Fourth-post-Newtonian-exact approximation to General Relativity

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An approximation to General Relativity is presented that agrees with the Einstein field equations up to and including the fourth post-Newtonian (PN) order. This approximation is formulated in a fully constrained scheme: all involved equations are explicitly elliptic except the wave equation that describes the two independent degrees of freedom of the gravitational field. The formalism covers naturally the conformal-flat-condition (CFC) approach by Isenberg, Wilson, and Mathews and the improved second PN-order exact approach CFC+. For stationary configurations, like Kerr black holes, agreement with General Relativity is achieved even through 5PN order. In addition, a particularly interesting 2PN-exact waveless approximation is analyzed in detail, which results from imposing more restrictive conditions. The proposed scheme can be considered as a further development on the waveless approach suggested by Schäfer and Gopakumar [Phys. Rev. D 69, 021501 (2004)].

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I. INTRODUCTION

The solution of the Einstein field equations for inspiraling compact binaries is one of the great challenges of Numerical Relativity, e.g., [1]. To fully succeed with this problem realistic initial data are needed which do not include incoming (spurious, “junk”) gravitational waves. If one could follow up the evolution from the very beginning of the inspiraling process, i.e., right from the infinite separation of the both objects and without gravitational waves present at this initial stage, the configuration to start with at any finite instant of time would be well under control just by calculating the history up to that instant of time. Lack of this piece of information requires the construction of initial value data by other means.

On the other hand, there are several astrophysically relevant scenarios, like supernovae explosions or initial stages of a binary black hole, which can be described very accurately with approximated versions of Einstein equations. These two issues (initial data generation and construction of truncated schemes of General Relativity) are deeply connected since by extending to all times the conditions imposed on the freely specifiable part of the initial data set, one can obtain an approximated version of Einstein equations.

As is well known, in the initial slice, we are free to choose four gravitational degrees of freedom from the initial data set (the two independent field degrees of freedom and their time derivatives or, their canonically conjugate momenta), whereas the remainder gravitational objects must be obtained through the resolution of the constraint equations and coordinate conditions. Resorting to a canonical formulation of General Relativity, e.g., the Arnowitt-Deser-Misner (ADM) one [2] – to be used throughout in this paper –, these independent gravitational degrees of freedom get clearly identified. The four degrees of freedom are encoded in the two independent metric-field components \( h_{ij}^{TT} \) and their canonical conjugate \( \pi^i_j \), where TT means transverse and traceless with respect to an auxiliary 3-dimensional flat metric \( \delta_{ij} \) with \( i, j = 1, 2, 3 \). Therefore, the initial value problem reduces to the fixation of the spatial dependence of these two objects at a finite initial time.

In the Isenberg-Wilson-Mathews conformal-flat condition (CFC) approach to General Relativity [3, 4], \( h_{ij}^{TT} \) and its time derivative \( h_{ij,0}^{TT} \) are chosen to be vanishing. Hence, this scheme is very appropriate to construct conformally flat initial data. On the other side, understood as an approximation to General Relativity so that the mentioned conditions \( (h_{ij}^{TT} = h_{ij,0}^{TT} = 0) \) are valid for all times, this formalism reproduces the evolution of the gravitating system at 1PN order, which implies deviations from General Relativity at 2PN.

Even though, near-zone post-Newtonian (PN) calculations show that for inspiraling binaries with no-incoming radiation, \( h_{ij}^{TT} \) can never be exactly zero, see, e.g., [5–9], but contain conservative and dissipative terms which destroy CFC. The first piece of deviation from CFC appears at 2PN order and is of conservative type. The higher-order pieces appearing in the cited papers are 2.5PN (dissipative), 3PN (conservative), and 3.5PN (dissipative). Therefore, many

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attempts have been made to construct conformally nonflat schemes, see for instance \cite{10, 11}. In particular, regarding works done based on PN corrections, the so-called CFC+ approach, as detailed in \cite{12}, generalizes the CFC framework by just correcting for the additional 2PN piece. In the scheme \cite{13} also 2.5PN terms have been considered and were recently further developed in \cite{14, 15}. The 2.5PN and 3.5PN pieces have been added within the CFC-type skeleton approach \cite{16} in \cite{17}. However, in this last reference the radiation reaction has been taken into account even through 6PN order. Herein, the 4PN, 5PN, and 6PN orders are both conservative and dissipative, where the dissipative parts result from the peculiar tail structure of the radiation process.

The separation in dissipative and conservative terms allows the separate calculation of the dissipative and conservative parts of the metric coefficients. The aim of the waveless approximations to General Relativity beyond CFC is the determination of conservative terms in the gravitational radiation process. Related with waveless approximations are the near-zone helically symmetric schemes introduced by \cite{18}, which also do not allow for emission of waves and have been intensively studied in the recent past, e.g., \cite{19–21}. The fully constraint formulation \cite{22} has found deeper investigations and waveless applications in \cite{23, 24}. A comparison between helically symmetric and waveless description of binary systems has been performed in, e.g., \cite{25}.

The article is organized as follows. In Sec. \ref{sec:adm} the Einstein field equations are shown in the context of the ADM formalism. Section \ref{sec:elliptic} presents explicit elliptic equations that are valid in full General Relativity and allows one to solve for all the geometric objects but the transverse and traceless part of the metric and of its conjugate momentum. In Sec. \ref{sec:waveless} the transverse and traceless part of the conjugate momentum is assumed to be vanishing, which leads to a 2PN-exact waveless approximation scheme. Section \ref{sec:4pn} generalizes the previous scheme to a 4PN-exact framework suggesting another 2PN-exact waveless approach. We show that this formalism reduces to the well-known CFC+ approach under the corresponding conditions in Sec. \ref{sec:4pn}. The particular matter model of a perfect fluid is presented in Sec. \ref{sec:fluid}. Finally the conclusions are drawn in Sec. \ref{sec:conclusions}.

