Local Anomalies, Local Equivariant Cohomology and the Variational Bicomplex

ROBERTO FERREIRO PÉREZ
Departamento de Economía Financiera y Contabilidad I
Facultad de Ciencias Económicas y Empresariales, UCM
Campus de Somosaguas, 28223-Pozuelo de Alarcón, Spain
E-mail: roferreiro@ccee.ucm.es

Part I
Local Cohomology and the Variational Bicomplex

Abstract
The differential forms on the jet bundle $J^\infty E$ of a bundle $E \to M$ over a compact $n$-manifold $M$ of degree greater than $n$ determine differential forms on the space $\Gamma(E)$ of sections of $E$. The forms obtained in this way are called local forms on $\Gamma(E)$, and its cohomology is called the local cohomology of $\Gamma(E)$. More generally, if a group $G$ acts on $E$, we can define the local $G$-invariant cohomology. The local cohomology is computed in terms of the cohomology of the jet bundle by means of the variational bicomplex theory. A similar result is obtained for the local $G$-invariant cohomology. Using these results and the techniques for the computation of the cohomology of invariant variational bicomplexes in terms of relative Gelfand-Fuchs cohomology introduced in [3], we construct non trivial local cohomology classes in the important cases of Riemannian metrics with the action of diffeomorphisms, and connections on a principal bundle with the action of automorphisms.

Key words and phrases: local cohomology, variational bicomplex, manifold of sections, space of metrics, space of connections.

Mathematics Subject Classification 2000: Primary 58A20; Secondary 55N99, 57R32, 58D17, 58E99.
1 Introduction

Let us recall some basic constructions on the jet bundle geometrical approach to the variational calculus. Let $p: E \rightarrow M$ be a bundle over a compact, oriented $n$-manifold $M$ without boundary and let $J^\infty E$ denote its infinite-jet bundle. If $\lambda \in \Omega^n(J^\infty E)$ is a Lagrangian density, it determines a function $A$ (the action functional) in the space $\Gamma(E)$ of sections of $E$ by setting $A(s) = \int_M (j^\infty s)^* \lambda$.

The exterior differential $dA$ of $A$ is determined in the following way. Let $s_t \in \Gamma(E)$ be a 1-parameter family of sections of $E$ with $s = s_0$ and let $X \in T_s \Gamma(E) \cong \Gamma(M, s^* V(E))$ be the vertical vector field along $s$ defined by $X(p) = \frac{d}{dt} \bigg|_{t=0}$.

Then we have

$$dA_s(X) = \frac{dA(s_t)}{dt} \bigg|_{t=0} = \int_M \frac{d(j^\infty s_t)^* \lambda}{dt} \bigg|_{t=0} = \int_M (j^\infty s)^* (L_{prX} \lambda) = \int_M (j^\infty s)^* (t_{prX} d\lambda).$$

We see that to the form $\lambda \in \Omega^n(J^\infty E)$ it corresponds the function $A \in \Omega^0(\Gamma(E))$, whereas to the form $d\lambda \in \Omega^{n+1}(J^\infty E)$ it corresponds the 1-form $dA \in \Omega^1(\Gamma(E))$. Generalizing this idea, we have defined in [18] an integration map $\mathcal{I}: \Omega^{n+k}(J^\infty E) \rightarrow \Omega^k(\Gamma(E))$, by setting

$$\mathcal{I}^\alpha(X_1, \ldots, X_k) = \int_M (j^\infty s)^* (t_{prX_1} \cdots t_{prX_k} \alpha).$$

for $\alpha \in \Omega^{n+k}(J^\infty E)$ and $X_1, \ldots, X_k \in T_s \Gamma(E)$. Then we have $\mathcal{I}[\lambda] = A$ and $\mathcal{I}[d\lambda] = dA = d\mathcal{I}[\lambda]$. We prove that we have $\mathcal{I}[d\alpha] = d\mathcal{I}[\alpha]$ for any form $\alpha \in \Omega^{n+k}(J^\infty E)$. The forms of the type $\mathcal{I}[\alpha]$ for $\alpha \in \Omega^{n+k}(J^\infty E)$ are called the local $k$-forms $\Omega^k_{loc}(\Gamma(E))$ on $\Gamma(E)$, and its cohomology $H^k_{loc}(\Gamma(E))$ the local cohomology of $\Gamma(E)$. Hence the local forms are those forms on $\Gamma(E)$ obtained by integration over $M$ of a form on the jet bundle (i.e. a form depending on a section and its derivatives). For $k = 0$ this notion of locality corresponds precisely to the notion of “local functional” needed in quantum field theory for the study of anomaly cancellation. Moreover, in [9] we show that the anomaly cancellation can be understood in terms of local cohomology of forms of degree 2, and we use some of the results obtained in the present paper solve the problem proposed in [31] consisting in explaining the topological nature of local anomalies.

The local cohomology can be studied in terms of the jet bundle by means of the variational bicomplex theory. Coming back to the example of the variational calculus, the Euler-Lagrange form $\mathcal{E}(\lambda) \in \Omega^{n+1}(J^\infty E)$ of $\lambda$ satisfies $\mathcal{I}[\mathcal{E}(\lambda)] = \mathcal{I}[d\lambda] = dA$, and $d\lambda = 0$ if and only if $\mathcal{E}(\lambda) = 0$. Note that both $\mathcal{E}(\lambda) = 0$ and $d\lambda$ determine the same local form $dA \in \Omega^1_{loc}(\Gamma(E))$. However, only $\mathcal{E}(\lambda)$ determines uniquely the properties of $dA$. We recall that from the point of view of the variational bicomplex theory the Euler-Lagrange operator is given by $\mathcal{E}(\lambda) = I(d_H \lambda)$ where $I$ is the interior Euler operator and $d_H$ the horizontal differential. In general, we show in Section $3$ that given $\alpha \in \Omega^{n+k}(J^\infty E)$ with $k > 0$, there are an infinite number of forms on $J^\infty E$ determining the same local form $\mathcal{I}[\alpha]$ on $\Gamma(E)$, but the interior Euler operator selects a canonical representative for it $I(\alpha_{n,k})$, that satisfies $\mathcal{I}[\alpha] = \mathcal{I}[I(\alpha_{n,k})]$, and $\mathcal{I}[\alpha] = 0$ if
and only if \( I(\alpha, k) = 0 \). Hence, if we denote by \( \mathcal{F}^k(J^\infty E) = I(\Omega^{n,k}(J^\infty E)) \) the space of functional forms, we have the isomorphisms \( \Omega^k_{\text{loc}}(\Gamma(E)) \cong \mathcal{F}^k(J^\infty E) \), and \( H^k_{\text{loc}}(\Gamma(E)) \cong H^k(\mathcal{F}^k(J^\infty E)) \cong H^{n+k}(J^\infty E) \cong H^{n+k}(E) \) for \( k > 0 \).

If a group \( \mathcal{G} \) acts on \( E \) by automorphisms, we can consider the local \( \mathcal{G} \)-invariant cohomology of \( \Gamma(E) \), \( H^k_{\text{loc}}(\Gamma(E))^\mathcal{G} \), and clearly we also have the isomorphisms \( \Omega^k_{\text{loc}}(\Gamma(E))^\mathcal{G} \cong \mathcal{F}^k(J^\infty E)^\mathcal{G} \), and \( H^k_{\text{loc}}(\Gamma(E))^\mathcal{G} \cong H^k(\mathcal{F}^k(J^\infty E)^\mathcal{G}) \) for \( k > 1 \). Under certain conditions analyzed in \([3, 3]\) the invariant cohomology of the Euler-Lagrange complex is isomorphic to the invariant cohomology of \( J^\infty E \), and in that case we have \( H^k_{\text{loc}}(\Gamma(E))^\mathcal{G} \cong H^{n+k}(J^\infty E)^\mathcal{G} \) for \( k > 1 \). Moreover, in \([3]\) it is shown that in certain cases the invariant cohomology of \( J^\infty E \) can be computed in terms of relative Lie algebra cohomology of formal vector fields.

Finally, we apply the preceding constructions to study the local invariant cohomology of the space of Riemannian metrics \( \text{Met} M \) with the action of the group \( \text{Diff} M \) of diffeomorphisms and the space of connections \( \mathcal{A}_P \) on a principal bundle \( P \) with the action of the group \( \text{Aut} P \) of automorphisms of \( P \). We construct some local invariant cohomology classes on these spaces and we prove the non-triviality of that classes, by relating them to relative Gelfand-Fuchs cohomology of formal vector fields. These results are used in \([9]\) to study the problem of local anomaly cancellation for gravitational and mixed anomalies.

## 2 Local cohomology

Let \( p : E \to M \) be a bundle over a compact, oriented \( n \)-manifold \( M \) without boundary. We denote by \( V(E) \) the vertical bundle and by \( J^r E \) its \( r \)-jet bundle. We have the projections \( p_r : J^r E \to M, \ p_{r,s} : J^r E \to J^s E \) for \( s < r \). Let \( \Gamma(E) \) be the space of global sections of \( E \), that we assume to be not empty. \( \Gamma(E) \) admits an structure of a Frechet manifold (see \([15, \text{Section I.4}]\) for details), and for any \( s \in \Gamma(E) \), the tangent space to the manifold \( \Gamma(E) \) is isomorphic to the space of vertical vector fields along \( s \), that is \( T_s \Gamma(E) \cong \Gamma(M, s^* V(E)) \).

A diffeomorphism \( \phi \in \text{Diff} E \) is said to be projectable if there exists \( \tilde{\phi} \in \text{Diff} M \) satisfying \( \phi \circ p = p \circ \tilde{\phi} \). We denote by \( \text{Proj} E \) the space of projectable diffeomorphism of \( E \), and we denote by \( \text{Proj}^+ E \) the subgroup of elements such that \( \tilde{\phi} \in \text{Diff}^+ M \), i.e., \( \tilde{\phi} \) is orientation preserving. The space of projectable vector fields on \( E \) is denoted by \( \text{proj} E \), and can be considered as the Lie algebra of \( \text{Proj} E \). We denote by \( \phi^{(r)} \) (resp. \( \phi^{*}X \)) the prolongation of \( \phi \in \text{Proj} E \) (resp. \( X \in \text{proj} E \)) to \( J^r E \). The group \( \text{Proj} E \) acts naturally on \( \Gamma(E) \) in the following way. If \( \phi \in \text{Proj} E \), we define \( \phi|_{\Gamma(E)} \in \text{Diff} \Gamma(E) \) by \( \phi|_{\Gamma(E)}(s) = \phi \circ s \circ \phi^{-1} \), for all \( s \in \Gamma(E) \). In a similar way, a projectable vector field \( X \in \text{proj} E \) induces a vector field \( X_{|\Gamma(E)} \in \mathfrak{X}(\Gamma(E)) \).

Let \( j^r : M \times \Gamma(E) \to J^r E, j^r(x,s) = j^{r}_x(s) \) be the evaluation map. We define a map \( \exists^r : \Omega^{n+k}(J^r E) \to \Omega^k(\Gamma(E)), \) by setting \( \exists^r[\alpha] = \int_{M} (j^r)^* \alpha, \) for \( \alpha \in \Omega^{n+k}(J^r E) \). If \( \alpha \in \Omega^k(J^r E) \) with \( k < n \), we set \( \exists^r[\alpha] = 0 \). The operator \( \exists \) satisfies the following properties (see \([18]\))

**Proposition 1** For any \( \alpha \in \Omega^{n+k}(J^r E) \) we have
1. \( \mathcal{S} r[\alpha] = d \mathcal{S} r[\alpha] \).
2. \( \mathcal{S} r[\phi(r)^{*}\alpha] = \phi_{\Gamma(E)}^{*} \mathcal{S} r[\alpha] \), for any \( \phi \in \text{Proj}^{+}E \).
3. \( \mathcal{S} r[t_{pr}^{*}X\alpha] = t_{X_{\Gamma(E)}} \mathcal{S} r[\alpha] \) for any \( X \in \text{proj}E \).

**Corollary 2** Let \( \alpha \in \Omega^{n+k}(J^{r}E) \) and \( X_{1}, \ldots, X_{k} \in T_{s}\Gamma(E) \). Then we have

\[
\mathcal{S} r[\alpha]_{s}(X_{1}, \ldots, X_{k}) = \int_{M} (j^{r}s)^{*}(t_{pr}^{*}X_{k} \ldots t_{pr}^{*}X_{1}\alpha).
\]

The forms of the type \( \mathcal{S} r[\alpha] \) for certain \( r \in \mathbb{N} \) and \( \alpha \in \Omega^{n+k}(J^{r}E) \) are called local \( k \)-forms, and the space of local \( k \)-forms on \( \Gamma(E) \) is denoted by \( \Omega^{n+k}_{loc}(\Gamma(E)) \). By Proposition \( \mathbb{N} \), \( \Omega^{n+k}_{loc}(\Gamma(E)) \) is closed under \( d \), and we denote by \( H^{n+k}_{loc}(\Gamma(E)) \) the cohomology of the complex \( (\Omega^{n+k}_{loc}(\Gamma(E)), d) \). We have an induced map in cohomology \( H^{n+k}_{loc}(\Gamma(E)) \rightarrow H^{n+k}(\Gamma(E)) \). The key point is that this map is not injective in general (e.g., see \( \mathbb{R} \) for an example). If \( \alpha \in \Omega^{n+k}_{loc}(\Gamma(E)) \) is closed, its cohomology class on \( H^{n+k}(\Gamma(E)) \) vanishes if and only if \( \alpha \) is the exterior differential of a form \( \beta \in \Omega^{k-1}_{loc}(\Gamma(E)) \), while its cohomology class on \( H^{n+k}_{loc}(\Gamma(E)) \) vanishes if and only if \( \alpha \) is the exterior differential of a local form \( \beta \in \Omega^{k-1}_{loc}(\Gamma(E)) \).

### 3 The variational bicomplex

We denote by \( J^{\infty}E \) the infinite jet bundle (see \([1][19][20]\) for the details on the geometry of \( J^{\infty}E \)). We have the projections \( p_{\infty}: J^{\infty}E \rightarrow M \), \( p_{\infty,r}: J^{\infty}E \rightarrow J^{r}E \), and \( \Omega^{k}(J^{r}E) = \lim_{\rightarrow} \Omega^{k}(J^{r}E) \). We denote by \( \phi_{\infty}(\text{pr}X) \) the prolongation of \( \phi \in \text{Proj}E \) (resp. \( X \in \text{proj}E \)) to \( J^{\infty}E \).

A local trivialization \( (U; x^{i}, y^{a}) \) of \( E \) induces a local coordinate system \((p_{\infty,0})^{-1}U; x^{i}, y^{a}, y^{a}_{j}), i = 1, \ldots, n, j = 1, \ldots, m, J \in \mathbb{N}^{k}, J \) symmetric \( k = 1, 2, \ldots \) on \( J^{\infty}E \), by setting \( y^{a}_{j}(s) = \frac{\partial(t_{\infty}^{*}(y^{a})^{k})}{\partial x^{j}}(x) \) for every local section \( s \) of \( p: E \rightarrow M \). If in local coordinates \( X = f^{i} \frac{\partial}{\partial x^{i}} + g^{a} \frac{\partial}{\partial y^{a}} \), then we have (\([13]\) Theorem 2.36)

\[
\text{pr}X = f^{i} \frac{\partial}{\partial x^{i}} + g^{a} \frac{\partial}{\partial y^{a}} + \sum_{i=1}^{k} \frac{d^{i}}{dx^{i}} \left( g^{a} - \sum_{j=1}^{n} f^{i} y^{a}_{j} \right) + \sum_{i=1}^{k} f^{i} y^{a}_{j+i} \tag{1}
\]

where \( \frac{d^{i}}{dx^{i}} = \frac{\partial}{\partial x^{i}} \ldots \frac{\partial}{\partial x^{i}}, \) and \( \frac{d}{dx^{i}} = \frac{\partial}{\partial x^{i}} + \sum_{k=1}^{\infty} y^{a}_{K+i} \frac{\partial}{\partial y^{a}_{k}} \).

The evolutionary vector fields are defined as the vertical fields on \( E \) with coefficients in \( J^{\infty}E \), i.e., \( \text{Ev}(E) = \Gamma(J^{\infty}E, V(E)) \). If \( X \in (\mathcal{X})(E) \) is a projectable vector field, the evolutionary part of \( X \) is given by \( \text{ev}(X)(j^{r}s) = X(s(x)) - s_{*}(X(x)) \). The total vector fields are the vector fields on \( M \) with coefficients in \( J^{\infty}E \), i.e., \( \text{Tot}(E) = \Gamma(J^{\infty}E, TM) \). Given a projectable vector field \( X \in (\mathcal{X})(E) \), we have \( \text{pr}X = \text{pr}(\text{ev}X) + \text{tot}X \), where \( \text{tot}X \) denotes the total part of \( X \). In local coordinates, if \( X = f^{i} \frac{\partial}{\partial x^{i}} + g^{a} \frac{\partial}{\partial y^{a}} \), then we have \( \text{tot}X = f^{i}d/dx^{i} \) and \( \text{ev}X = (g^{a} - y^{a}_{i} f^{i})\partial/\partial y^{a} \).
We define \( \gamma : \Gamma(E) \times \text{Ev}(E) \to \mathcal{T}(E) \) by setting \( \gamma(s, X) = (s, X \circ j^\infty s) \). We also use the notation \( \gamma_s(X) = \gamma(s, X) \). The following Proposition shows that the vector field \( X_{\Gamma(E)} \) is determined by the evolutionary part of \( X \).

**Proposition 3** If \( X \in \text{proj } E \) is a projectable vector field then \( X_{\Gamma(E)}(s) = \gamma_s(\text{ev } X) \). In particular, if \( X \) is a vertical vector field then \( X_{\Gamma(E)}(s) = X \circ s \).

**Proof.** Let \( X \in \text{proj } E \) be a projectable vector field with projection \( X \), and let \( \Phi_t \in \text{proj } E \) be its flux and \( \Phi_t \in \text{Diff } M \) its projection onto \( M \). Given \( s \in \Gamma(E) \) we have by definition \( X_{\Gamma(E)}(s) = \dot{s}_0 \), where \( s_t = \Phi_t \circ s \circ \Phi_{-t} \) and \( \dot{s}_0 = \frac{ds_1}{dt} |_{t=0} \). For every \( x \in M \) we have \( X_{\Gamma(E)}(s)(x) = \dot{s}_0(x) = \dot{\Phi}_0(s(x)) + Ds(x) \Phi_0 \circ Ds(-\dot{\Phi}_0) = X(s(x)) - s_\gamma \gamma_s(X)(x) = (\text{ev } X)(j^\infty s), \) where we have used that \( \Phi_0 \) is the identity.

