MARKOV DUALITY FOR STOCHASTIC SIX VERTEX MODEL

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Abstract. We prove a Schütz-type self-duality for the stochastic six vertex model. We introduce a new method of inductive argument to prove the Markov duality.

1. Introduction

1.1. Stochastic six vertex model. The stochastic six vertex model (S6V model) is a classical model in 2d statistical physics first introduced by Gwa and Spohn [GS92], as a special case of the six vertex (ice) model (see for example [Lie74] and [Bax16]). We associate each vertex in $\mathbb{Z}^2$ with six types of configuration with weights parametrized by $0 < b_1, b_2 < 1$, see Figure 1. The configurations chosen for two neighboring vertices need to be compatible in the sense that the lines keep flowing. We consider the lines from the south and the west as the inputs and the lines to the north and the east as the outputs. Each vertex is conservative in the sense that the number of input lines equals the number of output lines. The model is stochastic in the sense that when we fix the inputs, the weights of possible configurations sum up to 1.

| Type | I    | II   | III  | IV   | V    | VI   |
|------|------|------|------|------|------|------|
| Configuration |   |   |     |     |     |     |
| Weight  | 1   | 1   | $b_2$ | $1 - b_2$ | $b_1$ | $1 - b_1$ |

Figure 1. Six types of configuration for the vertex.

The S6V model is a member of the KPZ universality class (see [Cor12, Qua11] for a nice survey). We briefly review a few results that have been obtained for the S6V model. [BCG16] proves that under step initial condition, the one point fluctuation of the S6V model height function is asymptotically Tracy-Widom GUE. One point fluctuations of the S6V model under more general initial condition including stationary is obtained in [ABI0, Aag18]. In a slightly different direction, [CGST18] shows that under scaling $\frac{1}{b} \rightarrow 1$ with $b_1$ fixed, the fluctuation of the S6V model height function converges weakly to the solution of KPZ equation. More recently, [BG18, ST18] showed under a different scaling, the height fluctuation of the S6V model converges in finite dimensional distribution to the solution of stochastic telegraph equation.

In this paper, we consider the S6V model as an interacting particle system (see [GS92] or Section 2.2 of [BCG16]). Consider vertex configurations in the upper half-plane, we restrict ourselves to the boundary condition that there are no lines coming from the left boundary. Then the input lines coming from the bottom of the horizontal axis can be viewed as the trajectories of an exclusion-type particle system. We see the vertical axis as time variable and horizontal axis as space variable. The vertex configurations compose several paths, which can be viewed as trajectories of particles with the vertical lines denoting the particle location. As illustrated by Figure 2 we cut our plane by the line $y = t - \frac{1}{2}$ and the particle location at time $t$ is give by the intersections (red points in the figure) of these trajectories with $y = t - \frac{1}{2}$. To rigorously define our interacting particle system, we first introduce the following state spaces.

Definition 1.1. We define the space of left-finite particle occupation configuration $\mathcal{G}$ to be

$$\mathcal{G} = \left\{ \bar{g} = (\cdots, g_{-1}, g_0, g_1, \cdots) \in \{0, 1\}^\mathbb{Z} : \exists i \in \mathbb{Z} \text{ so that } g_x = 0, \forall x \leq i \right\},$$

where $g_x$ is understood as the particle number (either zero or one) at location $x$. We also define the space of left order particle location configuration to be

$$\mathcal{X} = \{ \bar{x} = (x_1 < x_2 < \cdots) : x_i \in \mathbb{Z} \cup \{+\infty\} \text{ for every } i \in \mathbb{N} \}.$$
We also define the S6V occupation process \( \vec{g} \) (a) When \( x_1 < x_2(t) < \cdots \) for the latter case, there exists some \( m \in \mathbb{N} \) so that \( x_i = +\infty \) for \( i \geq m \). It is straightforward that there is a bijection \( \varphi : \mathbb{X} \to \mathbb{G} \) defined by
\[
\vec{g} = \varphi(\vec{x}) \text{ such that } g_i = \mathbb{1}_{\left\{ \text{there exists } n \in \mathbb{N} \text{ so that } x_n = i \right\}} \text{ for every } i \in \mathbb{Z}.
\]
Having specified our state space, we proceed to define the particle interpretation of the S6V model as the following discrete time Markov processes. The following definition is similar as the one that appears in Section 2.1 of [CGST18].

**Definition 1.2.** We define the S6V location process, which is a discrete time \( \mathbb{X} \)-valued Markov process \( \vec{x}(t) = (x_1(t), x_2(t), \cdots) \) with the update rule (transition probability) from \( \vec{x}(t) \) to \( \vec{x}(t+1) \) specified as follows:

We denote \( x_0(t) = -\infty \) for any \( t \in \mathbb{Z}_{\geq 0} \), this is just a convention to simplify the notation. We sequentially consider \( i = 1, 2, \ldots \) and update as following independent probabilities:

(a) When \( x_i(t) > x_{i-1}(t+1) \), we update \( x_i(t) \) to \( x_i(t+1) \) via
\[
\begin{align*}
\mathbb{P}(x_i(t+1) - x_i(t) = n) &= \begin{cases} 
  b_1, & \text{if } n = 0; \\
  (1 - b_1)(1 - b_2)\mathbb{2}^{n-1}, & \text{if } 1 \leq n \leq x_{i+1}(t) - x_i(t) - 1; \\
  (1 - b_1)b_2^{n-1}, & \text{if } n = x_{i+1}(t) - x_i(t); \\
  0, & \text{else};
\end{cases}
\end{align*}
\]

(b) When \( x_i(t) = x_{i-1}(t+1) \), we update \( x_i(t) \) to \( x_i(t+1) \) via
\[
\begin{align*}
\mathbb{P}(x_i(t+1) - x_i(t) = n) &= \begin{cases} 
  (1 - b_2)b_2^{n-1}, & \text{if } 1 \leq n \leq x_{i+1}(t) - x_i(t) - 1; \\
  b_2^{n-1}, & \text{if } n = x_{i+1}(t) - x_i(t); \\
  0, & \text{else}.
\end{cases}
\end{align*}
\]

We also define the S6V occupation process \( \vec{g}(t) = (g_x(t))_{x \in \mathbb{Z}} \in \mathbb{G} \) by setting \( \vec{g}(t) = \varphi(\vec{x}(t)) \) i.e.
\[
g_x(t) = \mathbb{1}_{\left( \text{there exists } n \in \mathbb{N} \text{ so that } x_n(t) = x \right)} \text{ for every } x \in \mathbb{Z}.
\]

Clearly, \( \vec{g}(t) \) is a discrete time \( \mathbb{G} \)-valued Markov process. We remark that the occupation process \( \vec{g}(t) \) and location process \( \vec{x}(t) \) are just two ways to describe the particle interpretation of the S6V model.

