NON-SELF-ADJOINT RESOLUTIONS OF THE IDENTITY
AND ASSOCIATED OPERATORS

ATSUSHI INOUE AND CAMILLO TRAPANI

ABSTRACT. Closed operators in Hilbert space defined by a non-self-
adjoint resolution of the identity \( \{X(\lambda)\}_{\lambda \in \mathbb{R}} \), whose adjoints constitute
also a resolution of the identity, are studied. In particular, it is shown
that a closed operator \( B \) has a spectral representation analogous to the
familiar one for self-adjoint operators if and only if \( B = TAT^{-1} \) where
\( A \) is self-adjoint and \( T \) is a bounded operator with bounded inverse.

1. Introduction

In recent years there has been an increasing interest on non-self-adjoint
operators with real spectrum, because of the important role they play in the
so-called pseudo-hermitian quantum mechanics, an unconventional approach
to this branch of physics, based on the use of non-self-adjoint Hamiltonians
\[2, 4\]. Often self-adjointness can be restored by changing the environment:
in fact, if a closed operator \( H' \) can be expressed as \( H' = THT^{-1} \), where
\( H \) is self-adjoint and \( T \) is a bounded operator with bounded inverse then
\( H \) and \( H' \) have the same spectrum. If this condition is satisfied, \( H' \) and \( H \)
are said to be similar. This relation can be interpreted as the possibility
of defining a (possibly, indefinite) inner product which makes \( H' \) into a
self-adjoint operator with respect to the new metric. In this case, if \( H = \int_{\mathbb{R}} \lambda dE(\lambda) \) is the spectral representation of \( H \), then \( H' = \int_{\mathbb{R}} \lambda dX(\lambda) \) where
\( X(\lambda) = TE(\lambda)T^{-1} \). The family \( \{X(\lambda)\}_{\lambda \in \mathbb{R}} \) obtained in this way behaves
under many respects in analogous way to an ordinary spectral family with
the crucial difference that its elements are non-self-adjoint projections (this
means that, for every \( \lambda \in \mathbb{R} \), \( X(\lambda)^2 = X(\lambda) \) but \( X(\lambda)^* \neq X(\lambda) \), in general).

This observation is the starting point of this paper. In fact, we will con-
ider, as suggested by the previous example, a non-self-adjoint resolution
of the identity \( \{X(\lambda)\}_{\lambda \in \mathbb{R}} \) enjoying prescribed regularity properties (mono-
tonicity, uniform boundedness, etc.); in particular we will focus our attention
to the case where \( \{X(\lambda)^*\}_{\lambda \in \mathbb{R}} \) is a resolution of the identity too (we speak
in this case of a *-resolution of the identity) and study closed operators
that are associated to it. To be clearer, let us consider an ordinary spectral
family \( \{E(\lambda)\}_{\lambda \in \mathbb{R}} \) consisting of self-adjoint (or orthogonal) projections in
Hilbert space \( \mathcal{H} \). Then, as it is well-known, this family defines uniquely a
self-adjoint operator $A$ whose domain
\[ D(A) = \left\{ \xi \in \mathcal{H} : \int_{\mathbb{R}} \lambda^2 d \langle E(\lambda)\xi | \xi \rangle < \infty \right\} \]
can be expressed in several equivalent ways due to the equalities
\[ \langle E(\lambda)\xi | \xi \rangle = \| E(\lambda)\xi \|^2 = \left( (E(\lambda)^* E(\lambda))^{1/2} \xi | \xi \right), \quad \xi \in \mathcal{H}. \]
These equalities do not hold, in general, if we remove the assumption that each $E(\lambda)$ is self-adjoint, so that the corresponding spectral integrals may produce different operators and these operators are the main object of this paper. The main result of the paper consists in showing that a closed operator $B$ can be expressed as
\[ B = \int_{\mathbb{R}} \lambda dX(\lambda) \]
if, and only if it is similar to a self-adjoint operator $A$; i.e. $B = TAT^{-1}$ where $A$ is self-adjoint and $T$ is a bounded operator with bounded inverse.

2. Preliminaries

In this section we collect some definitions and facts concerning quasi-similarity and similarity for possibly unbounded linear operators [1].

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, $D(A)$ and $D(B)$ dense subspaces, respectively of $\mathcal{H}$ and $\mathcal{K}$; $A : D(A) \to \mathcal{H}, B : D(B) \to \mathcal{K}$ two linear operators. A bounded operator $T : \mathcal{H} \to \mathcal{K}$ is called an intertwining operator for $A$ and $B$ if
\begin{enumerate}
  \item[(i)] $T : D(A) \to D(B)$;
  \item[(ii)] $BT\xi = TA\xi$, $\forall \xi \in D(A)$.
\end{enumerate}

**Definition 2.1.** Let $A$ and $B$ be two linear operators in the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively.

We say that $A$ and $B$ are quasi-similar, and write $A \dashv B$, if there exists an intertwining operator $T$ for $A$ and $B$ which is invertible, with inverse $T^{-1}$ densely defined.

The operators $A$ and $B$ are said to be similar, and write $A \sim B$, if they are quasi similar and the inverse $T^{-1}$ of the intertwining operator $T$ intertwines $B$ and $A$.

**Remark 2.2.** We notice that $\sim$ is an equivalence relation. Moreover, if $A \sim B$, then $TD(A) = D(B)$.

If $A \dashv B$ (respectively, $A \sim B$) with intertwining operator $T$, then, $B^* \dashv A^*$ (respectively, $B^* \sim A^*$), with intertwining operator $T^*$.

**Remark 2.3.** Let $A$ and $B$ be linear operators in $\mathcal{H}$ and $\mathcal{K}$, respectively, with $A \sim B$. The following properties of similar operators are easily proved.
\begin{enumerate}
  \item[(i)] $A$ is closed if, and only if, $B$ is closed.
  \item[(ii)] $A^{-1}$ exists if, and only if, $B^{-1}$ exists. Moreover, $B^{-1} \sim A^{-1}$.
\end{enumerate}
If $A$ is a closed operator, we denote, as usual, by $\sigma(A)$ its spectrum. The parts in which the spectrum is traditionally decomposed, the point spectrum, the continuous spectrum and the residual spectrum, are denoted respectively by $\sigma_p(A)$, $\sigma_c(A)$, $\sigma_r(A)$.

