The clustered Sparrow algorithm

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Abstract. In this paper, we study an extension of Schöning’s algorithm [Schöning, 1991] for 3SAT, the clustered Sparrow algorithm. We also present strong arguments that this algorithm is polynomial.

Keywords. NP-complete problems, 3SAT, Markov chains

1 Introduction

The importance of efficiently solving NP-complete problems is due to the fact that this would imply that all the other problems belonging to the NP class can be efficiently solved in a constructive manner (the algorithms can be generated for all of them through polynomial-time reductions).

It is well known that 3SAT is NP-complete.

Proposition 1. (see [Papadimitriou, 94] and [Hopcroft, 1979]). 3SAT is NP-complete. Also, 3SAT remains NP-complete even for expressions in which each variable is restricted to appear at most three times, and each literal at most twice.

Proof. For the proof, see (see [Papadimitriou, 94] and [Hopcroft, 1979]).

We will first present a well-known randomized algorithm for 3SAT. This is Schöning’s algorithm from 1991 (see [Schöning, 1991]). We also note that Papadimitriou also discussed a similar algorithm for 2SAT in 1991 (see [Papadimitriou, 1991]).

It is known that Papadimitriou’s algorithm finds a solution in quadratic time with high probability for 2SAT. Only exponential bounds are known for Schöning’s algorithm for 3SAT.

Stochastic local search algorithms played an important role, related to SAT solvers. This includes the random walk algorithm of Papadimitriou for 2SAT, and Schöning’s extension of this algorithm for 3SAT.
Schöning’s algorithm for 3SAT

Input: a 3SAT expression in \( n \) variables.

Guess an initial truth assignment, uniformly at random.

Repeat \( 3 \cdot n \) times:

- If the expression is satisfied by the actual assignment, stop and accept.
- Let \( C \) be some clause not being satisfied by the actual assignment. Pick one of the three literals in the clause at random, and flip its truth value.
- Update.

Stop and reject, the expression is not satisfiable.

Note that these algorithms involve random walks on the boundary of the truth assignment hypercube (the unit hypercube where each vertex represents a truth assignment).

2 The clustered Sparrow algorithm

We are given a 3SAT expression \( E \). A positive literal represents the variable itself, a negative literal represents the negation of a variable. For each occurrence of a variable \( x \) (in a positive or negative literal) within the expression we introduce a new variable \( x_i \), and form the modified expression \( E^* \). We also have to include a conjunction of clauses that state that the variables \( x_i \) have the same truth assignment. The conjunction \( (x_i \lor \overline{x}_i) \land (\overline{x}_i \lor \overline{x}_j) \) has the truth value 1 if and only if the variables \( x_i \) and \( x_j \) have the same truth value. In general, the variables \( x_1, x_2, x_3, \ldots, x_m \) have the same truth value if and only if the conjunction \( (x_1 \lor \overline{x}_2) \land (x_2 \lor \overline{x}_3) \land (x_3 \lor \overline{x}_4) \land \ldots \land (x_{m-1} \lor \overline{x}_m) \land (x_m \lor \overline{x}_1) \) has the truth value 1. In the modified expression \( E^* \), each variable appears in at most three clauses, either in two positive literals and one negative, or in two negative literals and one positive. In other words, each variable \( x \) appears only in a 3-variable clause \( (x \lor \alpha \lor \beta) \) (or it could be \((\overline{x} \lor \alpha \lor \beta)\)), and in a couple of 2-variable clauses \((x \lor \gamma)\) and \((\overline{x} \lor \theta)\). Within the expression, each variable appears only in the conjunction of these clauses \((x \lor \alpha \lor \beta) \land (x \lor \gamma) \land (\overline{x} \lor \theta)\).

We start from a 3SAT expression \( E \), and we perform the conversion to the clustered SAT expression \( E^* \), where each variable appears in at most 3 clauses. For a detailed presentation of the procedure, see proposition 9.3, page 183, in [Papadimitriou, 1994]. We note that if the 3SAT expression \( E \) has \( n \) variables and \( c \) clauses (with each
variable appearing at least twice), then the clustered expression $E^*$ will contain $3c$ variables, and $4c$ clauses, out of which $c$ will be three variable clauses and $3c$ will be two variable clauses.

