SINGULAR RIEMANNIAN FOLIATIONS ON NONPOSITIVELY CURVED MANIFOLDS

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ABSTRACT. We prove the nonexistence of a proper singular Riemannian foliation admitting section in compact manifolds of nonpositive curvature. Then we give a global description of proper singular Riemannian foliations admitting sections on Hadamard manifolds. In addition by using the theory of taut immersions we provide a short proof of this result in the special case of a polar action.

1. INTRODUCTION

A singular Riemannian foliation with sections (see definition 3.1) can be seen as a generalization of an isoparametric family in a Euclidean space to arbitrary ambient spaces. Another important example is the orbit decomposition of a polar action (see definition 2.1). These actions are well understood in compact symmetric spaces; see [Ko] for a classification of hyperpolar actions and [PoTh] for a classification of polar actions in compact symmetric spaces of rank one. (For surveys on these topics, see [Th1] and [Th2]).

No classification of polar actions is known for symmetric spaces of noncompact type. Before we state our result, we give some examples. If \((G, K)\) is a symmetric pair of noncompact type, then the left action of \(K\) on \(G/K\) is hyperpolar as in the compact case and has compact, isoparametric orbits. If \(G = KAN\) denotes the Iwasawa decomposition, then the left action of \(N\) on \(G/K\) is hyperpolar with noncompact and regular orbits. Now let \(M\) be a closed submanifold of the hyperbolic space \(H^n\) such that its principal curvatures are not larger than 1. Then the set of distance tubes around \(M\) defines a singular Riemannian foliation \(F_M\) with normal geodesics of \(M\) as sections. One can easily choose \(M\) such that \(F_M\) is inhomogeneous, i.e. does not come from a polar action. This example can be easily generalized to Hadamard manifolds. The following theorem gives a global description of a proper singular Riemannian foliation admitting sections in an Hadamard manifold. Please note how the above examples are covered.

Theorem. Let \(F\) be a proper singular Riemannian foliation with sections in an Hadamard manifold \(X\). Then \(F\) is the product foliation of a compact isoparametric foliation and the trivial foliation of \(R^n\) with \(R^n\) as the only leaf.

This is Theorem 3.11 proved in section 3. The homogeneous counterpart is the following:

Theorem. Let \(G\) be a Lie group acting polarly and properly on an Hadamard manifold \(X\) and \(K\) be a maximal compact subgroup of \(G\). Then there is a natural
$G$-equivariant diffeomorphism $X \cong G \times_K \mathbb{R}^s$, $s = \dim X - \dim G/K$, where the action of $K$ on $\mathbb{R}^s$ is orbit equivalent to an $s$-representation.

This is Theorem 2.4. We give a short proof of it in section 2. Now what happens if the ambient space is a compact quotient of an Hadamard manifold?

**Theorem.** Except possibly for a regular Riemannian foliations there are no proper singular Riemannian foliations admitting sections on a compact, nonpositively curved Riemannian manifold $N$.

This is Theorem 3.10 proved in section 3.

I would like to thank Professor Thorbergsson for many helpful discussions and the Deutsche Forschungsgemeinschaft for their support by the Schwerpunktsprogramm "Globale Differentialgeometrie" (SPP 1154).

## 2. Polar Actions

**Definition 2.1.** An isometric action of a Lie group $G$ on a Riemannian manifold is polar, if there is an immersed submanifold $\Sigma$, called section, that meets any orbit and is orthogonal to them at each point of intersection.

Let $G$ be a Lie group acting by isometries on a complete Riemannian manifold $N$. The action is variationally complete, if for any orbit $M$ and any normal geodesic $\gamma$, an $M$-Jacobi field along $\gamma$ that is tangent to some other orbit is the restriction of a $G$-Killing field to $\gamma$. Now let $G$ be a Lie group acting properly and polarly on an Hadamard manifold $X$. For this case the following theorem applies.

**Theorem 2.2** (Conlon). A proper polar action whose sections have no conjugate points is variationally complete.

The proof of the theorem is essentially due to Conlon [Co]. He proves it for flat sections, but it also works for (not necessarily embedded) sections without conjugate points. Also the compact action group is replaced by a proper action.

