Some Continuous Analogs of Expansion in Jacobi Polynomials and \nVector Valued Hypergeometric Orthogonal Bases

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Vienna, Preprint ESI 1366 (2003)

September 29, 2003

Supported by the Austrian Federal Ministry of Education, Science and Culture
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Some continuous analogs of expansion in Jacobi polynomials and vector valued hypergeometric orthogonal bases

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1. Representation-theoretic motivation and formulation of results.

1.1. Continuous analogs of expansion in Jacobi polynomials. The present work is a counterpart on the level of special functions of Molchanov’s paper [14] on tensor products of unitary representations of the group SL₂(ℚ).

In the classical analysis, there are well-known expansions in Jacobi polynomials and also its continuous analogue. The latter is the index hypergeometric transform of H. Weyl, [28] (it is also called by Oleksy transform [21], Jacobi transform, generalized Fourier transform, see Koornwinder’s survey [10], see also [5], [9], [17]). These classical constructions have a transparent and important representation-theoretic interpretation (see [27]). Spherical functions of the projective spaces

\begin{align}
O(n+1)/O(n) \times O(1), \quad U(n+1)/U(n) \times U(1), \quad Sp(n+1)/Sp(n) \times Sp(1)
\end{align}

(1.1)

over ℜ, ℂ and the quaternion field H are the Jacobi polynomials \( P^\alpha_\beta \) for some special values of the parameters \( \alpha, \beta \). The problem of decomposition of \( L^2 \) on these spaces into a direct sum of irreducible representations is a corollary of the theorem on expansion of a function into a series in Jacobi polynomials.

For hyperbolic spaces

\begin{align}
O(n,1)/O(n) \times O(1), \quad U(n,1)/U(n) \times U(1), \quad Sp(n,1)/Sp(n) \times Sp(1)
\end{align}

(1.2)

the analogy of an expansion in the Jacobi polynomials is the index hypergeometric transform (see [21], [10])

\begin{align}
g(s) = \frac{1}{\Gamma(b+c)} \int_0^\infty f(x) \frac{1}{2} F_1(b+is, b-is; b+c; -x) x^{b+c-1}(1+x)^{b-c} dx
\end{align}

(1.3)

The problem of decomposition of \( L^2 \) on these spaces is reduced to the inversion formula

\begin{align}
f(x) = \frac{1}{\pi \Gamma(b+c)} \int_0^\infty g(s) \frac{1}{2} F_1(b+is, b-is; b+c; -x) \left( \frac{(b+is)(c+is)}{\Gamma(2is)} \right)^2 ds
\end{align}

for this integral transform.

Nevertheless, these two classical constructions (i.e., the expansion in the Jacobi polynomials and the index hypergeometric transform are not sufficient in the analysis on pseudoriemannian symmetric spaces of rank 1 (this class of spaces includes, in particular, other real forms of the spaces (1.1)-(1.2)). This
force to think that there exists some another analog (or analogs) of the index
hypergeometric transform (1.3). We construct one such transform\(^1\). 2.

1.2. The problem on vector-valued bases. The Askey–Wilson hierar-
chy of hypergeometric orthogonal polynomials is well known, see [8], [1], chapter
6. Almost all these polynomials (probably all) appear in the representation the-
ory of the group \(\text{SL}_2(\mathbb{R})\) (see, for instance, [27], [24]–[25]).

Consider the tensor product of a unitary highest weight representation of
\(\text{SL}_2(\mathbb{R})\) and a lowest weight representation. In each factor, there is a cano-
ical orthogonal basis consisting of \(\text{SO}(2)\)-eigenfunctions. Hence there exists a
canonical basis in the tensor product. This basis consists of continuos dual Hahn
polynomials ([29]). Recall (see [1], 6.10, [8]) that they are the polynomials
\[
p_n(s^2) = (a + b)_n(a + c)_n (a + is)(b + is)(c + is) \frac{\Gamma(2is)}{\Gamma(2is)} ds
\]
on the line \(s \in \mathbb{R}\); the explicit formula is
\[
p_n(s^2) = (a + b)_n(a + c)_n \mathbf{3}F_2\left[\begin{array}{c}
-n, a + is, a - is \\
a + b, a + c
\end{array}; 1\right].
\]

But the same problem has sense for each pair of unitary representations \(\text{SL}_2(\mathbb{R})\).
These tensor products (generally) have multiplicity 2 (see [20], [14]). Hence, we
must obtain some orthogonal bases consisting of \(\mathbb{C}^2\)-valued functions.

1.3. Notations. We use the standard notations for the Pochhammer sym-
bol \((a)_n = a(a + 1) \ldots (a + n - 1)\) and for hypergeometric functions
\[
\begin{align*}
\mathbf{2}F_1(a, b; c; z) & : = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n; \\
\mathbf{3}F_2(a, b; c; d, e; z) & : = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!} z^n.
\end{align*}
\]

1.4. Double index hypergeometric transform. Fix \(0 \leq \alpha \leq 1/2,
\beta \in \mathbb{R}\). Assume that \(\alpha + i\beta \neq 0\).

\(^1\)Molchanov [14], [15] [16] uses another method. Below we introduce the differential
operator (2.1). It arises if we restrict Laplace operator in tensor product of unitary representations
of \(\text{SL}_2(\mathbb{R})\) to eigenspaces of the rotation group. Molchanov [14] restrict the Laplace operator
to functions that are invariant with respect to the group of diagonal matrices. As a result, he
obtains the Legendre differential operator on some contour containing a singular point of the
Legendre equation.

