A bound on the scrambling index of a primitive matrix using Boolean rank

Mahmud Akelbek\textsuperscript{a,b,\ast}, Sandra Fital\textsuperscript{b}, Jian Shen\textsuperscript{a,1}

\textsuperscript{a}Department of Mathematics, Texas State University, San Marcos, TX 78666
\textsuperscript{b}Department of Mathematics, Weber State University, Ogden, UT 84408

Abstract

The scrambling index of an $n \times n$ primitive matrix $A$ is the smallest positive integer $k$ such that $A^k (A^t)^k = J$, where $A^t$ denotes the transpose of $A$ and $J$ denotes the $n \times n$ all ones matrix. For an $m \times n$ Boolean matrix $M$, its Boolean rank $b(M)$ is the smallest positive integer $b$ such that $M = AB$ for some $m \times b$ Boolean matrix $A$ and $b \times n$ Boolean matrix $B$. In this paper, we give an upper bound on the scrambling index of an $n \times n$ primitive matrix $M$ in terms of its Boolean rank $b(M)$. Furthermore we characterize all primitive matrices that achieve the upper bound.

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\textit{Key words}: Scrambling index; Primitive matrix; Boolean rank

1 Introduction

For terminology and notation used here we follow \cite{3}. A matrix $A$ is called \textit{nonnegative} if all its elements are nonnegative, and denoted by $A \geq 0$. A matrix $A$ is called \textit{positive} if all its elements are positive, and denoted by $A > 0$. For an $m \times n$ matrix $A$, we will denote its $(i,j)$-entry by $A_{ij}$, its $i$th entry...
row by $A_i$, and its $j$th column by $A_j$. For $m \times n$ matrices $A$ and $B$, we say that $B$ is dominated by $A$ if $B_{ij} \leq A_{ij}$ for each $i$ and $j$, and denote $B \leq A$.

We denote the $m \times n$ all ones matrix by $J_{m,n}$ (and by $J_n$ if $m = n$), The $m \times n$ all zeros matrix by $O_{m,n}$, the all zeros $n$-vector by $j_n$, the $n \times n$ identity matrix by $I_n$, and its $i$th column by $e_i(n)$. The subscripts $m$ and $n$ will be omitted whenever their values are clear from the context.

For an $n \times n$ nonnegative matrix $A = (a_{ij})$, its digraph, denoted by $D(A)$, is the digraph with vertex set $V(D(A)) = \{1, 2, \ldots, n\}$, and $(i, j)$ is an arc of $D(A)$ if and only if $a_{ij} \neq 0$. Then, for a positive integer $r \geq 1$, the $(i,j)$-th entry of the matrix $A^r$ is positive if and only if $i \rightarrow j$ in the digraph $D(A)$. Since most of the time we are only interested in the existence of such walks, not the number of different directed walks from vertex $i$ to vertex $j$, we interpret $A$ as a Boolean $(0,1)$-matrix, unless stated otherwise. A Boolean $(0,1)$-matrix is a matrix with only 0’s and 1’s as its entries. Using Boolean arithmetic, $(1 + 1 = 1, 0 + 0 = 0, 1 + 0 = 1)$, we have that $AB$ and $A + B$ are Boolean $(0,1)$-matrices if $A$ and $B$ are.

Let $D = (V, E)$ denote a digraph (directed graph) with vertex set $V = V(D)$, arc set $E = E(D)$ and order $n$. Loops are permitted but multiple arcs are not. A $u \rightarrow v$ walk in a digraph $D$ is a sequence of vertices $u, u_1, \ldots, u_t, v \in V(D)$ and a sequence of arcs $(u, u_1), (u_1, u_2), \ldots, (u_t, v) \in E(D)$, where the vertices and arcs are not necessarily distinct. We shall use the notation $u \rightarrow v$ and $u \leftrightarrow v$ to denote, respectively, that there is an arc from vertex $u$ to vertex $v$ and that there is no such an arc. Similarly, $u \not\rightarrow v$ and $u \not\leftrightarrow v$ denote, respectively, that there is a directed walk of length $k$ from vertex $u$ to vertex $v$, and that there is no such a walk.

