Accelerate iterated filtering

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Abstract: In simulation-based inferences for partially observed Markov process models (POMP), the by-product of the Monte Carlo filtering is an approximation of the log likelihood function. Recently, iterated filtering [14, 13] has originally been introduced and it has been shown that the gradient of the log likelihood can also be approximated. Consequently, different stochastic optimization algorithm can be applied to estimate the parameters of the underlying models. As accelerated gradient is an efficient approach in the optimization literature, we show that we can accelerate iterated filtering in the same manner and inherit that high convergence rate while relaxing the restricted conditions of unbiased gradient approximation. We show that this novel algorithm can be applied to both convex and nonconvex log likelihood functions. In addition, this approach has substantially outperformed most of other previous approaches in a toy example and in a challenging scientific problem of modeling infectious diseases.

Keywords and phrases: accelerate iterated filtering, accelerate inexact gradient method, sequential Monte Carlo, state space model, parameter estimation.

1. Introduction

The last decade has seen a great increase in the use of simulation-based inference where numerical approximations are based on either Markov chain Monte Carlo or sequential Monte Carlo sampling. These approaches have become popularized, in part, because of the increasing computational power and the emergence of efficient stochastic optimization algorithms. On the Bayesian paradigm, particle Markov chain Monte Carlo has been introduced and popularized by Doucet and collaborators [1, 2, 31]. Similar ideas have been developed previously [21, 7, 10, 16] but in different contexts than simulation-based inferences. On the frequentist paradigm, [14, 13] have introduced an original approach to perform simulation-based parameter inference in POMP models by combining stochastic gradient approximation and particle filtering. In this paper, we will focus on improving one of the most popular algorithm of this class, namely, iterated filtering (IF). Iterated filtering uses an approximation estimate of the gradient of the log likelihood computed from particle filters while proposing an artificial perturbation moves to update the parameters. This class of algorithm is attractive because it enables routine simulation-based parameter inferences in general POMP model, even in the cases of intractable likelihoods. Due to some interesting theoretical properties [13, 15, 29], its applications range in various fields such as biology, ecology, economics and engineering [24, 23, 20, 3, 5, 4].

Iterated filtering was later theoretically developed by Ionides et al. [13]. Recently, Lindström et al. [24] extended it to improve on numerical performance while Doucet et al. [8] expanded it to include filtering/smoothing with quite attractive theoretical
properties. Ionides et al. [15] generalized Lindström et al. [24]'s approach and combined the idea with data cloning [22], developing a Bayes map iterated filtering with an entirely different theoretical approach. Nguyen and Ionides [30] revisited the approach of Doucet et al. [8], using a different perturbation noise and computed both the gradient and the Hessian. Similar to intractable likelihood in the context of iterated filtering, Poyiadjis et al. [32], Nemeth et al. [26], Doucet et al. [8] showed that the gradient and Hessian information can also be computed from particle filter. In the same line, manifold Langevin Monte Carlo (mMALA) [12] exploits the Hessian information to simplify the tedious tuning method while improving on convergence rate. However, this relies on rather strong assumptions that the gradient, and Hessian information of transition density and observation density can be sampled from. This is quite unrealistic in many real world applications. We, therefore, followed the formal approaches, based solely on very weak assumptions of being able to sample from transition density and evaluate from observation density. Motivated from the fact that the gradient and Hessian information can be approximated using the first and the second moments [13, 8], we propose to use such approximations in the context of accelerate iterated filtering. Ionides uses score vector merely while Doucet includes the Hessian information for the independent white noise, which is not quite useful in the context of iterated filtering with natural random walk noise. Nguyen and Ionides [30] proposed to approximate the gradient and Hessian using random walk noise to efficiently explore the mode of the likelihood. Other than exploiting approximations of the Hessian under weak assumption, we chose an alternative approach. That is, we apply the accelerate gradient approach to the approximation of the gradient of the log likelihood for an effective estimation approach.

The key contributions of this paper are three folds. Firstly, we developed and showed that accelerate iterated filtering algorithm converges using a general non-increasing step size with bias approximation of the gradient. It is simple, elegant, and generalizable to faster algorithms. Secondly, we proved that it has a higher convergence rate in general convex and non-convex conditions of the objective log likelihood. Finally, we showed substantial improvements of the method on a toy problem and on a real world challenge problem of vivax malaria model compared to previous simulation-based inference approaches.

The paper is organized as follows. In the next section we introduce some notations and we develop the framework of accelerate iterated filtering. In Sections 3, we state the convergence of this approximation method to the true maximum likelihood estimation by iterating and accelerating noisy gradient of the log likelihood. We validate the proposed methodology by a toy example and a challenging inference problem of fitting a malaria transmission model to time series data in Section 4, showing substantial gains for our methods over current alternatives. We conclude in Section 5 with the suggesting of the future works to be extended. The proofs are postponed to the Appendix.

2. Background of simulation-based inferences

We are interested in a general latent variable model since this is an ubiquitous model for applied sciences. Let $\mathcal{X}$ be a latent state space with a density $q_0(x)$ parameterized
by \( \theta \in \Theta = \mathbb{R}^d \), and let \( \mathcal{Y} \) be an observation space equipped with a conditional density \( f_\theta(y|x) \). The observation \( y \in \mathcal{Y} \) are considered as fixed and we write the log-likelihood function of the data \( \ell(\theta) \leq \log \int q_\theta(x) f_\theta(y|x)dx \). We work with the maximum likelihood estimator, \( \hat{\theta} = \arg \max \ell(\theta) \) where \( \ell(\theta) \) is intractable but \( f_\theta(y|x) \) can be evaluated, by using samples where \( f_\theta(y|x) \) is also intractable. This process often uses the first order stochastic approximation [19], which involves a Monte Carlo approximation to a difference equation, \( \theta_m = \theta_{m-1} + \gamma_m \nabla \ell(\theta_{m-1}) \), where \( \theta_0 \in \Theta \) is an arbitrary initial estimate and \( \{ \gamma_m \}_{m \geq 1} \) is a sequence of step sizes with \( \sum_{m \geq 1} \gamma_m = \infty \) and \( \sum_{m \geq 1} \gamma_m^2 < \infty \). The algorithm converges to a local maximum of \( \ell(\theta) \) under regularity conditions. The term \( \nabla \ell(\theta) \), also called the score function, is shorthand for the \( \mathbb{R}^d \)-valued vector of partial derivatives, \( \nabla \ell(\theta) = \frac{\partial \ell(\theta)}{\partial \theta} \).

Sequential Monte Carlo (SMC) approaches have previously been developed to estimate the score function [32, 25, 6]. However, under the simulation-based setting, which does not require the ability to evaluate transition densities and their derivatives, these approaches are not applicable. As a result, [13], [8] used an artificial dynamics which does not require the ability to evaluate transition densities and their derivatives. Specifically, [30] considers a parametric model consisting of a density \( p_Y(y; \theta) \) with the log-likelihood of the data \( y^n \in \mathcal{Y} \) given by \( \ell(\theta) = \log p_Y(y^n; \theta) \). A stochastically perturbed model corresponding to a pair of random variables \((\tilde{\Theta}, \tilde{Y})\) having a joint probability density on \( \mathbb{R}^d \times \mathcal{Y} \) can be defined as \( p_{\tilde{\Theta} \tilde{Y}}(\tilde{\theta}, y; \tau, \gamma) = \tau^d \kappa \{ \tau^{-1} (\tilde{\theta} - \theta) \} p_Y(y; \tilde{\theta}) \). Suppose the following regularity conditions, identical to the assumptions of [8]:

**Assumption 1.** There exists \( C < \infty \) such that for any integer \( k \geq 1, 1 \leq i_1, \ldots, i_k \leq d \) and \( \beta_1, \ldots, \beta_k \geq 1 \),

\[
\int |u_{i_1}^{\beta_1} u_{i_2}^{\beta_2} \cdots u_{i_k}^{\beta_k}| \kappa(u) \, du \leq C ,
\]

where \( \kappa \) is a symmetric probability density on \( \mathbb{R}^d \) with respect to Lebesgue measure and \( \Sigma = (\sigma_{ij})_{i,j=1}^d \) is the non-singular covariance matrix associated to \( \kappa \).

**Assumption 2.** There exist \( \gamma, \delta, M > 0 \), such that for all \( u \in \mathbb{R}^d \), \( |u| > M \Rightarrow \kappa(u) < e^{-\gamma |u|^{\delta}} \).

