Microscopic theory of inhomogeneous ultradilute quantum droplets

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An ultradilute quantum droplet is a self-bound liquid-like state recently observed in ultracold Bose-Einstein condensates. In all previous theoretical studies, it is described by a phenomenological low-energy effective theory, termed as the extended Gross-Pitaevskii equation. Here, we microscopically derive the Gross–Pitaevskii equation for the condensate and also for a pairing field in an inhomogeneous quantum droplet realized by Bose-Bose mixtures with attractive inter-species interaction. We show that the inclusion of the pairing field is essential, in order to have a consistent description of the droplet state. We clarify that, the extended Gross–Pitaevskii equation used earlier should be understood as the equation of motion for the pairing field, rather than the condensate. The fluctuations of the pairing field give rise to low-energy collective excitations of the droplet. We also present the Bogoliubov equations for gapless phonon modes and gapped modes due to pairing in real space, which characterizes single-particle-like excitations of the droplet. The equations of motion derived in this work for the condensate and the pairing field serve an ideal starting point to understand the structure and collective excitations of non-uniform ultradilute quantum droplets in on-going cold-atom experiments.

I. INTRODUCTION

One of the recent breakthroughs in the ultracold atomic physics is the realization of a self-bound liquid-like droplet state [1]. In three spatial dimensions, this new quantum state of matter arises from a delicate balance between the attractive mean-field potential $\partial E_{\text{MF}}/\partial n$ and the repulsive potential $\partial E_{\text{LHY}}/\partial n$ provided by the celebrated Lee-Huang-Yang (LHY) term $E_{\text{LHY}}$ from quantum fluctuations [1], which scale like $n$ and $n^{3/2}$, respectively, as a function of the total density $n$ of the system. In almost all the theoretical studies of quantum droplets [10, 12, 26], the extended Gross–Pitaevskii (GP) equation ($E = E_{\text{MF}} + E_{\text{LHY}}$) [10],

$$i\hbar \frac{\partial}{\partial t} \psi = \left[-\frac{\hbar^2}{2m} \nabla^2 + \frac{\partial E}{\partial n} \left(n = |\psi|^2\right)\right] \psi,$$  

(1)

has been extensively used to describe the structure and dynamics of the condensate wave $\psi$ of the droplet. This phenomenological low-energy effective theory is often thought to be unavoidable, since the LHY term derived from the standard Bogoliubov theory [27] becomes complex due to an unstable, softening phonon mode and therefore should be amended in an empirical way [10]. As a priori assumption, the extended GP equation gives a useful description of the low-lying softening phonon mode, in view of the density functional theory.

It turns out that an amended LHY term is not the only choice. As demonstrated in our recent work [28, 29], the inconsistency of a complex LHY term in the Bogoliubov theory of quantum droplets could be alternatively removed by the inclusion of pairing between two bosons in different species. The bosonic pairing changes the unstable softening mode into stable gapped mode and leads to much more accurate ground-state energy for the droplet state, as benchmarked by the state-of-the-art diffusion Monte Carlo simulations [20, 21, 24]. Roughly speaking, however, the gapped mode should be considered as single-particle excitations with relatively high energy, as we learn from the conventional fermionic pairing theories [30–32]. Therefore, we lose the track of the low-lying collective excitations. Immediate questions then are, how can we find those collective excitations and how to physically interpret the phenomenological extended GP equation, Eq. (1), within the pairing theory?

In this Rapid Communication, we derive and discuss the equations of motion for the condensate, the pairing field and their fluctuations in a non-uniform quantum droplet realized with attractive Bose-Bose mixtures, following the framework of the previously developed microscopic pairing theory [28, 29]. We show that the existence of the pairing field implies additional $U(1)$ symmetry breaking for the many-body paired bosons, i.e., the bosonic Cooper pairs. Analogous to a Bardeen–Cooper–Schrieffer (BCS) fermionic superfluid [31, 32], the low-lying collective excitations of the droplet state then should be characterized by the pair fluctuations around the mean-field saddle-point for the pairing field. We confirm this picture, by microscopically derive the extended GP equation for a large droplet, where the local density approximation could be applied. We also present the Bogoliubov equations for the gapless phonon mode associated with the $U(1)$ symmetry breaking of the condensate and for the gapped mode due to the pairing. We interpret these two relatively high-lying modes as single-particle excitations of the droplet state. A self-consistent numerical solution of the inhomogeneous extended GP equation and Bogoliubov equations provides a useful description of the structure and collective oscillation of a self-bound quantum droplet in free space.