For a left-finite particle configuration \( \vec{g} \in \mathbb{G} \), we define the height function \( N_x(\vec{g}) \) to be the total number of particles in this particle configuration that is on the left or at location \( x \), i.e.
\[
N_x(\vec{g}) = \sum_{i \leq x} g_i.
\]

Our duality result is a self-duality between the S6V occupation process and its space reversal, which we define below:

\[\text{Figure 2. S6V model viewed as interacting particle system.}\]
Definition 1.3. Define the space of reversed $k$-particle location configuration $\mathcal{Y}^k = \{ \vec{y} = (y_1 > \ldots > y_k) : \vec{y} \in \mathbb{Z}^k \}$. The reversed $k$-particle S6V location process $\vec{g}(t) = (g_1(t) > \ldots > g_k(t))$ is a $\mathcal{Y}^k$-valued Markov process so that $(-y_k(t) < \ldots < -y_1(t))$ has the same update rule as the S6V location process.

1.2. Markov Duality.

Definition 1.4. Given two discrete (continuous) time Markov processes $X(t) \in U$ and $Y(t) \in V$ and a function $H : U \times V \to \mathbb{R}$, we say that $X(t)$ and $Y(t)$ are dual with respect to $H$ if for any $x \in U, y \in V$ and $t \in \mathbb{Z}_{\geq 0}$ ($t \in \mathbb{R}_{\geq 0}$ for continuous time), we have

$$\mathbb{E}^x [H(X(t), y)] = \mathbb{E}^y [H(x, Y(t))].$$

Here we use $\mathbb{E}^x$ to denote that we take the expectation under initial condition $X(0) = x$. Likewise, $\mathbb{E}^y$ represents the expectation with initial condition $Y(0) = y$.

Markov dualities have been found for different interacting particle systems including the contact process, voter process (ASEP), which is an interacting particle system on $\mathbb{Z}$ with at most one particle at each site. Each particle jumps to the left with rate $\ell$ and jumps to the right with rate $r$. If the site is already occupied by another particle, the jump is excluded.

We consider ASEP as a process $\vec{g}(t) = (g_x(t))_{x \in \mathbb{Z}} \in \{0,1\}^\mathbb{Z}$, where $g_x(t)$ is an indicator for the event that at time $t$, a particle is at site $x$. We call $\vec{g}(t)$ the ASEP occupation process. When the ASEP has finite $k$ particles, in terms of particle location, we also consider the $k$-particle ASEP location process $\vec{Y}(t) = (y_1(t) > \ldots > y_k(t)) \in \mathcal{Y}^k$ where $y_i(t)$ denotes the location of $i$-th particle counting from the right, at time $t$.

[Sch97] derived the following ASEP self-duality (using a spin chain representation): For any fixed $k \in \mathbb{N}$, the ASEP occupation process $\vec{g}(t)$ and the $k$-particle ASEP location process $\vec{Y}(t)$ with the jump rate $r$ and $\ell$ reversed are dual with respect to the duality functional

$$H(\vec{g}, \vec{y}) = \prod_{i=1}^k g_{y_i} q^{-N_{y_i}(\vec{g})},$$

where $q = \ell/r$. This generalizes the self-duality for the symmetric simple exclusion process [Lig12] where $\ell$ and $r$ are set to be equal. We call the self-duality with functional in (1.1) Schütz-type.

[BCS14] uses a different approach to prove Schütz’s result by directly applying the Markov generator on the duality functional. Further, they use this method to show that the ASEP is also self-dual with respect to the duality functional

$$G(\vec{g}, \vec{y}) = \prod_{i=1}^k q^{-N_{y_i}(\vec{g})}.$$  

The ASEP is a continuous time limit of the S6V model if we scale the parameter by $b_1 = c \ell, b_2 = c r$ and scale time by $c^{-1} t$ and shift the space to the right by $c^{-1} t$, see [BCG16, Agh17]. Given the ASEP is the limit of the S6V model and is self-dual with respect to the duality functional in (1.1) and (1.2), one might wonder if the S6V model is self-dual to these functional as well. Indeed, by setting $q = \frac{b_1}{b_2}$, [CPI15, Theorem 2.21] justifies that S6V model (and also its higher spin generalizations) is self-dual to the function in (1.2).

Our main theorem shows that the S6V model also enjoys a Schütz-type self-duality as in (1.1).

Theorem 1.5. Consider the S6V model with parameter $b_1, b_2$ and set $q = \frac{b_1}{b_2}$. For any $k \in \mathbb{N}$, the S6V occupation process $\vec{g}(t) \in \mathbb{G}$ and the reversed $k$-particle S6V location process $\vec{Y}(t) \in \mathcal{Y}^k$ are dual with respect to the function $H$ given in (1.1).

We remark that the duality in Theorem 1.5 has shown up in [CPI15, Theorem 2.23] and was later used in proving [CGST18, Proposition 5.3]. However, the proof that appeared in [CPI15, Theorem 2.23] is incorrect since it claims that the S6V duality (1.1) can be deduced by taking the discrete gradient of the duality functional in (1.2), which is not true in general when $k$ is larger than 1 [CP].

1 In fact, [CP15, Theorem 2.23] claims a duality for the stochastic higher spin vertex model, which is a generalization of the S6V model. However, the duality is not correct when the spin parameter therein is larger than 1.
Duality has been obtained for generalization of the ASEP and S6V model using algebraic methods. \cite{CGRS16} proves two ASEP\((q, j)\) (which is a higher spin generalization of ASEP) self-dualities based on the higher spin representations of \(U_q(sl_2)\). In the spirit of \cite{CGRS16}, self-duality has also been proved for multi-species version of ASEP \cite{BS15, Kua16}. \cite{Kua18} obtains a self-duality for the multi-species version of the stochastic higher spin vertex model, which is a generalization of the S6V model, via an algebraic construction.

Instead of using algebraic tools to prove duality, our proof of Theorem 1.5 follows a straightforward induction approach.

We remark that the duality functional from \cite{Kua18} Theorem 4.10 has a degeneration to S6V model. We state this degeneration here as a lemma.

**Lemma 1.6.** Consider the S6V model with parameter \(b_1, b_2\) and set \(q = \frac{b_1}{b_2}\). For any \(k \in \mathbb{N}\), the S6V occupation process \(\tilde{g}(t) \in \mathcal{G}\) and the reversed \(k\)-particle S6V location process \(\tilde{y}(t) \in \mathcal{Y}^k\) are dual with respect to the functional

\[
D(\tilde{g}, \tilde{y}) = \prod_{i=1}^{k} (1 - g_{y_i}) q^{-N_{y_i}(\tilde{g})}.
\]

