Relational evolution of observables for Hamiltonian-constrained systems

Andrea Dapor,1,∗ Wojciech Kamiński,1 Jerzy Lewandowski,1,2† and Jędrzej Świeżewski1,§

1Faculty of Physics, University of Warsaw, Hoża 69, 00-681 Warszawa, Poland
2Institute for Quantum Gravity (IQG), FAU Erlangen – Nurnberg, Staudtstr. 7, 91058 Erlangen, Germany

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Evolution of systems in which Hamiltonians are generators of gauge transformations is a notion that requires more structure than the canonical theory provides. We identify and study this additional structure in the framework of relational observables (“partial observables”). We formulate necessary and sufficient conditions for the resulting evolution in the physical phase space to be a symplectomorphism. We give examples which satisfy those conditions and examples which do not. We point out that several classic positions in the literature on relational observables contain an incomplete approach to the issue of evolution and false statements. Our work provides useful clarification and opens the door to studying correctly formulated definitions.

I. INTRODUCTION: RELATIONAL DIRAC OBSERVABLES

This paper is about physical systems in which the Hamiltonian is a generator of a gauge transformation. The most important example is general relativity. The physical evolution of those systems is a notion that requires more care than in a regular canonical theory and an additional structure. On one hand, theoretical physicists dealing with such systems usually know some methods which work in the examples they are interested in. On the other hand, there is a diversity of formulations of frameworks for the Hamiltonian-constrained systems [1–5], each one aiming to capture the peculiarity of this gauge-time evolution in a general way. The formulation we consider in this work originates from Rovelli’s idea of “partial observables” [6]. We prefer to call it “relational observables.” A systematic approach to the relational observables framework was developed in a series of papers and books [7–10]. Unfortunately, we found an error in those attempts [7]. The error has passed to the literature and still requires a correction. We would like to emphasise that, in specific examples, the idea of the relational observables is usually applied in a correct way [11]. It is the general theory that needs our correction. We present it in our current paper.

1. Kinematical phase space

In the (classical) canonical framework, the states form a phase space Γ. The phase space is a manifold endowed with a differential 2-form Ω – a symplectic form – which by definition is closed,

\[ d\Omega = 0, \]  
(I.1)

and not degenerate,

\[ \mathbf{X} \cdot \Omega = 0 \Rightarrow \mathbf{X} = 0, \]  
(I.2)

at every point of Γ.

2. Constrained systems and gauge transformations

In the case of a constrained system, the physical states are subject to constraints, and they form a constraint surface Γ_C contained in the phase space

\[ \Gamma_C \subset \Gamma. \]  
(I.3)

On the constraint surface, there is a naturally induced 2-form Ω_C, the pullback of Ω. Provided Ω_C is nondegenerate, the pair Γ_C and Ω_C becomes the physical phase space. Often, however, the 2-form Ω_C has null directions, i.e., there is a nonvanishing vector ℓ tangent to Γ_C, such that

\[ \ell \cdot \Omega_C = 0. \]  
(I.4)

In this case Ω_C is degenerate; it cannot be a symplectic form, and Γ_C cannot play the role of a physical phase space. The maps Γ_C → Γ which define the flow of a null vector field ℓ are considered gauge transformations. The flow of each null vector field Lie-drags the 2-form Ω_C,

\[ \mathcal{L}_\ell \Omega_C = 0, \]  
(I.5)

so Ω_C is gauge invariant.

