Self-Organized Criticality: Self-Organized Complexity?
The Disorder and “Simple Complexity” of Power Law Distributions

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The disorder and a simple convex measure of complexity are studied for rank ordered power law distributions, indicative of criticality, in the case where the total number of ranks is large. It is found that a power law distribution may produce a high level of complexity only for a restricted range of system size (as measured by the total number of ranks), with the range depending on the exponent of the distribution. Similar results are found for disorder. Self-organized criticality thus does not guarantee a high level of complexity, and when complexity does arise, it is self-organized itself only if self-organized criticality is reached at an appropriate system size.

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Self-organized criticality has been maintained to be “so far the only known general mechanism to generate complexity”. Actually, self-organized criticality says nothing about complexity itself directly. Rather, it can be taken as defined by two properties. The first is that of the critical state, indicated by a power law probability distribution. The second, self-organization, is that the critical state is reached spontaneously, without the influence of any external agent, to set parameter values for example. The question asked here is whether a power law distribution necessarily implies complexity.

Power law distributions usually occur in one of two forms. In the first the probability \( p(m) \) of an event of magnitude \( m \) is expressed as a power of the magnitude itself: \( p(m) \propto 1/m^\gamma \), where \( \gamma \) is a nonnegative constant. Examples include the Gutenberg-Richter law relating the frequency of earthquakes to the energy released, fluctuations in the stock and commodities markets, and the number of extinctions throughout biological evolution. It should be noted that some examples are not uncontroversial; for example, there are alternative explanations for the frequencies of biological extinctions.

In the second common form of power law distributions the distributions are rank \( r \) ordered. The most frequent or probable event or state has rank 1; the second most probable, rank 2; etc. The distribution is then written \( p(r) \propto 1/r^\gamma \). The first well known example was Zipf’s law for the distribution of word frequencies in the English language; others are the distribution of city populations and various aspects of the “web”.

Since the rank ordered form of power law distributions is the marginally appropriate maximum entropy. This measure for disorder has been applied to problems ranging from cosmology to biology, and can be generalized easily to measures based on higher order Rényi entropies, which are related to multifractals.

Based on the Landsberg “disorder” we have proposed a “simple measure of complexity” whose behavior encompasses both the monotonic and convex categories:

\[
\Delta = S/S_{\text{max}}, \quad S = -\sum_{i=1}^{N} p_i \ln p_i, \quad (1)
\]

where \( p_i \) is the probability of state \( i \) of \( N \) possible states, and \( S \) is therefore the information entropy. \( S_{\text{max}} \) is the appropriate maximum entropy. This measure for disorder has been applied to problems ranging from cosmology to biology, and can be generalized easily to measures based on higher order Rényi entropies, which are related to multifractals.

When \( \alpha > 0 \) and \( \beta \) vanishes, \( \Gamma_{\alpha \beta} \) is a monotonic complexity measure; when \( \alpha > 0, \beta > 0 \), \( \Gamma_{\alpha \beta} \) is a convex measure. Since the results here will not depend on the particular dependence of complexity on disorder, but rather only on whether the dependence is monotonic or convex, in the following only the simplest cases will be considered:

\[
\alpha = 1, \quad \beta = 0: \quad \Gamma_{10} = \Delta; \text{ monotonic}; \quad (3)
\]

\[
\alpha = 1, \quad \beta = 1: \quad \Gamma_{11} = \Delta(1-\Delta) \equiv \Gamma; \text{ convex}.
\]

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For convenience these will be referred to simply as “disorder” and “complexity”, respectively.

