ORACLE COMPLEXITY AND NONTRANSITIVITY IN PATTERN RECOGNITION

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ABSTRACT. Different mathematical models of recognition processes are known. In present paper we consider a recognizing algorithm as an oracle computation on Turing machine [Rogers]. Such point of view seems to be useful in pattern recognition as well as in recursion theory. Use of recursion theory in pattern recognition shows connection between a recognition algorithm comparison problem and complexity problems of oracle computation [Bulitko]. That is because in many cases we can take into account only the number of sign computations or in other words volume of oracle information needed. Therefore, the problem of recognition algorithm preference can be formulated as a complexity optimization problem of oracle computation. Furthermore, introducing a certain ”natural” preference relation on a set of recognition algorithms we discover it to be nontransitive. This relates to the well known nontransitivity paradox in probability theory [Székely].

KEYWORDS: Pattern Recognition, Recursion Theory, Nontransitivity, Preference Relation

Most of our notation follows [Rogers], also r.e. means recursively enumerable.

Let \((P_i)_{i \in \mathbb{N}}\) be a system of one-place predicates (basic signs) defined on a pattern set \(U\). We assume \((\forall a, b \in U) [a \neq b \Rightarrow \exists i [P_i(a) \neq P_i(b)]]\). Thus it is possible to represent every pattern \(a \in U\) as a subset \(\tilde{a} \subset \mathbb{N} : \tilde{a} = \{i \mid P_i(a)\}\). Also \(\tilde{D} = a \Rightarrow \tilde{a} = D\). So an image \(B\), which is a set of patterns from \(U\), can be represented as a subset of \(2^\mathbb{N}\). Let \(S\) be a set of all images. So the recognition problem for given, in general partial, function \(\tau : \mathbb{N} \to S\) and given pattern set \(U\) can be formulated as to find such \(z\) that for every pattern \(x \in U\):
\( \varphi^+_z(i) = \begin{cases} 
1, & \text{if } x \in \tau(i) \text{ and } \tau(i) \text{ is defined} \\
0, & \text{if } x \notin \tau(i) \text{ and } \tau(i) \text{ is defined} \\
\text{undefined}, & \text{if } \tau(i) \text{ is undefined} 
\end{cases} \)

If such \( z \) exists then we say the recognition problem is solvable for triple \( (U, \tau, (P_i)_{i \in \mathbb{N}}) \).

**Definition.** We use \( U^f \) for \( U \) if

1. \( \forall x \in U \ [|\bar{x}| < \infty] \);
2. \( \{p \mid D_p = \bar{x}, x \in U\} \) is recursive.

**Definition.** \( \tau \) is called computable \( \iff (\exists \text{ partial recursive } \gamma) \forall i \ [(\tau(i) \downarrow \Leftrightarrow \gamma(i) \downarrow) \& (\gamma(i) \downarrow \Rightarrow W_{\gamma(i)} = \{p \mid D_p = \bar{x}, x \in \tau(i)\}]]. \)

The following proposition can be easily proved.

**Proposition.**

1. There is a triple \( (U, \tau, (P_i)_{i \in \mathbb{N}}) \) such that \( \tau \) is total, \( \tau(i) \) is countable for every \( i \), and the recognition problem for \( (U, \tau, (P_i)_{i \in \mathbb{N}}) \) is unsolvable.

2. For given non-empty set \( U^f \) there is a computable total \( \tau \) such that the recognition problem for \( (U^f, \tau, (P_i)_{i \in \mathbb{N}}) \) is unsolvable.

We should consider unsolvability of a recognition problem as an indication of non-adequacy of the basic sign system to image set \( S \) and function \( \tau \). This means any algorithm \( z \) makes mistakes. However some of the algorithms might make more mistakes than others. The following is a formalization of that.

Let us consider \( (U^f, \tau, (P_i)_{i \in \mathbb{N}}) \), where \( \tau \) is computable. Define \( B = \{(k, i) \mid k \in W_{\gamma(i)} (i.e \ D_k \in \tau(i))\} \). \( B \) is r.e. If \( B \) is not recursive then \( (U^f, \tau, (P_i)_{i \in \mathbb{N}}) \) is unsolvable, however every algorithm \( z \) has a finite correctness domain \( [0, n] \) equal to such a maximal initial segment of \( \mathbb{N} \) that \( (\forall j \in [0, n]) \{(j = \langle k, i \rangle) \in B \Rightarrow \varphi^D_k(i) = 1\} \& (\exists j \neq \langle k, i \rangle \notin B \Rightarrow \varphi^D_k(i) = 0) \} \).