**Proof.** Taking the spin parameter \(m_x = 1\) for all \(x \in \mathbb{Z}\) and species number \(n = 1\) and substituting \(q\) by \(q^{1/2}\) in \cite{Kua18} Theorem 4.10, the multi-species higher spin vertex model considered in \cite{Kua18} Theorem 4.10 degenerates to the stochastic six vertex model. Referring to the duality functional \((\xi | D(u_0) | \eta)\) considered therein, substituting the symbol \(q = (\eta_x)_{x \in \mathbb{Z}}\) by \(\tilde{g} = (g_x)_{x \in \mathbb{Z}}\) and the particle occupation configuration \(\xi = (\xi_x)_{x \in \mathbb{Z}}\) by the \(k\)-particle location configuration \(\tilde{y} = (y_1, \ldots, y_k)\), we obtain that the reversed S6V occupation process \(\tilde{g}(t)\) (with particles jumping to the left) is dual to the \(k\)-particle S6V location process \(\tilde{y}(t)\) (with particles jumping to the right) with respect to the functional

\[
\tilde{D}(\tilde{g}, \tilde{y}) = \prod_{i=1}^{k} (1 - g_{y_i}) q^{-\sum_{z \geq y_i} g_z} = \prod_{i=1}^{k} (1 - g_{y_i}) q^{-\sum_{z \geq y_i} g_z}.
\]

In the last equality we used the fact that \(g_{y_i} \in \{0, 1\}\). Since \(\tilde{g}(t)\) and \(\tilde{y}(t)\) are nothing but the space reversal of \(\tilde{g}(t)\) and \(\tilde{y}(t)\) in the lemma. After swapping the role of left and right (then \(\sum_{z \geq y_i} g_z\) is exactly the height function of \(\tilde{g}\) at \(y_i\)), we readily obtain the duality in \((1.3)\). \(\square\)

When \(k = 1\), our duality can be derived by subtracting the functional \(G\) in \((1.2)\) by the functional \(D\) in \((1.3)\) (see Lemma 2.1). However, it appears that when \(k > 1\), there is no easy way to obtain our duality by combining the duality functionals in \((1.2)\) and \((1.3)\).

Finally, we explain several applications of our duality. Theorem 1.5 combined with the other S6V self-duality \((1.2)\) are the main tools for proving the self-averaging property of the specific quadratic function of the S6V height function in \cite{CGST18} Proposition 5.3, which is the crux in proving the convergence of stochastic six vertex model to KPZ equation. In a different direction, by using duality, we can compute the exact moment formula of certain observables of our model. \cite{BCS14} uses the Schütz-type duality of ASEP to derive the moment generating function of the ASEP height function under Bernoulli step initial data. Applying a similar approach, we expect by using our duality and the S6V Bethe ansatz eigenfunction given by \cite{CP16} Proposition 2.12, we can reprove the moment formula appears in \cite{BCG16} Theorem 4.12 and \cite{AB16} Theorem 4.4. Since this application of the duality is not related to our paper, we do not pursue to give the proof here.

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2. **Proof of Theorem 1.5**

In this section, we prove Theorem 1.5. We first introduce several notations for our proof. Define the space of \(\ell\)-particle location configuration

\[
\mathcal{X}^\ell = \{ \vec{x} = (x_1 < \cdots < x_\ell) : \vec{x} \in \mathbb{Z}^\ell \}.
\]
We also denote by $|\vec{g}|$, $|\vec{x}|$ and $|\vec{y}|$ the particle number in the particle configuration $\vec{g} \in \mathbb{G}$, $\vec{x} \in \mathbb{X}$ and $\vec{y} \in \mathbb{Y}^k$ (obviously $|\vec{y}| = k$ when $\vec{g} \in \mathbb{Y}^k$) respectively.

Referring to the Definition 4.12 of Markov duality, we need to show that for any $k \in \mathbb{N}$ and under any initial states $\vec{g} \in \mathbb{G}$, $\vec{y} \in \mathbb{Y}^k$, we have

$$E[\vec{y}(\vec{g}(t), \vec{y})] = E[\vec{y}(\vec{g}, \vec{y}(t))].$$

By Markov property, it suffices to prove that the preceding equation holds for $t = 1$, namely, for any $k \in \mathbb{N}$ and any $\vec{g} \in \mathbb{G}$, $\vec{y} \in \mathbb{Y}^k$, we have

$$E[\vec{y}(\vec{g}(1), \vec{y})] = E[\vec{y}(\vec{g}, \vec{y}(1))].$$

Observing that $|\vec{y}|$ is finite (since $\vec{y} \in \mathbb{Y}^k$) whereas $|\vec{g}|$ can either be finite or infinite. We claim that it suffices to prove (2.1) for all $\vec{g} \in \mathbb{G}$ such that $|\vec{g}|$ is finite, here is the reason: Suppose we have proved (2.1) for every $\vec{g} \in \mathbb{G}$ with $|\vec{g}| < \infty$. For $\vec{g} \in \mathbb{G}$ and $\vec{y} \in \mathbb{Y}^k$ such that $|\vec{g}| = \infty$ and $\vec{y} = (y_1, \ldots, y_k)$, we consider a particle configuration $\vec{g}' \in \mathbb{G}$ that corresponds with $\vec{g} \in \mathbb{G}$ in the following way:

$$g'_i = \begin{cases} g_i & \text{if } i \leq y_1; \\ 0 & \text{if } i > y_1. \end{cases}$$

Clearly, $|\vec{g}'| < \infty$ and hence $E[\vec{y}(\vec{g}(1), \vec{y})] = E[\vec{y}(\vec{g}', \vec{y}(1))].$ Additionally, observing that the particles in the configuration $\vec{g}'$ which are on the right of $y_1$ have no contribution to both the expectations $E[\vec{y}(\vec{g}(1), \vec{y})]$ and $E[\vec{y}(\vec{g}, \vec{y}(1))]$ (see Figure 3) and thus

$$E[\vec{y}(\vec{g}(1), \vec{y})] = E[\vec{y}(\vec{g}', \vec{y}(1))].$$

We conclude that $E[\vec{y}(\vec{g}(1), \vec{y})] = E[\vec{y}(\vec{g}', \vec{y}(1))]$ also holds for all $\vec{g} \in \mathbb{G}$ with $|\vec{g}| = \infty$.

**Figure 3.** The picture above shows an example for the initial states $\vec{g} \in \mathbb{G}$ and $\vec{y} \in \mathbb{Y}^3$. The blue particles in $\vec{g}$ that are on the right of $y_1$ have no affect on the computation of both expectation $E[\vec{y}(\vec{g}(1), \vec{y})]$ and $E[\vec{y}(\vec{g}, \vec{y}(1))]$.

It remains to prove (2.1) when $|\vec{g}|$ is finite. In other words, we need to prove (2.1) for all $\ell, k \in \mathbb{N}$ and all $\vec{g} \in \mathbb{Y}^k$, $\vec{g} \in \mathbb{G}$ satisfying $|\vec{g}| = \ell$. We apply induction on the particle number $\ell, k$. The first thing is to show that (2.1) holds when $\min(\ell, k) = 1$, as the induction basis.

**Lemma 2.1.** When $\min(\ell, k) = 1$, (2.1) holds.

**Proof of Lemma 2.1.** For $k > 1$ and $\ell = 1$, note

$$H(\vec{g}(1), \vec{y}) = \prod_{i=1}^{k} g_{y_i}(1)q^{-N_y(\vec{g}(1))}.$$

Since $\vec{g}(1)$ has only one non-zero component and $k > 1$, we have $\Pi_{i=1}^{k} g_{y_i}(1) = 0$ for any $\vec{g}(1)$ and thus $E[\vec{y}(\vec{g}(1), \vec{y})] = 0.$ Similarly, we have $E[\vec{y}(\vec{g}, \vec{y}(1))] = 0$ and hence the desired equality holds.

