Low Energy Dynamics in Ultradegenerate QCD Matter

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We study the low energy behavior of QCD Green functions in the limit that the baryon chemical potential is much larger than the QCD scale parameter $\Lambda_{\text{QCD}}$. We show that there is a systematic low energy expansion in powers of $(\omega/m)^{1/3}$, where $\omega$ is the energy and $m$ is the screening scale. This expansion is valid even if the effective quark-gluon coupling $g$ is not small. The expansion is purely perturbative in the magnetic regime $|\hat{k}| \gg k_0$. If the external momenta and energies satisfy $|\vec{k}| \approx k_0$, planar, abelian ladder diagrams involving the full quark propagator have to be resummed but the corresponding Dyson-Schwinger equations are closed.

Landau’s theory of Fermi liquids has been remarkably successful in many areas of physics \[1\]. Landau argued that although strongly interacting many-particle systems are very complicated in general, the situation greatly simplifies if the temperature is very low. In this limit the properties of the system can be understood in terms of weakly interacting excitations called “quasi-particles”. The complicated dynamics of the underlying many-body systems can be summarized in terms of just a few parameters that characterize the quasi-particles and their scattering amplitudes.

In dense quark matter the presence of unscreened magnetic interactions invalidates the basic assumptions of Landau Fermi liquid theory \[2\] \[3\] \[4\]. Weak coupling calculations suggest that non-Fermi liquid corrections to the specific heat in the normal phase \[2\] \[4\] or the gap in the superfluid phase \[3\] are substantial. However, weak coupling methods are restricted to densities that are much larger than the densities that can be achieved deep inside a neutron star.

This raises the question whether there is a systematic low-energy expansion that can be used in much the same way that Landau theory is used in nuclear and condensed matter physics. There has been a lot of progress in this direction, mostly based on effective theories of dense QCD \[3\] \[4\]. In this paper we show that QCD Green functions have low energy expansions in powers of $(\omega/m)^{1/3}$, where $\omega$ is the characteristic energy scale of the process and $m$ is the screening scale. This expansion is reliable as long as $\omega \ll m$ even if the gauge coupling $g$ is of order one. We provide a brief exposition of our main results. A detailed diagrammatic analysis will appear in a companion paper \[1\].

Our starting point is an effective field theory (EFT) for fermions near the Fermi surface interacting with unscreened gauge fields. We follow Hong \[2\] and expand the fermionic part of the effective action in derivatives of the fields,

\[
\mathcal{L}_f = \psi_{\pm \vec{v}}^\dagger \left( iZ_\parallel v_\pm \cdot D - Z_\perp D_\perp^2 + \delta \mu \right) \psi_{\pm \vec{v}} + \frac{V^\parallel_{\text{HDL}}}{\mu^2} (\psi_{\vec{v}}^\dagger \Gamma \psi_{\vec{v}}) (\psi_{\vec{v}}^\dagger \Gamma \psi_{\vec{v}}) + \frac{V^\parallel_{\text{HDL}}}{\mu^2} (\psi_{\vec{v}}^\dagger \Gamma \psi_{\vec{v}}) (\psi_{\vec{v}}^\dagger \Gamma \psi_{\vec{v}}) + \cdots.
\]

Here, $\psi_{\pm \vec{v}}$ describes particles and holes with momenta $p = \pm \mu \vec{v} + l$ and $v_\pm = (1, \pm \vec{v})$ labels the local Fermi velocity. We will write $l = l_0 + l_\parallel + l_\perp$ where $l_\parallel$ and $l_\perp$ are the components of $l$ parallel and orthogonal to $\vec{v}$. We concentrate on two patches on opposite sides of the Fermi surface as these are the only channels that lead to kinematic enhancements. The effect of fluctuations above the cutoff $\Lambda$ is encoded in low energy constants. The leading coefficients are the shift $\delta \mu$ of the Fermi energy, the Fermi velocity $v_F = |\vec{v}|$, the renormalization factors $Z_\parallel$ and $Z_\perp$, the gauge coupling $g$ at the scale $\Lambda$, and the four-fermion couplings $V^\parallel_{\text{HDL}}$ and $V^\parallel_{\text{BCS}}$ in the forward (Zero Sound) and back-to-back (BCS) channels.

