GROUP ACTIONS AND INVARIANTS IN ALGEBRAS OF GENERIC MATRICES

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Abstract. We show that the fixed elements for the natural GL_{m,n}-action on the universal division algebra UD(m,n) of m generic n \times n-matrices form a division subalgebra of degree n, assuming n \geq 3 and 2 \leq m \leq n^2 - 2. This allows us to describe the asymptotic behavior of the dimension of the space of SL_{m,n}-invariant homogeneous central polynomials p(X_1, \ldots, X_m) for n \times n-matrices. Here the base field is assumed to be of characteristic zero.

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1. INTRODUCTION

Let k be a field of characteristic zero, m and n be integers \geq 2, and \( G_{m,n} = k\{X_1, \ldots, X_m\} \) be the k-algebra of m generic n \times n-matrices. That

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is, $G_{m,n}$ is the $k$-subalgebra of $M_n(k[x_{ij}^{(h)}])$ generated by

$$X_1 = (x_{ij}^{(1)}), \ldots, X_m = (x_{ij}^{(m)}),$$

where the $x_{ij}^{(h)}$ are $mn^2$ independent commuting variables. By a theorem of Amitsur, $G_{m,n}$ is a domain of PI-degree $n$. There is a natural action of the general linear group $GL_m$ on $G_{m,n}$ given by

$$(1.1) \quad g = (g_{ij}): X_j \mapsto \sum_{i=1}^{m} g_{ij} X_i.$$ 

In this paper we will prove the following theorem.

1.2. **Theorem.** For $2 \leq m \leq n^2 - 2$, the domain $(G_{m,n})^{SL_m}$ has PI-degree $n$.

The trace ring $T_{m,n}$ of $G_{m,n}$ is defined as the subring of $M_n(k[x_{ij}^{(h)}])$ generated by elements of the form $Y$ and $tr(Y)$, as $Y$ ranges over $G_{m,n}$. The action (1.1) on $G_{m,n}$ naturally extends to $T_{m,n}$. Note that the algebras $G_{m,n}$ and $T_{m,n}$, and their centers $Z(G_{m,n})$ and $Z(T_{m,n})$ have a natural $\mathbb{Z}$-grading inherited from $M_n(k[x_{ij}^{(h)}])$ (each variable $x_{ij}^{(h)}$ has degree 1) and that this grading is preserved by the action (1.1). As a consequence of Theorem 1.2 we obtain the following result.

1.3. **Theorem.** Let $2 \leq m \leq n^2 - 2$, and let $R$ be one of the rings $G_{m,n}$, $T_{m,n}$, $Z(G_{m,n})$, or $Z(T_{m,n})$. Denote the degree $d$ homogeneous component of $R$ by $R[d]$. Then

$$\limsup_{d \to \infty} \frac{\dim_k R^{SL_m}[d]}{d^{(m-1)n^2-m^2+1}}$$

is a finite nonzero number.

One can think of the center of $G_{m,n}$ as consisting of the $m$-variable central polynomials for $n \times n$-matrices (over commutative $k$-algebras). Theorem 1.3 thus describes, for $R = Z(G_{m,n})$, the asymptotic behavior of the dimension of the space of $SL_m$-invariant homogeneous central polynomials $p(X_1, \ldots, X_m)$ for $n \times n$-matrices.

The $GL_m$-representations on $G_{m,n}, Z(G_{m,n}), T_{m,n}$ and $Z(T_{m,n})$ have been extensively studied; see, e.g., [1, 2, 7, 9, 17]. Once again, let $R$ be one of these rings. Recall that the irreducible polynomial representations of $GL_m$ are indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_s)$ with $s \leq m$ parts; cf., e.g., [1, Section 2]. Denote the multiplicity of the irreducible $GL_m$-representation corresponding to $\lambda$ in $R$ by $\text{mult}_{\lambda}(R)$. If $(r^m)$ is the partition $(r, \ldots, r)$ ($m$ times) then it is easy to show that

$$\dim R^{SL_m}[d] = \begin{cases} \text{mult}_{(r^m)}(R) & \text{if } d = rm, \\ 0 & \text{if } d \text{ is not a multiple of } m; \end{cases}$$
see Remark 9.2. The conclusion of Theorem 1.3 can thus be rephrased by saying that

$$\limsup_{r \to \infty} \frac{\text{mult}(r_m)(R)}{r(m-1)n^2 - m^2 + 1}$$

is a finite nonzero number. We also note that by the Berele-Drensky-Formanek correspondence, $\text{mult}(r_m)(R)$ equals the multiplicity of the $S_m$-character $\chi(d_m)$ in the cocharacter sequence of $R$; see [4, Section 4].

The division algebra of quotients of $G_{m,n}$ (or equivalently, of $T_{m,n}$) is called the universal division algebra of $m$ generic $n \times n$-matrices; we shall denote it by $UD(m,n)$. Note that the $GL_m$-action (1.1) on $G_{m,n}$ naturally extends to $UD(m,n)$. We shall deduce Theorem 1.2 from the following related result.

1.4. **Theorem.** If $2 \leq m \leq n^2 - 2$ and $n \geq 3$, then $UD(m,n)^{GL_m}$ is a division algebra of degree $n$.

For all other values of $m, n \geq 2$, $UD(m,n)^{GL_m}$ is a field; see Propositions 5.1 and 5.3. A brief summary of the properties of $UD(m,n)^{GL_m}$ and $UD(m,n)^{SL_m}$ is given in the two tables below.

**Table 1. Properties of $UD(m,n)^{GL_m}$**

| Case          | PI-Degree | Transcendence Degree/$k$ | Central in $UD(m,n)$? |
|---------------|-----------|--------------------------|-----------------------|
| $m \leq n^2 - 2, n \geq 3$ | $n$       | $(m - 1)n^2 - m^2 + 1$   | No                    |
| $m = n^2 - 1$  | 1         | $n - 1$                  | No                    |
| $m = n = 2$    | 1         | 1                        | No                    |
| $m \geq n^2$   | 1         | 0                        | Yes                   |

**Table 2. Properties of $UD(m,n)^{SL_m}$**

| Case          | PI-Degree | Transcendence Degree/$k$ | Central in $UD(m,n)$? |
|---------------|-----------|--------------------------|-----------------------|
| $m \leq n^2 - 2$ | $n$       | $(m - 1)n^2 - m^2 + 2$   | No                    |
| $m = n^2 - 1$  | 1         | $n$                      | No                    |
| $m = n^2$      | 1         | 1                        | Yes                   |
| $m \geq n^2 + 1$ | 1         | 0                        | Yes                   |

The assertions of the tables in the cases where $m \leq n^2 - 2$ and $n \geq 3$ are based on Theorems 1.4 and 3.4; the case where $m = n = 2$ is considered in [20, Section 14], and the cases where $m \geq n^2 - 1$ are treated in detail in Section 8.

It appears likely that Theorems 1.2 – 1.4 remain valid in prime characteristic (perhaps, not dividing $n$); we have not attempted to extend them
in this direction. Our arguments rely on the work of Richardson [21] and on our own prior papers [19, 20], all of which make the characteristic zero assumption.1

Conventions and Terminology. All central simple algebras in this paper are assumed to be finite-dimensional over their centers. All algebraic varieties, algebraic groups, group actions, morphisms, rational maps, etc., are assumed to be defined over the base field \( k \) (which we always assume to be of characteristic zero). By a point of an algebraic variety \( X \) we shall always mean a \( k \)-point. Throughout, \( G \) will denote a linear algebraic group. We shall refer to an algebraic variety \( X \) endowed with a regular \( G \)-action as a \( G \)-variety. We will say that a \( G \)-variety \( X \) (or the \( G \)-action on \( X \)) is generically free if \( \text{Stab}_G(x) = \{1\} \) for \( x \in X \) in general position. Finally, unless otherwise specified, \( m \) and \( n \) are integers \( \geq 2 \).

2. Preliminaries

Concomitants. Let \( \Gamma \) be an algebraic group and \( V \) and \( W \) be \( \Gamma \)-varieties. Then we shall denote the set of \( \Gamma \)-equivariant morphisms \( V \to W \) (also known as concomitants) by \( \text{Morph}_\Gamma(V,W) \) and the set of \( \Gamma \)-equivariant rational maps \( V \dashrightarrow W \) (also known as rational concomitants) by \( \text{RMaps}_\Gamma(V,W) \).

In the case where \( W \) is a finite-dimensional linear representation of \( \Gamma \), we also define a relative concomitant as a morphism \( f: V \to W \) satisfying the following condition (which is slightly weaker than \( \Gamma \)-equivariance): there is a character \( \chi: \Gamma \to k^* \) such that
\[
f(g \cdot v) = \chi(g) (g \cdot f(v))
\]
for all \( v \in V \) and \( g \in \Gamma \). For a rational map \( f: V \to W \) the notion of a relative rational concomitant is defined in a similar manner. If \( W = k \), with trivial \( \Gamma \)-action, then the term “invariant” is used in place of “concomitant”. For future reference we record the following:

2.1. Lemma. Suppose \( V \) and \( W \) are finite-dimensional linear representations of \( \Gamma \). Every rational concomitant \( f: V \to W \) can be written as \( a \cdot b \), where \( a \) is a relative concomitant and \( b \) is a relative invariant.

