A SOLUTION OF THE QUANTUM KNIZHNik
ZAMOLODCHIKOV EQUATION OF TYPE $C_n$

KATSUHISA MIMACHI

Abstract. We construct a solution of Cherednik’s quantum Knizhnik Zamolod-
chikov equation associated with the root system of type $C_n$. This solution is
given in terms of a restriction of a $q$-Jordan-Pochhammer integral. As its
application, we give an explicit expression of a special case of the Macdon-
ald polynomial of the $C_n$ type. Finally we explain the connection with the
representation of the Hecke algebra.

1. Introduction

We study the quantum Knizhnik Zamolodchikov (QKZ) equation ([2]) associated
with the root system of type $C_n$. A solution to this equation is found by means of
a restriction of the $q$-Jordan-Pochhammer integral.

A solution of the QKZ equation of type $A_{n-1}$ is given in [14]. Since the ap-
pearance of that work, however, there has been no progress in the study of the
QKZ equation for other types of root systems with regard to the determination of
solutions. This paper is devoted to such a task.

To construct our solution, we exploit a family of rational functions which would
 correspond to a basis of the $q$ de Rham cohomology attached to the integrand. This
turns out to be a natural basis for the representation of the Hecke algebra $H(W)$
through the Lusztig operator $T_i$.

Next, as a byproduct of our investigation, we obtain an integral representa-
on of the special case of an eigenfunction associated with the Macdonald operator of
the $C_n$ type. In particular, it is seen that, taking a suitable cycle, a restriction
of the $q$-Jordan-Pochhammer integral expresses the Macdonald polynomial of the
$C_n$ type parametrized by the partition $(\lambda, 0, \ldots, 0)$. This integral leads to a more
explicit expression.

We believe that the present paper represents a first step toward understanding
the $BC_n$ type QKZ equation and the $BC_n$ type Macdonald polynomial. It is
noteworthy that even in the classical ($q=1$) case was not previously known that such
an integral gives spherical functions associated with the root system $C_n$. For related
works on $BC_n$ type spherical functions, we refer the reader to [1] and references
therein.

Throughout this paper, $q$ is regarded as a real number satisfying $0 \leq q < 1$.

2. QKZ Equation of Type $C_n$

We first give a review of the QKZ equation associated with the root system of
type $C_n$ for the reader’s convenience, following Cherednik [2] and Kato [7].

Key words and phrases. Quantum Knizhnik Zamolodchikov equations, Macdonald polynomi-
als, $q$-Jordan-Pochhammer integrals.
Let $E = \oplus_{1 \leq i \leq n} \mathbb{R}e_i$ be the real Euclidean space with inner product $(\cdot, \cdot)$ such that $(\epsilon_i, \epsilon_j) = \delta_{ij}$. Let $\Delta = \{ \pm \epsilon_i \pm \epsilon_j (1 \leq i < j \leq n), \pm 2\epsilon_i (1 \leq i \leq n) \}$ be the root system of type $C_n$, $\Delta^+ = \{ \epsilon_i \pm \epsilon_j (1 \leq i < j \leq n), 2\epsilon_i (1 \leq i \leq n) \}$ the set of positive roots, $\Pi = \{ \alpha_i = \epsilon_i - \epsilon_{i+1} (1 \leq i \leq n-1), \alpha_n = 2\epsilon_n \}$ the set of simple roots, $P = \oplus_{1 \leq i \leq n} \mathbb{Z}\epsilon_i$ the weight lattice, and $P^\vee = \oplus_{1 \leq i \leq n} \mathbb{Z}\epsilon_i + \mathbb{Z}(\frac{1}{2} \sum_{i=1}^n \epsilon_i)$ the dual weight lattice for the root system $\Delta$. We frequently write $\alpha \in \Delta^+$ as $\alpha > 0$.

An element of the group algebra $A = \mathbb{C}[P]$ is denoted by $e^\lambda$, as is customary. Then the Weyl group $W = W(C_n) = \langle s_1, s_2, \ldots, s_n \rangle$ (where each $s_i$ is a standard generator corresponding to the simple root $\alpha_i$) acts on $A$ as $w(e^\lambda) = e^{w\lambda}$ ($w \in W$). The symbol $s_\alpha$ denoting the reflections is defined by $s_\alpha(x) = x - \langle x, \alpha \rangle \alpha^\vee$, with $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ for $x \in E$ and $\alpha \in \Delta$.

The set of affine roots associated with $\Delta$ is $\tilde{\Delta} = \{ \alpha + m\delta; \alpha \in \Delta, m \in \mathbb{Z} \}$, where $\delta$ denotes the constant function 1 on $E$. The simple roots are $\alpha_0 = -\theta + \delta$ with the highest root $\theta = 2\epsilon_1$ and $\alpha_i = \alpha_i \in \Delta$ for $1 \leq i \leq n$. We use the symbol introduced above, $s_i (0 \leq i \leq n)$ to also represent the generator for the corresponding affine Weyl group. We note that $s_0 = \tau(\theta^\vee)s_\theta = \tau(\epsilon_1)s_{2\epsilon_1}$, where $\tau(\mu)$ is a translation by $\mu$.

Let us introduce $V$ as the left free $A$–module of rank $|W| = 2^n n!$ with the free basis $h_w (w \in W)$; each element $F$ of $V$ can be written uniquely as $F = \sum_{w \in W} f_w h_w$ ($f_w \in A$). Then, let $A^\sim$ be a completion of the quotient field of $A$. We then have $V^\sim = A^\sim \otimes_A V$. The action $r_w$ of the Weyl group $W$ on $V^\sim$ is defined by the following:

$$r_w(fh_y) = w(f)h_{wy} \quad \text{for} \quad f \in A \quad \text{and} \quad w, y \in W.$$  

Moreover, the action of the translation $\tau(\mu)$ ($\mu \in P^\vee$) for a parameter $u \in E$ is given by

$$\tau(\mu)e^\lambda = q^{-\langle \lambda, \mu \rangle}e^\lambda \quad \text{for} \quad \lambda \in P, \quad \tau(\mu)h_w = q^{\langle \mu, wu \rangle}h_w \quad \text{for} \quad w \in W$$

and

$$r_{\tau(\mu)}(fh) = \tau(\mu)(f)q^{\langle \mu, wu \rangle}h_w \quad \text{for} \quad f \in A \quad \text{and} \quad w \in W.$$  

This is an evaluation representation for which $e^\delta$ is identified with $q$.

Hereafter the symbol $r_w$ is used also to represent the element $w$ from the extended affine Weyl group $W_{P^\vee} = W \rtimes P^\vee$ (the semidirect product of $W$ and $P^\vee$). Then $r_w \tau(\epsilon_i) = \tau(w(\epsilon_i))r_w$. Note also that, if $w = v\tau(\lambda), \quad \lambda \in P^\vee, \quad v \in W$, we have $w(\mu) = v\mu - \langle \lambda, \mu \rangle \delta$ for $\mu \in P$.

