The algebra of invariants of the adjoint action of the unitriangular group in the nilradical of a parabolic subalgebra

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Abstract. In the paper the algebra of invariants of the adjoint action of the unitriangular group in the nilradical of a parabolic subalgebra is studied. We set up a conjecture on the structure of the algebra of invariants. The conjecture is proved for parabolic subalgebras of special types.

Consider the general linear group $GL(n, K)$ defined over an algebraically closed field $K$ of characteristic 0. Let $B$ ($N$, respectively) be its Borel (maximal unipotent, respectively) subgroup, which consists of triangular matrices with nonzero (unit, respectively) elements on the diagonal. We fix a parabolic subgroup $P$ that contains $B$. Denote by $p$, $b$ and $n$ the Lie subalgebras in $gl(n, K)$ that correspond to $P$, $B$ and $N$, respectively. We represent $p = r ⊕ m$ as the direct sum of the nilradical $m$ and a block diagonal subalgebra $r$ with sizes of blocks $(n_1, \ldots, n_s)$. The subalgebra $m$ is invariant relative to the adjoint action of the group $P$, therefore $m$ is invariant relative to the adjoint action of the subgroups $B$ and $N$. We extend this action to the representation in the algebra $K[m]$ and in the field $K(m)$. The subalgebra $m$ contains a Zariski-open $P$-orbit, which is called the Richardson orbit (see [R]). Consequently, $K[m]^P = K$. In this paper, we study the structure of the algebra of invariants $K[m]^N$. In the case $P = B$, the algebra of invariants $K[m]^N$ is the polynomial algebra $K[x_{12}, x_{23}, \ldots, x_{n-1,n}]$. Let $r$ be the sum of two blocks; this case was considered by M. Brion in the paper [B]. A complete description of the field of invariants $K(m)^N$ for any parabolic subalgebra is a result of [S]. The question concerning the structure of the algebra of invariants $K[m]^N$ remains open and seems to be a considerable challenge. We do not know when the algebra of invariants $K(m)^N$ is finitely generated.

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In this paper, we consider the series of polynomials (see the notations (5), (6), (7)) in the algebra of invariants. We show that these polynomials generate the algebra of invariants in special cases (Theorems 4.1 and 5.2). We state a conjecture on the structure of the algebra of invariants $K[m]^N$ (Conjecture 5.4).

§1. The main definitions

We begin with definitions. Every positive root $\gamma$ in $\mathfrak{gl}(n, K)$ has the form (see [GG]) $\gamma = \varepsilon_i - \varepsilon_j$, $1 \leq i < j \leq n$. We identify a root $\gamma$ with the pair $(i, j)$ and the set of the positive roots $\Delta^+$ with the set of pairs $(i, j)$, $i < j$. The system of positive roots $\Delta^+_r$ of the reductive subalgebra $r$ is a subsystem in $\Delta^+$.

Let $\{E_{i,j} : i < j\}$ be the standard basis in $n$. By $E_{\gamma}$ denote the basis element $E_{i,j}$, where $\gamma = (i, j)$.

We define a relation in $\Delta^+$ such that $\gamma' \succ \gamma$ whenever $\gamma' - \gamma \in \Delta^+_r$. If $\gamma < \gamma'$ or $\gamma \succ \gamma'$, then the roots $\gamma$ and $\gamma'$ are said to be comparable. Denote by $M$ the set of $\gamma \in \Delta^+$ such that $E_{\gamma} \in m$. We identify the algebra $K[m]$ with the polynomial algebra in the variables $x_{i,j}$, $(i, j) \in M$.

**Definition 1.1.** A subset $S$ in $M$ is called a base if the elements in $S$ are not pairwise comparable and for any $\gamma \in M \setminus S$ there exists $\xi \in S$ such that $\gamma \succ \xi$.

**Definition 1.2.** Let $A$ be a subset in $S$. We say that $\gamma$ is a minimal element in $A$ if there is no $\xi \in A$ such that $\gamma \succ \xi$.

Note that $M$ has a unique base $S$, which can be constructed in the following way. We form the set $S_1$ of minimal elements in $M$. By definition, $S_1 \subset S$. Then we form a set $M_1$, which is obtained from $M$ by deleting $S_1$ and all elements

$$\{\gamma \in M : \exists \xi \in S_1, \gamma \succ \xi\}.$$ 

The set of minimal elements $S_2$ in $M_1$ is also contained in $S$, and so on. Continuing the process, we get the base $S$.

**Definition 1.3.** An ordered set of positive roots $\{\gamma_1, \ldots, \gamma_s\}$ is called a chain if $\gamma_1 = (a_1, a_2)$, $\gamma_2 = (a_2, a_3)$, $\gamma_3 = (a_3, a_4)$, and so on. The number $s$ is called the length of a chain.

**Definition 1.4.** We say that two roots $\xi, \xi' \in S$ form an admissible pair $q = (\xi, \xi')$ if there exists $\alpha_q \in \Delta^+_r$ such that the ordered set of roots $\{\xi, \alpha_q, \xi'\}$ is a chain. Note that the root $\alpha_q$ is uniquely determined by $q$. 

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We form the set \( Q := Q(p) \) that consists of admissible pairs of roots in \( S \). For every admissible pair \( q = (\xi, \xi') \) we construct a positive root \( \varphi_q = \alpha_q + \xi' \). Consider the subset \( \Phi = \{ \varphi_q : q \in Q \} \) in the set of positive roots.

**Definition 1.5.** The set \( S \cup \Phi \) is called an *expanded base*.

Using the given parabolic subgroup, we construct a diagram, which is a square matrix in which the roots from \( S \) are marked by the symbol \( \otimes \) and the roots from \( \Phi \) are labeled by the symbol \( \times \). The other entries in the diagram are empty.

