Obtaining hydrogen energy eigenstate wave functions using the Runge-Lenz vector

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Abstract
The Pauli method of quantizing the Hydrogen system using the Runge-Lenz vector is ingenious. It is well known that the energy spectrum is identical with the one obtained from the Schrödinger equation and the consistency contributed significantly to the development of Quantum Mechanics in the early days. Since the Runge-Lenz vector is a vector and it commutes with the Hamiltonian, it is natural to use it to connect energy eigenstate \( |n, l, m\rangle \) with other degenerate states \( |n, l \pm 1, m'\rangle \). Recursive relations can be obtained and the energy eigenstate wave functions of the whole spectrum can be obtained easily. Note that the recursive relations are consistent with those used in factorizing the Schrödinger equation. Nevertheless, this analysis provides a better reasoning originated from the conserved vector, the Runge-Lenz vector. As in the Pauli analysis, group theory or symmetry plays a prominent role in this analysis, while the rest of the derivations are elementary.

1. Runge-Lenz Vector and the Hydrogen energy spectrum

In the early days of the development of Quantum Mechanics, Pauli made use of the Runge-Lenz vector and successfully quantized the Hydrogen atom system in the Matrix Mechanics framework [1]. The energy spectrum obtained is identical with the one obtained from the Schrödinger equation [2]. His method is very interesting and ingenious and the consistency contributed greatly to the development of Quantum Mechanics. Some early development along this line can be found in [3], while the recent development is well summarised in [4].

Since the Runge-Lenz vector commutes with the Hamiltonian, it can connect different degenerate states. Furthermore, as a vector, it is natural to use it to connect energy eigenstate \( |n, l, m\rangle \) with other degenerate states \( |n, l \pm 1, m'\rangle \). It will be interesting to use it to obtain the corresponding wave functions and to show explicitly that they are identical to the results obtained from the Schrödinger equation. As we shall see recursive relations of radial wave functions can be obtained. They are consistent with the results obtained by factorized the Schrödinger equation [5–8] (see also [9]). Nevertheless we believe that it is more natural to use the Runge-Lenz vector, a conserved vector of the system, to obtain the factorization results. The wave functions of the whole spectrum can be obtained easily. We will also briefly discuss the \( E > 0 \) case and see that the corresponding wave functions can be verified. As in the Pauli analysis, we will see that group theory or symmetry plays a prominent role in the present analysis. Some early works somewhat related to this line of approach can be found in [10–12]. We will make a comparison later, as the discussion is rather technical. Nevertheless, as will be shown, it is worth noting that, in the present topics, this work goes beyond those early studies.

The lay out of this paper is as following. In the first section we briefly go through the derivations of the Hydrogen atom spectrum\(^1\) and will concentrate on obtaining wave function via the Runge-Lenz vector in the next section, which is followed by a discussion and conclusion section. An appendix is added for the derivation of some relevant matrix elements using group theory.

\(^1\) Our derivations follow closely to those in [13].
1.1. Runge-Lenz vector

The Hamiltonian of the Hydrogen atom is given by

\[ H = \frac{p^2}{2\mu} - \frac{Ze^2}{4\pi\varepsilon_0 r} = \frac{\vec{p}^2}{2\mu} - \kappa \frac{1}{r}, \]

with \( \kappa \equiv \frac{Ze^2}{4\pi\varepsilon_0} \). The Runge-Lenz vector is defined as:

\[ \vec{A} \equiv \frac{1}{2\mu}(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}) - \frac{\kappa^2}{r}. \]

Note that Jakob Hermann was the first to show that there exists a vector that is conserved for a special case of the inverse-square central force, and worked out its connection to the eccentricity of the orbital ellipse [14]. Hermann’s work was generalized to its modern form by Johann Bernoulli in 1710 [15].

The Runge-Lenz vector satisfies the following relations: [1]

\[ \vec{L} \cdot \vec{A} = \vec{A} \cdot \vec{L} = 0, \]

\[ [H, \vec{A}] = 0, \quad [H, \vec{L}] = 0, \]

\[ \{L_i, L_j\} = i\hbar\epsilon_{ijk}L_k, \]

\[ \{L_i, A_j\} = i\hbar\epsilon_{ijk}A_k, \]

\[ \{A_i, A_j\} = i\hbar\left( -\frac{2}{\mu}H \right)\epsilon_{ijk}L_k, \]

and

\[ \vec{A} \cdot \vec{A'} = \frac{2}{\mu}H(\vec{L}^2 + \hbar^2) + \kappa^2. \]

It should be noted that the Runge-Lenz vector is a conserved operator. The above relations will be useful in obtaining the Hydrogen atom energy spectrum [1].

1.2. Hydrogen atom energy spectrum

The eigenvalue equation of the Hydrogen atom Hamiltonian is given by

\[ H|E, \alpha\rangle = E|E, \alpha\rangle = -|E|\alpha\rangle, \]

where we only consider the \( E < 0 \) case here and \( \alpha \) is a possible quantum number. The set of the eigenstates \( \{|E, \alpha\rangle, \{E, \beta\}...\} \) with \( E \) fixed spans the degenerate space of the energy eigenstates all having the same energy.

From equation (4), we know that the Hamiltonian commutes with \( \vec{L} \) and \( \vec{A} \). Hence, it is useful to define the following matrices:

\[ (L)_{\alpha\beta} \equiv \langle E, \alpha|L|E, \beta\rangle, \quad (A)_{\alpha\beta} \equiv \langle E, \alpha|A|E, \beta\rangle, \]

and the relations in equations (3), (5) and (6) correspond to the following relations of matrices:

\[ (L \cdot A)_{\alpha\beta} = (A \cdot L)_{\alpha\beta} = 0, \]

\[ A^2_{\alpha\beta} = -\frac{2E}{\mu}(E^2 + \hbar^2)_{\alpha\beta} + \kappa^2l_{\alpha\beta}, \]

and

\[ [L_i, L_j]_{\alpha\beta} = i\hbar\epsilon_{ijk}(L_k)_{\alpha\beta}, \]

\[ [L_i, A_j]_{\alpha\beta} = i\hbar\epsilon_{ijk}(A_k)_{\alpha\beta}, \]

\[ [A_i, A_j]_{\alpha\beta} = i\hbar\epsilon_{ijk}\left( -\frac{2E}{\mu} \right)(L_k)_{\alpha\beta} \]

\[ = i\hbar\epsilon_{ijk}\left( \frac{2E}{\mu} \right)(L_k)_{\alpha\beta}. \]

With the following definition

\[ (A')_{\alpha\beta} \equiv \sqrt{\frac{\mu}{2E}}(A)_{\alpha\beta}, \]

the above equations become

\[ (L \cdot A')_{\alpha\beta} = (A' \cdot L)_{\alpha\beta} = 0, \]
\[(\mathcal{A}^2 + \mathbf{L})_{\lambda\beta} = \left(-\hbar^2 + \frac{\mu\kappa^2}{2[E]}\right)_{\lambda\beta},\]  
(14)

and

\[
\begin{align*}
\mathbf{L}_\ell \cdot \mathbf{L}_\ell &= i\hbar \varepsilon_{\ell\ell'}, \\
\mathbf{A}'_\ell \cdot \mathbf{A}'_\ell &= i\hbar \varepsilon_{\ell\ell'}, \\
\mathbf{A}_\ell \cdot \mathbf{A}'_\ell &= i\hbar \varepsilon_{\ell\ell}. 
\end{align*}
\]  
(15)

The above equation, equation (15), implies that \(\mathbf{L}\) and \(\mathbf{A}'\) are the generators of the \(O(4)\) group [1]. The quantization of the Hydrogen atom system can be achieved by using group theory [1].

