A Concentrated Capacity Model for Diffusion-Advection: Advection Localized to a Moving Curve

Colin Klaus
Mathematical Biosciences Institute, The Ohio State University
1735 Neil Avenue, Columbus OH 43212, USA
email: klaus.68@osu.edu

Abstract

In this work I show how a diffusion-advection equation in three space-dimensions may have its advection term weakly limited to a velocity field localized to a moving curve. This is rigorously accomplished through the technique of concentrated capacity, and the form of the concentrated capacity limit along with small time existence of solutions is determined. This problem is motivated by mathematical biology and the study of proteins in solvent where the latter is modeled as a diffusing quantity and the protein is taken to be a 1d object which advects the solvent by contact and its own motion. This work introduces a novel PDE’s framework for that interaction.

Key Words: Concentrated capacity, PDE’s with measure coefficients, mathematical biology.

1 Introduction

Consider a bounded domain $\Omega \subset \mathbb{R}^3$ and its parabolic cylinder $\Omega_T = [0, T) \times \Omega$. Initially, we consider a diffusion-advection equation with vanishing Neumann condition

\begin{align}
  u_t + \text{div} \left[ -K(x; t) \nabla u + u\vec{v} \right] &= 0 \text{ in } \Omega_T \quad \text{(1.1)}
  \\
  (-K(x; t) \nabla u + u\vec{v}) \cdot \vec{n} &= 0 \text{ on } \partial\Omega_T \quad \text{(1.2)}
\end{align}

We now would like to make precise the solution to such a problem when (1.1, 1.2) is interpreted weakly and $\vec{v} = \vec{v}(t, s)d\delta_{\Gamma(t)}$ is Dirac-mass localized to a moving curve. However, as stated, such a formulation lacks the necessary compactness to justify the existence of solutions in a natural function space. The technical difficulty is its producing uncontrolled energy terms of the form $\int_{\Gamma(t)} u\nabla u\vec{v}d\sigma$ after taking $u$ as a test function. It is natural to replace the distributional
velocity field $\vec{v}$ with an approximation $\vec{v}_\epsilon$ which recovers the distributional field weakly in the limit. In this paper, it is demonstrated how the technique of concentrating capacity makes this intuition precise.

Other instances of concentrated capacity in the literature may be found in \cite{1,2,3,4} for example. More recent examples are \cite{5,6}. A common theme is that a subset of the domain, which sites essential physics but is intrinsically lower-dimensional up to a small parameter $\epsilon$, is diminished by passing $\epsilon \to 0$. Sufficient compactness is included so that, in the limit, a new partial differential equation is recovered with the true lower dimensional domain and which preserves the physics of the original problem.

1.1 Novelty and Significance

The investigation presented here is novel for its applying concentrating capacity to a time-varying curve rather than a stationary hypersurface. In particular, the technique is shown to make precise a parabolic PDE whose equation coefficients are highly singular, time-varying measures.

In addition to its mathematical novelty, this problem takes inspiration from computational biology and the study of proteins in solvent. Proteins are chains of amino acids linked together in sequence along a polypeptide bond and are the molecular machines that drive many dynamics inside biological cells. In molecular dynamics simulations \cite{7,8}, resolving the motion of the explicit solvent which surrounds the protein is a great computational expense \cite[Sec 8.3]{7}. However, the localized interaction of solvent at different sites along the protein can strongly influence the conformation the protein takes \cite{9,10}, which in turn affects its ability to bind its targets or otherwise properly function. The implications for this extend to the molecular basis for disease as well as drug-design \cite{11}. This concentrated capacity model presents a novel PDE’s framework for considering these issues, while future investigations may incorporate other essential features such as amino acid sidechains and molecular force fields.

1.1.1 Diffusion-Advection Applied to Solvated Proteins

Note that in this section the variable symbols do not necessarily correspond with those in Sec 1.2. Also, this section’s argument is heuristic in nature. We proceed by first principles and suppose the solvent moves about the protein by conservation of mass, i.e. the equation of continuity.

$$u_t + \text{div} (u\vec{v}) = 0$$

As usual, the material flow will be comprised by diffusion and advection terms. However, to allow that at equilibrium the solvent may not be at uniform density owing to space occupied by the protein, we introduce an effective scaling $\rho(t; x)$ which is identically 1 away from the protein and greater than 1 about the protein. Now we proceed with the usual Fick’s Law assertion

$$u\vec{v} = -K \nabla (u\rho) + u\vec{v}_C$$
In the simplest conceptual case, the advection velocity $\vec{v}_C$ is the protein’s own motion, and the solvent moves by no-slip contact.

By combining these two equations and expanding the gradient, we arrive at the following model which is formally equivalent to (1.1).

$$u_t + \text{div} \left[ -K\rho \nabla u + u(\vec{v}_C - K\nabla \rho) \right] = 0$$

1.2 The Data for the Family of Approximating Problems

We now state more formally the data for the family of approximating diffusion-advection equations over which concentrating capacity is performed. In the process, we allow for the possibility that the diffusivity tensor – similarly the advection velocity – is functionally distinct across two regimes: one away from the curve and one at the curve.

The data for the problems consists of

1. Real numbers $T, \epsilon_0 > 0$

2. A smooth and time-varying curve:

   $$\Gamma : [0, T] \times [-\epsilon_0, 1 + \epsilon_0] \to \Omega \quad (1.3)$$

3. A positively oriented, and smoothly varying orthonormal triple:

   $$\vec{t}, \vec{n}, \vec{b} : [0, T] \times [-\epsilon_0, 1 + \epsilon_0] \to \mathbb{R}^3 \quad (1.4)$$

   $$(\vec{t}(t, s) = \partial_s \Gamma / |\partial_s \Gamma| \quad \vec{n}(t, s) \quad \vec{b}(t, s)) \in \text{SO}_3(\mathbb{R})$$

4. A map smoothly parameterizing a moving neighborhood of the curve:

   $$\mathcal{F} : [0, T] \times [-\epsilon_0, 1 + \epsilon_0] \times D_{\epsilon_0} \to [0, T] \times \mathcal{N}_{\epsilon_0}(t) \quad (1.5)$$

   $$\mathcal{F}(t, s, \nu, \omega) = \left( F(t, s, \nu, \omega) = \Gamma(t, s) + \nu \vec{n}(t, s) + \omega \vec{b}(t, s) \right)$$

   Here $D_{\epsilon_0}$ is a closed disc in $\mathbb{R}^2$ of radius $\epsilon_0$ centered at the origin. $\mathcal{N}_{\epsilon_0}(t)$ is the image of $F$ at time $t$. We will further denote

   $$D\mathcal{F} = \left( 1 \partial_t F \quad \nabla F (= D_{s,\nu,\omega} F) \right) \quad (1.6)$$

   with respective Jacobians $J_F$ and $J_{\mathcal{F}}$ as well as the Eulerian velocity field $\vec{w} : \text{Im} \mathcal{F} \to \mathbb{R}^3$ satisfying

   $$\vec{w} \circ \mathcal{F} = \partial_t F \quad (1.7)$$

5. Real numbers $\epsilon \in (0, \epsilon_0)$ and $\delta \in (0, \epsilon_0 - \epsilon)$
6. The geometric set and functions
\[ C_\epsilon = [0,1] \times D_\epsilon \]  
\[ \chi_\delta(t) = (1 - t/\delta)_+ \]  
\[ d_\epsilon(x) = \text{dist}(x,C_\epsilon) \]  
\[ \text{(1.8)} \]
\[ \text{(1.9)} \]
\[ \text{(1.10)} \]

**Remark 1** It is evident that \( \chi_\delta \) and \( d_\epsilon \) are Lipschitz.

We will also use the notation
\[ N_{\epsilon,\delta}(t) = F(t; [d_\epsilon(x) \leq \delta]) \]  
\[ \text{(1.11)} \]

7. The decay functions \( \zeta_{\epsilon,\delta} : [0,T] \times \Omega \rightarrow \mathbb{R} \) which are (smoothly) zero-extended outside \( N_{\epsilon_0}(t) \) and inside are determined by the relation
\[ \zeta_{\epsilon,\delta} \circ F = \chi_\delta \circ d_\epsilon \]  
\[ \text{(1.12)} \]

**Remark 2** By the regularity of \( F \), \( \zeta_{\epsilon,\delta} \) is Lipschitz because \( \chi_\delta \) and \( d_\epsilon \) are.

8. Concentrating capacity coefficients \( a_{\epsilon,\delta} : [0,T] \times \bar{\Omega} \rightarrow \mathbb{R} \) and given by
\[ a_{\epsilon,\delta}(t;x) = 1 + \left( \frac{\epsilon^2}{\epsilon^2} - 1 \right) \zeta_{\epsilon,\delta}(t;x) \]  
\[ \text{(1.13)} \]

9. Diffusivity tensors \( K_{\epsilon,\delta} : [0,T] \times \bar{\Omega} \rightarrow \mathcal{M}_{3 \times 3}(\mathbb{R}) \) given by
\[ K_{\epsilon,\delta}(t;x) = k_0 \mathbb{I}_3 + \left( \frac{\epsilon^2}{\epsilon^2} - 1 \right) \zeta_{\epsilon,\delta}(t;x) \]  
\[ \text{(1.14)} \]
\( \mathbb{I}_n \) is the n-dimensional identity matrix. We also assume that the principle directions of diffusion for \( K \) are governed by the curve geometry and that
- The functions \( k_s, k_n : [-\epsilon_0, 1+\epsilon_0] \times D_{\epsilon_0} \rightarrow (0,\infty) \) are assumed continuous.

10. A real number \( \vartheta \) such that
\[ 0 < \vartheta \leq \min \{k_0, k_s(s,\nu,\omega), k_n(s,\nu,\omega)\} \]  
\[ \text{(1.15)} \]

11. Advection fields \( \bar{v}_{\epsilon,\delta} : [0,T] \times \bar{\Omega} \rightarrow \mathbb{R}^3 \) given by
\[ \bar{v}_{\epsilon,\delta}(t;x) = \bar{v}(t;x) + \left( \frac{\epsilon^2}{\epsilon^2} - 1 \right) \zeta_{\epsilon,\delta}(t;x) \]  
\[ \text{(1.16)} \]
- The fields \( \bar{v}, \bar{v}_C : [0,T] \times \bar{\Omega} \rightarrow \mathbb{R}^3 \) are assumed continuous.

