INTEGRATION OF DIFFERENTIAL GRADED MANIFOLDS

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Abstract. We consider the problem of integration of $L_\infty$-algebroids (differential graded manifolds) to $L_\infty$-groupoids. We first construct a “big” Kan simplicial manifold (Fréchet or Banach) whose points are solutions of a (generalized) Maurer-Cartan equation. The main analytic trick in our work is an integral transformation sending the solutions of the Maurer-Cartan equation to closed differential forms.

Following ideas of Ezra Getzler we then impose a gauge condition which cuts out a finite-dimensional simplicial submanifold. This “smaller” simplicial manifold is (the nerve of) a local Lie $\ell$-groupoid. The gauge condition can be imposed only locally in the base of the $L_\infty$-algebroid; the resulting local $\ell$-groupoids glue up to a coherent homotopy, i.e. we get a homotopy coherent diagram from the nerve of a good cover of the base to the (simplicial) category of local $\ell$-groupoids.

Finally we show that a $k$-symplectic differential graded manifold integrates to a local $k$-symplectic Lie $\ell$-groupoid; globally these assemble to form an $A_\infty$-functor. As a particular case for $k = 2$ we obtain integration of Courant algebroids.

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1. Introduction

Let us recall that a Lie bracket on a finite-dimensional vector space $\mathfrak{g}$ is equivalent to a differential $Q$ on the graded-commutative algebra $\bigwedge \mathfrak{g}^* = S(\mathfrak{g}[1])^*$. One possible approach to the construction of the integrating simply-connected group $G$ is due to Dennis Sullivan [14]. Consider the simplicial set of morphisms of differential graded commutative algebras

$$K_\bullet^{\text{big}} = \text{Hom}_{\text{dgc}}\left((\bigwedge \mathfrak{g}^*, Q), (\Omega(\Delta^\bullet), d)\right)$$

where $\Delta^\bullet$ is the Euclidean simplex and $d$ is the de Rham differential. It can be shown that $K_\bullet^{\text{big}}$ is in fact a simplicial manifold and that its simplicial fundamental group is $\pi_1^\text{simpl}(K_\bullet^{\text{big}}) \cong G$.

This can be geometrically explained as follows. Let $\xi^i$ be a basis of $\mathfrak{g}^*$ and $c_{ij}^k$ the structure constants of $\mathfrak{g}$ in this basis, so that $Q\xi^i = c_{ij}^k \xi^j \xi^k / 2$. An $n$-simplex $\mu \in K_n^{\text{big}}$ is determined by

$$A^i := \mu(\xi^i), \quad i = 1, \ldots, \dim \mathfrak{g}.$$
which is a collection of 1–forms on \( \Delta^n \). Since \( \mu \) respects the differentials we get
\[
dA' = \frac{1}{2} v^i_{jk} A' A^k.
\]
This is the Maurer-Cartan equation so \( A \) is a flat \( g \)-connection on \( \Delta^n \). In other words, we can identify the elements of \( K_{\text{big}}^n \) with flat \( g \)-connections on \( \Delta^n \).

The integration of \( g \) to \( G \) can be described as follows. A connection \( A \) on an interval (automatically flat) will give rise to an element \( g_A \) of the integrating group (the holonomy of the connection along the interval). Two such elements \( g_A, g_{A'} \) are equal iff there exists a flat connection \( A_\Delta \) on a bigon which restricts on the boundary arcs to \( A \) and \( A' \) as on the left side of Figure 1. The group multiplication is given as \( g_A g_{A'} = g_{A''} \) iff there exists a flat connection \( A_\triangle \) on a triangle which restricts on the boundary in the manner of right side of Figure 1.

Let us note that the manifolds \( K_{\text{big}}^n \) are infinite dimensional. For any \( g \in G \) there are infinitely many \( A \)'s on the interval such that \( g = g_A \). Likewise, if \( g_A, g_{A'} = g_{A''} \), there are infinitely many \( A_\Delta \)'s proving this relation. However, \( K_{\text{big}}^n \) can be reduced by gauge-fixing to a “smaller” (finite dimensional) simplicial submanifold \( K^* \), which is a (local) simplicial deformation retract of \( K_{\text{big}}^n \) and which is (locally) isomorphic to the nerve of \( G \) (which means a unique \( A \) such that \( g = g_A \), and also a unique \( A_\Delta \) restricting to \( A_1, A_2, A_3 \) on the boundary). \( K^* \) is the nerve of the local Lie group integrating \( g \).

This paper generalizes this procedure to cases where \( (\bigwedge g^*, \mathcal{Q}) \) is replaced by a more general differential graded-commutative algebra. More precisely, we consider generalizations involving the introduction of generators in degrees \( i \geq 0 \) and allowing smooth functions of the degree-0 generators rather than just polynomials.

Let us thus consider non-negatively graded commutative algebras of the form
\[
\mathcal{A}_V = \Gamma(S(V^*)),
\]
where \( V \to M \) is a negatively graded vector bundle. Suppose that \( Q \) is a differential on the algebra \( \mathcal{A}_V \). We can say that \( \mathcal{A}_V \) is the algebra of functions on the graded manifold \( V \).

If the manifold \( M \) is a point then the differential \( Q \) is equivalent to an \( L_\infty \)-algebra structure on the non-positively graded vector space \( V'[-1] \). In the case of a general \( M \) and of \( V \) concentrated in degree \( -1 \), a differential \( Q \) is equivalent to a Lie algebroid structure on \( V[-1] \). In the general case, the differential \( Q \) is loosely called a \( L_\infty \)-algebroid structure on the vector bundle \( V'[-1] \), or more precisely a Lie \( \ell \)-algebroid, where \( -\ell \) is the lowest degree in the negatively graded bundle \( V \) (so that \( \ell \geq 1 \)). In this paper we will integrate these Lie \( \ell \)-algebroids to Lie \( \ell \)-groupoids.

Following Sullivan as above we shall study morphisms of differential graded algebras
\[
(\mathcal{A}_V, Q) \to (\Omega(\Delta^n), d),
\]
or equivalently, morphisms of differential graded manifolds
\[
T[1]|\Delta^n \to V.
\]
We can view \( L_\infty \)-algebroids as Lie algebroids with higher homotopies and the integration procedure as recovering the fundamental \( \infty \)-groupoid. This integration procedure was suggested in [13]. The motivation comes primarily from the problem of integration of Courant algebroids.

\[ \text{Figure 1. Left - two connections give the same group element. Right - group multiplication illustrated.} \]
Poisson manifolds are integrated to (local) symplectic groupoids and Courant algebroids should be integrated to symplectic 2-groupoids.

While these ideas are well known, in this work we finally overcome the long-standing analytic difficulties. Our first result (Theorem 6.2) says that the morphisms \( \kappa \) form naturally a (infinite-dimensional Fréchet) Kan simplicial manifold \( K^\text{big}_s \). This result is obtained via an integral transformation (similar to a transformation appearing in the work of Masatake Kuranishi [7]) taking morphisms \( \kappa \) to morphisms

\[
(A_V, 0) \to (\Omega(\Delta^n), d)
\]

which is interesting on its own.

In more detail, in the simplest case when \( M \) is an open subset of \( \mathbb{R}^\ell \) and \( V \to M \) is a trivial graded vector bundle, we choose generators \( \xi^i \) of the algebra \( A_V \) (\( \xi^i \)'s of degree 0 are the coordinates of \( M \subset \mathbb{R}^\ell \) and \( \xi^i \)'s of positive degree come from a basis of the fibre of \( V \)) then a morphism of dgcas \( \kappa \) is equivalent to a collection of differential forms \( A^i \in \Omega^{\deg \xi^i}(\Delta^n) \) satisfying a generalized Maurer-Cartan equation

\[
dA^i = F^i_Q(A)
\]

where \( F^i_Q := Q\xi^i \). The integral transformation is \( A^i \mapsto \kappa(A)^i := A^i - hF^i_Q(A) \), where \( h \) is the de Rham homotopy operator, and is transforms the equation \( \Box \) to

\[
d\kappa(A)^i = 0.
\]

We prove that \( \kappa \) is an open embedding (of Banach or Fréchet spaces), and thus show the regularity properties of the space of the solutions of \( \Box \).

As noted in the Lie algebra case, the integration can be reduced by gauge-fixing. Following ideas of Ezra Getzler [5] we define (locally in \( M \)) a finite-dimensional locally Kan simplicial manifold \( K^s_\bullet \subset K^\text{big}_\bullet \) by imposing a certain gauge condition \( s_\bullet \) on differential forms (see Theorem 7.1)

\[
K^s_\bullet = \{ (A^i \in \Omega^{\deg \xi^i}(\Delta^n)); dA^i = F^i_Q(A) \text{ and } s_\bullet A^i = 0 \}.
\]

The simplicial manifold \( K^s_\bullet \) can be seen as (the nerve of \( \ell \)) a local Lie \( \ell \)-groupoid (Definition 7.3) integrating the Lie \( \ell \)-algebroid structure on \( V \) (\( -\ell \) is the lowest degree in \( V \)). While \( K^s_\bullet \) depends on the choice of a gauge condition \( s_\bullet \), (in particular, on a choice of local coordinates and of a local trivialization of \( V \)), we construct a (local) simplicial deformation retraction of \( K^\text{big}_\bullet \) to \( K^s_\bullet \), i.e. show that \( K^\text{big}_\bullet \) and \( K^s_\bullet \) are equivalent as Lie \( \infty \)-groupoids. The deformation retraction also implies that \( K^s_\bullet \) is unique up to (non-unique) isomorphisms and that the local \( K^s_\bullet \)'s form a homotopy coherent diagram.

Let us relate our approach with the papers of Henriques [6] and Getzler [5], who solve closely related problems. André Henriques deals in [6] with the case when the base manifold \( M \) is a point. He defined Kan simplicial manifolds and Lie \( \ell \)-groupoids, which are central definitions in our work. His constructions are based on Postnikov towers: in his case it is enough to integrate a Lie algebra to a Lie group and then to deal with Lie algebra cocycles. This approach cannot be used for non-trivial \( M \), so there is little overlap between his and our methods.

The present paper is closer to the work of Ezra Getzler [5] who deals with nilpotent \( L_\infty \)-algebras (or, from the point of view of Rational homotopy theory [14], with finitely-generated Sullivan algebras). In particular, the idea of gauge fixing is simply taken from [5] and translated from formal power series to the language of Banach manifolds.

Our paper is also closely related to the work of Crainic and Fernandes [2] on integration of Lie algebroids (corresponding to \( \ell = 1 \)). Unlike in op. cit. we do not consider the truncation of \( K^\text{big}_\bullet \) at dimension \( \ell \) (keeping all simplices of dimension \( < \ell \) intact, replacing those of dimension \( \ell \) with their homotopy classes rel boundary, and adding higher simplices formally), as it leads, for \( \ell \geq 2 \), to infinite-dimensional spaces. Nonetheless our analytic results should be sufficient also for this kind of approach.

The plan of our paper is as follows. In Section 2 we recall some basic definitions concerning dg manifolds. In Section 3 we prove a technical result about homotopies of maps between dg manifolds, which is the basis for our analytic theorems. In Section 4 we define the Kuranishi map \( \kappa \) and show that it transforms the Maurer-Cartan equations \( \Box \) to linear equations. In this way we then prove that the spaces of solutions are manifolds. In Section 5 we prove that the
spaces of the solutions of the generalized Maurer-Cartan \( \mathfrak{m} \) on simplices form a Kan simplicial
manifold (i.e. a Lie \( \infty \)-groupoid) and in Section 6 we globalize the results of Sections 4 and 5. (Sections 5 and 6 are not needed for the rest of the paper, but they complete the picture.)
In Section 7 we show that gauge fixing produces a finite-dimensional local Lie \( \ell \)-groupoid. In
Section 8 we establish a deformation retraction from the big integration to the gauge integration,
which is then used in Section 9 to show that the gauge integration is functorial up to coherent
homotopies. Section 10 concerns integration of (pre)symplectic forms on a dg manifold. Closing
the paper with Section 11 we describe the \( A_\infty \)-functoriality of the local integration of symplectic
dg manifolds; as an example we will describe integration of Courant algebroids.

