Two Phases of Supersymmetric Gluodynamics

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Abstract

We argue that supersymmetric gluodynamics has two phases with equivalent infrared behavior, one of which is asymptotically free and another one is superstrongly coupled in the ultraviolet domain.
Supersymmetric (SUSY) gauge theories are unique examples of non-trivial four-dimensional theories where some dynamical aspects are exactly tractable. The first results of this type – calculation of the gluino condensate and the Gell-Mann-Low function – were obtained in the early eighties \[1, 2\]. The interest to the miraculous features of the supersymmetric theories was revived after the recent discovery \[3\] – \[6\] of a rich spectrum of various dynamical scenarios that may be realized with a special choice of the matter sector. The basic tools in unraveling these scenarios are: (i) instanton-generated superpotentials which may or may not lift degeneracies along classically flat directions \[7\]; (ii) the NSVZ \(\beta\) functions; (iii) the property of holomorphy in certain parameters \[8, 9\]; (iv) various general symmetry properties, i.e. the superconformal invariance at the infrared fixed points and its consequences \[1\]. A beautiful phenomenon revealed in this way is the existence of a generalized “electric-magnetic” duality in \(N = 2\) \[10\] and some versions of \(N = 1\) theories \[9\].

In this letter we use basically the same methods to argue the existence of two interrelated phases in the supersymmetric gluodynamics, the simplest SUSY gauge theory where we deal only with the gluons and gluinos. Unlike the examples mentioned above in this case parameters of the theory can not be adjusted in such a way that a weak coupling regime is ensured in a certain limit providing us with a clue to what happens in the strong coupling regime. Still some conclusions are possible.

The Lagrangian of the theory has the form

\[
\mathcal{L} = \frac{1}{g_0^2} W^2 |_F + \text{h.c.} \tag{1}
\]

where \(g_0\) is a complex parameter, the bare coupling constant at the ultraviolet cut off \(M_0\) (whose imaginary part is related to the \(\theta\) angle). Below we will use also \(\alpha \equiv g^2/4\pi\). The standard picture going with this Lagrangian is as follows: the theory is asymptotically free which means that, given a fixed correlation length \(\Lambda^{-1}\), the coupling \(\alpha_0\) at the ultraviolet cut off must be adjusted to be logarithmically small,

\[
\alpha_0 \sim \frac{2\pi}{3T(G)} \frac{1}{\ln(M_0/\Lambda)} \tag{2}
\]

where \(T(G)\) is the quadratic Casimir operator in the adjoint representation (e.g. \(T(G) = N\) for SU(N)). The theory is believed to be confining in the infrared domain. The choice \(\alpha_0 \ll 1\) guarantees that the mass scale developing in the infrared is small compared to \(M_0, \Lambda \ll M_0\).

As will be seen, the very same Lagrangian defines another phase of the theory which is superstrongly coupled in the ultraviolet domain, \(\alpha_0 \gg 1\); still it flows to the same infrared limit as the standard asymptotically free phase. In particular, the same correlation length \(\Lambda^{-1}\) is achieved provided that

\[
\alpha_0 \sim \kappa (M_0/\Lambda)^3 \tag{3}
\]

where

\[
\kappa = \frac{2\pi e}{T(G)}.
\]
Thus, a duality takes place – the phases with a weak and a superstrong cou-
plings (SSC) at the ultraviolet cut off evolve towards one and the same infrared
asymptotics.

Our starting point is the exact NSVZ $\beta$ function for SUSY gluodynamics,
\[ \beta(\alpha) = -\frac{\alpha^2}{2\pi} \frac{3T(G)}{1 - (T(G)\alpha/2\pi)}. \] (4)