3. Physical phase space

If ℓ_1 and ℓ_2 are two null vector fields, then so is their commutator,

\[ [\ell_1, \ell_2] \cdot \Omega_C = \mathcal{L}_{\ell_1} (\ell_2 \cdot \Omega_C) - \ell_2 \cdot \mathcal{L}_{\ell_1} \Omega_C = 0. \]  
(I.6)
Therefore, the null directions define a foliation of $\Gamma_C$. The set of the leaves of that foliation is the physical phase space $\bar{\Gamma}$. We have the natural projection

$$\Pi : \Gamma_C \rightarrow \bar{\Gamma}. \quad (I.7)$$

We will be assuming that $\bar{\Gamma}$ is a manifold and that the natural projection is smooth. Then, there is a 2-form $\Omega$ on $\bar{\Gamma}$ such that

$$\Pi^* \tilde{\Omega} = \Omega. \quad (I.8)$$

We will also be assuming that there is a global section of the projection $\Pi$, that is an embedding

$$\sigma : \bar{\Gamma} \rightarrow \Gamma_C, \quad (I.9)$$

such that combined with the natural projection $\Pi$, it is the identity

$$\Pi \circ \sigma = \text{id}. \quad (I.10)$$

In other words, our considerations will be local and will concern generic points. We will not address the issues of possible singular, or even non-Hausdorff points of $\bar{\Gamma}$, or possible topological nontriviality of the projection $\Pi$.

FIG. 1: The pictures illustrate the construction presented in the introduction. The picture (a) shows the phase space, the constraint surface, orbits of gauge transformations, the physical phase space, and the projection onto it. The picture (b) shows the physical phase space, the constraint surface, and the ($\tau$-dependent) sections which embed one into the other.

4. Hamiltonians

The dynamics of a canonical theory is defined by a Hamiltonian, a parameter $t$-dependent function $H(t)$ defined on the phase space $\Gamma$. At every value of $t$, the Hamiltonian $H(t)$ defines a vector field $X_{H(t)}$ tangent to $\Gamma$, such that

$$X_{H(t)} \cdot \Omega = dH(t). \quad (I.12)$$

For every function $F$ defined on $\Gamma$, the Hamiltonian defines its evolution $t \mapsto F(t)$ via

$$\frac{d}{dt} F(t) = X_{H(t)}(F(t)), \quad F(t_0) = F. \quad (I.13)$$

In the case of constraints, the vector field $X_{H(t)}$ is tangent to $\Gamma_C$ at every instant of $t$, and its flow preserves the $\Omega_C$ including its null directions. The Hamiltonian $H(t)$ is constant along the leaves of the null directions,

$$\ell(H(t)) = \ell \downarrow \Omega t \cdot \Omega t (X_{H(t)} \cdot \Omega t) = 0. \quad (I.14)$$

Therefore, the projection $\Pi_0 X_{H(t)}$ defines a unique vector field $\bar{X}_{H(t)}$ on $\bar{\Gamma}$, $H(t)$ defines a unique function $\bar{H}(t)$ on $\bar{\Gamma}$, and

$$\bar{X}_{H(t)} \cdot \tilde{\Omega} = d\bar{H}(t). \quad (I.15)$$

In fact, in a constrained system, the Hamiltonian vector field $X_{\tilde{H}(t)}$ is defined on $\Gamma_C$ modulated a null vector field by a nonunique Hamiltonian (that explains the plural ‘Hamiltonians’), but its projection onto the physical phase space $\bar{\Gamma}$ is unique. As a consequence, upon those smoothness assumptions, we end up with the physical phase space, symplectic form, and, respectively, Hamiltonian: $\Gamma$, $\bar{\Omega}$, $\bar{H}(t)$.

5. Hamiltonian-constrained systems

There is a catch, however. In some constrained systems, the Hamiltonian vector field $X_{\tilde{H}(t)}$ satisfies at every point of $\Gamma_C$,

$$X_{\tilde{H}(t)} \cdot \Omega t = 0. \quad (I.16)$$

Then, the projected Hamiltonian vector field is identically zero:

$$X_{\bar{H}(t)} = 0. \quad (I.17)$$

A system with this property is called Hamiltonian-constrained. An example is canonical general relativity, in which Hamiltonians (labelled by free functions – lapse and shift – representing the choice of space-time coordinates) are projected to 0 in $\bar{\Gamma}$.