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2. DG MORPHISMS AND DG MANIFOLDS

Let \( M \) be a manifold and \( V \to M \) a negatively-graded vector bundle. Throughout the paper
all vector bundles are finite dimensional. Let \( A_V \) be the graded commutative algebra
\[
A_V = \Gamma(S(V^*)).
\]
In particular \( \Omega(M) = A_{T[1]M} \). We can describe morphisms of graded-commutative algebras
\( A_V \to \Omega(N) \)
(where \( N \) is another manifold) in the following way.
If \( W \) is a non-positively graded vector space and \( N \) a manifold, let
\[
\Omega(N,W)^m = \bigoplus_{k \geq 0} \Omega^{m+k}(N) \otimes W^{-k}.
\]
Abusing the notation, if \( V \to M \) is a negatively graded vector bundle, let
\[
\Omega(N,V)|_U \subset \Omega(N,W)|_U
\]
is the set of those forms \( A \in \Omega(N,W)|_U \) whose function part (a \( W^0 \)-valued function on \( N \)) is a
map \( N \to U \subset W^0 \). Notice that
\[
\Omega(N,W|_U)^0 \subset \Omega(N,W)^0
\]
is the set of forms \( A \in \Omega(N,W)^0 \) whose function part (a \( W^0 \)-valued function on \( N \)) is a
map \( N \to U \subset W^0 \).

A morphism of graded algebras
\[
\mu : A_V \to \Omega(N)
\]
is equivalent to an element \( (f,\alpha) \in \Omega(N,V)|_U \) via
\[
\mu|_{C^\infty(M)} = f^* \quad \mu(s) = \langle \alpha, f^* s \rangle, \quad \forall s \in \Gamma(V^*).
\]
If \( W \) is a non-positively graded vector space and \( U \subset W^0 \) is an open subset, let \( W|_U \) be the
trivial vector bundle \( W^{-<0} \times U \to U \), i.e.
\[
\Omega(N,W|_U)^0 \subset \Omega(N,W)^0
\]
is the set of those forms \( A \in \Omega(N,W)^0 \) whose function part (a \( W^0 \)-valued function on \( N \)) is a
map \( N \to U \subset W^0 \). Notice that
\[
\Omega(N,W|_U)^0 \subset \Omega(N,W)^0
\]
is the set of those forms \( A \in \Omega(N,W)^0 \) whose function part (a \( W^0 \)-valued function on \( N \)) is a
map \( N \to U \subset W^0 \).

In this special case of \( V = W|_U \) a morphism \( \mu : A_{W|_U} \to \Omega(N) \) corresponds to \( A \in \Omega(N,W|_U)^0 \)
via
\[
\mu(\xi) = \langle A, \xi \rangle, \quad \forall \xi \in W^*.
\]
Let now \( Q \) be a differential on the graded algebra \( A_{W|_U} \). If \( \xi^i \) is a (homogeneous) basis of \( W^* \), let
\[
F^i_Q := Q \xi^i \in A_{W|_U}.
\]
The identity \( Q^2 = 0 \) is equivalent to
\[
F^i_Q \partial F^k_Q \partial \xi^i = 0.
\]
A morphism of graded algebras \( \mu : A_{W|_U} \to \Omega(N) \) is a differential graded (dg) morphism iff
\[
dA^i = F^i_Q(A)
\]
where $A^i := \mu(\xi^i) = \langle A, \xi^i \rangle$ and $F_Q^{\alpha}(A)$ is obtained from $F_Q^{\alpha}$ by substituting $A^k$'s for $\xi^k$'s. This equation generalizes the Maurer-Cartan equations of the Lie algebra case (1) where $F_Q^{\alpha}$ is quadratic. We can rewrite (7) as

\[ dA = F_Q(A) \]  

where $F_Q \in \mathcal{A}_{W|I} \otimes W$ is given by

\[ \langle F_Q, \xi \rangle = Q\xi \quad \forall \xi \in W^*. \]

The set of solutions $A$ of (8) will be denoted by

\[ \Omega(N, W|I)^{MC} \subset \Omega(N, W|I)^0. \]

More generally, if $Q$ is a differential on the algebra $\mathcal{A}_V$, the set of dg morphisms

\[ \mathcal{A}_V \rightarrow \Omega(N) \]

will be denoted by

\[ \Omega(N, V)^{MC} \subset \Omega(N, V)^0. \]

If $V \rightarrow M$ is a negatively-graded vector bundle, it is convenient to see the algebra $\mathcal{A}_V$ as the algebra of functions on the graded manifold (corresponding to) $V$.

**Definition 2.1** (e.g. Ševera [13]). An N-manifold (short hand for non-negatively graded manifold) is a supermanifold $Z$ with an action of the semigroup $(\mathbb{R}, \times)$ such that $-1 \in \mathbb{R}$ acts as the parity operator (i.e. just changes the sign of the odd coordinates).

Let $C^\infty(Z)^k$ be the vector space of smooth functions of degree $k$, where the degree is the weight with respect to the action of $(\mathbb{R}, \times)$, and let $C^\infty(Z) = \bigoplus_{k \geq 0} C^\infty(Z)^k$. It is a graded-commutative algebra.

Any negatively-graded vector bundle $V \rightarrow M$ gives rise to a N-manifold via

\[ C^\infty(Z) = \Gamma(S(V^*)), \]

such that $0 \cdot Z = M$. Any N-manifold (in the $C^\infty$-category) is of this type. By abuse of notation we shall denote the N-manifold corresponding to $V$ also by $V$. A morphism of graded algebras $\mathcal{A}_V \rightarrow \mathcal{A}_{V'}$ is thus equivalent to a morphism of N-manifolds $V' \rightarrow V$. We shall indicate morphisms of graded manifolds with wiggly arrows ($\rightsquigarrow$). This is to prevent confusion since the category of N-manifolds contains more morphisms then the category of negatively-graded vector bundles. In particular, the following are equivalent: a morphism of graded algebras

\[ \mathcal{A}_V \rightarrow \Omega(N), \]

an element of $\Omega(N, V)^0$, and a morphism of N-manifolds

\[ T[1]N \rightsquigarrow V. \]

A differential $Q$ on the graded algebra $\mathcal{A}_V$ then corresponds to the following notion.

**Definition 2.2.** An NQ-manifold (a differential non-negatively graded manifold) is an N-manifold equipped with a vector field $Q$ of degree 1 satisfying $Q^2 = 0$.

In particular, a morphism of dg-algebras $\mathcal{A}_V \rightarrow \Omega(N)$ is equivalent to a morphism of NQ-manifolds $T[1]N \rightsquigarrow V$.

3. A HOMOTOPY LEMMA

Let us suppose, as above, that $W$ is a non-positively graded finite-dimensional vector space, $U \subset W^0$ is an open subset and $Q$ a differential on the algebra $\mathcal{A}_{W|I}$, defined by (5). A morphism of graded algebras $\mathcal{A}_{W|I} \rightarrow \Omega(N)$ (or a map of N-manifolds $T[1]N \rightsquigarrow W|I$) is thus equivalent to a choice of a differential form $A \in \Omega(N, W|I)^0$, and it is a dg morphism iff $dA = F_Q(A)$, i.e. iff $A \in \Omega(N, W|I)^{MC}$ (see Section 2).

In this section we shall study “homotopies”, i.e. solutions of the MC equation $dA = F_Q(A)$ on $N \times I$, where $I$ is the unit interval.

If $\alpha \in \Omega(N \times I)$, let $\alpha_{t=0} \in \Omega(N)$ (where $t$ is the coordinate on $I = [0, 1]$) be the restriction of $\alpha$ to $N = N \times \{0\} \subset N \times I$. Any differential form $\alpha \in \Omega(N \times I)$ can be split uniquely into a horizontal and vertical part as

\[ \alpha = \alpha_h + d\alpha_v \]
\[
i_\partial \alpha_h = i_\partial \alpha_v = 0.
\]
The components \(\alpha_h\) and \(\alpha_v\) are given by
\[
\alpha_v = i_\partial \alpha, \quad \alpha_h = \alpha - dt \alpha_v.
\]
We can view \(\alpha_h\) and \(\alpha_v\) as forms on \(N\) parametrized by \(I\). For any \(\alpha\) we set
\[
d_\partial \alpha = (d\alpha)_h.
\]

**Proposition 3.1** (“Homotopy Lemma”). Let \(N\) be a manifold (possibly with corners) and let \(A \in \Omega(N \times I, W|_U)^0\), where \(I\) is the unit interval. If
\[
A|_{t=0} \in \Omega(N, W|_U)^{MC}
\]
and if
\[
i_\partial (dA - F_Q(A)) = 0,
\]
then
\[
A \in \Omega(N \times I, W|_U)^{MC}
\]
\[
N \times \{t = 1\} \Rightarrow A \in \Omega(N \times I, W)^{MC}
\]
\[
N \times I
\]
\[
i_\partial (dA - F_Q(A)) = 0
\]
\[
N \times \{t = 0\} \Rightarrow A|_{t=0} \in \Omega(N, W)^{MC}
\]

**Figure 2.** Homotopy Lemma.

**Proof.** Writing \(A = A_h + dt A_v\) we get
\[
dA - F_Q(A) = (d_h A_h - F_Q(A_h)) + dt \left( \frac{d}{dt} A_h - d_h A_v - A_v^i \frac{\partial F_Q}{\partial \xi^i}(A_h) \right)
\]
and since \(i_\partial (dA - F_Q(A)) = 0\), we get from here
\[
\frac{d}{dt} A_h = d_h A_v + A_v^i \frac{\partial F_Q}{\partial \xi^i}(A_h).
\]
We can now compute \(d\left(d_h A_h - F_Q(A_h)\right)\) using (11). We get
\[
\frac{d}{dt}(d_h A_h - F_Q(A_h)) = (-1)^{\deg \xi} (-1)^{\gamma} \left( d_h A_v A_h^k \frac{\partial F_Q}{\partial \xi^k}(A_h) - A_v^i \frac{\partial F_Q}{\partial \xi^i}(A_h) \frac{\partial F_Q}{\partial \xi^k}(A_h)\right).
\]
Since Equation (10) implies
\[
0 = \frac{\partial}{\partial \xi} \left( F_Q^k(A_h) \frac{\partial F_Q}{\partial \xi^k}(A_h) \right) = \frac{\partial F_Q^k}{\partial \xi^k}(A_h) \frac{\partial F_Q}{\partial \xi^k}(A_h) - (-1)^{\deg \xi} \xi^i \frac{\partial F_Q}{\partial \xi^i}(A_h) \frac{\partial F_Q}{\partial \xi^k}(A_h),
\]
we get
\[
\frac{d}{dt}(d_h A_h - F_Q(A_h)) = (-1)^{\deg \xi} (-1)^{\gamma} \left( d_h A_v A_h^k - F_Q^k(A_h) \right) \frac{\partial F_Q}{\partial \xi^k}(A_h).
\]
This is a linear differential equation for \(d_h A_h - F_Q(A_h)\), which together with the initial condition \(A|_{t=0} \in \Omega(N, W)^{MC}\) implies that
\[
d_h A_h - F_Q(A_h) = 0
\]
and thus, in view of (11), also
\[
dA - F_Q(A) = 0.
\]
This completes the proof. \(\square\)
Theorem 3.1. Let $N$ be a compact manifold (possibly with corners), let $A_0 \in \Omega(N, W|_{U})^{MC}$, and let $H \in \Omega(N \times I, W)^{-1}$ be a horizontal form (i.e. $\iota_0 H = 0$). Then there is $0 < \varepsilon \leq 1$ and a unique $A \in \Omega(N \times [0, \varepsilon], W|_{U})^{MC}$ such that $A|_{t=0} = A_0$ and $A_\varepsilon = H$. It is given by $A = A_h + dt \, H$, where $A_h$ is the solution of the differential equation

$$dA = dH + H^t \frac{\partial F_Q}{\partial \xi^i}(A_h)$$

with the initial condition $A_h|_{t=0} = A_0$. We can choose $\varepsilon = 1$ if the $C^0$-norm of $H^{(0)}$ is small enough, where $H^{(0)} : N \times I \to W^{-1}$ is the 0-form part of $H$.

Proof. By Proposition 3.1 and Equation (13) we see that $A \in \Omega(N \times I, W|_{U})^{MC}$ is a solution of $H^{(0)} = 0$ if the ODE (12) holds for $t \in [0, 1]$, let us split $A_h$ to its homogeneous parts

$$A_h = \sum_k A^{(k)}_h, \quad A^{(k)}_h \in \Omega^k(N \times I, W^{-k}).$$

In degree 0 Equation (12) is

$$\frac{d}{dt} A^{(0)}_h = (H^{(0)})^t \frac{\partial F_Q}{\partial \xi^i}(A^{(0)}_h)$$

with $i$ running only over the indices with deg $\xi^i = 1$, which has a solution under our hypothesis. Supposing that $A^{(m)}_h$'s are known for $m < k$, Equation (12) in degree $k$ is an inhomogeneous linear ODE for $A^{(k)}_h$, and thus always has a solution. We can thus find $A_h$ by solving (12) successively for $A^{(0)}_h, A^{(1)}_h, \ldots, A^{(\text{dim} \, N)}_h$.