A digraph $D$ is called primitive if for some positive integer $t$ there is a walk of length exactly $t$ from each vertex $u$ to each vertex $v$. If $D$ is primitive the smallest such $t$ is called the exponent of $D$, denoted by $\exp(D)$. Equivalently, a square nonnegative matrix $A$ of order $n$ is called primitive if there exists a positive integer $r$ such that $A^r > 0$. The minimum such $r$ is called the exponent of $A$, and denoted by $\exp(A)$. Clearly $\exp(A) = \exp(D(A))$. There are numerous results on the exponent of primitive matrices [3].

The scrambling index of a primitive digraph $D$ is the smallest positive integer $k$ such that for every pair of vertices $u$ and $v$, there exists some vertex $w = w(u,v)$ (dependent of $u$ and $v$) such that $u \rightarrow w$ and $v \rightarrow w$ in $D$. The scrambling index of $D$ is denoted by $k(D)$. For $u,v \in V(D) \ (u \neq v)$, we define the local scrambling index of $u$ and $v$ as

\[ k_{u,v}(D) = \min \{ k : u \rightarrow w \text{ and } v \rightarrow w \text{ for some } w \in V(D) \} \]
Then

\[ k(D) = \max_{u,v \in V(D)} \{ k_{u,v}(D) \}. \]

An analogous definition for scrambling index can be given for nonnegative matrices. The scrambling index of a primitive matrix \( A \), denoted by \( k(A) \), is the smallest positive integer \( k \) such that any two rows of \( A^k \) have at least one positive element in a coincident position. The scrambling index of a primitive matrix \( A \) can also be equivalently defined as the smallest positive integer \( k \) such that \( A^k A^t = J \), where \( A^t \) denotes the transpose of \( A \). If \( A \) is the adjacency matrix of a primitive digraph \( D \), then \( k(D) = k(A) \). As a result, throughout the paper, where no confusion occurs, we use the digraph \( D \) and the adjacency matrix \( A(D) \) interchangeably.

In [1] and [2], Akelbek and Kirkland obtained an upper bound on the scrambling index of a primitive digraph \( D \) in terms of the order and girth of \( D \), and gave a characterization of the primitive digraphs with the largest scrambling index.

**Theorem 1.1 [1]** Let \( D \) be a primitive digraph with \( n \) vertices and girth \( s \). Then

\[ k(D) \leq n - s + \left\{ \begin{array}{ll}
\frac{(s-1)n}{2}, & \text{when } s \text{ is odd}, \\
\frac{(n-1)s}{2}, & \text{when } s \text{ is even}.
\end{array} \right. \]

When \( s = n - 1 \), an upper bound on \( k(D) \) in terms of the order of a primitive digraph \( D \) can be achieved [1]. We state the theorem in terms of primitive matrices below.

**Theorem 1.2 [1]** Let \( A \) be a primitive matrix of order \( n \geq 2 \). Then

\[ k(A) \leq \left\lfloor \frac{(n-1)^2 + 1}{2} \right\rfloor. \quad (1) \]

Equality holds in (1) if and only if there is a permutation matrix \( P \) such that \( PAP^t \) is one of the following matrices

\[ W_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad J_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{when } n = 2, \]
The digraph $D(W_n)$ is called the Wielandt graph and denoted by $D_{n-1,n}$. It is a digraph with a Hamilton cycle $1 \to 2 \to \cdots \to n \to 1$ together with an arc from vertex $n-1$ to vertex 1. For simplicity, let $h_n = \left\lceil \frac{(n-1)^2+1}{2} \right\rceil$. The next proposition gives some information about the Wielandt graph $D_{n-1,n}$.

**Proposition 1.3** \[1\] For $D_{n-1,n}$, where $n \geq 3$,

(a) $k_{n,\lceil \frac{n}{2} \rceil}(D_{n-1,n}) = h_n$, and for all other pairs of vertices $u$ and $v$ of $D_{n-1,n}$, $k_{u,v}(D_{n-1,n}) < h_n$.

(b) There are directed walks from vertices $n$ and $\lfloor \frac{n}{2} \rfloor$ to vertex 1 of length $h_n$, that is $n \xrightarrow{h_n} 1$ and $\lfloor \frac{n}{2} \rfloor \xrightarrow{h_n} 1$.

For an $m \times n$ Boolean matrix $M$, we define its Boolean rank $b(M)$ to be the smallest positive integer $b$ such that for some $m \times b$ Boolean matrix $A$ and $b \times n$ Boolean matrix $B$, $M = AB$. The Boolean rank of the zero matrix is defined to be zero. $M = AB$ is called a Boolean rank factorization of $M$.

