The supergroup U(M/N) with regard to electronic Hamiltonians

Ko Okumura*

*Department of Physics, City College of the City University of New York, New York, NY10031

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Abstract

We study the U(M/N) supergroup keeping in mind its connection with electronic Hamiltonians. It is explicitly shown that the generators of the supergroup U(N/N) can be expressed by Clifford operators or Fermi operators. A multi-band supersymmetric electronic model is suggested.

*On leave from Department of Physics, Faculty of Science and Technology, Keio University, Yokohama 223, Japan
I Introduction

Supersymmetry was discovered and has been developed mainly in the context of high-energy physics. But recently the importance of supersymmetry in electronic lattice Hamiltonians has been noticed. It is known that the t-J Hamiltonian becomes supersymmetric at $2t=J$. This $U(1/2)$ supersymmetric model has been studied extensively partly because it can be solved at least in one dimension. A new electronic model for high-$T_c$ superconductivity has been recently proposed by Essler, Korepin and Schoutens. This model is also solvable and has a $U(2/2)$ supersymmetry. It was also found by the author in [4] (we call this "I" henceforward) that the above two supersymmetries of the models as well as the $U(2)$ symmetry of the Heisenberg model are manifest when expressed in terms of $U(2/2)$ operators.

With this in mind we study the $U(M/N)$ group in the present paper. As a straightforward generalization of I, we explicitly show that $U(N/N)$ group can be constructed both from the Clifford operators $\gamma^\mu$ and from the Fermi operators $c^\pm_{j\sigma}$. The electronic Hamiltonians are usually described by Fermi operators. On the one hand a transformation from Fermi operators to Clifford operators is known. On the other hand we show that we can construct the $U(N/N)$ supergroup from $\gamma^\mu$. Then it is natural that the $U(N/N)$ supergroup be realized directly by the Fermi operators as we show. A multi-band supersymmetric electronic Hamiltonian is also introduced.

The supergroup $U(M/N)$ is defined as a transformation group on vectors
\( \bar{x} \) on superspace \( V(M/N) \) which preserves the norm of these vectors. Here we mean, by vector on \( V(M/N) \), a complex vector taking the form

\[
\bar{x} = \begin{pmatrix}
x_1 \\
\vdots \\
x_M \\
x_{M+1} \\
\vdots \\
x_{M+N}
\end{pmatrix} \equiv \begin{pmatrix}
\bar{x}_B \\
\bar{x}_F
\end{pmatrix}
\]  

(1.1)

where the first \( M \) elements \( (\bar{x}_B) \) are ordinary complex numbers while the last \( N \) elements \( (\bar{x}_F) \) complex Grassmann numbers. Then, any element \( G \in U(M/N) \) satisfies \( G^+ G = 1 \). The condition that \( G \bar{x} \) must also be one of the elements of \( V(M/N) \) requires the following form for \( G \).

\[
G = \begin{pmatrix}
A & O \\
O & C
\end{pmatrix} + \begin{pmatrix}
O & D \\
E & O
\end{pmatrix}
\]  

(1.2)

where \( A \) (\( C \)) is a \( M \times M \) (\( N \times N \)) complex matrix while \( D \) (\( E \)) is a \( M \times N \) (\( N \times M \)) complex Grassmann matrix whose matrix elements are complex Grassmann numbers. Hereafter we call the block matrix like the first matrix in (1.2) a *Bose block matrix* while the matrix like the second, a *Fermi block matrix* (regardless of whether these matrix elements are ordinary or Grassmann numbers).

As in ordinary Lie groups, let us introduce generators \( H \) or \( B \) and \( F \) in the following way.

\[
G = e^{i\eta_j H_j} = e^{i(b_k B_k + f_l F_l)}
\]  

(1.3)

where \( b_k \)'s (\( f_l \)'s) are real (real Grassmann) parameters and \( B_k \)'s (\( F_l \)'s) are Bose (Fermi) block matrices. Notice here that all the matrix elements of \( B_j \) and \( F_j \)
are just complex numbers. Then the condition $G^+G = 1$ reduces to $B_j^+ = B_j$ and $F_j^+ = F_j$, that is, the hermicity of the generators. This condition makes the number of the independent generators or the dimensionality of the group to be $(M+N)^2$, $M^2+N^2$ of which are bosonic while $2MN$ are fermionic. Here we mention that in order to form a group the following relations are required.

$$\{F_j, F_k\} = f_{jkl}B_l, \quad [F_j, B_k] = if_{jkl}F_l, \quad [B_j, B_k] = if_{jkl}B_l$$ (1.4)

where $f$'s are real numbers. Hereafter we call commutation and anti-commutation relations like (1.4) graded commutation relations in which the anti-commutator occurs only when both of the elements in the bracket are fermionic.