A representation of \(O(4)\) can be expressed as a direct product of two \(SO(3)\) representations as following [2].

Defining two new sets of operators \(B_{\ell}^{(+)}\) and \(B_{\ell}^{(-)}\),

\[B_{\ell}^{(\pm)} = \frac{1}{2}(\mathbf{L}_\ell \pm \mathbf{A}'_\ell),\]  
(16)

equation (15) becomes

\[
\begin{align*}
[B_{\ell}^{(+)} - B_{\ell}^{(+)\dagger}, B_{\ell}^{(+)}]_{\lambda\beta} &= i\hbar \varepsilon_{\lambda\beta}, \\
[B_{\ell}^{(-)} - B_{\ell}^{(-)\dagger}, B_{\ell}^{(-)}]_{\lambda\beta} &= i\hbar \varepsilon_{\lambda\beta}, \\
[B_{\ell}^{(+)} + B_{\ell}^{(+)*}, B_{\ell}^{(-)*}]_{\lambda\beta} &= 0.
\end{align*}
\]  
(17)

It is clear that \(B_{\ell}^{(+)}\) commute with \(B_{\ell}^{(-)}\) and they satisfy the usual rotation group \([SO(3)]\) algebra. Conventionally the simultaneous eigenstates in the \(|\alpha\rangle\) space are chosen to be the eigenstates of the following mutual commuting matrices:

\[
(B_{\ell}^{(+)\dagger})^2, B_{\ell}^{(+)}, (B_{\ell}^{(-)*})^2, B_{\ell}^{(-)}. \]  
(18)

We can now return to the usual Dirac notation. The corresponding eigenvalue equations are

\[
\begin{align*}
\hbar^2 b^{(+)} b^{(-)} &= E^{1/2}[E, b^{(+)} + 1]^{1/2}[E, b^{(-)}], \\
B_{\ell}^{(+)\dagger} B_{\ell}^{(+)} &= E^{1/2} b^{(+)} b^{(-)} = E^{1/2} b^{(+)} b^{(-)}, \\
B_{\ell}^{(-)*} B_{\ell}^{(-)} &= E^{1/2} b^{(+)} b^{(-)} = E^{1/2} b^{(+)} b^{(-)}. 
\end{align*}
\]  
(19)

Hence, we have \((B_{\ell}^{(+)\dagger})^2 = (B_{\ell}^{(-)*})^2\). Equation (20) implies the following relation:

\[
\begin{align*}
(B_{\ell}^{(+)\dagger})^2[E, b, m^{(+)} + 1]^{1/2}[E, b, m^{(-)}] &= E^{1/2} b^{(+)} b^{(-)}, \\
&= \frac{1}{4}(-\hbar^2 + \frac{\mu}{2E^2})[E, b, m^{(+)} + 1]^{1/2}[E, b, m^{(-)}].
\end{align*}
\]  
(21)

with \(b^{(+)} = b^{(-)} = 0, 1/2, 1, 3/2, \ldots\) and \(-b^{(\pm)} \leq m^{(\pm)} \leq b^{(\pm)}.\) The energy eigenvalue \(E = E_n\) can now be obtained as [1]

\[
E_n = \frac{\kappa^2 \mu}{2\hbar^2 (2b + 1)^2} = \frac{-\kappa^2 \mu}{2\hbar^2 n^2} = -\frac{\hbar^2}{2\hbar^2 n^2} = -\frac{Z^2 e^4 \mu}{32\pi^2 \epsilon_0^2 \hbar^2 n^2},
\]  
(22)

with \(n\) defined as \(n \equiv 2b + 1 = 1, 2, 3, \ldots\) It will be useful to define the (reduced) Bohr radius, \(a_0 \equiv 4\pi \epsilon_0 \hbar^2 / e^2 \mu = (\hbar / \mu \alpha) = Z\hbar^2 / \kappa\mu\), where \(\alpha = e^2 / 4\pi \epsilon_0 \hbar c = 1/137\) is the fine structure constant. The energy spectrum obtained by Pauli [1] is consistent with the one obtained from the Schrödinger equation [2] and the consistency contributed significantly to the development of Quantum Mechanics in the early days.

2. Hydrogen atom energy eigenstate

2.1. Connecting degenerate states using the Runge-Lenz vector

Since the Runge-Lenz vector is a vector and it commutes with the Hamiltonian, it is natural to use it to connect energy eigenstate \(|n, l, m\rangle\) with other degenerate states, \(|n, l \pm 1, m'\rangle\) and \(|n, l, m'\rangle\). As shown in the previous

2 It is probable that a modern reader is more familiar with the case of the \(SO(3,1)\) Lorentz group, which can be analyzed using a similar manipulation, see, for example, [16].

3 The latter states \(|n, l, m'\rangle\) are, however, prohibited by parity. As we shall see the corresponding matrix elements are vanishing.
section the energy eigenstates of the Hydrogen atom system are \( |n, b, m^+, m^- \rangle \) with \( b = (n - 1)/2 \), which satisfy

\[
H \left| n, \frac{n - 1}{2}, m^+, m^- \right\rangle = E_n \left| n, \frac{n - 1}{2}, m^+, m^- \right\rangle ,
\]

\[
(B^{(\pm)})^2 \left| n, \frac{n - 1}{2}, m^+, m^- \right\rangle = \frac{n^2}{4} \hbar^2 \left| n, \frac{n - 1}{2}, m^+, m^- \right\rangle ,
\]

\[
B^z_{\pm} \left| n, \frac{n - 1}{2}, m^+, m^- \right\rangle = m \pm \hbar \left| n, \frac{n - 1}{2}, m^+, m^- \right\rangle ,
\]

(23)

with \(- (n - 1)/2 \leq m^\pm \leq (n - 1)/2 \). In general these eigenstates do not have specified angular momentum quantum numbers and are different from the energy eigenstates in a more familiar basis:

\[
H |n, l, m\rangle = E_n |n, l, m\rangle ,
\]

\[
\mathcal{L}_z^2 |n, l, m\rangle = \ell(\ell + 1) \hbar^2 |n, l, m\rangle ,
\]

\[
\mathcal{L}_z |n, l, m\rangle = m |n, l, m\rangle ,
\]

(24)

with \(-l \leq m \leq l \).

Since the angular momentum can be obtained through the following equation, see equation (16),

\[
\mathcal{L} = B^{(+) + B^{(-)}} ,
\]

(25)

with \( B^{(+) \ and \ B^{(-)} \)^} viewed as two independent spin operators, the \( |n, l, m\rangle \) state can be constructed as in the analysis of the addition of angular momentum:

\[
|n, l, m\rangle = \sum_{m', |m'|} \left| n, n - 1 \left| 2, m', m'\right\rangle \left| n - 1 \left| 2, m', m\right\rangle \right\rangle ,
\]

(26)

where \( \left< \frac{n-1}{2}, m^+, \frac{n-1}{2}, m^- \right| l, m \rangle \) is the Clebsch-Gordan coefficient. Note that the minimum of \( \ell \) is \( |(n - 1)/2 - (n - 1)/2| = 0 \), while the maximum is \( (n - 1)/2 + (n - 1)/2 = n - 1 \). These are consistent with the result obtained by using the Schrödinger equation.