12. Initial data \( u_0 \in \mathcal{C}(\bar{\Omega}) \)
1.3 Statement of the Concentrated Capacity Limit

In the next two sections, we present the equation to be satisfied by the concentrated capacity limit of \((1.1 \text{ to } 1.2)\) and the main theorem providing its small time existence.

**Definition 1** Defining the Limit

By the concentrated capacity solution of \((1.1 \text{ to } 1.2)\) with data prescribed in Sec \(1.2\), it is meant a solution pair \( (u, u_C) \in L^2(\Omega_h) \times L^2([0,1]_h) \), whose spatial gradients satisfy \( (\nabla u, \partial_s u_C) \in L^2(\Omega_h) \times L^2([0,1]_h) \), and satisfying for all smooth test functions \( \varphi \in C^\infty(\overline{\Omega}_h) \) such that \( \varphi(h; \cdot) \equiv 0 \)

\[
0 = \left( \int_{\Omega} \varphi(0; \cdot)u_0 dx - \int_{\Omega_h} \varphi_t u_0 dxdt + \int_{\Omega_h} \nabla \varphi \cdot k_0 \tilde{v}_i \nabla u dxdt - \int_{\Omega_h} \nabla \varphi \tilde{v}_i dxdt \right)_{\text{Vol}}
\]

(1.17)

\[
+ \pi \varepsilon_0^2 \left( \int_0^1 (\varphi u_0)(0,s,0,0) |\partial_s \Gamma| ds - \int_{[0,1]_h} \left[ \varphi_t - \nabla_{s,v,\omega} \varphi \cdot \nabla F^{-1} \partial_t F \right] (t,s,0,0)u_C |\partial_s \Gamma| ds dt 
- \int_{[0,1]_h} \left[ \nabla_{s,v,\omega} \varphi \cdot \nabla F^{-1} \tilde{v}_C \right] (t,s,0,0)u_C |\partial_s \Gamma| ds dt 
+ \int_{[0,1]_h} \left[ \nabla_{s,v,\omega} \varphi \cdot \left( \frac{k_s}{|\partial_s \Gamma|^2} \frac{\tilde{v}_C}{k_n \hat{n}_2} \right) \right] (t,s,0,0) \left( \xi_v \xi_v \xi_w \right) |\partial_s \Gamma| ds dt \right)_{\text{Line}}
\]

The functions \( \xi_v, \xi_w \) are given by

\[
\xi_v = \left[ \tilde{v}_2 \cdot \nabla F^{-1} (\tilde{v}_C - \partial_t F) \right] (t,s,0,0)u_C/k_n(s,0,0) \quad \xi_w = \left[ \tilde{v}_3 \cdot \nabla F^{-1} (\tilde{v}_C - \partial_t F) \right] (t,s,0,0)u_C/k_n(s,0,0)
\]

(1.18)

**Remark 3** As will be shown, the form of \((1.17)\) is a natural consequence of the compactness needed to make sense of the limit.

**Remark 4** Observe that the limit is independent of the diffusion coefficient \( k_n \). Also, as expected the 1d integrals can be regarded as line integrals taken over \( \Gamma \).

**Remark 5** For the volume integrals, \( \varphi \) and its gradient are with respect to rectangular coordinates \((t, x, y, z)\) of \( \Omega \). For the line integrals, \( \varphi \) and its gradient are with respect to local coordinates at the curve \((t, s, \nu, \omega)\).

**Remark 6** Since \( u_C \) belongs to \( L^2([0,1]_h) \) in the variables \((t, s)\), it cannot be differentiated in the variables \( \nu, \omega \) orthogonal to the curve. Nevertheless, the limiting process over the family of diffusion-advection equations \((2.1)\) retains a memory of the solution gradient in these directions. It will be shown in a weak sense that \((\xi_v, \xi_w)\) is the orthogonal gradient of \( u_C \) at the curve.
1.4 The Main Result

The main argument of this paper will concern the small time existence of concentrated capacity solutions for the diffusion-advection equation (1.1, 1.2).

**Theorem 1** For the data prescribed in Sec 1.2, there is an \( h = h(\text{data}) > 0 \) for which a concentrated capacity solution satisfying (1.17) exists.

Although the inherent regularity of (1.17) is not sufficient to rigorously justify its strong form, it is instructive to proceed formally with integration by parts and extract a corresponding but formal pointwise PDE from the weak one. We report this here over each subdomain as well as the global domain.

1.4.1 Formal Pointwise Limit: \( \Omega \setminus \Gamma(t) \)

By using an arbitrary test function in (1.17) which vanishes in a space-time neighborhood of the curve and integrating by parts twice, we arrive at a simple diffusion-advection process away from the curve.

\[
\partial_t u - k_0 \Delta u + \text{div} (u\vec{v}) = 0 \quad \text{in } \Omega \setminus \Gamma(t)
\]

1.4.2 Formal Pointwise Limit: \( \Gamma(t) \)

In (1.17) integrate by parts twice after taking a test function of type \( \phi \zeta_\epsilon, \delta \) with \( \phi \) arbitrary and letting \( \delta, \epsilon \to 0 \). Let \( D_\epsilon(\Gamma) \) denote the 2d cross section of \( N_\epsilon, 0(\Gamma) \) centered about \( \Gamma(t, s) \). Note that all derivatives below are with respect to the ambient \((t, x, y, z)\) coordinate system.

\[
\partial_t (u C d\delta_\Gamma(t)) - \text{div} (K \nabla u C d\delta_\Gamma(t)) + \text{div} (u C \vec{v} C d\delta_\Gamma(t)) = \left( \frac{1}{\pi \epsilon^2} \lim_{\epsilon \to 0} \int_{D_\epsilon(\Gamma)} k_0 \Delta u \right) d\delta_\Gamma(t) \quad \text{on } \Gamma(t)
\]

**Remark 7** Because \( \Delta u \notin L^1(\Omega_h) \), its limit above need not vanish.

1.4.3 Formal Pointwise Limit: Global

In (1.17) take an arbitrary test function and integrate by parts twice. Again all derivatives below are with respect to the ambient coordinate system.

\[
\partial_t (u + \pi \epsilon_0^2 u C d\delta_\Gamma(t)) - \text{div} \left( k_0 \nabla u + \pi \epsilon_0^2 K \nabla u C d\delta_\Gamma(t) \right) + \text{div} \left( u \vec{v} + \pi \epsilon_0^2 u C \vec{v} C d\delta_\Gamma(t) \right) = 0 \quad \text{in } \Omega
\]
2 Weak Compactness of Approximating Problems

2.1 The Family of Approximating Problems

Consider the following family of diffusion-advection equations with vanishing Neumann data and initial data $u_0$.

$$\partial_t \left( a_{\epsilon,\delta}(t;x) u_{\epsilon,\delta} \right) - \text{div} \left[ K_{\epsilon,\delta}(t;x) \nabla u_{\epsilon,\delta} \right] + \text{div} \left[ u_{\epsilon,\delta} \vec{v}_{\epsilon,\delta}(t;x) \right] = 0 \quad \text{in} \quad \Omega_T $$  (2.1)

The existence of solutions for (2.1) follows from standard theory \cite{12, Ch. 3} and properties of the data. Solutions can be taken to satisfy $u_{\epsilon,\delta} \in C \left( [0,T]; L^2(\Omega) \right) \cap L^2 \left( [0,T]; W^{1,2}(\Omega) \right)$.

Remark 8 The placement of the $(\epsilon^2/\delta^2)$ coefficients with respect to the partial derivatives in the first two, left-hand terms of (2.1) is not arbitrary. It has been chosen for mass balance and the requirement that there should exist a compactness estimate like Prop \[1\].

2.1.1 A priori Estimates

Proposition 1 For an $h = h(\text{data})$ and $\epsilon_i \to 0$, $\exists \kappa_i(\epsilon_i)(>0) \to 0$ such that if $\delta_i \leq \kappa_i$ then

$$\text{ess sup}_{t \leq h} \int_{\Omega} a_{\epsilon_i,\delta_i} u_{\epsilon_i,\delta_i}^2 (t; \cdot) \, dx + \int_{\Omega} a_{\epsilon_i,\delta_i} \left| \nabla u_{\epsilon_i,\delta_i} \right|^2 \, dx dt \leq C(\text{data})$$

Proof:

For a positive time $h > 0$, test by $u_{\epsilon,\delta}$ in (2.1) which is justified by Steklov averages \cite{13, Ch. 2} and obtain

$$0 = \int_{\Omega} u_{\epsilon,\delta} \partial_t \left( a_{\epsilon,\delta} u_{\epsilon,\delta} \right) \, dx dt + \int_{\Omega} \nabla u_{\epsilon,\delta} K_{\epsilon,\delta} \nabla u_{\epsilon,\delta} \, dx dt - \int_{\Omega} \nabla u_{\epsilon,\delta} u_{\epsilon,\delta} \vec{v}_{\epsilon,\delta} \, dx dt$$

$$= \int_{\Omega} \frac{1}{2} \left[ a_{\epsilon,\delta} u_{\epsilon,\delta}^2 \right] (h; \cdot) \, dx - \int_{\Omega} \frac{1}{2} \left[ a_{\epsilon,\delta} u_{\epsilon,\delta}^2 \right] (0; \cdot) \, dx + \int_{\Omega} \frac{1}{2} \partial_t a_{\epsilon,\delta} u_{\epsilon,\delta}^2 \, dx dt$$

$$+ \int_{\Omega} \nabla u_{\epsilon,\delta} K_{\epsilon,\delta} \nabla u_{\epsilon,\delta} \, dx dt - \int_{\Omega} \nabla u_{\epsilon,\delta} u_{\epsilon,\delta} \vec{v}_{\epsilon,\delta} \, dx dt$$

