SUBSPACES CONTAINING BIORTHOGONAL
FUNCTIONALS OF BASES OF DIFFERENT TYPES

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Abstract. The paper is devoted to two particular cases of the following general
problem. Let \( \alpha \) and \( \beta \) be two types of bases in Banach spaces. Let a Banach
space \( X \) has bases of both types and a subspace \( M \subset X^* \) contains the sequence
of biorthogonal functionals of some \( \alpha \)-basis in \( X \). Does \( M \) contain a sequence of
biorthogonal functionals of some \( \beta \)-basis in \( X \)?

The following particular cases are considered:
(\( \alpha, \beta \))=(Schauder bases, unconditional bases),
(\( \alpha, \beta \))=(Nonlinear operational bases, linear operational bases).

The paper contains an investigation of some of the spaces constructed by S. Belle-
nott in “The \( J \)-sum of Banach spaces”, J. Funct. Anal. 48 (1982), 95–106. (These
spaces are used in some examples.)

We use the standard Banach space notation as can be found in [LT2], [PP], [S2].

1. Definition 1. Let \( X \) be a Banach space with (unconditional) basis. A subspace
\( M \subset X^* \) is called (unconditionally) basic if it contains all biorthogonal functionals
of some (unconditional) basis of \( X \).

Basic subspaces have been studied in [DK], [O2].

Theorem 1. Let \( X \) be a non-reflexive Banach space with an unconditional basis.
There exists a subspace of \( X^* \) which is basic but is not unconditionally basic.

Proof. Let \( (x_i)_{i=1}^\infty \) be an unconditional basis of \( X \) and \( x_i^* (i \in \mathbb{N}) \) be its biorthog-
onal functionals. Then either \( (x_i) \) is boundedly complete or it is not. Suppose first
that \( (x_i) \) is boundedly complete. Then \( X \) can be equivalently renormed to become
the dual of the space \( N = [x_i^*]_{i=1}^\infty \), in natural duality. (We use square brackets to
denote the closure of linear span.) The space \( N \) is a non-reflexive Banach space
and \( (x_i^*) \) is an unconditional shrinking basis of it. So by James’ theorems [LT2,
p. 9, 22] this basis is not boundedly complete and the space \( N \) contains a sequence
of blocks

\[
m_i = \sum_{k=n_{i-1}+1}^{n_i} a_k x_k^*; \quad ||m_i|| = 1,
\]
equivalent to the unit vector basis of \( c_0 \). Let

\[
m_i^* = \sum_{n_{i-1}+1}^{n_i} b_k x_k
\]

1991 Mathematics Subject Classification. Primary 46B15, 46B20.
Key words and phrases. Banach Space, Basis, Biorthogonal functional.

This is an English translation of the paper published in “Teoriya Funktsii, Funktsional’nyi
Analiz i ikh Prilozheniya”, vol. 57 (1992), pp. 115–127.
be chosen so that \( m_i^*(m_i) = 1 \) and \( \sup_i ||m_i^*|| = C < \infty \). Let \( P : N \to N \) be defined by

\[
P(m) = \sum_{i=1}^{\infty} m_i^*(m)m_i.
\]

It is clear that \( P \) is a projection onto the isomorphic copy of \( c_0 \). One can modify Zippin’s arguments [Z, p. 76] to construct shrinking basis \( (q_i)_{i=1}^{\infty} \) in \( \ker P \). We have

\[
X = ([q_i]_{i=1}^{\infty})^* \oplus ([m_i]_{i=1}^{\infty})^*.
\]

It is clear that the second space is an isomorphic copy of \( l_1 \).

By \( c(\alpha) \) we denote the space of all continuous functions on the set of all ordinals not greater than \( \alpha \) provided with order topology. For countable ordinal \( \alpha \) we have \( (c(\alpha))^* = l_1 \) and \( c(\alpha) \) has a shrinking basis ([LT1, p. 177, 213], [LT2, p. 10]).

Let \( \{s_i\}_{i=1}^{\infty} \) be a shrinking basis of the space \( L := c(\omega^{<\omega}) \) and let \( \{s_i^*\} \) be the sequence of its biorthogonal functionals. The system

\[
\{q_i^*\}_{i=1}^{\infty} \cup \{s_i^*\}_{i=1}^{\infty}
\]

after any enumeration preserving the order in each of the sequences forms a boundedly complete basis of \( X \) [LT2, p. 9]. Let \( M \subset X^* \) be the closure of the linear span of the biorthogonal functionals of the system (1). Since the basis (1) is boundedly complete, it follows that \( M \) does not contain any proper closed total subspace. It is clear that \( M \) is isomorphic to \( L \oplus [q_i] \).

It is clear that \( M \) is a basic subspace. We shall prove that \( M \) is not unconditionally basic. Let us suppose that it is not the case and let \( \{u_i\}_{i=1}^{\infty} \) be an unconditional basis of \( X \) whose biorthogonal functionals \( \{u_i^*\}_{i=1}^{\infty} \) belong to \( M \). From the remark above we obtain that \( M = [u_i^*]_{i=1}^{\infty} \) and hence \( M \) have an unconditional basis. But by Maurey - Rosenthal theorem [MR], \( L \) and therefore \( M \) contains weakly null normalized sequences with no unconditional subsequence, a contradiction [LT2, p. 7, 19]. Thus, if \( \{x_i\}_{i=1}^{\infty} \) is boundedly complete, then we are done.