II. EINSTEIN FIELD EQUATIONS IN ADM FORMALISM

In the ADM formalism of General Relativity, the spacetime line element is split into the $(3 + 1)$ form,

$$\begin{align*}
\int ds^2 &= -c^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),
\end{align*}$$

where $\alpha$ is the lapse function, $\beta^i$ the shift vector, $\gamma_{ij}$ the induced metric on a three-dimensional spatial slice $\Sigma(t)$, parameterized by the time coordinate $t$, and $c$ is the speed of light. The three-metric $\gamma_{ij}$ and its canonical conjugate $\pi_{ij}$, which is a contravariant symmetric tensor density of weight +1, satisfy the Hamiltonian and momentum constraints \cite{2},

$$\begin{align*}
\gamma^{1/2} R &= \frac{1}{2\gamma^{1/2}} (2\pi^{ij} \pi_{ij} - \pi^2) + \kappa \gamma^{1/2} \alpha^2 T^{00},
\end{align*}$$

$$\begin{align*}
(-2\pi_{ij} = 0) - 2 \pi^{i}_{,j} + \pi^{kl} \gamma_{kl,i} &= \kappa \gamma^{1/2} \alpha T^0_i,
\end{align*}$$

where $R$ is the curvature scalar of $\Sigma(t)$, $\gamma$ the determinant of $\gamma_{ij}$, $\pi_{ij} = \gamma_{jk} \pi^{jk}$, and $\pi = \pi^i_i$. On the other hand, $T^{00}$ and $T^0_i$ are the components of the unspecified four-dimensional stress-energy tensor for the matter source $T^{\mu\nu}$. The canonical conjugate $\pi^{ij}$ is related to $\tilde{K}_{ij}$, the extrinsic curvature of $\Sigma(t)$, by $\pi^{ij} = -\gamma^{1/2}(\gamma^{ik}\gamma_{jm} - \gamma^{ij}\gamma_{km}) K_{im},$ where $\gamma^{ij}$ is the inverse metric of $\gamma_{ij}$. In the above equations, a partial derivative is denoted by a comma, whereas
stands for the three-dimensional covariant derivative, and \( \kappa \equiv \frac{\hbar c G}{c^4} \), with \( G \) the Newtonian gravitational constant. The three-metric and its canonical conjugate evolve in accordance with the following evolution Eqs. [2],

\[
\pi^{ij,0} = -\frac{1}{2} \alpha \gamma^{1/2}(2R^{ij} - \gamma^{ij} R) + \frac{1}{4} \alpha \gamma^{-1/2} \gamma^{ij} (2\pi_{kl} \pi_{kl} - \pi^2) - \alpha \gamma^{-1/2} (2\pi^{ik} \pi^j_k - \pi \pi^{ij}) + \gamma^{1/2} (\alpha \pi^{ij} - \gamma^{ij} \alpha |_k k) + (\pi^{ij} \beta^k) |_k - \pi^{ij} \beta^i |_k - \pi^{ik} \beta^j |_k + \frac{\kappa}{2} \alpha \gamma^{1/2} \gamma^{ij} \gamma^{kl} T_{kl},
\]

and

\[
\gamma_{ij,0} = \alpha \gamma^{-1/2} (2\pi_{ij} - \gamma_{ij} \pi) + \beta_{ij} + \beta_{ji},
\]

where \( R_{ij} \) is the Ricci tensor associated with \( \Sigma(t) \). In this paper, we raise and lower indices of three-dimensional objects with \( \gamma^{ij} \) and \( \gamma_{ij} \) respectively.

The ADM coordinate conditions, which generalize the isotropic Schwarzschild metric, read

\[
\pi^{ii} = 0,
\]

where, and from here onwards, repeated covariant or contravariant indices imply the usage of Einstein summation convention, and

\[
\gamma_{ij} = \Psi \delta_{ij} + h_{ij}^{TT},
\]

where \( \Psi \) is a conformal scalar and \( h_{ij}^{TT} \) the transverse and traceless part of the three-metric \( \gamma_{ij} \) with respect to the Euclidean 3-metric \( \delta_{ij} \). By definition, \( h_{ij}^{TT} \) satisfies \( h_{ii}^{TT} = h_{ij}^{TT} = 0 \). The conformally-flat condition \( h_{ij}^{TT} = 0 \) gives a simple expression for the three-dimensional curvature scalar, \( \gamma^{1/2} R = -8 \Psi^{1/4} \Delta \Psi^{1/4} \), where \( \Delta \) stands for the three-dimensional Euclidean Laplacian. The gauge fixing equation for \( \gamma_{ij} \) can be rewritten in a differential way,

\[
3\gamma_{ij,j} - \gamma_{jj,i} = 0.
\]

Equation (6) results in the covariant trace of \( \pi^{ij} \) of the form \( \pi = \pi^{ij} h_{ij}^{TT} \). Taking into account the space-asymptotic properties \( \pi^{ij} \sim 1/r^2 \) and \( h_{ij}^{TT} \sim 1/r \), the gauge condition (6) turns out to mean asymptotic maximal slicing \((K \sim 1/r^3)\). The gauge conditions (6) and (7), or (8), are very close to the well-known Dirac gauge conditions which include maximal slicing. In fact, condition (8) is identical with the corresponding Dirac gauge condition to linear order in \( \gamma_{ij} - \delta_{ij} \).

As will be made explicit in the next section, the function \( \Psi \), with \( \Psi - 1 \sim 1/r \) at asymptotic infinity, and the longitudinal part of the momentum \( \pi^{ij} \) are determined using the Hamiltonian and momentum constraints, given in Eqs. (2) and (3), by elliptic equations. On the other hand, the elliptic equations for the lapse \( \alpha \), with \( \alpha - 1 \sim 1/r \), and the shift \( \beta^i \), with \( \beta^i \sim 1/r \), result from the evolution Eqs. (4) and (5), respectively, after applying the coordinate conditions (6) and (8). Since we want to provide a set of equations that can be numerically solved, we also will take care that all the sources decay at least as \( 1/r^4 \) at asymptotic infinity, avoiding in this way convergence problems.

Finally, it is important to stress that our variables are those dictated by a canonical formulation. This fact clearly makes them form a fully consistent set of independent variables. Of course, other sets of variables may also be chosen, like the one presented in Ref. [22] or the Kol-Smolkin one [28], which is connected with the Landau-Lifshitz decomposition of the metric [29] and has extensively been used in, e.g., [30]. Even though, it should be pointed out that only within a radiation-type (“Coulomb-type”) set of coordinate conditions those variables fulfill a mixed set of elliptic and evolutionary (hyperbolic) field equations instead, e.g., a purely hyperbolic system.

### III. EXPLICIT ELLIPTIC EQUATIONS

In this section we will present explicit elliptic equations to solve for all the objects but the transverse and traceless parts of the three-metric \( \gamma_{ij} \) and of its corresponding momentum \( \pi^{ij} \). No assumptions will be made, hence these equations will be valid in full General Relativity.