Let us recall the basic definitions of the variational bicomplex theory (see [1] [19] for details). On \( J^\infty E \to M \) we have a bigraduation \( \Omega^k(J^\infty E) = \bigoplus_{p+q=k} \Omega^{p,q}(J^\infty E) \) into horizontal and contact (or vertical) degree. If \( \alpha \in \Omega^p(J^\infty E) \) we denote by \( \alpha_{p,q} \in \Omega^{p,q}(J^\infty E) \) its \( p \)-horizontal and \( q \)-contact component. We denote \( \Omega^{p,q}(J^\infty E) \) simply by \( \Omega^{p,q} \) when there is no risk of confusion. According to the preceding bigraduation we have a decomposition of the exterior differential \( d = d_H + d_V \).

We denote by \( I : \Omega^{n,k} \to \Omega^{n,k} \) the interior Euler operator. We recall that it satisfies the following properties: \( I^2 = I \), \( \ker I = d_H(\Omega^{n-1,k}) \), \( d_V = d_I \). The image of the interior Euler operator \( \mathcal{F}^k = I(\Omega^{n,k}) \) is called the space of functional \( k \)-forms. We have \( \Omega^{n,k} \cong \mathcal{F}^k \oplus d_H(\Omega^{n-1,k}) \), i.e. \( \mathcal{F}^k \cong \Omega^{n,k}/d_H(\Omega^{n-1,k}) \).

The vertical differential \( d_V \) induces a differential in the space of functional forms \( \delta_V : \mathcal{F}^k \to \mathcal{F}^{k+1} \), \( \delta_V \alpha = I(d_V \alpha) \). We have the usual diagram for the augmented variational bicomplex

\[
\begin{array}{cccccc}
\uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta_V \\
\Omega^{0,3} & \rightarrow & \Omega^{1,3} & \rightarrow & \Omega^{2,3} & \rightarrow & \Omega^{3,3} \\
\uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta_V \\
\Omega^{0,2} & \rightarrow & \Omega^{1,2} & \rightarrow & \Omega^{2,2} & \rightarrow & \Omega^{3,2} \\
\uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta_V \\
\Omega^{0,1} & \rightarrow & \Omega^{1,1} & \rightarrow & \Omega^{2,1} & \rightarrow & \Omega^{3,1} \\
\uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta_V \\
\Omega^{0,0} & \rightarrow & \Omega^{1,0} & \rightarrow & \Omega^{2,0} & \rightarrow & \Omega^{3,0} \\
\end{array}
\]

The Euler-Lagrange complex \( \mathcal{E}^\bullet(J^\infty E) \) is the following complex

\[
\begin{array}{cccccccc}
\Omega^{0,0} & \rightarrow & \Omega^{1,0} & \rightarrow & \ldots & \rightarrow & \Omega^{n-1,0} & \rightarrow & \Omega^{n,0} \\
\uparrow d_V & \rightarrow & \uparrow d_V & \rightarrow & \ldots & \rightarrow & \uparrow d_V & \rightarrow & \uparrow d_V \\
\mathcal{F}^1 & \rightarrow & \mathcal{F}^2 & \rightarrow & \ldots \\
\end{array}
\]

We recall that in the jet bundle formulation of the variational calculus a Lagrangian density is an element \( \lambda \in \Omega^{n,0} \), the map \( \delta_V : \Omega^{n,0} \to \mathcal{F}^1 \) is the Euler-Lagrange map (i.e. \( \delta_V \lambda \) is the Euler-Lagrange operator of \( \lambda \)), and the map \( \delta_V : \mathcal{F}^1 \to \mathcal{F}^2 \) is the Helmholtz-Sonin mapping characterizing locally variational
operators. A classical result in the variational bicomplex theory (see e.g. [12]) asserts that \( H^\bullet(E^\bullet(J^\infty E)) \cong H^\bullet(J^\infty E) \cong H^\bullet(E) \). This result is based on the fact that the interior rows of the variational bicomplex are exact.

\[
0 \to \Omega^0,k \xrightarrow{d_H} \Omega^1,k \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^n,k \xrightarrow{\mathcal{I}} F^k \to 0.
\]

## 4 Local forms and the variational bicomplex

The family of maps \( \mathcal{I} \) determine a map \( \mathcal{I} : \Omega^{n+k}(J^\infty(E)) \to \Omega^k(\Gamma(E)) \), and by definition we have \( \Omega_{\text{loc}}^k(\Gamma(E)) = \mathcal{I}(\Omega^{n+k}(J^\infty(E))) \). We now study the relation between the map \( \mathcal{I} \) and the variational bicomplex.

### Proposition 4

For \( \alpha \in \Omega^{n+k}(J^\infty E) \), \( k > 0 \), we have \( \mathcal{I}[\alpha] = \mathcal{I}[\alpha_{n,k}] = \mathcal{I}[I(\alpha_{n,k})] \).

**Proof.** That \( \mathcal{I}[\alpha] = \mathcal{I}[\alpha_{n,k}] \) follows from Proposition 2. As we have \( \alpha_{n,k} = I(\alpha_{n,k}) + d_H\eta \) for certain \( \eta \in \Omega^{n-1,k} \), to prove that \( \mathcal{I}[\alpha_{n,k}] = \mathcal{I}[I(\alpha_{n,k})] \) it is sufficient to prove that \( \mathcal{I}[d_H\eta] = 0 \). As \( d_V\eta \in \Omega^{n-1,k+1} \) we have \( \mathcal{I}[d_V\eta] = 0 \). Hence \( \mathcal{I}[d_H\eta] = \mathcal{I}[d_H\eta] = 0 \), where we have used that \( \mathcal{I}[\eta] = 0 \) because \( \eta \in \Omega^{n-1,k} \).

In [1] (see also [19] section 5.4) another more general interpretation of the functional forms is given. Precisely, every form \( \alpha \in \Omega^{n+k}(J^\infty(E)) \) determines a multilinear map on evolutionary vector fields \( W[\alpha] : \Gamma(E) \times \bigwedge^k \text{Ev}(E) \to \mathbb{R} \), \( W[\alpha](X_1, \ldots, X_k) = \int_{\mathcal{I}(s)} \mathcal{I}^{\alpha}(t_{pr}X_k \ldots t_{pr}X_1) \) for \( s \in \Gamma(E) \) and \( X_1, \ldots, X_k \in \text{Ev}(E) \). The relation between \( \mathcal{I} \) and \( W \) is that we have \( W[\alpha](X_1, \ldots, X_k) = \mathcal{I}[\alpha](\gamma_a(X_1), \ldots, \gamma_a(X_k)) \). Conversely, if \( X_1, \ldots, X_k \in T_s \Gamma(E) \cong \Gamma(s^*\text{Ev}(E)) \) and \( \bar{X}_1, \ldots, \bar{X}_k \) are vertical vector fields on \( E \) extending \( X_1, \ldots, X_k \) then we have \( \mathcal{I}[\alpha](\bar{X}_1, \ldots, \bar{X}_k) = W[\alpha](\bar{X}_1, \ldots, \bar{X}_k) \). Hence \( W[\alpha] \) is completely determined by \( \mathcal{I}[\alpha] \) and vice versa.

In [1] Proposition 3.1 (see also [19] Lemma 5.85) it is proved that \( W[\alpha] = 0 \) if and only if \( I(\alpha_{n,k}) = 0 \). By the preceding considerations we have \( W[\alpha] = 0 \) if and only if \( \mathcal{I}[\alpha] = 0 \). Hence we have the following

### Theorem 5

For \( \alpha \in \Omega^{n+k}(J^\infty E) \), \( k > 0 \), we have \( \mathcal{I}[\alpha] = 0 \) if and only if \( I(\alpha_{n,k}) = 0 \).

Hence the map \( \mathcal{I} \) is uniquely determined by its restriction to the space of functional forms \( F^k(J^\infty E) \) and this restriction \( F^k(J^\infty E) \hookrightarrow \Omega^k(\Gamma(E)) \) is injective.

### Corollary 6

For \( \alpha \in \Omega^{n+k}(J^\infty E) \), \( k > 0 \), we have \( d\mathcal{I}[\alpha] = \mathcal{I}[\delta_V I(\alpha_{n,k})] \), and \( d(\mathcal{I}[\alpha]) = 0 \) if and only if \( \delta_V I(\alpha_{n,k}) = 0 \).

### Corollary 7

For every \( k \geq 1 \) the integration map \( \mathcal{I} \) induces isomorphisms \( F^k(J^\infty E) \cong \Omega_{\text{loc}}^k(\Gamma(E)) \), and \( H^k_{\text{loc}}(\Gamma(E)) \cong H^{n+k}(\mathcal{I}^*E^\bullet(J^\infty E)) \cong H^{n+k}(E) \).
Let us analyze now what happens for $k = 0$. In this case we do not have an interior Euler operator, and hence we can not select a canonical lagrangian for a given local functional. We can define $\mathcal{F}^0(J^\infty E) = \Omega^{n,0}/d_\Omega \Omega^{n-1,0}$, and we have a map $\mathfrak{S} : \mathcal{F}^0(J^\infty E) \rightarrow \Omega^0(\Gamma(E))$. However, this map is not injective. For example (see [1]) we can consider the bundle $E = S^2 \times T^2 \rightarrow S^2$, and $\alpha, \beta \in \Omega^1(T^2)$ generators of $H^1(T^2)$. If we take $\omega = \alpha \wedge \beta \in \Omega^2(E)$, then $0 \neq [\omega] \in H^2(E)$ and hence $\omega$ defines a non trivial element in $\mathcal{F}^0(J^\infty E)$. However we have $\mathfrak{S}[\omega] = 0$, as for every $s \in \Gamma(E)$ we have $\mathfrak{S}[\omega]_s = \int_{S^2} (j^\infty s)^* \omega = \int_{S^2} (j^\infty s)^* \alpha \wedge (j^\infty s)^* \beta = 0$, where the last equality follows as we have $(j^\infty s)^* \alpha = d\eta$ for certain $\eta \in \Omega^1(S^2)$ because $H^1(S^2) = 0$.

Let $\ker(\delta_0) \subset \Omega^{n,0}$ denote the space of trivial lagrangian densities. Clearly we have by definition $\Omega^0_{\text{loc}}(\Gamma(E)) \cong \Omega^{n,0}/\ker(\delta_0)$. Moreover, if $\delta^0_V$ denotes the Euler-Lagrange operator $\delta^0_V : \Omega^{n,0} \rightarrow \mathcal{F}^1$, we have $d_\Omega(\Omega^{n-1,0}) \subset \ker(\delta_0) \subset \ker(\delta^0_V)$. If we define $\mathcal{N} = \ker(\delta_0)/d_\Omega(\Omega^{n-1,0})$ we have a natural inclusion $\mathcal{N} \subset H^n(\mathcal{E}^\bullet(J^\infty E) \cong H^n(E)$, and we have $H^0_{\text{loc}}(\Gamma(E)) \cong \ker(\delta^0_V)/\ker(\delta_0) \cong H^n(E)/\mathcal{N}$. In this way we can identify $\mathcal{N} \cong \bigcap_{s \in \Gamma(E)} \ker(s^* : H^n(E) \rightarrow H^n(M) \cong \mathbb{R}$). If $\mathcal{N}$ vanishes, we have $\Omega^0_{\text{loc}}(\Gamma(E)) \cong \Omega^{n,0}/d_\Omega(\Omega^{n-1,0})$ and $H^0_{\text{loc}}(\Gamma(E)) \cong H^n(E)$. This happens for example if $H^n(E) \cong \mathbb{R}$, as it is easily seen.

We can consider this computation of $H^0_{\text{loc}}(\Gamma(E))$ as a refinement of the inverse problem of the calculus of variations, that is, the computation of the variationally trivial lagrangian densities modulo divergences $\ker(\delta^0_V)/d_\Omega(\Omega^{n-1,0}) \cong H^n(E)$. If in place of lagrangian densities we consider local functionals, and we ask for the local functionals which are closed, then what we obtain is $H^0_{\text{loc}}(\Gamma(E)) = H^n(E)/\mathcal{N}$.

## 5 Local invariant cohomology

Let us assume now that a Lie group $\mathcal{G}$ acts on $E \rightarrow M$ by elements of $\text{Proj}^+ E$. Then we have an induced action of $\mathcal{G}$ on $J^\infty E$ and the variational bicomplex remains invariant under this action. By considering $\mathcal{G}$-invariant forms we obtain the $\mathcal{G}$-invariant variational bicomplex and the $\mathcal{G}$-invariant Euler-Lagrange complex. We define the space of local $\mathcal{G}$-invariant forms $\Omega^k_{\text{loc}}(\Gamma(E))^\mathcal{G}$ as the subspace of $\mathcal{G}$-invariant elements on $\Omega^k_{\text{loc}}(\Gamma(E))$, and the local $\mathcal{G}$-invariant cohomology as the cohomology of this complex. As the integration map $\mathfrak{S}$ is $\mathcal{G}$-equivariant, from Theorem 5 we obtain the following

**Corollary 8** For every $k \geq 1$ the integration map $\mathfrak{S}$ induces isomorphisms $\mathcal{F}^k(J^\infty E)^\mathcal{G} \cong \Omega^k_{\text{loc}}(\Gamma(E))^\mathcal{G}$ and we have $\Omega^0_{\text{loc}}(\Gamma(E))^\mathcal{G} = \mathfrak{S}(\Omega^{n+k}(J^\infty E)^\mathcal{G})$. Moreover, $\mathfrak{S}$ induces isomorphisms $H^k_{\text{loc}}(\Gamma(E))^\mathcal{G} \cong H^{n+k}(\mathcal{E}^\bullet(J^\infty E))^\mathcal{G}$ for $k > 1$.

Let us analyze now what happens for the local invariant cohomology of order 0 and 1. We have $\Omega^0_{\text{loc}}(\Gamma(E))^\mathcal{G} \cong (\Omega^{n,0}/\ker(\delta_0))^\mathcal{G}$, and hence $H^0_{\text{loc}}(\Gamma(E))^\mathcal{G} \cong (\ker(\delta^0_V)/\ker(\delta_0))^\mathcal{G} \cong \{H^n(E)/\mathcal{N}\}^\mathcal{G}$, where $\{H^n(E)/\mathcal{N}\}^\mathcal{G}$ denotes the space of $\mathcal{G}$-invariant elements in $H^n(E)/\mathcal{N}$. In particular, if $\mathcal{N} = 0$ we have $H^0_{\text{loc}}(\Gamma(E)) \cong \{H^n(E)\}^\mathcal{G}$. If $\mathcal{G}$ is connected we clearly have $\{H^n(E)\}^\mathcal{G} \cong H^n(E)$ and hence
Invariance lagrangians, then we have $\Omega^0_\lambda$ that if concrete example of gravitational Chern-Simons terms. The difference between $\lambda$ for $k > 0$, that a lagrangian density $\Omega$ are not $\text{Diff}_2$-invariant. Under very general conditions the interior rows of the $\mathcal{G}$-invariant variational bicomplex are exact. For example this happens if there exists a $\text{Diff}_2$-invariant homotopy operators which can be used to prove the exactness of the interior rows of the $\mathcal{G}$-invariant variational bicomplex

$$0 \rightarrow (\Omega^{0,k})^G \xrightarrow{d_H} (\Omega^{1,k})^G \xrightarrow{d_H} \cdots \xrightarrow{d_H} (\Omega^{n,k})^G \xrightarrow{f} (\mathcal{F}^k)^G \rightarrow 0$$

for $k > 0$. In that case we have isomorphisms for $k > 1$

$$H^0_{\text{loc}}(\Gamma(E))^G \cong H^0(\mathcal{E}^\bullet(J^\infty E))^G \cong H^0(\mathcal{J}^\infty E)^G.$$ (2)

In [3] it is shown that the invariant cohomology $H^{n+k}(J^\infty E)^G$ of the jet bundle can be determined in certain cases in terms of relative Lie algebra cohomology of formal vector fields. We apply this idea in the following sections in order to study the local invariant cohomology of the spaces of Riemannian metrics and connections on principal bundles.
6 Riemannian metrics and diffeomorphisms

6.1 Universal Pontryagin and Euler forms on $J^1\mathcal{M}_M$

Let $M$ be a compact and connected $n$-manifold without boundary, and $TM$ its tangent bundle. We define its bundle of Riemannian metrics $q: \mathcal{M}_M \to M$ by $\mathcal{M}_M = \{g_x \in S^2(T_x^*M) : g_x$ is positive defined on $T_xM\}$. Let $\mathfrak{R}et M = \Gamma(M, \mathcal{M}_M)$ denote the space of Riemannian metrics on $M$. We denote by $\text{Diff}M$ the diffeomorphism group of $M$, by $\text{Diff}^+M$ its subgroup of orientation preserving diffeomorphisms and by $\text{Diff}^sM$ the connected component of the identity in $\text{Diff}M$. We denote by $q_1: J^1\mathcal{M}_M \to M$ the 1-jet bundle of $\mathcal{M}_M$ and by $\pi: FM \to M$ the linear frame bundle of $M$. The pull-back bundle $q_1: q_1^*FM \to J^1\mathcal{M}_M$ is a principal $\text{GL}(n, \mathbb{R})$-bundle and we have the following commutative diagram

$$

\begin{align*}
q_1^*FM & \xrightarrow{\pi} FM \\
\downarrow & \downarrow \\
J^1\mathcal{M}_M & \xrightarrow{q_1} M
\end{align*}

$$

Every system of coordinates $(U; x^i)$ on $M$ induces a system of coordinates $(q^{-1}U; x^i, y_{ij})$ on $\mathcal{M}_M$ by setting $g_x = y_{ij}(g_x)(dx^i)_x \otimes (dx^j)_x, \forall g_x \in \mathcal{M}_M, x \in U$. We denote by $(y_{ij})$ the inverse matrix of $(y_{ij})$. Let $(q^{-1}U; x^i, y_{ij}, y_{ijk}, k)$, be the coordinate system on $J^1\mathcal{M}_M$ induced by $(q^{-1}U; x^i, y_{ij});$ i.e., $y_{ijk}(j^2_xg) = (\partial(y_{ij} \circ g)/\partial x^k)(x)$. Note that if $(U; x^i)$ is a normal coordinate system for the metric $g$ centered at $x$, then we have $y_{ij}(j^2_xg) = \delta_{ij}, y_{ijk}(j^2_xg) = 0$.

