The Birkhoff theorem and string clouds

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Abstract
We consider spherically symmetric space–times in GR under the unconventional assumptions that the spherical radius $r$ is either a constant or has a null gradient in the $(t, x)$ subspace orthogonal to the symmetry spheres (i.e., $(\partial r)^2 = 0$). It is shown that solutions to the Einstein equations with $r = \text{const}$ contain an extra (fourth) spatial or temporal Killing vector and thus satisfy the Birkhoff theorem under an additional physically motivated condition that the tangential pressure is functionally related to the energy density. This leads to solutions that directly generalize the Bertotti–Robinson, Nariai and Plebanski–Hacyan solutions. Under similar conditions, solutions with $(\partial r)^2 = 0$ but $r \neq \text{const}$, supported by an anisotropic fluid, contain a null Killing vector, which again indicates a Birkhoff-like behavior. Similar space–times supported by pure radiation (in particular, a massless radiative scalar field) contain a null Killing vector without additional assumptions, which leads to one more extension of the Birkhoff theorem. Exact radial wave solutions have been found (i) with an anisotropic fluid and (ii) with a gas of radially directed cosmic strings (or a ’string cloud’) combined with pure radiation. Furthermore, it is shown that a perfect fluid with isotropic pressure and a massive or self-interacting scalar field cannot be sources of gravitational fields with a null but nonzero gradient of $r$.

Keywords: general relativity, Birkhoff theorem, spherical symmetry, cosmic strings, anisotropic fluid, pure radiation

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1. Introduction

The original Birkhoff theorem [1–3] stated that in general relativity (GR) a spherically symmetric gravitational field in vacuum is necessarily static and thus reduces to the Schwarzschild solution.

It was later found that the full Schwarzschild space–time contains a T-region with a special Kantowski–Sachs type metric, and the theorem in fact tells us that a spherically symmetric vacuum field in a particular region is either static or homogeneous, i.e., the Einstein equations imply an additional space–time symmetry not postulated from the beginning. Theorems of this kind make easier the solution process in many physically important situations and provide their better understanding.

The Birkhoff theorem was later extended to different types of geometries (with spherical, planar, pseudospherical, cylindrical symmetries and diverse dimensions) and different material sources of gravity (the cosmological constant, linear and nonlinear electromagnetic fields, scalar fields, gauge fields, perfect fluids, etc), see, e.g., [4–6] and references therein.

In particular, in [7, 8] some general conditions were found under which the field equations lead to independence of the metric from a spatial or temporal coordinate, hence the existence of an additional Killing vector field, absent in the original problem setting. The approach of [7, 8] was extended in [5] to multidimensional GR with diverse geometries and material sources of gravity.

More general geometric conditions for the emergence of an additional Killing vector, connected with the existence of a conformal Killing-Yano tensor, were recently found in [9, 10].

Analogs of the Birkhoff theorem have also been found in a variety of extensions of GR, see, e.g., the recent papers [11–13] and references therein. Extensions of the theorem to quantum gravity models have also been considered [14].

In the present study we return to spherically symmetric space–times in GR. Our goal is to make clear what happens if we cancel one of the conditions used in a simple proof of the theorem, namely, that the gradient of the spherical radius \( r \) as a function of the temporal \( (x^0 = t) \) and radial \( (x^1 = \chi) \) coordinates should not be null (that is, \((\partial r)^2 \equiv \partial_a r \partial^a r \neq 0\), \( a = 0, 1 \)). This question is of certain interest even despite the existence of more advanced Birkhoff-like theorems (e.g., [9, 10, 15, 16]) and though the possible geometries have been generally classified in [17, 18]; meanwhile, the possible material sources of such geometries have not been described.

If \((\partial r)^2 = 0\) (in particular, if \( r = \text{const} \)), the Einstein equations seem to be of wave nature, and it is tempting to try to obtain wave solutions indicating a non-Birkhoff situation. We will consider a sufficiently general choice of material sources of gravity and really obtain some simple exact solutions which seem to be new to our knowledge. It turns out that many kinds of matter are incompatible with the condition \((\partial r)^2 = 0\) (e.g., a perfect fluid and a minimally coupled scalar field with a nonzero self-interaction potential), while others, in particular, some kinds of anisotropic fluids, pure radiation and a massless scalar field with a null gradient, combined with a ‘gas’ or ‘cloud’ of radially directed cosmic strings, lead to geometries with a null Killing vector, indicating a behavior in the spirit of the Birkhoff theorem.

The paper is organized as follows. Section 2 presents the basic equations, section 3 briefly reproduces the well-known proof of the Birkhoff theorem and discusses its straightforward extensions. Section 4 is devoted to consequences of the condition \( r = \text{const} \), which is a special case of \((\partial r)^2 = 0\), while section 5 deals with a null but nonzero gradient of \( r(x, t) \).
In the latter case, exact wave solutions are obtained with such sources of gravity as an anisotropic fluid, a cosmological constant, an electromagnetic field, a string cloud, a massless scalar field and pure radiation. Section 6 summarizes and discusses the results.

2. Basic relations

The general spherically symmetric metric can be written in the form

$$d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2,$$

$$dx^2 = e^{2\alpha} dx^2 - e^{2\gamma} dx^2 - r^2 dt^2,$$

where $\alpha$, $\gamma$, $r$ are functions of $x$ and $t$, and $r$ is the spherical radius, i.e., the curvature radius of a coordinate sphere $x = \text{const}$, $t = \text{const}$. In (1) there is a freedom of choosing a reference frame (RF) and specific coordinates in a given RF (a congruence of timelike world lines) by postulating certain relations between the functions $\alpha$, $\gamma$, $r$.

The Einstein equations can be written in the form

$$R_{\mu}^{\nu} = -T_{\mu}^{\nu} \equiv -(T_{\mu}^{\nu} - \frac{1}{2} \delta_{\mu}^{\nu} T),$$

where $T_{\mu}^{\nu}$ is the stress–energy tensor (SET) of matter, and $T \equiv T_{\mu}^{\mu}$ is its trace. In particular, we consider the cosmological constant as a special kind of matter.

