1. Introduction

The motion of the compressible fluids can be described by the isentropic compressible Navier-Stokes:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \text{div}(\rho v) &= 0, \\
\frac{\partial (\rho v)}{\partial t} + \text{div}(\rho v \otimes v) + \frac{\nabla P(\rho)}{\varepsilon} &= \mu \Delta v + (\lambda + \mu) \nabla \text{div} v,
\end{aligned}$$

where the unknowns $\rho = \rho(t,x,y,z), v = v(t,x,y,z)$ with $(x, y, z) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}$ are the density and velocity, respectively. $P = P(\rho)$ is the pressure, which is a smooth function of $\rho$. The parameter $\varepsilon$ is the Mach number, and the constants $\mu$ and $\lambda$ are the viscous coefficients satisfying the physical restriction

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0. $$

It is clear that $(\rho_s, v_s) := (1, U(y))$ with $U(y) := (y, 0, 0)^T$ the Couette flow is a stationary solution to the isentropic compressible Navier-Stokes equations (1.1). In this paper, we will investigate the stability of the stationary solution $(\rho_s, v_s)$ to (1.1) at the linear level. Let $(b, u)$ be the perturbation of $(\rho, v)$ around $(\rho_s, v_s)$, i.e.,

$$b := \rho - 1, \quad u = v - (y, 0, 0)^T.$$

Assume w.l.o.g. that $P'(1) = 1$, then the linearized system of (1.1) around $(\rho_s, v_s)$, the unknowns still denoted by $(b, u)$, read as follows:

$$\begin{aligned}
\frac{\partial b}{\partial t} + y \partial_y b + \text{div} u &= 0, \\
\frac{\partial u}{\partial t} + y \partial_y u + (u^2, 0, 0)^T + \frac{\nabla b}{\varepsilon} - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u &= 0.
\end{aligned}$$

1.1. Background and previous work. To begin with, let us give a short review of the extensive mathematical results on the compressible Navier-Stokes equations in the absence of shear flow. To the best of our knowledge, the global classical solutions for 3D flows were first obtained by Matsumura and Nishida [55] with initial data close to a constant state. The requirement of smoothness, but not smallness, was subsequently relaxed by Hoff [33, 34]. In the scaling invariant approach, Danchin [19] constructed the global solution with initial perturbation around constant state lying in critical space, see [15, 17, 20, 24, 31, 47] for further developments. For arbitrary initial data, the breakthrough was made by Lions [49], where the existence of large global weak solutions was proved for the first time, with the adiabatic constant $\gamma \geq \frac{9}{5}$. The restriction on $\gamma$ was later relaxed by Jiang and Zhang [36].
to $\gamma > 1$ for spherically symmetric data, and by Feireisl, Novotný and Petzeltová \cite{26} to $\gamma > \frac{3}{2}$ for general data. A new compactness approach was developed by Bresch and Jabin \cite{10} to deal with more general pressure laws and anisotropic viscous stress tensor.

The study of the linear stability of the Couette flow goes back to the classical results of Rayleigh \cite{60} and Kelvin \cite{42} for the incompressible fluid. Mathematically, the Couette flow is known to be spectrally stable for all Reynolds numbers \cite{22, 61}. On the contrary, experiments \cite{14, 29, 59, 62, 60} show instability of Couette flow and transition to turbulence for sufficient high Reynolds numbers. This paradox gained quite some attention in the fluid mechanics, see \cite{11, 56, 58, 63} for instance. In particular, the transient growth was explained by the non-normality of the linearized operator in \cite{63}. It is worthy pointing out that the velocity will tend to $0$ as $t \to \infty$ even for a time reversible system such as the incompressible Euler equation. This phenomenon is referred to as inviscid damping, due to the relationship with Landau damping in the Vlasov equations \cite{45, 57}. The nonlinear inviscid damping was first rigorously confirmed by Bedrossian and Masmoudi \cite{4, 5, 6} in this direction for the domain $\mathbb{T} \times \mathbb{R}$. Significant progress has been made by Bedrossian, Germain and Masmoudi \cite{4, 5, 6} in this direction for the domain $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$. More precisely, in \cite{6}, the authors proved that if the initial perturbation satisfies $||u_{\text{in}}||_{H^2} \leq \delta \text{Re}^{-\frac{3}{2}}$, with Re the Reynolds number, $\delta = \delta(s) > 0$, and $s > \frac{3}{2}$, then the solution remains within $O(\text{Re}^{-\frac{3}{2}})$ of the Couette flow $(y,0,0)^T$ in $L^2$ for all time, and converges to the streaks solution for $t \gg \text{Re}^\frac{1}{3}$. Recently, Wei and the second author \cite{64} proved the nonlinear stability of the Couette flow for the initial perturbation satisfying $||u_{\text{in}}||_{H^2} \leq \delta \text{Re}^{-1}$. This result implies that the transition threshold may be insensitive to the regularity of the perturbation above $H^2$. For the 2D incompressible Navier-Stokes equations, the threshold is smaller due to the absence of the lift-up effect. On the domain $\mathbb{T} \times \mathbb{R}$, it was shown in \cite{8} that the threshold is not larger than $\frac{1}{2}$ for the perturbation of the initial vorticity in $H^s$, $s > 1$. Very recently, Masmoudi and Zhao \cite{54} proved that the perturbation can lie in the almost critical space $H^s_{\text{loc}} L^2_{\gamma}$. Moreover, they showed \cite{53} that the threshold in \cite{8} can be improved to not larger than $\frac{1}{4}$ for more regular perturbation in Sobolev space. If the boundary is taken into consideration, the problem become much more complicated. We refer to \cite{9, 43, 50, 61} for some rough bounds on the transition threshold. In a recent work \cite{16}, Chen, Li, Wei and the second author proved that in the finite channel $\mathbb{T} \times [-1,1]$ if the initial perturbation satisfies $||u_{\text{in}}||_{H^2} \leq c_0 \text{Re}^{-\frac{1}{2}}$ for some $c_0 > 0$, then the solution remains within $O(\text{Re}^{-\frac{1}{2}})$ of the Couette flow in $L^\infty$ for all time. It is a challenging task to study the transition threshold problem of the Couette flow in the three dimensional finite channel $\mathbb{T} \times [-1,1] \times \mathbb{T}$. Despite the boundary layer effect, it was shown by Chen, Wei and the second author in a very recent work \cite{18} that the threshold is still not larger than $1$ (just as in the case without boundary \cite{64}), and thus the conjecture proposed by Trefethen et al \cite{63} was confirmed.

Despite the long history of the stability analysis of compressible flow \cite{46}, the compressible stability of Couette flow is less well understood than the corresponding incompressible case. Glatzel \cite{27, 28} studied the linear inviscid and viscous stability properties of the compressible plane Couette flow via a normal mode analysis in simplified flow model with constant viscous coefficients and a constant density profile. Then Duck et al. \cite{23} investigated more realistic compressible flow models where the temperature was taken into account. A transient growth mechanism was numerically studied by Hanifi et al. \cite{32}, where they found that the maximum transient growth scales with $\text{Re}^2$ and that the time at which this maximum is reached scales with $\text{Re}$. They also pointed out this transient growth mechanism may physically stems from the the lift-up effect as that in incompressible shear flows. More general results can be found in \cite{35, 52}, and see also \cite{30} for further developments recently. An nonmodal approach was given by Chagelishvili et al. \cite{36} to consider the inviscid stability of the 2D Couette flow in \cite{12, 13}. By means of some formal approximation, they showed that the energy of acoustic perturbations grows linear in time due to the transfer of energy from the mean
flow to perturbations. When the viscosity is present, Farrell and Ioannou [25] showed that the inviscid growth is not sustained as $t \to \infty$ because viscous damping then takes over. Nevertheless, the transient energy growth can last longer period of time than that in incompressible flow before being strongly affected by viscosity. As for the rigorous mathematical results, Kagei [41] proved that the plane Couette flow in an infinite layer is asymptotically stable if the Reynolds and Mach numbers are sufficiently small. He also showed that similar results hold for more general parallel flows [40, 39].

Later on, Li and Zhang [48] proved that the Navier-slip boundary condition at the bottom plays a stabilizing role, and can be used to relaxed the smallness restriction on the Reynolds number in [41]. Recently, Kagei and Teramoto [38] studied the spectrum of the linearized operator around the Couette flow of the compressible Navier-Stokes equations between two concentric rotating cylinders. Finally, we would like to remark that the inviscid damping and enhanced dissipation effects were obtained by Antonelli, Dolce and Marcati [1, 2] for the two dimensional inviscid and viscous compressible Couette flow on $T \times \mathbb{R}$. To our best knowledge, the stabilizing and destabilizing effects such as the enhanced dissipation and lift-up effects are still unclear for the 3D compressible Couette flow.

1.2. The main results. The aim of this paper is to rigorously study the stability properties of the Couette flow for the 3D linearized isentropic compressible Navier-Stokes equations on $T \times \mathbb{R} \times T$. We confirm the enhanced dissipation and the lift-up phenomena which have been deeply studied for the 3D incompressible fluids [6, 64].

To state our results, let us denote

$$f_0(y, z) = \frac{1}{2\pi} \int_T f(x, y, z) dx, \quad f_x = f - f_0,$$

and

$$P_{l=0} f(x, y) = \frac{1}{2\pi} \int_T f(x, y, z) dz, \quad P_{l\neq0} f = f - P_{l=0} f.$$

Moreover, we simply write

$$f_{00} = P_{l=0} f_0, \quad \text{and} \quad \tilde{u} = (u^2, u^3)^\top.$$

Our first theorem concerns the enhanced dissipation of the projection of the solution to the linearized isentropic Navier-Stokes equations (1.3) onto non-zero frequencies in $x$.

**Theorem 1.1.** Assume that the Mach number $\epsilon$ and the viscous coefficients $\mu$ and $\lambda$ satisfy (1.2) and

$$\lambda + 2\mu \leq 1, \quad \mu^\frac{1}{8} \epsilon \leq \frac{1}{4}, \quad \text{and} \quad \mu^\frac{1}{4} (\lambda + 2\mu) \epsilon^2 \leq 1,$$

and the initial data $(b_{in}, u_{in}) \in H^2(T \times \mathbb{R} \times T)$. Then there exists a constant $C_1$, depending on $\epsilon$, but independent of $\lambda$ and $\mu$, such that

$$\|b_\epsilon(t)\|_{L^2} \leq C_1 \mu^{-\frac{1}{8}} e^{-\frac{C_1}{4\mu} \epsilon^3} \|((\Delta b_{in})_\epsilon, (\Delta u_{in})_\epsilon)\|_{L^2},$$

(1.9)

$$\|u^1_\epsilon(t)\|_{L^2} \leq C_1 e^{-\frac{C_1}{4\mu} \epsilon^3} \|((\Delta b_{in})_\epsilon, (\Delta u_{in})_\epsilon)\|_{L^2},$$

(1.10)

$$\|u^2_\epsilon(t)\|_{L^2} \leq C_1 \mu^{-\frac{1}{8}} e^{-\frac{C_1}{4\mu} \epsilon^3} \|((\Delta b_{in})_\epsilon, (\Delta u_{in})_\epsilon)\|_{L^2},$$

and

(1.11)

$$\|u^3_\epsilon(t)\|_{L^2} \leq C_1 e^{-\frac{C_1}{4\mu} \epsilon^3} \|((\Delta b_{in})_\epsilon, (\Delta u_{in})_\epsilon)\|_{L^2}.$$
Remark 1.1. The uniform bounds in Theorem [1.7] are obtained via higher order estimates involving good derivatives. Indeed, (1.8) also holds for \( \nabla z \cdot b, \) (1.9) holds for \( \partial_{xz} u \) as well, and (1.10), (1.11) still hold with \( u_2^2 \) and \( u_3^3 \) replaced by \( \Delta_{xz} u_2^p \) and \( \Delta_{xz} u_3^p \), respectively. In particular, we would like to emphasize that the incompressibility of the quantity \( \Delta u - \nabla \text{div} u \) and the good structure possessed by the equation of the unknown \( \Delta u^2 - \partial_j \text{div} u + \partial_i b \) (see (2.3)\_5) play important roles in the estimate of \( u_3^p \).

The following theorem gives the enhanced dissipation of the second derivatives of the projection of the solution to (1.3) onto non-zero frequencies in \( x \). The estimates for \( \Delta u_\# \) are given in terms of the incompressible part \( \Delta u_\# - \nabla \text{div} u_\# \) and the compressible part \( \nabla \text{div} u_\# \).

**Theorem 1.2.** Under the condition of Theorem [1.7] there exists a constant \( C_2 \), depending on \( \epsilon \), but independent of \( \lambda \) and \( \mu \), such that

\[
\begin{align*}
\mu \left( \frac{\| \nabla z \cdot b_\#(t) \|}{\epsilon} \right)^2 + \frac{\| \nabla z \cdot \text{div} u_\#(t) \|^2}{\epsilon^2} + \left( \Delta u_\#^2 - \partial_j \text{div} u_\# \right)(t) \right)^2 \\
+ \mu^2 \left( \left( \Delta u_\#^2 - \partial_j \text{div} u_\# \right) \right)^2 \\
\leq C_2 e^{-\frac{\mu t}{2}} \left( \frac{\| \nabla z \cdot \text{div} u_\#(t) \|^2}{\epsilon^2} + \left( \Delta u_\#^2 - \partial_j \text{div} u_\# \right) \right)^2,
\end{align*}
\]

(1.12)

and

\[
\begin{align*}
\mu \left( \frac{\| \nabla \partial_j b_\#(t) \|}{\epsilon} \right)^2 + \left( \Delta u_\#^2 - \partial_j \text{div} u_\# \right)(t) \right)^2 \\
\leq C_2 e^{-\frac{\mu t}{2}} \left( \mu^2 \left( \frac{\| \nabla \partial_j b_\#(t) \|^2}{\epsilon^2} + \left( \Delta u_\#^2 - \partial_j \text{div} u_\# \right) \right)^2 + \| \nabla \text{div} u_\# \|^2 \right)^2,
\end{align*}
\]

(1.13)

Our last theorem is about the uniform bounds, decay estimates and the lift-up phenomenon of the projection of the solution to (1.3) onto zero frequencies in \( x \).

**Theorem 1.3.** Assume that the Mach number \( \epsilon \) and the viscous coefficients \( \mu \) and \( \lambda \) satisfy (1.2) and

\[
\lambda + 2\mu \leq 1, \quad (\lambda + 2\mu)\epsilon \leq 1, \quad \mu(\lambda + \mu)\epsilon^2 \leq 1,
\]

and the initial data \((b_{in}, u_{in}) \in H^2(\mathbb{T} \times \mathbb{R} \times \mathbb{T})\). Then there exists a constant \( C_3 \), independent of \( \epsilon, \lambda \) and \( \mu \), such that

\[
\begin{align*}
\sum_{0 \leq |\alpha| \leq 1} \left( \frac{\| \partial^\alpha b_0 \|^2}{\epsilon} \right)^2_{L^\infty H^1} + \| \partial^\alpha u_0 \|^2_{L^\infty H^1} + \mu \left( \left( \frac{\| \nabla \partial^\alpha b_0 \|}{\epsilon} \right)^2_{L^2 L^2} + \| \nabla \partial^\alpha u_0 \|^2_{L^2 L^2} \right)
\leq C_3 \sum_{0 \leq |\alpha| \leq 1} \left( \left( \frac{\| \partial^\alpha (b_{in})_0 \|}{\epsilon} \right)^2_{H^1} + \left( \frac{\| \partial^\alpha u_{in} \|}{\epsilon} \right)^2_{H^1} \right).
\end{align*}
\]

(1.15)

If, in addition, the projection of the initial data onto the zero frequencies in \( x \) and \( z \) satisfy \((b_{in})_{00}, (u_{in})_{00} \in L^1(\mathbb{R})\), then we have

\[
\begin{align*}
\frac{\| \partial^\alpha b_0(t) \|^2}{\epsilon} + \| \partial^\alpha u_0(t) \|^2_{L^2} \\
\leq C_3 e^{-\frac{\mu t}{2}} \left( \frac{\| \partial^\alpha (b_{in})_0 \|^2}{\epsilon} + \| \partial^\alpha (u_{in})_0 \|^2_{L^2} \right), \quad \text{if } \alpha_2 \neq 0,
\end{align*}
\]

(1.16)

\[
\begin{align*}
\frac{\| \partial^\alpha b_0(t) \|^2}{\epsilon} + \| \partial^\alpha u_0(t) \|^2_{L^2} \leq \left( \frac{\| \partial^\alpha (b_{in})_0 \|^2}{\epsilon} + \| \partial^\alpha (u_{in})_0 \|^2_{L^2} \right), \quad \text{if } \alpha_2 = 0,
\end{align*}
\]
and
\[ ||u_0^1(t)||_{L^2} + \mu^{\frac{1}{2}}||\nabla u_0^1||_{L^2} \leq \||u_{in}^1||_{L^2} + C_3 \mu^{-\frac{1}{2}} \left( \left\| \frac{P_{in0}(b_{in0})}{\epsilon} \right\|_{L^2} + \|P_{in0}(\tilde{u}_{in0})\|_{L^2} \right) \]
\[ + C_3 \mu^{-\frac{1}{4}}(t) \left\| \left( \frac{b_{in0}}{\epsilon}, \frac{u_{in0}^2}{\epsilon} \right) \right\|_{L^2 \cap L^4} \]
(1.17)

Several remarks are in order.

**Remark 1.2.** Our restrictions on the Mach number \( \epsilon \), and the viscoelastic coefficient \( \lambda \) and \( \mu \) are weaker than those in the result [2] for the 2D case. As a matter of fact, combining (1.7) and (1.14), the restrictions on \( \epsilon, \lambda \) and \( \mu \) in our results can be summarized as follows
\[ \lambda + 2\mu \leq 1, \quad \epsilon \max\{\mu^3, \lambda + 2\mu\} \leq \frac{1}{4}. \]

In [2], the power on the factor \( \lambda + 2\mu \) (the corresponding quantity should be \( \lambda + \mu \) in the 2D case) in the second inequality of (1.18) is \( \frac{1}{4} \). Moreover, in [2], a good unknown \( \Xi = -\nu M^2 \Lambda \) (see (4.4) in [2]) was introduced to overcome the difficulty caused by the failure of conservation of \( \Xi \). Similar circumstances will happen on \( w^1 \) and \( \tilde{w}^2 \) in our case, see the systems (2.9) and (2.14). However, we do not resort to extra auxiliary variables, since the dissipations for \( b^2 \) and \( b^1 \) in (2.9) and (2.14) are available, respectively. We believe our strategy is applicable to the 2D isentropic compressible Navier-Stokes equations.

**Remark 1.3.** Compared with the 2D result [2] obtained by Antonelli, Dolce and Marcati, there is no loss of derivative in the decay estimate (1.16) for \((b_0, \tilde{u}_0)\). As for \( u_0^1 \), the energy inequality (1.17) reveals the lift-up phenomenon. For the 3D incompressible Navier-Stokes equations, the last term in (1.17) can be neglected due to the incompressible condition, see [6], for example. We would like to point out that, different from the incompressible fluids, the lift-up phenomenon also happens in the 2D compressible Navier-Stokes equations around the Couette flow. Indeed, similar estimate to (1.17) holds for the 2D case if one ignores the \( z \) variable.

