Analysis of multivariate Gegenbauer approximation in the hypercube

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Abstract

In this paper, we are concerned with multivariate Gegenbauer approximation of analytic functions defined in the \(d\)-dimensional hypercube. Two new and sharper bounds for the coefficients of multivariate Gegenbauer expansion are presented based on two different extensions of the Bernstein ellipse. We then establish an explicit error bound for the multivariate Gegenbauer approximation associated with an \(\ell^q\) ball index set in the uniform norm. As an application, we extend our arguments to obtain some new tight bounds for the coefficients of tensorized Legendre expansions in the context of polynomial approximation of parameterized PDEs.

Keywords: hypercube, polyellipse, multivariate Gegenbauer approximation, \(\ell^q\) ball index set, error bound

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1 Introduction

Let \(f(x)\) be a smooth function defined in the \(d\)-dimensional hypercube

\[
\Omega_d := [-1,1]^d, \quad d \geq 1. \tag{1.1}
\]

An efficient and accurate approximation of \(f(x)\) is to expand it in terms of tensor products of orthogonal polynomials. Besides many well-known applications of such kind of expansions for the univariate case (i.e., \(d = 1\)), they have also been widely used in a variety of practical problems encountered in higher dimensions. For example, just to name a few, the tensorized Legendre expansion is an important tool to approximate the solutions of a large class of parametrized elliptic PDEs with stochastic coefficients.

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The bivariate Chebyshev expansion plays an important role in the fast solution method developed for Fredholm integral equation of the second kind [16] and the rapid evaluation of the Bessel functions of real orders and arguments [3], while the bivariate Jacobi expansions have been used to analyze the convergence of the $h$-$p$ version of the finite element solution on quasi-uniform meshes [11].

When using polynomial approximations, a fundamental issue is to estimate their convergence rates or establish some error bounds, which leads to intensive investigations in the literature. For the univariate case, the Chebyshev expansion was first considered in [3] (see also [9, 13, 20] for further studies), and has been considerably extended to other polynomial expansions since then (cf. [23, 24, 25, 28, 29] and the references therein). The multivariate case (i.e., $d \geq 2$), however, remains a research topic of great current interest, and some important progresses have been achieved over the past decades. Unlike the univariate case, a proper multi-index set has to be fixed for the multivariate polynomial approximation. Some popular choices include the hyperbolic cross index set and those induced from the 1- and $\infty$- norms of the multi-index. An error estimate of the tensorized Legendre expansion on the full grid (i.e., the index set induced from the $\infty$-norm of the multi-index) can be found in [17], evaluated in the Sobolev space. Shen and Wang in [17] analyzed the Jacobi approximations on the full grid and hyperbolic cross Jacobi approximations in the context of anisotropic Jacobi weighted Korobov spaces. More recently, based on a new observation, Trefethen introduced the Euclidean degree for the multivariate polynomial in [22], and further obtained the convergence rate of the tensorized Chebyshev expansions for the multi-index sets induced from 1-, 2-, and $\infty$- norms of the multi-index in [21]; see also the work of Bos and Levenberg [5] for the studies from the viewpoint of Bernstein-Walsh theory.

In this paper, we first establish some new and explicit bounds for the coefficients of multivariate Gegenbauer expansion. This can also be viewed as an extension of the results in [23] for the univariate case to the multivariate setting. We then apply these explicit bounds to derive an explicit error bound for the multivariate Gegenbauer approximation associated with an $\ell^q$ ball index set, which particularly include the approximations with the index sets induced from 1-, 2-, and $\infty$- norms of the multi-index as special cases. For isotropic functions which are rotationally invariant, we observe numerically that the error estimates obtained agree well with the empirical rates. Finally, as an application, we show that our arguments can be extended to obtain some new tight bounds on the coefficients of tensorized Legendre expansions arising from polynomial approximation for a family of parameterized PDEs.

The rest of this paper is organized as follows. In Section 2 we collect some basic properties of Gegenbauer polynomials and give an explicit bound for the weighted Cauchy transform of the Gegenbauer polynomials for later use. In Section 3, we derive two explicit bounds for the coefficients of multivariate Gegenbauer expansion based on two different assumptions on $f(x)$. In Section 4, we establish an explicit error bound for the multivariate Gegenbauer approximation associated with an $\ell^q$ ball index set, where the theoretical results are also illustrated in numerical experiments. In Section 5, we discuss an application of our results to polynomial approximation of parameterized PDEs.
We finish the paper with some concluding remarks in Section 6.

2 Preliminaries

It is the aim of this section to make some preparations for our later analysis. We first give a brief review of the basic properties of Gegenbauer polynomials $C_n^{(\lambda)}(x)$, and then present an explicit optimal upper bound of weighted Cauchy transform of $C_n^{(\lambda)}(x)$ on the Bernstein ellipse.

2.1 Gegenbauer polynomials

The Gegenbauer polynomials $C_n^{(\lambda)}(x)$ are polynomials of degree $n$ orthogonal over the interval $\Omega_1 = [-1, 1]$ with respect to the weight function

$$\omega_\lambda(x) = (1 - x^2)^{\lambda - \frac{1}{2}}, \quad \lambda > \frac{1}{2}. $$

More precisely, we have

$$\int_{\Omega_1} C_m^{(\lambda)}(x) C_n^{(\lambda)}(x) \omega_\lambda(x) dx = h_n^{(\lambda)} \delta_{m,n}, \quad (2.1)$$

where $\delta_{m,n}$ is the Kronecker delta and

$$h_n^{(\lambda)} = \frac{2^{1-2\lambda} \pi}{\Gamma(\lambda)^2} \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)(n+\lambda)}, \quad \lambda \neq 0, \quad (2.2)$$

with $\Gamma(z)$ being the usual gamma functions (cf. [15, Chapter 5]). The Gegenbauer polynomials are fixed by requiring

$$C_n^{(\lambda)}(1) = \frac{\Gamma(n+2\lambda)}{n! \Gamma(2\lambda)}, \quad \lambda > -\frac{1}{2}, \quad \lambda \neq 0, \quad (2.3)$$

If $\lambda = 0$, we have

$$C_0^{(0)}(x) = 1, \quad C_n^{(0)}(1) = \frac{2^n}{n!}, \quad n \geq 1.$$

Since the weight function $\omega_\lambda(x)$ is an even function, it is readily seen that

$$C_n^{(\lambda)}(x) = (-1)^n C_n^{(\lambda)}(-x), \quad n \geq 0, \quad (2.4)$$

which implies that $C_n^{(\lambda)}(x)$ is an even function for even $n$ and an odd function for odd $n$. Moreover, Gegenbauer polynomials satisfy the following inequality (cf. [15, Equation 18.14.4])

$$|C_n^{(\lambda)}(x)| \leq C_n^{(\lambda)}(1), \quad |x| \leq 1, \quad \lambda > 0, \quad n \geq 0. \quad (2.5)$$
Gegebauer polynomials include classical Chebyshev polynomials and Legendre polynomials as special cases. Recall that the Chebyshev polynomials of the first and second kinds are

\[ T_n(\cos \theta) = \cos(n\theta), \quad U_n(\cos \theta) = \frac{\sin((n + 1)\theta)}{\sin \theta}, \quad n \geq 0, \]

respectively. The relations are given by

\[ T_n(x) = n2 \lim_{\lambda \to 0} \frac{C^{(\lambda)}_n(x)}{\lambda}, \quad n \geq 1; \quad U_n(x) = C^{(1)}_n(x), \quad n \geq 0. \tag{2.6} \]

Note that the orthogonality for \( T_n(x) \) reads

\[ \int_{-1}^{1} T_n(x)T_m(x) \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} \, dx = \begin{cases} 0, & m \neq n, \\ \pi, & m = n = 0, \\ \pi/2, & m = n \geq 1. \end{cases} \tag{2.7} \]

The Legendre polynomials \( P_n(x) \) correspond to the case that \( \lambda = \frac{1}{2} \), i.e.,

\[ P_n(x) = C^{(\frac{1}{2})}_n(x), \quad n \geq 0. \tag{2.8} \]

2.2 An explicit optimal upper bound of weighted Cauchy transform of \( C^{(\lambda)}_n(x) \) on the Bernstein ellipse

For \( z \in \mathbb{C} \setminus \Omega_1 \), we define

\[ Q^{(\lambda)}_n(z) := \begin{cases} \frac{1}{2h^{(\lambda)}_n} \int_{\Omega_1} \frac{\omega_\lambda(x)C^{(\lambda)}_n(x)}{z-x} \, dx, & \lambda \neq 0, \\ \lim_{\lambda \to 0} Q^{(\lambda)}_n(z), & \lambda = 0, \quad n = 0, \\ \frac{2}{n} \lim_{\lambda \to 0} \lambda Q^{(\lambda)}_n(z), & \lambda = 0, \quad n \geq 1, \end{cases} \tag{2.9} \]

where \( h^{(\lambda)}_n \) is given in \( \text{[22]} \). When \( \lambda = 0 \), it is easily seen from \( \text{[26]} \) that

\[ Q^{(0)}_n(z) = \frac{1}{2h^{(0)}_n} \int_{\Omega_1} \frac{\omega_0(x)T_n(x)}{z-x} \, dx, \tag{2.10} \]

where \( h^{(0)}_0 = \pi \) and \( h^{(0)}_n = \pi/2 \) for \( n \geq 1 \). Thus, up to some constant term, \( Q^{(\lambda)}_n(z) \) is the weighted Cauchy transform of \( C^{(\lambda)}_n(x) \) (for \( \lambda \neq 0 \)) or \( T_n(x) \) (for \( \lambda = 0 \)), which is analytic in the whole complex plane with a cut along \( \Omega_1 \).

We need an explicit upper bound of \( Q^{(\lambda)}_n \) for \( z \) belonging to the so-called Bernstein ellipse, which is crucial in our subsequent analysis.
Definition 2.1. The Bernstein ellipse $E_\rho$ is defined by
\[
E_\rho = \left\{ z \in \mathbb{C} \mid z = \frac{1}{2}(u + u^{-1}), \ |u| = \rho > 1 \right\},
\]
which has the foci at $\pm 1$ with the major and minor semi-axes given by $\frac{1}{2}(\rho + \rho^{-1})$ and $\frac{1}{2}(\rho - \rho^{-1})$, respectively.

It comes out that an upper bound of $\left| Q_n^{(\lambda)}(z) \right|$ over $E_\rho$ is contained implicitly in recent work [23]. A key observation therein is that $Q_n^{(\lambda)}(z)$ admits an explicit representation in terms of the Gauss hypergeometric function $\, _2F_1$ defined by
\[
\, _2F_1 \left[ a, b; c; z \right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},
\]
where
\[
(a)_0 = 1, \quad (a)_k = a(a+1) \cdots (a+k-1), \quad k \geq 1,
\]
is the Pochhammer symbol. Indeed, by [23, Corollary 3.4], it follows that
\[
Q_n^{(\lambda)}(z) = c_{n,\lambda} (z \pm \sqrt{z^2 - 1})^{n+1} \, _2F_1 \left[ \lambda n + 1, 1 - \lambda; n + \lambda + 1; \frac{1}{(z \pm \sqrt{z^2 - 1})^2} \right],
\]
where the sign is chosen so that $|z \pm \sqrt{z^2 - 1}| > 1$ and
\[
c_{n,\lambda} = \frac{\Gamma(\lambda) \Gamma(n+1)}{\Gamma(n+\lambda)}.
\]
Based on the relation (2.12) and the results scattered in [23], we have the following explicit optimal upper bound of $Q_n^{(\lambda)}(z)$ over the Bernstein ellipse $E_\rho$.

