A Note on Bimodules and II₁-Subfactors

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Abstract

A brief introduction into bimodules of II₁-factors is presented. Furthermore a version of the following result due to M. Pimsner and S. Popa is derived: Let \( N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset \ldots \) denote the Jones tower of a II₁-factor \( N \subset M \) with finite index. Then the factor obtained by the basic construction from the pair \( N \subset M_{n-1} \) is equal to \( M_{2n-1} \).

Introduction

The theory of subfactors was established by V.F.R. Jones in his famous paper [6]. A. Ocneanu had the idea to use bimodules (also called correspondences) for the theory of subfactors (see [7] and [8]). In this note we present an easy access to the bimodules of II₁-factors and to their application to the theory of II₁-subfactors. Let

\[ N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \ldots \]

be the Jones tower of a II₁-subfactor \( N \subset M \) with finite index. The main goal of this paper is to construct a normal isomorphism from \( M_{2n-1} \) \((n \in \mathbb{N})\) onto the von Neumann algebra of the right \( N \)-linear operators on \( L^2(M_{n-1}) \). In particular we obtain a new version of the result due to M. Pimsner and S. Popa that the basic construction for \( N \subset M_{n-1} \) is equal to \( M_{2n-1} \). The author thinks that his result is more convenient for applications than Pimsner’s and Popa’s. We notice that Y. Denizeau and J.F. Havet developed a theory of correspondences of finite index for arbitrary von Neumann algebras (see [3] and [4]) from which our results follow likewise. S. Yamagami also proved some
results about bimodules and some parts of Ocneanu’s approach to subfactors (see [13] and [14]). This note contains the results of an introductionary section and of a part of the appendix of the author’s Habilitationsschrift [14].

1 Bimodules of \( \text{II}_1 \)-factors

In order to avoid subtleties we will assume that every von Neumann algebra appearing in this paper acts on a separable Hilbert space. For a \( \text{II}_1 \)-factor \( S \) let \( \text{tr}_S \) denote the unique normalized trace of \( S \). Let \( L, P, Q, \) and \( S \) be factors of type \( \text{II}_1 \). Let \( Q^{\text{op}} \) be the factor opposite to \( Q \). (\( Q^{\text{op}} \) is equal to \( Q \) as a complex vector space, the multiplication law is reversed, that means \( p \circ q := q \cdot p \) for \( p, q \in Q \), and the involution \( * \) is the same as in \( Q \).)

**Definition 1.1** (i) A (left) \( P \)-module is a Hilbert space \( \mathcal{H} \) endowed with a normal representation \( \lambda : P \rightarrow \mathcal{L}(\mathcal{H}) \) of \( P \). A right \( Q \)-module is a Hilbert space \( \mathcal{H} \) endowed with a normal representation \( \rho : Q^{\text{op}} \rightarrow \mathcal{L}(\mathcal{H}) \) of \( Q^{\text{op}} \). A \((P, Q)\)-bimodule \( \mathcal{H} \) is a left \( P \)-module and a right \( Q \)-module such that \( \lambda(P) \) and \( \rho(Q^{\text{op}}) \) commute. If we like to emphasize the Hilbert space \( \mathcal{H} \), we write \( \lambda_{\mathcal{H}} \) for \( \lambda \) and \( \rho_{\mathcal{H}} \) for \( \rho \).

(ii) For left \( P \)-modules \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) a continuous linear operator \( T : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) is called (left) \( P \)-linear, if \( T \lambda_{\mathcal{H}_1}(p) = \lambda_{\mathcal{H}_2}(p)T \) holds for every \( p \in P \). For right \( Q \)-modules \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) a continuous linear operator \( T : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) is called right \( Q \)-linear, if \( T \rho_{\mathcal{H}_1}(q) = \rho_{\mathcal{H}_2}(q)T \) for every \( q \in Q \). For \((P, Q)\)-bimodules \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) an operator \( T : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) is called \((P, Q)\)-linear, if \( T \) is left \( P \)- and right \( Q \)-linear.

We write \( p\mathcal{H}_Q \) for the \((P, Q)\)-bimodule \( \mathcal{H} \). For \( \xi \in \mathcal{H} \), \( p \in P \) and \( q \in Q \) we also write \( p \xi \) in place of \( \lambda(p)\xi \) and \( \xi q \) in place of \( \rho(q)\xi \). The commutant \( \lambda_{\mathcal{H}}(P)' \) is also denoted by \( \mathcal{L}_{P,-}(\mathcal{H}) \), the commutant \( \rho_{\mathcal{H}}(Q)' \) by \( \mathcal{L}_{-,Q}(\mathcal{H}) \), and the set \( \lambda_{\mathcal{H}}(P)' \cap \rho_{\mathcal{H}}(Q)' \) of the continuous \((P, Q)\)-linear operators on \( \mathcal{H} \) by \( \mathcal{L}_{P,Q}(\mathcal{H}) \).

Let \( L^2(Q) \) be the Hilbert space obtained by the completion of \( Q \) with respect to the inner product \( \langle x, y \rangle := \text{tr}_Q(y^*x) \) \( (x, y \in Q) \). We denote an element \( x \) of \( Q \) by \( x \), if \( x \) is regarded as an element of \( L^2(Q) \). \( L^2(Q) \) is a \((Q, Q)\)-bimodule, where the actions are given by left and right multiplication.

For \((P, Q)\)-bimodules we use the usual concepts of representation theory. For instance, a \((P, Q)\)-bimodule \( p\mathcal{H}_Q \) is called irreducible, if and only if there is no closed proper subspace \( \mathcal{K} \neq 0 \) of \( \mathcal{H} \) invariant under the left action of \( P \) and the right action of \( Q \), or equivalently, if and only if \( \mathcal{L}_{P,Q}(\mathcal{H}) = \mathbb{C} 1 \). Two \((P, Q)\)-bimodules \( \mathcal{H} \) and \( \mathcal{K} \) are called equivalent \( (\mathcal{H} \cong \mathcal{K}) \), if there is a unitary \((P, Q)\)-linear map \( U : \mathcal{H} \rightarrow \mathcal{K} \).
1.2 The \( \otimes_Q \)-tensor product

Let \( \mathcal{H} \) be a right \( Q \)-modules and \( \mathcal{K} \) a left \( Q \)-module. We describe J.-L. Sauvageot’s construction ([10]) of the tensor product \( \mathcal{H} \otimes_Q \mathcal{K} \) for our special case of \( \Pi_1 \)-factors. (It is not difficult to check that the definition in [12] is a special case of the definition in [10]). The Hilbert space \( \mathcal{H} \otimes_Q \mathcal{K} \) has similar properties like the algebraic tensor product \( \otimes_Q \) (see [10]).