Let $M$ be a closed and embedded submanifold in $X$ and let $q \in X$ be a point that is not focal point of $M$. Then the squared distance function $f_q := d^2(\cdot, q) : M \to \mathbb{R}$ is a proper Morse function. Let $M_s = M \cap f_q^{-1}([0, s])$ for any $s \in [0, \infty)$. The Morse inequalities say $b_i(M^s) \leq \mu_i(f_q|_{M_s})$, where $b_i(M^s)$ is the $i$-th Betti number of $M^s$, say over $\mathbb{Z}_2$, and $\mu_i(f_q|_{M^s})$ is the number of critical point of index $i$ of $f_q|_{M^s}$. We say that $M$ is taut in $X$, if all the Morse inequalities are equalities for any nonfocal point $q \in X$ and any $s$ (note that we replaced the energy functional $E_q$ defined on an infinite dimensional path space by $f_q$). By [BS], a variationally complete action has taut orbits:

**Theorem 2.3** (Bott-Samelson). Each orbit $M$ of $G$ is taut in $X$.

Using this result we can give a geometric description of a proper polar actions of a Lie group $G$ on an Hadamard manifold $X$. Let $K$ be a maximal compact subgroup of $G$. By Cartan’s Theorem $K$ fixes some point $p \in X$, so $K \subset G_p$ and by maximality $K = G_p$. It is known that $M := Gp \cong G/K$ is diffeomorphic to $\mathbb{R}^m$, where $m = \dim Gp$. Therefore $b_i(Gp)$ is 1 for $i = 0$, otherwise 0. As $M$ is taut, any nonfocal point of $M$ in $X$ has exactly one preimage under $\eta := \exp^\perp : \nu M \to X$. This implies that there are no focal points along a normal geodesic of $M$. So $M$ has no focal points at all. Thus there is exactly one normal geodesic segment ending in a given point $q \in X$. Therefore the normal exponential map $\eta$ is a diffeomorphism.
Let $S := \exp(\nu_p Gp)$ and $s := \dim S$. Now $X$ is a tube of $Gp$ with section $S$. By the Tube Lemma following two maps

$$G \times_K \nu_p Gp \longrightarrow \nu Gp \exp \longrightarrow X,$$

are $G$-equivariant diffeomorphisms. The action of $K$ on $\mathbb{R}^s$ is the linearization of the $K$-action on $S$; it is the polar slice representation of $G$ at $p$ and therefore by Dadok ([Da]) orbit equivalent to an $s$-representation, i.e. the isotropy representation of a symmetric space. We sum up:

**Theorem 2.4.** Under the above assumptions, there is a natural $G$-equivariant diffeomorphism $X \cong G \times_K \mathbb{R}^s$. The action of $K$ on $\mathbb{R}^s$ is orbit equivalent to an $s$-representation.

Abels has proven a topological version of the first statement in open manifolds, see Corollary 1.3 in [Ab], in which $\mathbb{R}^s$ is replaced by a slice for the $G$-action. He obtains his result by constructing this slice. It is natural to ask whether the slice is canonical, i.e. of the form $\exp(\nu_p Gp)$, in the case of an isometric action. This is not true as the one parameter group of glide rotations along an axis in $\mathbb{R}^3$ shows. But for polar actions these canonical slices exist. This is the main point of this theorem.

The theorem tells that by projection of $X = G \times_K \mathbb{R}^s$ onto the first factor $G/K = Gp$ every orbit $G(\exp v), v \in \nu_p Gp$ fibers over $Gp$ with fiber $Kv$ and structure group $K$. This fiber bundle is trivial, since $G/K \cong \mathbb{R}^s$, so $G(\exp v) \cong \mathbb{R}^s \times Kv$; this is a cylinder over an orbit of an $s$-representation, which is an isoparametric submanifold. We remark that is not difficult to show that this last cylindric decomposition is metric if $X$ is the Euclidean space.

### 3. Singular Riemannian Foliations admitting Sections

We want to prove an analogous result for singular Riemannian foliations admitting sections.

**Definition 3.1 ([Mo]).** Let $\mathcal{F}$ be a partition of injectively immersed submanifolds (the leaves) of a Riemannian manifold $N$. For any $p \in N$ let $L_p$ be the leaf through $p$ and let $T\mathcal{F} = \bigcup_{p \in N} T_p L_p$. We define $\Xi(\mathcal{F})$ as the module of (differentiable) vector fields on $N$ with values in $T\mathcal{F}$. We call $\mathcal{F}$ a singular foliation, if it satisfies the following differentiability condition, a singular Riemannian foliation, if it satisfies in addition the transnormality condition:

1. (Differentiability) $\Xi(\mathcal{F})$ acts transitively on $T\mathcal{F}$, i.e., for any $v \in T_p \mathcal{F}, p \in N$ there is $X \in \Xi(\mathcal{F})$ with $X_p = v$.
2. (Transnormality) a geodesic starting orthogonally to a leaf intersects the leaves it meets orthogonally;

A leaf of maximal dimension is called regular, and so each point of it, otherwise singular. If, in addition, for any regular $p$ there is an isometrically immersed complete totally geodesic submanifold $\Sigma_p$ (the section) with $T_p \Sigma = \nu_p L_p$, that meets any leaf and always orthogonally, $\mathcal{F}$ is a singular Riemannian foliation admitting sections.

