The composition series of ideals of the partial-isometric crossed product by the semigroup $\mathbb{N}^2$

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Received: 10 July 2023 / Accepted: 25 September 2023
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Abstract
Suppose that $\alpha$ is an action of the semigroup $\mathbb{N}^2$ on a $C^*$-algebra $A$ by endomorphisms. Let $A \times_{\alpha}^{\text{piso}} \mathbb{N}^2$ be the associated partial-isometric crossed product. By applying an earlier result which embeds this semigroup crossed product (as a full corner) in a crossed product by the group $\mathbb{Z}^2$, a composition series $0 \leq L_1 \leq L_2 \leq A \times_{\alpha}^{\text{piso}} \mathbb{N}^2$ of essential ideals is obtained for which we identify the subquotients with familiar algebras.

Keywords $C^*$-algebra · Endomorphism · Semigroup · Partial-isometry · Crossed product

Mathematics Subject Classification 46L55; 46L05

1 Introduction

It is shown in [10] that the partial-isometric crossed products (Nica-Toeplitz crossed products) by positive cones of abelian lattice-ordered groups are full corners in usual crossed products by groups. This actually generalizes the earlier result in [9], where the case of abelian totally ordered groups is treated. Now, in the present work, we consider the dynamical system $(A, \mathbb{N}^2, \alpha)$, where $\mathbb{N}^2$ denotes the positive cone of the abelian lattice-ordered group $\mathbb{Z}^2$, and $\alpha$ is an action of $\mathbb{N}^2$ on a $C^*$-algebra $A$ by endomorphisms. We would like to recall that we suppose that each endomorphism $\alpha_t$ of $A$ extends to a strictly continuous endomorphism $\overline{\alpha}_t$ of the multiplier algebra $\mathcal{M}(A)$ as we deal with non-unital $C^*$-algebras (see [10, §1] or [11, §1]). Therefore,
by [10, Theorem 4.1], the partial-isometric crossed product $A \times_{\alpha}^{\text{piso}} N^2$ of the system $(A, N^2, \alpha)$ is (isomorphic to) a full corner of a crossed product by the group $\mathbb{Z}^2$. We apply this corner realization to obtain a composition series of essential ideals $0 \leq L_1 \leq L_2 \leq A \times_{\alpha}^{\text{piso}} N^2$ and identify the subquotients with familiar algebras. In addition, when the action $\alpha$ on $A$ is given by automorphisms, we have simple identifications for the subquotients. Overall, we think that the present work contains useful information to understand the (ideal) structure of the semigroup crossed product $A \times_{\alpha}^{\text{piso}} N^2$.

Before we proceed, first, readers should be informed that the present work is essentially a revised version of section 5 of the earlier versions of [10] which are available in the pre-print server arXiv via links

https://arxiv.org/abs/1912.09682v1

and

https://arxiv.org/abs/1912.09682v2.

Since these versions were too long, a third version of [10] was then prepared by removing the section 5 and published separately (see its arXiv version at https://arxiv.org/abs/1912.09682v3). Therefore, the present work is indeed an application of the main theorem in [10] to the system $(A, N^2, \alpha)$. We hope that this clarifies the overlap of the present work with the earlier versions of [10] in arXiv. In addition, to see more on the theory of partial-isometric crossed products, readers may refer to [1–4, 6, 7, 11]. In particular, in [11], partial-isometric crossed products are studied for more general semigroups, namely, (left) LCM semigroups. However, as a preliminary background for the present work, [10, §2] should be enough for readers to see a quick recall on partial-isometric and isometric crossed products.

Here is the organization of the present work. It starts with a preliminary section in which the results in [10] are briefly recalled for the system $(A, N^2, \alpha)$. In Sect. 3, we show that for the partial-isometric crossed product $A \times_{\alpha}^{\text{piso}} N^2$ of the system, there is a composition series

$$0 \leq L_1 \leq L_2 \leq A \times_{\alpha}^{\text{piso}} N^2,$$

of essential ideals, and then identify the subquotients with familiar algebras. To do so, we apply the fact that the algebra $A \times_{\alpha}^{\text{piso}} N^2$ is a full corner in a crossed product by the group $\mathbb{Z}^2$ to import the information. The (essential) ideal $L_2$ is the kernel of the natural surjective homomorphism of $A \times_{\alpha}^{\text{piso}} N^2$ onto the isometric crossed product $A \times_{\alpha}^{\text{iso}} N^2$ of the system. We show that it is the sum of two essential ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ (corresponding to two generators of the group $\mathbb{Z}^2$), and hence, the composition series of essential ideals mentioned in above is

$$0 \leq \mathcal{I}_1 \cap \mathcal{I}_2 \leq \mathcal{I}_1 + \mathcal{I}_2 \leq A \times_{\alpha}^{\text{piso}} N^2.$$

Moreover, while clearly $(A \times_{\alpha}^{\text{piso}} N^2)/L_2 \simeq A \times_{\alpha}^{\text{iso}} N^2$, we show that the ideal $L_1$ is a full corner in the algebra $\mathcal{K}(\ell^2(N^2)) \otimes A$ of compact operators, and
Let $L_2/L_1 \cong \mathcal{A}_1 \oplus \mathcal{A}_2$, where each $\mathcal{A}_i$ is a full corner in an algebra of compact operators. Therefore, when the action $\alpha$ on $A$ is given by automorphisms, we simply have $(A \times_{\pi}^\text{piso} \mathbb{N}^2)/L_2 \cong A \times_{\pi}^\text{piso} \mathbb{Z}^2$, $L_1 \cong \mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A$, and $L_2/L_1 \cong \left[\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_{\alpha_1} \mathbb{Z})\right] \oplus \left[\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_{\alpha_2} \mathbb{Z})\right]$, where $\alpha_1$ and $\alpha_2$ are two automorphisms corresponding to two generators of the group $\mathbb{Z}^2$.

2 Preliminaries

Let $\mathbb{N}^2$ be the positive cone of the abelian lattice-ordered group $\mathbb{Z}^2$. Note that, here, sometimes an element of $\mathbb{Z}^2$ is simply denoted by $s$ instead of $(s_1, s_2)$ for convenience, where each $s_i$ belongs to $\mathbb{Z}$. Therefore, $0$ denotes the unit element $(0, 0)$ of $\mathbb{Z}^2$. Also, we use the additive notation “$+$” for the action of the group $\mathbb{Z}^2$, and hence, $-s$ denotes the inverse of an element $s$.

Let $(A, \mathbb{N}^2, \alpha)$ be a dynamical system consisting of a C*-algebra $A$, and an action $\alpha : \mathbb{N}^2 \to \text{End}(A)$ of $\mathbb{N}^2$ on $A$ by extendible endomorphisms, such that $\alpha_0 = \text{id}$. Suppose that $(A \times^\text{piso}_{\alpha} \mathbb{N}^2, i_A, i_{\mathbb{N}^2})$ and $(A \times^\text{iso}_{\alpha} \mathbb{N}^2, j_A, j_{\mathbb{N}^2})$ are the partial-isometric and isometric crossed product of the system, respectively. We recall from [10] that the pair $(j_A, j_{\mathbb{N}^2})$ induces a surjective homomorphism $q$ of $A \times^\text{piso}_{\alpha} \mathbb{N}^2$ onto $A \times^\text{iso}_{\alpha} \mathbb{N}^2$, such that

$$q(i_{\mathbb{N}^2}(m, n)^*i_A(a)i_{\mathbb{N}^2}(s, t)) = j_{\mathbb{N}^2}(m, n)^*j_A(a)j_{\mathbb{N}^2}(s, t),$$

for all $a \in A$ and $m, n, s, t \in \mathbb{N}$. Therefore, the following short exact sequence:

$$0 \longrightarrow \ker q \longrightarrow A \times^\text{piso}_{\alpha} \mathbb{N}^2 \xrightarrow{q} A \times^\text{iso}_{\alpha} \mathbb{N}^2 \longrightarrow 0 \quad (2.1)$$

is obtained, where by [10, Proposition 2.1], we have

$$\ker q = \text{span}\{i_{\mathbb{N}^2}(x)^*i_A(a)(1 - i_{\mathbb{N}^2}(s)^*i_{\mathbb{N}^2}(s))i_{\mathbb{N}^2}(y) : a \in A, x, y, s \in \mathbb{N}^2\}, \quad (2.2)$$

which is an essential ideal of $A \times^\text{piso}_{\alpha} \mathbb{N}^2$ (see [10, Proposition 4.3]). More importantly, the main theorem of [10] says that the algebra $A \times^\text{piso}_{\alpha} \mathbb{N}^2$ is a full corner in a crossed product by group. To be more precise, for every $s \in \mathbb{Z}^2$, a map $\phi_s : A \to \ell^\infty(\mathbb{Z}^2, A)$ is defined by

$$\phi_s(a)(t) = \begin{cases} \alpha_{t-s}(a) & \text{if } s \leq t \\ 0 & \text{otherwise,} \end{cases}$$

which is an injective $*$-homomorphism. Note that

$$\phi_s(a)\phi_t(b) = \phi_{s \vee t} \left(\alpha_{(s \vee t) - s}(a)\alpha_{(s \vee t) - t}(b)\right), \quad (2.3)$$
for all \( a, b \in A \) and \( s, t \in \mathbb{Z}^2 \). Now, for the \( C^* \)-subalgebra \( \mathcal{B} \) of \( \ell^\infty (\mathbb{Z}^2, A) \) generated by \( \{ \phi_s(a) : s \in \mathbb{Z}^2, a \in A \} \), we have

\[
\mathcal{B} = \overline{\text{span}} \{ \phi_s(a) : s \in \mathbb{Z}^2, a \in A \}.
\]

Also, each homomorphism \( \phi_s : A \to \mathcal{B} \) extends to a strictly continuous homomorphism \( \bar{\phi}_s : \mathcal{M}(A) \to \mathcal{M}(\mathcal{B}) \) of multiplier algebras (see [10, Lemma 3.2]), such that

\[
\bar{\phi}_s(m)\phi_t(n) = \bar{\phi}_s(\sigma_{s\lor t}^{-1}) \left( \bar{\alpha}(s\lor t)^{-1}m\bar{\alpha}(s\lor t)^{-1}n \right) \tag{2.4}
\]

for all \( s, t \in \mathbb{Z}^2 \) and \( m, n \in \mathcal{M}(A) \). Moreover, [10, Proposition 3.3] shows that the algebra \( \mathcal{B} \) contains an essential ideal \( \mathcal{J} \), such that

\[
\mathcal{J} = \overline{\text{span}} \{ \phi_s(a) - \phi_t(\alpha_{t-s}(a)) : s \leq t \in \mathbb{Z}^2, a \in A \} \tag{2.5}
\]

Next, the shift on \( \ell^\infty (\mathbb{Z}^2, A) \) induces an action \( \beta \) of \( \mathbb{Z}^2 \) on \( \mathcal{B} \) by automorphisms, such that \( \beta_t \circ \phi_s = \phi_{s+t} \) for all \( s, t \in \mathbb{Z}^2 \). Therefore, a group dynamical system \((\mathcal{B}, \mathbb{Z}^2, \beta)\) is obtained. If \((B \rtimes_\beta \mathbb{Z}^2, j_B, j_{\mathbb{Z}^2})\) is the group crossed product of the system, since \( \mathcal{J} \) is a \( \beta \)-invariant essential ideal of \( \mathcal{B} \), \( \mathcal{J} \rtimes_\beta \mathbb{Z}^2 \) sits in \( B \rtimes_\beta \mathbb{Z}^2 \) as an essential ideal (see [5, Proposition 2.4]). Also, recall that, if the maps \( \rho : \mathcal{B} \to \mathcal{L}(\ell^2(\mathbb{Z}^2) \otimes A) \) and \( U : \mathbb{Z}^2 \to \mathcal{L}(\ell^2(\mathbb{Z}^2) \otimes A) \) are defined by \( (\rho(\xi)f)(s) = \xi(s)f(s) \) and \( (U_tf)(s) = f(s-t) \), respectively, where \( \xi \in \mathcal{B} \) and \( f \in \ell^2(\mathbb{Z}^2) \otimes A \), then \( \rho \) is a nondegenerate representation and \( U \) is a unitary representation, such that \( \rho(\beta_t(\xi)) = U_t\rho(\xi)U_t^* \). Therefore, the pair \((\rho, U)\) is a covariant representation of \((\mathcal{B}, \mathbb{Z}^2, \beta)\) on \( \ell^2(\mathbb{Z}^2) \otimes A \simeq \ell^2(\mathbb{Z}^2, A) \). Now, if \( p = j_B \circ \phi_{(0,0)}(1) \), then by [10, Theorem 4.1], there is an isomorphism \( \Psi \) of \((A \rtimes_\alpha \mathbb{N}^2, j_A, j_{\mathbb{N}^2})\) onto the full corner \( p(B \rtimes_\beta \mathbb{Z}^2)p \) of the group crossed product \((B \rtimes_\beta \mathbb{Z}^2, j_B, j_{\mathbb{Z}^2})\), such that

\[
\Psi \left( i_{\mathbb{N}^2}(m, n)^* i_A(a) i_{\mathbb{N}^2}(s, t) \right) = pj_{\mathbb{Z}^2}(m, n)(j_B \circ \phi_{(0,0)}(a)) j_{\mathbb{Z}^2}(s, t)^* p, \tag{2.6}
\]

for all \( m, n, s, t \in \mathbb{N} \) and \( a \in A \). Also, by [10, Lemma 4.2], the ideal \( \ker q \) of \( A \rtimes_\alpha \mathbb{N}^2 \) is isomorphic to the full corner \( p(J \rtimes_\beta \mathbb{Z}^2)p \) of the algebra \( J \rtimes_\beta \mathbb{Z}^2 \) via the isomorphism \( \Psi \), such that

\[
\Psi \left( i_{\mathbb{N}^2}(x)^* i_A(a)(1 - i_{\mathbb{N}^2}(s)^*i_{\mathbb{N}^2}(y)) \right) = p \left[ j_{\mathbb{Z}^2}(x) j_B (\phi_0(a) - \phi_s(\alpha_s(a))) j_{\mathbb{Z}^2}(y)^* \right] p, \tag{2.7}
\]

for all \( x, y, s \in \mathbb{N}^2 \) and \( a \in A \).