For an affine root $\alpha + m\delta$ ($\alpha \in \Delta, m \in \mathbb{Z}$), define the $R$-matrix $R_{\alpha + m\delta}$ as an element of $End_{A^\sim}(V^\sim)$ by the formula

$$R_{\alpha + m\delta}h_y = \begin{cases} a_{\alpha + m\delta}h_y + q^m(\alpha^\vee, yu)b_{\alpha + m\delta}h_{sy}, & y^{-1}(\alpha) > 0, \\ c_{\alpha + m\delta}h_y + q^m(\alpha^\vee, yu)d_{\alpha + m\delta}h_{sy}, & y^{-1}(\alpha) < 0 \end{cases}$$

for $y \in W$, where

$$a_{\alpha + m\delta} = \frac{1 - q^m e^\alpha}{1 - t_\alpha q^m e^\alpha}, \quad b_{\alpha + m\delta} = \frac{1 - t_\alpha}{1 - t_\alpha q^m e^\alpha}, \quad c_{\alpha + m\delta} = \frac{t_\alpha (1 - q^m e^\alpha)}{1 - t_\alpha q^m e^\alpha}, \quad d_{\alpha + m\delta} = \frac{q^m e^\alpha (1 - t_\alpha)}{1 - t_\alpha q^m e^\alpha}.$$
and \( \alpha \mapsto t_\alpha \) is a \( W \)-invariant function taking positive values; there are two different \( t_\alpha \), which we may write as \( t_1 = t_{\pm \epsilon_i, \pm \epsilon_j} \), \( t_2 = t_{\pm 2\epsilon_j} \).

It is seen that

\[
\begin{align*}
    r_w R_\alpha &= R_{w(\alpha)} r_w \quad \text{for} \quad \alpha \in \tilde{\Delta} \, , \, w \in W^\nu \, , \\
    R_\beta &= R_{- \beta}^{-1} \quad \text{for} \quad \beta \in \tilde{\Delta}
\end{align*}
\]

and

\[
\begin{align*}
    R_{\epsilon_i - \epsilon_j} R_{\epsilon_i - \epsilon_k} R_{\epsilon_j - \epsilon_k} &= R_{\epsilon_j - \epsilon_k} R_{\epsilon_i - \epsilon_k} R_{\epsilon_i - \epsilon_j} \, , \quad 1 \leq i < j < k \leq n \\
    R_{\epsilon_i - \epsilon_j} R_{2\epsilon_i} R_{\epsilon_i + \epsilon_j} R_{2\epsilon_j} &= R_{2\epsilon_j} R_{\epsilon_i + \epsilon_j} R_{2\epsilon_i} R_{\epsilon_i - \epsilon_j} \, , \quad 1 \leq i < j \leq n.
\end{align*}
\]

The relations in (2.3) constitute the Yang-Baxter equation associated with the root system of type \( C_n \).

Then we can state the definition of the QKZ equation for the root system of type \( C_n \).

**Definition 2.1.** The QKZ equation for the root system \( C_n \) with a parameter \( u = (u_1, \ldots, u_n) \in \mathbb{R}^n \) is the following system of equations:

\[
r_{\tau(\epsilon_i)}^{-1} F = R_{\tau(\epsilon_i)} F, \quad 1 \leq i \leq n,
\]

and

\[
r_{\tau(\frac{\lambda}{2} + \epsilon_1 + \cdots + \epsilon_n)}^{-1} F = R_{\tau(\frac{\lambda}{2} + \epsilon_1 + \cdots + \epsilon_n)} F,
\]

for \( F \in V^\sim \), with

\[
R_{\tau(\epsilon_i)} = R_{\epsilon_i - \epsilon_{i-1} + \delta} \cdots R_{\epsilon_i - \epsilon_1 + \delta} R_{2\epsilon_i + \delta} R_{\epsilon_i + \epsilon_i} \cdots R_{\epsilon_i - 1 + \epsilon_i} \times R_{\epsilon_i + \epsilon_i + 1} \cdots R_{\epsilon_i + \epsilon_n} R_{2\epsilon_i} R_{\epsilon_i - \epsilon_n} \cdots R_{\epsilon_i - \epsilon_i + 1}
\]

for \( 1 \leq i \leq n \), and

\[
R_{\tau(\frac{\lambda}{2} + \epsilon_1 + \cdots + \epsilon_n)} = (R_{2\epsilon_i} R_{\epsilon_i + \epsilon_2} R_{\epsilon_i + \epsilon_3} \cdots R_{\epsilon_i + \epsilon_n}) \times (R_{2\epsilon_2} R_{\epsilon_2 + \epsilon_3} \cdots R_{\epsilon_2 + \epsilon_n}) \times \cdots \times (R_{2\epsilon_n-1} R_{\epsilon_n-1 + \epsilon_n}) R_{2\epsilon_n}.
\]

**Remark.** If we introduce the operators \( L_\mu (\mu \in P^\nu) \) and \( P^u_\mu \in \text{End}_{\mathcal{A}^\nu}(V^\sim) \) (\( \mu \in P^\nu, \ u \in E \)) defined by

\[
L_\mu (\sum f_w h_w) = \sum L_\mu(f_w) h_w \quad \text{with} \quad L_\mu(e^\lambda) = q^{(\mu, \lambda)} e^\lambda \quad (\lambda \in P)
\]

and

\[
P^u_\mu (h_w) = q^{(\mu, \mu)} h_w,
\]

then the equation above can be rewritten as

\[
L_{\epsilon_i} F = P^u_\epsilon R_{\tau(\epsilon_i)} F
\]

and

\[
L_{\frac{\lambda}{2} + \epsilon_1 + \cdots + \epsilon_n} F = P^u_{\frac{\lambda}{2} + \epsilon_1 + \cdots + \epsilon_n} R_{\tau(\frac{\lambda}{2} + \epsilon_1 + \cdots + \epsilon_n)} F.
\]

Fulfilment of the compatibility condition of the QKZ equation is guaranteed by the Yang-Baxter equation (2.3).
In the next section, we will construct a solution of the QKZ equation for the special case \( u = -\lambda \epsilon_1 (\lambda > 0) \) through application of the \( q \)-Jordan-Pochhammer integral.

3. Integrals and main result

We introduce the form

\[
\Phi = x^\lambda \prod_{1 \leq j \leq n} \frac{(ty_j/x)_\infty (ty_j^{-1}/x)_\infty}{(y_j/x)_\infty (y_j^{-1}/x)_\infty} \frac{dx}{x},
\]  

(3.1)

where \((a)_\infty = \prod_{s=0}^\infty (1 - a^s)\). This can be regarded as a form of a restriction of the \( q \)-Jordan-Pochhammer integral

\[
x^\lambda \prod_{1 \leq j \leq 2n} \frac{(ty_j/x)_\infty}{x},
\]

which is studied in \([14]\) and \([1]\).

