Very cost effective bipartition in $\Gamma(Z_n)$

Ravindra Kumar and Om Prakash

Abstract. Let $Z_n$ be the finite commutative ring of residue classes modulo $n$ and $\Gamma(Z_n)$ be its zero-divisor graph. The nilradical graph and non-nilradical graph of $Z_n$ are denoted by $N(Z_n)$ and $\Omega(Z_n)$ respectively. In 2012, Haynes et al. [5] introduced the concept of very cost effective graph. For a graph $G = (V, E)$ and a set of vertices $S \subseteq V$, a vertex $v \in S$ is said to be very cost effective if it is adjacent to more vertices in $V \setminus S$ than in $S$. A bipartition $\pi = \{S, V \setminus S\}$ is called very cost effective if both $S$ and $V \setminus S$ are very cost effective sets [3, 4]. In this paper, we investigate the very cost effective bipartition of $\Gamma(Z_n)$, where $n = p_1 p_2 \cdots p_m$, here all $p_i$'s are distinct primes. In addition, we discuss the cases in which $N(Z_n)$ and $\Omega(Z_n)$ graphs have very cost effective bipartition for different $n$. Finally, we derive some results for very cost effective bipartition of the Line graph and Total graph of $\Gamma(Z_n)$, denoted by $L(\Gamma(Z_n))$ and $T(\Gamma(Z_n))$ respectively.

1. INTRODUCTION

Let $R = Z_n$ be the finite commutative ring of residue classes modulo $n$ with identity $(1 \neq 0)$ and $\Gamma(Z_n)$ be its zero-divisor graph. The study of zero-divisor graphs of commutative rings reveals interesting relation between ring theory and graph theory. Because, algebraic tools help to understand graphs properties and vice versa. An element $z(\neq 0) \in R$ is said to be a zero-divisor if there exists a non-zero $r \in R$ such that $r z = 0$. The set of zero-divisors is denoted by $Z(R)$ and zero-divisor graph of $R$, $\Gamma(R)$, is the graph whose vertices are the zero-divisors of $R$ and its two vertices are connected by an edge if and only if their product is 0. In 1988, the concept of zero-divisor graph of a commutative ring was introduced by I. Beck [2] in context of coloring of rings and were redefined in 1999 by D. F Anderson and P. Livingston in [1]. For a graph $G = (V, E)$, the open neighborhood of a vertex $u \in V$ is the set

2010 Mathematics Subject Classification. 13M99, 05C25, 05C76.

Key words and phrases. Very cost effective bipartition, Zero divisor graph, nilradical graph, non-nilradical graph, line graph, Total graph.
$N(u) = \{ v \mid uv \in E \}$, and the closed neighborhood of $u$ is the set $N[u] = N(u) \cup \{ u \}$. In the same fashion, the open neighborhood of a set $S \subseteq V$ is the set $N(S) = \cup_{u \in S} N(u)$, and the closed neighborhood is the set $N[S] = N(S) \cup S$ respectively. The order of the open neighborhood of a vertex $u \in V$ is denoted by $|N(u)|$. The degree of a vertex $v$ in a graph $G$, denoted by $deg(v)$, is $|N(v)|$. For basic definitions and results on graph we refer [4].

In 2012, the concept of cost effective and very cost effective sets in graphs were introduced by Haynes et al. [5] and further studied in [6] for various graphs. A vertex $v$ in a set $S$ is said to be cost effective if it is adjacent to at least as many vertices in $V \setminus S$ as in $S$, that is, $|N(v) \cap S| \leq |N(v) \cap V \setminus S|$. A Vertex $v$ in a set $S$ is very cost effective if it is adjacent to more vertices in $V \setminus S$ than in $S$, that is, $|N(v) \cap S| < |N(v) \cap V \setminus S|$. A set $S$ is (very) cost effective if every vertex $v \in S$ is (very) cost effective. Moreover, very cost effective bipartition were also introduced in [5]. A bipartition $\pi = \{ S, V \setminus S \}$ is called cost effective if each of $S$ and $V \setminus S$ is cost effective, and $\pi$ is very cost effective if each of $S$ and $V \setminus S$ is very cost effective. Graphs that have a (very) cost effective bipartition are called (very) cost effective graphs. It was shown in [5] that every connected, non-trivial graph is cost effective. Also, they observed in [6] that all bipartite graphs with no isolated vertices are very cost effective.

