DEGENERATE C-DISTRIBUTION SEMIGROUPS IN LOCALLY
CONVEX SPACES

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Abstract. The main purpose of this paper is to investigate degenerate C-
distribution semigroups in the setting of barreled sequentially complete locally
convex spaces. In our approach, the infinitesimal generator of a degenerate
C-distribution semigroup is a multivalued linear operator and the regularizing
operator is not necessarily injective. We provide a few important theoretical
novelties, considering also exponential subclasses of degenerate C-distribution
semigroups.

1. Introduction and Preliminaries

In our recent paper [21], we have introduced and systematically analyzed the
classes of C-distribution semigroups and C-ultradistribution semigroups in locally
convex spaces (cf. [4]-[8], [11], [13], [17]-[19], [24]-[26], [30], [35]-[37] and references
cited therein). The main aim of this paper is to continue this research by investigat-
ging the classes of degenerate C-distribution semigroups in the setting of barreled
sequentially complete locally convex spaces (cf. [5], [12], [20], [30] and [36] for fur-
ther information about well-posedness of abstract degenerate differential equations
of first order). As mentioned in the abstract, we consider multivalued linear op-
erators as infinitesimal generators of such semigroups and allow the regularizing
operator C to be non-injective (cf. [3], [13], [24], [27] and [30]-[32] for the primary
source of information on degenerate distribution semigroups in Banach spaces). In
contrast to the analyses carried out in [30, Section 2.2] and [3, Section 3], we do
not use any decomposition of the state space E.

The organization of paper can be briefly described as follows. After explaining
the basic things about vector-valued generalized function spaces necessary for our
further work, in Section 2 we take a preliminary look at multivalued linear operators
in locally convex spaces. In Section 3, we repeat some known facts and definitions
about fractionally integrated C-semigroups in locally convex spaces and their sub-
genators (integral generators). Our main results are contained in Section 4, in
which we analyze various themes concerning degenerate C-distribution semigroups
in locally convex spaces and further generalize some of our recent results from [21].
The studies of differential and analytical properties of degenerate C-distribution
semigroups as well as degenerate q-exponential C-distribution semigroups in lo-
callly convex spaces is out of the scope of this paper.

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Development, Republic of Serbia.
1.1. Notation. Unless specified otherwise, we assume that $E$ is a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short. Our standing assumption henceforth will be that the state space $E$ is barreled. By $L(E)$ we denote the space consisting of all continuous linear mappings from $E$ into $E$. The symbol $\oplus_E$ (\oplus, if there is no risk for confusion) denotes the fundamental system of seminorms which defines the topology of $E$. The Hausdorff locally convex topology on $E^*$, the dual space of $E$, defines the system $(\|\cdot\|_B)_{B \in \mathcal{B}}$ of seminorms on $E^*$, where $|x^*|_B := \sup_{x \in B} |(x^*, x)|$, $x^* \in E^*$, $B \in \mathcal{B}$. The bidual of $E$ is denoted by $E^{**}$. Recall, the polars of nonempty sets $M \subseteq E$ and $N \subseteq E^*$ are defined as follows $M^o := \{y \in E^* : |y(x)| \leq 1 \text{ for all } x \in M\}$ and $N^o := \{x \in E : |y(x)| \leq 1 \text{ for all } y \in N\}$.

We need the following auxiliary lemma whose proof can be deduced as in the scalar-valued case.

**Lemma 1.1.** Suppose that $0 < \tau \leq \infty$, $n \in \mathbb{N}$. If $f : (0, \tau) \to E$ is a continuous function and
\[
\int_0^\tau \varphi^{(n)}(t) f(t) \, dt = 0, \quad \varphi \in D(0, \tau),
\]
then there exist elements $x_0, \ldots, x_{n-1}$ in $E$ such that $f(t) = \sum_{j=0}^{n-1} t^j x_j$, $t \in (0, \tau)$.
Following L. Schwartz \[34\], it will be said that a distribution \( G \in \mathcal{D}'(X) \) is of finite order on the interval \((-\tau, \tau)\) iff there exist an integer \( n \in \mathbb{N}_0 \) and an \( X \)-valued continuous function \( f : [-\tau, \tau] \to X \) such that

\[
G(\varphi) = (1)^n \int_{-\tau}^{\tau} \varphi^{(n)}(t) f(t) \, dt, \quad \varphi \in \mathcal{D}(-\tau, \tau), \; \tau > 0.
\]

\( G \) is of finite order iff \( G \) is of finite order on any finite interval \((-\tau, \tau)\). In the case that \( X \) is a quasi-complete (DF)-space, then it is well known that each \( X \)-valued distribution is of finite order.

We refer the reader to \[21\] for some characterizations of vector-valued distributions supported by a point. If the space \( E \) satisfies the property that any vector-valued distribution \( G \in \mathcal{D}'(E) \) with \( \text{supp}(G) \subseteq \{0\} \) can be represented as a finite sum of vector-valued distributions of form \( \delta^{(i)} \otimes x_i \), then we say that \( E \) is admissible.

## 2. Multivalued linear operators

In this section, we present some definitions and properties of multivalued linear operators that will be necessary for our further work (cf. the monographs \[9\] by R. Cross and \[12\] by A. Favini-A. Yagi for more details on the subject). The underlying SCLCS will be denoted by \( X \) and \( Y \); in the third section, we will coming back to our standing notation.

A multivalued map (multimap) \( A : X \to P(Y) \) is said to be a multivalued linear operator (MLO) iff the following holds:

\begin{enumerate}[(i)]
  \item \( D(A) := \{ x \in X : Ax \neq \emptyset \} \) is a subspace of \( X \);
  \item \( Ax + Ay \subseteq A(x+y), \; x,\; y \in D(A) \) and \( \lambda Ax \subseteq A(\lambda x), \; \lambda \in \mathbb{C}, \; x \in D(A) \).
\end{enumerate}

If \( X = Y \), then it is also said that \( A \) is an MLO in \( X \). An almost immediate consequence of the definition is that, for every \( x, \; y \in D(A) \) and for every \( \lambda, \; \eta \in \mathbb{C} \) with \( |\lambda| + |\eta| \neq 0 \), we have \( \lambda Ax + \eta Ay = A(\lambda x + \eta y) \). If \( A \) is an MLO, then \( A_0 \) is a linear manifold in \( Y \) and \( Ax = f + A_0 \) for any \( x \in D(A) \) and \( f \in Ax \). Set \( R(A) := \{ Ax : x \in D(A) \} \). The set \( A^{-1}0 = \{ x \in D(A) : 0 \in Ax \} \) is called the kernel of \( A \) and it is denoted by \( N(A) \). The inverse \( A^{-1} \) of an MLO is defined by \( D(A^{-1}) := R(A) \) and \( A^{-1}y := \{ x \in D(A) : y \in Ax \} \). It is easily seen that \( A^{-1} \) is an MLO in \( X \), as well as that \( N(A^{-1}) = A_0 \) and \( (A^{-1})^{-1} = A \). If \( N(A) = \{0\} \), i.e., if \( A^{-1} \) is single-valued, then \( A \) is said to be injective.

For any mapping \( A : X \to P(Y) \) we define \( \hat{A} := \{ (x,y) : x \in D(A), \; y \in Ax \} \). Then \( \hat{A} \) is an MLO iff \( \hat{A} \) is a linear relation in \( X \times Y \); \((x,\lambda y_1) + (x,\lambda y_2) = (x,\lambda y_1 + \lambda y_2),\) for each \( x \in X \) and \( y \in Y \) i.e., if \( \hat{A} \) is a subspace of \( X \times Y \). Since no confusion seems likely, we will sometimes identify \( \hat{A} \) with its graph.

If \( A, \; B : X \to P(Y) \) are two MLOs, then we define its sum \( A + B \) by \( D(A+B) := D(A) \cap D(B) \) and \( (A+B)x := Ax + Bx, \; x \in D(A+B) \). It can be simply checked that \( A + B \) is likewise an MLO.

Let \( A : X \to P(Y) \) and \( B : Y \to P(Z) \) be two MLOs, where \( Z \) is an SCLCS. The product of \( A \) and \( B \) is defined by \( D(BA) := \{ x \in D(A) : D(B) \cap Ax \neq \emptyset \} \) and \( BAx := B(D(B) \cap Ax) \). Then \( BA : X \to P(Z) \) is an MLO and \( (BA)^{-1} = A^{-1}B^{-1} \).

