M-branes and N=2 Strings

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The string field theory of N=(2,1) heterotic strings describes a set of self-dual Yang-Mills fields coupled to self-dual gravity in 2+2 dimensions. We show that the exact classical action for this field theory is a certain complexification of the Green-Schwarz/Dirac-Born-Infeld string action, closely related to the four dimensional Wess-Zumino action describing self-dual gauge fields. This action describes the world-volume of a 2+2d “M-brane”, which gives rise upon different null reductions to critical strings and membranes. We discuss a number of further properties of N=2 heterotic strings, such as the geometry of null reduction, general features of a covariant formulation, and possible relations to BPS and GKM algebras.

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1. Introduction

All known (weakly coupled) string theories, as well as eleven-dimensional supergravity, appear to be different manifestations of a unique theory, compactified on different manifolds and studied in various extreme limits in the moduli space of vacua. It is an outstanding problem to find a good presentation of this theory which is valid everywhere in moduli space. The degrees of freedom appropriate for such a presentation are unknown. The problem is especially challenging since the unified theory should have the property that:

a) in different limits it looks like strings with different worldsheet gauge principles, or even as an eleven-dimensional theory which doesn’t have a string description at all;

b) the dilaton (or string coupling) should play the role of a size of a compact manifold, and in particular appear on the same footing as other geometrical data ("U-duality").

All this is in sharp contrast with the conventional description of any single string vacuum, where the worldsheet gauge principle is chosen at the outset, and the string coupling appears in a very different way than the geometrical moduli.

In two recent papers \([1,2]\), it was pointed out that N=2 heterotic strings may provide important clues regarding the degrees of freedom appropriate for a more fundamental formulation of the theory. N=2 heterotic strings live in a 2+2 dimensional target space, with a null reduction restricting the dynamics to 1+1 or 2+1 dimensions. The main results of \([1,2]\) are:

1) The target space dynamics of critical N=2 heterotic strings describes critical string worldsheets and membrane worldvolumes in static gauge (see also \([3]\) for early work).

2) All types of ten-dimensional superstring theories, as well as the eleven-dimensional supermembrane, arise in different limits of the moduli space of N=2 strings. One can continuously interpolate between them by varying the moduli of the N=2 string. Thus, N=2 strings appear to be the ‘building blocks’ of critical (super)string/membrane theories.

3) The 1+1 dimensional dynamics on a superstring worldsheet and the 2+1 dimensional worldvolume dynamics of a supermembrane appear to be different manifestations of the 2+2 dimensional worldvolume dynamics of an ‘M-brane’ propagating in a 10+2 dimensional spacetime. The null reduction yields different string worldsheets and membrane worldvolumes.

The 2+2/10+2 dimensional perspective sheds new light on string dynamics. Scalars on the string worldsheet of the usual formalism describe the embedding of the string in spacetime;
on the M-brane, they are replaced by self-dual gauge fields. The abelian worldsheet gauge field familiar from D-strings is replaced by a self-dual four-dimensional metric. The principle of reparametrization invariance and the residual conformal invariance on the string worldsheet are replaced by a not yet fully understood symmetry principle, apparently related to four-dimensional self-duality. The appearance of extra dimensions allows relations among M-theory vacua that are hidden in other approaches. Since some of these relations are due to strong/weak coupling duality, the hope is that the study of M-branes will eventually lead to a non-perturbative formulation of string theory.

The 2+1 dimensional vacua of the theory describe membrane-like objects; because these objects appear in the theory on the same footing as strings and are embedded in the underlying 2+2 self-dual theory, it is a strong possibility that these membranes can be quantized, leading to a definition of ‘M-theory’ (see [4] for another suggestion in this regard).

There are two possible ways of viewing the role of the N=2 string in our construction, both of which are useful to keep in mind. One is to regard the N=2 string as merely a tool to probe the 2+2 dimensional dynamics on the M-brane worldvolume, which in this view is ‘fundamental’. If one adopts this point of view, the task is to extract as much information as possible about the 2+2 field theory; then study its quantization as a field theory (perhaps by using heterotic N=2 string field theory as a tool). Alternatively, one may take seriously the idea that N=2 heterotic strings are the ‘constituents’ out of which critical strings and membranes are made, and take seriously all physical fluctuations of N=2 strings. If one takes this stance, the problem of quantization of the unified theory turns into the problem of second (or third) quantization of the heterotic N=2 string.

At the current level of understanding, the second point of view appears to be more viable for two reasons:

a) As we’ll discuss below, the quantization of the 2+2d M-brane worldvolume theory implied by N=2 string field theory appears to be highly unconventional. For example, compactification of the 2+2d worldvolume reveals a rich spectrum of new degrees of freedom (arising from mixed momentum/winding excitations of the N=2 heterotic

1 More precisely, since the N=2 string so far only seems to give strings and membranes in static gauge, a more conservative interpretation is that N=2 strings are the quanta of the Goldstone modes; the constituents may be something else. A useful analogy might be two-dimensional non-critical strings, which are the quanta of a theory whose underlying constituents are free fermions.
string), unseen in the infinite radius limit, and absent in the 2+2d worldvolume theory. These degrees of freedom appear to be needed for a “microscopic” understanding of string duality; in fact, in the recent work of Dijkgraaf, Verlinde, and Verlinde, a similar construction of an ‘underlying string’ was claimed to shed light on the microscopic degrees of freedom needed for heterotic string S-duality in D=4.

b) In addition to the class of vacua describing supersymmetric 2+2 branes living in 10+2 dimensions, we’ll discuss below vacua of heterotic N=2 strings describing other kinds of critical strings in various dimensions (26d bosonic, 10d fermionic and various N=2 strings in target space). While the relation between the five known ten dimensional superstring theories and eleven dimensional supergravity can be understood in our framework directly on the 2+2d worldvolume of the M-brane, the other classes of strings can only be understood by appealing to the underlying heterotic N=2 string. The main argument against viewing N=2 heterotic strings as fundamental is that so far they have only given particularly symmetric points in the moduli space of string vacua, and it has proven difficult to construct the full moduli space this way. We will comment on this issue below.

In any case, the sort of ‘string miracles’ occurring in the N=2 heterotic string by now seem rather unlikely to be an extraneous coincidence, and strongly suggest that N=2 strings must be incorporated into the emerging picture of dualities in the master theory.

The purpose of this paper is to report on the status of the program of [1] of studying M-branes using their realization as collective excitations of N=2 strings, and in particular present new results regarding the target space dynamics of N=2 heterotic strings. Heterotic geometry is described by a gravity multiplet – metric, antisymmetric tensor, and dilaton – coupled to gauge fields. The geometrical setting is thus a vector bundle $E$ over a manifold $M$. As we describe in detail in sections 2 and 3, in N=(2,1) heterotic geometry the critical dimension is four, the geometry is self-dual, and $E$ is an adjoint bundle. The proposal of [1] is to regard the four-manifold (of signature 2+2) as the world-volume of an extended object embedded in the total space of the bundle $E \to M$ (see figure 1).
Consistency of the $(2,1)$ string requires a null reduction to be imposed on the geometry. Strings are obtained when the null vector field lies entirely in the base $M$; membranes when its timelike part lies in the base and its spacelike part lies in the fiber $E$. The ripples on the brane are described by a metric $e^\phi$ on the fibers, which specifies the self-dual gauge connection by the usual Yang ansatz. Our main result is that the dynamics of this arrangement is described by the geometrical action

$$S_{2+2} = \int d^4x \sqrt{\det[g_{i\bar{j}} + \partial_i \phi^a \partial_{\bar{j}} \phi^a]}$$

where $g_{i\bar{j}}$ is the self-dual metric with torsion on the base $M$, and the scalars $\phi^a$ parametrize the self-dual gauge connection in the case where the gauge group is abelian (the nonabelian extension is also described).

The plan of the paper is as follows. In section 2 we briefly review the technology of constructing $N=2$ heterotic string vacua. We emphasize the fact that the 1+1 dimensional vacua of $N=2$ heterotic strings seem always to describe in their target spaces critical string worldsheets in static gauge, and give a number of new examples of this phenomenon. There appears to be a close relation between the properties of the target worldsheets and the underlying $N=2$ one. We then discuss the 2+2 dimensional target space geometry of $N=2$ heterotic strings, which underlies the dynamics of the 1+1 and 2+1 dimensional string/membrane vacua, and describes a non-linear self-dual system of gauge fields coupled to gravity. We also describe the striking change the theory undergoes upon compactification, and point out the relations to the physics of BPS states and string duality.
Section 3 constructs the classical 2+2d dynamics on the worldvolume of the M-brane (for earlier work, see [1]). It is known that scattering amplitudes with more than three external particles vanish in N=2 string theory. Nevertheless, the target space action (1.1) is non-polynomial; we determine this action exactly in the classical limit of small N=2 string coupling constant (large M-brane tension). This is achieved by a combination of string S-matrix and beta-function techniques, which allow one to argue in this case that the one loop sigma model beta function is exact.

Section 4 contains discussions of a number of open issues, such as the geometrical role of the null reduction and its possible relation to conformal symmetry. We outline the ingredients required for a covariant formulation of the dynamics, and give a brief treatment of the generalizations necessary to include winding N=2 strings. The latter lead to intriguing connections to BPS algebras [7] and GKM algebras such as the monster Lie algebra. At various points, we comment on the possible relation of other recent work to ours, including F-theory [8] and matrix models of M-theory [4]. Two appendices discuss the different null reductions, and some of the reasons to expect the quantization of the 2+2d M-brane worldvolume theories to be subtle.

2. Some properties of N=2 heterotic strings.

Since their discovery twenty years ago [9], N=2 strings have been among the most enigmatic types of string theories. As we will review below (see e.g. [10], [11] for more on N=2 strings), unlike other critical string theories, critical N=2 strings live in a low-dimensional target space (two, three, or four dimensional depending on the particular model and background); they describe a finite number of field theoretic degrees of freedom in target space, whose dynamics has yet to be fully elucidated. We will focus on heterotic N=2 strings [12]; our interest in these theories stems from the observation of [12] mentioned above, that the target space dynamics of N=2 heterotic strings generates all known classes of M-theory vacua, and seems to offer an interesting perspective on the problem of finding a unified description of all such vacua.

The target space dynamics of N=2 heterotic strings appears to replace conformal invariance on the string worldsheet by self-duality on a 2+2 dimensional worldvolume (the target space of the N=2 string). The natural objects defined on the worldsheet – scalar fields describing the embedding of the worldsheet in space-time and the Born-Infeld gauge field – are replaced by self-dual gauge fields and metric on the worldvolume, respectively.
One of our primary tasks in this paper will be to study the dynamics of this non-linear 2+2d self-dual system.

Upon compactification on a circle one finds a qualitative change in the target space physics. The finite number of target space fields is replaced by a tower of states of arbitrarily high mass. The structure is very reminiscent of the construction of certain BPS states in string theory, and indeed we’ll see below that there are close parallels with recent discussions of the physics of BPS states in type II/heterotic string theories. The density of states with fixed momentum/winding around the circle grows as \( \rho(E) \sim E^\alpha \exp(\beta \sqrt{E}) \) (with \( \alpha \), \( \beta \) certain known constants), a fact that ensures that the dynamics is still relatively simple. The compactified N=2 heterotic string exhibits important connections between worldsheet and space-time physics, and its better understanding seems important for understanding string duality.

In this section we will summarize some of the known facts about N=2 heterotic strings, as well as make a few new points. We start with a review of the worldsheet construction of N=2 heterotic strings, followed by a discussion of the target space dynamics, and then an outline of the compactified case.

2.1. N=2 heterotic worldsheets.

Like the more familiar N=(1,0) heterotic string, N=2 heterotic strings have the property that left and right moving modes on the worldsheet couple to different (super) gravity theories. The right moving modes always couple to N=2 worldsheet supergravity. For N=(2,0) heterotic strings, the left movers describe a critical bosonic string, while for N=(2,1) strings the left movers couple to N=1 supergravity. The critical central charge for the right movers is \( \bar{c} = 6 \), which can be realized by four right moving superfields \( (x^\mu, \bar{\psi}^\mu) \), \( \mu = 0, 1, 2, 3 \) with signature \( (-,-,+,) \) living in a flat space-time \( \mathbb{R}^{2,2} \) with metric \( \eta_{\mu\nu} \) (we will discuss the interesting case of compact \( x^\mu \) later). The peculiar signature of space-time is dictated by the necessity of choosing a complex structure, which we will denote by \( I_{\mu\nu} \). The complex structure satisfies:

\[
I_{\mu\nu} = -I_{\nu\mu}; \quad I_{\mu\nu} I^{\nu\lambda} = \eta_{\mu}^{\lambda}.
\]  

\(^2\) See [1], [2] for additional discussion and references to earlier work.
The choice of complex structure breaks the Lorentz symmetry\footnote{One can think of the breaking as spontaneous and interpret $I_{\mu\nu}$ as an expectation value of a target space field.} from $O(2,2)$ to $U(1,1)$. The right moving $N=2$ superconformal gauge algebra, which consists of the stress tensor $\tilde{T}$, two superconformal generators $\tilde{G}^{\pm}$, and a $U(1)$ current $\tilde{J}$, is represented on $x^\mu, \bar{\psi}^\mu$ as:

$$
\begin{align*}
\tilde{T} &= -\frac{1}{2} \partial x \cdot \partial x - \frac{1}{2} \bar{\psi} \cdot \bar{\partial} \bar{\psi} \\
\tilde{G}^{\pm} &= (\eta_{\mu\nu} \pm I_{\mu\nu}) \bar{\psi}^\mu \partial x^\nu \\
\tilde{J} &= \frac{1}{2} I_{\mu\nu} \bar{\psi}^\mu \bar{\psi}^\nu.
\end{align*}
$$

(2.2)

The four non-compact scalar fields $x^\mu$ are shared by the left and right moving sectors. To describe the rest of the left moving structure it turns out to be essential \cite{12} to enlarge the left moving gauge principle by a $U(1)$ gauge symmetry. We next describe the resulting structure for the two kinds of $N=2$ heterotic strings, starting with the $N=(2,0)$ one.

$N=(2,0)$ heterotic strings.

