ON A HIDDEN SUPERSYMMETRY OF COSMOLOGICAL BILLIARDS

DIMITRY LEITES∗, OLEKSANDR LOZHECHNYK

Abstract. The hyperbolic Lie algebras with symmetrizable Cartan matrix are classified, there are 142 of them some of which can be “superized” to an almost affine Lie superalgebra. We list all 97 pairs (a hyperbolic Lie algebra $H$, its superized almost affine Lie superalgebra $S(H)$). Several (18 of the total 66) superizable hyperbolic Lie algebras have multiple superizations. The tracks of cosmological billiards corresponding to both terms of these pairs ($H \leftrightarrow S(H)$) are the same since the Weyl chambers of $H$ and of $S(H)$ differ only by a scale.

We also classified “superizations” of hyperbolic Lie algebras with non-symmetrizable Cartan matrix. No interpretations of these Lie (super)algebras are known to us.

1. Introduction

1.1. Cosmological billiards and hyperbolic Lie algebras. Two hidden supersymmetries. Studies of cosmological models that took superstrings into account revealed a remarkable connection (see [DH, DHJN, DHN]) between hyperbolic Lie algebras with symmetrizable Cartan matrices and the cosmological models demonstrating an oscillatory approximation to singularity, called Belinski-Khalatnikov-Lifshitz dynamics, see [BKhL, KhK1].

Recently, a hidden supersymmetry of these models was observed, see [KKN, DSp].

Here we demonstrate completely different hidden supersymmetries of 66 cosmological billiards of the total 142 corresponding to symmetrizable Cartan matrix. Namely, there are 97 superizations, 18 matrices (billiards) have several superizations each, see list (1). The tracks of cosmological billiards corresponding to these pairs (Cartan matrix $\leftrightarrow$ its superization) are the same since their Weyl chambers differ only by a scale. Since the supersymmetry we suggest has nothing in common, it seems, with the supersymmetry observed in [KKN, DSp], we do not overburden our elementary note reproducing the essence of [KKN, DSp].

1.2. Hyperbolic Lie algebras and almost affine Lie superalgebras. In what follows the ground field is $\mathbb{C}$; all Cartan matrices $A$ are supposed to be indecomposable and normalized so that $A_{ii} = 0$ or 1 or 2; moreover, $A_{ij} = 0 \iff A_{ji} = 0$. For details, and the construction of the Lie (super)algebras $g(A)$ from its Chevalley generators, see [CCLL, GL, BGL, BLS].

If the Lie algebra $g(A)$ grows polynomially in its Chevalley generators and is not finite-dimensional, it is called an affine Kac–Moody algebra. Let us explain why this definition can not be literally applied to Lie superalgebras.

To better visualize Lie superalgebras with Cartan matrix and of polynomial growth (unlike the non-super case, not all of them are affine Kac-Moody: some of them are “stringy”, see [CCLL]) we need the following relatively recently distinguished notion which elucidates which ingredients in the structure of affine Kac-Moody are important and what for. Recall (for examples, see [BLS, BKLS]) that a NIS-Lie (super)algebra $\mathfrak{a}$ is the one endowed with a Non-degenerate Invariant Symmetric bilinear form; and its double extension is the Lie (super)algebra $\hat{\mathfrak{a}} = (\mathfrak{c} \oplus \mathfrak{a}) \oplus \mathfrak{d}$, the semi-direct sum, where $\mathfrak{c}$ is a 1-dimensional center and $\mathfrak{d}$ is a 1-dimensional.

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space of derivations. The double extension is “interesting” (not isomorphic to the direct sum \(a \oplus (c \oplus b)\)) if the central extension is non-trivial and the derivation is outer, see [BKLS].

The affine Kac-Moody Lie algebra is a double extension of the loop algebra \(a^{(1)} := a \otimes \mathbb{C}[t, t^{-1}]\) with values in a simple finite-dimensional Lie algebra \(a\), or of the “twisted” loop algebra \(a_\varphi^{(k)}\) corresponding to the fixed points of the degree \(k\) automorphism \(\varphi\) of the target Lie algebra \(a\) (this \(\varphi\) corresponds to a symmetry of the Dynkin graph of \(a\)). Each of these affine Kac–Moody algebras has a symmetrizable Cartan matrix, its Dynkin graph is the extended Dynkin diagram, see [K].