The scalar multiplication of an MLO \( A : X \to P(Y) \) with the number \( z \in \mathbb{C}, \) for short, is defined by \( D(zA) := D(A) \) and \( (zA)x := zAx, \; x \in D(A) \). It is clear that \( zA : X \to P(Y) \) is an MLO and \( (\omega zA) = \omega (zA) = z(\omega A), \; z, \; \omega \in \mathbb{C} \).
The integer powers of an MLO \( A : X \to P(X) \) is defined recursively as follows: \( A^0 := I \); if \( A^{n-1} \) is defined, set \( D(A^n) := \{ x \in D(A^{n-1}) : D(A) \cap A^{n-1} x \neq \emptyset \} \), and \( A^n x := (A A^{n-1}) x = \bigcup_{y \in D(A) \cap A^{n-1}} A y, \ x \in D(A^n) \). It is well known that \( (A^n)^{-1} = (A^{n-1})^{-1} A^{-1} = (A^{-n}) n \in \mathbb{N} \) and \( D((\lambda - A)^n) = D(A^n) \).

If \( A : X \to P(Y) \) and \( B : X \to P(Y) \) are two MLOs, then we write \( A \subseteq B \) iff \( D(A) \subseteq D(B) \) and \( Ax \subseteq Bx \) for all \( x \in D(A) \). Assume now that a linear single-valued operator \( S : D(S) \subseteq X \to Y \) has domain \( D(S) = D(A) \) and \( S \subseteq A \), where \( A : X \to P(Y) \) is an MLO. Then \( S \) is called a section of \( A \); if this is the case, we have \( Ax = Sx + A0, x \in D(A) \) and \( R(A) = R(S) + A0 \).

We say that an MLO operator \( A : X \to P(Y) \) is closed if for any nets \( (x_\tau) \) in \( D(A) \) and \( (y_\tau) \) in \( Y \) such that \( y_\tau \in Ax_\tau \) for all \( \tau \in I \) we have that \( \lim_{\tau \to \infty} x_\tau = x \) and \( \lim_{\tau \to \infty} y_\tau = y \) imply \( x \in D(A) \) and \( y \in Ax \).

If \( A : X \to P(Y) \) is an MLO, then we define the adjoint \( A^* : Y \to P(X^*) \) of \( A \) by its graph
\[
A^* := \left\{ (y^*, x^*) \in Y^* \times X^* : \langle y^*, y \rangle = \langle x^*, x \rangle \right\} \text{ for all pairs } (x,y) \in A.
\]
It is simply verified that \( A^* \) is a closed MLO, and that \( \langle y^*, y \rangle = 0 \) whenever \( y^* \in D(A^*) \) and \( y \in A0 \).

Concerning the integration of functions with values in SCLCS, we follow the approach of C. Martínez and M. Sanz [23] pp. 99-102]. Denote by \( \Omega \) a locally compact and separable metric space and by \( \mu \) a locally finite Borel measure defined on \( \Omega \). Then the following fundamental lemma holds:

**Lemma 2.1.** Suppose that \( A : X \to P(Y) \) is a closed MLO. Let \( f : \Omega \to X \) and \( g : \Omega \to Y \) be \( \mu \)-integrable, and let \( g(x) \in AF(x), x \in \Omega \). Then \( \int_\Omega f d\mu \in D(A) \) and \( \int_\Omega g d\mu \in A \int_\Omega f d\mu \).

In [23], we have recently considered the C-resolvent sets of MLOs in locally convex spaces (where \( C \in L(X) \) is injective, \( CA \subseteq AC \)). The C-resolvent set of an MLO \( A \) in \( X \), \( \rho_C(A) \) for short, is defined as the union of those complex numbers \( \lambda \in \mathbb{C} \) for which \( R(C) \subseteq R(\lambda - A) \) and \( (\lambda - A)^{-1} C \) is a single-valued bounded operator on \( X \). The operator \( \lambda \mapsto (\lambda - A)^{-1} C \) is called the C-resolvent of \( A \) (\( \lambda \in \rho_C(A) \)). In this paper, we analyze the general situation in which the operator \( C \in L(X) \) is not necessarily injective. Then the operator \( (\lambda - A)^{-1} C \) is no longer single-valued, which additionally hinders our considerations and work.

### 3. Fractionally Integrated C-Semigroups in Locally Convex Spaces

In this section, we will collect the most important facts and definitions about (degenerate) fractionally integrated C-semigroups in locally convex spaces. Observe that we do not require the injectiveness of operator \( C \in L(E) \). Denote by \( g_\alpha(t) = t^{\alpha-1} \) for \( t > 0 \).

**Definition 3.1.** ([23]) Let \( 0 < \alpha < \infty \) and \( 0 < \tau \leq \infty \). A strongly continuous operator family \( (S_\alpha(t))_{t \in (0,\tau)} \subseteq L(E) \) is called a (local, if \( \tau < \infty \)) \( \alpha \)-times integrated C-semigroup iff the following holds:

1. \( S_\alpha(t)C = CS_\alpha(t), \ t \in [0, \tau], \) and
(ii) For all $x \in E$ and $t, \ s \in [0, \tau)$ with $t + s \in [0, \tau)$, we have

$$S_{\alpha}(t)S_{\alpha}(s)x = \left[ \int_{0}^{t+s} - \int_{0}^{t} - \int_{0}^{s} \right] g_{\alpha}(t + s - r)S_{\alpha}(r)Cxdr. $$

By a C-regularized semigroup (0-times integrated C-regularized semigroup) we mean any strongly continuous operator family $(S_{\alpha}(t))_{t \in [0, \tau)} \subseteq L(E)$ satisfying that $S(t)C = CS(t), \ t \in [0, \tau)$ and $S(t + s)C = S(t)S(s)$ for all $t, s \in [0, \tau)$ with $t + s \in [0, \tau)$. A global C-regularized semigroup $(S(t))_{t \geq 0}$ is said to be entire analytic iff, for every $x \in E$, the mapping $t \mapsto S(t)x, \ t \geq 0$ can be analytically extended to the whole complex plane. We refer the reader to [10] for the most important applications of non-degenerate C-regularized semigroups.

Let $0 < \alpha \leq \infty$. In the case $\tau = \infty$, $(S_{\alpha}(t))_{t \geq 0}$ is said to be exponentially equicontinuous (equicontinuous) iff there exists $\omega \in \mathbb{R} (\omega = 0)$ such that the family $\{e^{-\omega t}S_{\alpha}(t) : t \geq 0\}$ is equicontinuous. The integral generator $\hat{A}$ of $(S_{\alpha}(t))_{t \in [0, \tau)}$ is defined by its graph

$$\hat{A} := \left\{ (x, y) \in E \times E : S_{\alpha}(t)x - g_{\alpha}(t)Cx = \int_{0}^{t} S_{\alpha}(s)yds, \ t \in [0, \tau) \right\}. $$

The integral generator $\hat{A}$ of $(S_{\alpha}(t))_{t \in [0, \tau)}$ is a closed MLO in $E$. Furthermore, $\hat{A} \subseteq C^{-1}\hat{A}C$ in the MLO sense, with the equality in the case that the operator $C$ is injective.

By a subgenerator of $(S_{\alpha}(t))_{t \in [0, \tau)}$ we mean any MLO $A$ in $E$ satisfying the following two conditions:

(A) $S_{\alpha}(t)x - g_{\alpha+1}(t)Cx = \int_{0}^{t} S_{\alpha}(s)yds$, whenever $t \in [0, \tau)$ and $y \in Ax$.

(B) For all $x \in E$ and $t \in [0, \tau)$, we have $\int_{0}^{t} S_{\alpha}(s)xds \in D(A)$ and $S_{\alpha}(t)x - g_{\alpha+1}(t)Cx \in A \int_{0}^{t} S_{\alpha}(s)xds$.

