EKEDAHLOORT STRATA
AND KOTTWITZ-RAPPOPORT STRATA

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Abstract. We study the moduli space \( \mathcal{A}_g \) of \( g \)-dimensional principally polarized abelian varieties in positive characteristic, and its variant \( \mathcal{A}_I \) with Iwahori level structure. Both supersingular Ekedahl-Oort strata and supersingular Kottwitz-Rapoport strata are isomorphic to disjoint unions of Deligne-Lusztig varieties (see [4] and [2], resp.). Here we compare these isomorphisms. We also give an explicit description of Kottwitz-Rapoport strata contained in the supersingular locus in the general parahoric case. Finally, we show that every Ekedahl-Oort stratum is isomorphic to a parahoric Kottwitz-Rapoport stratum.

1. Introduction

Consider the moduli space \( \mathcal{A}_g \) of principally polarized abelian varieties of dimension \( g \) in characteristic \( p \), its variant \( \mathcal{A}_I \) with Iwahori level structure at \( p \), and more generally the spaces \( \mathcal{A}_J \) with parahoric level structure of type \( J \). In these moduli spaces the supersingular locus is of great interest. Whereas we have no hope to describe the whole moduli space explicitly, for the supersingular locus one can be more optimistic.

We can also look at interesting closed subsets of the supersingular locus. In the recent papers [2] and [3] C.-F. Yu and the first named author explicitly described supersingular Kottwitz-Rapoport strata in terms of Deligne-Lusztig varieties. In [4] the second named author did the same for supersingular Ekedahl-Oort strata. See section 2 for a summary of these results. Each EO stratum \( EO_w \) admits a finite \( \acute{e} \)tale cover \( \mathcal{A}_I,w \tau \rightarrow EO_w \) by a KR stratum (see [1], [3], section 9). Our main results are:

(1) The descriptions of supersingular KR-strata and supersingular EO-strata in terms of Deligne-Lusztig varieties are naturally compatible with respect to the projection \( \mathcal{A}_I,w \tau \rightarrow EO_w \) (theorem 3.4).
(2) A KR stratum is contained in the supersingular locus of \( \mathcal{A}_J \) if and only if it is superspecial (theorem 4.3) and in that case it is isomorphic to a disjoint union of Deligne-Lusztig varieties (theorem 4.3). This generalizes results in [2], [3].
(3) Each EO-stratum is isomorphic to a certain KR-stratum in the parahoric moduli space \( \mathcal{A}_J \) with \( J \) the type of the canonical filtration (theorem 5.3).

Let us mention two open questions. First of all, which EO strata occur in the image \( \pi(\mathcal{A}_x) \subseteq \mathcal{A}_g \) for \( x \in \operatorname{Adm}(\mu) \)? The answer to this question could depend on \( p \), but we don’t expect that. Ekedahl and van der Geer, who first posed this question, showed in [1], that if \( x = w \tau \) for a final element \( w \), then the image is just

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the single EO stratum corresponding to \( w \), while in general, it is a union of EO strata. Our theorem 5.3 says that for all \( y \in W_J wr W_J \cap \text{Adm}(\mu) \), the image of \( A_y \) under \( \pi \) is equal to \( EO_w \).

Secondly, what is the dimension of the supersingular locus in moduli spaces with a parahoric level structure? For general \( g \) we only know this for \( J = \{0\} \), or \( = \{g\} \), by the work of Li and Oort \cite{1}, or as a consequence of the purity of the Newton stratification shown by de Jong and Oort, and for \( J = I \) and \( g \) even (see \cite{3}, also for bounds in the case where \( g \) is odd). We expect that for \( J \neq \{0\}, \{g\} \) the supersingular locus is usually not equi-dimensional. The lower bound on the dimension obtained from KR strata in the supersingular locus cannot be sharp in the general parahoric case. Note also that the union of supersingular EO strata achieves only approximately half the dimension of the supersingular locus in \( A_g \).

2. Preliminaries and notation

In this section we recall the most important definitions and results from \cite{2}, \cite{3} and \cite{4}. Fix a prime \( p \) and let \( k \) be an algebraic closure of \( \mathbb{F}_p \). We only work with schemes over \( k \).