When $k = 1$, we note that $G(\vec{g}, \vec{y}) = H(\vec{g}, \vec{y}) - D(\vec{g}, \vec{y})$, where $H$ and $D$ are given by (1.2) and (1.3). Since the subtraction of two duality functional is still a duality functional, we obtain the desired (2.1). \qed

Before explaining how the induction works, we slightly reformulate (2.1). In order to keep track of the location of the particles, we utilize the S6V location process $\vec{x}(t)$ in Definition 1.2. Via the bijection $\varphi : \mathbb{X} \to \mathbb{G}$ we can identify a configuration $\vec{g} \in \mathbb{G}$ with $\vec{x} \in \mathbb{X}$ and define the function $\vec{H}$ as

$$\vec{H}(\vec{x}, \vec{y}) = H(\varphi(\vec{x}), \vec{y}).$$
By the relation $\vec{g}(t) = \varphi(\vec{x}(t))$ between the S6V occupation process $\vec{g}(t)$ and the S6V location process $\vec{x}(t)$, we can be paraphrased into the following:

For any $\ell, k \in \mathbb{N}$ and any initial states $\vec{x} \in \mathcal{X}^\ell$ and $\vec{y} \in \mathcal{Y}^k$, we have

$$E^F[H(\vec{x}(1), \vec{y})] = E^\vec{y}[H(\vec{x}, \vec{y}(1))].$$  \hspace{1cm} (2.2)

When $\min(\ell, k) = 1$, (2.2) is established via Lemma 2.1. When $\min(\ell, k) \geq 2$, we define our induction hypothesis as

$$\text{(HYP}_{\ell,k})$$

It suffices to prove (2.2) for any $\vec{x} \in \mathcal{X}^\ell$ and $\vec{y} \in \mathcal{Y}^k$ under (HYP$_{\ell,k}$). We briefly explain our strategy: We decompose the l.h.s. expectation of (2.2) into a combination of the expectations which are in the form of $E^F[H(\vec{x}(1), \vec{y})]$ with $|\vec{x}| + |\vec{y}| < \ell + k$. The decomposition which is in similar expression occurs for the r.h.s. expectation. By applying (HYP$_{\ell,k}$), we get the desired (2.2).

In the sequel, when there is only one particle in the S6V location process, we denote by $p(x, y)$ the one particle transition probability from location $x$ to $y$. Similarly $\hat{p}(x, y)$ denotes the one particle transition probability from $x$ to $y$ for the reversed S6V location process. Clearly, $p(x, y) = \hat{p}(y, x)$ and

$$p(x, y) = \begin{cases} b_1 & \text{if } y = x; \\ (1 - b_1)(1 - b_2)b_2^{y-x-1} & \text{if } y > x; \\ 0 & \text{else}. \end{cases}$$

**Proof of Theorem 1.1.** We denote the processes in (2.2) by $\vec{x}(t) = (x_1(t) < \cdots < x_\ell(t), y_1(t) > \cdots > y_k(t))$ and the initial states by $\vec{x} = (x_1 < \cdots < x_\ell), \vec{y} = (y_1 > \cdots > y_k)$. We split our proof into different cases depending on the relation fo $\vec{x}$ and $\vec{y}$.

**Case (1):** $y_k \notin \{x_1, \cdots, x_\ell\}$

Denote by $s$ the positive integer satisfying $x_s < y_k < x_{s+1}$. We consider the S6V location processes

$$\vec{x}^s(t) = (x_1^s(t) < \cdots < x_s^s(t)) \quad \text{with initial state } \vec{x}^s = (x_1 < \cdots < x_s),$$

$$\vec{x}^{\bar{s}}(t) = (x_{s+1}^{\bar{s}}(t) < \cdots < x_\ell^{\bar{s}}(t)) \quad \text{with initial state } \vec{x}^{\bar{s}} = (x_{s+1} < \cdots < x_\ell),$$

and the reversed S6V location processes

$$\vec{y}(t) \quad \text{with initial state } \vec{y} = y_k,$$

$$\vec{y}^{\bar{s}}(t) = (y_{s+1}^{\bar{s}}(t) > \cdots > y_k^{\bar{s}}(t)) \quad \text{with initial state } \vec{y}^{\bar{s}} = (y_1 > \cdots > y_{k-1}).$$

We observe that by the update rule of $\vec{x}(t)$ defined in Definition 1.2: $H(\vec{x}(1), \vec{y}) = 0$ if $x_s(1) \neq y_k$; thus

$$E^F[H(\vec{x}(1), \vec{y})] = E^\vec{y}[H(\vec{x}(1), \vec{y})1_{\{x_s(1) = y_k\}}].$$

Computing the expectation on the r.h.s gives

$$E^F[H(\vec{x}(1), \vec{y})] = E^F[H(\vec{x}(1), \vec{y})1_{\{x_s(1) = y_k\}}] = \sum_{\vec{x} = (z_1 < \cdots < z_\ell) \atop z_k = y_k} P^\vec{x}(\vec{x}(1) = \vec{z})H(\vec{z}, \vec{y}). \hspace{1cm} (2.3)$$

Given $\vec{z} = (z_1 < \cdots < z_\ell), \vec{z}^s = (z_1 < \cdots < z_s)$ and $\vec{z}^{\bar{s}} = (z_{s+1} < \cdots < z_\ell)$ so that $z_s = y_k$, we readily have

$$P^\vec{x}(\vec{x}(1) = \vec{z}) = P^\vec{x}(\vec{x}(1) = \vec{z}^s)P^\vec{z}^{\bar{s}}(\vec{x}^{\bar{s}}(1) = \vec{z}^{\bar{s}}), \quad H(\vec{z}, \vec{y}) = q^{-s(s-1)}H(\vec{z}^s, \vec{y})H(\vec{z}^{\bar{s}}, \vec{y}^{\bar{s}}). \hspace{1cm} (2.4)$$