A line graph $L(G)$ of a simple graph $G$ is obtained by associating a vertex with each edge of the graph $G$ and connecting two vertices by an edge if and only if the corresponding edges of $G$ have a vertex in common. The total graph $T(G)$ of the graph $G$ has a vertex for each edge and each vertex of $G$ and an edge in $T(G)$ for every edge-edge, vertex-edge, and vertex-vertex adjacency in $G$. Here, we illustrate an example of a Zero divisor graph and its line graph over ring $\mathbb{Z}_{16}$:

![Figure 1 $\Gamma(\mathbb{Z}_{16})$](image-url)
VERY COST EFFECTIVE BIPARTITION IN $\Gamma(Z_n)$

2. Bipartition in $\Gamma(Z_n)$ and $L(\Gamma(Z_n))$

Let $\pi = \{R, B\}$ be a bipartition of the graph $G$. If $\pi$ is a very cost effective bipartition of $G$, then we say that $G$ is very cost effective under $\pi$.

**Theorem 2.1.** If $n = p_1p_2...p_m, m \geq 1$ and $p_1 < p_2 < \ldots < p_m$ are primes, then $\Gamma(Z_n)$ is a very cost effective graph.

**Proof.** Let $n = p_1p_2...p_m, m \geq 1$ for distinct primes $p_1, p_2, \ldots, p_m$. Then all the zero divisor elements of $\mathbb{Z}_n$ are $p_1, 2p_1, \ldots, (p_2...p_m-1)p_1; p_2, 2p_2, \ldots, (p_1p_3 \ldots p_m - 1)p_2; \ldots, p_m, 2p_m, \ldots, (p_1p_2...p_{m-1} - 1)p_m$. Let $\pi = \{R, B\}$ be a bipartition of vertices in $\Gamma(Z_n)$. Suppose the set $R$ contains all the elements which are multiple of $p_m$ and the other set $B$ contains rest of elements. Then $B$ is an independent set because it does not contain any element which is multiple of $p_m$. Now, we take $u \in B$. Then there exists at least one element $v \in R$ such that $uv = 0$. Therefore, $|N(u) \cap B| = 0$ and $|N(u) \cap R| \geq 1$, for all $u \in B$. Hence, all the elements in set $B$ are very cost effective and thus the set $B$ is very cost effective set.

Now, we divide $R$ into two sets $R_1$ and $R_2$. $R_1$ is the subset of $R$ containing multiple of $p_m$ as well as some of $p_i's$ (not all at a time), for $i = 1, \ldots, m - 1$ and $R_2$ is containing those multiples of $p_m$ which are not multiple of any $p_i's, i = 1, \ldots, m - 1$. Then element $v \in R_2$ is not adjacent to any element in $R$. But, these elements are adjacent to the elements of set $B$, so $|N(v) \cap R| = 0$ and

![Figure 2 $L(\Gamma(Z_{16}))$](image_url)
$|N(v) \cap B| \geq 1$. Again, if $u \in R_1$, then number of vertices adjacent to $u$ in $R$ is $p_1 \cdot p_2 \cdots p_{m-1} = M$, where $p_i$ is the product of those $p_i$s $i = 1, \ldots, m - 1$ which are not available in $u$ and number of vertices which are adjacent to $u$ in $B$ is $(p_1p_2 \cdots p_m - 1 - M)$ where $p_i$ is the product of those $p_i$s $i = 1, \ldots, m - 1$ which are available in $u$. Since $|N(u) \cap R| = M$ and $|N(u) \cap B| > M$. So $|N(u) \cap R| < |N(u) \cap B|$. Therefore, the set $B$ is also very cost effective and the partition $\pi = \{R, B\}$ is very cost effective bipartition. Hence, the graph $\Gamma(Z_n)$ is very cost effective graph.