When these $\beta$ functions were originally derived [2] it was meant that they are ex-
act to all orders in perturbation theory; the question whether they are exact non-
perturbatively was not addressed. Later on it was shown that in some cases these $\beta$
functions do acquire additional non-perturbative parts [11]. In SUSY gluodynamics
(with no matter fields) Eq. (4) is exact non-perturbatively. The proof was actually
given in Ref. [12]. Let us reiterate the main steps considering for definiteness the
SU(2) gauge group. Introduce two auxiliary matter fields in the fundamental rep-
resentation (1 flavor), with a small mass term $m_0$. The theory is then in a weakly
coupled Higgs phase [7]. The existence of a conserved U(1) current unambiguously
dictates the form of the superpotential. It is crucial that this form is saturated
by one instanton, no multi-instanton contributions are allowed. The one-instanton
contribution above leads to the exact result for the gluino condensate $\langle \lambda\lambda \rangle$. One
then uses the property of the holomorphy to exactly extrapolate this result to large
values of $m_0$. At $m_0 \to \infty$ (or, more exactly, $M_0$) we return back to SUSY glu-
dynamics, and the exact expression for $\langle \lambda\lambda \rangle$ implies that the $\beta$ function (4) is
exact in both, perturbative and non-perturbative senses. (The one-instanton satu-
ration of the superpotential, precluding from getting non-perturbative corrections
in the $\beta$ function (4), can be inferred from [12] indirectly – any terms other than
the standard one-instanton would destroy the exact proportionality of $\langle \lambda\lambda \rangle$ to $\sqrt{m_0}$
established in [12]. A more general recent analysis [13] of all terms, perturbative
and non-perturbative, that can appear in the superpotential proves the very same
fact directly.)

A peculiar feature of this $\beta$ function is that it changes sign at $\alpha = 2\pi/T(G) \equiv \alpha_*$
ot through zero, as it happens at the regular fixed points, but rather through pole.
Nevertheless, changing the sign results in the fact that $\alpha = 2\pi/T(G)$ is an infrared
attractive point, the theory flows to it in the infrared from both sides, the small
and large $\alpha$ domains. As a matter of fact, the solution of the renormalization group
equation for $\alpha$ is a double-valued function of the normalization point $\mu$. If one starts
from $\alpha_*$ at $\mu = \Lambda$ one solution evolves to the standard asymptotically free phase,
with a small value of $\alpha$ at short distances, Eq. (2). Another solution evolves to
a superstrongly coupled (SSC) phase at short distances (see Eq. (3) and Fig. 1).
Near the critical point
\[ \alpha = \alpha_* \pm \sqrt{6\alpha_*^2\Lambda^{-1}(\mu - \Lambda)}. \]

If these two solutions are denoted by $\alpha_1$ and $\alpha_2$, respectively, the effective infrared
theory is invariant under the interchange $\alpha_1 \leftrightarrow \alpha_2$ at $M_0$. For large $M_0$ this invari-
ance reduces to

$$\alpha_0 \to \kappa \left(e^{\frac{2\pi}{\alpha_0}} - 1\right) + \frac{2\pi}{T} \frac{1}{\ln(\alpha_0/\kappa)}.$$  

If a lattice or a similar formulation of the SUSY gluodynamics existed one could develop a strong coupling expansion similar to Wilson’s [14], and the theory would trivially confine color in this phase. The similarity with Wilson’s argument ends quickly, though. Indeed the Wilson strong coupling expansion assumes that the correlation length in the strong coupling phase is of order of $M_0^{-1}$, while in our case the correlation length is much larger, $\sim (M_0 g_0^{-2/3})^{-1} \gg M_0^{-1}$.

If one studies only the vacuum condensates and other long distance characteristics both phases are indistinguishable from each other. Such quantities depend only on the Wilson coupling constant $\tilde{\alpha}$,

$$\alpha_W^{-1} = \alpha^{-1} - \frac{T}{2\pi} \ln \alpha^{-1}$$

which is the same in the both phases in the respective points $\alpha_{1,2}$. Incidentally, it is just the Wilson coupling $\alpha_W^{-1}$ on which the chiral quantities and $F$ terms depend holomorphically, not $\alpha^{-1}$. The distinction appears when one studies the short distance properties of different correlation functions. Consider for instance, the two-point function

$$\langle T\{\lambda(x)\lambda(x), \bar{\lambda}(0)\bar{\lambda}(0)\}\rangle.$$  

In the asymptotically free phase at $|x| \ll \Lambda$ it behaves as $x^{-6}(\ln x)^{-2}$ and is very singular at $x \to 0$. The corresponding spectral density at large $s$ grows linearly with $s$. In the SSC phase, where the interaction becomes strong at short distances, the bound states (except for a few low-lying ones) are generically expected to have sizes of order $M_0^{-1}$. One can expect something similar “fall to the center”. Then the spectral density must fall off rapidly above the lowest state (which is the same in both phases). Accordingly, the singularity at small $x$ in the correlation function (5) is much softer in the SSC phase.

