Differential geometry

The GBC mass for asymptotically hyperbolic manifolds

La masse de Gauss–Bonnet–Chern sur des variétés asymptotiquement hyperboliques

Yuxin Ge, Guofang Wang, Jie Wu

A Laboratoire d'analyse et de mathématiques appliquées, CNRS UMR 8050, Département de mathématiques, Université Paris-Est–Créteil–Val-de-Marne, 61, avenue du Général-de-Gaulle, 94010 Créteil cedex, France
B Albert-Ludwigs-Universität Freiburg, Mathematisches Institut, Eckerstr. 1, 79104 Freiburg, Germany
C School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, PR China

ABSTRACT

By using the Gauss–Bonnet curvature, we introduce a higher-order mass, the Gauss–Bonnet–Chern mass, for asymptotically hyperbolic manifolds and show that it is a geometric invariant. Moreover, we prove a positive mass theorem for this new mass for asymptotically hyperbolic graphs. Then, we prove the weighted Alexandrov–Fenchel inequalities in the hyperbolic space $\mathbb{H}^n$ for any horospherical convex hypersurface $\Sigma$. As an application, we obtain an optimal Penrose-type inequality for this new mass for asymptotically hyperbolic graphs with a horizon type boundary $\Sigma$, provided that a dominant energy condition $\tilde{L}_k \geq 0$ holds.

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RÉSUMÉ

En utilisant la courbure de Gauss–Bonnet, on introduit une nouvelle masse d'ordre supérieur – la masse de Gauss–Bonnet–Chern –, sur des variétés asymptotiquement hyperboliques. On montre qu'il s'agit d'un invariant géométrique. On démontre également le théorème de masse positive sur des graphes sur l'espace hyperbolique $\mathbb{H}^n$ et des inégalités d'Alexandrov–Fenchel à poids dans $\mathbb{H}^n$ pour toute hypersurface convexe de type horosphérique. Ainsi, on obtient une inégalité de type Penrose optimale pour cette masse sur toute variété asymptotiquement hyperbolique qui est graphe sur $\mathbb{H}^n$ avec un horizon au bord, à condition que la condition d'énergie dominante $\tilde{L}_k \geq 0$ soit satisfaite.

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1. Introduction

The Riemannian positive mass theorem (PMT), "Any asymptotically flat Riemannian manifold $M^n$ with a suitable decay order and with nonnegative scalar curvature has the nonnegative ADM mass", plays an important role in differential geometry. This theorem was first proved by Schoen and Yau [15] for manifolds of dimension $n \leq 7$ and later for spin manifolds by Witten [17] using spinors. A refinement of the PMT is the Riemannian Penrose inequality:
where $m_{\text{ADM}}$ is the ADM mass of the asymptotically flat Riemannian manifold with a horizon $\Sigma$ and $|\Sigma|$ denotes the area of $\Sigma$. (1.1), was proved by Huisken–Ilmanen [11] and Bray [1] for $n = 3$. Later, Bray and Lee [2] generalized Bray’s proof to the case $n \leq 7$. Recently, Lam [12] gave an elegant proof of PMT and (1.1) in all dimensions for an asymptotically flat manifold that can be realized as a graph in $\mathbb{R}^{n+1}$.

The ADM mass, together with the positive mass theorem, was generalized to asymptotically hyperbolic manifolds in [3,16,19]. For this asymptotically hyperbolic mass, the corresponding Penrose conjecture is: 

For asymptotically hyperbolic manifold $(\mathcal{M}^n, g)$ with an outermost horizon $\Sigma$, its mass satisfies:

$$m_0^H = m_{\mathcal{H}} \geq \frac{1}{2} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n}} + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n}{n+1}},$$

(1.2)

provided that the dominant energy condition:

$$R_g \geq -n(n-1),$$

(1.3)

holds”. Here $R_g$ denotes the scalar curvature of $g$. Recently, motivated by the work of Lam [12], Dahl, Gicquaud, and Sakovich [4], on the one hand, and de Lima and Girão [5], on the other hand, proved the Penrose inequality (1.2) for asymptotically hyperbolic graphs over $\mathbb{H}^n$ with the help of a weighted hyperbolic Minkowski inequality, or a weighted hyperbolic Alexandrov–Fenchel inequality:

$$\int \nabla H \, d\mu \geq (n-1)\omega_{n-1} \left\{ \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n}} + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n}{n+1}} \right\},$$

(1.4)

if $\Sigma$ is star-shaped and mean-convex (i.e. $H > 0$), which was proved by de Lima and Girão [5].