**Proposition 2.4.** Let $A$, $B$ be closed operators. Assume that $A \sim B$ and let $T$ be the corresponding intertwining operator. Then the spectra $\sigma(A)$ and $\sigma(B)$ coincide and

$$\sigma_p(A) = \sigma_p(B), \quad \sigma_c(A) = \sigma_c(B), \quad \sigma_r(A) = \sigma_r(B).$$

Moreover, if $\lambda \in \sigma_p(A)$, the multiplicity $m_A(\lambda)$ of $\lambda$ as eigenvalue of $A$ is the same of its multiplicity $m_B(\lambda)$ as eigenvalue of $B$.

The situation for quasi-similarity is more involved and it has been described in [1]. We summarize in the next proposition the main results.

**Proposition 2.5.** Let $A$, $B$ be closed operators. Assume that $A \dashv B$ with intertwining operator $T$. Then the following statements hold.

---

| Statement (sp.) | Condition | Description |
|-----------------|-----------|-------------|
| (sp.1) $\sigma_p(A) \subseteq \sigma_p(B)$ | $T^{-1}$ bounded and $T(D(A))$ is a core for $B$ | $\sigma_p(B) \subseteq \sigma(A)$ |
| (sp.2) $\sigma_p(A) \subseteq \sigma_p(B) \subseteq \sigma(B) \subseteq \sigma(A)$ | $T^{-1}$ everywhere defined and bounded and $TD(A)$ is a core for $B$. |

**Remark 2.6.** Suppose, for instance, that $A$ is self-adjoint, then any operator $B$ which is quasi-similar to $A$ by means on an intertwining operator $T$ whose inverse is bounded too, has real spectrum and, if $A$ has a pure point spectrum, then $B$ is isospectral to $A$.

3. Non-self-adjoint resolutions of the identity

To begin with, we fix some terminology. Let $\mathcal{H}$ be a Hilbert space. A bounded operator $X$ will be called a projection if $X^2 = X$ and a self-adjoint (or orthogonal) projection if $X = X^2 = X^*$. If $X$ is a nonzero projection, then $\|X\| \geq 1$, while if it is self-adjoint $\|X\| = 1$.

**Example 3.1.** Let us consider two biorthogonal Schauder bases $\Phi = \{\varphi_n, n \in \mathbb{N}\}$ and $\Psi = \{\psi_n, n \in \mathbb{N}\}$ of the Hilbert space $\mathcal{H}$, $\langle \varphi_i | \psi_j \rangle = \delta_{i,j}$ and let us consider an operator of the form

$$S = \sum_{k=1}^{\infty} \alpha_k (\psi_k \otimes \overline{\varphi_k})$$

with $\alpha_k \in \mathbb{C}$, $k \in \mathbb{N}$. The domain of $S$ is the following subspace of $\mathcal{H}$:

$$D(S) = \left\{ \xi \in \mathcal{H} : \lim_{n \to \infty} \left\| \sum_{k=n+1}^{n+p} \alpha_k \langle \xi | \varphi_k \rangle \psi_k \right\| = 0, \forall p \in \mathbb{N} \right\}.$$
This domain is dense, since it contains the vectors \( \psi_k, k \in \mathbb{N} \).

It is easy to see that every \( \alpha_k \) is an eigenvalue of \( S \) with eigenvector \( \phi_k \).

The spectrum \( \sigma(S) \) of \( S \) is the set \( \{ \alpha_k, k \in \mathbb{N} \} \). In particular, \( \sigma_p(S) = \{ \alpha_k, k \in \mathbb{N} \} \) and every limit point of \( \sigma_p(S) \), if any, lies in the continuous spectrum \( \sigma_c(S) \) of \( S \).

To simplify notations, we put \( R_k = \psi_k \otimes \overline{\phi_k} \). This family of rank one operators enjoys the following easy properties:

1. \( \| R_k \| \leq \| \phi_k \| \| \psi_k \| \);
2. \( R_k^* = R_k \); and \( R_k R_m = 0 \) if \( m \neq k \).

In particular, (iii) implies that \( R_k \) is a non-self-adjoint projection (unless \( \phi_k = \psi_k \)). Moreover, since \( \Psi = \{ \psi_n, n \in \mathbb{N} \} \) is a Schauder basis, one gets

\[
\xi = \sum_{k=1}^{\infty} R_k \xi, \quad \forall \xi \in \mathcal{H}.
\]

Thus, the family \( \{ R_k \} \) enjoys the property

\[
\sup_{n \in \mathbb{N}} \left\| \sum_{k=1}^{n} R_k \right\| < \infty.
\]

Let us now assume that the spectrum \( \sigma(S) \) of \( S \) is real. Then, we can define, for \( \lambda \in \mathbb{R} \) and \( \xi \in \mathcal{H} \),

\[
X(\lambda) \xi = \sum_{k \leq \lambda} R_k \xi.
\]

Then we can formally write

\[
S \xi = \int_{\mathbb{R}} \lambda dX(\lambda) \xi.
\]

Let us suppose that \( \{ \varphi_k \} \) and \( \{ \psi_k \} \) are biorthogonal Riesz bases. This means that there exists a symmetric bounded operator \( G \) with bounded inverse \( G^{-1} \) and an orthonormal basis \( \{ \chi_n \} \) such that \( \varphi_k = G^{-1} \chi_k \) and \( \psi_k = G \chi_k \), for every \( k \in \mathbb{N} \). Then, we get

\[
(\psi_k \otimes \overline{\varphi_k}) \xi = \langle \xi | \varphi_k \rangle \psi_k = \langle \xi | G^{-1} \chi_k \rangle G \chi_k = \langle G^{-1} \xi | \chi_k \rangle G \chi_k.
\]

Hence \( \psi_k \otimes \overline{\varphi_k} = G(\chi_k \otimes \overline{\chi_k})G^{-1} \).

Then, it is easily seen that the family of operators \( \{ X(\lambda) \} \lambda \in \mathbb{R} \) enjoys the properties \( q_1 \)-(\( q_4 \)) listed in Definition 3.2 below.

We remark that in finite dimensional spaces every family of projections whose sum is the identity operator is similar to a family of orthogonal projections; so that the situation discussed above is the more general possible.

