CYCLIC VECTORS OF SELF-ADJOINT OPERATORS IN HILBERT SPACE

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Introduction

Let $H$ be a separable Hilbert space with inner product $(.,.)$ and norm $\|\cdot\| = \sqrt{(.,.)}$, and let $A$ be a bounded or unbounded self-adjoint operator in this space with a domain $D(A)$. A vector $f \in C^\infty(A) = \bigcap_{n=1}^\infty D(A^n)$ is said to be a cyclic vector for $A$ if the closure $L$ of the span of the vectors $f, Af, A^2f, \ldots$ coincides with the space $H$, i.e. the system $\{f, Af, A^2f, \ldots\}$ is complete in the space $H$ (\cite{1}, \cite{11}).

The problem of finding conditions for a vector to be a cyclic vector for the given operator is a hard problem, but for some concrete operators even the criteria for a vector to be a cyclic vector are found (see survey paper \cite{9}).

In the present paper we obtain a criterion and sufficient conditions for a vector to be a cyclic vector for a class of self-adjoint operators, more precisely for self-adjoint operators $A$ satisfying the following condition:

a) the spectrum of the operator $A$ consists of simple eigenvalues $\lambda_j$: $\lambda_j < \lambda_{j+1}$, \( j = 0, \pm 1, \pm 2, \ldots \).

Then the system of eigenvectors $\{e_j\}_{j=-\infty}^{\infty}$ of the operator $A$ ($Ae_j = \lambda_j e_j$) forms an orthonormal basis in the space $H$.

Note that for self-adjoint operators the simplicity of the spectrum and the existence of at least one cyclic vector are equivalent (see, e.g., \cite{1}).

In order to formulate the main results of this paper we introduce the following notation:

$P_{2n+1}(\lambda) = \prod_{i=-n}^{n} (\lambda - \lambda_i)$, $\dot{P}_{2n+1}(\lambda) = \frac{d}{d\lambda} P_{2n+1}(\lambda)$;

$E_{2n+1}$ is a unit matrix in Euclidean space $\mathbb{C}_{2n+1}$;

$K_{2n+1}$ is a square matrix of order $(2n + 1)$ with elements

$$k^{(n)}_{ij} = \frac{1}{(f, e_i)(f, e_j)} \cdot \frac{1}{P_{2n+1}(\lambda_i) P_{2n+1}(\lambda_j)} \sum_{|s|>n} \frac{P_{2n+1}^2(\lambda_s)(f, e_s)^2}{(\lambda_s - \lambda_i)(\lambda_s - \lambda_j)};$$

$$k^{(n)}_{ji} = k^{(n)}_{ij}, \quad i, j = -n, -n+1, \ldots, n;$$

$\langle ., . \rangle$ is the inner product in Euclidean space $\mathbb{C}_{2n+1}$;

$e^{(k)}_{2n+1}$ is the $(2n + 1)$-dimensional column vector with components $\delta_{ik}$, $i = -n, -n + 1, \ldots, n$, $|k| < n$. Here $\delta_{ik}$ is the Kronecker delta.

Theorem 1. Let the self-adjoint operator $A$ satisfy the property a). In order that $f \in C^\infty(A)$ be a cyclic vector for the operator $A$ it is necessary and sufficient that

$$1$$
for each integer \( k \) the following conditions hold:

\[ 1^\circ. \quad (f, e_k) \neq 0; \]
\[ 2^\circ. \quad \lim_{n \to \infty} \langle (E_{2n+1} + K_{2n+1})^{-1} e_{2n+1}, e_{2n+1} \rangle = 1. \]

**Theorem 2.** Let the self-adjoint operator \( A \) satisfy the property a), and for each integer \( k \) let \( f \in C^\infty(A) \) satisfy the conditions

\[ 1^\circ. \quad (f, e_k) \neq 0; \]
\[ 2^\circ. \quad \lim_{n \to \infty} \frac{1}{P_{2n+1}(\lambda_k)} \sum_{|s|>n} |s|^{\lambda_k} (f, e_s)^2 = 0. \]

Then the vector \( f \) is a cyclic vector for the operator \( A \).

