A HOMOLOGICAL APPROACH TO FACTORIZATION

JIM COYKENDALL AND BRANDON GOODELL

Abstract. Mott noted a one-to-one correspondence between saturated multiplicatively closed subsets of a domain \( D \) and directed convex subgroups of the group of divisibility \( D \) \[5\]. With this, we construct a functor between inclusions of saturated localizations of \( D \) and projections onto partially ordered quotient groups of \( G(D) \). We use this functor to construct many cochain complexes of \( \alpha \)-homomorphisms of partially ordered groups. These complexes naturally lead to fundamental structure theorems and some natural cohomological results that provide insight into the factorization behavior of \( D \).

1. Introduction

Factorization is historically concerned with decomposing ring elements as products of irreducibles, but also with ordering group elements. Herein, we attempt to develop some homological tools to assess the structure of the group of divisibility for an integral domain, and to use these tools to inform our analysis of the factorization information in that domain. We bootstrap our way into the domain of homological algebra by way of studying partially ordered abelian groups.

Krull demonstrated a one-to-one correspondence between the prime ideals of a valuation ring \( R \) and convex subgroups of \( G(R) \) \[4\], and Sheldon demonstrated a one-to-one correspondence between the prime ideals of a Bezout domain \( D \) and prime filters of the positive cone of \( G(D) \), which are also convex subgroups \[7\]. Mott generalized these correspondences: there exists a one-to-one correspondence between convex and directed subgroups within \( G(D) \) and the saturated multiplicatively closed sets within \( D \) \[5\]. It has long been known that convex subgroups of a partially ordered group (po-group) \( G \) are the only subgroups for which the resulting quotient group is a po-group under the natural inherited quotient order \[3, 6\]. Rather, convex subgroups are precisely the kernels of \( \alpha \)-epimorphisms of po-groups. Hence, Mott’s correspondence connects saturated localizations of domains and quotient po-groups from the group of divisibility.

With Mott’s correspondence, we construct a functor between all saturated localizations of \( D \) with inclusions as morphisms and quotient po-groups of \( G(D) \) with projections as morphisms. In this way, factorization questions about saturated localizations of \( D \) may reduce to questions about the structure of the quotient po-groups of \( G(D) \). These sequences provide structure theorems and induce a menagerie of cochain complexes of order-preserving homomorphisms. We use these cochain complexes to extract homological information related to the factorization structure of the original domain \( D \).

If an element, \( x \), in an integral domain, \( D \), factors into a finite product of irreducibles (or atoms) of \( D \), then we say \( x \) is an atomic element. If \( D \) is an integral domain in which all elements are atomic, then we say \( D \) is an atomic domain. On the other hand, following Coykendall, Dobbs, and Mullins in \[2\], if an integral
domain $D$ contains no irreducibles whatsoever, then we say $D$ is an antimatter domain. These are two extreme examples of possible factorization behavior, but one expects the most general case to be a domain in which some elements are atomic and some elements are not atomic. This is the situation in which our techniques are most relevant. Indeed, in the body of literature that has amassed on the theory of factorization (especially since the paper [1]), the focus of the research concentrates on atomic domains.

This is natural, of course, since atomic domains are precisely the domains where every nonzero nonunit factors into irreducible elements. But to only consider atomic factorization is greatly limiting. As a simple example (which we discuss in more detail later) we consider a two-dimensional discrete valuation. In such a domain, the maximal ideal is principal and generated by the only irreducible element (up to associates). This demonstrates a sort of universal factorization property of this domain as every nonunit element is divisible by the generator of the maximal ideal. On the other hand, no element of the prime ideal with height one factors into a finite product of irreducibles, and localizing at this ideal provides a PID. Our homological techniques provide data that quantify these and other relationships with a factorization or “group of divisibility” flavor.

Divisibility information in the integral domain $D$ is encoded in the ordering information in the (abelian) group of divisibility $G(D)$. Indeed, $G(D)$ is an abelian, directed po-group that is ordered with the natural divisibility ordering. The minimal positive elements in a po-group are called atoms, the elements they generate over $\mathbb{N}$ are atomic, and the atomic elements of $G(D)$ correspond to the atomic elements of $D$. Hence, we develop definitions inspired by the ring-theoretic definitions above. If $G$ is an abelian po-group, we say $G$ is atomic if every positive element of $G$ is a product of atoms and we say $G$ is antimatter if it contains no atoms.

Let $D$ be an integral domain with quotient field $k$ and saturated multiplicatively closed subset $S \subseteq D$. Denote the multiplicative group of $k$ as $k^\times$, and denote the multiplicative unit group of $D$ as $U(D)$. We always presume $U(D) \subseteq S$ and $\not\in S$. Define $G(D) := k^\times / U(D)$ as the group of divisibility of $D$, following Mott [5]. In general, $G(D)$ is a po-group with ordering $aU(D) \leq bU(D)$ if and only if $\frac{b}{a} \in D$.

Following [5], given any po-group $G$ with identity $e$ and order $\leq$, we say that any non-identity $a \in G$ is positive if $e \leq a$ and we say that any non-identity $a \in G$ is an atom if $a$ is positive and $a$ is minimal in the sense that $a \leq b$ for any comparable $b \in G$. Denote the positive cone of $G$ as $G^+ = \{ g \in G | e \leq g \}$, and the positive cone of any subset $H \subseteq G$ as $H^+$; all groups of divisibility are generated by their positive cone, i.e. $G(D) = \langle G^+ \rangle$. We say a subgroup $H \subseteq G$ is convex whenever given any $h_1, h_2 \in H$, if $h_1 \leq g \leq h_2$ for some $g \in G$, then $g \in H$. We say $H$ is directed if any $h \in H$ factors as $h = h_2^{-1}h_1$ for some $h_1, h_2 \in H^+$. A group (subgroup) is directed if and only if it is generated by its positive cone.

If $G$ and $G'$ are po-groups and there exists some order-preserving group homomorphism $f : G \rightarrow G'$, we say $A$ and $B$ are $o$-homomorphic, and we refer to $f$ as an $o$-homomorphism. Any $o$-homomorphism satisfies $f(G^+) \subseteq G'^+$; if $f(G) = G'$ and $f(G^+) = G'^+$, we say $f$ is an $o$-epimorphism. Notice that all $o$-epimorphisms are surjective $o$-homomorphisms, but the converse is not necessarily true. If $f$ and its right inverse $f^{-1}$ are both $o$-epimorphisms, then we say $f$ is an $o$-isomorphism, $G$ and $G'$ are $o$-isomorphic, and we denote this $G \simeq G'$. 
Note that the group of divisibility is classically defined as the po-group of non-zero principal fractional ideals partially ordered by set containment \([3\) 6\). Since this group is \(o\)-isomorphic to \(k^\times/U(D)\), which is easier to handle, we simply use \(G(D) = k^\times/U(D)\).

2. Partially Ordered Abelian Groups

Let \(G\) be a multiplicative po-group with order \(\leq\), and let \(H \subseteq G\) be a nontrivial proper subgroup. Consider the induced partial order \(\leq'\) on \(G/H\)

\[
aH \leq' bH \text{ if and only if } \exists \alpha \in aH, \beta \in bH \text{ such that } \alpha \leq \beta
\]

which is always a quasi-ordering. We have the following theorems, in part due to Fuchs \([3\) and others.

**Theorem 2.1.** Let \(G\) be a po-group with subgroup \(H \subseteq G\). The following are equivalent:

(i) \(H\) is a convex subgroup of \(G\);

(ii) \(G/H\) is a po-group under the induced order;

(iii) \(H\) is the kernel of some \(o\)-epimorphism; and

(iv) for any \(g_1, g_2 \in G^+\), if \(g_1g_2 \in H\), then \(g_1 \in H\) and \(g_2 \in H\).

**Proof.** The equivalence of (i) and (iv) is easy. If \(H\) is convex and \(g_1g_2 \in H\), then \(1 \leq g_1 \leq g_1g_2\) and \(1 \leq g_2 \leq g_1g_2\). Convexity provides (iv). On the other hand, assume \(H\) satisfies (iv) and \(h_1 \leq g \leq h_2\) for some \(h_1, h_2 \in H\). Then \(g^{-1}h_1\) and \(g^{-1}h_2\) are both positive elements whose product is in \(H\). Since \(H\) satisfies (iv), we obtain that both \(h_1^{-1}g\) and \(g^{-1}h_2\) are in \(H\), yielding that \(g \in H\).

Fuchs proved the equivalency of (i), (ii), and (iii); for completeness we recapitulate the proof here. Assume \(H\) is a convex subgroup of \(G\). The induced order is always a quasi-ordering, so it suffices to show antisymmetry. Indeed, if \(aH \leq' bH\) and \(bH \leq' aH\) for some \(aH, bH \in G/H\), then there exists some \(\alpha_1, \alpha_2 \in aH\) and \(\beta_1, \beta_2 \in bH\) such that \(\alpha_1 \leq \beta_1\) and \(\beta_2 \leq \alpha_2\) in \(G\). Equivalently, we can find \(h_1, h_2 \in H\) such that \(a \leq bh_1\) and \(a \geq bh_2\). We obtain that \(h_2 \leq ab^{-1} \leq h_1\). By the convexity of \(H\), we have that \(ab^{-1} \in H\). Hence \(abH = bH\).

