The Hamiltonian BRST quantization of a noncommutative nonabelian gauge theory and its Seiberg-Witten map

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We consider the Hamiltonian BRST quantization of a noncommutative nonabelian gauge theory. The Seiberg-Witten map of all phase-space variables, including multipliers, ghosts and their momenta, is given in first order in the noncommutative parameter \( \theta \). We show that there exists a complete consistence between the gauge structures of the original and of the mapped theories, derived in a canonical way, once we appropriately choose the map solutions.

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I. INTRODUCTION

Since the pioneer paper \cite{1} about the noncommutative structure for spacetime coordinates as an attempt to introduce a natural ultraviolet cutoff for relativistic field theories, a great amount of work has been done concerning noncommutative geometry and noncommutative field theories \cite{2}. In recent years, strong motivations for further developing these subjects have appeared in the context of string theories. It was shown that noncommutativity then naturally arises in the effective action of open strings in the presence of magnetic fields \cite{3}. Due to noncommutativity, the spacetime coordinates \( \hat{x}^\mu \) are replaced by the Hermitian generators \( \hat{x}^\mu \) of a noncommutative \( C^* \)-algebra over spacetime functions satisfying

\[
[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}
\] (1.1)

where \( \theta^{\mu\nu} \), in the simplest description, is a constant antisymmetric \( D \times D \) matrix, \( D \) being the spacetime dimension. Constructing quantum field theories starting from these ingredients is a difficult program. However, it is possible to use the Weyl-Moyal ideas that relate operators to classical functions with the use of the so called Weyl transformations \cite{4}. Instead of working with non-commuting functions of the operators \( \hat{x}^\mu \), it is then possible to perform the appropriate calculations by using usual functions of \( x^\mu \). The price to be paid is the deformation of the usual commutative product to the noncommutative Moyal star product

\[
\phi_1(x) \star \phi_2(x) = \exp \left( \frac{i}{2} \theta^{\mu\nu} \partial_\mu \phi_1 \partial_\nu \phi_2 \right) \phi_1(x)\phi_2(y) |_{x=y}
\] (1.2)

As can be verified, the space-time integral of the Moyal product of two fields is the same as the usual one, provided we discard boundary terms. So the noncommutativity affects the vertices in the action. These possibilities imply in several interesting features of noncommutative quantum field theories \cite{2,5}.

It is not difficult to deform gauge theories in order to get actions which are invariant under gauge transformations associated with the Moyal structure. The form of the gauge transformations imply, however, that the algebra must close not only under commutation but also under anticommutation. This usually makes \( U(N) \) to be chosen as the symmetry group of noncommutative Yang-Mills theories in place of \( SU(N) \), although other symmetry groups can also be considered \cite{6}. It is possible, also, to let the connections take values in the enveloping algebra of an arbitrary symmetry group \cite{6}. Once we chose a representation for such a group, given for instance by \( n \times n \) matrices, the corresponding enveloping algebra can be properly spanned by the \( u(n) \) generators in the representation also given by \( n \times n \) hermitian matrices, since they form a basis for that vector space. In this way we can consider arbitrary symmetry groups, in a given representation, with enveloping algebra spanned by \( u(n) \).

The classical Lagrangian treatment of noncommutative Yang-Mills theories poses no formal problems regarding the specific values the components of \( \theta^{\mu\nu} \) can take. However, even a classical Hamiltonian treatment depends strongly if the noncommutative parameter \( \theta^{\mu\nu} \) vanishes or not \( \theta^{\mu\nu} = 0 \). In the last case we have an arbitrarily higher order derivative theory which has to be treated with non canonical means. At quantum level this same condition breaks unitarity. Due to the fact that \( \theta^{\mu\nu} \) is a constant matrix, Lorentz invariance is lost in any case.

In this work, after reviewing the Hamiltonian treatment of the gauge sector of a noncommutative gauge theory where the connections take values in a \( u(N) \) algebra, as previously presented in \cite{7}, and in the unitary case where \( \theta^{\mu\nu} = 0 \), we construct the Seiberg-Witten map for the phase space variables. We show that the specific solution for the map of the momentum found in \cite{8} is actually the one that permits us to consistently generate the constraints that give the appropriate gauge structure for the mapped theory. In this way we prove that the...
noncommutative $u(N)$ symmetry algebra can be actually generated from an underlying theory which presents a commutative $u(N)$ algebra, both structures generated canonically via gauge generators acting with the aid of a Poisson bracket structure. After that the BRST treatment of the noncommutative $u(N)$ gauge theory is considered. In this context, we prove that there are no structure functions of higher orders, which permits to construct in a simple way the appropriate extended phase space containing ghosts, their momenta, trivial pairs and all of the BFV-BRST machinery to generate nilpotent BRST transformations, a Hamiltonian path integral with convenient measure and appropriate gauge fixing. At this level, we consider again the Seiberg-Witten map including now the variables of the extended phase, showing its consistency also at the level of BRST transformations. Some results previously derived with the aid of cohomological techniques are reproduced, but new results are here presented as the mapping of ghost momenta, gauge generators, BRST charges and extended actions, as far we know, by the first time.

