THE COLLISION AVOIDANCE AND THE CONTROLLABILITY FOR \( n \) BODIES IN DIMENSION ONE

CHONG-KYU HAN* AND DONGHOON PARK

Abstract. We present a method of control system design for \( n \) bodies in the real line \( \mathbb{R}^1 \) and on the unit circle \( S^1 \), to be collision-free and controllable. The problem reduces to designing a control-affine system in \( \mathbb{R}^n \) and in \( n \)-torus \( T^n \), respectively, that avoids certain obstacles. We prove the controllability of the system by showing that the vector fields that define the control-affine system, together with their brackets of first order, span the whole tangent space of the state space, and then by applying the Rashevsky-Chow theorem.

1. Introduction and the statement of the main results

Let \( M^1 \) be a smooth \((C^\infty)\) connected manifold of dimension 1 without boundary, namely, either the real line \( \mathbb{R}^1 \) or the unit circle \( S^1 \). We are concerned in this paper with the design of control systems for \( n \) particles on \( M^1 \) to be collision-free and controllable, which we shall call the control \( n \) body problem on \( M^1 \). Let

\[
M^n := \underbrace{M^1 \times \cdots \times M^1}_{n \text{ times}}
\]

be the product. We shall see in §1.1 that the problem turns out to be constructing a control-affine system on \( M^n \) with the avoidance of a certain obstacle so that for any two points of \( M^n \) off the obstacle, one is reachable from the other by an admissible control. Thus we construct

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on $M^n$ a driftless control-affine system

$$\dot{x} = \sum_{\ell=1}^{m} \bar{f}_\ell(x)u_\ell, \quad x = (x_1, \cdots, x_n) \in M^n, \ m \leq n,$$

that satisfies the following conditions:

i) The motions avoid the obstacle.

ii) The Lie algebra generated by $\bar{f}_\ell(x)$, $\ell = 1, \cdots, m$, span the whole tangent space $T_xM$ at every point $x \in M^n$ off the obstacle. In terms of $n$-dimensional kinematics, this requirement is that the constraints on the velocity are completely non-holonomic (see [3]).

Then the controllability follows from the Rashevsky-Chow theorem (Theorem 2.4). We shall present our results separately; in §1.1 for the case $M^1 = \mathbb{R}^1$ and in §1.2 for the case $M^1 = S^1$, respectively. Since [8] was published there has been extensive study on control-affine systems and more general non-linear control systems. We mention some of the vast literature; [33] and [39] for non-holonomic constraints, [11, 9, 16, 17, 20, 21, 32, 35, 36, 44, 50, 51, 56, 58] for collision avoidance and obstacle avoidance, and [7, 15, 37, 40, 47, 52, 53, 61] for navigation and motion planning problems.

What we present in §2 as preliminaries are either basic calculus on manifolds or well known facts to the control theorists, and hence, experts may well skip this section. In §3 we present the proofs of our results. §4 is an epilogue newly added to the first draft of this paper mainly to answer the questions of the referees and possibly of other readers, as well. The authors thank the referees for their comments and critiques.

1.1. Control $n$ body problem on $\mathbb{R}^1$. We consider $n, n \geq 2$, particles in the real line $\mathbb{R}^1$. Let

$$x(t) := (x_1(t), \cdots, x_n(t)), \quad x_j < x_{j+1}, \ j = 1, \cdots, n-1,$$

be their position at time $t$. Once for all in this paper we fix a small positive number $\epsilon > 0$. We shall say that (1.2) is separated by $\epsilon > 0$ if the distance between two adjacent particles is greater than $\epsilon$, namely,

$$x_j + \epsilon < x_{j+1}, \quad \text{for each} \ j = 1, \cdots, n-1.$$
We want to design a control-affine system

\[
\begin{bmatrix}
\dot{x}_1 \\
\vdots \\
\dot{x}_n
\end{bmatrix} = \sum_{\ell=1}^{m} \vec{f}_\ell(x_1, \cdots, x_n)u_\ell(t), \quad m \leq n,
\]

where \( \vec{f}_\ell \) are smooth vector fields in \( \mathbb{R}^n \), expressed as \( n \)-dimensional column vectors, and \( u_\ell(t) \) are piecewise smooth, with the following requirements:

1) Collision free, that is, \( (1.2) \) is separated by \( \epsilon > 0 \).

2) Controllability, that is, for any sequences of real numbers \( p := (p_1, \cdots, p_n) \) and \( q := (q_1, \cdots, q_n) \) that are separated by \( \epsilon > 0 \), there exist piecewise smooth real-valued functions \( u_\ell(t), \ell = 1, \cdots, m \), in some interval \( 0 \leq t \leq T \), so that \( (1.4) \) has a solution \( x(t) \) with \( x_j(0) = p_j \) and \( x_j(T) = q_j \) for each \( j = 1, \cdots, n \). The terminal point \( x(T) = (x_1(T), \cdots, x_n(T)) \) of the initial value problem \( (1.4) \) with the initial condition \( x(0) = p \) shall be denoted by \( \Gamma(p, T, u) \), where \( u(t) = (u_1(t), \cdots, u_m(t)) \). See §2.1 for more details.

