CALABI-YAU ALGEBRAS AND THE SHIFTED NONCOMMUTATIVE SYMPLECTIC STRUCTURE

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Abstract. In this paper we show that for a Koszul Calabi-Yau algebra, there is a shifted bi-symplectic structure in the sense of Crawley-Boevey-Etingof-Ginzburg [15], on the cobar construction of its co-unitalized Koszul dual coalgebra, and hence its DG representation schemes, in the sense of Berest-Khachatryan-Ramadoss [3], have a shifted symplectic structure in the sense of Pantev-Toën-Vaquié-Vezzosi [29].

Keywords: Shifted noncommutative symplectic structure, Calabi-Yau algebra

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1. Introduction

The notion of Calabi-Yau algebras was introduced by Ginzburg in 2007. It is a noncommutative generalization of affine Calabi-Yau varieties, and due to Van den Bergh, admits the so-called “noncommutative Poincaré duality”. In a joint paper with Berest and Ramadoss [1] we showed that for a Koszul Calabi-Yau algebra, say \( A \), there is a version of noncommutative Poisson structure, called the shifted double Poisson structure, on its cofibrant resolution, and hence induces a shifted Poisson structure on the derived representation schemes of \( A \), a notion introduced by Berest, Khachatryan and Ramadoss in [3]. Here “shifted” means there is a degree shifting, depending on the dimension of the Calabi-Yau algebra, on the Poisson bracket. Such a shifted Poisson structure was later further studied in [9] in much detail.

A natural question that arises now is, given a Koszul Calabi-Yau algebra, whether or not the noncommutative Poisson structure associated to it is noncommutative symplectic. To
justify this question, let us remind the reader of a version of noncommutative symplectic structure introduced by Crawley-Boevey, Etingof and Ginzburg in [15], which they called the bi-symplectic structure in the paper. A bi-symplectic structure on an associative algebra is a closed 2-form in its Karoubi-de Rham complex that induces an isomorphism between the space of noncommutative vector fields (more precisely, the space of double derivations) and the space of noncommutative 1-forms. In [39, Appendix], Van den Bergh showed that any bi-symplectic structure naturally gives rise to a double Poisson structure, which is completely analogous to the classical case. However, the reverse is in general not true. Nevertheless, it is still interesting to ask if this is true in the special case of Calabi-Yau algebras.

The second motivation for the paper comes from the 2012 paper [29] of Pantev, Toën, Vaquié and Vezzosi, where they introduced the notion of shifted symplectic structure for derived stacks. This not only generalizes classical symplectic geometry to a much broader context, but also reveals many new features on a lot of geometric spaces, especially on various moduli spaces that mathematicians are now studying. As remarked by the authors, the shifted symplectic structure, if it exists, always comes from the Poincaré duality of the corresponding source spaces; they also outlined how to generalize shifted symplectic structures to noncommutative spaces, such as Calabi-Yau categories.

Note that Calabi-Yau algebras are highly related to Calabi-Yau categories. For example, a theorem of Keller (see [22, Lemma 4.1]) says that the bounded derived category of a Calabi-Yau algebra is a Calabi-Yau category. Applying the idea of [29], we would expect that the noncommutative Poincaré duality of a Calabi-Yau algebra shall also play a role in the corresponding shifted noncommutative symplectic structure if it exists.

The main results of the current paper may be summarized as follows. Let $A$ be a Calabi-Yau algebra of dimension $n$. Assume that $A$ is also Koszul, and denote its Koszul dual coalgebra by $A^!$. Let $\tilde{R} = \Omega(A^!)$ be the cobar construction of the co-unitalization $\tilde{A}^!$ of $A^!$. In this paper, we show that $\tilde{R}$, rather than $\Omega(A^!)$ as studied in [1, 9], has a $(2 - n)$-shifted bi-symplectic structure. Such a bi-symplectic structure comes from the volume form of the noncommutative Poincaré duality of $A$, and naturally induces a $(2 - n)$-shifted symplectic structure on the DG representation schemes $\text{Rep}_V(\tilde{R})$, for all vector spaces $V$ (see Theorem 4.6). By taking the corresponding trace maps we obtain a commutative diagram (see (38) for more details)

$$
\begin{align*}
\text{HH}^*(A) \xrightarrow{\text{Tr}} \text{H}^*(\text{Der}(\text{Rep}_V(\tilde{R}))^{\text{GL}}) \\
\text{HHH}_{n-*}(A) \xrightarrow{\text{Tr}} \text{H}_{n-*}(\Omega^1_{\text{com}}(\text{Rep}_V(\tilde{R}))^{\text{GL}}),
\end{align*}
$$

where the left hand side are the Hochschild cohomology and homology of $A$, with the isomorphism being Van den Bergh’s noncommutative Poincaré duality, and the right hand side are the cohomology and homology of the $GL(V)$-invariant complexes of vector fields and 1-forms on $\text{Rep}_V(\tilde{R})$ respectively.

By Van den Bergh’s result mentioned above, the shifted bi-symplectic structure induces a shifted double Poisson structure on $\tilde{R}$; therefore there is a shifted Poisson structure on $\text{Rep}_V(\tilde{R})$. In this paper, we will study the deformation quantization of such a shifted Poisson
structure, and show that it comes from the quantization of the “functions” $\tilde{R}_q := \tilde{R}/[\tilde{R}, \tilde{R}]$ of $\tilde{R}$ (see Theorem 7.5). By Koszul duality, the homology of $\tilde{R}_q$ minus the unit is isomorphic to the cyclic homology of $A$, and thus we obtain a quantization of the latter as well. This construction is inspired by the quantization of quiver representations.

This paper is organized as follows. In §2 we collect some basics of noncommutative geometry, such as the noncommutative 1-forms and vector fields, and some operations, such as the contraction and Lie derivative between them, then we recall the definition of bi-symplectic structure introduced by Crawley-Boevey et al. in [15].

In §3 we recall the notion of Koszul algebras and some of their basic properties. Let $A$ be a Koszul algebra over a field $k$. We give explicit formulas for the double derivations and 1-forms of $\tilde{R}$. The commutator quotient spaces of them are identified with the Hochschild cohomology and homology of $A$ respectively.

In §4 we first recall the definition of Calabi-Yau algebras, and then show that if the Calabi-Yau algebra, say $A$, is Koszul, then its noncommutative volume form gives a shifted bi-symplectic structure on $\tilde{R}$, where $\tilde{R}$ is given as above.

In §5 we first recall the DG representation schemes of a DG algebra, which was introduced by Berest et al. in [3]. Following the works [3, 15], we see that if a DG algebra admits a shifted bi-symplectic structure, then its DG representation schemes have a shifted symplectic structure. We then apply it to the Koszul Calabi-Yau algebra case.

In §6 and §7 we study the shifted Poisson structure on the representation schemes of $\tilde{R}$ and their quantization. We show that such quantizations are induced by the quantization of $\tilde{R}_q$ as a Lie bialgebra. This is completely analogous to the papers of Schedler [33] and Ginzburg-Schedler [19], where the quantization of the representation spaces of doubled quivers is constructed, which is compatible with the quantization of the necklace Lie bialgebra of the quivers via the canonical trace map.

In §8 we briefly discuss some relationships of the current paper with the series of papers by Berest and his collaborators [1, 2, 3, 4], where the derived representation schemes of associative algebras were introduced and studied.

In the last section, §9, we give the two examples of Calabi-Yau algebras, namely, the 3- and 4-dimensional Sklyanin algebras, and study the corresponding shifted bi-symplectic structure in some detail.

This paper is a sequel to [1, 9], where the shifted noncommutative Poisson structure associated to Calabi-Yau algebras was studied; however, in the current paper we try to be as self-contained as possible. When we were in the final stage of the paper, we learned that Y-T Lam in his thesis [24] as well as W.-K. Yeung in his paper [42] have obtained several results which are similar to ours. Nevertheless, the main results and methods of theirs and ours are quite different.

Convention 1.1. Throughout the paper, $k$ is a field of characteristic zero, though in a lot of cases it need not necessarily to be so. All morphisms and tensors are over $k$ unless otherwise specified. DG algebras (respectively DG coalgebras) are unital and augmented (respectively co-unital and co-augmented), with the degree of the differential being $-1$. For a chain complex, its homology is denoted by $H_\bullet(-)$, and its cohomology is given by $H^\bullet(-) := H_{-\bullet}(-)$. 
2. Some basics of noncommutative geometry

In this section, we recall some basic notions in noncommutative geometry. They are mostly taken from [15, 39, 40]; here we work in the differential graded (DG for short) setting.

2.1. Noncommutative differential forms. Suppose that \((R, \partial)\) is a DG associative algebra over \(k\), where \(\partial\) is the differential. The bimodule \(\Omega^1_{nc} R\) of noncommutative 1-forms of \(R\) is a DG \(R\)-bimodule generated by symbols \(dx\) of degree \(|x|\), linear in \(x\) for all \(x \in R\) and subject to the relations
\[
\begin{align*}
    d(xy) &= (dx)y + x(dy), \\
    \partial(dx) &= d(\partial x),
\end{align*}
\]
for all \(x, y \in R\). In this paper, we also assume \(d1 = 0\).

Alternatively, \(\Omega^1_{nc} R\) is the kernel of the multiplication \(R \otimes R \to R\), which is a subcomplex of \(R\)-bimodules in \(R \otimes R\), generated by \(1 \otimes x - x \otimes 1\) for all \(x \in R\), with differential induced from \(\partial\). The identification of two \(R\)-bimodules constructed above is given by \(dx \mapsto 1 \otimes x - x \otimes 1\).

Let \(\Omega^1_{nc} R[-1]\) be the suspension of \(\Omega^1_{nc} R\), i.e. the degrees of \(\Omega^1_{nc} R\) are shifted up by one. Sometimes we also write it in the form \(\Sigma \Omega^1_{nc} R\), which means the degree-shifting operator applies from the left. (In what follows, for a graded vector space \(V^*\), \(V[n]\) is a graded vector space with \(\langle V[n] \rangle_i = V_{i+n}\).)

Let \(\Omega^*_{nc} R := T_R(\Omega^1_{nc} R[-1])\) be the free tensor algebra generated by \(\Omega^1_{nc} R[-1]\) over \(R\). Let
\[
d : R \to \Omega^1_{nc} R[-1], \quad x \mapsto d(x) = \Sigma dx.
\]
Then by our sign convention, it is easy to check\(^\text{[4]}\)
\[
\begin{align*}
    d(xy) &= d(x) \cdot y + (-1)^{|x|} x \cdot d(y), \\
    d \circ \partial(x) &= -\partial \circ d(x),
\end{align*}
\]
for all \(x, y \in R\).

Now let \(d(d(x)) = 0\) for all \(x \in R\). Extend \(d\) to be a map \(d : \Omega^*_{nc} R \to \Omega^{*+1}_{nc} R\) by derivation. Also, extend \(\partial\) to \(\Omega^*_{nc} R\) by derivation. Then we have
\[
d^2 = 0, \quad \partial^2 = 0, \quad \text{and} \quad d \circ \partial + \partial \circ d = 0.
\]
In general, \(d\) is called the de Rham differential of \(\Omega^*_{nc} R\), and for convenience, \(\partial\) is called the internal differential of \(\Omega^*_{nc} R\).

Note that \(\partial\) and \(d\) have degrees \(-1\) and \(1\) respectively, which make \(\Omega^*_nc R\) into a mixed DG algebra and is called the set of noncommutative differential forms of \(R\). Let
\[
\text{DR}^*_{nc} R := (\Omega^*_{nc} R)_{\leq 1} = \Omega^*_{nc} R / [\Omega^*_{nc} R, \Omega^*_{nc} R]
\]
be the graded commutator quotient space. Since \(\partial\) and \(d\) are both derivations with respect to the product on \(\Omega^*_{nc} R\), \(\text{DR}^*_{nc} R\) with the induced differentials, still denoted by \((\partial, d)\), is a mixed complex, and is called the Karoubi-de Rham complex of \(R\).

\(^{[4]}\)Recall that for a graded \(R\)-bimodule \(M\), \(\Sigma M\) is a graded \(R\)-bimodule with \(a \cdot \Sigma m \cdot b = (-1)^{|a|i} \Sigma (amb)\), for homogeneous \(a, b \in R\) and \(m \in M\).
2.2. Noncommutative polyvector fields. Following [15, 39], the noncommutative vector fields on an associative algebra are given by the double derivations. By definition, the space of double derivations \( \text{Der} R \) of \( R \) is the set of derivations \( \text{Der}(R, R \otimes R) \). Since the map \( R \to \Omega_{\text{nc}}^1 R, x \mapsto dx = 1 \otimes x - x \otimes 1 \) is a universal derivation, meaning that every derivation of \( R \) factors through \( \Omega_{\text{nc}}^1 R \) (c.f. [32 Proposition 3.3]), we have that

\[
\text{Der} R \cong \text{Hom}_{R^e}(\Omega_{\text{nc}}^1 R, R \otimes R),
\]

where \( R^e \) is the enveloping algebra \( R \otimes R^{op} \). In the above notation, we have used the outer \( R \)-bimodule structure on \( R \otimes R \); namely, for any \( x, y, u, v \in R \),

\[
u \cdot (x \otimes y) \cdot v := ux \otimes yv.
\]

\( R \otimes R \) has also an inner \( R \)-bimodule structure, which is given by

\[
u \ast (x \otimes y) \ast v := (-1)^{|u||x|+|v||u|+|v|} xv \otimes uy.
\]

With the inner \( R \)-bimodule structure on \( R \otimes R \), \( \text{Der} R \) is a DG \( R \)-bimodule. Now let \( T_R(\text{Der} R[1]) \) be the free DG algebra generated by \( \text{Der} R[1] \) over \( R \), which is called the space of (noncommutative) polyvector fields of \( R \).

2.3. Actions of noncommutative vectors on noncommutative forms. Analogous to the classical case, the noncommutative vectors act on noncommutative forms by contraction and by Lie derivative, which together satisfy the noncommutative version of the Cartan identity (see Lemma 2.2).