In addition, see in [10, §5] that, if in the system \((A, \mathbb{N}^2, \alpha)\), the action \( \alpha \) is given by automorphisms of \( A \), then we have a simple picture for the algebra \( \mathcal{B} \). In this case, the action \( \alpha \) extends uniquely to an action of the group \( \mathbb{Z}^2 \) on \( A \) by automorphisms. Therefore, we obtain a group dynamical system \((B_{\mathbb{Z}^2} \otimes A, \mathbb{Z}^2, \tau \otimes \alpha^{-1})\), where \( B_{\mathbb{Z}^2} \) is the \( C^* \)-subalgebra of \( \ell^\infty (\mathbb{Z}^2) \) generated by the characteristic functions \( \{ 1_s \in \ell^\infty (\mathbb{Z}^2) : \)

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s ∈ \mathbb{Z}^2\}, such that

\[ 1_s(t) = \begin{cases} 
1 & \text{if } s \leq t, \\
0 & \text{otherwise}, 
\end{cases} \]

and the action \( \tau \) of \( \mathbb{Z}^2 \) on \( B_{\mathbb{Z}^2} \) is given by translation. Let \( B_{\mathbb{Z}^2, \infty} \) be the \( C^* \)-subalgebra of \( B_{\mathbb{Z}^2} \) generated by the elements \( \{ 1_s - 1_t : s \leq t \in \mathbb{Z}^2 \} \), which is actually a \( \tau \)-invariant essential ideal of \( B_{\mathbb{Z}^2} \). Now, the algebra \( \mathcal{B} \) is isomorphic to the tensor product \( (B_{\mathbb{Z}^2} \otimes A) \), where the isomorphism intertwines the actions \( \beta \) and \( (\tau \otimes \alpha^{-1}) \), and it maps the ideal \( J \) isomorphically onto the ideal \( (B_{\mathbb{Z}^2, \infty} \otimes A) \) of \( (B_{\mathbb{Z}^2} \otimes A) \). As a result, by [10, Corollary 5.3], the algebra \( B \) is isomorphic to \( (B_{\mathbb{Z}^2} \otimes A) \), where the isomorphism intertwines the actions \( \beta \) and \( (\tau \otimes \alpha^{-1}) \), and it maps the ideal \( J \) isomorphically onto the ideal \( (B_{\mathbb{Z}^2, \infty} \otimes A) \) of \( (B_{\mathbb{Z}^2} \otimes A) \). Therefore, we obtain a composition series

\[ 0 \leq \mathcal{I}_1 \cap \mathcal{I}_2 \leq \ker q \leq \mathcal{E}_{\mathbb{N}^2}, \quad (3.1) \]

of ideals, for which we identify the subquotients with familiar algebras. Of course, we already know that \( (A \times_{\alpha} \mathbb{N}^2) / \ker q \simeq A \times_{\alpha} \mathbb{N}^2 \).

To start, first, the action \( \alpha \) induces two actions \( \delta \) and \( \gamma \) of \( \mathbb{N} \) on \( A \) by extendible endomorphisms, such that

\[ \delta_n := \alpha_{(0,n)} \text{ and } \gamma_n := \alpha_{(n,0)}, \quad (3.2) \]

for every \( n \in \mathbb{N} \). Hence, two dynamical systems \( (A, \mathbb{N}, \delta) \) and \( (A, \mathbb{N}, \gamma) \) are obtained, which are actually generated by the single endomorphisms \( \delta := \delta_1 \) and \( \gamma := \gamma_1 \), respectively. We obviously have

\[ \alpha_{(m,n)} = \delta_m \gamma_n = \gamma_m \delta_n, \quad (3.3) \]

for all \( m, n \in \mathbb{N} \). Now, we define two subalgebras \( D \) and \( C \) of \( \ell^\infty(\mathbb{Z}, A) \), which are actually the corresponding algebra \( \mathcal{B} \) to the totally ordered abelian group \( (\mathbb{Z}, \mathbb{N}) \) and
the systems \((A, \mathbb{N}, \delta)\) and \((A, \mathbb{N}, \gamma)\), respectively (see also [9, §6]). Thus, we have

\[
D = \text{span}\{\varphi_n(a) : n \in \mathbb{Z}, a \in A\} \quad \text{and} \quad C = \text{span}\{\psi_n(a) : n \in \mathbb{Z}, a \in A\},
\]

where the maps \(\varphi_n : A \to \ell^\infty(\mathbb{Z}, A)\) and \(\psi_n : A \to \ell^\infty(\mathbb{Z}, A)\) are the (extendible) embeddings defined by

\[
\varphi_n(a)(m) = \begin{cases} 
\delta_{m-n}(a) & \text{if } n \leq m, \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
\psi_n(a)(m) = \begin{cases} 
\gamma_{m-n}(a) & \text{if } n \leq m, \\
0 & \text{otherwise}
\end{cases}
\]

for all \(m, n \in \mathbb{Z}\) and \(a \in A\), respectively. Note that the algebras \(D\) and \(C\) both contain the algebra \(C_0(\mathbb{Z}) \otimes A\) as an essential ideal, such that

\[
C_0(\mathbb{Z}) \otimes A = \text{span}\{\varphi_n(a) - \varphi_m(\delta_{m-n}(a)) : n \leq m \in \mathbb{Z}, a \in A\} = \text{span}\{\psi_n(a) - \psi_m(\gamma_{m-n}(a)) : n \leq m \in \mathbb{Z}, a \in A\}.
\]

Indeed, \(C_0(\mathbb{Z}) \otimes A\) is the corresponding (essential) ideal \(\mathcal{J}\) for the algebras \(D\) and \(C\). Next, for our purpose, we show that the algebra \(\mathcal{B} \subseteq \ell^\infty(\mathbb{Z}, D)\) and \(\ell^\infty(\mathbb{Z}, C)\) as a \(C^*\)-subalgebra via two families of fibers (column and row fibers). For every \(m, n \in \mathbb{Z}\) and \(a \in A\), define a map \(\Delta_{(m,n)} : A \to \ell^\infty(\mathbb{Z}, D)\) by

\[
\Delta_{(m,n)}(a)(r) = \begin{cases} 
\varphi_n(\gamma_{r-m}(a)) & \text{if } m \leq r, \\
0 & \text{otherwise}.
\end{cases}
\]

Each function \(\Delta_{(m,n)}(a)\) can actually be viewed as a sequence of column fibers. Also, each map \(\Delta_{(m,n)}\) is a norm-preserving \(*\)-homomorphism. Now, let \(\mathcal{B}_\delta\) be the \(C^*\)-subalgebra of \(\ell^\infty(\mathbb{Z}, D)\) generated by \(\{\Delta_{(m,n)}(a) : m, n \in \mathbb{Z}, a \in A\}\). Since calculation shows that

\[
\Delta_{(m,n)}(a)^* = \Delta_{(m,n)}(a^*) \quad \text{and} \quad \Delta_{(m,n)}(a)\Delta_{(t,u)}(b) = \Delta_{(x,y)}(c),
\]

where

\[
(x, y) = (m, n) \lor (t, u) = (m \lor t, n \lor u) \quad \text{and} \quad c = \delta_{y-n}(\gamma_{x-m}(a))\delta_{y-u}(\gamma_{x-t}(b)),
\]

it follows that:

\[
\mathcal{B}_\delta = \text{span}\{\Delta_{(m,n)}(a) : m, n \in \mathbb{Z}, a \in A\}.
\]
Similarly, for every $m, n \in \mathbb{Z}$ and $a \in A$, we define a map $\Gamma_{(m,n)} : A \to \ell^\infty(\mathbb{Z}, C)$ by

$$
\Gamma_{(m,n)}(a)(s) = \begin{cases} 
\psi_m(\delta_{s-n}(a)) & \text{if } n \leq s \\
0 & \text{otherwise},
\end{cases}
$$

which is an injective $*$-homomorphism. In this case, each function $\Gamma_{(m,n)}(a)$ can be viewed as a (columnar) sequence of row fibers. Then, for the $C^*$-subalgebra $B_\gamma$ of $\ell^\infty(\mathbb{Z}, C)$ generated by $\{\Gamma_{(m,n)}(a) : m, n \in \mathbb{Z}, a \in A\}$, we have

$$
B_\gamma = \text{span}\{\Gamma_{(m,n)}(a) : m, n \in \mathbb{Z}, a \in A\}.
$$

**Lemma 3.1** Each homomorphism $\Delta_{(m,n)} : A \to B_\delta$ extends to a strictly continuous homomorphism $\overline{\Delta}_{(m,n)} : \mathcal{M}(A) \to \mathcal{M}(B_\delta)$ of multiplier algebras as well as each homomorphism $\Gamma_{(m,n)} : A \to B_\gamma$.

**Proof** We skip the proof as it is similar to the proof of [10, Lemma 3.2]. In brief, this is due to the extendibility of the endomorphisms $\delta_n$ and $\gamma_n$, and homomorphisms $\varphi_n$ and $\psi_n$.

It, therefore, follows by Lemma 3.1 that:

$$
\overline{\Delta}_{(m,n)}(c)(r) = \begin{cases} 
\varphi_n(\gamma_{r-m}(c)) & \text{if } m \leq r \\
0 & \text{otherwise},
\end{cases}
$$

for all $m, n \in \mathbb{Z}$ and $c \in \mathcal{M}(A)$ (similarly for $\overline{\Gamma}_{(m,n)}(c)$).

**Lemma 3.2** There is an isomorphism $\Lambda_\delta$ of $\mathcal{B}$ onto $\mathcal{B}_\delta$, such that

$$
\Lambda_\delta(\phi_{(m,n)}(a)) = \Delta_{(m,n)}(a),
$$

for all $m, n \in \mathbb{Z}$ and $a \in A$. Similarly, the algebra $\mathcal{B}$ is isomorphic to the algebra $\mathcal{B}_\gamma$ via an isomorphism $\Lambda_\gamma$, such that

$$
\Lambda_\gamma(\phi_{(m,n)}(a)) = \Gamma_{(m,n)}(a),
$$

for all $m, n \in \mathbb{Z}$ and $a \in A$.

**Proof** We only prove the existence of the isomorphism $\Lambda_{\delta}$ as the existence of the isomorphism $\Lambda_{\gamma}$ follows similarly. Define a map

$$
\Lambda_{\delta} : \text{span}\{\phi_{(m,n)}(a) : (m, n) \in \mathbb{Z}^2, a \in A\} \to \mathcal{B}_\delta,
$$

by

$$
\Lambda_{\delta}\left(\sum_i \phi_{(m_i,n_i)}(a_i)\right) = \sum_i \Delta_{(m_i,n_i)}(a_i). \tag{3.4}
$$
Obviously, $\Lambda_\delta$ is linear. We show that it preserves the norm, from which it follows that it is a well-defined linear isometry. First, for any Hilbert space $H$, there is an isomorphism $U$ of the Hilbert space

$$\ell^2(\mathbb{Z}^2, H) \simeq \ell^2(\mathbb{Z}) \otimes H \simeq (\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})) \otimes H,$$

onto the Hilbert space

$$\ell^2(\mathbb{Z}) \otimes (\ell^2(\mathbb{Z}) \otimes H) \simeq \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}, H),$$

which induces the following isomorphism:

$$T \in B(\ell^2(\mathbb{Z}^2) \otimes H) \mapsto UTU^{-1} \in B(\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}, H)), \quad (3.5)$$

of $C^*$-algebras. Now, let $\pi : A \to B(H)$ be a faithful and nondegenerate represen-tation of $A$ on a Hilbert space $H$. Then, the map $\tilde{\pi} : B \to B(\ell^2(\mathbb{Z}^2) \otimes H)$ defined by

$$(\tilde{\pi}(\xi)f)(r, s) = \pi(\xi(r, s))f(r, s) \quad \text{for all } \xi \in B \text{ and } f \in \ell^2(\mathbb{Z}^2) \otimes H$$

is a nondegenerate and faithful representation of $B$ on the Hilbert space $\ell^2(\mathbb{Z}^2) \otimes H$. On the other hand, let $\rho : D \to B(\ell^2(\mathbb{Z}, H))$ be the nondegenerate and faithful representation defined by

$$(\rho(\xi)f)(s) = \pi(\xi(s))f(s) \quad \text{for all } \xi \in D \text{ and } f \in \ell^2(\mathbb{Z}, H).$$

Then, $\rho$, itself, induces a map $\tilde{\rho} : B_\delta \to B(\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}, H))$ defined by

$$(\tilde{\rho}(\eta)g)(r) = \rho(\eta(r))g(r),$$

for all $\eta \in B_\delta$ and $g \in \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}, H)$. It is not difficult to see that $\tilde{\rho}$ is indeed a nondegenerate and faithful representation of the algebra $B_\delta$ on the Hilbert space $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}, H)$. Now, calculation on spanning elements shows that

$$U\tilde{\pi}\left(\sum_i \phi(m_i, n_i)(a_i)\right) = \tilde{\rho}\left(\sum_i \Delta(m_i, n_i)(a_i)\right)U, \quad (3.6)$$

from which it follows that:

$$\left\|\Lambda_\delta \left(\sum_i \phi(m_i, n_i)(a_i)\right)\right\| = \left\|\sum_i \Delta(m_i, n_i)(a_i)\right\| = \left\|\tilde{\rho}\left(\sum_i \Delta(m_i, n_i)(a_i)\right)\right\|.$$
\[
\| U \tilde{\pi} \left( \sum_i \phi_{(m_i, n_i)}(a_i) \right) U^{-1} \| = \| \tilde{\pi} \left( \sum_i \phi_{(m_i, n_i)}(a_i) \right) \| = \left\| \sum_i \phi_{(m_i, n_i)}(a_i) \right\|.
\]

Therefore, \( \Lambda_\delta \) is a well-defined linear map which preserves the norm, and therefore, it extends to a linear isometry of \( B \) into \( B_\delta \). We use the same notation \( \Lambda_\delta \) for the extension, which is clearly onto by (3.4).

Finally, one can easily calculate on spanning elements to see that \( \Lambda_\delta \) preserves involution and multiplication, too. Thus, \( \Lambda_\delta \) is an isomorphism of \( B \) onto \( B_\delta \).

\[\Box\]

**Proposition 3.3** The algebras \( B_\delta \) and \( B_\gamma \) contain the algebras \( C_0(\mathbb{Z}) \otimes D \) and \( C_0(\mathbb{Z}) \otimes C \), respectively, as essential ideals, such that

\[C_0(\mathbb{Z}) \otimes D = \text{span}\{ \Delta_{(m,n)}(a) - \Delta_{(t,n)}(\gamma_t - m(a)) : m, t, n \in \mathbb{Z} \text{ with } m \leq t, a \in A \}, \tag{3.7}\]

and

\[C_0(\mathbb{Z}) \otimes C = \text{span}\{ \Gamma_{(m,n)}(a) - \Gamma_{(m,u)}(\delta_{u-n}(a)) : m, n, u \in \mathbb{Z} \text{ with } n \leq u, a \in A \}. \tag{3.8}\]

**Proof** We only prove for \( C_0(\mathbb{Z}) \otimes D \) and skip the proof on \( C_0(\mathbb{Z}) \otimes C \) as it follows by a similar discussion. First, the right-hand side of (3.7) is in fact equal to

\[\text{span}\{ \Delta_{(m,n)}(a) - \Delta_{(m+1,n)}(\gamma_{t-m}(a)) : m, n \in \mathbb{Z}, a \in A \}. \tag{3.9}\]

This is due to the following calculation:

\[
\Delta_{(m,n)}(a) - \Delta_{(t,n)}(\gamma_{t-m}(a)) \\
= \left[ \Delta_{(m,n)}(a) - \Delta_{(m+1,n)}(\gamma_{a}(a)) \right] \\
+ \left[ \Delta_{(m+1,n)}(\gamma_{a}(a)) - \Delta_{(m+2,n)}(\gamma_{2(a)}(a)) \right] \\
+ \cdots + \left[ \Delta_{(t-1,n)}(\gamma_{t-m-1}(a)) - \Delta_{(t,n)}(\gamma_{t-m}(a)) \right] \\
= \sum_{r=1}^{t-m} \left[ \Delta_{(m+r-1,n)}(\gamma_{r-1}(a)) - \Delta_{(m+r,n)}(\gamma_{r}(a)) \right]. \tag{3.10}\]

Thus, we only need to show that

\[C_0(\mathbb{Z}) \otimes D = \text{span}\{ \Delta_{(m,n)}(a) - \Delta_{(m+1,n)}(\gamma_{a}(a)) : m, n \in \mathbb{Z}, a \in A \}. \tag{3.11}\]

Since

\[\Delta_{(m,n)}(a) - \Delta_{(m+1,n)}(\gamma_{a}(a)) = (\ldots, 0, 0, 0, \varphi_n(a), 0, 0, 0, \ldots),\]
where \( \varphi_n(a) \) is in the \( m \)th slot, and elements \( \ldots, 0, 0, \varphi_n(a), 0, 0, \ldots \) span the algebra \( C_0(\mathbb{Z}) \otimes D \); it follows that (3.11) holds.