Now suppose that \(G/H\) is a po-group under the induced order. Since \(H\) is the kernel of the natural projection \(\pi : G \to G/H\), it suffices to show that this is an \(o\)-epimorphism and to this end it suffices to show that \(\pi(G^+) = (G/H)^+\). Note that the coset \(gH \geq H\) if and only if there exists \(h \in H\) such that \(gh \geq e_G\). Taking \(h = e_G\), we see that \(\pi(G^+) \subseteq (G/H)^+\). On the other hand, if \(gH\) is positive then there is an \(h \in H\) such that \(gh \in G^+\) and hence \(gH\) is in the image of \(G^+\). We conclude that \(\pi\) is an \(o\)-epimorphism.

Finally to show that (iii) implies (i), we suppose that the map \(\pi : G \to G/H\) is an \(o\)-epimorphism and that \(h_1 \leq g \leq h_2\). Since \(gh_1^{-1} \geq e_G\), we have that \(gH \geq H\) and the fact that \(g^{-1}h_2 \geq e_G\) implies that \(gH \leq H\). We conclude that \(gH = H\) and \(g \in H\), and the convexity of \(H\) is established.

Our interest in property (iv) of Theorem 2.1 is inspired by the definition of a saturated multiplicatively closed set of a ring. This theorem connects multiplicative saturation in the ring-theoretic setting to convexity in the group-theoretic setting. A proof of the First \(o\)-Isomorphism Theorem can be found in Fuchs \([3\).

**Theorem 2.2** (First \(o\)-Isomorphism Theorem). If \(H\) is the kernel of an \(o\)-epimorphism \(\phi : G \to G'\), then \(G' \simeq G/H\) \([3\).
Not all isomorphism theorems hold in partially ordered groups, so Theorem 2.2 is a small blessing, but Theorem 2.1 is as unfortunate a result as it is descriptive. Certainly, the statement provides an intuitive grasp of convex subgroups as the only subgroups for whom the natural quotient group is a po-group (or rather, the only kernels that provide $o$-epimorphisms). Not all subgroups of an arbitrary po-group $G$ are convex, unfortunately. Indeed, if $H$ is any infinite proper subgroup of $\mathbb{Z}$ then $\mathbb{Z}/H$ is finite, and hence does not admit a nontrivial partial order. Hence, $H$ is not convex.

Here is a nice application of the First o-Isomorphism Theorem that we apply in a later section.

**Proposition 2.3.** Let $\{G_\alpha\}_{\alpha \in \Lambda}$ be a family of po-groups with corresponding $o$-ideals $\{H_\alpha\}_{\alpha \in \Lambda}$. For each $\alpha \in \Lambda$, define $L_\alpha := G_\alpha/H_\alpha$. Further, define $G := \bigoplus_{\alpha \in \Lambda} G_\alpha$ and $H := \bigoplus_{\alpha \in \Lambda} H_\alpha$, both under the product order. Then $G/H \cong \bigoplus_{\alpha \in \Lambda} L_\alpha$ under the product order.

**Proof.** Consider first the map $\phi : \bigoplus_{\alpha} G_\alpha \longrightarrow \bigoplus_{\alpha} G_\alpha/H_\alpha$. Certainly $\phi$ is a surjective homomorphism. To see that $\phi$ is an $o$-epimorphism, consider a positive sequence $\{g_\alpha + H_\alpha\} \in \bigoplus_{\alpha} G_\alpha/H_\alpha$ (where almost every $g_\alpha \in H_\alpha$). For each $\alpha \in \Lambda$, $g_\alpha + H_\alpha \geq H_\alpha$ so there exists some $h_\alpha \in H_\alpha$ such that $g_\alpha + h_\alpha \geq 0$. However, $\{g_\alpha + H_\alpha\}_{\alpha \in \Lambda}$ is the image of $\{g_\alpha + h_\alpha\}_{\alpha \in \Lambda} \subset G^+$ under the map $\phi$.

Since $\phi$ is an $o$-epimorphism, we apply Theorem 2.2 to complete the proof. $\square$

In the sequel, we risk abusing notation by denoting all orderings as $\leq$ unless there is risk of confusion. In [3], Mott studied convex subgroups that are also directed (or, equivalently, generated by their positive cone), called $o$-ideals by Fuchs [3], Močkoř [6], and Mott [5]. Directed subgroups and convex subgroups are distinct.

**Example 2.4.** Let $D := \mathbb{Z}$, $G(D) := \mathbb{Q}^\times/U(\mathbb{Z})$ be the (multiplicative) group of divisibility, and $H := \langle 2/3 \rangle$. Then $H$ is convex. Indeed, if $\left(\frac{2}{3}\right)^n \leq x \leq \left(\frac{2}{3}\right)^m$ for some $n, m \in \mathbb{Z}$, then, in particular, $\left(\frac{2}{3}\right)^n \leq \left(\frac{2}{3}\right)^m$ under the natural divisibility order. In fact, we have that $\frac{2^m}{3^n} \in \mathbb{Z}$, i.e. $n = m$ and hence $H$ is (vacuously) convex.

Of course, simply consider $2\mathbb{Z} \subseteq \mathbb{Z}$ under the usual total ordering for an example of a directed but not convex subgroup.

The study of $o$-ideals lead naturally to Mott’s Correspondence Theorem [5], which generalized Krull’s correspondence [4] and Sheldon’s correspondence [7].

**Theorem 2.5** (Mott’s Correspondence Theorem). Let $D$ be an integral domain with quotient field $k$, unit group $U(D)$, and group of divisibility $G(D) = k^\times/U(D)$. Let $\nu : k^\times \to G(D)$ be the map defined by $x \mapsto xU(D)$. Let $S$ be the set of all saturated multiplicatively closed subsets of $D$ and let $O$ be the set of all $o$-ideals of $G(D)$.

The map $S \to \langle \nu(S) \rangle$ defines a one-to-one correspondence from $S$ to $O$. Further, $G(D)/\langle \nu(S) \rangle$ is precisely the group of divisibility of $D_S$.

This correspondence suggests a functor connecting inclusions between localizations of $D$ and $o$-epimorphisms between quotient po-groups $G(D)$. It is then natural to compare chains of inclusions of localizations of $D$ with chains of projections of $G(D)$. To this end, we have the following:

**Definition 2.6.** Fix an integral domain $D$. 

(a) Define $\mathcal{R}$ as the category with objects considered to be localizations of $D$ at saturated multiplicatively closed subsets, and with morphisms considered to be the natural inclusions

$$\text{Hom}(D_S, D_T) = \{ \epsilon_{S,T} : D_S \rightarrow D_T | \epsilon_{S,T}(r) = r/1 \}$$

when $S \subseteq T$ and $\text{Hom}(D_S, D_T) = \emptyset$ otherwise.

(b) Define $\mathfrak{S}$ as the category with objects considered to be quotient groups of $G(D)$ via $\mathfrak{o}$-ideals, and with morphisms considered to be the natural projections

$$\text{Hom}(G/H, G/L) = \{ \pi_{H,L} : G/H \rightarrow G/L | \pi_{H,L}(gH) = gL \}$$

when $H \subseteq L$ and $\text{Hom}(G/H, G/L) = \emptyset$ otherwise.

**Theorem 2.7.** Fix an integral domain $D$. Let $F : \mathcal{R} \rightarrow \mathfrak{S}$ be the a function defined by

$$F(D_S) = G(D)/\langle \nu(S) \rangle \text{ and}$$

$$F(\epsilon_{S,T}) = \pi(\nu(S)),(\nu(T)).$$

Then $F$ defines a covariant functor from the category $\mathcal{R}$ to the category $\mathfrak{S}$.

**Proof.** We have that $F(\epsilon_S,S) = \pi(\nu(S)),(\nu(S))$ is defined as the function taking $g(S) \rightarrow g(S)$. Hence $F(\epsilon_S,S) = \text{id}_{G(D)/\langle \nu(S) \rangle} = \text{id}_{D_S}$.