We call a singular Riemannian foliation proper if all leaves are properly immersed, i.e. closed and embedded.
Example 3.2. The orbit decomposition of a polar action is a singular Riemannian foliation admitting sections. We give another important example. Let $M$ be an isoparametric submanifold in $\mathbb{R}^n$. Then the partition of parallel and focal submanifolds of $M$ is a singular Riemannian foliation with sections. We call this partition an isoparametric foliation. This special kind of singular foliation occurs in every singular Riemannian foliation with sections as we will see in the next theorem. We remark that isoparametric submanifolds in $\mathbb{R}^n$ have been generalized to equifocal submanifolds in simply connected compact symmetric spaces in [TeTh], to submanifolds with parallel focal structure in general symmetric spaces in [Ew] and to arbitrary ambient spaces in [Tö].

Now let $\mathcal{F}$ be a singular Riemannian foliation with sections in a complete Riemannian manifold $N$. Let $x \in N$ and $P$ be an open, relatively compact neighborhood of $x$ in $L_x$. Then there is an $\varepsilon > 0$ such the restriction of $\exp^1 : \nu L_x \to N$ to $\{X \in \nu P \mid \|X\| < \varepsilon\}$ is a diffeomorphism onto its image $T$. Let $\pi : T \to P$ the orthogonal projection. Condition (1) of a singular Riemannian foliation implies that by eventually shrinking $\varepsilon$ we can assume that the leaves intersect the slices $S_y := \pi^{-1}(y), y \in P$ transversally. Then $T$ is a distinguished neighborhood in the sense of Molino ([Mö]). The transversal intersections induce a singular foliation on a slice $S_x$, such that each leaf has the same codimension as its restriction to $S_x$. We can lift this singular foliation to a singular foliation on a ball neighborhood $T S_x = \nu_x L_x$ of the origin via the exponential map. This singular foliation is homothety invariant by the Homothety Lemma (Lemma 6.2 in [Mo]) applied to $\mathcal{F}$. We therefore can extend it to $\nu_x L_x$. The singular foliation obtained is denoted by $\mathcal{F}_x$. The following theorem is due to Alexandrino (Theorem 2.10 in [Al]):

Theorem 3.3 (Slice Theorem). $\mathcal{F}_x$ is an isoparametric foliation. The sections are exactly of the form $T_x \Sigma$, where $\Sigma$ is a section of $\mathcal{F}$ through $x$.

Now consider a section $T_x \Sigma$ for $\mathcal{F}_x$. The set of singular points in $T_x \Sigma$ is a finite union of hyperplanes through the origin. The generalized Weyl group $W$ is generated by the reflections across these hyperplanes.

Let $\mathcal{F}$ be a singular Riemannian foliation admitting sections of an Hadamard manifold $X$. A section $\Sigma$ is also an Hadamard manifold. It is moreover closed and embedded in $X$. We want to introduce an analogue of the generalized Weyl group of a polar action. For details see [Tö]. Let $M$ be a regular leaf. Now let $\tau$ be an arbitrary curve in $M$ starting and ending in $M \cap \Sigma$. We obtain a map $T_{\tau(0)} \Sigma = \nu_{\tau(0)} M \to \nu_{\tau(1)} M = T_{\tau(1)} \Sigma$. By exponentiating we obtain an isometry from a ball neighborhood of $\tau(0)$ in $\Sigma$ to a ball neighborhood of $\tau(1)$ in $\Sigma$. This map preserves leaves. One can extend this map to an isometry $\phi_\tau : \Sigma \to \Sigma$ preserving leaves. The collection of these $\phi_\tau$ over all such curves $\tau$ is a group, called transversal holonomy group.

$$\Gamma \subset I(\Sigma) \quad \text{and} \quad \Gamma_p = L_p \cap \Sigma \text{ for all } p \in \Sigma.$$  

This group is independent of the choice of $M$. The linearization $d\Gamma_q$ of the isotropy group $\Gamma_q$ of a regular point $q \in \Sigma$ is the normal holonomy of $L_q$ and the orbit of any $q \in \Sigma$ describes the recurrence of $L_q$ to the section $\Sigma$. This property characterizes $\Gamma$ but also the generalized Weyl group of a polar action. This implies the equality of both notions for polar actions.