Proposition 2.2. For $z \in E_\rho$ and $\lambda > 0$, we have
\[
\left| Q_n^{(\lambda)}(z) \right| \leq \left\{ \begin{array}{ll}
D_\rho^{(\lambda)}, & n = 0, \\
D_\rho^{(\lambda)} n^{1-\lambda}, & n \geq 1,
\end{array} \right.
\]
where the $n$-independent constants $\overline{D}_\rho^{(\lambda)}$ and $D_\rho^{(\lambda)}$ are defined by
\[
\overline{D}_\rho^{(\lambda)} = \frac{1}{\rho} \times \left\{ \begin{array}{ll}
(1 + \rho^{-2})^{\lambda-1}, & \lambda \geq 1, \\
(1 - \rho^{-2})^{\lambda-1}, & 0 < \lambda < 1,
\end{array} \right.
\]
and
\[
D_\rho^{(\lambda)} = \frac{\Gamma(\lambda)}{\rho} \times \left\{ \begin{array}{ll}
\exp \left( \frac{1}{12} \right) (1 + \rho^{-2})^{\lambda-1}, & \lambda \geq 1, \\
\exp \left( \frac{1}{12} + \frac{1-\lambda}{2\lambda} \right) (1 - \rho^{-2})^{\lambda-1}, & 0 < \lambda < 1.
\end{array} \right.
\]
The bound in (2.14), apart from a constant factor, is optimal as $n \to \infty$ in the sense that it cannot be improved in any lower power of $n$ further.
Remark 2.3. If $\lambda = 0$, i.e., for the Chebyshev polynomials of the first kind, we have the following explicit formula for $Q^{(0)}(z)$:

\[
Q^{(0)}(z) = \begin{cases} \\
\frac{1}{\sqrt{z^2 - 1}(z \pm \sqrt{z^2 - 1})^n}, & n \geq 1, \\
\frac{1}{2\sqrt{z^2 - 1}}, & n = 0,
\end{cases}
\]  \hspace{1cm} (2.17)

which can be easily verified by combining (2.9) and (2.12).

If $\lambda = 1$, i.e., for the Chebyshev polynomials of the second kind, again by (2.12), we have

\[
Q^{(1)}(z) = \frac{1}{(z \pm \sqrt{z^2 - 1})^{n+1}}, \quad n \geq 0.
\]  \hspace{1cm} (2.18)

This particularly implies that

\[
\left|Q^{(1)}(z)\right| \leq \frac{1}{\rho^{n+1}}, \quad z \in \mathcal{E}_\rho,
\]  \hspace{1cm} (2.19)

i.e., the prefactor $D^{(1)}_{\rho}$ in (2.14) can be improved to be $1/\rho$.

3 Multivariate Gegenbauer expansion

In this section, we intend to estimate the coefficients of the multivariate Gegenbauer expansion based on two different assumptions on the function defined in the hypercube $\Omega_d$ given in (1.1).

3.1 Notations

We first introduce some notations to be used throughout the rest of this paper.

- We shall denote by $\mathbf{x}$ and $\mathbf{z}$ the point in $\mathbb{R}^d$ and $\mathbb{C}^d$, respectively, i.e.,

\[
\mathbf{x} = (x_1, \ldots, x_d), \quad \mathbf{z} = (z_1, \ldots, z_d).
\]  \hspace{1cm} (3.1)

- The notation $\mathbb{N}_0^d$ stands for the set of all $d$-tuples $\mathbf{k} = (k_1, k_2, \ldots, k_d)$, where $k_i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Such a $d$-tuple is called a multi-index. For any two multi-indices $\mathbf{k} = (k_1, \ldots, k_d)$ and $\mathbf{t} = (t_1, \ldots, t_d)$, we define the following componentwise operation

\[
\mathbf{k} + \mathbf{t} = (k_1 + t_1, \ldots, k_d + t_d),
\]

and use the convention

\[
\mathbf{k} \leq \mathbf{t} \iff k_j \leq t_j, \quad j = 1, 2, \ldots d.
\]
Let \( \mathbf{1} = (1, \ldots, 1) \in \mathbb{N}_0^d \). For a scalar \( t \in \mathbb{R} \), we define
\[
\mathbf{k} + t = \mathbf{k} + t \cdot \mathbf{1} = (k_1 + t, \ldots, k_d + t),
\]
and
\[
\mathbf{k}^t = \prod_{j=1}^d k_j^t.
\]

If \( \phi, \phi_{k,j}, j = 1, \ldots, d \), are functions of one variable, we define
\[
\phi(x) = \prod_{j=1}^d \phi(x_j), \quad \phi_k(x) = \prod_{j=1}^d \phi_{k,j}(x_j).
\]
Thus, \( x^k = \prod_{j=1}^d x_j^{k_j} \) is a multivariate monomial.

We define
\[
\|k\|_q := \begin{cases} (k_1^q + \cdots + k_d^q)^{1/q}, & 0 < q < \infty, \\ \max_{1 \leq i \leq d} k_i, & q = \infty. \end{cases}
\]

3.2 Multivariate Gegenbauer expansion

Let \( f(x) \) be an analytic function defined in the hypercube \( \Omega_d \). The multivariate Gegenbauer series expansion of \( f(x) \) is defined by
\[
f(x) = \sum_{k \in \mathbb{N}_0^d} a_k C_k^{(\lambda)}(x),
\]
where \( C_k^{(\lambda)}(x) = \prod_{i=1}^d C_{k_i}^{(\lambda)}(x_i) \) stands for the tensorized Gegenbauer polynomials, and by orthogonality \( 2.7 \),
\[
a_k = \frac{1}{h_k^{(\lambda)}} \int_{\Omega_d} f(x) C_k^{(\lambda)}(x) \omega(\lambda) \, dx
\]
with \( dx = \prod_{i=1}^d dx_i \) and \( h_k^{(\lambda)} = \prod_{i=1}^d h_{k_i}^{(\lambda)} \).

We are interested in the estimate of the expansion coefficients \( a_k \). The case of a single variable, i.e., \( d = 1 \), is well established. For example, by assuming that \( f \) is analytic in an open region bounded by the Bernstein ellipse \( \mathcal{E}_\rho \), it has been proved in \[23\] Theorem 4.3] that for \( n \geq 1 \),
\[
|a_n| \leq \Lambda(n, \rho, \lambda) \times \begin{cases} \left(1 - \frac{1}{\rho^2}\right)^{\lambda-1} \frac{n^{1-\lambda}}{\rho^{n+1}}, & 0 < \lambda < 1, \\ \left(1 + \frac{1}{\rho^2}\right)^{\lambda-1} \frac{n^{1-\lambda}}{\rho^{n+1}}, & \lambda \geq 1, \end{cases}
\]
where
\[ \Lambda(n, \rho, \lambda) = \frac{\Gamma(\lambda)}{\pi} \left[ 2 \left( \rho + \frac{1}{\rho} \right) + 2 \left( \frac{\pi}{2} - 1 \right) \left( \rho - \frac{1}{\rho} \right) \right] \exp \left( \frac{1 - \lambda}{2(n + \lambda - 1)} + \frac{1}{12n} \right), \]
and \( M = \max_{z \in E} |f(z)|. \)

To deal with the higher dimensional case \( d > 1 \), an essential issue here is to extend the Bernstein ellipse to a region in \( \mathbb{C}^d \). In what follows, we divide our discussions on the estimate of \( a_k \) into two cases, based on different extensions of the Bernstein ellipse.

### 3.3 Estimates of \( a_k \) under Assumption I on \( f \)

A natural extension of the Bernstein ellipse \( E_{\rho} \) to \( \mathbb{C}^d \) is the polyellipse, and we then make the following assumption on \( f \).

**Assumption I.** The function \( f \) is analytic inside the polyellipse

\[ E_{\rho} := \bigotimes_{j=1}^{d} E_{\rho_j}, \]

where \( E_{\rho} \) is defined in (2.11), and \( \rho = (\rho_1, \ldots, \rho_d) \) with \( \rho_i > 1, i = 1, \ldots, d \).

The main result of this section is the following theorem.

**Theorem 3.1.** Under Assumption I and for \( \lambda > 0 \), the multivariate Gegenbauer coefficients of \( f(x) \) satisfy

\[ |a_k| \leq \frac{B_f L(E_{\rho})}{\pi^d \rho^k} \prod_{1 \leq i \leq d, k_i=0}^{\nu} D_{\rho_i}^{(\lambda)} \prod_{1 \leq j \leq d, k_j \neq 0} D_{\rho_j}^{(\lambda)}, \]

where

\[ B_f = \max_{z \in E_{\rho}} |f(z)|, \]

\( L(E_{\rho}) := \prod_{i=1}^{d} L(E_{\rho_i}) \) with \( L(E_{\rho_i}) \) being the length of the circumference of the Bernstein ellipse \( E_{\rho_i} \), and the constants \( D_{\rho_i}^{(\lambda)} \), \( D_{\rho_j}^{(\lambda)} \) are defined in (2.14) and (2.16), respectively. In addition, apart from some constant factor, the bound (3.7) is optimal as \( k_j \to +\infty \) for \( j = 1, \ldots, d \).

**Proof.** Since \( f(z) \) is analytic inside the Bernstein polyellipse \( E_{\rho} \), thus, analytic in \( \Omega_d \). By Cauchy’s integral formula for the analytic function of several variables (cf. [4, Page 32]), we have

\[ f(x) = \left( \frac{1}{2\pi i} \right)^d \oint_{E_{\rho}} \frac{f(z)}{z-x} \, dz, \]

(3.9)
where \( \mathbf{z} - \mathbf{x} = \prod_{j=1}^{d}(z_j - x_j) \). Inserting (3.9) into (3.5), it then follows from interchanging the order of integration that

\[
a_k = \frac{1}{h_k^{(\lambda)}} \int_{\Omega_d} \omega_\lambda(\mathbf{x}) \left( \frac{1}{2\pi i} \oint_{\mathcal{E}_\rho} \frac{f(z)}{z - \mathbf{x}} \, dz \right) C_k^{(\lambda)}(\mathbf{x}) \, d\mathbf{x}
\]

\[
= \left( \frac{1}{2\pi i} \right)^d \oint_{\mathcal{E}_\rho} f(z) \left[ \frac{1}{h_k^{(\lambda)}} \int_{\Omega_d} \frac{\omega_\lambda(\mathbf{x})}{z - \mathbf{x}} C_k^{(\lambda)}(\mathbf{x}) \, d\mathbf{x} \right] \, dz
\]

\[
= \left( \frac{1}{\pi i} \right)^d \oint_{\mathcal{E}_\rho} f(z) Q_k^{(\lambda)}(z) \, dz,
\]

where recall that \( Q_k^{(\lambda)}(\mathbf{z}) = \prod_{i=1}^{d} Q_{k_i}^{(\lambda)}(z_i) \) with \( Q_{k_i}^{(\lambda)}(z) \) defined in (2.9).