\(^2\)It is interesting to understand, is it possible to give another proof of the Molchanov’s
Plancherel formula [15] on rank 1 symmetric spaces \(G/H\) using the classical index hyper-
geometric transform (in the variant [3], XIII or [5]) and the double index hypergeometric
transform constructed below in 1.4. Precisely, let \(K\) be a maximal compact subgroup in
\(G\), let \(V\) be an irreducible \(K\)-module. Consider the action of the Laplace operator on
\(\text{Hom}_K(V, L^2(G/H))\). Is it correct, that in all the cases we will obtain the hypergeometric
operator (2.3) on the segment \([0, 1]\) (the classical case) or on the contour \(\text{Re} z = 1/2\) (our
case)?
Consider the space of $\mathbb{C}^2$-valued functions of the half-line $s > 0$. An element of this space can be considered as a pair of scalar-valued functions $(\varphi_1(s), \varphi_2(s))$. Let us introduce the scalar product in this space by the formula

$$
\langle (\varphi_1, \varphi_2), (\psi_1, \psi_2) \rangle = \frac{1}{2\pi} \int_0^\infty \left[ r_{11}(s)\varphi_1(s)\overline{\psi_1(s)} + r_{12}(s)\varphi_1(s)\overline{\psi_2(s)} + r_{21}(s)\varphi_2(s)\overline{\psi_1(s)} + r_{22}(s)\varphi_2(s)\overline{\psi_2(s)} \right] ds \frac{ds}{|\Gamma(2is)|^2},
$$

where $r_{ij}(s)$ are given by

$$
R(s) = \begin{pmatrix} r_{11}(s) & r_{12}(s) \\ r_{21}(s) & r_{22}(s) \end{pmatrix} := \begin{pmatrix} \Gamma\left(\frac{1}{2} + i\beta - is\right) \Gamma\left(\frac{1}{2} - i\beta + is\right) & \Gamma\left(\frac{1}{2} - i\beta - is\right) \Gamma\left(\frac{1}{2} + i\beta + is\right) \\ \Gamma\left(\frac{1}{2} + i\beta - is\right) \Gamma\left(\frac{1}{2} + i\beta + is\right) & \Gamma\left(\frac{1}{2} - i\beta + is\right) \Gamma\left(\frac{1}{2} + i\beta - is\right) \end{pmatrix}.
$$

It is convenient to write this scalar product in the vector form

$$
\langle (\varphi_1, \varphi_2), (\psi_1, \psi_2) \rangle = \frac{1}{2\pi} \int_0^\infty (\varphi_1(s) \varphi_2(s)) R(s) \left( \begin{array}{c} \psi_1(s) \\ \psi_2(s) \end{array} \right) \frac{ds}{|\Gamma(2is)|^2}.
$$

Thus we obtain the Hilbert space of $\mathbb{C}^2$-valued functions. We denote it by $H_{\alpha, \beta}$.

Let $x \in \mathbb{R}$, $s \geq 0$. Consider two functions $Q_1(\alpha, \beta; x, s)$, $Q_2(\alpha, \beta; x, s)$, given by the formulae

$$
Q_1(\alpha, \beta; x, s) = \frac{1}{\Gamma(\alpha + i\beta)} \left( \frac{1}{2} + ix \right)^{-\alpha - i\beta/2} \left( \frac{1}{2} - ix \right)^{\alpha + i\beta/2 - is - 1/2} \times _2F_1\left[ \begin{array}{c} 1/2 - \alpha + is, 1/2 - i\beta + is \\ 1 - \alpha - i\beta \end{array} ; \frac{ix + 1/2}{ix - 1/2} \right],
$$

$$
Q_2(\alpha, \beta; x, s) = Q_1(-\alpha, -\beta; x, s).
$$

For a function $f \in L^2(\mathbb{R})$, we define the pair of functions $\varphi_1(s), \varphi_2(s)$ by

$$
\varphi_j(s) := \int_{-\infty}^\infty f(x)Q_j(\alpha, \beta; x, s) dx, \quad j = 1, 2.
$$

**Theorem 1.1.** Let $0 \leq 1/2 < \alpha, \alpha + i\beta \neq 0$.

a) The operator $f \mapsto (\varphi_1, \varphi_2)$ is a unitary operator from $L^2(\mathbb{R})$ to $H_{\alpha, \beta}$.

b) The inversion formula is

$$
f(x) = \frac{1}{2\pi} \int_0^\infty (Q_1(\alpha, \beta; x, s) Q_2(\alpha, \beta; x, s)) R(s) \left( \begin{array}{c} \varphi_1(s) \\ \varphi_2(s) \end{array} \right) \frac{ds}{|\Gamma(2is)|^2}.
$$

**1.5. Generalization for $\alpha > 1/2$.** Now, let $\alpha > 1/2$, $\beta \in \mathbb{R}$, and $\alpha + i\beta \neq 0, 1, 2, \ldots$. Denote by $n$ the integral part of $\alpha - 1/2$. Consider the finite
dimensional linear space $W_{\alpha,\beta}$, consisting of vectors $(c_0, c_1, \ldots, c_n)$. The scalar product in $W_{\alpha,\beta}$ is given by

$$\langle c, c' \rangle = \frac{1}{2\pi} \sum_{k=0}^{n} \frac{2\alpha - 2k - 1}{(2\alpha - k)k!} c_k c'_k.$$

