Broadcasting colourings on trees.
A combinatorial view.*

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Abstract
The broadcasting models on a $d$-ary tree $T$ arise in many contexts such as discrete mathematics, biology, information theory, statistical physics and computer science. We consider the $k$-colouring model, i.e. the root of $T$ is assigned an arbitrary colour and, conditional on this assignment, we take a random colouring of $T$. A basic question here is whether the information of the assignment at the root affects the distribution of the colourings at the leaves. This is the so-called reconstruction/non-reconstruction problem. It is well known that $d/\ln d$ is a threshold function for this problem, i.e.

- if $k \geq (1 + \epsilon)d/\ln d$, then the colouring of the root has a vanishing effect on the distribution of the colourings at the leaves, as the height of the tree grows
- if $k \leq (1 - \epsilon)d/\ln d$, then the colouring of the root biases the distribution of the colouring of the leaves regardless of the height of the tree.

However, there is no apparent combinatorial reason why such a result should be true.

When $k \geq (1 + \epsilon)d/\ln d$, the threshold implies the following: We can couple two broadcasting processes that assign the root different colours such that the probability of having disagreement at the leaves reduces with their distance from the root. It is natural to perceive such a coupling as a mapping from the colouring of the first broadcasting process to the colouring of the second one. In that terms, here, we study how can we have such a mapping “combinatorially”. Devising a mapping where the disagreements vanish as we move away from the root turns out to be a non-trivial task to accomplish for any $k \leq d$.

In this work we obtain a coupling which has the aforementioned property for any $k > 3d/\ln d$, i.e. much smaller than $d$. Interestingly enough, the decisions that we make in the coupling are somehow local. It is not clear clear whether such a coupling should be local for any $k$ down to $d/\ln d$. Finally, we relate our result to sampling $k$-colourings of sparse random graphs, with expected degree $d$ and $k \leq d$.

1 Introduction
The broadcasting models on trees and the closely related reconstruction problem were originally studied in statistical physics. Since then they have found applications in other areas including biology (in phylogenetic reconstruction [5, 12]), communication theory (in the study of noisy computation [6]). Very impressively, these models arise in computer science in the study of random constraints satisfaction problems such as random $k$-SAT, random graph colouring etc. That is, the models on trees seem to capture some of the most fundamental properties of the corresponding models on random (hyper)graphs, [10].

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The most basic problem in the study of broadcasting models is to determine the reconstruction/non-
reconstruction threshold. I.e. whether the configuration of the root affects the distribution of the configu-
ration of the leaves of the tree. The transition from non-reconstruction to reconstruction can be achieved
by adjusting appropriately the parameters of the model. Typically, this transition exhibits a threshold
behaviour. So far, the main focus of the study was to determine the precise location of this threshold for
various models.

In this work, we focus on the colouring model on a \(d\)-ary tree. The reconstruction/non-reconstruction
threshold for this model is known precisely \([13, 14, 15, 4]\). We investigate the phenomenon further by
searching for a combinatorial reason why the information decays in the non-reconstruction regime. Such
an explanation, somehow, has been elusive when \(k \leq d\). For the reconstruction regime combinatorial
explanation is already known \([13, 14]\).

Let us be more specific on what do we mean by combinatorial explanation. The threshold implies
that when \(k \geq (1 + \epsilon)d/\ln d\) we can couple two broadcasting processes that assign the root different
colours such that the probability of having disagreements at the leaves reduces as their distance from
the root increases. It is natural to perceive such coupling as a mapping from the colouring of the first
broadcasting process to the colouring of the second one. In that terms, here, we study how can we have
such a mapping combinatorially.

We provide a coupling between two broadcasting processes which implies non-reconstruction for
\(k\) well below \(d\), i.e. for \(k > 3d/\ln d\). It is based on describing a (combinatorial) mapping between
the colourings of two different broadcasting processes. It works inductively and considers two levels
of the underlying tree each time. E.g. given the colour assignments of the root in the two processes
the coupling considers only colour choices for the vertices up to two levels below. The basic idea is
to reveal partially some information for the decisions of the two processes and investigates for which
(small) subtrees of the root the colour assignments at their leaves are identically distributed (conditional
the revealed information).

Even though the coupling we present here is not optimal, a lot of its basic ideas are quite natural. It
seems reasonable to expect that an optimal coupling should adopt a lot of them. Finally, recent advances
in sampling colouring algorithms (see \([7]\)) relate this coupling to sampling \(k\)-colourings of random
graphs of expected degree \(d\) when \(k < d\) (see Section 1.2).

1.1 The model and the reconstruction problem

The broadcasting models on a tree \(T\) are models in which information is sent from the root over the
dges to the leaves. We assume that the edges represent noisy channels. For some finite set of spins
\(\Sigma = \{1, 2, \ldots, k\}\), a configuration on \(T\) is an element of \(\Sigma^T\), i.e. it is an assignment of spins to the
vertices of \(T\). The spin of the root \(r\) is chosen according to some initial distribution over \(\Sigma\). The
information propagates along the edges of the tree as follows: There is a \(k \times k\) stochastic matrix \(M\) such
that if the vertex \(v\) is assigned spin \(i\), then its child \(u\) is assigned spin \(j\) with probability \(M_{i,j}\).

Our focus is on the \(k\)-colouring model (or \(k\)-state Potts model at zero temperature). We assume that
the underlying tree \(T\) is a complete \(d\)-ary tree of height \(h\) and for the matrix \(M\) we have that

\[
M_{i,j} = \begin{cases} 
\frac{1}{k-1} & \text{for } i \neq j \\
0 & \text{otherwise.}
\end{cases}
\]

Broadcasting models give rise to Gibbs measures on trees. E.g. for the colouring model, assuming
that the broadcasting process over \(T\) starts with root \(r\) coloured \(i\), then the \(k\)-colouring we get after the
processes has finished is a random \(k\)-colouring of \(T\) conditional that \(r\) is coloured \(i\).

We let \(L_h\) denote the leaves of \(T\). Also, we let \(\mu_i\) denote the uniform distribution over the \(k\)-
colourings of \(T\) conditional that \(r\) is assigned colour \(i\). Reconstructibility is defined as follows:
**Definition 1** For any \(i, j \in [k]\) let \(\|\mu_i - \mu_j\|_{L_h}\) denote the total variation distance of the projections of \(\mu_i\) and \(\mu_j\) on \(L_h\). We say that a model is reconstructible on a tree \(T\) if there exists \(i, j \in [k]\) for which

\[
\lim_{h \to \infty} \|\mu_i - \mu_j\|_{L_h} > 0.
\]

When the above limit is zero for every \(i, j\), then we say that the model has non-reconstruction.

(Non)Reconstructibility expresses how information decays along the tree. As a matter of fact, non-reconstruction is equivalent to the mutual information between the colouring of root \(r\) and that of \(L_h\) is going to zero as \(h\) grows (see [11]).

When \(T\) is infinite \((h \to \infty)\) non-reconstruction is equivalent to the Gibbs measure being extremal. That is, the distribution of the colouring at the root \(r\) cannot be expressed as a convex combination of boundary conditions at the leaves of \(T\) (see [8]). For finite \(h\), non-reconstruction implies that typical colourings of the leaves have a vanishing bias on the distribution of the colouring of \(r\).

An early result about reconstruction/non-reconstruction problems on trees is the so-called “Kesten-Stigum bound” in [9]. The authors there show that reconstruction holds when \(\lambda^2d > 1\), where \(\lambda\) is the second largest eigenvalue of \(M\) in absolute value. This bound is sharp for a lot of models, e.g., Ising model (see [6]). In [11] it was shown that there are models where the Kesten-Stigum bound is not sharp, e.g., the binary models where \(M\) is sufficiently asymmetric or the ferromagnetic \(q\)-state Potts model with \(q\) large. As far as the \(k\)-colouring model is regarded the reconstruction threshold is known quite precisely. From [13, 14, 15, 4] we derive the following theorem:

**Theorem 1** For fixed \(\epsilon > 0\) and sufficiently large \(d\), the following is true for the \(k\)-colouring model on a \(d\)-ary tree \(T\):

- If \(k \geq (1 + \epsilon)d/\ln d\), then the model is non-reconstructible.
- If \(k \leq (1 - \epsilon)d/\ln d\), then the model is reconstructible.

**Remark 1** The reconstruction bound is from [13, 14] and is based on analyzing a simple reconstruction algorithm. As a matter of fact the reconstruction condition there is more precise than that in Theorem [1] i.e. it should hold \(d > k[\ln k + \ln \ln k + 1 + o(1)]\).

**Remark 2** The non-reconstruction bound is from [15, 4]. The result in [15] provides a very precise condition for non-reconstruction, i.e. \(d \leq k[\ln k + \ln \ln k + 1 - \ln 2 - o(1)]\). In [4] the reader can find further interesting results about the problem.

Using the Coupling Lemma (see [3]) with Theorem [1] we get the following corollary.

**Corollary 1** Consider a \(d\)-ary tree \(T\) of height \(h\). Assume that two broadcasting processes on \(T\) assign the root different colours. For \(\epsilon\) and \(d\) as in Theorem [7] and \(k = (1 + \epsilon)d/\ln d\) there is a coupling for the two processes such that the following holds: The probability that there are leaves with different colour assignments in the two processes reduces as \(h\) increases.

Somehow there is a rule which specifies how someone should correspond the choices of colourings in the first broadcasting process to the choices of the other one such that the probability of having the leaves taking different colours reduces with their distance from the root. Unfortunately, neither of [15, 4] casts a light on this question. It turns out that devising such a coupling is far from trivial for any \(k \leq d\).