The first equation above simply follows by the update rule of $\vec{x}(t)$ and the second equation follows by the readily checked relation $\sum_{i=1}^s N_{x_i}(\varphi(z)) = s(k-1) + N_{y_k}(\varphi(z^{\bar{s}})) + \sum_{i=s+1}^k N_{y_i}(\varphi(z^{\bar{s}}))$. Substituting (2.4) back to (2.3) allows us to factor the transition probability and duality functional by

$$E^F[H(\vec{x}(1), \vec{y})] = q^{-s(s-1)}\left(\sum_{\vec{z} = (z_1 < \cdots < z_s) \atop z_s = y_k} P^\vec{x}(\vec{x}(1) = \vec{z}^s)H(\vec{z}^s, \vec{y}^s)\right)\left(\sum_{\vec{z}^{\bar{s}} = (z_{s+1} < \cdots < z_{\ell}) \atop z_{s+1} = y_k} P^\vec{z}^{\bar{s}}(\vec{x}^{\bar{s}}(1) = \vec{z}^{\bar{s}})H(\vec{z}^{\bar{s}}, \vec{y}^{\bar{s}})\right),$$

$$= q^{-s(s-1)}E^F[H(\vec{x}(1), \vec{y})]E^\vec{y}[H(\vec{y}(1), \vec{y})]. \hspace{1cm} (2.5)$$
Likewise, we have
\[
\mathbb{E}^\theta [\bar{H}(\bar{x}, \bar{y}(1))] = \mathbb{E}^\theta [\bar{H}(\bar{x}, \bar{y}(1))1_{\{y_{k-1}(1) \geq x_{k+1}\}}] = \sum_{\bar{w} = (w_1 \geq \cdots > w_k)} \mathbb{P}^\theta (\bar{y}(1) = \bar{w}) \bar{H}(\bar{x}, \bar{w}),
\]

\[
= q^{-s(k-1)} \left( \sum_{w' \in \mathbb{Z}} \mathbb{P}^\theta (y'(1) = w') \bar{H}(x', w') \right) \left( \sum_{\bar{w}'' = (w_1 \geq \cdots > w_k)} \mathbb{P}^\theta (y''(1) = \bar{w}'') \bar{H}(x'', \bar{w}'') \right),
\]

\[
= q^{-s(k-1)} \mathbb{E}^{\theta'} [\bar{H}(\bar{x}', y'(1))] \mathbb{E}^{\theta''} [\bar{H}(\bar{x}'', y''(1))]. \tag{2.6}
\]

Observing \(|\bar{x}'| + |y'|\) and \(|\bar{x}''| + |y''|\) are both less than \(\ell + k\), hence via \([\text{HYP}_{\theta,k}]\)
\[
\mathbb{E}^{\theta'} [\bar{H}(\bar{x}'(1), y'(1))], \quad \mathbb{E}^{\theta''} [\bar{H}(\bar{x}''(1), y''(1))] = \mathbb{E}^{\theta'} [\bar{H}(\bar{x}', y'(1))].
\]

Combining the preceding equations with (2.5) and (2.6) gives us (2.2). We conclude our proof for Case (1).

Case (2): \(y_k \in \{x_1, \ldots, x_\ell\}\)

We divide our discussion into three sub-cases.

Case (2a): \(y_k = x_1\).

In this case, let us consider the S6V location process and the reversed S6V location process
\[
\bar{x}'(t) = (x'_1(t) < \cdots < x'_{k-1}(t)) \text{ with initial state } \bar{x}' = (x_2 < \cdots < x_k),
\]
\[
\bar{y}'(t) = (y'_1(t) > \cdots > y'_{k-1}(t)) \text{ with initial state } \bar{y}' = (y_1 > \cdots > y_{k-1}).
\]

We first expand the expectation l.h.s of (2.2). Since \(x_1 = y_k\) and \(\bar{x}(t)\) starts from \(\bar{x}\), it follows from the update rule that \(\bar{H}(\bar{x}(1), \bar{y}) = 0\) unless \(x_1(1) = x_1\). Thus,
\[
\mathbb{E}^\theta [\bar{H}(\bar{x}(1), \bar{y})] = \mathbb{E}^\theta [\bar{H}(\bar{x}(1), \bar{y})1_{\{x_1(1) = x_1\}}] = \sum_{\bar{x} = (z_1 < \cdots < z_\ell)} \mathbb{P}^\theta (\bar{x}(1) = \bar{z}) \bar{H}(\bar{z}, \bar{y}). \tag{2.7}
\]

Like (2.4), we see that given \(\bar{z} = (z_1 < \cdots < z_\ell)\) and \(\bar{z}' = (z_2 < \cdots < z_\ell)\) such that \(z_1 = x_1\),
\[
\mathbb{P}^\theta (\bar{x}(1) = \bar{z}) = p(x_1, x_1) \mathbb{P}^\theta (\bar{x}'(1) = \bar{z}'), \quad \bar{H}(\bar{z}, \bar{y}) = q^{-k} \bar{H}(\bar{z}', \bar{y}),
\]

where the left equation above follows from the update rule. Substituting the equations above back to (2.7) gives
\[
\mathbb{E}^\theta [\bar{H}(\bar{x}(1), \bar{y})] = q^{-k} p(x_1, x_1) \sum_{\bar{z}' = (z_2 < \cdots < z_\ell)} \mathbb{E}^{\theta'} [\bar{H}(\bar{x}'(1), \bar{y}') = q^{-k} b_1 \mathbb{E}^{\theta'} [\bar{H}(\bar{x}'(1), \bar{y}')]. \tag{2.8}
\]

Likewise,
\[
\mathbb{E}^\theta [\bar{H}(\bar{x}, \bar{y}(1))] = \mathbb{E}^\theta [\bar{H}(\bar{x}, \bar{y}(1))1_{\{y_1(1) = y_k\}}] = \sum_{\bar{w} = (w_1 \geq \cdots > w_k)} \mathbb{P}^\theta (\bar{y}(1) = \bar{w}) \bar{H}(\bar{x}, \bar{w}),
\]

\[
= q^{-k} \mathbb{P}(y_k, y_k) \sum_{\bar{w} = (w_1 \geq \cdots > w_k)} \mathbb{P}^\theta (y''(1) = \bar{w}') \bar{H}(\bar{x}, \bar{w}'),
\]

\[
= q^{-k} b_1 \mathbb{E}^\theta [\bar{H}(\bar{x}', \bar{y}'(1))]. \tag{2.9}
\]

Since \(|\bar{x}'| + |\bar{y}'| < \ell + k\), the induction hypothesis \(\text{[HYP}_{\theta,k}\text{]}\) gives \(\mathbb{E}^\theta [\bar{H}(\bar{x}'(1), \bar{y}')] = \mathbb{E}^\theta [\bar{H}(\bar{x}', \bar{y}'(1))].\) Combining this with (2.8) and (2.9) yields (2.2).

Case (2b): \(y_k = x_2 > x_1\)
To simplify our notation, we denote \( \tilde{x}(t) = (x_1(t) < \cdots < x_{\ell}(t)) \) with initial state \( \tilde{x} = (x_2 < \cdots < x_{\ell}) \), \( \tilde{y}(t) = (y_1(t) > \cdots > y_{k-1}(t)) \) with initial state \( \tilde{y} = (y_1 > \cdots > y_{k-1}) \). The left equation above follows straightforwardly from the update rule. Plugging (2.16) into (2.15) gives
\[
\mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y})], \quad L_2 = \mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y})], \quad L_3 = \mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y})],
\]
\[
R_1 = \mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y})], \quad R_2 = \mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y})], \quad R_3 = \mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y})].
\]

Since \(|\tilde{x}| + |\tilde{y}|, |\tilde{x}'| + |\tilde{y}'|, |\tilde{x}''| + |\tilde{y}'|\) are all less than \(\ell + k\), we have by induction hypothesis (HYP\(\ell,k\))
\[
L_1 = R_1, \quad L_2 = R_2, \quad L_3 = R_3.
\]

