LIE SUPERALGEBRA STRUCTURES IN $C^*(g;g)$ AND $H^*(g;g)$

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Abstract. Let $n = \text{vect}(M)$ be the Lie (super)algebra of vector fields on any connected (super)manifold $M$; let $\Pi$ be the change of parity functor, $C^i$ and $H^i$ the space of $i$-chains and $i$-cohomology. The Nijenhuis bracket makes $L_i = \Pi(C^{i+1}(n;n))$ into a Lie superalgebra that can be interpreted as the centralizer of the exterior differential considered as a vector field on the supermanifold $\hat{M} = (M, \Omega(M))$ associated with the de Rham bundle on $M$. A similar bracket introduces structures of DG Lie superalgebra in $L_0 = \Pi(C^{i+1}(n;n))$ and $L = \Pi(H^{i+1}(n;n))$ for any Lie superalgebra $n$. We explicitly describe (1) the algebras $L_0$ for the maximal nilpotent subalgebra $n$ of any simple finite dimensional Lie algebra $g$ and (2) the whole of $L$ for $g = g_2$ which conjecturally has an archetypal structure, and in the exceptional cases: $g = sl(2)$ and $sl(3)$; in $L_0$, we also list the homologic elements (odd elements $x$ such that $[x, x] = 0$). We observe that if the bracket in $n$ vanishes identically, as is the case for Hermitean symmetric spaces $X = G/P$, the differential $d$ is also zero and $L = \text{vect}(\dim \Pi(n))$, a simple Lie superalgebra.

We briefly review related results by Grozman, Penkov and Serganova, Poletaeva, and Tolpygo. We cite a powerful Premet’s theorem describing $H^i(n;N)$, where $N$ is an $n$-module which is not a $g$-module.

§1. Introduction

Though of 115 items of A. L. Onishchik’s works listed today by MathSciNet only a few are devoted to (co)homology and even these a few deal with topological questions (as [O]), rather than with “linear algebra” such as Lie algebra cohomology, the latter was one of the main topics of the Vinberg-Onishchik seminar and Onishchik’s own and his students’ studies some of which — partly unpublished for decades — we review below. It is at the above mentioned seminar that one of us (DL) became intrigued by the Nijenhuis bracket and deformation theory, a branch of which lately flourishes as the theory of $L_\infty$-algebras. The explicit definition of $L_\infty$-algebras is irrelevant here and we refer the reader to [JS1], [JS2], [JS3], [HS], [M]. Important for us is that various DG (differential graded) Lie superalgebras are examples of $L_\infty$-algebras. There are, however, not many explicit examples of $L_\infty$-algebras, even of DG Lie superalgebras.

Here we give several explicit examples of DG Lie superalgebras. The naturalist trying to list the cohomology species analytically is set back by the volume of the calculations. That is why computer-aided study, such as [Le], the one we used here, or [G2], [GL2], is indispensable. We also list three tasks we intend to consider shortly; our fourth task is to translate the theorems on Lie algebras mentioned in what follows into the realm of Lie superalgebras.

Date: October 31, 2003.

1991 Mathematics Subject Classification. Primary 17A70, 17B56; Secondary 17B01, 17B70.

Key words and phrases. Lie superalgebras, strongly homotopy Lie algebras, $L_\infty$-algebras, cohomology, Nijenhuis bracket, homological element.

For financial support and stimulating working conditions D.L. is thankful to MPIM-Bonn and IHES.
1. Origins of the problem: Cohomology in physics and mathematics. The electric charge and topological charges in gauge theories are given by de Rham cohomology — the pattern for formulation of various (co)homology theories, in particular, for Lie algebra cohomology, see [GM]. For further numerous (but still covering only selected trends) examples, see, e.g., proceedings of the International conference “Cohomological Physics” [CPH], [JST], [AK]. In short: cohomology is an important invariant. Here is one more application.

Curvature as cohomology. The case opposite in a sense to that considered in [B] is of particular interest. Denote by \( n_i \) the complement to a maximal parabolic subalgebra \( p \) generated by all simple coroots, both positive and negative, except the \( i \)-th negative one, in any simple finite dimensional Lie algebra \( g \). If \( n \) is commutative, the elements of \( H^2(n; g) \) can be interpreted as the values of the generalization of the Riemann curvature tensor at a point (see [LPS]) and they ARE such values if \( g \) is the Lie algebra of \( O(n) \) or \( O(p, q) \).

More generally, let \( P = P_{i_1 \ldots i_k} \) be any parabolic subgroup (generated by all simple coroots, both positive and negative, except the \( i_1 \)-st, \( \ldots \) and \( i_k \)-th negative ones) of a simple Lie group \( G \) and \( N \) the complementary Lie group (i.e., \( p \oplus n = g \), where \( p, n \) and \( g \) are the Lie algebras of \( P, N \) and \( G \), respectively). Observe that if \( n \) is not commutative, the coset space \( G/P \) is non-holonomic (here: is endowed with a non-integrable distribution) and various cohomology of \( n \) give invariants of this non-holonomic manifold. Some of such invariants (the ones belonging to \( H^2(n; g) \)) were recently interpreted as non-holonomic analogs of the curvature tensor, see [L1]. This interpretation is actually a lucid expression of Wagner’s description of the nonholonomic analog of the curvature tensor performed in components, see [DG] and refs therein.

For Lie algebras \( g \), the cohomology \( H^2(n; g) \) was explicitly calculated until now only when \( n \) is commutative – the cases of compact hermitian symmetric spaces [LPS]. Now that we have Premet’s theorem (below) we can use it to extract an explicit description of \( H^2(n; g) \) for noncommutative \( n \). This is our nearest task.

The cohomology \( H^i(n; g) \) for \( i = 1, 2 \) constitute a part of the Lie superalgebra \( H^*(n; g) \) whose bracket is given by point-wise bracketing of the skew-symmetric functions — cocycles. To describe this Lie superalgebra is our second task.

The above mentioned bracket in \( H^*(n; g) \) is given by a scalar operator, and therefore is not so exciting as the bracket given by a first order differential operator (on an appropriate supermanifold) considered in this paper.

There are two major types of cohomology of \( n \): with values in \( g \)-modules and with other type of values. Accordingly, there are two types of results.

Example 1: The BWB theorem. The Borel-Weil-Bott theorem (BWB) states [B] that,

for any simple finite dimensional Lie algebra \( g \), its maximal nilpotent subalgebra \( n_{\text{max}} \) and any (finite dimensional) irreducible \( g \)-module \( M \), the dimension \( H^i(n_{\text{max}}; M) \) does not depend on \( M \) and is equal to the cardinality of the set of elements of the Weyl group \( W(g) \) of length \( i \).

A generalization of the BWB theorem holds also for the nilpotent subalgebras complementary to any parabolic subalgebra of \( g \) (for references to ever more lucid and algebraic proofs of this algebraic theorem, see [I1]).

When the Lie superalgebra theory started to develop, being boosted by remarkable physical applications, to superize the BWB theorem was one of the first problems. It soon became clear that there is no hope to get a neat super analog of the BWB theorem. More precisely, for the maximal nilpotent subalgebras, the theorems are relatively concise and resemble the BWB theorem, though the notion of Weyl group becomes vague and has several analogs in
super setting (see \cite{Pe}, \cite{PS1}, \cite{PS2}). For the “opposite” case of the complements to the maximal parabolic superalgebras, the answer is rather complicated as Poletaeva’s results show \cite{Po1} (for the reader’s convenience, they are collected under one roof in \cite{Po2}).

Poletaeva’s results, as well as \cite{DPS}, demonstrate inevitability of computer-aided study (for more examples, see \cite{GL2}) of Lie algebra (co)homology too complicated to deal with with bare hands; especially so in super setting.