Remark. The hypothesis on the $C^0$-norm of $H^{(0)}$ was used to make sure that a solution $A^{(0)}_h$ of Equation (13) exists for $t \in I$. If $W^0 = 0$, no hypothesis is needed, since $A^{(0)}_h = 0$. When $W^0 \neq 0$ then $Q : C^\infty(U) \to C^\infty(U) \odot (W^{-1})^*$ can be seen as a linear map from $W^{-1}$ to the space of vector fields on $C^\infty(U)$, and defines an integrable distribution on $U$. If $A_0 \in \Omega(N, W|_{U})^{MC}$ then the image of $A^{(0)}_0 : N \to U$ is contained in a leaf of this distribution. If the distribution is compact, then again no hypothesis of the $C^0$-norm of $H^{(0)}$ is needed, since (13) is given by vector fields tangent to the leaf. The hypothesis is needed only if the closure of the leaf is non-compact.

Theorem 3.1 can be used to solve the generalized Maurer-Cartan equation $dA = F_Q(A)$ on cubes. We shall describe another method in the following section.

4. Solving the Generalized Maurer-Cartan Equation

Suppose now that $N \subset \mathbb{R}^n$ is a star-shaped $n$-dimensional submanifold with corners (typically we would take for $N$ a $n$-simplex with a vertex at the origin, or a ball centered at the origin). Let

$$h : \Omega^*(N) \to \Omega^{*-1}(N)$$

be the de Rham homotopy operator given by the deformation retraction

$$R : N \times I \to N, \quad R(x, t) = tx.$$ 

Let, as above, $U \subset W^0$ be an open subset and $Q$ a differential on the algebra $A_{W|_{U}}$.

Theorem 4.1. A form $A \in \Omega(N, W|_{U})^0$ satisfies

$$dA = F_Q(A)$$

if and only if the form

$$B = A - h(F_Q(A)) \in \Omega(N, W)^0$$

satisfies

$$dB = 0.$$
Proof. If \( dA = F_Q(A) \) then \( dhF_Q(A) = dhA = dA \), hence \( B \) is closed.

Suppose now that \( dB = 0 \). Let \( E \) be the Euler vector field on \( N \subset \mathbb{R}^n \). By construction of \( h \) we have \( i_Eh = 0 \), hence \( i_Edh = i_E \), and thus

\[
i_E(dA - F_Q(A)) = i_E(dA - dhF_Q(A)) = i_EdB = 0.
\]

Since the vector field \( E \) on \( N \) and the vector field \( t \partial_t \) on \( N \times I \) are \( R \)-related, we get

\[
i_\partial_t(dR^*A - F_Q(R^*A)) = 0.
\]

As \( (R^*A)|_{t=0} \) is a constant we have \( (R^*A)|_{t=0} \in \Omega(N,W)^{MC} \). Proposition 3.1 now implies that \( dR^*A - F_Q(R^*A) = 0 \), and therefore (by setting \( t=1 \)) \( dA = F_Q(A) \).

Let us show that the map \( A \mapsto B = A - h(F_Q(A)) \) is injective by describing explicitly its inverse.

**Proposition 4.1.** Let \( B \in \Omega(N,W)^0 \) be such that the 0-form part of \( B \) (a map \( N \to W^0 \)) sends \( 0 \in N \subset \mathbb{R}^n \) to an element of \( U \subset W^0 \). The equation

\[
(16) \quad \mathcal{L}_E a = F_Q(B + i_Ea)
\]

\((\mathcal{L}_E \) is Lie derivative along \( E \)) has a solution \( a \in \Omega(N',W)^1 \) on \( N' \subset N \) where \( N' \) is some star-shaped open neighborhood of 0 and the solution is unique if we demand \( N' \) to be maximal.

A form \( A \in \Omega(N,W|U)^0 \) such that

\[
B = A - h(F_Q(A))
\]

exists iff \( N' = N \), and in that case \( A \) is unique, \( A = B + i_Ea \).

**Proof.** We shall define a vector field \( \hat{E} \) on the total space of

\[
\tilde{N} := (\wedge T^*N \otimes W)^1
\]

with the following properties:

1. A (partial) section \( a : N' \to \tilde{N} \) (\( N' \subset N \)) of the bundle \( \tilde{N} \to N \) is a solution of \( (16) \) iff the vector field \( \hat{E} \) is tangent to the image of \( a \).
2. \( \hat{E} \) projects to the Euler vector field \( E \) on \( N \).
3. \( \hat{E} \) has a unique fixed point \( P \in \tilde{N} \) (lying over \( 0 \in N \)). The fixed point is hyperbolic: the stable subspace of \( T_P\tilde{N} \) is the vertical subspace, and the unstable subspace projects bijectively onto \( T_0N \).

![Figure 3. The section a is the unstable manifold of \( \hat{E} \)](image)

Once \( \hat{E} \) is defined, the proposition can be proven as follows. A partial section \( a : N' \to \tilde{N} \), where \( N' \subset N \) is a star-shaped neighbourhood of \( 0 \in N \), is a solution of \( (16) \) iff the image of \( a \) is a local unstable manifold of \( \hat{E} \). Since \( \hat{E} \) projects onto \( E \), the (full) unstable manifold of \( \hat{E} \) is the image of some section \( a_{max} \), which is the unique maximal solution of \( (16) \) we wanted to find.

The existence and uniqueness of \( A \) can then be proven as follows. If \( N' = N \) then \( A = B + i_Ea \) satisfies \( B = A - h(F_Q(A)) \). To get uniqueness, notice that the operator \( \mathcal{L}_E \) is invertible on \( \Omega^{>0}(N) \) (its inverse can be written explicitly as an integral), and \( h = i_E\mathcal{L}_E^{-1} \). If \( A \) satisfying
Let us suppose that \( \hat{N} \) is compact. For \( r \geq 1 \) let \( \Omega_r(N) \) denote the Banach space of \( C^r \)-forms. The previous two results (Theorems 4.1 and Proposition 4.1) remain valid when \( A, B \) and \( a \) are \( C^r \)-forms.

**Proposition 4.2.** The map

\[
\kappa : \Omega_r(N, W|_U)^0 \rightarrow \Omega_r(N, W)^0
\]

is a smooth open embedding of Banach manifolds.

**Proof.** The map \( \kappa \) is smooth. By Proposition 4.1 it is injective and its image is open. The inverse map (constructed in the proof of Proposition 4.1 via the unstable manifold of a vector field), defined on the image of \( \kappa \), is at least \( C^r \), by differentiable dependence of unstable manifold on parameters (Robbin [11, Theorem 4.1]). This implies that \( \kappa \) is a smooth open embedding. \( \square \)

A map similar to (17) appears in the work of Kuranishi [7] and we will refer to it as the Kuranishi map.

**Theorem 4.2.** The subset

\[
\{ A \in \Omega_r(N, W|_U)^0; dA - F_Q(A) = 0 \} =: \Omega_r(N, W|_U)^{MC} \subset \Omega_r(N, W|_U)^0,
\]

is a (smooth) Banach submanifold and it is closed in the \( C^0 \)-topology. The subset

\[
\Omega(N, W|_U)^{MC} \subset \Omega(N, W|_U)^0
\]

is a Fréchet submanifold closed in the \( C^0 \)-topology.

**Proof.** By Theorem 4.1 we have

\[
\Omega_r(N, W|_U)^{MC} = \kappa^{-1}(\Omega_r(N, W)^0, cl).
\]

Since \( \kappa \) is an open embedding, the theorem follows from the fact that the space of closed forms

\[
\Omega_r(N, W)^0, cl \subset \Omega_r(N, W)^0
\]

is a \( C^0 \)-closed subspace and from \( C^0 \)-continuity of \( \kappa \). Since the result is true for every \( r \geq 1 \), it also holds for \( C^\infty \)-forms. \( \square \)

The rest of this section is a preparation for the gauge-fixing procedure of Section 4. It is somewhat inconvenient that \( \Omega_r(N) \) is not a complex (i.e. that \( d \) is not an everywhere-defined operator \( \Omega_r(N) \rightarrow \Omega_r(N) \)). Following A. Henriques [5] let us consider the complex

\[
\Omega_{r+}(N) := \{ \alpha \in \Omega_r(N); d\alpha \in \Omega_r(N) \}
\]

which is a Banach space with the norm

\[
\| \alpha \|_{r+} := \| \alpha \|_{C^r} + \| d\alpha \|_{C^r}.
\]
We can identify $\Omega_{r+}(N)$ with the closed subspace
$$\Gamma := \{(\alpha, \beta); d\alpha = \beta\} \subset \Omega_r(N) \oplus \Omega_r(N),$$
i.e. with the graph of the unbounded operator $d : \Omega_r(N) \to \Omega_r(N)$. The isomorphism $\Gamma \to \Omega_{r+}(N)$ is given by the projection $(\alpha, \beta) \mapsto \alpha$.

We have
$$\Omega_r(N, W|_U)^MC \subset \Omega_{r+}(N, W)^0 \oplus \Omega_r(N, W)^1,$$
as for $A \in \Omega_r(N, W|_U)^MC$ we have $dA = F_Q(A) \in \Omega_r(N, W)$.

**Proposition 4.3.** $\Omega_r(N, W|_U)^MC \subset \Omega_{r+}(N, W)^0$ is a Banach submanifold.

**Proof.** Let us consider the embedding
$$e : \Omega_r(N, W)^0 \to \Omega_r(N, W)^0 \oplus \Omega_r(N, W)^1, \quad e(A) = (A, F_Q(A)).$$
For $A \in \Omega_r(N, W|_U)^MC$ we have $e(A) = (A, dA)$, hence $e$ embeds $\Omega_r(N, W|_U)^MC$ to $\Gamma \otimes W$. The isomorphism $\Gamma \otimes W \cong \Omega_{r+}(N, W)$ then implies that $\Omega_r(N, W|_U)^MC \subset \Omega_{r+}(N, W)^0$ is a submanifold.

If $A_c \in \Omega(N, W|_U)^0$ is a constant (i.e. if the 0-form component of $A_c$ is a constant map $N \to U$ and the higher-form components of $A_c$ vanish) then $A_c \in \Omega(N, W|_U)^MC$ and $\kappa(A_c) = A_c$. We shall identify constant $A_c$’s with elements of $U$, i.e. we have an inclusion $U \subset \Omega(N, W|_U)^MC$ and $\kappa|_U = \text{id}_U$.

Let us notice that the map
$$A \mapsto A - h dA$$
is a projection
$$\Omega_{r+}(N, W)^0 \to \Omega_{r+}(N, W)^0 = \Omega_r(N, W)^0,$$
which coincides with $\kappa$ on $\Omega_r(N, W|_U)^MC$. Let us now consider more general projections (equivalently, let us replace $h$ with another homotopy operator).

**Proposition 4.4.** Let $C \subset \Omega_r(N)$ be a graded Banach subspace such that $\Omega_{r+}(N) = \Omega_r(N)^0 \oplus C$. Then there is an neighbourhood $U \subset \mathcal{U} \subset C \otimes W)^0$

$$\pi : \Omega_{r+}(N, W)^0 \to \Omega_r(N, W)^0$$
restricts to an open embedding $\mathcal{U} \to \Omega_r(N, W)^0$ and to the identity on $U$.

**Proof.** Let $\pi^{res} := \pi|_{\Omega_r(N, W|_U)^MC}$. To show that there exists $\mathcal{U}$ with the desired property it is enough to show that the tangent map $T_{A_c}\pi^{res}$ is a linear isomorphism for each $A_c \in \mathcal{U}$, or equivalently, that the composition

$$(18) \quad \Omega_r(N, W)^0 \ni \kappa_{lin}^i A \mapsto T_{A_c} \Omega_r(N, W|_U)^MC \subset \Omega_{r+}(N, W)^0 \mapsto \Omega_{r+}(N, W)^0 \rightarrow \Omega_r(N, W)^0$$
is a linear isomorphism, where $\kappa_{lin} = T_{A_c}\kappa$ is the linearization of $\kappa$ at $A_c$. Explicitly,

$$\kappa_{lin}(A) = A - h F_Q(A),$$
where
$$F_Q(A) = \frac{\partial F_Q}{\partial \xi^i}(A) A^i$$
is the linearization of $F_Q$ at $A_c$.