Next, to construct our solution in case of \( u = -\lambda \epsilon_1 (\lambda > 0) \), we use the induced representation of the Weyl group \( W = W(C_n) \) from the trivial representation of a parabolic subgroup.

As a parabolic subgroup of \( W \), we choose a stabilizer \( W_{\epsilon_1} = \langle s_2, \ldots, s_n \rangle \) of \( \epsilon_1 \).

A representative of the quotient \( W/W_{\epsilon_1} \) is fixed to be

\[
\{ w_i = e, w_2 = s_1, w_3 = s_2s_1, \ldots, w_{n+1} = s_n \cdot \ldots \cdot s_2s_1, w_{n+2} = s_{n-1}w_{n+1}, w_{n+3} = s_n \cdot \ldots \cdot s_2s_1w_{n+1} = s_1 \cdot \ldots \cdot s_2 \cdot s_1w_{n+1}, \ldots, w_{2n} = s_1 \cdot \ldots \cdot s_2 \cdot s_1w_{n+1} \}.
\]

It is seen that the element \( h_e = \sum_{g \in W_{\epsilon_1}} h_g \) is invariant under the action of \( W_{\epsilon_1} \) and the induced representation of \( W \) from \( h_e \) is given by the elements

\[
\overline{h}_w = \sum_{g \in W_{\epsilon_1}} h_{wg} (1 \leq i \leq 2n).
\]

Using \( w_i \) as suffices, we define the following rational functions:

\[
\varphi_{w_i} = \begin{cases} 
\prod_{1 \leq \mu < i} \frac{1 - y_\mu^{-1}}{x}, & 1 \leq i \leq n, \\
\prod_{1 \leq \mu \leq i} \left( 1 - \frac{y_\mu}{x} \right), & n + 1 \leq i \leq 2n
\end{cases}
\]

Associated with the function \( \Phi \), we write

\[
\langle \psi \rangle = \int_C \psi \Phi
\]

for a rational function \( \psi \) and a fixed cycle \( C \), and define the element \( \Psi \) by

\[
\Psi = \sum_{1 \leq i \leq 2n} \langle \varphi_{w_i} \rangle \overline{h}_{w_i}.
\]
Then we obtain the following, which will be proven in the next section.

**Proposition 3.1.** \( r_i \Psi = R_{a_i} \Psi \) for \( 0 \leq i \leq n \).

We are now in a position to state our main result.

**Theorem 3.2.** The function

\[
\Psi = \sum_{1 \leq i \leq 2n} (\varphi_{w_i}) R_{w_i}
\]

satisfies the QKZ equation of type \( C_n \) with the parameter \( u = -\lambda \epsilon_1 (\lambda > 0) \) and \( t_1 = t_2 = t \):

\[
r^{-1}_{\tau(\epsilon_i)} \Psi = R_{\tau(\epsilon_i)} \Psi, \quad 1 \leq i \leq n, \tag{3.2}
\]

and

\[
r^{-1}_{\tau(\epsilon_1 \ldots + \epsilon_n)} \Psi = R_{\tau(\epsilon_1 \ldots + \epsilon_n)} \Psi. \tag{3.3}
\]

From this point we use the identification \( y_i = e^{\epsilon_i} \) for \( 1 \leq i \leq n \).

It is seen that a system of fundamental solutions is obtained by taking suitable linearly independent cycles.

**Proof.** We first note

\[
r^{-1}_{\tau(\epsilon_i)} \Psi = r_{s_0} s_0 \Psi.
\]

Proposition 3.1 and (2.1) imply

\[
r_{s_0} s_0 \Psi = r_{s_0} r_{s_0} \Psi = r_{s_0} R_{a_0} \Psi = R_{s_0(a_0)} r_{s_0} \Psi.
\]

Applying this process repeatedly, we finally obtain

\[
r_{s_0} s_0 \Psi = R_{s_0(a_0)} R_{(s_1 \ldots s_n)(s_n \ldots s_2)(a_1)} R_{(s_1 \ldots s_n)(s_2 \ldots s_3)(a_2)} \cdots R_{(s_1 \ldots s_n)(a_n-1)}
\]

\[
\times R_{(s_1 \ldots s_n-1)(a_n)} \cdots R_{s_1(\alpha_2)} R_{s_1(\alpha_1)} \Psi
\]

\[
= R_{2\epsilon_1+\delta} R_{\epsilon_1+\epsilon_2} \cdots R_{\epsilon_i+\epsilon_n} R_{2\epsilon_1} R_{\epsilon_1-\epsilon_n} \cdots R_{\epsilon_i-\epsilon_2} \Psi,
\]

since \( s_0 = (s_1 \ldots s_n-1)(s_n \ldots s_1) \). Thus we have

\[
r^{-1}_{\tau(\epsilon_i)} \Psi = R_{2\epsilon_1+\delta} R_{\epsilon_1+\epsilon_2} \cdots R_{\epsilon_i+\epsilon_n} R_{2\epsilon_1} R_{\epsilon_1-\epsilon_n} \cdots R_{\epsilon_i-\epsilon_2} \Psi. \tag{3.4}
\]

Next, let us apply \( r_{s_i-1 \cdots s_i} \) on both sides of (3.4). Then the left-hand side is

\[
r^{-1}_{\tau(s_i-1 \cdots s_i)} r^{-1}_{\tau(\epsilon_i)} \Psi = r^{-1}_{\tau(s_i-1 \cdots s_i)} r_{s_i-1 \cdots s_i} \Psi
\]

\[
= r^{-1}_{\tau(s_i-1 \cdots s_i)} r_{s_i-1 \# s_1 \# s_2} \Psi
\]

\[
= r^{-1}_{\tau(s_i-1 \cdots s_i)} r_{s_i-1 \# s_1 \# s_2} \Psi
\]

\[
= r^{-1}_{\tau(s_i-1 \cdots s_i)} r_{s_i-1 \# s_1 \# s_2} \Psi
\]

\[
= R_{\epsilon_1-\epsilon_i} R_{\epsilon_2-\epsilon_2} \cdots R_{\epsilon_{i-1}-\epsilon_{i-1}} R_{\epsilon_i-\epsilon_i} \Psi
\]

This follows from the relation \( \tau(-\epsilon_i)(\epsilon_j - \epsilon_i) = \epsilon_j - \epsilon_i - \delta \).

On the other hand, the right-hand side is

\[
r_{s_i-1 \cdots s_i} R_{2\epsilon_1+\delta} R_{\epsilon_1+\epsilon_2} \cdots R_{\epsilon_i+\epsilon_n} R_{2\epsilon_1} R_{\epsilon_1-\epsilon_n} \cdots R_{\epsilon_i-\epsilon_2} \Psi
\]

\[
= R_{2\epsilon_1} R_{\epsilon_1+\epsilon_i} R_{\epsilon_2+\epsilon_i} \cdots R_{\epsilon_{i-1}+\epsilon_i} R_{\epsilon_i+\epsilon_{i+1}} \cdots R_{\epsilon_i+\epsilon_n} R_{2\epsilon_1} \Psi.
\]

Here we have used

\[
r^{-1}_{\tau(s_i-1 \cdots s_i)} \Psi = R_{\epsilon_i-\epsilon_n} \cdots R_{\epsilon_{n-1}-\epsilon_n} \Psi.
\]
Therefore we reach the desired relation (3.2) by using (2.2).