Also, if $S \subseteq T \subseteq U$, then certainly $\epsilon_{S,U} = \epsilon_{T,U} \circ \epsilon_{S,T}$. Hence, $F(\epsilon_S,U) = F(\epsilon_{T,U} \circ \epsilon_{S,T})$. Of course, $F(\epsilon_S,U) = \pi(\nu(S)),(\nu(U))$. Hence, we obtain

$$F(\epsilon_S,U) = \pi(\nu(S)),(\nu(U)) = \pi(\nu(T)),(\nu(U)) \circ \pi(\nu(S)),(\nu(T))$$

$$= F(\epsilon_{T,U}) \circ F(\epsilon_{S,T})$$

Finally, for any $g(\nu(S)) \in G(D)/\langle \nu(S) \rangle$, we have

$$\pi(\nu(T)),(\nu(U)) \circ \pi(\nu(S)),(\nu(T)) (g(\nu(S))) = \pi(\nu(T)),(\nu(U)) (g(\nu(T)))$$

$$= g(\nu(U))$$

$$= \pi(\nu(S)),(\nu(U)) (g(\nu(S))).$$

Hence $F$ is a covariant functor. $\square$

**Theorem 2.7** provides us a valuable tool allowing us to translate between the group-theoretic setting and the ring-theoretic setting. However, our choice of saturated multiplicatively closed set remains to be determined. The next example demonstrates that problems arise if we localize $D$ at multiplicatively closed sets that contain non-atomic elements.

**Example 2.8.** In general, the set of nonatomic elements is not multiplicatively closed. In this example, this set is multiplicatively closed, but localizing at this set yields the quotient field. Let $R$ be a two dimensional discrete valuation domain with prime spectrum $(0) \subseteq p \subseteq m$.

Indeed, the maximal ideal, $m$, is principal. Disregarding units, any element of $m \setminus p$ uniquely factors into a power of the generator of $m$. However, $R$ is not atomic. Any element of $R$ that is not a power of the generator of $m$ is an antimatter element. Further, since $R$ is a two-dimensional discrete valuation domain, any proper overring of $R$ is atomic.

The set of nonatomic elements (i.e. $p \setminus 0$) is certainly multiplicatively closed, but its saturation is $R \setminus 0$. The corresponding localization is the quotient field of $R$ and we lose all factorization information.
Other problems arise if the set of non-atomic elements is not multiplicatively closed (or, equivalently, if the product of some antimatter elements is atomic).

**Example 2.9.** Let \( x \) be an indeterminate over some field \( k \), let \( X \) be the set \( X = \{ x^{2/3^n} \mid n \geq 1 \} \), and let \( R = k[x, X] \). We prove \( x \) is irreducible and \( x^{2/3} \) is not atomic. With this in hand, the factorization \( (x)(x) = (x^{2/3})(x^{2/3}) \) demonstrates that the set of non-atomic elements is not multiplicatively closed.

Let \( M = \{ 1, \frac{2}{3}, \frac{2}{9}, \ldots, \frac{2}{3^n}, \ldots \} \) be the additive monoid of exponents generated over \( \mathbb{N} \). We first claim that in \( M \), 1 is the only irreducible element. To this end, note that if 1 decomposes nontrivially, then

\[
1 = n_1 \frac{2}{3} + \cdots + n_t \frac{2}{3^t}
\]

where \( 0 \leq n_i \leq 2 \) for all \( 1 \leq i \leq t \) and \( n_t \neq 0 \). Multiplying both sides of this equation by \( 3^t \) we obtain

\[
3^t = 2n_1(3^{t-1}) + 2n_2(3^{t-2}) + \cdots + 2n_{t-1}(3) + 2n_t.
\]

Reducing this equation modulo 3 gives that \( n_t \) is divisible by 3, which is impossible. Hence 1 is irreducible. Of course, we see easily that any \( \frac{2}{3^t} \) is reducible. Indeed, since no nontrivial combination of the generators is irreducible, it suffices to observe that for all \( n \geq 1 \), \( \frac{2}{3^n} = \frac{2}{3} \frac{2}{3} \cdots + \frac{2}{3} \frac{2}{3} \cdots + \frac{2}{3} \frac{2}{3} \cdots + \frac{2}{3} \frac{2}{3} \cdots + \frac{2}{3} \frac{2}{3} \cdots + \frac{2}{3} \frac{2}{3} \cdots + \frac{2}{3} \frac{2}{3} \cdots \) and so is reducible.

Now in \( R \), the only possible divisors of \( x^{2/3} \) are monomials. It is easy to see that \( x \) is irreducible as 1 is irreducible in the monoid of exponents. In a similar vein, but the element \( \frac{2}{3} \) is not atomic in the monoid of exponents. The set of nonatomic elements of \( R \) is not closed under multiplication.

These examples suggest that we ought to localize at saturations of multiplicatively closed sets generated by atomic elements. Thus, our strategy is to project onto quotient groups by \( o \)-ideals generated by atoms. We turn our attention to the subgroup of \( G \) generated by its atoms, and two “saturations” of that subgroup.

**Definition 2.10.** Let \( G \) be an additive po-group. We define the **atomic subgroup** \( A(G) \subseteq G \) to be the subgroup of \( G \) generated by the atoms of \( G \).

We define the **almost atomic subgroup** \( AA(G) \subseteq G \) to be the subgroup of \( G \) generated by the set \( \{ g \in G^+ \mid \exists a \in A(G)^+ \text{ such that } g + a \in A(G) \} \).

We define the **quasi-atomic subgroup** \( Q(G) \), as the subgroup generated by the set \( \{ g \in G^+ \mid \exists h \in G^+ \text{ such that } g + h \in A(G) \} \).

Observe that \( A(G) \) and \( AA(G) \) need not be \( o \)-ideals. Indeed, in Theorem 2.13 we prove that \( Q(G) \) is the smallest \( o \)-ideal containing \( A(G) \), and notice that these definitions are distinct. Of course, we always have the containment

\[
0 \subseteq A(G) \subseteq AA(G) \subseteq Q(G).
\]

We now produce an example to show that these notions are distinct.

**Example 2.11.** Let \( k \) be a field with indeterminates \( t, x, y, z, w \), and consider the ring \( R = (k[t^a, x^a, y^{x^2}, z^y, w^z, \ldots, z, w, w^x, w^z, \ldots])_m \), where \( a \) ranges over positive rationals, \( m \geq 2 \), \( n \geq 1 \), and \( m \) is the maximal ideal generated by all monomials. Let \( G = G(R) \). Then the following chain of containment holds:

\[
0 \subset A(G(R)) \subset AA(G(R)) \subset Q(G(R)) \subset G(R)
\]
Proof. The elements $y, z,$ and $w$ are irreducible. In particular, $yU(R), zU(R),$ and $wU(R)$ are each elements of $A(G(R))$.

We also have that any $t^α$ is not a quasi-atomic element. Assume there exists some $f$ such that $ft^α$ is a product of irreducibles, say $ft^α = r_1r_2...r_n$. Since $(t^α)_{α ∈ Q^+}$ is a prime ideal, we have that some $r_i ∈ (t^α)$. But then we have that $r_i = st^β$ where $β ∈ Q^+$ and $s ∈ R$. Noting that $st^β = st^β/2k^β/2$ gives our contradiction. In particular, for all $α ∈ Q^+$, $t^α$ is an element of $G(R)$ but not $Q(G(R))$.

It is an easy check that for all $k ≥ 2$ and $n ≥ 1$, the element $\frac{x^k}{z^2}$ is not atomic but almost atomic.

To find examples of quasi-atomic elements that are not almost atomic is a more delicate task. To this end consider the prime ideal of $R$ generated by the set $\{x^α, y, \frac{x^k}{z^2} | α ∈ Q^+, k ≥ 2\}$ and focus on the domain $R_G$. Note that the integral closure of $R_G$ is a 2-dimensional valuation domain with non-principal maximal ideal. More precisely its value group is $Q ⊕ Z$ ordered lexicographically.

Note that in $R_G$, the element $\frac{x^k}{z^2}$ is quasi-atomic; we show that it is not almost atomic. Suppose that in $R_G$ we have irreducibles $r_i, s_j$ such that $\frac{x^k}{z^2}r_1r_2...r_n = s_1s_2...s_m$. As elements of the valuation overring, the values of each irreducible must be $(0, 1)$, and the above equation has value $(-1, n + 2)$ on the left and $(0, m)$ on the right. This establishes our claim.

Our choice of nomenclature becomes clear when we use our functor to gain inspiration from the ring-theoretic setting. For an integral domain $D$, we define an almost atomic element as any element $x ∈ D$ such that $xa$ is an atomic element for some atomic element $a ∈ D$. Naturally, any domain in which all elements are almost atomic elements is an almost-atomic domain. Similarly, we define a quasi-atomic element $x ∈ D$ as any element such that $xy$ is an atomic element for some element $y ∈ D$. Naturally, any domain in which all elements are quasi-atomic is known as a quasi-atomic domain.

Observe that all antimatter domains have groups of divisibility with no atomic elements (i.e. $A(G(D)) = 0$), and hence no quasi- or almost-atomic elements. It is also easy to demonstrate that $D$ is almost-atomic if and only if $G(D)$ is generated by its atoms.

