Family of Affine Quantum Group Invariant Integrable Extensions of Hubbard Hamiltonian

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Abstract

We construct the family of spin chain Hamiltonians, which have affine quantum group symmetry \( U_q\hat{g} \). Their eigenvalues coincide with the eigenvalues of the usual spin chain Hamiltonians, but have the degeneracy of levels, corresponding to affine \( U_q\hat{g} \). The space of states of these spin chains is formed by the tensor product of fully reducible representations of quantum group.

The fermionic representations of constructed spin chain Hamiltonians show that we have new extensions of Hubbard Hamiltonians. All of them are integrable and have affine quantum group symmetry. The exact ground state of such type model is presented, exhibiting superconducting behavior via \( \eta \)-pairing mechanism.
1 Introduction

The interest to strongly interacting electron systems was raised after discovery of high-$T_c$ superconductivity [1], where it is believed that the interplay between magnetism and insulating behavior has a crucial role. The Hubbard model [2] is the simplest example of strongly correlated electrons and was solved in one dimension via Bethe ansatz technique [3]. Subsequently numerous publications appeared investigating one-band (multi-band) Hubbard models in one and two dimensions, most of which used approximate or numerical methods and only few exact results are known. One of these exact results based on so called $\eta$-pairing mechanism of superconductivity, the idea of which was introduced by Yang [14, 15] for the Hubbard model. It appeared, that there is certain type of states exhibiting off-diagonal long-range order (ODLRO), the concept, which had been developed in [4]. In [4, 5, 6] was shown that ODLRO implies Meissner effect and flux quantization and thus can be regarded as definition of superconductivity. But ground state of simple Hubbard model is not superconducting [4]. The first model with truly superconducting ground state of $\eta$-pairing type is the supersymmetric Hubbard model, introduced in [16, 17]. This model is a certain extension of Hubbard model by additional nearest-neighbor interactions and some restriction on hopping mechanism.

It appeared that $\eta$-pairing mechanism is not an exotic phenomenon and recently some extended Hubbard models were defined, superconducting ground state of which could be found exactly [7, 8, 9, 11, 13]. The main property of these extensions is the fact, that they besides the ordinary electron hopping and Hubbard interaction terms contain also bond-charge interaction, pair-hopping, nearest-neighbor Coulomb interaction and $XXX$ spin interaction terms.

In this article we develop the new technique and define a family of other extensions of one dimensional Hubbard Hamiltonians, one of which exhibits superconductivity based on $\eta$-pairing. The defining property of this family is the fact, that they have $U_q\hat{sl}(2)$ affine quantum group symmetry.

Quantum group symmetry plays essential role in integrable statistical models [20, 21, 22] and conformal field theory [23, 24, 25].

It is well known that many integrable Hamiltonians have quantum group symmetry. For example, $XXZ$ Heisenberg Hamiltonian with particular boundary terms is $U_qsl_2$-invariant [24]. The infinite $XXZ$ spin chain has larger symmetry: affine $U_qsl_2$ [20]. The single spin site of most considered Hamiltonians forms an irreducible representation of Lie algebra or its quantum deformation.

In [19] we have constructed the family of spin chain Hamiltonians $\mathcal{H}$, which have affine quantum group symmetry $U_q\hat{g}$, using intertwining operators between tensor products of its spectral parameter dependent reducible representations. These intertwining operators commute with $U_q\hat{g}$ by definition, whereas Hamiltonians $\mathcal{H}$ commute with $U_q\hat{g}$ by construction. The space of states of these spin chains is formed by the tensor product of the fully reducible representations. We
have shown that the model, considered in [27], which corresponds to some extension of the Hubbard Hamiltonian in the strong repulsion limit, is a particular case of our general construction. The affine quantum group symmetry leads to a high degeneracy of energy levels.

The energy levels of these spin chains are formed on the states, constructed from the highest weight vectors of quantum group representations. In particular cases the restriction of the considered spin chains on these states gives rise to Heisenberg model or Haldane-Shastry long-range interaction spin chain. Hence, we have the generalizations of these Hamiltonians, which have affine quantum group symmetry.

It appeared that the fermionization of the simplest examples of these spin chain Hamiltonians gives the family of extensions of Hubbard model.

In the sec.2 we give the definitions of quantum Kac-Moody group $U_q\hat{g}$ and properties of $R$-matrix. In sec.3 we develop the technique for construction of nearest-neighbor spin chain Hamiltonians, which have $U_q\hat{sl}_2$ quantum group symmetry. In sec.4 we perform this construction to obtain Haldane-Shastry type long-range spin chain Hamiltonians.

In sec.5 we consider the fermionization of previously constructed integrable spin chains for some simplest particular cases, in particular, when the site is four dimensional and there is no dependence on quantum group deformation parameter $q$. We obtain in such way some integrable extensions of Hubbard model. In the case, when the space of states of each site is a direct sum of spin-1 and trivial multiplets, one obtain Hubbard model with doping, pair hopping and nearest-neighbor interaction of holes.

In the case, when the space of states of each site is a direct sum of two two dimensional representations, one obtain Hubbard model with bond-charge interaction, density-density interaction and 'boson-boson’ interaction between nearest-neighbor hole and double occupied site. We find the ground states in the fixed particle number sectors and phase space region, where this model exhibits $\eta$-pairing superconductivity. In the particular case, when the amplitude of last two interactions vanishes, one obtain well known extended Hubbard model with bond-charge interaction [12, 13].

## 2 Definitions

Let us recall the definition of quantum Kac-Moody group $U_q\hat{g}$ ($U_qg$) [20, 21, 22]. It is generated by elements $e_i$, $f_i$, $h_i$ satisfying the relations

\[
[h_i, e_j] = c_{ij}e_j, \quad [h_i, f_j] = -c_{ij}f_j \quad (2.1)
\]

\[
[e_i, f_j] = \delta_{ij}\{h\}_q
\]

and $q$-deformed Serre relations, which we don’t write. Here $i = 0, \ldots, n$ for $U_q\hat{g}$ and $i = 1, \ldots, n$ for $U_qg$, $q$ is a deformation parameter, $[x]_q := (x^q - x^{-q})/(q - q^{-1})$. 


\(c_{ij}\) is a Cartan matrix of corresponding affine Lie algebra \(\hat{g}\) (finite Lie algebra \(g\)). On \(U_q\hat{g}\) \((U_qg)\) there is a Hopf algebra structure:

\[
\Delta(e_i) = k_i \otimes e_i + e_i \otimes k_i^{-1}
\]
\[
\Delta(f_i) = k_i \otimes f_i + f_i \otimes k_i^{-1}
\]
\[
\Delta(k_i^{\pm1}) = k_i^{\pm1} \otimes k_i^{\pm1}
\]

(2.2)

where \(k_i := q^{h_i/2}\). This comultiplication can be extended to \(L\)-fold tensor product by

\[
\Delta^{L-1}(e_i) = \sum_{l=1}^{L} k_i \otimes \cdots \otimes k_i \otimes e_i \otimes k_i^{-1} \otimes \cdots \otimes k_i^{-1}
\]
\[
\Delta^{L-1}(f_i) = \sum_{l=1}^{L} k_i \otimes \cdots \otimes k_i \otimes f_i \otimes k_i^{-1} \otimes \cdots \otimes k_i^{-1}
\]
\[
\Delta^{L-1}(k_i^{\pm1}) = k_i^{\pm1} \otimes \cdots \otimes k_i^{\pm1}
\]

There is an opposite comultiplication \(\bar{\Delta}\), which is obtained from (2.2) by replacing \(k_i^{\pm1} \rightarrow k_i^{\mp1}\).