We consider the elements \( \eta \in \mathcal{K} \), for which there is a continuous linear operator \( R_l(\eta) : L^2(Q) \rightarrow \mathcal{K} \) such that \( R_l(\eta) \bar{x} = x \cdot \eta \) for every \( x \in Q \). These elements are called left bounded, they form a dense subspace \( D_l(\mathcal{K}) \) of \( \mathcal{K} \). For \( \eta_1, \eta_2 \in D_l(\mathcal{K}), \langle \eta_1, \eta_2 \rangle_N := JR_l(\eta_1)^* R_l(\eta_2)J \) is a right \( Q \)-linear continuous operator on \( L^2(Q) \) and thus belongs to \( Q \) (where \( J \) is the antilinear operator on \( L^2(Q) \) defined by \( J\overline{\eta} = \overline{\eta} \) for \( \eta \in Q \subset L^2(Q) \)). So a \( Q \)-valued inner product \( \langle \cdot, \cdot \rangle_N : D_l(\mathcal{K}) \times D_l(\mathcal{K}) \rightarrow Q \) is given.

For the right \( Q \)-module \( \mathcal{H} \) we introduce the set \( D_r(\mathcal{H}) \) of right-bounded elements and the operator \( R_r(\xi) : L^2(Q) \rightarrow \mathcal{H} \) is given by \( R_r(\xi) \bar{x} = \xi \cdot x \) for \( x \in Q \), and \( \langle \xi_1, \xi_2 \rangle_N := R_r(\xi_2)^* R_r(\xi_1) \) defines a \( Q \)-valued inner product on \( D_r(\mathcal{H}) \). Moreover, \( D_r(\mathcal{H}) \) is dense in \( \mathcal{H} \).

We use the symbol \( \odot \) for algebraic tensor products of \( \mathbb{C} \)-vector spaces. The Hilbert space \( \mathcal{H} \otimes_Q \mathcal{K} \) is defined as the Hausdorff completion of the algebraic tensor product \( D_r(\mathcal{H}) \odot \mathcal{K} \) with respect to the inner product

\[
\langle \xi_1 \odot \eta_1, \xi_2 \odot \eta_2 \rangle := \langle \langle \xi_1, \xi_2 \rangle_N, \eta_1, \eta_2 \rangle.
\]

(More precisely, \( \mathcal{H} \otimes_Q \mathcal{K} \) is the completion of the factor space \( D_r(\mathcal{H}) \odot \mathcal{K}/N \) with respect to the inner product \( \langle \cdot, \cdot \rangle \), where \( N \) is the subspace of \( D_r(\mathcal{H}) \odot \mathcal{K} \) consisting of all vectors \( \psi \), for which \( \langle \psi, \psi \rangle = 0 \).)

You also can consider \( \mathcal{H} \otimes_Q \mathcal{K} \) as the Hausdorff completion of \( \mathcal{H} \odot D_l(\mathcal{K}) \) with respect to

\[
\langle \xi_1 \odot \eta_1, \xi_2 \odot \eta_2 \rangle := \langle \langle \xi_1, \eta_1 \rangle_N, \eta_2 \rangle_N, \xi_2 \rangle.
\]

The inner products agree on \( D_r(\mathcal{H}) \odot D_l(\mathcal{K}) \) (see [10], Lemma 1.7), and one easily sees that the image of \( D_r(\mathcal{H}) \odot D_l(\mathcal{K}) \) is dense in \( \mathcal{H} \otimes_Q \mathcal{K} \) (for both the definitions of \( \mathcal{H} \otimes_Q \mathcal{K} \)).

**Lemma 1.3** The map \( \xi \in \mathcal{H} \mapsto \xi \otimes_Q \eta \) (resp. \( \eta \in \mathcal{K} \mapsto \xi \otimes_Q \eta \)) is continuous for every \( \eta \in D_r(\mathcal{K}) \) (resp. \( \xi \in D_l(\mathcal{H}) \)).

We call the pair \( (\xi, \eta) \in \mathcal{H} \times \mathcal{K} \) admissible, if and only if \( \xi \in D_r(\mathcal{H}) \) or \( \eta \in D_l(\mathcal{K}) \). For admissible \( (\xi, \eta) \) we denote the corresponding element of \( \mathcal{H} \otimes_Q \mathcal{K} \) by \( \xi \otimes_Q \eta \).

\( D_l(\mathcal{K}) \) is stable under the left action of \( Q \) and \( \langle q \cdot \eta_1, \eta_2 \rangle_N = q \cdot \langle \eta_1, \eta_2 \rangle_N \) for \( q \in Q \) and \( \eta_1, \eta_2 \in D_l(\mathcal{K}) \). Furthermore \( D_r(\mathcal{H}) \) is stable under the right action of \( Q \) and \( \langle \xi_1, q \cdot \xi_2 \rangle_N = \langle \xi_1, \xi_2 \rangle_N \cdot q \) for \( \xi_1, \xi_2 \in D_r(\mathcal{H}) \). It follows

\[
\xi \cdot q \otimes_Q \eta = \xi \otimes_Q q \cdot \eta
\]
for \((\xi, \eta)\) admissible and \(q \in Q\).

We have the following Lemmata:

**Lemma 1.4** (i) Let \(\mathcal{H}\) and \(\mathcal{H}'\) be right \(Q\)-modules and \(K\) and \(K'\) left \(Q\)-modules. If \(A : \mathcal{H} \rightarrow \mathcal{H}'\) is a continuous right \(Q\)-linear operator and \(B : K \rightarrow K'\) a continuous left \(Q\)-linear operator, then \((A \otimes_Q B) \xi \otimes_Q \eta = A \xi \otimes_Q B \eta\) for admissible \((\xi, \eta)\) defines a unique continuous linear operator \(A \otimes_Q B\) from \(\mathcal{H} \otimes_Q K\) onto \(\mathcal{H}' \otimes_Q K'\). One obtains \((A \otimes_Q B)^* = A^* \otimes_Q B^*\).

(ii) \(\mathcal{H} \otimes_Q K\) is a left \(Q\)-module with respect to the action \(\lambda(p) \otimes_Q 1\) \((p \in P)\), if \(\mathcal{H}\) is a \((P, Q)\)-bimodule. It is a right \(Q\)-module with respect to the right action \(1 \otimes_Q \rho(s)\) \((s \in S)\), if \(K\) is a \((Q, S)\)-bimodule, and a \((P, S)\)-bimodule, if both conditions are satisfied.

(iii) If \(\mathcal{H}\) and \(\mathcal{H}'\) are \((P, Q)\)-bimodules and \(A\) is \((P, Q)\)-linear, then \(A \otimes_Q B\) is left \(P\)-linear. The corresponding result for right actions on \(K\) and \(K'\) is satisfied likewise.

**Proof:** We note \(A(D_r(\mathcal{H})) \subset D_r(\mathcal{H})\) and \(B(D_l(\mathcal{K})) \subset D_l(\mathcal{K})\). Assertion (i) without the equation concerning the \(\ast\)-operation is Lemma 2.3 in [10]. Part (ii) and (iii) are easy consequences of part (i), hence we will only show \((A \otimes_Q B)^* = A^* \otimes_Q B^*\). If \(K = K'\) and \(B = 1\), then

\[
\langle (A \otimes_Q 1)(\xi \otimes_Q \eta), \xi' \otimes_Q \eta' \rangle = \langle (A \xi), \langle \eta, \eta' \rangle_N, \xi' \rangle = \\
\langle \xi, \langle \eta, \eta' \rangle_N, A^* \xi' \rangle = \langle \xi \otimes_Q \eta, (A^* \otimes_Q 1)(\xi' \otimes_Q \eta') \rangle
\]

for \(\xi \in \mathcal{H}, \xi' \in \mathcal{H}', \eta, \eta' \in D_l(\mathcal{K})\). So we obtain \((A \otimes_Q 1)^* = A^* \otimes_Q 1\). Similarly \((1 \otimes_Q B)^* = 1 \otimes_Q B^*\) follows. Using \(A \otimes_Q B = (A \otimes_Q 1) \cdot (1 \otimes_Q B)\), we get the assertion.