**Example 2: From Kostant and Leger–Luks to Tolpygo and Premet.** There are plenty of \(n\)-modules which is not a \(g\)-modules, the adjoint one is a most interesting. In what follows we will briefly write \(C^k\) instead of \(C^k(n; n)\) and similarly denote the spaces of cocycles, coboundaries and cohomology \((Z^k, B^k, H^k, \text{respectively})\). We denote \(H^* = \oplus H^k\), etc.

Kostant \cite{K} calculated \(\dim H^1\); Leger and Luks \cite{LL} calculated \(\dim H^2\). No general theorems on \(H^k\) existed until in mid-1970’s Tolpygo \cite{T} computed \(H^*\) and described the asymptotic behavior of \(\dim H^k\) as \(r \to \infty\) for \(k < 4\) and the maximal nilpotent subalgebra \(n\) of classical simple Lie algebra \(g\) of series \(sl(r), o(r)\) and \(sp(r)\).

To write supergravity equations, we need, at least, their left hand sides, which are (parts of) nonholonomic curvature tensors with values in \(H^2(n; g)\) for some nilpotent subalgebras \(n\) of \(sl(N|4)\), see \cite{GL1}, \cite{LPS}. It is natural to compute similar cohomology for nonholonomic manifolds as well, various flag varieties, to start with. DL discussed the matter with A. Premet and instead got the following beautiful general answer to a related question \cite{Pr}. The proof is regrettably still unpublished.

Hereafter the ground field is \(C\); let \(\alpha_i\) denote the \(i\)th simple root of a simple Lie algebra \(g\).

**Theorem.** (Premet) Let \(g\) be a simple finite dimensional Lie algebra, \(p_+\) its parabolic subalgebra, \(p = p_-\) the opposite algebra and let \(n\) be complementary to \(p_+\). Let \(E(\lambda) = L_\lambda\) be an irreducible (finite) \(g\)-module with highest weight \(\lambda\) such that \(E \simeq E^*\). Let \(V\) be a subspace in \(E\) which is \(p_-\)-invariant and contains \(E_− = \bigoplus_{\mu=k_0} \cdots \bigoplus_{\mu=k_0} E_\mu\). Let \(P(V)\) be the set of weights of \(V\) and \(w_0\) the longest element of the Weyl group and let \(\rho\) be the half sum of positive roots. Then, for any \(i < \rho g\), the following sequence is exact:

\[
0 \longrightarrow \bigoplus \{w \in W|\lambda(w) = i-1, w, w_0(\lambda) \notin P(V)\} \longrightarrow E(\lambda + \rho) \longrightarrow H^1(n; E(\lambda)/V) \longrightarrow 0.
\]

It is very interesting to compute \(\dim H^*\) and \(\dim H^k\) for various subalgebras \(n\) and all simple (finite dimensional, to start with) Lie algebras \(g\), and also Lie superalgebras. But the answer obtained in the form of dimensions of \(H^k\), even if known, is not quite satisfactory.

Indeed, the spaces \(\mathfrak{L}_i = \Pi(C^i+1)\) and \(\mathfrak{I}_i = \Pi(H^i+1)\) are endowed with natural structures of \(Z\)-graded Lie superalgebras (discovered by Nijenhuis in his, with co-authors, studies of tensor invariants used in physics, \cite{FN}, \cite{NR}). Now observe that nobody presents graded algebras by means of dimensions of their homogeneous components (e.g., we never introduce the polynomial algebra \(S^*(-)\) as “a graded algebra whose degree \(k\) component is of dimension \(\binom{n+k}{k}\)”), we usually present algebras in terms of generators and relations which is more graphic and more precise (the homogeneous components of \(U(-)\) have the same dimensions as those of \(S^*(-)\) for any deform of \(g\)). Besides, given a Lie (super)algebra, to “describe it” usually means to determine its semisimple part and the radical.

The Lie superalgebras \(\mathfrak{L}_i\) and \(\mathfrak{I}_i\) are graded and endowed with parity as follows: \(\mathfrak{L}_j = \Pi(C^j+1)\) and \(\mathfrak{I}_j = \Pi(H^j+1)\). In what follows we consider \(\mathfrak{I}_i\). The part \(\mathfrak{I}_{i-1}\) is obvious (especially if \(n\) is the maximal nilpotent subalgebra of a simple \(g\)), and every \(\mathfrak{I}_i\) is an \(\mathfrak{I}_0\)-module. In
particular, if \( n \) is a Lie algebra, then \( l_0 \) is a Lie algebra. The description of \( l_0 \) is a key step in the description of \( l \). The first examples (calculation of \( l \) for \( g = \mathfrak{sl}(2) \) and especially \( \mathfrak{sl}(3) \)) were very encouraging. Regrettably, for \( g \neq \mathfrak{sl}(3) \) and its maximal nilpotent subalgebra, the answer is not so neat.

2. Our result: examples of DG algebras and the list of their homologic elements.

The notion of \( L_\infty \)-algebras appeared as a slackening of the notion of Lie algebra and a formulation of the notions that vaguely lingered in various problems cf., e.g., [JS2], [JS3]. At the moment, there are not many explicit examples of \( L_\infty \)-algebras, though it is known that Lie algebra cohomology with values in the adjoint module, as well as the whole space of cochains, and any DG algebra, are such examples. Until this paper, nothing was known (as far as we know) about the structure of such DG algebras \( l \). To describe the whole of \( l \), in particular, for NON-maximal nilpotent sublagebras \( n \) of any simple (finite dimensional) Lie (super) algebra \( g \) is our third task.

Here we completely describe the Lie algebra \( l_0 = \Pi(H^1) \) for the maximal nilpotent subalgebra of any simple (finite dimensional) Lie algebra \( g \) and also the whole of \( l \) for \( g = \mathfrak{g}_2 \), \( \mathfrak{sl}(2) \) and \( \mathfrak{sl}(3) \). We conjecture that the structure of \( l \) is the same as for \( g = \mathfrak{g}_2 \) for any \( g \), except \( \mathfrak{sl}(2) \) and \( \mathfrak{sl}(3) \). The formulation of the result and conjecture (backed up with examples of \( g = \mathfrak{sl}(n) \) for \( n \leq 5 \)) was molded with the help of computer-aided experiments performed by one of us ([Le]).

In numerous problems, the homologic elements in every Lie superalgebra (odd elements \( x \) such that \([x,x] = 0\)), or rather the homology they determine, are vital, especially in deformation theory, see, e.g., [G2], [V2], [M]; they characterize algebroids [V1] and determine Lie algebras ([L2]). So we list them in \( l \) for \( g = \mathfrak{g}_2 \) and \( \mathfrak{sl}(3) \).