Suppose now that \( \{x_i\}_{i=1}^{\infty} \) is not boundedly complete. As before let us introduce \( N = [x_i^*]_{i=1}^{\infty} \). It is easy to see that there exists a functional \( x^{**} \in X^{**} \setminus X \) such that \( x(\alpha)|_N \neq 0 \). The space \( M = \ker x^{**} \cap N \) is a total subspace of \( X^* \). By [DK, Theorem 3] \( M \) is a basic subspace. By [O3, Theorem 1] \( M \) is not unconditionally basic.

2. Definition 2. A subspace \( M \subset X^* \) is said to be norming if there exists \( c > 0 \) such that

\[
(\forall x \in X)(\sup_{0 \neq f \in M} |f(x)|/||f|| \geq c||x||).
\]

Remark. M.I.Kadets [K] proved that if \( X \) is separable and \( M \subset X^* \) is a norming subspace, then \( X \) has a nonlinear operational basis all of whose biorthogonal functionals are in \( M \). V.P.Fonf [F] proved that every subspace with the last property is norming.

Definition 3. A subspace \( M \subset X^* \) is said to be quasibasic if there exists a sequence of continuous finite-dimensional linear operators \( v_n : X \to X (n \in \mathbb{N}) \) such that

\[
1)(\forall x \in X)(\lim_{n \to \infty} ||v_n(x) - x|| = 0);
2) (\forall n \in \mathbb{N})(\text{im}(v_n^*) \subseteq M).
\]
Remark. It is easy to see that a subspace $M \subset X^*$ is quasibasic if and only if $M$ contains all biorthogonal functionals of some linear operational basis of $X$.

Definition 4. A Banach space $X$ is said to have the total property of bounded approximation (TPBA in short) if every norming subspace $M$ of $X^*$ is quasibasic.

This property was introduced independently and almost in the same time by I.Singer [S1], F.S.Vakher [V1] and V.A.Vinokurov-A.N.Plichko [ViP] (we would like to note that [S1] is based on the lecture given in 1975). Later on this property was investigated by many authors (see [G], [GP], [MP], [O1], [V2], [VP], [VGP]), some of these results are discussed in [S2, pp. 776-779, 865]. The term TPBA appeared in [V2]. The purpose of the present paper is to make some additions to abovementioned works.

It is clear that if $X \in$TPBA then $X$ is separable and has the bounded approximation property (BAP). Our aim is to find conditions under which the converse is also true.

Definition 5. Let $X(1)$ and $X(2)$ be finite-dimensional subspaces of a Banach space $X$, such that $X(1) \subset X(2) \subset X$ and let $\lambda > 0$. The pair $(X(1), X(2))$ is said to be $\lambda$-approximatable if there exists a continuous linear operator $u : X \to X(2)$ satisfying the conditions $||u|| \leq \lambda$ and $u|_{X(1)} = I_{X(1)}$. A sequence

$$(X(1, i), X(2, i))_{i=1}^\infty$$

of pairs of subspaces of $X$ is said to be uniformly approximable if there exists $0 < \lambda < \infty$ such that all of the pairs $(X(1, i), X(2, i))$ are $\lambda$-approximable.

Definition 6. Let $U$ and $V$ be subspaces of a Banach space $X$. The number

$$\delta(U, V) = \inf \{||u - v|| : u \in S(U), v \in V\}$$

is called the inclination of $U$ to $V$.

Let $M$ be a subspace of $X^*$. We shall denote by $M^\perp$ the set \{\(x^* \in X^* : (\forall x^* \in M)(x^*(x^*) = 0)\). It is known [PP, p. 32] that $M$ is norming if and only if $\delta(M^\perp, X) > 0$. (We identify $X$ with its canonical image in $X^*$).

Let $\phi : X^*/M^\perp \to X^*/M^\perp$ be the natural quotient mapping. The space $X^*/M^\perp$ is naturally isometric to $M^*$. If $M$ is a norming subspace then $\phi|_X$ is an isomorphic embedding.

Theorem 2. Let $X$ be a separable Banach space (SBS) with the BAP. Let $M$ be a norming subspace of $X^*$. Subspace $M$ is quasibasic if and only if the sequence $(\phi(X(1, i)), \phi(X(2, i)))_{i=1}^\infty$ is uniformly approximable in $M^*$ for every uniformly approximable in $X$ sequence $(X(1, i), X(2, i))_{i=1}^\infty$.

Proof. Necessity. Let $M$ be a quasibasic subspace of $X^*$ and let

$$(X(1, i), X(2, i))_{i=1}^\infty$$

be a uniformly approximable sequence in $X$. Let $\{v_n\}$ be a sequence of operators for which the conditions of Definition 3 are satisfied. By Banach-Steinhaus theorem we have $\sup_n ||v_n|| = \beta < \infty$. Therefore we can select a subsequence $\{v_{n(i)}\}_{i=1}^\infty$ of $\{v_n\}$ such that

$$(\forall x \in X(1, i))(||v_{n(i)}(x) - x|| \leq ||v_{n(i)}(x)|| (\dim(X(1, i))))$$
Using standard reasoning (see [JRZ]) we can find operators \( A_i : X \rightarrow X \) \((i \in \mathbb{N})\) such that \( ||A_i|| \leq 2 \) and

\[
(\forall x \in X(1,i))(A_i v_{n(i)}(x) = x).
\]

Since the sequence \((X(1,i), X(2,i))_{i=1}^{\infty}\) is uniformly approximable, then for some \( \lambda < \infty \) there exists a sequence \( \{u_i\}_{i=1}^{\infty} \) of operators, \( u_i : X \rightarrow X \) such that

\[
(\forall i \in \mathbb{N})(im(u_i) \subset X(2,i));
\]

\[
\sup_i ||u_i|| = \lambda < \infty;
\]

\[
u_i|_{X(1,i)} = I_{X(1,i)}.
\]