**Example 1.6.** Below a diagram for a parabolic subalgebra with sizes of its diagonal blocks \((2, 1, 3, 2)\) is given.

![Diagram 1](image_url)

Consider the formal matrix \( X \) in which the variables \( x_{i,j} \) occupy the positions \((i, j) \in M\) and the other entries are equal to zero. For any root \( \gamma = (a, b) \in M \) we denote by \( S_\gamma \) the set of \( \xi = (i, j) \in S \) such that \( i > a \) and \( j < b \). Let \( S_\gamma = \{(i_1, j_1), \ldots, (i_k, j_k)\} \). Denote by \( M_\gamma \) a minor \( M_{IJ} \) of the matrix \( X \) with ordered systems of rows \( I \) and columns \( J \), where

\[
I = \text{ord}\{a, i_1, \ldots, i_k\}, \quad J = \text{ord}\{j_1, \ldots, j_k, b\}.
\]

For every admissible pair \( q = (\xi, \xi') \) such that \( q \) corresponds to \( \varphi \in \Phi \), we construct the polynomial

\[
L_\varphi = \sum_{\alpha_1, \alpha_2 \in \Delta^+_1 \cup \{0\}, \alpha_1 + \alpha_2 = \alpha_q} M_{\xi + \alpha_1} M_{\alpha_2 + \xi'}.
\]

**Theorem 1.7.** [PS] For an arbitrary parabolic subalgebra, the system of polynomials

\[
\{ M_\xi, \xi \in S, \ L_\varphi, \varphi \in \Phi, \}
\]

is contained in \( K[m]^N \) and is algebraically independent over \( K \).
Denote by $Y$ the subset in $m$ that consists of matrices of the form
\[ \sum_{\xi \in S} c_\xi E_\xi + \sum_{\varphi \in \Phi} c'_\varphi E_\varphi, \]
where $c_\xi \neq 0$ and $c'_\varphi \neq 0$.

**Definition 1.8.** The matrices from $Y$ are said to be canonical.

The proof of the following theorem is found in [S].

**Theorem 1.9.** There exists a nonempty Zariski-open subset $U \subset m$ such that the $N$-orbit of any $x \in U$ intersects $Y$ at a unique point.

Now let $S$ be the set of denominators generated by minors $M_\xi$, $\xi \in S$. Consider the localization $K[m]^N_S$ of the algebra of invariants $K[m]^N$ with respect to $S$. Since the minors $M_\xi$ are $N$-invariants, we have
\[ K[m]^N_S = (K[m]^N_S)^N. \]

The following results are consequences of Theorem 1.9.

**Theorem 1.10.** The ring $K[m]^N_S$ is the ring of polynomials in $M_\xi^{\pm 1}$, $\xi \in S$, and in $L_\varphi$, $\varphi \in \Phi$.

**Theorem 1.11.** The field of invariants $K(m)^N$ is the field of rational functions of $M_\xi$, $\xi \in S$, and $L_\varphi$, $\varphi \in \Phi$.

The polynomials $M_\xi$ and $L_\varphi$ do not generate the algebra of invariants. We prove in §4 and §5, respectively, that if the reductive subalgebra $\tau$ consists of three blocks or of four blocks with sizes $(1, 2, 2, 1)$, then the algebra of invariants $K[m]^N$ is not generated by the invariants $M_\xi$ and $L_\varphi$. Let the reductive subalgebra consist of blocks with sizes $(2, k, 2)$, $k > 3$, or with sizes $(1, 2, 2, 1)$. We construct a polynomial $D$ such that the algebra of invariants is generated by $D$ and by the invariants $M_\xi$ and $L_\varphi$.

**§2. The other definition of the invariant $L_\varphi$**

In the present chapter, we give the alternative definition of the invariant $L_\varphi$, where $\varphi \in \Phi$.

Let $I$ and $I'$ be two systems of rows, and let $J$ and $J'$ be two systems of columns:
\[ I = \{i, i + 1, \ldots, i + k\}, \]
\[ J = \{j, j + 1, \ldots, j + l\}, \]
\[ I' = \{i', i' + 1, \ldots, i' + k'\}, \]

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where \(i + k < j' \leq i' \) and \(j' + l' < i' + k' < j\). Suppose the minors \(X_{J'}^J\) and \(X_{J'}^{J'}\) are bordered by zeros from below and left in \(X\), where \(X_{J'}^J\) is the minor of the matrix \(X\) with systems of rows \(I\) and columns \(J'\). In other words, if for any positive root \((a, b)\) one of the following conditions holds

1) \(a \in I\) and \(b < j'\),
2) \(a \in I'\) and \(b < j\),
3) \(a > i + k\) and \(b \in J'\),
4) \(a > i' + k'\) and \(b \in J\),

then the root \((a, b)\) is not in the system \(M\). Let \(k + k' = l + l'\). The following determinant is one of order \(k + k' + 2\):

\[
D_{I,J'}^{I',J''} = \begin{vmatrix} X_{J'}^{J''} & (X^2)_{J'}^I \\ 0 & X_{I'}^J \end{vmatrix},
\]

where the minor \(X_{I'}^J\) of the matrix \(X\) is formed by systems of rows \(I\) and columns \(J\).

**Definition 2.1.** The determinant \(D_{I,J'}^{I',J''}\) is called a **combined minor**.

**Lemma 2.2.** A combined minor \((2)\) is an \(N\)-invariant.

**Proof.** A polynomial \(H\) is an \(N\)-invariant if \(H\) is an invariant under the adjoint action of all one-parameter subgroups \(g_m(t) = E + tE_{m,m+1}\), \(1 \leq m < n\), where \(E\) is a matrix unit. The adjoint action on the matrix \(X\) by \(g_m(t)\) reduces to the composition of two transformations:

1) the row with number \(m + 1\) multiplied by \(t\) is added to the row \(m\) of the matrix \(X\);

2) the column with the number \(m\) multiplied by \(-t\) is added to the column with the number \(m + 1\) of the matrix \(X\).

We show that the polynomial \((2)\) is an invariant under the adjoint action of \(g_{m,m+1}(t)\). Without loss of generality, it can be assumed that \(i + k = j' - 1\), \(i' + k' = j - 1\) and the minors \(X_{J'}^J\), \(X_{J'}^{J'}\), and \((X^2)_{J'}^I\) have the following form

\[
X_{I'}^{J'} = \begin{vmatrix} x_{i,j'} & x_{i,j'+1} & \ldots & x_{i,j'+l'} \\ x_{i+1,j'} & x_{i+1,j'+1} & \ldots & x_{i+1,j'+l'} \\ \ldots & \ldots & \ldots & \ldots \\ x_{i+k,j'} & x_{i+k,j'+1} & \ldots & x_{i+k,j'+l'} \end{vmatrix},
\]
the adjoint action of the element \( \tilde{g}_{m,m+1} \) doesn’t change the minor \( (2) \).