For general \(q\) the representations of quantum group \(U_qg\) are in one to one correspondence to the representations of nondeformed Lie algebra \(g\). Denote by \(V_\lambda\) the irreducible \(U_qg\)-multiplet with highest weight \(\lambda\). It possesses a highest weight vector \(v_\lambda^0\), such that

\[
e_i v_\lambda^0 = 0 \quad h_i v_\lambda^0 = \lambda(h_i)v_\lambda^0, \quad i = 1, \ldots, n
\]

(2.3)

Let \(\hat{g}\) be an affine algebra and \(g\) be its underlying finite algebra. Then for any complex \(x\) there is the \(q\)-deformation of loop homomorphism \(\rho_x:\ U_q\hat{g} \rightarrow U_qg\), which is given by \([20]\)

\[
\rho_x(e_0) = xf_0 \quad \rho_x(f_0) = x^{-1}e_0 \quad \rho_x(h_0) = -h_0
\]
\[
\rho_x(e_i) = e_i \quad \rho_x(f_i) = f_i \quad \rho_x(h_i) = h_i,
\]

where \(i = 1 \ldots n\) and \(\theta\) is a maximal root of \(U_qg\). Using \(\rho_x\) one can construct the spectral parameter dependent representation of \(U_q\hat{g}\) from the representation of \(U_qg\).

Let \(V_1(x_1)\) and \(V_2(x_2)\) are constructed in such way irreducible finite dimensional representations of \(U_q\hat{g}\) with parameters \(x_1\) and \(x_2\) correspondingly. The \(U_q\hat{g}\)-representations on \(V_1(x_1) \otimes V_2(x_2)\) constructed by means of both \(\Delta\) and \(\bar{\Delta}\) are irreducible, in general, and equivalent:

\[R(x_1, x_2)\Delta(g) = \bar{\Delta}(g)R(x_1, x_2), \quad \forall g \in U_q\hat{g}\]

The \(R\)-matrix \(R(x_1, x_2)\) depends only on \(x_1/x_2\) and is a Boltzmann weight of some integrable statistical mechanic system.
3 Quantum group invariant Hamiltonians for reducible representations

Let \( V = \bigoplus_{i=1}^{N} V_{\lambda_i} \) is a direct sum of finite dimensional irreducible representations of \( U_qg \). We denote by \( V(x_1, \ldots, x_N) \) corresponding affine \( U_q\hat{g} \) representation with spectral parameters \( x_i \):

\[
V(x_1, \ldots, x_N) = \bigoplus_{i=1}^{N} V_{\lambda_i}(x_i)
\]

(3.1)

We consider the intertwining operator

\[
H(x_1, \ldots, x_N) : V(x_1, \ldots, x_N) \otimes V(x_1, \ldots, x_N) \to V(x_1, \ldots, x_N) \otimes V(x_1, \ldots, x_N),
\]

\[
[H(x_1, \ldots, x_N), \Delta(a)] = 0, \quad \forall a \in U_q\hat{g}
\]

If \( V = V_{\lambda} \) consists of one irreducible component then \( H \) is a multiple of identity, because the tensor product is irreducible in this case. To carry out the general case let us gather all equivalent multiplets together:

\[
V(x_1, \ldots, x_N) = \bigoplus_{i=1}^{M} N_{\lambda_i} \hat{\otimes} V_{\lambda_i}(x_i),
\]

(3.2)

where all \( V_{\lambda_i}(x_i) \) are \( M \) nonequivalent irreps and \( N_{\lambda_i} \simeq C^{N_i} \) have a dimension equal to the multiplicity of \( V_{\lambda_i}(x_i) \) in \( V(x_1, \ldots, x_N) \). Note that \( \sum_{i=1}^{M} N_i = N \).

By the hat over the tensor product we mean that \( U_q\hat{g} \) doesn’t act on \( N_{\lambda_i} \hat{\otimes} V_{\lambda_i}(x_i) \) by means of \( \Delta \) but acts as \( \id \otimes g \).

So, we have:

\[
V(x_1, \ldots, x_M) \otimes V(x_1, \ldots, x_M)
\]

\[
= (\bigoplus_{i=1}^{M} N_{\lambda_i} \hat{\otimes} V_{\lambda_i}(x_i)) \otimes (\bigoplus_{i=1}^{M} N_{\lambda_i} \hat{\otimes} V_{\lambda_i}(x_i))
\]

\[
= \bigoplus_{i,j=1}^{M} N_{\lambda_i} \hat{\otimes} N_{\lambda_j} \hat{\otimes} \left( V_{\lambda_i}(x_i) \otimes V_{\lambda_j}(x_j) \right)
\]

Now, \( V_{\lambda_i}(x_i) \otimes V_{\lambda_j}(x_j) \) is equivalent only to itself and to \( V_{\lambda_j}(x_j) \otimes V_{\lambda_i}(x_i) \) (for \( i \neq j \)) by applying the intertwining operator \( \check{R}(x_i/x_j) = PR(x_i/x_j) \), where \( P \) is tensor product permutation: \( P(v_1 \otimes v_2) = v_2 \otimes v_1 \). So, the operator \( H(x_1, \ldots, x_M) \), commuting with \( U_q\hat{g} \) on \( V(x_1, \ldots, x_M) \otimes V(x_1, \ldots, x_M) \) has the following form:

\[
H|_{N_{\lambda_i} \hat{\otimes} N_{\lambda_j} \hat{\otimes} V_{\lambda_i} \otimes V_{\lambda_j}} = A_{ij} \hat{\otimes} \id_{V_{\lambda_i} \otimes V_{\lambda_j}} + B_{ij} \hat{\otimes} \check{R}_{V_{\lambda_i} \otimes V_{\lambda_j}}(x_i/x_j)
\]

(3.3)

\footnote{The \( U_q\hat{g} \)-equivalence of \( V_{\lambda_i}(x_i) \) requires that the spectral parameters \( x_i \) and highest weights \( \lambda_i \) are the same.}
where $A_{ij}$ and $B_{ij}$ are any operators on $N_\lambda \otimes N_\lambda$. Note that in one special case the formula (3.3) simplifies. Consider the situation if there is only one nontrivial representation $V_\lambda$, i.e. $\lambda \neq 0$, in the decomposition (3.2) of $V(x_1, \ldots, x_N)$. As $RV_{\lambda} \otimes V_0 = RV_0 \otimes V_\lambda = \text{id}$, the intertwining operator (3.3) doesn’t contain spectral parameters $x_i$ and deformation parameter $q$. So, $H$ is invariant under the action of quantum group for all values of deformation parameter. Of course, this is much larger symmetry and it can be written in another form without using of $q$.

To write down the action (3.3) more explicitly we introduce the projection operators

$$X_k^a = |a\rangle\langle b|,$$

where the vectors $|a\rangle$ span the space $V$. In accordance with the decomposition (3.2) we use the double index $n$ and $\lambda$ represented as

$$n \in \mathbb{N}, \lambda \in V_\lambda,$$

where $n_1$ and $n_2$ are indexes, which mention each $V_\lambda$. Then the formula (3.3) can be represented as

$$H(A, B) = \sum_{i,j=1}^M \left( \sum_{n_i, n_j, m_i, m_j} A_{ij}^{m_i m_j} \sum_{a_i, a_j} X_{(n_i, a_i)}^{(n_j, a_j)} \otimes X_{(m_j, a_j)}^{(m_i, a_i)} \right)$$

$$+ \sum_{n_i, n_j, m_i, m_j} B_{ij}^{m_i m_j} \sum_{a_i, a_j, a_i', a_j'} R_{ij}^{a_i a_j} (x_i/x_j) X_{(n_j, a_j')}^{(n_i, a_i')} \otimes X_{(m_j, a_j)}^{(m_i, a_i)} \right),$$

where $B_{ii} = 0$ and we use the matrix form of $R$-operator:

$$R_{\lambda_1 \otimes \lambda_2} (x_i/x_j) |a_i\rangle \otimes |a_j\rangle = \sum_{a_i', a_j'} R_{ij}^{a_i a_j} (x_i/x_j) |a_i'\rangle \otimes |a_j'\rangle.$$

Let us consider some particular cases of this general construction.