Let \( T_i = u_i A_i v_{n(i)} \). We have

\[
T_i|_{X(1,i)} = I_{X(1,i)}; \tag{2}
\]

\[
im(T_i) \subset X(2,i); \tag{3}
\]

\[
im(T_i^*) = \im(v_{n(i)}^* A_i^* u_i^*) \subset \im(v_{n(i)}^*) \subset M; \tag{4}
\]

\[
||T_i|| \leq 2\lambda\beta. \tag{5}
\]

Conditions (3) and (4) means that \( T_i \) can be represented in the form \( T_i(x) = \sum_{k=1}^{n(i)} f_k^i(x)x_k^i \), where \( x_k^i \in X(2,i), f_k^i \in M \). Let operators \( R_i : M^* \rightarrow M^* \) be given by the equalities \( R_i(m^*) = \sum_{k=1}^{n(i)} m^*(f_k^i)\phi(x_k^i) \) \((i \in \mathbb{N})\). It is clear that \( R_i \) are \( \sigma(M^*, M)\)-continuous and that \( \phi(B(X)) \) (where \( B(X) \) is the closed unit ball of \( X \)) is \( \sigma(M^*, M)\)-dense in some ball of non-zero radius of \( M^* \) [PP, p. 32]. By (5) it follows that \( R_i \) are uniformly continuous operators on \( M^* \). By (2) it follows that \( R_i|_{\phi X(1,i)} = I_{\phi X(1,i)} \), and by (3) it follows that \( \im(R_i) \subset \phi X(2,i) \). The necessity is proved.

Sufficiency. If \( X \) has the BAP and is separable then it is easy to find a sequence \( \{X(i)\}_{i=1}^{\infty} \) of subspaces of \( X \) such that \( X(1) \subset X(2) \subset \ldots \subset X(n) \subset \ldots \); \( \cl(\bigcup_{i=1}^{\infty} X(n)) = X \) and the pairs \( (X(1,i), X(2,i)) = (X(i), X(i+1)) \) forms a uniformly approximable sequence. Our supposition implies that the sequence \((\phi(X(1,i)), \phi(X(2,i)))\) is a uniformly approximable sequence in \( M^* \). Let the operators \( R_i : M^* \rightarrow M^* \) be such that \( \sup_i ||R_i|| < \infty; \im(R_i) \subset \phi X(2,i) \) and \( R_i|_{\phi X(1,i)} = I_{\phi X(1,i)} \).

**Lemma 1** [JRZ, p. 494]. Let \( L \) and \( N \) be Banach spaces with \( \dim(N) < \infty \). Let \( F \) be a finite dimensional subspace of \( L^* \), let \( Q \) be an operator from \( L^* \) into \( N \) and let \( \varepsilon > 0 \). Then there is a weak* continuous operator \( R \) from \( L^* \) to \( N \) such that \( R|_F = Q|_F \) and \( ||R|| \leq ||Q||(1 + \varepsilon) \).

By this lemma we may without loss of generality assume that operators \( R_i \) are weak* continuous, i.e.

\[
R_i(m^*) = \sum_{k=1}^{n(i)} m^*(f_k^i)\phi x_k^i,
\]

where \( f_k^i \in M, x_k^i \in X(2,i) \). Let operators \( T_i : X \rightarrow X \) be given by \( T_i(x) = \sum_{k=1}^{n(i)} f_k^i(x)x_k^i \). We have

\[
\sup_i ||T_i|| < \infty; \tag{6}
\]
(8) it follows that
\[ T_i|_{X(1,i)} = I_{X(1,i)}; \]
\[ \text{im}(T_i^*) \subset M \quad (8) \]
By (6), (7) and the equality \( \text{cl}(\cup_{n=1}^\infty X(n)) = X \) we obtain:
\[ (\forall x \in X)(\lim_{n \to \infty} ||T_n(x) - x|| = 0). \]

By (8) it follows that \( M \) is quasibasic. The theorem is proved.

Using this theorem we can obtain the following result of [MP].

**Corollary.** Let \( X \) be a SBS with the BAP. Let \( M \) be a norming subspace of \( X^* \), such that the subspace \( M^\perp \subset X^{**} \) has a complement, which contains \( X \). Then \( M \) is quasibasic.

**Proof.** Let us show that \( M \) satisfies the assumptions of theorem 2. Let
\[ (X(1,i), X(2,i))_{i=1}^\infty \]
be a uniformly approximable sequence in \( X \). Let \( Y \) be a complement of \( M^\perp \), such that \( Y \supset X \). It is clear that the restriction of \( \phi \) to \( Y \) is an isomorphism between \( Y \) and \( M^* \). Therefore, it is sufficient to show that the sequence \( (X(1,i), X(2,i))_{i=1}^\infty \) is uniformly approximable in \( Y \). But it is clear that the second conjugates of operators, which uniformly approximate pairs \( (X(1,i), X(2,i)) \) in \( X \), realize uniform approximation of pairs \( (X(1,i), X(2,i)) \) in \( X^{**} \) and, hence, in \( Y \).

**Remarks.** 1. Existence of the complement mentioned in the corollary is not necessary. It follows from the following result of [VP]: every \( L_\infty \)-space (in the sense of Lindenstrauss-Pelczyński) has the TPBA.