2.3.1. Contraction and Lie derivative. For any \( \Theta \in \text{Der} R[1] \), the contraction operator

\[
i_{\Theta} : \Omega_{\text{nc}}^* R \longrightarrow \Omega_{\text{nc}}^* R \otimes \Omega_{\text{nc}}^* R.
\]

is given as follows: first, let

\[
i_{\Theta}(a) = 0, \quad \text{for all } a \in R
\]

and

\[
i_{\Theta} : \Omega_{\text{nc}}^1 R[-1] \to R \otimes R, \quad d(a) \mapsto \Theta(a),
\]

where we have used the fact that \( \text{Der} R[1] = \text{Hom}_{R^e}(\Omega_{\text{nc}}^1 R[-1], R \otimes R) \). Second, extend \( i_{\Theta} \) to an \( R \)-linear operator on \( \Omega_{\text{nc}}^* R \) in the natural way, that is, if we write

\[
i_{\Theta}a = \sum i_{\Theta}' a \otimes i_{\Theta}'' a, \quad \text{for } a \in \Omega_{\text{nc}}^1 R,
\]

then

\[
i_{\Theta}(a_1 \cdots a_n) = \sum_{1 \leq k \leq n} (-1)^{\sigma_k} a_1 \cdots a_{k-1} (i_{\Theta}' a_k) \otimes (i_{\Theta}'' a_k) a_{k+1} \cdots a_n,
\]

for \( a_1, \ldots, a_n \in \Omega_{\text{nc}}^1 R \), where \( (-1)^{\sigma_k} \) is the Koszul sign. Recall that in general, the Koszul sign comes from switching the positions of graded elements: for two graded vector spaces \( V, W \), the isomorphism \( V \otimes W \to W \otimes V \) is given by \( v \otimes w \mapsto (-1)^{|v||w|} w \otimes v \); for example, in the formula above, \( (-1)^{\sigma_k} = (-1)^{|\Theta|(|a_1|+\cdots+|a_{k-1}|)} \). In what follows, we will sometimes omit its precise value if it is clear from the context.
Next, given $\Theta \in \text{Der} R[1]$ the Lie derivative of $\Theta$ on the noncommutative differential forms is given by
\[
L_\Theta(x_0 dx_1 \cdots dx_n) := \Theta'(x_0) \otimes \Theta''(x_0) dx_1 \cdots dx_n + \sum_{1 \leq k \leq n} (-1)^{\sigma_k}(x_0 dx_1 \cdots dx_{k-1} d\Theta'(x_k) \otimes \Theta''(x_k) dx_{k+1} \cdots dx_n) + (-1)^{\sigma_k} d\Theta'(x_k) \otimes d\Theta''(x_k) dx_{k+1} \cdots dx_n.
\]

2.3.2. Reduced contraction and Lie derivative. Now, for a graded algebra $A$ and $c = c_1 \otimes c_2 \in A \otimes A$, let $\circ c = (-1)^{|c_1||c_2|} c_2 c_1$, and given a map $\phi : A \to A \otimes A$, write $\circ \phi : A \to A : c \mapsto \circ(\phi(c))$.

Define the reduced contraction operator $\iota$ and the reduced Lie derivative $\mathcal{L}$ by
\[
\iota_\Theta(-) : \Omega^\bullet_{nc} R \to \Omega^\bullet_{nc} -1 R, \omega \mapsto \iota_\Theta \omega = \circ(i_\Theta(\omega)) \tag{2}
\]
and
\[
\mathcal{L}_\Theta(-) : \Omega^\bullet_{nc} R \to \Omega^\bullet_{nc} R, \omega \mapsto \mathcal{L}_\Theta \omega = \circ(L_\Theta(\omega)) \tag{3}
\]
respectively. More explicitly, for $a_1, a_2, \cdots, a_q \in \Omega^1_{nc} R$,
\[
\iota_\Theta(a_1 a_2 \cdots a_r) = \sum_k (-1)^{\sigma_k} (i_\Theta a_k) a_{k+1} \cdots a_r a_1 \cdots a_{k-1} (i_\Theta a_k),
\]
and similarly for $\mathcal{L}_\Theta(a_1 a_2 \cdots a_r)$ (see [13] (2.8.4) and (2.8.5)). The following two lemmas are proved in [13].

**Lemma 2.1** ([13] Lemma 2.8.6). \(1\) The reduced contraction defined above only depends on the image of $\omega$ in $\text{DR}^\bullet_{nc} R$; in other words, $\iota_\Theta$ descends to a well-defined map $\text{DR}^\bullet_{nc} R \to \Omega^\bullet_{nc} -1 R$.

\(2\) For $\omega \in \text{DR}^\bullet_{nc} R$, the map $\Theta \mapsto \iota_\Theta \omega$ gives an $R$-bimodule morphism $\text{Der} R[1] \to \Omega^\bullet_{nc} -1 R$.

**Lemma 2.2** ([13] Lemma 2.8.8(i)). Let $\iota$ and $\mathcal{L}$ be as above. Then for any $\Theta \in \text{Der} R[1]$, we have
\[
d \circ \iota_\Theta + \iota_\Theta \circ d = \mathcal{L}_\Theta \quad \text{and} \quad d \circ \mathcal{L}_\Theta = \mathcal{L}_\Theta \circ d.
\]

Moreover, from the definitions of $\iota$ and $\mathcal{L}$, it is clear that both of them respect $\partial$. With these preparations, we now recall the definition of bi-symplectic structures of Crawley-Boevey, Etingof and Ginzburg introduced in [13] (we here rephrase it for DG algebras):

**Definition 2.3** (Bi-symplectic structure). Suppose $R$ is a DG associative algebra. A closed form $\omega \in \text{DR}^2_{nc} R$ of total degree $2 - n$ is called an $n$-shifted bi-symplectic structure if the map
\[
\iota_{(-)} \omega : \text{Der} R[1] \to (\Omega^1_{nc} R[-1])[2 - n], \Theta \mapsto \iota_\Theta \omega \tag{4}
\]
is a quasi-isomorphism of complexes of $R$-bimodules.

The reader may refer to Remark 5.10 for some discussions on degree shifting in the above definition.

In general, it is difficult to check the existence of a bi-symplectic structure for a given DG algebra. The difficulty lies in the fact there is in general no closed formula for the
noncommutative vector fields and 1-forms. However, for free associative algebras, such as the path algebra of a quiver, both of them can be explicitly written down. This leads Crawley-Boevey et. al. [15] to give an explicit identification of them, and hence to give a bi-symplectic structure on the path algebra of a doubled quiver. In the following two sections, we generalize their construction to the DG case, where Koszul Calabi-Yau algebras naturally appear.

3. Koszul duality

Koszul duality was originally introduced by Priddy to compute the Hochschild homology and cohomology of associative algebras. Nowadays it plays an increasingly important role in the study of noncommutative algebraic geometry. Let $A$ be a Koszul algebra, and $A^!$ its Koszul dual coalgebra. Let $\tilde{\mathcal{R}}$ be the co-unitalization of $\tilde{A}$. In this section, we give explicit formulas for $\Omega_{\mathrm{nc}}^1 \tilde{\mathcal{R}}$ and $\mathrm{Der} \tilde{\mathcal{R}}$, which are very much related to the Hochschild homology and cohomology of $A$ (see Propositions 3.5 and 3.7 below).

3.1. Koszul algebra. Let $W$ be a finite-dimensional vector space over $k$. Denote by $T W$ the free algebra generated by $W$ over $k$. Let $Q$ be a subspace of $W \otimes W^*$, and let $(Q)$ be the two-sided ideal generated by $Q$ in $T W$. Then the quotient algebra $A := T W/(Q)$ is called a quadratic algebra.

Consider the subspace $U = \bigoplus_{n=0}^{\infty} U_n := \bigoplus_{n=0}^{\infty} \bigcap_{i+j+2=n} W^i \otimes Q \otimes W^j$ of $T W$, then $U$ is a coalgebra whose coproduct is induced from the de-concatenation of the tensor products. The Koszul dual coalgebra of $A$, denoted by $A^!$, is

$$A^!_n = \bigoplus_{n=0}^{\infty} \Sigma^\otimes n (U_n).$$

$A^!$ has a graded coalgebra structure induced from that of $U$ with

$$(A^!)_0 = k, \quad (A^!)_1 = \Sigma W, \quad (A^!)_2 = (\Sigma \otimes \Sigma)(Q), \quad \cdots \cdots$$

The Koszul dual algebra of $A$, denoted by $A^!$, is just the linear dual space of $A^!$, which is then a graded algebra. More precisely, let $W^* = \mathrm{Hom}(W, k)$ be the linear dual space of $W$, and let $Q^\perp$ denote the space of annihilators of $Q$ in $W^* \otimes W^*$. Shift the grading of $W^*$ down by one, denoted by $\Sigma^{-1} W^*$, then

$$A^! = T(\Sigma^{-1} W^*)/(\Sigma^{-1} \otimes \Sigma^{-1} \circ Q^\perp).$$

Choose a basis $\{e_i\}$ for $W$, and let $\{e^*_i\}$ be their duals in $W^*$. Let $\{\xi_i\}$ be the basis in $\Sigma^{-1} W^*$ corresponding to $\{e^*_i\}$, i.e. $\xi_i = \Sigma^{-1} e^*_i$. There is a chain complex associated to $A$, called the Koszul complex:

$$\cdots \xrightarrow{b'} A \otimes A^!_{i+1} \xrightarrow{b'} A \otimes A^!_i \xrightarrow{b'} \cdots \xrightarrow{b'} A \otimes A^!_0 \xrightarrow{b'} \cdots \xrightarrow{b'} k,$$

where for any $r \otimes u \in A \otimes A^!$, $b'(r \otimes u) = \sum_i r e_i \otimes \xi_i \triangleright u$. Here $\xi_i \triangleright u$ means the interior product (contraction) of $\xi_i$ with $u$, or in other words, the evaluation of $\xi_i$ on the first component of $u$. In what follows, we prefer to write it in the form $u \cdot \xi_i$ or simply $u_\xi_i$, since the interior product
gives a right $A^!$-module structure on $A^!$. (In what follows we will also use the contraction from the right, and in this case we write it in the form $\xi u$ for $\xi$ and $u$ as above.)

**Definition 3.1** (Koszul algebra). A quadratic algebra $A = TW/(Q)$ is called Koszul if the Koszul complex $(5)$ is acyclic.

Applying the graded version of Nakayama Lemma, we have the following (see [37, Proposition 3.1] for a proof):

**Proposition 3.2.** Suppose $A$ is a Koszul algebra. Then the following complex

$$
\cdots \longrightarrow A \otimes A^i_m \otimes A \stackrel{b}{\longrightarrow} A \otimes A^i_{m-1} \otimes A \stackrel{\mu}{\longrightarrow} A \otimes A \otimes A \otimes A \otimes A \otimes A \longrightarrow A, 
$$

where

$$
b(a \otimes c \otimes a') = \sum_i \left( ae_i \otimes c \xi_i \otimes a' + (-1)^m a \otimes \xi_i c \otimes e_i a' \right),
$$

for $a \otimes c \otimes a' \in A \otimes A^i_m \otimes A$, and $\mu$ is the multiplication on $A$, gives a resolution of $A$ as an $A^e$-module.

Let $K(A) := (A \otimes A^i \otimes A, b)$. Then

$$
\text{Tor}^A_e(A, A) = H_e(A \otimes A^e, K(A)) \quad \text{and} \quad \text{Ext}^A_e(A, A) = \text{H}^*(\text{Hom}_{A^e}(K(A), A)),
$$

which are also identical to the Hochschild homology $\text{HH}^*_e(A)$ and cohomology $\text{HH}^*(A)$ of $A$ respectively. This result is due to Priddy:

**Proposition 3.3** (Priddy [30]). The complexes $A \otimes A^e, K(A)$ and $\text{Hom}_{A^e}(K(A), A)$ are quasi-isomorphic to the Hochschild chain complex $\text{CH}^*_e(A)$ and the Hochschild cochain complexes $\text{CH}^*_e(A)$ respectively.

There are explicit formulas for the chain complexes $A \otimes A^e, K(A)$ and $\text{Hom}_{A^e}(K(A), A)$: as graded vector spaces, they are the same as $A \otimes A^i_e$ and $A \otimes A^i_e$, while the differentials are given by

$$
b(a \otimes u) = \sum_i \left( ae_i \otimes u \xi_i + (-1)^{|a|} e_i a \otimes \xi_i u \right),
$$

and

$$
b(a \otimes x) = \sum_i \left( ae_i \otimes x \xi_i + (-1)^{|a|} e_i a \otimes \xi_i x \right),
$$

respectively. Van den Bergh [37, Proposition 3.3] gave a formula for the quasi-isomorphism of these complexes.

Now for a Koszul algebra $A$, view it as a DG algebra concentrated in degree zero with trivial differential. Let $A^i$ be the Koszul dual coalgebra of $A$, and $\Omega(A^i)$ be the cobar construction of $A^i$, whose differential is denoted by $\partial$. Then from $(6)$ one can deduce that the natural surjective map

$$
(\Omega(A^i), \partial) \rightarrow (A, 0),
$$

is a quasi-isomorphism of DG algebras (c.f. [27, Theorem 3.4.4]), which then gives a cofibrant resolution of $A$ in the category of DG associative algebras.
3.2. Noncommutative geometry for Koszul algebras. Suppose \( A \) is a Koszul algebra. Let \( A^i \) and \( \bar{A}^i \) be its Koszul dual coalgebra and algebra respectively. Let \( \bar{A}^i \) be the counitalization of \( A^i \); that is, \( \bar{A}^i = A^i \otimes k \) with the coproduct \( \Delta \) being

\[
\Delta(a) = \Delta(a) + a \otimes 1 + 1 \otimes a, \quad \text{for} \quad a \in A^i,
\]

\[
\Delta(1) = 1 \otimes 1,
\]

where \( \Delta(a) \) is the coproduct of \( a \) in \( A^i \). Let \( \bar{R} = \Omega(\bar{A}^i) \) be the cobar construction of \( A^i \). The following is straightforward since \( \bar{R} \) is a quasi-free DG algebra:

**Proposition 3.4.** Let \( \bar{R} = \Omega(\bar{A}^i) \). Then

\[
\Omega_{nc}^1 \bar{R}[-1] \cong \bar{R} \otimes A^i \otimes \bar{R} \quad \text{and} \quad \text{Der} \bar{R}[1] \cong \bar{R} \otimes A^i \otimes \bar{R}. \tag{12}
\]

**Proof.** In fact, since \( \bar{R} = \Omega(\bar{A}^i) = T(A^i[1]) \) is quasi-free, we have the short exact sequence

\[
0 \longrightarrow \bar{R} \otimes A^i[1] \otimes \bar{R} \longrightarrow \bar{R} \otimes \bar{R} \longrightarrow \bar{R} \longrightarrow 0.
\]

It follows that \( \Omega_{nc}^1 \bar{R} = \bar{R} \otimes A^i[1] \otimes \bar{R} \), that is, \( \Omega_{nc}^1 \bar{R}[-1] = \bar{R} \otimes A^i \otimes \bar{R} \).