Recently motivated by the Gauss–Bonnet gravity, we have introduced the Gauss–Bonnet–Chern mass $m_{GBC}$ for asymptotically flat manifolds by using the following Gauss–Bonnet curvature:

$$L_k := \frac{1}{2k} \left( \sum_{j_1 \leq \cdots \leq j_k} R_{j_1 j_2 \cdots j_k} j_1 j_2 \cdots j_k \right),$$

(1.5)

where $R_{j_1 \cdots j_k}$ is the Riemannian curvature tensor. One can check that $L_1$ is just the scalar curvature $R$. For general $k$, it is just the Euler integrand in Chern’s proof of the Gauss–Bonnet–Chern theorem if $n = 2k$. See a survey of Zhang [18]. A systematic study of $L_k$ was first given by Lovelock [13]. The Gauss–Bonnet–Chern mass $m_{GBC}$ for the asymptotically flat flat manifolds is defined in [6] by:

$$m_k = m_{GBC} = \frac{(n-2k)!}{2k-1(n-1)!\omega_{n-1}} \lim_{r \to \infty} \int_{S_r} P_{(k)}^{ijlm} \partial_m g_{ij} \, d\mu,$$

(1.6)

where $\omega_{n-1}$ is the volume of $(n-1)$-dimensional standard unit sphere and $S_r$ is the Euclidean coordinate sphere, $d\mu$ is the volume element on $S_r$ induced by the Euclidean metric and $\nu$ is the outward unit normal to $S_r$ in $\mathbb{R}^n$. Here the $(0, 4)$-tensor $P_{(k)}$ is defined by:

$$P_{(k)}^{ijlm} := \frac{1}{2k} \left( \sum_{j_1 \leq \cdots \leq j_k} R_{j_1 j_2 \cdots j_k} j_1 j_2 \cdots j_k \right).$$

(1.7)

This $(0, 4)$-tensor $P_{(k)}$ has a crucial property that it is divergence-free, which guarantees that the Gauss–Bonnet–Chern mass is well defined and is a geometric invariant in [6]. In [6] and [7], we prove a positive mass theorem in the case where $\mathcal{M}$ is an asymptotically flat graph over $\mathbb{R}^n$ or $\mathcal{M}$ is conformal to $\mathbb{R}^n$, respectively. For our mass $m_{GBC}$, a corresponding Penrose conjecture was proposed in [6]:

$$m_k = m_{GBC} \geq \frac{1}{2k} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2k}{n+1}}.$$

(1.8)

Moreover, we proved in [6] that this conjecture is true for asymptotically flat graphs over $\mathbb{R}^n \setminus \Omega$ by using classical Alexandrov–Fenchel inequalities.
2. Hyperbolic Gauss–Bonnet–Chern mass and its Penrose inequality

In the paper [8], motivated by our previous work, by using the Gauss–Bonnet curvature we introduce a higher-order mass for asymptotically hyperbolic manifolds, which is a generalization of the mass introduced by Wang [16] and Cruțiel–Herzlitz [3]. See also [9,14,19]. However, if we use directly the Gauss–Bonnet curvature $L_k$, we can only obtain a mass proportional to the usual hyperbolic mass, rather than a new one. In order to define a higher-order mass for asymptotically hyperbolic manifolds, the crucial observation is a slight modification of the Gauss–Bonnet curvature. More precisely, on a Riemannian manifold $(\mathcal{M}^n, g)$, we consider a modified Riemann curvature tensor:

$$
\tilde{\text{Riem}}_{ijkl}(g) = \tilde{R}_{ijkl}(g) := R_{ijkl}(g) + g_{ij}g_{kl} - g_{il}g_{kj}
$$

and a new Gauss–Bonnet curvature with respect to this tensor $\tilde{\text{Riem}}$:

$$
\tilde{L}_k := \frac{1}{2k} \tilde{R}^{1i_2 \cdots i_{2k-1} 2k}_{1j_1 j_2 \cdots j_{2k-1} j_{2k}} \tilde{R}_{i_1 j_1} \cdots \tilde{R}_{i_{2k-1} j_{2k-1} j_{2k}} = \tilde{R}_{stlm} \tilde{P}^{stm}_{(k)},
$$

