Inference for parameters identified by conditional moment restrictions using a penalized Bierens maximum statistic

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Abstract

We develop an inference method for parameters identified by conditional moment restrictions, which are implied by economic models such as rational behavior and Euler equations. Building on Bierens (1990), we propose penalized maximum statistics and combine bootstrap inference with model selection. Our method is optimized to have nontrivial asymptotic power against a set of $n^{-1/2}$-local alternatives of interest by solving a data-dependent max-min problem for tuning parameter selection. Extensive Monte Carlo experiments show that our inference procedure becomes superior to those available in the literature when the number of irrelevant conditioning variables increases. We demonstrate the efficacy of our method by a proof of concept using two empirical examples: rational unbiased reporting of ability status and the elasticity of intertemporal substitution.

Keywords: conditional moments, economic restrictions, penalization, regularization, multiplier bootstrap, max-min.
1 Introduction

Conditional moment restrictions are ubiquitous in economics. There is now a mature literature on estimation and inference with conditional moment restrictions. A standard approach in the literature is firstly to estimate unknown parameters and secondly to develop suitable test statistics based on estimators (see, e.g., Chamberlain (1987), Donald, Imbens, and Newey (2003), Domínguez and Lobato (2004), Kitamura, Tripathi, and Ahn (2004), among many others). In this paper, we take a different path and aim to carry out inference directly by skipping the first step of estimating parameters of interest. To convey our main point succinctly, we focus on a simple version of conditional moment restrictions, namely, Chamberlain (1987)'s model:

\[ \mathbb{E} [g(X_i, \theta) | W_i] = 0 \text{ a.s. if and only if } \theta = \theta_0, \]

where \( g(x, \theta) \) is a known function with an unknown finite-dimensional parameter \( \theta_0 \).

Arguably, the most challenging data scenario in applying (1) is when the dimension (\( p \)) of \( W_i \in \mathbb{R}^p \) is high, due to the curse of dimensionality. For example, Benítez-Silva, Buchinsky, Chan, Cheidvasser, and Rust (2004) consider testing a conditional moment restriction with a dataset of sample size \( n \approx 350 \) and \( p \approx 20 \). They examine whether a self-reported disability status is an unbiased indicator of Social Security Administration (SSA)'s disability award decision. They fail to reject the null hypothesis of rational unbiased reporting of ability status with a battery of parametric and nonparametric tests; one may ask whether it is driven by a relatively large number of conditioning variables given the sample size, however. This is the motivation of our paper. Since \( p = 20 \) is much smaller than \( n = 350 \), we do not assume that \( p \) grows with \( n \). Nonetheless, it is still demanding to build an inference method by conditioning on a double-digit number of covariates in a fully nonparametric fashion.

To construct a confidence set for \( \theta_0 \) in a data scenario similar to the one mentioned
above, we propose to use Bierens (1990)’s maximum statistic in conjunction with a method of penalization. The original Bierens test is designed to test a functional form of nonlinear regression models and has been subsequently extended to time series (de Jong 1996) and to a more general form (Stinchcombe and White 1998), among other things. There are different specification tests based on conditional moment restrictions in the literature (e.g., Bierens, 1982; Fan and Li, 1996; Andrews, 1997; Bierens and Ploberger, 1997; Fan and Li, 2000; Horowitz and Spokoiny, 2001). All of the aforementioned papers, including Bierens (1990), aim to test the null hypothesis of a parametric (or semiparametric) null model against a nonparametric alternative. The motivation of this paper is different in the sense that it is concerned about testing the null hypothesis regarding $\theta_0$, assuming that the underlying model in (1) is correctly specified.

Recently, Antoine and Lavergne (2022) leverage Bierens (1982)’s integrated conditional moment statistic to develop an inference procedure for a finite-dimensional parameter in a linear instrumental variable model. In this paper, we modify Bierens (1990)’s maximum statistic to an $\ell_1$-penalized maximum statistic. Our idea of $\ell_1$-penalization resembles LASSO (Tibshirani, 1996); however, its use is fundamentally distinct.

In LASSO, parameter estimation and model selection are combined together to improve prediction accuracy. However, $\ell_1$-penalized estimators are irregular and inference based on them requires careful treatment (see, e.g., Taylor and Tibshirani (2015) among others). Our proposal of an $\ell_1$-penalized Bierens maximum statistic is motivated by the following research questions:

- Can we make use of penalization without distorting inference?
- Can we optimize a model selection procedure to improve the power of a test?

Our solution to these questions in the context of conditional moment models is to carry out inference directly based on Bierens (1990)’s maximum statistic without estimating $\theta_0$. That is, we propose to combine inference with model selection to improve the power of the
Bierens-max test. Our penalized inference method is asymptotically valid when the null hypothesis is true and can be optimized to have nontrivial asymptotic power against a set of $n^{-1/2}$-local alternatives of interest. Furthermore, the penalized test statistic is easier to compute than the one without penalization. The computational gains by penalization are practically important since the $p$-value is constructed by a multiplier bootstrap procedure. The penalization tuning parameter is selected by solving a data-dependent max-min problem. Specifically, we elaborate on the choice of the penalty parameter $\lambda$ under a limit of experiments, formally define optimal $\lambda$, and establish consistency of our proposed calibration method.

We demonstrate the usefulness of our method by a proof of concept. First, we revisit the test of Benítez-Silva, Buchinsky, Chan, Cheidvasser, and Rust (2004) and show that our method yields a rejection of the null hypothesis at the conventional level, in contrast to the original analysis. Second, we revisit Yogo (2004) and find that an uninformative confidence interval (resulting from unconditional moment restrictions) for the elasticity of intertemporal substitution, based on annual US series ($n \approx 100$ and $p = 4$), can turn into an informative one. Both empirical examples suggest that there is substantive evidence for the efficacy of our proposed method. We provide further supporting evidence via extensive Monte Carlo experiments.

The remainder of the paper is organized as follows. In Section 2, we define the test statistic and describe how to obtain bootstrap $p$-values. Section 3 establishes bootstrap validity. In Section 4, we derive consistency and local power and propose how to calibrate the penalization parameter to optimize the power of the test. In Section 5, we summarize our proposed inference procedure and provide pseudo-code. In Section 6, we extend our method to inference for $\theta_{10}$, for which we use plug-in estimation of $\theta_{20}$, where $\theta_0 = (\theta_{10}, \theta_{20})$. In Section 7, we examine finite sample properties of our proposed test and existing competing inference methods in a linear instrumental variable model. First, we compare our test with those of Antoine and Lavergne (2022) and also with other inference
procedures that are designed for efficient estimation in a linear IV model (Chamberlain, 1987; Donald, Imbens, and Newey, 2003; Domínguez and Lobato, 2004; Kitamura, Tri- 
pathi, and Ahn, 2004). The results of extensive Monte Carlo experiments indicate that our 
method performs well even if the number of irrelevant instruments increases but competing 
methods suffer from either trivial power or severe size distortion. Sections 8 and 9 
give two empirical examples and Section 10 gives concluding remarks. All the proofs are 
in Section A.

2 Test Statistic

In this section, we introduce our test statistic and describe how to carry out bootstrap 
inference.

Before we present our test statistic, we first assume the following conditions.

Assumption 1. (i) $\Theta$ is a compact subset of $\mathbb{R}^d$.

(ii) $\mathbb{E}[g(X_i, \theta) | W_i] = 0$ a.s. if and only if $\theta = \theta_0$, where $\theta_0 \in \Theta$.

(iii) $\mathbb{E} ||g(X_i, \theta)|| < \infty$ for each $\theta \in \Theta$.

(iv) $W_i$ is a bounded random vector in $\mathbb{R}^p$.

The boundedness assumption on $W_i$ is without loss of generality since we can take a 
one-to-one transformation to ensure that each component of $W_i$ is bounded (for instance, 
$x \mapsto \tan^{-1}(x)$ componentwise, as used in Bierens (1990)).

Define

$$M(\theta, \gamma) := \mathbb{E}[g(X_i, \theta) \exp(W_i^T \gamma)].$$

(2)

Let $\mu()$ denote the Lebesgue measure on $\mathbb{R}^p$. Bierens (1990) established the following 
result.
Lemma 1 (Bierens (1990)). Let Assumption 1 hold. Then, $M(\theta, \gamma) = 0$ under $\theta = \theta_0$ and 

$$M(\theta, \gamma) \neq 0 \quad \text{a.e.}$$

under $\theta \neq \theta_0$. That is, $\mu \{ \gamma \in \mathbb{R}^p : M(\theta, \gamma) = 0 \}$ is either 0 or 1.

To minimize notational complexity, we often abbreviate $M(\gamma) := M(\theta_0, \gamma)$ and $U_i := g(X_i, \theta_0)$ throughout this paper. In order to test a hypothesis

$$H_0 : \theta = \bar{\theta}$$

against its negation $H_1 : \theta \neq \bar{\theta}$, we construct a test statistic as follows. First, define

$$M_n(\theta, \gamma) := \frac{1}{n} \sum_{i=1}^{n} g(X_i, \theta) \exp(W_i'\gamma),$$

$$s_n^2(\theta, \gamma) := \frac{1}{n} \sum_{i=1}^{n} [g(X_i, \theta) \exp(W_i'\gamma)]^2,$$

$$Q_n(\theta, \gamma) := \sqrt{n} \frac{|M_n(\theta, \gamma)|}{s_n(\theta, \gamma)}.$$  \hspace{1cm} (3)

Note that $g(X_i, \theta_0) \exp(W_i'\gamma)$ is a centered random variable. Define the test statistic for the null hypothesis as

$$T_n(\bar{\theta}, \lambda) := \max_{\gamma \in \Gamma} \left[ Q_n(\bar{\theta}, \gamma) - \lambda \| \gamma \|_1 \right],$$  \hspace{1cm} (4)

where $\|a\|_1$ is the $\ell_1$ norm of a vector $a$, $\Gamma$ is a compact subset in $\mathbb{R}^p$ containing an open set, and $\lambda \geq 0$ is the penalization parameter. We regard $T_n(\bar{\theta}, \lambda)$ as a stochastic process indexed by $\lambda \in \Lambda$, where $\Lambda$ is a compact subset in $\mathbb{R}_+ := \{ \lambda \in \mathbb{R} | \lambda \geq 0 \}$.

We end this section by commenting that our test statistic $T_n(\bar{\theta}, \lambda)$ is substantially different from the LASSO’s criterion. First of all, the term $M_n(\bar{\theta}, \gamma)$ is not the least squares objective function; second, $\gamma$ is different from regression coefficients. Furthermore, if we
mimic LASSO more closely, the test statistic would be

\[ T_{n,\text{alt}}(\bar{\theta}, \lambda) := \max_{\gamma \in \Gamma} \left\{ \frac{1}{n} [M_n(\bar{\theta}, \gamma)]^2 - \lambda \|\gamma\|_1 \right\}. \tag{5} \]

We have opted to consider \( T_n(\bar{\theta}, \lambda) \) in the paper because it is properly studentized and comparable in scale to the \( \ell_1 \)-penalty term, unlike \( T_{n,\text{alt}}(\bar{\theta}, \lambda) \) in (5). In addition, it would be easier to specify \( \Gamma \) with \( T_n(\bar{\theta}, \lambda) \) because \( M_n(\bar{\theta}, \gamma) \) tends to get larger as the scale of \( \gamma \) increases but \( Q_n(\bar{\theta}, \gamma) \) may not.