Let us introduce a filtration $\mathcal{F}$ of the space $\Omega_{r+}(N, W)^0$ with
$$\mathcal{F}^i = \bigoplus_{k \leq i} \Omega^k_{r+}(N, W^k).$$
Then $h F_Q|_{\mathcal{F}^i} : \mathcal{F}^i \to \mathcal{F}^{i-1}$ and thus $\kappa_{lin}, \kappa_{lin}^{-1} : \mathcal{F}^i \to \mathcal{F}^i$.

To show that $T_{A_c}\pi^{res}$ is a linear isomorphism it is enough to verify that the filtered linear map $\mathcal{F}^i$ induces the identity map on the associated graded. Let $B \in \Omega_r(N, W)^0$ and $B \in \mathcal{F}^i$. Then

$$\kappa_{lin}^{-1}(B) = B + h F_Q(A) \kappa_{lin}^{-1}(B).$$
Since $h F_Q(A)(\kappa_{lin}^{-1}(B)) \in \mathcal{F}^{i-1}$ and $B$ is closed, we see that the associated graded is indeed the identity. \qed
5. Filling horns

Let $\Delta^n$ be the $n$-dimensional simplex. By Theorem 4.2, the sets

$$K^\big_{\nu} (A_{W_{|U}}, Q) := \Omega(\Delta^n, W_{|U})^{MC}$$

are naturally Fréchet manifolds. The affine maps between simplices then make the collection of $K^\big_{\nu}$’s to a simplicial Fréchet manifold.

For any simplicial set $S_\bullet$ and any $0 \leq k \leq n$ let $S_{n,k}$ denote the corresponding horn. The $k$-th horn of the geometric $n$-simplex $\Delta^n$ will be denoted $\Delta^n_k$.

Definition 5.1 (A. Henriques). A simplicial manifold $K_\bullet$ is Kan if the map $K_n \to K_{n,k}$ is a surjective submersion for any $0 \leq k \leq n$.

To make the definition meaningful, one needs to check (inductively in $n$) that $K_{n,k}$ are manifolds. This was done by Henriques [6].

Any simplicial topological vector space $X_\bullet$ is automatically a Kan simplicial manifold. Indeed, there is an explicit continuous linear map $X_{n,k} \to X_n$ due to Moore [9] (in the proof of his theorem stating that any simplicial group is Kan) which is right-inverse to the horn map $X_n \to X_{n,k}$.

Namely, supposing without loss of generality that $k = n$, if $(y_0, \ldots, y_{n-1}) \in X_{n,n}$ ($y_i \in X_{n-1}$), one defines $w_0, \ldots, w_{n-1} \in X_{n}$ via

$$w_0 = s_0 y_0, \quad w_i = w_{i-1} - s_id_iw_{i-1} + s_iy_i$$

for all $0 \leq i \leq n - 1$, i.e. $w_{n-1}$ fills the horn $(y_0, \ldots, y_{n-1}) \in X_{n,n}$.

Here is the principal result of this section (it is valid also for $C^r$-forms, when we get Kan simplicial Banach manifolds).

Theorem 5.1. $K^\big_{\nu} (A_{W_{|U}}, Q)$ is a Kan simplicial Fréchet manifold.

Proof. Let $K_\bullet := K^\big_{\nu} (A_{W_{|U}}, Q)$ and let $X_\bullet := \Omega(\Delta^*, W)^{0,cl}$. As $X_\bullet$ is a simplicial Fréchet vector space, it is a Kan simplicial manifold.

Let us first prove that the horn maps $K_n \to K_{n,k}$ are submersions. Let us place $\Delta^n$ to $\mathbb{R}^n$ so that the $k$’th vertex is at $0 \in \mathbb{R}^n$. By applying the Kuranishi map $\kappa$ we get the commutative square

$$\begin{array}{ccc}
K_n & \xrightarrow{\kappa} & X_n \\
\downarrow & & \downarrow \\
K_{n,k} & \xrightarrow{\kappa} & X_{n,k}
\end{array}$$

As the horizontal arrows are open embeddings and the vertical arrow $X_n \to X_{n,k}$ is a submersion, $K_n \to K_{n,k}$ is also a submersion.

It remains to prove that $K_n \to K_{n,k}$ is surjective. Let $A_\text{horn} \in K_{n,k}$, let $B_\text{horn} := \kappa(A) \in X_{n,k}$.

We choose $B \in X_n$ extending $B_\text{horn}$.

By Proposition 4.1 Equation (16) has a solution $a$ on an open subset $N' \subset \Delta^n$ such that $\Delta^n_k \subset N'$. The form

$$A' = B + i_{E,a}$$

thus satisfies

$$A' \in \Omega(N', W_{|U})^{MC}, \quad A'|_{\Delta^n_k} = A_\text{horn}.$$
Let now $\phi \in C^\infty(\Delta^n)$ satisfy $0 \leq \phi \leq 1$, $\phi|_{\Delta^k_n} = 1$, and $\phi|_{\Delta^k_n \setminus N'} = 0$. Let $h : \Delta^n \to N'$ be given by $x \mapsto \phi(x)x$. Then
$$A := h^*A'$$
satisfies
$$A \in \Omega(\Delta^n, W|_U)^{MC} = K_n, \ A|_{\Delta^k_n} = A_{\text{horn}}.$$ 
This proves surjectivity of $K_n \to K_{n,k}$, as the map sends $A \in K_n$ to $A_{\text{horn}} \in K_{n,k}$. □

6. Globalization

So far we considered dg algebras of the form $A_{W|_U}$. Let now $V \to M$ be a negatively-graded vector bundle, and let us consider the algebra $A_V$ with a chosen differential $Q$. In this section we shall prove that Theorems 4.2 and 5.1 hold also in this more general setting.

To prove these results we need to consider the following relative situation. Let $N \subset \mathbb{R}^n$ be a compact star-shaped $n$-dimensional submanifold with corners. The restriction $A_{T[1]|\mathbb{R}^n} = \Omega(\mathbb{R}^n) \to \Omega(N)$ is a morphism of dgca. The corresponding element $A_0 \in \Omega(N)_{T[1]|\mathbb{R}^n}$ is
$$A_0 = \sum_k E_kx^k + e_kdx^k$$
where $x^k$ are the coordinates on $N \subset \mathbb{R}^n$ and $E_k$ and $e_k$ is the standard basis of $\mathbb{R}^n$ and of $\mathbb{R}^n[1]$ respectively. Application of $\kappa$ gives
$$B_0 := \kappa(A_0) = \sum_k e_kdx^k \in \Omega(N)_{T[1]|\mathbb{R}^n}^{0,cl}.$$ 

Suppose now that $W$ is a non-positively graded vector space and $p : W \to T[1]|\mathbb{R}^n$ a surjective graded linear map. Let $U \subset W^0$ be open, $Q$ be a differential on $A_{W|U}$, and let us suppose that the pullback $p^* : A_{T[1]|\mathbb{R}^n} = \Omega(\mathbb{R}^n) \to A_{W|U}$ is a morphism of dgca (i.e. that $p : W|_U \to T[1]|\mathbb{R}^n$ is a morphism of NQ-manifolds). Let
$$\Omega(N, W|_U)^{0,A_0} := \{ A \in \Omega(N, W|_U)^0; p(A) = A_0 \}$$
and similarly
$$\Omega(N, W)^{0,B_0} := \{ B \in \Omega(N, W)^0; p(B) = B_0 \}.$$ 

The elements of
$$\Omega(N, W|_U)^{MC,A_0} := \Omega(N, W|_U)^{MC} \cap \Omega(N, W|_U)^{0,A_0}$$
correspond to those dgca morphisms $A_{W|_U} \to \Omega(N)$ for which the diagram
$$\begin{array}{ccc}
\Omega(N) & \xrightarrow{p^*} & A_{W|U} \\
\downarrow & & \downarrow p^* \\
\Omega(\mathbb{R}^n) & \xrightarrow{\kappa} & A_{W|U} \\
\end{array}$$
is commutative, i.e. of those morphisms $T[1]|N \to W|_U$ of NQ-manifolds for which
$$\begin{array}{ccc}
T[1]|N & \xrightarrow{p} & W|_U \\
\downarrow & & \downarrow p \\
T[1]|\mathbb{R}^n & \xrightarrow{\kappa} & T[1]|\mathbb{R}^n \\
\end{array}$$
is commutative.

**Proposition 6.1.**
$$\Omega(N, W|_U)^{MC,A_0} = \kappa^{-1}(\Omega(N, W)^{0,cl,B_0}).$$
In particular, $\Omega(N, W|_U)^{MC,A_0} \subset \Omega(N, W|_U)^{0,A_0}$ is a Fréchet submanifold closed in the $C^0$ topology.
Proof. The equality follows from Theorem 4.1 and from $B_0 = \kappa(A_0)$. The fact that
$$\Omega(N,W|_U)^{MC,A_0} \subset \Omega(N,W|_U)^{0,A_0}$$
is a Fréchet submanifold closed in the $C^0$ topology then follows from the fact that $\kappa$ is an open embedding and $C^0$-continuous, and from the fact that $\Omega(N,W)^{0,A_0}$ is a $C^0$-closed affine subspace of $\Omega(N,W)^0$.

We can now prove our first globalized result.

**Theorem 6.1.** Let $N \subset \mathbb{R}^n$ be a star-shaped compact $n$-dimensional submanifold with corners, $V \to M$ a negatively graded vector bundle, and $Q$ a differential on the gca $\mathcal{A}_V$. Then the subset
$$\{ dg\text{-morphisms } \mathcal{A}_V \to \Omega(N) \} \subset \{ \text{graded algebra morphisms } \mathcal{A}_V \to \Omega(N) \},$$
i.e.
$$\Omega(N,V)^{MC} \subset \Omega(N,V)^0,$$
is a smooth Fréchet submanifold closed in the $C^0$-topology.

**Proof.** For any smooth map $f : N \to M$ we can find an open subset $U' \subset \mathbb{R}^n \times M$ containing the graph of $f$ such that

1. There is an open subset $U \subset \mathbb{R}^n \times \mathbb{R}^m$ and a diffeomorphism $g : U' \to U$ such that $g$ is trivial over $\mathbb{R}^n$ (i.e. such that $\forall (x,y) \in U' \subset \mathbb{R}^n \times M$, $g(x,y) = (x,\tilde{y}(x,y))$ for some $\tilde{y}(x,y) \in \mathbb{R}^m$)
2. The graded vector bundle $(p_M^*V)|_{U'}$ is trivial (where $p_M : \mathbb{R}^n \times M \to M$ is the projection).

As a result, there is a non-positively graded vector space $\tilde{W}$ with $\tilde{W}^0 = \mathbb{R}^n$ (a local model of $V \to M$), an open subset $U \subset W^0$ where $W = T[1]|_{\mathbb{R}^n} \oplus \tilde{W}$ (so that $W^0 = \mathbb{R}^{n+m}$) and an isomorphism of graded vector bundles
$$t : (T[1]|_{\mathbb{R}^n} \times V)|_{U'} \to W|_U$$such that the triangle

$$\begin{array}{c}
(T[1]|_{\mathbb{R}^n} \times V)|_{U'} & \xrightarrow{t} & W|_U \\
p' \downarrow & & \downarrow p \\
T[1]|_{\mathbb{R}^n} & \xrightarrow{p} & W|_U
\end{array} \tag{20}$$

commutes, where $p$ and $p'$ are the projections. Under the resulting isomorphism of gcas
$$t^* : A_{W|_U} \cong A_{(T[1]|_{\mathbb{R}^n} \times V)|_{U'}}$$the differential $d + Q$ on $A_{(T[1]|_{\mathbb{R}^n} \times V)|_{U'}}$ is sent to a differential $\tilde{Q}$ on $A_{W|_U}$ such that
$$p^* : \Omega(R^n) \to A_{W|_U}$$is a dg morphism, as follows from the commutativity of (20).

By Proposition 6.1 we know that $\Omega(N,W|_U)^{MC,A_0} \subset \Omega(N,W|_U)^{0,A_0}$ is a $C^0$-closed submanifold. Let $\Omega(N,V)|_{U'} \subset \Omega(N,V)^0$ be the subset of those elements for which the graphs of their function parts $N \to M$ lie in $U'$. By construction $t^*$ restricts to a diffeomorphism $\Omega(N,W|_U)^{0,A_0} \cong \Omega(N,V)^0|_{U'}$ and the image of $\Omega(N,W|_U)^{MC,A_0}$ is $\Omega(N,V)^{MC} \cap \Omega(N,V)^0|_{U'}$. As a result $\Omega(N,V)^{MC} \cap \Omega(N,V)^0|_{U'}$ is a $C^0$-closed submanifold, and thus also $\Omega(N,V)^{MC} \subset \Omega(N,V)^0$ is a $C^0$-closed submanifold, as the subsets $\Omega(N,V)^0|_{U'} \subset \Omega(N,V)^0$ form a $C^0$-open cover of $\Omega(N,V)^0$.