Here we address the problem of constructing a coupling as specified in Corollary [1] based on local combinatorial rules. By local we mean that once the first process decides on the colouring of a fairly small set of vertices, then we should be able to know how the other process should colour the same set of vertices. In particular, we provide the following result:

**Main Result:** We construct a coupling of the processes in Corollary [1]. The coupling is combinatorial, local and implies non-reconstruction for any \(k \geq (3 + \epsilon)d/\ln d\), where \(\epsilon > 0\) is fixed and \(d\) is sufficiently large.
Notation. We use small letters of the greek alphabet for the colourings of $T$, e.g. $\sigma, \tau$. The capital letters denote random variables which take values over the colourings e.g. $X, Y$. We let $\sigma_v$ denote the colour assignment of the vertex $v$ under the colouring $\sigma$. Similarly, the random variable $X(v)$ is equal to the colour assignment that $X$ specifies for the vertex $v$. For an integer $k > 0$ we let $[k] = \{1, \ldots, k\}$.

1.2 Further Motivation - Non Reconstruction in Random Graphs & Sampling

It is believed that the non-reconstruction/reconstruction transition determines the dynamic phase transition for the $k$-colourings of the random graph $G(n, m)$. Where $G(n, m)$ denotes the random graph on $n$ vertices and $m$ edges with $d$ denoting the expected degree, i.e. $d = 2m/n$.

The dynamic phase transition is related to the geometry of $k$-colourings of $G(n, m)$ and it was predicted by statistical physicists in [10], based on ingenious but mathematically non-rigorous arguments. Let us be more specific. For typical instances of $G(n, m)$, the chromatic number $\chi$ is well known to be $\chi \sim \frac{d}{2 \ln d}$ (see [2]). The 1-step Replica Symmetry breaking hypothesis [10] considers the space of $k$-colourings of $G(n, m)$ as $k$ varies from large to small and predicted the following phenomenon: For $k = (1 + \epsilon)d / \ln d$ (i.e. greater than $2\chi$) all but a vanishing fraction of $k$-colourings form a giant connected ball. That is, starting from any colouring we can traverse the whole set of colourings in the ball by moving in steps. Each steps involves changing only a very small -constant- number of colour assignments. However, for $k = (1 - \epsilon)d / \ln d$ (e.g. smaller than $2\chi$) the set of $k$-colouring shatters into exponentially many connected balls with each ball containing an exponentially small fraction of all $k$-colourings. Any two colourings in different balls are separated with linear hamming distance (for rigorous result about shattering see in [11]).

It is believed that we can approximately randomly colour $G(n, m)$ efficiently for $k$ down to the dynamic phase transition threshold, i.e. $k = (1 + \epsilon)d / \ln d$. Recently, the author of this paper in [7] suggested a new algorithm for sampling colourings of $G(n, m)$ with constant expected degree. Interestingly enough the accuracy of the algorithm depends directly on non-reconstruction conditions. The idea there is that we first remove edges of $G(n, m)$ until it becomes so simple that we can take a random colouring in polynomial time. Then, we rebuild the graph by adding the deleted edges one by one while at the same time we update the colouring. I.e. whenever a new edge is inserted some vertices’ colouring is updated so that the colouring of the resulting graph remains random. This algorithm requires at least $(2 + \epsilon)d$ colours. However, since its accuracy depends on non-reconstruction conditions it is reasonable to expect that we can have an improvement by using even less colours. The algorithm does not exploit fully its dependency on non-reconstruction due to its colouring update rule. A new, improved, update rule is needed. Such an improvement could possibly reduce the minimum number of colours that the algorithm requires down to $(1 + \epsilon)d / \ln d$. Very good candidates for improved updating rules are couplings as the one we present here.

1.3 A basic description of the coupling.

Consider two broadcasting processes, the first one $k$-colours $T$ as $X$ and the second as $Y$. Assume that the root $r$ of $T$ is coloured such that $X(r) = c$ and $Y(r) = q$ while $c \neq q$, for some $c, q \in [k]$.

Consider, first, the following recursive naive coupling of the two processes. Start from the root $r$ down to the leaves. For each coloured vertex $u \in T$ we colour its descendant $w$ by using maximal coupling. I.e. minimize the probability of $w$ to be disagreeing. If $X(w) \neq Y(u)$, then we have $X(w) \neq Y(w)$ only if $X(w) = Y(w)$ and $Y(w) = X(u)$. On the other hand, if $X(w) = Y(u)$ then we always have $X(w) = Y(w)$. It is not hard to see that $Pr[X(w) \neq Y(w)] = 1/k$.

Clearly, when $k \leq d$, we expect that the naive coupling generates an ever increasing number of disagreeing vertices as it moves from the root down to the leaves. As a matter of fact the number of disagreeing vertices at each level grows as a supercritical branching process, i.e. the probability of having a disagreement at the leaves is strictly positive, regardless of their distance from the root.
Before introducing our coupling, consider the following notions. Let $N_i$ denote the 2 level subtree of $T$ rooted at the $i$-th child of the root $r$. In the same setting as that in the naive coupling, the colouring $X(N_i)$ is “bad” if $X(i) = q$ and $i$ has a child $j$ such that $X(j) = c$. Similarly, $Y(N_i)$ is bad if $Y(i) = c$ and $i$ has a child $j'$ such that $Y(j') = q$.

In the naive coupling, $X(N_i)$ is bad if and only if $Y(N_i)$ is bad. For such a pair the identity coupling is precluded and the creation of disagreements is inevitable. That is, the naive coupling handles the appearance of bad lists by coupling them together. Clearly this is not desirable. Especially, for $k \leq d$ the number of bad colourings $X(N_i), Y(N_i)$ are “too many”. This causes the ever increasing number of disagreements of the naive coupling.

The coupling we propose here uses the following, not so obvious, observation to handle the bad lists: Consider $X(N_j)$ conditional that (A) it is a bad and (B) there is at least one colour that is not used by $X(N_j)$. Then, it is highly likely that there is a child of $r$, e.g. the vertex $s$, where $Y(N_s)$ satisfies the following two conditions: (A’) The colour $Y(s)$ is not assigned to any child of $j$ under the colouring $X$ and (B’) the colour $c$ is assigned to at least one child of $s$ under the colouring $Y$. For such $X(N_j)$ and $Y(N_s)$, we can show the following: The colour assignment of the children of $j$ in the first process is identically distributed to that of the children of $s$ in the second process.

Based on the above observation, the target now is to couple the colourings $X(N_i)s$ and $Y(N_i)s$ such that if $X(N_i)$ satisfies the conditions (A) and (B), then $Y(N_i)$ satisfies (A’) and (B’) and vice versa. Then, clearly, we can couple the colouring of children of the vertex $i$ identically. Let us remark that it is not completely trivial to “aline” these two different kinds of colouring in the coupling.

Working as described in the previous paragraph, the number of disagreements drops dramatically, compared to the naive coupling. As a matter of fact the number of disagreeing vertices grows as a sub-critical branching process, i.e. the probability of having disagreement at the leaves drops exponentially with their distance from the root.

Remark 3 The update rule in the sampling algorithm in [7], somehow, is based on what we call here naive coupling.

2 Coupling

In this section we present the coupling in full detail. We let $\mu(\cdot)$ denote the uniform distribution over the $k$-colourings of $T$. We consider two broadcasting processes such that the first one assigns colour $c$ to the root while the second one assigns colour $q$. To avoid trivialities assume that $c \neq q$. Finally, we let $X, Y$ be the colourings that the two processes assign to $T$, respectively. We proceed by introducing some useful concepts.

2.1 Preliminaries

Let $c_1, c_2, c_3 \in [k]$ and, for $j = 1, 2, 3$, let $L_j$ be a $d$-dimensional list which contains colours in $[k] \setminus \{c_j\}$. For these three lists we have the following:

bad: The pair $(L_1, L_2)$ is called bad if and only if $c_1 \neq c_2$ and $c_2 \in L_1$ while $c_1 \in L_2$.

rescuable: A bad pair $(L_1, L_2)$ is called rescuable if there is at least one colour in $[k] \setminus \{c_1, c_2\}$ that does not appear in both $L_1$ and $L_2$.

special: Given that the pair $(L_1, L_2)$ is rescuable, the list $L_3$ is called “special w.r.t. $L_1$” if the following holds:

1I.e. the analogous conditions should hold for bad $Y(N_j)$.\[5\]
1. $c_3$ does not belong to any of $L_1, L_2, L_3$ while $c_3 \neq c_1$.
2. One of the following two holds:
   (a) $L_3$ contains $c_1$ but it does not contain $c_2$
   (b) $L_3$ contains $c_2$ but it does not contain $c_1$.

**good:** Given that the pair $(L_1, L_2)$ is rescuable, a list $L_3$ is called “good w.r.t. $L_1$” if it is special w.r.t.
$L_1$ and the condition 2(b) holds.

**Definition 2** For $c \in [k]$, we let $\lambda_c$ denote the uniform distribution over the $d$ dimensional lists of
colours which do not contain the color $c$.

**Lemma 1** Let $c, q, s \in [k]$ such that they are different with each other. Let $L_1, L_2$ and $L_3$ be distributed as in $\lambda_c$, $\lambda_q$, and $\lambda_s$, respectively. Conditional that the pair $(L_1, L_2)$ is rescuable and $L_3$ is good w.r.t.
$L_1$, then $L_1$ and $L_3$ are identically distributed.

For the proof of Lemma 1 see in Section 5.1

### 2.2 The coupling

The coupling works inductively. At each step it considers two consecutive levels of $T$. Here, we describe
how does it work for the two levels below the root. The coupling for the rest of the tree will be immediate.

We need to use the some auxiliary random variables defined w.r.t. $X, Y$. We let $L_X, L_Y \in [k]^d$ be
ordered lists which contain the colours that are assigned to the children of the root $r$ under the colour
assignments $X$ and $Y$, respectively. Additionally, for every $i \in [d]$ we let $L_X^{i}$ (and $L_Y^{i}$) be the
corresponding lists of the colour assignments of the children of the vertex that is going to be assigned
the colour $L_X(i)$ (and $L_Y(i)$), e.g. see Figure 1.