For the infinite dimensional case, an analogous statement was obtained by Mackey [3, Theorem 55]: every non-self-adjoint resolution of the identity is similar to a self-adjoint resolution of the identity (Mackey’s terminology is different: a non-self-adjoint resolution of the identity is a countably additive spectral measure on the Borel sets of the plane or of the real line); the...
resolution of the identity \( \{X(\lambda)\} \) of the next Definition \[3.2\] need not define a countably additive spectral measure on the Borel sets.

**Definition 3.2.** Let \( \mathcal{H} \) be a Hilbert space. A resolution of the identity of \( \mathcal{H} \) on the interval \( I := [\alpha, \beta], \) \( (-\infty \leq \alpha < \beta \leq +\infty) \) is a one parameter family of (non necessarily self-adjoint) bounded operators \( \{X(\lambda)\}_{\lambda \in I} \) satisfying the following conditions

\[
(qs_1) \quad \sup_{\lambda \in I} \|X(\lambda)\| := \gamma(X) < +\infty;

(qs_2) \quad X(\lambda)X(\mu) = X(\mu)X(\lambda) = X(\lambda)\text{ if } \lambda < \mu;

(qs_3) \quad \lim_{\lambda \to \alpha} X(\lambda)\xi = 0; \quad \lim_{\lambda \to \beta} X(\lambda)\xi = \xi, \quad \forall \xi \in \mathcal{H};

(qs_4) \quad \lim_{\epsilon \to 0^+} X(\lambda + \epsilon)\xi = X(\lambda)\xi, \quad \forall \lambda \in I; \quad \forall \xi \in \mathcal{H}.
\]

If the limits in (qs_3) and (qs_4) hold with respect to the weak topology only, then we say that \( \{X(\lambda)\}_{\lambda \in I} \) is a weak resolution of the identity.

If \( X(\lambda)^* = X(\lambda), \) for every \( \lambda \in I, \) we say that the resolution of the identity is self-adjoint.

**Remark 3.3.** Since the \( X(\lambda)'s \) are projections, \( \|X(\lambda)\| \geq 1, \) for every \( \lambda \in I. \) Hence \( \gamma(X) \geq 1. \)

**Proposition 3.4.** If \( \{X(\lambda)\}_{\lambda \in \mathbb{R}} \) is a resolution of the identity, \( \{X(\lambda)^*\}_{\lambda \in \mathbb{R}} \) is a weak resolution of the identity.

**Remark 3.5.** From (qs_2) and (qs_4) it follows that \( X(\lambda)^2 = X(\lambda), \) for every \( \lambda \in \mathbb{R}. \) Thus every \( X(\lambda) \) is a projection, but not an orthogonal projection, in general. If also \( X(\lambda)^* = X(\lambda), \) for every \( \lambda \in \mathbb{R}, \) then \( \{X(\lambda)\}_{\lambda \in \mathbb{R}} \) is a spectral family in the usual sense. In this case, weak limits, automatically become strong limits so every weak self-adjoint resolution of the identity is a resolution of the identity.

**Remark 3.6.** If \( \{X(\lambda)\}_{\lambda \in \mathbb{R}} \) is a self-adjoint resolution of the identity, then condition (qs_2) can be replaced with the following equivalent one:

\[
(qs_2') \quad X(\lambda) \leq X(\mu), \quad \lambda, \mu \in I, \quad \lambda < \mu.
\]

**Definition 3.7.** If both \( \{X(\lambda)\}_{\lambda \in I} \) and \( \{X(\lambda)^*\}_{\lambda \in I} \) are resolutions of the identity, we simply say that \( \{X(\lambda)\}_{\lambda \in I} \) is a *-resolution of the identity.

For reader’s convenience, we recall the definition of generalized resolution of the identity due to Naimark, \[5\] Appendix; \[6\] Vol.II, Appendix.

**Definition 3.8.** A *generalized resolution of the identity* is a one parameter family of bounded symmetric operators \( \{B(\lambda)\}_{\lambda \in I}, \) where \( I := [\alpha, \beta] \) is a bounded or unbounded interval of the real line, satisfying the following conditions

\[
(gr_i) \quad \lim_{\lambda \to \alpha} B(\lambda)\xi = 0; \quad \lim_{\lambda \to \beta} B(\lambda)\xi = \xi, \quad \forall \xi \in \mathcal{H};

(gr_2) \quad \lim_{\epsilon \to 0^+} B(\lambda + \epsilon)\xi = B(\lambda)\xi, \quad \forall \lambda \in I; \quad \forall \xi \in \mathcal{H};

(gr_3) \quad B(\lambda) \leq B(\mu), \text{ if } \lambda, \mu \in I, \lambda < \mu.
\]
Remark 3.9. The operators of a generalized resolution of the identity are self-adjoint but not necessarily idempotent, while in Definition 3.2 we require exactly the opposite.

Example 3.10. Let \( \{ E(\lambda) \}_{\lambda \in \mathbb{R}} \) be a self-adjoint resolution of the identity and \( G \) a bounded symmetric operator with bounded inverse, with \( G^2 \neq I \). Put \( X(\lambda) := GE(\lambda)G^{-1} \), for every \( \lambda \in \mathbb{R} \). Then \( \{ X(\lambda) \}_{\lambda \in \mathbb{R}} \) is a resolution of the identity in the sense of Definition 3.2. In particular, \( 1 \leq \| X(\lambda) \| \leq \| G \| \| G^{-1} \| \), for every \( \lambda \in \mathbb{R} \). Let us prove, for instance, the second equality in \((qs3)\). We have,
\[
\| GE(\lambda)G^{-1} \xi - \xi \| = \| GE(\lambda)G^{-1} \xi - GG^{-1} \xi \|
\leq \| G \| \| E(\lambda)G^{-1} \xi - G^{-1} \xi \| \to 0 \text{ as } \lambda \to +\infty.
\]
It is easily seen that in this case \( \{ X(\lambda)^* \}_{\lambda \in \mathbb{R}} \) is a (non-self-adjoint) resolution of the identity too.

If \( G^2 = I \) then every \( X(\lambda) \) is symmetric and so it is a self-adjoint resolution of the identity.

From now on, we will only consider the case
\[
\gamma(X) = \sup_{\lambda \in I} \| X(\lambda) \| = 1.
\]
This assumption does not imply that the \( X(\lambda) \)'s are self-adjoint projections. For instance, in the case considered in Example 3.10 one can easily find examples of operators \( G \) satisfying \( \| G \| \| G^{-1} \| = 1 \).