Next we proceed to derive (3.3).

For $1 \leq i \leq n$, we have

$$r_{(n)}^{-1}(\tau_{(\varepsilon_1, \ldots, \varepsilon_n)}) \langle \varphi_{w_i} \rangle$$

$$= \int_C x^\lambda \prod_{k=1}^{n} \left( q^{\frac{1}{2}} \frac{ty_k}{x} \right) \prod_{k=i+1}^{n} \left( q^{-\frac{1}{2}} \frac{y_k^{-1}}{x} \right) \prod_{k=1}^{i} \left( q^{\frac{1}{2}} \frac{y_k^{-1}}{x} \right) \infty \int dx$$

(3.5)

and

$$r_{(n)}^{-1}(\tau_{(\varepsilon_1, \ldots, \varepsilon_n)}) \langle \varphi_{w_{n+i}} \rangle$$

$$= \int_C x^\lambda \prod_{k=1}^{n-i} \left( q^{\frac{1}{2}} \frac{y_k}{x} \right) \prod_{k=n-i+1}^{n} \left( q^{\frac{1}{2}} \frac{y_k}{x} \right) \prod_{k=1}^{n-i+1} \left( q^{-\frac{1}{2}} \frac{y_k^{-1}}{x} \right) \infty \int dx.$$ (3.6)

By changing the integration variable such that $x \mapsto q^{-1/2}x$, from (3.5) we have

$$r_{(n)}^{-1}(\tau_{(\varepsilon_1, \ldots, \varepsilon_n)}) \langle \varphi_{w_i} \rangle$$

$$= q^{-\frac{1}{2}} \int_C x^\lambda \prod_{k=1}^{n} \left( q^{\frac{1}{2}} \frac{y_k}{x} \right) \prod_{k=i+1}^{n} \left( q^{-\frac{1}{2}} \frac{y_k^{-1}}{x} \right) \prod_{k=1}^{i} \left( q^{\frac{1}{2}} \frac{y_k^{-1}}{x} \right) \infty \int dx x$$

$$= q^{-\frac{1}{2}} \langle g \varphi_{w_{n+i}} \rangle$$

with

$$g = s_1(s_{n-1}s_n)(s_{n-2}s_{n-1}s_n) \cdots (s_1 \cdots s_n) \in W.$$ Here we note $g(\varepsilon_i) = -\varepsilon_{n-i+1}$ for each $1 \leq i \leq n$.

Similarly, as a result of the change $x \mapsto q^{1/2}x$, from (3.6) we have

$$r_{(n)}^{-1}(\tau_{(\varepsilon_1, \ldots, \varepsilon_n)}) \langle \varphi_{w_{n+i}} \rangle$$

$$= q^{\frac{1}{2}} \int_C x^\lambda \prod_{k=1}^{n-i} \left( \frac{ty_k}{x} \right) \prod_{k=n-i+1}^{n} \left( \frac{y_k}{x} \right) \prod_{k=1}^{n-i+1} \left( \frac{y_k^{-1}}{x} \right) \infty \int dx x$$

$$= q^{\frac{1}{2}} \langle g \varphi_{w_i} \rangle$$

with the same $g \in W$. 
As for this $g = s_n(s_{n-1}s_n)(s_{n-2}s_{n-1}s_n)\cdots(s_1\cdots s_n) \in W$, we have
\[ gw_i = w_{n+1} s_n(s_{n-1}s_n)(s_{n-2}s_{n-1}s_n)\cdots(s_2\cdots s_{n-1}s_n), \]
\[ gw_{n+1} = w_i s_n(s_{n-1}s_n)(s_{n-2}s_{n-1}s_n)\cdots(s_2\cdots s_{n-1}s_n) \]
for $1 \leq i \leq n$. These relations lead to
\[ g\overline{w}_i = \overline{w}_{n+1}, \]
\[ g\overline{w}_{n+1} = \overline{w}_i \]
for $1 \leq i \leq n$.

On the other hand, noting $u = -\lambda e_1$, we obtain
\[ \tau(-\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n))\overline{\Psi} = q^{-\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)}(\epsilon_1 + \cdots + \epsilon_n)\overline{\Psi} = q^{\frac{1}{2}}\overline{\Psi} \]
\[ \tau(-\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n))\overline{\Psi} = q^{\frac{1}{2}}\overline{\Psi} \]
for $1 \leq i \leq n$.

Combining these relations, we get
\[ \tau(-\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n))\Psi \]
\[ = \tau(-\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)) \sum_{1 \leq i \leq n} \left\{ (\varphi_{w_i})\overline{\Psi} + (\varphi_{w_{n+1}})\overline{\Psi} \right\} \]
\[ = \sum_{1 \leq i \leq n} \left\{ q^{-\frac{1}{2}}(g\varphi_{w_{n+1}})q^{\frac{1}{2}}\overline{\Psi} + q^{\frac{1}{2}}(g\varphi_{w_i})q^{-\frac{1}{2}}\overline{\Psi} \right\} \]
\[ = \sum_{1 \leq i \leq n} \left\{ (g\varphi_{w_{n+1}})\overline{\Psi} + (g\varphi_{w_i})\overline{\Psi} \right\} \]
\[ = \sum_{1 \leq i \leq n} \left\{ (g\varphi_{w_{n+1}})g\overline{\Psi} + (g\varphi_{w_i})g\overline{\Psi} \right\} \]
\[ = r_g \Psi. \]

At this stage, applying the relation
\[ r(s_{n}(s_{n-1}s_n)\cdots(s_{k+1}\cdots s_n))s_k\cdots s_{n-1} \Psi \]
\[ = R(s_{n}(s_{n-1}s_n)\cdots(s_{k+1}\cdots s_n))s_k\cdots s_{n-1}(\alpha_n)R(s_{n}(s_{n-1}s_n)\cdots(s_{k+1}\cdots s_n))s_k\cdots s_{n-2}(\alpha_{n-1}) \]
\[ \times \cdots \times R(s_{n}(s_{n-1}s_n)\cdots(s_{k+1}\cdots s_n))s_k(\alpha_{k+1})R(s_{n}(s_{n-1}s_n)\cdots(s_{k+1}\cdots s_n))(\alpha_k) \]
\[ \times r(s_{n}(s_{n-1}s_n)\cdots(s_{k+1}\cdots s_n)) \Psi \]
\[ = R_{2e_n} R_{\epsilon_{k+1}+\epsilon_{n-1}} R_{\epsilon_k+\epsilon_n} \]
\[ \times r(s_{n}(s_{n-1}s_n)\cdots(s_{k+1}\cdots s_{n-1})) \Psi, \quad (1 \leq k \leq n) \]
repeatedly, we finally obtain
\[ r_g \Psi = (R_{2\epsilon_1} R_{\epsilon_1+\epsilon_2} \cdots R_{\epsilon_1+\epsilon_n})(R_{2\epsilon_2} R_{\epsilon_2+\epsilon_3} \cdots R_{\epsilon_2+\epsilon_n}) \]
\[ \times \cdots \times (R_{2\epsilon_{n-1}} R_{\epsilon_{n-1}+\epsilon_n})(R_{2\epsilon_{n-1}} \Psi. \]

Therefore, we reach the desired result (3.3).
4. PROOF OF PROPOSITION 3.1

To prove Proposition 3.1, we start by considering the action of $s_i \in W$ on the $\varphi_{w_k}$.