Essentially, the list $L_X$ specifies the colours that are assigned to the children of the root by $X$ but
without providing exactly the information about which vertex takes which colour. The same holds for
the other lists $L_X^{i}$, $L_Y$ and $L_Y^{i}$, for every $i \in [d]$. We couple the colour assignments of $X, Y$ on the
vertices at levels 1 and 2 of $T$ by using these lists. I.e. we couple the entries of the lists, first, and then
we obtain the assignments of $X, Y$. There we need to use the following

**Remark 4** Given $L_X$, the colour assignments of $X$ to the children of $r$ can be obtained as follows:
Take $\pi$, a random permutation of the elements in $\{1, \ldots, d\}$. Then, for the $i$-th child of $r$ set $X(i) = L_X(\pi(i))$. Given $L_X$'s we obtain the colourings of the grandchildren of $r$ in an analogous way.

We use the notions of the “bad” or the “rescuable” pair for every $(L_X^{i}, L_Y^{i})$ such that $L_X(i) = q$
and $L_Y(i) = c$, with $i \in [d]$. That is, we consider a pair $(L_X^{i}, L_Y^{i})$ to be bad (or not) only if $L_X(i) = q$
and $L_Y(i) = c$. For $(L_X^{i}, L_Y^{i})$ “badness” is determined w.r.t the colours $c, q$. E.g. see Figure 2.
There $L_X^{i}$

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2There is a bijection between the elements $L_X$ and the colour assignments of $X$ at the children of the root $r$. The same holds for $Y$ and $L_Y$. 

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Figures 1 and 2: The lists $L_Y$ and $L_Y^{i}$. A “Bad” pair of lists.
does not contain $q$ due to the fact that $L_X(i) = q$ but it contains $c$, i.e. $L_X^j(j) = c$. On the other hand, $L_Y^j$ does not contain $c$ due to the fact that $L_Y(i) = c$ but it contains $q$, i.e. $L_Y^j(j') = q$.

The coupling works in three phases. In the first two it focuses on the list of colours. It considers $X$ and $Y$ only in the last phase.

**In Phase 1** only a certain part of information about $L_X$, $L_Y$, $L_X^j$ and $L_Y^j$, for $j \in [d]$, is revealed. That is, we reveal “bad” and “rescuable” pairs as well as which lists are “special”. Observe that the lists $L_X$, $L_Y$ are distributed as in $\lambda_c$ and $\lambda_q$, respectively. Also, given that $L_X(i) = c'$ (or $L_Y(i) = c'$) for some $c' \in [k]$, then $L_X^j$ (or $L_Y^j$) is distributed as in $\lambda_{c'}$. Phase 1 is as follows.

**Phase 1:**

1. **Reveal only for which $i \in [d]$ we have $L_X(i), L_Y(i) \in \{c, q\}$. Couple the choices of $L_X$ and $L_Y$ such that $L_X(i) = q$ if and only if $L_Y(i) = c$.**

2. **For each $i$ such that $L_X(i) = q$ and $L_Y(i) = c$ reveal whether $(L_X^i, L_Y^i)$ is “bad” or not.**

3. **If $(L_X^i, L_Y^i)$ is bad reveal whether it is “rescuable”. The coupling is so that the colours in $[k] \setminus \{c, q\}$ are chosen independently from the two lists.**

4. **If the number of rescuable pairs is $l$, partition the set of non-bad pairs $(L_X^i, L_Y^i)$ into $l$ parts which are as equal sized as possible. Each rescuable pair is associated to exactly one part in the partition.**

5. **For each non-bad pair $(L_X^i, L_Y^i)$ that is associated to the rescuable pair $(L_X^i, L_Y^i)$ do the following: Reveal if $(L_X^i, L_Y^i)$ consists of special lists, i.e. $L_X^i$ and $L_Y^i$ are special w.r.t $L_Y^i$ and $L_X^i$, respectively. We use coupling such that either both lists in the pair are special or both are not.**

We should recognize the bad pairs as the potential sources of disagreements in the coupling. Our attempt is to eliminate the disagreements caused by the rescuable pairs only\footnote{Here, we only ask if $L_X^i$ and $L_Y^i$ contain $c$ and $q$, respectively. It is trivial that $Pr[c \in L_X] = Pr[q \in L_Y]$.}. This eliminations uses Lemma\footnote{For the values of $k$ we consider it is highly unlikely that a bad pair is non-rescuable.} as follows: Consider the rescuable pair $(L_X^i, L_Y^i)$. We let $A_i$ be the set of indices such that if $j \in A_i$, then the pair $(L_X^j, L_Y^j)$ is associated to the bad pair $(L_X^i, L_Y^i)$ in step 4. Assume that the pair $(L_X^i, L_Y^i)$ is “$i$-good”, i.e. $L_X^i$ is good w.r.t $L_Y^i$ and $L_Y^i$ is good w.r.t $L_X^i$. Then, Lemma\footnote{Here, we only ask if $L_X^i$ and $L_Y^i$ contain $c$ and $q$, respectively.} implies that $L_X^i$ and $L_Y^i$ are identical distributed. The same holds for $L_Y^i$ and $L_X^i$. In this case, when we reveal all the information of the lists (which will be done in a subsequent phase) we couple $L_X^i$ with $L_Y^i$ and $L_Y^i$ and $L_X^i$. Clearly this eliminates all the potential disagreements generated by the rescuable pair $(L_X^i, L_Y^i)$.

**Remark 5** *For technical reasons which will become apparent soon, we do not reveal which pairs in $A_i$ are $i$-good. We only reveal if the pair $(L_X^j, L_Y^j)$, for $j \in A_i$, is $i$-special, i.e. $L_X^j$ and $L_Y^j$ are special w.r.t $L_Y^j$ and $L_X^j$, respectively.*

In **Phase 2**, we construct a mapping $f : [d] \to [d]$ with the following property: If $f(i) = j$, then when we reveal the full information about the lists we couple maximally $L_X(i)$ with $L_Y(j)$ and $L_X^i$ with $L_Y^j$. The mapping $f$ is constructed so as to minimize the number of disagreements between the lists $L_X^i$ and $L_Y^j$. In particular we have the following situation in mind. It is desirable that for each rescuable pair $(L_X^i, L_Y^j)$ to find an $i$-good pair among the $i$-special pairs in $A_i$. Once we have such a pair, e.g. $(L_X^j, L_Y^j)$ for some $j \in A_i$, we set $f(i) = j$ and $f(j) = i$.

The next step of the coupling reveals which $i$-special pairs $(L_X^j, L_Y^j)$ with $j \in A_i$ are also $i$-good. In order to reveal whether an $i$-special pair $(L_X^j, L_Y^j)$, for $j \in A_i$, is $i$-good we should couple the lists such that $L_X^j \neq L_Y^j$. This pair is $i$-good with probability $1/2$. With the remaining probability it is not and the lists in the pair $(L_X^j, L_Y^j)$ cannot be coupled identically.
Remark 6 For the $i$-special pair $(L^i_X, L^i_Y)$, we reveal whether “$c \in L^i_X$ and $q \notin L^j_X$” or “$c \notin L^i_X$ and $q \in L^j_X$.” E.g., assume that we have “$c \in L^i_X$ and $q \notin L^j_X$”, then the coupling should decide the opposite for $L^j_Y$, i.e. “$c \notin L^j_Y$ and $q \in L^i_Y$”. Revealing the lists in such a way it always holds $L^j_X \neq L^j_Y$.

Of course there is always the option of coupling an $i$-special pair identically. But then it is impossible to generate an $i$-good pair. The $i$-special pair $(L^i_X, L^j_Y)$ which is coupled so as to generate an $i$-good pair but it failed to do so is called $i$-fail (see example in Figure). The upper pair is $i$-fail). It is straightforward, now, that as we search for an $i$-good pair it is possible that we generate extra potential sources of disagreements. To this end we use the following lemma.

**Lemma 2** Assume that the $i$-special pairs $(L^i_X, L^i_Y)$ and $(L^s_X, L^s_Y)$ with $s, t \in A_i$ are revealed and $(L^i_X, L^i_Y)$ is $i$-good while $(L^s_X, L^s_Y)$ is $i$-fail. Then, $L^i_X$ is identically distributed to $L^s_X$ and $L^i_Y$ is identically distributed to $L^s_Y$.

For a proof of Lemma 2, see in Section 5.2. Figure 3 gives a schematic representation of what is stated in Lemma 2. The arrows show the pairs of lists that are identically distributed.

Lemma 2 suggests that $i$-good pairs can be used to eliminate the potential disagreements generated by $i$-fails. Thus, in the case we generate $i$-fails we (try) to reveal some extra $i$-good pairs. In particular, we work as follows:

**Phase 2.**

For each rescuable pair $(L^i_X, L^i_Y)$ do the following:

1. Reveal, sequentially, whether each $i$-special pair in $A_i$ is $i$-good or $i$-fail until either of the following two happens:
   - the number of $i$-good pairs exceeds the number of $i$-fails by one,
   - there are no other $i$-special pairs in $A_i$ to reveal.

2. The remaining unrevealed $i$-special pairs, if any, are coupled by using identity coupling.

3. If there is an $i$-good pair $(L^i_x, L^i_y)$ “match” it with the rescuable pair $(L^j_x, L^j_y)$, i.e. set $f(i) = j$ and $f(j) = i$.

4. Each of the remaining $i$-good pairs $(L^j_x, L^j_y)$ is matched to one $i$-fail pair $(L^s_x, L^s_y)$, i.e. set $f(j) = s$ and $f(s) = j$. No $i$-fail is matched to more than one $i$-good pairs and vice versa.

5. For each $j \in A_i$ such that $(L^j_x, L^j_y)$ is not matched to some other pair, match it to itself, i.e. set $f(j) = j$.

Ideally, Phase 2 generates a number of $i$-good pairs which exceed the number of $i$-fails by one. If this is the case, $f$ specifies pairs whose coupling generates no disagreement. That is, the rescuable pair $(L^i_x, L^j_y)$ and the $i$-fails are going to be coupled with an $i$-good pair each. Then, due to Lemma 1 and Lemma 2 no disagreement is going to be generated. Of course, it is possible that the number of the $i$-good pairs is not sufficiently large. Then, we end up with some $i$-fails which cannot be matched with any $i$-good pair (possibly with the rescuable pair $(L^i_x, L^i_y)$ as well). These pairs are matched to themselves and some disagreements are going to appear in the full revelation. However, we show that the expected number of disagreements vanishes as long as $k \geq (3 + \epsilon)d/\ln d$.