The Lie superalgebra structure on \( \Pi(H^*) \). Let \( R^+ \) be the set of positive roots, let the \( e_\beta \), where \( \beta \in R^+ \), be elements of the (weight) basis of \( g \) (for example, the Chevalley basis if \( g \) is simple), let the \( f^\beta \) be the elements of the dual basis. The basis of cochains is given by monomials \( a, b, c \in C^* \) of the form

\[
e^i \otimes f_1^{\beta_1} \wedge ... \wedge f_i^{\beta_i}, \text{ where (by abuse of notation) } i = a, b, c.
\]

Set

\[
a \cdot b = \sum_{k=1}^{n_a} (-1)^{n_a-k} f_a^{\beta_k} (e_a^b) e_a^i \otimes f_a^{\beta_1} \wedge ... \wedge f_a^{\beta_k} \wedge ... \wedge f_a^{\beta_{n_a}} \wedge f_b^{\beta_{n_1}} \wedge ... \wedge f_b^{\beta_{n_2}}
\]

and define the bracket on \( \Pi(C^*) \), where \( P(a) \) is the parity of \( a \) in \( P(C^*) \), so, for Lie algebras \( g \), it is equal to \( \deg a \pm 1 \)

\[
[a, b] = a \cdot b - (-1)^{P(a)P(b)} b \cdot a.
\]

It is subject to a direct verification that (1) defines the Lie superalgebra structure on \( \mathfrak{L}_{*+1} = \Pi(C^*)^{*+1} \), and

\[
d[a, b] = [da, b] + (-1)^{P(a)}[a, db].
\]

Therefore \( \Pi(Z^{*+1}) \) is a subalgebra of \( \Pi(C^*)^{*+1} \) and \( \Pi(B^{*+1}) \) is an ideal in \( \Pi(Z^{*+1}) \). Hence, we have a DG Lie superalgebras structure on \( \mathfrak{L} : = \Pi(C^*)^{*+1} \) and \( l : = \Pi(H^{*+1}) \), cf., e.g., [JS3]. The differential in \( l \) is zero and \( l \) often is very small, so \( \Pi(Z^{*+1}) \) might be more interesting than \( l \) in some questions, cf. sec. 3.

For any maximal nilpotent subalgebra \( n \) of any simple (finite dimensional) Lie algebra \( g \), the space \( l_{-1} \) is, clearly, one-dimensional.

Let us describe the Lie algebra \( l_0 = \Pi(H^1) \). Select a basis of \( n \) consisting of root vectors \( e_\beta \). It is convenient to select a basis in which the structure constants have the least possible
absolute values. The Chevalley basis is (the only) such basis and we select it for definiteness sake. Let \( n = \text{rk} \, g \). The \( \mathbb{Z}^n \)-grading of \( n \) by roots induces a \( \mathbb{Z}^n \)-grading of \( C^* \) and we denote by \( C^*_i \) the subspace of \( C^i \) of weight \( \gamma \).

2.1. To formulate our main result, we denote by \( \mu \) the (obviously, unique) maximal root in \( g \), and by \( l(\alpha) \) the level (i.e., the sum of coordinates with respect to the basis of simple roots) of the root \( \alpha \). For \( i = 1, \ldots, n = \text{rk} \, g \), let

\[
m_i = \max\{m \in \mathbb{Z}_+ \mid \mu - k\alpha_i \in R^+ \text{ for any } k = 0, 1, \ldots, m\}.
\]

Let \( \alpha = \sum_{1 \leq i \leq n} A_i(\alpha)\alpha_i \) be the decomposition of a weight \( \alpha \) with respect to the simple roots.

**Theorem.** If \( g \neq \mathfrak{sl}(2) \), then \( \text{dim} \, H^1 = 2n \) \([\mathbb{K}] \), and for a basis one can take the cocycles

\[
c_i = \sum_{\alpha \in R^+} A_i(\alpha)e_\alpha \otimes f^\alpha
\]

and

\[
b_i = e_{\gamma_i} \otimes f^{\alpha_i}, \quad \text{where } \gamma_i = \mu - m_i\alpha_i,
\]

where \( i = 1, \ldots, n \). Further, for \( g \neq \mathfrak{sl}(2), \mathfrak{sl}(3) \), we have

\[
[c_i, c_j] = [b_i, b_j] = 0,
\]

and

\[
[c_i, b_j] = w_i(b_j)b_j,
\]

where \( w(b) = (w_1(b), \ldots, w_n(b)) \) is the weight of the cochain \( b \).

2.2. **Proof.** We start with the following Lemma.

2.2.1. **Lemma.** a) \( \text{dim} \, H^1_0 = n \) and for a basis one can take the cocycles (3).

b) \( [c_i, c_j] = 0 \) for all \( i \) and \( j \).

**Proof.** a) Clearly, \( B^1_0 = 0 \), so \( H^1_0 = Z^1_0 \). Further, the cochain

\[
c = \sum_{\alpha \in R^+} a(\alpha)e_\alpha \otimes f^\alpha
\]

is a cocycle if and only if

\[
a(\alpha + \beta) = a(\alpha) + a(\beta) \text{ whenever } \alpha, \beta, \alpha + \beta \in R^+.
\]

It is evident that functions \( A_i \) on \( R^+ \) satisfy (6) and, by the induction on the level of the weight (in our case, the sum of the coefficients of the weight in the decomposition with respect to simple roots, see \([\mathbb{H}]\)) we prove that the set \( \{A_i\}_{i=1}^n \) forms a basis in the linear space of functions \( a : R^+ \rightarrow \mathbb{C} \) with property (6).

b) is straightforward. \( \square \)

2.2.2. **Lemma.** The cochains (4) are non-trivial linearly independent cocycles.

**Proof.** First, observe that the \( b_i \) are not co-boundaries because the weight \( \beta_i = \gamma_i - \alpha_i \) of \( b_i \) is not a root, and hence there are no co-boundaries of weight \( \beta_i \) in \( C^1 \).

Let us verify that \( db_i = 0 \). By definition, we have

\[
db_i = d(e_{\gamma_i} \otimes f^{\alpha_i}) = de_{\gamma_i} \wedge f^{\alpha_i} - e_\mu \otimes df^{\alpha_i}.
\]

It is clear that

\[
df^{\alpha_i}(e_\alpha, e_\beta) = f^{\alpha_i}([e_\alpha, e_\beta]) = N_{\alpha, \beta} f^{\alpha_i}(e_{\alpha + \beta}) = 0
\]
because \( \alpha_i \) is a simple root and equality \( \alpha + \beta = \alpha_i \) is impossible for any positive roots \( \alpha \) and \( \beta \). Further,

\[
de_{\alpha_i}(e_\alpha) = [e_\alpha, e_{\gamma_i}] = N_{\alpha_i \gamma_i} e_{\alpha + \gamma_i}.
\]

The right hand side of this expression is nonzero only if \( \beta = \alpha + \gamma_i \) is a positive root. Since \( \mu \) is the maximal root and \( k \alpha_i \) is not a root for any simple root \( \alpha_i \) and any integer \( k > 1 \), we see that

\[
de_{\gamma_i} = N_{\alpha_i \gamma_i} e_{\alpha_i + \gamma_i} \otimes f_{\alpha_i},
\]

where \( N_{\alpha \beta} = 0 \) if \( \alpha + \beta \) is not a root. Hence,

\[
d_{b_i} = d(e_{\gamma_i} \otimes f_{\alpha_i}) = de_{\gamma_i} \wedge f_{\alpha_i} - e_{\gamma_i} \otimes df_{\alpha_i} = N_{\alpha_i \gamma_i} e_{\gamma_i + \alpha_i} \otimes f_{\alpha_i} \wedge f_{\alpha_i} = 0.
\]

Since the weights of \( b_i \) are distinct, they are linearly independent. \( \square \)

Formula (6) is a particular case of (8), see sec. 2.4.3.

So we have \( 2n \) linearly independent nontrivial 1-cocycles; by Kostant’s result ([K]), they form a basis in \( H^1 \). To complete the proof of the theorem, we establish the following Lemma.