**Assumption 3.** \( \ell \) is four times continuously differentiable and \( \delta \) defined as in Assumption 2. For all \( \theta \in \mathbb{R}^d \), there exists \( 0 < \eta < \delta, \varepsilon, D > 0 \), such that for all \( u \in \mathbb{R}^d \), \( \mathcal{L}(\theta + u) \leq D \varepsilon |u|^\eta \), where \( \mathcal{L} : \mathbb{R}^d \to \mathbb{R} \) is the associated likelihood function \( \mathcal{L} = \exp \ell \).

Under these regularity assumptions, [8] show that

\[
|\tau^{-2} \Sigma^{-1} \mathbb{E} (\tilde{\Theta} - \theta \mid \tilde{Y} = y^n) - \nabla \ell(\theta)| < C \tau^2 . \tag{2.1}
\]

These approximations are useful for latent variable models, where the log-likelihood of the model consists of marginalizing over a latent variable, \( X \),

\[
\ell(\theta) = \log \int p_{X,Y}(x,y^n; \theta) \, dx .
\]

In this case, the expectations in equation 2.1 can be approximated by Monte Carlo importance sampling, as proposed by [13] and [8]. In [30], the POMP model is a specific latent variable model with \( X = X_{0:N} \) and \( Y = Y_{1:N} \). A perturbed POMP model
is defined to have a similar construction to our perturbed latent variable model with \( \tilde{X} = \tilde{X}_{0,N}, \tilde{Y} = \tilde{Y}_{1,N} \) and \( \tilde{\Theta} = \tilde{\Theta}_{0,N} \). [13] perturbed the parameters by setting \( \tilde{\Theta}_{0,N} \) to be a random walk starting at \( \theta \), whereas [8] took \( \tilde{\Theta}_{0,N} \) to be independent additive white noise perturbations of \( \theta \). We take advantage of the asymptotic developments of [8] while maintaining some practical advantages of random walk perturbations for finite computations, so we use the construct \( \tilde{\Theta}_{0,N} \) as in [30] as follows.

Let \( Z_0, \ldots, Z_N \) be \( N + 1 \) independent draws from a density \( \psi \). [30] introduces \( N + 2 \) perturbation parameters, \( \tau \) and \( \tau_0, \ldots, \tau_N \), and construct a process \( \tilde{\Theta}_{0,N} \) by setting \( \tilde{\Theta}_n = \theta + \tau \sum_{i=0}^n \tau_i Z_i \) for \( 0 \leq n \leq N \). We later consider a limit where \( \tau_0 \) as fixed and the scale factor \( \tau \) decreases toward zero, and subsequently another limit where \( \tau_0 \) is fixed but \( \tau_1 \) decrease toward zero together with \( \tau \). Let \( p_{\tilde{\Theta}_{0,N}}(\tilde{\Theta}_0; \theta, \tau, \tau_0) \) be the probability density of \( \tilde{\Theta}_{0,N} \). We define the artificial random variables \( \tilde{\Theta}_{0,N} \) via their density,

\[
p_{\tilde{\Theta}_{0,N}}(\tilde{\Theta}_0; \theta, \tau, \tau_0) = (\tau \tau_0)^{-d} \psi \left\{ (\tau \tau_0)^{-1} (\tilde{\Theta}_0 - \theta) \right\} \prod_{n=1}^N (\tau \tau_n)^{-d} \psi \left\{ (\tau \tau_n)^{-1} (\tilde{\Theta}_n - \tilde{\Theta}_{n-1}) \right\}. \]

We define the stochastically perturbed model with a Markov process \( (\tilde{X}_n, \tilde{\Theta}_n), 0 \leq n \leq N \), observation process \( \tilde{Y}_{1,N} \) and parameter \( (\theta, \tau, \tau_0) \) by the factorization of their joint probability density

\[
p_{\tilde{X}_{0,N}; \tilde{Y}_{1,N}; \tilde{\Theta}_{0,N}}(x_{0,N}, y_{1,N}; \tilde{\Theta}_{0,N}; \theta, \tau, \tau_0) = p_{\tilde{\Theta}_{0,N}}(\tilde{\Theta}_0; \theta, \tau, \tau_0) p_{\tilde{\Theta}_{0,N}; \tilde{Y}_{1,N}; \tilde{\Theta}_{0,N}}(x_{0,N}, y_{1,N}; \tilde{\Theta}_{0,N}), \]

where

\[
p_{\tilde{X}_{0,N}; \tilde{Y}_{1,N}; \tilde{\Theta}_{0,N}}(x_{0,N}, y_{1,N}; \tilde{\Theta}_{0,N}; \theta, \tau, \tau_0) = \mu(x_0; \tilde{\Theta}_0) \prod_{n=1}^N f_n(x_n|x_{n-1}; \tilde{\Theta}_n) \prod_{n=1}^N q_n(y_n|x_n; \tilde{\Theta}_n). \]

This extended model can be used to define a perturbed parameter log-likelihood function, defined as

\[
\tilde{\ell}(\tilde{\Theta}_{0,N}) = \log p_{\tilde{Y}_{1,N}|\tilde{\Theta}_{0,N}}(y_{1,N}|\tilde{\Theta}_{0,N}; \theta, \tau, \tau_0). \quad (2.2)
\]

Here, the right hand side does not depend on \( \theta, \tau \) or \( \tau_0 \). We have designed (2.2) so that, setting \( \tilde{\theta}^{[N+1]} = (\theta, \theta, \ldots, \theta) \in \mathbb{R}^{d(N+1)} \), the log-likelihood of the unperturbed model can be written as \( \ell(\theta) = \tilde{\ell}(\tilde{\theta}^{[N+1]}) \). For the perturbed likelihood, we need an additional assumption of the extended version.

**Assumption 4.** \( \tilde{\ell} \) is four times continuously differentiable. For all \( \theta \in \mathbb{R}^d \), there exist \( \varepsilon > 0, D > 0 \) and \( \delta \) defined as in Assumption 2, such that for all \( 0 < \eta < \delta \) and \( u_{0,N} \in \mathbb{R}^{d(N+1)} \), \( L(\tilde{\theta}^{[N+1]} + u_{0,N}) \leq De^\delta \sum_{n=1}^{N+1} |u_n|^{\eta} \), where \( L(\tilde{\theta}_{0,N}) = \exp\{\tilde{\ell}(\tilde{\Theta}_{0,N})\} \) is the perturbed likelihood.
Let \( \bar{\mathbb{E}}_{\theta, \tau, \tau_0}, \bar{\text{Cov}}_{\theta, \tau, \tau_0}, \bar{\text{Var}}_{\theta, \tau, \tau_0} \) denote the expectation, covariance and variance with respect to the associated posterior, \( p_{\bar{\Theta}_0|Y_{1:N}}(\bar{\Theta}_0; \theta, \tau, \tau_0) \). By using \( \bar{\mathbb{E}}, \bar{\text{Cov}}, \bar{\text{Var}} \) instead of \( \mathbb{E}_{\theta, \tau, \tau_0}, \text{Cov}_{\theta, \tau, \tau_0}, \text{Var}_{\theta, \tau, \tau_0} \) respectively, a theorem similar to theorem 4 of [8] but for random walk noise instead of independent white noise is derived.

**Theorem 1.** ([Theorem 2 of [30]]) Suppose Assumptions 1, 2 and 4, there exists a constant \( C \) independent of \( \tau, \tau_1, ..., \tau_N \) such that,

\[
|\nabla \ell(\theta) - \tau^{-2} \Psi^{-1} \{ \tau_0^{-2} \bar{\mathbb{E}} (\bar{\Theta}_0 - \theta)|_{Y_{1:N} = y^{*}_{1:N}} \}| < C \tau^2,
\]

where \( \Psi \) is the non-singular covariance matrix associated to \( \psi \).

Theorem 1 formally allows an approximation of \( \nabla \ell(\theta) \). [30] also presents an alternative variations on these results which lead to more stable Monte Carlo estimation.

**Theorem 2.** ([Theorem 3 of [30]]) Suppose Assumption 1, 2 and 4 hold. In addition, assume that \( \tau_n = O(\tau^2) \) for all \( n = 1, ..., N \), the following holds true,

\[
\left| \nabla \ell(\theta) - \frac{1}{N+1} \tau^{-2} \tau_0^{-2} \Psi^{-1} \sum_{n=0}^{N} \left\{ \bar{\mathbb{E}} (\bar{\Theta}_n - \theta)|_{Y_{1:N} = y^{*}_{1:N}} \right\} \right| = O(\tau^3).
\]

These theorems are useful for our approaches because we can approximate the gradient of the log-likelihood of the extended model to the second order of \( \tau \) which we will later show that it fits well with our accelerate simulation based setup.