The dominant effect in the gluon sector is due to hard dense quark loops (HDLs) in gluonic correlation functions. These fluctuations involve small external energies $\omega$ but hard loop momenta $p \sim \mu$ in a narrow interval of width $\omega$. Hard dense loops are non-analytic in the gluon energy and momentum and cannot be represented as local operators. However, as long as hard fluctuations do not change the symmetries of the groundstate all other corrections are analytic. We can then write the gluonic lagrangian as $\mathcal{L}_g = \mathcal{L}_0 + \mathcal{L}_{\text{HDL}} + \cdots$ where $\mathcal{L}_0$ is the free gluon lagrangian and $\mathcal{L}_{\text{HDL}}$ is the HDL generating functional \[12\].

\[
\mathcal{L}_{\text{HDL}} = -\frac{m^2}{2} \sum_V G_{\alpha \beta}^a v_\alpha v_\beta \left( (v \cdot D)^2 - \frac{4}{3} \right) G_{\mu \nu}^b.
\]

This term describes screening and damping of soft gluon modes due to particle-hole pairs on the entire Fermi surface. In perturbation theory the dynamical gluon mass is given by $m^2 = N_f g^2 \mu^2 / (4\pi^2)$, where $g$ is the QCD coupling constant.

We now analyze the dynamics of the theory governed by equ. \[12\]. We first note that particle-hole loops have already been integrated out and are represented by the HDL term. The effective theory describes the interaction of particles and holes with soft gluons which do not significantly change their velocity $\vec{v}$. Since electric fields are screened the interaction at low energies is dominated by the exchange of magnetic gluons. Magnetic gluons are weakly damped in the kinematic regime $|k_0| \ll |\vec{k}|$. In this regime

\[
D^{(n)}_{ij} (k) = \frac{\delta_{ij} - \vec{k}_i \vec{k}_j}{k_0^2 - \vec{k}^2 + i\frac{\pi}{2} m^2 |k_0|}.
\]
and we observe that the propagator becomes large if
\[ |\vec{k}| \sim (m^2 |k_0|)^{1/3} \gg |k_0|. \tag{4} \]

This implies that the gluon is very far off its energy shell and not a propagating state. We will compute a general diagram by picking up the pole in the quark propagator, and integrate over the cut in the gluon propagator using the kinematics dictated by equ. (4). In order for a quark to absorb the large momentum carried by a gluon and stay close to the Fermi surface this momentum has to be transverse to the momentum of the quark. This means that the term \( k_\perp^2 / (2\mu) \) in the quark propagator is relevant and has to be kept at leading order. Equation (4) shows that \( k_\perp^2 / (2\mu) \gg k_0 \) as \( k_0 \to 0 \). This means that the pole of the quark propagator is governed by the condition \( k_\parallel \sim k_\perp^2 / (2\mu) \). We conclude that quark and gluon momenta scale with respect to an external energy scale \( \omega \) as
\[ k_0 \sim \omega, \quad k_\parallel \sim m^2 \omega^2 / \mu, \quad k_\perp \sim m^2 \omega^4 / \mu^3. \tag{5} \]

We will refer to the regime in which all momenta, including external ones, satisfy the scaling relation (5) as the magnetic regime. A similar regime was identified in the context of gauge theories of condensed matter systems [14]. The scaling relations (5) are the basis of the low energy expansion in ultradegenerate matter.

In the low energy regime propagators and vertices can be simplified even further. The quark and gluon propagators are
\[ S_{\pm,\mp}^{\alpha\beta}(p) = \frac{i\delta_{\alpha\beta}}{Z_\parallel (p_0 \mp v_F p_\parallel) - Z_\perp p_\perp^2 / 2m + i\text{sgn}(p_0)}, \tag{6} \]
\[ \nu_+^\mu \nu_-^\nu D_{\mu\nu}^{(m)}(k) = \mp \frac{iv_F^2}{k_\perp^2 + i\epsilon m^2 |k_0| / k_\perp}, \tag{7} \]
and the quark gluon vertex is \( gZ || v_i (\lambda^\alpha / 2) \). Higher order corrections can be found by expanding the quark and gluon propagators as well as the HDL vertices in powers of the small parameter \( \epsilon \equiv \omega / m \) [11]. We observe that the transverse projector in the gluon propagator simplifies because \( k_\perp \gg k_\parallel \). We also note that in the magnetic regime the factor \( p_0 \) in the quark propagator can be dropped since \( p_0 \ll |p_\parallel| \).