2.2.3. Lemma. If the level of \( \mu \) is \( \geq 5 \), then \( [b_i, b_j] = 0 \) for all \( i, j \).

Proof. From the definition of the \( b_i \) we see that

\[
[b_i, b_j] = f_{\alpha_i}(e_{\gamma_j})e_{\gamma_i} \otimes f_{\alpha_j} - f_{\alpha_j}(e_{\gamma_i})e_{\gamma_j} \otimes f_{\alpha_i}.
\]

As is known ([FH]), in any simple finite dimensional Lie algebra \( \mathfrak{g} \), the “arithmetical sequence” of weights whose difference is a positive root can not have more than four roots. Hence, \( m_i \leq 3 \) and level of the root \( \gamma_i = \mu - m_i \alpha_i \) is \( \geq 2 \). So, \( \gamma_i \) is not a simple root and, in particular, \( \gamma_i \neq \alpha_j \). Hence, \( f_{\alpha_j}(e_{\gamma_i}) = 0 \) for all \( i, j \). \( \square \)

It is easy to see from the list of simple finite dimensional Lie algebras that the level of \( \mu \) is \( < 5 \) only for the following \( \mathfrak{g} \):

\[
A_1, A_2, A_3, A_4, B_2.
\]

Computer calculations show that (5) holds for all algebras in the above list, except \( A_1 = \mathfrak{sl}(2) \) and \( A_2 = \mathfrak{sl}(3) \). \( \square \)

2.3. The “exceptional” cases \( \mathfrak{sl}(2) \) and \( \mathfrak{sl}(3) \) are described as follows.

Lemma. \( \mathfrak{sl}(2) \): \( \mathfrak{l} = \mathfrak{l}_0 = \text{Span}(c_1) \) is commutative;

\( \mathfrak{sl}(3) \): \( \mathfrak{l}_0 \simeq \mathfrak{gl}(2) \) with \([b_1, b_2] = c_2 - c_1\).

Let \( L^n \) be the irreducible \( \mathfrak{gl}(2) \)-module with the highest weight \( n \) with respect to \( \mathfrak{sl}(2) \) and the value \( c \) on the center, \( 1 \in \mathfrak{gl}(2) \). Then, as \( \mathfrak{l}_0 \simeq \mathfrak{gl}(2) \)-modules,

\[
L_{-1} \simeq L^{0:2}; \quad L_1 \simeq L^{2:-2} \oplus L^{1:1}; \quad L_2 \simeq L^{1:-3}.
\]

Proof. The statement about \( \mathfrak{sl}(2) \) is obvious, that about \( \mathfrak{sl}(3) \) follows from the multiplication table 2.4.5. \( \square \)

2.4. Lie superalgebra \( \mathfrak{l} = \Pi(H^{*1}) \) for \( \mathfrak{g} = \mathfrak{g}_2 \) and \( \mathfrak{sl}(3) \).
2.4.1. The case of $g = g_2$.

**Theorem.** For $g = g_2$, the basis of $H^*$ is given by the following list. The multiplication table given in sec. 2.4.2 implies that $l$ is solvable and the nonzero terms of the derived series are as follows (where $l(0) = 1$, $l(i+1) = [l, l(i)]$):

$$
\begin{align*}
  l(1) &= H^0 \oplus \text{Span}(h_3^1, h_4^1) \oplus H^2 \oplus H^3 \oplus H^4 \oplus H^5 \oplus H^6; \\
  l(2) &= \text{Span}(h_3^1, h_4^1, h_2^1, h_2^2, h_6^1, h_3^2, h_3^3, h_3^4, h_3^5, h_3^6, h_4^3); \\
  l(3) &= \text{Span}(h_2^4, h_6^6).
\end{align*}
$$

2.4.2. The basis of $H^*$ for $g = g_2$. $H^0$: $h_0^0 = e_{2,3}$

$H^1$: $h_1^1 := c_1 = 2e_{0,1} \otimes f^{0,1} - 3e_{1,0} \otimes f^{1,0} - e_{1,1} \otimes f^{1,1} + e_{1,2} \otimes f^{1,2} + 3e_{1,3} \otimes f^{1,3}$

$H^2$: $h_2^2 = -68e_{0,1} \otimes f^{0,1} + 74e_{1,1} \otimes f^{1,0} + 39e_{1,2} \otimes f^{1,1} + 3e_{1,3} \otimes f^{1,2} + 3e_{2,3} \otimes f^{1,3}$

$H^3$: $h_3^3 = -18e_{0,1} \otimes f^{0,1} - 9e_{0,1} \otimes f^{1,0} + 60e_{1,0} \otimes f^{1,1} + 60e_{1,1} \otimes f^{1,2} + 21e_{1,1} \otimes f^{1,3}$

$H^4$: $h_4^4 = -4e_{0,1} \otimes f^{0,1} + 35e_{0,1} \otimes f^{1,0} + 31e_{1,0} \otimes f^{1,1} + 31e_{1,1} \otimes f^{1,2} + 27e_{1,1} \otimes f^{1,3} + 27e_{1,2} \otimes f^{1,3}$

$H^5$: $h_5^5 = -7e_{0,1} \otimes f^{0,1} + 2e_{0,1} \otimes f^{1,0} + 4e_{0,1} \otimes f^{1,1} + 4e_{0,1} \otimes f^{1,2} + 4e_{0,1} \otimes f^{1,3}$

$H^6$: $h_6^6 = 11e_{0,1} \otimes f^{0,1} + 11e_{0,1} \otimes f^{1,0} + 11e_{0,1} \otimes f^{1,1}$
\[ h^4_7 = e_{1,0} \otimes f^{0,1} \wedge f^{1,0} \wedge f^{1,2} \wedge f^{1,3} + e_{1,1} \otimes f^{0,1} \wedge f^{1,1} \wedge f^{1,2} \wedge f^{1,3} + e_{2,3} \otimes f^{0,1} \wedge f^{1,2} \wedge f^{1,3} \wedge f^{2,3} \]
\[ H^5: \]
\[ h^5_1 = e_{0,1} \otimes f^{1,0} \wedge f^{1,1} \wedge f^{1,2} \wedge f^{1,3} \wedge f^{2,3} \]
\[ h^5_2 = -e_{0,1} \otimes f^{0,1} \wedge f^{1,1} \wedge f^{1,2} \wedge f^{1,3} \wedge f^{2,3} + e_{1,0} \otimes f^{1,0} \wedge f^{1,1} \wedge f^{1,2} \wedge f^{1,3} \wedge f^{2,3} \]
\[ h^5_3 = e_{0,1} \otimes f^{0,1} \wedge f^{1,0} \wedge f^{1,1} \wedge f^{1,2} \wedge f^{1,3} \wedge f^{2,3} + e_{1,3} \otimes f^{1,0} \wedge f^{1,1} \wedge f^{1,2} \wedge f^{1,3} \wedge f^{2,3} \]
\[ h^5_4 = e_{1,0} \otimes f^{0,1} \wedge f^{1,0} \wedge f^{1,1} \wedge f^{1,2} \wedge f^{1,3} \wedge f^{2,3} \]
\[ h^5_5 = e_{1,0} \otimes f^{0,1} \wedge f^{1,0} \wedge f^{1,1} \wedge f^{1,2} \wedge f^{1,3} \wedge f^{2,3} + e_{2,3} \otimes f^{0,1} \wedge f^{1,1} \wedge f^{1,2} \wedge f^{1,3} \wedge f^{2,3} \]