(1.10)

Note that we have \(\tilde{x} = (x_2 < \cdots < x_{\ell}), \tilde{y} = (y_1 > \cdots > y_{k})\) with \(x_2 = y_k\). By the same argument as for (2.8), we have
\[
L_1 = b_1 q^{-k} L_3.
\]

(1.11)

Expanding the l.h.s. expectation of (2.2) as following (according to the update rule, \(x_1(1)\) can not exceed \(x_2\))
\[
\mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y})] = \mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y}) 1_{\{x_1(1) < x_2\}}] + \mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y}) 1_{\{x_1(1) = x_2\}}].
\]

(2.12)

For the first term on the r.h.s. of (2.12), we have
\[
\mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y}) 1_{\{x_1(1) < x_2\}}] = \sum_{\tilde{z} = (z_1 < \cdots < z_{\ell})} \mathbb{P}^\mathcal{F}(\tilde{x}(1) = \tilde{z}) \tilde{H}(\tilde{z}, \tilde{y}).
\]

(2.13)

Given \(\tilde{z} = (z_1 < \cdots < z_{\ell})\) and \(\tilde{z}' = (z_2 < \cdots < z_{\ell})\) with condition \(z_1 < x_2\) entails
\[
\mathbb{P}^\mathcal{F}(\tilde{x}(1) = \tilde{z}) = p(x_1, z_1) \mathbb{P}^\mathcal{F}(\tilde{x}'(1) = \tilde{z}'), \quad \tilde{H}(\tilde{z}, \tilde{y}) = q^{-k} \tilde{H}(\tilde{z}', \tilde{y}).
\]

Plugging these into (2.13) implies
\[
\mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y}) 1_{\{x_1(1) < x_2\}}] = q^{-k} \left( \sum_{z_1 < x_2} p(x_1, z_1) \right) \left( \sum_{\tilde{z}'' = (z_2 < \cdots < z_{\ell})} \mathbb{P}^\mathcal{F}(\tilde{x}'(1) = \tilde{z}'') \tilde{H}(\tilde{z}'', \tilde{y}) \right),
\]
\[
= (1 - (1 - b_1)b_2^{-x_2-x_1-1}) q^{-k} L_1,
\]

(2.14)

where in the second equality we compute \(\sum_{z_1 < x_2} p(x_1, z_1) = 1 - (1 - b_1)b_2^{x_2-x_1-1}\).

Let us compute the second term on the r.h.s. of (2.12),
\[
\mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y}) 1_{\{x_1(1) = x_2\}}] = \sum_{\tilde{z} = (z_1 < \cdots < z_{\ell})} \mathbb{P}^\mathcal{F}(\tilde{x}(1) = \tilde{z}) \tilde{H}(\tilde{z}, \tilde{y}).
\]

(2.15)

For \(\tilde{z} = (z_2 < \cdots < z_{\ell})\) and \(\tilde{z} = (z_1 < \cdots < z_{\ell})\) with \(z_1 = x_2\), we have
\[
\mathbb{P}^\mathcal{F}(\tilde{x}(1) = \tilde{z}) = b_2^{x_2-x_1-1} \mathbb{P}^\mathcal{F}(\tilde{x}'(1) = \tilde{z}'), \quad \tilde{H}(\tilde{z}, \tilde{y}) = q^{-k} \tilde{H}(\tilde{z}', \tilde{y}).
\]

(2.16)

The left equation above follows straightforwardly from the update rule. Plugging (2.16) into (2.15) gives
\[
\mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y}) 1_{\{x_1(1) = x_2\}}] = q^{-k} b_2^{x_2-x_1-1} \sum_{\tilde{z}' = (z_2 < \cdots < z_{\ell})} \mathbb{P}^\mathcal{F}(\tilde{x}'(1) = \tilde{z}') \tilde{H}(\tilde{z}', \tilde{y}),
\]
\[
= q^{-k} b_2^{x_2-x_1-1} \mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}'(1), \tilde{y}) 1_{\{x_1(1) \neq x_2\}}],
\]
\[
= q^{-k} b_2^{x_2-x_1-1} \left( L_2 - \mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y}) 1_{\{x_1(1) \neq x_2\}}]\right).
\]

(2.17)

Similar as the argument for (2.8), we have
\[
\mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y}) 1_{\{x_1(1) = x_2\}}] = q^{-(k-1)} p(x_2, x_2) \mathbb{E}^\mathcal{F}[\tilde{H}(\tilde{x}(1), \tilde{y})] = q^{-(k-1)} b_1 L_3.
\]
Substituting back to (2.17) gives
\[
\mathbb{E}^\tilde{q}[\tilde{H}(\tilde{x}(1), \tilde{y}) 1_{\{y_1(1)=x_2\}}] = q^{-k} b_2^{x_2-x_1-1} L_2 - q^{-(2k-1)} b_1 b_2^{x_2-x_1-1} L_3.
\] (2.18)

Plugging (2.14) and (2.18) into the r.h.s. of (2.12) yields
\[
\mathbb{E}^\tilde{q}[\tilde{H}(\tilde{x}(1), \tilde{y})] = (1 - (1 - b_1) b_2^{x_2-x_1-1}) q^{-k} L_1 + (q^{-k} b_2^{x_2-x_1-1} L_2 - q^{-(2k-1)} b_1 b_2^{x_2-x_1-1} L_3),
\]
\[
= q^{-k} L_1 + b_2^{x_2-x_1-1}(q^{-k} L_2 + q^{-(2k-1)}(b_1 b_2 - b_1 - b_2) L_3).
\] (2.19)

In the last line we used the relation \(L_1 = b_1 q^{-k} L_3\) provided by (2.11) and \(b_1 = q b_2\).

We turn our attention into computing \(\mathbb{E}^\tilde{q}[\tilde{H}(\tilde{x}, \tilde{y}(1))].\) Since \(y_k = x_2\), according to the update rule of \(\tilde{y}(t) = (y_1(t) > \cdots > y_k(t))\) with initial state \(\tilde{y}\), the only possible case for \(\tilde{H}(\tilde{x}, \tilde{y}(1)) \neq 0\) is either \(y_k(1) = x_1\) or \(y_k(1) = x_2\). Therefore, we have
\[
\mathbb{E}^\tilde{q}[\tilde{H}(\tilde{x}, \tilde{y}(1))] = \mathbb{E}^\tilde{q}[\tilde{H}(\tilde{x}, \tilde{y}(1)) 1_{\{y_k(1)=x_2\}}] + \mathbb{E}^\tilde{q}[\tilde{H}(\tilde{x}, \tilde{y}(1)) 1_{\{y_k(1)=x_1\}}].
\] (2.20)

For the first term on the r.h.s of (2.20), we readily have
\[
\mathbb{E}^\tilde{q}[\tilde{H}(\tilde{x}, \tilde{y}(1)) 1_{\{y_k(1)=x_2\}}] = q^{-k} \mathbb{E}^\tilde{q}[\tilde{H}(\tilde{x}, \tilde{y}(1)) 1_{\{y_k(1)=x_2\}}] = q^{-k} R_1.
\] (2.21)

The first equality above is due to the fact that given \(y_k(1) = x_2\) implies \(\tilde{H}(\tilde{x}, \tilde{y}(1)) = q^{-k} \tilde{H}(\tilde{x}, \tilde{y}(1))\).