The nontrivial components of the Einstein equations for the metric (1) are

$$R_{0}^{0} = e^{-2\gamma} [2\dot{r}/r + \dot{\alpha} + \alpha^2 - \gamma (2\dot{r}/r + \alpha)]$$

$$- e^{-2\gamma} [\gamma'' + \gamma'(2\dot{r}/r + \gamma' - \alpha')] = -\ddot{T}_{0},$$

(3a)

$$R_{1}^{1} = e^{-2\gamma} [\dot{\alpha} + \dot{\gamma} (2\dot{r}/r - \dot{\gamma} + \dot{\alpha})]$$

$$- e^{-2\gamma} [2\dot{r}/r + \gamma'' + \gamma'^2 - \alpha' (2\dot{r}/r + \gamma')] = -\ddot{T}_{1},$$

(3b)

$$R_{2}^{2} = 1/r^2 + e^{-2\gamma} [\ddot{r}/r + (\dot{r}/r)(\dot{r}/r - \dot{\gamma} + \dot{\alpha})]$$

$$- e^{-2\gamma} [\ddot{r}/r + (\dot{r}/r)(\dot{r}/r + \gamma' - \alpha')] = -\ddot{T}_{2},$$

(3c)

$$R_{01} = (2/r)[\dot{r}' - \dot{\alpha} r' - \dot{r}''] = -\ddot{T}_{01},$$

(3d)

where dots and primes stand for $\partial/\partial t$ and $\partial/\partial x$, respectively.

The spherical radius $r(x, t)$ may be considered as a scalar field in the 2D subspace of our space–time with the coordinates $x^a = (x^0, x^1) = (t, x)$. Its gradient $\partial_t r$ may be spacelike, timelike or null.

We are going to explore what happens if we omit one of the conditions often used in formulations of the Birkhoff theorem, the condition that $\partial_t r$ is not null. On the contrary, we will assume that in the whole 4D space–time or its region

$$(\partial_t r)^2 \equiv e^{-2\gamma} \dot{r}^2 - e^{-2\alpha} \dot{r}^2 = 0.$$  (4)

Since any 2D Riemannian metric is locally conformally flat, the coordinates $x$ and $t$ can always be chosen so that $\gamma(x, t) \equiv \alpha(x, t)$, and then the condition (4) reduces to $\dot{r} = \pm \dot{r}'$. Without loss of generality let us choose the plus sign (otherwise we can simply re-denote $x \rightarrow -x$). With $\alpha = \gamma$ and $\dot{r} = \dot{r}'$, the Einstein equation (3) are substantially simplified:

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6 Our conventions are: the metric signature (+ − − −); the curvature tensor $R_{\mu\nu\rho\sigma} = \partial_{\rho} \Gamma_{\mu\nu}^{\gamma} - \partial_{\nu} \Gamma_{\mu\rho}^{\gamma} - \partial_{\rho} \Gamma_{\nu\mu}^{\gamma} + \partial_{\mu} \Gamma_{\nu\rho}^{\gamma}$, so that the Ricci scalar $R > 0$ for de Sitter space–time and the matter-dominated cosmological epoch; the system of units $8\pi G = c = 1$. 
\[ R_0^0 = e^{-2\alpha}(4\alpha_{\mu\nu} + R_{00}) = -\bar{T}_0^0, \]  
\[ R_1^1 = e^{-2\alpha}(4\alpha_{\mu\nu} - R_{00}) = -\bar{T}_1^1, \]  
\[ R_2^2 = 1/r^2 = -\bar{T}_2^2, \]  
\[ R_{01} = (2/r)[\alpha_{\mu\nu} - 2u_\mu\alpha_{\nu}] = -\bar{T}_{01}, \]

where we have introduced the null coordinates

\[ u = t + x, \quad v = t - x, \]

and the subscripts \( u \) and \( v \) denote \( \partial/\partial u \) and \( \partial/\partial v \), respectively. Since \( \dot{r} = r' \), we deal with \( r = r(u) \). The metric takes the form

\[
dx^2 = e^{2\alpha(t,x)}(dt^2 - dx^2) - r^2 \, d\Omega^2 = e^{2\alpha(u,v)}du \, dv - r^2(u) \, d\Omega^2. \]  

(In what follows, we still prefer to write the Einstein equations and the SET components referred to the coordinates \( t \) and \( x \) for physical convenience, to be able to discuss RFs, densities and pressures. The tensor indices 0 and 1 will thus refer, as before, to \( t \) and \( x \).)

### 3. The Birkhoff theorem

Let us first, for completeness, formulate and prove the theorem in its simplest form, including into consideration vacuum and electrovacuum space–times with a cosmological constant.

**Theorem 1.** Consider the Einstein equations for the metric (1) with a cosmological constant \( \Lambda \) and a Maxwell electromagnetic field \( F_{\mu\nu} \). Suppose that the gradient of \( r(x,t) \) is spacelike or timelike in a space–time region \( \mathbb{D} \). Then, in a certain coordinate system in \( \mathbb{D} \) all metric coefficients in (1) are \( t \)-independent (the metric is static) or \( x \)-independent (the metric is homogeneous).

**Proof.** The only nonzero components of the Maxwell tensor \( F_{\mu\nu} \) compatible with spherical symmetry are \( F_{01} = -F_{10} \) (radial electric fields) and \( F_{23} = -F_{32} \) (radial magnetic fields), and the Maxwell equations without sources imply \( F_{01}F^{01} = q_e^2/r^4 \) and \( F_{23}F^{23} = q_m^2/r^4 \) in the general metric (1), the constants \( q_e \) and \( q_m \) being the electric and magnetic charges, respectively. So the SET of the Maxwell field is

\[
T_{\mu\nu}^m[F] = T_{\mu\nu}^m[F] = \frac{q^2}{r^4} \, \text{diag}(1, 1, -1, -1),
\]

where \( q^2 = q_e^2 + q_m^2 \). The contribution of the cosmological constant into the total SET \( T_{\mu\nu}^m \) is \( \Lambda \delta_{\mu\nu} \) (hence \( -\Lambda \delta_{\mu\nu} \) to \( \bar{T}_{\mu\nu}^m \)) in any coordinate system. The total SET component \( \bar{T}_{01} \), corresponding to a radial energy flow, is equal to zero.

