A note on local BRST cohomology of Yang-Mills type theories with free abelian factors

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Abstract

We extend previous work on antifield dependent local BRST cohomology for matter coupled gauge theories of Yang-Mills type to the case of gauge groups that involve free abelian factors. More precisely, we first investigate in a model independent way how the dynamics enters the computation of the cohomology for a general class of Lagrangians in general spacetime dimensions. We then discuss explicit solutions in the case of specific models. Our analysis has implications for the structure of characteristic cohomology and for consistent deformations of the classical models, as well as for divergences/counterterms and for gauge anomalies that may appear during perturbative quantization.
1 Introduction

The use of systematic algebraic methods has proved extremely useful in the context of renormalization of vector gauge models [1]. A subsequent reformulation in terms of the effective action [2] streamlines the analysis (see [3] for a review) and has paved the way for generalizations to generic gauge systems [4]. In this context, a detailed understanding of how the antifield dependent BRST differential encodes the information about gauge symmetries and dynamics [5] is a pre-requisite for the computation of the relevant cohomologies.

It turns out that it is the antifield dependent BRST cohomology computed in the space of local functionals that is needed for perturbative quantization and renormalization of gauge models. Indeed, even when no power counting restrictions are imposed [6], these cohomology classes control in a gauge independent way both anomalies (in ghost number 1) and non-trivial divergences/counterterms (in ghost number 0) that cannot be absorbed by a field-antifield redefinition, including a change of gauge [7].

The antifield dependent part of the cohomology depends on the dynamics of the model through (generalized) conservation laws and global symmetries and their interplay with the gauge structure [8, 9]. In the case of semi-simple gauge groups, this dependence can only occur through gauge invariant conserved currents, and is absent in ghost numbers 0 and 1. As a consequence, there is no non-trivial renormalization of the BRST symmetry itself since it is encoded in the antifield dependent part of the master action.

If there are abelian factors, the situation becomes more involved (see [10,11] for an early analysis) and additional antifield dependent cohomology may appear, both in ghost number 0 and in ghost number 1. If some of these abelian factors are free in the sense that the matter fields do not transform under the associated gauge transformations, the antifield dependence becomes even more complicated. From the point of view of renormalization of vector gauge theories, such models are usually not considered even though they should in case the model is not completely free. For instance, in the bosonic sector of supergravity models with gravity switched off, there typically are couplings of the curvatures of the vector gauge fields to the scalar fields.

Besides non-trivial divergences/counterterms, antifield dependent BRST cohomology in ghost number 0 also controls non-trivial deformations, so-called gaugings. The problem of systematically finding all such gaugings has recently attracted renewed interest in the context of supergravities (see [12] for a review). In this case, free abelian factors feature prominently from the very beginning. It is this question that constitutes the main motivation behind the present analysis (see [13] for more details and the companion paper [14] for detailed results in 4 dimensions with only free abelian factors).

It is thus of interest to have as complete results as possible on these cohomologies. Besides the illustration of how the complicated antifield dependence of the cohomology leads to standard non-abelian Yang-Mills models when starting from pure, free abelian vector fields, section 12 of reference [9] outlines how to proceed in more general cases with non-trivial couplings to matter fields. More
precisely, the equations that are affected by the modified dynamics have been identified, and a case by case analysis has been suggested.

The purpose of the present paper is to formulate in detail and in a model independent way the dynamical equations to be solved and how they appear as building blocks in the characterization of the cohomology, before working out explicit results in specific models.

In the case without free abelian vector fields, this had already been done in the review [15] by following a different strategy than the one adopted in [8,9]: instead of computing the cohomology by attacking the cocycle condition in top form degree at highest antifield number, one starts the analysis from the bottom of the descent equations. The advantages of this modified strategy are twofold: (i) The equivariant characteristic cohomology can be determined together with the antifield dependent local BRST cohomology in a unified and streamlined proof, and (ii) one may forgo the technical assumption of normality. More precisely, for the approach of [8,9] to be viable, one needs to prove that various expansions have bounded antifield number. This can be done by suitable assumptions on the number of derivatives in the interactions. Effective theories are however not normal in this sense. In this case, one needs to consider the space of formal power series in the couplings constants, the fields, antifields and their derivatives, so that the antifield number may be unbounded. The approach of [15] allows one to cover both situations simultaneously. That is why the terminology “normal theories” was extended so as to include effective theories with unbounded expansions and no additional assumptions on derivatives (see sections 5.3, 6.4.3 and 7.3 of [15] for details).

The present paper completes the results obtained in [15] to the case where there are free abelian gauge fields. The analysis is valid in all spacetime dimensions greater than or equal to three. In the present case, the dynamics enters the problem non-trivially already at form degree $n - 2$ and involves two additional building blocks.

In the next section, we summarize the ingredients of [15] needed for a self-contained presentation of the main results. In addition, we provide a new characterisation in terms of suitable Young tableaux of antifield independent local BRST cohomology [16–20] in the case where there are only abelian factors. Our analysis extends the considerations in [21] and allows one to make contact with the topological gaugings of [22].

Section 3 is devoted to the statement of the main theorem on the local BRST cohomology $H(s|d)$ and its implications for characteristic cohomology and for infinitesimal gaugings. In section 4, the general results are illustrated in the cases of vector and vector-scalar models in various dimensions. Finally, appendix A contains our conventions regarding forms, while appendix B contains the recursive proof of the main theorem.
2 BRST-BV differential for Yang-Mills type theories

Basic definitions. The generic Yang-Mills-type gauge algebra \( \mathfrak{g} \) that we consider is a direct sum of several abelian and simple Lie algebras. In a basis \( t_I \) of \( \mathfrak{g} \) we have \( \{ t_I, t_J \} = t_K f^{KIJ} \), \( I, J, K \in \{1, \ldots, \dim(\mathfrak{g})\} \). We denote by \( \mathcal{L} \) the space of \( \mathfrak{g} \)-invariant local functions in \( x^\mu, dx^\mu, F_{\mu\nu}^I, D_\mu F_{\mu\nu}^I, \ldots, \psi^i, D_\mu \psi^i, \ldots \), where

\[
\begin{align*}
F_{\mu\nu}^I &:= 2\partial_{[\mu} A_{\nu]}^I + f^{IJK} A_{\nu}^J A_{\mu}^K, \quad A_I := dx^\mu A_{\mu}^I, \\
D_\mu F_{\mu\nu}^I &:= \partial_\mu F_{\nu\rho}^I + f^{IJK} A_{\mu}^J F_{\rho\nu}^K, \quad D_\mu \psi^i := \partial_\mu \psi^i + T_{I}^{ij} A_{\mu}^I \psi^j
\end{align*}
\]

are, respectively, the components of the field strengths, the gauge one-forms, the \( \mathfrak{g} \)-covariant derivatives of the field strengths and of the (scalar or spinorial) matter fields \( \psi^i \). The latter transform in a \( \dim_\psi \)-dimensional representation of \( \mathfrak{g} \) with matrices \( \{ T_I^{ij} \}_{i=1, \ldots, \dim(\mathfrak{g})} \), \( i, j \in \{1, \ldots, \dim_\psi\} \).

The solution of the BV master equation for Yang-Mills theories is given by

\[
S^{(0)} = \int d^p x [\mathcal{L} + A_I^{\mu\nu} D_\mu C^I + \frac{1}{2} C_I^* f^{IJK} C_J C^K + C^I T_I^{ij} \psi^j \psi^i].
\]

When introducing the antifield number \( \text{antifd} \), which assigns antifield numbers 0 to the fields and ghosts, 1 to \( A_I^{\mu\nu} \), \( \psi^i \) and 2 to \( C_I^* \), the associated BRST differential decomposes into \( s = (S^{(0)}, \cdot) = \delta + \gamma \), where the longitudinal differential \( \gamma \) preserves the antifield number and the Koszul-Tate differential \( \delta \) decreases the antifield number by 1. For more details on the Koszul-Tate differential and its relevance for the conservation laws and rigid symmetries of a local theory, see [23] as well as sections 5 and 6 of [15].

Small algebra and descent of equations. The small algebra \( \mathcal{B} \) is defined to be the algebra of polynomials in the undifferentiated ghosts, the gauge field 1-forms and their exterior derivatives. In the review [15], following the original work [17], a basis of \( H^{*\cdot}(s, \mathcal{B}) \), adapted to the computation of \( H^{*\cdot}(s|d, \mathcal{B}) \), was constructed in terms of forms \( M \) and \( N \) satisfying

\[
s B^p = -d(B^{p-1} + M^{p-1}), \quad d B^p = -s(B^{p+1} + b^{p+1}) + N^{p+1},
\]

for suitably defined \( B^{p-1}, B^{p+1} \) and \( b^{p+1} \) in \( \mathcal{B} \). As a consequence, a basis of \( H^{*\cdot}(s|d, \mathcal{B}) \) is provided by the \( B \)'s and the \( M \)'s and 1. We will not reproduce explicit expressions here but refer instead to [15], sections 10.5-10.7.

If the gauge group \( G \) contains only \( N \) abelian factors, we now provide an alternative adapted basis, that is more suitable for our purpose below. Indeed, a basis of \( H^*(s, \mathcal{B}) \) is provided by 1 together with all the homogeneous polynomials \( p_{b,k} = t_{J(b),I(k)} F_{J(b)}^{I(k)} C_{I(k)} \) of degree \( b \) in the \( F \)'s and of degree \( k \) in the \( C \)'s. Here, we have used the notation \( F_{J(b)}^{I(k)} = F_{J_1} \cdots F_{J_b} \), and similarly for \( C_{I(k)} = C_{I_1} \cdots C_{I_k} \). The alternative basis of generators of \( H^*(s, \mathcal{B}) \), that is adapted to the computation of \( H^{*\cdot}(s|d, \mathcal{B}) \) in the purely abelian case, is given by

\[
\begin{align*}
\{ M_{b,k} \} = \{ \lambda_{J(b),I(k)} F_{J_1} \cdots F_{J_b} C_{I_1} \cdots C_{I_k} \}, & \quad k > 0, b \geq 0, \\
\{ N_{b+1,k-1} \} = \{ b+1 \choose b+1 \lambda_{J(b+1),I(k-1)} F_{J_1} \cdots F_{J_b} C_{I_1} \cdots C_{I_k} \}, & \quad k > 0, b \geq 0,
\end{align*}
\]
In terms of constant tensors \(\lambda_{J_1 \ldots J_{b+1} I_1 \ldots I_{k-1}}\) satisfying
\[
\lambda_{J_1 \ldots J_{b+1} I_1 \ldots I_{k-1}} = \lambda_{(J_1 \ldots J_{b+1}) I_1 \ldots I_{k-1}} = \lambda_{J_1 \ldots J_{b+1} [I_1 \ldots I_{k-1}]} ,
\]
\[
\lambda_{(J_1 \ldots J_{b+1} J_{b+2}) I_2 \ldots I_{k-1}} = 0 .
\]

In other words, the constants \(\lambda_{J_1 \ldots J_{b+1} I_1 \ldots I_{k-1}}\) transform in the \(GL(N)\) Young tableau associated with an upper row of length \(b + 1\) whose boxes are filled with the indices \(\{J_1, \ldots, J_{b+1}\}\), while all the lower rows are of length one and filled with the indices \(\{I_1, \ldots, I_{k-1}\}\). Based on these properties, one can check that:

(i) Together with 1, these \(M\)'s and \(N\)'s do define a basis of \(H^*(s, B)\). Indeed, any polynomial \(p_{b,k} = t_{J(b):I[k]} F^{J(b)} C^{I[k]}\) is represented by a \(GL(N)\) reducible tensor \(t_{J(b):I[k]}\). The latter decomposes in the direct sum of irreducible tensors \(\lambda_{J(b+1),I[k-1]}\) and \(\lambda_{J(b),I[k]}\). For example, the decomposition of the reducible tensor \(t_{J(2):I[3]} = t_{J_1 J_2 I_2 I_3}\) into \(\lambda_{J_1 J_2 I_1 I_2 I_3}^{(M)}\) and \(\lambda_{J_1 J_2 I_1 I_2 I_3}^{(N)}\) can be depicted as follows:

\[
\begin{array}{c}
J_1 \quad J_2 \\
I_1 \\
I_2 \\
I_3
\end{array} \otimes \begin{array}{c}
J_1 \quad J_2 \\
I_1 \\
I_2 \\
I_3
\end{array} = \begin{array}{c}
J_1 \\
I_1 \\
I_2 \\
I_3
\end{array} + \begin{array}{c}
J_1 \\
I_1 \\
I_2 \\
I_3
\end{array}
\]

(ii) The following descent equations hold
\[
s(\frac{1}{2} A^K A^L \partial_L \partial_K M_{b,k}) + d(A^K \partial_K M_{b,k}) = N_{b+1,k-1} ,
\]
\[
s(A^K \partial_K M_{b,k}) + dM_{b,k} = 0 ,
\]
\[
sM_{b,k} = 0 .
\]

Following the arguments given in [15], it then follows that a basis of \(H^{*,*}(s|d, B)\) is given by 1, \(M_{b,k}\) and \(A^K \partial_K M_{b,k}\).