II. MODEL HAMILTONIAN

We start by considering a three-dimensional homonuclear Bose-Bose mixture such as a cloud of $^{39}$K atoms.
in two hyperfine states (i.e., a $^{39}\text{K}-^{39}\text{K}$ mixture), as in recent experiments [4, 5]. For simplicity, we assume equal repulsive intra-species interactions with strengths $g_{11} = g_{22} = g$ and attractive inter-species interactions $g_{12} = g_{21}$, and also equal population in each species. The grand canonical Hamiltonian of the system can then be written as,

$$\hat{K} = \int dx [\mathcal{H}_0 + \mathcal{H}_\text{intra} + \mathcal{H}_\text{inter}],$$

(2)

where $\phi_i(x), i = 1, 2$ are creation and annihilation field operators for the $i$-species bosons with mass $m_1 = m_2 = m$ and with chemical potential $\mu_1 = \mu_2 = \mu$. We have explicitly included an external harmonic trap $V_T(x) = m\omega^2x^2/2$, in order to account for a possible residual weak potential in experiments. In three dimensions, the bare interaction strengths $g_{ij}$ are to be regularized, due to the well-known ultraviolet divergence of the contact inter-particle interaction. We shall rewrite them in terms of the three-dimensional $s$-wave scattering lengths $a_{11} = a_{22} = a$ and $a_{12}$,

$$\frac{1}{g_{ij}} = \frac{m}{4\pi\hbar^2a_{ij}} - \frac{1}{\mathcal{V}} \sum_k \frac{m}{\hbar^2k^2},$$

(6)

where $\mathcal{V}$ is the volume of the system. We note that in $\mathcal{H}_\text{inter}$ we have taken the Hubbard-Stratonovich transformation and have introduced a pairing field $\Delta(x)$ to decouple the inter-species interaction Hamiltonian [20]. In the weakly interacting regime, it suffices to take a static saddle-point solution for the pairing field. Hence, we treat $\Delta(x) = \Delta(x)$ as a variational $c$-number function. The dynamics of the pairing field can be added back later, when we consider the fluctuations around the saddle point.

### III. BOGOLIUBOV THEORY WITH PAIRING

We use the standard Bogoliubov theory to solve the model Hamiltonian, in the presence of a static pairing field $\Delta(x)$. Following Refs. [33–34], in the Bogoliubov approximation we rewrite the bosonic field operators,

$$\phi_i(x) = \phi_c(x) + \delta\phi_i(x),$$

(7)

where $\delta\phi_i$ is considered as a small correction to the condensate wave-function $\phi_c(x) = e^{i\theta(x)}|\phi_c(x)|$ and $\theta(x)$ is the phase of the condensate. For the ground state, we take $\theta(x) = 0$; while for a vortex state, we set $\theta(x) = e^{i\psi}$ with the polar coordinate $\varphi$. Our model Hamiltonian may then be expanded through second order in $\delta\phi_i^\dagger$ and $\delta\phi_i$, and the linear term vanish identically if $\phi_c(x)$ satisfies the GP equation,

$$\left[\frac{\hbar^2\nabla^2}{2m} + V_T(x) - \mu + g|\phi_c|^2\right]\phi_c - \Delta(x)\phi_c^* = 0.$$

(8)

By introducing the notations $C(x) = g|\phi_c|^2$ and

$$\hat{T}(x) = \frac{-\hbar^2(\nabla + i\nabla\theta(x))^2}{2m} + V_T(x) - \mu,$$

(9)

we may rewrite the GP equation into the relation for $\Delta(x)$,

$$\Delta(x) = C(x) + |C(x)|^{-1/2}\hat{T}(x)|C(x)|^{1/2}.$$

(10)

We then obtain the truncated Bogoliubov Hamiltonian,

$$\hat{K}_B = \sum_{i=1,2} \int dx \left[ \delta\phi_i^\dagger \mathcal{L}_0 \delta\phi_i + \left( \frac{C}{2} e^{i\theta}\delta\phi_i^\dagger \delta\phi_i^\dagger + \text{H.c.} \right) \right] - \int dx \left[ (\hat{\Delta}\delta\phi_i^\dagger \delta\phi_i^\dagger + \text{H.c.}) + \frac{C^2}{g} + \frac{|\Delta|^2}{g_{12}} \right],$$