Lemma 3.11. Let \( \{ X(\lambda) \}_{\lambda \in \mathbb{R}} \) be a resolution of the identity. If \( \lambda < \mu \) then
\[
\| X(\lambda) \xi \| \leq \| X(\mu) \xi \|, \quad \forall \xi \in \mathcal{H}.
\]
Hence, the nonnegative valued function \( \lambda \mapsto \| X(\lambda) \xi \| \) is increasing, for every \( \xi \in \mathcal{H} \).

Proof. Indeed, if \( \lambda \leq \mu \),
\[
\| X(\lambda) \xi \| = \| X(\lambda)X(\mu) \xi \| \leq \| X(\lambda) \| \| X(\mu) \| \| \xi \| \| X(\lambda) \xi \| \leq \gamma(X) \| X(\mu) \xi \| = \| X(\mu) \xi \|.
\]

3.1. Operators associated to a resolution of the identity. A resolution of the identity \( \{ X(\lambda) \} \) defines an operator valued function \( \lambda \mapsto F(\lambda) \), \( \lambda \in \mathbb{R} \), where \( F(\lambda) \) is the positive operator
\[
F(\lambda) = X(\lambda)^*X(\lambda), \quad \lambda \in \mathbb{R}.
\]
Of course, \( \langle F(\lambda) \xi, \xi \rangle = \| X(\lambda) \xi \|^2 \) for every \( \lambda \in \mathbb{R}, \xi \in \mathcal{H} \).

Lemma 3.12. Let \( \{ X(\lambda) \} \) be a \( * \)-resolution of the identity. Then, the operator valued function \( \lambda \mapsto F(\lambda) \) has the following properties:
\[\begin{align*}
(f_{s1}) & \sup_{\lambda \in \mathbb{R}} \| F(\lambda) \| = 1; \\
(f_{s2}) & F(\lambda) \leq F(\mu) \text{ if } \lambda < \mu;
\end{align*}\]
Lemma 3.13. The following properties hold:

\( \left( f_{33} \right) \lim_{\lambda \to -\infty} F(\lambda)\xi = 0; \lim_{\lambda \to +\infty} F(\lambda)\xi = \xi, \quad \forall \xi \in \mathcal{H}; \)
\( \left( f_{34} \right) \lim_{\epsilon \to 0^+} F(\lambda + \epsilon)\xi = F(\lambda)\xi, \quad \forall \lambda \in \mathbb{R}; \forall \xi \in \mathcal{H}. \)

Hence, \( \{F(\lambda)\}_{\lambda \in \mathbb{R}} \) is a generalized resolution of the identity in the sense of Definition 3.13.

Proof. \( (f_{31}) \) is an easy consequence of the \( C^* \)-property.

\( (f_{32}) \): Using Lemma 3.11 we have
\[ \langle F(\lambda)|\xi \rangle = \|X(\lambda)|\xi \|^2 \leq \|X(\mu)|\xi \|^2 = \langle F(\mu)|\xi \rangle. \]

\( (f_{33}) \) follows from the inequalities
\[ 0 \leq \|F(\lambda)|\xi \| = \|X(\lambda)^*X(\lambda)|\xi \| \leq \|X(\lambda)^*\|\|X(\lambda)|\xi \| \to 0 \text{ as } \lambda \to -\infty \]
and
\[ \|F(\lambda)|\xi - \xi \| = \|X(\lambda)^*X(\lambda)|\xi - \xi \|
\[ = \|X(\lambda)^*X(\lambda)|\xi - X(\lambda)|\xi + X(\lambda)|\xi - X(\lambda)|\xi - X(\lambda)|\xi - \xi \|
\[ \leq \|X(\lambda)^*\|\|X(\lambda)|\xi - \xi \| + \|X(\lambda)^* - X(\lambda)|\xi \| + \|X(\lambda)|\xi - \xi \| \to 0 \text{ as } \lambda \to +\infty \]

since, by the assumption \( \lim_{\lambda \to +\infty} X(\lambda)^*|\xi \| = \lim_{\lambda \to +\infty} X(\lambda)|\xi \| = \xi, \) for every \( \xi \in \mathcal{H}. \)

\( (f_{34}) \): In similar way, since
\[ \|F(\lambda + \epsilon)|\xi - F(\lambda)|\xi \| = \|X(\lambda + \epsilon)^*X(\lambda + \epsilon)|\xi - X(\lambda)^*X(\lambda)|\xi \|
\[ \leq \|X(\lambda + \epsilon)^*\|\|X(\lambda + \epsilon) - X(\lambda)|\xi \| + \|X(\lambda + \epsilon)^*X(\lambda)|\xi - X(\lambda)^*X(\lambda)|\xi \|, \]
both terms go to 0 by the assumption as \( \epsilon \to 0^+. \)

So under the assumptions of Lemma 3.12 one can define, in standard fashion, a positive operator valued measure on the Borel sets of the line: one begins with considering a bounded interval of the form \( \Delta = ]\lambda, \mu[ \) and defines
\[ F(\Delta) = F(\mu) - F(\lambda). \]

the measure of the closed interval \( [\lambda, \mu[ \) is then defined by \( F([\lambda, \mu[) = F(\{\lambda\}) + F(\Delta), \) the measure of the singleton \( \{\lambda\} \) being defined as
\[ F(\{\lambda\}) = \lim_{\epsilon \to 0^+} F([\lambda - \epsilon, \lambda[) \]
and then one extends the measure \( F(\cdot) \) to arbitrary Borel sets.

Lemma 3.13. The following properties hold:

\( (f_1) \) \( X(\mu)^*F(\lambda)X(\mu) = F(\lambda), \) if \( \lambda \leq \mu; \)
\( (f_2) \) \( F(\lambda) = X(\mu)^*F(\lambda), \) \( F(\lambda) = F(\lambda)X(\mu), \) if \( \lambda \leq \mu; \)
\( (f_2) \) If \( a < b, \)
\[ (X(b)^*-X(a)^*)F(\lambda)(X(b) - X(a)) \]
\[ = \begin{cases} 0 & \text{if } \lambda < a < b \\ (X(\lambda)^*-X(a)^*)(X(\lambda) - X(a)) & \text{if } a < \lambda < b \\ (X(b)^*-X(a)^*)(X(b) - X(a)) & \text{if } a < b < \lambda. \end{cases} \]
These properties come almost immediately from the definition and from (qs1).