So it is natural to choose new signs on patterns from \( U^f \) such that there is new recognition algorithm \( \varphi_z \) with correctness domain strictly including correctness domain of \( \varphi_z \). Definitely we have to keep doing that all the time and use new information about the images. It is possible to estimate the volume of the information using its Kolmogorov complexity [Loveland]. Namely given computable \( \tau \) the volume of information sufficient to define new signs \( P_i^* \) and construct recognition algorithm for \( (U^f, \tau, (P_i^*)_{i \in \mathbb{N}}) \), that has correctness domain with length \( n \), grows not faster than \( \ln n \).
Usually an image includes a given pattern along with its various modifications. They can be got by effective transformations of the pattern. In that more general case let \( B = ( \bigcup_{q \in \Theta} R_q) \cap W \) where \( \Theta \subset \mathbb{N}, R_q \) is a recursive set such that \( C_{R_q} = \phi_{g(q)} \); and \( \min R_q \geq h(q) \) for every \( q \in \Theta \); here \( g, h \) are total recursive functions and \( h \) is not a decreasing function. \( KR_\Theta \) is resolving complexity of \( \Theta \) [Loveland].

Proposition. Taking into account above made assumptions we do not need more than \( \ln n + KR_\Theta(h^{-1}(n)) \) of information about \( W \oplus \Theta \) to construct new signs \( P^*_n \) such that the recognition problem \( (U^I, (\tau(i_0)), (P^*_n)_{n \in \mathbb{N}}) \) is solvable.

In a training process a hypothesis about teacher’s signs can be transformed into a hypothesis about enumeration operator. So a recognition algorithm can be found using the Kleene recursion theorem. The last proposition allows to estimate frequency and volume of operator hypothesis changes.

So we can introduce time of the training. It would be interesting to introduce time in a recognition process. We propose the following method to do that.

As well known, there is a passage from r.e. set \( W_z \) defining partial recursive in \( A \) function \( \phi^A_z \) to tree diagram \( T_z \) of the Turing machine computation with oracle \( A \). Tree \( T_z \) can be equivalently represented as r.e. set \( W^f_1(z) \) containing the tree’s terminated branches. A passage from \( W_z \) to \( W^f_1(z) \) is called regularization. It can be done in such a way that \( f \) will be recursive and some additional conditions for \( W^f_1(z) \) will be satisfied. Notation \( \langle x, y, u, v \rangle \in^1 W^f_1(z) \) means there is a branch \( (x, n_1n_2\ldots n_k \sigma_k, y) \) belonging to \( W^f_1(z) \) such that \( D_u = \{ n_t \mid \sigma_t = 1 \} \) & \( D_v = \{ n_t \mid \sigma_t = 0 \} \). \( W_{z,t} \) denotes result of enumeration of \( W_z \) by algorithm \( z \) after \( t \) steps done.

Proposition. There is a recursive function \( f \) such that \( \phi^X_{f(z)}(x) = y \Rightarrow \phi^X_f(z)(x) = y; \forall x \forall y \left[ \bigcup_{\langle x, y, u, v \rangle \in W_{z,t}} (D_u \cup D_v) \supset \bigcup_{\langle x, y, u, v \rangle \in W^f_1(z), t} (D_u \cup D_v) \right] \) where \( W^f_{z,t} = \{ \langle x, y, u, v \rangle \mid \langle x, y, u, v \rangle \in^1 W^f_1(z) \} \).

So using this approach we can represent recognition algorithms as trees. Vertexes of these trees contain the signs to be computed. The end of a terminated branch of a tree contains \( \phi^A_z(i) \) (i.e. it indicates if \( A \in \tau(i) \) or not).

