the number of integers in $J_i$, so that $y_i \leq n/k^{2i-1}$. The number of subsets of $(n/k^{2i-1}, n]$ of cardinality $\leq y_i$ is less than

$$\sum_{u \leq y_i} \binom{n}{u} \leq 2^n \binom{n}{y_i} \leq \exp\left(\frac{n}{k^{2i-1}} (2^{i-1} \log k + 1)\right),$$

for $n$ sufficiently large, using the inequality $2/j! < (e/j)^j$ for $j \geq 3$. Multiplying these estimates we obtain that the number of choices for a sequence of sets $\{Y_i\}$ as described is

$$\prod_{j \in X_i} \tau_{k^{2^i}}(j) \frac{n}{j} \leq \prod_{j \leq n/k^{2i-1}} \tau_{k^{2^i}}(j) \frac{n}{j} \leq \exp\left(\frac{1}{[n/k^{2i-1}]} \sum_{j \leq n/k^{2i-1}} \tau_{k^{2^i}}(j) \right)^{n/k^{2i-1}} \frac{n^{n/k^{2i-1}}}{[n/k^{2i-1}]!},$$

by the AM-GM inequality (the arithmetic mean geometric mean inequality).
Since the harmonic sum $\sum_{d<z} 1/d$ is bounded above by $\log z + 1$, we have

$$\sum_{j \leq x} \tau_z(j) = \sum_{d<z} \left\lfloor \frac{x}{d} \right\rfloor = \sum_{d<z} \left\lfloor \frac{\lfloor x \rfloor}{d} \right\rfloor < \lfloor x \rfloor (\log z + 1).$$

We apply this above getting that the number of assignments for the numbers $j \in X_i$ is at most

$$\left( \log(k^{2^i}) + 1 \right)^{n/k^{2^i-1}} \exp \left( \frac{n}{k^{2^i-1}} (\log(k^{2^i-1}) + 1) \right) = \exp \left( \frac{n}{k^{2^i-1}} (\log(k^{2^i-1}) + \log(\log(k^{2^i}) + 1) + 1) \right).$$

Thus, multiplying these estimates we have that the number of assignments for numbers $j$ in the sets $X_i$, $1 \leq i \leq i_0$ is

$$\leq \exp(0.2269n)$$

for $n$ sufficiently large.

We next deal with the elements of the sets $Y_i$. Again referring to (1), for each $j \in Y_i$ we are to count pairs $a, c$ with $a \mid j$ and $a \leq n/j' < k^{2^i}$ and $c \leq n/j < k^{2^i-1}$. Assuming $Y_i$ is not empty, the number of assignments for elements of $Y_i$ is at most

$$\prod_{j \in Y_i} \left( \tau_{k^{2^i}}(j)k^{2^i-1} \right)^{y_i} \leq \left( \frac{1}{y_i} \sum_{j \in Y_i} \tau_{k^{2^i}}(j)k^{2^i-1} \right)^{y_i} \leq \left( \frac{1}{y_i} \sum_{j \leq n} \tau_{k^{2^i}}(j)k^{2^i-1} \right)^{y_i} \leq \left( \frac{n}{y_i} (\log(k^{2^i}) + 1)k^{2^i-1} \right)^{y_i},$$

using (5). We have $y_i < n/k^{2^i-1}$ and in this range, the above estimate is increasing as $y_i$ varies. So, the count is at most

$$\exp \left( \frac{n}{k^{2^i-1}} (\log(k^{2^i}) + \log(\log(k^{2^i}) + 1)) \right).$$

Multiplying these estimates we have that the number of assignments for numbers $j$ in the sets $Y_i$ is