where

$$
\tilde{P}^{stm}_{(k)} := \frac{1}{2k} \tilde{R}^{1i_2 \cdots i_{2k-1} 3j_1 j_2 \cdots j_{2k-2} j_{2k} 2}_{1j_1 j_2 \cdots j_{2k-1} j_{2k}} \tilde{R}_{i_1 j_1} \cdots \tilde{R}_{i_{2k-1} j_{2k-1} j_{2k}} \tilde{R}_{i_{2k-1} j_{2k}-1 j_{2k}} \cdots \tilde{R}_{i_{2k-2} j_{2k} j_{2k}} g_{i_{2k-3} j_{2k} j_{2k}} g_{i_{2k-4} j_{2k} j_{2k}} g_{i_{2k-6} j_{2k} j_{2k}}.
$$

The tensor $\tilde{P}^{stm}_{(k)}$ has also the crucial property of being divergence free, which enables us to define a new mass.

Let us assume now that $2 \leq k < \frac{n}{2}$. We first introduce a “higher-order” mass for asymptotically hyperbolic manifolds with slower decay.

**Definition 2.1.** Assume that $(\mathcal{M}^n, g)$ is an asymptotically hyperbolic manifold of decay order $\tau > \frac{n}{4+\gamma}$ and for $V \in \mathbb{N}_{b} := \{V \in C^\infty(\mathbb{H}^n) | \text{Hess}^b V = V b\}$, $V \tilde{L}_k$ is integrable on $(\mathcal{M}^n, g)$. We define the Gauss–Bonnet–Chern mass integral with respect to the diffeomorphism $\Phi$ by:

$$
H^\Phi_k (V) = \lim_{r \to \infty} \int \int_S ((V \tilde{\nabla} e_{ij} - e_{ij} \tilde{\nabla} V) \tilde{P}^{mij}_{(k)} V_{m} \text{d}\mu),
$$

where $e_{ij} := ((\Phi^{-1})^* g)_{ij} - b_{ij}$ and $\tilde{\nabla}$ denotes the covariant derivative with respect to the hyperbolic metric $b$.

This definition is motivated by the work of Cruțiel and Herzlitz [3]. See also [9,14,16,19].

**Theorem 2.2.** Suppose that $(\mathcal{M}^n, g)$ is an asymptotically hyperbolic manifold of decay order $\tau > \frac{n}{4+\gamma}$ and for $V \in \mathbb{N}_{b}$, $V \tilde{L}_k$ is integrable on $(\mathcal{M}^n, g)$, then the mass functional $H^\Phi_k (V)$ is well defined and does not depend on the choice of the coordinates at infinity used in the definition.

From the mass functional $H^\Phi_k$ on $\mathbb{N}_b$, we define a higher-order mass, the Gauss–Bonnet–Chern mass for asymptotically hyperbolic manifolds as follows:

$$
m_{\mathbb{H}}^k := c(n, k) \inf_{\mathbb{N}_b \cap \{V > 0, \text{Hess}^b V = V b\}} H^\Phi_k (V),
$$

where $c(n, k) = \frac{(n-2k)!}{2^{k-1} (n-1)! (n-2)!}$ is the normalization constant given in (1.6) and $\eta$ is a Lorentz inner product. One may assume that the infimum in (2.5) is achieved by:

$$
V = V(0) = \cosh r,
$$

where $r$ is the hyperbolic distance to a fixed point $x_0 \in \mathbb{H}^n$. Therefore, we fix $V = V(0) = \cosh r$.

**Theorem 2.3** (Positive Mass Theorem). Let $(\mathcal{M}^n, g) = (\mathbb{H}^n, b + V^2 df \otimes df)$ be the graph of a smooth asymptotically hyperbolic function $f : \mathbb{H}^n \to \mathbb{R}$ which satisfies $V \tilde{L}_k$ is integrable and the graph $(\mathcal{M}^n, g)$ is asymptotically hyperbolic of decay order $\tau > \frac{n}{4+\gamma}$. Then we have:

$$
m_{\mathbb{H}}^k = c(n, k) \int_{\mathcal{M}^n} \frac{\sqrt{V \tilde{L}_k} V_g}{\sqrt{1 + V^2 |\nabla f|^2}} \text{d}V_g.
$$

In particular, $\tilde{L}_k \geq 0$ implies $m_{\mathbb{H}}^k \geq 0$. 
The condition:
\[ L_k \geq 0, \]  
(2.7)
is a dominant energy condition, like (1.3). Such a beautiful expression (2.6) was found first by Lam for the scalar curvature \( R \) for asymptotically flat graphs over \( \mathbb{R}^n \), and was generalized for the Gauss–Bonnet curvature in [6]. Dahl, Gicquaud, and Sakovich [4] obtained this formula for asymptotically hyperbolic manifolds with a horizon boundary, as follows.