Proof. See the proof of [3, Chapter 1, Proposition 1]. Note that the characters associated to \( a \) and \( b \) are necessarily equal. \( \square \)

If \( W \) is a \( k \)-algebra and \( \Gamma \) acts on \( W \) by \( k \)-algebra automorphisms, then the algebra structure of \( W \) induces algebra structures on \( \text{Morph}_\Gamma(V,W) \) and \( \text{RMaps}_\Gamma(V,W) \) in a natural way. Namely, given \( a,b: V \to W \) (or

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1We remark that Richardson [21] worked over \( k = \mathbb{C} \), and his proofs are based on analytic techniques. The results we need (in particular, [21, Theorem 9.3.1]), remain valid over any algebraically closed field of characteristic zero by the Lefschetz principle. Extending [21, Theorem 9.3.1] to prime characteristic is an open problem of independent interest.
Let $\text{Theorem 14.16}$]

Moreover, the two isomorphisms identify the $i$-th projection $(M_n)^m \rightarrow M_n$ with the $i$-th generic matrix $X_i$. □

Here $T_{m,n}$ and $UD(m,n)$ are, respectively, the trace ring and the universal division algebra of $m$ generic $n \times n$-matrices, defined in the introduction. Note that part (b) of Theorem 2.2 follows from part (a) by Lemma 2.1, since the simple group $\text{PGL}_n$ does not have nontrivial characters (so that relative concomitants and invariants are actually concomitants and invariants, respectively).

We also remark that the construction of $T_{m,n}$ remains well-defined if $m = 1$. Theorem 2.2 also holds in this case, provided that one defines $UD(1,n)$ to be the field of quotients of $T_{1,n}$, rather than $G_{1,n}$. (For $m \geq 2$, $T_{m,n}$ and $G_{m,n}$ have the same division algebra of quotients, but this is not the case for $m = 1$.)

**Geometric actions.** For the rest of this section we will assume that $k$ is algebraically closed. If $X$ is a $\text{PGL}_n$-variety, then, as we mentioned above, $RMaps_{\text{PGL}_n}(X,M_n)$ has an algebra structure naturally induced from $M_n$. If the $\text{PGL}_n$-action on $X$ is generically free then $RMaps_{\text{PGL}_n}(X,M_n)$ is a central simple algebra of degree $n$, with center $k(X)^{\text{PGL}_n}$; cf. [18, Lemmas 8.5 and 9.1].

Suppose that $X$ is a $G \times \text{PGL}_n$-variety, and that the $\text{PGL}_n$-action on $X$ is generically free. Then the $G$-action on $X$ naturally induces a $G$-action on $RMaps_{\text{PGL}_n}(X,M_n)$. Following [20] we define the action of an algebraic group $G$ on a central simple algebra $A$ to be geometric if $A$ is $G$-equivariantly isomorphic to $RMaps_{\text{PGL}_n}(X,M_n)$ for some $G \times \text{PGL}_n$-variety $X$ as above. The $G \times \text{PGL}_n$-variety $X$ is then called the associated variety for the $G$-action on $A$; this associated variety is unique (as a $G \times \text{PGL}_n$-variety), up to birational isomorphism; cf. [20, Corollary 3.2].

Note that we defined geometric actions only if $k$ is algebraically closed. Also note that if an algebraic group acts geometrically on a central simple algebra $A$, then the center of $A$ is necessarily a finitely generated field extension of $k$.

Of particular interest to us will be the case where $X = (M_n)^m$ is the space of $m$-tuples of $n \times n$-matrices. Here $\text{PGL}_n$ acts on $(M_n)^m$ by simultaneous conjugation (since $m \geq 2$, this action is generically free) and $G = \text{GL}_n$ acts on $(A_1,\ldots,A_m) \in (M_n)^m$ by sending $(A_1,\ldots,A_m)$ to $(B_1,\ldots,B_m)$ where $B_j = \sum_{i=1}^m c_{ij} A_i$ and $g^{-1} = (c_{ij})$. The actions of $\text{GL}_n$ and $\text{PGL}_n$ commute,
and the GL<sub>m</sub>-action on (M<sub>n</sub>)<sup>m</sup> induces the GL<sub>m</sub>-action on UD(m, n). So (M<sub>n</sub>)<sup>m</sup> is the associated variety for the GL<sub>m</sub>-action on UD(m, n); see Theorem 2.2 (cf. also [21, Example 3.4]).

We conclude this section with a simple result which we will use repeatedly.

2.3. Lemma. Assume k is algebraically closed. Let X be a G × PGL<sub>n</sub>-variety which is PGL<sub>n</sub>-generically free. Denote by π: X → X/G the rational quotient map for the G-action. Then for x ∈ X in general position, the projection pr₂: G × PGL<sub>n</sub> → PGL<sub>n</sub> onto the second factor induces an isomorphism from Stab<sub>G×PGL<sub>n</sub></sub>(x)/Stab<sub>G</sub>(x) onto Stab<sub>PGL<sub>n</sub></sub>(π(x)).

Proof. Recall that by a theorem of Rosenlicht, π<sup>−1</sup>(x) is a single G-orbit for x ∈ X/G in general position; see [22, Theorem 2] or [14, Section 2.3]. Consequently, for x ∈ X in general position the projection pr₂ restricts to a surjective morphism Stab<sub>G×PGL<sub>n</sub></sub>(x) → Stab<sub>PGL<sub>n</sub></sub>(π(x)) of algebraic groups. The kernel of this morphism is clearly Stab<sub>G</sub>(x), and the lemma follows.

3. Geometric actions on division algebras

Throughout this section we will assume that k is algebraically closed. The main result of this section is the following theorem; after its proof, we will deduce several corollaries.

3.1. Theorem. Assume k is algebraically closed. Let G be a linear algebraic group acting geometrically on a division algebra D of degree n. Let X be the associated G × PGL<sub>n</sub>-variety. Then for x ∈ X in general position,

\[ S_x := \text{Stab}_{G×PGL_n}(x)/\text{Stab}_G(x) \]

is reductive.

Proof. Let X be the associated G × PGL<sub>n</sub>-variety for the G-action on D. Recall that the PGL<sub>n</sub>-action on X is generically free. We want to show that the group \( S_x = \text{Stab}_{G×PGL_n}(x)/\text{Stab}_G(x) \) is reductive for x ∈ X in general position. Assume the contrary. Denoting the unipotent radical by \( R_u \), this means that \( R_u(\text{Stab}_{G×PGL_n}(x)) \) is not contained in G. Since unipotent groups are connected, this is equivalent to

\[ \text{Lie}(R_u(\text{Stab}_{G×PGL_n}(x))) \not\subseteq \text{Lie}(G) \]

for x ∈ X in general position. Here and in the sequel Lie stands for the Lie algebra. To simplify notation, set \( H = G × PGL_n \), and for x ∈ X, set \( H_x = R_u(\text{Stab}_H(x)) \). Now define \( U_X \subseteq X × \text{Lie}(H) \) by

\[ U_X = \{ (x, a) \mid x ∈ X \text{ and } a ∈ \text{Lie}(H_x) \} . \]

We first show that \( U_X \) is a vector bundle over a dense open subset \( X_0 ⊂ X \). By [21, 6.2.1, 9.2.1, and 6.5.3], there is an H-stable dense open subset \( X_0 \) of X such that \( \{ H_x \mid x ∈ X_0 \} \) is an algebraic family of algebraic subgroups of H. Moreover, \( \dim(H_x) \) is constant for \( x ∈ X_0 \), say equal to d. Replacing X by \( X_0 \), we may assume that \( \{ H_x \mid x ∈ X \} \) is an algebraic family of algebraic
subgroups of \( H \). By [21, 6.2.2], \( x \mapsto \text{Lie}(H_x) \) defines a morphism of algebraic varieties from \( X \) to the Grassmannian of \( d \)-dimensional subspaces of \( \text{Lie}(H) \). Since the universal bundle over this Grassmannian is a vector bundle (see, e.g., [21, 3.3.1]), its pull-back \( U_X \) is a vector bundle over \( X \).