Suppose that the vector \( \partial_r \) is spacelike. Then \( r \) can be chosen as a radial coordinate, that is, \( r = x \). Equation (3d) then gives \( r}\ddot{\alpha} = 0 \), and since \( r' = 1 \), we have \( \alpha = \alpha(x) \). Thus only \( \gamma \) may depend on \( t \). However, equation (3c) expresses \( \gamma' \) via functions of \( x \) only, hence \( \gamma = \gamma_1(x) + \gamma_2(t) \). Lastly, \( \gamma_2 \) can be turned to zero by a coordinate transformation \( t \to \tilde{t}(t) \). Thus the metric is static.
If the vector $\partial_\alpha r$ is timelike, quite a similar reasoning shows that in the coordinate system where $r = t$ the metric is $x$-independent, in other words, the space–time is homogeneous and belongs to the class of Kantowski–Sachs cosmological models. This completes the proof. $\square$

As is well known, both static and homogeneous solutions to the Einstein equations are then unified in the Reissner–Nordström–de Sitter metric

$$ds^2 = A(r)dt^2 - \frac{dr^2}{A(r)} - r^2d\Omega^2,$$

$$A(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{\Lambda r^2}{3},$$

whose special cases are the Schwarzschild ($\Lambda = q = 0$), Reissner–Nordström ($\Lambda = 0$) and (anti-)de Sitter ($m = q = 0$) metrics. Regions with $A(r) > 0$ are static, those with $A(r) < 0$ are homogeneous Kantowski–Sachs regions, and regular zeros of $A(r)$ correspond to Killing horizons whose number and nature depend on the values of the three parameters $m$, $q$, and $\Lambda$.

A more general geometric formulation of the theorem is [15, 16]:

**Theorem 2.** Metrics with a group $G3$ of isometries on non-null orbits $V2$ and with Ricci tensors of Segre types $\{11\}$ $(1,1)$ and $\{111,1\}$ admit a group $G4$, provided that $Y_{,a} = 0$.

Some comments are in order. First, $G3$ with non-null orbits on $V2$ describe spherical, plane and pseudospherical symmetries, and $Y_{,a}$ corresponds in our notations to $\partial_\alpha r$. As already said, we here focus on spherical symmetry.

Second, any Ricci tensor types coincide with types of the SET $\bar{T}_{\mu}^\nu$ due to equation (2). Accordingly, type $\{111,1\}$ means a cosmological constant, while the more general type $\{11\}$ $(1,1)$ corresponds to any kind of matter with $T_0^0 = T_1^1$. This condition defines a generalized notion of vacuum as a kind of matter for which there is no unique comoving RF: instead, all RFs moving in a certain direction, say, $x^1$ (the radial one in our case) are comoving (the flux $T_0^0$ is zero) [21, 22]. We will call such a kind of matter Dybnikova’s vacuum or, shorter, $D$-vacuum. Special cases of $D$-vacuum are not only a cosmological constant and a Maxwell radial electromagnetic field but also, e.g., nonlinear electromagnetic fields with Lagrangians of the form $-L(F), F = F_{\mu\nu}F^{\mu\nu}$ (see, e.g., [23–25]). In [21] this kind of SET was argued to follow from the properties of quantum fields in curved space–time.

Third, the Birkhoff theorem can be extended to some SETs not mentioned in theorems 1 and 2 (e.g., perfect fluids, scalar fields, etc) and symmetries like the cylindrical one [7, 8], though under additional conditions that prevent the emergence of waves. Some cases of interest are listed in [7, 8]; such a result was obtained there by finding more general conditions under which, according to the Einstein equations, $\gamma'$ is $t$-independent or $t$ is $x$-independent (see the above proof of theorem 1).

Fourth, the conditions of theorem 2 exclude the assumption $r = \text{const}$ (to be discussed in section 4) but admit a null gradient $\partial_\alpha r = 0$ (to be discussed in section 5); evidently, in the latter case the extra (fourth) Killing vector, which exists according to the theorem, should be null. We shall see that such a null Killing vector also appears in solutions with a number of SETs other than those mentioned in theorem 2.

### 4. Systems with $r = r_0 = \text{const}$

The condition $r = \text{const}$ looks somewhat unusual, but the corresponding solutions are rather much discussed in the literature; examples of their possible application to certain regions of more plausible space–times are long wormhole throats [28] and ‘horned particles’ [29, 30].
If \( r = \text{const} \), we have \( R_{00} = 0 \) and \( R_{00}^2 = R_{11}^4 \) (see equation (5)), hence \( T_{00}^\mu = T_{11}^\mu \), i.e., only matter of D-vacuum type is admitted. The most general SET of such matter reads

\[
T_{00}^\mu = \text{diag}(\rho, \rho, -p_\perp),
\]

where \( \rho \) is the density and \( p_\perp \) the lateral (or tangential) pressure. For \( T_{11}^\mu \) it then follows

\[
\bar{T}_{11}^\mu = \text{diag}(p_1, p_1, -\rho, -\rho).
\]

Evidently, this ‘vacuum’ can consist of a few components, for example, a cosmological constant and a Maxwell field as mentioned above, and anything else with the same structure of the SET.