From this, we also see that

\[
\text{Der} \bar{R}[1] = \text{Hom}_{\bar{R}}(\Omega_{nc}^1 \bar{R}[-1], \bar{R} \otimes \bar{R})
\]

\[
= \text{Hom}_{\bar{R}}(\bar{R} \otimes A^i \otimes \bar{R}, \bar{R} \otimes \bar{R})
\]

\[
= \text{Hom}(A^i, \bar{R} \otimes \bar{R})
\]

\[
= \bar{R} \otimes A^i \otimes \bar{R},
\]

where in the last equality, we have identified \( A^i \otimes (\bar{R} \otimes \bar{R}) \) with \( \bar{R} \otimes A^i \otimes \bar{R} \) via \( a \otimes (u \otimes v) \mapsto (-1)^{|a||u|}v \otimes a \otimes u \). Thus \( \text{Der} \bar{R}[1] \cong \bar{R} \otimes A^i \otimes \bar{R} \) follows. \( \square \)

The following proposition was obtained in [3] (see also [9]), and therefore we will only sketch its proof. Given an associative algebra \( A \), denote by \( (\text{CC}_\bullet(A), b) \) and \( (\text{CH}_\bullet(A), b) \) the Connes cyclic complex and the Hochschild chain complex of \( A \) respectively.

**Proposition 3.5 ([3]).** Suppose \( A \) is a Koszul algebra, and \( \bar{R} = \Omega(\bar{A}^i) \) with differential \( \partial \). Then

\[
(\bar{R}_\partial, \partial) \simeq (\text{CC}_\bullet(A)[1], b) \quad \text{and} \quad (\text{DR}_{nc}^1 \bar{R}, \partial) \simeq (\text{CH}_\bullet(A), b) \tag{13}
\]

as chain complexes, where \( \bar{R} \) is the augmentation ideal of \( \bar{R} \).

To prove this proposition, we have to recall the definition of the cyclic homology of coalgebras. Suppose that \( C \) is a DG coalgebra, let \( \Omega(C) \) be the cobar construction of the counitalization of \( C \). Let \( \Omega(C) \) be the augmentation ideal of \( \Omega(C) \). Then by Quillen [32, §1.3] the cyclic complex of \( C \), denoted by \( (\text{CC}_\bullet(C), b) \), may take to be complex \( (\Omega(C)_2[-1], \partial) \).

**Sketch of proof of Proposition 3.5.** On the one hand, by (11) we have

\[
\text{CC}_\bullet(\Omega(A^i)) \simeq \text{CC}_\bullet(A) \tag{14}
\]
since quasi-isomorphic DG algebras have quasi-isomorphic cyclic chain complexes. On the other hand, by Jones-McCleary [20, Theorem 1] (see also [11, Lemma 17] for a proof from the Koszul duality point of view) we have

\[
\text{CC}_\bullet(\Omega(A)) \simeq \text{CC}_\bullet(A^i).
\]

(15)

Combining (14) and (15) we obtain \(\bar{\tilde{R}}_\natural \simeq \text{CC}_\bullet(A)[1]\).

Next, we show \(\text{DR}^1_{nc} \tilde{R} \simeq \text{CH}_\bullet(A)\). Since \(\Omega^1_{nc} \tilde{R}\natural[-1] = \tilde{R} \otimes A^i \otimes \tilde{R}\), we have

\[
\text{DR}^1_{nc} \tilde{R} = (\Omega^1_{nc} \tilde{R}\natural[-1])_\natural = A^i \otimes \tilde{R} = \text{CH}_\bullet(A^i),
\]

where \(\text{CH}_\bullet(A^i)\) is the underlying space of the Hochschild chain complex of \(A^i\). By a direct computation we also see that \(\partial\) on \(\Omega^1_{nc} \tilde{R}\natural\) coincides with the Hochschild boundary map on \(\text{CH}_\bullet(A^i)\).

Again by Koszul duality, the same argument as above yields \(\text{CH}_\bullet(A) \simeq \text{CH}_\bullet(A^i)\) (see [11, Theorem 15] for a complete proof), which implies \(\text{DR}^1_{nc} \tilde{R} \simeq \text{CH}_\bullet(A)\).

Convention 3.6. In the rest of the paper, as adopted by Berest et. al. in [3], when writing \(\text{CC}_\bullet(-)\), we always mean \(\text{CC}_\bullet(-)[1]\).

Proposition 3.7. Suppose \(A\) is a Koszul algebra and \(\tilde{R} = \Omega(\tilde{A}^i)\). Then \(((\text{Der} \tilde{R}[1])_\natural, \partial)\) is quasi-isomorphic to \((\text{Der} \tilde{R}[1], \partial)\), which is further quasi-isomorphic to the Hochschild cochain complex \(\text{CH}^\bullet(A)\).

Proof. Observe that as graded vector spaces

\[
\text{Der} \tilde{R}[1] = \text{Hom}(\tilde{R}\natural, (\Omega^1_{nc} \tilde{R}\natural[-1])_\natural = \text{Hom}(\tilde{R} \otimes A^i \otimes \tilde{R} \otimes \tilde{R}) = \text{Hom}(A^i, \tilde{R} \otimes \tilde{R}) = \tilde{R} \otimes A^i \otimes \tilde{R},
\]

thus

\[
(\text{Der} \tilde{R}[1])_\natural = \text{Der} \tilde{R}[1] \cong \tilde{R} \otimes A^i = \bigoplus_n \text{Hom}((A^i[-1])^\otimes n, A^i) = \text{CH}^\bullet(A^i).
\]

Under this identity, a direct calculation identifies the differential on \(\text{Der} \tilde{R}\) with the Hochschild coboundary on \(\text{CH}^\bullet(A^i)\).

Finally, by Keller [21, Theorem 3.5] (see also Shoikhet [34, Theorem 4.2]), for a Koszul algebra \(A\), \(\text{CH}^\bullet(A)\) is quasi-isomorphic to \(\text{CH}^\bullet(A^i)\) as DG Lie algebras, and hence in particular, as chain complexes. Thus combining it with the above quasi-isomorphisms, we have \((\text{Der} \tilde{R}[1])_\natural \simeq \text{CH}^\bullet(A)\).

4. Koszul Calabi-Yau algebras

The notion of Calabi-Yau algebras was introduced by Ginzburg [18] in 2007. Let \(A\) be a Koszul Calabi-Yau algebra and \(\tilde{R} = \Omega(\tilde{A}^i)\) be as before. In this section, we show that the volume form of the noncommutative Poincaré duality of \(A\) also gives the shifted bi-symplectic structure on \(\tilde{R}\).
Definition 4.1 (Ginzburg). Suppose $A$ is an associative algebra over $k$. Then $A$ is called **Calabi-Yau of dimension** $n$ (or $n$-**Calabi-Yau**) if

1. $A$ is homologically smooth, that is, $A$, viewed as a (left) $A^{e}$-module, has a bounded, finitely generated projective resolution, and
2. there is an isomorphism

$$\eta: \text{RHom}_{A^{e}}(A, A \otimes A) \cong A[n]$$

in the derived category $\mathcal{D}(A^{e})$ of (left) $A^{e}$-modules.

In the above definition, the $A^{e}$-module structure on $A \otimes A$ and $\text{RHom}_{A^{e}}(A, A \otimes A)$ is completely analogous to the case of $\text{Der} A$.

Suppose $A$ is a homologically smooth algebra, then Van den Bergh [38] showed that

$$\text{HH}_{i}(A) = H_{i}(A \otimes_{A^{e}} A) \cong \text{Hom}_{\mathcal{D}(A^{e})}(\text{RHom}_{A^{e}}(A, A \otimes A), A[i]), \text{ for all } i$$

and therefore for $A$ being Calabi-Yau, the isomorphism (16) corresponds to an element in $\text{HH}_{n}(A)$, which is still denoted by $\eta$ and is called the **volume class** of $A$.

In general, for an arbitrary Calabi-Yau algebra $A$, it is difficult to find its volume class. However, in the case when $A$ is also Koszul, this turns out to be very easy. Besides that, Koszul Calabi-Yau algebras have some other good features; for example, they form so far the most known and interesting examples of Calabi-Yau algebras, and they are all given by a superpotential (see [41]).

4.1. **The volume form.** Now suppose $A$ is Koszul Calabi-Yau of dimension $n$. First, by Proposition [38] we have

$$\text{Hom}_{\mathcal{D}(A^{e})}(\text{RHom}_{A^{e}}(A, A \otimes A), A[n]) = \text{Hom}_{\mathcal{D}(A^{e})}(\text{Hom}_{A^{e}}(A \otimes A^{i} \otimes A, A \otimes A^{i}[n] \otimes A), A \otimes A^{i}[n] \otimes A) = \text{Hom}_{\mathcal{D}(A^{e})}(A \otimes A^{i} \otimes A, A \otimes A^{i}[n] \otimes A).$$

Second, by (16), we get

$$A \otimes A^{i} \otimes A \cong A \otimes A^{i}[n] \otimes A$$

in $\mathcal{D}(A^{e})$. Since both sides of this identification are minimal free resolutions of $A$, we get an isomorphism of $A^{i} \cong A^{i}[n]$ as $A^{i}$-bimodules (see also [11 Proposition 28]). This implies that $A^{i}$ is a cyclic associative algebra of degree $n$. Let us recall its definition first.

Definition 4.2 (Cyclic associative algebra). Suppose $A$ is a graded associative algebra. It is called cyclic of degree $n$ if it admits a degree $n$, non-degenerate bilinear pairing $\langle -, - \rangle : A \times A \to k[n]$ such that

$$\langle a \cdot b, c \rangle = (-1)^{|a|+|b|}|c| \langle c \cdot a, b \rangle, \text{ for all } a, b, c \in A.$$

Proposition 4.3 (Van den Bergh). Suppose that $A$ is a Koszul algebra. Then $A$ is $n$-Calabi-Yau if and only if $A^{i}$ is cyclic of degree $n$. 

Proof. See Van den Bergh [41]. The interested reader may also refer to [11, Proposition 28] for a simple proof. □

In the literature, a cyclic associative algebra is sometimes also called a symmetric Frobenius algebra, or simply a symmetric algebra. If $A$ is a cyclic associative algebra of degree $n$, then its linear dual $A^* = \text{Hom}(A, k)$, which is a coassociative coalgebra, also has a pairing

$$\langle -, - \rangle : A^* \times A^* \to k[-n], \ (\alpha, \beta) \mapsto \langle \alpha^*, \beta^* \rangle,$$  

(18)

where $\alpha^*$ and $\beta^*$ are the images of $\alpha$ and $\beta$ under the map $A^* \cong A$ induced by the pairing. The cyclicity condition of the pairing becomes

$$\sum_{(\beta)} \langle \alpha, \beta^1 \rangle \cdot \beta^2 = \sum_{(\alpha)} (-1)^{|\alpha||\beta^1|} \langle \beta, \alpha^2 \rangle \cdot \alpha^1,$$  

(19)

for any $\alpha, \beta \in A^*$ with $\Delta(\alpha) = \sum_{(\alpha)} \alpha^1 \otimes \alpha^2$ and $\Delta(\beta) = \sum_{(\beta)} \beta^1 \otimes \beta^2$.

The above proposition in fact implies that the volume class of a Koszul Calabi-Yau algebra is represented by a(ny) nonzero top-degree element in $A^n$. More precisely, we have the following.

**Proposition 4.4.** Suppose $A$ is a Koszul $n$-Calabi-Yau algebra. Then any nonzero element $\eta \in A^n$, viewed as an element in $A \otimes A^i$ via the embedding $A^i \cong k \otimes A^i \subset A \otimes A^i$ and hence a chain in $(A \otimes A^i, b)$, is a cycle and represents the volume class of $A$.

**Proof.** First from the symmetric pairing

$$\langle x, y \rangle = (-1)^{|x||y|} \langle y, x \rangle$$

we see that in particular

$$\xi_i \eta = (-1)^n \eta \xi_i, \ \text{ for all } \xi_i \in \Sigma^{-1}W^*.$$ 

This means $\eta$ via the above embedding $A^i \cong k \otimes A^i \subset A \otimes A^i$ is a cycle, namely,

$$b(1 \otimes \eta) = \sum_i e_i \otimes \xi_i \eta - (-1)^n e_i \otimes \eta \xi_i = 0.$$ 

Also from the following operation

$$\begin{align*}
(A \otimes A^1, b) &\quad\longrightarrow\quad (A \otimes A^1, b) \\
a \otimes f &\quad\longmapsto\quad a \otimes (\eta \cap f)
\end{align*}$$  

(20)

it is direct to check that this is an isomorphism of chain complexes and induces an isomorphism on the homology

$$H^*(A \otimes A^1, b) \cong H_{n-\bullet}(A \otimes A^1, b).$$

Thus by Proposition 3.3 we have an isomorphism

$$\text{HH}^*(A) \cong \text{HH}_{n-\bullet}(A).$$  

(21)

We next need to check that the volume class represented above is identical to the one of Van den Bergh ([38]).
First, in [16] de Thanhoffer de Volcsey and Van den Bergh showed that the noncommutative Poincaré duality for Calabi-Yau algebras is given by a class $\eta \in \text{HH}_n(A)$ such that

$$\text{HH}^\bullet(A) \longrightarrow \text{HH}_{n-\bullet}(A) \quad f \mapsto \eta \cap f$$

is an isomorphism, where the “cap product” $\cap$ is given on the Hochschild chain level as follows:

$$\text{CH}_m(A) \times \text{CH}^n(A) \longrightarrow \text{CH}_{m-n}(A) \quad (\alpha, f) \mapsto \begin{cases} (\alpha_{\partial}(a_1, \ldots, a_n), a_{n+1}, \ldots, a_m), & \text{if } m \geq n \\ 0, & \text{otherwise}, \end{cases}$$

where $\text{CH}_*(A)$ and $\text{CH}^*(A)$ are the reduced Hochschild chain and cochain complexes respectively (c.f. Loday [25] for these notions).