In practice image system \( S \) and patterns are finite and every \( a_i \in S \) is a recursive subset of \( U^I \). In that case the recognition problem for \((U^I, (a_1, \ldots, a_k), n \in \mathbb{N})\) is definitely solvable however there is a complexity problem. Also we should mention the best algorithm problem comes up if we introduce a certain preference relation on the set of recognition algorithms. In order to consider that we assume sign
computation is an easy process (i.e. any sign computation takes one time unit).

Assume $a_i \cap a_j = \emptyset$ if $i \neq j$; $i, j \in \{1, \ldots, k\}$ and $U^f = \cup_{i} a_i$. Then evidently there is a recursive function $q$ such that $(\forall x \in U^f) \left[ \varphi^x_{g(q)}(0) = i \iff \varphi^x_{g}(i) = 1 \right]$. Under these assumptions one tree represents one recognition algorithm and any branch of such tree ends.

We call branch length (quantity of arcs) recognition time for pattern $x$ by algorithm $A$ and write it $T(A, x)$. Recognition time for a set $M \subset U^f$ by algorithm $A$ is $T(A, M) = \max T(A, x)$.

Let $R$ be a recognition algorithm set. Algorithm $A$ defeats algorithm $B$ recognition pattern $x$ iff $T(A, x) < T(B, x)$. Define $V(A, B) = |\{ x \in U \mid T(A, x) < T(B, x)\}|$.

Consider a situation when algorithms $A$ and $B$ are in process of recognizing random pattern sequences. Let $\pi(t), 1 \leq t \leq n$ be a random pattern sequence. Let $v(x, t)$ be the probability of event $\pi(t) = x$. So the mathematical expectation $m(A, B, n)$ of the number of members in $\pi$ on which $A$ defeats $B$ can be expressed as follows:

$$\sum_{x \in U^f} \sum_{t=1}^{n} v(x, t) sg(T(B, x) \geq T(A, x))$$

where $sg(a) = \begin{cases} 1, a > 0 \\ 0, \text{else} \end{cases}$, $a \equiv b = \begin{cases} a - b, a \geq b \\ 0, \text{else} \end{cases}$.

Assuming $v(x, t) = const$ we get $(\forall n \geq 1) \left[ \frac{m(A, B, n)}{n} = \frac{V(A, B)}{|U^f|} \right]$ and therefore recognizing the random pattern sequence algorithm $A$ wins more times in average than algorithm $B$ iff $V(A, B) > V(B, A)$. So we can introduce a preference relation on $R$ : we say algorithm $A$ is better than algorithm $B$ (write $A \ll B$) iff $V(A, B) > V(B, A)$. We call algorithm $A$ equivalent to $B$ iff $V(A, B) = V(B, A)$. It is clear that any two algorithms are comparable and the preference relation seems to be natural. Consider an example demonstrating nontransitivity of the introduced preference relation.

Let us define $\mathcal{B} = \{0, 1\}$, so $\mathcal{B}^9 = \{i_1i_2\ldots i_9 \mid i_s \in \mathcal{B}, s \in \{1, 9\}\}$ and \{i_1 \ldots i_{m_1-1}i_{m_1+1} \ldots i_{m_k-1}i_{m_k+1} \ldots i_9 \mid i_{m_s} \in \mathcal{B}; s = \{1, k\}; 0 \leq k \leq 9\} \subset \mathcal{B}^9$. Here and below $a, b$ means $\{a, \ldots, b\}$.

Now define $U^f \subset \mathcal{B}^9$ as $U^f = \alpha_0 \cup \alpha_1 \cup \alpha_2 \cup \alpha_3$ where $\alpha_0 = \{1\mathcal{B}\mathcal{B}01\mathcal{B}001\}$, $\alpha_1 = \{01\mathcal{B}0011\mathcal{B}\mathcal{B}\}$, $\alpha_2 = \{0011\mathcal{B}\mathcal{B}01\mathcal{B}\}$, $\alpha_3 = \{000000000\}$. Obviously $|\alpha_0| = |\alpha_1| = |\alpha_2| = 8, |\alpha_3| = 1, |U^f| = 25$. Set $S = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$.

Consider nine predicates on $U^f$ defined as $P_k(x) = i_k, k = 1, 9$. Also we consider
a set $\mathcal{R}$ of recognition algorithms which are binary trees in the form shown on [Fig. 1].