2.4.3. The multiplication table in \( H^5 \). Observe that the Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) naturally acts on \( \mathfrak{n} \) and \( C^* \). The image of \( \mathfrak{h} \) in \( C^1 = \mathfrak{gl}(\mathfrak{n}) \) is precisely \( \text{Span}(c_i \mid i = 1, \ldots, \text{rk} \, \mathfrak{g}) \), so we will save several lines in the multiplication tables \( H^1 \times H^k \) here and in sec. 2.4.5 by replacing them with a generalization of formula (6):
\[ [c_i, c] = w_i(c)c \text{ for any chain } c \text{ of weight } w(c). \]

\[ \mathbf{H}^0 \times \mathbf{H}^1: \]
\[
\begin{array}{cccc}
  & h^1_1 & h^1_2 & h^1_3 & h^1_4 \\
 h^0_1 & 0 & -2h^0_1 & 0 & 0
\end{array}
\]

\[ \mathbf{H}^0 \times \mathbf{H}^2: \]
\[
\begin{array}{cccccc}
  & h^2_1 & h^2_2 & h^2_3 & h^2_4 & h^2_5 & h^2_6 & h^2_7 \\
 h^0_1 & h^1_3 & 0 & 0 & 0 & 0 & 0 & 2h^1_4
\end{array}
\]

\[ \mathbf{H}^0 \times \mathbf{H}^3: \]
\[
\begin{array}{cccccccc}
  & h^3_1 & h^3_2 & h^3_3 & h^3_4 & h^3_5 & h^3_6 & h^3_7 & h^3_8 \\
 h^0_1 & -5 & h^2_2 & 0 & -h^2_4 & \frac{43}{35} & h^2_6 & 0 & 0 & 0
\end{array}
\]

\[ \mathbf{H}^0 \times \mathbf{H}^4: \]
\[
\begin{array}{cccccccc}
  & h^4_1 & h^4_2 & h^4_3 & h^4_4 & h^4_5 & h^4_6 & h^4_7 & h^4_7 \\
 h^0_1 & \frac{1}{22} & h^2_2 & 0 & -\frac{1}{3}h^2_3 & h^2_6 & h^2_7 & 0 & 0
\end{array}
\]

\[ \mathbf{H}^0 \times \mathbf{H}^5: \]
\[
\begin{array}{cccccccc}
  & h^5_1 & h^5_2 & h^5_3 & h^5_4 & h^5_5 & h^5_6 & h^5_7 & h^5_7 \\
 h^0_1 & -\frac{1}{93} & h^2_2 & \frac{5}{105} & h^4_6 & 0 & 0 & -\frac{2}{3}h^5_7
\end{array}
\]

\[ \mathbf{H}^0 \times \mathbf{H}^6 = 0; \]
\[ \mathbf{H}^1 \times \mathbf{H}^1 = 0 \text{ (except for (8))} \]
\[ \mathbf{H}^1 \times \mathbf{H}^2: \]
\[
\begin{array}{cccccccc}
  & h^2_1 & h^2_2 & h^2_3 & h^2_4 & h^2_5 & h^2_6 & h^2_7 & h^2_8 \\
 h^1_1 & 0 & 324h^2_4 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

\[ \mathbf{H}^1 \times \mathbf{H}^3: \]
\[ LIE \text{ SUPERALGEBRAS IN COHOMOLOGY} \]

\[
\begin{array}{cccccccc}
 & h_1^3 & h_2^3 & h_3^3 & h_4^3 & h_5^3 & h_6^3 & h_7^3 \\
\ h_3^4 & 90h_3^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
h_4^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

**H^1 \times H^4:**

\[
\begin{array}{cccccccc}
 & h_1^4 & h_2^4 & h_3^4 & h_4^4 & h_5^4 & h_6^4 & h_7^4 \\
\ h_3^5 & 0 & 0 & 0 & -\frac{3}{103}h_6^4 & 0 & 0 & 0 \\
h_4^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[ H^1 \times H^5 = 0 \text{ (except for (8))} \]

\[ H^1 \times H^6 = 0 \text{ (except for (8))} \]

**H^2 \times H^2:**

\[
\begin{array}{cccccccc}
 & h_1^2 & h_2^2 & h_3^2 & h_4^2 & h_5^2 & h_6^2 & h_7^2 \\
\ h_1^2 & 0 & -324h_3^3 & 0 & 0 & 0 & 0 & -\frac{2}{5}h_5^3 \\
h_2^2 & - & 0 & 144h_3^3 & 0 & 108h_7^3 & 0 & 0 \\
h_3^2 & - & - & 0 & 0 & \frac{10}{103}h_8^3 & 0 & 0 \\
h_4^2 & - & - & - & 0 & 0 & 0 & 0 \\
h_5^2 & - & - & - & - & 0 & 0 & 0 \\
h_6^2 & - & - & - & - & - & 0 & 0 \\
h_7^2 & - & - & - & - & - & - & 0 \\
\end{array}
\]

**H^2 \times H^3:**

\[
\begin{array}{cccccccc}
 & h_1^3 & h_2^3 & h_3^3 & h_4^3 & h_5^3 & h_6^3 & h_7^3 \\
\ h_1^2 & 0 & 0 & 0 & -7h_2^4 & 0 & -\frac{8}{103}h_6^4 & 0 \\
h_2^3 & \frac{4320}{31}h_2^4 & 0 & 0 & -\frac{14056}{103}h_6^4 & 0 & 0 & 0 \\
h_3^3 & -150h_3^4 & 0 & 0 & 0 & \frac{10}{7}h_7^4 & 0 & 0 \\
h_4^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h_5^3 & 30h_5^4 & -\frac{110}{103}h_6^4 & 0 & 0 & 0 & 0 & 0 \\
h_6^3 & \frac{1980}{103}h_6^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
h_7^3 & 0 & 0 & 0 & -434h_7^4 & 0 & 0 & 0 \\
\end{array}
\]

**H^2 \times H^4:**
$H^2 \times H^5$: 

| $h_1^2$ | $h_2^2$ | $h_3^2$ | $h_4^2$ | $h_5^2$ | $h_6^2$ |
|---------|---------|---------|---------|---------|---------|
| 0       | 0       | 0       | $h_2^5$ | 0       | 0       |
| 0       | 0       | 0       | 432$h_4^5$ | 0      | 0       |
| 0       | 0       | 0       | 5$h_5^5$   | 0       | 0       |
| 0       | 0       | 0       | 0       | 0       | 0       |
| $h_3^2$ | $h_4^2$ | 0       | 0       | 0       | 0       |
| 0       | 0       | 0       | 0       | 0       | 0       |
| $h_5^2$ | $h_6^2$ | 0       | 0       | 0       | 0       |
| $h_7^2$ | $-3h_3^5$ | 0       | 0       | 0       | 0       |

$H^3 \times H^3$: 

| $h_1^3$ | $h_2^3$ | $h_3^3$ | $h_4^3$ | $h_5^3$ | $h_6^3$ | $h_7^3$ | $h_8^3$ |
|---------|---------|---------|---------|---------|---------|---------|---------|
| -       | 0       | 0       | 13020$h_2^5$ | 0       | 120$h_4^5$ | 0       | 0       |
| -       | -       | 0       | 0       | 0       | 0       | 0       | 0       |
| -       | -       | -       | 0       | 0       | 0       | 0       | 0       |
| -       | -       | -       | -       | 0       | 0       | 0       | 0       |
| -       | -       | -       | -       | -       | 0       | 0       | 0       |
| $h_6^3$ | -       | -       | -       | -       | -       | 0       | 0       |
| $h_7^3$ | -       | -       | -       | -       | -       | -       | 0       |
| $h_8^3$ | -       | -       | -       | -       | -       | -       | -       |

$H^3 \times H^4$: 

| $h_1^3$ | $h_2^3$ | $h_3^3$ | $h_4^3$ | $h_5^3$ | $h_6^3$ | $h_7^3$ | $h_8^3$ |
|---------|---------|---------|---------|---------|---------|---------|---------|
| -       | 0       | 0       | 13020$h_2^5$ | 0       | 120$h_4^5$ | 0       | 0       |
| -       | -       | 0       | 0       | 0       | 0       | 0       | 0       |
| -       | -       | -       | 0       | 0       | 0       | 0       | 0       |
| -       | -       | -       | -       | 0       | 0       | 0       | 0       |
| -       | -       | -       | -       | -       | 0       | 0       | 0       |
| $h_6^3$ | -       | -       | -       | -       | -       | 0       | 0       |
| $h_7^3$ | -       | -       | -       | -       | -       | -       | 0       |
| $h_8^3$ | -       | -       | -       | -       | -       | -       | -       |
2.4.4. The basis of $H^*$ for $\mathfrak{g} = \mathfrak{sl}(3)$.