6. Relational Dirac observables

In order to introduce dynamics in the case of a Hamiltonian-constrained system, one needs some extra structure. Sometimes this extra structure is incorrectly characterized. We explain now this misunderstanding and identify the correct structure. We start by introducing the original construction used in the literature. Consider a global section of the natural projection $\Pi$,

$$\sigma_0 : \bar{\Gamma} \rightarrow \Gamma_0 \subset \Gamma_C \quad (I.18)$$
where we denoted $\Gamma_0 := \sigma_0(\bar{\Gamma})$. Every function $f$ defined on $\Gamma_C$ determines a function $\bar{f}$ defined on $\bar{\Gamma}$ by the restriction of $f$ to the slice $\Gamma_0$,

$$\bar{f} := \sigma_0^* f.$$  \hspace{1cm} (I.19)

Suppose $T$ is a family of functions $T^m, m \in \mathcal{M}$, on $\Gamma_C$ such that the slice $\Gamma_0$ is defined as their common zero set

$$\Gamma_0 = \{ \gamma \in \Gamma_C : T^m(\gamma) = 0, m \in \mathcal{M} \}. \hspace{1cm} (I.20)$$

We can then consider any other slice defined by

$$T^m = \tau^m \in \mathbb{R}; \hspace{1cm} (I.21)$$

that is,

$$\Gamma_\tau := \{ \gamma \in \Gamma_C : T^m(\gamma) = \tau^m, m \in \mathcal{M} \}. \hspace{1cm} (I.22)$$

In this way, for every family $\tau$ of numbers $\tau^m \in T^m := (\tau_0, \tau_1) \subset \mathbb{R}$, the clock system $T$ defines a global section $\sigma_\tau$. Functions $T^m$ are called in the literature clock functions. Then, for every $\tau$ we obtain from a function $f$ a function on $\bar{\Gamma}$; generalizing Eq. (I.19), it is

$$\bar{f}_\tau := \sigma_\tau^* f. \hspace{1cm} (I.23)$$

On the other hand, every function $\bar{f}$ on $\bar{\Gamma}$ can be equivalently expressed by the function $\Pi^* \bar{f}$ on $\Gamma_C$ extended arbitrarily to $\Gamma$ whenever it is useful. $\Pi^* \bar{f}$ is a gauge invariant (i.e., constant along each null direction), and we call it a (weak) Dirac observable. In the relational observables literature, the Dirac observable corresponding to such a function $\bar{f}$ is denoted by the symbol

$$F_{[f,T]}(\tau) := \Pi^* \bar{f}_\tau. \hspace{1cm} (I.24)$$

In this way, given a function $f$ on $\Gamma$ and a system $T$ of clock functions $T^m$, one defines for every family $\tau$ of numbers $\tau^m$ the Dirac observable [I.24].

7. **Gauge transforming a relational Dirac observable**

Given a function $f : \Gamma_C \rightarrow \mathbb{R}$, the dependence on $\tau$ of the Dirac observable $F_{[f,T]}(\tau)$ can be interpreted as an evolution,

$$\tau \rightarrow F_{[f,T]}(\tau), \hspace{1cm} (I.25)$$

or in terms of the physical phase space, as an evolution of the physical observable,

$$\tau \rightarrow \bar{f}_\tau. \hspace{1cm} (I.26)$$

More generally, for every gauge transformation, that is, a map $\alpha : \Gamma_C \rightarrow \Gamma_C$ which preserves each null leaf, we can define a transformed gauge invariant function:

$$\alpha \rightarrow F_{[\alpha^* f,T]}(\tau). \hspace{1cm} (I.27)$$

In this framework [I.3] one restricts to the gauge transformations preserving the foliation of $\Gamma_C$ defined by the clock system $T$, which is such that

$$\alpha^* T^m = T^m + \tau'^m, \hspace{1cm} \tau'^m \in \mathbb{R}, m \in \mathcal{M}. \hspace{1cm} (I.28)$$

Then, the gauge transformation amounts to

$$\tau' \rightarrow F_{[f,T]}(\tau + \tau'). \hspace{1cm} (I.29)$$

Therefore, this definition does not really add to Eq. (I.25) anything more; however, we keep it for the consistency with [I.7].