(i) Let $V(x) = V(x, x) = V_\lambda(x) \oplus V_\lambda(x)$. The second term in (3.3) is absent in this case and $H$ has factorized form:

$$H = A \otimes \text{id} V_\lambda \otimes \text{id} V_\lambda, \quad A = A_{n_1 n_2}^{m_1 m_2}$$

where $n_1, n_2, m_1, m_2 = 1, 2$ are indexes, which mention each $V_\lambda$. In the matrix form (3.6) can be written as

$$H = \sum_{n_i, n_j, m_i, m_j} A_{n_i n_j}^{m_i m_j} \sum_{a_i, a_j} X_{(n_i, a_i)}^{(n_j, a_j)} \otimes X_{(m_j, a_j)}^{(m_i, a_i)}$$

(ii) Let now $V(x_1, x_2) = V_{\lambda_1}(x_1) \oplus V_{\lambda_2}(x_2)$ ($V_{\lambda_1}(x_i)$ are mutually nonequivalent). Then $H$ acquires the following block-diagonal form

$$H(x_1, x_2) = \begin{pmatrix} a \cdot \text{id} & 0 & 0 & 0 \\ 0 & c \cdot \text{id} & d \cdot R_{12}(x_1/x_2) & 0 \\ 0 & e \cdot R_{21}(x_2/x_1) & f \cdot \text{id} & 0 \\ 0 & 0 & 0 & g \cdot \text{id} \end{pmatrix}$$

(3.8)
Here to be short we used the notation
\[ R_{12}(x_1/x_2) = R_{\lambda_1}^{\lambda_2} \otimes \nu_{\lambda_2}(x_1/x_2), \quad R_{21}(x_2/x_1) = R_{\lambda_2}^{\lambda_1} \otimes \nu_{\lambda_1}(x_2/x_1). \]

(iii) Let us choose \( g = sl(2) \) and \( V = V_s \oplus V_0 \), where \( V_s \) is \( 2s + 1 \)-dimensional spin-\( s \) representation of \( U_q sl_2 \) and \( V_0 \) is the trivial one dimensional representation. This case was considered in [27].

Following [27] from the operator \( H \) the following Hamiltonian acting on \( W = V \otimes L \) can be constructed:
\[
\mathcal{H} = \sum_{i=1}^{L-1} H_{ii+1}
\]
(3.9)

Here and in the following for the operator \( X = \sum_l x_l \otimes y_l \) on \( V \otimes V \) we denote by \( X_{ij} \) its action on \( W \) defined by
\[
X_{ij} = \sum_l \text{id} \otimes \ldots \otimes \text{id} \otimes x_i \otimes \text{id} \otimes \ldots \otimes y_j \otimes \text{id} \otimes \ldots \otimes \text{id}
\]
(3.10)

By the construction, \( \mathcal{H} \) is quantum group invariant:
\[
[\mathcal{H}, \Delta^{L-1}(g)] = 0, \quad \forall g \in U_q \hat{g}
\]

Let us define the projection operator \( P_i \) on \( V \) for each class of equivalent irreps \( (\lambda_i, x_i), i = 1, \ldots, M \):
\[
P_i v_j = \delta_{ij} v_j, \quad \forall v_j \in V_{\lambda_j}(x_j)
\]
\[
\sum_{i=1}^M P_i = \text{id}, \quad P_i^2 = P_i
\]

Their action on the tensor product \( W \) is given by
\[
\bar{P}_i = \sum_{l=1}^L \text{id} \otimes \ldots \otimes \text{id} \otimes P_i \otimes \text{id} \otimes \ldots \otimes \text{id}
\]

It is easy to see that these projections commute with Hamiltonian \( \mathcal{H} \) and quantum group \( U_q \hat{g} \):
\[
[\bar{P}_i, \mathcal{H}] = 0, \quad [\bar{P}_i, U_q \hat{g}] = 0
\]
(3.11)

Denote by \( W_{p_1\ldots p_M} \) the subspace of \( W \) with values \( p_i \) of \( \bar{P}_i \) on it. Then we have the decomposition
\[
W = \bigoplus_{p_1 + \ldots + p_M = L} W_{p_1\ldots p_M}
\]
(3.12)

\(^2\)Here and in the following we omit the dependence on \( x_i \).
Let $V^0$ is the linear space, spanned by the highest weight vectors of $U_q\mathfrak{g}$-module $V$:

$$V^0 := \bigoplus_{i=1}^N v^0_{\lambda_i},$$

where $v^0_{\lambda_i} \in V_{\lambda_i}$ is highest weight vector. We define also $W^0 := V^0 \otimes L$. The space $W^0$ is $\mathcal{H}$-invariant. This follows from (3.3). For general $q$ the action of $U_q\hat{\mathfrak{g}}$ on $W^0$ generate whole space $W$. Indeed, the $U_q\hat{\mathfrak{g}}$-action on each state of type $v^0_{\lambda_1} \otimes \ldots \otimes v^0_{\lambda_L}$ generates the whole space $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_L}$, because the tensor product of finite dimensional irreducible representations of affine quantum group is irreducible.

Consider now the subspace $W^0_{p_1 \ldots p_M} = W^0 \cap W_{p_1 \ldots p_M}$. We have the decomposition, which is inherited from (3.12)

$$W^0 = \bigoplus_{p_1 \ldots p_M, p_1 + \ldots + p_M = L} W^0_{p_1 \ldots p_M}. \quad (3.13)$$

Note that

$$d_{p_1 \ldots p_M} := \dim W^0_{p_1 \ldots p_M} = \left( \binom{L}{p_1 \ldots p_M} N_{1}^{p_1} \ldots N_{M}^{p_M} \right).$$

Let us define by $H_0$ and $\mathcal{H}_0$ the restrictions of $H$ and $\mathcal{H}$ on $V^0 \otimes V^0$ and $W^0$ correspondingly: $H_0 := H|_{V^0 \otimes V^0}$, $\mathcal{H}_0 := \mathcal{H}|_{W^0}$. It follows from (3.11) that Hamiltonians $\mathcal{H}$ and $\mathcal{H}_0$ have block diagonal form with respect to the decompositions (3.12) and (3.13) correspondingly. Every Hamiltonian eigenvector $w^0_{\alpha p_1 \ldots p_M} \in W^0_{p_1 \ldots p_M}$ with energy value $E_{\alpha p_1 \ldots p_M}$ gives rise to an irreducible $U_q\hat{\mathfrak{g}}$-multiplet $W_{\alpha p_1 \ldots p_M}$ of dimension

$$\dim W_{\alpha p_1 \ldots p_M} = \prod_{k=1}^M \left( \dim V_{\lambda_k} \right)^{p_k} \quad (3.14)$$