**Lemma 4.1.** (a) If $1 \leq i \leq n - 1$, $s_i \varphi_{w_k} = \varphi_{w_k}$ for each $1 \leq k \leq 2n$ such that $k \neq i, i + 1, 2n - i, 2n - i + 1$.
(b) $s_n \varphi_{w_k} = \varphi_{w_k}$ for each $1 \leq k \leq 2n$ such that $k \neq n, n + 1$.
(c) $s_0 \varphi_{w_k} = \varphi_{w_k}$ for each $1 \leq k \leq 2n$ such that $k \neq 1, 2n$.

**Proof.** These assertions follow from the definition of $s_i$ and $\varphi_{w_k}$.

Moreover we have

**Lemma 4.2.** (a) For $1 \leq i \leq n - 1$;\[
\begin{align*}
\left\{ \begin{array}{l}
    s_i \varphi_{w_{i+1}} &= a_{\alpha_i} \varphi_{w_i} + d_{\alpha_i} \varphi_{w_{i+1}}, \\
    s_i \varphi_{w_i} &= b_{\alpha_i} \varphi_{w_i} + c_{\alpha_i} \varphi_{w_{i+1}},
\end{array} \right. 
\tag{4.1}
\end{align*}
\]
and
\[
\begin{align*}
\left\{ \begin{array}{l}
    s_i \varphi_{w_{2n-i}} &= a_{\alpha_i} \varphi_{w_{2n-i}} + d_{\alpha_i} \varphi_{w_{2n-i+1}}, \\
    s_i \varphi_{w_{2n-i+1}} &= b_{\alpha_i} \varphi_{w_{2n-i}} + c_{\alpha_i} \varphi_{w_{2n-i+1}},
\end{array} \right. 
\tag{4.2}
\end{align*}
\]
(b)\[
\begin{align*}
\left\{ \begin{array}{l}
    s_n \varphi_{w_{n+1}} &= a_{\alpha_n} \varphi_{w_n} + d_{\alpha_n} \varphi_{w_{n+1}}, \\
    s_n \varphi_{w_n} &= b_{\alpha_n} \varphi_{w_n} + c_{\alpha_n} \varphi_{w_{n+1}}.
\end{array} \right. 
\tag{4.3}
\end{align*}
\]

**Proof.** By direct calculation or expansion of partial fractions, we find
\[
1 - \frac{y_{i+1}^{-1}}{x} = a_{\alpha_i} \frac{1}{1 - ty_{i+1}^{-1} x} + d_{\alpha_i} \frac{1 - y_i^{-1}}{x} \frac{1 - t y_{i+1}^{-1} x}{(1 - t y_{i+1}^{-1} x)(1 - ty_{i+1}^{-1} x)}. \tag{4.4}
\]

and
\[
1 - \frac{y_{i+1}^{-1}}{x} = b_{\alpha_i} \frac{1}{1 - ty_{i+1}^{-1} x} + c_{\alpha_i} \frac{1 - y_i^{-1}}{x} \frac{1 - t y_{i+1}^{-1} x}{(1 - t y_{i+1}^{-1} x)(1 - ty_{i+1}^{-1} x)}. \tag{4.5}
\]

Multiplying the factor
\[
\prod_{j=1}^{i-1} \frac{1 - y_j^{-1}}{1 - ty_j^{-1} x}
\]
on both sides of each equality (4.4) or (4.5), we get the desired relations (4.1).
While the change of variables $\epsilon_i \mapsto -\epsilon_{i+1}$ and $\epsilon_{i+1} \mapsto -\epsilon_i$ leave $\alpha_i$ unchanged, they produce the following from (4.4) and (4.5):

\[
1 - \frac{y_i}{x} \left( 1 - t \frac{y_{i+1}}{x} \right) \left( 1 - t \frac{y_i}{x} \right) = a_{\alpha_i} \frac{1}{1 - t \frac{y_i}{x}} + d_{\alpha_i} \frac{1}{1 - t \frac{y_{i+1}}{x}} \tag{4.6}
\]

and

\[
1 - \frac{y_{i+1}}{x} = b_{\alpha_i} \frac{1}{1 - t \frac{y_i}{x}} + c_{\alpha_i} \frac{1}{1 - t \frac{y_{i+1}}{x}}. \tag{4.7}
\]

Multiplying the factor

\[
\prod_{j=i+2}^{n} \frac{1 - y_j}{x} = b_{\alpha_n} \prod_{j=1}^{n-1} \frac{1 - y_j^{-1}}{x} \tag{4.8}
\]

on both sides of equalities (4.6) and (4.7), we obtain the desired relations (4.2).

Similarly, changing $\epsilon_i \mapsto \epsilon_n$ and $\epsilon_{i+1} \mapsto -\epsilon_n$ induces $\alpha_i \mapsto \alpha_n$ and leads from equalities (4.4) and (4.5) to

\[
1 - \frac{y_n}{x} \left( 1 - t \frac{y_n}{x} \right) \left( 1 - t \frac{y_{n-1}}{x} \right) = a_{\alpha_n} \frac{1}{1 - t \frac{y_n}{x}} + d_{\alpha_n} \frac{1}{1 - t \frac{y_{n-1}}{x}} \tag{4.8}
\]

and

\[
1 - \frac{y_{n-1}}{x} = b_{\alpha_n} \frac{1}{1 - t \frac{y_n}{x}} + c_{\alpha_n} \frac{1}{1 - t \frac{y_{n-1}}{x}}. \tag{4.9}
\]

Multiplying the factor

\[
\prod_{j=1}^{n-1} \frac{1 - y_j^{-1}}{x} \tag{4.8}
\]

on both sides of equalities (4.8) and (4.9), we get the desired relations (4.3).