A cluster associated to a variable $x$, can be of two types.  
Type 1 cluster: 
$$(x \lor y \lor z) \land (x \lor \neg u) \land (\neg x \lor w)$$

Here $y$ and $z$ are two literals (they can be positive or negative), and we write $\neg x$ for the negative literal that represents the negation of $x$.

Type 2 cluster: 
$$(\neg x \lor y \lor z) \land (x \lor u) \land (\neg x \lor w)$$

In the clusters above, we will call $x$ the primary variable associated to the cluster, and the other variables are called secondary (but they are primary variables as seen in their own clusters). In a cluster, the 3 - variable clause, and the 2 - variable clause where the primary variable is under the same literal (both positive or both negative) are called primary clauses. The two variable clause in the cluster in which the primary variable is under an opposite literal (compared to the three variable clause), is called the secondary clause. So a cluster has two primary clauses and a secondary clause.

The most general form of a cluster can be written in the form: 
$$(x \lor a) \land (x \lor b) \land (\neg x \lor c)$$

In this case, $b$, and $c$ can be positive or negative literals, and $a$ is a disjunction of literals, $a = y \lor z$ , $b = \neg u$ , $c = w$ , and $x$ can be a positive or negative literal.  
We construct the flip table of the cluster.

|   | x | \neg x | a | b | c | x \lor a | x \lor b | \neg x \lor c | 0 \rightarrow 1 | 1 \rightarrow 0 |
|---|---|-------|---|---|---|----------|----------|-------------|----------------|----------------|
| 1 | 1 | 0     | 0 | 0 | 0 | 0        | 0        | 1           | +1             |                |
| 2 | 1 | 0     | 0 | 0 | 0 | 1        | 1        | 0           | -1             |                |
| 3 | 0 | 1     | 1 | 0 | 0 | 1        | 0        | 1           |                | 0              |
| 4 | 1 | 0     | 0 | 0 | 0 | 1        | 1        | 0           | 0              |                |
| 5 | 0 | 1     | 0 | 1 | 0 | 0        | 1        | 1           |                | 0              |
| 6 | 1 | 0     | 0 | 1 | 0 | 1        | 1        | 0           | 0              |                |
| 7 | 0 | 1     | 0 | 0 | 1 | 0        | 1        | 0           |                | +2             |
| 8*| 1 | 0     | 0 | 0 | 1 | 1        | 1        | 1           |                |                |
| 9*| 0 | 1     | 1 | 1 | 0 | 1        | 1        | 1           |                |                |
|10 | 1 | 0     | 1 | 1 | 0 | 1        | 1        | 0           | +1             |                |
|11 | 0 | 1     | 1 | 1 | 0 | 1        | 0        | 1           |                |                |
|12*| 1 | 0     | 1 | 1 | 0 | 1        | 1        | 1           |                | +1             |
|13 | 0 | 1     | 0 | 1 | 1 | 0        | 1        | 1           |                |                |
|14*| 1 | 0     | 0 | 1 | 1 | 1        | 1        | 1           |                |                |
|15*| 0 | 1     | 1 | 1 | 1 | 1        | 1        | 1           |                |                |
|16*| 1 | 0     | 1 | 1 | 1 | 1        | 1        | 1           |                |                |
We note that the assignments marked with * satisfy the cluster (all clauses in the cluster are satisfied). The flips $0 \rightarrow 1$ and $1 \rightarrow 0$ refer to the primary variable $x$, when all the other literals do not change (under the current truth assignment). The last two columns represent the make minus break quantity $\Delta = \text{make} - \text{break}$, where “make” represents the number of clauses that change from unsatisfied to satisfied, as a result of the flip, and “break” represents the number of clauses that change from satisfied to unsatisfied, as a result of the flip. The algorithm only touches unsatisfied clauses (or clusters), so the satisfying assignments do not have a $\Delta$ value in the corresponding column (a possible exception is between assignments 15* and 16*, since the cluster is left satisfied and $\Delta = 0$).