It is important to understand that $\Delta$ is not equivalent to the entropy $S$ and that, therefore, $\Gamma$ is not simply a function of entropy. To say that $\Gamma$ is a function of $S$ alone is to misunderstand the reason for introducing $\Delta$ as an alternative to entropy as a measure of disorder in the first place [4, 15]. As $S$ varies, $S_{\text{max}}$ will in general also vary, but not proportionally to $S$, so that $\Delta$ will in general change in a manner distinct from the variation in $S$. $\Delta$ may even decrease as $S$ increases. Both $S$ and $S_{\text{max}}$ will generally change (nonproportionally) in several scenarios, including changes in system size, changes in the level of fine graining and changes in boundary conditions. Moreover, there may be several choices for $S$ and $S_{\text{max}}$ for a given system. Which ones are appropriate will depend on the questions being addressed. In the simplest cases $S_{\text{max}}$ can be taken to be the entropy of the equiprobable (all $p_i = 1/N$) distribution, $\ln N$. For nonequilibrium systems, it may be more appropriate to take $S_{\text{max}}$ to be the entropy of the corresponding equilibrium system, i.e. the equilibrium system with the same total energy, number of particles, etc. [27]. Even for a simple one-dimensional Ising spin system, several entropies and therefore several “disorders” can be introduced [26, 30]. Furthermore, multiple choices for $S$ and $S_{\text{max}}$ imply multiple $\Delta$’s and multiple $\Gamma$’s. When one considers that $\Gamma$ calculated from $\Delta$ is determined by $\Delta$. This is appropriate for distinguishing between complexity measures which increase monotonically with disorder and those which show a convex dependence on disorder, one purpose of this study.

The entropy, “disorder” and “complexity” of the rank ordered power law distribution, $p(r) \propto 1/r^\gamma$, with maximum rank $R$ are now

$$p(r) = 1/(\langle r \rangle^\gamma), \quad \sum_{r=1}^{R} p(r) = 1 \Rightarrow \Xi = \sum_{r=1}^{R} 1/(\langle r \rangle^\gamma),$$

$$S = -\sum_{r=1}^{R} p(r) \ln p(r) = \ln \Xi + (\gamma/\Xi) \sum_{r=1}^{R} (\ln r)/r^\gamma, \quad (4)$$

$$S_{\text{max}} = \ln R, \quad \Delta = S/S_{\text{max}}, \quad \Gamma = \Delta (1 - \Delta). \quad (5)$$

$R$ may have different interpretations. The most straightforward is that of $R$ as the maximum observed rank. $R$ may also be taken to be the size of the system, in the sense that a larger system has more states, i.e. more ranks. Finally, $R$ may be interpreted as a measure of the extent of fine graining. Examples of power law distributions are often presented as histograms, e.g. the number of biological extinctions $< 10\%$, between 10 and 20%, . . . [2]. If the distribution is more refined, say $< 1\%$, between 1 and 2%, . . ., then $R$ obviously increases. In the limit $R \rightarrow \infty$ we have the fine grained or thermodynamic limit.

Since the latter limit is of particular importance, I will now consider the case of large $R$. Replacing sums by integrals, we have [31]

$$\Xi = \left\{ \begin{array}{ll} p = \frac{1}{\Xi r^\gamma}, & \int_{1}^{R} p \, dr = 1 \\
\ln R, & R \rightarrow \infty \Rightarrow \Xi = \left\{ \begin{array}{ll} 1, & \gamma = 1, \\
\ln R, & \gamma \neq 1, \end{array} \right. \\
\left[\frac{R^{(1-\gamma)/\gamma}}{(1-\gamma)}\right] - \frac{\ln R}{1-\gamma}, & \gamma = 1, \gamma \neq 1, \gamma \neq 1, \end{array} \right. \quad (6)$$

The dependence of “disorder” and “complexity” on $R$ is shown in Fig. 3. The first point to note is

$$\lim_{R \rightarrow \infty} \Delta = \left\{ \begin{array}{ll} 1, & \gamma < 1 \\
1/2, & 1/2 < \gamma = 1 \\
0, & \gamma > 1 \end{array} \right., \quad \lim_{R \rightarrow \infty} \Gamma = \left\{ \begin{array}{ll} 0, & \gamma < 1 \\
1/4, & 1/4 < \gamma = 1 \\
0, & \gamma > 1 \end{array} \right.. \quad (6)$$

In the fine grained limit, the system is either completely disordered ($\gamma < 1$) or completely ordered ($\gamma > 1$), with the one exception of $\gamma = 1$, where the system is half maximally disordered (half maximally ordered). In the thermodynamic limit “complexity” vanishes except for the special case $\gamma = 1$, where “complexity” has its maximum possible value (with the choice of $\alpha = \beta = 1$). In this limit self-organized criticality does not imply “complexity” in general: “complexity” itself vanishes except for $\gamma = 1$, and “disorder” vanishes except for $\gamma \leq 1$.