Applying Theorem 2, we prove the following theorem.

**Theorem 3.** Let there exists \( C > 0 \) such that for all integers \( k \) the Fourier coefficients of the \( 2\pi \)-periodic function \( f(x) \) satisfy the conditions

\[ 0 < \left| \int_{-\pi}^{\pi} f(x)e^{-ikx}dx \right| \leq Ce^{-\delta|k|}, \]

where \( \delta > (6c_0^2 + 2)/3c_0^3 \), \( c_0 \) is the positive solution of the equation \( c^2 = e^{1/c^2} \). Then the system of successive derivatives of the function \( f(x) \), i.e. the system of functions

\[ f(x), f'(x), f''(x), \ldots \]

forms a complete system in the space \( L_2(-\pi, \pi) \).

(Note that \( c_0 = 1.328\ldots, (6c_0^2 + 2)/3c_0^3 = 1.79\ldots \)

**Example 1.** For the Fourier coefficients of the function \( f(x) = e^{a \cos x} \) \( (a \neq 0) \) we have (see, e.g., \[10\] Ch. 2, §10)

\[ \int_{-\pi}^{\pi} f(x)e^{-ikx}dx = 2 \int_{0}^{\pi} e^{a \cos x} \cos kxdx = 2\pi \left(\frac{a}{2}\right)^k \sum_{m=0}^{+\infty} \frac{|a/2|^{2m}}{m!(m+k)!}. \]

It follows that

\[ 0 < \left| \int_{-\pi}^{\pi} f(x)e^{-ikx}dx \right| \leq 2\pi \left| \frac{a}{2} \right|^k e^{a^2/4} \frac{1}{k!}. \]

Therefore the conditions of Theorem 3 hold and hence the system of functions

\[ e^{a \cos x}, (e^{a \cos x})', (e^{a \cos x})'', \ldots \]

is complete in the space \( L_2(-\pi, \pi) \).

Systems of derivatives of an analytic function were considered as complete systems in spaces of analytic functions in \[3, 4, 5, 8\] etc.
Proof of Theorem 1

Let \( f \in C^\infty(A) \) and assume that condition 1° of Theorem 1 holds. We denote by \( L_{2n+1}(f) \) the span of the vectors \( f, Af, \ldots, A^{2n}f \). Then the distance from the vector \( e_k \) to the subspace \( L_{2n+1}(f) \) is expressed as \( \Xi \) p. 20]

\[
\rho(e_k, L_{2n+1}(f)) = \sqrt{\frac{\Gamma(e_k, f, Af, \ldots, A^{2n}f)}{\Gamma(f, Af, \ldots, A^{2n}f)}},
\]

where \( \Gamma(g_1, g_2, \ldots, g_m) \) is the Gram determinant of the vectors \( g_1, g_2, \ldots, g_m \):

\[
\Gamma(g_1, g_2, \ldots, g_m) = \det \begin{pmatrix}
(g_1, g_1) & (g_1, g_2) & \cdots & (g_1, g_m) \\
(g_2, g_1) & (g_2, g_2) & \cdots & (g_2, g_m) \\
\vdots & \vdots & \ddots & \vdots \\
(g_m, g_1) & (g_m, g_2) & \cdots & (g_m, g_m)
\end{pmatrix}.
\]

