An infinite number of potentials surrounding 2d black hole

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Abstract

We found an infinite number of potentials surrounding 2d black hole. According to the transmission $T$ and reflection $R$ coefficients for scattering of string fields off 2d black hole, we can classify an infinite number of potentials into three: graviton-dilaton, tachyon and the other types. We suggest that the discrete states from all the Virasoro levels be candidates for new potentials (modes).
It is always possible to visualize the black hole as presenting an effective potential barrier (or well) to the on-coming waves [1]. In the case of 4d Schwarzschild black hole, two kinds of Schrödinger type equation arose from the metric perturbations. One is the Regge-Wheeler (RW) equation in the axial (odd-parity) perturbation [2],

$$\frac{d^2\Psi_o}{dr^*2} + (k^2 - V_{RW})\Psi_o = 0,$$

(1)

Here units are used in which \(G = c = 2M_4 = 1, r^* = r + \ln(r - 1)\), so that the horizon is at \(r^* = -\infty \) \((r = 1)\). The RW potential \(V_{RW}\) is given by

$$V_{RW} = \frac{2(n + 1)r - 3}{r^4(r - 1)}$$

with \(n = (l - 1)(l + 2)/2, l \geq 2\). The other is the Zerilli equation

$$\frac{d^2\Psi_e}{dr^*2} + (k^2 - V_Z)\Psi_e = 0.$$  

(2)

which differs only in the details of the potential

$$V_Z = \frac{2(n + 1)r^3 + 3r^2 + 9r/2n + 9/4n^2}{r^4(r + 3/2n)^2(r - 1)}.$$

The Zerilli equation arose in the study of polar (even-parity) perturbations in the same formalism. Although these potentials have different forms, they are equivalent. Chandrasekhar have showed that \(V_{RW}\) and \(V_Z\) are equivalent in the sense of producing the same reflection (\(R\)) and transmission (\(T\)) coefficients [1,3]. Further, Anderson and Price showed that there may exist an infinite number of equivalent potentials surrounding the 4d real black holes [4].

In the Schwarzschild black holes \(V_{RW}\) and \(V_Z\) are considered as only two realizations of an infinite number of possible potentials. Four potentials are realized for four graviton-Maxwell modes in the Reissner-Nordström black hole. However, the finding of an infinite number of potentials is a difficult problem in the real black holes. Moreover it is not clear what the candidates for new potentials (modes) are. We here consider a simpler toy model, 2d stringy black hole in which an analogous problem can be imposed and exactly solved [5,6]. The 2d dilaton gravity is far from being 4d realistic models in the sense that two propagating gravitons are missing.
In this paper we will find an infinite number of potentials in the 2d black hole. In analyzing two-dimensional stringy black hole, we begin with one graviton \((h)\), one dilaton \((\phi)\) and one tachyon \((t)\). These are, in turn, combined into one graviton-dilaton \((h - \phi)\), the other \((h + \phi)\) and tachyonic \((t)\) modes [7]. All of these modes satisfy the Schrödinger type equation (14). According to the transmission and reflection coefficients, we can classify an infinite number of potentials into three: graviton-dilaton, tachyon and the other types. Finally, we discuss the candidates for the new potentials (modes) except graviton, dilaton and tachyon.

The \(\sigma\)-model action of 2d critical string theory for graviton \((g_{\mu\nu})\), dilaton \((\Phi)\), and tachyon \((T)\) \((\mu, \nu = 0, 1)\) is given by [8]

\[
S_\sigma = \frac{1}{8\pi\alpha'} \int d^2z \sqrt{G}[g_{\mu\nu}(x)\nabla x^\mu \nabla x^\nu + \alpha' R\Phi(x) + 2T(x)]
\]  

(3)

with \(x^0 = \theta, x^1 = \phi\). The Minkowiski signature is recovered by the analytic continuation of \(\theta = i\tau\). The conformal invariance of \(S_\sigma\) requires the following \(\beta\)-function equations to be satisfied:

\[
R_{\mu\nu} + \nabla_\mu \nabla_\nu \Phi + \nabla_\mu T \nabla_\nu T = 0,
\]

(4)

\[
R + (\nabla \Phi)^2 + 2\nabla^2 \Phi + (\nabla T)^2 - 2T^2 - 8 = 0,
\]