The diffeomorphism group of $M$ acts in a natural way on $\mathcal{M}_M$. If $\phi \in \text{Diff}M$, its lift to the bundle of metrics $\bar{\phi}: \mathcal{M}_M \to \mathcal{M}_M$ is defined by $\bar{\phi}(g_x) = (\phi^*)^{-1}(g_x) \in (\mathcal{M}_M)_{\phi(x)}, \phi^*: S^2T^*_xM \to S^2T^*_xM$ being the induced homomorphism. Hence $q \circ \bar{\phi} = \phi \circ q$. In the same way, the lift of a vector field $X \in \mathfrak{X}(M)$ is denoted by $\bar{X} \in \mathfrak{X}(\mathcal{M}_M)$. If $X \in \mathfrak{X}(M)$ is given in local coordinates by $X = X^i \partial/\partial x^i$, then its lift $\bar{X} \in \mathfrak{X}(\mathcal{M}_M)$ to $\mathcal{M}_M$ is given by

$$

\bar{X} = X^i \frac{\partial}{\partial x^i} - \sum_{i \leq j} \left( \frac{\partial X^k}{\partial x^j} y_{kj} + \frac{\partial X^k}{\partial x^j} y_{ik} \right) \frac{\partial}{\partial y_{ij}}. \quad (3)

$$

We consider the principal $SO(n)$-bundle $O^+M \to J^1\mathcal{M}_M$ where $O^+M = \{(j^2_xg, u_x) \in q_1^*FM: u_x$ is $g_x$-orthonormal and positively oriented$\}$. In [20] it is shown that there exists a connection form $\omega \in \Omega^1(O^+M, \mathfrak{so}(n))$ (called the universal Levi-Civita connection) on $O^+M$ invariant under the natural action of the group $\text{Diff}^+M$. Let us recall how this connection is constructed. We define a $\text{Diff}M$-invariant connection $\omega_{\text{hor}} \in \Omega^1(q_1^*FM, \mathfrak{gl}(n))$ on $q_1^*FM \to J^1\mathcal{M}_M$ by setting $\omega_{\text{hor}}(X) = \omega^9((\bar{q}_1)_*X)$, for every $X \in T_{(j^2_xg, u_x)}(q_1^*FM)$, where $\omega^9$ denotes the Levi-Civita connection of the metric $g$. The connection $\omega_{\text{hor}}$ is not reducible to the $SO(n)$-bundle $O^+M \to J^1\mathcal{M}_M$, but in [20] it is shown that it is possible to obtain a reducible and $\text{Diff}^sM$-invariant connection $\omega$ by adding to $\omega_{\text{hor}}$ a contact form $\frac{1}{2} \theta \in \Omega^1(J^1\mathcal{M}_M, \text{End}TM)$. We denote by $\Omega$ and $\Omega_{\text{hor}}$ the
curvature form of $\omega$ and $\omega_{\text{hor}}$ respectively. In local coordinates, the expressions of $\Omega_{\text{hor}}$ and $\vartheta$ are given by (see \cite{20})

$$
(\Omega_{\text{hor}})_j^i = d\Gamma_{jk}^i + \Gamma_{ia}^j \Gamma_{jk}^a, \quad (4)
$$

$$
\vartheta_j^i = y^{ia} (dy_{aj} - y_{aj,k} dx^k), \quad (5)
$$

where $\Gamma_{jk}^i = \frac{1}{2} y^{ia} (y_{aj,k} + y_{ak,j} - y_{jk,a})$.

For any Weil polynomial $p \in \mathcal{P}^{SO(n)}$ the universal characteristic form $p(\Omega) \in \Omega^{4k}(J^1 \mathcal{M})$ corresponding to $p$ is defined as the form obtained by means of the Chern-Weil theory of characteristic classes by applying $p$ to the curvature $\Omega$ of the universal Levi-Civita connection $\omega$. In particular we have the universal $k$-th Pontryagin form of $\mathcal{M}$, $p_k(\Omega) \in \Omega^{4k}(J^1 \mathcal{M})$ and, for $n$ even, the universal Euler form $\chi(\Omega) = \frac{1}{2\pi} \text{Pf}(\Omega) \in \Omega^n(J^1 \mathcal{M})$, where $\text{Pf}$ denotes the Pfaffian. These forms are closed, $\text{Diff}^+ \mathcal{M}$-invariant and satisfy the following universal property (see \cite{20}): for every Riemannian metric $g$ we have $(j^1 g)^*(p(\Omega)) = p(\Omega^g)$, where $\Omega^g \in \Omega^{2}(\mathcal{M}, \text{End} \mathcal{TM})$ is the curvature form of the Levi-Civita connection of the metric $g$. Hence the Pontryagin forms of degree less or equal than $n$ determine the Pontryagin classes of $\mathcal{M}$, while the Pontryagin forms of degree greater than $n$ determine, by means of $\mathfrak{g}$, closed $\text{Diff}^+ \mathcal{M}$-invariant forms on $\mathfrak{Met} \mathcal{M}$.

6.2 Local invariant cohomology of $\mathfrak{Met} \mathcal{M}$

The local cohomology of $\mathfrak{Met} \mathcal{M}$ is easily computed. We have $H^k_{\text{loc}}(\mathfrak{Met} \mathcal{M}) \cong H^{n+k}(J^\infty \mathcal{M}) \cong H^{n+k}(\mathcal{M}) \cong H^{n+k}(M) = 0$ for $k > 0$. Moreover, as $H^n(\mathcal{M}) \cong H^n(M) = \mathbb{R}$, by the results explained in Section 5 we have $H^0_{\text{loc}}(\mathfrak{Met} \mathcal{M}) \cong \mathbb{R}$.

Hence $p(\Omega)$ is an exact form for every $p \in \mathcal{P}^{SO(n)}$ with $2r > n$. In fact it is easy to construct explicitly a form $\tau$ satisfying $p(\Omega) = d\tau$ by fixing a metric $g_0 \in \mathfrak{Met} \mathcal{M}$. Let $\omega^{g_0}$ be the Levi-Civita connection of $g_0$, considered as a connection on the frame bundle $F \mathcal{M}$. The connections $q_1^* \omega^{g_0}$ and $\omega$ are both connections on the same bundle $q_1^* F \mathcal{M} \to J^1 \mathcal{M}$, and hence we have $d(T p(\omega, q_1^* \omega^{g_0})) = p(\Omega) - p(q_1^* \Omega^{g_0}) = p(\Omega)$, where $T p(\omega, q_1^* \omega^{g_0}) \in \Omega^{n+1}(J^\infty \mathcal{M})^{\text{Diff}^+ \mathcal{M}}$ is the transgression form corresponding to $\omega$ and $\omega^{g_0}$ (see \cite{17}), and we have used that $p(q_1^* \Omega^{g_0}) = q_1^* p(\Omega^{g_0}) = 0$ by dimensional reasons.

As the connection $\omega_{\text{hor}}$ determines a torsion free connection on the space of total vector fields on $J^\infty \mathcal{M}$, by the results explained in Section 5 we have the following

**Proposition 9** For every $k > 1$ the integration map $\int$ induces isomorphisms

$$
H^k_{\text{loc}}(\mathfrak{Met} \mathcal{M})^{\text{Diff}^+ \mathcal{M}} \cong H^{n+k}(J^\infty \mathcal{M})^{\text{Diff}^+ \mathcal{M}}.
$$

This result is valid as well for the group $\text{Diff}^+ \mathcal{M}$. In Remark 12 we show that this result is not true for $k = 1$.

The invariant cohomology $H^{n+k}(J^\infty \mathcal{M})^{\text{Diff}^+ \mathcal{M}}$ of the jet bundle can be related to relative Gelfand-Fuchs cohomology of of formal vector fields. As
If a Lie group \( G \) acts on \( W \), the curvature form of \( g \) is a Diff(\( M \))-bundle. A connection form \( \omega \) on \( M \) we obtain a map \( \alpha : H(W(g), g) \to \Omega(M)pG \). Finally, by composing with the projection \( q_\infty,1 : J^\infty M \to J^1 M \) we obtain a map \( \alpha : H(W(g), g) \to \Omega(J^\infty M)Diff^rM \).}

**Theorem 10** The map \( \alpha : H(W(g), g) \to \Omega(J^\infty M)Diff^rM \) is injective.

The proof of Theorem 10 is given in Section 5.4 and is based on the ideas explained in 3 relating the cohomology of invariant variational bicomplexes to relative cohomology of formal vector fields.

The cohomology \( H(W(g), g) \) is well known due to its appearance in the cohomology of formal vector fields and characteristic classes of foliations (e.g. see 13, 16). Let \( W_{Ou} = \bigwedge (U_1, U_3, \ldots, U_{2k-1}) \otimes S_n[C_1, C_2, \ldots, C_n] \), where \( 2k-1 \) is the greater odd number \( \leq n \), \( \deg(U_i) = 2i-1 \), \( \deg(C_i) = 2i \), and \( S_n[C_1, C_2, \ldots, C_n] \) is the quotient of \( S[C_1, C_2, \ldots, C_n] \) by the ideal \( J \) generated by the elements of degree greater than \( 2n \). \( W_{Ou} \) is a differential graded algebra (DGA) with differential \( dU_i = C_i, dC_i = 0 \). We have (see 16) \( H(W(g), g) = H(W_{Ou}) \) for odd, and \( H(W_{Ou}) = H(W_{Ou})[T]/(T^2 - C_n) \) for even. If we set \( P_i = C_{2i} \), it is easy to see that for \( r \leq 2n \) we have \( H^r(W_{Ou}) \cong S[P_1, \ldots, P_{[n/2]}] \), the space of degree \( r \) elements on \( S[P_1, \ldots, P_{[n/2]}] \). Hence, we conclude that for \( r \leq 2n \) we have
The class of $\chi$ is the transgression form corresponding to $\omega$. Hence, for $p \leq n$, we have $\omega p$ is not zero, we conclude that the class of $\omega$ and $\omega$ form on $J^\infty M$ if and only if $p = 0$.

**Proof.** In $q_1^* FM$ we have considered two $Diff M$-invariant connections, $\omega_{\text{hor}}$ and $\omega$, the second one being a Riemannian connection. If $p \in I_k^{SO(n)}$ then we have $\omega(p) - p(\omega_{\text{hor}}) = d(Tp(\omega, \omega_{\text{hor}}))$, where $T(p(\omega, \omega_{\text{hor}})) \in \Omega^{2p-1}(J^\infty M)^{Diff M}$ is the transgression form corresponding to $\omega_{\text{hor}}$ and $\omega$ (see [17]). As $\omega_{\text{hor}}$ and $\omega$ are both $Diff M$-invariant, the form $T(p(\omega, \omega_{\text{hor}}))$ is also $Diff M$-invariant. Hence, the forms $p(\omega)$ and $p(\omega_{\text{hor}})$ determine the same cohomology class on $H^*(J^\infty M)^{Diff M}$. As we know that the class of $p(\omega_{\text{hor}})$ on $H^*(J^\infty M)^{Diff M}$ is not zero, we conclude that the class of $p(\omega)$ is also not zero. Moreover, for $n$ odd the class of $\chi(p)$ is also not zero as we have $\langle \chi(p) \rangle = [p_{n/2}(\omega)] \neq 0$. \hfill \blacksquare

**Remark 12** We can use Corollary [13] to show that, as commented before, in general the map $\exists: H^{n+1}(J^\infty M)^{Diff M} \to H^1_{loc}(\mathfrak{Met}M)^{Diff M}$ is not an isomorphism.

Let us suppose that $q = 4k - 1$ for an integer $k$. Let $p \in I_k^{SO(n)}$ and consider the corresponding universal Pontryagin form $p(\omega) \in \Omega^{n+1}(J^\infty M)^{Diff M}$. By Theorem [13] the class of $p(\omega)$ in $H^{n+1}(J^\infty M)^{Diff M}$ is not zero. However, the class of $\exists(p(\omega))$ in $H^1_{loc}(\mathfrak{Met}M)^{Diff M}$ vanishes.

This can be seen in the following way. Let $\alpha \in \Omega^p(J^\infty M)$ be a form satisfying $p(\omega(p)) = d\alpha$. Of course $\alpha$ is not $Diff M$-invariant, but it is weakly $Diff M$-invariant, as for every $X \in X(M)$ we have $L_{prX}\alpha = d(\mathfrak{J}(X) + prX \alpha)$, where we have used that $t_{prX} p(\omega(p)) = d(\mathfrak{J}(X))$ for certain $\mathfrak{J}(X) \in \Omega^{p-1}(J^\infty M)$. This fact follows from the existence of equivariant Pontryagin classes (see [11]). For example for $n = 3$ and $p = 1$ the first Pontryagin polynomial we can take (see [18] formula (8))), $\mathfrak{J}(X) = \frac{1}{12} \mathrm{tr} (\mathfrak{V}X) \circ \omega$, where $(\mathfrak{V}X) \circ \omega$ denotes the skew-symmetric part of $\mathfrak{V}X \in \Omega^3(J^\infty M, \text{End}T M)$ and in local coordinates we have $\mathfrak{V}X = (\frac{\partial X^i}{\partial x^j} - \Gamma^{i}_{jk} X^k) dx^j \otimes \frac{\partial}{\partial x^i}$.

**Remark 13** The map $\alpha$ of Theorem [13] is in fact bijective ([2]), i.e., all the cohomology classes on $H(J^\infty M)^{Diff M}$ come from $H(W(n)) \otimes \mathfrak{so}(n))$. Using this result, we obtain that $H^1_{loc}(\mathfrak{Met}M)^{Diff M} \cong H^{n+k}(W(n)) \otimes \mathfrak{so}(n))$ for $k > 1$. Also we have $H^0_{loc}(\mathfrak{Met}M)^{Diff M} = \mathbb{R}$, and for $k = 1$ the preceding remark shows that $H^1_{loc}(\mathfrak{Met}M)^{Diff M} = 0$. However, we confine ourselves to prove Theorem [13]. Note that this result is sufficient for the study of gravitational anomalies done in [9].

### 6.3 Gelfand-Fuchs cohomology

Let us recall some basic results about Gelfand-Fuchs cohomology of formal vector fields. We refer to [10] [13] for the details. Let $a_n = \{ X = X^i \partial/\partial x^i : X^i \}$
\[ \mathbb{R}[\{x_1, \ldots, x_n\}] \] be the Lie algebra of formal vector fields on \( \mathbb{R}^n \), with Lie bracket

\[
\left[ X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j} \right] = \left( X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}.
\]

The Lie algebra \( \mathfrak{gl}(n, \mathbb{R}) \) can be considered as a subalgebra of \( a_n \) by the map \( \mathfrak{gl}(n, \mathbb{R}) \to a_n, \) \( a_j \mapsto a_j \partial / \partial x^j \). We define \( a_n^* \) as the space generated by

\[
\theta^i(0) = X^i(0), \quad \theta^j_{\ell_1 \cdots \ell_k}(X) = (-1)^k \frac{\partial^r X^i}{\partial x^{\ell_1} \cdots \partial x^{\ell_k}}(0),
\]

and we set \( R^j_i = d\theta^i_j + \theta^i_{\ell_1} \wedge \theta^i_{\ell_2} \). It can be seen that we have \( R^j_i = \theta^r \wedge \theta^r_{j\ell} \).

A DGA’s homomorphism \( W(n) \mathfrak{gl}(n)) \to \bigwedge a_n^* \) is defined by mapping \( \lambda^j_i \) to \( \theta^j_i \) and \( \Lambda^j_i \) to \( R^j_i \). This map induces isomorphisms in cohomology \( H(W(n) \mathfrak{gl}(n)) \cong H(\mathfrak{a}_n) \) and \( H(W(n) \mathfrak{gl}(n)), \mathfrak{so}(n)) \cong H(\mathfrak{a}_n, \mathfrak{so}(n)). \)

**6.4 Proof of Theorem 10**

Let us consider a coordinate system \( (U; x^i) \). Using this local chart, for any \( x \in M, X \in \mathfrak{X}(M) \) we can identify \( j_x^\infty X \) with an element of \( a_n \). Let us consider a point \( \sigma = j_x^\infty g \in J^\infty \mathcal{M}_M \). For simplicity we take \( \sigma = j_x^\infty (g_0) \), where \( g_0 = \sum_i dx^i \otimes dx^i \), and hence we have \( y_{ij}(\sigma) = \delta_{ij}, y_{ij,k}(\sigma) = 0 \) for every multiindex \( J \). We define a map \( \nu_\sigma : a_n \to T_\sigma J^\infty \mathcal{M}_M \) in the following way. Given \( Y \in a_n \) let \( X \in \mathfrak{X}(M) \) be such that \( j_x^\infty X = Y \). Then we set \( \nu_\sigma(Y) = \text{pr}(X) \), which is well defined as \( \text{pr}(X) \) only depends on the derivatives of \( X \) at \( x \). From \( 10 \) and \( 8 \) it follows that the kernel of the map \( \nu_\sigma \) is identified with the Lie subalgebra \( \mathfrak{so}(n) \subset \mathfrak{a}_n\).

