GENERAL MULTIPLE DIRICHLET SERIES FROM PERVERSE SHEAVES

WILL SAWIN

ABSTRACT. We give an axiomatic characterization of multiple Dirichlet series over the function field $\mathbb{F}_q(T)$, generalizing a set of axioms given by Diaconu and Pasol. The key axiom, relating the coefficients at prime powers to sums of the coefficients, formalizes an observation of Chinta. The existence of multiple Dirichlet series satisfying these axioms is proved by exhibiting the coefficients as trace functions of explicit perverse sheaves and using properties of perverse sheaves. The multiple Dirichlet series defined this way include, as special cases, many that have appeared previously in the literature.

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1. Introduction

1.1. Background. Multiple Dirichlet series were originally defined as Dirichlet series in multiple variables satisfying twisted multiplicativity properties and certain groups of functional equations. These were first motivated by moments of $L$-functions [Siegel, 1955; Hoffstein and Goldfeld, 1985], and have since been successfully used to calculate several moments, with recent examples including [Diaconu, 2019; Diaconu and Whitehead, 2021; Gao and Zhao, 2023a,b]. If one defines a Dirichlet $L$-function where the Dirichlet character
is expressed as a Legendre symbol, as in
\[ L\left(s, \left( \frac{1}{m} \right) \right) = \sum_{n=1}^{\infty} \left( \frac{n}{m} \right) n^{-s} \]
then it is natural to consider moments like
\[ \sum_{m<X} \prod_{i=1}^{k} L\left(s_i, \left( \frac{1}{m} \right) \right), \]
which can be analyzed using the series
\[ \sum_{m=1}^{\infty} \prod_{i=1}^{k} L\left(s_i, \left( \frac{1}{m} \right) \right) m^{-s} = \sum_{n_1,\ldots,n_k,m=1}^{\infty} \left( \prod_{i=1}^{k} \left( \frac{n_i}{m} \right) \right) m^{-s} \prod_{i=1}^{k} n_i^{-s_i}. \]
A plausible strategy to analyze these moments is as follows. First, replace the coefficients \( \prod_{i=1}^{k} \left( \frac{n_i}{m} \right) \) by another set of coefficients \( a_{n_1,\ldots,n_k,m} \) which agree with them for \( n_1,\ldots,n_k,m \) squarefree and relatively prime, but may differ for other values. Also choose the coefficients \( a_{n_1,\ldots,n_k,m} \) to ensure the series has better analytic properties. Next, use these analytic properties to estimate suitable integrals of the series. Finally, use a sieve to extract information about the corresponding integral with the original set of coefficients. Since the coefficients \( \prod_{i=1}^{k} \left( \frac{n_i}{m} \right) \) satisfy a twisted multiplicativity analogous to the multiplicativity of the coefficients of classical Dirichlet series, one assumes the modified coefficients keep this twisted multiplicativity, i.e.
\[ a_{n_1n'_1,\ldots,n_kn'_k,mm'} = a_{n_1,\ldots,n_k,m}a_{n'_1,\ldots,n'_k,m'} \prod_{i=1}^{k} \left( \frac{n_i}{m} \right) \left( \frac{n'_i}{m'} \right) \]
as long as \( n_1,\ldots,n_k,m \) are relatively prime to \( n'_1,\ldots,n'_k,m' \). Generally, the better analytic properties one seeks to obtain are functional equations, and analytic continuation enabled by those functional equations.

The most desirable would be a meromorphic continuation to \( \mathbb{C}^k \) with an explicit description of the poles. This can be obtained when one has a functional equation in each variable generating a finite group of functional equations (typically a Weyl group). However, some recent work has studied multiple Dirichlet series with an infinite group of functional equations \[ \text{Whitehead, 2014} \], where one expects only meromorphic continuation to a certain region in \( \mathbb{C}^r \), and can only prove meromorphic continuation to a smaller region directly from the functional equations. Still, obtaining continuation to the larger region is sometimes possible \[ \text{Whitehead, 2016} \], and could hold the key to estimating higher moments of \( L \)-functions \[ \text{Diaconu, Goldfeld, and Hoffstein, 2003; Diaconu and Twiss, 2020} \].

Since the multiplicativity is twisted, one does not have an expression of the multiple Dirichlet series as an Euler product of local factors. However, twisted multiplicativity does still reduce the choice of coefficients for each tuple of numbers to the local choice of coefficients for each tuple of powers of a fixed prime. To obtain the desired functional equations, one needs that the generating series of these prime power coefficients satisfy certain analogous functional equations. Because these local functional equations were used to define the coefficients, the multiple Dirichlet series could only be uniquely defined when these functional
equations were sufficient to uniquely characterize the generating functions. Chinta [2008] first observed that, when working over the function field $\mathbb{F}_q(t)$, there was a local-to-global symmetry relating these generating functions to the multiple Dirichlet series. This could be proven by observing that they were both determined by their functional equations and then comparing their functional equations.

1.2. Summary of results. The goal of this paper is to provide a uniform construction of multiple Dirichlet series over the function field $\mathbb{F}_q(t)$, parameterized simply by the finite field $\mathbb{F}_q$, a character $\chi$ of $\mathbb{F}_q$, and a symmetric integer matrix $M$, that includes many multiple Dirichlet series previously constructed separately as well as new examples. Future work could investigate these new examples, finding functional equations they satisfy, regions to which they can be analytically continued, and applications to moments of $L$-functions. Furthermore, it may be possible to define new multiple Dirichlet series in the number field context by choosing the coefficients at tuples of powers of a prime $p$ to match the coefficients of the series defined here at powers of a polynomial over $\mathbb{F}_p$, and then to investigate their analytic properties also.

Our approach is inspired by Diaconu and Pasol [2018], who showed that the local-to-global properties observed by Chinta [2008], combined with the twisted multiplicativity, uniquely characterize the multiple Dirichlet series by an inductive argument, and thus could be used as a definition of multiple Dirichlet series. However, they were only able to show the existence of multiple Dirichlet series satisfying these local-to-global properties in one particular family of cases, the one relating to moments of quadratic Dirichlet $L$-functions, by a lengthy étale cohomology argument. In these cases, Whitehead [2014] was able to show that the functional equations follow from the local-to-global properties.

We propose a new approach. We define multiple Dirichlet series that satisfy quite general twisted multiplicativity relations involving arbitrary characters, which are uniquely characterized by local-to-global properties. Here the matrix $M$ and character $\chi$ determine the exact function we twist the multiplicativity relation by. However, we define and construct the multiple Dirichlet series coefficients as trace functions of certain perverse sheaves.

Using this local-to-global property, it is possible to show that our multiple Dirichlet series include as a special case some multiple Dirichlet series that appear before in the literature. We prove this for two series defined by Chinta and Mohler [2010] (Corollary 4.3 and (4.20)) and one defined by Chinta [2008] (Proposition 4.8). For those defined by Diaconu and Pasol [2018] the proof is automatic since their axioms are a special case of ours. It seems reasonable to expect, based on these examples, that every multiple Dirichlet series defined in the literature whose values at relatively prime tuples of squarefree numbers can be expressed in terms of Dirichlet characters, Jacobi symbols, and Gauss sums, are also special cases of ours. It seems reasonable to expect, based on these examples, that every multiple Dirichlet series defined in the literature whose values at relatively prime tuples of squarefree numbers can be expressed in terms of Dirichlet characters, Jacobi symbols, and Gauss sums, are also special cases of ours. It seems reasonable to expect, based on these examples, that every multiple Dirichlet series defined in the literature whose values at relatively prime tuples of squarefree numbers can be expressed in terms of Dirichlet characters, Jacobi symbols, and Gauss sums, are also special cases of ours.
“generic” values like tuples of relatively prime squarefree numbers to all values, and therefore that every extension that satisfies nice analytic properties likely comes from a suitable perverse sheaf.

The idea that the trace function of a perverse sheaf gives a well-behaved function in analytic number theory over function fields is most prominent in the geometric Langlands program, where automorphic forms are expected, and in many cases known, to arise in this way, but it can also be seen in more elementary situations. For example, the divisor function arises from a perverse sheaf. More generally, so do the coefficients of the $L$-function of a Galois representation.

The author also expects that these multiple Dirichlet series will satisfy functional equations analogous to those satisfied by existing series like the Weyl group multiple Dirichlet series [Brubaker, Bump, Chinta, Friedberg, and Hoffstein, 2006], and possibly more general ones, with the exact nature of the functional equations depending on the parameters $M, \chi$. The examples included in this paper give initial evidence for this: Proposition 4.3 covers a Weyl group multiple Dirichlet series that satisfies an interesting group of functional equations matching the Weyl group $S_3$, suggesting that further special cases of our construction may also satisfy similar functional equations. Furthermore (4.3) gives a relation between the coefficients of two multiple Dirichlet series that can be used to prove a functional equation relating the series themselves, with the Fourier transform in that equation playing the same crucial role it does in the classical functional equations of the zeta function and Dirichlet $L$-functions, again suggesting that more general functional equations of this type should exist. Work in progress by the author and Ian Whitehead, as well as by Matthew Hase-Liu, aims to prove these functional equations in greater generality. This work should also enable us to realize further previously-defined multiple Dirichlet series as special cases of the construction of this paper, as these series are uniquely determined by their functional equations so it suffices to check the newly-defined series satisfy the same functional equations.

1.3. Notation. Let $\mathbb{F}_q[t]$ be the ring of polynomials in one variable over a finite field $\mathbb{F}_q$. Let $\mathbb{F}_q[t]^+$ be the subset of monic polynomials. Let $f'$ be the derivative of $f$ with respect to $t$.

Fix a natural number $n$. We always let $\chi: \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a character of order $n$. Let $\chi_m: \mathbb{F}_q^m \to \mathbb{C}^\times$ be the composition of $\chi$ with the norm map $\mathbb{F}_q^m \to \mathbb{F}_q$.

Define a residue symbol

\[
\begin{pmatrix} f \\ g \end{pmatrix}_\chi
\]

for $(f, g) \in \mathbb{F}_q[t]$ coprime as the unique function that is separately multiplicative in $f$ and $g$ such that if $g$ is irreducible of degree $d$,

\[
\begin{pmatrix} f \\ g \end{pmatrix}_\chi = \chi(f^{q^d-1})
\]

where we use the fact that $f^{q^d-1}$ in $\mathbb{F}_q[T]/g = \mathbb{F}_{q^d}$ in fact lies in $\mathbb{F}_q$.

Let $\text{Res}(f, g)$ be the resultant of $f$ and $g$. For $g$ monic, as it will almost always be, this is the product of the values of $f$ at the roots of $g$. 

We define a “set of ordered pairs of Weil numbers and integers” to be a set $J$ consisting of ordered pairs $j$ of a Weil number $\alpha_j$ and an integer $c_j$, such that no $\alpha_j$ appears twice in the set, and $c_j$ is never zero.

For $J_1, J_2$ two sets of ordered pairs, we define $J_1 \cup J_2$ to be the union, except that if some Weil number $\alpha$ appears in both $J_1$ and $J_2$, we add the $c_j$s together, and if the sum is zero, we remove them. In other words, $J_1 \cup J_2$ is the unique set of ordered pairs of Weil numbers and integers such that

$$\sum_{j \in J_1 \cup J_2} c_j \alpha_j^e = \sum_{j \in J_1} c_j \alpha_j^e + \sum_{j \in J_2} c_j \alpha_j^e$$

for all integers $e$.

For a Weil number $\beta$, we take $\beta J$ to be the set of ordered pairs $(\beta \alpha_j, c_j)$, so that

$$\sum_{j \in \beta J} c_j \alpha_j^e = \beta^e \sum_{j \in J} c_j \alpha^e$$

for all integers $e$.

We say a $\mathbb{C}$-valued function $\gamma(q, \chi)$ on pairs of a prime power $q$ and character $\chi$ of $\mathbb{F}_q^\times$ is a compatible system of Weil numbers if

$$\gamma(q^e, \chi e) = \gamma(q, \chi)^e$$

for all $q, \chi, e$. For instance, the constant function 1 is a compatible system of Weil numbers.

We say that a function $J(q, \chi)$ from pairs of a prime power $q$ and a character $\chi$ of $\mathbb{F}_q^\times$ to sets of ordered pairs of Weil numbers and integers is a compatible system of sets of ordered pairs if, whenever $J(q, \chi) = \{(\alpha_j, c_j)\}$, we have $J(q^e, \chi e) = \{(\alpha_j^e, c_j)\}$, so that

$$\sum_{j \in J(q^e, \chi e)} c_j \alpha_j^e = \sum_{j \in J(q, \chi)} c_j \alpha_j^e.$$

We now define the general construction of sheaves that will be key for our paper. Fix once and for all a prime $\ell$ invertible in $\mathbb{F}_q$ and an isomorphism between $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ (or just the fields of algebraic numbers within each), with which we will freely identify elements of $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$. Let $X$ be an irreducible scheme of finite type over a field in which $\ell$ is invertible, generically smooth of dimension $d$, and $f$ a nonvanishing function on $X$. Let $U$ be the maximal smooth open set where $f$ is invertible and let $j : U \to X$ be the open immersion. We have a Kummer map $H^0(U, \mathbb{G}_m) \to H^1(U, \mu_{q-1})$. The image of $f$ under this map defines a $\mu_{q-1}$-torsor. We can twist the constant sheaf $\mathbb{Z}/\ell$ by the image of this torsor under $\chi : \mu_{q-1} = \mathbb{F}_q^\times \to \mathbb{Z}/\ell$, obtaining a lisse rank one sheaf $\mathcal{L}(f)$ on $U$. Because $U$ is smooth of dimension $d$, $\mathcal{L}(f)[d]$ is a perverse sheaf on $U$. Let $j_*!(\mathcal{L}(f)[d])$ be its middle extension from $U$ to $X$. Let

$$IC_{\mathcal{L}(f)} = j_*!(\mathcal{L}(f)[d])[-d]$$

be this middle extension, shifted so it lies generically in degree zero.

1.4. **Construction and main theorem.** Let $r$ be a natural number and let $M$ be a symmetric $r \times r$ matrix with integer entries. Let $d_1, \ldots, d_r$ be natural numbers and $q$ a prime power so that $\mathbb{F}_q$ is a finite field. View $\mathbb{A}^{d_i}$ over $\mathbb{F}_q$ as the moduli space of monic polynomials of degree $d_i$, so that $\prod_{i=1}^r \mathbb{A}^{d_i}$ is a moduli space of tuples $(f_1, \ldots, f_r)$ of monic polynomials. On $\prod_{i=1}^r \mathbb{A}^{d_i}$, define the polynomial
function

\[ F_{d_1, \ldots, d_r} = \prod_{i=1}^{r} \text{Res}(f'_i, f_i)^{M_{ii}} \prod_{1 \leq i < j \leq r} \text{Res}(f_i, f_j)^{M_{ij}}. \]

Let

\[ K_{d_1, \ldots, d_r} = \text{IC}_{L, \chi}(F_{d_1, \ldots, d_r}). \]

Given a tuple of polynomials \((f_1, \ldots, f_r)\) of degrees \(d_1, \ldots, d_r\) over \(\mathbb{F}_q\), let \(a(f_1, \ldots, f_r; q, \chi, M)\) be the trace of Frobenius acting on the stalk of \(K_{d_1, \ldots, d_r}\) at \((f_1, \ldots, f_r)\).

Define the multiple Dirichlet series

\[ Z(s_1, \ldots, s_r; q, \chi, M) = \sum_{f_1, \ldots, f_r \in \mathbb{F}_q[t]^+} \frac{a(f_1, \ldots, f_r; q, \chi, M)}{q^{\sum_{i=1}^{r} \deg(f_i)s_i}}. \]

The main theorem of this paper, giving an axiomatic characterization of the coefficients of the geometrically defined multiple Dirichlet series \(Z(s_1, \ldots, s_r; q, \chi, M)\), is as follows.

**Theorem 1.1.** For any fixed \(M\),

\[ a(f_1, \ldots, f_r; q, \chi, M) \]

is the unique function, that, together with a function \(J(d_1, \ldots, d_r; q, \chi, M)\) from tuples of natural numbers \(d_1, \ldots, d_r\), to compatible systems of sets of ordered pairs of Weil numbers and integers, satisfies the axioms

1. If \(f_1, \ldots, f_r\) and \(g_1, \ldots, g_r\) satisfy \(\gcd(f_i, g_j) = 1\) for all \(i, j\), then we have

\[ a(f_1g_1, \ldots, f_rg_r; q, \chi, M) = a(f_1, \ldots, f_r; q, \chi, M) a(g_1, \ldots, g_r; q, \chi, M) \prod_{1 \leq i \leq r} (f_i / g_i)^{M_{ii}} \prod_{1 \leq i < j \leq r} \left( \frac{f_i}{g_i} \right)^{M_{ij}} \left( \frac{g_i}{f_i} \right)^{M_{ii}}. \]

2. \(a(1, \ldots, 1; q, \chi, M) = 1\) and \(a(1, \ldots, 1, f, 1, \ldots, 1; q, \chi, M) = 1\) for all linear polynomials \(f\).

3. \(a(\pi^{d_1}, \ldots, \pi^{d_r}; q, \chi, M) = \left( \frac{\pi'}{\pi} \right)^{\sum_{i=1}^{r} d_i M_{ii}} \sum_{j \in J(d_1, \ldots, d_r; q, \chi, M)} c_j \alpha_j^{\deg \pi}. \)

4. \(\sum_{\sum_{i=1}^{r} d_i = \deg f_i} a(f_1, \ldots, f_r; q, \chi, M) = \sum_{j \in J(d_1, \ldots, d_r; q, \chi, M)} c_j q^{\sum_{i=1}^{r} d_i} \alpha_j. \)

5. \(|\alpha_j| < q^{\sum_{i=1}^{r} d_i - 1}\) as long as \(\sum_{i=1}^{r} d_i \geq 2\).

