Abstract: Let $k$ be an algebraically closed field of characteristic $p > 0$, and $V$ an $n$-dimensional $k$-vector space together with a non-degenerate symmetric bilinear form. Let $G$ denote one of the groups $G = \text{SL}(V)$ or $\text{SO}(V)$ where we assume that $p > 2$ if $G = \text{SO}(V)$. Let $R$ denote the coordinate ring of $V_{m,q} := V^\oplus m \oplus V^* \oplus q$ (resp. $V_m := V^\oplus m$) if $G = \text{SL}(V)$ (resp. if $G = \text{SO}(V)$), $V^*$ being the dual of $V$. The defining representation of $G$ on $V$ induces the diagonal action of $G$ on $V_{m,q}$ (resp. $V_m$). Let $S = R^G$. In this paper, we show that $S$ is Frobenius split.

1. Introduction

The concept of $F$-purity was introduced by Hochster-Roberts [6]; the $F$-purity for a noetherian ring of prime characteristic is equivalent to the splitting of the Frobenius map, when the ring is finitely generated over its subring of $p$-th powers. It is closely related to the Frobenius splitting property à la Mehta-Ramanathan [10] for algebraic varieties; to make it more precise, the $F$-split property for an irreducible projective variety $X$ over an algebraically closed field of positive characteristic is equivalent to the $F$-purity of the ring $\oplus_{n \geq 0} H^0(X; L^n)$ for any ample line bundle $L$ over $X$ (cf.[3],[13],[14]). We feel that it is but appropriate to dedicate this paper to Professor Hochster on the occasion of his sixty-fifth birthday and thus make a modest contribution to this birthday volume.

Let $k$ be an algebraically closed field of characteristic $p > 0$ and let $X$ be a $k$-scheme. One has the Frobenius morphism (which is only
an \( \mathbb{F}_p \)-morphism) \( F: X \rightarrow X \) defined as the identity map of the underlying topological space of \( X \), the morphism of structure sheaves \( F^\#: \mathcal{O}_X \rightarrow \mathcal{O}_X \) being the \( p \)-th power map. The morphism \( F \) induces a morphism of \( \mathcal{O}_X \)-modules \( \mathcal{O}_X \rightarrow F^* \mathcal{O}_X \). The variety \( X \) is called Frobenius split (or \( F \)-split or, merely, split) if there exists a splitting \( \varphi: F^* \mathcal{O}_X \rightarrow \mathcal{O}_X \) of the morphism \( \mathcal{O}_X \rightarrow F^* \mathcal{O}_X \). Equivalently, \( X \) is Frobenius split if there exists a morphism of sheaves of abelian groups \( \varphi: \mathcal{O}_X \rightarrow \mathcal{O}_X \) such that (i) \( \varphi(f^p g) = f \varphi(g) \), \( f, g \in \mathcal{O}_X \) and (ii) \( \varphi(1) = 1 \). Basic examples of varieties that are Frobenius split are smooth affine varieties, toric varieties (cf. \([1]\)), generalized flag varieties, and Schubert varieties \([10]\). Smooth projective curves of genus greater than 1 are examples of varieties that are not Frobenius split.

Frobenius splitting is an interesting property to study. If \( X \) is Frobenius split, then it is weakly normal (cf. \([1]\), Prop 1.2.5) and reduced (cf. \([1]\), Prop. 1.2.1). Indeed, projective varieties which are Frobenius split are very special. We refer the reader to \([1]\) for further details.

If \( X = \text{Spec}(R) \), then \( X \) is Frobenius split if and only if the Frobenius homomorphism \( R \rightarrow R \) defined as \( a \mapsto a^p \) admits a splitting \( \varphi: R \rightarrow R \) such that \( \varphi(a^p b) = a \varphi(b) \), and \( \varphi(1) = 1 \).

If a linearly reductive group \( G \) acts on a \( k \)-algebra \( R \) which is Frobenius split, then the invariant ring \( R^G \) is Frobenius split (see \([1]\) Exercise 1.1.E(5))). To quote Karen Smith \([13\), p. 571], “The story of \( F \)-splitting and global \( F \)-regularity for quotients by reductive groups in characteristics \( p \) that are not linearly reductive is much more subtle and complicated”. We shall show that although the groups \( \text{SO}(n), n \geq 3 \), and \( \text{SL}(n), n \geq 2 \), are not linearly reductive, it turns out that certain rings of invariants for these groups are Frobenius split.

We state below the main results of this paper.

Let \( k \) be an algebraically closed field of characteristic \( p > 2 \) and \( V \) an \( n \)-dimensional vector space over \( k \) with a symmetric non-degenerate bilinear form. Denote by \( A_m \) the the coordinate ring of \( V_m := V^\oplus m, m \geq 1 \), and consider the action of \( \text{SO}(V) \) on \( A_m \) induced by the diagonal action of \( \text{SO}(V) \) on \( V^\oplus m \). Then

**Theorem 1.1.** The invariant ring \( A_m^{\text{SO}(V)} \) is Frobenius split for all \( m \geq 1 \).

The group \( \text{SL}(V) \) acts on \( V \), as well as on the dual vector space \( V^* = \text{Hom}_k(V, k) \). Now consider the diagonal action of \( \text{SL}(V) \) on the vector space \( V_{m,q} := V^\oplus m \oplus V^{*\oplus q}, m, q \geq 1 \). This leads to an action of \( \text{SL}(V) \) on the coordinate ring \( A_{m,q} \) of \( V_{m,q} \).
Theorem 1.2. The invariant ring $A_{m,q}^{SL(V)}$ is Frobenius split for any $m, q \geq n$.