2.1. The symplectic group. From the perspective of Shimura varieties the algebraic group underlying the moduli spaces we study is the group \( \text{GSp}_{2g} \) of symplectic similitudes. We denote by \( W_g \) its finite Weyl group, generated by the simple reflections \( s_1, \ldots, s_g \), and by \( W^a_g \) its affine Weyl group, generated by \( s_0, s_1, \ldots, s_g \). For \( c \leq g \) we see \( W_c \) as a subgroup of \( W_g \) via the natural map induced by the inclusion of Dynkin diagrams, explicitly given by \( s_{c+1-i} \mapsto s_{g+1-i} \) for \( i = 1, \ldots, c \).

We identify \( I = \{0, \ldots, g\} \) with the set of simple affine reflections via \( i \mapsto s_i \). The subsets of \( I \) are the types of parahoric subgroups of \( \text{GSp}_{2g}(\mathbb{Q}_p) \). For \( J \subset I \) we let \( W_J \) be the subgroup of \( W^a_g \) generated by all \( s_i \) with \( i \notin J \). Warning: This differs from convention, where \( W_J \) is generated by \( s_j \) for \( j \in J \).

Let \( W_{g, \text{final}} \subset W_g \) be the set of final elements, i.e., the set of minimal length representatives for the cosets in \( W_g / S_g \), where \( S_g \) is the subgroup generated by \( s_1, \ldots, s_{g-1} \). See \cite{7} and \cite{1}. The inclusion \( W_c \subset W_g \) maps \( W_{c, \text{final}} \) to \( W_{g, \text{final}} \).

We write \( G' \) for the inner form (over \( \mathbb{Q}_p \)) of the derived group \( \text{Sp}_{2g} \) of \( \text{GSp}_{2g} \) that arises as the automorphism group of a superspecial abelian variety (together with a principal polarization), see \cite{2}, Section 6.1. For \( c \in \{0, \ldots, [g/2]\} \) the subset \( \{c, g - c\} \subset I \) gives a parahoric subgroup \( P'_{\{c,g-c\}} \) of \( G' \). We denote the maximal reductive quotient of \( P'_{\{c,g-c\}} \) by \( \overline{G'}_c \) (see loc. cit., where this group is denoted by \( \overline{G'}_{\{c,g-c\}} \)). This quotient is an algebraic group over \( \mathbb{F}_p \) which splits over \( \mathbb{F}_{p^2} \). Its Dynkin diagram is obtained from the extended Dynkin diagram of \( \text{Sp}_{2g} \) by removing the vertices \( c, g - c \). Frobenius acts on the Dynkin diagram by \( i \mapsto g - i \). Denote by \( \sigma' \) the Frobenius on \( \overline{G'}_{c,k} \) of the non-split form \( \overline{G'}_c \).

2.2. Deligne-Lusztig varieties. Let \( G \) be a connected reductive group over \( k \), defined over a finite field \( \mathbb{F}_q \) of characteristic \( p \). Let \( \sigma : G \to G \) be its \( q \)-power Frobenius. Fix a Borel subgroup \( B \) and a maximal torus in \( B \) defined over \( \mathbb{F}_q \). Let \( W \) be the Weyl group.

We denote with \( \mathcal{B}(G) \cong G / B \) the variety of Borel subgroups in \( G \). In \( \mathcal{B}(G) \) we have Deligne-Lusztig varieties

\[
X_G(w, \sigma)(k) = \{ B \in \mathcal{B}(G)(k) \mid \text{inv}(B, \sigma B) = w \}.
\]
a fine moduli space, we consider abelian varieties with a full level $A$. However, we suppress the level structure from the notation, because it plays only

We denote the supersingular locus in $X$ by $X(c)$ (denoted by $X(c)$ in [4]).

2.3. **Moduli spaces of abelian varieties.** Our main object of study is the moduli space $A_g$ of principally polarized abelian varieties of dimension $g$ over $k$. To get a fine moduli space, we consider abelian varieties with a full level $N$-structure for $N \geq 3$ and coprime to $p$, with respect to a fixed primitive $N$-th root of unity. However, we suppress the level structure from the notation, because it plays only a minor role.

On $A_g$ we have the *Ekedahl-Oort stratification*

$$A_g = \prod_{w \in W_{c, \text{final}}} EO_w.$$  

We denote the supersingular locus in $A_g$ by $S_g$.