When studying factorization, nonatomic divisors of atomic elements challenge conventional lines of attack. This leads naturally to the almost and the quasi-atomic subgroups, but we generalize the notion:

**Definition 2.12.** Let $G$ be a po-group with subgroups $H ⊆ L ⊆ G$. Then we call the subgroup $\langle \{g ∈ G^+ | ∃ℓ ∈ L^+ \text{ such that } ℓ + g ∈ H\} \rangle$ the $≤$-semisaturation of $H$ with respect to $L$. We denote this group $(H : L)$.

Using the notion of $≤$-semisaturation, we see $AA(G) = (A(G) : A(G))$ and $Q(G) = (A(G) : G)$. The following theorem demonstrates that $(H : G)$ is the $o$-ideal generated by $H$.

**Theorem 2.13.** If $G$ is a po-group with subgroup $H$ and $OH$ is the set of all $o$-ideals of $G$ containing $H$, then $(H : G) = ∩_{H' ∈ OH} H'$. In particular, $(H : G)$ is the smallest $o$-ideal containing $H$, and $Q(G)$ is the smallest $o$-ideal containing $A(G)$.

**Proof.** Observe $(H : G)$ is generated by its positive cone by definition and is therefore directed.
Let $x$ be a positive generator of $(H : G)$. By definition, there is an element $g \in G^+$ such that $x + g = h \in H$. Since $H \subseteq H'$ and $H'$ is convex, we have that $x \in H'$. Hence, $(H : G) \subseteq H'$ for any $H' \in \mathcal{O}_H$, so $(H : G) \subseteq \cap_{H' \in \mathcal{O}_H} H'$.

Now suppose that $0 \leq y \leq x$ where $x \in (H : G)^+$. We write $x = p - n$ where $p, n$ are two positive generators of $(H : G)$ (that is, there are positive elements $s, t$ such that both $p + s$ and $n + t$ are in $H$). Reworking the previous inequality, we have $n \leq y \leq n \leq p$ and so $p - (y + n)$ and $y + n + s$ are positive elements whose sum $p + s \in H$. Therefore, we have that $y + n + s \in (H : G)$ and since $n, s \in (H : G)$ as well, we have that $y \in (H : G)$. Thus, $(H : G)$ is convex by Theorem 2.14 and is therefore an $o$-ideal. Hence, $(H : G) \supseteq \cap_{H' \in \mathcal{O}_H} H'$.

Now, our factorization noses have led us to myriad $\leq$-semisaturations of subgroups of $G(D)$. In particular, given any subgroup, there exists an $o$-ideal associated with that subgroup, which is in turn associated with a saturated multiplicatively closed subset of $D$; with Theorem 2.14, this provides an $o$-epimorphism onto the associated quotient group, or rather the group of divisibility. Hence, we are now in a position to construct sequences of $o$-epimorphisms associated with $G(D)$.

**Definition 2.14.** Let $G_0 := G$ be a po-group, $G_1 := G_0/Q(G_0)$, and $\pi_0 : G_0 \to G_1$ be the natural $o$-epimorphism. Inductively define the po-group $G_{n+1} := G_n/Q(G_n)$ and the $o$-epimorphism $\pi_n$ for $n \in \mathbb{N}$. Call the sequence of po-groups together with the $o$-epimorphisms $G_0 \to G_1 \to G_2 \to \cdots$ the quasi-atomic quotient sequence of $G$.

Provided an arbitrary po-group, its quasi-atomic quotient sequence may terminate in the trivial group, it may stabilize to some nontrivial group, or neither.

**Example 2.15.** Let $G = \bigoplus_{i=1}^n \mathbb{Z}$ be ordered lexicographically. Then $G$ has a single atom, $(1, 0, \ldots, 0)$, and $A(G) = \mathbb{Z} \oplus (\bigoplus_{i=2}^n 0) \cong \mathbb{Z}$. Hence, by Proposition 2.3, $G_1 = \frac{\bigoplus_{i=1}^n \mathbb{Z}}{\mathbb{Z}(\bigoplus_{i=2}^n 0)} \cong \bigoplus_{i=1}^{n-1} \mathbb{Z}$ ordered lexicographically. We obtain the quasi-atomic quotient sequence

$$\bigoplus_{i=1}^n \mathbb{Z} \to \bigoplus_{i=1}^{n-1} \mathbb{Z} \to \cdots \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0 \to 0 \to \cdots$$

which terminates in the trivial group after $n$ steps.

Other examples are readily available. If $G = \bigoplus_{i=1}^n \mathbb{Z}$ in the product order, then after a single step, it terminates in the trivial group. Of course, $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}$ ordered lexicographically never stabilizes nor terminates, and $(\bigoplus_{i=1}^n \mathbb{Z}) \oplus \mathbb{Q}$ ordered lexicographically stabilizes to a nontrivial group after $n$ steps. We provide rings whose groups of divisibility yield some of these groups in Examples 3.5, 4.6, and 4.7.

These examples demonstrate nontrivial cases of each phenomena, and that any of the finite cases can become arbitrarily long, even for simple examples. We observe that a sequence terminates if and only if some quotient group $G_n$ is quasi-atomic, and we observe that a sequence stabilizes at a nontrivial group if and only if some quotient group $G_n$ has no quasi-atomic elements at all.

Classifying $G$ by the behavior of its quasi-atomic quotient sequence motivates the following definitions:

**Definition 2.16.** Given the quasi-atomic quotient sequence of $G$, define the following:

(a) we say $G$ is $n$-atomic if there exists some least positive integer $n$ such that $G_m = 0$ for all $m \geq n$;
(b) we say $G$ is $n$-antimatter if there exists some least positive integer $n$ such that $G_n \neq 0$ and $G_m = G_n$ for any $m \geq n$; and
(c) we say $G$ is asymptotically antimatter or $\infty$-antimatter if it is not $n$-antimatter or $n$-atomic for any positive integer $n$.

These definitions immediately provide some information. In [5], Mott defined, for a totally ordered po-group $G$ with a precisely $n$ distinct convex subgroups, $\dim(G) = n$. It is clear that if $D$ is an $n$-dimensional valuation domain, then $G(D)$ is $n$-dimensional totally ordered group. Mott also defined, for any po-group $G$, a prime subgroup $H \subseteq G$ to be any $o$-ideal such that $G/H$ is totally ordered. Let $P(G)$ be the set of all prime subgroups of $G$; Mott defined $\dim(G) = \sup_{H \in P(G)} \{\dim(G/H)\}$.

It is clear that if $\dim(G) = n < \infty$, then $G$ is not $\infty$-antimatter.

Further, the only $0$-atomic group is the trivial group; the group of divisibility of an antimatter domain is $0$-antimatter. A po-group is quasi-atomic if and only if it is $1$-atomic, and for any nontrivial group $G$, $Q(G)$ is $1$-atomic as a group unto itself. If $G$ satisfies the ascending chain condition on $o$-ideals, then $G$ is not $\infty$-antimatter.

It is immediately clear that Example 2.15 is $n$-atomic.

Our approach cannot distinguish between quasi-atomic, almost-atomic, and atomic domains. Indeed, each of these subgroups are contained inside the kernel of each projection in our sequence. On the contrary, rather than classifying the “niceness” of factorization behavior, these definitions classify the depth of pathological factorization behavior in a given domain.

In Section 3, we use the quasi-atomic quotient sequence to uncover some homological information about $G$. In Section 4, we use the quasi-atomic quotient sequence to reveal some structure within $G$.

3. Homology Theory for Quotient Sequences

This section develops homological tools to measure the depths to which the “atomicity” fails in a po-group $G$. To this end, let $G$ be a multiplicative po-group with identity 1.

Before continuing, we establish the following definitions set forth by Möcköf [6]. Any short exact sequence of $o$-homomorphisms of (multiplicative) po-groups $1 \to G \xrightarrow{\alpha} H \xrightarrow{\beta} J \to 1$ is called $o$-exact if $\alpha(G^+) = \alpha(G) \cap H^+$ and $\beta(H^+) = J^+$. Further, if the sequence is $o$-exact and $H^+ = \{ h \in H \mid \beta(h) \in J^+ \text{ or } h \in \alpha(G^+)\}$, then the sequence is lex-exact. We extend this definition naturally to complexes; define a short exact sequence of cochain complexes of $o$-homomorphisms of po-groups indexed by $n \geq 0$

$$1 \to G_\bullet \xrightarrow{\alpha_\bullet} H_\bullet \xrightarrow{\beta_\bullet} J_\bullet \to 1$$

to be $o$-exact if for each $n \geq 0$, $\alpha_n(G_n^+) = \alpha_n(G_n) \cap H_n^+$ and $\beta_n(H_n^+) = J_n^+$ analogously. We extend the definition of lex-exact similarly.

