Higher order homogenized boundary conditions for flows over rough and porous surfaces

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Abstract We derive a homogenized macroscopic model for fluid flows over ordered homogeneous porous surfaces. The unconfined free-flow is described by the Navier-Stokes equation, and the Darcy equation governs the seepage flow within the porous domain. Boundary conditions that accurately capture mass and momentum transport at the contact surface between these two domains are derived using the multiscale homogenization technique. In addition to obtaining the generalized version of the widely used Beavers-Joseph slip condition for tangential velocities, the present work provides an accurate formulation for the transpiration velocity and pressure jump at fluid-porous interfaces; these two conditions are essential for handling two- and three-dimensional flows over porous media. All the constitutive parameters appearing in the interface conditions are computed by solving a set of Stokes problems on a much smaller computational domain, making the formulations free of empirical parameters. The tensorial form of the proposed interface conditions makes it possible to handle flows over isotropic, orthotropic, and anisotropic media. A subset of interface conditions, derived for porous media, can be used to model flows over rough walls. The accuracy of the proposed macroscopic model is numerically quantified for flows over porous and rough walls by comparing the results from our homogenized model with those obtained from geometry-resolved mi-
croscopic simulations. In addition, this work provides a mathematical confirmation of the boundary conditions proposed by Lácis et al. (2018) to model flow over complex surfaces.

**Keywords** Multiscale homogenization · Beavers Joseph condition · Porous media · Rough wall flows

1 Introduction

Boundary conditions at the interface between a porous medium and the adjacent free-fluid region is a classical problem in fluid mechanics. Despite numerous analytical (Richardson, 1971; Saffman, 1971; Jones, 1973; Chandesris and Jamet, 2006; Jamet and Chandesris, 2009; Lácis and Bagheri, 2016), numerical (Larson and Higdon, 1986, 1987; Carraro et al., 2013; Kuwata and Suga, 2016, 2017), and experimental (Beavers and Joseph, 1967; Taylor, 1971; Gupte and Advani, 1997; Goharzadeh et al., 2005; Agelinchaab et al., 2006; Arthur et al., 2009; Morad and Khalili, 2009; Carotenuto and Minale, 2011; Terzis et al., 2019) efforts to shed light on the nature of such conditions, widely accepted formulations to accurately capture transport phenomena across fluid-porous interfaces are not available yet. In this work, we address this problem by deriving a macroscopic description of two- and three-dimensional flows over porous surfaces using multiscale homogenization approach. In addition, we will show that a reduced form of the formulations can be used to accurately model macroscopic flows over rough surfaces.

By macroscopic description of flows over porous media, we mean all of the following objectives: (1) Obtaining a macroscopic description that can model flows through porous media, and compute the associated constitutive parameters, (2) Deriving coupling conditions at the fluid-porous interface, and (3) Compute constitutive parameters of the interface.

Item (1) of the above list is relatively matured in its development, and various models have been put forward to describe flows through porous media (Lage, 1998). In the present work, we make use of Darcy’s law, which describes a linear relation between creeping velocity inside the porous medium \( \mathbf{u}^- \) and the pressure gradient \( \frac{\partial p^-}{\partial \mathbf{x}} \):

\[
\mathbf{u}^- = -\frac{K^i}{\mu} \cdot \frac{\partial p^-}{\partial \mathbf{x}},
\]

where \( \mu \) is the dynamic viscosity of the fluid and \( K^i \) is the permeability tensor of the medium, which can be computed for any ordered homogeneous porous medium by solving a Stokes problem on a periodic unit cell (Mei and Vernescu, 2010). The present work is focused on the items (2) and (3) i.e., derivation of interface conditions, and the computation of associated constitutive parameters.

In their classical work, Beavers and Joseph (1967) conducted experiments on Poiseuille flow over a porous block to understand the nature of fluid-porous
interface conditions. They observed that the viscous shear from the free-fluid region penetrates into the porous medium, and as a result, the fluid velocity changes from Darcy value in the interior of the porous medium ($u_1$) into a larger slip value at the interface ($u_1^-$). This boundary layer was modeled by the following linear relation between the slip velocity and the shear rate at the interface,

$$u_1^- - u_1 = \frac{\sqrt{K_i}}{\alpha} \frac{\partial u_1}{\partial x_2},$$  
(2)

where subscripts 1 and 2 denote quantities in the direction parallel and normal to the interface, respectively. Moreover, $K_i$ is the permeability of the isotropic porous medium and $\alpha$ is a constitutive parameter that describes the structure of fluid-porous interface. The presence of such shear-driven slip condition is verified by various studies (Agelinchaab et al., 2006; Arthur et al., 2009; Carotenuto and Minale, 2011), and it is the most widely used condition in the simulation of flows over porous media.

Due to the significant practical importance of understanding the flow behavior over porous media, numerous works have attempted to arrive at more accurate and more general fluid-porous interface conditions. Based on the nature of approximations, these works can be classified as either one- or two-domain approach.

The main advantage of the one-domain approach (Vafai and Kim, 1990; Basu and Khalili, 1999; Goyeau et al., 2003; Breugem et al., 2006; Chen and Wang, 2014) is that the flow in the free-fluid and the porous domains are governed by the same form of conservation equations. This is made possible by introducing a heterogeneous transition zone between these two regions. The properties of the flow as well as porous media (porosity, permeability, and apparent viscosity) are assumed to vary rapidly – but continuously – within this transition zone to match the values between the two regions. The thickness of the transition zone as well as the functional form of the spatial variation of macroscopic properties within this zone are essential for the complete mathematical description of the flow field. Although many studies have aimed at understanding the nature of the transition layer (Gupte and Advani, 1997; Goharzadeh et al., 2005; Morad and Khalili, 2009; Khalili et al., 2014), to the best of our knowledge, even for a simple unidirectional channel flow, there are no universally accepted guidelines to specify the properties of this layer. The inability to relate transition layer characteristics to the microstructural details of the porous medium results in an empirical formulation, and hence one-domain approaches require support from physical experiments.

In the two-domain approach, the free-fluid and the porous domains are separated by a sharp interface. Flows within the porous domain are modeled using the Darcy (or other forms such as the Darcy-Brinkmann or Forchheimer) equation; the Navier-Stokes equations describe the free-fluid region. Across the interface, properties may vary continuously or exhibit jumps (Ochoa-Tapia and Whitaker, 1995a,b; Chandesris and Jamet, 2006; Duman and Shavit, 2008). The crucial factor that dictates the accuracy of two-domain approaches is the
**Correct** specification of interface conditions that captures transport phenomena across the interface. Since the experiments of Beavers and Joseph (1967), several researchers have attempted to derive improved interface conditions by employing either boundary layer matching (Jäger et al., 2001) or homogenization methods (Lācis and Bagheri, 2016). The striking feature of this class of methods is the possibility of using microscale geometrical details of the porous medium for computing all coefficients appearing in the interface conditions. This leads to a non-empirical formulation. However, majority of the studies are limited to unidirectional flow over porous layers and are focused on predicting accurate interfacial tangential velocity alone. In the present work, we provide coupling conditions for tangential and transpiration velocities, as well as on the pressure. Two works that are relevant to the present formulations are Carraro et al. (2013) and Lācis and Bagheri (2016). In Carraro et al. (2013), a pressure jump condition across the fluid-porous interface is proposed, which is active only for anisotropic porous layers. They validated their formulation with microscopic simulations of unidirectional channel flows. Lācis and Bagheri (2016) studied two-dimensional flow over (isotropic) porous medium and showed the accuracy of predicting slip and transpiration velocity variation over the interface. Although both studies provide a non-empirical framework, transpiration velocity was absent in the former, while pressure condition was not investigated in the latter.

The present work falls into the category of two-domain approaches. We provide an accurate non-empirical macroscopic description of three-dimensional flow over isotropic as well as anisotropic porous media by employing the multiscale homogenization technique. While previous works (Marciniak-Czochra and Mikelić, 2012; Lācis and Bagheri, 2016) have focused on obtaining only leading order conditions at the fluid-porous interface, the present work provides higher-order terms that are essential to represent relevant physical features as discussed in Lācis et al. (2018). Moreover, higher order terms are essential to accurately capture transpiration velocity and pressure jump across fluid-porous interfaces. This will be demonstrated in section 5.

Similar to porous media, wall roughness also induces a shear-dependent slip velocity (Luchini et al., 1991; Miksis and Davis, 1994; Sarkar and Prosperetti, 1996; Stroock et al., 2002; Kamrin et al., 2010; Luchini, 2013; Bolanos and Vernescu, 2017). For a unidirectional flow, it can be deduced by simply dropping the creeping velocity term from Beavers-Joseph condition given in equation (2), yielding

\[ u_1 = \frac{\sqrt{K^2}}{\alpha} \frac{\partial u_1}{\partial x_2}. \]  

(3)

The goal of this paper is to rigorously derive interface conditions for flows over porous media. We will show that a reduced form of these fluid-porous conditions can be used to accurately model flow over rough surfaces. The main features of the derived results are:

- The proposed method is free of empirical parameters. All the coefficients introduced in the formulation are computed by solving a small set of inexpensive microscale problems.
Fig. 1 Descriptions of flows over a porous domain. (a) microscopic, and (b) macroscopic. Interface conditions on velocity and pressure must be specified in the macroscopic description.

- It is applicable to isotropic, anisotropic, and layered porous media that have different microscale geometry at the boundary from that of the interior.
- A subset of the derived conditions is directly applicable to accurately model the interaction of fluid flows with rough surfaces.

The paper is organized as follows. The macroscopic model for flows over porous and rough walls is described in section 2. For simplicity, §2 considers a flat interface aligned along one of the coordinate axes. The complete derivation of the interface conditions using the multiscale homogenization technique is presented in section 3. §4 elaborates on the auxiliary Stokes problems that are to be solved to compute the constitutive coefficients arising in the interface conditions. In §5, comparison between the proposed macroscopic model and geometry resolved microscopic simulations is presented to demonstrate the accuracy of the present model for rough and porous walls. Finally, conclusions are provided in section 6.