As a consequence, it is readily seen that

\[
|a_k| \leq \frac{B_f L(\mathcal{E}_\rho)}{\pi^d} \max_{x \in \mathcal{E}_\rho} |Q_k^{(\lambda)}(\mathbf{z})| = \frac{B_f L(\mathcal{E}_\rho)}{\pi^d} \max_{z \in \mathcal{E}_\rho} \left| Q_k^{(\lambda)}(z) \right|.
\]

The upper bound of \( a_k \) in (3.7) and its optimality follows directly by combining (3.11) with Proposition 2.2. This completes the proof of Theorem 3.1.

As mentioned at the end of Section 2.1, the classical Chebyshev polynomials and Legendre polynomials are special cases of Gegenbauer polynomials. Since these classical polynomials play important roles in practice, we next state the relevant results for these polynomials.

**Corollary 3.2.** Suppose that the multivariate function \( f \) satisfies Assumption I and consider the following tensorized Chebyshev expansion of the first kind:

\[
f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{N}_0^d} a_k^T T_k(\mathbf{x}), \quad a_k^T = \frac{2^{d-N(k)}}{\pi^d} \int_{\Omega_d} f(\mathbf{x}) T_k(\mathbf{x}) \omega_0(\mathbf{x}) \, d\mathbf{x},
\]

with \( N(k) := \# \{i : k_i = 0\} \). Then, we have

\[
|a_k^T| \leq 2^{d-N(k)} \frac{B_f}{\rho^k}.
\]

**Proof.** The proof is similar to that of Theorem 3.1. By Cauchy’s integral formula and (2.10), it is easily seen that

\[
a_k^T = \left( \frac{1}{\pi i} \right)^d \oint_{\mathcal{E}_\rho} f(z) Q_k^{(0)}(z) \, dz.
\]

We now make change of variables \( z_j = (u_j + u_j^{-1})/2 \) with \( u_j \in \mathcal{C}_{\rho_j} := \{z \in \mathbb{C} \mid |z| = \rho_j\} \) for each \( j = 1, \ldots, d \) in (2.17). A simple calculation shows that

\[
Q_k^{(0)}(z_j) = \begin{cases} \frac{2}{u_j^{k_j}(u_j - u_j^{-1})}, & k_j \geq 1, \\ \frac{1}{u_j - u_j^{-1}}, & k_j = 0. \end{cases}
\]
Consequently,
\[
    a_k^T = \left(\frac{1}{\pi i}\right)^d \oint_{C_\rho} f(z) \prod_{1 \leq i \leq d} Q_{k_i}^{(0)} (z_i) \prod_{1 \leq j \leq d \atop k_j \neq 0} Q_{k_j}^{(0)} (z_j) \, dz
\]
\[
    = \frac{1}{2^{N(k)}} \left(\frac{1}{\pi i}\right)^d \oint_{C_\rho} f(z(u)) \prod_{1 \leq i \leq d \atop k_i = 0} \frac{1}{u_i} \prod_{1 \leq j \leq d \atop k_j \neq 0} \frac{1}{u_{j+1}} \, du
\]
\[
    = \frac{1}{2^{N(k)}} \left(\frac{1}{\pi i}\right)^d \oint_{C_\rho} \frac{f(z(u))}{u^{k+1}} \, du,
\]
where $C_\rho := \bigotimes_{j=1}^d C_{\rho_j}$ is the polycircle.

The desired result (3.13) follows directly from the above formula. \hfill \square

**Remark 3.3.** If $d = 1$, the bound (3.13) reduces to
\[
    |a_k^T| \leq \begin{cases} 
        B_f, & k = 0, \\
        \frac{2B_f}{\rho^k}, & k \geq 1.
    \end{cases}
\]

Thus, we have recovered the sharpest bound which was first obtained by Bernstein in [3]. For $d \geq 2$, the bound (3.13) can also be found in [4, Page 95], up to the explicit prefactor.

On account of (2.19) and (3.11), the following corollary concerning tensorized Chebyshev expansion of the second kind is immediate.

**Corollary 3.4.** Suppose that the multivariate function $f$ satisfies Assumption 4 and consider the following tensorized Chebyshev expansion of the second kind
\[
    f(x) = \sum_{k \in \mathbb{N}_0^d} a_k^U U_k(x), \quad a_k^U = \frac{1}{h^{(1)}_k} \int_{\Omega_d} f(x) U_k(x) \omega_1(x) \, dx. \tag{3.17}
\]
Then, we have
\[
    |a_k^U| \leq \frac{B_f L(C_\rho)}{\pi^d \rho^{k+1}}. \tag{3.18}
\]

Finally, the tensorized Legendre expansion is defined by
\[
    f(x) = \sum_{k \in \mathbb{N}_0^d} a_k L_k(x), \quad a_k^L = \frac{1}{h^{(2)}_k} \int_{\Omega_d} f(x) L_k(x) \, dx. \tag{3.19}
\]
where $L_k(x) = \prod_{i=1}^d L_{k_i}(x_i)$, with $P_k(x)$ defined as in (2.8). Let $P_k(x)$ be the normalized Legendre polynomial of degree $k$, i.e.,
\[
    \overline{P}_k(x) = \sqrt{\frac{2k+1}{2}} P_k(x). \tag{3.20}
\]
The normalized Legendre expansion is defined by

\[ f(x) = \sum_{k \in \mathbb{N}_0} \overline{a}_k \overline{P}_k(x), \quad \overline{a}_k = \int_{\Omega_d} f(x) \overline{P}_k(x) dx, \tag{3.21} \]

where \( \overline{P}_k(x) = \prod_{i=1}^d \overline{P}_{k_i}(x_i) \). Both kinds of Legendre expansion are frequently used in practice. We state the estimate of \( a^L_k \) and \( \overline{a}_k \) in the following corollary.

**Corollary 3.5.** Under Assumption 7 we have

\[ |a^L_k| \leq \frac{B_f L(E)}{\pi^d \rho^k} \prod_{1 \leq i \leq d} D^{(1)}_{\rho_i} \prod_{1 \leq j \leq d} \sqrt{k_j D^{(1)}_{\rho_j}}, \tag{3.22} \]

and

\[ |\overline{a}_k| \leq \frac{2^{0(k)} B_f L(E)}{\pi^d \rho^k} \prod_{1 \leq i \leq d} D^{(1/2)}_{\rho_i} \prod_{1 \leq j \leq d} D^{(1/2)}_{\rho_j}, \tag{3.23} \]

where the constants \( D^{(1)}_{\rho_i} \), \( D^{(1/2)}_{\rho_j} \) are defined in (2.15) and (2.16), respectively.

**Proof.** For (3.22), it follows readily from (3.7) by setting \( \lambda = 1/2 \). As for (3.23), by (3.20), it follows that

\[ \overline{a}_k = \frac{a^L_k}{(k + \frac{1}{2})^{1/2}}. \tag{3.24} \]

This, together with (3.22), implies that

\[
|\overline{a}_k| \leq \frac{2^{0(k)} B_f L(E)}{\pi^d \rho^k (k + \frac{1}{2})^{1/2}} \prod_{1 \leq i \leq d} D^{(1/2)}_{\rho_i} \prod_{1 \leq j \leq d} \sqrt{k_j D^{(1/2)}_{\rho_j}}
\]

\[
= 2^{0(k)} B_f L(E) \prod_{1 \leq i \leq d} D^{(1/2)}_{\rho_i} \prod_{1 \leq j \leq d} D^{(1/2)}_{\rho_j} \sqrt{k_j D^{(1/2)}_{\rho_j}}
\]

\[
\leq 2^{0(k)} B_f L(E) \prod_{1 \leq i \leq d} D^{(1/2)}_{\rho_i} \prod_{1 \leq j \leq d} D^{(1/2)}_{\rho_j},
\]

which is (3.23). \( \square \)
3.4 Numerical experiments and Assumption II on $f$

Although we have derived an explicit bound for the coefficients of multivariate Gegenbauer expansion under Assumption I on $f$, it is unclear how to determine an optimal polyellipse such that the bound matches the decay rate of the coefficients well. To introduce our second assumption, we proceed to perform numerical experiments to the multivariate normalized Legendre coefficients $\overline{a}_L^k$ for the following two bivariate functions

$$f_1(x_1, x_2) = \sqrt{x_1^2 + x_2^2 + \frac{1}{2}}, \quad (3.25)$$

and

$$f_2(x_1, x_2) = \frac{1}{x_1^2 + x_2^2 + 1}. \quad (3.26)$$

Note that both functions are isotropic and are analytic for all real values of $x_1$ and $x_2$. Moreover, for complex values of $x_1$ and $x_2$, the former function has a branch point at $x_1^2 + x_2^2 = -1/2$ and the latter function has a pole at $x_1^2 + x_2^2 = -1$. Contour plots of $|\overline{a}_L^k|$ are shown in Figure 1. In both cases, we observe clearly that the contours look like circular arcs in the positive orthant. This phenomena was first reported in [22] for the multivariate Chebyshev coefficients of isotropic functions.

![Contour plots](image)

Figure 1: Contour plots of $|\overline{a}_L^k|$ for the bivariate functions $f_1$ (left) and $f_2$ (right). From inside out, the contours represent $10^{-1}, 10^{-2}, \ldots, 10^{-16}$.

To approximate a multivariate function $f$ in $\Omega_d$ by a multivariate polynomial, it is usual to use the so-called total degree $d_T$ or maximal degree $d_M$ of the multivariate polynomial. More precisely, for a multivariate monomial $x^k$, we set

$$d_T(x^k) := \|k\|_1, \quad d_M(x^k) := \|k\|_\infty, \quad (3.27)$$
and the degree of a multivariate polynomial is then defined as the maximum of the degrees of its nonzero monomials. The above observation, however, implies that any approximations based on these traditional notions might be suboptimal. This invokes Trefethen in [22] to introduce the following Euclidean degree for $x^k$:

$$d_E(x^k) := \|k\|_2,$$  \hspace{1cm} (3.28)

which also leads to the definition of Euclidean degree of a multivariate polynomial. Note that the Euclidean degree might not be an integer. The motivation behind this definition is the multivariate polynomials with prescribed Euclidean degree may provide approximations with uniform resolution in all directions for functions defined in the hypercube $\Omega_d$, as evidenced in Figure 1.

As an application of the Euclidean degree, it is used to establish the rate of decay of the multivariate Chebyshev coefficients in [21] by imposing some conditions on $f$. To some extent, this explains the aforementioned effect in a mathematical way. In particular, the following region is introduced therein to extend the Bernstein ellipse.

**Definition 3.6.** For any $s, a > 0$, we denote by $N_{s,a} \subseteq \mathbb{C}$ the open region bounded by the ellipse with foci 0 and $s$, and leftmost point $-a$.

