The Benefits of Segmentation in Trial-Offer Markets with Social Influence and Position Bias

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Abstract

The purchasing behaviour of consumers is often influenced by numerous factors, including the visibility of the products and the influence of other customers through their own purchases or their recommendations.

Motivated by trial-offer and freemium markets and a number of online markets for cultural products, leisure services, and retail, this paper studies the dynamics of a marketplace ran by a single firm and which is visited by heterogeneous consumers whose choice preferences can be modeled using a Mixed Multinomial Logit. In this marketplace, consumers are influenced by past purchases, the inherent appeal of the products, and the visibility of each product. The resulting market generalizes recent models already verified in cultural markets.

We examine various marketing policies for this market and analyze their long-term dynamics and the potential benefits of social influence. In particular, we show that the heterogeneity of the customers complicates the market significantly: Many of the optimality and computational properties of the corresponding homogeneous market no longer hold. To remedy these limitations, we explore a market segmentation strategy and quantify its benefits. The theoretical results are complemented by Monte Carlo simulations conducted on examples of interest.

1 Introduction

The effects of social influence on consumer behaviour have been observed in a wide range of settings (e.g., [11, 12, 15]). Social influence may appear through different social signals, such as the number of past purchases, consumer ratings, and consumer recommendations, depending on the market and/or the marketing platform. However, not all social signals are equally important. Indeed, two recent studies [3, 15] have been conducted to understand the relative importance of different social signals on consumer behaviour, one in the Android app platform and the other one in hotel selection. Both experiments arrived to the same conclusion, namely that the popularity signal (i.e., the number of purchases) has a much stronger impact on consumer behavior than the average consumer rating signal.

In addition to the impact of social influence, consumer preferences are also affected in significant ways by product visibilities. In digital markets, the impact of visibility on consumer behavior has been widely observed, including in internet advertisement where sophisticated mathematical models have been developed to determine the relative importance of the various product positions [2]. Positioning effects are also of high significance in online stores such as Expedia, Amazon, and iTunes, as well as physical retail stores (see, e.g., [6]).

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There are at least three key managerial decisions that naturally arise in markets with social influence and visibility:

1. How to assign products to the available positions?

2. How should social influence be used? Is it beneficial at all?

3. What are the benefits of market segmentation?

All these decisions may have a fundamental impact on market efficiency and predictability. The first type of decisions ponders how to rank the products and studies whether popularity or a concept of quality should drive the assignment of products to positions. The second type of decisions analyzes which social influence signal (e.g., popularity signals versus ratings) is beneficial (if any). Finally, the third type of decisions analyzes the benefits of presenting different information to different customers. This is particularly significant in online markets where rich information may have been gathered on an arriving customer and its relationship with others. For example, a recent analysis performed by the online travel agent Orbitz has shown that Mac users spend up to about 30% more in hotel bookings than their PC counterparts [7]. As a consequence, it is beneficial to show different rankings to customers depending on the computer they use.

Of course, these decisions do not take place in isolation. For instance, it is valuable to study how to combine social influence and market segmentation in order to determine which signal (if any) should be used for different segmentation strategies. While most online platforms display social signals about all past consumers, the major online travel agent Booking.com allows users to rank hotels according to the average consumers score of a particular segment such as couples, families, and solo travelers. See Figure 1 for an illustration of this feature.

The purpose of this paper is to address these questions by studying the dynamics of a marketplace run by a single firm and where the choice preferences of its heterogeneous consumers are modeled using a Mixed Multinomial Logit. In this marketplace, consumers are influenced by the past purchases, as well as by the inherent appeal of the products and how visible each product is in the marketplace. Our approach applies to multiple settings such as those arising in music and film online markets and trial-offer and freemium markets. It also generalizes recent models already verified in cultural markets.

Our work is related to the MusicLab experiment performed by Salganik et al. [11]. In that experiment, participants were presented a list of unknown songs from unknown bands, each song being described by its name and band. The participants were partitioned into two groups exposed to two different experimental conditions: the independent condition and the social influence condition.
In the independent group, participants were shown the songs in a random order and they were allowed to listen to each of them and then download them if they wish. In the second group (social influence condition), participants were shown the songs in popularity order, i.e., allocating the most popular songs to the most visible positions. Moreover, these participants were also shown a social signal, i.e., the number of times each song was downloaded too. In order to investigate the impact of social influence, participants in the second group were distributed in eight “worlds” evolving completely independently. In particular, participants in one world had no visibility about the downloads and the rankings in the other worlds. The MusicLab is an ideal experimental example of a trial-offer market where each song represents a product, and listening and downloading a song represent trying and purchasing a product respectively. The results by Salganik et al. [11] show that the different worlds evolve significantly differently from one another, providing evidence that social influence may introduce unpredictability in a market.

To explain these results, Krumme et al. [4] proposed a framework in which consumer choices are captured by a multinomial logit model whose product utilities depend on songs appeal, position bias, and social influence. Abeliuk et al. [1] provided a theoretical and experimental analysis of such trial-offer markets using different ranking policies following the framework of Krumme et al. [4]. They proved that social influence is beneficial in order to maximize the expected number of downloads when using a greedy heuristic known as performance ranking. In performance ranking selects greedily the ranking that maximizes the expected number of downloads at the next time period, i.e. it maximizes the short-term market efficiency. Abeliuk et al. [1] have also illustrated experimentally that the popularity ranking is outperformed by the performance ranking in a variety of settings. Still based on the model of Krumme et al. [4], Van Hentenryck et al. [13] have studied the performance of the quality ranking which ranks products by their intrinsic quality: They show that the quality ranking is in fact asymptotically optimal and has a considerably less unpredictability than the popularity ranking.

The main shortcoming of the models employed in [4, 1, 13] is the assumption that consumers are homogeneous: The product qualities and appeals are the same for all consumers. Given that current technology makes it possible for a website to obtain user information (including computer type), online firms may use this information to prioritize differently the product assortments for each incoming customer. Moreover, these firms may also decide to incorporate information about the past purchases of a specific consumer class as a social signal. For example, instead of displaying the total number of bookings for each hotel, the firm may only display the number of bookings by similar consumers (e.g., in age or income bracket). Pursuing such segmentation strategies has an associated cost as it requires to gather consumer information and to analyze it correctly. It is thus important to understand their potential benefits and to go beyond the models in [4, 1, 13].

In this paper, we remedy this limitation and study the dynamics of trial-offer markets with social influence and where consumers have heterogeneous preferences. More specifically, consumer preferences are modeled with a mixed multinomial logit [8]. Our main findings provide quantitative insights about the benefits of market segmentation in online markets and can be summarized as follows.

1. We first show that Computing the (optimal) performance ranking in a mixed multinomial logit model is NP-Hard under Turing reductions, indicating that moving from an homogeneous setting (MNL) to an heterogeneous setting (Mixed MNL) has significant computational

\*the quality of a product is here defined as the probability that a consumer would purchase/download the product once she has tried the product out
implications.

2. We then study a policy, the average quality ranking, which orders the items in decreasing order of average quality. We show that the average quality ranking converges to a unique equilibrium when consumers are shown the number of past purchases (the popularity signal). Unfortunately, we also show that the popularity signal may, perhaps surprisingly, decrease the expected market efficiency of the average quality ranking.

3. We consider a simple segmentation strategy, where customers are shown a quality ranking dedicated to their class and only observe the popularity signal for their own market segment (i.e., the past purchases of customers of the same class. We show that this segmented quality ranking always outperforms, asymptotically and in expectation, the average quality ranking and may improve the market efficiency by a factor $K$, where $K$ is the number of classes.

These theoretical results are complemented by an agent-based simulation performed on a number of settings. The simulation results highlight every theoretical results, suggesting that the segmented quality ranking with a popularity signal is an interesting avenue for trial-offer markets.