#### A. Equation for the shift

Taking the time derivative of condition (8) and making use of Eq. (5), we obtain an elliptic equation for the shift,

\[
(\beta_{ij})_j + (\beta_{ji})_j - \frac{2}{3} (\beta_{ij})_i = -(\alpha \gamma^{-1/2} (2\pi_{ij} - \gamma_{ij} \pi))_j + \frac{4}{3} (\alpha \gamma^{-1/2} (2\pi_{ij} - \gamma_{ij} \pi))_i.
\]
It can be more explicitly written down in the following way,
\[
\Delta \beta_i + \frac{1}{3} \beta_{j,j,i} = -\frac{2}{3} (\gamma^j \beta_j \Gamma_{kli},i) + 2(\gamma^k \beta_j \Gamma_{lji},j) + \frac{1}{3} (\alpha \gamma^{-1/2} \Psi_{i,j}) + \frac{1}{3} (\alpha \gamma^{-1/2} (2\pi_{ij} - \gamma_{ij} \pi))
\]
\[
- 2\pi_{ij}(\alpha \gamma^{-1/2}),j - \frac{2\alpha}{\gamma^{1/2}} \left[ \pi^j \Psi_{i,j} + h^j_{ki} \pi^j_{i,k} + \frac{\Psi}{2} \left( \pi^{kl} h_{ki}^{TT} - \kappa \gamma^{1/2} \alpha T^0_{ik} \right) \right],
\]
where we have applied the momentum constraint (3) to replace the term \(\pi_{ij,j}\) in order to guarantee an adequate fall-off behavior (~\(1/r^4\)) at asymptotic infinity of the right-hand side of this equation. Here one should use the usual definition for Christoffel symbols,
\[
\Gamma_{ijk} = \frac{1}{2} (\gamma_{ij,k} + \gamma_{ki,j} - \gamma_{jk,i}),
\]
and the exact relations (6),
\[
\gamma^{ij} = \chi_{ij} + H_{ij},
\]
\[
\gamma = \Psi^{\frac{3}{2}} - \frac{1}{2} \Psi h_{ij}^{TT} h_{ij}^{TT} + \frac{1}{3} h_{ij}^{TT} h_{jk}^{TT} h_{ki}^{TT},
\]
being \(\chi \equiv \Psi^2 - \frac{1}{2} h_{ij}^{TT} h_{ij}^{TT}\) and \(H_{ij} \equiv h_{ik}^{TT} h_{jk}^{TT} - \Psi h_{ij}^{TT}\), for the inverse and the determinant of the metric, respectively.

B. Equation for the lapse

Combining relations (4) and (6) it is straightforward to obtain an elliptic equation for \(\alpha\),
\[
\gamma^{ij} \alpha_{ij} - \alpha^{ij} = -\alpha R^{ij} + \left( \gamma^{mm} \delta_{jk} - 2\gamma^{ik} \right) \alpha \gamma^{-1} \pi_{ij} \pi_{jk} - \frac{\alpha}{2\gamma} \gamma^{ij} \pi^2 - 2\gamma^{-1/2} \pi^{im} \beta_m
\]
\[
\quad + \frac{\kappa}{2} \alpha \left( \gamma^{il} \gamma^{im} T_{lm} + \gamma^{ii} \alpha^{2} T^{00} \right),
\]
where we have made use of the Hamiltonian constraint (2) to get rid of the Ricci scalar. Note that the last term of the first line is corrected from Ref. [26]. The left-hand side of this equation can be given as,
\[
(\gamma^{jm} \gamma^{ij} - \gamma^{ji} \gamma^{mi}) (\alpha_{jm} - \gamma^{kl} \Gamma_{ijm} \alpha_{k}),
\]
where, the inverse metric combination can be rewritten making use of the formulas presented in the previous section,
\[
(\gamma^{jm} \gamma^{ij} - \gamma^{ji} \gamma^{mi}) = \frac{2\chi \Psi^2}{\gamma^2} \delta_{jm} + \frac{1}{\gamma^2} \left( \chi + H_{ii} \right) H_{jm} - H_{ij} H_{im}.
\]
Therefore, the equation for the lapse takes the explicit form,
\[
\Delta \alpha = -\frac{1}{2\chi \Psi^2} \left( \chi + H_{ii} \right) H_{jm} - H_{ij} H_{im} \left( \alpha_{jm} - \gamma^{kl} \Gamma_{ijm} \alpha_{k} \right) + \gamma^{kl} \Gamma_{ij}\alpha_{k}
\]
\[
\quad + \frac{\kappa}{2} \alpha \left( \gamma^{il} \gamma^{im} T_{lm} + \gamma^{ii} \alpha^{2} T^{00} \right).
\]
The only term that is left to be known in this equation in terms of metric components is the noncovariant trace of the Ricci tensor \(R^{ij}\). This tensor can be written in the following way,
\[
2\gamma R_{ij} = -\chi \left[ \Delta h_{ij}^{TT} + \delta_{ij} \Delta \Psi + \Psi_{ij} \right] - H_{kl} (\gamma_{kl,ij} + \gamma_{ij,kl} - \gamma_{jk,il} - \gamma_{il,kj})
\]
\[
+ 2\gamma^{kl} \gamma^{np} (\Gamma_{nil} \Gamma_{pkj} - \Gamma_{ni} \Gamma_{pkj}),
\]

(18)
and its trace is then given by,

$$R^{ii} = -\frac{2\chi^3}{\gamma^3} \Delta \Psi - \chi \left[ \frac{2\chi H_{ij} + H_{im} H_{jm}}{2\gamma^3} (\Delta h_{ij}^{TT} + \delta_{ij} \Delta \Psi + \Psi_{,ij}) \right]$$

$$- \frac{1}{2\gamma} \gamma^k \gamma^l \gamma^m \gamma^p \gamma^q (\Gamma_{ni} \Gamma_{pq} - \Gamma_{npi} \Gamma_{qij}).$$

(19)

In this expression one should replace the Laplacian of the conformal factor that appears in the first term of the right-hand side with its corresponding Eq. (21), which will be presented in the next subsection, in order to enforce a decay rate of $\sim 1/r^4$ of the right-hand side of Eq. (17).

C. Equation for the conformal factor

We will use the Hamiltonian constraint to solve for the conformal factor $\Psi$. The Ricci scalar can be written in the following way,

$$R = (\gamma^k \gamma^l \gamma^m \gamma^p \gamma^q \gamma^r n_{ij} n_{kl} + \gamma^k \gamma^l \gamma^m \gamma^p \gamma^q \gamma^r n_{ij} \gamma^s n_{kl} + \gamma^k \gamma^l \gamma^m \gamma^p \gamma^q \gamma^r n_{ij} n_{kl} H_{ij} H_{kl}) \Psi_{,ij},$$

(20)

where the first- and second-order derivatives of metric components are clearly separated. Expanding it we get the following more explicit expression,

$$R = -\frac{2\chi \Psi^2}{\gamma^2} \Delta \Psi - \chi \left[ \frac{2\chi H_{kk} H_{ij} - H_{ik} H_{jk}}{\gamma^2} \right] \Psi_{,ij}$$

$$+ \frac{1}{\gamma^2} \left[ H_{ik} H_{jl} - H_{ij} H_{kl} \right] h_{ij}^{TT} \Psi_{,ij} + \gamma^k \gamma^l \gamma^m \gamma^p \gamma^q \gamma^r (\Gamma_{ni} \Gamma_{pq} - \Gamma_{npi} \Gamma_{qij}).$$