### 3. Proposed accelerate iterated filtering

Our motivation comes from the accelerated gradient method for smooth non-linear stochastic programming literature. By using an approximation of the score function, it is possible to use an accelerated gradient method as in Nesterov acceleration scheme in optimization literature. One issue with the accelerated gradient approach is that it is not clear how the technique can be used in situations where both the likelihood and the gradient are intractable. These sorts of examples are common in scientific applications of state space models where the state process is a diffusion process or an ordinary differential equation (ODE) with stochastic coefficients. However, in these family of iterated filtering approaches, the score function can be approximated with noise under control without affecting the convergence rate. Specifically, applying an accelerated inexact gradient algorithm in the iterated filtering approach can obtain an optimal rate of convergence.

In this paper, \( \varepsilon_k \) denotes the error in the approximation of the gradient. Using the same notation as [11], denote the sequences of magnitudes of the errors in the gradient approximations \( \{||\varepsilon_k||\} \). Suppose the following assumptions:

**Assumption 5.** The function \( \ell : \Theta \rightarrow \mathbb{R} \) is differentiable, bounded from above and has a \( L \)-Lipschitz-continuous gradient, i.e. for all \( \theta, \vartheta \in \Theta \), \( ||\nabla \ell(\theta) - \nabla \ell(\vartheta)|| \leq L ||\theta - \vartheta|| \), where \( \nabla \ell \) denotes the gradient of \( \ell \). The function \( \ell \) attains its maximum at a certain \( \theta^* \in \Theta \).
In the sequel, $\Theta$ denotes a finite-dimensional Euclidean space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. It can be shown that (e.g. in [27]) Assumption 5 is equivalent to

$$\left|\ell(\vartheta) - \ell(\theta) - \langle \nabla \ell(\theta), \vartheta - \theta \rangle \right| \leq \frac{L}{2} \|\vartheta - \theta\|^2, \forall \theta, \vartheta \in \Theta \quad (3.1)$$

It is well-known that the gradient descent method converges for a general non-convex optimization problem but it does not achieve the optimal rate of convergence, in terms of the functional optimality gap, when $\ell(\cdot)$ is convex [11]. In contrast, the accelerated gradient method in [28] is optimal for solving convex optimization problems, but does not necessarily converge for solving nonconvex optimization problems. [11] proposed a modified accelerated gradient method which can converge in both convex and non-convex optimization problem. However, they assumed unbiased estimation of the gradient which is not satisfied for most simulation-based inferences. Below, we extend the approach of Ghadimi to an accelerated inexact gradient (AIG) method in the context of accelerate iterated filtering. That is, we allow bias in gradient approximation by properly specifying the stepsize policy. We prove that it not only achieves the same optimal rate of convergence for both convex and non-convex optimizations, but also exhibits the best-known rate of convergence for simulation-based inference problems.

Algorithm 1 Accelerate Inexact Gradient (AIG)

**Input:**

$\theta_0 \in \Theta$.

$\{\beta_k > 0\}, \{\lambda_k > 0\}$ for any $k \geq 2$.

$\{\alpha_k\} \in (0, 1)$ for $k > 1$ and $\alpha_1 = 1$.

1: $\theta_{ag}^0 = \theta_0$. \hspace{0.5cm} \triangleright$Initialize$

2: \text{for } k \in 1...N \text{ do}$

3: $\theta_{md}^k = (1 - \alpha_k)\theta_{ag}^{k-1} + \alpha_k \theta_{k-1}$ \hspace{0.5cm} (3.2)

4: $\theta_k = \theta_{k-1} - \lambda_k \left(\nabla \ell(\theta_{md}^k)\right)$ \hspace{0.5cm} (3.3)

5: $\theta_{ag}^k = \theta_{ag}^{k-1} - \beta_k \left(\nabla \ell(\theta_{ag}^k)\right)$ \hspace{0.5cm} (3.4)

$\triangleright$ where $\nabla \ell(\theta_{ag}^k)$ is an estimation of $\nabla \ell(\theta_{md}^k)$ with error $\varepsilon_k$.

6: $\text{end for}$

In addition to Assumption 5, we assume a noise control condition for Algorithm 1.

**Assumption 6.** $\Theta$ is bounded. There exists an $A < \infty$ such that $\sum_{k=1}^{N} \lambda_k \|\varepsilon_k\| < A$.

Given some mild conditions often satisfied by controlling the artificial noises, we have the following result.

**Theorem 3.** (Extension of Theorem 1 of [11]).

Suppose Assumptions 5 and 6 hold. In addition, let $\{\theta_k, \theta_{ag}^k\}$ $k \geq 1$ be computed by Algorithm 1.
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a) If sequences \( \{ \alpha_k \} \), \( \{ \beta_k \} \), \( \{ \lambda_k \} \) and \( \{ \Gamma_k \} \) satisfy

\[
\Gamma_k := \begin{cases} 
1 & k = 1 \\
(1 - \alpha_k)\Gamma_{k-1} & k \geq 2
\end{cases},
\]  
(3.5)

\[
C_k := 1 - L\lambda_k - \frac{L(\lambda_k - \beta_k)^2}{2\lambda_k\alpha_k\Gamma_k} \left( \sum_{\tau=k}^{N} \frac{1}{\Gamma_{\tau}} \right) > 0, \text{ for } 1 \leq k \leq N,
\]  
(3.6)

then for any \( N \geq 1 \), we have for some \( B < \infty \),

\[
\min_{k=1, \ldots, N} \left\| \nabla \ell(\theta_{md}^k) + \varepsilon_k \right\|^2 \leq \frac{\ell(\theta_b) - \ell^* + B}{\sum_{k=1}^{N} \lambda_k C_k},
\]  
(3.7)

b) Suppose that \( \ell(\cdot) \) is convex. If sequences \( \{ \alpha_k \} \), \( \{ \beta_k \} \), \( \{ \lambda_k \} \) and \( \{ \Gamma_k \} \) satisfy

\[
\alpha_k \lambda_k \leq \beta_k < \frac{1}{L},
\]  
(3.8)

\[
\frac{\alpha_1}{\lambda_1\Gamma_1} \geq \frac{\alpha_2}{\lambda_2\Gamma_2} \geq \ldots,
\]  
(3.9)

then for any \( N \geq 1 \), we have

\[
\min_{k=1, \ldots, N} \left\| \nabla \ell(\theta_{md}^k) + \varepsilon_k \right\|^2 \leq 2 \frac{\| \theta^* - \theta_b \|^2}{\lambda_1} + \frac{\sum_{k=1}^{N} \lambda_k^{-1}}{\Gamma_k^{-1}} \left[ \beta_k \| \varepsilon_k \| \left\| \nabla \ell(\theta_{md}^k) + \varepsilon_k \right\| + \alpha_k \| \varepsilon_k \| \| \theta_{k-1} - \theta_b \| \right],
\]  
(3.10)

\[
\ell(\theta_{md}^N) - \ell(\theta^*) \leq \Gamma_N \left[ \frac{\| \theta_0 - \theta^* \|^2}{\lambda_1} + \frac{\sum_{k=1}^{N} \lambda_k^{-1}}{\Gamma_k^{-1}} \left[ \beta_k \| \varepsilon_k \| \left\| \nabla \ell(\theta_{md}^k) + \varepsilon_k \right\| + \alpha_k \| \varepsilon_k \| \| \theta_{k-1} - \theta_b \| \right] \right].
\]  
(3.11)

There are various options for selecting \( \{ \alpha_k \} \), \( \{ \beta_k \} \), \( \{ \lambda_k \} \), \( \{ \Gamma_k \} \). By controlling error \( \varepsilon_k \), we can provide some of these selections below which guarantee the optimal convergence rate of the AIG algorithm for both convex and nonconvex problems.