$$\leq \exp(0.3134n).$$

For $X_0$ we directly look at all pairs $j, j'$ with $j, j' \in (n/k, n]$ with $\text{lcm}[j, j'] \leq n$. Take for example, the case $k = 3$. Then possibilities for $(a, c)$ in (1) are $(1, 1)$, $(1, 2)$, and $(2, 1)$. For each $j$ we can take $j' = j$, this corresponds to $(1, 1)$. For $j \in (n/3, n/2]$, we can also take $j' = 2j$, corresponding to $(1, 2)$. And for $j \in (2n/3, n]$ with $j$ even, we can take
\( j' = \frac{1}{2} j \), corresponding to \((2,1)\). Letting \( N_k(j) \) be the number of \( j' \) that can correspond to \( j \), we thus have \( N_k(j) = 2 \) for \( j \in (n/3, n/2] \) and for even \( j \) in \( (2n/3, n] \), so that
\[
\prod_{j \in (n/3, n]} N_3(j) \approx 2^{n/3}.
\]
(The symbol \( \asymp \) indicates the two sides are of the same magnitude up to a bounded factor.) However, we are taking \( k = 30 \), and this simple argument becomes more complicated, but nevertheless can be estimated. We have the number of assignments for numbers \( j \) in \( X_0 \) is
\[
\leq \exp(1.9115n).
\]
This estimate is arrived at as follows. We are interested in \( \prod_{j \in J_0} N_k(j) \), which is equal to
\[
\prod_{\substack{j \in J_0 \\mid N_k(j) > 1}} N_k(j).
\]
We have \( N_k(j) = 1 \) if and only if \( j \in (n/2, n] \) and \( j \) is not divisible by any prime \( < k \). Thus, up to an error of \( O(1) \), the number of factors in the above product is \( \nu n \), where
\[
\nu = 1 - 1/k - (1/2) \prod_{p < k} (1 - 1/p).
\]
By the AM-GM inequality,
\[
\prod_{\substack{j \in J_0 \\mid N_k(j) > 1}} N_k(j) \leq \left( \frac{1}{\nu n} \sum_{\substack{j \in J_0 \\mid N_k(j) > 1}} N_k(j) \right)^{\nu n + O(1)}.
\]
To compute the sum we refer to \((1)\). By reversing the order of summation, the sum is
\[
\sum_{a,c < k \\gcd(a,c) = 1} \left( \frac{1}{ac} - \max \left\{ \frac{1}{ka}, \frac{1}{kc} \right\} \right) - \frac{n}{2} \prod_{p < k} (1 - 1/p) + O(1).
\]
Computing this when \( k = 30 \) we arrive at the estimate \((8)\).
This leaves the contribution of numbers \( j \in J_{i_0+1} \). Note that \( J_{i_0+1} \) is a subset of \([A]\), where \( A = \lceil n/(\log n)^{1/2} \rceil \). As with \((4)\), \((5)\), this is at
most
\[ \prod_{j \leq A} (\tau(j) n/j) \leq \left( \frac{1}{A} \sum_{j \leq A} \tau(j) \right)^A \frac{n^A}{A!} \]
\[ \leq (\log n)^A \exp \left( A \log n - A \log A + A \right) \]
\[ \leq \exp \left( \frac{3}{2} A \log \log n + A \right). \]

This last estimate is of the form \( e^{o(n)} \), so it suffices to multiply the estimates in (3), (6), (7), and (8), getting that for all sufficiently large \( n \), we have \( \# S_{\text{lcm}}(n) \leq \exp(2.6071n) \). This completes the proof. \( \square \)

**Remark 1.** This argument gives up a fair amount in computing the contribution for \( j \in X_0 \), which is the estimate (8) with \( k = 30 \). Another way of estimating this count is to take some large numbers \( n \) and directly compute the product of the numbers \( N_k(j) \). It is seen that the \( n \)th root of this product hardly varies as \( n \) varies, and thus one can empirically arrive at a constant that is presumably more accurate than the one in (8). With \( k = 30 \), one gets in this way the number 1.5466, which leads to the estimate \( \# S_{\text{lcm}}(n) \leq \exp(2.2423n) \). In fact, if one is prepared to reason in this way, then one can do a little better by taking \( k = 100 \). This improves the numbers in (3), (6), and (7) to 0.0571, 0.0807, and 0.1175, with the number in (8) moving to 1.8709, which would give the estimate \( \# S_{\text{lcm}}(n) \leq \exp(2.1262n) \).

