Moduli and Kähler potential in fermionic strings

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ABSTRACT

We study the problem of identifying the moduli fields in fermionic four-dimensional string models. We deform a free-fermionic model by introducing exactly marginal operators in the form of Abelian Thirring interactions on the world-sheet, and show that their couplings correspond to the untwisted moduli fields. We study the consequences of this method for simple free-fermionic models which correspond to \(Z_2 \times Z_2\) orbifolds and obtain their moduli space and Kähler potential by symmetry arguments and by direct calculation of string scattering amplitudes. We then generalize our analysis to more complicated fermionic structures which arise in constructions of realistic models corresponding to asymmetric orbifolds, and obtain the moduli space and Kähler potential for this case. Finally we extend our analysis to the untwisted matter sector and derive expressions for the full Kähler potential to be used in phenomenological applications, and the target space duality transformations of the corresponding untwisted matter fields.

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1 Introduction

Superstring theory remains the only consistent theoretical framework which brings the expectation of unification of all fundamental interactions including gravity. The study of the low-energy effective field theories arising from four-dimensional string models, in particular those with $N = 1$ space-time supersymmetry [1], is of crucial importance in bridging the gap between string theory and the observed particle phenomenology, which at the moment is embodied in the Standard Model. It is generally believed that for energies well below the Planck scale, such string derived effective theory should take the form of an $N = 1$ locally supersymmetric quantum field theory, namely, $N = 1$ supergravity coupled to some $N = 1$ Yang-Mills supermultiplets and chiral supermultiplets. Therefore, its Lagrangian can be specified in terms of three standard functions [2]: (i) the gauge kinetic function $f_{ab}$, (ii) the superpotential $W$, and (iii) the K"ahler potential $K$. The ultimate goal would be to derive these functions entirely from the underlying string theory. Indeed, much progress towards this goal has been achieved in the past and there exists a great deal of information about the structure of the effective Lagrangian for various four-dimensional string models.

One of the advantages of string-derived effective field theories over conventional $N = 1$ supergravity theories, is the calculability of these three functions ($f_{ab}, W, K$) in string perturbation theory. This was demonstrated in the $S$-matrix approach in Refs. [3, 4, 5], where various string scattering amplitudes were computed using the techniques of conformal field theory. In addition, in the low-energy effective field theories of string, a special class of massless fields, called moduli, play a unique role. It is a distinct feature of string theory that some moduli fields always exist. In $N = 1$ supersymmetric string models, a modulus field is a special massless chiral superfield which has flat scalar potential to all orders in string perturbation theory, i.e., the vacuum expectation value (VEV) of its scalar component is completely unconstrained. The VEVs of the moduli parameterize a continuously connected family of string models. The study of moduli fields and their symmetry properties, most notably target space modular invariance or duality, has been found to be of great theoretical and phenomenological interest, with applications in string-derived supergravities, supersymmetry breaking, string threshold corrections to gauge couplings, string cosmology, etc [6].

In this paper, we consider the low-energy effective field theory for a class of four-dimensional heterotic string models constructed in the fermionic formulation [7, 8]. The first studies of the effective field theory for some simple models of this kind were performed sometime ago in Refs. [9, 10], by using a consistent truncation procedure similar to the dimensional reduction of the ten-dimensional supergravity Lagrangian [11]. This procedure is based on the following two observations: First, all $N = 1$ string models in the fermionic formulation can be obtained from an $N = 4$ model with a gauge group of rank 22, by adding non-trivial spin-structure vectors to reduce the space-time supersymmetry. In the bosonic language, this corresponds precisely to constructing orbifold models by introducing twists into the generalized toroidal compactification [12]. Second, the scalar couplings of $N = 4$ supergravity possess a
unique \([SO(6,m)/SO(6) \times SO(m)] \otimes [SU(1,1)/U(1)]\) non-linear sigma model structure \([3]\), the second factor corresponds to the dilaton field. It was found in Refs. \([4,5]\) that, in addition to the dilaton field which parameterizes \(SU(1,1)/U(1)\), the other scalar fields from the untwisted sector (the Neveu-Schwarz sector) typically parameterize a Kähler manifold which is a direct product of several factors of the form \(SO(2,n)/SO(2) \times SO(n)\).

This paper is organized as follows. In Sec. \(2\) we establish a procedure with which the untwisted moduli in fermionic models can be identified. We deform a free-fermionic model by introducing exactly marginal operators in the form of world-sheet Abelian Thirring interactions and show that their couplings correspond to the untwisted moduli fields. In Sec. \(3\) we illustrate the procedure presented in Sec. \(2\) with simple fermionic models. We identify explicitly the untwisted moduli fields, and obtain the corresponding moduli space and Kähler potential by symmetry arguments. The validity of the precise form of the Kähler potential is also demonstrated with explicit string perturbation theory calculations. In Sec. \(4\) we expand on the connection between the simple fermionic models in Sec. \(3\) and symmetric \(Z_2 \times Z_2\) orbifolds, focusing on the issue of untwisted moduli fields in the two approaches. In Sec. \(5\) we generalize our analysis to more complicated fermionic models corresponding to asymmetric orbifolds which often arise in constructions of realistic models. We show how the moduli space and Kähler potential can be changed for this case with a typical example. In Sec. \(6\) we extend our discussion to the untwisted matter fields and obtain the explicit form of the full Kähler potential for all the untwisted scalar fields, which is particularly useful in phenomenological studies of such fermionic string models. In Sec. \(7\) we study the target space duality invariance of the fermionic models, and as a by-product obtain the properties of the untwisted matter fields under duality transformations. Finally, we summarize our conclusions in Sec. \(8\).

### 2 Identification of the untwisted moduli

Given the importance of the moduli fields, first we would like to identify the untwisted moduli in fermionic models. One very special untwisted modulus field in such models, as in any string models, is the dilaton field. Since the properties of the dilaton field are well-known, in this paper we only discuss the other untwisted moduli fields in the four-dimensional fermionic heterotic string models. These models are described by two-dimensional internal conformal and superconformal field theories of central charges \(c_R = 22\) and \(c_L = 9\) respectively, which in the fermionic formulation are completely fermionized. It is particularly convenient to start with a model which makes use only of free world-sheet fermions, since such a free-fermionic model can be readily worked out using simple rules \([7,8]\) that ensure conformal as well as modular invariance. Once such a free-fermionic model is constructed, one would have found a special vacuum of the string theory, which should correspond to a special point in the moduli space. Suppose that one knows precisely what are the moduli fields in such free-fermionic models, then by changing their vacuum expectation values one
can obtain other (continuously connected) string vacua which may or may not be physically equivalent to the original vacuum specified by the free-fermionic construction. However, because the free-fermionic models have fixed (i.e., vanishing) values for the moduli scalars, as opposed to the orbifold construction [14] where arbitrary values of some moduli (such as the size of the orbifold) are explicit, it is not evident which massless scalars correspond to moduli fields in free-fermionic models.

For heterotic string models with (2, 2) internal superconformal theory, there is a standard procedure [15, 16, 5] by which the moduli fields can be unambiguously identified. This procedure can be applied to the (2, 2) models constructed in the free-fermionic formulation, and entails the existence of some massless scalar fields which can be viewed as the scalar components of the moduli fields. However, since we are interested in (2, 0) string models that can be constructed in the free-fermionic formulation, the above procedure is not applicable. In fact, there seems to be no universal way of identifying moduli in generic (2, 0) models [17]. Nevertheless, since free-fermionic models can be viewed as orbifold models with special values of the radii [7], and the untwisted moduli of various orbifold models have been worked out and are valid for both (2, 2) and (2, 0) cases, we are led to seek an analogy with the orbifold analysis.

It is crucial to recall that [18], in the context of conformal field theory (CFT), moduli fields correspond to exactly marginal operators which generate deformations of a CFT that preserve conformal invariance at the classical as well as quantum level. For symmetric orbifold models, the exactly marginal operators associated with the untwisted moduli fields take the general form \( \partial X^I \partial X^J \), where \( X^I \) for \( I = 1, \ldots, 6 \) are the coordinates of the six-torus \( T^6 \). Therefore, the untwisted moduli fields in such models admit the geometrical interpretation of background fields [19], which appear as couplings of the above exactly marginal operators in the non-linear sigma model action that is the generating functional for string scattering amplitudes [20]. Based on this interpretation, the untwisted moduli scalars are simply given by the background fields whose existence preserves the point group symmetry of the corresponding orbifolds, and the Kähler potential of these moduli fields or the metric of the moduli space can be determined accordingly using symmetry arguments [20, 21].

Note that in the Frenkel-Kac-Segal construction [22] of the Kac-Moody current algebra from chiral bosons, the operator \( i \partial X^I \) is nothing but a \( U(1) \) Cartan-subalgebra current. From the orbifold analogy, it is therefore natural for us to expect that the exactly marginal operators in the fermionic models should be given by Abelian Thirring operators of the form \( J_L(z) \bar{J}_R(\bar{z}) \), where \( J_L, \bar{J}_R \) are some \( U(1) \) chiral currents described by world-sheet fermions. Indeed, it has been shown [23] that Abelian Thirring interactions preserve conformal invariance, and a bosonic string model with general background fields [11] can be equivalently obtained via the fermionic formulation by introducing general Thirring interactions among the world-sheet fermions. In addition, a fermionic model with non-vanishing Thirring interactions may be re-

\[1\] We use the convention in which the left-moving sector is supersymmetric, whereas the right-moving sector is not.
formulated as a model with only free world-sheet fermions with highly non-trivial spin-structures [23], but in this case the requirement of modular invariance can not be simply solved as in Refs. [7, 8]. Thirring interactions have also been considered in the context of gauge symmetry-breaking at low energies in fermionic models [24, 25].