### 2.1 Bootstrap Critical Values

We consider the multiplier bootstrap to carry out inference. Define

\[ M_{n,*}(\theta, \gamma) := \frac{1}{n} \sum_{i=1}^{n} \eta_i^* g(X_i, \theta) \exp(W_i' \gamma), \]
\[ s_{n,*}^2(\theta, \gamma) := \frac{1}{n} \sum_{i=1}^{n} \left[ \eta_i^* g(X_i, \theta) \exp(W_i' \gamma) \right]^2, \]
\[ Q_{n,*}(\theta, \gamma) := \sqrt{n} \frac{|M_{n,*}(\theta, \gamma)|}{s_{n,*}(\theta, \gamma)}, \]
\[ T_{n,*}(\theta, \lambda) := \max_{\gamma \in \Gamma} \left[ Q_{n,*}(\theta, \gamma) - \lambda \|\gamma\|_1 \right], \tag{6} \]

where \( \eta_i^* \) is drawn from \( N(0, 1) \) and independent from data \( \{(X_i, W_i) : i = 1, \ldots, n\} \). For each bootstrap replication \( r \), let

\[ T_{n,*}^{(r)}(\theta, \lambda) := \max_{\gamma \in \Gamma} \left[ Q_{n,*}^{(r)}(\theta, \gamma) - \lambda \|\gamma\|_1 \right]. \tag{7} \]

For each \( \lambda \), the bootstrap \( p \)-value is defined as

\[ p_*(\bar{\theta}, \lambda) := \frac{1}{R} \sum_{r=1}^{R} 1\{T_{n,*}^{(r)}(\bar{\theta}, \lambda) > T_n(\bar{\theta}, \lambda)\} \]
for a large $R$. We reject the null hypothesis at the $\alpha$ level if and only if $p_\ast(\bar{\theta}, \lambda) < \alpha$. Then, a bootstrap confidence interval for $\theta_0$ can be constructed by inverting a pointwise test of \(H_0: \theta = \bar{\theta}\).

It is straightforward to extend our method to multiple conditional moment restrictions, although there may not be a unique way of doing so. For example, suppose that $\bar{\theta} = \theta_0$ for simplicity and that $U_i = (U_i^{(1)}, \ldots, U_i^{(J)})'$ be a $J \times 1$ vector. Let $Q_n^{(j)}(\gamma)$ denote $Q_n(\gamma)$ in (3) using $U_i^{(j)}$ for $j = 1, \ldots, J$. Then, we may generalize the test statistic in (4) by

$$T_n(\lambda) := \max_{\gamma \in \Gamma} \left[ \left( \sum_{j=1}^{J} Q_n^{(j)}(\gamma)^2 \right)^{1/2} - \lambda \|\gamma\|_1 \right]. \quad (8)$$

Alternatively, we may take a more general quadratic form for the first term inside the brackets in (8). For simplicity, in what follows, we focus on the case that $U_i$ is a scalar.

Finally, we conclude this section by commenting that our choice of the $\ell_1$ penalty is on an ad hoc basis. As is well known for LASSO, the $\ell_1$ penalty is more likely to exclude irrelevant instruments than the $\ell_2$ penalty. It is a topic for further research to study an optimal choice of the penalty term.

### 3 Bootstrap Validity

We now introduce some additional notation. Let $K(\theta, \gamma_1, \gamma_2) := \mathbb{E}[g(X_i, \theta)^2 \exp(W_i' (\gamma_1 + \gamma_2))]$ and $s^2(\theta, \gamma) := \mathbb{E}[g(X_i, \theta)^2 \exp(2W_i' \gamma)]$. As before, we suppress the dependence on $\theta$ when they are evaluated at $\theta_0$. We make the following regularity assumptions.

**Assumption 2.** 

(i) $\Gamma$ is a compact subset in $\mathbb{R}^p$ containing an open set, in which $Q_n(\gamma)$ is maximized.

(ii) The time series $\{X_i, W_i\}$ is strictly stationary and ergodic, where $W_i$ is adapted to the natural filtration $\mathcal{F}_{i-1}$ up to time $i - 1$, and $\{U_i\}$ is a martingale difference sequence (mds) with $0 < \mathbb{E}|U_i|^c < \infty$, for some $c > \max\{2, d\}$. 

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(iii) \( \Lambda \) is a compact subset in \( \mathbb{R}_+ := \{\lambda \in \mathbb{R} | \lambda \geq 0\} \) containing an open set.

Assumption 2(ii) is standard and ensures the weak convergence of \( \sqrt{n} M_n(\gamma) \) and \( s^2_n(\gamma) \). The boundedness of \( W_i \) and \( \Gamma \) and the moment condition for \( U_i \) together imply that

\[
\sup_{(\gamma_1, \gamma_2) \in \Gamma^2} K(\gamma_1, \gamma_2) < \infty \quad \text{and} \quad \inf_{\gamma \in \Gamma} s^2(\gamma) > 0.
\]

Recall that \( d = \dim(\Theta) \) is defined in Assumption 1.

Let \( \{M(\theta, \gamma) : \gamma \in \Gamma\} \) be a centered Gaussian process with the covariance kernel \( \mathbb{E}[M(\theta, \gamma_1) M(\theta, \gamma_2)] = K(\theta, \gamma_1, \gamma_2) \). Also, let \( \Rightarrow \) denote the weak convergence in the space of uniformly bounded functions on the parameter space that is endowed with the uniform metric.

We first establish the weak convergence of \( T_n(\lambda) \).

**Theorem 1.** Let Assumptions 1 and 2 hold. Then,

\[
\sqrt{n} M_n(\gamma) \Rightarrow M(\gamma) \quad \text{(9)}
\]

\[
s_n(\gamma) \xrightarrow{P} s(\gamma) \quad \text{uniformly in } \Gamma. \quad \text{(10)}
\]

Furthermore,

\[
T_n(\lambda) \Rightarrow \max_{\gamma \in \Gamma} \left[ \frac{|M(\gamma)|}{s(\gamma)} - \lambda \|\gamma\|_1 \right].
\]

We now show that the bootstrap analog \( T_{n,*}(\lambda) \) of \( T_n(\lambda) \) converges weakly to the same limit. The definition of the conditional weak convergence, \( \Rightarrow \) in \( P \), and conditional convergence in probability, \( \xrightarrow{P} \) in \( P \), employed in the following theorem is referred to e.g. Section 2.9 in van der Vaart and Wellner (1996).
Theorem 2. Let Assumptions 1 and 2 hold. Then, for any $\bar{\theta}$,

$$\sqrt{n} M_n(\bar{\theta}, \gamma) \Rightarrow M(\bar{\theta}, \gamma) \quad \text{in } P \quad (11)$$

$$s_n(\bar{\theta}, \gamma) \xrightarrow{p} s(\bar{\theta}, \gamma) \quad \text{in } P \quad \text{uniformly in } \Gamma. \quad (12)$$

Furthermore,

$$T_n(\bar{\theta}, \lambda) \Rightarrow \max_{\gamma \in \Gamma} \left[ \frac{|M(\bar{\theta}, \gamma)|}{s(\bar{\theta}, \gamma)} - \lambda \|\gamma\|_1 \right] \quad \text{in } P.$$

Recall that we abbreviate $M_n(\gamma) = M_n(\theta_0, \gamma)$ and $s_n^2(\gamma) := s_n^2(\theta_0, \gamma)$. On one hand Theorem 1 holds at $\theta = \theta_0$, but on the other hand Theorem 2 holds for any $\bar{\theta}$. Therefore, Theorems 1 and 2 imply that the bootstrap critical values are valid for any drifting sequence of $\lambda_n$ and thus for the calibration of $\Lambda$ described in Section 4.3. See the proof of Theorem 4 for more details.

4 Consistency, Local Power and Calibration of $\lambda$

4.1 Consistency

Suppose that the null hypothesis is set as $H_0 : \theta = \bar{\theta}$ for some $\bar{\theta} \neq \theta_0$. Then, being explicit about the null, we write

$$M_n(\bar{\theta}, \gamma) = \frac{1}{n} \sum_{i=1}^{n} g(X_i, \bar{\theta}) \exp(W_i'\gamma) \xrightarrow{p} \mathbb{E} \left[ g(X_i, \bar{\theta}) \exp(W_i'\gamma) \right],$$

$$s_n(\bar{\theta}, \gamma) \xrightarrow{p} \sqrt{\mathbb{E} \left[ g^2(X_i, \bar{\theta}) \exp(2W_i'\gamma) \right]}.$$

Therefore, for any $\lambda \in \Lambda$,

$$n^{-1/2} T_n(\bar{\theta}, \lambda) \xrightarrow{p} \sup_{\gamma \in \Gamma} \frac{|\mathbb{E} \left[ g(X_i, \bar{\theta}) \exp(W_i'\gamma) \right]|}{\sqrt{\mathbb{E} \left[ g^2(X_i, \bar{\theta}) \exp(2W_i'\gamma) \right]}}.$$
On the other hand, the bootstrap statistic is always $O_p(1)$, yielding the consistency of the test based on $T_n(\tilde{\theta}, \lambda)$, as in the following theorem.

**Theorem 3.** Let Assumptions 1 and 2 hold. Then, for $\tilde{\theta} \neq \theta_0$, $T_n(\tilde{\theta}, \lambda) \xrightarrow{P} +\infty$ for any $\lambda \in \Lambda$.

### 4.2 Local Power

Consider a sequence of local hypotheses of the following form: for some nonzero constant vector $B$,

$$
\theta_n := \theta_0 + B n^{-1/2},
$$

which leads to the following leading term after linearization:

$$
U_{i,n} := U_i + G(X_i, \theta_0) B n^{-1/2}, \quad (13)
$$

where $G(X_i, \theta) := \partial g(X_i, \theta) / \partial \theta'$, assuming the continuous differentiability of $g(\cdot)$ at $\theta_0$. Unless $g(X_i, \theta)$ is linear in $\theta$, the term $G(X_i, \theta_0)$ depends on $\theta_0$. However, under the null hypothesis, $G(X_i, \theta_0)$ is completely specified. For $B$, we may set it as, for example, a vector of ones times a constant. The form of (13) will be intimately related to our proposal regarding how to calibrate the penalization parameter $\lambda$.

As before, write $G_i := G(X_i, \theta_0)$. Under (13), we have

$$
\sqrt{n} M_n(\theta_n, \gamma) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i \exp(W_i' \gamma) + \frac{1}{n} \sum_{i=1}^{n} \exp(W_i' \gamma) G_i B + o_p(1) \Rightarrow \mathcal{M}(\gamma) + \mathbb{E} [\exp(W_i' \gamma) G_i B].
$$

Then, we can establish that

$$
T_n(\theta_n, \lambda) \Rightarrow \max_{\gamma \in \Gamma} \left[ \frac{\mathcal{M}(\gamma) + \mathbb{E} [\exp(W_i' \gamma) G_i B]}{s(\gamma)} - \lambda \| \gamma \|_1 \right], \quad (14)
$$

using arguments identical to those to prove Theorem 1.
Define the noncentrality term
\[ \kappa(\gamma) := \frac{E[exp(W_i'\gamma)G_iB]}{\sqrt{E[U^2 exp(2W_i'\gamma)]}}. \]

For the test to have a nontrivial power, we need that \( \kappa(\gamma^*(B)) \neq 0 \) with a positive probability, where \( \gamma^*(B) \) denotes a (random) maximizer of the stochastic process in (14). Since the penalty affects \( \gamma^*(B) \) in different ways under the null of \( B = 0 \) and alternatives of \( B \neq 0 \), its implication on power of the test is not straightforward to analyze. The subsequent section proposes a method to select \( \lambda \) in a more systematic way to increase power.

**Remark 1.** We discuss some sufficient conditions, under which the presence of penalty, \( -\lambda \|\gamma\|_1 \), may increase the power of the test. Let \( \tilde{\gamma}(\theta) \) denote the maximizer of \( Q_n(\theta, \gamma) \) for a given \( \theta \). Note that the derivative of \( T_n(\theta, \lambda) \) with respect to \( \lambda \) is \( -\|\tilde{\gamma}(\theta)\|_1 \). Suppose that
\[ \|\tilde{\gamma}(\theta_0)\|_1 > \|\tilde{\gamma}(\bar{\theta})\|_1 \]  
with probability approaching 1, where \( \theta_0 \) and \( \bar{\theta} \), respectively, denote the unknown true parameter value and the hypothetical value under \( H_0 \). Note that the derivative of \( Q_n(\theta, \gamma) \) with respect to \( \gamma \) is zero at its maximum for each \( \theta \). Then the power will be maximized at a strictly positive \( \lambda \), since \( T_n(\theta_0, \lambda) \) decreases more rapidly than \( T_n(\bar{\theta}, \lambda) \) at \( \lambda = 0 \).