One may now compute the dimensions of \(H^{*,*}(s|d, B)\) and compare to those given in [21]. From equations (2.5), (2.6), (2.11) and the theorems 9.1 and 9.2 of [15], \(H^{q,2p}(s|d, B)\) and \(H^{q-1,2p+1}(s|d, B)\) are isomorphic. The dimension of \(H^{q,2p}(s|d, B)\) is simply given by the independent components of the tensor \(\lambda_{J_1 \ldots J_p I_1 \ldots I_q}\) appearing in \(\lambda_{J_1 \ldots J_p I_1 \ldots I_q} F^{J_1} \ldots F^{J_p} C^{I_1} \ldots C^{I_q}\). In other words, \(h^{p,q} := \dim(H^{q,2p}(s|d, B))\) is equal to the dimension of the \(GL(N)\) Young tableau

\[
\begin{array}{c}
J_1 \\
I_1 \\
J_2 \\
I_2 \\
J_p \\
I_q
\end{array},
\]

which is [24]
\[
h^{p,q} = \frac{(N + p)!}{(N - p)! p! (q - 1)! (p + q)} .
\]
In [21], the dimensions $h^{p,q}$ were encoded in the generating function

$$h(x, y) = \sum_{p+q \geq 1} h^{p,q} x^p y^q = \frac{y}{1 - x} \sum_{r=0}^{N-1} \left( \frac{1 + y}{1 - x} \right)^r.$$

(2.14)

To show that these dimensions indeed agree, we first expand the geometric series,

$$h(x, y) = y \sum_{m=0}^\infty x^m + y(1+y) \sum_{m=0}^\infty (m+1) x^m + y(1+y)^2 \sum_{m=0}^\infty \frac{(m+1)(m+2)}{2!} x^m$$

$$+ \ldots + y(1+y)^{N-1} \sum_{m=0}^\infty \frac{(m+1)(m+2) \ldots (m+N-1)}{(N-1)!} x^m,$$

then expand $(1+y)^p$ and collect the coefficient of $x^p y^q$:

$$h^{p,q} = p+q-1 \binom{p+q}{p} \left[ 1 + (p+q) \frac{(p+q+1)}{2!} + \ldots + \frac{(p+q) \ldots (p+q+N-1)}{N!} \right],$$

where $\binom{n}{k} := \frac{n!}{(n-k)k!}$ are the binomial coefficients and $\tilde{N} := N - q \geq 0$. If we now view $h^{p,q}$ as a function of the non-negative integer $\tilde{N}$ and define $\sigma(\tilde{N}) := h^{p,q}/(p+q-1) \binom{p+q}{p+q-1}$, we can determine it by induction using

$$\sigma(\tilde{N} + 1) = \sigma(\tilde{N}) + \tilde{N} + p+q \binom{\tilde{N}+p+q}{\tilde{N}+1}, \quad \sigma(0) = 1.$$

(2.15)

It is then easy to see that the solution is $\sigma(\tilde{N}) = \tilde{N} + p+q \binom{\tilde{N}+p+q}{\tilde{N}}$. This gives $h^{p,q} = \frac{(\tilde{N}+p+q)!}{N!(p+q)!}$ and then (2.13) when using the definition of $\tilde{N}$.

### 3 The main theorem

In this section, theorem 11.1 of [15] is extended so as to include free abelian factors. As a consequence, there now exists non-trivial characteristic cohomology already in form degree $n-2$, $H^{n-2}_{\text{char}}(d) \cong H^{-2,n}(s|d)$ [8, 9], which in turn considerably enriches $H^{g,n}(s|d)$ in ghost numbers $g$ greater than $-2$.

The plan of this section is as follows. In subsection 3.1, we spell out the assumptions underlying the theorem. In subsection 3.2, we start by classifying elements of characteristic cohomology $H^{n-k}_{\text{char}}(d)$, for $k = 1, 2$ according to the shortest length of descent of the associated elements of $H^{-k,n}(s|d)$. In subsection 3.3, we list the main building blocks needed for the theorem. The theorem itself is stated in subsection 3.4 and proved by induction in appendix B. As a first corollary to the theorem, we provide a complete discussion of non-covariantizable characteristic cohomology and its relation to equivariant characteristic cohomology in subsections 3.5 and 3.6. Finally, we discuss implications for infinitesimal deformations in subsection 3.7.
3.1 Assumptions

Our assumptions are as follows:

a) There are free abelian gauge fields: the gauge group contains abelian factors such that the corresponding gauge transformations on the abelian gauge fields \( A^I_{\mu} \) and on the matter fields \( \psi^i \) are

\[
\delta_{\epsilon^{I_{fa}}} A^I_{\mu} = \partial_{\mu} \epsilon^{I_{fa}} \quad \text{and} \quad \delta_{\epsilon^{I_{fa}}} \psi^i = 0 .
\]

In other words, the matter fields are uncharged under the free abelian factors. This has the consequence that the action of the BRST differential \( s \) on the \( C^*_I_{fa} \) gives no contributions in the antifields \( \psi^*_i \) and only produces \( \delta C^*_I_{fa} = -\partial_{\mu} A^*_\mu_{I_{fa}} \).

b) The spacetime dimension is strictly greater than two, \( n > 2 \);

c) We consider only Lagrangians \( \mathcal{L} \) obeying the regularity conditions spelled out in the section 5.1.3 of [15] and defining normal theories, as explained in the section 6.4.3 of the same reference. The left-hand sides of the field equations \( \frac{\delta \mathcal{L}}{\delta \psi^i} \approx 0 \) and \( \frac{\delta \mathcal{L}}{\delta A^I_{\mu}} \approx 0 \) are assumed to be gauge covariant;

d) We assume that there is no nontrivial topology: the cohomology of \( d \) is trivial except for the constants and the inequivalent Lagrangians\(^1\);

e) There are only matter and vector fields and the only gauge invariance is of Yang-Mills type with a reductive\(^2\) gauge algebra \( g \).

3.2 Comments on characteristic cohomology

Under our assumptions, it is shown in [8, 9, 15] that characteristic cohomology in degree \( n - 2 \) is isomorphic to \( H^{-2,n}(s|d) \), a basis being given by \( d^n x C^*_I_{fa} \), where the index \( I_{fa} \) is restricted to run over the free abelian factors. The associated descent equations start with

\[
s d^n x C^*_I_{fa} + d * A^*_I_{fa} = 0 , \tag{3.1}
\]

\[
s * A^*_I_{fa} + dk_{I_{fa}}^{n-2} = 0 . \tag{3.2}
\]

In this case, characteristic cohomology in degree \( n - 2 \) is represented by \( k_{I_{fa}}^{n-2} \).

By applying \( s \) to the last equation, one then gets \( ds k_{I_{fa}}^{n-2} = 0 \), which implies

\[
s k_{I_{fa}}^{n-2} + dk_{I_{fa}}^{n-3} = 0 . \tag{3.3}
\]

From the general analysis of descent equations (see sections 9.2 and 9.3 of [15]), it follows that the shortest length of the descent, also called depth below, allows one to distinguish elements

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\(^1\) It would be interesting to relax this assumption in order to include magnetic-type charges.

\(^2\) This assumption could also be relaxed if needed. In the non-reductive case, functions in the covariant derivatives of the field strengths and the undifferentiated ghosts are not necessarily separately \( g \) invariant, see e.g. [25] for a discussion.
of $H^{-2,m}(s|d)$. More precisely, we split the elements into those for which the descent stops at form degree $n - 2$ and ghost number $0$, and those for which $\eta^{n-3}_I$ cannot be absorbed by allowed redefinitions. A basis for the former free abelian factors is denoted by $d^n x C^*_A$ and for the latter by $d^n x C^*_a$. Since $sk_a^{n-2} = 0$ and $k_a^{n-2}$ is of ghost number $0$ and does not involve antifields, it follows that $k_a^{n-2} \in \mathcal{I}$, while $k_a^{n-2}$ cannot belong to $\mathcal{I}$. For the latter class, one has

$$
\begin{align*}
    s d^n x C^*_A + d \ast A^*_A &= 0, \\
    s \ast A^*_A + dk^{n-2}_A &= 0, \\
    sk^{n-2}_A + d\eta^{n-3} &= 0, \\
    r^{n-3}_A \neq sp^{n-3}_A + dq^{n-4}_A.
\end{align*}
$$

For characteristic cohomology in degree $n - 1$, we consider the descent equations starting with $s\omega^{-1,n} + d\omega^{0,n-1} = 0$. In this case, characteristic cohomology in degree $n - 1$ is represented by the antifield independent part of $\omega^{0,n-1}$. For the shortest descent, one has $s\omega^{0,n-1} = 0$ and a basis for $(n - 1)$-forms at the bottom of the descent is denoted by $j_\Delta$.

Finally, the equation $P^n_A(F) \approx dI^{n-1}_A$ can be shown to imply $P^n_A(F) = dI^{n-1}_A + sK_A$ where the $K_A$ depend linearly on undifferentiated antifields and $\gamma K_A = 0$. More generally, if $\mathcal{I} \ni I^n \approx 0$, then $I^n + d(\delta R^{n-1}) = sK$, where $K$ can be assumed to depend linearly on undifferentiated antifields only and $\delta R^{n-1} \in \mathcal{I}$ — see the proof in section 12.5 of [15].

### 3.3 Ingredients

The theorem rests on the following “elementary” descent equations:

1. For all non-trivial solutions to the descent equations in $\mathcal{B}$, we have equation (2.4): $sB^p + d(B^{p+1} + M^{p-1}) = 0$, $dB^p = -s(B^{p+1} + B^{p+1}) + N^{p+1}$. In particular, for $\Theta_\alpha$ a basis of the $s$ cohomology in the space of undifferentiated ghosts, we have

$$
\begin{align*}
    s[\Theta^2_\alpha] + d[\Theta_\alpha]^1 &= N^2_\alpha = P^{2\beta}_\alpha(F)\Theta_\beta, \\
    s[\Theta_\alpha]^1 + d\Theta_\alpha &= 0, \\
    s\Theta_\alpha &= 0.
\end{align*}
$$

Note that the obstruction $N^2_\alpha$ exists only if $\Theta_\alpha$ involves an abelian factor.

2. A basis for gauge-invariant characteristic cohomology in form degree $n - 2$ is given by the $k_a^{n-2} \equiv -d^{n-2} x^{\mu}_{\lambda\nu} k^{\mu\nu}_a \in \mathcal{I}$. They satisfy $dk^{n-2}_a \approx 0$, or more precisely, $\delta \partial_{a\alpha} = \partial_{a\alpha} k^{\mu\mu}$. The $k_a^{n-2}$’s constitute a basis in the sense that $dI^{n-2} \approx 0 \Rightarrow I^{n-2} \approx \lambda^0 k_a + d\omega^{n-3}$, and $\lambda^0 k_a \approx d\eta^{n-3} \Rightarrow \lambda^0 = 0$. They are related to the $n$-forms $d^n x C^*_a$ through the chain:

$$
\begin{align*}
    s d^n x C^*_A + d \ast A^*_A &= 0, \\
    s \ast A^*_A + dk^{n-2}_A &= 0, \\
    sk^{n-2}_A &= 0.
\end{align*}
$$

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We then define the $(n - 1)$-forms $T_{aa}$ and the $n$-forms $U_{aa}$ via
\begin{align}
T_{aa} &:= \star A^*_a \Theta_a + k^{n-2} \Theta_a, \\
U_{aa} &:= d^n x C^*_a \Theta_a + \star A^*_a \Theta_a + k^{n-2} \Theta_a. 
\end{align}
They satisfy
\begin{align}
sU_{aa} + dT_{aa} &= (-)^n k^{n-2} N^2, \\
sT_{aa} + d(k^{n-2} \Theta_a) &= 0, \\
s(k^{n-2} \Theta_a) &= 0.
\end{align}

3. The characteristic classes\footnote{By an abuse of terminology, in the present context, characteristic classes are particular elements of $\mathcal{B}$ given by $g$-invariant polynomials in the curvature two-forms $F$.} in form degree $n - 1$ that are on-shell trivial in $\mathcal{I}$ can only exist for $n$ odd since the curvatures are two-forms. Let $P^{n-1}_A(F) \approx dI^{n-2}_A$, denote a basis of such characteristic classes: $P^{n-1}(F) \approx dI^{n-2}_A$, $I^{n-2}_A \in \mathcal{I}$, implies that $P^{n-1} = \lambda^A P_A$, with $\lambda^A P_A = 0 \Rightarrow \lambda^A = 0$. On the other hand, as for any characteristic class,
\begin{equation}
P^{n-1}_A = dq^{n-2}_A
\end{equation}
for an $(n - 2)$-form $q^{n-2}_A \notin \mathcal{I}$, of the “Chern-Simons” type. Together with the defining property that $P^{n-1}_A \approx dI^{n-2}_A$, one deduces that $q^{n-2}_A - I^{n-2}_A$ is a non-covariantizable element of $H^{-2,n}(s|d)$: $q^{n-2}_A - I^{n-2}_A = \lambda^A k_A =: k_A$, see subsection 3.2. When taking into account (3.4) and defining $\star A^*_A := \lambda^A \star A^*_a, d^n x C^*_A := \lambda^A d^n x C^*_a$, one gets the following descent equations
\begin{align}
sd^n x C^*_A + d \star A^*_A &= 0, \\
s \star A^*_A + d(q^{n-2}_A - I^{n-2}_A) &= 0, \\
sq^{n-2}_A + dr^{n-3}_A &= 0.
\end{align}
Note that every $P^{n-1}_A$ gives rise to non-covariantizable characteristic cohomology in degree $n - 2$, and is thus related to $d^n x C^*_A$, but at this stage there could be more of the latter. A consequence of the theorem will be that there is not and that the indices $A$ and $\bar{A}$ can be identified. This is analyzed in section 3.5.

4. Let $\{N_\gamma\}$ be a basis for linear combinations of the elements of the form $P^{n-1}_A(F) \Theta_a \in H(s)$ that are at the same time obstructions to lifts in form degree $n - 1$ in the small algebra. A basis of the latter is denoted by $\{N_i\}$, see [8], section 10.5. One can choose the basis such that the $N_\gamma$’s contain in particular the characteristic classes $P^{n-1}_A(F)$ themselves, in which case $\Theta_a = 1$. Indeed, every characteristic class is an obstruction to the lift of an element in $\mathcal{B}$, which amounts to saying that $\{N_i\}$ contains a basis of all $P(F)$’s. By the definition of $\{N_\gamma\}$, one has
\begin{align}
N_\gamma := k^\gamma A^A P^{n-1}_A \Theta_a &= k^\gamma A^A (dI^{n-2}_A - s \star A^*_A) \Theta_a = \\
&= k^\gamma A^A [d(I^{n-2}_A \Theta_a) - s \star A^*_A \Theta_a - I^{n-2}_A \Theta_a^1),
\end{align}
where the second equality requires (3.10) and (3.11) and the third one uses the invariance of the basis \( \{ \Theta_\alpha \} \) together with (3.5).

On the other hand, because \( N_\gamma \in \{ N_i \} \) one has \( k^A_\gamma P_A^{n-1} \Theta_\alpha \equiv N_\gamma = s b^{n-1}_\gamma + d B^{n-2}_\gamma \). There then exist \( b^n_\gamma \) such that

\[
k^A_\gamma P_A^{n-1} \Theta_\alpha = db^{n-1}_\gamma + s b^{n}_\gamma . \tag{3.13}
\]

This follows directly from the analysis of the descent equations in the small algebra of [8], subsection 10.6. One has \( (s + d) M_{r_1 \ldots r_k | s_1 \ldots s_N} (q(C + A, F), f(F)) = \sum_r f_r \frac{1}{c q_r} M_{r_1 \ldots r_k | s_1 \ldots s_N} (q(C + A, F), f(F)) \). The equation for \( N_\gamma \) is the one in form degree \( \frac{s}{2} + 2m(r_1) + 1 \), while the searched for relation is the one in form degree \( \frac{s}{2} + 2m(r_1) \).