The remaining of this paper is organized as follows. Section 2 introduces the model of the dynamic trial-offer market. The most relevant ranking policies for this model are described in Section 3, which also presents the NP-hardness results for performance ranking in mixed multinomial logit models. Section 4 describes the convergence and the impact of social influence for the quality ranking in the same setting. Section 5 presents our segmentation strategy and its benefits. Section 6 presents the results of the agent-based simulation and Section 7 concludes the paper. The proofs not given in the text are in the appendix.

2 The Model

Motivation We consider a firm running a marketplace that sells a set of products. Following [4, 11], we focus our attention to trial-offer markets, i.e., markets in which consumers can try the product for free before deciding to make a purchase. Consumers are position-biased in the following sense: The likelihood of trying a specific product is affected by the position of the product, as well as the position of the others products in the market. There is large experimental evidence for this type of bias in digital markets [9], as well as in traditional retailing [14]. It is therefore important for the firm to decide how to allocate products to locations since this affects the market efficiency, i.e., the total number of purchases. We also consider a marketplace where it is possible to display information about product popularity. In particular, we assume that the firm shows the total number of purchases for each item at each point in time. While there is considerable evidence and consensus among the researchers that the display of information about consumer preferences changes consumer behaviour, it remains a matter of discussion whether such display may benefit consumers and the market.

Unlike [4], we consider that there are different classes of consumers, meaning that the probability that a consumer would try a given item depends on the consumer class. Our model for consumer behaviour is therefore far more general than in [4]. Moreover, the probability that a given consumer tries an item will follow a Mixed Multinomial Logit (MMNL). Since McFadden and Train [8] proved that every random utility model can be well approximated by a MMNL, our model of consumer preferences is indeed very general.
Formalization  We now formally describe a dynamic model for this marketplace. Let $[N] = \{1, 2, \ldots, N\}$ denote the set of items in the marketplace and $S_N$ denote the set of the permutations of these items. At any point in time, the firm decides how to position the items in the market by selecting a permutation $\sigma \in S_N$ such that $\sigma(i) = j$ implies that item $i$ is placed in position $j$ ($j \in [N]$).

The consumer behaviour can be described as follows. There are $K$ different classes of consumers. The rate of consumers from class $k$ arriving in any given time interval of unitary length follows a Poisson process with a mean value $\lambda_k$. The weight $w_i$ of each class $i$ is then defined as

$$w_i = \frac{\lambda_i}{\sum_{i=1}^{N} \lambda_i}.$$  

When consumer $t$ enters the market, she observes all the items and a popularity vector $d_t = (d_{t1}, d_{t2}, \ldots, d_{tN}) \in \mathbb{N}^N$, where $d_{ti}$ is the number of times item $i$ has been bought for prior to her arrival at time $t$. When the consumer arrives, each of the $N$ items have been given a position through a permutation $\sigma \in S_N$. The consumer selects an item to try and then decides whether to buy it. If the consumer belongs to class $k$, the probability that she tries item $i$ is given by

$$p_{i,k}(\sigma, d_t) = \frac{v_{\sigma(i)}(a_{i,k} + d_{ti})}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_{tj}) + z}$$  

where $z \in \mathbb{R}_{\geq 0}$ is fixed to a constant for the duration of the process, $v_j \in \mathbb{R}_{\geq 0}$ represents the visibility of position $j \in [N]$ regardless of the consumer class (the higher the value $v_j$ the more visible the item in that position is), and $a_{i,k} \in \mathbb{R}_{> 0}$ captures the intrinsic appeal of item $i$ for consumer class $k$ for all $i \in [N]$ (higher values correspond to more appealing items).

If a consumer from class $k$ has selected item $i$ for a trial, the probability that she would purchase the item is given by $q_{i,k} \in [0, 1]$. Observe that this probability is independent of both the appeal vector $(a_{i,k}, \ldots, a_{N,k})$ and the visibility vector $v$. Intuitively, this assumption, which has been validated in the MusicLab experiment, captures the fact that it is more difficult to influence consumers after they have tested a product than before.

When the consumer decides to purchase item $i$, the popularity/sales vector $d$ is increased by one in position $i$. To analyze this process, we divide time into discrete periods such that each new period begins when a new consumer arrives. Hence, the length of each time period is not constant.

The objective of the firm running this market is to maximize the total expected number of purchases. To achieve this, the key managerial decision of the firm is what is known as the ranking policy $[\Pi]$, which consists in deciding at each point in time the permutation $\sigma \in S_N$ to display the items. The next section describes a number of relevant ranking policies for this model.

A key aspect of this paper is to study the potential benefits of the popularity signal on market efficiency and compare the ranking policies with and without this signal. In this paper, we always assume that the popularity signal is used as specified in Equation (1). When the popularity signal is not used, the probability of sampling a product is given by assuming that the popularity signal is simply the vector $(0, \ldots, 0)$.

3 Ranking Policies

Consider without loss of generality that the $N$ locations are sorted by their visibility such that $v_1 \geq v_2 \geq \ldots \geq v_N$. A ranking policy is a function $f : \mathbb{N}^N \to S_N$ which, given a vector of past purchases, returns a ranking of the items.
Ranking policies can be partitioned into two groups: static and dynamic. A ranking policy $g$ is said to be static if the output ranking does not depend on the popularity signal, i.e., if $g(d) = g(d')$ for all $d, d' \in \mathbb{N}^N$. On the other hand, a dynamic ranking policy is one in which the output ranking depends on this signal.

A widely used dynamic policy consists in ranking the items according to their current popularity: Select $\sigma \in S_N$ such that $\sigma(x) = i$ if item $x$ is currently the $i$th most purchased product at this point. This policy is called the popularity ranking and is widely used. Abeliuk et al. [1] have shown that this ranking policy does not maximize the expected number of purchases in the special case where $K = 1$.

Another dynamic ranking strategy, known as activity ranking, was studied in an experiment by Lerman and Hogg [5]. In activity ranking, products are sorted in chronological order of the last purchased time, i.e., the item last purchased appears first, while the item least recently purchased appears last. In their experiment conducted with individuals recruited with the Mechanical Turk, Lerman and Hogg [5] have observed that activity ranking improves consumer recommendations with respect to popularity ranking.

The performance ranking is a dynamic policy that greedily selects a ranking that maximizes the expected number of purchases in the following period. This strategy was first proposed by Abeliuk et al. [1] for the special case with $K = 1$ and we now generalize its definition for the more general model considered in this paper. Given the memory-less nature of the Poisson distribution, the probability that the next incoming consumer belongs to class $k$ is given by

$$ w_k = \frac{\lambda_k}{\sum_{k=1}^K \lambda_k}. $$

Therefore the performance ranking at time period $t$ consists of finding the permutation $\sigma^* \in S_N$ maximizing the probability of a purchase in the next time period, i.e.,

$$ \sigma^* = \arg\max_{\sigma \in S_n} \sum_{k=1}^K w_k \cdot \sum_{i=1}^N p_{i,k}(\sigma, d^t) \cdot q_{i,k}. \tag{2} $$

The probability $\Pi^{PR}$ of a purchase in the next time period is thus given by

$$ \Pi^{PR} = \max_{\sigma \in S_n} \left\{ \sum_{k=1}^K \left( w_k \cdot \sum_{i=1}^N \left( p_{i,k}(\sigma, d^t) \cdot q_{i,k} \right) \right) \right\}. \tag{3} $$

$$ = \max_{\sigma \in S_n} \left\{ \sum_{k=1}^K \left( w_k \cdot \sum_{i=1}^N \left( \frac{v_{\sigma(i)}(a_{i,k} + d^t_i) \cdot q_{i,k}}{\sum_{j=1}^N v_{\sigma(j)}(a_{j,k} + d^t_j) \cdot q_{j,k}} \right) \right) \right\}. \tag{4} $$

Abeliuk et al. [1] showed that the performance ranking can be computed efficiently, i.e., in strongly polynomial time: See Theorem 1 by Abeliuk et al. [1]. Moreover, despite the myopic focus of the performance ranking, a series of agent-based simulations performed by Abeliuk et al. [1] showed that, for the special case of $K = 1$, the performance ranking outperforms the popularity ranking by roughly 20 percent in the long run on experiments modeled after the MusicLab, and exhibits much less unpredictability. Unfortunately, the performance ranking cannot be computed efficiently when there are at least two classes of consumers. More precisely, we can show that the 2-Class Logit problem which is known to be NP-hard [10] can be reduced (under Turing reductions) to computing the performance ranking in our setting.