Next, to show that \( C_0(\mathbb{Z}) \otimes D \) is an ideal of \( B_\delta \), it is enough to calculate the product

\[
\Delta_{(r,s)}(b) \left[ \Delta_{(m,n)}(a) - \Delta_{(m+1,n)}(\gamma(a)) \right].
\]

on spanning elements to see that it belongs to \( C_0(\mathbb{Z}) \otimes D \). However, we skip the calculation as it is similar to the one in the proof of [10, Proposition 3.3]. To see that the ideal \( C_0(\mathbb{Z}) \otimes D \) of \( B_\delta \) is essential, note that it follows easily from the fact that \( B_\delta \subset \ell^\infty(\mathbb{Z}, D) = \mathcal{M}(C_0(\mathbb{Z}) \otimes D) \). \( \square \)

**Theorem 3.4** Let

\[
J_\delta := \overline{\text{span}} \{ \phi_{(m,n)}(a) - \phi_{(t,n)}(\gamma_{t-m}(a)) : m, t, n \in \mathbb{Z} \text{ with } m \leq t, a \in A \},
\]

and

\[
J_\gamma := \overline{\text{span}} \{ \phi_{(m,n)}(a) - \phi_{(m,u)}(\delta_{u-n}(a)) : m, n, u \in \mathbb{Z} \text{ with } n \leq u, a \in A \}.
\]

Then, \( J_\delta \) and \( J_\gamma \) are essential ideals of \( B \), such that \( J_\delta + J_\gamma = \mathcal{J} \) and \( J_\delta \cap J_\gamma = C_0(\mathbb{Z}^2) \otimes A \simeq C_0(\mathbb{Z}) \otimes C_0(\mathbb{Z}) \otimes A \), which is an essential ideal of \( B \). Moreover, \( J_\delta \) and \( J_\gamma \) are isomorphic to the algebras \( C_0(\mathbb{Z}) \otimes D \) and \( C_0(\mathbb{Z}) \otimes C \), respectively.

**Proof** By applying the isomorphism \( \Lambda_\delta : B \to B_\delta \) in Lemma 3.2, since \( C_0(\mathbb{Z}) \otimes D \) is an essential ideal of \( B_\delta \) by Proposition 3.3, \( J_\delta := \Lambda_\delta^{-1}(C_0(\mathbb{Z}) \otimes D) \) is an essential ideal of \( B \) which is clearly isomorphic to \( C_0(\mathbb{Z}) \otimes D \). Moreover, the following equation:

\[
\Lambda_\delta^{-1} \left( \Delta_{(m,n)}(a) - \Delta_{(t,n)}(\gamma_{t-m}(a)) \right) = \phi_{(m,n)}(a) - \phi_{(t,n)}(\gamma_{t-m}(a))
\]

along with (3.11) implies that

\[
J_\delta = \overline{\text{span}} \{ \phi_{(m,n)}(a) - \phi_{(t,n)}(\gamma_{t-m}(a)) : m, t, n \in \mathbb{Z} \text{ with } m \leq t, a \in A \}
\]

\[
= \overline{\text{span}} \{ \phi_{(m,n)}(a) - \phi_{(m+1,n)}(\gamma(a)) : m, n \in \mathbb{Z}, a \in A \}.
\]

The proof of \( J_\gamma \) follows similarly (using the isomorphism \( \Lambda_\gamma \)). Therefore, we skip it here.

Next, we show that \( \mathcal{J} = J_\delta + J_\gamma \). The inclusion \( J_\delta + J_\gamma \subset \mathcal{J} \) is immediate. For the other inclusion, take any spanning element \( \phi_{(m,n)}(a) - \phi_{(t,u)}(\alpha_{(t-m,u-n)}(a)) \) of \( \mathcal{J} \), where \( a \in A \) and \((m, n), (t, u) \in \mathbb{Z}^2 \) with \((m, n) \leq (t, u)\). We have

\[
\phi_{(m,n)}(a) - \phi_{(t,u)}(\alpha_{(t-m,u-n)}(a)) = \left[ \phi_{(m,n)}(a) - \phi_{(t,n)}(\gamma_{t-m}(a)) \right] + \left[ \phi_{(t,n)}(\gamma_{t-m}(a)) - \phi_{(t,u)}(\delta_{u-n}(\gamma_{t-m}(a))) \right]
\]

\[
= \left[ \phi_{(m,n)}(a) - \phi_{(t,n)}(\gamma_{t-m}(a)) \right] + \left[ \phi_{(t,n)}(b) - \phi_{(t,u)}(\delta_{u-n}(b)) \right] \in (J_\delta + J_\gamma),
\]
where \( b = \gamma_{t-m}(a) \in A \). This implies that \( \mathcal{J} \subset \mathcal{J}_\delta + \mathcal{J}_\gamma \). Therefore, \( \mathcal{J} = \mathcal{J}_\delta + \mathcal{J}_\gamma \), from which it follows that the ideal \( \mathcal{J} \) is actually spanned by the elements of the form
\[
\phi_{(m,n)}(a) - \phi_{(m+1,n)}(\gamma(a)) + \phi_{(t,u)}(b) - \phi_{(t,u+1)}(\delta(b)),
\]
where \( m, n, t, u \in \mathbb{Z} \) and \( a, b \in A \).

Now, to see that \( \mathcal{J}_\delta \cap \mathcal{J}_\gamma = C_0(\mathbb{Z}^2) \otimes A \), first recall that \( C_0(\mathbb{Z}^2) \otimes A \) is spanned by the elements (finitely supported functions) \( f_{(m,n)}^a : \mathbb{Z}^2 \to A \) defined by
\[
f_{(m,n)}^a(r, s) = \begin{cases} 
a & \text{if } (r, s) = (m, n) \\
0 & \text{otherwise} \end{cases},
\]
for every \( a \in A \) and \( (m, n) \in \mathbb{Z}^2 \). Then, it is not difficult to see that each function \( f_{(m,n)}^a \) is actually equal to the element
\[
\phi_{(m,n)}(a) = \phi_{(m+1,n)}(\gamma(a)) - [\phi_{(m,n+1)}(\delta(a)) - \phi_{(m+1,n+1)}(\alpha_{(1,1)}(a))],
\]
of \( \mathcal{J}_\delta \), which is also equal to
\[
\phi_{(m,n)}(a) = \phi_{(m,n+1)}(\delta(a)) - [\phi_{(m+1,n)}(\gamma(a)) - \phi_{(m+1,n+1)}(\alpha_{(1,1)}(a))] \in \mathcal{J}_\gamma,
\]
where \( \alpha_{(1,1)}(a) = \gamma(\delta(a)) = \delta(\gamma(a)) \). This implies that \( C_0(\mathbb{Z}^2) \otimes A \subset \mathcal{J}_\delta \cap \mathcal{J}_\gamma \). For the other inclusion, as \( \mathcal{J}_\delta \cap \mathcal{J}_\gamma = \mathcal{J}_\delta \mathcal{J}_\gamma \), it is enough to show that each product
\[
\phi_{(m,n)}(a) - \phi_{(m+1,n)}(\gamma(\delta(\gamma(a)))|\phi_{(t,u)}(b) - \phi_{(t,u+1)}(\delta(b))
\]
of the spanning elements of \( \mathcal{J}_\delta \) and \( \mathcal{J}_\gamma \) belongs to \( C_0(\mathbb{Z}^2) \otimes A \). To calculate the product (3.16) (of two functions in \( \ell^\infty(\mathbb{Z}^2, A) \)), think of the intersection point of two discrete rays in \( \mathbb{R}^2 \). One is vertical corresponding to \( \phi_{(m,n)}(a) - \phi_{(m+1,n)}(\gamma(a)) \) with the initial point \( (m, n) \), and the other one is horizontal corresponding to \( \phi_{(t,u)}(b) - \phi_{(t,u+1)}(\delta(b)) \) with the initial point \( (t, u) \). These two rays have only one intersection at the point \( (m, n) \) if \( m \geq t \) and \( n \leq u \). Otherwise, there is no intersection point. This is equivalent to saying that the product (3.16), as a function in \( \ell^\infty(\mathbb{Z}^2, A) \), is nonzero only when \( m \geq t \) and \( n \leq u \). Therefore, in this case, we have
\[
\phi_{(m,n)}(a) - \phi_{(m+1,n)}(\gamma(a))|\phi_{(t,u)}(b) - \phi_{(t,u+1)}(\delta(b))
\]
\[
= \phi_{(m,n)}(a)\phi_{(t,u)}(b) - \phi_{(m,n)}(a)\phi_{(t,u+1)}(\delta(b)) - \phi_{(m+1,n)}(\gamma(a))\phi_{(t,u)}(b) + \phi_{(m+1,n)}(\gamma(a))\phi_{(t,u+1)}(\delta(b))
\]
\[
= \phi_{(m,u)}(\delta_{u-n}(a)\gamma_{m-t}(b)) - \phi_{(m,u+1)}(\delta_{u-n+1}(a)\gamma_{m-t}(\delta(b))) - \phi_{(m+1,u)}(\delta_{u-n}(\gamma(a))\gamma_{m-t+1}(b)) + \phi_{(m+1,u+1)}(\delta_{u-n+1}(\gamma(a))\gamma_{m-t+1}(\delta(b))) \quad \text{[by (2.3)]}
\]
\[
= \left[ \phi_{(m,u)}(\delta_{u-n}(a)\gamma_{m-t}(b)) - \phi_{(m,u+1)}(\delta_{u-n+1}(a)\delta(\gamma_{m-t}(b))) \right]
\]
which is equal to the spanning element \( f^{e}_{(m,u)} \) of \( C_{0}(\mathbb{Z}^{2}) \otimes A \) (see 3.13–3.15). Therefore, the product (3.16) belongs to \( C_{0}(\mathbb{Z}^{2}) \otimes A \), and hence, \( J_{\delta} \cap J_{\gamma} \subset C_{0}(\mathbb{Z}^{2}) \otimes A \). At last, since the ideals \( J_{\delta} \) and \( J_{\gamma} \) are both essential, it follows that \( J_{\delta} \cap J_{\gamma} = C_{0}(\mathbb{Z}^{2}) \otimes A \) must be an essential ideal of \( B \). \( \square \)

The following are two remarks that are required for Theorem 3.7.

**Remark 3.5** Suppose that \((A \rtimes_{\alpha} \Gamma, i_A, i_{\Gamma})\) and \((B \rtimes_{\beta} G, i_B, i_{G})\) are the group crossed products of the dynamical systems \((A, \Gamma, \alpha)\) and \((B, G, \beta)\) by discrete groups, respectively. Recall that there is an action \( \alpha \otimes \beta \) of the group \( \Gamma \times G \) on the maximal tensor product \( A \otimes_{\max} B \) by automorphisms, such that \( (\alpha \otimes \beta)(s,t) = \alpha_{s} \otimes_{\max} \beta_{t} \) for every \((s,t) \in \Gamma \times G \). Then, the corresponding group crossed product \((A \otimes_{\max} B) \rtimes_{\alpha \otimes \beta} (\Gamma \times G) \) can be decomposed as the maximal tensor product of the crossed products \( A \rtimes_{\alpha} \Gamma \) and \( B \rtimes_{\beta} G \). More precisely, there is an isomorphism

\[
\Pi : ((A \otimes_{\max} B) \rtimes_{\alpha \otimes \beta} (\Gamma \times G), i) \to (A \rtimes_{\alpha} \Gamma) \otimes_{\max} (B \rtimes_{\beta} G),
\]

such that

\[
\Pi \left( i_{(A \otimes_{\max} B)}(a \otimes b)i_{\Gamma \times G}(s,t) \right) = i_{A}(a)i_{\Gamma}(s) \otimes i_{B}(b)i_{G}(t),
\]

for all \( a \in A, b \in B, \) and \((s,t) \in \Gamma \times G \). Now, if in particular \( \Gamma = G \), then one can see that the map \( \gamma : G \times G \to \text{Aut}(A \otimes_{\max} B) \) defined by

\[
\gamma_{(s,t)} = (\alpha \otimes \beta)(t,s) = \alpha_{t} \otimes_{\max} \beta_{s} \text{ for all } (s,t) \in G \times G
\]

is an action of the group \( G \times G \) on the algebra \( A \otimes_{\max} B \) by automorphisms. If \(((A \otimes_{\max} B) \rtimes_{\gamma} (G \times G), k)\) is the group crossed product of the system \((A \otimes_{\max} B, G \times G, \gamma)\), then it is not difficult to see that it is isomorphic to the crossed product \(((A \otimes_{\max} B) \rtimes_{\alpha \otimes \beta} (G \times G), i)\) via an isomorphism \( \Pi_{2} \), such that

\[
\Pi_{2} \left( k_{(A \otimes_{\max} B)}(a \otimes b)k_{G \times G}(s,t) \right) = i_{(A \otimes_{\max} B)}(a \otimes b)i_{G \times G}(t,s),
\]
for all \(a \in A, b \in B\), and \(s, t \in G\). Therefore, the composition

\[
(A \otimes_{\text{max}} B) \rtimes \gamma (G \times G) \xrightarrow{\Pi_2} (A \otimes_{\text{max}} B) \rtimes_{\alpha \otimes \beta} (G \times G) \\
\xrightarrow{\Pi} (A \rtimes_{\alpha} G) \otimes_{\text{max}} (B \rtimes_{\beta} G)
\]

of isomorphisms gives an isomorphism

\[
\Pi_3 : \left( (A \otimes_{\text{max}} B) \rtimes \gamma (G \times G), k \right) \rightarrow (A \rtimes_{\alpha} G, i) \otimes_{\text{max}} (B \rtimes_{\beta} G, j),
\]

such that

\[
\Pi_3 \left( k(a \otimes b)k_{G \times G}(s, t) \right) = i_A(a)i_G(t) \otimes j_B(b)j_G(s),
\]

for all \(a \in A, b \in B\), and \(s, t \in G\).