2 Main results

The main result of the present work is the derivation of accurate interface conditions at fluid-porous contact surfaces. A reduced set of these conditions are explained here for a 2D flow over porous- and rough-surfaces for clarity. The full derivation of more generalized conditions is detailed in section 3. It will be demonstrated later in section 5 that the reduced conditions introduced here can be used to model macroscopic flows over rough/patterned surfaces.

2.1 Flow over porous walls

Consider an incompressible viscous flow over a porous medium, as shown in figure 1(a). The configuration has two distinct length scales: microscale ($l$) denoting the length scale of voids in the porous medium, and macroscale ($L$) characterizing the relevant length scale of the fluid flow.
Numerical simulations employing microscopic description (figure 1a) mandates the resolution of all length scales of the problem. Owing to the inherent multiscale nature of such configurations, such simulations are computationally extremely demanding to achieve. This limitation of the microscopic description has motivated the development of macroscopic models (figure 1b) that govern the behavior of averaged quantities; such models do not require us to resolve all geometrical scales, and are thus feasible for addressing such multiscale problems.

In the macroscopic description (figure 1b), an interface is introduced above the porous medium, and the domain is split into a free-fluid region ($\Omega^+$) and the porous part ($\Omega^-$). The flow within the porous medium is upscaled to yield the Darcy equation, and the flow in the free-fluid region is governed by the Navier-Stokes equations. The key factor that decides the accuracy of macroscopic models is the appropriate specification of interface conditions that determine all essential features of transport phenomena across the two domains.

In this work, we derive the fluid-porous interface conditions by using multiscale homogenization approach, as detailed in section 3. We invoke the following assumptions,

1. The porous medium consists of ordered periodic solid inclusions of characteristic length scale $l$, also called as microscopic length scale.
2. The porous medium is homogeneous.
3. The characteristic macroscopic length scale ($L$) is much larger than $l$, i.e., $l \ll L$. This separation of scales is quantified by the parameter, $\epsilon = l/L \ll 1$. This implies that the nature of the flowfield near the patterned surface is unaffected by the details of macroscale flow, and vice versa.
4. Viscous effects dominate inertia near the surface.
5. Characteristic time scale of microscopic flow is much smaller than that of macroscopic flow. This implies that the microscopic flows readjust itself very quickly to any changes in the macroscopic flow. As a consequence, even when the macroscopic flow is unsteady, the behavior of effective microscopic flow need not contain time-dependent terms.

For flows over porous media, interface conditions on velocity and pressure are needed for complete specification of the problem. Generalized interface conditions that are derived using multiscale homogenization approach are given in equation (61). For clarity, we present here the interface conditions for 2D flows with horizontal interface. The condition for tangential velocity is

$$u_1 = L_{11L} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) - \frac{K_{11} l^2}{\mu} \frac{\partial p^-}{\partial x_1} - \frac{K_{12} l^2}{\mu} \frac{\partial p^-}{\partial x_2} + l^2 \left( \lambda_{11} + K_{11} \langle B_i^- \rangle \right) \left( \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_1^2} \right).$$

The combination of the slip-term and the Darcy-like term in equation (4) forms the classical Beavers-Joseph condition. The difference is that $K_{ij}$ appearing in
the above equations denote interface permeability tensor, which is different than the interior permeability used in the Darcy equation. Our model has one additional contribution for \( u_1 \), which accounts for variation of shear. The condition for interface normal velocity is

\[
 u_2 = -\frac{K_{21} l^2}{\mu} \frac{\partial p^-}{\partial x_1} - \frac{K_{22} l^2}{\mu} \frac{\partial p^-}{\partial x_2} + l^2 \left( M_{211} - K_{21} \langle B_1^- \rangle \right) \left( \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_1^2} \right).
\]

(5)

The transpiration velocity \( (u_2) \) has also a Darcy-like term, and an high-order term that accounts for the variation of shear along the interface. The latter term is a result of mass conservation, as elaborated in Lacis et al. (2018). The pressure jump across the interface \([p] = p - p^-\) is

\[
 [p] = \mu B_1 \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) + 2\mu \frac{\partial u_2}{\partial x_2} - A_{11} \frac{\partial p^-}{\partial x_1} - A_{22} \frac{\partial p^-}{\partial x_2} + \mu \left( C_{11} + A_1 \langle B_1^- \rangle \right) \left( \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_1^2} \right).
\]

(6)

The pressure jump derived in the present work has four terms, as marked in equation (6). The PJ-1 term accounts for interface normal force due to the slip velocity (Lacis et al., 2018); this term is non-zero only for anisotropic porous surfaces and it has been previous derived by Marciniak-Czochra and Mikelic (2012). The PJ-2 term is a direct consequence of normal stress balance at the interface. The PJ-3 term accounts for the normal force induced at the interface due to wall-normal velocity at the interface (Lacis et al., 2018). The PJ-4 term is the additional term that arises in our derivation. Here, PJ-3 and PJ-4 terms are the higher order terms, as will be shown in section 3.

In the above equations, \( L_{ij}, K_{ij}, M_{ijk}, B_i, A_i, C_{ij} \) and \( \langle B_1^- \rangle \) denote flow-independent constitutive coefficients that are dictated by the microscopic structure of fluid-porous interface. One of the main feature of this work is that all these coefficients can be computed by solving a number of Stokes problems in a reduced computational domain, as will be explained in section 4. This renders the entire formulation free of any empirical parameters.

2.2 Flow over rough walls

Macroscopic models in this case involve replacing the original rough wall with an equivalent smooth surface that is located slightly above the roughness elements, as shown in figure 2. The domain below the interface is cut off; the effect of geometrical perturbations is converted into perturbations in boundary conditions at the smooth wall, in such a way that these modified boundary conditions represent the averaged effect of roughness on the macroscale flow accurately.
Effective models developed for porous media, in general, contain all essential components for modeling flow over rough walls. The pressure jump condition given in equation (6) is redundant as there is no medium below the interface. Moreover, as will be discussed in section 4, $K_{12} = K_{21} = K_{22} = M_{111} = 0$. After performing the aforementioned simplifications, the interface conditions for 2D flows over rough walls are as follows:

$$u_1 = L_{11} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) - \frac{K_{11} l^2}{\mu} \frac{\partial p}{\partial x} + 2l^2 M_{121} \frac{\partial^2 u_2}{\partial x_1 \partial x_2} - K_{11} l^2 \frac{\partial p}{\partial x},$$

(7)

$$u_2 = l^2 M_{211} \left( \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_1^2} \right),$$

(8)

where $p$ is the free-fluid pressure. We can observe that when compared to interface conditions for porous media (equation 4), an additional term for $u_1$ involving $M_{121}$ appears for rough walls. This arises when rewriting the pore pressure gradient ($\partial p/\partial x_1$) in terms of the corresponding gradient of free-fluid pressure. The exact details can be found in Appendix 1.

The accuracy of these conditions are demonstrated by performing microscopic geometry resolved simulations and comparing the results with our macroscopic model in section 5.

3 Multiscale homogenization

In this section, we derive coupling conditions at the fluid-porous interface by employing multiscale homogenization technique. We consider isothermal incompressible single-phase flow of a Newtonian viscous fluid over an ordered homogeneous rigid porous medium. It is saturated by the same fluid of constant density $\rho$ and dynamic viscosity $\mu$ (figure 3). In the microscopic description, such a fluid flow is described by the Navier-Stokes equations:

$$u_{i,i} = 0,$$

(9a)

$$\rho (u_{i,i} + u_j u_{i,j}) = -p_{,i} + \mu u_{i,jj},$$

(9b)
\[ \Gamma \\
\begin{align*}
\text{ui} &= \text{Ui} + \text{u}^+ i \\
p &= P + p^+ \\
\text{Ui} &= 0; \\
\text{u}^+ i &= \text{u}^- i ; \\
\Sigma \text{U}_{ij} + \Sigma \text{u}^+ ij &= \Sigma \text{u}^- ij \\
\text{ui} &= \text{u}^- i \\
p &= p^- \\
\Omega^+ \\
\Omega^- \\
\text{l} \times \text{l}
\end{align*}
\]

Fig. 3 Flow over a porous block: Notation and decomposition of flow fields.

where \((\cdot)_t\) and \((\cdot)_i\) denote time- and spatial-derivates respectively; \((\cdot)_t = \frac{\partial (\cdot)}{\partial t}\), \((\cdot)_i = \frac{\partial (\cdot)}{\partial x_i}\), and \((\cdot)_{ij} = \frac{\partial (\cdot)}{\partial x_i \partial x_j}\). On the wetted surfaces of the porous medium, the flow obeys no-slip condition: \(u_i = 0\).

The physical microscopic configuration contains two distinct length scales: microscopic (\(l\)) and macroscopic (\(L\)). As per our assumptions, introduced in section 2, scale separation parameter is very small: \(\varepsilon = l/L \ll 1\). In the macroscopic description, (shown in figure 3) a virtual interface (\(\Gamma\)) is introduced, and the domain is decomposed into free-fluid part (\(\Omega^+\)) and the porous region(\(\Omega^-\)).

To denote the equations appearing in the same form on both domains, for example

\[ a^+_{i,j} = c^+_{ij} \quad \text{in } \Omega^+, \]
\[ a^-_{i,j} = c^-_{ij} \quad \text{in } \Omega^-, \]

we use the following notation for compactness

\[ a^\pm_{i,j} = c^\pm_{ij} \quad \text{in } \Omega^\pm. \]

The homogenization process employed in the present work involves the following six steps.

Step 1: Decomposition of the flow field

We decompose the flow field in the free fluid region into the sum of flow fields without the presence of porous medium (\(U_i, P\)) and perturbations introduced due to the porous medium (\(u_i^+, p^+\)):

\[ u_i = U_i + u_i^+; \quad p = P + p^+ \quad \text{in } \Omega^+. \]
Within the porous medium, flow quantities \((u_i^-, p^-)\) are only due to the porous block:

\[
u_i = u_i^-; \quad p = p^- \quad \text{in } \Omega^-.
\]

(11)

On the fluid-porous interface, the following conditions are met

\[
U_i = 0; \quad u_i^+ = u_i^-; \quad \Sigma_{ij}^U n_j + \Sigma_{ij}^u n_j = \Sigma_{ij}^- n_j \quad \text{on } \Gamma,
\]

(12)

where the stress tensors are defined as,

\[
\Sigma_{ij}^U = -P \delta_{ij} + \mu (U_{i,j} + U_{j,i}),
\]

\[
\Sigma_{ij}^u = -p^+ \delta_{ij} + \mu (u_{i,j}^+ + u_{j,i}^+),
\]

\[
\Sigma_{ij}^- = -p^- \delta_{ij} + \mu (u_{i,j}^- + u_{j,i}^-).
\]

(13)

This decomposition will be used in step 3 for obtaining coupled equations governing the perturbation quantities.

