Об одной сумме интегральных преобразований Ганкеля–Клиффорда функций Уиттекера

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Аннотация

В статье [11] авторами рассматривалась реализация $T$ представления группы $SO(2, 2)$ в одном пространстве однородных функций, заданных на $2 \times 4$-матрицах. Настоящее продолжение этой статьи посвящено вычислению матричных элементов тождественного оператора $T(e)$ и операторов представления $T(g)$ для подходящих элементов $g$ группы относительно смешанного базиса, соответствующего двум различным базисам пространства представления, и вычислению некоторых несобственных интегралов, содержащих произведение функций Бесселя–Клиффорда и Уиттекера. Полученные результаты могут быть переписаны на языке интегральных преобразований Ганкеля–Клиффорда и их аналогов. Первое и второе преобразования Ганкеля–Клиффорда, введенные соответственно Хейком и Перезом–Робайн, играют важную роль в теории дифференциональных операторов дробного порядка (см., например, [6, 8]). Близкий результат получен авторами недавно [12] для регулярной кулоноской функции.

Ключевые слова: группа $SO(2, 2)$, матричные элементы представления, интегральные преобразования Ганкеля–Клиффорда, интегральное преобразование Макдональд–Клиффорда, функции Уиттекера, функции Бесселя–Клиффорда.

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On one sum of Hankel–Clifford integral transforms of Whittaker functions

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Abstract

In [11], the authors considered the realization $T$ of $SO(2,2)$-representation in a space of homogeneous functions on $2 \times 4$-matrices. In this sequel, we aim to compute matrix elements of the identical operator $T(e)$ and representation operator $T(g)$ for an appropriate $g$ with respect to the mixed basis related to two different bases in the $SO(2,2)$-carrier space and evaluate some improper integrals involving a product of Bessel-Clifford and Whittaker functions. The obtained result can be rewritten in terms of Hankel-Clifford integral transforms and their analogue. The first and the second Hankel-Clifford transforms introduced by Hayek and Pérez-Robayna, respectively, play an important role in the theory of fractional order differential operators (see, e.g., [6, 8]). The similar result have been derived recently by the authors for the regular Coulomb function in [12].

Keywords: group $SO(2,2)$, matrix elements of representation, Hankel-Clifford integral transform, Macdonald-Clifford integral transform, Whittaker functions, Bessel-Clifford functions.

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1. Introduction and preliminaries

We recall the definitions and notations in [11]. The group $SO(2,2)$, which preserves the quadratic form $\mathcal{E}$ defined in $\mathbb{R}^4$ whose matrix with respect to the canonical basis is a diagonal matrix $e_{2,2} = \text{diag}(1,1,-1,-1)$, which is called the split orthogonal group and consists of real $4 \times 4$ matrices $g$ satisfying the equality $ge_{2,2}g^t = e_{2,2}$. Here and throughout, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, negative real numbers, integers and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $L$ be a real linear space
consisting of real $2 \times 4$ matrices. We define the cone $\Lambda$ in $L$ by the subset of matrices of rank 2 satisfying the equation $x e_{2,2} x^t = \text{diag}(0, 0)$. Let $\mathfrak{L}$ be the complex linear space consisting of infinitely differentiable functions defined on $\Lambda$ and satisfying the equality $f(bx) = |b_{11}|^\sigma_1 |b_{22}|^\sigma_2 f(x)$ for a fixed pair $(\sigma_1, \sigma_2) \in \mathbb{C}^2$ and arbitrary non-degenerate matrix $b = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}$. We consider the $SO(2, 2)$-representation $T$ in $\mathfrak{L}$ defined by formula $T(g)[f(x)] = f(xg)$. In [14], with a view to investigating some special functions of matrix argument, this construction has been used.

Shilin and Choi [11] dealt with the spherical section $\omega_1$ of $\Lambda$ consisting of matrices

$$\tilde{x}(\alpha_1, \beta_1) = \begin{pmatrix} \cos \alpha_1 & -\sin \alpha_1 & \cos \beta_1 & -\sin \beta_1 \\ \sin \alpha_1 & \cos \alpha_1 & \sin \beta_1 & \cos \beta_1 \end{pmatrix} \quad (\alpha_1, \beta_1 \in [0, 2\pi)). \quad (1)$$