2. The assertion of the corollary becomes wrong if we omit the condition \( Y \supset X \) (see Remark after Theorem 3).

Theorem 2 reduces the problem of characterization of the TPBA to the following one: for what SBS with the BAP there exist a weak* closed subspace \( H \) of \( X^{**} \) such that \( \delta(H, X) > 0 \), and the quotient mapping \( Q : X^{**} \to X^{**}/H \) maps some uniformly approximable sequence in \( X \) on the sequence which is not uniformly approximable in \( X^{**}/H \). We shall describe one of the approaches to this problem.

**Definition 7.** Let \( f : \mathbb{N} \to (0, +\infty) \). We shall say that a sequence
\[ (Z(1,i), Z(2,i))_{i=1}^\infty \]
of pairs of subspaces of a Banach space \( Z \) is \( f \)-approximable if there exists a sequence \( \{u_i\}_{i=1}^\infty \) of linear continuous operators \( u_i : Z \to Z(2,i) \) such that \( u_i|_{Z(1,i)} = I_{Z(1,i)} \) and \( \sup_i(||u_i||/f(i)) < \infty \).

**Proposition.** Let \( H \) be a weak* closed subspace of \( X^{**} \) and let \( \delta(H, X) > 0 \). Let \( \phi \) denote the quotient mapping \( \phi : X^{**} \to X^{**}/H \). Let us suppose that \( X \) contains a uniformly approximable sequence \( (X(1,i), X(2,i))_{i=1}^\infty \) such that for some sequence \( (Y(1,i), Y(2,i))_{i=1}^\infty \) of pairs of subspaces of \( X^{**} \) the following conditions are satisfied.
\[ \phi X(1,i) = \phi Y(1,i); \phi X(2,i) = \phi Y(2,i); \]
\[ (\forall i \in \mathbb{N})(\delta(Y(2,i), H) > 0); \]
\[ (9) \]
\[ (10) \]
and, furthermore, the sequence \((Y(1,i), Y(2,i))_{i=1}^{\infty}\) is not \(f\)-approximable in \(X^{**}\) for \(f\) defined by \(f(i) = 1/\delta(Y(2,i), H)\). Then subspace \(H^\top\) (where \(H^\top = \{x^* \in X^* : (\forall x^* \in H)(x^{**}(x^*) = 0)\}\) is a norming nonquasibasic subspace.

Proof. Let us suppose that it is not the case and apply Theorem 2. We obtain that the sequence \((\phi X(1,i), \phi X(2,i))\) is uniformly approximable in \(X^{**}/H\). This means that for some \(0 < \lambda < \infty\) there exist operators \(u_i : X^{**}/H \to \phi X(2,i)\) such that

\[
u_i|_{\phi X(1,i)} = I_{\phi X(1,i)} \quad (11)
\]

and \(\|u_i\| \leq \lambda\). Let operators \(v_i : X^{**} \to Y(2,i)\) be defined by

\[
u_i = (\phi|_{Y(2,i)})^{-1}u_i\phi.
\]

This operators are well-defined because \(\text{im}(u_i) \subset \phi X(2,i) = \phi Y(2,i)\), and the inequality \(\delta(Y(2,i), H) > 0\) implies that the inverse of \(\phi|_{Y(2,i)}\) exists. It is easy to see that \(\|((\phi|_{Y(2,i)})^{-1})\| = f(i)\). Therefore,

\[
(\forall i \in \mathbb{N})(\|v_i\| \leq \lambda f(i)).
\]

Furthermore, by (9) and (11) we have \(v_i|_{Y(1,i)} = I_{Y(1,i)}\). This contradicts the assumption that \((Y(1,i), Y(2,i))_{i=1}^{\infty}\) is not \(f\)-approximable. The proposition is proved.

The verification of the conditions of the proposition for concrete spaces is laborious. Therefore, the following criterion is of interest.

**Theorem 3.** Let \(X^{**}\) contains a reflexive uncomplemented subspace \(Y\) which is isomorphic to a complemented subspace \(Z\) of \(X\) and is such that \(\delta(Y, X) > 0\). Then \(X\) does not have the TPBA.

Proof. Let \(T : Y \to Z\) be an isomorphism. Let us consider the subspace \(H = \{y - Ty : y \in Y\}\) of \(X^{**}\). We shall check that it satisfies all the conditions of the proposition with \(f(i) \equiv C > 0\).

Since \(\delta(X, Y) > 0\), then \(H\) is isomorphic to \(Y\) and, hence, reflexive. Therefore, subspace \(H\) is weak* closed by Krein-Smulian theorem. It is easy to see that \(\delta(H, X) > 0\) and, therefore, [PP, p. 29–34] subspace \(M = H^\top \subset X^*\) is norming.

It is clear that we may restrict ourselves to the case when \(X\) is a SBS with the BAP. In this case \(Z\) is also a SBS with the BAP. Let \(Z(1) \subset Z(2) \subset \ldots \subset Z(n) \subset \ldots\) be a sequence of finite-dimensional subspaces of \(Z\) such that

\[
\text{cl}(\bigcup_{n=1}^{\infty} Z(n)) = Z,
\]

and the sequence \((Z(i), Z(i + 1))\) is uniformly approximable in \(Z\) and, hence, is uniformly approximable in \(X\).

Let us introduce the following sequences of subspaces: \(X(1,i) = Z(i), X(2,i) = Z(i + 1), Y(1,i) = T^{-1}Z(i), Y(2,i) = T^{-1}Z(i + 1)\).