The Runge-Lenz vector \( \mathcal{A} \) commutes with the Hamiltonian \( H \). It can connect eigenstate \( |n, lm\rangle \) with other degenerate energy eigenstates \( |n, l \pm 1, m', \rangle, \langle n, l, m'\rangle \). To proceed we define

\[
\mathcal{A}_\pm \equiv \mathcal{A}_x \pm i\mathcal{A}_y , \quad B^{(\pm)}_z \equiv B^{(\pm)}_x \pm iB^{(\pm)}_y .
\]

From equation (16), we have

\[
\frac{\mu}{\hbar^2} A_{|n, l\rangle} = \sum_{l' = |l| + 1} \left< n, l', l - 1 \right| \left< n, l', l - 1 \right| B^{(+) - B^{(-)}} |n, l, \ell\rangle \frac{2\mu E_n}{\sqrt{-\hbar^2}} ,
\]

(28)

where we apply \( A_- \) on the \( m \) state, which is found to be useful in obtaining the radial wave function in later discussion.

Using the familiar formula of non-vanishing matrix elements of lowering operator \( L_- = L_x - iL_y \), suitably, we have

\[
\left< n, l', l - 1 \right| B^{(+) \left| n, l, \ell\rangle} = \sum_{m'^+, m'^-} \left< n, l', l - 1 \right| \left< n, l', l - 1 \right| B^{(+) \left| n, l, \ell\rangle} \frac{2\mu E_n}{\sqrt{-\hbar^2}} ,
\]

(29)

and the above derivation is self-evident. Possible non-vanishing matrix elements are for \( l' = l, l \pm 1 \) as \( B^{(\pm)} \) are vector operators. Using group theory the above matrix elements (for \( l' = l, l \pm 1 \)) are found to be

\[
\left< n, l, l - 1 \right| B^{(\pm)}_z |n, l, \ell\rangle = \pm \sqrt{\frac{l(l^2 - 1)}{2(2l - 1)}} \hbar ,
\]

\[
\left< n, l, l + 1 \right| B^{(\pm)}_z |n, l, \ell\rangle = \pm \sqrt{\frac{n^2 - (l + 1)^2}{2(2l + 1)(2l + 3)}} \hbar ,
\]

\[
\left< n, l, l - 1 \right| B^{(\pm)}_z |n, l, \ell\rangle = \pm \sqrt{\frac{l}{2}} \hbar ,
\]

(30)

where the derivation are shown in appendix. Note that the above results also hold for the \( l = 0 \) case, where the equation implies that \( \left< n, l = 1 \right| B^{(\pm)}_z |n, l, \ell\rangle \) and \( \left< n, l, l = 1 \right| B^{(\pm)}_z |n, l, \ell\rangle \) are vanishing as they should.
Using the above equation, the corresponding matrix elements of $A_-$ are given by

$$\frac{\mu}{\hbar^2} \langle n', l' - 1 | A_- | n, l, l \rangle = \frac{\langle n, l' - 1, l | (B_z^{(+)})^* - B_z^{(-)} | n, l, l \rangle}{\hbar} \sqrt{\frac{2\mu E_n}{\hbar^2} - \frac{Z^2}{na_0^2} \left( \frac{2(n^2 - (l + 1)^2)}{(2l + 1)(2l + 3)} \right)} \begin{cases} 0, & l' = l \pm 1, \\
 Z \sqrt{\frac{2(n^2 - (l + 1)^2)}{(2l + 1)(2l + 3)}}, & l' = l + 1, \\
 -Z \sqrt{\frac{2(n^2 - l^2)}{(2l + 1)}}, & l' = l - 1, \end{cases}$$

(31)

where we have made use of $-2\mu E_n / \hbar^2 = \kappa^2 \mu^2 / \hbar^4 n^2 = (Z / na_0)^2$. Note that the matrix element for the $l' = l$ state is vanishing as required by parity. Substitute them into equation (28), we finally obtain our master formula:

$$\frac{\mu}{\hbar^2} A_- | n, l, l \rangle = \frac{Z}{na_0} \left[ \sqrt{\frac{2(n^2 - (l + 1)^2)}{(2l + 1)(2l + 3)}} | n, l + 1, l - 1 \rangle - \sqrt{\frac{2(n^2 - l^2)}{(2l + 1)}} | n, l - 1, l - 1 \rangle \right].$$

(32)

Note that as in the Pauli analysis, the above master formula follows from group theory or symmetry. As we shall see the above equation can provides recursive relations on degenerate states, and the relations are powerful enough to determine the wave functions.

2.2. Obtaining the radial wave function $R_{ad}(r)$

2.2.1. Recursive relations of $R_{ad}(r)$

We will obtain the recursive relations of the radial wave functions from the master formula here. Note that most of the derivations here are elementary. Using $\langle \vec{r} | A_- | n, l, l \rangle = R_{ad}(r) Y_{ll}(\theta, \phi)$ and

$$\langle \vec{r} | A_- | n, l, l \rangle = (A_-)_{op} \langle \vec{r} | n, l, l \rangle,$$

(33)

the master formula equation (32) can be expressed as:

$$\frac{\mu}{\hbar^2} (A_-)_{op} R_{nn,l,l}(r) Y_{l,l}(\theta, \phi) = \frac{Z}{na_0} \left[ \sqrt{\frac{2(n^2 - (l + 1)^2)}{(2l + 1)(2l + 3)}} R_{n,l+1,l-1}(r) Y_{l+1,l-1}(\theta, \phi) \\
- \sqrt{\frac{2(n^2 - l^2)}{(2l + 1)}} R_{n,l-1,l-1}(r) Y_{l-1,l-1}(\theta, \phi) \right].$$

(34)

To proceed we need to work out the left-hand-side of the above equation. Using $\vec{p}^* \times \vec{L}^* + \vec{L}^* \times \vec{p}^* = 2i\hbar \vec{p}^*$, it is convenient to express the ± components of the Runge-Lenz vector as

$$A_\pm = \frac{1}{\mu} (\vec{p} \times \vec{L} - i \hbar \vec{p}^*) = \kappa \frac{r_e}{r},$$

(35)

where we have defined $r_e = \pm \imath y$. Consequently, the operator $(A_-)_{op}$ in equation (34) is given by

$$(A_-)_{op} = -\frac{1}{\mu} (\imath \hbar \vec{p}_- \vec{L}_- + \vec{L}_- - \hbar \vec{p}_- \vec{L}_-) = \kappa \frac{r_e}{r},$$

(36)

with

$$(\vec{p}_{\pm})_{op} = \frac{\hbar}{i} (\partial_x \pm i \partial_y) = 2 \frac{\hbar}{i} \frac{\partial}{\partial r_e},$$

$$\left(\vec{L}_z\right)_{op} = \frac{\hbar}{i} (x \partial_y - y \partial_x) = \hbar \left( r_e \frac{\partial}{\partial r_e} - r \frac{\partial}{\partial r} \right),$$

$$\left(\vec{L}_\pm\right)_{op} = \mp \imath \hbar (\vec{p}_{\pm})_{op} = \mp \hbar \left( r_e \frac{\partial}{\partial z} - 2z \frac{\partial}{\partial r_e} \right).$$

(37)

Using $r^2 = r_+ r_- + z^2$ and

$$\frac{\partial}{\partial r_e} f(r) = \frac{r_e}{2r} df \frac{dr}{dr}, \quad \frac{\partial}{\partial z} f(r) = \frac{z}{r} df \frac{dr}{dr},$$

(38)

it can be shown from equation (37) that we must have $(\vec{L}_z)_{op} f(r) = (\vec{L}_\pm)_{op} f(r) = 0$, which are expected from the familiar forms of $(\vec{L}_\pm)_{op}$ in wave mechanics (as they only involve derivatives with respect to $\theta$ and $\phi$). Note that it is convenient to use the $(r_+, r_-, z)$ coordinate, as $Y_{l\pm,l}(\theta, \phi)$ in the left-hand-side of equation (34) have simple forms in terms of $r_\pm$ and $r$. 