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\footnote{Abeliuk et al. [1] assumed $z = 0$ but their proof can be easily generalized for any $z \in \mathbb{R}_{\geq 0}$.}
Theorem 1. Computing the performance ranking is \( NP \)-hard under Turing reductions. This is true even when \( K = 2 \) and the product qualities are the same for all consumer classes.

Finally, the quality ranking is a simple and natural static policy studied by Van Hentenryck et al. [13] for the case \( K = 1 \): It simply consists in ranking the products by quality, ignoring the appeals and popularity signal. The quality ranking can easily be generalized to \( K \) classes by taking the weighted average of the class qualities. The quality ranking for the MMNL model thus consists in placing in position \( j \) the item with the \( j \)th highest weighted average quality, where the weighted average quality of item \( i \in [N] \) is

\[
\bar{q}_i = \sum_{k=1}^{K} w_k q_{i,k}.
\]  

For the special case \( K = 1 \), Van Hentenryck et al. [13] proved that the quality ranking is optimal asymptotically and always benefits from the popularity signal used in our model. The next section will study whether this continues to hold \( K > 1 \). Note that, in the following, the ranking which orders the products by decreasing values of \( \bar{q}_i \) is called the average quality ranking.

4 Properties of Average Quality Ranking

This section studies the properties of the average quality ranking for the MMNL model. We first show that the average quality ranking converges to a monopoly (under weak conditions). We then study the benefits of popularity signal for the average quality ranking.

4.1 Convergence to a Monopoly

Given a ranking policy \( f \), the random variable

\[
\phi^t_i = \frac{d^t_i}{\sum_j d^t_j}
\]

is known as the market share of item \( i \) at time \( t \): It represents the ratio between the number of times that item \( i \) was purchased and the total number of purchases at time \( t \).

Definition 1. The MMNL model goes to a monopoly using a ranking policy \( f \) if, for each realization of the \( N \) random sequences \( \{\phi^t_i\}_{t \in \mathbb{N}} \ (i \in [N]) \), there exists a product \( i^* \) such that the realized sequence \( \{\phi^t_{i^*}\}_{t \in \mathbb{N}} \) converges to 1 as \( t \) goes to infinity. In this case, we also say that item \( i^* \) goes (predictably) to a monopoly.

We can now show that the MMNL model goes to a monopoly when using the average quality ranking. The proof is quite technical: Its key idea is to show that the Mixed Multinomial Logic Model (MMNL) can be reduced to a generalized case of the Multinomial Logic Model (MNL) where the appeal and the quality of an item at time \( t \) depend on the popularity signal at \( t \). The proof relies on the following lemma that generalizes the convergence result of the quality ranking for the MNL model by Van Hentenryck et al. [13] to the case where the appeal and quality of an item depend on the popularity ranking provided that that the resulting functions are bounded by above and below.
Lemma 1. Consider a Multinomial Logic Model, i.e., a setting with $K = 1$ where the appeal and quality of each item $i$ are functions of the purchases vector $d^t$, i.e., $\tilde{a}_i^t = \tilde{a}_i(d^t)$ and $\tilde{q}_i^t = \tilde{q}_i(d^t)$ respectively. Suppose that there exists a time period $t^*$ such that these two quantities are upper and lower bounded by constants for any period $t > t^*$ independent of the realizations of $\tilde{a}_i^t$ and $\tilde{q}_i^t$, i.e.,

$$q_{i, \min} \leq \tilde{q}_i^t \leq q_{i, \max} \quad \text{and} \quad a_{i, \min} \leq \tilde{a}_i^t \leq a_{i, \max} \quad \forall i \in [N], t > t^*.$$ 

Let $\sigma \in S_N$ denote a static ranking policy. If there exists an item $i^*$ and an instant $\hat{t}$ such that

$$v_{\sigma(i^*)}q_{i^*, \min} > v_{\sigma(i)}q_{i, \max} \quad \forall i \neq i^* \quad \text{and} \quad \forall t > \hat{t},$$

then item $i^*$ goes to a monopoly when using the ranking policy $\sigma$.

The main result of this section is about the convergence to a monopoly of a large class of static ranking policies. For simplicity, we assume a weak condition to break potential ties between items.

Definition 2. A static ranking policy $\sigma$ is tie-breaking for a MMNL model if there exists a unique item $i^*$ with the highest product of visibility and weighted average quality, i.e.,

$$\exists i^* \in [N] : \tilde{q}_i v_{\sigma(i)}^* > \tilde{q}_i v_{\sigma(i)} \quad \forall i \in [N], i \neq i^*. \quad (6)$$

We are now ready to prove the main result of this section.

Theorem 2. Consider a MMNL model $\mathcal{M}$ and a static, tie-breaking ranking policy $\sigma \in S_N$ for $\mathcal{M}$. Model $\mathcal{M}$ goes to a monopoly using $\sigma$ and the item $i^*$ that goes predictably to a monopoly using $\sigma$ in $\mathcal{M}$ is given by

$$i^* = \arg\max_{1 \leq i \leq N} v_{\sigma(i)}^* \tilde{q}_i.$$

The following corollary asserts that the average quality ranking converges to a monopoly for the product of highest average quality. This result is particularly interesting since it shows that the average quality ranking generalizes the quality ranking from the MNL to the MMNL model.

Corollary 1. (Predictability of Average Quality Ranking). Whenever the quality ranking is used, a MMNL model goes to a monopoly for the product with the highest weighted average quality.

4.2 The Impact of the Popularity Signal

The previous subsection has shown that the average quality ranking for the MMNL model inherits the asymptotic convergence of the quality ranking for the MNL. Van Hentenryck et al. [13] have also shown that The quality ranking always benefits in expectation from the popularity signal in the MNL model. Unfortunately, this result does not hold for the average quality ranking in the MMNL model in general. The proof uses the simple matrix inequality:

$$\sum_{j=1}^{m} \max_{1 \leq i \leq n} a_{i,j} \leq m \max_{1 \leq i \leq n} \sum_{j=1}^{m} a_{i,j}. \quad (7)$$

Theorem 3. When using the average quality ranking, the MMNL model can perform up to $K$ times better without showing the popularity signal, where $K$ is the number of classes.
Proof. At the limit, the probability that an item is purchased under the average quality ranking with the popularity signal is given by

\[ P_{AQGSI} = \max_{1 \leq i \leq N} \bar{q}_i. \]  

(8)

When no popularity signal is shown, this probability becomes

\[ P_{AQNSI} = \sum_{k=1}^{K} w_k \sum_{i=1}^{N} \frac{v_{\sigma(i)}a_{i,k}}{v_{\sigma(j)}a_{j,k} + z}. \]  

(9)

We can easily bound \( P_{AQNSI} \) as follows:

\[ 0 \leq \sum_{k=1}^{K} \min_{1 \leq i \leq N} (w_k q_{i,k}) \sum_{i=1}^{N} \frac{v_{\sigma(i)}a_{i,k}}{v_{\sigma(j)}a_{j,k} + z} \leq P_{AQNSI} \leq \sum_{k=1}^{K} \max_{1 \leq i \leq N} (w_k q_{i,k}) \]  

(10)

and hence, by inequality (7),

\[ 0 \leq \frac{P_{AQNSI}}{P_{AQGSI}} \leq K. \]  

(11)

Proposition 1. The bounds in Theorem 3 are tight.