For the second term on the r.h.s. of (2.20), via expanding the expectation, we have
\[
\mathbb{E}^\tilde{q}[\tilde{H}(\tilde{x}, \tilde{y}(1)) 1_{\{y_k(1)=x_1\}}] = \sum_{\tilde{w}=(w_1, \cdots, w_k)} \mathbb{P}^\tilde{q}(\tilde{y}(1) = \tilde{w}) \tilde{H}(\tilde{x}, \tilde{w}),
\] (2.22)

where the last line follows from the update rule, which implies the probability of the event \(\{y_k-1(1) < y_k\}\) is zero. Since \(x_2 = y_k\), it is easy to check that given \(\tilde{w} = (w_1 > \cdots > w_k)\) and \(\tilde{w}' = (w_1 > \cdots > w_{k-1})\) with condition \(w_k = x_1\) and \(w_{k-1} = x_2\) implies
\[
\mathbb{P}^\tilde{q}(\tilde{y}(1) = \tilde{w}) = \mathbb{P}(x_2, x_1) \mathbb{P}^\tilde{q}(\tilde{y}(1) = \tilde{w}), \quad \tilde{H}(\tilde{x}, \tilde{w}) = q^{-(2k-1)} \tilde{H}(x_2, \tilde{w}) - q^{-(2k-1)} R_3.
\] (2.23)

Similarly, given \(\tilde{w} = (w_1 > \cdots > w_k)\) and \(\tilde{w}' = (w_1 > \cdots > w_{k-1})\) with \(w_k = x_1\) and \(w_{k-1} = x_2\), we have
\[
\mathbb{P}^\tilde{q}(\tilde{y}(1) = \tilde{w}) = b_2^{x_2-x_1-1} \mathbb{P}^\tilde{q}(\tilde{y}(1) = \tilde{w}), \quad \tilde{H}(\tilde{x}, \tilde{w}) = q^{-k} \tilde{H}(\tilde{x}, \tilde{w}).
\]

Thus, the second term on the r.h.s. of (2.22) equals
\[
\sum_{\tilde{w}'=(w_1, \cdots, w_{k-1})} \mathbb{P}^\tilde{q}(\tilde{y}(1) = \tilde{w}') \tilde{H}(\tilde{x}, \tilde{w}') = q^{-k} \sum_{\tilde{w}'=(w_1, \cdots, w_{k-1})} \mathbb{P}^\tilde{q}(\tilde{y}(1) = \tilde{w}') \tilde{H}(\tilde{x}, \tilde{w}'),
\]
\[
= q^{-k} b_2^{x_2-x_1-1} \mathbb{E}^\tilde{q}[\tilde{H}(\tilde{x}, \tilde{y}(1)) 1_{\{y_{k-1}(1)=x_2\}}],
\]
\[
= q^{-k} b_2^{x_2-x_1-1} \left( R_2 - \mathbb{E}^\tilde{q}[\tilde{H}(\tilde{x}, \tilde{y}(1)) 1_{\{y_{k-1}(1)>x_2\}}]\right).
\] (2.24)

Under event \(\{y_{k-1}(1) > x_2\}\), we have \(\tilde{H}(\tilde{x}, \tilde{y}(1)) = q^{-(k-1)} \tilde{H}(x_2, \tilde{y}(1))\) and hence
\[
\mathbb{E}^\tilde{q}[\tilde{H}(\tilde{x}, \tilde{y}(1)) 1_{\{y_{k-1}(1)>x_2\}}] = q^{-(k-1)} \mathbb{E}^\tilde{q}[\tilde{H}(x_2, \tilde{y}(1)) 1_{\{y_{k-1}(1)>x_2\}}] = q^{-(k-1)} R_3.
\]
Combining (2.19) and (2.27) with (2.10) gives the desired (2.2).

The computation for this case is similar as for Case (2b). Let us consider the S6V location processes

\[ \vec{z} = (x'_1(t) < \cdots < x'_{\ell-1}(t)) \]

with initial state \( \vec{z} = (x_2 < \cdots < x_\ell) \), and denote

\[ L_1 = \mathbb{E}^{\vec{y}}[\tilde{H}(\vec{x}(1), \vec{y})], \quad L_2 = \mathbb{E}^{\vec{y}}[\tilde{H}(\vec{x}(1), \vec{y})], \]

\[ R_1 = \mathbb{E}^{\vec{y}}[\tilde{H}(\vec{x}, \vec{y})], \quad R_2 = \mathbb{E}^{\vec{y}}[\tilde{H}(\vec{x}, \vec{y})]. \]

By (HYP\textsubscript{rk}), we have

\[ L_1 = R_1, \quad L_2 = R_2. \tag{2.28} \]

As in Case (2b), we first write

\[ \mathbb{E}^{\vec{y}}[\tilde{H}(\vec{x}(1), \vec{y})] = \mathbb{E}^{\vec{y}}[\tilde{H}(\vec{x}(1), \vec{y})]|_{\{x_1(1) < x_2\}} + \mathbb{E}^{\vec{y}}[\tilde{H}(\vec{x}(1), \vec{y})]|_{\{x_1(1) = x_2\}}. \tag{2.29} \]

Similar as (2.13), the first term on the r.h.s of (2.29) can be expressed as

\[ \mathbb{E}^{\vec{y}}[\tilde{H}(\vec{x}(1), \vec{y})]|_{\{x_1(1) < x_2\}} = q^{-k}(1 - b_1)b_2^{z_2-x_1-1}L_1, \tag{2.30} \]

while the second term on the r.h.s of (2.29) equals

\[ \mathbb{E}^{\vec{y}}[\tilde{H}(\vec{x}(1), \vec{y})]|_{\{x_1(1) = x_2\}} = \sum_{\vec{z} = (z_1 < \cdots < z_\ell) \atop z_1 = x_2} \mathbb{P}^{\vec{x}}(\vec{z}(1) = \vec{z}) \tilde{H}(\vec{z}, \vec{y}). \tag{2.31} \]

Given \( \vec{z} = (z_1 < \cdots < z_\ell) \) and \( \vec{z}' = (z_2 < \cdots < z_\ell) \) with \( z_1 = x_2 \), we have

\[ \mathbb{P}^{\vec{x}}(\vec{z}(1) = \vec{z}) = b_2^{z_2-x_1-1} \mathbb{P}^{\vec{x}}(\vec{z}')(1) = \vec{z}', \quad \tilde{H}(\vec{z}, \vec{y}) = q^{-k} \tilde{H}(\vec{z}', \vec{y}). \]