An example that meets the condition (15) is the case where the noncentrality term \( \kappa(\gamma) \) under the alternative hypothesis induces a sparse solution. This happens when the set of instrumental variables \( W \) contains redundant elements. It is similar to the well-known fact that the presence of an irrelevant variable in the linear regression results in loss of power in the tests based on the OLS estimates. Specifically, suppose for simplicity that \( W_i = (Z_i, F_i) \) and \( F_i \) is a pure noise that is independent of everything else. Then, the noncentrality term
can be rewritten as

\[ \kappa(\gamma) = \frac{\mathbb{E} \left[ \exp(Z'_1 \gamma_1) G_i B \right]}{\sqrt{\mathbb{E}[U_i^2 \exp(2Z'_1 \gamma_1)]}} \cdot \frac{\mathbb{E} \left[ \exp(F'_1 \gamma_2) \right]}{\sqrt{\mathbb{E}[\exp(2F'_1 \gamma_2)]}} =: \kappa_1(\gamma_1) \kappa_2(\gamma_2). \]

Then, Jensen’s inequality yields that \( \kappa_2(\gamma_2) \leq 1 \) and the equality holds if and only if \( \gamma_2 = 0 \).

Figure 1: Graphical Representation of Power Improvements via Penalties

To provide visual representation of the discussion above, we generate a random sample with \( n = 1000 \) from a simple linear regression model with normal random variables. Specifically, we draw independent standard normal variables \( \varepsilon_{ji}, j = 1, \ldots, 4 \) and generate \( X_i = \varepsilon_{1i} + \varepsilon_{2i}, Y_i = X_i \theta_n + (\varepsilon_{2i} + \varepsilon_{3i})/2 \) with \( \theta_n = \theta_0 + B n^{-1/2} \) with some nonzero constant \( B \), and \( Z_i = (W_i, F_i) \), where \( W_i = \varepsilon_{1i} - \varepsilon_{4i} \), while \( F_i \) is a \((p - 1)\)-dimensional standard normal vector that is independent of all the others. Figure 1 plots the “theoretical” power functions of our proposed test, where the power curves are obtained via Monte Carlo simulations. There is only one endogenous regressor here and the three lines in the figure represent the power curves as a function of the penalty level \( \lambda \) for three different values of the dimension \( p \) of \( W_i \), whose first element is strongly correlated to \( U_i \) while the
others are irrelevant. Thus, \( p - 1 \) represents the number of irrelevant instruments. The power decreases as the number of irrelevant variables increases when the penalty \( \lambda = 0 \). This is analogous to the textbook treatment of hypothesis testing with the linear regression model. Next, the power increases gradually up to a certain point as the penalty grows for each \( p \). This is in line with the preceding discussion. In addition, the power gain from the penalization, that is, the difference between the maximum power and the power at \( \lambda = 0 \), gets bigger as \( p \) gets larger.

### 4.3 Calibration of \( \lambda \)

The penalty function works differently on how it shrinks the maximizer \( \hat{\gamma} \) under the alternatives. Ideally, it should induce sparse solutions that force zeros for the coefficients of the irrelevant conditioning variable to maximize the power of the test.

Although it is demanding to characterize the optimal choice of \( \lambda \) analytically, we can elaborate on the choice of the penalty parameter \( \lambda \) under the limit of experiments

\[
\frac{M(\gamma) + E[\exp(W_i'\gamma)G_iB]}{s(\gamma)},
\]

for which we parametrize the size of the deviation by \( B \). Then, our test becomes

\[
T(\lambda, B, \alpha) := 1 \left\{ \max_{\gamma \in \Gamma} \left[ \frac{M(\gamma) + E[\exp(W_i'\gamma)G_iB]}{s(\gamma)} - \lambda \|\gamma\|_1 \right] > c_\alpha(\lambda) \right\}
\]

for a critical value \( c_\alpha(\lambda) \), which is the \((1 - \alpha)\) quantile of \( \max_{\gamma \in \Gamma} \left[ \frac{M(\gamma)}{s(\gamma)} - \lambda \|\gamma\|_1 \right] \).

Let \( R(\lambda, B, \alpha) := E[T(\lambda, B, \alpha)] \) denote the power function of the test under the limit experiment for given \( \lambda, B \) and \( \alpha \), where \( 0 < \alpha < 1 \) is a prespecified level of the test. We propose to select \( \lambda \) by solving the max-min problem:

\[
\max_{\lambda \in \Lambda} \min_{B \in B} R(\lambda, B, \alpha),
\]

\[13\]
where \( \Lambda \) is a set of possible values of \( \lambda \) and and \( B \) is a set of possible values of \( B \). In some applications, the inner minimization over \( B \in B \) is simple and easy to characterize. For \( \Lambda \), we can take a discrete set of possible values of \( \lambda \), including 0, if suitable. The idea behind (17) is as follows. For each candidate \( \lambda \), the size of the test is constrained properly because \( R(\lambda, 0, \alpha) \leq \alpha \). Then we look at the least-favorable local power among possible values of \( B \) and choose \( \lambda \) that maximizes the least-favorable local power.

To sum up, we formally define optimal \( \lambda \), which is possibly set-valued, in the following way.

**Definition 1.** Let \( \Lambda_0 \subset \Lambda \) denote a set of the global solution in (17) so that

\[
\min_{B \in B} R(\lambda_0, B, \alpha) > \min_{B \in B} R(\lambda, B, \alpha) \quad \text{for any } \lambda \in \Lambda \setminus \Lambda_0 \text{ and } \lambda_0 \in \Lambda_0.
\]

The optimal set \( \Lambda_0 \) depends on the set \( \Lambda \) of possible values of \( \lambda \), the set \( B \) of possible values of \( B \) in (13), and the level \( \alpha \) of the test. To operationalize our proposal, we again rely on a multiplier bootstrap. Define

\[
M_{n,*}(\gamma, B) := \frac{1}{n} \sum_{i=1}^{n} \left( \eta_i^* U_i + \frac{B}{\sqrt{n}} G_i \right) \exp(W_i' \gamma),
\]

\[
s_{n,*}^2(\gamma, B) := \frac{1}{n} \sum_{i=1}^{n} \left[ \left( \eta_i^* U_i + \frac{B}{\sqrt{n}} G_i \right) \exp(W_i' \gamma) \right]^2,
\]

\[
Q_{n,*}(\gamma, B) := n \frac{[M_{n,*}(\gamma, B)]^2}{s_{n,*}^2(\gamma, B)},
\]

(18)

where \( \eta_i^* \) is drawn from \( N(0, 1) \) and independent from data \( \{(X_i, W_i) : i = 1, \ldots, n\} \). The quantities above are just shifted versions of (6). For each bootstrap replication \( r \), let

\[
T_{n,*}^{(r)}(\lambda, B) := \max_{\gamma \in \Gamma} \left[ \sqrt{Q_{n,*}^{(r)}(\gamma, B)} - \lambda \|\gamma\|_1 \right].
\]

(19)

Then the critical value \( c_\alpha(\lambda) \) is approximated by \( c_\alpha^*(\lambda) \), the \((1 - \alpha)\)-quantile of \( T_{n,*}^{(r)}(\lambda, 0) \).
Once $c^*_\alpha(\lambda)$ is obtained, $\mathcal{R}(\lambda, B, \alpha)$ is approximated by

$$
\frac{1}{R} \sum_{r=1}^{R} 1 \{ T_n^{(r)}(\lambda, B) > c^*_\alpha(\lambda) \}. 
$$

(20)

We conclude this section by commenting that we use a shifted version of $\hat{s}^2_{n,*}(\gamma, B)$ instead of $\hat{s}^2_{n,*}(\gamma, 0)$ when we define (19). This is because we would like to mimic more closely the finite-sample distribution of the test statistic under the alternative.

Similarly, let $\hat{\lambda}$ denote a maximizer of $\min_{B \in \mathcal{B}} \mathcal{R}_n(\lambda, B, \alpha)$, where

$$
\mathcal{R}_n(\lambda, B, \alpha) := \Pr^* \{ T_n,*(\lambda, B) > c^*_\alpha(\lambda) \},
$$

and $\Pr^*$ denotes the conditional probability of the bootstrap sample given the sample. Define $T(\lambda) := \max_{\gamma \in \Gamma} \left[ \frac{|M(\gamma)|}{s(\gamma)} - \lambda \| \gamma \|_1 \right]$. Let $F_\lambda$ and $F^*_\lambda$ denote the distribution function of $T(\lambda)$ and that of $T_n,*(\lambda)$ conditional on the sample $X_n$, respectively. We make the following regularity condition on $F_\lambda$.

**Assumption 3.** The partial derivatives $\{ \partial F_\lambda(x)/\partial x : \lambda \in \Lambda \}$ exist and are bounded away from zero for all $\lambda \in \Lambda$ and $x \in [-c + \min_\lambda c_\alpha(\lambda), \max_\lambda c_\alpha(\lambda) + c]$ for some $c > 0$.

The following theorem shows that the bootstrap critical values $c^*_\alpha(\lambda)$ are uniformly consistent for $c_\alpha(\lambda)$. Furthermore, it establishes consistency of our proposed calibration method in the sense that $d(\hat{\lambda}, \Lambda_0) \overset{p}{\rightarrow} 0$, where $d(x, X) := \inf \{|x - y| : y \in X\}$.

**Theorem 4.** Let Assumptions 1, 2 and 3 hold. Then, $c^*_\alpha(\lambda) \overset{p}{\rightarrow} c_\alpha(\lambda)$ uniformly in $\Lambda$. Furthermore, $d(\hat{\lambda}, \Lambda_0) \overset{p}{\rightarrow} 0$ and $\Pr\{T_n,*(\hat{\lambda}) \geq c^*_\alpha(\hat{\lambda})\} \rightarrow \alpha$.

### 4.4 Conditional vs. Unconditional Moment Restrictions

In this subsection, we provide a simple algebraic example for which the scalar parameter is identified by a conditional moment restriction but is not identified or only weakly
identified by an unconditional moment restriction. Suppose that data \( \{(X_{1i}, X_{2i}, W_i) : i = 1, \ldots, n\} \) are generated independently from

\[
\begin{align*}
X_{1i} &= \theta_0 X_{2i} + U_i, \\
X_{2i} &= c_n W_i + \bar{c}(W_i^2 - 1) + V_i,
\end{align*}
\]

(21)

where

\[
\begin{pmatrix}
U_i \\
V_i \\
W_i
\end{pmatrix}
\sim 
\mathcal{N}
\begin{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
1 & \rho & 0 \\
\rho & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\end{pmatrix}.
\]

Note that \( \mathbb{E}[U_i W_i] = 0 \) and \( \mathbb{E}[X_{2i} W_i] = c_n \). Hence, if \( c_n = 0 \), \( \theta_0 \) is unidentified by \( \mathbb{E}[U_i W_i] = 0 \); and if \( c_n = c_0 n^{-1/2} \) with some constant \( c_0 \neq 0 \), \( \theta_0 \) is only weakly identified by \( \mathbb{E}[U_i W_i] = 0 \).

However, if we consider the conditional moment restrictions, we have that \( \mathbb{E}[U_i | W_i] = 0 \) and \( \mathbb{E}[X_{2i} | W_i] = c_n W_i + \bar{c}(W_i^2 - 1) \), thereby suggesting that \( \theta_0 \) is identified with \( \bar{c} \neq 0 \) even if \( c_n = 0 \). We will carry out Monte Carlo experiments based on this kind of data-generating processes in Section 7.