In particular, they [11] showed that for any $x \in \Lambda$ there are a low triangular non-degenerate $2 \times 2$-matrix $b$ and $\tilde{x} \in \omega_1$ such that $x = b\tilde{x}$. If $\mathfrak{L}_1$ is the linear space of restrictions of functions $f \in \mathfrak{L}$ on $\omega_1$, we can realize the representations $T$ as the same representation in $\mathfrak{L}_1$. They also showed that the function $f = \exp(ip_1 \alpha_1) \exp(\overline{iq_1} \beta_1)$ defined on $\omega_1$ does not belong to $f = \exp(ip_1 \alpha_1) \exp(\overline{iq_1} \beta_1)$ if and only if the sum $p_1 + q_1$ is not divisible by 2, and defined the canonical basis

$$\tilde{B}_1 = \{ \tilde{f}_{p, q_1}(\alpha_1, \beta_1) = \exp(ip_1 \alpha_1) \exp(\overline{iq_1} \beta_1) \mid p_1, q_1 \in \mathbb{Z}, p_1 + q_1 \equiv 0(\text{mod } 2) \},$$

which is orthonormal with respect to the scalar product

$$f * g = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(\alpha_1, \beta_1) \overline{g(\alpha_1, \beta_1)} \, d\alpha_1 \, d\beta_1$$

in $\mathfrak{L}_1$. Writing Cartan decomposition $g = g_1 g_2 g_3$ for an arbitrary element $g$ of the group $SO(2, 2)$, where $g_1, g_3 \in SO(2) \times SO(2)$ and

$$g_2 \in \exp \left( \frac{0}{\text{diag}(\mu, \nu)} \begin{pmatrix} \text{diag}(\mu, \nu) & 0 \\ 0 & \text{diag}(\mu, \nu) \end{pmatrix} \right),$$

they showed that in case $|\nu| \neq |\mu|$ the matrix elements of the linear operator $T(g)$ with respect to $\tilde{B}_1$ can be written as a product of four exponential functions, depending respectively on four parameters of the rotations $g_1$ and $g_3$, and two Gaussian hypergeometric functions depending (respectively) on $(\tanh \frac{\pi \mu + \nu}{2})^2$.

The parabolic section $\omega_2$ of $\Lambda$ has been defined as the subset consisting of matrices

$$\tilde{x}(\alpha_2, \beta_2) = \begin{pmatrix} 1 & \alpha_2 & \cos \beta_2 - \alpha_2 \sin \beta_2 & \sin \beta_2 + \alpha_2 \cos \beta_2 \\ 0 & 1 & -\sin \beta_2 & \cos \beta_2 \end{pmatrix},$$

where $\alpha_2 \in \mathbb{R}$ and $\beta_2 \in [0, 2\pi)$. If $\mathfrak{L}_2$ is the linear space of restrictions of functions $f \in \mathfrak{L}$ on $\omega_2$, then the canonical basis in $\mathfrak{L}_2$ can be defined as follows:

$$\tilde{B}_2 = \{ \tilde{f}_{p_2, q_2}(\alpha_2, \beta_2) = \exp(ip_2 \alpha_2) \exp(\overline{iq_2} \beta_2) \mid p_2 \in \mathbb{R}, q_2 \in [0, 2\pi) \}.$$

They [11] established the one-to-one correspondence between the restrictions of $T(g)$ to $\mathfrak{L}_2$ and integral operators whose kernels can be described in terms of some Bessel functions.

2. Two bases in $\mathfrak{L}$ and our purpose

Let us denote the determinant $\det \begin{pmatrix} x_{1m} & x_{1n} \\ x_{2m} & x_{2n} \end{pmatrix}$ inside the matrix $x \in \Lambda$ by $\Delta_{m,n}$ and introduce the basis

$$B_1 = \{ f_{p_1, q_1}(x) \mid p_1, q_1 \in \mathbb{Z}, p_1 + q_1 \equiv 0(\text{mod } 2) \}$$
in \( L \), consisting of functions
\[
f_{p_1,q_1}(x) = \left( x_{11}^2 + x_{12}^2 \right)^{\frac{1}{2}} |\Delta_{1,2}|^{\sigma_2 - q_1} (x_{11} - i x_{12})^{p_1 - q_1} (\Delta_{1,3} + i \Delta_{1,4})^{q_1}.
\]
Obviously the restriction of \( f_{p_1,q_1} \) to \( \omega_1 \) coincides with \( i^{q_1} \tilde{f}_{p_1,-q_1} \):
\[
f_{p_1,q_1}|_{\omega_1} = i^{q_1} \tilde{f}_{p_1,-q_1}.
\] (2)

In this paper, we also use the basis
\[
B_2 = \{ f_{p_2,q_2}(x) \mid p_2 \in \mathbb{R}, q_2 \in \mathbb{Z} \},
\]
where
\[
f_{p_2,q_2}(x) = |x_{11}|^{\sigma_1 - \sigma_2} |\Delta_{1,2}|^{\sigma_2 - q_2} (\Delta_{1,3} + i \Delta_{1,4})^{q_2} \exp \frac{ip_2 x_{12}}{x_{11}}.
\]
It is easy to see that \( f_{p_2,q_2} \) is an extension of the function \( i^{q_2} \tilde{f}_{p_2,q_2} \) to \( \Lambda \).