\section*{Corollary 2.2} $\Gamma(Z_n)$ is very cost effective graph, for $n = pq$, where $p, q$ are distinct primes.

\begin{proof}
For $n = pq$, there are two independent sets of vertices in $\Gamma(Z_n)$. One set contains multiple of $p$ and other multiple of $q$ respectively. Therefore, the graph $\Gamma(Z_n)$ is a complete bipartite graph. Since every bipartite graph without isolated vertices is very cost effective. Thus, $\Gamma(Z_n)$ is very cost effective graph.
\end{proof}

\section*{Theorem 2.3} Let $p$ and $q$ be distinct primes and $n$ a positive integer.
(i) If $n = p^2q$, $p, q \geq 2$, then $\Gamma(Z_n)$ is very cost effective.
(ii) If $n = p^2q^2$, $p, q \geq 3$ and $p < q$, then $\Gamma(Z_n)$ is very cost effective.

\begin{proof}
(i) Let $n = p^2q$, where $p, q$ be distinct primes. Certainly, zero divisor elements of $Z_n$ are either multiples of $p$ or multiples of $q$ or multiples of $pq$. Let $R$ be the set of vertices contains all those elements which are multiple of $q$ and $B$ contains all those elements which are multiple of $p$ but not $q$. Let $\pi = \{R, B\}$ be the bipartition of $V(\Gamma(Z_n))$. Then $B$ is an independent set. Now, take $u \in B$, then $|N(u) \cap B| = 0$ and $|N(u) \cap R| \geq 1$. In the set $R$, take $v \in R$. If $v$ is multiple of $pq$, then $|N(v) \cap B| = p - 2$ and $|N(v) \cap B| > p - 2$. Again, if $v$ is a multiple of $q$ only, then $|N(v) \cap R| = 0$ and $|N(v) \cap B| \geq 1$. Hence, from both the condition $|N(v) \cap B| < |N(v) \cap B|$. Therefore, the sets $R$ and $B$ are very cost effective and thus, the graph $\Gamma(Z_n)$ is very cost effective.

(ii) Let $n = p^2q^2$, where $p, q$ are distinct odd primes. Then the zero divisor elements are either multiple of $p$ or $q$ or both. Now, take a bipartition $\pi = \{R, B\}$ in such a way that set $R = R_1 \cup R_2 \cup R_3$ where $R_1 = \{v \in V(\Gamma(Z_{p^2q^2})) : p^2 | v\}$, $R_2 = \{v \in V(\Gamma(Z_{p^2q^2})) : p | v \text{ and } q \nmid v\}$, and $R_3 = \{v \in V(\Gamma(Z_{p^2q^2})) : pq | v \text{ and } p^2 \nmid v \text{ and } q^2 \nmid v\}$. Here, $R_3$ contains $\frac{p(p-2)+1}{2}$ number of vertices. Now, set $B = B_1 \cup B_2 \cup B_3$ where $B_1 = \{v \in V(\Gamma(Z_{p^2q^2})) : q^2 | v\}$, $B_2 = \{v \in V(\Gamma(Z_{p^2q^2})) : q \text{ and } p \nmid v\}$ and $B_3 = \{v \in V(\Gamma(Z_{p^2q^2})) : pq | v \text{ and } p^2 \nmid v \text{ and } q^2 \nmid v\}$. Here, $B_3$ contains $\frac{q(q-2)+1}{2}$ number of vertices. Let $u \in R$. If $u$ is not a multiple of $q$, then $|N(u) \cap R| = 0$ and $|N(u) \cap B| \geq 1$. Again, if $u$ is a multiple of $p$ as well as $q$, then $|N(u) \cap R| = \frac{pq - 1}{2}$ and $|N(u) \cap B| \geq \frac{pq - 1}{2}$. 

So $|N(u) \cap R| < |N(u) \cap B|$ and the set $R$ is very cost effective. Similarly, set $B$ is also very cost effective and the bipartition $\pi$ is very cost effective bipartition. Thus, the graph $\Gamma(Z)$ is very cost effective. \qed