Here axioms (3) and (4) give the local-to-global principle, (1) is the twisted multiplicativity, and (2) and (5) are normalizations needed to ensure the axioms define a unique set of coefficients, with (5) also ensuring that individual coefficients are not so large that they dominate the series.
Note that the condition that $J$ be a compatible system relates different finite fields at a time, so it is not possible to check these axioms working only in a specific finite field $q$. Rather, one must calculate in all extensions of a fixed finite field $\mathbb{F}_q$.

In the case, $\chi$ is quadratic, when $M$ is the sum of a matrix with a row of ones and the rest of the entries zero and its transpose, the existence and uniqueness parts of Theorem 1.1 were obtained in [Diaconu and Pasol, 2018].

1.5. Perverse sheaves. The key geometric idea of this paper is that the local-to-global property described by axioms (3) and (4) is a consequence of duality properties of perverse sheaves. The local-to-global property relates the sum of many coefficients of the multiple Dirichlet series to a single coefficient, via the set of Weil numbers $J$. Geometrically, we interpret this as a relation between the sum of the trace of Frobenius on the stalk of a perverse sheaf over all the $\mathbb{F}_q$-points of a variety and the value at a single point. The Lefschetz fixed point formula relates the sum of the trace of Frobenius over all $\mathbb{F}_q$-points to the compactly supported cohomology of the variety with coefficients in the perverse sheaf. Because there is an action of the multiplicative group on the variety that fixes only that point, giving it a conical structure, a generalization of the result that the cohomology of a cone matches the cohomology of the point relates the stalk of that point to the usual cohomology. Verdier duality for perverse sheaves then relates the usual and compactly-supported cohomology.

Furthermore, axiom (1) will follow from a twisted multiplicativity property of the polynomial functions $F_{d_1,\ldots,d_r}$ used to construct the perverse sheaves $K_{d_1,\ldots,d_r}$. We then transform this identity involving the polynomials $F_{d_1,\ldots,d_r}$ to an isomorphism involving the perverse sheaves $K_{d_1,\ldots,d_r}$, using fundamental properties of the intermediate extension construction, which then implies an identity involving the trace functions $a(f_1,\ldots,f_r;q,\chi,M)$ of the perverse sheaves $K_{d_1,\ldots,d_r}$.

Axiom (5) follows from the theory of weights and purity for perverse sheaves, which gives bounds for the Frobenius eigenvalues in each degree.

Characteristic zero analogues of the perverse sheaves $IC_{\mathcal{L}_\chi}(F_{d_1,\ldots,d_r})$ used in our construction have been studied before from the perspective of quantum groups and Nichols algebras [Bezrukavnikov, Finkelberg, and Schechtman, 1998; Kapranov and Schechtman, 2020]. Some of our (brief) calculations with these sheaves in Section 3 are characteristic $p$ analogues of results previously obtained in the characteristic zero setting in those works. This connection between multiple Dirichlet series and quantum groups seems different from the usual one, as the coefficients of the multiple Dirichlet series correspond to traces of Frobenius on stalks of the sheaves that can be computed from the cohomology of the positive part of the small quantum group, but neither the Frobenius action nor the cohomology of the positive part appear in the usual picture. I learned of these connections thanks to helpful conversations with Jordan Ellenberg, Michael Finkelberg, Mikhail Kapranov, Tudor Pădurariu, and Vadim Schechtman.

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2. Preliminaries

2.1. Further notations. We use \( \xi \) to refer to, when \( q \) is odd, the unique character \( \xi : \mathbb{F}_q^\times \to \mathbb{C}^\times \) of order 2. If \( n \) is even, we have \( \xi = \chi^{n/2} \).

For a rational function \( f \), let \( \text{res}(f) \) be its residue at \( \infty \), normalized so that \( \text{res}(1/t) = 1 \), (i.e. the coefficient of \( t^{-1} \) when \( f \) is expressed as a formal Laurent series in \( t^{-1} \)).

For \( x \in \mathbb{F}_q \), let \( \psi(x) = e^{2\pi i \text{tr}_{\mathbb{F}_q/F_p} x/p} \). Let \( G(\chi, \psi) = \sum_{x \in \mathbb{F}_q^\times} \chi(x) \psi(x) \).

We say a function \( \gamma(q, \chi) \) on pairs of a prime power \( q \) and character \( \chi \) of \( \mathbb{F}_q^\times \) is a sign-compatible system of Weil numbers if

\[
-\gamma(q^e, \chi^e) = (-\gamma(q, \chi))^e
\]

for all \( q, \chi, e \). For instance, the Hasse-Davenport identities imply that \( G(\chi^r, \psi) \) is sign-compatible for any integer \( r \).

We let

\[
\lambda(d_1, \ldots, d_r; q, \chi, M) = \sum_{f_1, \ldots, f_r \in \mathbb{F}_q[t]} a(f_1, \ldots, f_r; q, \chi, M)
\]

so that

\[
Z(s_1, \ldots, s_r; q, \chi, M) = \sum_{d_1, \ldots, d_r \in \mathbb{N}} \lambda(d_1, \ldots, d_r; q, \chi, M) \prod_{i=1}^r q^{-d_i s_i}.
\]

For \( \pi \) a prime polynomial, we let \( v_\pi \) be the \( \pi \)-adic valuation of polynomials, i.e. \( v_\pi(f) \) is the maximum power of \( \pi \) dividing \( f \).

2.2. Function field evaluations. Certain functions important in classical number theory, such as the Möbius function, power residue symbol, and Gauss sums, admit alternate formulas in the function field \( \mathbb{F}_q(t) \), that make clear their relationship to the algebra of polynomials.

Lemma 2.1. We have

\[
\left( \frac{f}{g} \right)_\chi = \chi(\text{Res}(f, g)).
\]

Proof. Because the right side, by definition, is multiplicative in \( g \), it suffices to consider the case where \( g \) is prime. Then for \( \alpha \) a root of \( g \), the other roots are \( \alpha^q, \ldots, \alpha^{q^{d-1}} \). Hence the product of the values of \( f \) at these roots is

\[
\prod_{i=0}^{d-1} f(\alpha^i) = \prod_{i=0}^{d-1} f(\alpha)^{q^i} = f(\alpha)^{\frac{q^d-1}{q-1}}.
\]

Because \( \alpha \) is a root of \( g \), we can evaluate this by setting \( \alpha = T \) and reducing mod \( g(T) \), which matches the definition of \( \left( \frac{T}{g} \right)_\chi \). \( \square \)

Under this interpretation, the reciprocity law for power residue symbols is given by the following fact:
Lemma 2.2. For monic $f, g$,
\[
\text{Res}(f, g) = (-1)^{\deg f \deg g} \text{Res}(g, f).
\]

Proof. For $\alpha_1, \ldots, \alpha_{\deg f}$ the roots of $f$ and $\beta_1, \ldots, \beta_{\deg g}$ the roots of $g$,
\[
\text{Res}(f, g) = \prod_{i=1}^{\deg f} \prod_{j=1}^{\deg g} (\beta_j - \alpha_i)
\]
and
\[
\text{Res}(g, f) = \prod_{i=1}^{\deg f} \prod_{j=1}^{\deg g} (\beta_j - \alpha_i)
\]
so switching each term, we obtain $\deg f \deg g$ factors of $(-1)$. □

Let $\Delta(f)$ be the discriminant of $f$. Let $\mu$ be the Möbius function.

Lemma 2.3. For $q$ odd we have
\[
(2.1) \quad \mu(f) = (-1)^{\deg f} \xi(\Delta(f))
\]
and
\[
(2.2) \quad \Delta(f) = (-1)^{(\deg f)(\deg f - 1)/2} \text{Res}(f', f)
\]
so
\[
(2.3) \quad \mu(f) = (-1)^{\deg f} \left(-1\right)^{\deg f (\deg f - 1)/4} \left(\frac{f'}{f}\right) \xi(\Delta(f)).
\]

Proof. (2.1) is Pellet’s formula. (2.2) follows from noting that for $\alpha_1, \ldots, \alpha_{\deg f}$ the roots of $f$, we have $f'(\alpha_i) = \prod_{j \neq i} (\alpha_i - \alpha_j)$ so
\[
\text{Res}(f', f) = \prod_{1 \leq i \leq \deg f, j \neq i} (\alpha_i - \alpha_j) = \prod_{1 \leq i < j \leq \deg f} (\alpha_i - \alpha_j)(\alpha_j - \alpha_i)
\]
\[
= (-1)^{\deg f (\deg f - 1)/2} \prod_{1 \leq i < j \leq \deg f} (\alpha_i - \alpha_j)^2 = (-1)^{\deg f (\deg f - 1)/2} \Delta(f).
\]
(2.3) follows from combining (2.1), (2.2), and the fact that $\xi(-1) = (-1)^{\frac{q+1}{2}}$. □

Lemma 2.4. If $q$ is odd then for $f_2$ squarefree and $f_1$ prime to $f_2$ we have
\[
(2.4) \quad g_\chi(f_1, f_2) = (-1)^{\deg f_2 (\deg f_2 - 1)(q-1)/4} \left(\frac{f_1'}{f_2} \right)_\chi \left(\frac{f_1'}{f_2} \right)_\xi \left(\frac{f_1}{f_2} \right)^{-1} (G(\chi, \psi))^{\deg f_2}.
\]

Proof. We first evaluate the residue $\text{res} \left( \frac{bf_1}{f_2} \right)$ using the residue theorem. We define the residue of $\frac{bf_1}{f_2}$ at a root $\alpha$ of $f_2$ to be the coefficient of $\frac{1}{t-\alpha}$ in the Laurent series expansion of $\frac{bf_1}{f_2}$ around $\alpha$. The residue of $\frac{bf_1}{f_2}$ at $\infty$ is minus the coefficient of $1/t$ in the Laurent series expansion at $\infty$, i.e. $-\text{res} \left( \frac{bf_1}{f_2} \right)$. The residue theorem implies that the sum of the residues of $\frac{bf_1}{f_2}$ at each point on the projective line vanishes. It follows that $\text{res} \left( \frac{bf_1}{f_2} \right)$ is the sum of the residues of $\frac{bf_1}{f_2}$ at the roots of $f_2$. For $\alpha$ a root of $f_2$, necessarily of order 1, the residue of
\( \frac{hf_1}{f_2} \) at \( \alpha \) is the value of \( \frac{hf_2}{f_1} \) at \( \alpha \). Summing these over \( \alpha \) gives \( \text{tr} \ \frac{hf_1}{f_2} \) where \( \text{tr}: F_q[t]/g \to F_q \) is the trace. Thus

\[
g_\chi(f_1, f_2) = \sum_{h \in F_q[t]/f_2} \left( \frac{h}{f_2} \right)_x \psi \left( \text{res} \left( \frac{hf_1}{f_2} \right) \right)
\]

\[
= \sum_{h \in F_q[t]/f_2} \left( \frac{h}{f_2} \right)_x \psi \left( \text{tr} \ \frac{hf_1}{f_2} \right).
\]

If we change variables to \( h^* = f_1 h/f_2' \), we have

\[
\left( \frac{hf_2}{f_1} \right)_x = \left( \frac{h^*}{f_2} \right)_x \left( \frac{f_1'}{f_2} \right)_x^{-1}
\]

so

\[
g_\chi(f_1, f_2) = \left( \frac{f_1'}{f_2} \right)_x \left( \frac{f_1}{f_2} \right)_x \sum_{h^* \in F_q[t]/f_2} \left( \frac{h^*}{f_2} \right)_x \psi \left( \text{tr} h^* \right).
\]

The inner sum

\[
\sum_{h^* \in F_q[t]/f_2} \left( \frac{h^*}{f_2} \right)_x \psi \left( \text{tr} h^* \right)
\]

is multiplicative in \( f_2 \), and when \( f_2^2 \) is a prime \( \pi \) takes the value \((-G(\chi, \psi)^{\deg \pi})\) by the Hasse-Davenport relations. Hence the inner sum is equal to \((-G(\chi, \psi)^{\deg f_2 \mu(f_2)})\). (2.4) then follows from the last identity of Lemma 2.3.

\[\square\]

The term \( \text{Res}(f', f) \) that appears here has its own multiplicativity relation:

**Lemma 2.5.** We have

\[
\text{Res}((fg)', fg) = \text{Res}(f', f) \text{Res}(g', g) \text{Res}(f, g) \text{Res}(g, f).
\]

**Proof.**

\[
\text{Res}((fg)', fg)
\]

\[
= \text{Res}((fg' + f'g), f) \text{Res}((fg' + f'g), g)
\]

\[
= \text{Res}(f'g, f) \text{Res}(fg', g)
\]

\[
= \text{Res}(f', f) \text{Res}(g, f) \text{Res}(f, g) \text{Res}(g', g).
\]

\[\square\]

We record here also the multiplicativity relations for Gauss sums:

**Lemma 2.6.** If \( \gcd(f_2, f_3) = 1 \) then

\[
g_\chi(f_1f_3, f_2) = \left( \frac{f_3}{f_2} \right)_x^{-1} g_\chi(f_1, f_2).
\]

**Proof.**

\[
g_\chi(f_1f_3, f_2) = \sum_{h \in F_q[t]/f_2} \left( \frac{h}{f_2} \right)_x \psi \left( \text{res} \left( \frac{h_1f_3}{f_2} \right) \right).
\]

Letting \( h^* = hf_3 \), we have

\[
\left( \frac{h}{f_2} \right)_x = \left( \frac{h^*}{f_2} \right)_x \left( \frac{f_3}{f_2} \right)_x^{-1},
\]

\[\square\]
and we observe that this change of variables is a permutation, so
\[ \sum_{h \in \mathbb{F}_q[t]/f_2} \left( \frac{h}{f_2} \right)_x \psi \left( \text{res} \left( \frac{hf_1f_3}{f_2} \right) \right) = \sum_{h^* \in \mathbb{F}_q[t]/f_2} \left( \frac{h^*}{f_2} \right)_x \psi \left( \text{res} \left( \frac{hf_1}{f_2} \right) \right) = \left( \frac{f_3}{f_2} \right)_x^{-1} g_\chi(f_1, f_2). \]

Lemma 2.7. If \( \gcd(f_1, f_4) = \gcd(f_2, f_4) = \gcd(f_2, f_3) = 1 \) then
\[ g_\chi(f_1f_3, f_2f_4) = g_\chi(f_1, f_2) g_\chi(f_3, f_4) \left( \frac{f_2}{f_4} \right)_x \left( \frac{f_1}{f_4} \right)_x^{-1} \left( \frac{f_3}{f_2} \right)_x^{-1}. \]

Proof. \[ g_\chi(f_1f_3, f_2f_4) = \sum_{h \in \mathbb{F}_q[t]/(f_2f_4)} \left( \frac{h}{f_2f_4} \right)_x \psi \left( \text{res} \left( \frac{hf_1f_3}{f_2f_4} \right) \right). \]

As \( f_2 \) and \( f_4 \) are coprime, we can uniquely write \( h = h_2f_4 + h_4f_2 \) for \( h_2 \in \mathbb{F}_q[t]/f_2 \) and \( h_4 \in \mathbb{F}_q[t]/f_4 \). We then have
\[ \left( \frac{h}{f_2f_4} \right)_x = \left( \frac{h}{f_2} \right)_x \left( \frac{h}{f_4} \right)_x = \left( \frac{h_2f_4}{f_2} \right)_x \left( \frac{h_4f_2}{f_4} \right)_x = \left( \frac{h_2}{f_2} \right)_x \left( \frac{f_2}{f_4} \right)_x \left( \frac{h_4}{f_4} \right)_x \left( \frac{f_2}{f_4} \right)_x. \]

Furthermore, we have
\[ \psi \left( \text{res} \left( \frac{hf_1f_3}{f_2f_4} \right) \right) = \psi \left( \text{res} \left( \frac{h_2f_1f_3}{f_2} \right) \right) \psi \left( \text{res} \left( \frac{h_4f_1f_3}{f_4} \right) \right). \]

Hence
\[ g_\chi(f_1f_3, f_2f_4) = \left( \frac{f_2}{f_4} \right)_x \left( \frac{f_2}{f_4} \right)_x \left( \sum_{h_2 \in \mathbb{F}_q[t]/f_2} \left( \frac{h_2}{f_2} \right)_x \psi \left( \text{res} \left( \frac{h_2f_1f_3}{f_2} \right) \right) \right) \left( \sum_{h_4 \in \mathbb{F}_q[t]/f_4} \left( \frac{h_4}{f_4} \right)_x \right) \]
\[ = \left( \frac{f_2}{f_4} \right)_x \left( \frac{f_2}{f_4} \right)_x g_\chi(f_1, f_2) g_\chi(f_3, f_4). \]

Applying Lemma 2.6 to each factor, we get (2.5). \[ \Box \]

Another identity to evaluate Gauss sums will help compare with the work of Chinta and Mohler.

Lemma 2.8. For \( \chi \) of order \( n \) and \( \pi \) prime, we have
\[ g_\chi(\pi^{d_1}, \pi^{d_2}) = \begin{cases} 1 & \text{if } d_2 = 0 \\ (q^{\deg \pi} - 1)q^{(d_2 - 1)\deg \pi} & \text{if } d_2 \equiv 0 \text{ mod } n \text{ and } d_1 \geq d_2 \\ 0 & \text{if } d_2 \not\equiv 0 \text{ mod } n \text{ and } d_1 \geq d_2 \\ -q^{(d_2 - 1)\deg \pi} \left( \frac{\pi'}{\pi} \right)^{d_2} (-G(\chi^{d_2}, \psi))^{\deg \pi} & \text{if } d_1 = d_2 - 1 \\ 0 & \text{if } d_1 < d_2 - 1 \end{cases} \]
Proof. We begin by noting

\[(2.6) \quad g_\chi(\pi^{d_1}, \pi^{d_2}) = \sum_{h \in \mathbb{F}_q[t]/\pi^{d_2}} \left( \frac{h}{\pi^{d_2}} \right)_\chi \psi \left( \text{res} \left( h\pi^{d_1-d_2} \right) \right). \]

First, if \(d_2 = 0\), the sum \((2.6)\) has a single term and equals 1. Second, \(\left( \frac{h}{\pi^{d_2}} \right)_\chi\) depends only on \(h \mod \pi\), so if \(d_1 < d_2 - 1\), the \(\psi\) term cancels in each residue class \(\mod \pi\) and so the sum \((2.6)\) vanishes. If \(d_1 \geq d_2\), the \(\psi\) term can be ignored and the sum \((2.6)\) vanishes because the multiplicative character cancels, unless \(d_2 \equiv 0 \mod n\), in which case the summand is 1 if \(h\) is prime to \(\pi\) and 0 otherwise, and the value of the sum \((2.6)\) is simply the number of \(h\) prime to \(\pi\), which is \((q^{\deg \pi} - 1)q^{(d_2-1)\deg \pi}\).