**Definitions 1.** We will call a positive flip, a flip that has the value of delta strictly positive $\Delta > 0$. We will call a non - positive flip, a flip that has the value of delta negative or zero $\Delta \leq 0$. We will call a negative flip, a flip that has the value of delta negative $\Delta < 0$. We will call a null flip, a flip that has the value of the delta zero, $\Delta = 0$. We note that these definitions are all related to the expression $E^*$ that we have under consideration. We will also call a truth assignment positive, if it allows at least one positive flip. We will call a truth assignment non positive, if the only flips available in this state, are non - positive flips. We will call a truth assignment null, if it allows only null flips. A given truth assignment can be positive in relation to an expression, but non - positive in relation to another expression. Also, note that the definitions are not symmetrical, in relation to positive or non - positive flips. An important fact is that a non positive state (truth assignment) cannot have only negative flips available, null flips are also available.

In the following, we will describe what the algorithm does at every step. We consider that the clustered expression has $N$ variables. We will call a state, a truth assignment related to these $N$ variables. We will consider a small positive number $\alpha$, (smaller, but close to 1), that will represent the probability (at each step of the algorithm) that the algorithm will decide to look only for the higher value flip (positive flips take priority over null flips, which take priority over negative flips). When in a given state (positive state, that allows positive flips), the algorithm will decide to look for a positive flip with probability $\alpha$. Then, it will choose a positive flip at random (uniformly with equal probability), from all the positive flips available at this moment. With probability $1 - \alpha$, the algorithm will decide to look for non - positive flips, and then it will choose a non - positive flip at random (uniformly with equal probability), from all the non - positive flips available at the moment. When in a given state (non - positive state, that allows only null and negative flips), the algorithm will decide to look for a null flip with probability $\alpha$. Then, it will choose a null flip at random (uniformly with equal probability), from all the null flips available at this moment. With probability $1 - \alpha$, the algorithm will decide to look for negative flips, and then it will choose a negative flip at random (uniformly with equal probability), from all the negative flips available at the moment. We note that in a state that does not allow any positive flips, a null flip is always available. This can be easily proved by considering an expression in which each variable appears in at most three clauses, and considering the flip table of the chain of three clusters (associated to the occurrences of the variable considered).
We will model the evolution of the algorithm by a Markov chain. Each state of this Markov chain will be represented by a truth assignment for the \( N \) variables involved in our expression, so the Markov chain will have \( 2^N \) states.

Before we go forward, we need a few more definitions, and a short reminder of basic results from Markov chain theory.

**Definitions 2.** Two states in a Markov chain are said to be in the same class if we can go from one to another in a finite number of steps with positive probability. A Markov chain is said to be irreducible if all states are in the same class. A recurrent state is one for which the probability that the Markov chain will return to it after a finite time is one. We consider the set \( \{ n \geq 1; p_n(j,j) > 0 \} \), and let \( d \) be the greatest common divisor of the elements in this set. We call \( d \) the period of state \( j \). A chain with period one is said to be aperiodic. All states in the same recurrent class have the same period. A Markov chain is said to be ergodic if it is possible to go from every state to every state (not necessarily in one move). A Markov chain is called regular if some power of the transition probability matrix has only positive elements.

We are now ready to state the ergodic theorem.

**Theorem 1 (ergodic theorem for Markov chains).** We consider an irreducible Markov chain. Then, as the number of states performed by the Markov chain increases indefinitely \( (n \to \infty) \), we have \( p \left( \frac{V_j(n)}{n} \to \frac{1}{m_i} \right) = 1 \), where \( m_i \) is the expected return time to state \( i \), also \( V_j(n) \) represents the number of visits to state \( i \), before step \( n \). This will give us the long run proportion of time that the Markov chain spends in a given state. Moreover, in the positive recurrent case, for any bounded function \( f : I \to \mathbb{R} \), (where \( I \) is the state space of the Markov chain), we have

\[
p \left( \frac{1}{n} \cdot \sum_{k=0}^{n-1} f(X_k) \to \sum_{i \in I} w_i \cdot f(i) \right) = 1 ,
\]

as the number of steps performed by the Markov chain increases indefinitely \( (n \to \infty) \). Here \( w_i, i \in I \), is the unique invariant distribution of the chain (and in general we have \( w_i = \frac{1}{m_i} \)).