**Theorem 2.4.** $L(\Gamma(Z))$ is very cost effective, Where $n = pq$, $p < q$ and $p, q$ are primes.

**Proof.** If $n = pq$, then zero divisor graph $\Gamma(Z)$ is a complete bipartite graph. So, there are two independent sets of vertices in which each vertex of a set is adjacent to every vertex of the other set. We draw the line graph of $\Gamma(Z)$ with $(p-1)(q-1)$ vertices. This line graph is $(p+q-4)$ regular graph. Let $[u_i, v_j] \in V(L(\Gamma(Z)))$, where $u_i^t$ are multiple of $p$ i.e $u_i = p \cdot i$, $1 \leq i \leq q-1$ and $v_j^t$ are multiple of $q$ i.e $v_j = q \cdot j$, $1 \leq j \leq p-1$. Now, Let $\pi = \{R, B\}$ be a bipartition of vertices in $L(\Gamma(Z))$. Now we shall prove that $\pi$ is very cost effective bipartition. Since each set $R$ and $B$ contain $\frac{(p-1)(q-1)}{2}$ vertices. Therefore, set $R$ contains $\{[u_1, v_k], [u_2, v_L], [u_3, v_k], [u_4, v_L], ..., [u_{q-2}, v_k], [u_{q-1}, v_L]\}$ where $1 \leq k \leq \frac{q-1}{2}$ and $\frac{p-1}{2} \leq L \leq p-1$ and $B$ contains $\{[u_1, v_L], [u_2, v_k], [u_3, v_L], [u_4, v_k], ..., [u_{q-2}, v_L], [u_{q-1}, v_k]\}$ where $1 \leq k \leq \frac{p-1}{2}$ and $\frac{q-1}{2} \leq L \leq p-1$. Now, take $[u_i, v_j] \in R$. Then $|N([u_i, v_j]) \cap R| = \frac{p-2}{2} - 3$ and $|N([u_i, v_j]) \cap B| = \frac{q+2}{2} - 1$. Therefore, $|N([u_i, v_j]) \cap R| < |N([u_i, v_j]) \cap B|$. Hence, $R$ is very cost effective set. Similarly, if we take $[u_i, v_j] \in B$, then $|N([u_i, v_j]) \cap R| < |N([u_i, v_j]) \cap B|$. Thus, the line graph $L(\Gamma(Z))$ is very cost effective graph. \qed

### 3. Bipartition in $N(Z)$ and $\Omega(Z)$

In 2008, Bishop et al. [3] introduced the concept of Nilradical graph and Non-Nilradical graph and further some work appeared in [7]. They defined these graphs as follows:

**Definition 3.1.** [3] The nilradical graph, denoted $N(R)$, is the graph whose vertices are the nonzero nilpotent elements of $R$ and two vertices are connected by an edge if and only if their product is 0.

**Definition 3.2.** [3] The non-nilradical graph, denoted $\Omega(R)$, is the graph whose vertices are the non-nilpotent zero-divisors of $R$ and where two vertices are connected by an edge if and only if their product is 0.

Fig 1 is serve an example of $N(Z_{16})$. Also the example of $\Omega(Z_{18})$ is given below:
Theorem 3.3. Let \( p, q \) be distinct primes and \( n \), a positive integer.

(i) If \( n = p^2 \) and \( p > 2 \), then \( N(\mathbb{Z}_n) \) is very cost effective graph.

(ii) If \( n = p^2q^2 \) and \( p, q \geq 2 \), then \( N(\mathbb{Z}_n) \) is very cost effective graph.

(iii) If \( n = p^3 \) and \( p > 2 \), then \( N(\mathbb{Z}_n) \) is very cost effective graph.

(iv) If \( n = p^2q \) and \( p > 2 \), then \( N(\mathbb{Z}_n) \) is very cost effective graph.

Proof. (i) Let \( n = p^2 \), where \( p \) is a prime number and \( p > 2 \). Then the nilpotent elements in \( \mathbb{Z}_n \) are \( p, 2p, \ldots, (p - 1)p \). So, the number of nilpotent elements is \( p - 1 \) and every element is adjacent to the other element. So, these \( (p - 1) \) elements forms a complete graph. Since \( p \) is prime, then \( p - 1 \) is even and every complete graph of even order is very cost effective. Thus, \( N(\mathbb{Z}_n) \) is very cost effective graph.

(ii) Let \( n = p^2q^2 \), where \( p, q \) are distinct primes. Then the nilpotent elements in \( \mathbb{Z}_n \) are multiple of \( pq \) and number of nilpotent elements are \( pq - 1 \). Since all the nilpotent elements are multiple of \( pq \), so every vertex is adjacent to the all other vertices in \( N(\mathbb{Z}_n) \). Therefore, \( (pq - 1) \) elements form a complete graph with \( pq - 1 \) vertices. Since \( p \) and \( q \) are prime number and \( p, q \geq 2 \), so \( pq - 1 \) is even number. Hence, the graph of \( N(\mathbb{Z}_n) \) is very cost effective.