If $(S_{\alpha+1}(t))_{t \in [0, \tau)} \subseteq L(E)$, resp. $(S_{\alpha}(t))_{t \in [0, \tau)} \subseteq L(E)$, is strongly continuous and satisfies only (B), resp. (A), then we say that $(S_{\alpha}(t))_{t \in [0, \tau)}$, resp. $(S_{\alpha+1}(t))_{t \in [0, \tau)}$, is an $\alpha$-times integrated $C$-existence family with a subgenerator $A$, resp., $\alpha$-times integrated $C$-uniqueness family with a subgenerator $A$.

We denote by $\chi(S_{\alpha})$ the set consisting of all subgenerators of the $\alpha$-times integrated $C$-semigroup $(S_{\alpha}(t))_{t \in [0, \tau)}$. It is well known that $\chi(S_{\alpha})$ can have infinitely many elements; if $A \in \chi(S_{\alpha})$, then $A \subseteq \hat{A}$. In general, the set $\chi(S_{\alpha})$ can be empty and the integral generator of $(S_{\alpha}(t))_{t \in [0, \tau)}$ need not be a subgenerator of $(S_{\alpha}(t))_{t \in [0, \tau)}$ in the case that $\tau < \infty$. In global case, the integral generator $\hat{A}$ of $(S_{\alpha}(t))_{t \geq 0}$ is always its subgenerator. If $A$ is a closed subgenerator of $(S_{\alpha}(t))_{t \in [0, \tau)}$, defined locally or globally, then we know that $CA \subseteq AC, \ A \subseteq C^{-1}AC$ and that the injectivity of $C$ implies $\hat{A} = C^{-1}AC$. Suppose that $C$ is injective and $A$ is an MLO. Then there exists at most one $\alpha$-times integrated $C$-semigroup $(S_{\alpha}(t))_{t \in [0, \tau)}$ which do have $A$ as a subgenerator ([22]).

4. The basic properties of degenerate C-distribution semigroups in locally convex spaces

Throughout this section, we assume that $C \in L(E)$ is not necessarily injective operator. Since $E$ is barreled, the uniform boundedness principle [29], p. 273]
implies that each $G \in D'(L(E))$ is boundedly equicontinuous, i.e., that for every $p \in \otimes$ and for every bounded subset $B$ of $D$, there exist $c > 0$ and $q \in \otimes$ such that $p(G(\varphi)x) \leq cq(x)$, $\varphi \in B$, $x \in E$.

We start this section by introducing the following definition.

**Definition 4.1.** Let $G \in D'_0(L(E))$ satisfy $CG = GC$. Then it is said that $G$ is a pre-(C-DS) iff the following holds:

(C.S.1) \[ G(\varphi *_0 \psi)C = G(\varphi)G(\psi), \quad \varphi, \psi \in D. \]

If, additionally,

(C.S.2) \[ \mathcal{N}(G) := \bigcap_{\varphi \in D_0} N(G(\varphi)) = \{0\}, \]

then $G$ is called a $C$-distribution semigroup, (C-DS) in short. A pre-(C-DS) $G$ is called dense iff

(C.S.3) \[ \mathcal{R}(G) := \bigcup_{\varphi \in D_0} R(G(\varphi)) \text{ is dense in } E. \]

If $C = I$, then we also write pre-(DS),(DS), instead of pre-(C-DS), (C-DS).

Suppose that $G$ is a pre-(C-DS). Then $G(\varphi)G(\psi) = G(\psi)G(\varphi)$ for all $\varphi, \psi \in D$, and $\mathcal{N}(G)$ is a closed subspace of $E$.

The structural characterization of a pre-(C-DS) $G$ on its kernel space $\mathcal{N}(G)$ is described in the following theorem (cf. [18 Proposition 3.1.1] and the proofs of [24 Lemma 2.2], [15 Proposition 3.5.4]).

**Theorem 4.2.** Let $G$ be a pre-(C-DS), and let the space $L(\mathcal{N}(G))$ be admissible. Then, with $N = \mathcal{N}(G)$ and $G_1$ being the restriction of $G$ to $N$ ($G_1 = G|_N$), we have: There exists an integer $m \in \mathbb{N}$ for which there exist unique operators $T_0, T_1, \ldots, T_m \in L(\mathcal{N}(G))$ commuting with $C$ so that $G_1 = \sum_{j=0}^{m} \delta^{(j)} \otimes T_j$, $T_iC^1 = (-1)^iT_{i+1}^*, 0 \leq i \leq m - 1$ and $T_0T_m = T_0^{m+2} = 0$.

Let $G \in D'_0(L(E))$ and let $T \in \mathcal{E}'_0$ i.e., $T$ is a scalar-valued distribution with compact support contained in $[0, \infty)$. Define

\[ G(T) := \left\{ (x, y) \in E \times E : G(T * \varphi)x = G(\varphi)y \text{ for all } \varphi \in D_0 \right\}. \]

Then it can be easily seen that $G(T)$ is a closed MLO; furthermore, if $G \in D'_0(L(E))$ satisfy (C.S.2), then $G(T)$ is a closed linear operator. Assuming that the regularizing operator $C$ is injective, definition of $G(T)$ can be equivalently introduced by replacing the set $D_0$ with the set $D_{(0, \varepsilon)}$ for any $\varepsilon > 0$. In general case, for every $\psi \in D$, we have $\psi_+ := \psi 1_{[0, \infty)} \in \mathcal{E}'_0$, where $1_{[0, \infty)}$ stands for the characteristic function of $[0, \infty)$, so that the definition of $G(\psi_+)$ is clear. We define the (infinitesimal) generator of a pre-(C-DS) $G$ by $A := G(-\delta')$ (cf. [21] for more details about non-degenerate case, and [3] Definition 3.4] and [13] for some other approaches used in degenerate case). Then $\mathcal{N}(G) \times \mathcal{N}(G) \subseteq \mathcal{A}$ and $\mathcal{N}(G) = \mathcal{A}0$, which simply implies that $A$ is single-valued iff (C.S.2) holds. If this is the case, then we also have that the operator $C$ must be injective: Suppose that $Cx = 0$ for some $x \in E$. By (C.S.1), we get that $G(\varphi)G(\psi)x = 0, \varphi, \psi \in D$. In particular, $G(\psi)x \in \mathcal{N}(G) = \{0\}$ so that $G(\psi)x = 0, \psi \in D$. Hence, $x \in \mathcal{N}(G) = \{0\}$ and therefore $x = 0$.

Further on, if $G$ is a pre-(C-DS), $T \in \mathcal{E}'_0$ and $\varphi \in D$, then $G(\varphi)G(T) \subseteq G(T)G(\varphi)$, $CG(T) \subseteq G(T)C$ and $\mathcal{R}(G) \subseteq D(G(T))$. If $G$ is a pre-(C-DS) and $\varphi, \psi \in D$, then
the assumption \( \varphi(t) = \psi(t) \), \( t \geq 0 \), implies \( \mathcal{G}(\varphi) = \mathcal{G}(\psi) \). As in the Banach space case, we can prove the following (cf. [18] Proposition 3.1.3, Lemma 3.1.6): Suppose that \( \mathcal{G} \) is a pre-(C-DS). Then \((C\mathcal{G}(\psi)x), \psi \in \mathcal{D}, x \in E \) and \( A \subseteq C^{-1}AC \), while \( C^{-1}AC = A \) provided that \( C \) is injective. Furthermore, the following holds:

**Proposition 4.3.** Let \( \mathcal{G} \) be a pre-(C-DS), \( S, T \in \mathcal{E}_0', \varphi \in \mathcal{D}_0, \psi \in \mathcal{D} \) and \( x \in E \). Then we have:

\[
\begin{align*}
(1) & \quad (\mathcal{G}(\varphi)x, G(T \ast \cdots \ast T \ast \varphi)x) \in G(T)^m, \ m \in \mathbb{N}. \\
(2) & \quad G(S)G(T) \subseteq G(S \ast T) \text{ with } D(G(S)G(T)) = D(G(S \ast T)) \cap D(G(T)), \text{ and } G(S) + G(T) \subseteq G(S + T). \\
(3) & \quad G(\psi)x, G(-\psi')x - \psi(0)Cx) \in G(-\psi'). \\
(4) & \quad \text{If } \mathcal{G} \text{ is dense, then its generator is densely defined.}
\end{align*}
\]