Now, defining

\[
W_\gamma := b^{n-1}_\gamma + k^A_\gamma (-I_A^{n-2}[\Theta_\alpha]_1 + \star A^*_A \Theta_\alpha) , \tag{3.14}
\]

\[
R_\gamma := b^{n}_\gamma + k^A_\gamma (-I_A^{n-2}[\Theta_\alpha]_2 + \star A^*_A[\Theta_\alpha]_1 + \star C^*_A \Theta_\alpha) , \tag{3.15}
\]

it is direct to check that the following equations are satisfied:

\[
s R_\gamma + d W_\gamma = k^A_\gamma I_A^{n-2} N_\alpha^2 , \tag{3.16}
\]

\[
s W_\gamma + d(B^{n-2}_\gamma - k^A_\gamma I_A^{n-2} \Theta_\alpha) = 0 . \tag{3.17}
\]

5. We can repeat the analysis of the previous item in form degree \( n \) and consider characteristic classes \( P^n(F) \) that are on-shell \( d \)-trivial in \( \mathcal{I} \). They can only exist for even \( n \). We denote a basis of such classes by \( P_A(F) \): \( P_A(F) \approx d I_A^{n-1} \), \( I_A^{n-1} \in \mathcal{I} \) and \( P^n(F) \approx d I^{n-1} \), \( I^{n-1} \in \mathcal{I} \). For every \( \Delta \) there exists an element \( K_A = d q_A^{n-1} \). This allows us to construct the non-covariantizable on-shell conserved currents

\[
j_A := q_A^{n-1} - I_A^{n-1} \tag{3.18}
\]

with associated \( K_A \) linear in the antifields such that \( sK_A + dj_A = 0 \). Let \( N_\Gamma \) denote a basis for the elements of the form \( P_A(F) \Theta_\alpha \in H(s) \) that are at the same time obstructions to lifts in form degree \( n \) in the small algebra. We have \( N_\Gamma = k^A_\Gamma (d I^{n-1}_A - s K_A) \Theta_\alpha = k^A_\Gamma [d(I^{n-1}_A \Theta_\alpha) - s(K_A \Theta_\alpha - I^{n-1}_A \Theta_\alpha)] \). At the same time, \( N_\Gamma = s b^{n-1}_\Gamma + d B^{n-2}_\Gamma \). We then obtain

\[
s W_\Gamma + d(B^{n-1}_\Gamma - k^A_\Gamma I^{n-1}_A \Theta_\alpha) = 0 , \quad W_\Gamma := b^{n-1}_\Gamma - k^A_\Gamma (I^{n-1}_A \Theta_\alpha - K_A \Theta_\alpha) . \tag{3.19}
\]

6. A basis for gauge-invariant, non-trivial Noether currents is given by elements

\[
j_\Delta = d^{n-1} x_\mu j^\mu_\Delta \in \mathcal{I} . \tag{3.20}
\]

Each of these elements satisfies \( d j_\Delta \approx 0 \) and \( d I^{n-1} \approx 0 \Rightarrow I^{n-1} \approx \lambda^{\Delta} j_\Delta + d \omega^{n-2} \) with \( \lambda^{\Delta} j_\Delta \approx d \eta^{n-2} \Rightarrow \lambda^{\Delta} = 0 \). For every \( \Delta \) there exists an element \( K_\Delta \) at antifield number one such that \( d j_\Delta + s K_\Delta = 0 \). Defining \( V_\Delta := K_\Delta \Theta_\alpha + j_\Delta[\Theta_\alpha]_1 \), the following descent equations hold,

\[
s V_\Delta + d(j_\Delta \Theta_\alpha) = 0 , \quad s(j_\Delta \Theta_\alpha) = 0 . \tag{3.21}
\]
### 3.4 Statement of the theorem

The theorem generalizes theorem 11.1 of [15]. It decomposes into four items. The first characterizes the general solution of the cocycle condition \( s\omega^p + d\omega^{p-1} = 0 \), up to trivial ones. The second makes precise their linear dependence. The third item discusses the decomposition of an invariant form that is \( d \)-exact on-shell, while the fourth item parametrizes characteristic classes that are trivial in \( H_{\text{char}}^s(d, \mathcal{I}) \). Equivalence in \( H^{g,p}(s|d) \) is denoted by \( \sim; \omega^{g,p} \sim \omega^{g,p} \) if \( \omega^{g,p} = \omega^{g,p} + s\eta^{g-1,p} + d\eta^{g,p-1} \), for some \( \eta^{g-1,p}, \eta^{g,p-1} \).

**Theorem**

(i) The general solution of the cocycle condition \( s\omega^p + d\omega^{p-1} = 0 \) is given by

\[
\omega^p \sim I^{p\alpha}\Theta_\alpha + B^p + \delta_n^{p-1}(\lambda^{\alpha\alpha}T_{\alpha\alpha} + \lambda^\gamma W_\gamma) \\
+ \delta_n^n(I_{(\mu)}^{n-1\alpha}[\Theta_\alpha] + \hat{B}^n_{(\mu)} + b^n_{(\mu)} + \mu^{\alpha\alpha}U_{\alpha\alpha} + \mu^\gamma R_\gamma + K_{(\mu)}^{\alpha\alpha}\Theta_\alpha + \lambda^{\Delta\alpha}V_{\Delta\alpha} + \lambda^\Gamma W_\Gamma),
\]

(3.22)

where \( \mu^{\alpha\alpha}, \mu^\gamma \) is the most general solution to the obstruction equation,

\[
[\mu^{\alpha\alpha}(-)^n k^{n-2}_\alpha + \mu^\gamma k^{\Delta\alpha}_\gamma I^{n-2}_A]N^2 + N_{(\mu)}^n + dI_{(\mu)}^{n-1\alpha}\Theta_\alpha + s(K_{(\mu)}^{\alpha\alpha}\Theta_\alpha) = 0,
\]

(3.23)

and where elements with the subscript \( (\mu) \) vanish when the \( \mu \)'s vanish.

(ii) A solution as in (3.22) is trivial in \( H^{g,p}(s|d) \), \( \omega^p = s\eta^p + d\eta^{p-1} \), or more explicitly,

\[
0 \sim I^{p\alpha}\Theta_\alpha + B^p + \delta_n^{p-1}(\lambda^{\alpha\alpha}T_{\alpha\alpha} + \lambda^\gamma W_\gamma) \\
+ \delta_n^n(I_{(\mu)}^{n-1\alpha}[\Theta_\alpha] + \hat{B}^n_{(\mu)} + b^n_{(\mu)} + \mu^{\alpha\alpha}U_{\alpha\alpha} + \mu^\gamma R_\gamma + K_{(\mu)}^{\alpha\alpha}\Theta_\alpha + \lambda^{\Delta\alpha}V_{\Delta\alpha} + \lambda^\Gamma W_\Gamma)
\]

(3.24)

if and only if

\[
0 = \lambda^\Gamma = \lambda^{\Delta\alpha} = \lambda^{\alpha\alpha} = \lambda^\gamma = \mu^{\alpha\alpha} = \mu^\gamma = K_{(\mu)} = b^n_{(\mu)} = \hat{B}^n_{(\mu)} = I_{(\mu)}^{n-1\alpha} = B^p,
\]

\[
I^{p\alpha}\Theta_\alpha \approx N^p + dI_{(\mu)}^{p-1\alpha}\Theta_\alpha + \delta_n^n[\rho^{\alpha\beta}(-(\alpha) k^{n-2}_\alpha + \rho^\gamma k^{\Delta\alpha}_\gamma I^{n-2}_A]N^2.\]

(3.25)

(iii) If \( I^p \in \mathcal{I} \) \( (p > 0) \) is trivial in \( H_{\text{char}}^p(d, \Omega) \), then it is the sum of a characteristic class and a piece which is trivial in \( H_{\text{char}}^p(d, \mathcal{I}) \):

\[
p > 0 : \quad I^p \approx d\eta^{p-1} \iff I^p \approx P^p(F) + dI^{p-1};
\]

(3.26)

(iv) No non-trivial characteristic class in form degree strictly less than \( n-1 \) is trivial in \( H_{\text{char}}(d, \mathcal{I}) \),

\[
p < n - 1 : \quad P^p(F) \approx dI^{p-1} \implies P^p(F) = 0.
\]

(3.27)

Note that, by definition, in form degrees \( n-1 \) and \( n \), characteristic classes that are trivial in \( H_{\text{char}}(d, \mathcal{I}) \) can be written as linear combinations of the bases elements \( P_A, P_A : \)

\[
P^{n-1}(F) \approx dI^{n-2} \implies P^{n-1} = \lambda^A P_A, \quad P^{n}(F) \approx dI^{n-1} \implies P^{n} = \lambda^A P_A.
\]

(3.28)
3.5 Structure of characteristic cohomology in degree \( n-2 \)

We have shown that every characteristic class that is weakly \( d \)-exact in \( \mathcal{I} \) for form degrees \( n-1 \) and \( n \) gives rise to non-covariantizable characteristic cohomology in degrees \( n-2 \) and \( n-1 \). What remains to be analyzed is the converse. Characteristic cohomology in form degree \( n-2 \) is isomorphic to \( H^{-2,n}(s|d) \) (see e.g. Theorem 7.1 of [15]; there can be no constant since we assume \( n > 2 \)). Only \( U_{\alpha\alpha} \) and \( R_{\gamma} \) contain ghost number \(-2\) pieces, see (3.8) and (3.15), and this requires that the pure-ghost part of them should vanish, i.e., \( \Theta_\alpha = 1 \). In this case \( N_\beta^2 = 0 \) and equation (3.23) imposes no restrictions on \( \mu^a, \mu^\gamma \). It follows that \( \omega^{-2,n} \sim \mu^a d^n x C_a^\ast + \mu^\gamma k_\gamma^A d^n x C_A^\ast \). Since we can choose the basis \( N_\gamma \) to include the \( P_A \) and since, with \( \Theta_\alpha = 1 \), the characteristic classes \( P_A \)'s are actually equivalent to the \( N_\gamma \)'s from the very definition of the latter, we can take \( k_\gamma^A = \delta_\gamma^A \) and therefore

\[
\omega^{-2,n} \sim \mu^a d^n x C_a^\ast + \mu^\gamma d^n x C_A^\ast.
\] (3.29)

Since the first piece corresponds to covariantizable characteristic cohomology, we have shown that non-covariantizable characteristic cohomology in degree \( n-2 \) is exhausted by the classes related to \( d^n x C_A^\ast \) which are explicitly given by \( B_\gamma^{n-2} - \delta_\gamma^A I_A^{n-2} \), see equation (3.11). The former term satisfies \( P_A = dB_\gamma^{n-2} \) where \( B_\gamma^{n-2} = q_\gamma^{n-2} \). Indeed, since \( \Theta_\alpha = 1 \), \( k_\gamma^A P_A = N_\gamma = dB_\gamma^{n-2} + sb_\gamma^{n-1} \) is satisfied with \( b_\gamma^{n-1} = 0 \) and we recall that we chose \( k_\gamma^A = \delta_\gamma^A \). We have thus shown that non-covariantizable characteristic cohomology is exhausted by characteristic classes that become trivial on-shell and that, by a change of basis, the indices \( \bar{A} \) and \( A \) can be identified.

In form degree \( n-2 \), we thus have a direct sum decomposition of characteristic cohomology into \( U \)-type, which is covariantizable and related to \( \mu^a \) with associated elements of \( H^{-2,n}(s|d) \) of depth 2, and \( R \)-type, which is non-covariantizable and related to \( \mu^A \), with associated elements of \( H^{-2,n}(s|d) \) of depth at least 3 that can only exist in odd spacetime dimensions.

3.6 Structure of characteristic cohomology in degree \( n-1 \)

Characteristic cohomology in form degree \( n-1 \) is isomorphic to \( H^{-1,n}(s|d) \). From expression (3.22) one can see that the elements of \( \omega^n \) with antifield number 1 but not 2 should be multiplied by \( \Theta_\alpha = 1 \). In this case, it follows from the definition of \( N_\Gamma \) that the \( P_A \)'s are equivalent to the \( N_\Gamma \)'s and we can choose \( k_\Gamma^A = \delta_\Gamma^A \). Hence,

\[
\omega^{-1,n} \sim \lambda^\Delta K_\Delta + \lambda^A K_A + [\mu^{\alpha a} U_{\alpha a} + \mu^\gamma R_\gamma + K_\mu^{\alpha \alpha} \Theta_\alpha]^{-1}.
\] (3.30)

This leads to the following decomposition:

\( V \)-type corresponds to the elements \( \lambda^\Delta K_\Delta \) of \( H^{-1,n}(s|d) \) that have depth 1 and the associated covariantizable characteristic cohomology is \( \lambda^\Delta j_\Delta \).

\( W \)-type corresponds to the elements \( \lambda^A K_A \) of \( H^{-1,n}(s|d) \) that have depth at least 2 with associated non-covariantizable characteristic cohomology given by \( \lambda^A (B_\gamma^{n-1} - I_A^{n-1}) \). Here \( B_\gamma^{n-1} := q_\gamma^{n-1} \) coincides with \( B_\Gamma^{n-1} := k_\Gamma^A q_\Gamma^{n-1} \), see item 5 in section 3.3.
In order to work out the last terms, the ghost number $-1$ part of the $\mu$ part, we need to combine the pieces in $U_{aa}$ having strictly positive antifield number (1 and 2) with corresponding $\Theta_a$’s given by an abelian ghost $C^{Iab}$, so that $[\Theta_a]^2 = 0$ and $N_a^2 \equiv F^{Iab}$. For terms related to $\mu^\gamma$ to exist, one needs to be in odd spacetime dimensions so that the $N_\gamma := k^{\gamma}_\alpha P_\alpha \Theta_\alpha$ can exist. One should also have that $k^{\gamma}_\alpha \Theta_\alpha$ be in ghost number 1 — see the expression for $R_\gamma$ in (3.15). On the one hand, from eq. (10.18) of [8] and eq. (2.6),

$$k^i N_i = F^{Iab} \Theta^{Iab} \left[ P^{n-3}(F) \frac{1}{2} k_{Iab}^{Jab} C^{Iab} C^{Jab} \right] = P^{n-3}(F) F^{Iab} k_{Iab}^{Jab} C^{Jab}.$$ 

On the other hand, this needs to be equal to a linear combination of $P_\alpha C^{Iab}$. The gauge group thus needs to contain at least two abelian factors and hence two different $P_\alpha$’s containing abelian field strengths. In turn, this requires at least two different free abelian factors — recall that the $P_\alpha(F)$’s are related to the characteristic cohomology in degree $n - 2$ and hence to $C_A^\*$.