(5)

\[
\nabla^2 T + \nabla \Phi \nabla T + 2T = 0.
\]

(6)

These equations can also be derived from the 2d target space effective action in [9] with the substitutions

\[
-2\Phi_{DL} \to \Phi, T_{DL} \to T, -R_{DL} \to R.
\]

(7)

Let us begin with the static background solutions of the graviton-dilaton sector without the tachyonic condensation,

\[
\bar{\Phi} = 2Q\phi, \quad \bar{T} = 0, \quad \bar{g}_{\mu\nu} = \begin{pmatrix} -f & 0 \\ 0 & f^{-1} \end{pmatrix},
\]

(8)
where

\[ f = 1 - M e^{-2Q\phi}, \quad Q = \sqrt{2}. \]

(9)

Here the parameter \( M (> 0) \) is proportional to the mass of the black hole. All these solutions approach the linear dilaton vacuum in the asymptotically flat region \((\phi \to +\infty)\). The event horizon of the black hole occurs at \( \phi_{EH} = \frac{1}{2\sqrt{2}} \ln M \). We choose \( M = 1 \) and \( \phi_{EH} = 0 \) for simplicity. As a consequence, \( \phi_{EH} = 0 \) defines a null surface. Since the region interior to the horizon \((\phi < 0)\) is of no relevance to our consideration, the new coordinate \((\phi^*)\) is introduced as

\[ \phi^* \equiv \phi + \frac{1}{2\sqrt{2}} \ln(1 - e^{-2\sqrt{2}\phi}). \]

(10)

Note that \( \phi^* \) ranges from \(-\infty\) to \(+\infty\), while \( \phi \) ranges from the event horizon of the black hole \((\phi_{EH} = 0)\) to \(+\infty\).

To study the scatterings off black hole, let us introduce the small perturbed fields \( h_{\mu\nu}(\phi, \tau), \varphi(\phi, \tau) \) and \( t(\phi, \tau) \) as [7]

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} = \bar{g}_{\mu\nu}[1 - h(\phi, \tau)], \]

(11)

\[ \Phi = \bar{\Phi} + \varphi(\phi, \tau), \]

(12)

\[ T = \bar{T} + \tilde{t} \equiv \exp\left(-\frac{\Phi}{2}\right)[0 + t(\phi, \tau)]. \]

(13)

Substituting (11)-(13) into (4)-(6) and then keeping up to linear order of perturbed fields, one gets the linearized equations for \( \Psi = h - \varphi, h + \varphi, t \) modes.

Considering \( \Psi(\phi^*, \tau) = \psi(\phi^*)e^{-i\omega \tau} \), we find the Schrödinger type equation for 2d black hole

\[ \left[ \frac{d^2}{d\phi^*^2} + \omega^2 - \frac{V_0}{(\cosh \sqrt{2}\phi^*)^2} \right] \psi = 0. \]

(14)

Here \( V_0 = 0 \) for \((h - \varphi)\), \( V_0 = -4 \) for \((h + \varphi)\), and \( V_0 = 1/2 \) for \((t)\). The above equation governs the propagation of all perturbing string fields in the black hole background. In order to solve the equation (14), we make the substitution
\[ \psi = (\cosh \sqrt{2}\phi^*)^{-2\lambda} u \]  
\[ \text{with} \]
\[ \lambda = \frac{1}{4}(\sqrt{1 - 2V_0} - 1). \]

Then the equation for \( u \) takes the form
\[ \frac{d^2u}{d\phi^*^2} - 4\sqrt{2}\lambda \tanh \sqrt{2}\phi^* \frac{du}{d\phi^*} + 8(\lambda^2 + \frac{1}{8}\omega^2)u = 0. \]

Two kinds of solutions to the Schrödinger equation (14) correspond to scattering and bound states. Here we consider only the case of \( \omega^2 > 0 \) for scattering state. The other case \( (\omega^2 < 0) \) is used to study the bound state problem. If we introduce a new independent variable
\[ z = -(\sinh \sqrt{2}\phi^*)^2, \]
then the equation for \( u \) reduces to the hypergeometric equation
\[ z(1 - z)\frac{d^2u}{dz^2} + \left[ \frac{1}{2} - (1 - 2\lambda)z \right] \frac{du}{dz} - (\lambda^2 + \frac{1}{8}\omega^2)u = 0. \]