The conformal gauge generators for the left movers are the stress tensor $T(z)$, and the $U(1)$ current $J(z)$. The critical dimension is $c = 28$, which can be realized in terms of twenty-eight scalar fields $x^a, a = 0, 1, \cdots, 27$, with signature $(-, -, (+)^{26})$. The generators

$$
\begin{align*}
T(z) &= -\frac{1}{2} \partial_z x^a \partial_z x^a \\
J(z) &= v^a \partial_z x^a
\end{align*}
$$

(2.3)

involve a choice of a preferred null vector $v^2 = 0$. Physical states satisfy

$$
J_n|_{\textrm{phys}} = 0; \quad n \geq 0
$$

(2.4)

which effectively decouples two of the twenty-eight dimensions, giving back the familiar critical dimension of bosonic strings, twenty-six, and the usual Minkowski signature.

What kinds of vacua of $N=(2,0)$ strings are there? Keeping $(x^0, \cdots, x^3)$ non-compact, modular invariance requires the chiral scalars $(x^4, \cdots, x^{27})$ to live on an even, self-dual Euclidean torus, one of the twenty-four Niemeier tori. The worldsheet partition sums for the resulting theories appear in Appendix A. Additional possibilities involving orbifolds and twisting of the $N=2$ algebra will be mentioned below.
Depending on the choice of the null vector \( v \) (2.3), the theory can be thought of as living in 1+1 or 2+1d, corresponding, respectively, to \( v \) pointing in the first four directions or having its timelike component lie in the 0-1 plane and its spacelike component in the \((x^4, \cdots, x^{27})\) hyperplane. The spectrum of physical states for the two cases is:

1+1d: A massless scalar for every dimension one operator in the meromorphic \( c=24 \) conformal field theory (CFT) describing \((x^4, \cdots, x^{27})\). These states transform in the adjoint representation of a rank twenty-four simply laced group \( H \) characterizing the Niemeier lattice (e.g. \( U(1)^{24}, E_8^3, SU(25), \) etc). Their vertex operators are

\[
V_\phi^a = J^a(z) \int d^2\theta \ e^{ik \cdot x}; \quad a = 1, \cdots, \dim H
\]

where the \( \theta \) integrals schematically denote the raising operators \( \hat{G}^\pm \). The currents \( J^a \) are given by the standard construction; e.g. the Cartan subalgebra (CSA) generators correspond to \( J^a = \partial x^a \).

2+1d: Take for concreteness

\[
v = (1, 0, 0, 0, 1, 0, 0, \cdots).
\]

The theory now lives on the 2+1d space-time parametrized by \((x^1, x^2, x^3)\). On this space-time we find a massless gauge field with vertex operator

\[
V_K^\mu = \xi^\mu \partial x^\mu \int d^2\theta \ e^{ik \cdot x}
\]

with \( \xi \cdot k = 0, \xi \sim \xi + \epsilon k; k = (k^1, k^2, k^3), \) \( k^2 = 0 \). In addition we find scalars in the adjoint representation of a rank 23 group whose CSA is spanned by \( J^a = \partial x^a, \quad a = 5, \cdots, 27 \). States carrying non-zero charge \( Q_4 \) under \( J^4 = \partial x^4 \) become tachyonic, since

\[
E^2 \equiv k_1^2 = k_2^2 + k_3^2 - k_0^2
\]

and at the same time, the condition (2.4) relates \( k_0 \) to the charge \( Q_4 \): \( k_0 = Q_4 \). The tachyonic dispersion relation (2.8) should lead to destabilization of the perturbative vacuum (see also Appendix A), and it would be interesting to understand this in more detail.

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[4] One might have thought that a third possibility is to orient the null vector entirely along the Niemeier torus, with \( v \) a complex null vector, such as \( v = \hat{x}^4 + i\hat{x}^5 \). However, as we show in Appendix A, this option is inconsistent, leading to a non-modular invariant theory.
Note also that the vacua of N=(2,0) strings built on Niemeier lattices described above are not the only ones that exist. For the case of non-compact \((x^0, \cdots, x^3)\), vacua of the N=(2,0) string are in one to one correspondence with modular invariant meromorphic CFT’s with \(c=24\); there are quite a few of these (see e.g. [13]).

\(N=(2,1)\) heterotic strings.

Here, we find a left moving N=1 superconformal gauge algebra acting on the \(\hat{c} = 12\) system consisting of the superfields \((x^a, \psi^a)\), \(a = 0, \cdots, 11\), where, again, \((x^0, \cdots, x^3)\) are non-compact, while the rest of the \(\{x^a\}\) are chiral, compact scalars. The superstress tensor can be written as:

\[
T = -\frac{1}{2}(\partial x)^2 - \frac{1}{2}\psi \partial \psi \\
G = \psi \partial x
\]

and again it is convenient to add a gauged \(U(1)\) supercurrent of the form:

\[
J = v^a \partial x^a \\
\Psi = v^a \psi^a
\]

As in (2.4), \(J, \Psi\) must annihilate physical states, and depending on the direction of \(v\) the theory lives in 1+1 or 2+1d. Vacua of the uncompactified \(N=(2,1)\) string are in one to one correspondence with meromorphic Superconformal Field Theories (SCFT’s) with \(\hat{c} = 8\) \((c = 12)\), describing the “compact” chiral sector \((x^a, \psi^a)\), \(a = 4, \cdots, 11\). We are not aware of a classification of such SCFT’s, but it is easy to construct examples.

One example is obtained by compactifying the \(\{x^a\}\), \(a = 4, \cdots, 11\) on the \(E_8\) torus. This gives rise to the partition sum\(^5\)

\[
Z(\tau) = \frac{1}{4} \left[ \left( \frac{\theta_3}{\eta} \right)^4 - \left( \frac{\theta_4}{\eta} \right)^4 - \left( \frac{\theta_2}{\eta} \right)^4 \right] \left[ \left( \frac{\theta_3}{\eta} \right)^8 + \left( \frac{\theta_4}{\eta} \right)^8 + \left( \frac{\theta_2}{\eta} \right)^8 \right].
\]

with the two factors in brackets arising from the contributions of \(\{\psi^a\}\) and \(\{x^a\}\), respectively. The spectrum of the 1+1d theory includes in this case eight bosonic states with vertex operators

\[
V^a_\phi = \int d\theta \int d^2\bar{\theta} \psi^a e^{ik \cdot x}; \quad a = 4, \cdots, 11.
\]

\(^5\) Actually, this partition sum is appropriate to the 1+1d vacuum of the theory. See Appendix A for the modification needed for the 2+1d case.
We use $\int d\theta$ and $\int d^2 \bar{\theta}$ as shorthand for the action of the superconformal generators $G_{-\frac{1}{2}}$, $\bar{G}_{-\frac{1}{2}}$. In the 2+1d vacuum we find seven states of the form (2.12), respectively, however in addition to those and in analogy to (2.7), a massless gauge field appears:

$$V_K = \int d\theta \int d^2 \bar{\theta} \xi^\mu \psi^\mu e^{ik \cdot x}$$  \hspace{1cm} (2.13)

with the standard gauge invariance, $\xi \sim \xi + \epsilon k$, $\xi \cdot k = 0$. Note, in particular, that the total number of physical bosonic degrees of freedom is always eight.

A quick glance at the partition sum (2.11) reveals a new feature of the N=(2,1) string – the appearance of space-time fermions, with vertex operators

$$V_{\vartheta} = u^\alpha(k) \Sigma S_\alpha \int d^2 \bar{\theta} e^{ik \cdot x}$$  \hspace{1cm} (2.14)

where $S_\alpha$ is the dimension 3/4, 32 component spin field for the twelve fermions $\psi^a$, and $\Sigma = \exp(-\phi/2 + \rho/2)$ is a dimension 1/4 spin field for the superconformal and super $U(1)$ ghosts (see [2] for details). The null reduction (2.10) eliminates half of the components of $V_{\vartheta}$, while the Dirac equation eliminates half of the remaining ones. We are left with eight fermionic degrees of freedom, equal to the number of bosonic ones. In fact the system is supersymmetric [1,2]; the supercharges $Q_\alpha$ are given by

$$Q_\alpha = \oint dz \Sigma S_\alpha$$  \hspace{1cm} (2.15)

where, again, only half of the 32 components are physical due to the null constraints (2.10). The unbroken supercharges transform in the 16 of the Spin(10) subalgebra of Spin(12) preserving the form of (2.10).

Another type of vacuum is obtained by fermionizing the eight scalars $x^a$, $a = 4, \cdots, 11$, and realizing the meromorphic $\hat{\epsilon} = 8$ SCFT in terms of the resulting twenty-four fermions. A diagonal sum over spin structures of all twenty-four fermions

$$Z(\tau) = \frac{1}{2} \left[ \left( \frac{\theta_2}{\eta} \right)^{12} - \left( \frac{\theta_4}{\eta} \right)^{12} - \left( \frac{\theta_6}{\eta} \right)^{12} \right]$$  \hspace{1cm} (2.16)

leads [14], [1] to a theory with only bosonic physical states,

$$V_{\phi}^A = \int d\theta \int d^2 \bar{\theta} \psi^A e^{ik \cdot x}$$  \hspace{1cm} (2.17)
transforming in the adjoint of a twenty-four dimensional group $H$. The superconformal
generator for the twenty-four fermions is given in terms of the structure constants of $H$:

$$G = f_{ABC} \psi^A \psi^B \psi^C; \quad A, B, C = 1, \ldots, 24.$$ \hfill (2.18)

Note that the form (2.18) implies that the group $H$ can not contain any $U(1)$ factors.

An interesting question for all the vacua of $\tilde{N}=(2,0)$, $(2,1)$ strings described above
is what target space dynamics they describe. Before turning to that, we would like to
comment on the issue of $U(1)$ instantons and $\theta$ vacua for $N=2$ strings.

**Worldsheet instantons and $\theta$ vacua.**

$N=2$ heterotic strings have in fact two independent $U(1)$ gauge fields coupling to the
left and right movers. It is usually stated that the two are components of a single gauge
field $A_\mu = (A, \bar{A})$. However, it is possible to perform separate gauge transformations on $A$
and $\bar{A}$:

$$A \to A + \partial \epsilon; \quad \bar{A} \to \bar{A} + \bar{\partial} \bar{\epsilon}. \hfill (2.19)$$

This is because the gauge fields $A, \bar{A}$ couple to currents $\bar{J}, J$, which are separately anomaly
free. One can therefore try to add $\theta$ terms corresponding to both gauge fields to the
Lagrangian:

$$\mathcal{L} = \theta F + \bar{\theta} \bar{F}. \hfill (2.20)$$

In fact, since the left moving $U(1)$ current $J$ $(2.3)$, $(2.10)$ is non-compact, physics is
independent of $\bar{\theta}$ in $(2.20)$.

A natural question regards the dependence of the physics on $\theta$, i.e. how do $U(1)$
instantons modify the dynamics? In the matter sector, the effect of an instanton is to
induce spectral flow by one unit of $U(1)$ charge; for instance this shifts $\bar{J} = \eta_{ij} \bar{\psi}^i \bar{\psi}^j$
to $\bar{J}^+ = \epsilon_{ij} \bar{\psi}^i \bar{\psi}^j$. It is not difficult to show that the entire effect of instantons can be
incorporated as a modification of $I_{\mu\nu}$. This was observed in the context of $N=(2,2)$ strings
(and used to great effect) in $[15,16]$. In complex coordinates we can take:

$$I_{ij} = a \eta_{ij}$$
$$I_{ij} = b \epsilon_{ij} \hfill (2.21)$$
$$I_{\bar{i}\bar{j}} = b^* \epsilon_{\bar{i}\bar{j}}$$

\[6\] Note that in this case $H$ need not be simply laced.

\[7\] For $N=(2,2)$ strings, both $J$ and $\bar{J}$ are compact and amplitudes depend on both $\theta$ angles.
with $a$ real and $b$ complex, and $a^2 + |b|^2 = 1$ because of (2.1). One can then think of $\theta$ as the phase of $b$.

The choice of $I_{\mu\nu}$ breaks O(2,2) symmetry of the target space; if there were no other background data violating Lorentz invariance, all choices would be equivalent. However, N=2 heterotic string backgrounds also involve a choice of null vector $v$. Thus the physical data are the relative orientation of $v$ and $I_{\mu\nu}$. For instance, in the reduction to 1+1d, we can choose coordinates so that the null reduction is $x^i = x^{\bar{i}}$, and the complex structure is arbitrary as in (2.21). This is technically simpler than fixing, say, $b = 0$ in the choice of $I_{\mu\nu}$, and working with an arbitrary null reduction, and it is what we’ll do below.

2.2. Space-time features of N=2 heterotic strings.

As discussed in the previous section, N=2 heterotic strings describe in their two or three dimensional target space a field theory coupled to gravity. What is that theory?

Consider first the two dimensional case. We know of a consistent theory of matter coupled to gravity in 2d – worldsheet string theory. Thus one may ask what is the relation between the target space dynamics of N=2 heterotic strings in their 2d vacua and worldsheet string theory. In [1,2] evidence has been presented for the validity of the following conjecture:

The two dimensional target space dynamics of critical N=2 heterotic strings always describes critical string worldsheet dynamics in physical gauge.

It should be emphasized that there is at present no conceptual understanding of why this conjecture should hold; the fact that it does seems even in hindsight surprising. Since, as argued in [1,2], this presentation of string worldsheets appears to be closely related to string duality, it is important to verify and further understand this issue.

We next turn to a few examples of N=2 string vacua and the target space strings they describe.

Example 1: N=(2,0) strings on Niemeier tori.

In the previous subsection we saw that the physical excitations of an N=(2,0) string on a particular Niemeier torus characterized by a simply-laced group $H$ of rank twenty-four are scalars $g$ in the adjoint of $H$. What is the dynamics of these scalars? We will see that it is governed by the WZW lagrangian (coupled to gravity); for the above conjecture to hold, the level should be $k = 1$, since only then is the target string critical. We should emphasize that we are talking here about a WZW action in the 1+1d target space of the
N=2 string; there is a similar (but chiral) WZW action describing the worldsheet degrees of freedom, and in fact there is the possibility of an interesting ‘duality’ between the two; e.g. it is natural for the level $k$ of the affine Lie algebra $\hat{H}$ to be the same on the worldsheet and in target space. We should also stress that, at the moment, we have no idea why or how the level $k$ should be fixed at this particular value (likewise for the special value that seems appropriate for N=(2,1) strings, example 4 below). It could be that other values of $k$ are allowed, in which case the target worldsheet theory would be a noncritical string, with nontrivial dynamics for worldsheet gravity.