Now let $\mathcal{F}$ be proper, so any leaf is closed and embedded. This implies that $\Gamma$ is discrete. Let $M$ be a regular leaf with trivial normal holonomy. Let $\{p_i\}_{i \in I} :=$
M \cap \Sigma. We define the set \{D_{p_i}\}_{i \in I} of (closed) Dirichlet domains by
\[ D_{p_i} = \{q \in \Sigma | d(p_i, q) \leq d(p_j, q) \text{ for all } j \neq i\}. \]
The group \( \Gamma \) acts transitively on the set \( \{D_{p_i}\}_{i \in I} \). Since \( M \) has trivial holonomy we have \( \Gamma_{p_i} = \{e\} \), and \( \Gamma \) acts simply transitively on this set.

We take a look at the boundary of a Dirichlet domain. For \( x, y \in \Sigma \) let \( H_{x,y} := \{z \in \Sigma | d(x, z) = d(y, z)\} \) be the bisector/central hypersurface between \( x \) and \( y \). In an Hadamard it is a hypersurface that dissects \( \Sigma \). The set \( F = H_{p_i,p_j} \cap D_{p_i} \) is called a wall of \( D_{p_i} \), if it contains an open non-empty subset of \( H_{p_i,p_j} \). Two Dirichlet domains are called neighbors, if they contain a common wall.

Now we fix one \( p := p_i \) and let \( D = D_{p_i} \).

\[ \text{Proposition 3.4.} \quad \text{Let } F \text{ be a proper singular Riemannian foliation admitting sections on an Hadamard manifold } X. \text{ Let } \Sigma \text{ be a section of } F \text{ and } \Gamma \text{ be the transversal holonomy group acting on } \Sigma. \text{ Then } \Gamma \text{ is a Coxeter group.} \]

\[ \text{Proof.} \quad \text{By the Poincaré-Lemma (see for instance Lemma 2.5 in [AKLM]), } \Gamma \text{ is generated by the set of elements of } \Gamma \text{ that map } D \text{ to a neighboring domain. We want to show that these elements are reflections. This implies that } \Gamma \text{ is a Coxeter group by Theorem 3.5. in [AKLM] (see the definition of a Riemannian Coxeter manifold in subsection 3.2. of [AKLM]). Let } D' \text{ be a neighboring Dirichlet domain of } D \text{ and } g \in \Gamma \text{ be the unique element with } g(D) = D'. \text{ Then } D \text{ and } D' \text{ have a common wall } F. \text{ We claim that the wall } F \text{ consists of singular points of } F. \text{ If not, there is a regular point } q \in F. \text{ In [LG] the following was shown: singular points are focal points for any regular leaf, in particular for } M; \text{ conversely, a focal point } x, \text{ say in the section } \Sigma, \text{ that is not conjugate in } \Sigma \text{ to any point in } M \cap \Sigma, \text{ is a singular point of } F. \text{ Since in our case the sections have no conjugate points, the focal points of } M \text{ are exactly the singular points of } F. \text{ Thus } q \text{ is not a focal point and the squared distance function } f_q = d^2(\cdot, q) : L_p \to \mathbb{R} \text{ is a proper Morse function. As can be read for instance in [LG], } f_q \text{ has only one local minimum (regular leaves are 0-taut). We have two minimal normal geodesics } \gamma_1 \text{ and } \gamma_2 \text{ of } M \text{ from } p \text{ respectively } gp \text{ to } q. \text{ Thus } p \text{ and } gp \text{ are critical points of } f_q. \text{ The segment } \gamma_i \text{ does not meet any singular points, because the singular points are in the boundary of the Dirichlet domains as shown in [LG]. In particular there are no focal points of } M \text{ on } \gamma_i. \text{ Hence } p \text{ and } gp \text{ are critical points of } f_q \text{ of index 0, i.e. minima, contradiction. So } F \text{ only consists of singular points. Note that by using 0-tautness similarly as above one can show that } (X,F) \text{ has no exceptional leaves, i.e. regular leaves with nontrivial holonomy (see Lemma 1A.3 in [PoTh] and [To]).}

\[ \text{We choose } q \text{ from the interior of } F. \text{ Since } q \text{ is singular, there has to be a singular hyperplane in } T_q \Sigma \text{ for the isoparametric foliation } F_q. \text{ We denote the reflection across this hyperplane by } s. \text{ By definition this element is contained in the Weyl group of } F_q \text{ acting on } T_q \Sigma. \text{ As } q \text{ is in the interior of a wall, there cannot be other singular hyperplanes through the origin of } \nu_q L_q. \text{ Thus } W \text{ is generated by } s. \text{ Any element of } W \text{ can be extended to a leaf preserving isometry of } \Sigma \text{ in } \Gamma. \text{ In this sense we write } W \subset \Gamma. \text{ Then } H_{p, gp} = \text{Fix}(<s>) \text{ and } g = s. \text{ This shows that } \Gamma \text{ is a Coxeter group.} \]

\[ \square \]