Putting everything together, the expression for $[\mu^{\alpha} U_{aa} + \mu^\gamma R_\gamma + K_{\mu}^n \Theta^n]^{-1}$ looks as follows

$$\mu_{Iab}^\alpha \left( d^n x C_A^\ast C^{Iab} + \ast A_a^\ast A^{Iab} \right) + \mu^\gamma k^{\gamma}_{Iab} \left( d^n x C_A^\ast C^{Iab} + \ast A_a^\ast A^{Iab} \right) + K_{\mu}^n. \quad (3.31)$$

The condition that this defines additional $H^{-1,n}(s|d)$ cohomology and thus additional characteristic cohomology in form degree $n - 1$ is the existence of particular $P_\mu^n, I_{\mu}^{n-1}, K_{\mu}^n$ allowing one to solve the obstruction equation,

$$[(-)^n \mu_{Iab}^\alpha k_{\alpha}^{n-2} + \mu^\gamma k^{\gamma}_{Iab} I_{\alpha}^{n-2}] F^{Iab} + P_\mu^n + dI_{\mu}^{n-1} + sK_{\mu}^n = 0. \quad (3.32)$$

The associated characteristic cohomology in degree $n - 1$ is determined by the antifield independent part of

$$\mu_{Iab}^\alpha \left( k^{n-2}_{\alpha} A^{Iab} + \ast A_a^\ast C^{Iab} \right) + \mu^\gamma \left[ b^{n-1}_{\gamma} + k_\gamma^{Iab} \left( -I_{\alpha}^{n-2} A^{Iab} + \ast A_a^\ast C^{Iab} \right) \right], \quad (3.33)$$

depends explicitly on $A^{Iab}$. The argument that no allowed redefinition makes these additional characteristic cohomology classes invariant goes as follows: suppose that one of these classes could be made equivalent to a combination of $j_\Delta$. Then the BRST extension $\omega^{0,n-1}$ that contains that class would satisfy $\omega^{0,n-1} \sim \lambda^\Delta j_\Delta$. In turn, this implies that the associated $\omega^{-1,n} \sim \lambda^\Delta K_\Delta$. But part (ii) of the theorem then shows that this implies that the coefficients of all these terms vanish separately.

$U$-type corresponds to solutions with vanishing $\mu^\gamma$ but non-vanishing $\mu_{Iab}^\alpha$. They are thus related to covariantizable characteristic cohomology in degree $n - 2$ through the parameters $\mu_{Iab}^\alpha$. They have depth 2. Note that such solutions may involve $K_\mu$’s that vanish when the $\mu_{Iab}^\alpha$ do.

$R$-type corresponds to solutions with non-vanishing $\mu^\gamma$’s. They are related to non-covariantizable characteristic cohomology in degree $n - 2$ through the parameters $\mu^\gamma k_\gamma^{Iab}$. They have depth at least 3. Such solutions may involve $K_\mu$’s and $\mu_{Iab}^\alpha \left( d^n x C_A^\ast C^{Iab} + \ast A_a^\ast A^{Iab} \right)$’s that vanish when the $\mu^\gamma$ vanish.
In particular, for pure abelian Yang-Mills theory there is no $R$-type, so that the additional non-covariantizable characteristic cohomology is of $U$-type and reduces to $\mu^\alpha_{Ia}k^{-2}_iA_{ab}$. This class then contains the rigid symmetries corresponding to the rotation of free abelian vector fields among themselves, see section 4.3.1 and [14] for further discussions.

3.7 Structure of infinitesimal deformations

Infinitesimal deformations are described by $H^{0,n}(s|d)$ and are thus captured, cf. Item (i) of the theorem, by the nontrivial cocycles of $s$ modulo $d$, at ghost number zero and top form degree $n$:

$$\omega^{0,n} \sim I^n + B^{0,n} + V^{0,n} + W^{0,n} + [\mu^{\alpha\alpha}U_{\alpha\alpha} + \mu^\gamma R_\gamma + K^\mu_{\alpha\alpha}\Theta_\alpha + J^{-1}_\mu [\Theta_\alpha]_1 + \hat{\Theta}^n_\mu + b^n_\alpha],$$

where the ghost number is indicated in the superscript of the last term between square bracket; the form degree of the quantity between square bracket is $n$. According to item (ii) of the theorem, such a solution is non-trivial whenever $I^n \neq d\omega^{n-1}$, which is equivalent to saying that $I^n$ is not expressed as in (3.25), and the other terms in the expression of $\omega^{0,n}$ do not vanish.

More precisely, non-trivial solutions $\omega^{0,n}$ can be decomposed into the following linearly independent types:

- **I-type**: non trivial elements $I^n = d^n x I_{inv}([F, \psi]) \in \mathcal{I}$;
- **B-type**: $B^{0,n}$ is a linear combination of the independent Chern-Simons $n$ forms
  $$[M_{i_1\ldots i_n}]_{s+2m(r)-1} = [\theta_r]^{2m(r)-1} f_{s_1} \cdots f_{s_N},$$
  therefore they only arise in odd spacetime dimensions $n = s + 2m(r) - 1$;
- **V-type**: $V^{0,n} = \lambda^\Delta_{ab}(K^\alpha C^I_{ab} + j^\alpha A^I_{ab})$, they correspond to standard gaugings obtained from minimal coupling of abelian gauge fields to covariantizable conserved Noether currents;
- **W-type**: for $W^{0,n}$, the spacetime dimension needs to be even and one needs a relation $k^\Gamma N_\Gamma = k^i N^i_n = k^\alpha P^\alpha_{A}(F)\Theta_\alpha$ at ghost number 1, so that $b^n_\Gamma$ in the expression of $W_T$ in (3.19) has ghost number zero. The left hand side of the previous equation is of the form $F^k_{ab} \phi^k_{ab}(P^{n-2\Sigma}(F)^{1/2}k^{0}_{iab}^{\gamma}C^I_{ab}C^J_{ab} = P^{n-2\Sigma}(F)^{1/2}k^{0}_{iab}^{\gamma}C^I_{ab}C^J_{ab}$. This needs to be equal to a linear combination of $P_{A}(F)C^I_{ab}$. The gauge group thus needs to contain at least two abelian factors and two different $P_{A}(F)$'s containing abelian field strengths. For an example see equation (12.4) of [15]. Although it illustrates these classes, the Lagrangian used there is not covered by theorem 11.1 of [15] since it contains free abelian gauge fields. It fits perfectly however in the present context, and is treated in more detail in [14].
- **U-type** correspond to solutions to the obstruction equation (3.32) with vanishing $\mu^\alpha$ and non-vanishing $\mu^{\alpha\alpha}$. In ghost number 0, they are related to covariantizable characteristic cohomology in degree $n-2$ through the parameters $\mu^\alpha_{Ia}A_{ab}$ and contain
  $$[\mu^{\alpha\alpha}U_{\alpha\alpha}]^0 = (d^n x C^\alpha_a + \star A^\alpha_a A^I_{ab} \phi^k_{ab} + \frac{1}{2}k^{n-2}_a A^I_{ab} A^J_{ab} \phi^k_{ab} \phi^k_{ab}) \mu^\alpha_{Ia} A^I_{ab} C^J_{ab},$$

$$15$$
where \( \partial_{Iab} = \partial/\partial C^{Iab} \). In general, these terms might need to be supplemented by terms of the form \( K_n^\alpha I_{ab} C^{Iab} + I_{\mu I_{ab}} A^{Iab} + \hat{B}_\mu^n + \beta_\mu^p \) that vanish when the \( \mu_{[I_{ab},I_{ab}]} \) vanish.

- \( R \)-type correspond to solutions to the obstruction equation (3.32) with non-vanishing \( \mu^\gamma \)'s. The dimension \( n \) of spacetime needs to be odd for the existence of \( \gamma = N_\gamma \) and we must have that \( b_\gamma^n \) has ghost number zero, which implies that \( gh(b_\gamma^n) = 1 \) and hence \( gh(N_\gamma) = 2 \). Since the \( \theta_i \)'s have odd ghost number, the ghost degree of \( \gamma \) is carried by the product of two abelian ghosts. Therefore, from eq. (10.18) of [15] and from (2.6), the linear combination \( k_i^\gamma N_\gamma = k_i N_i \) should have the form

\[
F^{I_{ab}} \partial_{ab} \left( P^{n-2\Sigma} (F) \frac{1}{6} k_{\Sigma I_{ab}}^\alpha K_{ab} C^{I_{ab}} C^{J_{ab}} C^{K_{ab}} \right)
\]

and this needs to be equal to a linear combination of \( P_{\bar{A}}(F) C^{I_{ab}} C^{J_{ab}} \). The gauge group thus needs to contain at least three abelian factors and three different \( P_{\bar{A}}(F) \)'s containing abelian field strengths. In turn, this requires at least 3 free abelian factors.

Finally, we note from (3.15) that \( [\mu^\gamma R_{\gamma}]^0 \) terms contains a piece linear in the antighost \( C^{\bar{K}}_\alpha \), and that these terms might need to be supplemented by terms of the form \( [\mu^{\alpha a} U_{\alpha a}]^0 \) and \( K_{\mu I_{ab}} C^{I_{ab}} + I_{\mu I_{ab}} A^{I_{ab}} + \hat{B}_\mu^n + \beta_\mu^p \) that vanish when the \( \mu^\gamma \) vanish.

A similar analysis can be performed for anomaly candidates by spelling out the classes in ghost number 1.

## 4 Applications

### 4.1 No free abelian factors

If the gauge group contains no free abelian factors, there is no characteristic cohomology in degree \( n - 2 \), \( H_{\text{char}}^{n-2}(d) \approx H^{n-2}(s|d) \) vanishes. This implies the absence of \( k_{\alpha a}^n \) and the associated \( T_{\alpha a} \) and \( U_{\alpha a} \) on the one hand, and also the absence of \( P_{\bar{A}} \), and the associated \( W_\gamma \) and \( R_{\gamma} \), on the other. In other words, \( \lambda^{\alpha a} = \lambda^\gamma = \mu^{\alpha a} = \mu^\gamma = I_{\mu^{\alpha a}} = \hat{B}_\mu^n = b_\mu^p = 0 = \rho_{\alpha \beta} = \rho^\gamma \) in the theorem, which therefore reduces to theorem 11.1 of [15].

### 4.2 Chern-Simons theory

We consider pure Chern-Simons theory, as was done in the section 14 of [15], but now as a particular case of the present approach, instead of an AKSZ-type approach with complete ladder fields [26–28]. The action is given by

\[
S_{CS} = \int g_{IJ} \left( \frac{1}{2} A^I dA^J + \frac{1}{6} f_{KLM} A^K A^L A^L \right). \tag{4.1}
\]
The first question is about characteristic cohomology in degree $n - 2$: it can be represented as $d^n x C^\mu_{ab}$ and is associated with each (free\footnote{In pure Chern-Simons theory, there is no distinction between abelian and free abelian factors.}) abelian factor. Furthermore, as can be seen from $s A^*_I = \frac{1}{2} g_{IJ} e^{\mu \rho} F_{\mu \rho}^J + C^J f^K_A A^*_I$, every characteristic class in form degree $2 = 3 - 1$ associated with an abelian factor is trivial on-shell: $F^I_{ab} = \frac{1}{2} F^\mu_{ab} dx^\mu dx^\nu = -g^I_{ab} J_{ab} s \ C^*_I = -s \ A^*_I$. It follows that all abelian indices are $\tilde{A}$ indices. There is thus no $T_{aa}$ nor $U_{aa}$: $\Lambda_{aa} = 0 = \mu a = \rho a\beta$. We also see that $I_{\tilde{A}}^{-2} = 0$. The $M_{r_1 \ldots r_K | s_1 \ldots s_N}$ of equation (10.16) of [15], the $M$'s in (2.6) and the associated $N_{r_1 \ldots r_K | s_1 \ldots s_N} := \sum_{\nu: m(\nu) = m(\nu)} f_I \frac{\partial M_{r_1 \ldots r_K | s_1 \ldots s_N}}{\partial \theta_{\nu}}$ and $N$'s can be split according to whether they contain abelian factors or not, $m(r_1) = 1$ or $m(r_1) > 1$. The former constitute the $N_r$'s since they are given by $F^I_{ab} \partial_{I_{ab}} M_{r_1 \ldots r_K | s_1 \ldots s_N} = -s (\ast A^I_{ab} \partial_{I_{ab}} M_{r_1 \ldots r_K | s_1 \ldots s_N})$, cf. (3.12).

Now we need to address the question whether these $N_r$ can be lifted upon adding to $b^{-1}=2$ a contribution outside of $B$, thereby obtaining a $W_r$. Condition (3.23) is empty since there is no $\mu a$ and no $I_{\tilde{A}}^{-2}$ so all of the $W_r$'s can be lifted to $R_r$ without obstruction, see also (3.14)–(3.17).