5 Implementation

In this section, we summarize our inference procedure and provide pseudo-code in Algorithm 1 which describes how to conduct the pointwise test of \( H_0 : \theta = \bar{\theta} \).

In addition to the usual input such as the confidence level \( \alpha \) and the number of bootstrap replications \( R \), we need to specify the search space \( \Gamma \subset \mathbb{R}^p \), the grid for penalty levels \( \Lambda \subset \mathbb{R}_+ \), and the set of local alternatives \( \mathcal{B} \subset \mathbb{R}^d \). In our numerical work, we let \( \Gamma = [-C, C]^p \) with some constant (e.g., \( C = 5, 10 \)). For \( \Lambda \), we recommend excluding \( \lambda = 0 \) if \( p \) is somewhat large (e.g., \( p > 5, 10 \)). Regarding \( \mathcal{B} \), it is necessary to know the structure of the inference problem in hand. In our Monte Carlo experiments as well as empirical appli-
Algorithm 1: Inference for $H_0 : \theta_0 = \bar{\theta}$ versus $H_1 : \theta_0 \neq \bar{\theta}$ under the conditional moment model $\mathbb{E}[g(X_i, \theta_0)|W_i] = 0$

**Input:** the data set $\{(X_i, W_i) : i = 1, \ldots, n\}$, the hypothesized parameter $\bar{\theta}$, the search space $\Gamma \in \mathbb{R}^p$, the grid for penalty levels $\Lambda \subset \mathbb{R}_+$.

1. **for** $k = 1, 2, \ldots, p$ **do**
2.   * Calculate $\bar{W}_k = \frac{1}{n} \sum_{i=1}^{n} W_{ki}$ and $s_n(W_k) = \frac{1}{n-1} \sum_{i=1}^{n} (W_{ki} - \bar{W}_k)^2$.
3.   * Transform $W_{ki} \leftarrow \tan^{-1}\left((W_{ki} - \bar{W}_k)/s_n(W_k)\right)$. 
4. **end**

5. Declare variables $U_i \leftarrow g(X_i, \bar{\theta})$.

6. **for** each $\lambda$ in $\Lambda$ **do**
7.   * Declare a function $\gamma \mapsto Q_n(\bar{\theta}, \gamma)$ according to (3).
8.   * Calculate $T_n(\bar{\theta}, \lambda) = \max_{\gamma \in \Gamma} \left[ Q_n(\bar{\theta}, \gamma) - \lambda \|\gamma\|_1 \right]$.
9. **end**

**Output:** the set of test statistics $\{T_n(\bar{\theta}, \lambda) : \lambda \in \Lambda\}$.

**Input for bootstrap test:** the number of bootstrap replications $R$, the set of local alternatives $\mathcal{B} \subset \mathbb{R}^d$, the significance level $\alpha$.

10. **for** $r = 1, 2, \ldots, R$ **do**
11.   * Generate $\{n^*_i : i = 1, \ldots, n\}$ from i.i.d. $N(0, 1)$ independently of the data.
12.   **for** each $\lambda$ in $\Lambda$ **do**
13.     * Declare a function $\gamma \mapsto Q_{n^*_r}(\bar{\theta}, \gamma)$ according to (6).
14.     * Calculate $T^{(r)}_{n^*_r}(\bar{\theta}, \lambda) = \max_{\gamma \in \Gamma} \left[ Q_{n^*_r}(\bar{\theta}, \gamma) - \lambda \|\gamma\|_1 \right]$.
15.   **for** each $B$ in $\mathcal{B}$ **do**
16.     * Declare a function $\gamma \mapsto Q_{n^*_r}(\gamma, B)$ according to (18).
17.     * Calculate $T^{(r)}_{n^*_r}(\lambda, B) = \max_{\gamma \in \Gamma} \left[ Q_{n^*_r}(\gamma, B) - \lambda \|\gamma\|_1 \right]$.
18. **end**
19. **end**

20. **for** each $\lambda$ in $\Lambda$ **do**
21.   * Declare $cv^*_\alpha(\lambda) \leftarrow [(1 - \alpha)\text{-quantile of } \{T^{(r)}_{n^*_r}(\bar{\theta}, \lambda) : r = 1, \ldots, R\}]$.
22.   **for** each $B$ in $\mathcal{B}$ **do**
23.     * Compute the simulated local power as
24.       $\mathcal{R}_n(\lambda, B, \alpha) = \frac{1}{R} \sum_{r=1}^{R} 1\{T^{(r)}_{n^*_r}(\lambda, B) > cv^*_\alpha(\lambda)\}$.
25. **end**
26. **end**

27. Select the optimal penalty level as $\hat{\lambda}(\bar{\theta}) \in \arg\max_{\lambda \in \Lambda} \min_{B \in \mathcal{B}} \mathcal{R}_n(\lambda, B, \alpha)$.
28. Compute the p-value as $p_\alpha(\bar{\theta}, \hat{\lambda}(\bar{\theta})) = \frac{1}{R} \sum_{r=1}^{R} 1\{T^{(r)}_{n^*_r}(\bar{\theta}, \hat{\lambda}(\bar{\theta})) > T_n(\bar{\theta}, \hat{\lambda}(\bar{\theta}))\}$.

**Output:** Reject $H_0$ at the significance level $\alpha$ iff $p_\alpha(\bar{\theta}, \hat{\lambda}(\bar{\theta})) < \alpha$. 

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cations, $d = 1$ and we need to choose $\mathcal{B}$ as a subset of $\mathbb{R}$. We provide details in Sections 7, 8 and 9.

We now make remarks on computation of $T_n(\bar{\theta}, \lambda)$, $T_{n,*}^{(r)}(\bar{\theta}, \lambda)$, and $T_{n,*}^{(r)}(\lambda, B)$ in Algorithm 1. On one hand, in the Monte Carlo experiments reported in Section 7, we apply a random grid search on $\Gamma$. On the other hand, in empirical applications in Sections 8 and 9, we use the particleswarm solver available in Matlab. Particle swarm optimization (PSO) is a stochastic population-based optimization method proposed by Kennedy and Eberhart (1995). It conducts gradient-free global searches and has been successfully used in economics (for example, see Qu and Tkachenko (2016)). Hence, PSO can be viewed as a more refined approach to global optimization than simple random grid search. Nonetheless, we did not use the particleswarm solver in the Monte Carlo experiments because PSO is based on a heuristic procedure and requires careful monitoring to check whether it produces reasonable solutions. It was too costly to monitor the particleswarm solver in the Monte Carlo experiments; however, it was possible with empirical applications because we did not have to regenerate data.

We end this section by recalling that the confidence interval for $\theta_0$ can be constructed by inverting a pointwise test of $H_0 : \theta = \bar{\theta}$. For this purpose, one could generate $R$ collections of $\{\eta_i^* : i = 1, \ldots, n\}$ and use the same collections across different values of $\bar{\theta}$ to reduce the random noise in bootstrap inference.

1Specifically, the particleswarm solver is included in the global optimization toolbox software of Matlab. We use the default option of the particleswarm solver.

2It is possible to adopt the two-step approach used in Qu and Tkachenko (2016). That is, we start with the PSO solver, followed by multiple local searches. Further, the genetic algorithm (GA) can be used in the first step instead of PSO and both GA and PSO methods can be compared to check whether a global solution is obtained. We do not pursue these refinements in our paper to save the computational times of bootstrap inference.
6 Inference with Pre-Estimated Parameters

Partition \( \theta = (\theta_1', \theta_2') \) and \( \theta_0 = (\theta_{10}', \theta_{20}') \). We now consider the inference for \( \theta_{10} \). We assume that for each fixed \( \theta_1 \), there exists some prior estimator \( \hat{\theta}_2(\theta_1) = \psi_n(\{X_i, W_i\}_{i=1}^n) \) of \( \theta_2(\theta_1) \), so that \( \theta_{20} = \theta_2(\theta_{10}) \). For example, suppose that \( g(X_i, \theta) \) can be written as \( g(X_i, \theta) = g_1(X_i, \theta_1) - \theta_2 \). Then, \( \theta_{20} = \mathbb{E}[g_1(X_i, \theta_{10})] = 0 \), thereby yielding the following estimator of \( \theta_2 \) given \( \theta_1 \):

\[
\hat{\theta}_2(\theta_1) = n^{-1} \sum_{i=1}^{n} g_1(X_i, \theta_1) .
\] (22)

In what follows, we assume standard regularity conditions on \( \hat{\theta}_2(\theta_1) \). Let \( \Theta_1 \) denote the parameter space for \( \theta_1 \), which is a compact set whose interior is non-empty. Let \( \|a\| \) denote the Euclidean norm of a vector \( a \).

Assumption 4. Suppose that there exists a \( \sqrt{n} \)-consistent estimator \( \hat{\theta}_2(\theta_1) \) of \( \theta_2(\theta_1) \) that has the following representation: uniformly in \( \theta_1 \in \Theta_1 \), which is compact and has non-empty interior,

\[
\sqrt{n} \left( \hat{\theta}_2(\theta_1) - \theta_2(\theta_1) \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_{ni}(\theta_1) + o_p(1) ,
\]

where \( \{\psi_{ni}(\theta_1) = (\eta_{ni}(\theta_1), U_i)\} \) is a strictly stationary ergodic mds array and

\[
V_{\psi(\theta_1)} := \lim_{n \to \infty} \mathbb{E}[\psi_{ni}(\theta_1)\psi_{ni}(\theta_1)']
\]

is positive definite. Furthermore, assume that there exists \( G(x, \theta) \) such that \( \mathbb{E}\|G(X_i, \theta_0)\| < \infty \), \( \theta \mapsto G(X_i, \theta) \) is continuous at \( \theta_0 \) almost surely, and

\[
g(X_i, \theta) - g(X_i, \theta_0) - G(X_i, \theta_0)'(\theta - \theta_0) = o_p(\|\theta - \theta_0\|) ,
\] (23)

and that \( \theta_1 \mapsto \theta_2(\theta_1) \) is continuously differentiable.
Define \( \widehat{U}_i(\theta_1) := g[X_i, \{\theta_1, \hat{\theta}_2(\theta_1)\}] \), \( \widehat{U}_i := \widehat{U}_i(\theta_{10}) \), and accordingly define the statistics

\[
\widehat{M}_n(\theta_1, \gamma) := \frac{1}{n} \sum_{i=1}^{n} \widehat{U}_i(\theta_1) \exp(W_i' \gamma),
\]

\[
\widehat{s}_n^2(\theta_1, \gamma) := \frac{1}{n} \sum_{i=1}^{n} \left[ \widehat{U}_i(\theta_1) \exp(W_i' \gamma) \right]^2,
\]

\[
\widehat{Q}_n(\theta_1, \gamma) := \sqrt{n} \frac{|\widehat{M}_n(\theta_1, \gamma)|}{\widehat{s}_n(\theta_1, \gamma)},
\]

and the test statistic

\[
\widehat{T}_n(\theta_1, \lambda) := \max_{\gamma \in \Gamma} \left[ \widehat{Q}_n(\theta_1, \gamma) - \lambda \| \gamma \|_1 \right].
\]

(24)

Partition \( G(X_i, \theta) = [G_1(X_i, \theta), G_2(X_i, \theta)]' \) corresponding to the partial derivatives with respect to \( \theta_1 \) and \( \theta_2 \). Suppressing the dependence on \( \theta_1 \) in the notation when it is evaluated at \( \theta_{10} \), we obtain the following result.

**Theorem 5.** Let Assumptions 1, 2 and 4 hold. Then,

\[
\widehat{T}_n(\lambda) \Rightarrow \max_{\gamma \in \Gamma} \left[ \frac{\| M(\gamma) + Z' \mathbb{E}[G_2(X_i, \theta_0) \exp(W_i' \gamma)] \|}{s(\gamma)} - \lambda \| \gamma \|_1 \right],
\]

where \( (Z, M(\gamma)) \) is a centered Gaussian random vector with \( \mathbb{E}[ZZ'] = \lim_{n \to \infty} \mathbb{E}[\eta_{ni} \eta_{ni}'] \) and \( \mathbb{E}[ZM(\gamma)] = \lim_{n \to \infty} \mathbb{E}[U_i \eta_{ni} \exp(W_i' \gamma)] \).