If \( \{X(\lambda)\} \) is a *-resolution of the identity, then the family \( \{F(\lambda)\} \) defined above is a generalized spectral family in the sense of Naimark [5, Appendix]. Hence, there exists a self-adjoint resolution of the identity \( H \) in a possibly larger Hilbert space \( \mathcal{H} \), containing \( \mathcal{H} \) as a closed subspace, such that \( F(\lambda) = P \mathbf{E}(\lambda) \upharpoonright \mathcal{H} \), where \( P \) is the projection of \( \mathcal{H} \) onto \( \mathcal{H} \) and by requiring that \( \mathcal{H} \) is spanned by the vectors of the form \( E(\lambda)\xi, \xi \in \mathcal{H} \), then \( \mathcal{H} \) is (essentially) unique.

Let \( A \) be the self-adjoint operator, with dense domain \( PD(A) \) in \( \mathcal{H} \) whose spectral family is \( \{E(\lambda)\} \). Then the operator \( T_X = PA \upharpoonright \mathcal{H} \) on the dense domain \( D(T_X) = PD(A) \) of \( \mathcal{H} \) is closed and symmetric.

Then one easily proves that

**Theorem 3.14.** Assume that \( \{X(\lambda)\} \) is a *-resolution of the identity and let \( F(\cdot) = X(\cdot)^*X(\cdot) \) be the positive operator valued function defined above.

Set

\[
D(T_X) = \left\{ \xi \in \mathcal{H} : \int_{\mathbb{R}} \lambda^2 \langle F(\lambda)\xi | \xi \rangle < \infty \right\}.
\]

Then, \( D(T_X) \) is dense in \( \mathcal{H} \) and there exists a unique closed symmetric operator \( T_X \), defined on \( D(T_X) \) such that

\[
\langle Tx \xi | \eta \rangle = \int_{\mathbb{R}} \lambda d \langle F(\lambda)\xi | \eta \rangle, \quad \forall \xi \in D(T_X), \eta \in \mathcal{H}.
\]

**Remark 3.15.** We remark that one can prove that \( D(T_X) \) is dense in \( \mathcal{H} \) directly. Indeed, if \( \xi \in \mathcal{H} \), we put \( \xi_n = (X(n) - X(-n))\xi, n \in \mathbb{N} \). From (qs3) it follows that \( \|\xi - \xi_n\| \to 0 \). It remains to prove that \( \xi_n \in D(T_X) \). Taking into account the properties given in Lemma 3.13 one has

\[
\int_{\mathbb{R}} \lambda^2 d \langle F(\lambda)\xi_n | \xi_n \rangle = \int_{\mathbb{R}} \lambda^2 d \langle F(\lambda)(X(n) - X(-n))\xi | (X(n) - X(-n))\xi \rangle
\]

\[
= \int_{\mathbb{R}} \lambda^2 d \langle (X(n)^* - X(-n)^*)F(\lambda)(X(n) - X(-n))\xi | \xi \rangle
\]

\[
= \int_{-n}^n \lambda^2 d\|X(\lambda) - X(-n)\|\xi\|)^2
\]

\[
\leq 2 \int_{-n}^n \lambda^2 d\|X(\lambda)\xi\|^2
\]

\[
\leq 2n^2 \int_{\mathbb{R}} d\|X(\lambda)\xi\|^2 = 2n^2\|\xi\|^2 < \infty.
\]

Of course, since we have supposed a full symmetry of \( \{X(\lambda)\} \) and \( \{X(\lambda)^*\} \), we can also define \( F_*(\lambda) := X(\lambda)X(\lambda)^* \), \( \lambda \in \mathbb{R} \) and apply the previous statements to the family \( \{F_*(\lambda)\} \). Then, by Theorem 3.14 one defines a second closed symmetric operator \( T_X^* \), related to the resolution of the identity \( \{X(\lambda)\} \). But there is more.
Assume, in fact, that the resolution of the identity \( \{X(\lambda)\} \) is of bounded variation, by which we mean that, for every \( \xi \in \mathcal{H} \) the complex valued function \( \lambda \in \mathbb{R} \rightarrow \langle X(\lambda)\xi | \xi \rangle \) is of bounded variation on the line. Then there exists a complex Borel measure \( \mu_\xi \) on \( \mathbb{R} \) such that 
\[
\langle X(\lambda)\xi | \xi \rangle = \mu_\xi((-\infty, \lambda)), \quad \lambda \in \mathbb{R}.
\]
By the elementary properties of measures, it follows that 
\[
\mu_\xi((\lambda, \mu)) = \langle X(\mu)\xi - X(\lambda)\xi | \xi \rangle, \quad \lambda < \mu.
\]
It is clear that if \( \{X(\lambda)\} \) is of bounded variation so it is also \( \{X(\lambda)^*\} \), and the corresponding measure is nothing but the complex conjugate of \( \mu_\xi \).

**Proposition 3.16.** Let \( \{X(\lambda)\} \) be a \(*\)-resolution of the identity. Then the function \( \lambda \rightarrow X(\lambda) \) is of bounded variation.

**Proof.** Let \( \lambda_0, \ldots, \lambda_n \) be a finite set of points with \(-\infty < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda \). We shorten \( \omega = (-\infty, \lambda] \), \( \omega_k = [\lambda_k - \lambda_{k-1}] \). Then we have
\[
\sum_{k=1}^n |\langle X(\omega_k)\xi | \eta \rangle| = \sum_{k=1}^n |\langle X(\omega_k)^2\xi | \eta \rangle| \\
= \sum_{k=1}^n |\langle X(\omega_k)\xi | X(\omega_k)^*\eta \rangle| \\
\leq \sum_{k=1}^n \|X(\omega_k)\xi\| \|X(\omega_k)^*\eta\| \\
= \sum_{k=1}^n \langle F(\omega_k)\xi | \xi \rangle^{1/2} \langle F^*_k(\omega_k)\eta | \eta \rangle^{1/2} \\
\leq \left( \sum_{k=1}^n \langle F(\omega_k)\xi | \xi \rangle \right)^{1/2} \left( \sum_{k=1}^n \langle F^*_k(\omega_k)\eta | \eta \rangle \right)^{1/2} \\
= \langle F(\omega)\xi | \xi \rangle^{1/2} \langle F^*_k(\omega)\eta | \eta \rangle^{1/2}.
\]
Hence the supremum over all possible decompositions of \( \omega \) is finite and so is the limit when \( \lambda \rightarrow +\infty \).