(11)

where $\mathcal{L}_0 = -\hbar^2\nabla^2/(2m) + V_T(x) - \mu + 2C(x)$. The Bogoliubov Hamiltonian consists of a $c$-number condensate part and a quadratic form in $\delta\phi_i$ and $\delta\phi_i^\dagger$. This quadratic form could be diagonalized with the linear Bogoliubov transformation,

$$\delta\phi_i(x) = e^{+i\theta(x)} \sum_n \left[ u_{ni}(x) \hat{a}_n + v_{ni}^*(x) \hat{a}_n^\dagger \right],$$

(12)

$$\delta\phi_i^\dagger(x) = e^{-i\theta(x)} \sum_n \left[ v_{ni}^*(x) \hat{a}_n^\dagger + u_{ni}(x) \hat{a}_n \right],$$

(13)

where $\hat{a}_n^\dagger$ and $\hat{a}_n$ are creation and annihilation field operators of Bogoliubov quasiparticles satisfying the usual Bose commutation relations. We can show that the truncated Bogoliubov Hamiltonian reduces to

$$\hat{K}_B = -\int dx \left[ \frac{C^2(x)}{g} + \frac{|\Delta(x)|^2}{g_{12}} \right] - \int dx \sum_{ni} E_n |v_{ni}(x)|^2 + \sum_n E_n \hat{a}_n^\dagger \hat{a}_n,$$

(14)

provided that the quasiparticle wave-functions $u_{ni}(x)$ and $v_{ni}(x)$ obey the coupled Bogoliubov eigenvalue equations ($i = 1, 2$),

$$\mathcal{L}u_{ni} + C(x) v_{ni} - \Delta(x) v_{n,i-1} = E_n u_{ni},$$

(15)

$$\mathcal{L}^*v_{ni} + C(x) u_{ni} - \Delta^*(x) u_{n,i-1} = -E_n v_{ni},$$

(16)

where $\mathcal{L} \equiv \hat{T}(x) + 2C(x)$ is a Hermitian operator satisfying $\int dx u^* \mathcal{L} v = \int dx v^* \mathcal{L}^* u$ [33]. It is easy to check
that the wave-functions satisfy the normalization and orthogonality conditions,
\[
\sum_{i=1,2} \int dx [u_{ni}^* u_{mi} - v_{ni}^* v_{mi}] = \delta_{nm},
\]
\[
\sum_{i=1,2} \int dx [u_{ni} v_{mi} - v_{ni} u_{mi}] = 0.
\]

The Bogoliubov equations (15) and (16) have a well-known particle-hole symmetry: if \(u_{ni}\) and \(v_{ni}\) are a solution with energy \(E_n\), then there is always another solution \(u_{ni}^*\) and \(u_{ni}^*\) with energy \(-E_n\). In the diagonalized Bogoliubov Hamiltonian Eq. (14), therefore, we choose the positive eigenvalues \(E_n \geq 0\). As a result, the thermodynamic potential at zero temperature takes the form,
\[
\Omega = -\int dx \left[ \frac{C^2}{g} + \frac{\Delta^2}{g_{12}} + \sum_{ni} E_n |v_n(x)|^2 \right],
\]
from which, we may determine the variational pairing field \(\Delta(x)\) through the functional minimization, i.e.,
\[
\frac{\delta \Omega [\mu, \Delta(x)]}{\delta \Delta(x)} = 0.
\]

It is worth noting that the LHY contribution \(\Omega_{LHY} = -\int dx \sum_n E_n |v_n(x)|^2\) is formally divergent. This divergence can be removed by regularizing the bare interaction strengths \(g\) and \(g_{12}\). In practice, we introduce a high-energy cut-off energy \(E_c\), above which the discreteness of the energy spectrum \(E_n\) is no longer important and we semi-classically solve the Bogoliubov equations under the local density approximation to obtain \(u_k(x), v_k(x)\) and \(E_k\) [32]. We then rewrite the thermodynamic potential into two parts, \(\Omega = \int dx [\Omega_d(x) + \Omega_c(x)]\), where
\[
\Omega_d = -\frac{m}{4\pi \hbar^2} \left[ \frac{C^2}{a} + \frac{\Delta^2}{a_{12}} \right] - \sum_{i, E_n < E_c} E_n |v_{ni}|^2,
\]
\[
\Omega_c = \sum_k \frac{m}{\hbar^2 k^2} \left[ \frac{C^2}{a} + \frac{\Delta^2}{a_{12}} \right] - \sum_{i, E_k \geq E_c} E_k |v_{ki}|^2.
\]