In the super case, there are simple finite-dimensional superalgebras \(b\) without Cartan matrices, but a double extension of \(b_\varphi^{(k)}\) may have a Cartan matrix; and the other way round: even if \(b\) has a Cartan matrix, some of the twisted loops \(b_\varphi^{(k)}\), or their double extensions, have no Cartan matrix, see [BLS].

For the results of classifications of Lie superalgebras with symmetrizable Cartan matrix, see [Se3] and [vdL] for finite-dimensional Lie superalgebras and infinite-dimensional Lie superalgebras of polynomial growth, and [HS] for Lie superalgebras of polynomial growth with non-symmetrizable Cartan matrices. A general conjecture on the list of \(\mathbb{Z}\)-graded Lie superalgebras of polynomial growth (see [LSS, BLS]) analogous to Kac’s conjecture proved by O. Mathieu (see [M]) is still open.

In the indescribable sea of Lie (super)algebras with Cartan matrix of exponential growth there are islands consisting of interesting algebras; moreover, they can be completely classified. We will consider two of such sets.

- A given Cartan matrix \(A\) with entries in the ground field is said to be almost affine if the Lie (super)algebra determined by it is not finite-dimensional or affine Kac–Moody, but the Lie (super)algebra determined by any main submatrix of \(A\), i.e., the submatrix obtained by striking out any row and any column intersecting on the main diagonal, is a sum of finite-dimensional or affine Lie (super)algebras.

- A given Lie (super)algebra \(g(A)\) is called almost affine if all its Cartan matrices are almost affine. Note that a given Lie (super)algebra, though not almost affine, may have an almost affine Cartan matrix which goes into a not almost affine matrix under an odd reflection. We are interested in the properties of Lie (super)algebra, not in those of one or several of its Cartan matrices. Ignoring the difference between the two problems (classification of Cartan matrices vs. classification of Lie (super)algebras) have twice lead to mistakes corrected in [CCLL].

The almost affine Lie algebras are usually called hyperbolic (for a reason, see [CCLL]), or “over extended”. The latter term is somewhat confusing since it is applied, actually, not to the Lie algebra, which is not an extension of any Lie algebra, but to the analog of the Dynkin graph corresponding to the Cartan matrix of this Lie algebra. The Dynkin graph of an affine Kac–Moody (super)algebra is called an extension of the Dynkin graph of a finite-dimensional Lie (super)algebra, and the Dynkin graph of any hyperbolic Lie algebra is, obviously, an extension of an extended Dynkin graph.

Li Wang Lai was the first to classify all hyperbolic Lie algebras. He gave his answer in terms of analogs of Dynkin graphs, but he left for the reader to guess the way his graphs correspond to Cartan matrices, and the method of getting the answer. Earlier, Kobayashi and Morita classified hyperbolic Lie algebras with symmetrizable Cartan matrix, see [KM]. These results for symmetrizable matrices were double-checked in [BS], where an incomplete but very interesting paper [S] was corrected; for the classification of any, in particular, non-symmetrizable, Cartan matrices of Lie algebras reproducing Li Wang Lai’s result, see the arXiv
version of [CCLL]. In [CCLL], almost affine Lie superalgebras are classified; the answer is given in terms of Cartan matrices and clearly defined analogs of Dynkin graphs.

1.3. Non-uniqueness of the “superization” procedure. In this note, we intend to find out which of the hyperbolic Lie algebras allow a “superization” of the same type as $\mathfrak{sp}(2n)$ does: it is the only simple finite-dimensional Lie algebra which serves as the even part of a Lie superalgebra with Cartan matrix, namely, of $\mathfrak{osp}(1|2n)$.