Now, from equation (5c) it follows that

\[
\rho = 1/r_0^2 = \text{const}.
\]

The conservation law \( \nabla_\nu T_{\nu}^\mu = 0 \) then holds automatically, leaving \( p_\perp \) quite an arbitrary function of \( u \) and \( v \). Furthermore, equation (5a) has the form of a nonlinear wave equation for \( \alpha(u, v) \):

\[
4 \alpha_{uu} = -p_\perp e^{2\alpha}.
\]

With an arbitrary \( p_\perp \), \( \alpha(u, v) \) is also arbitrary, and no extra symmetry is observed. So the case \( r = \text{const} \) is correctly excluded from theorem 2 (as remarked in [16]). However, it is physically reasonable to suppose that \( p_\perp \) is connected with \( \rho \) by a kind of equation of state, then \( p_\perp = \text{const} \), and equation (13) is a Liouville equation whose solutions are well known: according to [26], the general solution for \( p_\perp = 0 \) is

\[
2\alpha(u, v) = f(u) + g(v) + 2 \ln \left| k \int e^{f(u)} du - \frac{p_\perp}{4k} \int e^{g(v)} dv \right|
\]

with arbitrary functions \( f(u) \) and \( g(v) \) and an arbitrary constant \( k \). There are, in addition, five special solutions [26]:

\[
e^{2\alpha} = \frac{-4ab}{p_\perp S^2(z)}, \quad z = au + bv,
\]

with four variants of \( S(z) \):

\( S(z) = \{z, \cos z, -\cosh z, \sinh z\} \), and also

\[
e^{2\alpha} = \frac{-4C}{p_\perp (uv - C)^2},
\]

where \( a, b, C \) are arbitrary nonzero constants. Note that \( p_\perp \) can have any sign: for example, if matter in question consists of a cosmological constant and a Maxwell field, then

\[
\rho = \Lambda + q^2/r^4, \quad p_\perp = -\Lambda + q^2/r^4.
\]

In the case \( p_\perp = 0 \), equation (13) simply gives \( 2\alpha(u, v) = f(u) + g(v) \), and then the substitution \( e^{f(u)} du = dU, \ e^{g(v)} dv = dV \) reduces the metric (7) to the simplest form

\[
d\bar{s}^2 = dU \ dV - r_0^2 d\Omega^2 = dT^2 - dX^2 - r_0^2 d\Omega^2,
\]

where \( 2T = U + V \) and \( 2X = U - V \), i.e., the geometry is simply \( M^2 \times S^2 \), where \( M^2 \) stands for 2D Minkowski space. This solution not only corresponds to the special case \( \Lambda = q^2/r^4 \) of (15), but also to a kind of matter with a SET having the structure.
which can be ascribed to a distribution of cosmic strings aligned to the direction \( x = x^1 \), a 'string cloud' for brevity (see, e.g., [31] and references therein).

In the pure vacuum case with \( T^\mu_\nu = 0 \), due to (12), there is no solution, i.e., for pure vacuum the condition \( \partial_r r = 0 \) in theorem 2 is un-necessary (as was mentioned, e.g., in [27]).

Returning to the solutions (14), we notice that the same substitution \( Uu = u \), \( dv = v \) applied to the solution (14a) and a change in the notations \( \alpha Uu \), \( \delta Vv \) result in

\[
e^{2\alpha} du \, dv = du \, dv \left( \frac{\mu - \frac{\beta}{4k}}{\nu} \right)^{-2}.
\]

It coincides with the special solution (14b) with \( S(z) = z \) and \( a/b = -p_\mu/(4k^2) \). A curious observation is that a general solution from the viewpoint of differential equations reduces to a special one from the viewpoint of space–time geometry.

Furthermore, in all cases of (14b) a further substitution \( au \mapsto u \) and \( bv \mapsto v \) (or \( bv \mapsto -v \) since \( a \) and \( b \) can be positive or negative), leads to \( au + bv \mapsto u \pm v \). Returning to \( t \) and \( x \) according to equation (6), we see that in the new, thus obtained coordinates the function \( \alpha \) depends either on \( t \) only or on \( x \) only.

In (14c), assuming \( u > 0, v > 0 \), we substitute

\[
u = \sqrt{C + y \ e^{-\alpha/2}}, 
\]

and obtain

\[
e^{2\alpha} du \, dv = \frac{C}{p_\mu y^2} \left( \frac{dy^2}{C + y \ e^{-\alpha/2}} + dz^2 \right).
\]

Thus the metric depends on the single coordinate \( y \) which can be spatial or temporal depending on the sign of the factor \( C/p_\mu \). If \( v \) or/and \( u \) are negative, the same substitution will work with \( -u \) or/and \( -v \) instead of \( u \) and \( v \), respectively.

Thus in all such cases the resulting 2D metric only depends on one of the coordinates, spatial or temporal, indicating the existence of the fourth, temporal or spatial Killing vector. In other words, the Birkhoff theorem holds under the assumption \( r = \text{const} \) under the additional (physically motivated) assumption \( p_\mu = \text{const} \), but one need not assume the Ricci tensor structure as in theorem 2 since this structure now follows from the Einstein equations.

As to the specific form of static and homogeneous (Kantowski–Sachs) solutions for \( \alpha(x) \) or \( \alpha(t) \), it is more convenient, instead of transforming different cases of equation (14), to find them directly from equation (5a) (coinciding with (5b)). We thus have the Liouville equation

\[
\alpha'' = p_\mu e^{2\alpha}, \quad \text{or} \quad \alpha' = -p_\mu e^{2\alpha}.
\]

With \( p_\mu = \text{const} \neq 0 \). Its solutions are well known and lead to straightforward generalizations of the Bertotti–Robinson [32, 33], Nariai [34] and Plebański–Hacyan [35] space–times to more general cases of \( D \)-vacuum. Different cases and properties of these solutions are quite well described in the literature (see, e.g., [37] and references therein) and are beyond the scope of this paper.
5. Systems with null nonzero $\partial_a r$

5.1. Comoving matter

Let us try to reveal which sources of gravity are compatible with variable $r$ satisfying $(\partial r)^2 = 0$. We will first try to simplify the problem assuming that the corresponding coordinate system $(t, x)$ belongs to the comoving RF of matter under consideration. It is manifestly the case for any kind of $D$-vacuum, including the cosmological constant and the radial Maxwell fields, but it is an additional assumption for other kinds of matter\(^7\). As mentioned before, we have $r = r(u)$.