**Theorem 2.4.** Let \( \Omega \) be a bounded open set in \( \mathbb{H}^n \) with boundary \( \Sigma = \partial \Omega \). Assume \( (\mathcal{M}^n, g) = (\mathbb{H}^n \setminus \Omega, b + V^2 d f \otimes d f) \) is an asymptotically hyperbolic manifold with a horizon \( \Sigma \) (i.e. \( \partial \mathcal{M} = \partial \Omega \subset M \) is minimal) which satisfies that \( V L_k \) is integrable. Moreover, assume that each connected component of \( \Sigma \) is in a level set of \( f \) and \( |\nabla f(x)| \to \infty \) as \( x \to \Sigma \). Then:

\[
m^n_k = c(n, k) \left( \frac{1}{2} \int_{\mathcal{M}^n} \frac{\sqrt{L_k}}{1 + V^2 |\nabla f|^2} \, dV_g + \frac{(2k - 1)!}{2} \int_{\Sigma} V \sigma_{2k-1} \, d\mu \right).
\]

where \( \sigma_k \) denotes \( k \)-th mean curvature of \( \Sigma \) induced by the hyperbolic metric \( b \).

In order to obtain a Penrose-type inequality for the hyperbolic mass \( m^n_k \) for asymptotically hyperbolic graphs with a horizon, we need to establish a “weighted” hyperbolic Alexandrov–Fenchel inequality. A hypersurface in \( \mathbb{H}^n \) is horospherical convex if all principal curvatures are larger than or equal to 1.

**Theorem 2.5.** Let \( \Sigma \) be a horospherical convex hypersurface in the hyperbolic space \( \mathbb{H}^n \). We have:

\[
\int_{\Sigma} V \sigma_{2k-1} \, d\mu \geq C_{n-1}^{2k-1} \omega_{n-1} \left( \frac{\left| \Sigma \right|}{\omega_{n-1}} \right)^{\frac{n}{n-1}} + \left( \frac{\left| \Sigma \right|}{\omega_{n-1}} \right)^{\frac{n-2k}{n-1}} k.
\]

(2.8)

Equality holds if and only if \( \Sigma \) is a centered geodesic sphere in \( \mathbb{H}^n \).

When \( k = 1 \), inequality (2.8) is just (1.4), which was proved by de Lima and Girão in [5]. These inequalities have their own interest in integral geometry as well as in differential geometry.

As a consequence of Theorems 2.4 and 2.5, the Penrose inequality for the Gauss–Bonnet–Chern mass \( m^n_k \) for asymptotically hyperbolic graphs with horizon boundaries follows.

**Theorem 2.6 (Penrose Inequality).** Let \( \Omega \) be a bounded open set in \( \mathbb{H}^n \) and \( \Sigma = \partial \Omega \). Assume \( (\mathcal{M}^n, g) = (\mathbb{H}^n \setminus \Omega, b + V^2 d f \otimes d f) \) is an asymptotically hyperbolic manifold with a horizon \( \Sigma \) which satisfies that \( V L_k \) is integrable. Moreover, suppose that each connected component of \( \Sigma \) is in a level set of \( f \) and \( |\nabla f(x)| \to \infty \) as \( x \to \Sigma \). Assume that each connected component of \( \Sigma \) is horospherical convex, then:

\[
m^n_k \geq \frac{1}{2^k} \left( \frac{\left| \Sigma \right|}{\omega_{n-1}} \right)^{\frac{n}{n-1}} + \left( \frac{\left| \Sigma \right|}{\omega_{n-1}} \right)^{\frac{n-2k}{n-1}} k,
\]

(2.9)

provided that

\[
L_k \geq 0.
\]

Moreover, equality is achieved by the anti-de Sitter Schwarzschild type metric:

\[
g_{\text{adS-Sch}} = \left( 1 + \rho^2 - \frac{2m}{\rho^2} \right)^{-1} \, d\rho^2 + \rho^2 \, d\Omega^2,
\]

(2.10)

which is a generalization of the ordinary one. Here \( \rho = \sinh r \) and \( d\Omega^2 \) is the round metric on \( S^{n-1} \).

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