In \[1\], Gregory, Kirkland and Pullman obtained an upper bound on the exponent of primitive Boolean matrix in terms of Boolean rank.

**Proposition 1.4** \[4\] Suppose that $n \geq 2$ and that $M$ is an $n \times n$ primitive Boolean matrix with $b(M) = b$. Then

$$\exp(M) \leq (b - 1)^2 + 2.$$  \[2\]

In \[4\], Gregory, Kirkland and Pullman also gave a characterization of the matrices for which equality holds in (2). In \[5\], Liu, You and Yu gave a characterization of primitive matrices $M$ with Boolean rank $b$ such that $\exp(M) = (b - 1)^2 + 1$.

In this paper, we give an upper bound on the scrambling index of a primitive matrix $M$ using Boolean rank $b = b(M)$, and characterize all Boolean primitive matrices that achieve the upper bound.
2 Main Results

We start with a basic result.

**Lemma 2.1** Suppose that $A$ and $B$ are $n \times m$ and $m \times n$ Boolean matrices respectively, and that neither has a zero line. Then

(a) $AB$ is primitive if and only if $BA$ is primitive.

(b) If $AB$ and $BA$ are primitive, then

$$|k(AB) - k(BA)| \leq 1.$$ (3)

**Proof.** Part (a) was proved by Shao [6]. We only need to show part (b). Since $AB$ and $BA$ are primitive matrices, $A$ and $B$ has no zero rows. Then $AA^t \geq I_n$ and $BJ_nB^t = J_m$. Suppose $k(AB) = k$. By the definition of scrambling index

$$(AB)^k((AB)^t)^k = J_n.$$ 

Then

$$(BA)^k((BA)^t)^k + 1 = B(AB)^kAA^t((AB)^t)^kB^t \geq B(AB)^kI_n((AB)^t)^kB^t$$

$$= B(AB)^k((AB)^t)^kB^t = BJ_nB^t = J_m.$$ 

Thus $k(BA) \leq k + 1 = k(AB) + 1$. The result follows by exchanging the roles of $A$ and $B$. □

**Proposition 2.2** [5] Let $M$ be an $n \times n$ primitive Boolean matrix, and $M = AB$ be a Boolean rank factorization of $M$. Then neither $A$ nor $B$ has a zero line.

**Theorem 2.3** Let $M$ be an $n \times n$ ($n \geq 2$) primitive matrix with Boolean rank $b(M) = b$. Then

$$k(M) \leq \left\lceil \frac{(b-1)^2 + 1}{2} \right\rceil + 1.$$ (4)

**Proof.** Let $M = AB$ be a Boolean rank factorization of $M$, where $A$ and $B$ are $n \times b$ and $b \times n$ Boolean matrices respectively. Then by Lemma 2.2 neither $A$ nor $B$ has a zero line. By lemma 2.1 we have

$$k(M) = k(AB) \leq k(BA) + 1.$$
Since $BA$ is primitive and $BA$ is a $b \times b$ matrix, by Theorem 1.2

$$k(BA) \leq \left\lceil \frac{(b-1)^2 + 1}{2} \right\rceil,$$

from which Theorem 2.3 follows. □

From (1) we see that no matrix of full Boolean rank $n$ can attain the upper bound in (4). Further, since the only $n \times n$ primitive Boolean matrix with Boolean rank 1 is $J_n$, no matrix of Boolean rank 1 can attain the upper bound in (4). Thus we may assume that $2 \leq b \leq n - 1$.

For simplicity, let

$$h = \left\lceil \frac{(b-1)^2 + 1}{2} \right\rceil.$$

Recall from Theorem 1.2 that $k(W_b) = h$. We first make some observations about $W_b$. Recall that $D = D(W_b)$ is the Wielandt graph $D_{b-1,b}$ with $b$ vertices.

**Lemma 2.4** If $b \geq 3$, then the zero entries of $(W_b)_{h-1}^h(W_b^t)^{h-1}$ occur only in the $(b, \lfloor \frac{b}{2} \rfloor)$ and $(\lfloor \frac{b}{2} \rfloor, b)$ positions.