The proof shows that the bisector \( H_{p, gp} \) as the fixed point set of a disecting reflection is a connected and totally geodesic hypersurface. Moreover it only consists of singular points, otherwise they would belong to exceptional leaves. Let \( \{A_i\}_{i \in I} \)
be the set of reflection hyperplanes of $\Gamma$. Then the union of the $A_i$’s is the set of singular points in $\Sigma$. We call $A_i$ a singular hyperplane. Each $A_i$ as a bisector of a dissecting reflection is totally geodesic and complete in the Hadamard manifold $\Sigma$, therefore closed and embedded (and itself an Hadamard manifold). The connected components of $\Sigma \setminus \bigcup_{i \in I} A_i$ are by definition the chambers of $\Gamma$. By Corollary 3.8 of [AKLM] the chambers coincide with the Dirichlet domains of regular $\Gamma$-orbits.

The Coxeter groups have been classified for the Euclidean space but not for the hyperbolic space. A priori we can expect complicated Coxeter groups for the transversal holonomy group. But not every Coxeter group can occur.

**Proposition 3.5.** Let $\mathcal{F}$ be a proper singular Riemannian foliation admitting sections on an Hadamard manifold $X$. Let $\Sigma$ be a section of $\mathcal{F}$ and $\Gamma$ be the transversal holonomy group acting on $\Sigma$. Then $\Gamma$ is isomorphic to a (finite) Coxeter group of Euclidean space.

**Proof.** We want to show that $|\Gamma|$ is finite. Then it follows from Cartan’s Theorem that $\Gamma$ has a fixed point $p_0$. Thus $(T_{p_0} \Sigma, d\Gamma)$ is a finite Coxeter group of Euclidean space and completely describes $(\Sigma, \Gamma)$.

Therefore we assume $|\Gamma| = \infty$. Let $C_0 = D$ be a chamber. Let $S$ be the set of generators of $\Gamma$ given by the reflections across the walls of $C$. We claim that the set $R = \{l(g) \mid g \in \Gamma\} \subset \mathbb{N}$ is not bounded, where $l(g)$ denotes the length of $g$ with respect to $S$.

**Case 1:** $|S| < \infty$. If $R$ were bounded, say by a number $N$, then $|\Gamma| \leq |S|^N < \infty$, contradiction.

**Case 2:** $|S| = \infty$. Let us give some definitions first. A sequence of chambers $C_0, \ldots, C_l$, for which any two consecutive chambers have a common wall, is called a gallery of length $l$. It is called minimal if there is no gallery from $C_0$ to $C_l$ of length smaller than $l$. For any chamber $C$, there is a unique $g_C \in \Gamma$ with $g_C(C_0) = C$. This correspondence between chambers and elements of $\Gamma$ is bijective. The length of a minimal gallery from $C_0$ to a chamber $C$ is equal to $l(g_C)$ (see [Gu] for the simple relation between a gallery and a word in $\Gamma$ with respect to $S$). Now let $C_0, \ldots, C_l$ be a minimal gallery. The chamber $C_l$ is given by the reflection of $C_{l-1}$ across the singular hyperplane $A_i$ that contains a common wall. A gallery is minimal if and only if the $A_i, i = 1, \ldots, l$, are pairwise different, since another gallery with the same first and the same last chamber would have to pass each $A_i$ at least once (see Corollary 2 in [Gu]). As the set of walls of a chamber is infinite by assumption we can choose a singular hyperplane $A_{l+1}$ that contains a wall of $C_l$ and is different from the $A_i, i = 1, \ldots, l$. Reflecting $C_l$ across $A_{l+1}$ gives a chamber $C_{l+1}$ which extends the given gallery to a minimal gallery of length $l + 1$. This implies that $R$ is unbounded.

So in any case $R$ is unbounded; thus there is a $g \in \Gamma$ of length larger than $n$, the dimension of a regular leaf. Let $\Gamma$ be a geodesic from a regular point in $C_0$ to a point in $gC_0$. By slightly perturbing the endpoints, we can assume that $\gamma$ meets the singular hyperplanes only one at a time. Then $\gamma$ intersects at least $l(g) > n$ singular hyperplanes. This contradicts the following proposition, since $\gamma$ is an $L_{\gamma(0)}$-geodesic.

**Proposition 3.6.** Let $M$ be a closed and embedded submanifold of an Hadamard manifold $X$ and $\gamma$ be a normal ray of $M$. Then the number of focal points along $\gamma$, counted with multiplicity is not larger than $\dim M$. 

\[\Box\]
Proof. We assume that $\gamma$ is a unit speed geodesic. Let $x \in M$ be the initial point of $\gamma$. We define $f_t : M \to \mathbb{R}$ by $f_t(y) = d(y, \gamma(t))$. The sum of the focal points of $M$ along $\gamma[0,t]$, counted with multiplicity, is equal to the index of $\text{Hess} f_t(x) \subset T_x M$, which is not larger than $\dim M$.