1. Suppose \( i \leq m < i+k \); then the action of the element \( g_{m,m+1} \) changes the variables \( x_{a,b} \) of the minor \( D^{J,J'}_{I,I'} \) such that \( a = m \) and doesn’t change the other variables. Therefore

\[
g_{m,m+1}(t) x_{m,p} = x_{m,p} - tx_{m+1,p}, \quad j' \leq p \leq j'+k';
\]

\[
g_{m,m+1}(t) \sum_{p=j'}^{j'+k'} x_{m,p} x_{p,b} = \sum_{p=j'}^{j'+k'} x_{m,p} x_{p,b} - t \sum_{p=j'}^{j'+k'} x_{m+1,p} x_{p,b}.
\]

Then we have

\[
g_{m,m+1}(t) X^J_{I'} = X^J_{I'},
\]

\[
g_{m,m+1}(t) X^J_{I} = X^J_{I} - tD_1.
\]

The order of \( D_1 \) is equal to the order of \( X^J_{I'} \). The minor \( D_1 \) is formed by the columns \( J' \) and \( D_1 \) has two identical rows with numbers \( m+1 \) and \( m \). Further,

\[
g_{m,m+1}(t) (X^2)^J_I = (X^2)^J_I - tD_2,
\]

where the order of \( D_2 \) is equal to the order of \( (X^2)^J_I \). The minor \( D_2 \) has two identical rows with numbers \( m \) and \( m+1 \).

Thus,

\[
g_{m,m+1}(t) D^{J,J'}_{I,I'} = g_{m,m+1}(t) \begin{vmatrix} X^J_{I} & (X^2)^J_I \\ 0 & X^J_{I'} \end{vmatrix} =
\]
\[
\begin{vmatrix}
X_{I}^\prime
\frac{(X^2)_{I}^\prime}{t}
\frac{D_1}{0}
\frac{D_2}{X_{I^\prime}^\prime}
\end{vmatrix}.
\]

The minor that consists of \(D_1\), \(D_2\), and \(X_{I^\prime}^\prime\) has two identical rows; therefore this minor is equal to zero. We have

\[g_{m,m+1}(t)D_{I,I^\prime}^{J,J^\prime} = D_{I,I^\prime}^{J,J^\prime}.\]

2. Let \(i + k \leq m < j\). For any \(a \in I\) and \(b \in J\) we have

\[g_{m,m+1}(t) \sum_{p=j}^{i+k} x_{a,p}x_{p,b} = \sum_{p=j}^{m-1} x_{a,p}x_{p,b} + x_{a,m}(x_{m,b} - tx_{m+1,b}) + \sum_{p=m+2}^{i+k} x_{a,p}x_{p,b},\]

i.e., the adjoint action of the element \(g_{m,m+1}(t)\) does not change the minor \((X^2)_{I}^\prime\). Besides, the action of the element \(g_{m,m+1}(t)\) adds the column with the number \(m\) to the \((m+1)\)th column of the minor \(X_{I^\prime}^\prime\) and the \((m+1)\)th row to the \(m\)th row of the minor \(X_{I^\prime}^\prime\). Thus the minor \(D_{I,I^\prime}^{J,J^\prime}\) is a \(g_{m,m+1}(t)\)-invariant.

3. For \(j \leq m < j + l\), the proof is similarly. \(\square\)

We shall show that the invariant \(L_\phi\) is a combined minor \((2)\) for any systems \(I, I^\prime, J, J^\prime\).

**Definition 2.3.** Any root \((i, j) \in M\) satisfying the conditions

\[R_{k-1} < i \leq R_k \text{ and } R_k < j \leq n,\]

is called a root lying to the right of the \(k\)th block in \(r\).

**Definition 2.4.** Any root \((i, j) \in M\) satisfying the conditions

\[1 \leq i \leq R_{k-1} \text{ and } R_{k-1} < j \leq R_k,\]

is called a root lying to the above of the \(k\)th block in \(r\).

Let \(\phi\) be any root from \(\Phi\). The root \(\phi\) corresponds to an admissible pair \((\xi, \xi^\prime)\). Let \(\xi = (i, j)\), \(\xi^\prime = (l, m)\), and \(i < l\). Assume that \(\phi\) lies to the right of the \(k\)th block in \(r\), i.e., \(R_{k-1} < j \leq R_k\). Let the set \(S_\xi\) has \(p\) roots and
Let the set $S_{\xi}$ has $q$ roots. We have $i + p = R_{k-1}$, i.e., the minimal root $\gamma$ in $S_{\xi}$, which has the greatest number of row, lies in the row with number $R_{k-1}$. Similarly, the root of the base that has the least number of column is also minimal; so we have $l - q = R_k + 1$.

Denote

$$I = \{i, i + 1, \ldots, i + p\}, \quad J = \{m - q, m - q + 1, \ldots, m\},$$

$$I' = \{l + 1, l + 2, \ldots, l + q\}, \quad J' = \{j - p, j - p + 1 \ldots, j - 1\}.$$  

Then $|I| = p + 1$, $|J| = q + 1$, $|I'| = q$, $|J'| = p$. Consider the following combined minor

$$\widetilde{L}_\varphi = D_{I,J,I',J'} = \begin{vmatrix} X_{I,J} & (X^2)_I^J \\ 0 & X_{I',J'} \end{vmatrix},$$

where $X_{I,J}$ is the minor of $X$ with the systems of rows $I$ and columns $J$. The order of $\widetilde{L}_\varphi$ is equal to $(p + q + 1)$.

**Proposition 2.5.** We have the equation $\widetilde{L}_\varphi = L_\varphi$.

**Proof.** By Lemma 2.2, the polynomial $\widetilde{L}_\varphi$ is an invariant. Let $p : K[m]^N \to K[Y]$, $p(f) = f|_Y$ be a restriction homomorphism. We calculate the invariants $L_\varphi$ and $\widetilde{L}_\varphi$ on $Y$.

Let $Y$ be the formal matrix in which the variables $x_{i,j}$ occupy the positions $(i, j) \in S \cup \Phi$ and the other entries are equal to zero.