2.4. **Moduli spaces with parahoric level structure.** We also study variants of $A_g$ with parahoric level structure. Fix a subset $J \subseteq I$, or in other words the type of a parahoric subgroup. We get a moduli space $A_J$ with parahoric level structure of type $J$ at $p$, which roughly speaking classifies chains $A_{j_0} \to A_{j_1} \to \cdots \to A_{j_r}$ of isogenies (of fixed $p$-power-degree) with $j_i \in J$. On each $A_j$ in the chain we have a polarization $\lambda_j$; if $j = 0$ or $j = g$, then this polarization is principal. We denote by $S_J$ the supersingular locus in $A_J$.

The *Kottwitz-Rapoport stratification* of $A_J$ is given as

$$A_J = \coprod_{x \in \text{Adm}_J(\mu)} A_{J,J},$$

where $\text{Adm}_J(\mu)$ is the image under $\tilde{W} \to W \backslash \tilde{W}/W$ of the admissible set $\text{Adm}(\mu)$ (the set of all elements less than some element in the $W$-orbit of $\nu$, where $\mu$ is the dominant minuscule coweight $(1^{(g)}, 0^{(g)})$). For $x \in \text{Adm}(\mu)$, we denote by $\pi$ its image in $\text{Adm}_J(\mu)$.

We have the natural projection $\pi_{J,J} : A_J \to A_{J,J}$, and for each $x \in \text{Adm}(\mu)$,

$$\pi_{J,J}^{-1}(A_{J,J}) = \coprod_{v \in W_J, xW_{J,J} \cap \text{Adm}(\mu)} A_{J,J,v},$$

We identify $A_g = A_{\{0\}}$, and write $\pi$ for the projection $A_J \to A_g$.

2.5. **Supersingular Kottwitz-Rapoport strata.** We summarize the results of [2]. Let $\tau$ be the minimal element of $\text{Adm}(\mu)$, and let $w$ be in $W_{\{c,g-c\}}$ so that $A_{J,J}$ is $c$-superspecial. Suppose that $S$ is a connected scheme over $k$ and $y = (A_0 \to \cdots \to A_g, \lambda_0, \lambda_g, \eta)$ is an $S$-valued point of $A_{J,J}$.

Now consider the projection to $A_{\{c,g-c\},J}$. The point $x = \pi_{\{c,g-c\},J}(y)$ is in $\pi_{\{c,g-c\},J}(A_{J,J})$. Because $A_{J,J}$ is 0-dimensional, there are trivializations $A_c = S \times_k$
A'_c$ and $A'_{g-c} = S \times_k A'_{g-c}$ with $A'_c$ and $A'_{g-c}$ both superspecial abelian varieties over $k$.

Let $\omega_i \in H^1_{\text{DR}}(A_i)$ be the Hodge filtration of $A_i$ and $\omega'_i$ that of $A'_i$. We get flags

$$0 \subseteq \alpha(\omega_{j_1-1}) \subseteq \cdots \subseteq \alpha(\omega_{j_0+1}) \subseteq \omega_{j_0}/\omega_{j_1} = \mathcal{O}_S \otimes (\omega'_{j_0}/\omega'_{j_1})$$

where $(j_0, j_1)$ is equal to $(-c,c), (c, g-c)$ or $(g-c, g+c)$ and we abusively write $\alpha$ for all the maps induced by $A_i \to A_{j_0}$. These flags define a point $\phi(y)$ of $\mathcal{B}(\overline{A}_e)$.

**Theorem 2.1.** For $w$ in $W_{(c,g-c)}$ the map $y \mapsto (\pi_{(c,g-c)}, t(y), \phi(y))$ is an isomorphism

$$\mathcal{A}_{I,w} \sim \pi_{(c,g-c),t}(\mathcal{A}_{I,\tau}) \times X_{\overline{A}_e}(w^{-1}, \sigma').$$

**Proof.** This is corollary 6.5 in [2]. Also see the proof of proposition 6.1 there for the definition of the morphism.

Since $\pi_{(c,g-c),t}(\mathcal{A}_{I,\tau})$ is a finite set of points, $\mathcal{A}_{I,w}$ is a finite disjoint union of Deligne-Lusztig varieties.

### 2.6. Supersingular Ekedahl-Oort strata

Now we summarize the results of [3]. In the case $g = 1$, which is more or less trivial from this point of view, one has to make some obvious modifications, so we exclude it from the discussion. Suppose that $w$ is in $W_{c,\text{final}} \subset W_{g,\text{final}}$ for some $c \leq g/2$. Let $S$ be a connected scheme over $k$ and $x = (A, \lambda)$ be an $S$-valued point of EO$_w$.