In this case, the non-self-adjoint resolution of the identity \( \{X(\lambda)\} \) defines a countably additive measure \( X(\omega) \), on the Borel sets \( \omega \) of the line. By the quoted result of Mackey [3, Theorem 55] one gets the following statement.

**Lemma 3.17.** Let \( \{X(\lambda)\} \) be a \(*\)-resolution of the identity. Then there exists a self-adjoint resolution of the identity \( \{E(\lambda)\} \) and an invertible operator \( T \) with bounded inverse such that
\[
X(\lambda) = TE(\lambda)T^{-1}, \quad \forall \lambda \in \mathbb{R}.
\]
Let \( \{X(\lambda)\} \) be a \(*\)-resolution of the identity. By Proposition 3.16 we can define the integral
\[
\int_{\mathbb{R}} \lambda dX(\lambda).
\]

Let \( \mathcal{D} \) denote the set of all \( \xi \in \mathcal{H} \) such that the integral \( \int_{\mathbb{R}} \lambda d \langle X(\lambda)\xi | \eta \rangle \) exists, for every \( \eta \in \mathcal{H} \), and the conjugate linear functional
\[
\Theta_\xi(\eta) := \int_{\mathbb{R}} \lambda d \langle X(\lambda)\xi | \eta \rangle, \quad \eta \in \mathcal{H}
\]
is bounded on \( \mathcal{H} \).

Then, by the Riesz theorem there exists an operator \( L \) on \( \mathcal{D} \) such that
\[
\langle L\xi | \eta \rangle = \int_{\mathbb{R}} \lambda d \langle X(\lambda)\xi | \eta \rangle, \quad \eta \in \mathcal{H}.
\]
Here we denote the operator \( L \) by the integral \( \int_{\mathbb{R}} \lambda dX(\lambda) \).

**Lemma 3.18.** Let \( \{X(\lambda)\} \) be a \(*\)-resolution of the identity and let \( \{\Delta_k\} \) be a finite or countable family of disjoint intervals of the real line. Then, for every \( \eta \in \mathcal{H} \),
\[
\left( \sum_{k=1}^{n} \|X(\Delta_k)\eta\|^2 \right)^{1/2} \leq \|\eta\|.
\]

**Proof.** Since \( \{X(\lambda)\} \) is a resolution of the identity, \( X(\lambda)^*X(\lambda) \) is a generalized resolution of the identity. Then we can write \( X(\lambda)^*X(\lambda) = PE(\lambda) \upharpoonright \mathcal{H} \) where, as before, \( E(\cdot) \) is an ordinary (i.e. self-adjoint) resolution of the identity in a possibly larger Hilbert space \( \mathcal{H} \) and \( P \) the projection onto \( \mathcal{H} \). Then we have
\[
\sum_{k=1}^{n} \|X(\Delta_k)\eta\|^2 = \sum_{k=1}^{n} \langle X(\Delta_k)\eta | X(\Delta_k)\eta \rangle = \sum_{k=1}^{n} \langle X(\Delta_k)^*X(\Delta_k)\eta | \eta \rangle
\]
\[
= \sum_{k=1}^{n} \langle PE(\Delta_k)\eta | \eta \rangle = \sum_{k=1}^{n} \langle PE(\Delta_k)\eta | \eta \rangle
\]
\[
= \sum_{k=1}^{n} \langle E(\Delta_k)\eta | P\eta \rangle = \sum_{k=1}^{n} \langle E(\Delta_k)\eta | \eta \rangle
\]
\[
= \sum_{k=1}^{n} \langle E(\Delta_k)\eta | E(\Delta_k)\eta \rangle = \sum_{k=1}^{n} \|E(\Delta_k)\eta\|^2 \leq \|\eta\|^2.
\]

**Lemma 3.19.** We have \( D(T_X) = \mathcal{D} \).

**Proof.** For every \( \xi \in D(T_X) \), \( \eta \in \mathcal{H} \)
(3.5) \[ \int_{\mathbb{R}} \lambda d \langle X(\lambda) \xi \mid \eta \rangle \leq \left( \int_{\mathbb{R}} \lambda^2 d \langle F(\lambda) \xi \mid \xi \rangle \right)^{1/2} \| \eta \| \].

Indeed, let \([\alpha, \beta]\) be a bounded interval, and \(\{\Delta_k; k = 1, \ldots, n\}\) a family of disjoint intervals whose union is \([\alpha, \beta]\). For every \(k\), choose \(\lambda_k \in \Delta_k\). Then, for the Cauchy sums defining the integrals we get

\[
\sum_{k=1}^{n} \lambda_k \langle X(\Delta_k) \xi \mid \eta \rangle \leq \sum_{k=1}^{n} |\lambda_k| \langle X(\Delta_k) \xi \mid X(\Delta_k)^* \eta \rangle \]

\[
\leq \left( \sum_{k=1}^{n} \lambda_k^2 \|X(\Delta_k)\| \right)^{1/2} \left( \sum_{k=1}^{n} \|X(\Delta_k)^* \eta\| \right)^{1/2}
\leq \left( \sum_{k=1}^{n} \lambda_k^2 \langle F(\Delta_k) \xi \mid \xi \rangle \right)^{1/2} \| \eta \|.
\]

Hence, the inequality (3.5) holds on every finite interval and, by taking limits also on the real line, and so \(D(T_X) \subseteq \mathcal{D}\).