The $(M+N)^2$ generators $H$ can be $(M+N) \times (M+N)$ independent hermitian matrices $T^\alpha$ ($\alpha = 1, \ldots, (M+N)^2$) satisfying

$$tr(T^\alpha T^\beta) = \delta_{\alpha\beta} tr 1,$$

(1.5) $M^2+N^2$ of which are Bose block matrices while $2MN$ are Fermi block matrices. $T^\alpha$ forms a closed superalgebra with the graded commutator.

In this case any element of $G$ of $U(M/N)$ takes the form

$$G = e^{i\eta_\alpha T^\alpha}$$

(1.6)

where $\eta$'s are real parameters.

Another matrix representation of the $U(M/N)$ generators is given by $X^{ac}$;

$$(X^{ac})_{ij} = \delta_{ia}\delta_{jc}.$$ (1.7)

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1A way of constructing these matrices and explicit examples are given in Sec. II
where $a, c = 1, \ldots, M+N$. In other words $X$ is another basis of the $U(M/N)$ generators, different from $T$. Indeed $T$ and $X$ are related by a linear transformation:

$$T_{ij}^\alpha = T_{ac}^\alpha X_{ij}^{ac} \quad \text{or} \quad X_{ac}^{ac} = \frac{1}{tr_1} T_{ca}^\alpha T^\alpha. \quad (1.8)$$

$X$ satisfies the following graded commutation rule which is a natural generalization of the ordinary $U(N)$ algebra.

$$[X_{ac}^{ac}, X_{a'c'}^{a'c'}]_{\pm} = X_{a'c'}^{a'c'} \delta_{a'a} \pm X_{a'c'}^{a'c'} \delta_{ac} \quad (1.9)$$

where $[,]_{\pm}$ denotes the graded commutator, that is, commutator($-$) or anti-commutator($+$) the latter occurring only if both $X$’s in the bracket are Fermi block matrices.

In this case any element of $G$ of $U(M/N)$ is given by

$$G = e^{i\zeta_{ac} X_{ac}^{ac}} \quad (1.10)$$

where complex parameter $\zeta$’s are defined as $\zeta_{ac} = \eta_{a} T_{ac}^{\alpha} T^{\alpha}$ with $a, c = 1, \ldots, M+N$.

In the $X$ basis the algebra has the same form for any $M$ and $N$ while in the $T$ basis it doesn’t. Notice here that in the $(M+N) \times (M+N)$ matrix representation any form $[M_1, M_2]$ or $\{M_1, M_2\}$ where $M_1, M_2 \in T^\alpha$ or $X^{ac}$ is a $(M+N) \times (M+N)$ matrix so that it can be expressed in terms of a linear combination of $T^\alpha$ or $X^{ac}$. Thus we can form $U(M+N), U(M+N-1/1), \ldots, U(1/M+N-1)$ from $T^\alpha$ or $X^{ac}$.
II The construction of U(N/N) generators from Clifford or Fermi operators

In this section we explicitly show that the matrix representation of the U(N/N) generators $T$ and $X$ can be realized by the $\hat{T}$ and $\hat{X}$ operators which are constructed from the Clifford($\gamma$) and the Fermi($c$ and $c^+$) operators respectively. These $\hat{T}$ and $\hat{X}$ operators are related through

$$\hat{X}^{ac} = \frac{1}{\text{tr}1} T^\alpha_a T^\alpha_c \quad \text{or} \quad \hat{T}^\alpha = T^\alpha_{ac} \hat{X}^{ac} \quad (2.1)$$

where $T^\alpha_{ac}$ is the $(a, c)$ element of the matrix $T^\alpha$. Eqs. (2.1) implies the following relation\cite{5};

$$\gamma^{2l-1} = c^l + (c^l)^+, \quad \gamma^{2l} = -i(c^l - (c^l)^+) \quad (2.2)$$

which is the transformation between the $\gamma$ (O(N)) operators and the Fermi (U(N)) operators.