Fix a supersingular elliptic curve $E$. There is a unique isogeny

$$\rho: S \times E^g \to A$$

such that the kernel of the restriction to each geometric point is $\alpha'_p$ ([3] theorem 1.2). The pull-back of $\lambda$ gives a polarization $\mu$ on $E^g$. The pair $(E^g, \mu)$ is a point of $A_{g,c}$, the set of isomorphism classes of superspecial abelian varieties with a polarization with kernel $\alpha'^{2c}$ (with level structure).

Let $\omega(-)$ denote the Hodge filtration of an abelian variety. We get a subbundle

$$\alpha(\omega(A)) \subset \mathcal{O}_S \otimes (\omega(E^g)/\mu(\omega((E^g)^{\vee}))).$$

In fact this is a bundle of $c$-dimensional isotropic subspaces in a $2c$-dimensional symplectic vector space. Let $\psi(x)$ be the stabilizer of this subbundle in $\text{Sp}_{2c} \times S$. This is an $S$-valued point in a variety of parabolic subgroups of $\text{Sp}_{2c}$.

**Theorem 2.2.** Suppose that $w$ is in $W_{c,\text{final}}$ for some $c \leq g/2$ and $w \notin W_{c-1}$. Then the morphism $x \mapsto ((E^g, \mu), \psi(x))$ is an isomorphism

$$\text{EO}_w \sim A_{g,c} \times X'_{\text{Sp}_{2c}}(w^{-1}, \sigma^2).$$

**Proof.** This is theorem 1.2 in [3], except for a few superficial differences.

First of all in [3] lemma 3.1 the isomorphism is constructed with Dieudonné theory. If $M(A[p])$ is the Dieudonné module of $A[p]$, then there is a natural isomorphism $M(A[p]) \cong H^1_{\text{DR}}(A)$. The Verschiebung gives an isomorphism

$$\omega(E^g)/\mu(\omega((E^g)^{\vee})) \cong M(E^g[p]) / \mu((E^g)^{\vee}[p]) = M(\ker(\mu)).$$

Under this isomorphism the image $\omega(A)$ corresponds to the image of $M(A[p])$, to which $x$ is sent in [3]. So the two constructions are equivalent.

Secondly, we get $w^{-1}$ instead of $w$, because we use a different indexation. See [3] section 2.4 for the different indexations for the EO-stratifications. In [3] the
We start with the connected components. How this map relates to the above descriptions of supersingular KR- and EO-strata.

□

This action is trivial because of the level structures.

3. Comparison between supersingular EO strata and supersingular KR strata

Remember that for \( w \in W_{g, \text{final}} \) the projection \( \pi: \mathcal{A}_{g,I} \to \mathcal{A}_g \) restricts to a finite étale surjective map \( \mathcal{A}_{I,w} \to \mathcal{E}_w \), see [1], [3] section 9. In this section we show how this map relates to the above descriptions of supersingular KR- and EO-strata. We start with the connected components.

Lemma 3.1. The map \( \beta_c: \pi(c,g,c)(\mathcal{A}_{I,r}) \to \Lambda_{g,c} \) that sends a point \( ((A_c, \lambda_c) \to (A_{g-c}, \lambda_{g-c})) \) to \( (A^c, \lambda^c) \) is a bijection.

Proof. We construct an inverse as follows. Given \( (A, \lambda) \in \Lambda_{g,c} \) we put \( A_c = A^c', \ A_{g-c} = A^{(p)} \) factors through \( A^c \), and we get an isogeny \( A_c \to A_{g-c} \). This, together with the natural polarizations on \( A_c, A_{g-c} \), is the desired point in \( \pi(c,g,c)(\mathcal{A}_{I,r}) \).

Next we look at the index sets of the stratifications.

Lemma 3.2. We have:

\[
W_{g, \text{final}} \cap W_{l(c,g,-c)} = W_{c, \text{final}}.
\]

Proof. Since \( W_c \subset W_{l(c,g,-c)} \), the right hand side is included in the left hand side.

For the other inclusion, suppose that \( w \) is in \( W_{g, \text{final}} \cap W_{l(c,g,-c)} \). The set of simple reflections in a reduced expression of \( w \) is equal to \( \{ s_i, s_{i+1}, \ldots, s_{g-1}, s_g \} \) for some \( i \) (see the proof of lemma 7.1 in [3]). Since \( s_{g-c} \) is not in \( W_{l(c,g,-c)} \), we must have \( i > g - c \). Because \( W_c \subset W_g \) is generated by all \( s_j \) with \( j > g - c \), it contains \( w \).