It is well-known that the level one WZW models for all simply laced groups lead to theories with $c = \text{rank}(H)$, which in the present case rank$(H) = 24$ are equivalent to twenty-four free scalars living on a torus. Therefore, the 1+1d target space theory of the N=(2,0) string corresponds in this case to a critical bosonic string compactified to two dimensions on a twenty-four dimensional torus (presumably the same Niemeier torus appearing in the worldsheet construction). Equivalently, one can think of these vacua as describing critical bosonic strings compactified on a rank twenty-four group manifold.

The simplest example of a Niemeier torus is the Leech torus which has no vectors of length $\sqrt{2}$, and thus has twenty-four physical states corresponding to $H = U(1)^{24}$. These twenty-four scalar fields are governed by the Nambu-Goto action and describe the transverse dimensions of a critical string.

An interesting point concerns the N=(2,0) string coupling $\lambda$, which determines the coupling of the target space WZW model. It is well known that the WZW model describes an RG flow from a free UV fixed point with central charge $c_{UV} = \dim H$ at $\lambda = 0$ to an interacting IR fixed point with $c_{IR} = \text{rank } H$ (for level $k = 1$) at $\lambda \sim 1$. Since the target bosonic string worldsheet we find is critical only at the IR fixed point (for nonabelian $H$), we conclude that for the conjecture articulated above to be valid, the N=2 string coupling $\lambda$ should be fixed (presumably by nonperturbative N=2 string considerations) to its IR value. One can argue that $\lambda$ is quantized, by noting that $1/\lambda^2$ multiplies the Wess Zumino term in target space. It would be interesting to understand what fixes it uniquely.

**Example 2:** N=(2,0) strings on the monster module.

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8 Actually, this point is somewhat confusing. The 2+2d theory has a Wess-Zumino term which is quantized via the Picard lattice of the four-manifold; but it also induces the 1+1d Wess-Zumino term via null reduction, whose quantization is more restricted. We thank G. Moore for discussions of this issue.
A well known example of a meromorphic \( c=24 \) CFT is the “monster module” \cite{Friedan}, which has no dimension one operators. When used to construct a vacuum of the \( N=(2,0) \) string, it has \textit{no} physical states (before compactification). The target space dynamics is compatible (albeit in a rather trivial way) with an \( N=(2,2) \) string worldsheet in physical gauge, which also has no transverse excitations.

\textbf{Example 3}: Type IIB target space strings from the \( N=(2,1) \) string on the \( E_8 \) torus.

The vacuum described by (2.11) has eight bosonic and eight fermionic physical states (2.12), (2.14), and \( N=(8,8) \) global SUSY on the 2d target space. The matter content and symmetry structure is that of the type IIB string in static gauge \cite{Friedan}. We’ll verify later that the dynamics agrees as well.

It is also worth mentioning that the type IIB construction belongs to a larger class of critical 10d superstring target worldsheets, which can be obtained starting from the underlying \( N=(2,1) \) string. In particular, in \cite{Friedan} it has been shown how to get heterotic \( SO(32) \) and type I’ strings by orientifolding.

\textbf{Example 4}: Fermionic strings on group manifolds from the \( N=(2,1) \) string.

The vacuum described by (2.10)-(2.18) provides an interesting example of the conjecture discussed above. In \cite{Friedan} it has been argued that this system describes in target space a bosonic string worldsheet. However, this appeared problematic, since the twenty-four scalars one finds transform in the adjoint of the group \( H \), and it is natural, as in example 1, to expect the dynamics to be described by a WZW model for the group \( H \) at some level \( k \). For any finite \( k \), the central charge \( c<24 \), in apparent contradiction with the fact that we expect critical strings to emerge.

The resolution of this contradiction could be the following: as in example 1, it is natural to conjecture that the level of the target space affine Lie algebra \( \hat{H} \) is equal to that of the worldsheet one; in this case it is \( k = Q_H \), the quadratic Casimir of \( H \) in the adjoint representation. The target space central charge is thus \( c = D_H/2 = 12 \), which indeed seems non-critical.

However, precisely for that level, the WZW model is equivalent to a collection of (twenty-four) free fermions, \( \{\theta^A\} \) in the adjoint representation of \( H \), and furthermore, that collection of fermions is invariant under a non-linear SUSY transformation

\[
\delta \theta^A = \epsilon f^{ABC} \theta^B \theta^C. \tag{2.22}
\]
This symmetry is not visible in the worldsheet construction of \[1\] since it acts in a highly non-linear fashion on the vertex operators of the twenty-four scalars in target space \(2.17\).

We conclude that it is possible that \(2.16\) describes a critical ten dimensional fermionic string, rather than a non-critical bosonic one, in agreement with the conjecture. Note that as in Example 1, there is close similarity between the worldsheet and target space structures. In particular, an analog of \(2.18\) for the \(\theta^A\) generates the target SUSY transformations \(2.22\). The \(N=(2,1)\) string coupling must be fixed to the value for which the WZW model is at its IR fixed point in order that the target fermionic string is critical.

It is natural to expect the sum over spin structures of the target space fermions \(\theta^A\) to be the same as that for the worldsheet fermions of the underlying \(N=2\) heterotic string, \(2.16\), separately for left and right movers in target space. Thus, this vacuum describes a non-supersymmetric space-time compactification of the ten dimensional fermionic string to two dimensions.

**Example 5**: \(N=(2,2), (2,1)\) strings from the \(N=(2,1)\) string.

It is amusing to note that the \(N=(2,1)\) string is “self-reproducing”, sometimes giving target space spectra which are consistent with the \(N=(2,1)\) string itself or other \(N=2\) strings. To get \(N=(2,2)\) strings in target space, we split the twenty-four fermions \(\psi^A\) \(2.18\) into three groups of eight, and sum over the spin structures of the three groups independently:

\[
Z(\tau) = \frac{1}{8} \left[ \left( \frac{\theta_3}{\eta} \right)^4 - \left( \frac{\theta_4}{\eta} \right)^4 - \left( \frac{\theta_2}{\eta} \right)^4 \right]^3. \tag{2.23}
\]

Since the coefficient of \(q^0\) in \(2.23\) vanishes sector by sector, it is clear that this model has no physical states before compactification, just like the \(N=(2,0)\) string vacuum in Example 2. As there, we interpret the dynamics as describing the worldsheet of an \(N=(2,2)\) string in physical gauge.\footnote{Off shell, or after compactification, the theory described by \(2.23\) and that of Example 2 are clearly inequivalent (e.g. the former is supersymmetric in target space while the latter is not), and possess non-trivial dynamics. It would be interesting to understand them better.}
To get the N=(2,1) string in target space, we start with the supersymmetric vacuum of Example 3, and orbifold by the $Z_2$ symmetry:

$$ (x^a, \psi^a) \rightarrow -(x^a, \psi^a); \quad a = 4, \ldots, 11. \quad (2.24) $$

The partition sum of the model is:

$$ Z(\tau) = -\frac{3}{2} \left( \frac{\theta_2 \theta_3 \theta_4}{\eta^3} \right)^4 = -24. \quad (2.25) $$

The spectrum includes the Ramond fields

$$ V_\alpha = \Sigma S_\alpha e^{ik \cdot x}; \quad \alpha = 1, \ldots, 8 $$

$$ W_i = \Sigma \sigma_i e^{ik \cdot x}; \quad i = 1, \ldots, 16 \quad (2.26) $$

where $S_\alpha$ are spin fields for $\psi^a$ (and transform in the 8 of Spin(8)); $\sigma_i$ are the chiral twist fields for the $\{x^a\}$, and $\Sigma$ is the dimension 1/2 spin field for ghosts and longitudinal fermions ($\psi^0, \ldots, \psi^3$). The target space Dirac equation imposes the condition $k^+ = 0$. Thus the spectrum of the model consists of twenty-four free left moving fermions (2.26) on the two dimensional target space, precisely the right spectrum for interpreting the target space as an N=(2,1) string worldsheet (again, in physical gauge). One expects a nonlinearly realized SUSY to act on these fermions, but it clearly shouldn’t be visible via a linear action on the vertex operators (2.26). Finally, note that these vacua (as well as the monster module, example 2) have the property that boundary conditions on the fields require that the null reduction always lie within the 2+2 longitudinal dimensions; one has only target space strings and not membranes.

Hence, the 1+1d vacua of N=2 heterotic strings describe critical string worldsheets in their target space, and their quantization is presumably equivalent to the standard string one. As mentioned in the introduction, the N=2 heterotic string construction appears to yield particularly symmetric points in the moduli space of target strings, and it is not clear how to construct the full moduli space of vacua in this framework. Consider, as an

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10 An alternative fermionic formulation of the same model follows more closely the discussion of the N=(2,2) target string above. Split the twenty-four fermions $\psi^A$ into three groups of eight, and sum over boundary conditions such that the Ramond sector in one of the groups is correlated with NS boundary conditions in the other two. This makes it easier to evaluate the partition sum of the model, given in the text.
example, the construction of Example 1. It describes a critical bosonic string compactified to 1+1 space-time dimensions. It is well known that the moduli space of such theories is $\mathcal{M} \simeq SO(24,24)/SO(24) \times SO(24)$ (modulo the discrete T duality group), while we are finding a unique theory with no moduli, corresponding, presumably, to a separate compactification of the left and right movers on an even self-dual twenty-four dimensional Niemeier lattice.

In fact, this example is typical. Apparently, N=2 heterotic strings lead to target string compactifications to 1+1 dimensions for which left and right movers on the target worldsheet are decoupled. Thus, example 3 is consistent with a superstring compactified on the $E_8$ torus, and in example 4 we find a 1+1 dimensional fermionic string whose compact SCFT describes twenty-four left and right moving fermions whose spin structures are identified, and summed separately for left and right movers. The target partition sums in all the above cases appear to be the absolute values squared of the worldsheet partition sums (A.3), (2.11), (2.16). This is another example of the close connection of N=2 worldsheet and target physics mentioned above.

If this interpretation is correct, perhaps the absence of Narain moduli of the target strings is due to a symmetry that N=2 heterotic strings possess which is only unbroken in the highly symmetric vacua described above. It would be interesting to understand the symmetry and find a more powerful formalism that is capable of describing the full moduli space of vacua.

What about the 2+1d vacua? There, it is not known how to quantize theories of gravity coupled to matter, which makes it interesting to understand the target dynamics of N=2 heterotic strings. In particular, choosing the 2+1 dimensional reduction of the model in Example 3 above, yields a supersymmetric eleven dimensional theory – M-theory (type IIA string theory at finite string coupling). By orbifolding this theory by a symmetry that acts as the nontrivial outer automorphism of the N=2 algebra, one can construct vacua of M-theory compactified on $S^1/Z_2$ and thus study $E_8 \times E_8$ heterotic strings at finite coupling, as well as heterotic SO(32) and type I’ strings.

11 Of course, in 2d compactifications of string theory one can not think of the Narain moduli as fixed; their dynamics is described by a sigma model with target space $\mathcal{M}$. One can still describe states in which the wave function of the moduli is centered at some point in the classical moduli space.

12 As discussed in [1], questions regarding sums over spin structures and the shape of the compact manifolds in target space are non-perturbative in the N=2 string coupling and thus are difficult to address directly at present.
2.3. The target space geometry.

Since all vacua of N=2 heterotic strings arise from 2+2 dimensional theories by different null reductions, it is natural to study the target space dynamics by first relaxing the constraint (2.4), understanding the resulting 2+2 dimensional dynamics, and then restoring the null reduction.

The first question one needs to address is what is the geometrical meaning of the scalars (2.5), (2.12), (2.17) and gauge fields (2.7), (2.13) in 2+2d. The answer to that question is known [18,19] (see also [12]): Performing the $\theta, \bar{\theta}$ integrals one finds that the scalars $V^a_\phi$ can be thought of as parametrizing a self-dual gauge field:

$$A_j = i \partial_j \Omega^{-1} \Omega$$
$$A_{\bar{j}} = i \partial_{\bar{j}} \Omega \Omega^{-1}$$

where $\Omega = \exp (\phi^a t^a)$, and $t^a$ are Hermitian generators of $H$. In the abelian case we have:

$$A^a_j = -i \partial_j \phi^a$$
$$A^a_{\bar{j}} = i \partial_{\bar{j}} \phi^a$$

The gauge field $V_K$ (2.13) parametrizes a Hermitian metric with torsion. Its contribution to the N=(2,1) string worldsheet Lagrangian is described by the deformation

$$\int d^2z \int d\theta \int d^2 \bar{\theta} K_{\mu} (x) DX^\mu.$$  

(2.29)

Defining

$$F_{\mu\nu} = \partial_\mu K_\nu - \partial_\nu K_\mu$$

(2.30)

and performing the $\theta, \bar{\theta}$ integrals, one finds that $F_{\mu\nu}$ parametrizes a metric with torsion:

$$g_{\mu\nu} = \frac{1}{2} \{I, F\}_{\mu\nu}; \quad B_{\mu\nu} = \frac{1}{2} \{I, F\}_{\mu\nu}.$$  

(2.31)

In complex coordinates, $I_{i\bar{j}} = \delta_{i\bar{j}}$ we have the metric and $B$ field:

$$g_{ij} = \partial_i K_j - \partial_j K_i$$
$$B_{i\bar{j}} = \partial_i K_{\bar{j}} + \partial_{\bar{j}} K_i$$

(2.32)

We will typically expand around a flat background, $g_{ij} = \eta_{ij} + F_{ij}$. We see that the CFT on the target worldsheet describing different kinds of strings is generalized in an interesting way in passing to 2+2d, to a non-linear self-dual system of gauge fields (2.27), and gravity (2.32). In some cases, this system should be supersymmetrized (2.15).

It should be very interesting to understand the dynamics of this system directly in 2+2d. Therefore we will examine in the next section the 2+2d classical dynamics describing the target space fields of N=2 heterotic strings. We conclude this section with some brief comments on compactification of N=2 heterotic strings.
2.4. Compactified N=2 heterotic strings.

Consider an N=2 heterotic string compactified on a circle of radius $R$, $\mathbb{R}^{2,2} \rightarrow \mathbb{R}^{2,1} \times S^1$. One of the most striking features of N=2 heterotic strings is the qualitative difference between the spectra of the finite and infinite $R$ theories. As explained in previous sections, for infinite $R$ the spectrum includes a finite collection of field theoretic degrees of freedom (self-dual gauge fields and metric). For any finite $R$, we find a rich string theoretic spectrum with an exponential density of states\(^{13}\).

