Most Bell Operators do not Significantly Violate Locality

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Abstract

The worst violation of Bell’s inequality for $n$ qbits is of size $2^{\frac{n-1}{2}}$ and it is obtained by a specific operator acting on a specific state. We show, to the contrary, that for a vast majority of Bell operators the worst violation is bounded by $O((n \log n)^{\frac{1}{2}})$, below experimental detection. With respect to the extremal operators, introduced by Werner and Wolf [Phys. Rev. A 64, 032112 (2001)], we show that a large majority of them have a norm bounded by $O(n^{\frac{1}{2}})$.

It is commonly believed that a quantum system that is composed of a large number of particles behaves “nearly classically”. A dramatic example to the contrary is the violation of a Bell-type inequality by a system of $n$ qbits. Mermin [1] showed that the violation not only persists as $n$ grows, but actually increases exponentially. Mermin’s result has since been strengthened in various respects [2–4]. Meanwhile, further sets of Bell inequalities have been determined, most notably, the set of all inequalities for $n$-partite systems with two dichotomic observable each. [5–7].

However, the question still remains how prevalent Mermin’s phenomenon is. In other words, how crucially it depends on the use of an exotic quantum state (the generalized Greenberger Horne Zeilinger GHZ state) and a specific operator (the Mermin Klyshko MK operator, or one of its variants). Here we show that Mermin’s phenomenon is the exception, not the rule, and it becomes more isolated as $n$ grows.

Consider $r \times n$ directions in physical space

$$a_1^1, a_2^1, ..., a_r^1$$

$$a_1^2, a_2^2, ..., a_r^2$$

. 

$$a_1^n, a_2^n, ..., a_r^n$$

Let $c(k_1, k_2, ..., k_n), \ 1 \leq k_j \leq r$, be real, non-negative numbers such that $\sum c^2(k_1, k_2, ..., k_n) = 1$. Consider the operator
\[ Q = \sum_{k_1, k_2, \ldots, k_n} \pm c(k_1, k_2, \ldots, k_n) \sigma(a_{k_1}^1) \otimes \sigma(a_{k_2}^2) \otimes \cdots \otimes \sigma(a_{k_n}^n) \quad (2) \]

Where \( \sigma(a) \) is the spin operator in the \( a \)-direction (with eigenvalues \(-1\) and \(+1\)), and the \( \pm \) signs of the coefficients \( c(k_1, k_2, \ldots, k_n) \) have been chosen randomly. We shall call such \( Q \) a random Bell operator. Let \( \|Q\| \) be the norm of \( Q \), that is, the maximum over the absolute values of its eigenvalues. We shall prove the following results:

**Proposition 1** For a vast majority of choices of signs \( \pm \) in (2) we have \( \max \|Q\| \leq 9(rn \log n)^{\frac{3}{2}} \). Where the maximum is taken over directions in (1) such that for each \( 1 \leq j \leq n \) the directions \( a_{k_j}^j \), \( k = 1, 2, \ldots, r \) are in the same plane.

Note that for \( r = 2 \), the restriction on the directions to be in the same plane does not limit the generality of the Bell operator in (2), since any two directions determine a plane.

In this case the set of all Bell inequalities has been derived in a particularly convenient form by Werner and Wolf [5]. We shall see later (proposition 3) how this allows a more stringent estimation of \( \|Q\| \) than that given in proposition 1.

The next result does not involve any restrictions.

**Proposition 2** Let \( |\Phi\rangle \) be a fixed arbitrary \( n \) qbits state. Then for a vast majority of choices of \( \pm \) signs in (2) we have \( \max |\langle \Phi |Q| \Phi\rangle| \leq 36(rn \log n)^{\frac{3}{2}} \). The maximum is taken over all choices of directions \( a_{k_j}^j \).