Let \( \mathbb{R}_\epsilon^n \) be the set of points \( x = (x_1, \cdots, x_n) \in \mathbb{R}^n \) that satisfy \( (1.3) \). For each \( j = 1, \cdots, n-1 \), let

\[
\rho_j(x) := x_{j+1} - x_j - \epsilon.
\]

Then

\[
\mathbb{R}_\epsilon^n = \bigcap_{j=1}^{n-1} \{ \rho_j > 0 \}.
\]

The requirement 1); system’s being collision-free, is equivalent to that \( \mathbb{R}_\epsilon^n \) is invariant under the flow of the vector fields \( \pm \vec{f}_\ell, \ell = 1, \cdots, m \). Our basic observations are the following: A smooth vector field \( \vec{f} = \sum_{i=1}^{n} a_j(x)\partial/\partial x_i \) on \( \mathbb{R}^n \) is tangent to a hypersurface given by \( \{ \rho = 0 \} \), with \( d\rho \neq 0 \), if and only if

\[
(\vec{f} \rho)(x) = 0, \quad \text{for all } x \text{ with } \rho(x) = 0.
\]

A function \( \rho \) that satisfies \( (1.7) \) is called a generalized first integral of the vector field \( \vec{f} \) (see §2.2). For a smooth function \( \rho \) on \( \mathbb{R}^n \) let us denote by \( (\rho) \) the ideal generated by \( \rho \), namely, the collection of all the smooth functions on \( \mathbb{R}^n \) that are divisible by \( \rho \). Then \( (1.7) \) is saying
that \( f_\rho \) belongs to the ideal \(((\rho))\); see Lemma 2.5 for the details. So, in order to design a control-affine system (1.4) we construct vector fields \( f_\ell, \ell = 1, \ldots, m \), on \( \mathbb{R}^n \) so that

i) each \( f_\ell \) is tangent to the boundary of \( \mathbb{R}^n \) (collision-free),

ii) the Lie algebra generated by \( f_1, \ldots, f_m \) spans the whole tangent space at every point of \( \mathbb{R}^n \) (controllability).

Now we state our main results:

**Theorem 1.1.** Given \( \epsilon > 0 \) and a positive integer \( n, n \geq 2 \), let \( \rho_j, j = 1, \ldots, n-1 \), be smooth real valued functions on \( \mathbb{R}^n \) as in (1.5) and \( \mathbb{R}^n_\epsilon \) is the connected open subset of \( \mathbb{R}^n \) given by (1.6). Then there exist smooth vector fields \( f_1, \ldots, f_m, m \leq n \), on \( \mathbb{R}^n \) having the following properties:

i) The hypersurfaces \( \rho_j(x) = 0, j = 1, \ldots, n-1 \), and the submanifold \( \mathbb{R}^n_\epsilon \) are invariant under the flow of the vector fields \( \pm f_\ell, \ell = 1, \ldots, m \).

ii) The control-affine system \( \dot{x} = \sum_{\ell=1}^{m} f_\ell(x)u_\ell(t) \) is controllable in \( \mathbb{R}^n_\epsilon \) (see Definition 2.2).

Such \( f_\ell, \ell = 1, \ldots, m \), are constructible for \( m \) with

\[
2m \geq n + 1,
\]

so that \( f_\ell \) depends only on \( x_{\ell-1}, x_\ell, x_{m+\ell-2} \) and \( x_{m+\ell-1} \).

**Example 1.2.** The case \( n = 2 \) : Let \( \rho(x_1, x_2) = x_2 - x_1 - \epsilon \) and \( \mathbb{R}^2_\epsilon = \{\rho > 0\} \) be the open half plane. Then \( m = 2 \) is the smallest integer that satisfies (1.8) and the vector fields

\[
\begin{align*}
\bar{f}_1 &:= \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \\
\bar{f}_2 &:= (1 + \rho) \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2},
\end{align*}
\]

as shown in Figure 1, are linearly independent in \( \mathbb{R}^n_\epsilon \).

Moreover, \( \bar{f}_\ell \rho = 0 \) on the set \( \{\rho = 0\} \), for each \( \ell = 1, 2 \), therefore, \( \bar{f}_\ell \) is tangent to the zero set of \( \rho \), which is the boundary of \( \mathbb{R}^2_\epsilon \). It follows that \( \mathbb{R}^2_\epsilon \) is invariant under the flow of \( \bar{f}_\ell \) and the controllability of (1.4) follows from the Rashevsky-Chow theorem (Theorem 2.4).

Recalling that \( n \) bodies in \( \mathbb{R}^1 \) were identified as a point of \( \mathbb{R}^n \), from Theorem 1.1 it follows
Corollary 1.3. Given $\epsilon > 0$ and an integer $n$, $n \geq 2$, the control $n$-body problem in the real line $\mathbb{R}$ with separation by $\epsilon$ is solvable by constructing $n$-dimensional vector fields $\vec{f}_1, \cdots, \vec{f}_m$, $2m \geq n + 1$. Furthermore, for each $\ell = 1, \cdots, m$, $\vec{f}_\ell$ depends only on $x_{\ell-1}, x_\ell, x_{m+\ell-2}$ and $x_{m+\ell-1}$.

1.2. Control $n$-body problem on $S^1$. We consider now $n, n \geq 2$, particles moving on $S^1$. Let

$$e^{ix_1(t)}, \cdots, e^{ix_n(t)}$$

be their position at time $t$, where $x_j, j = 1, \cdots, n$, is the angle measured in radians from a reference point of $S^1$. Assuming $\epsilon < 2\pi/n$, we shall say that these points are separated by $\epsilon$ if

$$x_1 + \epsilon < x_2 < x_2 + \epsilon < x_3 < \cdots < x_n + \epsilon < x_1 + 2\pi. \tag{1.10}$$

Let $\mathbf{x} := (x_1, \cdots, x_n)$ and let

$$\rho_j(\mathbf{x}) := x_{j+1} - x_j - \epsilon, \quad \text{for } j = 1, \cdots, n-1$$

$$\rho_n(\mathbf{x}) := x_1 + 2\pi - x_n - \epsilon. \tag{1.11}$$

Let

$$\tilde{\mathbb{R}}^n_\epsilon := \{ \mathbf{x} \in \mathbb{R}^n : \rho_j(\mathbf{x}) > 0, \ j = 1, \cdots, n \}, \tag{1.12}$$

that is, the set of points where (1.10) holds. Now we denote by $2\pi\mathbb{Z}$ the set of all integer multiples of $2\pi$ and consider the $n$-torus

$$T^n := \mathbb{R}^n/(2\pi\mathbb{Z})^n.$$ 

Since the projection

$$p : \mathbb{R}^n \to T^n$$

is a local diffeomorphism and the subset $\tilde{\mathbb{R}}^n_\epsilon \subset \mathbb{R}^n$ is open and connected, we have
Proposition 1.4. \( T^n_\epsilon := p(\tilde{T}^n_\epsilon) \) is an open connected subset of \( T^n \).