**Remark.** Another proof of Theorem 6.1 avoiding Proposition 6.1 is as follows. If $\Omega(N,V)^{MC}|_{U'}$ is non-empty, one can prove that (after possibly decreasing $U'$) there is a NQ-manifold $\tilde{W}|_{\tilde{G}}$ and an open embedding of NQ-manifolds
$$(T[1]|_{\mathbb{R}^n} \times V)|_{U'} \to T[1]|_{\mathbb{R}^n} \times \tilde{W}|_{\tilde{G}}$$commuting with the projections to $T[1]|_{\mathbb{R}^n}$. As a result we can identify $\Omega(N,V)^{MC}|_{U'} \subset \Omega(N,V)^0|_{U'}$ with $\Omega(N,\tilde{W}|_{\tilde{G}})^{MC} \subset \Omega(N,\tilde{W}|_{\tilde{G}})^0$, which is a submanifold by Theorem 4.2.

Our second globalized result will be proved by similar methods.

**Theorem 6.2.** The Fréchet simplicial manifold $\Omega(\Delta^*,V)^{MC}$ is Kan.
Proof. Let us use the notation $K_n := \Omega(\Delta^n, V)^{MC}$. Let us first prove that the map $K_n \to K_{n,k}$ is surjective. Any element of $K_{n,k}$ gives us, in particular, a map $f_{\text{horn}} : \Delta^n_k \to M$. Let us extend $f_{\text{horn}}$ to a map $f : \Delta^n \to M$. As in the proof of Theorem 6.1 let $U' \subset \mathbb{R}^n \times M$ be an open subset such that $U'$ contains the graph of $f$, and such that

1. $U'$ is diffeomorphic over $\mathbb{R}^n$ to an open subset $U''$ of $\mathbb{R}^n \times \mathbb{R}^m$ ($m = \dim M$)
2. the graded vector bundle $p_M^*V|_{U'} \to U'$ is trivial.

This ensures the existence of an isomorphism
t : $(T[1]\mathbb{R}^n \times V)|_{U'} \cong (T[1]\mathbb{R}^n \times \tilde{W})|_{U'}$

for some non-positively graded vector space $\tilde{W}$ with $\tilde{W}^0 = \mathbb{R}^m$ and some open subset $U \subset \tilde{W}^0$.

Let us choose an element $A_{\text{horn}} \in K_{n,k}$, i.e.

$$A_{\text{horn}} \in \Omega(\Delta^n_k, W|_{U'})^{MC,A_0}.$$  

After we apply the map $\kappa$ to $A_{\text{horn}}$, we get a closed form

$$B_{\text{horn}} \in \Omega(\Delta^n_k, W)^{0,cl}.$$  

and extend $B_{\text{horn}}$ to a closed form

$$B \in \Omega(\Delta^n, W)^{0,cl}.$$  

By Proposition 4.1, Equation (16) has a solution $a$ on an open subset $N' \subset \Delta^n$ such that $\Delta^n_k \subset N'$. The form

$$A' = B + i_E a$$

thus satisfies

$$A' \in \Omega(N', W|_{U'})^{MC}, \quad A'|_{\Delta^n_k} = A_{\text{horn}}.$$  

Let now $\phi \in C^\infty(\Delta^n)$ satisfy $0 \leq \phi \leq 1$, $\phi|_{\Delta^n_k} = 1$, and $\phi|_{\Delta^n \setminus N'} = 0$. Let $h : \Delta^n \to N'$ be given by $x \mapsto \phi(x)x$. Then

$$A := h^*A'$$

satisfies

$$A \in \Omega(\Delta^n, W|_{U'})^{MC}, \quad A|_{\Delta^n_k} = A_{\text{horn}}.$$  

The isomorphism $t$ then allows us to see $A$ as an element of $\Omega(\Delta^n, (T[1]\mathbb{R}^n \times V)|_{U'})^{MC}$. We project it to get an element of $\Omega(\Delta^n, V)^{MC} = K_n$, which finally shows that the horn map $K_n \to K_{n,k}$ is surjective.

Let us now prove that $K_n \to K_{n,k}$ is a submersion. For an open $U'' \subset \mathbb{R}^n \times M$ satisfying the conditions (1) and (2) above, let $K_n' := \Omega(\Delta^n, V)^{MC}|_{U''} \subset \Omega(\Delta^n, V)^{MC} = K_n$ be the subset of those elements for which the graph of their function part lies in $U''$. The isomorphism of graded vector bundles gives us a diffeomorphism

$$\text{id}_{\Omega(\Delta^n)} \otimes t : K_n' = \Omega(\Delta^n, V)^{MC}|_{U''} \cong \Omega(\Delta^n, W|_{U'})^{MC,A_0}$$

and so we have a commutative square

$$\begin{array}{ccc}
K_n' & \xrightarrow{\kappa \circ (\text{id}_{\Omega(\Delta^n)} \otimes t)} & \Omega(\Delta^n, W)^{0,cl,B_0} \\
\downarrow & & \downarrow \\
K_{n,k} & \xrightarrow{\kappa \circ (\text{id}_{\Omega(\Delta^n)} \otimes t)} & \Omega(\Delta^n_k, W)^{0,cl,B_0}
\end{array}$$

where the horizontal arrows are open embeddings and the right vertical arrow is a submersion, hence also the left vertical arrow is a submersion. Thus $K_n \to K_{n,k}$ is a submersion. \qed
7. Gauge fixing

The simplicial manifold $K^\bigbullet_{\bigbullet}$ is infinite-dimensional. We follow Getzler’s [5] approach and define a finite-dimensional simplicial submanifold $K^\bullet_{\bigbullet} \subset K^\bigbullet_{\bigbullet}$. In the case of a Lie algebra (or algebroid) $K^\bullet_{\bigbullet}$ is the nerve of the corresponding local Lie group (or groupoid). In general $K^\bullet_{\bigbullet}$ is a local Lie $\ell$-groupoid (Definition 7.3). $K^\bullet_{\bigbullet}$ depends on a choice of a gauge condition; different gauges lead to isomorphic (by non-canonical isomorphisms) local (weak) Lie $\ell$-groupoids. This construction is local in nature. In particular, we will not need the results of Sections 5 and 6.

Let us fix an integer $r \geq 1$. Consider the cochain complex of elementary Whitney forms in $\Omega_{r+}(\Delta^n)$ on the geometric $n$--simplex

$$E(\Delta^n) = \bigoplus_{k \geq 0} E^k(\Delta^n) \subset \Omega_{r+}(\Delta^n).$$

It is given in each degree by the linear span of differential forms defined using standard coordinates $(t_0, \ldots, t_n, \sum_k t_k = 1)$ on $\Delta^n$ by the formulas

$$\omega_{i_0 \ldots i_k} = k! \sum_{j=0}^k (-1)^j t_{i_j} dt_{i_0} \cdots \widehat{dt_{i_j}} \cdots dt_{i_k}$$

where $\widehat{dt_{i_j}}$ means the omission of $dt_{i_j}$. The $k!$ factor is prescribed so that the integral over the $k$--dimensional sub-simplex given by the sequence of $k + 1$ vertices $(e_{i_0}, \ldots, e_{i_k})$ of $\Delta^n$ is

$$\int_{(e_{i_0}, \ldots, e_{i_k})} \omega_{i_0 \ldots i_k} = 1$$

(the integral of $\omega_{i_0 \ldots i_k}$ over any other sub-simplex vanishes).

$E(\Delta^n)$ is a sub-complex of $\Omega_{r+}(\Delta^n)$, as

$$d \omega_{i_0 \ldots i_k} = \sum_{i=0}^n \omega_{i_{i_0} \ldots i_k}.$$  

There is an explicit projection $p^\bullet$ given by Whitney [16]

$$p_n : \Omega_{r+}(\Delta^n) \rightarrow E(\Delta^n)$$

$$p_n \alpha = \sum_{k=0}^n \sum_{i_0 < \ldots < i_k} \omega_{i_0 \ldots i_k} \int_{(e_{i_0}, \ldots, e_{i_k})} \alpha$$

compatible with the simplicial (cochain complex) structures of $\Omega_{r+}(\Delta^\bullet)$ and $E(\Delta^\bullet)$. $E(\Delta^\bullet)$ is furthermore isomorphic (with the isomorphism respecting the simplicial structures) to the complex of simplicial cochains on $\Delta^\bullet$.

Following Getzler [5] let us make the following Definition.

**Definition 7.1.** A gauge on $\Omega_{r+}(\Delta^\bullet)$ is a continuous simplicial linear map

$$s^\bullet : \Omega_{r+}(\Delta^\bullet) \rightarrow \Omega_{r+}(\Delta^\bullet)$$

of degree $-1$ satisfying

$$id_{\Omega_{r+}(\Delta^\bullet)} - p^\bullet = [d, s^\bullet]$$

and

$$s^2 = 0, \quad p^\bullet s^\bullet = s^\bullet p^\bullet = 0,$$

which is furthermore invariant under the action of the symmetric group $S_{n+1}$ on $\Delta^\bullet$ (by permuting the vertices).

Restricting to $\ker s^\bullet$ can be thought of as gauge fixing. An important example of a a gauge is given by Dupont [3] in the proof of the simplicial de Rham theorem. Its construction is as follows. Let us denote $h_i$ the de Rham homotopy operator associated to the retraction of $\Delta^\bullet$ to the $i$-th vertex. Then the operators

$$s^n = \sum_{k=0}^{n-1} \sum_{i_0 < \ldots < i_k} \omega_{i_0 \ldots i_k} h_{i_k} \cdots h_{i_0}$$

where $n \geq 0$ form a gauge $s^\bullet$ (it is $\parallel \cdot \parallel_{r+}$-continuous for every $r$).
Notation. We shall set
\[ \Omega_{r+}(\Delta^*)^p := \ker s_\bullet \subset \Omega_{r+}(\Delta^*), \]
\[ \Omega_{r+}(\Delta^*)^{s,p} := \ker p_\bullet \subset \Omega_{r+}(\Delta^*), \]
\[ \Omega_{r+}(\Delta^*)^{s,p} := \ker s_\bullet \cap \ker p_\bullet \subset \Omega_{r+}(\Delta^*). \]

More generally, if \( W \) is a graded vector space, we let
\[ \Omega_{r+}(\Delta^*, W)^s := \ker (s_\bullet \otimes \id_W) \subset \Omega_{r+}(\Delta^*, W) \]

etc.

The meaning of a gauge is summed-up by the following proposition.

**Proposition 7.1.** A gauge is equivalent to a choice of a closed and \( S_{n+1} \)-invariant simplicial subspace
\[ s_\bullet \in \Omega_{r+}(\Delta^*), \]
such that
\[ d : s_\bullet \in \Omega_{r+}(\Delta^*), \]

is a bijection, i.e. such that \( \Omega_{r+}(\Delta^*)^{p,cl} = \Omega_{r}(\Delta^*)^{p,cl} \oplus s_\bullet \). If \( s_\bullet \) is given then
\[ s_\bullet = \Omega_{r+}(\Delta^*)^{s,p}. \]

If \( s_\bullet \) is given then \( s_{\bullet} \) is
\[ s_{\bullet} = d^{-1}. \]

**Proof.** The map \( s_{\bullet} \) defined by (21) clearly satisfies
\[ s_\bullet p_\bullet = 0, \quad p_\bullet s_\bullet = 0, \quad \id_{\Omega_{r+}(\Delta^*)} - p_\bullet = [d, s_{\bullet}], \quad s_{\bullet}^2 = 0 \]
so it is a gauge, and \( s_\bullet \in \Omega_{r+}(\Delta^*)^{s,p} \).

Conversely given a gauge \( s_{\bullet} \) we set \( s_\bullet \in \Omega_{r+}(\Delta^*)^{s,p} \) and we easily see that \( d : s_\bullet \in \Omega_{r+}(\Delta^*) \)

is an isomorphism, and that the gauge defined by (21) coincides with the original \( s_{\bullet} \).