If \(d_1 = d_2 - 1\), the sum \((2.6)\) is equal to

\[q^{(d_2-1)\deg \pi} \sum_{h \in \mathbb{F}_q[t]/\pi} \left( \frac{h}{\pi} \right)_\chi^{d_2} \psi \left( \text{res} \left( \frac{h}{\pi} \right) \right) = q^{(d_2-1)\deg \pi} g_\chi^{d_2}(1, \pi)\]

by Lemma 2.4 and (2.3), verifying the last remaining case. \(\square\)

2.3. \(\ell\)-adic sheaves. We have the following basic properties of \(\text{IC}_{L_\chi}(f)\).

Lemma 2.9. \quad (1) For \(f\) a function on \(X\) and \(g\) an invertible function on \(X\),

\[\text{IC}_{L_\chi}(fg) \cong \text{IC}_{L_\chi}(f) \otimes L_\chi(g).\]

(2) For \(s: X \to Y\) a smooth map and \(f\) a function on \(Y\),

\[\text{IC}_{L_\chi}(f \circ s) \cong s^* \text{IC}_{L_\chi}(f).\]

(3) For \(X\) and \(Y\) two varieties, \(f\) a function on \(X\) and \(g\) a function on \(Y\),

\[\text{IC}_{L_\chi}(x,y) \to f(x), g(y)) \cong \text{IC}_{L_\chi}(f) \boxtimes \text{IC}_{L_\chi}(g).\]

(4) For \(f\) a function on \(X\), with \(X\) of dimension \(d\), and \(D\) the Verdier dual,

\[\text{DIC}_{L_\chi}(f) \cong \text{IC}_{L_{\chi-1}(f)}[2d](d).\]

Proof. These all are proved by combining a basic property of middle extension with a property of the sheaves \(L_\chi\) that follows in a straightforward way from their definition.

(1) follows from the fact that middle extension is compatible with tensor product with lisse sheaves, and the fact that \(L_\chi(f) \otimes L_\chi(g) = L_\chi(fg)\).

(2) follows from the fact that middle extension is compatible with smooth pullback (once shifts are taken into account) and \(s^*L_\chi(f) = L_\chi(f \circ s)\).

(3) follows from the fact that both middle extension and \(L_\chi\) are compatible with \(\boxtimes\).

(4) follows from the fact that middle extension is compatible with Verdier duality and \(L_\chi\) is dual to \(L_{\chi-1}\) as a lisse sheaf, hence \(DL_\chi(f) = L_{\chi^{-1}}(f)[2d](d)\).

These middle extension compatibilities follow from the, even more standard, compatibilities of \(j_!\) and \(j_*\) with these operations. \(\square\)
We need also a slightly more complicated observation along the same lines. First, we define and describe the notion of a tensor direct image of sheaves, building on the notion of a tensor direct image of sheaves defined by Rojas-León [2020].

**Definition 2.10.** Let $k'/k$ be a finite Galois field extension. Let $X$ be a variety over $k'$. The Weil restriction $WR_{k'}^k X$ is defined as the variety over $k$ whose $R$ points for a $k$-algebra $R$ are the $R \otimes_k k'$-points of $X$.

For $R$ a $k'$-algebra, the natural map $R \otimes_k k' \to R$ defines a map from $R$-points of $WR_{k'}^k X$ to $R$-points of $X$, defining a map $\rho: (WR_{k'}^k X)_{k'} \to X$.

Let $\tau: (WR_{k'}^k X)_{k'} \to WR_{k'}^k X$ be the natural map.

For $K'$ a complex on $(WR_{k'}^k X)_{k'}$, Rojas-León [2020, Definition 2 on p. 133] defines the tensor direct image $\pi_{\otimes_k^*} K'$ as the unique complex on $WR_{k'}^k X$ whose pullback to $(WR_{k'}^k X)_{k'}$ is isomorphic to $\bigotimes_{\tau \in \text{Gal}(k'/k)} \tau^* K'$ where the natural action of $\text{Gal}(k'/k)$ on the pullback is equal to the natural action of $\text{Gal}(k'/k)$ permuting the factors (which exists and is unique by Rojas-León [2020, Proposition 8 on p. 133]).

For $K$ a complex on $X_{k'}$, define the Weil restriction $WR_{k'}^k K$ by

$$WR_{k'}^k K = \pi_{\otimes_k^*} \rho^* K.$$ 

**Remark 2.11.** Note that this definition uses complexes of sheaves rather than the derived category of sheaves because the descent argument needed to prove existence and uniqueness would, in the derived category, require checking higher compatibilities of the action. If $K$ is an ordinary sheaf, or a perverse sheaf, up to shift, these subtleties can be avoided, as these categories satisfy étale descent. We will only apply this in the case of perverse sheaves up to shift.

**Lemma 2.12.** Let $X$ be a variety over $\mathbb{F}_{q^d}$. Let $K$ be a perverse sheaf on $X$. Then the trace of Frobenius on the stalk of $WR_{\mathbb{F}_{q^d}}^\mathbb{F}_{q^d} K$ at a $\mathbb{F}_{q^d}$-point is equal to the trace of Frobenius on the stalk of $K$ at the corresponding $\mathbb{F}_{q^d}$-point, using the natural bijection $X(\mathbb{F}_{q^d}) = WR_{\mathbb{F}_{q^d}}^\mathbb{F}_{q^d} X(\mathbb{F}_{q^d})$.

**Proof.** By definition and Rojas-León [2020, Proposition 9 on p. 133], the trace of Frobenius on the stalk of $WR_{\mathbb{F}_{q^d}}^\mathbb{F}_{q^d} K$ at a $\mathbb{F}_{q^d}$-point $x$ is the trace of Frobenius on the stalk of $\rho^* K$ on the $\mathbb{F}_{q^d}$-point $\pi^{-1}(x)$. The stalk of $\rho^* K$ at $\pi^{-1}(x)$ is the stalk of $K$ at $\rho(\pi^{-1}(x))$, which is the corresponding $\mathbb{F}_{q^d}$-point of $X$. \qed

**Lemma 2.13.** Let $k'/k$ be a finite Galois field extension of fields containing $\mu_n$. Let $X$ be a variety over $k'$ and $f$ a function on $X$. Let $WR_{k'}^k X$ be the Weil restriction from $k'$ to $k$ of $X$. The function $f$ on $X$ induces a map $WR_{k'}^k X \to WR_{k'}^k \mathbb{A}^1$, which we can compose with the norm map $WR_{k'}^k \mathbb{A}^1 \to \mathbb{A}^1$ to obtain a function $N f$ on $WR_{k'}^k X$. Let $WR_{k'}^k IC_{\mathcal{L}_{\chi}(f)}$ be the Weil restriction of $IC_{\mathcal{L}_{\chi}(f)}$. Then

$$WR_{k'}^k IC_{\mathcal{L}_{\chi}(f)} \cong IC_{\mathcal{L}_{\chi}(N f)}.$$ 

**Proof.** Since $k'/k$ is Galois, we have an isomorphism

$$(WR_{k'}^k X)_{k'} = \prod_{\tau \in \text{Gal}(k'/k)} X$$
with the projection onto the \(\tau\)’th factor given by \(\rho \circ \tau\).

By definition the pullback of \(WR^k_\kappa IC_{\mathcal{L}_\chi(f)}\) to \(k’\) is given by

\[
\bigotimes_{\tau \in \text{Gal}(k'/k)} \tau^* \rho^* IC_{\mathcal{L}_\chi(f)} = \bigotimes_{\tau \in \text{Gal}(k'/k)} IC_{\mathcal{L}_\chi(f)} = IC_{\mathcal{L}_\chi(k' \circ \rho \circ \tau)} = IC_{\mathcal{L}_\chi(Nf)}
\]

by Lemma 2.7(3) and the identity \(\prod_{\tau \in \text{Gal}(k'/k)} f \circ \rho \circ \tau = Nf\) on \((WR^k_\kappa X)_{k’}\). So the two complexes in 2.7 are isomorphic after pullback to \(k’\).

Since \(IC_{\mathcal{L}_\chi(Nf)}\) is the middle extension of a lisse sheaf of rank one, it follows that \(WR^k_\kappa IC_{\mathcal{L}_\chi(f)}\) is the middle extension of a lisse sheaf of rank one as well. To check they are isomorphic over \(k\), it suffices to check that the lisse sheaves are isomorphic, for which, because they are isomorphic over \(k’\), it suffices to check that their stalks at a single point are isomorphic as Galois representations.

For \(WR^k_\kappa IC_{\mathcal{L}_\chi(f)}\), the stalk at a geometric point \(x \in WR^k_\kappa X\) where \(Nf\) is nonzero is naturally the tensor product of a one-dimensional vector space for each \(\tau \in \text{Gal}(k'/k)\), and on each one-dimensional vector space the action is the same as on an \(n’\)th power of \(f(\rho(\tau(x)))\). For \(IC_{\mathcal{L}_\chi(Nf)}\), the Galois action is the same as the Galois action on the \(n\)th root of \(Nf(x)\). Because \(Nf(x) = \prod_{\tau \in \text{Gal}(k'/k)} f(\rho(\tau(x)))\), and the \(n\)th root of the product is the product of the \(n\)th roots of the factors, these are the same. \(\Box\)

**Lemma 2.14.** Let \(X\) be a variety with an action of \(\mathbb{G}_m\) described by a map \(a: X \times \mathbb{G}_m \to X\). Let \(f\) be a function on \(X\) and \(r\) an integer such that \(f(a(x, \lambda)) = f(x)\lambda^r\) for all \(x \in X\) and \(\lambda \in \mathbb{G}_m\).

If \(r\) is divisible by \(n\), then \(IC_{\mathcal{L}_\chi(f)}\) is \(\mathbb{G}_m\)-invariant, in the sense that \(a^* IC_{\mathcal{L}_\chi(f)} = IC_{\mathcal{L}_\chi(f)} \boxtimes \mathbb{Q}_\ell\). In particular, this always happens if we compose \(a\) with the \(n\)th power homomorphism \(\mathbb{G}_m \to \mathbb{G}_m\).

If \(r\) is not divisible by \(n\), then the stalk of \(IC_{\mathcal{L}_\chi(f)}\) vanishes at every \(\mathbb{G}_m\)-invariant point.

**Proof.** Because \(a\) is smooth, we have by Lemma 2.7(2,3)

\[a^* IC_{\mathcal{L}_\chi(f)} \cong IC_{\mathcal{L}_\chi(f(a) \circ a)} \cong IC_{\mathcal{L}_\chi(f(x)\lambda^r)} \cong IC_{\mathcal{L}_\chi(f)} \boxtimes IC_{\mathcal{L}_\chi(\lambda^r)}.\]

If \(r\) is divisible by \(n\), then \(IC_{\mathcal{L}_\chi(\lambda^r)}\) is the middle extension of the constant sheaf, hence is simply the constant sheaf.

If \(r\) is not divisible by the order of \(\chi\), then restricting this identity to \(P \times \mathbb{G}_m\) for a \(\mathbb{G}_m\)-fixed point \(P\), we have \((IC_{\mathcal{L}_\chi(f)})_P \boxtimes \mathbb{Q}_\ell = (IC_{\mathcal{L}_\chi(f)})_P \boxtimes \mathcal{L}_\chi(\lambda^r)\). Because one side has trivial monodromy and the other nontrivial, they cannot be isomorphic unless they both vanish. \(\Box\)

**Lemma 2.15.** Let \(B\) be a scheme of finite type over a field, \(Y = B \times \mathbb{A}^1\), \(u: B \times \mathbb{G}_m \to B \times \mathbb{A}^1\) the inclusion, \(\pi: B \times \mathbb{A}^1 \to B\) the projection, \(K\) a complex on \(B \times \mathbb{G}_m\), and \(N \neq 0\) an integer.

Assume that \(K\) is invariant for the action of \(\mathbb{G}_m\) on \(B \times \mathbb{G}_m\) given by \(a((b, \lambda_1), \lambda_2) = (b, \lambda_1 \lambda_2^N)\) for all \(b \in B, \lambda_1, \lambda_2 \in \mathbb{G}_m\).

Then \(\pi_* u_! K = 0\).

**Proof.** Let \(\pi: \mathbb{A}^1 \to \text{pt}\) be the projection and \(\pi: \mathbb{G}_m \to \mathbb{A}^1\) the inclusion, so that \(u = \text{id} \times \pi\) and \(\pi = \text{id} \times \pi\). Let \(\rho: \mathbb{G}_m \to \mathbb{G}_m\) be the \(N\)th power map. Let \(i: \text{pt} \to \mathbb{G}_m\) be the inclusion of the identity.
Restricting the $\mathbb{G}_m$-invariance property to the locus where $\lambda = 1$, we see that $(id \times \rho)^* K = ((id \times i)^* K) \boxtimes \mathbb{Q}_\ell$. Since $\rho$ is finite, it follows that $K$ is a summand of $(id \times \rho)_*((id \times i)^* K) \boxtimes \mathbb{Q}_\ell$, so it suffices to prove the vanishing of

$$\pi_* u_! (id \times \rho)_*(((id \times i)^* K) \boxtimes \mathbb{Q}_\ell) = (id \times \pi)_* (id \times u_!) (id \times \rho)_*(((id \times i)^* K) \boxtimes \mathbb{Q}_\ell).$$

But by the Künneth formula in the form [Fu, 2015, Corollary 9.3.5], together with its compactly supported version [Fu, 2015, Corollary 7.4.9], we have

$$(id \times \pi)_* (id \times u_!) (id \times \rho)_*(((id \times i)^* K) \boxtimes \mathbb{Q}_\ell) = (id \times \pi)_* (id \times u_!) (id \times \rho)_*(((id \times i)^* K) \boxtimes \rho_* \mathbb{Q}_\ell)$$

$$= (id \times \pi)_* (((id \times i)^* K) \boxtimes \pi_* u_! \rho_* \mathbb{Q}_\ell) = ((id \times i)^* K) \boxtimes \pi_* \pi_* \rho_* \mathbb{Q}_\ell.$$ 

Thus it suffices to prove

$$\pi_* \pi_* \rho_* \mathbb{Q}_\ell = 0,$$

but $\pi_* \pi_* \rho_* \mathbb{Q}_\ell$ is a complex on a point, given by the cohomology groups $H^*(\mathbb{A}^1, \pi_* \rho_* \mathbb{Q}_\ell)$.

By Artin’s theorem [Fu, 2015, Corollary 7.5.2], this cohomology vanishes in all degrees but zero and one. All global sections of $\pi_* \rho_* \mathbb{Q}_\ell$ vanish at zero, hence vanish in a neighborhood of zero, hence vanish everywhere because $\pi_* \rho_* \mathbb{Q}_\ell$ is lisse away from zero, so $H^0$ vanishes. By the Grothendieck-Ogg-Shafarevich Euler characteristic formula [Illusie, 1977, Theorem 7.1], the Euler characteristic of $\pi_* \rho_* \mathbb{Q}_\ell$ is zero, so $H^1$ vanishes as well. \qed

It is a classical fact that the ring of functions on an affine scheme with a $\mathbb{G}_m$-action is a $\mathbb{Z}$-graded ring (and the graded structure is equivalent to the $\mathbb{G}_m$ action). Indeed, for $R$ the ring of functions, the action of $\mathbb{G}_m$ defines a ring homomorphism $a^*: R \to R[\lambda, \lambda^{-1}]$ and one can define $R_d$ for $d \in \mathbb{Z}$ as the set of $x$ with $a^*(x) = x\lambda^d$. Then if we write $a^*(x) = \sum_{d \in \mathbb{Z}} x_d \lambda^d$ with all but finitely many of the $x_d$ zero, associativity of the action implies $x_d \in R_d$ and identity implies $x = \sum_d x_d$ so that $R = \bigoplus_d R_d$, and the fact that $a^*$ is a ring homomorphism implies that $R_{d_1} \cdot R_{d_2} \subseteq R_{d_1 + d_2}$.

**Lemma 2.16.** Let $X$ be an affine scheme of finite type over a field with a $\mathbb{G}_m$-action. Equivalently, the ring of functions on $X$ is a graded ring.

Assume that all nonconstant homogeneous functions on $X$ have positive degree. Let $P$ be the unique $\mathbb{G}_m$-fixed point of $X$ (corresponding to the ideal generated by homogenous functions of positive degree). Let $K$ be a $\mathbb{G}_m$-invariant complex on $X$. Then

$$H^*(X, K) \cong K_P.$$

**Proof.** Let $x_1, \ldots, x_n$ be generators of the ring of functions on $X$ of degrees $d_1, \ldots, d_n$. Let $d$ be the least common multiple of $d_1, \ldots, d_n$. Then $(x_1^{d/d_1}, \ldots, x_n^{d/d_n})$ defines a finite $\mathbb{G}_m$-equivariant map from $X$ to $\mathbb{A}^n$, where $\mathbb{G}_m$ acts on $\mathbb{A}^n$ by multiplying all coordinates by the $d$th power. Because the map is finite, and $P$ is the unique point in the inverse image of $0 \in \mathbb{A}^n$, both $H^*(X, K)$ and $K_P$ are preserved by pushing forward along this map, and because this map is $\mathbb{G}_m$-equivariant, the $\mathbb{G}_m$-invariance is preserved. So we can reduce to the case where $X = \mathbb{A}^n$.