We shall now sketch the proofs of the main results (assuming $m, q > n$). Let $S$ be the invariant ring in question. Let $R$ be the ring of invariants under the larger group $\widetilde{G} = GL(V)$ (resp. $\widetilde{G} = O(V)$), we have (cf. [2, 8]) that $R$ is the coordinate ring of a certain determinantal variety in $M_{m,q}$, the space of $m \times q$ matrices (resp. $SymM_m$, the space of symmetric $m \times m$ matrices) with entries in $k$. Now a determinantal variety in $M_{m,q}$ (resp. $SymM_m$) can be canonically identified (cf. [8]) with an open subset in a certain Schubert variety in $G_{q,m+q}$, the Grassmannian variety of $q$-dimensional subspaces of $k^{m+q}$ (resp. the symplectic Grassmannian variety, the variety of all maximal isotropic subspaces of a $2m$-dimensional vector space over $k$ endowed with a non-degenerate skew-symmetric bilinear form). Hence we obtain that $R$ is Frobenius split (since Schubert varieties are Frobenius split). Let $X = \text{Spec}(S), Y = \text{Spec}(R)$, and $\pi: X \to Y$, the morphism induced by the inclusion $R \subset S$. When $G = SO(V)$, we show that $\pi$ is a double cover which is étale over a ‘large’ open subvariety – that is a subvariety whose complement has codimension at least 2. When $G = SL(n)$, we show that restricted to a large open subvariety, $\pi$ is a $G_m$ bundle. The main theorems are then deduced using normality of $S$.

Theorem 1.1 can also be deduced from Hashimoto’s work [4], wherein it is shown that if a reductive group $G$ acts on a polynomial ring $A$ over $k$ (of positive characteristic) with good filtration, then the ring $A^G$ of invariants is strongly $F$-regular. Our Theorem 1.2 does not seem to follow from the results of [4]. Granting the results of [9] and [7]—we don’t need all the results of these papers, only some of the relatively easier ones—the arguments used in our proofs are straightforward and quite elementary; the techniques used in [4] are representation theoretic.

Theorem 1.1 will be proved in §2 and Theorem 1.2 in §3.

2. Splitting $SO(n)$-invariants

Let $k$ be an algebraically closed field of characteristic $p > 0$. Suppose that $S$ is an affine $k$-algebra which is Frobenius split and that a finite group $\Gamma$ acts on $S$ as $k$-algebra automorphisms. Then the invariant ring $R = S^\Gamma$ is Frobenius split provided the order of $\Gamma$ is not divisible by $p$ (cf. [1] Ex. 1.1.E(5)]). We first obtain a partial converse to this statement in the case when $\Gamma$ is of order 2.
Assume that char$(k) > 2$. Let $S$ be an affine $k$-domain and let $\Gamma = \{1, \gamma\} \cong \mathbb{Z}/2\mathbb{Z}$ act effectively on $S$. Denote by $R$ the invariant subalgebra $S^\Gamma$. Then $R$ is an affine $k$-algebra and $S$ is quadratic and integral over $R$. Indeed, any $b \in S$ can be expressed as $b = b_0 + b_1$ where $b_0 = (1/2)(b + \gamma(b)) \in R$ and $b_1 = (1/2)(b - \gamma(b))$ satisfies $\gamma(b_1) = -b_1$. Thus, we can choose generators $u_1, \ldots, u_s$ for the $R$-algebra $S$ to be in the $-1$ eigenspace of $\gamma$. Clearly $u_i^2 = -u_i\gamma(u_i) =: p_i \in R$ for all $i \leq s$. Furthermore,

$$\gamma(u_iu_j) = u_iu_j =: p_{i,j} \in R \text{ for all } i, j \leq s \text{ (with } p_{i,i} = p_i).$$

Observe that $p_{i,j}^2 = p_ip_j$.

We shall assume that $S$ is reduced so that $p_i \neq 0$, for all $i$. Now let $R_i = R[1/p_i], 1 \leq i \leq n$. Let $S_i = S[1/u_i]$. We claim that $S_i = R_i[u_i]$. To see this, first observe that $R_i[u_i] \subset S[1/u_i]$, since $1/p_i = (1/u_i)^2 \in S[1/u_i]$. To show that $S[1/u_i] \subset R_i[u_i]$, it suffices to show that $u_j \in R_i[u_i]$ for all $j$ and $(1/u_i) \in R_i[u_i]$. Indeed, $1/u_i = u_i/u_i^2 = u_i/p_i \in R_i[u_i]$ and so $u_j = p_{i,j}/u_i \in R_i[u_i]$.

Write $X = \text{Spec}(S), Y = \text{Spec}(R)$ and let $\pi : X \longrightarrow Y$ be the morphism (induced by the inclusion $R \subset S$). As above, let $S_i = S[1/u_i]$, and let $U_i = \text{Spec}(S_i) \subset X$ and let $U := \bigcup_{1 \leq i \leq s} U_i$; it is the full inverse image under $\pi$ of $\bigcup_{1 \leq i \leq s} \text{Spec}(R_i)$. It is readily verified that $\pi|U : U \longrightarrow \pi(U)$ is étale. Indeed, $S_i$ is a free $R_i$ module with basis $\{1, u_i\}$ and $\det(u_i) = -p_i \neq 0$ and so $\pi|U_i$ is étale. Hence $\pi|U$ is étale.