Comoving means that the energy flow is zero, hence due to the Einstein equations $R_{01} = 0$, and by equation (5d)

$$\mu_{\mu\nu} = 2\alpha_u \rho_{\mu\nu}, \quad (22)$$

The remaining Einstein equation (3) lead to the following relations:

$$\tilde{T}_0^0 = \tilde{T}_1^1 = 4 e^{-2\alpha_u} \rho_{\mu\nu},$$

$$\tilde{T}_2^2 = -e^{-2\beta} = -\frac{1}{u^2}(u). \quad (24)$$

Since by assumption $r \equiv \text{const}$, according to (22) $\alpha_u$ is a function of $u$ only, and

$$e^{2\alpha} = |\alpha_u| \cdot A_t(v). \quad (25)$$

Choosing properly the coordinate $v$, we achieve $A_t \equiv 1$, while a transformation $u = u(U), U \mapsto u$ allows us to put $r = u$, hence $e^\alpha = 1$. Thus the metric is completely known and has the simple form

$$ds^2 = du dv - u^2 dt^2 - (t + x)^2 d\Omega^2.$$  

(26)

Furthermore, by (23) $R_0^0 = R_1^1 = 0$, and the total SET necessarily has the form (17), where, according to (24)

$$\rho = 1/u^2. \quad (27)$$

We see that the only admissible kind of matter able to support this gravitational field has the SET (17) and is interpreted as a cloud of radially aligned cosmic strings\(^8\).

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\(^7\) The form (7) of the metric selects a certain class of RFs, connected by Lorentz boosts in the $(s, t)$ Minkowski plane and their conformal extensions corresponding to substitutions $u = u(U), v \mapsto v(V)$.

\(^8\) As mentioned above, rather a general form of $D$-vacuum is represented by nonlinear electromagnetic fields with Lagrangians of the form $-L(F), F \equiv F_{\mu\nu}F^{\mu\nu}$. It is of interest to know which $L(F)$ corresponds to a 'stringy' $D$-vacuum with $p_{\text{vac}} = 0$. Assuming that there are both electric and magnetic fields with charges $q_e$ and $q_m$, respectively, it can be shown that the condition $p_{\text{vac}} = -T_2^2 = 0$ leads to the following differential equation for $L(F)$:

$$L(q_e^2 L_e^2 - q_m^2) = 2q_e^2 F L_e^2,$$

where $L_e \equiv dL/dF$. It is hard to solve in general, but a simple solution is obtained in the case $q_e = 0$ (a pure magnetic field): $L = \sqrt{F}/R_0$, $R_0 = \text{const}$, a kind of Lagrangian used, e.g., in [29]. If, on the contrary, $q_m = 0$, we obtain the condition $L = 0$, which can only hold for a nonzero electric field if this field is constant, which in turn requires $r = \text{const}$. 

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5.2. A noncomoving fluid

Now let us assume that there is matter compatible with \((\partial r)^2 = 0\) and having a nonzero SET component \(T_{01}\). We will consider a few kinds of such matter. (Recall that we consider the SET components corresponding to the coordinates \(t\) and \(x\), for example, \(T_{01} \equiv T_{t}^{x}\).) Since \(r = r(u) \approx \text{const}\), we again choose the null coordinate \(u\) so that \(r \equiv u\) (at least in a certain range of \(u\)).

In all cases we will admit that, in addition to the sort of matter under consideration, the sources of gravity include some sort of \(D\)-vacuum with the SET (10), or even a combination of different \(D\)-vacua not interacting with matter or with each other. As already mentioned, all of them are comoving to any radially moving RF and do not contribute to \(T_{01}\) and \(T_{0}^{0} - T_{1}^{1}\). Their density and tangential pressure will be denoted \(\rho_{\text{vac}}\) and \(p_{\perp\text{vac}}\), respectively, to distinguish them from the noncomoving matter.

Let us begin with an (in general) anisotropic fluid. The SET has the form [36]

\[
T_{\mu\nu} = (\rho + p_{\perp})u_{\nu}u^{\mu} - \delta_{\mu}^{\nu}\pi_{\perp} + (p_{r} - p_{\perp})\chi^{\mu}\chi^{\nu},
\]

(28)

where \(\rho, p_{r}, u^{\mu}\) are the density, pressure and (timelike) 4-velocity, respectively, and \(\chi^{\mu}\) is a spacelike unit vector in the velocity direction. By symmetry of the problem, \(u^{2} = u^{3} = 0\), and the normalization condition \(u^{\mu}u_{\mu} = 1\) can be written in the form

\[
u_{0}\nu^{0} + u_{1}\nu^{1} = e^{-2\alpha}(u_{0}^{2} - u_{1}^{2}) = 1.
\]

(29)

Also, \(\chi^{\nu} = (0, e^{-\alpha}, 0, 0)\). Now, independent combinations of the Einstein equations (5) can be written as follows:

\[
4\frac{\partial u}{u} = (\rho + p_{\perp})u_{0}u_{1}, \quad (30a)
\]

\[
8\frac{\partial u}{u} = (\rho + p_{\perp})(u_{0}^{2} + u_{1}^{2}) + e^{2\alpha}(p_{r} - p_{\perp}), \quad (30b)
\]

\[
4\alpha_{uu} = -e^{2\alpha}(p_{\perp} + p_{\perp\text{vac}}), \quad (30c)
\]

\[
\frac{2}{u^{2}} = \rho - p_{r} + 2\rho_{\text{vac}}, \quad (30d)
\]

where equation (30a) is the \(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\) component, equations (30b) and (30c) are a difference and a sum of \(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\) and \(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\) components, and (30d) is the \(\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)\) component of (5a). Subtracting doubled (30a) from (30b), we obtain

\[
e^{2\alpha}(p_{\perp} - p_{\perp}) = (\rho + p_{\perp})(u_{0} - u_{1})^{2}.
\]

(31)

The opportunities \(\rho + p_{\perp} = 0\) or \(p_{\perp} = p_{r}\) should be rejected since they lead either to the SET of a cosmological constant (which is comoving to any RF) or to \(u_{0} = u_{1}\), making \(u^{\mu}\) a null vector. In both cases it is not a noncomoving fluid.

Thus, in particular, a Pascal perfect fluid with \(p_{\perp} = p_{r}\) cannot be a source of the metric under consideration. This conclusion is independent of a possible presence of any kind of \(D\)-vacuum since the latter does not contribute to equations (30a) and (30b).