As before, let $G = G_0$ be a po-group. We consider the quasi-atomic quotient sequence

$$G_0 \to G_1 \to G_2 \to \cdots$$

We consider a detailed view of the quasi-atomic quotient sequence including the atomic, almost-atomic, and quasi-atomic subgroups in the following commutative diagram:
This immediately yields some cochain complexes such as

\[ A(G_0) \rightarrow A(G_1) \rightarrow A(G_2) \rightarrow \cdots \]

which can be padded on the left with the trivial groups in the usual manner to obtain a bi-infinite sequence. However, these complexes are trivial in the sense that each map is the trivial \( o \)-homomorphism.

To construct cochain complexes that do not consist of the trivial \( o \)-homomorphisms in each degree, we consider the pullbacks of the atomic, almost atomic, and quasi-atomic subgroups.

**Definition 3.1.** Let \( A_n := A(G_n) \) be the atomic subgroup of \( G_n \), let \( AA_n := AA(G_n) \) be the almost atomic subgroup of \( G_n \), and \( Q_n := Q(G_n) \) be the quasi-atomic subgroup. Define \( \overline{A}_n = \pi_n^{-1}(A_{n+1}) \), \( \overline{AA}_n = \pi_n^{-1}(AA_{n+1}) \), and \( \overline{Q}_n = \pi_n^{-1}(Q_{n+1}) \).
With these definitions in hand, we resolve the quasi-atomic quotient sequence in a yet more detailed commutative diagram:

\[
\begin{array}{cccccc}
\hat{Q}_n & \hat{Q}_n & \hat{Q}_{n+1} \\
\downarrow & \downarrow & \downarrow \\
\hat{A}_n & \hat{A}_n & \hat{A}_{n+1} \\
\downarrow & \downarrow & \downarrow \\
Q_n & Q_n & Q_{n-1} \\
\downarrow & \downarrow & \downarrow \\
A_n & A_n & A_{n+1} \\
\downarrow & \downarrow & \downarrow \\
A_{n-1} & A_{n-1} & A_{n-1} \\
\downarrow & \downarrow & \downarrow \\
1 & 1 & 1 \\
\end{array}
\]

in which each map to the right is the natural \(o\)-homomorphism \(\pi_n\) restricted appropriately, and the vertical maps are the natural inclusions (which are obviously \(o\)-homomorphisms). Commutativity is easily verified.

It is also easy to verify that, if \(H' \subseteq G'\) is an \(o\)-ideal and \(\phi : G \to G'\) is an \(o\)-epimorphism, then \(\phi^{-1}(H')\) is an \(o\)-ideal of \(G\) (following from the definition of an \(o\)-epimorphism). We also have the obvious containment

\[
A_n \subseteq AA_n \subseteq Q_n \subseteq \hat{A}_n \subseteq \hat{A}A_n \subseteq \hat{Q}_n.
\]

The subgroups defined above form many cochain complexes indexed by the natural numbers with maps induced by \(\{\pi_n | n \geq 0\}\) restricted appropriately, leading to the following lemma.
Lemma 3.2. Each of the following is a cochain complex of \( o \)-homomorphisms of po-groups:

\[
\begin{align*}
A_\bullet &= \cdots \rightarrow 1 \rightarrow A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \\
AA_\bullet &= \cdots \rightarrow 1 \rightarrow AA_0 \rightarrow AA_1 \rightarrow AA_2 \rightarrow \cdots \\
Q_\bullet &= \cdots \rightarrow 1 \rightarrow Q_0 \rightarrow Q_1 \rightarrow Q_2 \rightarrow \cdots \\
\hat{A}_\bullet &= \cdots \rightarrow 1 \rightarrow \hat{A}_0 \rightarrow \hat{A}_1 \rightarrow \hat{A}_2 \rightarrow \cdots \\
\hat{AA}_\bullet &= \cdots \rightarrow 1 \rightarrow \hat{AA}_0 \rightarrow \hat{AA}_1 \rightarrow \hat{AA}_2 \rightarrow \cdots \\
\hat{Q}_\bullet &= \cdots \rightarrow 1 \rightarrow \hat{Q}_0 \rightarrow \hat{Q}_1 \rightarrow \hat{Q}_2 \rightarrow \cdots \\
\hat{A}/Q_\bullet &= \cdots \rightarrow 1 \rightarrow \hat{A}/Q_0 \rightarrow \hat{A}/Q_1 \rightarrow \hat{A}/Q_2 \rightarrow \cdots \\
\hat{AA}/Q_\bullet &= \cdots \rightarrow 1 \rightarrow \hat{AA}/Q_0 \rightarrow \hat{AA}/Q_1 \rightarrow \hat{AA}/Q_2 \rightarrow \cdots \\
\hat{Q}/Q_\bullet &= \cdots \rightarrow 1 \rightarrow \hat{Q}/Q_0 \rightarrow \hat{Q}/Q_1 \rightarrow \hat{Q}/Q_2 \rightarrow \cdots \\
\end{align*}
\]

in which the differentials are induced by the \( o \)-epimorphisms \( \pi_n \). Furthermore, the sequences \( A_\bullet, AA_\bullet, Q_\bullet, \hat{A}_\bullet, \hat{AA}_\bullet, \hat{Q}_\bullet \) are each trivial in the sense that each differential is the trivial homomorphism, and the sequence \( \hat{Q}_\bullet \) is exact.

Proof. We have that the above sequences form cochain complexes by the construction of the groups, verified by an easy diagram chase. Triviality and exactness follow immediately from the fact that \( \text{Ker} \pi_n = Q_n \) and hence \( A_n, AA_n \subseteq \text{Ker} \pi_n \). \( \square \)

Notice the induced maps are not \( o \)-epimorphisms in general. Using the above complexes as a starting point, we form lex-exact sequences of cochain complexes of \( o \)-homomorphisms. We form appropriate cohomology groups and demonstrate relationships between them.

Lemma 3.3. The following three sequences are lex-exact sequences of cochain complexes of \( o \)-homomorphisms of po-groups:

\[
\begin{align*}
1 \rightarrow Q_\bullet &\rightarrow \hat{A}_\bullet \rightarrow \frac{\hat{A}}{\hat{Q}}_\bullet \rightarrow 1 \\
1 \rightarrow Q_\bullet &\rightarrow \hat{AA}_\bullet \rightarrow \frac{\hat{AA}}{Q}_\bullet \rightarrow 1 \\
1 \rightarrow Q_\bullet &\rightarrow \hat{Q}_\bullet \rightarrow \frac{\hat{Q}}{Q}_\bullet \rightarrow 1 \\
\end{align*}
\]

where the chain maps between the complexes are induced by inclusion or projection where appropriate.

Proof. An easy diagram chase provides that each cochain complex is short exact, and that each cochain complex is \( o \)-exact follows immediately from the fact that the chain maps are induced from order-preserving inclusions and projections. We show the first complex is also lex-exact; the proof of the other two are similar, \textit{mutatis mutandis}.\footnote{Mutatis mutandis: Latin for “With changes made.” It is used when discussing similar processes or concepts with different specific details.}

Indeed, take \( x \in \hat{A}^+_n \); either \( x \) is injected from \( Q_n \), or not. If \( x \) is injected from \( Q_n \), then it is positive, in which case we are done. If not, then \( x \) is not in the kernel of the projection, in which case \( \pi_n(x) \in A^+_{n+1} \). Hence, the sequence is lex-exact. \( \square \)
**Definition 3.4.** Let $X_\bullet$ denote any of the above complexes where $\delta_i : X_i \to X_{i+1}$ is the differential map. For $n \in \mathbb{Z}$, define the $n^{th}$ cohomology group of $X_\bullet$ by $H^n(G, X_\bullet) := \ker(\delta_n)/\text{im}(\delta_{n-1}).$

Definition 3.4 together with the cochain complexes in Lemma yields the following cohomology groups for $n = 0$

\begin{align*}
(3.13) & \quad H^0(G, A_\bullet) = A_0 \\
(3.14) & \quad H^0(G, AA_\bullet) = AA_0 \\
(3.15) & \quad H^0(G, Q_\bullet) = Q_0 \\
(3.16) & \quad H^0(G, \hat{A}_\bullet) = Q_0 \\
(3.17) & \quad H^0(G, \hat{AA}_\bullet) = Q_0 \\
(3.18) & \quad H^0(G, \hat{Q}_\bullet) = Q_0 \\
(3.19) & \quad H^0 \left( G, \frac{\hat{A}_\bullet}{Q_\bullet} \right) = \hat{A}_0/Q_0 \simeq A_1 \\
(3.20) & \quad H^0 \left( G, \frac{\hat{AA}_\bullet}{Q_\bullet} \right) = \hat{AA}_0/Q_0 \simeq AA_1 \\
(3.21) & \quad H^0 \left( G, \frac{\hat{Q}_\bullet}{Q_\bullet} \right) = \hat{Q}_0/Q_0 \simeq Q_1
\end{align*}

and the following cohomology groups for $n \geq 1$:

\begin{align*}
(3.22) & \quad H^n(G, A_\bullet) = A_n \\
(3.23) & \quad H^n(G, AA_\bullet) = AA_n \\
(3.24) & \quad H^n(G, Q_\bullet) = Q_n \\
(3.25) & \quad H^n(G, \hat{A}_\bullet) = Q_n \quad \hat{A}_n \\
(3.26) & \quad H^n(G, \hat{AA}_\bullet) = Q_n \quad \hat{AA}_n \\
(3.27) & \quad H^n(G, \hat{Q}_\bullet) = 1 \\
(3.28) & \quad H^n \left( G, \frac{\hat{A}_\bullet}{Q_\bullet} \right) = \hat{A}_n/Q_n \simeq A_{n+1} \\
(3.29) & \quad H^n \left( G, \frac{\hat{AA}_\bullet}{Q_\bullet} \right) = \hat{AA}_n/Q_n \simeq AA_{n+1} \\
(3.30) & \quad H^n \left( G, \frac{\hat{Q}_\bullet}{Q_\bullet} \right) = \hat{Q}_n/Q_n \simeq Q_{n+1}
\end{align*}

At this point, we may temporarily abandon our concern for preserving the partial order, for we are projecting onto quotient subgroups via subgroups that may not be convex. Indeed, since $\text{im}(\delta_n)$ is usually not convex, some of these quotient groups are not po-groups under the induced order, although they are certainly quasi-ordered groups. However, we are still capable of drawing interesting
factorization-based conclusions from these homological results, as this next example demonstrates.