It is easy to show that

\[
\rho^2(e_k, L_{2n+1}(f)) = 1 - \langle A_{2n+1}^{-1}b_{2n+1}^{(k)}, b_{2n+1}^{(k)} \rangle,
\]

where \( A_{2n+1} \) is the Gram matrix and \( b_{2n+1}^{(k)} \) is the vector from \( C_{2n+1} \):

\[
A_{2n+1} = \begin{pmatrix}
(f, f) & (f, Af) & \cdots & (f, A^{2n}f) \\
(Af, f) & (Af, Af) & \cdots & (Af, A^{2n}f) \\
\vdots & \vdots & \ddots & \vdots \\
(A^{2n}f, f) & (A^{2n}f, Af) & \cdots & (A^{2n}f, A^{2n}f)
\end{pmatrix},
\]

\[
b_{2n+1}^{(k)} = \begin{pmatrix} 1 \\ \lambda_k \\ \vdots \\ \lambda_k^{2n} \end{pmatrix} (f, e_k).
\]

Indeed, if we denote \( \vec{0} = (0, 0, \ldots, 0)^T \in C_{2n+1} \), then

\[
\rho^2(e_k, L_{2n+1}(f)) = \det \left\{ \begin{pmatrix} 1 \\ \vec{0}^* \end{pmatrix} A_{2n+1}^{-1} \begin{pmatrix} 1 \\ b_{2n+1}^{(k)} \end{pmatrix} \right\}
\]

\[
= \det \left( A_{2n+1}^{-1} b_{2n+1}^{(k)} \right) = 1 - \langle A_{2n+1}^{-1} b_{2n+1}^{(k)}, b_{2n+1}^{(k)} \rangle.
\]

Using the eigenvector expansion for the self-adjoint operator \( A^\ell \),

\[
A^\ell f = \sum_{i=-\infty}^{\infty} \lambda_i^\ell (f, e_i)e_i, \quad \ell = 0, 1, 2, \ldots,
\]

it is easy to check that

\[
A_{2n+1} = \sum_{j=-\infty}^{+\infty} B_{2n+1,j}B_{2n+1,j}^*,
\]
where $B^*$ denotes the adjoint matrix of $B$ and the matrices $B_{s,j}$ $(s = 2n + 1)$ have the form

$$
B_{s,j} = \begin{pmatrix}
(f, e_{sj-n}) & (f, e_{sj-n+1}) & \cdots & (f, e_{sj+1}) \\
\lambda_{sj-n}(f, e_{sj-n}) & \lambda_{sj-n+1}(f, e_{sj-n+1}) & \cdots & \lambda_{sj+1}(f, e_{sj+1}) \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{2n}(f, e_{sj-n}) & \lambda_{2n+1}(f, e_{sj-n+1}) & \cdots & \lambda_{2n+1}(f, e_{sj+1})
\end{pmatrix}.
$$

(4)

In the sequel $k$ is a fixed integer and $n$ is a natural number such that $|k| < n$. We put $c^{(k)}_{2n+1} = A^{-1}_{2n+1} b^{(k)}_{2n+1}$. Then $A_{2n+1} c^{(k)}_{2n+1} = b^{(k)}_{2n+1}$ and according to (3)

$$
\sum_{j=-\infty}^{+\infty} B_{2n+1,j} B^{*}_{2n+1,j} c^{(k)}_{2n+1} = b^{(k)}_{2n+1},
$$
or

$$
B_{2n+1,0} B^{*}_{2n+1,0} c^{(k)}_{2n+1} + \sum_{|j|>0} B_{2n+1,j} B^{*}_{2n+1,j} c^{(k)}_{2n+1} = b^{(k)}_{2n+1}.
$$

If we put $\hat{c}^{(k)}_{2n+1} = B^{*}_{2n+1,0} c^{(k)}_{2n+1}$ we obtain

$$
\hat{c}^{(k)}_{2n+1} + K_{2n+1} \hat{c}^{(k)}_{2n+1} = B^{-1}_{2n+1,0} b^{(k)}_{2n+1},
$$
where the self-adjoint matrix $K_{2n+1}$ has the form

$$
K_{2n+1} = \sum_{|j|>0} B^{-1}_{2n+1,0} B_{2n+1,j} B^{*}_{2n+1,j} B^{-1}_{2n+1,0}.
$$

(6)