Let \( \text{span}(f_{p_1,q_1}, f_{p_1,\dot{q}_1}) \) be the subspace in \( \mathbb{L}_1 \). It is invariant with respect to the linear operator \( T(g) \) (for some fixed \( g \)) and its basis vectors \( f_{p_1,q_1} \) and \( \tilde{f}_{p_1,\dot{q}_1} \) which are not eigenfunctions of this operator. In this paper, we aim to establish dependence between the matrix elements of the operator \( T(g) \) with respect to the ordinary basis \( B_2 \) and the mixed basis \( B_2 | B_1 \) and matrix elements of the operator \( \text{id} \equiv T(e) \) with respect to \( B_2 | B_1 \). Choosing here the group element as follows:
\[
h^* = \text{diag} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \text{diag}(1,1),
\]
we will show that the above dependence can be rewritten as a representation of Whittaker function of the second kind in the form of integral involving Whittaker and Bessel-Clifford functions. The Bessel-Clifford functions are used, for example, for solution of wave equation [1] and are a particular case of more generalized so-called Bessel-Maitland functions (see [8]).

The above-described approach, together with other methods, was used by Shilin and Choi [10] who considered another realization of the representation of the group \( SO(2,2) \) and representation operators corresponding to some diagonal and block-diagonal matrices which belong to the split orthogonal group.

3. Transitive subgroups and invariant measures

It is obvious that \( \omega_1 \) is an orbit of the subgroup \( H_1 \cong SO(2) \times SO(2) \), consisting of the matrices
\[
h_1(\varphi_1, \psi_1) = \begin{pmatrix}
\cos \varphi_1 & -\sin \varphi_1 & 0 & 0 \\
\sin \varphi_1 & \cos \varphi_1 & 0 & 0 \\
0 & 0 & \cos \psi_1 & -\sin \psi_1 \\
0 & 0 & \sin \psi_1 & \cos \psi_1
\end{pmatrix}.
\]

Let us consider the matrices
\[
h(\varrho) = \begin{pmatrix}
2 & \varrho & 0 & 0 \\
-\varrho & 2 & \varrho & 0 \\
0 & \varrho & 2 & \varrho \\
\varrho & 0 & -\varrho & 2
\end{pmatrix}
\]
and the points \( \tilde{x}(\alpha_2, \beta_2) \) and \( \tilde{x}(\dot{\alpha}_2, \dot{\beta}_2) \) belong to the subset \( \omega_2 \) of \( \Lambda \). Since the matrix elements \( h_{i,j}(\varrho) \) of the matrix \( h(\varrho) \) satisfy the equalities
\[
h_{i1}(\varrho) + h_{i2}(\varrho) - h_{i3}(\varrho) - h_{i4}(\varrho) = 4 \text{ sign}(2.5 - i),
\]
we get $\frac{1}{2}h(g) \in SO(2, 2)$. It is easy to see that

$$\tilde{x}(\alpha_2, \beta_2)h_1(0, -\beta_2) = \tilde{x}(\alpha_2, 0) \equiv \begin{pmatrix} 1 & \alpha_2 & 0 & \alpha_2 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$\frac{1}{2} \tilde{x}(\alpha_2, 0)h(\hat{\alpha}_2 - \alpha_2) = \tilde{x}(\hat{\alpha}_2, 0)$$

and

$$\tilde{x}(\hat{\alpha}_2, 0)h_1(0, \hat{\beta}_2) = \tilde{x}(\hat{\alpha}_2, \hat{\beta}_2).$$

Thus the matrix

$$h_2(\hat{\alpha}_2 - \alpha_2, \hat{\beta}_2 - \beta_2) = \frac{1}{2}h_1(0, -\beta_2)h(\hat{\alpha}_2 - \alpha_2)h_1(0, \hat{\beta}_2)$$

transforms the point $\tilde{x}(\alpha_2, \beta_2)$ into the point $\tilde{x}(\hat{\alpha}_2, \hat{\beta}_2)$. It means that the subgroup

$$H_2 = \{h_2(\varphi_2, \psi_2) \mid \varphi_2 \in \mathbb{R}, \psi_2 \in [0, 2\pi)\}$$

acts transitively on $\omega_2$. Also we find that $d\omega_2 = d\alpha_1 d\beta_2$ is an $H_2$-invariant measure on $\omega_2$. It is found that $f_{p_2, q_2}$ is an eigenfunction of the linear operator $T(h_2(\varphi_2, \psi_2))$, more exactly,

$$T(h_2(\varphi_2, \psi_2))[f_{p_2, q_2}] = \exp(i)f_{p_2, q_2}.$$

Similarly $d\omega_1 = d\alpha_1 d\beta_1$ is an $H_1$-invariant measure on the spherical section $\omega_1$ and $f_{p_1, q_1}$ is an eigenfunction of the operator $T(h_1(\varphi_1, \psi_1))$, namely

$$T(h_1(\varphi_1, \psi_1))[f_{p_1, q_1}] = \exp(ip_1 \varphi_1) \exp(iq_1 \psi_1)f_{p_1, q_1}.$$

4. Functionals $F_1$ and $F_2$ and assorted spaces

Let us introduce the following bilinear functionals defined on the direct product $\mathfrak{L} \times \mathfrak{L}^*$ of two representation spaces:

$$F_i : (u, v^*) \mapsto \int \int_{\omega_i} u(\alpha_1, \beta_1) v^*(\alpha_i, \beta_i) d\omega_i \quad (i = 1, 2),$$

where the functions on $\mathfrak{L}^*$ are $(\sigma_1^*, \sigma_2^*)$-homogeneous.