$H^0 : h_1^0 = e_{1,1}$

$H^1$ (where $c_i = E_{ii}$, $h_4^1$ and $h_1^1$ are the raising and lowering operators of $\mathfrak{sl}(2)$):

$h_1^1 = e_{0,1} \otimes f^{1,0}$

$h_2^1 := c_1 - c_2 = -e_{0,1} \otimes f^{0,1} + e_{1,0} \otimes f^{1,0}$

$h_3^1 := c_1 + c_2 = e_{0,1} \otimes f^{0,1} + e_{1,0} \otimes f^{1,0} + 2e_{1,1} \otimes f^{1,1}$

$h_4^1 = e_{1,0} \otimes f^{0,1}$

$H^2$:

$h_1^2 = e_{0,1} \otimes f^{1,0} \wedge f^{1,1}$

$h_2^2 = -e_{0,1} \otimes f^{0,1} \wedge f^{1,1} + e_{1,0} \otimes f^{1,0} \wedge f^{1,1}$

$h_3^2 = e_{0,1} \otimes f^{0,1} \wedge f^{1,0} - e_{1,1} \otimes f^{1,0} \wedge f^{1,1}$

$h_4^2 = e_{1,0} \otimes f^{0,1} \wedge f^{1,1}$

$h_5^2 = e_{1,0} \otimes f^{0,1} \wedge f^{1,0} + e_{1,1} \otimes f^{0,1} \wedge f^{1,1}$

$H^3$:

$h_1^3 = e_{0,1} \otimes f^{0,1} \wedge f^{1,0} \wedge f^{1,1}$

$h_2^3 = e_{1,0} \otimes f^{0,1} \wedge f^{1,0} \wedge f^{1,1}$

2.4.5. The multiplication table in $l$. for $\mathfrak{g} = \mathfrak{sl}(3)$.

$H^0 \times H^1$:

$$
\begin{array}{c|cccc}
 & h_1^0 & h_2^0 & h_3^0 & h_4^0 \\
\hline
h_1^0 & 0 & 0 & -2h_1^0 & 0 \\
\end{array}
$$

$H^0 \times H^2$:

$$
\begin{array}{c|cccc}
 & h_1^0 & h_2^0 & h_3^0 & h_4^0 \\
\hline
h_1^0 & h_1^0 & h_2^0 & h_3^0 & h_4^0 \\
\end{array}
$$

$H^0 \times H^3$:

$$
\begin{array}{c|c}
 & h_1^0 \\
\hline
h_1^0 & -\frac{1}{2}h_3^0 \\
\end{array}
$$

$H^1 \times H^1$:

$$
\begin{array}{c|c}
 & h_1^0 \\
\hline
h_1^0 & 0 \\
\end{array}
$$

$H^1 \times H^2$:

$$
\begin{array}{c|cccc}
 & h_1^0 & h_2^0 & h_3^0 & h_4^0 \\
\hline
h_1^0 & 0 & 2h_1^0 & h_3^0 & h_4^0 \\
\end{array}
$$
\[ \mathbf{H}^1 \times \mathbf{H}^3: \]

\[
\begin{array}{cccc}
  h_1^0 & h_1^1 & h_1^2 & h_1^3 \\
  h_1^1 & 0 & h_1^3 & h_1^1 \\
  h_1^2 & h_1^3 & 0 & h_1^1 \\
  h_1^3 & h_1^1 & h_1^3 & 0 \\
\end{array}
\]

\[ \mathbf{H}^2 \times \mathbf{H}^2: \]

\[
\begin{array}{ccccc}
  h_2^0 & h_2^1 & h_2^2 & h_2^3 & h_2^4 & h_2^5 \\
  h_2^1 & 0 & 0 & 0 & 0 & -2h_2^1 \\
  h_2^2 & 0 & 0 & 2h_2^3 & 0 & -2h_2^5 \\
  h_2^3 & 0 & 2h_2^1 & 0 & -2h_2^3 & 0 \\
  h_2^4 & 0 & 2h_2^3 & 0 & 0 & 0 \\
  h_2^5 & -2h_2^1 & -2h_2^5 & 0 & 0 & 0 \\
\end{array}
\]

2.5.1. The homologic elements in \( \mathfrak{l} \) for \( \mathfrak{g} = \mathfrak{g}_2 \). Let \( x \in \mathfrak{l} \) be a homologic element (odd and such that \( [x,x] = 0 \)). Then \( x \) is of the form

\[ x = x^0 + x^2 + x^4 + x^6, \quad \text{where} \quad x^i = \sum_j k_j^ih_j^i \in \mathbf{H}^i. \]

a) \( x^0 = 0 \). In this case the condition \( [x,x] = 0 \) takes the form

\[ [x^2,x^2] = 0 \quad \text{and} \quad [x^2,x^4] = 0. \]

Let us find first the form of \( x^2 \). From the multiplication table we deduce that \( [x^2,x^2] = 0 \) if and only if the support of (the indices of) nonzero coefficients in the sum \( x^2 = \sum_j k_j^2h_j^2 \) — a subset of the set \( \{1, 2, \ldots, 7\} \) — is of the form \( A \cup B \), where \( A \subset \{4, 6\} \) and \( B \) is only one of following

\[ 0; \{1\}; \{2\}; \{3\}; \{5\}; \{7\}; \{1, 3\}; \{1, 5\}; \{2, 7\}; \{3, 7\}; \{5, 7\}. \]

We have \( [x^2,x^4] = 0 \) if and only if the following conditions hold:

\[ k_1^2k_4^1 = k_3^2k_4^1 = 0; \quad k_2^4k_4^1 = 0; \quad k_5^3k_4^1 = 0; \quad k_2^7k_4^1 = 0. \]

So, the homologic elements with \( x^0 = 0 \) are of form

\[ x = y + z, \quad \text{where} \quad y \in \text{Span}(h_4^2, h_6^2, h_2^4, h_3^4, h_5^4, h_6^4, h_7^4, h_1^0, h_2^0) \]

and \( z \) is one of the following:

\[ k_1^2h_1^2 + k_3^2h_5^2 + k_4^1h_4^1 + k_4^1h_4^4 \quad \text{with} \quad k_7^2k_4^1 + k_3^2k_1^4 = 0; \]

\[ k_2^4h_2^2 + k_4^1h_4^4; \]

\[ k_2^3h_2^2 + k_2^4h_2^4; \]

\[ k_2^3h_3^2 + k_2^4h_2^2; \]

\[ k_2^3h_1^2 + k_3^3h_3^3 + k_4^1h_4^1; \]

\[ k_2^3h_5^2 + k_2^3h_2^2 + k_4^1h_4^4. \]

b) \( x^0 \neq 0 \), i.e., \( k_1^0 \neq 0 \). Then since \( [\mathbf{H}^0,\mathbf{H}^6] = 0 \), the condition \( [x,x] = 0 \) holds if and only if

\[ [x^0,x^2] = 0; \quad [x^0,x^4] + [x^2,x^2] = 0; \quad [x^2,x^4] = 0. \]