There is a considerable freedom in choosing the form of matter variables which can lead to metrics with \(\alpha = \alpha(u, v)\). Let us, however, show that under some natural assumptions there is no \(v\) dependence, and we again obtain the fourth (\(t\)-directed) Killing field, i.e., a Birkhoff-like situation.

Suppose that (i) there are equations of state \(p_{r} = p_{r}(\rho)\) and \(p_{\perp} = p_{\perp}(\rho)\) and (ii) \(\rho_{\text{vac}}\) and \(p_{\perp\text{vac}}\) are functions of \(r\), hence of \(u\). (Note that it is true for a cosmological constant and an
electromagnetic field.) From (30d) it is then clear that \( \rho, p_\parallel, \) and \( p_\perp \) are functions of \( u \), and from (31) with (29) it follows that \( e^{-\alpha}u_0 \) and \( e^{-\alpha}u_1 \) are functions of \( u \). Further, from (30a) we find that \( (e^{-2\alpha})_u \) is a function of \( u \), hence

\[
(e^{-2\alpha})_u = 0 \Rightarrow \alpha_{uv} = 2\alpha_u \alpha_v, \tag{32}
\]

\[
e^{-2\alpha} = U(u) + V(v), \quad U, V = \text{arbitrary}. \tag{33}
\]

Substituting (32) into equation (30c), we obtain

\[
4U_u \alpha_v = p_\perp + p_\parallel \text{vac}. \tag{34}
\]

If \( U(u) = \text{const} \), we have \( \alpha = \alpha(v) \) which is converted to \( \alpha \equiv 0 \) by rescaling of \( v \), and then by (34), \( p_\perp + p_\parallel \text{vac} = 0 \). If, on the contrary, \( U(u) = \text{const} \), then we can equate two different expressions for \( \alpha \) that follow from (33) and (34), to obtain

\[
V_v = -\frac{p_\perp + p_\parallel \text{vac}}{2U_u} (U + V). \tag{35}
\]

Since \( U_u = 0 \), applying \( \partial_v \) to both parts of this equality, we see that it can hold only with \( p_\perp + p_\parallel \text{vac} = 0 \) and \( V = \text{const} \).

Thus under the assumptions made we find that the field equations inevitably lead to (i) \( \alpha = \alpha(u) \) and (ii) \( p_\parallel + p_\parallel \text{vac} = 0 \).

To verify that such solutions do really exist, let us give a simple example, assuming that there is no ‘vacuum’, \( p_\parallel = 0 \), and \( p_\perp = -w\rho \) with \( w = \text{const} \in (0, 1) \) (the inequality \( p_\perp < 0 \) follows from (31) with \( p_\parallel = 0 \)). Then from equations (30d), (31) and (29) we obtain

\[
\rho = \frac{2}{(1 + w)u^2}, \quad u_0 = e^{\frac{1 + w}{2\sqrt{w}}}, \quad u_1 = e^{\frac{1 - w}{2\sqrt{w}}}. \tag{36}
\]

It remains to find \( e^\alpha(u) \), which can be done using (30a) with \( \rho \) substituted from (36). Integrating it, we obtain

\[
e^{-2\alpha} = -\frac{1 - w}{4w} \ln \frac{u}{u_\parallel}, \quad u_\parallel = \text{const}. \tag{37}
\]

It is of interest that at \( u = u_\parallel \) there is an apparent singularity, where \( e^\alpha \to \infty \); however, one can verify that all curvature invariants remain finite.

Other solutions with anisotropic fluids, for example, those with a cosmological constant or a string cloud added, can also be found. However, due to equation (30c), in all such cases the total tangential pressure is equal to zero.

### 5.3. Pure radiation

We have shown that there are solutions with a noncomoving anisotropic fluid. Let us now consider such matter that has no comoving RF at all, namely, pure radiation whose SET has the form \( T_\mu^\nu = \Phi(u, v)k_\mu k^\nu \), where \( \Phi \) is a scalar function and \( k_\mu \) is a radial null vector, which, since there are two possible directions, can be chosen in one of the two forms

\[
k_\mu^{(\pm)} = (e^{-\alpha}, \pm e^{-\alpha}, 0, 0),
\]

\[
k_\mu^{(\pm)} = (e^{\alpha}, \mp e^{\alpha}, 0, 0), \tag{38}
\]

where the upper sign corresponds to a flow directed to larger values of \( x \) and the lower sign to smaller ones. A general situation is the existence of two opposite radiation flows, with the full SET
where \( \Phi_u \) are scalar functions. In the metric (7), this SET has the following nonzero components:

\[
\begin{pmatrix}
T^0_0 & T^0_1 \\
T^1_0 & T^1_1
\end{pmatrix} = \begin{pmatrix}
\Phi + \Psi & \Phi - \Psi \\
-\Phi + \Psi & -\Phi - \Psi
\end{pmatrix}
\]

\( a, b = 0, 1 \equiv t, x, \) (40)
while all other \( T^\mu_\nu \) are zero. Note that the trace \( T^\mu_\mu \) is zero, therefore, (see (2)) \( T^\mu_\mu = T^\nu_\nu \). With this SET, equation (5d) and the difference of (5a) and (5b) lead to the following relations:

\[
4\alpha_u / u = e^{2\alpha} (\Phi - \Psi),
\]
\[
4\alpha_u / u = e^{2\alpha} (\Phi + \Psi),
\]
whence it immediately follows \( \Psi = 0 \). Thus the metric (7) is only compatible with a radiation flow directed to decreasing values of \( x \), which is in fact the propagation direction of the wave of the quantity \( r \) since \( \Phi = + \).