Lemma [1.3] implies:

**Lemma 1.5** (i) If \(\mathcal{H}\) is a right \(Q\)-module, \(K\) an \((Q, P)\)-bimodule and \(\eta \in D_r(\mathcal{K})\), then \(\xi \otimes_Q \eta \in D_r(\mathcal{H} \otimes_Q K)\) for \(\xi \in \mathcal{H}\).

(ii) If \(K_0\) is a dense linear subspace of \(D_r(\mathcal{K})\) and \(\mathcal{H}_0\) a dense linear subspace of \(\mathcal{H}\), then

\[
\mathcal{H}_0 \otimes_Q K_0 := \text{span} \{\xi \otimes_Q \eta : \xi \in \mathcal{H}_0, \eta \in K_0\}
\]

is a dense subset of \(D_r(\mathcal{H} \otimes_Q K)\).

(iii) If \(\mathcal{H}\) is a \((P, Q)\)-bimodule and \(K\) a left \(Q\)-module, one obtains the analogous results.

**Lemma 1.6** If \(\mathcal{H}\) is a right \(Q\)-module, \(K\) a \((Q, P)\)-module, and \(\mathcal{L}\) a left \(P\)-module, there is a unique unitary operator

\[
a = a(\mathcal{H}, K, \mathcal{L}) : \mathcal{H} \otimes_Q (K \otimes_P \mathcal{L}) \rightarrow (\mathcal{H} \otimes_Q K) \otimes_P \mathcal{L}
\]

such that \(a \xi \otimes_Q (\eta \otimes_P \rho) = (\xi \otimes_Q \eta) \otimes_P \rho\) for \(\xi \in D_r(\mathcal{H}), \eta \in K\) and \(\rho \in D_l(\mathcal{L})\). If \(\mathcal{H}\) is an \((L, Q)\)-bimodule and \(\mathcal{L}\) a \((P, S)\)-bimodule, then \(a\) is \((L, S)\)-linear.
**Proof:** We apply the Lemma 1.3 and see that it suffices to check
\[
\langle (\xi \otimes Q \eta) \otimes_P \rho, (\xi' \otimes Q \eta') \otimes_P \rho' \rangle = \langle \xi \otimes_Q (\eta \otimes_P \rho), \xi' \otimes_Q (\eta' \otimes_P \rho') \rangle
\]
for \(\xi, \xi' \in D_r(\mathcal{H}), \eta, \eta' \in \mathcal{K}\) and \(\rho, \rho' \in D_l(\mathcal{L})\). We compute
\[
\langle (\xi \otimes Q \eta) \otimes_P \rho, (\xi' \otimes Q \eta') \otimes_P \rho' \rangle = \langle (\xi \otimes Q \eta), (\rho, \rho')^l_{\mathcal{P}}, \xi' \otimes Q \eta' \rangle = \langle (\xi, \xi')^r_{\mathcal{Q}}, Q \eta, (\rho, \rho')^l_{\mathcal{P}}, \eta' \rangle
\]
and get the same result, if we carry out similar steps for the right inner product in (2). □

For every \((P, Q)\)-bimodule \(\mathcal{P} \mathcal{H} Q\) there is the conjugate bimodule \(\mathcal{Q} \mathcal{H} P\), where \(\mathcal{H}\) is equal to \(\mathcal{K}\) as a real vector space, but the inner product and the scalar multiplication are conjugate. The left action \(\lambda\) of \(Q\) and the right action \(\rho\) of \(P\) on \(\mathcal{H}\) are given by \(\lambda(q) = \rho(q^*)\) for \(q \in Q\) and \(\rho(p) = \lambda(p^*)\) for \(p \in P\). Obviously, \(\mathcal{H} \cong \mathcal{K}\) entails \(\mathcal{H} \cong \mathcal{K}\).

The \((Q, Q)\)-bimodule \(L^2(Q)\) is a unit in the following meaning: For a \((Q, S)\)-bimodule \(\mathcal{H}\) there is a unitary \((Q, S)\)-linear map \(l_\mathcal{H}\) from \(L^2(Q) \otimes_Q \mathcal{H}\) onto \(\mathcal{H}\) determined by
\[
l_\mathcal{H} \xi \otimes_Q q = q \cdot \xi \quad \text{for} \quad \xi \in \mathcal{H} \quad \text{and} \quad q \in Q.
\]
(3)
The existence of \(l_\mathcal{H}\) allows us to identify the bimodules \(L^2(Q) \otimes_Q \mathcal{H}\) and \(\mathcal{H}\). Similarly for a \((P, Q)\)-bimodule \(\mathcal{H}\) there is a unitary \((P, Q)\)-linear map \(r_\mathcal{H}\) from \(\mathcal{H} \otimes_Q L^2(Q)\) onto \(\mathcal{H}\) determined by
\[
r_\mathcal{H} \xi \otimes q = \xi \cdot q \quad \text{for} \quad \xi \in \mathcal{H} \quad \text{and} \quad q \in Q.
\]
(4)

**Definition 1.7** A \((Q, P)\)-bimodule \(\mathcal{H}\) is called regular, if \(D(\mathcal{H}) := D_l(\mathcal{H}) \cap D_r(\mathcal{H})\) is dense in \(\mathcal{H}\).

A subbimodule of a regular bimodule, a finite direct sum of regular bimodules, the conjugate of a regular bimodule and the tensor product of regular bimodules are regular. (The proof is easy, Lemma 1.3 is used for the tensor product.) Let \(Q_i, i = 0, 1 \ldots, n\), be factors of type \(\Pi_1\), and \(Q_{i-1} \mathcal{H}_i Q_i, i = 1, \ldots, n\) be regular bimodules. There are different possibilities to put the the brackets in \(\mathcal{H}_1 \otimes Q_1 \mathcal{H}_2 \otimes Q_2 \cdots \otimes Q_{n-1} \mathcal{H}_n\). From Lemma 1.6 we conclude that the \((Q_0, Q_n)\)-bimodules corresponding to different choices of the brackets are equivalent in a canonical way and we may identify these \((Q_0, Q_n)\)-bimodules. If \(D_i\) is a dense linear subset of \(D(\mathcal{H}_i)\) for \(i = 1, \ldots, n\), then
\[
D_1 \otimes Q_1 D_2 \cdots \otimes Q_{n-1} D_n := \text{span} \{ \xi_1 \otimes Q_1 \xi_2 \otimes Q_2 \cdots \otimes Q_{n-1} \xi_n : \xi \in D_i, i = 1, \ldots, n \}
\]
is dense in \(\mathcal{H}_1 \otimes Q_1 \mathcal{H}_2 \otimes Q_2 \cdots \otimes Q_{n-1} \mathcal{H}_n\).
2 Bimodules and the Jones Tower