**Theorem 4.** Suppose Assumptions 5 and 6 hold. In addition, suppose that \( \{ \beta_k \} \) in the accelerated gradient method are set to \( \beta_k = \frac{1}{2L} \).

a) If sequences \( \{ \alpha_k \} \) and \( \{ \lambda_k \} \) satisfy

\[
\lambda_k \in \left[ \beta_k, (1 + \frac{1}{k})\beta_k \right], \text{ for } \forall k \geq 1,
\]  
(3.12)

then for any \( N \geq 1 \), we have

\[
\min_{k=1, \ldots, N} \left\| \nabla \ell(\theta_{md}^k) + \varepsilon_k \right\|^2 \leq O \left( \frac{1}{N} \right).
\]  
(3.13)
Suppose that $\epsilon_k = O\left(\frac{1}{N^2}\right)$, then the AIG method can find a solution $\hat{\theta}$ such that $\|\nabla \ell(\hat{\theta})\|^2 \leq \epsilon$ in at most $O(1/\epsilon^2)$ iterations.

b) Suppose that $\ell(\cdot)$ is convex and $\epsilon_k = O\left(\frac{1}{k^{1+\delta}}\right)$ for some $\delta_1 > 0$. If $\{\lambda_k\} \text{ satisfies}$

$$\lambda_k = \left(k^{1+\delta} - (k-1)^{1+\delta}\right) \forall k \geq 1,$$

then for any $N \geq 1$, we have

$$\min_{k=1,\ldots,N} \left\|\nabla \ell(\theta_{k}^{\text{opt}}) + \epsilon_k\right\|^2 \leq O\left(\frac{1}{N^{2+\delta}}\right),$$

and

$$\ell(\theta_{k}^{\text{opt}}) - \ell(\theta^*) \leq O\left(\frac{1}{N^{1+\delta}}\right),$$

then the AIG method can find a solution $\hat{\theta}$ such that $\|\nabla \ell(\hat{\theta})\|^2 \leq \epsilon$ in at most $O\left(\frac{1}{\epsilon^{\frac{1}{1+\delta}}}\right)$ at most.

**Algorithm 2 Accelerate Iterated Filtering (AIF)**

**Input:**
- Starting parameter, $\theta_0 = \theta_0^{\text{MLE}}$, sequences, $\alpha_n, \beta_n, \lambda_n, \Gamma_n$
- Simulator for $f_{X|Y}(x|\theta)$, $f_{X|X_{n-1},x_n}(x_n|x_{n-1}|\theta)$, evaluator for $f_{Y|X}(y|\theta)$
- Data, $Y^\ast_n$, labels designating IVPs, $I \subset \{1,\ldots,p\}$, initial scale multiplier, $C > 0$
- Number of particles, $J$, number of iterations, $M$, cooling rate, $0 < a < 1$, perturbation scales, $\sigma_{i,p}$

**Output:**
- Maximum likelihood estimate $\hat{\theta}_{\text{MLE}}$

1: $\theta_{0}^{\text{opt}} = \theta_0$ \hfill $\triangleright$ Initialize
2: $[\theta_{0,j}]_i \sim N([\theta_{0,j}],(C\alpha^{-1}\sigma_i)^2)$ for $i$ in $1..p$, $j$ in $1..J$. \hfill $\triangleright$ Initialize filter means for parameters
3: simulate $X_{i,j}^{F} \sim f_{X_0}(\cdot;\theta_{0,j})$ for $j$ in $1..J$. \hfill $\triangleright$ Initialize states
4: for $n$ in $1..M$ do
5: $\theta_{n}^{\text{opt}} = (1 - \alpha_n)\theta_{n-1}^{\text{opt}} + \alpha_n \theta_{n-1}$. \hfill $\triangleright$ Initialize
6: for $n$ in $1..N$ do
7: $[\theta_{n}^{F}]_i \sim \mathcal{N}([\theta_{n}^{F}],(e^{-\alpha_n}\sigma_i)^2)$ for $i$ in $1:J$. \hfill $\triangleright$ Perturb
8: $X_{n,j}^{F} \sim f_{X_n}(\cdot;\theta_{n,j}^{F},\theta_{n}^{F})$ for $j$ in $1:J$. \hfill $\triangleright$ Simulate prediction particles
9: $w(n,j) = g_{n}(y_{n,j}|X_{n,j}^{F},\theta_{n}^{F})$ for $j$ in $1:J$. \hfill $\triangleright$ Evaluate weights
10: $\tilde{w}(n,j) = w(n,j)/\sum_{j=1}^{J} w(n,j)$. \hfill $\triangleright$ Normalize weights
11: $k_{i,j}$ with $P(k_{i,j} = j) = \tilde{w}(n,j)$. \hfill $\triangleright$ Apply systematic resampling to select indices
12: $X_{n,j}^{P} = X_{n,j}^{F}$ and $\theta_{n,j}^{P} = \theta_{n,j}^{F}$ for $j$ in $1:J$. \hfill $\triangleright$ Resample particles
13: end for
14: $S_m = e^{-2(m-1)\gamma^2} \sum_{m=1}^{M} \left(\hat{\theta}_m - \theta_{m}^{\text{opt}}\right)$ \hfill $\triangleright$ Update Parameters
15: $[\theta_{m}]_i = \theta_{m-1} - \lambda_m [S_m]_i$ for $i$ in $1:J$. \hfill $\triangleright$ Update Parameters
16: $[\theta_{m}^r]_i = \theta_{m}^{opt} - \beta_m [S_m]_i$ for $i$ in $1:J$. \hfill $\triangleright$ Update Parameters
17: $[\theta_{m}]_i = \frac{1}{J} \sum_{j=1}^{J} [\theta_{m,j}]_i$ for $i$ in $1:J$. \hfill $\triangleright$ Update Parameters
18: end for

We now add a few remarks about the extension results obtained in Theorem 4. First, if the problem is convex, by choosing more aggressive stepsizes $\{\lambda_k\}$ in (3.14), the AIG method exhibits the optimal rate of convergence in (3.16). It is also worth noting that with such a selection of $\{\lambda_k\}$, the AIG method can find a solution $\hat{\theta}$ such that
\[ \| \nabla \ell(\tilde{\theta}) \| \leq \varepsilon \] in at most \( O(1/\varepsilon^{1/2+\delta}) \) iterations. The latter result has been shown by [27], [11] but only for the accelerate unbiased gradient method. Second, observe that \( \{\lambda_k\} \) in (3.12) for general nonconvex problems is in the order of \( O(1/L) \), while the one in (3.14) for convex problems are more aggressive (in \( O(k/L) \)). The value \( \delta \) is optimal at 1 for convergence rate. However, it may not be optimal for computation of controlling the noises. Finally, we show that we can apply the stepsize policy in (3.12) for solving general inexact gradient problems for both convex and nonconvex optimization. The sequential Monte Carlo filter can be arbitrarily approximated to the exact filter by choosing sufficiently large number of particles [13]. It can be seen that we can choose the perturbation sequence so that the gradient noise satisfies condition in Theorem 4. For completeness, we present the pseudo code of the proposed algorithm as in Algorithm 2.

4. Numerical examples

To measure the performance of the new inference algorithm, we evaluate our accelerate iterated filtering on some benchmark examples and compare it to the existing simulation-based approaches. We make use of well tested and maintained code of R [33] packages such as pomp [17]. Specifically, models are coded using C snippet declarations [17]. New algorithm is written in R package is2, which provides user friendly interfaces in R and efficient matrix operations in the highly optimized Rcpp [9]. All the simulation-based approaches mentioned above use sequential Monte Carlo algorithm (SMC), implemented using bootstrap filter. Experiments were carried out on a cluster of 32 cores Intel Xeon E5-2680 2.7 Ghz with 256 GB memory. For a fair comparison, we try to use the same setup and assessment for every inference method. A public Github repository containing scripts for reproducing our results may be found at https://github.com/nxdao2000/AIFcomparisons.

4.1. Toy example: A linear, Gaussian model

In this subsection, we compare our accelerate iterated filtering algorithm to the original iterated filtering algorithm IF1 [14], Bayes map iterated filtering (IF2) [15] and the second-order iterated smoothing (IS2) [30]. It has been shown in [30] and [15] that the second-order iterated smoothing with white noise (IS1) [8] and particle Markov chain Monte Carlo (PMCMC) [1] do not perform as well as Bayes map iterated filtering so we leave them out. For a computationally convenient setting, simple models provide an opportunity to test the basic features of inference algorithms. Therefore, we first consider a bivariate discrete time Gaussian autoregressive process, a relatively simple mechanistic model. This model is chosen so that the Monte Carlo calculations can be verified using a Kalman filter. For this example, there are some alternatives to iterated filtering class. For example, EM and MCMC algorithms would be practical in this case although they do not scale well to large dynamic models, so we do not include them here. The model is given by the state space forms: \( X_n | X_{n-1} = x_{n-1} \sim \mathcal{N}(\alpha x_{n-1}, \sigma \sigma^T) \), \( Y_n | X_n = x_n \sim \mathcal{N}(x_n, I_2) \) where \( \alpha, \sigma \) are \( 2 \times 2 \) matrices and \( I_2 \) is \( 2 \times 2 \) identity matrix.
The data are simulated from the following parameters:

$$\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.5 \\ 0.3 & 0.9 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 3 & 0 \\ -0.5 & 2 \end{bmatrix}.$$  