In what follows we consider \( T^\mu_\mu = \Phi(u, v)k_\mu k^\nu \) with \( k^\mu = k^\nu_{(\perp)} \). From (41) we now have

\[
4\alpha_u / u = e^{2\alpha} \Phi(u, v),
\]
and the other Einstein equations, (5c) and a sum of (5a) and (5b) give

\[
1 / u^2 = \rho_{\text{vac}},
\]
\[
4 e^{-2\alpha} \alpha_{uv} = -p_{\text{vac}}.
\] (44)

Equation (43) shows, in particular, that the inclusion of \( \rho_{\text{vac}} \) is quite necessary for the existence of a solution. Even more than that: not each kind of ‘vacuum’ is suitable, for example, a combination of a cosmological constant and a Maxwell field gives \( \rho_{\text{vac}} = \Lambda + q^2 / u^2 \), and equation (43) then leads to \( u = \text{const} \) which is meaningless. On the contrary, a string cloud (17) is suitable, with it equation (43) simply expresses \( \rho_{\text{vac}} = \rho_{\text{string}} \) in terms of \( u \).

Lastly, the conservation equation \( \nabla_\mu T^\mu_\nu \) applied to the SET under consideration gives

\[
\Phi_v = -2\Phi\alpha_v = 0 \Rightarrow \Phi = F(u) e^{-2\alpha}
\] (45)
with an arbitrary function \( F(u) \). Its comparison with (42) gives \( \alpha_u = u F(u) / 4 \), a function of \( u \) only; its further integration leads to \( \alpha \) with an additive arbitrary function of \( v \) which can be, as usual, absorbed by rescaling of the coordinate \( v \), so that \( \alpha = \alpha(u) \) without loss of generality.

This completes the solution. We have a single arbitrary function \( \alpha(u) \), the radiation flux density \( \Phi \) is then obtained from (45) with \( F = 4\alpha_u / u \). Furthermore, from (44) and (43) it follows \( p_{\text{vac}} = 0 \) and \( \rho_{\text{vac}} = 1 / u^2 \), so that again the only admissible kind of \( D \)-vacuum is a string cloud.

We also conclude that pure radiation as a source of gravity with the metric (7) automatically leads to a Birkhoff situation with a null (\( v \)-directed) additional Killing vector.

### 5.4. Scalar fields

Consider a scalar field \( \phi(x, t) = \phi(u, v) \) with the Lagrangian

\[
L_\phi = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2V(\phi),
\] (46)
where $\varepsilon = \pm 1$, $\varepsilon = 1$ corresponds to a normal, canonical scalar field, $\varepsilon = -1$ to a phantom one, and $V(\phi)$ is its self-interaction potential. The modified SET $\tilde{T}_\mu^\nu$, entering into the right-hand side of equation (3), has the form

$$\tilde{T}_\mu^\nu = \varepsilon \partial_\mu \phi \partial_\nu \phi - \delta_\mu^\nu V(\phi).$$

(47)

The scalar field equation following from (46) is

$$\varepsilon \Box \phi + \frac{dV}{d\phi} = 0,$$

(48)

where $\Box = \nabla^\mu \nabla_\mu$ is the d’Alembert operator. In the metric (7) it takes the form

$$4\varepsilon \ e^{-2\alpha} \left( \phi_{,\nu} + \frac{1}{u} \phi_\nu \right) + \frac{dV}{d\phi} = 0.$$

(49)

Now, equation (5d) and the difference of (5a) and (5b) take a form similar to (41)

$$4\alpha_u/u = \varepsilon (\phi_u^2 - \phi_v^2),$$

$$4\alpha_u/u = \varepsilon (\phi_u^2 + \phi_v^2),$$

(50)

whence it follows $\phi_v = 0$, so that $\phi = \phi(u)$. Then $\Box \phi = 0$ (see (49) and (48)), and consequently $dV/d\phi = 0$. Thus the scalar field can only be massless and $u$-dependent. (A possible constant potential is simply an addition to the cosmological constant.)

A further substitution to equations (5c) and a sum of (5a) and (5b) (the latter takes the form (44)) lead to expressions for $\rho_{\text{vac}}$ and $p_{\text{vac}}$.

Since the conservation equation $\nabla_\mu T_\mu^\nu = 0$ for the scalar field SET is automatically satisfied, we conclude that in this solution the function $\phi(u)$ is arbitrary, while after choosing it, the metric function $\alpha$ is obtained from (50), that is, $4\alpha_u = \varepsilon u \phi_u^2$. We see that actually a scalar field is of radiative nature and represents a special case of a radiation flow considered in the previous subsection: we just have

$$\Phi = \Phi_v = \varepsilon \ e^{-2\alpha} \phi_v^2.$$

(51)

Therefore, again, $\alpha = \alpha(u)$ under a proper choice of the $v$ coordinate. As in the previous case, the only admissible form of vacuum is again a string cloud, and a Birkhoff situation automatically occurs.

One should also notice that canonical and phantom scalar fields can form such a solution on equal grounds.

### 6. Conclusion

We have considered some unconventional spherically symmetric geometries in connection with the Birkhoff theorem and obtained the following:

1. In the case $r = \text{const}$, the Ricci tensor necessarily belongs to the Segre types mentioned in theorem 2, but the Birkhoff theorem holds under an additional physically motivated condition that the tangential pressure is functionally related to the energy density. The fourth Killing vector (in addition to those due to spherical symmetry) is spacelike or timelike.

The corresponding solutions are the Bertotti–Robinson, Nariai and Plebanski–Hacyan solutions and their straightforward generalizations which consist in possible inclusion of Dymnikova’s vacuum ($D$-vacuum) in other forms than a Maxwell field and a cosmological constant.
(2) In the case \( r = \text{const} \) but \( (\partial r)^2 = 0 \), if we require that the geometry (7) is supported by some matter comoving to one of the RFs realized by the coordinates \( t = (u + v)/2 \) and \( x = (u - v)/2 \), then the only kind of such matter is a cloud of radially aligned cosmic strings, and the metric has the form (26).

(3) In the case \( r = \text{const} \) but \( (\partial r)^2 = 0 \), admitting noncomoving matter, there exists a much wider set of solutions with the metric (7) supported by either an anisotropic fluid or a combination of a cosmic string cloud and null matter in the form of pure radiation, in particular, a radiative massless scalar field \( \phi(u) \).

In the case of an anisotropic fluid, an additional null Killing vector that manifests a ‘Birkhoff property’ of the system exists under extra (though natural) assumptions that pressures and densities are functionally related by some equations of state.