Next, define the functions

$$R(\alpha, \beta; x; k) := \frac{1}{\Gamma(\alpha + i\beta)} \left(\frac{1}{2} + ix\right)^{-\alpha-i\beta/2} \left(\frac{1}{2} - ix\right)^{-\alpha+i\beta/2} \, _2F_1\left[-k, k - 2\alpha + 1; 1 - \alpha - i\beta; \frac{1}{2} + ix\right].$$

Now, consider the linear operator

$$J_{\alpha,\beta} : L^2(\mathbb{R}) \to H_{\alpha,\beta} \oplus W_{\alpha,\beta},$$

given by

$$f \mapsto (\varphi_1, \varphi_2, \theta),$$

where $\varphi_1, \varphi_2$ is defined by (1.7) as above, and the coordinates of the vector $\theta \in W_{\alpha,\beta}$ have the form

$$\theta_k = \int_{-\infty}^{\infty} f(x) R(\alpha, \beta; x; k) \, dx.$$

**Theorem 1.2.** The operator $J_{\alpha,\beta}$ is unitary.

**1.6. Romanovski polynomials.** The Romanovski polynomials [23] are the polynomials on $\mathbb{R}$ orthogonal with respect to the weight

$$\left(\frac{1}{2} + ix\right)^{-\alpha-i\beta/2} \left(\frac{1}{2} - ix\right)^{-\alpha+i\beta/2} \, dx.$$

This weight decreases as $x^{-2\alpha}$, and hence it is possible to orthogonalize only finite number of power functions $1, x, x^2, \ldots$. Romanovski polynomials are defined by the formula

$$\, _2F_1[-k, k - 2\alpha + 1; 1 - \alpha - i\beta; \frac{1}{2} + ix],$$

i.e., they coincide with (1.9) up to an elementary factor.

Recall that the Jacobi polynomials are given by

$$P_n^{\gamma,\delta}(x) = \text{const} \cdot \, _2F_1[-n, n + \gamma + \delta; \frac{1}{2} + (1 + x)/2].$$

We observe that the Romanovski polynomials are analytic continuations of the Jacobi polynomials with respect to the superscripts.

Many orthogonal systems of this kind are known, see [2], [11]–[13], [18]. Since these systems are eigenfunctions of some second order differential or difference operators, we obtain a good collection of spectral problems (and hence this
must give a collection of new integral transforms). Recently, one such problem related a family of $F_3$-polynomials from [18] was solved by Groenevelt [6].

1.7. Vector-valued bases. Fix parameters $0 < \alpha < 1/2$, $\beta, p, q \in \mathbb{R}$. We consider the Hilbert space $Y(\alpha, \beta; q)$, consisting of $C^2$-valued functions on half-line $s \geq 0$ with the scalar product

$$
\langle (\varphi_1, \varphi_2), (\psi_1, \psi_2) \rangle =
\frac{1}{2} \int_0^\infty (\varphi_1(s) \quad \varphi_2(s)) R(s) \left( \frac{\psi_1(s)}{\psi_2(s)} \right) \frac{\Gamma(1+iq+is)\Gamma(1-iq+is)}{\Gamma(2is)}^2 \, ds,
$$

where the matrix $R$ is the same as above (1.6).

Next, define functions $\Xi_n^{(1)}(\alpha, \beta; p, q; s)$, $\Xi_n^{(2)}(\alpha, \beta; p, q; s)$ in the variable $s$, given by the formula

$$
\Xi_n^{(1)}(\alpha, \beta; p, q; s) = \frac{\cos(p-\alpha-iq+ib)\pi/2\Gamma(1-\alpha+ib)}{\Gamma(\alpha-i\beta)\Gamma(1+iq+ib)\Gamma((1+iq))(1+iq-\alpha)} \times 3F_2 \left[ \frac{(1-\alpha+i\beta-p+aq)/2-n, 1/2+iq+is, 1/2+iq-is}{1+iq+i\beta, 1+iq-\alpha} ; 1 \right];
\Xi_n^{(2)}(\alpha, \beta; p, q; s) = \Xi_n^{(1)}(-\alpha,-\beta;p,q;x).
$$

**REMARK.** The hypergeometric series $3F_2(a_1, a_2, a_3; b_1, b_2; 1)$ is absolutely convergent for $\sum a_i < \sum b_j$ and admits an analytic continuation as a meromorphic (single-valued) function to arbitrary values of the parameters $a_1$, $a_2$, $a_3$, $b_1$, $b_2$. We understand (1.12) as a value of this meromorphic function. Below (3.4), we represent the function $\Xi_n^{(1)}$ as a hypergeometric series that converges for all interesting for us values of the parameters.

**THEOREM 1.3.** The system $(\Xi_n^{(1)}, \Xi_n^{(2)})$, where $n$ ranges in $\mathbb{Z}$, is an orthonormal basis of the Hilbert space $Y(\alpha, \beta;p)$.