Now it is straightforward to introduce this expression in the Hamiltonian constraint obtaining, in this way, an elliptic equation to solve for the conformal factor,

$$\Delta \Psi = -\frac{1}{2\chi \Psi^2} H_{ij} \Delta h_{ij}^{TT}$$

$$- \frac{1}{\gamma^2} \left[ (\chi + H_{kk}) H_{ij} - H_{ik} H_{jk} \right] \Psi_{,ij}$$

$$+ \frac{1}{\gamma^2} \left[ H_{ik} H_{jl} + H_{ij} H_{kl} \right] h_{ij}^{TT} \Psi_{,ij} + \gamma^k \gamma^l \gamma^m \gamma^p \gamma^q \gamma^r (\Gamma_{ni} \Gamma_{pq} - \Gamma_{npi} \Gamma_{qij})$$

$$- \frac{\gamma}{4\chi \Psi^2} (2\pi_{ij} \pi_{ij} - \pi^2) - \frac{\kappa}{2\chi \Psi^2} \alpha^2 \gamma^2 T^{00}.$$

(21)

D. Equation for the longitudinal field momentum

Because of the gauge condition (6), we can perform the following decomposition for $\pi^{ij}$,

$$\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij}_{TT},$$

(22)

where the longitudinal part $\tilde{\pi}^{ij}$ can be given in terms of a vector field,

$$\tilde{\pi}^{ij} = \pi^{j, i} + \pi^{i, j} - \frac{2}{3} \delta^{ij} \pi^{m, m}.$$

(23)

An elliptic equation for this vector can be obtained from Eq. (3), since $\pi^{j, i} = \Delta \pi^i + \frac{1}{2} \pi^j , ij$,

$$\Delta \pi^i + \frac{1}{3} \pi^j, ij = -\gamma^{ik} \Gamma_{kj} \pi^{jl} - \frac{\kappa}{2} \gamma^{lj} \alpha \gamma^{ij} T^{00}.$$

(24)

In summary, we have fixed the eight (out of 16) degrees of freedom in the initial slice through gauge conditions (6) and (7) together with the constraint equations, which are implemented by the elliptic equations for the conformal factor (21) and for the longitudinal part of the momentum (24). Finally, Eqs. (10) and (17) for the shift and lapse guarantee that the mentioned gauge conditions are conserved throughout the evolution.

The only objects that are then left to be fixed are the transverse and traceless part of the metric $h_{ij}^{TT}$ and its conjugate momentum $\pi^{ij}_{TT}$. The four components of these tensors describe, in this setting, the two physical degrees of freedom of the theory. This means that they are freely specifiable on the initial slice but the conditions imposed on them will restrict the physical problem we are dealing with.
IV. A 2PN-EXACT WAVELESS CONDITION: $\pi^{ij}_{TT} \equiv 0$

Thus, in order to close the set of equations and obtain an approximated version of Einstein theory, we will introduce the following restrictions,

$$\pi^{ij}_{TT} \equiv 0 \quad \text{and thus,} \quad \pi^{ij}_{TT,0} \equiv 0.$$  \hspace{1cm} (25)

This assumption is exact at 2PN order since, in the general scenario, $\pi^{ij}_{TT}$ is vanishing up to $O(c^{-5})$ and it can be understood as the conjugate condition to that of the CFC case, where $h^{ij}_{TT}$ and its time derivative are assumed to be vanishing. Therefore, in this approach, the conjugate momentum $\pi^{ij}$ will be purely longitudinal and, hence, the vector field $\pi^{i}$ will contain all the needed information in order to reconstruct it \cite{22}. Thereby now the only missing equation is that for $h^{ij}_{TT}$ and it will be obtained from the TT part of Eq. \cite{4}. Let us denote by $A^{ij}$ the right-hand side of that equation,

$$A^{ij} \equiv -\alpha \gamma^{1/2}R^{ij} + Y^{ij} + \alpha,ij,$$

where we have defined,

$$Y^{ij} \equiv \frac{\alpha}{2\gamma^3} (2\gamma^{ij} \pi^{kl} \pi_{kl} - 4\pi^{ik} \pi^{j}_k + 2\pi \pi^{ij} - \pi^2 \gamma^{ij})$$

$$+ \gamma^{1/2}(\gamma^{ik,\gamma^{jm}} - \gamma^{ij,\gamma^{km}})(\alpha_{km} - \frac{\gamma^{ln}}{\gamma^{nm}} \Gamma_{nkm} - \alpha,ij)$$

$$+ (\pi^{ij} \beta^m)_{,m} - \pi^{mij} \beta^i_{,m} - \pi^{mij} \beta^j_{,m} + \frac{K}{2} \alpha \gamma^{1/2}(\gamma^{ij} \gamma^{jm} T_{lm} + \gamma^{ij} \alpha^2 T^{00}),$$  \hspace{1cm} (26)

and use has been made of the Hamiltonian constraint \cite{22} in order to remove the Ricci scalar from this expression. The term $\alpha,ij$ has been added and subtracted from the previous definition in order to ensure a decay rate of $1/r^4$ for $Y^{ij}$ at spacelike infinity.

On the other hand, raising indices in formula \cite{15}, the contravariant Ricci tensor is given by,

$$R^{ij} = -\frac{\chi^3}{2\gamma^3} (\Delta h^{TT}_{ij} + \Psi,ij + \delta_{ij} \Delta \Psi) + X^{ij},$$  \hspace{1cm} (27)

with

$$X^{ij} \equiv -\frac{\chi}{2\gamma^3} \left[ \Delta h^{TT}_{km} + \delta_{km} \Delta \Psi + \Psi_{,km} \right] \left\{ \chi \delta_{mi} H_{jk} + \chi \delta_{kj} H_{im} + H_{im} H_{jk} \right\}$$

$$- \frac{1}{2\gamma} \gamma^{ik,\gamma^{jm}} (\gamma_{np,km} + \gamma_{km,np} - \gamma_{nm,kp} - \gamma_{kp,nm}) H_{np}$$

$$+ \gamma^{ik,\gamma^{jm}} \gamma^{rs} \gamma^{np} (\Gamma_{nks} \Gamma_{prm} - \Gamma_{nkm} \Gamma_{prs}).$$  \hspace{1cm} (28)