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From equations (37), (38) and the fact that \((L_{z})_{op}\) and \((L_{-})_{op}\) do not act on any function of \(r\), it can be easily shown that we must have

\[
(i\hbar p_{z})_{op}(L_{z})_{op} - \hbar \frac{\partial}{\partial r_{+}}f(r) = 2\hbar^{2}(l - 1) \frac{\partial}{\partial r_{+}} \left( \frac{r_{+}}{r} \right) f(r),
\]

\[
(i\hbar p_{z})_{op}(L_{-})_{op} = -\hbar \frac{\partial}{\partial r_{+}} \left( \frac{2Z}{r} \right) f(r),
\]

and, consequently, the left-hand-side of equation (34) becomes

\[
\frac{\mu(A_{-})_{op} (r_{z})_{op}}{\hbar^{2} r} R_{al}(r) = \left( r_{+} \frac{2l + 2z z}{r_{+} r} + \frac{2l(l - 1)}{r_{+}} + (l - 1) \frac{d}{dr} - \frac{Z}{a_{0}} \right) \frac{R_{al}(r)}{r^{l}}.
\]

It can be further expressed as

\[
\frac{\mu(A_{-})_{op} (r_{z})_{op}}{\hbar^{2} r} R_{al}(r) = (r_{+} \frac{2l + 2z z}{r_{+} r} + \frac{2l(l - 1)}{r_{+}} + (l - 1) \frac{d}{dr} - \frac{Z}{a_{0}} \right) \frac{R_{al}(r)}{r^{l}},
\]

where we have made use of equation (38) and \(\kappa\mu/\hbar^{2} = Z/a_{0}\). Use again \(z^{2} = r^{2} - r_{+} r_{-}\), we obtain

\[
\frac{\mu(A_{-})_{op} (r_{z})_{op}}{\hbar^{2} r} R_{al}(r) = (r_{+} \frac{2l + 2z z}{r_{+} r} + \frac{2l(l - 1)}{r_{+}} + (l - 1) \frac{d}{dr} + \frac{Z}{a_{0}} \right) \frac{R_{al}(r)}{r^{l}},
\]

or, equivalently, [see equation (39)]

\[
\frac{\mu(A_{-})_{op} (r_{z})_{op}}{\hbar^{2} r} Y_{l,l} R_{al}(r) = \left[ r^{l-1} \frac{d}{dr} Y_{l,l} \frac{2l + 2z z}{r_{+} r} + \frac{2l(l - 1)}{r_{+} r} + (l - 1) \frac{d}{dr} + \frac{Z}{a_{0}} \right] \frac{R_{al}(r)}{r^{l}}.
\]

It is useful to note that the spherical harmonics \(Y_{l,l}\) has the following properties:

\[
\frac{r}{r_{+}} Y_{l,l} = -\sqrt{\frac{(2l + 1)}{2l}} Y_{l-1,l-1},
\]

\[
\frac{r}{r_{+}} Y_{l,l} = \sqrt{\frac{8\pi}{3}} Y_{l-1,l} = -\sqrt{\frac{2l + 1}{2l + 3}} Y_{l-1,l-1} + \sqrt{\frac{2}{(2l + 1)(2l + 3)}} Y_{l+1,l+1},
\]

and the above equation can be expressed as

\[
\frac{\mu(A_{-})_{op} Y_{l,l} R_{al}(r)}{\hbar^{2} r} = -\sqrt{\frac{2}{(2l + 1)(2l + 3)}} r^{l-1} Y_{l+1,l+1} \left( \frac{d}{dr} + \frac{Z}{a_{0}} \right) \frac{R_{al}(r)}{r^{l}}
\]

\[
-\frac{2l + 2z z}{r_{+} r} \frac{d}{dr} Y_{l-1,l-1} \left( \frac{2l + 1}{2l + 3} + \frac{Z}{a_{0}} \right) \frac{R_{al}(r)}{r^{l}}.
\]

Compare the above equation to the master formula equation (34), we finally obtain

\[
\frac{Z\sqrt{\mu^{2} - \frac{l(l + 1)}{r^{2}}}}{na_{0}} R_{al}(r) \left( \frac{d}{dr} + \frac{Z}{a_{0} r} \right) \frac{R_{al}(r)}{r^{l}} = \left( 2l + 1 \right) \frac{d}{dr} + \frac{Z}{a_{0}} \frac{R_{al}(r)}{r^{l}}.
\]

These are the recursive relations of the radial wave function \(R_{al}(r)\) and they are important results of this section.

The above relations are consistent to the results found in [5–7] (see also [9]) using the factorization method. Nevertheless we believe that according to the properties of the Runge-Lenz vector, which is a conserved vector, the above derivation is the most natural way to obtain them. As we shall see shortly they are powerful enough to determine the radial wave functions.

\footnote{The first relation follows from equation (39), while the second relation follows from the familiar relation \(Y_{l,m} Y_{l',m'} = \delta_{ll'} \delta_{mm'} \sum_{m'} \sqrt{\frac{(2l + 1)(2l + 3)}{4\pi (2l + 1)}} \).}
2.2.2. Solve for $R_{n,n-1}(r)$ using the recursive relation

For $l = n - 1$, the first relation of the recursive relations in equation (47) gives

$$
\left( \frac{d}{dr} + \frac{Z}{na_0} \right) \frac{R_{n,n-1}(r)}{r^{n-1}} = 0.
$$

(48)

Its solution is

$$
R_{n,n-1}(r) = c_n \left( \frac{Zr}{a_0} \right)^{n-1} e^{-Zr/(na_0)},
$$

(49)

with the normalization constant,

$$
c_n = \frac{2^n}{n^{n-1/2} \sqrt{2(2n)!}} \left( \frac{Z}{na_0} \right)^{3/2},
$$

(50)

obtained from the usual normalization condition

$$
\int_0^\infty dr \ r^2 \ R_{n,n-1}^2(r) = 1.
$$

In fact, the above result can be quickly obtained by noting

$$
\left| n, l = n - 1, m = \pm(n - 1) \right\rangle = \begin{cases} \psi_{n,n-1}^{(n-1)}(r, \theta) e^{i \pm(n-1) \phi} & \text{for} \ \pm \phi = \mp \frac{\pi}{2}, \\ -i \frac{n \hbar}{\mu} \left( \mp \frac{Zr}{a_0} \right) \left| n, l = n - 1, m = \pm(n - 1) \right\rangle, \end{cases}
$$

(51)

and $A_{\pm} \propto B_{\pm}^{(n-1)} - B_{\pm}^{(n-1)}$, which imply:

$$
A_{\pm} \left| n, l = n - 1, \pm(n - 1) \right\rangle = 0.
$$

(52)

From

$$
A_{\pm} = \frac{1}{\mu} \left( \pm i p_\parallel L_\parallel \right) + \frac{1}{\mu} \left( \mp i p_\parallel L_\parallel - i p_\perp \right) - \kappa \frac{r_{\parallel}}{r},
$$

(53)

we have

$$
0 = A_{\pm} \left| n, l = n - 1, \pm(n - 1) \right\rangle = \left( \mp i \frac{n \hbar}{\mu} - \kappa \frac{r_{\parallel}}{r} \right) \left| n, l = n - 1, \pm(n - 1) \right\rangle.
$$

(54)

Using \( \langle \tilde{r} \ | n, l, m \rangle = R_{nl}(r) Y_{lm}(\theta, \phi) \) the above equation takes the following form:

$$
\left( \frac{n \hbar}{\mu} \left( p_\parallel \right)_{ho} + \kappa \frac{r_{\parallel}}{r} \right) R_{n,n-1}(r) Y_{n-1,\pm(n-1)}(\theta, \phi) = 0.
$$

(55)

With the help of $Y_{l\pm\ell} \propto (r_\perp)^{l+\ell}/r^{\ell l}$ and

$$
(p_\parallel)_{ho} (r_\perp)^{l} f(r) = 2 \frac{\hbar}{i} \frac{\partial}{\partial r_{\parallel}} (r_\perp)^{l} f(r) = (r_\perp)^{l} \frac{\partial}{\partial r_{\parallel}} \left( \frac{\partial}{\partial r} f(r) \right),
$$

(56)

we clearly see that the it is equivalent to equation (48).