Let $N \subset M$ be a subfactor of a $\text{II}_1$-factor $M$ with finite Jones’ index $\beta := [M : N]$. $N$ and $M$ act on $L^2(M)$ by left multiplication. The von Neumann algebra $M_1 := L_{-N}(L^2(M))$ is a $\text{II}_1$-factor containing $M$ and is called the basic construction for $N \subset M$. The orthogonal projection $e_0$ from $L^2(M)$ onto $L^2(N) \subset L^2(M)$ is called the Jones projection. It is known that $M_1$ is generated by $M$ and $e_0$ as a $\ast$-algebra. $e_0$ maps $M \subset L^2(M)$ onto $N \subset L^2(M)$, the restriction 

$$E_0 = e_0|_{M : M} \longrightarrow N$$

of $e_0$ is a normal faithful conditional expectation from $M$ onto $N$. For $m \in M$, $E_0(m)$ is the unique element of $N$ satisfying 

$$\text{tr}_M(E_0(m)n) = \text{tr}_M(mn) \quad \text{for every } n \in N.$$ 

$E_0$ is called the conditional expectation from $M$ to $N$ corresponding to the trace $\text{tr}_M$.

We obtain $[M_1 : M] = [M : N] < \infty$, so we are able to repeat the basic construction infinitely many times and get the so called Jones tower 

$$N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \ldots,$$

where $M_{k+1}$ is the basic construction for the subfactor $M_{k-1} \subset M_k$. For $k \in \mathbb{N} \cup \{0\}$ let $e_k \in M_{k+1}$ denote the orthogonal projection from $L^2(M_k)$ onto $L^2(M_{k-1})$ and $E_k : M_k \longrightarrow M_{k-1}$ the corresponding conditional expectation. We have the following relations:

$$e_k e_{k \pm 1} e_k = \beta^{-1} e_k \quad \text{and} \quad e_k e_l = e_l e_k \quad \text{for } |k - l| \geq 2. \quad (5)$$

Moreover we have 

$$e_k x e_k = E_k(x)e_k \quad \text{for } x \in M_k \quad \text{and} \quad E_k(e_{k-1}) = \beta^{-1}1. \quad (7)$$

The trace $\text{tr}_{M_{k+1}}$ on $M_{k+1}$ satisfies the Markov property 

$$\beta \text{ tr}_{M_{k+1}}(xe_k) = \text{tr}_{M_{k+1}}(x) \quad \text{for } x \in M_k. \quad (9)$$

$L^2(M_k)$ ($k \in \mathbb{N} \cup \{0\}$) is a regular $(M,M)$-bimodule (as well as a regular $(N,N)$-, $(N,M)$- and $(M,N)$-bimodule), as $M_k \subset D_l(L^2(M_k)) \cap D_r(L^2(M_k))$ shows.
Lemma 2.1 There is a unique unitary \((M, M)\)-linear operator \(U_2\) from \(L^2(M) \otimes_N L^2(M)\) onto \(L^2(M_1)\) such that \(U_2 m_1 \otimes_N m_2 = \beta^{1/2} m_1 e_0 m_2\) for \(m_1, m_2 \in M\).

Lemma 2.1 is a Hilbert space version of the following result (see [5], Corollary 3.6.5): There is a canonical isomorphism \((\text{of algebraic } (M, M)\text{-bimodules})\) from the algebraic \(N\)-tensor product \(M \otimes_N M\) onto \(M_1\) given by \(m_1 \otimes_N m_2 \mapsto m_1 e_0 m_2\).

**Proof:** According to Lemma 1.3 it suffices to verify

\[
\langle m_1 \otimes_N m_2, \bar{m}_1 \otimes_N \bar{m}_2 \rangle = \beta \text{tr}_{M_2}(l_1^* m_1 E_0(m_2 l_2^*) e_0) = \langle m_1 \otimes_N m_2, \bar{m}_1 \otimes_N \bar{m}_2 \rangle
\]

for \(l_1, l_2, m_1, m_2 \in M\) (observe the equations (7) and (8)). The right side of equation (10) is equal to

\[
\beta \text{tr}_{M_1}(l_1^* m_1 E_0(m_2 l_2^*) e_0) = \text{tr}_{M_1}(l_1^* m_1 E_0(m_2 l_2^*)) = \langle \bar{m}_1, E_0(m_2 l_2^*), \bar{m}_1 \rangle.
\]

So the Lemma is proved, if we show \(\langle \bar{m}_2, l_2 \rangle^l_N \equiv E_0(m_2 l_2^*)\). We get

\[
\langle R_i(\bar{m}_2^*) \bar{x}, \bar{n} \rangle = \text{tr}_M(x (nm_2)^*) = \text{tr}_M(x m_2^* n^*) = \text{tr}_M(E_0(x m_2^*) n^*)
\]

for all \(n \in N\) and \(x \in M\). It follows \(R_i(\bar{m}_2^*) \bar{x} = E_0(x m_2^*) \in L^2(N)\) and

\[
\langle \bar{m}_2, l_2 \rangle^l_N, \bar{n} \equiv J R_i(\bar{m}_2^*) \bar{n} l_2 = J E_0(n^* l_2 m_2^*) = E_0(m_2 l_2^*). \bar{n}
\]

for every \(n \in N\).

We define unitary \((M, M)\)-linear operators

\[
U_k = U(k; N, M) : L^2(M) \otimes_N \cdots \otimes_N L^2(M) \longrightarrow L^2(M_{k-1})
\]

for \(k \in \mathbb{N}\) by induction. (Lemma 1.3 shows that it is not necessary to use brackets in \(L^2(M) \otimes_N \cdots \otimes_N L^2(M)\).) Set \(U_1 := id_{L^2(M)}\) and let \(U_2\) be as in Lemma 2.1.

For \(k \geq 2\) let \(V_k = U(k-1; M, M_1) : L^2(M_1) \otimes_M^{k-1} \longrightarrow L^2(M_{k-1})\). Observing that the \((M, M)\)-bimodules \(L^2(M)\) and \(L^2(M) \otimes_M L^2(M)\) are equivalent in a canonical way according to (3) (or (4)), we use the following identification of \((M, M)\)-bimodules:

\[
L^2(M) \otimes_N (L^2(M) \otimes_M L^2(M)) \otimes_N \cdots \otimes_N (L^2(M) \otimes_M L^2(M)) \otimes_N L^2(M) = (L^2(M) \otimes_N L^2(M)) \otimes_M^{k-1}.
\]

We put

\[
U_k := V_k \circ U_2^{\otimes_{M}^{k-1}}.
\]
(Concerning the definition of the operator $U_2^{\otimes k-1}$ see Lemma [4].) Using the notation

$$e_{n,m} := \begin{cases} \quad e_n \cdot e_{n+1} \cdot \ldots \cdot e_{m-1} \cdot e_m & \text{for } n < m, \\ \quad e_n & \text{for } n = m, \\ \quad e_n \cdot e_{n-1} \cdot \ldots \cdot e_{m+1} \cdot e_m & \text{for } n > m, \end{cases} \tag{12}$$

we get

$$U_k \overline{\mathbf{x}_1} \otimes_N \ldots \otimes_N \overline{\mathbf{x}_k} = \beta^{k(k-1)/4} \frac{x_1 e_0 x_2 e_1 x_3 e_2 x_4 \ldots x_{k-1} e_{k-2} x_k}{x_1 e_{0,k-2} x_2 e_{0,k-3} x_3 \ldots x_{k-2} e_{0,1} x_{k-1} e_0 x_k} \tag{13}$$

for $x_1, \ldots, x_k \in M$. (The first equation follows, if we identify $\overline{\mathbf{x}}_i \in L^2(M)$ with $\overline{\mathbf{x}}_i \otimes_M 1 \in L^2(M) \otimes_M L^2(M)$ in (14), and so does the second equation, if we put $\overline{\mathbf{x}}_i = 1 \otimes_M \overline{\mathbf{x}}_i$.) Observe that the considerations after Definition [4] imply that $\{\overline{\mathbf{x}}_1 \otimes_N \ldots \otimes_N \overline{\mathbf{x}}_k : x_1, \ldots, x_k \in M\}$ is dense in $L^2(M)^\otimes N$.