Indeed, while the right movers are forced by the N=2 physical state conditions to remain in their ground state, one can for finite radius excite the whole tower of left moving oscillators, in mixed momentum/winding sectors. The mass spectrum satisfies:

$$M^2 = \left( \frac{n}{R} + \frac{mR}{2} \right)^2 = \left( \frac{n}{R} - \frac{mR}{2} \right)^2 + 2N \quad (2.33)$$

where $n, m$ are the momentum and winding around the $S^1$, and $N$ the level of left-moving oscillator excitation\(^{14}\). By (2.33),

$$N = nm. \quad (2.34)$$

In the $R \rightarrow \infty$ limit all states with $m \neq 0$ go to infinite mass, and we are left with the field theoretic spectrum described above. For any finite $R$, the density of states is exponential. In a sector with given $n, m$, (2.33), (2.34) imply that the mass $M$ is fixed, $M \sim N$, and since the number of states at level $N$ is of order $N^\alpha \exp(\beta \sqrt{N})$, we have for given $n, m$ of the order of $M^\alpha \exp(\beta \sqrt{M})$ states. It is important for the dynamics – in particular the triviality of the S-matrix – that this number of states, while of course very large by field theoretic standards, is smaller than that in ordinary string theory (where the number of states with mass $M$ grows like $\exp(\beta M)$).

The total density of states is string theoretic. Indeed, for states with a certain large mass $M \gg R, 1/R$, the density of states is dominated by the sectors with $n/R \approx mR/2 \approx M/2$, and by (2.34), $N \approx M^2/2$. The density of states grows with mass as $\rho(M) \approx M^\alpha e^{\beta M}$.

Thus, N=2 heterotic strings have the remarkable property that at infinite radius physics is field theoretic, while for any finite radius the system exhibits a Hagedorn transition and has a much richer dynamics. There are two other contexts in recent discussions

\(^{13}\) The role of winding N=2 heterotic strings has been also discussed in [20].

\(^{14}\) We are suppressing momenta along the chiral, left-moving torus, which do not lead to a significant modification of the following discussion. We will return to them later in the paper.
of string theory that exhibit similar features; both are probably related to N=2 heterotic strings.

1) **The physics of BPS states in string theory.** In ten dimensional uncompactified (perturbative) superstring theory, the only states which are killed by half of the supercharges (and thus belong to short representations of space-time SUSY) are massless. Compactification on one or more circles gives rise to a rich spectrum of Dabholkar-Harvey states [21] with a spectrum analogous to (2.33). As we’ll discuss below, this analogy is not accidental – one can think of N=2 heterotic strings as describing the sector of BPS states of type II / (1,0) heterotic strings.

2) **String duality.** If one replaces the radius by the string coupling in the above considerations, one arrives at a picture familiar from string duality. As the string coupling goes to zero, the magnetic and dyonic strings go to infinite mass and decouple. For any finite coupling, a duality symmetric formulation of any string theory would presumably include electric, magnetic and dyonic degrees of freedom. It is not understood what those are in general (and it is one of the main motivations of our work to gain a better understanding of them), but a recent proposal for understanding S-duality in heterotic string theory on $T^6$ [5] provides a concrete realization in which magnetic and dyonic strings are realized as winding and mixed momentum-winding modes of an underlying string, whose target space is (roughly) the heterotic worldsheet.

To summarize, we propose that the peculiar behavior of N=2 heterotic strings upon compactification is related to similar phenomena in the context of BPS physics and string duality. We will return to both later.

### 3. The target space action

The physical degrees of freedom of the metric, $B$-field, and dilaton are encoded in the one-form potential $K$ (2.29)-(2.32); those of the gauge fields in a Yang scalar $\phi^a$ (2.27), (2.28); and the Rarita-Schwinger fields are reduced to the derivative of a set of spin one-half fields $\vartheta$. In this section we will derive all of the purely bosonic terms in the target space action governing the dynamics of these fields, as well as a few of the terms involving fermions. We start by reviewing some known features of the dynamics of N=2 strings.

One of the remarkable features of N=2 strings is the fact that while they generically have non-vanishing three-point couplings, higher-point functions vanish on-shell. This
phenomenon is due to a clash between the following two properties: $N=2$ string amplitudes exhibit the usual string theoretic exponential fall off at large transverse momentum transfer; at the same time they have a finite number of field theoretic degrees of freedom (before compactification). The incompatibility of the two forces all scattering amplitudes that depend non-trivially on kinematic invariants ($N \geq 4$ point functions) to vanish.

After compactification, one might have expected to find non-trivial higher-point functions since the spectrum is no longer field theoretic – in fact, as we’ve seen in section 2.4 the total density of states is string theoretic. However, the density of states that is relevant for the behavior of amplitudes at large momentum transfer is the density of states with \textit{fixed momentum and winding}. That density of states was shown in section 2.4 to grow like $\exp(\sqrt{M})$ and hence, is insufficient to give Regge behavior of amplitudes. Thus, despite the string theoretic spectrum, compactified $N=2$ strings also have vanishing $N \geq 4$ point functions.

The fact that most $N$ point functions vanish does not in general mean that the target space action is cubic. Iterating the three-point function to calculate an on-shell S-matrix element generically gives a non-zero local four-point coupling. To reproduce the vanishing four-point S-matrix of the string, one then has to add an explicit local four-point interaction to cancel the iterated three-point coupling. Repeating this procedure, one generically finds a non-polynomial action, which in principle can be determined recursively from the three-point function as indicated above. In practice, there are subtleties having to do both with calculational difficulties at high orders, and the ambiguity of inferring an off shell irreducible vertex from the knowledge of an on-shell counterterm (the two difficulties are of course related). As we’ll see, symmetries provide an important guide and sometimes allow one to solve the problem.

The simplest case is that of the $N=(2,2)$ string \cite{11}. The iteration of the three-point function vanishes on-shell due to kinematic identities in 2+2 dimensions; there is no local four-point contact term, and all higher-point contact interactions vanish as well. At the same time, the $N=(2,2)$ sigma model in four dimensions has global $N=(4,4)$ supersymmetry (one easily constructs the additional currents by spectral flow \cite{22}), for which the beta functions are exact at one loop. All of this information points to the cubic Plebanski action for self-dual gravity as the target space theory \cite{11}:

$$S_{N=(2,2)} = \int I \wedge \partial \phi \wedge \partial \phi + \partial \phi \wedge \partial \phi \wedge \partial \bar{\partial} \phi,$$

(3.1)
where the Kähler form is $k = I + \partial \bar{\partial} \phi$. The variation of this action yields the equation $\det[g] = 1$; the beta function equations follow from $R = \partial \bar{\partial} \log \det[g] = 0$.

The situation for $\text{N}=(2,1)$ and $\text{N}=(2,0)$ strings is more complicated. The analysis of Ooguri and Vafa [12] indicates that, in order to enforce the vanishing of the S-matrix, contact terms must be introduced at each order to cancel the iteration of lower-point S-matrices. This is in agreement with the fact that the sigma-model has only global $\text{N}=(4,1)$ or $\text{N}=(4,0)$ supersymmetry, so the beta-functions might receive contributions to all orders in perturbation theory (c.f. [23]). In this section we will determine these counterterms. Our analysis will concentrate on $\text{N}=(2,1)$ strings; our methods are less powerful for $\text{N}=(2,0)$ strings, and we will highlight a few of the differences along the way.

The basic idea is to use the complementary information about the target space action encoded in the S-matrix elements and the sigma model beta functions; both encode the target space action, but in different ways. The $n$-point S-matrix yields terms with any number of derivatives and $n$ fields, while the $m^{\text{th}}$ order in the loop expansion of the beta-functions gives all powers in fields with $m$ derivatives. We will see that by comparing the two at low orders in the expansion, and in particular by using the fact that the full S-matrix vanishes, one can in some cases deduce the entire structure. We start with a discussion of the gravitational sector.

3.1. The gravitational sector

The three-point S-matrix of the gravitational modes (2.29), with momenta $k_r$ and polarizations $\xi_r$, $r = 1, 2, 3$, is given by

\[ \langle V_K(1)V_K(2)V_K(3) \rangle = [k_1 \cdot I \cdot k_3] \times [(\xi_1 \cdot \xi_2)(k_1 \cdot \xi_3) + (\xi_1 \cdot \xi_3)(k_2 \cdot \xi_2) + (\xi_2 \cdot \xi_3)(k_2 \cdot \xi_1)]. \quad (3.2) \]

Equation (3.2) factorizes into a right-moving contribution which is the standard one for $\text{N}=2$ strings, and a left-moving one which is identical to that of a gauge field in $\text{N}=1$ string theory. Indeed, the left-moving part of (2.29) is just that of an $\text{N}=1$ gauge boson, with the standard gauge invariance $K_\mu \rightarrow K_\mu + \partial_\mu \epsilon$; here this symmetry corresponds to a redundancy of the description of the metric in terms of $K_\mu$ (2.31)\footnote{In [12], a nonabelian form of this symmetry is suggested; this transformation is not compatible with the vanishing of the S-matrix. One can also see that the symmetry is abelian by an examination of the sigma-model action.}. The form of the
three-point interaction that gives (3.2) on-shell and preserves the above gauge invariance is:

\[ L_3 = \frac{1}{2} F_{\mu \nu} F_{\nu \lambda} I_{\lambda \rho} - \frac{1}{8} F_{\alpha \beta} F_{\beta \alpha} I_{\mu \nu} F_{\nu \mu}, \quad (3.3) \]

where \( F_{\mu \nu} \) determines the geometry via (2.30)-(2.32). A useful observation for the subsequent discussion is that, in coordinates that diagonalize \( I_{\mu \nu} \), \( L_3 \) can be rewritten as

\[ L_3 = F_{i \bar{j}} F_{j \bar{k}} F_{k \bar{i}} - \frac{1}{2} F_{i \bar{i}} F_{j \bar{j}} F_{k \bar{k}}. \quad (3.4) \]

In particular, it is independent of \( F_{ij}, F_{i \bar{j}} \) when \( I \) is diagonal. For simplicity, let us temporarily work with such a complex structure.

Another important fact about (3.2), (3.3) is that the cubic vertex has the qualitative form \( L_3 \sim \xi^3 k^3 \), i.e. it is cubic in fields and momenta. Power counting then implies that iterating the cubic vertex leads to a four-point coupling that behaves as \( \xi^4 k^4 \) (two factors of \( k^3 \) from the vertices and \( 1/k^2 \) from the propagator), and more generally the \( n \)-point coupling will go like \( \xi^nk^n \). As mentioned above, the \( 3 < n \)-point S-matrix elements all vanish; hence the above \( n \)-point couplings must be cancelled by explicit, local higher order contributions to the effective action. The power-counting argument implies that the necessary higher-order couplings go like \( L_n \sim (F_{\mu \nu})^n \) and hence are given exactly by a one-loop calculation in the \( N=(2,1) \) sigma model (2.29) (recall that the sigma model metric is essentially \( g_{\mu \nu} = \eta_{\mu \nu} + F_{\mu \nu} \)). Fortunately, the necessary calculations have already been done \([19,24]\); let us recall the results. The connection with torsion \( \Gamma^\mu_{\nu \lambda} \) for (2.32) takes the form

\[ \Gamma_{j k}^i = g^{i \bar{\ell}} g_{k \bar{\ell}, j}, \quad \Gamma_{\bar{j} \bar{k}}^i = g^{i \bar{\ell}} [g_{j \bar{\ell}, \bar{k}} - g_{j k, \bar{\ell}}]. \quad (3.5) \]

The standard beta-function equations \( R_{\mu \nu} [\Gamma] = \nabla_\mu \nabla_\nu \Phi \) can be integrated once or twice, and yield the set of equations

\[ \Gamma_{\mu} = 0 \]

\[ \log \det [g_{ij}] = 2\Phi, \quad (3.6) \]

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16 In contrast, the analogous three-point function in the \( N=(2,0) \) string has a piece proportional to \( \xi^3 k^5 \), which would invalidate the simplest version of the subsequent discussion. It is not difficult to see that the origin of this term is the gravitational Chern-Simons contribution to the sigma-model anomaly, which is cancelled in the (2,1) string by the left-moving superpartners \( \psi^\mu \) of the target space coordinates \( x^\mu \); on the other hand, this anomaly does contribute to the (2,0) theory by the usual shift of the \( B \)-field.
where $\Gamma_\mu = \Gamma_{\nu\rho}^\rho I_\nu$, and $\Phi$ is the dilaton. The second equation in (3.6) fixes the dilaton in terms of the metric. The first equation serves as an equation of motion for (2.32). One can show that, treating $K_\mu$ (2.29) as independent variables, this equation follows from the action

$$L_g = T\sqrt{\det[\eta_{ij} + 2F_{ij}]} . 
$$

(3.7)

The expression for a general complex structure is easily deduced by an $O(2,2)$ rotation. The tension $T$ is related to the N=2 string coupling, $T = 1/g_{\text{str}}^2$. Note that the determinant here is of a $2 \times 2$ matrix, not a $4 \times 4$ matrix. Another form of the action (3.7) incorporates the dilaton equation (the second of equations (3.6)), and will be useful in our discussion of the fermionic terms below:

$$L_g = e^{-\Phi}\det[g_{ij}] + T^2 e^\Phi . 
$$

(3.8)

Eliminating $\Phi$ by its algebraic equation of motion gives back (3.7). Expanding $L_g$ (equation (3.7)) in $F$, setting $T = 1$, we find for the first few nontrivial orders (and up to total derivatives)

$$L_g = -\frac{1}{4} F_{\mu
u} F_{\mu\nu} - \frac{1}{4} F_{\mu
u} \tilde{F}^{\mu\nu} + f A_{ij}A_{ji} - \frac{1}{2} (A_{ij}A_{ji})^2 - f^2 A_{ij}A_{ji} + \ldots , 
$$

(3.9)

where we have introduced the notation

$$f = F_{ii}$$

$$A_{ij} = F_{ij} - \frac{1}{2} \eta_{ij} f . 
$$

(3.10)

and $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$. The kinetic term in (3.9) gives the correct propagator for $K_\mu$, while the cubic interaction $f A^2$ can be checked to be equivalent to (3.4). We will check later that the quartic terms in (3.9) are exactly what is needed to cancel the effect of iterating the three-point coupling.