**Remark 3.6** Let \((A \rtimes_{\alpha} G, i)\) be the group crossed product of a dynamical system \((A, G, \alpha)\) by discrete group. If \(I\) and \(J\) are two \(\alpha\)-invariant ideals of the algebra \(A\), then one can compute on spanning elements to see that we have

\[
(I \rtimes_{\alpha} G) + (J \rtimes_{\alpha} G) = (I + J) \rtimes_{\alpha} G,
\]

(3.18)

and

\[
(I \rtimes_{\alpha} G) \cap (J \rtimes_{\alpha} G) = (I \cap J) \rtimes_{\alpha} G.
\]

(3.19)

**Theorem 3.7** Consider the (essential) ideals \(J_\delta, J_\gamma\) and \(J\) of the algebra \(B\) in the group dynamical system \((B, \mathbb{Z}^2, \beta)\) induced by the system \((A, \mathbb{N}^2, \alpha)\). The ideals \(J_\delta\) and \(J_\gamma\) are \(\beta\)-invariant, such that

\[
(J_\delta \rtimes_{\beta} \mathbb{Z}^2) + (J_\gamma \rtimes_{\beta} \mathbb{Z}^2) = J \rtimes_{\beta} \mathbb{Z}^2,
\]

(3.20)

and

\[
(J_\delta \rtimes_{\beta} \mathbb{Z}^2) \cap (J_\gamma \rtimes_{\beta} \mathbb{Z}^2) \simeq \mathcal{K} (\ell^2 (\mathbb{Z}^2)) \otimes A,
\]

(3.21)

which is an essential ideal of \((B \rtimes_{\beta} \mathbb{Z}^2, j)\). In addition, the ideals \(J_\delta \rtimes_{\beta} \mathbb{Z}^2\) and \(J_\gamma \rtimes_{\beta} \mathbb{Z}^2\) of \(B \rtimes_{\beta} \mathbb{Z}^2\) are essentails, and isomorphic to the algebras \(\mathcal{K} (\ell^2 (\mathbb{Z})) \otimes (D \rtimes_{\tau} \mathbb{Z})\) and \(\mathcal{K} (\ell^2 (\mathbb{Z})) \otimes (C \rtimes_{\tau} \mathbb{Z})\) of compact operators, respectively, where \(\tau\) denotes the action of \(\mathbb{Z}\) on the subalgebras \(D\) and \(C\) of \(\ell^\infty (\mathbb{Z}, A)\) by the left translation.

**Proof** First, one can easily see that the essential ideals \(J_\delta\) and \(J_\gamma\) of \(B\) are \(\beta\)-invariant, and hence, the algebras \(J_\delta \rtimes_{\beta} \mathbb{Z}^2\) and \(J_\gamma \rtimes_{\beta} \mathbb{Z}^2\) are essential ideals of \(B \rtimes_{\beta} \mathbb{Z}^2\) (see [5, Proposition 2.4]). Now, Eq. (3.20) follows immediately by (3.18) in Remark 3.6 as
$\mathcal{J}_\delta + \mathcal{J}_\gamma = \mathcal{J}$ (see Theorem 3.4). It thus follows that $\mathcal{J} \rtimes_\beta \mathbb{Z}^2$ is indeed spanned by the elements of the form:

$$
\begin{align*}
& \left[ j_B \left( \phi_{(m,n)}(a) - \phi_{(m+1,n)}(\gamma(a)) \right) j_{\mathbb{Z}^2}(x, y) \right] \\
& \quad + \left[ j_B \left( \phi_{(t,u)}(b) - \phi_{(t,u+1)}(\delta(b)) \right) j_{\mathbb{Z}^2}(r, s) \right],
\end{align*}
$$

where $a, b \in A$ and $(m, n), (x, y), (t, u), (r, s) \in \mathbb{Z}^2$. To see (3.21), we apply (3.19) in Remark 3.6, and since $\mathcal{J}_\delta \cap \mathcal{J}_\gamma = C_0(\mathbb{Z}^2) \otimes A$ (see Theorem 3.4), we have

$$(\mathcal{J}_\delta \rtimes_\beta \mathbb{Z}^2) \cap (\mathcal{J}_\gamma \rtimes_\beta \mathbb{Z}^2) = (\mathcal{J}_\delta \cap \mathcal{J}_\gamma) \rtimes_\beta \mathbb{Z}^2 = (C_0(\mathbb{Z}^2) \otimes A) \rtimes_\beta \mathbb{Z}^2,$$

where it is known that the crossed product $(C_0(\mathbb{Z}^2) \otimes A) \rtimes_\beta \mathbb{Z}^2$ is isomorphic to the algebra

$$\mathcal{K}(\ell^2(\mathbb{Z}^2) \otimes A) \simeq \mathcal{K}(\ell^2(\mathbb{Z})) \otimes \mathcal{K}(\ell^2(\mathbb{Z})) \otimes A,$$

of compact operators. Moreover, since $C_0(\mathbb{Z}^2) \otimes A$ is an essential ideal of $B$, $(C_0(\mathbb{Z}^2) \otimes A) \rtimes_\beta \mathbb{Z}^2 \simeq \mathcal{K}(\ell^2(\mathbb{Z})) \otimes A$ is an essential ideal of $B \rtimes_\beta \mathbb{Z}^2$ (again by [5, Proposition 2.4]).

Next, let $\text{lt}$ denote the action of $\mathbb{Z}$ on $C_0(\mathbb{Z})$ by the left translation. Then, consider the action $\text{lt} \otimes \tau$ of $\mathbb{Z}^2$ on the algebra $C_0(\mathbb{Z}) \otimes D$, and let $\sigma$ be the action of $\mathbb{Z}^2$ on the algebra $C_0(\mathbb{Z}) \otimes C$ defined by

$$\sigma_{(m,n)} = (\text{lt} \otimes \tau)(n,m) = \text{lt}_n \otimes \tau_m,$$

for all $(m, n) \in \mathbb{Z}^2$ (see Remark 3.5). Now, we have

$$(\text{lt} \otimes \tau) \circ \Lambda_\delta = \Lambda_\delta \circ \beta \text{ and } \sigma \circ \Lambda_\gamma = \Lambda_\gamma \circ \beta,$$

from which it follows that the systems $(\mathcal{J}_\delta, \mathbb{Z}^2, \beta)$ and $(C_0(\mathbb{Z}) \otimes D, \mathbb{Z}^2, \text{lt} \otimes \tau)$ are equivariantly isomorphic as well as $(\mathcal{J}_\gamma, \mathbb{Z}^2, \beta)$ and $(C_0(\mathbb{Z}) \otimes C, \mathbb{Z}^2, \sigma)$. Therefore, by [8, Lemma 2.65], the crossed products $\mathcal{J}_\delta \rtimes_\beta \mathbb{Z}^2$ and $\mathcal{J}_\gamma \rtimes_\beta \mathbb{Z}^2$ are isomorphic to $(C_0(\mathbb{Z}) \otimes D) \rtimes_{\text{lt} \otimes \tau} \mathbb{Z}^2$ and $(C_0(\mathbb{Z}) \otimes C) \rtimes_\sigma \mathbb{Z}^2$, respectively. To be more precise, there are isomorphisms

$$\Psi_1 : (\mathcal{J}_\delta \rtimes_\beta \mathbb{Z}^2, j) \rightarrow ((C_0(\mathbb{Z}) \otimes D) \rtimes_{\text{lt} \otimes \tau} \mathbb{Z}^2, k),$$

and

$$\Psi_2 : (\mathcal{J}_\gamma \rtimes_\beta \mathbb{Z}^2, j) \rightarrow ((C_0(\mathbb{Z}) \otimes C) \rtimes_\sigma \mathbb{Z}^2, i),$$

such that

$$\Psi_1(j_B(\xi)j_{\mathbb{Z}^2}(x, y)) = k_{(C_0(\mathbb{Z}) \otimes D)}(\Lambda_\delta(\xi))k_{\mathbb{Z}^2}(x, y),$$

$\otimes$ Birkhäuser
and
\[
\Psi_2(j_B(\eta)j_{\mathbb{Z}^2}(x, y)) = i_{(C_0(\mathbb{Z}) \otimes D)(\Lambda_\gamma(\eta))}i_{\mathbb{Z}^2}(x, y),
\]
for all $\xi \in \mathcal{F}_\delta$, $\eta \in \mathcal{F}_\gamma$, and $(x, y) \in \mathbb{Z}^2$. Moreover, the crossed products $(C_0(\mathbb{Z}) \otimes D) \rtimes_{\text{lt}} \mathbb{Z}^2$ and $(C_0(\mathbb{Z}) \otimes C) \rtimes_{\sigma} \mathbb{Z}^2$ are decomposed as the tensor products $(C_0(\mathbb{Z}) \rtimes_{\text{lt}} \mathbb{Z}) \otimes (D \rtimes \mathbb{Z})$ and $(C_0(\mathbb{Z}) \rtimes_{\text{lt}} \mathbb{Z}) \otimes (C \rtimes \mathbb{Z})$, respectively (see Remark 3.5), via the isomorphisms
\[
\Psi_3 : (C_0(\mathbb{Z}) \otimes D) \rtimes_{\text{lt}} \mathbb{Z}^2 \rightarrow (C_0(\mathbb{Z}) \rtimes_{\text{lt}} \mathbb{Z}, i) \otimes (D \rtimes \mathbb{Z}, k)
\]
and
\[
\Psi_4 : (C_0(\mathbb{Z}) \otimes C) \rtimes_{\sigma} \mathbb{Z}^2 \rightarrow (C_0(\mathbb{Z}) \rtimes_{\text{lt}} \mathbb{Z}, i) \otimes (C \rtimes \mathbb{Z}, \tilde{i}),
\]
such that
\[
\Psi_3(k_{(C_0(\mathbb{Z}) \otimes D)}(f \otimes \xi)k_{\mathbb{Z}^2}(x, y)) = [i_{C_0(\mathbb{Z})}(f)i_{\mathbb{Z}}(x)] \otimes [k_{D}(\xi)k_{\mathbb{Z}}(y)]
\]
and
\[
\Psi_4(i_{(C_0(\mathbb{Z}) \otimes C)}(f \otimes \eta)i_{\mathbb{Z}^2}(x, y)) = [i_{C_0(\mathbb{Z})}(f)i_{\mathbb{Z}}(y)] \otimes [\tilde{i}_{C}(\eta)\tilde{i}_{\mathbb{Z}}(x)],
\]
for all $f \in C_0(\mathbb{Z}), \xi \in D, \eta \in C$, and $x, y \in \mathbb{Z}$. Also, the crossed product $C_0(\mathbb{Z}) \rtimes_{\text{lt}} \mathbb{Z}$ is isomorphic to the algebra $\mathcal{K}(\ell^2(\mathbb{Z}))$ of compact operators on $\ell^2(\mathbb{Z})$. The isomorphism is given by
\[
\Psi_5 : (C_0(\mathbb{Z}) \rtimes_{\text{lt}} \mathbb{Z}, i) \rightarrow \mathcal{K}(\ell^2(\mathbb{Z})),
\]
such that
\[
\Psi_5(i_{C_0(\mathbb{Z})}(f)i_{\mathbb{Z}}(x)) = e_r \otimes e_{\bar{r}-x},
\]
where $f = (..., 0, 0, 1, 0, 0, ...) \in C_0(\mathbb{Z})$ with 1 in $r$th slot, $\{e_r : r \in \mathbb{Z}\}$ is the usual orthonormal basis of $\ell^2(\mathbb{Z})$, and $e_r \otimes e_{\bar{r}-x}$ is the rank-one operator on $\ell^2(\mathbb{Z})$ defined by $h \mapsto \langle h | e_{\bar{r}-x}\rangle e_r$ for all $h \in \ell^2(\mathbb{Z})$. Therefore, the composition
\[
\mathcal{J}_{\delta} \rtimes_{\beta} \mathbb{Z}^2 \xrightarrow{\Psi_1} (C_0(\mathbb{Z}) \otimes D) \rtimes_{\text{lt}} \mathbb{Z}^2 \xrightarrow{\Psi_3} (C_0(\mathbb{Z}) \rtimes_{\text{lt}} \mathbb{Z}) \otimes (D \rtimes \mathbb{Z})
\]
\[
\xrightarrow{\Psi_5 \otimes \text{id}} \mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D \rtimes \mathbb{Z})
\]
of isomorphisms gives an isomorphism
\[
\Psi_7 : \mathcal{J}_{\delta} \rtimes_{\beta} \mathbb{Z}^2 \rightarrow \mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D \rtimes \mathbb{Z}),
\]
such that

\[ \Psi_7 \left[ j_B (\phi_{(m,n)}(a) - \phi_{(m+1,n)}(y(a))) \right] j_{\mathbb{Z}_2}(x, y) = \left[ e_m \otimes e_{m-x} \right] \otimes [k_B (\varphi_n(a)) k_{\mathbb{Z}}(y)], \]

for all \( a \in A \) and \((m, n), (x, y) \in \mathbb{Z}_2\). Note that the restriction of \( \Psi_7 \) to the ideal \((C_0(\mathbb{Z}_2) \otimes A) \rtimes_\beta \mathbb{Z}_2\) of \( J_\delta \rtimes_\beta \mathbb{Z}_2\) is the canonical isomorphism of the crossed product \((C_0(\mathbb{Z}_2) \otimes A) \rtimes_\beta \mathbb{Z}_2\) onto the algebra \( K(\ell^2(\mathbb{Z})) \otimes (K(\ell^2(\mathbb{Z})) \otimes A)\) of compact operators.

Similarly, the composition \((\Psi_5 \otimes \text{id}) \circ \Psi_4 \circ \Psi_2\) of isomorphisms gives an isomorphism

\[ \Psi_6 : \mathcal{J}_Y \rtimes_\beta \mathbb{Z}_2 \to K(\ell^2(\mathbb{Z})) \otimes (C \rtimes_\tau \mathbb{Z}), \]

such that

\[ \Psi_6 \left[ j_B (\phi_{(t,u)}(a) - \phi_{(t,u+1)}(\delta(a))) \right] j_{\mathbb{Z}_2}(r, s) = \left[ e_u \otimes e_{u-s} \right] \otimes \left[ i_c (\psi_1(a)) i_{\mathbb{Z}_2}(r) \right], \]

for all \( a \in A \) and \((t, u), (r, s) \in \mathbb{Z}_2\).

We are now ready to import the information from \( \mathcal{B} \rtimes_\beta \mathbb{Z}_2\) to the crossed product \( A \rtimes_\alpha^{\text{piso}} \mathbb{N}_2\) to get the desired composition series (3.1) of ideals and identify the subquotients with familiar algebras.

