The equivalence of the Szemerédi and Petruska conjecture and the maximum order of 3-uniform $\tau$-critical hypergraphs

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Abstract
The long-standing Szemerédi and Petruska conjecture (Studia Sci Math Hungar 7:363–374, 1972) was recently resolved asymptotically (Kézdy and Lehel in Discrete Math 346:113469, 2023). Several decades ago Gyárfás et al. (J Combin Theory Ser B 33:161–165, 1982) observed, via a straightforward but unpublished argument, that this conjecture is equivalent to the problem of determining the maximum order of a 3-uniform $\tau$-critical hypergraph. Consequently, an asymptotically tight upper bound for the maximum order of a 3-uniform $\tau$-critical hypergraph follows from our recent work, reawakening interest in this equivalence. In this companion paper we supply a simple proof of this equivalence. We also present related background with open problems, and mention combinatorial geometry applications of the Szemerédi and Petruska conjecture.

Keywords  Hypergraphs · Transversal number · Extremal set theory

Mathematics Subject Classification 05D05 · 05D15 · 05C65

1 Introduction

The two extremal problems indicated in the title are essentially the same through a straightforward complementary argument observed by Gyárfás et al. [4] decades ago. The long-standing Szemerédi and Petruska conjecture [10] was recently resolved asymptotically in [9]. As a corollary, an asymptotically tight upper bound follows for
the maximum order of a 3-uniform $\tau$-critical hypergraph. Weaker bounds were given earlier by using the theory of $\tau$-critical hypergraphs. In contrast, Kézdy and Lehel [9] apply the iterative technique introduced by Szemerédi and Petruska; the iterative private pair technique there is reconsidered, substantially refined, then ultimately combined with the skew version of Bollobás theorem [1] on cross-intersecting set pair systems. This strategy asymptotically resolves the Szemerédi and Petruska conjecture. The success of this approach has reawakened interest in the equivalence of the two problems mentioned in the title. This paper completes its companion [9] by supplying a simple proof of this equivalence.

In Sect. 2 we begin by recalling the Hajnal–Folkman lemma [5] which can be considered the forefather of all the extremal problems considered here, including the very general family of “arrow symbol” problems introduced by Erdős [2]. A special case of these latter problems was investigated by Szemerédi and Petruska [10] leading to their conjecture, see Sect. 5 (Conjecture 5.1). Sections 3 and 4 show that the two problems mentioned in the title are equivalent for general $r$-uniform hypergraphs (Proposition 4.3). It is important to emphasize that these ideas are not original. The observation that the two problems are equivalent goes back to the early work of Gyárfás et al. [4] and Tuza [11]. The equivalence was exploited, for $r = 3$, with a combinatorial geometry application by Jobson et al. [7, Lemma 3]. We conclude the note with open problems in Sect. 6. In particular, we propose a question that generalizes the Hajnal–Folkman lemma, interpreting it as a problem on the maximum cliques of $r$-uniform hypergraphs (Problem 6.2).

2 The Hajnal–Folkman lemma

The Hajnal–Folkman lemma states (see [2, 5]): If a graph has at most $2k - 1$ vertices, where $k$ is the maximum clique size, then its maximum cliques share a common vertex. Generalizing this lemma to set systems, Erdős [2] introduced an “arrow notation” for a class of extremal problems which we now describe.

For $1 \leq \ell \leq 3$ and $3 \leq r \leq k \leq n$, let $\mathcal{K} = \{N_1, \ldots, N_\ell\}$ be a family of sets containing at least $k$ elements and let $|\bigcup_{i=1}^\ell N_i| = n$. The family $\mathcal{K}$ generates an $r$-uniform hypergraph $H$ on vertex set $V = \bigcup_{i=1}^\ell N_i$ such that an $r$-element subset $R \subset V$ is an edge of $H$ if and only if $R \subset N_i$, for some $N_i \in \mathcal{K}$. In particular, each $N_i \in \mathcal{K}$ becomes a complete $r$-uniform subhypergraph of $H$ called a “clique”. Erdős’s arrow symbol $(n, k, t) \rightarrow u$ denotes the claim that for every $\mathcal{K}$, if the sets of $\mathcal{K}$ have no $t$-element transversal (a $t$-set meeting each $N_i$), then the hypergraph generated by $\mathcal{K}$ contains a $u$-clique (a clique with $u$ vertices).1