According to (2) and (4)

$$
B^{-1}_{2n+1,0} b^{(k)}_{2n+1} = e^{(k)}_{2n+1},
$$
where $e^{(k)}_{2n+1}$ is the column vector from $C_{2n+1}$ with components $\delta_{ik}$, $i = -n, -n + 1, \ldots, n$. Therefore the equation (5) can be written in the form

$$
\hat{c}^{(k)}_{2n+1} + K_{2n+1} \hat{c}^{(k)}_{2n+1} = \hat{e}^{(k)}_{2n+1}.
$$

From here, we have

$$
\hat{c}^{(k)}_{2n+1} = (E_{2n+1} + K_{2n+1})^{-1} \hat{e}^{(k)}_{2n+1}.
$$

Now we can express the right-hand side of the formula (11) in terms of the matrix $K_{2n+1}$ and the vector $\hat{e}^{(k)}_{2n+1}$:

$$
\rho^2(e_k, L_{2n+1}(f)) = 1 - \langle A^{-1}_{2n+1} b^{(k)}_{2n+1}, b^{(k)}_{2n+1} \rangle = 1 - \langle e^{(k)}_{2n+1}, b^{(k)}_{2n+1} \rangle
$$

$$
= 1 - \langle B^{-1}_{2n+1,0} \hat{c}^{(k)}_{2n+1}, b^{(k)}_{2n+1} \rangle = 1 - \langle \hat{c}^{(k)}_{2n+1}, B^{-1}_{2n+1,0} b^{(k)}_{2n+1} \rangle
$$

$$
= 1 - \langle (E_{2n+1} + K_{2n+1})^{-1} e^{(k)}_{2n+1}, e^{(k)}_{2n+1} \rangle.
$$

So we obtain the following main formula which will play an important role in the sequel:

$$
\rho^2(e_k, L_{2n+1}(f)) = 1 - \langle (E_{2n+1} + K_{2n+1})^{-1} e^{(k)}_{2n+1}, e^{(k)}_{2n+1} \rangle.
$$

(7)
We express the elements of the matrix $K_{2n+1}$ in terms of the eigenvalues $\lambda_j$ and the Fourier coefficients $(f, e_j)$ of the element $f$. First we consider the matrix $B_{2n+1,0}^{-1}B_{2n+1,j}$. Using the form (4) for the matrix $B_{s,j}$ we can write

\[
B_{2n+1,0}^{-1} = \begin{pmatrix}
(f, e_{-n}) & (f, e_{-n+1}) & \cdots & (f, e_{n}) \\
\lambda_n(f, e_{-n}) & \lambda_n(f, e_{-n+1}) & \cdots & \lambda_n(f, e_{n}) \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{2n}(f, e_{-n}) & \lambda_{2n}(f, e_{-n+1}) & \cdots & \lambda_{2n}(f, e_{n})
\end{pmatrix}^{-1}
\]

\[
= \frac{1}{\det B_{2n+1,0}} \begin{pmatrix}
\hat{B}_{-n,-n} & \hat{B}_{-n+1,-n} & \cdots & \hat{B}_{n,-n} \\
\hat{B}_{-n,-n+1} & \hat{B}_{-n+1,-n+1} & \cdots & \hat{B}_{n,-n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{B}_{-n,n} & \hat{B}_{-n+1,n} & \cdots & \hat{B}_{n,n}
\end{pmatrix},
\]

where $\det B_{2n+1,0} = \prod_{j=-n}^{n} (f, e_j) W(\lambda_{-n}, \ldots, \lambda_n)$, $W(\lambda_{-n}, \ldots, \lambda_n)$ is the Vandermonde determinant

\[
W(\lambda_{-n}, \ldots, \lambda_n) = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\lambda_{-n} & \lambda_{-n+1} & \cdots & \lambda_n \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{2n} & \lambda_{2n+1} & \cdots & \lambda_{2n}
\end{pmatrix},
\]

and $\hat{B}_{ij}$ are the algebraic complements of the elements of the matrix $B_{2n+1,0}$. Therefore