For a given chemical potential \(\mu\), the GP equation (10), the Bogoliubov equations (15) and (16), and the thermodynamic potential Eqs. (21) and (22) form a closed set of equations to determine \(C(x)\) and \(\Delta(x)\), and consequently the condensate wave-function \(\phi_c(x) \propto \sqrt{C(x)}\) and the Bogoliubov spectrum \(E_n\). This is the first key result of our work. The total number of atoms can be calculated from the thermodynamic potential relation \(N = -\partial \Omega / \partial \mu\), which provides the normalization to \(\phi_c(x)\). We note that, if we neglect the quantum depletion, which is small in the weakly interacting regime, we may write down directly \(\phi_c(x) = \sqrt{mC(x)/(2\pi\hbar^2a)}\).

For simplicity, from now on we focus on the ground state with a phase \(\theta(x) = 0\). Our discussion given below can be easily extended to the general case with a nonzero phase factor \(\theta(x) \neq 0\) and we will put \(\theta(x)\) back when it is needed.

\section{IV. Bulk Properties of Quantum Droplets}

For a large droplet in the absence of any external potential and in the ground state, the function \(C(x)\) and the pairing field \(\Delta(x)\) are essentially real constant, except at the edge of the droplet. Thus, if we neglect the edge effect, the GP equation (8) gives the relation \(\mu = C + \Delta\). The Bogoliubov equations (14) and (16) in momentum space take the form,
\[
\begin{bmatrix}
B_k & 0 & C & -\Delta \\
0 & B_k & -\Delta & C \\
C & -\Delta & B_k & 0 \\
-\Delta & C & 0 & B_k
\end{bmatrix}
\begin{bmatrix}
u_{k1} \\
u_{k2} \\
u_{k1} \\
u_{k2}
\end{bmatrix} = E_k
\begin{bmatrix}
+\nu_{k1} \\
+\nu_{k2} \\
-\nu_{k1} \\
-\nu_{k2}
\end{bmatrix},
\]
where \(B_k \equiv \varepsilon_k + C + \Delta\) with \(\varepsilon_k = \hbar^2 k^2/(2m)\). We then obtain two Bogoliubov spectra,
\[
E_-(k) = \sqrt{\varepsilon_k(\varepsilon_k + 2C + 2\Delta)},
\]
\[
E_+(k) = \sqrt{\varepsilon_k(\varepsilon_k + 2C)}(\varepsilon_k + 2\Delta).
\]

For both dispersions, the wave-functions \(u_{k1}(x) = u_{k1} e^{ikx}/\sqrt{V}\) and \(v_{k2}(x) = v_{k2} e^{ikx}/\sqrt{V}\) are given by,
\[
u_{k1}^2 = u_{k1}^2 = \frac{1}{4} \left[ \frac{B_k}{E_k} + 1 \right],
\]
\[
u_{k2}^2 = v_{k2}^2 = \frac{1}{4} \left[ \frac{B_k}{E_k} - 1 \right],
\]
from which, we determine the thermodynamic potential,
\[
\frac{\Omega}{V} = -\frac{m}{4\pi \hbar^2} \left[ \frac{C^2}{a} + \frac{\Delta^2}{a_{12}} \right] + \frac{1}{2} \sum_k \left[ E_+(k) + E_-(k) \right] - 2(\varepsilon_k + C + \Delta) + \frac{C^2 + \Delta^2}{\varepsilon_k}.
\]