**Remark 2.** Since \( S_{\text{div}}(n) \subset S_{\text{lcm}}(n) \), the upper bound counts of this section hold as well for \( \# S_{\text{div}}(n) \). However, we can do a little better than this. The savings comes from \( X_0 \), namely the estimate in (8). For \( j \in J_0 \), let \( N'_k(j) \) denote the number of \( j' \in J_0 \) with either \( j' \mid j \) or \( j \mid j' \). We are interested in

\[ \sum_{n/k < j \leq n} N'_k(j). \]

The number of \( j' \mid j \) with \( j' \in J_0 \) is the number of \( a \mid j \) with \( j/a > n/k \), and the number of \( j' \in J_0 \) with \( j \mid j' \) is the number of \( c \leq n/j \). We should subtract 1 since otherwise \( j' = j \) would be counted twice. We have up to an error of \( +O(1) \),

\[ \sum_{n/k < j \leq n} \sum_{a \mid j} 1 = \sum_{a < k} \left( \frac{n}{a} - \frac{n}{k} \right). \]
and
\[ \sum_{n/k<j \leq n} \sum_{c \leq k} \frac{1}{c} = \sum_{c<k} \left( \frac{n}{c} - \frac{n}{k} \right). \]

Thus, the sum in (10), up to \(+O(1)\), is
\[ 2 \sum_{a<n} \frac{n}{a} - 3n \left( 1 - \frac{1}{k} \right) - \frac{n}{2} \prod_{p<k} \left( 1 - \frac{1}{p} \right). \]

Using this when \(k = 100\) in place of (9) and the \(k = 100\) estimates for (3), (6), and (7) as mentioned in Remark 1, we obtain \(\#S_{\text{div}}(n) < \exp(2.1745n)\) for all large \(n\). Doing the numerical experiment analogous to the one at the end of Remark 1, the number in (8) moves to 1.6161, giving \(\#S_{\text{div}}(n) < \exp(1.8714n) < 6.5^n\).

3. Lower bounds

Let \(b\) denote a positive integer, and for \(a \mid b\), let
\[ s(a, b) = \{ d \mid b : d \leq a \}. \]

Further, let \(P(a, b)\) denote the set of permutations \(\pi\) of \(s(a, b)\) such that for each \(d \in s(a, b)\), we have \(\text{lcm}[d, \pi(d)] \leq a\). Write the divisors \(a\) of \(b\) in increasing order: \(1 = a_1 < a_2 < \cdots < a_k = b\), where \(k = \tau(b)\). Let
\[ c(b) = \frac{\log(\tau(b)!)}{b} + \sum_{i=1}^{\tau(b)-1} \left( \frac{1}{a_i} - \frac{1}{a_{i+1}} \right) \log(#P(a_i, b)). \]

**Theorem 2.** For any positive integer \(b\) we have
\[ \#S_{\text{lcm}}(n) \geq \exp((c(b)\varphi(b)/b + o(1))n) \]
as \(n \to \infty\).

We illustrate Theorem 2 in the first interesting case: \(b = 2\). Then \(p(1, 2) = 1\) and \(p(2, 2) = 2\), so that \(c(2) = \log(2)/2\) and the theorem asserts that \(\#S_{\text{lcm}}(n) \geq \exp((\log(2)/4 + o(1))n)\) as \(n \to \infty\). To see why this is true, look at sets \(\{j, 2j\}\) where \(j \leq n/2\) and \(j\) is odd. There are \(n/4 + O(1)\) of these pairs and any permutation \(\pi\) of \([n]\) for which \(\pi(\{j, 2j\}) = \{j, 2j\}\) for each \(j\), and \(\pi\) otherwise acts as the identity, is in \(S_{\text{lcm}}(n)\). Since the sets \(\{j, 2j\}\) are pairwise disjoint, this shows that \(S_{\text{lcm}}(n)\) contains at least \(2^{n/4+O(1)}\) elements. The sets are pairwise disjoint since we are taking \(j\) odd. But a weaker condition also insures this. Let \(v_p(j)\) be the exponent on \(p\) in the canonical prime factorization of \(j\). Then we take sets \(\{j, 2j\}\) where \(j \leq n/2\) and \(v_2(j)\) is even. This insures that the sets \(\{j, 2j\}\) are pairwise disjoint, and now there are
$n/3 + O(\log n)$ pairs, leading to $\#S_{\text{lcm}}(n) \geq \exp((\log(2)/3 + o(1))n)$ as $n \to \infty$.