The assertions (ii)-(vi) of [18] Proposition 3.1.2 can be reformulated for pre-(C-DS)’s in locally convex spaces; here it is only worth noting that the reflexivity of state space \( E \) implies that the spaces \( E^* \) and \( E^{**} = E \) are both barreled and sequentially complete:

**Proposition 4.4.** Let \( \mathcal{G} \) be a pre-(C-DS). Then the following holds:

\[
\begin{align*}
(i) & \quad C(\overline{\mathcal{R}(\mathcal{G})}) \subseteq \overline{\mathcal{R}(\mathcal{G})}, \text{ where } \overline{\mathcal{R}(\mathcal{G})} \text{ denotes the linear span of } \mathcal{R}(\mathcal{G}). \\
(ii) & \quad \text{Assume } \mathcal{G} \text{ is not dense and } C\overline{\mathcal{R}(\mathcal{G})} = \overline{\mathcal{R}(\mathcal{G})}. \text{ Put } R := \overline{\mathcal{R}(\mathcal{G})} \text{ and } H := \mathcal{G}_R. \text{ Then } H \text{ is a dense pre-}(C^*_1-\text{DS}) \text{ on } R \text{ with } C_1 = C_1|_R. \\
(iii) & \quad \text{The dual } \mathcal{G}(\cdot)^* \text{ is a pre-}(C^*_1-\text{DS}) \text{ on } E^* \text{ and } N(\mathcal{G}^*) = \overline{\mathcal{R}(\mathcal{G})^*}. \\
(iv) & \quad \text{If } E \text{ is reflexive, then } N(\mathcal{G}) = \overline{\mathcal{R}(\mathcal{G})^*}. \\
(v) & \quad \text{The } \mathcal{G}^* \text{ is a } (C^*_1-\text{DS}) \text{ in } E^* \text{ iff } \mathcal{G} \text{ is a dense pre-}(C-\text{DS}). \text{ If } E \text{ is reflexive, then } \mathcal{G}^* \text{ is a dense pre-}(C^*_1-\text{DS}) \text{ in } E^* \text{ iff } \mathcal{G} \text{ is a } (C-\text{DS}).
\end{align*}
\]

The following proposition has been recently proved in [21] in the case that the operator \( C \) is injective (cf. [18] Proposition 2 for a pioneering result in this direction). The argumentation contained in [21] shows that the injectivity of \( C \) is superfluous:

**Proposition 4.5.** Suppose that \( \mathcal{G} \in \mathcal{D}_0'(L(E)) \) and \( \mathcal{G}(\varphi)C = C\mathcal{G}(\varphi), \varphi \in \mathcal{D} \). Then \( \mathcal{G} \) is a pre-(C-DS) iff

\[
\mathcal{G}(\varphi')\mathcal{G}(\psi) - \mathcal{G}(\varphi)\mathcal{G}(\psi') = \psi(0)\mathcal{G}(\varphi)C - \varphi(0)\mathcal{G}(\psi)C, \quad \varphi, \psi \in \mathcal{D}.
\]

In [21], we have recently proved that every (C-DS) in locally convex space is uniquely determined by its generator. Contrary to the single-valued case, different pre-(C-DS)’s can have the same generator. To see this, we can employ [24] Example 2.3: Let \( C = I, E \) is a Banach space and \( T \in L(E) \) is nilpotent of order \( n \geq 2 \). Then the pre-(C-DS)’s \( \mathcal{G}_1(\cdot) \equiv \sum_{i=0}^{n-2}T^i(0)T^{i+1} \) and \( \mathcal{G}_2(\cdot) \equiv 0 \) have the same generator \( A \equiv E \times E \).

In Theorem 4.6 and Theorem 4.8 we clarify connections between degenerate \( C \)-distribution semigroups and degenerate local integrated \( C \)-semigroups. For the proof of first theorem, we need some preliminaries from our previous research study of distribution cosine functions (see e.g. [18] Section 3.4): Let \( \eta \in \mathcal{D}_{[-2, -1]} \) be a fixed test function satisfying \( \int_{-\infty}^{\infty} \eta(t) \, dt = 1 \). Then, for every fixed \( \varphi \in \mathcal{D} \), we
define $I(\varphi)$ as follows

$$I(\varphi)(x) := \int_{-\infty}^{x} \left[ \varphi(t) - \eta(t) \right] \int_{-\infty}^{\infty} \varphi(u) \, du \, dt, \ x \in \mathbb{R}. $$

It can be simply verified that, for every $\varphi \in \mathcal{D}$ and $n \in \mathbb{N}$, we have $I(\varphi) \in \mathcal{D}$, $I^n(\varphi^{(n)}) = \varphi$, $\frac{d^n}{dx^n} I(\varphi)(x) = \varphi(x) - \eta(x) \int_{-\infty}^{\infty} \varphi(u) \, du$, $x \in \mathbb{R}$ as well as that, for every $\varphi \in \mathcal{D}_{[a,b]} (-\infty < a < b < \infty)$, we have: $\text{supp}(I(\varphi)) \subseteq [\min(-2, a), \max(-1, b)]$. This simply implies that, for every $\tau > 2$, $-1 < b < \tau$ and for every $m, n \in \mathbb{N}$ with $m \leq n$, we have:

$$I^n \left(\mathcal{D}_{(-\tau, b]}\right) \subseteq \mathcal{D}_{(-\tau, b]} \text{ and } \frac{d^m}{dx^m} I^n(\varphi)(x) = I^{n-m}(\varphi)(x), \ \varphi \in \mathcal{D}, \ x \geq 0,$$

where $I^0 \varphi := \varphi, \ \varphi \in \mathcal{D}$.

Now we are ready to show the following extension of [24, Proposition 4.3 a)] (E is a Banach space, $C = I$, given here with a different proof.

**Theorem 4.6.** Let $\mathcal{G}$ be a pre-(C-DS) generated by $\mathcal{A}$, and let $\mathcal{G}$ be of finite order. Then, for every $\tau > 0$, there exist a number $n_{\tau} \in \mathbb{N}$ and a local $n_{\tau}$-times integrated $C$-semigroup $(S_{n_{\tau}}(t))_{t \in [0, \tau)}$ such that

$$\mathcal{G}(\varphi)x = (-1)^{n_{\tau}} \int_{0}^{\infty} \varphi^{(n_{\tau})}(s) S_{n_{\tau}}(s)x \, ds, \ \varphi \in \mathcal{D}_{(-\tau, \tau)}, \ x \in \mathcal{E}.$$

Furthermore, $(S_{n_{\tau}}(t))_{t \in [0, \tau)}$ is an $n_{\tau}$-times integrated $C$-existence family with a subgenerator $\mathcal{A}$, and the admissibility of space $L(\mathcal{N}(\mathcal{G}))$ implies that $S_{n_{\tau}}(t)x = 0, t \in [0, \tau)$ for some $x \in \mathcal{N}(\mathcal{G})$ iff $T_{i}x = 0$ for $0 \leq i \leq n_{\tau} - 1$; see Theorem 4.21 with $m \geq n_{\tau} - 1$.