With pure radiation, an additional null Killing vector exists without any extra assumptions. In all these cases, the structure of the Ricci tensor is more general than assumed in theorem 2, so these results make one more extension of the Birkhoff theorem.

It turns out that in all such cases the total tangential pressure \( p_t \) is equal to zero.

Some kinds of matter with non-null behavior, such as an isotropic perfect fluid or scalar fields with potentials \( V(\phi) \neq \text{const} \), are incompatible with the type of geometry under consideration.

(4) All solutions in the case \( r = \text{const} \), \( (\partial r)^2 = 0 \) contain a singularity at \( u = 0 \), as is clear from the expression for the Kretschmann invariant for all of them: for the metric (7) with \( \alpha = \alpha(u) \) and \( r \equiv u \) it is equal to \( 4/u^4 \).

The solutions obtained here, with such kinds of matter as an anisotropic fluid, string clouds, scalar fields and pure radiation, seem to be new, although the corresponding geometries have been previously classified from a mathematical viewpoint [17, 18].

These solutions are not asymptotically flat, hence they cannot describe the gravitational fields of isolated bodies, but one can speculate that they can represent a limiting form of the fields in a neighborhood of a forming horizon in the process of gravitational collapse.

The present study can be easily extended to plane and pseudospherical symmetries, to an arbitrary number of dimensions, and to nonminimally coupled scalar fields.

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References

[1] Birkhoff G D and Langer R 1923 Relativity and Modern Physics (Cambridge, MA: Harvard University Press)
[2] Jebsen J T 1921 Ark. Mat. Astron. Fys. 15 1
[3] Alexandrow W 1923 Ann. Phys. 72 141
[4] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 Exact Solutions to Einstein’s Field equations 2nd edn (Cambridge: Cambridge University Press)
[5] Bronnikov K A and Melnikov V N 1995 Gen. Relativ. Gravit. 27 465
[6] Schmidt H-J 2013 The tetralogy of Birkhoff theorems Gen. Relativ. Gravit. 45 395–410
[7] Bronnikov K A and Kovalchuk M A 1979 Problems in Gravitation Theory and Particle Theory ed K P Staniukovich (Moscow: Atomizdat) (in Russian) 10th issue
[8] Bronnikov K A and Kovalchuk M A 1980 J. Phys. A: Math. Gen. 13 187
[9] Ferrando J J and Sáez J A 2015 Birkhoff theorem and conformal Killing-Yano tensors Gen. Relativ. Gravit. 47 66
[10] Ferrando J J and Sáez J A 2016 Null conformal Killing-Yano tensors and Birkhoff theorem Gen. Relativ. Gravit. 48 40
[11] Kunstatter G, Maeda H and Taves T New 2D dilaton gravity for nonsingular black holes Class. Quantum Grav. 33 105005
[12] Ray S 2015 Birkhoff’s theorem in Lovelock gravity for general base manifolds Class. Quantum Grav. 32 195022
[13] Nzioki A M, Goswami R and Dunsby P K S 2014 Jebsen–Birkhoff theorem and its stability in f(R) gravity Phys. Rev. D 89 064030
[14] Cavaglià M, de Alfaro V and Filippov A T 1998 Quantization of the string inspired dilaton gravity and the Birkhoff theorem Phys. Lett. B 424 265–70
[15] Barnes A 1973 On Birkhoff’s theorem in general relativity Commun. Math. Phys. 33 75–82
[16] Bona C 1988 A new proof of the generalized Birkhoff theorem J. Math. Phys. 29 1440
[17] Goenner H 1970 Einstein tensor and generalizations of Birkhoff’s theorem Commun. Math. Phys. 16 34–47
[18] Foyster J M and McIntosh C B G 1972 A class of solutions of Einstein’s equations which admit a 3-parameter group of isometries Commun. Math. Phys. 27 241–6
[19] Landau L D and Lifshitz E M 1993 The Classical Theory of Fields (Oxford: Pergamon)
[20] Bronnikov K A and Rubin S G 2012 Black Holes, Cosmology, and Extra Dimensions (Singapore: World Scientific)
[21] Dymnikova I G 1992 Gen. Relativ. Gravit. 24 235
[22] Bronnikov K A, Dobosz A and Dymnikova I G 2003 Class. Quantum Grav. 20 3797
[23] Peres A 1961 Nonlinear electrodynamics in general relativity Phys. Rev. 122 273
[24] Plebanski J 1966 Non-Linear Electrodynamics–A Study (Mexico: C.I.E.A. del I.P.N.)
[25] Bronnikov K A 2001 Regular magnetic black holes and monopoles from nonlinear electrodynamics Phys. Rev. D 63 044005
[26] Polyanin A D and Zaitsev V F 2004 Handbook of Nonlinear Partial Differential equations (Boca Raton, FL: CRC Press)
[27] Hawking S W and Ellis G F R 1973 The Large Scale Structure of Space–Time (Cambridge: Cambridge University Press)
[28] Popov A A 2010 Self-force on a scalar point charge in the long throat Phys. Lett. B 693 180–3
[29] Guendelman E and Vasilhou M 2012 Fully explorable horned particles hiding charge Gen. Relativ. Gravit. 46 123001
[30] Zaslavskii O B 2009 Regular black holes with $\mathbb{R}$ gravity Commun. Math. Phys. 76 95
[31] Bertotti B 1959 Phys. Rev. 116 1331
[32] Robinson I 1959 Bull. Acad. Pol. Sci. 7 351
[33] Nariai H 1951 On a new cosmological solution of Einstein’s field equations of gravitation Sci. Rep. Tōhoku Univ., I. Ser. 35 62–7
[34] Reprinted in Nariai H 1951 Gen. Relativ. Gravit. 31 963–71
[35] Plebański J F and Hacyan S 1979 Some exceptional electroweak type $D$ metrics with cosmological constant J. Math. Phys. 20 1004–10
[36] Letelier P S and Verdauger E 1987 J. Math. Phys. 28 2431
[37] Griffiths J B and Podolsky J 2009 Exact Space–Times in Einstein’s General Relativity (Cambridge: Cambridge University Press)