**REMARK.** Emphasis, that the expression (1.12) has the structure

$$
\text{const} \cdot \, 3F_2 \left[ \frac{-n+h, a+is, a-is}{a+h, a+c} ; 1 \right] \quad (1.13)
$$

with

$$
\text{Re } a = 1/2, \quad \text{Re } h = 1/2, \quad c \in \mathbb{R}.
$$

Also, compariong with the formula (1.4) for Hahn polynomials, the parameter $n$ is shifted by some complex value $h$.

---

3It seems, now only difference operators remain interesting. There are 3 types of difference operators related to

a) Shift operator on the lattice $\mathbb{Z}$: $Tf(n) = f(n+1)$.

b) Shift operator on the line $\mathbb{R}$: $Tf(x) = f(x+1)$.

c) Shift operator on the line $\mathbb{R}$ in the imaginary direction $Tf(x) = f(x+i)$.

It seems, that for the case b), we have infinite multiplicities. In any case, variants a), c) are interesting for our purposes.
Remark. Similar perturbations for several more simple hypergeometric systems (Laguerre, Jakobi, Meixner–Pollachek, Meixner) are constructed in [19].

1.8. Further structure of the paper. Theorems 1.1–1.2 are proved in §2, Theorem 1.3 is obtained in §3.

§2. Spectral decomposition of the hypergeometric differential operator on the contour $\Re z = 1/2$.

A proof of Theorems 1.1–1.2 proposed below is direct but very tedious. We directly apply the Weyl–Titshmarsh–Kodaira theorem for an appropriate differential operator. This theorem with lot of examples is analyzed in Chapter XIII of Dunford–Schwartz’s book [3], see also the Titchmarsh’s book [26].

2.1. Hypergeometric operator. Let $\alpha \geq 0$, $\beta \in \mathbb{R}$. We consider the differential operator

$$D := \left( \frac{1}{4} + x^2 \right) \frac{d^2}{dx^2} + 2x \frac{d}{dx} + \frac{(\alpha + i\beta)^2}{4(1/2 + ix)} + \frac{(\alpha - i\beta)^2}{4(1/2 - ix)} + \frac{1}{4}. \quad (2.1)$$

This operator is formally self-adjoint in $L^2(\mathbb{R})$. Its resolvent $(D - \lambda)^{-1}$ is explicitly evaluated below. It is well defined for $\Im \lambda \neq 0$. Hence the deficiency indices of $D$ are 0; this implies the self-adjointness of the operator $D$. Our purpose is to construct the spectral decomposition of the operator $D$.

Let

$$r(x) := \left( \frac{1}{2} + ix \right)^{-(\alpha+i\beta)/2} \left( \frac{1}{2} - ix \right)^{-(\alpha-i\beta)/2}.$$ 

Evaluating directly the differential operator

$$Bf := r^{-1}D(rf),$$

we obtain

$$B = \left( \frac{1}{4} + x^2 \right) \frac{d^2}{dx^2} + \left( \beta + x(2 - 2\alpha) \right) \frac{d}{dx} + \left( \alpha - \frac{1}{2} \right)^2$$

Passing to the complex variable

$$z = 1/2 + ix,$$

we obtain the operator

$$A := -z(1 - z) \frac{d^2}{dz^2} - (1 - \alpha - i\beta - z(2 - 2\alpha)) \frac{d}{dz} - \left( \alpha - \frac{1}{2} \right)^2. \quad (2.2)$$

Hence the equation $Af = \mu^2 f$ became the hypergeometric equation

$$\left[ z(1 - z) \frac{d^2}{dz^2} + (c - (a + b + 1)z) \frac{d}{dz} - ab \right] f = 0 \quad (2.3)$$

with

$$a = \frac{1}{2} - \alpha + \mu, \quad b = \frac{1}{2} - \alpha - \mu, \quad c := 1 - \alpha - i\beta$$

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Now, we can use the Kummer series for solutions of the equation \( DF = \mu^2 f \). Below we use the standard notations \( u_1, \ldots, u_6 \) of [4] for the Kummer solutions of the standard hypergeometric equation (2.3) and use explicit formulae [4], (2.9.1)–(2.9.24) for their expansions in series.

2.2. Bases in the space of solutions of the hypergeometric equation.

We will use four bases in the space of solutions of the equation \((D - \mu^2)f = 0\).

Basis \( S_1, S_2 \). Writing the Kummer solutions \( u_1, u_5 \) of the usual hypergeometric equation (2.3), we obtain the following pair of solutions \( S_1, S_2 \) of the equation (2.1):

\[
S_1(\alpha, \beta; \mu; x) := \left( \frac{1}{2} + ix \right)^{-\alpha+i\beta/2} \left( \frac{1}{2} - ix \right)^{\alpha+i\beta/2-\mu-1/2} \times \nonumber
2F1 \left[ \frac{1}{2} - \alpha + \mu, 1/2 - i\beta + \mu; \frac{ix + 1/2}{ix - 1/2} \right], \quad (2.4)
\]

\[
S_2(\alpha, \beta; \mu; x) = S_1(-\alpha, -\beta; \mu; x).
\]

The hypergeometric series in (2.4) is absolutely convergent for \( \text{Im} x > 0 \), or equivalently, for \( \text{Re} z < 1/2 \). On the line \( x \in \mathbb{R} \) the series is conditionally convergent. Nevertheless it admits analytical continuation through the \( \text{Im} x = 0 \), and it is more convenient to think that we consider its analytic continuation. Further, assume that for \( x = iy \) with \(-1/2 < y < 1/2\) (or, equivalently, for \( 0 < z < 1 \))

\[
\left( \frac{1}{2} + ix \right)^\lambda := e^{\lambda \ln(1/2+ix)}, \quad \left( \frac{1}{2} - ix \right)^\nu := e^{\nu \ln(1/2-ix)}.
\]

Now we can assume that our solutions \( S_1, S_2 \) are defined in the domain \( \{ \text{Re} z \leq 1/2 \} \setminus \{ -\infty, 0 \} \).