**Example 3.5.** Herein, we revisit Example 2.9 to demonstrate that sometimes these cohomology groups have torsion, in contrast with previous examples such as Example 2.15. Torsion in the cohomology groups provides interesting factorization information. Let $k$ be a field with some indeterminates $y$ and $x$.

Let $Y = \{ \frac{x^2}{y}, x^{2/3^n}, \frac{x^{2/3^n}}{y^n} \mid n \geq 1, j \geq 1 \}$ and consider $T = (k[y, x, Y])_m$, where $m = (y, x, Y)$. It is clear that, up to associates, $y$ is an irreducible; in fact, $T$ has only one irreducible. Indeed, $y$ divides every generator of the maximal ideal $m$, and therefore every monomial in $T$. Hence, $y$ divides every element of $T$. Thus, any irreducible in $T$ must be an associate of $y$.

In the first stage of localization, we localize at the saturated multiplicative subset generated by $y$, resulting in a ring analogous to the ring in Example 2.9. Indeed, $x$ becomes irreducible in the localization and we have some quasi-atomic elements whose powers become atomic. In Example 2.9, we saw that $(x^{2/3})^3$ was atomic but $x^{2/3}$ was not atomic, and a similar proof demonstrates that $(x^{2/3})^2$ is also not atomic. Hence, $H^1(G(T), \hat{A}) = Q_1/A_1$ has at least two 3-torsion elements, $x^{2/3}$ and $x^{4/3}$, completing the example.

Of course, we can extend Example 3.5 inductively so that we can draw factorization conclusions based on torsion phenomenon on the $n$th step for any $n \geq 1$.

We now construct the usual long exact sequences for the previous short exact sequences of cochain complexes.

**Theorem 3.6.** Let $1 \to X_* \to Y_* \to Z_* \to 1$ denote any of the lex-exact sequences of cochain complexes from Lemma 3.3. Then we have the following induced long exact sequence of group homomorphisms in cohomology:

$$H^0(G, X_*) \to H^0(G, Y_*) \to H^0(G, Z_*) \to H^1(G, X_*) \to \cdots$$

**Corollary 3.7.** The following sequences of group homomorphisms are exact:

$$1 \to Q_0 \to Q_0 \to \frac{\hat{A}_0}{Q_0} \to Q_1 \to \frac{\hat{A}_1}{Q_1} \to Q_2 \to \frac{\hat{A}_2}{Q_2} \to \cdots$$

$$1 \to Q_0 \to Q_0 \to \frac{A A_0}{Q_0} \to Q_1 \to \frac{A A_1}{Q_1} \to Q_2 \to \frac{A A_2}{Q_2} \to \cdots$$

$$1 \to Q_0 \to Q_0 \to \frac{\hat{Q}_0}{Q_0} \to Q_1 \to 1 \to \frac{\hat{Q}_1}{Q_1} \to Q_2 \to 1 \to \cdots$$

In particular, we recover the obvious:

(i) for any $n \geq 1$, $1 \to A_n \to Q_n \to \frac{\hat{Q}_n}{Q_n} \to 1$ is exact;

(ii) for any $n \geq 1$, $1 \to A A_n \to Q_n \to \frac{Q_n}{A A_n} \to 1$ is exact; and

(iii) for any $n \geq 1$, $\frac{\hat{Q}_n}{Q_n} \simeq Q_{n+1}$.

The kernel of the projections are not o-ideals. And, of course, $\frac{\hat{Q}_n}{Q_n} = Q_{n+1}$ precisely because of the definition of $\pi_n$ and Theorem 2.2.
Example 3.8. Let $k$ be a field and consider the 4-dimensional valuation domain

$$R = \left( k \left[ x_1, x_2, x_3, x_4 \frac{x_2}{x_1}, \frac{x_4}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_3} \right] \right)_m$$

where $j$ ranges over all integers $\geq 1$ and $m$ is the maximal ideal generated by all indeterminates over $k$. This domain has group of divisibility $\bigoplus_{i=1}^4 \mathbb{Z}$ ordered lexicographically; in particular, we have a single irreducible, $x_1$. After the first localization, we have only one irreducible, $x_2$. Localizing twice yields a single irreducible, $x_3$, localizing a third time yields a single irreducible, $x_4$, and localizing a fourth time yields the quotient field.

Observe, however, that our quasi- and almost atomic subgroups coincide with our atomic subgroup, which is precisely $\bigoplus_{i=1}^4 \mathbb{Z}$. Hence, our quasi-atomic quotient sequence is $\bigoplus_{i=1}^4 \mathbb{Z}$-isomorphic to the sequence

$$\bigoplus_{i=1}^4 \mathbb{Z} \to \bigoplus_{i=1}^3 \mathbb{Z} \to \bigoplus_{i=1}^2 \mathbb{Z} \to \mathbb{Z} \to 0 \to 0 \to \cdots$$

We resolve this sequence in detail:

$$\hat{G}_0 \cong \bigoplus_{i=1}^4 \mathbb{Z} \to \bigoplus_{i=1}^3 \mathbb{Z} \to \bigoplus_{i=1}^2 \mathbb{Z} \to \mathbb{Z} \to 0 \to 0 \to \cdots$$

Hence, Corollary 3.7 only yields one distinct long exact sequence $\bigoplus_{i=1}^4 \mathbb{Z}$-isomorphic to the sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z}} \to \mathbb{Z} \to 0 \to \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z}} \to \mathbb{Z} \to 0 \to 0 \to \cdots$$

4. Structure within Partially Ordered Abelian Groups

Earlier, we provided examples of arbitrary po-groups (not necessarily groups of divisibility) that realized our definitions of $n$-atomic, $n$-antimatter, and $\infty$-antimatter. In this section, we develop some theorems that allows us to demonstrate these examples are not vacuous in the factorization sense. Those po-groups had structures that depended sensitively on the chosen order; our goal in this section is to describe how those choices relate to the ring-theoretic phenomena. In this section, we use additive notation for groups unless otherwise stated since our primary concern is group structure in terms of direct sums of po-groups.

For each of these examples, the groups easily split into atomic and antimatter pieces. Unfortunately, we cannot in general expect this, or even expect $G$ to split as an abelian group.

Example 4.1. Let $G = \mathbb{Q}$ with partial ordering $a \leq b$ if and only if $b - a \in \mathbb{N}$. Then $G^+ = \mathbb{N}$. Hence $A(G) = Q(G) = \mathbb{Z}$. No direct sum decomposition is possible, since that would require $\mathbb{Z}$ to be a direct summand of $\mathbb{Q}$. 

$$\hat{A}_0 \cong \bigoplus_{i=1}^2 \mathbb{Z} \to \bigoplus_{i=1}^2 \mathbb{Z} \to \mathbb{Z} \to 0 \to \cdots$$

Hence, Corollary 3.7 only yields one distinct long exact sequence $\bigoplus_{i=1}^4 \mathbb{Z}$-isomorphic to the sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z}} \to \mathbb{Z} \to 0 \to \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z}} \to \mathbb{Z} \to 0 \to 0 \to \cdots$$
Certainly, Example 4.1 is of little interest to the study of factorization, since it cannot appear as a group of divisibility with the given ordering since it is not generated by its positive elements. In the sequel, we assume that $G$ is generated by its positive elements, i.e. we assume $G$ is directed. The motivation for this assumption is ring-theoretic; in particular, the group of divisibility of any integral domain is generated by its positive cone. Whereas Example 4.1 is degenerate in the sense that it is not directed, the next example is degenerate in the sense that the group does not split neatly into atomic and antimatter pieces.