Under the presence of estimated quantity in the test statistic, the multiplier bootstrap in (6) is not valid. To develop valid inference, we now describe how to modify the multiplier bootstrap by exploiting the influence function \( \eta_{ni} \). Define

\[
\widehat{G}_{2i} := G_2(X_i, \left(\theta_{10}, \hat{\theta}_2\right)).
\]
Let
\[
\hat{M}_{n,*}(\gamma) := \frac{1}{n} \sum_{i=1}^{n} \eta_i^* \left\{ \hat{U}_i \exp(W_i' \gamma) + \eta_{ni}^* \frac{1}{n} \sum_{j=1}^{n} \hat{G}_{2j} \exp(W_j' \gamma) \right\},
\]
\[
\hat{s}_{n,*}^2(\gamma) := \frac{1}{n} \sum_{i=1}^{n} \left[ \eta_i^* \left\{ \hat{U}_i \exp(W_i' \gamma) + \eta_{ni}^* \frac{1}{n} \sum_{j=1}^{n} \hat{G}_{2j} \exp(W_j' \gamma) \right\} \right]^2.
\] (25)

Then, we proceed with these modified quantities, as in Section 3. That is, to implement the bootstrap we need to obtain an explicit formula for the influence function \( \eta_{ni}(\theta_1) \) in Assumption 4 such as \( g_1(X_i, \theta_1) \) in 22.

### 6.1 Choice of Penalty

We start with a sequence of local alternatives \( \theta_{1n} = \theta_{10} + B/\sqrt{n} \). Then, expressing the hypothesized value of \( \theta_{1n} \) explicitly, we write the corresponding statistics by

\[
\hat{M}_n(\theta_{1n}, \gamma) := \frac{1}{n} \sum_{i=1}^{n} \hat{U}_i(\theta_{1n}) \exp(W_i' \gamma),
\]
\[
\hat{s}_n^2(\theta_{1n}, \gamma) := \frac{1}{n} \sum_{i=1}^{n} \left[ \hat{U}_i(\theta_{1n}) \exp(W_i' \gamma) \right]^2,
\]
\[
\hat{Q}_n(\theta_{1n}, \gamma) := \sqrt{n} \frac{[\hat{M}_n(\theta_{1n}, \gamma)]}{\hat{s}_n(\theta_{1n}, \gamma)},
\]

and the test statistic

\[
\hat{T}_n(\theta_{1n}) := \max_{\gamma \in \Gamma} \left[ \hat{Q}_n(\theta_{1n}, \gamma) - \lambda_n \|\gamma\|_1 \right].
\] (26)

Define \( g_i(\theta_1, \theta_2) := g[X_i, \{\theta_1, \theta_2\}] \) and partition \( G(X_i, \{\theta_1, \theta_2\}) = [G_{1i}(\theta_1, \theta_2)', G_{2i}(\theta_1, \theta_2)']' \) as before. The limit of the test statistic \( \hat{T}_n(\theta_{1n}) \) can be easily obtained by modifying the
proof of Theorem 5. Specifically, we note that

\[
\sqrt{n} \hat{M}_n(\theta_{1n}, \gamma) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_i(\theta_{1n}, \theta_2(\theta_{1n})) \exp(W'_i \gamma) \\
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta'_{ni} \frac{1}{n} \sum_{j=1}^{n} G_{2j}(\theta_{1n}, \theta_2(\theta_{1n})) \exp(W'_j \gamma) + o_p(1)
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_i(\theta_0) \exp(W'_i \gamma) + B' \frac{1}{n} \sum_{i=1}^{n} \left(G_{1i}(\theta_0) + \frac{\partial^2 (\theta_{10})'}{\partial \theta_1} G_{2i}(\theta_0)\right) \exp(W'_i \gamma) \\
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta'_{ni} \frac{1}{n} \sum_{j=1}^{n} G_{2j}(\theta_0) \exp(W'_j \gamma) + o_p(1),
\]

Thus, the noncentrality term is determined by \( B' \mathbb{E}[\omega_i] \), where

\[
\omega_i(\gamma) := \left(G_{1i}(\theta_0) + \frac{\partial^2 (\theta_{10})'}{\partial \theta_1} G_{2i}(\theta_0)\right) \exp(W'_i \gamma).
\]

As shorthand notation, let \( G_{1i} := G_1(X_i, \theta_0) \) and \( G_{2i} := G_2(X_i, \theta_0) \). We now adjust (16) in Section 4.3 as follows: let

\[
\mathcal{T}(\lambda, B) = 1 \left\{ \max_{\gamma \in \Gamma} \left| \frac{M(\gamma) + Z' \mathbb{E}[G_{2i} \exp(W'_i \gamma)] + B' \mathbb{E}[^{\hat{\omega}_i}(\gamma)]}{s(\gamma)} \right| - \lambda \|\gamma\|_1 > c_\alpha(\lambda) \right\}
\]

(27)

for a critical value \( c_\alpha(\lambda) \) and \( \mathcal{R}(\lambda, B) = \mathbb{E}[\mathcal{T}(\lambda, B)] \). Then, as before, choose \( \lambda \) by solving (17). To implement this procedure, we modify the steps in Section 4.3 with

\[
\hat{M}_{n,*B}(\gamma) := \frac{1}{n} \sum_{i=1}^{n} \eta_i \left\{ \hat{U}_i \exp(W'_i \gamma) + \eta'_{ni} \frac{1}{n} \sum_{j=1}^{n} \hat{G}_{2j} \exp(W'_j \gamma) \right\} + \frac{1}{n} \sum_{i=1}^{n} \frac{B}{\sqrt{n}} \hat{\omega}_i(\gamma),
\]

\[
\hat{s}_{n,*B}^2(\gamma) := \frac{1}{n} \sum_{i=1}^{n} \left[ \eta_i \left\{ \hat{U}_i \exp(W'_i \gamma) + \eta'_{ni} \frac{1}{n} \sum_{j=1}^{n} \hat{G}_{2j} \exp(W'_j \gamma) \right\} + \frac{1}{n} \sum_{i=1}^{n} \frac{B}{\sqrt{n}} \hat{\omega}_i(\gamma) \right]^2,
\]

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where
\[
\hat{\omega}_i(\gamma) := \left( G_{1i} + \frac{\partial \hat{\theta}_2(\theta_{10})'}{\partial \theta_1} \hat{G}_{2i} \right) \exp(W'_i \gamma).
\]

Then, the remaining steps are identical to those in Section 4.3.

7 Monte Carlo Experiments

In this section, we examine finite sample properties of the proposed test in a linear instrumental variable model. We consider the data-generating process introduced in Section 4.4. That is, data \(\{(X_{1i}, X_{2i}, W_i) : i = 1, \ldots, n\}\) are generated independently from \(X_{1i} = \theta_0 X_{2i} + U_i\) and \(X_{2i} = c_n W_i + c(W_i^2 - 1) + V_i\), where

\[
\begin{pmatrix}
U_i \\
V_i \\
W_i
\end{pmatrix} \sim \mathcal{N}
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
1 & \rho & 0 \\
\rho & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

In the experiments, we set \(\theta_0 = 0, \rho = 0.8, c_n = 3n^{-1/2}\), and \(c = 1\). Recall that we have weak identification of \(\theta_0\) using the linear first stage, whereas \(c \neq 0\) induces strong identification in the conditional moment model regardless of the value of \(c_n\).

To see how the presence of irrelevant instruments affects performance of the tests, we introduce i.i.d. random vectors \(\{F_i : i = 1, \ldots, n\}\) in \(\mathbb{R}^\ell\), where \(F_i\) is a \(N(0, I_\ell)\) vector independent of \(\{(U_i, V_i, W_i) : i = 1, \ldots, n\}\) and form an augmented set of instruments \(Z_i = (W_i, F'_i) \in \mathbb{R}^{1+\ell}\), where \(\ell \geq 1\) represents the dimension of irrelevant information. For each value of \(\bar{\theta}\), the pointwise test of \(H_0 : \theta = \bar{\theta}\) against a two-sided alternative \(H_1 : \theta \neq \bar{\theta}\) is performed without a priori information on the irrelevance of \(F_i\), ruling out any preliminary process on the model selection. Thus, the conditional moment condition \(\mathbb{E}[X_{1i} - X_{2i}\theta_0 | Z_i] = 0\) is employed for the identification of \(\theta_0\).
To compute the test statistic defined in (4), we apply a random grid search on a search space \( \Gamma = [-5, 5]^{1+\ell} \). Specifically, we approximate \( \Gamma \) with a set of \( m = \max\{10^\ell, 10^3\} \) uniformly random draws from \( \Gamma \), called \( \Gamma_m \subset \Gamma \), and find the maxima over the smaller search space \( \Gamma_m \). To achieve the boundedness and normalization of \( Z_t \), we standardize each instrument, apply a one-to-one transformation \( \tan^{-1}(\cdot) \) as in Bierens (1990) and standardize again the transformed instruments. In each replication, the critical values are computed from 300 simulations with the multiplier bootstrapping scheme. A grid for the penalty level \( \lambda \) is taken as \( \Lambda = \{0.005j : j = 0, 1, \ldots, 24\} \). The sample size is set as \( n = 200 \). Throughout the experiments, the nominal level of the test is set at \( \alpha = 0.1 \) and there were 5,000 replications for each experiment. To choose the optimal \( \lambda \), we set \( B = \{2\} \) as the twice standard deviation of reduced form error \( u_i \). Since \( R(\lambda, B, \alpha) \) is an increasing function of \( |B| \) when \( d = 1 \), it is sufficient to consider a singleton set of local alternatives. For more details, see the related discussion in Section 8.

7.1 Comparison with Antoine and Lavergne (2022)

In this subsection, we compare our test with those of Antoine and Lavergne (2022). We report the size and power properties of these tests as the dimension of irrelevant instruments increases and show that \( \ell_1 \)-penalization leads to substantial improvement in the power under the presence of irrelevant conditioning variables.

The integrated conditional mean (ICM) and conditional ICM (CICM) tests proposed by Antoine and Lavergne (2022), which we call the AL tests. Both ICM and CICM tests are obtained in their heteroscedasticity-robust version. The product triangular kernel \( w(z) = \prod_{k=1}^{\dim(z)} \max\{1-|z_k|, 0\} \) is adopted for construction of a weight matrix \( W \). The other settings for the AL tests are as in Section 6 in Antoine and Lavergne (2022).

In Figure 2, we display the power curves of our test with \( \lambda = 0 \) and the AL tests as \( \ell \) varies from 0 to 4. The main finding of this simulation study is that the Bierens’ maximum statistic gains more robust power to sparse relevant instruments than the integrated statis-
tics. The power curves shift downward gradually as $\ell$ increases, aligned with the fact that irrelevant instruments cause noisy estimation and inference. The ICM and CICM tests exhibit the same pattern, but the behaviors were qualitatively distinct, featuring a transition between low and high values of $\ell$. When there are no irrelevant instruments, both the ICM and CICM tests are slightly undersized and have better statistical powers compared to our test. In contrast, as $\ell$ reaches 4, the AL tests become severely undersized below 0.01 and generate trivial power.