**Lemma 1.** The functional $F_1$ coincides with $F_2$ if and only if

$$\sigma_1^* - \sigma_2^* = \sigma_2 - \sigma_1 - 4. \quad (3)$$

**Доказательство.** It was shown in [11] that for any point $x \in \Lambda$ there are a low triangular non-degenerate $2 \times 2$-matrix $b_x$ and the point $\tilde{x}(\alpha_2, \beta_2)_x \in \omega_2$ such that $x = b_x \tilde{x}(\alpha_2, \beta_2)_x$, and $b_{11} = x_{11}, b_{21} = x_{21}, \alpha_2 = \frac{x_{21}}{x_{11}}$. In particular, for an arbitrary point (1) belonging to $\omega_1$, we have

$$b_{\tilde{x}(\alpha_1, \beta_1)} = \begin{pmatrix} \cos \alpha_1 & 0 \\ \sin \alpha_1 & \sec \alpha_1 \end{pmatrix},$$

and, therefore, the correspondence $\tilde{x}(\alpha_1, \beta_1) \mapsto \tilde{x}(\alpha_2, \beta_2)_{\tilde{x}(\alpha_1, \beta_1)}$ is one-to-one. Since the operands $u \in \mathfrak{L}$ and $v^* \in \mathfrak{L}^*$ of the functional $F_1$ are $(\sigma_1, \sigma_2)$- and $(\sigma_1^*, \sigma_2^*)$-homogeneous, respectively, we have

$$u(\alpha_1, \beta_1) v^*(\alpha_1, \beta_1) = (\cos \alpha_1)^{\sigma_1 + \sigma_2^* - \sigma_2} u(\alpha_2, \beta_2) v^*(\alpha_2, \beta_2).$$
Considering that $\omega_1$-coordinates depend on $\omega_2$-coordinates according to the formulae
\[
\alpha_1 = \arccos\left(\pm \frac{1}{\sqrt{1 + \alpha_2^2}}\right), \quad \beta_1 = \arctan\frac{\alpha_2 \cos \beta_2 + \sin \beta_2}{\alpha_2 \sin \beta_2 - \cos \beta_2}, \quad \frac{\partial(\alpha_1, \alpha_2)}{\partial(\alpha_2, \alpha_2)} = (1 + \alpha_2^2)^{-1},
\]
we get
\[
F_1(u, v^*) = \int_{\omega_1} \int_{\omega_2} u(\alpha_1, \beta_1) v^*(\alpha_1, \beta_1) \, d\alpha_1 \, d\beta_1 = \int_{\omega_2} \int_{\omega_2} \frac{u(\alpha_2, \beta_2) v^*(\alpha_2, \beta_2) \, d\alpha_2 \, d\beta_2}{(1 + \alpha_2^2)^{\sigma_1 + \sigma_2 - \sigma_2^* + 4}}.
\]
It is clear that the equality $F_1 = F_2$ is equivalent to $\sigma_1 + \sigma_1^* - \sigma_2 - \sigma_2^* + 4 = 0$. □

Further we assume that representation spaces $\mathcal{L}$ and $\mathcal{L}^*$ are mutually assorted, i.e., the pair $(\sigma_1^*, \sigma_2^*)$ for the representation space $\mathcal{L}^*$ is connected with the pair $(\sigma_1, \sigma_2)$ for $\mathcal{L}$ by the equality (3).

5. Matrix elements of the $B_1^* \longrightarrow B_2^*$ basis transformation

Let us express the function $f_{p_2, q_2}$ as a linear combination of the functions belonging to the basis $B_2^*$:
\[
f_{p_2, q_2}^*(x) = \sum_{p_1, q_1 \in \mathbb{Z}} c_{p_1, q_1, p_2, q_2} f_{p_1, q_1}^*(x). \quad (4)
\]
In view of Lemma 1, we have
\[
F_2(f_{p_2, q_2}^*, f_{p_1, q_1}) = \sum_{p_1, q_1 \in \mathbb{Z}} c_{p_1, q_1, p_2, q_2} F_1(f_{p_1, q_1}^*, f_{p_1, q_1}).
\]
Since
\[
F_1(f_{p_2, q_2}^*, f_{p_1, q_1}) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \exp(i[p_1 + \hat{p}_1]q_1) \exp(i[q_1 + \hat{q}_1]p_1) \, d\alpha_1 \, d\beta_1,
\]
we obtain
\[
c_{p_1, q_1, p_2, q_2} = \frac{1}{4\pi^2} F_2(f_{p_2, q_2}^*, f_{-p_2, -q_2}).
\]

We compute the matrix elements of the linear operator acting in $\mathcal{L}^*$ and transforming the basis $B_1^*$ into $B_2^*$, asserted by the following theorem.