On the other hand, if $y \in Y$ is a closed point such that $p_i(y) = 0$ for all $i \leq s$, then the fibre $f^{-1}(y) = \text{Spec}(S_y \otimes_{R_y} k)$ is the scheme whose coordinate ring is $S_y \otimes_{R_y} k = k[u_1, \ldots, u_s]/(u_i^2, 1 \leq i \leq s)$. Here $R_y$ is the local ring at $y$. Thus $f^{-1}(y)$ is non-reduced. It follows that the ramification locus of $\pi$ equals $Y \setminus \pi(U)$. (See [12], §III.10, Theorem 3.)

**Proposition 2.1.** Let $k$ be an algebraically closed field of characteristic $p > 2$. Let $S$ be an affine normal domain over $k$ acted on by $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ and let $R := S^\Gamma$ be Frobenius split. Suppose that the ramification locus of the double cover $\pi : \text{Spec}(S) \longrightarrow \text{Spec}(R)$ has codimension at least 2 in $\text{Spec}(R)$. Then, any splitting $\varphi : R \longrightarrow R$ extends uniquely to a splitting $\psi : S \longrightarrow S$.

**Proof.** We use the notations introduced above.

Since $X$ is normal and the codimension of the the ramification locus of $\pi$ is at least 2, it suffices to show that $U = \bigcup_{1 \leq i \leq s} U_i$ is Frobenius split (cf. [1], Lemma 1.1.7,(iii)).
Let \( \varphi : R \to R \) be a splitting of \( Y = \text{Spec}(R) \). First, we shall extend \( \varphi \) to a splitting \( \psi_i : S_i \to S_i \) of \( U_i = \text{Spec}(S_i)(= \text{Spec}(R_i[u_i])) \) for each \( i \) and verify that these splittings agree on the overlaps \( U_i \cap U_j \) for \( 1 \leq i, j \leq s \). Thus we will obtain a splitting of \( U = \bigcup_{1 \leq i \leq s} U_i \). By normality of \( X \) and the hypothesis on the codimension of the ramification locus, we will conclude that this splitting extends to a splitting of \( X \). Next, we shall establish the uniqueness of the extension.

Recall that \( \{1, u_i\} \) is an \( R_i \)-basis for \( S_i \). Since \( u_i = u_i^{-p}p_i^{(1+p)/2} \) on \( U_i \), if \( \psi_i : S_i \to S_i \) is any splitting of \( U_i \) which extends the splitting \( \varphi_i \) of \( R_i \) defined by \( \varphi \), we must have \( \psi_i(au_i) = \psi_i((1/u_i)p_i(p+1)/2) = (1/u_i)\varphi_i(p_i^{(p+1)/2}) \). By additivity, we must have

\[
\psi_i(au_i + b) = (1/u_i)\varphi_i(p_i^{(p+1)/2}) + \varphi_i(b) = (u_i/p_i)\varphi_i(p_i^{(p+1)/2}) + \varphi_i(b)
\]

where \( a, b \in R_i \). Thus the extension \( \psi_i \), if it exists, is unique.

We now define \( \psi_i \) by the above equation and claim that \( \psi_i \) is indeed a splitting of \( S_i \). First, observe that \( \psi_i(1) = 1 \), by the very definition of \( \psi_i \).

Now, for any \( x, y, a \in R_i \), we have

\[
\psi_i((xu_i + y^paui) = \psi_i(x^pp_i^{(p+1)/2}a + y^paui) = x\varphi(p_i^{(p+1)/2})a + y\varphi(aui) = x(p_i/u_i)\psi_i(aui) + y\varphi(aui) = xu_i\psi_i(aui) + y\psi(aui) = (xu_i + y)\varphi(aui)
\]

An entirely similar (and easier) computation shows that

\[
\psi_i((xu_i + y)^p) = (xu_i + y)\psi_i(b),
\]

completing the verification that \( \psi_i \) is a splitting.

We verify, by another straightforward computation, that these \( \psi_i \) agree on the overlaps \( U_i \cap U_j \). Indeed, writing \( u_j = u_ip_{i,j}/p_i \), we have

\[
\psi_i(u_j) = \psi_i(u_ip_{i,j}/p_i) = (u_i/p_i)\varphi((p_{i,j}/p_i)p_i^{(p+1)/2}) = (u_i/p_i)\varphi(p_{i,j}p_i^{(p+1)/2})
\]

Since \( p_i = p_{i,j}^2/p_j \) on \( U_i \cap U_j \), we have

\[
\varphi(p_{i,j}p_i^{(p+1)/2}) = \varphi(p_{i,j}^p(p_i/p_{i,j})^{(p-1)/2}) = p_{i,j}\varphi(p_j^{(1-p)/2}) = (p_{i,j}/p_j)\varphi(p_j^{(p+1)/2})
\]
Substituting in the above expression for \( \psi_i(u_j) \) we get

\[
\psi_i(u_j) = (u_ip_{i,j}/(p_{i}p_{j})) \varphi(p_j^{(p+1)/2}) = (u_j/p_j) \varphi(p_j^{(p+1)/2}) = \psi_j(u_j)
\]

This implies that the extensions \( \{\psi_i\} \) patch to yield a well-defined splitting on \( U \) as claimed. As observed above, the normality of \( X \) and the hypothesis on the codimension of the ramification locus implies that this splitting extends to a unique splitting \( \psi : S \to S \).