Using these expressions we can show that the power of \( \epsilon \) associated with a Feynman diagram always increases with the number of loops and the number of higher-order vertices. One way to see this is to rescale the fields in the effective lagrangian so that the kinetic terms are scale invariant under the transformation \( (x_0, x_||, x_\perp) \to (\epsilon^{-1} x_0, \epsilon^{-2/3} x_||, \epsilon^{-1/3} x_\perp) \). The scaling behavior of the fields is \( \psi \to \epsilon^{5/6} \psi \) and \( A_i \to \epsilon^{5/6} A_i \). We find now that the scaling dimension of all interaction terms is positive. The quark gluon vertex scales as \( \epsilon^{1/6} \), the HDL three gluon vertex scales as \( \epsilon^{1/2} \), and both the quark-two-gluon vertex induced by the \( D_{\perp}^2 \) term in equ. (11) and the four gluon vertex scale as \( \epsilon \). Since higher order diagrams involve at least one pair of quark gluon vertices

the expansion involves positive powers of \( \epsilon^{1/3} \) and the magnetic regime is completely perturbative.

As a simple example we consider the fermion self energy. The one-loop diagram is
\[ \Sigma(p) = g^2 C_F \int \frac{dk_0}{2\pi} \int \frac{dk_\perp}{(2\pi)^2} \frac{k_\perp}{k_\perp^2 + i\epsilon m^2 |k_0|} \times \int \frac{dk_\parallel}{2\pi} \frac{\Theta(p_0 + k_0)}{k_\parallel + p_\parallel} + \frac{Z_\perp^2 (k_\parallel + p_\parallel)}{2Z_\parallel v_F \mu / 2\pi} + i\epsilon, \tag{8} \]
with \( C_F = (N_c^2 - 1) / (2N_c) \). This expression shows a number of interesting features. First we observe that the longitudinal and transverse momentum integrations factorize. The longitudinal momentum integral is performed by picking up the pole in the quark propagator. The result is independent of the external momenta and only depends on the external energy. The transverse momentum integral is logarithmically divergent. We find
\[ \Sigma(p) = g^2 \frac{\omega}{9\pi^2} \left( \omega \log \left( \frac{\Lambda^2}{|\omega|} \right) + \omega + \frac{i\pi}{2} |\omega| \right) + O \left( \epsilon^4 \right), \tag{9} \]
where \( \omega \equiv p_0 \). We have absorbed the logarithmic cutoff dependence into the low energy constant \( Z_\parallel \). In general this result depends on two unknown parameters \( g \) and \( \Lambda_\Sigma \) where \( \Lambda_\Sigma = \Lambda \exp(9\pi^2 (Z_\parallel - 1) / (g^2 Z_\parallel v_F)) \) and \( \Lambda = 2\Lambda_\Sigma^3 / (\pi m^2) \) is related to the transverse momentum cutoff. If the coupling is small, the scale is determined by the exchange of electric gluons and we find \( \Lambda_\Sigma = 2\Lambda_0 \pi / m \). We observe that the self energy correction is large and has to be included in the propagator whenever its energy dependence is relevant. We showed previously that rainbow diagrams do not give corrections of the form \( g^2\omega \log(\omega)^n \) [7]. Equ. (9) shows that higher order corrections are suppressed by powers of \( \epsilon^{2/3} \) [13].