---

5This implies that the mapping $f$ is a bijection.
We now, proceed with **Phase 3**. There we reveal the full information about the lists by coupling the pairs as specified by \( f \). Given the full information for the lists we reveal the assignments of \( X, Y \) for the (grand)children of \( r \). Note that if \( f(i) = j \), then the child of \( r \) that gets \( L_X(i) \) under \( X \) will get \( L_Y(j) \) under \( Y \). Additionally, the grand child of \( r \) that is assigned the colour \( L_X(i) \) under \( X \) is going to take the colour \( L_Y(j) \) under \( Y \).

**Phase 3:**

1. For every \( s, t \) such that \( f(s) = t \), couple optimally \( L_X(s) \) with \( L_Y(t) \) as well as \( L_X(s) \) with \( L_Y(t) \).
2. Reveal which element of the list \( L_X \) is assigned to which child of \( r \) and which element of \( L_X \) goes to which grandchild of \( r \), as Remark 4 specifies.
3. Assuming that \( v \), child of \( r \), is such that \( X(v) \) is set \( L_X(s) \), then we set \( Y(v) \) equal to \( L_Y(t) \), where \( t = f(s) \). Also, for \( u \), child of \( v \), such that \( X(u) \) set \( L_X(j) \) we set \( Y(u) \) equal to \( L_Y(j) \).

Applying the coupling inductively, i.e. for the grandchildren of the root and so on, at the end we get the full colourings \( X \) and \( Y \). A very basic result is the following theorem.

**Theorem 2** For \( c, q \in [k] \), assume that in the above coupling it holds \( X(r) = c \) and \( Y(r) = q \), where \( r \) is the root vertex of \( T \). Then at the end of the coupling, \( X \) and \( Y \) are distributed as in \( \mu(\cdot | X(r) = c) \) and \( \mu(\cdot | Y(r) = q) \), respectively.

**Proof:** Theorem follows by noting that for every list, conditional on the information that is already known to us, we reveal some information by using the appropriate distribution.

Furthermore, from the description of the coupling the following corollary is direct.

**Corollary 2** The disagreements in the coupling have three different sources:

1. Pairs of bad lists which are not rescuable.
2. Pairs of rescuable lists for which it was impossible to find a good pair.
3. Pairs of \( i \)-fail lists, for some \( i \), which are not matched to an \( i \)-good pair.
**Proposition 1** Consider the above coupling between $X$ and $Y$ and let $W_l$ be the number of vertices $u$ at level $l$ such that $X(u) \neq Y(u)$. For fixed $\epsilon > 0$, sufficiently large $d$, $k = (1 + \epsilon) \frac{d}{\ln d}$ and every even integer $l > 0$ it holds that

$$E[W_l] \leq \left( d^{-0.1 \frac{1}{d+1}} \right)^{l/2}. $$

Using Proposition 1, it is direct to see that our combinatorial construction implies the following theorem.

**Theorem 3** For fixed $\epsilon > 0$ and sufficiently large $d$, the following is true for the $k$-colouring model on a $d$-ary tree $T$: If $k = (3 + \epsilon) d/ \ln d$, then the model is non-reconstructible.

**Proof:** Take $k = (3 + \epsilon) d/ \ln d$. Let $X$ and $Y$ be distributed as in $\mu(\cdot|X(r) = c)$ and $\mu(\cdot|Y(r) = q)$, respectively, while their joint distribution is specified by the coupling we presented. Let the set $L_h$ contain all the vertices of $T$ at level $h$. We take $h$ to be even. By Coupling Lemma we have

$$||\mu(\cdot|X(r) = c) - \mu(\cdot|Y(r) = q)||_{L_h} \leq Pr[X(L_h) \neq Y(L_h)].$$

(1)

Let $W_h$ be the number of vertices $u \in L_h$ such that $X(u) \neq Y(u)$. It holds that

$$Pr[X(L_h) \neq Y(L_h)] = Pr[W_h > 0] \leq E[W_h] \quad \text{[by Markov’s inequality]}$$

$$\leq \left( d^{-0.1 \frac{1}{d+1}} \right)^{h/2}. \quad \text{[from Proposition 1]}$$

(2)

The theorem follows by combining (1) and (2). □

### 3 Proof of Proposition 1

Consider in the coupling two vertices $v, w \in T$ at the same level $l$, where $l$ is even. Consider, also, the colourings $X(v), Y(v)$ and $X(w)$ and $Y(w)$ while w.l.o.g assume that $X(v) \neq Y(v)$ and $X(w) \neq Y(w)$. Clearly, whether the descendants of $v$ disagree or not does not dependent on what happens at the descendants of $w$ and vice versa. This observation yields the following: In the coupling, for each vertex $v \in T$, let $D_v$ be the number of disagreements two levels below $v$. It holds that

$$E[W_l | W_{l-2}] = E[D_v] \cdot W_{l-2} \quad \text{for even } l > 0. $$

Taking the average from both sides and working out the recursion we get that

$$E[W_l] = (E[D_v])^{1/2}. $$

The proposition will follow by bounding appropriately $E[D_v]$. To this end, we need to bound the number of disagreements that are generated by each of the three sources of disagreement specified in Corollary 2. It, always, holds that $D_v \leq d^2$, since $T$ is a $d$-ary tree.

Consider the following quantities related to the vertex $v$: Let $\beta_v$ denote the number of bad pairs of lists two levels below $v$. Let $\delta_v$ be the probability for a bad pair to be resucuable, for a given number of colours $k$. Finally, given some resucuable pair $(L'_X, L'_Y)$ let $h'_v$ be the number of $j$-special lists in the associated partition. Let the event $A_v$ denote that at least one of the following three occurs

1. $\beta_v \geq 100 \ln d$.
2. There is at least one bad pair which is not in a resucuable pair.
3. There is a resucuable pair $(L'_X, L'_Y)$ that is associated to a partition with less than $d^{\frac{4}{d+1} - 2}$ $j$-special lists.
It is direct to get that
\[ E[D_v] \leq d^2 \Pr[A] + E[D_v|A^c], \tag{3} \]
where we use the rather crude overestimate that conditional on the event \( A \) occurs all the \( d^2 \) descendants of \( v \) are disagreeing. It suffices to bound appropriately \( \Pr[A] \) and \( E[D_v|A^c] \). To this end, we use the following propositions.

**Proposition 2** For \( k = (1 + \epsilon)d/\ln d \) and for sufficiently large \( d \), we have that
\[ E[D_v|A^c] \leq d^{-0.102 \frac{-d^2}{1+\epsilon}}. \]

**Proposition 3** For \( k = (1 + \epsilon)d/\ln d \) and for sufficiently large \( d \), we have that
\[ \Pr[A] \leq 5d^{-250}. \]

Plugging into (3) the bounds from Proposition 2 Proposition 3 we get that
\[ E[D_v] \leq d^{-0.1 \frac{-d^2}{1+\epsilon}}. \]

The proposition follows.

### 3.1 Proof of Proposition 2

Since we have conditioned on \( A^c \), we have that A) \( \beta_v \), the number of bad lists, is less than 100 \( \ln d \), B) all the bad lists are rescuable and C) every rescuable pair \((L_X^i, L_Y^i)\) is associated to a partition which contains at least \( d^{\frac{1}{3}+\epsilon + \frac{1}{1+\epsilon}} \) \( i \)-special lists. From (A) and (B), we deduce that the number of rescuable pairs is equal to \( \beta_v \).

In this setting, consider the rescuable pair \((L_X^i, L_Y^i)\). We remind the reader that during the second phase of the coupling, in the partition associated to \((L_X^i, L_Y^i)\), we reveal which of the \( i \)-special pairs are \( i \)-good or not, i.e. during the steps 1 and 2. During these revelations it is possible that we introduce pairs which are \( i \)-fails which my end up being coupled together (due to lack of \( i \)-good pairs). Let \( \Delta_i \) be the indices of these \( i \)-fails.

We remind the reader that we denote with \( A_i \) the set of indices of the pairs that are associated to the rescuable pair \((L_X^i, L_Y^i)\).

Consider \((L_X^t, L_Y^t)\) for some \( t \in \Delta_i \). We can couple \( L_X(t), L_Y(t) \) such that \( L_X(t) = L_Y(t) \). Also, it holds that \( c \in L_X^t \) and \( q \notin L_Y^t \) while \( q \in L_Y^t \) and \( c \notin L_X^t \). Given that \( L_X(t) = L_Y(t) \), all the colours in \([k] \setminus \{c, q, L_X(t)\} \) are symmetric for both \( L_X^t \) and \( L_Y^t \). Clearly, we can couple \( L_X^t \) and \( L_Y^t \), such that if \( L_X^t(s) = c \) then \( L_Y^t(s) = q \) while if \( L_X^t(s) \neq c \), then \( L_X^t(s) = L_Y^t(s) \) for any \( s \in [d] \).