Accordingly, the form $(n, k, t) \rightarrow u$ means: there exists a family $\mathcal{K} = \{N_1, \ldots, N_\ell\}$ as above having no $t$-transversal, and such that the $r$-uniform hypergraph $H$ generated by $\mathcal{K}$ has $n$ vertices and contains no $u$-clique. One can consider this hypergraph $H$ as a “witness” for $(n, k, t) \rightarrow u$.

Referring to the negative form of the arrow symbol for $u = k + 1$ and $t = 1$, an $r$-uniform hypergraph $H$ of order $n$ is defined to be an $r$-uniform $(n, k)$-witness

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1 To emphasize fixed $r$ we abbreviated the syntax $(n, k, t, r) \rightarrow u$ introduced by Erdős [2] to $(n, k, t) \rightarrow u$. Springer
hypergraph (a witness for \((n, k, 1)^r \rightarrow k + 1\)) provided its clique number \(\omega(H) = k\) and the \(k\)-cliques of \(H\) have no common vertex.

The Hajnal–Folkman lemma states that \((2k - 1, k, 1)^2 \rightarrow k + 1\); consequently, a witness graph for \((n, k, 1)^2 \rightarrow k + 1\) has \(n \geq 2k\) vertices. Rewriting this in terms of \(m = n - k\), the lemma says that the order of an \((n, k)\)-witness graph is at most \(2m\), which bound is actually tight. The Szemerédi and Petruska conjecture concerns the case \(r = 3\). It states that, in terms of \(m = n - k\), the maximum order of a 3-uniform \((n, k)\)-witness hypergraph is \(\binom{m + 2}{2}\).

3 Complementarity

Here all definitions assume a fixed positive integer \(r \geq 2\). Let \(H = (V, E)\) be an \(r\)-uniform hypergraph. A set \(T \subset V\) meeting all edges of \(H\) is called a transversal set, a set \(S \subset V\) containing no edge of \(H\) is called an independent set, and a set \(N \subset V\) such that its every \(r\)-element subset is an edge of \(H\) defines a clique in \(H\).

The independence number \(\alpha(H)\) and the transversal number \(\tau(H)\) of \(H\) are defined as

\[
\alpha(H) = \max \{ |S| : S \subset V, \text{ if } R \subset S, |R| = r, \text{ then } R \notin E \},
\]

\[
\tau(H) = \min \{ |T| : T \subset V, \text{ each } e \cap T \neq \emptyset \text{ for each } e \in E \}.
\]

A transversal \(T\) such that \(|T| = \tau(H)\) is a minimum transversal of \(H\). The clique number of \(H\) is defined to be

\[
\omega(H) = \max \{ |N| : N \subset V, \text{ if } R \subset N, |R| = r, \text{ then } R \in E \}.
\]

Notice that, by definition, if \(T\) is a minimum transversal, then \(S = V \setminus T\) is a largest independent set. Therefore, we obtain \(\tau(H) = |V| - \alpha(H)\), a hypergraph extension of one of the graph identities due to Gallai [3].