This can also be seen differently, as in section 14 of [15]: Introduce $C^J = J^J + A^J + \ast A^J + d^n x C^J$ so that $(s + d) C^J = f_{JK}^I C^K C^J$ and replace in $M_{r_1 \ldots r_K | s_1 \ldots s_N}$ every $\theta_r(C)$ by $\theta_r(C)$. It follows that $(s + d) M_{r_1 \ldots r_K | s_1 \ldots s_N} (\theta(C), f) = 0$. When splitting according to the form degree, we have

\[ M_{r_1 \ldots r_K | s_1 \ldots s_N} (\theta(C), f) = M_{r_1 \ldots r_K | s_1 \ldots s_N}^{L+1} + M_{r_1 \ldots r_K | s_1 \ldots s_N}^{L+2} + \ldots \]

where

\[
M_{r_1 \ldots r_K | s_1 \ldots s_N}^{L+1} = A_{ab}^I \partial_{I_{ab}} M_{r_1 \ldots r_K | s_1 \ldots s_N} + \sum_{r : m(\nu) > 1} (A_{I_{ab}}^I \partial_{I_{ab}} \theta_{\nu}) \frac{\partial M_{r_1 \ldots r_K | s_1 \ldots s_N}}{\partial \theta_{\nu}},
\]

\[
M_{r_1 \ldots r_K | s_1 \ldots s_N}^{L+2} = \frac{1}{2} A_{ab}^I A_{ab}^J \frac{\partial J_{ab}}{\partial I_{ab}} M_{r_1 \ldots r_K | s_1 \ldots s_N} + \sum_{r : m(\nu) > 1} (A_{I_{ab}}^I \partial_{I_{ab}} \theta_{\nu}) A_{ab}^J \frac{\partial M_{r_1 \ldots r_K | s_1 \ldots s_N}}{\partial \theta_{\nu}}
\]

\[
+ \frac{1}{2} \sum_{r : m(\nu) > 1} \sum_{r' : m(\nu') > 1} (A_{I_{ab}}^I \partial_{I_{ab}} \theta_{\nu}) (A_{I_{ab}}^I \partial_{I_{ab}} \theta_{\nu'}) \frac{\partial M_{r_1 \ldots r_K | s_1 \ldots s_N}}{\partial \theta_{\nu} \partial \theta_{\nu'}} + \ast A_{I_{ab}}^I \partial_{I_{ab}} M_{r_1 \ldots r_K | s_1 \ldots s_N},
\]

with $s M^{L+2} + d M^{L+1} = 0$ and $s M^{L+1} + d M = 0$. As was pointed out in [15], the antifield dependent term in $M^{L+2}$ which contains $\ast A_{I_{ab}}^I$ can be replaced by a term in the small algebra $B$. We know that $\theta_r(C)$ obeying $s \theta_r(C) = 0$ can be completed to $q_r(C, f)$ that obeys the cocycle relation $s q_r(C, f) = 0$ for $s := s + d$ in three dimensions, since the $f_r$'s with $m(r) > 1$ are forms of degree higher than 3, as recalled in section 10 of [15]. Therefore, if instead of the $\tilde{s}$-cocycle $M_{r_1 \ldots r_K | s_1 \ldots s_N} (\theta(C), f)$ one uses the $\tilde{s}$-cocycle obtained by replacing in $M_{r_1 \ldots r_K | s_1 \ldots s_N} (\theta, f)$ every $\theta_r$ having $m(r) > 1$ by the corresponding $q_r(C, f)$, still keeping the abelian $C_{I_{ab}}$'s unaffected, one obtains an equivalent $\tilde{s}$-cocycle without any non-abelian $\ast A_{I_{ab}}^I$ nor any $\ast C_{I_{ab}}^I$. When this
replacement is made, $W_{\gamma}$ is given by $M_{r_1...r_K|s_1...s_N}^{L+2}$, where all terms correspond to $b_{\gamma}^{n-1=2}$ except for the last one in (4.2). There will be no obstruction to a lift since $M_{L+3}$ exists. In other words, $b_{\gamma}^{n-1=2}$ can be lifted once more when supplemented by the last term in (4.2), where the lift of this last term brings in $d^n x C^*_{Iab}$. Note that, on account of the form degree, $M_{L+2}$ must correspond to $W_2^2$ and hence $L = 0$.

Because $n = 3$ is odd there are no characteristic classes $P^n_a(F)$. There are also no non-trivial $j_\Delta$ since all of them depend on field strengths and their covariant derivatives, which vanish weakly. For the same reason $I^n \alpha \Theta_\alpha$ reduce to polynomials in the $\theta_r$ times the volume form $d^3 x$, which is what remains of the $M_{r_1...r_K|s_1...s_N}$ in agreement with [15].

For ghost number 0 and form degree $n = 3$, we need $M^2$ and in particular $b_{\gamma}^{n-1=2}$ to be of ghost number 1, so that $b_{\gamma}^n$ has ghost number zero and qualifies for an infinitesimal deformation entering $R_{\gamma}$. This means that $M$ is of ghost number 3 and the only possibility then is 3 abelian ghosts in order to reproduce $N_\gamma = F^Iab \partial _{Iab} M_{r_1...r_K|s_1...s_N}$. Therefore one must have $M = \frac{1}{3} k_{[I_{ab},j_{ab}]K_{ab}} C^I_{I_{ab}} J^I_{ab} C^J_{J_{ab}} K_{ab}$. We thus remain with $S^{(1)} = \int \omega^{0,3}$ with $\omega^{0,3} = B^{0,3} + [\frac{1}{3} k_{[I_{ab},j_{ab}]K_{ab}} C^I_{I_{ab}} J^I_{ab} C^J_{J_{ab}} K_{ab}]^3$ and where $B^{0,3}$ are just the Chern-Simons forms for the non-abelian factors. The second term is explicitly given by

$$[d^n x C^*_{Iab} \partial _{Iab} + *A^*_{Iab} A^I_{Iab} \partial _{Iab} + \frac{1}{3} A^I_{Iab} A^J_{Iab} A^K_{Iab} \partial _{Iab} \partial _{Iab} + \frac{1}{3} k^I_{Iab} k^J_{Iab} k^J_{Iab} C^I_{Iab} C^J_{Iab} C^K_{Iab}].$$

Hence, by construction, $k_{[I_{ab},j_{ab}]K_{ab}}$ is completely skew-symmetric. When asking for the infinitesimal deformation $S^{(1)}$ to be consistent at second order, $\frac{1}{2} (S^{(1)}, S^{(1)}) + (S^{(0)}, S^{(2)}) = 0$, one finds the Jacobi identity for $I_{I_{ab},j_{ab}} K_{ab}$ by asking that the term in $C^*_{Iab}$, which is the tail of a non trivial cohomology class, be absent. There are no further conditions and there is no need for an $S^{(2)}$.

### 4.3 Abelian gauge theories

The gauge group contains only free abelian factors without any specification of the Lagrangian at this stage, so that abelian Chern-Simons theory is covered and so is the coset model$^5$. The specificity of the models under consideration is that the $k^I \Theta_\alpha$ are given by $\Theta(C^I)$, polynomials in undifferentiated free abelian ghosts $C^I$ — in this section we drop the subscript $fab$ on the index $I$.

It also follows that $\{C_I^I\} = \{C_\alpha^I, C_\bar{\alpha}^I\}$ whose indices are lifted with Kronecker deltas. To define $W_\gamma$, we need a relation $N = P^\Sigma(F) F^I \partial _{I} \Theta_\Sigma = P_A \Theta^A$. In ghost number 0, we find $P^\Sigma(F) F^I K_{\Sigma I} = k^A P_A$, in ghost number 1, $P^\Sigma(F) F^I K_{[\Sigma I]} = k^A P_A$ ... with $N_\gamma$ the associated basis. This basis has been explicitly given in (2.5).

If $W_\gamma = b_{\gamma}^{n-1} + (\star A^* \Theta_\gamma \bar{A}^\gamma - I^{n-1 \bar{A}} A^I \partial _{I} \Theta_\gamma \bar{A})$, we have

$$\omega^{n-1} \sim I^{n-1 \alpha} \Theta_\alpha + B^{n-1} + \star A^* \Theta_\alpha + k^{n-2 \alpha} A^I \partial _{I} \Theta_\alpha + \lambda^n W_\gamma$$

$^5$The analysis here makes more precise the remark on how to extend the analysis for a free abelian Lagrangian of Maxwell type to more general Lagrangians at the end of section 13.1 of [15].
and the obstruction equation (3.23) becomes

\[ (-)^n k_a^{-2a} F^I \delta_I \Theta_a + \mu^\gamma k_\alpha^\gamma I_a^{-2} F^I \partial_I \Theta_a + P^\Sigma(F) F^I \partial_I \Theta_\Sigma + dI^{n-1} \Theta_a \approx 0. \tag{4.3} \]

### 4.3.1 Pure abelian Yang-Mills theory

Abelian Yang-Mills theory is treated in [9, 29] and in section 13 of [15]. From the current perspective, there are no \( \vec{A} \) indices and all indices \( I \) are \( a \) indices, because all characteristic cohomology in degree \( n - 2 \) is covariantizable, \( s \star A^* + d \star F^I = 0 \). The obstruction equation becomes

\[ (-)^n \star F^I F^I \delta_I \Theta_I + P^\Sigma(F)^{n-2} F^I \partial_I \Theta_\Sigma + dI^{n-1} \Theta_a \approx 0. \]

In the non-independent case, this implies

\[ (-)^n \star F^I F^I \delta_I \Theta_I + P^\Sigma(F)^{n-2} F^I \partial_I \Theta_\Sigma = 0 \]

by putting to zero derivatives of the \( F \)'s. Then both terms have to vanish separately. This can be seen by taking an Euler-Lagrange derivative with respect to \( A^I_\mu \) and using the fact that \( P^\Sigma(F)^{n-2} F^I \) is a total derivative. This implies \( \partial_I \Theta_I = 0 \) which gives in turn \( \Theta_I = \delta _I \Theta \) (fermionic Poincaré lemma). In this case, there are no obstructions and the cubic vertex comes from \([d^n x C^{a1} \delta_I + \star A^* I A^I \partial_I + 1/2 \star F^I A^K \partial_K \partial_I \partial_I] \Theta \) with \( \Theta \) in ghost number 3. There can be no \( P_A \) classes since \( P^n \approx dI^{n-1} \) implies \( P^n = 0 \) when putting to zero all derivatives of the \( F^I_\mu \).

There is thus only \( U, B \) or \( I \)-type cohomology, and the most general infinitesimal deformation is described by \( n \) forms \( U^{0,n} + B^{0,n} + I^n \). With only abelian factors, the \( B^{0,n} \) exists only in odd spacetime dimensions \( n \) and are characterized by completely symmetric tensors of rank \( m \) with \( n = 2m - 1 \).

### 4.3.2 Higher dimensional Yang-Mills-Chern-Simons theory

Abelian Yang-Mills-Chern-Simons theory in dimension \( n = 2m - 1 \) is determined by the action \( \int_{M^{2m-1}} L^{2m-1} \), where \( L^{2m-1} = 1/2 F^I \star F^I \delta I J + 1/4 d_I J_1 \ldots J_{m-1} A^I F^{J_1} \ldots F^{J_{m-1}} \) and \( d_I J_1 \ldots J_{m-1} = d_{I_1 \ldots I_{m-1}} \) is a totally symmetric, constant symbol. We assume linear independence of the \( n_v \) symmetric tensors of rank \( m - 1 \) obtained by letting the index \( I \) in \( d_I J_1 \ldots J_{m-1} \) run over its \( n_v \) values. This is equivalent to requiring that the \( d \) symbol does not possess any null vector, in the sense that \( V^I d_I J_1 \ldots J_{m-1} = 0 \Leftrightarrow V^I = 0 \) for all \( I \). For \( n = 3, m = 2 \), this means that \( d_I J \) is a nondegenerate bilinear form on \( \mathbb{R}^{n_v} \) that we take to be \( \delta I J \). Below, abelian indices are lowered and raised with \( \delta I J \) and its inverse.

The field equations read

\[ 0 \approx \frac{\delta I J^{2m-1}}{\delta A^I} = d_I J^{(m-1)} F^{J(m-1)} + d \star F_I, \]

where we use the notation \( d_I J^{(m-1)} = d_{I_1 J_1} \ldots J_{m-1} \) and \( F^{J(m-1)} = F_{J_1} \ldots F_{J_{m-1}} \). This is equivalent to \( s \star A^* + d \star F_I + d_I J^{(m-1)} F^{J(m-1)} = 0 \). There thus exist characteristic classes that are trivial in \( H^{n-1}_\text{char}(d, \mathcal{Z}) : P_A^{n-1} = d_{I_1 \ldots I_{m-1}} F^{J(m-1)} \). Indeed, our assumption on the symbol \( d_I J_1 \ldots J_{m-1} \) implies that there are exactly \( n_v \) independent characteristic classes that are trivial in \( H^{n-1}_\text{char}(d, \mathcal{Z}) \), so that the index \( \vec{A} \) can be identified with the index \( I \) running over the \( n_v \) abelian factors. In other words, there is no covariantizable characteristic cohomology in degree \( n - 2 \), there are no \( k_a^{n-2} \), and all indices \( I \) are again \( \vec{A} \) indices. Note also that the invariant \( I^{n-2}_A \) from the definition \( P_A^{n-1} \approx dI^{n-2}_A \) is equal to \(- \star F_I \). There is thus no \( U \)-type cohomology. There is no \( W \)-type cohomology either since the spacetime dimension is odd.
In order to construct cohomology classes of $R$-type as in (3.15), we need to determine (a basis of) the intersection $k^{\alpha a} P_{A}^{n-1} \Theta_{\alpha} = N^{n-1}$, where $N^{n-1} = sb^{n-1} + dB^{n-2}$. From the characterization of $H(s|d, B)$ in (2.5), this intersection is determined by constants $k^{\alpha I} = \frac{1}{k!} f^{I}_{J_{1}...J_{k}}$ such that

$$
\frac{1}{k!} f^{K}_{I_{1}...I_{k}} d_{K, I^{(m-1)}} F^{J(m-1)} C^{J_{1}} \ldots C^{J_{k}} = \frac{m+k-1}{m-1} \lambda_{j(m-1), I_{1}...I_{k}} F^{J(m-1)} C^{J_{1}} \ldots C^{J_{k}} .
$$

(4.4)

In other words, the constants $f^{K}_{I_{1}...I_{k}}$ should be such that $\frac{1}{k!} f^{K}_{I_{1}...I_{k}} d_{K, I^{(m-1)}} = \frac{m+k-1}{m-1} \lambda_{j(m-1), I_{1}...I_{k}}$. The symmetry (2.8) then implies the constraint

$$
d_{K(J_{1}...J_{m-1}) J_{m} I_{2}...I_{k}} = 0 .
$$

(4.5)