Second, on the Koszul complexes we have an analogous cap product given by the following (note that (20) is just a special case)

$$(A \otimes A^i) \times (A \otimes A^j) \longrightarrow A \otimes A^i$$

$$(a \otimes u, b \otimes f) \mapsto ab \otimes (uf).$$

It has been shown by Berger et. al. [5] that for Koszul algebras, these two versions of cap product on the homology level, via the isomorphism given in Proposition 3.3 are the same. Thus the isomorphism (21) is a version of noncommutative Poincaré duality in the sense of Van den Bergh.

Third, by Van den Bergh [41] the volume class of the noncommutative Poincaré duality, if it exists, is unique up to an inner automorphism of $A$.

Thus by the above three arguments, the noncommutative Poincaré duality of (21) is identical to the one of Van den Bergh given in [38], possibly up to an inner automorphism of $A$. This completes the proof. \qed

Now consider the coproduct

$$\Delta(\eta) = \sum \eta^i_1 \otimes \eta^i_2 \in A^i \otimes A^i.$$ 

Observe that we have an embedding

$$A^i \cong k \otimes A^i \otimes k \subset \tilde{R} \otimes A^i \otimes \tilde{R} = \Omega^1_{\text{nc}} \tilde{R}[-1], \quad a \mapsto d([a]).$$

Via this embedding, $\Delta(\eta)$ corresponds to an element

$$\omega := \sum d([\eta^i_1]) \otimes d([\eta^i_2]) \in \Omega^2_{\text{nc}} \tilde{R}. $$

**Lemma 4.5.** $\omega$ descends to a $\partial$- and $d$-closed cycle of degree $n$ in $\text{DR}^2_{\text{nc}} \tilde{R}$.

**Proof.** Denote by $[\omega]$ the image of $\omega$ in $\text{DR}^2_{\text{nc}} \tilde{R}$. We show $[\omega]$ is closed with respect to both $d$ and $\partial$. First, $[\omega]$ is automatically $d$-closed. Second, applying $\partial$ to $\omega$, we have

$$\partial(\omega) = \sum \partial \circ d([\eta^i_1]) \otimes d([\eta^i_2]) + (-1)^{|\eta^i_1|} d([\eta^i_1]) \otimes \partial \circ d([\eta^i_2])$$

$$= -\sum d \circ \partial([\eta^i_1]) \otimes d([\eta^i_2]) + (-1)^{|\eta^i_1|} d([\eta^i_1]) \otimes d \circ \partial([\eta^i_2])$$

$$= -\sum (-1)^{|\eta^i_1|} d([\eta^i_{11}] | [\eta^i_{22}] ) \otimes d([\eta^i_2]) + (-1)^{|\eta^i_1|+|\eta^i_2|} d([\eta^i_2]) \otimes d([\eta^i_{21}] | [\eta^i_{11}]).$$
\[ = - \sum (-1)^{|\eta_1|} d(\eta_1^{11}) \cdot [\eta_1^{12}] \otimes d(\eta_1^2) - [\eta_1^{11}] \cdot d([\eta_1^{12}]) \otimes d([\eta_1^2]), \]

where we write \(\Delta(\eta_1) = \sum \eta_1^{11} \otimes \eta_1^{12}\) and \(\Delta(\eta_1^2) = \sum \eta_1^{21} \otimes \eta_1^{22}\). In the last equality, after descending to \(\text{DR}^1_{\mathcal{R}}\), the first and the last summands cancel with each other due to the co-associativity of \(A^1\), while the second and the third summands cancel with each other due to the cyclic condition of the pairing on \(A^1\), which is equivalent to (19). This proves the statement.

\[ \square \]

**Theorem 4.6.** Let \(A\) be a Koszul Calabi-Yau algebra of dimension \(n\). Let \(\tilde{R} = \Omega(\tilde{A})\). Then \(\tilde{R}\) has a \((2-n)\)-shifted bi-symplectic structure.

**Proof.** Since \(\Omega^1_{\mathcal{R}}\tilde{R}[-1] = \tilde{R} \otimes A^1 \otimes \tilde{R}\), we have

\[ \text{Der} \tilde{R}[1] = \text{Hom}_{\tilde{R}^e}(\Omega^1_{\mathcal{R}}\tilde{R}[-1], \tilde{R} \otimes \tilde{R}) = \text{Hom}_{\tilde{R}^e}(\tilde{R} \otimes A^1 \otimes \tilde{R}, \tilde{R} \otimes \tilde{R}) = \text{Hom}(A^1, \tilde{R} \otimes \tilde{R}) = \text{Hom}(A^1, k) \otimes (\tilde{R} \otimes \tilde{R}). \]

For any \(f \otimes r_1 \otimes r_2 \in \text{Hom}(A^1, k) \otimes (\tilde{R} \otimes \tilde{R})\), by (2) its reduced contraction with \(\omega\) is

\[ 2 \cdot \sum f(\eta_1) \cdot r_2 \otimes d\eta_1^2 \otimes r_1 \in \Omega^1_{\mathcal{R}}\tilde{R}[-1]. \]

In other words, the reduced contraction is essentially given by \(A^1 \xrightarrow{\eta \circ (-)} A^1[n]\). Since \(\eta\) and thus \(\omega\) are non-degenerate of total degree \(n\), we thus have

\[ \text{Der} \tilde{R}[1] \cong (\Omega^1_{\mathcal{R}}\tilde{R}[-1])[2-n] \]

as \(\tilde{R}\)-bimodules. This proves the theorem.

\[ \square \]

By taking the commutator quotient space of both sides of the above (26) and Propositions 3.5 and 3.7, we once again obtain the noncommutative Poincaré duality:

\[ \text{HH}^*(A) \cong \text{HH}_{n-\bullet}(A), \]

which coincides with Van den Bergh’s one. In general, suppose \(R\) has a (shifted) bi-symplectic structure \(\omega\), then Crawley-Boevey et. al. showed in [15] Lemma 2.8.6] that there is a commutative diagram

\[ \begin{array}{ccc} 
\text{Der} R & \xrightarrow{\iota} & (\text{Der} R)_{\#} \\
\iota(-)\omega & \cong & (\iota(-)\omega)_{\#} \\
\Omega^1_{\mathcal{R}}R & \xrightarrow{\iota} & \text{DR}^1_{\mathcal{R}}R. 
\end{array} \]

5. **Representation schemes and the shifted symplectic structure**

In this section, we briefly go over the relationship between the shifted bi-symplectic structure of a DG algebra and the shifted symplectic structure on its DG representations.
5.1. **Representation functors.** Let $\text{DGA}$ be the category of associative, unital DG $k$-algebras, and $\text{CDGA}$ its subcategory of DG commutative $k$-algebras. Fix a finite dimensional vector space $V$, and consider the following functor:

$$\text{Rep}_V(A) : \text{CDGA} \to \text{Sets}, \ B \mapsto \text{Hom}_{\text{DGA}}(A, B \otimes \text{End } V).$$

The following result generalizes the result of Bergman [6] and Cohn [12] for associative algebras:

**Theorem 5.1 (3 Theorem 2.2).** The functor $\text{Rep}_V(A)$ is representable, that is, there exists an object in $\text{CDGA}$, say $A_V$, depending only on $A$ and $V$, such that

$$\text{Hom}_{\text{DGA}}(A, B \otimes \text{End } V) = \text{Hom}_{\text{CDGA}}(A_V, B).$$

More precisely, in [3] the authors considered the following two functors:

$$\sqrt{\cdot} : \text{DGA} \to \text{DGA}, \ A \mapsto (A \ast_k \text{End } V)^{\text{End } V},$$

and

$$(\cdot)_V : \text{DGA} \to \text{CDGA}, \ A \mapsto A/\langle[A, A] \rangle,$$

where $\langle[A, A] \rangle$ is the ideal of $A$ generated by the commutators. Then the functor

$$(\cdot)_V : \text{DGA} \to \text{CDGA}, \ A \mapsto A_V$$

is given by the composition of the above two functors, namely, $A_V = (\sqrt{A})_{2\zeta}$. In what follows we sometimes also write $A_V$ as $\text{Rep}_V(A)$.

In (27), if we take $B = A_V$, then we have

$$\text{Hom}(A, A_V \otimes \text{End } V) = \text{Hom}(A_V, A_V).$$

The identity map on the right hand side corresponds to a map

$$\pi_V : A \to A_V \otimes \text{End } V$$

on the left hand side, which is usually called the **universal representation map**.

**Example 5.2 (Rep for quasi-free algebras).** For a quasi-free algebra $(R, \partial)$ and $V = k^n$, the DG commutative algebra $R_V$ can be described explicitly: Let $\{x^\alpha\}_{\alpha \in I}$ be a set of generators of $R$. Consider a free graded algebra $R'$ on generators $\{x^\alpha_{ij} : 1 \leq i, j \leq n, \alpha \in I\}$, where $|x^\alpha_{ij}| = |x^\alpha|$ for all $i, j$. Form matrices $X^\alpha := (x^\alpha_{ij})$, and define the algebra map

$$\pi : R \to M_n(R'), \ x^\alpha \mapsto X^\alpha,$$

where $M_n(R')$ is the algebra of $n \times n$-matrices with entries in $R'$. Let

$$\partial(x^\alpha_{ij}) := (\partial x^\alpha)_{ij},$$

and extend it to $R'$ by linearity and the Leibniz rule. We thus obtain a DG algebra $(R', \partial)$, and $R_V$ is $R'_V$ with the differential induced from $\partial$ (see [3 Theorem 2.8] for a proof).
5.2. GL-invariants and the trace map. Observe that GL(V) acts on Rep_V(A) by conjugation. More precisely, then any \( g \in \text{GL}(V) \) gives a unique automorphism of \( A_V \) which, under the identity (27), corresponds to the composition

\[
A \xrightarrow{\pi_V} \text{End}(A_V) \xrightarrow{\text{Ad}(g)} \text{End}(A_V).
\]

This action is natural in \( A \), and hence defines a functor

\[
\text{Rep}_V(-)^{\text{GL}} : \text{DGA} \rightarrow \text{CDGA}, \ A \mapsto (A_V)^{\text{GL}},
\]

where \((-)^{\text{GL}}\) means the GL(V)-invariants.

Now consider the following composite map

\[
A \xrightarrow{\pi_V} \text{End}(A_V) \xrightarrow{\text{Tr}} A_V,
\]

which is GL(V)-invariant and factors through \( A_2 \), we get a map

\[
\text{Tr} : A_2 \rightarrow (A_V)^{\text{GL}}.
\]

If \( A \) is an associative algebra (i.e., a DG algebra concentrated in degree 0), then the famous result of Procesi says that the image of \( \text{Tr} \) generates \( \text{Rep}_V(A)^{\text{GL}} \); in other words, if we extend \( \text{Tr} \) to be a commutative algebra map

\[
\text{Tr} : \Lambda^\bullet A_2 \rightarrow (A_V)^{\text{GL}},
\]

then it is surjective. However, for an arbitrary DG algebra, Berest and Ramadoss showed in [4] that this is in general not true on the homology level; there are some homological obstructions for \( \text{Tr} \) to be so.

5.3. Van den Bergh’s functor. Let \((R, \partial)\) be a DG algebra. Suppose \((M, \partial_M)\) is a DG \( R \)-bimodule. Let \( \pi : R \rightarrow R_V \otimes \text{End} V \) be the universal representation of \( R \), which means the map on the left hand side of (27) that corresponds to the identity map on the right hand side. Then \( \pi \) gives an \( R \)-bimodule structure on \( R_V \otimes \text{End} V \). Denote

\[
M_V := M \otimes_{R^e} (R_V \otimes \text{End} V),
\]

which is now a DG \( R_V \)-module. More specifically, let \( V = k^n \), then \( M_V \) is generated by symbols \( m_{ij} \), \( 1 \leq i, j \leq n \), for each \( m \in M \), with the action of \( R_V \) given by

\[
(r \cdot m)_{ij} = \sum_k r_{ik} \cdot m_{kj}, \quad (m \cdot r)_{ij} = \sum_k r_{kj} \cdot m_{ik},
\]

and with the differential, denoted by \( \partial_{M_V} \), given by

\[
\partial_{M_V}(m_{ij}) = (\partial_M m)_{ij}, \quad \text{for all } m \in M.
\]

The assignment from the category of DG \( R \)-bimodules to the category of DG \( R_V \)-modules

\[
\text{DGBimod} \ R \rightarrow \text{DGMmod} \ R_V, \ M \mapsto M_V
\]

is a well-defined functor, and is first introduced by Van den Bergh in [40]. Next, we apply Van den Berg’s idea to the case of noncommutative differential forms and poly-vectors.

Let \((R, \partial)\) be a DG commutative algebra over \( k \). Let

\[
I := \ker(R \otimes R \xrightarrow{\mu} R)
\]
be the kernel of the multiplication map and let $\Omega^1_{\text{com}} R := I/I^2$, which is the set of Kähler differentials of $R$. Let

$$\Omega^p_{\text{com}} R = \Lambda^p_R(\Omega^1_{\text{com}} R[-1]).$$

Similarly to the DG algebra case, we have the degree 1 de Rham differential

$$d : \Omega^\bullet_{\text{com}} R \to \Omega^{\bullet+1}_{\text{com}} R,$$

which makes $(\Omega^\bullet_{\text{com}} R, d)$ into a DG cochain algebra. The differential $\partial$ on $R$ also gives a degree $-1$ differential on $\Omega^\bullet_{\text{com}} R$, which also respects the product and commutes with $d$.

The dual space of the cotangent space $\text{Hom}_R(\Omega^1_{\text{com}} R, R)$ is called the complex of vector fields of $R$, and is identified with $\text{Der}_R$.

**Proposition 5.3** ([40] Proposition 3.3.4). Suppose $R$ is a DG algebra. Then

1. $(\Omega^1_{\text{nc}} R)_V = \Omega^1_{\text{com}} (R_V)$;
2. $(\mathcal{D}\text{er} R)_V = \text{Der}_V R$.