In fact, even without referring to the orbifold analogy, it is not hard to see that the Abelian Thirring operators \( J_L(z) \bar{J}_R(\bar{z}) \) satisfy the necessary and sufficient condition for integrability established in Ref. [26], and hence they are exactly marginal. It is interesting to note that the result of Ref. [26] also implies the existence of additional exactly marginal operators in certain fermionic models, and one should regard the Abelian Thirring operators as a minimal set of the exactly marginal operators generally available in fermionic models. In this paper, we confine ourselves to the untwisted moduli associated with this minimal set of exactly marginal operators, and we will use the orbifold analogy throughout to illuminate our results.

Our focus here is to show how can one use the Abelian Thirring interactions to identify the untwisted moduli in a class of fermionic models. Contrary to the orbifold case where the modular invariant solutions in the presence of non-trivial background fields and Wilson lines are known [27], we do not attempt to find the most general modular invariant solution with non-vanishing Thirring interactions in the fermionic language. Instead, we follow the strategy of Refs. [24, 25]. We start with a free-fermionic model with vanishing Thirring interactions, and then perturb around this particular vacuum by turning on some Thirring interactions. What we need in order to identify the untwisted moduli fields are the symmetry properties of the Thirring interactions. These can be determined in our perturbative analysis and are expected to hold even for vacua far away from the original one. Our main observation is that the untwisted moduli scalars in fermionic models correspond to the Abelian Thirring interactions which, if turned on, are compatible with the spin-structure of the free world-sheet fermions. In essence, this is similar to the starting point of Ref. [20], i.e., that a background action becomes the generating functional of string scattering amplitudes of a particular symmetric orbifold model, if and only if it is invariant under the action of the corresponding orbifold point group. We note that the analysis of Ref. [20] cannot be directly used for the case of asymmetric orbifolds [28], whereas our approach using Thirring interactions seems to offer a viable method. In fact, in this paper we use this approach to discuss the untwisted moduli in certain realistic fermionic models which can only be interpreted as asymmetric orbifolds.

3 Simple models: moduli and Kähler potential

To be concrete, in what follows we restrict ourselves to the fermionic models whose spin-structure-vector bases \( \mathcal{B} \) all contain a sub-basis \( \mathcal{B}_{N=4} \), which by itself would generate an \( N = 4 \) model with an \( SO(44) \) gauge group arising from the right-movers and an \( U(1)^6 \) gauge group from the left-movers. For the sake of simplicity, we consider models in which one can use six sets of left-moving real fermions \( (\chi^I, y^I, \omega_I) \) \((I = 1, \ldots, 6)\), each transforming in the adjoint representation of \( SU(2) \), to describe
the left-moving $N = 2$ superconformal symmetry dictated by $N = 1$ space-time supersymmetry. For instance, we can choose $\mathcal{B}_{N=4} = \{1, S\}$, where in vector 1 all world-sheet fermions are periodic, whereas in the “supersymmetry generating” vector $S$ only the two transverse $\psi^\mu$ and six $\chi^I$ world-sheet fermions are periodic and the rest are antiperiodic. It is helpful to view these four-dimensional fermionic models as the result of certain compactifications of the ten-dimensional heterotic string. In this view, with the choice of $S$ just given, we can identify the six $\chi^I$ with the fermionic superpartners of the six compactified bosonic coordinates $X^I$ on the six-torus $T^6$. Therefore each pair $(y^I, \omega^I)$ is nothing but the fermionized version of the left-moving mode of the corresponding $X^I$ itself, i.e., $i\partial X^I_L \sim y^I \omega^I$. Clearly, the sector produced by the sub-basis $\mathcal{B}_{N=4} = \{1, S\}$ corresponds to Narain’s generalized toroidal compactification \[12\], and the sub-sector which is invariant under orbifold twists gives rise to the untwisted sector of orbifold models.

Let us now consider the following two-dimensional action for the Abelian Thirring interactions

$$S = \int d^2 z h_{ij}(X) J^i_L(z) \bar{J}^j_R(\bar{z}),$$

where $J^i_L$ ($i = 1, \ldots, 6$) are the chiral currents of the left-moving $U(1)^6$, and $\bar{J}^j_R$ ($j = 1, \ldots, 22$) are the chiral currents of the right-moving $U(1)^{22}$. The couplings $h_{ij}(X)$, as functions of the space-time coordinates $X^\mu$, are four-dimensional scalar fields which we will identify with the scalar components of the untwisted moduli fields. In fact, for the simplest model with only the $N = 4$ sub-basis $\mathcal{B}_{N=4}$, the $6 \times 22$ fields $h_{ij}(X)$ in Eq. (1) are in one-to-one correspondence with the background field metric $G_{IJ}$, the antisymmetric tensor $B_{IJ}$ ($I, J = 1, \ldots, 6$), and the Wilson lines $A_{Ia}$ ($a = 1, \ldots, 16$). These $h_{ij}(X)$ fields are precisely the moduli scalars of toroidal compactification, which parameterize the coset space $SO(6, 22)/SO(6) \times SO(22)$ \[13\]. This simple case was first discussed in Ref. \[7\].

We now point out that all these moduli scalars are indeed present in the massless spectrum of the free-fermionic model given by the basis $\{1, S\}$. The massless scalar states of this model are those from the Neveu-Schwarz sector, which transform in the adjoint representation of $SO(44)$ and can be given in terms of 22 right-moving complex fermions $\Psi^+ A$ and their complex conjugates $\Psi^- A$ as ($1 \leq A \neq B \leq 22$)

$$|\chi^I_J \rangle \otimes |\Psi^+ A \Psi^- A \rangle;$$

$$|\chi^I_J \rangle \otimes |\Psi^\pm A \Psi^\mp B \rangle.$$  

The $6 \times 22$ states in the Cartan subalgebra (those in (2)) are the massless moduli fields $h_{ij}(X)$ of this model ($i = I$ and $j = A$), and the corresponding marginal operators can be given as (up to normalization constants)

$$J^i_L(z) \bar{J}^j_R(\bar{z}) : = y^i(z) \omega^j(\bar{z}) : : \Psi^+ i(\bar{z}) \Psi^- j(\bar{z}) :.$$  

In Eq. (4) the form of the left-moving chiral current $J^i_L(z)$ is obtained from the fermionization of the six compactified coordinates $i\partial X^i_L \sim y^i \omega^i$, which is also dictated
by the local $N = 1$ internal world-sheet supercurrent

$$T_F^{\text{int}} = i \sum_I \chi^I y^I \omega^I. \quad (5)$$

In fact, from the following OPE

$$T_F(w)\chi^I(z) \sim \frac{1}{w - z} y^I(z)\omega^I(z), \quad (6)$$

one can see that the marginal operator $J_L^I(z)\bar{J}_R^I(\bar{z})$ in Eq. (4) is the same as the zero-momentum vertex operator for the associated moduli scalars $h_{ij}$ of Eq. (2) in the 0-ghost picture.

We next consider a model with basis $B = \{1, S, b_1, b_2, b_3\}$, where

$$S = (1 100 100 100 100 100 100 : 000000 000000 000000 000000 000000 000000 000000), \quad (7)$$

$$b_1 = (1 100 100 010 010 010 010 : 001111 000000 11111 100 000000), \quad (8)$$

$$b_2 = (1 010 010 100 100 001 001 : 110000 000011 11111 010 000000), \quad (9)$$

$$b_3 = (1 001 001 001 001 100 100 : 000000 111100 11111 001 000000). \quad (10)$$

In writing these spin-structure vectors we have separated the right-moving fermions into 12 real fermions consisting of six pairs $\bar{y}^I, \bar{\omega}^I$ ($I = 1, \ldots, 6$), and the rest are treated as 16 complex fermions $\bar{\Psi}^{\pm a}$ ($a = 1, \ldots, 16$). This separation amounts to a decomposition of the group $SO(44)$ into its subgroup $SO(12) \times SO(32)$. With this separation we can now use the bosonic analogy and consider the six pairs $\bar{y}^I, \bar{\omega}^I$ as the fermionization of the right-moving modes of $X^I$. Therefore we can define

$$\bar{J}_R^j(\bar{z}) =: \bar{y}^j(\bar{z})\bar{\omega}^j(\bar{z}) : \quad (j = 1, \ldots, 6), \quad (11)$$

and

$$\bar{J}_R^j(\bar{z}) =: \bar{\Psi}^{+(j-6)}(\bar{z})\bar{\Psi}^{-(j-6)}(\bar{z}) : \quad (j = 7, \ldots, 22). \quad (12)$$

We can also rewrite the massless states in the Cartan subalgebra (2) as two sets:

$$(a) \quad |\chi^I\rangle \otimes |\bar{y}^I\bar{\omega}^I\rangle \quad (I = 1, \ldots, 6); \quad (13)$$

$$(b) \quad |\chi^I\rangle \otimes |\bar{\Psi}^{a}\bar{\Psi}^{-a}\rangle \quad (a = 1, \ldots, 16); \quad (14)$$

which are the massless states in the Cartan subalgebras of $SO(12)$ and $SO(32)$ respectively. Clearly, using right-moving currents of the form in Eq. (11) and (12), we would get the same result for the simple $N = 4$ model as we did using (4). In the models with additional spin-structure basis vectors, some chiral currents ($J_L^I$ or $\bar{J}_R^I$) become antiperiodic, and as a result certain terms in the general Thirring action (4) are not invariant when the world-sheet fermions are parallel-transported around the

\[^2\text{See Ref. [29] for our notation.}\]
noncontractible loops of the world-sheet. Such terms are inconsistent with the spin-structure of the fermions and are therefore forbidden. We can view each additional spin-structure vector as introducing a “GSO-projection” on the Thirring action (1), just as what these vectors do for physical string states, and only those terms in action (1) which survive such “GSO-projections” lead to the compatible exactly marginal operators and thus the untwisted moduli fields of the model.