Теорема 1. Let $p_1, q_1, q_2 \in \mathbb{Z}$, $p_2 \in \mathbb{R} \setminus \{0\}$, and $\text{Re}(\sigma_1 - \sigma_2) > -3$. Then
\[
c_{p_1, q_1, p_2, q_2} = 2^\frac{\sigma_2 - \sigma_1}{2} - 3 \pi^{-1} \delta_{q_1, -q_2} |p_2|^\frac{\sigma_1 - \sigma_2}{2} + 1
\times \Gamma\left(\frac{\sigma_1 - \sigma_2 + (q_1 - p_1) \text{sign} p_2}{2}\right)^{-1} W_{(q_1 - p_1) \text{sign} p_2, \frac{\sigma_1 - \sigma_2}{2} - \sigma_2 - 3} (2|p_2|), \quad (5)
\]
where $\Gamma$ is the gamma function, $W_{\mu, \nu}$ is the Whittaker function of the second kind, and $\delta_{s,t}$ is the Kronecker symbol.

Доказательство. Using iterated integrals for
\[
F_2(f_{p_2, q_2}^*, f_{-p_1, -q_1}) = i^{q_1 + q_2} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} (1 + \alpha_2^2)^\frac{\sigma_2 - \sigma_1 - p_1 + q_1}{2} - 2 (1 - i\alpha_2)^{p_1 - q_1}
\times \exp\left(i[p_2\alpha_2 + (q_1 + q_2)\beta_2]\right) \, d\alpha_2 \, d\beta_2,
\]
we find that $c_{p_1,q_1,p_2,q_2}$ can be expressed as an exponential Fourier transform:

$$F_2(f_{p_2,q_2}, f^*_{-p_1,-q_1}) = 2\pi \delta_{q_1,-q_2} \int_{\infty}^{\infty} (1 + \imath \alpha_2)^{\frac{\alpha_2 - \alpha_1 + q_1 - p_1}{2} - 2} \times (1 - \imath \alpha_2)^{\frac{\alpha_2 - \alpha_1 + q_1 - p_1}{2} - 2} \exp(\imath p_2 \alpha_2) \, d\alpha_2.$$  

The integral in (6) can be evaluated by the following known formulae (see, e.g., [4, Entry 3.2.(12)])

$$\int_{-\infty}^{\infty} (\alpha + \imath \beta)^{-2\mu} (\beta - \imath \alpha)^{-2\nu} \exp(-\imath \alpha y) \, dx = 2\pi (\alpha + \beta)^{-\nu - \mu} \Gamma(2\nu)^{-1}$$

$$\times \exp \left( \frac{(\beta - \alpha)y}{2} \right) y^{\nu + \mu - 1} W_{\nu - \frac{1}{2}, \nu - \mu - \mu}([\alpha + \beta]y)$$

$$\left( \Re(\mu + \nu) > \frac{1}{2}, \min\{\Re(\alpha), \Re(\beta)\} > 0; \, y \in \mathbb{R}^+ \right)$$

and

$$\int_{-\infty}^{\infty} (\alpha + \imath \beta)^{-2\mu} (\beta - \imath \alpha)^{-2\nu} \exp(-\imath \alpha y) \, dx = 2\pi (\alpha + \beta)^{-\nu - \mu} \Gamma(2\mu)^{-1}$$

$$\times \exp \left( \frac{(\alpha - \beta)y}{2} \right) (-y)^{\nu + \mu - 1} W_{\nu - \frac{1}{2}, \nu - \mu - \mu}([-\alpha + \beta]y)$$

$$\left( \Re(\mu + \nu) > \frac{1}{2}, \min\{\Re(\alpha), \Re(\beta)\} > 0; \, y \in \mathbb{R}^- \right)$$

\[ \Box \]

6. Matrix elements of the operator $T^*(h^*)$ with respect to $B_2^*$

For any $g \in SO(2,2)$, let $t_{p_2,q_2,\tilde{p}_2,\tilde{q}_2}(g)$ be a matrix element of the linear operator $T^*(g)$ with respect to the basis $B_2^*$, that is,

$$T^*(g)[f_{p_2,q_2}] = \sum_{\tilde{p}_2,\tilde{q}_2} \int_{\mathbb{Z}^+} t_{p_2,q_2,\tilde{p}_2,\tilde{q}_2}(g) f_{p_2,q_2} \, d\tilde{p}_2.$$  