This work is organized as follows: In Section II we present a brief review of some aspects already treated in order to establish notation and conventions. Section III is devoted to construct the Hamiltonian Seiberg-Witten map of the noncommutative $u(N)$ theory, and the canonical gauge structure of the mapped theory is displayed, showing that it is canonically consistent with the one of the original theory. In Section IV the BRST formalism is applied to the original noncommutative theory, discussing its gauge structure, BRST transformations and the path integral quantization, with appropriate gauge fixing. We consider the Seiberg-Witten map of that extended theory in Section V. At last, in Section VI, we present some final remarks. In an appendix we collect some results useful for calculations performed through the work.

II. HAMILTONIAN DESCRIPTION

Accordingly to what has been discussed in the previous section, we start, without lost of generality, from the Lagrangian action which describes the gauge sector of a noncommutative Yang-Mills theory, which can be written as

$$S = -\frac{1}{2} tr \int d^3x F_{\mu \nu} \star F^{\mu \nu} \quad (2.1)$$

Here the curvature tensor is defined by

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i (A_\mu \star A_\nu - A_\nu \star A_\mu)$$

and the connections take values in the $u(N)$ algebra, with generators $T^A$, assumed to be normalized as

$$tr(T^A T^B) = \frac{1}{2} \delta^{AB} \quad (2.3)$$

They not only form a Lie algebra but also close under anticommutation:

$$[T^A, T^B] = i f^{ABC} T^C$$

$$\{T^A, T^B\} = d^{ABC} T^C \quad (2.4)$$

In the above expressions, we take $f^{ABC}$ and $d^{ABC}$ as completely antisymmetric and completely symmetric respectively. From the above equations, it is possible to derive the momenta conjugate to $A_\mu^i$ as

$$\Pi^B_\mu = \frac{\partial L}{\partial \dot{A}_\mu^B} = F^B_\mu \quad (2.5)$$

There are primary constraints

$$T^A_i = \Pi^A_i \quad (2.6)$$

and the primary Hamiltonian is given by

$$H_p = \int d^3 x \left( \frac{1}{2} \Pi^B \Pi^B - \frac{1}{4} F^B_{ij} F^{ij} - (D_i \Pi^i)^B A^{0B} + \Lambda^{1B} T^B_1 \right) \quad (2.7)$$

where

$$(D_i \Pi^i)^B = (\partial_i \Pi^i - i [A_i, \Pi^i])^B$$

$$= \partial_i \Pi^B + \frac{1}{2} f^{BCD} [A^C_i, \Pi^D]$$

$$- \frac{i}{2} d^{BCD} [A^C_i, \Pi^D] \quad (2.8)$$

With the aid of the Poisson brackets

$$\{X(x), Y(y)\}_{PB} = \int d^3 z \left( \frac{\delta X(x)}{\delta A^C_{\mu}(z)} \frac{\delta Y(y)}{\delta \Pi^C(z)} - \frac{\delta Y(y)}{\delta A^C_{\mu}(z)} \frac{\delta X(x)}{\delta \Pi^C(z)} \right) \quad (2.9)$$

with $x^0 = y^0 = z^0$, the time evolution of the primary constraints imply, as usual, the secondary constraint

$$\{T_1, H_p\}_{PB} = (D_i \Pi^i) \equiv T_2 \quad (2.10)$$

It is easy to verify that the constraint $T^A_1$ satisfies the abelian algebra
\begin{align}
\{T^A_1(x), T^B_1(y)\}_{PB} & = 0 \\
\{T^A_1(x), T^B_2(y)\}_{PB} & = 0
\end{align}
(2.11)

but only after a bit longer calculation it is possible to show that $T^A_2$ closes in an algebra with itself 3:

\begin{align}
\{T^A_2(x), T^B_2(y)\}_{PB} & = \frac{1}{2} f^{ABC} \{\delta(x - y) \ast T^C_2(x)\} \\
& \quad - \frac{i}{2} \delta^{ABC} [\delta(x - y) \ast T^C_2(x)]
\end{align}
(2.12)

One can also prove that

\begin{align}
\{T^A_2, H\}_{PB} & = \frac{1}{2} f^{ABC} \{A^{2B} - A^{0B} \ast T^C_2\} \\
& \quad - \frac{i}{2} \delta^{ABC} [A^{2B} - A^{0B} \ast T^C_2]
\end{align}
(2.13)

and consequently no more constraints are produced. In the above equation,

\begin{align}
H & = H_p + 2 Tr \int d^3 x \Lambda^2 T_2
\end{align}
(2.14)

is the first class Hamiltonian.