Note also that \( U_X \) is, by definition, an \( H \)-invariant subbundle of the trivial bundle \( X \times \text{Lie}(H) \rightarrow X \); here \( H \) acts on its Lie algebra by the adjoint action. Since the \( \text{PGL}_n \)-action on \( X \) is generically free, the no-name lemma tells us that there is a \( \text{PGL}_n \)-equivariant birational isomorphism \( U_X \rightarrow X \times k^d \) such that the following diagram commutes

\[
\begin{array}{ccc}
U_X & \rightarrow & X \times k^d \\
\downarrow & & \downarrow \\
X & \rightarrow & \\
\end{array}
\]

(For a proof and a brief discussion of the history of the no-name lemma, see [4, Section 4.3].) In other words, the vector bundle \( U_X \rightarrow X \) has \( d \) \( \text{PGL}_n \)-equivariant rational sections \( \beta_1, \ldots, \beta_d : X \rightarrow U_X \) such that \( \beta_i(x) \), \( \ldots, \beta_d(x) \) are linearly independent for \( x \in X \) in general position. We identify here \( \beta_i(x) \) with \( a \) if \( \beta_i(x) = (x, a) \in \{ x \} \times \text{Lie}(H_x) \). In view of (3.2), some \( k \)-linear combination \( \beta = c_1\beta_1 + \cdots + c_d\beta_d \) has the property that \( \beta(x) \notin \text{Lie}(G) \) for \( x \in X \) in general position.

Now recall that the natural projection \( \text{SL}_n \rightarrow \text{PGL}_n \) induces a Lie algebra isomorphism \( \text{sl}_n \rightarrow \text{Lie}(\text{PGL}_n) \), allowing us to identify the two Lie algebras. Hence

\[
U_X \subseteq X \times \text{Lie}(G) \times \text{sl}_n.
\]

Let \( f = \text{pr} \circ \beta : X \rightarrow \text{sl}_n \), where \( \text{pr} : U_X \rightarrow \text{sl}_n \) denotes the natural projection. Note that \( \text{sl}_n \hookrightarrow \text{gl}_n = \text{M}_n \), so that \( f \) may be viewed as a \( \text{PGL}_n \)-equivariant rational map \( X \rightarrow \text{M}_n \), i.e., as an element of \( D \). The condition that \( \beta(x) \notin \text{Lie}(G) \) ensures that \( f \neq 0 \). On the other hand, we will show below that for \( x \in X \) in general position, \( f(x) \) is a nilpotent \( n \times n \)-matrix, so that \( f^n = 0 \). This means that \( D \) contains a non-zero nilpotent element \( f \), contradicting our assumption that \( D \) is a division algebra.

It remains to be shown that for \( x \in X \) in general position, \( f(x) \) is a nilpotent matrix. The natural projection \( G \times \text{PGL}_n \rightarrow \text{PGL}_n \) maps the unipotent group \( H_x \) to a unipotent subgroup \( U \) of \( \text{PGL}_n \). Denote by \( K \) the preimage of \( U \) in \( \text{SL}_n \). It is a solvable group, so its subset \( K_u \) of unipotent elements is a closed subgroup. The surjection \( K_u \rightarrow U \) is finite-to-one, so their Lie algebras are isomorphic. In particular, \( f(x) \) belongs to \( \text{Lie}(K_u) \subseteq \text{sl}_n \subset \text{gl}_n = \text{M}_n \). Finally, since \( K_u \) is a unipotent subgroup of \( \text{GL}_n \), its Lie algebra in \( \text{M}_n \) consists of nilpotent matrices, see, e.g., [3, I.4.8]. This completes the proof of Theorem 3.1.

We now proceed with the corollaries. Recall that a subgroup \( S \subset \Gamma \) is said to be a stabilizer in general position for a \( \Gamma \)-variety \( X \) if there exists a
dense $\Gamma$-invariant subset $U \subset X$ such that $\text{Stab}(x)$ is conjugate to $S$ for any $x \in U$. For a detailed discussion of this notion, see [14, Section 7].

3.3. Corollary. Assume $k$ is algebraically closed. Let $G$ be a linear algebraic group acting geometrically on a division algebra $D$ of degree $n$. Let $X$ be the associated $G \times \text{PGL}_n$-variety.

(a) The induced $\text{PGL}_n$-action on the rational quotient $X/G$ has a stabilizer $S$ in general position. Moreover, $S$ is reductive, and $S \cong S_x = \text{Stab}_{G \times \text{PGL}_n}(x)/\text{Stab}_G(x)$ for $x \in X$ in general position.

(b) If the $G$-action on $X$ is generically free, then

$$\text{trdeg}_k(\mathbb{Z}(D^G)) = \text{trdeg}_k(\mathbb{Z}(D)) = \dim(X) - \dim(G) + \dim(S) - n^2 + 1.$$ 

Proof. (a) It follows from Theorem 3.1 and Lemma 2.3 that points in $X/G$ in general position have a reductive stabilizer. A theorem of Richardson (see [21, Theorem 9.3.1] or [14, Theorem 7.1]) now implies that the $\text{PGL}_n$-action on $X/G$ has a stabilizer $S$ in general position. By Lemma 2.3, $S \cong S_x = \text{Stab}_{G \times \text{PGL}_n}(x)/\text{Stab}_G(x)$ for $x \in X$ in general position.

(b) The first equality follows from the fact that $\mathbb{Z}(D^G)$ is an algebraic extension of $\mathbb{Z}(D)$. Indeed, the minimal polynomial of any element of $D^G$ over $\mathbb{Z}(D)$ is unique and must therefore have coefficients in $\mathbb{Z}(D)$. To prove the second equality, note that, $\mathbb{Z}(D) = k(X/\text{PGL}_n) = k(X)^{\text{PGL}_n}$ and thus

$$\mathbb{Z}(D^G) \cong (k(X)^{\text{PGL}_n})^G = k(X)^{G \times \text{PGL}_n} = k(X/(G \times \text{PGL}_n)).$$ 

Since we are assuming that $G$ acts generically freely on $X$, part (a) implies that $S \cong \text{Stab}_{G \times \text{PGL}_n}(x)$ for $x \in X$ in general position. Hence the dimension of the general fiber of the rational quotient map $X \rightarrow X/(G \times \text{PGL}_n)$ is equal to the dimension of $(G \times \text{PGL}_n)/S$. The fiber dimension theorem now tells us that the transcendence degree of $\mathbb{Z}(D^G)$ is

$$\dim X/(G \times \text{PGL}_n) = \dim(X) - \dim(G) - \dim(\text{PGL}_n) + \dim(S). \quad \Box$$

3.4. Corollary. Assume $k$ is algebraically closed. Let $G$ be a unipotent linear algebraic group acting geometrically on a division algebra $D$ of degree $n$. Then $D^G$ is a division algebra of degree $n$.

This was proved for algebraic actions in [20, Proposition 12.1].

Proof. By [20, Lemma 7.1], for $x \in X$ in general position, $\text{Stab}_{G \times \text{PGL}_n}(x)$ is isomorphic to a subgroup of $G$, so is unipotent. On the other hand, by Theorem 3.1, the projection $S_x$ of this group to $\text{PGL}_n$ is reductive. Thus $S_x$ is both unipotent and reductive, which is only possible if $S_x = \{1\}$. In other words,

$$\text{Stab}_{G \times \text{PGL}_n}(x) \subset G \times \{1\}.$$

The desired conclusion now follows from [20, Theorem 1.4]. \quad \Box
4. Dimension counting in the Grassmannian

In preparation for the proof of Theorem 1.4 in the next section, we will now establish the following:

4.1. Proposition. Assume $k$ is algebraically closed. Let $V$ be an $N$-dimensional $k$-vector space and let $V = V_1 \oplus \cdots \oplus V_r$, where $\dim(V_i) = N_i \geq 1$. Let $Z$ be the subset of the Grassmannian $\text{Gr}(m, N)$ consisting of $m$-dimensional subspaces $W$ of $V$ of the form

$$W = W_1 \oplus \cdots \oplus W_r,$$

where $W_i \subseteq V_i$. (Here we allow $W_i = (0)$ for some $i$.) Then $Z$ is a closed subvariety of $\text{Gr}(m, N)$. If $2 \leq m \leq N - 2$, then each irreducible component of $Z$ has codimension $\geq N - \max_{i=1,\ldots,r}(N_i)$ in $\text{Gr}(m, N)$. Moreover, equality holds (for some irreducible component of $Z$) only if (i) $r = 1$ or (ii) $r = 2$, $m = 2$ and $N = 4$.