Example 4.2. Let $R := k[x, X]$ be the domain from Example 2.9. In this domain, the element $x^3$ is an antimatter element. But since $x$ is an atom and $(x^3)^3 = (x)(x)$ we see that the subgroup $\langle x^3 \rangle$ intersects $Q(G(R))$ nontrivially. Hence, $Q(G(R))$ is not a direct summand of $G(R)$.

To avoid these pathologies, we discuss some conditions that split a po-group into atomic and antimatter pieces. The following structure theorem, observed by Mocek [6], comes quite naturally.

**Theorem 4.3.** Let $1 \to G \xrightarrow{\alpha} H \xrightarrow{\beta} J \to 1$ be a lex-exact sequence of po-groups. If the sequence splits, then the sequence splits lexicographically. That is, there exists an $\alpha$-isomorphism $\tau : G \oplus J \xrightarrow{} H$ ordered lexicographically such that the following diagram is commutative

$$
\begin{array}{ccc}
1 & \longrightarrow & G \\
\downarrow & & \downarrow \tau \\
G \oplus J & \longrightarrow & J \\
\downarrow \pi & & \downarrow = \\
1 & \longrightarrow & G \\
\end{array}
\quad \begin{array}{ccc}
1 & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
G \oplus J & \longrightarrow & J \\
\downarrow \beta & & \downarrow = \\
1 & \longrightarrow & 1 \\
\end{array}
$$

in which $\iota : G \to G \oplus J$ is defined by $g \mapsto (g, 0)$, and $\pi : G \oplus J \to J$ is the natural projection $(g, j) \mapsto j$.

Under some circumstances, this theorem allows us, given a po-group $G$ and its quasi-atomic quotient sequence, to write $G_0$ as a direct sum of a 1-atomic part, $Q(G_0)$, and an antimatter part, $G_1$. Note, however, the terminology “antimatter part” refers to the properties that $G_1$ has inside the group $G_0$. It is, of course, possible for $G_1$ to be 1-atomic when considered as a group unto itself (e.g. if $G_0 = \mathbb{Z} \oplus \mathbb{Z}$ with the lexicographic ordering). To mitigate ambiguity, we formalize the notion of an antimatter part of $G$ later.

**Corollary 4.4.** Let $G$ be a po-group and consider the quasi-atomic quotient sequence of $G$, and let $\iota_n : Q(G_n) \longrightarrow G_n$ be the natural inclusion. If a group homomorphism $\phi_{n+1} : G_{n+1} \longrightarrow G_n$ exists such that $\phi_{n+1} \pi_n = 1_{G_n}$, then $G_n \simeq Q(G_n) \oplus G_{n+1}$ ordered lexicographically. In particular, we have the following:

(i) if $G$ is $n$-atomic and for all $0 \leq i < n$, $G_i \simeq Q(G_i) \oplus G_{i+1}$ ordered lexicographically, then $G \simeq \bigoplus_{i=0}^{n-1} Q(G_i)$ ordered lexicographically, and

(ii) if $G$ is $n$-antimatter and for all $0 \leq i < n$, $G_i \simeq Q(G_i) \oplus G_{i+1}$ ordered lexicographically, then $G \simeq \bigoplus_{i=0}^{n-1} Q(G_i)$ ordered lexicographically.

In other words, when the quasi-atomic quotient sequence terminates and every term is a direct sum of a quasi-atomic part and antimatter part, then every element of $G$ is (uniquely) a sum of terms, one from each quasi-atomic part of the sequence. We have a similar result for an $n$-antimatter group: if the quasi-atomic sequence
stabilizes at $G_n$ rather than terminates, the same result holds, except every element has a term from the antimatter $G_n$.

As promised, the following examples realize rings that have groups of divisibility such as $\oplus_{i=1}^n \mathbb{Z}$ and $(\oplus_{i=1}^n \mathbb{Z}) \oplus \mathbb{Q}$ in the product and lexicographic orders, and the countably infinite versions of those groups.

**Example 4.5.** Let $k$ be a field and consider the 2-dimensional valuation domain

$$R := \left(k \left[ x_1, x_2, \frac{x_2}{x_1^j} \right] \right)_m$$

where $j$ ranges over all integers $j \geq 1$ and we localize at the maximal ideal $m$ generated by the set of all indeterminates. The group of divisibility is $G(R) \simeq \mathbb{Z} \oplus \mathbb{Z}$ in the lexicographic order. In particular, $R$ has precisely one irreducible, $x_1$: by localizing at $x_1$, we obtain a new ring in which $\frac{x_1}{x_2}$ is the only irreducible, and localizing again yields the quotient field. Hence, the ring is 2-atomic. After the first stage of localizing, the new group of divisibility is 1-atomic and the atomic subgroup is now $\sigma$-isomorphic to $\mathbb{Z}$ since it, too, has only one irreducible.

Hence, the quasi-atomic quotient sequence is $\sigma$-isomorphic to the sequence

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0 \to 0 \to \cdots$$

Of course, this example can be extended inductively so as to terminate after an arbitrary number of steps. On the other hand, if we adjoin to $k$ a countably infinite set of indeterminates, this example never terminates nor stabilizes.

**Example 4.6.** Let $k$ be a field and consider the infinite dimensional valuation domain

$$R := \left(k \left[ x_1, x_2, x_3, \ldots, \frac{x_2}{x_1^j}, \ldots, \frac{x_m}{x_{m-1}^j}, \ldots \right] \right)_m$$

where $j$ ranges over all integers $j \geq 1$ and we localize at the maximal ideal $m$ generated by the set of all indeterminates. The group of divisibility is $G(R) \simeq \oplus_{i \in \mathbb{N}} \mathbb{Z}$. In particular, $R$ has precisely one irreducible, $x_1$, which is prime; by localizing at $x_1$, we obtain a new ring in which $\frac{x_2}{x_1}$ is the only irreducible, which is prime, and so on. The ring is $\infty$-antimatter.

Since each stage has one and only one prime, $A(G) \simeq \mathbb{Z} \oplus (\oplus_{i \geq 1} \mathbb{Z}) \simeq \mathbb{Z}$, and each localization has $\sigma$-isomorphic groups of divisibility. The quasi-atomic quotient sequence is

$$\oplus_{i \geq 1} \mathbb{Z} \to (\oplus_{i=1}^1 \mathbb{Z}) \oplus (\oplus_{i \geq 2} \mathbb{Z}) \to (\oplus_{i=1}^2 \mathbb{Z}) \oplus (\oplus_{i \geq 3} \mathbb{Z}) \to \cdots$$

This is clearly $\sigma$-isomorphic to the sequence

$$\oplus_{i \in \mathbb{N}} \mathbb{Z} \to \oplus_{i \in \mathbb{N}} \mathbb{Z} \to \oplus_{i \in \mathbb{N}} \mathbb{Z} \to \cdots$$

which neither terminates nor stabilizes.

**Example 4.7.** Let $k$ be a field and consider the ring

$$R := \left(k \left[ y^n, x, \frac{y^n}{x^j} \right] \right)_m$$
where $j$ ranges over all integers $j \geq 1$, $\alpha$ ranges over all strictly positive rationals, and we have localized at the maximal ideal $m$ generated by the set of all indeterminates. In particular, we have precisely one irreducible, $x$, but after localization, we have an antimatter ring. That is, $R$ is 1-antimatter and $G(R) \simeq \mathbb{Z} \oplus \mathbb{Q}$ in the lexicographic order.

Hence, the quasi-atomic quotient sequence is $$\mathbb{Z} \oplus \mathbb{Q} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q} \rightarrow \cdots$$ which stabilizes to a nontrivial group after the first step. Of course, we can extend this example so that it stabilizes to a nontrivial group after as many localizations as we like.

To complete this section, we develop a fundamental structure theorem for po-groups in the product order. To this end, we formalize our notion of “antimatter part” as the quasi-atomic complement, and we develop some elementary group-theoretic results.

Definition 4.8. Let $G$ be a po-group with $Q(G)$ be the quasi-atomic subgroup of $G$. We define a quasi-atomic complement of $G$ to be any subgroup $H'$ such that $H' \cap Q(G) = 0$ and for all $g \in G$ there is an $n \in \mathbb{N}$ such that $n > 0$ and $ng \in H' \oplus Q(G)$. We define a maximal quasi-atomic complement of $G$ to be any quasi-atomic complement maximal among the set of all quasi-atomic complements of $G$ with respect to the inclusion partial order.

It is straightforward to demonstrate that quasi-atomic complements exist.

Lemma 4.9. Let $G$ be a po-group with quasi-atomic subgroup $Q(G)$. Then a maximal quasi-atomic complement, $H'$, of $G$ exists.