Let $j$ be the inclusion from $\mathbb{A}^n - \{0\}$ to $\mathbb{A}^n$. From the excision exact sequence $j_! j^* K \to K \to K_P$, it suffices to prove $H^*(\mathbb{A}^n, j_! j^* K) = 0$. Let $Y$ be the blowup of $\mathbb{A}^n$ at the origin,
let \( u: \mathbb{A}^n - \{0\} \to Y \) be the inclusion, \( b: Y \to \mathbb{A}^n \) the blowup map, and \( \pi: Y \to \mathbb{P}^{n-1} \) the projection onto the exceptional fiber. We have \( j = b \circ u \) and \( b \) is proper so
\[
H^*(\mathbb{A}^n, j_! j^* K) = H^*(\mathbb{A}^n, b_* u_! j^* K) = H^*(\mathbb{A}^n, b_! u_! j^* K) = H^*(Y, u_! j^* K) = H^*(\mathbb{P}^{n-1}, \pi_* u_! j^* K).
\]

Thus it suffices to show that \( \pi_* u_! K' \) is zero for a \( \mathbb{G}_m \)-equivariant sheaf \( K' \) on \( \mathbb{A}^n - \{0\} \). Locally on \( \mathbb{P}^{n-1} \), \( Y \) is an \( \mathbb{A}^1 \)-bundle, \( \pi \) the structure map, \( u \) the inclusion of the complement of the 0 section, and the \( \mathbb{G}_m \) action is by multiplication by the \( d \)th power. To prove this vanishing, we work locally on \( \mathbb{P}^{n-1} \), where we are in the setting of Lemma 2.15. We take \( B \) to be an open subset of \( \mathbb{P}^{n-1} \) where this bundle can be trivialized and let \( K \) be the pullback of \( K' \) along this trivialization. By Lemma 2.15 \( \pi_* u_! K' = 0 \). \( \square \)

### 3. Proofs of the Axioms

We are now ready to check that the function \( a \) satisfies the axioms of Theorem 1.1.

**Lemma 3.1.** If \( f_1, \ldots, f_r \) and \( g_1, \ldots, g_r \) satisfy \( \gcd(f_i, g_j) = 1 \) for all \( i \) and \( j \), then we have
\[
a(f_1 g_1, \ldots, f_r g_r; q, \chi, M) = a(f_1, \ldots, f_r; q, \chi, M) a(g_1, \ldots, g_r; q, \chi M) \prod_{1 \leq i \leq r} \left( \frac{f_i}{g_i} \right)^{M_{ii}} \prod_{1 \leq i < j \leq r} \left( \frac{f_i}{g_j} \right)^{M_{ij}} \prod_{1 \leq i < j \leq r} \left( \frac{g_j}{f_i} \right)^{M_{ij}}.
\]

**Proof.** Let \( d_i = \deg f_i \) and \( e_i = \deg g_i \). Consider the map \( \mu: \prod_{i=1}^r \mathbb{A}^{d_i} \times \prod_{i=1}^r \mathbb{A}^{e_i} \to \prod_{i=1}^r \mathbb{A}^{d_i + e_i} \) defined by polynomial multiplication.

Observe that
\[
\text{Res}(f_i g_i; f_j g_j) = \text{Res}(f_i, f_j) \text{Res}(g_i, g_j) \text{Res}(f_i, f_j) \text{Res}(g_i, g_j)
\]
and by Lemma 2.5
\[
\text{Res}((f_i g_i)', f_j g_i) = \text{Res}(f_i', f_i) \text{Res}(g_i', g_i) \text{Res}(f_i, g_i) \text{Res}(g_i, f_i).
\]
Hence, letting
\[
G = \prod_{1 \leq i \leq r} \text{Res}(f_i, g_i)^{M_{ii}} \text{Res}(g_i, f_i)^{M_{ii}} \prod_{1 \leq i < j \leq r} \text{Res}(f_i, g_j)^{M_{ij}} \text{Res}(g_i, f_j)^{M_{ij}}
\]
we have
\[
(3.1) \quad F(f_1 g_1, \ldots, f_r g_r) = F(f_1, \ldots, f_r) F(g_1, \ldots, g_r) G.
\]

Let \( U \) be the open set \( U \subseteq \prod_{i=1}^r \mathbb{A}^{d_i} \times \prod_{i=1}^r \mathbb{A}^{e_i} \) where \( \gcd(f_i, g_j) = 1 \) for all \( i \) and \( j \). Certainly, \( G \) has no zeroes or poles on \( U \) as the resultants are all nonvanishing on \( U \).

Let us check that \( \mu \) is étale on \( U \). The derivative of \( \mu \) at a point \( f_1, \ldots, f_r, g_1, \ldots, g_r \) is the linear map that sends a tangent vector \( df_1, \ldots, df_r, dg_1, \ldots, dg_r \), where \( df_i \) is a polynomial of degree \( < d_i \) and \( dg_i \) is a polynomial of degree \( < e_i \), to the vector \( g_1 df_1 + f_1 dg_1, \ldots, g_r df_r + f_r dg_r \). The derivative is injective unless there exist polynomials \( df_1, \ldots, df_r, dg_1, \ldots, dg_r \), not all zero, such that \( g_i df_i = -f_i dg_i \) for all \( i \). If \( f_i \) and \( g_i \) do not share a common root, this equation implies that \( f_i \) divides \( df_i \) which, since \( \deg df_i < d_i = \deg f_i \), implies that \( df_i = 0 \), and similarly with \( dg_i \). Thus the derivative is injective on \( U \). Since, restricted to \( U \), \( \mu \) is a map between smooth varieties of the same dimension with injective derivative, it is étale.
In particular, \( \mu \) is smooth on \( U \). Hence by applying Lemma 2.9(1,2,3) to (3.1), we obtain an isomorphism

\[
\mu^*K_{d_1+e_1,\ldots,d_r+e_r} \cong (K_{d_1,\ldots,d_r} \boxtimes K_{e_1,\ldots,e_r}) \otimes \mathcal{L}_\chi(G)
\]
on \( U \). Taking trace functions of both sides, and applying Lemma 2.11 to evaluate the trace function \( \chi(G) \) of \( \mathcal{L}_\chi(G) \), we get the stated identity. \( \square \)

**Lemma 3.2.** \( a(1,\ldots,1,f,1,\ldots,1;q,\chi,M) = 1 \) for all linear polynomials \( f \).

**Proof.** In this case, all resultants and discriminants are 1, so \( F = 1 \), thus \( K_{0,\ldots,0,1,0,\ldots,0} \) is the constant sheaf \( \mathbb{Q}_\ell \), hence its trace function is 1. \( \square \)

To check the remaining axioms, it will be useful to describe the translation and dilation symmetries of the function \( F \).

**Lemma 3.3.**

1. We have

\[
F(\lambda^{d_1}f_1(x/\lambda),\ldots,\lambda^{d_r}f_r(x/\lambda)) = \lambda^{\sum_{i=1}^r d_i(d_i-1)M_{ii}+\sum_{1 \leq i < j \leq r} d_id_jM_{ij}} F(f_1,\ldots,f_r).
\]

2. All nonconstant polynomials on \( \prod_{i=1}^r \mathbb{A}^{d_i} \), which are homogeneous for the action of \( \mathbb{G}_m \) on \( \prod_{i=1}^r \mathbb{A}^{d_i} \) which acts by dilation of polynomials, i.e. \( f_i \to \lambda^{d_i} f_i(x/\lambda) \), have positive degree in \( \lambda \).

**Proof.** (1) follows from the definition of the resultant of two monic polynomials as a product of differences of their roots, since dilation multiplies each root by \( \lambda \), hence each difference of roots by \( \lambda \), and thus a product of \( N \) differences of roots by \( \lambda^N \).

(2) holds because the ring of functions is generated by the coefficients of \( f_i \), which all have positive degree in \( \lambda \). \( \square \)

**Lemma 3.4.** The complex \( K_{d_1,\ldots,d_r} \) is invariant under the action of \( \mathbb{G}_a \) on \( \prod_{i=1}^r \mathbb{A}^{d_i} \) given by \( ((f_1,\ldots,f_r),\alpha) \mapsto (f_1(T+\alpha),\ldots,f_r(T+\alpha)) \).

**Proof.** This follow from Lemma 2.9 and the identity

\[
F_{d_1,\ldots,d_r}(f_1(T+\alpha),\ldots,f_r(T+\alpha)) = F_{d_1,\ldots,d_r}(f_1,\ldots,f_r)
\]

which is immediate from the definition of \( F_{d_1,\ldots,d_r} \). \( \square \)

For each finite field \( \mathbb{F}_q \), character \( \chi \), and natural numbers \( d_1,\ldots,d_r \), let \( J(d_1,\ldots,d_r;q,\chi,M) \) be the finite set of ordered pairs of Weil numbers and integers given by the eigenvalues of \( \text{Frob}_q \) on the stalk of \( K_{d_1,\ldots,d_r} \) at \( (T^{d_1},\ldots,T^{d_r}) \), together with their signed multiplicities.

**Lemma 3.5.** For each tuple of natural numbers \( d_1,\ldots,d_r \), the function \( (d_1,\ldots,d_r) \mapsto J(d_1,\ldots,d_r;q,\chi,M) \) is a compatible system of sets of ordered pairs of Weil numbers and integers.

**Proof.** Let \( \mathbb{F}_q \) be a finite field, \( \chi \) a character, \( \mathbb{F}_{q^m} \) a field extension, and \( \chi_m \) the composition of \( \chi \) with the norm map. We can construct \( K_{d_1,\ldots,d_r} \) on \( \prod_{i=1}^r \mathbb{A}^{d_i}_{\mathbb{F}_q} \) using the character \( \chi \) and then pull back to \( \prod_{i=1}^r \mathbb{A}^{d_i}_{\mathbb{F}_{q^m}} \), or we can construct the sheaf directly on \( \prod_{i=1}^r \mathbb{A}^{d_i}_{\mathbb{F}_{q^m}} \) using the character \( \chi_m \). These two sheaves are naturally isomorphic because the \( q^{m-1} \)-th-power map \( \mu_{q^m-1} \to \mu_{q-1} \) used to compare the Kummer sheaves matches the norm map \( \mathbb{F}_{q^m}^\times \to \mathbb{F}_q^\times \) given by

\[
N(x) = x \cdot \text{Frob}_q(x) \cdot \ldots \cdot \text{Frob}_q^{m-1}(x) = x \cdot x^q \cdot \ldots \cdot x^{q^{m-1}} = x^{q^m-1}
\]
used to convert \( \chi \) to \( \chi_m \), and because forming the intermediate extension commutes with change of base field.

The elements of \( J(d_1, \ldots, d_r; q^m, \chi_m, M) \) are the eigenvalues of \( \text{Frob}_{q^m} \), with multiplicity, on the stalk of \( K_{d_1, \ldots, d_r} \), constructed over \( \mathbb{F}_{q^m} \) using \( \chi_m \), and hence also the eigenvalues of \( \text{Frob}_{q^m} \) on the stalk of \( K_{d_1, \ldots, d_r} \), constructed over \( \mathbb{F}_q \) and pulled back to \( \mathbb{F}_{q^m} \). Since \( \text{Frob}_{q^m} = \text{Frob}_q \), the elements of \( J(d_1, \ldots, d_r; q^m, \chi_m, M) \) are the \( m \)'th powers, with multiplicity, of the eigenvalues of \( \text{Frob}_q \) on the stalk of \( K_{d_1, \ldots, d_r} \), i.e. the \( m \)'th powers of the elements of \( J(d_1, \ldots, d_r; q, \chi, M) \), as desired. \( \square \)

**Lemma 3.6.** For each finite field \( \mathbb{F}_q \), character \( \chi \), natural numbers \( d_1, \ldots, d_r \), and prime polynomial \( \pi \) over \( \mathbb{F}_q \), we have

\[
(3.2) \quad a(\pi^{d_1}, \ldots, \pi^{d_r}; q, \chi, M) = \left( \frac{\pi'}{\pi} \right)^{\sum_{i=1}^r d_i M_{ii}} \sum_{j \in J(d_1, \ldots, d_r; q, \chi, M)} c_j \alpha_j^{\deg \pi}.
\]

**Proof.** This follows from the definition when \( \pi = T \), and then follows from Lemma 3.4 when \( \pi = T - x \) for \( x \in \mathbb{F}_q \).

Let us handle the case when \( \pi \) has a higher degree. To do this, let \( e \) be the degree of \( \pi \), and consider the Weil restriction \( W \mathbb{F}_q^R \prod_{i=1}^r A_i^{d_i} \) of \( \prod_{i=1}^r A_i^{d_i} \) from \( \mathbb{F}_{q^e} \) to \( \mathbb{F}_q \). This Weil restriction admits a map \( \text{norm} \) to \( \prod_{i=1}^r A_i^{ed_i} \) given by taking norms of polynomials. For \( x \) a root of \( \pi \), the image of \( ((T - x)^{d_1}, \ldots, (T - x)^{d_r}) \) under \( \text{norm} \) is \( (\pi^{d_1}, \ldots, \pi^{d_r}) \). Thus

\[
(3.3) \quad a(\pi^{d_1}, \ldots, \pi^{d_r}; q, \chi, M) = \text{tr} \left( \text{Frob}_q, (K_{\mathbb{F}_{q^e}} K_{d_1, \ldots, d_r})((T - x)^{d_1}, \ldots, (T - x)^{d_r}) \right)
\]

On the other hand, by Lemma 2.12

\[
(3.4) \quad \text{tr} \left( \text{Frob}_q, (W \mathbb{F}_q^R K_{d_1, \ldots, d_r})((T - x)^{d_1}, \ldots, (T - x)^{d_r}) \right) = \text{tr} \left( \text{Frob}_q, (K_{d_1, \ldots, d_r})((T - x)^{d_1}, \ldots, (T - x)^{d_r}) \right) = \sum_{j \in J(d_1, \ldots, d_r; q, \chi, M)} c_j \alpha_j^{\deg \pi}.
\]

To finish the argument, we will compare the stalks of \( W \mathbb{F}_q^R K_{d_1, \ldots, d_r} \) and \( \text{norm}^* K_{d_1, \ldots, d_r} \) at \( ((T - x)^{d_1}, \ldots, (T - x)^{d_r}) \). To do this, note from Lemma 2.13 that

\[
(3.5) \quad W \mathbb{F}_q^R K_{d_1, \ldots, d_r} \cong \text{IC}_{\mathbb{L}_x}(NF(f_1, \ldots, f_r)).
\]

The restriction of \( \text{norm} \) to the open set where none of the polynomials share any roots with their Galois conjugates is étale, so

\[
\text{norm}^* \text{IC}_{\mathbb{L}_x}(F_{d_1, \ldots, d_r}) = \text{IC}_{\mathbb{L}_x}(F_{d_1, \ldots, d_r} \circ \text{norm})
\]

by Lemma 2.3(2). Let \( N \) be the norm map on polynomials. By definition, we have identities

\[
(3.6) \quad F(Nf_1, \ldots, NF) = \prod_{1 \leq i \leq r} \left( \text{Res}(((Nf_j)')))_{M_{ii} \prod_{1 \leq i < j \leq r} (\text{Res}(Nf_i, Nf_j))}^{M_{ij}}
\]

\[
(3.7) \quad NF(f_1, \ldots, f_r) = \prod_{1 \leq i \leq r} (N \text{Res}(f_i'))^{M_{ii}} \prod_{1 \leq i < j \leq r} (N \text{Res}(f_i, f_j))^{M_{ij}}.
\]
For compactness, we will here write \( \text{Fr}_q \) for \( \text{Frob}_q \). The multiplicativity of the resultant gives

\[
(3.8) \quad \text{Res}(N f_i, N f_j) = N \text{Res}(f_i, f_j) \prod_{0 \leq t_1, t_2 \leq e-1 \atop t_1 \neq t_2} \text{Res}(\text{Fr}_q^{t_1} f_i, \text{Fr}_q^{t_2} f_j)
\]

and the multiplicativity property of \( \text{Res}(f', f) \) gives

\[
(3.9) \quad \text{Res}((N f_i)', N f_i) = N \text{Res}(f_i', f_i) \prod_{0 \leq t_1, t_2 \leq e-1 \atop t_1 \neq t_2} \text{Res}(\text{Fr}_q^{t_1} f_i, \text{Fr}_q^{t_2} f_i).
\]

Plugging (3.8) and (3.9) into (3.6) and then applying (3.7) we obtain

\[
(3.10) \quad F(N f_1, \ldots, N f_r)
= NF(f_1, \ldots, f_r) \prod_{1 \leq i \leq r} \left( \prod_{0 \leq t_1, t_2 \leq e-1 \atop t_1 \neq t_2} \text{Res}(\text{Fr}_q^{t_1} f_i, \text{Fr}_q^{t_2} f_i) \right)^{M_{ii}} \prod_{1 \leq i < j \leq r} \left( \prod_{0 \leq t_1, t_2 \leq e-1 \atop t_1 \neq t_2} \text{Res}(\text{Fr}_q^{t_1} f_i, \text{Fr}_q^{t_2} f_j) \right)^{M_{ij}}.
\]