Observe that the properties of $\mathfrak{osp}(1|2n)$ resemble properties of the simple Lie algebras, see [DLZ], where the mechanism of this resemblance found in earlier works is clarified; for the same reasons we conjecture that the properties of almost affine Lie superalgebras we consider here are close to similar properties of hyperbolic Lie algebras.

The passage $\mathfrak{sp}(2n) \leftrightarrow \mathfrak{osp}(1|2n)$ is clear: the bottom lines of their Cartan matrices are, respectively,

$$0 \ldots 0 \leftrightarrow 0 \ldots 0 \leftarrow 1 \rightarrow 1$$

For affine Kac–Moody superalgebras, these procedures “desuperization” $\mapsto$ “superization”, first described by J. van de Leur [vdL], are one-to-one in both directions for Lie superalgebras with Cartan matrices without zeros on the main diagonal, except for the matrix $
abla (\begin{array}{ll}
2 & -2 \\
-2 & 2
\end{array})$

which obviously allows 2 superizations.

For the following hyperbolic Lie algebras, the “desuperization” $\mapsto$ “superization” correspondences are not one-to-one: they have at least 2 (the number in parentheses if > 2) superizations each:

$$H3_{27}, H3_{87}, H3_{93}(3), H3_{98}(3), H3_{108}(3), H3_{113}(5), H3_{115}(3), H3_{117}(3),$$

$$H3_{120}(3), H3_{123}, H4_{5}(3), H4_{16}(3), H4_{24}, H4_{24}, H4_{45}, H4_{46}(3), H5_{22}(3), H6_{9}.$$ (1)

2. The pairs “almost affine Lie superalgebra $\rightarrow$ hyperbolic Lie algebra” for symmetrizable Cartan matrices

In this section, we list all pairs “almost affine Lie superalgebra $\rightarrow$ hyperbolic Lie algebra”. In the list below, the first column contains the name of an almost affine Lie superalgebra $\mathfrak{g}$, see [CCLL], the second one contains its Cartan matrix, the third one contains the name of the hyperbolic Lie algebra, see [CCLL], equal to $\mathfrak{g}_6$. The penultimate column contains the Cartan matrix of a hyperbolic Lie algebra. The last column contains the permutation of rows/columns of the given $S$-matrix after its rows with 1 on the main diagonal are multiplied by 2. In the process we spotted a typo in [CCLL]: the matrix $S3_{35}$ is non-symmetrizable, should be denoted $NS3_{35}$. We mark the Lie algebras with several superizations by a [III] in front of a superization.

$$S3_{44} \left( \begin{array}{cc}
2 & -1 \\
-2 & -1
\end{array} \right) H3_{34} \left( \begin{array}{cc}
2 & -2 \\
-2 & 2
\end{array} \right) \{1,3,2\}$$

$$S3_{55} \left( \begin{array}{cc}
2 & -1 \\
-2 & -1
\end{array} \right) H3_{56} \left( \begin{array}{cc}
2 & -1 \\
-2 & -2
\end{array} \right) \{1,3,2\}$$

$$S3_{66} \left( \begin{array}{cc}
2 & -1 \\
-2 & -1
\end{array} \right) H3_{14} \left( \begin{array}{cc}
2 & -1 \\
-2 & -2
\end{array} \right) \{1,3,2\}$$

[III] $S3_{19} \left( \begin{array}{cc}
2 & -2 \\
-1 & -1
\end{array} \right) H3_{27} \left( \begin{array}{cc}
2 & -2 \\
-2 & -2
\end{array} \right) \{1,3,2\}$
$$S_{325} \begin{pmatrix} 2 & -1 & -2 \\ -2 & 1 & 2 \\ -2 & 1 & 2 \end{pmatrix} \quad H_{382} \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ -3 & -4 & 2 \end{pmatrix} \{1, 3, 2\}$$

$$S_{326} \begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \quad H_{383} \begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & 2 \\ -1 & 2 & -1 \end{pmatrix} \{1, 3, 2\}$$

$$S_{334} \begin{pmatrix} 2 & -2 & -4 \\ -1 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \quad H_{387} \begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & 1 \\ -3 & -4 & 2 \end{pmatrix} \{1, 3, 2\}$$