According to \( 9 \) we define a map

\[
\psi_\sigma : \Omega^k(J^\infty \mathcal{M}_M)^\text{Diff}^+ M \to \Omega^k(\mathfrak{a}_n, \mathfrak{so}(n))
\]

\[
\psi_\sigma(\alpha)(Y_1, \ldots, Y_k) = (-1)^k \alpha(\nu_\sigma(Y_1), \ldots, \nu_\sigma(Y_k)),
\]

for \( \alpha \in \Omega^{n+2}(J^\infty \mathcal{M}_M)^\text{Diff}^+ M \), and \( Y_1, \ldots, Y_{n+2} \in a_n \). It is a cochain map and induces a map in cohomology \( \psi_\sigma : H^k(J^\infty \mathcal{M}_M)^\text{Diff}^+ M \to H^k(\mathfrak{a}_n, \mathfrak{so}(n)) \). We have the following

**Proposition 14** The following diagram is commutative

\[
\begin{array}{ccc}
H(W(n) \mathfrak{gl}(n)), \mathfrak{so}(n)) & \xrightarrow{\alpha} & H(\mathfrak{so}(n)) \\
\downarrow^{\beta} & & \downarrow^{\psi} \\
H(J^1 \mathcal{M}_M)^{\text{Diff}^+ M} & \xrightarrow{\psi} & H(\mathfrak{a}_n, \mathfrak{so}(n)).
\end{array}
\]

As the map \( \beta \) is an isomorphism, we conclude that \( \alpha \) is injective and \( \psi \) is surjective, proving Theorem 10.

**Proof.** As vector spaces we have \( \mathfrak{gl}(n) \cong \mathfrak{so}(n) \oplus \text{sym}(n) \), where \( \text{sym}(n) \) denotes the space of symmetric matrices of order \( n \). Hence we have \( W(\mathfrak{gl}(n)) \cong \mathfrak{so}(n) \oplus \mathfrak{gl}(n) \).
\[\wedge so(n)^* \otimes \wedge sym(n)^* \otimes gl(n)^*.\] By definition, the space of \(so(n)\)-horizontal elements of \(W(gl(n))\) is \(\wedge sym(n)^* \otimes gl(n)^*\). If \(\lambda^S\) denotes the symmetric part of \(\lambda\), the map \(\beta\) maps \(q(\lambda^S, \Lambda) \in W(gl(n))_{\text{basic}} \) to \(q(\theta^S, R)\)

We denote by \(\omega^A_{\text{hor}} \in \Omega^1(q_1^* FM, so(n))\) and \(\omega^S_{\text{hor}} \in \Omega^1(q_1^* FM, \text{sym}(n))\) the skew-symmetric and symmetric parts of the connection \(\omega_{\text{hor}} \in \Omega^1(q_1^* FM, gl(n))\) respectively. As we have \(\omega_{\text{hor}} = \omega - \frac{1}{2} \vartheta\) and \(\omega \in \Omega^1(q_1^* FM, so(n)), \vartheta \in \Omega^1(q_1^* FM, \text{sym}(n))\), we clearly have \(\omega^A_{\text{hor}} = \omega, \omega^S_{\text{hor}} = -\frac{1}{2} \vartheta\).

By the definition of \(\alpha\), \(q(\lambda^S, \Lambda) \in W(gl(n))_{\text{basic}}\) is mapped by \(\alpha\) to \(q(\omega^S_{\text{hor}}, \Omega_{\text{hor}})\). From formulas (5), (4) and (3) it follows the following

**Lemma 15** If \(X = X^i \partial/\partial x^i, Y = Y^i \partial/\partial x^i\) are the local expressions of two vector fields on \(M\), then we have

\[(\Omega_{\text{hor}})_{j \leftarrow \vartheta}(pr \bar{X}, pr \bar{Y}))^i_j = X^r(x) \frac{\partial^2 Y^i}{\partial x^j \partial x^r}(x) - Y^r(x) \frac{\partial^2 X^i}{\partial x^j \partial x^r}(x) = R^i_j(X, Y).
\]

\[(\omega^S_{\text{hor}})_{j \leftarrow \vartheta}(pr \bar{X}))^i_j = -\frac{1}{2} (\vartheta_{j \rightarrow \vartheta}(pr \bar{X}))^i_j = -\frac{1}{2} \left( \frac{\partial X^j}{\partial x^i}(x) + \frac{\partial X^i}{\partial x^j}(x) \right) = -(\theta^S)^i_j(X).
\]

We conclude from the preceding Lemma that \(\psi_\sigma(q(\omega^S_{\text{hor}}, \Omega_{\text{hor}})) = q(\theta^S, R)\), and this proves Proposition 13.

7 Connections and metrics

7.1 The universal characteristic forms on the bundle of connections

Let \(\pi: P \rightarrow M\) a principal \(G\)-bundle over a compact \(n\)-manifold \(M\). We denote by \(A_P\) the space of principal connections on \(P\). In order to apply our general constructions about local cohomology to the case of connections on principal bundles we consider a bundle (the bundle of connections) \(p: C(P) \rightarrow M\) whose global sections correspond to principal connections on \(P\), i.e., we have \(A_P \cong \Gamma(M, C(P))\). Let us recall the definition of this bundle (see [14], [23], [27] for details). Let \(\bar{p}: J^1 P \rightarrow P\) be the first jet bundle of \(P\). The action of \(G\) on \(P\) lifts to an action on \(J^1 P\). We denote by \(p: C(P) = J^1 P/G \rightarrow M = P/G\) the quotient bundle, called the bundle of connections of \(P\). The projection \(\bar{\pi}: J^1 P \rightarrow C(P)\) is a principal \(G\)-bundle, isomorphic to the pull-back bundle \(p^* P \rightarrow C(P)\), that we denote by \(\bar{\pi}: \bar{P} \rightarrow C(P)\). We have the following commutative diagram

\[
\begin{array}{ccc}
P & \xrightarrow{\bar{p}} & P \\
\bar{\pi} \downarrow & & \downarrow \pi \\
C(P) & \xrightarrow{p} & M
\end{array}
\]

The map \(\bar{p}\) is \(G\)-equivariant, i.e., is a principal \(G\)-bundle morphism.

The group \(\text{Aut} P\) of principal \(G\)-bundle automorphisms is denoted by \(\text{Aut} P\). If \(\phi \in \text{Aut} P\), we denote by \(\phi \in \text{Diff} M\) its projection to \(M\). We denote by
Aut^+ P the subgroup of elements \( \phi \in Aut P \) such that \( \phi \in \text{Diff}^+ M \). The kernel of the projection \( Aut P \rightarrow \text{Diff} M \) is the gauge group of \( P \), denoted by \( \text{Gau} P \).

The Lie algebra of \( Aut P \) can be identified with the space \( \text{aut} P \subset \mathfrak{X}(P) \) of \( G \)-invariant vector fields on \( P \). The subspace of \( G \)-invariant vertical vector fields is denoted by \( \text{gau} P \) and can be considered as the Lie algebra of \( \text{Gau} P \). We have an exact sequence of Lie algebras \( 0 \rightarrow \text{gau} P \rightarrow \text{aut} P \rightarrow \mathfrak{X}(M) \rightarrow 0 \).

The action of \( Aut P \) on \( P \) induces actions on \( J^1 P \) and \( C(P) \), and the maps \( \tilde{\pi} \) and \( \bar{\rho} \) are \( Aut P \)-invariant. At the infinitesimal level, if \( X \in \text{aut} P \), we denote by \( X \in \mathfrak{X}(M) \) its projection to \( M \), and by \( X_\pi \in \mathfrak{X}(P) \), \( X_{C(P)} \in \mathfrak{X}(C(P)) \) its lifts to \( P = J^1 P \) and \( C(P) \) respectively.

Let \((U, x^i)\) be a local coordinate system on \( M \), \((B^\alpha)\) a basis for \( \mathfrak{g} \). If \( \tilde{B}_\alpha \) denotes the \( G \)-invariant vector field \((B_\alpha)_P\), then for every \( X \in \text{aut} P \) we have \( X = f^i \partial/\partial x^i + g^\alpha \tilde{B}_\alpha \), with \( f^i, g^\alpha \in C^\infty(U) \). Let \((p^{-1}U, x^i, A^\gamma_\alpha)\) be the induced coordinate system on \( C(P) \) (see [14] Section 3.2). If \( X \in \text{aut} P \) is given in local coordinates by \( X = f^i \partial/\partial x^i + g^\alpha \tilde{B}_\alpha \) then we have

\[
X_{C(P)} = f^j \frac{\partial}{\partial x^j} - \left( \frac{\partial f^i}{\partial x^j} A^\gamma_i + \frac{\partial g^\alpha}{\partial x^j} - c^\beta_{\gamma\alpha} g^\beta A^\gamma_j \right) \frac{\partial}{\partial A^\gamma_j},
\]  

where \( c^\beta_{\gamma\alpha} \) are the structure constants of \( \mathfrak{g} \).

The principal \( G \)-bundle \( \tilde{\pi} : P \rightarrow C(P) \) is endowed with a canonical \( Aut P \)-invariant connection \( \mathfrak{a} \in \Omega^1(P, \mathfrak{g}) \). This connection can be identified to the contact form on \( J^1 P \). Alternatively, it can be defined by setting \( \mathfrak{a}_{(\sigma_A(x), u)}(X) = A_u(p_* X) \), for every connection \( A \) on \( P \), \( x \in M \), \( u \in \pi^{-1}(x) \), \( X \in T_{(\sigma_A(x), u)} P \), and where \( \sigma_A : M \rightarrow C(P) \) is the section of \( C(P) \) corresponding to \( A \). Let \( \mathcal{F} \) be the curvature of \( \mathfrak{a} \). In local coordinates we have (see [14])

\[
\mathcal{F} = \left( dA^\gamma_j \wedge dx^j + c^\beta_{\gamma\alpha} A^\beta_k dx^j \wedge dx^k \right) \otimes \tilde{B}_\alpha.
\]  

If \( f \in I^G_k \) is a Weil polynomial of degree \( k \) for \( G \), we define the universal characteristic form associated to \( f \) as the 2-form on \( C(P) \) defined by \( f(\mathcal{F}, \ldots, \mathcal{F}) \in \Omega^{2k}(C(P)) \).

### 7.2 Local cohomology of \( \text{Met}_M \times A_P \)

Now we consider the product bundle \( \mathcal{M}_M \times_M C(P) \), whose space of sections is the product \( \text{Met}_M \times A_P \). The group \( Aut(P) \) acts on \( C(P) \) as explained above, and acts on \( \mathcal{M}_M \) through its projection \( Aut(P) \rightarrow \text{Diff}(M) \).

The connection \( \omega_{\text{hor}} \) determines a torsion free \( Aut^+ P \)-invariant connection on the space of total vector fields on \( J^\infty(\mathcal{M}_M \times_M C(P)) \), and by the results explained in Section 5 we have for \( k > 1 \) the isomorphism

\[
H^k_{\text{loc}}(\text{Met}_M \times A_P)^{\text{Aut}^+ P} \cong H^{n+k}(J^\infty(\mathcal{M}_M \times_M C(P)))^{\text{Aut}^+ P}.
\]  

As the connection \( \omega_{\text{hor}} \times \mathfrak{a} \) is \( Aut(P) \)-invariant, it determines a homomorphism \( W(\mathfrak{g}(n) \times \mathfrak{g}) \rightarrow \Omega(q_1^*FM \times P)^{\text{Aut}^+ P} \). By formulas 1 and 7 it factors to a map \( W(\mathfrak{g}(n) \times \mathfrak{g}) \rightarrow \Omega(q_1^*FM)^{\text{Aut}^+ M} \). By composing with the inclusion of
Corollary 18 The map
\[
\alpha: H(W(n) \langle gl(n) \times g \rangle, \mathfrak{so}(n) \times g) \rightarrow H(J^\infty(\mathcal{M}_M \times M C(P)))^{Aut_P}
\]
\[\text{is injective.}\]

The proof of this Theorem is similar to that of Theorem 13 and is given in Section 7.3.

The cohomology \(H(W(n) \langle gl(n) \times g \rangle, \mathfrak{so}(n) \times g)\) is computed in [14]. In particular for \(k \leq n\) we have \(H^{2k}(W(n) \langle gl(n) \times g \rangle, \mathfrak{so}(n) \times g) \cong \bigoplus_{r+s=k} I_r^G \otimes I_s^G\).

Hence we have the following

Corollary 17 The map \(\bigoplus_{r+s=k} I_r^G \otimes I_s^G \rightarrow H^{2k}(J^\infty(\mathcal{M}_M \times M C(P)))^{Aut_P}\), \(p \otimes f \mapsto [p(\Omega) \wedge f(\mathcal{F})]\) is injective for \(k \leq n\).

We have also the following

Corollary 18 The map \(I_k^G \rightarrow H^{2k}(J^\infty(C(P)))^{Aut_P}\), \(f \mapsto [f(\mathcal{F})]\) is injective for \(k \leq n\).

### 7.3 Cohomology of formal \(G\)-invariant vector fields

Let \(G\) be a Lie group with Lie algebra \(\mathfrak{g}\). We consider a basis \(B^a\) of \(\mathfrak{g}\), and we denote by \(c^a_{\beta\gamma}\) the structure constants of \(\mathfrak{g}\) in this basis.

Let \(\mathfrak{a}_{n,\mathfrak{g}} = \{f^i \partial/\partial x^i + g^a B_a : f^i, g^a \in \mathbb{R}[[x_1, \ldots, x_n]]\}\) be the Lie algebra of formal \(G\)-invariant vector fields on \(\mathbb{R}^n \times G\), with Lie bracket given by

\[
\begin{align*}
[f^i \frac{\partial}{\partial x^i}, k^j \frac{\partial}{\partial x^j}] &= \left( f^j \frac{\partial k^i}{\partial x^j} - k^j \frac{\partial f^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}, \\
\left[ f^i \frac{\partial}{\partial x^i}, g^a B_a \right] &= f^i \frac{\partial g^a}{\partial x^i} B_a, \quad [g^a B_a, h^\alpha B_\alpha] = c^a_{\beta\gamma} g^\beta h^\gamma B_\alpha,
\end{align*}
\]

We define \(\mathfrak{a}_{n,\mathfrak{g}}^*\) as the space generated by

\[
\begin{align*}
\theta^i(X) &= f^i(0), \quad \theta^i (x^i) = (-1)^k \frac{\partial^k f^i}{\partial x^i \partial x^{i_1} \cdots \partial x^{i_k}}(0), \\
\sigma^a(X) &= g^a(0), \quad \sigma^a (x^i) = (-1)^k \frac{\partial^k g^a}{\partial x^i \partial x^{i_1} \cdots \partial x^{i_k}}(0),
\end{align*}
\]

and we set \(R_j^i = d\theta^i + \theta^i \wedge \theta_j^k\) and \(S^a = d\sigma^a + \frac{1}{2} c^a_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma\). We have (see [13])

\[
\begin{align*}
R_j^i &= \theta^k \wedge \theta_{jk}^i, \quad dR_j^i + \theta^i \wedge R_j^k - R_k^j \wedge \theta^i = 0, \quad (10) \\
S^a &= \theta^i \wedge \sigma^a_i, \quad dS^a + c^a_{\beta\gamma} \wedge \sigma^\beta \wedge \sigma^\gamma = 0. \quad (11)
\end{align*}
\]
The Lie algebra $\mathfrak{gl}(n)$ is considered as a Lie subalgebra of $\mathfrak{a}_{n,\mathfrak{g}}$ through the map $\mathfrak{gl}(n) \to \mathfrak{a}_{n,\mathfrak{g}}$, $A^i_j \mapsto A^i_j x^j \partial / \partial x^i$, and similarly $\mathfrak{g}$ is considered as a Lie subalgebra of $\mathfrak{a}_{n,\mathfrak{g}}$ with the obvious map.

The Lie algebra cohomology of $\mathfrak{a}_{n,\mathfrak{g}}$ relative to a Lie subalgebra $\mathfrak{h}$ is the cohomology of the subcomplex of $\mathfrak{h}$-basic elements in $\wedge \mathfrak{a}_{n,\mathfrak{g}}$, and is denoted by $H(\mathfrak{a}_{n,\mathfrak{g}}, \mathfrak{h})$. As usual, the cohomology of $\mathfrak{a}_{n,\mathfrak{g}}$ can be computed in terms of the cohomology of truncated Weil algebras. We consider the Weil algebra $W(\mathfrak{gl}(n) \times \mathfrak{g})$ of the Lie algebra $\mathfrak{gl}(n) \times \mathfrak{g}$. By formulae (10) and (11) we have a map $\beta : W(n)\mathfrak{gl}(n, \mathbb{R}) \times \mathfrak{g} \to \wedge \mathfrak{a}_{n,\mathfrak{g}}$ defined by setting $\beta(\lambda^j_i) = \theta^j_i$, $\beta(\Lambda^i_{ji}) = R^i_j$, $\beta(\lambda^n) = \sigma^n$ and $\beta(\Lambda^n) = S^n$. In [14] it is proved that the induced map on relative cohomology $\beta : H^k(\mathfrak{a}_{n,\mathfrak{g}}, \mathfrak{so}(n) \times \mathfrak{g}) \to H^k(W(n)\mathfrak{gl}(n, \mathbb{R}) \times \mathfrak{g}), \mathfrak{so}(n) \times \mathfrak{g}$ is an isomorphism.

### 7.4 Proof of Theorem 16

Let us fix a local trivialization $(\pi^{-1}(U), x^i, u^\alpha)$ of $P$ such that $x^i(x) = 0$ and a point $\sigma = (j^\infty \mathfrak{g}, j^\infty \mathfrak{h}) \in J^\infty(\mathcal{M}_M \times_M C(P))$. Again for simplicity we assume that $y_{ij}(j^\infty \mathfrak{g}) = \delta_{ij}$, $y_{ij}(j^\infty \mathfrak{h}) = 0$, and $A_{ij}^k(j^\infty \mathfrak{h}) = 0$, for every multiindex $J$. Using this trivialization, for any $X \in \operatorname{aut} P$ we can identify $j^\infty \mathfrak{h}$ with an element of $\mathfrak{a}_{n,\mathfrak{g}}$. We define a map $\nu_{\sigma} : \mathfrak{a}_{n,\mathfrak{g}} \to T_{\sigma}J^\infty(\mathcal{M}_M \times_M C(P))$ in the following way. Given $Y \in \mathfrak{a}_{n,\mathfrak{g}}$ let $X \in \operatorname{aut} P$ be such that $j^\infty X = Y$. Then we set $\nu_{\sigma}(Y) = \operatorname{pr}(X, X_{C(P)})(\sigma)$, which is well defined as $\operatorname{pr}(X, X_{C(P)})(\sigma)$ only depends on the derivatives of $X$ at $x$. From (11), (3) and (6) if follows that the kernel of the map $\nu_{\sigma}$ is identified with the Lie subalgebra $\mathfrak{so}(n) \times \mathfrak{g} \subset \mathfrak{a}_{n,\mathfrak{g}}$.