On $W_{\alpha p_1 \ldots p_M}$ the Hamiltonian $\mathcal{H}$ is diagonal with eigenvalue $E_{\alpha p_1 \ldots p_M}$. In particular case when all $V_{\lambda_i}$ are equivalent, the degeneracy levels are the same for all $E_{\alpha p_1 \ldots p_M}$ and are equal to $(\dim V_{\lambda})^L$. Note that

$$\dim W = \sum_{p_1 \ldots p_M, p_1 + \ldots + p_M = L} \sum_{\alpha p_1 \ldots p_M = 1}^{d_{p_1 \ldots p_M}} \dim W_{\alpha p_1 \ldots p_M}$$

$$= \sum_{p_1 \ldots p_M, p_1 + \ldots + p_M = L} \left( \binom{L}{p_1 \ldots p_M} \right) \prod_{k=1}^M N_{k}^{p_k} \prod_{k=1}^M \left( \dim V_{\lambda_k} \right)^{p_k}$$

$$= \left( \sum_{k=1}^M N_k \dim V_{\lambda_k} \right)^L$$
according to the decomposition

\[ W = \bigoplus_{p_1, \ldots, p_M} \bigoplus_{\alpha_{p_1 \ldots p_M} = 1} W_{\alpha_{p_1 \ldots p_M}} \]

Now, we suggest that one can obtain the energy spectrum \( E_{\alpha_{p_1 \ldots p_M}} \) for \( \mathcal{H}_0 \). Then for the statistical sum we have

\[ Z_{\mathcal{H}_0}(\beta) = \sum_{p_1, \ldots, p_M} \sum_{\alpha_{p_1 \ldots p_M} = 1} \exp(\beta E_{\alpha_{p_1 \ldots p_M}}) \]  

(3.15)

Then the statistical sum of \( \mathcal{H} \) has the following form:

\[ Z_{\mathcal{H}}(\beta) = \sum_{p_1, \ldots, p_M} \prod_{k=1}^{M} (\dim V_{\lambda_k})^{p_k} \sum_{\alpha_{p_1 \ldots p_M} = 1} \exp(\beta E_{\alpha_{p_1 \ldots p_M}}) \]  

(3.16)

So, if the underlying Hamiltonian \( \mathcal{H}_0 \) is integrable and its eigenvectors and eigenvalues can be obtained, then this problem is solved for \( \mathcal{H} \) also. Performing the quantum group on all eigenvectors of an energy level of \( \mathcal{H}_0 \) one can obtain the whole eigenspace \( \mathcal{H} \) for this level.

Consider now some particular cases of our general construction. If we choose two equivalent representations (the first example above), then \( \dim V^0 = 2 \) and there is one term in decomposition (3.13). \( \mathcal{H}_0 \) now is the most general action on \( V^0 \otimes V^0 \). As a particular case, the XYZ Hamiltonian in the magnetic field can be obtained. The correspondence between the spectrum of \( \mathcal{H} \) and \( \mathcal{H}_0 \) is the simplest in this case because the additional degeneracy of all energy levels is the same and for the statistical sums (3.15), (3.16) we have

\[ Z_{\mathcal{H}}(\beta) = (\dim V_{\lambda})^L Z_{\mathcal{H}_0}(\beta) = (\dim V_{\lambda})^L Z_{XXZ}(\beta) \]

Consider now the second example of \( H \)-operator above. Let us choose for parameters in (3.8)

\[ a = g = c = d = 1 \quad c = f = 0 \]  

(3.17)

One can renormalize \( R \)-matrices in (3.8) to satisfy the unitarity condition

\[ R_{12}(z) R_{21}(z^{-1}) = \text{id} \]

. Together with (3.17) this leads to

\[ H(x_1, x_2)^2 = \text{id} \]  

(3.18)

The restriction of \( \mathcal{H} \) on \( W^0 \) coincides with the XXX Heisenberg spin chain

\[ \mathcal{H}_0 = H_{XXX} = \sum_i P_{i i+1} = \frac{1}{2} \sum_i (1 + \vec{\sigma}_i \vec{\sigma}_{i+1}) \]

The space \( W^0_{pp_2} \) corresponds to all states with \( s_z = p_1/2 \) value of spin projection \( S^z = 1/2 \sum_i \sigma_i^z \). If we return to \( \mathcal{H} \), the energy level degeneracy of each eigenstate with the spin projection \( s_z \) is multiplied by \( (\dim V_{\lambda_1})^2 s_z (\dim V_{\lambda_2})^{L-2s_z} \).
4 Generalization to long-range interaction spin chains

Let us consider now the generalization of the construction above for the case of long-range interacting Hamiltonians. Recall the Haldane-Shastry spin chain is given by \cite{29, 30, 31}

\[
\mathcal{H}_{0}^{HS} = \sum_{i<j} \frac{1}{d_{i-j}^2} P_{ij},
\]

(4.1)

Here the spins take values in the fundamental representation of \(sl_n\) and \(P_{ij}\) permutes the spins at \(i\)-th and \(j\)-th positions. It is well known that the Hamiltonian (4.1) is integrable if \(d_i\) has one of the following values

\[
d_j = \begin{cases} 
  j, & \text{rational case} \\
  (1/\alpha) \sinh(\alpha j), \quad \alpha \in \mathbb{R}, & \text{hyperbolic case} \\
  (L/\pi) \sin(\pi j/L), & \text{trigonometric case} 
\end{cases}
\]

The trigonometric model is defined on periodic chain and the sum in (4.1) is performed over \(1 \leq i, j \leq L\). Rational and hyperbolic models are defined on infinite chain.

For simplicity we consider here \(sl_2\) Haldane-Shastry model only. This means that the space \(V^0\) should be two dimensional, so we should have two terms in the decomposition (3.1).

Now our goal is to construct a long-range interaction model, which on the highest weight space \(W^0\) coincides with (4.1). So, we take \(V = V_{\lambda_1} \oplus V_{\lambda_2}\). This is the second example considered in sec.3. To achieve an permutation action on the subspace \(V^0 \otimes V^0\) let us impose the conditions (3.17) on parameters of \(H\)-operator (3.8). One can try to generalize the Hamiltonian (4.1) in this way by

\[
\mathcal{H}^{HS} = \sum_{i<j} \frac{1}{d_{i-j}^2} H_{ij}.
\]

But it is easy to see that then \(\mathcal{H}^{HS}\) isn’t invariant with respect to quantum group \(U_q\hat{g}\). This is because the equation

\[
\hat{R}_{ij}(x_1, x_2) \Delta^{L-1}(g) = \Delta^{L-1}(g) \hat{R}_{ij}(x_1, x_2), \quad g \in U_q\hat{g}
\]

is valid only for \(i = j \pm 1\).

To overcome this difficulty let us use instead of \(H_{ij}\) the operator \(\hat{F}_{ij}\)

\[
\hat{F}_{ij} = G_{[ij]} H_{j-1} G^{-1}_{[ij]}, \quad \text{where} \quad G_{[ij]} = H_{i+1} H_{i+1+i+2} \cdots H_{j-2j-1} \tag{4.2}
\]

To be short, we omit \(x_i, x_j\)-dependence of operators \(H, G, F\)-operators in this section.
We remark that $F_{ij}$ and $G_{ij}$ act nontrivially on all indexes $i, i+1, \ldots, j$. So we include them into bracket do not confuse with the definition (3.10). The ‘nonlocal’ term like $F_{ij}$ appeared earlier in the construction of quantum group invariant periodic spin chains as a boundary term [32, 33, 35].

The operators $H_{ii+1}$ satisfy the permutation group relations (3.18) and

$$H_{i-1i}H_{ii+1}H_{i-1i} = H_{ii+1}H_{i-1i}H_{ii+1} \quad (4.3)$$

These relations obey on $W^0$, because the restriction of $H_{ij}$ on it gives rise to permutations $P_{ij}$

$$H_{ij}|_{W^0} = P_{ij}$$

On the whole space $W$ the equations (4.3) are continued using the affine quantum group symmetry of $H_{ii+1}$. In contrast to the standard realization by $P_{ij}$, the relation

$$P_{i-1i}P_{ii+1}P_{i-1i} = P_{i-1i+1} \quad (4.4)$$

isn’t fulfilled. It is easy to see from (4.2) and (4.4) that

$$F_{ij}|_{W^0} = P_{ij}$$

So, the spin chain defined by

$$H^{HS} = \sum_{i<j} \frac{1}{d_{i-j}^2} F_{ij}, \quad (4.5)$$

is quantum group invariant and its restriction on the space $W^0$ it coincides with the Haldane-Shastry spin chain (4.1).