The number of time points \(N\) is set to 100 and initial starting point \(X_0 = (-3, 4)\). For each method mentioned above, we estimate parameters \(\alpha_2\) and \(\alpha_3\) for this model using \(J = 1000\) particles and run our estimation for \(M = 25\) iterations. We start the initial search uniformly on a large rectangular region \([-1, 1] \times [-1, 1]\). As can be seen from Fig. 1, all of the distributions of estimated maximized log likelihoods touch the true MLE (computed from Kalman filter) at the vertical broken line, implying that they all successfully converged. The results show that AIF is the most efficient method of all because using AIF the results have higher mean and smaller variance compared to other approaches, indicating a higher empirical convergence rate. Algorithmically, AIF has similar computational costs with the first order approaches IF1, IF2, and is cheaper than the second order approach IS2. Indeed, average computational time of twenty independent runs of each approach is given in Table 1. Additional overheads for estimating score make the computation time of AIF a bit larger compared to computational time of IF2. However, with complex models and large enough number of particles, these overheads become negligible and computational time of AIF will be similar to other first order approaches. The fact that it has the convergence rate of second order with computation complexity of first-order shows that it is a very promising algorithm. In addition, the results also imply that AIF is robust to initial starting guesses.

To see how the final MLEs clustered around the true MLE, we only show 40 Monte Carlo replications for this toy example. As can be observed from Fig. 2, most of the replications clustered near the true MLE for AIF approach, while none of them stays in a lower likelihood region. It can be interpreted as a statistical summary of Fig. 2, with 200 Monte Carlo replications. These results indicate that AIF is clearly the best of the investigated methods for this test compared to others. Given additional computational resources, we also checked how the results of each method compared. Specifically, we set \(M = 100\) iterations and \(J = 10000\) particles, with the random walk standard deviation decreasing geometrically from 0.02 down to 0.0018 for each method. In this situation, we confirm that AIF is the best among other IF1, IF2 and IS2. All methods have comparable computational demands for given \(M\) and \(J\).
4.2. Malaria benchmark

Many real world dynamic systems are highly nonlinear, partially observed and even weakly identifiable. To demonstrate the capabilities of accelerate iterated filtering for such situations, we apply it to evaluate the likelihood in a stochastic differential equation for vivax malaria model of Roy et al. [34]. The reason to choose this challenging model is that it provides a rigorous performance benchmark for our verification. The model $SEIH^3QS$ we consider splits up the study population of size $P(t)$ into seven classes: susceptible individuals, $S(t)$, exposure $E(t)$, infected individuals, $I(t)$, dormant classes $H_1(t), H_2(t), H_3(t)$ and recovered individuals, $Q(t)$. This strain of malaria characterized by relapse following initial recovery from symptoms [30]. Therefore the last $S$ in the model name indicates the possibility that a recovered person can return to the class of susceptible individuals. The data, denoted by $y_{1:N}^t$, are in the form of monthly time series over a 20-year period, counting the malaria morbidity. $\delta$ denotes the mortality rate, $\kappa(t)$ a delay stage, $\mu_{SE}(t)$ the current force of infection, and $\tau_D$ the mean latency time. The state process is

$$X(t) = (S(t), E(t), I(t), Q(t), H_1(t), H_2(t), H_3(t), \kappa(t), \mu_{SE}(t)),$$

where transition rates from stage $H_1$ to $H_2$, $H_2$ to $H_3$ and $H_3$ to $Q$ are specified to be $3\mu_{HI}$ while infected population to dormancy transition rate is $\mu_{IH}$. The model satisfies
FIG 2. Comparison of different estimators. The likelihood surface for the linear model, with the location of the MLE is marked with a green cross. The crosses show final points from 40 Monte Carlo replications of the estimators: (A) Original iterated filtering method; (B) Bayes map iterated filtering method; (C) Accelerate iterated filtering method; (D) Second-order iterated filtering method. Each method was started uniformly over the rectangle shown, with $M = 25$ iterations, $N = 1000$ particles, and a random walk standard deviation decreasing from 0.02 geometrically to 0.011 for both $\alpha_2$ and $\alpha_3$. 
the following stochastic differential equation system
\[
\begin{align*}
\frac{dS}{dt} &= \delta P + dP/dt + \mu IS + \mu QS Q \\
&\quad + a\mu HI + b\mu EI - \mu SE(t)S - \delta S, \\
\frac{dE}{dt} &= \mu SE(t)S - \mu EI - \delta E, \\
\frac{dI}{dt} &= (1-b)\mu EI + 3\mu HI - (\mu HI + \mu IS + \mu IQ)I - \delta I, \\
\frac{dH_i}{dt} &= (1-a)\mu HI - n\mu HI H_i - \delta H_i, \quad \text{for } i \in \{2, 3\}, \\
\frac{dQ}{dt} &= \mu IQ - \mu QS - \delta Q.
\end{align*}
\]
In addition, the malaria pathogen reproduction within the mosquito vector is given by
\[
\begin{align*}
\frac{d\kappa}{dt} &= [\lambda(t) - \kappa(t)]/\tau_D, \\
\frac{d\mu SE}{dt} &= [\kappa(t) - \mu SE(t)]/\tau_D,
\end{align*}
\]
where \( \lambda(t) \) is the latent force of infection and \( \lambda(t), \kappa(t) \) and \( \mu SE(t) \) satisfies
\[
\mu SE(t) = \int_{-\infty}^{t} \gamma(t-s)\lambda(s)ds, \tag{4.1}
\]
with \( \gamma(s) = \frac{(2/\tau_D)^2\gamma^2-1}{(2-1)!} \exp(-2s/\tau_D) \), a gamma distribution with shape parameter 2.
Since the latent force of infection is constrained by rainfall covariate \( R(t) \) and some Gamma white noise, from Roy et al. [34] we have:
\[
\lambda(t) = \left(1 + \frac{qQ}{P}\right) \times \exp\left\{ \sum_{i=1}^{N_{t}} b_i S_i(t) + b_i R(t) \right\} \times \left[ \frac{d\Gamma(t)}{dt} \right].
\]
In this equation, \( q \) denotes a reduced infection risk from humans in the Q class and \( \{s_i(t), i = 1, \ldots, N_s\} \) is a periodic cubic B-spline basis, with \( N_s = 6 \). The observation model for \( Y_n \) is a negative binomial distribution with mean \( M_n \) and variance \( M_n + M_n^2\sigma^2_{obs} \), where \( M_n = \rho \int_{t_{n-1}}^{t_n} [\mu_{EI}(s) + 3\mu_{HI}(s)] ds \) is the number of new cases observed from time \( t_{n-1} \) to time \( t_n \) and \( \rho \) the mean age. The coupled system of stochastic differential equations is solved using an Euler-Maruyama scheme [18] with a time step of 1/20 month in our case. Given the data obtained from National Institutes of Malaria Research [34], we carried out simulation-based inference via the original iterated filtering (IF1), the perturbed Bayes map iterated filtering (IF2), the second order iterated smoothing (IS2), and the new accelerate iterated filtering (AIF). The inference goal used to assess all of these methods is to find high likelihood parameter values starting from randomly drawn values in a large hyperrectangle. In the presence of possible multi-modality, weak identifiability, and considerable Monte Carlo error of this model, we start 200 random searches. The random walk standard deviation is initially set to 0.1 for estimated parameters while the cooling rate \( c \) is set to 0.1^{0.02} \approx 0.95. These corresponding quantities for initial value parameters are 2 and 0.1^{0.02}, respectively, but they are applied only at time zero. We run our experiment on a cluster computers with \( M = 50 \) iterations and with \( J = 1000 \) particles. The reason to choose these values for this model is that increasing the iterations to 100 and the number of particles to 10000 does not improve the results much but it takes significant longer time. Figure 4 shows the distribution of the MLEs estimated by IF1, IF2, IS2 and AIF. All distributions touch the global maximum as expected and the higher mean and smaller variance of IF2, AIF estimation clearly demonstrate that they are considerably more effective than IF1. Note that the computational times for IF1, IF2, IS2 and AIF are 44.86, 43.92, 53.10 and 52.25 minutes respectively, confirming that accelerate iterated filtering has
essentially the same computational cost as first order methods IF1, IF2 and is cheaper a bit than IS2, for a given Monte Carlo sample size and number of iterations. In this hard problem, while IF1 reveal their limitations, we have shown that IF2 and AIF can still offer a substantial improvement. A natural heuristic idea to further improve the method is hybridizing IF2 and AIF but we leave it for the future work.