Define \(\widehat{H} = (V, \widehat{E})\), where \(\widehat{E}\) contains all \(r\)-element subsets of \(V\) not in \(E\). Obviously, a clique of \(H\) is an independent set in \(\widehat{H}\), furthermore,

\[
\widehat{H} = H.
\]

Summarizing these complementarity properties we obtain:

**Observation 3.1** If \(H\) is an \(r\)-uniform hypergraph of order \(n\), then

\[
n = \alpha(H) + \tau(H) = \omega(\widehat{H}) + \tau(H) = \omega(H) + \tau(\widehat{H}).
\]
4 r-critical hypergraphs

A hypergraph $H = (V, E)$ is $r$-critical if it has no isolated vertex (that is, $\bigcup_{e \in E} e = V$) and $\tau(H - e) = \tau(H) - 1$, where $H - e$ is the partial hypergraph with vertex set $V$ and edge set $E \setminus e$.

Observation 4.1 In an $r$-uniform $\tau$-critical hypergraph ($r \geq 2$), for every vertex $v$ there is a minimum transversal containing $v$, and there is a minimum transversal not containing $v$.

Proof Recall that $v$ is not an isolated vertex and $r > 1$; let $e \in E$ be an arbitrary edge containing $v$, and let $v' \in e \setminus \{v\}$. Because $H$ is $\tau$-critical, $H - e$ has a transversal set $T \subset V$, $|T| = \tau(H - e) = \tau(H) - 1$. Obviously, $e \cap T = \emptyset$, therefore, $T \cup \{v\}$ and $T \cup \{v'\}$ are two minimum transversals of $H$, one containing $v$ and the other one avoiding $v$. \qed

A hypergraph $H = (V, E)$ is vertex critical if every $v \in V$ belongs to some minimum transversal of $H$. Notice that, due to Observation 4.1, for fixed transversal number $t$ the family of $\tau$-critical hypergraphs is contained in the family of vertex critical hypergraphs.

Proposition 4.2 (Gyárfás et al. [4]) A hypergraph $H = (V, E)$ is vertex critical if and only if every $\tau$-critical partial hypergraph $H' = (V', E')$ defined by $E' \subset E$, $V' = \bigcup_{e \in E'} e$, and such that $\tau(H') = \tau(H)$, satisfies $V' = V$.

Szemerédi and Petruska [10] proved a bound on the maximum order of $3$-uniform $(n, k)$-witness hypergraphs. Gyárfás et al. [4] investigated the maximum order, $v_{\text{max}}(r, t)$, of an $r$-uniform $\tau$-critical hypergraph $H$ with $\tau(H) = t$. It was observed in [4] that these two extremal problems are essentially the same, so both works address the same function $g(r, t)$ using different techniques. Here is a proof of this equivalence.

Proposition 4.3 $v_{\text{max}}(r, t) \leq g(r, t)$ for some function $g(r, t)$ if and only if each $r$-uniform $(n, k)$-witness hypergraph with $k = n - t$ satisfies $n \leq g(r, t)$.

Proof Let $H = (V, E)$ be an $r$-uniform $(n, k)$-witness hypergraph with $|V| = n$, and $\omega(H) = k = n - t$. By Observation 3.1, $k = \omega(H) = \alpha(\hat{H}) = n - \tau(\hat{H})$, hence $\tau(\hat{H}) = n - k = t$. Since the $k$-cliques of $H$ have no common vertex, every $x \in V$ belongs to the complement of some $k$-clique of $H$, that is to a minimum transversal of $\hat{H}$. Therefore $\hat{H}$ is vertex critical. Due to Proposition 4.2, $\hat{H}$ has an $r$-uniform $\tau$-critical partial hypergraph spanning $V$. Thus $n = |V| \leq v_{\text{max}}(r, t) \leq g(r, t)$ follows.