According to [3] we define a map

$$\psi_{\sigma} : H^k(J^\infty(\mathcal{M}_M \times_M C(P)))^{\operatorname{Aut} P} \to H^k(\mathfrak{a}_{n,\mathfrak{g}}, \mathfrak{so}(n) \times \mathfrak{g})$$

by setting $\psi_{\sigma}(\alpha)(Y_1, \ldots, Y_k) = (-1)^k \alpha(\nu_{\sigma}(Y_1), \ldots, \nu_{\sigma}(Y_k))$ for $Y_1, \ldots, Y_k \in \mathfrak{a}_{n,\mathfrak{g}}$, and $\alpha \in \Omega^k(\mathcal{M}_M \times_M C(P)))^{\operatorname{Aut} P}$.

**Proposition 19** The following diagram is commutative

$$
\begin{array}{ccc}
H(W(n)\mathfrak{gl}(n) \times \mathfrak{g}), \mathfrak{so}(n) \times \mathfrak{g}) \\
\alpha \downarrow \quad \beta \\
H(J^\infty(\mathcal{M}_M \times C(P)))^{\operatorname{Aut} P} & \xrightarrow{\psi} & H(\mathfrak{a}_{n,\mathfrak{g}}, \mathfrak{so}(n) \times \mathfrak{g})
\end{array}
$$

As the map $\beta$ is an isomorphism, we conclude that $\alpha$ is injective, proving Theorem [16].

The proof of Proposition [19] is the same than that of Proposition [14] using Lemma [17] and the following lemma, that shows that $\psi_{\sigma}([p(\Omega_{n,\mathfrak{g}}) \wedge f(E)]) = [p(R) \wedge f(S)]$.

**Lemma 20** If $X = f_1 \partial / \partial x^i + g_1^\alpha \bar{B}_\alpha$, $Y = f_2 \partial / \partial x^i + g_2^\alpha \bar{B}_\alpha$, is the local expression of $X, Y \in \operatorname{aut} P$, we have

$$F(X_{C(P)}, Y_{C(P)}) = (f_1 g_2^\alpha / \partial x^i - f_2 g_1^\alpha / \partial x^i) \otimes \bar{B}_\alpha.$$
Acknowledgement 21 I would like to thank I. Anderson for letting me know some of his unpublished results on the cohomology of invariant variational bicomplexes, and to P. Martínez Gadea for calling my attention to reference [14]. This work is supported by Ministerio de Educación y Ciencia of Spain, under grant #MTM2005–00173.

References

[1] Anderson, I.: The Variational Bicomplex, preprint.
[2] —, Private communication.
[3] Anderson, I., Pohjanpelto, J.: The cohomology of invariant variational bicomplexes, Acta Appl. Math. 41 3–19, (1995).
[4] —, Infinite dimensional Lie algebra cohomology and the cohomology of invariant Euler-Lagrange complexes: A preliminary report, Differential geometry and applications (Brno, 1995), 427–448, Masaryk Univ., Brno, 1996.
[5] Bott, R.: Notes on Gel’fand Fuks cohomology and Characteristic Classes, Raoul Bott: Collected Papers, Vol. 3, Birkhäuser Boston, 288–356, (1995).
[6] Castrillón López, M., Muñoz Masqué, J.: The geometry of the bundle of connections, Math. Z. 236 797–811, (2001).
[7] Ferreiro Pérez, R.: Equivariant characteristic forms in the bundle of connections, J. Geom. Phys. 54 197–212, (2005).
[8] —, On the equivariant variational bicomplex, Proc. Conf. Differential Geometry and its Applications (Prague, 2004), Charles University Prague (Czech Republic) 2005, 587–596.
[9] —, Local anomalies and local equivariant cohomology, to appear in Comm. Math. Phys.
[10] Ferreiro Pérez, R., Muñoz Masqué, J.: Natural connections on the bundle of Riemannian metrics, Monatsh. Math. 155, 67–78 (2008).
[11] —, Pontryagin forms on $(4k - 2)$-manifolds and symplectic structures on the spaces of Riemannian metrics, preprint (arXiv: math.DG/0507076).
[12] García Pérez, P.L.: Gauge algebras, curvature and symplectic structure, J. Differential Geom. 12 209–227, (1977).
[13] Godbillon, C.: Cohomologies d’algèbres de Lie de champs de vecteurs formels, Séminaire Bourbaki, Vol.1972/1973, No. 421, Lecture Notes in Math., 383 69–87, (1974).
[14] Hamasaki, A.: Continuous cohomologies of Lie algebras of formal $G$-invariant vector fields and obstructions to lifting foliations, Publ. Res. Inst. Math. Sci. 20 401–429, (1984).

[15] Hamilton, R.: The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. (N.S.) 7 65–222, (1982).

[16] Kamber, F., Tondeur, P.: Foliated bundles and characteristic classes. Lecture Notes in Mathematics, Vol. 493. Springer-Verlag, Berlin-New York, 1975.

[17] Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, John Wiley & Sons, Inc. (Interscience Division), New York, Volume I, 1963; Volume II, 1969.

[18] Margiarotti, L., Sardanashvily, G.: Connections in Classical and Quantum Field Theory, World Scientific, 2000.

[19] Olver, P.: Applications of Lie groups to differential equations. Second edition. Graduate Texts in Mathematics, 107. Springer-Verlag, New York, 1993.

[20] Saunders, D.J.: The Geometry of Jet Bundles, London Mathematical Society Lecture Notes Series 142, Cambridge University Press, 1989.

[21] Singer, I.M.: Families of Dirac operators with applications to physics, The mathematical heritage of Élie Cartan (Lyon, 1984). Astérisque Numero Hors Serie, 323–340, (1985).

[22] Takens, F.: A global version of the inverse problem to the calculus of variations, J. Differential Geometry 14, 543–562, (1979).
Part II
Local Anomalies and Local Equivariant Cohomology

Abstract
The locality conditions for the vanishing of local anomalies in field theory are shown to admit a geometrical interpretation in terms of local equivariant cohomology. This interpretation allows us to solve the problem proposed by Singer in [31], and consisting in defining an adequate notion of local cohomology to deal with the problem of locality in the geometrical approaches to the study of local anomalies based on the Atiyah-Singer index theorem. Moreover, using the relation between local cohomology and the cohomology of jet bundles studied in [19] we obtain necessary and sufficient conditions for the cancellation of local gravitational and mixed anomalies.

Key words and phrases: local equivariant cohomology, local anomalies, equivariant characteristic classes, BRST cohomology.
Mathematics Subject Classification 2000: Primary 81T50; Secondary 55N91, 57R20, 58D17, 58A20, 70S15.

1 Introduction

An anomaly appears in a theory when a classical symmetry is broken at the quantum level. One fundamental concept in the study of local anomalies is locality. In order to cancel the anomaly, only local terms are allowed, “local” meaning terms obtained integrating forms depending on the fields and its derivatives. In the algebraic approaches to local anomalies (local BRST cohomology, descent equations) only local terms are considered. However, in the geometric and topological approaches based on the Atiyah-Singer index theorem it is not clear how to deal with the problem of locality. The aim of the present paper is to solve the old problem, suggested by Singer in [31], and consisting in determining an adequate notion of “local cohomology” which allows to deal with the problem of locality in that geometric approaches.

Let us briefly recall some basic ideas about the problem of locality in the study of local anomalies (e.g. see [8]). In this paper we consider only local anomalies, and hence we can assume that we are dealing with a connected group \( \mathcal{G} \), with Lie algebra \( \mathfrak{g} \). We consider an action of \( \mathcal{G} \) on a bundle \( E \to M \) over a compact \( n \)-manifold \( M \). Let \( \{ D_s : s \in \Gamma(E) \} \) be a \( \mathcal{G} \)-equivariant family of elliptic operators acting on fermionic fields \( \psi \in \Gamma(V) \) and parametrized by \( \Gamma(E) \). Then the lagrangian density \( \mathcal{L}(\psi, s) = \bar{\psi} i D_s \psi \) is \( \mathcal{G} \)-invariant, and hence the classical action \( S_{\mathcal{L}}(\psi, s) = \int_M \mathcal{L}(\psi, s) \), is a \( \mathcal{G} \)-invariant function on \( \Gamma(V) \times \Gamma(E) \). However, at the quantum level, the corresponding effective action \( W(s) \), defined in terms of the fermionic path integral by \( \exp(-W(s)) = \)
\[ \int D\psi D\bar{\psi} \exp \left( - \int_M \bar{\psi} D\phi \psi \right) \] could fail to be \( \mathcal{G} \)-invariant if the fermionic measure \( D\psi D\bar{\psi} \) is \( \mathcal{G} \)-invariant. To measure this lack of invariance we define \( \mathcal{A} \in \Omega^1(\mathfrak{g}, \Omega^0(\Gamma(E))) \) by \( \mathcal{A} = \delta W, \) i.e. \( \mathcal{A}(X)(s) = L_X W(s) \) for \( X \in \mathfrak{g}, \) \( s \in \Gamma(E). \) Although \( W \) is clearly a non-local functional, \( \mathcal{A} \) is local in \( X \) and \( s, \) i.e. we have \( \mathcal{A} \in \Omega^1_{\text{loc}}(\mathfrak{g}, \Omega^0_{\text{loc}}(\Gamma(E))). \) It is clear that \( \mathcal{A} \) satisfies the condition \( \delta \mathcal{A} = 0 \) (the Wess-Zumino consistency condition). Moreover, if \( \mathcal{A} = \delta \Lambda \) for a local functional \( \Lambda = \int_M \lambda \in \Omega^1_{\text{loc}}(\Gamma(E)) \) then we can define a new lagrangian density \( \hat{\mathcal{L}} = \mathcal{L} + \lambda, \) such that the new effective action \( \hat{W} \) is \( \mathcal{G} \)-invariant, and in that case the anomaly cancels. If \( \mathcal{A} \neq \delta \Lambda \) for every \( \Lambda \in \Omega^0_{\text{loc}}(\Gamma(E)) \) then we say that there exists an anomaly in the theory. Hence the anomaly is measured by the cohomology class of \( \mathcal{A} \) in the BRST cohomology \( H^1_{\text{loc}}(\mathfrak{g}, \Omega^0_{\text{loc}}(\Gamma(E))). \) In this way the problem of anomaly cancellation can be reduced to the pure algebraic computation of the BRST cohomology (e.g. see \cite{[11] [12] [15] [17] [22] [26] [30] [31]}).

Local anomalies also admit a nice geometrical interpretation in terms of the Atiyah-Singer index theorem for families of elliptic operators (see \cite{[1] [2] [4] [22] [31]}). The first Chern class \( c_1(\det \text{Ind} D/\mathcal{G}) \) of the (quotient) determinant line bundle \( \det \text{Ind} D/\mathcal{G} \to \Gamma(E)/\mathcal{G} \) represents an obstruction for anomaly cancellation. The Atiyah-Singer index theorem for families provides an explicit expression for \( c_1(\det \text{Ind} D/\mathcal{G}) \) and more precisely, of the curvature \( \Omega^1(\det \text{Ind} D/\mathcal{G}) \) of its natural connection. Now the problem of locality appears again. The condition \( c_1(\det \text{Ind} D/\mathcal{G}) = 0 \) is a necessary but not a sufficient condition for local anomaly cancellation. For example (see \cite{[1]}), for \( M = S^6 \) although \( c_1(\det \text{Ind} \mathfrak{g}/\text{Diff}^0 M) = 0, \) the local gravitational anomaly does not cancel. Moreover we recall (see \cite{[4] [9] [28]}) that the BRST and index theory approaches are related by means of the transgression map (see Section 2) \( t: H^2(\Gamma(E)/\mathcal{G}) \to H^1(\mathfrak{g}, \Omega^0(\Gamma(E))) \) i.e. \( \mathcal{A} = t(c_1(\det \text{Ind} D/\mathcal{G})). \) As the transgression map \( t \) is injective, the condition \( c_1(\det \text{Ind} D/\mathcal{G}) = 0 \) on \( H^2(\Gamma(E)/\mathcal{G}) \) is equivalent to \( \mathcal{A} = 0 \) on \( H^1(\mathfrak{g}, \Omega^0(\Gamma(E))). \) However, the condition for local anomaly cancellation is \( \mathcal{A} = 0 \) on the BRST cohomology \( H^1_{\text{loc}}(\mathfrak{g}, \Omega^0_{\text{loc}}(\Gamma(E))). \) Hence, in order to cancel the local anomaly, \( \Omega^1(\det \text{Ind} D/\mathcal{G}) \) should be the exterior differential of a “local” form on \( \Gamma(E)/\mathcal{G}, \) and the local anomaly cancellation should be expressed in terms of an adequate notion of “local cohomology of \( \Gamma(E)/\mathcal{G}, \) \( H^1_{\text{loc}}(\Gamma(E)/\mathcal{G}). \) Note however that it is by no means clear how to define \( H^1_{\text{loc}}(\Gamma(E)/\mathcal{G}), \) as the expression of \( \Omega^1(\det \text{Ind} D/\mathcal{G}) \) itself contains non-local terms (Green operators). The problem of defining this notion of “local cohomology” was proposed in \cite{[31]}. In \cite{[1]} a paper studying the preceding problem is announced to be in preparation, but to the best of our knowledge, this paper has not been published.

Let us explain how local \( \mathcal{G} \)-equivariant cohomology solves that problem. The \( \mathcal{G} \)-equivariant cohomology of \( \Gamma(E) \) and the cohomology of \( \Gamma(E)/\mathcal{G} \) are related by the generalized Chern-Weil homomorphism \( \text{ChW}: H^2_\mathcal{G}(\Gamma(E)) \to H^2(\Gamma(E)/\mathcal{G}). \) We define another injective transgression map \( \tau: H^2_\mathcal{G}(\Gamma(E)) \to H^1(\mathfrak{g}, \Omega^0(\Gamma(E))) \) in such a way that \( t \circ \text{ChW} = \tau \) (see Section 2).

Now, to deal with the problem of locality, we define the local \( \mathcal{G} \)-equivariant cohomology \( H^1_{\mathcal{G}, \text{loc}}(\Gamma(E)) \) in a natural way, and we prove that the restriction of \( \tau \) to \( H^1_{\mathcal{G}, \text{loc}}(\Gamma(E)) \) takes values on \( H^1_{\text{loc}}(\mathfrak{g}, \Omega^0_{\text{loc}}(\Gamma(E))). \) We set \( H^1_{\text{loc}}(\Gamma(E)/\mathcal{G}) = \)
ChW(\(H^2_{\text{G,loc}}(\Gamma(E))\)) and we have the following commutative diagram

\[
\begin{array}{ccc}
H^2_{\text{G,loc}}(\Gamma(E)) & \xrightarrow{\text{ChW}} & H^2_\text{loc}(\Gamma(E)/\mathcal{G}) \\
\tau \searrow & & \nearrow t \\
H^1_\text{loc}(\mathfrak{g}, \Omega^0_\text{loc}(\Gamma(E))) & & \\
\end{array}
\]

Moreover, as \(t\) and \(\tau\) are injective, if \(\omega \in \Omega^2_{\mathcal{G},\text{loc}}(\Gamma(E))\) is closed and \([\omega] = \text{ChW}([\omega])\) then the following conditions are equivalent

(a) \([\omega] = 0\) on \(H^2_{\text{G,loc}}(\Gamma(E))\),
(b) \([\omega] = 0\) on \(H^2_\text{loc}(\Gamma(E)/\mathcal{G})\),
(c) \([\tau(\omega)] = [t(\omega)] = 0\) on \(H^1_\text{loc}(\mathfrak{g}, \Omega^0_\text{loc}(\Gamma(E)))\).

Hence our definition of \(H^2_{\text{G,loc}}(\Gamma(E)/\mathcal{G})\) solves the problem. It is important to note that if \(\omega \in \Omega^2_{\mathcal{G},\text{loc}}(\Gamma(E))\) is closed, the form \(\omega \in \Omega^2(\Gamma(E)/\mathcal{G})\) determining the class \(\text{ChW}(\omega)\) could contain non-local terms, as \(\omega\) depends on the curvature of a connection \(\Theta\) on the principal \(\mathcal{G}\)-bundle \(\Gamma(E) \to \Gamma(E)/\mathcal{G}\), and \(\Theta\) usually contains non-local terms. However, the form \(t(\omega)\) obtained by applying the transgression map \(t\) to \(\omega\) is local.

In this paper we prefer to work with local \(\mathcal{G}\)-equivariant cohomology in place of the cohomology of the quotient for several reasons. Generally, in order to have a well defined quotient manifold, it is necessary to restrict the group \(\mathcal{G}\) to a subgroup acting freely on \(\Gamma(E)\). However, the equivariant cohomology is well defined for arbitrary actions. Furthermore, the local \(\mathcal{G}\)-equivariant cohomology can be related to the cohomology of jet bundles, thus providing new tools for the study of local anomalies. In terms of local \(\mathcal{G}\)-equivariant cohomology the conditions for anomaly cancellation can be expressed in the following way. Let \(\Omega^\text{detIndD}_{\mathcal{G}} \in \Omega^2_{\mathcal{G}}(\Gamma(E))\) be the \(\mathcal{G}\)-equivariant curvature of the determinant line bundle \(\det \text{IndD} \to \Gamma(E)\) with respect to its natural connection. For free actions we have \(\text{ChW}(\Omega^\text{detIndD}_{\mathcal{G}}) = [\Omega^\text{detIndD}_{\mathcal{G}}/\mathcal{G}]\). Hence, our preceding considerations can be resumed by saying that if \(\Omega^\text{detIndD}_{\mathcal{G}} \in \Omega^2_{\mathcal{G},\text{loc}}(\Gamma(E))\), then the local anomaly is measured by the cohomology class of the \(\mathcal{G}\)-equivariant curvature \(\Omega^\text{detIndD}_{\mathcal{G}}\) of the determinant line bundle \(\det \text{IndD} \to \Gamma(E)\) on the local \(\mathcal{G}\)-equivariant cohomology \(H^2_{\text{G,loc}}(\Gamma(E))\).