Let us show that the sequence \((Y(1,i), Y(2,i))_{i=1}^{\infty}\) is not uniformly approximable in \(X^{**}\). In fact, if we assume that for some operators \(u_i : X^{**} \to Y(2,i)\) we have \(\sup_i \|u_i\| < \infty\) and

\[
u_i|_{Y(1,i)} = I_{Y(1,i)} \quad (13)
\]

then by reflexivity of \(Y\) we can define the operator \(u : X^{**} \to Y\) by the equality \(u(x) = w - \lim_i u_i(x)\), where \(A\) is some ultrafilter on \(\mathbb{N}\). By (12) and (13) this operator is a projection onto \(Y\). This contradicts the fact that \(Y\) is uncomplemented.
It is easy to check that all the other conditions of the proposition are also satisfied. The theorem is proved.

**Corollary.** There exists a SBS $X$ with a basis which is isometric to its bidual but does not have the TPBA.

Proof. Let $X = (\sum_{i=1}^{\infty} \oplus J)_p$ ($p \neq 1, 2, \infty$), where $J$ is the James’ space (non-reflexive space with a basis, such that $J^{**}$ is isometric to $J$, and $J$ has codimension one in $J^{**}$ (see [LT2, p. 25])). It is clear that $X$ has a basis and is isometric to its second dual. Furthermore, we have $X^{**} = X \oplus l_p$. By well-known results ([BDGJN], [R]) $l_p$ contains an uncomplemented subspace isomorphic to $l_p$. On the other hand, $X$ contains a complemented subspace isomorphic to $l_p$. We are in the conditions of Theorem 3.

**Remark.** If we develop the construction of Theorem 3 for the space $X$ from the corollary, then the subspace $H$ would be complemented in $X^{**}$.

In fact, let $P : X \to Z$ be the projection, whose existence is supposed and let $Q : X^{**} \to X$ be the projection corresponding to the decomposition $X^{**} = X \oplus l_p$. Then $PQ$ is a projection of $X^{**}$ on $Z$ and $PQ|_Y = 0$. Therefore, the operator $(I_{X^{**}} - T^{-1})PQ$ is a projection of $X^{**}$ onto $H$.

It turns out that a SBS $X$ with the BAP but without the TPBA need not satisfy the conditions of Theorem 3.

**Theorem 4.** There exists a SBS $X$ with a basis such that $X^{**} = X \oplus Y$ and $Y$ does not contain infinite-dimensional subspaces which are isomorphic to subspaces of $X$, but $X \not\in \text{TPBA}$.

Proof. We need to use the variant of the proof of James-Lindenstrauss theorem ([J], [L]), which is due to S.F. Bellenot [B]. We use the following particular case of the construction of [B].

Let $(X_n)_{n=0}^{\infty}$ be an increasing sequence of finite-dimensional subspaces of a Banach space $Z$, such that $\text{cl}(\bigcup_{n=0}^{\infty} X_n) = Z$. For ease of notation we adopt the convention that $X_0 = \{0\}$.

Let $(x_i)_{i=0}^{\infty}$ be a sequence with $x_i \in X_i$. If $(x_i)$ is finitely non-zero, then we define the norm $|| \cdot ||_J$ by

$$2|| (x_i)_{i=0}^{\infty} ||_J^2 = \sup (\sum_{i=1}^{k-1} || x_p(i) - x_p(i+1) ||^2 + || x_p(k) ||^2),$$

where the sup is over all integer sequences $(p(i))_{i=1}^{k}$ with $0 \leq p(1) < p(2) < \ldots < p(k)$. The completion of the space of all finitely non-zero sequences under this norm will be denoted by $J(X_n)$.

We shall call the sequence $(x_i)_{i=0}^{\infty}, x_i \in X_i$, eventually constant, if for some $n \in \mathbb{N}$ we have $x_n = x_{n+1} = x_{n+2} = \ldots$. We endow the space of all eventually constant sequences with the semi-norm

$$||(x_i)_{i=0}^{\infty} ||_\Omega = \lim_{k \to \infty} || x_k ||,$$

and denote this space by $\Omega(X_n)$. We denote by $K(X_n)$ the space of all sequences $(x_i)$ with $x_i \in X_i$ and whose norm

$$||(x_i)_{i=0}^{\infty} ||_K = \sup_n || (x_0, \ldots, x_n, 0, \ldots, 0, \ldots) ||_J$$

is finite. It is clear that $K(X_n) \supset \Omega(X_n)$. 
**Theorem 5** [B]. Let the sequence \((X_n)_{n=0}^\infty\) be as above. Then
(I) \(\Omega(X_n)\) is dense in \(K(X_n)\).
(II) \((J(X_n))^{**} = K(X_n)\) and \((J(X_n))^{**}/J(X_n)\) is isometric to \(Z\).
(III) If the spaces \(X_n\) \((n \in \mathbb{N})\) have uniformly bounded basic constants, then \(J(X_n)\) has a basis.

Let us turn to the proof of theorem 4. We fix some \(1 < p < 2\) and let \(Z = l_p\). For \(X_n\) take the linear spans of the first \(n\) vectors of the unit vector basis of \(l_p\). Let us introduce the space \(X = J(X_n) \oplus l_2\). The space \(X\) has a basis by part III of Theorem 5.

**Lemma 2.** The space \(X^{**}\) can be represented in the form: \(X^{**} = X \oplus l_p\).

Proof. Denote by \(\{e_n^i\}_{i=1}^n\) the unit vector basis of \(X_n\). Let us introduce the vectors
\[ f_i = (0, \ldots, 0, e_i^1, e_i^{i+1}, \ldots) \in K(X_n). \]

Let us show that the sequence \(\{f_i\}_{i=1}^\infty\) is equivalent to the unit vector basis of \(l_p\).