For the converse, assume to the contrary that $H$ is an $r$-uniform $\tau$-critical hypergraph of order $n = v_{\text{max}}(r, t) > g(r, t)$. By Observation 3.1, $\tau(H) = t$ implies $\omega(\hat{H}) = n - t$. By Observation 4.1, the minimum transversals of $H$ have no common vertex and their union covers $V$, therefore, the union of the $k$-cliques of $\hat{H}$ covers $V$ and these $k$-cliques have no common vertex. In other words, $\hat{H}$ is an $r$-uniform $(n, k)$-witness hypergraph, where $k = n - t$. Thus we obtain $g(r, t) < v_{\text{max}}(r, t) = |V| = n \leq g(r, t)$, a contradiction. \qed
5 The case $r = 3$

Szemerédi and Petruska [10] obtained the estimation $n \leq 8t^2 + 3t$ for the order of a 3-uniform $(n, k)$-witness hypergraph, which is equivalent with $v_{\text{max}}(3, t) \leq 8t^2 + 3t$, by Proposition 4.3. They gave a lower bound construction and made the tight conjecture that can be phrased as follows.

**Conjecture 5.1 (Szemerédi and Petruska [10])** For $n = k + t$, if $H$ is a 3-uniform $(n, k)$-witness hypergraph, then $n \leq \binom{t+1}{2}$.

The upper bound was improved by Gyárfás et al. [4] to $v_{\text{max}}(3, t) \leq 2t^2 + t$, and later by Tuza² to $v_{\text{max}}(3, t) \leq \frac{3}{4}t^2 + t + 1$. We have recently proved [9] that the Szemerédi and Petruska conjecture is asymptotically correct, which, due to Proposition 4.3, immediately implies the asymptotically tight bound $v_{\text{max}}(3, t) = \binom{t+2}{2} + O(t^{5/3})$.

The Szemerédi and Petruska conjecture was verified for $t = 2, 3$ and 4 by Jobson et al. [6]. The resolution of the conjecture has applications in combinatorial geometry problems related to a question posed by Petruska and another one by Eckhoff involving convex sets in the plane, see [7, 8].

6 Further problems

The early results due to Szemerédi and Petruska [10] motivated further research and led to unexpected applications. In the introduction to their 1972 paper (that is titled part I.) they write, “... for larger $r$ we get a more general problem and we are to return to it in a forthcoming paper”. As we learned³ a few years ago, they never revisited this work. So the innovative iterative approach introduced in their paper was never extended for $r > 3$; the technique was forgotten for a while. Their conjecture survived a decade later in a different setting as a problem on $r$-critical hypergraphs.

Gyárfás et al. [4] proved general bounds pertaining to $r$-uniform hypergraphs:

$\left(\frac{t + r - 2}{r - 1}\right) + (t + r - 2) \leq v_{\text{max}}(r, t) \leq t^{r-1} + t\left(\frac{t + r - 2}{r - 2}\right)$.

The question was asked there whether the correct value for $t \geq r$ (or the asymptotic) of $v_{\text{max}}(r, t)$ is the lower bound above (cf. Tuza [11, Problem 18]).

**Problem 6.1** Is $v_{\text{max}}(r, t) = \binom{t+r-2}{r-1} + (t+r-2) (t \geq r)$ true (or true asymptotically)?

We conclude the note by returning to the Hajnal–Folkman lemma, and extend it from graphs to a problem on hypergraphs. The original form of the lemma according to Hajnál [5] is as follows. If $k$ is the size of the maximum cliques in a graph on $n$ vertices, and $\{N_1, \ldots, N_\ell\}$ is the family of its $k$-cliques, then $\left|\bigcap_{i=1}^\ell N_i\right| \geq 2k - n$.

² Zs. Tuza, personal communication (2019).
³ Gy. Petruska, personal communication (2018).
Problem 6.2 If $k$ is the size of the maximum cliques in an $r$-uniform hypergraph on $n = k + m$ vertices, and $\{N_1, \ldots, N_\ell\}$ is the family of its $k$-cliques, then

$$\left| \bigcap_{i=1}^\ell N_i \right| \geq n - \left[ \binom{m + r - 2}{r - 1} + (m + r - 2) \right] ?$$

Notice that for $r = 2$ the solution of Problem 6.2 is the Hajnai–Folkman lemma; for $r = 3$ it is open, and implies the Szemerédi and Petruska conjecture.

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