Rearranging we have

$$\int_{\Omega} \frac{1}{2} \left[ a_{\epsilon,\delta} u_{\epsilon,\delta}^2 \right] (h; \cdot) \, dx + \int_{\Omega} \nabla u_{\epsilon,\delta} K_{\epsilon,\delta} \nabla u_{\epsilon,\delta} \, dx dt$$

$$\leq \int_{\Omega} \frac{1}{2} \left[ a_{\epsilon,\delta} u_{\epsilon,\delta}^2 \right] (0; \cdot) \, dx - \int_{\Omega} \frac{1}{2} \partial_t a_{\epsilon,\delta} u_{\epsilon,\delta}^2 \, dx dt + \int_{\Omega} \nabla u_{\epsilon,\delta} u_{\epsilon,\delta} \vec{v}_{\epsilon,\delta} \, dx dt$$
From the definitions of $K_{\varepsilon, \delta}$ \(^{(1.14)}\) and $\vartheta$ \(^{(1.15)}\), we may estimate

$$\nabla u_{t, \delta} K_{\varepsilon, \delta} \nabla u_{t, \delta} = k_0 |\nabla u_{t, \delta}|^2 (1 - \zeta_{\varepsilon, \delta}) + \frac{\varepsilon_0^2}{\varepsilon^2} \nabla u_{t, \delta} K \nabla u_{t, \delta} \zeta_{\varepsilon, \delta}$$

$$\geq \vartheta |\nabla u_{t, \delta}|^2 (1 - \zeta_{\varepsilon, \delta}) + \vartheta \frac{\varepsilon_0^2}{\varepsilon^2} |\nabla u_{t, \delta}|^2 \zeta_{\varepsilon, \delta}$$

$$= \vartheta a_{\varepsilon, \delta} \nabla u_{t, \delta}^2 \zeta_{\varepsilon, \delta}$$

Similarly, we may use the definition of $\vec{v}_{t, \delta}$ \(^{(1.16)}\) to estimate

$$|\vec{v}_{t, \delta}| \leq |\vec{v}| (1 - \zeta_{\varepsilon, \delta}) + \frac{\varepsilon_0^2}{\varepsilon^2} |\vec{v}_C| \zeta_{\varepsilon, \delta}$$

$$\leq \|\langle \vec{v}, \vec{v}_C \rangle\|_\infty a_{\varepsilon, \delta}$$

We make these substitutions into (2.2) and arrive at

$$\int_{\Omega} \frac{1}{2} \left[ a_{\varepsilon, \delta} u_{t, \delta}^2 \right] \langle h; \cdot \rangle dx + \vartheta \int_{\Omega_h} a_{\varepsilon, \delta} |\nabla u_{t, \delta}|^2 dx \tag{2.3}$$

$$\leq \int_{\Omega} \frac{1}{2} \left[ a_{\varepsilon, \delta} u_t^2 \right] \langle 0; \cdot \rangle dx - \int_{\Omega_h} \frac{1}{2} \partial_t a_{\varepsilon, \delta} u_{t, \delta}^2 dx + \|\langle \vec{v}, \vec{v}_C \rangle\|_\infty \int_{\Omega_h} a_{\varepsilon, \delta} |\nabla u_{t, \delta} u_{t, \delta}| dx \tag{2.2}$$

The time derivative of the concentrating capacity coefficients requires some attention

**Lemma 1** The $\partial_t a_{\varepsilon, \delta}$ term

$$\partial_t a_{\varepsilon, \delta} \circ F = \left( \frac{1}{\delta} \left( \frac{\varepsilon_0^2}{\varepsilon^2} - 1 \right) \nabla d_{\varepsilon} \nabla F^{-1} \partial_t F \right) \chi_{[0 < d_{\varepsilon} < \delta]}$$

**Proof:** From \(^{(1.14)}\), we see that

$$\partial_t a_{\varepsilon, \delta} = \left( \frac{\varepsilon_0^2}{\varepsilon^2} - 1 \right) \partial_t \zeta_{\varepsilon, \delta}$$

We may compute $\partial_t \zeta_{\varepsilon, \delta}$ from chain-rule and **Lemma 6**

$$\partial_t a_{\varepsilon, \delta} \circ F = \left( \frac{\varepsilon_0^2}{\varepsilon^2} - 1 \right) \partial_t \zeta_{\varepsilon, \delta} \circ F = \left( \frac{\varepsilon_0^2}{\varepsilon^2} - 1 \right) D \left[ \zeta_{\varepsilon, \delta} \circ F \right] \left( -\nabla F^{-1} \partial_t F \right)$$

$$= \left( \frac{\varepsilon_0^2}{\varepsilon^2} - 1 \right) D \left[ \xi_{\delta} \circ d_{\varepsilon} \right] \left( -\nabla F^{-1} \partial_t F \right)$$

We conclude once we observe that

$$D \left[ \xi_{\delta} \circ d_{\varepsilon} \right] = \left( 0 - \frac{1}{\delta} \chi_{[0 < d_{\varepsilon} < \delta]} \nabla d_{\varepsilon}^2 \right)$$

Using **Lemma 1**, a change of variables as in **Lemma 5** and the coarea formula
\( \int_{\Omega_h} \partial_t a_{\epsilon, \delta} u_{\epsilon, \delta}^2 dx dt = \int_0^h \int_{N_{\epsilon,0}(t)} \partial_t a_{\epsilon, \delta} u_{\epsilon, \delta}^2 dx dt \)

\( = \int_0^h \int_{[0, \epsilon, \delta]} \left( \frac{c_0^2}{c^2} - 1 \right) \int_0^\epsilon \int_{[d, \eta]} J_F \nabla F^{-1} \partial_t F u_{\epsilon, \delta}^2 \cdot \frac{\nabla d_e}{|\nabla d_e|} d\sigma d\eta dt \)

In the last line, we used that \( \nabla d_e / |\nabla d_e| \) is the exterior normal to \( [d_e \leq \eta] \) and applied the divergence theorem. Next, recall that for a vector field \( \vec{q} \) defined on \( \Omega_T \) coordinates, and which may be shown from a weak formulation,

\( \text{div}_x y z \vec{q} \circ F = \frac{1}{J_F} \text{div}_{s, \nu, \omega} [J_F \nabla F^{-1} (\vec{q} \circ F)] \quad (2.4) \)

In particular, we may use this change of variables for the motion field \( \vec{w} \), defined by (1.7). Making this substitution and changing variables back to ambient \( x, y, z \) coordinates we have

\( \int_{\Omega_h} \partial_t a_{\epsilon, \delta} u_{\epsilon, \delta}^2 dx dt = \left( \frac{c_0^2}{c^2} - 1 \right) \int_0^h \int_{N_{\epsilon,0}(t)} \text{div} (\vec{w} u_{\epsilon, \delta}^2) d\sigma d\eta dt \)

We now distribute the divergence, majorize terms, then majorize the integral over \( N_{\epsilon, \delta}(t) \), and lastly breakup the integral over disjoint domains.

\[ \int_{\Omega_h} \partial_t a_{\epsilon, \delta} u_{\epsilon, \delta}^2 dx dt \leq \frac{2c_0^2}{c^2} \| \vec{w}, \nabla \vec{w} \|_\infty \int_0^h \int_{N_{\epsilon,0}(t)} \left( u_{\epsilon, \delta}^2 + 2 |u_{\epsilon, \delta} \nabla u_{\epsilon, \delta}| \right) dx dt \]

\( \leq C(\text{data}) \left( \int_0^h \int_{N_{\epsilon,0}(t)} a_{\epsilon, \delta} u_{\epsilon, \delta}^2 dx dt + \int_0^h \int_{N_{\epsilon,0}(t)} a_{\epsilon, \delta} |u_{\epsilon, \delta} \nabla u_{\epsilon, \delta}| dx dt \right. \)

\( + \left. \frac{c_0^2}{c^2} \int_0^h \int_{[N_{\epsilon, \delta} \setminus N_{\epsilon,0}] (t)} \left( u_{\epsilon, \delta}^2 + |u_{\epsilon, \delta} \nabla u_{\epsilon, \delta}| \right) dx dt \right) \)

We focus now on the terms within \( [N_{\epsilon, \delta} \setminus N_{\epsilon,0}] (t) \). Specifically we will look at the \( |u_{\epsilon, \delta} \nabla u_{\epsilon, \delta}| \) term as the other may be handled similarly. For a still to be
determined $\sigma < 2$, apply H"older’s inequality and Lemma 8

\[
\frac{c^2}{\epsilon^2} \int_0^h \int_{[N_{\epsilon,\delta} \setminus N_{\epsilon,0}](t)} |u_{\epsilon,\delta} \nabla u_{\epsilon,\delta}| \, dx \, dt \leq \frac{c^2}{\epsilon^2} \left\| u_{\epsilon,\delta} \right\|_{\sigma,\Omega_h} \left\| \chi_{[0,h]} \times [N_{\epsilon,\delta} \setminus N_{\epsilon,0}(t)] \right\|_{2,\sigma,\Omega_h} \\
\leq C(\text{data}) \frac{\delta^{\sigma-1}}{\epsilon^2} \left\| u_{\epsilon,\delta} \right\|_{\sigma,\Omega_h} \\
\leq C(\text{data}) \frac{\delta^{\sigma-1}}{\epsilon^2} \left( \left\| \nabla u_{\epsilon,\delta} \right\|_{2,\sigma,\Omega_h} \left\| u_{\epsilon,\delta} \right\|_{\sigma,\Omega_h} \right)^{1/\sigma} \\
= C(\text{data}) \frac{\delta^{\sigma-1}}{\epsilon^2} \left\| \nabla u_{\epsilon,\delta} \right\|_{2,\Omega_h} \left\| u_{\epsilon,\delta} \right\|_{\sigma,\Omega_h}
\]

Recall from the parabolic Sobolev embedding [13] Ch. 1, $\exists \gamma(\Omega, T)$, which is increasing with respect to $T$, so that

$$
\left\| u_{\epsilon,\delta} \right\|_{10/3,\Omega_h} \leq \gamma(\Omega, T) \left( \text{ess sup}_{t \leq h} \left\| u_{\epsilon,\delta}(t; \cdot) \right\|_{2,\Omega_h} + \left\| \nabla u_{\epsilon,\delta} \right\|_{2,\Omega_h} \right)
$$

We now choose $\sigma$ to satisfy $2\sigma/(2 - \sigma) = 10/3$ and a posteriori verify that $\sigma = 5/4 < 2$ and $(\sigma - 1)/\sigma = 1/5$. For this choice we next apply Young’s inequality and majorize with $1 \leq a_{\epsilon,\delta}$.