The following lemma is probably well-known.

**Lemma 3.7.** Let $\mathcal{F}$ be an isoparametric foliation on $\mathbb{R}^n$ with compact leaves and $\Sigma$ be a section on which the generalized Weyl group $W$ acts. Then $\text{Fix}(W)$ is equal to the intersection $I$ of all sections through the origin and to the stratum $S$ of zero-dimensional leaves.

**Proof.** We can assume that the leaves are contained in spheres around the origin. We can decompose $(\mathbb{R}^n, \mathcal{F}) = (V, \mathcal{F}_1) \oplus (V^\perp, \mathcal{F}_2)$, where $V$ is a subspace of $\mathbb{R}^n$ and $\mathcal{F}_1$ is an isoparametric foliation, in which one and therefore all regular leaves are full (i.e. not contained in a proper subspace), and where $\mathcal{F}_2$ is the trivial foliation of $V^\perp$ by points. We want to show the chain of inclusions

$$ V^\perp \subset I \subset S \subset \text{Fix}(W) = V^\perp. $$

Let $M$ be a regular leaf of $\mathcal{F}_1$; then $M$ is also a regular leaf of $\mathcal{F}$. The section of $\mathcal{F}_1$ through $q \in M$ has the form $q + \nu_q V$, where $\nu_q V$ is the normal space of $M$ in $V$ at $q$; the section of $\mathcal{F}$ through $q \in M$ has the form $q + \nu_q \mathbb{R}^n M$, where $\nu_q \mathbb{R}^n M$ is the normal space of $M$ in $\mathbb{R}^n$ at $q$. We have the direct orthogonal sum $\nu_q \mathbb{R}^n M = \nu_q V \oplus V^\perp$. Since $M$ meets every section, $V^\perp \subset I$. Now $I \subset S$ since the dimension of the family of sections through a point $x \in \mathbb{R}^n$ considered as a submanifold in $G_k(\mathbb{R}^n)$, $k = \dim \Sigma$, gives the defect $\dim M - \dim L_x$. The zero-dimensional leaves meet $\Sigma$ exactly once. Therefore $S \subset \text{Fix}(W)$. As $W$ acts trivially on $V^\perp \subset \Sigma$ and $\text{Fix}(W|V \cap \Sigma) = \{0\}$ by Proposition 6.2.3. in [PaTe], we have $\text{Fix}(W) = V^\perp$.

**Remark 3.8.** The proof shows that $\mathcal{F}_1$ has only one zero-dimensional leaf, namely the origin, which is the intersection of all sections of $\mathcal{F}_1$.

Now we come back to the case $(X, \mathcal{F})$.

**Proposition 3.9.** Let $S$ be the stratum of leaves of $\mathcal{F}$ with least dimension, say $m$, let $\Sigma$ be a section and $\Gamma$ be the transversal holonomy group acting on $\Sigma$. Then $S \cap \Sigma = \text{Fix}(\Gamma)$ and consequently this is a connected, totally geodesic submanifold. Any leaf $M$ in this stratum is diffeomorphic to $\mathbb{R}^m$ and $\exp^\perp : \nu M \to X$ is a diffeomorphism.

**Proof.** Let $p \in \text{Fix}(\Gamma)$ and $x \in S \cap \Sigma$. We know that $\mathcal{F}$ induces an isoparametric foliation $\mathcal{F}_x$ on $\nu_x L_x$. Let $W$ be its generalized Weyl group acting on $T_x \Sigma$ and extend each element of $W$ to an isometry of $\Sigma$. We have $W \subset \Gamma_x \subset \Gamma_p = \Gamma$. Thus the intersection of all sections through $x$, being equal to $\text{Fix}(W)$ by the second statement of the Slice Theorem and Lemma 3.7, also contains the geodesic $\gamma_p$. Therefore the family of sections through $p$ has the same dimension as that of $x$, so $p \in S$. We have shown $\text{Fix}(\Gamma) \subset S \cap \Sigma$. We prove $\text{Fix}(\Gamma) \supset S \cap \Sigma$ later.

