The GS String Action on $AdS_5 \times S^5$

Renata Kallosh$^a$ and J. Rahmfeld$^b$

Department of Physics, Stanford University, Stanford, CA 94305-4060

ABSTRACT

We present a simple form of the Type IIB string action on $AdS_5 \times S^5$. The result is achieved by fixing $\kappa$-symmetry in the Killing spinor gauge defined by the projector of the Killing spinor of the D3 brane. We show explicitly that in this gauge the superspace is greatly simplified which is the crucial ingredient for the simple string action.

$^a$ e-mail: kallosh@physics.stanford.edu.
$^b$ e-mail: rahmfeld@leland.stanford.edu.
Recently, research in worldvolume theories of strings and extended objects on $AdS_5 \times S^5$ backgrounds has attracted attention, motivated by the conjectured duality between $D = 4$ Large $N$ Yang-Mills theory and Type IIB string theory compactified on $AdS_5 \times S^5$ [1, 2, 3]. Clearly, it is of great importance to obtain the string action in this background in a reasonably simple form, which is the purpose of this note.

In flat superspace, the string action with manifest ten dimensional super Poincare symmetry was discovered by Green and Schwarz [4]. The classical action is non-linear. Upon gauge-fixing in the light-cone gauge the gauge-fixed theory is free as the action is quadratic. The GS string action was generalized in [5] to generic Type IIB backgrounds. In such backgrounds in which even after gauge-fixing the action is not free.

More recently, the classical string action was presented in a closed form in $AdS_5 \times S^5$ supercoset construction [6, 7, 8]:

$$S = -\frac{1}{2} \int d^2 \sigma \left[ \sqrt{g} g^{ij} L_i \dot{L}_j + 4i \epsilon^{ij} \int_0^1 ds L_i \dot{S}^{IJ} \Theta^I \Gamma^a L^J \right],$$

where $S^{IJ}$ has non-vanishing elements $S^{11} = -S^{22} = 1$ and where $\dot{a} = 0, \ldots, 9$.

$$L^I_s = \left[ \left( \frac{\sinh (s M)}{M} \right) D \Theta \right]^I$$

and

$$L^\dot{a} = \epsilon_{\dot{m}}^\dot{a}(x) dx^{\dot{m}} - 4i \Theta^I \gamma^\dot{a} \left( \frac{\sinh^2 (s M/2)}{M^2} D \Theta \right)^I,$$

where

$$(M^2)^{IL} = \left[ \epsilon^{IJ} (-\gamma^a \Theta^I \Theta^L \gamma^a + \gamma^{a'} \Theta^I \Theta^L \gamma^{a'}) + \frac{1}{2} \epsilon^{KL} (\gamma^{ab} \Theta^I \Theta^K \gamma^{ab} - \gamma^{a'b'} \Theta^I \Theta^K \gamma^{a'b'}) \right],$$

and $L^I = L^I_{s=1}, \quad L^{\dot{a}} = L^{\dot{a}}_{s=1}$. Here,

$$(D \Theta)^I = \left( d + \frac{1}{4} (\omega^{ab} \gamma_{ab} + \omega^{a'b'} \gamma_{a'b'}) \right) \Theta^I - \frac{1}{2} i \epsilon^{IJ} (\epsilon^a \gamma_a + i \epsilon^{a'} \gamma_{a'}) \Theta^J$$

with $d = dx \partial_x + d \Theta \partial_\Theta$. Also, we use a 5+5 split [8] of $\dot{a}$ into $\dot{a} = (a, a')$. This presents a closed form of a $\kappa$- and reparametrization symmetric string action in $AdS_5 \times S^5$.
background. Although closed, the classical action as it is depends on even powers of \( \Theta \) up to \( \Theta^{32} \).

Since both symmetries are local, they have to be gauge-fixed. As shown in [9] there exists a gauge which utilizes the Killing spinors of the background which leads to a significant simplification of the action. The argument was based on the properties of a supersolvable subalgebra of the full superconformal algebra. The connection between gauge fixing \( \kappa \)-symmetry and supersolvable subalgebras was discovered in [10] in the context of the \( M2 \)-brane.