**Proof.** Let $\tau > 2$ and $\rho \in \mathcal{D}_{[0,1]}$ with $\int \rho \, dm = 1$ be fixed. Set $\rho_n(\cdot) := n\rho(n\cdot), n \in \mathbb{N}$. Then, for every $t \in [0, \tau)$, the sequence $\rho_n(\cdot) := \rho_n(-t)$ converges to $\delta_t$ as $n \to +\infty$ (in the space of scalar-valued distributions). Since $\mathcal{G} \in \mathcal{D}_{[0, \tau]}(L(E))$ and $\mathcal{G}$ is of finite order, we know that there exist a number $n_{\tau} \in \mathbb{N}$ and a strongly continuous operator family $(S_{n_{\tau}}(t))_{t \in [0, \tau)} \subseteq L(E)$ such that (4.2) holds good. We will first prove that $(S_{n_{\tau}}(t))_{t \in [0, \tau)}$ is a local $n_{\tau}$-times integrated $C$-existence family commuting with $C$ and having $\mathcal{A}$ as a subgenerator. In order to do that, observe that the commutation of $\mathcal{G}(\cdot)$ and $C$ yields

$$\int_{0}^{\infty} \varphi^{(n_{\tau})}(s) C S_{n_{\tau}}(s)x \, ds = \int_{0}^{\infty} \varphi^{(n_{\tau})}(s) S_{n_{\tau}}(s)Cx \, ds, \ \varphi \in \mathcal{D}_{(-\tau, \tau)}, \ x \in \mathcal{E}. $$

Plugging $\varphi = I^n(\rho_n^t)$ in this expression (cf. also (4.1)), we get that

$$\int_{0}^{\infty} \rho_n^t(s) CS_{n_{\tau}}(s)x \, ds = \int_{0}^{\infty} \rho_n^t(s) S_{n_{\tau}}(s)Cx \, ds, \ \varphi \in \mathcal{D}_{(-\tau, \tau)}, \ x \in \mathcal{E}, \ t \in [0, \tau).$$

Letting $n \to +\infty$ we obtain $CS_{n_{\tau}}(t)x = S_{n_{\tau}}(t)Cx, x \in \mathcal{E}, t \in [0, \tau).$ Now we will prove that the condition (B) hold with the number $\alpha$ replaced with the number $n_{\tau}$ therein. By Proposition 4.3 iii), we have $(\mathcal{G}(\varphi)x, \mathcal{G}(-\varphi'x - \varphi(0)Cx) \in \mathcal{A}, \varphi \in \mathcal{D},$
$x \in E$. Applying integration by parts and multiplying with $(-1)^{n+1}$ after that, the above implies
\[
\left( \int_{0}^{t} \rho_{n}(s) \int_{0}^{s} S_{n}(r) x \, dr \, ds, \int_{0}^{\infty} \rho_{n}(s) S_{n}(s) x \, ds \right) \in \mathcal{A},
\]
for any $\varphi \in \mathcal{D}_{[-\tau,\tau]}$ and $x \in E$. Plugging $\varphi = I^{n+1}(\rho_{n})$ in this expression, we get that
\[
(4.3) \quad \left( \int_{0}^{t} \rho_{n}^{t}(s) \int_{0}^{s} S_{n}(r) x \, dr \, ds, \int_{0}^{\infty} \rho_{n}^{t}(s) S_{n}(s) x \, ds \right) \in \mathcal{A},
\]
for any $t \in [0,\tau]$ and $x \in E$. Let us prove that
\[
(4.4) \quad \lim_{n \to +\infty} I^{n+1}(\rho_{n})(x) = (-1)^{n+1} g_{n+1}(t-x), \quad t \in [0,\tau], \; 0 \leq x \leq t.
\]
Let $t \in [0,\tau]$ and $x \in [0,t]$ be fixed. Then a straightforward integral computation shows that
\[
I^{n+1}(\varphi)(x) = (-1)^{n+1} \int_{x}^{\infty} \int_{x}^{\infty} \int_{x}^{\infty} \cdots \int_{x}^{\infty} \varphi(x_{1}) \, dx_{1} \, dx_{2} \cdots dx_{n+1}
\]
for any $\varphi \in \mathcal{D}$. For $\varphi = I^{n+1}(\rho_{n}^{t})$, we have
\[
I^{n+1}(\rho_{n}^{t})(0) = (-1)^{n+1} \int_{x}^{\infty} \int_{x}^{\infty} \int_{x}^{\infty} \cdots \int_{x}^{\infty} \varphi(x_{1}) \, dx_{1} \, dx_{2} \cdots dx_{n+1}
\]
\[
\times \rho_{n}^{t}(x_{1}) \, dx_{1} \, dx_{2} \cdots dx_{n+1}
\]
\[
= (-1)^{n+1} \int_{x}^{\infty} \int_{x}^{\infty} \int_{x}^{\infty} \cdots \int_{x}^{\infty} \varphi(x_{1}) \, dx_{1} \, dx_{2} \cdots dx_{n+1}
\]
\[
\times \left[ 1 - \int_{0}^{nx_{2} - nt} \rho(x_{1}) \, dx_{1} \right] \, dx_{2} \cdots dx_{n+1}
\]
\[
= (-1)^{n+1} \int_{x}^{\infty} \int_{x}^{\infty} \int_{x}^{\infty} \cdots \int_{x}^{\infty} \varphi(x_{1}) \, dx_{1} \, dx_{2} \cdots dx_{n+1}
\]
\[
\times \int_{0}^{nx_{2} - nt} \rho(x_{1}) \, dx_{1} \, dx_{2} \cdots dx_{n+1}
\]
\[
:= (-1)^{n+1} [I_{1}(t,x) - I_{2}(t,x)], \quad t \in [0,\tau].
\]
Since
\[
\int_{t}^{t+1/n} \int_{0}^{nx_{2} - nt} \rho(x_{1}) \, dx_{1} \, dx_{2} \leq 1/n, \quad t \in [0,\tau], \; n \in \mathbb{N},
\]
we have that $\lim_{n \to +\infty} I_{2}(t,x) = 0, \; t \in [0,\tau]$. Clearly,
\[
\lim_{n \to +\infty} I_{1}(t,x) = \int_{x}^{t} \int_{x}^{t} \int_{x}^{t} \cdots \int_{x}^{t} \rho(x_{2}) \, dx_{2} \cdots dx_{n+1} = g_{n+1}(t-x).
\]
This gives (4.4). Keeping in mind this equality and letting \( n \to +\infty \) in (4.3), we obtain (B). It remains to be proved the semigroup property of \((S_{n_x}(t))_{t \in [0, \tau]}\). Toward this end, let us recall that

(4.5)
\[
(\varphi \ast_0 \psi)^{(n_\tau)}(u) = (\varphi^{(n_\tau)} \ast_0 \psi)(u) + \sum_{j=0}^{n_\tau-1} \varphi^{(j)}(0)\psi^{(n_\tau-1-j)}(u), \quad \varphi, \psi \in D, \ u \in \mathbb{R}.
\]

Fix \( x \in E \) and \( t, s \in [0, \tau] \) with \( t + s \in [0, \tau] \). Using (4.5), (C.S.1) and the foregoing arguments, we get that, for every \( m, n \in \mathbb{N} \) sufficiently large:

\[
\int_0^t \int_0^s \rho^t_n(u)\rho^s_m(v)S_{n_x}(u)S_{n_x}(v)x \, du \, dv
\]

\[
= (-1)^{n_\tau} \int_0^{t+s} \left[ \left( \rho^t_n \ast_0 I^{n_\tau}(\rho^s_m) \right)(u) + \sum_{j=0}^{n_\tau-1} I^{n_\tau-j}(\rho^t_n)(0)I^{j+1}(\rho^s_m)(u) \right] S_{n_x}(u)Cx \, du.
\]

Letting \( n \to +\infty \), we obtain with the help of (4.4) that

\[
\int_0^{t+s} \rho^s_m(v)S_{n_x}(t)S_{n_x}(v)x \, dv
\]

\[
= (-1)^{n_\tau} \lim_{n \to +\infty} \int_0^{t+s} \left[ \left( \rho^t_n \ast_0 I^{n_\tau}(\rho^s_m) \right)(u) + \sum_{j=0}^{n_\tau-1} I^{n_\tau-j}(\rho^t_n)(0)I^{j+1}(\rho^s_m)(u) \right] S_{n_x}(u)Cx \, du
\]

\[
= (-1)^{n_\tau} \int_0^{t+s} \left[ \sum_{j=0}^{n_\tau-1} (-1)^{n_\tau-j} g_{n_{\tau-j}}(t)I^{j+1}(\rho^s_m)(u) \right] S_{n_x}(u)Cx \, du
\]

\[
+ (-1)^{n_\tau} \int_0^{t+s} \left[ I^{n_\tau}(\rho^s_m)(u-t) + \sum_{j=0}^{n_\tau-1} (-1)^{n_\tau-j} g_{n_{\tau-j}}(t)I^{j+1}(\rho^s_m)(u) \right] S_{n_x}(u)Cx \, du
\]

\[
= \sum_{j=0}^{n_\tau-1} (-1)^j g_{n_{\tau-j}}(t) \int_0^s I^{j+1}(\rho^s_m)(u)S_{n_x}(u)Cx \, du
\]

\[
+ (-1)^{n_\tau} \int_0^{t+s} I^{n_\tau}(\rho^s_m)(u-t)S_{n_x}(u)Cx \, du.
\]