The solutions \( S_1, S_2 \) have asymptotics

\[
S_1(x) \sim (\frac{1}{2} + ix)^{\alpha+i\beta/2}, \quad S_2(x) \sim (\frac{1}{2} + ix)^{-\alpha+i\beta/2}; \quad \text{for } x \to i/2.
\]

(2.5)

We also can define \( S_1 \) by the formula

\[
(\frac{1}{2} + ix)^{-\alpha+i\beta/2} (\frac{1}{2} - ix)^{-\alpha-i\beta/2} 2F1 \left[ \frac{1}{2} - \alpha + \mu, 1/2 - \alpha - \mu; \frac{ix + 1/2}{ix - 1/2} \right].
\]

(2.6)

Here the hypergeometric series converges for \( |1/2 - ix| < 1/2 \), i.e., we do not obtain an explicit formula on the whole line \( x \in \mathbb{R} \).

We mention that

\[
S_1(\alpha, \beta; -\mu; x) = S_1(\alpha, \beta; \mu; x).
\]

Basis \( T_1, T_2 \). Writing the pair of the Kummer solutions \( u_2, u_6 \), we obtain the following pair of solutions \( T_1, T_2 \) of the equation \((D - \mu^2)f = 0\)

\[
T_{1,2}(\alpha, \beta; \mu; x) := S_{1,2}(\alpha, -\beta; \mu; -x).
\]

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These solutions are defined in the domain \( \{ \Re z \geq 1/2 \} \setminus [1, \infty) \).

Bases \( V_-, V_+ \) and \( W_-, W_+ \). The Kummer solutions \( u_3, u_4 \) are defined outside the circular lune \( |z| < 1, |z - 1| < 1 \); in particular, this lune contains the segment \(-\sqrt{3}/2 < x < \sqrt{3}/2\). We define the solution \( V_- \) of the equation \((D - \mu^2)f = 0\) as the solution that for \( x > \sqrt{3}/2 \) is given by the formula

\[
V_-(\alpha, \beta; \mu; x) = e^{(-1/2 + \alpha + \beta - \mu)\pi i/2} (\frac{1}{2} + ix)^{-(\alpha + i\beta)/2} \left( \frac{1}{2} - ix \right)^{-1/2 + (\alpha + i\beta)/2 - \mu} \times 2F1 \left[ 1/2 - \alpha + \mu, 1/2 - i\beta + \mu; \frac{1}{1/2 - ix} \right].
\]

Further, assume

\[
V_+(\alpha, \beta; \mu; x) = V_-(\alpha, \beta; -\mu; x).
\]

The asymptotics of \( V_\pm \) for \( x \to +\infty \) has the form

\[
V_\pm \sim x^{1/2 \pm \mu}, \quad x \to +\infty.
\]

The solutions \( W_\pm \) are defined by the condition

\[
W_\pm \sim x^{1/2 \pm \mu}, \quad x \to -\infty.
\]

To obtain a formula for \( W_- \) for \( x < \sqrt{3}/2 \) we must change the sign in the argument of the exponential function in (2.7). Next, \( W_+(\alpha, \beta; \mu; x) = W_-(\alpha, \beta; -\mu; x) \).

2.3. Transition matrices. Define a constant

\[
C(\alpha, \beta; \mu) := \frac{\Gamma(\alpha + i\beta)\Gamma(1 + 2\mu)}{\Gamma(1/2 + \alpha + \mu)\Gamma(1/2 + i\beta + \mu)};
\]

\[
\chi(\alpha, \beta; \mu) := e^{-(1/2 + \alpha + \beta - \mu)\pi i/2}.
\]

In this notations (we use \[4\], (2.9.37), (2.9.39)),

\[
V_-(\alpha, \beta; \mu; x) = C(\alpha, \beta; \mu)\chi(\alpha, \beta; \mu)S_1(\alpha, \beta; \mu; x) + C(-\alpha, -\beta; \mu)\chi(-\alpha, -\beta; \mu)S_2(\alpha, \beta; \mu; x);
\]

\[
V_+(\alpha, \beta; \mu; x) = C(\alpha, \beta; -\mu)\chi(\alpha, \beta; -\mu)S_1(\alpha, \beta; \mu; x) + C(-\alpha, -\beta; -\mu)\chi(-\alpha, -\beta; -\mu)S_2(\alpha, \beta; \mu; x).
\]

Formulæ expressing \( W_-, W_+ \) in \( S_1, S_2 \) can be obtain from (2.12) - (2.13) by the change of the variables \( \chi \mapsto \chi^{-1} \).