8. **Dirac bracket**

One more element bridging the kinematical phase space $\Gamma$ with the physical phase space $\bar{\Gamma}$ with the help of the system $T$ of the clock functions is the Dirac bracket,

$$C(\Gamma) \ni f, g \rightarrow \{ f, g \}^* \in C(\Gamma),$$

which may be defined using Eq. (I.23) by the following equality:

$$\{ f, g \}^*_\tau = \{ \bar{f}_\tau, g_\tau \}^\text{phys}$$

which is required to be true for every $\tau$. The relational construction of observables $F_{[\cdot,\cdot]}(\cdot)$, the Dirac bracket $\{ \cdot, \cdot \}^*$, the Poisson bracket in $\Gamma$, and the “gauge transformations” (I.29) are consistent in the following way [I.7]:

$$\{ F_{[f,T]}(\tau+\tau'), F_{[g,T]}(\tau+\tau') \} = F_{[\{ f,g \},T]}(\tau+\tau') \hspace{1cm} (I.30)$$

[Of course $\tau'$ could be erased without any loss of information, but we keep it for the consistency with Ref. [I.7] and Eq. (8.20) therein].

9. **Incorrect statements**

It is often stated in the relational observables literature about this action of the gauge transformations (I.25) and the evolution (I.25) (I.26) that:

1. They naturally pass to a map defined in the set $\mathcal{D} \subset C(\Gamma_C)$ of the gauge invariant functions on $\Gamma_C$,

$$\hat{\alpha} : \mathcal{D} \ni F_{[f,T]}(\tau) \rightarrow F_{[\alpha^* f,T]}(\tau) \in \mathcal{D} \hspace{1cm} (I.31)$$

by a fixed gauge transformation $\alpha$, or, respectively, to a map

$$\hat{\alpha} : \mathcal{D} \ni F_{[f,T]}(\tau) \rightarrow F_{[f,T]}(\tau + \tau') \in \mathcal{D}, \hspace{1cm} (I.32)$$

where $\tau'$ is defined by Eq. (I.28) (see Eq. (8.15) in Ref. [I.7]).

2. For every $\alpha$ or $\tau'$, the corresponding maps (I.31) (I.32) preserve the Poisson bracket $\{ \cdot, \cdot \}$ restricted to $\mathcal{D}$ (see the interpretation of Eq. (8.20) in Ref. [I.7]).
Let us explain why item 1 above is not true. Given a function $F_{[f,\tau]}(\tau)$ on $\Gamma_C$, the function $f$ is neither known nor unique. The would-be evolution would be well-defined if the right-hand side of Eq. (II.31) were independent of that ambiguity. Unfortunately, the right-hand side does depend on that choice; given the function $F_{[f,\tau]}(\tau)$, we can choose functions $f_1$ and $f_2$ such that

$$F_{[f_1,\tau]}(\tau) = F_{[f_2,\tau]}(\tau) = F_{[f,\tau]}(\tau);$$

nonetheless,

$$F_{[f_1,\tau]}(\tau + \tau') \neq F_{[f_2,\tau]}(\tau + \tau').$$

The maps (II.31) are therefore ill-defined.

Since item 1 above is not true, item 2 is pointless. But it is even worse than that. Itself, the consistency Eq. (II.30) is true, and one could think that it will eventually imply 2 as soon as the definition 1 is fixed. In this paper we correct the idea of item 1 of the action of the maps (II.5) de-...
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III.3

has been cured by the
ties. In this case, however, owing to Eq. (II.9).