For example, from the form of $b_1$ it is easy to get the following boundary conditions of the chiral currents:

$$J_{L}^{1,2} \rightarrow J_{L}^{1,2}, \quad J_{L}^{3,4,5,6} \rightarrow -J_{L}^{3,4,5,6};$$

$$J_{R}^{1,2} \rightarrow J_{R}^{1,2}, \quad J_{R}^{3,4,5,6} \rightarrow -J_{R}^{3,4,5,6},$$

whereas $\bar{J}_{R}^{j}(j = 7, \ldots, 22)$ are always periodic. Hence the Thirring terms consistent with $b_1$ are simply

$$J_{L}^{1,2} \bar{J}_{R}^{1,2}, \quad J_{L}^{3,4,5,6} \bar{J}_{R}^{3,4,5,6}, \quad J_{L}^{5,6} \bar{J}_{R}^{5,6},$$

Following the symmetry argument of Ref. [21], we see that for model $B = \{1, S, b_1\}$, the untwisted moduli scalars $h_{ij}^{(1)} (i = 1, 2; j = 1, 2$ and $7, \ldots, 22)$ and $h_{ij}^{(2)} (i, j = 3, 4, 5, 6)$ span a coset space

$$M = \frac{SO(2, 2 + 16)}{SO(2) \times SO(2 + 16)} \otimes \frac{SO(4, 4)}{SO(4) \times SO(4)}.$$

Working out the massless string states in Eqs. (13) and (14) that survive the GSO-projection due to $b_1$, it is easy to see that these states are indeed in one-to-one correspondence with the marginal operators in Eq. (17), as it should be. Again, these marginal operators give the 0-ghost picture vertex operators of the corresponding string states at zero momentum.

Carrying out the same analysis for $b_2$ and $b_3$, we find that the model $B = \{1, S, b_1, b_2, b_3\}$ has only the following allowed Thirring terms

$$J_{L}^{1,2} \bar{J}_{R}^{1,2}, \quad J_{L}^{3,4} \bar{J}_{R}^{3,4}, \quad J_{L}^{5,6} \bar{J}_{R}^{5,6},$$

and thus we get the following moduli space

$$M = \frac{SO(2, 2)}{SO(2) \times SO(2)} \otimes \frac{SO(2, 2)}{SO(2) \times SO(2)} \otimes \frac{SO(2, 2)}{SO(2) \times SO(2)};$$

parameterized by the following three sets of untwisted moduli scalars

$$h_{ij} = |\chi^{i} \otimes \bar{\omega}^{j}|$$

with

$$\begin{align*}
(1) & \quad i, j = 1, 2 \\
(2) & \quad i, j = 3, 4 \\
(3) & \quad i, j = 5, 6
\end{align*}$$

Thus far our results are consistent with those of Refs. [9, 10]. However, in Refs. [4, 10], apart from the dilaton field, all the other scalar fields from the Neveu-Schwarz
sector were treated collectively, whereas we have been able to identify a subset of these scalar fields (those given in Eq. (21)) as the untwisted moduli scalars, and the moduli space (21) is just a subspace of the full Kähler manifold underlying the non-linear sigma model of the untwisted sector. In order to work out the low-energy effective field theory for a string model, for instance the one given by $\mathcal{B} = \{1, S, b_1, b_2, b_3\}$, it is necessary to know not only what Kähler manifold the scalars span, but also how the scalars actually parameterize the corresponding Kähler manifold. In other words, one would like to know precisely the form of the Kähler potential or equivalently the Kähler metric in terms of the scalar fields of concern. We next investigate this issue for the untwisted moduli scalars given in Eq. (21).

Let us first briefly mention one property of the coset space $SO(2,2)/SO(2) \times SO(n)$ ($n \geq 1$). One special parameterization of this coset space, among many different possibilities [30], is to consider it as a bounded subdomain of $\mathbb{C}^n$ that obeys the conditions

$$\left| \sum_{i}^{n} \alpha_i^2 \right| < \frac{1}{2}, \quad 1 - \sum_{i}^{n} \alpha_i \bar{\alpha}_i + \frac{1}{4} \left| \sum_{i}^{n} \alpha_i^2 \right|^2 > 0.$$  \hspace{1cm} (22)

In this parameterization, the coset space $SO(2,2)/SO(2) \times SO(n)$ assumes the following standard Kähler potential [31]

$$K(\alpha_i, \bar{\alpha}_i) = -\log \left( 1 - \sum_{i}^{n} \alpha_i \bar{\alpha}_i + \frac{1}{4} \left| \sum_{i}^{n} \alpha_i^2 \right|^2 \right).$$  \hspace{1cm} (23)

For what follows it is helpful to expand the Kähler potential in powers of the fields. Up to fourth-order we get

$$K(\alpha_i, \bar{\alpha}_i) \approx \sum_{i}^{n} \alpha_i \bar{\alpha}_i + \frac{1}{4} \sum_{i}^{n} \alpha_i^2 \bar{\alpha}_i^2 + \sum_{i<j}^{n} (\alpha_i \bar{\alpha}_j \alpha_j \bar{\alpha}_j - \frac{1}{4} \alpha_i^2 \bar{\alpha}_j^2 - \frac{1}{4} \alpha_j^2 \bar{\alpha}_i^2),$$  \hspace{1cm} (24)

and the second and fourth derivatives become

$$K_{\alpha_i \bar{\alpha}_i} = 1 + \sum_{j}^{n} \alpha_j \bar{\alpha}_j, \quad K_{\alpha_i \bar{\alpha}_j} = -\alpha_i \bar{\alpha}_j + \bar{\alpha}_i \alpha_j, \quad (i \neq j)$$  \hspace{1cm} (25)

and $(i \neq j)$

$$K_{\alpha_i \alpha_i, \bar{\alpha}_i \alpha_i} = K_{\alpha_i \bar{\alpha}_i, \alpha_j \bar{\alpha}_j} = K_{\alpha_i \bar{\alpha}_i, \alpha_j \alpha_j} = 1, \quad K_{\alpha_i \bar{\alpha}_i, \alpha_j \bar{\alpha}_j} = -1.$$  \hspace{1cm} (26)

We now would like to show that the four real moduli scalars of each set in Eq. (21) provide precisely the very special parameterization of $SO(2,2)/SO(2) \times SO(2)$ in

\footnote{Here we have chosen the complex coordinate system $\alpha_i (i = 1, \ldots, n)$ such that it is canonical at the origin.}

\footnote{We adopt the notation that subscripts on the Kähler potential denote derivatives with respect to the corresponding fields, \textit{e.g.}, $K_\Phi = \partial_\Phi K$.}
Eq. (23). Because of the symmetric structure of the moduli space (20), we can just consider the first set as an example. As it is often done in string calculations of this class of models [32], we define naturally the following two complex fields

\[ H^{(1)}_1 = \frac{1}{\sqrt{2}} (h_{11} + ih_{21}) = \frac{1}{\sqrt{2}} |\chi^1 + i\chi^2\rangle \otimes |\bar{y}^1\bar{\omega}^1\rangle, \]

(27)

\[ H^{(1)}_2 = \frac{1}{\sqrt{2}} (h_{12} + ih_{22}) = \frac{1}{\sqrt{2}} |\chi^1 + i\chi^2\rangle \otimes |\bar{y}^2\bar{\omega}^2\rangle. \]

(28)

Then, using the methods of Ref. [32], we obtain the following non-vanishing string scattering amplitudes

\[ \mathcal{A}(H_1, H_2, \bar{H}_2, H_1) = \mathcal{A}(H_2, H_1, \bar{H}_1, \bar{H}_2) \]

\[ = -\frac{g^2}{4} \frac{\Gamma(-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(s/8)\Gamma(t/8)\Gamma(u/8)} \left\{ \frac{s}{1 + t/8} + \frac{su}{t(1 + t/8)} \right\}, \]

(29)

\[ \mathcal{A}(H_1, H_2, \bar{H}_1, \bar{H}_2) = \mathcal{A}(H_2, H_1, \bar{H}_2, \bar{H}_1) \]

\[ = -\frac{g^2}{4} \frac{\Gamma(-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(s/8)\Gamma(t/8)\Gamma(u/8)} \left\{ \frac{s}{1 + u/8} + \frac{st}{u(1 + u/8)} + \frac{su}{t(1 + t/8)} - \frac{s}{1 + s/8} \right\}, \]

(30)

\[ \mathcal{A}(H_1, H_2, \bar{H}_1, \bar{H}_1) = \mathcal{A}(H_2, H_2, \bar{H}_2, \bar{H}_2) \]

\[ = -\frac{g^2}{4} \frac{\Gamma(-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(s/8)\Gamma(t/8)\Gamma(u/8)} \left\{ \frac{s}{1 + s/8} \right\}. \]

(31)

\[ \mathcal{A}(H_1, H_2, \bar{H}_2, \bar{H}_2) = \mathcal{A}(H_2, H_2, \bar{H}_1, \bar{H}_1) \]

\[ = g^2 \frac{\Gamma(-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(s/8)\Gamma(t/8)\Gamma(u/8)} \left\{ \frac{s}{1 + s/8} \right\}. \]

(32)