**Proof.** For the proof of this theorem, see [Durrett, 2010], or [Feller, 1968].

The invariant distribution satisfying \( w \cdot M = w \), tells us the proportion of time that the Markov chain spends in various states (here \( M \) is the transition probability matrix of the Markov chain). Also \( M^k \) converges quickly to \( W \) (the matrix with each row the invariant vector), in fact, for a finite chain the convergence is exponential. The \( j \)-th component \( w_j \) of the fixed probability row vector \( w \) is the proportion of time the chain is in state \( j \), in the long run. The expected value of the random variable \( V_j(n) \), as \( n \to \infty \), is asymptotic to \( n \cdot w_j \). We are also interested in the limiting distribution of the random variable \( V_j(n) \). The following central limit theorem for Markov chains will prove that the limiting distribution of the random variables \( V_j(n) \) is the normal distribution.
Theorem 2 (central limit theorem for Markov chains). For an ergodic chain, for any real number \( r < s \), we have

\[ p \left( r < \frac{V_j(n) - n \cdot w_j}{\sqrt{n} \cdot \sigma_j^2} < s \right) \rightarrow \frac{1}{\sqrt{2\pi}} \cdot \int_r^s e^{-\frac{x^2}{2}} \, dx \],

for any choice of the starting state (and for \( n \to \infty \)), where \( \sigma_j^2 = 2 \cdot w_j \cdot z_{jj} - w_j - w_j^2 \). Here \( z_{jj} \) is one of the diagonal elements of the matrix \( Z = (I - P + W)^{-1} \), where \( I \) is the identity matrix, \( P \) is the probability transition matrix of the chain, and \( W \) is the invariant distribution matrix, the matrix with all rows the same probability vector, representing the invariant distribution of the chain.

Proof. For the proof of this theorem, see [Grinstead, 1997].

In general, our algorithm will make the following choices. When in a given state (under a given truth assignment), the algorithm will decide to look for a higher value flip with probability \( \alpha \), if the state allows different types of flips. Then, it will choose a higher value flip at random (uniformly with equal probability), from all the higher value flips available at this moment. With probability \( 1 - \alpha \), the algorithm will decide to look for lower value flips, and then it will choose a lower value flip at random (uniformly with equal probability), from all the lower value flips available at the moment.

Observation 1. For the presentation of the algorithm, we will consider \( \alpha = \frac{3}{4} \).

There are probably optimum values for these parameters, but in principle, the results hold for any proper values of these parameters. We also note that working with the clustered expression is essential, since as we will see in the following, we have to find proper bounds for the expectations and variances of the random variables considered.
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We start with a 3SAT expression $E$. We construct the clustered expression $E^*$, where each variable appears in at most 3 clauses, as described above.

Initialization – start with a random truth assignment.

Repeat $9m^2$ times ($m$ is the number of clauses of $E^*$)

1. If the current truth assignment is a solution, return the solution and stop.
2. Analyze the current state, find the unsatisfied clusters and all the types of flips available.
   - Decide at random the type of flip to be performed (higher value flips, with probability $\frac{3}{4}$ or lower value flips with probability $\frac{1}{4}$).
   - Choose one of the clusters at random uniformly, that allow the type of flip chosen (with equal probability). Perform the flip. Update.

Return no solution was found.

We note that at each step, only the variables that have unsatisfied clusters are taken into consideration (only variables that appear in unsatisfied clauses). At each step, the algorithm gives probabilistic priority to higher value flips, thus avoiding negative flips as much as possible (they cannot be completely avoided).

Also note that at every step, the algorithm temporarily flips each variable involved in unsatisfied clauses, in order to identify the positive, null, or negative flips available, after which it commits to a certain flip, according to the probability function described above.

So at every step, the algorithm looks at all the variables that have unsatisfied clusters, and chooses a higher value flip, according to the probability functions described above.

Observation 2. Basically, the algorithm searches and gives priority to higher value flips, whatever the state it is in. The make – break idea is discussed in [Schöning, 2012]. The idea to work with clustered expressions $E^*$ and the application of the central limit theorem for Markov chains, are original.