Proof. Consider first the upper bound. Choose a MMNL model where \( z = 0, K = N \), the quality matrix is diagonal with a value of 1 for the first element and 1 - \( \epsilon \) for all others, the appeal matrix is the identity, and the classes have the same weights \( w_i = \frac{1}{K} \). Then,

\[ P_{AQNSI} = \sum_{1 \leq i \leq N} \frac{1}{K} (1 - \epsilon(1 - \delta_{ii})) \]  

and

\[ P_{AQGSI} = \frac{1}{K}. \]

and

\[ \lim_{\epsilon \to 0} \frac{P_{AQNSI}}{P_{AQGSI}} = \lim_{\epsilon \to 0} \sum_{1 \leq i \leq N} (1 - \epsilon \delta_{ii}) = \lim_{\epsilon \to 0} (K - \epsilon(K - 1)) = K. \]  

(12)

Consider now the lower bound. Choose a MMNL model where \( z = 0, K = N \) with the same quality matrix as before, the same weights, and an appeal matrix filled with ones except in its diagonal where each element has a value of \( \epsilon_A \). Then,

\[ P_{AQNSI} = \sum_{1 \leq i \leq N} \frac{1}{K} (1 - \epsilon(1 - \delta_{ii})) \frac{v_{\sigma(i)}\epsilon_A}{\epsilon_A v_{\sigma(i)} + \sum_{j \neq i} v_{\sigma(j)}} \]

\[ P_{AQGSI} = \frac{1}{K}. \]

and

\[ \lim_{\epsilon_A \to 0} \frac{P_{AQNSI}}{P_{AQGSI}} = \lim_{\epsilon_A \to 0} \sum_{1 \leq i \leq N} (1 - \epsilon(1 - \delta_{ii})) \frac{v_{\sigma(i)}\epsilon_A}{\epsilon_A v_{\sigma(i)} + \sum_{j \neq i} v_{\sigma(j)}} = 0. \]  

(13)

\]
The result shows that, in the worst case, using the popularity signal decreases the number of purchases by the average quality ranking by a factor of $K$. The direct consequence of these results is the lack of clarity about the benefits of the popularity signal in MMNL models. The popularity signal may be beneficial to the average quality ranking but it may also significantly degrade the performance of the market. This is in contrast with the MNL model where the quality ranking always benefits from the popularity signal (in expectation). As a result, having a heterogeneous set of customers complicates, once again, the managerial decisions in the marketplace.

5 Market Segmentation and its Benefits

In the previous sections, we have shown a number of negative results for the MMNL model. In particular, we have shown that, in MMNL models, computing the performance ranking is intractable when $K > 1$ and that displaying the popularity signal to customers may significantly reduce the asymptotic market efficiency of the average quality ranking. In this section, we show that the widely used marketing strategy known as market segmentation remedies these limitations, while retaining the original benefits of quality ranking for the Multinomial Logit Model.

The market segmentation considered here assumes that the firm has the ability to know the class of each arriving consumer. This is a natural assumption in a number of online markets (e.g., Amazon, online retail stores, ITunes, and Netflix) where firms are able to learn information about their customers over time. Armed with this information, the firm will now propose item rankings dedicated to each customer class. Moreover, and equally important, the popularity signal will be tailored to each class. In other words, the firm will only show the popularity signal derived from purchases of customers of the same class as the incoming customer, not the popularity obtained from the entire customer pool. As shown in Figure 1, websites such as Booking.com already give customers the option of selecting their peer groups to refine the site recommendations. Under this new strategy where each class of consumers has its own quality ranking and observes the past purchases of its own class only, the policy is called the segmented quality ranking. The firm uses $K$ permutations $\sigma_k \in S_N (k \in [K])$, where $\sigma_k$ sorts the products in decreasing order according to their quality for class $k$. In addition, the probability of sampling item $i$ for a customer of class $k$ is given by

$$p_{i,k}(\sigma, d^t_{i,k})$$

where $d^t_{i,k} = (d^t_{1,k}, \ldots, d^t_{N,k})$ and $d^t_{i,k}$ denotes the number of purchases of item $i$ by customers from class $k$ up to time $t$.

We now study the benefits of this market segmentation. Observe first that each market segment can be viewed as evolving independently and hence directly inherits the original benefits identified by Van Hentenryck et al. [13] for the quality ranking: It is asymptotically optimal and predictably goes to a monopoly. This observation will enable us to prove the benefits of market segmentation.

**Definition 3.** The segmented quality ranking policy $\sigma_k \in S_N \ (k \in [K])$ is tie-breaking if, for each class $k$, there exists a unique item $i^*_{k}$ with the highest quality:

$$\forall k \in [K] \ \exists i^*_{k} \in [N] \ \forall j \in [N], j \neq i^*_{k} : q_{i^*_{k},k} > q_{j,k}.$$ (14)

Assuming a MMNL model for which the average quality ranking and the segmented quality ranking are tie-breaking, we compare the probability of a purchase at time $t$ in both settings. More precisely, we compare two quantities:
- $P_{AQGSi}^t$: the probability of a purchase at time $t$ when the firm uses the average quality ranking and the “global” popularity signal $d^t$;
- $P_{SQSSI}^t$: the probability of a purchase at time $t$ when the firm uses the segmented quality ranking with the class popularity signal $d^t_k$.

The probabilities $P_{AQGSi}^t$ and $P_{SQSSI}^t$ concern the behaviour of the consumer arriving to the market at time $t$ independently from the customer class. Comparing $P_{AQGSi}^t$ and $P_{SQSSI}^t$ for any time $t$ is a very challenging task. Instead, we compare both variables in the limit.

**Theorem 4.** Assume that the average quality ranking and its segmented version are tie-breaking for a MMNL model. Then,

$$1 \leq \lim_{t \to \infty} \frac{P_{SQSSI}^t}{P_{AQGSi}^t} \leq K.$$  \hspace{1cm} (15)

**Proof.** By Theorem 2, we have

$$P_{AQGSi}^t \equiv \lim_{t \to \infty} P_{AQGSi}^t = \max\{q_1, q_2, \ldots, q_N\}.$$  

As mentioned earlier, for the segmented quality ranking, each class is independent from each other and all of them will converge to a monopoly for the product with the highest quality in that class. We have that

$$P_{SQSSI}^t \equiv \lim_{t \to \infty} P_{SQSSI}^t = \sum_{k=1}^{K} w_k \max_i q_{i,k}.$$  

As a result,

$$\frac{P_{SQSSI}^t}{P_{AQGSi}^t} = \frac{\sum_{k=1}^{K} w_k \max_{1 \leq i \leq N} q_{i,k}}{\max_{1 \leq i \leq N} \sum_{k=1}^{K} w_k q_{i,k}} = \frac{\sum_{k=1}^{K} \max_{1 \leq i \leq N} w_k q_{i,k}}{\max_{1 \leq i \leq N} \sum_{k=1}^{K} w_k q_{i,k}}.$$  

The lower bound is obviously valid and the upper bound follows from inequality (7). \hfill \square

**Proposition 2.** The upper bound of Theorem 4 is tight.

**Proof.** Consider a model with $K$ items and $K$ consumer classes. Without loss of generality, let the class 1 be the class with the lowest weight, i.e., $w_1 \leq w_k \forall k \in [K]$. Then, for any set of positive appeals in each class, define the elements $q_{i,k}$ as follows:

$$q_{i,k} = \begin{cases} 
\frac{\min_{j \in [K]} w_j}{w_k} & \text{if } i = k = 1 \\
\frac{\min_{j \in [K]} w_j}{w_k} - \epsilon & \text{if } i = k \neq 1 \\
0 & \text{otherwise}
\end{cases}$$

where $\epsilon$ is a positive number ensuring that the model is tie-breaking for the quality rankings. Then

$$\lim_{\epsilon \to 0} \frac{P_{SQSSI}^t}{P_{AQGSi}^t} = \lim_{\epsilon \to 0} \frac{K \min_{j \in [K]} w_j - \epsilon \sum_{k=2}^{K} w_k}{\min_{j \in [K]} w_j} = K.$$  \hfill \square
These results show that the segmented quality ranking always outperforms the average quality ranking in expectation and that the improvement in market efficiency can be up to a factor of $K$. Note that the segmented quality ranking is optimal asymptotically, since the market segment are operating independently and each are optimal asymptotically. It does not necessarily mean that the segmented quality ranking is always better, since the popularity signal is weaker early in the market evolution. This will be illustrated in the agent-based simulations presented in the next section.