Note that $z \in E_\rho \iff z^2 \in \partial N_{1,h^2}$,  \hspace{1cm} (3.29)

where $\partial U$ denotes the boundary of a region $U$ and

$$h = \frac{\rho - \rho^{-1}}{2}, \hspace{0.5cm} \rho = h + \sqrt{1 + h^2}.$$ \hspace{1cm} (3.30)

It is then required in [21] that $f$ is analytic in the $d$-dimensional region defined by $\sum_{i=1}^d x_i^2 \in N_{d,h^2}$ for some $h > 0$, which clearly extends the analyticity of $f$ in the Bernstein ellipse to a higher dimensional space.

To deal with the case of multivariate Gegenbauer expansion, we will adopt the following assumption, which is a slight generalization of the one just mentioned.

**Assumption II.** There exists some $h > 0$ such that $f(z)$ is analytic in the $d$-dimensional region $D_{h,\epsilon}$ defined by

$$D_{h,\epsilon} := \left\{ z \in \mathbb{C}^d \left| \sum_{i=1}^d z_i^2 \in N_{d,h^2+\epsilon} \right. \right\},$$ \hspace{1cm} (3.31)

where the region $N_{d,h^2+\epsilon}$ is defined in Definition 3.6, $\epsilon > 0$ is an arbitrarily small fixed constant when $0 < \lambda < 1$, and $\epsilon = 0$ when $\lambda \geq 1$ or $\lambda = 0$.

As we shall see later, the region $D_{h,\epsilon}$ actually contains some polyellipses. The reason why we need $\epsilon > 0$ for $0 < \lambda < 1$ will be explained in Remark 3.8 below. We next show the upper bound of multivariate Gegenbauer coefficients under Assumption II which extends the results for the Chebyshev case.
3.5 Estimates of $a_k$ under Assumption [II]

The main result of this section is the following theorem.

**Theorem 3.7.** Under Assumption [II] and $\lambda > 0$, the multivariate Gegenbauer coefficients of $f(x)$ satisfy

$$|a_k| \leq \frac{\hat{B}_f L(\mathcal{E}_{\hat{\rho}})}{\pi^d |\hat{\rho}|^{1/2}} \prod_{1 \leq i \leq d} D^{(\lambda)}_{\hat{\rho}_i} \prod_{1 \leq j \leq d} k_j^{1-\lambda} D^{(\lambda)}_{\hat{\rho}_j},$$  \hspace{1cm} (3.32)

where $\rho = h + \sqrt{1 + h^2}$,

$$\hat{\rho}_j = \sqrt{(c_j h)^2 + \epsilon + \sqrt{1 + (c_j h)^2 + \epsilon}}, \quad j = 1, \ldots, d,$$ \hspace{1cm} (3.33)

with $c_j = k_j/\|k\|_2$, and the constants $D^{(\lambda)}_{\hat{\rho}_i}$, $D^{(\lambda)}_{\hat{\rho}_j}$ are defined in (2.15) and (2.16), respectively. Moreover, $\mathcal{E}_{\hat{\rho}} := \bigotimes_{j=1}^d \mathcal{E}_{\hat{\rho}_j}$ and the constant $\hat{B}_f$ is defined by

$$\hat{B}_f = \max_{z \in \mathcal{E}_{\hat{\rho}}} |f(z)|.$$ \hspace{1cm} (3.34)

**Proof.** We follow the idea in [21], which deals with the multivariate Chebyshev coefficients.

For each $k \in N_0^d$, we define $h_j = c_j h$ with $c_j = k_j/\|k\|_2$, $j = 1, \ldots, d$. It is then easily seen that $h_1^2 + \cdots + d^2 = h^2$. From [21] Lemma 5.2, we have

$$N_1 h_1^2 + \epsilon \bigoplus \cdots \bigoplus N_{d,h_1^2+\epsilon} \subseteq N_d \sum_{i=1}^d h_i^2 + \epsilon = N_d h^2 + \epsilon,$$

where $\bigoplus$ denotes the Minkowski sum of sets. This, together with Assumption [II] on $f$, implies that $f$ is analytic in the region defined by \{\(z \in \mathbb{C}^d \mid z_j^2 \in N_{1,h_1^2+\epsilon}, j = 1, \ldots, d\}\}. On account of (2.16) and (3.30), we further conclude that $f$ is analytic in the polyellipse $\mathcal{E}_{\hat{\rho}} = \bigotimes_{j=1}^d \mathcal{E}_{\hat{\rho}_j}$, where each $\hat{\rho}_j$ is defined in (3.33). Hence, by Theorem 3.1 it follows that

$$|a_k| \leq \frac{\hat{B}_f L(\mathcal{E}_{\hat{\rho}})}{\pi^d |\hat{\rho}|^{1/2}} \prod_{1 \leq i \leq d} D^{(\lambda)}_{\hat{\rho}_i} \prod_{1 \leq j \leq d} k_j^{1-\lambda} D^{(\lambda)}_{\hat{\rho}_j}. \hspace{1cm} (3.35)$$

To this end, we see from [21] Lemma 5.3 that

$$\hat{\rho}_j \geq c_j h + \sqrt{1 + (c_j h)^2} \geq (h + \sqrt{1 + h^2}) c_j = \rho^{1/2},$$ \hspace{1cm} (3.36)

which implies

$$\hat{\rho}_k^k = \prod_{j=1}^d \hat{\rho}_j^k \geq \rho^{k_1^2 + \cdots + k_d^2} = \rho^{\|k\|_2^2}. \hspace{1cm} (3.37)$$

Combining (3.37) and (3.35) then gives us the the bound of $|a_k|$ given in (3.32). This completes the proof of Theorem 3.7. \hfill \Box
Remark 3.8. When $0 < \lambda < 1$, we note that the $\hat{\rho}_j$-dependent constants $D^{(\lambda)}_{\hat{\rho}_j}$ and $\overline{D}^{(\lambda)}_{\hat{\rho}_j}$ would be infinity as $\hat{\rho}_j \to 1$; see (2.15) and (2.16). By (3.33), this is indeed the case if $\epsilon = 0$ and $c_j = 0$ for some $j$. This explains why we have assumed that $\epsilon > 0$ when $0 < \lambda < 1$, so that $\hat{\rho}_j > 1$ for all $j = 1, \ldots, d$.

Since the cases $\lambda = 0$ and $\lambda = 1$ are of particular interest, we conclude this section with the relevant results in the following corollary.

Corollary 3.9. Let $f$ be analytic in the $d$-dimensional region $D_{h,0}$ defined in (3.31) for some $h > 0$. Then, the multivariate Chebyshev coefficients of the first kind for $f$ satisfy

$$|a^T_k| \leq \frac{2^{d-L(k)} \hat{B}_f}{\hat{\rho}^{\|k\|_2}},$$

(3.38)

and the multivariate Chebyshev coefficients of the second kind for $f$ satisfy

$$|a^U_k| \leq \frac{\hat{B}_f L(E_{\hat{\rho}})}{\pi d \hat{\rho}^{\|k\|_2+1}},$$

(3.39)

where $\hat{\rho}_j = c_j h + \sqrt{1 + (c_j h)^2}$ with $c_j = k_j/\|k\|_2$ for $j = 1, \ldots, d$.

Proof. To show (3.39), we note that, as in the proof of Theorem 3.7, $f$ is analytic in the polyellipse $E_{\hat{\rho}} := \bigotimes_{j=1}^d E_{\hat{\rho}_j}$, where $\hat{\rho}_j = c_j h + \sqrt{1 + (c_j h)^2}$. This, together with Corollary 3.4 implies that

$$|a^U_k| \leq \frac{\hat{B}_f L(E_{\hat{\rho}})}{\pi d \hat{\rho}^{\|k\|_2+1}}.$$

(3.40)

In view of (3.33), we have

$$\hat{\rho}_j^{k_j+1} = \prod_{j=1}^d \hat{\rho}_j^{k_j+1} \geq \prod_{j=1}^d \rho^{k_j(k_j+1)} = \rho^{\|k\|_2 + \|k\|_1} \geq \rho^{\|k\|_2+1},$$

(3.41)

where we have made use of the fact that $\|k\|_1 \geq \|k\|_2$ in the last step (cf. Lemma 4.2 below). Combining (3.41) and (3.40) then gives (3.39).

The proof of (3.38) is similar, where one needs to use the estimate (3.13). We omit the details here. This completes the proof of Corollary 3.9.

4 Multivariate Gegenbauer approximation

In this section, we investigate the error bound of the multivariate Gegenbauer approximation with the multi-indices chosen from a specified index set.
4.1 Multivariate Gegenbauer approximation with an $\ell^q$ ball index set

We are interested in the multivariate Gegenbauer approximation corresponding to an $\ell^q$ ball index set in $N_0^d$ defined by

$$\Lambda_N^q = \left\{ k \in N_0^d \mid \|k\|_q \leq N \right\},$$  \hspace{1cm} (4.1)

where $q > 0$ and $\|k\|_q$ is defined as in (3.3). Note that such an index set is a lower set and includes some important index sets as special cases. For example, the total, Euclidean and maximal degrees of a multivariate polynomial at most $N$ correspond to $q = 1$, $2$, $\infty$ in (4.1), respectively. To gain some intuition regarding the distribution of the grids in $\Lambda_N^q$, we plot in Figure 2 the index set $\Lambda_{30}^q$ for $d = 2$ and three different values of $q$.

![Figure 2: Illustration of the index set $\Lambda_{30}^q$ for $q = 1/2$ (left), $q = 1$ (middle) and $q = 2$ (right) in dimension $d = 2$.](image)

We now consider the finite-dimensional polynomial space $P_N^q$ corresponding to the $\ell^q$ ball index set, namely,

$$P_N^q := \text{span} \left\{ C^{(\lambda)}_k(x) \mid k \in \Lambda_N^q \right\}.$$  \hspace{1cm} (4.2)

Let $\Pi_N^\lambda$ be the orthogonal projection from the space $L^2(\omega_{\lambda}(x)(\Omega_d))$ to $P_N^q$ such that

$$\int_{\Omega_d} ((\Pi_N^\lambda f)(x) - f(x))\omega_{\lambda}(x)Q(x)dx = 0, \quad \forall Q(x) \in P_N^q.$$  \hspace{1cm} (4.3)

It is well-known that $(\Pi_N^\lambda f)(x)$ can be written explicitly as

$$(\Pi_N^\lambda f)(x) = \begin{cases} \sum_{k \in \Lambda_N^q} a_k C^{(\lambda)}_k(x), & \lambda > 0, \\ \sum_{k \in \Lambda_N^q} a_k^T T_k(x), & \lambda = 0, \end{cases}$$  \hspace{1cm} (4.4)

where the coefficients $a_k$ and $a_k^T$ are given in (3.5) and (3.12), respectively. The main result of this section is the following theorem regarding the explicit error bound of the multivariate Gegenbauer approximation $(\Pi_N^\lambda f)(x)$ in the uniform norm.
Theorem 4.1. Let \( \lambda \geq 0 \) and let \( \Pi_{\lambda N} f(x) \) defined in (4.4) be the multivariate Gegenbauer approximation associated with the index set \( \Lambda_q^N \). Suppose that \( f \) satisfies Assumption II. Then, we have,

\[
\max_{x \in \Omega_d} \left| f(x) - \Pi_{\lambda N} f(x) \right| \leq K \rho^{-\frac{N}{d}}, \quad N > \frac{\lambda \gamma d}{\ln \rho}, \quad (4.5)
\]

where \( K \) is a constant independent of the index set (see (4.29) and (4.31) below for explicit representations), \( \rho = h + \sqrt{1 + h^2} \), and

\[
\gamma = \begin{cases} 
1, & q \geq 2, \\
\frac{1}{2} \frac{q}{q - 1}, & 0 < q < 2.
\end{cases} \quad (4.6)
\]

Some comments regarding Theorem 4.1 are given below.