We also need in an expression of \( W_\pm \) in terms of \( T_1, T_2 \). Similarly, applying \[4\], (2.9.38), (2.9.40), we obtain

\[
W_-(\alpha, \beta; \mu; x) = -C(\alpha, -\beta; \mu)\chi^{-1}(-\alpha, -\beta; -\mu)T_1(\alpha, \beta; \mu; x) - C(-\alpha, -\beta; \mu)\chi^{-1}(\alpha, -\beta; -\mu)T_2(\alpha, \beta; \mu; x)
\]

To obtain a formula for \( W_+ \) we must change sign in from of \( \mu \) in \( C(\alpha, \beta; \mu) \) and in \( \chi(\alpha, \beta; \mu) \).
2.4. Evaluation of Wronskians. Denote by \( \text{wr}(P, Q) := \det \begin{pmatrix} P & Q \\ P' & Q' \end{pmatrix} \) the Wronskian of two arbitrary solutions \( P, Q \) of the equation \((D - \mu^2)f = 0\). This expression must have a form
\[
\text{wr}(P, Q) = \frac{\sigma(P, Q)}{1/4 + x^2},
\]
where \( \sigma(P, Q) \) is a constant, see [7], I.17.1.

The value \( \sigma(S_1, S_2) \) can be easily evaluated using asymptotics (2.5), we obtain
\[
\sigma(S_1, S_2) = i(\alpha + i\beta).
\]
Below, we need in \( \sigma(V_-, W_-) \). The determinant \( \Delta \) of the transition matrix from the basis \((S_1, S_2)\) to the basis \((V_-, W_-)\) can be easily evaluated, this gives
\[
\sigma(V_-, W_-) = \Delta \cdot \sigma(S_1, S_2) = \frac{2\pi iC(\alpha, \beta; \mu)C(-\alpha, -\beta; \mu)}{\Gamma(\alpha + i\beta)\Gamma(-\alpha - i\beta)}.
\]

2.5. Kernel of resolvent. Now we are ready to write the kernel \( K(x, y; \lambda) \) of the resolution \( R(\lambda) := (D - \lambda)^{-1} \) of the operator \( D \),
\[
R(\lambda)f(x) = \int_{-\infty}^{\infty} K(x, y; \lambda)f(y) \, dy.
\]
Assume \( \lambda \in \mathbb{C} \setminus (-\infty, 0) \). Then the solution \( V_- \) is an element of \( L^2(0, +\infty) \) and \( W_- \in L^2(0, -\infty) \), see (2.8)–(2.9). Hence (see [3], XIII.3.6),
\[
K(x, y; \lambda) = \begin{cases} \frac{V_-(y)W_-(x)}{\sigma(V_-, W_-)}, & \text{for } y < x \\ \frac{V_-(x)W_-(y)}{\sigma(V_-, W_-)}, & \text{for } x < y, \end{cases}
\]
the value \( \sigma(V_-, W_-) \) was evaluated above (2.15).

We intend to write an expansion of the differential operator \( R \) in eigenfunctions. According prescription rising to Weyl’s work [28] (see [26] and detailed presentation in Dunford, Schwartz [3], XIII.5.18), we must evaluate the jump of the resolution on the real line
\[
\frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} (R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)) \, d\lambda,
\]
and represent it in the form
\[
Lf(x) = \sum_{i=1,2} \sum_{j=1,2} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \sigma_i(x, \lambda)\sigma_j(y, \lambda) f(y) \, d\mu_{ij}(\lambda) \right) dy
\]
(2.17)
for some solutions $\sigma_1, \sigma_2$ of the equation $Df = \lambda f$. Here $\mu_{ij}(\lambda)$ are (complex-valued) measures on $\mathbb{R}$. Then $\mu_{ij}$ is the spectral measure. Precisely, the operator

$$f \mapsto \left( \int_{-\infty}^{\infty} f(x)\sigma_1(x)\,dx, \int_{-\infty}^{\infty} f(x)\sigma_2(x)\,dx \right)$$

is a unitary operator from $L^2(\mathbb{R})$ to the space of $\mathbb{C}^2$-valued functions $(\varphi_1(\lambda), \varphi_2(\lambda))$ with the scalar product

$$\langle (\varphi_1(\lambda), \varphi_1(\lambda), (\psi_1(\lambda), \psi_1(\lambda)) = \sum_{i=1,2} \sum_{i=1,2} \int_{-\infty}^{\infty} \varphi_i(\lambda)\varphi_j(\lambda)\,d\mu_{ij}(\lambda)$$

(in particular, the right-hand side of this equality is a positive definite scalar product).

2.6. Formula for resolvent. To evaluate the jump of the resolvent for $\lambda \geq 0$ and for $\lambda \leq 0$ we need two explicit expressions for the kernel $K(x, y; \lambda)$ of the resolvent.

Expressing $V_-$ via $S_1, S_2$ and $W_-$ via $T_1, T_2$, we obtain the following expression for the resolvent

$$K(x, y; \lambda) = \frac{1}{2\pi} \Gamma(\alpha + i\beta) \Gamma(-\alpha - i\beta) \left[ \frac{C(\alpha, -\beta; \sqrt{\lambda})}{C(-\alpha, -\beta; \sqrt{\lambda})} e^{-(1/2 + \alpha - \sqrt{\lambda}) \pi i} S_1(x)T_1(y) + \frac{C(-\alpha, \beta; \sqrt{\lambda})}{C(\alpha, \beta; \sqrt{\lambda})} e^{-(1/2 - \beta + \sqrt{\lambda}) \pi i} S_2(x)T_2(y) \right]$$

where $C(\ldots)$ is defined by (2.10).