In contrast to the action of $s_i$ for $1 \leq i \leq n$, the action of $s_0$ is understood as it acts on the $q$ de Rham cohomology, not on the rational functions.
Lemma 4.3.
\[
\begin{align*}
q^\lambda \langle s_0 \varphi_{w_1} \rangle &= a_{\delta - \theta} \langle \varphi_{w_{2n}} \rangle + q^\lambda d_{\delta - \theta} \langle \varphi_{w_1} \rangle, \\
q^{-\lambda} \langle s_0 \varphi_{w_2} \rangle &= q^{-\lambda} b_{\delta - \theta} \langle \varphi_{w_{2n}} \rangle + c_{\delta - \theta} \langle \varphi_{w_1} \rangle.
\end{align*}
\]

Proof. Make the change of variables \( \epsilon_i \mapsto -\epsilon_1 \) and \( \epsilon_{i+1} \mapsto \epsilon_1 - \delta \) (i.e. \( y_i^{-1} \mapsto y_1, y_{i+1}^{-1} \mapsto qy_1^{-1} \)) in (4.4) and (4.5). Then we have
\[
1 - qy_1^{-1} x (1 - ty_1 x) (1 - t qy_1^{-1} x) = a \delta - \theta \int \frac{1 - y_1 x}{1 - y_1 x} + d \delta - \theta \int \frac{1 - y_1 x}{1 - y_1 x} (1 - t qy_1^{-1} x) (1 - t y_1 x)
\]
and
\[
\frac{1}{1 - t qy_1^{-1} x} = b \delta - \theta \int \frac{1}{1 - y_1 x} + c \delta - \theta \int \frac{1 - y_1 x}{1 - y_1 x} (1 - t y_1 x).
\]
Integration after multiplying the factor
\[
\prod_{j=2}^{n} \frac{1 - y_j x}{1 - t y_j x} \prod_{j=1}^{n} \frac{1 - y_j^{-1} x}{1 - t y_j^{-1} x} \Phi
\]
on both sides of equalities (4.11) and (4.12) gives the following:
\[
\int_C x^\lambda \prod_{k=1}^{n} \left( q^{-1} y_k x \right)^{\infty} \prod_{k=2}^{n} \left( q^{-1} y_k x \right)^{\infty} \prod_{k=2}^{n} \left( q^{-1} y_k^{-1} x \right)^{\infty} \prod_{k=2}^{n} \left( q^{-1} y_k^{-1} x \right)^{\infty} dx
\]
\[
= a \delta - \theta \int_C x^\lambda \prod_{k=1}^{n} \left( q^{-1} y_k x \right)^{\infty} \prod_{k=1}^{n} \left( q^{-1} y_k^{-1} x \right)^{\infty} \prod_{k=1}^{n} \left( q^{-1} y_k^{-1} x \right)^{\infty} dx
\]
\[
+ d \delta - \theta \int_C x^\lambda \prod_{k=1}^{n} \left( q^{-1} y_k x \right)^{\infty} \prod_{k=1}^{n} \left( q^{-1} y_k^{-1} x \right)^{\infty} \prod_{k=1}^{n} \left( q^{-1} y_k^{-1} x \right)^{\infty} dx
\]
QKZ EQUATION 11

and

\[
\int_C x^\lambda \left( \frac{ty_1}{x} \right)_\infty \prod_{k=2}^n \left( q \frac{ty_k}{x} \right)_\infty \frac{q^2 y_1^{-1}}{x} \prod_{k=2}^n \left( q y_k^{-1} \right)_\infty \frac{q t y_k^{-1}}{x} \right)_\infty \frac{dx}{x} = \left( b_{\delta, \theta} \right)_C x^\lambda \left( \frac{ty_k}{x} \right)_\infty \prod_{k=2}^n \left( q \frac{ty_k}{x} \right)_\infty \frac{q y_k^{-1}}{x} \prod_{k=2}^n \left( q y_k^{-1} \right)_\infty \frac{dx}{x} \\
+ c_{\delta, \theta} x^\lambda \prod_{k=2}^n \left( q \frac{ty_k}{x} \right)_\infty \frac{q^2 y_1^{-1}}{x} \prod_{k=2}^n \left( q t y_k^{-1} \right)_\infty \frac{dx}{x} = q^\lambda \left( s_0 \varphi_{w_1} \right)
\]

Here, changing the integration variable such that \( x \mapsto qx \), we have

\[
\int_C x^\lambda \left( \frac{ty_k}{x} \right)_\infty \prod_{k=2}^n \left( q \frac{ty_k}{x} \right)_\infty \frac{q^2 y_1^{-1}}{x} \prod_{k=2}^n \left( q t y_k^{-1} \right)_\infty \frac{dx}{x} = q^\lambda \left( \varphi_{w_1} \right)
\]

and

\[
\int_C x^\lambda \prod_{k=1}^n \left( q \frac{ty_k}{x} \right)_\infty \frac{q^2 y_1^{-1}}{x} \prod_{k=2}^n \left( q t y_k^{-1} \right)_\infty \frac{dx}{x} = q^\lambda \left( \varphi_{w_1} \right)
\]

Therefore, it is seen that (4.13) and (4.14) are equivalent to the desired relations (4.10).

Next, we consider the action of \( W \) on the \( \tau_{w_k} \).
Lemma 4.4. (a) If $1 \leq i \leq n - 1$, $\overline{h}_{s_i w_k} = \overline{h}_{w_k}$ for $k \neq i, i+1, 2n-i, 2n-i+1$.
(b) $\overline{h}_{s_i w_k} = \overline{h}_{w_k}$ for $k \neq n, n+1$.
(c) $\overline{h}_{s_i w_k} = \overline{h}_{w_k}$ for $k \neq 1, 2n$.

Proof. In the case that $1 \leq i \leq n - 1$, we have $s_i w_k = w_k s_i$ for $1 \leq k \leq i - 1$ or $2n - i + 2 \leq k \leq 2n$, and $s_i w_k = w_k s_{i+1}$ for $i + 2 \leq k \leq 2n - i - 1$. These lead to the desired equalities in (a).

In the same way, the relations $s_n w_k = w_k s_n$ (for $k \neq n, n+1$) and $s_\delta w_k = w_k (s_\delta \cdots s_{n-1}) (s_n \cdots s_2)$ (for $k \neq 1, 2n$) lead to the relations in (b) and (c).

Next we consider the action of $R_{\alpha_i}$ on the $\overline{h}_{w_k}$.

Lemma 4.5. (a) If $1 \leq i \leq n - 1$, $R_{\alpha_i} \overline{h}_{w_k} = \overline{h}_{w_k}$ for each $1 \leq k \leq 2n$ such that $k \neq i, i+1, 2n-i, 2n-i+1$.
(b) $R_{\alpha_i} \overline{h}_{w_k} = \overline{h}_{w_k}$ for each $1 \leq k \leq 2n$ such that $k \neq n, n+1$.
(c) $R_{\delta - \theta} \overline{h}_{w_k} = \overline{h}_{w_k}$ for each $2 \leq k \leq 2n - 1$.