Step 2: Order estimates

It is, both experimentally and numerically, well established that the porous medium introduces a non-zero slip velocity \((U_s)\) at the interface \(\Gamma\). Since this slip velocity is the manifestation of the fluctuating velocities of porous medium near the interface, it is natural to assume that

\[
u^± \sim U_s.
\]

(14)

Using this estimate, and also assuming that the perturbation shear stress above the interface is much smaller than the free fluid shear stress \(u^+/l \ll U_f/L\), we can write the following estimate based on the stress balance (equation 12),

\[
u^- l \sim U_s l \sim U_f L,
\]

(15)

where \(U_f\) is the characteristic fast flow velocity. The above expression yields

\[
u^s \sim \epsilon U_f.
\]

(16)

The pressure fluctuations scales naturally with the macroscopic pressure difference \((\Delta P)\)

\[
p^± \sim \Delta P.
\]

(17)

Moreover, we assume that the pressure forces balances viscous effects

\[
\frac{\Delta P}{l} \sim \frac{\mu U_s}{l^2},
\]

(18)

which provides the following estimate

\[
U_s \sim \frac{l \Delta P}{\mu}.
\]

(19)

This inherently leads to the assumption of negligible inertia.
Step 3: Non-dimensionalization of the governing equations

Using the above estimates, we introduce the following non-dimensionalization of variables in the whole domain, with overbar denoting dimensionless quantities,

\[ x_i = \bar{x}_i l, \quad u_i = \bar{u}_i U^s, \quad p = \bar{p} \Delta P = \frac{\bar{p} U_s \mu}{l}, \quad t = \bar{t} T. \]  \( \text{(20)} \)

Here, \( T \) denotes the time scale associated with \((U_s, P)\), and we have used equation (19) for pressure. Introducing the above non-dimensionalization and removing overbar for convenience, the Navier-Stokes equations become

\[ Re_s (St u_{i,t} + u_j u_{i,j}) = -p_{i,i} + u_{i,jj}, \quad u_{i,t} = 0, \]  \( \text{(21)} \)

where \( Re_s = \rho U^s l / \mu \) is the slip Reynolds number, and \( St = l / U^s T \) is the Strouhal number, which relates time scales associated with fast flow and fluctuations. Introducing the decomposition from step 1, we get the governing equations for the fast flow

\[ U_{i,i} = 0 \quad \text{in } \Omega^+, \]
\[ Re_s (St \cdot U_{i,t} + U_j U_{i,j}) = -P_{i,i} + U_{i,jj} \quad \text{in } \Omega^+, \]
\[ U_i = 0 \quad \text{on } \Gamma, \]  \( \text{(22)} \)

and the following set of coupled equations for perturbation quantities

\[ u^\pm_{i,i} = 0 \quad \text{in } \Omega^\pm, \]
\[ Re_s (St \cdot u^+_{i,t} + u^+_j u^+_{i,j} + u^-_j U_{i,j} + U_j u^-_{i,j}) = -p^+_j + u^+_{i,jj} \quad \text{in } \Omega^+, \]
\[ Re_s (u^-_{i,t} + u^-_j u^-_{i,j}) = -p^-_j + u^-_{i,jj} \quad \text{in } \Omega^-, \]
\[ u^+ = u^- \quad \text{on } \Gamma, \]
\[ \Sigma^U_{ij} n_j + \Sigma^u^+_{ij} n_j = \Sigma^u^-_{ij} n_j \quad \text{on } \Gamma, \]  \( \text{(23)} \)

where the stress tensors are defined as,

\[ \Sigma^U_{ij} = -P \delta_{ij} + (U_{ij} + U_{ji}), \]
\[ \Sigma^u^+_{ij} = -p^+ \delta_{ij} + (u^+_{ij} + u^+_{ji}), \]
\[ \Sigma^u^-_{ij} = -p^- \delta_{ij} + (u^-_{ij} + u^-_{ji}). \]  \( \text{(24)} \)

The solution of the coupled system of PDEs given in equation (23) will enable us to quantify the effect of the porous medium on the overlying free flow. In the next steps, we employ formal homogenization to analytically quantify the averaged effect of perturbation introduced by the porous medium.
Step 4: Multiscale expansion of perturbation quantities

In most practical applications involving porous media, there exists a clear scale separation, $\epsilon = l/L \ll 1$. This enables us to employ the method of multiple scales by introducing fast and slow coordinates $x$ and $X = \epsilon x$, respectively. The perturbation quantities are assumed to be functions of both coordinates i.e., $u^\pm_i = u^\pm_i(x, X)$ and $p^\pm = p^\pm(x, X)$. According to the chain rule, the derivative is written as,

$$\frac{\partial (\cdot)}{\partial x} = \frac{\partial (\cdot)}{\partial x} + \epsilon \frac{\partial (\cdot)}{\partial X} = (\cdot)_i + \epsilon (\cdot)_I.$$

Here, the first and second term on the right side denote derivative with respect to microscale and macroscale coordinates. We assume that fluctuation quantities can be written in the form of perturbation expansion as follows:

$$u^\pm_i(x, X) = \epsilon^0 u_i^{\pm(0)}(x, X) + \epsilon^1 u_i^{\pm(1)}(x, X) + \epsilon^2 u_i^{\pm(2)}(x, X) + \ldots$$

$$p^\pm(x, X) = \epsilon^0 p^{\pm(0)}(x, X) + \epsilon^1 p^{\pm(1)}(x, X) + \epsilon^2 p^{\pm(2)}(x, X) + \ldots$$

(26)

Substituting the above expansion into the governing equations of fluctuation quantities (equation 23), and collecting the terms corresponding to different orders of $\epsilon$, we obtain the following hierarchy of equations. We make use of assumptions (4) and (5) presented in section 2.1, which leads to $Re_s \leq \epsilon^2$ and $St \leq 1$, respectively.

**Continuity equation:**

$$O(\epsilon^0): \quad u_i^{\pm(0)} = 0,$$

$$O(\epsilon^1): \quad u_i^{\pm(1)} + u_i^{\pm(0)} = 0.$$  (27)  (28)

**Momentum equations:**

$$O(\epsilon^0): \quad -p_i^{\pm(0)} = -u_i^{\pm(0)}_{i,jj},$$

$$O(\epsilon^1): \quad -p_i^{\pm(1)} + u_i^{\pm(1)}_{i,jj} = -\left[-p_i^{\pm(0)} + 2u_i^{\pm(0)}_{i,jj}\right].$$

(29)  (30)

**Interface conditions:**

$$O(\epsilon^0): \quad u_i^{\pm(0)} = u_i^{\mp(0)},$$

$$O(\epsilon^1): \quad u_i^{\pm(1)} = u_i^{\mp(1)},$$

$$O(\epsilon^0): \quad \Sigma_{ij}^+ n_j + \Sigma_{ij}^{-} n_j = \Sigma_{ij}^{-},$$

$$O(\epsilon^1): \quad \left[-p^{(1)} \delta_{ij} n_j + \left(u_i^{(1)} + u_j^{(1)}\right) n_j + \left(u_i^{(0)} + u_j^{(0)}\right) n_j\right] = 0,$$

(31)  (32)  (33)  (34)

where the stress tensors $\Sigma_{ij}$ are defined in equation (24), and the jump identity across the interface is defined as, $[(\cdot)] = (\cdot)^+ - (\cdot)^-$. 


Step 5: \( \mathcal{O}(\varepsilon^0) \)-formulation and microscale problem

The leading order governing equations for perturbation quantities are given by equations (27) and (29) together with coupling conditions (31) and (33). They constitute a Stokes problem in terms of \( (u_{\pm}^{(0)}, p_{\pm}^{(0)}) \) as shown below:

\[
\begin{align*}
u_{i,i}^{\pm(0)} &= 0 \quad \text{in } \Omega^\pm, \quad (35a) \\
-p_j^{\pm(0)} + u_{i,jj}^{\pm(0)} &= 0 \quad \text{in } \Omega^\pm, \quad (35b) \\
u_i^{\pm(0)} &= u_i^{\mp(0)} \quad \text{on } \Gamma, \quad (35c) \\
\Sigma_{ij} n_j + \Sigma_{ij}^{u^{+}(0)} n_j &= \Sigma_{ij}^{u^{-}(0)} n_j \quad \text{on } \Gamma. \quad (35d)
\end{align*}
\]

We can infer from equation (35d) that the flow is driven by the known traction induced at the interface due to fast flow \( (\Sigma_{ij}^U n_j) \). The fast flow is decoupled from the perturbation quantities, and the traction can be obtained by solving equation (22). Using this fact, and owing to the linearity of the problem, the solution to this system of equations can be constructed as follows

\[
\begin{align*}
u_i^{\pm(0)} &= L_{ij}^\pm(x) S_j(X), \\
p^{\pm(0)} &= B_{ij}^\pm(x) S_i(X), \\
p^{\mp(0)} &= B_{ij}^{\mp}(x) S_i(X) + P^{-}(X), \quad (36)
\end{align*}
\]

where \( S_i = (U_{i,j} |_\Gamma + U_{j,i} |_\Gamma) n_j \) is the traction due to viscous stresses of fast flow, in which \( n_j \) denotes unit normal vector to the interface. The integration constant \( P^{-}(X) \) is dependent only on macroscale coordinates, such that at the interface it reaches the value of fast flow pressure: \( P^{-}(X)|_\Gamma = P|_\Gamma \). Such constants in first two of the above equations are set to zero respectively by (i) no-slip condition on the wetted surfaces of porous solids, and (ii) zero-stress condition at the top of the domain in microscale problems, as will be discussed with section 4. In principle, one could retain all constants and all forcing terms in the solution; this would, however, lead to trivial microscale problems confirming that these terms indeed can be neglected.