The scaling arguments apply to arbitrary Green functions in the magnetic regime, see Fig. 1. Exceptions occur if the external fields have small momenta of the order of the external energy scale. This situation can occur in quark-quark scattering amplitudes or in vertex functions for external currents like the weak interaction [17]. Consider the one-loop vertex correction for a color singlet vertex \( \Gamma_\mu = e Z || v_\mu \). In the magnetic regime the graph scales like \( \epsilon^{1/3} \). This is confirmed by an explicit calcula-
In the limit \( \delta \rightarrow 0 \) the scattering amplitude. Consider a one-loop correction to the vertex of an external current and the BCS interaction. Both diagrams are kinetically enhanced and scale as \( \log(\omega) \) and \( \log^2(\omega) \), respectively.

In the time-like regime \( p_0 - q_0 > |\vec{p} - \vec{q}| \)

\[
\Gamma^\mu(p, q) = \frac{e g^2 C_F}{2\pi} \int \frac{d^2 k}{(2\pi)^2} v_+^\mu D_{\rho\sigma}^{(m)}(k) \times \int \frac{dk_\parallel}{2\pi} Z\parallel^2 S_\parallel(k + p) S_\parallel(k + q),
\]

(10)

where \( p, q \) are the momenta of the external quarks. The important point is that if we combine the fermionic propagators using Fermi’s trick in order to resolve the pole in the longitudinal momentum integration, the large components \( k_\parallel \) and \( k_\perp^2 / (2\mu) \) of the propagators cancel and the result becomes sensitive to the small scales \( p, q \)

\[
S_\parallel(k + p) S_\parallel(k + q) = \frac{S_\parallel(k + p) - S_\parallel(k + q)}{S_\parallel^{-1}(p) - S_\parallel^{-1}(q) - 2 k_\perp^2 / (2\mu k_\parallel)},
\]

where constant factors are suppressed. As a consequence the result is enhanced by a factor \( 1/\epsilon^{1/2} \). This enhancement is analogous to the one occurring in the HDL case. In the limit \( p_0 - q_0 \rightarrow 0 \) the \( k_\parallel \) integral gives a factor \( \delta(p_0 - k_0) \) and the vertex correction is \( \Gamma \)

\[
\Gamma^\mu(p, q) = \frac{e g^2}{2\pi} \delta^\mu_+ \log \left( \frac{\Lambda_{\text{BS}}}{|\omega|} \right),
\]

(11)

where \( \omega = (p_0 + q_0)/2 \). The logarithmic divergence was removed by adding the contribution from the four-fermion vertex in the zero sound channel. If the coupling is weak the scale inside the logarithm is again determined by electric gluon exchange. We find that in this case the scale is equal to the scale in the quark self energy.

The cancellation that occurs in the one-loop diagram repeats itself at any loop order if additional gluon ladders are added. This implies that ladder diagrams have to be summed. We also note that quark propagators in the ladders are sensitive to the small scale \( \omega \) and thereby the full fermion self energy has to be included. A detailed analysis shows that all other corrections like crossed ladders, vertex corrections, interconnections between the gluon ladders, etc. introduce extra transverse momenta and follow the scaling relations in the magnetic regime [11].

The same phenomenon occurs in the quark-quark scattering amplitude. Consider a one-loop correction to the scattering amplitude in the BCS-channel, see Fig. 2.

\[
\delta \Gamma_{\text{BCS}}(p, q) = g^2 C_F \int \frac{dk_0}{2\pi} \int \frac{d^2 k_\perp}{(2\pi)^2} v_+^\mu v_+^\nu D^{(m)}_{\rho\sigma}(k - q)
\]

\[
\begin{array}{|c|c|c|c|}
\hline
g(\Lambda) & m & \delta \mu & v_F \\
\hline
\frac{\Lambda_{\text{BS}}}{|k_0 - q_0|} & \frac{\Lambda_{\text{BS}}}{|k_0 - q_0|} & \frac{\Lambda_{\text{BS}}}{|k_0 - q_0|} & \frac{\Lambda_{\text{BS}}}{|k_0 - q_0|} \\
\hline
\end{array}
\]

TABLE I: Leading order expressions for the low energy constants appearing in the effective Lagrangian for magnetic modes equ. (11) in the weak coupling limit.