Let \( Z_t \) be the number of disagreements that are generated by the coupling of the pair \((L_X^t, L_Y^t)\) with \( t \in \Delta_i \). Also, let \( Q_i = \sum_{t \in \Delta_i} Z_t \). It holds that
\[ E[Q_i|A^c] = E[|\Delta_i||A^c| \cdot E[Z_j|A^c]. \tag{4} \]

Apart from the pairs in \( \Delta_i \), it is possible that the lists in the rescuable pair \((L_X^i, L_Y^i)\) are coupled together. This happens when there is no \( i \)-good pair among the \( i \)-specials. The probability of having no \( i \)-good pairs at most \( 2^{-d^{\frac{1}{3}+\epsilon + \frac{1}{1+\epsilon}}} \), as every special pair is \( i \)-good with probability \( 1/2 \) and we have at least \( d^{\frac{1}{3}+\epsilon + \frac{1}{1+\epsilon}} \) \( i \)-special pairs. Let \( W_i \) be the number of disagreements that are generated by the rescuable pair. It holds that
\[ E[W_i|A^c] \leq d^{\Pr[\text{No } i \text{-good pair in } A_i|A^c]} \leq 2^{-d^{\frac{3}{4}+\epsilon + \frac{1}{1+\epsilon}}}. \tag{5} \]
Conditional on $\mathcal{A}^c$, $D_v$ is the sum of disagreements generated by the rescuable pairs and the $i$-fails, for various $i$. By the linearity of expectation we get that

$$E[D_v|\mathcal{A}^c] \leq (100 \ln d) (E[W_1|\mathcal{A}^c] + E[Q_0|\mathcal{A}^c])$$

[as $\mathcal{A}^c$ assumes that $\beta_v < 100 \ln d$]

$$\leq 2d^{1/\lceil 2d \rceil} + (100 \ln d) \cdot E[|\Delta_i| |\mathcal{A}^c] \cdot E[Z_j|\mathcal{A}^c].$$  [from (5) and (4)]  (6)

The proposition will follow by bounding appropriately $E[|\Delta_i| |\mathcal{A}^c]$ and $E[Z_j|\mathcal{A}^c]$.

As far as $E[Z_j|\mathcal{A}^c]$ is concerned we have the following: For any $t \in \Delta_i$, the number of disagreements is exactly the number of occurrences of $c$ in $L_X^i$. Conditional on $\mathcal{A}^c$, the number of entries in $L_X^i$ with colour $c$ is binomially distributed with parameters $d, 1/(k-1)$, conditional that it is positive. It follows that

$$E[Z_j|\mathcal{A}^c] = \sum_{s=0}^{d} s \cdot Pr[c \text{ appears } s \text{ times in } L_X^i | c \text{ appears at least once in } L_X^i]$$

$$= \left(1 - \left(1 - \frac{1}{k-1}\right)^d\right)^{-1} \sum_{s=1}^{d} s \cdot \binom{d}{s} \left(\frac{1}{k-1}\right)^s \left(1 - \frac{1}{k-1}\right)^{d-s}$$

$$\leq \frac{2d}{k-1} \quad \text{[since } 1 - \left(1 - \frac{1}{k-1}\right)^d > 1/2 \text{]}$$

$$\leq 2 \ln d. \quad \text{[since } k = (1 + \epsilon)d/\ln d \text{]}$$  (7)

As far as $E[|\Delta_i| |\mathcal{A}^c]$ is concerned, we work as follows: Let $S_i$ be the set of indices of all the $i$-special pairs in $A_i$ as well as of the rescuable pair $(L_X^i, L_Y^i)$. W.l.o.g. assume that $i = 1$ while the indices of the $i$-special pairs in $S_i$ are from 2 to $|S_i|$. Let the 0-1 matrix $S = |S_i| \times 2$ be defined as follows: $S(1, t) = 1$, if $c \in L_X^i$ and $q \notin L_Y^i$, otherwise, i.e. $c \notin L_X^i$ and $q \in L_Y^i$, $S(1, t) = 0$. Similarly, $S(2, t) = 1$ if $c \notin L_X^i$ and $q \in L_Y^i$, otherwise $S(2, t) = 0$. If the i-special pair $(L_X^i, L_Y^i)$ is i-good, the it holds that $(S(1, t), S(2, t)) = (0, 1)$, otherwise, i.e. the pair is i-fail, then $(S(1, t), S(2, t)) = (1, 0)$.

**Remark 7** The second phase of the coupling specifies how $S(1, j)$ and $S(2, j)$ are correlated with each other, i.e. the following holds: if $\sum_{j=1}^{t-1} (S(1, j) - S(2, j)) > 0$, then $S(1, i)$ and $S(2, i)$ get complementary values. Otherwise, i.e. if $\sum_{j=1}^{t-1} (S(1, j) - S(2, j)) = 0$, they are identical.

Since we have assumed that the values in $(S(1, 1), (2, 1))$ are specified by the rescuable pair $(L_X^i, L_Y^i)$, by definition, it holds that $(S(1, 1), (2, 1)) = (1, 0)$. Furthermore, for each $t = 2 \ldots |S_i|$ and as long as $\sum_{j=1}^{t-1} (S(1, j) - S(2, j)) > 0$ we have

$$(S(1, t), S(2, t)) = \begin{cases} (1, 0) & \text{with probability } 1/2 \\ (0, 1) & \text{with probability } 1/2 \end{cases}.$$  

For the matrix $S$ we have the following lemma.

**Lemma 3** Let $N$ be the number of columns of the matrix $S$. It holds that

$$|\Delta_i| \leq \sum_{t=1}^{N} S(1, t) - S(2, t).$$

**Proof:** First notice that $S(1, 1) - S(2, 1) = 1$. The coupling during the second phase assigns complementary values to each pair $S(1, t)$, $S(2, t)$ as long as $R_t = \sum_{t=1}^{t-1} |S(1, t) - S(2, t)| > 0$. Once $R_t = 0$ it sets $S(1, t) = S(2, t)$, i.e. $R_t$ remains zero for the rest values of $t$.  

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Let $T$ be the maximum $t$ such that $S(1, t) \neq S(2, t)$. It suffices to show that

$$|\Delta_i| \leq \sum_{t=1}^{T} S(1, t) - S(2, t).$$

For $t < T$, the fact that $S(1, t) = 1$ (and consequently $S(2, t) = 0$) suggests that we have revealed an $i$-fail. On the other hand, if $S(1, t) = 0$ (and consequently $S(2, t) = 1$), then it suggests that it has been revealed an $i$-good pair. This observation implies that the sum $\sum_{t=1}^{T} S(1, t)$ is equal to the number of $i$-fails we have revealed plus one, while $\sum_{t=1}^{T} S(2, t)$ is equal to the number of $i$-good pairs.

Since we can match an $i$-fail with an $i$-good pair to avoid generating disagreements, the number of pairs which do not admit identical coupling, i.e. the $i$-fail and possibly the rescuable pair, is equal to

$$\sum_{t=1}^{T} S(1, t) - S(2, t) = N \sum_{t=1}^{T} S(1, t) - S(2, t).$$

The lemma follows.

**Proposition 4** Let $N$ be the number of columns of $S$. Then for sufficiently large $N$ it holds that

$$E\left[\sum_{j=1}^{N} (S(1, j) - S(2, j))\right] \leq \left(\frac{2.3}{\pi}\right)^{0.43 \ln N}.$$

For a proof of Proposition 4 see in Section 4.

Using Proposition 4 and Lemma 3 and the assumption that the number of $i$-special pairs in $A_i$ is at least $d^{\frac{\epsilon - 2}{4}}$, we get

$$E[|\Delta_i||A_c] \leq \left(\frac{2.3}{\pi}\right)^{0.43 \frac{\ln d}{\ln d}} \ln d \leq d^{-0.344^{\frac{\epsilon - 2}{4}} \ln (\frac{\pi}{2})} \leq d^{-0.107^{\frac{\epsilon - 2}{4}}}.$$ (8)

Plugging the inequalities (7) and (8) into (6) we get that

$$E[D_v|A_c] \leq 200(\ln d)^2 d^{-0.107^{\frac{\epsilon - 2}{4}}}.$$

The proposition follows.

### 3.2 Proof of Proposition 5

For the quantities, $\beta_v$, $\delta_k$ and $h_v$ we defined in Section 3 we have the following proposition.

**Proposition 5** For $k = (1 + \epsilon)d/\ln d$ the following are true:

$$Pr \left[ \beta_v \geq \left(1 + x\right) \frac{d}{k-1} \right] \leq d^{-\frac{3\phi(x)}{4k+1}},$$ (9)

where $\phi(x) = (1 + x) \ln(1 + x) - x$, for real $x > 0$. Also, it holds that

$$\delta_k \geq 1 - 2 \exp\left(-\frac{(1 + \epsilon)}{24 \ln d} \frac{d^{\frac{1}{4\pi}}}{}\right).$$ (10)

Finally, for any $c > 0$ it holds that

$$Pr \left[ h_v \leq \frac{d^{\frac{1}{4\pi}}}{16c \ln d} | \beta_v \leq c \ln d \right] \leq \exp\left(-\frac{d^{\frac{1}{4\pi}}}{64c \ln d}\right).$$ (11)
The proof of Proposition 5 appears in Section 3.3.

Let the events $E_1 = \{\beta_v > 100 \ln d\}$, $E_2 = \{\text{there is at least one bad pair of lists which is not rescuable}\}$ and $E_3 = \{\text{there is a pair rescuable lists} (L_X^j, L_Y^j) \text{ that is associated to a partition with less than } d^{\frac{4}{1+\epsilon}} \text{ } j\text{-special pairs}\}$. From a simple union bound we get that

$$Pr[A] = Pr \left[ \bigcup_{i=1}^{3} E_i \right] \leq \sum_{i=1}^{3} Pr[E_i]. \tag{12}$$

The proposition will follow by bounding appropriately the probability terms $Pr[E_1], Pr[E_2]$ and $Pr[E_3]$. As far as $Pr[E_1]$ is regarded it holds that

$$Pr[E_1] \leq Pr \left[ \beta_v > (1 + x_0) \frac{d}{k - 1} \right], \tag{13}$$

where $1 + x_0 = 98(1 + \epsilon)$. The above inequality holds since $\frac{d}{k - 1} \leq \ln d + 2 \ln^2 d$. We use Proposition 5 (i.e. (9)) to bound the r.h.s of (13). In particular, for $x_0 = 98(1 + \epsilon) - 1$ it holds that $\phi(x_0) \geq 343(1 + \epsilon) + 98(1 + \epsilon) \ln(1 + \epsilon)$. Then, from (13) we get that

$$Pr[E_1] \leq d^{-250}. \tag{14}$$

As far as $Pr[E_2]$ is regarded, we let $J_v$ be the number of non-rescuable pairs. It holds that

$$Pr[E_2] = Pr[J_v > 0] \leq E[J_v], \tag{15}$$

where the last inequality follows from Markov’s inequality. Using (10), we get that

$$E[J_v] \leq (1 - \delta) d \leq \exp \left( -\frac{3(1 + \epsilon)}{8 \ln d} d^{\frac{4}{1+\epsilon}} \right) d \leq \exp \left( -d^{\frac{4}{1+\epsilon}} \right).$$