Identifying $L^2(M_1)$ and $L^2(M) \otimes_N L^2(M)$, we may regard the orthogonal projection $e_1$ from $L^2(M_1)$ onto $L^2(M) \subset L^2(M_1)$ as a continuous linear operator of $L(L^2(M) \otimes_N L^2(M))$ and denote it by $F_1$.

**Theorem 2.2** For $n \in \mathbb{N}$ there is a normal isomorphism $J_n$ from $M_{2n-1} = \mathcal{L}_{-M_{2n-3}}(L^2(M_{2n-2}))$ onto $\mathcal{L}_{-N}(L^2(M)^\otimes N)$ satisfying the following properties:

$$J_n(m) = \lambda(m) \quad \text{for } m \in M,$$

$$J_n(e_{2k} \overline{\mathbf{x}}_1 \otimes_N \overline{\mathbf{x}}_2 \ldots \otimes_N \overline{\mathbf{x}}_n) = \overline{\mathbf{x}}_1 \otimes_N \ldots \otimes_N \overline{E_0(x_{k+1})} \otimes_N \ldots \otimes_N \overline{E_n}$$

for $x_1, \ldots, x_n \in M$ and $k = 0, 1, \ldots, n - 1$,

$$J_n(e_{2k+1} \overline{\mathbf{x}}_1 \otimes_N \overline{\mathbf{x}}_2 \ldots \otimes_N \overline{\mathbf{x}}_n) = \overline{\mathbf{x}}_1 \otimes_N \ldots \otimes_N \overline{F_1(\overline{\mathbf{E}_{k+1}} \otimes_N \overline{\mathbf{x}}_{k+2})} \otimes_N \ldots \otimes_N \overline{\mathbf{x}}_n$$

for $x_1, \ldots, x_n \in M$ and $k = 0, 1, \ldots, n - 2$ ($n \geq 2$).

As $L^2(M)^\otimes N$ and $L^2(M_{n-1})$ can be identified, Theorem 2.2 especially states that the basic construction $\mathcal{L}_{-N}(L^2(M_{n-1}))$ for $N \subset M_{n-1}$ is the same as $M_{2n-1}$. In [4] M. Pimsner and S. Popa proved a version of this result, which is somewhat weaker than Theorem 2.2.

Before we prove Theorem 2.2, we note some useful consequences.

Using $L^2(M) = L^2(M) \otimes_M L^2(M)$ $n$ times, we can identify $L^2(M)^\otimes N$ with

$$L^2(M) \otimes_M L^2(M) \otimes_N L^2(M) \otimes_M L^2(M) \otimes_N \ldots \otimes_N L^2(M) \otimes_M L^2(M). \tag{14}$$

Let

$$\epsilon(k) = \begin{cases} \quad 0 & \text{for } k \text{ even}, \\ \quad -1 & \text{for } k \text{ odd}. \end{cases}$$

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For $0 \leq k < l \leq 2n$ let $\mathcal{H}_k^l$ denote the tensor product

$$L^2(M) \otimes_{M_{(k)}} L^2(M) \otimes_{M_{(k+1)}} \cdots \otimes_{M_{(l-2)}} L^2(M),$$

which consists of the $(k+1)$-th, $(k+2)$-th, and $l$-th factor in the tensor product (14), moreover let $\mathcal{H}_0^0 := \mathcal{H}_{2n}^2 := L^2(N)$.

**Corollary 2.3** $J_n : M_{2n-1} \to \mathcal{L}_{-N}((L^2(M) \otimes_M L^2(M)) \otimes_N)$ satisfies the following relations:

(i) $J_n(M_k) = \mathcal{L}_{-N}((\mathcal{H}_0^{k+1}) \otimes_{M_{(k)}} \mathcal{C}1_{\mathcal{H}_0^{k+1}})$ for $-1 \leq k \leq 2n - 1$,

(ii) $J_n(M_k \cap M_l) = \mathcal{C}1_{\mathcal{H}_0^{k+1}} \otimes_{M_{(k)}} \mathcal{L}_{M_{(k)},M_{(l)}}((\mathcal{H}_0^{l+1}) \otimes_{M_{(l)}} \mathcal{C}1_{\mathcal{H}_0^{l+1}})$ for $-1 \leq k < l \leq 2n - 1$.

**Proof** (i) The case $k = 2r - 1$ ($r \in \mathbb{N}$, $r \leq n$) follows from $J_n(M_k) = J_r(M_k) \otimes_{N} \mathcal{C}1_{\mathcal{H}_0^{k+1}}$ and $J_n(M_k) = \mathcal{L}_{-N}(\mathcal{H}_0^{k+1})$ (according to Theorem 2.2).

In the case $k = 2r$, $(r \in \mathbb{N}$, $r < n$) we know that $J_n(M_{k-1}) \cup J_n(e_{k-1})$ and consequently $J_n(M_k)$ is contained in $\mathcal{L}_{-M}(\mathcal{H}_0^{k+1}) \otimes_{M} \mathcal{C}1_{\mathcal{H}_0^{k+1}}$. The remaining inclusion is a consequence of

$$[J_n(M_{k+1}) : J_n(M_k)] = [M : N] = [\mathcal{L}_{-N}(L^2(M) \otimes_N^{r+1}) : \mathcal{L}_{-M}(L^2(M) \otimes_N^{r+1})] = [J_n(M_k) : \mathcal{L}_{-M}(\mathcal{H}_0^{k+1}) \otimes_{M} \mathcal{C}1_{\mathcal{H}_0^{k+1}}].$$

(ii) follows from (i) and from the formula

$$\mathcal{L}_{-Q}(\mathcal{H}) \otimes_{Q} \mathcal{C}1_{\mathcal{K}} = \mathcal{C}1_{\mathcal{H}} \otimes_{Q} \mathcal{L}_{Q,-}(\mathcal{K})$$

for a II$_1$-factor $Q$, a right $Q$-module $\mathcal{H}$ and a left $Q$-module $\mathcal{K}$.

Corollary 2.3 gives a useful presentation of the standard invariant

$$\mathcal{C}1 = N' \cap N \subset N' \cap M \subset N' \cap M_1 \subset N' \cap M_2 \subset \cdots$$

$$\mathcal{C}1 = M' \cap M \subset M' \cap M_1 \subset M' \cap M_2 \subset \cdots$$

of the subfactor $N \subset M$.