Assume that the root $\varphi$ corresponds to the admissible pair $(\xi, \xi')$, where

$$\xi = (i, j), \quad \xi' = (l, m),$$

\[ S_\xi = \{\xi_1, \xi_2, \ldots, \xi_p\}, \quad S_{\xi'} = \{\xi'_1, \xi'_2, \ldots, \xi'_q\}, \]

respectively.

Then using elementary transformations of columns, the minor $Y_{I,J}''$ of the matrix $Y$ is reduced to the following minor of order $(p + 1) \times p$:

\[
\begin{vmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & x_{\xi_p} \\
\ldots & \ldots & \ldots & \ldots \\
0 & x_{\xi_2} & \ldots & 0 \\
x_{\xi_1} & 0 & \ldots & 0
\end{vmatrix}
\]

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The minor $\mathcal{Y}^I_p$ is reduced to the following minor of order $q \times (q + 1)$ by elementary transformations of rows:

\[
\begin{array}{cccc}
0 & 0 & \ldots & x_{\xi_q} \\
\vdots & \vdots & \ddots & \vdots \\
0 & x_{\xi_2} & \ldots & 0 \\
x_{\xi_1} & 0 & \ldots & 0 \\
\end{array}
\]

We have

\[
\widetilde{L}_\varphi = \pm \begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & x_{\xi_p} \\
\vdots & \vdots & \ddots & \vdots \\
0 & x_{\xi_2} & \ldots & 0 \\
x_{\xi_1} & 0 & \ldots & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \ldots & x_{\xi_q} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 \\
x_{\xi_1} & 0 & \ldots & 0 \\
\end{pmatrix}
= \pm x_\xi \prod_{s=1}^p x_{\xi_s} \prod_{t=1}^q x_{\xi_t}.
\]

(3)

Direct calculations show that the invariant $L_\varphi$ on $\mathcal{Y}$ is equal to (3).

Let we show that the mapping $p$ is an embedding. Indeed, if $f \in \ker p$, then $f(\text{Ad}_N \mathcal{Y}) = 0$. By Theorem 1.9, we have $\text{Ad}_N \mathcal{Y}$. Then $f \equiv 0$ and $p$ is an embedding.

The invariant $L_\varphi$ is equal to $\widetilde{L}_\varphi$ on $\mathcal{Y}$. Therefore $L_\varphi$ is equal to $\widetilde{L}_\varphi$ on the pre-image $p^{-1}(\mathcal{Y})$. Finally, we get $L_\varphi = \widetilde{L}_\varphi$. \square

§3. The additional set of $N$-invariants

Let $\mathfrak{p}$ be any parabolic subalgebra.

Now we introduce a system of $N$-invariants that are not in the algebra $K[M_{\xi}, L_\varphi]_{\varphi \in \Phi}$.

Let $(i, j) \in S \cup \Phi$. Denote

\[
L_{i,j} = \begin{cases} 
L_{(i,j)} & \text{if } (i, j) \in \Phi; \\
M_{(a,i)} \cdot M_{(i,j)} & \text{if } (i, j) \in S \text{ and there is a number } a \text{ such that } (a, i) \in S.
\end{cases}
\]
Assume that for any numbers \( m < l \) and \( i < j \) the roots
\[
(m, i), (l, i), (m, j), (l, j)
\]
are contained in the expanded base. Obviously, the root \((m, i)\) lies in the system \( \Phi \). Suppose that the root \((m, i)\) corresponds to some admissible pair \((\xi, \xi') \in Q, \xi, \xi' \in S\).

**Notation 3.1.** Denote
\[
A_{m}^{i,j} = \frac{L_{m+1,i}L_{m,j} - L_{m+1,j}L_{m,i}}{M_{\xi}}.
\]

Now assume that the following condition holds for the roots \((4)\): there is not a number \( \tilde{i} \), \( i < \tilde{i} < j \), such that \((m, \tilde{i}) \in S \cup \Phi\).

**Notation 3.2.** Denote
\[
B_{m,l}^{i} = \frac{L_{m,j}L_{i,l} - L_{l,j}L_{m,i}}{M_{\xi'}},
\]
\[
C_{m}^{i} = \frac{L_{m+1,i}L_{m,j} - L_{m+1,j}L_{m,i}}{M_{\xi} \cdot M_{\xi'}}.
\]

Clearly, all rational function \((5), (6), (7)\) are contained in the localization \( K[m]_{S}^{N} \), where the set \( S \) is generated by the minors \( M_{\xi}, \xi \in S \). We shall prove that the functions \((5), (6), (7)\) are polynomials.

Let the root \((m, i)\) be to the right of the \( k \)th block in \( r \), i.e., \( R_{k-1} < m \leq R_{k} \). Let the invariant \( C_{m}^{i} \) is constructed by roots
\[
(m, i), (m + 1, i), (m, j), (m + 1, j),
\]
which contained in the expanded base. Then there exists no number \( \tilde{i} \) such that \( i < \tilde{i} < j \) and \((m, \tilde{i}) \in \Phi\). If the root \((m + 1, j)\) lies in the system \( \Phi \), then \((m + 1, j)\) corresponds to some admissible pair \((\xi, \xi')\). If \((m + 1, j) \in S\), then there exists a root \( \xi \) in the column with number \( m + 1 \). Indeed, \((m + 1, i) \in \Phi\) corresponds to an admissible pair that contains \( \xi \). In any case, we have that there exist the roots \( \xi = (a, m + 1) \) and \( \xi' = (b, j) \) for some numbers \( a \) and \( b \). Consider the sets \( S_{\xi} \) and \( S_{\xi'} \). Suppose \( S_{\xi} \) contains \( p \) roots and \( S_{\xi'} \) contains \( q \) roots. Since the root \( \xi \) lies to the above of the \( k \)th block in \( r \), then we have \( a + p = R_{k-1} \), i.e., the root from \( S_{\xi} \) that has the greatest number of row is contained in the row \( R_{k-1} \). Similarly, \( j - q = R_{k} + 1 \).

**Example 3.3.** Suppose the reductive subalgebra \( r \) is formed by the blocks \((3, 1, 4, 1, 2, 3)\). We have the following diagram.
Consider the invariant $C_{5}^{11}$. The invariant corresponds to the roots 

$$(5, 11), (6, 11), (5, 14), (6, 14)$$

from the expanded base. By the above notations, we have $\xi = (2, 6)$ and $\xi' = (6, 14)$, $p = 2$, and $q = 5$.