**Remark 1.4.** If the domain \( \mathbb{T} \times \mathbb{R} \times \mathbb{T} \) is replaced by \( \mathbb{T} \times \mathbb{R}^2 \), our proofs are still valid. In particular, the estimates for the zero mode \((b_0, u_0)\) can be improved, since in that case \((b_0, \tilde{u}_0)\) decays as fast as the solution to the heat equation on \( \mathbb{R}^2 \). More precisely, (1.16) and (1.17) can be replaced by
\[ \left\| \frac{\partial^\alpha b_0(t)}{\epsilon} \right\|_{L^2} + \left\| \partial^\alpha \tilde{u}_0(t) \right\|_{L^2} \leq \frac{C}{(\mu(t))^{\alpha+\frac{1}{2}}} \left\| \frac{\partial^\alpha (b_{in0})}{\epsilon}, \partial^\alpha (\tilde{u}_{in0}) \right\|_{L^2 \cap L^4}, \]
and
\[ ||u_0^1(t)||_{L^2} + \mu^{\frac{1}{2}}||\nabla u_0^1||_{L^2} \leq \||u_{in}^1||_{L^2} + C \mu^{-\frac{1}{2}} \log(t) \left\| \left( \frac{b_{in0}}{\epsilon}, \frac{u_{in0}^2}{\epsilon} \right) \right\|_{L^2 \cap L^4}, \]
respectively, for some positive constant \( C \) independent of \( \epsilon, \lambda \) and \( \mu \). Furthermore, we would like to remark that these improvements on the estimates for the zero mode may be helpful to solve the nonlinear stability problem.

**Remark 1.5.** The dependence of the Mach number \( \epsilon \) of \( C_1 \) and \( C_2 \) mainly stems from the definitions of \( m_1 \) and \( m_3 \), where a constant \( N \) depending on \( \epsilon \) is involved. In this paper, it suffices to choose \( N = O(\epsilon^2) \). See (2.39) and (4.37) for more details.

**Remark 1.6.** The results in Theorems 1.3 and 1.2 can be regard as preparations for the nonlinear problem. Note that the Sobolev embedding \( H^2 \hookrightarrow L^\infty \) ensures that the perturbed density \( b \) and velocity \( u \) are bounded, which will be useful for the nonlinear estimates. This partially explains why the initial data \((b_{in0}, u_{in0})\) are assumed to lie in \( H^2 \) in this paper. We will study the nonlinear transition threshold problem in our future work.

**Remark 1.7.** We believe that our approach can be used to deal with more general case, for example, the non-isentropic flow that the temperature is taken into account.
Notations

1. Throughout the paper, we denote by $C$ various “harmless” positive constants independent of the viscous coefficients $\mu, \lambda$, and the time $t$. Sometimes we use the notation $A \lesssim B$ as an equivalent to $A \leq CB$. We would like to point out that $C$ may be different from line to line.

2. The Fourier transform $\hat{f}(k, \eta, l)$ of a function $f(x, y, z)$ is defined by

$$\hat{f}(k, \eta, l) := \frac{1}{(2\pi)^2} \int_T \int_{\mathbb{R}} \int_{\mathbb{T}} f(x, y, z)e^{-i(kx+\eta y+lz)} \, dx \, dy \, dz.$$ 

Then

$$f(x, y, z) = \sum_{(k, \eta, l) \in \mathbb{Z}^3} \hat{f}(k, \eta, l) e^{i(kx+\eta y+lz)}.$$ 

3. The Sobolev space $H^s(\mathbb{T} \times \mathbb{R} \times \mathbb{T})$, $s \geq 0$ is defined by

$$H^s(\mathbb{T} \times \mathbb{R} \times \mathbb{T}) := \left\{ f \in L^2(\mathbb{T} \times \mathbb{R} \times \mathbb{T}) : \langle D \rangle^s f \in L^2(\mathbb{T} \times \mathbb{R} \times \mathbb{T}) \right\},$$

where

$$\langle D \rangle^s f(k, \eta, l) = \left(1 + (k^2 + \eta^2 + l^2)^{\frac{s}{2}} \right) \hat{f}(k, \eta, l).$$

4. For $1 \leq p, q \leq \infty$, we simply write

$$L^p = L^p(\Omega), \quad H^s = H^s(\Omega),$$

and

$$\|f\|_{L^p} := \|f\|_{L^p(\Omega)}.$$ 

for a function of space and time $f = f(t, \cdot)$, and the domain $\Omega$ may be taken as $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$, $\mathbb{R}$ or $\mathbb{R}^2$.

5. For a vector $x$, we denote $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$. For $f, g \in L^2$, we use $\langle f, g \rangle$ to denote the $L^2$ inner product of $f$ and $g$.

6. For $a \in \mathbb{C}$, we use $\Re a$ and $\bar{a}$ to denote the real part and the conjugation of $a$, respectively.

The rest part of this paper is organized as follows. In section 2, we show the main ideas of proof. We will introduce some good unknowns and give a toy model to demonstrate the ingredients we shall use in what follows. In section 3, we bound the $x$-average of the solution $(b_0, u_0)$ and show the lift-up effect. The enhanced dissipation effect of the non-zero mode $(b_\neq, u_\neq)$ is shown in section 4, and section 5 is devoted to the proof of Theorems 1.1–1.3.

2. Ingredients of the proof

In this section, we show the ingredients of the proof. We shall give a reformulation of the linearized compressible Navier-Stokes equations (1.3). The basic idea is to decompose $\Delta u$ into the compressible part $\nabla \div u$ and the incompressible part $\Delta u - \nabla \div u$. Several good unknowns will be introduced to utilize the intrinsic cancellation structure. The proof relies on a Fourier multiplier method in the spirit of Bedrossian and Masmoudi [3]. A toy model will be given to explain how to capture the dissipation of the perturbed density $b$ and how to design the multipliers to balance the linear stretching.

2.1. Derivation the equations of the quantity $\Delta u - \nabla \div u$. For the stability of 3D Couette flow in the incompressible Navier-Stokes equations, an important unknowns is $\Delta u^i$ which was first introduced by Kelvin [42]. Moreover, it is convenient to work with the set of of unknowns $q^i := \Delta u^i, i = 1, 2, 3$ as observed in [4]. In our case, applying the $\Delta$ operator to the momentum equation in (1.3), and using the fact

$$\Delta (y \partial_x u^i) = y \partial_x \Delta u^i + 2 \partial_y u^i,$$

we find that the equation of $q$ takes the form

$$\partial_t q + y \partial_x q + 2 \partial_y u + (q^2, 0, 0)^T + \frac{\nabla \Delta b}{e^2} - \mu \Delta q - (\lambda + \mu) \nabla \div u = 0.$$
One can not see any evidence from (2.1) that $\Delta u^i$ behaves better than $u^i$ itself even for $i = 2$. On the other hand, we infer from (1.3) that the equation of $\nabla \div u$ satisfies

$$\partial_t \nabla \div u + y \partial_x \nabla \div u + (0, \partial_x \div u, 0)^\top + 2\partial_x \nabla u^2 + \frac{\nabla \Delta b}{\epsilon^2} - (\lambda + 2\mu)\nabla \Delta \div u = 0.$$ 

Set

$$w := q - \nabla \div u = \Delta u - \nabla \div u,$$

then the different between (2.1) and (2.2) shows that the equation of $w$ takes the form of

$$\begin{align*}
\partial_tw^1 + y\partial_xw^1 + w^2 + \partial_x\div u + 2\partial_{xy}u^1 - 2\partial_{xx}u^2 - \mu \Delta w^1 &= 0, \\
\partial_tw^2 + y\partial_xw^2 - \partial_x\div u - \mu \Delta w^2 &= 0, \\
\partial_tw^3 + y\partial_xw^3 - 2\partial_{xy}u^3 - 2\partial_{xx}u^2 - \mu \Delta w^3 &= 0.
\end{align*}$$

This is noting but the system of the incompressible part of the linearized compressible Navier-Stokes equations (1.3). Owing to the coupling between the compressible and incompressible part, we need further reformulation of system (1.3).

### 2.2. Reformulation of the equations.

Let us denote

$$d := \div u.$$ 

Then from (1.3), (2.4), we find that $(b, d, w^2)$ is not coupled with other components, and the system of $(b, d, w^2)$ satisfies

$$\begin{align*}
\partial_tb + y\partial_xb + d &= 0, \\
\partial_td + y\partial_xd + 2\partial_xu^2 + \frac{\Delta b}{\epsilon^2} - (\lambda + 2\mu)\Delta d &= 0, \\
\partial_tw^2 + y\partial_xw^2 - \partial_xd - \mu \Delta w^2 &= 0,
\end{align*}$$

with

$$\begin{align*}
\partial_xu^2 &= \partial_x\Delta^{-1}w^2 + \partial_{xy}\Delta^{-1}d.
\end{align*}$$

It’s just reminiscent of the 2D linearized isentropic compressible fluids around the Couette flow $(y, 0)$ in terms of the density, divergence of the velocity and vorticity, see [1, 2]. In order to cancel out the coupling between $w^2$ and $d$ in (2.5), we derive the equation of $\partial_xb$ as follows

$$\partial_\tau \partial_xb + y\partial_x\partial_xb + \partial_xd = 0,$$

and define

$$w^2 := w^2 + \partial_xb.$$ 

Adding (2.7) to (2.5), we find that the equation of $w^2$ takes the form of

$$\partial_t w^2 + y\partial_xw^2 - \mu \Delta w^2 + \mu \partial_xb = 0.$$ 

Denote

$$b^1 := \partial_xb, \quad \text{and} \quad d^1 := \partial_xd.$$ 

Now we give the system of $(b^1, d^1, w^2)$

$$\begin{align*}
\partial_tb^1 + y\partial_xb^1 + d^1 &= 0, \\
\partial_td^1 + y\partial_xd^1 + 2\partial_x\Delta^{-1}d^1 + 2\partial_{xy}\Delta^{-1}(w^2 - b^1) + \frac{\Delta b^1}{\epsilon^2} - (\lambda + 2\mu)\Delta d^1 &= 0, \\
\partial_tw^2 + y\partial_xw^2 - \mu \Delta w^2 + \mu \Delta b^1 &= 0.
\end{align*}$$

Next, denote

$$b^3 := \partial_xb, \quad \text{and} \quad d^3 := \partial_xd.$$ 

Note that

$$\partial_{xx}u^2 = \partial_{xx}\Delta^{-1}(w^2 - b^1) + \partial_{xy}\Delta^{-1}d^1,$$

and

$$\partial_{xy}u^3 = \partial_{xy}\Delta^{-1}(w^3 + d^3).$$
Therefore,
\[ \partial_{xy} u^3 - \partial_{xz} u^2 = \partial_{xy} \Delta^{-1} w^3 - \partial_{xz} \Delta^{-1} \left( w^2 - b^1 \right). \]

Then it is not difficult to verify that \((b^3, d^3, w^3)\) satisfies
\[
\begin{cases} 
\partial_t b^3 + y \partial_y b^3 + d^3 = 0, \\
\partial_t d^3 + y \partial_y d^3 + 2 \partial_{xy} \Delta^{-1} d^3 + 2 \partial_{xz} \Delta^{-1} \left( w^2 - b^1 \right) + \frac{\Delta b^3}{\epsilon} - (\lambda + 2\mu) \Delta d^3 = 0, \\
\partial_t w^3 + y \partial_y w^3 + 2 \partial_{xy} \Delta^{-1} w^3 - 2 \partial_{xz} \Delta^{-1} \left( w^2 - b^1 \right) - \mu \Delta w^3 = 0.
\end{cases}
\]

(2.10)

Now we turn to reformulate the system of \((\partial_t b, \partial_t d, w^1)\). The term \(\partial_t d\) in the equation (2.11) of \(w^1\) may cause growth and hence we try to cancel out this bad term. To this end, applying \(\partial_y\) to (2.5) yields
\[
(2.11) \quad \partial_t \partial_y b + y \partial_y \partial_y b + \partial_y b + \partial_y d = 0.
\]

Let us define
\[ w^1 := w^1 - \partial_y b, \]
and denote
\[ b^2 := \partial_t b, \] and \[ d^2 := \partial_t d. \]

Taking the difference between (2.4) and (2.11), we are led to
\[
(2.12) \quad \partial_t w^1 + y \partial_y w^1 + w^2 - b^1 + 2 \partial_{xy} u^1 - 2 \partial_{xz} u^2 - \mu \Delta w^1 - \mu \Delta b^2 = 0.
\]

Noting that
\[ \partial_{xy} u^1 = \partial_{xy} \Delta^{-1} \left( w^1 + b^2 + d^1 \right), \]
and using (2.6), we obtain
\[
(2.13) \quad \partial_{xy} u^1 - \partial_{xx} u^2 = \partial_{xy} \Delta^{-1} \left( w^1 + b^2 \right) - \partial_{xx} \Delta^{-1} \left( w^2 - b^1 \right).
\]

Then we find that \((b^2, d^2, w^1)\) solves
\[
\begin{cases} 
\partial_t b^2 + y \partial_y b^2 + b^1 + d^2 = 0, \\
\partial_t d^2 + y \partial_y d^2 + d^1 + 2 \partial_{xy} \Delta^{-1} d^2 + 2 \partial_{xy} \Delta^{-1} \left( w^2 - b^1 \right) + \frac{\Delta b^2}{\epsilon} - (\lambda + 2\mu) \Delta d^2 = 0, \\
\partial_t w^1 + y \partial_y w^1 - \mu \Delta w^1 + w^2 - 2 b^1 + 2 \partial_{xy} \Delta^{-1} \left( w^1 + b^2 \right) - 2 \partial_{xx} \Delta^{-1} \left( w^2 - b^1 \right) - \mu \Delta b^2 = 0.
\end{cases}
\]

(2.14)

2.3. **Change of coordinates.** As mentioned above, we will use the Fourier multiplier method in this paper. Thereby it is convenient to switch to new variables defined below
\[
(2.15) \quad X = x - ty, \quad Y = y, \quad Z = z.
\]

For \(i \in \{1, 2, 3\}\), and \(j \in \{1, 2\}\), we define functions
\[ B^i(t, X, Y, Z) = b^i(t, X + tY, Y, Z), \]
\[ D^i(t, X, Y, Z) = d^i(t, X + tY, Y, Z), \]
\[ W^i(t, X, Y, Z) = w^i(t, X + tY, Y, Z), \]
and
\[ W^3(t, X, Y, Z) = w^3(t, X + tY, Y, Z). \]

Then the systems (2.9), (2.10) and (2.14) can be rewritten as
\[
\begin{cases} 
\partial_t B^1 + D^1 = 0, \\
\partial_t D^1 + 2 \partial_{xy} \Delta L^{-1} D^1 + 2 \partial_{xx} \Delta L^{-1} \left( W^2 - B^1 \right) + \frac{\Delta B^1}{\epsilon} - (\lambda + 2\mu) \Delta L D^1 = 0, \\
\partial_t W^2 - \mu \Delta L W^2 + \mu \Delta L B^1 = 0,
\end{cases}
\]

(2.16)
In order to eliminate the coupling term $-\epsilon$ following toy model (taking $\epsilon = 1$ for simplicity):

$$\begin{align*}
\partial_t B^3 + D^3 &= 0, \\
\partial_t D^3 + 2\partial_{XY} D_t^3 + 2\partial_{XZ} \Delta_L^{-1} \left( W^2 - B^1 \right) + \frac{\Delta B^3}{\epsilon} - (\lambda + 2\mu)\Delta_L D^3 &= 0, \\
\partial_t W^3 + 2\partial_{XY} \Delta_L^{-1} W^3 - 2\partial_{XZ} \Delta_L^{-1} \left( W^2 - B^1 \right) - \mu\Delta_L W^3 &= 0,
\end{align*}$$

and

$$\begin{align*}
\partial_t B^2 + D^2 &= F, \\
\partial_t D^2 + 2\partial_{XY} \Delta_L^{-1} \left( W^1 + B^2 \right) + \Delta_B^2 - (\lambda + 2\mu)\Delta_L D^2 &= G, \\
\partial_t W^1 + 2\partial_{XY} \Delta_L^{-1} \left( W^1 + B^2 \right) - \mu\Delta_L W^1 - \mu\Delta_L B^2 &= H,
\end{align*}$$

with

$$F := -B^1,$$ 

$$G := -D^1 - 2\partial_{XY} \Delta_L^{-1} \left( W^2 - B^1 \right),$$

$$H := 2B^1 - W^2 + 2\partial_{XX} \Delta_L^{-1} \left( W^2 - B^1 \right).$$

Denote

$$p = k^2 + (\eta - kt)^2 + \ell^2, \quad p' = -2k(\eta - kt).$$

Then

$$2\partial_{XY}^2 = -2k(\eta - kt) = p', \quad 2\partial_{XY} \Delta_L^{-1} = \frac{2k(\eta - kt)}{p} = \frac{p'}{p}.$$

### 2.4. The toy model and Fourier multipliers.

The systems (2.16)–(2.18) lead us to consider the following toy model (taking $\epsilon = 1$ for simplicity):

$$\begin{align*}
\partial_t \phi + \psi &= 0, \\
\partial_t \psi + 2\partial_{XY} \Delta_L^{-1} \psi + \Delta_L \phi - (\lambda + 2\mu)\Delta_L \psi &= 0.
\end{align*}$$

In Fourier variables (2.21) can be rewritten as

$$\begin{align*}
\partial_t \hat{\phi} + \hat{\psi} &= 0, \\
\partial_t \hat{\psi} - \frac{2k}{p} \hat{\psi} - p\hat{\phi} + (\lambda + 2\mu)p\hat{\psi} &= 0.
\end{align*}$$

The evolution of $|\hat{\psi}|^2$ reads

$$\partial_t \left( \frac{|\hat{\psi}|^2}{2} \right) - \frac{p'}{p} |\hat{\psi}|^2 - p\Re(\hat{\phi}\hat{\psi}) + (\lambda + 2\mu)p|\hat{\psi}|^2 = 0.$$ 

In order to eliminate the coupling term $-p\Re(\hat{\phi}\hat{\psi})$, which equals to $-\Re(\sqrt{p}\hat{\phi} \sqrt{p}\hat{\psi})$, let us investigate the evolution of $\sqrt{p}\hat{\phi}$

$$\partial_t \left( \sqrt{p}\hat{\phi} \right) - \frac{1}{2} \frac{p'}{p} \left( \sqrt{p}\hat{\phi} \right) + \sqrt{p}\hat{\psi} = 0.$$ 

Consequently,

$$\partial_t \left( \frac{|\sqrt{p}\hat{\phi}|^2}{2} \right) - \frac{1}{2} \frac{p'}{p} |\sqrt{p}\hat{\phi}|^2 + p\Re(\hat{\phi}\hat{\psi}) = 0.$$ 

Combining (2.23) with (2.24) yields

$$\partial_t \left( \frac{|\hat{\psi}|^2 + |\sqrt{p}\hat{\phi}|^2}{2} \right) - \frac{p'}{p} \left( |\hat{\psi}|^2 + \frac{1}{2} |\sqrt{p}\hat{\phi}|^2 \right) + (\lambda + 2\mu)p|\hat{\psi}|^2 = 0.$$
We can see from this equality that \((\sqrt{\rho}\phi, \psi)\) is partially dissipative. Fortunately, the damping effect of \(\sqrt{\rho}\phi\) hidden in the system (2.21) can be shown by the classical method which goes back to Matsmura and Nishida [55]. Indeed, we infer from (2.22) that the product \(\psi_\beta\phi\) of \(\psi\) and \(\phi\) satisfies

\[
(\psi_\beta\phi)_t - \sqrt{\rho}\phi_t + |\psi_\beta\phi|^2 = p'\frac{\mu}{p} \psi_\beta\phi - (\lambda + 2\mu)p\psi_\beta\phi.
\]

A linear combination of (2.25) and (2.26) gives

\[
(\psi_\beta\phi)_t - \sqrt{\rho}\phi_t + |\psi_\beta\phi|^2 = \frac{p'}{p} |\psi_\beta\phi|^2 - c\psi_\beta\phi + p'\frac{\mu}{p} \psi_\beta\phi - (\lambda + 2\mu)p\psi_\beta\phi = error \ terms,
\]

where \(c\) is to be determined. Clearly, one can choose \(c = \delta\mu\) with \(\delta \ll 1\) to ensure \((\lambda + 2\mu)p - c)|\psi_\beta\phi|^2 \geq \mu p|\psi_\beta\phi|^2\).

and

\[
\frac{|\psi_\beta\phi|^2 - |\sqrt{\rho}\phi_t|^2}{2} - c\psi_\beta\phi \approx |\psi_\beta\phi|^2 + |\sqrt{\rho}\phi_t|^2,
\]

for \(k \neq 0\). However, if \(c\) were chosen in this way, it means that we only used the dissipation of \(\psi\), but ignored the enhanced dissipation of \(\psi\) which is easily to be seen if one just focuses on \((\psi_{\beta\phi} - (\lambda + 2\mu)\Delta L \psi)\) in (2.21), see [6], for instance. In fact, if the enhanced dissipation of \(\psi\) is taken into account, we can regard (2.27) as follows at least formally

\[
(\psi_{\beta\phi})_t - \sqrt{\rho}\phi_t + |\psi_{\beta\phi}|^2 = \frac{p'}{p} |\psi_{\beta\phi}|^2 + c|\sqrt{\rho}\phi_t|^2 + (\alpha(\lambda + 2\mu)p + \beta\mu^\frac{1}{2} - c)|\psi_{\beta\phi}|^2 = error \ terms,
\]

with some positive constants \(\alpha\) and \(\beta\) satisfying \(\alpha + \beta \leq 1\). At this stage, it is natural to choose

\[
c = \delta\mu^\frac{1}{2}, \quad \text{with} \quad \delta \ll 1.
\]

As a result, both the enhanced dissipation of \(\psi\) and \(-\sqrt{-\Delta L}\phi\) are expected to be generated.