From this proposition, we immediately have:

**Proposition 5.4** ([40] Corollary 3.3.5). Suppose $R$ is a quasi-free DG algebra. Then

$$\left(T_R(\Omega^1_{\text{nc}} R[-1])\right)_V = \Omega^\bullet_{\text{com}} (R_V) \quad \text{and} \quad \left(T_R(\mathcal{D}\text{er} R[1])\right)_V = \Lambda^\bullet(\text{Der}_V R_V[1]).$$

(33)

Now applying (31) to (33), we have the trace maps

$$\left(T_R(\Omega^1_{\text{nc}} R[-1])\right)_z = \text{DR}^\bullet_{\text{nc}} R \to \Omega^\bullet_{\text{com}} (R_V)^{\text{GL}}$$

and

$$\left(T_R(\mathcal{D}\text{er} R[1])\right)_z \to \Lambda^\bullet(\text{Der}_V R_V[1])^{\text{GL}}.$$

When restricting to the first component, we have the trace map

$$\text{DR}^\bullet_{\text{nc}} R \to \Omega^1_{\text{com}} (R_V)^{\text{GL}} \quad \text{and} \quad (\text{Der} R[1])_z \to (\text{Der}_V R_V[1])^{\text{GL}}.$$

(35)

**5.4. The shifted symplectic structure.** The notion of shifted symplectic structure is introduced by Pantev-Toën-Vaqué-Vezzosi in [29]; see also [8, 28, 31] for some further studies. In the following we only consider the affine case, which is enough for our purpose.

Suppose $(R, \partial)$ is a DG algebra. For any closed form $\omega \in \Omega^2_{\text{com}} R$ of total degree $2 - n$, the contraction map

$$\iota(-) \omega : \text{Der} R[1] \to (\Omega^1_{\text{com}} R[-1])[2 - n], \alpha \mapsto \omega(\alpha, -)[2 - n]$$

is a map of $\partial$-complexes. The following is a slightly stronger version of the shifted symplectic structure introduced in [29].

**Definition 5.5** (Shifted symplectic structure). Suppose $(R, \partial)$ is a DG commutative algebra over $k$. An $n$-shifted symplectic structure on $R$ is a 2-form $\omega \in \Omega^2_{\text{com}} R$ of total degree $2 - n$, closed under $\partial$ and $d$, such that the contraction

$$\iota(-) \omega : \text{Der} R[1] \to (\Omega^1_{\text{com}} R[-1])[2 - n],$$

(36)

is a quasi-isomorphism.
Remark 5.6. The original definition of shifted symplectic structure in [29] requires $\omega$ to be $\partial$-closed, which can be extended to be a closed form in the negative cyclic complex associated to the mixed complex $\Omega^\bullet_{\text{com}}(R)$.

In both [41] and [30] we have shifted the degrees on the right hand side of the equations, namely on the (noncommutative) differential 1-forms, up by $n-2$, which looks different from [29]. However, they are the same in the following sense: [41] and [30] can be alternatively written as

$$\text{Der} R \to \Omega^1_{\text{nc}} R[-n] \quad \text{and} \quad \text{Der} R \to \Omega^1_{\text{com}} R[-n]$$

respectively, which coincides with [29]. Such a degree shifting guarantees that an $n$-shifted symplectic structure gives an $n$-shifted Poisson structure, whose Poisson bracket has degree $n$.

Theorem 5.7 ([15]; see also [40] §2.4). Suppose $R$ is a DG algebra which admits an $n$-shifted bi-symplectic structure. Then $R_V$ has an $n$-shifted symplectic structure.

Proof. Follows from Proposition 5.3 and the functoriality of Van den Bergh’s functor. More precisely, applying Van den Bergh’s functor to

$$\text{Der}(R_V)[1] \overset{\iota(-)\omega}{\cong} (\Omega^1_{\text{com}}(R_V)[-1])[n]$$

and then using Proposition 5.3, we obtain

$$\text{Der}((R_V)\text{GL})[1] \overset{\cong}{\cong} (\Omega^1_{\text{com}}(R_V)^\text{GL}[-1])[n],$$

where the isomorphism, according to [15] Theorem 6.4.3, is given by $\iota(-)\text{Tr}(\omega)$.

Combining the above theorem with Theorem 4.6, we immediately have the following:

Corollary 5.8. Suppose $A$ is a Koszul Calabi-Yau algebra of dimension $n$, and let $\tilde{R} = \Omega(\tilde{A})$ as before. Then $\tilde{R}_V$ has an $(2-n)$-shifted symplectic structure.

5.5. Identification of $\text{GL}$-invariant 1-forms and vectors. In this subsection, we show that $\text{Rep}_V(\tilde{R})^{\text{GL}}$ is “symplectic”. What we mean is the following:

Theorem 5.9. Let $A$ be a Koszul Calabi-Yau algebra of dimension $n$. Then

$$\text{Der}(\tilde{R}_V)^{\text{GL}}[1] \overset{\cong}{\cong} (\Omega^1_{\text{com}}(\tilde{R}_V)^{\text{GL}}[-1])[n].$$

Proof. Observe that in (37), $\text{Tr}(\omega)$ in fact lies in $\Omega^2_{\text{com}}(\tilde{R}_V)^{\text{GL}}$ (see (34)). Thus by taking the $\text{GL}(V)$-invariant vector fields and 1-forms, we get the desired isomorphism.

Combining this proposition with Propositions 3.5 and 3.7, we obtain that the trace map (35) gives the following commutative diagram of chain complexes

$$\begin{array}{ccc}
\text{CH}^*(A) & \xrightarrow{\text{Tr}} & \text{Der}(\tilde{R}_V)^{\text{GL}}[1] \\
\cong & & \cong \\
\text{CH}_{n-*}(A) & \xrightarrow{\text{Tr}} & (\Omega^1_{\text{com}}(\tilde{R}_V)^{\text{GL}}[-1])[n].
\end{array}$$
Taking the (co)homology on both sides, we get the commutative diagram stated in §1. As we remarked before, the vertical maps are both isomorphisms, while the two horizontal maps are neither surjective nor injective in general (for see [1] more details).

6. The shifted double Poisson structure

Shifted bi-symplectic structures are intimately related to shifted double Poisson structures. Let us remind the work [39] of Van den Bergh (here we rephrase it in the DG case; see also [1]).

**Definition 6.1** (Double bracket). Suppose $R$ is a DG algebra over $k$. A *double bracket* of degree $n$ on $R$ is a DG map $\{\{\cdot, \cdot\}\}$ of degree $n$ which is a derivation in its second argument and satisfies

$$\{\{a, b\}\} = -(-1)^{|a|+|b|+n} \{\{b, a\}\},$$

where $(u \otimes v)^\circ = (-1)^{|u||v|} v \otimes u$.

**Definition 6.2** (Double Poisson structure). Suppose that $\{\{\cdot, \cdot\}\}$ is a double bracket of degree $n$ on $R$. For $a, b_1, \ldots, b_n \in R$, let

$$\{\{a, b_1 \otimes \cdots \otimes b_n\}\} := \{\{a, b_1\}\} \otimes b_2 \otimes \cdots \otimes b_n,$$

and for $s$ is a permutation of $\{1, 2, \ldots, n\}$, let

$$\sigma_s(b_1 \otimes \cdots \otimes b_n) := (-1)^{\sigma(s)} b_{s^{-1}(1)} \otimes \cdots \otimes b_{s^{-1}(n)},$$

where

$$\sigma(s) = \sum_{i < j; s^{-1}(j) < s^{-1}(i)} |a_{s^{-1}(i)}||a_{s^{-1}(j)}|.$$

If furthermore $R$ satisfies the following *double Jacobi identity*

$$\{\{a, \{b, c\}\}\} = (-1)^{(|a|+|b|+|c|)} \sigma_{(132)} \{\{\{b, c\}, a\}\} - (-1)^{(|b|+|c|+|a|+|b|)} \sigma_{(123)} \{\{\{a, b\}, c\}\} = 0,$$

then $R$ is called a *double Poisson algebra* of degree $n$ (or $n$-shifted double Poisson algebra).

Van den Bergh [39] showed that, if $R$ is equipped with a double Poisson structure, then there is a Poisson structure on the affine scheme $\text{Rep}_V(R)$ of all the representations of $R$ in $V$. Independently and simultaneously, Crawley-Boevey gave in [14] the *weakest* condition for $\text{Rep}_V(R)\!/\!\text{GL}(V)$ of $R$ to have a Poisson structure. He called such condition the $H_0$-Poisson structure, since it involves the zeroth Hochschild/cyclic homology of $R$. It turns out Van den Bergh’s double Poisson structure satisfies this condition, and is so far the most interesting example therein.
6.1. From shifted bi-symplectic to shifted double Poisson. In [39] Appendix] Van den Bergh showed that a bi-symplectic structure gives a double Poisson structure. We rephrase it in the Koszul Calabi-Yau case. Suppose $A$ is a Koszul $n$-Calabi-Yau algebra, and let $\tilde{R} = \Omega(A)$. For any $r \in \tilde{R}$, since
\[
\iota_r : \text{Der } \tilde{R}[1] \to (\Omega_{nc}[1])[n]
\]
is an isomorphism of chain complexes, there exists an element $H_r \in \text{Der } \tilde{R}$ such that
\[
\iota_r \omega = d(r).
\]
$H_r$ is called the bi-Hamiltonian vector field associated to $r$. Now consider the following bracket
\[
\ll [\cdot, \cdot] : \tilde{R} \times \tilde{R} \to \tilde{R} \otimes \tilde{R}, (r_1, r_2) \mapsto H_{r_1}(r_2),
\]
then we have:

**Proposition 6.3.** $\ll [\cdot, \cdot]$ gives a $(2 - n)$-shifted double Poisson structure on $\tilde{R}$.

**Proof.** By Van den Bergh [39 Lemma A.3.3], the $(2 - n)$-shifted double Poisson bracket on $\tilde{R} = \Omega(A)$ is given by the following formula:
\[
\ll [x, y] := \sum_{i=1}^{k} \sum_{j=1}^{\ell} (-1)^{\sigma_{ij}} (x_i, y_j) \cdot (y_1 \cdots y_{j-1} x_{i+1} \cdots x_k) \otimes (x_1 \cdots x_{i-1} y_j \cdots y_\ell),
\]
for $x = (x_1 \cdots x_k)$ and $y = (y_1 \cdots y_\ell)$ in $\tilde{R}$, where $(-1)^{\sigma_{ij}}$ is the Koszul sign. Here $(x_i, y_j) \in k$ for $x_i, y_j \in A[i][1]$ is the graded skew-symmetric pairing induced from the pairing on $A[i]$; more precisely, $(x_i, y_j) = (-1)^{|x_i||x_j|}(\sum x_i, \sum y_j)$.

Formula (41) also appeared in [1 Theorem 15] (see also [9 Lemma 4.4]) for $\tilde{R} = \Omega(A)$, where we also showed that $\ll [\cdot, \cdot]$ given above commutes with the differential on $R$. The only difference between $\tilde{R}$ and $R$ is that the generators of $\tilde{R}$ contain one more element, namely the desuspension of the co-unit of $A$. The sufficient condition for $\ll [\cdot, \cdot]$ commuting with the differentials on $R$ and on $\tilde{R}$ is the cyclic condition (19) for $A$. Thus the proof of [1 Theorem 15] applies to the above proposition, too. \qed

Suppose $R$ has a double Poisson structure, then Van den Bergh gave an explicit formula for the Poisson structure on $\text{Rep}_V(R)$ (see [39 Propositions 7.5.1 and 7.5.2]). For a Koszul Calabi-Yau algebra $A$, the Poisson structure on $\text{Rep}_n(\tilde{R})$ is given as follows: suppose $A$ has a set of basis $\{x^\alpha\}_{\alpha \in I}$, and $\Delta(x^\alpha) = \sum x^\alpha_1 \otimes x^\alpha_2$. Then $\text{Rep}_n(\tilde{R})$ is the quasi-free DG commutative algebra generated by
\[
\{x^\alpha_{ij} | \alpha \in I, 1 \leq i, j \leq n, |x^\alpha_{ij}| = |x^\alpha| - 1\},
\]
with
\[
d(x^\alpha_{ij}) = \sum_{\alpha} (-1)^{|x^\alpha_{ij}|} \sum_{k=1}^{n} x^{\alpha_1}_{ik} \cdot x^{\alpha_2}_{kj}.
\]
The $(2 - n)$-Poisson bracket is given by
\[
\{x^\alpha_{ij}, x^\beta_{kl}\} = (-1)^{|x^\alpha|} \delta_{ij \delta_{kl}} (x^\alpha, x^\beta)
\]
on the generators, which extends to the whole $\text{Rep}_n(\tilde{R})$ by the Leibniz rule.
6.2. The work of Crawley-Boevey. In \cite{14} Crawley-Boevey introduced what he called the $H_0$-Poisson structure. Let us recall its definition:

**Definition 6.4** ($H_0$-Poisson structure). Suppose $R$ is an associative algebra. An $H_0$-Poisson structure on $R$ is a Lie bracket $\{ -, - \} : R^\# \times R^\# \to R^\#$ such that the adjoint action $\text{ad}_{\bar{u}} : R^\# \to R^\#$, $\bar{v} \mapsto \{ \bar{u}, \bar{v} \}$ can be lifted to be a derivation $d_u : R \to R$, for all $u \in R$.

Crawley-Boevey proved that if $R$ admits an $H_0$-Poisson structure, then there is a unique Poisson structure on $\text{Rep}_V(R)^{\text{GL}}$ such that the trace map

$$\text{Tr} : R^\# \to \text{Rep}_V(R)^{\text{GL}}$$

is a map of Lie algebras, or in other words, $\text{Tr} : \Lambda^\bullet R^\# \to \text{Rep}_V(R)^{\text{GL}}$ is a map of Poisson algebras. Here the Poisson bracket on $\Lambda^\bullet R^\#$ is the extension of the bracket on $R^\#$ by derivation.

Now suppose $R$ has a double Poisson bracket $\{ \{ - , - \} \}$. Then $\{ -, - \} : R \times R \to R$, $(u, v) \mapsto \mu \circ \{ u, v \}$ descends to a well-defined Lie bracket (see \cite{39}, Lemma 2.4.1)

$$\{ -, - \} : R^\# \times R^\# \to R^\#,$$

which is an $H_0$-Poisson structure. The restriction of the Poisson structure on $\text{Rep}_V(R)$ gives the one on $\text{Rep}_V(R)^{\text{GL}}$.

This result was later generalized to the DG setting in \cite{1}. For the Koszul Calabi-Yau algebra case, we have the following.