We can use \( s_{\bullet} \) to specify the closed graded subspace \( C \subset \Omega_{r+}(\Delta^*) \) of Proposition 4.4. Let
\[ D(\Delta^*) := \ker (E(\Delta^*)) = E(\Delta^*), \]

where \( h_0 \) is the de Rham homotopy operator given by the contraction to the vertex 0, so that
\[ E(\Delta^*) = E(\Delta^*)^{cl} + D(\Delta^*). \]

We set \( C = D(\Delta^*) \oplus s_\bullet \in \Omega_{r+}(\Delta^*) \) and consider the projection
\[ s_{\bullet} : \Omega_{r+}(\Delta^*) \to \Omega_{r}(\Delta^*)^{cl} \]
w.r.t. \( C \).

Notice that \( D(\Delta^*) \subset E(\Delta^*) \) is not a simplicial subspace, but it is compatible with the maps between simplices that preserve the vertex 0. Likewise, \( s_{\bullet} \) is not a simplicial map, but it is 0-simplicial in the following sense:

**Definition 7.2.** If \( X_\bullet \) and \( Y_\bullet \) are simplicial sets then a sequence of maps \( f_\bullet : X_\bullet \to Y_\bullet \) is a 0-simplicial map if it is functorial under the order-preserving maps \( \{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\} \)
sending 0 to 0.

As before let \( W \) be a finite-dimensional non-positively graded vector space, \( U \subset W^0 \) be an open subset, and let \( Q \) be a differential on the algebra \( A_{W|U} \) and \( \kappa \) be the corresponding Kuranishi map given by (17). Furthermore let \( U_\bullet \) be the open neighborhood \( U \subset \Omega_\bullet \subset \Omega_{r+}(\Delta^*, W_{|U})^{MC} \)
from Proposition 4.4, where \( C = D(\Delta^*) \oplus s_\bullet \). We can demand \( U_\bullet \) to be a simplicial submanifold such that \( U_\bullet \) is invariant under the action of the symmetric group \( S_{n+1} \) for every \( n \).

We will denote the simplicial set of gauge-fixed dg-morphisms in \( U_\bullet \) by
\[ K^0_{\bullet}(A_{W|U}, Q) := \Omega_{r}(\Delta^*, W_{|U})^{MC,s} := U_\bullet \cap \Omega_{r+}(\Delta^*, W)^{0,s}. \]
Its elements are (sufficiently small) forms $A \in \Omega_r(\Delta^\bullet, W|_{U})^0$ satisfying the equations
\[ dA = F_Q(A), \quad s\cdot A = 0. \]

Let us now recall the definition of higher Lie groupoids. As is usual we actually define higher groupoids as nerves.

**Definition 7.3.** (Getzler [5], Henriques [1], Zhu [17]) A Kan simplicial manifold $K^\bullet$ is called a Lie $\ell$-groupoid if the maps $K_n \to K_{n,k}$ are diffeomorphisms for all $n > \ell$ and all $0 \leq k \leq n$. A finite dimensional simplicial manifold $K^\bullet$ is called a local Lie $\ell$-groupoid if the maps $K_n \to K_{n,k}$ are submersions for all $n \geq 1$, $0 \leq k \leq n$ and open embeddings for all $n > \ell$, $0 \leq k \leq n$.

**Theorem 7.1.** The simplicial set $K^\bullet_s(\mathcal{A}_{\mathcal{U}|_{U}}, Q)$ is a finite-dimensional local Lie $\ell$-groupoid with $-\ell$ being the lowest degree of $W$.

**Proof.** Let
\[ \pi_s : \Omega_r(\Delta^\bullet, W)^0 \to \Omega_r(\Delta^\bullet, W)^{0,cl} \]
be the projection w.r.t. $(C \otimes W)^0$ (where $C = D(\Delta^\bullet) \cap \mathcal{M}(\Delta^\bullet)$) and let
\[ \pi^{res}_s : \mathcal{U}_s \to \Omega_r(\Delta^\bullet, W)^{0,cl} \]
be the restriction of $\pi_s$ to $\mathcal{U}_s$. By Proposition 4.4, $\pi^{res}_s$ is an open embedding. Let us recall that $\pi^{res}_s$ is a 0-simplicial map.
Let $K_s^\bullet := K^\bullet_s(\mathcal{A}_{\mathcal{U}|_{U}}, Q)$. We have
\[ K^\bullet_s = \mathcal{U}_s \cap \Omega_r(\Delta^\bullet, W)^{0,s} = (\pi^{res}_s)^{-1}(\Omega_r(\Delta^\bullet, W)^{0,cl} \cap \Omega_r(\Delta^\bullet, W)^{0,s}) = (\pi^{res}_s)^{-1}(E(\Delta^\bullet, W)^{0,cl}) \]
as $\Omega_r(\Delta^\bullet, W)^{0,cl} \cap \Omega_r(\Delta^\bullet, W)^{0,s} = E(\Delta^\bullet, W)^{0,cl}$. This implies that each degree of the simplicial set $K^\bullet_s$ is a finite-dimensional smooth manifold and that
\[ (22) \quad \pi^{res}_s : K^\bullet_s \to E(\Delta^\bullet, W)^{0,cl} \]
is an open embedding.

It remains to prove that it is a local Lie $\ell$-groupoid. Since $E(\Delta^\bullet, W)^{0,cl}$ is a Lie $\ell$-groupoid and $\pi^{res}_s$ a 0-simplicial map, $K^\bullet_s$ satisfies the required conditions for the projections $K^\bullet_s \to K^\bullet_{s,0}$. The action of the symmetric group $S_{n+1}$ on $K^\bullet_n$ then ensures that the conditions are satisfied for all horn projections $K^\bullet_n \to K^\bullet_{n,k}$, i.e. that $K^\bullet_s$ is a local Lie $\ell$-groupoid.

The simplicial manifold $K^\bullet_s(\mathcal{A}_{\mathcal{U}|_{U}}, Q) = \Omega_r(\Delta^\bullet, W|_{U})^{MC,s}$ was constructed using infinite-dimensional techniques. We can now see it as a simplicial submanifold of a finite-dimensional simplicial vector space:

**Theorem 7.2.** The projection
\[ E(\Delta^\bullet, W)^0 \oplus \mathcal{M}(\Delta^\bullet, W)^0 \to E(\Delta^\bullet, W)^0 \]
restricts to an embedding of simplicial manifolds
\[ K^\bullet_s(\mathcal{A}_{\mathcal{U}|_{U}}, Q) \to E(\Delta^\bullet, W)^0. \]

**Proof.** The projection is a simplicial map. It restricts to an embedding since (22) is an (open) embedding. \qed

We can thus identify $K^\bullet_s(\mathcal{A}_{\mathcal{U}|_{U}}, Q)$ with
\[ \{ A \in E(\Delta^\bullet, W)^0 ; (\exists A' \in \mathcal{M}(\Delta^\bullet, W)^0) \ d(A + A') = F_Q(A + A') \} \]
intersected with a suitable open subset of $E(\Delta^\bullet, W)^0$. 
8. Deformation retraction

In this section we shall prove that $K^*_{\bullet}$ and $K^*_{big}$ are equivalent as local Lie $\infty$-groupoids, namely we shall construct a local simplicial deformation retraction of $K^*_{big}$ onto $K^*_{\bullet}$.

If $A_0 \in \Omega_r(\Delta^\bullet, W|_U)^{MC}$ and $G \in \Omega_r(\Delta^\bullet, W)^{0,cl,p}$ so that $d(s_{\bullet}G) = G$, and if the $C^0$-norm of $s_{\bullet}G$ is small enough, then Theorem 5.11 gives us a form

$$\tilde{A} \in \Omega_r(\Delta^\bullet \times I, W|_U)^{MC}$$

such that $\tilde{A}|_0 = A_0$ and $\tilde{A}_t := i_{\tilde{A}} \tilde{A} = q^*s_{\bullet}G$, where $q : \Delta^\bullet \times I \to \Delta^\bullet$ is the projection. It is given by Equation (12) with $H = q^*s_{\bullet}G$, i.e., by

$$\frac{d}{dt} \tilde{A}_t = G + (s_{\bullet}G')\frac{\partial F_Q}{\partial \xi_t}(\tilde{A}_t).$$

Let us use the notation $\tilde{A}(A_0, G)$ for the form $\tilde{A}$ constructed in this way.

**Proposition 8.1.** There is an open neighbourhood $V_\bullet$ of $U \times \{0\}$ in

$$\Omega_r(\Delta^\bullet, W|_U)^{MC,s} \times \Omega_r(\Delta^\bullet, W)^{0,cl,p}$$

such that the map $(A_0, G) \mapsto \tilde{A}(A_0, G)|_{t = 1}$ is an open embedding $V_\bullet \to \Omega_r(\Delta^\bullet, W|_U)^{MC}$. \hspace{1cm} \square

**Proof.** Since $\psi$ restricts to the identity on $U \times \{0\}$, it’s enough to show that for each $A_c \in U$ the tangent map $\psi_{lin} := T_{(A_c, 0)}\psi$ is invertible. We have

$$\psi_{lin}(A_0, G) = A_0 + G + (s_{\bullet}G')\frac{\partial F_Q}{\partial \xi_t}(A_c)$$

(for $A_0$ tangent to $\Omega_r(\Delta^\bullet, W|_U)^{MC,s}$ at $A_c$ and $G \in \Omega_r(\Delta^\bullet, W)^{0,cl,p}$).

Let us recall that the projection $\pi_\bullet$ w.r.t. $(C \otimes W)^0$ (where $C = D(\Delta^\bullet) \oplus \mathcal{M}(\Delta^\bullet)$) gives us isomorphisms

$$T_{A_c}\Omega_r(\Delta^\bullet, W|_U)^{MC,s} \cong E(\Delta^\bullet, W)^{0,cl}$$

and

$$T_{A_c}\Omega_r(\Delta^\bullet, W|_U)^{MC} \cong \Omega_r(\Delta^\bullet, W)^{0,cl} \cong E(\Delta^\bullet, W)^{0,cl} \oplus \Omega_r(\Delta^\bullet, W)^{0,cl,p}.$$ 

Since the term $(s_{\bullet}G')\frac{\partial F_Q}{\partial \xi_t}(A_c) \in \mathcal{M}(\Delta^\bullet, W)^0$ is removed by this projection, we see that $\psi_{lin}$ is indeed an isomorphism. \hspace{1cm} \square

**Theorem 8.1.** There is an open neighbourhood $W_\bullet$ of $U$ in

$$\Omega_r(\Delta^\bullet, W|_U)^{MC}$$

which is a simplicial submanifold, and a (smooth) simplicial map

$$\Psi^\bullet : W_\bullet \to \Omega_r(\Delta^\bullet \times I, W|_U)^{MC}$$

with these properties:

1. $\Psi^\bullet(A)|_{t = 1} = A$ (for every $A \in W_\bullet$)
2. $\Psi^\bullet(A)|_{t = 0} \in \Omega_r(\Delta^\bullet, W|_U)^{MC,s}$
3. if $A \in \Omega_r(\Delta^\bullet, W|_U)^{MC,s}$ then $\Psi^\bullet(A) = q^*A$ where $q : \Delta^\bullet \times I \to \Delta^\bullet$ is the projection.

![Figure 5. Deformation retraction](image-url)
Proof. We set $W_\bullet := \psi(Y_\bullet)$ (we might then have to decrease $W_\bullet$ to ensure that it is a simplicial submanifold), and $\Psi^\bullet(A) := \tilde{A}(\psi^{-1}(A))$. 

Let us recall that a simplicial homotopy is a simplicial map $X_\bullet \times I_\bullet \to Y_\bullet$ where the simplicial set $I_\bullet$ (the simplicial interval) is the set of non-decreasing maps $f : \{0, \ldots, \bullet\} \to \{0, 1\}$. We can now formulate the main result of this section.

**Theorem 8.2.** There is a local simplicial deformation retraction

$$R^\bullet_s : \Omega_r(\Delta^\bullet, W_\bullet | U)^{MC} \times I_\bullet \to \Omega_r(\Delta^\bullet, W_\bullet | U)^{MC}$$

of $\Omega_r(\Delta^\bullet, W_\bullet | U)^{MC}$ to $\Omega_r(\Delta^\bullet, W_\bullet | U)^{MC, s} \subset \Omega_r(\Delta^\bullet, W_\bullet | U)^{MC}$, where “local” means that it is defined on an open neighbourhood of $U$ in $\Omega_r(\Delta^\bullet, W_\bullet | U)^{MC}$.