\[
\frac{c^2}{\epsilon^2} \int_0^h \int_{[N_{\epsilon,\delta} \setminus N_{\epsilon,0}](t)} |u_{\epsilon,\delta} \nabla u_{\epsilon,\delta}| \, dx \, dt \leq C(\text{data}) \frac{\delta^{1/5}}{\epsilon^2} \left( \text{ess sup}_{t \leq h} \int_{\Omega} a_{\epsilon,\delta} u_{\epsilon,\delta}^2 (t; \cdot) \, dx + \int_{\Omega} a_{\epsilon,\delta} \left\| \nabla u_{\epsilon,\delta} \right\|^2 \, dx \right)
\]

By writing $u_{\epsilon,\delta}^2 = |u_{\epsilon,\delta} u_{\epsilon,\delta}|$ and majorizing $\left\| u_{\epsilon,\delta} \right\|^2_{2,\Omega_h} \leq h \left( \text{ess sup}_{t \leq h} \int_{\Omega} u_{\epsilon,\delta}^2 (t; \cdot) \, dx \right)$, we may proceed similarly as when it was $|u_{\epsilon,\delta} \nabla u_{\epsilon,\delta}|$.

\[
\frac{c^2}{\epsilon^2} \int_0^h \int_{[N_{\epsilon,\delta} \setminus N_{\epsilon,0}](t)} u_{\epsilon,\delta}^2 \, dx \, dt \leq C(\text{data}) \frac{\delta^{1/5}}{\epsilon^2} \left( \text{ess sup}_{t \leq h} \int_{\Omega} a_{\epsilon,\delta} u_{\epsilon,\delta}^2 (t; \cdot) \, dx + \int_{\Omega} a_{\epsilon,\delta} \left\| \nabla u_{\epsilon,\delta} \right\|^2 \, dx \right)
\]

Now we return to the energy terms of (2.5) in $N_{\epsilon,0}(t)$. These are simpler to estimate. Let $\alpha > 0$ and use Young’s inequality.

\[
\int_0^h \int_{N_{\epsilon,0}(t)} a_{\epsilon,\delta} u_{\epsilon,\delta}^2 \, dx \, dt \leq h \left( \text{ess sup}_{t \leq h} \int_{\Omega} a_{\epsilon,\delta} u_{\epsilon,\delta}^2 (t; \cdot) \, dx \right) \tag{2.8}
\]

\[
\int_0^h \int_{N_{\epsilon,0}(t)} a_{\epsilon,\delta} |u_{\epsilon,\delta} \nabla u_{\epsilon,\delta}| \, dx \, dt \leq \frac{\alpha}{2} \int_{\Omega} a_{\epsilon,\delta} \left\| \nabla u_{\epsilon,\delta} \right\|^2 \, dx + \frac{h}{2\alpha} \left( \text{ess sup}_{t \leq h} \int_{\Omega} a_{\epsilon,\delta} \left\| u_{\epsilon,\delta} \right\|^2 (t; \cdot) \, dx \right) \tag{2.9}
\]
We finally combine (2.6, 2.7, 2.8, 2.9) into (2.5).

\[
\int_{\Omega_h} \partial_t a_{\epsilon, \delta} u_{\epsilon, \delta}^2 dx dt \leq C(\text{data}) \left[ \left( h + \frac{h}{2\alpha} + \frac{\delta^{1/5}}{\epsilon^2} \right) \left( \text{ess sup}_{t \leq h} \int_{\Omega} a_{\epsilon, \delta} |u_{\epsilon, \delta}|^2 (t; \cdot) dx \right) \right. \\
\left. \left( \frac{\alpha}{2} + \frac{\delta^{1/5}}{\epsilon^2} \right) \int_{\Omega_h} a_{\epsilon, \delta} |\nabla u_{\epsilon, \delta}|^2 dx dt \right]
\]

(2.10)

Having estimated the $\partial_t a_{\epsilon, \delta} u_{\epsilon, \delta}^2$ energy term, we next look at the advection term of (2.3). This is estimated by a simple Young's inequality.

\[
\int_{\Omega_h} a_{\epsilon, \delta} |u_{\epsilon, \delta} \nabla u_{\epsilon, \delta}| dx dt \leq \left( \frac{\alpha}{2} \int_{\Omega_h} a_{\epsilon, \delta} |\nabla u_{\epsilon, \delta}|^2 dx dt + \frac{h}{2\alpha} \text{ess sup}_{t \leq h} \int_{\Omega} [a_{\epsilon, \delta} u_{\epsilon, \delta}^2] (t; \cdot) dx \right)
\]

(2.11)

We are now ready to return to (2.3) and begin concluding the uniform Sobolev estimate by applying (2.10, 2.11).

\[
\int_{\Omega} \frac{1}{2} [a_{\epsilon, \delta} u_{\epsilon, \delta}^2] (h; \cdot) dx + \int_{\Omega_h} a_{\epsilon, \delta} |\nabla u_{\epsilon, \delta}|^2 dx dt \leq C(\text{data}) \left[ \left( h + \frac{h}{2\alpha} + \frac{\delta^{1/5}}{\epsilon^2} \right) \left( \text{ess sup}_{t \leq h} \int_{\Omega} a_{\epsilon, \delta} |u_{\epsilon, \delta}|^2 (t; \cdot) dx \right) \right. \\
\left. \left( \frac{\alpha}{2} + \frac{\delta^{1/5}}{\epsilon^2} \right) \int_{\Omega_h} a_{\epsilon, \delta} |\nabla u_{\epsilon, \delta}|^2 dx dt \right] + \int_{\Omega} \frac{1}{2} [a_{\epsilon, \delta} u_{\epsilon, \delta}^2] (0; \cdot) dx
\]

(2.12)

As the right side is an increasing function of $h$, we may add an ess sup$_{t \leq h}$ to the time trace with argument $(t; \cdot)$ on the left-hand side at the small expense of increasing $C(\text{data})$. It is now clear that we may pick $\alpha = \alpha(\text{data})$, $h = h(\alpha, \text{data})$, and $\delta \leq C(\alpha, h, \text{data}) \epsilon^{11}$, so that for some new $\tilde{C}(\text{data})$.

\[
\text{ess sup}_{t \leq h} \int_{\Omega} [a_{\epsilon, \delta} u_{\epsilon, \delta}^2] (t; \cdot) dx + \int_{\Omega_h} a_{\epsilon, \delta} |\nabla u_{\epsilon, \delta}|^2 dx dt \leq \tilde{C}(\text{data}) \int_{\Omega} [a_{\epsilon, \delta} u_{\epsilon, \delta}^2] (0; \cdot) dx
\]

By Lemma 9 we see that once a sequence $\epsilon_i \to 0$ is chosen that there is a $\kappa_i$ so that if $\delta_i \leq \kappa_i$, the right hand integral has a limit and so the whole sequence is bounded. In particular, the $\delta = O(\epsilon^{11})$ choice is sufficient.

2.2 Extracting Weakly Convergent Subsequences

From Prop 11 we may conclude the existence of weak limits in both the volume and at the curve.

Remark 9 By abuse of notation, we will continue to index the convergent subsequences with the same single index notation for which the energy estimates were shown.
Proposition 2 For \( \epsilon_i \to 0 \) and \( \delta_i < \kappa_i \) as in Prop \([\text{H}]) \) we may find a further subsequence (redundantly notated by \( i \)) such that weakly
\[
\|u_{\epsilon_i, \delta_i} \|_2 \to \|u\|_2 \quad \text{and} \quad \|\nabla u_{\epsilon_i, \delta_i} \|_2 \to \|\nabla u\|_2 \quad \text{in} \quad L^2(\Omega_h)
\]
\[
\int_{D_{\epsilon_i}} u_{\epsilon_i, \delta_i}(t, s, \nu, \omega) d\nu d\omega \to u_C \quad \text{and} \quad \int_{D_{\epsilon_i}} \nabla_{s, \nu, \omega} u_{\epsilon_i, \delta_i} d\nu d\omega \to \left( \begin{array}{c} \partial_s u_C \\ \xi_{\nu} \\ \xi_{\omega} \end{array} \right) \quad \text{in} \quad L^2([0,1]_h)
\]

Remark 10 The weak limits \( \xi_v, \xi_w \in L^2([0,1]_h) \) with initially unknown relationship to \( u_C \) emerge because the curve is intrinsically one dimensional, two fewer dimensions than the ambient 3d space. Their final determination is deferred to Sec. \( \text{[\text{A}]} \). See also Remark \([\text{H}]\) and Remark \([\text{I}2] \) below.

2.2.1 Convergence Over the Volume

Since \( a_{\epsilon, \delta} \geq 1 \), we have the basic estimate
\[
\text{ess sup}_{t \leq h} \int_{\Omega} u_{\epsilon_i, \delta_i}^2(t; \cdot) dx + \int_{\Omega_h} |\nabla u_{\epsilon_i, \delta_i}|^2 dx dt \leq C(\text{data})
\]
Accordingly, the \( u_{\epsilon_i, \delta_i} \) and \( |\nabla u_{\epsilon_i, \delta_i}| \) are uniformly bounded in \( L^2(\Omega_h) \) and we may extract weakly convergent subsequences. In particular
\[
u_{\epsilon_i, \delta_i} \to u \quad \text{and} \quad \nabla u_{\epsilon_i, \delta_i} \to \nabla u \quad \text{in} \quad L^2(\Omega_h)
\]

Remark 11 Recall that the weak limit of the gradients tends to the gradient of the weak limit by a simple integration by parts argument. If \( u_{\epsilon_i, \delta_i} \to u \) and \( \partial_x u_{\epsilon_i, \delta_i} \to \phi \), then for an arbitrary, compactly supported and smooth test function \( \varphi \) it holds
\[
\int_{\Omega_h} \varphi \phi dx dt = \lim_i \int_{\Omega_h} \varphi \partial_x u_{\epsilon_i, \delta_i} dx dt = \lim_i \int_{\Omega_h} \partial_x \varphi u_{\epsilon_i, \delta_i} dx dt = -\int_{\Omega_h} \partial_x \varphi u dx dt
\]
It follows \( \phi = \partial_x u \). Observe this argument is not possible unless the domain \( \Omega \) contains the direction for which integration by parts is to be performed.