\[
B_{2n+1,0}^{-1}B_{2n+1,j} = \frac{1}{\prod_{j=-n}^{n} (f, e_j) W(\lambda_{-n}, \ldots, \lambda_n)}
\]

\[
\left\{ \sum_{s=-n}^{n} \hat{B}_{s,-n} \lambda_{(2n+1)j-n}^s (f, e_{(2n+1)j-n}) \cdots \sum_{s=-n}^{n} \hat{B}_{s,-n} \lambda_{(2n+1)j+n}^s (f, e_{(2n+1)j+n}) \right\}.
\]

Since

\[
\sum_{s=-n}^{n} \hat{B}_{s,i} \mu^s (f, e) = \prod_{s=-n}^{n} (f, e_s) (f, e) W(\lambda_{-n}, \ldots, \mu, \ldots, \lambda_n),
\]

where $W(\lambda_{-n}, \ldots, \mu, \ldots, \lambda_n)$ denotes the Vandermonde determinant which is obtained from $[\text{8}]$ by replacing its $i$-th column by the vector $(1, \mu, \mu^2, \ldots, \mu^{2n})^T$, we
have

\[
\sum_{s=-n}^{n} \tilde{B}_{s,i} \mu^{s+n}(f, e) = \frac{\prod_{s=-n}^{n} (f, e)_{s}(f, e) W(\lambda_{-n}, \ldots, \mu, \ldots, \lambda_{n})}{\det B_{2n+1,0}}
\]

\[
= \frac{(f, e)_{i} (\lambda_{i} - \lambda_{-n}) \ldots (\lambda_{i} - \lambda_{-1}) (\lambda_{i+1} - \mu) \ldots (\lambda_{n} - \lambda_{i})}{(f, e)_{i} (\lambda_{i} - \lambda_{-n}) \ldots (\lambda_{i} - \lambda_{-1}) (\lambda_{i+1} - \lambda_{i}) \ldots (\lambda_{n} - \lambda_{i})}
\]

\[
= \frac{f, e_{i}}{P_{2n+1}(\mu)} \frac{1}{\mu - \lambda_{i}}, \quad i = -n, -n + 1, \ldots, n,
\]

where \( P_{2n+1}(\mu) = \prod_{i=-n}^{n} (\mu - \lambda_{i}) \) is the polynomial of degree \( 2n+1 \) and \( \tilde{P}_{2n+1}(\mu) = \frac{d}{d\mu} P_{2n+1}(\mu) \). Therefore

\[
B_{2n+1,0}^{-1} B_{2n+1,j} = \left( \begin{array}{ccc}
(f, e_{(2n+1)_{j-1}}) P_{2n+1}(\lambda_{(2n+1)_{j-1}}) & \cdots & (f, e_{n}) P_{2n+1}(\lambda_{n}) \\
(f, e_{(2n+1)_{j-1}}) & \cdots & (f, e_{n}) P_{2n+1}(\lambda_{n}) \\
(f, e_{(2n+1)_{j-1}}) P_{2n+1}(\lambda_{n}) & \cdots & (f, e_{n}) P_{2n+1}(\lambda_{n})
\end{array} \right).
\]

Then from (8) we have the following expression for the elements \( k_{ij}^{(n)} \) of the matrix \( K_{2n+1}^{(n)} \):

\[
k_{ij}^{(n)} = \frac{1}{(f, e_{i})(f, e_{j})} \cdot \frac{1}{P_{2n+1}(\lambda_{i}) P_{2n+1}(\lambda_{j})} \sum_{|s|>n} \frac{P_{2n+1}^{2}(\lambda_{s})}{(\lambda_{s} - \lambda_{i})(\lambda_{s} - \lambda_{j})} |(f, e_{s})|^{2},
\]

\[
\sum_{i,j=-n}^{n} k_{ij}^{(n)} = k_{ij}^{(n)}, \quad i, j = -n, -n + 1, \ldots, n.
\]