Specializing to \( f_i = (T - x)^{d_i} \) we have

\[
(3.11) \quad \prod_{0 \leq t_1, t_2 \leq e-1 \atop t_1 \neq t_2} \text{Res}(\text{Fr}_q^{t_1} f_i, \text{Fr}_q^{t_2} f_j) = \prod_{0 \leq t_1, t_2 \leq e-1 \atop t_1 \neq t_2} \text{Res}(\text{Fr}_q^{t_1} (T - x)^{d_i}, \text{Fr}_q^{t_2} (T - x)^{d_j})
\]

\[
= \prod_{0 \leq t_1, t_2 \leq e-1 \atop t_1 \neq t_2} (\text{Fr}_q^{t_2} x - \text{Fr}_q^{t_1} x)^{d_id_j} = \text{Res}(\pi', \pi)^{d_id_j}.
\]

By similar logic

\[
(3.12) \quad \prod_{0 \leq t_1, t_2 \leq e-1 \atop t_1 \neq t_2} \text{Res}(\text{Fr}_q^{t_1} f_i, \text{Fr}_q^{t_2} f_i) = \text{Res}(\pi', \pi)^{d_i^2}.
\]

In combination, (3.10), (3.11), and (3.12) demonstrate that \( F(N f_1, \ldots, N f_r) \) is equal to \( NF(f_1, \ldots, f_r) \) times a polynomial function whose value at \( ((T - x)^{d_1}, \ldots, (T - x)^{d_r}) \) is

\[
(3.13) \quad \text{Res}(\pi', \pi)^{\sum_{i=1}^r M_{ii}d_i^2 + \sum_{1 \leq i < j \leq r} M_{ij}d_id_j} \neq 0.
\]

Thus, by Lemma 2.9(1) and (3.13) we have

\[
(3.14) \quad \text{tr} \left( \text{Frob}_q, \left( \text{norm}^* K_{e_1, \ldots, e_r} \right)_{(T - x)^{d_1}, \ldots, (T - x)^{d_r}} \right)
= \text{tr} \left( \left( \text{Frob}_q, \left( \text{Res}_q^{r} K_{d_1, \ldots, d_r} \right)_{(T - x)^{d_1}, \ldots, (T - x)^{d_r}} \right) \left( \frac{\pi'}{\pi} \right)^{\sum_{i=1}^r M_{ii}d_i^2 + \sum_{1 \leq i < j \leq r} M_{ij}d_id_j} \right). 
\]

By Lemma 3.3 and Lemma 2.14 unless

\[
(3.15) \quad \sum_{i=1}^r M_{ii}d_i(d_i - 1) + \sum_{1 \leq i < j \leq r} M_{ij}d_id_j \equiv 0 \mod n,
\]
the stalk of $K_{d_1, \ldots, d_r}$ at $((T - x)^{d_1}, \ldots, (T - x)^{d_r})$ vanishes, which by (3.4) means the right side of (3.14) vanishes, so the left side of (3.14) vanishes as well. It follows that

$$\text{tr} \left( \text{Frob}_q, \left( \text{norm}^* K_{e_1, \ldots, e_r} \right)_{(T - x)^{d_1}, \ldots, (T - x)^{d_r}} \right)$$

(3.16)

$$= \text{tr} \left( \left( \text{Frob}_q, (W \mathcal{R}G_{\pi, q}^0 K_{d_1, \ldots, d_r})_{(T - x)^{d_1}, \ldots, (T - x)^{d_r}} \right) \left( \frac{\pi'}{\pi} \right)^{\sum_{i=1}^r M_i d_i} x \right)$$

since if (3.15) is satisfied may subtract $\sum_{i=1}^r M_i d_i (d_i - 1) + \sum_{1 \leq i < j \leq r} M_{ij} d_i d_j$ from the exponent of $\left( \frac{\pi'}{\pi} \right)^x$ without changing the value because $\left( \frac{\pi'}{\pi} \right)^x$ is an $n$'th root of unity, and if (3.15) is not satisfied, then both sides are zero.

Combining (3.3), (3.16), and (3.4), we obtain (3.2). $\square$

**Lemma 3.7.** For each finite field $\mathbb{F}_q$, character $\chi$, natural numbers $d_1, \ldots, d_r$, setting $d = \sum_{i=1}^r d_i$ we have

$$\lambda(d_1, \ldots, d_r, q^m, \chi, M) = \sum_{j \in J(d_1, \ldots, d_r, q^m, \chi, M)} c_j \frac{q^d}{\alpha_j^j}.$$ 

**Proof.** If we compose the $G_m$ action by dilation (Lemma 3.3) with the $n$th power map $G_m \to G_m$, the factor $\lambda \sum_{i=1}^r d_i (d_i - 1)^M \sum_{1 \leq i < j \leq r} d_i d_j$ becomes an $n$th power, and so $K_{d_1, \ldots, d_r}$ is preserved by this $G_m$ action by Lemma 2.14. Hence the Verdier dual $DK_{d_1, \ldots, d_r}$ is also preserved.

By Verdier duality, $H^i \left( \prod_{i=1}^r \mathbb{A}^d_{\mathbb{F}_q}, K_{d_1, \ldots, d_r} \right)$ is dual to $H^{-i} \left( \prod_{i=1}^r \mathbb{A}^d_{\mathbb{F}_q}, DK_{d_1, \ldots, d_r} \right)$ which by Lemma 2.16 using Lemma 3.3(2) to check the condition, is $H^{-i} \left( (DK_{d_1, \ldots, d_r})_{(\pi, 1, \ldots, T^{nr})} \right)$.

Because $K_{d_1, \ldots, d_r}$ is pure of weight zero on the open set where it is lisse, and $K_{d_1, \ldots, d_r} [d]$ is perverse, $K_{d_1, \ldots, d_r} [d]$ is perverse and pure of weight $d$, so by a theorem of Gabber [Fujisawa, 2002], the trace of Frobenius on each stalk of $DK_{d_1, \ldots, d_r}$ is the complex conjugate of the trace of Frobenius on the stalk of $K_{d_1, \ldots, d_r}$ divided by $q^d$. Because this applies over each finite field extension, the Frobenius eigenvalues on the stalk of $DK_{d_1, \ldots, d_r}$ at any point are equal to the complex conjugates, divided by $q^d$, of the Frobenius eigenvalues of $K_{d_1, \ldots, d_r}$ at the same point, at least up to signed multiplicity. So the eigenvalues of Frobenius on $H^{-i} \left( (DK_{d_1, \ldots, d_r})_{(\pi, 1, \ldots, T^{nr})} \right)$ are $\overline{\frac{\pi'}{\pi}}^d$, with signed multiplicities $c_j$.

By definition, the Grothendieck-Lefschetz fixed point formula, and the above isomorphisms, dualities, and eigenvalue calculations, we have

$$\lambda(d_1, \ldots, d_r; q, \chi, M) = \sum_{f_1, \ldots, f_r \in \mathbb{F}_q[t]^{\frac{1}{d}}} \sum_{\deg f_1 = 1, \ldots, \deg f_r = d_r} (-1)^i \text{tr}(\text{Frob}_q, H^i(K_{d_1, \ldots, d_r} f_1, \ldots, f_r))$$

$$= \sum_i (-1)^i \text{tr} \left( \text{Frob}_q, H^i \left( \prod_{i=1}^r \mathbb{A}^d_{\mathbb{F}_q} K_{d_1, \ldots, d_r} \right) \right) = \sum_i (-1)^i \text{tr} \left( \text{Frob}_q^{-1}, H^{-i} \left( \prod_{i=1}^r \mathbb{A}^d_{\mathbb{F}_q} DK_{d_1, \ldots, d_r} \right) \right)$$

$$= \sum_i (-1)^i \text{tr} \left( \text{Frob}_q^{-1}, H^{-i} \left( (DK_{d_1, \ldots, d_r})_{(\pi, 1, \ldots, T^{nr})} \right) \right) = \sum_j c_j \left( \frac{\alpha_j}{q^d} \right)^{-1} = \sum_j c_j \frac{q^d}{\alpha_j}. \quad \square
Lemma 3.8. For each finite field $\mathbb{F}_q$, character $\chi$, natural numbers $d_1, \ldots, d_r$, and $(\alpha_j, c_j) \in J(d_1, \ldots, d_r; q, \chi, M)$ we have $|\alpha_j| < q^{\frac{d_j^2}{2}}$ as long as $d \geq 2$ where $d = \sum_{i=1}^r d_i$.

Proof. Because $K_{d_1, \ldots, d_r}$ is the IC sheaf of a lisse sheaf, its $H^i$ is supported in codimension at least $i + 1$ for all $i > 0$ [Beilinson et al., 1982, Proposition 2.1.11]. By Lemma 3.4, any stalk cohomology at a point must also occur at its one-dimensional orbit under the action of $G_a$ by translation, hence with codimension $\leq d - 1$, thus in degree $\leq d - 2$, as long as $d \geq 2$. So because intersection cohomology complexes are pure, any Frobenius eigenvalues that appear are $\leq q^{\frac{d_j^2}{2}}$.

Proof of Theorem (1.1). In view of Lemmas 3.1, 3.2, 3.5, 3.6, 3.7 and 3.8 it suffices to prove that the function $a$ is uniquely determined by these axioms.

In fact we will show that $J(d_1, \ldots, d_r; q, \chi, M)$ is determined by these axioms whenever $d_1 + \cdots + d_r \leq d$, for all $d$. This will then determine $a$ by axioms (1) and (3).

We do this by induction on $d$. The cases $d = 0$ and $d = 1$ are determined by axiom (2) and the fact that there is at most one way of expressing a given function of a natural number $m$ as a finite signed sum of $m$th powers.

For the induction step, assume that $J(d_1, \ldots, d_r; q, \chi, M)$ is determined by these axioms whenever $d_1 + \cdots + d_r < d$. From axiom (3), this determines $a(\pi^{d_1}, \ldots, \pi^{d_r}; q, \chi, M)$ whenever $d_1 + \cdots + d_r < d$. From axiom (1), this determines $a(f_1, \ldots, f_r; q, \chi, M)$ whenever each prime factor of $\prod_{i=1}^r f_i$ occurs with multiplicity less than $d$.

Thus, if $\deg f_i = d_i$ and $d_1 + \cdots + d_r = d$, the axioms determine $a(f_1, \ldots, f_r; q, \chi, M)$ when $\prod_{i=1}^r f_i$ is not a $d$th power of a linear prime, i.e. in all cases but when $f_i$ is of the form $(T - x)^{d_i}$ for all $i$. By axioms (3) and (4) applied to $\mathbb{F}_{q^m}$ and $\chi_m$ we have

$$\sum_{f_1, \ldots, f_r \in \mathbb{F}_{q^m}[t]^+_{\deg f_i = d_i}} a(f_1, \ldots, f_r; q^m, \chi_m, M) - \sum_{x \in \mathbb{F}_{q^m}} a((T - x)^{d_1}, \ldots, (T - x)^{d_r}; q^m, \chi_m, M)$$

$$= \sum_{j \in J(d_1, \ldots, d_r; q^m, \chi_m, M)} c_j q^{\sum_{i=1}^r d_i} - q^m \sum_{j \in J(d_1, \ldots, d_r; q^m, \chi_m, M)} c_j \alpha_j,$$

However, by the compatibility of $J$, $J(d_1, \ldots, d_r; q^m, \chi_m, M)$ consists of the $m$th powers of $J(d_1, \ldots, d_r; q, \chi, M)$ so we obtain

$$(3.17)$$

$$\sum_{f_1, \ldots, f_r \in \mathbb{F}_{q^m}[t]^+_{\deg f_i = d_i}} a(f_1, \ldots, f_r; q^m, \chi_m, M) - \sum_{x \in \mathbb{F}_{q^m}} a((T - x)^{d_1}, \ldots, (T - x)^{d_r}; q^m, \chi_m, M)$$

$$= \sum_{j \in J(d_1, \ldots, d_r; q, \chi, M)} c_j \left(\frac{q^{\sum_{i=1}^r d_i}}{\alpha_j}\right)^m - \sum_{j \in J(d_1, \ldots, d_r; q, \chi, M)} c_j (q \alpha_j)^m.$$

We have already shown the left side of (3.17) is determined by the axioms for all $m$. The right side of (3.17) is a finite sum of $m$th powers of Weil numbers, so the Weil numbers appearing, and their multiplicity, are uniquely determined by the left side of (3.17). The only difficulty is whether any given Weil number occurs in the first term or the second term.
However, by axiom (5), \( q\alpha_j \) appears in the second term only if \(|\alpha_j| < q^{(d-1)/2} \), so \(|q\alpha_j| < q^{(d+1)/2} \), while \( q^d/\alpha_j \) appearing in the first term satisfies \(|q^d/\alpha_j| > q^{(d+1)/2} \), so each Weil number can only appear in one of the two terms, thus both terms are uniquely determined.

\[ \square \]

**Corollary 3.9.** Fix \( M, w_1, \ldots, w_r \in \mathbb{Z}, \epsilon_1, \ldots, \epsilon_r \in \{0,1\} \). Fix for each \( i \) with \( \epsilon_i = 0 \) a compatible system of Weil numbers \( \gamma_i \) and for each \( i \) with \( \epsilon_i = 1 \) a sign-compatible system of Weil numbers \( \gamma_i \). In either case, assume that \(|\gamma_i(q, \chi)| = q^{|w_i|/2} \). Let

\[
a^*(f_1, \ldots, f_r; q, \chi, M) = a(f_1, \ldots, f_r; q, \chi, M) \prod_{i=1}^r \gamma_i(q, \chi)^{\deg f_i}.
\]

Then

\[
a^*(f_1, \ldots, f_r; q, \chi, M)
\]

is the unique function that, together with a function \( J^*(d_1, \ldots, d_r; q, \chi, M) \) from tuples of natural numbers \( d_1, \ldots, d_r, \) to compatible systems of sets of ordered pairs of Weil numbers and integers, satisfies the axioms

1. If \( f_1, \ldots, f_r \) and \( g_1, \ldots, g_r \) satisfy \( \gcd(f_i, g_j) = 1 \) for all \( i \) and \( j \), then we have

\[
a^*(f_1g_1, \ldots, f_rg_r; q, \chi, M) = a^*(f_1, \ldots, f_r; q, \chi, M)a^*(g_1, \ldots, g_r; q, \chi, M).
\]

2. \( a^*(1, \ldots, 1; q, \chi, M) = 1 \) and \( a^*(1, \ldots, 1, f, 1, \ldots, 1; q, \chi, M) = \gamma_i(q, \chi) \) for all linear polynomials \( f \).

3. \( a^*(\pi^{d_1}, \ldots, \pi^{d_r}; q, \chi, M) = \left( \frac{\pi^d}{\pi} \right)_\chi (1+\epsilon d_i)^{\deg \pi}
\]

4. \( \sum_{\begin{subarray}{c} f_1, \ldots, f_r \in \mathbf{F}_q[t]^+ \\ \deg f_i = d_i \end{subarray}} a^*(f_1, \ldots, f_r; q^m, \chi, M) = \sum_{j \in \{d_1, \ldots, d_r; q, \chi, M\}} c_j q^{\sum_{i=1}^r (1+w_i)d_i}/\alpha_j
\]

5. \(|\alpha_j| < q^{\sum_{i=1}^r (1+w_i)d_i-1} \) as long as \( \sum_{i=1}^r d_i \geq 2 \).

**Proof.** This follows from Theorem [1.1] once we check that \( a^*(f_1, \ldots, f_r; q, \chi, M) \) satisfies these axioms with a given \( J(d_1, \ldots, d_r; q, \chi, M) \) if and only if

\[
\tilde{a}(f_1, \ldots, f_r; q, \chi, M) = \frac{a^*(f_1, \ldots, f_r; q, \chi, M)}{\prod_{i=1}^r \gamma_i(q, \chi)^{\deg f_i}}
\]

satisfies the axioms of Theorem [1.1] after adjusting \( J^*(d_1, \ldots, d_r; q, \chi, M) \) by dividing each \( \alpha_j \) by \( \prod_{i=1}^r (-1)^{\epsilon_i \gamma_i(q, \chi)}d_i \) and each \( c_j \) by \( (-1)^{\sum_{i=1}^r \epsilon_i d_i} \).
This can be checked one axiom at a time by plugging these expressions into each axiom of Theorem 1.1, simplifying, and observing that they match the corresponding axiom here, as well as deducing the compatibility of \( J \) from the (sign-)compatibility of \( \gamma_i \).

In each case, this is relatively straightforward. In (4) it requires the identity \( \gamma_i(q, \chi) \gamma_i(q, \chi) = q^{\omega_i} \).

\[ \square \]

4. Examples

For some special values of \( M \), we can calculate \( a \) by exhibiting an explicit function and checking that it satisfies the axioms of Theorem 1.1. In fact, these will be functions \( a \) that have essentially appeared in the literature already as coefficients of multiple Dirichlet series, and most of the properties described in Theorem 1.1 were previously observed (but in slightly different language, so we will have to do some work to match it up). In some cases, it will also be convenient to use additional geometric techniques to calculate \( a \).

One reason for the difference in language is that prior work has tended to define twisted multiplicative functions as the product of a multiplicative function with a Dirichlet character. We have found it more convenient to define twisted multiplicative functions all at once.

We will always use \( \tilde{a} \) to refer to a function we are trying to prove satisfies the axioms of Theorem 1.1, but haven’t yet.