The parameters \( \alpha, \beta, \gamma \) which occur in the general form of the hypergeometric equation,
\[ z(1 - z)\frac{d^2u}{dz^2} + [\gamma - (\alpha + \beta + 1)z] \frac{du}{dz} - \alpha\beta u = 0, \]
take in our case the following values :
\[ \gamma = \frac{1}{2}, \quad \alpha = -\lambda + \frac{i\omega}{2\sqrt{2}}, \quad \beta = -\lambda - \frac{i\omega}{2\sqrt{2}}. \]

Two exact solutions of equation (18) are of the forms
\[ u_1 = F(-\lambda + \frac{i\omega}{2\sqrt{2}}, -\lambda - \frac{i\omega}{2\sqrt{2}}, \frac{1}{2}; z), \]  
\[ u_2 = \sqrt{z}F(-\lambda + \frac{i\omega}{2\sqrt{2}} + \frac{1}{2}, -\lambda - \frac{i\omega}{2\sqrt{2}} + \frac{3}{2}; z). \]

The general form of the wavefunction is
Here the coefficients $C_1$ and $C_2$ will be determined by requiring the boundary conditions at $\phi^* \rightarrow \pm\infty$ \cite{10}.

For $V_0 > \frac{1}{2}$, we have

$$
T = \frac{(\sinh(\pi\omega/\sqrt{2}))^2}{(\sinh(\pi\omega/\sqrt{2}))^2 + (\cosh(\pi \sqrt{2V_0 - 1}))^2},
$$
(23)

$$
R = \frac{(\cosh(\pi \sqrt{2V_0 - 1}))^2}{(\sinh(\pi\omega/\sqrt{2}))^2 + (\cosh(\pi \sqrt{2V_0 - 1}))^2}.
$$
(24)

In this case, one cannot obtain an infinite number of potentials because of the non-periodic nature of the hyperbolic functions.

For $V_0 \leq \frac{1}{2}$, we obtain the transmission coefficient

$$
T = \frac{(\sinh(\pi\omega/\sqrt{2}))^2}{(\sinh(\pi\omega/\sqrt{2}))^2 + (\cos \pi(2\lambda + \frac{1}{2}))^2}.
$$
(25)

Similarly one can derive the reflection coefficient

$$
R = \frac{(\cos \pi(2\lambda + \frac{1}{2}))^2}{(\sinh(\pi\omega/\sqrt{2}))^2 + (\cos \pi(2\lambda + \frac{1}{2}))^2}.
$$
(26)

We classify these forms into the following types.

**A. graviton-dilaton type :** $2\lambda = n$

This comes from the condition : $\cos \pi(2\lambda + \frac{1}{2}) = 0$. Using the relation (16), we obtain an infinite series

$$
V_0 = 0(n = 0), \quad V_0 = -4(n = 1), \quad V_0 = -12(n = 2), \quad V_0 = -24(n = 3), \cdots,
$$
(27)

which give us the same transmission ($T^{g-d}$) and reflection ($R^{g-d}$) coefficients. Note that $h + \varphi$ ($h - \varphi$) recover from this series when $n = 0(n = 1)$ respectively. The transmission coefficient is given by
\[ T^{g-d} = |T^{g-d}|^2 = 1 \]  

This means that there is no reflection, i.e. \( R^{g-d} = |R^{g-d}|^2 = 0 \). Even though the different potential wells have arisen from the black hole, all modes which belong to this series propagate freely from \(+\infty\) to \(-\infty\). This corresponds to the total transmission \([11]\). For example, there is anomalously large transmission in the low energy electrons (0.1 eV) scatterings off noble gases such as neon or argon. For the electron scatterings the prototype potential is the square well, instead of \(-\frac{1}{\cosh^2 x}\). This type of scattering can thus be understood by noting the analogies: graviton-dilaton modes \( \leftrightarrow \) electrons, 2d black hole \( \leftrightarrow \) neon or argon.