Finally we look at the Deligne-Lusztig varieties. The group \( \overline{G}_c \) splits over \( k \) (in fact already over \( \mathbb{F}_p \)) as

\[
\overline{G}_c \cong \text{Sp}_{2c} \times \text{SL}_{g-2c} \times \text{Sp}_{2c}.
\]

as can be seen from the Dynkin diagram of \( \overline{G}_c \). It follows from the description of \( \sigma' \) in [2] that with respect to this decomposition

\[
\sigma'(g_1, g_2, g_3) = (\sigma(g_3), \overline{\sigma}(g_2), \sigma(g_1)),
\]

where \( \sigma \) is the Frobenius of \( \text{Sp}_{2c} \) and \( \overline{\sigma} \) is the Frobenius of a \( \text{SU}_{g-2c} \).

Anything related to \( \overline{G}_c \) splits in a similar way. For instance, its absolute Weyl group splits as

\[
W_{l(c,g,-c)} = W_c \times S_{g-2c} \times W_c,
\]

where \( S_{g-2c} \) is the symmetric group, which is the Weyl group of \( \text{SL}_{g-2c} \). The inclusion \( W_c \subset W_g \) we are using, is the inclusion of \( W_c \) in \( W_{l(c,g,-c)} \) as the last factor above, followed by \( W_{l(c,g,-c)} \subset W_g \).

Proposition 3.3. For \( w \in W_c \subset W_g \), the projection

\[
\mathcal{B}(\overline{G}_c) = \mathcal{B}(\text{Sp}_{2c}) \times \mathcal{B}(\text{SL}_{g-2c}) \times \mathcal{B}(\text{Sp}_{2c}) \to \mathcal{B}(\text{SL}_{g-2c}) \times \mathcal{B}(\text{Sp}_{2c})
\]
to the final two factors induces an isomorphism
\[ X_{w,c}^\tau (w, \sigma') \cong X_{St_{w-2c}}(1, \hat{\sigma}) \times X_{Sp_{2c}}(w, \sigma^2). \]

Proof. The Deligne-Lusztig variety \( X_{w,c}^\tau (w, \sigma') \) consists of triples \((g_1, g_2, g_3)\) such that
\[ \text{inv}(g_1, \sigma(g_3)) = 1, \text{inv}(g_2, \hat{\sigma}(g_2)) = 1, \text{inv}(g_3, \sigma(g_1)) = w'. \]
The first and last equations are equivalent to \( g_1 = \sigma(g_3) \) and \( \text{inv}(g_3, \sigma^2(g_3)) = w' \), which is the equation for \( X_{Sp_{2c}}(w, \sigma^2) \).

The proposition gives a morphism
\[ f : X_{w,c}^\tau (w, \sigma') \to X_{Sp_{2c}}(w, \sigma^2) \to X'_{Sp_{2c}}(w, \sigma^2). \]

Theorem 3.4. For \( w \in W_{c,\text{final}} \) there is a commutative diagram
\[
\begin{array}{ccc}
\mathcal{A}_{I,w} & \sim & \pi_{(c,g-c)}(A_{I,w}) \times X_{w,c}^\tau (w^{-1}, \sigma') \\
\pi & \downarrow & \sim \times f \\
EO_w & \sim & \Lambda_{g,c} \times X'_{Sp_{2c}}(w^{-1}, \sigma^2)
\end{array}
\]
where the horizontal maps are the isomorphisms from \([21]\) and \([22]\).

Proof. Suppose \((A_i)_{i \in I}\) is an \( S\)-valued point in \( \mathcal{A}_{I,w} \). Since the morphism \( \rho : S \times E^g \to A_0 \) from theorem \([22]\) is unique, it must be equal to \( A_{-c} \to A_0 \). (Here we extend the chain \((A_i)_{i}\) to a chain indexed by \( Z \) by duality and periodicity as usual.)