5. Conclusion

In this paper, we have proposed a novel class of iterated filtering theory using an accelerated inexact gradient approach. We have shown that choosing perturbation sequence and number of particles carefully results in an algorithm which has led to many advances including the statistical and computational efficiency. This is also very fruitful as it is extendable to a more generalized class of algorithm, based on proximal theory. Previous proof of iterated filtering class require some difficult conditions, which is not easily verifiable. However, in this article, we use only general standard gradient conditions. We are going further down the road of a more systematic approach which could be easily generalized to the state of the art algorithm in the optimization literatures. The convergence rate is also explicitly stated and it is better than standard theory. From a theoretical point of view, it could be an interesting perspective and insight.

In addition, from practical point of view, we have provided an efficient framework, applicable to a general class of nonlinear, non-Gaussian non-standard POMP models, especially suitable in the control feedback system. There are a lot of such systems, which are not well-treated by current available modeling framework. We simultaneously present the performance of our open source software package is2 to facilitate the needs of the community. The performance of this new approaches surpass the other frameworks by a large margin of magnitude.

It may be surprising that this simple accelerated inexact gradient approach has the needed convergence properties, and can easily be generalized, at least in some asymptotic sense. It is not hard to show that the accelerated inexact proximal gradient iterated filtering theory can be adapted to apply with iterated smoothing and with either independent white noise or random walk perturbations while our empirical results still show strong evidences of the improvements. In principle, different simulation-based inference methods can readily be hybridized to build on the strongest features of multiple algorithms. Our results could also be applied to develop other simulation-based methodologies which can take advantage of proximal map. For example, it may be possible to use our approach to help design efficient proposal distributions for particle Markov chain Monte Carlo algorithms. The theoretical and algorithmic innovations of this paper will help to build a new direction for future developments on this frontier. Applying this approach to methodologies like Approximate Bayesian Computation (ABC), Liu-West Particle Filter (LW-PF), Particle Markov chain Monte Carlo (PM-CMC), with different samplers scheme, e.g. forward backward particle filter, forward smoothing or forward backward smoothing are foreseeable extensions.
Appendix A: Proofs

We first need a simple technical result (see Lemma 1 of [11]). The proof is the same as that of Lemma 1 of [11] but we provide it here for completeness.

**Lemma 1.** *(Lemma 1 of [11].)*

Assume sequences \( \{\alpha_k\} \in (0, 1) \) for \( k > 1 \) and \( \alpha_1 = 1 \) and sequences \( \{a_k\}, \{\eta_k\} \)

satisfy

\[
a_k \leq (1 - \alpha_k)a_{k-1} + \eta_k, \; k = 1, 2, \ldots
\]

(A.1)

If we define a positive sequence \( \{\Gamma_k\} \) as in 3.5 then for any \( k \geq 1 \), we have

\[
a_k \leq \Gamma_k \sum_{i=1}^{k} \left( \eta_i / \Gamma_i \right).
\]

**Proof.** Since \( \alpha_1 = 1 \) and \( \Gamma_1 = 1 \), from 3.5 we have

\[
a_1 \leq \eta_1
\]

or

\[
a_1 \leq \frac{\eta_1}{\Gamma_1}.
\]

Since \( \Gamma_k > 0 \) for every \( k > 1 \), dividing both sides of A.1 by \( \Gamma_k \),

\[
\frac{a_k}{\Gamma_k} \leq \frac{(1 - \alpha_k)a_{k-1} + \eta_k}{\Gamma_k} = \frac{a_{k-1}}{\Gamma_{k-1}} + \frac{\eta_k}{\Gamma_k}, \; \forall k \geq 2.
\]

Summing up the above inequalities and rearranging the terms, the conclusion follows. \( \square \)

**Lemma 2.**

\[
\sum_{\tau=1}^{k} \frac{\alpha_{\tau}}{\Gamma_\tau} = \frac{1}{\Gamma_k}.
\]

(A.2)

**Proof.** We have

\[
\sum_{\tau=1}^{k} \frac{\alpha_{\tau}}{\Gamma_\tau} = \frac{\alpha_1}{\Gamma_1} + \sum_{\tau=2}^{k} \frac{1}{\Gamma_\tau} (1 - (1 - \alpha_{\tau}))
\]

\[
= \frac{1}{\Gamma_1} + \sum_{\tau=2}^{k} \left( \frac{1}{\Gamma_\tau} - \frac{1}{\Gamma_{\tau-1}} \right) = \frac{1}{\Gamma_k}.
\]

\( \square \)

### A.1. Proof of Theorem 3

**Proof.** The proof follows closely to the proof of theorem 1 of [11] except we consider bias estimate of the gradient. We first prove part a.
By 3.1 and 3.3, we have

\[
\ell(\theta_k) \leq \ell(\theta_{k-1}) + \langle \nabla \ell(\theta_{k-1}), \theta_k - \theta_{k-1} \rangle + \frac{L}{2} \|\theta_k - \theta_{k-1}\|^2 \\
= \ell(\theta_{k-1}) + \left( \langle \nabla \ell(\theta_{k-1}) - \nabla \ell(\theta_k^{md}), \omega_k \rangle + \langle \nabla \ell(\theta_k^{md}), \varepsilon_k \rangle, -\lambda_k \langle \nabla \ell(\theta_k^{md}), \varepsilon_k \rangle \right) \\
+ \frac{L\lambda_k^2}{2} \|\nabla \ell(\theta_k^{md}) + \varepsilon_k\|^2
\]

\[
= \ell(\theta_{k-1}) - \lambda_k \left(1 - \frac{L\lambda_k}{2}\right) \|\nabla \ell(\theta_k^{md}) + \varepsilon_k\|^2 \\
+ L(1 - \alpha_k) \lambda_k \|\nabla \ell(\theta_k^{md}) + \varepsilon_k\| \cdot \|\theta_k - \theta_{k-1}\| + \lambda_k \|\varepsilon_k\| \cdot \|\nabla \ell(\theta_k^{md}) + \varepsilon_k\|
\]

\[
\leq \ell(\theta_{k-1}) - \lambda_k \left(1 - \frac{L\lambda_k}{2}\right) \|\nabla \ell(\theta_k^{md}) + \varepsilon_k\|^2 \\
+ \frac{L\lambda_k^2}{2} \|\nabla \ell(\theta_k^{md}) + \varepsilon_k\|^2 + \frac{L(1 - \alpha_k)^2}{2} \|\theta_k - \theta_{k-1}\|^2 + \lambda_k \|\varepsilon_k\| \cdot \|\nabla \ell(\theta_k^{md}) + \varepsilon_k\|^2 \\
= \ell(\theta_{k-1}) - \lambda_k \left(1 - \frac{L\lambda_k}{2}\right) \|\nabla \ell(\theta_k^{md}) + \varepsilon_k\|^2 \\
+ \frac{L(1 - \alpha_k)^2}{2} \|\theta_k - \theta_{k-1}\|^2 + \lambda_k \|\varepsilon_k\| \cdot \|\nabla \ell(\theta_k^{md}) + \varepsilon_k\|^2 \tag{A.3}
\]

The second inequality is from triangular inequality and the Cauchy-Schwarz inequality while the second inequality is due to the Lipschitz of gradient assumption (1.2) and last equality comes from 3.2. We have the last inequality follows from $ab \leq (a^2 + b^2)/2$. From 3.2, 3.3, and 3.4, it follows that

\[
\theta_k^{ag} - \theta_k = (1 - \alpha_k)\theta_k^{ag} + \alpha_k \theta_{k-1} - \beta_k \left( \nabla \ell(\theta_k^{md}) + \varepsilon_k \right) - \left( \theta_{k-1} - \lambda_k \left( \nabla \ell(\theta_k^{md}) + \varepsilon_k \right) \right)
\]

\[
= (1 - \alpha_k)(\theta_k^{ag} - \theta_{k-1}) + (\lambda_k - \beta_k) \left( \nabla \ell(\theta_k^{md}) + \varepsilon_k \right).
\]