Let me make precise what I mean by “vast majority”. There are \( L = 2^{r^n} \) possible choices of signs \( \pm \) to the coefficients \( c(k_1, k_2, \ldots, k_n) \). Consider \( \{-1, 1\}^L \) as a probability space with each sequence having identical probability \( L^{-1} = 2^{-r^n} \). Denote the probability measure in this space by \( P \), and assume that \( \pm \) are assigned independently and with identical distribution. The magnitude \( \max \|Q\| \) can then be considered as a random variable on the probability space (as we vary the \( \pm \) signs in (2)). We shall prove the following estimation on the distribution of \( \|Q\| \):

\[
P \left( \max \|Q\| \leq 9(rn \log n)^{\frac{3}{2}} \right) > 1 - \frac{1}{n^2 e^{rn}} \quad (3)
\]
And a similar expression for proposition 2.

The bound $9(n \log n)^{\frac{1}{2}}$ should be compared with the maximum $2^{\frac{n-1}{2}}$ which is achieved by the MK operator in the generalized GHZ state \[1\]–\[3\]. The bound should also be compared with the predictions of local hidden variable theories. In such theories we replace (2) with the expression

$$C = \sum_{k_1, k_2, \ldots, k_n} \pm c(k_1, k_2, \ldots, k_n) X_{k_1}^1 X_{k_2}^2 \ldots X_{k_n}^n$$

(4)

Where the $X_{k_i}^j$ are $r \times n$ variables, each with two possible values $\pm 1$. The quantum value $\|Q\|$ is then compared with the classical value $\|C\|_{\infty} = \max |C|$, where the maximum is taken over all $2^{rn}$ values of the $X_{k_i}^j$. We shall see that for a large majority of choices of $\pm$ signs in (4) the number $9(n \log n)^{\frac{1}{2}}$ is also an upper bound for $\|C\|_{\infty}$. This means that for most choices of signs, $\|Q\|$ violates the conditions of local realism only slightly, if at all. Such violation cannot have an observable significance in the presence of small noise or measurement error.

The two propositions are direct consequences of a theorem in Fourier analysis due to Salem, Zygmund and Kahane (chapter 6 in \[8\]). Consider the multivariable trigonometric polynomial $P(t_1, t_2, \ldots, t_s) = \sum b(k_1, k_2, \ldots, k_s) e^{i(k_1 t_1 + k_2 t_2 + \ldots + k_s t_s)}$, where the sum is taken over all negative and nonnegative integers $k_1, k_2, \ldots, k_s$ which satisfy $|k_1| + |k_2| + \ldots + |k_s| \leq N$.

The integer $N$ is called the degree of the polynomial. Denote $\|P\|_{\infty} = \max_{t_1, \ldots, t_s} |P(t_1, t_2, \ldots, t_s)|$.

Now, consider the random polynomial $R = \sum_{j \in J} \pm c_j P_j(t_1, t_2, \ldots, t_s)$ where the sum is taken over a finite index set $J$, the $c_j$’s are real or complex, the $\pm$ signs are chosen at random in the sense explained above, and each $P_j$ is a trigonometric polynomial of degree $\leq N$. Then

$$\mathcal{P} \left( \|R\|_{\infty} \leq 9 \left( s \sum_{j \in J} |c_j|^2 \|P_j\|_{\infty}^2 \log N \right)^{\frac{1}{2}} \right) \geq 1 - \frac{1}{N^2 e^s}$$

(5)

Consider the random operator $Q$ in (2) with the restriction that the directions in each raw in (1) are in the same plane. In this case we can calculate explicitly the eigenvalues of $Q$ using the technique in \[8\]. (Scarani and Gisin assume that $r = 2$, but their argument depends
just on coplanarity). Let $z_j$ be the direction orthogonal to the vectors in the $j$-th row of (1). Denote by $|-1\rangle_j$ and $|1\rangle_j$ the states spin-down and spin-up in the $z_j$ direction. Let $x_j$ be orthogonal to $z_j$ and let $t^j_k$ be the angle between $a^j_k$ and $x_j$. Then the vectors $|\omega_1, \omega_2, ..., \omega_n\rangle$, $\omega = (\omega_1, \omega_2, ..., \omega_n) \in \{-1, 1\}^n$ form a basis for the $n$-qbits space. The eigenvectors of $Q$ then have the form

$$|\Psi(\omega)\rangle = \frac{1}{\sqrt{2}}(e^{i\theta(\omega)}|\omega_1, \omega_2, ..., \omega_n\rangle + |-\omega_1, -\omega_2, ..., -\omega_n\rangle)$$