II.1 The construction of $T^\alpha$ from Clifford operators

From the $\gamma^\mu$ operators which form the 2D-dimensional Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta_{\mu\nu} \quad (\mu, \nu = 1, \ldots, 2D), \quad (2.3)$$

we can construct through multiplication of $\gamma$’s, $2^{2D}$ independent hermitian operators $\hat{T}^\alpha$ ($\alpha = 1, \ldots, 2^{2D}$) which satisfy

$$\text{tr}(\hat{T}^\alpha \hat{T}^\beta) = \delta_{\alpha\beta} \text{tr}1. \quad (2.4)$$

\footnote{We can get U(M/N) generators (M\#N) in terms of these operators by applying appropriate projection operators as the examples are shown in Sec. III.}
We give an explicit way for this construction in the following table.

| $T^\alpha$ | # of indep. operators | $T^\alpha$ ($D = 1$) | $T^\alpha$ ($D = 2$) |
|------------|------------------------|------------------------|------------------------|
| 1          |                        | 1                      | 1                      |
| $\gamma^\mu$ |                        | $\gamma^\mu$          | $\gamma^\mu$          |
| $(\gamma^{\mu_1} \gamma^{\mu_2} - \gamma^{\mu_2} \gamma^{\mu_1})/(2i)$ | | | |
| $\langle\langle\gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3}\rangle\rangle$ | | | |
| : | | : | : |
| $\langle\langle\gamma^{\mu_1} \ldots \gamma^{\mu_l}\rangle\rangle$ | $2D C_{2l} = 1$ | $\sigma^{\mu\nu}$ | $\gamma^5$ |
| : | | : | : |
| $\langle\langle\gamma^{\mu_1} \ldots \gamma^{\mu_{2D}}\rangle\rangle$ | $\sum_{l=0}^{2D} 2D C_l = 2^{2D}$ | | |
| total # of indep. operators | $\sum_{l=0}^{2D} 2D C_l = 2^{2D}$ | 4 | 16 |

where $nC_r = \frac{n!}{(n-r)! r!}$ and

$$\langle\langle\gamma^{\mu_1} \ldots \gamma^{\mu_l}\rangle\rangle = \begin{cases} -\gamma^{\mu_1} \ldots \gamma^{\mu_l} & l = 4m, 4m + 1 \\ -i\gamma^{\mu_1} \ldots \gamma^{\mu_l} & l = 4m + 2, 4m + 3 \end{cases} \quad (2.5)$$

where $m$ is an integer. Note that the operators in the even (odd) lines are Bose (Fermi) operators and both are $2^{2D-1}$ in number ($\sum_{l=0}^{D-1} 2D C_{2l} = \sum_{l=0}^{D-1} 2D C_{2l+1} = 2^{2D-1}$).

In this construction it is easy to see that (2.4) actually hold. So $\hat{T}^\alpha$ is an abstract representation of the generators of $U(N/N)$ with $N = 2^{D-1}$. In order to get a $2N \times 2N$ matrix representation or a defining representation of $T^\alpha$ from $\hat{T}^\alpha$, we first write down $2D$ $2N \times 2N$ independent Fermi block matrices and identify them with $\gamma^\mu$ ($\mu = 1, \ldots, 2D$). Then through multiplication of these $2N \times 2N \gamma^\mu$ matrices we get the other $T^\alpha$'s. An example for $D = 2$ case is given in Sec. III.

Notice here that the $\hat{T}^\alpha$ thus obtained can form $U(2N)$ etc. other than
U(N/N) with appropriate redefinition of the Fermi and Bose operators. Nonetheless if we introduce local $\gamma_j^\mu (j = 1, \ldots, L)$ at each site and require anticommutation rule among different site operators, that is, $\{\gamma_j^\mu, \gamma_k^\nu\} = 2\delta_{jk}\delta_{\mu\nu}$ and construct $T_j^\alpha$ then the global $\tilde{T}^\alpha (= \sum_{j=1}^L T_j^\alpha)$ forms only U(N/N).