In view of Lemma 1, we get

$$F_i(T^*(g)[f_{p_2,q_2}], f_{\tilde{p}_2,\tilde{q}_2}) = \sum_{\tilde{p}_2,\tilde{q}_2} \int_{\mathbb{Z}^+} t_{p_2,q_2,\tilde{p}_2,\tilde{q}_2}(g) F_2(f_{p_2,q_2}, f_{\tilde{p}_2,\tilde{q}_2}) \, d\tilde{p}_2$$

$$= 4\pi^2 \int_{\mathbb{Z}^+} t_{p_2,q_2,\tilde{p}_2,-\tilde{q}_2}(g) \delta(\tilde{p}_2 + \tilde{p}_2) \, d\tilde{p}_2,$$

where $\delta(\tilde{p}_2 + \tilde{p}_2)$ is the $(-\tilde{p}_2)$-delayed Dirac delta function. We therefore have

$$t_{p_2,q_2,\tilde{p}_2,\tilde{q}_2}(g) = \frac{1}{4\pi^2} F_i(T^*(g)[f_{p_2,q_2}], f_{\tilde{p}_2,-\tilde{q}_2}).$$

In Theorem 2, we show that the matrix elements $t_{p_2,q_2,\tilde{p}_2,\tilde{q}_2}(h^*)$ can be described in terms of either Bessel–Clifford functions of the first kind

$$C_\nu(z) = z^{-\frac{\nu}{2}} J_\nu(2\sqrt{z})$$

or modified Bessel-Clifford functions of the second kind

$$K_\nu(z) = z^{-\frac{\nu}{2}} K_\nu(2\sqrt{z})$$

depending on sign($p_2\tilde{p}_2$) in both cases (see [2]). Here $J_\nu$ and $K_\nu$ are Bessel functions of the first kind and modified Bessel functions of the second kind, respectively, (see, e.g., [13, Chapter 9]).
Theorem 2. Let \( p_2, \hat{p}_2 \in \mathbb{R} \setminus \{0\}, q_2, \hat{q}_2 \in [0, 2\pi) \), and \( 2 < \text{Re}(\sigma_2 - \sigma_1) < 4 \). Then
\[
 t_{p_2, q_2, \hat{p}_2, \hat{q}_2}^\ast (h^\ast) = -\frac{2i^{q_2} \delta_{q_2, \hat{q}_2}}{\pi} |p_2|^{\sigma_2 - \sigma_1 - 3} \sin \left( \frac{(\sigma_2 - \sigma_1)\pi}{2} \right) K_{\sigma_2 - \sigma_1 - 3}(-p_2 \hat{p}_2)
\]
(12)
and
\[
 t_{p_2, q_2, \hat{p}_2, \hat{q}_2}^\ast (h^\ast) = \frac{i^{q_2} \delta_{q_2, \hat{q}_2}}{2} \cos \left( \frac{(\sigma_2 - \sigma_1)\pi}{2} \right)
\times [\hat{p}_2]^{\sigma_1 - \sigma_2 + 3} C_{\sigma_1 - \sigma_2 + 3}(p_2 \hat{p}_2) - |p_2|^{\sigma_2 - \sigma_1 - 3} C_{\sigma_2 - \sigma_1 - 3}(p_2 \hat{p}_2)]
\]
(13)
\[
 (p_2 \hat{p}_2 \in \mathbb{R}^-)
\]