Plugging the above inequality into (15) we get that

$$Pr[E_2] \leq \exp \left( -d^{\frac{4}{1+\epsilon}} \right). \tag{16}$$

Finally, for $Pr[E_3]$ we work as follows:

$$Pr[E_3] \leq Pr[E_3 | \beta_v < 100 \ln d] + Pr[\beta_v \geq 100 \ln d]. \tag{17}$$

We let $M_v$ be the number of bad pairs which are associated to a partition with less than $d^{\frac{4}{1+\epsilon}}$ special pairs. Clearly, it holds that

$$Pr[E_3 | \beta_v \leq 100 \ln d] = Pr[M_v > 0 | \beta_v \leq 100 \ln d] = Pr[|\beta_v| \leq 100 \ln d].$$

We remind the reader that $h^j_v$ denotes the number of $j$-special pairs that appear in the partition that is associated to the rescuable pair $(L_X^j, L_Y^j)$. It holds that

$$Pr[h^j_v \leq d^{\frac{4}{1+\epsilon}} | \beta_v \leq 100 \ln d] \leq Pr \left[ \frac{h^j_v}{100 \ln d} \leq d^{\frac{4}{1+\epsilon}} | \beta_v \leq 100 \ln d \right] \leq \exp \left( -d^{\frac{4}{1+\epsilon}} \right). \quad \text{[from (11)]}$$
It is direct that
\[ E[M_v | \beta_v \leq 100 \ln d] \leq (100 \ln d) \Pr[h^i_v \leq d^{3 + \frac{2}{t+1}} | \beta_v \leq 100 \ln d] \leq \exp \left( -d^{\frac{3}{t+1}} \right). \]

Using Markov’s inequality we get that
\[ \Pr[M_v > 0 | \beta_v \leq 100 \ln d] \leq E[M_v | \beta_v \leq 100 \ln d] \leq \exp \left( -d^{\frac{3}{t+1}} \right). \]

Plugging the above inequality and (14) to (17) we get that
\[ \Pr[E_3] \leq \exp \left( -d^{\frac{3}{t+1}} \right) + d^{-250} \leq 2d^{-250}. \tag{18} \]

Plugging (14), (16) and (18) into (12) we get that \( \Pr[\delta] \leq 5d^{-250}. \) The proposition follows.

### 3.3 Proof of Proposition 5

The inequality in (10) follows from the following two lemmas.

**Lemma 4** Let \( L_X^i \) be a list which belongs to a bad pair, for some \( i \in [d] \). For \( k = (1 + \epsilon) d/ \ln d \) and for any colour \( s \in [k] \backslash \{c, q\} \) it holds that
\[ |\Pr[s \notin L_X^i] - d^{-\frac{1}{t+1}}| \leq 3d^{-\frac{2}{t+1}}. \]

**Proof:** It holds that \( c \in L_X^i \). Let \( t \) be the number of the appearances of \( c \) in the \( L_X^i \). Then, it holds that \( \Pr[s \notin L_X^i | t] = \left( 1 - \frac{1}{k-1} \right)^{d-t} \). The random variable \( t \) is binomially distributed with parameters \( 1/(k-1) \) and \( d \), conditional that it is positive. It is direct that
\[ p_0 = \Pr[B(1/(k-1), d) = 0] = \left( 1 - \frac{1}{k-1} \right)^d \leq \exp \left( -\frac{d}{k} \right) \leq d^{-\frac{1}{t+1}}. \tag{19} \]

Thus, it holds that
\[
\Pr[s \notin L_X^i] = \sum_{i=1}^{d} \left( 1 - \frac{1}{k-2} \right)^{d-i} \Pr[t = i]
\]
\[
= \frac{1}{1 - p_0} \sum_{i=1}^{d} \binom{d}{i} \left( \frac{1}{k-1} \right)^i \left( 1 - \frac{1}{k-1} \right)^{d-i} \left( 1 - \frac{1}{k-2} \right)^{d-i}
\]
\[
\leq \frac{1}{1 - d^{-\frac{1}{t+1}}} \left( 1 - \frac{1}{k-2} + \frac{1}{(k-1)(k-2)} \right)^d
\]
\[
\leq \left( 1 + 2d^{-\frac{1}{t+1}} \right) \exp \left( -\frac{d}{k} + \frac{d}{(k-2)^2} \right) \quad \text{[as \( 1 - x \leq e^{-x} \) and \( d^{-\frac{1}{t+1}} < 1/2 \)]}
\]
\[
\leq d^{-\frac{1}{t+1}} \left( 1 + 2d^{-\frac{1}{t+1}} \right) \left( 1 + \frac{2d}{(k-2)^2} \right) \quad \text{[as \( e^x < 1 + 2x \) for \( 0 < x < 0.1 \)]}
\]
\[
\leq d^{-\frac{1}{t+1}} \left( 1 + 3d^{-\frac{1}{t+1}} \right) \quad \text{[as \( d/k^2 = o_d(d^{-1/(1+\epsilon)}) \)]}
\]

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We get a lower bound on the \( Pr[c \notin L^i_X] \) by working similarly. In particular, we have that

\[
Pr[s \notin L^i_X] \geq \frac{1}{1 - p_0} \sum_{i=1}^{d} \left( \frac{d}{i} \right) \left( 1 - \frac{1}{k-1} \right)^i \left( 1 - \frac{1}{k-2} \right)^{d-i} \geq \left( 1 - \frac{1}{k-2} + \frac{1}{(k-1)(k-2)} \right)^d - \left( 1 - \frac{1}{k-1} \right)^d \left( 1 - \frac{1}{k-2} \right)^d \quad [\text{as } \frac{1}{1-p_0} \geq 1] \\
\geq \exp \left( - \frac{d}{k-2} \left( 1 - \frac{1}{k-2} \right)^{-1} \right) - \exp (-2d/k) \quad [\text{as } 1 - x \geq \exp(-\frac{x}{1-x}) \text{ for } 0 < x < 0.1] \\
\geq \exp \left( - \frac{d}{k} - \frac{6d}{k^2} \right) - \exp (-2d/k) \\
\geq d^{-\frac{1}{1+\epsilon}} \left( 1 - \frac{6d}{k^2} \right) - d^{-\frac{2}{1+\epsilon}} \\
\geq d^{-\frac{1}{1+\epsilon}} \left( 1 - 3d^{-\frac{1}{1+\epsilon}} \right). 
\]

The lemma follows. \( \square \)

**Lemma 5**  Let \( H_i \) denote the number of colours in \([k] \setminus \{c, q\}\) that do not appear in both lists of the bad pair \((L^i_X, L^i_Y)\). For \( k = (1 + \epsilon)d/\ln d \) and for any \( y \in (0, 1) \) it holds that

\[
Pr \left[ H_i \leq \frac{1 - y \left( 1 + \epsilon \right)}{3 \ln d} d^{-\frac{1}{1+\epsilon}} \right] \leq 2 \exp \left( - \frac{y^2 \left( 1 + \epsilon \right)}{6 \ln d} d^{-\frac{1}{1+\epsilon}} \right).
\]

**Proof:** Since we have assumed that \((L^i_X, L^i_Y)\) is a bad pair, for \( L^i_X \) we have that \( c \in L^i_X \) and \( q \notin L^i_X \), while for \( L^i_Y \) we have that \( q \in L^i_Y \) and \( c \notin L^i_Y \).

Let \( f_X, f_Y \) be the number of colours that do not appear in the lists \( L^i_X \) and \( L^i_Y \), respectively. Using Lemma 4 we have that

\[
E[f_X] \geq (k-2)d^{-\frac{1}{1+\epsilon}} \left( 1 - 3d^{-\frac{1}{1+\epsilon}} \right) \\
\geq (1 + \epsilon) \frac{d^{1+\epsilon}}{\ln d} \left( 1 - 4d^{-\frac{1}{1+\epsilon}} \right) \geq \frac{3}{4} (1 + \epsilon) d^{-\frac{1}{1+\epsilon}}. \quad (20)
\]

Using a ball and bins argument, we can show that Chernoff bounds apply for \( f_X \). In particular, for any \( y \in (0, 1) \) it holds that

\[
Pr[f_X \leq (1-y)E[f_X]] \leq \exp \left( - \frac{y^2}{2} E[f_X] \right) \leq \exp \left( - \frac{y^2}{2} \frac{3(1+\epsilon)}{4 \ln d} d^{-\frac{1}{1+\epsilon}} \right). \quad [\text{from } (20)].
\]

Let the event \( R_X = \{ f_X > \frac{3}{8} \frac{(1+\epsilon)}{\ln d} d^{-\frac{1}{1+\epsilon}} \} \).

\[
1 - Pr[R_X] = Pr \left[ f_X \leq \frac{3}{8} \frac{(1+\epsilon)}{\ln d} d^{-\frac{1}{1+\epsilon}} \right] \leq \exp \left( - \frac{\frac{3}{32} \frac{(1+\epsilon)}{\ln d} d^{-\frac{1}{1+\epsilon}} \right), \quad (21)
\]

where the last inequality follows from Chernoff bounds by setting \( y = 1/2 \).
Any information for $f_X$ does not affect the distribution of the colourings in $L^*_Y$. This holds since the choice of colours in the two lists are independent with each other (Step 3 in Phase 1 of the coupling). That is, $E[H_i|f_X] \geq f_X \cdot d^{- \frac{i}{1+\epsilon}} \left(1 - 3d^{- \frac{i}{1+\epsilon}} \right)$. Also, we get that

$$E[H_i|R_X] \geq \frac{3}{8} \frac{(1 + \epsilon)}{\ln d} d^{\frac{i+1}{3}} (1 - 3d^{\frac{1}{1+\epsilon}}) \geq \frac{1}{3} \frac{(1 + \epsilon)}{\ln d} d^{\frac{i+1}{3}}. \quad (22)$$