For a projection $p$ in $L^2(M) \otimes_N^k$ the Hilbert space $pL^2(M) \otimes_N^k$ is an $(N,N)$-submodule of $L^2(M) \otimes_N^k$, if and only if $p \in J_k(N' \cap M_{2k-1})$, and $pL^2(M) \otimes_N^k$ is an irreducible $(N,N)$-bimodule, if and only if $J_k^{-1}(p)$ is a minimal projection of $N' \cap M_{2k-1}$. Now let $f$ and $g$ denote two minimal projections of $N' \cap M_{2k-1}$. One easily sees that the $(N,N)$-bimodules $J_k(f)L^2(M) \otimes_N^k$ and $J_k(g)L^2(M) \otimes_N^k$ are equivalent, if and only if $f$ and $g$ belong to the same simple direct summand of $N' \cap M_{2k-1}$. So we get a bijective correspondence between the equivalence classes of the irreducible $(N,N)$-bimodules contained in $L^2(M) \otimes_N^k$ and the simple direct summands of $N' \cap M_{2k-1}$.

If one considers the principal graph of the subfactor $N \subset M$, which contains the information about the tower of the finite dimensional von Neumann algebras in the upper line of (13), then one usually identifies the simple direct
summands of $N' \cap M_{2k-1}$ with simple direct summands of $N' \cap M_{2k+1}$ such that the following holds: If $f$ is a minimal projection of a simple direct summand of $N' \cap M_{2k-1}$, then $f e_{2k}$ is a minimal projection of the corresponding direct summand of $N' \cap M_{2k+1}$ (compare [5], Section 4.6).

Using the description of the algebras $N' \cap M_{2k-1}$ and $N' \cap M_{2k+1}$ in Corollary 2.3, we see that corresponding simple direct summands of $M$ by Theorem 2.2. Given a minimal projection

one obtains an analogous connection between the algebras $M$ and $Q$.

(1) Let $L \in \mathcal{L}_{-,L}(H)$ and $Q \mathcal{K} \subseteq S$ be two bimodules. We assume that there is an isomorphism $I$ from $\mathcal{L}_{-,L}(H)$ onto $\mathcal{L}_{-,S}(K)$ such that $I(\lambda_{H}(q)) = \lambda_{K}(q)$ for every $q \in Q$. Then $H$ is also an $\mathcal{L}_{-,S}(K)$-module, where the action is given by $I^{-1}$. Let us suppose that the $\mathcal{L}_{-,S}(K)$-module $K$ is equivalent to a submodule of $H$.

A linear isometry $W : \mathcal{K} \rightarrow H$ is said to be associated with $I$, if $W$ is $\mathcal{L}_{-,S}(K)$-linear. Now let us fix a linear isometry $W$ associated with $I$. Let $p$ be the projection $W \cdot W^{*} \in \mathcal{L}_{-,L}(H)' = \rho(L)$. We have

$$W a W^{*} = p I^{-1}(a)$$

for every $a \in \mathcal{L}_{-,S}(K)$. By restricting $W$ to its image $pH$, we obtain a unitary operator $\tilde{W} : \mathcal{K} \rightarrow pH$. If we endow $pH$ with the right action $\rho_{pH}$ of $S$ defined by $\rho_{pH}(s) := \tilde{W} \cdot \rho(s) \cdot \tilde{W}^{*}$ for $s \in S$, then $pH$ is a $(Q, S)$-bimodule, equivalent to $K$. Using (1) we get

$$\mathcal{L}_{-,S}(pH) = \tilde{W} \mathcal{L}_{-,S}(K) \tilde{W}^{*} = W I(\mathcal{L}_{-,L}(H)) W^{*} = pH$$

which implies

$$\rho_{pH}(S) = p \rho_{H}(L) p.$$
(2) Additionally, let $pL_Q$ be a bimodule. Then $L \otimes_Q H$ is a $(P, L)$-bimodule and $L \otimes_Q K$ is a $(P, S)$-bimodule. Starting with the isomorphism $I$ and the linear isometry $W$, we will define an isomorphism

$$1 \otimes_Q I : L_{-L}(L \otimes_Q H) \to L_{-S}(L \otimes_Q K)$$

with the following properties:

(a) $(1 \otimes_Q I)(\lambda_{L \otimes_Q H}(x)) = \lambda_{L \otimes_Q K}(x)$ for every $x \in P$ and

(b) The $L_{-S}(L \otimes_Q K)$-module $L \otimes_Q K$ is equivalent to a submodule of the $L_{-S}(L \otimes_Q K)$-module $L \otimes_Q H$, and $1 \otimes_Q W$ is a linear isometry associated with $1 \otimes_Q I$.

$1 \otimes_Q p$ is a projection of $\rho_{L \otimes_Q H}(L)$, hence $x \mapsto (1 \otimes_Q p)x$ defines an isomorphism $J_1$ from $\rho_{L \otimes_Q H}(L)' = L_{-L}(L \otimes_Q H)$ onto the commutant $((1 \otimes_Q p) \rho_{L \otimes_Q H}(L)(1 \otimes_Q p))'$ (in $(1 \otimes_Q p) L \otimes_Q H = L \otimes_Q pH$).

In (1) $pH$ was endowed with a $(Q, S)$-bimodule structure, hence $L \otimes_Q pH$ is a $(P, S)$-bimodule. By applying equation (18) we get

$$\rho_{L \otimes_Q pH}(S) = 1 \otimes_Q \rho_pH(S) = 1 \otimes_Q p \rho_pH(L)p = (1 \otimes_Q p) \rho_{L \otimes_Q H}(L)(1 \otimes_Q p)$$

with the consequence that

$$((1 \otimes_Q p) \rho_{L \otimes_Q H}(L)(1 \otimes_Q p))' = L_{-S}(L \otimes_Q pH).$$

Now $x \mapsto (1 \otimes_Q \tilde{W})^* x (1 \otimes_Q \tilde{W})$ defines an isomorphism $J_2$ from $L_{-S}(L \otimes_Q pH)$ onto $L_{-S}(L \otimes_Q K)$. We put $1 \otimes_Q I := J_2 \circ J_1$. Property (a) is obvious, for the proof of Property (b) we note that $(1 \otimes_Q W) \cdot (1 \otimes_Q W)^* = 1 \otimes_Q p$ belongs to $\rho_{L \otimes_Q H}(L)$. From

$$(1 \otimes_Q W)(1 \otimes_Q I)(a) = (1 \otimes_Q W)(1 \otimes_Q \tilde{W})^*(1 \otimes_Q p)a(1 \otimes_Q \tilde{W}) = (1 \otimes_Q p)a(1 \otimes_Q W) = a(1 \otimes_Q p)(1 \otimes_Q W) = a(1 \otimes_Q p)$$

for every $a \in L_{-L}(L \otimes_Q H)$ we conclude that the linear isometry $1 \otimes_Q W$ is associated with $1 \otimes_Q I$.

We point out that the definition of $1 \otimes_Q I$ does not depend on the choice of $W$.