\[
x \Gamma_{\text{BCS}}(p, q) = \frac{g^2}{18\pi^2} \int \frac{dk_0}{k_0} \frac{\Lambda_{\text{BCS}}}{|k_0 - q_0|} \log \left( \frac{\Lambda_{\text{BCS}}}{|k_0 - q_0|} \right),
\]

(13)

which again depends to leading order on \( \omega \equiv p_0 \). The cutoff dependence was removed by adding the one-loop graph with the BCS four-fermion operator. We can compute the scale in the weak coupling limit. In the quark self energy and vertex correction the logarithmic UV divergence cancels between electric and magnetic gluon exchanges and the scale is determined by transverse momenta \( k_\perp \sim m \). In the BCS channel the two terms add and the logarithm is sensitive to larger scales \( k_\perp \sim \mu \). Matching to the full tree level scattering amplitude at this scale gives \( \Lambda_{\text{BCS}} = 2^{11/2} \mu^6 / (\pi m^3) \) as shown in [17].

The double logarithmic structure of the quark-quark scattering amplitude in QCD was discovered by Son [8]. Again we find that all planar, abelian ladder diagrams contribute at the same order but other corrections are perturbative and follow the scaling rules in the magnetic regime. The ladder sum can be determined by solving a Bethe-Salpeter equation, see Fig. 3. Since the quark propagators are sensitive to the low energy scale the quark self energy has to be included to all orders. We saw, however, that the one-loop result for the quark self energy is exact in the low energy limit. The situation simplifies further if the coupling is small. In this case we find that the low energy scale in the BCS channel is \( \omega \sim \Delta \sim \mu \exp(-1/g) \). This means that the quark self energy correction never becomes large and can be treated perturbatively, too.
The damping scale $m$ of all gluonic correlation functions and can be used to determine equation is obtained for the forward channel.

The analysis in the forward scattering channel is identical to the case of an external current. Green functions with more than four external quark lines do not have kinematic enhancements except for those that occur in four-particle reducible graphs. The reason is that whenever a quark line is connected to a given graph via a gluon an extra transverse momentum is introduced which leads to suppression factors at low energy consistent with the scaling rules.

In summary, we have shown that QCD Green functions in high density QCD have a systematic low energy expansion in powers of $(\omega/m)^{1/3}$. The results are valid in any phase in which the transverse components of the gauge field are dynamically damped, $\text{Im}\Pi(k_0,\vec{k}) \sim m^2 k_0/|\vec{k}|$. The damping scale $m$ determines the asymptotic behavior of all gluonic correlation functions and can be used to define $m$ non-perturbatively. We note, in particular, that the low energy interaction between gluons is weak and no magnetic screening mass is generated in the normal phase. Our results apply likewise to ordinary condensed matter systems described by a strong effective gauge interaction.\footnote{11}

The low energy expansion is directly applicable to the normal phase of dense quark matter, and to modes that remain ungapped below the (largest) critical temperature. Such ungapped modes play a vital role for a potential detection of quark matter in compact stars. The effective theory can be used to study transport properties, neutrino emissivities\footnote{13}, etc. Under conditions appropriate to neutron star cores we have $(T/\mu)^{1/3} \sim 1/10$ and the expansion is expected to converge well. This provides robust QCD results on the generic temperature dependence of low energy observables which due to the huge ratio $\mu/T$ are not impaired by the ignorance of low energy constants of order one.

For $T < T_c \sim \Delta$ dense quark matter becomes superfluid, and the Fermi liquid description breaks down completely. For energies less than the gap, the non-Fermi liquid EFT has to be matched against an effective theory of the superfluid phase\footnote{13}. The approximation scheme described here can be used to compute the parameters of the superfluid EFT in an effective expansion controlled by parameters of magnitude $(\Delta/m)^{1/3}$ and $1/|\log(m/\Delta)|$. We note, in particular, that $g$ decreases only logarithmically with $\mu$, while the low energy expansion contains inverse powers of $\mu$. Therefore, this expansion represents a significant improvement over the mere use of perturbation theory in $g$ and extends the range of validity of perturbative QCD to considerably lower densities.

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