The semigroup property now easily follows by letting \( m \to +\infty \) in the above expression, with the help of (4.4) and the identity

\[
\sum_{j=0}^{n_\tau-1} g_{n_{\tau-j}}(t)g_{j+1}(s-u) = g_{n_\tau}(t+s-u), \quad u > 0.
\]

Let \( x \in \mathcal{N}(\mathcal{G}) \). Then there are \( x_0, x_1, \ldots, x_{n_\tau-1} \in E \), such that \( S_{n_x}(t)x = \sum_{i=0}^{n_\tau-1} \frac{t^i}{i!}x_i \), for \( t \in [0, \tau] \) and \( x \in E \). For \( \varphi \in D \), such that \( \varphi = 1 \) on a neighborhood of zero and integrating by parts \( n_\tau \)-times we have

\[
T_i x = \mathcal{G}(\varphi)x = (-1)^n \int_0^\infty \varphi^{(n_\tau)}(t)S_{n_x}(t)x \, dt = \varphi(0)\left(S_{n_x}(t)\right)x^{(n_\tau-1)}|_{t=0} = x_{n_{\tau}-1}.
\]
Now, for \( x \) is not an element in \( \ker T_i, \; i = 0, 1, ..., n_r - 1, \; m \geq n_r - 1, \) we have that \( x \) is not an element in \( \ker S_{n_r}(t) \). But for \( x \in \ker T_i, \; i = 0, 1, 2, ..., n_r - 1, \) we have that \( G(\varphi)x = 0 \) holds for all \( \varphi \in D_{(\infty, \tau]} \) and this implies that \( S_{n_r}(t)x = 0, \; t \in [0, \tau) \).

\( \square \)

**Remark 4.7.**

(i) We have already seen that \( G(\cdot) = 0 \) is a degenerate pre-distribution semigroup with the generator \( A \equiv E \times E \). Then, for every \( \tau > 0 \) and for every number \( n_r \in \mathbb{N} \), there exists only one local \( n_r \)-times integrated semigroup \( (S_{n_r}(t) \equiv 0)_{t \in [0, \tau]} \) so that (4.2) holds. It is clear that the condition (B) holds and that condition (A) does not hold here. Denote by \( \mathcal{A}_r \) the integral generator of \( (S_{n_r}(t) \equiv 0)_{t \in [0, \tau]} \). Then \( \mathcal{A}_r = \{0\} \times E \) is strictly contained in the integral generator \( \mathcal{A} \) of \( G \). Furthermore, if \( C \neq 0 \), then there do not exist \( \tau > 0 \) and \( n_r \in \mathbb{N} \) such that \( A \) is the integral generator (subgenerator) of a local \( n_r \)-times integrated \( C \)-semigroup.

(ii) A similar line of reasoning as in the final part of the proof of [18, Theorem 3.1.9] shows that for each \( (x, y) \in A \) there exists elements \( x_0, x_1, \cdots, x_{n_r} \) in \( E \) such that

\[
S_{n_r}(t)x - g_{n_r+1}(t)Cx - \int_0^t S_{n_r}(s)y \, ds = \sum_{j=0}^{n_r} a_{j+1}(t)x_j, \quad t \in [0, \tau)
\]

and \( x_j \in A x_{j-1} \) for \( 1 \leq j \leq n_r \). In purely multivalued case, it is not clear how we can prove that \( x_j = 0 \) for \( 0 \leq j < n_r \) without imposing some additional unpleasant conditions.

(iii) Using dualization, we can simply reformulate the second equality appearing on the second line after the equation [24, (11)] in our context.

The proof of subsequent theorem can be deduced by using the argumentation contained in the proof of [18, Theorem 3.1.8].

**Theorem 4.8.** Suppose that there exists a sequence ((\( p_k, \tau_k \))\( k \in \mathbb{N}_0 \)) in \( \mathbb{N}_0 \times (0, \infty) \) such that \( \lim_{k \to \infty} \tau_k = \infty \), \( (p_k)_{k \in \mathbb{N}_0} \) and \( (\tau_k)_{k \in \mathbb{N}_0} \) are strictly increasing, as well as that for each \( k \in \mathbb{N}_0 \) there exists a local \( p_k \)-times integrated \( C \)-semigroup \( (S_{p_k}(t))_{t \in [0, \tau_k]} \) on \( E \) so that

\[
S_{p_m}(t)x = (g_{p_m-p_k} \ast_0 S_{p_k} (\cdot))x(t), \quad x \in E, \; t \in [0, \tau_k),
\]

provided \( k < m \). Define

\[
G(\varphi)x := (-1)^p \int_0^\infty \varphi^{(p_k)}(t)S_{p_k}(t)x \, dt, \quad \varphi \in D_{(-\infty, \tau_k]}, \; x \in E, \; k \in \mathbb{N}_0.
\]

Then \( G \) is well-defined and \( G \) is a pre-(\( C \)-DS).

**Remark 4.9.**

(i) Denote by \( \mathcal{A}_k \) the integral generator of \( (S_{p_k}(t))_{t \in [0, \tau_k]} \) \( (k \in \mathbb{N}_0) \). Then \( \mathcal{A}_k \subseteq \mathcal{A}_m \) for \( k > m \) and \( \bigcap_{k \in \mathbb{N}_0} \mathcal{A}_k \subseteq \mathcal{A} \), where \( \mathcal{A} \) is the integral generator of \( G \). Even in the case that \( C = I \), \( \bigcup_{k \in \mathbb{N}_0} \mathcal{A}_k \) can be a proper subset of \( \mathcal{A} \).

(ii) Suppose that \( \mathcal{A} \) is a subgenerator of \( (S_{p_k}(t))_{t \in [0, \tau_k]} \) for all \( k \in \mathbb{N}_0 \). Then (4.6) automatically holds.
In the case that $C = I$, then it suffices to suppose that there exists an MLO $\mathcal{A}$ such that $\mathcal{A}$ is a subgenerator of a local $p$-times integrated semigroup $(S_p(t))_{t \in [0, \tau)}$ for some $p \in \mathbb{N}$ and $\tau > 0$ ([22]).

Let $\alpha \in (0, \infty) \setminus \mathbb{N}$, $f \in \mathcal{S}$ and $n = \lceil \alpha \rceil$. Let us recall that the Weyl fractional derivative $W_\alpha^n$ of order $\alpha$ is defined by

$$W_\alpha^n f(t) := \frac{(-1)^n}{\Gamma(n - \alpha) dt^n} \int_0^t (s - t)^{n-\alpha-1} f(s) \, ds, \quad t \in \mathbb{R}.$$ 

If $\alpha = n \in \mathbb{N}_0$, then we set $W_\alpha^n := (-1)^n \frac{df}{dt^n}$. It is well known that the following equality holds: $W_\alpha^{\alpha + \beta} f = W_\alpha^\alpha W_\alpha^\beta f$, $\alpha, \beta > 0$, $f \in \mathcal{S}$.

Suppose now that $\alpha \in (0, \infty) \setminus \mathbb{N}$ and $\mathcal{A}$ is the integral generator of a global $\alpha$-times integrated $C$-semigroup $(S_\alpha(t))_{t \geq 0}$ on $E$. Then $\mathcal{A}$ is the integral generator of a global $n$-times integrated $C$-semigroup $(S_n(t))_{t \geq 0}$ on $E$, where $n = \lceil \alpha \rceil$ and $S_n(t)x := g_{n-\alpha} \ast S_\alpha(x)(t)$, $x \in E$, $t \geq 0$ ([22]). Arguing as in [21], we have that:

$$\int_0^\infty W_\alpha^n \varphi(t) S_\alpha(t)x \, dt = (-1)^n \int_0^\infty \varphi^{(n)}(t) S_n(t)x \, dt, \quad x \in E, \; \varphi \in \mathcal{D}.$$ 

Keeping in mind the proof of [18, Theorem 3.1.8], we obtain the following:

**Theorem 4.10.** Assume that $\alpha \geq 0$ and $\mathcal{A}$ is the integral generator of a global $\alpha$-times integrated $C$-semigroup $(S_\alpha(t))_{t \geq 0}$ on $E$. Set

$$\mathcal{G}_\alpha(\varphi)x := \int_0^\infty W_\alpha^n \varphi(t) S_\alpha(t)x \, dt, \quad x \in E, \; \varphi \in \mathcal{D}.$$ 

Then $\mathcal{G}$ is a pre-(C-DS) whose integral generator contains $\mathcal{A}$.