(3) We suppose the assumptions from (1). Additionally, let $Q \mathcal{M}_T$ be a bimodule and $J : L_{-S}(K) \to L_{-T}(\mathcal{M})$ an isomorphism such that $\mathcal{M}$ is an $L_{-T}(\mathcal{M})$-submodule of $K$. Considering the isomorphism $1 \circ I : L_{-L}(H) \to L_{-T}(\mathcal{M})$ we also may regard $\mathcal{M}$ as an $L_{-T}(\mathcal{M})$-subbimodule of $H$. If $V : \mathcal{M} \to K$ is a linear isometry associated with $J$, then $W \circ V$ is a linear isometry associated with $J \circ I$. The proof of this fact is easy.

**Lemma 2.5** (i) $\beta e_{n,0}e_{n+1,0} = e_{n,0}e_{n+1,2}$ for $n \in \mathbb{N}$.  

(ii) $\beta^{k(k+1)/2} e_{n,0}e_{n+1,0} \cdots e_{n+k,0} = e_{n,0}e_{n+1,2}e_{n+2,4} \cdots e_{n+k,2k}$ for $0 < k \leq n$.

(iii) $e_{n,0}e_{n+1,0} \cdots e_{2n-2,0}e_{2n-1,2n-2} = \beta^{n-1} e_{n,0}e_{n+1,0} \cdots e_{2n-1,0}$ for $n \in \mathbb{N}$.
Proof: $\beta e_{n,0}e_{n+1,0} = \beta e_{n,1}e_{n+1,2}e_0e_1e_0 = e_{n,1}e_{n+1,2}e_0 = e_{n,0}e_{n+1,2}$ shows (i). We get (ii) by applying (i) several times and (iii) by applying (ii) twice (with $k = n - 1$ to the right side of the equation in (iii) and with $k = n - 2$ to the left side).

2.6 The isomorphisms $J_n$

(1) We introduce an isomorphism $I : M_3 = \mathcal{L}_{-M_1}(L^2(M_2)) \rightarrow \mathcal{L}_{-M}(L^2(M_1))$ satisfying $I(m) = \lambda(m)$ for every $m \in M$. Let $K$ be the canonical isomorphism from $\mathcal{L}_{-M_1}(L^2(M_1)) = \lambda(M_1)$ onto $\mathcal{L}_{-M}(L^2(M)) = M_1$. $K$ is an isomorphism as in Section 2.4 (1) (with $Q = M$) and

$$V : L^2(M) \rightarrow L^2(M_1), \overline{m} \mapsto \beta^{1/2} \overline{me_0},$$

is a linear isometry associated with $K$. (The Markov property (3) of $\text{tr}_{M_1}$ implies

$$\langle \beta^{1/2} \overline{me_0}, \beta^{1/2} \overline{le_0} \rangle = \beta \text{tr}_{M_1}(me_0l^*) = \text{tr}_{M}(ml^*) = \langle \overline{m}, \overline{l} \rangle$$

for $m, l \in M$, hence $V$ is isometric. $V$ is $M$-linear and $V \circ e_0 = \lambda(e_0) \circ V$ holds by equation (7), thus $V$ is $M_1$-linear.)

Applying Section 2.4 (2) with the $(M, M)$-bimodule $\mathcal{L} = L^2(M_1)$ we get an isomorphism

$$1 \otimes_M K : \mathcal{L}_{-M_1}(L^2(M_1) \otimes_M L^2(M_1)) \rightarrow \mathcal{L}_{-M}(L^2(M_1) \otimes_M L^2(M)).$$

We identify $L^2(M_1) \otimes_M L^2(M_1)$ and $L^2(M_2)$ according to Lemma 2.1 as well as $L^2(M_1) \otimes_M L^2(M)$ and $L^2(M_1)$ according to relation (3). After these identifications, $1 \otimes_M K$ is the desired isomorphism $I$, and the linear isometry $1 \otimes_M V : L^2(M_1) \rightarrow L^2(M_2)$ is associated with $I$ and satisfies the relation

$$(1 \otimes_M V) \overline{x} = \beta \overline{xe_1,0} \quad \text{for } x \in M_1. \quad (19)$$

(2) Inductively we define isomorphisms

$$J_n : M_{2n-1} = \mathcal{L}_{-M_{2n-3}}(L^2(M_{2n-2})) \rightarrow \mathcal{L}_{-M}(L^2(M)^{\otimes^n_N}) \quad (n \geq 1)$$

with the following properties:

(a) $J_n(m) = \lambda(m)$ for every $m \in M$ and

(b) $W_n : L^2(M)^{\otimes^n_N} \rightarrow L^2(M_{2n-2}), \overline{x_1 \otimes_N x_2 \otimes_N \ldots \otimes_N x_n} \mapsto$

$$\beta^{(n-1)(2n-1)/2} x_1 e_{0,1} x_2 e_{1,0} x_3 e_{0,1} \ldots x_{n-1} e_{n-2,0} x_n e_{n-1,0} e_0 e_1 e_{0,1} e_{0,2} e_{0,2n-4} e_{0,2n-4} \ldots e_{2n-3,0} = \beta^{(n-1)(2n-1)/2} x_1 e_{0,2n-3} x_2 e_{0,2n-4} x_3 e_{0,2n-4} \ldots x_{n-1} e_{n-2,0} x_n e_{n-1,0} e_0 e_1 e_{0,1} e_{0,2} e_{0,2n-4} \ldots e_{2n-3,0}$.$
is a linear isometry associated with $J_n$ for $n \geq 2$.

Let $J_1 = id_{M_1}$ and let $W_1 = id_{L^2(M)}$.

Now for $n \geq 2$ we assume that $J_{n-1}$ is defined and that $W_{n-1}$ is associated with $J_{n-1}$. Let

$$I_n : M_{2n-1} = \mathcal{L}_{-n,M_{2n-3}}(L^2(M_{2n-2})) \longrightarrow \mathcal{L}_{-n,M_{2n-5}}(L^2(M_{2n-3}))$$

denote the isomorphism $I$ from Part (1) and $V_n$ the linear isometry $V$ from (1), where $N$ is replaced by $M_{2n-5}$, $M$ by $M_{2n-4}$ and so on. We write $\hat{V}_n$ in place of $1 \otimes_{M_{2n-4}} V_n$.