Now, it is not difficult to show that the gauge invariance of the first order action

\begin{align}
S_{FO} & = \int d^4 x \Pi^{\mu B} \hat{A}^B_\mu - \int dx^\alpha H \\
& = Tr \int d^4 x (2 \Pi^{\mu A} \dot{\Lambda}_\mu - \Pi^{\mu A} - \frac{1}{2} F_{ij} F^{ij} \\
& \quad - 2 T_2 (\Lambda^2 - A^0) - 2 T_1 A^1)
\end{align}
(2.15)

can be achieved with the aid of the gauge generator

\begin{align}
G & = -2 Tr \int d^3 x (\epsilon^1 T_1 + \epsilon^2 T_2)
\end{align}
(2.16)

which acts canonically on the phase space variables $Y$ through $\delta Y = \{Y, G\}_{PB}$ to produce the gauge transformations

\begin{align}
\delta A^0 & = \epsilon^1 \\
\delta A_i & = D_i \epsilon^2 \\
\delta \Pi^0 & = 0 \\
\delta \Pi_i & = i [\epsilon^2, \Pi_i]
\end{align}
(2.17)

Indeed (2.15) is invariant under (2.17) if we also assume that

\begin{align}
\delta \Lambda^1 & = \epsilon^1 \\
\delta \Lambda^2 & = \epsilon^1 - \epsilon^2 + i[\Lambda^2 - A^0, \epsilon^2]
\end{align}
(2.18)

As expected, the redefinition

\begin{align}
A^0 & \rightarrow \hat{A}^0 = A^0 - \Lambda^2
\end{align}
(2.19)

permits to write the gauge transformations involving $\hat{A}_0$ and $A_i$ in the covariant way, the gauge transformation of the connections defined as $\delta A_\mu = D_\mu \epsilon^2$. It is useful to note that the transformation of $\Pi_\mu$ is consistent with the identification (2.10), since from (2.20) and (2.14) we arrive directly to

\begin{align}
\delta F_{\mu \nu} & = i [\epsilon^2 \ast F_{\mu \nu}]
\end{align}
(2.20)

when one uses (2.10).

III. THE SEIBERG-WITTEN MAP

The gauge transformations appearing in (2.17), here generically written as $\delta Y$, close in the algebra

\begin{align}
[\delta_1, \delta_2] Y = \delta_3 Y
\end{align}
(3.1)

where $Y$ represents any one of the fields appearing in those equations. As can be verified, the composition rule for the parameters is given by

\begin{align}
\epsilon^1_3 = 0 \\
\epsilon^2_3 = i [\epsilon^2_1, \epsilon^2_2]
\end{align}
(3.2)

which means that the gauge sector involving $\epsilon^1$ is abelian and the noncommutativity is actually associated with the gauge sector involving $\epsilon^2$. This means, for instance, that $[\delta_1, \delta_2] A^0 = 0$, or $[\delta_1, \delta_2] A_i = D_i \epsilon^2_3$, with $\epsilon^2_3$ given by (3.2).

Now, for a possibly underlying commutative gauge theory, where the corresponding phase space variables written here with small letters, we would have gauge transformations $\delta y$ and algebras similar to those listed above, but replacing the Moyal commutators by usual ones. Specifically,

\begin{align}
[\delta_1, \delta_2] y = \delta_3 y
\end{align}
(3.3)

with $y$ representing the commutative fields and the corresponding gauge parameters designated by $\alpha$ in place of $\epsilon$. They must obey the composition rule

\begin{align}
\alpha^1_3 = 0 \\
\alpha^2_3 = i [\alpha^1_1, \alpha^2_2]
\end{align}
(3.4)
since it naturally follows from the commutative limit of (3.2).

The basic idea in the Seiberg-Witten map is to write the noncommutative fields $Y$ as functions of the commutative fields $y$. It is assumed as well in the noncommutative parameters $\epsilon$ a dependence on the commutative fields $y$ and parameters $\alpha$ in such a way that $\delta Y[y] = \delta Y$. The form of the dependence of $\epsilon$ on the commutative fields $y$ and parameters $\alpha$ is determined when one also assumes that

$$[\delta_1, \delta_2] Y[y] = \tilde{\delta}_3 Y[y]$$

using the composition rule given by (3.4). With these considerations and taking in account (2.19), we can write (3.6) in detail as

$$[\delta_1, \delta_2] \Lambda^2[y] = 0$$

$$[\delta_1, \delta_2] \Pi_0[y] = 0$$

$$[\tilde{\delta}_1, \tilde{\delta}_2] \Pi_i[y] = i \left[ \tilde{\delta}_1 \epsilon^2_2[y] - \tilde{\delta}_2 \epsilon^2_1[y] + i \left[ \epsilon^2_2[y] \right] \right] + \Pi_i[y]$$

$$= \left[ \epsilon^2_2[y] \right] + \Pi_i[y]$$

$$[\tilde{\delta}_1, \tilde{\delta}_2] A_\mu[y] = D_\mu \left( \tilde{\delta}_1 \epsilon^2_2[y] - \tilde{\delta}_2 \epsilon^2_1[y] + i \left[ \epsilon^2_2[y] \right] \right)$$

$$= D_\mu \epsilon^2_2[y]$$

(3.6)

where $\epsilon^2_3[y]$ is a shorthand notation for $\epsilon^2[\alpha^2_3, y]$.