**Proposition 4.1.** Take \( r = 2 \), \( M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

Then

\[
a(f_1, f_2; q, \chi, M) = \begin{cases} \left( \frac{f_1/g^n}{f_2/g^n} \right) \chi^{(n-1) \deg g} q^{\deg g} & \text{if } \gcd(f_1, f_2) = g^n \text{ for some } g \\ 0 & \text{if } \gcd(f_1, f_2) \text{ is not an } n \text{th power} \end{cases}
\]

We prove this after making some definitions. Let

\[
\tilde{a} \left( f_1, f_2; q, \chi, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \begin{cases} \left( \frac{f_1/g^n}{f_2/g^n} \right) \chi^{(n-1) \deg g} q^{\deg g} & \text{if } \gcd(f_1, f_2) = g^n \text{ for some } g \\ 0 & \text{if } \gcd(f_1, f_2) \text{ is not an } n \text{th power} \end{cases}
\]

In Chinta and Mohler 2010, (1.2)], a function \( a \) is defined to be the unique multiplicative function such that

\[
a(\pi^j, \pi^k) = \begin{cases} p^{(n-1) \min(j,k)/n} & \text{if } \min(j, k) = 0 \text{ mod } n \\ 0 & \text{otherwise} \end{cases}
\]

Furthermore they define \( f_{2,0} \) as quotient of \( f_2 \) by its maximal \( n \)th power divisor and \( \hat{f}_1 \) as the greatest divisor of \( f_1 \) coprime to \( f_{2,0} \). They define a Dirichlet series with coefficients

\[
\left( \frac{\hat{f}_1}{f_2,0} \right) \chi a(f_1, f_2).
\]
Lemma 4.2. For all finite fields $\mathbb{F}_q$, characters $\chi$, and monic polynomials $f_1, f_2$ over $\mathbb{F}_q$, we have

$$\tilde{a}\left(f_1, f_2; q, \chi, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \left(\frac{\hat{f}_1}{\hat{f}_2, 0}\right)_\chi a(f_1, f_2).$$

Proof. First we note that $a(f_1, f_2)$ vanishes unless $\gcd(f_1, f_2) = g^n$ for some $g$ and is $q^{(n-1)\deg g}$ in that case. So it suffices to check, when $\gcd(f_1, f_2) = g^n$, that

$$\left(\frac{f_1/g^n}{f_2/g^n}\right)_\chi = \left(\frac{\hat{f}_1}{\hat{f}_2, 0}\right)_\chi.$$

First note that $f_{2,0}$ divides $f_2/g^n$ and the ratio is an $n$th power which is prime to $f_1/g^n$, so we have

$$\left(\frac{f_1/g^n}{f_2/g^n}\right)_\chi = \left(\frac{\hat{f}_1}{\hat{f}_2, 0}\right)_\chi.$$

Now $\hat{f}_1$ is the quotient of $f_1$ by a product of $\pi^{v_\pi(f_1)}$, where $\pi$ are some primes. Each such $\pi$ divides $f_{2,0}$, so $v_\pi(f_2)$ cannot be multiple of $n$. Since $v_\pi(\gcd(f_1, f_2)) = \min(v_\pi(f_1), v_\pi(f_2))$ is a multiple of $n$, we must have $v_\pi(f_1)$ a multiple of $n$ strictly less than $v_\pi(f_2)$. Thus $f_1/f_1$ is an $n$th power and divides $g^n$, so $\hat{f}_1$ is a multiple of $f_1/g^n$ by an $n$th power prime to $f_{2,0}$. Thus

$$\left(\frac{f_1/g^n}{f_2, 0}\right)_\chi = \left(\frac{\hat{f}_1}{\hat{f}_2, 0}\right)_\chi$$

and we are done. \hfill $\square$

Proof of Proposition 4.1. It suffices to prove that $\tilde{a}$ satisfies the axioms of Theorem 1.1. Axiom (2) is immediate. To check $\tilde{a}$ satisfies axiom (1), observe that if $\gcd(f_i, g_j) = 1$ for all $i, j$ then $\gcd(f_1 g_1, f_2 g_2) = \gcd(f_1, f_2) \gcd(g_1, g_2)$, and moreover the two gcds on the right are coprime, so $\gcd(f_1 g_1, f_2 g_2)$ is an $n$th power if and only if both $\gcd(f_1, f_2)$ and $\gcd(g_1, g_2)$ are.

We next choose $J(d_1, d_2; q, \chi, M)$. We observe that $\tilde{a}(\pi^{d_1}, \pi^{d_2}; q, \chi, M)$ vanishes unless $\min(d_1, d_2)$ is divisible by $n$ and equals $q^{(n-1)\deg \pi \min(d_1, d_2)/n}$ in that case. Hence we can take $J(d_1, d_2; q, \chi, M)$ to be empty unless $\min(d_1, d_2)$ is divisible by $n$ and to consist of the ordered pair $(q^{(n-1)\min(d_1, d_2)/n}, 1)$ if it is divisible.

This makes (3) immediate. (5) is similarly clear. With this value of $J$, (4) is equivalent to the statement that

$$\sum_{f_1, f_2 \in \mathbb{F}_q[t] \atop \deg(f_1) = d_1 \atop \deg(f_2) = d_2} \tilde{a}(f_1, f_2; q, \chi, M) = \begin{cases} q^{d_1 + d_2 - \frac{n-1}{n} \min(d_1, d_2)} & \text{if } n \mid \min(d_1, d_2) \\ 0 & \text{otherwise} \end{cases}.$$

We now use [Chinta and Mohler, 2010, (1.7)], which is

$$\sum_{f_1, f_2 \in \mathbb{F}_q[t] \atop \deg(f_1) = d_1 \atop \deg(f_2) = d_2} \left(\frac{\hat{f}_1}{\hat{f}_2, 0}\right)_\chi a(f_1, f_2) x^{\deg f_1} y^{\deg f_2} = \frac{1 - q^2 xy}{(1 - qx)(1 - qy)(1 - q^n x^n y^n)},$$

and hence for some other $J$.

□
Thus, by Lemma 4.2,

\[
\sum_{f_1, f_2 \in \mathbb{F}_q[t]^+} \tilde{a}(f_1, f_2; q, \chi, M) x^{\deg f_1} y^{\deg f_2} = \frac{1 - q^2 xy}{(1 - qx)(1 - qy)(1 - q^{n+1} x^n y^n)}.
\]

Then

\[
\sum_{f_1, f_2 \in \mathbb{F}_q[t]^+} \tilde{a}(f_1, f_2)
\]

is simply the coefficient of \(x^{d_1} y^{d_2}\) in (4.1). Hence to verify (4) it suffices to check that

\[
\frac{1 - q^2 xy}{(1 - qx)(1 - qy)(1 - q^{n+1} x^n y^n)} = \sum_{d_1, d_2 \in \mathbb{N}} q^{d_1 + d_2 - (n-1) \min(d_1, d_2)/n} x^{d_1} y^{d_2}
\]

which is straightforward. \(\square\)

**Corollary 4.3.** For all finite fields \(\mathbb{F}_q\), characters \(\chi\), and monic polynomials \(f_1, f_2\) over \(\mathbb{F}_q\), we have

\[
a(f_1, f_2; q, \chi, \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)) = \left(\frac{\hat{f}_1}{f_2, 0}\right)_\chi a(f_1, f_2).
\]

**Proof.** This follows from combining Lemma 4.2 and Proposition 4.1. \(\square\)

**Proposition 4.4.** Assume \(n\) even.

Take \(r = 2\), \(M = \left(\begin{array}{cc} 0 & -1 \\ -1 & \frac{n}{2} + 1 \end{array}\right)\).

Then

\[
a(f_1, f_2; q, \chi, M) = (-1)^{\deg f_2 (\deg f_2 - 1)(q-1)}/G(\chi, \psi)^{\deg f_2} \sum_{u \in \mathbb{F}_q[t]^+, u^n \mid f_2} q^{(n-1) \deg u g_{\chi}(f_1, f_2/u^n)}.
\]

Let

\[
\tilde{a} \left( f_1, f_2; q, \chi, \left(\begin{array}{cc} 0 & -1 \\ -1 & \frac{n}{2} + 1 \end{array}\right)\right) = (-1)^{\deg f_2 (\deg f_2 - 1)(q-1)}/G(\chi, \psi)^{\deg f_2} \sum_{u \in \mathbb{F}_q[t]^+, u^n \mid f_2} q^{(n-1) \deg u g_{\chi}(f_1, f_2/u^n)}.
\]

We give two proofs of Proposition 4.4. The first uses geometric properties of perverse sheaves, while the second relies on Theorem 1.1 and \cite{Chinta and Mohler, 2010}.

The geometric proof proceeds by a series of lemmas that establish (4.2) in successively more cases.

**Lemma 4.5.** (4.2) holds when \(f_2\) is squarefree and \(f_1\) and \(f_2\) are coprime.

**Proof.** The trace function of \(\mathcal{L}_\chi(f)\) is \(\chi(f)\) on any open set where \(f\) is a nonvanishing polynomial function. On the open set where \(f_2\) is squarefree and \(f_1\) and \(f_2\) are coprime,
$F_{d_1, d_2}$ is a nonvanishing polynomial (the condition that $f_1$ is squarefree being unneeded since $M_{11} = 0$) so

$$a(f_1, f_2; q, \chi, M) = \chi(F_{d_1, d_2}(f_1, f_2)) = \left( \frac{f_2}{f_2} \right)^{n/2 + 1} \left( \frac{f_2}{f_1} \right)^{-1}$$

by Lemma 2.1.

On the other hand, when $f_2$ is squarefree we have

$$\sum_{u \in F_q[t]^+ \atop u^n|f_2} q^{(n-1)\deg u} g_\chi(f_1, f_2/u^n) = g_\chi(f_1, f_2)$$

and by Lemma 2.4

$$\frac{(-1)^{\deg f_2/(\deg f_2 - 1)(q-1)}}{G(\chi, \psi)^{\deg f_2}} g_\chi(f_1, f_2) = \left( \frac{f_2'}{f_2} \right)^{\deg f_2} \left( \frac{f_2'}{f_2} \right) \xi \left( \frac{f_1}{f_2} \right)^{-1}$$

so $4.2$ follows upon noting that

$$\left( \frac{f_2'}{f_2} \right)^{\deg f_2} \left( \frac{f_2'}{f_2} \right) \xi = \left( \frac{f_2'}{f_2} \right)^{n/2 + 1}.$$

\[ \square \]

**Lemma 4.6.** $4.2$ holds when $\deg f_1 \geq \deg f_2$.

**Proof.** Let $M' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Observe that

$$\sum_{u \in F_q[t]^+ \atop u^n|f_2} q^{(n-1)\deg u} g_\chi(f_1, f_2/u^n) = \sum_{u \in F_q[t]^+ \atop u^n|f_2} \sum_{h \in F_q[t]/f_2} \frac{h/u^n}{f_2/u^n} \chi \left( \frac{h/f_1}{f_2} \right)$$

$$= \sum_{h \in F_q[t]^+ \atop \deg h = \deg f_2} \sum_{u \in F_q[t]^+ \atop u^n|h, f_2} q^{(n-1)\deg u} \left( \frac{h/u^n}{f_2/u^n} \right) \chi \left( \frac{h/f_1}{f_2} \right)$$

using the fact that there is a unique monic $h$ of degree $\deg f_2$ in each residue class mod $f_2$, the fact that $\left( \frac{h/u^n}{f_2/u^n} \right) = 0$ unless $u^n = \gcd(h, f_2)$, and Lemma 4.1. Thus

$$\begin{pmatrix} 0 & 1 \\ -1 & n/2 + 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(4.3)

$$\tilde{a} \left( f_1, f_2; q, \chi, \begin{pmatrix} 0 & 1 \\ -1 & n/2 + 1 \end{pmatrix} \right) = \frac{(-1)^{\deg f_2/(\deg f_2 - 1)(q-1)}}{G(\chi, \psi)^{\deg f_2}} \sum_{h \in F_q[t]^+ \atop \deg h = \deg f_2} a(h, f_2; q, \chi, M') \psi \left( \frac{h/f_1}{f_2} \right).$$

Let $d_1 = \deg f_1$ and $d_2 = \deg f_2$.

We now make a geometric argument. To distinguish IC sheaves constructed with the matrix $M'$ from those constructed with the matrix $M$, we put the matrix as an additional subscript. Thus $K_{d_2, d_2, M'}$ is a complex of sheaves on $\mathbb{A}^{d_2} \times \mathbb{A}^{d_2}$ whose trace function is $a(h, f_2; q, \chi, M')$ and $K_{d_1, d_2, M}$ is a complex of sheaves on $\mathbb{A}^{d_1} \times \mathbb{A}^{d_2}$ whose trace function is $a(h, f_2; q, \chi, M).$
We now recall the ℓ-adic Fourier transform defined by [Katz and Laumon 1985]. We define two maps $A^{d_2} \times A^{d_2} \times A^{d_2} \to A^{d_2} \times A^{d_2}$, namely $pr_{13}$ and $pr_{23}$, given respectively by projection onto the first and third factors, and projection onto the second and third factors. We also define a map $\mu: A^{d_2} \times A^{d_2} \times A^{d_2} \to A^1$ given by taking the dot product of the first and second factors. Precisely in coordinates, we think of points of the second $A^{d_2}$ as parameterizing monic polynomials $t^n + \sum_{i=0}^{d_2-1} c_i t^i$, points of the first $A^{d_2}$ as simply tuples $b_0, \ldots, b_{d_2-1}$, and $\mu((b_0, \ldots, b_{d_2-1}), (c_0, \ldots, c_{d_2-1}), f_2) = \sum_{i=0}^{d_2-1} b_i c_i$.

Katz and Laumon 1985, (2.1.1)] define the Fourier transform $\mathcal{F}_\psi$ by the formula

$$\mathcal{F}_\psi K_{d_2,d_2,M'} = pr_{132}(pr_{23} K_{d_2,d_2,M'} \otimes \mu^* \mathcal{L}_\psi)[d_2].$$

The operations of pullback, compactly supported pushforward, tensor product, and shift each transform the trace function in a predictable way. Using this, it is immediate that the trace function of $\mathcal{F}_\psi K_{d_2,d_2,M'}$ at a point $(b, f_2)$ of $A^{d_2} \times A^{d_2}$ is given by the formula

$$(-1)^{d_2} \sum_{h \in \mathcal{F}_\psi[t]^+ \atop \deg h = \deg f_2} \alpha(h, f_2; q, \chi, M') \psi(h \cdot b).$$

Let $\sigma: A^{d_1} \times A^{d_2} \to A^{d_1} \times A^{d_2}$ be the map sending $(f_1, f_2)$ to $(b, f_2)$ where $b_i = \text{res} \left( \frac{t^i f_1}{f_2} \right)$. Let $\alpha: A^{d_1} \times A^{d_2} \to A^1$ send $(f_1, f_2)$ to $\text{res} \left\{ \frac{t^i f_1}{f_2} \right\}$. We have chosen these so that for $(b, f_2) = \sigma(f_1, f_2)$, we have

$$\alpha(f_1, f_2) + h \cdot b = \text{res} \left( \frac{t^i f_1}{f_2} \right) + \sum_{i=0}^{d_2-1} c_i \text{res} \left( \frac{t^i f_1}{f_2} \right) = \text{res} \left( \frac{t^{d_2} + \sum_{i=0}^{d_2-1} c_i t^i}{f_2} \right) = \text{res} \left( \frac{h f_1}{f_2} \right).$$

Thus the trace function of

$$\sigma^* \mathcal{F}_\psi K_{d_2,d_2,M'} \otimes \alpha^* \mathcal{L}_\psi$$

given by

$$(-1)^{d_2} \sum_{h \in \mathcal{F}_\psi[t]^+ \atop \deg h = \deg f_2} \alpha(h, f_2; q, \chi, M') \psi \left( \text{res} \left( \frac{h f_1}{f_2} \right) \right) = \frac{(-G(\chi, \psi))^{d_2}}{(-1)^{d_2(d_2-1)(q-1)}} \tilde{a}(f_1, f_2; q, \chi, M).$$

By the Hasse-Davenport relations, the quantity $-G(\chi, \psi)$ is a compatible system of Weil numbers, and the same is true for $(-1)^{d_2-1} = \xi(-1)$, so there exists a sheaf $\mathcal{L}_G$ on $\text{Spec} \mathcal{F}_p$ whose trace of Frobenius is $\frac{(-G(\chi, \psi))^{d_2}}{(-1)^{d_2(d_2-1)(q-1)}}$ for all finite fields $\mathcal{F}_q$. It follows that the trace function of $\sigma^* \mathcal{F}_\psi K_{d_2,d_2,M'} \otimes \alpha^* \mathcal{L}_\psi \otimes \mathcal{L}_G$ is $\tilde{a}(f_1, f_2; q, \chi, M)$.

Next let’s check that $\sigma^* \mathcal{F}_\psi K_{d_2,d_2,M'} \otimes \alpha^* \mathcal{L}_\psi \otimes \mathcal{L}_G[d_1 + d_2]$ is an irreducible perverse sheaf. The complex $K_{d_2,d_2,M'}[2d_2]$ is perverse by construction. Fourier transform preserves perversity by the same argument as Katz and Laumon 1985, Corollary 2.1.5(iii)], which shows Fourier transform preserves relative perversity, and preserves irreducibility by an immediate consequence of [Kiehl and Weissauer, 2001 III, Theorem 8.1(3)]. We can check that $\sigma$ is smooth because for each fixed value of $f_2$, $\sigma$ is given by a linear map of vector spaces, and this linear map is surjective because $(h, f_1) \mapsto \text{res} \left( \frac{h f_1}{f_2} \right)$ gives a perfect pairing on polynomials.
modulo $f_2$. This also shows that $\sigma$ has nonempty, geometrically connected fibers. Since the source of $\sigma$ has dimension $d_1 + d_2$ and the target has dimension $2d_2$, the map $\sigma$ must be smooth of relative dimension $d_1 - d_2$, so $\sigma$ preserves perversity after a shift by $d_1 - d_2$. Because $\sigma$ has nonempty, geometrically connected fibers, this pullback and shift functor is fully faithful and thus preserves irreducibility [Beilinson, Bernstein, and Deligne, 1982, Corollary 4.2.6.2]. Finally, $L_\psi$ is lisse of rank one so its pullback under $\alpha$ is lisse of rank one and tensor product with it preserves perversity and irreducibility, and the same is true for $L_G$.