In some cases nonlocal expression for $F_{ij}$ (4.2) becomes depending on sites $i$ and $j$ only. For example, if $V_{\lambda_1}$ is trivial ($V_{\lambda_1} = V_0$, $V_{\lambda_2} = V_\lambda$), then $R_{12}(x_1/x_2) = R_{21}(x_2/x_1) = id$ and $H$-matrix (3.8) has the following block-diagonal form on $V \otimes V$ with respect to the decomposition $V = V_0 \oplus V_\lambda$

$$H = \begin{pmatrix} id & 0 & 0 & 0 \\ 0 & id & 0 & 0 \\ 0 & 0 & id & 0 \\ 0 & 0 & 0 & id \end{pmatrix} \quad (4.6)$$

So, the block-diagonal form of $H$ is permutation and, hence, obeys the relations (4.4). Then the constructing blocks $F_{ij}$ act nontrivially on two sites $i, j$ only, where they are the block-diagonal permutations (4.6). Now all the terms in (4.3) are pairwise interactions between two different sites and it can be written in explicit form

$$H^{HS} = \sum_{i<j} \frac{1}{d_{i-j}^2} \left[ \sum_{\alpha=1}^{\text{dim}V_\lambda} \left( X_{i0}^a X_{j0}^0 + X_{ja}^0 X_{i0}^a \right) + X_{i0}^0 X_{j0}^0 + \sum_{a,a'=1}^{\text{dim}V_\lambda} X_{ia}^a X_{ja'}^{a'} \right]$$

The energy levels of $H^{HS}$ coincide with the levels of (1.4). The degeneracy degree with respect to the later is defined by (3.14). The relations (3.13), (3.16) between the statistical sums of $H_0^{HS}$ and $H^{HS}$ remain the same.
5 Fermionic representations

In this section we consider the representations of some Hamiltonians derived in previous sections in terms of \( c_{i,\sigma}^\pm \), which are the annihilation-creation operators of spin \( \sigma = \uparrow, \downarrow \) fermion at site \( i \).

Below we are dealing with the quantum group \( U_q\hat{sl}_2 \) only. We consider the simplest cases, where in the decomposition of \( V \) there is only one representation of spin \( > 0 \). Then \( R \)-matrices, appearing in Hamiltonian are permutations and the Hamiltonian doesn’t depend on \( q \). So, it will be \( U_q\hat{sl}_2 \)-invariant for all values of deformation parameter \( q \).

We identify the space \( V \) with the space of spin-\( \sigma \) \( (\sigma = \uparrow, \downarrow) \) fermionic wavefunctions. Let us denote by \( |0\rangle \) the fermionic vacuum: \( c_\sigma |0\rangle = 0 \). Also we use \( |\sigma\rangle = c_\uparrow^\dagger c_\downarrow|0\rangle \), \( |\uparrow\downarrow\rangle = -|\downarrow\uparrow\rangle = c_\downarrow^\dagger c_\uparrow|0\rangle \).

To construct the fermionic representations of Hamiltonians, introduced in the previous sections, it is convenient to use the fermionic representation of projection operators (3.4), introduced earlier. Here we denote \( a, b = 0, \pm 1, 2 \), where 0 means the empty site, \( \pm 1 \) mean the sites \( |\uparrow\rangle \) and \( |\downarrow\rangle \) correspondingly and 2 denotes the site \( |\downarrow\uparrow\rangle \) with double occupation. Creation and annihilation operators can be expressed through the projection operators as follows

\[
c_\sigma^+ = X_0^\sigma - \sigma X_2^\sigma, \quad c_\sigma = X_0^\sigma - \sigma X_2^{-\sigma}, \quad n_\sigma = X_0^\sigma + X_2^\sigma, \quad \sigma = \pm 1 \tag{5.1}
\]

and vice versa

\[
X_0^\sigma = (1 - n_-\sigma)c_\sigma^+, \quad X_2^\sigma = n_\sigma c_\sigma^-, \quad X_0^0 = c_\uparrow c_\downarrow, \\
X_0^\sigma = (1 - n_-\sigma)n_\sigma, \quad X_0^\sigma = (1 - n_-\downarrow)(1 - n_-\uparrow) = n^h, \quad X_2^\sigma = n_\downarrow n_\uparrow = d \tag{5.2}
\]

Other formulae are obtained using \( (X_0^\sigma)^+ = X_0^b \). Here we introduced the local Hubbard interaction operator \( d \), which counts the double occupation of site and the hole number operator \( n^h \).

5.1 \( t-J \) model for zero spin-spin coupling and Hubbard model in infinite repulsion limit

Consider the simplest case, when the space of states decomposes into direct sum of fundamental spin-1/2 and trivial spin-0 multiplets \( V = V_0 \oplus V_{1/2} \) [27]. We associate \( V_0 \) with empty state \( |0\rangle \) and \( V_{1/2} \) with \( |\uparrow\rangle, |\downarrow\rangle \) one electron states. Using (3.8) for the present case, we obtain

\[
\mathcal{H}(t, W_1, W_2) = \sum_{i=1}^{L-1} \left[ -t \sum_{\sigma, \sigma' = \pm 1} (X_{i0}^\sigma X_{i+10}^0 + X_{i+10}^\sigma X_{i0}^0) + W_1 X_{i0}^0 X_{i+10}^0 \right. \\
\left. + W_2 \sum_{\sigma, \sigma' = \pm 1} X_{i0}^\sigma X_{i+10}^\sigma \right]
\]
After substitution of fermionic representation (5.2) of projection operators the Hamiltonian above (by canceling out nonessential in the thermodynamic limit boundary and constant terms) transforms into

\[
\mathcal{H}_{t-J}(t, W, \mu) = \sum_{i=1}^{L-1} \left[ -t \sum_{\sigma=\pm1} (1 - n_{i,\sigma})(c_{i,\sigma}^+ c_{i+1,\sigma} + c_{i+1,\sigma}^+ c_{i,\sigma})(1 - n_{i+1,\sigma}) \right.
+ W n_i n_{i+1}] + \mu \sum_{i=1}^{L} n_i, \tag{5.3}
\]

where \( W = W_1 + W_2 \) and \( \mu = -2W_1 \). As \( \mathcal{H}(t, W, \mu) \) conserves the particle number, it remains integrable without last constrain on \( \mu \). This Hamiltonian forbids double occupied sites and coincides with one of \( t-J \) model for vanishing spin-spin coupling \( J = 0 \). Integrability of this model and its correspondence to XXZ spin chain is well known \[34\]. For vanishing density-density interaction \( V = 0 \) \[5.3\] reduces to Hubbard model in infinite repulsion limit.