**Proof.** If $f \in I_\bullet$, $f : \{0, \ldots, \bullet\} \to \{0, 1\}$, let $g : \Delta^\bullet \to \Delta^\bullet \times I$ be the affine map given on the vertices by $g(\nu_i) = (\nu_i, 1 - f(i))$ ($i = 0, \ldots, \bullet$). Given $A \in W_\bullet \subset \Omega(\Delta^\bullet, W_\bullet | U)^{MC}$ we set $R^\bullet_s(A, f) := g^*\Psi^\bullet(A)$.

9. **Functoriality and naturality**

If $\phi : V \rightsquigarrow V'$ is a morphism of NQ-manifolds, i.e. if we have a morphism of dg algebras $\phi^* : \mathcal{A}_V \to \mathcal{A}_V$, by composition we get a morphism of Banach simplicial manifolds

$$(\phi_s)_* : \Omega_r(\Delta^\bullet, V)^{MC} \to \Omega_r(\Delta^\bullet, V')^{MC}$$

(or of Fréchet simplicial manifolds if we remove the subscript $r$). In this way $V \mapsto \Omega_r(\Delta^\bullet, V)^{MC}$ is a functor from the category of NQ-manifolds to the category of Banach simplicial manifolds.

The situation is more complicated when we consider the finite-dimensional simplicial manifolds $K^\circ_r(W_\bullet | U) := \Omega_r(\Delta^\bullet, W_\bullet | U)^{MC, s}$. To avoid mentioning new and new open subsets, let us make the following definition.

**Definition 9.1.** If $X_\bullet$ and $Y_\bullet$ are simplicial manifold, a local homomorphism $X_\bullet \to Y_\bullet$ is a smooth simplicial map $U_\bullet \to Y_\bullet$, where $U_\bullet \subset X_\bullet$ is an open simplicial submanifold containing all fully degenerate simplices. Two local homomorphisms are declared equal if they coincide on some $U_\bullet$.

Let

$$i^\bullet_s : \Omega_r(\Delta^\bullet, W_\bullet | U)^{MC, s} \to \Omega_r(\Delta^\bullet, W_\bullet | U)^{MC}$$

be the inclusion, and

$$p^\bullet_s : \Omega_r(\Delta^\bullet, W_\bullet | U)^{MC} \to \Omega_r(\Delta^\bullet, W_\bullet | U)^{MC, s}$$

the projection given by $p^\bullet_s(A) = \Psi^\bullet(A)|_{t=0}$ ($p^\bullet_s$ is a local homomorphism).

If $\phi : W_\bullet | U \rightsquigarrow W'_\bullet | U'$ is a map of dg manifolds, i.e. if we have a morphism $\phi^* : \mathcal{A}_{W'_\bullet | U'} \to \mathcal{A}_{W_\bullet | U}$ of dg algebras, we get a local morphism

$$K^\circ_s(\phi) : \Omega_r(\Delta^\bullet, W_\bullet | U)^{MC, s} \to \Omega_r(\Delta^\bullet, W'_\bullet | U')^{MC, s}$$

of local Lie $\ell$-groupoids, defined as the composition

$$\Omega_r(\Delta^\bullet, W_\bullet | U)^{MC, s} \xrightarrow{i^\bullet_s} \Omega_r(\Delta^\bullet, W_\bullet | U)^{MC} \xrightarrow{(\phi_s)_*} \Omega_r(\Delta^\bullet, W'_\bullet | U')^{MC} \xrightarrow{p^\bullet_s} \Omega_r(\Delta^\bullet, W'_\bullet | U')^{MC, s}.$$  

Let us observe that

$$K^\circ_s(\phi \circ \phi') \neq K^\circ_s(\phi) \circ K^\circ_s(\phi')$$

in general, i.e. $K^\circ_s$ is not a functor. It is, however, a “functor up to homotopy”, i.e. a homotopy coherent diagram in the sense of Vogt [15], or, using a more recent terminology, a quasi-functor. Let us describe this quasi-functor in pedestrian terms. An $n$-simplex $f$ in the nerve of the category of dg manifolds of the type $W_\bullet | U$ is a chain of composable morphisms

$$f := \left( W_0 | U_0 \xrightarrow{\phi_0} W_1 | U_1 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{n-1}} W_n | U_n \right).$$

Combining the maps $(\phi_s)_\bullet$, $i = 0, \ldots, n - 1$ with the local simplicial deformation retractions $R^\bullet_s$ we obtain simplicial maps

$$K^\circ_s(f) : K^\circ_s(\{W_0 | U_0\} \times I^{n-1}) \to K^\circ_s(\{W_n | U_n\})$$

in the nerve of the category of dg manifolds of the type $W_\bullet | U$. A more formal description of this quasi-functor is given in Section 8.3. In the next section we will show how to associate a local homomorphism

$$\phi^* : \tilde{\Omega}_r(\Delta^\bullet, \tilde{Y} \otimes W_\bullet | U)^{MC, s} \to \tilde{\Omega}_r(\Delta^\bullet, \tilde{X} \otimes W_\bullet | U)^{MC, s}$$

of local Lie $\ell$-groupoids to a morphism of dg manifolds $\phi : X_\bullet \to Y_\bullet$. In Section 9 we will show that $\phi^*$ is a local homomorphism of NQ-manifolds.
defined (in the case of \( n = 3 \), as an illustration) as the composition

\[
K^{\ast}_s(W_0|U_0) \times (I_3)^2 \xrightarrow{i^* \times id} K^{big}_s(W_1|U_1) \times (I_2)^2 \xrightarrow{R^* \times id} K^{big}_s(W_2|U_2) \times I_1 \xrightarrow{R^* \times id} K^{big}_s(W_3|U_3)
\]

By construction, they define a homotopy coherent diagram. To explain what it means, let us use Vogt’s notation for

\[
\phi \in W \text{ manifold of the form } K \text{ invariant symplectic (resp. closed) } 2\text{-form}
\]

Definition 10.1. A NQ-manifold

\[
(K, \varphi) \in \text{NQ-manifolds with local morphisms)
\]

then a homotopy coherently diagram from

\[
\text{local } 2\text{-symplectic } 2\text{-groupoids, and more generally, NQ-manifolds with}
\]

we have (id

\[
\text{id}^* \ast \text{id}^* \ast \text{id}^* = \text{id}^* \ast \text{id}^* \ast \text{id}^* = \text{id}^* \ast \text{id}^* \ast \text{id}^*
\]

More compactly, a homotopy coherent diagram can be described as follows (an introduction to the subject can be found in Porter [10]). If \( C \) is a category (in our case the category of NQ-manifolds of the form \( W^{|U|} \)) and if \( \mathcal{D} \) is a simplicially enriched category (in our case the category of simplicial manifolds with local morphisms) then a homotopy coherent diagram from \( C \) to \( \mathcal{D} \) is a simplicially enriched functor \( S(C) \rightarrow \mathcal{D} \), where \( S(C) \) is the simplicially enriched category defined as the free simplicial resolution of \( C \), introduced by Dwyer and Kan [3].

We thus have the following result.

**Theorem 9.1.** \( K^{\ast}_s \) is a homotopy coherent diagram from the category of dg manifolds of type \( W^{|U|} \) to the simplicially enriched category of local simplicial manifolds with the simplicial enrichment given by

\[
\text{Hom}(L, M)_n := \text{Hom}(L \times \Delta[n], M)
\]

where \( \Delta[n] \) is the simplicial set representing the n-simplex. Here “local” means that morphisms are defined to be local homomorphisms of simplicial manifolds.

For a general dg manifold given by a negatively graded vector bundle \( V \rightarrow M \) (not of the form \( W^{|U|} \)) one can apply the quasi-functor \( K^{\ast}_s \) on the nerve \( N_{\ast}(U) \) of a good cover \( U \) of \( M \). We can say a bit more in this situation: the homomorphisms \( K^\ast_s(\phi) \) on overlaps are actually isomorphisms.

Indeed, for any dg-map \( \phi : W^{|U|} \rightarrow W^{|U|} \) the linearization \( (\phi_{\ast})_{\text{lin}} \) of \( (\phi_{\ast}) \) commutes with \( s_{\ast} \cdot p_{\ast} \). Therefore the linearization \( (\phi_{\ast})_{\text{lin}} \) of \( (\phi_{\ast}) \) is functorial, i.e. if \( \phi' : W^{|U|} \rightarrow W^{|U|} \) is another map of dg manifolds then \( (\phi' \circ \phi_{\ast})_{\text{lin}} = (\phi'_{\ast})_{\text{lin}} \circ (\phi_{\ast})_{\text{lin}} \). For \( id : W^{|U|} \rightarrow W^{|U|} \) we have \( (id_{\ast})_{\text{lin}} = id_{\ast, (\Delta \cdot W^{|U|})_{\text{MC}}} \). Together with functorality this implies that if \( \phi \) is an isomorphism of dg manifolds then \( K^\ast_s(\phi) \) is an isomorphism of local Lie \( \ell \)-groupoids.

**10. Integration of (pre)symplectic forms**

One of the most remarkable facts of Poisson geometry is that if \( M \) is a Poisson manifold then \( T^*M \) is a Lie algebroid (the corresponding differential on \( \Gamma(TM) \) is given by \( [\pi, \cdot] \) where \( \pi \) is the Poisson structure and \( [\cdot, \cdot] \) the Schouten bracket) and that the corresponding (local) Lie groupoid is symplectic. In [13] it was suggested that Courant algebroids should be integrated to (local) 2-symplectic 2-groupoids, and more generally, NQ-manifolds with \( Q \)-invariant symplectic form of degree \( k \) should integrate to (local) \( k \)-symplectic \( \ell \)-groupoids (where \( k \geq \ell \) and \( k = \ell \) whenever the base \( M \) is non-trivial). In this section we shall see that it is indeed the case.

**Definition 10.1.** A NQ-manifold \( Z \) is \( k \)-symplectic (resp. \( k \)-presymplectic) if it carries a \( Q \)-invariant symplectic (resp. closed) 2-form \( \pi \) of degree \( k \) w.r.t. the grading on \( Z \).
As observed in [13], if $\omega$ is symplectic then $k \geq \ell$ and $k = \ell$ whenever $\dim M \geq 1$. Another observation from [13] is that any 1-symplectic NQ-manifold is naturally of the form $T^*[1]M$ and the differential $Q$ is given by a Poisson structure on $M$, and that any 2-symplectic NQ-manifold is equivalent to a Courant algebroid. (Courant algebroids were introduced by Liu, Weinstein and Xu in [5] and their connection with 2-symplectic NQ-manifolds is explained in detail in Roytenberg [12].)

To discuss symplectic forms on Lie $\ell$-groupoids, let us recall that if $K_\bullet$ is a simplicial manifold then $\bigoplus_{m,n} \Omega^m(K_n)$ is a bicomplex. The first differential is de Rham’s $d$ and the second differential $\delta$ is given by the simplicial structure

$$\delta \alpha := \sum_{p=0}^{n+1} (-1)^p d_p^* \alpha \in \Omega^m(K_{n+1}) \text{ for } \alpha \in \Omega^m(K_n)$$

where $d_p : K_{n+1} \to K_n$ are the face maps.

A Lie groupoid is called symplectic if its nerve $K_\bullet$ is endowed with a symplectic form $\omega \in \Omega^2(K_1)$ such that $\delta \omega = 0$.

**Definition 10.2.** A (local) Lie $\ell$-groupoid $K_\bullet$ is strictly $k$-symplectic (resp. presymplectic) if it is endowed with a symplectic (resp. closed) 2-form $\omega \in \Omega^2(K_k)$ such that $\delta \omega = 0$.

Let now $Z$ be a $k$-presymplectic NQ-manifold. We shall construct a closed 2-form $\omega^{big}$ on $K_k^{big}(Z)$ satisfying $\delta \omega^{big} = 0$. Moreover, if $\omega$ is symplectic and if $Z$ is of the form $W|U$ (or if $W|U$ is a local piece of $Z$) we shall see that the 2-form $\omega^{big}$ restricts to a symplectic form on $K_k^1(W|U)$, i.e. that $K_k^0(W|U)$ is a local $k$-symplectic $\ell$-groupoid.

Suppose that $N$ is a compact oriented manifold (possibly with corners) and $f : T[1]N \to Z$ a NQ-map. Following the AKSZ construction [1], if $u, v \in \Gamma(f^*TZ)$, let us define

$$\omega_{N,f}(u,v) := \int_N \omega(u,v)$$

(this expression makes sense since $\omega(u,v)$ is a function on $T[1]N$, i.e. a differential form on $N$).