**Proof.** By Proposition 1.3 we know that $k_{\lfloor \frac{b}{2} \rfloor, b}(D_{b-1,b}) = h$, and for all other pairs of vertices $u$ and $v$, $k_{u,v}(D_{b-1,b}) < h$. Therefore in $W_{b}^{h-1}$ every pair of rows intersect with each other except rows $b$ and $\lfloor \frac{b}{2} \rfloor$. Thus the only zero entries of $(W_b)_{h-1}^h(W_b^t)^{h-1}$ are in the $(b, \lfloor \frac{b}{2} \rfloor)$ and $(\lfloor \frac{b}{2} \rfloor, b)$ positions. □

For an $n \times n$ ($n \geq 2$) matrix $A$, let $A(\{i_1, i_2\}, \{j_1, j_2\})$ be the submatrix of $A$ that lies in the rows $i_1$ and $i_2$ and the columns $j_1$ and $j_2$.

**Lemma 2.5** For $b \geq 3$, $W_b^{h-1}(\{\lfloor \frac{b}{2} \rfloor, \{b-1, b\})$ is either $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

**Proof.** By Proposition 1.3 we know that $k_{\lfloor \frac{b}{2} \rfloor, b}(D_{b-1,b}) = h$ and $\lfloor \frac{b}{2} \rfloor \rightarrow 1$ and $b \rightarrow 1$. From the digraph $D_{b-1,b}$, we know that the directed walks of length $h$ from vertices $\lfloor \frac{b}{2} \rfloor$ and $b$ to vertex 1 is either

$$\lfloor \frac{b}{2} \rfloor \rightarrow 1 \rightarrow 1,$$

$$b \rightarrow b \rightarrow 1,$$

or

$$\lfloor \frac{b}{2} \rfloor \rightarrow b \rightarrow 1,$$

$$b \rightarrow b \rightarrow 1.$$
For the first case, if \( \left[ \frac{b}{2} \right] \) \( b \rightarrow b - 1 \) and \( b \rightarrow b - 1 \), then \( b \rightarrow b - 1 \) and \( \left[ \frac{b}{2} \right] \) \( b \rightarrow b - 1 \). Otherwise it contradicts to \( k(\frac{1}{2}, b) = h \). Similarly, for the second case if \( \left[ \frac{b}{2} \right] \) \( b \rightarrow b - 1 \) and \( b \rightarrow b - 1 \), then \( b \rightarrow b - 1 \) and \( \left[ \frac{b}{2} \right] \) \( b \rightarrow b - 1 \). The result follows by applying these to the matrix \( W_b^{h-1} \).

**Theorem 2.6** Suppose \( M \) is an \( n \times n \) primitive Boolean matrix with \( 3 \leq b = b(M) \leq n - 1 \). Then

\[
k(M) = \left\lceil \frac{(b-1)^2 + 1}{2} \right\rceil + 1
\]

if and only if \( M \) has a boolean rank factorization \( M = AB \), where \( A \) and \( B \) have the following properties:

(i) \( BA = W_b \),
(ii) some row of \( A \) is \( e_i^{(\frac{b}{2})}(b) \), some row of \( A \) is \( e_i(b) \), and
(iii) no column of \( B \) is \( e_{b-1}(b) e_b(b) \).

**Proof.** First suppose \( M \) is primitive with \( k(M) = h + 1 \), and \( M = \tilde{A}\tilde{B} \) is a Boolean rank factorization of \( M \). By Lemma 2.1 \( \tilde{B}\tilde{A} \) is primitive and \( k(\tilde{B}\tilde{A}) \geq h \). But \( \tilde{B}\tilde{A} \) is a \( b \times b \) matrix. By Theorem 1.2, \( k(\tilde{B}\tilde{A}) \leq h \). Therefore \( k(\tilde{B}\tilde{A}) = h \). Also by Theorem 1.2 there is a permutation matrix \( P \) such that \( P\tilde{B}\tilde{A}P^t = W_b \). Let \( B = \tilde{P}\tilde{B} \) and \( A = \tilde{A}P^t \). Then \( AB = \tilde{A}P^t\tilde{P}\tilde{B} = \tilde{A}\tilde{B} = M \). Thus \( A \) and \( B \) satisfy condition (i).