Under the horizontal map in the upper row, \((A_i)_{i}\) is mapped to the flags of the images of the Hodge filtrations \( \omega(A_i) \) inside \( \omega(A_c)/\omega(A_{-c}), \omega(A_{g-c})/\omega(A_c), \) and \( \omega(Sp_{2c})/\omega(g_{-c}) \). Because the stratum is \( c \)-superspecial, these quotients can actually be identified with quotients of the de Rham cohomology, or in other words of lattices in the standard lattice chain, however with a shift of the indices by \( g \). See the proof of Prop. 6.1 in \([2]\). This shows that when we identify the Weyl group of \( \mathcal{G}_c \) with \( W_{(c,g-c)} \) and decompose it as in \([20,22]\), then the subgroup generated by \( s_0, \ldots, s_{c-1} \) corresponds to the flags in \( \omega(A_{g+c})/\omega(A_{g-c}) \); the middle factor, i.e., the subgroup generated by \( s_{c+1}, \ldots, s_{g-c-1} \) corresponds to the flags in \( \omega(A_{g-c})/\omega(A_c) \); and finally the part generated by \( s_{g-c+1}, \ldots, s_g \) corresponds to flags in \( \omega(A_c)/\omega(A_{-c}) \). The projection considered in the previous proposition projects onto the latter two factors, and in particular, the map \( f \) defined takes the flag \( \omega(A_i) \) to \( \omega(A_0) \) which is the maximal totally isotropic member of the flag in the third factor. Since the map \( \pi \) which gives the left column maps our chain \((A_i)_{i}\) to \( A_0 \), and the lower horizontal map maps \( A_0 \) to \( \omega(A_0) \), we see that the diagram commutes.

\[ \square \]

4. Supersingular KR strata in the parahoric case

Let \( J \subseteq I \) be the type of a standard parahoric subgroup. We call a KR-stratum in \( \mathcal{A}_J \) supersingular, if it is contained in the supersingular locus. We want to describe these strata. As one would hope, there is a description completely analogous to the one in \([2]\), and it can in fact be derived from the Iwahori version with relatively little additional work.

We make the following definitions, analogous to the Iwahori case.

Definition 4.1. Let \( J \subseteq I \), and let \( c \in I \).
A stratum $\mathcal{A}_{J,x}$ is called $c$-superspecial, if $\pi_{J,1}^{-1}(\mathcal{A}_{J,x})$ is a union of $c$-superspecial KR strata in $\mathcal{A}_I$, i.e., if

$$W_JxW_J \cap \text{Adm}(\mu) \subseteq W_{\{c,g-c\}} \tau.$$  

(2) A stratum $\mathcal{A}_{J,x}$ is called superspecial, if it is superspecial for some $c \in I$.

Lemma 4.2. There exists a $c$-superspecial stratum in $\mathcal{A}_I$ if and only if $c$ and $g-c$ are in $J$.

Proof. If $c, g-c \in J$ then $W_J \tau W_J \subseteq W_{c,g-c} \tau$ which shows that the minimal KR stratum in $\mathcal{A}_I$ is $c$-superspecial. To show the converse, assume that $c \not\in J$ or $g-c \not\in J$. Because the union of all $c$-superspecial strata is closed, it is enough to show that the minimal stratum in $\mathcal{A}_I$ is not $c$-superspecial, i.e., that

$$W_J \tau W_J \cap \text{Adm}(\mu) \not\subseteq W_{\{c,g-c\}} \tau.$$  

By assumption we have $s_c \tau, s_{g-c} \tau = \tau s_c \in W_J \tau W_J$. The lemma follows, because for all simple reflections $s$ we have $s \tau \in \text{Adm}(\mu)$. This can be shown by checking that all these elements are permissible. Another (much more roundabout) way to prove this is to say that for every $c$, either $s_c \tau \leq t^\mu$ or $s_{g-c} \tau \leq t^\mu$, since clearly $\mathcal{A}_{I,\nu}$ is not superspecial. The claim follows because conjugating by $\tau$ preserves the Bruhat order, interchanges $s_c$ and $s_{g-c}$ and maps $t^\mu$ to $t^{w_0 \mu}$, where $w_0$ is the $W$-component of $\tau \in \bar{W} = X_s(T) \times W$. □

In view of (2.4.1), the lemma says that for $x \in \text{Adm}(\mu)$ and $c \not\in J$, we can find a point $(A_j)_{j \in J} \in \mathcal{A}_{J,\pi}$ which can be extended to a chain $(A_{j})_{j \in J}$ which does not lie in the $c$-superspecial locus, i.e., where we do not have $A_{c_j}$, $A_{g-c}$ superspecial and $A_z \rightarrow A_{g-c}$ the Frobenius.