In [19] we have shown that, using the variational bicomplex theory, the local cohomology can be computed in terms of the cohomology of the jet bundle. By definition, a local functional \(\Lambda \in \Omega^0_\text{loc}(\Gamma(E))\) is given by integration over \(M\) of a function \(L(s, \partial s)\) depending of the section \(s \in \Gamma(E)\) and its derivatives \(\Lambda(s) = \int_M L(s, \partial s) \text{vol}_M\). The jet bundle \(J^\infty(E)\) is the space of Taylor series \(j^\infty_x s\) of sections \(s \in \Gamma(E)\) at points \(x \in M\). Hence, the function \(L(s, \partial s)\) can be considered as a function \(L \in \Omega^0(J^\infty(E))\) such that \((j^\infty_x s)^* L = L(s, \partial s)\) for every \(s \in \Gamma(E)\), and the Lagrangian density \(\lambda = L \text{vol}_M \in \Omega^\infty(J^\infty(E))\) can be considered as an \(n\)-form on \(J^\infty(E)\). We define a map \(\mathfrak{Z} : \Omega^n(J^\infty(E) \to \Omega^0(\Gamma(E))\) by setting \(\mathfrak{Z}[\lambda] = \int_M (j^\infty_x s)^* \lambda\) and we have \(\Omega^n_\text{loc}(\Gamma(E)) = \mathfrak{Z}(\Omega^n(J^\infty(E)))\).

For our study of anomalies we need to consider not only local functionals, but also local \(k\)-forms of degree \(k > 0\). For this reason we extend the map
forms of degree greater than \( n \). Let \( \mathcal{G}^{n+k}(J^\infty E) \to \Omega^k(\Gamma(E)) \) and we set \( \Omega^k_{\text{loc}}(\Gamma(E)) = \mathcal{G}(\Omega^{n+k}(J^\infty E)) \). This map can be studied completely in terms of the jet bundle by means of the variational bicomplex theory. For \( k > 1 \) the interior Euler operator \( I: \Omega^{n+k}(J^\infty E) \to \Omega^{n+k}(J^\infty E) \) (a generalization of the Euler-Lagrange operator) satisfies \( I^2 = I \), \( \mathcal{G}[\alpha] = \mathcal{G}[I(\alpha)] \) and \( \mathcal{G}[\alpha] = 0 \) if and only if \( I(\alpha) = 0 \), for \( \alpha \in \Omega^{n+k}(J^\infty E) \). The image of the interior Euler operator \( \mathcal{F}^k(J^\infty E) = I(\Omega^{n+k}(J^\infty E)) \) is called the space of functional forms, and clearly we have \( \mathcal{F}^k(J^\infty E) \cong \Omega^k(\Gamma(E)) \), \( H^k_{\text{loc}}(\Gamma(E)) \cong H^k(\mathcal{F}^k(J^\infty E)) \) for \( k > 0 \). Standard results on the variational bicomplex theory can be used to show that \( H^k(\mathcal{F}^k(J^\infty E)) \cong H^{n+k}(J^\infty E) \), and in this way the local cohomology is computed in terms of the cohomology of jet bundles. In a similar way, for the invariant cohomology, under very general conditions we have \( H^k_{\text{loc}}(\Gamma(E))^{\mathcal{G}} \cong H^{n+k}(J^\infty E)^{\mathcal{G}} \) for \( k > 1 \) (see [19] for details). Although we do not have a similar result for equivariant cohomology (see Section 3), we can use these results in order to study local anomalies in the following way. A necessary condition for anomaly cancellation is that \( \Omega^{\text{det Ind}} \) should be the exterior differential of a local \( \mathcal{G} \)-invariant 1-form. We call \( [\Omega^{\text{det Ind}}] \in H^2_{\text{loc}}(\Gamma(E))^{\mathcal{G}} \) the first obstruction for anomaly cancellation.

We apply these results to gravitational and mixed anomalies in Sections 5 and 6 and we show that in these cases the first obstruction for anomaly cancellation provides necessary and sufficient conditions for anomaly cancellation.

We conclude that, when the locality conditions are taken into account, the anomaly cancellation is not related to the topology of \( \Gamma(E)/\mathcal{G} \) or \( \mathcal{G} \), but to the geometry of the jet bundle.

\section{The transgression maps}

First we recall some results of equivariant cohomology in the Cartan model (e.g. see [6],[24]). We consider a left action of a connected Lie group \( \mathcal{G} \) on a manifold \( \mathcal{N} \), i.e. a homomorphism \( \rho: \mathcal{G} \to \text{Diff}\mathcal{N} \). We have an induced Lie algebra homomorphism \( \mathfrak{g} \to \mathfrak{X}(\mathcal{N}) \), \( X \to X_N = \frac{d}{dt}|_{t=0}(\exp(-tX)) \).

The space of \( \mathcal{G} \)-invariant \( r \)-forms is denoted by \( \Omega^r(\mathcal{N})^{\mathcal{G}} \), and the \( \mathcal{G} \)-invariant cohomology by \( H^\bullet(\mathcal{N})^{\mathcal{G}} \). We denote by \( \mathcal{P}^k(\mathfrak{g},\Omega^r(\mathcal{N})^{\mathcal{G}}) \) the space of degree \( k \) \( \mathcal{G} \)-invariant polynomials on \( \mathfrak{g} \) with values in \( \Omega^r(\mathcal{N}) \). We recall that \( \alpha \in \mathcal{P}^k(\mathfrak{g},\Omega^r(\mathcal{N})) \) is \( \mathcal{G} \)-invariant if for every \( X \in \mathfrak{g} \) and every \( g \in \mathcal{G} \) we have \( \alpha(\text{Ad}_gX) = \rho(g^{-1})^*(\alpha(X)) \). The infinitesimal version of this condition is

\begin{equation}
L_{X_N}(\alpha(X)) = k\alpha([Y,X],X,(k-1),X), \quad \forall X,Y \in \mathfrak{g}.
\end{equation}

If \( \mathcal{G} \) is connected, then condition (12) is equivalent to the \( \mathcal{G} \)-invariance of \( \alpha \).

We assign degree \( 2k+r \) to the elements of \( \mathcal{P}^k(\mathfrak{g},\Omega^r(\mathcal{N})^{\mathcal{G}}) \). The space of \( \mathcal{G} \)-equivariant differential \( q \)-forms is \( \Omega^q_{\mathcal{G}}(\mathcal{N}) = \bigoplus_{2k+r=q}(\mathcal{P}^k(\mathfrak{g},\Omega^r(\mathcal{N})^{\mathcal{G}})) \).

The Cartan differential \( d_c: \Omega^q_{\mathcal{G}}(\mathcal{N}) \to \Omega^{q+1}_{\mathcal{G}}(\mathcal{N}) \) is defined by \( (d_c\alpha)(X) = d(\alpha(X)) - \iota_{X_N}\alpha(X) \), and we have \( (d_c)^2 = 0 \). The \( \mathcal{G} \)-equivariant cohomology (in the Cartan model) of \( \mathcal{N} \), \( H^k_{\mathcal{G}}(\mathcal{N}) \), is the cohomology of the complex \( (\Omega^0_{\mathcal{G}}(\mathcal{N}),d_c) \).
Let $\omega \in \Omega^2(N)$ be a $G$-equivariant 2-form. Then we have $\omega = \omega_0 + \mu$ where $\omega_0 \in \Omega^2(N)^G$, and $\mu \in \text{Hom}(\mathfrak{g}, C^\infty(N))^G$, i.e., $\mu$ is a $G$-equivariant linear map $\mu: \mathfrak{g} \to C^\infty(N)$. We have $d_\omega = 0$ if and only if $d\omega_0 = 0$, and $\iota_X^*\omega_0 = d(\mu(X))$, for every $X \in \mathfrak{g}$. Hence a closed $G$-equivariant 2-form is the same as a $G$-invariant pre-symplectic form and a moment map for it.

We recall the Berline-Vergne construction of equivariant characteristic classes (see [7, 8]). Let $\pi: P \to N$ a principal $G$-bundle and $G$ a Lie group acting (on the left) on $P$ by automorphisms. If $A$ is a $G$-invariant connection on $P$ with curvature $F$, we define the equivariant curvature of $A$ by $F_G(X) = F - A(X_P)$. Then for every Weil polynomial $f \in I^G_k$, the $G$-equivariant characteristic form associated to $f$ and $A$ is $f(F_G) \in \Omega^G_{2k}(N)$. It can be seen that $d_\omega(f(F_G)) = 0$ and that the equivariant cohomology class $f_G(P) = [f(F_G)] \in H^G_{2k}(N)$ is independent of the $G$-invariant connection $A$.

Finally we recall (e.g. see [6]) that if $N \to N/G$ is a principal $G$-bundle we have the (generalized) Chern-Weil homomorphism $ChW: H^*_G(N) \to H^*(N/G)$. If $A$ is an arbitrary connection on $N \to N/G$ with curvature $F$, and $\alpha \in \Omega^G_k(N)$, then we have $ChW(\alpha) = [\text{hor}_A(\alpha(F))]$, where $\text{hor}_A$ is the horizontalization with respect to the connection $A$. We also use the notation $\alpha = ChW(\alpha)$. A direct computation shows that we have the following result, that provides a direct proof of the fact that the Chern-Weil map $ChW: H^*_G(N) \to H^*(N/G)$ is an isomorphism.

**Proposition 1** Let $N \to N/G$ be a principal $G$-bundle, and let $A \in \Omega^1(N, \mathfrak{g})$ be a connection form, with curvature $F$. If $\omega = \omega_0 + \mu \in \Omega^G_2(N)$ is a closed $G$-equivariant 2-form and we define $\alpha \in \Omega^1(N)^G$ by $\alpha = \mu(A)$ then we have $\text{hor}_A(\omega(F)) = \omega + d_\omega \alpha$.

Let us assume that $H^1(N) = H^2(N) = 0$. We denote by $H^*(\mathfrak{g}, \Omega^0(N))$ the cohomology of the Lie algebra $\mathfrak{g}$ with values in $\Omega^0(N)$. The following Proposition can be proved using Formula (12).

**Proposition 2** Let $\omega = \omega_0 + \mu \in \Omega^G_2(N)$ be a closed $G$-equivariant form. If $\rho \in \Omega^1(N)$ satisfies $\omega_0 = d\rho$, then the map $\tau_\rho \in \Omega^1(\mathfrak{g}, \Omega^0(N))$ given by $\tau_\rho(X) = \rho(X_N) + \mu(X)$ determines a linear map $\tau: H^2_G(N) \to H^1(\mathfrak{g}, \Omega^0(N))$ which is independent of the form $\rho$ chosen, and that we call the transgression map $\tau$. If the group $G$ is connected, then the transgression map $\tau$ is injective.

Now we assume that the action of $G$ on $N$ is free, and $\pi: N \to N/G$ is a principal $G$-bundle. Then we can consider the more familiar transgression map defined as follows.

**Proposition 3** Let $\omega \in \Omega^2(N/G)$ be a closed 2-form. If $\eta \in \Omega^1(N)$ is a form such that $\pi^*\omega = d\rho$, then the map $t_\eta: \mathfrak{g} \to \Omega^0(N)$, $t_\eta(X) = \eta(X_N)$ determines a linear map $t: H^2(N/G) \to H^1(\mathfrak{g}, \Omega^0(N))$, which is independent of the form $\eta$ chosen, and that we call the transgression map $t$. If the group $G$ is connected, then the transgression map $t$ is injective.
The following Proposition relates the two transgression maps. We use this result in order to relate our approach to anomalies with the BRST approach.

**Proposition 4** Let \( \omega \in H^2_0(N) \) and \( \underline{\omega} = \text{ChW}(\omega) \in H^2(N/G) \). We have \( \tau(\omega) = t(\underline{\omega}) \).

**Proof.** If \( \omega = \omega_0 + \mu \), by Proposition [1] we have \( \omega = \pi^* \omega + d_c \alpha \) for some \( \alpha \in \Omega^1(N) = \Omega^1(N)^G \), i.e. \( \omega_0 = \pi^* \omega + d \alpha \) and \( \mu(X) = -\alpha(X_N) \).

Let \( \eta \in \Omega^1(N) \) be a form such that \( \pi^* \omega = d \eta \). If we set \( \rho = \eta + \alpha \) then \( \omega_0 = d \rho \) and for every \( X \in \text{Lie } G \) we have \( \tau_\rho(X) = \rho(X_N) + \mu(X) = t_\eta(X) \). 

### 3 Local equivariant cohomology

Let \( p : E \to M \) be a bundle over a compact, oriented \( n \)-manifold \( M \) without boundary. We denote by \( J^rE \) its \( r \)-jet bundle, and by \( J^\infty E \) the infinite jet bundle (see [29] for the details on the geometry of \( J^\infty E \)). We recall that the points on \( J^\infty E \) are the Taylor series of sections of \( E \) and that \( \Omega^k(J^\infty E) = \lim_k \Omega^k(J^rE) \).

A diffeomorphism \( \phi \in \text{Diff} E \) is said to be projectable if there exists \( \underline{\phi} \in \text{Diff} M \) satisfying \( \phi \circ p = p \circ \underline{\phi} \). We denote by \( \text{Proj} E \) the space of projectable diffeomorphisms of \( E \), and we denote by \( \text{Proj}^+ E \) the subgroup of elements such that \( \phi \in \text{Diff}^+ M \), i.e. \( \phi \) is orientation preserving. The space of projectable vector fields on \( E \) is denoted by \( \text{proj} E \), and can be considered as the Lie algebra of \( \text{Proj} E \). We denote by \( \text{pr} \phi \) (resp. \( \text{pr} X \)) the prolongation of \( \phi \in \text{Proj} E \) (resp. \( X \in \text{proj} E \)) to \( J^\infty E \).

Let \( \Gamma(E) \) be the manifold of global sections of \( E \), that we assume to be not empty. For any \( s \in \Gamma(E) \), the tangent space to the manifold \( \Gamma(E) \) is isomorphic to the space of vertical vector fields along \( s \), that is \( T_s \Gamma(E) \simeq \Gamma(M, s^* V(E)) \).

Let \( j^\infty : M \times \Gamma(E) \to J^\infty E \), \( j^\infty(x, s) = j^\infty_s x \) be the evaluation map. We define a map \( \exists : \Omega^{n+k}(J^\infty E) \to \Omega^k(\Gamma(E)) \), by \( \exists[\alpha] = \int_M (j^\infty)^* \alpha \) for \( \alpha \in \Omega^{n+k}(J^\infty E) \). If \( \alpha \in \Omega^k(J^\infty E) \) with \( k < n \), we set \( \exists[\alpha] = 0 \). We define the space of local \( k \)-forms on \( \Gamma(E) \) by \( \Omega^k_{\text{loc}}(\Gamma(E)) = \exists(\Omega^{n+k}(J^\infty E)) \subset \Omega^k(\Gamma(E)) \). The local cohomology of \( \Gamma(E) \), \( H^k_{\text{loc}}(\Gamma(E)) \), is the cohomology of \( (\Omega^k_{\text{loc}}(\Gamma(E)), d) \). The map \( \exists \) induces isomorphisms \( H^k_{\text{loc}}(\Gamma(E)) \cong H^{n+k}(E) \) for \( k > 0 \) (see [19] for details). Note that \( \Omega^0_{\text{loc}}(\Gamma(E)) \) is precisely the space of local functions on \( \Gamma(E) \).

The group \( \text{Proj} E \) acts naturally on \( \Gamma(E) \) as follows. If \( \phi \in \text{Proj} E \), we define \( \phi_{\Gamma(E)} \in \text{Diff}(\Gamma(E)) \) by \( \phi_{\Gamma(E)}(s) = \phi \circ s \circ \phi^{-1} \), for all \( s \in \Gamma(E) \). In a similar way, a projectable vector field \( X \in \text{proj} E \) induces a vector field \( X_{\Gamma(E)} \in \mathfrak{X}(\Gamma(E)) \).

Let \( G \) be a Lie group acting on \( E \) by elements \( \text{Proj}^+ E \). We define the space of local \( G \)-invariant forms \( \Omega^k_{\text{loc}}(\Gamma(E))^G \) as the subspace of \( G \)-invariant elements on \( \Omega^k_{\text{loc}}(\Gamma(E)) \), and the local \( G \)-invariant cohomology, \( H^k_{\text{loc}}(\Gamma(E))^G \), as the cohomology of \( (\Omega^k_{\text{loc}}(\Gamma(E))^G, d) \). In [19] it is shown that we have \( \Omega^k_{\text{loc}}(\Gamma(E))^G = \exists(\Omega^{n+k}(J^\infty E)^G) \) for \( k > 0 \) and that under certain conditions \( \exists \) induces isomorphisms \( H^k_{\text{loc}}(\Gamma(E))^G \cong H^{n+k}(E)^G \) for \( k > 1 \).
We assume that
\[ \text{det Ind} \]
with
\[ 2 \]
\[ G \]
For example, if we consider the trivial action of a group
\[ \text{Let} \]
\[ \text{4.1 Conditions for anomaly cancellation} \]
\[ \text{4 Local anomalies and local equivariant cohomology} \]
\[ \text{Remark 5} \]
\[ \text{Local anomalies and local equivariant cohomology} \]
\[ \text{We define the space of local} \]
\[ \text{G-equivariant cohomology of} \]
\[ \text{Remark 5} \]
\[ \text{Remark 5} \]
\[ \text{Local anomalies and local equivariant cohomology} \]
\[ \text{Let} \]
\[ \text{Let} \]
\[ \text{2r > n, we have by definition} \]
\[ \text{We assume that} \]
\[ \text{We assume that} \]
\[ \text{We make the following assumption} \]
\[ \text{We made the following assumption} \]
\[ \text{We also assume that the map} \]
In Sections 5 and 6 we show that for the classical cases of gravitational and mixed anomalies, assumption (A2) follows from the Atiyah-Singer Index theorem for families and the results on [18] and [21].

**Definition 6** We say that the local anomaly corresponding to the \( G \)-equivariant family \( \{ D_s : s \in \Gamma(E) \} \) cancels if the cohomology class of \( \Omega^\text{det Ind} \) on the local \( G \)-equivariant cohomology \( H^2_{G, \text{loc}}(\Gamma(E)) \) vanishes.