Using once more the multiplication table, we deduce that \( x \) is homologic if and only if

\[ x = y + z, \quad \text{where} \quad y \in \text{Span}(h_4^0, h_6^0, h_2^0, h_3^0, h_5^0, h_6^0, h_7^0, h_1^0, h_2^0) \]
and \( z \) is only one of the following (for \( \alpha \neq 0 \) and any \( \beta, \gamma \in \mathbb{C} \)):
\[
\begin{align*}
\alpha h_1^{0} + \beta h_2^{0} + \gamma (h_3^{2} + 540\frac{2}{\alpha}h_4^{3}) ; \\
\alpha h_1^{0} + \beta h_2^{0} + \gamma (h_5^{2} - \frac{108\beta}{5\alpha}h_6^{3}) .
\end{align*}
\]

2.5.2. The homologic elements in \( \mathfrak{l} \). for \( \mathfrak{g} = \mathfrak{sl}(3) \). Let \( x \in \mathfrak{l} \) be a homologic element. Then \( x \) is of the form \( x = x^0 + x^2 \), cf. (7), and hence \( x \) is homologic if and only if
\[
[x^0, x^2] = 0 \quad \text{and} \quad [x^2, x^2] = 0 .
\]

a) \( x^0 \neq 0 \). One can see from the multiplication table, that \([x^0, x^2] = 0 \) if and only if
\[
k_1^0 k_1^2 = k_1^0 k_2^2 = k_1^0 k_4^2 = 0 ,
\]
e.g., in this case \( k_2^2 = k_3^2 = k_4^2 = 0 \). So, we need \( x \in \text{Span}(h_1^0, h_3^2, h_5^2) \). One can also see from the table, that
\[
[h_1^0, h_3^2] = [h_1^0, h_5^2] = [h_3^2, h_5^2] = 0 ,
\]
i.e., any \( x \in \text{Span}(h_1^0, h_3^2, h_5^2) \) is homologic. So, the final answer is:
\[
x = ah_1^0 + bh_3^2 + ch_5^2 \quad \text{for} \ a \neq 0 \text{ and any} \ b, c \in \mathbb{C} .
\]

b) \( x^0 = 0 \). Clearly, the elements that belong to either \( L^{2,-2} \) or \( L^{1,-1} \) are homologic. But there are also “mixed” elements \( x^2 \) satisfying \([x^2, x^2] = 0 \). From the multiplication table we deduce that \([x^2, x^2] = 0 \) if and only if
\[
k_1^2 k_2^2 - k_2^2 k_3^2 = k_2^2 k_4^2 + k_3^2 k_4^2 = 0 .
\]
So, the answer in this case is:
\[
x = ah_1^2 + bh_2^2 + ch_3^2 + dh_4^2 + eh_5^2 , \quad \text{where} \ ae - bc = be + cd = 0 .
\]

3. A discussion. After a nice example of \( \mathfrak{l} \). for \( \mathfrak{g} = \mathfrak{sl}(3) \) it was rather discouraging to discover the lack of simple components in \( \mathfrak{l}_0 \) for all other algebras \( \mathfrak{g} \). Could it be that the “big” DG Lie superalgebra \( \mathfrak{L} \). = \( \Pi(C^{*+1}) \) is more interesting than \( \mathfrak{l} \).? Let us consider two examples.

3.1. The Nijenhuis bracket. Recall that the Nijenhuis bracket, with which all similar examples started, is defined on the space of sections of \( \Omega^* \otimes \mathcal{F} \text{Vect} \), where \( \mathcal{F} \), \text{Vect} and \( \Omega^* \) are the sheaves of functions, vector fields and differential forms on a given manifold. Hereafter tensoring is performed over \( \mathcal{F} \) and sheaves are replaced by modules and rings of sections.

The Nijenhuis bracket is defined, for any \( \omega^k \in \Omega^k \), \( \omega^l \in \Omega^l \) and \( \xi, \eta \in \text{Vect} \), to be
\[
\begin{align*}
\omega^k \otimes \xi, \omega^l \otimes \eta &\mapsto (\omega^k \wedge \omega^l) \otimes [\xi, \eta] + \\
(\omega^k \wedge L_\xi(\omega^l) + (-1)^kd\omega^k \wedge \iota(\omega^l)) \otimes \eta + \\
(-L_\eta(\omega^k) \wedge \omega^l + (-1)^l\iota(\eta)(\omega^k) \wedge d\omega^l) \otimes \xi ,
\end{align*}
\]
where \( \iota \) is the inner product and \( L_X \) is the Lie derivative with respect to the field \( X \). The Nijenhuis bracket has the following interpretation which implies its invariance: the invariant operator \( D : (\Omega^k \otimes \mathcal{F} \text{Vect}(M), \Omega^*) \rightarrow \Omega^* \) given by the formula
\[
D(\omega^k \otimes \xi, \omega) = \\
d(\omega^k \wedge \iota(\xi)(\omega)) + (-1)^k\omega^k \wedge \iota(\xi)(d\omega) = d\omega^k \wedge \iota(\xi)(\omega) + (-1)^k\omega^k \wedge L_\xi(\omega)
\]
is, for a fixed \( \omega^k \otimes \xi \), a superderivation of the supercommutative superalgebra \( \Omega^* \) and the Nijenhuis bracket is just the supercommutator of these superderivations, see [G1]. So we
can identify \( \Omega^* \otimes \mathcal{F} \mathbf{Vect}(\hat{M}) \), with the centralizer \( C(d) \) of the exterior differential considered as a vector field on \( \hat{M} \), where \( \hat{M} \) is the supermanifold \( (M, \Omega^*(M)) \):

\[
C(d) = \{ D \in \mathbf{vect}(\hat{M}) \mid [D, d] = 0 \}.
\]

(Here by \( \mathbf{vect} \) we denote the Lie superalgebra on the space of sections of the sheaf \( \mathbf{Vect} \)).

In contradistinction with a rich algebra of chains \( \mathbf{C} \), we have \( \mathbf{I} = 0 \) (at least, locally). The above applies to supermanifolds \( M \) as well.

### 3.2. On formula (1) and the codifferential \( d \)

Formula (1) shows that \( \mathbf{C} \) is isomorphic to the simple Lie superalgebra \( \mathbf{vect}(\dim \Pi(n)) \) of polynomial vector fields on the superspace \( \Pi(n)^* \) regardless of multiplication in \( n \).

The exterior differential in the cochain complex \( C^* \) is just a homological vector field of degree 1 in \( \mathbf{vect}(\dim \Pi(n)) \). The definition of the codifferential \( d \) shows that if the bracket in \( n \) vanishes identically, as is the case for Hermitean symmetric spaces \( X = G/P \), the differential \( d \) is also zero and \( \mathbf{I} = \mathbf{vect}(\dim \Pi(n)) \).

In Example 3.1, \( d = \sum \xi_i \frac{\partial}{\partial x_i} \), where \( \xi_i = dx_i \). In this realization, \( d \) is a degree 0 “maximally nondegenerate” (see [V2]) vector field.

If \( n \) is a simple Lie algebra, the degree 1 vector field in \( \mathbf{vect}(\dim \Pi(n)) \) is also “maximally nondegenerate” but, unlike degree 0 fields, its explicit form is rather complicated, except for \( \mathfrak{sl}(3) \) and \( \mathfrak{gl}(n) \), cf. [L2].