Denote

$$I = \{a, a + 1, \ldots, a + p\}, \quad J = \{j - q, j - q + 1, \ldots, j\},$$

$$I' = \{b + 2, b + 3, \ldots, b + q\}, \quad J' = \{m - p + 1, m - p + 2 \ldots, m - 1\}.$$  

Then $|I| = p + 1$, $|J| = q + 1$, $|I'| = q - 1$, $|J'| = p - 1$. Consider the following combined minor of order $(p + q)$:

$$\tilde{C}_m^i = D_{I,J,I',J'}^{J',J} = \begin{vmatrix}
X_{I,J}' & (X^2)^J_I \\
0 & X_{I,J}'
\end{vmatrix},$$

where the minor $X^I_J$ of the formal matrix is formed by the intersection of rows $I$ and columns $J$.

**Example 3.4.** The polynomial $\tilde{C}_5^{11}$ of Example 3.3 is formed in the following way.

$$\begin{vmatrix}
X_{2,3,4}^4 & (X^2)_{0,10,11,12,13,14}^{0,10,11,12,13,14} \\
0 & X_{8,9,10,11}^{9,10,11,12,13,14}
\end{vmatrix}.$$
The reader will easily prove that \( \tilde{C}_{11}^{11} = C_{11}^{11} \).

**Proposition 3.5.** We have \( C_m^i = \tilde{C}_m^i \) for any numbers \( m \) and \( i \).

**Proof.** Let us show that \( \tilde{C}_m^i = C_m^i \).

Let
\[ p : K[m]^N \to K[Y], \quad p(f) = f|_Y \]
be the the restriction homomorphism. We show that \( p \) is an embedding. Indeed, if \( f \in \text{Ker} \, p \), then \( f(\text{Ad}_N \, Y) = 0 \). Since by Theorem 1.9 \( \text{Ad}_N \, Y \) is contained a Zariski-open subset, then \( f \equiv 0 \). Therefore \( p \) is an embedding.

By Lemma 2.2, the minor \( \tilde{C}_m^i \) is an invariant. We show that the invariant \( L_\phi \) is equal to \( \tilde{L}_\phi \) on \( Y \).

Now assume that the roots
\[ \varphi_1 = (m, i), \varphi_2 = (m + 1, i), \varphi_3 = (m, j), \varphi_4 = (m + 1, j) \]
are in the expanded base and there exists no number \( \tilde{i} \) such that \( i < \tilde{i} < j \) and \( (m, \tilde{i}) \in \Phi \). Suppose the root \( \varphi = (m+1, j) \) corresponds to the admissible pair
\[ \xi = (a, m+1), \xi' = (b, j). \]

Assume that the sets \( S_\xi \) and \( S_{\xi'} \) consist of the roots
\[ S_\xi = \{\xi_1, \xi_2, \ldots, \xi_p\}, \quad S_{\xi'} = \{\xi'_1, \xi'_2, \ldots, \xi'_q\}, \]
where the roots \( \xi_p \) and \( \xi'_q \) lie in the \( m \)th column and in the \((b+1)\)th row, respectively. Then by elementary transformations of columns, the minor \( Y_{I'}^{I'} \) of the matrix \( Y \) is reduced to the following minor of order \((p+1) \times (p-1)\):
\[
\begin{vmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & x_{\xi_p-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & x_{\xi_3} & \ldots & 0 \\
x_{\xi_1} & 0 & \ldots & 0
\end{vmatrix}
\]

By elementary transformations of rows, the minor \( Y_{I'}^{I'} \) is reduced to the following minor of order \((q-1) \times (q+1)\):
\[
\begin{vmatrix}
0 & 0 & \ldots & x_{\xi_q-1} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & x_{\xi_2} & \ldots & 0 & 0 & 0 \\
x_{\xi'_1} & 0 & \ldots & 0 & 0 & 0
\end{vmatrix}
\]
We have

$$\widetilde{C}_m^i = \pm \begin{vmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & x_{\xi_2} & \ldots & 0 \\
x_{\xi_1} & 0 & \ldots & 0 \\
0 & 0 & \ldots & x_{\xi_{q-1}}^\prime \\
\vdots & \vdots & \ddots & \vdots \\
x_{\xi_1}^\prime & 0 & \ldots & 0
\end{vmatrix} = (Y^2)_{1}^j$$

$$= \pm \prod_{s=1}^{p-1} x_{\xi_s} \cdot \prod_{t=1}^{q-1} x_{\xi_t}^\prime \cdot \begin{vmatrix}
x_{\xi_{p+1}} x_{\varphi_2} & x_{\xi_{p+1}} x_{\varphi_3} \\
x_{\xi_p} x_{\varphi_1} & x_{\xi_p} x_{\varphi_3}
\end{vmatrix} = \pm x_\xi \prod_{s=1}^{p} x_{\xi_s} \cdot \prod_{t=1}^{q-1} x_{\xi_t}^\prime \cdot (x_{\varphi_2} x_{\varphi_3} - x_{\varphi_1} x_{\varphi_4}). \quad (8)$$

By direct calculations, $C_m^i$ coincides with (8) on $Y$. Thus $C_m^i$ coincides with (8) on $\mathcal{Y}$, and we have $\widetilde{C}_m^i$ is equal to $C_m^i$ on the pre-image $p^{-1}(\mathcal{Y})$. So, $C_m^i = \widetilde{C}_m^i$. □

Thus the invariant $C_m^i$ is a polynomial.

**Proposition 3.6.** We have $A_{m}^{i,j}, \; B_{m,l}^{i,j} \in K[m]^N$.

**Proof.** Let us prove that $A_{m}^{i,j}$ is a polynomial. The proof for $B_{m,l}^{i,j}$ is similarly.

Suppose the roots $(m, i)$, $(m, j)$, $(m + 1, i)$, and $(m + 1, j)$, $i < j$, are contained in the expanded base and these roots lie to the right of the $k$th block in the subalgebra $r$. We show that $A_{m}^{i,j}$ is a polynomial. The proof is by induction.