Proof. We first demonstrate that quasi-atomic complements always exist, and then we demonstrate the existence of maximal quasi-atomic complements. We construct a quasi-atomic complement $H$ via Zorn’s Lemma. If $\mathfrak{J}$ denotes the set of all subgroups $J \subseteq G$ such that $J \cap Q(G) = 0$ (clearly $\mathfrak{J}$ is nonempty), then the union of any chain in $\mathfrak{J}$ suffices as an upper bound. Applying Zorn’s Lemma, we obtain a maximal element which we call $H$. It suffices to show that for all $g \in G$, there is a strictly positive $n \in \mathbb{N}$ such that $ng \in H \oplus Q(G)$.

If $g \in H \oplus Q(G)$ then we are done. If not, then the maximality of $H$ implies that $\langle g, H \rangle$ has nontrivial intersection with $Q(G)$. Hence we have the existence of $n > 0$, $h \in H$, and $\alpha \in Q(G)$ such that $ng + h = \alpha$, so $ng \in H \oplus Q(G)$.

Hence, a quasi-atomic complement always exists. It remains to be shown that a maximal quasi-atomic complement always exists. To this end, take any quasi-atomic complement $H \subseteq G$ and consider $\mathfrak{Q}$ as the set of all quasi-atomic complements of $G$ containing $H$ partially ordered by inclusion. If $\mathfrak{C}$ is a chain in $\mathfrak{Q}$, we claim that $H' := \cup_{J \in \mathfrak{C}} J \in Q$ is an upper bound for $\mathfrak{C}$. Indeed,

$$H' \cap Q(G) \subseteq \cup_{J \in \mathfrak{C}} (J \cap Q(G)) = 0.$$ Further, if $g \in G$, then for each $J \in \mathfrak{C}$, there exists some $n_J$ such that $n_J g \in J \oplus Q(G) \subseteq H' \oplus Q(G)$. Hence, $H' \in \mathfrak{Q}$ and so Zorn’s Lemma gives us a maximal element. □

We remark that the maximal quasi-atomic complement to $Q(G)$ (which we have called $H'$) is not unique even with respect to order considerations. For example, if
we consider \( \mathbb{Z} \oplus \mathbb{Z} \) under the lexicographic ordering, \( Q(G) \) is uniquely determined (it is the subgroup \( \mathbb{Z} \oplus 0 \)) but we have many choices for \( H' \). For example, the subgroups generated by \((0,1)\) and \((1,1)\) are two choices. This leads naturally to a corollary:

**Corollary 4.10.** Let \( G \) be a directed po-group. If \( G/Q(G) \) is torsion free and there exists some \( H' \subseteq G \) that is a divisible quasi-atomic complement of \( G \), then \( G = H' \oplus Q(G) \).

**Proof.** It is sufficient to show positive elements of \( G \) are elements of \( H' \oplus Q(G) \).

Let \( g \in G \) to be positive. Since \( H' \) is a divisible quasi-atomic complement, we have some positive integer \( n \) and a pair of elements \( h, s \in Q(G) \) such that \( ng = h + s \).

Of course, \( h \) is divisible, so \( h = nh_0 \) for some \( h_0 \in H' \). Thus, \( ng = nh_0 + s \) or rather, \( n(g - h_0) + Q(G) \) is trivial in \( G/Q(G) \). Since \( G/Q(G) \) is torsion free, we conclude \( g - h_0 \in Q(G) \), or rather \( g = h_0 + s_0 \) for some \( s_0 \in Q(G) \). Hence \( g \in H' \oplus Q(G) \).

Of course, if \( G \) is not generated by its positive elements (as in Example 4.1), the corollary fails miserably. We now complete this section with a structure theorem regarding the atomic and quasi-atomic subgroups of a direct sum of a partially ordered abelian group in the product order.

**Theorem 4.11.** Let \( \{G_\alpha\}_{\alpha \in \Lambda} \) be a family of po-groups. Define \( G = \bigoplus_\alpha G_\alpha \) and \( H = \prod_\alpha G_\alpha \), both in the product order. Then \( A(G) = A(H) = \bigoplus_\alpha A(G_\alpha) \) and \( Q(G) = Q(H) = \bigoplus_\alpha Q(G_\alpha) \).

**Proof.** It suffices to observe that in both the direct product and the direct sum, the minimal positive elements are sequences of the form \( m_\beta = \{\epsilon_\alpha\}_{\alpha \in \Lambda} \) where

\[
\epsilon_\alpha = \begin{cases} 0, & \text{if } \alpha \neq \beta; \\ p_\beta, & \text{if } \alpha = \beta. \end{cases}
\]

where \( p_\beta \) is some atom in \( G_\beta \), and these sequences generate the subgroup \( \bigoplus_\alpha A(G_\alpha) \).

This leads immediately to the following corollary:

**Corollary 4.12.** Let \( \{G_\alpha\}_{\alpha \in \Lambda} \) be a family of po-groups, \( G = \bigoplus_\alpha G_\alpha \) in the product order, let \( \{n_\alpha\}_{\alpha \in \Lambda} \) be a net of natural numbers, and let \( N = \text{sup}_{\alpha \in \Lambda} \{n_\alpha\} < \infty \).

The following hold:

(i) if for every \( \alpha \in \Lambda \), \( G_\alpha \) is \( n_\alpha \)-atomic, then \( G \) is \( N \)-atomic; and

(ii) if for every \( \alpha \in \Lambda \), \( G_\alpha \) is \( n_\alpha \)-antimatter, then \( G \) is \( N \)-antimatter.

**Proof.** We prove the first statement; the second is proven similarly.

We define \( G_0 := \bigoplus_\alpha G_\alpha \) as usual and \( G_0^{(0)} := G_\alpha \). Define \( G_0^{(1)} := G_\alpha / Q(G_\alpha) \) and inductively define \( G_0^{(n)} := G_\alpha / Q(G_\alpha^{(n-1)}) \) for \( n \geq 1 \). Also define as usual \( G_1 := G_0 / Q(G_0) \) and, inductively define \( G_n := G_{n-1} / Q(G_{n-1}) \) for \( n \geq 1 \). By Proposition 2.3, \( G_n = \bigoplus_\alpha G_\alpha^{(n)} \) for any \( n \geq 0 \).

Observe that \( G_n = \bigoplus_\alpha G_\alpha^{(n)} \), and since each \( G_\alpha \) is \( n_\alpha \)-atomic, each \( G_\alpha^{(n_\alpha)} = 0 \). Hence, we obtain the quasi-atomic quotient sequence \( G_0 \to G_1 \to G_2 \to \cdots \) or rather

\[
\bigoplus_\alpha G_\alpha^{(0)} \to \bigoplus_\alpha G_\alpha^{(1)} \to \bigoplus_\alpha G_\alpha^{(2)} \to \cdots
\]
Since each \( n_\alpha \leq N \), we have that this sequence terminates by the \( N^{th} \) step and no earlier. \( \square \)

More corollaries also follow quite easily from this theorem. For example, if \( H \) is a po-group that is not \( n \)-atomic for any \( n \in \mathbb{N} \) and \( G \) is any po-group, then \( H \oplus G \) in the product order is not \( n \)-atomic for any \( n \in \mathbb{N} \).

5. Further Questions

As we have seen, by adding a partial order, many “nice” properties such as isomorphism theorems are often lost. We defined the cohomology groups as usual, i.e. if \( X_* \) denotes any of the cochain complexes in Lemma 3.2 we defined the \( n^{th} \) cohomology group of \( X_* \) by \( H^n(G, X_*) := \ker(\delta_n)/\text{im}(\delta_{n-1}) \) where \( \delta_i : X_i \to X_{i+1} \) is the induced map. This presents problems because, in general, \( \text{im}(\delta_{n-1}) \) is not an \( o \)-ideal; under the induced order, this is not a po-group.

Provisionally, an interested reader may whimsically define a so-called \( n^{th} \) \( o \)-cohomology group by recalling that \((H:G)\) is the smallest \( o \)-ideal generated by \( H \) and defining \( H^n_o(G, X_*):= \frac{\ker(\delta_n)}{(\text{im}(\delta_{n-1}):G)} \). By inflating the kernel of the projection to the smallest associated \( o \)-ideal, the resulting cohomologies may provide interesting factorization information. However, this may cause some cohomology groups to become trivial.

On the other hand, the \( n^{th} \) cohomology group we have already defined, \( H^n(G, X_*) \) is always a quasi-ordering under the induced order. Further, a quasi-ordering \( \preceq \) always admits a partial order \( a \preceq b \) if and only if \( a \preceq b \) and \( b \preceq a \). Hence, we have another provisional method of defining the \( n^{th} \) \( o \)-cohomology group by imposing this equivalence relation on the usual cohomology group \( H^n(G, X_*) \). It may be the case that theses two provisional definitions of an \( o \)-cohomology coincide, or at least may coincide in some particular cases.

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Department of Mathematics, North Dakota State University, Fargo, ND 58108

E-mail address, J. Coykendall: jim.coykendall@ndsu.edu

Department of Mathematics, North Dakota State University, Fargo, ND 58108

E-mail address, B. Goodell: brandon.goodell@ndsu.edu