II.2 The construction of $X^{ac}$ from Fermi operators

From $2n$ Fermi operators $(c^l)^\pm$ where $(c^l)^- = c^l$ with $l = 1, \ldots, 2n$ which satisfy

$$\{c^l, (c^l)^+\} = \delta_{ll}$$

we can construct $2^{2n} \times 2^{2n}$ independent real operators $X^{ac} (a, c = 1, \ldots, 2^{2n})$ which satisfy

$$[\hat{X}^{ac}, \hat{X}^{a'c'}]_\pm = \hat{X}^{ac}\delta_{a'c} \pm \hat{X}^{a'c}\delta_{ac'}.$$  (2.7)

Let us first introduce vacuum state $|0\rangle$ which is annihilated by $c^l; c^l|0\rangle = 0$. Then all the possible independent state are as follows.

| Type of states | # of indep. states |
|----------------|---------------------|
| $|0\rangle$    | $2nC_0 = 1$         |
| $(c^l_1)^+$| $(c^l_2)^+$| $|0\rangle$ | $2nC_1 = 2n$ |
| $\vdots$      | $\vdots$           |
| $(c^l_1)^+ \cdots (c^l_r)^+$| $|0\rangle$ | $2nC_r$         |
| $\vdots$      | $\vdots$           |
| $(c^l_1)^+ \cdots (c^{l_{2n}})^+$| $|0\rangle$ | $2nC_{2n} = 1$ |
| total # of indep. states | $\sum_{r=0}^{2n} 2nC_r = 2^{2n}$ |

$^3l$ tacitly denotes band- and spin-indices.
Now we can introduce $2^{2n} \times 2^{2n}$ independent operators $\hat{X}^{ac}$:

$$\hat{X}^{ac} = |a\rangle \langle c| \quad (a, c = 1, \ldots, 2^{2n}) \quad (2.8)$$

It is easy to check that (2.7) holds. Noting that

$$|0\rangle\langle 0| = : e^{-(c^\dagger)^{+} c^\dagger} : = : \prod_l (1 - (c^\dagger)^{+} c^\dagger) : , \quad (2.9)$$

Eq. (2.1) may be derived directly. $X^{ac}$ is called Fermi (Bose) type if the total number of Fermi operators in $|a\rangle \langle c|$ is odd (even). Notice here that the number of independent Bose and Fermi operators is $2^{4n-1}$ each. So $\hat{X}^{ac}$ is an abstract representation of the generators of $U(N/N)$ with $N=2^{2n-1}$. The $2N \times 2N$ matrix representation is clearly given by (1.7).

Notice here that the $\hat{X}^{ac}$ thus obtained can form $U(2N)$ etc. other than $U(N/N)$ with appropriate redefinition of the Bose and Fermi operators. If we introduce local $c^l_j$ at each site ($j = 1, \ldots, L$), require ordinary fermion anti-commutation relation

$$\{c^l_j, (c^{l'}_{j'})^{+}\} = \delta_{jj'} \delta_{ll'} \quad (2.10)$$

on it and define local $X^{ac}_j = |a_j\rangle \langle c_j|$ then the $X^{ac}_j$ satisfies

$$[X^{ac}_j, X^{a'd'}_{j'}]_\pm = \delta_{jj'} (X^{ac}_j \delta^{a'd'}_{a'c} \pm X^{a'd'}_{j'} \delta^{ac}_{a'c}). \quad (2.11)$$

Here we have introduced the local vacuum $|0\rangle_j$ which is annihilated by $c^l_j$ and $|a_j\rangle = (c^{l_1}_j)^{+} \cdots (c^{l_k}_j)^{+} |0\rangle_j$. In this case the vacuum is defined as

$$|0\rangle = \otimes_j |0\rangle_j \quad (2.12)$$
which satisfies $c_j |0\rangle = 0$. Now if we define the global $\tilde{X}^{ac} = \sum_{j=1}^{L} X_j^{ac}$, then $\tilde{X}^{ac}$ forms only $U(N/N)$.

### III Examples from condensed matter physics

Let us consider a one-band electronic lattice Hamiltonian like Hubbard model which has $L$ lattice points. Then the Hilbert space is spanned by the states of the form

$$|a_1\rangle \otimes \cdots \otimes |a_j\rangle \otimes \cdots \otimes |a_L\rangle$$

where $|a_j\rangle$ is $|1_j\rangle = |0\rangle_j$, $|2_j\rangle = c_{j\uparrow}^\dagger c_{j\downarrow}^\dagger |0\rangle_j$, $|3_j\rangle = c_{j\uparrow}^\dagger |0\rangle_j$, or $|4_j\rangle = c_{j\downarrow}^\dagger |0\rangle_j$. Here $|0\rangle_j$ is defined as $c_{j\sigma} |0\rangle_j = 0$ and the vacuum $|0\rangle = \otimes |0\rangle_j$ satisfies $c_{j\sigma} |0\rangle = 0$. Then we can introduce 16 $X_j^{ac}$ operators at each site and construct the global $\tilde{X}^{ac}$ which forms $U(2/2)$.