### 5.2 Extended Hubbard with doping, pair hopping and hole density-density interaction

Let consider now the case \( V = V_0 \oplus V_1 \). This corresponds to \( R_{12}(x) = R_{21}(x) = \text{id} \) in \[3.8\]. Note that in this case \( H \) doesn’t depend on \( x_i \) and \( q \) and we choose the coefficients there as follows

\[
H = \begin{pmatrix}
W_1 \cdot \text{id} & 0 & 0 & 0 \\
0 & 0 & -t \cdot \text{id} & 0 \\
0 & -t \cdot \text{id} & 0 & 0 \\
0 & 0 & 0 & W_2 \cdot \text{id}
\end{pmatrix} \tag{5.4}
\]

We identify the empty state \(| 0 \rangle \) with \( V_0 \) and the space spanned by \(| \uparrow \rangle \), \(| \downarrow \rangle \) with \( V_1 \). The general Hamiltonian \[3.9\] in our case has the form

\[
\mathcal{H}(t, W_1, W_2) = \sum_{i=1}^{L-1} \left[ -t \sum_{\sigma=\pm1} (X_{i0}^\sigma X_{i+1,0}^\sigma + X_{i+10}^\sigma X_{i+2}^\sigma) - t(X_{i0}^2 X_{i+2}^0) + X_{i+10}^2 X_{i+2}^0 \right.
+ W_1 X_{i0}^0 X_{i+10}^0 + W_2 \sum_{\sigma,\sigma'=\pm1} (X_{i0}^\sigma + X_{i+2}^\sigma)(X_{i+10}^{\sigma'} + X_{i+2}^{\sigma'} \tag{5.5}
\]

After substituting (5.2) into (5.3) one obtain

\[
\mathcal{H}(t, W_1, W_2) = \sum_{i=1}^{L-1} \left[ -t \sum_{\sigma}(1 - n_{i,\sigma})(c_{i,\sigma}^+ c_{i+1,\sigma} + c_{i+1,\sigma}^+ c_{i,\sigma})(1 - n_{i+1,\sigma}) \right.
- t(c_{i,\uparrow}^+ c_{i,\downarrow} c_{i+1,\downarrow}^+ c_{i+1,\uparrow} + c_{i+1,\uparrow}^+ c_{i+1,\downarrow} c_{i,\uparrow} c_{i,\downarrow}) + W_1 n_i^h n_{i+1}^h \tag{5.6}
+ W_2(n_i - d_i)(n_{i+1} - d_{i+1}) - 2W_2 \sum_{i=1}^{L} n_i + W_2(n_1^h + n_L^h) + W_2 L.
\]
The quantum group generators (2.1) (after some renormalization) can be written in terms of fermionic operators also:

\[ e_1 = [2]^{-1/2}(n_\uparrow c_\uparrow + n_\downarrow c_\downarrow) \]
\[ f_1 = [2]^{-1/2}(n_\uparrow c_\downarrow + n_\downarrow c_\uparrow) \]
\[ e_0 = f_1 \]
\[ f_0 = e_1 \] (5.7)

\[ h_1 = 2(n_\uparrow - n_\downarrow) \]
\[ h_0 = -h_1 \]

The first term in (5.6) is known as "doping" term, the second one is pair hopping term. Using \[ n_{h} = 1 - n + d \] and omitting unessential boundary and constant terms in (5.6) one obtain

\[ H(t, W, U, \mu) = \sum_{i=1}^{L-1} \left[-t \sum_{\sigma}(1 - n_{i,-\sigma})(c_{i,\sigma}^+ c_{i+1,\sigma} + c_{i+1,\sigma}^+ c_{i,\sigma})(1 - n_{i+1,\sigma}) \right. \\
\left. - t(c_{i,\uparrow}^+ c_{i+1,\downarrow} c_{i+1,\uparrow} + c_{i+1,\uparrow}^+ c_{i,\downarrow} c_{i,\uparrow}) + W n_{h}^i n_{h}^{i+1} \right] + U \sum_{i=1}^{L} n_{i,\uparrow} n_{i,\downarrow} + \mu \sum_{i=1}^{L} n_{i}, \] (5.8)

where \( W = W_1 + W_2 \), \( \mu = 2W_2 \), \( U = -2W_2 \).

In view of the consideration, carried out in sec.3, the energy levels of obtained fermionic model coincide with the levels of spin chain model \( H_0(t, W_1, W_2) \) on \( W^0 \).

It is easy to see that its two site interaction term

\[ H_0 = \begin{pmatrix} W_1 & 0 & 0 & 0 \\ 0 & 0 & -t & 0 \\ 0 & -t & 0 & 0 \\ 0 & 0 & 0 & W_2 \end{pmatrix} \] (5.9)

leads to XXZ model in external magnetic field (see (5.15), (5.16) below).

We note that the Hamiltonian (5.6) preserves the number of holes \( N_{h} = \sum_{i=1}^{L} n_{i,\downarrow} \) and the number of double occupied states \( D = \sum_{i=1}^{L} d_{i} \). So, we can choose the parameters of corresponding terms in (5.8) to be arbitrary without loose of integrability. Of course, the Hamiltonian (5.8) is not invariant with respect to quantum group (5.7) now.

### 5.3 Hubbard model with bond-charge interaction and additional density-density and boson-boson interactions

Let us consider \( V(x_a, x_b) = V_a(x_a) \oplus V_b(x_b) \) and identify the first multiplet in this decomposition \( V_a \) with the space spanned by \( | \uparrow \rangle, | \downarrow \rangle \) and the second one \( V_b \) with the space spanned by \( | 0 \rangle, | \uparrow \rangle \). We consider two different commuting quantum group actions on \( V \). The first one \( U_q s\hat{l}_2^{(1)} \) acts on \( V_a \) as spin-1/2 representation \( V_a \sim V_{1/2} \) and on \( V_b \) as two spin singlets \( V_b \sim V_0 \oplus V_0 \). The second one \( U_q s\hat{l}_2^{(2)} \) acts vice versa: \( V_a \sim V_0 \oplus V_0 \) and \( V_b \sim V_{1/2} \).
Now by use of the form of intertwining operator $H$ (3.5) for the decomposition $V = V_{1/2} \oplus V_0 \oplus V_0$ and taking vanishing matrixes $A_{04}$ and $A_{40}$, one obtain

$$\mathcal{H}(A, B) = \sum_{(i,j)} \left[ \sum_{\delta', \delta, \delta', \delta''} A_{00, \delta, \delta'} X_{i, \delta} X_{j, \delta'} + \sum_{\sigma', \sigma} A_{\delta, \sigma, \delta', \sigma} X_{i, \sigma} X_{j, \delta'} \right]$$

$$+ \sum_{\sigma, \delta', \delta} \left( B_{02, \delta, \delta'} X_{i, \delta} X_{j, \delta'} + B_{02, \delta, \delta'} X_{i, \sigma} X_{j, \delta'} \right)$$

(5.10)

Note that the indexes $i, j$ here denote the nearest neighbors but not nonequivalent representations $\lambda_i$, as in (3.3). So we used the upper index for them not to mix with (3.5). Also, we changed the double indexes in (3.5) on single indexes $\sigma = \pm 1$, $\delta = 0, 2$ by $(1, a_{12}) \sim \sigma$, $(n_0, 1) \sim \delta$. Let us choose the parameters in (5.10) in the following way

$$A_{00, \delta, \delta'} = \delta_{\delta, \delta'} \delta_{\delta, \delta'} W_2, \quad A_{\delta, \sigma, \delta', \sigma} = W_1, \quad B_{02, \delta, \delta'} = B_{\delta, \delta'} = -i \delta_{\delta, \delta'}.$$

After these simplifications we have

$$\mathcal{H}(t, W_1, W_2) = \sum_{i=1}^{L-1} \left[ -t \sum_{\sigma, \sigma' = \pm 1} \sum_{\delta, \delta' = 0, 2} (X_{i, \sigma} X_{i+1, \delta} + X_{i+1, \sigma} X_{i, \delta}) \right]$$

$$+ W_1 \sum_{\sigma, \sigma' = \pm 1} X_{i, \sigma} X_{i+1, \sigma'} + W_2 \sum_{\delta, \delta' = 0, 2} X_{i, \delta} X_{i+1, \delta'}$$

(5.11)

The nearest neighbor interaction of this Hamiltonian conserves its form with respect to index exchange $\delta \leftrightarrow \sigma$. This means that $\mathcal{H}(t, W_1, W_2)$ is invariant with respect to both quantum groups $U_q \widehat{sl}_2(1)$ and $U_q \widehat{sl}_2(2)$. So, (5.11) has $U_q \widehat{sl}_2 \otimes U_q \widehat{sl}_2$-symmetry. The highest weight space at each site is formed by the two vectors $|\uparrow\rangle$ and $|0\rangle$ of $V_a$ and $V_b$ correspondingly.