Due to the Frobenius theorem, locally, there is a local
solution $\theta^l$ such that

$$\theta^l' = \tilde{\theta}^l(\theta^l). \quad (II.10)$$

The evolution (II.9) defined by $\theta^l$ is the same as that defined by $\theta$. The relevant part of the choice of $\theta$ is encoded in the vector fields $\partial_m$ corresponding to the coordinate system set by the functions $\theta^l, T^m$ on the constraint surface $\Gamma_C$. Conversely, for a given system $T$ of clock functions, let us consider a set of vector fields $\partial_m$ on $\Gamma_C$ satisfying

$$\partial_m(T^m) = \delta^N_m, \quad [\partial_m, \partial_n] = 0. \quad (II.11)$$

Due to the Frobenius theorem, locally, there is a local solution $\theta^l$ to the equations

$$\partial_m \theta^l = 0, \quad I \in \mathcal{I}.$$

That is enough if we keep ignoring the global nontrivialities. In this case, however, owing to Eq. (II.10), our definition of the evolution (II.7) extends consistently from one local chart $\tilde{\theta}_\tau$ to another local chart $\tilde{\theta}_\tau$.

III. SYMPLECTOMORPHICITY CONDITIONS

15. (Non)preserving of $(\cdot, \cdot)_{\text{phys}}$

As shown, item 1 of Sec. II.9 has been cured by the introduction of a system $\theta$ of reference functions. What about item 2? Does the map (II.7) preserve the physical Poisson bracket (II.11) defined on $C(\Gamma)$? For every fixed family of numbers $\tau$, we have defined coordinates on $\Gamma$,

$$\tilde{\theta}_\tau^l := \sigma_{\tau*} \theta^l. \quad (III.1)$$

In terms of them, the evolution (II.7) reads

$$\tilde{\theta}_\tau^l \mapsto \tilde{\theta}^l_{\tau'}. \quad (III.2)$$

The 2-form $\Omega_C$ used in the previous section to define the physical Poisson bracket can be written as

$$\Omega_C = \Omega_{II, J}(\theta^K, T^m) d\theta^I \wedge d\theta^J + dT^m \wedge (\omega_{mI} d\theta^I + \omega_{mn} dT^n). \quad (III.3)$$

On the physical phase space $\tilde{\Gamma}$, the formula for the physical symplectic form $\bar{\Omega}$ reads

$$\bar{\Omega} = \Omega_{II, J} (\tilde{\theta}^K, \tau^m) d\tilde{\theta}^I \wedge d\tilde{\theta}^J. \quad (III.4)$$

The physical Poisson bracket between two functions $\tilde{\theta}^l$ and $\tilde{\theta}^J$ reads

$$(\tilde{\theta}^l, \tilde{\theta}^J)_{\text{phys}} = (\Omega^{-1})^{IJ} (\tilde{\theta}^K, \tau^m). \quad (III.5)$$

Therefore, the Poisson bracket is preserved if and only if the 2-form $\Omega_C$ decomposed according to Eq. (III.3) satisfies

$$\frac{\partial}{\partial T^m} \Omega_{II, J}(\theta^K, T^m) = 0. \quad (III.6)$$

Generically, this condition is not satisfied, and the map (II.7) is not a symplectomorphism; hence, item 2 is not true. In fact, in example 2 we will see that every arbitrarily chosen family of maps $C(\Gamma) \to C(\Gamma)$ labelled by $\tau$, $\tilde{f} \mapsto f_{\tau}$ can be obtained as the evolution (II.9).

16. Identities

The condition $d\Omega_C = 0$ implies conditions on the terms of the decomposition (III.3) $\Omega_C$. In particular we have

$$d^{(\theta)} (\Omega_{II, J} d\theta^I \wedge d\theta^J) = 0 \quad (III.7)$$

where by $d^{(\theta)}$ we denoted the part of the exterior derivative involving only the derivatives $\frac{\partial}{\partial \theta^l}$. This condition is equivalent to simply

$$d\bar{\Omega} = 0. \quad (III.8)$$

Another condition is

$$d^{(\theta)} (\Omega_{II, J} d\theta^I \wedge d\theta^J) = d^{(\omega_{mI} d\theta^I)} \quad (III.9)$$