Note that (3.7) can be written as $L_g = [\det g_{\mu\nu}]^{1/4}$, where $g$ is the $4 \times 4$ Hermitian metric. It is curious that $L_g$ is not a density; the reason is that one has fixed complex coordinates, and furthermore one must make a holomorphic coordinate transformation to remove integration ‘constants’ of the form $h(x) + \bar{h}(\bar{x})$ in (3.9). The residual symmetry group of (3.7) is thus holomorphic area-preserving diffeomorphisms

$$x^i \rightarrow f^i(x^j) ; \quad |\det \partial_j f^i| = 1 . 
$$

(3.11)

17 This action for the gravitational dynamics was independently arrived at by Chris Hull [25].
It is interesting to note the progression of target space theories as the worldsheet supersymmetry is increased:

\[ R_{\mu\nu} = \nabla_{\mu} \nabla_{\nu} \Phi + \ldots , \quad N = (1, 1) \]
\[ \Gamma_{\mu} = \Gamma_{\nu} I_{\mu}^\nu = 0 \quad , \quad N = (2, 1) \]
\[ \det[g] = 1 \quad , \quad N = (2, 2) . \]  

(3.12)

3.2. Adding the self-dual gauge fields

The vertex operators probing the ‘internal space’ of N=2 heterotic strings describe self-dual gauge fields (c.f. section 2.3) [12]. The background field with canonical kinetic energy in the target space dynamics is the Yang scalar \( \phi^a \) of eq. (2.27), which parametrizes self-dual configurations. The linearized self-duality equations then reduce to \( \partial_i \partial_{\bar{i}} \phi^a = O(\phi^2) \).

The first nonlinear term on the RHS arises from the cubic S-matrix element

\[ \langle V_\phi(1)V_\phi(2)V_\phi(3) \rangle = [k_2 \cdot I \cdot k_3] \times [\zeta^a_1 \zeta^b_2 \zeta^c_3 f_{abc}] , \]  

(3.13)

corresponding to the interaction Lagrangian

\[ L_{3}^{ gauge} = \phi^a \partial_\mu \phi^b \partial_\nu \phi^c I_{\mu\nu} f_{abc} . \]  

(3.14)

The gauge fields also couple to the gravitational sector; the leading interaction is

\[ \langle V_\phi(1)V_\phi(2)V_K(3) \rangle = [k_2 \cdot I \cdot k_3] \times [\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot k_1] , \]  

(3.15)

arising from the interaction Lagrangian

\[ L_{3}^{ gauge-grav} = (F_{\mu\lambda} I_{\lambda\nu} - \frac{1}{2} \eta_{\mu\nu} F_{\alpha\beta} I_{\beta\alpha}) \partial_\nu \phi^a \partial_\mu \phi^a . \]  

(3.16)

Note that this last expression can be obtained from (3.3) by extending all indices to twelve dimensions, while keeping the coordinates four-dimensional (and of course \( I_{\mu\nu} \) nonzero only in four dimensions).

Ooguri and Vafa [12] analyzed the dynamics of the gauge field, ignoring gravity. This is self-consistent since the gauge-gravitational interactions are higher-order in derivatives. The vanishing of the four-point S-matrix (see also [28]) and the self-duality equations
(which are equivalent to the one-loop gauge field beta-function equations in four dimensions) lead to the effective action

\[ S_{\text{gauge}} = \int I \wedge \omega_{WZ} \]

\[ \omega_{WZ} = \int_0^1 dt \text{Tr}[\bar{\partial} \phi \wedge e^{-t\phi} \partial e^{t\phi}] \]

\[ = \text{Tr}\left[ \frac{1}{2} \bar{\partial} \phi \wedge \partial \phi + \sum_{n=3}^\infty \frac{1}{n!} \bar{\partial} \phi \wedge \cdots \left[ [\partial \phi, \phi], \phi \right] \cdots \right] , \]

which is the four-dimensional Wess-Zumino action [26,27]. To discern the coupling of the gauge sector to gravity, it is simpler to consider the abelian case (2.28). A power-counting argument similar to that employed in the previous subsection implies that n-point functions must depend on \( \phi^a \) only via \((\partial_\mu \phi^a)^n\), i.e. \((A^a_\mu)^n\). Thus, to determine the contribution of the \( \{\phi^a\} \) to the target space action, it is sufficient to consider the case of constant (or very slowly varying) gauge fields \( A^a_\mu \). In that limit, the only effect of the gauge fields is a shift (due to the sigma model anomaly) [18,29]

\[ g_{ij} \rightarrow g_{ij} + A^a_i A^a_j \]

\[ H = dB \rightarrow H + \omega^{YM}_{3}. \]

Thus the action for all propagating bosonic fields in this theory is

\[ \mathcal{L} = T \sqrt{\det[g_{ij}]} \]

\[ g_{ij} = \eta_{ij} + 2F_{ij} + 2\partial_i \phi^a \partial_j \phi^a. \]

This is our main result: The action of 2+2 M-branes bears an intriguing resemblance to the Nambu-Goto/Dirac-Born-Infeld actions describing D-branes. Specifically, it looks like a complexification of the effective action describing a D-string, but with two differences: 1) (3.19) is the exact target space action of the N=(2,1) string, while its analog for D-strings receives corrections both from higher orders in \( \alpha' \), and from interactions with closed string states that live in the bulk of spacetime; 2) the interpretation of the fields is different – the gauge field on the worldsheet of a D-string is replaced here by a Hermitian metric potential \( K_\mu \), while the scalars \( \phi^a \) parametrizing transverse motion of the D-string here arise as self-dual \( U(1)^8 \) gauge fields (2.28).

Expanding (3.19) to quartic order in the fields, we find (\( \mathcal{L}_g \) is given in equation (3.9)):

\[ \mathcal{L}_{\text{full}} = \mathcal{L}_g + \frac{1}{2} (\partial_\mu \phi^a \partial_\mu \phi^a) - 2A_{ij} \partial_j \phi^a \partial_i \phi^a \]

\[ + \frac{1}{2} (\partial_i \phi^a \partial_i \phi^a)^2 - \partial_i \phi^a \partial_j \phi^a \partial_j \phi^b \partial_i \phi^b + 2f A_{ij} \partial_j \phi^a \partial_i \phi^a + A_{ij} A_{ji} \partial_i \phi^a \partial_j \phi^a. \]
The cubic interaction in (3.20) is the same as that obtained from a direct examination of the S-matrix (3.4), (3.16). The quartic terms will be verified below as well.

The generalization of (3.19) to the nonabelian case is straightforward; one simply replaces $g_{ij}$ by

$$g_{ij} = \eta_{ij} + 2F_{ij} + 4\omega_{ij}^{WZ},$$

(3.21)

where $\omega_{ij}^{WZ}$ are the components of the Wess-Zumino two-form of (3.17). The lagrangian $L = T\sqrt{|g_{ij}|}$ then reproduces the two-derivative term (3.17) deduced by [12], as well as the correct cubic S-matrix and four-point contact terms. It is important to emphasize that the simple analysis performed above has led us to a rather striking prediction – that the lagrangian $L$ has in fact a vanishing S-matrix to all orders in the fields! As we’ll see momentarily, checking this statement even to quartic order is rather non-trivial, and we haven’t been able to prove this claim to all orders by a direct evaluation of Feynman diagrams. It is important to understand the reason for the vanishing; perhaps the symmetries (3.11) can be utilized to explain this remarkable result (c.f. [11,30] for a discussion of the N=(2,2) case).

The canonical dimensional reduction of the action (3.21) to two dimensions obtained by setting the imaginary part of $x_i$ to zero, gives the Nambu-Goto action for a string whose transverse coordinates lie in a group manifold. In the discussion of section 2 we saw that the Wess-Zumino term, determining the level of the resulting affine Lie algebra, played an important role. Can there be such a two-dimensional Wess-Zumino term in the target space theory? One can see such a term directly from the four-dimensional Wess-Zumino action of (3.17), using the general complex structure (2.21). The gauge vertex three-point function (3.14) in a theory with a non-trivial complex structure (2.21) (i.e. with $b \neq 0$) has a term:

$$b \phi^a \partial_i \phi^b \partial_j \phi^c \epsilon_{ij} f_{abc} + h.c.;$$

(3.22)

with the canonical null reduction $x^i = x^\tilde{i}$, this is precisely the leading term in the expansion of the two-dimensional Wess-Zumino term in powers of $\phi$.

3.3. A check of the action

We will now perform a check of the action (3.19) to quartic order. The cubic interactions in the Lagrangian are (see (3.10) for the notation)

$$L^{(3)} = f A_{i\bar{j}} A_{j\bar{i}} - 2A_{i\bar{j}} \partial_j \phi^a \partial_i \phi^a .$$

(3.23)
Iterating the three-point coupling yields four-point couplings of the form $(K_{\mu})^4$, $(K_{\mu})^2(\phi^n)^2$, and $(\phi^a)^4$. If the Lagrangian (3.19) is correct, two things must happen. First, all poles in the S-matrix must cancel on-shell. Second, any remaining local terms in the iterated three-point function must cancel against the explicit four-point couplings in (3.9), (3.20), such that the total four-point S-matrix vanishes.

To perform the check, one uses the propagators

\[
\langle K_{\mu}(k) K_{\nu}(-k) \rangle = \frac{\eta_{\mu\nu}}{k^2} \quad \text{(3.24)}
\]

\[
\langle \phi^a(k) \phi^b(-k) \rangle = \frac{\delta^{ab}}{k^2} .
\]

Using these propagators, one can show that for any two traceless $2 \times 2$ matrices $M_{ij}$, $N_{ij}$,

\[
M_{ij} A_{j\ell i} A_{\ell m} N_{m\ell} = \frac{1}{2} M_{ij} N_{ji} . \quad \text{(3.25)}
\]

The poles in the propagator (3.24) cancel and one gets a local expression (3.25). Using the identity (3.25), one can show that all the terms in $\langle \mathcal{L}^{(3)} \mathcal{L}^{(3)} \rangle$ which involve contractions of $A_{ij}$ with itself add up to

\[
\langle \mathcal{L}^{(3)} \mathcal{L}^{(3)} \rangle_{(AA)\text{terms}} = -2 \mathcal{L}^{(4)} , \quad \text{(3.26)}
\]

where

\[
\mathcal{L}^{(4)} = -\frac{1}{2} (A_{ij} A_{j\ell i})^2 - f^2 A_{ij} A_{j\ell i} + \frac{1}{2} (\partial_i \phi^a \partial_\ell \phi^a)^2 - \partial_i \phi^a \partial_\ell \phi^a \partial_j \phi^b \partial_\ell \phi^b + 2 f A_{ij} \partial_j \phi^a \partial_\ell \phi^a + A_{ij} A_{j\ell i} \partial_\ell \phi^a \partial_\ell \phi^a \quad \text{(3.27)}
\]

is the quartic interaction in (3.9), (3.20). This leaves terms obtained by contracting $\langle f A_{i\ell} \rangle$ or $\langle \phi^a \phi^b \rangle$. These all turn out to cancel amongst themselves. For example, consider the terms

\[
2 A_{ij} A_{j\ell i} \frac{1}{2} \partial_k \partial_\ell \phi^a \partial_k \partial_\ell \phi^a + A_{ij} \partial_\ell \partial_j \phi^a \frac{1}{2} A_{k\ell} \partial_k \partial_\ell \phi^a \quad \text{(3.28)}
\]

that appear in the scattering of two gravitons and two scalars. The first term comes from a contraction $\langle f A_{k\ell} \rangle$, the second from $\langle \phi^a \phi^b \rangle$. One can show that the combination of these terms vanishes on-shell, by repeated use of the kinematic identities [11,28]

\[
\frac{(k_1 \cdot \bar{k}_2) (k_4 \cdot \bar{k}_3)}{s_{12}} = -\frac{(k_1 \cdot \bar{k}_3) (k_4 \cdot \bar{k}_2)}{s_{13}} = -\frac{(k_1 \cdot \bar{k}_4) (k_2 \cdot \bar{k}_3)}{s_{14}} - k_1 \cdot \bar{k}_3 , \quad \text{(3.29)}
\]

and other identities obtained by antisymmetrizing on three holomorphic or three antiholomorphic indices. Here $s_{\alpha\beta} = k_{\alpha} \cdot \bar{k}_{\beta} + k_{\beta} \cdot \bar{k}_{\alpha}$ are the Mandelstam variables. This sort of highly nontrivial cancellation is one of the seemingly miraculous properties of self-dual gauge systems in four dimensions. Given this result, equation (3.26) then implies that the quartic terms obtained by iterating the three-point function cancel the explicit four-point couplings in (3.19).
3.4. Fermionic terms

So far we have concentrated on the bosonic terms in the action. The interaction arising from the cubic S-matrix involving fermions\footnote{The origin of the fermions is a four-dimensional Rarita-Schwinger field $\chi_\mu^a = I_\mu^a \partial_\nu \vartheta^a$. The fermion $\vartheta$ is a potential for it, just as $K_\mu$, $\phi^a$ are potentials for $g$, $b$, and $A$.}

\[
\langle \vartheta(1) V_K(2) \vartheta(3) + \vartheta(1) \vartheta(2) \vartheta(3) \rangle = [k_2 \cdot I \cdot k_3] \times [\bar{u}_1 (\xi_2^a \Gamma_\mu + \zeta_3^a \Gamma_a) u_3]
\]  

(3.30)

can be obtained from the cubic Lagrangian:

\[
\mathcal{L}^{(3)} = \bar{\vartheta} \Gamma^\mu \partial_\nu \vartheta I_{\lambda \mu} + \bar{\vartheta} \Gamma^a \partial_\nu \vartheta I^{\nu \lambda} \partial_\lambda \phi^a .
\]  

(3.31)

By analogy with D-strings and D-twobranes, we would expect the fermions to lead to two modifications:

1) In (3.19) we should replace:

\[
\partial_i \phi^a \rightarrow \Pi_i^a \equiv \partial_i \phi^a - \bar{\vartheta} \Gamma^a \partial_i \vartheta
\]

\[
\bar{\partial}_i \phi^a \rightarrow \Pi_i^a \equiv \bar{\partial}_i \phi^a - \bar{\vartheta} \Gamma^a \partial_i \vartheta
\]  

(3.32)

2) The volume term (3.19) should be corrected by a Wess-Zumino term, which will be needed for the $\kappa$-symmetry expected in a covariant formulation.

The cubic interaction (3.31) is compatible with the extension (3.32); making this substitution in (3.19) (or its expansion (3.20)), one finds complete agreement to this order for the interaction of the fermions with the transverse scalars $\phi^a$. The first term in (3.31) also appears in the expansion of (3.20) about a general complex structure, with the assumption that the $n_{ij}$ term in (3.19) arises as the expectation value of a covariant $\Pi_{\mu i}^a$ \textit{(i.e.} including longitudinal scalars $\phi^\mu$) in static gauge, $\langle \Pi_{\mu i}^a \rangle = \delta_{\mu i}^a$. However, we have not checked the precise coefficients.