In all the analyses above, the bad term \(-\frac{p'}{p} \left(|\psi_{\beta\phi}|^2 + \frac{1}{2} |\sqrt{\rho}\phi_t|^2\right)\) is overlooked. Now we turn to deal with this term. First of all, based on the choice of \(c\), it is necessary to study the following simplified toy model

\[
(\psi_{\beta\phi})_t - \frac{p'}{p} \psi_{\beta\phi} + \mu \psi_{\beta\phi} = 0.
\]

This equation can be seen as a competition between the linear stretching term \(-\frac{p'}{p} \psi_{\beta\phi}\) and the damping term \(\mu \psi_{\beta\phi}\). Motivated by [6], we introduce a multiplier \(m\) to balance the linear stretching if the damping is not dominated. Roughly speaking, if the stretching overcomes the damping, we define the multiplier \(m\) in such a way that \(m^{-1} \psi_{\beta\phi}\) solves

\[
(\psi_{\beta\phi})_t - \frac{p'}{p} \psi_{\beta\phi} + \mu^\frac{1}{2} \psi_{\beta\phi} = 0.
\]

To this end, let us first compare the sizes of the two factors \(\frac{p'}{p}\) and \(\mu^\frac{1}{2}\). Indeed, if \(k \neq 0\), then

\[
\frac{|k(\eta - kt)|}{k^2 + (\eta - kt)^2 + l^2} = \frac{|t - \frac{\eta}{k}|}{1 + \left(t - \frac{\eta}{k}\right)^2 + \frac{l^2}{L^2}} \leq \frac{1}{|t - \frac{\eta}{k}|} \ll \mu^\frac{1}{2},
\]

uniformly in \((k, \eta, l)\), as long as

\[
|t - \frac{\eta}{k}| \gg \mu^\frac{1}{2}.
\]
This coincides with the comparison between \( \frac{\|u\|_1}{p} \) and \( \mu p \) for the 3D incompressible case in [6]. Accordingly, the corresponding multiplier constructed by Bedrossian, Germain and Masmoudi in [6] still applies in this paper. We will rewrite it in a form for our convenience, see also [2].

**Definition 2.1.** Let us define \( m = m(t, k, \eta, l) \) by the exact formulas below:

(i) if \( k = 0 \): \( m(t, 0, \eta, l) = 1 \);
(ii) if \( k \neq 0, \frac{\eta}{k} \leq -64\mu^{-\frac{3}{4}} \): \( m(t, k, \eta, l) = 1 \);
(iii) if \( k \neq 0, -64\mu^{-\frac{3}{4}} < \frac{\eta}{k} \leq 0 \):
   (iii.1) if \( 0 \leq t < \frac{\eta}{k} + 64\mu^{-\frac{3}{4}}, m(t, k, \eta, l) := \frac{k^2 + (\eta - kt)^2 + l^2}{k^2 + \eta^2 + l^2} \);
   (iii.2) if \( t > \frac{\eta}{k} + 64\mu^{-\frac{3}{4}}, m(t, k, \eta, l) := \frac{k^2 + (64\mu^{-\frac{3}{4}}k)^2 + l^2}{k^2 + l^2} \);
(iv) if \( k \neq 0, \frac{\eta}{k} > 0 \):
   (iv.1) if \( 0 \leq t < \frac{\eta}{k}, m(t, k, \eta, l) = 1 \),
   (iv.2) if \( \frac{\eta}{k} \leq t < \frac{\eta}{k} + 64\mu^{-\frac{3}{4}}, m(t, k, \eta, l) := \frac{k^2 + (\eta - kt)^2 + l^2}{k^2 + l^2} \);
   (iv.3) if \( t > \frac{\eta}{k} + 64\mu^{-\frac{3}{4}}, m(t, k, \eta, l) = \frac{k^2 + (64\mu^{-\frac{3}{4}}k)^2 + l^2}{k^2 + l^2} \).

**Remark 2.1.** For \( k \neq 0 \), the value of the multiplier \( m \) can be classified into the following three situations:

- In case (ii), sub-case (iii.2), and sub-case (iv.3), we have \( t - \frac{\eta}{k} \geq 64\mu^{-\frac{3}{4}}, \) and thus
  \[
  (2.31) \quad \frac{|k(\eta - kt)|}{k^2 + (\eta - kt)^2 + l^2} < \frac{\mu^\frac{3}{4}}{64}.
  \]

- In sub-cases (iii.1) and (iv.2), there holds \( k(\eta - kt) \leq 0 \), which means that the linear term \(-\frac{\mu^\frac{3}{4}}{p} \hat{f}\) in (2.29) corresponds to an amplification of \( \hat{f} \). Indeed, in these two sub-cases, \( m \) satisfies the ODE
  \[
  (2.32) \quad \frac{\partial m}{m} = \frac{p'}{p},
  \]

  - In the sub-case (iv.1), it holds that
  \[
  (2.33) \quad \frac{p'}{p} \leq 0,
  \]

  and thus the linear term \(-\frac{\mu^\frac{3}{4}}{p} \hat{f}\) in (2.29) amounts to a damping effect.

Furthermore, noting that \( m \) is nondecreasing in \( t \), we have
\[
1 \leq m(t, k, \eta, l) \leq \frac{k^2 + (64\mu^{-\frac{3}{4}}k)^2 + l^2}{k^2 + l^2}.
\]

Accordingly, for \( 0 < \mu \leq 1 \), one easily deduces that
\[
(2.34) \quad 1 \leq m(t, k, \eta, l) \leq \mu^{-\frac{3}{4}}.
\]

Finally, using again the fact that \( m(t, k, \eta, l) \) is nondecreasing in \( t \), we find that
\[
(2.35) \quad m(t, k, \eta, l) \leq \frac{k^2 + (\eta - kt)^2 + l^2}{k^2 + l^2}.
\]

Several additional multipliers are also needed in this paper. In fact, we will use multipliers \( m_1 \) and \( m_3 \) to balance the growth due to the linear coupling between \( \text{div}u \) and \( w \). Similar multipliers can be found in [1] [6] [66]. In [6], Bedrossian, Germain and Masmoudi constructed a multiplier \( M^2 \) to compensate for the transient slow-down of the enhanced dissipation near the critical times. The multiplier \( m_2 \) below, already used in [2] [8], is an adaptation of \( M^2 \) in [6].

**Definition 2.2.** Assume that \( N \) is a positive constant. Let us define \( m_i, i = 1, 2, 3 \) as follows: if \( k = 0 \), \( m_i(t, 0, \eta) = 1 \) and
• if \( k \neq 0 \),
\[
\partial_t m_1 = N \frac{k^2}{p} m_1, \quad m_1|_{t=0} = 1,
\]

• if \( k \neq 0 \),
\[
\partial_t m_2 = \frac{\mu^\frac{1}{3}}{1 + (\mu^\frac{1}{3}|t - \eta/k|^2)} m_2, \quad m_2|_{t=0} = 1,
\]

• if \( k \neq 0 \),
\[
\partial_t m_3 = N \frac{kl}{p} m_3, \quad m_3|_{t=0} = 1.
\]

**Remark 2.2.** The multipliers \( m_i, i = 1, 2, 3 \) are bounded from above and below uniformly in \( \mu \) and \( (t, k, \eta, l) \in [0, \infty) \times \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \). In particular, the upper bounds of \( m_1 \) and \( m_3 \) depend on the constant \( N \). More precisely, direct calculations yield
\[
1 \leq m_i \leq e^{N\pi}, \quad i = 1, 3, \quad \text{and} \quad 1 \leq m_2 \leq e^\pi.
\]

Moreover, for \( (t, k, \eta, l) \in [0, \infty) \times \mathbb{Z} \setminus \{0\} \times \mathbb{R} \times \mathbb{Z} \), the multiplier \( m_2 \) satisfies
\[
\frac{\partial_t m_2}{m_2} + \mu p \geq \frac{1}{2} \mu^\frac{1}{3}.
\]

In fact, if \( |t - \eta/k| \geq \mu^{-\frac{1}{3}} \), and \( k \neq 0 \)
\[
\mu p = \mu^\frac{1}{3} \mu^\frac{2}{3} (k^2 + (\eta - kt)^2 + \tilde{l}^2) = \mu^\frac{1}{3} k^2 \left( \mu^\frac{2}{3} + \mu^\frac{2}{3} |t - \eta/k|^2 + \frac{\tilde{l}^2}{k^2} \right) > \mu^\frac{1}{3}.
\]

On the other hand, if \( |t - \eta/k| \leq \mu^{-\frac{1}{3}} \), and \( k \neq 0 \)
\[
\frac{\partial_t m_2}{m_2} = \frac{\mu^\frac{1}{3}}{1 + (\mu^\frac{1}{3}|t - \eta/k|^2)} \geq \frac{\mu^\frac{1}{3}}{2},
\]
so the inequality \( (2.40) \) is true.

3. Estimates of zero mode

We begin this section by giving the \( x \)-average of the system \( (1.3) \):
\[
\begin{cases}
\partial_t b_0 + \text{div}\tilde{u}_0 = 0, \\
\partial_t \tilde{u}_0 + \frac{\text{div}b_0}{e} - \mu \Delta \tilde{u}_0 - (\lambda + \mu) \text{div}\tilde{u}_0 = 0, \\
\partial_t u_0^1 + u_0^2 - \mu \Delta u_0^1 = 0.
\end{cases}
\]

Moreover, it will also be convenient to consider the projection of the sub-system of \((b_0, \tilde{u}_0)\) in \( (3.1) \) onto the zero frequencies in \( z \):
\[
\begin{cases}
\partial_t b_{00} + \partial_z u_{00}^2 = 0, \\
\partial_t u_{00}^2 + \frac{\partial b_{00}}{e^2} - (\lambda + 2\mu) \partial_{yy} u_{00}^2 = 0, \\
\partial_t u_{00}^3 - \mu \partial_{yy} u_{00}^3 = 0,
\end{cases}
\]
where we have used \((b_{00}, u_{00}^2, u_{00}^3)\) to denote \((P_{t=0} b_0, P_{t=0} u_0^2, P_{t=0} u_0^3)\).

The aim of this section is to study the long-time dynamics of \((b_0, u_0)\). One can see from \( (3.1) \) that \((b_0, \tilde{u}_0)\) satisfies a parabolic-hyperbolic system, while \( u_0 \) satisfies an inhomogeneous heat equation. Thus, \((b_0, u_0)\) and \( u_0^1 \) should be estimated in different ways.
3.1. Estimates of \((b_0, \bar{u}_0)\). To begin with, we establish the energy estimates of \((b_0, \bar{u}_0)\).

**Proposition 3.1.** Assume that the Mach number \(\epsilon\) and the viscous coefficients \(\lambda\) and \(\mu\) satisfy

\[
(3.3) \quad \max \left\{ (\lambda + \mu)\epsilon^2, (\lambda + \mu)\epsilon \right\} \leq 1.
\]

Then there exists a positive constant \(C\), independent of \(\epsilon, \lambda\) and \(\mu\), such that for any multi-index \(\alpha = (\alpha_1, \alpha_2)\), there holds

\[
(3.4) \quad \left\| \frac{\partial^\alpha b_0}{\epsilon} \right\|_{H^1}^2 + \left\| \partial^\alpha \bar{u}_0 \right\|_{H^1}^2 + \mu \left\| \nabla \partial^\alpha \bar{u}_0 \right\|_{L^2 H^1}^2 + \frac{\mu}{\epsilon} \left\| \nabla \partial^\alpha b_0 \right\|_{L^2 H^1}^2 \leq C \left( \left\| \frac{\partial^\alpha (b_0 u_0)}{\epsilon} \right\|_{H^1}^2 + \left\| \partial^\alpha (\bar{u}_0 u_0) \right\|_{H^1}^2 \right),
\]

where \(\partial^\alpha = \partial_{\alpha_1}^\alpha \partial_{\alpha_2}^\alpha\).

**Proof.** Standard energy estimates show that

\[
(3.5) \quad \frac{1}{2} \frac{d}{dt} \left( \left\| \frac{\partial^\alpha b_0}{\epsilon} \right\|_{H^1}^2 + \left\| \partial^\alpha \bar{u}_0 \right\|_{H^1}^2 \right) + \mu \left\| \nabla \partial^\alpha \bar{u}_0 \right\|_{H^1}^2 + (\lambda + \mu) \left\| \text{div} \partial^\alpha \bar{u}_0 \right\|_{H^1}^2 = 0,
\]

and

\[
(3.6) \quad \frac{d}{dt} \langle \partial^\alpha \bar{u}_0, \nabla \partial^\alpha b_0 \rangle + \left\| \frac{\nabla \partial^\alpha b_0}{\epsilon} \right\|_{L^2}^2 - \left\| \text{div} \partial^\alpha \bar{u}_0 \right\|_{L^2}^2 = \mu \langle \Delta \partial^\alpha \bar{u}_0, \nabla \partial^\alpha b_0 \rangle + (\lambda + \mu) \langle \nabla \text{div} \partial^\alpha \bar{u}_0, \nabla \partial^\alpha b_0 \rangle.
\]

Consequently,

\[
(3.7) \quad \frac{1}{2} \frac{d}{dt} \left( \left\| \frac{\partial^\alpha b_0}{\epsilon} \right\|_{H^1}^2 + \left\| \partial^\alpha \bar{u}_0 \right\|_{H^1}^2 \right) + \mu \left\| \nabla \partial^\alpha \bar{u}_0 \right\|_{H^1}^2 + \frac{\lambda + \mu}{2} \left\| \text{div} \partial^\alpha \bar{u}_0 \right\|_{H^1}^2 + \frac{\lambda + \mu}{2} \left\| \frac{\nabla \partial^\alpha b_0}{\epsilon} \right\|_{L^2}^2 = \frac{\mu(\lambda + \mu)}{2} \langle \Delta \partial^\alpha \bar{u}_0, \nabla \partial^\alpha b_0 \rangle + \frac{\lambda + \mu}{2} \langle \nabla \text{div} \partial^\alpha \bar{u}_0, \nabla \partial^\alpha b_0 \rangle.
\]

Clearly,

\[
(3.8) \quad \left\| \frac{\partial^\alpha b_0}{\epsilon} \right\|_{H^1}^2 + \left\| \partial^\alpha \bar{u}_0 \right\|_{H^1}^2 + (\lambda + \mu) \langle \partial^\alpha \bar{u}_0, \nabla \partial^\alpha b_0 \rangle \lesssim \left\| \frac{\partial^\alpha b_0}{\epsilon} \right\|_{H^1}^2 + \left\| \partial^\alpha \bar{u}_0 \right\|_{H^1}^2.
\]

On the other hand, if \((\lambda + \mu)\epsilon^2 \leq 1\), there holds

\[
(3.9) \quad \frac{\mu(\lambda + \mu)}{2} \langle \Delta \partial^\alpha \bar{u}_0, \nabla \partial^\alpha b_0 \rangle \leq \frac{\lambda + \mu}{8} \left\| \nabla \partial^\alpha b_0 \right\|_{L^2}^2 + \frac{(\lambda + \mu)\epsilon^2}{2} \left\| \Delta \partial^\alpha \bar{u}_0 \right\|_{L^2}^2 \leq \frac{\lambda + \mu}{8} \left\| \nabla \partial^\alpha b_0 \right\|_{L^2}^2 + \frac{\mu}{2} \left\| \Delta \partial^\alpha \bar{u}_0 \right\|_{L^2}^2,
\]

and if \((\lambda + \mu)\epsilon \leq 1\), we have

\[
(3.10) \quad \frac{(\lambda + \mu)^2}{2} \langle \nabla \text{div} \partial^\alpha \bar{u}_0, \nabla \partial^\alpha b_0 \rangle \leq \frac{\lambda + \mu}{8} \left\| \nabla \partial^\alpha b_0 \right\|_{L^2}^2 + \frac{(\lambda + \mu)^2 \epsilon^2}{2} \left\| \nabla \text{div} \partial^\alpha \bar{u}_0 \right\|_{L^2}^2 \leq \frac{\lambda + \mu}{8} \left\| \nabla \partial^\alpha b_0 \right\|_{L^2}^2 + \frac{\lambda + \mu}{2} \left\| \nabla \text{div} \partial^\alpha \bar{u}_0 \right\|_{L^2}^2.
\]

It follows that

\[
(3.11) \quad \frac{d}{dt} \left( \left\| \frac{\partial^\alpha b_0}{\epsilon} \right\|_{H^1}^2 + \left\| \partial^\alpha \bar{u}_0 \right\|_{H^1}^2 \right) + \mu \left\| \nabla \partial^\alpha \bar{u}_0 \right\|_{H^1}^2 + \frac{\lambda + \mu}{2} \left\| \frac{\nabla \partial^\alpha b_0}{\epsilon} \right\|_{L^2}^2 \leq 0.
\]
Combining this with (3.8), and using (1.2), we get (3.4). The proof of Proposition 3.1 is completed.

\[ \square \]

It is worth pointing out that the differential inequality (3.9) implies the exponential decay of the projection of \((b_0, \tilde{u}_0)\) onto the nonzero frequencies in \(z\). We shall give a more accurate estimate below.