Proof. Let $m_1, \ldots, m_r$ be non-negative integers such that $m_1 + \cdots + m_r = m$ and such that $m_i \leq N_i$ for all $i$. Let $Z_{m_1,\ldots,m_r}$ be the image of the map

$$\phi_{m_1,\ldots,m_r} : \text{Gr}(m_1, N_1) \times \cdots \times \text{Gr}(m_r, N_r) \to \text{Gr}(m, N)$$

given by $(W_1, \ldots, W_r) \mapsto W_1 \oplus \cdots \oplus W_r$. (Here $\text{Gr}(m_i, N_i)$ is the Grassmannian of $m_i$-dimensional vector subspaces of $V_i$.) Since the domain of the map $\phi_{m_1,\ldots,m_r}$ is projective, its image is closed in $\text{Gr}(m, N)$. Thus each $Z_{m_1,\ldots,m_r}$ is a closed irreducible subvariety of $\text{Gr}(m, N)$ birationally isomorphic to

$$\text{Gr}(m_1, N_1) \times \cdots \times \text{Gr}(m_r, N_r)$$

and $Z$ is the union of the $Z_{m_1,\ldots,m_r}$. It remains to show that

$$\dim \text{Gr}(m, N) - \sum_{i=1}^r \dim \text{Gr}(m_i, N_i) \geq N - \max_{i=1,\ldots,r}(N_i),$$

and that equality is only possible if $r = 1$ or $r = 2$, $N_1 = N_2 = 2$ and $m_1 = m_2 = 1$ (and thus $N = N_1 + N_2 = 4$ and $m = m_1 + m_2 = 2$). Recall that $\dim \text{Gr}(m, N) = (N - m)m$. Letting $l_i = N_i - m_i$ and $l = N - m = l_1 + \cdots + l_r$, we can rewrite (12) as

$$lm - \sum_{i=1}^r l_im_i \geq l + m - \max_{i=1,\ldots,r}(l_i + m_i),$$

or, equivalently,

$$(l - 1)(m - 1) - 1 \geq \sum_{i=1}^r l_im_i - \max_{i=1,\ldots,r}(l_i + m_i).$$

Proposition 4.1 is thus a consequence of the following elementary lemma. □
4.3. Lemma. Let \((l_1, m_1), \ldots, (l_r, m_r)\) be \(r\) pairs of non-negative integers and let \(l = \sum_{i=1}^r l_i\) and \(m = \sum_{i=1}^r m_i\). Assume that \(l_i + m_i \geq 1\) for every \(i = 1, \ldots, r\) and \(l, m \geq 2\). Then

\[
(l - 1)(m - 1) - 1 \geq \sum_{i=1}^r l_i m_i - \max_{i=1,\ldots, r} (l_i + m_i).
\]

Moreover, equality holds if and only if either (i) \(r = 1\) or (ii) \(r = 2\) and \((l_1, m_1) = (l_2, m_2) = (1, 1)\).

Proof. We consider two cases.

Case 1: Suppose that for every \(i = 1, \ldots, r\), either \(l_i = 0\) or \(m_i = 0\). Since \(l, m \geq 2\), we have \((l - 1)(m - 1) - 1 \geq 0\). On the other hand, \(\sum_{i=1}^r l_i m_i - \max_{i=1,\ldots, r} (l_i + m_i) = -\max_{i=1,\ldots, r} (l_i + m_i) < 0\). Hence, in this case \((4.4)\) holds and is a strict inequality.

Case 2: Now suppose that \(l_i, m_i \geq 1\) for some \(i \geq 1, \ldots, r\). After renumbering the pairs \((l_1, m_1), \ldots, (l_r, m_r)\), we may assume \(i = 1\). Now set

\[
l'_j = \begin{cases} l_1 - 1, & \text{if } j = 1, \\ l_j, & \text{otherwise}; \end{cases} \quad \text{and} \quad m'_j = \begin{cases} m_1 - 1, & \text{if } j = 1 \\ m_j, & \text{otherwise}. \end{cases}
\]

Note that \(l'_j, m'_j \geq 0\) for every \(j = 1, \ldots, r\). Thus

\[
(l - 1)(m - 1) - 1 = (\sum_{i=1}^r l'_i)(\sum_{j=1}^r m'_j) - 1 = \sum_{i=1}^r l'_i m'_i + \sum_{i\neq j} l'_i m'_j - 1
\]

\[
\geq \sum_{i=1}^r l'_i m'_i - 1 = \sum_{i=1}^r l_i m_i - (l_1 + m_1)
\]

\[
\geq \sum_{i=1}^r l_i m_i - \max_{i=1,\ldots, r} (l_i + m_i).
\]

This completes the proof of the inequality \((4.4)\).

It is easy to see that equality holds in cases (i) and (ii). It remains to show that the inequality \((4.4)\) is strict for all other choices of \((l_1, m_1), \ldots, (l_r, m_r)\). Indeed, a closer look at the above argument shows that equality in \((4.4)\) can hold if and only if we are in Case 2 and

(a) \(l'_i m'_j = 0\) whenever \(i \neq j\) and

(b) \(l_1 + m_1 = \max_{i=1,\ldots, r} (l_i + m_i)\).

Assume that conditions (a) and (b) are satisfied. Since \(\sum_{i=1}^r l'_i = l - 1 \geq 1\), we cannot have \(l'_i = 0\) for all \(i = 1, \ldots, r\). In other words, \(l'_{i_0} \geq 1\) for some \(i_0 \in \{1, \ldots, r\}\). Then condition (a) says that \(m'_j = 0\) for every \(j \neq i_0\). On the other hand, \(m'_{i_0} = \sum_{j=1}^r m'_j = m - 1 \geq 1\), and applying condition (a) once again, we conclude that \(l'_i = 0\) for every \(i \neq i_0\). To sum up, there exists an \(i_0 \in \{1, \ldots, r\}\) such that \(l'_{i_0} = l - 1\), \(m_{i_0} = m - 1\) and \(l'_i = m'_i = 0\) for every \(i \neq i_0\).
In particular, for every \( i \neq 1, i_0 \), we have \( l_i = l_i' = 0 \) and \( m_i = m_i' = 0 \), contradicting our assumption that \( l_i + m_i \geq 1 \). Thus \( i \in \{1, i_0\} \) for every \( i = 1, \ldots, r \). In other words, either \( i_0 = 1 \) and \( r = 1 \) (in which case (i) holds, and we are done) or \( i_0 = 2 \) and \( r = 2 \). In the latter case \( l_1' = m_1' = 0 \), \( l_2' = l - 1 \) and \( m_2' = m - 1 \), i.e., \((l_1, m_1) = (1, 1)\) and \((l_2, m_2) = (l-1, m-1)\). Condition (b) now tells us that \( l = m = 2 \), so that (ii) holds.

This completes the proof of Lemma 4.3 and thus of Proposition 4.1. \( \square \)

5. PROOF OF THEOREM 1.4 OVER AN ALGEBRAICALLY CLOSED FIELD

Recall from Section 3 that \( X = (M_n)^m \) is the associated \( GL_m \times PGL_n \)-variety for the \( GL_m \)-action on \( UD(m, n) \). Here \( PGL_n \) acts on \((M_n)^m\) by simultaneous conjugation (since \( m \geq 2 \), this action is generically free) and \( GL_m \) acts on \((M_n)^m\) by sending \((A_1, \ldots, A_m)\) to \((B_1, \ldots, B_m)\), where \( B_j = \sum_{i=1}^m c_{ij} A_i \) and \( g^{-1} = (c_{ij}) \).

We shall assume throughout this section that \( k \) is an algebraically closed field of characteristic zero. Our goal is to prove Theorem 1.4 over such \( k \).

In view of [20, Theorem 1.4(a)], we only need to establish the following.

5.1. Theorem. Assume \( k \) is algebraically closed. If \( n \geq 3 \) and \( 2 \leq m \leq n^2 - 2 \), then the \( GL_m \times PGL_n \)-action on \((M_n)^m\) is generically free.

Proof. The linear action of \( GL_m \) on \((M_n)^m\) is easily seen to be the direct sum of \( n^2 \) copies of the natural \( m \)-dimensional representation of \( GL_m \), i.e., to be isomorphic to the \( GL_m \)-action on \( n^2 \)-tuples of vectors in \( k^m \). Since \( n^2 > m \), this action is generically free. Corollary 3.3(a) with \( G = GL_m \) and \( X = (M_n)^m \) tells us that the \( PGL_n \)-action on \((M_n)^m/GL_m \) has a reductive stabilizer \( S \) in general position, and that \( S \simeq Stab_{GL_m \times PGL_n}(x)/Stab_{GL_m}(x) = Stab_{GL_m \times PGL_n}(x) \) for \( x \in X \) in general position.

Recall that \((M_n)^m/GL_m \) is \( PGL_n \)-equivariantly birationally isomorphic to the Grassmannian \( Gr(m, n^2) \) of \( m \)-dimensional subspaces of \( M_n \). Thus the \( PGL_n \)-action on \( Gr(m, n^2) \) has a stabilizer \( S \) in general position, where \( S \) is a reductive subgroup of \( PGL_n \). (Recall that \( S \) is only well-defined up to conjugacy in \( PGL_n \).)

To prove Theorem 5.1, it suffices to show that \( S \) is trivial.