Доказательство. Since the right shift of the subset \( \omega_2 \) by the matrix \( h^\ast \) permutes the first and the second columns of any point belonging to \( \omega_2 \), we have
\[
 T(h^\ast)[f_{p_2, q_2}(x)] = |x_{12}|^{\sigma_1 - \sigma_2} |\Delta_{1,2}|^{\sigma_2 - q_2} (\Delta_{2,3} + i\Delta_{2,4})^{q_2} \exp \left( -\frac{i p_2 x_{11}}{x_{12}} \right).
\]
In view of Lemma 1, the \( T^\ast(h^\ast) \)-image of the restriction of \( f_{p_2, q_2}^\ast \) to \( \omega_2 \) is given as follows:
\[
 T^\ast(h^\ast)[\tilde{f}_{p_2, q_2}^\ast](\alpha_2, \beta_2) = (-1)^{q_2} |\alpha_2|^{\sigma_2 - \sigma_1 - 4} \exp(iq_2 \beta_2) \exp \left( -\frac{i p_2}{\alpha_2} \right).
\]
We obtain
\[
 t_{p_2, q_2, \hat{p}_2, \hat{q}_2}^\ast (h^\ast) = \frac{1}{4\pi^2} F_2(T^\ast(h^\ast)[f_{p_2, q_2}^\ast], f_{-\hat{p}_2, -q_2}^\ast)
\]
\[
 = \frac{i^{q_2 - \hat{q}_2}}{4\pi^2} \int_{-\pi}^{\pi} \exp(i[q_2 - \hat{q}_2] \beta_2) \, d\beta_2 \int_{-\infty}^{\infty} |\alpha_2|^{\sigma_2 - \sigma_1 - 4} \exp \left( -i \left[ \hat{p}_2 \alpha_2 + \frac{p_2}{\alpha_2} \right] \right) \, d\alpha_2.
\]
Considering here the principle value of the last integral and using the following known formulae (see, e.g., [9, Entires 2.5.24.4 and 2.5.24.7])
\[
 \int_0^\infty x^{\alpha - 1} \cos \left( ax + \frac{b}{x} \right) \, dx = \frac{\pi}{2} \left( \frac{b}{a} \right)^{\frac{\alpha}{2}} \sin \left( \frac{\alpha\pi}{2} \right) \left[ J_{-\alpha}(2\sqrt{ab}) - J_{\alpha}(2\sqrt{ab}) \right]
\]
and
\[
 \int_0^\infty x^{\alpha - 1} \cos \left( ax - \frac{b}{x} \right) \, dx = 2 \left( \frac{b}{a} \right)^{\frac{\alpha}{2}} \cos \left( \frac{\alpha\pi}{2} \right) K_{\alpha}(2\sqrt{ab})
\]
\[
 (a, b \in \mathbb{R}^+; |\text{Re}(\alpha)| < 1),
\]
with the aid of (10) and (11), we complete the proof. \( \Box \)

7. Matrix elements of the operator \( T^\ast(h^\ast) \) with respect to the mixed basis \( f_{p_2, q_2} | f_{p_1, q_1} \)

From (2) and (9), we find
\[
 T^\ast(g)[f_{p_2, q_2}^\ast] = \sum_{p_1, q_1 \in \mathbb{Z}} \left( \sum_{\hat{p}_2, \hat{q}_2} \int_{-\infty}^{\infty} t_{p_2, q_2, \hat{p}_2, \hat{q}_2}^\ast (g) c_{p_1, q_1, \hat{p}_2, \hat{q}_2} \, d\hat{p}_2 \right) f_{p_1, q_1}^\ast.
\]
(14)
The expression in the brackets in (14) can be characterised as a matrix element of the operator \( T^*(h^*) \) with respect to the so-called mixed basis \( f_{p_2,q_2} f_{p_1,q_1} \) (see, e.g., [3, p. 204]).

On the other hand, these matrix elements may be obtained in the following way. It is not hard to check that the linear subspace \( \text{span}(f_{p_1,q_1}, f_{p_2,q_2}) \) is invariant with respect to the linear operator \( T^*(h^*) \), namely, in view of (2),

\[
T^*(h^*)|f_{p_1,q_1}|_{\omega_1} = T(h^*)|f_{p_1,q_1}| = i^{p_1+q_1} \exp(i[p_1\omega_1 - q_1\beta_1]) = i^{p_1} f_{p_1,q_1}|_{\omega_1}.
\] (15)

**Theorem 3.** Let \( p_1, q_1 \in \mathbb{Z} \) such that \( p_1 + q_1 \equiv 0 (\text{mod } 2) \), \( p_2 \in \mathbb{R}^+ \), and \( 1 < \text{Re}(\theta) < 2 \). Then

\[
\int_0^\infty \left[ \frac{1}{2} \cos(\theta \pi) p_2^{\theta - 1} \hat{p}_2^{4 - 3\theta} C_{3 - 2\theta}(p_2 \hat{p}_2) W_{\frac{1 - p_1}{2}, \frac{3 - \theta}{2}}(\hat{p}_2)
- \frac{1}{2} \cos(\theta \pi) p_2^3 \hat{p}_2^{2 - \theta} C_{2\theta - 3}(p_2 \hat{p}_2) W_{\frac{1 - p_1}{2}, \frac{3 - \theta}{2}}(\hat{p}_2)
+ (-1)^\frac{\theta}{2} 2 \Gamma \left( \frac{q_1 - p_1 - \theta}{2} \right) \sin(\theta \pi) p_2^{3 - \theta} \hat{p}_2^{2 - \theta} K_{2\theta - 3}(p_2 \hat{p}_2) W_{\frac{1 - p_1}{2}, \frac{3 - \theta}{2}}(\hat{p}_2) \right] \, d\hat{p}_2
= (-1)^{p_1 + q_1} W_{\frac{1 - p_1}{2}, \frac{3 - \theta}{2}}(2p_2).
\]

**Proof.** From (14) and (15) we have

\[
T^*(h^*)|f_{p_2,q_2}^*| = \sum_{p_1,q_1 \in \mathbb{Z}} c_{p_1,q_1,p_2,q_2} T^*(g)|f_{p_1,q_1}^*| = \sum_{p_1,q_1 \in \mathbb{Z}} i^{p_1} c_{p_1,q_1,p_2,q_2} f_{p_1,q_1}^*.
\] (16)