2.2.3. Obtaining other $R_{nl}(r)$

Once $R_{n,n-1}(r)$ is known, other $R_{nl}(r)$ can be obtained readily by applying the second relation of the recursive relations given in equation (47):

$$
\frac{R_{n,l-1}(r)}{r^{l-1}} = \frac{n}{\sqrt{n^2 - l^2}} \left( \frac{2l + 1}{Z} \right) \frac{(2l + 1)l a_0}{Z} + \frac{l a_0}{Z} \frac{d}{dr} \left( \frac{r}{Z} \right) R_{nl}(r).
$$

(57)

For illustration, we apply the above equation on the $l = n - 1$ radial wave function and obtain

$$
R_{n,n-2}(r) = c_n n(n - 1) \sqrt{2n - 1} \left( \frac{Zr}{a_0} \right)^{n-2} \left( \frac{1}{n(n-1)a_0} - \frac{Zr}{2(n-1)a_0} \right) e^{-Zr/a_0},
$$

(58)

and, sequentially, applying the relation on $l = n - 2$, we have

$$
R_{n,n-3}(r) = \frac{1}{2} c_n \left( \frac{Zr}{a_0} \right)^{n-3} \left( \frac{2(n-3)(n-2)n^2 \sqrt{2n-1}(n-1)}{n(n-2)a_0} + \frac{1}{n^2(6 - 7n + 2n^2)a_0^2} \right) e^{-Zr/a_0}.
$$

(59)

In principle, the procedure can be carried out to obtain all $R_{nl}(r)$. 
It will be useful to show explicitly some of the radial wave functions obtained:

\[
R_{00}(r) = 2 \left( \frac{Z}{a_0} \right)^{3/2} e^{-Zr/a_0},
\]

\[
R_{31}(r) = \frac{1}{\sqrt{3}} \left( \frac{Z}{2a_0} \right)^{3/2} \frac{Zr}{a_0} e^{-Zr/2a_0},
\]

\[
R_{20}(r) = 2 \left( \frac{Z}{2a_0} \right)^{3/2} \left( 1 - \frac{Zr}{2a_0} \right) e^{-Zr/2a_0},
\]

\[
R_{32}(r) = \frac{2\sqrt{2}}{27\sqrt{3}} \left( \frac{Z}{3a_0} \right)^{3/2} \frac{Zr}{a_0} e^{-Zr/3a_0},
\]

\[
R_{31}(r) = \frac{4\sqrt{2}}{9} \left( \frac{Z}{3a_0} \right)^{3/2} \left( \frac{Zr}{a_0} \right) \left( 1 - \frac{Zr}{6a_0} \right) e^{-Zr/3a_0},
\]

\[
R_{30}(r) = 2 \left( \frac{Z}{3a_0} \right)^{3/2} \left( 1 - \frac{2Zr}{3a_0} + \frac{2(Zr)^2}{27a_0^2} \right) e^{-Zr/3a_0},
\]

and compare them to those obtained by solving the Schrödinger equation directly, see for example, [18]. Indeed, it is clear that the wave functions obtained in the two approaches are consistent. It is interesting that even the phase conventions match.

2.3. The radial wave function \( R_{nl} \) satisfies the radial Schrödinger equation

Applying the recursive relations, equation (47), on \( R_{nl}/r^n \) can bring it to \( R_{n+1,l}/r^{n+1} \) and suitably apply the relations again can bring them back to \( R_{nl}/r^n \). These procedures produce the following identities on \( R_{nl}/r^n \), \( ^6,7 \)

\[
\frac{1}{(l+1)^2} \left[ (2l+3)(l+1) + (l+1)r \frac{d}{dr} - r \frac{Z}{a_0} \right] R_{nl}(r) = - \frac{Z^2(n^2 - (l+1)^2)}{n^2a_0^2(l+1)^2} R_{nl}(r),
\]

\[
\frac{1}{l^2} \left( \frac{Z}{a_0} + \frac{l}{r} \frac{d}{dr} \right) \left[ (2l+1)(l+1) + l \frac{d}{dr} - r \frac{Z}{a_0} \right] R_{nl}(r) = - \frac{Z^2(n^2 - l^2)}{n^2a_0^2l^2} R_{nl}(r),
\]

giving,

\[
\left( \frac{d^2}{dr^2} + \frac{2(l+1)}{r} \frac{d}{dr} + \frac{2Z}{a_0 r} - \frac{Z^2}{l^2a_0^2} \right) R_{nl}(r) = - \frac{Z^2(n^2 - (l+1)^2)}{n^2a_0^2(l+1)^2} R_{nl}(r),
\]

or, equivalently,

\[
\left( \frac{d^2}{dr^2} + \frac{2(l+1)}{r} \frac{d}{dr} + \frac{2Z}{a_0 r} - \frac{Z^2}{n^2a_0^2} \right) R_{nl}(r) = \frac{Z^2(n^2 - l^2)}{n^2a_0^2l^2} R_{nl}(r) = 0.
\]

It can be compared to the radial equation from the Schrödinger equation, see for example [18],

\[
\left( \frac{d^2}{dr^2} + \frac{2}{r} + \frac{2\mu}{\hbar^2} E_n - \frac{l(l+1)}{2r^2} \right) R_{nl}(r) = 0,
\]

or, equivalently,

\[
\left( \frac{d^2}{dr^2} + \frac{2(l+1)}{r} \frac{d}{dr} + \frac{2Z}{a_0 r} + \frac{2\mu}{\hbar^2} E_n \right) R_{nl}(r) = 0.
\]

It is clear that equation (63) is same as the above equation with \( E_n = -Z^2\hbar^2/2\mu a_0^2 n^2 \).

\(^5\) Note that there is a typo in the normalization factor of \( R_{nl}(r) \) in [18].

\(^6\) The additional factors \( 1/(l+1)^2 \) and \( 1/l^2 \) are designed to remove the coefficients of \( d^2/dr^2 \).