Finally, if \( \psi' \) is another splitting of \( X \) which also extends \( \varphi \), then \( \psi' \) and \( \psi \) agree on \( U_i \) (for any \( i \)) as already observed. As \( X \) is irreducible, \( U_i \) is dense in \( X \) and we conclude that \( \psi' = \psi \).

As a corollary, we obtain the following

**Theorem 2.2.** Let \( \pi : X \to Y \) be a double cover of a Noetherian scheme whose ramification locus has codimension at least 2. Suppose that \( X \) is reduced, irreducible and normal and that \( Y \) is Frobenius split, then \( X \) is Frobenius split.

**Proof.** Cover \( X \) by finitely many affine patches \( X_\alpha \). Let \( Y_\alpha := \pi X_\alpha \). Then each \( \pi|_{X_\alpha} \) satisfies the hypotheses of the above proposition. Let \( \varphi \) be a splitting of \( Y \) and let \( \psi_\alpha \) be the unique splitting of \( X_\alpha \) that extends the splitting \( \varphi|_{Y_\alpha} \). The \( \psi_\alpha \)'s agree on overlaps and hence define a unique splitting of \( X \) which 'extends' \( \varphi \).

We now turn to proof of Theorem 1.1.

**Proof of Theorem 1.1.** Denote by \( S \) the ring of \( \text{SO}(V) \)-invariants of \( A_m \), where \( A_m \) is the coordinate ring of \( V_m \). Let \( R \) be the ring of \( \text{O}(V) \)-invariants.

We shall assume that \( m > n \). By [2, 8] we have that \( Y := \text{Spec}(R) \) is the determinantal variety \( D_n(Sym M_m) \) consisting of all matrices in \( Sym M_m \) (the space of symmetric \( m \times m \) matrices with entries in \( k \)) of rank at most \( n \); further, we have (cf. [8]) an identification of \( D_n(Sym M_m) \) with an open subset of a certain Schubert variety in the Lagrangian Grassmannian variety (of all maximal isotropic subspaces of a \( 2m \)-dimensional vector space over \( k \) endowed with a non-degenerate skew-symmetric bilinear form). Hence we obtain that \( Y \) is Frobenius split (since Schubert varieties are Frobenius split (cf. [10])).

Observe that \( \Gamma := \text{O}(n)/\text{SO}(n) \cong \mathbb{Z}/2\mathbb{Z} \) acts on \( S \) (the subring of \( \text{SO}(V) \)-invariants of \( A_m \)) and that \( S^\Gamma \) equals \( R \). As above, let \( X := \text{Spec}(S) \), and \( \pi : X \to Y \) be the morphism induced by the inclusion \( R \subset S \). We need only verify the hypotheses of Theorem 2.2. It is well-known that \( S \) is an affine normal domain. It remains to verify that
the codimension of the branch locus is at least two. This was proved in [7]. In fact, the ramification locus of $Y$ equals the singular locus of $Y$, but this more refined assertion is not relevant here. Since $Y$ is normal it follows that the codimension of the ramification locus is at least 2. Therefore, by Theorem 2.2, $X$ is Frobenius split.

The case $m = n$ is isolated separately as Lemma 2.3 below. When $m < n$, it is easy to see that $S = R$. Again, $R$ is a polynomial algebra over $k$ and hence $S$ is Frobenius split. □

Assume that $m = n$. In this case $R = k[y_{i,j} : 1 \leq i \leq j \leq n]$ is a polynomial ring, being the ring of polynomial functions on the space of $n \times n$ symmetric matrices. As an $R$-algebra, $S = R[u]/(u^2 - f)$, where $f$ denotes the determinant function of the symmetric $n \times n$ matrix whose entry in position $(i, j)$ for $1 \leq i \leq j \leq n$ is $y_{i,j}$.

Lemma 2.3. Let $m = n$. The ring $S$ of $SO(V)$-invariants is Frobenius split in this case also.

Proof. There is a natural identification of Spec($R$) with an affine patch of the symplectic Grassmannian and the vanishing locus of $f$ under this identification becomes an open part of a Schubert variety [8, 7]. Thus by [10] (see also [1]), there exists a splitting of Spec($R$) that compatibly splits Spec($R/(f)$). Let $\varphi$ be such a splitting. Continue to denote by $\varphi$ the restriction of $\varphi$ to the open part Spec($R/(f)$). Arguing as in the proof of Proposition 2.1 above, we may ‘lift’ the restriction $\varphi$ to a splitting (also denoted $\varphi$) of Spec($R[1/f]$). We claim that $\varphi$ maps $S$ to $S$ and hence extends to a splitting of Spec($S$). Indeed, a general element of $S$ is of the form $au + b$ with $a, b$ in $R$, so that $\varphi(au + b) = \varphi(\frac{au^{p+1}}{u} + b) = \frac{\varphi(a f^{(p+1)/2})}{u} + \varphi(b)$. Since $\varphi$ compatibly splits the vanishing locus of $f$, it follows that $\varphi(a f^{(p+1)/2})$ belongs to the ideal $(f)$. Writing $\varphi(a f^{(p+1)/2}) = c f$, we have $\varphi(au + b) = \frac{cf}{u} + \varphi(b) = cu + \varphi(b) \in S$. □

We conclude this section with the following remarks.