There $F_i \in \{P_k \mid k = 1, 9\}, \alpha_{l_q} \in \{\alpha_s \mid s = 0, 3\}$. We call $F_1$ the first predicate in the algorithm or predicate in the root.

Every pattern $x$ corresponds to a branch in tree of algorithm $A$ from the root to a leaf. The leaf is output of algorithm $A$ recognizing pattern $x$. We assume jump to the left subnode of node $F_i$ iff $F_i(x)$ is true.

Finally consider three algorithms shown on [Fig. 2].

[Tab. 1] shows their recognition time.

Theorem 1.

(1) $A \ll B, B \ll C, C \ll A$. Therefore the preference relation is nontransitive.
(2) There is no such $\mathcal{X} \in \mathcal{R} \setminus \{A, B, C\}$ that $(A \ll \mathcal{X}) \lor (B \ll \mathcal{X}) \lor (C \ll \mathcal{X})$.

Proof.

The first statement follows from [Tab. 1].

Consider the second statement. Let $\mathcal{X}$ be an arbitrary recognition algorithm. We have to investigate two cases:

1. $\forall x \in U \lceil T(\mathcal{X}, x) \geq 2]$;
2. $\exists x \in U \lceil T(\mathcal{X}, x) = 1]$.

For the first case we will go through all the predicates which can be situated in the root of the tree of $\mathcal{X}$. Those predicates are $P_2$, $P_3$, $P_5$, $P_6$, $P_8$, $P_9$ because $P_1$, $P_4$, $P_7$ separate an entire image. We will pay attention only to $P_2$ because the other cases are similar.

So suppose $P_2$ is the first predicate computed in $\mathcal{X}$. It will "divide" all the patterns into two sets: $\{1_2\alpha_0, \alpha_1\}$ and $\{1_2\alpha_0, \alpha_2, \alpha_3\}$. Here and below $\frac{1}{2}\alpha_0$ is a subset of $\alpha_0$ such that $\frac{1}{2}\alpha_0 = \frac{1}{2}\lceil \alpha_0 \rceil$. It is clear we can easily split the first of those two sets. However we cannot split the second set by one predicate because the second set contains patterns belonging to the three images. So we get two sub cases:

1. we separate patterns belonging to $\alpha_0$ from the second set and split $\alpha_2, \alpha_3$ by another predicate. The scheme on [Fig. 3] shows that sub case.
2. we separate $\alpha_2$ from the second set first and then split $1_2\alpha_0$ and $\alpha_3$ [see Fig. 4].

However comparison of these algorithms with $A, B, C$ shows that the first one is equivalent to $B$ but worse than $A$ and $C$. The second algorithm is equivalent to $A$ but worse than $B$ and $C$. Obviously the other algorithms with $P_2$ as the first predicate (i.e. in the root of tree) are even worse.

Consider the second case. Obviously we have to put $P_2$ or $P_4$ or $P_7$ in the root if we want to have an image separated (recognized) in one time unit. So we have all patterns belonging to one image if the first predicate is true and patterns from the other three images otherwise. The scheme on [Fig. 5] illustrates that.

| $M \setminus \mathcal{Y}$ | A | B | C |
|---------------------------|---|---|---|
| $\alpha_0$                | 1 | 2 | 3 |
| $\alpha_1$                | 2 | 3 | 1 |
| $\alpha_2$                | 3 | 1 | 2 |
| $\alpha_3$                | 3 | 3 | 3 |

Table 1. Recognition time of algorithms $A$, $B$, $C$
Figure 3. The first sub case

Figure 4. The second sub case

Figure 5. The second case

$X$, where $X$ is $P_2$ or $P_4$ or $P_7$. 
Figure 6. The second sub case

Figure 7. The first case for $Z$

Here and below $\alpha, \beta, \gamma, \delta$ mean entire images $\alpha_i$. There are three sub cases:

(1) The second predicate ($Y$ on [Fig. 5]) separates one entire image;
(2) $Y$ separates patterns belonging to half an image;
(3) $Y$ doesn’t separate either an image or half an image.

Let us consider the second sub case only (i.e. where $Y$ separates half an image) [see Fig. 6].

Now we have to consider possible cases for $Z$. There are two of them. In the first one $Z$ separates the rest of image $\beta$ [see Fig. 7].