On very general grounds, one would expect the target theory to have a local fermionic invariance. String theory always contains gravity in its target space; supersymmetric string theories always have a local fermionic symmetry whose generators close on the diffeomorphism generators. Usually this local fermionic symmetry is gauged supersymmetry; in the present case, the natural candidate is $\kappa$-symmetry\footnote{String theory presents these symmetries in a fixed gauge. One is used to conformal gauge on the worldsheet admitting a linearized gauge symmetry in target space; in the N=2 string, the additional BRST constraints on the gauge parameter eliminate such a transformation.}. A concrete indication that the
action results from the gauge-fixing of a $\kappa$-symmetry comes from the linear spacetime supersymmetry of the N=(2,1) string. A superspace completion of the form (3.32) results in a theory with global fermionic invariance

$$\delta \vartheta = \epsilon$$

$$\delta \phi^a = \epsilon \Gamma^a \vartheta$$ \hspace{1cm} (3.33)

in static gauge, this symmetry can be used to cancel the inhomogeneous part of the global remnant of the $\kappa$-symmetry transformation

$$\delta \vartheta = (1 + \Gamma_{p+1})\kappa$$

$$\delta \phi^a = (\delta \bar{\vartheta})\Gamma^a \vartheta$$ \hspace{1cm} (3.34)

leading to a linear supersymmetry on the worldvolume. Here $\Gamma_{p+1} = \Pi^{a_1} \cdots \Pi^{a_{p+1}} \Gamma_{a_1 \cdots a_{p+1}}$ is the static gauge worldvolume chirality operator. Such a linear supersymmetry appears in the Green-Schwarz string [31]; a general argument is given in [32]. On the static gauge M-brane, we indeed see the linear supersymmetry transformations generated by (2.15),

$$\delta K^\mu = \bar{\eta} \Gamma^\mu \vartheta$$

$$\delta \phi^a = \bar{\eta} \Gamma^a \vartheta$$

$$\delta \vartheta = (F_{\mu \nu} \Gamma^{\mu \nu} + \partial_\mu \phi^a \Gamma^\mu \Gamma^a) \eta$$ \hspace{1cm} (3.35)

These transformations receive nonlinear corrections at higher order in field strengths (see [32]).

Since we do not at the moment possess a fully covariant formalism for the theory, and sigma models with Ramond backgrounds are not as well understood as ones with Neveu-Schwarz backgrounds, we haven’t analyzed the fermionic terms to all orders in the fields. A cursory examination of the sort of four-point contact terms that must arise indicates that the extension of the Goldstone fields to their superspace counterparts is indeed occurring.

Another clue to the structure of the fermionic terms comes from the expected relation between the null reduction to 1+1 dimensions and the type IIB string [12]. Since the M-brane world-volume theory contains a Born-Infeld gauge field, we should look for a Dirac-Born-Infeld formulation of the Green-Schwarz string. Fortunately, this has been worked out by Bergshoeff, London, and Townsend [33] (BLT). The action takes the form

$$\mathcal{L}_{\text{IIB str}} = e^{-\Phi} \det[\Pi^2 \Pi^a + 2 F_{ij}]$$ \hspace{1cm} (3.36)
in terms of the superspace forms

\[ \Pi^a = d\phi^a - \bar{\vartheta}^r \Gamma^a d\vartheta^r \]

\[ F = dA + \left[ \frac{1}{2} d\phi^a \bar{\vartheta}^r \Gamma^a \tau^3_{rs} d\vartheta^s - \frac{1}{4} (\bar{\vartheta} \Gamma^a d\vartheta)(\bar{\vartheta} \Gamma^a \tau^3 d\vartheta) \right]. \]  

(3.37)

Here \( r, s = 1, 2 \) labels the two like-chirality (in spacetime) spinors of IIB supersymmetry, and \( \tau^3 \) is the usual Pauli matrix. Solving the equation of motion for \( A \) (which is not dynamical in 2d) gives

\[ \mathcal{L} = e^{-\Phi} \det[\Pi_i^a \Pi_j^a] - T^2 e^\Phi + T(F - dA) ; \]  

(3.38)

eliminating the auxiliary field \( \Phi \), one finds the conventional Green-Schwarz string action; (3.38) is a superspace version of (3.8), restricted to two dimensions.

An interesting feature of the action (3.36), to which we shall return below, is that it is spacetime scale invariant under the transformations \( \phi \to \lambda \phi, \vartheta \to \lambda^{1/2} \vartheta, A \to \lambda^2 A, \) and \( e^\Phi \to \lambda^4 e^\Phi \) \[33\]. The string tension \( T \) arises from the expectation value of \( F \), spontaneously breaking this symmetry.

Since the M-brane action is only known so far in static gauge, we should compare it to the action (3.38) gauged fixed via \( \partial_i \phi^a = \delta_i^a \), and \( \Gamma_2 \vartheta^r = \tau^3_{rs} \vartheta^s \) (for the definition of \( \Gamma_2 \), see the discussion following (3.34)). The superspace Wess-Zumino term of the Green-Schwarz string action has a cubic part

\[ \mathcal{L}_{3}^{WZ} = T e^{ij} \partial_i \phi^a \bar{\vartheta} \Gamma^a \partial_j \vartheta. \]  

(3.39)

In (3.38), this expression appears in the quantity \( F - dA \); in (3.19) and (3.31), it appears when expanding about a general complex structure. Note its similarity to (3.22); this means that instanton amplitudes of the N=2 string are a source of the 2d Green-Schwarz WZ term of the target worldsheet. Thus one might obtain this term from the superspace extension of (3.19) à la BLT, or alternatively from the complexification of (3.36); however, we do not yet know what the precise form of the action is that results in vanishing four-dimensional S-matrix as required by N=2 string dynamics.

The vanishing of the (2,1) string S-matrix together with the known ‘volume term’ (3.19) in principle determines the Wess-Zumino terms inductively, but the algebra looks complicated. Clearly a better geometrical understanding of the relation between the Rarita-Schwinger potential \( \vartheta \) and the sorts of ‘free differential algebras’ appearing in the dynamics of \( p \)-branes \[34,33\] would be helpful. The gauge corrections to the gravitational
action (3.7) were quite simply derived from the sigma-model anomaly. This is an anomaly in \( A \to A + d\Lambda \); in the self-dual case, one has \( \phi \to \phi + \Lambda \) at the linearized level, so the anomaly is connected to spontaneous breaking of translation symmetry by the extended object in our reinterpretation of the geometry. It is quite plausible that a similar approach will give the fermionic terms via a contribution of the \( \theta \)'s to the sigma-model anomaly, this time for local supersymmetry transformations in spacetime (i.e. (3.33) in the self-dual case). One needs to understand properly the spacetime supersymmetry current algebra on the (2,1) string worldsheet. It may be that some variant of the Green-Schwarz-Berkovits sigma model [35], which manifests more of the spacetime supersymmetry, might allow one to determine this structure more easily.

It is important to complete the determination of the fermionic terms in the M-brane worldvolume action; in particular, the Wess-Zumino terms should be the flat spacetime expectation values of superspace antisymmetric tensor fields, hence could help determine what sort of fields couple to the M-brane. Moreover, these terms will help in uncovering the \( \kappa \)-symmetry of a more covariant formulation of the M-brane worldvolume dynamics.

4. Discussion

The main result of this paper is the derivation of the exact target space action for the bosonic excitations of N=(2,1) heterotic strings in uncompactified space-time, \( \mathbb{R}^{2,2} \), given in equations (3.19), (3.21). The form of this action, and in particular its relation to the Nambu-Goto/Dirac-Born-Infeld action for strings, provides new evidence for the proposal of [3,1,2] that the target space dynamics of heterotic N=2 strings in their two dimensional vacua describes critical string worldsheets. It also opens the way for a more thorough investigation of the unification of string worldsheet and membrane worldvolume dynamics in the framework of self-dual gauge theory coupled to self-dual gravity in 2+2 dimensions, as described by N=2 heterotic strings.

In the course of the discussion, we have encountered a few interesting features of strings which our construction points to, and are worth summarizing briefly:

1) The string target space dynamics seems to mirror rather closely (and in some cases exactly) the worldsheet structure\(^\text{20}\). This may have important consequences for understanding target space dynamics in general, in particular the necessity to include target spaces of different topologies, the cosmological constant problem, etc.

\(^{20}\) Similar behavior appears to be exhibited by certain topological string theories [30].
2) In some of our constructions, the N=2 string coupling constant appears to be fixed by non-perturbative N=2 string considerations. This may be of interest to the question of vacuum selection in more realistic string theories.

3) The target space strings one gets from N=2 heterotic string are fixed at particularly symmetric points in the moduli spaces of vacua. This too may hint at the mechanism for vacuum selection in string theory. For instance, the initial state of the universe near the big bang might involve such an exceptional vacuum of string theory, compactified down to spatial dimensions of order the Planck scale and possessing a very stringy sort of enhanced symmetry.

4) The target space dynamics of N=2 heterotic strings possesses unbroken symmetries that act nonlinearly on the vertex operators (the small oscillations of target space fields), and lead to crucial dynamical effects. It is important to understand whether there are such symmetries in more realistic string theories, and what is their role in the dynamics.

In summary, N=(2,1) heterotic strings provide a simple model in which one might be able to address important issues in string dynamics in a controlled setting.

The 2+2d M-brane, as given to us by the target space dynamics of N=2 heterotic strings, seems to contain the appropriate degrees of freedom needed to provide a unified presentation of all classes of string vacua. In addition, one discovers a window into aspects of string theory that are not seen by other probes – hidden dimensions, hidden symmetries such as spacetime conformal invariance, and underlying structure on the worldsheet, to name a few. These may be features of string theory that are hard to detect except at the very symmetric vacua forced upon us by self-consistency of the N=(2,1) string. Let us discuss a few of them now.

4.1. Null reduction and conformal symmetry

The geometrical role of the null reduction (2.3), (2.4) has been a persistent puzzle in our construction. Is it a feature of spacetime or of the worldsheet? Is O(10,2) symmetry restored in some dynamical regime? Recently, B.E.W. Nilsson suggested to us [37] that null reduction is related to a conformally covariant formulation [38]. Dynamics of a point particle in (d–1)+1 dimensions can be reformulated in a d+2 dimensional spacetime which

\[\text{In previous work [23,39], we speculated that the extension of O(9,1) to O(10,2) is related to conformal symmetry; the present suggestion provides a concrete mechanism.}\]
linearly realizes the conformal group $O(d,2)$. To maintain equivalence to the usual particle dynamics, one needs additional first-class constraints:

\begin{align}
  p^2 &= 0 \\
  x \cdot p &= 0 \\
  x^2 &= 0.
\end{align} \tag{4.1}

The first of these is the usual mass-shell condition; the second constraint forces dynamics to be orthogonal to the $d+2$ null cone selected by the third constraint (for details of the gauge fixing, see [38]). Together these constraints form an $Sp(2)$ algebra in phase space. The symmetry under $x \leftrightarrow p$ is reminiscent of T-duality. The incorporation of spin follows similarly [38]; one adds a set of worldline fermions $\psi$ to represent the algebra of Dirac matrices, together with the constraints

\begin{align}
  p \cdot \psi &= 0 \\
  x \cdot \psi &= 0,
\end{align} \tag{4.2}

enlarging the algebra to $OSp(2|1)$. The first constraint is the Dirac equation; the second reduces the spinor content from $d+2$ to $d$ dimensions. Vector [38] and higher antisymmetric tensor fields [40] are similarly incorporated (for instance, the Lorentz gauge $\partial \cdot A = 0$ is augmented by $x \cdot A = 0$). Mass is represented in this framework as momentum in the hidden dimensions (the eigenvalue of the dilation operator).

The null reduction in $N=2$ heterotic string theory is remarkably similar to (4.1), (4.2); the main difference being that $x^2 = 0$ has been eliminated, perhaps by gauge fixing. What then remains are the Virasoro and super-Virasoro constraints $P^2 = 0$, $P \cdot \psi = 0$, and the null current and supercurrent conditions $v \cdot P = 0$, $v \cdot \psi = 0$.

This connection between null reduction and conformal invariance resonates with the results of Bergshoeff et al. described in section 3 (equation (3.36) and subsequent discussion), where the breaking of spacetime scale invariance is related to the expectation value of the Born-Infeld vector field strength. It also might provide a deeper explanation for the strong resemblance of the dilaton to a conformal compensator in all critical string theories, as well as the close connection between conformal and Poincare supergravity theories. In (4.1), [38], the mass scale is momentum in the hidden dimensions; in (3.36) [33], it is the expectation value of the Born-Infeld vector field; are the two related? We will argue shortly that they are indeed. It remains an open problem to formulate M-brane dynamics in a way that respects the above $OSp(2|1)$ symmetry, as well as $\kappa$-symmetry and reparametrization invariance.
4.2. Covariant formulation

What are the ingredients required for a covariant formulation of M-brane dynamics? N=(2,1) heterotic strings present the M-brane in a fixed (static) gauge, describing only the dynamics of the physical degrees of freedom. These consist of eight bosons and eight fermions on-shell. A covariant formulation requires reintroduction of the four longitudinal bosonic degrees of freedom eliminated via reparametrization gauge fixing. There are sixteen Green-Schwarz fermion fields appearing in the physical gauge M-brane dynamics (the Dirac equation reducing this to eight on-shell); covariant type II branes have 32 (two spinors of O(9,1)), compensated by a pair of eight-component local fermionic ‘κ-symmetries’. A further doubling to 64 is needed to have O(10,2) covariance before the null reduction.

Sixty-four component spinors are also the minimum required such that the supersymmetry algebra realizes either IIA or IIB supersymmetry with the same degrees of freedom, projected in different ways \[4,33\]. In other words, one null reduction of the covariant formalism should yield the D2-brane and IIA supersymmetry; another should yield the D-string and IIB supersymmetry.