In particular, for $n = 3$ with $d_{I J} = \delta_{I J}$, this requires $f^{I}_{J_{1}...J_{k}} := \delta_{I L} f^{L}_{J_{1}...J_{k}}$ to be completely antisymmetric, so that $k^{\alpha a} \Theta_{\alpha} = \frac{1}{k!} f^{I}_{J_{1}...J_{k}} C^{J_{1}} \ldots C^{J_{k}}$ is equal to $\partial^{I} \Theta$ with

$$
\Theta = \frac{1}{(k+1)!} f_{J_{1}...J_{k+1}} C^{J_{1}} \ldots C^{J_{k+1}} .
$$

(4.6)

More generally, this also implies that all $N$'s of form degree $n - 1 = 2m$ are $N_{s}$'s for particular constants $f^{I}_{J_{1}...J_{k}}$ and $d_{I J_{1}...J_{m-1}}$ such that (4.5) is true. Indeed, because a tensor $\lambda_{j(m-1), I^{[k]}}$ is an irreducible $GL(n_{v})$ tensor with symmetry given by the following Young tableau

$$
\begin{array}{ccccccc}
1 & 2 & \cdots & n-1 \\
\vdots & \ddots & & & \\
k & & & & & & \\
\end{array}
\cong
\begin{array}{cccc}
1 \\
\vdots \\
k & & & & & & \\
\end{array}
\otimes
\begin{array}{cccc}
1 & 2 & \cdots & m \\
\vdots & & & & \\
k & & & & & & \\
\end{array},
$$

(4.7)

it can be parametrised as

$$
\lambda_{J_{1}...J_{m-1}, I_{1}...I_{k}} = d_{J_{1}...J_{m-1} I_{1}...I_{k}} f_{I_{1}...I_{k}} , \quad \text{with} \quad d_{J_{1}...J_{m-1} I_{1}...I_{k}} f_{J_{m} I_{2}...I_{k}} = 0 ,
$$

(4.8)

where the latter constraint expresses the subtraction term on the right-hand side of (4.7). Therefore, a fortiori an $GL(n_{v})$-irreducible tensor $\lambda_{J_{m-1}, I^{[k]}}$ (with $k \leq n_{v}$) can be written as a sum of terms of the type given in (4.8):

$$
\lambda_{J_{1}...J_{m-1}, I_{1}...I_{k}} = d_{K J_{1}...J_{m-1}} f^{K}_{I_{1}...I_{k}} , \quad \text{with} \quad d_{K(J_{1}...J_{m-1}) J_{m} I_{2}...I_{k}} = 0 ,
$$

(4.9)

which is what we wanted to show. According to (3.14) and (2.11), this gives

$$
W^{n-1} = \frac{1}{2} A^{K} A^{L} \partial_{L} \partial_{K} \left[ \frac{m-1}{k!(m+k-1)} d_{M J_{1}...J_{m-2} I_{1}} f_{M I_{2}...I_{k+1}} F^{J_{1}} \ldots F^{J_{m-2}} C^{I_{1}} \ldots C^{I_{k+1}} \right] \\
+ \frac{1}{k!} f^{K}_{I_{1}...I_{k}} \left( * F_{K} A^{L} \partial_{L} + \ast A^{K} \right) C^{I_{1}} \ldots C^{I_{k}} .
$$

(4.10)

As we explained above, in the 3-dimensional case $n = 3$ (i.e., $m = 2$), the constraint (4.5) can be solved explicitly, which allows to rewrite (4.10) in a more compact way:

$$
W^{2} = (\frac{1}{2} A^{K} A^{L} + * F_{K} A^{L}) \partial_{L} \partial_{K} + \ast A^{K} \partial_{K} \left[ \Theta , \quad \Theta = \frac{1}{(k+1)!} f_{J_{1}...J_{k+1}} C^{J_{1}} \ldots C^{J_{k+1}} .
$$

(4.11)
In dimension $n = 2m - 1$, we know that $dW^{n-1} + sR^n = k^A F^I \partial_I \Theta$, with
\[
R^n = \frac{1}{3!} A^K A^L A^M \partial_M \partial_L \partial_K \left[ \frac{m-1}{k!(m-k)!} d_{N,(m-2)J} f^N_{I[K} F^{J_1} \ldots F^{J_{m-2}} C^{I_1} \ldots C^{I_{k+1}} \right] \\
+ \frac{1}{k!} f^M_{I_1 \ldots I_k} \left( \ast F_M A^K A^L \partial_L \partial_K + \ast A^*_M A^K \partial_K + \ast C^*_M \right) C^{I_1} \ldots C^{I_k}. \tag{4.12}
\]

In dimension $n > 3$, the obstruction in lifting $W^{n-1}$ is due to the fact that $k^A F^I \partial_I \Theta = k^A F^I \partial_I \Theta$ does not vanish because the constant symbol $f_{K|J[k]}$ is not necessarily antisymmetric. In dimension $n = 3$, the constraint (4.5) entails antisymmetry of the $f_{K|J[k]}$ and therefore no obstruction arises in the lift of the 2-form $W^2$.

Furthermore, because $n$ is odd, the obstruction equation (4.3) reduces to
\[
k^A F^I \partial_I \Theta + N^I_{(\mu)} + dI_{(\mu)} \Theta + s(K_{(\mu)} \Theta) = 0, \quad \Theta = C^{I_1} \ldots C^{I_k}, \tag{4.13}
\]
or, more explicitly,
\[
-\frac{1}{k!} f^K_{I_1 \ldots I_k} \ast F_K F^{I_1} C^{I_2} \ldots C^{I_k} + N^I_{(\mu)} + dI_{(\mu)} \Theta + s(K_{(\mu)} \Theta) = 0. \tag{4.14}
\]

In the case where the $g$-invariant local functions in $\mathcal{I}$ do not explicitly depend on $x^\mu$, there is no solution to the obstruction equation if the first term does not vanish identically. It does vanish if and only if the symbol $f_{K|J[k]} = \delta_{KM} f^M_{I[k]}$ is completely antisymmetric, on account of $\ast F^K F^L = \ast F^L F^K$. In this case, there is no obstruction and no need for $I_{n-1}^\alpha$ nor $K^{n}$. This allows us to write $R^n$ and $W^{n-1}$ as follows:
\[
R^n = \frac{1}{3!} A^K A^L A^M \partial_M \partial_L \partial_K \left[ \frac{m-1}{k!(m-k-1)!} d_{N,(m-2)J} f^N_{I[K} F^{J_1} \ldots F^{J_{m-2}} C^{I_1} \ldots C^{I_{k+1}} \right] \\
+ (\frac{1}{2} \ast F^K A^L A^M \partial_M \partial_L \partial_K + \ast A^*_M A^K \partial_K + \ast C^*_K \partial_K) \Theta, \quad \Theta = \frac{1}{(k+1)!} f^K_{I_1 \ldots I_k} C^{I_1} \ldots C^{I_{k+1}}, \tag{4.15}
\]
with the descent equations $sR^n + dW^{n-1} = 0$, $sW^{n-1} + dQ^{n-2} = 0$, $sQ^{n-2} = 0$, where
\[
Q^{n-2} = A^K \partial_K \left( \frac{m-1}{k!(m-k-1)!} d_{M,(m-2)J} f^M_{I_1 \ldots I_{k+1}} F^{I_1} \ldots F^{I_{m-2}} C^{I_1} \ldots C^{I_{k+1}} \right) + \ast F^K \partial_K \Theta. \tag{4.16}
\]

In particular, in $n = 3$ dimensions (i.e. $m = 2$), it yields
\[
R^3 = \left( \frac{1}{2} [\ast F^K A^L A^M + \frac{1}{3} A^K A^L A^M] \partial_M \partial_L \partial_K + \ast A^*_M A^K \partial_K + \ast C^*_K \partial_K \right) \Theta, \tag{4.17}
\]
with $W^2$ already given in (4.11) and $Q^1 = (A^I + \ast F^I) \partial_I \Theta$. Note that $R^3$ contains the expected cubic vertices for the 3-dimensional non-abelian Yang-Mills-Chern-Simons theory.

An R-type infinitesimal deformation requires ghost number 0 and thus $k = 2$, so that (4.4) reduces to
\[
\frac{1}{2} f^K_{I_1 I_2} d_{KJ} = \frac{m+1}{m-1} \lambda_{J(m-1), I_1 I_2}, \tag{4.18}
\]
the zero ghost number component of $R^n$ being
\[
R^{0,2m-1} = \frac{1}{2} f^I_{IJ} t_{IJK} \cdot \star C^*_{IJ} C^I C^J + f^I_{IJ} (\star A^*_{IJ} A^I A^J + \frac{1}{2} \star F^I A^I A^J)
+ \frac{m-1}{m+1} \frac{1}{2} f^K \cdot dI_{IJ} dI_{K^m} F^{J(m-2)} A^I A^J A^K.
\] (4.19)

As before, when asking for the infinitesimal deformation $S^{(1)}$ to be consistent at second order, \(\frac{1}{2}(S^{(1)}, S^{(1)}) + (S^{(0)}, S^{(2)}) = 0\), one finds the Jacobi identity for $f^{K}_{IJ}$ by asking that the term in $C^*_I$, which is the tail of a non-trivial cohomology class, be absent. There are no further conditions. The constraint (4.5) can then be interpreted as an invariance condition for the totally symmetric
\[
\text{for some constants}
\]
\[
\text{the additional antifield independent terms add up}
\]
\[
\text{to give the non-abelian Yang-Mills action in dimension} \ n = 2m - 1 \text{ plus the standard non-abelian}
\]
\[
\text{Chern-Simons Lagrangian given by the homotopy integral}
\]
\[
\mathcal{L}_{na}^{2m-1} = \int_0^1 dt \left[ dA + t^2 A^2 \right]^{I(m-1)}.
\] (4.20)

### 4.3.3 Axion models in even dimensions

We now consider even spacetime dimension $n = 2m$ with Lagrangian
\[
\mathcal{L} = -\frac{1}{2} \partial_\mu \phi^I \partial^\mu \phi^I \delta_{IJ} + \frac{1}{2} \delta_{IJ} F^I \star F^J + \phi^I t_{I|J_1...J_m} F^{J_1} \ldots F^{J_m},
\] (4.21)
for some constants $t_{I_1...J_m}$, and concentrate on $W$-type cohomology classes.

The field equations for the $n_s = n_v$ scalars are responsible for the existence of characteristic classes $P^m_I = t_{I|J_1...J_m} F^{J_1} \ldots F^{J_m}$ such that $P^m_I = dI_{I} - s K_I$, where $K_I = \star \phi^*_I$. In order to find $W$-type cohomology classes as in (3.19), $W = b^p - k^I \theta_\alpha + I^I \theta_\alpha$, we need to determine the intersection $s b^p + db^p = N_{m,k} = k^A P_A \theta_\alpha$ where $N = \frac{m+k}{m} \lambda_{J(m),l[k]} F^{J_1} \ldots F^{J_m} C^{I_1} \ldots C^{I_k}$, $b^p = \frac{1}{\bar{A}_L} A^L \partial^I_{J} (\lambda_{J(m-1),l[k]} F^{I_1} \ldots F^{I_m} C^{I_1} \ldots C^{I_k})$ and $I^I = \star d \phi_I$. The intersection condition is satisfied provided
\[
\frac{m+k}{m} \lambda_{J(m),l[k]} t_{I|J_1...J_m} k^K I_{1...I_k}.
\] (4.22)

The symmetry properties of $\lambda_{J(m),l[k]}$ imply the following constraint: $t_{K|J_1...J_m} k^K I_{1...I_k} = 0$. Similarly to the analysis leading to (4.8), any $GL(n_v)$-irreducible tensor $\lambda_{J_1...J_m,I_{1...I_k}}$ with $k \leq n_v$ can be written as in (4.22) provided the latter constraint is fulfilled.

This condition is fulfilled by taking $k_{L|l[k]} = \delta_{KL} k^K l[k] = f_{L|l_1...l_k}$, for $f_{L|l_1...l_k}$ an antisymmetric tensor of rank $k + 1$ and $t_{I|J|l(m)} = \delta_{IJ} \hat{t}_{J|l(m-1)}$ for $\hat{t}_{J|l(m-1)}$ a symmetric rank-$(m - 1)$ tensor $\hat{t}$. Introducing $\Theta = \frac{1}{k+1} f_{I_1...I_{k+1}} C^{I_1} \ldots C^{I_{k+1}}$, the $W$-type cohomology classes are then given by
\[
W^{k-1,2m} = \frac{1}{2^{(m-k)}} A^K A^L \partial^I_{J} \partial_K \left[ \hat{t}_{J(m-1)} f_{I[k]} + (-)^k (m - 1) \hat{t}_{J(m-2)} f_{I[k]} \right] F^{J(m-2)} C^{I_1} \ldots C^{I_{k+1}}
- \left( \star \alpha_J A^K \partial_K \partial_J \Theta \right).
\] (4.23)
Infinitesimal deformations require ghost number 0 and thus \( k = 1 \). The first term on the right-hand side then exactly reproduces the first order deformation in the Lagrangian (35) of [22], provided one identifies the constants \( C_{I_1...I_m} \) therein with \( \lambda_{J_1...J_m,I} \).

The examples of this section generalise to arbitrary even dimensions the \( W \)-type cohomology classes in 4 spacetime dimensions discussed in section 6.4 of [14].

### 4.4 Coupled abelian vector-scalar models in four dimensions

The results for this case are discussed in detail in section 3.2 of [14].

### Acknowledgments

The authors thank M. Henneaux, B. Julia, V. Lekeu and A. Ranjbar for useful discussions. GB is grateful to F. Brandt for earlier collaborations on this subject. This work was partially supported by FNRS-Belgium (convention FRFC PDR T.1025.14 and convention IIS N 4.4503.15). NB is Senior Research Associate of the F.R.S.-FNRS.

### A Conventions

The components of the Minkowski metric are given, in inertial coordinates in which we work, by the mostly plus expression \( \eta_{\mu\nu} = \text{diag}(-1, +1, \ldots, +1) \). The symbol \( \epsilon_{\mu_1...\mu_n} \) denotes the completely antisymmetric Levi-Civita density with the convention that \( \epsilon^{01...n-1} = 1 \) so that \( \epsilon_{01...n-1} = -1 \). A local basis of anticommuting exterior differential 1-forms is given by the family \((dx^\mu)_{\mu=0,...,n-1}\). The wedge product symbol \( \wedge \) will always be omitted.