Let $M$ be a leaf through a fixed point $p$ of $\Gamma$. We already know that $M$ has the lowest dimension among all leaves. We want to show that its normal exponential map $\eta := \exp^\perp : \nu M \to X$ is a diffeomorphism. Surjectivity follows from properness. We prove injectivity. Assume $\eta(v_1) = \eta(v_2) = q$ for $v_1, v_2 \in \nu M$ with foot points...
Let $\Sigma_1$ respectively $\Sigma_2$ be a section to which $v_1$ respectively $v_2$ is tangential. Both sections contain $q$. Let $\Gamma'$ denote the transversal holonomy group acting on $\Sigma'$; then $x_1$ is a fixed point of $\Gamma'$, as two transversal holonomy groups acting on $\Sigma$ respectively $\Sigma'$ are conjugate by a leaf-preserving isometry $\Sigma \to \Sigma'$. Let $W$ be the generalized Weyl group of $F_q$ acting on $T_q\Sigma_1$. As before, we consider $W \subset \Gamma'_1$. Now $W \subset \Gamma'_1 \subset \Gamma'_{x_1} = \Gamma_1$. This means that the geodesic $\gamma_{x_1,q}$ is fixed under $\Gamma'_1$ and therefore also under $W$. But $\text{Fix}(W)$ is the intersection of all sections through $q$. In particular $\gamma_{x_1,q}$ is contained in both $\Sigma_1$ and $\Sigma_2$. Thus $x_1 \in \Sigma_2$. So $M$ intersects $\Sigma_2$ in $x_1$ and $x_2$. But $M \cap \Sigma_2 = \Gamma^2(x_2) = \{x_2\}$, hence $x_1 = x_2$. Since $\Sigma_2$ is an Hadamard manifold, $\gamma_{x_1,q} = \gamma_{x_2,q}$ and therefore $v_1 = v_2$ and we have proved injectivity of $\varsigma$.

Injectivity of $\varsigma$ also implies that $M$ has no focal points: assume $v \in \nu M$ is a singular point for $\varsigma$, then $\gamma_{v,0}([0,t])$ is not a minimal normal geodesic of $M$ for $t > 1$. Fix such a $t$. Thus there is also a minimal normal geodesic of $M$ ending in $\gamma_{v}(t)$ which contradicts that $\varsigma$ is injective.

This means that $\varsigma : \nu M \to X$ is a diffeomorphism. We fix a point $q \in X$. Then the squared distance function $f_q = d^2(\cdot, q) : M \to \mathbb{R}$ has only one minimum, say $x$, and no other critical points. By standard Morse theory we can use the flow of the negative gradient field to define a diffeomorphism of $M$ to a ball neighborhood of $x$. Therefore $M''$ is diffeomorphic to $\mathbb{R}^m$.

Now we want to show $S \cap \Sigma \subset \text{Fix}(\Gamma)$. Let $q \in S \cap \Sigma$. We choose $p \in \text{Fix}(\Gamma) \subset S \cap \Sigma$ and define $M = L_p$. As we have shown the normal exponential map of $M$ is a diffeomorphism. In the words of Molino $X$ is a (global) distinguished neighborhood of $M$. Therefore any leaf meets the slices of $M$ transversally. The restriction $\rho$ of the orthogonal projection $X \to M$ to $L_q$ is a surjective diffeomorphism. We want to show that $\rho$ is a diffeomorphism. For $p' \in M$ we have $\rho^{-1}(p') = S_{p'} \cap L_q$, where $S_{p'}$ is the global slice of $M$ through $p'$. By Proposition 2.1.(a) in [Al] $S_{p'}$ is the union of all sections through $p'$. Let $\Sigma'$ be such a section. Then $L_q \cap \Sigma' = \Gamma'(q')$, where $\Gamma'$ is the transversal holonomy acting on $\Sigma'$ and $q' \in L_q \cap \Sigma'$. If a point of $\Gamma'(q')$ were not in the intersection of all sections through $p'$, then $\dim(S_{p'} \cap L_q) > 0$, contradiction. So $S_{p'} \cap L_q$ is in this intersection and is therefore equal to $\Gamma'(q')$. This implies that $\rho : L_q \to M$ is a finite covering with degree $\lvert \Gamma'(q') \rvert = \lvert \Gamma(q') \rvert$. But since $M \cong \mathbb{R}^m$ the map $\rho$ must be a diffeomorphism. In particular $\Gamma(q) = \{q\}$, so $q \in \text{Fix}(\Gamma)$.

Let $N$ be compact Riemannian manifold of negative curvature. Then its isometry group is trivial and in particular $N$ admits no polar actions. In [Wa] it was shown that $N$ admits no Riemannian foliation. Is this also true for singular Riemannian foliations? The theorem below gives a partial answer.

**Theorem 3.10.** Except possibly for regular Riemannian foliations there are no proper singular Riemannian foliations admitting sections on a compact, nonpositively curved Riemannian manifold $N$.