So $\sigma^*F_\psi K_{d_1,d_2,M} \otimes \alpha^*L_\psi \otimes L_G[d_1 + d_2]$ and $K_{d_1,d_2,M}[d_1 + d_2]$ are two irreducible perverse sheaves. Furthermore, by Lemma 4.5 the trace functions of these two perverse sheaves agree on the open set where $f_1$ is squarefree and $f_1$ and $f_2$ are coprime by Lemma 4.5. Restricting to a possibly smaller open set where both are lisse, we get two irreducible lisse sheaves with the same trace function, which must be isomorphic. Since $K_{d_1,d_2,M}$ is lisse of nonzero rank on an open set, both sheaves are lisse of nonzero rank, and because they are irreducible, must be middle extensions from that open set. Since both are the middle extension of the same lisse sheaf from the same open set, they are isomorphic as perverse sheaves. It follows that these two irreducible perverse sheaves have the same trace function, giving (4.2). □

Conclusion of geometric proof of Proposition 4.4. Given $f_1, f_2$, find $v$ coprime to $f_2$ and such that $\deg f_1 + \deg v \geq \deg f_2$, and compute using axiom (1) and the fact that $K_{\deg v,0}$ is the constant sheaf that

\[
a(f_1v, f_2; q, \chi, M) = a(f_1, f_2; q, \chi, M) a(v, 1; q, \chi, M) \left(\frac{v}{f_2}\right)^{-1} = a(f_1, f_2; q, \chi, M) \left(\frac{v}{f_2}\right)^{-1}.
\]

By Lemma 4.6 we have

\[
a(f_1v, f_2; q, \chi, M) = \frac{(-1)^{\deg f_2(\deg f_2-1)(q-1)}}{G(\chi, \psi)^{\deg f_2}} \sum_{u \in \mathbb{F}_q[t]^+} q^{(n-1)\deg v} g_\chi(f_1v, f_2/u^n).
\]

But by Lemma 2.6

\[
g_\chi(f_1v, f_2/u^n) = g_\chi(f_1, f_2/u^n) \left(\frac{v}{f_2/u^n}\right)^{-1} = g_\chi(f_1, f_2/u^n) \left(\frac{v}{f_2}\right)^{-1}
\]

so combining (4.4), (4.5), and (4.6), we get

\[
a(f_1, f_2; q, \chi, M) \left(\frac{v}{f_2}\right)^{-1} = \frac{(-1)^{\deg f_2(\deg f_2-1)(q-1)}}{G(\chi, \psi)^{\deg f_2}} \sum_{u \in \mathbb{F}_q[t]^+} q^{(n-1)\deg u} \left(\frac{v}{f_2}\right)^{-1} g_\chi(f_1, f_2/u^n).
\]

and dividing both sides by $\left(\frac{v}{f_2}\right)^{-1}$ we get (1.2) in general. □

Before performing our proof using [Chinta and Mohler, 2010], we will explain the relationship of the Gauss sums we work with to the formula defined by Chinta and Mohler [2010]. To do this, we use notation from [Chinta and Mohler, 2010]. They define a function $b$ as the unique multiplicative function satisfying
involving \( \hat{f}_2 \) divided by the greatest \( n \)th power that divides \( f_2 \). Let \( f_{2,0} \) be the largest squarefree divisor of \( f_{2,0} \) and let \( \hat{f}_1 \) be the largest divisor of \( f_1 \) prime to \( f_{2,0} \). Equivalently,

\[
f_{2,0} = \prod_{\pi \text{ prime } \pi \mid f_{2,0}} \pi^{v_\pi(f_{2,0}) - n[\varphi(f_{2,0})/\varphi(\pi)]} \ 	ext{ and } \hat{f}_1 = \prod_{\pi \text{ prime } \pi | f_1, n | v_\pi(f_1)} \pi^{v_\pi(f_1)}.
\]

Let

\[
g\left( \left( \frac{\cdot}{f_2} \right)_\chi \right) = \sum_{h \in \mathbb{F}_{q}[t]/f_{2,0}} \left( \frac{h}{f_2} \right)_\chi \psi\left( \text{res}\left( \frac{h}{f_{2,0}} \right) \right).
\]

**Lemma 4.7.** We have

\[
g_\chi(f_1, f_2) \frac{1}{q^{\deg f_2/2}} = b(f_1, f_2) g\left( \left( \frac{\cdot}{f_2} \right)_\chi \right) \left( \frac{\hat{f}_1}{f_{2,0}} \right)^{-1} \frac{1}{q^{\deg f_{2,0}/2}}.
\]

**Proof.** We define \( \hat{f}_2 \) as the largest divisor of \( f_2 \) prime to \( f_{2,0} \), in other words

\[
\hat{f}_2 = \prod_{\pi \text{ prime } \pi | f_2, n | v_\pi(f_2)} \pi^{v_\pi(f_2)},
\]

as well as \( \hat{f}_1 = f_1/\hat{f}_1 \) and \( \hat{f}_2 = f_2/\hat{f}_2 \). It is immediate from the definitions that \( \hat{f}_2 \) is an \( n \)th power, and that \( \hat{f}_1 \) and \( \hat{f}_2 \) are coprime to \( \hat{f}_1 \) and \( \hat{f}_2 \).

By Lemma 2.7 we have

\[
g_\chi(f_1, f_2) = g_\chi(\hat{f}_1, \hat{f}_2) g_\chi(\bar{f}_1, \bar{f}_2) \left( \frac{\hat{f}_2}{f_2} \right)_\chi \left( \frac{\hat{f}_2}{f_2} \right)_\chi \left( \frac{\hat{f}_1}{f_{2,0}} \right)^{-1} \left( \frac{\hat{f}_1}{f_{2,0}} \right)^{-1}.
\]

Because \( \hat{f}_2 \) is an \( n \)th power and prime to \( \hat{f}_1 \) and \( \hat{f}_2 \), we may ignore all the residue symbols involving \( f_2 \), obtaining

\[
g_\chi(f_1, f_2) = g_\chi(\hat{f}_1, \hat{f}_2) g_\chi(\bar{f}_1, \bar{f}_2) \left( \frac{\hat{f}_1}{f_2} \right)^{-1}
\]

Similarly, we split the right side of (4.7) into \( \hat{f} \) and \( \bar{f} \) parts. We note that \( f_{2,0} \) is also \( \hat{f}_2 \) divided by the greatest \( n \)th power that divides \( \hat{f}_2 \), in other words, \( f_{2,0} = f_{2,0} \), so that
\( f_{2,\nu} = \hat{f}_{2,\nu} \) and

\[
g \left( \frac{\cdot}{f_2} \right)_\chi = g \left( \frac{\cdot}{\hat{f}_2} \right)_\chi. \tag{4.9}
\]

The multiplicativity of \( b \) gives

\[
b(f_1, f_2) = b(\hat{f}_1, \hat{f}_2) b(\hat{f}_1, \hat{f}_2). \tag{4.10}
\]

Combining (4.8), (4.9), and (4.10), we see that (4.7) is equivalent to

\[
g_\chi(\hat{f}_1, \hat{f}_2) g_\chi(\hat{f}_1, \hat{f}_2) \left( \frac{\hat{f}_1}{f_2} \right)^{-1}_\chi \frac{1}{q^{\deg f_2/2 + \deg \hat{f}_2/2}} = b(\hat{f}_1, \hat{f}_2) g_\chi \left( \frac{\cdot}{f_2} \right)_\chi \left( \frac{\hat{f}_1}{f_{2,0}} \right)^{-1}_\chi \frac{1}{q^{\deg f_2,\nu/2}}
\]

and therefore would follow from the triple of equations

\[
g_\chi(\hat{f}_1, \hat{f}_2) \frac{1}{q^{\deg f_2/2}} = b(\hat{f}_1, \hat{f}_2) \tag{4.11}
\]

\[
\left( \frac{\hat{f}_1}{f_2} \right)^{-1}_\chi = \left( \frac{\hat{f}_1}{f_{2,0}} \right)^{-1}_\chi \tag{4.12}
\]

\[
g_\chi(\hat{f}_1, \hat{f}_2) \frac{1}{q^{\deg f_2/2}} = b(\hat{f}_1, \hat{f}_2) g_\chi \left( \frac{\cdot}{f_2} \right)_\chi \frac{1}{q^{\deg f_2,\nu/2}}. \tag{4.13}
\]

We now verify these three equations. (4.12) follows immediately from the fact that \( f_{2,0} \) and \( \hat{f}_2 \) differ by an \( n \)th power prime to \( \hat{f}_1 \).

For (4.11), we note from Lemma 2.7 that \( g_\chi(f_1, f_2) \) is multiplicative when restricted to \( f_2 \) that are \( n \)th powers. Since both sides are multiplicative when restricted to this set, we can reduce to the case that \( f_1 \) and \( f_2 \) are prime powers (because any \( n \)th power can be factored into prime powers that are \( n \)th powers). In this case, it follows from the definition of \( b \) and Lemma 2.8 noting that \( G(\chi^{d_2}, \psi) = -1 \) if \( d_2 \) is divisible by \( n \).

For (4.13), we note that \( v_\pi(f_2) \) is never a multiple of \( n \) for any \( \pi \) dividing \( \hat{f}_2 \). It follows from this and the definition of \( b \) that \( b(\hat{f}_1, \hat{f}_2) \) vanishes unless \( v_\pi(\hat{f}_1) = v_\pi(\hat{f}_2) - 1 \) for each such \( \pi \). In other words, the right side of (4.13) vanishes unless \( \hat{f}_1 = \hat{f}_2/\hat{f}_{2,\nu} \). From Lemmas 2.7 and 2.8 we see that \( g(\hat{f}_1, \hat{f}_2) \) vanishes under the same condition.

Thus, we may assume that \( \hat{f}_1 = \hat{f}_2/\hat{f}_{2,\nu} \). In this case,

\[
b(\hat{f}_1, \hat{f}_2) = q^{(\deg f_2 - \deg \hat{f}_{2,\nu}) / 2} \tag{4.14}
\]

since only the second-to-last case of the definition of \( b \) occurs. Furthermore, we have

\[
g_\chi(\hat{f}_1, \hat{f}_2) = \sum_{h \in \mathbb{F}_q[t]/f_2} \left( \frac{h}{f_2} \right)_\chi \psi \left( \operatorname{res} \left( \frac{h\hat{f}_1}{f_2} \right) \right)
\]

\[
= \sum_{h \in \mathbb{F}_q[t]/f_2} \left( \frac{h}{f_2} \right)_\chi \psi \left( \operatorname{res} \left( \frac{h}{f_{2,\nu}} \right) \right) = g \left( \left( \frac{\cdot}{f_2} \right)_\chi \right) q^{\deg f_2 - \deg \hat{f}_{2,\nu}}
\]
which together with (4.14) gives (4.13).

Proof of Proposition 4.4 using Chinta-Mohler. Let

\[ \tilde{a}^*(f_1, f_2; q, \chi, M) = G(\chi, \psi)^{\deg f_2} \tilde{a}(f_1, f_2; q, \chi, M) = (-1)^{\deg f_2 (\deg f_2 - 1)(q - 1)/4} \sum_{u \in \mathbb{F}_q[t]^+ / u^n f_2} q^{(n-1)\deg u} g_x(f_1, f_2/u^n).
\]

We prove that \( \tilde{a}^* \) satisfies the axioms of Corollary 3.9 with \( w_1 = \epsilon_1 = 0, \ w_2 = \epsilon_2 = 1, \ \gamma_1(q, \chi) = 1, \ \gamma_2(q, \chi) = G(\chi, \psi). \)

For axiom (1) we have

\[ \tilde{a}^*(f_1 f_3, f_2 f_4; q, \chi, M) = \frac{\deg f_2 (\deg f_2 - 1)(q - 1)}{4} + \frac{\deg f_2 \deg f_4 (q - 1)}{2} \sum_{u \in \mathbb{F}_q[t]^+ / u^n f_2} q^{(n-1)\deg u} g_x(f_1 f_3, f_2 f_4/u^n).
\]

Because \( f_2 \) and \( f_4 \) are coprime, we can write any \( u \) where \( u^n | f_2 f_4 \) uniquely as \( u_2 u_4 \) where \( u_2^n \) divides \( f_2 \) and \( u_4^n \) divides \( f_4 \)

From Lemma 2.7, we get

\[ g_x(f_1 f_3, f_2 f_4/(u_2^n u_4^n)) = g_x(f_1, f_2/u_2^n) g_x(f_3, f_4/u_4^n) \left( \frac{f_2}{u_2^n f_4/u_4^n} \right)_x \left( \frac{f_4}{u_4^n f_2/u_2^n} \right)_x \left( \frac{f_1}{f_4/u_4^n} \right)_x^{-1} \left( \frac{f_3}{f_2/u_2^n} \right)_x^{-1}.
\]

However, we can ignore the \( u_2^n \) and \( u_4^n \) factors in the power residue symbols as they are \( n \)th powers and because \( u_2 \), dividing \( f_2 \), is prime to \( f_3 \) and \( f_4 \) and similarly \( u_4 \) is prime to \( f_1 \) and \( f_2 \). Thus

\[ g_x(f_1 f_3, f_2 f_4/(u_2^n u_4^n)) = g_x(f_1, f_2/u_2^n) g_x(f_3, f_4/u_4^n) \left( \frac{f_2}{f_4} \right)_x \left( \frac{f_4}{f_2} \right)_x \left( \frac{f_1}{f_4} \right)_x^{-1} \left( \frac{f_3}{f_2} \right)_x^{-1}.
\]

Plugging (4.16) into (4.15) gives

\[ \tilde{a}^*(f_1 f_3, f_2 f_4; q, \chi, M) = \tilde{a}^*(f_1, f_2; q, \chi, M) \tilde{a}^*(f_3, f_4; q, \chi, M) (-1)^{\deg f_2 \deg f_4 (q - 1)/2} \left( \frac{f_2}{f_4} \right)_x \left( \frac{f_4}{f_2} \right)_x \left( \frac{f_1}{f_4} \right)_x^{-1} \left( \frac{f_3}{f_2} \right)_x^{-1}.
\]

We have

\[ (-1)^{\deg f_2 \deg f_4 (q - 1)/2} = \left( \frac{f_2}{f_4} \right)_x^{n/2} \left( \frac{f_4}{f_2} \right)_x^{n/2}
\]

by Lemma 2.2, which, plugged into (4.17), verifies axiom (1).

For axiom (2), we have

\[ \tilde{a}^*(T - x, 1; q, \chi, M) = g_x(T - x, 1) = 1
\]

and

\[ \tilde{a}^*(1, T - x; q, \chi, M) = g_x(1, T - x) = G(\chi, \psi),
\]

both using Lemma 2.8.
Next, let

\[ J_1(d_1, d_2; q, \chi, M) = \begin{cases} 
(1, 1) & \text{if } d_2 = 0 \\
\{(q^{d_2}, 1), (q^{(d_2-1)}, -1)\} & \text{if } d_2 \equiv 0 \mod n \text{ and } d_1 \geq d_2 \\
\emptyset & \text{if } d_2 \not\equiv 0 \mod n \text{ and } d_1 \geq d_2 \\
\{(-q^{(d_2-1)}G(\chi^{d_2}, \psi), -1)\} & \text{if } d_1 = d_2 - 1 \\
\emptyset & \text{if } d_1 < d_2 - 1 
\end{cases} \]

Then by Lemma 2.8 we have

\[ g_\chi(\pi^{d_1}, \pi^{d_2}) = \left(\frac{\pi'}{\pi}\right)^{d_2} \sum_{\chi \in J_1(d_1, d_2; q, \chi, M)} c_j \alpha_j^{\deg \pi} \]

noting that the \( (\pi')^{d_2} \) term can be ignored in the cases where \( d_2 \) is divisible by \( n \).

Furthermore, we have

\[ \sum_{u \in \mathbb{F}_q[t]^+} q^{(n-1)\deg u} g_\chi(\pi^{d_1}, \pi^{d_2}/u^n) = \sum_{c=0}^{[d_2/n]} q^{(n-1)c \deg \pi} g_\chi(\pi^{d_1}, \pi^{d_2-nc}). \]

So letting

\[ J(d_1, d_2; q, \chi, M) = (-1)^{d_2(d_2-1)(q-1)/4} \bigcup_{c=0}^{[d_2/n]} q^{(n-1)c} J_1(d_1, d_2 - nc; q, \chi, M) \]

we have

\[ \tilde{a}^*(\pi^{d_1}, \pi^{d_2}; q, \chi, M) \]

= \( (-1)^{d_2 \deg \pi (d_2 \deg \pi - 1)(q-1)/4} \sum_{u \in \mathbb{F}_q[t]^+} q^{(n-1)\deg u} g_\chi(\pi^{d_1}, \pi^{d_2}/u^n) \)

= \( (-1)^{d_2 \deg \pi (d_2 \deg \pi - 1)(q-1)/4} \left(\frac{\pi'}{\pi}\right)^{d_2} \sum_{\chi \in J_1(d_1, d_2; q, \chi, M)} c_j \alpha_j^{\deg \pi} \)

verifying axiom (3) because \( d_2 \equiv d_2^2 \mod 2 \) and thus

\[ (-1)^{d_2 \deg \pi (d_2 \deg \pi - 1)(q-1)/4} \left(\frac{\pi'}{\pi}\right)^{d_2 (n/2)} (-1)^{d_2 (\deg \pi + 1)} \]

by Lemma 2.3.
Next, to verify axiom (4), it suffices to show that
\[
(-1)^{d_2(d_2-1)(q-1)/4} \sum_{f_1, f_2 \in \mathbb{F}_q[t]^+} \sum_{u \in \mathbb{F}_q[t]^+} q^{(n-1)\deg u} g_\lambda(f_1, f_2/u^n)
\]
\[
= \sum_{j \in J(d_1, d_2, q, \chi, M)} c_j q^{d_1+2d_2}/\alpha_j.
\]
(4.18)

To do this, it suffices to show the identity of formal power series in \( q^{-s} \) and \( q^{-(w+1)/2} \)
\[
\sum_{f_1, f_2 \in \mathbb{F}_q[t]^+} \sum_{u \in \mathbb{F}_q[t]^+} q^{(n-1)\deg u} g_\lambda(f_1, f_2/u^n) q^{-s \deg f_1} q^{-(w+1)/2} \deg f_2
\]
\[
= \sum_{d_1, d_2} q^{-sd_1} q^{-(w+1)/2} d_2 \sum_{j \in J(d_1, d_2, q, \chi, M)} (-1)^{d_2(d_2-1)(q-1)/4} q^{d_1+2d_2}.
\]
(4.19)

The change of variables \( f_2 \mapsto u^n f_2 \), the identity \( \zeta(nw-n/2+1) = \sum_{u \in \mathbb{F}_q[t]^+} q^{-(nw+n/2-1)\deg u} \), and finally Lemma 4.7 together transform the left side of (4.19) into
\[
\zeta(nw-n/2+1) \sum_{f_1, f_2 \in \mathbb{F}_q[t]^+} g_\lambda(f_1, f_2) q^{-s \deg f_1} q^{-(w+1/2) \deg f_2}
\]
\[
= \zeta(nw-n/2+1) \sum_{f_1, f_2 \in \mathbb{F}_q[t]^+} b(f_1, f_2) g \left( \left( \frac{f_1}{f_2} \right) \chi \left( \frac{f_1}{f_2,0} \right) \right)^{-1} \frac{1}{q^{\deg f_2+n/2}} q^{-s \deg f_1} q^{-w \deg f_2} = Z_2(s, w)
\]
as defined in [Chinta and Mohler, 2010, (1.6)].