Remark 2.4. (i) The condition on codimension of $U$ in Proposition 2.1 will be satisfied if $S$ is generated over $R$ by two or more elements $u_i$ such that there exist $u_i, u_j$ such that the supports $D_i$ and $D_j$ of the reduced scheme defined by $u_i = 0$ and $u_j = 0$ do not have any component in common.

(ii) Theorem 2.2 is not valid when the hypothesis on the codimension of the ramification locus is not satisfied. For example, if $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the involution of a hyperelliptic curve $X$ of genus $g \geq 2$, then the quotient is a smooth projective curve which is Frobenius
split. However, $X$ is not split since $\omega_X$ is ample but $H^1(X; \omega) \cong k$, whereas higher cohomologies for ample line bundles over Frobenius split projective varieties vanish.

(iii) We do not know if Theorem 2.2 remains valid if $\Gamma$ is any finite group whose order is prime to the characteristic $p$ of $k$, even in the case when $\Gamma$ is cyclic.

(iv) One has an isomorphism of $\text{SL}(2)$ with $\text{SO}(3)$ such that the $\text{SO}(3)$ action on $V = k^3$ corresponds to the conjugation action of $\text{SL}(2)$ on trace zero $2 \times 2$ matrices. In this case the Frobenius splitting property of $A_m$ was proved by Mehta-Ramadas [11, Theorem 6]. It should be noted that when $\dim V = 3$, the completion of the stalk at the origin in $A_m$ is isomorphic to the completion of the stalk at the point corresponding to the class of the trivial rank 2 vector bundle in the moduli space of equivalence classes of semi-stable, rank 2 vector bundles with trivial determinant on a smooth projective curve of genus $m > 2$ (see [11]).

3. Splitting $\text{SL}(n)$ invariants

In this section we shall establish Theorem 1.2. Let $V$ be an $n$ dimensional vector space over an algebraically closed field $k$ of characteristic $p \geq 2$ and let $V^*$ denote its dual. Let $V_{m,q} := V^{\oplus m} \oplus V^{\ast \oplus q}$, and let $A$ denote the ring of regular functions on $V_{m,q}$. By fixing dual bases for $V$ and $V^*$, we shall view elements of $V$ and $V^*$ as row and column vectors respectively, so that $V^{\oplus m}$ (resp. $V^{\ast \oplus q}$) is identified with the space $M_{m,n}$ of $m \times n$ matrices (resp. the space $M_{n,q}$ of $n \times q$ matrices) over $k$; further, $\text{GL}(V)$ gets identified with $\text{GL}_n(k)$ (the group of invertible $n \times n$ matrices over $k$). In the sequel, we shall denote $\text{GL}_n(k)$ by just $\text{GL}(n)$. Then the action of $\text{GL}(V)$ on $V^{\oplus m}$ gets identified with the multiplication on the right of $M_{m,n}$ by $\text{GL}(n)$. Similarly, the action of $g \in \text{GL}(V)$ on $V^{\ast \oplus q}$ gets identified with the multiplication on the left of $M_{n,q}$ by $g^{-1}$. The diagonal action of $\text{GL}(V)$ on $V^{\oplus m} \oplus V^{\ast \oplus q}$ is therefore defined as $(u, \xi).g = (ug, g^{-1}\xi)$ where $g \in \text{GL}(n)$ and $(u, \xi) \in M_{m,q} := M_{m,n} \times M_{n,q}$. We identify $A$ with the coordinate ring of $M_{m,q}$.

We denote by $R$ and $S$ the rings of invariants $A^{\text{GL}(n)}$ and $A^{\text{SL}(n)}$ respectively. Let $Y = \text{Spec}(R)$ and $X = \text{Spec}(S)$. Note that $Y$ and $X$ are the GIT quotients $M_{m,q}/\!/ \text{GL}(n)$ and $M_{m,q}/\!/ \text{SL}(n)$ respectively.

Let $m, q \geq n$. By [2, 8] we have that $Y$ is the determinantal variety $D_n(M_{m,q})$ consisting of all matrices in $M_{m,q}$ (the space of $m \times q$ matrices with entries in $k$) of rank at most $n$; further, we have (cf. [8])
an identification of $D_u(M_{m,q})$ with an open subset of a certain Schubert variety in the Grassmannian variety (of $q$-dimensional subspaces of $k^{m+q}$). Hence we obtain that $Y$ is Frobenius split (since Schubert varieties are Frobenius split). The multiplication map $\mu: M_{m,q} \to M_{m,q}$ factors through $Y$; further, under $\pi: X \to Y$ (induced by the inclusion $R \subset S$), we have, $\pi([u, \xi]) = u\xi \in M_{m,q}$ where $[u, \xi] \in X$ is the image of $(u, \xi) \in M_{m,q}$ under the GIT quotient $M_{m,q} \to X$.

Let $I(n, m)$ denote the set of all $n$-element subsets $I$ of $\{1, 2, \cdots, m\}$. Any such $I$ determines a regular function $u_I: M_{m,q} \to k$ which maps $(u, \xi)$ to the determinant of the $n \times n$ submatrix $u(I)$ of $u \in M_{m,n}$ with column entries given by $I$. Clearly $u_I$ is invariant under the action of $\text{SL}(n)$ on $M_{m,q}$ and hence yields a regular function $u_I$ on $X$.