In fact, this improvement generalizes. For a prime power $p^i$ let

$$\alpha(p^i) = \frac{p^{i+1} - p^i}{p^{i+1} - 1},$$

and extend $\alpha$ as a multiplicative function on the positive integers. Note that $\alpha(b)$ is the density of the set of integers $j$ such that for all primes $p \mid b$, $v_p(j) \equiv 0 \pmod{v_p(b) + 1}$. (Steve Fan pointed out to me that $\alpha(b) = b/\sigma(b)$, where $\sigma$ is the sum-of-divisors function.)

**Theorem 3.** For any positive integer $b$ we have

$$\#S_{\text{lcm}}(n) \geq \exp((c(b)\alpha(b) + o(1))n)$$

as $n \to \infty$.

**Proof.** For $1 \leq i \leq \tau(b)$ consider the intervals $I_i := (n/a_{i+1}, n/a_i]$ and $I_{\tau(b)} = (0, n/b]$. For $j \in I_i$ with $v_p(j) \equiv 0 \pmod{v_p(b) + 1}$ for each prime $p \mid b$, we have the set $T(i, j) := \{d_j : d \in s(a_i, b)\}$ as a subset of $[n]$. Moreover, the sets $T(i, j)$ are pairwise disjoint for all pairs $i, j$ with $j \in I_i$ and $v_p(j) \equiv 0 \pmod{v_p(b) + 1}$ for each prime $p \mid b$. For $j \in I_i$, we can view a permutation $\pi \in P(a_i, b)$ as a permutation on $T(i, j)$, where $d_j$ gets sent to $\pi(d)_j$. So, consider a permutation $\pi$ on $[n]$ such that for each $i, j$ with $i \leq \tau(b)$, $j \in I_i$, and $v_p(j) \equiv 0 \pmod{v_p(b) + 1}$, it acts on $T(i, j)$ like a permutation in $P(a_i, j)$, and otherwise acts as the identity. Then $\pi \in S_{\text{lcm}}(n)$.

For a given value of $i < \tau(b)$ there are $\sim (1/a_i - 1/a_{i+1})\alpha(b)n$ values of $j$, and for $i = \tau(b)$, there are $\sim n\alpha(b)/a_{\tau(b)}$ values of $j$. We conclude that $\#S_{\text{lcm}}(n)$ is at least

$$(\#P(a_{\tau(b)}, b))^{(1 + o(1))n\alpha(b)/a_{\tau(b)}} \prod_{i=1}^{\tau(b)-1} (\#P(a_i, b))^{(1 + o(1))n\alpha(b)(1/a_i - 1/a_{i+1})}.$$

Since $a_{\tau(b)} = b$ and $p(b, b) = \tau(b)!$, the result follows. \[\square\]

We have an analogous result for $S_{\text{div}}(n)$. Let $p_d(a, b)$ denote the number of permutations $\pi$ of $s(a, b)$ such that for each $d \in s(a, b)$, we have $d \mid \pi(d)$ or $\pi(d) \mid d$.

$$c_d(b) = \frac{\log(p_d(b, b))}{b} + \sum_{i=1}^{\tau(b)-1} \left(\frac{1}{a_i} - \frac{1}{a_{i+1}}\right) \log(p_d(a_i, b)).$$

**Corollary 1.** For any positive integer $b$ we have

$$\#S_{\text{div}}(n) \geq \exp((c_d(b)\alpha(b) + o(1))n)$$

as $n \to \infty$. 
Table 2. Some values of $c(b)\alpha(b)$ and $c_d(b)\alpha(b)$ to 6 places with their exponentials rounded down to 4 places.