17. Symplectomorphy and Hamiltonians

Therefore, a reference system $\theta$ and a clock system $T$ do satisfy the symplectomorphism condition (III.6) if and only if

$$d^{(\theta)} (\omega_{mI} d\theta^I) = 0. \quad (III.10)$$

In this case, there are (locally) defined Hamiltonians $H_m$,

$$\omega_{mI} d\theta^I = d^{(\theta)} H_m (\theta, T), \quad (III.11)$$

which project to $\tau$-dependent Hamiltonians on $\Gamma$:

$$\tilde{H}_m (\tau) = \sigma_{\tau*} H_m, \quad m \in \mathcal{M}. \quad (III.12)$$

We will go back to this and specifically to how the corresponding Hamiltonian vector fields fit into this approach after the following three examples.
IV. EXAMPLES

18. Example 1: Trivial evolution

Let $\tilde{\theta}_I$ be coordinates on $\bar{\Gamma}$. Define the system $\theta$ of reference functions to consist of the functions

$$\theta^I = \Pi^* \tilde{\theta}^I.$$  \hspace{1cm} (IV.1)

In terms of those functions

$$\Omega_C = \Omega_{I,\bar{I}}(\theta^K) d\theta^I \wedge d\tilde{\theta}^J.$$  \hspace{1cm} (IV.2)

The resulting evolution is the identity

$$\bar{\theta}^I_t = \tilde{\theta}^I.$$  \hspace{1cm} (IV.3)

In this case we even did not need to fix on $\Gamma_C$ any system $T$ of clock functions. The result is independent of that choice.

19. Example 2: Arbitrary evolution

Suppose there is given an arbitrary family of maps $C(\tilde{\Gamma}) \to C(\bar{\Gamma})$,

$$f \mapsto \tilde{f}_\tau$$  \hspace{1cm} (IV.4)

labelled by the family $\tau$ of numbers $\tau^m$ such that:

- the family $\tau$ is consistent with a system $T$ of clock functions $T^m$ on $\Gamma_C$;
- for $\tau = 0$, the map (IV.4) is the identity;
- $\tilde{f}_{\tau}(\bar{\gamma})$ is differentiable in each $\tau^m$ for every $\bar{\gamma}$ and every differentiable function $\tilde{f}$.

We construct now a system $\theta$ of reference functions $\theta^I$ on $\Gamma_C$, such that the corresponding evolution (IV.3) will coincide with

$$(\tilde{f}, \tau) \mapsto \tilde{f}_\tau$$

given by Eq. (IV.3).

Let $\bar{\theta}^I$ be coordinates in $\bar{\Gamma}$. The map (IV.4) maps each of them appropriately:

$$\bar{\theta}^I \mapsto \tilde{\theta}^I.$$  \hspace{1cm} (IV.5)

The suitable reference functions are defined at each point $\gamma \in \Gamma_C$ as follows:

$$\theta^I(\gamma) := \bar{\theta}^I(\Pi(\gamma))|_{\tau = T(\gamma)}.$$  \hspace{1cm} (IV.6)

20. Example 3: Standard choice

In practice, we have at our disposal the auxiliary kine-

mical phase space $\Gamma$ endowed with coordinate system $(p_x,q^\chi)$, where $\chi$ runs over a labelling set $\mathcal{X}$, such that

$$\Omega = \sum_\chi dp_\chi \wedge dq^\chi.$$  \hspace{1cm} (IV.7)

Suppose that on the constraint surface $\Gamma_C$ the coordinates

$$p_m, \quad m \in \mathcal{M} \subset \mathcal{X}$$  \hspace{1cm} (IV.8)

are determined by the remaining coordinates. Split the set of coordinates accordingly:

$$p_\chi = p_i, \quad p_m = q_i, \quad q^\chi.$$  \hspace{1cm} (IV.9)