Now, if all the conditions of Theorem 1 hold, then it follows from (7) that

\[
\lim_{n \to \infty} \rho(e_{k}, L_{n+1}(f)) = 0 \quad \text{for any fixed integer } k, \text{ i.e. all eigenvectors } e_{k} \text{ of the operator } A \text{ belong to the closure } L \text{ of the span of the vectors } f, Af, A^{2}f, \ldots .
\]

Indeed, according to the definition of the limit, for any \( \varepsilon > 0 \), we can find \( n_{\varepsilon} \) such that

\[
\alpha = \inf_{c_{i}} \|e_{k} - \sum_{i=0}^{n_{\varepsilon}} c_{i} A^{i} f\| < \varepsilon.
\]

By the definition of the infimum, we also have that there are \( c_{i}^{\varepsilon} \) such that

\[
\|e_{k} - \sum_{i=0}^{n_{\varepsilon}} c_{i}^{\varepsilon} A^{i} f\| < \alpha + \varepsilon < 2\varepsilon.
\]

Since \( \{e_{k}\} \) forms an orthonormal basis in the Hilbert space \( H \) and \( e_{k} \in L \), we have \( L = H \).

The necessity of conditions 1° and 2° for the equality \( L = H \) is obvious. \( \Box \)
Proof of Theorem 2

It is sufficient to prove the inequality (|k| < n)

\[
\rho^2(e_k, L_{2n+1}(f)) \leq \frac{1}{|(f, e_k)|^2} \cdot \frac{1}{P_{2n+1}^2(\lambda_k)} \sum_{|s|>n} \frac{P_{2n+1}^2(\lambda_s)|(f, e_s)|^2}{(\lambda_s - \lambda_k)^2}. \tag{10}
\]

We use the following easy fact (see, e.g., [7]): if \( M \) is a positive definite Hermitian square matrix of order \( m \times m \), \( (\cdot, \cdot)_m \) is the inner product in Euclidean space \( \mathbb{C}_m \) and \( e \in \mathbb{C}_m, \|e\|_m = 1 \), then the following inequality holds:

\[
\langle M^{-1}e, e \rangle_m \cdot \langle Me, e \rangle_m \geq 1. \tag{11}
\]

If we put \( M = E_{2n+1} + K_{2n+1}, m = 2n + 1, e = e_{2n+1}^{(k)} \) from (11) we obtain

\[
\langle (E_{2n+1} + K_{2n+1})^{-1}e_{2n+1}^{(k)}, e_{2n+1}^{(k)} \rangle \geq \frac{1}{1 + \langle K_{2n+1}e_{2n+1}^{(k)}, e_{2n+1}^{(k)} \rangle}.
\]

Then from (11) we have

\[
\rho^2(e_k, L_{2n+1}(f)) \leq \frac{\langle K_{2n+1}e_{2n+1}^{(k)}, e_{2n+1}^{(k)} \rangle}{1 + \langle K_{2n+1}e_{2n+1}^{(k)}, e_{2n+1}^{(k)} \rangle}. \tag{12}
\]

On the other hand, according to (9)

\[
\langle K_{2n+1}e_{2n+1}^{(k)}, e_{2n+1}^{(k)} \rangle = k_{kk}^{(n)} = \frac{1}{|(f, e_k)|^2} \cdot \frac{1}{P_{2n+1}^2(\lambda_k)} \sum_{|s|>n} \frac{P_{2n+1}^2(\lambda_s)|(f, e_s)|^2}{(\lambda_s - \lambda_k)^2}. \tag{13}
\]

Relations (12) and (13) imply inequality (10).

\[\square\]