with the corresponding eigenvector

$$\lambda(\omega) = e^{i\theta(\omega)}\sum_{k_1, k_2, ..., k_n} \pm c(k_1, k_2, ..., k_n) \exp i (\omega_1 t^1_{k_1} + \omega_2 t^2_{k_2} + ... + \omega_n t^n_{k_n})$$

where the angle $\theta(\omega)$ in the phase factor of (7) makes $\lambda(\omega)$ real. Hence we have

$$\max \|Q\| \leq \max \left| \sum_{k_1, k_2, ..., k_n} \pm c(k_1, k_2, ..., k_n) \exp i (t^1_{k_1} + t^2_{k_2} + ... + t^n_{k_n}) \right|$$

Where the maximum is taken over all values of $t^j_k$, $k = 1, 2, ..., r$, $j = 1, 2, ..., n$. Now, apply (3) to (8). We take the index set $J = \{1, 2, ..., r\}^n$, the trivial polynomials $P_{(k_1, k_2, ..., k_n)} = \exp i (t^1_{k_1} + t^2_{k_2} + ... + t^n_{k_n})$ considered as polynomials in (a part of ) the $rn$ variables $t^j_k$. We have $\|P_{(k_1, k_2, ..., k_n)}\|_\infty = 1$, $s = rn$, $N = n$. Proposition 1 follows since $\sum c^2(k_1, k_2, ..., k_n) = 1$.

Now, consider the classical local hidden variable expression (4). The value of $\|C\|_\infty$ is certainly bounded by the right hand side of (8). Hence, with high probability, $\|C\|_\infty \leq 9(rn \log n)^{1/2}$. Therefore, this bound does not give us any information about the likelihood of (a slight) violation of locality by $Q$. In any case, even if such a violation exists, it is too small to be detected by any real experiment.

Let us relax the assumption of coplanarity of the directions and assume that the $a^j_k$’s are arbitrary. In a fixed polar coordinates assume $a^j_k = (\theta^j_k, \phi^j_k)$. Let $|\Phi\rangle$ be an arbitrary unit vector in the $n$ qbits space. We have

$$\langle \Phi |Q| \Phi \rangle = \sum_k \pm c(k)P_k(\theta^1_{k_1}, ..., \theta^n_{k_n}, \phi^1_{k_1}, ..., \phi^n_{k_n})$$

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where $P_k = \langle \Phi | \sigma(a^1_{k_1}) \otimes \sigma(a^2_{k_2}) \otimes \cdots \otimes \sigma(a^n_{k_n}) | \Phi \rangle$ is considered as a trigonometric polynomial in (a part of) the $2rn$ variables $\theta^i_{k_j}, \phi^i_{k_j}$. The degree of $P_k$ is $2n$ and $\|P_k\|_\infty \leq 1$. Hence we can use (5) with $s = 2rn$, $N = 2n$ to obtain proposition 2.

In the case $r = 2$ we can derive a better bound than indicated by proposition 1. In this case the complete set of inequalities has been derived [5,6]. The set is complete in the sense that all other valid Bell inequalities for that case are convex combinations of elements of this set. Finding such a set for larger $r$ is highly unlikely, as the problem becomes intractable [9–11]. I shall use the characterization of Werner and Wolf. For convenience, let the row indices in (1) range over 0 and 1 (instead of 1 and 2). Then the classical inequalities are

$$-1 \leq \sum_{s_1, ..., s_n = 0, 1} \beta_f(s_1, ..., s_n)X_{s_1}^1X_{s_2}^2...X_{s_n}^n \leq 1 \quad (10)$$

There are $2^{2n}$ such inequalities, each determined by an arbitrary function $f : \{0, 1\}^n \rightarrow \{-1, 1\}$ by

$$\beta_f(s_1, ..., s_n) = \frac{1}{2^n} \sum_{\varepsilon_1, ..., \varepsilon_n = 0, 1} (-1)^{\varepsilon_1s_1 + ... + \varepsilon_ns_n}f(\varepsilon_1, ..., \varepsilon_n) \quad (11)$$