$$S_{338} \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -1 & -1 & 2 \end{pmatrix} \quad H_{383} \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & 2 & -2 \end{pmatrix}$$

$$S_{339} \begin{pmatrix} 2 & -1 & -1 \\ -1 & -2 & 2 \\ -1 & -2 & 2 \end{pmatrix} \quad H_{394} \begin{pmatrix} 2 & -1 & -2 \\ -1 & -2 & 2 \\ -2 & -2 & 2 \end{pmatrix} \{1, 3, 2\}$$

$$S_{340} \begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad H_{327} \begin{pmatrix} 2 & -2 & -2 \\ -2 & -2 & 2 \\ -2 & -2 & 2 \end{pmatrix} \{2, 1, 3\}$$

$$S_{343} \begin{pmatrix} 2 & -2 & -2 \\ -1 & 1 & 1 \\ -2 & -1 & 1 \end{pmatrix} \quad H_{387} \begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & 1 \\ -2 & -4 & 2 \end{pmatrix} \{2, 1, 3\}$$

$$S_{345} \begin{pmatrix} 2 & -2 & -2 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \quad H_{393} \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

$$S_{346} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \quad H_{393} \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

$$S_{347} \begin{pmatrix} 2 & -1 & 0 \\ -2 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad H_{397} \begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_{348} \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad H_{398} \begin{pmatrix} 2 & -2 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_{349} \begin{pmatrix} 2 & -1 & 0 \\ -2 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad H_{3107} \begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_{350} \begin{pmatrix} 2 & -2 & 0 \\ -1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad H_{3108} \begin{pmatrix} 2 & -2 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_{351} \begin{pmatrix} 2 & -2 & 0 \\ -1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad H_{3113} \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_{352} \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad H_{3104} \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \{3, 2, 1\}$$

$$S_{353} \begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad H_{3110} \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \{3, 2, 1\}$$

$$S_{354} \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad H_{3115} \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \{3, 2, 1\}$$

$$S_{355} \begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad H_{3119} \begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_{356} \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad H_{3120} \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_{357} \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad H_{3100} \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \{3, 2, 1\}$$

$$S_{358} \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad H_{3106} \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \{3, 2, 1\}$$

$$S_{359} \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad H_{3112} \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \{3, 2, 1\}$$
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| $S_{360}$ | $H_{3117}$ | \{3, 2, 1\} |
| $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} |

| $S_{361}$ | $H_{3118}$ | \{3, 2, 1\} |
| $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} |

| $S_{362}$ | $H_{3120}$ | \{3, 2, 1\} |
| $\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} |

| $S_{363}$ | $H_{3122}$ | \{3, 2, 1\} |
| $\begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} |

| $S_{364}$ | $H_{3123}$ | |
| $\begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} |

| $S_{365}$ | $H_{3098}$ | \{3, 2, 1\} |
| $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix} |

| $S_{366}$ | $H_{3103}$ | \{3, 2, 1\} |
| $\begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix} |

| $S_{367}$ | $H_{3108}$ | \{3, 2, 1\} |
| $\begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix} |

| $S_{368}$ | $H_{3113}$ | |
| $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix} |

| $S_{369}$ | $H_{3114}$ | |
| $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix} |

| $S_{370}$ | $H_{3115}$ | |
| $\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix} |

| $S_{371}$ | $H_{3116}$ | |
| $\begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix} |

| $S_{372}$ | $H_{3117}$ | |
| $\begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix} |

| $S_{373}$ | $H_{3120}$ | |
| $\begin{pmatrix} 1 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} |

| $S_{374}$ | $H_{3115}$ | \{3, 2, 1\} |
| $\begin{pmatrix} 1 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} |

| $S_{375}$ | $H_{3123}$ | |
| $\begin{pmatrix} 1 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix} |

| $S_{376}$ | $H_{3117}$ | \{3, 2, 1\} |
| $\begin{pmatrix} 1 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} |

| $S_{377}$ | $H_{3113}$ | |
| $\begin{pmatrix} 1 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} |

| $S_{378}$ | $H_{3098}$ | \{3, 2, 1\} |
| $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} |

| $S_{379}$ | $H_{3108}$ | \{3, 2, 1\} |
| $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} |

| $S_{380}$ | $H_{3113}$ | |
| $\begin{pmatrix} 2 & -2 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} |

| $S_{381}$ | $H_{3113}$ | |
| $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ | \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} |