6 Agent-Based Simulation

This section presents the results of an agent-based simulation to illustrate the theoretical results and complement them by depicting how the markets evolve over time for different types of rankings.

6.1 The Experimental Setting

The Agent-Based Simulation The experimental setting uses an agent-based simulation to emulate the MusicLab [11]. It generalizes prior results which simulated the MusicLab through the use of a MNL model (e.g., [4, 1]) to a MMNL model. Each simulation consists of $N$ iterations and, at each iteration $t$ ($1 \leq t \leq N$), the simulator

1. randomly selects a customer class $k$ according to the classes weights $w_k$;

2. randomly selects an item $i$ for the incoming customer according to the probabilities $p_{i,k}(\sigma, d)$, where $\sigma$ is the ranking proposed by the policy under evaluation and $d$ is the popularity signal;

3. randomly determines, with probability $q_{k,i}$, whether selected item $i$ is downloaded. In the case of a download, the simulator increases the popularity signal for item $i$, i.e., $d_{i,t+1} = d_{i,t} + 1$. Otherwise, $d_{i,t+1} = d_{i,t}$.

The experimental setting aims at being close to the MusicLab experiments and it considers 50 items and simulations with 20,000 steps. The reported results in the graphs are the average of 1,000,000 simulations.

Qualities and Appeals To highlight and complement the theoretical results, we consider four different schemes. The schemes share the following characteristics: They have two customer classes with the same weight and they use 50 products. They differ in how the values for the item appeals and qualities are chosen. The schemes are depicted visually in Figures 2 and 3 and were obtained as follows:

1. Scheme 1: The product qualities for each class were chosen uniformly at random ($q_1$ and $q_2$ are independent). Appeals were negatively correlated with quality, i.e., $a_{i,k} = 1 - q_{i,k}$.

2. Scheme 2: Product qualities are similar to Scheme 1. Appeal vectors are now correlated with the quality vectors. More precisely, the appeal vector for each class was set to 0.8 times the quality plus a random uniform vector between -0.4 and 0.4, i.e., $a_{i,k} = q_{i,k}(0.8 + 0.4 \text{rand}(1,50))$.

3. Scheme 3: The product quality for class 1 is a random vector, while $q_{i,2} = 1 - q_{i,1} + 0.01 \ast \text{rand}(1,50)$. Appeals are negatively correlated with quality, i.e., $a_{i,k} = 1 - q_{i,k}$.
The Quality $q_i$ (blue) and Appeal $A_i$ (green and yellow) of product $i$ for settings 1 and 2 for both classes of consumers. The settings only differ in the appeal of items, and not in the quality of items. In Setting 1, the appeals are negatively correlated with quality, so that the sum between them is always 1. In Setting 2, the appeal is correlated to the quality with a small noise.

4. Scheme 4: The product qualities are the same as in Scheme 3 but the appeals are correlated with qualities, $a_{i,k} = q_{i,k}(0.8 + 0.4 \ \text{rand}(1, 50))$.

Observe that, in Schemes 3 and 4, customers in the two classes associate fundamentally different qualities with the products.

The Policies The simulations compare the average and segmented quality rankings with and without the popularity signal. We use the following notations:

- SQSSI: Segmented quality ranking with segmented popularity signal;
- SQNSI: Segmented quality ranking without popularity signal;
- AQGSI: Average quality ranking with “global” popularity signal;
- AQNSI: Average quality ranking without popularity signal.
6.2 Market Efficiency

Figure 4 depicts the results for Schemes 1 and 2. For Scheme 1, the popularity signal is beneficial for both the segmented and average quality rankings. SQSSI is the most efficient ranking policy. Interestingly, without the popularity signal, the segmentation policy (i.e., the SQNSI policy) performs the worst. Scheme 2 exhibits similar results but the benefits of popularity signal is extremely significant. It is also interesting to observe that AQGSI outperforms SQSSI early on before being overtaken as highlighted in Figure 5.

Figure 6 depicts the results for Schemes 3 and 4 and they are particularly interesting. Recall that, in Schemes 3 and 4, the two classes of customers associate opposite qualities to items. For Scheme 3, SQSSI is again the best ranking policy but the second best policy is AQNSI, the average quality ranking with no popularity signal. The worst policy is AQGSI, providing a compelling illustration of Theorem 3. The popularity signal may be detrimental to the average quality ranking. For Scheme 4, the popularity signal is again beneficial for the segmented and average quality rankings. SQSSI is almost twice as efficient than AQGSI, nicely illustrating Theorem 15 since the improvement is close to the best possible ratio. Once again, SQNSI performs the worst.

These results can be summarized as follows:
Figure 4: The Number of Purchases over Time for the Various Rankings. The x-axis represents the number of item samplings and the y-axis represents the average number of purchases over all experiments. The left figure depicts the results for Scheme 1 and the right figure for Scheme 2.

1. SQSSI (segmentation with the popularity signal) is clearly the best policy and it dominates all other policies. Market segmentation with the popularity signal is very effective in these trial-offer markets.

2. SQNSI (segmentation with no popularity signal) is almost always the worst policy and is dominated by AQNSI (average quality with no popularity signal). In these trial-offer settings, segmentation with no popularity signal is not an effective policy.

3. The popularity signal may be beneficial or detrimental to the average quality ranking. It is detrimental when the market has customers are from two classes with opposite tastes.

6.3 Purchase Profiles

We now illustrate the customer and market behaviors for the SQSSI and AGGSI rankings, which exhibit some significant differences. For Scheme 3, the results are presented in Figures 7, 8, and 9. Figure 7 depicts the purchase profiles of customers of Classes 1 and 2 for policy SQSSI. The products are sorted by increasing quality for each class: i.e., the products of highest quality for customers of class 1 (resp. class 2) is in the rightmost position in the left (resp. right) picture. Since the market is segmented, the results are not surprising and consistent with past results: The number of purchases is strongly correlated with quality. Figure 8 is more interesting and depicts the same information for policy AQGSI. Here the number of purchases is no longer correlated with quality for a specific customer class. Figure 9 compares SQSSI and AQGSI over all customers and the products are sorted by average quality. The figure highlights a fundamental difference in market behavior between the two policies, with very different products emerging as the “best sellers”.

Schemes 3 and 4 feature customer classes with opposite tastes. It is thus interesting to report the results on Scheme 1 where the tastes (qualities) were generated independently for the two classes. Figure 10 depicts these results. We already know from Figure 4 that policy SQSSI outperforms AQGSI but it is interesting to see how different the market behaves under these two
policies. For AQGSI, as expected, the products of best average quality receives the most purchases: Asymptotically the market goes to a monopoly for that product. For SQSSI, the purchases at this stage of the market are distributed through a larger number of products, each of which have fewer purchases. Asymptotically, the market will go to a monopoly for two products (one for class 1 and one for class 2) but the popularity signal is weaker for SQSSI since it is spread across the two classes. It is interesting to observe that the segmentation policy SQSSI is still more efficient than policy AGGSI despite this weaker popularity signal. Figure 11 depicts the profiles for policy SQSSI and nicely highlights that many products are receiving significant purchases.