**Remark 7** If the local anomaly cancels, then clearly \( c_1(G \text{det Ind} D) = 0 \). However, the converse is not true, as the condition for anomaly cancellation involves local equivariant cohomology. Furthermore, if the action of \( G \) on \( \Gamma(E) \) is free, then we have \( \text{ChW}([\Omega^\text{det Ind} D]) = c_1(\text{det Ind} D/\mathcal{G}) \). Hence, if the local anomaly cancels then we have \( c_1(\text{det Ind} D/\mathcal{G}) = 0 \), but again, this condition is not sufficient.

We have \( \Omega^\text{det Ind} D = \Omega^\text{det Ind} D + \mu \), where \( \mu \) is a moment map for the action of \( G \) on the pre-symplectic manifold \( (\Gamma(E), \Omega^\text{det Ind} D) \). By definition, the local anomaly cancels if and only if there exists a local \( G \)-invariant 1-form \( \rho \in \Omega^1_{\text{loc}}(\Gamma(E)) \) satisfying the conditions \( \Omega^\text{det Ind} D = dp \), and \( \mu(X) = -\rho(X_{\Gamma(E)}) \), \( \forall X \in \mathcal{G} \). Hence a necessary condition for the anomaly cancellation is that \( \Omega^\text{det Ind} D \) should be the exterior differential of a \( G \)-invariant 1-form. For this reason we made the following

**Definition 8** The first obstruction for anomaly cancellation is defined as the cohomology class \( [\Omega^\text{det Ind} D] \in H^2_{\text{loc}}(\Gamma(E))^G \) of the curvature of the determinant line bundle in the local \( G \)-invariant cohomology.

The first obstruction for anomaly cancellation involves local \( G \)-invariant cohomology, which in [19] is shown to be isomorphic to the cohomology of the \( G \)-invariant variational bicomplex. Moreover under certain conditions we have \( H^2_{\text{loc}}(\Gamma(E))^G \cong H^{n+2}(J^\infty E)^G \), and then the first obstruction for anomaly cancellation can be expressed directly in terms of the jet bundle as follows. If \( \eta \in \Omega^{n+2}(J^\infty E)^G \) is a closed form such that \( \mathcal{Z}[\eta] = \Omega^\text{det Ind} D \) and the class of \( \eta \) on \( H^{n+2}(J^\infty E)^G \) does not vanish, then the anomaly does not cancel. In this way, the techniques developed in [3] for computing the invariant cohomology of the variational bicomplex in terms of Gel’fand-Fuks cohomology can be applied to study the problem of anomaly cancellation. We apply these results in sections 5 and 6 to the case of gravitational and mixed anomalies.

### 4.2 Anomaly cancellation and BRST cohomology

In this section we show that our definition for anomaly cancellation can be expressed in terms of BRST cohomology. We recall (see [11], [30]) that the BRST cohomology \( H^*_{\text{loc}}(\mathfrak{g}, \Omega^0_{\text{loc}}(\Gamma(E))) \) is the Lie algebra local cohomology of \( \mathfrak{g} \) with values in \( \Omega^0_{\text{loc}}(\Gamma(E)) \), that is, the cohomology of \( (\Omega^*_{\text{loc}}(\mathfrak{g}, \Omega^0_{\text{loc}}(\Gamma(E))), \delta) \). Now we assume that \( H^2(\Gamma(E)) = H^1(\Gamma(E)) = 0 \) and also that \( H^2_{\text{loc}}(\Gamma(E)) = H^1_{\text{loc}}(\Gamma(E)) = 0 \).
Proposition 9 The restriction of the transgression map $\tau$ to $H^2_{G,loc}(\Gamma(E))$ takes values on the BRST cohomology $H^1_{loc}(\mathcal{G}, \Omega^0_{loc}(\Gamma(E)))$ and the map $\tau: H^2_{G,loc}(\Gamma(E)) \to H^1_{loc}(\mathcal{G}, \Omega^0_{loc}(\Gamma(E)))$ is injective for $G$ connected.

Proof. Let $\omega = \omega_0 + \mu \in \Omega^2_{G,loc}(\Gamma(E))$ be a closed local $G$-equivariant 2-form. As $H^2_{loc}(\Gamma(E)) = 0$, we have $\omega_0 = d\rho$, for certain $\rho \in \Omega^1_{loc}(\Gamma(E))$. By the definition of local equivariant cohomology and assumption (A1) the map $\tau_{\rho}: \mathcal{G} \to \Omega^0_{loc}(\Gamma(E))$, $\tau_{\rho}(X) = \rho(X_{\Gamma(E)}) + \mu(X)$ is a local map. The injectiveness of $\tau$ follows from Proposition 2. Note that we can assume that the group is connected as we are dealing with local anomalies.

If the action of $G$ on $\Gamma(E)$ is free, by Proposition 4 we have the following

Proposition 10 Let $\omega \in \Omega^2_{G,loc}(\Gamma(E))$ be a closed local $G$-equivariant 2-form and let $\omega = \text{ChW}(\omega) \in H^2(\Gamma(E)/G)$. Then we have $\tau(\omega) = \tau(\omega)$, and in particular $\tau(\omega) \in H^1_{loc}(\mathcal{G}, \Omega^0_{loc}(\Gamma(E)))$. Moreover, $\tau(\omega) = 0$ if and only if the cohomology class of $\omega$ on $H^2_{G,loc}(\Gamma(E))$ vanishes.

With the preceding results, our condition for anomaly cancellation can be expressed in terms of BRST cohomology in the following way

Theorem 11 Let $\{D_s : s \in \Gamma(E)\}$ be a $G$-equivariant family of elliptic operators parametrized by the space $\mathfrak{M} M$ of Riemannian metrics on $M$, and the action of diffeomorphisms. First we recall the definition of the equivariant Pontryagin and Euler forms on the 1-jet bundle of the bundle of metrics given in [20] and [21]. Then we show how the equivariant curvature of the determinant line bundle can be obtained from these constructions on the jet bundle, and that assumptions (A1) and (A2) hold in this case. Finally, we use our characterization of local anomaly cancellation in terms of local equivariant cohomology and the results in [19] to obtain necessary and sufficient conditions for local gravitational anomaly cancellation.

5 Riemannian metrics and gravitational anomalies

In this section we apply the preceding considerations to the case of gravitational anomalies (see [11, 22, 25]). We consider the family of Dirac operators $\mathcal{D}_g$ parametrized by the space $\mathfrak{M} M$ of Riemannian metrics on $M$, and the action of diffeomorphisms. First we recall the definition of the equivariant Pontryagin and Euler forms on the 1-jet bundle of the bundle of metrics given in [20] and [21]. Then we show how the equivariant curvature of the determinant line bundle can be obtained from these constructions on the jet bundle, and that assumptions (A1) and (A2) hold in this case. Finally, we use our characterization of local anomaly cancellation in terms of local equivariant cohomology and the results in [19] to obtain necessary and sufficient conditions for local gravitational anomaly cancellation.

5.1 Equivariant Pontryagin and Euler forms on $J^1 \mathcal{M}_M$

Let $M$ be a compact and connected $n$-manifold without boundary, and $TM$ its tangent bundle. We define its bundle of Riemannian metrics $q: \mathcal{M}_M \to M$
by \(\mathcal{M}_M = \{g_x \in S^2(T_x^*M) : g_x \text{ is positive defined on } T_xM\}\). Let \(\mathfrak{Met}M = \Gamma(M,\mathcal{M}_M)\) denote the space of Riemannian metrics on \(M\). We denote by \(\text{Diff}M\) the diffeomorphisms group of \(M\), and by \(\text{Diff}^+M\) its subgroup of orientation preserving diffeomorphisms. We denote by \(\varrho_1 : J^1\mathcal{M}_M \rightarrow M\) the 1-jet bundle of \(\mathcal{M}_M\) and by \(\pi : FM \rightarrow M\) the linear frame bundle of \(M\). The pull-back bundle \(\varrho_1^*FM \rightarrow J^1\mathcal{M}_M\) is a principal \(GL(n,\mathbb{R})\)-bundle.

We consider the principal \(SO(n)\)-bundle \(O^+M \rightarrow J^1\mathcal{M}_M\) where \(O^+M = \{(j^*_x g, u_x) \in q_1^*FM : u_x \text{ is } g_x\text{-orthonormal and positively oriented}\}\). In [20] it is shown that there exists a unique connection form \(\omega \in \Omega^1(O^+M,\mathfrak{so}(n))\) (called the universal Levi-Civita connection) on \(O^+M\) invariant under the natural action of the group \(\text{Diff}^+M\). We denote by \(\Omega\) the curvature form of \(\omega\).

As the universal Levi-Civita connection \(\omega\) is \(\text{Diff}^+M\)-invariant, the Berline-Vergne construction of equivariant characteristic classes (see Section 2) can be applied. For any Weil polynomial \(p \in I^r_{SO(n)}\) we have the \(\text{Diff}^+M\)-equivariant characteristic form \(p(\Omega_{\text{Diff}^+M}) \in \Omega^2_{\text{Diff}^+M}(J^1\mathcal{M}_M)\) corresponding to \(p\). In particular we have the equivariant Pontryagin and Euler forms. If \(2r > n\), by applying the integration map \(\mathfrak{S}\) to \(\varrho(\Omega_{\text{Diff}^+M})\), we obtain a closed \(\text{Diff}^+M\)-equivariant form on \(\mathfrak{Met}M\), \(\mathfrak{S}[p(\Omega_{\text{Diff}^+M})] \in \Omega^{2r-n}_{\text{Diff}^+M}(\mathfrak{Met}M)\).

Now let us assume that \(n = 4k - 2\) for some integer \(k\), and let \(p \in I^{2k}_{SO(n)}\). Then \(\omega = \mathfrak{S}[p(\Omega_{\text{Diff}^+M})] \in \Omega^2_{\text{Diff}^+M}(\mathfrak{Met}M)\) is a closed \(\text{Diff}^+M\)-equivariant 2-form on \(\mathfrak{Met}M\). The explicit expression of \(\omega = \omega_0 + \mu\) can be found in [21] where some geometrical properties of these equivariant 2-forms are studied. In particular \(\mu : \mathfrak{X}(M) \rightarrow \Omega^0(\mathfrak{Met}M)\) is given for \(g \in \mathfrak{Met}M\) and \(X \in \mathfrak{X}(M)\) by \(\mu(X)_g = -2k \int_M p((\nabla^g X)_\lambda, \Omega^g, (2k-1)^{\Lambda}, \Omega^g)\), where \(\Omega^g \in \Omega^2(M,\text{End}TM)\) is the curvature of the Levi-Civita connection of \(g\), and \((\nabla^g X)_\lambda\) denote the skew-symmetric part of \(\nabla^g X \in \Omega^0(M,\text{End}TM)\) with respect to \(g\). It follows from this expression of \(\mu\) that \(\omega \in \Omega^2_{\text{Diff}^+M,\text{loc}}(\mathfrak{Met}M)\).

### 5.2 Gravitational anomalies

In this section we apply the preceding considerations to the case of local gravitational anomalies (see [1, 22, 25]), and hence we consider the action of \(\text{Diff}^+M\), the connected component with the identity on \(\text{Diff}^+M\) on the space of Riemannian metrics \(\mathfrak{Met}M\). Let \(M\) be a compact spin \(n\)-manifold, with \(n = 4k - 2\) for some integer \(k\), and let \(\rho\) be a representation of \(\text{Spin}(n)\). We consider the \(\text{Diff}^+M\)-equivariant family of chiral Dirac operators \(\{\vartheta_g : g \in \mathfrak{Met}M\}\) coupled to a vector bundle \(V\) associated to the spin frame bundle. The curvature of the determinant line bundle \(\text{det } \text{Ind } \vartheta \rightarrow \mathfrak{Met}M\) is given by the Atiyah-Singer index theorem for families in the following way.

Let us consider the principal \(SO(n)\)-bundle \(O^+M \rightarrow M \times \mathfrak{Met}M\), where \(O^+M = \{(u_x, g) \in FM \times \mathfrak{Met}M : u_x \text{ is } g_x\text{-orthonormal and positively oriented}\}\). The evaluation map \(j^1 : M \times \mathfrak{Met}M \rightarrow J^1\mathcal{M}_M\), admits a lift to the corresponding orthonormal frame bundles \(\tilde{j}^1 : O^+M \rightarrow O^+M\), \(\tilde{j}^1(u_x, g) = (u_x, j^1_x g)\). The map \(\tilde{j}^1\) is a morphism of principal \(SO(n)\)-bundles and is \(\text{Diff}^+M\)-equivariant. The pull-back of the universal Levi-Civita connection \(\omega \in \Omega^1(O^+M,\mathfrak{so}(n))\) by
is a $\text{Diff}^+ M$-invariant connection form $\omega = \overline{j}^* \omega$ on $O^+ M$, with curvature $\Omega = j^* (\Omega)$, and $j^* (p_k (\Omega_{\text{Diff}^+ M}))$ is the $\text{Diff}^+ M$-equivariant $k$-th Pontryagin form of $\omega$. By the Atiyah-Singer index theorem for families we have

$$\Omega_{\text{Diff}^+ M}^\text{det Ind} = \int_M [\hat{A}(\Omega_{\text{Diff}^+ M}) \text{ch}^\rho(\Omega_{\text{Diff}^+ M})]_{n+2}$$

$$= \exists [P(\Omega_{\text{Diff}^+ M})] \in \Omega_{\text{Diff}^+ M, loc}^2 (\text{Met}_M$$

where $P = [\hat{A}\chi^\rho]_{n/2+1} \in I_n^{\text{O}(n)}$ is the component of $\hat{A}\chi^\rho$ of polynomial degree $n/2 + 1$, $\hat{A}$ is the $\hat{A}$-genus and $\text{ch}^\rho$ denotes the Chern character with respect to the representation $\rho$. Hence the condition of assumption (A2) is satisfied. That assumption (A1) is also satisfied follows from the local expression of the lift of $X \in \mathcal{X}(M)$ to $\mathcal{M}_M$ (see e.g. [20]).

Remark 12 If we prefer to work with the quotient bundle, we restrict to the subgroup $\text{Diff}^0 M$ of diffeomorphisms $\phi \in \text{Diff} M$ such that $\phi(x_0) = x_0$ and $\phi_{\ast x_0} = \text{id}_x_{\ast x_0}$ for certain $x_0 \in M$. Then the action of $\text{Diff}^0 M$ on $\mathcal{M}_M$ is free and we have a well defined quotient manifold $\mathcal{M}_M/\text{Diff}^0 M$. The first Chern class of the quotient bundle is given by $c_1 (\det \text{Ind} \phi)/\text{Diff}^0 M = \text{ChW}([\Omega_{\text{Diff}^0 M}^{\text{Ind}}]) \in H^2 (\mathcal{M}_M/\text{Diff}^0 M)$. As remarked in the introduction, in this paper we prefer to work with equivariant cohomology rather than with the cohomology of the quotient.

According to Definition 8 the first obstruction for anomaly cancellation is the class $[\Omega_{\text{Diff}^0 M}] \in H^2_{\text{loc}} (\mathcal{M}_M/\text{Diff}^0 M)$. In [19] it is proved that we have $H^2_{\text{loc}} (\mathcal{M}_M/\text{Diff}^0 M) \cong H^{n+2} (J^\infty \mathcal{M}_M/\text{Diff}^0 M)$, and hence, the cohomology class $[\Omega_{\text{Diff}^0 M}] \in H^2_{\text{loc}} (\mathcal{M}_M/\text{Diff}^0 M)$ vanishes if and only if the class of $P(\Omega)$ on $H^{n+2} (J^\infty \mathcal{M}_M/\text{Diff}^0 M)$ vanishes. We have the following result (see [19]).

**Theorem 13** The map $I_k^{SO(n)} \to H^{2k} (J^\infty \mathcal{M}_M/\text{Diff}^0 M), p \mapsto p(\Omega)$ is injective for $k \leq n$. Hence a form $p(\Omega)$ is the exterior differential of a $\text{Diff}^0 M$-invariant form on $J^\infty \mathcal{M}_M$ if and only if $p = 0$.

Hence, we conclude that the local gravitational anomaly vanishes if and only if $P = 0$. Note that the condition for anomaly cancellation is independent of the manifold $M$ and of the topology of $\text{Diff}^+ M$ or $\mathcal{M}_M/\text{Diff}^0 M$. It only depends on the dimension $n$ and the Spin representation $\rho$. This result is in accordance with the universality character of anomalies expressed in [10] [13].

Remark 14 The preceding Corollary tell us that if $P \neq 0$ it is impossible to find a local counterterm to cancel the anomaly. However, it could be possible to obtain a non-local counterterm. For example (see [7]), for $M = S^6$ we have $c_1 (\det \text{Ind} \phi/\text{Diff}^0 M) = 0$, and hence there exist a non-local counterterm.

As the space $\mathcal{M}_M$ is contractible and we have $H^k_{\text{loc}} (\mathcal{M}_M) \cong H^{n+k} (\mathcal{M}_M) \cong H^{n+k} (M) = 0$ for $k > 0$, from Theorems 13 and 10 we obtain the following
Corollary 15 Given \( p \in J_{n/2+1} \), let \( \omega = \omega_0 + \mu \in \Omega^2_{\text{Diff}^+M,\text{loc}}(\mathcal{M},\mathcal{R}M) \) be the \( \text{Diff}^+M \)-equivariant two form \( \omega = \Im[p(\Omega_{\text{Diff}^+M})] \). For any \( \alpha \in \Omega^1_{\text{loc}}(\mathcal{R}M) \) such that \( \omega_0 = d\alpha \), the cohomology class of \( \tau_\alpha \) in the local BRST cohomology \( H^1_{\text{loc}}(\mathcal{X}(M),\Omega^0_{\text{loc}}(\mathcal{R}M)) \) does not vanish.

6 Connections and mixed Anomalies

In this section we made an study of mixed anomalies similar to that of Section 5 for gravitational anomalies. We consider the family of Dirac operators \( \{ \nabla_{g,A} : g \in \mathcal{M}, A \in \mathcal{A}_P \} \) parametrized by metrics on \( M \) and connections on a principal bundle \( P \), and the action of the group \( \text{Aut}P \) of automorphisms of \( P \) (we consider that \( \text{Aut}P \) acts on \( \mathcal{M} \) through its projection to \( \text{Diff}M \)). First we recall the definition of the equivariant characteristic forms on the bundle of connections introduced on \([18]\), and using that construction and those in Section 5 we show that assumptions (A1) and (A2) also hold in the case of mixed anomalies. Finally, we obtain necessary and sufficient conditions for local mixed anomaly cancellation.