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\begin{bmatrix}
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-1 & 2 & -1 & -1 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 0 & 2
\end{bmatrix}
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-1 & 2 & -1 & -1 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 2 & 0
\end{bmatrix}
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\{1, 2, 4, 3\}

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0 & 1 & 1 & 0 \\
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-1 & 2 & -1 & -1 \\
0 & 1 & 1 & 0 \\
0 & -1 & 0 & 2
\end{bmatrix}
\]

\{3, 2, 1, 4\}
$S_{510} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$H_{514} \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$S_6 \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$H_{621} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$S_{62} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$H_{67} \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$H_{60} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$S_{63} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$H_{69} \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$S_{64} \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$H_{69} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$S_{65} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$H_{69} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$S_{66} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$H_{618} \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$S_{67} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$H_{617} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$S_{71} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$H_{71} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$S_{81} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$H_{82} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$S_{91} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$H_{92} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$S_{101} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$H_{102} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
3. The pairs “almost affine Lie superalgebra $\rightarrow$ hyperbolic Lie algebra” for non-symmetrizable Cartan matrices

This section is added for completeness; we do not know of any of its possible interpretations. In this section, we desuperize non-symmetrizable Cartan matrices of almost affine Lie superalgebras. The matrices with multiple (in parentheses) superizations:

\[(2) \quad NH3_{25}(3), \ NH3_{29}(3), \ NH3_{35}(3)\]

Total NS matrices: 36, total NH matrices admitting superization: 30 of 96.