Expressing $V_-, W_-$ by $S_1, S_2$, we obtain

$$K(x, y; \lambda) = \frac{\Gamma(\alpha + i\beta) \Gamma(-\alpha - i\beta)}{2\pi C(\alpha, \beta; \sqrt{\lambda}) C(-\alpha, -\beta; \sqrt{\lambda})} \times \left[ C(\alpha, \beta; \sqrt{\lambda}) \chi(\alpha, \beta; \sqrt{\lambda}) S_1(x) + C(-\alpha, -\beta; \sqrt{\lambda}) \chi(-\alpha, -\beta; \sqrt{\lambda}) S_2(x) \right] \times \left[ C(\alpha, \beta; \sqrt{\lambda}) \chi^{-1}(\alpha, \beta; \sqrt{\lambda}) S_1(x) + C(-\alpha, -\beta; \sqrt{\lambda}) \chi^{-1}(-\alpha, -\beta; \sqrt{\lambda}) S_2(x) \right],$$

where $\chi$ is given by (2.11).

These two expressions are meromorphic in the plane $\lambda \in \mathbb{C}$, with a cat on the negative semiaxis $\lambda < 0$.

2.7. Jump of resolvent for $\lambda > 0$. Evaluation of

$$L^{[0, \infty)} := \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{0}^{\infty} (R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)) \,d\lambda$$

(220)
is reduced to an evaluation of the residues. Expression (2.19) has simple poles at the points \( \lambda = (\alpha - k - 1/2)^2 \) for integer \( k \), satisfying the condition \( 0 \leq k \leq \alpha - 1/2 \). In addition, only the coefficient at \( S_1(x)S_1(y) \) has nonzero residues, three other coefficients are holomorphic at these points.

Thus the integral operator (2.20) equals

\[
L^{[0, \infty)} f(x) = \frac{1}{2\pi} \Gamma(\alpha + i\beta) \Gamma(\alpha - i\beta) \sum_{0 \leq k < \alpha - 1/2} \frac{(2\alpha - 2k - 1)(1 - \alpha - i\beta)_k}{\Gamma(2\alpha - k)(1 - \alpha + i\beta)_k k!} \times \int_{-\infty}^{\infty} S_1(\alpha, \beta; \alpha - k - 1/2; x) S_1(\alpha, \beta; \alpha - k - 1/2; y) f(y) dy
\]

Writing \( S_1 \) at the form (2.6), we obtain its representation in the terms of the Romanovski polynomials. Applying identity [4], (10.8.16) for Jacobi polynomials, we convert our expression to the form

\[
\frac{1}{2\pi} \Gamma(\alpha + i\beta) \Gamma(\alpha - i\beta) \sum_{0 \leq k < \alpha - 1/2} \frac{2\alpha - 2k - 1}{\Gamma(2\alpha - k) k!} S_1(\alpha, \beta; \alpha - k - 1/2; x) \times \int_{-\infty}^{\infty} S_1(\alpha, \beta; \alpha - k - 1/2; y) f(y) dy
\]

and this gives the required expression.

2.8. Jump of resolvent for \( \lambda < 0 \). We must evaluate

\[
K(x, y; \sqrt{\lambda}) - K(x, y; -\sqrt{\lambda}),
\]

assuming that \( \lambda \) is negative real and \( \text{Im} \sqrt{\lambda} > 0 \).

For definiteness, we present the calculation of the coefficient at \( S_1(x)T_1(y) \).

In three remaining cases, the evaluation is almost identical\(^4\).

Thus, (see (2.18)), we intend to find

\[
\frac{1}{2\pi} \Gamma(\alpha + i\beta) \Gamma(-\alpha - i\beta) \times \left\{ \frac{C(\alpha, -\beta; \sqrt{\lambda}) e^{-(1/2 - \alpha + \sqrt{\lambda}) \pi i}}{C(-\alpha, -\beta; \sqrt{\lambda})} - \frac{C(\alpha, -\beta; -\sqrt{\lambda}) e^{-(1/2 - \alpha - \sqrt{\lambda}) \pi i}}{C(-\alpha, -\beta; -\sqrt{\lambda})} \right\}.
\]

After direct cancellations, we obtain

\[
\frac{1}{2\pi} \Gamma(\alpha + i\beta) \Gamma(\alpha - i\beta) \times \left\{ \frac{\Gamma(1/2 - \alpha + \sqrt{\lambda}) e^{-(1/2 - \alpha + \sqrt{\lambda}) \pi i}}{\Gamma(1/2 + \alpha + \sqrt{\lambda})} + \frac{\Gamma(1/2 - \alpha - \sqrt{\lambda}) e^{-(1/2 - \alpha - \sqrt{\lambda}) \pi i}}{\Gamma(1/2 + \alpha - \sqrt{\lambda})} \right\}.
\]