Using the fermionic representation of $X$-operators (5.2) we obtain

$$\mathcal{H}(t, W_1, W_2) = \sum_{i=1}^{L-1} \sum_{\sigma = \uparrow, \downarrow} \left\{ -t(c_{i, \sigma}^+ c_{i+1, \sigma} + c_{i+1, \sigma}^+ c_{i, \sigma}) \right.$$

$$+ t(c_{i, \sigma}^+ c_{i+1, \sigma} + c_{i+1, \sigma}^+ c_{i, \sigma})(n_{i, -\sigma} + n_{i+1, -\sigma}) \right\}$$

$$+ (W_1 + W_2)(n_{i+1} - 2n_{i+1}d_{i+1} - 2n_{i+1}d_i + 4d_i d_{i+1})$$

$$+ 2W_1(d_i + d_{i+1}) - W_1(n_i + n_{i+1}) + W_1$$

(5.12)

Two types of hopping are allowed here. First, fermion from single occupied side can hope to empty site and vice versa. Second, from double occupied site fermion can hope to single occupied site, which contains fermion with opposite
spin. Such hopping term appeared in [12] after projection of Hubbard model on a fixed occupation number subspace. Thus, the Hamiltonian (5.12) preserves the occupation number of fermions: number of holes $N_h$ and number of double occupied sites $D$. So, corresponding terms without affecting on the integrability an be rewritten with arbitrary coefficients. Also, note that the term

$$2(W_1 + W_2) \sum_{i=1}^{L-1} 2d_i d_{i+1} - n_i d_{i+1} - n_{i+1} d_i$$

can be written as hole and double occupation site interaction and Hubbard interaction in the following way

$$2(W_1 + W_2) \sum_{i=1}^{L-1} n_i^h d_{i+1} + n_{i+1}^h d_i - d_i - d_{i+1}$$

Thus one can generalize (5.12) by

$$H(t, W_1, W_2, U, \mu) = \sum_{i=1}^{L-1} \sum_{\sigma=\uparrow, \downarrow} \left[ -t(c_{i,\sigma}^+ c_{i+1,\sigma} + c_{i+1,\sigma}^+ c_{i,\sigma}) + t(c_{i,\sigma}^+ c_{i+1,\sigma} + c_{i+1,\sigma}^+ c_{i,\sigma})(n_i - \sigma + n_{i+1} - \sigma) \right]$$

$$+ W \sum_{i=1}^{L-1} (n_i n_{i+1} + 2n_i^h d_{i+1} + 2n_{i+1}^h d_i) + U \sum_{i=1}^{L} d_i + \mu \sum_{i=1}^{L} n_i$$

$$+ W_1 (n_1 + n_L) + 2W_2 (d_1 + d_L)$$

The term with coefficient $W = W_1 + W_2$ consists of standard fermion density-density interaction between nearest neighbors and interaction between empty site (hole) and nearest neighbor site occupied by two fermions with opposite spins. The later interaction can be considered as interaction between two different bosonic sites $|0\rangle$, $|\downarrow\uparrow\rangle$. The boundary terms, which appeared in (5.13), do not affect on the spectra of (5.13) in the thermodynamic limit and can be omitted. So, in this limit the Hamiltonian above depends only on $W = W_1 + W_2$.

The connection between Hamiltonians (5.12) and (5.13) is given by

$$H(t, W_1, W_2, U, \mu) = H(\psi, t, W_1, W_2) + (U+4W_2)D + (\mu+2W_1)N - W_1 (L-1)$$

Now we consider the restriction of (5.12) on the highest weight space, which is generated by the empty sites and sites occupied by the single $\sigma = \uparrow$ spin. Looking on the matrix form of $H(t, W_1, W_2) = A$ we recognize $XXZ$ spin chain. It follows from our previous investigation that the obtained model is exactly solvable and has the same energy levels as $XXZ$ Heisenberg model in the external homogeneous magnetic field $\vec{B} = B\hat{z}$ along $z$ axis

$$H_0(t, W_1, W_2) = \mathcal{H}_{XXZ}(t, \Delta, B)$$

$$= -\frac{t}{2} \sum_{i=1}^{L-1} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z + \frac{B}{2} \sigma_i^z \right)$$

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where
\[ \Delta = -\frac{W}{2t}, \quad B = \frac{2}{t}(W_1 - W_2) \] (5.16)

All energy levels have the same degeneracy. Recall that we identified the spin states \(|+\rangle, |-\rangle\) of (5.13) with the highest weight states \(|\uparrow\rangle\) and \(|0\rangle\) of \(V\) correspondingly.

It is easy to check that the Hamiltonian (5.11), (5.12) commutes with \(\eta\)-pairing operators \([14, 15, 16, 17]\):

\[ \eta = \sum_{i=1}^{L} c_{i\uparrow} c_{i\downarrow} = \sum_{i=1}^{L} X_{i2}^0, \quad \eta^+ = \sum_{i=1}^{L} c_{i\uparrow} c_{i\downarrow}^{\dagger} = \sum_{i=1}^{L} X_{i0}^2 \]

\[ \eta^z = N - L = \sum_{i=1}^{L} (X_{i2}^2 - X_{i0}^0), \] (5.17)

which generate \(sl_2\) algebra. Using this fact and (5.14) one can compute the commutation rules of (5.17) with generalized Hamiltonian (5.13)

\[ [\mathcal{H}(t,W_1,W_2,U,\mu),\eta] = -(U + 2\mu + 4W)\eta \]

\[ [\mathcal{H}(t,W_1,W_2,U,\mu),\eta^+] = (U + 2\mu + 4W)\eta^+ \] (5.18)

So, we have \([\mathcal{H}(t,W_1,W_2,U,\mu),\eta^\pm] = 0\) if parameters satisfy the condition \(\mu = -U/2 - 2W\).

We are interested in thermodynamic behavior at \(T = 0\). Denote by \(E_N = E_N(t,W_1,W_2,U,\mu)\) the ground state energy of \(\mathcal{H}(t,W_1,W_2,U,\mu)\) in the \(N\)-particle sector. Also, let \(\mathcal{E}_{N_1} = \mathcal{E}_{N_1}(\Delta)\) is the ground state of \(XXZ\) spin chain (5.13) without external magnetic field \((B = 0)\) in the sector with \(N_1\) upturned spins. Recall that single occupied sites of particle space correspond to positive spin in \(XXZ\) model whereas double and empty sites correspond to negative spin. Thus, in the sector, which consists of states with fixed occupation number \(N_1 = N_1, D = (N - N_1)/2\) the minimal energy \(E_{N_1}^{N_1}\) of \(\mathcal{H}(t,W_1,W_2,U,\mu)\) is (see (5.14))

\[ E_{N_1}^{N_1} = \mathcal{E}_{N_1} - \left(\frac{U}{2} + W\right)N_1 + \left(\frac{U}{2} + \mu + 2W\right)N + \frac{L}{2}(W_1 - W_2) \] (5.19)

To obtain ground state \(E_N\) one should minimize (5.19) among all possible values of \(N_1\)

\[ E_N = \min_{0 \leq N_1 \leq N} E_{N_1}^{N_1} = \min_{0 \leq N_1 \leq N} \left[ \mathcal{E}_{N_1} - \left(\frac{U}{2} + W\right)N_1 \right] \]

\[ + \left(\frac{U}{2} + \mu + 2W\right)N + \frac{L}{2}(W_1 - W_2) \] (5.20)

Note that the Hamiltonians \(\mathcal{H}_{XXZ}(t,\Delta,0)\) and \(\mathcal{H}_{XXZ}(-t,-\Delta,0)\) are related by similarity transformation \([36]\). So, the spectrum of \(\mathcal{H}_{XXZ}(t,\Delta,0)\) in the ferromagnetic regions \((-t > 0, \Delta \leq -1)\) and \((-t < 0, \Delta \geq 1)\) coincides. Hereafter we dial
with the first region and choose the parametrization $\Delta = -\cosh \gamma, \gamma > 0$. Note that (5.13) in zero magnetic field has two $Z_2$ symmetric ferromagnetic ground states $|\pm, \text{vac}\rangle_{XXZ} = |\pm, \pm, \ldots, \pm\rangle$ with energy $\mathcal{E}_0 = \mathcal{E}_L = -t/2(L - 1)\Delta$.