Arguing in the same manner as above, we apply Chernoff bounds for $\phi$ where $\Pr(H_i \leq (1-y)E[H_i|R_X]|R_X)$ holds that

$$Pr[H_i \leq (1-y)E[H_i|R_X]|R_X] \leq \exp \left(- \frac{y^2}{2} E[H_i|R_X]\right) \leq \exp \left(- \frac{y^2}{6} \frac{(1 + \epsilon)}{\ln d} d^{\frac{i+1}{3}}\right). \quad \text{[from (22)]} \quad (23)$$

It holds that

$$Pr \left[H_i \leq \frac{1-y}{3} \frac{(1 + \epsilon)}{\ln d} d^{\frac{i+1}{3}} \right] \leq Pr \left[H_i \leq \frac{1-y}{3} \frac{(1 + \epsilon)}{\ln d} d^{\frac{i+1}{3}} |R_X\right] + (1-Pr[R_X]) \leq Pr[H_i \leq (1-y)E[H_i|R_X]|R_X] + 1 - Pr[R_X] \quad \text{[from (22)]} \leq \exp \left(- \frac{y^2}{6} \frac{(1 + \epsilon)}{\ln d} d^{\frac{i+1}{3}}\right) + \exp \left(- \frac{3}{32} \frac{(1 + \epsilon)}{\ln d} d^{\frac{i+1}{3}}\right). \quad \text{[from (23),(21)]}$$

The lemma follows.

Using Lemma 5 where we set $y = 1/2$, we get (10), i.e.

$$\delta_k \geq 1 - Pr \left[H_i \leq \frac{(1 + \epsilon)}{6 \ln d} d^{\frac{i+1}{3}} \right] \geq 1 - 2 \exp \left(- \frac{(1 + \epsilon)}{24 \ln d} d^{\frac{i+1}{3}}\right).$$

Also, for proving (9) we use the following lemma.

**Lemma 6** For $k = (1 + \epsilon)d/\ln d$, it holds that

$$Pr \left[\beta_v \geq (1 + x) \frac{d}{k-1} \right] \leq d^\left(- \frac{3\phi(x)}{4^{\frac{1}{3+\epsilon}}}\right),$$

where $\phi(x) = (1 + x) \ln(1 + x) - x$, for $x > 0$.

**Proof:** There are $d$ different pairs of lists and each of them is bad independently of the others. Let $p_{bad}$ be the probability for the pair $(L^*_X, L^*_Y)$ to be bad. It suffices to have that $L_Y(i) = c$ while $q \in L^*_Y$. It holds that

$$p_{bad} = \frac{1}{k-1} \left(1 - \left(1 - \frac{1}{k-1}\right)^d\right) \leq \frac{1}{k-1},$$

as $\left(1 - \left(1 - \frac{1}{k-1}\right)^d\right) \leq 1$. By the linearity of expectation we get that

$$E[\beta_v] \leq dp_{bad} \leq d/(k-1). \quad (24)$$

Also, using Lemma 4, we get that

$$p_{bad} \geq \frac{1}{k-1} \left(1 - d^{\frac{1}{1+\epsilon}} \left(1 + \frac{4d}{k^2}\right)\right) \geq \frac{3}{4k^4}.$$
In turn, we get that
\[ E[\beta_u] \geq dp_{bad} \geq \frac{3 \ln d}{4(1 + \epsilon)}. \] (25)

Applying Chernoff bounds, for any \( x > 0 \), we have that
\[ Pr[\beta_u \geq (1 + x)E[\beta_u]] \leq \exp(-\phi(x) \cdot E[\beta_u]), \]
where \( \phi(x) = (1 + x) \ln(1 + x) - x \). The lemma follows by plugging the bounds from (24) and (25) into the above inequality. The lemma follows.

The next two lemmas show that (11) holds.

**Lemma 7** Let \((L^j_X, L^j_Y)\) be a rescuable pair and let \(A_j\) be the set of indices of the pairs where we check for \(j\)-special lists. Assume that \(A_j\) is non empty. Let \(k = (1 + \epsilon)d/\ln d\). For any \(i \in A_j\), it holds that
\[ Pr[L^j_Y \text{ is special w.r.t. } L^j_X | H_j] \geq \frac{H_j}{k} \cdot d^{-\frac{1}{1+\epsilon}}, \]
where \(H_j\) is the number of colours that do not appear in both \(L^j_X, L^j_Y\).

**Proof:** Since \((L^j_X, L^j_Y)\) is rescuable, it means that \(L_X(j) = c\) and \(q \in L^j_X\). Also, there is non-empty set of colours \(U_i \in [k] \setminus \{c, q\}\) which contains colours that do not appear in \(L^j_X \cup L^j_Y\). So as to have \(L^j_Y\) special special w.r.t. \(L^j_X\), it should hold that \(L_Y(i) \in U_j\) and either of the following two holds A) \(q \in L^j_Y\) and \(c \notin L^j_Y\) or B) \(q \notin L^j_Y\) and \(c \in L^j_Y\). Let the event \(Q = \{L_Y(i) \in U_j\}\). It holds that
\[ \Pr \left[ q \notin L^j_Y | c \in L^j_Y, Q \right] \geq \frac{H_j}{k} \cdot d^{-\frac{1}{1+\epsilon}}, \] (26)

where \(H_j = |U_j|\). Also, working as in the proof of Lemma 4 we get that
\[ Pr[c \in L^j_Y | Q] \geq 1 - d^{-\frac{1}{1+\epsilon}} \left(1 + 3d^{-\frac{1}{1+\epsilon}}\right), \] (27)
\[ Pr[q \notin L^j_Y | c \in L^j_Y, Q] \geq \frac{3}{4} d^{-\frac{1}{1+\epsilon}}. \] (28)

The lemma follows by substituting the bounds from (27) and (28) into (26).

**Lemma 8** Let \(h^j_v\) be the number of the non-bad pairs that correspond to the rescuable pair \((L^j_X, L^j_Y)\). For \(k = (1 + \epsilon)d/\ln d\) and fixed \(c > 0\), it holds that
\[ Pr \left[ h^j_v \leq \frac{d^{\frac{1+\epsilon}{2}}}{16c \ln d} | \beta_v \leq c \ln d \right] \leq 2 \exp \left(-\frac{d^{\frac{1+\epsilon}{2}}}{64c \ln d}\right). \]

**Proof:** The number of lists that are associated to each rescuable pair depends on the actual number of bad lists. Conditioning that the number of bad pairs \(\beta_v \leq c \ln d\), for some fixed \(c > 0\), the rescuable pair \((L^j_X, L^j_Y)\), is assigned a set of at least \(\lfloor \frac{d}{c \ln d} \rfloor\) non-bad lists. Let \(H_j\) denote the number of colours that do not in \(L^j_X \cup L^j_Y\). Let the event \(\mathcal{H} = \{H_j > \frac{1 + \epsilon}{6 \ln d} d^{1+\epsilon}\}\). From Lemma 5 we get that
\[ 1 - Pr[\mathcal{H}] = Pr \left[ H_j \leq \frac{1 + \epsilon}{6 \ln d} d^{1+\epsilon} \right] \leq 2 \exp \left(-\frac{1 + \epsilon}{24 \ln d} d^{1+\epsilon}\right). \]
From Lemma 7 we get that

\[ E \left[ h_v^t | \mathcal{H}, \beta_v \leq c \ln d \right] \geq \frac{d^{1+\epsilon}}{8c \ln d}. \]

We can apply Chernoff bounds and get the following

\[ Pr \left[ h_v^t \leq (1 - y) \frac{d^{1+\epsilon}}{8c \ln d} | \mathcal{H}, \beta_v \leq c \ln d \right] \leq \exp \left( -\frac{y^2 d^{1+\epsilon}}{16c \ln d} \right). \]

From the law of total probability it holds that

\[ Pr \left[ h_v^t \leq (1 - y) \frac{d^{1+\epsilon}}{8c \ln d} | \mathcal{H}, \beta_v \leq c \ln d \right] \leq Pr \left[ h_v^t \leq (1 - y) \frac{d^{1+\epsilon}}{8c \ln d} | \mathcal{H}, \beta_v \leq c \ln d \right] + Pr [\mathcal{H}^c | \beta_v \leq c \log d] \]

\[ \leq \exp \left( -\frac{y^2 d^{1+\epsilon}}{16c \ln d} \right) + 2 \exp \left( -\frac{1 + \epsilon}{24 \ln d} d^{1+\epsilon} \right). \]

We used the fact that the events \( \mathcal{H} \) and \( \beta_v < c \ln d \) are independent with each other. The lemma follows by setting \( y = 1/2 \).

\[ \square \]

4 Proof of Proposition 4

A way of constructing \( S \), which is equivalent to the one described in Remark 7, is the following one: Consider some sufficiently large positive integer \( l \ll N \). We construct \( S \) in rounds. Assume that after round \( i - 1 \) we have constructed \( S \) up to some column \( t \), for some \( t \ll N \). Additionally, let \( X_t = \sum_{j=1}^t S(1, j) - S(2, j) \). Then, during the round \( i \) we proceed as described in the following paragraph.

If \( X_t = 0 \), then we use identical coupling for \( S(1, j), S(2, j) \) for all \( t < j \leq N \). If \( X_t > 0 \), then we consider \( X_t \) many sets of columns of \( S \) whose values has not been set yet. Each of these \( X_t \) many sets contains at most \( l \) columns. More specifically, the first set \( R_1^t \) starts from column \( t + 1 \) up to column \( T \), the value of \( T \) will be defined in what follows. We set the values in each column \( j \in R_1^t \) by coupling \( S(1, j), S(2, j) \) such that \( S(1, j) = 1 - S(2, j) \). \( T \) is either the first time that \( \sum_{j=t+1}^T S(1, j) - S(2, j) = -1 \) or if this is not possible up to column \( t + l \), then we have \( T = t + l \). Continue with the second set of columns \( R_2^t \) and so on. Round \( i \) ends after having finished with all these \( X_t \) sets of columns. Then we continue in the same manner with the round \( i + 1 \).