We have
\[ \| \sum_{i=1}^\infty a_if_i \|_K = \|(0, a_1e_1^1, a_1e_1^2 + a_2e_2^2, \ldots, \sum_{i=1}^n a_ie_i^n, \ldots)\|_K. \]

Recall the definition of \(K\)-norm and choose \(p(1) = 0\) and \(p(2) = n\). We obtain
\[ \| \sum_{i=1}^\infty a_if_i \|_K \geq \| \sum_{i=1}^n a_ie_i^n \| = \left( \sum_{i=1}^n |a_i|^p \right)^{1/p}. \]

Since this inequality is valid for every \(n \in \mathbb{N}\), then we have
\[ \| \sum_{i=1}^\infty a_if_i \|_K \geq \left( \sum_{i=1}^\infty |a_i|^p \right)^{1/p}. \]

On the other hand, we have
\[ \| \sum_{i=1}^\infty a_if_i \| = 2^{-1/2} \sup_{(p(i))} \left( \sum_{s=p(i)+1}^{p(k)} |a_s|^p \right)^{2/p} + \]
\[ \left( \sum_{s=1}^{p(k)} |a_s|^p \right)^{1/2} \leq 2^{-1/2} \sup_{(p(i))} \left( \sum_{s=p(i)+1}^{p(k)} \sum_{i=1}^{p(i)} |a_s|^p \right)^{1/p} + \]
\[ \sum_{s=1}^{p(k)} |a_s|^p \leq 2^{1/2} \sup_{(p(i))} \left( \sum_{s=1}^{p(k)} |a_s|^p \right)^{1/p} = 2^{1/2} \sum_{s=1}^{p(k)} |a_s|^p. \]

From here and from the proof of Theorem 5 in [B] it follows that the restriction of the quotient mapping \(K(X_n) \to K(X_n)/J(X_n)\) to the closure of the linear span of \(\{f_i\}\) is an isomorphism. The lemma is proved.

**Lemma 3.** Every infinite-dimensional subspace of \(X\) contains a subspace isomorphic to \(l_2\).

Proof. Since \(X = J(X_n) \oplus l_2\), then it is sufficient to show that every infinite-dimensional subspace of \(J(X_n)\) contains a subspace isomorphic to \(l_2\).
It is easy to see (it is shown in the proof of part III of Theorem 5 in [B]) that the vectors \( f_i^n = (0, \ldots, 0, e_i^n, 0, \ldots) \) form a basis of \( J(X_n) \).

The equality \( (J(X_n))^{**} = J(X_n) \oplus l_p \) implies separability of \( (J(X_n))^{*} \). Therefore, every infinite-dimensional subspace of \( J(X_n) \) contains a weakly null sequence \((x_k)_{k=1}^\infty \) which is bounded away from zero. By the well-known arguments [LT2, p. 7] it follows that we can select a subsequence \((x_{n(k)})_{k=1}^\infty \) of \((x_k)\), which is equivalent to the sequence of the form

\[
    h_k = \sum_{n=r(k)+1}^{r(k+1)-1} \left( \sum_{i=1}^n a_i^n f_i^n \right).
\]

It can be directly verified that the sequence \((h_k)\) is equivalent to the unit vector basis of \( l_2 \). The lemma is proved.

**Lemma 4.** \( X \notin TPBA \).

Proof. It is known [BDGJN] that for every \( 1 < p < 2 \) there exists a sequence \( \{W_i\}_{i=1}^\infty \) of finite-dimensional subspaces of \( l_p \) such that the following conditions are satisfied: \( \dim(W_i) = i \);

\[
    \sup_i d(W_i, l_2^i) = C < \infty,
\]

\[
    (\exists 0 < c_1 \leq c_2 < \infty)(\forall\{w_i\}_{i=1}^\infty; \ w_i \in W_i)
    \]

\[
    (c_1(\sum ||w_i||^p)^{1/p} \leq ||\sum w_i|| \leq c_2(\sum ||w_i||^p)^{1/p});
\]

and the sequence \((W_i, W_i^2)_{i=1}^\infty \) is not uniformly approximable.

Let \( W = \text{cl}(\text{lin}(\cup_{i=1}^\infty W_i)) \). By Lemma 2 we have \( X^{**} = X \oplus l_p = J(X_n) \oplus l_2 \oplus l_p \). Let us represent \( l_2 \) as an infinite direct sum: \( l_2 = (\sum_{i=1}^\infty U_i)_2 \), where \( U_i \) are subspaces isometric to \( l_2^i \). Let the isomorphisms \( T_i : W_i \to U_i \ (i \in \mathbb{N}) \) are such that

\[
    ||T_i|| \leq 1; \ ||T_i^{-1}|| \leq C. \quad (14)
\]

Let us introduce the operator \( T : W \to l_2 \) by the equality \( T((w_i)_{i=1}^\infty) = (T_i w_i)_{i=1}^\infty \).

It is clear that \( T \) is a bounded operator and that

\[
   TW_i = U_i. \quad (15)
\]

Let \( H = \{x - T x : x \in W\} \subset X^{**} \). Let us check that \( H \) satisfies all the conditions of the proposition.

The subspace \( H \) is weak* closed by reflexivity of \( W \). It is clear that \( \delta(H, X) > 0 \).

Let \( X(1, i) = X(2, i) = U_i, \ Y(1, i) = Y(2, i) = W_i \). It is clear that

\[
    (X(1, i), X(2, i))_{i=1}^\infty
\]

is uniformly approximable and \((Y(1, i), Y(2, i))_{i=1}^\infty \) is not. Condition (15) implies (9), and (14) implies (10). Moreover, we have \( \inf_i \delta(Y(2, i), H) > 0 \). The lemma is proved.