2.2.2 Convergence Over the Curve

Restrict attention to the subdomain \( N_{\epsilon_i, 0}(t) \) where \( \epsilon_i, \delta_i \downarrow N_{\epsilon_i, 0}(t) = \frac{\epsilon_0}{\epsilon_i} \). Here the estimates of Prop \([\text{I}] \) yield
\[
\text{ess sup}_{t \leq h} \frac{\epsilon_0^2}{\epsilon_i^2} \int_{N_{\epsilon_i, 0}(t)} u_{\epsilon_i, \delta_i}^2(t; \cdot) dx + \frac{\epsilon_0^2}{\epsilon_i^2} \int_0^h \int_{N_{\epsilon_i, 0}(t)} |\nabla u_{\epsilon_i, \delta_i}|^2 dx dt \leq C(\text{data})
\]
Now perform a change of variables to a common domain and conclude with elementary manipulations

\[
\text{ess sup}_{t \leq h} \int_{s=0}^{1} \int_{D_{\epsilon_i}} u_{\epsilon_i, \delta_i}(t, s, \nu, \omega) F dx + \int_{0}^{h} \int_{s=0}^{1} \nabla_{s, \nu, \omega} u_{\epsilon_i, \delta_i} : (\nabla F^t \nabla F)^{-1} \nabla_{s, \nu, \omega} u_{\epsilon_i, \delta_i} F dx dt \\
\leq C(\text{data})
\]

(2.13)

**Lemma 2** \( \exists \beta(\text{data}) > 0 \) such that

\[
\beta |q|^2 \leq \bar{q} \cdot \left[ \left( \nabla F^t \nabla F \right)^{-1} F \right] \bar{q}
\]

**Proof:** It is sufficient to show for a unit vector \( \bar{q} \). From **Lemma 6**, we see that the matrix is symmetric and so must have an orthonormal basis of eigenvectors. Because the matrix is invertible, these eigenvalues are nonzero. Since further, the quadratic form may be written as \( \bar{p} \cdot \bar{p} F \geq 0 \) for \( p = (\nabla F^{-1})^t \bar{q} \), the eigenvalues are nonnegative. Altogether the eigenvalues are all strictly positive.

With the variational characterization of the minimum eigenvalue, \( \lambda_{\text{min}} \), for a symmetric matrix \( A \)

\[
\lambda_{\text{min}} = \min_{|u|=1} u^t Au
\]

we may consider the map

\[
(q, x) \rightarrow q^t \left( \left[ \nabla F^t \nabla F \right](x) \right)^{-1} q : [\bar{q}] = 1 \times [-\epsilon_0, 1 + \epsilon_0] \times D_{\epsilon_0} \rightarrow (0, \infty)
\]

The map is continuous, its domain is compact, and is strictly positive due to the considerations of the last paragraph. Hence, its global minimum is also strictly positive, being attained. Since \( F(x) \geq \gamma > 0 \) for some \( \gamma \) by assumptions made on the data, the result is concluded.

Now minorize \( F \), combine **Lemma 2** with (2.13), and apply Jensen’s inequality.

\[
\text{ess sup}_{t \leq h} \int_{s=0}^{1} \left( \int_{D_{\epsilon_i}} u_{\epsilon_i, \delta_i} d\nu d\omega \right)^2 ds + \int_{0}^{h} \int_{s=0}^{1} \left( \int_{D_{\epsilon_i}} |\nabla_{s, \nu, \omega} u_{\epsilon_i, \delta_i}| d\nu d\omega \right)^2 ds dt \\
\leq C(\text{data})
\]

(2.14)

It now follows that the \( \int_{D_{\epsilon_i}} u_{\epsilon_i, \delta_i} d\nu d\omega \) and \( \int_{D_{\epsilon_i}} \nabla_{s, \nu, \omega} u_{\epsilon_i, \delta_i} d\nu d\omega \) are uniformly bounded in \( L^2([0, 1], h) \), and we may extract weakly convergent subsequences

\[
\int_{D_{\epsilon_i}} u_{\epsilon_i, \delta_i} d\nu d\omega \rightarrow u_C \text{ and } \int_{D_{\epsilon_i}} \nabla_{s, \nu, \omega} u_{\epsilon_i, \delta_i} d\nu d\omega \rightarrow \left( \frac{\partial s u_C}{\xi} \right) \frac{\xi}{\xi_\omega} \text{ in } L^2([0, 1], h)
\]
Remark 12 Observe that the domain $[0, 1]_h$ in variables $(t, s)$ does contain the unit $s$-direction. Consistent with Remark 11, the corresponding weak limit may be identified with $\partial_s u$. However, it does not contain the unit directions for $\nu$ and $\omega$. For these cases compactness only implies the membership of $\xi_\nu, \xi_\omega$ in $L^2([0, 1]_h)$.

3 Determining the Concentrated Capacity Limit

We now proceed with computing the equation satisfied by the concentrated capacity limit. First, we will prove Theorem 1 without yet determining the unknown weak limits $\xi_\nu, \xi_\omega$.

Remark 13 To not overburden notation, hereafter we will drop the subscript $i$ from solutions but understand we work with a sequence given by Prop 2.

Proposition 3 The solution pair $(u, u_C)$ provided by Prop 2 satisfies the integral, weak formulation (1.17).

Proof: For a smooth test function $\varphi \in C^\infty(\bar{\Omega}_h)$ and vanishing at time $t = h$,

$$0 = \int_{\Omega_h} \varphi \left( \partial_t (a_{\nu, \delta} u_{\nu, \delta}) - \text{div} [K_{\nu, \delta} \nabla u_{\nu, \delta}] + \text{div} [u_{\nu, \delta} \vec{v}_{\nu, \delta}] \right) dx dt$$

$$= -\int_{\Omega_h} \varphi a_{\nu, \delta} u_{\nu, \delta} dx dt - \int_{\Omega} \varphi a_{\nu, \delta}(0; \cdot) u_0 dx$$

$$+ \int_{\Omega_h} \nabla \varphi \cdot K_{\nu, \delta} \nabla u_{\nu, \delta} dx dt - \int_{\Omega_h} \nabla \varphi \cdot a_{\nu, \delta} \vec{v}_{\nu, \delta} dx dt$$

Next we group these terms by those in the volume and those about the curve.

$$0 = \left( -\int_{\Omega_h} \varphi (1 - \zeta_{\nu, \delta}) u_{\nu, \delta} dx dt \right) - \int_{\Omega_h} \left[ \varphi (1 - \zeta_{\nu, \delta}) \right] (0; \cdot) u_0 dx$$

$$+ \int_{\Omega_h} \nabla \varphi \cdot K_{\nu, \delta} \nabla u_{\nu, \delta} dx dt - \int_{\Omega_h} \nabla \varphi a_{\nu, \delta} \vec{v} (1 - \zeta_{\nu, \delta}) dx dt \right)_{\text{Vol}}$$

$$+ \left( -\int_{\Omega_h} \varphi \left( \frac{\epsilon^2}{\epsilon^2 \zeta_{\nu, \delta}} \right) u_{\nu, \delta} dx dt \right) - \int_{\Omega_h} \left[ \varphi \left( \frac{\epsilon^2}{\epsilon^2 \zeta_{\nu, \delta}} \right) \right] (0; \cdot) u_0 dx$$

$$+ \int_{\Omega_h} \nabla \varphi \cdot \left( \frac{\epsilon^2}{\epsilon^2 \zeta_{\nu, \delta}} \right) \nabla u_{\nu, \delta} dx dt - \int_{\Omega_h} \nabla \varphi a_{\nu, \delta} \left( \frac{\epsilon^2}{\epsilon^2 \zeta_{\nu, \delta}} \vec{v}_{\nu, \delta} \right) dx dt \right)_{\text{Line}}$$

We will now examine the terms in subsets.
3.1 Limits in the Volume

Because $1 - \zeta_{\epsilon,\delta} \to 1$ strongly in $L^2(\Omega_h)$, we may apply this in tandem with the volume weak limit of Proposition 2. The limit becomes

$$\left(-\int_{\Omega_h} \varphi \, u \, dx \, dt - \int_{\Omega_h} \varphi(0 \cdot u_0 \, dx + \int_{\Omega_h} \nabla \varphi \cdot k_0 I \nabla u \, dx \, dt - \int_{\Omega_h} \nabla \varphi \, u \, dx \, dt \right)_{\text{Vol}}$$  \hspace{1cm} (3.2)

3.2 Limits on the Curve

3.2.1 The Initial Time Term

As a direct application of Lemma 9, we may determine the limit of the initial time data on the curve.

$$-\int_{\Omega} \left[ \varphi \left( \frac{c_0}{c} \zeta_{\epsilon,\delta} \right) \right] (0 \cdot u_0 \, dx \to -\pi \epsilon_0^2 \int_{0}^{1} (\varphi u_0)(0, s, 0, 0) |\partial_s \Gamma| \, ds$$  \hspace{1cm} (3.3)

Remark 14 Lemma 9 was stated for the $a_{\epsilon,\delta}$ coefficients, but its proof showed convergence of this term as well.

3.2.2 Controlling the Remaining $[N_{\epsilon,\delta} \setminus N_{\epsilon,0}](t)$ Terms

For the remaining integrals in $(\ldots)_{\text{Line}}$, the contribution due to $[N_{\epsilon,\delta} \setminus N_{\epsilon,0}](t)$ is vanishing. More precisely, we have the following.