\[
\begin{align*}
\text{NS31} & \quad \left( \begin{array}{ccc} 2 & -1 & -1 \\ -2 & 1 & -1 \\ -1 & -1 & 2 \end{array} \right) \\
\text{NH31} & \quad \left( \begin{array}{ccc} 2 & -1 & -1 \\ -2 & 1 & -1 \\ -1 & -1 & 2 \end{array} \right) \{1,3,2\} \\
\text{NS32} & \quad \left( \begin{array}{ccc} 2 & -1 & -1 \\ -1 & -1 & 2 \end{array} \right)
\end{align*}
\]
\[
\begin{align*}
\text{NS321} & \quad \left( \begin{array}{ccc} 2 & -2 & -1 \\ 1 & 4 & -1 \\ -1 & 1 & 2 \end{array} \right) \quad \text{NH329} & \quad \left( \begin{array}{ccc} 2 & -4 & -2 \\ -1 & 2 & -2 \\ -1 & 2 & 2 \end{array} \right) \quad \{1, 3, 2\} \\
\text{NS322} & \quad \left( \begin{array}{ccc} 2 & -1 & -1 \\ -2 & 1 & -2 \\ -2 & -1 & 2 \end{array} \right) \quad \text{NH325} & \quad \left( \begin{array}{ccc} 2 & -4 & -4 \\ 1 & 2 & 1 \\ -1 & -2 & 2 \end{array} \right) \quad \{2, 1, 3\} \\
\text{NS323} & \quad \left( \begin{array}{ccc} 2 & -1 & -1 \\ -2 & 1 & -2 \\ -3 & 1 & 2 \end{array} \right) \quad \text{NH326} & \quad \left( \begin{array}{ccc} 2 & -4 & -4 \\ 1 & 2 & 1 \\ -1 & -3 & 2 \end{array} \right) \quad \{2, 1, 3\} \\
\text{NS324} & \quad \left( \begin{array}{ccc} 2 & -1 & -1 \\ -2 & 1 & -2 \\ -4 & 1 & 2 \end{array} \right) \quad \text{NH327} & \quad \left( \begin{array}{ccc} 2 & -4 & -1 \\ -1 & -4 & -2 \\ -1 & -4 & 2 \end{array} \right) \quad \{1, 3, 2\} \\
\text{NS325} & \quad \left( \begin{array}{ccc} 2 & -2 & -1 \\ -1 & 1 & 2 \\ -1 & 1 & 2 \end{array} \right) \quad \text{NH328} & \quad \left( \begin{array}{ccc} 2 & -2 & -1 \\ -1 & 2 & -2 \\ -1 & 3 & 2 \end{array} \right) \quad \{2, 1, 3\} \\
\text{NS329} & \quad \left( \begin{array}{ccc} 2 & -2 & -1 \\ -1 & 1 & 2 \\ -4 & 1 & 2 \end{array} \right) \quad \text{NH330} & \quad \left( \begin{array}{ccc} 2 & -2 & -2 \\ -1 & 2 & -2 \\ -2 & -4 & 2 \end{array} \right) \quad \{1, 3, 2\} \\
\text{NS331} & \quad \left( \begin{array}{ccc} 2 & -2 & -1 \\ -1 & 1 & 2 \\ -1 & 1 & 2 \end{array} \right) \quad \text{NH332} & \quad \left( \begin{array}{ccc} 2 & -4 & -2 \\ -1 & 2 & -2 \\ -2 & -2 & 2 \end{array} \right) \quad \{3, 1, 2\} \\
\text{NS333} & \quad \left( \begin{array}{ccc} 2 & -2 & -3 \\ -1 & 1 & 2 \\ -1 & 1 & 2 \end{array} \right) \quad \text{NH334} & \quad \left( \begin{array}{ccc} 2 & -4 & -2 \\ -1 & 2 & -2 \\ -2 & -3 & 2 \end{array} \right) \quad \{3, 1, 2\} \\
\text{NS335} & \quad \left( \begin{array}{ccc} 2 & -2 & -1 \\ -1 & 1 & 2 \\ -2 & -2 & 2 \end{array} \right) \quad \text{NH336} & \quad \left( \begin{array}{ccc} 2 & -2 & -2 \\ 2 & -2 & -1 \\ 2 & -3 & 2 \end{array} \right) \quad \{2, 1, 3\} \\
\text{NS337} & \quad \left( \begin{array}{ccc} 2 & -2 & -1 \\ 1 & 4 & 2 \\ -1 & 1 & 2 \end{array} \right) \quad \text{NH338} & \quad \left( \begin{array}{ccc} 2 & -2 & -2 \\ 2 & -4 & 2 \\ 2 & -4 & 2 \end{array} \right) \quad \{2, 1, 3\} \\
\text{NS339} & \quad \left( \begin{array}{ccc} 2 & -1 & -1 \\ -2 & 1 & -1 \\ -2 & 1 & 1 \end{array} \right) \quad \text{NH330} & \quad \left( \begin{array}{ccc} 2 & -4 & -2 \\ -1 & -2 & -2 \\ -1 & -2 & 2 \end{array} \right) \quad \{2, 1, 3\} \\
\text{NS340} & \quad \left( \begin{array}{ccc} 2 & -2 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \quad \text{NH335} & \quad \left( \begin{array}{ccc} 2 & -2 & -2 \\ 2 & -2 & -1 \\ 2 & -2 & 2 \end{array} \right) \quad \{2, 1, 3\} \\
\text{NS341} & \quad \left( \begin{array}{ccc} 2 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right) \quad \text{NH336} & \quad \left( \begin{array}{ccc} 2 & -2 & -2 \\ 2 & -2 & -1 \\ -2 & -3 & 2 \end{array} \right) \quad \{2, 1, 3\} \\
\text{NS342} & \quad \left( \begin{array}{ccc} 2 & -2 & -1 \\ -1 & 1 & 1 \\ -2 & 1 & 2 \end{array} \right) \quad \text{NH338} & \quad \left( \begin{array}{ccc} 2 & -2 & -2 \\ 2 & -2 & -1 \\ -2 & -4 & 2 \end{array} \right) \quad \{2, 1, 3\} \\
\text{NS343} & \quad \left( \begin{array}{ccc} 2 & -2 & -1 \\ -1 & 1 & 1 \\ -2 & 1 & 1 \end{array} \right) \quad \text{NH339} & \quad \left( \begin{array}{ccc} 2 & -2 & -2 \\ 2 & -2 & -1 \\ -2 & -4 & 2 \end{array} \right) \quad \{2, 1, 3\} \\
\text{NS415} & \quad \left( \begin{array}{ccc} 2 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & -1 \end{array} \right) \quad \text{NH420} & \quad \left( \begin{array}{ccc} 2 & -1 & 0 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{array} \right) \quad \{2, 3, 4, 1\} \\
\text{NS416} & \quad \left( \begin{array}{ccc} 2 & 1 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & -1 \end{array} \right) \quad \text{NH421} & \quad \left( \begin{array}{ccc} 2 & -2 & 0 \\ -1 & 2 & -1 \\ -1 & 0 & -1 \end{array} \right) \quad \{2, 3, 4, 1\} \\
\text{NS418} & \quad \left( \begin{array}{ccc} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & -1 \end{array} \right) \quad \text{NH423} & \quad \left( \begin{array}{ccc} 2 & -2 & 0 \\ -1 & 1 & -1 \\ -2 & 2 & -2 \end{array} \right) \quad \{2, 3, 4, 1\}
\end{align*}
\]
3.1. Remark. For completeness, observe that there are 2 series of simple finite-dimensional Lie superalgebras \( g \) without Cartan matrix, such that \( g_0 \) is a simple Lie algebra.