Assume that there are $p$ roots to the above of the $k$th block in $S$. It is easily shown that these roots are the following ones:

$$(i_1, R_{k-1} + 1), (i_2, R_{k-1} + 2), \ldots, (i_p, R_{k-1} + p)$$

for some $i_1 > i_2 > \ldots > i_p$. Note that we have $i_1 = R_{k-1}$ for the root $(i_1, R_{k-1} + 1)$. Suppose there are $q$ roots in $S$ to right of the $k$th block:

$$(R_k, j_1), (R_k - 1, j_2), \ldots, (R_k - q + 1, j_q)$$

for some $j_1 < j_2 < \ldots < j_q$. We have $j_1 = R_k + 1$. There exist the numbers $v$ and $w$ such that $i = j_v$ and $j = j_w$. Assume that $w - v > 2$. Suppose the root
corresponds to the admissible pair \((\xi, \xi')\). We fix the number \(m\). We have the following equation

\[
L_{m+1,j_w} M_{\xi} C_{j_w-1} = A_{m}^{j_w} L_{m+1,j_w-1} - A_{m}^{j_w} L_{m+1,j_w}
\]

for any \(1 < w \leq q\). By Proposition 3.5, the left part of the equation (9) lies in \(K[m]\). But \(A_{m}^{j_w}\) is contained in the ring \(K[m]\) provided \(A_{m}^{j_w-1} \in K[m]\). Finally from Proposition 3.5 the function

\[
A_{m}^{j_w+1} = C_{m}^{j_w} \cdot M_{\xi'}
\]

lies in the ring \(K[m]\). \(\square\)

**Conjecture 3.7.** Suppose the reductive subalgebra \(\mathfrak{r}\) consists of three blocks; then the algebra of invariants \(K[m]^{N}\) is generated by the polynomials \(M_{\xi}, \xi \in S, L_{\varphi}, \varphi \in \Phi\), and the elements \((5), (6), (7)\).

§4. The algebra of invariants in the case \((2, k, 2)\)

Assume that the reductive subalgebra of the parabolic subalgebra consists of three blocks with sizes \((2, k, 2), k > 3\). We shall prove Conjecture 3.7 in this case.

The roots

\[
S = \{(1, 4), (2, 3), (k + 1, k + 4), (k + 2, k + 3)\},
\]

\[
\Phi = \{(3, k + 3), (3, k + 4), (4, k + 3), (4, k + 4)\}
\]

form the base and the system \(\Phi\), respectively. We have the following diagram for the parabolic subalgebra in the case \((2, k, 2)\).

![Diagram 3](attachment:image.png)
Denote
\[ M_1 = M_{(2,3)}, \quad M_2 = M_{(1,4)}, \quad N_1 = M_{(k+2,k+3)}, \quad N_2 = M_{(k+1,k+4)}; \]
\[ L_{11} = L_{((2,3),(k+2,k+3))}, \quad L_{12} = L_{((2,3),(k+1,k+4))}, \]
\[ L_{21} = L_{((1,4),(k+2,k+3))}, \quad L_{22} = L_{((1,4),(k+1,k+4))}. \]

The nonzero polynomials (5), (6) are not defined; the invariant (7) is the following one:
\[ D = C^k_{3} = \frac{L_{12}L_{21} - L_{11}L_{22}}{M_1N_1}. \quad (10) \]

We show that the invariants \(M_i, N_j\), and \(L_{ij}\), \(i, j = 1, 2\), do not generate the algebra of invariants \(K[m]^N\). Denote
\[ B_0 = K[M_1, M_2, N_1, N_2, L_{11}, L_{12}, L_{21}, L_{22}]. \]

Assume the converse. Then \(K[m]^N = B_0\). Therefore from Theorem 1.10 it follows that there exists a polynomial \(f\) in 8 variables and there are numbers \(l_i, i = 1, \ldots, 4\), such that
\[ D = \frac{f(M_1, M_2, N_1, N_2, L_{11}, L_{12}, L_{21}, L_{22})}{M_1^{l_1}M_2^{l_2}N_1^{l_3}N_2^{l_4}}. \]

Using (10), we get the identity, which contradicts the algebraic independence of the polynomials \(M_i, N_j\), and \(L_{ij}\) (see Theorem 1.7).

**Theorem 4.1.** For any parabolic subalgebra \(p\) such that the reductive subalgebra of \(p\) consists of three block with sizes \((2, k, 2)\), \(k > 3\), it follows that the algebra of invariants \(K[m]^N\) is generated by the polynomials \(M_i, N_i, L_{ij}\), and \(D\), where \(i, j = 1, 2\).

**Proof.** Denote the set
\[ B = K[M_i, N_j, L_{ij}, D]_{i,j=1,2} \subset K[m]^N. \]

We show that \(K[m]^N \subset B\).

As above, let \(S\) be the set of denominators generated by the minors \(M_i, N_i\), where \(i = 1, 2\). From Theorem 1.10 it follows that the localization \(K[m]^N_S\) coincides with the algebra of Laurent polynomials
\[ K \left[(M_i)^{\pm 1}, (N_i)^{\pm 1}, L_{ij}\right]_{i,j=1,2}. \]

If \(f \in K[m]^N\), then there exist natural numbers \(k_1, k_2, l_1, l_2\) such that
\[ f \cdot M_1^{k_1}M_2^{k_2}N_1^{l_1}N_2^{l_2} \in B_0. \]

Suppose for any minor $M$ from the collection $\{M_i, N_i\}_{i=1,2}$ we have that
\[ \text{if } f \in K[m] \text{ and } M \cdot f \in B, \text{ then } f \in B; \]
then $f \in B$.