Figure 2: Power Comparisons

(a) Our Test with $\lambda = 0$

(b) ICM

(c) CICM

Heuristically, the loss of statistical power of integral-type tests may be attributed to the concentration of power-generating region. For simplicity of presentation, let us assume that a product kernel $w(z) = \prod_{k \leq \text{dim}(z)} \tilde{w}(z_k)$ is used for a weight matrix. The ICM statistic
has nontrivial local power depending on an integrated noncentrality parameter

$$\Delta = \int \left| \mathbb{E} \left[ (X_1 - X_2 \theta) e^{i\gamma W} e^{i\delta' F} \right] \right|^2 \mu(\gamma) \prod_{k=1}^{\ell} \mu(\delta_k) \ d\gamma d\delta_1 \cdots d\delta_{\ell},$$

where $\mu$ defines a probability measure via the inverse Fourier transform of $\tilde{w}(\cdot)$ which is assumed to be non-negative everywhere. Since $F$ is $N(0, I_\ell)$ independent of $(X_1, X_2, W)'$, the expression reduces to $\nu^\ell \cdot \int \left| \mathbb{E}[(X_1 - X_2 \theta)e^{i\gamma W}] \right|^2 d\mu(\gamma)$ where $\nu = \int \exp(-\delta^2) \mu(\delta) d\delta < 1$ unless $\tilde{w} \equiv 1$, implying that $\Delta$ decays exponentially in the dimension of irrelevant instruments. This is because the noncentrality parameter is small in a large region where $|\delta|^2 = \delta' \delta > 0$ with $\delta = (\delta_1, \ldots, \delta_\ell)'$. Hence, after being averaged, the detection power is diluted by the factor $\nu^\ell$. The maximum Bierens statistic is relatively free from the issue since the maximum noncentrality parameter remains unchanged with the dimension of irrelevant instruments.

Table 1: Rejection Probabilities by Penalty

| $\lambda$ | $\ell = 6$ | $\ell = 7$ |
|-----------|------------|------------|
| $\theta_2$ | 51.0 | 42.7 |
| $\theta_1$ | 37.8 | 31.5 |
| $\theta_0$ | 11.7 | 11.5 |
| $\theta_2$ | 84.9 | 80.4 |
| $\theta_1$ | 80.6 | 75.0 |
| $\lambda$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ |
| 0.02 | 54.8 | 41.1 | 12.0 | 87.9 | 83.8 |
| 0.04 | 56.7 | 42.6 | 12.2 | 88.8 | 84.6 |
| 0.06 | 56.6 | 43.1 | 12.4 | 88.6 | 84.7 |
| 0.08 | 55.8 | 42.6 | 12.4 | 88.0 | 84.5 |
| 0.10 | 54.7 | 42.3 | 12.7 | 87.4 | 83.7 |
| 0.12 | 53.7 | 41.9 | 12.9 | 86.2 | 82.6 |
| opt | 55.7 | 42.5 | 13.1 | 88.7 | 84.5 |

Notes: Reported are rejection probabilities of $H_0 : \theta = \theta_j$ versus $H_1 : \theta \neq \theta_j$ for $\theta_j = 16j/\sqrt{n}$ at the nominal level 10%. The opt row represents the usage of optimal lambda $\hat{\lambda}(\theta_j)$ for the testing. The numbers are based on 5000 replications from the DGP in Section 4.4.

Table 1 reports simulation results for our test with $\ell = 6$ and $\ell = 7$. Reported are rejection probabilities of $H_0 : \theta = \theta_j$ versus $H_1 : \theta \neq \theta_j$ for $\theta_j = 16j/\sqrt{n}$ at the nominal level 10%. When positive penalty levels were chosen instead of 0, there is a global improvement in powers, especially when $\ell = 7$, holding onto good size control, albeit slight over-rejection.
Table 2: Distribution of Estimated Optimal Penalty

| Statistics | \(\ell = 6\) | \(\ell = 7\) |
|------------|---------------|---------------|
|            | \(\theta_{-2}\) | \(\theta_{-1}\) | \(\theta_0\) | \(\theta_1\) | \(\theta_2\) | \(\theta_{-2}\) | \(\theta_{-1}\) | \(\theta_0\) | \(\theta_1\) | \(\theta_2\) |
| Mean       | 0.041          | 0.041          | 0.043          | 0.044          | 0.043          | 0.040          | 0.041          | 0.043          | 0.043          | 0.042          |
| Pr(\(\hat{\lambda} > 0\)) | 84.4          | 83.1          | 85.4          | 82.9          | 84.1          | 84.0          | 83.6          | 86.7          | 83.7          | 84.4          |
| Pr(\(\lambda \geq 0.02\)) | 62.8          | 61.9          | 65.4          | 63.6          | 63.6          | 61.3          | 62.2          | 66.5          | 64.2          | 63.1          |
| Pr(\(\lambda \geq 0.04\)) | 44.0          | 44.3          | 46.9          | 46.9          | 46.5          | 43.1          | 44.0          | 47.0          | 46.6          | 45.9          |
| Pr(\(\lambda \geq 0.06\)) | 30.7          | 31.6          | 33.1          | 34.8          | 33.5          | 30.6          | 32.1          | 32.5          | 33.8          | 32.7          |
| Pr(\(\lambda \geq 0.08\)) | 17.9          | 19.1          | 19.9          | 21.7          | 20.4          | 17.7          | 19.2          | 19.7          | 21.0          | 19.2          |
| Pr(\(\lambda \geq 0.10\)) | 8.9           | 9.6           | 10.7          | 12.2          | 10.6          | 9.3           | 10.1          | 10.7          | 11.8          | 10.0          |
| Pr(\(\lambda = 0.12\)) | 2.6           | 3.2           | 3.9           | 4.3           | 3.0           | 2.9           | 3.1           | 3.3           | 4.2           | 3.4           |

Notes: The probabilities are displayed as percentages.

This verifies our basic motivation that \(\ell_1\)-regularization can improve powers by picking over irrelevant information.

For low values of \(\ell \in [1, 3]\), adding an \(\ell_1\)-penalization with \(\lambda\) less than 0.01 leads to a marginal increase in powers on a local region of the parameter space. Apparently, there was no clear advantage of penalization for low values of \(\ell\). We do not report details for brevity.

Table 2 shows the Monte Carlo distributions of estimated optimal \(\lambda\) reported in Table 1. The distributions of the optimal \(\lambda\) is similar across different values of \(\theta\) and also between \(\ell = 6\) and \(\ell = 7\). The average value is about 0.04 and a positive value is selected with probability greater than 0.82 across all cases.

7.2 Comparison with Estimation-Based Tests

In this section, we compare our test with other competing procedures that are designed for efficient estimation in a linear IV model. Specifically, we consider alternative inference procedures based on Chamberlain (1987), Donald, Imbens, and Newey (2003), Domínguez and Lobato (2004), and Kitamura, Tripathi, and Ahn (2004). The experimental design is the same as in the previous subsection. Some remarks on the methods are in order,
followed by some details on implementation.

Chamberlain (1987) establishes that a complete sequence (sieve) of functions $\mathcal{F} = \{f_j(z) : j = 1,2,\ldots\}$ can replace the conditional moment condition with a countable set of unconditional moment conditions, and an optimal GMM estimation using increasingly many of these moment restrictions gains asymptotic efficiency. The idea behind Donald, Imbens, and Newey (2003) is similar to Chamberlain (1987), but it also allows for a triangular sequence of sieve $\mathcal{F}_n$ and verifies growth conditions on $K_n = |\mathcal{F}_n|$, the user-chosen number of unconditional moment restrictions, required for a class of estimators to be consistent and asymptotically normal. In conducting our experiments, we choose the GMM estimator over the generalized empirical-likelihood (GEL) and 2SLS estimators for its efficiency and popularity in applications. Examples of commonly used sieves include power series, b-spline functions, and Fourier series, and many others. We take $\mathcal{F}_n$ as either a sequence of power series with increasing degrees or b-splines with increasing knots as in Chamberlain (1987) and Donald, Imbens, and Newey (2003). Domínguez and Lobato (2004) propose an estimator $\hat{\theta}_{DL}$ that does not require any selection of parameters or moment conditions by users and is consistent and asymptotically normal. They also offer an iterative procedure to obtain an efficient GMM estimator $\hat{\theta}^E$ from $\hat{\theta}_{DL}$ under local identification only. Kitamura, Tripathi, and Ahn (2004) suggest a smoothed empirical-likelihood (SEL) based estimator and test procedure. Unlike the GEL estimator proposed by Donald, Imbens, and Newey (2003), their approach depends on a sequence of localizing constants $b_n$, which determines the extent of information pooling from nearby data points.

We compare our test with four inference procedures based on this strand of the literature: FGMM with the sieve-type approximations of conditional moment restriction, the Domínguez-Lobato estimator, and the SEL ratio test. We use power series, linear and quadratic b-splines as a sieve, each indexed by the degree $m$ or the order of b-spline $d$ and number of knots $T$. A basis b-spline is formed by taking a tensor product of the 1-dimensional bases in each instrument, where knots are placed internally and evenly in
Table 3: Rejection Probabilities by Competing Methods

| Methods       | Bandwidths | $\ell = 0$ | $\ell = 1$ | $\ell = 2$ | $\ell = 3$ | $\ell = 4$ |
|---------------|------------|------------|------------|------------|------------|------------|
|               |            | $\theta_{-1}$ | $\theta_0$ | $\theta_1$ | $\theta_{-1}$ | $\theta_0$ | $\theta_1$ | $\theta_{-1}$ | $\theta_0$ | $\theta_1$ | $\theta_{-1}$ | $\theta_0$ | $\theta_1$ |
| Our test      | optimal $\lambda$ | 80.4 | 10.8 | 99.4 | 64.7 | 11.9 | 97.7 | 53.9 | 12.6 | 94.2 | 40.9 | 12.5 | 81.4 | 36.3 | 13.3 | 76.5 |
| AL ICM        | .          | 86.5 | 13.3 | 100.0 | 67.5 | 9.7 | 100.0 | 37.5 | 5.2 | 99.1 | 4.7 | 0.6 | 52.4 | 0.4 | 0.0 | 0.3 |
| AL CICM       | $m = 2$    | 93.4 | 9.6 | 100.0 | 88.0 | 8.4 | 100.0 | 72.2 | 3.8 | 99.8 | 11.7 | 0.3 | 67.2 | 0.0 | 0.0 | 0.2 |
| FGMM          | $d = 1, T = 2$ | 93.5 | 8.5 | 93.9 | 92.1 | 15.1 | 88.3 | 99.1 | 59.1 | 91.6 | 99.9 | 81.2 | 91.6 | 99.8 | 89.7 | 93.9 |
|               | $d = 2, T = 1$ | 93.0 | 10.6 | 93.6 | 92.1 | 24.4 | 87.0 | 99.0 | 71.8 | 90.0 | 99.5 | 87.2 | 93.5 | 99.7 | 92.9 | 96.6 |
| DL            | .          | 98.3 | 16.2 | 96.9 | 98.1 | 20.5 | 94.2 | 98.6 | 27.8 | 90.5 | 98.9 | 38.6 | 85.5 | 99.4 | 53.6 | 77.8 |
| SELR          | $b_{ROT}$ | 99.7 | 13.5 | 100.0 | 6.7 | 2.6 | 5.8 | 5.9 | 3.4 | 2.5 | 31.3 | 16.5 | 14.7 | 68.1 | 43.6 | 36.1 |
|               | $b_{ROT}/2$ | 99.8 | 10.1 | 100.0 | 5.8 | 3.1 | 2.2 | 6.0 | 3.6 | 2.5 | 55.7 | 33.2 | 30.5 | 98.5 | 74.9 | 39.9 |

Notes: Reported are rejection probabilities of $H_0 : \theta = \theta_j$ versus $H_1 : \theta \neq \theta_j$ for $\theta_j = 8j/\sqrt{n}$ at the nominal level 10%. $m$ denotes the degree of power series and $d$ and $T$ are the order and the number of knots in b-splines. DL represents Dominguez and Lobato’s (2004) test. A fixed bandwidth $b_{ROT}$ is set in reference to Silverman’s rule of thumb and ranges over $[0.6, 0.72]$ as $\ell$ goes from 0 to 4.

terms of the quantile. In SEL estimation and test, we use the product Epanechnikov kernel $\kappa(z) = \prod_{k=1}^{\dim(z)} (1 - z_k^2) 1(|z_k| \leq 1)$ as a smoother, and the bandwidths $b_n$ are chosen in reference to Silverman’s rule of thumb and their halved values.