To quadratic order in the momenta, Eqs. (31)-(32) become

\[ \mathcal{A}(H_1, H_2, \bar{H}_2, \bar{H}_1) = \mathcal{A}(H_2, H_1, \bar{H}_1, \bar{H}_2) = \frac{g^2}{4} \left\{ \frac{su}{t} + s \right\}, \]

(33)

\[ \mathcal{A}(H_1, H_2, \bar{H}_1, \bar{H}_2) = \mathcal{A}(H_2, H_1, \bar{H}_2, \bar{H}_1) = \frac{g^2}{4} \left\{ \frac{st}{u} + s \right\}, \]

(34)

\[ \mathcal{A}(H_1, H_1, \bar{H}_1, \bar{H}_1) = \mathcal{A}(H_2, H_2, \bar{H}_2, \bar{H}_2) = \frac{g^2}{4} \left\{ \frac{st}{u} + \frac{su}{t} + s \right\}, \]

(35)

\[ \mathcal{A}(H_1, H_1, \bar{H}_2, \bar{H}_2) = \mathcal{A}(H_2, H_2, \bar{H}_1, \bar{H}_1) = \frac{g^2}{4} \left\{ -s \right\}. \]

(36)

We now compare the string scattering amplitudes (33)-(36) with those that would be obtained from $N = 1$ supergravity calculations [3, 33]. For this purpose, we only need
the scattering amplitudes due to sigma-model interactions as well as gravity, namely
\[ A(H_i, H_j, \bar{H}_k, \bar{H}_l) \propto \left( \frac{su}{t} \delta_{ik} \delta_{jl} + \frac{st}{u} \delta_{ik} \delta_{jl} + sK_{H_i, \bar{H}_k, H_l, \bar{H}_l} \right). \]  
(37)

This comparison yields
\begin{align*}
K_{H_1, \bar{H}_1, H_2, \bar{H}_2} &= K_{H_2, \bar{H}_2, H_1, \bar{H}_1} = K_{H_1, \bar{H}_1, H_2, \bar{H}_2} = 1, \\
K_{H_1, \bar{H}_2, H_2, \bar{H}_1} &= K_{H_2, \bar{H}_1, H_1, \bar{H}_2} = 1, \\
K_{H_1, \bar{H}_2, H_1, \bar{H}_2} &= K_{H_2, \bar{H}_1, H_2, \bar{H}_1} = -1,
\end{align*}
(38-40)
which is consistent with Eq. (26) for the case of \( n = 2 \), and \( \alpha_{1,2} = H_{1,2} \). We note in passing that, since in our current case the exactly marginal operators corresponding to the untwisted moduli fields are known, one can also perturbatively compute the Kähler metric of the moduli space by calculating the Zamolodchikov metric \([34] \) with these exactly marginal operators, which we expect will lead to the same results.

From the above analysis, for the model \( \mathcal{B} = \{1, S, b_1, b_2, b_3\} \), the Kähler potential of the untwisted moduli takes the following explicit form
\[ K(H, \bar{H}) = -\sum_{i=1}^{3} \log \left( 1 - \sum_{j=1,2} H_j^{(i)} \bar{H}_j^{(i)} + \frac{1}{4} \left| \sum_{j=1,2} H_j^{(i)} \bar{H}_j^{(i)} \right|^2 \right), \]
(41)
where the complex fields \( H_{1,2}^{(2)} \) and \( H_{1,2}^{(3)} \) are defined in a similar fashion as Eqs. (27) and (28).

4 Comparison with \( Z_2 \times Z_2 \) orbifolds

It is instructive to revisit the results in Sec. 3 for the fermionic string models in light of the orbifold analogy. First of all, the coset space \( SO(2,2)/SO(2) \times SO(2) \) is a reducible Kähler manifold, since \([34]\)
\[ \frac{SO(2,2)}{SO(2) \times SO(2)} \simeq SU(1,1) \times SU(1,1). \]
(42)

Therefore, the untwisted moduli space \([20]\) for the model \( \mathcal{B} = \{1, S, b_1, b_2, b_3\} \) can be written in the following more familiar form
\[ \mathcal{M} = \left[ \frac{SU(1,1)}{U(1)} \otimes \frac{SU(1,1)}{U(1)} \right]^3. \]
(43)
This is exactly the untwisted moduli space for the symmetric \( Z_2 \times Z_2 \) orbifold model \([35]\), which can be obtained by using the method of Ref. [20]. Furthermore, we can
make a linear transformation which maps each pair of the original complex string fields $H_{1,2}^{(i)}$ into another pair $\Phi_{T,U}^{(i)}$. For the first set we have

$$\Phi_T^{(1)} = \frac{1}{\sqrt{2}} (H_1^{(1)} - i H_2^{(1)}) = \frac{1}{\sqrt{2}} (\chi^1 + i \chi^2) \otimes \frac{1}{\sqrt{2}} |\bar{y}^1 \bar{\omega}^1 - i \bar{y}^2 \bar{\omega}^2\rangle,$$

$$\Phi_U^{(1)} = \frac{1}{\sqrt{2}} (H_1^{(1)} + i H_2^{(1)}) = \frac{1}{\sqrt{2}} (\chi^1 + i \chi^2) \otimes \frac{1}{\sqrt{2}} |\bar{y}^1 \bar{\omega}^1 + i \bar{y}^2 \bar{\omega}^2\rangle. \quad (44)$$

The other two sets, $\Phi_{T,U}^{(2)}$ and $\Phi_{T,U}^{(3)}$, are defined analogously. Note that the $T$-type fields are not the complex conjugates of the corresponding $U$-type fields; they are two independent complex fields.

In terms of the new complex fields $\Phi_{T,U}^{(i)}$, the Kähler potential of the untwisted moduli fields of the model $B = \{1, S, b_1, b_2, b_3\}$, given in Eq. (41) in Sec. 3, can be rewritten as

$$K(\Phi, \bar{\Phi}) = -\sum_{i=1}^{3} \log \left(1 - \Phi_T^{(i)} \bar{\Phi}_T^{(i)}\right) - \sum_{i=1}^{3} \log \left(1 - \Phi_U^{(i)} \bar{\Phi}_U^{(i)}\right). \quad (46)$$

In Eq. (46) the $T$-type and $U$-type fields are completely separated, and each $T$-type (or $U$-type) field provides a bounded parameterization of the coset space $SU(1,1)/U(1)$.

Of course, the reason that we can define complex moduli fields $\Phi_{T,U}^{(i)}$ with this property for our fermionic model is precisely the isomorphism [12].

Once again, Eq. (46) can be confirmed by computing the string scattering amplitudes of four moduli fields, as we did in Sec. 3. This time we find that the non-vanishing string scattering amplitude is the one involving the same four $T$-type (or $U$-type) fields, given by

$$A(\Phi, \Phi, \bar{\Phi}, \bar{\Phi}) = -\frac{g^2}{4} \frac{\Gamma(-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(s/8)\Gamma(t/8)\Gamma(u/8)} \times \left\{ \frac{s}{1 + u/8} + \frac{s}{1 + t/8} + \frac{st}{u(1 + u/8)} + \frac{su}{t(1 + t/8)} \right\}, \quad (47)$$

which in the low-energy limit becomes (to quadratic order in the momenta)

$$A(\Phi, \Phi, \bar{\Phi}, \bar{\Phi}) = \frac{g^2}{4} \left( \frac{st}{u} + \frac{su}{t} + 2s \right). \quad (48)$$

From Eq. (48) one can infer that the metric of the moduli space has only the following non-vanishing component [33]

$$K_{\Phi\bar{\Phi}} = 1 + 2\Phi\bar{\Phi} \approx \frac{1}{(1 - \Phi\bar{\Phi})^2}, \quad (49)$$

which is exactly the standard Fubini–Study metric of $SU(1,1)/U(1)$ derived from the Kähler potential (46).
The above results are expected because our fermionic model with \( b_1, b_2 \) and \( b_4 \) can be regarded as a symmetric \( Z_2 \times Z_2 \) orbifold model. It was first noted in Ref. [10] that in the type of fermionic models we are considering there is a \( Z_2 \) orbifold structure.

The connection between such fermionic models and the \( Z_2 \times Z_2 \) orbifold model was explained in Ref. [46] (see also Ref. [47]). To clearly see this, let us first carry out explicitly the bosonization described in Sec. 3.

\[
e^{i\chi_L} = \frac{1}{\sqrt{2}}(y^I + i\omega^I), \quad e^{i\chi_R} = \frac{1}{\sqrt{2}}(\bar{y}^I + i\bar{\omega}^I) \quad (I = 1, \ldots, 6).
\]

We then form the three complex planes as follows (\( X = X_L + X_R \))

\[
Z^\pm_k = \frac{1}{\sqrt{2}}(X^{2k-1} \pm iX^{2k}), \quad \psi^\pm_k = \frac{1}{\sqrt{2}}(\chi^{2k-1} \pm i\chi^{2k}) \quad (k = 1, 2, 3),
\]

where the \( Z^\pm_k \) are the complex coordinates of the six compactified dimension now viewed as three complex planes, and \( \psi^\pm_k \) are the corresponding superpartners. From (50) and (51), one see that \( b_1 \) can be interpreted as the twist \( \theta \) of a symmetric \( Z_2 \times Z_2 \) orbifold which keeps the first complex plane unrotated, but rotates the second and the third ones simultaneously by \( \pi \). Indeed, under \( b_1 \) we have \( y^{1,2} \to -y^{1,2}, \omega^{1,2} \to -\omega^{1,2}, \) and thus \( e^{i\chi_L^{1,2}} \to -e^{i\chi_L^{1,2}} = e^{i(X_L^{1,2} + \pi)} \). Whereas \( y^{3,4,5,6} \to y^{3,4,5,6}, \omega^{3,4,5,6} \to -\omega^{3,4,5,6}, \) and thus \( e^{i\chi_L^{3,4,5,6}} \to e^{-i\chi_L^{3,4,5,6}}. \)

The right-moving modes have the same transformations, because of the form of \( b_1 \) chosen. This fact indicates that the orbifold is symmetric. Therefore, the first complex plane is only shifted, \( i.e., Z_1^\pm \to Z_1^\pm + \text{shift} \), but the other two are rotated, \( i.e., Z_2^\pm \to -Z_2^\pm = e^{i\pi}Z_2^\pm. \) Also, from \( \chi^{1,2} \to \chi^{1,2} \) and \( \chi^{3,4,5,6} \to -\chi^{3,4,5,6} \) one gets \( \psi_1^\pm \to \psi_1^\pm \) and \( \psi_2^\pm \to -\psi_2^\pm. \)

Analogously, one can show that \( b_2 \) is the twist \( \omega \) which rotates the first and the third complex planes by \( \pi \), and finally, \( b_3 \) is the twist \( \theta \omega \) that keeps the third complex plane fixed. As mentioned in Sec. 3, it is this connection between fermionic models and orbifold models that inspired our approach.