With these definitions at hand, one can write down $A^{ij}$ as,

$$A^{ij} = \frac{1}{2} \left( \frac{\alpha \chi^3}{\gamma^{5/2}} - 1 \right) (\Delta h^{TT}_{ij} + \Psi,ij + \delta_{ij} \Delta \Psi) - \alpha \gamma^{1/2} X^{ij} + Y^{ij} + \alpha,ij$$

$$+ \frac{1}{2} \left( \Delta h^{TT}_{ij} + \Psi,ij + \delta_{ij} \Delta \Psi \right),$$  \hspace{1cm} (29)

where, again, terms in the second line have been added and subtracted for future convenience. Because of the gauge condition \cite{3}, $A^{ij}$ is tracefree on shell. Even though, in order to make an eventual numerical evolution of this system more stable, we will make it explicitly tracefree. After that we will need to obtain the transverse part of the resulting object. In order to do so, we take into account that full-derivative terms do not contribute to the transverse and traceless part, thus both $\alpha,ij$ and $\Psi,ij$ can be eliminated, whereas explicit pure-trace terms (those proportional to $\delta_{ij}$) can be included in the trace part $A$. Therefore, the evolution equation for $\pi^{ij}_{TT}$ \cite{4} is now rewritten as,

$$\pi^{ij}_{TT,0} = \frac{1}{2} \left( \Delta h^{TT}_{ij} - B^{ij}_{TT} \right),$$  \hspace{1cm} (30)

where the source term is given by,

$$B^{ij}_{TT} \equiv B^{ij} - \frac{1}{3} \delta_{ij} B - V^{j,i} - V^{i,j} + \frac{2}{3} \delta_{ij} V^{l,l},$$  \hspace{1cm} (31)
Finally, the vector field $V^i$ encodes the longitudinal part of $B^{ij}$ and hence obeys the following equation,

$$\Delta V^i + \frac{1}{3} V^j \delta_{ji} = B^{ij} - \frac{1}{3} B_{ij}.$$  

(33)

Imposing now our assumption ($\pi_{TT}^{ij} \equiv \pi_{TT,0}^{ij} \equiv 0$) on Eq. (30), it is straightforward to get an elliptic equation for $h_{ij}^{TT}$.

$$\Delta h_{ij}^{TT} = B_{ij}^{TT}.$$  

(34)

This assumption must also apply on other elliptic equations presented in last section, obtaining in this way a closed system of equations which does not contain $\pi_{TT}^{ij}$. Because of the terms that have been added and subtracted previously, the source $B_{ij}^{TT}$ decays properly ($\sim 1/r^4$) for an eventual numerical resolution of this equation. In particular, note that the terms contained inside the first brackets of the definition of $B^{ij}$ (32) have been constructed such that their combination decays as $\sim 1/r$ at asymptotic infinity and, moreover, it is of order $\mathcal{O}(c^{-2})$, as can be verified using the expressions (35) below.

In summary, the gravitational metric functions are encoded in the objects $\{\beta, \alpha, \Psi, h_{ij}^{TT}\}$ which are connected with the auxiliary variables $\{\pi^i, V^i\}$ and the corresponding elliptic equations are, once imposed the condition $\pi_{TT}^{ij} = 0$ in all of them, (10), (17), (21), (31), and (24), (33), respectively.

This set of equations has two main applications. On the one hand, it can be used to obtain initial data, such that $\pi_{TT}^{ij} = \pi_{TT,0}^{ij} = 0$, for a subsequent numerical evolution of the full Einstein equations. On the other hand, it could be solved as a self-contained theory of gravitation, which would not have waves, but would approximately describe the motion of the astrophysical bodies.

Regarding this last application, we would like to end this section by analyzing the accuracy of our theory, that is, which is the error one would commit when solving our approximated set of equations with respect to solving the full Einstein equations. Since there is no approximation in the equations for the rest of the objects, the only error is introduced in the system via the assumption on $\pi_{TT}^{ij}$. Looking at their respective equations, and recalling their leading order post-Newtonian series, namely,

$$\alpha = 1 + \mathcal{O}(c^{-2}) , \quad \beta_i = \mathcal{O}(c^{-3}) , \quad \pi^{ij} = \mathcal{O}(c^{-3}) ,$$

$$\Psi = 1 + \mathcal{O}(c^{-2}) , \quad h_{ij}^{TT} = \mathcal{O}(c^{-4}) ,$$  

(35)

one can check that if $\pi_{TT}^{ij}$ were computed properly, where the subscript $(n)$ denotes the coefficient of $1/c^{n}$ in the corresponding post-Newtonian expansion, the metric components $\{\alpha_{(n+3)}, \beta_{(n+2)}^{ij}, \gamma_{(n+1)}^{ij}\}$ would be obtained correctly. In fact the conformal factor is calculated up to $\Psi_{(n+3)}$, otherwise we could not compute the lapse up to the mentioned order. As we have already commented, the assumption we have proposed is correct up to 2PN order, i.e., we are computing $\pi_{(3)}^{ij}$ correctly thus, by using the proposed scheme, $\{\alpha_{(6)}, \beta_{(5)}^{ij}, \gamma_{(4)}^{ij}\}$ can be properly determined.

V. A 4PN-EXACT APPROXIMATION

In this section we will generalize the scheme we have presented in the previous section, which will give rise to a formalism that agrees with General Relativity up to 4PN order. Contrary to our 2PN scheme, this more general approach will indeed contain gravitational waves since the main equation to obtain $h_{ij}^{TT}$ will be of hyperbolic nature.

Let us write the evolution equation for the spatial metric (36) in the form

$$\gamma_{ij,0} = C_{ij},$$  

(36)

where

$$C_{ij} \equiv \alpha \gamma^{-1/2}(2\pi_{ij} - \gamma_{ij} \pi) + \beta_{ij} + \beta_{ji}. $$  

(37)

Using the gauge conditions on $\gamma_{ij}$, that are given by Eq. (7), we obtain the evolution equation for the transverse and traceless part of the metric,

$$h_{ij,0}^{TT} = C_{ij} - \frac{1}{3} C_{ll} \delta_{ij}, $$  

(38)
where \( C_{ij} = 3\eta(C^{ij} - \gamma^{ij}\gamma_{kl}h_{ij,kl}^{\mathrm{TT}}) \) with \( C^{ij} = -\alpha \gamma^{-1/2} \gamma^{ij} \pi + 2\beta^{ij} \) and \( \eta \equiv (\gamma^{ij} \gamma_{ij})^{-1} \). This equation can then be rearranged so that \( \pi^{ij} \) follows in terms of \( h_{ij,0}^{\mathrm{TT}} \) in the form

\[
M_{ij}^{kl} \pi^{kl} = \frac{\gamma^{1/2}}{2\alpha} \left[ 2\eta \beta^{m|n} \gamma^{ij} \gamma^{jn} - \beta^{ij} - \beta^{ji} \right] \\
+ \left( \gamma^{im} \gamma^{jn} - \eta \gamma^{ik} \gamma^{jk} \gamma^{lm} \gamma^{jn} \right) h_{mn,0}^{\mathrm{TT}},
\]