Since \( M \) is primitive, we have \( \sum_{i=1}^{b} A_{i} = j_n = \sum_{i=1}^{b} B_i \). Since \( k(M) = h + 1 \), the matrix \( M^h \) must have two rows that do not intersect. Without lost of generality, suppose rows \( p \) and \( q \) of \( M^h \) do not intersect. Then entries in the \( (p, q) \) and \( (q, p) \) positions of \( M^h(M^t)^h \) are zero. Since matrix \( B \) has no zero row, we have \( BB^t \geq I_b \). Thus

\[
M^h(M^t)^h = (AB)^h((AB)^t)^h = A(AB)^h-1BB^t((BA)^t)^h-1A^t
\]

\[
= A(W_b)^h-1BB^t(W_b^t)^h-1A^t
\]

\[
\geq A(W_b)^h-1I_b(W_b^t)^h-1A^t = A(W_b)^h-1(W_b^t)^h-1A^t
\]

\[
= AZA^t
\]

\[
= \left[ j_{n,\left[ \frac{b}{2} \right]-1} \left( \sum_{i=1}^{b-1} A_{i} \right) j_{n,\left[ \frac{b}{2} \right]-1} \left( \sum_{i=1}^{b-1} A_{i} \right) \right] A^t
\]

\[
= j_n \left( \sum_{i=1}^{\left[ \frac{b}{2} \right]-1} A_{i} \right)^t + j_n \left( \sum_{i=1}^{\left[ \frac{b}{2} \right]+1} A_{i} \right)^t + j_n \left( \sum_{i=1}^{\left[ \frac{b}{2} \right]+1} A_{i} \right)^t + j_n \left( \sum_{i=1}^{\left[ \frac{b}{2} \right]+1} A_{i} \right)^t
\]

\[
\]
where $Z = (W_b)^{h-1}(W_b^t)^{h-1}$ is the $b \times b$ matrix which has zero entries only in the $(\lfloor \frac{b}{2} \rfloor, b)$ and $(b, \lfloor \frac{b}{2} \rfloor)$ positions. Since $AZA^t$ is dominated by $M^h(M^t)^h$ and $M^h(M^t)^h$ has zero entries in the $(p, q)$ and $(q, p)$ positions, the entries in the $(p, q)$ and $(q, p)$ positions of $AZA^t$ are also zero. Thus

$$
\sum_{i=1}^{\lfloor \frac{b}{2} \rfloor - 1} A_{qi} + \left( \sum_{i=1}^{b-1} A_{pi} \right) A_{q\lfloor \frac{b}{2} \rfloor} + \sum_{i=\lfloor \frac{b}{2} \rfloor + 1}^{b-1} A_{qi} + \left( \sum_{i=1}^{b} A_{pi} \right) A_{qb} = 0 \quad (5)
$$

and

$$
\sum_{i=1}^{\lfloor \frac{b}{2} \rfloor - 1} A_{pi} + \left( \sum_{i=1}^{b-1} A_{qi} \right) A_{p\lfloor \frac{b}{2} \rfloor} + \sum_{i=\lfloor \frac{b}{2} \rfloor + 1}^{b-1} A_{pi} + \left( \sum_{i=1}^{b} A_{qi} \right) A_{pb} = 0. \quad (6)
$$

Then $A_{qi} = 0$ and $A_{pi} = 0$ for $i = 1, \ldots, b - 1$ and $i \neq \lfloor \frac{b}{2} \rfloor$. Substitute these back to (5) and (6), we have

$$A_{q\lfloor \frac{b}{2} \rfloor}A_{p\lfloor \frac{b}{2} \rfloor} + A_{qb}A_{pb} = 0. \quad (7)
$$

If $A_{q\lfloor \frac{b}{2} \rfloor} \neq 0$, then $A_{p\lfloor \frac{b}{2} \rfloor} = 0$. Since every row of $A$ is nonzero, we have $A_{pb} \neq 0$. By (7), $A_{qp} = 0$. Therefore some rows of $A$ is $e_{\lfloor \frac{b}{2} \rfloor}(b)$ and some row of $A$ is $e_1(b)$. This concludes (ii).

We claim $B$ cannot have a column which is equal to $u$. Otherwise, suppose some column of $B$ is $u$. Since $B$ has no zero row, by Proposition 2.2, $BB^t \geq I_b + uu^t$. Thus