(iii) If \( n = p^3 \), where \( p \) is prime number, then all the nilpotent elements are multiple of \( p \) and total number of nilpotent elements are \( p^2 - 1 \). Now, we take bipartition \( \pi = \{R, B\} \) of vertex set in \( N(\mathbb{Z}_n) \) such that \( R \) contains only those elements which are divisible by \( p \) but not by \( p^2 \) and the set \( B \) contains those elements which are divisible by only \( p^2 \). Then the set \( R \) has \( p(p - 1) \) elements and the set \( B \) has \( p - 1 \) elements. Now, set \( R \) is independent set and in \( B \), each vertex is adjacent to every other vertices in \( B \). Since all the elements of \( R \) is adjacent to all elements of set \( B \). Therefore, \( R \) is very cost effective set. Now, take \( u \in B \), then \( |N(u) \cap B| = p - 2 \) and \( |N(u) \cap R| = p(p - 1) \). Since \( |N(u) \cap B| < |N(u) \cap R| \), so set \( B \) is very cost effective. Hence, \( \pi = \{R, B\} \) is
VERY COST EFFECTIVE BIPARTITION IN $\Gamma(Z_n)$

A very cost effective bipartition and graph $N(Z_n)$ is very cost effective.

(iv) Let $n = p^2q$, where $p$ and $q$ are distinct prime number and $p \neq q$. Then the nilpotent elements of $N(Z_n)$ are multiples of $pq$ and the number of nilpotent elements are $p-1$. These $p-1$ elements are connected to each other so these $p-1$ vertices forms a complete graph. Since $p$ is an odd prime so $p-1$ is even and complete graph of even number is very cost effective. Hence, $N(Z_n)$ is very cost effective graph.

\[ \square \]

Theorem 3.4. If $p$ and $q$ are distinct prime number and $n$ is a positive integer, then $\Omega(Z_n)$ is not very cost effective graph, where $n = p^2q$.

Proof. Let $n = p^2q$, where $p$ and $q$ be a distinct primes. Then the non-nilradical elements are all zero-divisors which are not divisible by $pq$. Since these elements are not adjacents to themselves but the vertices which are multiple of $p^2$ is adjacent to multiple of $q$. Here, $p$ is also another vertex which is not adjacent to any other vertices. Therefore, $p$ is an isolated vertices. Hence $\Omega(Z_n)$ is not very cost effective graph.

\[ \square \]

Theorem 3.5. Let $p_1, p_2, ..., p_m$ be distinct prime and $n = p_1p_2...p_m, m \geq 1$. Then $\Omega(Z_n)$ is very cost effective graph.

Proof. Here, $\Omega(Z_n) = \Gamma(Z_n)$. So $\Omega(Z_n)$ is very cost effective graph.

\[ \square \]

4. Bipartition in $T(\Gamma(Z_n))$

In this section, we have studied the very cost effective properties of $T(\Gamma(Z_n))$ for $n = 2p$ and $n = pq$.

Theorem 4.1. Let $n = 2p$, $p$ be an odd prime. Then the total graph $T(\Gamma(Z_n))$ is not very cost effective.

Proof. If $n = 2p$, then $\Gamma(Z_n)$ is a star graph with $p$ vertices. Now, total graph of $\Gamma(Z_n)$ is a graph with $2p-1$ vertices in which $p-1$ vertices have degree two, $p-1$ vertices have degree $p$ and one vertex which is itself $p$ has degree $2p-2$ respectively. In order to prove $T(\Gamma(Z_n))$ is not very cost effective, let $T(\Gamma(Z_n))$ be a very cost effective graph. Then it has a very cost effective bipartition $\pi = \{R, B\}$. Take $u \in R$ and suppose $u$ is a vertex whose degree is 2. Then $|N(u) \cap R| = 0$ and $|N(u) \cap B| = 2$. Now, the vertex $p \in B$ because $u$ is adjacent to the vertex $p$. So, all the vertices whose degree are 2 belong to the set $R$ because these vertices are adjacent to vertex $p$. Also, rest of the $p-1$ vertices which are of the form $[2, p], [2, 2, p], ..., [2(p-1), p]$ will be in $B$. If any one of these vertices belongs to $R$, then $R$ is not very
cost effective set. So, take \([u_i, p] \in B\), where \(u_i = 2i, 1 \leq i \leq p - 1\). Then 
\(|N([u_i, p]) \cap B| = p - 1\) and \(|N([u_i, p]) \cap R| = 1\). Hence, the set \(B\) is not very 
cost effective set, which contradicts our assumption. Thus, the graph \(T(\Gamma(Z_n))\) is not very cost effective.