In Table 3 we report the rejection probabilities of the inference procedures in the baseline models under the true parameter $\theta_0 = 0$ and two local alternatives, $\theta_{-1}$ and $\theta_1$, where $\theta_j = 8j/\sqrt{n}$. For each of the compared methods, there was significant size distortion as $\ell$ increases, suggesting that credible inference is difficult to attain as $\ell$ gets large. The only exception was the SELR test when $\ell$ is either 1 or 2. Yet, in these cases, the test shows trivial powers. It appears that there are at least two issues with the estimation-based approaches.

First, tests based on the aforementioned estimators become seriously biased as the dimension of irrelevant instruments increases, causing the power curves to skew away from the origin. Second, normal approximation theory fits poorly to the finite sampling distribution. In particular, consistent estimates of standard errors were hard to obtain when $\ell > 0$.

8 Testing Rational Unbiased Reporting of Ability Status

Benítez-Silva, Buchinsky, Chan, Cheidvasser, and Rust (2004, BBCCR hereafter) examine whether a self-reported disability status is a conditionally unbiased indicator of Social
Security Administration (SSA)’s disability award decision. Specifically, they test if \( U_i = \tilde{A}_i - \tilde{D}_i \) has mean zero conditional on covariates \( W_i \), where \( \tilde{A}_i \) is the SSA disability award decision and \( \tilde{D}_i \) is a self-reported disability status indicator. Their null hypothesis is \( H_0 : \mathbb{E}[\tilde{A}_i - \tilde{D}_i|W_i] = 0 \), which is termed as the hypothesis of rational unbiased reporting of ability status (RUR hypothesis). They use a battery of tests, including a modified version of Bierens (1990)’s original test, and conclude that they fail to reject the RUR hypothesis. In fact, their Bierens test has the smallest \( p \)-value of 0.09 in their test results (see Table II of their paper). In this section, we revisit this result and apply our testing procedure.

Table 4 shows a two-way table of \( \tilde{A}_i \) and \( \tilde{D}_i \) and Table 5 reports the summary statistics of \( \tilde{A}_i \) and \( \tilde{D}_i \) along those of covariates \( W_i \). After removing individuals with missing values in any of covariates, the sample size is \( n = 347 \) and the number of covariates is \( p = 21 \).

| SSA award decision (\( \tilde{A}_i \)) | Self-reported disability (\( \tilde{D}_i \)) | Total |
|--------------------------------------|--------------------------------------|-------|
|                                      | 0                                   | 1     | 86    |
| 0                                    | 35                                  | 51    | 86    |
| 1                                    | 61                                  | 200   | 261   |
| Total                                | 96                                  | 251   | 347   |

Before computing the penalized maximum test statistic, we first studentize each of covariates and transform them by \( x \mapsto \tan^{-1}(x) \) componentwise. This step ensures that each of the components of \( W_i \) is bounded and they are comparable among each other. The space \( \Gamma \) is set as \( \Gamma = [-10, 10]^p \). To compute \( T_n \) in (4), we use the \texttt{particleswarm} solver available in \texttt{Matlab}. It is computationally easier to obtain \( T_n \) in (4) when \( \lambda \) is relatively larger. This is because a relevant space for \( \Gamma \) is smaller with a larger \( \lambda \).

To choose an optimal \( \lambda \) as described in Section 4.3, we parametrize the null hypothesis \( H_0 : \mathbb{E}[U_i|W_i] = 0 \) by \( g(X_i, \theta) = U_i - \theta \) in (1) with \( \theta_0 = 0 \). First note that \( G_i = 1 \) in

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3 According to Table I in BBCCR, the sample size is 393 before removing observations with the missing values; however, there are only 388 observations in the data file archived at the Journal of Applied Econometrics web page. After removing missing values, the size of the sample extract we use is \( n = 347 \), whereas the originally reported sample size is \( n = 356 \) in Table 2 in BBCCR.
Table 5: Summary Statistics

| Variable                                      | Mean  | Stan. Dev. | Min. | Max. | $\hat{\gamma}$ |
|-----------------------------------------------|-------|------------|------|------|-----------------|
| SSA award decision ($A$)                      | 0.75  | 0.43       | 0    | 1    |                 |
| Self-reported disability ($\hat{D}$)          | 0.72  | 0.45       | 0    | 1    |                 |
| Covariates                                    |       |            |      |      |                 |
| White                                         | 0.57  | 0.50       | 0    | 1    | 0.07            |
| Married                                       | 0.58  | 0.49       | 0    | 1    | -0.16           |
| Prof./voc. training                           | 0.36  | 0.48       | 0    | 1    | 0.17            |
| Male                                          | 0.39  | 0.49       | 0    | 1    | 0.02            |
| Age at application to SSDI                    | 55.97 | 4.81       | 33   | 76   | 0.33            |
| Respondent income/1000                        | 6.19  | 10.28      | 0    | 52   | 0.12            |
| Hospitalization                               | 0.88  | 1.44       | 0    | 14   |                 |
| Doctor visits                                 | 13.12 | 13.19      | 0    | 90   |                 |
| Stroke                                        | 0.07  | 0.26       | 0    | 1    | -0.90           |
| Psych. problems                               | 0.25  | 0.43       | 0    | 1    |                 |
| Arthritis                                     | 0.40  | 0.49       | 0    | 1    |                 |
| Fracture                                      | 0.13  | 0.33       | 0    | 1    |                 |
| Back problem                                  | 0.59  | 0.49       | 0    | 1    | -0.13           |
| Problem with walking in room                  | 0.15  | 0.36       | 0    | 1    |                 |
| Problem sitting                               | 0.48  | 0.50       | 0    | 1    | -0.03           |
| Problem getting up                            | 0.59  | 0.49       | 0    | 1    | -0.03           |
| Problem getting out of bed                    | 0.24  | 0.43       | 0    | 1    | -0.13           |
| Problem getting up the stairs                 | 0.45  | 0.50       | 0    | 1    |                 |
| Problem eating or dressing                    | 0.07  | 0.26       | 0    | 1    |                 |
| Prop. worked in $t - 1$                       | 0.32  | 0.41       | 0    | 1    | 1.32            |
| Avg. hours/month worked                       | 2.68  | 8.85       | 0    | 60   |                 |
this example. Therefore, for each \( \lambda \in \Lambda \), \( R(\lambda, B) \) is an increasing function of \(|B|\). Thus, it suffices to evaluate the smallest value of \(|B|\) satisfying \( B \in B \). Here, we take it to the sample standard deviation of \( U_i \). For \( \lambda \), we take \( \Lambda = \{1, 0.9, \ldots, 0.2\} \). This range of \( \lambda \)'s is chosen by some preliminary analysis. When \( \lambda \) is smaller than 0.2, it is considerably harder to obtain stable solutions; thus, we do not consider smaller values of \( \lambda \). Since \( \lambda \mapsto T_n(\lambda) \) is a decreasing function, we first start with the largest value of \( \lambda \) and then solves sequentially by lowering the value of \( \lambda \), while checking whether the newly obtained solution indeed is larger than the previous solution. This procedure results in a solution path by \( \lambda \).

Top-left panel of Figure 3 shows the solution path \( \lambda \mapsto T_n(\lambda) \) along with the number of selected covariates, which is defined to be ones whose coefficients are no less than 0.01 in absolute value. For the latter, 4 covariates are selected with \( \lambda = 1 \), whereas 12 are chosen with \( \lambda = 0.2 \). Top-right panel displays the rejection probability defined in (20) when \( B = 0 \) (size) and \( B = \hat{\sigma}(U_i) \), where \( \hat{\sigma}(U_i) \) is the sample standard deviation of \( U_i \). The level of the test is 0.1 and there are 100 replications to compute the rejection probability. The power is relatively flat up to \( \lambda = 0.4 \), increase a bit at \( \lambda = 0.3 \) and is maximized at \( \lambda = 0.2 \). The bottom panel visualizes each of 21 coefficients as \( \lambda \) decreases. It can be seen that the proportion of working in \( t - 1 \) (\texttt{worked prev} in the legend of the figure) has the largest coefficient (in absolute value) for all values of \( \lambda \) and an indicator of stroke has the second largest coefficient, followed by age at application to Social Security Disability Insurance (SSDI). The coefficients for selected covariates are given in the last column of Table 5 for \( \lambda = 0.2 \).

| \( \lambda \) | Test statistic | No. of selected cov.s | Bootstrap p-value |
|----------|----------------|-----------------------|------------------|
| 0.2      | 3.525          | 12                    | 0.021            |
| 0.3      | 3.213          | 11                    | 0.020            |

Since the power in the top-right panel of Figure 3 is higher at \( \lambda = 0.2 \) and 0.3, we report bootstrap test results for \( \lambda \in \{0.2, 0.3\} \) in Table 6. There are \( R = 1,000 \) bootstrap
Figure 3: Testing Results
replications to obtain the bootstrap p-values. Interestingly, we are able to reject the RUR hypothesis at the 5 percent level, unlike BBCCR. Furthermore, our analysis suggests that the employment history, captured by the proportion of working previously, stroke, and the age at application to SSDI are the three most indicative covariates that point to the departure from the RUR hypothesis.

9 Inference of Itinerant Substitution

In this section, we revisit Yogo (2004) and conduct inference on the elasticity of intertemporal substitution (EIS). We look at the case of annual US series (1891–1995) used in Yogo (2004) and focus on $U_t(\theta) = \Delta c_{t+1} - \theta_2 - \theta_1 r_t$, where $\Delta c_{t+1}$ is the consumption growth at time $t + 1$ and $r_t$ is the real interest rate at time $t$. The instruments $W_t$ are the twice lagged nominal interest rate, inflation, consumption growth, and log dividend-price ratio. As in the previous section, each instrument is studentized and is transformed by $\tan^{-1}(\cdot)$. The main parameter of interest is EIS, denoted by $\theta_1$. In this example, we have data $\{(\Delta c_{t+1}, r_t, W_t) : t = 1, \ldots, n\}$, where $n = 105$.