Applying Lemma 1 where $\theta_k^{ag} - \theta_k := a_k$ and $\eta_k := (\lambda_k - \beta_k) \left( \nabla \ell(\theta_k^{md}) + \varepsilon_k \right)$, we obtain

\[
\theta_k^{ag} - \theta_k = \Gamma_k \sum_{\tau=1}^{k} \left( \frac{\lambda_{\tau} - \beta_{\tau}}{\Gamma_{\tau}} \right) \left( \nabla \ell(\theta^m_{\tau}) + \varepsilon_{\tau} \right).
\]
Replacing the above bound in A.3, and the fact that \( \Gamma \) is in 3.6 and summing up the above inequalities, we have

\[
\ell(\theta_k) \leq \ell(\theta_{k-1}) - \lambda_k (1 - L \lambda_k) \left\| \nabla \ell(\theta_k^m) + \varepsilon_k \right\|^2
\]

\[
+ \frac{L \Gamma_{k-1}(1 - \alpha_k)^2}{2} \sum_{t=1}^{k-1} \frac{(\lambda_t - \beta_t)^2}{\Gamma_t \alpha_t} \left\| \nabla \ell(\theta_t^m) + \varepsilon_t \right\|^2 + \lambda_k \| \varepsilon_k \| \cdot \left\| \nabla \ell(\theta_k^m) + \varepsilon_k \right\|
\]

\[
\leq \ell(\theta_{k-1}) - \lambda_k (1 - L \lambda_k) \left\| \nabla \ell(\theta_k^m) + \varepsilon_k \right\|^2
\]

\[
+ \frac{L \Gamma_k}{2} \sum_{t=1}^{k} \frac{(\lambda_t - \beta_t)^2}{\Gamma_t \alpha_t} \left\| \nabla \ell(\theta_t^m) + \varepsilon_t \right\|^2 + \lambda_k \| \varepsilon_k \| \cdot \left\| \nabla \ell(\theta_k^m) + \varepsilon_k \right\| \quad (A.5)
\]

for every \( k \geq 1 \). Using the definition of \( C_k \) in 3.6 and summing up the above inequalities, we have

\[
\ell(\theta_N) \leq \ell(\theta_0) - \sum_{k=1}^{N} \lambda_k (1 - L \lambda_k) \left\| \nabla \ell(\theta_k^m) + \varepsilon_k \right\|^2
\]

\[
+ \frac{L}{2} \sum_{k=1}^{N} \Gamma_k \sum_{t=1}^{k} \frac{(\lambda_t - \beta_t)^2}{\Gamma_t \alpha_t} \left\| \nabla \ell(\theta_t^m) + \varepsilon_t \right\|^2 + \sum_{k=1}^{N} \lambda_k \| \varepsilon_k \| \cdot \left\| \nabla \ell(\theta_k^m) + \varepsilon_k \right\|
\]

\[
= \ell(\theta_0) - \sum_{k=1}^{N} \lambda_k C_k \left\| \nabla \ell(\theta_k^m) + \varepsilon_k \right\|^2 + \sum_{k=1}^{N} \lambda_k \| \varepsilon_k \| \cdot \left\| \nabla \ell(\theta_k^m) + \varepsilon_k \right\| \quad (A.6)
\]
Rearranging the terms in the above inequality
\[
\sum_{k=1}^{N} \lambda_k C_k \left\| \nabla \ell(\theta_k^{md}) + \varepsilon_k \right\|^2 \leq \ell(\theta_0) - \ell(\theta^*) + \sum_{k=1}^{N} \lambda_k \left\| \varepsilon_k \right\| \left\| \nabla \ell(\theta_k^{md}) + \varepsilon_k \right\|
\]
By assumption 4 that \( \| \nabla \ell(\cdot) \| \) and \( \sum_{k=1}^{N} \lambda_k \left\| \varepsilon_k \right\| \) are bounded. Since \( \ell(\theta_N) \geq \ell(\theta^*) \) and in view of the assumption that \( C_k > 0 \), we obtain for some constant \( B \),
\[
\min_{k=1, N} \left\| \nabla \ell(\theta_k^{md}) + \varepsilon_k \right\|^2 \leq \frac{\ell(\theta_0) - \ell(\theta^*) + B}{\sum_{k=1}^{N} \lambda_k C_k}
\]
which clearly implies 3.7.
We now prove part b).
First, from L-Lipschitz-continuous gradient property 3.4, we have
\[
\ell(\theta_k^{ag}) \leq \ell(\theta_k^{md}) + \frac{L}{2} \left\| \theta_k^{ag} - \theta_k^{md} \right\|^2
\]
\[
\leq \ell(\theta_k^{md}) - \beta_k \left\| \nabla \ell(\theta_k^{md}) + \varepsilon_k \right\|^2 + \left\| \varepsilon_k \right\| \left\| \nabla \ell(\theta_k^{md}) + \varepsilon_k \right\| + \frac{L\beta_k^2}{2} \left\| \nabla \ell(\theta_k^{md}) + \varepsilon_k \right\|^2.
\]
By the assumption that \( \ell(\cdot) \) is convex and 3.2,
\[
\ell(\theta_k^{md}) - \left[ (1 - \alpha_k) \ell(\theta_k^{ag}) + \alpha_k \ell(\theta) \right]
= \alpha_k \left[ \ell(\theta_k^{md}) - \ell(\theta) \right] + (1 - \alpha_k) \left[ \ell(\theta_k^{md}) - \ell(\theta_{k-1}^{ag}) \right]
\leq \alpha_k \left\langle \nabla \ell(\theta_k^{md}), \theta_k^{md} - \theta \right\rangle + (1 - \alpha_k) \left\langle \nabla \ell(\theta_k^{md}), \theta_k^{md} - \theta_k^{ag}^{md} \right\rangle
= \left\langle \nabla \ell(\theta_k^{md}), \alpha_k (\theta_k^{ag} - \theta) + (1 - \alpha_k)(\theta_k^{md} - \theta_k^{ag}^{md}) \right\rangle
= \alpha_k \left\langle \nabla \ell(\theta_k^{md}), \theta_{k-1} - \theta \right\rangle. \tag{A.7}
\]
From 3.3, we have
\[
\left\| \theta_k - \theta \right\|^2 = \left\| \theta_{k-1} - \lambda_k \nabla \ell(\theta_k^{md}) - \theta \right\|^2
= \left\| \theta_{k-1} - \theta \right\|^2 - 2\lambda_k \left\langle \nabla \ell(\theta_k^{md}), \theta_{k-1} - \theta \right\rangle + \lambda_k^2 \left\| \nabla \ell(\theta_k^{md}) \right\|^2,
\]
\[
= \left\| \theta_{k-1} - \theta \right\|^2 - 2\lambda_k \left\langle \nabla \ell(\theta_k^{md}) + \varepsilon_k, \theta_{k-1} - \theta \right\rangle + \lambda_k^2 \left\| \nabla \ell(\theta_k^{md}) + \varepsilon_k \right\|^2,
\]
which implies
\[
\alpha_k \left\langle \nabla \ell(\theta_k^{md}) + \varepsilon_k, \theta_{k-1} - \theta \right\rangle = \frac{\alpha_k}{2\lambda_k} \left[ \left\| \theta_{k-1} - \theta \right\|^2 - \left\| \theta_k - \theta \right\|^2 \right]
+ \frac{\alpha_k \lambda_k}{2} \left\| \nabla \ell(\theta_k^{md}) + \varepsilon_k \right\|^2.
\]
Hence we obtain
\[
\alpha_k \left\langle \nabla \ell (\theta_k^{md}), \theta_{k-1} - \theta \right\rangle \leq \frac{\alpha_k}{2\lambda_k} \left[ \| \theta_{k-1} - \theta \|^2 - \| \theta_k - \theta \|^2 \right] + \frac{\alpha_k \lambda_k}{2} \left\| \nabla \ell (\theta_k^{md}) + \xi_k \right\|^2 + \alpha_k \| \xi_k \| \| \theta_{k-1} - \theta \| \tag{A.9}
\]
Using the results of A.7, A.8, and A.9, we get
\[
\ell (\theta_k^{ag}) \leq (1 - \alpha_k) \ell (\theta_{k-1}^{ag}) + \alpha_k \ell (\theta) + \frac{\alpha_k}{2\lambda_k} \left[ \| \theta_{k-1} - \theta \|^2 - \| \theta_k - \theta \|^2 \right] + \alpha_k \| \xi_k \| \| \theta_{k-1} - \theta \|
- \beta_k (1 - L \beta_k) \left\| \nabla \ell (\theta_k^{md}) + \xi_k \right\|^2 + \| \xi_k \| \beta_k \left\| \nabla \ell (\theta_k^{md}) + \xi_k \right\| + \alpha_k \| \xi_k \| \| \theta_{k-1} - \theta \| , \tag{A.10}
\]
where the last inequality follows from the assumption in 3.8. Subtracting \( \ell (\theta) \) from both sides of the above inequality and using Lemma 1, we conclude that
\[
\ell (\theta_k^{ag}) - \ell (\theta) \leq \Gamma_N \left[ \sum_{k=1}^N \frac{\alpha_k}{2\lambda_k} \left[ \| \theta_{k-1} - \theta \|^2 - \| \theta_k - \theta \|^2 \right] \right]
- \frac{\sum_{k=1}^N \beta_k}{2\Gamma_k} \left( 1 - L \beta_k \right) \left\| \nabla \ell (\theta_k^{md}) + \xi_k \right\|^2 + \frac{1}{\Gamma_k} \left[ \| \xi_k \| \beta_k \left\| \nabla \ell (\theta_k^{md}) + \xi_k \right\| + \alpha_k \| \xi_k \| \| \theta_{k-1} - \theta \| \right]
\leq \Gamma_N \frac{\| \theta_0 - \theta \|^2}{2\lambda_1} - \Gamma_N \sum_{k=1}^N \frac{\beta_k}{2\Gamma_k} \left( 1 - L \beta_k \right) \left\| \nabla \ell (\theta_k^{md}) + \xi_k \right\|^2
+ \Gamma_N \sum_{k=1}^N \frac{1}{\Gamma_k} \left[ \| \xi_k \| \beta_k \left\| \nabla \ell (\theta_k^{md}) + \xi_k \right\| + \alpha_k \| \xi_k \| \| \theta_{k-1} - \theta \| \right] \tag{A.11}
\]
for every \( \theta \in \mathbb{R}^n \). By our contraction 3.9 that sequence \( \left\{ \frac{\alpha_k}{\lambda_k \Gamma_k} \right\} \) is decreasing and the fact that \( \alpha_1 = \Gamma_1 = 1 \), we have
\[
\sum_{k=1}^N \frac{\alpha_k}{\lambda_k \Gamma_k} \left[ \| \theta_{k-1} - \theta \|^2 - \| \theta_k - \theta \|^2 \right] \leq \frac{\alpha_1 \| \theta_0 - \theta \|^2}{\lambda_1 \Gamma_1} = \frac{\| \theta_0 - \theta \|^2}{\lambda_1} \tag{A.12}
\]
which immediately implies the last inequality of A.11.