For $R$ large, but $R < \infty$, significant points to be noted are:

- For $\gamma < 1$, $\Delta$ goes through a relative minimum and $\Gamma$ through a relative maximum as $R$ increases;
- For $\gamma > 1$, $\Delta$ decreases monotonically with $R$, and $\Gamma$ first increases to its absolute maximum to then vanish in the limit $R \rightarrow \infty$.

Values of $\gamma$ other than 1 may yield nonvanishing values of “complexity”, but only for some range of intermediate values of $R$. It should be noted that these results are not limited to the particular forms for disorder and complexity used here. Qualitatively the complexity results apply
for all convex complexity measures, since they all vanish for minimum and maximum disorder. Similarly, the disorder results are valid qualitatively for all monotonic complexity measures.

Even if the power law probability distribution of self-organized criticality does lead to a high level of “complexity”, this does not imply that the “complexity” itself is self-organized. For self-organized complexity, the system must not only evolve to a power law probability distribution spontaneously, it must also do so so that $\gamma$ is exactly 1 in the limit $R \to \infty$. Or, if $R$ is large but finite, the system must spontaneously evolve to a size for which the complexity is large for the operative value of $\gamma$. Thus self-organized complexity requires much more stringent conditions than self-organized criticality. Whether the many systems which are known to be examples of self-organized criticality also show self-organized complexity remains to be seen.

Two additional points — (1) Mandelbrot’s finding, that a monkey pounding on a typewriter at random produces a rank ordered power law distribution for the “words” typed, is consistent with the result of maximum “disorder” and vanishing “complexity” in the fine grained limit for $\gamma < 1$. (2) Bak alludes to the “blind watchmaker” argument of Dawkins. This is essentially an anti-vitalist argument, namely that the order and complexity produced by evolution apparent to us do not require the intervention of an external agent, i.e. “divine” intervention. In the case of self-organized criticality the argument is that nature can produce criticality and therefore complexity without the intervention of an external agent. However, self-organized criticality produces a high level of “complexity” in the thermodynamic limit only for $\gamma = 1$, or for $R$ large but finite, only for certain system sizes. For the “blind watchmaker” argument to apply to “complexity”, one must show not only that criticality is reached in a self-organized fashion but also that it is reached at an appropriate system size and $\gamma$. Otherwise, if the system size necessary for a high level of “complexity” is not reached spontaneously but must be set by an outside agent, the “watchmaker” must have sufficient visual acuity to do so.

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the discrete case can be represented exactly in terms of the Riemann zeta function, \( \zeta(x) \equiv \lim_{R \to \infty} \sum_{r=1}^{R} 1/r^x \):

\[
S = \ln \zeta(\gamma) - \gamma \left[ \partial \zeta(\gamma) / \partial \gamma \right] / \zeta(\gamma).
\]

This equation yields the same results for \( \Delta \) as does the continuous approximation of eq. (2). [32] B. Mandelbrot, in Structures of Language and Its Mathematical Aspects, edited by R. Jacobson (American Mathematical Society, New York, 1961). (See also [34].)

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FIG. 1. Classification of complexity measures based on dependence on disorder. Left: complexity increases monotonically with disorder; right: complexity displays a convex dependence on disorder.

FIG. 2. The dependence of “disorder” and “complexity” on the maximum rank \( R \). The curves are labeled with the values of the exponent \( \gamma \).
Fig. 1
Fig. 2