Proof. Since $w_k^i \alpha_i = \alpha_i > 0$ for $1 \leq k \leq i - 1$ (then $i \geq 2$), we have

$$R_{\alpha_i} h_{w_k} = a_{\alpha_i} h_{w_k} + b_{\alpha_i} h_{s_i w_k}$$

These imply

$$R_{\alpha_i} (h_{w_k} + h_{s_i w_k}) = h_{w_k} + h_{s_i w_k},$$

following from the relations $a_{\alpha_i} + b_{\alpha_i} = b_{\alpha_i} + c_{\alpha_i} = 1$. Hence, noting $s_i w_k = w_k s_i$, we obtain $R_{\alpha_i} \overline{h}_{w_k} = \overline{h}_{w_k}$. Other cases are similarly derived.

Lemma 4.6. (a) For $1 \leq i \leq n - 1$,

$$R_{\alpha_i} \overline{h}_{w_i} = a_{\alpha_i} \overline{h}_{w_i} + b_{\alpha_i} \overline{h}_{w_{i+1}},$$

$$R_{\alpha_i} \overline{h}_{w_{i+1}} = c_{\alpha_i} \overline{h}_{w_{i+1}} + d_{\alpha_i} \overline{h}_{w_i}.$$

(b) $R_{\alpha_n} \overline{h}_{w_n} = a_{\alpha_n} \overline{h}_{w_n} + b_{\alpha_n} \overline{h}_{w_{n+1}},$

$$R_{\alpha_n} \overline{h}_{w_{n+1}} = c_{\alpha_n} \overline{h}_{w_{n+1}} + d_{\alpha_n} \overline{h}_{w_n}.$$

(c) $R_{\delta - \theta} \overline{h}_{w_{2n}} = a_{\delta - \theta} \overline{h}_{w_{2n}} + q^{-\lambda} b_{\delta - \theta} \overline{h}_{w_1},$

$$R_{\delta - \theta} \overline{h}_{w_1} = c_{\delta - \theta} \overline{h}_{w_1} + q^{\lambda} d_{\delta - \theta} \overline{h}_{w_{2n}}.$$

Proof. This follows almost immediately from the definitions.

At this stage, by combination of the above lemmas, we obtain the following:

In case of $1 \leq i \leq n - 1$, we have

$$r_{s_i} \Psi = \sum_{1 \leq k \leq 2n} s_i \langle \varphi_{w_k} \rangle \overline{h}_{s_i w_k} = \sum_{k \neq i, i+1, 2n-i, 2n-i+1} \sum_{k=1, i+1}^{2n-i, i+1} \sum_{2n-i, 2n-i+1} \langle s_i \varphi_{w_k} \rangle \overline{h}_{s_i w_k}$$

$$= \sum_{k \neq i, i+1, 2n-i, 2n-i+1} \langle \varphi_{w_k} \rangle \overline{h}_{w_k}$$

$$+ \{ b_{\alpha_i} \langle \varphi_{w_i} \rangle + c_{\alpha_i} \langle \varphi_{w_{i+1}} \rangle \} \overline{h}_{s_i w_i} + \{ a_{\alpha_i} \langle \varphi_{w_i} \rangle + d_{\alpha_i} \langle \varphi_{w_{i+1}} \rangle \} \overline{h}_{s_i w_{i+1}}$$
This completes the proof of Proposition 3.1. □
5. Macdonald Polynomials

Macdonald introduced the $q$-difference operators \([10]\) to define his orthogonal polynomials associated with root systems. In the case of a root system of type $C_n$, the $q$-difference operator to define such a polynomial is given by

$$E = \sum_{a_1, \ldots, a_n = \pm 1} \prod_{1 \leq i < j \leq n} \frac{1 - ty_i a_i y_j a_j}{1 - y_i a_i y_j a_j} \prod_{1 \leq i \leq n} \frac{1 - ty_i 2a_i}{1 - y_i 2a_i} T_{a_i}^{\frac{1}{2}}$$

where

$$(T_y f)(y_1, \ldots, y_n) = f(y_1, \ldots, qy_i, \ldots, y_n).$$

Its eigenvalue is known to be

$$c_\mu = \sum_{a_1, \ldots, a_n = \pm 1} \prod_{j=1}^n q^{\frac{1}{2}(\lambda_j + \cdots + \lambda_n)} \prod_{j=1}^n (1 + t^j q^{\lambda_j - 1})$$

with the parameter $\mu = (\lambda_1, \ldots, \lambda_n)$. (We consider only the special case corresponding to the condition $t_1 = t_2 = t$).

As for the eigenfunction of the operator $E$, we easily find the following:

**Corollary 5.1.** The sum

$$\sum_{i=1}^{2n} t^{i-1} \langle \varphi_{w_i} \rangle$$

is a solution of the equation attached to the parameter $(\lambda, 0, \ldots, 0)$:

$$E \psi = c_{(\lambda,0,\ldots,0)} \psi.$$  \hspace{1cm} (5.2)

**Proof.** This is proven by applying the result of Kato (Theorem 4.6 in \([7]\)) to our Theorem 3.2.

We next proceed to simplify the sum (5.1).

We note the equality

$$t^{2n} \prod_{j=1}^n \left( 1 - t \frac{y_j}{x} \right) \left( 1 - t \frac{y_j}{x}^{-1} \right) = 1 + (t-1) \left\{ \sum_{j=1}^{2n} t^{i-1} \varphi_{w_i} \right\},$$

which is demonstrated by using the partial fractions.

On the other hand, we have

$$\left\langle \prod_{j=1}^n \left( 1 - t \frac{y_j}{x} \right) \left( 1 - t \frac{y_j}{x}^{-1} \right) \right\rangle = q^\lambda \int_\mathcal{C} \Phi = q^\lambda \langle 1 \rangle,$$
which is demonstrated by changing the integration variable such that $x \mapsto qx$. Hence, combination of (5.3) and (5.4) gives the relation

$$\sum_{j=1}^{2n} t^{i-1} \langle \varphi_{w_i} \rangle = \frac{1 - q^2t^{-2n}}{1-t} \int_\mathcal{C} \Phi.$$  

Therefore we reach

**Proposition 5.2.** The function $\int_\mathcal{C} \Phi$ is a solution to the equation (5.2).

It should be remarked that this is valid for arbitrary cycle $\mathcal{C}$ and that linearly independent solutions are obtained by choosing several cycles. This situation is similar to that studied in [15].

In case that the parameter $\mu$ is from the set of partitions, the eigenfunction of the form

$$P_{\mu}(y|q,t) = m_\mu + \sum_{\nu < \mu} a_{\mu \nu} m_\nu,$$

is the Macdonald polynomial for the root system $C_n$. Here $m_\mu = \sum_{\nu \in W} e^\nu$, and $\nu < \mu$ is defined to be $\mu - \nu \in Q^+$ with $Q^+$ the positive cone of the root lattice.

In our case, to get the Macdonald polynomial, it is enough to consider the case that $\lambda$ is a positive integer and take the cycle, with the counterclockwise direction, which encircles the sequence of poles such that $y_i, y_iq, y_iq^2, \ldots$, for $1 \leq i \leq n$ and $y_i^{-1}, y_i^{-1}q, y_i^{-1}q^2, \ldots$, for $1 \leq i \leq n$. This is an integral representation of the Macdonald polynomial $P_{(\lambda,0,\ldots,0)}(y|q,t)$.