The program of gauge-fixing consists of these steps:

- The choice of an algebraic projector which eliminates 1/2 of the 32 spinorial degrees of freedom of the classical action. This projector will be suggested by the full \( D3 \)-brane Killing spinor. The surviving fermionic directions will be precisely those Killing directions of the geometry.

- A change of the fermionic coordinates to accommodate the space-time dependence of the Killing spinors.

- Ensuring the consistency of the gauge.

The split of the 32 component spinor of the classical string action is suggested by the Killing spinor of the full \( D3 \)-brane background which preserves 1/2 of the supersymmetries. Define the projectors

\[
\mathcal{P}_\pm = \frac{1}{2} \left( \delta^{IJ} \pm \Gamma_{0123} \epsilon^{IJ} \right),
\]

where \( \Gamma_{0123} \) is the product of \( \Gamma \) matrices in the direction of the brane. The gauge is defined by

\[
\mathcal{P}_- \Theta = 0.
\]

For future convenience we introduce besides the 5+5 split \( \hat{a} = (a, a') \), \( \hat{m} = (m, m') \) also the 4+6 split \( \hat{a} = (\bar{p}, \bar{t}) \), \( \hat{m} = (p, t) \) with \( p, \bar{p} = 0, 1, 2, 3 \) and \( t, \bar{t} = 4, \ldots, 9 \). This reflects the separation of the space-time coordinates into directions along the \( D3 \)-brane and transverse to it. With

\[
\Theta_\pm = \mathcal{P}_\pm \Theta
\]
we can define a convenient basis
\[\Theta^1_\pm \equiv (\Theta_\pm)^1 \equiv \frac{1}{2} (\Theta^1 \pm \Gamma_{0123}\Theta^2)\]
\[\Theta^2_\pm \equiv (\Theta_\pm)^2 \equiv \frac{1}{2} (\Theta^2 \mp \Gamma_{0123}\Theta^1) = \mp \Gamma_{0123}\Theta^{1}_\pm.\] (9)

In this basis the gauge reads
\[\Theta^1_\pm = \Theta^2_\pm = 0.\] (10)

A tremendous simplification of the supervielbeins occurs if we apply this gauge to (2) and (3). Notice that in these objects all fermionic contributions appear in the form
\[(M^{2n})D\Theta.\] (11)

We will show below that
\[(M^2)D\Theta_+ = 0\] (12)
which means that all terms of the type (11) vanish for \(n > 0\). For simplicity, we prove this in the 5+5 split.

First, rewrite \(\Theta_\pm = 0\) as
\[\Theta^f_+ = i\varepsilon^{IJ}\gamma^4\Theta^J_+\] (13)
where we used
\[\Gamma_{0123} = i\gamma^4 \times 1_{2 \times 2}.\] (14)

In general, the relation between ten dimensional Dirac matrices \(\Gamma\) and 5+5 dimensional ones \(\gamma^a\) and \(\Gamma^{a'}\) are \(\Gamma^a = \gamma^a \times \sigma_1,\) \(\Gamma^{a'} = \gamma^{a'} \times \sigma_2,\) (15)
where \(\gamma^a\) and \(\gamma^{a'}\) obey the (anti-)commutation relations
\[\{\gamma^a, \gamma^b\} = 2\eta^{ab},\] \[\{\gamma^{a'}, \gamma^{b'}\} = 2\delta^{a'b'},\] \([\gamma^a, \gamma^{b'}] = 0.\] (16)