Theorem 4 follows immediately from Lemmas 2, 3 and 4.

3. Definition 8. Let \( X \) be a subspace of a Banach space \( Z \) and let \( M \) be a subspace of \( X^* \). The subspace \( M \) is said to be *boundedly extendable* onto \( Z \) if there exists an isomorphic embedding \( \pi : M \to Z^* \) such that

\[
    (\forall f \in M)(\forall x \in X)((\pi(f))(x) = f(x)).
\]
Theorem 6. Let $X$ be a SBS with the BAP. The space $X$ does not have the TPBA if and only if there exist a Banach space $Z$ such that $X$ is a subspace of $Z$ and the following conditions are satisfied:

(a) $X^\perp$ is uncomplemented in $Z^*$;

(b) $X^*$ contains a norming subspace $M$ which is boundedly extendeable onto $Z$.

Proof. Sufficiency is proved in [V2]. Here is a shorter proof of it.

Let us show that the subspace $M$ from the formulation of the theorem is not quasibasic. Let us assume the contrary. Let finite-dimensional continuous operators $u_n : X \to X$ be such that

$$(\forall x \in X)(\lim_{n \to \infty} ||u_n(x) - x|| = 0);$$

$$(\forall n \in \mathbb{N})(\text{im}(u_n^*) \subset M).$$

Therefore, the operators $u_n$ can be represented in the following form: $u_n(x) = \sum_{i=1}^{p(n)} x_{i,n}^*(x)x_{i,n}$, where $x_{i,n}^* \in M$; $x_{i,n} \in X$. Let us denote by $r : Z^* \to X^*$ the operator of the restriction and by $\pi : M \to Z^*$ the operator, whose existence follows from the definition of a boundedly extendeable subspace. Let us introduce the operators $\alpha_n : Z^* \to Z^*$ ($n \in \mathbb{N}$) by the equalities $\alpha_n(z^*) = \sum_{i=1}^{p(n)} z^*(x_{i,n})\pi(x_{i,n}^*) = \pi u_n^*r(z^*)$ ($n \in \mathbb{N}$).

It is easy to see that the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is uniformly bounded. Let the operator $Q : Z^* \to Z^*$ be defined by $Q(z^*) = w^* - \lim_A \alpha_n(z^*)$, where $A$ is some ultrafilter on the set of natural numbers. Let us show that $Q$ is a projection and that $\ker(Q) = X^\perp$.

The relation $X^\perp \subset \ker(Q)$ follows immediately from the definition of $Q$. Furthermore, we have

$$rQz^* = w^* - \lim_A r\alpha_n(z^*) = w^* - \lim_A u_n r(z^*) = r(z^*).$$

Therefore, $\ker(Q) = X^\perp$. The equality $Q^2 = Q$ follows by (*) and the fact that $Q(z^*)$ depends only on $r(z^*)$. Therefore, $X^\perp$ is a complemented subspace of $Z^*$. This contradiction completes the proof.

Necessity. Let $X \not\in$TPBA and $M$ be a norming nonquasibasic subspace of $X^*$. Let $Z = M^*$. There is a natural isomorphic embedding of $X$ into $Z$. Therefore (after corresponding renorming) we may consider $X$ as a subspace of $Z$. The subspace $M$ is a norming subspace of $X$. Furthermore, $M$ is boundedly extendeable to $Z$ in a natural way. It remains to prove that $X^\perp$ is uncomplemented subspace of $Z^*$. Assume the contrary. In this case $M^{**} = Z^*$ can be represented in the form $X^\perp \oplus U$, moreover $U$ is isomorphic to $X^*$ in a natural way. Since $X \in$BAP then there exists vectors $\{x_{i,n}\}_{i=1}^{p(n)} \subset X$ and $\{x_{i,n}^*\}_{i=1}^{p(n)} \subset X^*$ such that

$$(\forall x \in X)(x = \lim_{n \to \infty} \sum_{i=1}^{p(n)} x_{i,n}^*(x)x_{i,n}).$$

We denote by $S : X^* \to U$ the natural isomorphism. Let us introduce the sequence $\{T_n\}_{n=1}^{\infty}$ of the operators, $T_n : Z \to Z$ by the equalities:

$$T_n(z) = \sum_{i=1}^{p(n)} (Sx_{i,n}^*)(z)x_{i,n}.$$
This sequence is uniformly bounded. It converges to the identity operator on $X \subset Z$. By Lemma 1 and separability of $X$ we can find a sequence of weak* continuous operators on $M^* = Z$, such that their restrictions to $X$ converge to the identity operator. Hence, $M$ is a quasibasic subspace of $X^*$. The theorem is proved.

4. The result of [MP] cited after Theorem 2 implies that if a SBS $X$ with the BAP is such that every closed norming subspace $M$ of $X^*$ has a finite codimension, then $X \in \text{TPBA}$. Therefore, it is useful to study the class of such spaces and to compare it with the class of quasireflexive SBS with the BAP. (Recall that a Banach space $X$ is called quasireflexive if $\dim(X^*/X) < \infty$).

W.J. Davis and W.B. Johnson [DJ] gave examples of nonquasireflexive SBS such that every closed norming subspace $M$ of $X^*$ is of finite codimension. The argument in [DJ] is based on the following observation. If $M$ is a norming subspace of $X^*$ then, on the one hand, $M^\perp \subset X^*$ is isomorphic to a subspace of $X^{**}/X$ and, on the other hand, $M^\perp$ is isometric to $(X^*/M)^*$. Therefore, if $X^{**}/X$ does not contain infinite-dimensional subspaces which are isomorphic to dual spaces, then $X^*$ does not contain closed norming subspaces of infinite codimension. The purpose of the final part of the present paper is to show that the converse statement is false.