**Lemma 3.1.** Assume that the Mach number \(\varepsilon\) and the viscous coefficients \(\lambda\) and \(\mu\) satisfy

\[ (\lambda + 2\mu)\varepsilon \leq 1. \]

Then there exists a positive constant \(C\), independent of \(\varepsilon, \lambda\) and \(\mu\), such that for any multi-index \(\alpha = (\alpha_1, \alpha_2)\), there holds

\[ \left\| \partial^\alpha P_{t\neq 0} b_0 \right\|_{L^2(z)}^2 + \left\| \partial^\alpha P_{t\neq 0} \tilde{u}_0 \right\|_{L^2(z)}^2 \leq C e^{-\frac{1}{2} \mu t} \left( \left\| \partial^\alpha P_{t\neq 0} (b_{00}) \right\|_{L^2(z)}^2 + \left\| \partial^\alpha P_{t\neq 0} (\tilde{u}_{00}) \right\|_{L^2(z)}^2 \right). \]

**Proof.** Clearly, the equality (3.5) still holds for \((P_{t\neq 0} b_0, P_{t\neq 0} \tilde{u}_0)\) with the \(H^1\) norm replaced by \(L^2\) norm

\[ \frac{1}{2} \frac{d}{dt} \left( \left\| \partial^\alpha P_{t\neq 0} b_0 \right\|_{L^2(z)}^2 + \left\| \partial^\alpha P_{t\neq 0} \tilde{u}_0 \right\|_{L^2(z)}^2 \right) + \mu \left\| \nabla \partial^\alpha P_{t\neq 0} \tilde{u}_0 \right\|_{L^2(z)}^2 + (\lambda + \mu) \left\| \text{div} \partial^\alpha P_{t\neq 0} \tilde{u}_0 \right\|_{L^2(z)}^2 = 0. \]

Apply the operator \(\Delta^{-1} \text{div} P_{t\neq 0}\) to the equation (3.12) yields

\[ \partial_t \Delta^{-1} \text{div} P_{t\neq 0} \tilde{u}_0 + \frac{P_{t\neq 0} b_0}{\varepsilon^2} - (\lambda + 2\mu) \text{div} P_{t\neq 0} \tilde{u}_0 = 0. \]

Then it is easy to verify that

\[ \frac{d}{dt} \langle \partial^\alpha P_{t\neq 0} b_0, \Delta^{-1} \text{div} P_{t\neq 0} \tilde{u}_0 \rangle + \left\| \partial^\alpha P_{t\neq 0} b_0 \right\|_{L^2(z)}^2 - \left\| (\Delta^{-1} \text{div} \partial^\alpha P_{t\neq 0} \tilde{u}_0 \right\|_{L^2(z)}^2 = (\lambda + 2\mu) \langle \text{div} \partial^\alpha P_{t\neq 0} \tilde{u}_0, \partial^\alpha P_{t\neq 0} b_0 \rangle. \]

Combining (3.12) with (3.13), and using (3.10), we are led to

\[ \frac{1}{2} \frac{d}{dt} \left( \left\| \partial^\alpha P_{t\neq 0} b_0 \right\|_{L^2(z)}^2 + \left\| \partial^\alpha P_{t\neq 0} \tilde{u}_0 \right\|_{L^2(z)}^2 + \mu \left\| \nabla \partial^\alpha P_{t\neq 0} \tilde{u}_0 \right\|_{L^2(z)}^2 \right) \]

\[ \leq \frac{(\lambda + 2\mu)\mu}{2} \left\| \text{div} \partial^\alpha P_{t\neq 0} \tilde{u}_0 \right\|_{L^2(z)} \left\| \partial^\alpha P_{t\neq 0} b_0 \right\|_{L^2(z)} \leq \frac{\mu}{4} \left( \left\| \partial^\alpha P_{t\neq 0} b_0 \right\|_{L^2(z)}^2 + \left\| \nabla \partial^\alpha P_{t\neq 0} \tilde{u}_0 \right\|_{L^2(z)}^2 \right). \]

Noticing that

\[ \mu \left( \left\| \nabla \partial^\alpha P_{t\neq 0} \tilde{u}_0 \right\|_{L^2(z)}^2 + \left\| \partial^\alpha P_{t\neq 0} b_0 \right\|_{L^2(z)}^2 \right) \]

\[ \geq \frac{2\mu}{3} \left( \left\| \partial^\alpha P_{t\neq 0} b_0 \right\|_{L^2(z)}^2 + \left\| \partial^\alpha P_{t\neq 0} \tilde{u}_0 \right\|_{L^2(z)}^2 + \mu \left( \partial^\alpha P_{t\neq 0} b_0, \Delta^{-1} \text{div} \partial^\alpha P_{t\neq 0} \tilde{u}_0 \right) \right), \]

then (3.11) follows immediately.

\[ \square \]

Recalling that the projection of \((b_0, \tilde{u}_0)\) onto the zero frequencies in \(z\) satisfies the system (3.2), it is well known (see [44], for instance) that the linear operator in (3.2) behaves like the heat semigroup, so that \((b_{00}, \tilde{u}_{00})\) decays as fast as the solution to the corresponding heat equation on \(\mathbb{R}\) with viscous coefficient \(\lambda + 2\mu\). For the convenience of reads, we shall give a proof of this fact in the following lemma.
Lemma 3.2. Let $k \in \mathbb{N}$. Assume that the Mach number $\epsilon$ and the viscous coefficients $\lambda$ and $\mu$ satisfy

$$\lambda + 2\mu \leq 1, \quad \text{and} \quad (\lambda + 2\mu)\epsilon \leq 1,$$

and the initial data $((b_{00})_{00}, (u^2_{00})_{00}) \in H^k \cap L^1$. Then there exists a positive constant $C$, depending on $k$, but independent of $\epsilon$, $\lambda$ and $\mu$, such that

$$\| \partial^k_t \left( \frac{b_{00}}{\epsilon} \right) - \left( \frac{b_{00}}{\epsilon} \right) \|_{L^2}^2 \leq C((\lambda + 2\mu)(t)\epsilon)^{-(k + \frac{1}{2})} \| \left( \frac{(b_{in})_{00}}{\epsilon}, (u^2_{in})_{00} \right) \|_{H^k \cap L^1}^2,$$

and

$$\| \partial^3_t u^2_{00} \|_{L^2}^2 \leq C(\mu(t))^{-2} \| u^2_{00} \|_{L^2}^2.$$

Proof. We use the method in [67] to avoid subtle analysis on the eigenvalues of the system (3.2). Since $u^2_{00}$ satisfies a heat equation, we only focus on the sub-system for $(b_{00}, u^2_{00})$ below. Similar to (3.6) and (3.7), direction calculations in Fourier variable show that

$$\frac{1}{2} \frac{d}{dt} \left( \frac{b_{00}}{\epsilon} \right)^2 + \left( \frac{b_{00}}{\epsilon} \right)^2 + (\lambda + 2\mu)\epsilon^2 \left( \frac{b_{00}}{\epsilon} \right)^2 = 0,$$

and

$$\frac{d}{dt} \left( \frac{b_{00}}{\epsilon} \right)^2 + \left( \frac{b_{00}}{\epsilon} \right)^2 + (\lambda + 2\mu)\epsilon^2 \left( \frac{b_{00}}{\epsilon} \right)^2 = 0.$$

For $|\eta| \leq 1$, a linear combination of (3.18) and (3.19) yields

$$\frac{1}{2} \frac{d}{dt} \left( \frac{b_{00}}{\epsilon} \right)^2 + \left( \frac{b_{00}}{\epsilon} \right)^2 + (\lambda + 2\mu)\epsilon^2 \left( \frac{b_{00}}{\epsilon} \right)^2 = \frac{(\lambda + 2\mu)^2\epsilon^2}{2} \left( \frac{b_{00}}{\epsilon} \right)^2 + \left( \frac{b_{00}}{\epsilon} \right)^2,$$

where we have used (3.15). Using again the fact $|\eta| \leq 1$ and (3.15), one deduces that

$$\frac{1}{2} \left( \frac{b_{00}}{\epsilon} \right)^2 + \left( \frac{b_{00}}{\epsilon} \right)^2 \leq \frac{3}{4} \left( \frac{b_{00}}{\epsilon} \right)^2 + \left( \frac{b_{00}}{\epsilon} \right)^2.$$

It follows from (3.20) and (4.14) that

$$\left( \frac{b_{00}}{\epsilon} \right)^2 + \left( \frac{b_{00}}{\epsilon} \right)^2 \leq Ce^{-\frac{\lambda + 2\mu\epsilon^2}{\lambda\epsilon^2} \left( \frac{(b_{00})_{00}}{\epsilon} \right)^2 + \left( \frac{(b_{00})_{00}}{\epsilon} \right)^2}, \quad \text{if} \quad |\eta| \leq 1.$$

For $|\eta| > 1$, multiplying (3.19) by $\frac{\lambda + 2\mu}{2\eta^2}$, and adding the resulting equality to (3.18), we arrive at

$$\frac{1}{2} \frac{d}{dt} \left( \frac{b_{00}}{\epsilon} \right)^2 + \left( \frac{b_{00}}{\epsilon} \right)^2 - \frac{(\lambda + 2\mu\epsilon^2)}{\eta} \left( \frac{b_{00}}{\epsilon} \right)^2 + (\lambda + 2\mu)\epsilon^2 \left( \frac{b_{00}}{\epsilon} \right)^2 = \frac{(\lambda + 2\mu)^2\epsilon^2}{2} \left( \frac{b_{00}}{\epsilon} \right)^2 + \left( \frac{b_{00}}{\epsilon} \right)^2.$$

Consequently,

$$\left( \frac{b_{00}}{\epsilon} \right)^2 + \left( \frac{b_{00}}{\epsilon} \right)^2 \leq Ce^{-\frac{(\lambda + 2\mu\epsilon^2)}{\lambda\epsilon^2} \left( \frac{(b_{00})_{00}}{\epsilon} \right)^2 + \left( \frac{(b_{00})_{00}}{\epsilon} \right)^2}, \quad \text{if} \quad |\eta| > 1.$$

Next, by virtue of Plancherel’s theorem, (3.22) and (3.23), we find that for all $k \in \mathbb{N}$, there holds

$$\left\| \partial^k_t \left( \frac{b_{00}}{\epsilon} \left( u^2_{00} \right) \right) \right\|_{L^2}^2 = \int_{\mathbb{R}} \eta^{2k} \left( \frac{b_{00}}{\epsilon} \right)^2 + \left( \frac{b_{00}}{\epsilon} \right)^2 d\eta.
Nevertheless, in this paper the spectral gap made via the divergence free condition does not hold and the projection of the initial data onto the zero frequencies in $x$ and $z$ satisfy

$$
\|u\|_{L^\infty} \text{ decay of } t
$$

Therefore, in view of (3.15), the inequality (3.16) holds, and (3.17) can be obtained in the same manner as (3.24). This completes the proof of Lemma 3.2.

3.2. The lift-up effect. Now we are in a position to investigate $u^1_0$. Noting that $u^1_0$ satisfies (3.13), which can be solved explicitly

$$
(3.25) \quad u^1_0(t) = e^{it\Delta}(u^1_0) - \int_0^t e^{i(t-s)\Delta}u^2_0(s)ds.
$$

We would like to remark that in the corresponding 3D incompressible case, $u^2_0(s) = e^{it\Delta}(u^2_0)\text{in}$, and the formula (3.25) reduces to

$$
(3.26) \quad u^1_0(t) = e^{it\Delta}(u^1_0) - t(u^2_0).
$$

The linear in time growth predicted by this formula for $t \leq \mu^{-1}$ is known as the lift-up effect. Moreover, in that case the divergence free condition implies that $u^2_0 = P_{F\neq 0}u^2_0$, and the following estimate for $u^1_0$ is available

$$
(3.27) \quad \|u^1_0(t)\|_{L^2} + \mu^{1/2}\|\nabla u^1_0\|_{L^2} \leq \|u^1_0\|_{L^2} + C\mu^{-1}\left(\left\|\frac{P_{F\neq 0}(b^1_0)}{\epsilon}\right\|_{L^2} + \|P_{F\neq 0}(\bar{a}_0)\|_{L^2}\right)
+ C(\lambda + 2\mu)^{-1/4}\|\hat{u}^2_0\|_{L^2_{L^2}}.
$$

Proof. From (3.25), (3.11) and (3.16), we find that

$$
(3.28) \quad \|u^1_0(t)\|_{L^2} \leq \|e^{it\Delta}(u^1_0)\|_{L^2} + \int_0^t \|e^{i(t-s)\Delta}P_{F\neq 0}u^2_0(s)\|_{L^2} + \|u^2_0(s)\|_{L^2} ds
$$

$$
\leq \|u^1_0\|_{L^2} + \int_0^t e^{-\mu(t-s)} ds\|P_{F\neq 0}u^2_0(s)\|_{L^2} + C \int_0^t \left|\lambda + 2\mu\right|^{-1/4}\|\hat{u}^2_0\|_{L^2_{L^2}}
$$

$$
(3.29) \quad \leq C\left|\lambda + 2\mu\right|^{-1/4}\|\hat{u}^2_0\|_{L^2_{L^2}} + C\mu^{-1}\left(\left\|\frac{P_{F\neq 0}(b^1_0)}{\epsilon}\right\|_{L^2} + \|P_{F\neq 0}(\bar{a}_0)\|_{L^2}\right)
+ C(\lambda + 2\mu)^{-1/4}\|\hat{u}^2_0\|_{L^2_{L^2}}
$$

and

$$
\mu^{1/2}\|\nabla u^1_0\|_{L^2} \leq \|\nabla\left(\frac{P_{F\neq 0}(b^1_0)}{\epsilon}\right)\|_{L^2} + \mu^{1/2}\left\|\int_0^t \nabla\left(\left(\frac{P_{F\neq 0}(b^1_0)}{\epsilon}\right)\right) ds\right\|_{L^2} =: I + II.
$$
Using Plancherel’s theorem, we have
\[
I = \left( \sum_{l \in \mathbb{Z}} \int_0^t \int_0^t \mu(\eta^2 + \eta^4) e^{-2\mu(\tau-s)\eta^2} d\eta \right)^{\frac{1}{2}} \left( \int_0^t \left( \sum_{l \in \mathbb{Z}} \int_0^t \mu(\eta^2 + \eta^4) e^{-2\mu(\tau-s)\eta^2} \right)^{\frac{1}{2}} d\tau \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2}} \|\mu(0)\|_{L^2}. \]

In view of Minkowski’s inequality, and employing (3.11) and (3.16) again, one deduces that
\[
II \leq \left( \int_0^t \left( \sum_{l \in \mathbb{Z}} \int_0^t \mu(\eta^2 + \eta^4) e^{-2\mu(\tau-s)\eta^2} \right)^{\frac{1}{2}} d\tau \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2}} \int_0^t \|\mu(0)\|_{L^2} d\tau \leq C \mu^{-1} \left( \frac{P_{\leq 0}^0(b_{m0})}{e} + \|P_{>0}^0(\tilde{u}_{m0})\|_{L^2} \right) + (\lambda + 2\mu)^{-\frac{1}{2}} \left( \frac{(b_{m0})_0}{e} + \|\mu(0)\|_{L^2} \right). \]
We complete the proof of Proposition 3.2.

4. ESTIMATES OF NON-ZERO MODE

The purpose of this section is to obtain the enhanced dissipation of the non-zero mode of \((B^i, D^i, W^i)\) for \(i = 1, 2, 3\). More precisely, we will establish the following proposition.

**Proposition 4.1.** Let \(s \geq 0\). Assume that the Mach number \(e\) and the viscous coefficients \(\mu\) and \(\lambda\) satisfy (1.2) and (1.7). Then there exists a constant \(C\), depending on \(e\), but independent of \(\lambda\) and \(\mu\), such that
\[
\sum_{i=1,3} \left( \left\| m^{-\frac{3}{4}} \sqrt{-\Delta} B_{e,i} \right\|_{H^s}^2 + \left\| m^{-\frac{3}{4}} D_{e,i} \right\|_{H^s}^2 \right) + \left\| m^{-\frac{3}{4}} W_e \right\|_{H^s}^2 + \left\| m^{-1} W^3 \right\|_{H^s}^2 \leq Ce^{-\frac{s}{42}} \left( \left\| \frac{\Delta \nabla \div \frac{b_{m0}}{e} \nabla} \right\|_{H^s}^2 + \left\| \frac{\nabla \div \nabla b_{m0}}{e} \right\|_{H^s}^2 \right) + \left\| \frac{\nabla \div \nabla b_{m0}}{e} \right\|_{H^s}^2 + \left\| \frac{\nabla \div \nabla b_{m0}}{e} \right\|_{H^s}^2 + \left\| \frac{\nabla \div \nabla b_{m0}}{e} \right\|_{H^s}^2 + \left\| \frac{\nabla \div \nabla b_{m0}}{e} \right\|_{H^s}^2 + \left\| \frac{\nabla \div \nabla b_{m0}}{e} \right\|_{H^s}^2 + \left\| \frac{\nabla \div \nabla b_{m0}}{e} \right\|_{H^s}^2 + \left\| \frac{\nabla \div \nabla b_{m0}}{e} \right\|_{H^s}^2 + \left\| \frac{\nabla \div \nabla b_{m0}}{e} \right\|_{H^s}^2 \right).
\]

**Remark 4.1.** Since the system (2.9) of \((b^1, d^1, w^2)\) is self-closed, if one just focus on the system (2.9), the proof in subsection 4.1 below actually gives a more accurate estimate on \((b^1, d^1, w^2)\):
\[
\left( \left| m^{-\frac{3}{4}} \sqrt{-\Delta} B^2_{e} \right|_{H^s}^2 + \left| m^{-\frac{3}{4}} D^1_{e} \right|_{H^s}^2 \right) + \left| m^{-\frac{3}{4}} W^2 \right|_{H^s}^2 \leq Ce^{-\frac{s}{42}} \left( \left| \frac{\nabla \div \frac{b_{m0}}{e} \nabla} \right|_{H^s}^2 + \left| \frac{\nabla \div \nabla b_{m0}}{e} \right|_{H^s}^2 \right) + \left| \frac{\nabla \div \nabla b_{m0}}{e} \right|_{H^s}^2 + \left| \frac{\nabla \div \nabla b_{m0}}{e} \right|_{H^s}^2 + \left| \frac{\nabla \div \nabla b_{m0}}{e} \right|_{H^s}^2 + \left| \frac{\nabla \div \nabla b_{m0}}{e} \right|_{H^s}^2 + \left| \frac{\nabla \div \nabla b_{m0}}{e} \right|_{H^s}^2 + \left| \frac{\nabla \div \nabla b_{m0}}{e} \right|_{H^s}^2 + \left| \frac{\nabla \div \nabla b_{m0}}{e} \right|_{H^s}^2 + \left| \frac{\nabla \div \nabla b_{m0}}{e} \right|_{H^s}^2 \right).
Moreover, one can see from system (2.10) that \((b^3, d^3)\) and \(w^3\) are linearly independent of each other, but both depend on \(b^1\) and \(w^2\). Thus \((b^1, d^1, w^2, b^3, d^3)\) and \((b^1, d^1, w^2, w^3)\) can be bounded separately. As for the system (2.14), it is clear that \((b^2, d^2, w^1)\) does not depend on \((b^3, d^3, w^3)\). Therefore, in order to get the bounds for \((B^2, D^2, W^1)\), \((B^3, D^3, W^3)\) is not necessarily involved in subsection 4.2. Accordingly, the derivatives of the initial data on the \(z\)-direction and \(w^3\) can be removed on the right hand side of (4.2).

Before proceeding any further, let us denote

\[
M := (k, \eta, \lambda)^{m+1} \frac{1}{m} M_{k \neq 0}, \\
M_1 := (k, \eta, \lambda)^{m+1} \frac{1}{m} M_{k \neq 0}, \\
h^2 := c M^2, \\
g^2 := \frac{1}{4m} M^2,
\]

with the constant \(N\) appearing in the definitions of \(m_1\) and \(m_3\) (see Definition 2.2) and the constant \(c\) above to be determined below. The power \(\frac{3}{4}\) of \(m^{-1}\) in \(M\) and the factor \(\frac{1}{4}\) in \(g^2\) are closely related to each other, which was first observed in [1] for the 2D inviscid flow. The proof of Proposition 4.1 will be achieved in the following two subsections.

### 4.1. Estimates of \((B^1, D^1, W^2)\) and \((B^3, D^3, W^3)\)

Define

\[
E^1_i(t) := \sum_{i=1,3} \left( \left\| M \sqrt{-\Delta_L} B^i \right\|_{L^2}^2 + \left\| \frac{\partial m}{m} M B^i \right\|_{L^2}^2 + \left\| M D^i \right\|_{L^2}^2 + \left\| M W^2 \right\|_{L^2}^2 + \left\| M_1 W^3 \right\|_{L^2}^2 \right),
\]

and

\[
E^3_i(t) := E^1_i(t) + 2 \sum_{i=1,3} \left( \langle g B^i, g D^i \rangle - \langle h B^i, h D^i \rangle \right).
\]

We would like to remark that if the Mach number \(\epsilon \in (0, 1]\), the second term in (4.5), first introduced by Antonelli, Dolce and Marcati in [2], is not needed.