7 Conclusion

This paper considered a trial-offer market with social influence and position bias, where customers follow a mixed multinomial logit model (MMNL). It proved that, for such a market, finding the best ranking at every step is a computationally hard problem.

The paper then studied the performance of a ranking policy (AQGSI) based on average product qualities. The paper proved that the trial-offer market converges predictably to a monopoly by transforming the MMNL model into a traditional MNL model whose appeals and qualities depends on the popularity signal at each time step but are bounded from below and above. Unfortunately, this average quality ranking is no longer guaranteed to benefit from the popularity signal in all cases.

The paper also studied a market segmentation policy which produces a different quality ranking for each class of customers and uses a popularity signal aggregating only customers in the specific segment. The resulting policy (SQSSI) is optimal asymptotically in expectation and may improve the market efficiency by a factor $K$ over AQGSI, where $K$ is the number of customer classes.

Agent-based simulation results have been presented to illustrate the theoretical results. They present settings in which the popularity signal is indeed detrimental to policy AQGSI and in which SQSSI improves AQGSI by a factor of about 2 with two classes. They also highlight that SQSSI
outperforms all other policies in these settings, complementing the fact that it is asymptotically optimal. Moreover, they highlight the fact that AQGSI and SQSSI produce very different market behavior, even in settings where the overall market efficiency is relatively close.

These results seem to indicate that a market segmentation policy, together with a popularity signal and position bias, is an interesting avenue to make markets more efficient, both for the market place and the customers.
Figure 8: The Purchase Profiles of AQGSI on Scheme 3 for Class 1 (left) and Class 2 (right).

Figure 9: The Purchase Profiles of SQSSI and AQGSI on Scheme 3 for both Classes of Customers.
Figure 10: The Purchase Profiles of SQSSI and AQGSI on Scheme 1 for both Classes of Customers.

Figure 11: The Purchase Profiles of SQSSI on Scheme 1 for Class 1 (left) and Class 2 (right).
References

[1] A. Abeliuk, Berbeglia G, M. Cebrian, and P. Van Hentenryck. The benefits of social influence in optimized cultural markets. *PLOS One*, 2015.

[2] Nick Craswell, Onno Zoeter, Michael Taylor, and Bill Ramsey. An experimental comparison of click position-bias models. In *Proceedings of the 2008 International Conference on Web Search and Data Mining*, pages 87–94. ACM, 2008.

[3] Per Engstrom and Eskil Forsell. Demand effects of consumers’ stated and revealed preferences. *Available at SSRN 2253859*, 2014.

[4] Coco Krumme, Manuel Cebrian, Galen Pickard, and Sandy Pentland. Quantifying social influence in an online cultural market. *PloS one*, 7(5):e33785, 2012.

[5] Kristina Lerman and Tad Hogg. Leveraging position bias to improve peer recommendation. *PloS one*, 9(6):e98914, 2014.

[6] Andrew Lim, Brian Rodrigues, and Xingwen Zhang. Metaheuristics with local search techniques for retail shelf-space optimization. *Management Science*, 50(1):117–131, 2004.

[7] Dana Mattioliand. On orbitz, mac users steered to pricier hotels. *The Wall Street Journal*, 2012.

[8] Daniel McFadden and Kenneth Train. Mixed mnl models for discrete response. *Journal of applied Econometrics*, 15(5):447–470, 2000.

[9] Justin M Rao. Experiments as instruments: Heterogeneous position effects in sponsored search auctions. *Available at SSRN*, 2014.

[10] Paat Rusmevichientong, David Shmoys, Chaoxu Tong, and Huseyin Topaloglu. Assortment optimization under the multinomial logit model with random choice parameters. *Production and Operations Management*, 2014.

[11] Matthew J Salganik, Peter Sheridan Dodds, and Duncan J Watts. Experimental study of inequality and unpredictability in an artificial cultural market. *Science*, 311(5762):854–856, 2006.

[12] Catherine Tucker and Juanjuan Zhang. How does popularity information affect choices? a field experiment. *Management Science*, 57(5):828–842, 2011.

[13] P. Van Hentenryck, A. Abeliuk, Berbeglia F., and Berbeglia G. On the optimality and predictability of cultural markets with social influence. 2015.

[14] Erjen Van Nierop, Dennis Fok, and Philip Hans Franses. Interaction between shelf layout and marketing effectiveness and its impact on optimizing shelf arrangements. *Marketing Science*, 27(6):1065–1082, 2008.

[15] Giampaolo Viglia, Roberto Furlan, and Antonio Ladrón-de Guevara. Please, talk about it! when hotel popularity boosts preferences. *International Journal of Hospitality Management*, 42:155–164, 2014.
Proofs

Proof of Theorem 1. The proof uses the 2-Class Logit problem which is known to be NP-hard \[10\]. The inputs to a 2-Class Logit instance are \(N\) products, two sequences \(V^1 = (V^1_1, V^1_2, \ldots, V^1_N)\) and \(V^2 = (V^2_1, V^2_2, \ldots, V^2_N)\) with \(V^1, V^2 \in \mathbb{Q}_+^N\), and a number \(\alpha \in [0, 1]\). Each product \(i\) has a revenue \(r_i \in \mathbb{Z}_+\). Each sequence \(V^i\) represents a realization of the product utilities under a multinomial logit model. Sequence \(V^1\) (resp. \(V^2\)) has a realization probability of \(\alpha\) (resp. \(1 - \alpha\)). The problem consists in finding a product assortment \(S \subseteq [N]\) maximizing the expected revenue \(\Pi^{Logit}\), i.e.,

\[
\Pi^{Logit} = \max_{S \subseteq [N]} \alpha \frac{\sum_{i \in S} r_i V^1_i}{1 + \sum_{i \in S} V^1_i} + (1 - \alpha) \frac{\sum_{i \in S} r_i V^2_i}{1 + \sum_{i \in S} V^2_i}.
\]

The proof shows that, if there exists an oracle to compute the performance ranking for the MMNL with two classes of consumers (i.e., Equation (2) with \(K = 2\)), then the 2-Class logit problem can be solved in polynomial time.

Given an instance of the 2-Class Logit problem, the idea is to create \(N\) different instances of the performance-ranking problem in order to capture the various possible assortments. The \(N\) instances have a common core. Each of them has the same \(N\) items and two classes of consumers (i.e., \(K = 2\)). For each consumer class \(j \in \{0, 1\}\) and each item \(i \in [N]\), we set the appeal of item \(i\) for class \(j\) to satisfy \(a_{i,j} = V^j_i\). Similarly, for each consumer class \(j \in \{0, 1\}\) and each item \(i \in [N]\), we set the quality of \(i\) for class \(j\) to satisfy \(q_{i,j} = r_i\). Note that the quality of item \(i\) is the same for both classes. The weights of classes 0 and 1 are \(\alpha\) and \(1 - \alpha\) respectively. We also set \(z = 1\) and \(t = 0\) which implies that \(d^t = 0\). The \(N\) instances differ in the position visibilities. In instance \(i\) (\(i \in [N]\)), the visibility of position \(j \in [N]\) is:

\[
v_j = \begin{cases} 
1 & \text{if } j \leq i \\
0 & \text{otherwise}.
\end{cases}
\]