$$M^h(M^t)^h = (AB)^h((AB)^t)^h = A(BA)^{h-1}BB^t((BA)^t)^{h-1}A^t$$

$$= A(W_b)^{h-1}BB^t(W_b^t)^{h-1}A^t$$

$$\geq A(W_b)^{h-1}(I_b + uu^t)(W_b^t)^{h-1}A^t$$

$$= A[(W_b)^{h-1}(W_b^t)^{h-1} + (W_b)^{h-1}u(W_b^t)^{h-1}u]^tA^t.$$  

By lemma 2.3, $W_b^{h-1}([\lfloor \frac{b}{2} \rfloor, b], \{b - 1, b\})$ is either \[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \] or \[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]. Then $W_b^{h-1}u \geq e_{\lfloor \frac{b}{2} \rfloor}(b) + e_b(b)$. By Lemma 2.3, the zero entries of $W_b^{h-1}(W_b^t)^{h-1}$ are in the $(b, \lfloor \frac{b}{2} \rfloor)$ and $(\lfloor \frac{b}{2} \rfloor, b)$ positions. Therefore $W_b^{h-1}(W_b^t)^{h-1} + (W_b)^{h-1}u(W_b^t)^{h-1}u^t = J_b$. Since $A$ has no zero lines, we have $M^h(M^t)^h = AJ_bA^t = J_n$, which is a contradiction to $k(M) = h + 1$. This proves (iii).
Finally, suppose that \( M = AB \) is a Boolean rank factorization of \( M \) and \( A \) and \( B \) satisfy (i), (ii) and (iii). By Lemma 2.1(a) and Theorem 1.2, the matrix \( M \) is primitive and \( k(M) \leq h + 1 \) by Lemma 2.1(b) and . But it follows from Lemma 2.4 and conditions (i), (ii) and (iii) that \( M^h \) has zero entries. So we conclude that \( k(D) = h + 1 \).

Next we will reinterpret conditions (i), (ii) and (iii) of Theorem 2.6 to show that if \( k(M) = h + 1 \), then \( M \) is one of the three basic types of matrices in Theorem 2.7.

\[ \begin{array}{c}
M_1 = \\
\begin{bmatrix}
0 & J & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & J & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & J & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & J & 0 \\
J & 0 & \cdots & 0 & 0 & 0 & 0 \\
J & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\end{array} \]

\[ \begin{array}{c}
M_2 = \\
\begin{bmatrix}
0 & J & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & J & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & J & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & J & 0 \\
0 & 0 & \cdots & 0 & 0 & J & 0 \\
J & 0 & \cdots & 0 & 0 & 0 & 0 \\
J & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\end{array} \]

\[ \begin{array}{c}
M_3 = \\
\begin{bmatrix}
0 & J & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & J & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & J & 0 & J \\
0 & 0 & \cdots & 0 & 0 & J & 0 \\
J & 0 & \cdots & 0 & 0 & 0 & 0 \\
J & 0 & \cdots & 0 & 0 & 0 & 0 \\
J & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\end{array} \]

**Theorem 2.7** Suppose \( M \) is an \( n \times n \) Boolean matrix with \( b(M) = b \), where \( 3 \leq b \leq n - 1 \). Then \( M \) is primitive with \( k(M) = h + 1 \) if and only if there is a permutation matrix \( P \) such that \( PMP^t \) has one of the forms in Table 1.
In Table 1 the rows and columns of $M_1$, $M_2$ and $M_3$ are partitioned conformally, so that each diagonal block is square, and the top left hand submatrix common to each has $b$ blocks in its partitioning.

**Proof.** Suppose $M$ is primitive, $b \geq 3$, and $k(M) = h + 1$. Then by Theorem 2.6(i), $M$ has a Boolean rank factorization $M = AB$ such that $BA = W_b$. Since $A$ has no zero row, each column of $B$ is dominated by a column of $W_b$. Similarly, each row of $A$ is dominated by a row of $W_b$. Thus each column of $B$ is in the set $S_1 = \{e_1(b), e_2(b), \cdots, e_b(b), u\}$, where $u = e_{b-1}(b) + e_b(b)$. Similarly, each row of $A$ is in the set $S_2 = \{e_1'(b), e_2'(b), \cdots, e_b'(b), v\}$, where $v = e_1(b) + e_b(b)$. But by Theorem 2.6(iii), no column of $B$ is $u$. Hence each column of $B$ is in the set $S'_1 = \{e_1(b), e_2(b), \cdots, e_b(b)\}$.