(39)

where the matrix \( M_{ij}^{kl} \) is given by

\[
M_{ij}^{kl} = \frac{1}{2} \begin{bmatrix} \delta_{ij}\delta_{jk} + \delta_{ik}\delta_{jl} - (\gamma^{ij} - \eta \gamma^{in} \gamma^{jn} \gamma^{mn}) h_{kl}^{\mathrm{TT}} \end{bmatrix}.
\]

(40)

Remarkably, the matrix \( M_{ij}^{kl} \) deviates from the unit identity matrix in the quadratic order of \( h_{ij}^{\mathrm{TT}} \) only [26]. Taking into account that \( h_{ij}^{\mathrm{TT}} \) is of \( 2\mathrm{PN} \) order (35), neglecting the quadratic terms of \( h_{ij}^{\mathrm{TT}} \) in \( M_{ij}^{kl} \) is a truncation at \( 5\mathrm{PN} \) order only. Thus, we can write down

\[
\pi^{ij} = \frac{\gamma^{1/2}}{2\alpha} \left[ 2\eta \beta^{m|n} \gamma^{ij} \gamma^{jn} - \beta^{ij} - \beta^{ji} \right] \\
+ \left( \gamma^{im} \gamma^{jn} - \eta \gamma^{ir} \gamma^{jr} \gamma^{sm} \gamma^{sn} \right) h_{mn,0}^{\mathrm{TT}} + \mathcal{O}(c^{-11}),
\]

(41)

which is a \( 4.5\mathrm{PN} \)-exact relation. Keeping only the leading order of the time derivative of \( h_{ij}^{\mathrm{TT}} \) and calculating the transverse and tracefree part of this relation, we obtain,

\[
\pi_{TT}^{ij} = \frac{1}{2} h_{ij,0}^{\mathrm{TT}} + D^{ij} - D_{L}^{ij} + \mathcal{O}(c^{-9}),
\]

(42)

where

\[
D^{ij} = -\frac{\gamma^{1/2}}{2\alpha} \left( \beta^{ij} + \beta^{ji} - 2\eta \beta^{m|n} \gamma^{ij} \gamma^{jn} \right),
\]

(43)

and

\[
D_{L}^{ij} = W^{i}_{,j} + W^{j}_{,i} - \frac{2}{3} W^{k}_{,l} \delta_{ij},
\]

(44)

hold with

\[
\Delta W^{i} + \frac{1}{3} W^{j}_{,ji} = D^{ij}_{-,j}.
\]

(45)

Note that the decay rate of the right-hand side of the last equation is \( 1/r^3 \) at asymptotic infinity. As we have commented, this could give problems when solving this equation numerically. Even though, since the term in question is a full divergence, the vector \( W^{i} \) can be redefined including in it the terms that goes like \( \sim 1/r^3 \), so that the source for the equation for the new vector decays properly.

In order to check that Eq. (42) is valid up to \( \mathcal{O}(c^{-9}) \) one has to take into account the fact that \( \alpha^{2n}\Psi^n = 1 + \mathcal{O}(c^{-4}) \), as can be verified with the first terms of their PN expansions: \( \Psi = 1 + 2U/c^2 \) and \( \alpha = 1 - U/c^2 \), \( U \) being the Newtonian potential.

The (nonapproximated) evolution equation for \( \pi_{TT}^{ij} \) (39) and the (approximated) one for \( h_{ij}^{TT} \) (42),

\[
h_{ij,0}^{\mathrm{TT}} = 2\pi_{TT}^{ij} - 2 \left( D^{ij} - D_{L}^{ij} \right) + \mathcal{O}(c^{-9}),
\]

(46)

define our system of truncated evolution equations of the Einstein theory put into canonical form. Combining the mentioned two equations, and thus making an implicit change from a Hamiltonian to a Routhian framework (see, e.g., [3]), results into a second-order hyperbolic equation for \( h_{ij}^{TT} \),

\[
h_{ij,00}^{TT} + \Delta h_{ij}^{TT} = 2 \left( D^{ij} - D_{L}^{ij} \right) + B_{TT}^{ij} + \mathcal{O}(c^{-10}),
\]

(47)
which describes the propagation of the two gravitational degrees of freedom. Defining the source $S_{ij}^{TT} = 2 \left( D^{ij} - D_{L}^{ij} \right)_{\partial} + B_{TT}^{ij}$, the no-incoming radiation formal solution of this equation is given by the standard retarded integral (see below for an iterative solution),

$$h_{ij}^{TT}(t, \vec{x}) = -\frac{1}{4\pi} \int \frac{d^3y}{|\vec{x} - \vec{y}|} S_{ij}^{TT}(t_{ret}, \vec{y}),$$

(48)

with the retarded time $t_{ret} = t - |\vec{x} - \vec{y}|/c$; for another representation of the retarded solution see, e.g., [31].

In summary, the system of equations to be solved is composed by the elliptic Eqs. (10), (17), (21), (24), (33) and the wave Eq. (47) for $h_{ij}^{TT}$. This set of equations can be understood as a truncated version of Einstein equations so, by assuming certain initial data for the hyperbolic equation, they provide an approximated solution of General Relativity. On the other hand, they can also be used to obtain initial data for a subsequent evolution via full Einstein equations. Here we propose an iterative scheme to construct such initial data for the metric coefficients. As a first step, one imposes the assumptions $\pi_{ij}^{TT} = 0$ and $h_{ij}^{TT} = 0$ in all elliptic equations and solves them including the equations for the matter variables for all past times. Then, with this information, one can solve the hyperbolic Eq. (47) for $h_{ij}^{TT}$ via the retarded integral (and hence without any “instant-of-time” initial conditions) dropping the time derivatives of the right-hand side since they are of $O(c^{-6})$. In a second step, by solving again the elliptic system with the previously computed $h_{ij}^{TT}$, and still $\pi_{ij}^{TT} = 0$, a 2.5PN-exact solution $\{\alpha(7), \beta(6), \gamma(5)\}$ is obtained. Note that this also 2PN-exact solution is essentially different from that obtained by solving the scheme proposed in Sec. IV This 2.5PN-exact solution can now be used to compute $\pi_{ij}^{TT}(5)$ properly through its truncated definition (12). With this information at hand, we can calculate the sources (all the right-hand sides) of our system of equations to a PN-level of precision higher than in the second step. Therefore, by solving this system, a 3PN (and thus also 3.5PN)-exact solution can be obtained, $\{\alpha(8), \beta(7), \gamma(6)\}$. Repeating again this process, the metric components can be computed up to 4PN-exact order, $\{\alpha(10), \beta(9), \gamma(8)\}$. The limit of this procedure is given by the order up to which Eq. (42) is valid. This iterative scheme generalizes a procedure which is under development in [14, 15].