If we try to analyze the situation substituting predicates $P_i$ into $X, Y, Z$ we get [Tab. 2] that shows recognition time for all the cases except unrealizable ones. A fraction means that half the image is recognized in time shown in the numerator and the other half is recognized in time shown in the denominator.

[Tab. 3] shows the result of comparison those three cases with algorithms $A, B, C$
Sign ‘+’ means that algorithm in the row is better than algorithm in the column.
So we see there is no algorithm better than A or B or C. Let us find out about
the second case for Z, i.e. when Z separates an entire image γ [see Fig. 8].
Again we get two tables [Tab. 4] which have the same meaning as [Tab. 2, 3].
It gives us the same result.
We finished the second sub case. The other two sub cases can be investigated similarly.
The theorem is proved.

There is also a more general theorem on nontransitivity.

Theorem 2. Let \( n \geq 3 \). Then there are such \( U^f, S, \) and \( P_i \) that there exist \( n \) recognition algorithms \( A_j \) forming a nontransitive sequence: \( A_0 \ll A_1 \ll A_2 \ll \ldots \ll A_{n-1} \ll A_0 \).

Proof.

| image \ case | 1 | 2 | 3 | 4 | 5 | 6 |
|--------------|---|---|---|---|---|---|
| \( \alpha_0 \) | 1 | \( \frac{4}{3} \) | \( \frac{3}{2} \) | 3 | 1 | 1 |
| \( \alpha_1 \) | \( \frac{4}{3} \) | 3 | 1 | 1 | \( \frac{1}{2} \) | 3 |
| \( \alpha_2 \) | 3 | \( \frac{3}{2} \) | 3 | 1 | \( \frac{1}{2} \) | 3 |
| \( \alpha_3 \) | \( \geq 4 \) | \( \geq 4 \) | \( \geq 4 \) | \( \geq 4 \) | \( \geq 4 \) | \( \geq 4 \) |

Table 2. Recognition time for the first case

| case | 1 | 2 | 3 |
|------|---|---|---|
| A    | + | + | + |
| B    | + | + | + |
| C    | + | + | + |

Table 3. The result of comparison

| case | 1 | 2 | 3 | 4 | 5 | 6 |
|------|---|---|---|---|---|---|
| A    | + | + | + | + | + | + |
| B    | + | + | + | + | + | + |
| C    | + | + | + | + | + | + |

Table 4. Comparison time and result for the second case
Define \( v_i = \overbrace{0 \ldots 0}^{i-1} \bar{B} \ldots \overbrace{B}^{n-i} \), 1 \leq i \leq n. We introduce \( n + 1 \) images as follows: \( \alpha_0 = \{v_1v_2 \ldots v_n\} \), \( \alpha_1 = \{v_2v_3 \ldots v_nv_1\} \), \( \alpha_2 = \{v_3v_4 \ldots v_nv_1v_2\} \), \ldots, \( \alpha_{n-1} = \{v_nv_1 \ldots v_{n-1}\} \), \( \alpha_n = \{0 \ldots 0\} \). Each image is a subset of \( \mathcal{B}^{n^2} \). \( U^f = \bigcup_{0 \leq i \leq n} \alpha_i \).

\( S = \{\alpha_i \mid 0 \leq i \leq n\} \). Then we define the signs: \( P_m^i(x) = a_{ni+m} \), where \( x = a_1 \ldots a_na_{n+1} \ldots a_{2n} \ldots \ldots a_{n(n-1)+1} \ldots a_n^2; 0 \leq i \leq n-1, 1 \leq m \leq n \). The above mentioned recognition algorithms can be defined as shown on [Fig. 9].

They have recognition time \( T(Y, M) \) shown in [Tab. 5].

The theorem statement follows from that table obviously. Note the example from theorem 1 can be got from the last theorem when \( n = 3 \).
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\[ \begin{array}{c|cc|c}\hline M \setminus Y & A_0 & A_1 & \ldots & A_{n-1} \\ \hline \alpha_0 & 1 & 2 & \ldots & n \\ \alpha_1 & 2 & 3 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \alpha_{n-1} & n & 1 & \ldots & n-1 \\ \alpha_n & n & n & \ldots & n \\ \hline \end{array} \]

Table 5. Recognition time of $A_j$

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