In the light of Rashevsky-Chow’s theorem (Theorem 2.4) we want to construct smooth vector fields \( \vec{f}_\ell, \ell = 1, \ldots, m, m \leq n \), on \( T^n \), that satisfy the following conditions:

i) The submanifold \( T^n_\epsilon \subset T^n \) is invariant under the motions subject to the control-affine system (1.4).

ii) \( \vec{f}_\ell, \ell = 1, \ldots, m \), together with their Lie-brackets of first order span the whole tangent space of \( T^n_\epsilon \).

A vector field on \( \mathbb{R}^n \)

(1.13) \[
\vec{f} = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j}
\]

shall be said to be periodic if each \( a_j \) is a periodic function in each variable \( x_1, \ldots, x_n \). A periodic vector field on \( \mathbb{R}^n \) with period \( 2\pi \) in each variable may be regarded as a vector field on \( T^n \). Thus we state our result as follows:

Theorem 1.5. Given \( \epsilon > 0 \) and a positive integer \( n, n \geq 2 \), with \( \epsilon < 2\pi/n \), there exist smooth vector fields \( \vec{f}_1, \ldots, \vec{f}_m, m \leq n \), on \( \mathbb{R}^n = \{ x := (x_1, \ldots, x_n) \} \), which are periodic with period \( 2\pi \), having the following properties:

i) The hypersurfaces \( \rho_j(x) = 0, j = 1, \ldots, n - 1 \), the hypersurface \( \rho_n(x) = 0 \), and the open submanifold \( \tilde{T}^n_\epsilon \) are invariant under the flow of vector fields \( \pm \vec{f}_\ell \), where \( \rho_j \) and \( \rho_n \) are as defined by (1.11).

ii) \( \vec{f}_\ell, [\vec{f}_\ell, \vec{f}_k], \ell, k = 1, \ldots, m \), span the whole tangent space of \( \tilde{T}^n_\epsilon \).

Such \( \vec{f}_\ell, \ell = 1, \ldots, m \), are constructible in the following cases:

Case 1; both \( n \) and \( m \) are odd numbers and \( 2m \geq n + 1 \).

Case 2; \( n \) is even and \( 2m \geq n + 2 \).

2. PRELIMINARIES

2.1. Orbits and controllability. Let \( M \) be a connected smooth manifold of dimension \( n \) and \( \mathcal{U} \) be a set of admissible controls \( u : [0, \infty) \rightarrow \mathbb{R}^m, m \leq n \). Let

(2.1) \[
\dot{x} := \frac{dx}{dt} = \sum_{j=1}^{m} \vec{f}_\ell(x) u_\ell
\]
be a control-affine system on $M$, where $\vec{f}_\ell, \ell = 1, \cdots, m$, are smooth vector fields on $M$, and $u = (u_1, \ldots, u_m) \in U$. Here the control $u(t)$ can be chosen variously, for instance, to be piecewise continuous, measurable, smooth, and so forth. In this paper we shall assume that $U$ is the set of all piecewise smooth functions with finitely many discontinuities.

For a point $p \in M$ let $x(t)$ be the solution of (2.1) subjected to the initial condition $x(0) = p$. The solution $x(t)$ is continuous, piecewise smooth and uniquely determined by the choice of $p$ and $u(t)$, which we shall denote by $\Gamma(p, t, u)$.

**Definition 2.1.** A submanifold $N$ of $M$ is said to be invariant under controls of (2.1) if $x_0 \in N$ implies that $\Gamma(x_0, t, u) \in N$ for all possible choices of $u \in U$ and for all $t \geq 0$.

Now we recall some basics of non-linear control systems. For other definitions and theorems we refer the reader to our basic references [3, 8]. We consider the set of points that are reachable from a point in some non-negative time.

**Definition 2.2.** A point $q \in M$ is said to be reachable from $p$ if $q = \Gamma(p, t, u)$, for some $u \in U$ and for some $t \geq 0$. The reachable set $R_p$ of the control system (2.1) from a point $p \in M$ is a subset of $M$ defined by

$$R_p = \{\Gamma(p, t, u) : t \geq 0, \ u \in U\}.$$

The control system (2.1) is said to be controllable from $p \in M$ if

$$(2.2) \quad R_p = M$$

and it is called controllable if (2.2) holds for every $p \in M$. The orbit $O_p$ of the control system (2.1) through a point $p \in M$ is the set of points $q \in M$ such that either $q$ is reachable from $p$ or $p$ is reachable from $q$.

A basic theorem is the following

**Theorem 2.3** (Nagano-Sussmann theorem, [3]). $O_p$ is a connected immersed submanifold of $M$.

See [3] for the proof of Theorem 2.3. Now let $\mathcal{G}$ be the Lie algebra of vector fields generated by $\vec{f}_1, \ldots, \vec{f}_m$. Consider the vector space $\mathcal{G}(p) \subset$
$T_pM$, which is the linear span at $p \in M$ of the left iterated Lie brackets

$$[\vec{f}_1, [\vec{f}_2, \cdots, [\vec{f}_{i_k-1}, \vec{f}_{i_k}] \cdots]]$$

of the vector fields $\vec{f}_1, \ldots, \vec{f}_m$. Then we have

**Theorem 2.4** (Rashevsky-Chow theorem, [3]). Let $M$ and $G$ be as above. If $M$ is connected and $G(p) = T_pM$ at every point $p \in M$, then $O_p = M$, that is (2.1) is controllable.