Substituting the velocity Ansatz given by equation (36) into the multiscale expansion given in equation (26), the perturbation velocity in the free-fluid domain can be written as quantities can be written as

\[
u_i^+ = L_{ij}^+ S_j + \mathcal{O}(\varepsilon), \quad (37)
\]

and the perturbation pressure in both domains as

\[
\begin{align*}
p^+ &= B_{ij}^+ S_j + \mathcal{O}(\varepsilon), \\
p^- &= B_{ij}^- S_j + P^- + \mathcal{O}(\varepsilon). \quad (38)
\end{align*}
\]

Plugging the equation (37) into the decomposition given in equations (10), we get

\[
u_i^+ = U_i + L_{ij}^+ S_j + \mathcal{O}(\varepsilon). \quad (39)
\]
Note that $u_i^-$ is not considered in the above steps because we are interested in the effect of the porous medium on the free-fluid region, which can be obtained by considering $u_i^+$ alone. A complete specification of coupling conditions at the fluid-porous interface requires conditions on the free-fluid velocities, and a condition relating the stresses in both domains.

Taking average of equations (39) and (38), and using the fact that $U_i|_{\Gamma} = 0$ and $P^-|_{\Gamma} = P|_{\Gamma}$, we get

$$\langle u_i^+ \rangle = \langle L_{ij}^+ (U_{j,k}|_{\Gamma} + U_{k,j}|_{\Gamma})n_k + \mathcal{O}(\epsilon), \quad (40)$$

$$[p] = \left[ \langle B_i^+ \rangle - \langle B_i^- \rangle \right] (U_{j,k}|_{\Gamma} + U_{k,j}|_{\Gamma})n_k + \mathcal{O}(\epsilon), \quad (41)$$

where the pressure jump is given by $[p] = P + (p^+) - (p^-)$.

From the decomposition introduced in step 1, it is clear that $P^+$ and $P^-$ represent pressure in free-fluid and porous domain respectively. The averaging operator used in the above equations is defined as,

$$\langle a^\pm \rangle (X) = \frac{1}{l^3} \int_{l/2}^{l/2} \int_{l/2}^{l/2} \int_{l/2}^{l/2} a(x - X)d\Omega^\pm. \quad (42)$$

Detailed discussion about the location of the averaging volume is provided in section 4.1.

In order to make the equations (40) and (41) into useful conditions at the fluid-porous interface, the fast scale quantities $(U_i, P)$ should be written in terms of the actual flow field quantities $(u_i, p)$. This can be obtained by differentiating the following

$$\langle u_i^+ \rangle = U_i + \langle u_i^{+(0)} \rangle + \mathcal{O}(\epsilon), \quad (43)$$

which yields

$$\langle u_i^+ \rangle_{,j} = U_{i,j} + \langle u_i^{+(0)} \rangle_{,j} + \mathcal{O}(\epsilon). \quad (44)$$

Since averaged quantities are independent of microscale coordinates, we can write

$$U_{i,j} = u_{i,j} + \mathcal{O}(\epsilon).$$

Substituting the above expression into equations (40) and (41), we get the final form of the leading order interface conditions:

$$\langle u_i^+ \rangle = L_{ij} (u_{j,k}|_{\Gamma} + u_{k,j}|_{\Gamma})n_k + \mathcal{O}(\epsilon),$$

$$[p] = B_i (u_{j,k}|_{\Gamma} + u_{k,j}|_{\Gamma})n_k + \mathcal{O}(\epsilon), \quad (45)$$

where $L_{ij} = \langle L_{ij}^+ \rangle$ and $B_i = \langle B_i^+ \rangle - \langle B_i^- \rangle$. The second order tensor $L_{ij}^\pm$ and vector $B_i^\pm$ are constitutive coefficients of the fluid-porous interface, that are dependent only on the geometry of the porous material near the interface. Computation of these coefficients are discussed in detail in section 4.

The aforementioned interface condition for velocity and a part of pressure jump condition have already been reported in Mikelić and Jäger (2000); Jäger and Mikelić (2009); Marciniak-Czochra and Mikelić (2012); Carraro et al.
(2018). As will be explained in section 4, \( L_{2j} = 0 \), and as a result \( u_2 = 0 \). As a consequence, the flow within the porous medium and in the overlying region are completely decoupled from each other; the effect of porous medium is only to introduce a slip velocity at the interface. This suggests that existing models do not capture full physical features of the transport phenomena at the fluid-porous interface.

Based on mass conservation arguments, it has been shown in Lacis et al. (2018) that \( u_2 = 0 \) only when the tangential velocity is constant along the entire length of the interface. When \( \partial u_1 / \partial x_1 \neq 0 \), then mass conservation requires that \( u_2 \neq 0 \). In order to rigorously introduce this physics in our model, we will solve next order perturbation problem in the following.

Step 6: \( \mathcal{O}(\epsilon^1) \) formulation and microscale problem

Equations (28) and (30) together with interface conditions (32) and (34) form the perturbation problem for next order:

\[
\begin{align*}
  u_{i,i}^{\pm(1)} &= -u_{i,1}^{(0)} & \text{in } \Omega^{\pm}, \\
  -p_i^{\pm(1)} + u_{i,jj}^{\pm(1)} &= -\left[-p_i^{(0)} + 2u_{i,jj}^{(0)}\right] & \text{in } \Omega^{\pm}, \\
  u_i^{+ (1)} &= u_i^{- (1)} & \text{on } \Gamma, \\
  \Sigma^{u^+ (1)} n_j + \left(u_{i,j}^{(0)} + u_{j,i}^{(0)}\right) n_j &= \Sigma^{u^- (1)} n_j + \left(u_{i,j}^{(0)} + u_{j,i}^{(0)}\right) n_j & \text{on } \Gamma.
\end{align*}
\]

This is a Stokes problem for the unknowns \((u^{\pm(1)}, p^{\pm(1)})\), forced by the solution of \( \mathcal{O}(\epsilon^0) \)–problem.

Substituting the Ansatz for \( \mathcal{O}(\epsilon^0) \) problem (equation 36) in to the above equation and using the fact that (i) \( L_{ij} \) and \( B_i \) are independent of macroscale coordinate \( X \), and (ii) \( S_j \) is independent of microscale coordinate \( x \), we obtain the following system of equations for \( \mathcal{O}(\epsilon^1) \) problem:

\[
\begin{align*}
  u_{i,i}^{\pm(1)} &= -L_{ik}^\pm D_{ki} & \text{in } \Omega^{\pm}, \\
  -p_i^{\pm(1)} + u_{i,jj}^{\pm(1)} &= -\Sigma_{ij}^L D_{kj} & \text{in } \Omega^{\pm}, \\
  -p_i^{- (1)} + u_{i,jj}^{- (1)} &= \delta_{ij} P_{i,j} - \Sigma_{ij}^{L^-} D_{kj} & \text{in } \Omega^-, \\
  u_i^{+ (1)} &= u_i^{- (1)} & \text{on } \Gamma, \\
  \Sigma_{ij}^{u^+ (1)} n_j &= \Sigma_{ij}^{u^- (1)} n_j & \text{on } \Gamma.
\end{align*}
\]

where \( D_{ij} = S_{i,j} \) and \( \Sigma_{ij}^{L,\pm} = -B_{i,j} \delta_{ij} + 2L_{ik,j}^{\pm} \). Due to linearity, the solution of the above problem can be written as

\[
\begin{align*}
  u_i^{\pm (1)} &= -K_{ij}^{\pm} P_{i,j} + M_{ijk}^{\pm} D_{jk}, & (47a) \\
  p^{\pm (1)} &= -A_{ij}^{\pm} P_{i,j} + C_{ij}^{\pm} D_{ij}, & (47b)
\end{align*}
\]
where $K_{ij}$, $L_{ij}$, $A_i$ and $C_{ij}$ are constitutive tensors that depend on the structure of the porous medium at the interface.

In order arrive at the $\mathcal{O}(\epsilon)$ interface conditions, we use the decomposition and multiscale expansion introduced in equations (10) and (26), respectively,

\begin{align*}
  u_i &= U_i + u_i^{+ (0)} + \epsilon u_i^{+ (1)} + \mathcal{O}(\epsilon^2) \quad \text{on } \Omega^+, \quad (48) \\
  p^+ &= P_i + p_i^{+ (0)} + \epsilon p_i^{+ (1)} + \mathcal{O}(\epsilon^2) \quad \text{on } \Omega^+, \quad (49) \\
  p^- &= p_i^{- (0)} + \epsilon p_i^{- (1)} + \mathcal{O}(\epsilon^2) \quad \text{on } \Omega^- \quad (50).
\end{align*}

Averaging equation (48) at the interface, substituting the Ansatz of the problem, using the fact that $L_{ij}$ and $\epsilon$ are independent of the microscale coordinates, we get

\begin{align*}
  \langle u_i \rangle^+ &= L_{ij} S_j |r + \epsilon M_{ijk} D_{jk} |r - \epsilon K_{ij} P_j |r + \mathcal{O}(\epsilon^2), \quad (51) \\
  \lbrack p \rbrack &= B_i S_j + \epsilon C_{ij} D_ij - \epsilon A_i P_j |r + \mathcal{O}(\epsilon^2).
\end{align*}

The above pressure jump condition is obtained by taking the average of difference between equations (49) and (50). The notations introduced in these equations are $L_{ij} = \langle L_{ij}^+ \rangle$, $M_{ijk} = \langle M_{ijk}^+ \rangle$, $K_{ij} = \langle K_{ij}^+ \rangle$, $B_i = \langle B_i^+ \rangle - \langle B_i^- \rangle$, $C_{ij} = \langle C_{ij}^+ \rangle - \langle C_{ij}^- \rangle$, and $A_i = \langle A_i^+ \rangle - \langle A_i^- \rangle$. They are constitutive coefficients that depend only on the geometry of fluid-porous interface. In equation (51), all terms of the RHS contain fast flow quantities ($U_i$, $P_i$). In order to transfer the equation into interface conditions, we need to express these terms ($S_j$, $D_{ij}$, and $P_j$) in terms of their counterparts defined with actual fields ($u_i$, $p_i$). These expressions are obtained here as follows.