Proposition 2. The algorithm eventually finds a solution, but the average waiting time for a solution to be found is not necessarily polynomial.

Proof. At each step, we select the variable to be flipped from the subset of variables involved in unsatisfied clauses (or unsatisfied clusters), and each such variable has a
nonzero probability to be selected. Clearly, some of these variables do not have the correct truth assignment. Based on the design of the algorithm, that means that with probability $p > \frac{1}{4n}$ (where $n$ is the number of variables), a variable will be set to the correct truth assignment, at every step. This implies that with probability at least $\frac{1}{(4n)^n}$, the algorithm will find a solution in at most $n$ steps. Note that this lower bound can be greatly improved, but the point of proposition 2 is to prove that the algorithm will find a solution, eventually, it will not be trapped in a subset of states not containing the solution.

**Proposition 3.** Any SAT problem in which each variable appears in at most two places is solved by the algorithm in polynomial time, with high probability.

**Proof.** When we make the conversion to the clustered expression, in this case a cluster will have the general form:

$$(x \lor a) \land (x \lor \bar{b}) \land (\bar{x} \lor b)$$

In this case, $b$ can be a positive or negative literal, and $a$ is a disjunction of literals, $a = y \lor z$, and $x$ can be a positive or negative literal. We construct the flip table of the cluster.

|   | $x$ | $\bar{x}$ | $A$ | $b$ | $x \lor a$ | $x \lor \bar{b}$ | $\bar{x} \lor b$ | $0 \rightarrow 1$ | $1 \rightarrow 0$ |
|---|-----|----------|-----|-----|-------------|-----------------|-----------------|----------------|----------------|
| 1 | 0   | 0        | 0   | 0   | 0           | 1               | 1               | 0              | 0              |
| 2 | 1   | 1        | 0   | 0   | 1           | 1               | 1               | 0              | 0              |
| 3*| 0   | 1        | 1   | 0   | 0           | 1               | 1               | 1              | 0              |
| 4 | 1   | 0        | 1   | 0   | 1           | 1               | 0               | 0              | 0              |
| 5 | 0   | 1        | 0   | 1   | 0           | 0               | 1               | +2             | 0              |
| 6*| 1   | 0        | 0   | 1   | 1           | 1               | 1               | 0              | +1             |
| 15| 0   | 1        | 1   | 1   | 0           | 0               | 1               | +1             | 0              |
| 16*| 1  | 0        | 1   | 1   | 1           | 1               | 1               | 1              | 1              |

We note that there are no negative flips, only positive and null flips. We consider now the birth and death chain on the set of states $\{0, 1, 2, \ldots, m\}$, with transition probabilities $p(i, i + 1) = \frac{3}{4}$, $p(i, i) = \frac{1}{4}$, for $i$ greater than 0, with a reflecting barrier at 0, $p(0, 1) = 1$. Then it can be proved that, starting in any state, the chain will hit the state $m$ in linear time (linear time in $m$), with high probability. We write $S_k$ for the set of truth assignments that put the expression in a state with $k$ satisfied clauses. If we are in a positive state (a truth assignment under which positive flips are possible), then the transition $S_k \rightarrow S_{k+1}$ will occur with probability $\frac{3}{4}$.

Some truth assignments might allow only null flips, but these are transitory states (and the number of satisfied clauses does not decrease), that do not greatly affect the waiting time to hit a solution (we do not prove that here, but it can be done). As a conclusion, the algorithm will hit a solution in at most polynomial time in $m$ (in fact, linear time).
Theorem 3. The clustered Sparrow algorithm is polynomial. Any SAT problem in which each variable appears in at most three places is solved by the algorithm in polynomial time, with high probability.

Sketch of proof. We write $S_+$ for the set of positive states (truth assignments), and $S_-$ for the set of non-positive states of the Markov chain. Here $w_i, i \in I$, is the unique invariant distribution of the chain (for the chain with $2^N$ states described above), and $p_{ij}$ are the transition probabilities. Note that we assume that our chain is ergodic. In fact, we can modify the algorithm (only for this theoretical argument), such that in each state, the algorithm will choose a completely random state to jump to (with a very small probability that does not affect any other arguments), making the chain regular, thus ergodic. The assumption of ergodicity does not restrict the general character of the following arguments.