Lemma 3

$$\left| \int_{0}^{h} \int_{[N_{\epsilon,\delta} \setminus N_{\epsilon,0}](t)} \varphi_t \left( \frac{c_0}{c} \zeta_{\epsilon,\delta} \right) u_{\epsilon,\delta} \, dx \, dt \right| + \left| \int_{0}^{h} \int_{[N_{\epsilon,\delta} \setminus N_{\epsilon,0}](t)} \nabla \varphi \cdot \left( \frac{c_0}{c} \zeta_{\epsilon,\delta} \right) \nabla u_{\epsilon,\delta} \, dx \, dt \right| \leq C(\text{data}) \left\langle \varphi, \varphi_t, \nabla \varphi \right\rangle_{L^2} \left( \frac{\delta^{1/2}}{\epsilon^2} \right)$$

Proof: We handle the first term, the others being similar.

$$\left| \int_{0}^{h} \int_{[N_{\epsilon,\delta} \setminus N_{\epsilon,0}](t)} \varphi_t \left( \frac{c_0}{c} \zeta_{\epsilon,\delta} \right) u_{\epsilon,\delta} \, dx \, dt \right| \leq \frac{c_0^2}{c^2} \| \varphi_t \|_{L^\infty} \int_{0}^{h} \int_{[N_{\epsilon,\delta} \setminus N_{\epsilon,0}](t)} |u_{\epsilon,\delta}| \, dx \, dt$$

$$\leq \frac{c_0^2}{c^2} \| \varphi_t \|_{L^\infty} \| u_{\epsilon,\delta} \|_{L^2(\Omega_h)} \left\| X_{[0,h] \times [N_{\epsilon,\delta} \setminus N_{\epsilon,0}](t)} \right\|_{L^2(\Omega_h)}$$

We may use that $1 \leq a_{\epsilon,\delta}$ and Proposition 1 to bound the energy norm of $u_{\epsilon,\delta}$ by a constant depending on the data. Further, by Lemma 8 the $L^2$ space time norm of the gap must be bounded like $C(\text{data}) \delta^{1/2}$. \hfill \blacksquare
3.2.3 The $\varphi_t$ and Advection Terms

Use Lemma 3 and a change of variables to obtain

$$- \int_{\Omega_h} \varphi_t \left( \frac{\mu_2}{\epsilon^2} \zeta_{\epsilon,\delta} \right) u_{\epsilon,\delta} dx \, dt - \int_{\Omega_h} \nabla \varphi u_{\epsilon,\delta} \left( \frac{\mu_2}{\epsilon^2} \bar{v}_C \zeta_{\epsilon,\delta} \right) dx \, dt$$

$$= - \int_0^h \int_{N_{s,\omega}(t)} \varphi_t \left( \frac{\mu_2}{\epsilon^2} \right) u_{\epsilon,\delta} dx \, dt - \int_0^h \int_{N_{s,\omega}(t)} \nabla \varphi u_{\epsilon,\delta} \left( \frac{\mu_2}{\epsilon^2} \bar{v}_C \right) dx \, dt + O \left( \frac{\delta^{1/2}}{\epsilon^2} \right)$$

$$= - \frac{\pi \epsilon_0^2}{\Omega_h} \int_0^h \int_{D_h} \varphi_t \|u_{\epsilon,\delta} - \nabla F^{-1} t \| \nabla F^{-1} u_{\epsilon,\delta} \| \bar{v}_C \| F \| \, dv \| ds \, dt + O \left( \frac{\delta^{1/2}}{\epsilon^2} \right)$$

Note that the $\varphi_t, \nabla \varphi$ here are derivatives of $\varphi(t, x, y, z)$ and then composed with $F$ but not yet the derivatives of $\varphi(t, s, \nu, \omega)$. We now apply chain-rule with the help of Lemma 3 to revert to the coordinate system at the curve.

$$- \int_{\Omega_h} \varphi_t \left( \frac{\mu_2}{\epsilon^2} \zeta_{\epsilon,\delta} \right) u_{\epsilon,\delta} dx \, dt - \int_{\Omega_h} \nabla \varphi u_{\epsilon,\delta} \left( \frac{\mu_2}{\epsilon^2} \bar{v}_C \zeta_{\epsilon,\delta} \right) dx \, dt$$

$$= - \pi \epsilon_0^2 \int_0^h \int_{D_h} \varphi_t \left( \nabla s_{s,\nu,\omega} \bar{v}_C \right) \nabla F^{-1} u_{\epsilon,\delta} \| \bar{v}_C \| F \| \, dv \| ds \, dt + O \left( \frac{\delta^{1/2}}{\epsilon^2} \right)$$

From the uniform continuity over Dom $F$ of all terms except for $u_{\epsilon,\delta}$ whose integral average converges weakly by Prop 2 and recalling $\delta = O(\epsilon^{11})$, we have

$$- \int_{\Omega_h} \varphi_t \left( \frac{\mu_2}{\epsilon^2} \zeta_{\epsilon,\delta} \right) u_{\epsilon,\delta} dx \, dt - \int_{\Omega_h} \nabla \varphi u_{\epsilon,\delta} \left( \frac{\mu_2}{\epsilon^2} \bar{v}_C \zeta_{\epsilon,\delta} \right) dx \, dt \quad (3.4)$$

$$\to \pi \epsilon_0^2 \left[ \int_{[0,1]^h} \left[ \varphi_t - \nabla s_{s,\nu,\omega} \bar{v}_C \right] F^{-1} t \nu, 0, 0 \nu C \| \partial_s \Gamma \| ds \, dt \right]$$

$$+ \int_{[0,1]^h} \left[ \nabla s_{s,\nu,\omega} \bar{v}_C \| \nabla F^{-1} \bar{v}_C \| F \| \, dv \| ds \, dt \right]$$

3.2.4 The Diffusion Term

By Lemma 3 and a change of variables we have

$$\int_{\Omega_h} \nabla \varphi \left( \frac{\mu_2}{\epsilon^2} K \zeta_{\epsilon,\delta} \right) u_{\epsilon,\delta} dx \, dt = \int_0^h \int_{N_{s,\omega}(t)} \nabla \varphi \left( \frac{\mu_2}{\epsilon^2} K \right) u_{\epsilon,\delta} dx \, dt + O \left( \frac{\delta^{1/2}}{\epsilon^2} \right)$$

$$= \pi \epsilon_0^2 \int_0^h \int_{D_h} \nabla s_{s,\nu,\omega} \varphi \left[ \nabla F^{-1} K \nabla F^{-1} \right] \nabla s_{s,\nu,\omega} u_{\epsilon,\delta} F \| dv \| \| ds \, dt + O \left( \frac{\delta^{1/2}}{\epsilon^2} \right)$$

Appealing to the uniform continuity over Dom $F$ of all terms save $\nabla s_{s,\nu,\omega} u_{\epsilon,\delta}$ whose integral average converges weakly by Prop 2 and recalling that $\delta = O(\epsilon^{11})$, we have
$O(\epsilon^{11})$, we have

\[
\int_{\Omega_h} \nabla \varphi \cdot \left( \frac{\epsilon_0^2 \mathcal{K}_{\epsilon,\delta}}{\epsilon^2} \nabla u_{\epsilon,\delta} \right) d\Omega \cdot d\tau \\
\rightarrow \pi \epsilon_0^2 \int_0^1 \int_0^1 \left[ \nabla_{s,\nu,\omega} \varphi \cdot \nabla F^{-1} \mathcal{K} \left( \nabla F^{-1} \right) \right] \left( t, \nu, 0, 0 \right) \left( \frac{\partial u_{\nu,\omega}}{\xi_{\nu}, \xi_{\omega}} \right) |\partial_{s, \nu, \omega}| |\partial_{t, \nu, \omega}| d\Omega \cdot d\tau
\]

From Lemma 6 and the structure conditions of $\mathcal{K}$ (1.14) we have

\[
\left[ \nabla F^{-1} \right] \left( t, s, 0, 0 \right) = \begin{pmatrix} |\partial_{s, \nu, \omega}|^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

from whence it follows

\[
\left[ \nabla F^{-1} \mathcal{K} \left( \nabla F^{-1} \right) \right] \left( t, s, 0, 0 \right) = \begin{pmatrix} k_{s}(s, 0, 0)/|\partial_{s, \nu, \omega}|^2 & 0 & 0 \\ 0 & k_{n}(s, 0, 0) & 0 \\ 0 & 0 & k_{n}(s, 0, 0) \end{pmatrix}
\]

Combining we have

\[
\int_{\Omega_h} \nabla \varphi \cdot \left( \frac{\epsilon_0^2 \mathcal{K}_{\epsilon,\delta}}{\epsilon^2} \nabla u_{\epsilon,\delta} \right) d\Omega \cdot d\tau \\
\rightarrow \pi \epsilon_0^2 \int_0^1 \int_0^1 \left[ \nabla_{s,\nu,\omega} \varphi \cdot \left( k_{s}/|\partial_{s, \nu, \omega}|^2 \begin{pmatrix} 0 \\ 0 \\ \partial_{t, \nu, \omega} \end{pmatrix} \right) \right] \left( t, s, 0, 0 \right) \left( \frac{\partial u_{\nu,\omega}}{\xi_{\nu}, \xi_{\omega}} \right) |\partial_{s, \nu, \omega}| |\partial_{t, \nu, \omega}| d\Omega \cdot d\tau
\]

**3.3 Concluding the Limit**

We now conclude using (3.2, 3.3, 3.4, 3.5) that (3.1) limits to

\[
0 = \left( - \int_{\Omega} \varphi(0, \lambda) u_0 dx - \int_{\Omega_h} \varphi \cdot u_0 \omega dx + \int_{\Omega_h} \nabla \varphi \cdot k_{0,\lambda} \nabla u_0 dx - \int_{\Omega_h} \nabla \varphi \cdot u_0 \omega dx \right)_{\text{Vol}}
\]

\[
+ \pi \epsilon_0^2 \left( - \int_0^1 \varphi u_0 (0, s, 0, 0) |\partial_{s, \nu, \omega}| ds - \int_{[0,1]} \left[ \varphi - \nabla_{s,\nu,\omega} \varphi \cdot \nabla F^{-1} |\partial_{s, \nu, \omega}| \right] \left( t, s, 0, 0 \right) u_{\nu,\omega} |\partial_{s, \nu, \omega}| ds dt \\
- \int_{[0,1]} \left[ \nabla_{s,\nu,\omega} \varphi \cdot \nabla F^{-1} |\partial_{s, \nu, \omega}| \right] \left( t, s, 0, 0 \right) u_{\nu,\omega} |\partial_{s, \nu, \omega}| ds dt \\
+ \int_{[0,1]} \left[ \nabla_{s,\nu,\omega} \varphi \cdot \left( k_{s} |\partial_{s, \nu, \omega}|^2 \begin{pmatrix} 0 \\ 0 \\ \partial_{t, \nu, \omega} \end{pmatrix} \right) \right] \left( t, s, 0, 0 \right) \left( \frac{\partial u_{\nu,\omega}}{\xi_{\nu}, \xi_{\omega}} \right) |\partial_{s, \nu, \omega}| |\partial_{t, \nu, \omega}| ds dt \right)
\]
4 Determining the Weak Limits \( \xi_v, \xi_w \)

To conclude the expressions for \( \xi_v, \xi_w \), we will need to use test functions of the form

\[
\varphi_{\epsilon, \delta} = \psi + (\psi_C - \psi) \zeta_{\epsilon, \delta}
\]

for \( \psi, \psi_C \in C^\infty(\overline{\Omega}_h) \) and vanish at time \( t = h \) in (3.6). The use of \( \zeta_{\epsilon, \delta} \) is justified by a simple mollification argument. Because \( \varphi_{\epsilon, \delta} \equiv \psi_C \) in a neighborhood of the curve, we may directly substitute \( \psi_C \) for \( \varphi \) in (3.6) of (3.6). The corresponding (3.6)_Vol terms need consideration. Note that \( \varphi_{\delta} := \varphi_{\epsilon, \delta} \).