Assume the contrary. Since \( S \) is reductive, it contains a non-trivial element \( g \) of finite order. Then every \( L \in \text{Gr}(m, n^2) \) in general position is fixed by some conjugate of \( g \). In other words, the map

\[
PGL_n \times \text{Gr}(m, n^2)^g \to \text{Gr}(m, n^2) \quad (h, L) \mapsto h(L)
\]

is dominant; here \( \text{Gr}(m, n^2)^g \) denotes the fixed points of \( g \) in \( \text{Gr}(m, n^2) \).

Denote by \( C(g) \) the centralizer of \( g \) in \( PGL_n \). Note that \( \text{Gr}(m, n^2)^g \) is \( C(g) \)-stable. Hence the fiber of \((5.2)\) over \( h(L) \) contains \((hc, c^{-1}(L))\) for every \( c \in C(g) \). So by the fiber dimension theorem,

\[
\dim \text{Gr}(m, n^2) + \dim C(g) \leq \dim PGL_n + \dim \text{Gr}(m, n^2)^g.
\]
Since \( g \) has finite order, it is diagonalizable. So we may assume that 
\[
g = \text{diag}(\lambda_1, \ldots, \lambda_1, \ldots, \lambda_s, \ldots, \lambda_s) = \text{diag}(\alpha_1, \ldots, \alpha_n),
\]
where \( \lambda_1, \ldots, \lambda_s \) are the (distinct) eigenvalues of \( g \). Note that \( s \geq 2 \), because \( g \neq 1 \) in \( \text{PGL}_n \) and that \( g \) acts on the matrix units \( E_{ij} \) by \( g \cdot E_{ij} = \alpha_i \alpha_j^{-1} E_{ij} \). The matrix algebra \( M_n \) naturally decomposes as a direct sum of character spaces

\[
V_\mu = \text{Span}(E_{ij} \mid \alpha_i \alpha_j^{-1} = \mu).
\]
In particular, \( V_1 \) is the commutator of \( g \) in \( M_n \). Now (5.3) implies

\[
\dim \text{Gr}(m, n^2) - \dim \text{Gr}(m, n^2)^g \leq \dim(\text{PGL}_n) - \dim C(g)
\]
\[
= n^2 - \dim(V_1).
\]
So if \( n \geq 3 \), part (b) of the following lemma gives the desired contradiction, which completes the proof of Theorems 5.1, and thus of Theorem 1.4 in the case that \( k \) is algebraically closed.

\[\Box\]

5.4. Lemma. Let \( n \geq 2 \), and \( 2 \leq m \leq n^2 - 2 \).

(a) \( \dim V_1 \geq \dim V_\mu \) for any \( \mu \neq 1 \).

(b) If \( n \geq 3 \) (or \( n = 2 \) and there are more than two distinct nonzero \( V_\mu \)), then \( \dim \text{Gr}(m, n^2) - \dim \text{Gr}(m, n^2)^g > n^2 - \dim(V_1) \).

Proof. (a) Note that \( \dim V_1 = l_1^2 + \cdots + l_s^2 \) and

\[
\dim V_\mu = \sum_{\lambda_i \lambda_j^{-1} = \mu} l_i l_j
\]
Since the eigenvalues \( \lambda_1, \ldots, \lambda_s \) of \( g \) are distinct, the last sum has at most one term for each \( i = 1, \ldots, s \). Thus there is a permutation \( \sigma \) of \( \{1, \ldots, s\} \) such that

\[
\dim V_\mu \leq l_1 l_\sigma(1) + \cdots + l_s l_\sigma(s).
\]
So for \( v = (l_1, \ldots, l_s) \) and \( w = (l_\sigma(1), \ldots, l_\sigma(s)) \), \( \dim V_\mu \leq v \cdot w \). Hence by the Cauchy-Schwarz inequality,

\[
\dim V_\mu \leq v \cdot w \leq |v| |w| = |v|^2 = l_1^2 + \cdots + l_s^2 = \dim(V_1).
\]

(b) Since \( g \) is semisimple, every \( L \in \text{Gr}(m, n^2)^g \) is a direct sum of its character subspaces spaces, i.e., a direct sum of vector subspaces of the \( V_\mu \). Part (b) now follows from Proposition 5.1 with \( V = M_n \), \( N = n^2 \), \( N_\mu = \dim(V_\mu) \), \( Z = \text{Gr}(m, n^2)^g \), and \( r \) the number of distinct nonzero \( V_\mu \).

\[\Box\]

5.5. Remark. We assumed throughout this section that \( n \geq 3 \). If \( n = 2 \) then the above argument still goes through provided there are more than two distinct non-zero character subspaces \( V_\mu \); see Lemma 5.1(b). In particular, this will always be the case if \( g^2 \neq 1 \) in \( \text{PGL}_n \); indeed, in this case \( g = (\lambda_1, \lambda_2) \), where \( \mu = \lambda_1 / \lambda_2 \neq \pm 1 \) and the three spaces \( V_1, V_\mu \) and \( V_\mu^{-1} \) are distinct. Thus the above argument also shows that for \( n = m = 2 \), either
|S| = 1 or S has exponent 2. It turns out that, in fact, in this case |S| = 2; see [21, Lemma 14.2].

5.6. Remark. An alternative approach to proving Theorem 5.1 would be to appeal to the classification, due to A. G. Elashvili [8] and A. M. Popov [13], of pairs (G, φ), where G is a semisimple algebraic group and φ: G ↪ GL(V) is an irreducible linear representation of G such that the G-action on V has a non-trivial stabilizer in general position. Since this classification is rather involved, and since additional work would be required to apply it in our situation (note that the group GL_m × PGL_n is not semisimple, and that its representation on (M_n)^m is not irreducible), we opted instead for the self-contained direct proof presented in this section.

6. SL_m-invariant generic matrices

The goal of this section is to relate the rings of SL_m-invariants in G_m,n and UD(m, n).

6.1. Lemma. (a) Every element of UD(m, n)^GL_m can be written in the form \( \frac{a}{b} \), where a is a homogeneous element of \( (T_{m,n})^{SL_m} \), and b is a non-zero homogeneous element of \( Z(T_{m,n})^{SL_m} \) of the same degree as a.

(b) Assume that a subgroup G of GL_m has no non-trivial characters. Then every element of UD(m, n)^G can be written as \( \frac{a}{b} \) where \( a \in (T_{m,n})^G \) and \( 0 \neq b \in Z(T_{m,n})^G \).

Proof. Both parts follow from Lemma 2.1. In part (a), we take \( \Gamma = GL_m \times PGL_n \), V = \( (M_n)^m \) (with the \( \Gamma \)-action defined in the beginning of Section 5) and W = M_n (where GL_m acts trivially on W and PGL_n acts by conjugation). Here the relative concomitants \( (M_n)^m \rightarrow M_n \) are the homogeneous elements of \( (T_{m,n})^{SL_m} \) and the relative invariants \( (M_n)^m \rightarrow k \) are the homogeneous elements of \( Z(T_{m,n})^{SL_m} \); cf. Theorem 2.2.

If G has no non-trivial characters then relative concomitants are (absolute) concomitants, i.e., elements of \( (T_{m,n})^G \). Similarly, relative invariants are elements of \( Z(T_{m,n})^G \), and part (b) is thus simply a restatement of Lemma 2.1 in this special case. \( \Box \)

6.2. Proposition. Let G be a subgroup of GL_m such that G has no non-trivial characters. Then the following conditions are equivalent:

(a) UD(m, n)^G has PI-degree n.

(b) \( (T_{m,n})^G \) has PI-degree n.

(c) \( (G_{m,n})^G \) has PI-degree n.

Proof. The equivalence of (a) and (b) follows from Lemma 6.1(b). The implication (c) \( \Rightarrow \) (b) is obvious, since \( G_{m,n} \subset T_{m,n} \). It thus remains to prove that (b) \( \Rightarrow \) (c).

Let \( g_n \) be the multilinear central polynomial for \( n \times n \)-matrices in [23, 13.5.11] (or [23, p. 26]). If R is a prime PI-algebra of PI-degree n, denote
by \( g_n(R) \) the set of all evaluations of \( g_n \) in \( R \), and denote by \( Rg_n(R) \) the nonzero ideal of \( R \) generated by \( g_n(R) \). Denote by \( T \) the trace ring of \( R \).

(Since we are working in characteristic zero, the (noncommutative) trace ring in \([12, 13.9.2]\) is the same as the one we are using, see \([12, 13.9.4]\).) Then \( Rg_n(R) \) is a common ideal of \( R \) and \( T \), see \([12, 13.9.6]\) (or \([23, 4.3.1]\)).