Such a supersymmetry algebra has been investigated by Bars \[11\] (for earlier work, see for example \[42\]). In fact, the starting point of \[11\] is in some sense ‘11+2 dimensional’ in order to have manifest eleven-dimensional Lorentz covariance under null reduction to IIA/M-theory; IIB supersymmetry results from a 9+1/2+1 splitting of the algebra. An 11+2 superalgebra is not explicitly written; only its various type IIA/type IIB projections are discussed. These are

\[
\begin{align*}
\text{IIA:} & \quad \{Q_\alpha, Q_\beta\} = (C\Gamma^m)_{\alpha\beta}P_m + (C\Gamma^{m_1m_2})_{\alpha\beta}Z_{m_1m_2} + (C\Gamma^{m_1\cdots m_5})_{\alpha\beta}Z_{m_1\cdots m_5} \\
\text{IIB:} & \quad \{Q^{\bar{a}}, Q^{\bar{b}}\} = \gamma^{\bar{a}\bar{b}}(ct_i)_{\bar{a}\bar{b}} Z_i^{(1)} + \gamma^{\bar{m_1}\cdots\bar{m_3}} c^{\bar{a}\bar{b}} Z_{\bar{m_1}\cdots\bar{m_3}} + \gamma^{\bar{m_1}\cdots\bar{m_5}} (ct_i)_{\bar{a}\bar{b}} Z_i^{(1)} \cdots Z_i^{(5)}.
\end{align*}
\]

\[(4.3)\]

Here $\Gamma$, $C$ are the $32\times32$ Dirac matrices and charge conjugation matrix for eleven dimensional supersymmetry; $\gamma$ are the chirally projected Dirac matrices relevant to ten dimensional IIB supersymmetry; and $\tau_i$ are Pauli matrices, and $c$ the charge conjugation matrix, acting on the flavor indices $\bar{a}, \bar{b} = 1, 2$ (the spinor indices of the ‘hidden dimensions’; see \[11\] for details). The $Z_{m_1\cdots m_p}$ are the various $p$-form central charges carried by the states of the theory. In the IIB reduction, the momentum is $P_m = Z_m^{(2)}$; the remaining one-form charges $Z_m^{(1)}$, $Z_m^{(3)}$ couple to the NS and RR $B$-fields of the IIB $F$- and D-strings, respectively. The algebra appears to admit an O(2,1) symmetry on the indices $i, \bar{a}, \bar{b}$; in string theory one only sees the O(2) that preserves the momentum $P_m$, and even this is
broken to the $\mathbb{Z}_2$ of electric-magnetic duality $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ inside the $\text{SL}(2,\mathbb{Z})$ S-duality symmetry of type IIB.

To facilitate comparison to our work, for the remainder of this subsection let us adopt the conventions and notation of [41]: Timelike indices are labelled 0,0'; spacelike indices 1,2,...,11. As an ansatz, identify coordinates $0',0,1,...,9,11$ as the 10+2 spacetime dimensions of the M-brane formalism, and coordinate 10 as part of the field space of the Born-Infeld vector field. Orient a 2+2 brane in the hyperplane spanned by 0,0',9,11; let the space transverse to the brane be spanned by indices 1,2,...,8. This leaves the 10 direction, which carries the IIA interpretation as the tenth spatial coordinate of ‘visible’ spacetime, related to the Born-Infeld vector field by a duality transformation on the two-brane worldvolume. The zero mode of $F$ is momentum/winding in this extra dimension via $*F = d\phi$. We cannot have such an interpretation in the IIB theory, where the gauge field has no dual – only its flux is a physical degree of freedom. In order to maintain a uniform language for both IIA and IIB, we drop manifest 10+1 Lorentz symmetry in M-theory and use the vector field description, remembering that its flux represents motion/wrapping in an eleventh dimension which is sometimes physical and sometimes a gauge coordinate apart from its zero-mode.\textsuperscript{22} Thus this ‘dimension’ (the one with index 10) is real or hidden depending on the reduction. In what follows, we always associate the 10 ‘direction’ with the flux of the Born-Infeld vector field.

With these conventions, the D2-brane of the IIA theory results from null reduction along (for example) the 0'-8 plane; the two-brane lies along 0,9,11, with transverse space 1,2,...,7 (and one more from the vector field, \textit{i.e.} the 10 direction):

\[
\begin{pmatrix} 0' \\ B \end{pmatrix} \begin{array}{c|c} B & TTTTTT \begin{pmatrix} 0' \\ B \end{pmatrix} \end{array} \begin{pmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{array} \end{pmatrix} \begin{array}{c} B \\ F \\ B \end{array} \]

(4.4)

Here $B$ denotes a brane direction, $T$ a transverse direction, and $F$ the flux direction; parenthesis denote directions spanned by the null vector. On the other hand, the D1-brane of IIB results from splitting 0',10,11 from 0,1,...,9, and regarding the former as ‘hidden dimensions’. In the M-brane formalism, two of these three hidden IIB directions (0',11) are eliminated by the null reduction; the 10 coordinate is eliminated (apart from its

\textsuperscript{22} One might be able to regard this extra coordinate as the fiber of the line bundle whose connection is the Born-Infeld vector potential $K$, much as the transverse directions to the brane comprise the bundle $E$ described in the introduction.
zero mode) by the gauge symmetry of the Born-Infeld vector field. With the null reduction along the 0'-11 plane, the D-string is along 0, 9 with transverse space 1, 2, . . . , 8:

\[
\begin{pmatrix}
0' \\
B
\end{pmatrix} \begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
T & T & T & T & T & T & T & T & B & F & B
\end{array}
\]

(4.5)

The square brackets denote the ‘hidden’ nature of the 10 direction. As mentioned above, in the IIB algebra (4.3), the SO(2) that rotates the IIB string into the D-string is rotation in the 10-11 plane. Both of these directions are hidden (i.e., gauge coordinates) in the M-brane; one corresponds to the spatial coordinate of the M-brane worldvolume eliminated by the null reduction, the other to the Born-Infeld vector field on the effective D-string (for example, its spatial component in \( A_0 = 0 \) gauge). Just as 10+1d Lorentz symmetry is non-manifest in the IIA reduction, the SL(2,Z) S-duality of the type IIB reduction is non-manifest as well. Nevertheless, if the above identification is correct, we have found the relation of our framework to F-theory [8]. Note that the two hidden dimensions of F-theory are indeed spacelike here, and not the same as the extra 1+1 dimensions introduced via null reduction (one spatial coordinate is common to both). The identification of the Born-Infeld flux with one of the coordinates of the F-theory torus (whose modulus is the complex IIB coupling constant) is consistent with the fact that the electric charge of \((p, q)\) strings is represented on the D-string by this flux [43], while in F-theory this charge is supposed to be represented by the winding number of some brane (conjectured to be the self-dual IIB 3-brane) along the A-cycle of the hidden torus. In the work of BLT described in section 3, and in the formula for the string tension of \((p, q)\) strings [43,44,45], flux of the Born-Infeld vector field generates a shift of the string tension. Here we have regarded it as momentum/winding in a ‘hidden dimension’, very much in concert with the remarks of the previous subsection. In other words, in particle dynamics mass is momentum in the hidden directions, spontaneously breaking scale transformations in O(d,2); in string theory the analog is the string tension, which might again be interpreted as momentum/winding in hidden dimensions. Finally, note that we see hints of extra coordinates, and non-manifest symmetries beyond O(10,2), as advocated in [39].

4.3. Dynamics in the compactified case

When one (or more) of the spatial directions on \( \mathbb{R}^{2,2} \) is compactified, the theory starts probing the chiral, left-moving CFT that, on \( \mathbb{R}^{2,2} \), is frozen by the physical state conditions. This means, as discussed in section 2.4, that oscillations in these “internal”
directions can be excited and lead to the exponential density of states described after
equation (2.33). In addition, the effective dimensionality of the target space increases; to
see that, one notes that it is possible to generalize the states (2.12), (2.13) to:

\[ V_\xi = \int d\theta \int d^2 \bar{\theta} \xi_\mu \psi_\mu e^{ik \cdot x + i\bar{k} \cdot \bar{x}} \]  (4.6)

where \( \xi, k \) are 10+2 dimensional vectors satisfying the usual physical state and null reduc-
tion conditions, and the level matching condition \( k^2 = \bar{k}^2 \) now has solutions for arbitrary
momenta in the eight left-moving directions. Thus, the theory becomes effectively ten di-
menisonal! Of course, the additional allowed momenta lie on the \( E_8 \) torus, supporting the
picture advocated in section 2 – the internal space of the (2,1) string mirrors the spacetime
geometry.

Consider for concreteness the type IIB “target worldsheet” construction of Example
3 in section 2.2. Take the IIB worldsheet to be parametrized by \( (x^1, x^2) \) (i.e. gauge
\( J = \partial x^0 + \partial x^3 \)); compactify \( x^2 \) on a circle of radius \( R \). Denoting the momentum in the
\( (x^4, \ldots, x^{11}) \) directions (which lies in the \( E_8 \) root lattice) by \( \vec{p} \), and recalling that the left
and right moving momenta in the compact \( x^2 \) direction are \( k_2 = \frac{n}{R} - \frac{mR}{2} \); \( \bar{k}_2 = \frac{n}{R} + \frac{mR}{2} \),
level matching implies:

\[ (\vec{p})^2 = 2nm. \]  (4.7)

we see that the vertex operators \( V_\xi \) describe a 9 + 1 dimensional \( U(1) \) gauge field on a
compact space (the \( E_8 \) torus \( \times S^1 \), with the extra peculiarity that the \( S^1 \) momentum \( k^2 \)
actually corresponds to mixed momentum and winding). Of course, in addition to this
gauge field the spectrum also includes the excited states (2.33).

The fact that compactification leads to a higher dimensional “underlying” theory is
reminiscent of a similar phenomenon in \[ \] where compactification of M-theory on a \( p \)-
torus is described in terms of a \( p + 1 \) dimensional worldvolume (related by T-duality to
zero-brane quantum mechanics).

The dynamics of the 9+1 dimensional “gauge field” (4.6) and its massive relatives
(2.33) provides a highly non-trivial generalization of the dynamics studied earlier in section
3. As mentioned above, the only non-vanishing S-matrix elements are still the three-point
couplings. There is presumably a non-polynomial generalization of (3.19), (3.21), which
would be very interesting to find. To leading order, the dynamics of the gauge field (4.7)
is described by a 10+2 dimensional supersymmetric gauge theory with a null reduction to
9+1d. Such a theory has been recently studied in \[ \]
As discussed in section 2.4, compactified N=2 heterotic strings exhibit similarities to some properties of BPS states and string duality. We next comment further on these relations, starting with properties of BPS states.

Consider the heterotic string toroidally compactified on an even self-dual Narain torus, \( \Gamma_{24,8} \). From our discussion of N=2 strings we expect a special role to be played by the compactification for which \( \Gamma_{24,8} \) decomposes into separate even self-dual tori for the left and right movers, \( \Gamma_{24} \) and \( \Gamma_{8} \) respectively. The former is a Niemeier torus; the latter, the \( E_8 \) torus.

It is well known [21] that this system has an infinite number of BPS states of arbitrarily high mass, for which the (N=1 superconformal) right movers are in their ground state (with momentum \( \vec{p}_r \in \Gamma_{8} \)), while the (bosonic) left movers are excited at level \( N \) (and with \( \vec{p}_l \in \Gamma_{24} \)), such that:

\[
M^2 = (\vec{p}_r)^2 = (\vec{p}_l)^2 + 2(N - 1). \quad (4.8)
\]

One can focus on the physics of BPS states by noting that the right moving ground state in (4.8) in fact preserves N=2 worldsheet supersymmetry; gauging the global right moving N=2 superconformal symmetry leaves the BPS states while projecting out all non-BPS ones. In order for the resulting N=(2,0) string to be critical, one has to do two things:

1) Add two more (non-compact) dimensions to make the non-compact system 2+2 dimensional. Of course, the introduction of the additional coordinates is harmless, since they disappear upon the left moving null gauging discussed in section 2.

2) Remove the right moving \( \hat{c} = 8 \) system corresponding to the \( E_8 \) torus to make the N=2 string critical. One way to accomplish this is by a chiral topological twist [47]. The latter condition means that only states with \( \vec{p}_r = 0 \) survive the gauging. Clearly, from (4.8), there is a finite number of such states, however upon further compactification one finds a large subset of the BPS states, namely those with vanishing charges in the \( \Gamma_{8} \) directions but arbitrary left moving excitation.

The N=(2,0) heterotic string can be thought of describing the physics of those states\(^{23}\). In particular, the algebra of BPS states \([4]\) can be thought of (for the appropriate subset of BPS states) as the OPE algebra of physical N=(2,0) string states. In other words, the BPS algebra is merely the three-point S-matrix of the N=2 string, its only nontrivial scattering amplitude! The kinematic condition that two BPS states make another on-shell

\[^{23}\] The remarks in the remainder of this subsection derive from discussions with G. Moore.
amounts in the N=2 string to the requirement that their momenta lie in the same self-dual null plane. The vanishing of the S-matrix for the tower of states \([4,6]\) should be related to a symmetry group, much larger than the usual symmetries of self-dual gravity and self-dual Yang-Mills (discussed for example in [11,27,30]). It is tempting to speculate, along the lines of [18,19], that these S-matrices are related to the structure constants of a large broken symmetry group of spacetime (\(e.g\). for the N=(2,0) string, self-dual Yang-Mills with gauge algebra the Fake Monster Lie Algebra). In this connection, it is interesting that denominator formulae corresponding to generalized Kac-Moody algebras may be obtained by evaluating the one loop partition sums of various N=(2,0) strings, using formulae of [7]. Consider the N=(2,0) string with internal sector compactified on the Leech torus, and four-dimensional spacetime \(\mathbb{R}^2 \times T^2\). It was pointed out in [1] (section 6.1) that the one-loop partition function is [50,7]

\[
Z_1(T, U) = \frac{1}{2} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \left( \sum_{(p_L, p_R)} q^{\frac{1}{2}p_L^2} q^{\frac{1}{2}p_R^2} \right) (J(\tau) + 24)
\]

\[
= -24 \log \left( \sqrt{T_2} |\eta(T)|^2 \sqrt{U_2} |\eta(U)|^2 \right) - \log |J(T) - J(U)|^2 .
\]

Here \(T, U\) are the complex structure and Kähler moduli of the \(T^2\); \(J(\tau)\) is the unique modular form with both a simple pole of unit residue and no constant term in its \(q\)-expansion. This particular compactification is a special point in the Narain moduli space \(\Gamma^{2,26}\) of the N=(2,0) string. From [31], [7], we know that this result extends uniquely to an automorphic form over the full Narain moduli space; this automorphic form is the denominator formula for the so-called Fake Monster Lie Algebra. Repeating the same exercise with the Monster module as the internal space yields the denominator product for the Monster Lie Algebra.