We will sometimes use the notation \((d^{p-\nu}x)_{\mu_1...\mu_n} := -\frac{1}{p!(n-p)!} dx^{\mu_1} \cdots dx^{\mu_{n-p}} \epsilon_{\mu_1...\mu_n} \) for \( 1 \leq p \leq n \), and \( d^nx := dx^0 \cdots dx^{n-1} \). The Hodge dual of a differential \( p \)-form \( \omega^p \equiv \frac{1}{p!} dx^{\mu_1} \cdots dx^{\mu_p} \omega_{\mu_1...\mu_p} \), is the \( n-p \)-form given, in our convention, by

\[
\star \omega^p = \frac{1}{p!(n-p)!} dx^{\nu_1} \cdots dx^{\nu_{n-p}} \epsilon_{\nu_1...\nu_{n-p} \mu_{n-p+1}...\mu_n} \omega^{\mu_{n-p+1}...\mu_n} = -(d^{n-p}x)_{\mu_{n-p+1}...\mu_n} \omega^{\mu_{n-p+1}...\mu_n}.
\]

As a consequence, the exterior differential of the dual of a \( p \)-form reads

\[
d \star \omega^p = -(-)^{n-p} (d^{n-p+1}x)_{\nu_1...\nu_{n-p-1}} \partial_\nu \omega^{\mu_{n-p+1}...\mu_n}.
\]

### B Proof of the main theorem

The proof proceeds by induction on the form degree. At \( p = 0 \) and due to our assumption \( n > 2 \), items (i) and (ii) reduce respectively to \( s \omega^0 = 0 \Leftrightarrow \omega^0 = I^0 \Theta_0 + s \eta^0 \) and \( I^0 \Theta_0 = s \eta^0 \Leftrightarrow I^0 \Theta_0 \equiv 0 \).
Both hold on account of corollary 11.2 of [15]. Item (iii) has no content for \( p = 0 \). Item (iv) holds: if a constant vanishes weakly it vanishes by assumption on the equations of motion.

We will further decompose the induction into three main steps. In step [A], we consider \( p = m - 1 < n - 2 \) and go to \( p = m < n - 1 \). For step [B], we consider the induction going from \( p = n - 2 \) to \( p = n - 1 \) and in step [C] we will finally reach \( p = n \).

**Proof**

A. Assume now that (i), (ii), (iii), (iv) hold at form degrees \( p = m - 1 < n - 2 \) and let us show that they hold at form degree \( m < n - 1 \). In this case, the proof proceeds exactly like in theorem 11.1 of [15]:

(i) The cocycle condition \( s\omega^m + d\omega^{m-1} = 0 \) implies that \( \omega^{m-1} \) satisfies the similar equation

\[
\omega^{m-1} = I^{m-1\alpha}\Theta_\alpha + B^{m-1}
\]

since we assume that (i) holds at degree \( m - 1 < n - 2 \) and since trivial contributions can be neglected. This implies \( d\omega^{m-1} = dI^{m-1\alpha}\Theta_\alpha - s(I^{m-1\alpha}[\Theta_\alpha]) + P^{m-1\alpha}(F) \approx 0 \). When inserted in the equation \( d\omega^{m-1} = dI^{m-1\alpha}\Theta_\alpha - s(I^{m-1\alpha}[\Theta_\alpha]) + P^{m-1\alpha}(F) \approx 0 \) and since trivial contributions can be neglected, this implies \( I^{m-1\alpha} \approx d\omega^{m-2\alpha} \) (or constant for \( m = 1 \)) since there is no non-trivial characteristic cohomology in degree \( m - 1 < n - 2 \). Since (iii) is assumed to hold for \( p = m - 1 \), we find \( I^{m-1\alpha} \approx dI^{m-2\alpha} + P^{m-1\alpha} \). By using corollary 11.1 of [15], this yields

\[
I^{m-1\alpha} = sK^{m-1\alpha} + dI^{m-2\alpha} + P^{m-1\alpha}
\]

\[
\iff \omega^{m-1} = [P^{m-1\alpha} + dI^{m-2\alpha} + sK^{m-1\alpha}]\Theta_\alpha + B^{m-1}. \tag{B.2}
\]

The part \( P^{m-1\alpha}\Theta_\alpha \) can be decomposed in terms of the basis of \( H(s, B) \) consisting of the \( M \)'s and \( N \)'s: \( P^{m-1\alpha}\Theta_\alpha = N^{m-1} + M^{m-1} + \delta^{m-1}_0 \). Neither the constant nor \( N^{m-1} \) will contribute when used to calculate \( \omega^m \). Since \( [dI^{m-2\alpha} + sK^{m-1\alpha}]\Theta_\alpha = d(I^{m-2\alpha}\Theta_\alpha) + \]
s(I^{m-2\alpha}\Theta_\alpha] + K^{m-1\alpha}\Theta_\alpha), these terms will not contribute either. We can thus assume that \(\omega^{m-1} = M^{m-1} + B^{m-1}\). Since \(dM^{m-1} = -sB^m\) for some \(B^m \in \mathcal{B}\), if we denote \(B^m := \bar{B}^m + \tilde{B}^m\), we have shown that \(d\omega^{m-1} + sB^m = 0\). Finally, the initial cocycle condition \(s\omega^{m-1} + d\omega^{m-1} = 0\) now reduces to \(s(\omega^m - B^m) = 0\) and thus \(\omega^m = I^{m}\Theta_\alpha + B^m + s\eta^m\) by corollary 11.2 of [15]. Taking into account the trivial terms that we have neglected (in particular \(N^{m-1}\)), we find \(\omega^m \sim I^{m}\Theta_\alpha + B^m\), which is (i) at form degree \(m < n - 1\).

(ii) Consider \(\omega^m = I^{m}\Theta_\alpha + B^m\) (we recall that \(m < n - 1\)) where \(sB^m + d(B^{m-1} + M^{m-1}) = 0\). The cocycle condition \(s\omega^m + d\omega^{m-1} = 0\) is satisfied with \(\omega^{m-1} = B^{m-1} + I^{m-1}\Theta_\alpha\) and where \(I^{m-1}\Theta_\alpha = M^{m-1}\). Assume now that \(\omega^m = I^{m}\Theta_\alpha + B^m \sim 0\). This implies that \(\omega^{m-1} = B^{m-1} + I^{m-1}\Theta_\alpha \sim 0\). Since we assume that (ii) holds at form degree \(m - 1\), we find \(B^{m-1} = 0\), \(M^{m-1} = I^{m-1}\Theta_\alpha \approx N^{m-1} + dI^{m-2}\Theta_\alpha\). This implies \(P^{m-1}\Theta_\alpha := M^{m-1} - N^{m-1} \approx dI^{m-2}\Theta_\alpha\), which gives on account of (iv) that \(P^{m-1}\Theta_\alpha = 0\) and thus that \(M^{m-1} = N^{m-1}\) which means that both have to vanish separately, since they are independent elements of the basis of \(H(s, B)\). It then follows that \(sB^m = 0\), which implies that \(B^m = 0\) by definition of \(B^m\). We thus find \(I^{m}\Theta_\alpha = s\eta^m + d\eta^{m-1}\). Applying \(s\) to this equation gives \(s\eta^{m-1} + d\eta^{m-2} = 0\) and then, on account of (i), that \(\eta^{m-1} = \bar{B}^{m-1} + I^{m-1}\Theta_\alpha + s\eta^{m-1} + d\eta^{m-2}\). This yields \(I^{m}\Theta_\alpha = s(\eta^{m-1} - \bar{B}^{m-1} + \tilde{B}^{m-1} - \bar{I}^{m-1}[\Theta_\alpha]) + N^{m} + dI^{m-1}\Theta_\alpha\), which implies \(I^{m}\Theta_\alpha = \tilde{P}^{m}\Theta_\alpha\). Together with the relation \(B^m = 0\) obtained above, this shows (ii) at form degree \(m < n - 1\).

(iii) \(I^m \approx d\omega^{m-1}\) implies \(I^m \sim 0\). Using (ii) at \(m\), which we have proved, gives \(I^m \approx dI^{m-1} + P^m\). The converse is trivial since \(P^m = dq^{m-1}\).

B. Let us now proceed to form degree \(p = n - 1\).

(iv) For \(p = n - 1\), there is nothing to be proved. Indeed, the condition \(dq^{n-2} = P^{n-1}(F) \approx dI^{n-2}\) means that \(I^{n-2} - q^{n-2}\) is an element of characteristic cohomology in form degree \(n - 2\). But we have assumed that the set of \(P^{n-1}\)'s form a basis of characteristic classes that are weakly \(d\)-exact in \(\mathcal{T}\), \(P^{n-1} = \lambda A P^{n-1}\). For later use, we note that it follows from equation (3.11) that \(P^{n-1} = \lambda^2(dI^{n-2} - sA^\alpha)\).

(i) Following the same proof as before when \(p < n - 1\), we get from \(s\omega^{n-1} + d\omega^{n-2} = 0\) that

\[
\omega^{n-2} = I^{n-2}\Theta_\alpha + B^{n-2} \quad \text{(B.3)}
\]

with \(dB^{n-2} = -s(\bar{B}^{n-1} + \tilde{B}^{n-1}) + N^{n-1}\) for some \(\bar{B}^{n-1}, \tilde{B}^{n-1}\) and \(N^{n-1} = P^{n-1}\Theta_\alpha\), which is the obstruction to the lift of an element in \(\mathcal{B}\). The polynomial in the curvatures \(P^{n-1}\Theta_\alpha = P^{n-1}\alpha(F)\) are themselves also obstructions to a lift in the small algebra, being a linear combination of the \(N\)'s only — the \(M\)'s explicitly depend on the undifferentiated ghosts. Computing \(d\omega^{n-2}\) and plugging back into the initial cocycle relation, we find

\[
dI^{n-2}\Theta_\alpha + P^{n-1}\alpha(F) \approx 0 \quad \text{(B.4)}
\]
where Lemma 1 was used, as before. It follows from (iv) at \( p = n - 1 \) that \( P^{n-1,\alpha} = \lambda^\alpha A P^{n-1} \approx \lambda^\alpha dI^{n-2}_A \). Combining this with relation (B.4) leads to \( d(I^{n-2,\alpha} + \lambda^\alpha I^{n-2}_A) \approx 0 \). This means that \( I^{n-2,\alpha} + \lambda^\alpha I^{n-2}_A \) defines a covariant element of characteristic cohomology in form degree \( n - 2 \). Therefore \( I^{n-2,\alpha} + \lambda^\alpha I^{n-2}_A \approx \lambda^\alpha k^{n-2}_a + d\omega^{n-3,\alpha} \), when taking into account the definitions before the descent equations (3.6). Using (iii) for \( p = n - 2 \) yields

\[
I^{n-2,\alpha} + \lambda^\alpha I^{n-2}_A - \lambda^\alpha k^{n-2}_a = P^{n-2,\alpha}(F) + dI^{n-3,\alpha} + sK^{n-2,\alpha},
\]

where corollary 11.1 of [15] has been used.

From the definition of \( N_\gamma = k_\gamma^\alpha P A^{-1} \Theta_\alpha \) in item 4 of the list of Ingredients and the fact that \( N^{n-1} = P^{n-1,\alpha} \Theta_\alpha = \lambda^\alpha A P^{n-1} \Theta_\alpha \), we can decompose \( N^{n-1} = \lambda^\gamma N_\gamma \) and identify

\[
\lambda^\gamma k^{\alpha,\gamma}_\gamma = \lambda^\alpha.
\]

Consider then

\[
\hat{\omega}^{n-1} := \omega^{n-1} - \lambda^\alpha T^{aa} - \lambda^\gamma W_\gamma,
\]

\[
\hat{\omega}^{n-2} := \omega^{n-2} - \lambda^\alpha k^{n-2}_a - \lambda^\gamma (B^{n-2}_\gamma - k^{\alpha,\gamma}_\gamma I^{n-2}_A \Theta_\alpha).
\]

They are related by the cocycle relation \( s \hat{\omega}^{n-1} + d\hat{\omega}^{n-2} = 0 \), as can be seen from (3.9) and (3.17). By using (B.3) and (B.5), we find that

\[
\hat{\omega}^{n-2} = [P^{n-2,\alpha} + dI^{n-3,\alpha} + sK^{n-2,\alpha}] \Theta_\alpha + (B^{n-2} - \lambda^\gamma B^{n-2}_\gamma). \tag{B.8}
\]

Following the same reasoning as the one used after the equation (B.2), this then implies \( \hat{\omega}^{n-1} \sim B^{n-1} + I^{n-1,\alpha} \Theta_\alpha \) and therefore proves (i) for \( p = n - 1 \).

(ii) Consider \( \omega^{n-1} = I^{n-1,\alpha} \Theta_\alpha + B^{n-1} + \lambda^\alpha T^{aa} + \lambda^\gamma W_\gamma \) with \( s\omega^{n-1} + d\omega^{n-2} = 0 \) and \( \omega^{n-2} = \hat{\omega}^{n-2} = \hat{I}^{n-2,\alpha} \Theta_\alpha \), with \( \hat{B}^{n-2} = B^{n-2} + \lambda^\gamma B^{n-2}_\gamma \), and \( \hat{I}^{n-2,\alpha} \Theta_\alpha = M^{n-2} + \lambda^\alpha k^{n-2}_a \Theta_\alpha - \lambda^\gamma k^{\alpha,\gamma}_\gamma I^{n-2}_A \Theta_\alpha \). The assumption that \( \omega^{n-1} \) is trivial implies that so is \( \omega^{n-2}, \omega^{n-2} \sim 0 \). We have shown that (ii) holds at form degree \( n - 2 \), so we get \( B^{n-2} = -\lambda^\gamma B^{n-2}_\gamma \), and \( \hat{I}^{n-2,\alpha} \Theta_\alpha \approx N^{n-2} + dI^{n-3,\alpha} \Theta_\alpha \).

Since \( N_\gamma \) exists only if the form degree \( n - 1 \) is even and thus \( n \) is odd, there is no \( N_\gamma \), nor \( B^{n-2}_\gamma \) for \( n \) even, and thus both \( B^{n-2} = 0 \) and \( B^{n-2}_\gamma = 0 \) for \( n \) even. For \( n \) odd, \( M^{n-2} = 0 \).