Now let \( R = G_{m,n} \). Then its trace ring is \( T = T_{m,n} \). Recall that we are assuming (b) holds, i.e., \( T^G \) has PI-degree \( n \). Let \( s \) be a non-zero evaluation of \( g_n \) on \( T^G \). Then \( s \) is a nonzero \( G \)-invariant, and a central element of \( T \) (since it is also an evaluation of \( g_n \) on \( T \)). Since \( g_n \) is multilinear, and since \( T \) is generated as an \( R \)-module by central elements, \( s \) belongs to the ideal of \( T \) generated by \( g_n(R) \), so that \( sT \subseteq Rg_n(R) \subseteq R \). Since \( s \) is a \( G \)-invariant, it follows that \( sT^G \subseteq R^G \). Consider the central localization \( R^G[s^{-1}] \subseteq UD(m,n) \). Since it contains \( T^G \), \( R^G[s^{-1}] \) must have PI-degree \( n \), implying that also \( R^G \) must have PI-degree \( n \). This completes the proof of the implication (b) \( \iff \) (c) and thus of Proposition 6.2.

6.3. Remark. The same argument also shows that if the three equivalent conditions in Proposition 6.2 are true, then the division algebras of fractions of \((G_{m,n})^G\) and \((T_{m,n})^G\) are both equal to \( UD(m,n)^G \).

7. Proof of Theorems 1.2 and 1.4

Proof of Theorem 1.2. Proposition 6.2 tells us that \((G_{m,n})^{SL_m}\) has PI-degree \( n \) if and only if so does \( UD(m,n)^{SL_m} \). Thus in order to prove Theorem 1.2 it suffices to show that \( UD(m,n)^{SL_m} \) has PI-degree \( n \) whenever \( 2 \leq m \leq n^2 - 2 \).

For \( n = m = 2 \) we showed this in \([20, \text{Remark} \, 14.4]\) (in fact, the argument we gave there remains valid over any base field \( k \) of characteristic \( \neq 2 \)). For \( n \geq 3 \) and \( 2 \leq m \leq n^2 - 2 \), Theorem 1.4 tells us that \( UD(m,n)^{GL_m} \) has PI-degree \( n \) (and consequently, so does \( UD(m,n)^{SL_m} \)). In summary, we have shown that \( \text{Theorem 1.2 follows from Theorem 1.4} \).

Proof of Theorem 1.4. We have already proved Theorem 1.4 in the case where the base field \( k \) is algebraically closed; see Section 5. We will now reduce the general case to this one by using Lemma 6.1(a).

We begin with a simple lemma.

7.1. Lemma. Let \( K \) be an extension field of \( k \), let \( V \) be a finite-dimensional \( k \)-vector space, and \( V_K = V \otimes_k K \). Given a linear representation of \( SL_m(k) \) on \( V \), we have

\[
(V_K)^{SL_m(K)} = V^{SL_m(k)} \otimes_k K.
\]

Proof. Since \( SL_m(k) \) is dense in \( SL_m(K) \), the subspace \((V_K)^{SL_m(K)}\) is defined inside \( V_K \) by a system of homogeneous linear equations with coefficients in \( k \). Clearly finitely many of these equations suffice. Since the dimension of the solution space of such a system is the rank of the corresponding matrix (which has coefficients in \( k \)), \((V_K)^{SL_m(K)}\) has a \( K \)-basis consisting of elements of \( V^{SL_m(k)} \).
For the remainder of this section, we will write \( G_{m,n}(K) \), \( T_{m,n}(K) \) and \( UD(m,n)(K) \) to denote the generic matrix algebra, trace ring and universal division algebra defined over the field \( K \). Denote the algebraic closure of \( k \) by \( \overline{k} \). Since the \( \text{SL}_m \)-action on \( UD(m,n) \) preserves degree, Lemma 7.1 immediately implies the following fact, which we record for later use.

7.2. Corollary. \( G_{m,n}(\overline{k})^{\text{SL}_m(\overline{k})} = G_{m,n}(k)^{\text{SL}_m(k)} \otimes_k \overline{k} \), and \( T_{m,n}(\overline{k})^{\text{SL}_m(\overline{k})} = T_{m,n}(k)^{\text{SL}_m(k)} \otimes_k \overline{k} \).

We are now ready to complete the proof of Theorem 1.4 over an arbitrary field \( k \) of characteristic zero. In Section 5 we showed that Theorem 1.4 holds for \( \text{SL}_m \). Corollary. (a) \( \text{SL}_m \text{-invariant elements of } UD(m,n) \) exist elements \( c_1, \ldots, c_r \in UD(m,n)(\overline{k})^{\text{GL}_m(\overline{k})} \) which span \( UD(m,n)(\overline{k}) \) as a vector space over its center. By Lemma 6.1 we can write \( c_i = a_i/b_i \), where \( a_i \in T_{m,n}(\overline{k})[d_i]^{\text{SL}_m} \) and \( 0 \neq b_i \in \mathbb{Z}(T_{m,n}(\overline{k}))[d_i]^{\text{SL}_m} \) for some \( d_i \geq 0 \), \( i = 1, \ldots, r \). By Lemma 7.1, with \( K = \overline{k} \) and \( V = \mathbb{Z}(T_{m,n}(k))[d_i] \), we have \( \mathbb{Z}(T_{m,n}(k))[d_i]^{\text{SL}_m} \neq 0 \). We may now replace \( b_i \) by a non-zero element of \( \mathbb{Z}(T_{m,n}(k))[d_i]^{\text{SL}_m} \). The new \( c_i = a_i/b_i \) are still \( \text{GL}_m \)-invariant elements of \( UD(m,n)(\overline{k}) \), and they still generate \( UD(m,n)(\overline{k}) \) as a vector space over its center.

We now apply Lemma 7.1 once again (this time with \( V = T_{m,n}(k)[d_i] \)) to write each \( a_i \) as a finite sum \( \sum \gamma_{ij} a_{ij} \), where each \( \gamma_{ij} \in \overline{k} \) and each \( a_{ij} \in T_{m,n}(k)[d_i]^{\text{SL}_m} \). Now replace our collection of \( \text{GL}_m \)-invariant elements \( \{c_i = a_i/b_i\} \) in \( UD(m,n)(\overline{k}) \) by \( \{c_{ij} = a_{ij}/b_i\} \). By construction, the elements \( c_{ij} \) lie in \( UD(m,n)(k)^{\text{GL}_m} \) and span \( UD(m,n)(\overline{k}) \) over its center. Hence, these elements generate a \( k \)-subalgebra of \( UD(m,n)(k)^{\text{GL}_m} \) of PI-degree \( n \). Consequently, \( UD(m,n)(k)^{\text{GL}_m} \) itself has PI-degree \( n \). This completes the proof of Theorem 1.4 (and of Theorem 1.2).

8. The case \( m \geq n^2 - 1 \)

Theorems 1.2 and 1.4 assume that \( m \leq n^2 - 2 \). We will now describe \( UD(m,n)^{\text{GL}_m} \) and \( UD(m,n)^{\text{SL}_m} \) for \( m \geq n^2 - 1 \).

Recall the definition of the discriminant of \( n^2 \) matrices of size \( n \times n \), say \( A_1, \ldots, A_{n^2} \): it is the determinant of the \( n^2 \times n^2 \)-matrix whose \( i \)-th row consists of the entries of \( A_i \), cf. (8.5). When viewed as a function \( (M_n)^{n^2} \to k \), \( \Delta \) is the unique multilinear alternating function such that \( \Delta(e_{11}, e_{12}, \ldots, e_{nn}) = 1 \); cf., e.g., [8, Lemma 3]. Here the \( e_{ij} \) are the matrix units.

8.1. Proposition. (a) If \( m > n^2 \), then \( UD(m,n)^{\text{GL}_m} = UD(m,n)^{\text{SL}_m} = k \).
Now let \( m = n^2 \), and denote by \( \Delta \) the discriminant of the generic matrices \( X_1, \ldots, X_m \).

(b) \( UD(m,n)^{\text{GL}_m} = k \).
(c) \( (T_{m,n})^{\text{SL}_m} = k[\Delta] \).
(d) \( UD(m,n)^{\text{SL}_m} = k(\Delta) \).
Proof. (a) We may clearly assume that \( k \) is algebraically closed. In this case \( \text{SL}_m \) has a dense orbit in the associated variety \( X = (M_n)^m \). Thus the rational quotient \( X/\text{SL}_m \) is a single point (with trivial \( \text{PGL}_n \)-action), and
\[
UD(m, n)^{\text{SL}_m} = \text{RMaps}_{\text{PGL}_n}(pt, M_n) = k.
\]
Now suppose \( m = n^2 \). Then \( \text{GL}_m \) has a dense orbit in \( X = (M_n)^m \). Arguing as in part (a), we prove (b); cf. [20, Proposition 13.1(a)]. (c) is proved in [9, p. 210], and (d) follows from (c) by Lemma 6.1(b).

8.2. Remark. Let \( m = n^2 \). Formanek showed that \( \Delta \notin (G_{m,n})^{\text{SL}_m} \) ([9, p. 214]) but \( \Delta^i \in G_{m,n} \) for every integer \( i \geq 2 \) (this follows from [10, Theorem 16]). Consequently for \( m = n^2 \),
\[
(G_{m,n})^{\text{SL}_m} = k[\Delta^2, \Delta^3].
\]

8.3. Proposition. Suppose \( m = n^2 - 1 \), and let
\[
Y = \sum_{i,j=1}^n \Delta(X_1, \ldots, X_m, e_{ji})e_{ij},
\]
where the \( e_{ij} \) are the matrix units.