We define $\xi(J)$ and $\xi_J$ for $J \in I(n, q)$ analogously; $\xi_J$ is also an $\text{SL}(n)$-invariant.

We have, $u_I\xi_J = := p_{I,J} \in R$ for all $I \in I(n, m), J \in I(n, q)$; indeed $p_{I,J}([u, \xi])$ is just the determinant of the $n \times n$ submatrix $u\xi \in M_{m,q}$ with row and column indices given by $I$ and $J$ respectively. It is shown in [9], among other things, that $S$ is generated as an $R$-algebra by $u_I, \xi_J, I \in I(n, m), J \in I(n, q)$, the ideal of relations being generated by $u_I\xi_J - p_{I,J}$, $I \in I(n, m), J \in I(n, q)$ together with certain quadratic relations among the $u_I$'s and certain quadratic relations among the $\xi_J$'s. Further, in [9], a standard monomial basis is constructed for $S$: as a particular consequence, we have that each $u_I$ (resp. $\xi_J$) is algebraically independent over $R$ for $I \in I(n, m)$ (resp. $J \in I(n, q)$).

For $K \in I(n, m), L \in I(n, q)$, let

$$R_{K,L} = R[1/p_{K,L}], Y_{K,L} = \text{Spec}(R_{K,L})$$

For a given $I \in I(n, m), J \in I(n, q)$, let

$$Y_I = \bigcup_{J' \in I(n, q)} Y_{I,J'}, Y_J = \bigcup_{I' \in I(n, m)} Y_{I',J}$$

Note that for $I \in I(n, m)$, any $Y_{I,J'}$ is contained in $Y_I$; similarly, for $J \in I(n, q)$, any $Y_{I',J}$ is contained in $Y_J$.

Set $X_I = \pi^{-1}(Y_I) \subset X$ and $X_J = \pi^{-1}(Y_J) \subset X$. Note that $u_I$ (resp. $\xi_J$) is non-zero on $X_I$ (resp. $X_J$). Denote by $f_I: X_I \to Y_I \times k^*$ the morphism $f_I = (\pi|X_I, u_I|X_I)$, and by $f_J: X_J \to Y_J \times k^*$ the morphism $f_J = (\pi|X_J, \xi_J|X_J)$.

**Lemma 3.1.** The morphisms $f_I: X_I \to Y_I \times k^*$ and $f_J: X_J \to Y_J \times k^*$ are isomorphisms for any $I \in I(n, m), J \in I(n, q)$.
Proof. We shall prove that $f_I$ is an isomorphism, the proof in the case of $f_J$ being the same.

Let $X_{I,J} = \pi^{-1}(Y_{I,J})$; then $X_{I,J}$ equals $\text{Spec}(S_{I,J})$ (where $S_{I,J} = S[1/p_{I,J}]$) and $X_{I,J}$ is contained in $X_I$. The morphism $f_{I,J}: X_{I,J} \to Y_{I,J}$ defined by the restriction of $f_I$ is induced by the $R_{I,J}$-algebra map $f_{I,J}^*: R_{I,J}[t, t^{-1}] \to S_{I,J}$ which maps $t$ to $u_I$. Note that $p_{I,J} = u_I \xi_J$ implies that $u_I$ is invertible in $S_{I,J}(= S[1/p_{I,J}])$.

We must show that

1. $f_{I,J}^*$ is an isomorphism of $k$-algebras
2. $f_{I,J}$ and $f_{I,J'}$ agree on the overlap $X_{I,J} \cap X_{I,J'}$ for any two $J, J' \in I(n,m)$.

1. Note that $u_{I'} = u_I u_{I'} \xi_J / p_{I,J} = u_I p_{I,J'} / p_{I,J} = f_{I,J}^*(p_{I,J'} / p_{I,J} t)$. Hence $u_{I'}$ is in the image of $f_{I,J}^*$ for any $I' \in I(n,m)$. Similarly $\xi_{I'}$ is also in the image of $f_{I,J}^*$ for any $J' \in I(n,m)$. Therefore $f_{I,J}^*$ is surjective. Now suppose that $P(t) \in R_{I,J}[t, t^{-1}]$ is in the kernel of $f_{I,J}^*$. We may assume that $P(t)$ is a polynomial in $t$ and that the coefficients of $P(t)$ are actually in $R$. Then $0 = f_{I,J}^*(P(t)) = P(u_I)$. Since $X_{I,J}$ is open in $X$, which is irreducible, the we see that the equation $P(u_I) = 0$ must hold in $S$. This contradicts the fact that $u_I$ is algebraically independent over $R$ (cf. [9],Theorem 6.06,(3)). Hence $f_{I,J}^*$ is an isomorphism.

2. It is evident that $f_{I,J}^*(t) = u_I \in S_{I,J}$ and $f_{I,J'}^*(t) = u_I \in S_{I,J'}$ both restrict to the same regular function, namely $u_I|_{X_{I,J} \cap X_{I,J'}}$, on the overlap $X_{I,J} \cap X_{I,J'} = \text{Spec}(S[1/p_{I,J}, 1/p_{I,J'}])$. It follows that $f_{I,J}$ and $f_{I,J'}$ agree on $X_{I,J} \cap X_{I,J'}$. This completes the proof that $f_I$ is an isomorphism.