Hence,

$$p_m|_{\Gamma_C} = H_m, \quad m \in \mathcal{M}$$  \hspace{1cm} (IV.10)

where $H_m : \Gamma \to \mathbb{R}$ is for every $m \in \mathcal{M}$ a function such that

$$\partial_{p_m} H_m = 0, \quad n \in \mathcal{M}.$$  \hspace{1cm}

Suppose that the index $m$ ranging the subset $\mathcal{M}$ labels also the null directions tangent to $\Gamma_C$. This is what happens when $\Gamma_C$ is defined by first class constraints. Choose for a system of clock functions

$$T^m = q^m.$$  \hspace{1cm} (IV.11)

Let us choose for reference functions

$$\theta^I = q^i, p_j.$$  \hspace{1cm} (IV.12)

The 2-form $\Omega_C$ expressed by those functions is

$$\Omega_C = dp_i \wedge dq^i + dT^m \wedge dH_m.$$  \hspace{1cm} (IV.13)

Remarkably, somewhat for free, it satisfies the condition (III.6) for the corresponding evolution in $\bar{\Gamma}$ to be Hamiltonian. The corresponding Hamiltonians are the functions $p_m$ restricted to slices of $\Gamma_C$ such that

$$T^m = \text{const}, \quad m' \in \mathcal{M}.$$  \hspace{1cm}

This method is often used in practice (e.g., Refs. [11, 12]) due to its simplicity and efficiency.

21. Example 4: Nontrivial clock function

The starting point is similar to the previous example, namely

$$\Omega = dp \wedge dq + dP \wedge dQ.$$  \hspace{1cm} (IV.14)
and \( \Gamma_C \) is defined by the equation
\[
P - h(q,p) = 0. \tag{IV.15}
\]
Let the clock function
\[
T = \tilde{T}(q,p,Q,P),
\]
be defined in the whole \( \Gamma \), and, conversely,
\[
Q = \tilde{Q}(q,p,T,P).
\]
Finally, let
\[
\theta^I = q,p.
\]
Then, in the new coordinates \( q,p,T,P \),
\[
\Omega_C = (1 + h.p\tilde{Q},q - h.q\tilde{Q},p)dp \wedge dq + \tilde{Q},Tdh \wedge dT.
\]
The symplectomorphicity condition
\[
(h.p\tilde{Q},q - h.q\tilde{Q},p)_T = 0,
\]
generically is not satisfied.

22. Message

We learn from the examples that our relational evolution can be trivial (example 1) or arbitrary (example 2). If the gauge fixing functions are just some of the coordinates and for the physical degrees of freedom we choose another subset of the coordinate system in which the symplectic 2-form had the canonical form, then, in a case of first class constraints, the corresponding evolution is symplectomorphic (example 3). Finally, if in example 3 we make a nontrivial choice of clock function but leave the coordinates parametrizing the physical degrees of freedom, then, generically, the symplectomorphicity condition will be violated (example 4).

V. FROM \((T,\theta)\) TO THE GENERATORS OF EVOLUTION

23. Generators

Given a reference system \( \theta \) and a clock system \( T \) on the constraint surface \( \Gamma_C \), the corresponding evolution defined by the relational framework is generated just by the vector fields \( \frac{\partial}{\partial T^m} \), \( m \in M \). What is nontrivial about those vector fields are their \( (\tau\text{-dependent}) \) projections onto the physical phase space \( \Gamma \). They can be calculated by using the decomposition of \( \frac{\partial}{\partial T^m} \) into the null part and the part tangent to the slices
\[
T^m' = \tau^m', \quad m' \in M, \tag{V.1}
\]
that is, by
\[
\frac{\partial}{\partial T^m} = \frac{\partial}{\partial T^m} - X_m + X_m, \tag{V.2}
\]
such that
\[
(\frac{\partial}{\partial T^m} - X_m) \odot \Omega_C = 0, \quad X_m = \tilde{X}_m^I\frac{\partial}{\partial \theta^I}. \tag{V.3}
\]
The evolution defined by \( (\theta,T) \) in \( C(\tilde{\Gamma}) \) is generated by the \( \tau \)-dependent vector fields on \( \tilde{\Gamma} \) labelled by \( m \in M \),
\[
\tilde{X}_m(\tau) = \Pi_x\frac{\partial}{\partial T^m} |_{T=\tau} = \Pi_x X_m |_{T=\tau} = \tilde{X}_m^I(\tau)\frac{\partial}{\partial \theta^I}. \tag{V.4}
\]