**Step (I): Estimates of the functional \(E^1_i(t)\)**. First of all from (2.16) and (2.17), we find that for \(i = 1, 3\), there hold

\[
\frac{1}{2} \frac{d}{dt} \left( M \sqrt{-\Delta_L} B^i \right)_{L^2}^2 = -\frac{3}{4} \left\| \frac{\partial m}{m} M \sqrt{-\Delta_L} B^i \right\|_{L^2}^2 - \sum_{j=1,2,3} \left\| \frac{\partial m}{m_j} M \sqrt{-\Delta_L} B^j \right\|_{L^2}^2
\]

\[
+ \frac{1}{2} \sum_{k,l} \int_{\mathbb{R}} M^2 p^l \left\| \frac{\partial }{\epsilon} \right\|_{L^2}^2 + \frac{1}{\epsilon^2} \left\langle M \Delta_L B^i, M D^i \right\rangle,
\]

and

\[
\frac{1}{2} \frac{d}{dt} \left\| M D^i \right\|_{L^2}^2 = -\frac{3}{4} \left\| \frac{\partial m}{m} M D^i \right\|_{L^2}^2 - \sum_{j=1,2,3} \left\| \frac{\partial m_j}{m_j} M D^j \right\|_{L^2}^2
\]

\[
- (\lambda + 2\mu) \left\| M \sqrt{-\Delta_L} D^i \right\|_{L^2}^2 + \sum_{k,l} \int_{\mathbb{R}} M^2 D^l \left| \frac{\partial }{p} \right|_{L^2}^2 d\eta
\]

\[
- \frac{1}{\epsilon^2} \left\langle M \Delta_L B^i, M D^i \right\rangle - \left\{ \begin{array}{ll}
2 \sum_{k,l} \int_{\mathbb{R}} M^2 k^2 (\hat{W}^2 - \hat{B}^1) \hat{D}^1 d\eta, & \text{if } i = 1, \\
2 \sum_{k,l} \int_{\mathbb{R}} M^2 k l (\hat{W}^2 - \hat{B}^1) \hat{D}^3 d\eta, & \text{if } i = 3.
\end{array} \right.
\]
Similar to (4.7), we get the evolution of \( \| \sqrt{\frac{\partial m}{m}} MB \|_{L^2} \),
\[
\frac{1}{2} \frac{d}{dt} \left( \| \sqrt{\frac{\partial m}{m}} MB \|_{L^2} \right)^2 = -\frac{3}{2} \| \sqrt{\frac{\partial m}{m}} MB \|_{L^2}^2 - \sum_{j=1,2,3} \left( \frac{\partial m}{m} \right)^j MB \|_{L^2}^2,
\]
(4.9)

The evolutions of the last two terms in the energy functional \( E_1(t) \) are given as follows:
\[
\frac{1}{2} \frac{d}{dt} \left( \| MW^2 \|_{L^2} \right)^2 = -\frac{3}{4} \| \sqrt{\frac{\partial m}{m}} MW^2 \|_{L^2}^2 - \sum_{j=1,2,3} \left( \frac{\partial m}{m} \right)^j MW^2 \|_{L^2}^2
\]
(4.10)

and
\[
\frac{1}{2} \frac{d}{dt} \left( \| M_1 W^3 \|_{L^2} \right)^2 = -\left( \| \sqrt{\frac{\partial m}{m}} M_1 W^3 \|_{L^2}^2 - \sum_{j=1,2,3} \left( \frac{\partial m}{m} \right)^j M_1 W^3 \|_{L^2}^2 \right) - \mu \left( \| M \sqrt{-\Delta_L} W^2 \|_{L^2}^2 + \mu \langle M \sqrt{-\Delta_L} B^1, M \sqrt{-\Delta_L} W^2 \rangle \right),
\]
(4.11)

From (4.7)–(4.11), one deduces the evolution of the energy functional \( E_1(t) \),
\[
\frac{1}{2} \frac{d}{dt} E_1(t) + (\lambda + 2\mu) \sum_{i=1,3} \left( \| M \sqrt{-\Delta_L} D^i \|_{L^2}^2 + \mu \| M \sqrt{-\Delta_L} W^2 \|_{L^2}^2 \right) + \mu \| M \sqrt{-\Delta_L} W^3 \|_{L^2}^2
\]
\[
+ \sum_{i=1,3} \sum_{j=1,2,3} \left( \| \sqrt{\frac{\partial m}{m}} MB \|_{L^2}^2 + \sum_{i=1,3} \sum_{j=1,2,3} \left( \frac{\partial m}{m} \right)^j MB \|_{L^2}^2 \right),
\]
(4.12)

\[
-2 \sum_{k,l} \int_R M^2 \left( \frac{k^2}{p} \right) (W^2 - \hat{B}^1 \hat{D}^3) d\eta - 2 \sum_{k,l} \int_R M^3 \left( \frac{k^1}{p} \right) (W^2 - \hat{B}^1) \hat{D}^3 d\eta
\]
\[
+ 2 \sum_{k,l} \int_R M^2 \left( \frac{k^1}{p} \right) (W^2 - \hat{B}^1) \hat{D}^3 d\eta + \mu \langle M \sqrt{-\Delta_L} B^1, M \sqrt{-\Delta_L} W^2 \rangle
\]
\[
+ \frac{1}{2} \sum_{i=1,3} \sum_{k,l} \int_{\mathbb{R}} \partial_t \left( \frac{\partial m}{m} \right) M^2 |\tilde{B}|^2 \, d\eta - \sum_{i=1,3} \sum_{k,l} \int_{\mathbb{R}} \frac{\partial m}{m} M^2 \tilde{B}^i \tilde{D}^i \, d\eta.
\]

Now we go to bound the right hand side of (4.12). Firstly, we infer from (2.31)–(2.33) in Remark 2.1 that

\[
(4.13) \quad \leq \sum_{i=1,3} \left( \frac{1}{2} \left\| \sqrt{\frac{\partial m}{m}} M \sqrt{-\Delta} \frac{\tilde{B}^i}{e} \right\|^2_{L^2} + \left\| \sqrt{\frac{\partial m}{m}} M D \frac{\tilde{B}^i}{e} \right\|^2_{L^2} + \left\| \sqrt{\frac{\partial m}{m}} M W^3 \right\|^2_{L^2}
\]

Recalling the definitions of \( m^1 \) and \( m^3 \), and using the fact \( M_1 \leq M \), we are led to

\[
-2 \sum_{k,l} \int_{\mathbb{R}} M^2 \frac{k^2}{p} (\tilde{W}^2 - \tilde{B}^1) \tilde{D}^3 \, d\eta - 2 \sum_{k,l} \int_{\mathbb{R}} M^2 \frac{k^2}{p} (\tilde{W}^2 - \tilde{B}^1) \tilde{D}^3 \, d\eta
\]

\[
+2 \sum_{k,l} \int_{\mathbb{R}} M^2 \frac{k^2}{p} (\tilde{W}^2 - \tilde{B}^1) \tilde{D}^3 \, d\eta
\]

\[
\leq \frac{2}{N} \left( \left\| \sqrt{\frac{\partial m}{m}} M W^2 \right\|_{L^2} + e \left\| \sqrt{\frac{\partial m}{m}} M B^1 \right\|_{L^2} \right) \left\| \sqrt{\frac{\partial m}{m}} M D \right\|_{L^2}
\]

\[
+ \frac{2}{N} \left( \left\| \sqrt{\frac{\partial m}{m}} M W^2 \right\|_{L^2} + e \left\| \sqrt{\frac{\partial m}{m}} M B^1 \right\|_{L^2} \right) \left\| \sqrt{\frac{\partial m}{m}} M D \right\|_{L^2}
\]

\[
+ \frac{2}{N} \left( \left\| \sqrt{\frac{\partial m}{m}} M W^2 \right\|_{L^2} + e \left\| \sqrt{\frac{\partial m}{m}} M B^1 \right\|_{L^2} \right) \left\| \sqrt{\frac{\partial m}{m}} M W^3 \right\|_{L^2}
\]

\[
(4.14) \quad \leq \frac{2}{N} \sum_{j=1,3} \left\| \sqrt{\frac{\partial m}{m}} M W^2 \right\|_{L^2}^2 + \frac{2}{N} \sum_{j=1,3} \left\| \sqrt{\frac{\partial m}{m}} M B^1 \right\|_{L^2}^2
\]

\[
+ \frac{1}{N} \left( \left\| \sum_{j=1,3} \sqrt{\frac{\partial m}{m}} M D \right\|_{L^2}^2 + \left\| \sqrt{\frac{\partial m}{m}} M W^3 \right\|_{L^2}^2 \right).
\]

Next, by the definition of \( m \) and \( m_1 \), thanks to the fact \( \frac{|p'|}{p} \leq \frac{2k^2}{Np^2} \), one deduces that

\[
0 \leq \frac{\partial m}{m} \leq \frac{|p'|}{p} \leq 1,
\]

\[
0 \leq \frac{\partial m}{m \sqrt{p}} \leq \frac{|p'|}{p^2} \leq \frac{2k^2}{Np} \leq \frac{2}{N} \frac{\partial m_1}{m_1},
\]

and

\[
\left| \partial_t \left( \frac{\partial m}{m} \right) \right| \leq \frac{2k^2}{p} + \frac{|p'|^2}{p^2} \leq \frac{6k^2}{Np} \leq \frac{6}{N} \frac{\partial m_1}{m_1}.
\]
Consequently,

\begin{equation}
\frac{1}{2} \sum_{i=1,3} \sum_{k,l} \int_{\mathbb{R}} \partial_t \left( \frac{\partial_i m}{m} \right) M^2 |\tilde{B}'|^2 \, d\eta \leq \frac{3\epsilon^2}{N} \sum_{i=1,3} \left\| \sqrt{\frac{\partial_i m}{m_1}} M \sqrt{-\Delta_L} \frac{B'}{\epsilon} \right\|_{L^2}^2,
\end{equation}

and

\begin{equation}
- \sum_{i=1,3} \sum_{k,l} \int_{\mathbb{R}} \frac{\partial_i m}{m} M^2 B' \tilde{d}' \, d\eta \leq \frac{2\epsilon}{N} \sum_{i=1,3} \left\| \sqrt{\frac{\partial_i m}{m_1}} M \sqrt{-\Delta_L} \frac{B'}{\epsilon} \right\|_{L^2} \left\| \sqrt{\frac{\partial_i m}{m_1}} M D' \right\|_{L^2} \leq \frac{\epsilon}{N} \sum_{i=1,3} \left( \left\| \sqrt{\frac{\partial_i m}{m_1}} M \sqrt{-\Delta_L} \frac{B'}{\epsilon} \right\|_{L^2}^2 + \left\| \sqrt{\frac{\partial_i m}{m_1}} M D' \right\|_{L^2}^2 \right).
\end{equation}

Finally, by virtue of the Cauchy-Schwarz inequality, we have

\begin{equation}
\mu \left\langle M \sqrt{-\Delta_L} B^1, M \sqrt{-\Delta_L} W^2 \right\rangle \leq \frac{\mu}{2} \left\| M \sqrt{-\Delta_L} W^2 \right\|_{L^2}^2 + \frac{\mu\epsilon^2}{2} \left\| M \sqrt{-\Delta_L} \frac{B^1}{\epsilon} \right\|_{L^2}^2.
\end{equation}

Substituting the above inequalities into (4.12) yields

\begin{align*}
\frac{1}{2} \frac{d}{dt} E^1(t) + (\lambda + 2\mu) \sum_{i=1,3} \left\| M \sqrt{-\Delta_L} D' \right\|_{L^2}^2 + \frac{\mu}{2} \left\| M \sqrt{-\Delta_L} W^2 \right\|_{L^2}^2 & + \frac{\mu}{2} \sum_{i=1,3} \left\| M \sqrt{-\Delta_L} \frac{B'}{\epsilon} \right\|_{L^2}^2 + \frac{3\epsilon^2}{N} \sum_{i=1,3} \left\| \sqrt{\frac{\partial_i m}{m_1}} M \sqrt{-\Delta_L} \frac{B'}{\epsilon} \right\|_{L^2}^2 \\
& + \sum_{i=1,3} \left( \left\| \sqrt{\frac{\partial_i m}{m_2}} M \sqrt{-\Delta_L} \frac{B'}{\epsilon} \right\|_{L^2}^2 + \left\| \sqrt{\frac{\partial_i m}{m_2}} M W^3 \right\|_{L^2}^2 \right) \leq \mu^2 \frac{1}{32} \sum_{i=1,3} \left( \left\| M \sqrt{-\Delta_L} \frac{B'}{\epsilon} \right\|_{L^2}^2 + \left\| M D' \right\|_{L^2}^2 + \left\| M W^3 \right\|_{L^2}^2 \right) \\
& + \frac{\mu^2}{2} \left\| M \sqrt{-\Delta_L} \frac{B^1}{\epsilon} \right\|_{L^2}^2 + \frac{1}{4} \sum_{i=1,3} \left\| \sqrt{\frac{\partial_i m}{m_1}} M D^i \right\|_{L^2}^2,
\end{align*}

where

\begin{equation}
\epsilon_1 := \max \left\{ 2, 1 + 2\epsilon, 3\epsilon + 3\epsilon^2 \right\}.
\end{equation}
Step (II): Estimates of the modified energy functional $E_1^i(t)$. To this end, we turn to derive the evolutions of the cross terms appearing in the modified energy functional $E_1^i(t)$. Note first that

\begin{equation}
\partial_t (h^2) = -2c \sum_{j=1,2,3} \frac{\partial_t m_j}{m_j} M^2 - \frac{3c}{2} \frac{\partial_t m}{m} M^2,
\end{equation}

and

\begin{equation}
\partial_t (g^2) = -\frac{1}{2} \sum_{j=1,2,3} \frac{\partial_t m \partial_t m_j}{m} M^2 - \frac{3}{8} \left( \frac{\partial_t m}{m} \right)^2 M^2 + \frac{1}{4} \partial_t \left( \frac{\partial_t m}{m} \right) M^2.
\end{equation}

Then we have

\begin{equation}
\frac{d}{dt} \langle hB^i, hD^j \rangle = c \left\| M \sqrt{-\Delta_L} \frac{B^i}{e} \right\|^2_{L^2} - c \| MD^j \|^2_{L^2} - c(\lambda + 2\mu) \sum_{k,l} \int_{\mathbb{R}} p M^2 \hat{B}^i \hat{D}^j d\eta + c \sum_{k,l} \int_{\mathbb{R}} M^2 \frac{\partial_t m_j}{m_j} \hat{B}^i \hat{D}^j d\eta
\end{equation}

\begin{equation}
-2c \sum_{j=1,2,3} \sum_{k,l} \int_{\mathbb{R}} M^2 \frac{\partial_t m_j}{m_j} \hat{B}^i \hat{D}^j d\eta - \frac{3c}{2} \sum_{k,l} \int_{\mathbb{R}} M^2 \frac{\partial_t m_j}{m_j} \hat{B}^i \hat{D}^j d\eta
\end{equation}

\begin{equation}
\leq \left\{ \begin{array}{ll}
2c \sum_{k,l} \int_{\mathbb{R}} M^2 \frac{k^2}{p} \hat{B}^i (\hat{\tilde{W}^2} - \hat{\tilde{B}^i}) d\eta, & \text{if } i = 1, \\
2c \sum_{k,l} \int_{\mathbb{R}} M^2 \frac{k^2}{p} \hat{B}^i (\hat{\tilde{W}^2} - \hat{\tilde{B}^i}) d\eta, & \text{if } i = 3,
\end{array} \right.
\end{equation}

and

\begin{equation}
\frac{d}{dt} \langle gB^i, gD^j \rangle
\end{equation}

\begin{equation}
= \frac{1}{4} \left\| \sqrt{\frac{\partial_t m}{m}} M \sqrt{-\Delta_L} \frac{B^i}{e} \right\|^2_{L^2} - \frac{1}{4} \left\| \sqrt{\frac{\partial_t m}{m}} MD^j \right\|^2_{L^2}
\end{equation}

\begin{equation}
- \frac{\lambda + 2\mu}{4} \sum_{k,l} \int_{\mathbb{R}} \frac{\partial_t m_j}{m_j} M^2 \hat{B}^i \hat{D}^j d\eta + \frac{1}{4} \sum_{k,l} \int_{\mathbb{R}} \frac{\partial_t m_j}{m_j} M^2 \frac{\partial_t m_j}{m_j} \hat{B}^i \hat{D}^j d\eta
\end{equation}

\begin{equation}
- \frac{1}{2} \sum_{j=1,2,3} \sum_{k,l} \int_{\mathbb{R}} \frac{\partial_t m_j}{m_j} \frac{\partial_t m_j}{m_j} M^2 \hat{B}^i \hat{D}^j d\eta - \frac{3}{8} \sum_{k,l} \int_{\mathbb{R}} \left( \frac{\partial_t m}{m} \right)^2 M^2 \hat{B}^i \hat{D}^j d\eta
\end{equation}

\begin{equation}
\leq \left\{ \begin{array}{ll}
\frac{1}{2} \sum_{k,l} \int_{\mathbb{R}} \frac{\partial_t m_j}{m_j} M^2 \frac{k^2}{p} \hat{B}^i (\hat{\tilde{W}^2} - \hat{\tilde{B}^i}) d\eta, & \text{if } i = 1, \\
\frac{1}{2} \sum_{k,l} \int_{\mathbb{R}} \frac{\partial_t m_j}{m_j} M^2 \frac{k^2}{p} \hat{B}^i (\hat{\tilde{W}^2} - \hat{\tilde{B}^i}) d\eta, & \text{if } i = 3,
\end{array} \right.
\end{equation}

Using the Cauchy-Schwarz inequality and (4.15), we obtain

\begin{equation}
2c \sum_{j=1,2,3} \sum_{k,l} \int_{\mathbb{R}} M^2 \frac{\partial_t m_j}{m_j} \hat{B}^i \hat{D}^j d\eta \leq 2c \sum_{j=1,2,3} \left\| \sqrt{\frac{\partial_t m_j}{m_j}} MB^i \right\|_{L^2} \left\| \sqrt{\frac{\partial_t m_j}{m_j}} MD^j \right\|_{L^2}
\end{equation}

\begin{equation}
\leq c \epsilon \sum_{j=1,2,3} \left( \left\| \sqrt{\frac{\partial_t m_j}{m_j}} M \sqrt{-\Delta_L} \frac{B^i}{e} \right\|^2_{L^2} + \left\| \sqrt{\frac{\partial_t m_j}{m_j}} MD^j \right\|^2_{L^2} \right),
\end{equation}

and

\begin{equation}
\frac{1}{2} \sum_{j=1,2,3} \sum_{k,l} \int_{\mathbb{R}} \frac{\partial_t m_j \partial_t m_j}{m_j} M^2 \hat{B}^i \hat{D}^j d\eta
\end{equation}
Moreover, in view of (4.15)–(4.17), we find that
\[
\frac{3c}{2} \sum_{k,l} \int_{\mathbb{R}} M^2 \frac{\partial m}{m} B^i \tilde{D}^j \, d\eta - c \sum_{k,l} \int_{\mathbb{R}} M^2 \frac{p'}{p} B^i \tilde{D}^j \, d\eta \\
- \frac{3}{8} \sum_{k,l} \int_{\mathbb{R}} M^2 \left( \frac{\partial m}{m} \right)^2 B^i \tilde{D}^j \, d\eta + \frac{1}{4} \sum_{k,l} \int_{\mathbb{R}} \frac{\partial m}{m} M^2 \frac{p'}{p} B^i \tilde{D}^j \, d\eta \\
+ \frac{1}{4} \sum_{k,l} \int_{\mathbb{R}} M^2 \partial_i \left( \frac{\partial m}{m} \right) B^i \tilde{D}^j \, d\eta \\
\leq \frac{3}{2} (c + 1) \sum_{k,l} \int_{\mathbb{R}} M^2 \frac{\partial m}{m} \sqrt{p} (\sqrt{\rho} B^i) |\tilde{D}^j| \, d\eta \\
+ \frac{1}{4} \int_{\mathbb{R}} M^2 \frac{p'}{p^2} (\sqrt{\rho} B^i) |\tilde{D}^j| \, d\eta + \frac{3}{2N} \left\| \frac{\partial m}{m} M B^i \right\|_{L^2} \left\| \frac{\partial m}{m} M D^i \right\|_{L^2} \\
\leq \frac{10(c + \frac{1}{2}) e + 3e}{2N} \left\| \frac{\partial m}{m} M \sqrt{-\Delta_L} B^i \right\|_{L^2} \left\| \frac{\partial m}{m} M D^i \right\|_{L^2} \\
(4.28) \quad \leq \frac{(5c + 3)e}{2N} \left\| \left( \frac{\partial m}{m} M \sqrt{-\Delta_L} B^i \left\|_{L^2} \right\| \frac{\partial m}{m} M D^i \right\|_{L^2} \right\|_{L^2} \right)^2.
\]