**Corollary 6.5.** Suppose that $A$ is a Koszul Calabi-Yau algebra, and let $\tilde{R} = \Omega(\tilde{A}^!)$ be as above. The trace map

$$\text{Tr} : \tilde{R}^\# \to \text{Rep}_V(\tilde{R})^{\text{GL}}$$

is a map of degree $(2 - n)$ DG Lie algebras. As a consequence,

$$\text{Tr} : \Lambda^\bullet \tilde{R}^\# \to \text{Rep}_V(\tilde{R})^{\text{GL}}$$

is a map of $(2 - n)$-shifted Poisson algebras.

**Proof.** See \cite{1}, Corollary 3. Again we emphasize that in \cite{1} we proved the statement for $R = \Omega(A^!)$, but the same proof applies to the current case. \hfill $\Box$

Alternatively, one can prove the above corollary by considering the $\text{GL}(V)$-invariant elements in $\text{Rep}_V(\tilde{R})$ (see \cite{1} Theorem 3.1) and then applying \cite{43} to them directly. We leave it to the interested readers.
6.3. Relations with quivers and quiver representations. The shifted bi-symplectic and double Poisson structures on $\tilde{R}$ generalize the ones of quivers given in [15, 39].

Let $Q$ be a quiver. For simplicity let us assume $Q$ has only one vertex. Let $\tilde{Q}$ be the double of $Q$. Then the path algebra $k\tilde{Q}$ is an associative algebra over $k$; viewing $\tilde{Q}$ as a 1-dimensional CW complex, then $k\tilde{Q}$ is exactly the cobar construction of the coalgebra of the chain complex of $\tilde{Q}$.

Denote the set of the edges of $Q$ by $\{e_i\}$ and their duals by $\{e_i^*\}$, then there is a graded symmetric pairing on $\{e_i\} \cup \{e_i^*\}$ given by

$$\langle e_i, e_j \rangle = \langle e_i^*, e_j^* \rangle = 0, \quad \langle e_i, e_j^* \rangle = -\langle e_j^*, e_i \rangle = \delta_{ij}$$

Such pairing is non-degenerate and cyclically invariant, and therefore by results in previous sections, the cobar construction of $\tilde{Q}$, that is, $k\tilde{Q}$, has a 0-shifted bi-symplectic and double Poisson structure. The bi-symplectic and double Poisson structures are exactly the ones obtained in [15, 39]. The corresponding symplectic structure on $\text{Rep}_V(k\tilde{Q})$ as well as the Lie bracket on $k\tilde{Q}_2$ was also previously studied by Ginzburg in [17]; see also Bocklandt-Le Bruyn [7], where the Lie algebra on $k\tilde{Q}_2$ is called the necklace Lie algebra.

Now assign the gradings of the edges $\tilde{Q}$ other than one and obtain a DG coalgebra, say $C$. To obtain the shifted bi-symplectic and double Poisson structures on $\Omega(C)$ (respectively $\Omega(\tilde{C})$, where $\tilde{C}$ is the co-unitalization of $C$), then a sufficient condition is that $\tilde{C} = C \setminus k$ (respectively $C$) is cyclic, in other words, the dual space $\text{Hom}(\tilde{C}, k)$ (respectively $\text{Hom}(C, k)$) is a cyclic associative and not necessarily unital algebra.

7. Quantization

In this section, we study the quantization problem. In [33], Schedler proved that the necklace Lie algebra of a doubled quiver is in fact an involutive Lie bialgebra, and constructed a Hopf algebra which quantizes this Lie bialgebra. He also showed that the Hopf algebra is mapped to the Moyal-Weyl quantization of the quiver representation spaces as associative algebras.

Later in [19], he together with Ginzburg constructed a Moyal-Weyl type quantization of the necklace Lie bialgebra, and showed that such quantization is isomorphic to the Hopf algebra constructed in [33].

The purpose of this section is to generalize their results to the Koszul Calabi-Yau case. The main result is the commutative diagram (56). Some partial results have been previously obtained in [10].

7.1. Quantization of $\text{Rep}_n(\tilde{R})$. In the bivector form, the shifted Poisson structure on $\text{Rep}_n(\tilde{R})$ is given by

$$\pi = \sum_{\alpha, \beta \in I} \sum_{i,j} (-1)^{n-|x^\alpha|} \langle x^{\alpha}, x^{\beta} \rangle \frac{\partial}{\partial x^\alpha_{ij}} \wedge \frac{\partial}{\partial x^\beta_{ji}}.$$  \hspace{1cm} (47)

Observe that $\pi$ is of constant coefficients, and thus we have the Moyal-Weyl quantization of $\text{Rep}_n(\tilde{R})$ which is given by

$$f \ast g := \mu \circ e^{\frac{\hbar}{2}\pi} (f \otimes g) \in \text{Rep}_n(\tilde{R})[\hbar], \quad \text{for any } f, g \in \text{Rep}_n(\tilde{R}),$$  \hspace{1cm} (48)
where “µ” is the original multiplication on Rep_n(\(\tilde{R}\)), and \(\hbar\) is a formal parameter of degree \(n - 2\).

**Proposition 7.1.** (Rep_n(\(\tilde{R}\))[\(\hbar\], ∗)) is a DG associative algebra over \(k[\hbar]\), which quantizes Rep_n(\(\tilde{R}\)).

**Proof.** We only need to show that the differential \(d\) commutes with ∗, or equivalently, \(\partial\) commutes with \(\mu \circ \pi^r(\cdot, \cdot)\), for all \(r \in \mathbb{N}\).

In fact, since \(\pi^r(f, g)\) is of \(r\)-th order for both arguments, we only need to check the case when \(f\) and \(g\) are degree \(r\) monomials. In this case, \(\mu \circ \pi^r(f, g)\) is a number, whose differential is zero, and hence we need to check

\[
\mu \circ \pi^r(\partial f, g) + \mu \circ \pi^r(f, \partial g) = 0.
\]

(49)

Suppose \(f = x_{i_1j_1}^{a_1} \cdots x_{i_rj_r}^{a_r}\), \(g = y_{k_1\ell_1}^{\beta_1} \cdots y_{k_r\ell_r}^{\beta_r}\). Then up to sign, \(\mu \circ \pi^r(\partial f, g)\) and \(\mu \circ \pi^r(f, \partial g)\) both contain a common scalar factor which is obtained by applying \(\pi^{r-1}\) to \(x_{i_1j_1}^{a_1} \cdots x_{i_rj_r}^{a_r}\) and \(y_{k_1\ell_1}^{\beta_1} \cdots y_{k_r\ell_r}^{\beta_r}\), where \(\widetilde{\cdot}\) means the corresponding component is omitted. The rest factors are just \(\pi(d(x_{i_rj_r}^{a_r}), y_{k_r\ell_r}^{\beta_r})\) and \(\pi(x_{i_rj_r}^{a_r}, d(y_{k_r\ell_r}^{\beta_r}))\) respectively. This means, to prove (49) it is sufficient to show \(\pi(f, g)\) commutes with the boundary, which is already done. This proves the statement. \(\square\)

### 7.2. Quantization of \(\tilde{R}_2\)

In this subsection we study the quantization of the Lie bialgebra on \(\tilde{R}_2\). Let us start with several definitions.

**Definition 7.2 (Lie bialgebra).** Suppose that \((L, \{-, -\})\) is a graded Lie algebra with the bracket \(\{-, -\}\) having degree \(m\) and \((L, \delta)\) is a graded Lie coalgebra with the cobracket \(\delta\) having degree \(n\). The triple \((L, \{-, -\}, \delta)\) is called a Lie bialgebra of degree \((m, n)\) if the following Drinfeld compatibility (also called the cocycle condition) holds: for all \(a, b \in L\),

\[
\delta(\{a, b\}) = (ad_a \otimes id + id \otimes ad_a)\delta(b) + (ad_b \otimes id + id \otimes ad_b)\delta(a),
\]

(50)

where \(ad_a(b) = \{a, b\}\) is the adjoint action. If furthermore, \(\{-, -\} \circ \delta(g) \equiv 0\) for any \(g \in L\), then the Lie bialgebra is called involutive.

In the above definition, if \(L\) is equipped with a differential which commutes with both the Lie bracket and Lie cobracket, then it is called a DG Lie bialgebra.

Now let \(A\) be a Koszul Calabi-Yau algebra. Recall that the DG Lie bracket on \(\tilde{R}_2\) is given by the following formula (c.f. (41) and (44)):

\[
\{x, y\} := \sum_{i=1}^{k} \sum_{j=1}^{\ell} (-1)^{\alpha_j} (x_i, y_j) \cdot pr(y_{1} \cdots y_{j-1} x_{i+1} \cdots x_{k} x_{1} \cdots x_{i-1} y_{j+1} \cdots y_{\ell}),
\]

(51)

for \(x, y \in \tilde{R}_2\) represented by \((x_1 \cdots x_k)\) and \(y = (y_1 \cdots y_{\ell})\) in \(\tilde{R}\), where \(pr(-)\) means the projection of \(\tilde{R}\) to \(\tilde{R}_2\). Now define

\[
\delta : \tilde{R}_2 \rightarrow \tilde{R}_2 \otimes \tilde{R}_2
\]

by
\[
\delta(x_1 x_2 \cdots x_n) := \sum_{i,j: i < j} (-1)^{\sigma_{ij}} (x_i, x_j) \cdot \text{pr}(x_1 \cdots x_{i-1} x_{i+1} \cdots x_{j-1} x_j x_{j+1} \cdots x_n) \otimes \text{pr}(x_{i+1} \cdots x_{j-1} x_j).
\]

**Theorem 7.3** ([10] Theorem 1(i)). \((\tilde{R}, \{-,-\}, \delta)\) forms an involutive DG Lie bialgebra of degree \((2 - d, 2 - d)\).

This Theorem, together with the following Theorem 7.5, is proved in [10]. They are directly inspired by [33]; what is new there is that the Lie bracket and cobracket thus defined are compatible with the differential. As we remarked in the proof of Proposition 6.3, the cyclic condition (19) guarantees that all constructions respect it. We thus omit the proof and refer the interested reader to [10] for more details.

**Definition 7.4** (Quantization of Lie bialgebra). Suppose \((L, \{-,-\}, \delta)\) is a DG Lie bialgebra of degree \((m, n)\) over the field \(k\). A quantization of \(L\) is a Hopf algebra \((A, \ast, \Delta)\), flat over \(k[h, h]\), where \(h\) and \(\bar{h}\) are formal parameters of degree \(-m\) and \(-n\) respectively, together with a surjective map

\[
\Psi : A \rightarrow \Lambda^* L
\]

such that for all \(x, y \in A\),

\[
\Psi \left( \frac{x \ast y - y \ast x}{h} \right) = \{\Psi(x), \Psi(y)\}, \quad \Psi \left( \frac{\Delta(x) - \Delta^{\text{op}}(x)}{h} \right) = \delta(\Psi(x)),
\]

(52)

where \(\Delta^{\text{op}}(-)\) means the opposite coproduct, and \(\delta : \Lambda^* L \rightarrow \Lambda^* L \otimes \Lambda^* L\) is the cobracket induced by \(\delta : L \rightarrow L \otimes L\).

Let us remind that the cobracket on \(\Lambda^* L\) is defined as follows: for any Lie coalgebra \((L, \delta)\), its graded symmetric product \(\Lambda^* L\) admits a co-Poisson algebra structure which is induced by the Lie coalgebra structure \(\delta\) via the formula:

\[
\delta \circ \mu = (\mu \otimes \mu) \circ (1 \otimes \tau \otimes 1) \circ (\delta \otimes \Delta + \Delta \otimes \delta),
\]

where \(\mu\) is the product of \(\Lambda^* L\), \(\tau : a \otimes b \mapsto (-1)^{[a][b]} b \otimes a\) is the switching operator, and \(\Delta\) is the coproduct of \(\Lambda^* L\):

\[
\Delta(x^I) = \sum_{I_1 \cup I_2 = I} x^{I_1} \otimes x^{I_2}.
\]

In the rest of this subsection, we study the quantization of \(\tilde{R}\). Recall that \(R = T(A[1])\) and \(A^i\) is a cyclic coalgebra.

Now let \(L\Lambda^i = A^i \otimes k[\nu, \nu^{-1}]\), where \(\nu\) is a formal parameter of degree 0. An element \(a \otimes \nu^r\) in \(L\Lambda^i\) is denoted by \((a, r)\). Let \(T(L\Lambda^i[1])\) be the free tensor algebra of the desuspension of \(L\Lambda^i\), where its elements are written in the form \([[(a_1, r_1)] \cdots [(a_p, r_p)]\), for \((a_1, r_1), \cdots, (a_p, r_p) \in L\Lambda^i\). Let \(LH\) be the commutator quotient space of \(T(L\Lambda^i[1])\), namely \(LH = T(L\Lambda^i[1])_2\).

To avoid complicated notations, in the following we write elements in \(LH\) again in the form \([[(a_1, r_1)] \cdots [(a_p, r_p)]\); in other words we omit the symbol \(\text{pr}(-)\).
Let $SLH$ be the graded symmetric algebra generated by $LH$, whose product is denoted by $\bullet$. Define a differential $\partial$ on $SLH$ given by the following formula:

$$\partial([(a_{1,1}, r_{1,1})] \cdots [(a_{1,p_1}, r_{1,p_1})]) \cdots \bullet [(a_{s,1}, r_{s,1})] \cdots [(a_{s,p_s}, r_{s,p_s})])$$

$$:= \sum_{i=1}^s \sum_{j=1}^{p_i} \sum_{a_{i,j}} (-1)^{\sigma_{ij}} [(a_{1,1}, r_{1,1})] \cdots [(a_{i,j}, r_{i,j})] [(a_{i,j}, 1 + r_{i,j})] \cdots [(a_{s,p_s}, r_{s,p_s})] \bullet \cdots$$

$$\bullet [(a_{i,j}, \tilde{r}_{i,j})] \cdots [(a'_{i,j}, r_{i,j})] [(a''_{i,j}, 1 + r_{i,j})] \cdots [(a'_{s,p_s}, \tilde{r}_{i,j})] \bullet \cdots ,$$

(53)

where $(-1)^{\sigma_{ij}}$ is the Koszul sign, $a'_{i,j}$ and $a''_{i,j}$ come from $\Delta(a_{i,j}) = \sum (a_{i,j} a'_{i,j} \otimes a''_{i,j})$, and for $(i', j') \neq (i, j)$,

$$\tilde{r}_{i', j'} = \begin{cases} r_{i', j'}, & \text{if } r_{i', j'} \leq r_{i,j} \\ 1 + r_{i', j'}, & \text{if } r_{i', j'} > r_{i,j} \end{cases}$$

The coassociativity of $A^i$ implies that $b^2 = 0$.