Remark 1. In the proof of (10), we assumed that all Fourier coefficients \((f, e_s)\) of the vector \( f \) differ from zero. Now consider the case when some of the Fourier coefficients are equal to zero. Denote by \((f, e_{m_s})\), \( s = 0, \pm 1, \pm 2, \ldots \) all nonzero Fourier coefficients. Then eigenvector expansion formula for the vector \( f \) has the form

\[
A^\ell f = \sum_{i=-\infty}^{\infty} \lambda_i^\ell (f, \overline{e}_i)\overline{e}_i, \quad \ell = 0, 1, 2, \ldots, \text{ where } \lambda_s = \lambda_{m_s}, \overline{e}_s = e_{m_s}.
\]

Repeating the arguments used in the proofs of Theorems 1 and 2 we obtain the following analogue of inequality (10) (|k| < n):

\[
\rho^2(\overline{e}_k, L_{2n+1}(f)) \leq \frac{1}{|(f, \overline{e}_k)|^2} \cdot \frac{1}{\overline{P}_{2n+1}^2(\overline{\lambda}_k)} \sum_{|s|>n} \frac{\overline{P}_{2n+1}^2(\lambda_s)|(f, \overline{e}_s)|^2}{(\lambda_s - \overline{\lambda}_k)^2},
\]

where

\[
\overline{P}_{2n+1}(\lambda) = \prod_{i=-n}^{n} (\lambda - \lambda_i).
\]
Remark 2. In the proof of Theorem 2 we actually obtained the following estimate:

$$\rho^2(e_k, L_{2n+1}(f)) \leq \frac{k_k^{(n)}}{1 + k_k^{(n)}}.$$ 

It can be shown that a more accurate estimate holds:

$$\rho^2(e_k, L_{2n+1}(f)) \leq \frac{k_k^{(n)}}{1 + k_k^{(n)}} - \frac{\left(\sum_{i=1}^{n} |k_{ik}^{(n)}|^2\right)^2}{\sum_{i=1}^{n} |k_{ik}^{(n)}|^2 + \sum_{i=1}^{n} \sum_{j \neq k} k_{ik}^{(n)} k_{jk}^{(n)} k_{jk}^{(n)}}, \frac{1}{\left(1 + k_k^{(n)}\right)^2}.$$ 

Proof of Theorem 3

Consider the operator $A$ generated by the differential expression $id/dx$ in the space $L_2(-\pi, \pi)$:

$$Ay = i\frac{dy}{dx},$$
$$D(A) = \{y \in L_2(-\pi, \pi) | y \in \mathcal{C}(-\pi, \pi), y(-\pi) = y(\pi), y' \in L_2(-\pi, \pi)\}.$$ 

It is known (see, e.g., [1]) that $A$ is a self-adjoint operator in the Hilbert space $H = L_2(-\pi, \pi)$, $\lambda_k = k$ ($k = 0, \pm1, \pm2, \ldots$) are the eigenvalues and $e_k = \frac{1}{\sqrt{2\pi}}e^{-ikx}$ are the eigenfunctions of the operator $A$. So the operator $A$ satisfies the property a). Let the function $f(x)$ satisfy the conditions of Theorem 3. Let us show that all the conditions of Theorem 2 are satisfied. From the estimate for the Fourier coefficients we obtain that $f(x)$ is an analytic function on the segment $[-\pi, \pi]$ (see, e.g., [2, p. 90]) and since this function is $2\pi$-periodic we have $f^{(m)}(-\pi) = f^{(m)}(\pi), m = 0, 1, 2, \ldots$, i.e. $f \in C^\infty(A)$. Now it is sufficient to prove the fulfilment of condition 2° of Theorem 2.