We would like to add a speculative remark on the nature of the point $\alpha_*$ which may be a bifurcation point of the renormalization group (RG) flow. Formally the running coupling constant corresponding to the $\beta$ function (4) satisfies the relation

$$\frac{\kappa}{\alpha(\mu)} \exp \left[-\frac{2\pi}{\alpha(\mu)T}\right] = \left(\frac{\Lambda}{\mu}\right)^3.$$  

It is easy to see that Eq. (6) has no real solutions for $\mu < \Lambda$. However, one can find a complex solution. For example if $\mu$ is slightly smaller than $\Lambda$, i.e. $(\Lambda/\mu)^3 = (1 + \epsilon)$ (where $\epsilon \ll 1$) one can continue the RG trajectories

$$\frac{1}{\alpha(\mu)} = \frac{1}{\alpha_*} \left[(1 - \frac{2}{3}\epsilon) \pm i\sqrt{2\epsilon}\right].$$  

3
This looks like generation of a $\theta$ term, $2\pi/\alpha \rightarrow 2\pi/\alpha + i\theta$, which is unobservable, anyway.

Such a behavior is rather unusual in quantum field theories, but as an example of such a strange behavior let us remind the reader about a peculiar RG flow in the two-dimensional O(3) $\sigma$ model with the $\theta$ term [15], i.e. in the theory with the action

$$S = \frac{1}{2g^2} \int d^2x \partial_\mu \vec{n} \partial_\mu \vec{n} + i \frac{\theta}{8\pi} \int d^2x \epsilon_{\mu\nu} (\vec{n} [\partial_\mu \vec{n} \times \partial_\nu \vec{n}]) .$$

The RG flow here is as shown on Fig. 2; one can see that at $\theta = \pi$ there is an unstable fixed point ($\beta$ function for $g$ has a double zero at some $g_\ast$) which can be considered either as an infrared fixed point for the trajectories starting at small $g$ and $\theta = \pi$ or as an ultraviolet fixed point leading to strong coupling phase with all possible values for $\theta$. One can observe some analogy between this behavior and the $\alpha_\ast$ point arising because of the pole in the NSVZ $\beta$ function. To reveal this analogy it is more appropriate to consider the zero charge case rather than the asymptotically free theory. This is achieved by adding $N_f$ massless matter flavors ($2N_f$ chiral superfields). The $\beta$ function in this case is (for the gauge group SU(N))

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi} \frac{3N - N_f T(1 - \gamma(\alpha))}{1 - (N\alpha/2\pi)} .$$

where $\gamma(\alpha)$ is the anomalous dimension of the matter superfields in the fundamental representation. For $N_f$ large enough the weak coupling phase is actually infrared free (the Landau zero charge), and the pole point $\alpha_\ast$ becomes an ultraviolet attractor unless the behavior of $\gamma(\alpha)$ screens it (i.e. the numerator of Eq. (7) develops zero before the denominator). In other words, we assume that, unlike the situation in Ref. [3], the conformal point does not develop at $N_f > 3N$. Then again, the critical value $\alpha_\ast$ is achieved at a finite value of the normalization point; the question arises as to what happens when $\mu$ evolves further, to higher values. In this context the problem is even more striking – if in the case of the asymptotically free theory one can insist that because of confinement one simply can not go below $\Lambda$, in the latter case we have to define the theory in the ultraviolet limit. Presumably one can think about this further evolution along the lines suggested by the $\sigma$ model above, i.e. considering this point as a bifurcation of the RG flow. This question deserves further investigation.

**Acknowledgments**

We thank A. Rosly and A. Vainshtein for discussions. I. K. is grateful to all members of TPI for hospitality during his visit in March-April 1995. This work was supported in part by DOE under the grant number DE-FG02-94ER40823, PPARC grant GR/J 21354 and by Balliol College, University of Oxford.
Figure Captions

Fig. 1.
The double-valued solution of the renormalization group equation for the running coupling constant in supersymmetric gluodynamics.

Fig. 2.
The renormalization group flow in the two-dimensional $O(3) \sigma$ model with the $\theta$ term.
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