Observe that, if $J, J' \in I(n,m)$, then $\xi_J / \xi_{J'} \in S[1/\xi_{J'}]$ defines a regular function on $Y_{J'}$. This is because, on $Y_{I,J'}$, $\xi_J / \xi_{J'} = (u_I \xi_J) / (u_I \xi_{J'}) = p_{I,J} / p_{I,J'}$. It is immediately seen that, on $Y_{I,J} \cap Y_{I,J'}$, the two regular functions $p_{I,J} / p_{I,J'}$ and $p_{I,J'} / p_{I,J'}$ agree. Therefore we conclude that $\xi_J / \xi_{J'}$ is a well-defined regular function on $Y_{J'}$. Clearly it is invertible on $Y_J \cap Y_{J'}$. Similar statements concerning $u_I / u_{I'}$ hold for any $I, I' \in I(n,m)$.

Notation: Let $m, q \geq n$.

Denote by $\mathcal{I}$ the disjoint union $I(n,m) \bigsqcup I(n,q)$. We set

$$\lambda_{\beta,\alpha} = \begin{cases} u_\alpha / u_\beta & \text{if } \alpha, \beta \in I(n,m), \\ \xi_\beta / \xi_\alpha & \text{if } \alpha, \beta \in I(n,q), \\ p_{\alpha,\beta} & \text{if } \beta \in I(n,q), \alpha \in I(n,m), \\ 1 / p_{\beta,\alpha} & \text{if } \beta \in I(n,m), \alpha \in I(n,q). \end{cases}$$
Consider the covering \( \{Y_\alpha\}_{\alpha \in I} \) of the open subvariety \( Y_0 := \bigcup_{\alpha \in I} Y_\alpha \subset Y \). The cocycle condition \( \lambda_{\alpha, \beta} \lambda_{\beta, \gamma} = \lambda_{\alpha, \gamma} \) is readily verified for any \( \alpha, \beta, \gamma \in I \). Thus we obtain a \( \mathbb{G}_m \)-bundle over \( Y_0 \); call it \( E \).

Let \( X_0 := \bigcup_{\alpha \in I} X_\alpha \).

**Lemma 3.3.** Assume that \( m, q \geq n \). With the above notations, the total space of the \( \mathbb{G}_m \)-bundle \( E \) over \( Y_0 \) is isomorphic to the open subvariety \( X_0 := \bigcup_{\alpha \in I} X_\alpha \subset X \).

**Proof.** The total space of the \( \mathbb{G}_m \)-bundle corresponding to \( D \) is \( \prod_{\alpha \in I} Y_\alpha \times k^*/\sim \) where \( (\pi([u, \xi]), t) \in Y_\alpha \times k^* \) is identified with \( (\pi([u; \xi]), \lambda_{\beta, \alpha}(\pi([u, \xi]), t)) \in Y_\beta \times k^* \) whenever \( \pi([u, \xi]) \in Y_\alpha \cap Y_\beta \). One has the following commuting diagram for any \( \alpha, \beta \in I \):

\[
\begin{array}{ccc}
Y_\alpha \times k^* & \supset & (Y_\alpha \cap Y_\beta) \times k^* \\
X_\alpha & \supset & X_\alpha \cap X_\beta \\
& \uparrow f_\alpha & \uparrow f_\beta \\
& X_\alpha \cap X_\beta & \subset X_\beta
\end{array}
\]

where \( f_\alpha \) is the restriction of \( f_\alpha \). Since, by Lemma 3.1, the \( f_\alpha \)'s are isomorphism of varieties, it follows that that the total space of the \( \mathbb{G}_m \)-bundle over \( Y_0 \) is isomorphic to the union \( X_0 := \bigcup_{\alpha \in I} X_\alpha \subset X \). 

We shall now compute the codimension of \( Z := X - X_0 \). We give the reduced scheme structure on \( Z \). It is evident that \( Z \) is defined by the equations \( p_{1,I} = 0, \forall I \in I(n,m), J \in I(n,q) \). We claim \( Z = Z_u \cup Z_\xi \) where \( Z_u \) is the closed subvariety with reduced scheme structure defined by the equations \( u_I = 0, \forall I \in I(n,m) \) and \( Z_\xi \), by the equations \( \xi_J = 0, \forall J \in I(n,q) \). Clearly \( Z_u \cup Z_\xi \subset Z \). On the other hand, if \( [u, \xi] \) is not in \( Z_u \cup Z_\xi \), then \( u_I([u, \xi]) \neq 0 \) for some \( I \) and \( \xi_J([u, \xi]) \neq 0 \) for some \( J \). This implies that \( p_{1,I}([u, \xi]) \neq 0 \). Hence \( [u, \xi] \in X_0 \). Thus \( Z_u \cup Z_\xi = Z \).

**Lemma 3.3.** Let \( m > n \) (resp. \( q > n \)). Then the codimension of \( Z_u \) (resp. \( Z_\xi \)) in \( X \) is at least 2.