With (13) we arrive at
\[(M^{2L})^{I^L}_K D\Theta^L_+ = \left[ (i\gamma^a\gamma^4\Theta^L_+\bar{\Theta}^L_+\gamma^a D\Theta^L_+ - i\gamma^{a'}\gamma^4\Theta^L_+\bar{\Theta}^L_+\gamma^{a'} D\Theta^L_+ ) \right.\]
\[+ \frac{1}{2} \left. ( -i\gamma^{ab}\Theta^L_+\bar{\Theta}^{K}\gamma^{ab}\gamma^4 D\Theta^K + i\gamma^{a'b'}\Theta^L_+\bar{\Theta}^{K}\gamma^{a'b'}\gamma^4 D\Theta^K ) \right].\] (17)
To show that this expression vanishes we use the fact that all terms of the structure
\[ \bar{\Theta}_I^L \hat{\Gamma} D\Theta^L \] with \[ [\gamma^4, \hat{\Gamma}] = 0 \] (18)
vanish. This leads to
\[ (M_{\text{fix}}^2)^{IL} D\Theta^L_+ = [(i\gamma^p \gamma^4 \Theta^L_+ \Theta^L_+ \gamma^p D\Theta^L_+) + \frac{1}{2}(-i\gamma^4 \Theta^L_+ \Theta^L_+ \gamma^p \gamma^4 D\Theta^L_+)] = 0, \] (19)
as can be easily verified. We are left after gauge-fixing with the following supervielbeins:
\[
\begin{align*}
(L^I_s)_+ &= s D\Theta^I_+ \\
(L^I_s)_- &= 0 \\
L^p_s &= e^p_m(x) dx^m - is^2 \bar{\Theta}^I_+ \gamma^p D\Theta^I_+ \\
L^t_s &= e^t_m(x) dx^m,
\end{align*}
\] (20)
where the indices \(p\) and \(t\) refer to directions parallel and orthogonal to the \(D3\)-brane.

We now turn to the change of fermionic variables. At this point it is useful to remind ourselves of the explicit \(AdS_5 \times S^5\) metric. Since in the following we use both notations, \(5 + 5\) and \(6 + 4\) split, we give it in spherical/AdS coordinates as well as in Cartesian coordinates:

spherical/AdS:
\[
ds^2 = r^2(dx^p dx_p) + \frac{dr^2}{r^2} + d\Omega^2
\] (21)
Cartesian:
\[
ds^2 = y^2(dx^p dx_p) + \frac{1}{y^2}(dy^i dy_i).
\] (22)

To simplify the expression \(D\Theta_+\) the following change of variables (in Cartesian coordinates) is in order\(^1\):
\[
\Theta^I_+ = g^\pm(|y|) \theta^I_+ \] (24)
\[\text{In spherical coordinates the suitable change of variables is}\]
\[
\Theta^I_+ = f(\eta) g^\pm(\nu) \theta^I_+,
\] (23)
where \(f(\eta)\) is a function of the 5 Euler angles of the 5-sphere\(^1\).
With this definition $D\Theta^1_+$ simply becomes

$$D\Theta^1_+ = (g(|y|))^{1/2}d\theta_+ = \sqrt{|y|}d\theta_+. \quad (25)$$

This can be seen by enforcing the constraint (7) on (5) which reduces to solving the Killing spinor equation for the full D3-brane Killing spinor in the near horizon regime.

In the coordinates (24) the supervielbeins take the form (at $s = 1$)

$$L^I_+ = \sqrt{|y|}d\theta^I_+$$
$$L^I_- = 0$$
$$L^p = |y|(dx^p - i\tilde{\theta}^I_+\Gamma^p d\theta^I_+)$$
$$L^t = \frac{1}{|y|}dy^t \quad (26)$$