We now describe $c$-superspecial KR-strata in terms of Deligne-Lusztig varieties. Suppose that $J$ is a subset of $I$ with $c, g-c \in J$ (so that $c$-superspecial strata exist in $\mathcal{A}_I$). Let $P_{\sigma'(J)}(G'_{c,k})$ be the variety of parabolic subgroups of type $I/\sigma'(J)$ in $G'_{c,k}$. Note that it is not defined over $\mathbb{F}_p$ if $J$ is not Frobenius invariant. There are coarse Deligne-Lusztig varieties

$$X_{G'_{c,k}}(\sigma'(J))(\overline{w}, \sigma') = \{ P \in P_{\sigma'(J)}(G'_{c,k}) | \text{inv}(P, \sigma'(P)) = \overline{w} \}$$

for all double cosets $\overline{w} \in W_{\sigma'(J)} \backslash W_{\{c,g-c\}}/W_J$.

Theorem 4.3. Let $\mathcal{A}_{J,\pi}$ be a $c$-superspecial stratum inside $\mathcal{A}_I$. Write $x = w \tau$, so that $w \in W_{\{c,g-c\}}$. There is an isomorphism

$$\mathcal{A}_{J,\pi} \cong \pi_{\{c,g-c\},1}(\mathcal{A}_{I,\tau}) \times X_{G'_{c,k}}(\sigma'(J))(\overline{w^{-1}}, \sigma').$$

Proof. Taking into account the remarks above, the proof is the same as in the Iwahori case, see [2], section 6. □

It is evident that every superspecial stratum is supersingular. We will show below that the converse is true, as well. We first prove a connectedness result in the parahoric case, analogous to [3], Thm. 7.3.

Proposition 4.4. If a KR-stratum is not superspecial, then it is irreducible.

The converse holds if the level structure is large enough.

Proof. Let $x \in \text{Adm}(\mu)$ and suppose $\mathcal{A}_{J,\pi}$ is not superspecial, i.e.

$$W_JxW_J \cap \text{Adm}(\mu) \not\subseteq W_{\{c,g-c\}} \tau,$$  

for all $c \in \{0, \ldots, [g/2] \}$.  

Then there exists for each \( c \in \{0, \ldots, \lfloor g/2 \rfloor \} \) an element in \( W_J x W_J \cap \text{Adm}(\mu) \) larger than \( s_{c, r} \) or than \( s_{g-c, r} \).

Therefore by \( \mathbb{K} \) Thm. 7.2 the closure of the union of all 1-dimensional strata in \( \pi_{J,I}^{-1}(A_J, \mathfrak{a}) \),

\[
\bigcup_{v \in W_J x W_J \cap \text{Adm}(\mu)} \bigcup_{s \leq v \tau^{-1} \ell(s)=1} A_{I,s}
\]

is connected. Now every connected component of \( \pi_{J,I}^{-1}(A_J, \mathfrak{a}) \) meets the previous set (because every connected component of a KR stratum in \( A_I \) has a point of the minimal stratum in its closure, \( \mathbb{K} \) Thm. 6.2). This implies that \( \pi_{J,I}^{-1}(A_J, \mathfrak{a}) \) is connected, but then its image \( A_J, \mathfrak{a} \) is connected, too. This closure is normal, so connectedness implies irreducibility, and the proposition follows. \( \square \)

**Theorem 4.5.** A KR-stratum is supersingular if and only if it is superspecial.

**Proof.** Suppose that \( A_J, \mathfrak{a} \) is contained in \( S_J \). Its image in \( A_g \) is a union of supersingular EO strata. We may assume that the level structure outside \( p \) is large enough, so that this union, and hence \( A_J, \mathfrak{a} \), is reducible. By the previous proposition, this implies the desired statement. \( \square \)

**Remark 4.6.** Let us discuss the combinatorial meaning of the theorem. Start with \( x \in \text{Adm}(\mu) \), such that \( A_{J,x} \subseteq S_J \). The latter condition is equivalent to

\[
\pi_{J,I}^{-1}(A_J, \mathfrak{a}) \subseteq S_J,
\]

or in other words to

\[
W_J x W_J \cap \text{Adm}(\mu) \subseteq \bigcup_{c=0}^{\lfloor g/2 \rfloor} W_{\{c, g-c\}}.
\]

By the theorem there exists \( c \in J \) such that

\[
W_J x W_J \cap \text{Adm}(\mu) \subseteq W_{\{c, g-c\}}
\]

It seems hard to prove this statement combinatorially because it is not easy to understand what happens when one intersects the double coset \( W_J x W_J \) with the admissible set.