| $b$  | $c(b)\alpha(b)$ | $e^{c(b)\alpha(b)}$ | $c_d(b)\alpha(b)$ | $e^{c_d(b)\alpha(b)}$ |
|------|------------------|----------------------|-------------------|----------------------|
| 4    | .354987          | 1.4261               | .354987           | 1.4261               |
| 12   | .536243          | 1.7095               | .479872           | 1.6158               |
| 24   | .602065          | 1.8258               | .542689           | 1.7206               |
| 48   | .638300          | 1.8932               | .578122           | 1.7826               |
| 60   | .646856          | 1.9095               | .552061           | 1.7368               |
| 96   | .658201          | 1.9313               | .597849           | 1.8182               |
| 120  | .707611          | 2.0291               | .610358           | 1.8410               |
| 144  | .704928          | 2.0237               | .631752           | 1.8809               |
| 210  | .609981          | 1.8239               | .496559           | 1.6430               |
| 240  | .740127          | 2.0962               | .648821           | 1.9132               |
| 288  | .723607          | 2.0618               | .650371           | 1.9162               |
| 420  | .716176          | 2.0465               | .597383           | 1.8173               |
| 480  | .757765          | 2.1335               | .660864           | 1.9364               |

So, by Table 2 and Theorem 3 with $b = 480$, we have $\# S_{\text{lcm}}(n) \geq 2.1335^n$ for all large values of $n$ and by Corollary 1 we have $\# S_{\text{div}}(n) \geq 1.9364^n$ for all large $n$.

4. Comparing $\# S_{\text{div}}(n)$ and $\# S_{\text{lcm}}(n)$

Let

$$R(n) = \# S_{\text{lcm}}(n)/\# S_{\text{div}}(n).$$

It appears from a glance at Table 1 that $R(n)$ grows at least geometrically. Here we prove this.

**Theorem 4.** There is a constant $c > 1$ such that for all large values of $n$ we have $R(n) > c^n$.

**Proof.** Let $A$ denote the set of integers $a$ with $n/7 < a \leq n/6$ and with $a$ not divisible by any prime $< 10^4$. Note that

$$\# A \geq \frac{n}{42} \prod_{p < 10^4} \left(1 - \frac{1}{p}\right) + O(1),$$

so that $\# A > 14n/10^4$ for all large $n$. For $a \in A$, let

$$B_a = \{a, 2a, 3a, 4a, 5a, 6a\}.$$

Any divisor of a member of $B_a$ that is not itself in $B_a$ must be $<(n/6)/10^4$. Since clearly each member of $B_a$ has no multiple in $[n]$ that is not in $B_a$, we have that each $\pi \in S_{\text{div}}(n)$ has $\pi(B_a) \neq B_a$ for at most $n/10^4$ values of $a \in A$. We conclude that each $\pi \in S_{\text{div}}(n)$ has $\pi(B_a) = B_a$ for at least $13n/10^4$ values of $a \in A$. 
Let $\pi \in S_{\text{div}}(n)$ with $\pi(B_a) = B_a$, and let $\pi_0$ be $\pi$ restricted to $[n] \setminus B_a$. There are exactly 36 permutations $\pi \in S_{\text{div}}(n)$ which give rise to the same $\pi_0$ corresponding to the 36 permutations $\sigma$ of $B_a$ with each $ja | \sigma(ja)$ or $\sigma(ja) | ja$ (since $\#S_{\text{div}}(6) = 36$). However, for a given $\pi_0$ here, there are exactly 56 permutations $\pi \in S_{\text{lcm}}(n)$ with $\pi$ restricted to $[n] \setminus B_a$ equal to $\pi_0$ (since $\#S_{\text{lcm}}(6) = 56$).

It thus follows that $R(n) > (56/36)^{13n/10^4}$, so the theorem is proved with $c = (56/36)^{13/10^4} > 1.00057$. □

5. AN UPPER BOUND FOR ANTI-COPRIME PERMUTATIONS

Let $A(n)$ denote the number of anti-coprime permutations $\pi$ of $[n]$. We prove the following theorem.

Theorem 5. We have $A(n) = n!/(\log n)^{(\log \log n)n}$ as $n \to \infty$.

Proof. In light of the lower bound from [7], it suffices to prove that $A(n) \leq n!/(\log n)^{(\log \log n)n}$ as $n \to \infty$.