- Up to the algebraic pre-factor, the rate of convergence of the multivariate Chebyshev approximation established in (4.5) was first obtained by Trefethen in [21] for \( q = 1, 2, \infty \). Here we have extended his result to a more general setting.

- For the multivariate normalized Legendre approximation associated with the index set \( \Lambda_q^N \), our analysis will also lead to the same error bound as shown in (4.5), although the constant \( K \) might be different.

- For the bivariate Runge function

\[
f(x) = \frac{1}{x_1^2 + x_2^2 + h^2}, \quad h > 0,
\]

it is readily seen from [5, Theorem 3.1 and Lemma 4.6] that

\[
\limsup_{N \to \infty} D_{N,q}(f)^{1/N} \leq 1/\rho, \quad (4.7)
\]

where \( q \geq 2 \) and

\[
D_{N,q}(f) := \inf \left\{ \max_{x \in \Omega_2} \left| f - \sum_{k \in \Lambda_q^N} c_k x^k \right| , \; c_k \in \mathbb{C} \right\}.
\]

Hence, by comparing (4.5) with (4.7) we can conclude that the multivariate Gegenbauer and Chebyshev approximations with an \( \ell^q \) ball index set \( q \geq 2 \) achieve the best possible rate of convergence of polynomial approximations in this case.

We next present the proof of Theorem 4.1 and start with some auxiliary results to be used later.
4.2 Some auxiliary lemmas

**Lemma 4.2.** Let $\|k\|_q$ be defined in (3.3). For $r \geq s > 0$ and $k \in \mathbb{N}^d_0$, we have

$$\|k\|_r \leq \|k\|_s \leq d^{s-\frac{s}{r}} \|k\|_r.$$  \hspace{1cm} (4.8)

Moreover, it is worthwhile to point out that the above inequalities are optimal in the sense that there are no smaller constants such that they still hold for all $k \in \mathbb{N}^d_0$.

We note that, if $q \geq 1$, $\| \cdot \|_q$ defines a norm in $\mathbb{R}^d$ and the inequalities (4.8) are well-known (cf. [27, Proposition 2.10]). It comes out this result can be extended to the case $q > 0$. For the convenience of the reader, we include a proof in what follows.

**Proof.** We first show that $\|k\|_r \leq \|k\|_s$. Note that if $k = 0 = (0, \ldots, 0)$, the inequality is trivial, and we may assume that $k \neq 0$. By setting $\kappa = k/\|k\|_s = (k_1, \ldots, k_d)$, it follows that

$$\|\kappa\|_s = 1, \quad |\kappa_i| \leq 1, \quad i = 1, \ldots, d.$$  

Thus,

$$\|\kappa\|_r = (|\kappa_1|^r + \cdots + |\kappa_d|^r)^{\frac{1}{r}} \leq (|\kappa_1|^s + \cdots + |\kappa_d|^s)^{\frac{1}{r}} = 1,$$

which gives the first inequality $\|k\|_r \leq \|k\|_s$ in (4.8). Moreover, this inequality is sharp since it becomes an equality for $k = (1, 0, \ldots, 0)$.

Next, we show that $\|k\|_s \leq d^{s-\frac{s}{r}} \|k\|_r$. Since $r \geq s > 0$, we obtain from the Hölder’s inequality that

$$\|k\|_s^s = \sum_{j=1}^d |k_j|^s \leq \left( \sum_{j=1}^d (|k_j|^s)^{\frac{s}{s}} \right)^{\frac{s}{s}} \left( \sum_{j=1}^d 1 \right)^{1-\frac{s}{s}} = d^{1-\frac{s}{s}} \left( \sum_{j=1}^d |k_j|^r \right)^{\frac{s}{r}},$$

or equivalently,

$$\|k\|_s \leq d^{s-\frac{s}{r}} \|k\|_r,$$

as required. Note that for $k = (1, 1, \ldots, 1)$, the above inequality becomes an equality, which implies that the constant $d^{s-\frac{s}{r}}$ can not be improved any more. This completes the proof of Lemma 4.2.

The second lemma is about the upper bound of an integral over an unbounded interval.

**Lemma 4.3.** Let $a, b > 0$ and $M > 0$. We have

$$\int_M^\infty e^{-ax}x^b \, dx \leq e^{-aM} \left( \sum_{j=1}^{mh+1} \frac{M^{b-j+1} j^{-2}}{a^j} \prod_{i=0}^{j-2} (b-i) \right),$$  \hspace{1cm} (4.9)

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where $m_b$ is a positive integer depending on $b$ that is uniquely defined by
\[
m_b = \begin{cases} 
  b, & b \in \mathbb{N}, \\
  \lfloor b \rfloor + 1, & b \notin \mathbb{N},
\end{cases}
\] (4.10)
and the product in the right-hand side of (4.9) is assumed to be one for $j < 2$. In (4.10), $\lfloor x \rfloor$ denotes the integral part of a real number $x$.

**Proof.** With $m_b$ given in (4.10), we obtain from integration by parts $m_b$ times that
\[
\int_{M}^{\infty} e^{-ax}x^b dx = -\frac{1}{a} \int_{M}^{\infty} x^{b-1} d\left( e^{-ax} \right) = M^b e^{-aM} + \frac{b}{a} \int_{M}^{\infty} e^{-ax}x^{b-1} dx = \ldots
\]
\[
= e^{-aM} \sum_{j=1}^{m_b} \frac{M^{b-j+1}}{a^j} \prod_{i=0}^{j-2} (b-i) + \frac{1}{a^{m_b}} \prod_{i=0}^{m_b-1} (b-i) \int_{M}^{\infty} x^{b-m_b} e^{-ax} dx. \tag{4.11}
\]
Note that $-1 < b - m_b \leq 0$, it is easily seen that
\[
\int_{M}^{\infty} x^{b-m_b} e^{-ax} dx \leq M^{b-m_b} \int_{M}^{\infty} e^{-ax} dx = \frac{M^{b-m_b}}{a} e^{-aM}. \tag{4.12}
\]
Combining (4.11) and (4.12) gives us the desired result. \hfill \square

Finally, we also need the following lemma which gives us an explicit upper bound of the ratio of Gamma functions.

**Lemma 4.4.** Let $k \geq 1$ and $a, b \in \mathbb{R}$. For $k + a > 1$ and $k + b > 1$, we have
\[
\frac{\Gamma(k + a)}{\Gamma(k + b)} \leq \Upsilon_k^{a,b} k^{a-b}, \tag{4.13}
\]
where
\[
\Upsilon_k^{a,b} = \exp \left( \frac{a-b}{2(k+b-1)} + \frac{1}{12(k+a-1)} + \frac{(a-1)(a-b)}{k} \right). \tag{4.14}
\]
**Proof.** See [29, Lemma 2.1]. \hfill \square

We are now ready to prove Theorem 4.1

### 4.3 Proof of Theorem 4.1

By (4.1) and (4.4), it follows that, for $\lambda > 0$,
\[
\max_{x \in \Omega_d} \left| f(x) - (\Pi_N^d f)(x) \right| = \max_{x \in \Omega_d} \left| \sum_{k \in \mathbb{N}_0^d \setminus \Lambda_N^q} a_k C^{(\lambda)}_k(x) \right| \leq \sum_{k \in \mathbb{N}_0^d \setminus \Lambda_N^q} |a_k| \max_{x \in \Omega_d} |C^{(\lambda)}_k(x)|
\]
\[
= \sum_{k \in \mathbb{N}_0^d \setminus \Lambda_N^q} |a_k| C^{(\lambda)}_k(1), \tag{4.15}
\]
where we have made use of (2.37) in the last step.

To this end, with \( \hat{\rho}_j, j = 1, \ldots, d \), defined in (3.33), it is readily seen that
\[
\hat{\rho}_j \leq \rho := \sqrt{\eta^2 + \epsilon + \gamma} + \sqrt{1 + \eta^2 + \epsilon},
\]
and from Assumption II on \( f \) and the proof of Theorem 3.7 that
\[
\mathcal{E}_{\rho} = \bigotimes_{j=1}^d \mathcal{E}_{\hat{\rho}_j} \subseteq \mathcal{D}_{h,e}.
\]
Thus, we conclude from (3.32) that, for any multi-index \( k \in \mathbb{N}_0^d \),
\[
|a_k| \leq \max_{x \in \mathcal{D}_{h,e}} |f(x)| L(\mathcal{E}_{\rho_\epsilon}) \frac{1}{d^d \rho \|k\|_2} \prod_{1 \leq i \leq d, k_i=0}^d D_{\rho_i}^{(\lambda)} \prod_{1 \leq j \leq d, k_j \neq 0}^d k_j^{1-\lambda} D_{\hat{\rho}_j}^{(\lambda)},
\]
where we emphasize that the constant \( \max_{x \in \mathcal{D}_{h,e}} |f(x)| L(\mathcal{E}_{\rho_\epsilon}) \) is independent of \( k \). This, together with (4.15) and (2.3), implies that
\[
\max_{x \in \Omega_d} \left| f(x) - (\Pi_N f)(x) \right| \leq \max_{x \in \mathcal{D}_{h,e}} |f(x)| L(\mathcal{E}_{\rho_\epsilon}) \frac{1}{d^d \rho \|k\|_2} \prod_{1 \leq i \leq d, k_i=0}^d D_{\rho_i}^{(\lambda)} \prod_{1 \leq j \leq d, k_j \neq 0}^d D_{\hat{\rho}_j}^{(\lambda)} \times \left( \prod_{1 \leq j \leq d, k_j \neq 0}^d k_j^{1-\lambda} \frac{\Gamma(k_j + 2\lambda)}{\Gamma(2\lambda) \Gamma(k_j + 1)} \right) \frac{1}{\rho \|k\|_2}.
\]
For the product in the last line of the above formula, we obtain from Lemma 44 that
\[
\prod_{1 \leq j \leq d, k_j \neq 0}^d k_j^{1-\lambda} \frac{\Gamma(k_j + 2\lambda)}{\Gamma(2\lambda) \Gamma(k_j + 1)} \leq \prod_{1 \leq j \leq d, k_j \neq 0}^d k_j^{2\lambda} \frac{\Gamma(2\lambda) \Gamma(k_j + 1)}{\Gamma(2\lambda) \Gamma(k_j + 1)} = \left( \prod_{1 \leq j \leq d, k_j \neq 0}^d \frac{\Gamma(2\lambda) \Gamma(k_j + 1)}{\Gamma(2\lambda)} \right) \left( \prod_{1 \leq j \leq d, k_j \neq 0}^d k_j \right)^\lambda,
\]
where \( \Upsilon_{ab}^k \) is defined in (4.14). A further appeal to the arithmetic geometric mean inequality shows that
\[
\prod_{1 \leq j \leq d, k_j \neq 0}^d k_j \leq \left( \frac{k_1 + \cdots + k_d}{d - N(k)} \right)^{d-N(k)} \left( \frac{\|k\|_1}{d - N(k)} \right)^{d-N(k)}.
\]
Thus, it follows from (4.17), (4.18) and (4.19) that