To describe these superizations, recall (see, e.g., [CCLL]) that \( q(n) \), called the queer Lie superalgebra, is a another, in addition to \( gl(a|b) \), super analog of \( gl(n) \). It can be interpreted as preserving a complex structure given by an odd operator \( J \) in an \((n|n)\)-dimensional superspace \( V \). Having selected a basis in \( V \) so that the matrix of \( J \) is 
\[
\begin{pmatrix}
0 & 1_n \\
-1_n & 0
\end{pmatrix}
\]
we can realize the supermatrices of \( q(n) \) in the form \((A, B) := \begin{pmatrix} A & B \\ B & A \end{pmatrix} \), where \( A, B \in gl(n) \). The queer trace \( qtr : (A, B) \mapsto \text{tr} B \) singles out a queertraceless subalgebra \( sq(n) \); set \( psq(n) := sq(n)/C_{12n} \).

The Lie superalgebra \( pe(n) \) preserving a non-degenerate odd symmetric bilinear form in an \((n|n)\)-dimensional superspace is called periplectic. Set \( spe(n) := \{ X \in pe(n) \mid \text{str} X = 0 \} \), where \( \text{str} \) denotes the supertrace.

The two series spoken above are for \( n \geq 3 \) (here \( \Pi \) is the inversion of parity functor):
\[
\begin{align*}
psq(n) & \text{ with } psq(n)_0 \simeq sl(n) \text{ and } psq(n)_1 \simeq \Pi(sl(n)); \\
\end{align*}
\[
spc(n) \text{ with } spc(n)_0 \simeq sl(n) = sl(V) \text{ and } spc(n)_1 \simeq \begin{cases} 
\Pi(A^2(V) \oplus S^2(V^*)) & \text{or} \\
\Pi(A^2(V^*) \oplus S^2(V)) 
\end{cases}.
\]

4. Open question

The Weyl chamber of the almost affine Lie superalgebra with symmetrizable Cartan matrix is the same as the Weyl chamber of any of the “desuperizations”. The Weyl chambers of superizations are a bit more spacious than the Weyl chambers of Lie algebras since, together with some roots \( \alpha \), the Lie superalgebra has roots \( 2\alpha \). How does this “superization” of the conventional billiard affect the model, cf. [BS, KKN, DSp]?

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