5. **EO strata as parahoric KR strata**

In this section, we fix a final element \( w \in W_{g, \text{final}} \), and denote by \( J \subseteq I \) the type of the corresponding canonical filtration. Since the canonical filtration is a flag in \( \mathbb{H} \) (rather than a lattice chain), we have \( 0 \in J \) “automatically”. Furthermore, since the canonical filtration always contains a maximal totally isotropic subspace, we also have \( g \in J \). Furthermore, \( J \) is \( \sigma \)-stable.

**Example 5.1.** On the EO-stratum of abelian varieties with \( a \)-number 1 and \( p \)-rank \( f \), the type of the canonical filtration is \( \{0, f, f+1, \ldots, 2g-f-1, 2g-f, 2g\} \), see \( \mathbb{P} \) example 3.4. On the stratum of abelian varieties with \( a \)-number \( a \) and \( p \)-rank \( g-a \) the canonical type is \( \{0, g-a, g, g+a, 2g\} \).

**Example 5.2.** In the case \( g = 2 \), we have four final elements, corresponding to the superspecial locus (id), the supersingular locus without the superspecial points
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\((s_2)\), the \(p\)-rank 1 locus \((s_1s_2)\) and the \(p\)-rank 2 locus \((s_2s_1s_2)\). The corresponding canonical filtrations are given by

\[ J = \{0, 2\}, \{0, 1, 2\}, \{0, 1, 2\}, \{0, 2\}, \]

respectively.

In the next theorem, we view \(A_J\) as the moduli space of principally polarized abelian schemes \(A\) together with a flag \(F^*\) of finite flat subgroup schemes of \(A[p]\) of appropriate ranks, as determined by \(J\).

From the flag \(F^*\) we can make a conjugate flag \(F_c^*\) by

\[ F_c^{g+1} = V^{-1}(F_i^p) \quad \text{and} \quad F_c^{g-i} = (F_c^{g-i})^\perp \quad (i = 1, \ldots, g). \]

The stratum \(A_{J^{wτ}}\) consists of all points \((A, (F_j)_{j \in J})\) such that \(F^*\) and \(F_c^*\) (or rather their Dieudonné modules) are in relative position \(w\).

**Theorem 5.3.** The natural map \(π_J : A_J \to A_g\) restricts to an isomorphism \(A_{J^{wτ}} \to EO_w\). Its inverse maps a point \(A \in EO_w\) to \((A, F_{can}^* \subset A[p])\), where \(F_{can}^*\) denotes the canonical filtration of \(A[p]\).

**Proof.** Because over \(EO_w\) the canonical filtration can be constructed globally (see [7], Prop. (3.2)) we have a section \(s : EO_w \to A_J\) of \(π_J\). By definition its image is in \(A_{J^{wτ}}\).

To finish the proof, we show that for each point \((A, (F_j)_{j \in J}) \in A_{J^{wτ}}(k)\), \((F_j)\) is the canonical filtration of \(A[p]\). Let \(ν\) be the final type of \(w\). We know that for all points in the image of \(s\) we have \(V(F_j) = F_{ν(j)}\). But then this must hold for all points in \(A_{J^{wτ}}(k)\), since the relative position of \(V(F^*)\) and \(F^*\) is constant on \(A_{J^{wτ}}(k)\). This implies that \(F^*\) is the canonical filtration. \(\square\)

There is a commutative diagram

\[
\begin{array}{ccc}
A_{J^{wτ}} & \xrightarrow{π_J} & A_{J^{wτ}} \\
π \downarrow & & \downarrow \cong \\
EO_w & \xrightarrow{wτ} & EO_w
\end{array}
\]

So for \(w\) final and \(J\) as above the horizontal map is surjective. Since all KR strata in \(π_{1/J}^{-1}(A_{J^{wτ}})\) map (necessarily surjectively) to \(EO_w\), \(wτ\) is the unique element of minimal length in \(W_{JwτW_J}\). So we are in a very special situation.

**Remark 5.4.** In case \(EO_w\) is supersingular, i.e., contained in \(S_g\), we have theorems [2.2] and [4.3] at our disposal, and the isomorphism above corresponds to the identification of fine Deligne-Lusztig varieties with coarse Deligne-Lusztig varieties for a different parabolic subgroup, see [3] Cor. 2.7. Also, from [2] theorem 6.3 we know that the fibres of \(π\) and of the horizontal map in the diagram are Deligne-Lusztig varieties.

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