For $i = 1$, by the definition of $m_1$, we have
\[
2c \sum_{k,l} \int_{\mathbb{R}} M^2 \frac{k^2}{p} B^1 (\tilde{W}^2 - \tilde{B}^1) \, d\eta - \frac{1}{2} \sum_{k,l} \int_{\mathbb{R}} \frac{\partial m}{m} M^2 \frac{k^2}{p} B^1 (\tilde{W}^2 - \tilde{B}^1) \, d\eta \\
\leq \frac{2(c + \frac{1}{4})}{N} \left\| \left( \frac{\partial m}{m} M W^2 \right) \right\|_{L^2} \left\| \frac{\partial m}{m} M B^1 \right\|_{L^2} \\
\leq \frac{c + \frac{1}{4}}{N} \left\| \frac{\partial m}{m} M W^2 \right\|_{L^2} \left\| \frac{\partial m}{m} M B^1 \right\|_{L^2}^2.
(4.29)
\]

For $i = 3$, recalling the definition of $m_3$, there holds
\[
2c \sum_{k,l} \int_{\mathbb{R}} M^2 \frac{k^2l}{p} B^3 (\tilde{W}^2 - \tilde{B}^1) \, d\eta - \frac{1}{2} \sum_{k,l} \int_{\mathbb{R}} \frac{\partial m}{m} M^2 \frac{k^2l}{p} B^3 (\tilde{W}^2 - \tilde{B}^1) \, d\eta \\
\leq \frac{2(c + \frac{1}{4})}{N} \left( \frac{\partial m}{m} M W^2 \right) \left\| \frac{\partial m}{m} M B^1 \right\|_{L^2} \\
\leq \frac{c + \frac{1}{4}}{N} \left( \frac{\partial m}{m} M W^2 \right)^2 + \frac{e^2}{N} + \frac{\partial m}{m} M B^1 \right\|_{L^2}^2 + \frac{2(c + \frac{1}{4}) e^2}{N} \left\| \frac{\partial m}{m} M B^1 \right\|_{L^2}^2.
(4.30)
\]

Finally, by virtue of the Cauchy-Schwarz inequality and (4.15), we are led to
\[
c(\lambda + 2\mu) \sum_{k,l} \int_{\mathbb{R}} M^2 p \hat{B}^i \hat{D}^j \, d\eta \leq \frac{c}{4} \left\| M \sqrt{-\Delta_L} B^i \right\|_{L^2}^2 + c(\lambda + 2\mu)^2 e^2 \left\| M \sqrt{-\Delta_L D^i} \right\|_{L^2}^2.
(4.31)
\]
On the other hand, using the fact that

\[
\frac{\partial m}{m} \leq \frac{|p'|}{p} \leq \frac{2|k|}{\sqrt{p}} = \frac{2}{\sqrt{N}} \sqrt{\frac{\partial m_1}{m_1}}.
\]

we arrive at

\[
-\frac{\lambda + 2\mu}{4} \sum_{i,j} \int_{\mathbb{R}} \frac{\partial m}{m} M^2 p \hat{B}^i \hat{D}^j d\eta \leq \frac{\lambda + 2\mu}{2N} \sum_{i,j} \int_{\mathbb{R}} \frac{\partial m_1}{m_1} M^2 p \hat{B}^i \|\hat{D}^j d\eta \leq \frac{(\lambda + 2\mu)e^2}{2N} \left( \| \frac{\partial m_1}{m_1} M \sqrt{-\Delta_L} B^i \|_{L^2} \right)^2 + \frac{\lambda + 2\mu}{8} \| M \sqrt{-\Delta_L} D^j \|_{L^2}^2.
\]

(4.33)

It follows from (4.25)–(4.33) that

\[
\sum_{j=1,3} \frac{d}{dt} \left( \langle gB^i, gD^j \rangle - \langle hB^i, hD^j \rangle \right)
\leq \frac{3c}{4} \sum_{i=1,3} \left( \| M \sqrt{-\Delta_L} B^i \|_{L^2}^2 + (\lambda + 2\mu)(c(\lambda + 2\mu)e^2 + \frac{1}{8}) \sum_{i=1,3} \| M \sqrt{-\Delta_L} D^j \|_{L^2}^2 
+ \frac{1}{4} \sum_{i=1,3} \left( \| \frac{\partial m}{m} M \sqrt{-\Delta_L} B^i \|_{L^2}^2 - \frac{1}{4} \sum_{i=1,3} \left( \| \frac{\partial m}{m} M \sqrt{-\Delta_L} B^i \|_{L^2}^2 + c \sum_{i=1,3} \| MD^j \|_{L^2}^2 
+ ce \sum_{i=1,3} \left( \frac{\partial m_1}{m_1} M \sqrt{-\Delta_L} B^i \|_{L^2}^2 + \left( ce + \frac{1}{16} \right) \left( \frac{\partial m_2}{m_2} M \sqrt{-\Delta_L} B^i \|_{L^2}^2 + \frac{\partial m_1}{m_1} \left( \frac{\partial m_j}{m_j} M \sqrt{-\Delta_L} B^i \|_{L^2}^2 + \sqrt{-\Delta_L} D^j \|_{L^2}^2 \right) \right) 
+ \left( ce + \frac{1}{16} + \frac{e_2}{2N} \right) \sum_{i=1,3} \sum_{j=1,3} \frac{1}{i+1,3} \left( \| \frac{\partial m_2}{m_2} M \sqrt{-\Delta_L} B^i \|_{L^2}^2 + \| \frac{\partial m_1}{m_1} \frac{\partial m_j}{m_j} M \sqrt{-\Delta_L} B^i \|_{L^2}^2 \right) \right) 
+ \frac{c}{N} \sum_{j=1,3} \left( \| \frac{\partial m_j}{m_j} MW^2 \|_{L^2}^2 + \sum_{i=1,3} \sum_{j=1,2,3} \left( \| \frac{\partial m_1}{m_1} \frac{\partial m_j}{m_j} MB^i \|_{L^2}^2 \right) \right),
\]

(4.34)

where

\[
e_2 := (5c + 3)e + (6c + 2)e^2 + (\lambda + 2\mu)e^2.
\]

Combining this with (4.21), we obtain

\[
\frac{1}{2} \frac{d}{dt} \mathcal{E}^1(t) + \frac{3c}{4} \sum_{i=1,3} \left( \| M \sqrt{-\Delta_L} B^i \|_{L^2}^2 
+ \left( \lambda + 2\mu \right) \left( \frac{7}{8} - \frac{c(\lambda + 2\mu)e^2}{8} \right) \sum_{i=1,3} \| M \sqrt{-\Delta_L} D^j \|_{L^2}^2 
+ \frac{\mu}{2} \| M \sqrt{-\Delta_L} W^2 \|_{L^2}^2 + \mu \| M \sqrt{-\Delta_L} W^2 \|_{L^2}^2 
+ \left( \frac{15}{16} - ce \right) \sum_{i=1,3} \left( \frac{\partial m_2}{m_2} M \sqrt{-\Delta_L} B^i \|_{L^2}^2 + \frac{\partial m_1}{m_1} \frac{\partial m_j}{m_j} M \sqrt{-\Delta_L} B^i \|_{L^2}^2 \right) \right) 
+ \left( \frac{\partial m_2}{m_2} MW^2 \|_{L^2}^2 + \frac{\partial m_2}{m_2} MW^2 \|_{L^2}^2 + \frac{3}{4} \sum_{i=1,3} \| \frac{\partial m_1}{m_1} MB^i \|_{L^2}^2 \right).
\]
Let us take
\[ c = \frac{\mu^3}{8}, \quad \text{and} \quad N = 9(1 + \epsilon)^2. \]

Then recalling the definitions of \( \epsilon_1 \) and \( \epsilon_2 \) in (4.22) and (4.35), respectively, and using (4.7), the following inequalities hold:
\[
\begin{align*}
(\lambda + 2\mu) \left( \frac{7}{8} - c(\lambda + 2\mu)\epsilon^2 \right) & \geq \frac{1}{2} (\lambda + 2\mu) \geq \frac{3}{4} \mu, \\
\frac{15}{16} - ce & \geq \frac{1}{2}, \\
\min\{c_1, c_2\} & \geq \frac{1}{4},
\end{align*}
\]

where we have used the fact that
\[
\lambda + \frac{2}{3}\mu \geq 0, \quad \text{i.e.} \quad \mu \leq \frac{3}{4} (\lambda + 2\mu).
\]

Then we infer from (4.36) that
\[
\begin{align*}
\frac{d}{dt} E_1(t) & + \frac{3\mu^3}{16} \sum_{i=1,3} \left\| M \sqrt{-\Delta_L} B'_j \epsilon \right\|_{L^2}^2 \\
& + \mu \left( \sum_{i=1,3} \left\| M \sqrt{-\Delta_L} D' \epsilon \right\|_{L^2}^2 + \left\| M \sqrt{-\Delta_L} W^2 \right\|_{L^2}^2 + \left\| M_1 \sqrt{-\Delta_L} W^3 \right\|_{L^2}^2 \right) \\
& + \sum_{i=1,3} \left( \left\| \sqrt{\frac{\partial m_j}{m_j}} M \sqrt{-\Delta_L} B'_i \epsilon \right\|_{L^2}^2 + \left\| \sqrt{\frac{\partial m_j}{m_j}} M_1 D' \epsilon \right\|_{L^2}^2 \right) \\
& + \left\| \sqrt{\frac{\partial m_j}{m_j}} M W^2 \right\|_{L^2}^2 + \left\| \sqrt{\frac{\partial m_j}{m_j}} M_1 W^3 \right\|_{L^2}^2 \\
& + \frac{1}{2} \sum_{i=1,3} \sum_{j=1,3} \left( \left\| \sqrt{\frac{\partial m_j}{m_j}} M \sqrt{-\Delta_L} B'_i \epsilon \right\|_{L^2}^2 + \left\| \sqrt{\frac{\partial m_j}{m_j}} M D' \epsilon \right\|_{L^2}^2 \right).
\end{align*}
\]
\[ + \frac{1}{2} \sum_{j=1,3} \left( \left\| \sqrt{\frac{\partial m_j}{m_j}} MW^j \right\|_{L^2}^2 + \left\| \sqrt{\frac{\partial m_j}{m_j}} M W^j \right\|_{L^2}^2 \right) \]
\[ \leq \frac{\mu^\frac{1}{4}}{16} \left( \sum_{i=1,3} \left( \frac{1}{2} \left\| \sqrt{-\Delta_L} B_i^j \right\|_{L^2}^2 + \left\| M D_i^j \right\|_{L^2}^2 + \left\| M W_i^3 \right\|_{L^2}^2 \right) \right) \]
\[ + \mu_\varepsilon^2 \left\| \sqrt{-\Delta_L} B_1^j \right\|_{L^2}^2 \frac{\mu^\frac{1}{4}}{4} \sum_{i=1,3} \left\| M D_i^j \right\|_{L^2}^2. \]

Thanks to (2.40), one deduces that
\[ \frac{3\mu}{4} \left( \sum_{i=1,3} \left\| M \sqrt{-\Delta_L} D_i^j \right\|_{L^2}^2 + \left\| M \sqrt{-\Delta_L} W^j \right\|_{L^2}^2 + \left\| M_1 \sqrt{-\Delta_L} W^3 \right\|_{L^2}^2 \right) \]
\[ + \frac{3\mu}{4} \left( \sum_{i=1,3} \left\| \sqrt{\frac{\partial m_2}{m_2}} M D_i^j \right\|_{L^2}^2 + \left\| \sqrt{\frac{\partial m_2}{m_2}} M W_i^2 \right\|_{L^2}^2 + \left\| \sqrt{\frac{\partial m_2}{m_2}} M W_i^3 \right\|_{L^2}^2 \right) \]
\[ \geq \frac{3\mu}{8} \sum_{i=1,3} \left\| M D_i^j \right\|_{L^2}^2 + \left\| M W_i^2 \right\|_{L^2}^2 + \left\| M_1 W_i^3 \right\|_{L^2}^2. \]

On the other hand, thanks to (4.16), we find that
\[ \frac{1}{4} \sum_{i=1,3} \left\| \sqrt{\frac{\partial m_1}{m_1}} M \sqrt{-\Delta_L} B_i^1 \right\|_{L^2}^2 \leq \frac{N}{8\varepsilon^2} \sum_{i=1,3} \left\| \sqrt{\frac{\partial m_1}{m_1}} M B_i^1 \right\|_{L^2}^2 \geq \frac{\mu^\frac{1}{4}}{16} \sum_{i=1,3} \left\| \sqrt{\frac{\partial m_1}{m_1}} M B_i^1 \right\|_{L^2}^2. \]

Noting that (1.7) implies that \( \mu\varepsilon^2 \leq \frac{\mu^\frac{1}{4}}{16} \), then the right hand side of (4.40) can be bounded as follows
\[ \frac{\mu^\frac{1}{4}}{16} \left( \sum_{i=1,3} \left( \frac{1}{2} \left\| \sqrt{-\Delta_L} B_i^j \right\|_{L^2}^2 + \left\| M D_i^j \right\|_{L^2}^2 \right) + \left\| M W_i^3 \right\|_{L^2}^2 \right) \]
\[ + \mu_\varepsilon^2 \left\| \sqrt{-\Delta_L} B_1^j \right\|_{L^2}^2 \frac{\mu^\frac{1}{4}}{4} \sum_{i=1,3} \left\| M D_i^j \right\|_{L^2}^2 \]
\[ \leq \frac{3\mu}{32} \sum_{i=1,3} \left\| M \sqrt{-\Delta_L} B_i^1 \right\|_{L^2}^2 + \frac{5\mu^\frac{1}{4}}{16} \sum_{i=1,3} \left\| M D_i^j \right\|_{L^2}^2 + \left\| M W_i^3 \right\|_{L^2}^2. \]

Now substituting (4.41), (4.42) and (4.43) into (4.40), we conclude that
\[ \frac{d}{dt} E_1^j(t) + \frac{\mu^\frac{1}{4}}{16} E_s(t) \]
\[ + \frac{1}{4} \left( \sum_{i=1,3} \left\| M \sqrt{-\Delta_L} D_i^j \right\|_{L^2}^2 + \left\| M \sqrt{-\Delta_L} W^2 \right\|_{L^2}^2 + \left\| M_1 \sqrt{-\Delta_L} W^3 \right\|_{L^2}^2 \right) \]
\[ + \frac{1}{4} \sum_{i,j=1,2,3} \left( \left\| \sqrt{\frac{\partial m_j}{m_j}} M \sqrt{-\Delta_L} B_i^j \right\|_{L^2}^2 + \left\| \sqrt{\frac{\partial m_j}{m_j}} M D_i^j \right\|_{L^2}^2 \right) \]
\[ \leq 0. \]
Recalling the choice of $c$, using (4.15) and (1.7), we see that
\[
2 \sum_{i=1,3} \left( (gB^i, gD^i) - \langle hB^i, hD^i \rangle \right) \\
\leq \frac{1}{4} \sum_{i=1,3} \left( \left\| \frac{\partial m}{m} MB \right\|_{L^2}^2 + \|MD\|_{L^2}^2 \right) + \frac{\mu^4}{8} \left( \left\| M \sqrt{-\Delta_L} \frac{B}{e} \right\|_{L^2}^2 + \|MD\|_{L^2}^2 \right) \\
\leq \frac{3}{8} \sum_{i=1,3} \left( \left\| M \sqrt{-\Delta_L} \frac{B}{e} \right\|_{L^2}^2 + \left\| \frac{\partial m}{m} MB \right\|_{L^2}^2 + \|MD\|_{L^2}^2 \right) .
\]
Accordingly,
\[
(4.45) \quad \frac{5}{8} E_s^1(t) \leq E_s^1(t) \leq \frac{11}{8} E_s^1(t).
\]
Combining (4.43) with (4.44), the inequality (4.1) in Proposition 4.1 follows immediately.

4.2. Estimates of $\langle B^2, D^2, W^1 \rangle$. Define
\[
E_s^2(t) := \left\| M \sqrt{-\Delta_L} \frac{B^2}{e} \right\|_{L^2}^2 + \left\| \frac{\partial m}{m} MB \right\|_{L^2}^2 + \left\| MD \right\|_{L^2}^2 + \left\| M_1 W^1 \right\|_{L^2}^2,
\]
and
\[
E_s^3(t) := E_s^2(t) + 2 \left( (gB^2, gD^2) - \langle hB^2, hD^2 \rangle \right),
\]
where the parameter $c$ in the weight $h$ is chosen as before. Similar to (4.12), it is not difficult to verify that
\[
\frac{1}{2} \frac{d}{dt} E_s^2(t) + (\lambda + 2\mu) \left\| M \sqrt{-\Delta_L} D^2 \right\|_{L^2}^2 + \mu \left\| M_1 \sqrt{-\Delta_L} W^1 \right\|_{L^2}^2 \\
+ \frac{3}{4} \left\| \frac{\partial m}{m} MB \right\|_{L^2}^2 + \sum_{j=1,2,3} \left\| \frac{\partial m}{m} \frac{\partial m_j}{m_j} MB \right\|_{L^2}^2 \\
+ \frac{3}{4} \left( \left\| \frac{\partial m}{m} M \sqrt{-\Delta_L} \frac{B^2}{e} \right\|_{L^2}^2 + \left\| \frac{\partial m}{m} MD \right\|_{L^2}^2 \right) + \left\| \frac{\partial m}{m} M_1 W^1 \right\|_{L^2}^2 \\
+ \sum_{j=1,2,3} \left( \left\| \frac{\partial m_j}{m_j} M \sqrt{-\Delta_L} \frac{B^2}{e} \right\|_{L^2}^2 + \left\| \frac{\partial m_j}{m_j} MD \right\|_{L^2}^2 + \left\| \frac{\partial m_j}{m_j} M_1 W^1 \right\|_{L^2}^2 \right)
\]
\[
= \frac{1}{2} \sum_{k,l} \int_{\mathbb{R}} M^2 \rho' \left\| \frac{B^2}{e} \right\|^2 d\eta + \sum_{k,l} \int_{\mathbb{R}} M^2 \rho' \left| \Delta \frac{B^2}{e} \right| d\eta + \sum_{k,l} \int_{\mathbb{R}} M^2 \rho' \left| \frac{\dot{W}^1}{p} \right| d\eta \\
+ \sum_{k,l} \int_{\mathbb{R}} M^2 \rho' \dot{B} \dot{W}^1 d\eta - \mu \left( M_1 \sqrt{-\Delta_L} B^2, M_1 \sqrt{-\Delta_L} W^1 \right) \\
+ \frac{1}{2} \sum_{k,l} \int_{\mathbb{R}} \frac{\partial m_j}{m_j} M^2 \frac{\dot{B}^2}{e} d\eta - \sum_{k,l} \int_{\mathbb{R}} \frac{\partial m_j}{m_j} M^2 \frac{\dot{B} \dot{\tilde{W}}^1}{e} d\eta + \sum_{k,l} \int_{\mathbb{R}} \frac{\partial m_j}{m_j} M^2 \frac{\dot{B} \dot{\tilde{W}}^1}{e} d\eta \\
+ \frac{1}{2} \sum_{k,l} \int_{\mathbb{R}} \frac{\partial m_j}{m_j} M^2 \frac{\dot{B} \dot{\tilde{W}}^1}{e} d\eta + \left( MG, MD^2 \right) + \left( M_1 H, M_1 W^1 \right).
\]
Using the facts $\frac{\left| \rho' \right|}{p^2} \leq \frac{\left| \rho \right|}{p^2} \leq \frac{\left| \partial m \right|}{m}$, and $M_1 \leq M$, we arrive at
\[
\sum_{k,l} \int_{\mathbb{R}} M^2 \rho' \dot{B} \dot{W}^1 d\eta \leq \sum_{k,l} \int_{\mathbb{R}} M^2 \rho' \left( \sqrt{p} \left| \dot{B}^2 \right| \right) \dot{W}^1 d\eta
\]
\[
\leq \frac{2}{N} \left\| \frac{\partial m_1}{m} M_1 \left( \sqrt{-\Delta_L} B^2 \right) \right\|_{L^2} + \left\| \frac{\partial m_1}{m} M_1 W^1 \right\|_{L^2} 
\]

\[
(4.49)
\]