Let \(\Pi^{PR}_i\) denote the optimal value of the performance ranking for problem instance \(i\) and let \(\mathcal{S}_i\) denote the collection of all possible subsets of products whose size is \(i\), i.e., \(\mathcal{S}_i = \{S \subseteq [N] : |S| = i\}\). Define \(\Pi^{Logit}_i\) as the following optimization problem:

\[
\Pi^{Logit}_i = \max_{S \in \mathcal{S}_i} \alpha \frac{\sum_{i \in S} r_i V^1_i}{1 + \sum_{i \in S} V^1_i} + (1 - \alpha) \frac{\sum_{i \in S} r_i V^2_i}{1 + \sum_{i \in S} V^2_i}.
\]

It follows that

\[
\Pi^{Logit} = \max_{i=1,\ldots,N} \Pi^{Logit}_i.
\]

We now show that \(\Pi^{PR}_i\) is equal to \(\Pi^{Logit}_i\).
\[ \Pi_i^{PR} = \max_{\sigma \in S_n} \left\{ \sum_{c=1}^{2} \left( w_c \cdot \sum_{\ell=1}^{N} \left( p_{i}(\sigma,0) \cdot q_{\ell,c} \right) \right) \right\}, \tag{17} \]

\[ = \max_{\sigma \in S_n} \left\{ \sum_{c=1}^{2} \left( w_c \cdot \sum_{\ell=1}^{N} \left( \frac{v_{\sigma(\ell)}(a_{\ell,k})}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,c}) + 1} \cdot q_{\ell,k} \right) \right) \right\}, \tag{18} \]

\[ = \max_{\sigma \in S_n} \left\{ \sum_{c=1}^{2} \left( w_c \cdot \sum_{\ell=1}^{N} \left( \frac{v_{\sigma(\ell)}V_1^r}{\sum_{j=1}^{N} v_{\sigma(j)}V_1^j + 1} \right) \right) \right\} + (1 - \alpha) \cdot \sum_{\ell=1}^{N} \left( \frac{v_{\sigma(\ell)}V_2^r}{\sum_{j=1}^{N} v_{\sigma(j)}V_2^j + 1} \right) \right\}, \tag{19} \]

\[ = \max_{S \in S_n} \left\{ \alpha \cdot \sum_{\ell \in S} \left( \frac{V_1^r}{\sum_{j \in S} V_1^j + 1} \right) + (1 - \alpha) \cdot \sum_{\ell \in S} \left( \frac{V_2^r}{\sum_{j \in S} V_2^j + 1} \right) \right\} \tag{20} \]

\[ = \Pi_i^{Logit} \tag{21} \]

where the equivalence between (19) and (20) follows from the fact that the first \( i \) positions have visibility of 1 and the remaining ones have a visibility of 0 and therefore selecting a permutation \( \sigma \in S_n \) reduces to deciding which \( i \) items should be assigned the top \( i \) positions. As a consequence, using (16), we have

\[ \Pi^{Logit} = \max_{i=1,\ldots,N} \Pi_i^{Logit} \tag{22} \]

\[ = \max_{i=1,\ldots,N} \Pi_i^{PR}. \tag{23} \]

We have shown that, by using an oracle to solve \( N \) instances of the performance-ranking problem, it is possible to solve the original 2-class logit problem instance in polynomial time. Hence, the performance ranking is NP-hard under Turing reductions.

**Proof of Lemma 1.** The market share of item \( i^* \) at any period of time \( t > \hat{t} \) for this system would be underestimated by considering the following set of qualities and appeals:

\[ q_{i,\text{new}} \begin{cases} q_{i,\text{min}} & \text{if } i = i^* \\ q_{i,\text{max}} & \text{if } i \neq i^* \end{cases} \]

\[ a_{i,\text{new}} \begin{cases} a_{i,\text{min}} & \text{if } i = i^* \\ a_{i,\text{max}} & \text{if } i \neq i^* \end{cases}. \]

If this new set of qualities satisfies that \( v_{\sigma(i^*)}q_{i^*,\text{new}} > v_{\sigma(i)}q_{i,\text{new}} \) for all \( i \in [N] \setminus \{i^*\} \), it follows from the convergence result in [13] (Theorem 4.3) that the system goes to a monopoly for item \( i^* \). Therefore, the original system also goes to a monopoly for item \( i^* \).

**Proof of Theorem 2.** The proof first shows that the MMNL model can be reduced to a Multinomial Logic Model whose item appeals and qualities are functions of the vector of purchases at each time \( t \). It then shows that these functions stay in the bounded range, so that it is possible to apply Lemma 1.

When the same ranking \( \sigma \) and popularity signals are shown to all consumers, the probability that item \( i \) is downloaded in time period \( t \) is given by

\[ P_i(\sigma, d_t) = \sum_{k=1}^{K} \left( w_k \cdot \left( \frac{a_{i,k} + d_{i}^t}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_{j}^t) + z} \cdot q_{i,k} \right) \right). \tag{24} \]
By rearranging the previous expression, it comes

\[ P_i(\sigma, d^t) = \sum_{k=1}^{K} \frac{w_k q_{i,k} v_{\sigma(i)}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z} \alpha_{i,k} + \sum_{k=1}^{K} \frac{w_k q_{i,k} v_{\sigma(i)}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z} d_i^t \]

\[ = \sum_{k=1}^{K} \frac{w_k q_{i,k} v_{\sigma(i)}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z} \alpha_{i,k} + d_i^t \left( \frac{\sum_{k=1}^{K} \frac{w_k q_{i,k} v_{\sigma(i)}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z}}{\sum_{k=1}^{K} \frac{w_k q_{i,k} v_{\sigma(i)}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z}} \right) \]

\[ = \left( \sum_{k=1}^{K} \frac{w_k q_{i,k} v_{\sigma(i)}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z} \right) \left( \frac{\sum_{k=1}^{K} \frac{w_k q_{i,k} a_{i,k}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z}}{\sum_{k=1}^{K} \frac{w_k q_{i,k}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z}} \right) \left( \frac{v_{\sigma(i)} \left( \sum_{k=1}^{K} \frac{w_k q_{i,k} a_{i,k}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z} \right)}{v_{\sigma(i)} \left( \sum_{k=1}^{K} \frac{w_k q_{i,k}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z} \right)} \right) + d_i^t \]

Now, for each item \( i \) and each time period \( t \), define the function

\[ \tilde{a}_i(t) = \frac{\sum_{k=1}^{K} \frac{w_k q_{i,k} a_{i,k}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z}}{\sum_{k=1}^{K} \frac{w_k q_{i,k}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z}} \] (25)

which depends on the total number of purchases at time \( t \). Using this definition, we have that:

\[ P_i(\sigma, d^t) = \left( \sum_{k=1}^{K} \frac{w_k q_{i,k} v_{\sigma(i)}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z} \right) \left( \frac{v_{\sigma(i)} \left( \sum_{k=1}^{K} \frac{w_k q_{i,k} a_{i,k}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z} \right)}{v_{\sigma(i)} \left( \sum_{k=1}^{K} \frac{w_k q_{i,k}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z} \right)} \right) \left( \tilde{a}_i(t) + d_i^t \right) . \]

By dividing and multiplying by \( \sum_{j=1}^{N} v_{\sigma(j)}(\tilde{a}_j(t) + d_j^t) + z \), \( P_i(\sigma, d^t) \) becomes

\[ \left( \sum_{k=1}^{K} \frac{w_k q_{i,k}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z} \right) \left( \sum_{j=1}^{N} v_{\sigma(j)}(\tilde{a}_j(t) + d_j^t) + z \right) \left( \frac{v_{\sigma(i)} (\tilde{a}_i(t) + d_i^t)}{\sum_{j=1}^{N} v_{\sigma(j)}(\tilde{a}_j(t) + d_j^t) + z} \right) . \]