Hence, we can conclude 3.11 from the above inequality and the assumption in 3.8:
\[
\ell (\theta_k^{ag}) - \ell (\theta^*) \leq \Gamma_N \left[ \frac{\| \theta_0 - \theta^* \|^2}{\lambda_1} + \sum_{k=1}^N \Gamma_k^{-1} \left[ \| \xi_k \| \beta_k \left\| \nabla \ell (\theta_k^{md}) + \xi_k \right\| + \alpha_k \| \xi_k \| \| \theta_{k-1} - \theta \| \right] \right]
\]
Finally, noting the fact that \( \ell(\theta_{ag}^N) \geq \ell(\theta^*) \), substitute \( \theta := \theta^* \), re-arranging the terms in A.11 we obtain

\[
\sum_{k=1}^{N} \frac{\beta_k}{2L_k} (1 - L\beta_k) \left\| \nabla \ell(\theta_k^{md}) + \varepsilon_k \right\|^2 \quad k = 1, \ldots, N
\]

\[
\leq \frac{\|\theta^* - \theta_0\|^2}{2\lambda_k} + \sum_{k=1}^{N} \frac{1}{\Gamma_k} \left[ \|\varepsilon_k\| \beta_k \left\| \nabla \ell(\theta_k^{md}) + \varepsilon_k \right\| + \alpha_k \|\varepsilon_k\| \|\theta_{k-1} - \theta\| \right],
\]

or

\[
\min_{k=1, \ldots, N} \left\| \nabla \ell(\theta_k^{md}) + \varepsilon_k \right\|^2 \leq 2 \frac{\|\theta^* - \theta_0\|^2}{2\lambda_k} + \sum_{k=1}^{N} \frac{1}{\Gamma_k} \left[ \|\varepsilon_k\| \beta_k \left\| \nabla \ell(\theta_k^{md}) + \varepsilon_k \right\| + \alpha_k \|\varepsilon_k\| \|\theta_{k-1} - \theta\| \right]
\]

which together with 3.8, clearly imply 3.10.

\[\square\]

**A.2. Proof of Theorem 4**

**Proof.** We first prove part a). Note that by choosing

\[
\beta_k = \frac{1}{2L_k},
\]

\[
\Gamma_k = \frac{1}{k^{1+\delta}}, \quad (A.13)
\]

which implies that for sufficient large \( k \)

\[
\sum_{\tau=k}^{N} \Gamma_{\tau} = \frac{N}{k^{1+\delta}} = O\left(\frac{1}{k^{\delta}}\right)
\]

We also have

\[
1 - \alpha_k = \frac{(k-1)^{1+\delta}}{k^{1+\delta}} \quad (A.14)
\]

for every \( k > 1 \), or \( \alpha_k = \frac{(1+\delta)(k-1)^\delta}{k^{1+\delta}} = O\left(\frac{(1+\delta)\delta}{k^{1+\delta}}\right) = O\left(\frac{1}{k}\right) \). If we choose \( \lambda_k \) such that \( \lambda_k - \beta_k = o(k^{-1}) \) then

\[
\frac{(\lambda_k - \beta_k)^2}{2\alpha_k \Gamma_k \lambda_k} \left( \sum_{\tau=k}^{N} \Gamma_{\tau} \right) = \frac{o(k^{-2})}{k^{-1}k^{-(1+\delta)}k^{\delta}} = o(1)
\]

so for sufficiently large \( k \) we have

\[
C_k = 1 - L[\lambda_k + \frac{(\lambda_k - \beta_k)^2}{2\alpha_k \Gamma_k \lambda_k} (\sum_{\tau=k}^{N} \Gamma_{\tau})] > \frac{1}{4}
\]

Hence, it can also be seen from 3.7 that for some positive bounded constant \( B_2 \),

\[
\min_{k=1, \ldots, N} \left\| \nabla \ell(\theta_k^{md}) + \varepsilon_k \right\|^2 \leq \frac{\ell(\theta_k) - \ell^* + B}{NB_2} = O\left(\frac{1}{N}\right).
\]
which concludes the first part of the proof. Since \( \| \epsilon_k \| = O(\tau^2) \leq O(\frac{1}{k}) \), we have \( \nabla \ell(\theta_k^{md}) \) converge to 0 at the rate of

\[
\min \left\{ O(\frac{1}{\sqrt{N}}), O(\| \epsilon_k \|) \right\} = O(\frac{1}{\sqrt{N}}),
\]

which gives us the desired result.

We now show part b). Let \( \lambda_k = (k^{1+\delta} - (k-1)^{1+\delta}) \) for some constant \( c \)

\[
\frac{\alpha_1}{\lambda_1 \Gamma_1} = \frac{\alpha_2}{\lambda_2 \Gamma_2} = \cdots = \frac{\alpha_k}{\lambda_k \Gamma_k}.
\]

Observe that

\[
\alpha_k \lambda_k = c^2 (k^{1+\delta} - (k-1)^{1+\delta})^2 = c^2 (1+\delta)^2 O(k^{2\delta}) \to 0
\]

for \( \delta < 1 \) so \( \frac{(1+\delta)^2 c^2}{k^{1+\delta}} < \beta_k = \frac{1}{2\pi} \) for sufficient large \( k \), which implies that conditions 3.8 and 3.9 hold. Moreover, it can also be easily seen from A.3 that

\[
\min_{k=1,N} \| \nabla \ell(\theta_k^{md}) + \epsilon_k \|^2 \leq \frac{2\lambda_k}{2\lambda_k} \sum_{k=1}^{N} \Gamma_k^{-1} \left[ \| \epsilon_k \|^2 + O(\frac{1}{k}) \| \epsilon_k \| \right] \leq O(N^{-2-\delta}).
\]

The last equality is due to the fact that \( \sum_{k=1}^{N} \Gamma_k^{-1} = \sum_{k=1}^{N} k^{1+\delta} = O(N^{2+\delta}) \). Combining the above relation with 3.7, and since \( \| \epsilon_k \| = O(\tau^2) \leq O(\frac{1}{k^{1+\delta+\delta_1}}) \) for some \( \delta_1 > 0 \), we have \( \nabla \ell(\theta_k^{md}) \) converge to 0 at the rate of \( O\left(\frac{1}{N^{2+\delta_1}}\right) \).

Since \( \alpha_k \lambda_k < \beta_k = \frac{1}{2\pi} \), we have \( \delta \leq 1 \) which implies that the best convergence rate is \( O(N^{-3}) \).

\[\square\]

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