The average number of transitions the chain makes from a set $A$ to a set $B$, per unit time, conditioned by the current state being in $A$, is:

$$\lambda(A, B) = \frac{1}{\sum_{i \in A} w_i} \cdot \sum_{i \in A} (w_i \cdot \sum_{j \in B} p_{ij}).$$

This expression represents the mean number of transitions the chain makes from $A$ to $B$, per unit time, when the chain is in equilibrium, or the rate at which the chain moves from $A$ to $B$, or the probability of a transition from $A$ to $B$, conditioned by the current state being in $A$.

We will construct a Markov chain with only two states, $S_+$ (the set of positive states), and $S_-$ (the set of negative states). For a positive state at least one positive flip is available, but there could be many. For a non-positive state only null and negative flips are available. We construct the transition probability matrix for this chain. We can consider the transition probabilities as follows (even if the conclusions do not depend on their actual values):

$$p(S_+, S_+) = \frac{1}{\sum_{i \in S_+} w_i} \cdot \sum_{i \in S_+} (w_i \cdot \sum_{j \in S_+} p_{ij})$$

$$p(S_+, S_-) = \frac{1}{\sum_{i \in S_+} w_i} \cdot \sum_{i \in S_+} (w_i \cdot \sum_{j \in S_-} p_{ij})$$

$$p(S_-, S_+) = \frac{1}{\sum_{i \in S_-} w_i} \cdot \sum_{i \in S_-} (w_i \cdot \sum_{j \in S_+} p_{ij})$$

$$p(S_-, S_-) = \frac{1}{\sum_{i \in S_-} w_i} \cdot \sum_{i \in S_-} (w_i \cdot \sum_{j \in S_-} p_{ij})$$

Asymptotically, the behavior of our Markov chain (associated to the algorithm) on the sets $S_+$ and $S_-$ will be the same as the evolution of this two state Markov chain, described above.
We also consider the expected change in the number of satisfied clauses, for each step performed by the algorithm. For a positive state, and $\alpha = \frac{3}{4}$, this expected change in the number of satisfied clauses is greater than $\alpha - (1 - \alpha) = 2 \cdot \alpha - 1 = \frac{1}{2}$. It can be proved that in a non-positive state, there are always null flips available (as noted before). For such a non-positive state, the expected change in the number of satisfied clauses is at least $-(1 - \alpha) = -\frac{1}{4}$, and at most 0 (if only null flips are available).

In general, the two state chain described above will have the transition matrix

$$P = \begin{pmatrix} 1 - a & a \\ b & 1 - b \end{pmatrix},$$ where $a$ and $b$ are not zero, and are less than 1.

We have the defining relations:

$$1 - a = p(S_+, S_+) = \frac{1}{\sum_{i \in S_+} w_i} \cdot \sum_{i \in S_+} (w_i \cdot \sum_{j \in S_+} p_{ij})$$

$$a = p(S_+, S_-) = \frac{1}{\sum_{i \in S_+} w_i} \cdot \sum_{i \in S_+} (w_i \cdot \sum_{j \in S_-} p_{ij})$$

$$b = p(S_-, S_+) = \frac{1}{\sum_{i \in S_-} w_i} \cdot \sum_{i \in S_-} (w_i \cdot \sum_{j \in S_+} p_{ij})$$

$$1 - b = p(S_-, S_-) = \frac{1}{\sum_{i \in S_-} w_i} \cdot \sum_{i \in S_-} (w_i \cdot \sum_{j \in S_-} p_{ij})$$

It can be proved that for this two state chain, the invariant probability vector (the invariant distribution) $w$ will have the form:

$$u = \left( \frac{b}{a + b}, \frac{a}{a + b} \right) \quad (1)$$

The invariant matrix $W$ (the matrix with each row the invariant probability vector) is then:

$$W = \frac{1}{a + b} \cdot \begin{pmatrix} b & a \\ b & a \end{pmatrix} \quad (2)$$

More generally, we consider a Markov chain with transition matrix $P$, fixed probability $u$ (this is the invariant distribution, as above), and a payoff function $f$, which assigns to each state $i$, and amount $f_i$, which can be positive or negative. As the number of steps performed by the chain increases, every time you are in state $i$, you receive an amount $f_i$. We want to find the expected winnings (per unit step), when the number of steps performed increases indefinitely.