**Theorem 4.2.** Let \(n = pq\), \(p, q \geq 3\) and \(p < q\) are primes. Then total 
graph \(T(\Gamma(Z_n))\) is very cost effective.

**Proof.** If \(n = pq\), then \(\Gamma(Z_n)\) is a bipartite graph with \(p + q - 2\) vertices and \((p - 1)(q - 1)\) edges. Now, in total graph \(T(\Gamma(Z_n))\), there is \(pq - 1\) vertices in which \(q - 1\) vertices are multiples of \(p\) and each has degree \(2(p - 1)\). Also \(p - 1\) vertices which are multiples of \(q\) have degree \(2(q - 1)\). The vertices \([u_i, v_j] \in V(T(\Gamma(Z_n))), u_i = p,i, 1 \leq i \leq q - 1\) and \(v_j = q,j, 1 \leq j \leq p - 1\) have degree \(p + q - 2\). Suppose set \(R\) contains the element \(\{2, 2p, \ldots, (q - 1)p, u_1, v_k, u_2, v_k, \ldots, u_{q-1}, v_k\}\) and \(B\) contains \(\{q, 2q, \ldots, (p - 1)q, u_1, v_k, u_2, v_k, \ldots, u_{q-1}, v_k\}\) where \(1 \leq k \leq \frac{pq}{2}\), \(\frac{pq}{2} \leq L \leq p - 1\). Take bipartition \(\pi = \{R, B\}\). We have to show that this 
bipartition is very cost effective.

Now, consider \(u_i \in R\), then \(|N(u_i) \cap R| = p - 2\) and \(|N(u_i) \cap B| = p\). Also, 
for \([u_i, v_j] \in R\), we have \(|N([u_i, v_j]) \cap R| = \frac{pq - 4}{2}\) and \(|N([u_i, v_j]) \cap B| = \frac{pq + 1}{2}\). 
So, for every vertex \(v \in R\), \(|N(v) \cap R| < |N(v) \cap B|\) and the set \(R\) is very 
cost effective set. In the set \(B\), take \(v_j \in B\), then \(|N(v_j) \cap B| = q - 2\) and \(|N(v_j) \cap R| = q\). Again, take \([u_i, v_j] \in B\), we have \(|N([u_i, v_j]) \cap B| = \frac{pq - 4}{2}\) 
and \(|N([u_i, v_j]) \cap R| = \frac{pq + 1}{2}\). Therefore, \(|N(v) \cap B| < |N(v) \cap R|\) for every 
vertex \(v\) in \(B\). So set \(B\) is also very cost effective. Hence, the bipartition 
\(\pi\) is very cost effective bipartition and the total graph \(T(\Gamma(Z_n))\) is very cost 
effective graph.

**References**

[1] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, J. 
Algebra 217 (1999), 434-447.

[2] I. Beck, Coloring of commutative rings, J. Algebra 116 (1988), 208-226.

[3] A. Bishop, T. Cuchta, K. Lokken and O. Pechenik, The nilradical and non-nilradical 
graphs of commutative rings, Int. J. Algebra 2(20) (2008), 981-994.

[4] F. Harary, Graph Theory, Addison-Wesley Publishing Company Inc., Boston, 1969.

[5] T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi and T. L. McCoy, I. Vasylieva, Cost 
effective domination in graphs, Congr. Numer. 211 (2012), 197-209.

[6] T. W. Haynes, S. T. Hedetniemi and I. Vasylieva, Very cost effective bipartitions in 
graphs, AKCE Int. J. Graphs Comb. 12(2-3) (2015), 155-60.
[7] O. Prakash, S. Suthar and S. Chandra, Some properties of the nilradical and non-nilradical graphs over finite commutative ring $\mathbb{Z}_n$, Algebra Discrete Math. (Preprint-2016).

Department of Mathematics, IIT Patna, Bihta campus, Bihta-801 106
E-mail address: ravindra.pma15@iitp.ac.in

Department of Mathematics, IIT Patna, Bihta campus, Bihta-801 106
E-mail address: om@iitp.ac.in