(a) \( Y \in (T_{m,n})^{\text{SL}_m} \).
(b) The eigenvalues of \( Y \) are algebraically independent over \( k \) (and, in particular, distinct).
(c) \( (T_{m,n})^{\text{SL}_m} = k[c_1, \ldots, c_{n-1}, Y] \) is a polynomial ring in \( n \) independent variables over \( k \). Here \( c_1 = -\text{tr}(Y), \ldots, c_n = (-1)^n \text{det}(Y) \).
(d) \( UD(m, n)^{\text{SL}_m} = k(c_1, \ldots, c_{n-1}, Y) \).
(e) \( UD(m, n)^{\text{GL}_m} = k\left(\frac{c_2}{c_1}, \ldots, \frac{c_{n-1}}{c_1}, \frac{1}{c_1}Y\right) \).

Proof. For the proof of (a)—(c), we may assume that \( k \) is algebraically closed (cf. Corollary 7.2). (a) We view \( Y \) as a regular map \( (M_n)^m \to M_n \). We want to show that this map is \( \text{PGL}_n \)-equivariant, i.e., \( Y \in T_{m,n} \). Since \( Y \) is clearly \( \text{SL}_m \)-equivariant (recall that \( \text{SL}_m \) acts trivially on \( M_n \)), this will imply part (a).

We begin by observing that for any \( (A_1, \ldots, A_m) \in (M_n)^m \), and \( Z \in M_n \),
\[(8.4) \quad \text{tr}(Y(A_1, \ldots, A_m)Z) = \Delta(A_1, \ldots, A_m, Z).
\]
Indeed, both sides are linear in \( Z \), so we only need to check (8.4) for the elementary matrices \( Z = e_{ij} \), where it is easy to see directly from the definition of \( Y \).

Fix an \( m \)-tuple \( (A_1, \ldots, A_m) \in (M_n)^m \) of \( n \times n \)-matrices. Since the trace form on \( M_n \) is non-singular, \( Y(A_1, \ldots, A_m) \) is the unique matrix satisfying (8.4) for every \( Z \in M_n \). The \( \text{PGL}_n \)-equivariance of \( Y : (M_n)^m \to M_n \) is an easy consequence of this and the fact that \( \Delta \) is \( \text{PGL}_n \)-invariant (see [1, p. 209]). This concludes the proof of (a).
Our proof of parts (b) and (c) relies on the following claim: \( Y : (M_n)^m \rightarrow M_n \) is the categorical quotient map for the \( SL_m \)-action on \( (M_n)^m \). In other words, we claim that the \( n^2 \) elements \( \Delta(X_1, \ldots, X_m, e_{ij}) \) \((i, j = 1, \ldots, n)\) generate \( k[(M_n)^m]_{SL_m} \) as a \( k \)-algebra. To prove this claim we will temporarily write \( (A_1, \ldots, A_m) \in (M_n)^m \) as an \( m \times n^2 \)-matrix

\[
(8.5) \quad A = \begin{pmatrix}
(a^{(1)}_{11}) & (a^{(1)}_{12}) & \cdots & (a^{(1)}_{ij}) & \cdots & (a^{(1)}_{mn}) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
(a^{(m)}_{11}) & (a^{(m)}_{12}) & \cdots & (a^{(m)}_{ij}) & \cdots & (a^{(m)}_{mn})
\end{pmatrix}.
\]

That is, we write each \( n \times n \) matrix \( A_h = (a^{(h)}_{ij}) \) as a single row of \( A \). In this notation, \( g \in SL_m \) acts on \( (M_n)^m \) by multiplication by the transpose of \( g^{-1} \) on the left; that is, \( g(A) = (g^{-1})^{\text{transpose}} \cdot A \) for every \( g \in SL_m \). Let \( \delta_{ij}(A_1, \ldots, A_m) \) be the \( m \times m \)-minor of this matrix obtained by removing the \( ij \)-column from \( A \) and taking the determinant of the resulting \( m \times m \)-matrix. The first theorem of classical invariant theory (see [27, Theorem II.6.A] or [13, Theorem 2.1]) says that the elements \( \delta_{ij}(X_1, \ldots, X_m) \) generate \( k[(M_n)^m]_{SL_m} \) as a \( k \)-algebra. On the other hand, it is easy to see that \( \delta_{ij}(X_1, \ldots, X_m) = \pm \Delta(X_1, \ldots, X_m, e_{ij}) \). This proves the claim.

Now observe that since \( m = n^2 - 1 \),

\[
\dim((M_n)^m//SL_m) = m n^2 - (m^2 - 1) = n^2 = \dim(M_n).
\]

This means that the \( n^2 \) \( SL_m \)-invariant functions

\[
\Delta(X_1, \ldots, X_m, e_{ji}) : (M_n)^m \rightarrow k
\]

are algebraically independent over \( k \). In other words, \( Y \) (viewed as a matrix in \( T_{m,n} \subset M_n(k[x^{(h)}_{ij}]) \) has algebraically independent entries. Part (b) easily follows from this assertion; cf. [15, Lemma II.1.4].

Furthermore,

\[
(T_{m,n})_{SL_m} = \text{Morph}_{SL_m \times PGL_n}((M_n)^m, M_n) \\
\simeq \text{Morph}_{PGL_n}((M_n)^m//SL_m, M_n) \\
\simeq \text{Morph}_{PGL_n}(M_n, M_n) = T_{1, n},
\]

where \( T_{1, n} \) is the trace ring of one generic \( n \times n \)-matrix. Here the last equality is a special case of Procesi’s Theorem [22, (a)] (with \( m = 1 \)). Since the chain of isomorphisms identifies \( Y \) with the identity map \( M_n \rightarrow M_n \), we conclude that

\[
(T_{m,n})_{SL_m} = k[c_1, \ldots, c_n, Y].
\]

Since \( Y^n + c_1 Y^{n-1} + \cdots + c_n = 0 \), \( k[c_1, \ldots, c_{n-1}, Y] = k[c_1, \ldots, c_n, Y] \). This proves the first assertion in (c).

To show that \( c_1, \ldots, c_{n-1}, Y \) are algebraically independent over \( k \), denote the eigenvalues of \( Y \) by \( \lambda_1, \ldots, \lambda_n \). By part (b), \( \lambda_1, \ldots, \lambda_n \) are algebraically
independent over $k$. Since $Y$ is algebraic over $k(c_1, \ldots, c_n)$, we have
\[ \text{trdeg}_k k(c_1, \ldots, c_{n-1}, Y) = \text{trdeg}_k k(c_1, \ldots, c_n, Y) \]
\[ = \text{trdeg}_k k(c_1, \ldots, c_n) = \text{trdeg}_k k(\lambda_1, \ldots, \lambda_n) = n. \]
This shows that $c_1, \ldots, c_{n-1}, Y$ are algebraically independent over $k$, thus completing the proof of (c).

(d) is an immediate consequence of (c) and Lemma 6.1(b). To prove (e), denote the central torus of $\text{GL}_m$ by $\mathbb{G}_m$. Then
\[ UD(m, n)^{\text{GL}_m} = (UD(m, n)^{\text{SL}_m})^{\mathbb{G}_m} = k(c_1, \ldots, c_{n-1}, Y)^{\mathbb{G}_m}, \]
where $\mathbb{G}_m$ acts on the purely transcendental extension $k(c_1, \ldots, c_{n-1}, Y)$ as follows: $t \cdot c_i \mapsto t^{im}c_i$ for $i = 1, \ldots, n - 1$, and and $t \cdot Y \mapsto t^mY$. Part (e) easily follows from this description.

8.6. Remark. Note that $c_1 = -\Delta(X_1, \ldots, X_{n^2-1}, I_n)$, where $I_n$ is the $n \times n$ identity matrix. By a theorem of Formanek, $(c_1)^2$ is an element of $Z(G_{m,n})^{\text{SL}_m}$ for $m = n^2 - 1$, see [10, Theorem 16].

9. Proof of Theorem 1.3

By Corollary 3.2, we may assume that $k$ is algebraically closed. Set $A = (G_{m,n})^{\text{SL}_m}$ and $B = (T_{m,n})^{\text{SL}_m}$. By Theorem 1.2, $A$ and $B$ both have PI-degree $n$. Thus $Z(A) = (Z(G_{m,n}))^{\text{SL}_m}$ and $Z(B) = (Z(T_{m,n}))^{\text{SL}_m}$. Since $\text{SL}_m$ is a reductive group, and since $T_{m,n}$ is a finitely generated $k$-algebra and a finite module over its center, $B$ is a finite $Z(B)$-module, and both $B$ and $Z(B)$ are finitely generated Noetherian $k$-algebras, cf. [26, Proposition 4.2]. Moreover, $B$ is an FBN ring, cf. [12, 13.6.6].