6.1 The equivariant characteristic forms on the bundle of connections

We consider a principal \( G \)-bundle \( \pi : P \to M \) over a compact \( n \)-manifold \( M \). We denote by \( \mathcal{A}_P \) the space of principal connections on \( P \). Let us recall the definition of the bundle of connections of \( P \) (see \([14, 23, 27]\) for details).

Let \( \bar{p} : J^1P \to P \) be the first jet bundle of \( P \). The action of \( G \) on \( P \) lifts to an action on \( J^1P \). We denote by \( p : C(P) = J^1P/G \to M = P/G \) the quotient bundle, called the bundle of connections of \( P \). We have a natural identification \( \Gamma(C(P)) \cong \mathcal{A}_P \), and we denote by \( \sigma_A \) the section of \( C(P) \) corresponding to \( A \in \mathcal{A}_P \). The projection \( \bar{\pi} : J^1P \to C(P) \) is a principal \( G \)-bundle, isomorphic to the pull-back bundle \( p^*P \to C(P) \), that we denote by \( \bar{\pi} : \mathbb{P} \to C(P) \). We have the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{P} & \xrightarrow{\bar{p}} & P \\
\bar{\pi} \downarrow & & \downarrow \pi \\
C(P) & \xrightarrow{p} & M
\end{array}
\]

The map \( \bar{p} \) is \( G \)-equivariant, i.e., is a principal \( G \)-bundle morphism.

The group \( \text{Aut}P \) of principal \( G \)-bundle automorphisms is denoted by \( \text{Aut}P \). If \( \phi \in \text{Aut}P \), we denote by \( \hat{\phi} \in \text{Diff}M \) its projection onto \( M \). We denote by \( \text{Aut}^+P \) the subgroup of elements \( \phi \in \text{Aut}P \) such that \( \hat{\phi} \in \text{Diff}^+M \). The kernel of the projection \( \text{Aut}P \to \text{Diff}M \) is the gauge group of \( P \), denoted by \( \text{Gau}P \).

The Lie algebra of \( \text{Aut}P \) can be identified with the space \( \text{aut}P \subset \mathfrak{X}(P) \) of \( G \)-invariant vector fields on \( P \). The subspace of \( G \)-invariant vertical vector fields is denoted by \( \text{gau}P \) and can be considered as the Lie algebra of \( \text{Gau}P \). We have an exact sequence of Lie algebras \( 0 \to \text{gau}P \to \text{aut}P \to \mathfrak{X}(M) \to 0 \).
The action of $\text{Aut}P$ on $P$ induces actions on $J^1P$ and $C(P)$, and the maps $\bar{\pi}$ and $\bar{p}$ are $\text{Aut}P$-invariant. At the infinitesimal level, if $X \in \text{aut}P$, we denote by $\mathbf{X} \in \mathfrak{X}(M)$ its projection to $M$, and by $X_p \in \mathfrak{X}(P)$, $X_{C(P)} \in \mathfrak{X}(C(P))$ its lift to $P = J^1P$ and $C(P)$ respectively. That the action of $\text{Aut}P$ on $C(P)$ satisfies assumption (A1) follows from the natural identification $\text{aut}P \cong \Gamma(M, TP/G)$ and the local expression of $X_{C(P)}$ (e.g. see [14]).

The principal $G$-bundle $\bar{\pi}: P \to C(P)$ is endowed with a canonical $\text{Aut}P$-invariant connection $\mathbb{A} \in \Omega^1(P, g)$ of curvature $\hat{\text{Met}}$. Weil polynomial of degree $k$ for $G$, we denote by $f(\mathcal{F}_{\text{Aut}P}) \in \Omega^2_{\text{Aut}P}(C(P))$ the $\text{Aut}P$-equivariant characteristic form associated to $f$.

If $2k > n$, by applying the map $\exists$ to $f(\mathcal{F}_{\text{Aut}P})$ we obtain closed $\text{Aut}^+P$-equivariant form on $A_P$. In particular if $n = 2r$ is even and $f \in \mathcal{I}_r^2$ then $\omega = \exists[f(\mathcal{F}_{\text{Aut}P})] \in \Omega^2_{\text{Aut}^+P}(A_P)$. We have $\omega = \omega_0 + \mu$, and the expression of $\mu : \text{aut}P \to \Omega^1(A_P)$ is given for $X \in \text{aut}P$ and $A \in A_P$ by $\mu(X)_A = \int_M f(A(X), F_A, \ldots, F_A)$, and from this expression we conclude that $\omega \in \Omega^2_{\text{Aut}^+P, \text{loc}}(A_P)$.

As usual (see [4]), we consider the principal $G$-bundle $P \times A_P \to M \times A_P$. The evaluation map $\text{ev} : M \times A_P \to C(P)$, $\text{ev}(x, A) = \sigma_A(x)$ extends to an $\text{Aut}P$-equivariant map $\mathbf{ev} : P \times A_P \to P$, by setting $\mathbf{ev}(u_x, A) = (u_x, \sigma_A(x))$ for every $x \in M$. Then $\hat{\mathbb{A}} = \mathbf{ev}^* \mathbb{A}$ is an $\text{Aut}P$-invariant connection on $P \times A_P$, with curvature $\hat{\mathbb{F}} = \mathbf{ev}^* \mathbb{F}$, and for every $f \in \mathcal{I}_r^2$, $\mathbf{ev}^* f(\mathcal{F}_{\text{Aut}P})$ is the $\text{Aut}P$-equivariant characteristic form of $\hat{\mathbb{A}}$ associated to $f$.

### 6.2 Mixed anomalies

Now we consider the product bundle $\mathcal{M}_M \times_M C(P) \to M$. The group $\text{Aut}P$ acts on $C(P)$ as explained above, and acts on $\mathcal{M}_M$ through its projection on $\text{Diff}M$, and hence $\text{Aut}P$ acts on the product $\mathcal{M}_M \times M C(P)$ and on $J^\infty(\mathcal{M}_M \times M C(P))$. The two projections $J^\infty(\mathcal{M}_M \times M C(P)) \to J^\infty \mathcal{M}_M$, $J^\infty(\mathcal{M}_M \times M C(P)) \to J^\infty C(P)$ are $\text{Aut}P$-equivariant. We denote by the same letter the forms on these spaces and their pull-backs to $J^\infty(\mathcal{M}_M \times M C(P))$. In particular, on $\Omega^*_{\text{Aut}P}(J^\infty(\mathcal{M}_M \times M C(P)))$ we have the $\text{Aut}^+P$-equivariant Pontryagin forms $p(\Omega_{\text{Aut}P})$ coming from $J^\infty \mathcal{M}_M$, and the $\text{Aut}P$-equivariant characteristic forms $f(\mathcal{F}_{\text{Aut}P})$, coming from $J^\infty C(P)$.

Let $\beta : G \to \text{Gl}(E)$ be a linear representation of $G$ and let $E \to M$ be the vector bundle associated to $P$ and $\beta$. We denote by $\text{Aut}^+P$ the connected component with the identity in $\text{Aut}P$, and we consider the $\text{Aut}^+P$-equivariant family of Dirac operators $\{ \nabla_{g,A} : g \in \text{Met}M, A \in A_P \}$. Let us consider the bundle $Q = \pi_1^* (P \times A_P) \times \pi_2^* (\mathcal{O}^+ M) \to M \times \text{Met}M \times A_P$, where $\pi_1 : M \times \text{Met}M \times A_P \to M \times A_P$ and $\pi_2 : M \times \text{Met}M \times A_P \to M \times \text{Met}M$ are the
projections. We have the following commutative diagram

\[
\begin{array}{cccc}
P \times A_P & \xleftarrow{\pi_1} & Q & \xrightarrow{\pi_2} \mathcal{O}^+(M) \\
\downarrow & & \downarrow & \downarrow \\
M \times A_P & \xleftarrow{\pi_1} & M \times \text{Met} \times A_P & \xrightarrow{\pi_2} M \times \text{Met} M
\end{array}
\]

The bundle \( Q \) is a principal \((SO(n) \times G)\)-bundle, with \( \text{Aut}^+ P \)-invariant connection \( \hat{\Omega} = \pi_1^* \hat{\Lambda} + \pi_2^* \hat{\omega} \) and curvature \( \hat{\Omega}_n = \pi_1^* \hat{\varphi} + \pi_2^* \hat{\Omega} \). By the Atiyah-Singer index theorem for families, the \( \text{Aut}^+ P \)-equivariant curvature of the determinant line bundle is given by

\[
\Omega_{\text{det Ind}}^{\text{Aut}^+ P} = \int_M \left( \hat{A}(\hat{\Omega}_{\text{Aut}^+ P}) \wedge \text{ch}^P(\hat{\Omega}_{\text{Aut}^+ P}) \wedge \text{ch}^\beta(\hat{\Omega}_{\text{Aut}^+ P}) \right)_{n+2}
\]

\[
= \int_M \left( \pi_2^* \left( \hat{A}(\hat{\Gamma}_{\text{Aut}^+ P}) \wedge \text{ch}^P(\hat{\Omega}_{\text{Aut}^+ P}) \right) \wedge \pi_1^* \left( \text{ch}^\beta(\hat{\varphi}_{\text{Aut}^+ P}) \right) \right)_{n+2}
\]

\[
= \exists \left[ \hat{A}(\hat{\Omega}_{\text{Aut}^+ P}) \wedge \text{ch}^P(\hat{\Omega}_{\text{Aut}^+ P}) \wedge \text{ch}^\beta(\hat{\varphi}_{\text{Aut}^+ P}) \right]_{n+2},
\]

and hence \( \Omega_{\text{det Ind}}^{\text{Aut}^+ P} \in \Omega_{\text{Aut}^+ P \text{ loc} \text{ Met} \times A_P}^{\text{Aut}^+ P} \) and assumption (A2) is satisfied.

By Definition 8 the first obstruction for anomaly cancellation is

\[
[p, f] = \exists \left[ \hat{A}(\hat{\Omega}) \wedge \text{ch}^P(\hat{\Omega}) \wedge \text{ch}^\beta(\hat{\varphi}) \right]_{n+2} \in H_{\text{loc} \text{ Met} \times A_P}^{\text{Aut}^+ P}.
\]

Again (see [19]) the map \( \exists \) induces an isomorphism \( H_{\text{loc} \text{ Met} \times A_P}^{\text{Aut}^+ P} \cong H^{n+2}(J^\infty(\text{Met}_M \times M C(P)))^{\text{Aut}^+ P} \). Under that isomorphism the first obstruction for anomaly cancellation corresponds to the cohomology class of the form \( \hat{A}(\hat{\Omega}) \wedge \text{ch}^P(\hat{\Omega}) \wedge \text{ch}^\beta(\hat{\varphi}) \) on \( H^{n+2}(J^\infty(\text{Met}_M \times M C(P)))^{\text{Aut}^+ P} \). We have the following result (see [19]).

**Theorem 16** The map

\[
\bigoplus_{r+s=k} I_{rSO(n)} \otimes I_s^G \longrightarrow H^{2k}(J^\infty(\text{Met}_M \times M C(P)))^{\text{Aut}^+ P}
\]

\[
p \otimes f \mapsto [p(\hat{\Omega}) \wedge f(\hat{\varphi})]
\]

is injective for \( k \leq n \).

Hence, if \( Q \) is the component of polynomial degree \( n/2 + 1 \) of \( \hat{A} \text{ch}^P \otimes \text{ch}^\beta \in I_{SO(n) \times G} \cong I_{SO(n)} \otimes I_G \), then the mixed anomaly cancels if and only if \( Q = 0 \). In particular the gauge and gravitational anomalies cannot cancel between them. Again the condition for anomaly cancellation does not depend on the particular manifold \( M \) or bundle that we have. It only depends on the structure group \( G \) of \( P \) and the dimension \( n \) of \( M \).

As the space \( \text{Met} \times A_P \) is contractible and we have \( H_{\text{loc} \text{ Met} \times A_P}^{k} \cong H^{n+k}(\text{Met}_M \times M C(P)) \cong H^{n+k}(M) = 0 \) for \( k > 0 \), by Theorems 10 and 13 we have the following
Corollary 17 Let \( Q = \sum p_i \otimes f_i \in I^{SO(n)} \otimes I^G \) be a Weil polynomial of degree \( n/2 + 1 \), and let \( \omega = \omega_0 + \mu \in \Omega^2_{\text{Aut}^*M, \text{loc}}(\mathfrak{m} \times \mathcal{A}_P) \) be the \( \text{Aut}^*M \)-equivariant two form \( \omega = \sum \Im[p_i(\Omega_{\text{Aut}^*M}) \wedge f_i(\mathbb{F}_{\text{Aut}^*P})] \). For any \( \alpha \in \Omega^1_{\text{loc}}(\mathfrak{m} \times \mathcal{A}_P) \) such that \( \omega_0 = d\alpha \), the cohomology class of \( \tau_\alpha \) in the local BRST cohomology \( H^1_{\text{loc}}(\text{aut}P, \Omega^0_{\text{loc}}(\mathfrak{m} \times \mathcal{A}_P)) \) does not vanish.

Acknowledgement 18 This work is supported by Ministerio de Educación y Ciencia of Spain, under grant #MTM2008-01386.

References

[1] O. Álvarez, I. Singer, B. Zumino, Gravitational Anomalies and the Family’s Index Theorem, Commun. Math. Phys. 96 (1984) 409–417.

[2] L. Álvarez-Gaumé, P. Ginsparg, The structure of gauge and gravitational anomalies, Ann. Phys. 161, (1985) 423–490.

[3] I. Anderson, J. Pohjanpelto, Infinite dimensional Lie algebra cohomology and the cohomology of invariant Euler-Lagrange complexes: A preliminary report, Differential geometry and applications (Brno, 1995), 427–448, Masaryk Univ., Brno, 1996.

[4] M.F. Atiyah, I. Singer, Dirac operators coupled to vector potentials, Proc. Natl. Acad. Sci. USA 81 (1984), 2597–2600.

[5] G. Barnich, F. Brandt, M. Henneaux, Local BRST cohomology in gauge theories, Phys. Rep. 338 (2000), no. 5, 439–569.

[6] N. Berline, E. Getzler, M. Vergne, Heat Kernels and Dirac Operators, Springer Verlag Berlin Heidelberg 1992.

[7] N. Berline, M. Vergne, Classes caractéristiques équivariantes. Formules de localisation en cohomologie équivariante, C. R. Acad. Sci. Paris 295 (1982) 539–541.

[8] R.A. Berthmann, Anomalies in Quantum Field Theory, Oxford University Press 2000.

[9] M. Blau, Wess-Zumino Terms and the Geometry of the Determinant Line Bundle, Phys. Lett. B 209 (1988), 503–506.

[10] L. Bonora, C.S. Chu, M. Rinaldi, Anomalies and locality in field theories and M-theory. Secondary calculus and cohomological physics (Moscow, 1997), 39–52, Contemp. Math., 219, Amer. Math. Soc., Providence, RI, 1998.

[11] L. Bonora, P. Cotta-Ramusino, Some Remarks on BRS Transformations, Anomalies and the Cohomology of the Lie Algebra of the Group of Gauge Transformations, Commun. Math. Phys. 87 (1983), 589–603.
[12] —, Consistent and covariant anomalies and local cohomology, Phys. Rev. D (3) 33 (1986), 3055–3059.

[13] L. Bonora, P. Cotta-Ramusino, M. Rinaldi, J. Stasheff. The evaluation map in Field Theory and strings I, Commun. Math. Phys. 112, (1987) 237–282.

[14] M. Castrillón López, J. Muñoz Masqué, The geometry of the bundle of connections, Math. Z. 236 (2001), 797–811.

[15] M. Dubois-Violette, M. Henneaux, M. Talon and C. Viallet, General solution of the consistency equation, Phys. Lett. B 289 (1992), 361–367.

[16] M. Dubois-Violette, M. Talon, C. Viallet, BRS algebras. Analysis of the consistency equations in gauge theory, Comm. Math. Phys. 102 (1985), 105–122.

[17] —, Results on BRS cohomologies in gauge theory, Phys. Lett. B 158 (1985), 231–233.

[18] R. Ferreiro Pérez, Equivariant characteristic forms in the bundle of connections, J. Geom. Phys. 54 (2005), 197–212.

[19] —, Local cohomology and the variational bicomplex, Int. J. Geom. Methods Mod. Phys. 5, 587–604 (2008).

[20] R. Ferreiro Pérez, J. Muñoz Masqué, Natural connections on the bundle of Riemannian metrics, Monatsh. Math. 155, 67–78 (2008).

[21] —, Pontryagin forms on (4k−2)-manifolds and symplectic structures on the spaces of Riemannian metrics, preprint (arXiv: math.DG/0507076).

[22] D.S. Freed, Determinants, torsion, and strings, Comm. Math. Phys. 107 (1986), no. 3, 483–513.

[23] P.L. García Pérez, Gauge algebras, curvature and symplectic structure, J. Differential Geom. 12 (1977), 209–227.

[24] V. Guillemin, S. Sternberg, Supersymmetry and Equivariant de Rham Theory, Springer-Verlag, Berlin Heidelberg, 1999.

[25] G. Kelnhofer, Universal bundle for gravity, local index theorem, and covariant gravitational anomalies, J. Math. Phys. 35 (1994), no. 11, 5945–5968.

[26] J. Mañes, R. Stora, B. Zumino, Algebraic study of chiral anomalies, Comm. Math. Phys. 102 (1985),157-174.

[27] L. Margiarotti, G. Sardanashvily, Connections in Classical and Quantum Field Theory, World Scientific, 2000.
[28] M. Martellini, C. Reina, *Some remarks on the index theorem approach to anomalies*, Ann. Inst. H. Poincarè 113 (1985), 443-458.

[29] D.J. Saunders, *The Geometry of Jet Bundles*, London Mathematical Society Lecture Notes Series 142, Cambridge University Press, 1989.

[30] R. Schmid, *Local cohomology in gauge theories, BRST transformations and anomalies*, Differential Geom. Appl. 4 (1994), no. 2, 107–116.

[31] I.M. Singer, *Families of Dirac operators with applications to physics*, The mathematical heritage of Élie Cartan (Lyon, 1984). Astérisque (1985), Numero Hors Serie, 323–340.