Next, we note that for each $1 \leq i \leq b$, the product $B_iA_i$ is dominated by $W_b$. Since each $B_j$ and $A_i$ must be in $S'_1$ and $S_2$ respectively and $(B_j, A_i)$ must be one of the following pairs: $(e_i, e_i^t)$, $1 \leq i \leq b-1$, $(e_{b-1}, e_{b-1}^t)$, $(e_b, e_1^t)$, or $(e_b, v^t)$, where $e_i = e_i(b)$ for any $i \in \{1, 2, \cdots, b\}$. Thus, for each $i$, $1 \leq i \leq b-1$, $(e_i, e_{i+1}^t) = (B_{k_i}, A_{k_i})$ for some $k_i$. Some outer product $B_jA_j$ has a 1 in the $(1,1)$ position, hence $(B_{k_b}, A_{k_b}) = (e_b, e_1^t)$ for some $k_b$. Finally some outer product $B_jA_j$ must have a 1 in the $(b-1,1)$ position, hence for some $k_{b+1}$, $(B_{k_{b+1}}, A_{k_{b+1}})$ is one of $(e_b, e_1^t)$ or $(e_b, v^t)$. It follows from the above argument that there is an $n \times n$ permutation matrix $Q$ such that

$$BQ^t = [\tilde{B} | \tilde{B}]$$

and

$$QA = \begin{bmatrix} \bar{A} \\ \bar{A} \end{bmatrix},$$

where

$$\tilde{B} = [e_1 j_{n_1}^t | e_2 j_{n_2}^t | \cdots | e_b j_{n_b}^t]$$

and

$$\bar{A} = \begin{bmatrix} j_{n_1} e_2^t \\ j_{n_2} e_3^t \\ \cdots \\ j_{n_{b-1}} e_b^t \\ j_{n_b} e_1^t \end{bmatrix}$$

for some $n_1, \cdots, n_b \geq 1$, and where each $(\tilde{B}_i, \bar{A}_i)$ is one of $(e_{b-1}, e_1^t)$ or $(e_{b-1}, v^t)$. Thus $\tilde{B}$ and $\bar{A}$ can be one of the following pairs of matrices:

$$\tilde{B}_1 = e_{b-1} j_{m_1}^t, \quad \bar{A}_1 = j_{m_1} e_1^t \text{ for some } m_1 \geq 1;$$

$$\tilde{B}_2 = e_{b-1} j_{m_2}^t, \quad \bar{A}_2 = j_{m_2} v^t \text{ for some } m_2 \geq 1;$$

$$\tilde{B}_3 = [e_{b-1} j_{m_3}^t | e_{b-1} j_{p_2}^t], \quad \bar{A}_3 = \begin{bmatrix} j_{m_3} e_1^t \\ j_{p_3} v^t \end{bmatrix} \text{ for some } m_3, p_3 \geq 1.$$
It is now readily verified that

\[
\begin{bmatrix}
\overline{A} \\
\bigg| \\
\overline{A}_i
\end{bmatrix}
\begin{bmatrix}
\overline{B} |
\overline{B}_i
\end{bmatrix} = M_i \quad \text{for } 1 \leq i \leq 3,
\]

so that \(QMQ^t\) is one of the matrices in Table 1.

Finally, since the Boolean rank factorization

\[
M_i = \begin{bmatrix}
\overline{A} \\
\bigg| \\
\overline{A}_i
\end{bmatrix}
\begin{bmatrix}
\overline{B} |
\overline{B}_i
\end{bmatrix}
\]

satisfies conditions (i), (ii) and (iii) of Theorem 2.6, each \(M_i\) is primitive and \(k(M) = h + 1\).

When \(b(M) = 2\), we have the following result.

**Theorem 2.8** Suppose \(M\) is an \(n \times n\) primitive Boolean matrix with \(b(M) = b = 2\). Then \(k(M) = 2\) if and only if \(M\) has a boolean rank factorization \(M = AB\), where \(A\) and \(B\) have the following properties:

(i) \(BA = W_2\) or \(BA = J_2\),

(ii) some row of \(A\) is \(e_1^t(2)\), some row of \(A\) is \(e_2^t(2)\), and

(iii) no column of \(B\) is \(e_1(2) + e_2(2)\).