Now let $h, h$ be formal parameters both of degree $n - 2$, and let $SLH[h, h]$ be the $k[h, h]$-module generated by $SLH$, on which $\partial$ extends $k[h, h]$-linearly. Let $\widetilde{SLH}$ be the subcomplex of $SLH[h, h]$ spanned by

$$[(a_{1,1}, r_{1,1})] \cdots [(a_{1,p_1}, r_{1,p_1})] \bullet \cdots \bullet [(a_{s,1}, r_{s,1})] \cdots [(a_{s,p_s}, r_{s,p_s})]$$

(54)

where $r_{i,j}$ are all distinct.

Consider the quotient space of $\widetilde{SLH}$ by identifying

$$[(a_{1,1}, r_{1,1})] \cdots [(a_{1,p_1}, r_{1,p_1})] \bullet \cdots \bullet [(a_{s,1}, r_{s,1})] \cdots [(a_{s,p_s}, r'_{s,p_s})]$$

with

$$[(a_{1,1}, r'_{1,1})] \cdots [(a_{1,p_1}, r'_{1,p_1})] \bullet \cdots \bullet [(a_{s,1}, r'_{s,1})] \cdots [(a_{s,p_s}, r'_{s,p_s})]$$

under the condition that $r_{i,j} < r'_{i', j'}$ if and only if $r_{i', j'} < r'_{i,j}$. Denote this quotient space by $\tilde{A}$. Pick an element in $\tilde{A}$, suppose it is represented by $[(a_{1,1}, r_{1,1})] \cdots [(a_{1,p_1}, r_{1,p_1})] \bullet \cdots \bullet [(a_{s,1}, r_{s,1})] \cdots [(a_{s,p_s}, r_{s,p_s})]$, without loss of generality, we may assume all $r_{i,j}$ are even, then the image of $[\square]$ represents an element in $\tilde{A}$. This means that $\tilde{A}$ with the differential induced by $b$ is a chain complex.

Now let $\tilde{B}$ be the subdmodule of $\tilde{A}$ generated by elements of the following form:

1. $X - X'_{i,j,i',j'} - h \cdot X''_{i,j,i',j'}$, where $i \neq i'$, $r_{i,j} < r'_{i', j'}$, and there does not exist $(i'', j'')$ with $r_{i,j} < r'_{i'', j''}$ and $r_{i,j} < r'_{i', j'}$.

2. $X - X'_{i,j,i',j'} - h \cdot X''_{i,j,i',j'}$, where $r_{i,j} < r'_{i', j'}$, and there does not exist $(i'', j'')$ with $r_{i,j} < r'_{i'', j''} < r_{i,j}$.

where $X'_{i,j,i',j'}$ and $X''_{i,j,i',j'}$ are given as follows: if $i \neq i'$, then $X'_{i,j,i',j'}$ is the same as $X$ except that $r_{i,j}$ and $r_{i', j'}$ are interchanged, while $X''_{i,j,i',j'}$ replaces the factors $[(a_{i,j}, r_{i,j})] \cdots [(a_{i,p_i}, r_{i,j})]$ and $[(a_{i', j'}, r_{i', j'})] \cdots [(a_{i', p_{i'}}, r_{i', j'})]$ by

$$(-1)^{\sigma_{ij, i', j'}} (a_{i,j}, a_{i', j'}) [(a_{i,j+1}, r_{i,j+1})] \cdots [(a_{i,j-1}, r_{i,j-1})] [(a_{i', j'+1}, r_{i', j'+1})] \cdots [(a_{i', j'-1}, r_{i', j'-1})]$$

similiarly, $X''_{i,j,i',j'}$ is the same as $X$ but with $r_{i,j}$ and $r_{i', j'}$ interchanged, while $X''_{i,j,i',j'}$ replaces the factor with the following factor:

$$(-1)^{\sigma_{ij, i', j'}} (a_{i,j}, a_{i', j'}) [(a_{i,j'+1}, r_{i,j'+1})] \cdots [(a_{i,j-1}, r_{i,j-1})] \bullet [(a_{i', j+1}, r_{i', j+1})] \bullet [(a_{i', j-1}, r_{i', j-1})].$$
It is proved in [10, Lemma 14] that $\tilde{B}$ is a subcomplex of $\tilde{A}$. Let $H = \tilde{A}/\tilde{B}$.

**Theorem 7.5** ([10] Theorem 15). There is a DG Hopf algebra structure on $H$ over $k[h, h]$, which quantizes $\tilde{R}_2$.

The product on $H$, denoted by $\star$, is easy to describe (as we will only use it): for two elements in $H$, say $X$ and $Y$, suppose they are both represented by elements in the form $(54)$, raise those $r_{i,j}$ in $Y$ such that they are all greater than those in $X$, then the product of $X$ and $Y$, $X \star Y$, is represented by $X \cdot Y$.

7.3. **Lifting the trace map.** In this subsection we relate the quantization of $\tilde{R}_2$ with the one of $\text{Rep}_V(\tilde{R}_2)$. We set $h = h$.

Extending the trace map (46) by $k[h]$-linearity, we obtain a $k[h]$-linear map

$$\overline{\text{Tr}} : \Lambda^\bullet \tilde{R}_2[h] \to \text{Rep}_V(\tilde{R})[h].$$

Clearly $\overline{\text{Tr}}$ commutes with the differential. We have the following.

**Theorem 7.6.** For a Koszul Calabi-Yau algebra $A$, the following map

$$\overline{\text{Tr}} : (\Lambda^\bullet \tilde{R}_2[h], \star) \to (\text{Rep}_V(\tilde{R})[h], \star).$$

is a map of DG algebras over $k[h]$.

**Proof.** Since $\overline{\text{Tr}}$ commutes with the differential on both sides, we only need to show it is a graded algebra map; however, this is already done in Schedler [33, §3.4].

More precisely, for a quiver $Q$, if we denote its double by $\bar{Q}$ then what Schedler constructed in [33] is the following:

1. a Lie bialgebra structure, called the necklace Lie bialgebra, on the commutator quotient space $(k\bar{Q})_2$;
2. a Hopf algebra $H$ over $k[h]$, completely analogous to the construction in previous subsection, quantizing the necklace Lie bialgebra $(k\bar{Q})_2$;
3. an algebra map from this Hopf algebra to the Weyl algebra of differential operators on $\text{Rep}_V(kQ)$.

Note that $\text{Rep}_V(k\bar{Q})$ is the cotangent space of $\text{Rep}_V(kQ)$, the algebra of differential operators is exactly the Moyal-Weyl quantization of $\text{Rep}_V(kQ)$.

As we remarked in [6.3] if we assume the the number of vertices of $Q$ is one, and the edges of $Q$ are graded, then $\tilde{R}$ in the current paper is exactly the path algebra $k\bar{Q}$. Therefore Schedler’s construction and proof hold in our case. □

Later Ginzburg and Schedler showed in [19] that the quantization constructed above is also of Moyal-Weyl type.

In summary, we obtain the following commutative diagram

$$\begin{array}{ccc}
(\Lambda^\bullet \tilde{R}_2[h], \star) & \xrightarrow{\overline{\text{Tr}}} & (\text{Rep}_V(\tilde{R})[h], \star) \\
\text{quantization} & \downarrow & \text{quantization} \\
\Lambda^\bullet \tilde{R}_2 & \xrightarrow{\text{Tr}} & \text{Rep}_V(\tilde{R}).
\end{array}$$
Recall that the images of $\text{Tr}$ in fact lie in $\text{Rep}_V(\tilde{R})^{\text{GL}}$, and the restriction of the Poisson structure on $\text{Rep}_V(\tilde{R})$ gives the Poisson structure on $\text{Rep}_V(\tilde{R})^{\text{GL}}$. From \((56)\) we thus obtain the following commutative diagram

\[
\begin{array}{ccc}
\Lambda^*\tilde{R} & \xrightarrow{\text{quantization}} & \text{Rep}_V(\tilde{R})^{\text{GL}} \\
\downarrow \text{quantization} & & \downarrow \text{quantization} \\
\Lambda^*\tilde{R} & \xrightarrow{\text{Tr}} & \text{Rep}_V(\tilde{R})^{\text{GL}}
\end{array}
\]

Recall that $\tilde{R} \simeq C_*^e(A)$ (see Proposition 3.5 and Convention 3.6) and observe that the bracket of any element in $\tilde{R}$ with unit of $\tilde{R}$ vanishes, we have an embedding $HC_0(A) \hookrightarrow H_*(\tilde{R})$ as graded vector spaces, where $HC_0(A)$ is the cyclic homology of $A$. By pulling the Lie bialgebra structure on $H_*(\tilde{R})$, $HC_0(A)$ thus has a Lie bialgebra structure of degree $(2-n, 2-n)$. Thus taking the homology on both sides of \((57)\) and combining the above argument we in fact obtain a commutative diagram

\[
\begin{array}{ccc}
\Lambda^*HC_0(A)[h] & \xrightarrow{\text{quantization}} & (H_*(\text{Rep}_V(\tilde{R}))^{\text{GL}})[h] \\
\downarrow \text{quantization} & & \downarrow \text{quantization} \\
\Lambda^*HC_0(A) & \xrightarrow{\text{Tr}} & H_*(\text{Rep}_V(\tilde{R}))^{\text{GL}},
\end{array}
\]

where on the right hand side, we have used the fact that $H_*(\text{Rep}_V(\tilde{R})^{\text{GL}}) \cong H_*(\text{Rep}_V(\tilde{R}))^{\text{GL}}$ (see [3, Theorem 2.6(b)] for a proof).

8. Derived representation schemes

In this section, we briefly discuss the results in previous sections with the derived representation schemes, introduced by Berest, Khachatryan and Ramadoss. The interested reader may refer to [1, 2, 3, 4] for more details.

8.1. Derived representation schemes. In algebraic geometry, there is an equivalence of categories between affine schemes and commutative algebras, and many geometric structures on affine schemes have an algebraic description, and vice versa. However, for associative algebras, this correspondence does not exist. In 1998, Kontsevich and Rosenberg [23] proposed a heuristic principle to study the non-commutative geometry on associative, not-necessarily commutative, algebras, which is roughly stated as follows: for an associative algebra over a field $k$, say $A$, any non-commutative geometric structure, such as non-commutative Poisson, non-commutative symplectic, etc., should induce its classical counterpart in a natural way on its representation scheme $\text{Rep}_V(A)$, for all $k$-vector space $V$.

During the past decade, much progress has been made in the study of non-commutative geometry under the guidance of the Kontsevich-Rosenberg Principle (see [15, 17, 40]). However, for a general algebra $A$, $\text{Rep}_V(A)$ is very singular. In 2011 Berest et. al. [3] suggested that one should instead consider the derived representation schemes of $A$. In this case, one replaces $A$ with its cofibrant resolution, say $QA$, in the category of differential graded algebras, and then considers the DG representation schemes of $QA$, which are then smooth in the DG
sense. The cofibrant resolution of a DG algebra is not unique, but unique up to homotopy. Correspondingly, $\text{Rep}_V(QA)$ is also unique up to homotopy. By modulo such ambiguity, the DG representation scheme of $QA$ in $V$ in the homotopy category of DG commutative algebras, denoted by $\text{DRep}_V(A)$ and called the derived representation scheme of $A$ in $V$, is a very good object that we can successfully apply the Kontsevich-Rosenberg principle. Formally, their result is stated as follows.

**Theorem 8.1** ([3]). For any two objects $A, B \in \text{DGA}$ and any $f \in \text{Hom}(A, B)$, let $QA$ and $QB$ be any cofibrant replacement of $A$ and $B$ respectively, and $Qf \in \text{Hom}(QA, QB)$ be the corresponding cofibrant lifting of $f$. The functor (28) has a total left derived functor

$$L(-)_V : \text{Ho}(\text{DGA}) \to \text{Ho}(\text{CDGA}), \quad A \mapsto (QA)_V, \quad f \mapsto (Qf)_V.$$  

According to [3], $L(A)_V$ is called the derived representation scheme (or DRep for short) of $A$ in $V$, and is also denoted by $\text{DRep}_V(A)$; its homology $H_*(\text{DRep}_V(A))$ is called the representation homology of $A$, and is sometimes denoted by $H_*(A, V)$.

**Example 8.2** (DReps of Koszul algebras). Suppose $A$ is a Koszul algebra. Then $A$ has an explicit cofibrant resolution $\Omega(A^i) \simeq A$. Thus $\text{DRep}_n(A)$ is explicit in the Koszul case: suppose $\hat{A}^i$ has a set of basis $\{x^\alpha\}_{\alpha \in I}$, and the reduced coproduct $\hat{\Delta}(x^\alpha) = \sum (x^\alpha) x^{\alpha_1} \otimes x^{\alpha_2}$. Then $\text{DRep}_n(A)$ is the quasi-free DG commutative algebra generated by

$$\{x^\alpha_{ij} \mid \alpha \in I, \, 1 \leq i, j \leq n, |x^\alpha_{ij}| = |x^\alpha| - 1\},$$

with

$$\partial(x^\alpha_{ij}) = \sum_{\{x^\alpha\}} (-1)^{|x^{\alpha_1}|} \sum_{k=1}^n x_{ik}^{\alpha_1} \cdot x_{kj}^{\alpha_2}.$$  

8.2. **Comparison of $\text{DRep}_n(A)$ and $\text{Rep}_V(\hat{R})$.** Suppose $A$ is a Koszul algebra. Then one obtains $R = \Omega A^i$ from $\hat{R} = \Omega \hat{A}^i$ by identifying the counit of $A^i$ with zero. Thus correspondingly, if we identify the coordinates of $\text{Rep}_V(\hat{R})$ corresponding the co-unit of $\hat{A}^i$ with zero, we obtain $\text{Rep}_V(R)$, which is isomorphic to $\text{DRep}_V(A)$ in the homotopy category of DG algebras.