\[
\max_{x \in \Omega_d} |f(x) - (\Pi_N f)(x)| \leq \max_{z \in D_{\eta, \epsilon}} |f(z)| L(E_{\rho_0})^d \sum_{k \in N_0 \setminus \Lambda_N^q} \left( \prod_{1 \leq i \leq d} D_{\rho_i}^{(\lambda)} \prod_{1 \leq j \leq d, k_j \neq 0} \frac{\Gamma(2\lambda)}{\Gamma(2\lambda)} \right) 
\]

\[
\times \left( \frac{\|k\|_1}{d - \kappa(k)} \right)^{\lambda(d - \kappa(k))} \frac{1}{\rho \|k\|_2} 
\]

\[
\leq \max_{z \in D_{\eta, \epsilon}} |f(z)| L(E_{\rho_0})^d \max_{k \in N_0 \setminus \Lambda_N^q} \left( \prod_{1 \leq i \leq d} D_{\rho_i}^{(\lambda)} \prod_{1 \leq j \leq d, k_j \neq 0} \frac{\Gamma(2\lambda)}{\Gamma(2\lambda)} \right) 
\]

\[
\times \sum_{k \in N_0 \setminus \Lambda_N^q} \frac{\|k\|_1^d}{\rho \|k\|_2} 
\]

(4.20)

where in the last step we have made use of the fact that

\[ 1 \leq d - \kappa(k) \leq d \]

for any \( k \in N_0 \setminus \Lambda_N^q \). The remaining task is then to estimate the two factors \( \sum_{k \in N_0 \setminus \Lambda_N^q} \frac{\|k\|_1^d}{\rho \|k\|_2} \) and \( \max_{k \in N_0 \setminus \Lambda_N^q} \left( \prod_{1 \leq i \leq d} D_{\rho_i}^{(\lambda)} \prod_{1 \leq j \leq d, k_j \neq 0} \frac{\Gamma(2\lambda)}{\Gamma(2\lambda)} \right) \), respectively.

To estimate \( \sum_{k \in N_0 \setminus \Lambda_N^q} \frac{\|k\|_1^d}{\rho \|k\|_2} \), we first observe from Lemma 4.2 that

\[ \|k\|_1 \leq \sqrt{d} \|k\|_2, \quad \|k\|_q \leq \gamma \|k\|_2, \]

where the constant \( \gamma \) depending on \( d \) and \( q \) is given in (1.6). Thus, it is readily seen that

\[ \sum_{k \in N_0 \setminus \Lambda_N^q} \frac{\|k\|_1^d}{\rho \|k\|_2} = \sum_{\|k\|_q > \lambda} \frac{\|k\|_1^d}{\rho \|k\|_2} \leq \lambda^d \sum_{\|k\|_2 > \lambda} \frac{\|k\|_1^d}{\rho \|k\|_2}. \]

Note that \( \frac{\|k\|_2^d}{\rho \|k\|_2} \) decreases strictly for \( \|k\|_2 > \frac{\lambda}{\ln p} \), the last term can be further
bounded as

\[ \sum_{||k||_2 > \frac{N}{\gamma}} \frac{||k||_{\lambda d}^2}{\rho ||k||_2^2} \leq \int \cdots \int_{x_1^2 + \cdots + x_d^2 \geq \frac{N}{\gamma}^2} \frac{(x_1^2 + \cdots + x_d^2)^{\lambda d}}{\rho \sqrt{x_1^2 + \cdots + x_d^2}} \, dx_1 \cdots dx_d \]

\[ \leq C_d \int_{x}^{\infty} \frac{\lambda^d d - 1}{\rho^t} \, dt, \quad N > \frac{\lambda d}{\ln \rho}, \]  

(4.21)

where we have evaluated the integral with the aid of spherical coordinates and

\[ C_d = \begin{cases} 1, & \text{if } d = 1, \\ \frac{(\pi/2)^{d/2}}{(d-2)!}, & \text{if } d \geq 2. \end{cases} \]  

(4.22)

Next, by setting \( a = \ln \rho \), \( b = \lambda d + d - 1 \) and \( M = N/\gamma \) in Lemma 4.3 it follows that the last integral in (4.21) admits the following upper bound:

\[ \int_{x}^{\infty} \frac{\lambda^d d - 1}{\rho^t} \, dt = \int_{x}^{\infty} e^{-t \ln \rho} \lambda^d d - 1 \, dt \]

\[ \leq \rho^{-x} \left( \sum_{j=1}^{m_d d - 1 + 1} \frac{N_j \lambda^d d - j - 2}{(\ln \rho)^j} \prod_{i=0}^{\lambda d + d - i - 1} \right) \],

where recall that the constant \( m_b \) is defined in (4.10). As a consequence, we finally arrive at

\[ \sum_{k \in N_0^d \setminus \Lambda_N^\lambda} \frac{||k||_{\lambda d}^2}{\rho ||k||_2} \leq C_d \left( \sum_{j=1}^{m_d d - 1 + 1} \frac{N_j \lambda^d d - j - 2}{(\ln \rho)^j} \prod_{i=0}^{\lambda d + d - i - 1} \right) \rho^{-x}. \]  

(4.23)

To find an upper bound of \( \max_{k \in N_0^d \setminus \Lambda_N^\lambda} \left( \prod_{1 \leq i \leq d} D^{(\lambda)}_{\hat{\rho}_i} \prod_{1 \leq j \leq d} \frac{\tau^{2 \lambda, 1}_{k_j} D^{(\lambda)}_{\hat{\rho}_j}}{\Gamma(2 \lambda)} \right) \), on one hand, we observe from (4.14) that, for \( \lambda > 0 \) and \( k_j \geq 1 \),

\[ \tau^{2 \lambda, 1}_{k_j} = \exp \left( \frac{2 \lambda - 1}{2 k_j} + \frac{1}{12 (k_j + 2 \lambda - 1)^2} + \frac{(2 \lambda - 1)^2}{k_j} \right) \leq \exp \left( \max \left\{ \left( \frac{2 \lambda - 1}{2} \right) \right\} + \frac{1}{24 \lambda} + (2 \lambda - 1)^2 \right). \]  

(4.24)

On the other hand, in view of (3.33), we have

\[ \hat{\rho}_j \begin{cases} = \sqrt{\epsilon} + \sqrt{1 + \epsilon}, & k_j = 0, \\ \geq \sqrt{\epsilon} + \sqrt{1 + \epsilon}, & k_j \neq 0. \end{cases} \]  

(4.25)
As it can be easily seen from (2.10) that \(D^{(\lambda)}_{\hat{\rho}}\) is a strictly decreasing function of \(\rho > 1\) for fixed \(\lambda > 0\), thus, it follows from (4.28) and (2.15) that, for \(k_j \neq 0\),

\[
D^{(\lambda)}_{\hat{\rho}_j} \leq D^{(\lambda)} = \left\{ \begin{array}{ll}
\Gamma(\lambda)e^{1/2}D^{(\lambda)}_{\sqrt{1+\epsilon}}, & \lambda \geq 1,
\Gamma(\lambda)e^{1/2+\lambda/2\epsilon}D^{(\lambda)}_{\sqrt{1+\epsilon}}, & 0 < \lambda < 1,
\end{array} \right.
\]

\[
\leq \Gamma(\lambda)e^{1/2+\max\{0,1/2\epsilon\}}D^{(\lambda)}_{\sqrt{1+\epsilon}}. \tag{4.26}
\]

Combining (4.24) and (4.26), we obtain that

\[
\max_{k \in \mathbb{N}_0^d \setminus \Lambda_N^d} \left( \prod_{1 \leq i \leq d} \frac{D^{(\lambda)}_{\hat{\rho}_i}}{\Gamma(2\lambda)} \right)^{\frac{2\lambda-1}{\lambda}} \leq \kappa^d, \tag{4.27}
\]

where

\[
\kappa := \max\left\{ 1, \frac{\Gamma(\lambda)}{\Gamma(2\lambda)} e^{2\max\{0,\lambda/2,1/2\epsilon\} + 1/2\epsilon} \right\} D^{(\lambda)}_{\sqrt{1+\epsilon}}. \tag{4.28}
\]

Substituting the estimates (4.23) and (4.27) into (4.29) then gives us (4.5) with

\[
K = \max_{z \in D_{h,0}} |f(z)|L(E_{\rho,0})^d \left( \frac{\pi}{2} \right)^d C_d \left( \sum_{j=1}^{m_n, d-d-j+1} \frac{(N_j)^{\lambda_d+\lambda_j+1}}{(\ln \rho)^j} \prod_{i=0}^{\lambda_j} (d-i+1) \right) . \tag{4.29}
\]

For the multivariate Chebyshev approximation of the first kind, i.e., \(\lambda = 0\), we note from (3.12) and (3.38) that

\[
\max_{x \in \Omega_d} |f(x) - (\Pi_N f)(x)| \leq \sum_{k \in \mathbb{N}_0^d \setminus \Lambda_N^d} |a_k| \leq 2^d \max_{z \in D_{h,0}} |f(z)| \sum_{k \in \mathbb{N}_0^d \setminus \Lambda_N^d} \frac{1}{\rho||k||_2}.
\]

Similar to the derivation of (4.23), it is readily seen that

\[
\sum_{k \in \mathbb{N}_0^d \setminus \Lambda_N^d} \frac{1}{\rho||k||_2} \leq C_d \sum_{j=1}^d \frac{(N_j)^{d-j}}{(\ln \rho)^j} \prod_{i=0}^{d-j} (d-i+1) \rho^{-N_j}, \quad N > 0. \tag{4.30}
\]

Hence, a combination of the above two inequalities shows that, for \(\lambda = 0\), we still have (4.5) but with the constant \(K\) replaced by

\[
K = \max_{z \in D_{h,0}} |f(z)|^2 \left( \sum_{j=1}^d \frac{(N_j)^{d-j}}{(\ln \rho)^j} \prod_{i=0}^{d-j} (d-i+1) \right). \tag{4.31}
\]

This completes the proof of Theorem 4.1. \(\Box\)
4.4 Numerical experiments and further discussions

From Theorem 4.1, it is readily seen that the error bound of the multivariate Gegenbauer approximation is $O(\rho^{-N})$ for $q \geq 2$. If $0 < q < 2$, the error bound is $O(\rho^{-N/2})$, which deteriorate gradually as $q \to 0^+$. Our results match numerical experiments very well for isotropic functions, as illustrated in what follows.