Since $P_{2n+1}(\lambda) = \prod_{i=-n}^{n}(\lambda - \lambda_i) = \prod_{i=-n}^{n}(\lambda - i) = \lambda(\lambda^2 - 1^2) \ldots (\lambda^2 - n^2)$, then using the inequality $1 - x^2 < e^{-x^2}$, $0 < x < 1$ for $s > n$ we obtain

$$P_{2n+1}(\lambda_s) = s(s^2 - 1^2) \ldots (s^2 - n^2) = s^{2n+1} \left(1 - \left(\frac{1}{s}\right)^2\right) \ldots \left(1 - \left(\frac{n}{s}\right)^2\right) < s^{2n+1}e^{-(1^2 + \ldots + n^2)/s^2} = s^{2n+1}e^{-n(n+1)(2n+1)/6s^2} \leq s^{2n+1}e^{-n^3/3s^2}.$$ 

Finally, taking into account that $|(f, e_s)| \leq Ce^{-\delta|s|}$ and

$$P_{2n+1}(\lambda_k) = \prod_{i=-n}^{n} (k - i) = (-1)^{n-k}(n + k)!(n - k)!,$$ 

$|k| < n.$
and putting $\delta = \sigma + \delta_1$, where $\sigma = (6c_0^2 + 2)/3c_0^3$, $\delta_1 > 0$ we have

$$\frac{1}{P_{2n+1}(\lambda_k)} \sum_{|\lambda| > n} \frac{P^2_{2n+1}(\lambda_s)|\langle f, e_s \rangle|^2}{(\lambda - \lambda_k)^2}$$

$$\leq \frac{2C^2}{((n + k)!(n - k)!)^2} \sum_{s=n+1}^{+\infty} \frac{s^{4n+2}e^{-2n^3/3s^2}e^{-2\delta s}}{(s - k)^2}$$

$$\leq \frac{2C^2}{((n + k)!(n - k)!)^2} \max_{n < s < \infty} \left( s^{4n}e^{-2n^3/3s^2}e^{-2\sigma s} \right) \sum_{s=n+1}^{+\infty} \frac{s^2e^{-2\delta_1 s}}{(s - k)^2}$$

$$= \frac{2C^2}{((n + k)!(n - k)!)^2} \left( \sum_{s=0}^{+\infty} s^2e^{-2\delta_1 s} \right)$$

$$\leq \frac{2C^2}{((n + k)!(n - k)!)^2} \frac{1}{(n + 1 - k)^2} e^{-2\delta_1(n + 1)} \frac{2}{(\delta_1)^2}.$$  

From here it follows that for each fixed $k$

$$\lim_{n \to \infty} \frac{1}{P_{2n+1}(\lambda_k)} \sum_{|\lambda| > n} \frac{P^2_{2n+1}(\lambda_s)|\langle f, e_s \rangle|^2}{(\lambda - \lambda_k)^2} = 0,$$

i.e. condition $2^o$ of Theorem 2 holds. \hfill $\square$

**Remark 3.** Theorem 3 shows that the function $f(x)$ satisfying the conditions of this theorem is a cyclic vector for the self-adjoint operator $A$ generated in the space $L_2(-\pi, \pi)$ by the boundary value problem

$$iy' = \lambda y,$$

$$y(-\pi) = y(\pi).$$

Let us show that an infinitely differentiable finite function $\varphi(x)$ with support $\text{supp} \varphi(x) \subset [-\pi, \pi]$ is not a cyclic vector for the operator $A$. Assume the contrary, i.e., assume that such a function is a cyclic vector for the operator $A$.

Consider the Fourier transform of the function $\varphi(x)$:

$$\Phi(\lambda) = \int_{-\pi}^{\pi} \varphi(x)e^{-i\lambda x} dx.$$  

It is known (see, e.g., [6, p. 22]) that the function $\Phi(\lambda)$ has an infinite number of zeros. Let $\lambda_0$ be a zero of this function:

$$\int_{-\pi}^{\pi} \varphi(x)e^{-i\lambda_0 x} dx = 0.$$
Integrating by parts we obtain
\[ \int_{-\pi}^{\pi} \varphi^{(j)}(x)e^{-i\lambda_0 x} \, dx = 0, \quad j = 0, 1, 2, \ldots. \]

But this is impossible since by assumption the system of functions \( \{\varphi^{(j)}(x)\}_{j=0}^{\infty} \) is complete in the space \( L_2(-\pi, \pi) \).

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