Similar to (4.18) and (4.19), we have

\[
\frac{1}{2} \sum_{k,l} \int_R \partial_t \left( \frac{\partial m}{m} \right) M^2 |B^2|^2 d\eta \leq \frac{3e^2}{N} \left\| \frac{\partial m_1}{m_1} M \sqrt{-\Delta_L} B^2 \right\|_{L^2},
\]

and

\[
- \sum_{k,l} \int_R \frac{\partial m}{m} M^2 B^2 \tilde{D} \tilde{D} d\eta + \sum_{k,l} \int_R \frac{\partial m}{m} M^2 \tilde{F} \tilde{B} \tilde{D} d\eta
\]

\[
\leq \frac{2}{N} \left\| \frac{\partial m_1}{m_1} M \sqrt{-\Delta_L} B^2 \right\|_{L^2} + \left\| \frac{\partial m_1}{m_1} M \sqrt{-\Delta_L} F \right\|_{L^2}
\]

\[
(4.51)
\]

The Cauchy-Schwarz inequality implies that

\[
\frac{1}{e} \left( \left\langle M \sqrt{-\Delta_L} F, M \sqrt{-\Delta_L} B^2 \right\rangle + \left\langle MG, MD^2 \right\rangle + \left\langle M_1 H, M_1 W^1 \right\rangle \right)
\]

\[
\leq \frac{\mu^2}{256} \left( \left\| M \sqrt{-\Delta_L} B^2 \right\|_{L^2}^2 + \left\| MD^2 \right\|_{L^2}^2 + \left\| M_1 W^1 \right\|_{L^2}^2 \right)
\]

\[
+ \frac{64}{\mu^2} \left( \left\| M \sqrt{-\Delta_L} F \right\|_{L^2}^2 + \left\| MG \right\|_{L^2}^2 + \left\| M_1 H \right\|_{L^2}^2 \right)
\]

Then arguing as (4.13), one deduces from (4.48)–(4.52) that

\[
\frac{1}{2} \frac{d}{dt} E_2^2(t) + (\lambda + 2\mu) \left\| M \sqrt{-\Delta_L} D^2 \right\|_{L^2}^2 + \frac{\mu}{2} \left\| M_1 \sqrt{-\Delta_L} W^1 \right\|_{L^2}^2
\]

\[
+ \frac{3}{4} \left\| \frac{\partial m}{m} MB^2 \right\|_{L^2}^2 + \sum_{j=1,2,3} \left\| \frac{\partial m}{m} \frac{\partial m}{m} M B^2 \right\|_{L^2}^2
\]

\[
+ \left( \left\| \frac{\partial m}{m} M \sqrt{-\Delta_L} B^2 \right\|_{L^2}^2 + \left\| \frac{\partial m}{m} M \sqrt{-\Delta_L} D^2 \right\|_{L^2}^2 + \left\| \frac{\partial m}{m} M_1 W^1 \right\|_{L^2}^2 \right)
\]

\[
+ \left( 1 - \frac{e_3}{N} \right) \sum_{j=1,2,3} \left( \left\| \frac{\partial m}{m} M \sqrt{-\Delta_L} B^2 \right\|_{L^2}^2 + \left\| \frac{\partial m}{m} M \sqrt{-\Delta_L} D^2 \right\|_{L^2}^2 + \left\| \frac{\partial m}{m} M_1 W^1 \right\|_{L^2}^2 \right)
\]

\[
\leq \frac{1}{4} \left\| \frac{\partial m}{m} MD^2 \right\|_{L^2}^2 - \left\| \frac{\partial m}{m} M \sqrt{-\Delta_L} B^2 \right\|_{L^2}^2
\]

\[
+ \left( \frac{5\mu^2}{256} + \frac{\mu e_3}{2} \right) \left\| M \sqrt{-\Delta_L} B^2 \right\|_{L^2}^2 + \frac{9\mu^3}{256} \left( \left\| MD^2 \right\|_{L^2}^2 + \left\| M_1 W^1 \right\|_{L^2}^2 \right)
\]
Recalling that

(4.54)
\[ e_3 := 4\epsilon^2 + 2\epsilon. \]

Next, direct calculations show that the evolutions of \( \langle hB^2, hD^2 \rangle \) and \( \langle gB^2, gD^2 \rangle \) are given as follows

\[
\frac{d}{dt} \langle hB^2, hD^2 \rangle = c \left( \frac{M \sqrt{-
abla_L - \frac{F}{\epsilon}}}{e} \right)_L^2 - \sum_{k,l} \int_R p M^2 B^2 \beta^2 \text{d}\eta + \frac{c}{2} \sum_{k,l} \int_R \frac{\partial m}{m} M^2 B^2 \beta^2 \text{d}\eta
\]

\[
-(\lambda + 2\mu) \sum_{k,l} \int_R p M^2 B^2 \beta^2 \text{d}\eta + \frac{c}{2} \sum_{k,l} \int_R \frac{\partial m}{m} M^2 B^2 \beta^2 \text{d}\eta
\]

(4.55)
\[
2c \sum_{j=1,2,3} \sum_{k,l} \int_R M^2 \frac{\partial m_j}{m} \beta^2 Z^2 \text{d}\eta + c \left( MF, MD^2 \right) + c \left( MB^2, MG \right),
\]

and

(4.56)
\[
\frac{d}{dt} \langle gB^2, gD^2 \rangle
\]

\[
= \frac{1}{4} \left( \frac{M \sqrt{-
abla_L - \frac{F}{\epsilon}}}{e} \right)_L^2 + \sum_{k,l} \left( \frac{\partial m}{m} M \sqrt{-\Delta_L} \right)_L^2 + \sum_{k,l} \left( \frac{\partial m}{m} M \sqrt{-\Delta_L} \right)_L^2
\]

\[
2c \sum_{j=1,2,3} \sum_{k,l} \int_R M^2 \frac{\partial m_j}{m} \beta^2 Z^2 \text{d}\eta + c \left( MF, MD^2 \right) + c \left( MB^2, MG \right).
\]

Recalling that \( c = \frac{\mu^4}{256} \), using the Cauchy-Schwarz inequality, we have

(4.57)
\[
-c \left( MF, MD^2 \right) - c \left( MB^2, MG \right)
\]

Using (4.16) again, we obtain

(4.58)
\[
\frac{d}{dt} \left( \langle gB^2, gD^2 \rangle - \langle hB^2, h\bar{D}^2 \rangle \right)
\]
\[\frac{1}{4} \left( \left\| \frac{\partial_m}{m} M \sqrt{-\Delta_L} B^2 \epsilon \right\|_{L^2} + \left\| \int_{\Omega} \frac{\partial_m}{m} \nabla M \nabla \frac{\partial_m}{m} \nabla B \right\| \right) \]

\[-\frac{23\mu^+}{256} \left\| M \sqrt{-\Delta_L} B^2 \epsilon \right\|_{L^2} + (\lambda + 2\mu)(c(\lambda + 2\mu)\epsilon^2 + \frac{1}{8})\left\| M \sqrt{-\Delta_L} D^2 \right\|_{L^2}\]

\[+ c \epsilon \left( \left\| \frac{\partial_m}{m_2} M \sqrt{-\Delta_L} B^2 \epsilon \right\|_{L^2} + (c + \frac{1}{16}) \left\| \frac{\partial_m}{m_2} \nabla M \right\|^2_{L^2} + \sum_{j=1,2,3} \left\| \frac{\partial_m}{m_j} \frac{\partial_m}{m_j} MB \right\|^2_{L^2}\]

\[+ \frac{33\mu^+}{256} \left\| MD^2 \right\|^2_{L^2} + \frac{(\mu^+ \epsilon^2)^2}{\mu^7} \left( \left\| M \sqrt{-\Delta_L} F \epsilon \right\|^2_{L^2} + \left\| MG \right\|^2_{L^2}\right)\]

\[(4.59) \quad \frac{\epsilon}{4N} \left( \left\| \frac{\partial_m}{m_1} M \sqrt{-\Delta_L} F \epsilon \right\|^2_{L^2} + \left\| \frac{\partial_m}{m_1} MG \right\|^2_{L^2}\right),\]

where

\[(4.60) \quad \epsilon_4 := 2(5c + 3)\epsilon + 2(\lambda + 2\mu)\epsilon^2 + \epsilon.\]

Noting that

\[\frac{15}{16} - c\epsilon - \frac{4\epsilon_3 + \epsilon_4}{4N} \geq c_1,\]

then (4.38) holds. Thus, we infer from (4.53) and (4.59) that

\[\frac{d}{dt} E^2_{t}(t) + \frac{9\mu^+}{64} \left\| M \sqrt{-\Delta_L} B^2 \epsilon \right\|^2_{L^2} + \mu \left( \left\| M \sqrt{-\Delta_L} D^2 \right\|^2_{L^2} + \left\| M_1 \sqrt{-\Delta_L} W^1 \right\|^2_{L^2}\right)\]

\[+ \left\| \frac{\partial_m}{m_2} M \sqrt{-\Delta_L} B^2 \epsilon \right\|^2_{L^2} + \left\| \frac{\partial_m}{m_2} \nabla M \right\|^2_{L^2} + \left\| \frac{\partial_m}{m_2} M_1 W^1 \right\|^2_{L^2}\]

\[+ \frac{1}{2} \sum_{j=1,3} \left( \left\| \frac{\partial_m}{m_j} M \sqrt{-\Delta_L} B^2 \epsilon \right\|^2_{L^2} + \left\| \frac{\partial_m}{m_j} \nabla M \right\|^2_{L^2} + \left\| \frac{\partial_m}{m_j} M_1 W^1 \right\|^2_{L^2}\right)\]

\[\leq \mu \epsilon^2 \left\| M \sqrt{-\Delta_L} B^2 \epsilon \right\|^2_{L^2} + \frac{21\mu^+}{64} \left( \left\| MD^2 \right\|^2_{L^2} + \left\| M_1 W^1 \right\|^2_{L^2}\right)\]

\[+ \frac{130}{\mu^7} \left( \left\| M \sqrt{-\Delta_L} F \epsilon \right\|^2_{L^2} + \left\| MG \right\|^2_{L^2} + \left\| M_1 H \right\|^2_{L^2}\right)\]

\[(4.61) \quad + \epsilon + \frac{4\epsilon^2}{2N} \left( \left\| \frac{\partial_m}{m_1} M \sqrt{-\Delta_L} F \epsilon \right\|^2_{L^2} + \left\| \frac{\partial_m}{m_1} MG \right\|^2_{L^2}\right).\]

Then it follows from (2.40), (1.7) and (4.61) that

\[\frac{d}{dt} E^2_{t}(t) + \frac{\mu^4}{32} E^2_{t}(t) + \frac{\mu^4}{4} \left( \left\| M \sqrt{-\Delta_L} D^2 \right\|^2_{L^2} + \left\| M_1 \sqrt{-\Delta_L} W^1 \right\|^2_{L^2}\right)\]

\[+ \frac{1}{4} \sum_{j=1,2,3} \left( \left\| \frac{\partial_m}{m_j} M \sqrt{-\Delta_L} B^2 \epsilon \right\|^2_{L^2} + \left\| \frac{\partial_m}{m_j} \nabla M \right\|^2_{L^2} + \left\| \frac{\partial_m}{m_j} M_1 W^1 \right\|^2_{L^2}\right)\]
then adding the resulting inequality to (4.44), and using (1.7) again, one deduces that

\[
\frac{130}{\mu^3} \left( \left\| M \sqrt{-\Delta_L} \frac{F}{\epsilon} \right\|_{L^2}^2 + \left\| MG \right\|_{L^2}^2 + \left\| M_1 H \right\|_{L^2}^2 \right)
+ \frac{e + 4\epsilon^2}{2N} \left( \left\| \frac{\partial_m}{m_1} m \sqrt{-\Delta_L} \frac{F}{\epsilon} \right\|_{L^2}^2 + \left\| \frac{\partial_m}{m_1} M G \right\|_{L^2}^2 \right),
\]

where we have used (4.42) with \( i = 1, 3 \) replaced by \( i = 2 \). Obviously, by the definition of the multiplier \( m_1 \), for \( k \neq 0 \), there holds \( \frac{\mu}{N} \leq \frac{1}{\mu} m_{i1} \). Thus,

\[
\left\| MB^1 \right\|_{L^2}^2 = \sum_{k,n} \int_0^1 \frac{1}{p} M^2 |\sqrt{p} B^1|^2 \, dp \leq \frac{\epsilon^2}{N} \left\| \frac{\partial_m}{m_1} M \sqrt{-\Delta_L} B^1 \right\|_{L^2}^2.
\]

Accordingly, recalling the definitions of \( F, G \) and \( H \), it is easy to verify that

\[
\left\| M \sqrt{-\Delta_L} \frac{F}{\epsilon} \right\|_{L^2}^2 + \left\| MG \right\|_{L^2}^2 + \left\| M_1 H \right\|_{L^2}^2
\leq \left\| M \sqrt{-\Delta_L} \frac{B^1}{\epsilon} \right\|_{L^2}^2 + 35 \left( \left\| MD^1 \right\|_{L^2}^2 + \left\| MW^2 \right\|_{L^2}^2 + \left\| MB^1 \right\|_{L^2}^2 \right)
\]

\[
\leq 35 \left( \left\| M \sqrt{-\Delta_L} \frac{B^1}{\epsilon} \right\|_{L^2}^2 + \left\| MD^1 \right\|_{L^2}^2 + \left\| MW^2 \right\|_{L^2}^2 + \frac{\epsilon^2}{N} \left\| \frac{\partial_m}{m_1} M \sqrt{-\Delta_L} \frac{B^1}{\epsilon} \right\|_{L^2}^2 \right).
\]

Substituting this into (4.62), and recalling that \( N \) is chosen to be \( 9(1 + \epsilon)^2 \) in (4.37), we are led to

\[
\frac{d}{dt} E^2(t) + \frac{\mu^2}{32} E^2(t) + \frac{\mu}{4} \left( \left\| M \sqrt{-\Delta_L} D^2 \right\|_{L^2}^2 + \left\| M_1 \sqrt{-\Delta_L} W^1 \right\|_{L^2}^2 \right)
\leq \frac{1}{4} \sum_{j=1,2,3} \left( \left\| \frac{\partial_m}{m_j} M \sqrt{-\Delta_L} \frac{B^2}{\epsilon} \right\|_{L^2}^2 + \left\| \frac{\partial_m}{m_j} MD^2 \right\|_{L^2}^2 + \left\| \frac{\partial_m}{m_j} M W^2 \right\|_{L^2}^2 \right)
\leq \frac{4550}{\mu^3} \left( \left\| M \sqrt{-\Delta_L} \frac{B^1}{\epsilon} \right\|_{L^2}^2 + \left\| MD^1 \right\|_{L^2}^2 + \left\| MW^2 \right\|_{L^2}^2 + \left\| \frac{\partial_m}{m_1} M \sqrt{-\Delta_L} \frac{B^1}{\epsilon} \right\|_{L^2}^2 \right)
\]

\[
\left. + \left( 1 + \epsilon \right)^2 \left\| \frac{\partial_m}{m_1} M \sqrt{-\Delta_L} \frac{B^1}{\epsilon} \right\|_{L^2}^2 + \left\| \frac{\partial_m}{m_1} MD^1 \right\|_{L^2}^2 + \left\| \frac{\partial_m}{m_1} MW^2 \right\|_{L^2}^2 \right).
\]

Multiplying (4.64) by \( c_0 \mu^\frac{3}{2} \) with positive constant \( c_0 \) so small that

\[
c_0 \leq \frac{1}{32 \times 4550},
\]

then adding the resulting inequality to (4.44), and using (1.7) again, one deduces that

\[
\frac{d}{dt} \left( E^1(t) + c_0 \mu^\frac{3}{2} E^2(t) \right) + \frac{\mu^2}{32} \left( E^1(t) + c_0 \mu^\frac{3}{2} E^2(t) \right)
\leq \frac{1}{4} \sum_{i=1,3} \left( \left\| M \sqrt{-\Delta_L} D^2 \right\|_{L^2}^2 + \left\| M \sqrt{-\Delta_L} W^2 \right\|_{L^2}^2 + \left\| M_1 \sqrt{-\Delta_L} W^2 \right\|_{L^2}^2 \right)
\leq \frac{4550}{\mu^3} \left( \left\| M \sqrt{-\Delta_L} \frac{B^1}{\epsilon} \right\|_{L^2}^2 + \left\| MD^1 \right\|_{L^2}^2 + \left\| MW^2 \right\|_{L^2}^2 + \left\| \frac{\partial_m}{m_1} M \sqrt{-\Delta_L} \frac{B^1}{\epsilon} \right\|_{L^2}^2 \right)
\]

\[
\left. + \left( 1 + \epsilon \right)^2 \left\| \frac{\partial_m}{m_1} M \sqrt{-\Delta_L} \frac{B^1}{\epsilon} \right\|_{L^2}^2 + \left\| \frac{\partial_m}{m_1} MD^1 \right\|_{L^2}^2 + \left\| \frac{\partial_m}{m_1} MW^2 \right\|_{L^2}^2 \right).
\]
\[ + \frac{1}{8} \sum_{i=1,5} \sum_{j=1,2,3} \left( \left\| \frac{\partial_i m_j}{m_j} M_1 \sqrt{u^2} \right\|_{L^2}^2 + \left\| \frac{\partial_i m_j}{m_j} M_1 \right\|_{L^2}^2 \right) \]
\[ + \frac{1}{8} \sum_{j=1,2,3} \left( \left\| \frac{\partial_j m_j}{m_j} M W^2 \right\|_{L^2}^2 + \left\| \frac{\partial_j m_j}{m_j} M_1 W^3 \right\|_{L^2}^2 \right) \]
\[ + \frac{c_0\mu^2}{4} \sum_{j=1,2,3} \left( \left\| \frac{\partial_j m_j}{m_j} M_1 \sqrt{u^2} \right\|_{L^2}^2 + \left\| \frac{\partial_j m_j}{m_j} M_1 W^2 \right\|_{L^2}^2 + \left\| \frac{\partial_j m_j}{m_j} M_1 W^3 \right\|_{L^2}^2 \right) \leq 0. \]

Obviously, the relation (4.45) holds for \( E_3(t) \) and \( E_3(t) \) as well, and thus one easily obtain (4.2). We complete the proof Proposition 4.1.