Let us consider with some detail the gauge sector which is non trivial. To simplify the notation, let us suppress the superior index 2 denoting the class of $\alpha$ or $\epsilon$ parameters in the forthcoming equations. We see that the last two equations in (3.6), involving the transformation of $\Pi_i$ and $A_\mu$, imply that

$$\epsilon_3[y] = \tilde{\delta}_1 \epsilon_2[y] - \tilde{\delta}_2 \epsilon_1[y] + i \left[ \epsilon_2[y] \right]$$

(3.7)

in place of (3.2). Now, (3.6) is exactly the equation appearing for instance in [6] whose solution, at first order in $\theta$, is given by

$$\epsilon_3[y] = \alpha + \epsilon^{(1)} + O(\theta^2)$$

$$= \alpha + \frac{1}{4} \delta^{ij} \{\partial_\alpha, a_j\} + O(\theta^2)$$

(3.8)

which defines $\epsilon^{(1)}$, and so it is enough to consider a field dependence of the noncommutative connections $a_\mu$. The map of $A_\mu$, which also has been worked out in the literature, comes from (2.17), when one also considers (3.8) and imposes that

$$\delta A_\mu[y] = \delta A_\mu.$$ We get

$$A_\mu[a] = a_\mu - \frac{1}{4} \theta^{kl} \{a_k, \partial_\alpha a_\mu + f_{\mu\nu} \} + O(\theta^2)$$

(3.9)

where

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - i [a_\mu, a_\nu]$$

(3.10)
\[
\Pi_i[y] = \pi_i + c_1 \theta^{kl} \{ f_{ik}, \pi_i \} \\
+ c_2 \theta^{kl} \{ f_{kl}, \pi_i \} + c_3 \theta^{ik} \{ f^{kl}, \pi_i \} \\
- \frac{1}{4} \theta^{kl} \{ a_k, (\partial_l + D_l) \pi_i \} 
\]

(3.14)

for the Seiberg-Witten map of the momentum. Now, it is useful to observe that from (2.19), at first order in \(\theta\),

\[
F_{\mu\nu}[y] = f_{\mu\nu} + \frac{1}{2} \theta^{kl} \{ f_{\mu k}, f_{\nu l} \} - \frac{1}{4} \theta^{kl} \{ a_k, (\partial_l + D_l) f_{\mu\nu} \}
\]

(3.15)

so there exists a map for \(\Pi_i\), which is consistent with (2.5) and the above definition, given by

\[
\Pi_i[y] = \pi_i - \frac{1}{2} \theta^{kl} \{ f_{ik}, \pi_i \} - \frac{1}{4} \theta^{kl} \{ a_k, (\partial_l + D_l) \pi_i \} \\
+ 0(\theta^2)
\]

(3.16)

This is also in accordance with (3.14), although this is not the only solution found in literature.

The map for \(\Pi_i\) obtained in Ref. 8, for instance, which is different from (3.14), is also a particular case of (3.14). It is given by

\[
\Pi_i[y] = \pi_i + \frac{1}{4} \theta^{kl} \{ f_{ik}, \pi_i \} + \frac{1}{2} \theta^{ik} \{ f^{kl}, \pi_i \} \\
- \frac{1}{4} \theta^{kl} \{ a_k, (\partial_l + D_l) \pi_i \} + 0(\theta^2)
\]

(3.17)

Soon we will show that with this last choice we get a consistent canonical formulation, when we also take into consideration de map of the Lagrange multipliers.

The map for \(\Lambda^1\) and \(A^0\) is trivial, since the sector involving \(\epsilon^1\) is abelian. The same occurs with \(\Pi^0\), since it is invariant. We get

\[
\Lambda^0[y] = \lambda^0 \\
\Pi^0[y] = \pi^0 \\
\Lambda^1[y] = \lambda^1
\]

(3.18)

Naturally (3.14) and the spatial part of (3.19) are still valid for the map of \(A_i[y]\) and \(\Pi_i[y]\). Of course, from (3.19) we get (explicitly writing the tilde)

\[
\tilde{A}_0[a] = \tilde{a}_0 - \frac{1}{4} \theta^{kl} \{ a_k, \partial_l \tilde{a}_0 + \tilde{f}_{0l} \} + O(\theta^2)
\]

(3.19)

By using (2.19), which implies not only \(\tilde{A}_0 = A_0 - \Lambda_2\) but also that \(\tilde{a}_0 = a_0 - \lambda_2\), and remembering (3.18), we arrive at

\[
\Lambda^2[y] = \lambda^2 - \frac{1}{4} \theta^{kl} \{ a_k, (\partial_l + D_l) \lambda^2 - \partial_0 a - \pi_l \} + O(\theta^2)
\]

(3.20)

which completes the Hamiltonian Seiberg-Witten map for all the pertinent variables. The corresponding Hamiltonian action can then be written in first order in \(\theta\), from the full action (2.15) and the map described above as

\[
S_{FO}[y] = tr \int d^4x \{ (2 IP^0[y] \tilde{A}_0[y] - \Pi^0[y]) \Pi^0[y] \\
- \frac{1}{2} F_{ij}[y] F^{ij}[y] - 2 T_2[y](\Lambda^2[y] - A^0[y]) \\
- 2 T_2[y] \Lambda^1[y] + 0(\theta^2)
\]

(3.21)

It is, by construction, invariant under the \(\bar{\delta}\) variations, since the original Noether identities are not altered, by construction, under the Seiberg-Witten map.