A similar relation exists between the sector of BPS states of type II superstrings compactified on \(\Gamma^{8,8} = \Gamma^8 \oplus \Gamma^8\) and the N=(2,1) string. The (2,1) string on the spatial Narain torus \(\Gamma^{2,10}\) involves the appropriate GKM superalgebra related to \(II^{1,9}\). If one factors out the trivial zero due to boson-fermion cancellation, one should obtain a denominator product for this superalgebra.

\[24\] The 24 here arises from the oscillator states at level one, the only massless states in a compactification on the Leech torus. Compactification on some other Niemeier torus would replace this number by \(C_0\), the dimension of the Niemeier group.
In the context of string duality, an interesting question concerns the relation of winding N=2 strings and dyonic strings (see [5] for a related discussion). To discuss this, we return to Example 3 of section 2.2. If the spatial direction of the IIB worldsheet, $x^2$, is compact, the eight bosonic and fermionic target space massless fields describing the IIB worldsheet acquire “magnetic” partners obtained by exchanging momentum and winding on the N=2 string. In addition we find a tower of mixed momentum winding states (2.33). It is natural to interpret the pure winding N=2 string states as describing a “magnetic” type IIB string, and the mixed momentum/winding ones as dyonic type IIB strings. Such an interpretation would provide a link between the physics of BPS states in space-time and string duality, and in particular explain the appearance of GKM algebras in both contexts.

4.4. Relation to other work

(a) Self-duality

As was mentioned in section 3, N=2 worldsheet supersymmetry in two complex target dimensions automatically extends to N=4 supersymmetry, with the additional currents built by U(1) spectral flow (see the discussion before equation (2.21)). Thus backgrounds of N=2 strings admit a triplet of complex structures (2.21); N=2 heterotic strings possess a torsion background $H$ that obstructs closure, $dI = H$ (c.f. [24]), and hence the hypercomplex geometry is not Kähler (although by abuse of language such geometries have been called ‘hyper-Kähler with torsion’).

The field-theoretic action (3.17) that describes the M-brane in the absence of self-dual gravity has been studied by Nair and Schiff [26], and by Losev, Moore, Nekrasov and Shatashvili (LMNS) [27]. This action is a natural lifting of the two-dimensional WZW theory to four dimensions; indeed, we have seen that null reduction of (3.17) reduces to the 2d chiral model (with a WZ term, depending on the relative orientation of the null reduction and the complex structure). Nair and Schiff and LMNS have shown that much of rational conformal field theory has a counterpart in higher dimensions. Many two-dimensional features lift to four dimensions – the Polyakov-Wiegmann identity, holomorphic anomalies, holomorphic factorization, current algebra Ward identities, $b$-$c$ systems, and so on. Symmetry considerations determine the form of an ‘algebraic sector’ of the theory (a certain subset of correlations [27]), although these do not provide enough data to reconstruct the full theory as is the case in two dimensions. Roughly speaking, the holomorphic gauge transformations one uses to derive the Ward identities depend on functions of two variables $(x^i, i = 1, 2)$, whereas the boundary data specifying the dynamics depend on functions of
three variables. In the case at hand, however, one is interested in the dimensional reduction of the dynamics to 3d or 2d, in which case the algebraic sector – with a few global considerations – might be sufficient.

LMNS have derived the target space action of open and closed N=(2,2) strings using the holomorphic anomaly of free fields in the presence of a background gauge and gravitational connection. The (2,2) string was obtained because the background geometry was chosen to be Kähler; one might hope that an extension to the hyper-complex case will generate the action (3.19). One might look for an analog of the equivalence between Nambu-Goto and Polyakov actions that exists in the two-dimensional quantum theory. The work of [27] shows this to be the case for the algebraic sector of the target space field theory of the N=(2,2) string. One caveat to this approach is the above-mentioned incompleteness of the algebraic sector; there might not be complete equivalence between the free-field dynamics and that of the effective gauge action (3.17). However, one property that one might be able to extract from the results of LMNS, appropriately extended to Hermitian metrics, is a determination of the critical dimension directly from anomalies in the M-brane field theory. Clearly the N=(2,1) string gives a 12d spacetime as the target space of the M-brane; one would like to see this as a consistency condition for the quantization of reparametrization and $\kappa$-symmetries, just as anomaly cancellation on the string generates the critical dimension d=10.

Apart from considerations of quantization, there is a new and unexpected integrability underlying membrane dynamics in special backgrounds, derived from self-duality. A standard technique for generating classical solutions to self-dual dynamics is the twistor transform. A wrinkle in the application of twistor ideas to the present case is again the presence of torsion in the geometry, which is involved in the integrability conditions on the triplet of complex structures. This situation has been considered by Howe and Papadopoulos [52], who have given a construction of the twistor space for the relevant hyper-complex geometry; it is again essentially a fibration of the four-manifold over the $\mathbb{CP}^1$ of complex structures (2.21). It would be interesting to apply the twistor transform to construct exact solutions of the 2+2d and 2+1d dynamics of brane waves governed by (3.19).

(b) M-theory as a matrix model

Banks et.al. [3] have proposed a definition of M-theory as a matrix model. In the regime where the matrices are approximately commutative, their eigenvalues describe the
dynamics of zero-brane ‘partons’ of an infinite momentum frame formulation of eleven-dimensional supergravity. When the commutators are large, the theory provides a regularized description of membranes. How might this proposal be related to the N=2 string?

There are a number of similarities between N=2 string theory and matrix models of (two dimensional) noncritical strings. Both have dynamics in only the center-of-mass degrees of freedom of the string. Both give rise to integrable field theories in target space. The S-matrix is largely trivial. Area-preserving diffeomorphisms seem to play a role in each. The noncritical string field theory and the matrix collective field are related by a nontrivial transform similar to a Backlund transformation; similarly, the integrable theory (3.19) is nontrivially related to the usual supermembrane field theory.

The N=(2,1) string describes a particular state in M-theory – a long, stretched membrane (in static gauge); N=(2,1) strings are the quanta of small excitations about this state. In a matrix model, the long stretched membrane would correspond to a particular master field of the large N limit. One might imagine that N=2 strings describe the perturbative quantization (in powers of the inverse membrane tension) of the collective field theory expanded about this master field, much as noncritical strings encode the perturbative expansion about the collective field of matrix quantum mechanics (the scaling limit of the eigenvalue distribution). It may well be that N=(2,1) strings also have a nonperturbative ambiguity like that of noncritical strings, here related to the fingering instability of membranes – something that would not be seen in the perturbation expansion about a long stretched membrane. It would also be interesting to see if, as one might expect on the basis of the general arguments of Shenker [53], there are large $O(\exp[-1/g_{str}])$ nonperturbative effects in the (2,1) string, which in this interpretation might be zero-brane/eigenvalue tunnelling processes.

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**Note added:** At the time of completion of the present work, there appeared a paper of Hewson and Perry [54] investigating a theory of 2+2 branes in 10+2 dimensions. It would be interesting to understand the relation of their work to ours. In particular, their construction is a ‘p-brane’, having no world-volume vector field, whereas ours is more along the lines of the D-brane.

**Appendix A. Null reductions and partition sums.**

The purpose of this Appendix is to comment on the different null reductions to 1+1 and 2+1 dimensions, and in particular on the question of modular invariance of the corresponding N=2 worldsheet theories. For simplicity, we discuss the vacua of the N=(2,0) string on Niemeier tori described in Example 1 of section 2.2. Generalization to other cases is straightforward.

Since we are gauging a null (anomaly free) symmetry, the only effect of the gauging on the N=2 heterotic string path integral on a worldsheet torus, is the imposition of the constraint \( \delta(p \cdot v) \) on the momentum \( p \) flowing around the torus. This constraint can be thought of as arising from the integral over the zero mode of the gauge field which couples to \( J \) (2.3). One might be worried that for the 2+1d null reduction, the procedure could break modular invariance, since it eliminates one of the twenty-four chiral scalars \( x^a \) (2.3), and there are of course no 23 dimensional even self-dual lattices. To examine this issue we study here in turn the partition sums corresponding to the 1+1 and 2+1 dimensional cases.

Consider first the null reduction (2.3), (2.4) to 1+1d, achieved by gauging e.g.:

\[
J = \partial x_1 + \partial x_3. \tag{A.1}
\]

The effect of (A.1) on the zero mode part of the torus path integral is the replacement:

\[
\int dp_1 dp_3 (q \bar{q})^{\frac{1}{2}(p_3^2 - p_1^2)} \rightarrow \int dp_1 dp_3 (q \bar{q})^{\frac{1}{2}(p_3^2 - p_1^2)} \delta(p_3 - p_1) \tag{A.2}
\]

Before the replacement, the zero mode integral over \( p_1, p_3 \) gave (after an appropriate Wick rotation) \( 1/\tau_2 \). After imposing the constraint, we can perform the integral over \( p_1, p_3 \).
are left with: $\int dp_3$. States corresponding to different $p_3$ are identified by spectral flow in the null $U(1)$, and thus this infinite degeneracy should be discarded. The net effect of the null reduction is to eliminate the zero modes of $x_1, x_3$. It is easy to see that the non-zero modes are eliminated as well. Hence, the full partition sum of the $\text{N}=\!(2,0)$ string whose momenta live on a Niemeier lattice $\Lambda$ is:

$$Z(\tau) = \frac{1}{\eta^{24}(\tau)} \sum_{p \in \Lambda} q^{\frac{1}{2}p^2}.$$  

(A.3)

The spectrum of physical states one reads off from (A.3) agrees with the analysis of section 2.

Reduction to 2+1d is achieved by gauging (say):

$$J = \partial x_1 + \vec{v} \cdot \vec{\partial} x.$$  

(A.4)

where $\vec{v}$ is a unit vector which points along the Niemeier torus associated with the twenty four chiral scalars $x^a$. After imposing the delta function $\delta(p_1 - \vec{v} \cdot \vec{p})$ in the path integral, and performing the integral over $p_1$, we now find the partition sum:

$$Z_0 = \frac{1}{\eta^{24}(\tau)} \sum_{p \in \Lambda} q^{\frac{1}{2}[p^2 - (\vec{p} \cdot \vec{v})^2]} \bar{q}^{-\frac{1}{2}(\vec{p} \cdot \vec{v})^2}$$  

(A.5)

The form in square brackets implies that one combination of the twenty-four scalars $x^a$ decouples, and we are left with an effectively 23 dimensional left moving lattice. However, the $\bar{q}$ dependence indicates that the missing combination reappears on the right moving side, with negative norm. Thus, overall, we find that momenta live on a 23+1 dimensional lattice where all twenty-four dimensions have positive norm, the first twenty-three trivially, and the twenty-fourth because of a cancellation of two minus signs. One is due to its right moving nature (as in Narain compactifications); the other to its being timelike (A.3).

Hence, the 2+1 dimensional vacuum leads to a modular invariant theory; however, the partition sum (A.3) makes it clear that the vacuum is unstable, due to a tachyonic divergence arising from states with $\vec{p} \cdot \vec{v} \neq 0$ (see the discussion in the text following eq. (2.8)).

Note that, as mentioned in the text, a potential 2+2 dimensional vacuum of $\text{N}=2$ strings is rendered inconsistent by the above analysis. Indeed, if we try to gauge, e.g.,

$$J = \partial x_4 + i \partial x_5,$$  

(A.6)

45
the constraint $p_4 + ip_5 = 0$ imposed on the path integral implies that we are summing over the sublattice of $\Lambda$ with $p_4 = p_5 = 0$. This twenty-two dimensional lattice does not lead to a modular invariant partition sum, and hence corresponds to an inconsistent theory.

Appendix B. Comments on quantization.

While (uncompactified) $N=2$ heterotic strings have a field theoretic spectrum, their quantization might nevertheless be subtle. We have seen one aspect of this in this paper: compactifying $N=2$ heterotic strings on a circle reveals a rich spectrum of states absent from the target field theoretic description. A related subtlety arises when one attempts to study the system at finite temperature by Euclideanizing time and making it compact.

In addition, it has been pointed out in the past (see e.g. [11]) that loop amplitudes in $N=2$ string theory appear to differ from those one would write down in the target space field theory with the same classical spectrum and interactions. As an example, at one loop, in addition to the usual Schwinger parameter, which corresponds to the imaginary part of the modulus of the torus $\tau$, one finds in $N=2$ string theory a second modulus $u$ related to the $U(1)$ gauge field (see section 2.1). Since this $U(1)$ modulus lives on the Jacobian of the torus, the measure for integrating over $u$ involves a factor of $\text{Im} \tau$, which affects power-counting of momentum integrals [11]. Moreover, modular invariance cuts off ultraviolet divergences in a very non field-theoretic way.

One can in fact see some of these issues classically. Consider an $N=2$ string in a linear dilaton background, $\Phi(x) = Q_\mu x^\mu = Q^i x^i + Q^\bar{i} x^\bar{i}$. Criticality requires that $Q^2 = Q \cdot \bar{Q} = 0$. As is well known, the linear dilaton modifies the superconformal generators (2.2) to:

$$\bar{T} = -\frac{1}{2} (\bar{\partial} x \cdot \bar{\partial} x + Q \cdot \bar{\partial}^2 x + \bar{\psi} \cdot \bar{\partial} \psi)$$

$$\bar{G}^\pm = (\eta_{\mu\nu} \pm I_{\mu\nu}) (\bar{\psi}^\mu \bar{\partial} x^\nu + Q^\mu \bar{\partial} \psi^\nu)$$

$$\bar{J} = \frac{1}{2} I_{\mu\nu} \bar{\psi}^\mu \bar{\psi}^\nu + I_{\mu\nu} Q^\mu \bar{\partial} x^\nu.$$  \hspace{1cm} (B.1)

The modified form of $\bar{J}$ leads to a change in the physical state conditions. While before, physical states such as (2.3), (2.7) etc. had to satisfy the physical state condition $k^2 = 0$, now one finds two conditions which can be written as:

$$k_\mu(k^\mu + Q^\mu) = 0; \quad k_\mu Q_\nu I^{\mu\nu} = 0$$  \hspace{1cm} (B.2)
or in complex coordinates:

\[ \mathbf{k} \cdot (\tilde{\mathbf{k}} + \tilde{\mathbf{Q}}) = 0 \; ; \; \tilde{\mathbf{k}} \cdot (\mathbf{k} + \mathbf{Q}) = 0. \]  
(B.3)

The theory behaves as if the kinetic operator itself has been complexified in going from 1+1 to 2+2d. Target space field theory in a linear dilaton background would reproduce naturally the first of equations (B.2). It remains to be understood how it knows about the second.
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