For each set of columns \( R_i^t \), \( R_i^t \) is submatrix of \( S \), we have the following lemma which is going to be useful in the proof of Proposition 4.

Lemma 9 Let \( l \geq 10 \), the maximum number of columns of \( R_i^t \). If the entries are such that \( R_i^t(1, s) \neq R_i^t(2, s) \) for any column \( s \) of \( R_i^t \), then it holds that

\[ E \left[ 1 + \sum_{t=1}^T R_i^t(1, t) - R_i^t(2, t) \right] \leq \frac{2.3}{\pi}, \]

where \( T \) is the actual number of columns of \( R_i^t \).

\[ ^6 R_2^t \] starts from the column \( T + 1 \)
Proof: For every $t$ it holds that $R_j^1(1, t) - R_j^1(2, t)$ is equal to $-1$ with probability $1/2$ or it is equal to $1$ with probability $1/2$. It is direct to see that the partial sums $W_s = \sum_{i=1}^{s} R_j^1(1, t) - R_j^1(2, s)$, for $s \leq T$ constitute a symmetric random walk on the integers which starts from position zero and stops either when it hits $-1$ or after $l$ steps, whatever happens first. We can simplify the analysis and remove the dependency from the random variable $T$, by assuming that $W_s$ continues always for $l$ steps and the state $-1$ is absorbing. Then, the lemma follows by just computing $E[W_l + 1]$. In particular, we have that

$$E[W_l + 1] = E[W_l + 1|W_l \neq -1] \cdot Pr[W_l \neq -1].$$

Let $T$ be the step that $W_l$ hits $-1$ for first time. By the Reflection Principle we have that for any nonnegative integer $i$ it holds that

$$Pr[T = 2i + 1] = 2^{-2(i+1)} \binom{2i}{i}.$$  

(30)

It is direct that the $W_l$ cannot be $-1$ for $t$ even, i.e. $Pr[T = 2i] = 0$, for every positive integer $i$. It is direct to see that it holds that

$$Pr[W_l = -1] = Pr[T \leq l] = 1 - \sum_{i \geq (l-1)/2} 2^{-2(i+1)} \binom{2i}{i}.$$  

(31)

To this end we use Stirling approximation, i.e. for a sufficiently large $n$ it holds that $n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\lambda_n}$, with $\frac{1}{2n+1} \leq \lambda_n \leq \frac{1}{2n}$. Then we have that

$$\sum_{i \geq (l-1)/2} 2^{-2(i+1)} \binom{2i}{i} \leq \frac{1}{2\sqrt{\pi}} \sum_{i \geq (l-1)/2} \frac{1}{i^{3/2}} \leq \sqrt{\frac{2}{\pi l}}.$$  

Thus, we get that

$$Pr[W_l = -1] \geq 1 - \sqrt{\frac{2}{\pi l}}.$$  

(32)

On the other hand, it is direct to see that given that the walk $W_l$ does not hit $-1$ it is just a random walk on the positive integers and it is a folklore result that

$$E[Z_l|Z_l \neq -1] \leq \sqrt{\frac{2}{\pi l}} \cdot \left( 1 + \frac{3}{2l} \right).$$  

(33)

The lemma follows by plugging (31) and (32) into (29) and taking $l \geq 10$.  \hfill $\square$

Proof of Proposition 4. Consider the revelation of the values of the matrix $S$ we gave above. Let $t_i$ be the index of the column we have revealed up to round $i$. I.e. at round $i + 1$ we check whether $X_{t_i} = \sum_{j=1}^{t_i} S(1, j) - S(2, j)$ is zero or not. Let $l$ the maximum number of columns in each submatrix $R_j^i$ be equal to 10.

Given $X_{t_i}$ and assuming that the coupling continuous, i.e. $t_i$ the number of columns we have revealed so far is much smaller than $N$, we show that it holds that

$$E[X_{t_{i+1}}|X_{t_i}] \leq \frac{2.3}{\pi} X_{t_i}.$$  

(33)

However, before showing the above let us see which are its consequences. Taking the average from both sides, we get

$$E[X_{t_i}] \leq \frac{2.3}{\pi} E[X_{t_{i-1}}] \leq \left( \frac{2.3}{\pi} \right)^i.$$  

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since \( X_{t_i} = 1 \) (it always holds that \( S(1, 1) - S(2, 1) = 1 \)). It is also direct to see that it always holds that \( X_{t_i} \leq l \cdot X_{t_{i-1}} \leq l^i \). That is, in round \( i \) we will need to reveal at most \( l^i \) columns of the matrix. This fact implies that the maximum \( j \) which satisfies the condition that \( \sum_{j=0}^{l^i} t \leq N \) is a lower bound for the number of rounds we can have. Direct calculations suggest that the number of rounds \( j_0 \geq \frac{99}{100} \ln N = 0.43 \ln N \), since \( l = 10 \). Clearly, the proposition follows once we show (33). For this we are going to use Lemma \( 9 \). Notice that given that at round \( i \) we have \( X_{t_i} = \left| \sum_{j=1}^{t_i} (S(1, j) - S(2, j)) \right| \), for \( X_{t_{i+1}} \) the following holds:

\[
X_{t_{i+1}} = \sum_{s=1}^{X_{t_i}} \left( 1 + \sum_{j=1}^{T_s} R_s^i(1, j) - R_s^i(2, j) \right),
\]

where \( T_s \) is the length of the submatrix \( R_s^i \). From Lemma \( 9 \) we have that for any \( i, s \) it holds

\[
E \left[ 1 + \sum_{j=1}^{T_s} R_s^i(1, j) - R_s^i(2, j) \right] \leq \frac{2.3}{\pi}.
\]

Combining the above two relations and by linearity of expectation we get that

\[
E[X_{t_{i+1}} | X_{t_i}] = \sum_{s=1}^{X_{t_i}} E \left[ 1 + \sum_{j=1}^{T_s} R_s^i(1, j) - R_s^i(2, j) \right] \leq \frac{2.3}{\pi} X_{t_i}.
\]

The proposition follows.

\[\square\]

5 Rest of the proofs

5.1 Proof of Lemma \( 1 \)

Since \((L_1, L_2)\) is a rescuable (thus bad) pair, we have the following information for the lists. For \( L_1 \) we know that the colour \( q \in L_1 \), and \( c \notin L_1 \). For \( L_2 \), we know that \( q \in L_2 \), \( c \notin L_2 \). Also, there is a non-empty set of colours \( U \subseteq [k] \setminus \{c, q\} \) such that for each \( c' \in U \) it holds that \( c' \notin L_1 \cup L_2 \). Finally, since \( L_3 \) good with respect to \( L_1 \), this implies that \( s \in U \) while \( q \in L_3 \) and \( c \notin L_3 \).

Let the event \( A = “L_3 \) is good w.r.t. \( L_1”\). For any \( S \in [k]^d \) it holds that

\[
Pr[L_3 = S | A] = \lambda_S(S | B),
\]

where \( B = “there exists \ t \in [d] \ such that \ S(t) = q \ and \ there \ is \ no \ t \in [d] \ such that \ S(t) = c”\).

Let \( Q = |U| \). It suffices to show that,

\[
Pr[L_1 = S | c \notin L_1, q \in L_1, s \in U, Q > 0] = \lambda_S(S | B). \tag{34}
\]

Clearly we have that

\[
Pr[L_1 = S | c \notin L_1, q \in L_1, s \in U, Q > 0] = \frac{Pr[L_1 = S, c \notin L_1, q \in L_1, s \in U, Q > 0]}{Pr[c \notin L_1, q \in L_1, s \in U, Q > 0]}
= \frac{Pr[L_1 = S, c \notin L_1, q \in L_1, s \in U]}{Pr[c \notin L_1, q \in L_1, s \in U]}
= Pr[L_1 = S | s, c \notin L_1, q \in L_1]. \tag{35}
\]

In the penultimate derivation we eliminated the event \( Q > 0 \) from both probability terms, in the numerator and denominator, since whenever \( s \in U \) holds it also holds that \( Q > 0 \). Then, it is straightforward that the r.h.s. of (35) is equal to \( \lambda_S(S | B) \), i.e. (34) holds.
5.2 Proof of Lemma

The lemma follows by just examining the information we have for each of the four lists. As far as the \(i\)-good pair \((L^i_X, L^i_Y)\) is concerned we have the following: \(L_X(t)\) is distributed uniformly at random among the colours \([k]\setminus\{c, q\}\) that do not appear in \(L_X \cup L_Y\), while \(c \notin L_X^i\) and \(q \in L_Y^i\). Also, \(L_Y(t)\) is distributed uniformly at random among the colours \([k]\setminus\{c, q\}\) that do not appear in \(L_Y \cup L_X\), while \(q \notin L_Y^i\) and \(c \in L_X^i\).

As far as the \(i\)-fail pair \((L^*_X, L^*_Y)\) is concerned we have the following: \(L_X(s)\) is distributed uniformly at random among the colours \([k]\setminus\{c, q\}\) that do not appear in \(L_X^i \cup L_Y^i\) while \(q \notin L_X^i\) and \(c \in L_X^i\). Additionally, \(L_Y(s)\) is distributed uniformly at random among the colours \([k]\setminus\{c, q\}\) that do not appear in \(L_Y^i \cup L_X^i\) while \(c \notin L_Y^i\) and \(q \in L_Y^i\).

Thus, we can couple identically \(L_X(t)\) with \(L_Y(s)\) and \(L_X(s)\) with \(L_Y(t)\). Then, it is direct that we can couple identically \(L_X^i\) with \(L_Y^i\) and \(L_X^*\) with \(L_Y^*\).

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