\(^7\) The equation is said to be factorizable \([5–8]\).
2.4. Wave functions in the $E > 0$ case
We briefly discuss the $E > 0$ case here. For the $E > 0$ case, as in the $E < 0$ case, one can still define

$$B_i^{(\pm)} \equiv \frac{1}{2} (L_i \pm iA_i') \equiv \frac{1}{2} \left( L_i \pm \frac{i}{\sqrt{2E}} A_i' \right),$$

(66)

which satisfy

$$[B_i^{(\pm)}, B_j^{(\pm)}]_{\alpha\beta} = i\hbar g \delta(B_i^{(\pm)})_{\alpha\beta},$$

$$[B_i^{(-)}, L_{(±)}]_{\alpha\beta} = i\hbar g \delta(B_i^{(-)})_{\alpha\beta}$$

and

$$\left( B_{(-)}^{(±)} \right)^2 = \frac{1}{4} (L^2 \pm i\mathbf{L} \cdot \mathbf{A}' \pm i\mathbf{A}' \cdot \mathbf{L} - \mathbf{A}'^2) = \frac{1}{4} (L^2 - \mathbf{A}'^2) = \frac{1}{4} \left( -\hbar^2 - \frac{\mu}{2E} \right).$$

(68)

However, now the situation is different. The above relations cannot be used to obtain energy eigenvalue as in the $E < 0$ case. The reason is that $B^{(±)}$ are no longer hermitian. The usual procedure of obtaining quantum numbers in $L_L$ and $L_\nu$ cannot be used in $(B^{(±)})^2$ and $B^{(±)}$. For example, the above equation show that $(B^{(±)})^2$ are negative matrices and the usual steps in quantizing angular momentum break down in quantizing this system.

Nevertheless some results obtained in the previous section can still be used. In particular the recursive relations similar to equation (47) can be used to find the wave function in the $E > 0$ case. Replacing $n$ by $\nu$ in equation (47), we have the following recursive relations

$$\frac{-Z\sqrt{\nu^2 + (l+1)^2} R_{l+1}(r)}{\nu a_0} = \left( \frac{l+1}{r} \frac{d}{dr} + \frac{Z}{a_0} \right) R_l(r),$$

$$\frac{Z\sqrt{\nu^2 + l^2} R_{l-1}(r)}{\nu a_0} = \left( 2l + 1 + \frac{d}{dr} - \frac{Z}{a_0} l \right) R_l(r),$$

(69)

which lead to

$$\left( \frac{d^2}{dr^2} + \frac{2(l+1)}{r} \frac{d}{dr} + \frac{2Z^2}{r a_0} - \frac{Z^2 (l+1)^2}{a_0^2} \right) R_l(r) = \frac{Z^2 (-\nu^2 - (l+1)^2) R_l(r)}{r^2},$$

$$\left( \frac{d^2}{dr^2} + \frac{2(l+1)}{r} \frac{d}{dr} + \frac{2Z^2}{r a_0} - \frac{Z^2 l^2}{a_0^2} \right) R_l(r) = \frac{Z^2 (-\nu^2 - l^2) R_l(r)}{r^2},$$

(70)

or, simply,

$$\left( \frac{d^2}{dr^2} + \frac{2(l+1)}{r} \frac{d}{dr} + \frac{2Z^2}{r a_0} + \frac{Z^2}{\nu^2 a_0^2} \right) R_l(r) = 0.$$

(71)

The above equation can match to the Schrödinger equation (with $E > 0$):

$$\left( \frac{d^2}{dr^2} + \frac{2(l+1)}{r} \frac{d}{dr} + \frac{2Z^2}{r a_0} + \frac{2\mu}{\hbar^2 E} \right) R_l(r) = 0,$$

(72)

by taking

$$\nu = \frac{Z\hbar}{\sqrt{2\mu E a_0}} = \frac{Z}{a_0} k,$$

(73)

with $k \equiv \sqrt{2\mu E / \hbar}$. Solving the Schrödinger equation now reduces to finding functions that satisfy the recursive relations in equation (70).

From [19], we find that the Coulomb functions, $u_\nu(\eta, \rho) = F_\nu(\eta, \rho)$ and $G_\nu(\eta, \rho)$, have the following recursive relations:

$$\frac{l}{d\rho} u_\nu(\eta, \rho) + \left( \frac{l^2}{\rho} + \eta \right) u_\nu = \sqrt{l^2 + \eta^2} u_{l-1},$$

$$\frac{(l+1)}{d\rho} u_\nu(\eta, \rho) - \left( \frac{(l+1)^2}{\rho} + \eta \right) u_\nu = -\sqrt{(l+1)^2 + \eta^2} u_{l+1},$$

(74)
or, equivalently,
\[
\frac{d}{d \rho} \frac{u_l}{\rho^{l+1}} + l(l + 1) \frac{u_l}{\rho^{l+1}} + 2 \sqrt{\frac{l^2 + \eta^2}{\rho^2}} u_{l-1} = \sqrt{\frac{l^2 + \eta^2}{\rho^2}} u_l,
\]
\[
\frac{1}{\rho} \frac{d}{d \rho} \frac{u_l}{\rho^{l+1}} - \frac{\eta}{\rho} \frac{u_l}{\rho^{l+1}} = -\sqrt{(l + 1)^2 + \eta^2} u_{l+1}.
\]
(75)

Compare the above relations with those in equation (69), we see that by taking
\[
\eta = -\nu = - \frac{Z}{a_0 k}, \quad \rho = kr,
\]
the following \( R_l/r^l \),
\[
R_l(r) = a \frac{F_l(-Z/a_0 k, kr)}{r^{l+1}} + b \frac{G_l(-Z/a_0 k, kr)}{r^{l+1}},
\]
with constants \( a, b \), satisfies the recursive relations in equation (69). Indeed, the above result on \( R_l \) is consistent with those obtained by solving the Schrödinger equation directly, see for example, [20].

3. Discussion and conclusions

We now compare our results with those obtained in [10–12].

In [10], Stahlhofen and Bleuler, pointed out the relation of factor (in the factorization approach) and the Runge-Lenz vector. In their two one-page discussions on this issue [10], they defined
\[
A_l^\pm \equiv \frac{d}{dr} + \frac{l}{r} - \frac{1}{l},
\]
in section 3,
\[
\vec{S} \equiv \vec{A}^\pm(\rho + 1) - \frac{1}{2} [\vec{L}^2, \vec{A}^\pm], \quad \rho \equiv \left( \vec{L}^2 + \frac{1}{4} \right)^{1/2} - \frac{1}{2},
\]
in section 4 (matched to our notation, but with \( \hbar = 1 \)), and obtained [10]
\[
S_y \psi_{alm} = Y_{l-1,m} \left( \frac{d}{dr} + \frac{l}{r} - \frac{1}{l} \right) R_{al}(r),
\]
\[
S_y \psi_{alm} = Y_{l-1,m} R_{a,l-1}(r).
\]
(80)

Note that the choice of \( \vec{S} \) is not a good one, since the operator \( \vec{L}^2 \) in wave mechanics is rather complicated. It will be easier and more natural to use \( \vec{A} \) directly instead. Note that the two relations in the above equation imply
\[
A_l^- R_{al}(r) = \left( \frac{d}{dr} + \frac{l}{r} - \frac{1}{l} \right) R_{al}(r) \equiv R_{a,l-1}(r).
\]
(81)

It should be compared with the second equation in equation (47):
\[
Z \sqrt{n^2 - l^2} R_{a,l-1}(r) = \left( \frac{l}{r} + \frac{1}{l+1} \right) R_{al}(r) - \frac{Z}{a_0 r} R_{al}(r).
\]
(82)

We see that our result does not agree with theirs. Furthermore, the raising operator \( A_l^+ \) in equation (78) is also different from the raising operator used in the first equation in equation (47),
\[
- Z \sqrt{n^2 - (l + 1)^2} R_{a,l+1}(r) = \left( \frac{1}{l+1} \frac{d}{dr} - \frac{l+1}{r} \right) R_{al}(r).
\]
(83)

Since the master formulas are different, the comparison should stop here.