We also have \( d(\lambda^\gamma B^{n-2}_\gamma) + s(\lambda^\gamma b^{n-1}_\gamma) = \lambda^\gamma N_\gamma \). Using this, we find that \( s(B^{n-1} + \lambda^\gamma b^{n-1}_\gamma) + d(M^{n-2} + B^{n-2} + \lambda^\gamma B^{n-2}_\gamma) = \lambda^\gamma N_\gamma \). Since the \( d \) exact term on the left hand side vanishes, we get that \( \lambda^\gamma = 0 \) since \( \lambda^\gamma N_\gamma \) is exact only if \( \lambda^\gamma \) vanishes. This then also shows that \( B^{n-2} = 0 \) if \( n \) is odd. This then implies that \( N^{n-2} - N^{n-2} = P^{n-2,\alpha} \Theta_\alpha = dq^{n-3,\alpha} \Theta_\alpha \), we find \( \lambda^\alpha k^{n-2}_a \approx d\omega^{n-3} \) which implies \( \lambda^\alpha = 0 \) since they represent non trivial characteristic cohomology classes. We then remain with \( P^{n-2,\alpha} \approx dI^{n-3,\alpha} \).

Using (iv) at \( p = n - 2 \) then gives \( P^{n-2,\alpha} = 0 \) and then that \( M^{n-2} = 0 = N^{n-2} \). It follows that \( sB^{n-1} = 0 \) which implies \( B^{n-1} = 0 \) since \( B^{n-1} \) needs to have a non trivial descent. We remain with \( I^{n-1,\alpha} \Theta_\alpha = s\eta^{n-1} + d\eta^{n-2} \) and the rest of the proof goes through as in the proof of (ii) before since one only has to use (i) at form degree \( n - 2 \) which is of the same form than at lower form degrees.
(iii) $I^{n-1} \approx d\omega^{n-1}$ implies $I^{n-1} \sim 0$. Using (ii) at $p = n - 1$, which we have already shown, gives $I^{n-1} \approx dI^{n-2} + P^{n-1}$. The converse obviously holds since $P^{n-1} = dq^{n-2}$.

C. Finally, let us conclude by dealing with form degree $p = n$.

(iv) For $p = n$, there is again nothing to be proved. Indeed, we now find that $I^{n-1} - q^{n-1}$ is equivalent to an element of non-covariantizable characteristic cohomology in degree $n - 1$. But we have assumed that the set of $P_A$'s form a basis of characteristic classes that are weakly $d$-exact in $\mathcal{I}$, so that $P^n = \lambda^A P_A$. For later use, we now note that $P^n = \lambda^A dq_A^{n-1} = \lambda^A(dI_A^{n-1} + sK_A)$.

(i) We can assume in $s\omega^n + dw^{n-1} = 0$ that $\omega^{n-1}$ is of the form

$$\omega^{n-1} = I^{n-1\alpha} \Theta_\alpha + B^{n-1} + \lambda^{\alpha\beta} T_{\alpha\beta} + \lambda^\gamma W_\gamma,$$

with $dB^{n-1} = -s(\dot{B}^n + b^n) + N^n$, for some $\dot{B}^n, b^n$ and $N^n = P^{\alpha\alpha} \Theta_\alpha$. We then get $dw^{n-1} = (dI^{n-1\alpha}) \Theta_\alpha - s(I^{n-1\alpha} [\Theta_\alpha]^1 + \dot{B}^n + b^n + \lambda^{\alpha\beta} U_{\alpha\beta} + \lambda^\gamma R_\gamma) + N^n + [\lambda^{\alpha\beta} (-)^n k_{a}^{n-2} + \lambda^\gamma k_{a}^{\alpha\beta} I_A^{n-2}] N_\beta^2$. Inserting this into $s\omega^n + dw^{n-1} = 0$ gives

$$\left(dI^{n-1\alpha} + P^{\alpha\alpha} + [\lambda^{\alpha\beta} (-)^n k_{a}^{n-2} + \lambda^\gamma k_{a}^{\alpha\beta} I_A^{n-2}] P_{\beta}^{2\alpha}\right) \Theta_\alpha + s(\omega^n - I^{-1\alpha} [\Theta_\alpha]^1 - \dot{B}^n - b^n - \lambda^{\alpha\beta} U_{\alpha\beta} - \lambda^\gamma R_\gamma) = 0$$

with $N_\alpha^2 = P_\alpha^2 \Theta_\beta$. This implies

$$P^{\alpha\alpha} + dI^{-1\alpha} + [\lambda^{\alpha\beta} (-)^n k_{a}^{n-2} + \lambda^\gamma k_{a}^{\alpha\beta} I_A^{n-2}] P_{\beta}^{2\alpha} \approx 0.$$  \hspace{1cm} (B.9)

This equation is equivalent to the obstruction equation (3.23), formulated without antifields and ghosts. Indeed, the weak equality can be replaced by a strong equality with zero on the right-hand side replaced by $s(-K^{\alpha a})$ with $K^{\alpha a}$ depending linearly on undifferentiated antifields and such that $\gamma K^{\alpha a} = 0$, — the required modification of $I^{-1\alpha}$ corresponds to a trivial term in $\omega^{n-1}$. Those $\lambda^{\alpha\beta}, \lambda^\gamma$ for which one cannot find a solution to this equation correspond to parts of $\omega^{n-1}$ that cannot give a solution to $s\omega^n + dw^{n-1} = 0$. Let us denote by $\mu^{\alpha\beta}, \mu^\gamma$ the general solution with non-vanishing first term and particular $P^{\alpha\alpha}_\mu$ and $I^{-1\alpha}_\mu$ solving this equation. It means that both these quantities vanish when $\mu^{\alpha\beta}$ and $\mu^\gamma$ vanish. We can thus assume that

$$\omega^{n-1} = (I_R^{-1\alpha} + I^{-1\alpha}_\mu) \Theta_\alpha + B^{n-1}_R + B^{n-1}_\mu + \mu^{\alpha\beta} T_{\alpha\beta}^{n-1} + \mu^\gamma W_\gamma^{n-1},$$

and have

$$P^{\mu\alpha} + dI^{-1\alpha}_\mu + [\mu^{\alpha\beta} (-)^n k_{a}^{n-2} + \mu^\gamma k_{a}^{\alpha\beta} I_A^{n-2}] P_{\beta}^{2\alpha} \approx s K^{\alpha a}_\mu \Theta_\alpha + I^{-1\alpha}_\mu [\Theta_\alpha]^1 = 0.$$  

Upon multiplying by $\Theta_\alpha$, this yields

$$[\mu^{\alpha\beta} (-)^n k_{a}^{n-2} + \mu^\gamma k_{a}^{\alpha\beta} I_A^{n-2}] N_\beta^2 + N^n + d(I^{-1\alpha}_\mu \Theta_\alpha) + s(K^{\alpha a}_\mu \Theta_\alpha + I^{-1\alpha}_\mu [\Theta_\alpha]^1) = 0.$$
We then define
\[
\bar{\omega}^{n-1} := \omega^{n-1} - I_{\mu}^{n-1} \Theta_\alpha - B^{n-1}_\mu - \mu^{\alpha\beta} T_{\alpha\beta}^{n-1} - \mu^\gamma W^{n-1}_\gamma = I_{\mu}^{n-1} \Theta_\alpha + B^{n-1}_R ,
\]
\[
\bar{\omega}^n = \omega^n - I_{\mu}^{n-1} [\Theta_\alpha] - \hat{B}^{n}_\mu - b^{n}_\mu - \mu^{\alpha\alpha} U_{\alpha\alpha} - \mu^\gamma R_\gamma - K^{n\alpha}_\mu \Theta_\alpha ,
\]
where \( dB^{n-1}_R = - s(\hat{B}^{n}_R + b^{n}_R) + N^n_R \) and we denote \( N^n_R = P^{n\alpha}_R \Theta_\alpha \). Similarly, we have that \( dB^{n-1}_\mu = - s(\hat{B}^{n}_\mu + b^{n}_\mu) + N^n_\mu \) with \( N^n_\mu = P^{n\alpha}_\mu \Theta_\alpha \). The cochains \( \bar{\omega}^n \) and \( \bar{\omega}^{n-1} \) satisfy
\[
\bar{s} \bar{\omega}^n + d\bar{\omega}^{n-1} = 0 \quad \text{and in the last equality we used (B.11), the cocycle relation to be obeyed.}
\]
As in the proof of (i) for lower form degree above, on \( e \) then concludes that
\[
\text{We have shown that (ii) holds at form degree } \mu \text{.}
\]

(ii) Consider \( \omega^n \) as in (3.22), so that \( s \omega^n + d\omega^{n-1} = 0 \) with
\[
\omega^{n-1} = B^{n-1} + M^{n-1} + I_{\mu}^{n-1} \Theta_\alpha + B^{n-1}_\mu + \mu^{\alpha\alpha} T_{\alpha\alpha} + \mu^\gamma W_{\alpha} + \lambda^\Delta j_\Delta \Theta_\alpha + \lambda^\Gamma (B^{n-1}_\Gamma - k^{\alpha\alpha}_\Gamma I^{n-1}_A \Theta_\alpha) , \]
or equivalently, \( \omega^{n-1} = \mu^{\alpha\alpha} T_{\alpha\alpha} + \mu^\gamma W_{\alpha} + \hat{B}^{n-1} + I^{n-1} \Theta_\alpha \), with \( \hat{B}^{n-1} = B^{n-1} + B^{\mu}_\mu + \lambda^\Gamma B^{n-1}_\Gamma \), and \( I^{n-1} \Theta_\alpha = M^{n-1} + I^{n-1} \Theta_\alpha + \lambda^\Delta j_\Delta \Theta_\alpha - \lambda^\Gamma k^{\alpha\alpha}_\Gamma I^{n-1}_A \Theta_\alpha \). The assumption that \( \omega^n \) is trivial implies that so is \( \omega^{n-1} \), \( \omega^{n-1} \sim 0 \).

We have shown that (ii) holds at form degree \( n - 1 \), so we get \( \mu^{\alpha\alpha} = 0 = \mu^\gamma \) which also implies \( I_{\mu}^{n-1} = B^{n-1}_\mu = \hat{B}^{n}_\mu = b^{n}_\mu = 0 \), \( B^{n-1} = - \lambda^\Gamma B^{n-1}_\Gamma \), and \( I^{n-1} \Theta_\alpha \approx N^{n-1} + dI^{n-2} \Theta_\alpha \).

Since \( N_\Gamma \) exists only if the form degree \( n \) is even, there is no \( N_\Gamma \), nor \( B^{n-1}_\Gamma \) for odd, and thus \( B^{n-1} = 0 = B^{n-1}_\Gamma = 0 \) for odd.

For \( n \) even, \( M^{n-1} = 0 \). We also have \( d(\lambda^\Gamma B^{n-1}_\Gamma) + s(\lambda^\Gamma b^{n}_\mu) = \lambda^\Gamma N_\Gamma \). Using this, we find that
\[
\lambda^\Gamma b^{n}_\mu + d(M^{n-1} + B^{n-1} + \lambda^\Gamma B^{n-1}_\Gamma) = \lambda^\Gamma N_\Gamma .
\]
Since the second term on the left hand side vanishes, we get that \( \lambda^\Gamma = 0 \). Indeed, \( \lambda^\Gamma N_\Gamma \) is \( s \) exact only if it vanishes. This shows that \( B^{n-1} = 0 \) if \( n \) is even as well.

The equation for \( I^{n-1} \) with \( \lambda^\Gamma = 0 \) now gives \( \lambda^\Delta j_\Delta \Theta_\alpha \approx N^{n-1} + dI^{n-2} \Theta_\alpha \). Since \( M^{n-1} - N^{n-1} = P^{n-1} \Theta_\alpha = dq^{n-2} \Theta_\alpha \), we find \( \lambda^\Delta j_\Delta \approx d\omega^{n-2} \) which implies \( \lambda^\Delta = 0 \) since
they represent non trivial characteristic cohomology classes. We then remain with $P^{n-1\alpha} \approx d\bar{n}^{n-2\alpha}$. Using (iv) at $p = n - 1$ then gives $P^{n-1\alpha} = 0$ and then that $M^{n-1} = 0 = N^{n-1}$.

It follows that $sB^n = 0$ which implies $B^n = 0$ since $B^n$ needs to have a non trivial descent. We remain with $I^{\alpha\alpha}\Theta_\alpha = s\eta^n + d\eta^{n-1}$. This implies that $s\eta^{n-1} + d\eta^{n-2} = 0$. Using (i) at $p = n - 1$ we can assume that $s\eta^{n-2} = B^{n-1} + \bar{I}^{n-1\alpha}\Theta_\alpha + \rho^{\alpha\alpha}T_{aa} + \rho^nW_\gamma + s\bar{\eta}^{n-1} + d\bar{\eta}^{n-2}$. Using $d\bar{B}^{n-1} = -s(B^n + b^n) + \tilde{N}^n$ and $d(\bar{I}^{n-1\alpha}\Theta_\alpha) = d\bar{I}^{n-1\alpha}\Theta_\alpha - d(I^{n-1\alpha}[\Theta_\alpha])$ gives $[I^{\alpha\alpha} - d\bar{I}^{n-1\alpha} - \bar{P}^{\alpha\alpha} - (\rho^{\beta\gamma}(-)^n h^{\alpha\gamma}_\alpha + \rho^n k^{\alpha\beta}_\gamma)P^{2\alpha}_\beta]\Theta_\alpha = s(\eta^n - \bar{B}^n - b^n - d\bar{\eta}^{n-1} - \bar{I}^{n-1\alpha}[\Theta_\alpha])$. This reduces to $I^{\alpha\alpha} \approx d\bar{I}^{n-1\alpha} + \bar{P}^{\alpha\alpha} + (\rho^{\alpha\beta}(-)^n k^{\alpha\gamma}_\alpha + \rho^n k^{\alpha\beta}_\gamma)P^{2\alpha}_\beta$ and proves (ii) for $p = n$.

(iii) $\bar{I}^{\alpha\alpha} \approx d\omega^{n-1}$ implies $\bar{I}^{\alpha\alpha} \sim 0$. Using (ii) at $p = n$, which we have already proved, gives $\bar{I}^{\alpha\alpha} \approx P^n + dI^{n-1}$ since when $\Theta_\alpha$ reduces to 1, $N^{2\alpha}_\alpha = 0$. The converse is again trivial since $P^n = dq^{n-1}$.

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