At this stage some remarks are in order.

In this second-order formalism, $\pi_{ij}^{TT}$ has to be replaced by $h_{ij}^{TT}$ in all our equations using its truncated definition (12). This time derivative, as well as the first-order time derivatives on the right-hand side of the wave Eq. (47), have to be vanishing. This context suggests another 2PN-exact waveless approach through dropping $h_{ij}^{TT}$ in Eq. (41) and also all terms in Eq. (47) with time derivatives, which is equivalent to the assumption $\pi_{ij}^{TT,0} = 0$ in Eq. (30). In this way, a quasistationary approximation scheme is achieved which is 5PN exact for stationary configurations and covers both the 1PN-exact CFC and the 2PN-exact CFC+ approach (see next section). This latter approach is closest to [26], where $\pi_{ij}^{TT,0}$ is also put equal to zero in Eq. (30) but Eq. (59) is kept exact.

VI. RELATION TO 2PN-EXACT CFC+ APPROACH

In this section we want to reduce our scheme to CFC+ order in order to compare with previous results in the literature. As has already been explained, the CFC case is nothing but allowing $h_{ij}^{TT}$ to be vanishing. This is the
only approximation that is done in this approach, so all the equations that we will present in this section will be fully
correct in the conformal flat case just by removing terms with \( h_{ij}^{TT} \). On the other hand, the CFC+ approach considers
also linear terms in \( h_{ij}^{TT} \) hence, in this case, the following definitions apply,

\[
\gamma^{ij} = \Psi^{-1} \delta^{ij} - h_{ij}^{TT}, \quad \gamma = \Psi^3, \quad \chi = \Psi^2, \quad \text{and} \quad \eta = \frac{\Psi^2}{3}.
\] (49)

The time derivative of \( h_{ij}^{TT} \) in Eq. (41) can also be neglected, including however a nonvanishing \( \pi^{ij} \) which makes it
a 2PN-exact approach different from Sec. IV, so for \( \pi^{ij} \) the shift, we introduce relation (50) in the momentum constraint an d obtain,

\[
\pi^{ij} = -\frac{\gamma^{1/2}}{2\alpha} \left( \frac{\beta^{ij} + \beta^{ij}}{2} - 2\eta \beta^{imn} \gamma^{inj} \right)
\]
\[
= -\frac{\Psi^{1/2}}{2\alpha} \left( \beta^{ij} + \beta^{ij} - \frac{2}{\delta_{ij}\beta^{jk}} \right),
\] (50)
is needed. In fact, this equation is only used in the CFC case since in the CFC+ scheme, it is only valid up to 1PN
order \( c^{-3} \) counting in terms of integer PN orders.

At this linearized order, our equation for the shift,

\[
\Delta \beta_i + \frac{1}{3} \beta_j, ij = -\frac{2}{3} ((\gamma^{jk} \beta_j) \Gamma_{kkl}), i + 2((\gamma^{jk} \beta_j) \Gamma_{kkl}), j - 2((\alpha \gamma^{-1/2} \pi_{ij})), j,
\] (51)
is trivially obeyed when taking into account last relation (50). Hence, in order to obtain a meaningful equation for
the shift, we introduce relation (50) in the momentum constraint and obtain,

\[
\Delta \beta^i + \frac{1}{3} \beta^j, ij = -2\Psi \left( \frac{\alpha}{\Psi^{1/2}} \right) \pi^{ij} + \Psi \kappa \alpha^2 \gamma^{ij} T^0_0.
\] (52)

In the equations for the lapse (17) and the conformal factor (21), the terms that survive at this order are given by,

\[
\Delta \alpha = \frac{1}{2} h_{ij}^{TT} \alpha_{ij} + \gamma^{kl} \Gamma_{lji} \alpha_{ik} + \frac{\gamma^2}{2\chi \Psi^2} \left[ \kappa R^{ii} + \frac{\alpha}{\gamma} \pi^{ij} \pi_{ij} - 2\gamma^{-1/2} \pi^{im} \beta^i, m \right]
\]
\[
+ \frac{\kappa}{2} \alpha (\gamma^{il} \gamma^{im} \Gamma_{km} + \gamma^{ii} \alpha^2 T^{000}),
\] (53)

\[
\Delta \Psi = \frac{\gamma^2}{2\chi \Psi^2} \gamma^{kl} \gamma^{lm} \pi^{ij} (\Gamma_{nkl} \Gamma_{pkl} - \Gamma_{nkj} \Gamma_{pkl}) - \frac{\gamma}{2\chi \Psi^2} \pi^{ij} \pi_{ij} - \frac{\kappa}{2\Psi^2} \gamma^2 \alpha^2 T^{000},
\] (54)

which can be written in a simpler way as

\[
\Delta \alpha = -\frac{1}{2\Psi} \pi^{ij} \beta_{ij} + \frac{\kappa \alpha}{4} (T_{ii} + \Psi \alpha^2 T^{000}) + \frac{1}{2} h_{ij}^{TT} \alpha_{ij} - \frac{1}{4} h_{ij}^{TT} \Psi, ij,
\] (55)

\[
\Delta \Psi = \frac{3}{4\Psi} \pi^{ij} \pi_{ij} - \frac{\kappa}{2} \Psi \pi^{ij} \pi_{ij} - \frac{\kappa}{2} \Psi \alpha^2 T^{000}.
\] (56)

Here, we have used the fact that the noncovariant trace of the Ricci tensor at this order is given just by,

\[
R^{ii} = \frac{\kappa \alpha^2}{4} T^{000} + \frac{1}{\Psi^2} \pi^{ij} \pi_{ij} + \frac{1}{2} h_{ij}^{TT} \Psi, ij.
\] (57)

Finally, the set of CFC+ equations is closed by the elliptic equation for the transverse and traceless part of the
spatial metric,

\[
\Delta h_{ij}^{TT} = B_{ij} - \frac{1}{3} \delta_{ij} B - V^i, j - V^j, i + \frac{2}{3} \delta_{ij} V^m, m,
\] (58)

\[
\Delta V^i + \frac{1}{3} V^j, ji = B_{ij} - \frac{1}{3} B, j + \kappa T_{ij},
\] (59)

where \( B = B_{ii} \) and now the following definition for \( B_{ij} \) should be applied,

\[
B_{ij} = \Psi, i \alpha_{j} + \Psi, j \alpha_{i} - \kappa T_{ij}.
\] (60)
In order to arrive to this expression, we have integrated by parts terms with second derivatives of the lapse and the conformal factor, taking into account that the full-derivative terms are eliminated when computing the transverse and tracefree part, as it is done in the right-hand side of Eq. \[33\]. These CFC+ equations have been compared to those presented in Ref. \[12\] and obtained exact agreement. For such comparison, one has to consider that, as already shown in the previous section, $\Psi = 1 + 2U/c^2$ and $\alpha = 1 - U/c^2$ for the 2PN terms beyond CFC.