**Relation between $U_{i,j}$ and $u_{i,j}$**

Averaging equation (48), and taking derivative, we get

\begin{equation}
  \langle u_i \rangle_{,j} = U_{i,j} + \langle u_i^{+ (0)} \rangle_{,j} + \epsilon \langle u_i^{+ (0)} \rangle_{,j} + \epsilon \langle u_i^{+ (1)} \rangle_{,j} + \mathcal{O}(\epsilon^2). \quad (52)
\end{equation}

Since averaged quantities are independent of microscale coordinates, $\langle i \rangle_{,i} = 0$, the above equations simplify to

\begin{equation}
  \langle u_i \rangle_{,j} = U_{i,j} + \epsilon \langle u_i^{+ (0)} \rangle_{,j} + \mathcal{O}(\epsilon^2). \quad (53)
\end{equation}

Substituting the Ansatz for $u_i^{+ (0)}$ given by equation (36) in the above equation,

\begin{equation}
  \langle u_i \rangle_{,j} = U_{i,j} + \epsilon (L_{ik} S_k)_{,j} + \mathcal{O}(\epsilon^2), \quad (54)
\end{equation}

and using the fact that $L_{ij}$ is macroscale independent, we can write

\begin{equation}
  \langle u_i \rangle_{,j} = U_{i,j} + \epsilon L_{ik} D_{kj} + \mathcal{O}(\epsilon^2). \quad (55)
\end{equation}

Substituting the above expression in the definition of $S_j = (U_{j,k} + U_{k,j}) n_k$, we get

\begin{equation}
  S_j = s_j - \epsilon L_{km} D_{mj} n_k - \epsilon L_{jm} D_{mk} n_k + \mathcal{O}(\epsilon^2), \quad (56)
\end{equation}

where $s_j = (u_{j,k} + u_{k,j}) n_k$ denotes traction due to viscous stresses of the physical fluid variables.
Relation between $U_{i,jk}$ and $u_{i,jk}$

Taking derivative of equation (53), we obtain
\[
\langle U_{i} \rangle_{jk} = \langle u_{i} \rangle_{jk} + \mathcal{O}(\epsilon^2),
\]
which will be used to express velocity diffusion tensor $D_{ij}$ in terms of physical fluid variables.

Relation between $P_{j}^{-}$ and $p_{i}$

Consider the multiscale expansion for pressure perturbation inside the porous medium,
\[
p^{-} = p^{-(0)} + \epsilon p^{-(1)} + \mathcal{O}(\epsilon^2).
\]

Averaging the above equation at the interface, and taking its derivative we get
\[
\langle p^{-} \rangle_{i} = \epsilon \langle p^{-(0)} \rangle_{j} + \mathcal{O}(\epsilon^2).
\]

Substituting the Ansatz for $p^{-(0)}$ from equation (36), we get the required relation,
\[
\epsilon P_{j}^{-} = \langle p^{-} \rangle_{i} - \epsilon B_{k}^{-} D_{ki} + \mathcal{O}(\epsilon^2).
\]

Final interface conditions

By substituting equations (56), (57), and (60) into equation (51), we get the higher order interface conditions:
\[
\langle u_{i}^{+} \rangle = L_{ij} (u_{j,k} + u_{k,j}) n_{k} + \epsilon \hat{M}_{ijk} (u_{j,ik} + u_{i,jk}) n_{l} - K_{ij} p_{j}^{-} + \mathcal{O}(\epsilon^2),
\]
\[
[p] = B_{i} (u_{i,j} + u_{j,i}) n_{j} + \epsilon \hat{C}_{jk} (u_{j,ik} + u_{i,jk}) n_{l} - A_{i} p_{i}^{-} + \mathcal{O}(\epsilon^2),
\]
where,
\[
\hat{M}_{ijk} = M_{ijk} - L_{ik} L_{mj} n_{m} - L_{im} L_{mj} n_{k} + K_{ik} \langle B_{j}^{-} \rangle,
\]
\[
\hat{C}_{ij} = C_{ij} - B_{j} L_{ki} n_{k} - B_{m} L_{mi} n_{j} + A_{j} \langle B_{i}^{-} \rangle.
\]

In the dimensional setting, the interface conditions presented in equation (61) become:
\[
\langle u_{i}^{+} \rangle = L_{ij} l (u_{j,k} + u_{k,j}) n_{k} + \hat{M}_{ijk} l^2 (u_{j,ik} + u_{i,jk}) n_{l} - \frac{K_{ij} l^2}{\mu} p_{j}^{-},
\]
\[
[p] = \mu B_{i} (u_{i,j} + u_{j,i}) n_{j} + \mu l \hat{C}_{jk} (u_{j,ik} + u_{i,jk}) n_{l} - \mu A_{i} p_{i}^{-}.
\]

In section 2, a simplified form of the above boundary condition is presented for the case of two-dimensional flows over rough and porous wall. The computation of all the constitutive coefficients are elaborated in the following section.
4 Computation of constitutive coefficients

The interface conditions presented in the previous section contain a number of constitutive coefficients, which carry all information about the microscale geometry of complex surfaces. This section details the computation of these coefficients for a porous medium example. The method of computing interior permeability tensor ($K$) (as appearing in the Darcy equation given in equation 1) by solving a Stokes problem on a periodic unit cell (figure 4a) is well established (Mei and Vernescu, 2010). Hence we focus only on the procedure to compute interface coefficients in equation (61).

In order to compute the interface coefficients, we construct an interface cell (figure 4b), which contains one slice of periodic repetitive elements in horizontal direction. In the vertical direction, the interface cell consists of four repetitive solid inclusions of porous media below the interface; the domain extent above the interface is 5$l$. The general methodology of computing coefficients, as will be shown below, requires to solve Stokes system in the interface cell with appropriate coupling conditions at the interface. The flow is driven either by forcing on the interface, or by volume forcing within the porous domain. As far as the boundary conditions on the interface cell are concerned, we enforce zero traction at the top boundary, and periodicity on the vertical sides of the interface cell. The general guideline for boundary condition on the bottom surface is as follows: no-slip condition is applied if the problem is interface-forced; solution obtained at the top surface of the interior problem (figure 4a) is enforced, if the flow is driven by volume forcing.

To compute all the constitutive coefficients appearing in equation (61), we need to solve three sets of Stokes problems, as explained below. The configu-
A macroscopic model for flows over porous media

(a) Microscale problem to compute $L_{11}$, $L_{21}$ and $B_{1}$

(b) Microscale problem to compute $M_{111}$, $M_{211}$ and $C_{11}$

Fig. 5 Computation of coefficients $L_{ij}$, $B_{i}$, $M_{ijk}$, and $C_{ij}$. (a) Solution with unit interface forcing enables computation of $L_{ij}$ and $B_{i}$ (b) Fields of $L_{ij}$ and $B_{i}$ act as source terms in the computation of $M_{ijk}$ and $C_{ij}$.

Rations of first and third problem are discussed in Läcis and Bagheri (2016), however in that study the focus was only on computation of coefficients arising in the velocity boundary conditions.
4.1 Computation of $L_{ij}$ and $B_i$

Substituting the Ansatz of $O(\epsilon^0)$, given in equation (36), into the respective governing equations (35) and requiring the resulting equations to hold for any arbitrary surface shear forcing, we obtain the Stokes problem for $L_{ij}$ and $B_i$:

\begin{align}
L_{ik,i}^\pm &= 0 \quad \text{in} \; \hat{\Omega}^\pm, \quad (63a) \\
-B_{k,j}^\pm + L_{ik,jj}^\pm &= 0 \quad \text{in} \; \hat{\Omega}^\pm, \quad (63b) \\
[L_{ik}] &= 0 \quad \text{on} \; \hat{\Gamma}, \quad (63c) \\
\left[ -B_k \delta_{ij} n_j + (L_{ik,j} + L_{jk,i}) n_j \right] &= -\delta_{ik} \quad \text{on} \; \hat{\Gamma}. \quad (63d)
\end{align}

Equation (63d) states that the flow is driven by a unit forcing applied at the interface. Owing to the interface forcing, on the bottom side of the interface cell, no-slip conditions are enforced. It can be directly inferred from the above equation system that $L_{ik}$ physically denotes the velocity component in $i$-direction due to interface forcing in direction $k$. Moreover, $B_k$ can be interpreted as the pressure jump produced in the microscale problem due to unit interface forcing in direction $k$.

The microscale problem (equation 63) with unit forcing applied parallel to the interface is illustrated in figure 5(a). The solution of this problem provides the fields of $L_{i1}$ (velocity solution) and $B_1$ (pressure solution) on the entire domain. From these fields, the constitutive coefficients can be computed by employing appropriate averaging as explained below.

\begin{align}
\mathcal{L}_{ij} &= \frac{1}{\Omega^+_{\text{avg}}} \int_{\hat{\Omega}^+_{\text{avg}}} L_{ij}^+ d\hat{\Omega}, \quad (64a) \\
B_i &= \langle B_i^+ \rangle - \langle B_i^- \rangle = \frac{1}{\Omega^-_{\text{avg}}} \int_{\hat{\Omega}^-_{\text{avg}}} B_i d\hat{\Omega} - \frac{1}{\Omega^+_{\text{avg}}} \int_{\hat{\Omega}^+_{\text{avg}}} B_i d\hat{\Omega}, \quad (64b)
\end{align}

where $\Omega^+_{\text{avg}}$ and $\Omega^-_{\text{avg}}$ is defined in figure 4(b). The above averaging is justified due to the following reason. The interface introduced in the macroscopic representation (figure 1b) is a lumped representation of the interface cell used in this section. Hence, the relevant quantities averaged over $\Omega^+_{\text{avg}}$ and $\Omega^-_{\text{avg}}$ (in the microscopic interface cell) represent interface constitutive coefficients on the free-fluid and porous domain respectively. The interface cell and the associated averaging can also be interpreted from the point of view of matched asymptotic solutions: if the cell is viewed as the boundary layer (inner solution) between two macroscopic outer solutions (free fluid and porous domain), then the matching should be done at $+\infty$ for the free fluid and at $-\infty$ for the porous domain. These are analogous, respectively, to averaging in $\Omega^+_{\text{avg}}$ and $\Omega^-_{\text{avg}}$ for an interface cell of a finite size.