**Proof.** Consider the closed subvariety \( M_u := D_{n-1}(M_{m,n}) \times M_{n,q} \subset \mathcal{M}_{m,q} \) (with reduced scheme structure). We have,

\[
dim M_u = (n-1)(m+1) + nq \quad \text{(note the dimension of the determinantal variety \( D_t(M_{r,s}) \) (consisting of \( r \times s \) matrices of rank at most \( t \)) equals \( t(r+s-t) \) (cf. \[8\])).}
\]

Clearly \( M_u \) is stable under the \( SL(n) \)-action and \( M_u/SL(n) = Z_u \). We shall find an open subset \( Z_{u,0} \) of \( Z_u \) such that \( SL(n) \) acts freely on the inverse image of \( Z_{u,0} \) under the quotient morphism \( \eta: M_u \longrightarrow Z_u \) and \( \eta^{-1}(Z_{u,0})/SL(n) = Z_{u,0} \). It would then follow that \( \dim(Z_u) = \dim(\eta^{-1}(Z_u)) - \dim(SL(n)) = (n-1)(m+1) + nq - (n^2 - 1) = (m+n)q - (n^2 - 1) - (m-n+1) = \dim(X) - (m-n+1) \leq \dim(X) - 2 \) (note that \( \dim X = (m+n)q - (n^2 - 1) \) (cf. \[9\])).
Define
\[ W_u = D_n(M_{m,n}) \times M_{n,q}^0 \]
where \( M_{n,q}^0 := \{ \xi \in M_{n,q} \mid \xi_J(\xi) \neq 0, \text{ for some } J \in I(n,q) \} \). Then \( W_u \) is the inverse image of
\[ Z_{u,0} := \{ [u, \xi] \mid \xi_J(\xi) \neq 0 \} \]
under the quotient morphism \( \eta: M_u \to Z_u \). The assertion that the \( \text{SL}(n) \)-action is free on \( W_u \) follows from the fact that the \( \text{SL}(n) \)-action on \( M_{n,q}^0 \) is free.

An entirely similar argument shows that \( Z_\xi \) has codimension at least 2, and consequently codimension of \( Z \) in \( X \) is at least 2.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let \( m, q > n \). As already observed, we have that \( Y = D_n(M_{m,q}) \) can be identified with an open subset of a certain Schubert variety in the Grassmann variety \( \text{SL}(m + q)/P_q \) of \( q \)-dimensional vector subspaces in \( k^{m+q} \). Since Schubert varieties in the Grassmann variety are Frobenius split, it follows that \( Y \) is Frobenius split. Since \( Y_0 \) is open in \( Y \), it follows that it is also Frobenius split. The variety \( X_0 \), being the total space of a \( \mathbb{G}_m \)-bundle over \( Y_0 \), is Frobenius split by [1], Lemma 1.1.11. Now \( X \) being normal and codimension of \( X_0 \) in \( X \) being at least 2, it follows that \( X \) is Frobenius split (cf. [1], Lemma 1.1.7, (iii)).

If \( m, q < n \), then \( X = Y = M_{m,q} \) and hence Frobenius split. The case \( m = n \) is isolated separately as Lemma 3.4 below.

Assume that \( q = n = m \). In this case \( Y = M_{n,n} \). Denote the \((i,j)\)-th coordinate function on \( Y \) by \( y_{i,j} \). The set \( I(n,m) \) and \( I(n,q) \) are singletons and so \( S = R[u, \xi]/(u\xi - f) \) where \( f \) is the determinant function on \( Y = M_{m,q} \).

**Lemma 3.4.** Let \( q = n = m \). The ring \( S \) of \( \text{SL}(V) \)-invariants is Frobenius split in this case also.

**Proof.** Let \( \varphi \) be a splitting of \( \text{Spec}(R) \). Continue to denote by \( \varphi \) the restriction of \( \varphi \) to the open part \( \text{Spec}(R[1/f]) \). We can ‘lift’ \( \varphi \) to the \( \mathbb{G}_m \)-bundle \( \text{Spec}(R[1/f][u, u^{-1}]) \) (over \( \text{Spec}(R[1/f]) \)) as follows: define \( \varphi(a + \sum b_iu^i + \sum c_ju^{-j}) := \varphi(a) + \sum \varphi(b_i)u^{i/p} + \sum \varphi(c_j)u^{-j/p} \), where the summations are over positive integers and \( u^{i/p} \) (respectively \( u^{-j/p} \)) is interpreted to be 0 unless \( i \) (respectively \( -j \)) is an integral multiple of \( p \). Observe that \( R[1/f][u, u^{-1}] = S[1/f] \), so we have a splitting of \( \text{Spec}(S[1/f]) \) which we still denote \( \varphi \). We claim that \( \varphi \) maps \( S \) to \( S \) and hence extends to a splitting of \( \text{Spec}(S) \). Indeed, a general element \( s \) of \( S \)
is of the form $a + \sum b_i u^i + \sum c_j \xi^j$ with $a$, $b_i$, and $c_j$ in $R$, so that $\varphi(s) = 
abla(a + \sum b_i u^i + \sum c_j f^j u^{-j}) = \varphi(a) + \sum \varphi(b_i) u^{i/p} + \sum \varphi(c_j f^j) u^{-j/p}$. Rewriting $\varphi(c_j f^j) u^{-j/p}$ as $\varphi(c_j) f^{j/p} u^{-j/p} = \varphi(c_j) \xi^{j/p}$, we see that $\varphi(s)$ belongs to $S$.

**Remark 3.5.** In the case when one of $\{m, q\}$ being $< n$, and the other $\geq n$, we expect the ring of invariants to be Frobenius split though at the moment, we do not have a proof of this assertion!

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