### 2.2. Generalized first integrals.

Let $O$ be a germ of a smooth ($C^\infty$) manifold of dimension $n$, or a small open ball of $\mathbb{R}^n$ centered at the origin. A set of real-valued smooth functions $r_1, \cdots, r_d$ of $O$ is said to be **non-degenerate** if

$$dr_1 \wedge \cdots \wedge dr_d \neq 0.$$  

The common zeros of a non-degenerate set of $d$ real-valued functions form a submanifold of codimension $d$.

**Lemma 2.5.** A smooth real-valued function $\rho$ vanishes on the common zero set of a non-degenerate system $r_1, \cdots, r_d$ of smooth real-valued functions if and only if

$$\rho = \sum_{k=1}^d \alpha_k r_k,$$

for some smooth functions $\alpha_k$s.

**Proof.** The sufficiency of the condition is obvious. To show the necessity of the condition (2.3) notice that the non-degeneracy implies that $r_1, \cdots, r_d$ can be part of the coordinate system of $O$. Thus we may assume that $r_j = x_j$ for $j = 1, \cdots, d$. We have

$$\rho(x_1, \cdots, x_d, x_{d+1}, \cdots, x_n) - \rho(0, \cdots, 0, x_{d+1}, \cdots, x_n)$$

$$= [\rho(sx_1, \cdots, sx_d, x_{d+1}, \cdots, x_n)]_0^1$$

$$= \int_0^1 \frac{d}{ds} \rho(sx_1, \cdots, sx_d, x_{d+1}, \cdots, x_n) ds$$

$$= \sum_{k=1}^d x_k \int_0^1 \frac{\partial \rho}{\partial x_k} (sx_1, \cdots, sx_d, x_{d+1}, \cdots, x_n) ds.$$
Setting the integral of the last line $\alpha_k$ and recalling $x_k = r_k$, for $k = 1, \cdots, d$, we obtain (2.3).

Lemma 2.5 is a local version in the smooth category of Hilbert’s Nullstellensatz and the condition (2.3) is that $\rho$ is in the ideal $((r^1, \cdots, r^d))$.

**Definition 2.6.** Let $\vec{f}_1, \cdots, \vec{f}_m$ be smooth vector fields on $\mathcal{O}$ that are linearly independent at every point of $\mathcal{O}$. A real-valued smooth function $\rho$ is a first integral if

$$\vec{f}_\ell \rho = 0, \text{ for all } \ell = 1, \cdots, m.$$  

For $m$ vector fields on $n$-manifold there can be at most $(n - m)$ non-degenerate first integrals. This is the case that the Frobenius integrability condition holds (see [6]).

**Definition 2.7.** A real-valued smooth function $\rho$ is a generalized first integral of $\vec{f}_1, \cdots, \vec{f}_m$ if

$$\vec{f}_\ell \rho = 0, \text{ on } \{\rho = 0\}, \text{ for all } \ell = 1, \cdots, m.$$  

The notion of the generalized first integral was first defined in [2]. We have

**Proposition 2.8.** Suppose that $\rho$ is a smooth real-valued function with $d\rho \neq 0$ on $\mathcal{O}$ and $X \subset \mathcal{O}$ is the zero set of $\rho$. Then the following are equivalent:

i) $\rho$ is a generalized first integral of the vector fields $\vec{f}_1, \cdots, \vec{f}_m$.

ii) For each $\ell = 1, \cdots, m$, $\vec{f}_\ell \rho$ is divisible by $\rho$, that is, there is a smooth function $\alpha(x)$ so that

$$\vec{f}_\ell \rho = \alpha \rho.$$  

iii) For each $\ell = 1, \cdots, m$, $\vec{f}_\ell$ is tangent to $X$.

iv) $X$ is invariant under the flow of $\pm \vec{f}_\ell, \ell = 1, \cdots, m$.

**Proof.** $i) \implies ii)$ follows from Definition 2.5 and Lemma 2.5 The other implications are easy to prove or obvious.  

Using the observations of Proposition 2.8 the partial integrability of systems of vector fields, or equivalently, the partial holonomy of (2.1) has been studied in [22, 23, 24] and their applications to invariant submanifolds in [25, 26, 27].
3. Proof of the theorems

3.1. Proof of Theorem 1.1. We shall construct $\vec{f}_\ell$, $\ell = 1, \cdots, m$, for each of the following three cases:

1) Case $m = n$. Let $\vec{f}_\ell$ be the $\ell$-th column of the following $m \times m$ matrix:

$$
\begin{bmatrix}
1 & 1 + \rho_1 & 1 + \rho_2 & 1 + \rho_3 & \cdots & 1 + \rho_{m-1} \\
1 & 1 & 1 + \rho_2 & 1 + \rho_3 & \cdots & 1 + \rho_{m-1} \\
1 & 1 & 1 & 1 + \rho_3 & \cdots & 1 + \rho_{m-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & 1 & \cdots & 1 + \rho_{m-1} \\
1 & 1 & 1 & 1 & \cdots & 1 \\
\end{bmatrix},
$$

(3.1)

namely,

$$
\vec{f}_1 = \sum_{j=1}^{n} \partial / \partial x_j ,
$$

(3.2)

$$
\vec{f}_\ell = \sum_{j=1}^{\ell-1} (1 + \rho_{\ell-1}) \partial / \partial x_j + \sum_{j=\ell}^{n} \partial / \partial x_j , \quad \ell = 2, \cdots, m.
$$

Then (1.5) implies that

$$
\vec{f}_1 \rho_j = 0, \quad j = 1, \cdots, n - 1,
$$

$$
\vec{f}_\ell \rho_{\ell-1} = -\rho_{\ell-1}, \quad \ell = 2, \cdots, m,
$$

$$
\vec{f}_\ell \rho_j = 0, \quad \text{for the other pairs } (\ell, j).
$$

Therefore, by Proposition 2.8, $\vec{f}_\ell$ are tangent to $\{\rho_j = 0\}$, for each pair $(\ell, j)$. This implies that the boundary of $\mathbb{R}^n_c$ is invariant, and therefore the interior of $\mathbb{R}^n_c$ is invariant, under the affine control system (1.4). Furthermore, the square matrix (3.1) is invertible on $\mathbb{R}^n_c = \bigcap_{j=1}^{n-1} \{\rho_j > 0\}$. Hence, by Theorem 2.4, the control-affine system (1.4) is controllable.