To conduct inference for $\theta_1$ developed in Section 6, we use a demeaned version:

$$\hat{U}_t(\theta_1) = \left(\Delta c_{t+1} - \frac{1}{n} \sum_{t=1}^{n} \Delta c_{t+1}\right) - \theta_1 \left(r_t - \frac{1}{n} \sum_{t=1}^{n} r_t\right).$$

Using the notation in Section 6, we have that in this example,

$$\theta_2(\theta_1) = \mathbb{E}\left[\Delta c_{t+1}\right] - \theta_1 \mathbb{E}\left[r_t\right],$$

$$\hat{\theta}_2(\theta_1) = \frac{1}{n} \sum_{t=1}^{n} \Delta c_{t+1} - \theta_1 \frac{1}{n} \sum_{t=1}^{n} r_t,$$

$$\eta_{nt}(\theta_1) = \left(\Delta c_{t+1} - \mathbb{E}[c_{t+1}]\right) - \theta_1 \left(r_t - \mathbb{E}[r_t]\right),$$

$$\hat{\eta}_{nt}(\theta_1) = \hat{U}_t(\theta_1).$$
Then, because $G_{2t} = -1$, adopting (25) yields the following multiplier bootstrap:

$$\hat{M}_{n,*}^2(\gamma) = \frac{1}{n} \sum_{t=1}^{n} \eta_t^* \hat{U}_t \left\{ \exp(W_t' \gamma) - \frac{1}{n} \sum_{t=1}^{n} \exp(W_t' \gamma) \right\},$$

$$\hat{s}_{n,*}^2(\gamma) = \frac{1}{n} \sum_{t=1}^{n} \left[ \eta_t^* \hat{U}_t \left\{ \exp(W_t' \gamma) - \frac{1}{n} \sum_{t=1}^{n} \exp(W_t' \gamma) \right\} \right]^2.$$

Furthermore, since $G_{1t} = -r_t$, we can use the following quantities to calibrate the optimal $\lambda$:

$$\hat{M}_{n,*B}^2(\gamma) = \frac{1}{n} \sum_{t=1}^{n} \eta_t^* \hat{U}_t \left\{ \exp(W_t' \gamma) - \frac{1}{n} \sum_{t=1}^{n} \exp(W_t' \gamma) \right\} - \frac{1}{n} \sum_{t=1}^{n} \frac{B r_t}{\sqrt{n}} \exp(W_t' \gamma),$$

$$\hat{s}_{n,*B}^2(\gamma) = \frac{1}{n} \sum_{t=1}^{n} \left[ \eta_t^* \hat{U}_t \left\{ \exp(W_t' \gamma) - \frac{1}{n} \sum_{t=1}^{n} \exp(W_t' \gamma) \right\} - \frac{1}{n} \sum_{t=1}^{n} \frac{B r_t}{\sqrt{n}} \exp(W_t' \gamma) \right]^2.$$

To solve for $\lambda$ in (17), we take $\Lambda = \{1, 0.8, 0.6, 0.4, 0.2, 0\}$ and $B = \{0.2\}$. The latter is taken be a singleton set since the rejection probability is an increasing function of $|B|$ and $\eta_t^* \sim N(0, 1)$ is symmetrically distributed about zero. In computing the optimal $\lambda$, the level of the test is 0.1 and a grid $\{-0.5, -0.4, \ldots, 0.5\}$ for $\theta_{10}$ is used. For each case, the rejection frequency in (20) is obtained out of 100 replications with the steps modified as in Section 6.1. Figure 4 shows a heatmap of the rejection probability with $B = 0.2$, where the rejection probability is shown in each cell. The first noticeable result is that setting $\lambda = 0$ results in substantial loss of power. This indicates that the sample is not large enough to accommodate all four instruments without penalization. The power is above 0.6, as long as $\lambda > 0$. There are some differences across different values of $\theta_{10}$. For simplicity, we will use the common tuning parameter across different $\theta_{10}$ and set $\lambda = 0.4$ at which the power seems to be maximized or nearly maxmized.

The top panel of Figure 5 shows the bootstrap p-values for each $\theta_{10}$ that are constructed with 1000 bootstrap replications. The resulting 95% confidence interval is $[-0.31, 0.15]$. Yogo (2004) commented that “there appears to be identification failure for the annual
Figure 4: Empirical Results (EIS): Selection of $\lambda$
Figure 5: Empirical Results (EIS): US Annual Data, $\lambda = 0.4$
U.S. series.” Indeed, the confidence interval from the Anderson-Rubin (AR) test was $[-0.49, 0.46]$ and those from the Lagrange multiplier (LM) test and the conditional likelihood ratio (LR) tests were $[-\infty, \infty]$ (see Table 3 of Yogo, 2004). Our confidence interval is tighter than any of these similar tests based on unconditional moments, thereby suggesting that the conditional moment restrictions could provide a more informative confidence interval without arbitrarily creating a particular set of unconditional moment restrictions. The bottom panel of Figure 5 displays the estimated $\gamma$. The estimates of $\gamma$ vary over $\theta_{10}$, even change the signs, and seem to have large impacts when $\theta_{10}$ ranges from 0.1 to 0.2. There is no particular instrument that stands out in terms of the estimated magnitude. In a nutshell, we demonstrate that a seemingly uninformative set of instruments can provide an informative inference result if one strengthens unconditional moment restrictions by making them the infinite-dimensional conditional moment restrictions with the aid of penalization.

10 Conclusions

We have developed an inference method for a vector of parameters using an $\ell_1$-penalized maximum statistic. Our inference procedure is based on the multiplier bootstrap and combines inference with model selection to improve the power of the test. We have recommended solving a data-dependent max-min problem to select the penalization tuning parameter. We have demonstrated the efficacy of our method using two empirical examples.

There are multiple directions to extend our method. First, we may consider a panel data setting where the number of conditioning variables may grow as the time series

\footnote{If each instrument is used separately for the AR test, the resulting 95% confidence intervals are as follows: (i) $[-\infty, \infty]$ with the nominal interest rate; (ii) $[-0.29, 0.28]$ with inflation; (iii) $[-\infty, \infty]$ with consumption growth; (iv) $[-\infty, -0.12] \cup [0.02, \infty]$ with log dividend-price ratio, where the instruments are twice lagged. The confidence interval using inflation is similar to ours, but its length is more than 20% larger than our confidence interval $[-0.31, 0.15]$.}
dimension increases. Second, unknown parameters may include an unknown function (e.g., Chamberlain 1992; Newey and Powell 2003; Ai and Chen 2003; Chen and Pouzo, 2012). In view of results in Breunig and Chen (2020), Bierens-type tests without penalization might not work well when the parameter of interest is a nonparametric function. It would be interesting to study whether and to what extent our penalization method improves power for nonparametric inference. Third, a continuum of conditional moment restrictions (e.g., conditional independence assumption) might be relevant in some applications. Fourth, it would be interesting to extend our method for empirical industrial organization. For instance, Gandhi and Houde (2019) proposed a set of relevant instruments from conditional moment restrictions to avoid the weak identification problem. It is an intriguing possibility to combine our approach with their insights into Berry, Levinsohn, and Pakes (1995). All of these extensions call for substantial developments in both theory and computation.

A Proofs

Proof of Theorem 1. First, we show the stochastic equicontinuity of the processes \( \sqrt{n}M_n(\gamma) \) and \( s_n(\gamma) \) for (9) and (10). Due to the boundedness of \( \Gamma \) and \( W_i, U_i \exp(W_i'\gamma) \) is Lipschitz continuous with a bound \( K|U_i|\|\gamma_1 - \gamma_2\| \) for some \( K \) and for any \( \gamma_1 \) and \( \gamma_2 \). Then, this Lipschitz property and the existence of moment of some \( c > d \) implies due to Theorem 2 in Hansen (1996) that the empirical process \( \sqrt{n}M_n(\gamma) \) is stochastically equicontinuous. The Lipschitz continuity and the ergodic theorem also imply that \( s_n(\gamma) \) is stochastically equicontinuous.

Next, the martingale difference sequence central limit theorem and the ergodic theorem yield the desired finite-dimensional convergence for (9) and (10) under Assumptions 1 and 2; see e.g. Davidson (1994)'s Section 24.3 and 13.4.

Finally, for the convergence of \( T_n(\lambda) \), note that both \( \Lambda \) and \( \Gamma \) are bounded, implying
\( \lambda \| \gamma \|_1 \) is uniformly continuous. Thus, the process \( \frac{M(\gamma)}{n(\gamma)} - \lambda \| \gamma \|_1 \) converges weakly in \( \ell^\infty (\Gamma \times \Lambda) \), the space of bounded functions on \( \Gamma \times \Lambda \), and the weak convergence of \( T_n(\lambda) \) follows from the continuous mapping theorem since (elementwise) \( \max \) is a continuous operator.

**Proof of Theorem 2** For the same reason as in the proof of Theorem 1, it is sufficient to verify the conditional finite dimensional convergence. As \( \eta^*_i g(X_i, \bar{\theta}) \exp(W'_i \gamma) \) is a martingale difference sequence, we verify the conditions in Hall and Heyde (1980)'s Theorem 3.2, a conditional central limit theorem for martingales. Their first condition that

\[
n^{-1/2} \max_i \left| \eta^*_i g(X_i, \bar{\theta}) \exp(W'_i \gamma) \right| \overset{p}{\to} 0
\]

and the last condition \( \mathbb{E} \left[ \max_i \eta^2_i g(X_i, \bar{\theta})^2 \exp(2W'_i \gamma) \right] = O(n) \) are straightforward since \( \exp(W'_i \gamma) \) is bounded and \( |\eta^*_i g(X_i, \bar{\theta})| \) has a finite \( c \) moment for \( c > 2 \). Next,

\[
n^{-1} \sum_{i=1}^{n} \eta^2_i g(X_i, \bar{\theta})^2 \exp(2W'_i \gamma) \overset{p}{\to} \mathbb{E} \left[ g(X_i, \bar{\theta})^2 \exp(2W'_i \gamma) \right]
\]

by the ergodic theorem. This completes the proof.

**Proof of Theorem 3** It follows from Lemma 1 that

\[
\frac{\mathbb{E} \left[ g(X_i, \bar{\theta}) \exp(W'_i \gamma) \right]}{\sqrt{\mathbb{E} \left[ g^2(X_i, \bar{\theta}) \exp(2W'_i \gamma) \right]}} > 0
\]

for almost every \( \gamma \in \Gamma \). Then, the result follows from the ergodic theorem.

**Proof of Theorem 4** We begin with showing that \( c^*_\alpha(\lambda) \overset{p}{\to} c_\alpha(\lambda) \) uniformly in \( \Lambda \). First, recall that the inverse map on the space of the distribution function \( F \) that assigns its \( \alpha \)th quantile is Hadamard-differentiable at \( F \) provided that \( F \) is differentiable at \( F^{-1}(\alpha) \) with a strictly positive derivative; see e.g. Section 3.9.4.2 in van der Vaart and Wellner (1996). Therefore, it is sufficient for the uniform consistency of the bootstrap to show that \( F^*_\lambda(x) \overset{p}{\to} F_\lambda(x) \)
uniformly \( x \in [\min_{\lambda} F^{-1}_{\lambda}(1 - \alpha), \max_{\lambda} F^{-1}_{\lambda}(1 - \alpha)] \) and \( \lambda \in \Lambda \). However, this is a direct consequence of the conditional stochastic equicontinuity and the convergence of the finite-dimensional distributions established in Theorem 2.

Next, the preceding step implies that \( T_{n,\ast}(\lambda, B) - c_{\alpha}^*(\lambda) \) converges weakly to

\[
\max_{\gamma \in \Gamma} \left\{ \frac{M(\gamma) + \mathbb{E}[\exp(W_i'\gamma)G_iB]}{s(\gamma)} - \lambda \|\gamma\|_1 \right\} - c_{\alpha}(\lambda),
\]

which is the limit in (16). This in turn implies the uniform convergence of \( R_n(\lambda, B) \) in probability to \( R(\lambda, B) \). Since \( R \) is continuous on a compact set, the usual consistency argument yields that \( d(\hat{\lambda}, \Lambda_0) \overset{p}{\to} 0 \).

For the same reason, \( T_n(\lambda) - c^*_n(\lambda) \) converges weakly to \( T(\lambda) - c_{\alpha}(\lambda) \) and thus the probability that \( T_n(\hat{\lambda}) \geq c^*_n(\hat{\lambda}) \) converges to \( \alpha \) for any sequence \( \hat{\lambda} \) due to the weak convergence.

\[ \square \]

**Proof of Theorem 5** Write \( g_i(\theta) \) and \( G_2i(\theta) \) for \( g(X_i, \theta) \) and \( G_2(X_i, \theta) \), respectively. Note that for \( \theta_1 = \theta_{10} \),

\[
\sqrt{n}M_n(\gamma) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_i(\theta_0) \exp(W_i'\gamma) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_i n^{-1} \sum_{j=1}^{n} G_{2j}(\theta_0) \exp(W_j'\gamma) + o_p(1)
\]

due to Assumption 4. Then, \( \frac{1}{n} \sum_{j=1}^{n} G_{2j}(\theta_0) \exp(W_j'\gamma) \) converges uniformly in probability and \( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} a_1 g_i(\theta_0) \exp(W_i'\gamma) + a_2 \eta_i n^{-1} \) is P-Donker for any real \( a_1 \) and \( a_2 \) for the same reasoning as in the proof of Theorem 1. Similarly, the uniform convergence of \( s_n^2(\gamma) \) follows since \( g() \) is Lipschitz in \( \theta \) by (23). \[ \square \]
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