This result was obtained in our previous works using a path-integral functional approach [28, 29]. By integrating over the momentum \(k\), we arrive at the expression,
\[
\frac{\Omega}{V} = -\frac{m}{4\pi \hbar^2} \left[ \frac{C^2}{a} + \frac{\Delta^2}{a_{12}} \right] + \frac{8m^{3/2}C^{5/2}}{15\pi^2 \hbar^3} G_3 \left( \frac{\Delta}{C} \right),
\]
where \(G_3(\alpha) = (1 + \alpha)^{3/2} + h_3(\alpha)\) with \(h_3(\alpha) = (15/4) \int_0^\infty dt(1/t)^2 \left( t + (t + 1)(t + a) - (t + 1/2 + a/2) + (1 - a)\right)^2/(8t)\) . Near the equilibrium density of quantum droplets, the chemical potential \(\mu\) is typically much smaller than \(C\) and \(\Delta\) [28, 29]. We may then expand \(\Omega(\mu)\) in powers of \(\mu, \Omega(\mu) = \Omega^{(0)} + \mu \Omega^{(1)} + \cdots\), where
\[
\frac{\Omega^{(0)}}{V} = -\frac{m}{4\pi \hbar^2} \left[ \frac{1}{a} + \frac{1}{a_{12}} \right] \Delta^2 + \frac{32 \sqrt{2} m^{3/2}}{15\pi^2 \hbar^3} \Delta^{5/2},
\]
\[
\frac{\Omega^{(1)}}{V} = -\frac{m}{2\pi \hbar^2} \Delta + \frac{8 \sqrt{2} m^{3/2}}{3\pi^2 \hbar^3} \Delta^{3/2}.
\]
By taking the derivative \(-\partial \Omega / \partial \mu\) \[36\], we obtain the pairing gap
\[
\Delta \simeq \frac{2\pi \hbar^2 a}{m} n [1 + \eta], \tag{32}
\]
and the total energy per particle \(E/V = \mu n + \Omega/V\),
\[
\frac{E}{V} \simeq -\frac{\pi \hbar^2}{m} \left( a + \frac{a^2}{a_{12}} \right) n^2 \left( 1 + 2\eta \right)
+ \frac{256\sqrt{\pi}}{15} \frac{\hbar^2 a^{5/2}}{m} n^{5/2} \left( 1 + \frac{5}{2} \eta \right), \tag{33}
\]
where the correction factor \(\eta \equiv 32\sqrt{\pi}a^3 / (3\sqrt{\pi}) \ll 1\) comes from the second term in \(\partial \Omega^{(1)}/V\). This small correction is absent if we approximate \(C \simeq \Delta\) in the LHY thermodynamic potential (i.e., the second term on the right-hand-side of Eq. \[23\]).

V. LARGE DROPLETS WITHIN THE LOCAL DENSITY APPROXIMATION

To take into account the edge effect for a large quantum droplet, we may take the local density approximation, by assuming very slowly varying \(C(x)\) and \(\Delta(x)\) in real space. This amounts to setting the cut-off energy \(E_c = 0\) in Eqs. \[21\] and \[22\]. In this case, we may treat \(\Delta(x)\) as a real and non-negative function, since it only differs slightly from \(C(x)\). Therefore, we obtain,
\[
\Omega = -\int dx \frac{m}{4\pi \hbar^2} \left[ \frac{C^2(x)}{a} + \frac{\Delta^2(x)}{a_{12}} \right]
+ \int dx \frac{8m^{3/2}}{15\pi \hbar^3} \left( C + \Delta(x) \right)^{5/2}. \tag{34}
\]
Here, we have used the fact that \(C(x) \simeq \Delta(x)\), so the function \(h_3(C/\Delta) \propto |C(x) - \Delta(x)|^2 / \Delta^2(x) \ll 1\) can be safely neglected \[23\]. By taking the functional derivative \(\delta \Omega(\Delta(x))/\delta \Delta = 0\), we find that
\[
\left[ \frac{C \delta C}{a \delta \Delta} + \frac{\Delta}{a_{12}} \right] - \frac{16\sqrt{m}}{3\pi \hbar} |C + \Delta|^{3/2} \left( \delta C / \delta \Delta + 1 \right) = 0.
\]
From the GP equation \[11\], the function \(C(x)\) is related to the pairing field \(\Delta(x)\),
\[
C(x) \simeq \Delta(x) - |\Delta(x)|^{-1/2} \hat{T}(x) |\Delta(x)|^{1/2}. \tag{36}
\]
As \(\Delta(x)\) is a very slowly varying function for a large droplet, to a good approximation we may take \(\delta C / \delta \Delta = 1\) and also set \(C(x) \simeq \Delta(x)\) in the second term of Eq. \[33\]. The latter is equivalent to neglecting the small correction factor \(\eta\) in Eq. \[32\] and Eq. \[33\], as we discussed earlier. By further inspired by the relation \[32\] to introduce \(\Phi^2(x) \equiv [m/(2\pi \hbar^2 a)] e^{2i\theta(x)} / \Delta(x) = [m/(2\pi \hbar^2 a)] \hat{\Delta}(x)\), we rewrite Eq. \[35\] into the form,
\[
\hat{T} - \frac{2\pi \hbar^2}{m} \left( a + \frac{a^2}{a_{12}} \right) |\Phi|^2 + \frac{128\sqrt{\pi}}{3} \frac{\hbar^2 a^{5/2}}{m} |\Phi|^3 \Phi = 0. \tag{37}
\]
By recalling \(\hat{T} = -\hbar^2 \nabla^2 / (2m) - \mu = -\hbar^2 \nabla^2 / (2m) - i\hbar \partial_t\) in the absence of external potential, the above equation is precisely the extended GP equation \[11\], once we take Eq. \[33\] with \(\eta = 0\) as the density functional \(E(n)\). Hence, we have microscopically derived the extended GP equation, under the condition (i.e., within the local density approximation) that it is applicable.