For each choice of function $f$ there corresponds a choice of coefficients $\beta_f$. Since $\beta_f$ is the inverse Fourier transform of $f$ on the group $\mathbb{Z}_2^n$ we have by Plancherel’s theorem [12]:

$$\sum |\beta_f(s)|^2 = \frac{1}{2^n} \sum |f(\varepsilon)|^2 = 1 \quad (12)$$

The set of quantum operators corresponding to the functions in (10) are the Werner Wolf operators

$$W_f = \sum_s \beta_f(s_1, ..., s_n)\sigma(a^1_{s_1}) \otimes \cdots \otimes \sigma(a^n_{s_n}) \quad (13)$$

and, as above, their eigenvalues have the form

$$\lambda_f = \sum_s \beta_f(s_1, ..., s_n) \exp i (\theta^1_{s_1} + \theta^2_{s_2} + ... + \theta^n_{s_n}) \quad (14)$$

Substituting in (14) the values of $\beta_f$ from (11) we get after changing the order of summation
\[ \lambda_f = \frac{1}{2^n} \sum_{\varepsilon} f(\varepsilon) \sum_{s} (-1)^{\varepsilon_1 s_1 + \ldots + \varepsilon_n s_n} \exp i (\theta_{s_1}^1 + \theta_{s_2}^2 + \ldots + \theta_{s_n}^n) \]  
\[ = \frac{1}{2^n} \sum_{\varepsilon} f(\varepsilon) \prod_{j=1}^{n} (\exp i\theta_j^j + (-1)^{\varepsilon_j} \exp i\theta_0^j) \]  

(15)

Assume that the \( \theta_j^j \) are arbitrary and fixed and denote \( c(\varepsilon) = 2^{-n} \prod_j (\exp i\theta_0^j + (-1)^{\varepsilon_j} \exp i\theta_1^j) \), then \( \sum_\varepsilon |c(\varepsilon)|^2 = 1 \). To see that note that \( |c(\varepsilon)|^2 = 2^{-2n} \left| \prod_j (1 + (-1)^{\varepsilon_j} \exp i\phi_j) \right|^2 \), \( \phi_j = \theta_1^j - \theta_0^j \). Now, \( |1 + \exp i\phi_j|^2 = 4 \cos^2 \left( \frac{\phi_j}{2} \right) \) and \( |1 - \exp i\phi_j|^2 = 4 \sin^2 \left( \frac{\phi_j}{2} \right) \) and therefore \( \sum_\varepsilon |c(\varepsilon)|^2 = \prod_j \left( \cos^2 \left( \frac{\phi_j}{2} \right) + \sin^2 \left( \frac{\phi_j}{2} \right) \right) = 1 \).

Consider \( \lambda_f = \sum_\varepsilon f(\varepsilon)c(\varepsilon) \), the signs \( f(\varepsilon) = \pm 1 \) are completely arbitrary and we can take them as independent, identically distributed random variables on \( \{-1, 1\}^{2^n} \). Put \( n(\varepsilon) = \sum_j \varepsilon_j 2^{j-1} \) and define the random trigonometric polynomial in a single variable \( t \)

\[ R(t) = \sum_\varepsilon \pm c(\varepsilon) \exp(in(\varepsilon)t) \]  

(16)

From the previous discussion we know that \( \|W_f\| = \max |\lambda_f| \leq \|R\|_{\infty} \), with \( f \) corresponding to the choice of signs in \( R \). The degree of \( R \) is \( 2^n \) and therefore by a straightforward application of (13) we get.

**Proposition 3** For each choice of directions in (13) a vast majority of the resulting Werner Wolf operators satisfy \( \|W_f\| \leq 13\sqrt{n} \).

**Conclusion** As the number of particles grows the violation of Bell’s inequality increases exponentially. If this is the case why don’t we observe entanglement on the macroscopic scale? Initially there are two possible answers. The first, which puts the blame on us, states that our thermodynamic observables are so crude that they cancel all interesting interference effects (this crudeness is assumed to include decoherence). The second possible answer is that the Mermin type violations require very exotic quantum states and very specific operators; we are very unlikely to run into them by chance. Here we have shown that the second answer is true, so the blame is not entirely on us.

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