Now one should be able to show that the natural constraints coming from the action described above not only are first class but generate the commutative \(u(N)\) gauge transformations, designated by \(\delta\), for instance, in (3.17). Actually, it is easy to show that

\[
\bar{\delta}^1_1 = \frac{\delta S_{FO}[y]}{\delta \lambda^1} = \pi^0 \\
\bar{\delta}^2_2 = \frac{\delta S_{FO}[y]}{\delta \lambda^2} = T_2[y] + \frac{1}{4} \theta^{kl} (\partial_k + D_k) \{ t_2, a_k \} + O(\theta^2)
\]

(3.22)

where \(T_2[y] = \partial_i \Pi^0[y] - i [ A_i[y], \Pi^0[y] ] \) and \(t_2 = \partial_0 \pi^0 - i [ a_0, \pi^0 ]\), as in (2.11). At this stage, after a long calculation with the use of Bianchi and Jacobi identities and many cancellations, we arrive at a simple relation involving both quantities above, which is given by

\[
T_2[y] = t_2 + \frac{1}{4} \theta^{kl} (\partial_k + D_k) \{ a_l, t_2 \}
\]

(3.23)

when (3.17) is chosen as the Seiberg-Witten map for the momentum. This fact implies, via (3.22), that \(\bar{\delta}^2_2 = t_2\), which guarantees that the underlying gauge structure actually exists and is the one given by the commutative \(u(N)\) symmetry. Indeed, by defining the gauge generator

\[
g = -2 tr \int d^4x (\alpha^1 \bar{t}_1 + \alpha^2 \bar{t}_2)
\]

(3.24)

it is trivial to show, via \(\bar{\delta}y = \{ y, g \}_{PB}\), that

\[
\bar{\delta} a^0 = \alpha^1 \\
\bar{\delta} a_i = D_i \alpha^2 \\
\bar{\delta} \pi^0 = 0 \\
\bar{\delta} \pi_i = i [\alpha^2, \pi_i].
\]

(3.25)
and a complete consistency between the canonical Hamiltonian formalism of the original noncommutative theory and the one of the mapped commutative theory is achieved in \( O(\theta^2) \). Although the results we have derived are strictly valid in this order in \( \theta \), we conjecture that the identity between \( t_2 \) and \( \bar{t}_2 \) probably is valid at all orders in \( \theta \), since only with this identity we would guarantee that the underlying gauge symmetry of the mapped theory presents the desired structure. This fact has actually been proved, in higher orders in \( \theta \), in a Lagrangian formalism context, exploring directly the form of the transformations \( \bar{t}_2 \). 

IV. BRST QUANTIZATION

Once we have reviewed the classical aspects of the Hamiltonian treatment of the noncommutative \( u(N) \) gauge theory, we are ready to consider its Hamiltonian BRST formulation \([12]\). This is the first step to derive the functional quantization of the theory from a constructive point of view. Let us first consider the full theory treated in section II. In the next section we will consider the BRST quantization of the mapped theory obtained in the previous section. Accordingly to the usual procedure adopted in the BFV-BRST quantization of usual Yang-Mills (Y-M) theory \([10]\), here we also discard the \( N^2 \) pairs \((A_0, \Pi_0)\) absorbing \( A_0 \) in \( \Lambda^2 \) so that \((A_1, \Pi_1)\) and the multipliers \( \Lambda^2 \) and their canonical momenta can be taken as the dynamical variables of the theory. The relevant algebraic structure to be considered is then the one given by \([2, 12]\). If we rewrite Eq. \([2.12]\) as

\[
\{ T^A_2(x), T^B_2(y) \}_{PB} = 2 \int d^3 \xi \ U^{ABC}_1(x, y, z) T^C_2(z) \tag{4.1}
\]

the first order structure function \( U^{ABC}_1 \) is identified with

\[
U^{ABC}_1(x, y, z) = \frac{1}{4} f^{ABC} \{ \delta(x-z) ; \delta(z-y) \} + \frac{i}{4} d^{ABC} [\delta(x-z) ; \delta(z-y)] \tag{4.2}
\]

By using the Jacobi identity, it can be proved that the existence of non trivial second order structure functions depends on the quantity

\[
D^{ABCD}_1(x, y, z, w) = 2 \ U^{[AB][D][C]ED}_1 \tag{4.3}
\]

where \( U^A \equiv T^A_2 \) and the integrations over intermediary variables are implicit.