This allows us to reduce the complicated classical action to a much simpler gauge-fixed action. With the obvious replacements $dZ^M \to \partial_i Z^M d\sigma$ we find

$$S = -\frac{1}{2} \int d^2\sigma \left[ \sqrt{g} g^{ij} \left( y^2(\partial_i x^p - 2i\tilde{\theta}^I_+\Gamma^p \partial_i \theta_+)(\partial_j x^p - 2i\tilde{\theta}^I_+\Gamma^p \partial_j \theta_+) + \frac{1}{y^2} \partial_i y^t \partial_j y^t \right) 
+ 4i\epsilon^{ij} \partial_i y^t (\tilde{\theta}^I_+\Gamma^t \partial_j \theta_+) \right] \quad (28)$$

which constitutes our main result\footnote{If we would work instead in spherical coordinates, we would change the $y^2$ to $r^2$, modify the second term to $\frac{1}{r^2} \partial_i r \partial_i r + \ldots$ (where the dots denote derivatives on angular coordinates), and include the angular dependence of the spinors as in (23).}. Here, for compactness $\theta_+$ denotes $\theta^I_+$. The last term originates from the WZ term.

The action still remains reparametrization invariant. There are various possibilities to fix this symmetry which have to be studied of the $AdS_5 \times S^5$ background.

Finally, one has to ensure that the $\kappa$-symmetry gauge (7) is an acceptable one, which requires the differential operator in the quadratic part of the fermions to be invertible. The relevant term in the action is of the form

$$\mathcal{L} \sim \tilde{\theta}_+ \left[ (\Pi_p \Gamma^p + \Pi_t \Gamma^t) \partial_z + (\Pi_p \Gamma^p - \Pi_t \Gamma^t) \partial_\bar{z} \right] \theta_+. \quad (29)$$
The zero modes of the classical action in this notation are
\[(\Pi_p \Gamma^p + \Pi_\ell \Gamma^\ell)_z\quad \text{and} \quad (\Pi_p \Gamma^p + \Pi_\ell \Gamma^\ell)_{\bar{z}}.\] (30)

None of them remains a zero mode in our gauge fixed action, hence the gauge we have used is acceptable.

In summary, we have shown, following [9], that \(\kappa\)-symmetry of the Green-Schwarz string can be used to remove fermionic degrees of freedom in a way that simplifies the \(AdS_5 \times S^5\) superspace geometry and therefore the action.

We had stimulating discussion with Arvind Rajaraman and Arkady Tseytlin. The work of R.K and J.R is supported by the NSF grant PHY-9219345.

**Note Added:**
After completion of this work we became aware of the paper by I. Pesando [12] which displays the string action with a simple fermionic sector. The action was obtained via the technique of supersolvable algebras, and the relation to gauge fixing procedure still needs to be fully understood.

**References**

[1] J. Maldacena, *The large n limit of superconformal field theories and supergravity*, \texttt{hep-th/9711200} (1997).

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, Phys. Lett. B428 (1998) 105.

[3] E. Witten, *Anti-de sitter space and holography*, \texttt{hep-th/9802150} (1998).
[4] M. B. Green and J. H. Schwarz, *Covariant description of superstrings*, Phys. Lett. 136B (1984) 367.

[5] M. T. Grisaru, P. Howe, L. Mezincescu, B. Nilsson and P. K. Townsend, *N=2 superstrings in a supergravity background*, Phys. Lett. 162B (1985) 116.

[6] R. R. Metsaev and A. A. Tseytlin, *Type iib superstring action in ads(5) x s**5 background*, hep-th/9805028 (1998).

[7] R. Kallosh, J. Rahmfeld and A. Rajaraman, *Near horizon superspace*, hep-th/9805217 (1998).

[8] R. R. Metsaev and A. A. Tseytlin, *Supersymmetric d3-brane action in ads(5) x s(5)*, hep-th/9806095 (1998).

[9] R. Kallosh, *Superconformal actions in killing gauge*, hep-th/9807206 (1998).

[10] G. Dall’Agata et al., *The osp(8|4) singleton action from the supermembrane*, hep-th/9807115 (1998).

[11] H. Lu, C. N. Pope and J. Rahmfeld, *A construction of killing spinors on s(n)*, hep-th/9805151 (1998).

[12] I. Pesando, *A kappa gauge fixed type iib superstring action on ads5 x s5*, hep-th/9808020 (1998).