**Lemma 3.8** Let

\[ \mathcal{I}_\delta = \overline{\text{span}} \left\{ \xi_{(m,n)}^{(x,y)}(a) : m, n, x, y \in \mathbb{N}, a \in A \right\}, \]

and

\[ \mathcal{I}_Y = \overline{\text{span}} \left\{ \eta_{(m,n)}^{(x,y)}(a) : m, n, x, y \in \mathbb{N}, a \in A \right\}, \]

where

\[ \xi_{(m,n)}^{(x,y)}(a) := i_{\mathbb{N}_2}(m, n)^* i_A(a) [1 - i_{\mathbb{N}_2}(1, 0)^* i_{\mathbb{N}_2}(1, 0)] i_{\mathbb{N}_2}(x, y) \]

and

\[ \eta_{(m,n)}^{(x,y)}(a) := i_{\mathbb{N}_2}(m, n)^* i_A(a) [1 - i_{\mathbb{N}_2}(0, 1)^* i_{\mathbb{N}_2}(0, 1)] i_{\mathbb{N}_2}(x, y). \]

Then, \( \mathcal{I}_\delta \) and \( \mathcal{I}_Y \) are essential ideals of \( A \rtimes_\alpha^{\text{piso}} \mathbb{N}_2\), such that \( \mathcal{I}_\delta + \mathcal{I}_Y = \ker q\). Moreover, we have

\[ \xi_{(m,n)}^{(x,y)}(a) - \xi_{(m,n+1)}^{(x,y+1)}(\delta(a)) = \eta_{(m,n)}^{(x,y)}(a) - \eta_{(m+1,n)}^{(x+1,y)}(\gamma(a)), \tag{3.22} \]

for all \( a \in A \) and \( m, n, x, y \in \mathbb{N} \), and

\[ \mathcal{I}_\delta \cap \mathcal{I}_Y = \overline{\text{span}} \left\{ \xi_{(m,n)}^{(x,y)}(a) - \xi_{(m,n+1)}^{(x,y+1)}(\delta(a)) : m, n, x, y \in \mathbb{N}, a \in A \right\}. \tag{3.23} \]

\[ \text{Birkhäuser} \]
which is also an essential ideal of \( A \times_{\alpha}^{\text{piso}} \mathbb{N}^2 \).

**Proof** First, it follows from Theorem 3.7 that:

\[
p(J \times_{\beta} \mathbb{Z}^2)p = p(J_\delta \times_{\beta} \mathbb{Z}^2 + J_\gamma \times_{\beta} \mathbb{Z}^2)p = p(J_\delta \times_{\beta} \mathbb{Z}^2)p + p(J_\gamma \times_{\beta} \mathbb{Z}^2)p, \tag{3.24}
\]

where \( p = \overline{\psi_1(\bar{f}(0,0)(1))} \) (see §2). Then, by exactly the same computation done in [10, Lemma 4.2], one can see that the corners \( p(J_\delta \times_{\beta} \mathbb{Z}^2)p \) and \( p(J_\gamma \times_{\beta} \mathbb{Z}^2)p \) are full. Moreover, they are indeed essential ideals of the algebra (full corner) \( p(B \times_{\beta} \mathbb{Z}^2)p \).

Now, by applying the isomorphism \( \Psi \) of \( A \times_{\alpha}^{\text{piso}} \mathbb{N}^2 \) onto \( p(B \times_{\beta} \mathbb{Z}^2)p \) (see [10, Theorem 4.1] or §2), it follows that:

\[
I_\delta := \Psi^{-1}(p(J_\delta \times_{\beta} \mathbb{Z}^2)p) \quad \text{and} \quad I_\gamma := \Psi^{-1}(p(J_\gamma \times_{\beta} \mathbb{Z}^2)p),
\]

are two essential ideals of \( A \times_{\alpha}^{\text{piso}} \mathbb{N}^2 \), such that

\[
\ker q = \Psi^{-1}(p(J \times_{\beta} \mathbb{Z}^2)p) = \Psi^{-1}(p(J_\delta \times_{\beta} \mathbb{Z}^2)p) + \Psi^{-1}(p(J_\gamma \times_{\beta} \mathbb{Z}^2)p) = I_\delta + I_\gamma.
\]

Next, we show that the ideal \( I_\delta \) is spanned by the elements \( \xi^{(x,y)(m,n)}_p(a) \) and skip the discussion on \( I_\gamma \), as it follows similarly. To do so, first note that the algebra \( p(J_\delta \times_{\beta} \mathbb{Z}^2)p \) is spanned by elements of the form

\[
p\left[j_B(\phi_{(m,n)}(a) - \phi_{(m+1,n)}(\gamma(a))) j_{\mathbb{Z}^2}(x,y)\right]p,
\]

where \( a \in A \) and \((m, n)\), \((x, y)\) \( \in \mathbb{Z}^2 \). However, again by exactly the same computation available in [10, Lemma 4.2] along with applying the covariance equation of the pair \((j_B, j_{\mathbb{Z}^2})\), it follows that \( p(J_\delta \times_{\beta} \mathbb{Z}^2)p \) is precisely spanned by the elements:

\[
p\left[j_{\mathbb{Z}^2}(m,n)j_B(\phi_{(0,0)}(a) - \phi_{(1,0)}(\gamma(a))) j_{\mathbb{Z}^2}(x,y)^*\right]p, \tag{3.25}
\]

where \( a \in A \) and \((m, n)\), \((x, y)\) \( \in \mathbb{N}^2 \). Therefore, elements of the form

\[
\Psi^{-1}\left( p[j_{\mathbb{Z}^2}(m,n)j_B(\phi_{(0,0)}(a) - \phi_{(1,0)}(\gamma(a))) j_{\mathbb{Z}^2}(x,y)^*]p \right)
= i_{\mathbb{N}^2}(m,n)^*i_A(a)[1 - i_{\mathbb{N}^2}(1,0)^*i_{\mathbb{N}^2}(1,0)]i_{\mathbb{N}^2}(x,y)
= \xi^{(x,y)}_{(m,n)}(a) \quad \text{(see [10, Lemma 4.2])}
\]

span the ideal \( \Psi^{-1}(p(J_\delta \times_{\beta} \mathbb{Z}^2)p) = I_\delta \). Consequently, the ideal \( \ker q \) is indeed spanned by the elements

\[
\xi^{(x,y)}_{(m,n)}(a) + \eta^{(i,j)}_{(r,s)}(b). \tag{3.26}
\]
To see Eq. (3.22), for convenience, let

\[ P_{(1,0)} = 1 - i_{N^2}(1, 0)^*i_{N^2}(1, 0) \quad \text{and} \quad P_{(0,1)} = 1 - i_{N^2}(0, 1)^*i_{N^2}(0, 1), \]

which are two projections in \( \mathcal{M}(A \times_{\alpha} N^2) \). Now, by applying the covariance equation of the pair \((i_A, i_{N^2})\), we have

\[
\xi_{(m,n)}^{(x,y)}(a) - \xi_{(m,n+1)}^{(x,y+1)}(\delta(a)) \\
= i_{N^2}(m, n)^*i_A(a)P_{(1,0)}i_{N^2}(x, y) - i_{N^2}(m, n + 1)^*i_A(\delta(a))P_{(1,0)}i_{N^2}(x, y + 1) \\
= i_{N^2}(m, n)^*i_A(a)P_{(1,0)}i_{N^2}(x, y) \\
- i_{N^2}(m, n)^*i_{N^2}(0, 1)^*i_A(\delta(a))P_{(1,0)}i_{N^2}(0, 1)i_{N^2}(x, y) \\
= i_{N^2}(m, n)^*i_A(a)P_{(1,0)}i_{N^2}(x, y) \\
- i_{N^2}(m, n)^*i_A(0, 1)^*i_{N^2}(0, 1)^*P_{(1,0)}i_{N^2}(0, 1)i_{N^2}(x, y) \\
= i_{N^2}(m, n)^*i_A(a)\left[P_{(1,0)} - i_{N^2}(0, 1)^*P_{(1,0)}i_{N^2}(0, 1)\right]i_{N^2}(x, y).
\]

Then, in the bottom line, for \( \left[ P_{(1,0)} - i_{N^2}(0, 1)^*P_{(1,0)}i_{N^2}(0, 1) \right] \), we have

\[
P_{(1,0)} - i_{N^2}(0, 1)^*P_{(1,0)}i_{N^2}(0, 1) \\
= [1 - i_{N^2}(1, 0)^*i_{N^2}(1, 0)] - i_{N^2}(0, 1)^*[1 - i_{N^2}(1, 0)^*i_{N^2}(1, 0)]i_{N^2}(0, 1) \\
= 1 - i_{N^2}(1, 0)^*i_{N^2}(1, 0) - i_{N^2}(0, 1)^*i_{N^2}(0, 1) + i_{N^2}(1, 1)^*i_{N^2}(1, 1) \\
= [1 - i_{N^2}(0, 1)^*i_{N^2}(0, 1)] - [i_{N^2}(1, 0)^*i_{N^2}(1, 0) - i_{N^2}(1, 1)^*i_{N^2}(1, 1)] \\
= [1 - i_{N^2}(0, 1)^*i_{N^2}(0, 1)] - i_{N^2}(1, 0)^*[1 - i_{N^2}(0, 1)^*i_{N^2}(0, 1)]i_{N^2}(1, 0) \\
= P_{(0,1)} - i_{N^2}(1, 0)^*P_{(0,1)}i_{N^2}(1, 0).
\]

Therefore, it follows that:

\[
\xi_{(m,n)}^{(x,y)}(a) - \xi_{(m,n+1)}^{(x,y+1)}(\delta(a)) \\
= i_{N^2}(m, n)^*i_A(a)\left[P_{(0,1)} - i_{N^2}(1, 0)^*P_{(0,1)}i_{N^2}(1, 0)\right]i_{N^2}(x, y) \\
= i_{N^2}(m, n)^*i_A(a)P_{(0,1)}i_{N^2}(x, y) \\
- i_{N^2}(m, n)^*i_{N^2}(1, 0)^*P_{(0,1)}i_{N^2}(1, 0)i_{N^2}(x, y) \\
= i_{N^2}(m, n)^*i_A(a)P_{(0,1)}i_{N^2}(x, y) \\
- i_{N^2}(m, n)^*i_{N^2}(1, 0)^*i_{N^2}(y(\gamma))P_{(0,1)}i_{N^2}(x, y + 1) \\
= i_{N^2}(m, n)^*i_A(a)P_{(0,1)}i_{N^2}(x, y) - i_{N^2}(m + 1, n)^*i_A(y(\gamma))P_{(0,1)}i_{N^2}(x, y + 1) \\
= \eta_{(m,n)}^{(x,y)}(a) - \eta_{(m+1,n)}^{(x+1,y)}(\gamma(\gamma)).
\]

Finally, to see (3.23), it follows immediately by (3.22) that the right-hand side of (3.23) is contained in \( \mathcal{I}_{\delta} \cap \mathcal{I}_{\gamma} \). To see the other inclusion, since \( \mathcal{I}_\delta \cap \mathcal{I}_\gamma = \mathcal{I}_{\delta}\mathcal{I}_{\gamma} \), it is
enough to see that each product
\[ \xi_{(m,n)}^{(x,y)}(a)\eta_{(r,s)}^{(t,u)}(b), \]
of the spanning elements of \( I_\delta \) and \( I_\gamma \) is in the right-hand side of (3.23). For convenience, first, let
\[ \tilde{\xi}_{(m,n)}^{(x,y)}(a) := p[j_{\mathbb{Z}^2}(m, n) j_B (\phi_{(0,0)}(a) - \phi_{(1,0)}(\gamma(a))) j_{\mathbb{Z}^2}(x, y)^*]p \]
and
\[ \tilde{\eta}_{(r,s)}^{(t,u)}(b) := p[j_{\mathbb{Z}^2}(r, s) j_B (\phi_{(0,0)}(b) - \phi_{(0,1)}(\delta(b))) j_{\mathbb{Z}^2}(t, u)^*]p. \]
Then, since
\[ \xi_{(m,n)}^{(x,y)}(a)\eta_{(r,s)}^{(t,u)}(b) = \Psi^{-1} \left( \tilde{\xi}_{(m,n)}^{(x,y)}(a)\tilde{\eta}_{(r,s)}^{(t,u)}(b) \right), \]
we need to compute the product
\[ \tilde{\xi}_{(m,n)}^{(x,y)}(a)\tilde{\eta}_{(r,s)}^{(t,u)}(b), \]
for which we can use Eq. (3.17) in the proof of Theorem 3.4. Therefore, by applying the covariance equation of \((j_B, j_{\mathbb{Z}^2})\) and (2.4), we have
\[ \tilde{\xi}_{(m,n)}^{(x,y)}(a)\tilde{\eta}_{(r,s)}^{(t,u)}(b) = p[j_{\mathbb{Z}^2}(m, n) j_{\mathbb{Z}^2}(x, y)^* j_B(\mu) j_{\mathbb{Z}^2}(r, s) j_{\mathbb{Z}^2}(t, u)^*]p, \]
where
\[ \mu = [\phi_{(x,y)}(a\overline{a}_{(x,y)}(1)) - \phi_{(x+1,y)}(\gamma(a)\overline{a}_{(x+1,y)}(1))] [\phi_{(r,s)}(b) - \phi_{(r,s+1)}(\delta(b))]. \]
Now, as it was discussed in the proof of Theorem 3.4, if \( x \geq r \) and \( y \leq s \), then \( \mu \) is nonzero. Otherwise, it is zero, and therefore, the product (3.27) becomes zero, which is clearly in the right-hand side of (3.23). However, if \( x \geq r \) and \( y \leq s \), by applying Eq. (3.17), we get
\[ \mu = [\phi_{(x,s)}(c) - \phi_{(x,s+1)}(\delta(c))] - [\phi_{(x+1,s)}(\gamma(c)) - \phi_{(x+1,s+1)}(\alpha(1)_1(1))], \]
where \( c = \delta_{x-y}(a\overline{a}_{(x,y)}(1))\gamma_{x-r}(b) \). Hence, for (3.28), we have
\[ \tilde{\xi}_{(m,n)}^{(x,y)}(a)\tilde{\eta}_{(r,s)}^{(t,u)}(b) = p[j_{\mathbb{Z}^2}(m, n) j_{\mathbb{Z}^2}(x, y)^* j_B(\phi_{(x,s)}(c) - \phi_{(x,s+1)}(\delta(c))) j_{\mathbb{Z}^2}(r, s) j_{\mathbb{Z}^2}(t, u)^*]p - p[j_{\mathbb{Z}^2}(m, n) j_{\mathbb{Z}^2}(x, y)^* j_B(\phi_{(x+1,s)}(\gamma(c)) - \phi_{(x+1,s+1)}(\alpha(1)_1(1))) j_{\mathbb{Z}^2}(r, s) j_{\mathbb{Z}^2}(t, u)^*]p, \]
where again by applying the covariance equation of \((j_B, j_{\mathbb{Z}^2})\), it follows that:

\[
\tilde{\xi}_{(m,n)}(a)\tilde{\eta}_{(r,s)}(b) = p \left[ j_{\mathbb{Z}^2}(m, n + s - y) j_{\mathbb{Z}^2}(x, s) j_B \left( \phi(0,0)(c) - \phi(0,1)(\delta(c)) \right) \right.
\]
\[
- j_{\mathbb{Z}^2}(x, s) j_{\mathbb{Z}^2}(t - r, u - s)^* \left. \right] p
\]
\[
- p \left[ j_{\mathbb{Z}^2}(m, n + s - y) j_{\mathbb{Z}^2}(x + 1, s) j_B \left( \phi(0,0)(\gamma(c)) - \phi(0,1)(\alpha(1,1)(c)) \right) \right.
\]
\[
- j_{\mathbb{Z}^2}(x + 1, s) j_{\mathbb{Z}^2}(t - r, u - s)^* \left. \right] p
\]
\[
= p \left[ j_{\mathbb{Z}^2}(m, n + s - y) j_B \left( \phi(0,0)(c) - \phi(0,1)(\delta(c)) \right) \right]
\]
\[
- j_{\mathbb{Z}^2}(x + t - r, u)^* \left. \right] p
\]
\[
= p \left[ j_{\mathbb{Z}^2}(m + 1, k) j_B \left( \phi(0,0)(c) - \phi(0,1)(\delta(c)) \right) \right]
\]
\[
- j_{\mathbb{Z}^2}(i + 1, u)^* \left. \right] p
\]
\[
= \tilde{\eta}_{(m,k)}(c) - \tilde{\eta}_{(m+1,k)}(\gamma(c)),
\]
where \(k = n + s - y\) and \(i = x + t - r\), which belong to \(\mathbb{N}\). Consequently, for the product \((3.27)\), we get

\[
\tilde{\xi}_{(m,n)}(a)\tilde{\eta}_{(r,s)}(b) = \Psi^{-1} \left( \tilde{\xi}_{(m,n)}(a)\tilde{\eta}_{(r,s)}(b) \right)
\]
\[
= \Psi^{-1} \left( \tilde{\eta}_{(m,k)}(c) - \tilde{\eta}_{(m+1,k)}(\gamma(c)) \right)
\]
\[
= \xi_{(m,k)}(c) - \xi_{(m+1,k)}(\delta(c)) \quad \text{(by (3.22))},
\]
which belongs to the right-hand side of \((3.23)\). Thus, \((3.23)\) is valid. Note that \(\mathcal{I}_\delta \cap \mathcal{I}_\gamma\)

is an essential ideal of \(A \rtimes_{\alpha}^\text{piso} \mathbb{N}^2\) as \(\mathcal{I}_\delta\) and \(\mathcal{I}_\gamma\) both are. This completes the proof. \(\square\)

**Proposition 3.9** Suppose that \((A \rtimes_{\alpha}^\text{piso} \mathbb{N}, j_A, \nu)\) and \((A \rtimes_{\gamma}^\text{piso} \mathbb{N}, i_A, \omega)\) are the partial-isometric crossed products of the dynamical systems \((A, \mathbb{N}, \delta)\) and \((A, \mathbb{N}, \gamma)\), respectively. Then, the (essential) ideals \(\mathcal{I}_\delta\) and \(\mathcal{I}_\gamma\) of \(A \rtimes_{\alpha}^\text{piso} \mathbb{N}^2\) are full corners in algebras \(\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \rtimes_{\alpha}^\text{piso} \mathbb{N})\) and \(\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \rtimes_{\gamma}^\text{piso} \mathbb{N})\) of compact operators, respectively.

**Proof** We only provide the proof on the ideal \(\mathcal{I}_\delta\) as the proof on \(\mathcal{I}_\gamma\) follows similarly. First, by Lemma 3.8

\[
\mathcal{I}_\delta \cong p(A \rtimes_{\alpha}^\text{iso} \mathbb{Z}^2)p,
\]
where \(p = j_B(\phi(0,0)(1))\). Let the map

\[
\Pi : \mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D \rtimes_{\tau} \mathbb{Z}) \to \mathcal{K}\left(\ell^2(\mathbb{Z}) \otimes (D \rtimes_{\tau} \mathbb{Z})\right)
\]
be the canonical isomorphism, such that
\[ \Pi((e_m \otimes e_n) \otimes \xi \eta^*) = \Theta_{e_m \otimes \xi, e_n \otimes \eta}, \]
for all \( m, n \in \mathbb{Z} \) and \( \xi, \eta \in (D \rtimes \mathbb{Z}, k) \), where \( \{e_m : m \in \mathbb{Z}\} \) is the usual orthonormal basis of \( \ell^2(\mathbb{Z}) \). Then, \( \Omega := (\Pi \circ \Psi_\gamma) \) (see the isomorphism \( \Psi_\gamma \) in Theorem 3.7) is an isomorphism of \( \mathcal{J}_\delta \rtimes \beta \mathbb{Z}^2 \) onto \( \mathcal{K}(\ell^2(\mathbb{Z}) \otimes (D \rtimes \mathbb{Z})) \), such that it maps each spanning element
\[ j_B(\phi_{(m,n)}(ab^*) - \phi_{(m+1,n)}(\gamma(ab^*))) j_{\mathbb{Z}^2}(x, y), \]
of \( \mathcal{J}_\delta \rtimes \beta \mathbb{Z}^2 \) to the (spanning) element
\[ \Theta_{[e_m \otimes k_D(\varphi_n(a))], [e_{m-\gamma} \otimes k_{\mathbb{Z}}(\gamma)k_D(\varphi_n(b))]}, \]
where \( a, b \in A \). Therefore, we have
\[ \mathcal{I}_\delta \Psi_p(\mathcal{J}_\delta \rtimes \beta \mathbb{Z}^2) \sim \Omega(p) \mathcal{K}(\ell^2(\mathbb{Z}) \otimes (D \rtimes \mathbb{Z})) \]
where \( \Omega(p) \) is a projection in \( \mathcal{M}(\mathcal{K}(\ell^2(\mathbb{Z}) \otimes (D \rtimes \mathbb{Z}))) \) which we denote by \( P_\delta \). We claim that
\[ (P_\delta f)(m) = \begin{cases} k_D \circ \varphi_0(\gamma_m(1)) f(m) & \text{if } m \geq 0, \\ 0 & \text{if } m < 0, \end{cases} \]
for all \( f \in \ell^2(\mathbb{Z}) \otimes (D \rtimes \mathbb{Z}) \). To prove our claim, it suffices to see that
\[ P_\delta (e_m \otimes k_D(\varphi_n(ab^*))\xi) = \begin{cases} e_m \otimes [k_D \circ \varphi_0(\gamma_m(1))]k_D(\varphi_n(ab^*))\xi & \text{if } m \geq 0, \\ 0 & \text{if } m < 0, \end{cases} \]
on the spanning element \([e_m \otimes k_D(\varphi_n(ab^*))\xi]\) of \( \ell^2(\mathbb{Z}) \otimes (D \rtimes \mathbb{Z}) \), where \( \xi \in D \rtimes \mathbb{Z} \). Since
\[ e_m \otimes k_D(\varphi_n(ab^*))\xi = \Theta_{[e_m \otimes k_D(\varphi_n(a))], [e_m \otimes k_D(\varphi_n(b))]}(e_m \otimes \xi) = \Omega(j_B(\phi_{(m,n)}(ab^*) - \phi_{(m+1,n)}(\gamma(ab^*)))) (e_m \otimes \xi), \]
it follows that:
\[ P_\delta (e_m \otimes k_D(\varphi_n(ab^*))\xi) \]
\[ = \Omega(p) \Omega(j_B(\phi_{(m,n)}(ab^*) - \phi_{(m+1,n)}(\gamma(ab^*)))) (e_m \otimes \xi) \]
\[ = \Omega(p j_B(\phi_{(m,n)}(ab^*) - \phi_{(m+1,n)}(\gamma(ab^*)))) (e_m \otimes \xi). \]
Now, in the bottom line, as
\[
pj_B \left( \phi_{(m,n)}(ab^*) - \phi_{(m+1,n)}(\gamma(ab^*)) \right)
= \overline{j_B} (\overline{\alpha}(0,1)) j_B \left( \phi_{(m,n)}(ab^*) - \phi_{(m+1,n)}(\gamma(ab^*)) \right)
= j_B \left( \overline{\phi}(0,1)[\phi_{(m,n)}(ab^*) - \phi_{(m+1,n)}(\gamma(ab^*))] \right),
\]
we need to compute the product
\[
\overline{\phi}(0,1)[\phi_{(m,n)}(ab^*) - \phi_{(m+1,n)}(\gamma(ab^*))]. \tag{3.33}
\]

To do so, we consider two cases \( m \geq 0 \) and \( m < 0 \) separately. If \( m \geq 0 \), calculation by applying (2.4) shows that
\[
\overline{\phi}(0,1)[\phi_{(m,n)}(ab^*) - \phi_{(m+1,n)}(\gamma(ab^*))] = \phi_{(m,t)}(cd^*) - \phi_{(m+1,t)}(\gamma(cd^*)�(3.34)),
\]
where \( c = \overline{\alpha}_{(m,t)}(1)\delta_{1-n}(a), \) \( d^* = \delta_{1-n}(b^*) \), and \( t = 0 \lor n \). Therefore, by applying (3.34) to (3.32), we get
\[
P_\delta \left( e_m \otimes k_D(\varphi_n(ab^*))\xi \right)
= \Omega \left( j_B \left( \phi_{(m,t)}(cd^*) - \phi_{(m+1,t)}(\gamma(cd^*)) \right) \right) (e_m \otimes \xi)
= \Theta[e_m \otimes k_D(\varphi_n(c))\xi](e_m \otimes \xi) = e_m \otimes k_D(\varphi_i(cd^*))\xi.
\]

Moreover, in the bottom line, for \( \varphi_i(cd^*) \), we have
\[
\varphi_i(cd^*) = \varphi_i(\overline{\alpha}_{(m,t)}(1)\delta_{1-n}(ab^*))
= \varphi_i(\overline{\delta}_{1}(\overline{\varphi}_{m}(1))\delta_{1-n}(ab^*))
= \overline{\varphi}_{0}(\overline{\varphi}_{m}(1))\varphi_n(ab^*) \text{ (recall that } t = 0 \lor n)�(3.35).
\]

Consequently
\[
P_\delta \left( e_m \otimes k_D(\varphi_n(ab^*))\xi \right) = e_m \otimes k_D(\overline{\varphi}_{0}(\overline{\varphi}_{m}(1))\varphi_n(ab^*))\xi
= e_m \otimes k_D(\overline{\varphi}_{0}(\overline{\varphi}_{m}(1)))k_D(\varphi_n(ab^*))\xi
= e_m \otimes \left( k_D \circ \varphi_0(\overline{\varphi}_{m}(1)) \right) k_D(\varphi_n(ab^*))\xi.
\]

If \( m < 0 \), again by applying (2.4), one can calculate to see that the product (3.33) equals zero, and hence, for (3.32), it follows that:
\[
P_\delta \left( e_m \otimes k_D(\varphi_n(ab^*))\xi \right) = \Omega(j_B(0))(e_m \otimes \xi) = 0.
\]

Thus, (3.31) is indeed valid. Next, we show that \( P_\delta K(\ell^2(\mathbb{Z}) \otimes (D \rtimes \tau \mathbb{Z})) P_\delta \) is actually equal to the corner
\[
Q_\delta K(\ell^2(\mathbb{N}) \otimes (A \rtimes \delta \mathbb{N})),
\]
\(\Box\)
of the algebra $\mathcal{K}\left(\ell^2(\mathbb{N}) \otimes (A \times^\text{piso}_\delta \mathbb{N})\right) \simeq \mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times^\text{piso}_\delta \mathbb{N})$, where $Q_\delta$ is a projection in $\mathcal{M}\left(\mathcal{K}\left(\ell^2(\mathbb{N}) \otimes (A \times^\text{piso}_\delta \mathbb{N})\right)\right) \simeq \mathcal{L}\left(\ell^2(\mathbb{N}) \otimes (A \times^\text{piso}_\delta \mathbb{N})\right)$ defined by

$$(Q_\delta f)(m) = \overline{f_A(\mathcal{V}_m(1))} f(m),$$

for all $f \in \ell^2(\mathbb{N}) \otimes (A \times^\text{piso}_\delta \mathbb{N})$. To do so, recall that first, by [9, Theorem 4.1] (or [10, Theorem 4.1]), the crossed product $r(D \rtimes_\tau \mathbb{Z})r$, where $r$ is the projection $k_D \circ \varphi_0(1)$ in the multiplier algebra $\mathcal{M}(D \rtimes_\tau \mathbb{Z})$. Then, since $\mathcal{K}(\ell^2(\mathbb{Z}) \otimes (D \rtimes_\tau \mathbb{Z}))$ is spanned by the elements (compact operators) $\{\Theta_{e_m \otimes \xi, e_n \otimes \eta} : m, n \in \mathbb{Z}, \xi, \eta \in (D \rtimes _\tau \mathbb{Z})\}$, we have

$$P_\delta \mathcal{K}\left(\ell^2(\mathbb{Z}) \otimes (D \rtimes_\tau \mathbb{Z})\right) P_\delta = \text{span}\{P_\delta(\Theta_{e_m \otimes \xi, e_n \otimes \eta}) P_\delta : m, n \in \mathbb{Z}, \xi, \eta \in (D \rtimes_\tau \mathbb{Z})\}$$

$$= \text{span}\{\Theta_{P_\delta(e_m \otimes \xi), P_\delta(e_n \otimes \eta)} : m, n \in \mathbb{Z}, \xi, \eta \in (D \rtimes_\tau \mathbb{Z})\}.$$ 

However, if $m < 0$ or $n < 0$, then $P_\delta(e_m \otimes \xi) = 0$ or $P_\delta(e_n \otimes \eta) = 0$, and hence, $\Theta_{P_\delta(e_m \otimes \xi), P_\delta(e_n \otimes \eta)} = 0$. It thus follows that:

$$P_\delta \mathcal{K}\left(\ell^2(\mathbb{Z}) \otimes (D \rtimes_\tau \mathbb{Z})\right) P_\delta = \text{span}\{\Theta_{P_\delta(e_m \otimes \xi), P_\delta(e_n \otimes \eta)} : m, n \in \mathbb{N}, \xi, \eta \in (D \rtimes_\tau \mathbb{Z})\}.$$ 

Moreover, since

$$P_\delta(e_m \otimes \xi) = e_m \otimes k_D \circ \varphi_0(\mathcal{V}_m(1))\xi$$

$$= e_m \otimes k_D \circ \varphi_0(\mathcal{V}_m(1))k_D \circ \varphi_0(1)\xi$$

$$= e_m \otimes k_D \circ \varphi_0(\mathcal{V}_m(1))r\xi = P_\delta(e_m \otimes r\xi),$$

we have

$$\Theta_{P_\delta(e_m \otimes \xi), P_\delta(e_n \otimes \eta)} = \Theta_{P_\delta(e_m \otimes r\xi), P_\delta(e_n \otimes r\eta)} = P_\delta(\Theta_{e_m \otimes r\xi, e_n \otimes r\eta}) P_\delta,$$  

(3.35)

for all $m, n \in \mathbb{N}$ and $\xi, \eta \in D \rtimes_\tau \mathbb{Z}$. However, each element $\Theta_{[e_m \otimes r\xi], [e_n \otimes r\eta]}$ is actually a compact operator in $\mathcal{K}\left(\ell^2(\mathbb{N}) \otimes (A \times^\text{piso}_\delta \mathbb{N})\right)$. This is due to the facts that

$$\Theta_{[e_m \otimes r\xi], [e_n \otimes r\eta]} = \Pi((e_m \otimes e_n) \otimes r\xi \eta^* r),$$

where $r\xi \eta^* r \in r(D \rtimes_\tau \mathbb{Z})r \simeq A \times^\text{piso}_\delta \mathbb{N}$, and the restriction of the isomorphism $\Pi$ to the subalgebra $\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times^\text{piso}_\delta \mathbb{N})$ of $\mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D \rtimes_\tau \mathbb{Z})$ gives the canonical
since for every \( m / \Omega_1 \).