We will accept the following definition an exponential pre-(C-DS).

**Definition 4.11.** Let $\mathcal{G}$ be a pre-(C-DS). Then $\mathcal{G}$ is said to be an exponential pre-(C-DS) iff there exists $\omega \in \mathbb{R}$ such that $e^{-\omega t} \mathcal{G} \in \mathcal{S}'(L(E))$. We use the shorthand pre-(C-EDS) to denote an exponential pre-(C-DS).

We have the following fundamental result:

**Theorem 4.12.** Assume that $\alpha \geq 0$ and $\mathcal{A}$ generates an exponentially equicontinuous $\alpha$-times integrated $C$-semigroup $(S_\alpha(t))_{t \geq 0}$. Define $\mathcal{G}$ through $\mathcal{G}_\alpha(\varphi)x := \int_0^\infty W_\alpha^n \varphi(t) S_\alpha(t)x \, dt$, $x \in E, \; \varphi \in \mathcal{D}$. Then $\mathcal{G}$ is a pre-(C-EDS) whose integral generator contains $\mathcal{A}$.

**Remark 4.13.**

(i) Suppose that $\mathcal{G}$ is a pre-(C-EDS) generated by $\mathcal{A}$, $\omega \in \mathbb{R}$ and $e^{-\omega t} \mathcal{G} \in \mathcal{S}'(L(E))$. Suppose, further, that there exists a non-negative integer $n$ and a continuous function $V : \mathbb{R} \to L(E)$ satisfying that

$$\langle e^{-\omega t} \mathcal{G}, \varphi \rangle = (-1)^n \int_{-\infty}^\infty \varphi^{(n)}(t)V(t) \, dt, \quad \varphi \in \mathcal{D},$$ 

and that there exists a number $r \geq 0$ such that the operator family $\{(1 + t^r)^{-1} V(t) : t \geq 0\} \subseteq L(E)$ is equicontinuous. Since $e^{-\omega t} \mathcal{G}$ is a pre-(C-EDS) generated by $\mathcal{A} - \omega$, the proof of Theorem 4.10 shows that $(V(t))_{t \geq 0}$ is an
exponentially equicontinuous $n$-times integrated $C$-semigroup; by Theorem 4.12 the integral generator $\hat{A}^\omega$ of $(V(t))_{t \geq 0}$ is contained in $A - \omega$. Define

$$S_n(t)x := e^{\omega t}V(t)x + \int_0^t \sum_{k=1}^n \binom{n}{k} \frac{(-1)^k \omega^k(t-s)^{k-1}}{(k-1)!} e^{\omega s}V(s)x\,ds.$$ 

Arguing as in the proof of [18 Theorem 2.5.1, Theorem 2.5.3], we can prove that the MLO $\hat{A}^\omega + \omega (\subseteq A)$ is the integral generator of an exponentially equicontinuous $n$-times integrated $C$-semigroup $(S_n(t))_{t \geq 0}$.

(ii) The conclusions from Theorem 4.12 and the first part of this remark can be reword for the classes of $q$-exponentially equicontinuous integrated $C$-semigroups and $q$-equicontinuous pre-$(C$-DS)'s; cf. [21] for the notion.

Remark 4.14. Suppose that $G \in \mathcal{D}_0(L(E))$, $G(\varphi)C = CG(\varphi)$, $\varphi \in D$ and $A$ is a closed MLO on $E$ satisfying that $G(\varphi)A \subseteq AG(\varphi)$, $\varphi \in D$ and

$$(4.7) \quad G(-\varphi')x - \varphi(0)Cx \in AG(\varphi)x, \quad x \in E, \quad \varphi \in D.$$ 

In [21], we have proved the following:

(i) If $A = A$ is single-valued, then $G$ satisfies (C.S.1).

(ii) If $G$ satisfies (C.S.2) holds, $C$ is injective and $A = A$ is single-valued, then $G$ is a (C-DS) generated by $C^{-1}AC$.

(iii) If $E$ is admissible and $A = A$ is single-valued, then the condition (C.S.2) automatically holds for $G$.

As we have already seen, the conclusion from (ii) immediately implies that $A = A$ must be single-valued and that the operator $C$ must be injective.

Concerning the assertion (i), its validity is not true in multivalued case: Let $C = I$, let $A = E \times E$, and let $G \in \mathcal{D}_0(L(E))$ be arbitrarily chosen. Then $G$ commutes with $A$ and (4.7) holds but $G$ need not satisfy (C.S.1).

Concerning the assertion (iii) in multivalued case, we can prove that the admissibility of state space $E$ implies that for each $x \in \mathcal{N}(G)$ there exist an integer $k \in \mathbb{N}$ and a finite sequence $(y_i)_{0 \leq i \leq k-1}$ in $D(A)$ such that $y_i \in A y_{i+1}$ $(0 \leq i \leq k-1)$ and $Cx \in A y_0 \subseteq A^{k+2} y_0$.

Now we will reconsider some conditions introduced by J. L. Lions [20] in our new framework. Suppose that $G \in \mathcal{D}_0(L(E))$ and $G$ commutes with $C$. We analyze the following conditions for $G$:

$$(d_1) \quad G(\varphi \ast \psi)C = G(\varphi)G(\psi), \quad \varphi, \psi \in \mathcal{D}_0,$$

$$(d_3) \quad R(G) \text{ is dense in } E,$$

$$(d_4) \quad \text{for every } x \in R(G), \text{ there exists a function } u_x \in C([0, \infty) : E) \text{ so that }$$

$$u_x(0) = Cx \text{ and } G(\varphi)x = \int_0^\infty \varphi(t)u_x(t)\,dt, \quad \varphi \in \mathcal{D},$$

$$(d_5) \quad (Cx, G(\psi)x) \in G(\psi_+), \quad \psi \in \mathcal{D}, \quad x \in E.$$ 

Suppose that $G \in \mathcal{D}_0(L(E))$ is a pre-(C-DS). Then it is clear that $G$ satisfies $(d_1)$, our previous considerations shows that $G$ satisfies $(d_5)$; by the proof of [18 Proposition 3.1.24], we have that $G$ also satisfies $(d_4)$. On the other hand, it is well known that $(d_1)$, $(d_3)$ and (C.S.2) taken together do not imply (C.S.1), even in the case that $C = I$; see e.g. [18 Remark 3.1.20]. Furthermore, let $(d_1)$, $(d_3)$ and $(d_4)$ hold. Then $(d_5)$ holds, as well. In order to see this, fix $x \in R(G)$ and $\varphi \in \mathcal{D}$; then it suffices to show that $(Cx, G(\varphi)x) \in G(\varphi_+)$. Suppose that $(\rho_n)$ is a regularizing
sequence and \( u_x(t) \) is a function appearing in the formulation of the property \((d_4)\). The arguments contained in the proof of [13 Proposition 3.1.19] shows that, for every \( x, y \in \mathcal{D} \), one has

\[
\mathcal{G}(\rho_n)\mathcal{G}(\varphi_+ \ast \eta)x = \mathcal{G}((\varphi_+ \ast \rho_n) \ast \eta)Cx = \mathcal{G}(\eta)\mathcal{G}(\varphi_+ \ast \rho_n)x
\]

\[
= \mathcal{G}(\eta) \int_0^\infty (\varphi_+ \ast \rho_n)(t)u_x(t) \, dt
\]

\[
\to \mathcal{G}(\eta) \int_0^\infty \varphi(t)u_x(t) \, dt = \mathcal{G}(\eta)\mathcal{G}(\varphi)x, \quad n \to \infty;
\]

\[
\mathcal{G}(\rho_n)\mathcal{G}(\varphi_+ \ast \eta)x = \mathcal{G}(\varphi_+ \ast \eta)\mathcal{G}(\rho_n)x \to \mathcal{G}(\varphi_+ \ast \eta)\mathcal{G}(\rho_n)x, \quad n \to \infty.
\]