\(^4\)This is explained by natural symmetries \((\alpha, \beta) \sim (-\alpha, -\beta) \sim (i\beta, -i\alpha)\) of the equation (2.1).
We convert this expression to
\[
\frac{1}{-2\pi} \Gamma(\alpha + i\beta) \Gamma(\alpha - i\beta) \Gamma(1/2 - \alpha + \sqrt{\lambda}) \Gamma(1/2 - \alpha - \sqrt{\lambda}) \times \\
\times \left\{ \frac{e^{-(1/2 - \alpha + \sqrt{\lambda})\pi i}}{\Gamma(1/2 + \alpha + \sqrt{\lambda}) \Gamma(1/2 - \alpha + \sqrt{\lambda})} - \frac{e^{-(1/2 - \alpha - \sqrt{\lambda})\pi i}}{\Gamma(1/2 + \alpha - \sqrt{\lambda}) \Gamma(1/2 - \alpha - \sqrt{\lambda})} \right\}.
\]

Now, we will transform only the expression in the curly brackets
\[
\pi \left\{ \ldots \right\} = \cos(\pi(\alpha + \sqrt{\lambda})) e^{-(1/2 - \alpha + \sqrt{\lambda})\pi i} - \cos(\pi(\alpha - \sqrt{\lambda})) e^{-(1/2 - \alpha - \sqrt{\lambda})\pi i}.
\]

Next, we apply the Euler formulae for cos, collect similar terms in the linear combination of exponential functions, and again apply the Euler formulae. We obtain
\[
i \sin(2\sqrt{\lambda} \pi),
\]
and this finishes the evaluation of the coefficient at $S_1(x) T_1(y)$.

After this, for the jump of the resolvent $L^{(-\infty,0]}$ on the semiaxis $(-\infty,0]$, we obtain the expression of the form
\[
L^{(-\infty,0]} f(x) = \sum_{i=1,2} \sum_{j=1,2} \int_{-\infty}^{0} d\lambda \left\{ \theta_{ij}(\lambda) \int_{-\infty}^{\infty} S_i(x) T_j(y) f(y) dy \right\}
\]
where $\theta_{ij}$ are some explicit products of $\Gamma$-functions.

Further, observe that for $\lambda < 0$ and $y \in \mathbb{R}$ we have $T_j(y) = \overline{S_j(y)}$. We obtain an expression of the form (2.17), and this finishes the evaluation of the spectral decomposition of the operator (2.1).

The parameter $s$ from 1.4 is $\sqrt{\lambda}$.

§ 3. Construction of bases

3.1. One basis in $L^2(\mathbb{R})$. Fix real parameters $p$ and $q$.

**Lemma 3.1.** The system of functions
\[
r_{p,q}^{(n)}(x) := \left( \frac{1}{2} + ix \right)^{-1/2 - n - (p+iq)/2} \left( \frac{1}{2} - ix \right)^{-1/2 + n + (p-iy)/2},
\]
where $n$ ranges in $\mathbb{Z}$, is an orthonormal basis of $L^2(\mathbb{R}, dx/2\pi)$.

**Proof.** Pass to the new variable $\psi \in [0, 2\pi]$ by the formula
\[
e^{i\psi} = \frac{1/2 + ix}{1/2 - ix}, \quad d\psi = \frac{dx}{1/4 + x^2}.
\]

Our system of functions transforms to
\[
e^{-i\psi} \cdot e^{-ip\psi/2(2 \cos \psi/2)^{iq}}.
\]

We obtain the standard orthogonal system $e^{-ip\psi}$ up to multiplication by a function whose absolute value is 1. \qed
Emphasis also that orthogonality of our system can be obtained directly (since the scalar products can be easily evaluated using residues).

3.2. Construction of bases. The bases (3.13) can be easily obtained, if we apply double index hypergeometric transform to the orthogonal system (3.1).

Let us explain, how to perform the calculation. We must find

$$\int_{-\infty}^{\infty} r_{p,q}^{(n)}(x)Q_1(\alpha, \beta; x; s) \, dx$$

where the function $Q_1$ is defined by the formula (1.6). For this purpose, we expand $3F_1$ from formula (1.6) in the hypergeometric series in powers of

$$\frac{ix + 1/2}{ix - 1/2}$$

Then we integrate it termwise using the Cauchy beta-integral (see [22], v.1, (2.2.6.31) or [1], chapter 1, ex. 13)

$$\int_{-\infty}^{\infty} \frac{dx}{(1/2 + ix)^\sigma(1/2 - ix)^\tau} = \frac{2\pi \Gamma(\sigma + \tau - 1)}{\Gamma(\sigma)\Gamma(\tau)}$$

As a result, we obtain the following expression for (3.3)

$$\frac{2\pi \Gamma(1/2 + iq - is)}{\Gamma(\alpha - i\beta)\Gamma((p + iq - \alpha + i\beta)/2 + n + 1 - is)\Gamma((-p + iq + \alpha - i\beta + 1)/2 - n)} \times$$

$$\times \, 3F_2\left[\frac{1/2 - \alpha - is, 1/2 + i\beta - is, 1/2 + (p - iq - \alpha + i\beta)/2 + n}{1 - \alpha + i\beta, (p + iq - \alpha + i\beta)/2 + n + 1 - is}; 1\right]$$

This expression is a variant of the final answer, its more symmetric form (1.13) can be obtained by Thomae transformation see [1], Corollary 3.3.6, [22], v.3, (7.4.4.2).

$$3F_2(a,b,c|d,e;1) = \frac{\Gamma(d)\Gamma(e)\Gamma(r)}{\Gamma(a)\Gamma(b+r)\Gamma(c+r)} 3F_2\left[\frac{d-a, e-a, r}{b+r, c+r}; 1\right],$$

where $r = d + e - a - b - c.$

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