More precisely, the composition it follows that [see (3.35)]:

\[
P_\delta(e_m \otimes a) = e_m \otimes (\overline{k_D \circ \varphi_0})(\overline{\gamma_m}(1))a = e_m \otimes \overline{f_A(\overline{\gamma_m}(1))}a = Q_\delta(e_m \otimes a),
\]

(3.36)

it follows that [see (3.35)):

\[
P_\delta K\left(\ell^2(\mathbb{Z}) \otimes (D \times_\tau \mathbb{Z})\right) P_\delta
\]

\[
= \text{span} \left\{ \varphi_0 \left( e_m \otimes r_\xi e_n \otimes r_\eta \right) \mid m, n \in \mathbb{N}, \xi, \eta \in (D \times_\tau \mathbb{Z}) \right\}
\]

\[
= \text{span} \left\{ \varphi_0 \left( e_m \otimes r_\xi e_n \otimes r_\eta \right) 
\right\}
\]

\[
= \text{span} \left\{ \varphi_0 \left( e_m \otimes r_\xi e_n \otimes r_\eta \right) \right\}
\]

\[
= \text{span} \left\{ \varphi_0 \left( e_m \otimes r_\xi e_n \otimes r_\eta \right) \right\}
\]

\[
= Q_\delta K\left(\ell^2(\mathbb{N}) \otimes (A \times_\delta \mathbb{N})\right) Q_\delta.
\]

Consequently

\[
T_\delta \cong p(J_\delta \times_B \mathbb{Z}^2) p \cong P_\delta K\left(\ell^2(\mathbb{Z}) \otimes (D \times_\tau \mathbb{Z})\right) P_\delta
\]

\[
= Q_\delta K\left(\ell^2(\mathbb{N}) \otimes (A \times_\delta \mathbb{N})\right) Q_\delta.
\]

More precisely, the composition \( \varphi_0 \circ \psi \) of isomorphisms gives an isomorphism

\[
\psi_\delta : T_\delta \rightarrow Q_\delta K\left(\ell^2(\mathbb{N}) \otimes (A \times_\delta \mathbb{N})\right) Q_\delta,
\]

such that

\[
\psi_\delta \left( i_{\mathbb{N}^2}(m, n)^* i_A(ab^*)[1 - i_{\mathbb{N}^2}(1, 0)^* i_{\mathbb{N}^2}(1, 0)] i_{\mathbb{N}^2}(x, y) \right)
\]

\[
= Q_\delta \left( e_m \otimes r_\xi e_n \otimes r_\eta \right) v_\psi \left( j_A(ab^*)v_y \right) Q_\delta
\]

\[
= Q_\delta \left( \varphi_0 \left( e_m \otimes r_\xi e_n \otimes r_\eta \right) \right) Q_\delta.
\]

(3.37)

Note that, by applying the covariance equations of the pairs \( (J_B, j_{\mathbb{Z}^2}) \) and \( (k_D, k_{\mathbb{Z}}) \), (3.35), and (3.36), one can calculate on the spanning elements of \( T_\delta \) to see (3.37).
To see that $\mathcal{I}_\delta$ is a full corner in $\mathcal{K}(\ell^2(\mathbb{N}) \otimes (A \times^\text{iso}_\delta \mathbb{N}))$, first, note that the algebra $\mathcal{K}(\ell^2(\mathbb{N}) \otimes (A \times^\text{iso}_\delta \mathbb{N}))$ is spanned by the elements

$$\Theta[v_m \otimes v_n^* j_A(a) v_r], [e_s \otimes v_r^* j_A(bc^*) v_s].$$

(3.38)

Now, if $\{a_i\}$ is an approximate unit in $A$, then the spanning element (3.38) is the norm-limit of the net

$$\Theta[v_m \otimes v_n^* j_A(a_i) v_r], [e_s \otimes v_r^* j_A(bc^*) v_s].$$

(3.39)

in $\mathcal{K}(\ell^2(\mathbb{N}) \otimes (A \times^\text{iso}_\delta \mathbb{N}))$. However, calculation shows that, for (3.39), we have

$$\Theta[v_m \otimes v_n^* j_A(a) v_r], [e_s \otimes v_r^* j_A(bc^*) v_s]$$

$$= \left( \Theta[v_m \otimes v_n^* j_A(a)], [e_s \otimes v_r^* j_A(a)] \right) \left( \Theta[v_0 \otimes v_j^* j_A(c)], [e_s \otimes v_r^* j_A(b)] \right)$$

$$= \left( \Theta[v_m \otimes v_n^* j_A(a)], [e_s \otimes v_r^* j_A(a)] \right) Q_\delta \left( \Theta[v_0 \otimes v_j^* j_A(c)], [e_s \otimes v_r^* j_A(b)] \right),$$

which belongs to

$$\mathcal{K}(\ell^2(\mathbb{N}) \otimes (A \times^\text{iso}_\delta \mathbb{N})) Q_\delta \mathcal{K}(\ell^2(\mathbb{N}) \otimes (A \times^\text{iso}_\delta \mathbb{N})).$$

(3.40)

It therefore follows that each spanning element (3.38) of $\mathcal{K}(\ell^2(\mathbb{N}) \otimes (A \times^\text{iso}_\delta \mathbb{N}))$ must belong to (3.40), and hence, we have

$$\mathcal{K}(\ell^2(\mathbb{N}) \otimes (A \times^\text{iso}_\delta \mathbb{N})) = \mathcal{K}(\ell^2(\mathbb{N}) \otimes (A \times^\text{iso}_\delta \mathbb{N})) Q_\delta \mathcal{K}(\ell^2(\mathbb{N}) \otimes (A \times^\text{iso}_\delta \mathbb{N})).$$

At last, it follows by a similar discussion that there is an isomorphism $\Psi_\gamma$ of the ideal $\mathcal{I}_\gamma$ onto the full corner $Q_\gamma \mathcal{K}(\ell^2(\mathbb{N}) \otimes (A \times^\text{iso}_\gamma \mathbb{N})) Q_\gamma$, where $Q_\gamma$ is a projection in $\mathcal{M} \left( \mathcal{K} \left( \ell^2(\mathbb{N}) \otimes (A \times^\text{iso}_\gamma \mathbb{N}) \right) \right) \simeq \mathcal{L} \left( \ell^2(\mathbb{N}) \otimes (A \times^\text{iso}_\gamma \mathbb{N}) \right)$ defined by

$$(Q_\gamma h)(n) = \mathcal{A}(\bar{\delta}_n)(1) h(n),$$

for all $h \in \ell^2(\mathbb{N}) \otimes (A \times^\text{iso}_\gamma \mathbb{N})$. The isomorphism $\Psi_\gamma$ maps each spanning element

$$i_{\mathbb{N}^2}(r, s)^* i_A(ab^*) [1 - i_{\mathbb{N}^2}(0, 1)^* i_{\mathbb{N}^2}(0, 1)] i_{\mathbb{N}^2}(t, u),$$

of $\mathcal{I}_\gamma$ to the (spanning) element

$$Q_\gamma \left( \Theta[e_i \otimes v_j^* j_A(a)], [e_s \otimes v_r^* j_A(b)] \right) Q_\gamma.$$

This completes the proof. □
Theorem 3.10 Let \( A \times^\text{piso}_\alpha N^2 \) be the partial-isometric crossed product of the system \((A, N^2, \alpha)\) in which the action \( \alpha \) on \( A \) is given by extendible endomorphisms. Then, there is a composition series

\[
0 \leq L_1 \leq L_2 \leq A \times^\text{piso}_\alpha N^2,
\]

of essential ideals, such that
(i) the ideal \( L_1 \) is (isomorphic to) a full corner in the algebra \( \mathcal{K}(\ell^2(N^2)) \otimes A \) of compact operators,
(ii) \( L_2/L_1 \simeq A_\delta \oplus A_\gamma \), and
(iii) \( (A \times^\text{piso}_\alpha N^2)/L_2 \simeq A \times^\text{iso} N^2 \),

where the algebras \( A_\delta \) and \( A_\gamma \) are full corners in algebras \( \mathcal{K}(\ell^2(N)) \otimes (A \times^\text{iso}_\delta N) \) and \( \mathcal{K}(\ell^2(N)) \otimes (A \times^\text{iso}_\gamma N) \) of compact operators, respectively.

**Proof** The composition series (3.41) is

\[
0 \leq \mathcal{I}_\delta \cap \mathcal{I}_\gamma \leq \ker q \leq A \times^\text{piso}_\alpha N^2.
\]

Therefore, (iii) is indeed true [see the exact sequence (2.1)].

To see (i), first, note that we have

\[
\mathcal{I}_\delta \cap \mathcal{I}_\gamma \simeq \Psi(\mathcal{I}_\delta \cap \mathcal{I}_\gamma) = \Psi(\mathcal{I}_\delta) \cap \Psi(\mathcal{I}_\gamma) = [p(\mathcal{J}_\delta \times_\beta \mathbb{Z}^2)p] \cap [p(\mathcal{J}_\gamma \times_\beta \mathbb{Z}^2)p] = [p(\mathcal{J}_\delta \times_\beta \mathbb{Z}^2)p][p(\mathcal{J}_\gamma \times_\beta \mathbb{Z}^2)p].
\]

Then, it follows by inspection on spanning elements that (see also Theorem 3.7):

\[
[p(\mathcal{J}_\delta \times_\beta \mathbb{Z}^2)p][p(\mathcal{J}_\gamma \times_\beta \mathbb{Z}^2)p] = p((C_0(\mathbb{Z}^2) \otimes A) \times_\beta \mathbb{Z}^2)p,
\]

and therefore

\[
\mathcal{I}_\delta \cap \mathcal{I}_\gamma \simeq p((C_0(\mathbb{Z}^2) \otimes A) \times_\beta \mathbb{Z}^2)p.
\]

Now, if \((\rho, U)\) is the covariant representation of the system \((B, \mathbb{Z}^2, \beta)\) in \( \mathcal{L}(\ell^2(\mathbb{Z}^2) \otimes A) \) mentioned in Sect. 2, the restriction of the corresponding (nondegenerate) representation \( \rho \times U \) of \( B \times_\beta \mathbb{Z}^2 \) to \((C_0(\mathbb{Z}^2) \otimes A) \times_\beta \mathbb{Z}^2 \) is the canonical isomorphism of \((C_0(\mathbb{Z}^2) \otimes A) \times_\beta \mathbb{Z}^2 \) onto the algebra \( \mathcal{K}(\ell^2(\mathbb{Z}^2) \otimes A) \simeq \mathcal{K}(\ell^2(\mathbb{Z}^2)) \otimes A \) of compact operators. Therefore, we have

\[
\mathcal{I}_\delta \cap \mathcal{I}_\gamma \simeq p((C_0(\mathbb{Z}^2) \otimes A) \times_\beta \mathbb{Z}^2)p \overset{(\rho \times U)}{\simeq} RK(\ell^2(\mathbb{Z}^2) \otimes A)R,
\]

where \( R = \overline{\rho \times U}(p) \) is a projection in \( \mathcal{M}(\mathcal{K}(\ell^2(\mathbb{Z}^2) \otimes A)) \simeq \mathcal{L}(\ell^2(\mathbb{Z}^2) \otimes A) \). By taking an approximate unit \( \{a_\lambda\} \) in \( A \) and using the equation

\[
(\rho \times U)(j_B(\phi(0,0)(a_\lambda))) = \rho(\phi(0,0)(a_\lambda)),
\]
one can see that $R = \overline{\varrho(\Theta(0,0)(1))}$. Moreover, calculation on spanning elements of $\ell^2(\mathbb{Z}^2) \otimes A$ shows that we have

$$(Rf)(m, n) = \begin{cases} \overline{\alpha(m,n)}(1)f(m,n) & \text{if } (m,n) \in \mathbb{N}^2, \\ 0 & \text{otherwise} \end{cases}$$

for all $f \in (\ell^2(\mathbb{Z}^2) \otimes A)$. Each $Rf$ is obviously in $\ell^2(\mathbb{N}^2) \otimes A$. Now, define a map $Q : \ell^2(\mathbb{N}^2) \otimes A \to \ell^2(\mathbb{N}^2) \otimes A$ by

$$(Qh)(m, n) = \overline{\alpha(m,n)}(1)h(m,n) \text{ for all } h \in (\ell^2(\mathbb{N}^2) \otimes A).$$

One can see that $Q$ is a projection in $\mathcal{M}\left(\mathcal{K}\left(\ell^2(\mathbb{N}^2) \otimes A)\right)\right) \simeq \mathcal{L}\left(\ell^2(\mathbb{N}^2) \otimes A)\right)$. Then, by a similar calculation done in Proposition 3.9, we get

$$RK(\ell^2(\mathbb{Z}^2) \otimes A)R = QK(\ell^2(\mathbb{N}^2) \otimes A)Q,$$

and consequently

$$\mathcal{I}_8 \cap \mathcal{I}_y \simeq \Psi[(\mathcal{C}_0(\mathbb{Z}^2) \otimes A) \times_\rho \mathbb{Z}^2]p (\rho \times U)R \simeq R K(\ell^2(\mathbb{Z}^2) \otimes A)R = QK(\ell^2(\mathbb{N}^2) \otimes A)Q.$$

Therefore, the composition $(\rho \times U) \circ \Psi$ gives an isomorphism

$$\phi : \mathcal{I}_8 \cap \mathcal{I}_y \to QK(\ell^2(\mathbb{N}^2) \otimes A)Q,$$

such that

$$\phi \left(\xi(x,y)_{(m,n)}(ab^*) - \xi(x,y+1)_{(m,n+1)}(ab^*)\right) = Q\left(\Theta[e(m,n) \otimes a],[e(x,y) \otimes b]\right)Q,$$

for all $a, b \in A$ and $m, n, x, y \in \mathbb{N}$. To see that $L_1 = \mathcal{I}_8 \cap \mathcal{I}_y$ is a full corner, first note that $\mathcal{K}(\ell^2(\mathbb{N}^2) \otimes A)$ is spanned by the elements of the form $\Theta[e(m,n) \otimes ab],[e(r,s) \otimes c]$. Now, if $\{a_\lambda\}$ is any approximate unit in $A$, then

$$\Theta[e(m,n) \otimes aa_\lambda b],[e(r,s) \otimes c] \to \Theta[e(m,n) \otimes ab],[e(r,s) \otimes c], \tag{3.43}$$

in the norm topology. However, calculation shows that

$$\Theta[e(m,n) \otimes aa_\lambda b],[e(r,s) \otimes c]$$

$$= \Theta[e(m,n) \otimes a],[e(0,0) \otimes a_\lambda b]\left(\Theta[e(0,0) \otimes b],[e(r,s) \otimes c]\right)$$

$$= \Theta[e(m,n) \otimes a],[e(0,0) \otimes a_\lambda b]Q\left(\Theta[e(0,0) \otimes b],[e(r,s) \otimes c]\right). \tag{3.44}$$