**Theorem 7.** There exists a Banach space $Y$ with a basis, such that the quotient space $Y^{**}/Y$ is an infinite-dimensional reflexive SBS, but $Y^*$ does not contain closed norming subspaces of infinite codimension.

Proof. We use the construction due to S.F. Bellenot [B], which is described above. Let $p > 2$, $Z = l_p$ and $X_n$ be the linear span of the first $n$ elements of the unit vector basis of $l_p$. Let $Y = J(X_n)$.

**Lemma 5.** The space $Y^{**}$ does not contain isomorphic copies of $l_p$.

Proof. Assume the contrary. Let $\{f_i\}_{i=1}^\infty$ be a sequence in $Y$, that is equivalent to the unit vector basis of $l_p$. Since $Y^{**}$ is isomorphic to $K(X_n)$, then we can represent $f_i$ in the form

$$f_i = (x_{i0}, x_{i1}, \ldots, x_{in}, \ldots), \quad (16)$$

where $x_{in} \in X_n$. By part I of theorem 5 we may assume that the sequence $(16)$ is eventually constant. Let $f_1 = (x_{10}, x_{11}, \ldots, x_{1(n-1)}, x_1, x_1, \ldots, x_1, \ldots)$.

Since the sequence $\{f_i\}_{i=1}^\infty$ is weakly null, then for every $\varepsilon_2 > 0$ we can find a natural number $m(2)$ such that $f_{m(2)}$ satisfies the condition $\|x_{0m(2)}^2\| + \ldots + \|x_{n(1)+1}^m\| < \varepsilon_2$. We have

$$f_{m(2)} = (x_{0m(2)}, \ldots, x_{n(2)-1m(2)}, x_{m(2)}, \ldots, x_{m(2)}, \ldots).$$

It is clear that we may suppose that $n(2) - 1 > n(1) + 1$. Let $\varepsilon_3 > 0$. We can find a natural number $m(3)$ such that $f_{m(3)}$ satisfies the condition $\|x_{0m(3)}^1\| + \ldots + \|x_{n(2)+1m(3)}^m\| < \varepsilon_3$. We have $f_{m(3)} = (x_{0m(3)}, \ldots, x_{n(3)-1m(3)}, x_{m(3)}, \ldots, x_{m(3)}, \ldots)$.

We continue in an obvious manner.

We choose the sequence $\{\varepsilon_k\}_{k=1}^\infty$ quickly converging to zero in order that the sequence

$$g_1 = f_1;$$

$$g_2 = (0, 0, x_{0m(2)}, \ldots, x_{0m(2)}, x_{1m(2)}, x_{1m(2)}, \ldots);$$

$$g_3 = (0, 0, 0, 0, x_{0m(2)}, \ldots, x_{0m(2)}, x_{1m(2)}, x_{1m(2)}, \ldots);$$

$$g_4 = (0, 0, 0, 0, 0, 0, 0, x_{0m(2)}, \ldots, x_{0m(2)}, x_{1m(2)}, x_{1m(2)}, \ldots);$$

and so on.

We have $g_k = 0$ for $k < n(1)$. Hence, for $k$ large enough, $g_k$ is equivalent to the basis of $X$. Therefore, $g_k$ is not contained in $l_p$. This completes the proof.
$g_3 = (0, \ldots, 0, x_{n(2)+2}^m, \ldots, x_{n(3)-1}^m, \ldots, x_m^m, \ldots);$

is equivalent to the unit vector basis of $l_p$. Later on for the sake of convenience we let $m(1) = 1$, $n(0) = -1$.

Let \{p(k, i)\}_{k=1}^\infty \text{s(k)} be a collection of natural numbers such that $n(k-1) + 1 = p(k, 1) < \ldots < p(k, s(k)) = n(k)$ and

$$\sum_{i=1}^{s(k)-1} \|x_{p(k,i+1)}^m - x_{p(k,i)}^m\|^2 \geq \|g_k\|^2_K.$$ 

Let us estimate $\|\sum_k a_k g_k\|_K$. For this let us consider the sequence $(p(i))_{i=1}^\infty$ that consists of the following integers:

$$p(1, 1) < p(1, 2) < \ldots < p(1, s(1)) <$$

$$p(2, 1) < p(2, 2) < \ldots < p(2, s(2)) < \ldots <$$

$$p(r, 1) < p(r, 2) < \ldots < p(r, s(r)).$$

We obtain

$$2\|\sum_{k=1}^r a_k g_k\| \geq \sum_{k=1}^r a_k^2 \|g_k\|^2_K.$$ 

Since $p > 2$, then this estimate contradicts the fact that \{g_k\}_{k=1}^\infty is equivalent to the unit vector basis of $l_p$. The lemma is proved.

Proof of Theorem 7. The space $Y$ has a basis by the part III of theorem 5. Part II of theorem 5 implies that $Y^{**}/Y$ is isometric to $l_p$.

Let us assume that $Y^*$ contains a closed norming subspace of infinite codimension. Then $M^\perp \subset Y^{**}$ is isomorphic to a subspace of $Y^{**}/Y$, i.e., to a subspace of $l_p$. Hence, $M^\perp$ contains a subspace isomorphic to $l_p$. By Lemma 5 this is impossible. The theorem is proved.

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