**Proof.** First suppose \(M\) is primitive with \(k(M) = 2\), and \(M = \overline{A} \overline{B}\) is a Boolean rank factorization of \(M\). By Lemma 2.4, \(\overline{B} \overline{A}\) is primitive and \(k(\overline{B} \overline{A}) \geq 1\). But \(\overline{B} \overline{A}\) is a \(2 \times 2\) matrix. By Theorem 1.2, \(k(\overline{B} \overline{A}) \leq 1\). Therefore \(k(\overline{B} \overline{A}) = 1\). Also by Theorem 1.2, there is a permutation matrix \(P\) such that \(P \overline{B} \overline{A} P^t = W_2\) or \(P \overline{B} \overline{A} P^t = J_2\). Let \(B = \overline{P} \overline{B}\) and \(A = \overline{P} \overline{A}^t\). Then \(AB = \overline{A} \overline{P}^t \overline{P} \overline{B} = \overline{A} \overline{B} = M\). Thus \(A\) and \(B\) satisfy condition (i).

Proof of the conditions (ii) and (iii) are similar to the proof of Theorem 2.6.

\(\square\)

By a similar argument, we can reinterpret conditions (i), (ii) and (iii) of Theorem 2.8 to show that if \(M\) satisfies \(k(M) = 2\), then \(M\) is one of the 21 basic types of matrices which we will show in the following.

**Theorem 2.9** Suppose \(M\) is an \(n \times n\) Boolean matrix with \(b(M) = b = 2\). Let \(M = AB\) be a Boolean rank factorization. Then \(M\) is primitive with \(k(M) = 2\) if and only if there is a permutation matrix \(P\) such that \(PMP^t\) has one of the
forms in Table 2 if $BA = W_2$ or $PMP^t$ has one of the forms in Table 3 if $BA = J_2$.

In Table 2 and Table 3 the rows and columns of each matrix are partitioned conformally, so that each diagonal block is square.

Table 2 ($b = 2$)

\[
\begin{bmatrix}
0 & J & 0 \\
J & 0 & J \\
0 & J & J
\end{bmatrix},
\begin{bmatrix}
0 & J & 0 \\
J & 0 & J \\
J & J & J
\end{bmatrix},
\begin{bmatrix}
0 & J & 0 \\
J & 0 & J \\
J & J & J
\end{bmatrix},
\begin{bmatrix}
0 & J & 0 \\
J & 0 & J \\
J & J & J
\end{bmatrix}.
\]

Table 3 ($b = 2$)

\[
\begin{bmatrix}
J & J & 0 & 0 \\
0 & 0 & J & J \\
J & J & 0 & 0 \\
0 & 0 & J & J
\end{bmatrix},
\begin{bmatrix}
J & J & 0 & 0 \\
J & J & 0 & 0 \\
J & J & 0 & 0 \\
J & J & 0 & 0
\end{bmatrix},
\begin{bmatrix}
J & J & 0 & 0 \\
J & J & 0 & 0 \\
J & J & 0 & 0 \\
J & J & 0 & 0
\end{bmatrix},
\begin{bmatrix}
J & J & 0 & 0 \\
J & J & 0 & 0 \\
J & J & 0 & 0 \\
J & J & 0 & 0
\end{bmatrix},
\begin{bmatrix}
J & J & 0 & 0 \\
J & J & 0 & 0 \\
J & J & 0 & 0 \\
J & J & 0 & 0
\end{bmatrix},
\begin{bmatrix}
J & J & 0 & 0 \\
J & J & 0 & 0 \\
J & J & 0 & 0 \\
J & J & 0 & 0
\end{bmatrix},
\begin{bmatrix}
J & J & 0 & 0 \\
J & J & 0 & 0 \\
J & J & 0 & 0 \\
J & J & 0 & 0
\end{bmatrix}.
\]
References

[1] M. Akelbek, S.J. Kirkland, Coefficients of ergodicity and the scrambling index, *Linear Algebra Appl.* 430 (2009), 1111–1130.

[2] M. Akelbek, S.J. Kirkland, Primitive digraphs with the largest scrambling index, *Linear Algebra Appl.* 430 (2009), 1099–1110.

[3] R.A. Brualdi, H.J. Ryser, *Combinatorial Matrix Theory*, Encyclopedia of Mathematics and its Applications 39, Cambridge University Press, Cambridge, 1991.

[4] D.A. Gregory, S.J. Kirkland, N.J. Pullman, A bound on the exponent of a primitive matrix using Boolean rank, *Linear Algebra Appl.* 217 (1995), 101–116.

[5] B.L. Liu, L.H. You, G.X. Yu, On extremal matrices of second largest exponent by Boolean rank, *Linear Algebra Appl.* 422 (2007), 186-197.

[6] J.Y. Shao, On the exponent of primitive digraph, *Linear Algebra Appl.* 64 (1985), 21-31.