Whatever the form of \( d \), here is an implicit description of various DG Lie superalgebras: in \( \mathbf{vect}(m|n) = \mathfrak{der}C[X] \), where \( X = (x_1, \ldots, x_m, \xi_1, \ldots, \xi_n) \) and where \( \deg X_i = 1 \) for all \( i \), fix an element \( d \) of degree 1. Then the centralizer \( C(d) \) is a subalgebra in \( \mathbf{vect}(m|n) \), is isomorphic to \( \mathbf{vect}(C(d)/\text{Im } d) \) and \( C(d)/\text{Im } d \) are DG Lie algebras. In sec. 2 we explicitly described \( \mathbf{I} = C(d)/\text{Im } d \) related with the nilpotent Lie algebra structure on \( \Pi(\mathbf{vect}(0|n)_{-1}) \).

### References

[Ak] M. Alexandrov, M. Kontsevich, A. Schwarz, O. Zaboronsky, The geometry of the master equation and topological quantum field theory, Internat. J. Modern Phys. A **12** (1997), no. 7, 1405–1429

[JS2] G. Barnich, R. Fulp, T. Lada, J. Stasheff, The sh-Lie structure of Poisson brackets in field theory, Comm. Math. Phys. **191** (1998), no. 3, 585–601

[B] R. Bott, Homogeneous vector bundles, Ann. Math. **66**, 1957, 203–248

[CPh] M. Henneaux, J. Krasilshchik, A. Vinogradov (eds.), Secondary calculus and cohomological physics. Proceedings of the conference held at Moscow State University, Moscow, August 24–31, 1997. Contemporary Mathematics, **219**, American Mathematical Society, Providence, RI, 1998. xiv+287 pp.

[DG] V. Dragovic, B. Gajic, The Wagner curvature tensor in nonholonomic mechanics, math-ph/0304018

[GM] S. Gelfand, Yu. Manin, Methods of homological algebra. Second edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. xx+372 pp.

[GL1] P. Grozman, D. Leites, From supergravity to ballbearings. In: J. Wess, E. Ivanov (eds.), Supersymmetries and quantum symmetries, (SQS’97, 22–26 July, 1997), Lecture Notes in Phys., **524**, 1999, 58–67

[GL2] P. Grozman, D. Leites, SuperLie and problems (to be) solved with it, MPIM-Bonn preprint (www.mpim-bonn.mpg.de) MPI-2003-39
[Gs] C. Gruson, *Sur l'idéal du cone auto-commutant des super algèbres de Lie basiques classiques et étranges.* (French) [On the ideal of the self-commuting cone of basic classical and strange Lie superalgebras], Ann. Inst. Fourier (Grenoble) **50** (2000), no. 3, 807–831.

[HS] V. Hinich, V. Schechtman, *Homotopy Lie algebras.* In: S. Gelfand, S. Gindikin (eds.), *I. M. Gelfand Seminar*, Adv. Soviet Math., **16, Part 2**, Amer. Math. Soc., Providence, RI, 1993, 1–28

[K] B. Kostant, *Lie algebra cohomology and the generalized Borel–Weil theorem*, Ann. Math. **74**, 1961, 329–387

[Le] A. Lebedev, *Invariants of nonholonomic systems and Lie superalgebra structures in cohomology of nilpotent Lie algebras*, B.S. Diploma, Department “High School of General and Applied Physics”, Nizhny Novgorod University, 2003.

[LL] G. Leger, E. Luks, *Cohomology of nilradicals of Borel subalgebras*, Trans. Amer. Math. Soc. **195**, 1974, 305–316

[L1] D. Leites, *The Riemann tensor for nonholonomic manifolds*. Homology, Homotopy and Applications, vol. 4 (2), 2002, 397–407; [math.RT/0202213]

[L2] D. Leites, *On unconventional integration on supermanifolds and cross ratio on classical superspaces*. In: E. Ivanov, S. Krivonos, J. Lukierski, A. Pashnev (eds.) *Proceedings of the International Workshop “Supersymmetries and Quantum Symmetries”*, September 21-25, 2001, Karpacz, Poland. JINR, Dubna, 2002, 251–262; [math.RT/0202194]

[LPS] D. Leites, E. Poletaeva, V. Serganova, *On Einstein equations on manifolds and supermanifolds*, J. Nonlinear Math. Physics, vol. 9, 2002, no. 4, 394–425; [math.DG/0306209]

[M] S. A. Merkulov, *Operads, deformation theory and F-manifolds*, *math.AG/0210478*

[NR] A. Nijenhuis, R. Richardson, *Cohomology and deformations in graded Lie algebras*, Bull. Am. Math. Soc. **72**, 1966, 1–29

[O] A. L. Onishchik *About derivations and vector-valued differential forms*. Algebraic geometry, 6. J. Math. Sci. (New York) 90 (1998), no. 4, 2274–2286

[Pe] I. Penkov, *Borel-Weil-Bott theory for classical Lie supergroups*. Itogi Nauki i Tekhniki, Current problems in mathematics. Newest results, Vol. 32, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1988, 71–124. (Russian) Translated in J. Soviet Math. **51** (1990), no. 1, 2108–2140

[PS1] I. Penkov, V. Serganova, *Character formulas for some classes of atypical g(m + nε)- and p(m)-modules*, Lett. Math. Phys. **16** (1988), no. 3, 251–261

[PS2] I. Penkov, V. Serganova, *Cohomology of G/P for classical complex Lie supergroups G and characters of some atypical G-modules*, Ann. Inst. Fourier (Grenoble) **39** (1989), no. 4, 845–873

[Po1] E. Poletaeva, *Spencer cohomology connected with some Lie superalgebras*. In: A. Onishchik et al. (eds.) *Problems in group theory and homological algebra* (Russian), Matematika, Yaroslav. Gos. Univ., Yaroslavl, 1988, 162–167 MR 1 175 010; E. Poletaeva, *Structure functions on the usual and exotic symplectic and periplectic supermanifolds*. In: Bartocci C. et al. (eds.), *Differential geometric methods in theoretical physics* (Rapallo, 1990), Lecture Notes in Phys., **375**, Springer, Berlin, 1991, 390–395. MR 92h:58207; E. Poletaeva, *Spencer cohomology of Lie superalgebras of vector fields*. In: A. Onishchik et al. (eds.), *Problems in group theory and homological algebra* (Russian), Matematika, Yaroslav. Gos. Univ., Yaroslavl, 1990, 168–169. MR 93c:17036; E. Poletaeva, *Analogues of Riemann tensors for the odd metric on supermanifolds*, Acta Appl. Math. **31** (1993), no. 2, 137–169. MR 94d:58166

[Po2] E. Poletaeva, *Penrose’s tensors on super-Grassmannians*, Math. Scand. **72** (1993), no. 2, 161–190. MR 94m:58008a; E. Poletaeva, *Penrose’s tensors. II*, Math. Scand. **72** (1993), no. 2, 191–211. MR 94m:58008b

[pr1] A. Premet, *A letter to D. Leites*, Nov. 1990

[JS1] J. Stasheff, *Cohomological physics. Algebraic topology—rational homotopy* (Louvain-la-Neuve, 1986), Lecture Notes in Math., **1318**, Springer, Berlin, 1988, 228–237

[T] A. Tolpygo, *Lie algebra cohomology and generating functions*. In: D. Leites (ed.), *Seminar on supermanifolds* **29**, **1988-11**, 34pp; MPIM-Bonn preprint (www.mpim-bonn.mpg.de) MPI-2003-17

[V1] A. Vaintrob, *Lie algebroids and homological vector fields*. (Russian) Uspekhi Mat. Nauk **52** (1997), no. 2(314), 161–162; translation in Russian Math. Surveys **52** (1997), no. 2, 428–429

[V2] A. Vaintrob, *Normal forms of homological vector fields*. Algebra, 3. J. Math. Sci. **82** (1996), no. 6, 3865–3868.
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