Hence, \( \mathcal{G}(\varphi_+ \ast \eta)Cx = \mathcal{G}(\eta)\mathcal{G}(\varphi)x \) and \((d_5)\) holds, as claimed. On the other hand, \((d_1)\) is a very simple consequence of \((d_5)\); to verify this, observe that for each \( \varphi \in \mathcal{D} \) and \( \psi \in \mathcal{D} \) we have \( \psi_+ \ast \varphi = \psi_0 \ast \varphi = \varphi_0 \ast \psi \), so that \((d_5)\) is equivalent to say that

\[
\mathcal{G}(\varphi_+ \ast \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi) \quad (\varphi \in \mathcal{D}, \psi \in \mathcal{D}).
\]

In particular,

\[
(4.8) \quad \mathcal{G}(\varphi)\mathcal{G}(\psi) = \mathcal{G}(\psi)\mathcal{G}(\varphi), \quad \varphi \in \mathcal{D}, \quad \psi \in \mathcal{D}.
\]

Suppose now that \((d_5)\) holds. Let \( \varphi \in \mathcal{D} \) and \( \psi, \eta \in \mathcal{D} \). Observing that \( \psi_+ \ast \eta_+ \ast \varphi = (\psi_0 \ast \eta)_+ \ast \varphi \), we have (cf. also [23, Remark 3.13]):

\[
(4.9) \quad \mathcal{G}(\varphi)\mathcal{G}(\eta)\mathcal{G}(\psi) = C\mathcal{G}(\eta_+ \ast \varphi)\mathcal{G}(\psi)
\]

\[
= C\mathcal{G}(\psi_+ \ast \eta_+ \ast \varphi) = C\mathcal{G}(\psi_0 \ast \eta)_+ \ast \varphi)C
\]

\[
= C\mathcal{G}(\varphi)\mathcal{G}(\psi_0 \ast \eta) = \mathcal{G}(\varphi)\mathcal{G}(\psi_0 \ast \eta)C.
\]

By \((4.8) - (4.10)\), we get

\[
(4.10) \quad \mathcal{G}(\eta)\mathcal{G}(\psi)\mathcal{G}(\varphi) = \mathcal{G}(\psi_0 \ast \eta)C\mathcal{G}(\varphi).
\]

Due to \((4.8) - (4.10)\), we have the following:

(i) \((d_5)\) and \((d_3)\) together imply \((C.S.1)\); in particular, \((d_1), (d_3)\) and \((d_4)\) together imply \((C.S.1)\). This is an extension of [13 Proposition 3.1.19].

(ii) \((d_5)\) and \((d_2)\) together imply that \( \mathcal{G} \) is a \((C.DS)\); in particular, \( \mathcal{A} = \mathcal{A} \) must be single-valued and \( \mathcal{C} \) must be injective.

On the other hand, \((d_5)\) does not imply \((C.S.1)\) even in the case that \( \mathcal{C} = \mathcal{I} \). A simple counterexample is \( \mathcal{G} \in \mathcal{D}'(L(E)) \) given by \( \mathcal{G}(\varphi)x := \varphi(0)x, \quad x \in \mathcal{E}, \quad \varphi \in \mathcal{D} \).

The exponential region \( E(a, b) \) has been defined for the first time by W. Arendt, O. El-Menmouati and V. Keyantuo in [4]:

\[
E(a, b) := \left\{ \lambda \in \mathbb{C} : \Re \lambda \geq a, \quad |\Im \lambda| \leq e^{\alpha \Re \lambda} \right\} \quad (a, \ b \ > 0).
\]

Now we are able to state the following theorem:

**Theorem 4.15.** Let \( a > 0, \ b > 0 \) and \( \alpha > 0 \). Suppose that \( \mathcal{A} \) is a closed MLO and, for every \( \lambda \) which belongs to the set \( E(a, b) \), there exists an operator \( F(\lambda) \in L(E) \) so that \( F(\lambda)A \subseteq AF(\lambda), \ \lambda \in E(a, b), \ F(\lambda)x \in (\lambda - A)^{-1}Cx, \ \lambda \in E(a, b), \ x \in \mathcal{E}, \ F(\lambda)C = CF(\lambda), \ \lambda \in E(a, b), \ F(\lambda)x - Cx = F(\lambda)y, \ \text{whenever} \ \lambda \in E(a, b) \) and \( (x, y) \in \mathcal{A} \), and that the mapping \( \lambda \mapsto F(\lambda)x \) is analytic on \( \Omega_{a,b} \) and continuous on \( \Gamma_{a,b} \), where \( \Gamma_{a,b} \) denotes the upwards oriented boundary of \( E(a, b) \) and \( \Omega_{a,b} \) the open
region which lies to the right of $\Gamma_{a,b}$. Let the operator family $\{(1+|\lambda|)^{-\alpha}F(\lambda) : \lambda \in E(a,b)\} \subseteq L(E)$ be equicontinuous. Set

$$G(\varphi)x := (-i) \int_{\Gamma_{a,b}} \hat{\varphi}(\lambda)F(\lambda)x d\lambda, \quad x \in E, \ \varphi \in D.$$ 

Then $G$ is a pre-$(C-DS)$ generated by an extension of $A$.

**Proof.** Arguing as in non-degenerate case [21], we can prove with the help of Lemma 2.3.17 that $G \in \mathcal{D}_b(L(E))$ as well as that $G$ commutes with $C$ and $A$. The prescribed assumptions imply by [22] Theorem 3.23 (cf. also [18] Theorem 2.7.2(iv)) that for each $n \in \mathbb{N}$ with $n > \alpha + 1$ the MLO $\mathcal{A}$ subgenerates a local $n$-times integrated $C$-semigroup $(S_n(t))_{t \in [0,a(n-\alpha-1))]$. It is straightforward to prove [21] that

$$G(\varphi)x = (-1)^n \int_{-\infty}^{t} \varphi^{(n)}(t)S_n(t)x dt, \quad x \in E, \ \varphi \in D_{(-\infty,a(n-\alpha-1))}.$$ 

Now the conclusion directly follows from Theorem 4.8 and Remark 4.9(i)-(ii). □

**Remark 4.16.** (i) If $C$ is injective, $A = A$ is single-valued, $\rho_C(A) \subseteq E(a,b)$ and $F(\lambda) = (\lambda - A)^{-1}C$, $\lambda \in E(a,b)$, then $G$ is a $(C-DS)$ generated by $C^{-1}AC$ (21). Even in the case that $C = I$, the integral generator $A$ of $G$, in multivalued case, can strictly contain $C^{-1}AC$; see Remark 4.14(i).

(ii) Let $\mathcal{A}$ be a closed MLO, let $C$ be injective and commute with $\mathcal{A}$, and let $\rho_C(A) \subseteq E(a,b)$. Then the choice $F(\lambda) = (\lambda - A)^{-1}C$, $\lambda \in E(a,b)$ is always possible; in this case, we have $\mathcal{A}0 \subseteq N(G(\varphi))$, $\varphi \in D$ (20).

Local integrated semigroups generated by multivalued linear operators (see e.g. [20] Example 3.2.11(i)) can be used for construction of pre-$(DS)$’s. In [20] Theorem 3.2.21 and [20] Example 3.2.23], we have investigated the entire solutions of backward heat Poisson equation, showing the existence of an entire $C$-regularized semigroup $(C \in L(L^p(\Omega)))$ non-injective) generated by the multivalued linear operator $\Delta \cdot m(x)^{-1}$ in $L^p(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$. This example can serve us to construct an important example of a pre-$(C-DS)$; cf. also [22] Example 3.24]. Examples of exponentially bounded integrated semigroups generated by multivalued linear operators can be found in [12] Chapter II-III, Section 5.8 and these examples can be used for construction of exponential pre-$(DS)$’s. Also by Proposition 4.4(iii) the duals of non-dense pre-$(C-DS)$’s are pre-$(C^{*}-DS)$’s on $E^*$, so this is another way of constructing of degenerate $C$-distribution semigroups.

By Proposition 4.4(iii), the duals of non-dense $(C-DS)$’s.

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