In conclusion, we have shown that our proposed 4PN-exact formalism covers naturally both CFC and CFC+ approximations. Regarding the 2PN-exact approach presented in Sec. \[IV\], it also coincides with the CFC and CFC+ cases up to the mentioned PN-level. The key difference between the CFC+ scheme and the one of Sec. \[IV\] is that whereas we have kept all the nonlinear terms present in all the equations, in CFC+ approach one just keeps those terms that are of the corresponding (2PN) order. Even though, our 2PN elliptic framework of Sec. \[IV\] can not “fully” cover the CFC case since we have assumed that $\pi_{TT}^i = 0$ whereas in the standard CFC scheme the definition \[50\] is used. Only under spherical symmetry conditions both $h_{ij}^{TT}$ and $\pi_{TT}^i$ vanish.

VII. A SPECIFIC MATTER MODEL

In this section we specify the matter model to the particular case of a barotropic perfect fluid [in this section, $c = 1$]. Therefore, the stress-energy tensor will be given in terms of the fluid four-velocity $u^\mu$, pressure $p$, proper mass density $\rho$, and specific enthalpy $h$ in the following way,

$$ T^{\mu\nu} = \rho (1 + h) u^\mu u^\nu + pg^{\mu\nu}, $$

where the conservation law $\nabla_\mu (\rho u^\mu) = 0$ holds with $\nabla_\mu$ the four-dimensional covariant derivative.

We need the following components of the stress-energy tensor density $(-g)^{1/2}T^{\mu\nu}$ that appear in the equations,

$$ \alpha \gamma^{1/2} T^{00} = \rho_s (1 + h) u^t - \gamma^{1/2} \rho,$$

$$ \alpha \gamma^{1/2} T^{0j} = \rho_s (1 + h) u^j, $$

$$ \alpha \gamma^{1/2} T^i_j = \rho_s (1 + h) u_i u^j + \gamma^{1/2} p \gamma^i_j, $$

where the four-velocity is decomposed as $u^\mu = (u^t, u^i)$, $u_\mu = (u_t, u_i)$ and the definition $\rho_s = \alpha \gamma^{1/2} u^t \rho$ has been made. From the normalization of the four-velocity $u^\mu u_\mu = -1$, it is easy to see that $u^t$ is given by $u^t = (1 + \gamma^{ij} u_i u_j)^{1/2}/\alpha$.

In the case of a barotropic perfect fluid, where $dp = \rho dh$ holds, the equation of state $p = p(\rho)$ and the conservation of stress-energy tensor,

$$ \nabla_\mu T^{\mu\nu} = 0, $$

give rise to all the equations of motion for the matter. The independent equations of motions can be cast into the form \[33\] \[54\],

$$ \partial_t \rho_s = - \partial_i (\rho_s v^i), $$

$$ \partial_t P_i = - \partial_j (P_i v^j) - \partial_i (\alpha \gamma^{1/2} p) + \frac{\alpha}{2} \gamma^{1/2} T^{\mu\nu} \partial_i g_{\mu\nu}, $$

where $v^i \equiv u^i/u^t$ and $P_i \equiv \rho_s (1 + h) u_i$. Introducing $w_i \equiv P_i/\rho_s$, i.e., $w_i = (1 + h) u_i$, the latter equation of motion can be written

$$ \partial_t w_i = -v^j \partial_j w_i - \frac{1}{\rho_s} \partial_i (\alpha \gamma^{1/2} p) + \frac{\alpha \gamma^{1/2} T^{\mu\nu}}{2 \rho_s} \partial_i g_{\mu\nu}. $$

With the aid of the relation

$$ v^i = \frac{\alpha \gamma^{ij} w_j}{[(1 + h)^2 + \gamma^{ij} w_i w_j]^{1/2}} - \beta^i, $$

all matter variables can be reduced to the independent ones ($\rho_s, w_i$) or ($\rho_s, P_i$).
VIII. CONCLUSIONS

In this paper we have developed a 4PN-exact approximation to the field equations of General Relativity which turns out to be fully exact in the conformal flat case and 5PN exact for stationary configurations. The elliptic equations for the lapse $\alpha$, the shift $\beta^i$, and the conformal factor $\Psi$ are exact (just the well-known equations from the ADM formalism but in some more explicit form): Eqs. (17), (19), and (21), respectively. Only the wave equation for $h_{ij}^{\ TT}$ and the definition for its conjugate momentum $\pi^{ij}_{TT}$ are approximated. In order to obtain the mentioned hyperbolic equation, an implicit transition from a Hamiltonian to a Routhian framework, regarding the independent gravitational degrees of freedom, has taken place [8]. This equation has to be solved under the condition of no incoming radiation. The construction of various transverse-traceless objects resulted in several auxiliary vectors $\pi^i$, $V^i$, and $W^i$ obeying the Eqs. (24), (33), and (45), respectively.

Our proposal is to solve the commented set of equations iteratively. In particular, the first step is to solve the elliptic equations and the equations of motion of the matter for all times prior to the initial value slice imposing $h_{TT}^{ij} \equiv 0$ and $\pi_{TT}^{ij} \equiv 0$. Then, using this result, the hyperbolic equation for $h_{TT}^{ij}$ as well as the elliptic equations again are being solved which gives rise to a 2[2.5]PN-exact approximation to the Einstein field equations. The obtained 2PN-exact solution can be used to correctly compute $\pi_{TT}^{ij}$ through its truncated definition (42) up to 3PN order. With this information at hand, the sources of the mentioned system of equations can be calculated to one PN-order.

Finally, we have also analyzed the 2PN-exact waveless approximation to General Relativity that is obtained by assuming $\pi_{TT}^{ij} = \pi_{TT}^{ij,0} = 0$. These conditions are quite interesting since they can be considered as the conjugate assumptions to the well-known conformal flat conditions ($h_{TT}^{ij} = h_{TT}^{ij,0} = 0$). A maximum elliptic system without any partial time derivatives of the gravitational field variables that one can obtain within General Relativity is dropping $h_{TT}^{ij,0}$ in Eq. (30) and $\pi_{TT}^{ij,0}$ in Eq. (42). Hereof, the simplified waveless approach suggested in Sec. V results by just dropping $h_{TT}^{ij,0}$ in Eq. (42). A further truncation is obtained by putting $\pi_{TT}^{ij} \equiv 0$ which is the waveless approximation of Sec. V. All these waveless approximations are 2PN exact where the latter one is the simplest of them. In order to obtain more accurate truncated versions of General Relativity, hyperbolic equations should be considered, as it is done in the above commented 4PN-exact approach.

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