In [11], Burkhardt and Leventhal obtained (with \( \hbar = 1 \))
\[
A_2|n, l, \pm l\rangle = \frac{1}{n} \sqrt{\frac{n^2 - (l + 1)^2}{2l + 3}} |n, l + 1, \pm l\rangle,
\]
\[
A_+|n, l, l\rangle = - \frac{1}{n} \sqrt{\frac{2(l + 1)(n^2 - (l + 1)^2)}{2l + 3}} |n, l + 1, l + 1\rangle,
\]
\[
A_-|n, l, -l\rangle = - \frac{1}{n} \sqrt{\frac{2(l + 1)(n^2 - (l + 1)^2)}{2l + 3}} |n, l + 1, -l - 1\rangle.
\]
(84)
To compare, we note that in this work we consider $A_{\pm}|n, l, l\rangle$, which give $|n, l + 1, l - 1\rangle$ and $|n, l - 1, l - 1\rangle$ states, see equation (32), and we also work out the left hand side of the equation as linear differential operators acting on $R_{nl}/r^l$, see equation (47). It is clear that their work [11] only has the raising part, which raises $l$ to $l + 1$, and does not have the lowering part. Furthermore, no attempt is shown to obtain the energy eigenstate wave function, as the left hand sides of the above equations are not developed in [11]. Note that the famous paper on factorization by Infeld and Hull, [6], was not mentioned in [11].

No attempt of obtaining bound state energy eigenstate wave function have been made in [12], and the famous paper on factorization by Infeld and Hull, [6], was not mentioned as well. Instead, the author was interested in establishing a relation of $\vec{A}$ and $\vec{r}$ in the degenerate space. By comparing the effect of $A_2$ and $z$ on the state $|n, l, m\rangle$, Flamand proved the following theorem [12] (in our notation),

$$\vec{r}^2 = \frac{3}{2} \frac{1}{2E_n} \vec{A},$$

(85)
in the degenerate space. To verify this we note that using equations (32) and (45), we have

$$r_+ y_{i\pm} R_{nl} = -\sqrt{\frac{2l}{2l + 1}} y_{i-1, l-1}^{-r} R_{nl} + \sqrt{2} \frac{2}{(2l + 1)(2l + 3)} y_{i+1, l-1}^{-r} R_{nl}$$

(86)

and

$$\frac{\mu^2}{2Z} A_{\pm}|n, l, l\rangle = \frac{Z}{n!} \left( -\frac{2l(n^2 - l^2)}{(2l + 1)} |n, l - 1, l - 1\rangle + \frac{2(n^2 - l(l + 1))}{(2l + 1)(2l + 3)} |n, l + 1, l - 1\rangle \right).$$

(87)

Making use of the following identities [21] (in our phase convention and notation, where additional factor of $-a_0/Z$ are added),

$$\int_0^{\infty} dr \ r^2 R_{n-1, l-1}^{-r} R_{nl} = -\frac{3a_0}{2Z} n\sqrt{n^2 - l^2}, \int_0^{\infty} dr \ r^2 R_{n+1, l+1}^{-r} R_{nl} = -\frac{3a_0}{2Z} n\sqrt{n^2 - (l + 1)^2},$$

(88)

and comparing the above equations, we obtain

$$\langle n, l \pm 1, l - 1 | A_{\pm}|n, l, l\rangle = -\frac{2}{3} \frac{Z^2 \hbar^2}{\mu a_0 n^2} \langle n, l \pm 1, l - 1 | r | n, l, l\rangle = \frac{2}{3} E_n \langle n, l \pm 1, l - 1 | r | n, l, l\rangle,$$

(89)

which can be easily extended to the following results using the Wigner–Eckart theorem,

$$\langle n', l', m'| A_n^+ | n, l, m\rangle = \frac{\mu}{\hbar} \langle n, l', m' | \vec{r}^2 | n, l, m\rangle.$$}

(90)

Note that we have replaced one of the $n$ in the matrix element of $A_2$ to $n'$ as $A_2$ commutes with the Hamiltonian. The above equation agrees with Flamand’s result, equation (85), given that both sides of equation (85) are viewed as $n^2 \times n^2$ matrices.

It will be interesting to see if it is possible to replace $\vec{r}$ by $\vec{p}$ (with a suitable coefficient) in the above relation. It can easily shown that this cannot be done, since we always have

$$\langle n, l', m'| \vec{p} | n, l, m\rangle = \frac{\mu}{\hbar} \langle n, l', m'| [ \vec{r}^2, H] | n, l, m\rangle = 0.$$}

(91)

This is the reason that $\vec{p}$ does not play any role in equation (85). Nevertheless, as we shall see, the above equation can lead to an identity in the integral involving $R_{nl}$ and $R_{nl}$. The easiest way to show this is applying $(p_z)_{op}$ on $\psi_{nl}$. Using the second equation in equation (38) and recalling $y_{i\pm} \propto (r_i)^{l/r}$, we have

$$\langle n, l', m' | (p_z)_{op} | n, l, m\rangle = \frac{\mu}{\hbar} (n, l', m' | [ \vec{r}^2, H] | n, l, m\rangle = 0.$$

This is the reason that $\vec{p}$ does not play any role in equation (85). Nevertheless, as we shall see, the above equation can lead to an identity in the integral involving $R_{nl}$ and $R_{nl}$. The easiest way to show this is applying $(p_z)_{op}$ on $\psi_{nl}$. Using the second equation in equation (38) and recalling $y_{i\pm} \propto (r_i)^{l/r}$, we have

$$\langle n, l', m' | (p_z)_{op} | n, l, m\rangle = \frac{\mu}{\hbar} (n, l', m' | [ \vec{r}^2, H] | n, l, m\rangle = 0.$$

(91)

Using $z/r \propto Y_{00}$ and the formulas of additional of angular momentum, we see that $(z/r)_i (r_i)^{l/r} \propto Y_{i\pm} Y_{00}$ is proportional to $Y_{i+1,l}$, (as this is the only viable term, see also footnote 4) and, hence, equation (91) implies

$$\int_0^{\infty} dr \ r^2 R_{n,l+1}^{-r} \left( \frac{d}{dr} - \frac{l}{r} \right) R_{nl} = 0.$$}

(93)

As we checked, the radial wave functions shown in equation (60) satisfy the above identity. The above identity can be further developed by using the raising operator in the first equation in equation (47) and can be related to other identities in integral involving $R_{nl}$ and $R_{nl}$. We shall stop here and will not go any further on this issue.

In extending the relation in equation (85) to the $E > 0$ case, Flamand made use of a mathematical theorem on Lorentz group taken from [22] and obtained the recursive relations for scattering states: (taking $\hbar = Z = m = 1, \kappa = c^2$) [12]
\[ (l^2 - n^2)\frac{\partial}{\partial \rho} f_{l-1}(\rho) = \left\{ -2l \left[ \frac{d}{d\rho} + \frac{l + 1}{\rho} \right] + n \right\} f_{l}(\rho), \]
\[ ((l + 1)^2 - n^2)\frac{\partial}{\partial \rho} f_{l+1}(\rho) = \left\{ -2(l + 1) \left[ \frac{d}{d\rho} - \frac{l}{\rho} \right] - n \right\} f_{l}(\rho). \quad (94) \]

with \( \psi = Y_{lm}^\nu \rho = 2i(2E)^{1/2}r \) and \( n = -ie^2/(2E)^{1/2} \). The above results can be compared with our results in sections 2, 3, see equation (65):

\[ \sqrt{l^2 + \nu^2} iR_{l-1}(r) = \left\{ -2l \left[ \frac{d}{d(2iZr/va_0)} + \frac{(l + 1)}{(2iZr/va_0)} \right] - i\nu \right\} R_{l}(r), \]
\[ \sqrt{(l + 1)^2 + \nu^2} (-i)R_{l+1}(r) = \left\{ -2(l + 1) \left[ \frac{d}{d(2iZr/va_0)} - \frac{l}{(2iZr/va_0)} \right] + i\nu \right\} R_{l}(r). \quad (95) \]

These two results can be matched by taking \( \rho = 2iZr/va_0, n = -i\nu; f_{l} = R_{l}, f_{l-1} = i R_{l-1} \) and \( f_{l+1} = -i R_{l+1} \) in the above equation.