Let us make use of this connection a bit further. In terms of the complex coordinates \( Z^\pm_k \), we can rewrite the allowed Thirring interaction terms for the untwisted moduli of Eq. (21) in bosonic form. For instance, we have for the first set

\[
\sum_{i,j=1,2} h_{ij} J^i_L J^j_R = \Phi_T^{(1)} \partial Z_1^- \bar{\partial} Z_1^+ + \bar{\Phi}_T^{(1)} \partial Z_1^+ \bar{\partial} Z_1^- + \Phi_U^{(1)} \partial Z_1^- \bar{\partial} Z_1^- + \bar{\Phi}_U^{(1)} \partial Z_1^+ \bar{\partial} Z_1^+,
\]

where \( \Phi_T^{(1)}, \bar{\Phi}_T^{(1)} \) are the complex fields defined in Eqs. (14) and (15). Obviously the same can be done for the other two sets. From Eq. (52), we immediately see that the \( T \)-type moduli field \( \Phi_T \) is associated with the exactly marginal operator \( \partial Z^- \bar{\partial} Z^+ \) which deforms the Kähler class of the compact space, and the \( U \)-type moduli field \( \Phi_U \) is associated with the operator \( \partial Z^- \bar{\partial} Z^- \) which deforms the complex structure of the compact space. Therefore, the above orbifold analogy allows us to assign the geometrical meaning to the \( T \)-type moduli as those corresponding to the harmonic \((1,1)\) forms of the compact space, and the \( U \)-moduli as those corresponding to the
forms. The number of such fields are given by the non-trivial Hodge numbers of
the compact space, which for the case of the symmetric $Z_2 \times Z_2$ orbifold are $h^{(1,1)} = 3$
and $h^{(2,1)} = 3 \ [35]$. This is the reason why we get precisely three sets of $\Phi^{(i)}_T$ and
three sets of $\Phi^{(i)}_U$ in the model $\mathcal{B} = \{1, S, b_1, b_2, b_3\}$. In fact, from the previous discussion
it should be clear that the properties of the untwisted moduli which we have derived
are not only valid in this simple model, but remain valid in any fermionic model
with additional spin-structure vectors, as long as these new vectors do not spoil the
symmetric $Z_2 \times Z_2$ orbifold structure. Examples of such fermionic models can be
found, e.g., in Ref. [38].

5 A new feature in realistic models

Phenomenologically realistic fermionic models have been constructed by adding more
spin-structure vectors to the basis $\mathcal{B}$ \ [39, 40, 41, 29]. In these models, there are
spin-structure vectors which assign asymmetrically the boundary conditions for the
left-moving real fermions $y^I, \omega^I$ relative to the right-moving ones $\bar{y}^I, \bar{\omega}^I$, so that the
left-moving mode of some compactified coordinate $X^I_L$ and the corresponding right-
moving mode $X^I_R$ will be twisted differently. From the discussion in Sec. 4, it is clear
that such models should be interpreted as asymmetric orbifolds \ [28]. We now address
this new feature of such models in connection with the determination of the untwisted
moduli fields.

It is convenient to consider a concrete and typical example, for which we choose the “revamped” flipped $SU(5)$ model \ [39]. In this model, the “asymmetry” of the twist is introduced by the following basis vector

$$\alpha = (0 000 000 000 101 001 1101 011101 \ 111111 \ 111111 \ 111111 \ 111111 \ 1100).$$  

Now consider the remaining Thirring terms in Eq. (13). Under this vector $\alpha$, the
left-moving current $J^2_L$ is periodic but its right-moving counterpart $J^2_R$ is antiperiodic;
the same holds for $J^3_L$ and $J^3_R$. Note also that currents $J^1_L$ and $J^4_L$ remain periodic. As
a result, operators $J^1_L J^2_R$ and $J^3_L J^4_R$ and no longer consistent with the spin-structure
of the model that contains the vector $\alpha$. In fact, it is easy to check that the string
states $h_{12}, h_{33}, h_{43}$ and $h_{43}$ that were present in Eq. (21) do not exist in the massless
spectrum of this model any more. They have been projected out of the spectrum by
precisely the GSO-projection due to $\alpha$.

In this model, therefore, the remaining Thirring terms are

$$J^{1,2}_{L} J^{1}_{R}, \ J^{3,4}_{L} J^{4}_{R}, \ J^{5,6}_{L} J^{5,6}_{R},$$  

and thus the untwisted moduli space reduces to

$$M = \frac{SO(2, 1)}{SO(2)} \otimes \frac{SO(2, 1)}{SO(2)} \otimes \frac{SO(2)}{SO(2) \times SO(2)},$$  

13
which because of the isomorphisms \((12)\) and \([\,]\)
\[
\frac{SO(2,1)}{SO(2)} \simeq \frac{SU(1,1)}{U(1)} ,
\]
is isomorphic to
\[
\mathcal{M} = \frac{SU(1,1)}{U(1)} \otimes \frac{SU(1,1)}{U(1)} \otimes \left[ \frac{SU(1,1)}{U(1)} \otimes \frac{SU(1,1)}{U(1)} \right].
\]

Using the complex notation \(H_1^{(i)}\) defined in Sec. 3, we see that the untwisted moduli fields for this model are just
\[
H_1^{(1)}, \quad H_2^{(2)}, \quad H_{1,2}^{(3)},
\]
and they give the following Kähler potential (see Eq. (11))
\[
K(H, \bar{H}) = -2 \log \left( 1 - \frac{1}{2} H_1^{(1)} \bar{H}_1^{(1)} \right) - 2 \log \left( 1 - \frac{1}{2} H_2^{(2)} \bar{H}_2^{(2)} \right)
- \log \left( 1 - \sum_{j=1,2} H_j^{(3)} \bar{H}_j^{(3)} + \frac{1}{4} \left| \sum_{j=1,2} H_j^{(3)} H_j^{(3)} \right|^2 \right).
\]

In this model, the third set in Eq. (21) is intact, which gives one \(T\)-type field \(\Phi_{(3)}^T\)
and one \(U\)-type field \(\Phi_{(3)}^U\). But the first and second sets each give only one complex modulus field, and one can no longer attribute to them the geometrical meaning of either \(T\)-type ((1,1) form) or \(U\)-type ((2,1) form), simply because now the argument which leads to Eq. (52) does not apply. However, because of the isomorphism \((56)\), it should still be possible to find a new parameterization of \(SO(2,1)/SO(2)\), such that the first two terms in Eq. (59) can be recast into the Fubini-Study form of \(SU(1,1)/U(1)\). The desired parameterization is provided by the following transformation
\[
\Phi = H \left( 1 - \frac{1}{4} H \bar{H} \right)^{-\frac{1}{2}}.
\]
Transforming \(H_1^{(1)} (H_2^{(2)})\) into \(\Phi_1 (\Phi_2)\) according to \((60)\), and also denoting \(\Phi_{(3)}^{T,U}\) simply by \(\Phi_{3,4}\), the Kähler potential \((59)\) becomes
\[
K(\Phi, \Phi) = -\sum_{i=1}^{4} \log \left( 1 - \Phi_i \bar{\Phi}_i \right),
\]
to be contrasted with that obtained before introducing the vector \(\alpha\), \(i.e., Eq. (41)\).
6 The untwisted matter fields

In addition to the untwisted moduli which correspond to the Cartan subalgebra at the $N = 4$ level (see Eq.(4)), there are other massless scalar fields in the Neveu-Schwarz sector which correspond to the non-zero roots of $D_{22}$ at the $N = 4$ level (see Eq.(3)). Contrary to the untwisted moduli, these scalar fields in general are not associated with exactly marginal operators, and for this reason they should be treated as untwisted matter fields. For the class of $N = 1$ fermionic string models we are considering, as we have shown in previous sections, the untwisted moduli fields split up into three sets, each of which parameterizes a coset space: $SO(2, 2)/SO(2) \times SO(2)$ or $SO(2, 1)/SO(2)$. Similarly, in such $N = 1$ models the untwisted matter fields that survive the various GSO-projections also fall into three sets. It is interesting to investigate the non-linear sigma model structure of these untwisted matter fields.