The definition of \( J \) gives
\[
\sum_{j \in J(d_1, d_2, q, \chi, M)} (-1)^{d_2(d_2-1)(q-1)/4} c_j q^{d_1+2d_2}/\alpha_j = \sum_{c=0}^{d_2/n} \sum_{j \in J_1(d_1, d_2 - nc, q, \chi, M)} q^{d_1+2d_2}/q^{(n-1)c}/\alpha_j.
\]

This, followed by the substitution \( d_2 \mapsto d_2 + nc \) and the evaluation of a geometric series in \( c \), implies that the right side of (4.19) is equal to
\[
\sum_{d_1, d_2} q^{-sd_1} q^{-(w+1)/2} d_2 \sum_{c=0}^{d_2/n} \sum_{j \in J(d_1, d_2 - nc, q, \chi, M)} q^{d_1+2d_2}/q^{(n-1)c}/\alpha_j
\]
\[
= \sum_{d_1, d_2} \sum_{c=0}^{\infty} q^{-sd_1} q^{-(w+1/2)(d_2+n_c)} \sum_{j \in J(d_1, d_2, q, \chi, M)} q^{d_1+2d_2+2nc}/q^{(n-1)c}/\alpha_j
\]
\[
= \frac{1}{1 - q^{-n(w+1/2)} q^{n_d}} \sum_{d_1, d_2} q^{-sd_1} q^{-(w+1/2)} d_2 \sum_{j \in J_1(d_1, d_2, q, \chi, M)} c_j q^{d_1+2d_2}/\alpha_j.
\]
We have
\begin{equation}
\frac{1}{1 - q^{-n(w+1/2)d_i/d_n}} = \frac{1}{1 - q^{2+1-nw}}
\end{equation}
and
\begin{align*}
\sum_{d_1,d_2} q^{-sd_1} q^{-(w+1/2)d_2} & \sum_{j \in J_1(d_1,d_2;q,\chi,M)} c_j q^{d_1+2d_2} \\
= \sum_{d_1 \in \mathbb{N}} d_1 - d_1s & \sum_{d_2 \in \mathbb{N}^+ \mod n} q^{-sd_1-wd_2} q^{d_1+d_2/2}(1 - q) \\
+ \sum_{i=1}^{n-1} \sum_{d_2 \in \mathbb{N}} q^{-sd_1-wd_2} q^{d_1+d_2/2} G(\chi^i,\psi) & - \sum_{d_2 \in \mathbb{N}} q^{-(n-1)s-nw+3n/2} q^{-nsw+nw+3n/2} \frac{1}{1 - q^{nsw+nw+3n/2}}.
\end{align*}

Introducing the variables \(x = q^{-s} \) and \(y = q^{-w} \), we can rewrite (4.22) as
\begin{align*}
\frac{1}{1 - qx} + \frac{1}{1 - qx} - \frac{(1 - q) q^{3n/2} x^n y^n}{1 - q^{3n/2} x^n y^n} & + \sum_{i=1}^{n-1} \frac{q^{3i/2-1} G(\chi^i,\psi)x^i-y^i}{1 - q^{3n/2} x^n y^n} - \frac{q^{3n/2} x^n y^n}{1 - q^{3n/2} x^n y^n} \\
= \frac{1 - q^{3n/2+1} x^n y^n}{(1 - qx)(1 - q^{3n/2} x^n y^n)} & + \sum_{i=1}^{n-1} \frac{q^{3i/2-1} G(\chi^i,\psi)x^i-y^i}{1 - q^{3n/2} x^n y^n} - \frac{q^{3n/2} x^n y^n}{1 - q^{3n/2} x^n y^n} \\
= 1 - q^{3n/2+1} x^n y^n + & \sum_{i=1}^{n-1} q^{3i/2-1} G(\chi^i,\psi)x^i-y^i(1 - qx) - q^{3n/2} x^n y^n(1 - qx) \\
(1 - qx) & (1 - q^{3n/2} x^n y^n)
\end{align*}

So bringing in the initial factor (4.21), (4.19) is equivalent to
\begin{equation}
Z_2 = \frac{1 - q^{3n/2} x^n y^n}{(1 - q^{2+1} y^n)(1 - q^{-3n/2} x^n y^n)}.
\end{equation}

Noting that \(\tau(\epsilon) = G(\chi^i,\psi)\), (4.23) is precisely [Chinta and Mohler, 2010, (1.8)], finishing the proof of axiom (4).

For axiom (5), we first check that \(J_1(d_1,d_2;q,\chi,M)\) has all \(|\alpha_j| < q^{d_1+2d_2-1}\) unless \(d_1, d_2 = (0,0)\) or \((0,1)\), case-by-case. In the \(d_2 \equiv 0 \mod n\) and \(d_1 \geq d_2\) case, the key is that \(d_1 \geq d_2 \geq n \geq 2\) so \(q^{d_1+2d_2-1} > q^{d_2}\), and in the \(d_1 = d_2 - 1\) case, we have \(q^{d_1+2d_2-1} > q^{d_2-1}\) as long as \(d_1 > 0\). Furthermore, in the \((0,0)\) and \((0,1)\) cases, we have \(|\alpha_j| \leq q^{d_1+2d_2-1}\).

By the definition of \(J\) in terms of \(J_1\), it follows that each \(\alpha_j\) appearing either has \(c = 0\) and thus satisfies \(|\alpha_j| < q^{d_1+2d_2-1}\) since \(d_1 + d_2 \geq 2\) implies \((d_1, d_2) \neq (0,0), (0,1)\), or has \(c > 0\).
Proposition 4.8. Assume

\[ a \leq q^{(n-1)c} \]

since \(2c \geq 2 > 1\), verifying (5).

We now describe a third case where we can relate \(a(f_1, f_2; q, \chi, M)\) to prior work. First, following Chinta [2008, (3.2), (3.3)], let \(H(f_1, f_2)\) be the unique function satisfying

1. If \(\gcd(f_1, f_2, g_1, g_2) = 1\) then

\[ H(f_1, f_2) = \left( \frac{f_1}{g_1} \right)_\chi \left( \frac{f_2}{g_2} \right)_\chi H(f_1, f_2)H(g_1, g_2). \]

2. For \(\pi\) prime,

\[ H(\pi^{d_1}, \pi^{d_2}) = \begin{cases} 1 & \text{if } (d_1, d_2) = (0, 0) \\ g_\chi(1, \pi) & \text{if } (d_1, d_2) = (1, 0) \text{ or } (0, 1) \\ g_\chi(\pi, \pi^2)g_\chi(1, \pi) & \text{if } (d_1, d_2) = (2, 1) \text{ or } (1, 2) \\ g_\chi(\pi, \pi^2)g_\chi(1, \pi)^2 & \text{if } (d_1, d_2) = (2, 2) \\ 0 & \text{otherwise} \end{cases} \]

Proposition 4.8. Assume \(n\) even and \(q \equiv 1 \pmod{4}\).

Take \(r = 2\), \(M = \left( \begin{array}{cc} \frac{n}{2} + 1 & -1 \\ -1 & \frac{n}{2} + 1 \end{array} \right) \).

Then

\[ a(f_1, f_2; q, \chi, M) = \frac{1}{G(\chi, \psi)^{\deg f_1 + \deg f_2}} \sum_{a, b, c \in F_q[\epsilon]} q^{(n-1)\deg a + (2n-1)\deg b + (n-1)\deg c} H(f_1/a^n b^n, f_2/b^n c^n). \]

Proof. To prove this, we verify the axioms of Theorem 3.9 are satisfied for

\[ \tilde{a}^\ast(f_1, f_2; q, \chi, M) = \sum_{a, b, c \in F_q[\epsilon]^+} q^{(n-1)\deg a + (2n-1)\deg b + (n-1)\deg c} H(f_1/a^n b^n, f_2/b^n c^n) \]

with \(\epsilon_1 = \epsilon_2 = 1, w_1 = w_2 = 1, \gamma_1(q, \chi) = \gamma_2(q, \chi) = G(\psi, \chi)\).

The multiplicativity axiom (1) follows immediately from the multiplicativity axiom of \(H\), noting that the factors \(a^n b^n\) and \(b^n c^n\) have degree divisible by \(n\) and can be ignored, and that the term \(\left( \frac{a}{g_1} \right)^{n/2} \left( \frac{g_1}{f_1} \right)^{n/2} \) is 1 by Lemma 2.2 because \(q \equiv 1 \pmod{4}\), and so can be ignored.

Axiom (2) is straightforward. In the case when \((f_1, f_2) = (T - x, 1)\) or \((1, T - x)\), the sum over \(a, b, c\) is trivial, and \(H(f_1, f_2) = g_\chi(1, \pi) = G(\chi, \psi)\).

We have that

\[ \tilde{a}^\ast(\pi^{d_1}, \pi^{d_2}; q, \chi, M) = \sum_{j_1, j_2 \in \mathbb{N}} q^{(n-1)j_1 + (2n-1)j_1 + (n-1)j_2 + n(j_1 + j_2) = d_1 + d_2} \deg \pi H(\pi^{d_1-nj_1-nj_2}, \pi^{d_2-nj_1-nj_2}). \]
From Lemma 2.8 and the Hasse-Davenport identities, we have \( g_\chi(1, \pi) = -(-G(\chi, \psi))^{\text{deg } \pi} \left( \frac{\pi'}{\pi} \right)_\chi \) and \( g_\chi(\pi, \pi^2) = -(-qG(\chi^2, \psi))^{\text{deg } \pi} \left( \frac{\pi'}{\pi} \right)_\chi ^2 \), so we can write \((1.24)\) as

\[
\left( \frac{\pi'}{\pi} \right)_{\chi} d_1 + d_2 \sum_{(j_1, j_{12}, j_{22}, r_1, r_2) \in \mathbb{N}^5 \atop n_{j_1} + n_{j_{12}} + r_1 = d_1 \atop n_{j_{12}} + n_{j_{22}} + r_2 = d_2} c_{(j_1, j_{12}, j_{22}, r_1, r_2)} G_{(j_1, j_{12}, j_{22}, r_1, r_2)}^{\text{deg } \pi} \]

where

\[
\alpha_{(j_1, j_{12}, j_{22}, r_1, r_2)} = q^{(n-1)j_1 + (2n-1)j_{12} + (n-1)j_2} \begin{cases} 
1 & \text{if } (r_1, r_2) = (0, 0) \\
-G(\chi, \psi) & \text{if } (r_1, r_2) = (1, 0) \text{ or } (0, 1) \\
qG(\chi^2, \psi)G(\chi, \psi) & \text{if } (r_1, r_2) = (2, 1) \text{ or } (1, 2) \\
-qG(\chi^2, \psi)G(\chi, \psi)^2 & \text{if } (r_1, r_2) = (2, 2) 
\end{cases}
\]

and

\[
c_{(j_1, j_{12}, j_{22}, r_1, r_2)} = \begin{cases} 
1 & \text{if } (r_1, r_2) = (0, 0), (2, 1), \text{ or } (1, 2) \\
-1 & \text{if } (r_1, r_2) = (1, 0), (0, 1), \text{ or } (2, 2) 
\end{cases}.
\]

So we may take

\[
J(d_1, d_2; q, \chi, M) = \left\{ (j_1, j_{12}, j_{22}, r_1, r_2) \in \mathbb{N}^5 \mid \begin{array}{c} n_{j_1} + n_{j_{12}} + r_1 = d_1 \\ n_{j_{12}} + n_{j_{22}} + r_2 = d_2 \\ (r_1, r_2) \in \{(0,0),(1,0),(0,1),(2,1),(1,2),(2,2)\} \end{array} \right\}
\]

and take these \( \alpha_j \) and \( c_j \). By \((2.3)\), because \( q \equiv 1 \text{ mod } 4 \), we have \((-1)^{(d_1 + d_2)\text{deg } \pi + 1} = \left( \frac{\pi'}{\pi} \right)^{(d_1 + d_2)(n/2)} \). This, and the definition of \( J \), implies \( \bar{a}^* \) satisfies axiom (3).

\( J \) is a manifestly a compatible system of sets of ordered pairs. For axiom (4), we must check

\[
\sum_{f_1, f_2 \in \mathbb{F}_q[t]^+} \bar{a}^*(f_1, f_2; q, \chi, M) x^\text{deg } f_1 y^\text{deg } f_2
\]

\[
= \sum_{j_1, j_{12}, j_{22} \in \mathbb{N}} \sum_{(r_1, r_2) \in \{(0,0),(0,1),(1,0),(1,2),(2,1),(2,2)\}} c_{(j_1, j_{12}, j_{22}, r_1, r_2)} q^{2d_1 + 2d_2} x^{n_{j_1} + n_{j_{12}} + r_1} y^{n_{j_{12}} + n_{j_{22}} + r_2}.
\]

We have

\[
\frac{q^{2d_1 + 2d_2}}{\bar{a}^*(j_1, j_{12}, j_{22}, r_1, r_2)} = q^{(n+1)j_1 + (2n+1)j_{12} + (n+1)j_2} \begin{cases} 
1 & \text{if } (r_1, r_2) = (0, 0) \\
-qG(\chi, \psi) & \text{if } (r_1, r_2) = (1, 0) \text{ or } (0, 1) \\
q^3G(\chi^2, \psi)G(\chi, \psi) & \text{if } (r_1, r_2) = (2, 1) \text{ or } (1, 2) \\
-q^4G(\chi^2, \psi)G(\chi, \psi)^2 & \text{if } (r_1, r_2) = (2, 2) 
\end{cases}.
\]

Here we use \( G(\chi, \psi)G(\chi, \psi) = q \) to calculate the inverse conjugate of \( \alpha \).

Hence we have

\[
\sum_{j_1, j_{12}, j_{22} \in \mathbb{N}} \sum_{(r_1, r_2) \in \{(0,0),(0,1),(1,0),(1,2),(2,1),(2,2)\}} c_{(j_1, j_{12}, j_{22}, r_1, r_2)} q^{2d_1 + 2d_2} x^{n_{j_1} + n_{j_{12}} + r_1} y^{n_{j_{12}} + n_{j_{22}} + r_2} =
\]
According to [Chinta, 2008, Theorem 4.2], upon observing that $\tau < \tau_j$, which is equivalent to $\log q^{\frac{1}{2}}(x,y)G(x,y)x^2y = q^2G(x,\psi)G(\chi,\psi)xy^2 + q^4G(\chi,\psi)G(\chi,\psi)^2x^2y^2$ which is exactly the desired identity.

By the definition of the series $Z(x,y)$ in [Chinta, 2008], we have

$$Z(x,y) = \sum_{f_1, f_2 \in \mathbb{F}_q[t^+]^*} a^*(f_1, f_2; q, \chi, M)x^{\deg f_1}y^{\deg f_2}.$$ 

According to [Chinta, 2008, Theorem 4.2], upon observing that $\tau_1 = G(\chi,\psi)$ and $\tau_2 = G(\chi,\psi)$, we have

$$Z(x,y) = \frac{1 + qG(\chi,\psi)x + qG(\chi,\psi)y + q^3G(\chi,\psi)^2G(\chi,\psi)x^2y + q^5G(\chi,\psi)G(\chi,\psi)xy^2 + q^7G(\chi,\psi)G(\chi,\psi)^2x^2y^2}{(1 - q^{n+1}x^n)(1 - q^{2n+1}x^n y^n)(1 - q^{n+1}x^n y^2)}$$

which is exactly the desired identity.

For axiom (5), note that

$$\log_q |\alpha(j_1, j_2, j_3, r_1, r_2)| = (n - 1) j_1 + (2n - 1) j_2 + (n - 1) j_3 + \begin{cases} 0 & \text{if } (r_1, r_2) = (0, 0) \\ \frac{1}{2} & \text{if } (r_1, r_2) = (1, 0) \text{ or } (0, 1) \\ 2 & \text{if } (r_1, r_2) = (2, 1) \text{ or } (1, 2) \\ \frac{3}{2} & \text{if } (r_1, r_2) = (2, 2) \end{cases}$$

which is $< nj_1 + 2nj_2 + nj_3 + r_1 + r_2 - \frac{1}{2}$ as long as $nj_1 + 2nj_2 + nj_3 + r_1 + r_2 \geq 2$. 

\[\square\]

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Department of Mathematics, Columbia University, New York, NY

Email address: sawin@math.columbia.edu