It can be rigorously proved that the expected winnings per unit step (when $n$ increases indefinitely) is $E(f) = \sum_i u_i \cdot f_i$ (here the sum is performed after all states $i$).
In our case, the quantity \( f_i \) represents the expected change in the number of satisfied clauses, when the chain is in state \( i \), and a flip is performed. We will write \( f_1 \) for the expected gain in the number of satisfied clauses, when the chain is in a positive state, and we will write \( f_2 \) for the expected gain in the number of satisfied clauses, when the chain is in a non–positive state.

We proved above that \( 1 \geq f_1 \geq \frac{1}{4} \), and \( f_2 = -\frac{1}{4} \).

We also want to calculate the variance that appears in the central limit theorem. The fundamental matrix can be easily calculated, and we have:

\[
Z = (I - P + W)^{-1} = \frac{-1}{(a+b)^2} \begin{pmatrix} b^2 + ab - a & a^2 + ab + a \\ b^2 + ab + b & a^2 + ab - b \end{pmatrix}
\] (3)

The asymptotic variances that appear in the central limit theorem above are then (see Grinstead, 1997):

\[
\sigma_j^2 = 2 \cdot u_j \cdot z_{jj} - u_j - u_j^2 , \text{ where } j = 1,2
\] (4)

From (1), (2), (3), and (4), when we can calculate the variances involved in the central limit theorem for Markov chains.

\[
\sigma_1^2 = \frac{b}{(a+b)^3} (4b^2 - 5ab + 2a - a^2)
\]

\[
\sigma_2^2 = \frac{a}{(a+b)^3} (4a^2 - 5ab + 2b - b^2)
\] (5)

We know that \( u_1 = \frac{b}{a+b} \), \( u_2 = \frac{a}{a+b} \). We make a distinction here between the invariant distribution of the chain with \( 2^N \) states and this invariant distribution of the 2 state chain that we constructed. We also note that every time when a negative flip is performed in a non positive state (and that happens with probability \( \frac{1}{4} \)), the new state is a positive state, because the reverse flip is positive. But if we are in a positive state, we could have more than one positive flips available. So when in a positive state, we can perform a positive flip (with probability \( \frac{3}{4} \) and still remain in a positive flip. So we have \( b = \frac{1}{4} \) and \( a \leq \frac{3}{4} \). So when we are in a positive state, the expected change in the number of satisfied clauses satisfies the relation :

\[
E(S_+) \geq \frac{3}{4} \cdot \frac{1}{4} = \frac{1}{2}
\]

When we are in a non positive state, the expected change in the number of satisfied clauses satisfies the relation :

\[
E(S_-) = -\frac{1}{4}
\]
The expected change in the number of satisfied clauses, when a step is performed by
the Markov chain is then:

\[ E \geq u_1 \cdot \frac{1}{2} - u_2 \cdot \frac{1}{4} = \frac{b}{a+b} \cdot \frac{1}{2} - \frac{a}{a+b} \cdot \frac{1}{4} = \frac{1}{4(a+b)} (2b - a) \]  

(6)

When the algorithm performs \( m^2 \) steps (where \( m \) is the number of clauses in the
clustered expression), either the expected change in the number of satisfied clauses, or
the standard deviation is of order of magnitude \( m \) (these are the expectation and
variances after \( m^2 \) steps ). From relations (5) and (6) we can see that we cannot have
both the mean and the variance taking negligible values in the central limit theorem
for Markov chains. That means that after a number of steps of order \( m^2 \) the Markov
chain will find a solution with high probability.

**Observation 3.** This phenomenon (polynomial time until a solution was found) has
been empirically observed (through experiment) by Schoning (see [Schöning, 2012] ),
for a different (but related) class of algorithms (I emphasized the differences above),
but no theoretical justification has been given. In other words, there is empirical
evidence that this type of algorithms are extremely efficient.

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