By Corollary 3.3(b) and Remark 7.3, the transcendence degrees of both $B$ and $Z(B)$ are $t = (m - 1)n^2 - m^2 + 2$. For notational simplicity, set
\[ \mu(S) = \limsup_{d \to \infty} \frac{\dim_k S[d]}{d^{t-1}} \]
for any graded $k$-algebra $S = \oplus_{d \geq 0} S[d]$. By [25, Lemma 6.1] (cf. also [11, §12.6]), $f(d) = \dim_k B[d]$ is eventually periodically polynomial, i.e., there are polynomials $f_1, \ldots, f_s$ with rational coefficients such that $f(d) = f_i(d)$ whenever $d$ is large enough and congruent to $i$ modulo $s$; moreover, the maximum of the degrees of the $f_i$ is $t - 1$. Consequently $\mu(B)$ exists and is equal to the largest among the leading coefficients of those $f_i$ of degree $t - 1$. A similar argument shows that $\mu(Z(B))$ exists and is a nonzero number.

Consider the multilinear central polynomial $g_n$ for $n \times n$ matrices used in the proof of Proposition 6.2. Since it is multilinear and nonzero on $A$, we can find a nonzero evaluation $c$ of $g_n$ at homogeneous elements of $A$; this $c$ is homogeneous. Since $c$ is also an evaluation of $g_n$ on $G_{m,n}$, $cT_{m,n} \subseteq G_{m,n}$, so that $cB \subseteq A$ and $cZ(B) \subseteq Z(A)$. Then for all integers $d \geq j$, $cB[d-j] \subseteq A[d] \subseteq B[d]$, where $j = \deg c$. Replacing $c$ by $c^a$ if necessary, we may assume
that \( s \) divides \( j \). Consequently, whenever \( d \) is large enough and congruent to \( i \) modulo \( s \),
\[
f_i(d - j) \leq \dim_k A[d] \leq f_i(d).
\]
It follows easily that \( \mu(A) \) exists and is equal to the largest among the leading coefficients of those \( f_i \) of degree \( t - 1 \). A similar argument shows that \( \mu(Z(A)) \) exists and is a nonzero number. \( \square \)

9.1. **Remark.** The above proof shows that \( \mu((G_{m,n})^{SL_m}) = \mu((T_{m,n})^{SL_m}) \), and that \( \mu(Z((G_{m,n})^{SL_m})) = \mu(Z((T_{m,n})^{SL_m})) \).

9.2. **Remark.** Consider the \( GL_m \)-representation on \( R \), where \( R = G_{m,n}, T_{m,n}, Z(G_{m,n}) \) or \( Z(T_{m,n}) \). Recall that irreducible polynomial representations of \( GL_m \) are indexed by partitions \( \lambda = (\lambda_1, \ldots, \lambda_s) \) with \( s \leq m \) parts; cf. [9, Section 2]. Denote the multiplicity of the irreducible representation corresponding to \( \lambda \) in \( R \) by \( \text{mult}_\lambda(R) \). Writing \( (r^m) \) for the partition \( \lambda = (r, \ldots, r) \) \((m \text{ times})\), we have

\((a)\) \( \dim_k R^{SL_m}[d] = 0 \) if \( d \) is not a multiple of \( m \), and
\((b)\) \( \dim_k R^{SL_m}[rm] = \text{mult}_{(r^m)}(R) \) for any integer \( r \geq 1 \).

**Proof.** (a) Assume \( R^{SL_m}[d] \) is nonzero. Then it is a direct sum of one-dimensional representations of \( GL_m \) of the form \( M = \text{Span}_k(v) \). Moreover, any such representation is given by \( g(v) = \det(g)^r v \) for some integer \( r \); cf., e.g., [9, Theorem 3(a)]. On the other hand, substituting \( g(tI_m) \), where \( t \in k \) and \( I_m \) is the \( m \times m \) identity matrix, we obtain, \( g(v) = t^d v \). Since \( \det(tI_m) = t^m \), we see that \( d = rm \), as claimed.

(b) If \( d = rm \) and \( 0 \neq v \in R^{SL_m}[rm] \) then the partition associated to the 1-dimensional irreducible \( GL_m \)-module \( M = \text{Span}(v) \) is \((r^m)\); cf., e.g., [9, Theorem 2]. Now consider the direct sum decomposition \( R = \oplus R_\lambda \), where \( R_\lambda \) is the sum of all irreducible \( GL_m \)-submodules of \( R \) with associated partition \( \lambda \). The argument of part (a) shows that \( R_{(r^m)} = R^{SL_m}[rm] \). Moreover, since \( \dim(M) = 1 \), we have
\[
\dim_k R^{SL_m}[rm] = \dim_k R_{(r^m)} = \text{mult}_{(r^m)}(R),
\]
as claimed. \( \square \)

10. **Standard polynomials**

Let \( G_{m,n} \) be the ring of \( m \) generic \( n \times n \)-matrices. By Theorem 1.2, \((G_{m,n})^{SL_m}\) is a PI domain of degree \( n \), whenever \( 2 \leq m \leq n^2 - 2 \). We will now describe one particular element of this ring. Let
\[
F_m(x_1, \ldots, x_m) = \sum_{\sigma \in S_m} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} \in k\{x_1, \ldots, x_m\}
\]
be the standard polynomial. Since \( F_m \) is multilinear and alternating, one checks easily that for \( g \in GL_m \),
\[(10.1)\quad g(F_m) = \det(g) \cdot F_m;\]
see, e.g., [23, 1.4.12]. Substituting $m$ generic $n \times n$-matrices $X_1, \ldots, X_m$ into $F_m$, we obtain
\[
f_{m,n} = F_m(X_1, \ldots, X_m) = \sum_{\sigma \in S_n} (-1)^\sigma X_{\sigma(1)} \cdots X_{\sigma(n)} \in G_{m,n}.
\]
From (10.1), we see that $f_{m,n} \in (G_{m,n})_{SL_m}$. By the Amitsur-Levitzki Theorem, $f_{m,n} = 0$ iff $m \geq 2n$.

Fix $m, n \geq 2$ and let $K$ be the center of $UD(m,n)$.

10.2. Proposition. For $2 \leq m < 2n$, $K(f_{m,n})$ generates a $GL_m$-stable maximal subfield of $UD(m,n)$.

The proof is algebraic in nature and works in characteristic $\neq 2$.

Proof. The fact that $K(f_{m,n})$ is a $GL_m$-stable subfield follows from (10.1). In order to prove that this subfield is maximal, it suffices to verify that $f_{m,n}$ has an eigenvalue of multiplicity 1. (Indeed, if, say, $n = d \cdot [K(f_{m,n}) : K]$, then the characteristic polynomial $p(t)$ of $f_{m,n}$ in $UD(m,n)$ has the form $p(t) = q(t)^d$, where $q(t)$ is the minimal polynomial of $f_{m,n}$ over $K$. This shows that the multiplicity of each eigenvalue of $f_{m,n}$ is divisible by $d$.)

Since the multiplicity of eigenvalues cannot decrease when evaluating $f_{m,n}$ in $M_n$, it suffices now to show that $f_{m,n}$ (or equivalently, $F_m$) has some evaluation in $M_n$ with an eigenvalue of multiplicity one. We now proceed to construct such an evaluation. Since $F_m(1, x_2, \ldots, x_m) = F_{m-1}(x_2, \ldots, x_m)$ for $m$ odd (cf. [23, Exercise 1.2.3]), we may assume that $m$ is even, say $m = 2r - 2$, with $1 < r \leq n$. In $M_n$, consider the sequence of $m$ matrix units
\[e_{1,2}, e_{2,3}, e_{3,4}, \ldots, e_{r-2,r-1}, e_{r-1,r}, e_{r,r-1}, e_{r-r-1}, e_{r-1,r}, e_{r,1}.\]

When permuting these matrix units cyclically, their product is nonzero; for any other permutation, their product is zero. Since an $m$-cycle is odd, it follows that $F_m$ evaluated at these matrix units is
\[e_{1,1} - e_{2,2} + e_{2,2} - \cdots - e_{r-1,r-1} + e_{r-1,r-1} - e_{r,r} = e_{1,1} - e_{r,r},\]
which has 1 as an eigenvalue of multiplicity one (since $\text{char}(k) \neq 2$).

We do not know an explicit expression for any non-constant element of $(G_{m,n})_{SL_m}$ (as a polynomial in the generic $n \times n$-matrices $X_1, \ldots, X_m$) in the case where $2n \leq m \leq n^2 - 2$; we leave this as an open question. Note that for $m = n^2$ and $m = n^2 - 1$, such elements are exhibited in Remarks 8.4 and 8.6.

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