5. Proof of the Main Results

In this section, we complete the proof of Theorems 1.1, 1.2, and 1.3 one by one. As pointed out in Remark 1.1, we shall prove more than we need in Theorem 1.1 below. To begin with, recalling the change of coordinates (2.15), and using (2.34), (2.35), and (4.1), we arrive at
\[ \| \partial_x b \|_{L^2}^2 = \| B_{\infty} \|_{L^2}^2 \leq \| m^{-\frac{1}{2}} \sqrt{-\Delta} B_{\infty}^1 \|_{L^2}^2 \leq C \mu^{-\frac{1}{4}} \| m^{-\frac{1}{2}} \sqrt{-\Delta} B_{\infty} \|_{L^2}, \]
and the estimate for \( \partial_z b \) is similar, thus (1.8) holds. Next, we postpone the estimate of \( u_{\infty}^2 \) and prove (1.10) and (1.11) first, since \( u_{\infty}^2 \) and \( u_{\infty}^3 \) are easy to treat. Recalling that \( W = \Delta u_{\infty}^2 - \partial_y \text{div} v + \partial_x b \), then using (2.35), (4.1), and (4.2), we find that
\[ \| \Delta_{x_z} u_{\infty}^2 \|_{L^2} = \| \Delta_{x_z} \Delta^{-1} (W_{\infty}^2 - \partial_y b_{\infty} + \partial_y \text{div} v) \|_{L^2} \]
\[ = \| \Delta_{x_z} \Delta^{-1} (W_{x_z}^2 - B_{\infty}^1 + D_{\infty}^2) \|_{L^2} \]
\[ \leq \| m^{-1} (W_{x_z}^2 - B_{\infty}^1 + D_{\infty}^2) \|_{L^2} \]
\[ \leq C \left( \| m^{-\frac{1}{2}} W_{x_z}^2 \|_{L^2} + \| m^{-\frac{1}{2}} \sqrt{-\Delta} B_{\infty} \|_{L^2} + \| m^{-\frac{1}{2}} D_{\infty}^2 \|_{L^2} \right) \]
\[ \leq C \mu^{-\frac{1}{4}} e^{-\frac{1}{4\mu}} \| (\Delta b_{\infty})_{x_z} (\Delta u_{\infty})_{x_z} \|_{L^2}, \]
thus (1.10) holds. We would like to point out that (1.2) does not need to be involved in the estimates of \( \Delta_{x_z} u_{\infty}^2 \) and \( \partial_x u_{\infty}^1 \). More precisely, it follows from (2.35) and (4.1) that
\[ \| \Delta_{x_z} u_{\infty}^3 \|_{L^2} = \| \Delta_{x_z} \Delta^{-1} (W_{x_z}^3 + \partial_y \text{div} v) \|_{L^2} \]
\[ = \| \Delta_{x_z} \Delta^{-1} (W_{x_z}^3 + D_{\infty}^3) \|_{L^2} \]
\[ \leq \| m^{-1} (W_{x_z}^3 + D_{\infty}^3) \|_{L^2} \]
\[ \leq C \mu^{-\frac{1}{4}} e^{-\frac{1}{4\mu}} \| (\Delta b_{\infty})_{x_z} (\Delta u_{\infty})_{x_z} \|_{L^2}, \]
this gives the proof of (1.11). For the proof of (1.9), we shall use the incompressibility of \( w = \Delta u - \nabla \text{div} v \) to avoid the appearance of \( w_{\infty}^1 \) and give a refined estimate of \( W_{x_z}^3 \). In fact, by virtue of \( \text{div} w = 0 \), (2.35) and (4.1), there holds
\[ \| \partial_x u_{\infty}^1 \|_{L^2} = \| \partial_x \Delta^{-1} (w_{\infty}^1 + \partial_y \text{div} v) \|_{L^2} \]
\[ = \| \partial_x \Delta^{-1} d^1 - \partial_{x_y} \Delta^{-1} w_{\infty}^2 - \partial_{x_x} \Delta^{-1} w_{\infty}^3 \|_{L^2} \]
\[= \|\partial_{xx} \Delta^{-1} d^1 - \partial_{xy} \Delta^{-1} w^2 + \partial_{xy} \Delta^{-1} b^1 - \partial_{xz} \Delta^{-1} w^3\|_{L^2} \]
\[= \|\partial_{XX} \Delta_L^{-1} D_x - \partial_{XY} \Delta_L^{-1} W_x^2 + \partial_{XY} \Delta_L^{-1} B^1_x - \partial_{XZ} \Delta_L^{-1} W_x^3\|_{L^2} \]
\[\leq \|m^{-1}(D_x^1 + W_x^3 + \sqrt{\Delta} B_x^1)\|_{L^2} + \|m^{-\frac{1}{2}} W_x^2\|_{L^2} \]
\[(5.2) \quad \leq \operatorname{Ce}^{-\mu \frac{1}{4} \epsilon} \|((\Delta b_{in})_x, (\Delta u_{in})_x)\|_{L^2} + \|m^{-\frac{1}{2}} W_x^2\|_{L^2}.
\]
We proceed via the equation (2.4) to bound \(\|m^{-\frac{1}{4}} W_x^2\|_{L^2}\). To this end, let us define
\[\tilde{M} := e^{-\mu \frac{1}{4} \epsilon} m^{-\frac{1}{2}} m_2^{-1} 1_{k \neq 0}.
\]
Similar to (4.10), we have
\[
\frac{1}{2} \frac{d}{dt} \|\tilde{M} W^2\|_{L^2}^2 + \frac{1}{4} \left\| \sqrt{\frac{\partial}{m}} \tilde{M} W^2 \right\|_{L^2}^2
\]
\[+ \left\| \sqrt{\frac{\partial}{m^2}} m \tilde{M} W^2 \right\|_{L^2}^2 + \mu \left\| \tilde{M} \sqrt{-\Delta_L} W^2 \right\|_{L^2}^2 \]
\[
\frac{\mu^2}{44} \|\tilde{M} W^2\|_{L^2}^2 + \mu \left\{ \tilde{M} \sqrt{-\Delta_L} B^1, \tilde{M} \sqrt{-\Delta_L} W^2 \right\}
\]
\[(5.3) \quad \leq \frac{\mu^2}{44} \|\tilde{M} W^2\|_{L^2}^2 + \frac{\mu}{4} \|\tilde{M} \sqrt{-\Delta_L} W^2\|_{L^2}^2 + \mu \|\tilde{M} \sqrt{-\Delta_L} B^1\|_{L^2}^2.
\]
Thanks to (2.40), the first two terms on the right hand side of (5.3) can be absorbed by the left hand side. On the other hand, in view of (2.34) and (4.1), we find that
\[\|\tilde{M} \sqrt{-\Delta_L} B^1\|_{L^2}^2 \leq C \mu^{-\frac{1}{2}} \left\| e^{-\mu \frac{1}{4} \epsilon} m^{-\frac{1}{2}} \sqrt{-\Delta_L} B_x^1 \right\|_{L^2}^2 \leq C \mu^{-\frac{1}{2}} e^{-\mu \frac{1}{4} \epsilon} \|((\Delta b_{in})_x, (\Delta u_{in})_x)\|_{L^2}.
\]
Substituting this into (5.3), and integrating with respect to the time variable, one deduces that
\[\|\tilde{M} W^2(t)\|_{L^2}^2 + \frac{\mu}{4} \|\tilde{M} W^2\|_{L^2}^2 + \mu \left\| \tilde{M} \sqrt{-\Delta_L} W^2 \right\|_{L^2}^2 \]
\[\leq C \|W^2(0)\|_{L^2}^2 + C \|((\Delta b_{in})_x, (\Delta u_{in})_x)\|_{L^2}^2 \]
\[(5.4) \quad \leq C \|((\Delta b_{in})_x, (\Delta u_{in})_x)\|_{L^2}^2,
\]
which implies that
\[\|m^{-\frac{1}{4}} W_x^2(t)\|_{L^2}^2 \leq C e^{-\mu \frac{1}{4} \epsilon} \|((\Delta b_{in})_x, (\Delta u_{in})_x)\|_{L^2}.
\]
It follows from this and (5.2) that (1.9) holds. We complete the proof of Theorem 1.2.

Now we turn to the proof of (2.2). To this end, recalling the definitions of \((B^i, D^i, W^i), i = 2, 3\), we have
\[
\left\| \frac{\nabla \epsilon_{x_i} b_x^i}{\epsilon} \right\|_{L^2}^2 + \|\nabla \epsilon_{x_i} \div u_x^i\|_{L^2}^2 + \|w_x^2\|_{L^2}^2 + \|w_x^3\|_{L^2}^2
\]
\[= \sum_{i=1,3} \left\| \sqrt{-\Delta_L} B^i_x \right\|_{L^2}^2 + \|D_x^i\|_{L^2}^2 + \|W_x^2\|_{L^2}^2 + \|W_x^3\|_{L^2}^2.
\]
Then from (2.34), (4.2), the fact \(w^2 = \Delta u^2 - \partial_y \div u + \partial_y b\), and
\[\|\Delta u_x^2 - \partial_y \div u_x^2\|_{L^2} \leq \|\Delta u_x^2 - \partial_y \div u_x^2 + \partial_y b\|_{L^2} + \epsilon \left\| \frac{\nabla \epsilon}{\epsilon} b \right\|_{L^2}^2,
\]
we find that (1.12) holds. Similarly, (1.13) is a consequence of (2.34), (4.2) and Remark 4.1. This completes the proof of Theorem 1.2.
Finally, we are left to prove Theorem 1.3. In fact, (1.15) is nothing but (3.4) in Proposition 3.1. Combining (3.11) with (3.16) and (3.17), and using (1.2), one easily deduces that (1.16) holds. From (3.27) and (1.2), we get (1.17). The proof of Theorem 1.3 is completed.

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References

[1] P. Antonelli, M. Dolce, and P. Marcati, Linear stability analysis for 2d shear flows near Couette in the isentropic compressible euler equations. arXiv preprint arXiv:2003.01694, (2020).
[2] P. Antonelli, M. Dolce, and P. Marcati, Linear stability analysis of the homogeneous Couette flow in a 2D isentropic compressible fluid. arXiv preprint arXiv:2101.01696, (2021).
[3] J. Bedrossian and N. Masmoudi, Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. Publ. Math. Inst. Hautes Etudes Sci., 122(2015), 195–300.
[4] J. Bedrossian, P. Germain, and N. Masmoudi, Dynamics near the subcritical transition of the 3D Couette flow I: Below threshold case. Mem. Amer. Math. Soc. 266 (2020), no. 1294, v+158 pp.
[5] J. Bedrossian, P. Germain and N. Masmoudi, Dynamics near the subcritical transition of the 3D Couette flow II: Above threshold case, arXiv 1506.03721(2015).
[6] J. Bedrossian, P. Germain, N. Masmoudi, On the stability threshold for the 3D Couette flow in Sobolev regularity. Ann. of Math.,185(2017), 541–608.
[7] J. Bedrossian, N. Masmoudi, and V. Vicol, Enhanced dissipation and inviscid damping in the inviscid limit of the Navier-Stokes equations near the two dimensional Couette flow. Arch. Ration. Mech. Anal. 219 (2016), 1087–1159.
[8] J. Bedrossian, V. Vicol, and F. Wang, The Sobolev stability threshold for 2D shear flows near Couette. J. Nonlinear Sci., 28(2018), 2051–2075.
[9] P. Braz e Silva, Nonlinear stability for 2 dimensional plane Couette flow. Rev. Integr. Temas Mat. 22 (2004), 67–81.
[10] D. Bresch, P. E. Jabin, Global existence of weak solutions for compressible Navier-Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor. Ann. of Math.,188 (2018), 577–684.
[11] K. M. Case, Stability of inviscid plane Couette flow. Phys. Fluids, 3 (1960), 143–148.
[12] G. Chagelishvili, A. Rogava, and I. Segal, Hydrodynamic stability of compressible plane Couette flow. Phys. Rev. E, 50(1994), 4283–4285.
[13] G. Chagelishvili, A. Tevzadze, G Bodo, and S. Moiseev, Linear mechanism of wave emergence from vortices in smooth shear flows, Phys. Rev. Lett., 79 (1997), 3178–3181.
[14] S. J. Chapman, Subcritical transition in channel flows. J. Fluid Mech., 451(2002), 35–97.
[15] F. Charve, R. Danchin, A global existence result for the compressible Navier-Stokes Navier-Stokes equations in the critical L^p framework. Arch. Rational Mech. Anal., 198(2010), 233–271.
[16] Q. Chen, T. Li, D.Y. Wei, and Z. F. Zhang, Transition threshold for the 2-D Couette flow in a finite channel. Arch. Ration. Mech. Anal., 238 (2020), 125–183.
[17] Q. L. Chen, C. X. Miao, Z. F. Zhang, Global well-posedness for compressible Navier-Stokes equations with highly oscillating initial velocity. Comm. Pure Appl. Math., 63(2010), 1173–1224.
[18] Q. Chen, D.Y. Wei, and Z. F. Zhang, Transition threshold for the 3D Couette flow in a finite channel. arXiv:2006.00721, (2020).
[19] R. Danchin, Global existence in critical spaces for compressible Navier-Stokes equations. Invent. Math., 141(2000), 579–614.
[20] R. Danchin, P. B. Mucha, Compressible Navier-Stokes system: large solutions and incompressible limit. Adv. Math., 320(2017), 904–925.
[21] Y. Deng and N. Masmoudi, Long time instability of the Couette flow in low Geyvre spaces, arXiv:1803.01246(2018).
[22] P. G. Drazin, W. H. Reid. Hydrodynamic stability. Cambridge University Press, Cambridge, 1981.
[23] P. W Duck, G. Erlebacher, and M Y. Hussaini, On the linear stability of compressible plane Couette flow. J. Fluid Mech., 258(1994), 131–165.
[24] D. Y. Fang, T. Zhang and R. Z. Zi, Global solutions to the isentropic compressible Navier-Stokes equations with a class of large initial data. SIAM J. Math. Anal. 50 (2018), no. 5, 4983–5026.
[25] B. Farrell and P. Ioannou, Transient and asymptotic growth of two-dimensional perturbations in viscous compressible shear flow. Phys. Fluids, 12(2000), 3021–3028.
[26] E. Feireisl, A. Novotný, H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations. J. Math. Fluid Mech., 3(2001), 358–392.
[27] W. Glatzel, Sonic instability in supersonic shear flows. Mon. Not. R. Astron. Soc., 233(1988), 795–821.
[28] W. Glatzel, The linear stability of viscous compressible plane Couette flow. J. Fluid Mech., 202(1989), 515–541.
[29] T. Gebhardt, S. Grossmann, Chaos transition despite linear stability. Phys. Rev. E, 50(1994), 3705–3711.
[30] Y. N. Grigoryev, I. V. Ershov, Stability and suppression of turbulence in relaxing molecular gas flows. Fluid Mechanics and Its Applications, 117, Springer, 2017.
[31] B. Haspot, Existence of global strong solutions in critical spaces for barotropic viscous fluids. Arch. Rational Mech. Anal., 202(2011), 427–460.
[32] A. Hanifi, P. J Schmid, and D. S Henningson, Transient growth in compressible boundary layer flow. Phys. Fluids 8 (1996), 826–837.
[33] D. Hoff, Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. J. Differential Equations, 120(1995), 215–254.
[34] D. Hoff, Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids. Arch. Rational Mech. Anal., 139(1997), 303–354.
[35] S. Hu and X. Zhong, Linear stability of viscous supersonic plane Couette flow. Phys. Fluids, 10(1998), 709–729.
[36] S. Jiang and P. Zhang, On spherically symmetric solutions of the compressible isentropic Navier-Stokes equations. Commun. Math. Phys., 215(2001), 559–581.
[37] A. Ionescu, H. Jia, Inviscid damping near the Couette flow in a channel. Comm. Math. Phys. 374 (2020), 2015–2096.
[38] Y. Kagei, Y. Teramoto, On the spectrum of the linearized operator around compressible Couette flows between two concentric cylinders. J. Math. Fluid Mech. 22 (2020), no. 2, Paper No. 21, 23 pp.
[39] Y. Kagei, Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a parallel flow. Arch. Ration. Mech. Anal. 205 (2012), 585–650.
[40] Y. Kagei, Global existence of solutions to the compressible Navier-Stokes equation around parallel flows. J. Differential Equations, 251(2011), 3248–3295.
[41] Y. Kagei, Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow. J. Math. Fluid Mech., 13(2011), 1–31.
[42] L. Kelvin, Stability of fluid motion-rectilinear motion of viscous fluid between two parallel plates. Phil. Mag., 24(1887), 188–196.
[43] G. Kreiss, A. Lundbladh, and D.S. Henningson, Bounds for the threshold amplitudes in subcritical shear flows. J. Fluid Mech., 270(1994), 175–198.
[44] T. Kobayashi, Some estimates of solutions for the equations of motion of compressible viscous fluid in an exterior domain in $\mathbb{R}^3$. J. Differential Equations, 184(2002), 587–619.
[45] L. Landau, On the vibration of the electronic plasma. J. Phys. USSR, 10(1946), 25.
[46] L. Lees, C. C. Lin, Investigation of the stability of the laminar boundary layer in a compressible fluid. NACA Tech. Note, No. 1115, 1946.
[47] J. K. Li, Global small solutions of heat conductive compressible Navier-Stokes equations with vacuum: smallness on scaling invariant quantity. Arch. Ration. Mech. Anal., 237(2020), 899–919.
[48] H. L. Li, X. W. Zhang, Stability of plane Couette flow for the compressible Navier-Stokes equations with Navier-slip boundary. J. Differential Equations 263 (2017), 1160–1187.
[49] P. L. Lions, “Mathematical topics in fluid mechanics”. Vol. 2. Compressible models. Oxford University Press, New York, 1998.
[50] M. Liefvendahl, G. Kreiss, Bounds for the threshold amplitude for plane Couette flow. J. Nonlinear Math. Phys., 9(2002), 311–324.
[51] Z. Lin and C. Zeng, Inviscid dynamic structures near Couette flow. Arch. Rat. Mech. Anal., 200(2011), 1075–1097.
[52] M. Malik, J. Dey, and M. Alam, Linear stability, transient energy growth, and the role of viscosity stratification in compressible plane Couette flow. Phys. Rev. E, 77(2008), 036322.
[53] N. Masmoudi, W. R. Zhao, Stability threshold of the 2D Couette flow in Sobolev spaces. arXiv:1908.11042, (2019).
[54] N. Masmoudi, W. R. Zhao, Enhanced dissipation for the 2D Couette flow in critical space. Comm. Partial Differential Equations, 45 (2020), 1682–1701.
[55] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ., 20(1980), 67–104.
[56] P. S. Marcus, W. H. Press, On Green’s functions for small disturbances of plane Couette flow. J. Fluid Mech., 79(1977), 525–534.
[57] C. Mouhot and C. Villani, On Landau damping. Acta Math., 207 (2011), 29–201.
[58] W. Orr, The stability or instability of steady motions of a perfect liquid and of a viscous liquid, Part I: a perfect liquid, Proc. R. Ir. Acad., A Math. Phys. Sci., 27 (1907), 9–68.
[59] S. Orszag, L. Kells, Transition to turbulence in plane Poiseuille and plane Couette flow. J. of Fluid Mech., 96(1980), 159–205.
[60] L. Rayleigh, On the stability or instability of certain fluid motions. Proc. London Math. Soc., 9(1880), 57–70.
[61] V. A. Romanov, Stability of plane-parallel Couette flow. Funkcional. Anal. i Priložen 7(1973), 62–73.
[62] P. Schmid, D. Henningson, Stability and Transition in Shear Flows, Applied Mathematical Sciences 142, Springer-Verlag, New York, 2001.
[63] L. Trefethen, A. Trefethen, S. Reddy and T. Driscoll, Hydrodynamic stability without eigenvalues. Science, 261(1993), 578–584.
[64] D. Y. Wei, Z. F. Zhang, Transition threshold for the 3D Couette flow in Sobolev space. Comm. Pure Appl. Math., (2020), 0001–0082 (Preprint).
[65] A. M. Yaglom, Hydrodynamic Instability and Transition to Turbulence. Fluid Mech. Appl. 100, Springer-Verlag, New York, 2012.
[66] C. Zillinger, Linear inviscid damping for monotone shear flows. Trans. Amer. Math. Soc., 369 (2017), 8799–8855.
[67] Z. S. Zhou, C. J. Zhu, and R. Z. Zi, Global well-posedness and decay rates for the three dimensional compressible Oldroyd-B model. J. Differential Equations 265 (2018), 1259–1278.

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