2) Case $m < n$ and $2m = n + 1$: For $\ell = 1, \cdots, m$, let $\vec{f}_\ell$ be the $\ell$-th column of the following matrix:
Thus $\vec{f}_\ell, \ell = 1, \ldots, m$, is an $n$-dimensional vector given by

\begin{equation}
\vec{f}_1 = \sum_{j=1}^{n} \frac{\partial}{\partial x_j},
\end{equation}

\begin{equation}
\vec{f}_\ell = \sum_{j=1}^{\ell-1} (1 + \rho_{\ell-1}) \frac{\partial}{\partial x_j} + \sum_{j=\ell}^{m-2} \frac{\partial}{\partial x_j} + \sum_{j=m+\ell-1}^{n} (1 + x_{m+\ell-2}\rho_{m+\ell-2}) \frac{\partial}{\partial x_j}, \quad \text{for } \ell = 2, \ldots, m.
\end{equation}

Then by easy computations we have

\begin{equation}
\vec{f}_1 \rho_j = 0, \quad \text{for all } j = 1, \ldots, n - 1,
\end{equation}

\begin{equation}
\vec{f}_\ell \rho_{\ell-1} = -\rho_{\ell-1}, \quad \text{for all } \ell = 2, \ldots, m,
\end{equation}

\begin{equation}
\vec{f}_\ell \rho_{m+\ell-2} = x_{m+\ell-2}\rho_{m+\ell-2}, \quad \text{for all } \ell = 2, \ldots, m,
\end{equation}

\begin{equation}
\vec{f}_\ell \rho_j = 0, \quad \text{for all other pairs } (\ell, j).
\end{equation}

Then by Proposition 2.8 and (3.5) the vector fields $\vec{f}_1, \ldots, \vec{f}_m$ are tangent to the boundary of $\mathbb{R}^n_{\epsilon}$, which implies that the interior of $\mathbb{R}^n_{\epsilon}$ is invariant, under the control-affine system (1.4). On the other hand
\begin{equation}
[f_1, \bar{f}_\ell] = \rho_{m+\ell-1} \sum_{j=m+\ell-1}^{n} \frac{\partial}{\partial x_j}, \text{ for all } \ell = 2, \ldots, m,
\end{equation}
so that \([f_1, \bar{f}_\ell]\) is the \((\ell - 1)\)th column of the following matrix:

\begin{equation}
\begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\rho_m & 0 & 0 & \cdots & 0 \\
\rho_m & \rho_{m+1} & 0 & \cdots & 0 \\
\rho_m & \rho_{m+1} & \rho_{m+2} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_m & \rho_{m+1} & \rho_{m+2} & \rho_{m+3} & \cdots & \rho_{n-1}
\end{bmatrix}.
\end{equation}

Notice that the bottom part of (3.7) is a \((m - 1) \times (m - 1)\) square matrix, which is invertible on \(\mathbb{R}^n\). Thus \(m\) columns of the matrix (3.1) and \((m - 1)\) columns of (3.7), so that their totality \(m + (m - 1) = n\) vectors are independent on \(\mathbb{R}^n\). Now conclusion follows from Theorem 2.4.

3) Case \(2m > n + 1\): Proof is same as the previous case, except for that we need only the columns of (3.7) up to \((n + 1 - m)\)-th in order to span the whole tangent space of \(\mathbb{R}^n\).

3.2. Proof of Theorem 1.5. Let \(\rho_j\) be the function on \(\mathbb{R}^n\) as in (1.11). We define a function \(\sigma(s)\) in a real variable \(s\) as in Figure 2 by

\[
\sigma(s) := \sin(s - \pi/2) + 1.
\]

![Figure 2](image-url)
Now we define

\[(3.8) \quad \sigma_j(x) := \sigma(\rho_j(x)), \quad j = 1, \cdots, n.\]

Immediately we have the following

**Proposition 3.1.** For each \( j = 1, \cdots, n \), \( \sigma_j(x) \) is a smooth function of \( \mathbb{R}^n \) with the following properties:

i) \( \sigma_j \) is periodic in each variable with period \( 2\pi \).

ii) For \( j = 1, \cdots, n-1 \), \( \sigma_j \) depends only on \( x_{j+1} \) and \( x_j \), and \( \sigma_n \) depends only on \( x_1 \) and \( x_n \).

iii) \( \sigma_j(x) > 0 \), for all \( x \in \mathbb{R}^n \).

iv) \( \sigma_j(x) = 0 \), for all \( x \) with \( \rho_j(x) = 0 \), that is, \( \sigma_j \in ((\rho_j)) \).

Now we prove Theorem 1.1 for the following three cases:

1) case \( m = n \); the largest choice of \( m \). Let \( \vec{f}_j \) be the \( j \)-th column of the following matrix:

\[(3.9) \quad \begin{bmatrix}
1 & 1 + \sigma_1 \sigma_n & 1 + \sigma_2 \sigma_n & \cdots & 1 + \sigma_{n-2} \sigma_n & 1 + \sigma_{n-1} \sigma_n \\
1 & 1 & 1 + \sigma_2 \sigma_n & \cdots & 1 + \sigma_{n-2} \sigma_n & 1 + \sigma_{n-1} \sigma_n \\
1 & 1 & 1 & \cdots & 1 + \sigma_{n-2} \sigma_n & 1 + \sigma_{n-1} \sigma_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 + \sigma_{n-2} \sigma_n & 1 + \sigma_{n-1} \sigma_n \\
1 & 1 & 1 & \cdots & 1 & 1 + \sigma_{n-1} \sigma_n \\
1 & 1 & 1 & \cdots & 1 & 1 
\end{bmatrix} .\]