Conversely, take an arbitrary \(\xi \in \mathcal{D}\). Then we have, taking into account Lemma 3.13

\[
\|L \xi\| = \int_{\mathbb{R}} \lambda d \int_{\mathbb{R}} \mu d \langle X(\lambda) \xi \mid X(\mu) \xi \rangle
= \int_{\mathbb{R}} \lambda d \int_{\mathbb{R}} \mu d \langle X(\mu)^* X(\lambda) \xi \mid \xi \rangle
= \int_{\mathbb{R}} \lambda d \int_{-\infty}^{\lambda} \mu d \langle X(\mu)^* X(\lambda) \xi \mid \xi \rangle + \int_{\mathbb{R}} \lambda d \int_{\lambda}^{\infty} \mu d \langle X(\mu)^* X(\lambda) \xi \mid \xi \rangle
= \int_{\mathbb{R}} \lambda d \int_{-\infty}^{\lambda} \mu d \langle X(\mu)^* F(\lambda) \xi \mid \xi \rangle + \int_{\mathbb{R}} \lambda d \int_{\lambda}^{\infty} \mu d \langle F(\mu) X(\lambda) \xi \mid \xi \rangle
= \int_{\mathbb{R}} \lambda d \int_{-\infty}^{\lambda} \mu d \langle F(\mu) \xi \mid \xi \rangle + \int_{\mathbb{R}} \lambda d \int_{\lambda}^{\infty} \mu d \langle F(\lambda) \xi \mid \xi \rangle
= \int_{\mathbb{R}} \lambda^2 d \langle F(\lambda) \xi \mid \xi \rangle.
\]

Hence \(\mathcal{D} \subseteq D(T_X)\). Thus, we have \(\mathcal{D} = D(T_X)\). \(\Box\)

**Theorem 3.20.** Let \(B\) be closed operator in \(\mathcal{H}\). The following statements are equivalent.

(i) \(B\) is similar to a self-adjoint operator \(A\), that is, \(B = T A T^{-1}\), with an intertwining operator \(T\) satisfying \(\| T E(\lambda) T^{-1} \| = 1\), for every \(\lambda \in \mathbb{R}\), where \(\{E(\lambda)\}\) is the spectral resolution of \(A\).
(ii) There exists a *-resolution of the identity \( \{X(\lambda)\} \) such that

\[
B = \int_{\mathbb{R}} \lambda dX(\lambda).
\]

Proof. (i)⇒(ii): Let \( A = \int_{\mathbb{R}} \lambda dE(\lambda) \) be the spectral resolution of \( A \) and put \( X(\lambda) = TE(\lambda)T^{-1} \). Then it is easily shown that \( \{X(\lambda)\} \) is a *-resolution of the identity (with \( \|X(\lambda)\| = 1 \), for every \( \lambda \in \mathbb{R} \)). We show that \( D(B) = TD(A) = D(T_X) \).

Let \( \xi \in D(T_X) \). Since

\[
\int_{\mathbb{R}} \lambda^2 d \langle E(\lambda)T^{-1}\xi \mid T^{-1}\xi \rangle = \int_{\mathbb{R}} \lambda^2 d \langle E(\lambda)T^{-1}\xi \mid E(\lambda)T^{-1}\xi \rangle = \int_{\mathbb{R}} \lambda^2 d \langle T^{-1}X(\lambda)\xi \mid T^{-1}X(\lambda)\xi \rangle \leq \|T^{-1}\|^2 \int_{\mathbb{R}} \lambda^2 d \langle X(\lambda)\xi \mid X(\lambda)\xi \rangle = \|T^{-1}\|^2 \int_{\mathbb{R}} \lambda^2 d \langle F(\lambda)\xi \mid \xi \rangle < \infty,
\]

we have, \( T^{-1}\xi \in D(A) \) or, equivalently \( \xi \in D(B) \).

On the other hand, let \( \xi \in TD(A) \). Then,

\[
\int_{\mathbb{R}} \lambda^2 d \langle F(\lambda)\xi \mid \xi \rangle = \int_{\mathbb{R}} \lambda^2 d \langle TE(\lambda)T^{-1}\xi \mid TE(\lambda)T^{-1}\xi \rangle \leq \|T\|^2 \int_{\mathbb{R}} \lambda^2 d \langle E(\lambda)T^{-1}\xi \mid T^{-1}\xi \rangle < \infty,
\]

and so \( \xi \in D(T_X) \). Thus, \( D(B) = D(T_X) \) and by Lemma 3.19 \( D(B) = D(\int_{\mathbb{R}} \lambda dX(\lambda)) \). Furthermore, we have

\[
\langle B\xi \mid \eta \rangle = \left( \int_{\mathbb{R}} \lambda dX(\lambda)\xi \right) \left( \int_{\mathbb{R}} \lambda dX(\lambda)\xi \right)
\]

for every \( \xi \in D(B) \) and \( \eta \in \mathcal{H} \).

(ii)⇒(i): By Lemma 3.17, there exists a self-adjoint resolution of the identity \( \{E(\lambda)\} \) and an invertible operator \( T \) with bounded inverse such that \( X(\lambda) = TE(\lambda)T^{-1} \), for every \( \lambda \in \mathbb{R} \).

Let now \( A \) be the self-adjoint operator \( A = \int_{\mathbb{R}} \lambda dE(\lambda) \). As shown above we have \( D(T_X) = TD(A) \) and furthermore \( D(B) = D(\int_{\mathbb{R}} \lambda dX(\lambda)) \) and

\[
\langle B\xi \mid \eta \rangle = \left( \int_{\mathbb{R}} \lambda dX(\lambda)\xi \right) \left( \int_{\mathbb{R}} \lambda dX(\lambda)\xi \right) = \langle TAT^{-1}\xi \mid \eta \rangle,
\]

for every \( \xi \in D(B) \) and \( \eta \in \mathcal{H} \).

\[\square\]

Remark 3.21. The operator \( B \) has real spectrum and empty residual spectrum, since \( \sigma(B) = \sigma(A) \) and \( \sigma_r(B) = \sigma_r(A) \). Moreover, \( B \) is a \textit{pseudo-hermitian} operator; i.e. \( B \) is a spectral operator of scalar type.
Lemma 3.23. Let $B$ be a closed operator with positive spectrum. The following statements are equivalent.

(i) $B$ is similar to a positive self-adjoint operator $A$, with an intertwining operator $T$ satisfying $\|TE(\lambda)T^{-1}\| = 1$, for every $\lambda \in \mathbb{R}$, where $\{E(\lambda)\}$ is the spectral resolution of $A$.

(ii) There exists a *-resolution of the identity $\{X(\lambda)\}_{\lambda \in \mathbb{R}^+}$ on $\mathbb{R}^+ := [0, \infty)$ such that

$$B = \int_0^\infty \lambda dX(\lambda).$$

If one of the equivalent conditions (i) or (ii) holds, then there exists a closed operator $B_2$ with positive spectrum such that $B_2^2 = B$.