Observe that $R_{2n-4} := (id_{L^2(M)} \otimes N U_{2n-4}) \circ U_{2n-3}^*$ is an $(M,M)$-linear unitary operator from $L^2(M_{2n-3})$ onto $L^2(M) \otimes N L^2(M_{2n-4})$ and that

$$R_{2n-4} \beta^{2n-3/2} x e_{0,2n-4} y = x \otimes N y$$

holds for $x \in M$ and $y \in M_{2n-4}$. Hence it is possible to identify the bimodules $L^2(M_{2n-3})$ and $L^2(M) \otimes N L^2(M_{2n-4})$. Applying Section 2.4 (2) we get an isomorphism $1 \otimes N J_{n-1}$ from $\mathcal{L}_{-n,M_{2n-5}}(L^2(M) \otimes N L^2(M_{2n-4})) = \mathcal{L}_{-n,M_{2n-5}}(L^2(M_{2n-3}))$ onto $\mathcal{L}_{-n}(L^2(M) \otimes N)$ and put

$$J_n := (1 \otimes N J_{n-1}) \circ I_n.$$

Obviously $J_n$ satisfies Property (a). From Section 2.4 (3) we know that $W_n := \hat{V}_n \circ (1 \otimes N W_{n-1})$ is associated with $J_n$. The following computation shows that $W_n$ fulfills Property (b) for $n \geq 3$:

$$(\hat{V}_n \circ (1 \otimes N W_{n-1})) x_1 \otimes N x_2 \otimes N \ldots \otimes N x_n =$$

$$\beta^{(n-2)(2n-3)/2} \hat{V}_n x_1 \otimes N x_2 e_0 x_3 \ldots x_{n-1} e_{n-3,0} x_n \epsilon_{n-2,0} \epsilon_{n-1,0} \ldots \epsilon_{2n-5,0} =$$

$$\beta^{(n-1)(2n-3)/2} + \beta^{(n-1)} \epsilon_{n-1,0} \epsilon_{n-3,0} x_n \otimes N \epsilon_{n-2,0} \epsilon_{n-1,0} \ldots \epsilon_{2n-5,0} \epsilon_{2n-3,0} =$$

$$(\text{We used equation } (13) \text{ in line 3, Lemma } 2.5 \text{ (iii) in line 4 and equation } (13) \text{ in line 6.)}$$

The case $n = 2$ follows from $W_2 = 1 \otimes_M V$.

### 2.7 Proof of Theorem 2.2

We prove that $J_n$ satisfies the properties required in Theorem 2.2. It suffices to show

$$e_{l} W_n(x_1 \otimes N x_2 \otimes N \ldots \otimes N x_n) =$$

$$= \left\{ \begin{array}{ll}
W_n(x_1 \otimes N \ldots \overline{E_0(x_{k+1})} \otimes N \ldots \otimes N x_n) & (l = 2k \text{ even}) \\
W_n(x_1 \otimes N \ldots F_1(x_{k+1} \otimes N x_{k+2}) \otimes N \ldots \otimes N x_n) & (l = 2k + 1 \text{ odd})
\end{array} \right. \quad (20)$$
for \( l \in \mathbb{N} \cup \{0\} \), \( 2n - 2 \geq l \) and \( x_1, \ldots, x_n \in M \). We show (20) by induction over \( l \) (simultaneously for every \( n \in \mathbb{N} \) with \( 2n - 2 \geq l \): 

\( l = 0 \) is a simple consequence of \( e_0x_1e_0 = E_0(x_1)e_0 \).

\( l = 1 \): We get

\[
\beta^{(n-1)(2n-1)/2} x_1 e_1 \cdot W_n x_1 \otimes_N x_2 \otimes_N \ldots \otimes_N x_n = \beta^{(n-1)(2n-1)/2} x_1 e_1 \cdot x_1 e_2 e_1,0 \ldots = \beta^{(n-1)(2n-1)/2} E_1(x_1 e_0 x_2) e_1,0 \ldots = W_n F_1(x_1 \otimes_N x_2) \otimes_N \ldots \otimes_N x_n.
\]

Concerning the last \( '=' \) observe that there are an \( m \in \mathbb{N} \) and \( y_j, z_j \), \( j = 1, \ldots, m \), such that \( E_1(x_1 e_0 x_2) = \sum_{j=1}^m y_j e_0 z_j \) and consequently \( F_1(x_1 \otimes_N x_2) = \sum_{j=1}^n y_j \otimes_N z_j \) holds.

\( 2 \leq l \leq 2n - 2 \): The left side of Equation (20) is equal to

\[
\beta^{(n-1)(2n-1)/2} e_l \cdot x_1 e_0 x_2 x_3 \ldots x_{n-1} e_n x_n e_n e_{n-1} \ldots e_{2n-5} \ldots e_{2n-4} = \beta^{(n-1)(2n-1)/2} x_1 e_0 x_2 x_3 \ldots x_{n-1} e_n x_n e_n e_{n-1} \ldots e_{2n-5} \ldots e_{2n-4}
\]

(see the computation from Section 2.4 (2)). For the right side we use the facts that \( W_n = \hat{V}_n \circ (1 \otimes_N W_{n-1}) \) holds and that the assertion is fulfilled for \( l - 2 \) and \( n - 1 \) by induction hypothesis. For \( l < 2n - 2 \) the right side is equal to

\[
\beta^{(n-1)(2n-1)/2} \hat{V}_n x_1 \otimes_N e_{l-2} \cdot W_{n-1} x_2 \otimes_N \ldots \otimes_N x_n = \beta^{(n-1)(2n-1)/2} \hat{V}_n x_1 \otimes_N e_{l-2} \cdot x_2 e_0 x_3 \ldots x_{n-1} e_n x_n e_n e_{n-1} \ldots e_{2n-5} \ldots e_{2n-4}
\]

(The computation is similar to that in Section 2.4 (2)). So the left side and the right side of equation (20) coincide, if \( e_l e_{0,2n-3} = e_{0,2n-3} e_{l-2} \), which, however, the following computation shows:

\[
e_l e_{0,2n-3} = e_{0,l-2} e_{l-1} e_{l-2} e_{l-3} = \beta^{-1} e_{0,l-2} e_{l-1} e_{l-2} = e_{0,2n-3} e_{l-2}.
\]

At last we deal with the case \( l = 2n - 2 \). Using the induction hypothesis and similar arguments as before, we see that the right side of (20) is equal to

\[
\beta^{(n-1)(2n-1)/2} \hat{V}_n x_1 \otimes_N E_{2n-4}(x_2 e_0 x_2 x_3 \ldots x_{n-1} e_n x_n e_n e_{n-1} \ldots e_{2n-5} \ldots e_{2n-4}) = \beta^{(n-1)(2n-1)/2} \hat{V}_n x_1 \otimes_N E_{2n-4}(x_2 e_0 x_2 x_3 \ldots x_{n-1} e_n x_n e_n e_{n-1} \ldots e_{2n-5} \ldots e_{2n-4})
\]

\[
\beta^{(n-1)(2n-1)/2} x_1 e_0 x_2 e_{1,0} x_3 \ldots x_{n-1} e_n x_n e_n e_{n-1} \ldots e_{2n-5} e_{2n-4} = \beta^{(n-1)(2n-1)/2} x_1 e_0 x_2 e_{1,0} x_3 \ldots x_{n-1} e_n x_n e_n e_{n-1} \ldots e_{2n-5} e_{2n-4}
\]

\[
\beta^{(n-1)(2n-1)/2} \hat{E}_{2n-2}(x_1 e_0 x_2 e_{1,0} x_3 \ldots x_{n-1} e_n x_n e_n e_{n-1} \ldots e_{2n-5} e_{2n-4}) = \beta^{(n-1)(2n-1)/2} \hat{E}_{2n-2}(x_1 e_0 x_2 e_{1,0} x_3 \ldots x_{n-1} e_n x_n e_n e_{n-1} \ldots e_{2n-5} e_{2n-4})
\]

(We used (8) in line 3, (9) in line 4, (10) in line 5, and Lemma 2.5 (iii) in line 6.)
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