By easy computation we see that

\[(3.10) \quad \begin{align*}
\vec{f}_i \rho_j & = 0, \quad \text{for all } j = 1, \cdots, n \\
\vec{f}_i \rho_{\ell-1} & = -\sigma_{\ell-1} \sigma_n, \quad \text{for } \ell = 2, \cdots, m \\
\vec{f}_i \rho_{\ell} & = \sigma_{\ell-1} \sigma_n, \quad \text{for } \ell = 2, \cdots, m \\
\vec{f}_i \rho_j & = 0, \quad \text{for all other pairs } (\ell, j) .
\end{align*}\]

From the second equation of (3.10) and Proposition 3.1 (iv) we have

\[(3.11) \quad \begin{align*}
\vec{f}_i \rho_{\ell-1} & \in ((\rho_{\ell-1})) \\
\vec{f}_i \rho_{\ell} & \in ((\rho_{\ell})).
\end{align*}\]
It follows from (3.10)-(3.11) that the vector fields \( \vec{f}_1, \cdots, \vec{f}_m \) are tangent to the zero sets \( \{ \rho_j = 0 \} \), which implies that the boundary of \( \tilde{R}^n \) is invariant, and also, the interior of \( \tilde{R}^n \) is invariant under the control-affine system (1.4). Furthermore, by Proposition 3.1-(iii) the matrix (3.9) is invertible on \( \tilde{R}^n \), which implies that \( \vec{f}_\ell, \ell = 1, \cdots, m, \) span the whole tangent space of \( \tilde{R}^n \).

2) Case that \( n \) is odd, \( m \) is odd, and \( 2m = n + 1 \).

Let us consider the following \( n \times m \) matrix, whose top part is the same as the first \( m \) columns and the first \( m \) rows of (3.9) and the bottom part has \( (n - m) \) rows. For \( \ell = 1, \cdots, m \), let \( \vec{f}_\ell \) be the \( \ell \)-th column of the following matrix:

\[
\begin{bmatrix}
1 & 1 + \sigma_1 \sigma_n & 1 + \sigma_2 \sigma_n & 1 + \sigma_3 \sigma_n & \cdots & 1 + \sigma_{m-2} \sigma_n & 1 + \sigma_{m-1} \sigma_n \\
1 & 1 & 1 + \sigma_2 \sigma_n & 1 + \sigma_3 \sigma_n & \cdots & 1 + \sigma_{m-2} \sigma_n & 1 + \sigma_{m-1} \sigma_n \\
1 & 1 & 1 & 1 + \sigma_3 \sigma_n & \cdots & 1 + \sigma_{m-2} \sigma_n & 1 + \sigma_{m-1} \sigma_n \\
1 & 1 & 1 & 1 & \cdots & 1 + \sigma_{m-2} \sigma_n & 1 + \sigma_{m-1} \sigma_n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & 1 & \cdots & 1 + \sigma_{m-2} \sigma_n & 1 + \sigma_{m-1} \sigma_n \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 + \sigma_{m-1} \sigma_n \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & 1 & \cdots & 1 + A_{(m-1)/2} & 1 + B_{(m-1)/2} \\
1 & 1 & 1 & 1 & \cdots & 1 - B_{(m-1)/2} & 1 + A_{(m-1)/2} \\
\end{bmatrix}.
\]

The bottom part of (3.12) has \( 2 \times 2 \) square submatrices

\[
\begin{bmatrix}
1 + A_\lambda & 1 - B_\lambda \\
1 + B_\lambda & 1 + A_\lambda \\
\end{bmatrix}, \quad \lambda = 1, 2, \cdots, (m - 1)/2,
\]

where
\[ A_\lambda = \sin x_{m+2(\lambda-1)}\sigma_{m+2(\lambda-1)}\sigma_{m+2(\lambda-1)+1} \]
\[ B_\lambda = \cos x_{m+2(\lambda-1)}\sigma_{m+2(\lambda-1)}\sigma_{m+2(\lambda-1)+1}, \]

so that the last block with \( \lambda = (m-1)/2 \) is
\[
\begin{bmatrix}
1 + A_{(m-1)/2} & 1 - B_{(m-1)/2} \\
1 + B_{(m-1)/2} & 1 + A_{(m-1)/2}
\end{bmatrix},
\]

where
\[
A_{(m-1)/2} = \sin x_{n-2}\sigma_{n-2}\sigma_{n-1},
\]
\[
B_{(m-1)/2} = \cos x_{n-2}\sigma_{n-2}\sigma_{n-1}.
\]

Since \( \vec{f}_1 = \sum_{i=1}^{n} \partial / \partial x_i \) calculation shows that for each \( j = 1, \cdots, n \), and each \( \lambda = 1, \cdots, (m-1)/2 \) we have
\[
\vec{f}_1 \sigma_j = \sigma'(\rho_j) \vec{f}_1 \rho_j = 0,
\]
\[
\vec{f}_1 A_\lambda = \cos x_{m+2(\lambda-1)}\sigma_{m+2(\lambda-1)}\sigma_{m+2(\lambda-1)+1}
\]
\[
\vec{f}_1 B_\lambda = -\sin x_{m+2(\lambda-1)}\sigma_{m+2(\lambda-1)}\sigma_{m+2(\lambda-1)+1}.
\]