Now define the following function for each item \( i \) at each time period \( t \):

\[ \tilde{q}_i(t) = \left( \sum_{k=1}^{K} \frac{w_k q_{i,k}}{\sum_{j=1}^{N} v_{\sigma(j)}(a_{j,k} + d_j^t) + z} \right) \left( \sum_{j=1}^{N} v_{\sigma(j)}(\tilde{a}_j(t) + d_j^t) + z \right) . \] (26)
The probability of purchasing product $i$ in the next iteration becomes:

$$P_i(\sigma, d^t) = \left( \frac{v_{\sigma(i)}(\bar{a}_i(t) + d_i^t)}{\sum_{j=1}^{N} v_{\sigma(j)}(a_j(t) + d_j^t) + z} \right) \tilde{q}_i(t).$$

This is almost a multinomial logit model, except that the quality and appeal vectors that depend on time. When the number of iterations $t$ tends to infinity, the total number of purchases $\sum_{j=1}^{N} d_j^t$ also goes to infinity. Moreover, as $t$ goes to infinity, the generalized appeal ($\bar{a}_i(t)$) and quality ($\tilde{q}_i(t)$) for every item converges to

$$\bar{a}_i \equiv \lim_{t \to \infty} \bar{a}_i(t) = \frac{\sum_{k=1}^{K} w_k a_{i,k} q_{i,k}}{\sum_{k=1}^{K} w_k q_{i,k}} \quad \text{and} \quad \tilde{q}_i \equiv \lim_{t \to \infty} \tilde{q}_i(t) = \sum_{k=1}^{K} w_k q_{i,k}.$$

In addition, observe that $\bar{Q}_i(t) \equiv v_{\sigma(i)}\bar{q}_i(t)$ also converges when $t$ goes to infinity:

$$\bar{Q}_i \equiv \lim_{t \to \infty} v_{\sigma(i)}\tilde{q}_i(t) = v_{\sigma(i)}\tilde{q}_i.$$

The tie-breaking condition (Equation [6]) guarantees that there exists only one item $i^*$ such that $i^* = \arg \max_{i \in [N]} \bar{Q}_i$. Let $i^{**}$ be the item with the second highest value $\bar{Q}_i$, i.e., $\bar{Q}_{i^{**}} \geq \bar{Q}_j$ for all $j \in [N], j \neq i^*$. Consider now the following difference $\Delta \bar{Q} = \bar{Q}_{i^*} - \bar{Q}_{i^{**}}$. Equation [25] can be seen as a weighted average on $k$ for $a_{i,k}$ and hence

$$\min_{1 \leq k \leq K} a_{i,k} \leq \bar{a}_i(t) \leq \max_{1 \leq k \leq K} a_{i,k} \quad \forall i \in [N], t \in \mathbb{N} \quad (27)$$

Moreover, by applying this result to Equation [26], we obtain the following bounds for $\tilde{q}_i$:

$$\tilde{q}_i(t) \geq \sum_{k=1}^{K} \frac{w_k q_{i,k}}{\sum_{j=1}^{N} v_{\sigma(j)}(\max_{1 \leq k \leq K} a_{i,k} + d_j^t) + z} \left( \sum_{j=1}^{N} v_{\sigma(j)}(-\max_{1 \leq k \leq K} a_{i,k} + \max_{1 \leq k \leq K} a_{i,k} + d_j^t) + z \right)$$

$$= \sum_{k=1}^{K} \frac{w_k q_{i,k} \left( \sum_{j=1}^{N} v_{\sigma(j)}(\max_{1 \leq k \leq K} a_{i,k} + d_j^t) + z \right) - w_k q_{i,k} \max_{1 \leq k \leq K} a_{i,k}}{\sum_{j=1}^{N} v_{\sigma(j)}(\max_{1 \leq k \leq K} a_{i,k} + d_j^t) + z}$$

$$= \left( 1 - \frac{\sum_{j=1}^{N} v_{\sigma(j)}(\max_{1 \leq k \leq K} a_{i,k})}{\sum_{j=1}^{N} v_{\sigma(j)} d_j^t + z} \right) \tilde{q}_i.$$

Since $z \geq 0$, we have that ($\forall i \in [1, N], t \in \mathbb{N}$)

$$\tilde{q}_i(t) \geq \left( 1 - \frac{\sum_{j=1}^{N} v_{\sigma(j)}(\max_{1 \leq k \leq K} a_{i,k})}{\sum_{j=1}^{N} v_{\sigma(j)} d_j^t + z} \right) \tilde{q}_i$$

$$\tilde{q}_i(t) \leq \frac{\sum_{k=1}^{K} w_k q_{i,k}}{\sum_{j=1}^{N} v_{\sigma(j)}(\max_{1 \leq k \leq K} a_{i,k})} \left( \sum_{j=1}^{N} v_{\sigma(j)}(\max_{1 \leq k \leq K} a_{i,k} + d_j^t) + z \right)$$

$$\leq \left( 1 + \frac{\sum_{j=1}^{N} v_{\sigma(j)}(\max_{1 \leq k \leq K} a_{i,k})}{\sum_{j=1}^{N} v_{\sigma(j)} d_j^t} \right) \tilde{q}_i.$$
As a result, the bounds for \( \tilde{Q}_i(t) \) (\( \forall i \in [1, N], t \in \mathbb{N} \)) are given by

\[
\tilde{Q}_i(t) \geq \left( 1 - \frac{\sum_{j=1}^{N} v_{\sigma(j)} \max_{1 \leq k \leq K} a_{j,k} }{\sum_{j=1}^{N} v_{\sigma(j)} d^*_j} \right) \tilde{Q}_i,
\]

\[
\tilde{Q}_i(t) \leq \left( 1 + \frac{\sum_{j=1}^{N} v_{\sigma(j)} \max_{1 \leq k \leq K} a_{j,k} }{\sum_{j=1}^{N} v_{\sigma(j)} d^*_j} \right) \bar{Q}_i.
\]

To conclude the proof, we need to estimate the total number of purchases \( \hat{d}_{tot} \) that guarantees that

\[
\forall t > t^* : \tilde{Q}_{i^*}(t) > \tilde{Q}_{i^{**}}(t)
\]

where \( t^* \) is the time period in which the total number of purchases becomes \( \hat{d}_{tot} = \sum_{i \in [N]} d^*_i \). The value \( \hat{d}_{tot} \) and its associated vector of purchases \( d^* \) must satisfy the following condition

\[
\Delta \bar{Q} > \frac{\sum_{j=1}^{N} v_{\sigma(j)} \max_{1 \leq k \leq K} a_{j,k} }{\sum_{j=1}^{N} v_{\sigma(j)} d^*_j} (\bar{Q}_{i^*} + \bar{Q}_{i^{**}}).
\] (28)

To verify inequality (28), it suffices to choose \( \hat{d}_{tot} \) to satisfy So with the following condition we guarantee the validity of equation

\[
\hat{d}_{tot} > \frac{\max_j v_{\sigma(j)} \sum_{j=1}^{N} \max_{1 \leq k \leq K} a_{j,k} }{\min_j v_{\sigma(j)} \Delta \bar{Q}} (\bar{Q}_{i^*} + \bar{Q}_{i^{**}}).
\] (29)

We can now apply Lemma 1 using ranking policy \( \sigma \) to prove that the model goes to a monopoly for item \( i^* \), which maximizes the product of its visibility and its weighted average quality, i.e., \( v_{\sigma(i)} \bar{q}_i \).

**Proof of Corollary 1.** From Theorem (2), a MMNL model goes to a monopoly for the item \( i \) that maximizes \( v_{\sigma(i)} \bar{q}_i \). When the quality ranking is used, the product \( i^* \) that goes to a monopoly is

\[
i^* = \arg \max_i v_{\sigma(i)} \bar{q}_i = \arg \max_i \bar{q}_i = \bar{q} \]

\[\Box\]