Next we consider the restricted Hilbert space $P|a_1\rangle \otimes \cdots \otimes |a_L\rangle$ in which doubly occupied states $|2_j\rangle$ are missing. Here we have introduced the Gutzwiller projection operator $P$ (defined in I). On this space $X_j^{ac}$ becomes $P X_j^{ac} P$ (or $X_j^{ac}$ with only $a, c = 1, 3, 4$). This restricted $X_j^{ac}$ is the Hubbard projection operator and $\tilde{X}^{ac}$ forms $U(1/2)$.

Further, consider other restricted spaces $P_F|a_1\rangle \otimes \cdots \otimes |a_L\rangle$ and $P_B|a_1\rangle \otimes \cdots \otimes |a_L\rangle$ in which $|1_j\rangle$, $|2_j\rangle$ and $|3_j\rangle$, $|4_j\rangle$ states are missing respectively. Here $P_F$ and $P_B$ are also defined in Eqs.(2.12) and (2.13) of I. It is clear that both $P_F X^{ac} P_F$ and $P_B X^{ac} P_B$ satisfy the $U(2)$ algebra. Notice here that in these cases all the Fermi operators of $X_j^{ac}$ are missing and thus the superalgebra
reduces to an ordinary algebra.

In terms of $T_\alpha$ or $\gamma$ operators, we first define $\gamma$ through Eq.(2.1) of I and then the 16 $T_\alpha$'s ($= (1, \Gamma^\mu, \Sigma^{\mu\nu}, iA^\mu, \Gamma^5)$) are explicitly given by (2.5) of I and the $U(2/2)$ algebra becomes

\[
\begin{align*}
[\Sigma^{\mu\nu}, \Sigma^{\rho\tau}] &= 4i(\delta_{\mu\tau}\Sigma^{\rho\nu} + \delta_{\nu\rho}\Sigma^{\mu\tau} - \delta_{\mu\nu}\Sigma^{\rho\tau} - \delta_{\mu\tau}\Sigma^{\nu\rho}), \quad [\Sigma^{\mu\nu}, \Gamma^5] = [\Gamma^5, \Gamma^5] = 0 \\
\{\Gamma^\mu, \Gamma^\nu\} &= \{iA^\mu, iA^\nu\} = 2\delta_{\mu\nu}, \quad \{\Gamma^\mu, iA^\nu\} = \varepsilon^{\mu\nu\rho\tau}\Sigma^{\rho\tau} \\
[\Gamma^\mu, \Sigma^{\rho\tau}] &= 2i(\delta_{\mu\rho}\Gamma^\tau - \delta_{\mu\tau}\Gamma^\rho), \quad [\Gamma^\mu, \Gamma^5] = 2i(iA^\mu) \\
[iA^\mu, \Sigma^{\rho\tau}] &= 2i(\delta_{\mu\rho}iA^\tau - \delta_{\mu\tau}iA^\rho), \quad [iA^\mu, \Gamma^5] = -2i(\Gamma^\mu)
\end{align*}
\]

Next we consider a 2-band electronic system where $d$ and $f$ electrons are described by canonical Fermi operator $d_{j\sigma}^\pm$ and $f_{j\sigma}^\pm$ which satisfy the usual anti-commutation rule. Introducing 16 independent states following the table in Sec. II. 2 we define 256 independent $X_{j}^{ac}$'s at each site. Then the global $\tilde{X}^{ac}$ form $U(8/8)$ and appropriate projection operators reduce $U(8/8)$ to other $U(M/N)$.

In terms of the $\gamma$ operator we introduce 8 $\gamma^\mu$ matrix as real and imaginary parts of $d^\pm$ and $f^\pm$ (see (2.2)) and construct the $T$ operators.

Finally we would like to mention that the $U(M/N)$ symmetry of the model manifests itself if we rewrite Hamiltonians in terms of the $X$ operators[3]. As a straightforward generalization of the result given in I the electronic lattice
Hamiltonian given by

\[ H = -t \sum_{<jk>} \sum_{ac} X^a_j X^c_k (-1)^{F(c)} \]  \hspace{1cm} (3.3)

for \( n \)-band electronic system \((a, c = 1, 2, \ldots, 2^{n-1})\) will be interesting. This model has a \( U(2^{2n-1}/2^{2n-1}) \) symmetry.

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