Thus, it follows from (3.43) and (3.44) that:

$$\Theta[e(m,n) \otimes ab],[e(r,s) \otimes c] \in \mathcal{K}(\ell^2(\mathbb{N}^2) \otimes A)QK(\ell^2(\mathbb{N}^2) \otimes A),$$
and hence, we have

\[ K(\ell^2(\mathbb{N}^2) \otimes A) Q K(\ell^2(\mathbb{N}^2) \otimes A) = K(\ell^2(\mathbb{N}^2) \otimes A). \]

To see (ii), first

\[ L_2/L_1 = (\mathcal{I}_\delta + \mathcal{I}_\gamma)/(\mathcal{I}_\delta \cap \mathcal{I}_\gamma) \simeq \mathcal{I}_\delta/(\mathcal{I}_\delta \cap \mathcal{I}_\gamma) \oplus \mathcal{I}_\gamma/(\mathcal{I}_\delta \cap \mathcal{I}_\gamma). \] (3.45)

Next, we show that the quotients in the above direct sum are isomorphic to full corners in algebras \( K(\ell^2(\mathbb{N})) \otimes (A \times^{iso}_{\delta} \mathbb{N}) \) and \( K(\ell^2(\mathbb{N})) \otimes (A \times^{iso}_{\gamma} \mathbb{N}) \), respectively. We only do this for

\[ \mathcal{I}_\delta/(\mathcal{I}_\delta \cap \mathcal{I}_\gamma), \] (3.46)
as the other one follows similarly. First, by Proposition 3.9

\[ \mathcal{I}_\delta \overset{\Psi_{\delta}}{\simeq} Q_{\delta} K \left( \ell^2(\mathbb{N}) \otimes (A \times^{iso}_{\delta} \mathbb{N}) \right) Q_{\delta}. \]

Let \( I \) be the kernel of the natural surjective homomorphism \( \varphi \) of \( (A \times^{iso}_{\delta} \mathbb{N}, j_A, v) \) onto the isometric crossed product \( (A \times^{iso}_{\delta} \mathbb{N}, k_A, u) \) of the system \( (A, \mathbb{N}, \delta) \), where

\[ \varphi(v_m^* j_A(a)v_n) = u_m^* k_A(a)u_n \quad \text{for all } a \in A, \; m, n \in \mathbb{N}. \]

See in [2] that \( I \) is spanned by the elements \( \{v_m^* j_A(a)(1 - v^*v)v_n : a \in A, \; m, n \in \mathbb{N}\} \), which is an essential ideal of \( A \times^{iso}_{\delta} \mathbb{N} \). Moreover, it is a full corner in algebra \( K(\ell^2(\mathbb{N})) \otimes A \). Now, calculation on the spanning elements of the ideal \( \mathcal{I}_\delta \cap \mathcal{I}_\gamma \) shows that

\[
\Psi_{\delta} \left( \xi^{(x,v)}_{(m,n)}(ab^*) - \xi^{(x,v+1)}_{(m,n+1)}(\delta(ab^*)) \right) \\
= Q_{\delta} \prod \left( (e_m \otimes e_x) \otimes v_n^* j_A(ab^*)(1 - v^*v)v_y \right) Q_{\delta} \\
= Q_{\delta} \left( \Theta_{[e_m \otimes v_n^* j_A(a)(1-v^*v)],[e_x \otimes v_y^* j_A(b)(1-v^*v)]] \right) Q_{\delta},
\]

which implies that

\[ \mathcal{I}_\delta \cap \mathcal{I}_\gamma \simeq \Psi_{\delta}(\mathcal{I}_\delta \cap \mathcal{I}_\gamma) = Q_{\delta} K(\ell^2(\mathbb{N}) \otimes I) Q_{\delta}. \]

Then, consider the following diagram:
The partial-isometric crossed product $\mathcal{K}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N)) \to \mathcal{K}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N))$

where the vertical arrows denote the canonical isomorphisms. It induces a surjective homomorphism

$$\varphi : \mathcal{K}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N)) \to \mathcal{K}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N)),$$

such that

$$\varphi(\Theta e_m \otimes \xi, e_n \otimes \eta) = \Theta e_m \otimes \varphi(\xi, e_n \otimes \varphi(\eta)),$$

for all $\xi, \eta \in A \times^{\text{iso}}_\delta N$ and $m, n \in \mathbb{N}$. One can see that indeed $\ker \varphi = \mathcal{K}(\ell^2(N) \otimes I)$. Now, the restriction of $\varphi$ to the (full) corner $Q_\delta \mathcal{K}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N)) Q_\delta$ gives a surjective homomorphism $\varphi|_\lambda$ of it onto the subalgebra

$$\varphi|_\lambda \left( Q_\delta \mathcal{K}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N)) Q_\delta \right) = R_\delta \mathcal{K}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N)) R_\delta, \quad (3.47)$$

of $\mathcal{K}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N))$, where $R_\delta = \overline{\varphi(Q_\delta)}$ is a projection in

$$\mathcal{M} \left( \mathcal{K}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N)) \right) \simeq \mathcal{L}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N)).$$

By calculating on the spanning elements of $\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N)$, it follows that:

$$(R_\delta f)(m) = \overline{\varphi_m(1)} f(m) \text{ for all } f \in \ell^2(N) \otimes (A \times^{\text{iso}}_\delta N).$$

Let us denote the corner $R_\delta \mathcal{K}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N)) R_\delta$ by $A_\delta$. Since $I_\delta$ is a full corner in $\mathcal{K}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N))$, one can apply the surjection $\varphi$ to see that $A_\delta$ is actually a full corner in $\mathcal{K}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N))$. Now, it follows from:

$$\ker \varphi|_\lambda = \ker \varphi \cap \left[ Q_\delta \mathcal{K}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N)) \right] Q_\delta = \mathcal{K}(\ell^2(N) \otimes I) \cap \left[ Q_\delta \mathcal{K}(\ell^2(N) \otimes (A \times^{\text{iso}}_\delta N)) \right] Q_\delta,$$

that $\ker \varphi|_\lambda = Q_\delta \mathcal{K}(\ell^2(N) \otimes I) Q_\delta$. Consequently, we get
\[
I_\delta/(I_\delta \cap I_\gamma) \simeq [Q_\delta K(\ell^2(N) \otimes (A \times_{\delta}^{\text{piso}} N)) Q_\delta]/[Q_\delta K(\ell^2(N) \otimes I) Q_\delta]
\]
\[
\simeq R_\delta K(\ell^2(N) \otimes (A \times_{\delta}^{\text{iso}} N)) R_\delta = A_\delta.
\] (3.48)

Similarly, there is a surjective homomorphism of \(Q_\gamma K(\ell^2(N) \otimes (A \times_{\gamma}^{\text{piso}} N)) Q_\gamma\) onto the full corner \(A_\gamma = R_\gamma K(\ell^2(N) \otimes (A \times_{\gamma}^{\text{iso}} N)) R_\gamma\) whose kernel is

\[
Q_\gamma K(\ell^2(N) \otimes J) Q_\gamma = \Psi_\gamma (I_\delta \cap I_\gamma) \simeq I_\delta \cap I_\gamma,
\]
where \(R_\gamma\) is a projection in \(\mathcal{M}(K(\ell^2(N) \otimes (A \times_{\gamma}^{\text{iso}} N))) = L(\ell^2(N) \otimes (A \times_{\gamma}^{\text{iso}} N))\), and \(J\) is the essential of \(A \times_{\gamma}^{\text{piso}} N\) such that \((A \times_{\gamma}^{\text{piso}} N)/J \simeq A \times_{\gamma}^{\text{iso}} N\) (see again [2]). Therefore

\[
I_\gamma/(I_\delta \cap I_\gamma)
\]
\[
\simeq [Q_\gamma K(\ell^2(N) \otimes (A \times_{\gamma}^{\text{piso}} N)) Q_\gamma]/[Q_\gamma K(\ell^2(N) \otimes J) Q_\gamma]
\]
\[
\simeq R_\gamma K(\ell^2(N) \otimes (A \times_{\gamma}^{\text{iso}} N)) R_\gamma = A_\gamma.
\] (3.49)

Consequently, it follows by (3.48) and (3.49) that [see (3.45)]:

\[
L_2/L_1 \simeq I_\delta/(I_\delta \cap I_\gamma) \oplus I_\gamma/(I_\delta \cap I_\gamma) \simeq A_\delta \oplus A_\gamma.
\]

This completes the proof. \(\square\)

**Remark 3.11** If in the system \((A, N^2, \alpha)\), the action \(\alpha\) on \(A\) is given by automorphisms, then since \(A \times_{\alpha}^{\text{iso}} N^2 \simeq A \times_{\alpha} \mathbb{Z}^2\), the short exact sequence (2.1) is

\[
0 \longrightarrow \ker q \longrightarrow A \times_{\alpha}^{\text{piso}} N^2 \overset{q}{\longrightarrow} A \times_{\alpha} \mathbb{Z}^2 \longrightarrow 0.
\] (3.50)

Moreover, since the systems \((A, N, \delta)\) and \((A, N, \gamma)\) are obviously given by automorphic actions, the algebras \(D\) and \(C\) are isomorphic to \(B_\mathbb{Z} \otimes A\), where \(B_\mathbb{Z}\) is the subalgebra of \(\ell^\infty(\mathbb{Z}, A)\) generated by the characteristic functions \(\{1_n : n \in \mathbb{Z}\}\) (see [9, Proposition 5.1]). Therefore, by [9, Corollary 5.2] or [2, Corollary 5.3], the algebras \(A \times_{\delta}^{\text{piso}} N\) and \(A \times_{\gamma}^{\text{piso}} N\) are full corners in the group crossed products

\[
(B_\mathbb{Z} \otimes A) \rtimes_{\mathbb{Z}}^{\mathbb{Z}} \text{ and } (B_\mathbb{Z} \otimes A) \rtimes_{\mathbb{Z}}^{\mathbb{Z}} \text{,}
\]
respectively.

**Corollary 3.12** Let \((A, N^2, \alpha)\) be a system in which the action \(\alpha\) on \(A\) is given by automorphisms. Then, the (essential) ideals \(I_\delta\) and \(I_\gamma\) of \(A \times_{\gamma}^{\text{piso}} N^2\) are isomorphic to the algebras \(K(\ell^2(N)) \otimes (A \times_{\delta}^{\text{piso}} N)\) and \(K(\ell^2(N)) \otimes (A \times_{\gamma}^{\text{piso}} N)\) of compact operators, respectively.
Proof This is due to the fact that, since each $\delta_n$ as well as each $\gamma_n$ is an automorphism, the projections $Q_\delta$ and $Q_\gamma$ in Proposition 3.9 become just identity operators.

Corollary 3.13 Let $(A, N^2, \alpha)$ be a system in which the action $\alpha$ on $A$ is given by automorphisms. Then, there is a composition series

$$0 \leq L_1 \leq L_2 \leq A \times_{\alpha} piso N^2,$$

of essential ideals, such that

(i) $L_1 \simeq \mathcal{K}(\ell^2(N^2)) \otimes A$,

(ii) $L_2/L_1 \simeq \left[\mathcal{K}(\ell^2(N)) \otimes (A \times_\delta \mathbb{Z})\right] \oplus \left[\mathcal{K}(\ell^2(N)) \otimes (A \times_\gamma \mathbb{Z})\right]$, and

(iii) $(A \times_{\alpha} piso N^2)/L_2 \simeq A \times_{\alpha} \mathbb{Z}^2$.

Proof Since each $\alpha_{(m,n)}$ is an automorphism, the projections $Q$, $R_\delta$, and $R_\gamma$ in Theorem 3.10 are just identity operators. Moreover, we have $A \times_{\alpha} piso N^2 \simeq A \times_\alpha \mathbb{Z}^2$, $A \times_{\delta} piso N \simeq A \times_\delta \mathbb{Z}$, and $A \times_{\gamma} piso N \simeq A \times_\gamma \mathbb{Z}$. Therefore, the rest follows from Theorem 3.10.

Example 3.14 Note that, for the trivial system $(C, N^2, \text{id})$, the short exact sequence (3.50) is just the well-known exact sequence

$$0 \longrightarrow C_{\mathbb{Z}^2} \longrightarrow T(\mathbb{Z}^2) \longrightarrow C(\mathbb{T}^2) \longrightarrow 0,$$

where $C_{\mathbb{Z}^2}$ is the commutator ideal of the Toeplitz algebra $T(\mathbb{Z}^2) \simeq T(\mathbb{Z}) \otimes T(\mathbb{Z})$ (see the remark prior to [10, Corollary 5.5]). Moreover, the essential ideals $I_\delta$ and $I_\gamma$ of $C \times_{\text{id}} piso N^2 \simeq T(\mathbb{Z}^2)$ are both isomorphic to the algebra

$$\mathcal{K}(\ell^2(N)) \otimes (C \times_{\text{id}} piso N) \simeq \mathcal{K}(\ell^2(N)) \otimes T(\mathbb{Z}),$$

where $C \times_{\text{id}} piso N \simeq T(\mathbb{Z})$ is known by [2, Example 4.3]. Also, in this case, we have

$$L_1 = I_\delta \cap I_\gamma \simeq \mathcal{K}(\ell^2(N^2)) \simeq \mathcal{K}(\ell^2(N)) \otimes \mathcal{K}(\ell^2(N)),$$

and since $C \times_{\text{id}} \mathbb{Z} \simeq C^*(\mathbb{Z}) \simeq C(\hat{\mathbb{Z}}) \simeq C(\mathbb{T})$

$$L_2/L_1 \simeq C_{\mathbb{Z}^2}/\mathcal{K}(\ell^2(N^2))$$

$$\simeq \left[\mathcal{K}(\ell^2(N)) \otimes (C \times_{\text{id}} \mathbb{Z})\right] \oplus \left[\mathcal{K}(\ell^2(N)) \otimes (C \times_{\text{id}} \mathbb{Z})\right]$$

$$\simeq \left[\mathcal{K}(\ell^2(N)) \otimes C(\mathbb{T})\right] \oplus \left[\mathcal{K}(\ell^2(N)) \otimes C(\mathbb{T})\right].$$

Acknowledgements The author would like to thank the anonymous reviewer for valuable suggestions and comments on the earlier version of the paper.

Funding This work (Grant No. RGNS 64-102) was financially supported by Office of the Permanent Secretary, Ministry of Higher Education, Science, Research and Innovation, Thailand.

Data availability No datasets generated or analysed during the current study.
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