**Lemma 4** For the choice of test function (4.1) with \( \epsilon = \delta \) and two smooth test functions \( \psi, \psi_C \) which uniformly satisfy for all times \( t \), \( \|\psi - \psi_C\|_{\infty, N_{\epsilon, \delta}(t)} = O(\delta) \), it holds that as \( \delta \to 0 \) the volume terms in (3.6) have the limit

\[
- \int_{\Omega_h} \partial_t \varphi_{\delta} u dx dt - \int_{\Omega_h} \varphi_{\delta} (0; \cdot) u_0 dx + \int_{\Omega_h} \nabla \varphi_{\delta} \cdot k_0 I_3 \nabla u dx dt - \int_{\Omega_h} \nabla \varphi_{\delta} \vec{w} dx dt
\]

\[
\to - \int_{\Omega_h} \partial_t \psi u dx dt - \int_{\Omega_h} \psi (0; \cdot) u_0 dx + \int_{\Omega_h} \nabla \psi \cdot k_0 I_3 \nabla u dx dt - \int_{\Omega_h} \nabla \psi \vec{w} dx dt
\]

**Proof:** Examine first the diffusion term.

\[
\int_{\Omega_h} \nabla \varphi_{\delta} \cdot k_0 I_3 \nabla u dx dt = \int_{\Omega_h} \left[ \nabla \psi + \nabla (\psi_C - \psi) \zeta_{\epsilon, \delta} + (\psi_C - \psi) \nabla \zeta_{\epsilon, \delta} \right] \cdot k_0 I_3 \nabla u dx dt
\]

\[
= \int_{\Omega_h} \left[ \nabla \psi + \nabla (\psi_C - \psi) \zeta_{\epsilon, \delta} \right] \cdot k_0 I_3 \nabla u dx dt
\]

\[
- \frac{1}{\delta} \int_{0}^{h} \int_{|0<d_\delta|<\delta} \left[ (\psi_C - \psi) \right] (t, s, \nu, \omega) \nabla d_\epsilon \cdot k_0 \left( \nabla F^t \nabla F \right)^{-1} \nabla s, \nu, \omega u J_F dx dt
\]

By assumption \( |\psi_C - \psi| = O(\delta) \) over \( [d_\delta < \delta] \), and so the \( \delta^{-1} \) singularity is balanced. The remaining integral vanishes in the limit from the absolute continuity of \( \nabla u \) and the sup bounds on the remaining terms. Since also \( \zeta_{\epsilon, \delta} \to 0 \) a.e., this concludes the first term.

The time derivative and advection terms follow similarly using the absolute continuity of \( u \). For the former we have below.

\[
- \int_{\Omega_h} \partial_t \varphi_{\delta} u dx dt = - \int_{\Omega_h} \left[ \partial_t \psi + (\partial_t \psi_C - \partial_t \psi) \zeta_{\epsilon, \delta} + (\psi_C - \psi) \partial_t \zeta_{\epsilon, \delta} \right] u dx dt
\]

\[
= - \int_{\Omega_h} \left[ \partial_t \psi + (\partial_t \psi_C - \partial_t \psi) \zeta_{\epsilon, \delta} \right] u dx dt
\]

\[
- \frac{1}{\delta} \int_{0}^{h} \int_{|0<d_\delta|<\delta} \left[ (\psi_C - \psi) u \right] (t, s, \nu, \omega) \nabla d_\epsilon \cdot \nabla F^{-1} \partial_t F J_F dx dt
\]
For the latter we have below.

\[
- \int_{\Omega_h} \nabla \varphi \bar{u} \tilde{v} dx dt = - \int_{\Omega_h} \left[ \nabla \psi + \nabla (\psi_C - \psi) \zeta_{\delta,\delta} + (\psi_C - \psi) \nabla \zeta_{\delta,\delta} \right] \bar{u} \tilde{v} dx dt \\
= - \int_{\Omega_h} \left[ \nabla \psi + \nabla (\psi_C - \psi) \zeta_{\delta,\delta} \right] \bar{u} \tilde{v} dx dt \\
+ \frac{1}{\delta} \int_0^h \int_{[0<d<\delta]} \left[ (\psi_C - \psi) \bar{u} \right] (t, s, \nu, \omega) \nabla d \cdot \nabla F^{-1} \tilde{v}(t, s, \nu, \omega) J_F dx dt
\]

The integral for the initial time trace follows again because \( \zeta_{\delta,\delta} \to 0 \) a.e. \( \blacksquare \)

**Remark 15** If \( \bar{v} \) is assumed smooth, then both the time derivative and advection terms could be handled by coarea arguments that circumvent the need for \( \psi_C - \psi = O(\delta) \). However, the diffusion term could not since \( \nabla u \) is not sufficiently regular to have traces on hypersurfaces amenable to the divergence theorem.

**Corollary 1** For a given smooth \( \psi \in C^\infty (\bar{\Omega}_h) \), the choices

\[
\psi_C = \psi(t, s, 0, 0), \psi(t, s, \nu, 0), \psi(t, s, 0, \omega), \psi(t, s, \nu, \omega)
\]

all satisfy \( |\psi - \psi_C| = O(\delta) \) over \( N_{\delta,\delta}(t) \). Accordingly, for these choices \( (3.6) \) holds with \( \varphi \) replaced by \( \psi \) in the terms \( (\ldots)_{Vol} \) and replaced by \( \psi_C \) in the terms \( (\ldots)_{Line} \).

**Proof:** Since the image of \( F \) is compact and contained in \( \Omega \), \( \nabla \psi \) takes a global maximum which may be used in conjunction with a line integral over the convex domain of \( F \) to bound the difference. We thus satisfy the conditions of Lemma 4 \( \blacksquare \)

### 4.1 Concluding

**Proposition 4** The weak limits \( \xi_{\nu}, \xi_{\omega} \in L^2 ([0, 1]_h) \) are given by \( (4.16) \).

**Proof:** Take the weak formulation supplied by **Corollary 1** and subtract the result for the pair \( (\psi, \psi(t, s, 0, 0)) \) from the result for the pair \( (\psi, \psi(t, s, \nu, 0)) \). Since both pairs produce the same terms in the volume, we need only examine the terms at the curve. After canceling we are left with

\[
0 = \int_{[0,1]_h} \left( \partial_v \psi \tilde{e}_2 \cdot \nabla F^{-1} \partial_t u_C - \partial_v \psi \tilde{e}_2 \cdot \nabla F^{-1} \tilde{v}_C u_C + \partial_v \psi k_n \xi_{\nu} \right) |\partial_s \Gamma| ds dt \\
= \int_{[0,1]_h} \partial_v \psi \left( \tilde{e}_2 \cdot \nabla F^{-1} (\partial_t F - \tilde{v}_C) u_C + k_n \xi_{\nu} \right) |\partial_s \Gamma| ds dt
\]

From the arbitrariness of \( \psi \) in the volume, \( \partial_v \psi \) can recover all smooth testing functions on \([0,1]_h\) in the variables \((t, s)\) which vanish at time \( t = h \). We conclude the integrand is identically 0, and thus \( \xi_{\nu} \) must be

\[
\xi_{\nu}(t, s) = \frac{[\tilde{e}_2 \cdot \nabla F^{-1} (\tilde{v}_C - \partial_t F)] (t, s, 0, 0) u_C}{k_n(s, 0, 0)}
\]
The function \( \xi_\omega \) may be derived analogously.

\[
\xi_\omega(t, s) = \left[ \vec{\epsilon}_3 \cdot \nabla F^{-1} (\vec{v}_C - \partial_t F) \right] (t, s, 0, 0) u_C \overline{k_n(s, 0, 0)}
\]

5 Appendix

5.1 Coordinate Transformations and Geometric Properties

In this appendix we collect several elementary but helpful observations regarding changes of variable and the geometry used in this investigation. The quantities in Lemma 5 and Lemma 6 are required for administration of chain-rule.