Substituting back to (2.31) yields

\[ \mathbb{E}^{\vec{y}}[\tilde{H}(\vec{x}(1), \vec{y})]|_{\{x_1(1) = x_2\}} = q^{-k}b_2^{z_2-x_1-1} \sum_{\vec{z}' = (z_2 < \cdots < z_\ell) \atop z_2 > x_2} \mathbb{P}^{\vec{x}}(\vec{z}'(1) = \vec{z}') \tilde{H}(\vec{z}', \vec{y}), \]

\[ = q^{-k}b_2^{z_2-x_1-1} \mathbb{E}^{\vec{x}}[\tilde{H}(\vec{x}'(1), \vec{y})]|_{\{x'_1(1) > x_2\}}, \]

\[ = q^{-k}b_2^{z_2-x_1-1} \left( L_1 - \mathbb{E}^{\vec{y}}[\tilde{H}(\vec{x}(1), \vec{y})]|_{\{x_1(1) = x_2\}} \right). \tag{2.32} \]

As a consequence by (2.28),

\[ \mathbb{E}^{\vec{y}}[\tilde{H}(\vec{x}(1), \vec{y})]|_{\{x_1(1) = x_2\}} = q^{-k}b_2^{z_2-x_1-1}(L_1 - q^{-k}b_1L_2). \tag{2.33} \]
Plugging (2.30) and (2.33) into the r.h.s. of (2.29) yields

\[
\mathbb{E}^\xi [\tilde{H}(\tilde{x}(1), \tilde{y})] = q^{-k_1}(1 - (1 - b_1) b_2^2 - x_1 - 1) L_1 + q^{-k_2} b_2^2 - x_2 - 1 (L_1 - q^{-k_1} L_2),
\]

\[
= q^{-k_1} L_1 + q^{-(k_1)} b_2^2 - x_1 (L_1 - q^{-k_1} L_2),
\]

(2.34)

where in the last line we used the relation \( b_1 = q b_2 \).

We turn our attention to compute \( \mathbb{E}^\xi [\tilde{H}(\tilde{x}, \tilde{y}(1))] \). As given \( y_k(1) < x_1 \) yields \( \tilde{H}(\tilde{x}, \tilde{y}(1)) = 0 \), we have

\[
\mathbb{E}^\xi [\tilde{H}(\tilde{x}, \tilde{y}(1))] = \mathbb{E}^\xi [\tilde{H}(\tilde{x}, \tilde{y}(1)) \mathbb{I}_{\{y_k(1) > x_1\}}] + \mathbb{E}^\xi [\tilde{H}(\tilde{x}, \tilde{y}(1)) \mathbb{I}_{\{y_k(1) = x_1\}}],
\]

\[
= q^{-k_1} R_1 + \mathbb{E}^\xi [\tilde{H}(\tilde{x}, \tilde{y}(1)) \mathbb{I}_{\{y_k(1) = x_1\}}].
\]

(2.35)

The last equality is due to that given \( y_k(1) > x_1 \) implies \( \tilde{H}(\tilde{x}, \tilde{y}(1)) = q^{-k_1} \tilde{H}(\tilde{x}', \tilde{y}(1)) \) and thus

\[
\mathbb{E}^\xi [\tilde{H}(\tilde{x}, \tilde{y}(1)) \mathbb{I}_{\{y_k(1) > x_1\}}] = q^{-k_1} \mathbb{E}^\xi [\tilde{H}(\tilde{x}', \tilde{y}(1)) \mathbb{I}_{\{y_k(1) > x_1\}}] = q^{-k_1} \mathbb{E}^\xi [\tilde{H}(\tilde{x}', \tilde{y}(1))] = q^{-k_1} R_1.
\]

(2.36)

For the second term on the r.h.s. of (2.35), we have

\[
\mathbb{E}^\xi [\tilde{H}(\tilde{x}, \tilde{y}(1)) \mathbb{I}_{\{y_k(1) = x_1\}}] = \sum_{\tilde{w}^2 = (w_1, \ldots, w_k)} \mathbb{P}^\xi (\tilde{y}(1) = \tilde{w}) \tilde{H}(\tilde{x}, \tilde{w}),
\]

\[
= \sum_{\tilde{w}^2 = (w_1, \ldots, w_k)} \mathbb{P}^\xi (\tilde{y}(1) = \tilde{w}) \tilde{H}(\tilde{x}, \tilde{w}).
\]

(2.37)

Since \( y_k > x_2 \), given \( \tilde{w} = (w_1 > \cdots > w_k) \) satisfying \( w_k \geq y_k \), we have

\[
\mathbb{P}^\xi (\tilde{y}(1) = \tilde{w}) = b_2^2 - x_1 \mathbb{E}^\xi (\tilde{y}(1) = \tilde{w}), \quad \tilde{H}(\tilde{x}, \tilde{w}) = q^{-k_1} \tilde{H}(\tilde{x}', \tilde{w}).
\]

Substituting back to (2.30) yields

\[
\mathbb{E}^\xi [\tilde{H}(\tilde{x}, \tilde{y}(1)) \mathbb{I}_{\{y_k(1) = x_1\}}] = q^{-(k_1)} b_2^2 - x_1 \sum_{\tilde{w}^2 = (w_1, \ldots, w_k)} \mathbb{E}^\xi (\tilde{y}(1) = \tilde{w}) \tilde{H}(\tilde{x}, \tilde{w}),
\]

\[
= q^{-(k_1)} b_2^2 - x_1 \mathbb{E}^\xi [\tilde{H}(\tilde{x}', \tilde{y}(1)) \mathbb{I}_{\{y_k(1) = x_1\}}],
\]

\[
= q^{-(k_1)} b_2^2 - x_1 (R_1 - \mathbb{E}^\xi [\tilde{H}(\tilde{x}', \tilde{y}(1)) \mathbb{I}_{\{y_k(1) > x_1\}}]).
\]

(2.38)

Therefore, we have by (2.37)

\[
\mathbb{E}^\xi [\tilde{H}(\tilde{x}, \tilde{y}(1)) \mathbb{I}_{\{y_k(1) = x_1\}}] = q^{-(k_1)} b_2^2 - x_1 R_1 - q^{-(2k_1)} b_2^2 - x_1 R_2.
\]

(2.39)

We conclude \( \mathbb{E}^\xi [\tilde{H}(\tilde{x}(1), \tilde{y})] = \mathbb{E}^\xi [\tilde{H}(\tilde{x}(1), \tilde{y}(1))] \) by combining (2.34), (2.39) and (2.28).

As all the possible cases for \( x \in \mathbb{X}^k \) and \( \tilde{y} \in \mathbb{Y}^k \) were discussed, we conclude our proof. □

**Remark 2.2.** It appears that the proof of Theorem 1.3 also adapts to the space inhomogeneous stochastic six vertex model, where we allow the parameters \( b_1, b_2 \) in Figure 1 to vary at different location \( x \in \mathbb{Z} \) and are expressed by \( b_{1,x} \) and \( b_{2,x} \). Under the condition that there exists \( q > 0 \) such that \( b_{1,x} = q b_{2,x} \) for all \( x \in \mathbb{Z} \), Theorem 1.3 holds for this space inhomogeneous stochastic six vertex model as well. To avoid extra notation, we have opted not to state and prove this more general result here.
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