**Proof.** Assume there is a singular Riemannian foliation $F$ on $N$ that admits sections and has singular leaves. Then we can lift it to $\tilde{F}$ to the Hadamard manifold $\tilde{N}$ along the Riemannian universal covering $\pi : \tilde{N} \to N$. Clearly $\tilde{F}$ is a singular Riemannian foliation admitting sections. Let $M$ be a leaf of $\tilde{F}$ of least dimension and $p \in M$. Let $B$ be a ball around $p$ whose radius $r$ is larger than the diameter of $N$; this implies that the translations of $B$ by the action of $\pi_1(N)$ exhausts $\tilde{N}$. The
deck transformation group $\pi_1(N)$ respects $\tilde{F}$ by definition. This implies that $M$ is
mapped onto a leaf of the singular stratum $S$ of leaves of lowest dimension $m$. Let
$\Sigma$ be a section through $p$. By Proposition 3.11 $S \cap \Sigma = \text{Fix}(\Gamma)$ is a connected, totally
geosic submanifold of $\Sigma$. It has at least codimension one because $F$ has regular
and singular leaves. Thus there is a regular point $q \in \Sigma$ with $d_2(q, S \cap \Sigma) > r$. The
distance from $q$ to the orbit $\pi_1(N)p$ attains its minimum at a point $p' \in \pi_1(N)p$.
We have
\[ d(q, \pi_1(N)p) = d(q, p') > d(q, L_{p'}) \]
There is a point $p'' \in L_{p'}$ realizing this distance, i.e. $d(q, p'') = d(q, L_{p'})$. This
point $p''$ has to be in $\Sigma$; it is the unique point of the intersection $L_{p'} \cap \Sigma \subset S \cap \Sigma$.
Now $d(q, p'') > d_2(q, S \cap \Sigma) > r$. Altogether $d(q, \pi_1(N)p) > r$, so the action of $\pi_1(N)$ on $B$ does not cover $q$, contradiction.
\[ \square \]

Now we prove the analogue of Theorem 2.11 for singular Riemannian foliations
admitting sections.

**Theorem 3.11.** Let $F$ be a proper singular Riemannian foliation with sections in
an Hadamard manifold $X$. Then $F$ is the product foliation of a compact isoparametric
foliation and the trivial foliation of $\mathbb{R}^m$ with $\mathbb{R}^m$ as the only leaf.

**Proof.** Let $M$ be a leaf of lowest dimension $m$. The normal exponential map $\exp^\perp : \nu M \to X$ is a diffeomorphism. We denote the orthogonal projection onto $M$ by
$\rho : X \to M$. Let $\mathcal{H}$ be the horizontal distribution of $\rho$. We now want to change the
metrics on $M$ and $X$ such that $\rho$ becomes a Riemannian submersion. First, since
$M \cong \mathbb{R}^m$, we can introduce a flat metric on $M$. Moreover we change the metric
on $X$ by keeping its induced metric on the fibres of $\rho$. We demand that the fibres
are orthogonal to $\mathcal{H}$. Now we change the metric on the horizontal distribution $\mathcal{H}$
of $\rho$ such that $\rho$ becomes a Riemannian submersion. For any point $p \in M$ let $S_p$
be the global slice $\exp^\perp(\nu_p M)$ of $F$. For any $q \in S_p$ let $S_q$ be a local slice. Then
$S_q \subset S_p$ (see Proposition 2.1.1(b) in [Al], also [To]), so $\mathcal{H}_q \subset T_q L_q$. This means that
horizontal lifts of curves in $M$ remain in leaves of $F$. We fix $p \in M$. We identify $M$ and
$T_p M$ and we define a map $T_p M \times \nu_p M \to X$ by mapping $(v, w)$ to the endpoint
of the lift of the straight line $\gamma_w$ in $T_p M$ along $\rho$ to the starting point $\exp^\perp((v) \in S_p$.
This map $\phi$ is a diffeomorphism. On $\nu_p M$ we have the isoparametric foliation $F_p$
and on $T_p M$ the trivial foliation by $\{T_p M\}$. By construction $\phi$ maps leaves of the
product foliation on $T_p M \times \nu_p M$ diffeomorphically onto leaves of $F$. \[ \square \]

**Remark 3.12.** Following the proof we can easily show the above product is metric if
$X$ is Euclidean. Let $X$ be the Euclidean space. Then a leaf $M$ of lowest dimension
is an affine subspace, because its normal exponential map is a diffeomorphism, and
its global slices are the orthogonal, complementary affine subspaces. Let $F'$ denote
the foliation by parallel affine subspaces of $M$. As stated in the proof each other leaf
is a union of leaves of $F'$. This means the orthogonal projection of $X$ onto a given
slice $S$ of $M$ preserves the leaves of $F'$ and therefore the leaves of $F$. Together
with the orthogonal projection onto $M$ we have a metric foliated decomposition
$X = M \times S$. This recovers the result about isoparametric splitting; see Corollary
6.3.12. in [PaTe].

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