The energy levels of $XXZ$ spin chain are determined by means of Bethe Ansatz [37, 38]. In the thermodynamic limit $L \to \infty$ there it has string type solutions. It was shown in [18] that the minimal energy in the sector with $N_1$ upturned spins is

$$E_{N_1} = -2t \sinh \gamma \frac{\sinh N_1 \gamma}{\cosh N_1 \gamma + 1} + \mathcal{E}_0 = -2t \sinh \gamma \tanh \frac{N_1}{2} \gamma + \mathcal{E}_0$$

(5.21)

and corresponds to single string of length $N_1$. Here the magnetization is restricted by $0 \leq N_1/L \leq 1/2$, i.e. we are dealing with excitations near ferromagnetic vacuum $|+, \text{vac}\rangle$. Due to $Z_2$ symmetry we have $\mathcal{E}_{N_1} = \mathcal{E}_{L-N_1}$ and the excitations near $|+, \text{vac}\rangle$ ($1/2 \leq N_1/L \leq 1$) can be considered in a similar way by exchanging $N_1 \to L - N_1$ in (5.21).

For infinite number of turned spins $\lim_{L \to \infty} N_1/L = 0, L$ the minimal energy (5.21) is $\mathcal{E}_\infty = -2t \sinh \gamma + \mathcal{E}_0$. So, all ground states for fixed magnetizations, which differ from both ferromagnetic ground states by infinite number of turned spins, have the same energy.

It is well known that $XXZ$ spin chain for $|\Delta| > 1$ is massive. The energy gap between ground state and elementary excitations in the limit $N_1 \to \infty$ is $\Delta E = -2t(\cosh \gamma - 1)$.

Substituting (5.21) into (5.20) one obtain the value of $N_1$, which minimize $E_{N_1}^\mathcal{E}$ in (5.20)

$$N_{1\min} = \begin{cases} 0 & (U/2 + W \leq 0) \text{ or } (0 < U/2 + W \leq \gamma/2, N \leq N_c) , \\ N & (U/2 + W > 0) \text{ or } (0 < U/2 + W \leq \gamma/2, N \geq N_c) , \end{cases}$$

(5.22)

where $N_c$ is solution of

$$\tanh \frac{N_c}{2} \gamma = \left(\frac{U}{2} + W\right) N_c, \quad \frac{U}{2} - W > 0, \quad N_c > 0$$

Consider the regions $(U/2 + W \leq 0, W \leq 2t)$ and $(0 < U/2 + W \leq \gamma/2, N \leq N_c, W \leq 2t)$ in (5.22), where there are no single occupied states. The ground state of Hamiltonian in the $N$-particle sector ($N$ is even) is any state with $D = N/2$ double occupied sites and $L - N/2$ empty sites. Let us write down all these states in the form

$$\psi_P^N = (\eta_P^+)^{N/2}|\text{vac}\rangle, \quad |\text{vac}\rangle = |0, 0, \ldots, 0\rangle$$

(5.23)

Here $\eta_P^+$ are generalized $P$-momentum $\eta$-pairing operators

$$\eta_P = \sum_{j=1}^L e^{iPj}c_{j,\uparrow}c_{j,\downarrow} \quad \eta_P^+ = \sum_{j=1}^L e^{-iPj}c_{j,\uparrow}c_{j,\downarrow}$$
Note that $\eta_0^\pm = \eta^\pm$.

All the states $\psi^N_P$ obey ODLRO, i.e.

$$\lim_{|i-j| \to \infty} \frac{\langle \psi^N_P | c^+_i c^+_j c_i c_j | \psi^N_P \rangle}{\langle \psi^N_P | \psi^N_P \rangle} \neq 0$$

(5.24)

Note also that $\eta^N_P$ operators obey $\eta^N_P |\text{vac}\rangle = 0$. As a consequence the ground states $\psi^N_P$ are superconducting.

In the phase space regions ($U/2 + W > \gamma/2$, $W \leq 2t$) and ($0 < U/2 + W \leq \gamma/2$, $N \geq N_c$, $W \leq 2t$) there are no double occupied sites in the ground states of (5.13). These states are linear combinations of states with $N$ single fermions of same spin. Of course, they do not exhibit ODLRO (5.24).

The projection of (5.13) on states without double occupation ($D = 0$) gives rise to $t - J$ Hamiltonian for zero spin-spin coupling $J = 0$ (5.3) with additional boundary term $W_1(n_1 + n_L)$. As it was mentioned above, this model is integrable and equivalent to Heisenberg magnet. Note that this equivalence can be seen from (5.3). If we restrict $H(t, W, \mu)$ on the states with only one sort of fermion (spin-$\uparrow$ or spin-$\downarrow$), we obtain the fermion representation of $XXZ$ model

$$H_{XXZ}(t, W_1, W_2, \mu) = \sum_{i=1}^{L-1} -t(c^+_i c_{i+1}^+ + c^+_i c_i + c^+_i + c^+_i c_i) + W \sum_{i=1}^{L-1} n_i n_{i+1} + \mu \sum_{i=1}^{L} n_i$$

(5.25)

For the special values of interaction potentials $W_1 = -W_2 = W$ the Hamiltonian under consideration simplifies drastically. We have up to boundary terms

$$H(t, W) = \sum_{i=1}^{L-1} \sum_{\sigma = \uparrow, \downarrow} [-t(c^+_i c_{i+1, \sigma} + c^+_i + c_{i+1, \sigma}) + t(c^+_i c_{i+1, \sigma} + c^+_i + c_{i+1, \sigma})(n_{i, -\sigma} + n_{i+1, -\sigma})] + U \sum_{i=1}^{L} d_i + \mu \sum_{i=1}^{L} n_i$$

(5.25)

This is Hubbard model with bond-charge interaction. On the highest weight space it coincides with $XY$ model in external magnetic field (or equivalently free fermion model with chemical potential term added) ($\Delta = 0$ in (5.13)). So, $H(t, W)$ is exactly solvable and have the same energy levels as free fermion model with potential term with degeneracy degree of each level.

The Hamiltonian (5.25) $D$ was obtained and solved in [12, 13]. In our construction its integrability is obvious, because the restriction $H^0(t, W = 0)$ of (5.25) on the highest weight space coincides with ordinary $XY$ spin chain and, consequently, it is equivalent to free fermion spin chain. It was shown in [13] that (5.25) has superconducting ground states. Note that here we didn’t consider the antiferromagnetic region, which corresponds to $\Delta = 0$ case. We suggest that in this region (5.13) has superconducting ground state too, obeying $\eta$-pairing mechanism.
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