When the forcing is in interface normal direction, since the bottom boundary carries no-slip condition, the only possibility to maintain the periodicity along vertical sides is that $L_{i2} = 0$, and hence $\mathcal{L}_{i2} = 0$ (Läcis and Bagheri, 2016). Since there is no flow, and a zero stress condition is applied at the
top boundary $\langle B_1^+ \rangle = 0$. Moreover, stress balance on the interface cell lead to $\langle B_2^- \rangle = -1$, which gives us $B_2 = 1$. These values are independent of the type of solid inclusions. In summary, for a 2D interface, the only microscale problem required to compute all effective non-zero $L_{ij}$ and $B_i$ coefficients is the one presented in figure 5(a). Moreover, this is the only microscale problem that need to be solved if we employ $O(\epsilon^0)$ interface conditions presented in equation (45).

To compute additional constitutive coefficients appearing in the higher order interface conditions presented in equation (61), two additional sets of microscale problems need to be solved. We can observe from equation (47) that $O(\epsilon)$-problem is driven by two sources: velocity diffusion tensor $D_{ij}$ and the macroscopic pore pressure gradient $(P^-_j)$. Due to linearity of the equations the effects of these sources can be computed separately, and can be superimposed to find their combined effect. This fact will be used to arrive at the two corresponding microscale problems, as described below.

### 4.2 Computation of $\mathcal{M}_{ijk}$ and $\mathcal{C}_{ij}$

Microscale equations governing the coefficients $\mathcal{M}$ and $\mathcal{C}$ can be found by neglecting $P^-_j$ from Ansatz, given by equation (47), and substituting $u^{\pm(1)}_i = M^{\pm}_{ijk} D_{jk}$ and $p^{\pm(1)} = C^{\pm}_{ij} D_{ij}$ into the $O(\epsilon^1)$ governing equations. This yields the following system of equations:

\begin{align}
M^{\pm}_{ilm,i} &= -L^{\pm}_{ilm} \quad \text{in } \hat{\Omega}^\pm, \quad (65a) \\
C^{\pm}_{ilm,i} + M^{\pm}_{ilm,jj} &= -\left[ -B^{\pm}_{im}\delta_{il} + 2L^{\pm}_{ilm,j} \right] \quad \text{in } \hat{\Omega}^\pm, \quad (65b) \\
\lbrack M_{ijk} \rbrack &= 0 \quad \text{on } \hat{F}, \quad (65c) \\
\lbrack \Sigma^{M\pm}_{ij} n_j \rbrack &= 0 \quad \text{on } \hat{F}, \quad (65d)
\end{align}

where $\Sigma^{M\pm}_{ij} = -C^{\pm}_{ilm} \delta_{ij} + \left( M^{\pm}_{ilm,j} + M^{\pm}_{jlm,i} \right)$. The solution of the $O(\epsilon^0)$ microscale problem acts as source/sink in the continuity equation as well as the body force term in the momentum equation.

Physically $M_{ijk}$ can be interpreted as the velocity components obtained due to $L_{jk}$ acting as the source/sink terms, and $C_{jk}$ quantifies the corresponding pressure jump. As presented in section 2, for a 2D problem, non-zero terms appearing in the interface conditions contain only three coefficients: $M_{111}$, $M_{211}$ and $C_{11}$, whose computation is schematically illustrated in figure 5(b). Since these problems are driven by volume forcing, at the bottom boundary, the corresponding solution obtained from an interior cell is enforced. After solving this microscale problem, $M_{ijk}$ and $C_{jk}$ are computed by appropriate volume integration discussed for the previous microscale problem.

While all coefficients are computed by appropriate domain-averaging (for example equation 64), $M_{211}$ requires special attention. It is a general trend
that all line-averaged coefficients, computed as

$$\langle q \rangle|_y = \frac{1}{l} \int q|_y dx$$

remains a constant in the free-fluid domain. This is in general true for all coefficients, and a representative example is shown in figure 6(a). However, $\langle M_{211} \rangle$ shows linear variation with $y$, due to the non-zero divergence of the corresponding velocity field (equation 65a). This source/sink, driven by the solution field of $L$-problem, together with the periodicity condition along vertical walls does not influence the $M_{111}$ field. For the linearly varying $\langle M_{211} \rangle$, if regular averaging procedure is employed, the constitutive coefficient will be a function of domain length, and convergence can never be obtained. In order to eliminate this issue, $M_{211}$ is obtained by interface averaging

$$M_{211} = \frac{1}{l} \int_{\hat{\Gamma}} M_{211} dx.$$  

(67)

4.3 Computation of $K_{ij}$ and $A_i$

These microscale problems are obtained by neglecting $D_{ij}$, and substituting $u^{(1)}_i = -K^{\pm}_{ij} P_j$ and $p^{(1)} = -A^\pm_i P_j$ in equation (46),

$$K_{ik,i} = 0 \quad \text{in} \hat{\Omega}^\pm,$$  

(68a)

$$A^+_{ik} + K^+_{ik,j} = 0 \quad \text{in} \hat{\Omega}^+,$$  

(68b)

$$A^-_{ik} + K^-_{ik,j} = -\delta_{ik} \quad \text{in} \hat{\Omega}^-,$$  

(68c)

$$K^+_{ik} = K^-_{ik} \quad \text{on} \hat{\Gamma}^+,$$  

(68d)

$$\Sigma^K_{ij} n_j = \Sigma^K_{ij} n_j \quad \text{on} \hat{\Gamma}^-.$$  

(68e)
Fig. 7 Computation of coefficients $K_{ij}$ and $A_i$.

Here $\Sigma K^\pm_{ij} = -A_i \pm \delta_{ij} + \left( K_{ik,j}^\pm + K_{jk,i}^\pm \right)$. The tensor $K_{ij}$ represents the velocity field in direction $i$ due to volume forcing within the porous medium in the direction $j$. Details of the microscale problems necessary to compute all terms of $K_{ij}$ and $A_i$ are depicted in figure 7. After the solving these microscale problems, the necessary constitutive coefficients can be obtained by performing
appropriate averaging:

\[
K_{ij} = \frac{1}{L^2} \int_{\hat{\Omega}_{avg}^+} K_{ij}^+ \, d\hat{\Omega},
\]

(69a)

\[
A_i = \langle A_i^+ \rangle - \langle A_i^- \rangle = \frac{1}{L^2} \int_{\hat{\Omega}_{avg}^+} A_i \, d\hat{\Omega} - \frac{1}{L^2} \int_{\hat{\Omega}_{avg}^-} A_i \, d\hat{\Omega}.
\]

(69b)

As explained in this section, by solving these three sets of microscale problems, we can compute all the coefficients appearing in our formulations apriori.

4.4 Salient points

A few notable points in the computation of the interface coefficients are summarized here. These points are used in arriving at the 2D interface conditions presented in section 2 from the general conditions given in equation (61).

- When the problem is driven by forcing at the interface, at the bottom wall of the interface cell, no-slip conditions are specified. When volume forcing is applied at the porous part of the interface cell, then the interior solution is prescribed.
- \( \mathcal{L}_{12} = 0 \) because due to the combination of no-slip condition at the bottom and periodicity of vertical boundaries, mass conservation requires zero velocity everywhere. This is elaborated in Lacis and Bagheri (2016).
- Irrespective of the geometry of the considered porous medium, \( \langle B_{1}^- \rangle = -1 \) because the forcing is applied in the vertical direction at the interface, and since flow does not occur, the pressure difference balances the applied interface forcing. This leads to \( B_2 = 1 \), because \( \langle B_i^+ \rangle = 0 \).
- \( \mathcal{M}_{12i} = K_{1i} \) and \( \mathcal{M}_{22i} = K_{2i} \) (corresponding pressure terms \( \mathcal{C}_{21} = A_1 \) and \( \mathcal{C}_{22} = A_2 \)). Although these problems are governed by different equations, effectively these are driven by corresponding unit volume forcing in the porous domain. Together with these conditions, the fact that \( \langle B_{2}^- \rangle = -1 \) is used to set the following terms to zero:

\[
\mathcal{M}_{12i} + K_{1i} \langle B_{2}^- \rangle = 0,
\]

\[
\mathcal{M}_{22i} + K_{2i} \langle B_{2}^- \rangle = 0,
\]

\[
\mathcal{C}_{2i} + A_i \langle B_{2}^- \rangle = 0.
\]

- Based on several tests we performed over various geometries, we observe that numerically the following expressions always give zero result:

\[
\mathcal{M}_{112} - \mathcal{C}_{21}^2 + K_{12} \langle B_{1}^- \rangle = 0,
\]

\[
\mathcal{M}_{212} + K_{22} \langle B_{1}^- \rangle = 0,
\]

\[
\mathcal{C}_{12} - \mathcal{L}_{11} B_1 + A_2 \langle B_{1}^- \rangle = 0.
\]

- The interface permeability tensor \( K_{ij} \) is in general non-symmetric.
The following coefficients are non-zero only for anisotropic porous media: \( K_{12}, K_{21}, B_1, \langle B_1^{-} \rangle, A_1, \) and \( M_{111} \).

In order to compute constitutive coefficients for rough surfaces, the bottom boundary of the interface cell will be the physical geometry of rough surface. No-slip condition will be applied irrespective of the type of forcing applied. As a result of the no-slip condition, the mass conservation yields \( K_{21} = 0 \). Moreover, for 2D rough surfaces, also due to mass conservation \( K_{21} = 0 \) and \( M_{111} = 0 \).

5 Results and discussion

In this section, we demonstrate the accuracy of our macroscopic model for flow over rough and porous surfaces. For all test cases, we perform geometry-resolved microscopic simulations, denoted as DNS, which are used as reference values to compare the accuracy of the macroscopic model. We consider three macroscopic models in all the examples: (i) \( O(\epsilon^0) \) model given in equation (45), (ii) Transpiration resistance model presented in Lacis et al. (2018), and (iii) \( O(\epsilon^1) \) model given in equation (61). In the macroscopic models, the averaged influence of microscopic flow fields is represented by the interface conditions. The microscopic geometrical details of the surfaces are lumped into the constitutive coefficients appearing in the interface conditions. Solution process of the macroscopic flow field involves two steps: computation of coefficients, and the solution of the coupled two-domain problem. All the simulations presented in this paper are performed using Freefem++ (Hecht, 2012), an open-source finite element package.