Therefore, expressing the brackets \([\vec{f}_1, \vec{f}_\ell] \) for \( \ell = 2, 3, 4, 5 \) as column vectors, we have
\[
(3.16)
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 \\
\cos x_m\sigma_m\sigma_{m+1} & \sin x_m\sigma_m\sigma_{m+1} & 0 & 0 & 0 \\
-\sin x_m\sigma_m\sigma_{m+1} & \cos x_m\sigma_m\sigma_{m+1} & 0 & 0 & 0 \\
0 & 0 & \cos x_{m+2}\sigma_{m+2}\sigma_{m+3} & \sin x_{m+2}\sigma_{m+2}\sigma_{m+3} & \cos x_{m+2}\sigma_{m+2}\sigma_{m+3} \\
0 & 0 & -\sin x_{m+2}\sigma_{m+2}\sigma_{m+3} & \cos x_{m+2}\sigma_{m+2}\sigma_{m+3} & \cos x_{m+2}\sigma_{m+2}\sigma_{m+3} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Notice that \( \sigma_j \)'s are positive valued on \( \tilde{\mathbb{R}}^n = \bigcap_{i=1}^{n} \{ \rho_i > 0 \} \). Thus for each \( \ell = 2, \cdots, m \), \([\vec{f}_1, \vec{f}_\ell] \) divided by a positive scalar, is the \((\ell - 1)\)-th column of the following matrix;
The above \((m - 1)\) vectors are linearly independent and we see that \(\vec{f}_\ell\), \(\ell = 1, \cdots, m\) and \([\vec{f}_1, \vec{f}_\ell]\), \(\ell = 2, \cdots, m\), span the tangent space of \(\mathbb{R}^n\).

3) Case that \(n\) is even, \(m\) is even, and \(2m \geq n + 2\): It is enough to construct for the case \(2m = n + 2\). Since \(m\) is even and we want \(\vec{f}_1\) to be a constant vector field \(\sum_{j=1}^{n} \partial / \partial x_j\) as in the previous case. Since \(m - 1\) is odd, we can construct \(m\) blocks of size \(2 \times 2\) in the same way as in the previous case. Construction is same except for the brackets \([\vec{f}_1, \vec{f}_\ell]\) ends up with \(\ell = m - 1\).

4. **The Authors' Motivation and Some Future Problems**

The present paper is an outcome of the authors’ attempt to find applications to control theory of the generalizations of the Frobenius theorem on involutivity \[22, 23, 24\]. The quasi-linear systems of first order partial differential equations that arise from various problems in geometry and physics are often the problem of deciding the partial integrability of the associated Pfaffian systems. In particular, the notion of the generalized first integral and its construction are useful for finding the partition of the configuration space into invariant submanifolds of dynamical systems, see \[25, 26, 27, 28\]. The collision avoidance problems belong to this category. It turns out that even 1-dimensional cases are not obvious, for which we constructed vector fields \(\vec{f}_\ell\), so that the associated obstacle and its complement in the product space are
invariant of the dynamics of the control-affine system \([1,1]\). In general dimensions, however, the methods of the exterior differential system are more convenient for finding the generalized first integrals. We shall explain the ideas briefly now.

4.1. **Exterior differential system approach to the first integrals and the generalized first integrals.** Let

\[
\vec{f}_1, \ldots, \vec{f}_m
\]

be a system of linearly independent smooth vector fields defined on a small open set \(\mathcal{O}\) of an \(n\)-dimensional smooth manifold. Let

\[
\theta := (\theta^1, \ldots, \theta^s), \quad s := n - m,
\]

be a system of independent 1-forms that annihilate \(\vec{f}_j, j = 1, \ldots, m\), namely,

\[
\theta^\alpha(\vec{f}_\ell) = 0, \quad \text{for all } \alpha = 1, \ldots, s, \forall \ell = 1, \ldots, m.
\]

We shall call (4.2) the **Pfaffian system** associated with (4.1). Let

\[
\Omega^*(\mathcal{O}) := \bigoplus_{k=0}^{n} \Omega^k(\mathcal{O})
\]

be the exterior algebra of smooth differential forms of \(\mathcal{O}\), where \(\Omega^k(\mathcal{O})\) is the module of differential \(k\)-forms for \(k = 1, \ldots, n\), and \(\Omega^0(\Omega) := C^\infty(\Omega)\) is the ring of smooth real-valued functions of \(\mathcal{O}\). We denote by \(((\cdots))\) an algebraic ideal of \(\mathcal{O}^*(\mathcal{O})\) generated by what are inside the double parenthesis \(((\ ))\). Then Definition 2.6 and Definition 2.7 are equivalent to the following

**Definition 4.1.** A smooth function \(\rho\) is a first integral of (4.1) if

\[
d\rho \in ((\theta)),
\]

where \(((\theta)) = ((\theta^1, \ldots, \theta^s))\). A smooth function \(\rho\) is a generalized first integral of (4.1) if

\[
d\rho \in ((\rho, \theta)).
\]

**Definition 4.2.** The system of 2-forms

\[
d\theta, \text{ modulo } ((\theta))
\]

is called the **torsion** of (4.2).
Torsion is the obstruction to the integrability of (4.2) and by analyzing its exterior-algebraic properties we can determine various partial integrabilities. Among them we can find the generalize first integrals explicitly, as we shall now explain briefly for the cases $n = 3$ and $m = 2$.

Suppose that $\vec{f}_1$ and $\vec{f}_2$ are smooth vector fields that are linearly independent on a small connected open subset $\mathcal{O} \subset \mathbb{R}^3 = \{(x, y, z)\}$. Let $\theta$ be a nowhere vanishing 1-form that annihilates $\vec{f}_\ell, \ell = 1, 2$. Choose 1-forms $\omega^1, \omega^2$ so that $(\theta, \omega^1, \omega^2)$ is a coframe of $\mathcal{O}$. Let $T$ be the coefficient of the torsion:

$$d\theta \equiv T\omega^1 \wedge \omega^2, \mod (\theta).$$

By (4.4)

$$d\rho = \rho\psi + \lambda\theta, \quad \lambda \neq 0,$$

for some 1-form $\psi$ and for some function $\lambda$. Now apply $d$ to (4.6) and mod out by $(\theta)$ and then substitute (4.6) for $d\rho$:

$$0 = d\rho \wedge \psi + \rho d\psi + d\lambda \wedge \theta + \lambda d\theta$$

$$\equiv \lambda T\omega^1 \wedge \omega^2, \mod (\rho, \theta).$$

Since $\lambda \neq 0$ and $\theta, \omega^1, \omega^2$ are independent, (4.7) implies

$$T \in (\rho),$$

which means that $T$ is divisible by $\rho$. Thus for a smooth function $\rho$ to be a generalized first integral of $\theta$, a necessary condition is that $\rho$ is a smooth non-degenerate factor of $T$. Now we pass $\rho$ through the test: if (4.4) holds this $\rho$ is indeed a generalized first integral.