We again consider the functions given in (3.25) and (3.26), respectively. Note that both functions satisfy Assumption II with $h^2 = 0.5$ and $\rho \approx 1.931851652578136$ for the former function, and $h^2 = 1$ and $\rho \approx 2.414213562373095$ for the latter function. We then use multivariate Legendre expansion (i.e., $\lambda = \frac{1}{2}$) on $\Lambda^q_N$ to approximate these functions. In our computations, the maximum error

$$\max_{x \in \Omega_2} \left| f(x) - (\Pi^q_N f)(x) \right|$$

is measured by using a finer grid in $\Omega_2$. The results are shown in Figure 3 as a function of $N$ for three different moderate values of $q$. For each $q$, we clearly observe that the decay rate of the maximum error is consistent with the one predicted in Theorem 4.1.

![Figure 3: Maximum errors of multivariate Legendre approximation for the functions (3.25) (left) and (3.26) (right) as a function of $N$ in $\Omega_2$ with $q = \frac{1}{2}, 1, 2$. Straight lines exhibit the convergence rates predicted by Theorem 4.1.](image)

A further numerical illustration of our results are shown in Figure 4 where we plot the maximum error of the multivariate Legendre approximation for the function (3.25) with several smaller and larger values of $q$. Again, the results of numerical experiments fit the predicted error bound in a satisfactory way.

Finally, it is worthwhile to point out that a direct comparison of the rates of convergence of $(\Pi^q_N f)(x)$ established in Theorem 4.1 for different $q$ is not fair, since the
number of terms in \((\Pi_N f)(x)\) also depends on \(q\). Indeed, for large \(N\), we could estimate this number denoted by \(N_q\) via a continuum approximation and obtain that

\[ N_q \approx N^d V_q, \] \hspace{1cm} (4.32)

where

\[ V_q = \frac{\Gamma\left(\frac{1}{q} + 1\right)^d}{\Gamma\left(\frac{d}{q} + 1\right)}, \hspace{1cm} q > 0, \]

is the volume of the unit \(\ell^q\) ball restricted to the positive orthant (cf. [26] and the references therein). Thus, to evaluate the efficiency of multivariate Gegenbauer approximation with two different \(\ell^q\) index sets, it is much more reasonable to compare \(N_q\) under the condition that the same convergence rate is achieved. Assume that the predicted rate of convergence of \((\Pi_N f)(x)\) given in (4.5) is sharp, we compare two different \(\ell^q\) ball index sets: (i) \(q \neq 2\) and (ii) \(q = 2\). If \(q \in (0, 2)\), to achieve the same convergence rate, say, \(O(\rho^{-N})\), it follows from (4.5) that the number of terms in \((\Pi_N f)(x)\) corresponding to the set (i) is equal to the number of the multi-indices satisfying \(\|k\|_q \leq d^{\frac{1}{q} - \frac{1}{2}} N\), while that corresponding to the set (ii) is equal to the number of the multi-indices satisfying \(\|k\|_2 \leq N\). By (4.32), it is easily seen that the ratio of these two numbers admits the following estimate:

\[
\frac{(d^{\frac{1}{q} - \frac{1}{2}} N)^d \Gamma\left(\frac{1}{q} + 1\right)^d}{N^d \frac{\Gamma\left(\frac{1}{q} + 1\right)^d}{\Gamma\left(\frac{d}{2} + 1\right)}} = d^{\frac{d}{q} - \frac{d}{2}} \left(\frac{\Gamma\left(\frac{1}{q} + 1\right)}{\Gamma\left(\frac{d}{2} + 1\right)}\right)^d \frac{\Gamma\left(\frac{2}{q} + 1\right)}{\Gamma\left(\frac{d}{2} + 1\right)}. \] \hspace{1cm} (4.33)
Numerical experiments show that the above ratio is always greater than one for \( d \geq 2 \) and grows exponentially fast as \( d \) increases. This means that the index set induced from an \( \ell^q \) ball with \( q \in (0, 2) \) may be less efficient compared with \( q = 2 \). If \( q > 2 \), from (4.3) we see that the predicted rate of convergence is always the same, and one only needs to compare \( N_q \) and \( N_2 \). By (4.32), it is easily seen that \( N_q \) is strictly increasing as \( q \) increases and thus \( N_q > N_2 \) for \( q > 2 \). As a consequence, we conclude that the multivariate Gegenbauer approximation based on the Euclidean degree of multivariate polynomial, i.e., on the index set \( \Lambda^q_N \) with \( q = 2 \), provides an optimal choice among the multivariate Gegenbauer approximation with an \( \ell^q \) ball index set, if the convergence rate of \((\Pi^2_N, f)(x)\) established in Theorem 1.1 is sharp. For the multivariate Chebyshev approximation with an \( \ell^q \) ball index set and \( q = 1, 2, \infty \), this viewpoint was first proposed by Trefethen in [22]. Here, we have extended his conclusion to a general setting.

5 An extension to polynomial approximation of parameterized PDEs

In this section, we will apply an extension of Theorem 3.1 with emphasis on tensorized Legendre expansions to the polynomial approximation for parameterized PDEs.

The extension deals with a function \( f(x, y) \) defined in a bounded regular domain \( D \subset \mathbb{R}^n \) with the parameters \( y \in \Omega_d \). Suppose that \( f \in L^\infty(\Omega_d, V, \Pi_{i=1}^d dx_i) \), where \( V = V(D) \) is certain Banach space equipped with the norm \( \| \cdot \|_{V(D)} \). Then, \( f \) admits the following tensorized Legendre expansions

\[
f(x, y) = \sum_{k \in \mathbb{N}_0^d} a_k(x) P_k(y) = \sum_{k \in \mathbb{N}_0^d} \tilde{a}_k(x) \overline{P}_k(y), \quad (5.1)
\]

where the convergence is understood in \( L^2(\Omega_d, V, \Pi_{i=1}^d dx_i) \), and, as in (3.19) and (3.24),

\[
a_k(x) = \frac{1}{h_k^{(\frac{1}{2})}} \int_{\Omega_d} f(x, y) P_k(y) dy, \quad \tilde{a}_k(x) = \frac{a_k(x)}{k + \frac{1}{2}}. \quad (5.2)
\]

By assuming that the dependence of the parameters \( y \) is analytically smooth, we have the following estimates of the coefficients \( a_k(x) \) and \( \tilde{a}_k(x) \).

**Proposition 5.1.** Let \( f(x, y) \) be a function defined in a bounded regular domain \( D \subset \mathbb{R}^n \) with the parameters \( y \in \Omega_d \). Suppose that \( f \in L^\infty(\Omega_d, V, \Pi_{i=1}^d dx_i) \), where \( V = V(D) \) is certain Banach space equipped with the norm \( \| \cdot \|_{V(D)} \), and the analytic continuation \( f(x, z) \) of \( f(x, y) \) satisfies Assumption 1, we have the following estimates of the coefficients in \( (5.1): \)

\[
\|a_k\|_{V(D)} \leq \frac{\sup_{x \in E_P} \|f\|_{V(D)} L(E_P)}{\prod_{\rho=d} \prod_{1 \leq i \leq d} \rho_{k_i}^{(\frac{1}{2})} \prod_{1 \leq j \leq d} \sqrt{k_j D_{j_i}^{(\frac{1}{2})}}}, \quad (5.3)
\]
and
\[
\left\| \alpha_k \right\|_{V(D)} \leq \frac{2^{N(k)}}{\pi^d} \frac{\sup_{\mathbf{z} \in \mathcal{E}_\rho} \left\| f \right\|_{V(D) L(E_{\rho})}}{p^k} \prod_{1 \leq i \leq d} D_{\rho_i}^{(\frac{1}{2})} \prod_{1 \leq j \leq d} D_{\rho_j}^{(\frac{1}{2})},
\]  
(5.4)

where the constants \( D_{\rho_i}^{(\frac{1}{2})}, D_{\rho_j}^{(\frac{1}{2})} \) are defined in \((2.15)\) and \((2.16)\), respectively.

**Proof.** Since the proof is similar to that of Theorem 3.1, we only sketch the proof of (5.3). Thanks to the analytic dependence of \( y \), as in the derivation of (3.10), we obtain from Cauchy’s integral formula that
\[
a_k(x) = \left( \frac{1}{\pi} \right)^d \oint_{\mathcal{E}_\rho} f(x, z) Q^{(\frac{1}{2})}_k(z) dz.
\]  
(5.5)

Hence, it is readily seen that
\[
\left\| a_k \right\|_{V(D)} \leq \frac{\sup_{\mathbf{z} \in \mathcal{E}_\rho} \left\| f \right\|_{V(D) L(E_{\rho})}}{\pi^d} \frac{\max_{\mathbf{z} \in \mathcal{E}_\rho} \left\| Q^{(\frac{1}{2})}_k(z) \right\|}{d^d} \prod_{i=1}^d \max_{z_i \in \mathcal{E}_{\rho_i}} \left\| Q^{(\frac{1}{2})}_{\rho_i}(z_i) \right\|.
\]  
(5.6)

This, together with Proposition 2.2, gives us (5.3).

This completes the proof of Proposition 5.1. \( \square \)

As an application of the above proposition, let us consider a family of elliptic PDEs of the form
\[
\begin{cases} 
- \nabla \cdot (a(x, y) \nabla u(x, y)) = f(x), & \forall (x, y) \in D \times \Gamma, \\
u(x, y) = 0, & \forall (x, y) \in \partial D \times \Gamma,
\end{cases}
\]  
(5.7)

where \( D \subset \mathbb{R}^n \) is a bounded Lipschitz domain with \( n \in \{1, 2, 3\} \), the diffusion coefficient \( a(x, y) \) is a function of \( x \) and of parameters \( y = \{y_1, \ldots, y_d\} \in \Gamma = \Omega_d \), and the function \( f \) is a fixed function on \( D \). The gradient operator \( \nabla \) is taken with respect to \( x \). It is assumed that \( a \) and \( f \) are chosen such that the system (5.7) is well-defined in the Sobolev space \( V(D) := H^1_0(D) \) equipped with the energy norm \( \| \cdot \|_{V(D)} := \| \nabla (\cdot) \|_{L^2(D)} \). Parameterized linear elliptic PDEs of this type arise in a variety of stochastic and deterministic modeling of complex systems; cf. \[12, 14].

Since the solution of (5.7) depends smoothly on the coefficient \( a \), a major method to find it is based on a polynomial approximation, which leads to an approximation to the solution \( u \) of the form
\[
uu(x, y) = \sum_{k \in \Lambda} c_k(x) \Psi_k(y),
\]  
(5.8)
where $\Lambda \in \mathbb{N}^d_0$ is a finite index set, $\Psi_k(y)$ is a multivariate polynomial, and $c_k(x) \in V(D)$ is the coefficient to be computed. Suppose that $\|\Psi_k(y)\|_{L^\infty(\Gamma)} = 1$, the error of the approximation (5.8) can be bounded by

$$\sup_{y \in \Gamma} \left\| u(x, y) - u_{\Lambda}(x, y) \right\|_{V(D)} \leq \sum_{k \in \Lambda^c} \|c_k(x)\|_{V(D)},$$

(5.9)

where $\Lambda^c$ denotes the complement of $\Lambda$ in $\mathbb{N}^d_0$.