This problem in fact has already been partially solved. In Refs. [9, 10], by using a truncation method, it was shown that each set of scalar fields from the Neveu-Schwarz sector admits a non-linear sigma model structure of $SO(2, n)/SO(2) \times SO(n)$, where $n$ counts the total number of the scalar fields in the set, which we now know includes both moduli and matter fields. Furthermore, the Kähler potential of the form (23) was also written down in Ref. [10] by solving the constraints satisfied by the representative fields for the $N = 4$ matter scalar manifold [13], although the complex coordinate system used there is not canonical at the origin. Ref. [10] obtained another form of the Kähler potential by solving the constraints in a slightly different way. What was not addressed in these papers is the issue of how the canonical complex coordinates $\alpha_i$ ($i = 1, \ldots, n$) which appear in the Kähler potential (23) are related to the actual massless string states. We now elucidate this relation.

In Sec. 3, we established this relation for the moduli fields by calculating various string scattering amplitudes. In this case the complex coordinates $\alpha_{1,2}$ are simply given by the string states $H_{1,2}$ defined in Eqs. (27) and (28). For the matter fields one would expect that such direct relations also hold. That is, one can just write down a string state in the most natural way, and then identify it with a coordinate $\alpha$. Interestingly enough, we found that this is not always the case. For matter states whose right-moving part consists of only real fermionic oscillators, e.g., a state with $|\tilde{y}^I\tilde{\omega}^J\rangle$ ($I \neq J$), the string states themselves give the coordinates $\alpha$'s. However, it is quite common for matter states to have the right-moving part consist of complex fermionic oscillators, of the form $|\tilde{\psi}^{\pm a}\tilde{\psi}^{\pm b}\rangle$ ($a \neq b$), $|\tilde{y}^I\tilde{\psi}^{\pm a}\rangle$, or $|\tilde{\omega}^I\tilde{\psi}^{\pm a}\rangle$. Such matter fields always come in pairs, such that the right-moving oscillators in each pair are complex conjugates of each other. (The fields themselves are distinct since their left-moving oscillators are the same.) We found that for each such pair of matter fields, the two corresponding canonical complex coordinates entering in the Kähler potential (23) are given by the real and imaginary parts of the right-moving oscillators respectively, with the complex left-moving part untouched. In this sense, one can regard such pairs of string matter states as corresponding to the pair $\Phi_{T,U}$ in (44) and (45), and then the procedure of finding the canonical complex coordinates simply corresponds to finding $H_{1,2}$ in terms of $\Phi_{T,U}$ from Eqs. (44) and (45).
We now illustrate the above discussion in the “revamped” flipped $SU(5)$ model \[39\]. In the notation of Ref. \[39\], the matter fields in the first set of this model are: 

\[
\Phi_{23} = \frac{1}{\sqrt{2}} |\chi^1 + i\chi^2| \otimes |\bar{\Psi}^{-7}\bar{\Psi}^{-8}>, \quad \bar{\Phi}_{23} = \frac{1}{\sqrt{2}} |\chi^1 + i\chi^2| \otimes |\bar{\Psi}^7\bar{\Psi}^8>,
\]

\[
\bar{h}_1 = \frac{1}{\sqrt{2}} |\chi^1 + i\chi^2| \otimes |\bar{\Psi}^{-a}\bar{\Psi}^6>, \quad \tilde{h}_1 = \frac{1}{\sqrt{2}} |\chi^1 + i\chi^2| \otimes |\bar{\Psi}^{a}\bar{\Psi}^{-6}>
\]

Note that $\tilde{h}_1$ is not the complex conjugate of $h_1$: under $SU(5) \times U(1)$ $h_1$ transforms as $(5, 1)$, whereas $\tilde{h}_1$ transforms as $(5, -1)$. Similarly, $\Phi_{23}$ and $\bar{\Phi}_{23}$ are different fields, which transform as singlets of $SU(5) \times U(1)$, but carry additional $U(1)$ charges. In what follows, when discussing specifically about this model, we denote the complex conjugate of field $\Phi$ by $\Phi^\dagger$ (instead of $\bar{\Phi}$) to avoid possible confusions with the notation.

The untwisted matter fields in the first set consist of $12 = 5+5+1+1$ complex degrees of freedom, which combined with the one modulus field $\Phi_1 = H_1^{(1)}$ (see Sec. \[39\]), altogether parameterize a coset space $SO(2,1+12)/[SO(2) \times SO(1+12)]$.

Let us consider the matter fields $\Phi_{23}$ and $\bar{\Phi}_{23}$ in Eq. (62). To quadratic order in the momenta, the non-vanishing string scattering amplitudes involving only these two fields are:

\[
A(\Phi_{23}, \Phi_{23}, \bar{\Phi}_{23}, \bar{\Phi}_{23}) = A(\Phi_{23}, \Phi_{23}, \Phi_{23}, \Phi_{23}) = \frac{g^2}{4} \left( \frac{su}{t} - 16\frac{s}{t} \right),
\]

\[
A(\Phi_{23}, \Phi_{23}, \bar{\Phi}_{23}, \bar{\Phi}_{23}) = A(\Phi_{23}, \Phi_{23}, \bar{\Phi}_{23}, \bar{\Phi}_{23}) = \frac{g^2}{4} \left( \frac{st}{u} - 16\frac{s}{u} \right),
\]

\[
A(\Phi_{23}, \Phi_{23}, \Phi_{23}, \Phi_{23}) = A(\Phi_{23}, \Phi_{23}, \bar{\Phi}_{23}, \bar{\Phi}_{23}) = \frac{g^2}{4} \left( \frac{st}{u} + \frac{su}{t} + 2s + 16\frac{s}{u} + 16\frac{s}{t} \right).
\]

In Eqs. (64)–(66), the terms proportional to $s/t$ and $s/u$ are “D-terms” due to the relevant gauge interactions ($\Phi_{23}$ and $\bar{\Phi}_{23}$ are charged under two $U(1)$’s) \[4\]. The presence of such terms indicates that the scalar potential in the $\Phi_{23}$ and $\bar{\Phi}_{23}$ directions is not flat, which is consistent with our observation that these fields are not moduli. A simple comparison of Eqs. (64)–(66) with Eq. (67) does not yield something like Eq. (60), hence $\Phi_{23}$ and $\bar{\Phi}_{23}$ do not correspond to the canonical coordinates $\alpha_i$ of the coset space $SO(2,1+12)/[SO(2) \times SO(1+12)]$; the “telltale” sign is the “$2s$” in Eq. (60). In this case, as we stated above, the correct canonical coordinates associated with $\Phi_{23}$ and $\bar{\Phi}_{23}$ are ($\alpha_1 = \Phi_1$)

\[
\alpha_2 = \frac{1}{\sqrt{2}} (\Phi_{23} + \bar{\Phi}_{23}), \quad \alpha_3 = \frac{i}{\sqrt{2}} (\Phi_{23} - \bar{\Phi}_{23}).
\]

Indeed, in terms of $\alpha_2$ and $\alpha_3$ the string scattering amplitudes (64)–(66) become

\[
A(\alpha_2, \alpha_3, \alpha_3^\dagger, \alpha_2^\dagger) = A(\alpha_3, \alpha_2, \alpha_2^\dagger, \alpha_3^\dagger) = \frac{g^2}{4} \left( \frac{su}{t} + s + 16\frac{s}{u} \right).
\]
\[ A(\alpha_2, \alpha_3, \alpha_2^\dagger, \alpha_3^\dagger) = A(\alpha_3, \alpha_2, \alpha_2^\dagger, \alpha_3^\dagger) = \frac{g^2}{4} \left( \frac{st}{u} + s + 16 \frac{s}{t} \right), \] (69)

\[ A(\alpha_2, \alpha_3, \alpha_3^\dagger, \alpha_2^\dagger) = A(\alpha_3, \alpha_2, \alpha_3^\dagger, \alpha_2^\dagger) = \frac{g^2}{4} \left( \frac{st}{u} + su + s \right), \] (70)

\[ A(\alpha_2, \alpha_3, \alpha_3^\dagger, \alpha_2^\dagger) = A(\alpha_3, \alpha_2, \alpha_3^\dagger, \alpha_2^\dagger) = \frac{g^2}{4} \left( -s - 16 \frac{s}{u} - 16 \frac{s}{t} \right). \] (71)

Note that the overall factor \((-i)\) in the definition of \(\alpha_3\) in Eq. (67) is crucial in order to get the right sign in front of the “s” term in Eq. (71) (c.f. Eq. (30)). In addition to (68)–(71), we also have the following non-vanishing string scattering amplitudes involving the modulus field \(\alpha_1\) and the matter fields \(\alpha_i\) \((i = 2, 3)\):

\[ A(\alpha_1, \alpha_i, \alpha_i^\dagger, \alpha_1^\dagger) = A(\alpha_i, \alpha_1, \alpha_1^\dagger, \alpha_i^\dagger) = \frac{g^2}{4} \left( \frac{su}{t} + s \right), \] (72)

\[ A(\alpha_1, \alpha_i, \alpha_i^\dagger, \alpha_1^\dagger) = A(\alpha_i, \alpha_1, \alpha_1^\dagger, \alpha_i^\dagger) = \frac{g^2}{4} \left( \frac{st}{u} + s \right), \] (73)

\[ A(\alpha_1, \alpha_i, \alpha_i^\dagger, \alpha_1^\dagger) = A(\alpha_i, \alpha_1, \alpha_1^\dagger, \alpha_i^\dagger) = \frac{g^2}{4} (-s). \] (74)

The comparison between Eqs. (68)–(74) with Eq. (37) yields Eq. (26), which demonstrates that \(\alpha_{2,3}\) defined in (67) are indeed the correct canonical coordinates.