Proof. The equivalence of (i) and (ii) follow from Proposition 2.4 and Theorem 3.20. Suppose that $B = TAT^{-1}$ for some positive self-adjoint operator $A$ and an invertible bounded operator $T$ with bounded inverse. Then, putting $B_2 = T^1/2A^{1/2}T^{-1}$, $B_2$ is a closed operator with positive spectrum such that $B_2^2 = B$. \hfill \Box

In order to go further, we need the following lemma

Corollary 3.22. Let $B$ be a closed operator with positive spectrum. The following statements are equivalent.

(i) $B$ is similar to a positive self-adjoint operator $A$, with an intertwining operator $T$ satisfying $\|TE(\lambda)T^{-1}\| = 1$, for every $\lambda \in \mathbb{R}$, where $\{E(\lambda)\}$ is the spectral resolution of $A$.

(ii) There exists a *-resolution of the identity $\{X(\lambda)\}_{\lambda \in \mathbb{R}^+}$ on $\mathbb{R}^+ := [0, \infty)$ such that

$$B = \int_0^\infty \lambda dX(\lambda).$$

Proof. The equivalence of (i) and (ii) follow from Proposition 2.4 and Theorem 3.20. Suppose that $B = TAT^{-1}$ for some positive self-adjoint operator $A$ and an invertible bounded operator $T$ with bounded inverse. Then, putting $B_2 = T^1/2A^{1/2}T^{-1}$, $B_2$ is a closed operator with positive spectrum such that $B_2^2 = B$. \hfill \Box

In order to go further, we need the following lemma

Lemma 3.23. The function $K \mapsto \sqrt{K}$ is strongly continuous on the set $\mathcal{M} := \{K \in \mathcal{B}(\mathcal{H}); K \geq 0, \|K\| \leq M\}$.

Proof. Let $K_\alpha \rightharpoonup K$, $K, K_\alpha \in \mathcal{M}$. By the Weierstrass theorem, there is a sequence of polynomials $\{p_n(x)\}$ such that $p_n(x) \to \sqrt{x}$, uniformly on $[0, M]$. This implies that $\|p_n(Z) - \sqrt{Z}\| \to 0$, for every $Z \in \mathcal{M}$. Since

$$\|(\sqrt{K_\alpha} - \sqrt{K})\| \leq \|(\sqrt{K_\alpha} - p_n(K_\alpha))\| + \|p_n(K_\alpha) - p_n(K)\|$$

Now choose, $n$ large enough to make the first and third term in the right hand side smaller than $\epsilon > 0$ and, fixed this $n$, take $\alpha$ big enough to make the second term smaller than $\epsilon$. The latter is possible since the multiplication is jointly strongly continuous on every norm bounded ball of $\mathcal{B}(\mathcal{H})$; thus, if $K_\alpha \rightharpoonup K$ then, for every polynomial $p$, $p(K_\alpha) \to p(K)$. \hfill \Box

By Lemma 3.12 and Lemma 3.23 we obtain the following result.

Proposition 3.24. Let $\{X(\lambda)\}$ be a *-resolution of the identity. Then, the operator valued function $\lambda \mapsto \Phi(\lambda)$, where $\Phi(\lambda) := F(\lambda)^{1/2} = (X(\lambda)X(\lambda)^*)^{1/2}$, has the following properties:

(fs1) $\sup_{\lambda \in \mathbb{R}} \|\Phi(\lambda)\| = 1$;

(fs2) $\Phi(\lambda) \leq \Phi(\mu)$ if $\lambda < \mu$;

(fs3) $\lim_{\lambda \to -\infty} \Phi(\lambda)\xi = 0$; $\lim_{\lambda \to +\infty} \Phi(\lambda)\xi = \xi$, $\forall \xi \in \mathcal{H}$;

(fs4) $\lim_{\epsilon \to 0^+} \Phi(\lambda + \epsilon)\xi = \Phi(\lambda)\xi$, $\forall \lambda \in \mathbb{R}$; $\forall \xi \in \mathcal{H}$.

Hence, in analogy to Theorem 3.14 we have
Theorem 3.25. Let $\Phi(\cdot) = (X(\cdot)^*X(\cdot))^{1/2}$ be the positive operator valued function defined by the $*$-resolution of the identity $\{X(\lambda)\}$ on the real line. Set,

$$D(S_X) = \left\{ \xi \in \mathcal{H} : \int_{\mathbb{R}} \lambda^2 \, d \langle \Phi(\lambda)\xi | \xi \rangle < \infty \right\}.$$ 

Then there exists a unique closed symmetric operator $S_X$, defined on $D(S_X)$ such that

$$\langle S_X \xi | \eta \rangle = \int_{\mathbb{R}} \lambda \, d \langle \Phi(\lambda)\xi | \eta \rangle, \quad \forall \xi \in D(S_X), \, \eta \in \mathcal{H}.$$ 

Question: Is there any relationship between the operators $B$, $T_X$ and $S_X$?

Clearly, if $\{X(\lambda)\}$ is a self-adjoint resolution of the identity, then the three operators coincide. So far we know that $B$ and $T_X$ have the same domain but we do not know if and how $B$ can be expressed in terms of $T_X$. About the relationship between $B$ or $T_X$ we do not know almost anything, so we leave this question open.

Acknowledgements – The authors thank the referee for pointing out a serious inaccuracy in a previous version of this paper.

References

[1] J.-P. Antoine and C. Trapani, Partial inner product spaces, metric operators and generalized hermiticity, J. Phys. A: Math. Theor. 46 (2013) 025204 (21pp)
[2] C.M. Bender, A. Fring, U. Günther and H. Jones, Quantum physics with non-Hermitian operators, J. Phys. A: Math. Theor. 45 (2012) 440301
[3] G.W. Mackey, Commutative Banach Algebras, Notas de Matematica n. 17, Rio de Janeiro, 1959.
[4] A. Mostafazadeh, Pseudo-Hermitian representation of quantum mechanics, Int. J. Geom. Methods Mod. Phys. 7 (2010) 1191–1306
[5] F. Riesz and B. Sz. Nagy, Lecons d’Analyse fonctionelle, Gauthier-Villars, Paris (1972)
[6] N.I. Akhiezer and I.M. Glazman, Theory of Linear Operators in Hilbert space, II, Dover Publ. (1993)

Department of Applied Mathematics, Fukuoka University, Fukuoka 814-0180, Japan
E-mail address: a-inoue@fukuoka-u.ac.jp

Dipartimento di Matematica e Informatica, Università di Palermo, I-90123 Palermo, Italy
E-mail address: camillo.trapani@unipa.it