**Lemma 5** Jacobian Matrices

It holds that

\[
DF = \begin{pmatrix}
\partial_t F & \nabla F (= D_{s,\nu,\omega} F) \\
\partial_s \Gamma & \nu \partial_s \vec{n} + \omega \partial_s \vec{b} & \vec{n} & \vec{b}
\end{pmatrix}
\]

\[
\nabla F = \begin{pmatrix}
\partial_s \Gamma + \nu \partial_s \vec{n} + \omega \partial_s \vec{b} & \vec{n} & \vec{b}
\end{pmatrix}
\]

\[
\nabla F^T \nabla F = \begin{pmatrix}
\| \partial_s \Gamma + \nu \partial_s \vec{n} + \omega \partial_s \vec{b} \|^2 & \omega \langle \partial_s \vec{b}, \vec{n} \rangle & \nu \langle \partial_s \vec{n}, \vec{b} \rangle \\
\omega \langle \partial_s \vec{b}, \vec{n} \rangle & 1 & 0 \\
\nu \langle \partial_s \vec{n}, \vec{b} \rangle & 0 & 1
\end{pmatrix}
\]

\[
J_F := \det DF = \det \nabla F
\]

\[
= \sqrt{\| \partial_s \Gamma + \nu \partial_s \vec{n} + \omega \partial_s \vec{b} \|^2 - \nu^2 \langle \partial_s \vec{n}, \vec{b} \rangle - \omega^2 \langle \partial_s \vec{b}, \vec{n} \rangle} =: J_F
\]

**Proof:** Use (1.6) and elementary manipulations.

**Lemma 6** Inverses of Jacobian Matrices
It holds that

\[ DF^{-1} = \begin{pmatrix} 1 & 0 \\ -\nabla F^{-1} \partial_t F & \nabla F^{-1} \end{pmatrix} \]

\[ (\nabla F^t \nabla F)^{-1} = \begin{pmatrix} 1 & \frac{-\omega \langle \partial_s \tilde{b}, \tilde{n} \rangle}{\langle \partial_s \tilde{b}, \tilde{n} \rangle} \\ \frac{-\omega \langle \partial_s \tilde{b}, \tilde{n} \rangle}{\langle \partial_s \tilde{b}, \tilde{n} \rangle} & \frac{-\nu \langle \partial_s \tilde{n}, \tilde{b} \rangle}{\langle \partial_s \tilde{n}, \tilde{b} \rangle} \end{pmatrix} \]

\[ \nabla F^{-1} = (\nabla F^t \nabla F)^{-1} \nabla F^t \]

\[ = (\nabla F^t \nabla F)^{-1} \begin{pmatrix} \partial_s \Gamma + \nu \partial_s \tilde{n} + \omega \partial_s \tilde{b} \\ \frac{\sqrt{\nu^2 + \omega^2} - \epsilon}{\langle \partial_s \tilde{n}, \tilde{b} \rangle} \end{pmatrix} \]

**Proof:** Use Lemma 5 and elementary manipulations.

The distance function \( d_\epsilon(x) \) may be computed explicitly due to the simplicity of \( C_\epsilon \).

**Lemma 7** The distance function

For \( x = (s, \nu, \omega) \)

\[ d_\epsilon(x) = \sqrt{s^2 + (s - 1)^2 + \left( \sqrt{\nu^2 + \omega^2} - \epsilon \right)^2} \]

**Proof:** We may decompose the distance function and write

\[ \min_{y \in C} \| x - y \|^2 = \text{dist}(s, [0, 1])^2 + \text{dist}(\nu, \omega, D_\epsilon)^2 \]

**Lemma 8** Measuring \([N_\epsilon, \delta \setminus N_\epsilon, 0](t)\)

There exists a constant \( C = C(\epsilon_0) \) s.t.

\[ \mu\left( [0 < d_\epsilon < \delta] \right) \leq C(\epsilon_0) \delta \]

\[ \mu\left( [N_\epsilon + \delta \setminus N_\epsilon, 0](t) \right) \leq (\text{ess sup}_{t \leq T} \| J_F \|_\infty (t)) C(\epsilon_0) \delta \]

**Proof:** Begin by applying the coarea formula.

\[ \mu\left( [0 < d_\epsilon < \delta] \right) = \int_{[0 < d_\epsilon < \delta]} 1 dx = \int_0^\delta \int_{[d_\epsilon = \eta]} \frac{1}{|\nabla d_\epsilon|} d\sigma d\eta \]
From Lemma 7 one verifies that \( |\nabla d_\epsilon(x)| = 1 \) when \( d_\epsilon(x) \neq 0 \). From the geometry of \( C_\epsilon \), we have \( \exists C(\epsilon_0) \text{ s.t. } \forall \tau \in (0, \delta) \)

\[
\int_{[d_\epsilon=\tau]} d\sigma \leq C(\epsilon_0)
\]

This concludes the first assertion. The second follows from a simple change of variables.

\[
\int_{[N_{\epsilon+1} \setminus N_{\epsilon,0}(t)]} dx = \int_{[0 < \epsilon, \delta]} J_F dx \leq (\text{ess sup}_{t \leq T} \|J_F\|_\infty (t)) \mu \left( \{0 < \epsilon, \delta \} \right)
\]

\[\blacksquare\]

### 5.2 The Distributional Limit of \( a_{\epsilon,\delta}(t; x) \)

In this appendix, we prove a simple lemma regarding the weak convergence of \( a_{\epsilon,\delta}(t; x) \) as distributions on the space of continuous functions.

**Lemma 9** The weak limit of \( a_{\epsilon,\delta} \)

Let \( f \in C(\Omega) \) and \( \epsilon_i \to 0 \). Then \( \exists \kappa_i (>0) \to 0 \), so that if \( \delta_i < \kappa_i \) it holds

\[
\lim_{i} \int_{\Omega} a_{\epsilon_i,\delta_i}(t; \cdot) f dx = \int_{\Omega} f dx + \pi \epsilon_i^2 \int_{0}^{1} [(f \circ \Gamma)(\partial_s \Gamma)](s; t) ds
\]

**Proof:** Begin by writing

\[
\int_{\Omega} a_{\epsilon_i,\delta_i} f dx = \int_{\Omega} (1 - \zeta_{\epsilon_i,\delta_i}) f dx + \int_{\Omega} \epsilon_i^2 \zeta_{\epsilon_i,\delta_i} f dx
\]

As \( \epsilon_i \to 0 \), \( \chi_{N_{\epsilon_i,0}(t)} \to 0 \) a.e. and so does strongly as well. Since \( \left| \zeta_{\epsilon_i,\delta_i} - \chi_{N_{\epsilon_i,0}(t)} \right| \leq 2 \) and Lemma 8 shows the set where they differ has measure \( O(\delta_i) \), we have

\[
\left| \int_{\Omega} (1 - \zeta_{\epsilon_i,\delta_i}) f dx - \int_{\Omega} f dx \right| \leq 2C(\text{data}) \|f\|_\infty \delta_i + \int_{\Omega} \left(1 - \chi_{N_{\epsilon_i,0}(t)}\right) f dx - \int_{\Omega} f dx
\]

This proves the result for the first term. We proceed as follows to handle the second.

\[
\int_{\Omega} \frac{\epsilon_i^2}{\epsilon_i^2} \zeta_{\epsilon_i,\delta_i} f dx = \int_{N_{\epsilon_i,0}(t)} \frac{\epsilon_i^2}{\epsilon_i^2} \zeta_{\epsilon_i,\delta_i} (f \circ F) dx = \int_{[-\epsilon_0,1+\epsilon_0] \times D_\epsilon} \frac{\epsilon_i^2}{\epsilon_i^2} (X_{\delta_i \circ d_\epsilon})(f \circ F) J_F dx
\]

\[
= \int_{\epsilon_i^2} \frac{\epsilon_i^2}{\epsilon_i^2} (f \circ F) J_F dx + \int_{[0 < \epsilon_i < \delta_i]} \frac{\epsilon_i^2}{\epsilon_i^2} (X_{\delta_i \circ d_\epsilon})(f \circ F) J_F dx
\]

\[
= \pi \epsilon_0^2 \int_{D_\epsilon} (f \circ F) J_F dx + \int_{[0 < \epsilon_i < \delta_i]} \frac{\epsilon_i^2}{\epsilon_i^2} (X_{\delta_i \circ d_\epsilon})(f \circ F) J_F dx
\]

From Lemma 8 we may write

\[
\int_{\Omega} \frac{\epsilon_i^2}{\epsilon_i^2} \zeta_{\epsilon_i,\delta_i} f dx = \pi \epsilon_0^2 \int_{D_\epsilon} (f \circ F) J_F dx + C(\text{data}) \|J_F\|_\infty \|f\|_\infty \frac{\delta_i}{\epsilon_i^2}
\]

22
The result is concluded using $f$'s and $J_F$'s continuity. Note that Lemma 5 implies that $J_F \rightarrow |\partial_s \Gamma|$.

Remark 16 We observe for concreteness that we may take $\kappa_i = \epsilon_i^3$.

References

[1] R. E. Showalter and Xiangsheng Xu. Convergence of diffusion with concentrating capacity. *J. Math. Anal. Appl.*, 137(1):132–147, 1989.

[2] Pierluigi Colli and José-Francisco Rodrigues. Diffusion through thin layers with high specific heat. *Asymptotic Anal.*, 3(3):249–263, 1990.

[3] E. Magenes. Stefan problems in a concentrated capacity. In *Advanced mathematics: computations and applications (Novosibirsk, 1995)*, pages 82–90. NCC Publ., Novosibirsk, 1995.

[4] E. Magenes. On a Stefan problem in a concentrated capacity. In *Partial differential equations and applications*, volume 177 of *Lecture Notes in Pure and Appl. Math.*, pages 237–253. Dekker, New York, 1996.

[5] D. Andreucci, P. Bisegna, and E. DiBenedetto. Homogenization and concentrated capacity for the heat equation with non-linear variational data in reticular almost disconnected structures and applications to visual transduction. *Ann. Mat. Pura Appl. (4)*, 182(4):375–407, 2003.

[6] Laura Gioia Andrea Keller. Homogenization and concentrated capacity for the heat equation with two kinds of microstructures: uniform cases. *Ann. Mat. Pura Appl. (4)*, 196(3):791–818, 2017.

[7] S. A. Adcock and J. A. McCammon. Molecular dynamics: survey of methods for simulating the activity of proteins. *Chem Rev*, 106(5):1589–1615, May 2006.

[8] R. O. Dror, R. M. Dirks, J. P. Grossman, H. Xu, and D. E. Shaw. Biomolecular simulation: a computational microscope for molecular biology. *Annu Rev Biophys*, 41:429–452, 2012.

[9] M. C. Bellissent-Funel, A. Hassanali, M. Havenith, R. Hencman, P. Pohl, F. Sterpone, D. van der Spoel, Y. Xu, and A. E. Garcia. Water determines the structure and dynamics of proteins. *Chem Rev*, 116(13):7673–7697, 07 2016.

[10] D. Laage, T. Elsaesser, and J. T. Hynes. Water Dynamics in the Hydration Shells of Biomolecules. *Chem Rev*, 117(16):10694–10725, Aug 2017.

[11] J. R. Wagner, C. T. Lee, J. D. Durrant, R. D. Malmstrom, V. A. Feher, and R. E. Amaro. Emerging Computational Methods for the Rational Discovery of Allosteric Drugs. *Chem Rev*, 116(11):6370–6390, 06 2016.
[12] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural’ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[13] Emmanuele DiBenedetto. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, 1993.

[14] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[15] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.