24. Calculation of \( \tilde{X}_m \)

The vector field \( X_m \), can be calculated from Eqs. III.3 \( \text{V.3} \) by inverting the equality
\[
X_m^I\Omega_{IJ} = \omega_{m,J} \tag{V.5}
\]
(notice that \( \Omega_{IJ} \) is necessarily invertible).

25. Additional conditions

There are additional consistency conditions on \( \Omega_{IJ}\), \( \omega_{m,I} \), and \( \omega_{mn} \), namely Eq. V.3 implies
\[
\omega_{nn} = 0, \quad \omega_{mn} - \omega_{nm} = X_n \odot \omega_m \tag{V.6}
\]
and \( d\Omega_C = 0 \) in addition to Eqs. III.7 \( \text{V.9} \) imply
\[
\frac{\partial}{\partial T^m}\omega_{nI} - \frac{\partial}{\partial T^I}\omega_{mI} - 2\frac{\partial}{\partial \theta^I}\omega_{[mn]} = 0 = \omega_{[m'n',m'n]}, \tag{V.7}
\]

26. Symplectomorphisms once again

Now, in the case \( \text{III.11} \) considered before when the corresponding evolution \( C(\Gamma) \to C(\tilde{\Gamma}) \) preserved the Poisson bracket, we have
\[
\tilde{X}_m(\tau) \odot \tilde{\Omega} = \omega_{m,I}(T = \tau)d\frac{\partial}{\partial \theta^I} = d\tilde{H}_m(\tau), \tag{V.8}
\]
where the functions \( \tilde{H}_m \) were defined on \( \tilde{\Gamma} \) in Eq. \( \text{III.12} \). Hence, indeed, in this case \( \tilde{X}_m(\tau) \) are Hamiltonian vector fields, and the Hamiltonians are the functions we had already defined before.

VI. SUMMARY

We have fixed the inconsistent definition of a relational evolution of the Dirac observables of Hamiltonian-constrained systems [Ref. 1, Eq. (8.15)]. To this end, in addition to a system, say, \( T \) of clock functions used in the relational observables framework, we introduced a system of reference functions, say, \( \theta \). With this structure, the gauge transformations preserving the constant-value surfaces of the clock functions do induce a family of movements of the physical phase space \( \Gamma \). In general and in
fact even generically, the movements are not symplectomorphisms of $\widehat{\Gamma}$ (unlike what could be concluded from Ref. [7], Eq. (8.20), assuming that Ref. [7], Eq. (8.15) is true). While the induced movements are defined by pairs $(T, \theta)$, all the clock function systems $T$ are gauge equivalent to each other. Therefore, one can fix any system of clock functions on $\Gamma_C$ and vary only the reference function systems. The reference function systems which do induce symplectomorphisms in the physical phase space $\widehat{\Gamma}$ are characterized by some special form taken by the 2-form $\Omega_C$. One can read off the corresponding Hamiltonians in that case. On the other hand, every family of maps $\widehat{\Gamma} \rightarrow \Gamma$ (not necessarily symplectomorphic) labelled by the labelling set of the clock function system can be obtained as the relational evolution corresponding to a suitable reference function system. One of the examples we illustrate our construction with shows in what way a naive choice of the clock and, respectively, reference systems provides a symplectomorphic relational evolution. Another example shows that a nontrivial choice of a clock function system induces in the physical phase space a nonsymplectic evolution.

Our work provides the correctly defined evolution of Hamiltonian-constrained systems. The issue of the physical evolution of those systems is still an outstanding problem of general relativity. We hope that our correction of the relational approach will help to solve this problem.

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