Since the matrix element \( c_{p_1,q_1,p_2,q_2} \) is equal to zero in case \( q_1 \neq -q_2 \) and the matrix element \( f_{p_2,q_2}^*(h^*) \) is equal to zero in case \( q_2 \neq \hat{q}_2 \), considering (14) and (16), we have

\[
\int_{-\infty}^{\infty} f_{p_2,-q_1,p_2,-q_1}^*(h^*) c_{p_1,q_1,p_2,-q_1} \, d\hat{p}_2 = i^{p_1} c_{p_1,q_1,p_2,-q_1}.
\] (17)

Using, for (16), the results from Theorems 1 and 2, and letting \( \theta = \frac{\sigma_2 - \sigma_1}{2} \), we complete the proof. \( \square \)

**8. Concluding Remarks**

Setting \( p_1 = q_1 = 0 \) in (14) and considering the following relation between Macdonald functions \( K_\nu \) and Whittaker functions \( W_{0,\nu} \) (see, e.g., Entry [15, 7.8.8]):

\[
K_\nu \left( \frac{z}{2} \right) = \left( \frac{\pi}{z} \right)^\frac{1}{2} W_{0,\nu}(z),
\] (18)

from the result in Theorem 3, we obtain the following integral formula for the \( K \)-transform (see [4]) of the linear combination of the Bessel–Clifford functions:

\[
\int_0^\infty \left[ \frac{1}{2} \cos(\theta \pi) p_2^{-\frac{3}{2}} \hat{p}_2^{-4.5 - 3\theta} C_{3 - 2\theta}(p_2 \hat{p}_2)
- \frac{1}{2} \cos(\theta \pi) p_2^{3 - \theta} \hat{p}_2^{-\frac{9}{2} - \theta} C_{2\theta - 3}(p_2 \hat{p}_2)
- \frac{2}{\pi} (-1)^\frac{\theta}{2} \sin(\theta \pi) p_2^{3 - \theta} \hat{p}_2^{-\frac{9}{2} - \theta} K_{2\theta - 3}(p_2 \hat{p}_2) \right] K_{\frac{1}{2}}(\hat{p}_2) \, d\hat{p}_2 = K_{\frac{1}{2}}(p_2)
\] (19)

\( (p_2 \in \mathbb{R}^+, \ 1 < \text{Re}(\theta) < 2) \).

Some similar results to those in Theorem 3 and formula (19) can be obtained from (17) in case \( p_2 \in \mathbb{R}^- \).

Using the following three integral transformations:
(i) The first Hankel-Clifford integral transform (see, e.g., [7, Eq. (2.7)])

\[ H_1^{(1)}[f](\lambda) = \lambda^\sigma \int_0^\infty C_\sigma(\lambda \lambda') f(\lambda') \, d\lambda' \quad (\lambda \in \mathbb{R}^+) ; \]

(ii) The second Hankel-Clifford integral transform (see, e.g., [5]; see also [7, Eq. (2.9)])

\[ H_2^{(2)}[f](\lambda) = \int_0^\infty \lambda^\sigma C_\sigma(\lambda \lambda') f(\lambda') \, d\lambda' \quad (\lambda \in \mathbb{R}^+) ; \]

(iii) The Macdonald-Clifford transform (see [12])

\[ K_\sigma[f](\lambda) = \int_0^\infty \lambda^\sigma K_\sigma(\lambda \lambda') f(\lambda') \, d\lambda' \quad (\lambda \in \mathbb{R}^+) , \]

we can rewrite the identity in Theorem 3 in the following form:

\[
\frac{\cos(\theta \pi)}{2} \left[ H_3^{(1)} \left[ \hat{p}_2^{4-3\theta} W_{\frac{p_1}{2} - \frac{p_1}{2} - \theta} \left(2\hat{p}_2\right) \right] (p_2) - H_2^{(2)} \left[ \hat{p}_2^{4-3\theta} W_{\frac{p_1}{2} - \frac{p_1}{2} - \theta} \left(2\hat{p}_2\right) \right] (p_2) \right] \\
+ \frac{2}{\pi} \frac{(-1)^\theta \Gamma \left( \frac{p_1}{2} - \frac{p_1}{2} - \theta \right)}{\Gamma \left( \frac{p_1}{2} - \frac{p_1}{2} - \theta \right)} K_{2\theta-3} \left[ \hat{p}_2^{4-3\theta} W_{\frac{p_1}{2} - \frac{p_1}{2} - \theta} \left(2\hat{p}_2\right) \right] (p_2) \\
= (-1)^{p_1+q_1} \hat{p}_2^{4-3\theta} W_{\frac{p_1}{2} - \frac{p_1}{2} - \theta} \left(2\hat{p}_2\right) .
\]

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