Since \( U^{ABC}_1 \) does not depend on the phase space variables, the first term in the right side of the above expression is trivially zero. Therefore it follows that

\[
D^{ABCD}_1(x, y, z, w) = 0 \tag{4.6}
\]

and it is possible to choose the higher order structure functions to vanish:

\[
D^{ABCD}_1(x, y, z, w) = 0 \tag{4.6}
\]
At this point, we extend the original phase space by introducing the ghosts $C^A$ and their momenta $P^A$ in order to construct the BRST operator as

$$
\Omega = \int d^3 x C^A (x) T_2^A (x) + \int d^3 x d^3 y d^3 z C^B (y) C^C (x) \times U^{ABC} (x, y, z) P^A (z)
$$

$$
= 2 \text{Tr} \int d^3 x (C (x) T_2 (x) - i \{ C (x) ; C (x) \} P (x))
$$

To generate the BRST transformations and the dynamics in the extended phase space, it is necessary to extend the former definition of the Poisson brackets in order to include Grassmannian variables. As usual, we can write that

$$
\{ X, Y \}_{PB} = \frac{\partial X}{\partial Z^A} \frac{\partial Y}{\partial Z^B} \quad (4.9)
$$

where $Z^A = \{ A_i, \Pi_i, C, P \}$ and intermediary integrations are assumed. The symplectic matrix $C^{AB} = \{ z^A, z^B \}_{PB}$ and the equal time Poisson Brackets for the Grassmannian sector which do not vanish are given by

$$
\{ \Pi (x), \Pi (y) \}_{PB} = \{ P (x), C (y) \}_{PB} = -\delta (x - y).
$$

Now, a BRST transformation of an arbitrary quantity $X$ is generated via

$$
sX = \{ X, \Omega \}_{PB} \quad (4.10)
$$

giving for the phase space variables, including ghosts,

$$
sA_i = -D_i C
$$

$$
s\Pi^i = i \{ C ; \Pi^i \} \quad (4.11)
$$

$$
sC = \frac{i}{2} \{ C ; C \}
$$

$$
sP = -T_2 + i \{ C ; P \}
$$

By noting that $s$ is an odd derivative acting from the right, it is easy to demonstrate that actually it is nilpotent. For instance,

$$
s^2 A_i = -s (\partial_i C - i [A_i ; C])
$$

$$
= -D_i (s C) + i D_i C ; C
$$

$$
= 0
$$

Also

$$
\sum (n) U = 0 \quad \text{for} \quad n \geq 2 \quad (4.7)
$$

We follow the canonical approach which absorbs $A_0$ in $\Lambda_2$ (it is the reverse of (2.19)), discarding also the variable $\Pi_0$, since the pair $A_0, \Pi_0$ is non-dynamical. The momenta conjugate to $A_2$ are also introduced and generate the constraints

$$
\Gamma^A = \frac{\partial \mathcal{L}}{\partial \dot{A}^A} \approx 0 \quad (4.13)
$$

To simplify the notation, let us suppress in what follows the subscript 2 in $\Lambda_2^A$. With these considerations, we describe the phase space with the variables $\{ A_i, \Pi_i, \Lambda, \Gamma, C, P \}$ and an additional pair of canonically conjugate ghosts $\Theta$ and $\bar{C}$ in order to implement the constraints in the path integral.

The BRST operator for the non minimal space is redefined as

$$
\Omega = 2 \text{Tr} \int d^3 x (C (x) T_2 (x)
$$

$$
+ i \{ C (x) ; C (x) \} P (x) - i \Theta (x) \Gamma (x)) \quad (4.14)
$$

and the extended action rewrites as below

$$
S_k = \int d^4 x \left( \dot{A}^A \Pi^A + \dot{\Lambda}^A \Gamma^A + \dot{C}^A P^A + \dot{\Theta} \bar{C}^A - \mathcal{H}_{eff} \right)
$$

where $\mathcal{H}_{eff} = \mathcal{H} + \{ K, \Omega \}_{PB}$ and the Hamiltonian density $\mathcal{H}$ being obtained from (2.7) by letting $A_0$ and $\Lambda$ vanish. $K$ is the gauge fixing fermion.
The BRST invariance of the action $S_k$ is demonstrated by the invariance of the kinetic term since $\mathcal{H} + \{K, \Omega\}_P$ is an extended invariant for $\mathcal{H}$ and so is BRST invariant. As can be verified, besides (4.11) we get the BRST transformations of the two sets of trivial pairs

\[
\begin{align*}
\alpha \Lambda &= -i \Theta \\
\alpha \Theta &= 0 \\
\alpha \Pi &= i \Gamma \\
\alpha \Gamma &= 0
\end{align*}
\]

which leads to

\[
s \text{Tr} \left( \hat{A}^i \Pi_i + \hat{A}^i \Gamma + \hat{C} \mathcal{P} + \hat{\Theta} \hat{C} \right) = -\partial_i \text{Tr} \left( \hat{C} \Pi^i \right).
\]

which is a boundary term and can be discarded under the integral sign.