We here give two remarks regarding the role of the co-unit of $A^i$. On the one hand, since $R = \Omega(A^i) \simeq A$ is a cofibrant resolution of $A$, $\text{Rep}_V(R)$ is a derived functor of $A$.

On the other hand, considering $\text{Rep}_V(\hat{R})$ has the advantage when considering the GL-invariants; namely the GL($V$)-invariant functions, the cotangent vector space and the tangent vector space on $\text{Rep}_V(\hat{R})$ is directly related to the cyclic complex, the Hochschild chain and cochain complexes of $A$ via the trace maps (see [33] and [34]). Sometimes, keeping the unit (respectively co-unit) of an algebra (respectively coalgebra) is important in the study of Noncommutative Geometry; see for example the early works of Connes and Quillen in this field (c.f. [13] [32]).

Now suppose $A$ is also Calabi-Yau. In [1] [9], we showed that $R$ has a shifted Poisson structure, while in the current paper we deduced the shifted Poisson structure on $\hat{R}$. We may understand them in this way: geometrically, $\text{DRep}_V(A)$ can be understood as a DG subscheme of $\text{Rep}_V(\hat{R})$. It is direct to see that the Poisson bracket (28) on $\text{Rep}_V(\hat{R})$ restricts to a Poisson bracket on $\text{Rep}_V(R)$ studied in [1] and [9]; in other words, $\text{Rep}_V(R)$ is a shifted Poisson subscheme of $\text{Rep}_V(\hat{R})$. 
9. Example: The Sklyanin algebras

In this section, we give two examples of Koszul Calabi-Yau algebras, namely the 3- and 4-dimensional Sklyanin algebras, and study the corresponding shifted bi-symplectic structure with some detail. In general the geometry of the representations of Sklyanin algebras are complicated, however, both the derived representation schemes DRep$_V$(A) and the representation schemes Rep$_V$(R) are easy to describe; here A is a Sklyanin algebra and R = Ω(Ā) as before.

**Example 9.1 (3-dimensional Sklyanin algebras).** Let $a, b, c ∈ k$ satisfying the following two conditions:

1. $[a : b : c] ∈ P^2_k \setminus D$, where
   $$D = \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\} \cup \{[a : b : c]|a^3 = b^3 = c^3 = 1\};$$

2. $abc ≠ 0$ and $(3abc)^3 ≠ (a^3 + b^3 + c^3)^3$.

The 3-dimensional Sklyanin algebra $A = A(a, b, c)$ is the graded $k$-algebra with generators $x, y, z$ of degree one, and relations

- $cx^2 + bzy + ayz = 0,$
- $azx + cy^2 + bxz = 0,$
- $byx + ayz + cz^2 = 0.$

Smith showed in [35, Example 10.1] that A is Koszul, whose dual algebra $A^!$ is generated by $ξ_1, ξ_2, ξ_3$ with relations

- $cξ_2ξ_3 - bξ_3ξ_2$, $bξ_1^2 - aξ_2ξ_3$, $cξ_3ξ_1 - bξ_1ξ_3$,
- $bξ_2^2 - aξ_3ξ_1$, $cξ_1ξ_2 - bξ_2ξ_1$, $bξ_3^2 - aξ_1ξ_2$.

In degree 3, we have relations

- $ξ_i^3 = \frac{a}{b}ξ_1ξ_2ξ_3 = ξ_i^3 = \frac{a}{b}ξ_2ξ_3ξ_1 = ξ_3ξ_2ξ_1 = \frac{a}{c}ξ_3ξ_2ξ_1 = \frac{a}{c}ξ_1ξ_3ξ_2$, $i = 1, 2, 3$,
- $ξ_iξ_j^2 = ξ_jξ_i^2 = 0$, $i, j = 1, 2, 3$ and $i ≠ j$.

From these relations, we obtain that $A^!$ is isomorphic to a graded coalgebra spanned by

- $A^!_0 : 1$,
- $A^!_1 : ξ_1^*, ξ_2^*, ξ_3^*$,
- $A^!_2 : ξ_1^*ξ_1^*, ξ_2^*ξ_2^*, ξ_3^*ξ_3^*$,
- $A^!_3 : ξ_1^*ξ_2^*ξ_1^*$ (59)

with the coproducts given by

- $Δ(1) = 1 ⊗ 1$,
- $Δ(ξ_i) = 1 ⊗ ξ_i + ξ_i ⊗ 1$, $i = 1, 2, 3$,
- $Δ(ξ_i^*ξ_j^*) = 1 ⊗ ξ_i^*ξ_j^* + ξ_i^* ⊗ ξ_j^* + \frac{a}{b}ξ_2^* ⊗ ξ_3^* + \frac{a}{c}ξ_3^* ⊗ ξ_2^* + ξ_1^*ξ_i^* ⊗ 1$,
- $Δ(ξ_i^*ξ_j^*ξ_k^*) = 1 ⊗ ξ_2^*ξ_3^* + ξ_2^* ⊗ ξ_3^* + \frac{a}{c}ξ_1^* ⊗ ξ_3^* + \frac{a}{b}ξ_3^* ⊗ ξ_1^* + ξ_2^*ξ_3^* ⊗ 1$,
- $Δ(ξ_i^*ξ_j^*ξ_k^*ξ_l^*) = 1 ⊗ ξ_3^*ξ_3^* + ξ_3^* ⊗ ξ_3^* + \frac{a}{b}ξ_2^* ⊗ ξ_1^* + \frac{a}{c}ξ_1^* ⊗ ξ_1^* + ξ_3^*ξ_3^* ⊗ 1$,
\[ \Delta(\xi_0^i \xi_2^j \xi_3^k) = 1 \otimes \xi_2^j \xi_3^k \xi_1^i + \frac{b}{a} \xi_0^i \otimes \xi_1^i \xi_2^j \xi_3^k + \frac{b}{a} \xi_2^j \otimes \xi_2^j \xi_3^k + \frac{b}{a} \xi_3^k \otimes \xi_3^k \xi_1^i. \]  

(60)

The pairing on \( A' \) given by \((60)\) is cyclic, and therefore \( A \) is 3-Calabi-Yau.

Now, \( \tilde{R} \) is the DG algebra freely generated by those elements in \((59)\) with degree shifted down by one. From \((60)\) we see that the \((-1)\)-shifted bi-symplectic structure \( \omega \in \text{DR}^2_{\omega} \tilde{R} \) is given by

\[ \omega = d(1) \otimes d(\xi_3^j \xi_2^j \xi_1^i) + \frac{b}{a} d(\xi_1^i) \otimes d(\xi_1^i \xi_2^j) + \frac{b}{a} d(\xi_2^j) \otimes d(\xi_2^j \xi_3^k) + \frac{b}{a} d(\xi_3^k) \otimes d(\xi_3^k \xi_1^i) \]

\[ + \frac{b}{a} d(\xi_1^i \xi_1^i) \otimes d(\xi_1^i) + \frac{b}{a} d(\xi_2^j \xi_2^j) \otimes d(\xi_2^j) + \frac{b}{a} d(\xi_3^k \xi_3^k) \otimes d(\xi_3^k) + \frac{b}{a} d(\xi_3^k \xi_1^i) \otimes d(1). \]

Note that here (as well as in the next example) \( d(1) \neq 0 \).

**Example 9.2** (4-dimensional Sklyanin algebras). Let \( \alpha, \beta, \gamma \in k \) such that

\[ \alpha + \beta + \gamma + \alpha \beta \gamma = 0, \quad \{\alpha, \beta, \gamma\} \cap \{0, \pm 1\} = \emptyset. \]

The 4-dimensional Sklyanin algebra \( A = A(\alpha, \beta, \gamma) \) is the graded \( k \)-algebra with generators \( x_0, x_1, x_2, x_3 \) of degree one, and relations \( f_i = 0 \), where

\[ f_1 = x_0 x_1 - x_1 x_0 - \alpha(x_2 x_3 + x_3 x_2), \quad f_2 = x_0 x_1 + x_1 x_0 - (x_2 x_3 - x_3 x_2), \]
\[ f_3 = x_0 x_2 - x_2 x_0 - \beta(x_3 x_1 + x_1 x_3), \quad f_4 = x_0 x_2 + x_2 x_0 - (x_3 x_1 - x_1 x_3), \]
\[ f_5 = x_0 x_3 - x_3 x_0 - \gamma(x_1 x_2 + x_2 x_1), \quad f_6 = x_0 x_3 + x_3 x_0 - (x_1 x_2 - x_2 x_1). \]

As proved by Smith and Stafford (\([36] \) Propositions 4.3-4.9), \( A \) is Koszul, whose Koszul dual algebra \( A' \) is generated by \( \xi_0, \xi_1, \xi_2, \xi_3 \) with the following relations:

\[ \xi_0^3 = \xi_1^3 = \xi_2^3 = \xi_3^3 = 0, \]
\[ 2\xi_2 \xi_3 + (\alpha + 1)\xi_0 \xi_1 - (\alpha - 1)\xi_1 \xi_0 = 0, \]
\[ 2\xi_3 \xi_1 + (\alpha - 1)\xi_0 \xi_1 - (\alpha + 1)\xi_1 \xi_0 = 0, \]
\[ 2\xi_1 \xi_2 + (\beta + 1)\xi_0 \xi_2 - (\beta - 1)\xi_2 \xi_0 = 0, \]
\[ 2\xi_1 \xi_3 + (\beta - 1)\xi_0 \xi_3 - (\beta + 1)\xi_3 \xi_0 = 0, \]
\[ 2\xi_2 \xi_1 + (\gamma + 1)\xi_0 \xi_3 - (\gamma - 1)\xi_3 \xi_0 = 0, \]
\[ 2\xi_1 \xi_2 + (\gamma - 1)\xi_0 \xi_1 - (\gamma + 1)\xi_1 \xi_0 = 0. \]

Smith and Stafford also showed that \( A' \) admits a non-degenerate symmetric pairing, and hence \( A \) is 4-Calabi-Yau. In particular, in degree 4, we have the following identities:

\[ \xi_0 \xi_i \xi_0 \xi_i = -\xi_i \xi_0 \xi_i \xi_0 \quad \text{for } 1 \leq i \leq j, \quad \xi_0 \xi_i \xi_0 \xi_j = 0 \quad \text{for } i \neq j, \]
\[ \xi_0 \xi_2 \xi_0 \xi_2 = \frac{1 + \alpha + \gamma}{1 - \beta - \gamma} \xi_0 \xi_1 \xi_0 \xi_1, \]
\[ \xi_0 \xi_3 \xi_0 \xi_3 = \frac{1 + \alpha}{1 - \gamma} \xi_0 \xi_1 \xi_0 \xi_1. \]
From the above two groups of identities, we obtain that the Koszul dual coalgebra $A^i$ is isomorphic to a coalgebra spanned by

$$
A_0^i : 1
A_1^i : \xi_0^* \xi_1^* \xi_2^* \xi_3^* \\
A_2^i : \xi_1^* \xi_0^* \xi_1^* \xi_2^* \xi_3^* \xi_4^* \xi_5^* \xi_6^* \xi_7^* \\
A_3^i : \xi_0^* \xi_1^* \xi_2^* \xi_3^* \xi_4^* \xi_5^* \xi_6^* \xi_7^* \xi_8^* \xi_9^* \xi_{10}^* \\
A_4^i : \xi_1^* \xi_0^* \xi_1^* \xi_2^* \xi_3^*
$$

with the coproduct of $\xi_1^* \xi_2^* \xi_3^* \xi_4^* \xi_5^*$ given by

$$
\Delta(\xi_1^* \xi_2^* \xi_3^* \xi_4^* \xi_5^*) = 1 \otimes \xi_1^* \xi_2^* \xi_3^* \xi_4^* \xi_5^* + \xi_1^* \otimes \xi_2^* \xi_3^* \xi_4^* \xi_5^* - \xi_1^* \otimes \xi_2^* \xi_3^* \xi_4^* \xi_5^* \\
\quad - \frac{1 - \beta \gamma}{1 + \alpha} \xi_2^* \otimes \xi_3^* \xi_4^* \xi_5^* - \frac{1 - \gamma}{1 + \alpha} \xi_3^* \xi_4^* \xi_5^* - \xi_3^* \xi_4^* \xi_5^*
$$

Thus the $(-2)$-shifted bi-symplectic structure $\omega \in DR^2_{\text{nc}} \tilde{R}$ is given by

$$
d(1) \otimes d(\xi_1^* \xi_2^* \xi_3^* \xi_4^*) + d(\xi_2^*) \otimes d(\xi_1^* \xi_2^* \xi_3^* \xi_4^*) - d(\xi_1^* \xi_2^* \xi_3^*) \otimes d(\xi_1^* \xi_2^* \xi_3^* \xi_4^*) \\
\quad - \frac{1 - \beta \gamma}{1 + \alpha} d(\xi_2^*) \otimes d(\xi_3^*) \xi_4^* \xi_5^* - \frac{1 - \gamma}{1 + \alpha} d(\xi_3^*) \otimes d(\xi_3^*) \xi_4^* \xi_5^* - d(\xi_3^*) \otimes d(\xi_3^*) \xi_4^* \xi_5^* \\
\quad - \frac{1 - \beta \gamma}{1 + \alpha} d(\xi_3^*) \otimes d(\xi_4^*) \xi_5^* - \frac{1 - \gamma}{1 + \alpha} d(\xi_4^*) \otimes d(\xi_4^*) \xi_5^* - d(\xi_4^*) \otimes d(\xi_4^*) \xi_5^* \\
\quad + \frac{1 - \beta \gamma}{1 + \alpha} d(\xi_4^*) \otimes d(\xi_5^*) \xi_6^* - \frac{1 - \gamma}{1 + \alpha} d(\xi_5^*) \otimes d(\xi_5^*) \xi_6^* - d(\xi_5^*) \otimes d(\xi_5^*) \xi_6^* \\
\quad + d(\xi_5^*) \otimes d(\xi_5^*) \xi_6^* \xi_7^* \xi_8^* \xi_9^* \xi_{10}^* + \frac{1 - \beta \gamma}{1 + \alpha} d(\xi_6^*) \xi_7^* \xi_8^* \xi_9^* \xi_{10}^* \xi_1^* + \frac{1 - \gamma}{1 + \alpha} d(\xi_7^*) \xi_8^* \xi_9^* \xi_{10}^* \xi_1^* \xi_2^* \xi_3^*
$$

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