In practice, the polynomials $\Psi_k(y)$ are often chosen to be the monomials or the tensorized Legendre polynomials (cf. [2, 8, 19]), which correspond to Taylor and Legendre approximations, respectively. For the latter case, both the Legendre and the normalized Legendre expansions, i.e.,

$$u(x, y) = \sum_{k \in \mathbb{N}^d_0} u_k(x) P_k(y) = \sum_{k \in \mathbb{N}^d_0} v_k(x) \tilde{P}_k(y),$$

(5.10)

have been discussed. In view of the truncation error given in (5.9), an effective way of computation requires using the multi-index set largest the norm of the coefficients among all the multi-index sets with fixed cardinality. This is usually a difficult task in implementation. Alternatively, one could relax the condition by performing the so-called quasi-optimal approximation, that is, the multi-index set is chosen so that the upper bounds of the coefficients are maximized. A general strategy for convergence analysis of quasi-optimal polynomial approximations for parameterized PDEs (5.7) was presented in [19], and a key ingredient of the analysis therein is the upper bounds of the Legendre coefficients $u_k(x)$ and $v_k(x)$ given in (5.10). In what follows, we will provide sharper bounds of $u_k(x)$ and $v_k(x)$ with the aid of Proposition 5.1, which improve those used in [19]; see also [2, 8].

Following the framework proposed in [19], we make the following two assumptions on the diffusion coefficient $a$ in (5.7).

- There exist two positive constants $0 < a_{\min} < a_{\max} < \infty$ such that for all $x \in D$ and $y \in \Gamma$,

$$a_{\min} \leq a(x, y) \leq a_{\max}.$$  

(5.11)

- The complex continuation $a(x, z)$ of $a(x, y)$ is a $L^\infty(D)$-valued holomorphic function on $\mathbb{C}^d$.

**Proposition 5.2.** Assume that the coefficient $a(x, y)$ in the parameterized PDEs (5.7) satisfies the above two assumptions. If we further require that $\Re(a(x, z)) \geq \delta$ for some $0 < \delta < a_{\min}$, $x \in \overline{D}$ and $z \in \mathcal{E}_\rho$ with $\rho_j > 1$ for each $j = 1, \ldots, d$. Then, the coefficients of tensorized Legendre expansions of $u$ given in (5.10) admit the following estimates.

$$\|u_k(x)\|_{V(D)} \leq \hat{C}_{\rho, \delta} \rho^{-k} \prod_{k_j \neq 0} \sqrt{k_j}, \quad \|v_k(x)\|_{V(D)} \leq \hat{C}_{\rho, \delta} 2^{\frac{R(k)}{2}} \rho^{-k},$$

(5.12)

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where
\[ \hat{C}_{\rho, \delta} = \frac{\|f\|_{V^*(D)}}{\delta} \left( \prod_{j=1}^{d} \frac{L(E_{\rho_j})}{\pi} \right) \prod_{k=0}^{d} \prod_{k_j \neq 0} D_{\rho_j}^{(k)} \]
with \( V^*(D) \) being the dual of the space \( V(D) \).

Proof. By [19, Theorem 1], it follows that the conditions satisfied by \( a(x, y) \) ensure that \( z \to u(x, z) \) is analytic in an open neighborhood of \( E_{\rho} \) and this solution satisfies a priori estimate
\[ \|u\|_{V(D)} \leq \frac{\|f\|_{V^*(D)}}{\delta}. \] (5.13)
As a consequence, the solution \( u \) satisfies the conditions of Proposition 5.1, and the estimates (5.12) follow directly from (5.3), (5.4) and (5.13).

Remark 5.3. The following estimates of \( \|u_k(x)\|_{V(D)} \) and \( \|v_k(x)\|_{V(D)} \) are reported in [19, Proposition 2]:
\[ \|u_k(x)\|_{V(D)} \leq C_{\rho, \delta} \rho^{-k} \prod_{j=1}^{d} (2k_j + 1), \quad \|v_k(x)\|_{V(D)} \leq C_{\rho, \delta} \rho^{-k} \prod_{j=1}^{d} \sqrt{2k_j + 1}, \] (5.14)
where \( C_{\rho, \delta} = \frac{\|f\|_{V^*(D)}}{\delta} \prod_{j=1}^{d} \frac{L(E_{\rho_j})}{\pi} \), where \( C \) is some constant independent of the multi-index \( k \). A comparison of (5.14) and (5.12) shows our results (5.12) are sharper, especially when \( k_j \to +\infty \) for some \( j \in \{1, \ldots, d\} \).

6 Concluding remarks

In this paper, we have derived some new and sharper bounds for the coefficients of multivariate Gegenbauer expansion based on two different extensions of the Bernstein ellipse. These bounds also allow us to establish an explicit error bound for the multivariate Gegenbauer approximation associated with an \( \ell^q \) ball index set in the uniform norm. For isotropic functions, the predicted rates of convergence agree well with the empirical rates observed in the numerical experiments. Moreover, our analysis suggests that the multivariate Gegenbauer approximation based on the index set \( \Lambda_N^2 \) is an optimal choice among that of the \( \ell^q \) ball index set, provided that the convergence rate established in Theorem 4.1 is sharp. As an application, we improve the estimates of the coefficients of tensorized Legendre expansions arising from polynomial approximation for a family of parameterized PDEs.

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A Proof of Proposition 2.2

By (2.11) and (2.12), it is readily seen that if \( z \in \mathcal{E}_\rho \),

\[
\left| Q_n^{(\rho)}(z) \right| \leq \left| c_{n,\lambda} \right| \max_{z \in \mathcal{E}_\rho} \left| 2F_1 \left[ \begin{array}{c} n + 1, 1 - \lambda; \\ n + \lambda + 1; \frac{1}{(z \pm \sqrt{z^2 - 1})^2} \end{array} \right] \right| \\
= \left| c_{n,\lambda} \right| \max_{u \in \mathcal{C}_\rho} \left| 2F_1 \left[ \begin{array}{c} n + 1, 1 - \lambda; \\ n + \lambda + 1; \frac{1}{u^2} \end{array} \right] \right|,
\]

(A.1)

where \( \mathcal{C}_\rho \) is the circle \(|u| = \rho\).

We now estimate the two absolute values on the right hand side of the above formula, respectively. On one hand, if \( n = 0 \), it follows from (2.13) that

\[
c_{0,\lambda} = 1.
\]

(A.2)

If \( n \geq 1 \), we see from [29, Lemma 2.1] that

\[
|c_{n,\lambda}| \leq \Gamma(\lambda) \exp \left( \frac{1 - \lambda}{2(n + \lambda - 1)} + \frac{1}{12n} \right) n^{1-\lambda} \\
\leq \Gamma(\lambda)n^{1-\lambda} \times \begin{cases} 
\exp \left( \frac{1}{12} \right), & \lambda \geq 1, \\
\exp \left( \frac{1}{12} + \frac{1-\lambda}{2\lambda} \right), & 0 < \lambda < 1.
\end{cases}
\]

(A.3)

On the other hand, the Gauss hypergeometric function admits the following Euler integral representation (cf. [1, Theorem 2.2.1])

\[
2F_1 \left[ \begin{array}{c} n + 1, 1 - \lambda; \\ n + \lambda + 1; \frac{1}{u^2} \end{array} \right] = \frac{\Gamma(n + \lambda + 1)}{\Gamma(n + 1)\Gamma(\lambda)} \int_0^1 t^n(1-t)^{\lambda-1} \left( 1 - \frac{t}{u^2} \right)^{\lambda-1} dt,
\]

(A.4)

which is valid for \( \lambda > 0 \). Hence, it is straightforward to check that

\[
\max_{u \in \mathcal{C}_\rho} \left| 2F_1 \left[ \begin{array}{c} n + 1, 1 - \lambda; \\ n + \lambda + 1; \frac{1}{u^2} \end{array} \right] \right| \\
= \begin{cases} 
2F_1 \left[ \begin{array}{c} n + 1, 1 - \lambda; \\ n + \lambda + 1; \frac{1}{\rho^2} \end{array} \right], & 0 < \lambda < 1, \\
2F_1 \left[ \begin{array}{c} n + 1, 1 - \lambda; \\ n + \lambda + 1; \frac{1}{\rho^2} \end{array} \right], & \lambda \geq 1,
\end{cases}
\]

\[
= \begin{cases} 
\frac{\Gamma(n + \lambda + 1)}{\Gamma(n + 1)\Gamma(\lambda)} \int_0^1 t^n(1-t)^{\lambda-1} \left( 1 - \frac{t}{\rho^2} \right)^{\lambda-1} dt, & 0 < \lambda < 1, \\
\frac{\Gamma(n + \lambda + 1)}{\Gamma(n + 1)\Gamma(\lambda)} \int_0^1 t^n(1-t)^{\lambda-1} \left( 1 + \frac{t}{\rho^2} \right)^{\lambda-1} dt, & \lambda \geq 1,
\end{cases}
\]

\[
\leq \begin{cases} 
(1 - \rho^{-2})^{\lambda-1}, & 0 < \lambda < 1, \\
(1 + \rho^{-2})^{\lambda-1}, & \lambda \geq 1.
\end{cases}
\]

(A.5)
A combination of (A.2), (A.3) and (A.5) then gives us (2.14).

Finally, to see the bound is optimal, we note from [18, Equation (15)] that as \( n \to \infty \),

\[
\begin{align*}
2F_1 \left[ \frac{n + 1}{n + \lambda + 1}, \frac{1}{u^2} \right] &= \left( 1 - \frac{1}{u^2} \right)^{\lambda - 1} \left[ 1 + O(n^{-1}) \right], \quad u \neq 1. \quad (A.6)
\end{align*}
\]

Moreover, the ratio asymptotics of Gamma functions (cf. [15, Equation 5.11.12])

\[
\frac{\Gamma(n + a)}{\Gamma(n + b)} \sim n^{a - b}, \quad n \to \infty,
\]

implies that

\[
e_{n, \lambda} = O(n^{1-\lambda}), \quad (A.7)
\]
as \( n \to \infty \). This, together with (A.6), yields that for large \( n \),

\[
\left| Q_n^{(\lambda)}(z) \right| = O(n^{1-\lambda} \rho^{-n-1}), \quad z \in E_\rho.
\]

Comparing the above formula with (2.14), it is clear that, up to some constant factor, the bound (2.14) is optimal in the sense that it can not be improved in any lower power of \( n \).

This completes the proof of Proposition 2.2. \( \Box \)

Remark A.1. If \(-1/2 < \lambda < 0\), we still have

\[
\max_{u \in \mathbb{C}_\rho} \left| 2F_1 \left[ \frac{n + 1}{n + \lambda + 1}, \frac{1}{u^2} \right] \right| = 2F_1 \left[ \frac{n + 1}{n + \lambda + 1}, \frac{1}{\rho^2} \right]. \quad (A.8)
\]

This follows from the fact that the coefficients of \( z^k \) in \( 2F_1 \left[ \frac{n + 1}{n + \lambda + 1}, \frac{1}{u^2} \right] \) are non-negative for \(-1/2 < \lambda < 0\) and the triangle inequality. In this case, the integral representation (A.4) for the hypergeometric function, however, is not valid anymore. Thus, in view of (A.6) and (A.7), it is expected a result similar to (2.14) still holds for \(-1/2 < \lambda < 0\), but with an explicit bound unknown.

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