The gauge fixing procedure can be done as usually. Writing the gauge fixing fermion as

\[
\omega = 2 \text{Tr} \int d^3 x (c(x) \tilde{t}_2(x)) - i \{c(x), c(x)\} p(x) - i \theta \gamma
\]

which generates the BRST transformations

\[
\begin{align*}
\bar{s} a_i &= -D_i c \\
\bar{s} \pi^i &= -i \{c, \pi^i \}
\end{align*}
\]

\[
\begin{align*}
\bar{s} c &= \frac{i}{2} \{c, c\} \\
\bar{s} p &= -t_2 + i \{c, p\} \\
\bar{s} \lambda &= -i \theta \\
\bar{s} \theta &= 0 \\
\bar{s} e &= i \gamma \\
\bar{s} \gamma &= 0
\end{align*}
\]

Now we are in a position appropriate to construct the Seiberg-Witten map in the extended phase space. By denoting by $Z^\Xi$ the variables $A_\mu, \Pi_\mu, C, \mathcal{P}, \Lambda, \Gamma, \hat{C}$ and $\Theta$, we need to solve the relations

\[
s Z = \bar{s} Z[z]
\]

in the same spirit of the one found in Section III. Consistency demands that the subset $Y$ of $Z$ must be mapped according the results already found in Section III. This implies that (see (3.8)

\[
C[z] = c + \frac{1}{4} \partial^i \partial^j \{ \partial_i c, a_j \} + O(\theta^2)
\]

keeping the maps (3.9) for $A_1[y], \Pi_1[y]$ and (3.11) for $A_0[y]$ and $\Pi_0[y]$. As can be verified, solution (3.4) also consistently solve

\[
s \mathcal{P} = \bar{s} \mathcal{P}[z]
\]

The next step is to found the map for $\mathcal{P}$. From (4.11) and (3.7),

\[
\begin{align*}
s \mathcal{P} &= -T_2 + i \{C ; \mathcal{P}\} \\
&= -t_2 - \frac{1}{4} \theta^{kl} (\partial_k + D_k) \{a_l, t_2\} + i \left\{ c + \frac{1}{4} \theta^{kl} \{ \partial_k c, a_l \} ; \mathcal{P} \right\} + O(\theta^2)
\end{align*}
\]

By writing $\mathcal{P} = p + \mathcal{P}^{(1)}$ and remembering (5.2) and (5.5), we arrive at an equation for $\mathcal{P}^{(1)}$ given by

\[
K = 2 \text{Tr}(i \hat{C} \chi + \mathcal{P} \Lambda)
\]

where

\[\chi = \partial^k A_k\]

we get

\[\{K, \Omega\}_P = -2 \text{Tr}(\Gamma \chi + i \hat{C} \partial_k D^k \hat{C}) + (-T_2 + i \{C ; \mathcal{P}\}) \Lambda + i \mathcal{P} \Theta
\]

This permits to obtain an explicit form for $S_k$ according to (4.11), giving

\[
S_{\text{eff}} = \text{Tr} \int d^4 x \left( -\frac{1}{2} F_\mu F^{\mu \nu} + 2i \hat{C} \partial^\mu D_\mu \hat{C} + 2 \Gamma (\partial^\mu A_\mu) \right)
\]

after functionally integrating over $\Pi, \mathcal{P}$ and $\Theta$ and identifying $\Lambda$ and $A_0$. Other gauge choices are implemented in a similar way, reproducing the results obtained if one starts directly from the Lagrangian formalism [12].

V. MAPPING THE EXTENDED THEORY

As we have seen in Section III, the algebraic structure of the mapped theory is the one given by the $u(N)$ commutative theory. This implies that we can reproduce every step developed in the previous section only by making trivial the structure due to the Moyal product, which is obtained by letting $\theta$ vanish. Following an obvious notation, the extended phase space in now spanned by the commutative variables $a_i, \pi_i, c, p, \lambda, \gamma, \bar{e}$ and $\theta$, here generically denoted by $z^\Xi$. The BRST operator is obviously written as

\[
\omega = 2 \text{Tr} \int d^3 x (c(x) \tilde{t}_2(x)) - i \{c(x), c(x)\} p(x) - i \theta \gamma
\]
\[ \tilde{s} \mathcal{P}^{(1)}[z] - i \{ c, \mathcal{P}^{(1)}[z] \} = -\frac{1}{4} \theta^{kl}(\partial_k + D_k)\{ a_l, t_2 \} + 2 [\partial_k c, \partial_l p] + i \{ \partial_k c, a_l \}, p \} + O(\theta^2) \] (5.7)

As can be verified after a long calculation, the solution of the above equation gives a simple expression for \( \mathcal{P}^{(1)}[z] \) and implies that

\[ \mathcal{P}[z] = p + \frac{1}{4} \theta^{kl}(\partial_k + D_k)\{ a_l, p \} \] (5.8)

in \( O(\theta^2) \). We observe, as expected, that all the obtained maps respect ghost and antighost degrees.