**B. tachyon type: \( 2\lambda = n - 1/2 \)**

Requiring that \( \cos \pi(2\lambda + \frac{1}{2}) = \pm 1 \), we obtain the following infinite series

\[
V_0 = \frac{1}{2}(n = 0), \quad V_0 = -\frac{3}{2}(n = 1), \quad V_0 = -\frac{15}{2}(n = 2), \quad V_0 = -\frac{35}{2}(n = 3), \ldots. \tag{29}
\]

The first one corresponds to the tachyon mode. All of these lead to the same transmission (\( T^t \)) and reflection (\( R^t \)) coefficients

\[
T^t = \frac{(\sinh(\pi\omega/\sqrt{2}))^2}{1 + (\sinh(\pi\omega/\sqrt{2}))^2}, \quad \tag{30}
\]

\[
R^t = \frac{1}{1 + (\sinh(\pi\omega/\sqrt{2}))^2}. \quad \tag{31}
\]

As might be expected, one finds that \( T^t + R^t = 1 \). Given the energy of mode \( (E = \omega^2) \), this case has the maximum reflection and minimum transmission.

**C. the other types**

For \( 0 < |\cos \pi(2\lambda + \frac{1}{2})| < 1 \), we have the transmission and reflection coefficients which belong to

\[
T^t < T < 1, \quad 0 < R < R^t. \quad \tag{32}
\]
Here we can find an infinite number of potentials which give us the same physical consequences.

Now let us discuss our results. The number of degrees of freedom for the gravitational field \( h_{\mu \nu} \) in \( d \)-dimensions is \([12]\)

\[
\frac{1}{2} d(d+1) - 1 - d - (d-1) = \frac{1}{2} d(d-3).
\]

For 4d Schwarzschild case, we obtain two propagating physical gravitons (RW in (1) and Zerilli in (2) cases). There is no two dimensional analog of the Schwarzschild solution since the Einstein equation is trivial for \( d=2 \) \( (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 0) \). Also this counting is \(-1\) for \( d=2 \). This means that in two dimensions, the contribution of graviton is equal and opposite to that of a spinless particle (dilaton). We recognize from (11) that the gauge-fixing for graviton is lack on the basis of 4d gravity theory. The combined modes of graviton and dilaton \( (h - \varphi, h + \varphi) \) should have zero physical degree of freedom \([13]\). Even though the potential well for \( h - \varphi \) has arisen from the black hole, this mode can be removed by the coordinate transformation (translation) \([14]\). In view of the above, it is obvious that two graviton-dilaton modes cannot be realized into the physically propagating modes. Thus the net physical degrees of freedom should be given by

\[-1(\text{graviton}) + 1(\text{dilaton}) + 1(\text{tachyon}) = 1(\text{tachyon}).\]

The above implies that the tachyon is only a physical degree of freedom. If additional modes with \( 2\lambda = n - 1/2 \) except tachyon may exist, they are all the physical degrees of freedom. It is clear that the unexplored modes with \( 2\lambda = n \) except two graviton-dilaton modes are the physical degrees of freedom. For example, we consider the CGHS model with \( N \) conformal matters \( (f_i) \) \([15]\) instead of tachyon. Although all of these belong to the \( 2\lambda = n \) case \( (V_0 = 0, n = 0) \), they correspond to the physical fields. This is because \( f_i \) are external matter fields. By the similar way it is also suggested that all external modes including the other types correspond to the physical degrees of freedom. In the 2d dilaton black hole, the modes \( (h - \varphi, h + \varphi, t) \) are only three realizations of an infinite number of modes. Among
these the tachyon (external matter) turns out to be a physically propagating one. By the similar way, the modes \((h - \varphi, h + \varphi, f_i)\) in CGHS model are the \((2 + N)\) realizations of an infinite number of modes. Here \(f_i\) are physical modes.

What is the origin of the infinite number of new potentials (equations)? In order to answer to this question, one has to recognize that the key equations in (14) are derived from the string perturbations propagating under the 2d black hole. The new equations may come from another string fields except graviton, dilaton and tachyon. There are an infinite number of physical states in the string theories, the discrete states arising from all the Virasoro levels including graviton-dilaton sector [16]. However, having application of string theories to 2d black hole, two graviton-dilatons turn out to be nonpropagating modes. Thus the remaining discrete states from all the Virasoro levels may be candidates for new potentials (modes) surrounding 2d black hole.

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