In order to compare the results of DNS with macroscopic simulations, we eliminated the microscopic flow field details by performing ensembled averaging of the flow field. Samples for the ensembled averaging are generated by moving the solid inclusions in the porous medium (or roughness elements). For each successive sampling, the solid inclusions are moved to a small horizontal distance, in such a way that after the required number of sampling, the inclusions are traversed by one microscopic length \( l \). We emphasize here that although DNS can be performed for simple configurations considered here, for real-life porous media applications, DNSs are prohibitively expensive.

5.1 Stokes flow over porous media

In order to quantify the accuracy of our macroscopic description, we consider a steady 2D incompressible Stokes flow within a modified lid driven cavity, in which the bottom half of the cavity is filled with an ordered rigid porous bed. The geometry is similar to that considered in Sipp and Lebedev (2007), but added with a porous bed (figure 8). It consists of a channel ABGI to which a cavity is attached. The fluid flows through the channel inlet AB with a uniform velocity, and a laminar boundary layer develops from point C. The
Fig. 8 Geometry used to study flow over porous media. Cavity flow is driven by the flow through the channel. \(H_p\) is set in such a way that the porous medium contains 5 solid inclusions, and the interface is located at 0.1\(H\) from the top surface of solid inclusions. Boundary conditions in \(AB\) is \(u_1 = 1\) and \(u_2 = 0\). \(CD\), \(EF\), and cavity surfaces are no-slip walls. On \(BC\), \(FG\) and \(AI\), \(u_2 = 0\) and zero shear stress conditions are enforced. (i) isotropic porous medium with \(d = 0.5642\), (ii) anisotropic porous medium with \(h = 0.6\), and (iii) isotropic porous medium whose inclusions at the interface are different from that of interior, \(d_o = 0.5046\) and \(d_i = 0.2523\).

The tensorial form of our interface conditions make it possible to use our macroscopic model for flows over isotropic as well as anisotropic surfaces. In order to demonstrate this point, as shown in figure 8, we consider three different classes microscale geometries: (i) isotropic, consisting of circular solid inclusions, (ii) anisotropic, consisting of triangular inclusions, and (iii) layered porous medium, whose interface geometry is different than that of the interior. Auxiliary microscopic problems, described in section 4, are solved to compute the constitutive coefficients for all these geometries. The coefficients are listed in table 1.

The vortex generated within the cavity induces a negative shear along the entire length of the interface, and as a consequence it introduces a negative slip velocity at the interface. This is illustrated in figure 9(a), which shows tangential velocity along the interface for the layered porous case. It can be seen that all three macroscopic models approximate the tangential velocity accurately, while \(O(\epsilon^1)\) model performing slightly better than the other two; this is quantified in table 2.

For the considered configuration, the slip velocity is spatially varying along the interface, and hence non-zero transpiration velocity arises owing to mass conservation (Lācis et al., 2018). By comparing figures 9(a) and (b), three
Table 1: Constitutive coefficients $L_{11}$, $B$, $K$, and $A$ for the geometries considered in flows over porous media, represented in figure 8. Except $L_{11}$, all other terms in $L_{ij}$ are zero.

|         | $L_{11}$ | $B$       | $K$         | $A$         | $M_{111}$ | $M_{211}$ | $C_{11}$ |
|---------|----------|-----------|-------------|-------------|-----------|-----------|----------|
| isotropic | 0.1516   | (0)       | [0.01295 0] | (0 0.14389) | 0         | -0.01295 | -0.2912  |
| anisotropic | 0.1807   | (0.3831)  | [0.01943 0.00472] | (0.01942 0.05122) | 0.000354 | -0.01755 | -0.4185  |
| layered  | 0.1538   | (0)       | [0.01329 0] | (0 2.569)   | 0         | -0.01329 | -0.30142 |

regions in the transpiration velocity curve can be seen (i) $u_2 > 0$ when $\frac{du_1}{dx_1} < 0$, (ii) $u_2 < 0$ when $\frac{du_1}{dx_1} > 0$, and (iii) $u_2 = 0$ when $\frac{du_1}{dx_1} = 0$. These observations are consistent with the physical arguments presented in Lâcis et al. (2018). As far as the performance of macroscale models are concerned, the $O(\epsilon^0)$ model predicts zero transpiration, owing to the fact that $L_{2j} = 0$. Hence, in certain cases, this model will not be able to capture transport phenomena across fluid-porous interfaces. It has been shown in Lâcis et al. (2018) that owing to this limitation, the $O(\epsilon^0)$ model produces unphysical results in a turbulent flow simulation. While the TR model and the $O(\epsilon^1)$ model captures the trend in variation of transpiration velocity along the entire length of the interface, the $O(\epsilon^1)$ model produces more accurate predictions.

In order to quantify the errors introduced in all macroscopic models, we compare minimum tangential velocity ($u_{1\text{min}}$) and maximum transpiration velocity ($u_{2\text{max}}$) of the interface quantities. This comparison is presented for all three porous cases in table 2. It can be seen that the tangential velocity is accurately predicted by all three macroscopic models. While transpiration velocity is well-approximated by both TR and $O(\epsilon^1)$ models, the later produces more accurate results.

It has been shown that the $O(\epsilon^1)$ model can predict both velocity components accurately. For capturing transport phenomena across fluid-porous interfaces, the model needs to predict the pressure variation across the interface, in addition to velocity components. In order to show this, in figure 9(c), we plot pressure variation along the dashed vertical line shown in figure 8. It is evident that pressure jump exists across the fluid-porous interfaces. While the $O(\epsilon^0)$ model poorly approximate pressure jump, both TR and $O(\epsilon^1)$ model provide very good prediction of pressure jump.
Fig. 9 Cavity with channel: layered porous inclusions.

Table 2 Stokes flow over cavity with porous bed.

| Isotropic     | Anisotropic | Layered     |
|---------------|-------------|-------------|
| $u_{1}^{\text{min}}/U_0$ | $u_{1}^{\max}/U_0$ | $u_{2}^{\text{max}}/U_0$ |
| $-2.3502 \times 10^{-3}$ | $1.4104 \times 10^{-3}$ | $2.38923 \times 10^{-4}$ |
| $-2.77153 \times 10^{-3}$ | $1.710 \times 10^{-4}$ | $3.28$ |
| $-2.36893 \times 10^{-3}$ | $2.38923 \times 10^{-4}$ | $3.28$ |

| Error in $u_{1}^{\text{min}}$ (%) | Error in $u_{1}^{\max}$ (%) | Error in $u_{2}^{\text{max}}$ (%) |
|----------------------------------|-----------------------------|-------------------------------|
| $\mathcal{O}(\epsilon^0)$ model | TR model | $\mathcal{O}(\epsilon^1)$ model |
| $3.71$ | $3.85$ | $0.12$ |
| $4.85$ | $5.00$ | $0.21$ |
| $3.28$ | $3.59$ | $1.61$ |

DNS

O($\epsilon^0$) model

TR model

O($\epsilon^1$) model

(a) Tangential velocity

(b) Transpiration velocity

(c) Pressure variation along the vertical dashed line shown in figure 8
5.2 Cavity with rough bottom:

In this section, we show that the interface conditions introduced in section 2 can accurately capture the effect of roughness elements on the overlying free-fluid region.

The configuration considered is same as that in the previous section: Stokes flow through a channel, which drives the flow inside the cavity. However, instead of the porous block at the bottom, a solid wall with roughness elements is placed (figure 10). The scale separation parameter (\( \epsilon \)) is 0.1, and we consider three different roughness geometries (semi-elliptical, triangular, and square elements) as shown in the figure 10.

**Semi-elliptical roughness**

The nature of flow field within the cavity is same as that described in the previous section. The flow through the channel drives the fluid flow inside the cavity within which a large vortex is generated. This vorticity induces a negative shear on the interface. As in the previous example, consistent with our boundary conditions, it induces a negative tangential velocity at the interface, as shown in figure 11(a). The shape of transpiration velocity curve (figure 11b) is same as that of previous section, for the same physical reason. We can see that the present model, and the transpiration resistance model very accurately capture the distribution of velocity components along the entire length of the interface.

In table 3, we compare minimum tangential velocity (\( u_{1 \min} \)) and maximum transpiration velocity (\( u_{2 \max} \)) produced by different models against the DNS
values, for various interface locations. The presented values indicate that the error incurred in the transpiration velocity is usually larger than that of the tangential velocity. Also, it is evident that the higher order interface conditions produce results with superior accuracy when compared to the other two models for all interface locations.

The influence of the interface location is taken into account in the macroscopic model by the corresponding change in the values of constitutive coefficients. This is quantified in Table 4, which shows that as the interface is moved away from the top surface of solid inclusions, the value of constitutive coefficients increases. This observed behavior is due to the fact that as the interface moves up, it experiences reduced resistance offered by the rough wall to the overlying fluid flow.

Influence of roughness geometry

In order to show that the effective models produce consistent and predictive results, we present the results for the cavity problem with square and triangular inclusions. The quantitative comparison of $u_{1}^{\text{min}}$ and $u_{2}^{\text{max}}$, presented earlier for

![Diagram](image-url)
Table 4: Constitutive coefficients for semi-elliptic roughness elements for various interface locations. As discussed in section 4, $M_{121} = K_{11}$.

| $\frac{\varepsilon^2}{L}$ | $C_{11}$ | $K_{11}$ | $M_{211}$ |
|--------------------------|--------|--------|----------|
| 0.0                      | 0.0969765 | 0.0045239 | -0.0045253 |
| 0.1                      | 0.169692 | 0.0164874 | -0.0164876 |
| 0.2                      | 0.269692 | 0.0384566 | -0.0384569 |
| 0.3                      | 0.369692 | 0.0704258 | -0.0704261 |
| 0.4                      | 0.469692 | 0.1123950 | -0.1123950 |
| 0.5                      | 0.569692 | 0.1643640 | -0.1643650 |

Semi-elliptical inclusions, are repeated here for square and triangular elements, respectively in tables 5 and 6. It can be seen from the tables that similar to the semi-elliptic geometries, the $O(\epsilon^1)$ model produces more accurate results when compared to the other two models. This confirms that the higher order interface conditions produces more accurate results in a consistent manner, by introducing more physical features of the problem into the formulations.