The above analysis can be readily extended to the matter fields \(h_1\) and \(\bar{h}_1\), with similar results, i.e., the canonical coordinates are \(\frac{1}{\sqrt{2}}(h_1 + \bar{h}_1)\) and \(\frac{1}{\sqrt{2}}(h_1 - \bar{h}_1)\) (five components each). To write down the Kähler potential for the coset space \(SO(2,1+12)/[SO(2) \times SO(1+12)]\) spanned by the scalar fields \(\Phi_1, \Phi_{23}, \bar{\Phi}_{23}, h_1, \bar{h}_1\), we begin with the canonical form (23) using the properly defined complex coordinates \(\alpha_i\), and then rewrite all the \(\alpha_i\) in terms of the original string states. In our current example we obtain

\[ K^{(1)}(\Phi, \Phi^\dagger) = -\frac{1}{4} \left| \Phi_1^2 + 2\Phi_{23}\bar{\Phi}_{23} + 2h_1\bar{h}_1 \right|^2, \] (75)

where we have suppressed the \(SU(5)\) group indices for \(h_1\) and \(\bar{h}_1\). Thus we see that those string scattering amplitudes in terms of the original string states, i.e., Eqs. (54)–(60) are indeed consistent with (73). The canonical coordinates \(\alpha_i\) introduced above help provide a systematic proof of this result, and can be discarded once this is accomplished. In practice, by simply following the above example, one can easily work out the Kähler potential for all the untwisted scalar matter fields.

In the “revamped” flipped \(SU(5)\) model, the second set of scalar fields from the Neveu-Schwarz sector \(\Phi_{2, \Phi_{31}, \bar{\Phi}_{31}, \bar{h}_3, \bar{h}_2}\) parameterize a coset space \(SO(2,1+12)/[SO(2) \times SO(1+12)]\), whose Kähler potential is given by

\[ K^{(2)}(\Phi, \Phi^\dagger) = -\frac{1}{4} \left| \Phi_2^2 + 2\Phi_{31}\bar{\Phi}_{31} - 2\Phi_2\bar{\Phi}_2 + 2h_2\bar{h}_2 \right|^2, \] (76)
where $\Phi_2 = H_2^{(2)}$ is the modulus field (see Sec. [3]). The third set $[\Phi_3, \Phi_4, \Phi_5, \Phi_{12}, \bar{\Phi}_{12}, h_3, \bar{h}_3]$ contains two moduli fields $\Phi_{4, 5} = H_{1, 2}^{(3)}$ (see Sec. [3]) and thirteen matter fields, which parameterize a coset space $SO(2, 2+13)/[SO(2) \times SO(2+13)]$ with the following Kähler potential

$$K^{(3)}(\Phi, \Phi^\dagger) = - \log \left( 1 - \Phi_4 \Phi_4^\dagger - \Phi_5 \Phi_5^\dagger - \Phi_3 \Phi_3^\dagger - \Phi_{12} \Phi_{12}^\dagger - \Phi_{12} \bar{\Phi}_{12}^\dagger - h_3 h_3^\dagger - \bar{h}_3 \bar{h}_3^\dagger + \frac{1}{4} |\Phi_4^2 + \Phi_5^2 + \Phi_3^2 + 2\Phi_{12} \bar{\Phi}_{12} + 2 h_3 \bar{h}_3|^2 \right),$$

(77)

where $\Phi_3$ is a matter field whose right-moving fermionic oscillator is real and transforms as a singlet under $SU(5) \times U(1)$ and has no $U(1)$ charges (see Ref. [39]).

Finally we note that we can confirm in string perturbation theory that the potential is flat in the moduli directions. In this case, since the moduli are neutral under all gauge symmetries they do not appear in $D$-terms. Moreover, restricting the superpotential of the model to terms involving only products of three untwisted fields, one can verify that the moduli do not appear as $F$-terms either: the only such terms are $(\Phi_{12} \Phi_{23} \Phi_{31} + \bar{\Phi}_{12} \bar{\Phi}_{23} \bar{\Phi}_{31})$ [39].

Although here we only discuss explicitly the “revamped” flipped $SU(5)$ model, it is a straightforward exercise to apply our method to any fermionic string model of this class, such as those derived in Refs. [41, 42, 29].

7 Target space duality invariance

When the moduli fields move around in the moduli space $\mathcal{M}$, their associated exactly marginal operators generate deformations of the underlying CFT of the string model. One well-known stringy phenomenon is that a subset of these deformations leads to new CFTs which are physically equivalent to the original one. Such deformations correspond to some discrete reparameterizations of the moduli space $\mathcal{M}$, which are referred to as the target space duality transformations, and form the discrete duality group $\Gamma$ under which the string spectrum is invariant [14, 15, 16]. The target space duality invariance strongly restricts the Kähler potential and the superpotential of the string-derived effective field theory [46]. In this section, based on the results obtained in previous sections, we discuss the target space duality invariance in the context of fermionic string models, in particular, we establish the properties of the untwisted matter fields under the target space duality transformations.

As we have shown, for the class of fermionic string models that we are considering, the total untwisted moduli space factorizes into three distinct subspaces because the moduli fields separate into three sets. The sub-moduli space is either $SO(2, 2)/SO(2) \times SO(2)$ when there are two moduli fields, or $SO(2, 1)/SO(2)$ when there is only one modulus field (see Eqs. (20) and (55)). In what follows, we examine these two cases separately.

We first consider the case of moduli space $SO(2, 2)/SO(2) \times SO(2)$. According to the discussion in Sec. [1], one can replace the two moduli fields $H_{1, 2}$ by a $T$-type moduli
Φ_T and a U-type moduli Φ_U, such that each separately parameterizes a coset space SU(1,1)/U(1). The target space duality group in this case is given by PSL(2, Z)_T × PSL(2, Z)_U [14, 15], acting on the moduli fields T, U in the so-called “supergravity basis” as

\[ T \rightarrow \frac{a_T T - i b_T}{i c_T T + d_T}, \quad U \rightarrow \frac{a_U U - i b_U}{i c_U U + d_U}, \quad (78) \]

where \( a, b, c, d \in \mathbb{Z} \) and \( ad - bc = 1 \) for the coefficients of the T-transformation and the U-transformation. The moduli fields Φ_T,U in the “supergravity basis” are related to the moduli fields T, U in the “string basis” through the following equations [47]

\[ \Phi_T = T - T_c \bar{T} + T_c, \quad \Phi_U = U - U_c \bar{U} + U_c, \quad (79) \]

where \( T_c(U_c) \) is an unspecified complex number which can be viewed as the vacuum expectation value of field \( T \) in the free fermionic model that we start with, which corresponds to a critical point in the moduli space. From Eq. (79) we see that at this critical point \( \langle \Phi_T \rangle = \langle \Phi_U \rangle = 0 \), which is consistent with the fact that all Thirring interactions are turned off in the free fermionic model.

Now suppose that in addition to the moduli fields Φ_T and Φ_U there are \( n \) associated untwisted matter fields, which for convenience we express in terms of the corresponding canonical coordinates \( \alpha_i \). Starting from Eq. (23) for the case with \( n + 2 \) coordinates, but replacing two of them (the moduli \( H_{1,2} \)) with fields \( \Phi_{T,U} \) according to Eqs. (44) and (45), we can write the full Kähler potential for all these untwisted fields as

\[
K = - \log \left\{ (1 - \Phi_T \Phi_T^\dagger)(1 - \Phi_U \Phi_U^\dagger) - \sum_i \alpha_i \bar{\alpha}_i + \frac{1}{4} \left| \sum_i \alpha_i^2 \right|^2 
+ \frac{1}{2} \Phi_T \Phi_U \sum_i \bar{\alpha}_i^2 + \frac{1}{2} \bar{\Phi}_T \Phi_U \sum_i \alpha_i^2 \right\}. \quad (80)
\]

We now show that the Kähler potential (80) is target space duality invariant. First we recall that the physical content of target space duality not only includes the transformations (78) for the moduli fields \( T \) and \( U \), which are the generalizations of the famous \( R \rightarrow 1/2R \) duality transformation, but also requires the simultaneous interchange of the Kaluza-Klein (momentum) modes with the winding modes. This interchange is equivalent [17] to transforming the critical values \( T_c \) and \( U_c \) in the same way as the fields \( T \) and \( U \) according to (78). Therefore, from (79) we see that the moduli fields \( \Phi_{T,U} \) in the “string basis” simply transform under target space duality group \( PSL(2, \mathbb{Z})_T \times PSL(2, \mathbb{Z})_U \) by a field-independent phase, namely,

\[ \Phi_T \rightarrow e^{-2i \arg(i c_T T_c + d_T)} \Phi_T, \quad \Phi_U \rightarrow e^{-2i \arg(i c_U U_c + d_U)} \Phi_U. \quad (81) \]

Given the transformations (81) for the moduli \( \Phi_{T,U} \), in order for the Kähler potential (80) to be target space duality invariant, it is not hard to see that the matter fields \( \alpha_i \) have to transform universally as

\[ \alpha_i \rightarrow e^{-i \arg(i c_T T_c + d_T)} e^{-i \arg(i c_U U_c + d_U)} \alpha_i. \quad (82) \]
Since the canonical coordinates $\alpha_i$ are related to the string matter fields through linear transformations (see e.g., Eq. (67)), this argument shows that the string matter fields transform under target space duality also according to (82).

Two remarks are in order. First, we see that the matter fields do not transform as modular forms. This is because we are working in the “string basis”, whereas concepts such as modular forms only come into play if the duality properties are analyzed in the so-called “supergravity basis”, as it has been done traditionally [46]. Second, we have demonstrated the target space duality invariance of (80) without performing any additional Kähler transformations. Again, this is a result of working in the “string basis”, where the fields are basically “inert” under modular transformations (up to field-independent phases). In the “string basis”, since the Kähler potential $K$ is itself target space duality invariant, the requirement of an invariant Kähler function $G = K + \log |W|^2$ implies that the superpotential $W$ (written in terms of the “string basis” fields) can only be allowed to have a phase transformation, i.e., $W \rightarrow e^{i\phi} W$.