Let us prove that if $M \cdot f \in B$, then $f \in B$ in the case $M = M_1$. The proof for the other cases is similarly. Suppose $f \in K[m]^N$ and $M_1 f \in B$. The polynomial $M_1 f$ is equal to zero on $\text{Ann } M_1$. Let $a_i, b_i, c_{ij}, i, j = 1, 2$, be any numbers in $K$. We construct a matrix
\[
Q_1 = \begin{pmatrix}
0 & 0 & a_2 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & a_1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & c_{11} & c_{12} \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & c_{21} & c_{22} \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & b_2 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & b_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\end{pmatrix}
\]
By direct calculation, we have
\[
M_1(Q_1) = 0, \quad M_2(Q_1) = a_1a_2, \quad N_1(Q_1) = b_1, \quad N_2(Q_1) = -b_1b_2;
\]
\[
L_{11}(Q_1) = a_1c_{21}, \quad L_{12}(Q_1) = -a_1b_1c_{22},
\]
\[
L_{21}(Q_1) = a_1a_2c_{21}, \quad L_{22}(Q_1) = a_1a_2b_1c_{22},
\]
\[
D(Q_1) = a_1a_2(c_{12}c_{21} - c_{11}c_{22}).
\]
Consider an algebra of polynomials
\[
\mathcal{A} = K[u_1, u_2, v_1, v_2, w_{11}, w_{12}, w_{21}, w_{22}, z]
\]
in 9 variables. Since the polynomial $f M_1$ is contained in $B$, we have there exists a polynomial $p(u_1, u_2, v_1, v_2, w_{11}, w_{12}, w_{21}, w_{22}, z)$ in the algebra $\mathcal{A}$ such that
\[
f M_1 = p(M_1, M_2, N_1, N_2, L_{11}, L_{12}, L_{21}, L_{22}, D).
\] (11)
Since $f M_1 = 0$ on $\text{Ann } M_1$, we obtain $(f M_1)(Q_1) = 0$. It now follows that
\[
0 = (f M_1)(Q_1) = p\left(0, a_1a_2, b_1, -b_1b_2, a_2c_{21}, -a_2b_1c_{22}, a_1a_2c_{21}, -a_1a_2b_1c_{22}, a_1a_2(c_{12}c_{21} - c_{11}c_{22})\right).
\]

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By \( Z \) denote a subset in \( K^q \) that consists of the following collections
\[
\left(0, a_1a_2, b_1, -b_1b_2, a_2b_1c_{21}, a_1a_2c_{21},
-a_1a_2b_1c_{22}, a_1a_2(c_{12}c_{21} - c_{11}c_{22})\right),
\]
where \( a_i, b_j, c_{ij} \in K \).

It can easily be checked that the polynomials \( u_1 \) and \( w_{12}w_{21} - w_{11}w_{22} \) are equal to zero on \( Z \). We show that the ideal \( I_Z = \{ \varphi \in A : \varphi(Z) = 0 \} \) is generated by \( u_1 \) and \( w_{12}w_{21} - w_{11}w_{22} \).

Let the ideal \( I \) be generated by the polynomials \( u_1, w_{12}w_{21} - w_{11}w_{22} \). Note that \( I \) is a relevant prime ideal. Indeed, the polynomial \( w_{12}w_{21} - w_{11}w_{22} \) is a irreducible one. Therefore the algebra
\[
A/I = A/\langle u_1, w_{12}w_{21} - w_{11}w_{22} \rangle = K[u_2, v_1, v_2, w_{11}, w_{12}, w_{21}, w_{22}, z]/\langle w_{12}w_{21} - w_{11}w_{22} \rangle \tag{12}
\]
is a domain of integrity. Hence the ideal \( I \) is a relevant prime one.

Since the polynomials \( u_1, w_{12}w_{21} - w_{11}w_{22} \) vanish on the set \( Z \), it follows that \( I \subset I_Z \) and \( \text{Ann} I \supset Z \). The dimension of the variety \( \text{Ann} I \) is equal to the transcendence degree of the quotient field of the algebra \( \langle \rangle \) over \( K \), i.e., \( \text{dim} \text{Ann} I = 7 \).

Evidently, \( \text{dim} Z = 7 \), then \( \text{dim} \text{Ann} I = \text{dim} Z \). Since \( \text{Ann} I \supset Z \), we have \( \text{Ann} I = \overline{Z} \). We show that \( I = I_Z \). Suppose \( g \in I_Z \). By the Hilbert’s theorem on zeros, there exists a natural number \( N \) such that \( g^N \in I \). Since \( I \) is a relevant prime ideal, we have \( g \in I \). Hence, \( I_Z \subset I \). Using \( I \subset I_Z \), we obtain \( I_Z = I \).

Thus since \( p|Z = 0 \), it follows that \( p \in I \). Therefore there exist polynomials \( p_1 \) and \( p_2 \) in the algebra \( A \) such that
\[
p = p_1u_1 + p_2(w_{12}w_{21} - w_{11}w_{22})..
\]
Combining this with \( \langle \rangle \), we obtain
\[
fM_1 = p_1M_1 + p_2(L_{12}L_{21} - L_{11}L_{22}) = p_1M_1 + p_2M_1N_1D.
\]
Further, we have \( f = p_1 + p_2N_1D \). So, \( f \in B \). \( \Box \)

**Corollary 4.2.** The algebra of invariants \( K[m]^N \) is isomorphic onto the factor algebra
\[
K[X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4, Z]/(X_2X_4Z - Y_2Y_3 + Y_1Y_4)
\]
in the case \((2, k, 2), k > 3\).
§5. The algebra of invariants in the case \((1, 2, 2, 1)\)

Suppose the reductive subalgebra \(\mathfrak{r}\) of \(\mathfrak{p}\) consists of the blocks with sizes \((1, 2, 2, 1)\). We describe the algebra of invariants in this case. The expanded base consists of the following roots:

\[ S = \{(1, 2), (3, 4), (2, 5), (5, 6)\}, \quad \Phi = \{(2, 4), (4, 6)\}. \]

The diagram has the form

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\otimes & & & & & 1 \\
1 & \times & & & & 2 \\
1 & \otimes & & & & 3 \\
1 & \times & & & & 4 \\
1 & \otimes & & & & 5 \\
& & & & \otimes & 1 \\
& & & & & 6 \\
\end{array}
\]

Diagram 4

The roots of the expanded base correspond to the following invariants.

\[
\begin{align*}
M_1 &= M_{(1,2)} = x_{1,2}, \quad M_2 = M_{(3,4)} = x_{3,4}, \\
M_3 &= M_{(2,5)} = \begin{bmatrix} x_{2,4} & x_{2,5} \\ x_{3,4} & x_{3,5} \end{bmatrix}, \quad M_4 = M_{(5,6)} = x_{5,6}; \\
L_1 &= L_{(2,4)} = x_{1,2}x_{2,4} + x_{1,3}x_{3,4}, \\
L_2 &= L_{(4,6)} = x_{3,4}x_{4,6} + x_{3,5}x_{5,6}.
\end{align*}
\] (13)