Let $\psi(n) \to \infty$ arbitrarily slowly, but with $\psi(n) = o(\log \log n)$. Let $\alpha = 1 + 1/\psi(n)$, so that $\alpha \to 1^+$ as $n \to \infty$. Let the integer variable $i$ satisfy

$$\psi(n) < i < \log \log n / \log \alpha - \psi(n).$$

Thus, $e^{\alpha i} \to \infty$ and $e^{\alpha i} = n^{o(1)}$ as $n \to \infty$. For each $i$ satisfying (11), let

$$I_i := (e^{\alpha i-1}, e^{\alpha i}], \quad J_i := \{j \in [n] : P^-(j) \in I_i\},$$

where $P^-(j)$ is the least prime factor of $j$. By the Fundamental Lemma of the Sieve (see [3, Theorem 6.12]) and Mertens’ theorem, we have

$$\#J_i \sim \frac{n}{e^{\gamma \alpha i-1}} - \frac{n}{e^{\gamma \alpha i}} = \frac{n(\alpha - 1)}{e^{\gamma \alpha i}}$$

uniformly in $i$ satisfying (11), as $n \to \infty$.

For $j \in [n]$ the number of $j' \in [n]$ with $\gcd(j', j) > 1$ is $\leq \sum_{p | j} n/p$. (This is a poor bound for most integers $j$, but fairly accurate for most $j$’s without small prime factors, as is the case for members of $J_i$.) Thus, the total number of assignments for the numbers $j \in J_i$ in an anti-coprime permutation of $[n]$ is

$$\leq \prod_{j \in J_i} \left( \frac{n}{\sum_{p | j} 1/p} \right) \leq \left( \frac{n}{\#J_i \sum_{j \in J_i} \sum_{p | j} 1/p} \right)^{\#J_i},$$
by the AM-GM inequality. The double sum here is
\[
\sum_{p > e^{\alpha_i - 1}} \sum_{i \in J_i \atop p \mid j} \frac{1}{p} \ll n \sum_{p > e^{\alpha_i - 1}} \frac{1}{p^{2} \alpha_i} \ll \frac{n}{e^{\alpha_i - 1} \alpha_i^2},
\]
uniformly, using an upper bound for the sieve. Thus, for \( n \) large,
\[
\frac{n}{\#J_i} \sum_{i \in J_i} \sum_{p \mid j} \frac{1}{p} \leq \frac{n}{(\alpha - 1)e^{\alpha_i - 1}} \leq \frac{n}{e^{\alpha_i - 2}}.
\]
Hence, using (12),
\[
\prod_{j \in J_i} \left( n \sum_{p \mid j} \frac{1}{p} \right) \leq \left( \frac{n}{e^{\alpha_i - 1}} \right)^{\#J_i} = \exp(\#J_i \log n - \#J_i \alpha_i^{-2})
\]
\[
= \exp(\#J_i \log n - (1 + o(1))n(\alpha - 1)/e^\gamma)
\]
uniformly.

Let \( N \) be the number of \( j \in [n] \) not in any \( J_i \), so that \( N = n - \sum_i \#J_i \). Further the number of values of \( i \) is at most \( \log n / \log (\alpha - 2\psi(n)) \). After assignments have been made for the values of \( j \in \bigcup J_i \), there are at most \( N! \) remaining assignments for \( j \notin \bigcup J_i \) and multiplying this by the product of the previous estimate for all \( i \) is at most
\[
\exp \left( n \log n - (1 + o(1)) \frac{n(\alpha - 1)}{e^\gamma} \left( \frac{\log \log n}{\log \alpha} - 2\psi(n) \right) \right).
\]
It remains to note that \( (\alpha - 1)/\log \alpha \sim 1 \) as \( n \to \infty \), which completes the proof of the theorem. \( \square \)

**Dedication and Acknowledgments**

Eduard Wirsing is not primarily known for his work in combinatorial number theory, yet one of his papers that influenced me a great deal is his joint work with Hornfeck, later improved on his own (see [11]), on the distribution of integers \( n \) with \( \sigma(n)/n = \alpha \), for a fixed rational number \( \alpha \), where \( \sigma \) is the sum-of-divisors function. In a survey I wrote with Sárközy [8] on combinatorial number theory, we singled out this particular work for being a quintessential exemplar of the genre. It is in this spirit that I offer this note on combinatorial number theory in remembrance of Eduard Wirsing.

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