In fact, one can show [32] that in non-vanishing cubic terms in $W$ which are products of three untwisted fields, each field comes from a different set, i.e.,

$$W = \sum \lambda_{ijk} \alpha_i^{(1)} \alpha_j^{(2)} \alpha_k^{(3)}.$$  

Thus, suppose the sub-moduli space of the first set is $SO(2,2)/SO(2) \times SO(2)$, then from Eq. (82) we see that under target space duality group $PSL(2,\mathbb{Z})_T \times PSL(2,\mathbb{Z})_U$ associated with this set,

$$W \rightarrow \sum \lambda_{ijk} e^{-i\varphi_T^{(1)}} e^{-i\varphi_U^{(1)}} \alpha_i^{(1)} \alpha_j^{(2)} \alpha_k^{(3)} = e^{-i\varphi_T^{(1)} - i\varphi_U^{(1)}} W,$$

as required by duality invariance of $G$. Here $\varphi_T^{(1)} = \arg(ic c \bar{T}t + d_T)$ and $\varphi_U^{(1)} = \arg(ic U c + d_U)$. This assumption also shows that moduli (which transform as in Eq. (53)) are not allowed in all-untwisted-field cubic couplings, as expected from their flatness properties. Extension of this argument to non-renormalizable terms leads to non-trivial constraints on the corresponding couplings, which acquire a non-trivial moduli dependence [15].

To make contact with previous results, we also study this problem in the “supergravity basis”. To this end, one starts with the moduli fields $T, U$ instead of $\Phi_{T, U}$, and then it is necessary to make a holomorphic field redefinition for the matter fields such that the “supergravity basis” matter fields ($A_i$) are given by their “string basis” counterparts ($\alpha_i$) as follows

$$A_i = \frac{T + T_c}{\sqrt{T + T_c \sqrt{U + U_c}} \alpha_i}.$$  

In this “supergravity basis”, up to a Kähler transformation, one can bring (80) into the following form

$$K' = -\log \left\{ (T + \bar{T})(U + \bar{U}) - \sum_i A_i \bar{A_i} + \frac{1}{4} \frac{(T + T_c)(U + U_c)}{|T + T|^2 |U + U|^2} \left| \sum_i A_i^2 \right|^2 \right\}.$$
\[ \frac{1}{2} \left( \frac{T_c - T}{T_c + T} \right) \left( \frac{U_c - U}{U_c + U} \right) \sum_i \bar{A}_i^2 + \frac{1}{2} \left( \frac{\bar{T}_c - \bar{T}}{\bar{T}_c + \bar{T}} \right) \left( \frac{\bar{U}_c - \bar{U}}{\bar{U}_c + \bar{U}} \right) \sum_i A_i^2 \], \quad (86)

which under target space duality transformations \((78)\) for the moduli fields \(T, U\) and the critical values \(T_c, U_c\) transforms to

\[ K' \rightarrow K' + \log(|ic_T T + d_T|^2|ic_U U + d_U|^2), \quad (87) \]

accompanied by the following universal transformation for matter fields \(A_i\),

\[ A_i \rightarrow \frac{1}{(ic_T T + d_T)(ic_U U + d_U)} A_i. \quad (88) \]

Note that the last two terms in \((86)\) fix completely the phase in \((88)\), and thus the \(A_i\) transform as a modular form of weight \((-1)\). The duality transformation \((88)\) can also be readily obtained from Eq.\((85)\). In this “supergravity basis” the superpotential \(W\) must also transform as a modular form of weight \((-1)\) in order to cancel the second term in \((87)\). This is the usual analysis of Refs. \([46, 35]\). However, in our case, at least for the discussion of target space duality, such an approach does not seem to be necessary. In fact, the presence of the second term in \((87)\) is because one has neglected a Kähler transformation when bringing \(K\) of \((80)\) into \(K'\) of \((86)\); this piece automatically cancels the second term in \((87)\) so that \(K\) is completely invariant, as we have shown above in terms of the “string basis”.

We next discuss the case of moduli space \(SO(2,1)/SO(2)\). In this case, according to \((60)\), we can replace the moduli field \(H\) by \(\Phi_t\) which leads to the Fubini-Study metric of \(SU(1,1)/U(1)\). The target space duality group is simply \(PSL(2,\mathbb{Z})\), under which the modulus field \(t\), in the “supergravity basis”, transforms as

\[ t \rightarrow \frac{a_t t - ib_t}{ic_t t + d_t}, \quad (89) \]

with \(a_t, b_t, c_t, d_t \in \mathbb{Z}\) and \(a_t d_t - b_t c_t = 1\). The field \(\Phi_t\), although not a string state in this case, can still be related to \(t\) via

\[ \Phi_t = \frac{t_c - t}{t_c + t}, \quad (90) \]

where \(t_c\) is the critical value of \(t\) in the free fermionic model. Therefore, under the target space duality group \(PSL(2,\mathbb{Z})\), \(\Phi_t\) simply goes through a field-independent phase transformation,

\[ \Phi_t \rightarrow e^{-2i\arg(ic_t t + d_t)} \Phi_t. \quad (91) \]

This is also the transformation for the original string state \(H\), as can be seen from \((60)\). In this sense, we can roughly refer to \(\Phi_t\) as the modulus field in the “string basis”.

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In this case, starting from Eq. (23) for the case with \( n+1 \) coordinates, but replacing one of them (the modulus \( H \)) by \( \Phi_t \) using the inverse of Eq. (60), one can obtain the following full Kähler potential

\[
K = -\log \left\{ 1 - \Phi_t \bar{\Phi}_t - \sum_{i}^{n} \alpha_i \bar{\alpha}_i + \frac{1}{4} \left| \sum_{i}^{n} \alpha_i^2 \right|^2 \right.
\]
\[
+ \frac{1}{2} \frac{1}{1 + \sqrt{1 - \Phi_t \bar{\Phi}_t}} \left( \Phi_t^2 \sum_{i}^{n} \bar{\alpha}_i^2 + \bar{\Phi}_t^2 \sum_{i}^{n} \alpha_i^2 \right),
\]

(92)

which is target space duality invariant provided the matter fields transform universally as

\[
\alpha_i \rightarrow e^{-2i \arg(ic t_c + dt)} \alpha_i.
\]

(93)

We note that in this case the matter fields \( \alpha_i \) transform in the same way as the modulus field \( \Phi_t \) (see Eq. (91)), whereas in the case of moduli space \( SO(2,2)/SO(2) \times SO(2) \) this is not true (see Eqs. (81) and (82)). One can also carry out the usual analysis of Refs. 46, 35 for the case of moduli space \( SO(2,1)/SO(2) \), as we did for the case of moduli space \( SO(2,2)/SO(2) \times SO(2) \). The only difference is that the holomorphic field redefinition which defines the matter fields in the “supergravity basis” here becomes

\[
A_i = \frac{\bar{t}_c + t_c}{\sqrt{t_c + t_c}} \alpha_i,
\]

(94)

and the \( A_i \) transform as modular forms of weight \((-1)\) with a non-trivial phase

\[
A_i \rightarrow e^{i \arg(ic t_c + dt)} \left( \frac{i c t_c + d t}{i c t + d t} \right) A_i,
\]

(95)

which should be contrasted with (88) for the case of moduli space \( SO(2,2)/SO(2) \times SO(2) \).

8 Conclusions

Identifying the moduli fields and their symmetries in string-derived models constitutes the first step in the determination of the low-energy effective field theory. This knowledge allows one to calculate the Kähler potential, which together with the superpotential and gauge kinetic functions determine completely the effective supergravity theory. This effective theory can then be used for various phenomenological studies, such as string threshold corrections in gauge coupling unification, supersymmetry breaking, string cosmology, etc.

We have presented a general procedure by which the untwisted moduli fields in fermionic models can be identified. The crucial element of our procedure is the deformation of the free-fermionic model by the exactly marginal operators which take the form of world-sheet Abelian Thirring interactions, thus getting away from
the free-fermionic point to appreciate the duality symmetries embodied in the moduli fields. We also back all of our generic symmetry-based arguments by explicit string perturbation theory calculations.

Previous phenomenological studies in free-fermionic models \cite{49, 50, 48, 51} assumed that the usual symmetric orbifold analysis was applicable. Our results show that this assumption holds only for the simplest free-fermionic models, but fails in the case of realistic models where the asymmetric nature of the equivalent orbifold formulation is essential. Furthermore, with several examples we have demonstrated the method to obtain the full Kähler potential for the non-linear sigma model of the untwisted sector in such fermionic models, in terms of the actual massless string states; this problem had remained largely obscure until now.

The results derived in this paper concerning the moduli space and the Kähler function, e.g., Eqs. (43), (46) and (57), (61), bear close resemblance to results obtained early on in the context of no-scale supergravity \cite{52, 53}. In particular, the ever-present moduli fields in string models lead to flat potentials which are characteristic of no-scale supergravity. The corresponding expressions for the Kähler function in our string-derived supergravity are more complex than those obtained in traditional no-scale supergravity, because of the rich string-theory structure underlying the effective supergravity model. Nonetheless, the original motivations embodied in no-scale supergravity, i.e., the vanishing of the cosmological constant and the flat potentials which allow the dynamical determination of mass scales \cite{54}, remain valid and we would hope that such a physics program could also be pursued successfully in the context of string no-scale supergravity.

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