Moreover, applying the $q$-binomial theorem

$$\sum_{m \geq 0} \frac{(a)_m}{(q)_m} z^m = \frac{(az)_\infty}{(z)_\infty} \quad (|z| < 1), \quad (a)_m = \prod_{0 \leq k \leq m-1} (1 - aq^k)$$

and the residue calculus to our integral, we obtain an exact expression of the Macdonald polynomial for the root system $C_n$.

**Theorem 5.3.**

$$P_{(\lambda,0,\ldots,0)}(y|q,t) = \frac{(q)_\lambda}{(t)_\lambda} \sum_{i_1+\cdots+i_{2n}=\lambda \atop i_1, \ldots, i_{2n} \geq 0} (t)_{i_1} \cdots (t)_{i_{2n}} \frac{(q)_{i_1} \cdots (q)_{i_{2n}}}{y_{i_1}^{i_1-i_2} \cdots y_{i_{2n}}^{i_{2n}-i_{2n-1}} \cdots y_{i_n}^{i_n-i_{n+1}}}.$$

**Remark.** We also have a direct way to obtain the integral representation of the eigenfunction for (5.2). This will appear in a future paper. For the related work, we also refer the reader to [16].

6. **Final Comment**

We finally make a comment on the meaning of our elements $\varphi_{w_i}$ from the viewpoint of the Hecke algebra.

Set

$$T_i = t + \frac{1 - te^{s_i}}{1 - e^{s_i}}(s_i - 1), \quad \text{for} \quad 1 \leq i \leq n.$$
where $\alpha_i$ is an element of the simple roots and $s_i$ a corresponding generator of the Weyl group $W$. This is the Lusztig operator associated with the root system $C_n$ (in the special case $t_1 = t_2 = t$), which satisfies the following:

\[
(T_i - t)(T_i + 1) = 0 \quad (1 \leq i \leq n),
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \leq i \leq n - 2),
\]

\[
T_{n-1} T_n T_{n-1} = T_n T_{n-1} T_n,
\]

\[
T_i T_j = T_j T_i \quad (|i - j| > 2).
\]

These are the fundamental relations for the Hecke algebra $H(W)$ associated with the root system of type $C_n$. The action of the Lusztig operator on our $\varphi_{w_i}$ is given as follows.

**Proposition 6.1.** For $1 \leq k \leq n$;

\[
\begin{aligned}
T_i \varphi_{w_k} &= t \varphi_{w_k}, \quad i \neq k - 1, k, \\
T_{k-1} \varphi_{w_k} &= (t - 1) \varphi_{w_k} + \varphi_{w_{k-1}}, \\
T_k \varphi_{w_k} &= t \varphi_{w_{k+1}}, \\
T_i \varphi_{w_{n+k}} &= t \varphi_{w_{n+k}}, \quad i \neq n - k, n - k + 1, \\
T_{n-k+1} \varphi_{w_{n+k}} &= (t - 1) \varphi_{w_{n+k}} + \varphi_{w_{n+k-1}}, \\
T_{n-k} \varphi_{w_{n+k}} &= t \varphi_{w_{n+k+1}}.
\end{aligned}
\]

This shows that the vector space $\bigoplus_{i=1}^{2n} C \varphi_{w_i}$ gives the representation of the Hecke algebra $H(W)$ for the $C_n$ type. Moreover, we can also obtain the representation of the affine Hecke algebra in the space of the $q$ de Rham cohomology. See [16] for $A_{n-1}$ case.

In any case, we expect that such a basis attached to the action of the Hecke algebras could be generalized to the case of higher representations. This is our future problem.

**Acknowledgement.** The author wishes to thank Professor Shin-ichi Kato for valuable suggestion.

**References**

[1] Aomoto, K., Kato, Y., Mimachi, K.: A solution of the Yang-Baxter equation as connection coefficients of a holonomic $q$-difference system. Internat. Math. Res. Notices 1992, No.1, 7-15

[2] Cherednik, I.: Quantum Knizhnik-Zamolodchikov equations and affine root systems. Commun. Math. Phys. 150, 109-136 (1992)

[3] Cherednik, I.: Double affine Hecke algebras, Knizhnik-Zamolodchikov equations, and Macdonald’s operators. Internat. Math. Res. Notices 1992, No.9, 171-180

[4] Cherednik, I.: Double affine Hecke algebras, and Macdonald’s conjectures. Ann. of Math. 141, 191-216 (1995)

[5] Cherednik, I.: Induced representations of double affine Hecke algebras and applications. Math. Res. Lett. 1, 319-337 (1994)

[6] Debiard, A. and Gaveau, B.: Integral formulas for the spherical polynomials of a root system of type $BC_2$. Journ. de Funct. Anal. 119, 401-454 (1994)

[7] Kato, S.: R-matrix arising from affine Hecke algebras and its application to Macdonald’s difference operators. Commun. Math. Phys. 165, 533-553 (1994)

[8] Koornwinder, T. H.: Askey-Wilson polynomials for root systems of type $BC$, in Hypergeometric functions on domains of positivity, Jack polynomials and applications, D.St.P.Richards(ed.), Contemp. Math. 138, Amer. Math. Soc. (1992), pp.189-204
[9] Lusztig, G.: Affine Hecke algebras and their graded version. J. Am. Math. Soc. 2, 599-635 (1989)
[10] Macdonald, I.G.: A new class of symmetric functions, in Actes Séminaire Lotharingen, Publ. Inst. Rech. Math. Adv., Strasbourg, 1988, 131-171
[11] Macdonald, I.G.: Affine Hecke algebras and orthogonal polynomials. Séminaire BOURBAKI, 47ème année, 1994-95, n°797
[12] Macdonald, I.G.: Symmetric Functions and Hall Polynomials (Second Edition), Oxford Mathematical Monographs, Clarendon Press, Oxford, 1995.
[13] Mimachi, K.: Connection problem in holonomic q-difference system associated with a Jackson integral of Jordan-Pochhammer type. Nagoya Math. J. 116, 149-161 (1989)
[14] Mimachi, K.: A solution to quantum Knizhnik-Zamolodchikov equations and its application to eigenvalue problems of the Macdonald type, Duke Math. J. 85, 635-658 (1996).
[15] Mimachi, K.: Rational solutions to eigenvalue problems of the Macdonald type, in preparation.
[16] Mimachi, K. and Noumi, M.: An integral representation of eigenfunctions for Macdonald’s q-difference operators, Tôhoku Math. J. 49, 517-525 (1997).
[17] Mimachi, K. and Noumi, M.: Representaions of the Hecke algebra on a family of rational functions, preprint 1997.

KATSUSHIMA MIMACHI
DEPARTMENT OF MATHEMATICS
KYUSHU UNIVERSITY 33
HAKOZAKI, FUKUOKA 812-81
JAPAN
E-mail address: mimachi@math.kyushu-u.ac.jp