**Example 4.3.** Let $\mathcal{O} = \mathbb{R}^3 = \{(x, y, z)\}$ and

$$\vec{f}_1 = \frac{\partial}{\partial x},$$

$$\vec{f}_2 = \frac{\partial}{\partial y} + xyz \frac{\partial}{\partial z}.$$  

Then

$$\theta = dz - xyz dy$$
annihilates $\vec{f}_\ell$, $\ell = 1, 2$. In terms of coframe $(\theta, dx, dy)$ we have

$$d\theta \equiv -yz \ dx \wedge dy, \mod ((\theta)),$$

thus $T(x, y, z) := -yz$ and a single 2-form

$$T dx \wedge dy$$

is the torsion of $\theta$. Now we check whether (2.7) holds for each of the factors of $T$:

$$dy \notin ((y, dz - xyz \ dy))$$

$$dz \in ((z, dz - xyz \ dy)),$$

$z$ is a generalized first integral and $y$ is not.

In this paper we went through a reverse process: For a prescribed function $\rho$ we constructed a Pfaffian system $\theta$, or equivalently, vector fields $\vec{f}_\ell$, $\ell = 1, 2, \cdots$ for which $\rho$ is a generalized first integral. We refer to [19] for the partial integrability as a generalization of the Frobenius theorem, to [41, 48, 49, 57] for the applications of exterior differential systems to control theory, and to [6, 10, 34] for general references.

4.2. Number of vector fields. We constructed $m$ vector fields $\vec{f}_\ell$, $\ell = 1, \cdots, m$, so that they span, together with their brackets of first order, the whole tangent space of dimension $n$. However, the controllability can be achieved with a much smaller number $m \in O(n^{1/2})$ of vector fields and their bracket of first order. If the Lie brackets up to order $k$ are to be used to span the whole tangent spaces of the configuration space, the number of vector fields can be as small as $m \in O(n^{1/2k})$.

4.3. Optimality. We did not discuss the optimality in this paper. In dimension 1, minimizing the total distance traveled is not a problem. In higher dimensions, however, minimizing distance, or more generally, minimizing a certain action is at the heart of path planning. Results on optimal control of collision avoidance and obstacle avoidance are found in [5, 30, 46, 58]. When the obstacle and the configuration space off the obstacle have symmetries then the variational symmetry gives rise to conservation laws due to E. Noether’s theorem, see [18, 59].
4.4. **Monotonicity.** When the configuration space is endowed with a partial order, a dynamical system \( \dot{x} = F(x, u) \), with input \( u \), is said to be monotone if \( \Gamma(x, t, u) \) preserves the order for all \( t \geq 0 \), where \( \Gamma \) is the flow map as defined in \( \S 2 \). We refer to \([4, 30]\) for monotone dynamical systems, and to \([13, 31, 45, 54, 55]\) for monotone control systems. The controllability and reachability for monotone control systems and the relevant results are found in \([12, 14, 43, 60]\). Observe that the monotonicity or the order structure plays no role in the present paper. The circle \( S^1 \) has no natural order and hence the corresponding dynamics on \( T^n \) have no natural notion of monotonicity. However, \( \mathbb{R}^1 \) has a natural order and hence our configuration space \( \mathbb{R}^n \) has partial order defined by the orthant \( \mathbb{R}^n_+ \). Our input functions \( u = (u^1, \cdots, u^m) \) are allowed to take negative values and \((1.4)\) is not necessarily monotone in general. It seems to the authors that for the particular affine control systems in Theorem 1.1, we can determine the monotonicity by using the criteria given by Corollary III.3 of \([4]\), to conclude that

i) The case \( m < n \) and \( 2m = n + 1 \) given by \((3.3)\) is not monotone.

ii) The case \( m = n \), if the input functions \( u \) satisfy

\[ u_2 \geq u_3 \geq \cdots \geq u_m \geq 0, \]

the dynamical system \((3.1)\) is cooperative.

4.5. **Some further problems.** 1) Collision avoidance in \( M^2 \) of dimension 2, namely, \( \mathbb{R}^2, S^2, T^2 \), and so forth: Construct a control-affine system \((1.1)\), so as to avoid collision and be controllable.

2) Construct a control-affine system \((1.1)\) on \( M^2 \) that avoids prescribed obstacles. If the configuration space has symmetry, for example, the annulus bounded by two concentric circles, find the necessary conditions for optimality that come from the variational symmetry and Noether’s theorem. Can one give a complete metric on the annulus so that the geodesics are optimal in any practical sense?
3) Systems of Kolmogorov type equations for three species: Let $x_j$, $j = 1, 2, 3$, be the population of three species. We consider a control system $\dot{x} = \vec{F}(x, u)$, with control parameter $u$. For a prescribed open subset $\mathcal{O}$ with smooth boundary of the configuration space, determine the control $u(t)$ so that $\mathcal{O}$ is invariant under the flow maps. Control problems of Kolmogorov type equations have been studied in [11, 29, 38, 42, 62].

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(C.-K. Han) *Research institute of Mathematics, Seoul National University, 1 Gwanak-ro, Gwanak-gu, Seoul 08826, Republic of Korea*

*Email address*: ckhan@snu.ac.kr

(D. Park) *Department of Mathematics, Yonsei University, 50 Yonsei-Ro, Seodaemun-gu, Seoul 03722, Republic of Korea*

*Email address*: dhpark12@gmail.com