Consider a polynomial

\[ D = x_{1,2}x_{2,4}x_{4,6} + x_{1,2}x_{2,5}x_{5,6} + x_{1,3}x_{3,4}x_{4,6} + x_{1,3}x_{3,5}x_{5,6}. \]

**Lemma 5.1.** The polynomial \(D\) is an \(N\)-invariant, \(D\) is equal to the entry \((1, 6)\) of the matrix \(X^3\), and we have

\[ M_2D = L_1L_2 - M_1M_3M_4. \] (14)

**Proof.** This lemma can be proved by direct calculations. \(\Box\)

Note that the algebra of invariants \(K[m]^N\) does not coincide with the algebra

\[ B_0 = K[M_1, M_2, M_3, M_4, L_1, L_2]. \]

Indeed, if \(K[m]^N = B_0\), then by Theorem \([\text{??}]\) there exists a polynomial \(f(t_1, \ldots, t_6)\) and there are natural numbers \(l_i, i = 1, \ldots, 4\), such that

\[ D = \frac{f\left(M_1, M_2, M_3, M_4, L_1, L_2\right)}{M_1^{l_1}M_2^{l_2}M_3^{l_3}M_4^{l_4}}. \]

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Hence using the equation (14), we obtain that the invariants (13) are algebraically depended. This contradicts Theorem 1.7.

**Theorem 5.2.** Let the reductive subalgebra consists of the blocks (1, 2, 2, 1). Then the algebra of invariants is generated by the invariants (13) and by the polynomial \( D \).

The proof is similarly to the proof of Theorem 4.1.

**Corollary 5.3.** The algebra of invariants \( K[\mathbf{m}]^N \) is isomorphic onto the factor algebra

\[
K[X_1, X_2, X_3, X_4, Y_1, Y_2, Z]/(X_2Z - Y_1Y_2 + X_1X_3X_4)
\]

in the case \((1, 2, 2, 1)\).

So, we have proved that the invariant \( D \) is the minor of order 1 of the matrix \( X^3 \) in the case \((1, 2, 2, 1)\) (see Lemma 5.1). By Proposition 3.5 the invariants \( C_m^1 \) can be determined as combined minors, which are constructed by the formal matrices \( X \) and \( X^2 \). Similarly, the basic invariants \( M_\xi, \xi \in S \), and \( L_\varphi, \varphi \in \Phi \), are combined minors (see Proposition 2.5). Conjecture 5.4 generalizes all cases.

Let \( \mathfrak{p} \) be any parabolic subalgebra. Let \( I_1, I_2, \ldots, I_k \) and \( J_1, J_2, \ldots, J_k \) be any collections of rows and columns, respectively, such that we have the following properties.

1. \(|I_1| + |I_2| + \ldots + |I_k| = |J_1| + |J_2| + \ldots + |J_k|\).
2. Let \( I \) be a set in the collection \( \{I_1, I_2, \ldots, I_k, J_1, J_2, \ldots, J_k\} \). If \( \min I < i < \max I \), then \( i \in I \).
3. For any \( l = 1, \ldots, k - 1 \) we have
   \[
   \max I_l < \min I_{l+1}, \ \min J_l > \max J_{l+1}.
   \]
4. Suppose for a positive root \((a, b)\) and a number \( l \) such that \( 1 \leq l \leq k \), we have one of the following conditions:
   a) \( a > \max I_l \) and \( b \in J_{k-l+1} \),
   b) \( a \in I_l \) and \( b < \min J_{k-l+1} \);

then the root \((a, b)\) is not contained in \( M \). In other words, the minor \( X_{I_l}^{J_{k-l+1}} \) is bordered of zeros to the right and to the below in the matrix \( X \) for any \( i = 1, \ldots, k \).
We form a determinant, which is defined by the minors of the formal matrix $X, X^2, \ldots, X^k$:

$$\begin{vmatrix}
X_{J_1 I_1} & (X^2)^{J_2 - I_1}_{I_1} & \cdots & (X^k)^{J_k - I_1}_{I_1} \\
0 & X_{J_2 - 1 I_2} & \cdots & (X^{k-1})^{J_k - 1}_{I_2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & X^{J_k}_{I_k}
\end{vmatrix}.$$

**Conjecture 5.4.** The algebra of invariants $K[m]^N$ is generated by the combined minors $D_{I_1, I_2, \ldots, I_k}^{J_1, J_2, \ldots, J_k}$, where the conditions 1–4 hold for $I_1, I_2, \ldots, I_k$ and $J_1, J_2, \ldots, J_k$.

**References**

[B] M. Brion, Representations exceptionnelles des groups semi-simple, *Ann. Scient. Ec. Norm. Sup.* **18** (1985), pp. 345–387.

[GG] M. Goto and F. Grosshans, Semisimple Lie algebras, Lect. Notes in Pure Appl. Math., vol. 38 (1978).

[K] H. Kraft, Geometrische Methoden in der Invariantentheorie, Friedr. Vieweg and Sohn, Braunschweig/Wiesbaden (1985).

[PS] A. N. Panov and V. V. Sevostyanova, Regular $N$-orbits in the nilradical of a parabolic subalgebra, *Vestnik SamGU*, **7**(57) (2007), pp. 152–161. See also [http://arxiv.org/abs/1203.2754](http://arxiv.org/abs/1203.2754).

[PV] V. L. Popov and E. B. Vinberg, Invariant theory, in: *Progress in Science and Technology*, VINITI, Moscow (1989), pp. 137–309.

[R] R. W. Richardson, Conjugacy classes in parabolic subgroups of semisimple algebraic groups, *Bull. London Math. Soc.* **6** (1974), pp. 21–24.

[S] V. V. Sevostyanova, The field of invariants of the adjoint action of the unitriangular group in the nilradical of a parabolic subalgebra, *Zapiski nauchn. seminarov POMI*, vol. 375, 2010, pp. 167–194. (English translation: *Journal of Math. Sciences*, vol. 171, No. 3, 2010, pp. 400–415.) See also [http://arxiv.org/abs/1203.3000](http://arxiv.org/abs/1203.3000).

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