The construction of the remaining maps for \( \Lambda, \Gamma, \bar{C} \) and \( \Theta \) is immediate, since they form trivial pairs. It is enough to identify these last quantities with the corresponding ones with small letters. Putting everything together, we see that we have succeeded in solving (5.3).

This implies that the extended action \( S_k \) appearing in (4.15) can be mapped as well. Due to (5.3), the mapped action is BRST invariant, when the BRST transformations are given (5.2). Remaining points as the gauge fixing fermion, the measure and the external sources to properly define the generating functional of the mapped theory will not be considered here.

VI. CONCLUSION

In this work we have considered the Hamiltonian formalism concerning the gauge sector of a generic noncommutative gauge theory, whose enveloping algebra structure is embedded in a noncommutative \( u(N) \) algebra. We have succeeded in constructing, at first order in the noncommutative parameter \( \theta \), its appropriate Hamiltonian Seiberg-Witten map, showing the algebraic consistence between the gauge transformations of both descriptions, the original and the mapped theory, generated canonically. To achieve this goal, it was necessary to choose the proper solutions of the Seiberg-Witten map between the phase space variables. We also have presented the BRST extensions of both theories, generated via the action of BRST charges. The Seiberg-Witten map between these descriptions has been then constructed with the aid of the BRST transformations, being in accordance with the results previously found. The map has been consistently given for all the variables of the extended phase space, including trivial pairs, ghosts and their momenta, gauge generators, BRST charges, extended actions etc.. Some points regarding gauge fixing in the Hamiltonian path integral formalism of the original theory have also been discussed.

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APPENDIX A: SOME IDENTITIES RELATED TO THE MOYAL PRODUCT

\[ \int d^4x \phi_1 \ast \phi_2 = \int d^4x \phi_1 \phi_2 = \int d^4x \phi_2 \ast \phi_1 \]
\[ (\phi_1 \ast \phi_2) \ast \phi_3 = \phi_1 \ast (\phi_2 \ast \phi_3) = \phi_1 \ast \phi_2 \ast \phi_3 \]
\[ \int d^4x \phi_1 \ast \phi_2 \ast \phi_3 = \int d^4x \phi_2 \ast \phi_3 \ast \phi_1 = \int d^4x \phi_3 \ast \phi_1 \ast \phi_2 \]
\[ \phi(x) \ast \delta(x-y) = \delta(x-y) \ast \phi(y) \]
\[ \phi(x) \ast \partial^\alpha_x \delta(x-y) = -\partial^\alpha_x \delta(x-y) \ast \phi(y) \]
\[ [\phi_1, [\phi_2, \phi_3]] + [\phi_2, [\phi_3, \phi_1]] + [\phi_3, [\phi_1, \phi_2]] = 0 \]
\[ [\phi_1(x), [\phi_2(x), \delta(x-y)]] = [\phi_2(y), [\phi_1(y), \delta(x-y)]] \]

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[1] H.S. Snyder Quantized Space-Time, Phys. Rev. 71 (1947) 38.
[2] See R.J. Szabo, Phys. Rept. 378 (2003) 207 and references therein.
[3] N. Seiberg and E. Witten, JHEP 09 (1999) 32.
[4] S.R. de Groot and L. G. Suttorp, Foundations of Electrodynamics, North-Holland Publishing Company - Amsterdam, 1972.
[5] L. Bonora and L. Salizzoni, Phys. Lett. B 504 (2001) 80; A. Armoni, Nucl. Phys. B593 (2001) 229; M.M. Sheikh-Jabbari, Nucl. Phys. Proc. Suppl. 108 (2002) 113-117.
[6] B. Jurco, J. Moller, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C 21 (2001) 383; D. Brace, B. L. Cerchiai, A. F. Pasqua, U. Varadarajan and B. Zumino, JHEP 0106 (2001) 047.
[7] J. Gomis and T. Mehem, Nucl. Phys. B 591 (2000) 263; O.F. Dayi, Phys. Lett. B 481 (2000) 408; J. Gomis, K. Kamimura, and J. Llosa, Phys. Rev. D 63 (2001) 045003; J. Gomis, K. Kamimura and T. Mateos, JHEP 0130 (2001) 040; R. Amorim and J. Barcelos-Neto, Phys. Rev. D 63 (2001) 045003.
[8] R. Banerjee, Phys. Rev. D 67 (2003) 105002.
[9] R. Amorim and F. A. Farias, Phys. Rev. D 65 (2002) 065009.
[10] M. Henneaux and C. Teitelboim, Quantization of Gauge
[11] G. Barnich, M. Grigoriev and M. Henneaux, JHEP 0110 (2001) 004; V. E. R. Lemos, M. Picariello, M. S. Sarandy and S. P. Sorella, J. Phys. A35 (2002) 3703.

[12] For a Lagrangian BRST quantization, see M. Soroush, Phys. Rev. D67 (2003) 105005.