Our derivation clearly shows that the wave-function in the extended GP equation \[11\] represents the pairing field \(\Delta(x)\), rather than the condensate wave-function \(\phi_c(x)\), as one may naively anticipate. The latter satisfies instead the ordinary GP equation, as shown in Eq. \[8\]. This clarification is another key result of our work.

In our microscopic pairing theory, the quantum droplet could be viewed as a mixture of bosonic atoms and loosely-bound many-body bosonic Cooper pairs, analogous to a two-component Fermi superfluid where fermions and loosely-bound fermionic Cooper pairs co-exist \[31, 32\]. Eq. \[37\] derived here can therefore be regarded as the bosonic counterpart of the well-known Ginzburg-Landau equation for the BCS pairing order parameter \[37\]. Physically, there are two \(U(1)\) symmetry breakings, one is associated with the condensate wavefunction \(\phi_c(x)\) and another with the pairing field \(\Delta(x)\). As in a fermionic superfluid, low-energy collective excitations correspond to the fluctuations around the static saddle-point solution \(\Delta(x)\) and can be studied by linearizing the extended GP equations \[11\] or \[37\] with
\[
\Phi(t) = e^{-i\mu t} \left[ \Phi_0 + \sum_n \left( U_n e^{-i\omega_n t} + V_n^* e^{+i\omega_n t} \right) \right], \tag{38}
\]
\(t\) to the first order in fluctuations \(U_n(x)\) and \(V_n(x)\). This leads to the Bogoliubov equations for pairing fluctuations. In contrast, the quasi-particles described by the Bogoliubov equations in \[15\] and \[16\] should be understood as single-particle excitations of bosonic atoms.

It is worth noting that, in a scalar weakly-interacting Bose gas collective excitations and single-particle excitations are strongly correlated, due to the existence of the condensate, so the Bogoliubov quasi-particles are generally treated as collective excitations \[38, 39\]. The situation in quantum droplets is less clear, as a result of the bosonic pairing. A careful identification of collective excitations and single-particle excitations is therefore needed, by calculating of density-density correlation functions, presumably within the random-phase-approximation \[39\].

VI. CONCLUSIONS

In summary, we have constructed a microscopic pairing theory for describing the non-uniform states of quantum droplets realized with an attractive Bose-Bose mixture in three dimensions. We have pointed out that the droplet can be viewed as a coherent mixture of bosonic atoms and loosely-bound bosonic Cooper pairs, both of which
are Bose-condensed. We have presented the equations of motion for the atomic condensate and the pairing field of Cooper pairs, as well as the fluctuations around them. The resulting closed set of equations (10), (15), (16), (21) and (22) forms the basis to investigate the structure and dynamics of quantum droplets in future studies.

By using these equations for a large quantum droplet, where the spatial variation in the condensate wave-function and the pairing field is small, we have microscopically derived the extended Gross-Pitaevskii equation, which has been previously used as a phenomenological low-energy effective theory. We have clarified that the extended Gross-Pitaevskii equation describes the pairing field, instead of the condensate wave-function as one may naively expect. This clarification would be important for properly distinguishing the collective excitations and single-particle behaviors in the intriguing new quantum state of self-bound liquid-like droplets.

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