In summary of the comparison, the connection of raising and lowering factor and Runge-Lenz vector was pointed out in [10], but their discussion was very brief and their master formulas are different from ours. The study of [11] is just half way of ours, as they did not work out the \( \langle A \rangle_{l\rho n} \) in the wave mechanics framework and only studied the raising case (bringing \( |n, l, m\rangle \) to \( |n, l + 1, m'\rangle \)). Although some of the technics used in [12] were similar to those used here, it is clear that the topics of this work is not the concern of the author of [12].

In conclusion, in this work we follow the Pauli method of quantizing the Hydrogen atom system using the Runge-Lenz vector. Since the Runge-Lenz vector is a vector and it commutes with the Hamiltonian, it is natural to use it to connect energy eigenstate \( |n, l, m\rangle \) with other degenerate eigenstates \( |n, l \pm 1, m'\rangle \). Recursive relations for the radial wave functions are found. They are consistent with the results found in [5–8] using the factorization method. The wave functions of the whole spectrum can be obtained easily. These radial wave functions are shown to satisfy the Schrödinger equation. In addition, using the recursive relations the energy eigenstate wave functions in the \( E > 0 \) case can also be verified. As in the Pauli analysis, group theory plays a prominent role in this analysis, while the rest of the derivations are mostly elementary. It will be interesting to extend the present study to the relativistic case.

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**Appendix. Obtaining the matrix elements of \( B^{(\pm)} \) as shown in equation (30)**

In section 2.1, we need to evaluate the following matrix elements of \( B^{(\pm)} \):

\[
\langle n, l' , l - 1 | B^{(\pm)} | n, l \rangle = \sum_{m'^{\pm}, m^{\mp}} \langle l', l - 1 | b, m'^{\pm} - 1 , b, m^{\mp} \rangle \sqrt{b + m'^{\pm})(b - m'^{\pm} + 1) \hbar} \times \langle b, m'^{\pm} , b, m^{\mp} | l , l \rangle ,
\]
\[
\langle n, l' , l - 1 | B^{(-)} | n, l \rangle = \sum_{m'^{\pm}, m^{\mp}} \langle l', l - 1 | b, m'^{\pm} + 1 , b, m^{\mp} - 1 \rangle \sqrt{b + m'^{\pm})(b - m'^{\pm} + 1) \hbar} \times \langle b, m'^{\pm} , b, m^{\mp} | l , l \rangle . \quad (A1)
\]

First we note that from the known symmetry properties of Clebsch–Gordan coefficients,

\[
\langle l' , l - 1 | b, m'^{\pm} , b, m^{\mp} - 1 \rangle = (-)^{2b - l'} \langle l' , l - 1 | b, m'^{\mp} - 1 , b, m^{\pm} \rangle ,
\]
\[
\langle b, m'^{\pm} , b, m^{\mp} | l , l \rangle = (-)^{2b - l} \langle b, m'^{\mp} , b, m^{\pm} | l , l \rangle , \quad (A2)
\]

we have

\[
\langle n, l' , l - 1 | B^{(-)} | n, l \rangle = (-)^{-l'} \langle n, l' , l - 1 | B^{(+)} | n, l \rangle , \quad (A3)
\]

for integers \( l, l' \), which is certainly the case here.
Using the known formula in the study of addition of angular momenta [23] (see, also [24]),
\[
\langle j_1'j_2'; j'm'|T_{kl}^m(1)| j_1j_2; jm \rangle = (-)^{j_2'j_1' - j - k} \langle j, m, k, \mu| j', m'\rangle (2j + 1)\frac{1}{2} (2j'_k + 1)\frac{1}{2} \times \frac{1}{\sqrt{(2j + 1)(2j' + 1)(2j + 1)(2j' + 1) - 1}} \langle j'_k\| T_{kl}^m(1)\| j_k \rangle,
\]
(A4)

where \( \{ \) is the Wigner 6-\( j \) symbol and \( \langle j'_k\| T_{kl}^m(1)\| j_k \rangle \) is the reduced matrix element, we must have
\[
\langle n, l', l - 1|B^{(+)\| n, l, l \rangle = (-)^{2b - l + 1} (l, l, 1, -1|l' l - 1) (2l + 1)\frac{1}{2}(2b + 1)\frac{1}{2} \times \langle b\| B^{(+)\| b \rangle}

\]
(A5)

with \( j_1 = j_2 = j'_1 = j'_2 = b = (n - 1)/2 \). For \( l' = l, l \pm 1 \), the above formula and equation (A3) give
\[
\langle n, l - 1, l - 1|B^{(+)\| n, l, l \rangle = \frac{l(n^2 - l^2)}{\sqrt{(2l + 1)(n^2 - 1)}} \langle b|B^{(+)\| b \rangle = - \langle n, l - 1, l - 1|B^{(-)\| n, l, l \rangle},
\]
\[
\langle n, l + 1, l - 1|B^{(+)\| n, l, l \rangle = \frac{n^2 - (l + 1)^2}{(2l + 1)(2l + 3)(n^2 - 1)} \langle b|B^{(+)\| b \rangle = - \langle n, l + 1, l - 1|B^{(-)\| n, l, l \rangle},
\]
\[
\langle n, l, l - 1|B^{(+)\| n, l, l \rangle = \frac{l}{n^2 - 1} \langle b|B^{(+)\| b \rangle = \langle n, l - 1, l - 1|B^{(-)\| n, l, l \rangle}.
\]
(A6)

The reduced matrix element \( \langle b|B^{(+)\| b \rangle \) can be determined by using
\[
\langle n, l - 1, l - 1|B^{(+)\| n, l, l \rangle + \langle n, l - 1, l - 1|B^{(-)\| n, l, l \rangle = \langle n, l - 1, l - 1|L^z| n, l, l \rangle = \sqrt{2l + 1} \hbar,
\]
(A7)

and the third relation in equation (A6), giving
\[
\langle b|B^{(+)\| b \rangle = \sqrt{\frac{n^2 - 1}{2}} \hbar,
\]
(A8)

and, consequently, we obtain
\[
\langle n, l - 1, l - 1|B^{(+)\| n, l, l \rangle = \pm \frac{l(n^2 - l^2)}{2(2l + 1)} \hbar,
\]
\[
\langle n, l + 1, l - 1|B^{(+)\| n, l, l \rangle = \pm \frac{n^2 - (l + 1)^2}{2(2l + 1)(2l + 3)} \hbar,
\]
\[
\langle n, l, l - 1|B^{(+)\| n, l, l \rangle = \frac{l}{2} \hbar.
\]
(A9)

These are the results shown in equation (30). They follow from group theory.

Note that the above results also hold for the \( l = 0 \) case, where the equation implies that \( \langle n, l - 1, l - 1|B^{(+)\| n, l, l \rangle \) and \( \langle n, l, l - 1|B^{(-)\| n, l, l \rangle \) are vanishing as they should. Furthermore, in the above derivation we encountered \( 1/(n^2 - 1)^{1/2} \) factors in equation (A6), and, hence, \( n \) needs to be greater than 1. Nevertheless, using equation (A1) and by direct computation, it is easy to check that the above equation also holds for the \( n = 1 \) case.

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