As discussed earlier, the constitutive coefficients contain all details about the geometrical details of rough/porous interfaces. The difference in the geometrical details of roughness accordingly modified the values of the coefficients (not shown here).

Table 5: Stokes flow over cavity with rough walls with square inclusions

| $\frac{\varepsilon^2}{L}$ | $u_1^{min}$ DNS | $u_2^{max}$ DNS | $u_1^{min}$ | $u_2^{max}$ |
|--------------------------|-------------------|-------------------|-------------|-------------|
| 0.0                      | -3.0948 $\times$ 10^{-3} | 3.324 $\times$ 10^{-6} | 15.17 $\pm$ 0.14 | 15.17 $\pm$ 0.14 |
| 0.1                      | -1.92885 $\times$ 10^{-3} | 4.227 $\times$ 10^{-5} | 2.97 | 3.01 |
| 0.2                      | -3.46361 $\times$ 10^{-3} | 1.3432 $\times$ 10^{-4} | 4.33 | 4.44 |
| 0.3                      | -4.91824 $\times$ 10^{-3} | 2.7559 $\times$ 10^{-4} | 5.32 | 5.55 |
| 0.4                      | -6.29809 $\times$ 10^{-3} | 4.6247 $\times$ 10^{-4} | 5.96 | 6.33 |
| 0.5                      | -7.69808 $\times$ 10^{-3} | 6.9176 $\times$ 10^{-4} | 6.28 | 6.82 |

5.3 Backward facing step with rough bottom:

The previous two examples demonstrated the improved accuracy of $O(\epsilon^1)$ interface conditions for Stokes flow over rough and porous surfaces. As a further validation, in this section, we report results of a finite Reynolds number incompressible flow over a backward facing step with rough bottom. The configuration and boundary conditions are depicted in figure 12. The height of the step is $H$, and the length is taken to be 30 times that of height. A parabolic velocity profile is specified at the inlet with averaged velocity given as $U_{avg}$, and the boundary conditions at outlet are $u_2 = 0$ and zero shear. On
Table 6 Stokes flow over cavity with rough walls with triangular inclusions.

| $x_2$ | $u_1^{min}$ DNS | $u_2^{max}$ DNS | Error in $u_1^{min}$ (%) | Error in $u_2^{max}$ (%) |
|-------|-----------------|-----------------|------------------------|------------------------|
| 0.0   | $-1.36365 \times 10^{-3}$ | $3.753 \times 10^{-4}$ | $7.61 \pm 0.1$ | $4.35 \pm 0.1$ | $21.13 \pm 0.39$ | $17.04 \pm 0.42$ |
| 0.1   | $-2.88761 \times 10^{-3}$ | $1.1178 \times 10^{-4}$ | $5.57$ | $6.65$ | $15.42$ | $8.47$ |
| 0.2   | $-4.33469 \times 10^{-3}$ | $2.3427 \times 10^{-4}$ | $5.97$ | $6.15$ | $14.14$ | $4.22$ |
| 0.3   | $-5.70736 \times 10^{-3}$ | $4.0237 \times 10^{-4}$ | $6.38$ | $6.69$ | $14.19$ | $1.37$ |
| 0.4   | $-7.01077 \times 10^{-3}$ | $6.1287 \times 10^{-4}$ | $6.61$ | $7.09$ | $14.37$ | $1.07$ |
| 0.5   | $-8.24953 \times 10^{-3}$ | $8.6292 \times 10^{-4}$ | $6.62$ | $7.29$ | $14.55$ | $3.36$ |

Fig. 12 Flow over a backward facing step with rough walls.

Fig. 13 Streamlines for flow over a backward facing step. (a) showing separated vortex, (b) recirculation region within the roughness elements.

all other surfaces, no slip condition is enforced. The geometry of triangular-shaped roughness elements and the interface definition are exactly same as that of the previous example. Scale separation parameter $\epsilon = 0.1$ and the Reynolds number, $Re = U_{avg} H/\nu = 100$. It is well-known that the step induces flow separation, and as a consequence forms a large vortex, as shown in figure 13(a). A small distance downstream of the vortex reattachment point, the flow behaves as if it is a fully developed flow through a channel. The vortex behind the step interacts with the triangular-shaped roughness elements at the bottom surface. Due to inertial effects, as shown in figure 13(b), a local recirculation is formed within each roughness elements. The velocity magnitude at the crest plane of roughness elements is $U_s \sim 10^{-2}$, and the slip Reynolds number $Re_s \sim 1$.

In order to quantify the accuracy of our interface conditions, as in the previous examples, we report tangential and transpiration velocities along the
interface located at $x_2^l = 0.2l$. It can be seen from figure 14(a) that the $O(\epsilon^0)$ and TR model produces results of same level of accuracy, while the $O(\epsilon^1)$ model’s accuracy is better. The accuracy of all three effective models for tangential velocity with respect to interface height is presented in table 7. It can be seen that even for this problem, $O(\epsilon^1)$-model is more accurate than other two alternatives. Moreover, as can be seen from figure 14(b), the transpiration velocity is also very accurately predicted by TR and $O(\epsilon^1)$-model with the later producing slightly better results.

These results are surprising because in theory our $O(\epsilon^1)$-model is valid only when $Re_s < \epsilon^2$; for the present example, $Re_s = 1$, in addition to having a small recirculation region within the roughness elements. However, additional numerical examples must be carried out before concluding that our model can produce accurate results in the presence of separation within the roughness elements.

**Table 7** Backward facing step with triangular inclusions.

| $x_2^l$ | $u_2^{lin}$ | Error in $u_2^{lin}$ (%) | $O(\epsilon^0)$ model | TR model | $O(\epsilon^1)$ model |
|---------|-------------|--------------------------|------------------------|----------|----------------------|
| 0.1     | $-2.66089 \times 10^{-2}$ | 7.34 | 7.54 | 0.55 |
| 0.2     | $-3.88864 \times 10^{-2}$ | 7.60 | 8.11 | 0.27 |
| 0.3     | $-5.11105 \times 10^{-2}$ | 8.57 | 9.27 | 0.14 |
| 0.4     | $-6.26855 \times 10^{-2}$ | 9.52 | 10.56 | 0.04 |
| 0.5     | $-7.36112 \times 10^{-2}$ | 10.40 | 11.84 | 0.08 |

6 Conclusions

In this paper, we presented the derivation of a macroscopic model for flows over porous surfaces. The proposed model enables the simulation of three-
dimensional flows over porous and rough surfaces. The tensorial nature of the proposed interface conditions makes it possible to deal with isotropic, anisotropic, and layered porous structures in 3D. For clarity of exposition, we provided simplified formulations to deal with 2D flows over flat rough/porous interfaces that are aligned along one of the coordinate axes. By solving a set of auxiliary Stokes problems, we can compute all the constitutive coefficients appearing in the macroscopic interface conditions, making the formulations free of any empirical parameters. Thus, it involves neither trial-and-error procedure nor empirical curve fitting. Moreover, since the domain to solve the auxiliary equations is only a tiny fraction of the total computational domain, the solution process is very efficient, and hence it does not add noticeable computational overhead to the solution of the equations describing the macroscopic description. Accuracy of the proposed formulation is studied by comparing the results from our macroscopic model to those obtained from geometry resolved simulations of microscopic texture. These comparisons show that the proposed macroscopic model is accurate in modeling flows over rough/porous surfaces in the Stokes regime. An additional example shows the predictive capability of the derived interface conditions even when including inertial effects. These aspects are crucial and make the present formulation a viable tool to map the microstructural details of complex porous/rough surfaces to the associated transport phenomena, without inputs from expensive experiments.

Appendix 1: Relation between porous and rough wall interface conditions

The velocity boundary condition for the porous wall, given in equation (61), reads

\[ u_i = \mathcal{L}_{ij} S_j + \epsilon \left( \mathcal{M}_{ijk} - \mathcal{L}_{ik} \mathcal{L}_{mj} n_m - \mathcal{L}_{im} \mathcal{L}_{mj} n_k + \mathcal{K}_{ik} \langle B_j^- \rangle \right) D_{jk} 
- \mathcal{K}_{ij} \langle p^- \rangle_j + O(\epsilon^2). \]  

In order to use this condition for a rough wall, we need write quantities defined in porous domain (\( \langle p^- \rangle_j \) and \( \langle B_j^- \rangle \)) in terms of their equivalent free-fluid quantities. This is because in the effective simulations of rough wall flows, we cut off the domain below the interface, and hence these quantities are undefined.

In order to do so, we consider the pressure jump condition given in equation (61),

\[ [p] = \mathcal{B}_i (u_{i,j} + u_{j,i}) n_j + \epsilon \mathcal{C}_{jk} (u_{j,ik} + u_{i,jk}) n_l - \mathcal{A}_i \mathcal{P}^-_l + O(\epsilon^2). \]  

Taking derivative along the interface tangential direction of the above expression, we obtain

\[ (P + \langle p^+ \rangle)_j - \langle p^- \rangle_j = \mathcal{B}_k D_{kj} + O(\epsilon^2). \]
Substituting the above expression in equation (70), we get the desired result,
\[ u_i = \mathcal{L}_{ij} S_j + \epsilon \left( \mathcal{M}_{ijk} - \mathcal{L}_{ik} \mathcal{L}_{mj} n_m - \mathcal{L}_{im} \mathcal{L}_{mj} n_k + K_{ik} \langle B_j^+ \rangle \right) D_{jk} \]
\[ - K_{ij} \langle p^+ \rangle_j \right) + O(\epsilon^2), \] (73)

where \( p = P + \langle p^+ \rangle \), which is the quantity defined in the free-fluid region. The above expression is used to arrive at the boundary conditions for rough walls given in equation (8).

We stress that the derivative of pressure, defined in equation (72) is valid only along the direction tangent to the interface. Derivative of \( B \) along the interface normal direction is undefined, which means it is not possible to relate wall-normal pressure gradient across either sides of the interface using such an expression.

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