If $Z = W|U$ then $u, v \in \Omega(N,W)$. In this case, if $\omega = \omega_{ij}(\xi) d\xi^i d\xi^j$ ($\omega_{ij}(\xi) \in A_{W|U}$) and $A \in \Omega(N,W|U)^{MC}$ then

$$\omega_{N,A}(u,v) = \int_N \omega_{ij}(A) u^i v^j.$$  

The differentials $d$ on $T[1]N$ and $Q$ on $Z$ turn $\Gamma(f^*TZ)$ to a differential graded module over $\Omega(N) = C^\infty(T[1]N)$, with a differential $d_{tot}$. When $Z = W|U$ and thus $\Gamma(f^*TZ) = \Omega(N,W)$, the differential $d_{tot}$ can be computed as

$$d_{tot} u = du - u^i \frac{\partial F_Q}{\partial N^i}(A) \quad \forall u \in \Omega(N,W).$$

The $Q$-invariance of $\omega$ implies the identity (with signs unimportant for what follows)

$$d\left(\omega(u,v)\right) = \pm \omega(d_{tot} u, v) \pm \omega(u, d_{tot} v) \in \Omega(N).$$

As a consequence, we get the following result (Lemma 3 of [13]).

**Proposition 10.1.** Let $i_p : \Delta^{n-1} \to \Delta^n$ be the inclusion of the p-th face of $\Delta^n$. Then

$$\sum_{p=0}^{n} (-1)^p \omega_{\Delta^{n-1},i_p}(i_p^* u, i_p^* v) = \pm \omega_{\Delta^n,f}(d_{tot}u,v) \pm \omega_{\Delta^n,f}(u,d_{tot}v)$$

*Proof.* The claim follows from the Stokes theorem applied to (24) with $N = \Delta^n$. \hfill $\square$

The tangent space of $K_k^{big}(Z)$ at $f : T[1] \Delta^k \to Z$ is $\Gamma(f^*TZ)^{0,cl}$. Following [1] we now define a closed 2-form $\omega^{big}$ on the manifold $K_k^{big}(Z)$ by

$$\omega^{big}(u,v) := \omega_{\Delta^n,f}(u,v).$$

As a consequence of Proposition 10.1 we get the following result.

**Theorem 10.1.** The closed 2-form $\omega^{big} \in \Omega^2(K_k^{big}(Z))$ satisfies $\delta \omega^{big} = 0$.

*Proof.* Let us use Equation (25) when $u, v \in T_f K_k^{big}(Z) = \Gamma(f^*TZ)^{0,cl}$, the RHS vanishes as $d_{tot}u = d_{tot}v = 0$, and the LHS is $(\delta \omega^{big})(u,v)$. As a result $\delta \omega^{big} = 0$. \hfill $\square$
Suppose now that $Z$ is of the form $W|U$. Let us define the closed 2-form
\[ \omega^*_{W|U} \in \Omega^2(K^*_k(W|U)) \]
as the restriction of $\omega^{\text{big}}$ to the simplicial submanifold $K^*_k(W|U) \subset K^{\text{big}}_k(W|U)$. As $\omega^{\text{big}}$ it satisfies the relation
\[ \delta \omega^*_{W|U} = 0. \]

**Theorem 10.2.** If the closed 2-form $\varpi$ on $W|U$ is symplectic then $\omega^*_{W|U} \in \Omega^2(K^*_k(W|U))$ is symplectic on an open neighbourhood of the totally degenerate simplices $U \subset K^*_k(W|U)$. The local Lie $\ell$-groupoid $K^*_k(W|U)$ is thus strictly $k$-symplectic (after we possibly replace it with an open neighbourhood of the totally degenerate simplices).

**Proof.** It is enough to prove that $\omega^*_{W|U}$ is non-degenerate at the points of $U \subset K^*_k(W|U)$. If $x \in U$, the linearization of $Q$ at $x$ gives us a differential $Q_{\text{lin}} : W \to W$ making $W$ to a cochain complex; explicitly, if $w \in W$,
\[ Q_{\text{lin}} w = -w^i \frac{\partial F}{\partial \xi^i}(x). \]
The tangent space $T_x K^*_k(W|U)$ is
\[ T_x K^*_k(W|U) = E(\Delta^n, W)^{0,cl_{\text{tot}}} \]
where the subscript $cl_{\text{tot}}$ means closed w.r.t. the total differential $d + Q_{\text{lin}}$.

Let us introduce Grassmann parameters $\epsilon_0, \ldots, \epsilon_n$ of degree 1 and consider the graded vector space $\tilde{E}_n := \wedge(\epsilon_0, \ldots, \epsilon_n)$. On $\tilde{E}_n$ there is a differential
\[ d = \sum \epsilon_i \]
of degree 1, and a differential
\[ \partial = \sum \frac{\partial}{\partial \epsilon_i} \]
of degree -1, and
\[ d\partial + \partial d = n + 1. \]
Both $d$ and $\partial$ are thus acyclic and $\tilde{E}_n$ is the direct sum of its $d$-closed part and it $\partial$-closed part.

Moreover, we have a morphism of chain complexes
\[ \chi : (\tilde{E}_n[1], d) \to (E(\Delta^n), d), \quad \epsilon_i, \ldots, \epsilon_m \mapsto \omega_{\epsilon_i \ldots \epsilon_m}, \quad 1 \mapsto 0 \]
which is an isomorphism with the exception of degree $-1$. We can use $\chi$ to identify $T_x K^*_n(W|U) = E(\Delta^n, W)^{0,cl_{\text{tot}}}$ with $(\tilde{E}_n \otimes W)^{1,cl_{\text{tot}}}$. Let us use the non-degenerate pairing $\tilde{E}_n \otimes \tilde{E}_n \to \mathbb{R}$
\[ \langle \sigma, \tau \rangle := \text{the coefficient of } \epsilon_0 \ldots \epsilon_n \text{ in } \sigma \tau \]
and the pairing $\tilde{E}_n \otimes \tilde{E}_n \to \mathbb{R}$
\[ \langle \sigma, \tau \rangle = \int_{\Delta^n} \chi(\sigma) \chi(\tau). \]
A straightforward calculation shows that
\[ \langle \sigma, \tau \rangle = \frac{(i-1)!(j-1)!}{(n+1)!} \langle \sigma, \partial \tau \rangle \quad \forall \sigma \in \tilde{E}_i, \tau \in \tilde{E}_j(i, j \geq 1). \]
As a consequence, the kernel of $(,)$ is the $\partial$-closed part of $\tilde{E}_n$.

We can now prove that $\omega^*_{W|U}$ is non-degenerate at $x$. The symplectic form $\varpi$ at $x$ gives us a non-degenerate pairing $\varpi_x : W \otimes W \to \mathbb{R}$ of degree $k$, and $\omega^*_x$ on
\[ T_x K^*_k(W|U) \cong (\tilde{E}_k \otimes W)^{1,cl_{\text{tot}}} \]
is, by definition, the restriction of $(, \otimes \varpi_x) : (\tilde{E}_k \otimes W)^{1,cl_{\text{tot}}} \subset \tilde{E}_k \otimes W$. The kernel of $(, \otimes \varpi_x)$ is the $\partial$-closed part of $\tilde{E}_k \otimes W$ and $(\tilde{E}_k \otimes W)^{\text{cl}_{\text{tot}}}$ is its complement (as $d_{\text{tot}} \theta + \partial d_{\text{tot}} = k + 1$), hence $(, \otimes \varpi_x)$ is non-degenerate on $(\tilde{E}_k \otimes W)^{1,cl_{\text{tot}}}$, as we wanted to show. \qed
11. $A_\infty$-Functiorality of $\omega^s$

If $Y$ is a simplicial set and $\tau \in Y_n$, let $\hat{\tau} : \Delta[n] \to Y$ be the morphism sending the non-degenerate $n$-simplex of $\Delta[n]$ to $\tau$. If $X$ is another simplicial set and if $\sigma \in (\Delta[n] \times X)_N$ (for some $N \in \mathbb{N}$), let

$$\sigma^2 : Y_n \to Y_N \times X_N$$

be the map defined via

$$\sigma^2(\tau) = (\hat{\tau} \times id_X)(\sigma).$$

If $K$ is a simplicial manifold, we thus get the map (where we understand $X_N$ as discrete)

$$\sigma^* := (\sigma^2)^* : \Omega(K_N \times X_N) = \Omega(K_N) \times X_N \to \Omega(K_n).$$

More generally, if $c = \sum_i a_i\sigma_i$ ($a_i \in \mathbb{R}$) is a $N$-chain in $\Delta[n] \times X$, we set

$$c^* := \sum_i a_i\sigma_i^*.$$ 

By construction we have

$$ (\partial c)^* \alpha = \delta(c^* \alpha)$$

for every $\alpha \in \Omega(K_n)$.

Let us now consider the special case $X = I^m$ and the chain

$$c_{m,n} = [I^m \times \Delta^n] \in C_{m+n}(\Delta[n] \times I^m)$$

giving the fundamental class (rel boundary) of $I^m \times \Delta^n$; it is the signed sum of all non-degenerate $m + n$-simplices of $\Delta[n] \times I^m$ with the signs given by comparing with the orientation of the space $I^m \times \Delta^n$. Let us use the notation

$$\mathcal{J}^m := c_{m,n} : \Omega(K_{n+m} \times (I_{n+m})^m) \to \Omega(K_n).$$

Proposition 11.1. For any $\alpha \in \Omega(K_{n+m} \times (I_{n+m})^m)$ we have

$$ (\delta, \mathcal{J}^m - (-1)^m \mathcal{J}^m \delta) \alpha = \sum_{r=1}^{m} (-1)^{r-1} \left( \mathcal{J}^m - (-1)^{r-1} \mathcal{J}^{m-1} \alpha \right) \in \Omega(K_{n+1})$$

where $\alpha|_{I_0}, \alpha|_1 \in \Omega(K_n \times (I_m)^{m-1})$ is obtained from $\alpha$ by restricting the $r$'th $I_n$ to 0 and to 1 respectively.

Proof. The boundary of $c_{m,n} = [I^m \times \Delta^n] \in C_{m+n}(\Delta[n] \times I^m)$ is

$$\partial c_{m,n} = \partial[I^m] \times [\Delta^n] + (-1)^m[I^m] \times \partial[\Delta^n],$$

so Equation (26) gives us the identity (for every $\alpha \in \Omega(K_{n+m} \times (I_{m+n})^m)$)

$$\delta(c_{m,n} \alpha) = \sum_{r=1}^{m} (-1)^{r-1} \left( c_{m-1,n}^* \alpha \right) + (-1)^m c_{m,n-1}^* \delta \alpha$$

which is the identity we wanted to prove. \qed

We can now deal with the problem of functoriality of $\omega^s$. Let us consider the category $C_{\infty,k}$ of $k$-presymplectic NQ-manifolds of the form $W|_{U^k}$. Morphisms of this category are NQ-maps $\phi : W|_{U} \leadsto W'|_{U'}$ such that $\phi^* \varpi_{W}|_{U'} = \varpi_{W|_{U'}}$.

For every chain of morphisms of $C_{\infty,k}$ (i.e. for every simplex of the nerve of $C_{\infty,k}$)

$$f := (W_0|_{U_0} \xrightarrow{\phi_0} W_1|_{U_1} \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_n} W_n|_{U_n})$$

let us define $\omega^s_{W_n|_{U_n}}(f) \in \Omega^2(\mathcal{J}_{K_{n-1}}(W_0|_{U_0}))$ as $\mathcal{J}^n$ of the pullback of $\omega^s_{W_n|_{U_n}}$ via the composition (with $n = 2$ as an illustration)

$$K^s(W_0|_{U_2}) \times (I_2)^2 \xrightarrow{\iota^s \times id} K_{K^s(W_1|_{U_1}) \times (I_1)^2} \xrightarrow{K_{\phi^s} \times id} K_{K_{\phi^s}(W_2|_{U_2}) \times (I_2)^2} \xrightarrow{K_{\phi^s} \times id} K_{K^s(W_2|_{U_2}) \times (I_2)^2}$$
Theorem 11.1. The closed 2-forms $\omega^s(f)$ satisfy the identities

\begin{equation}
\delta \omega^s_{\Omega|\Omega}(\phi_0, \phi_1, \ldots, \phi_{n-1}) = \omega^s_{\Omega|\Omega}(\phi_0 \circ \phi_2, \ldots, \phi_{n-1}) - \omega^s_{\Omega|\Omega}(\phi_0, \phi_1 \circ \phi_3, \ldots, \phi_{n-1}) + \cdots + (-1)^{n-1} \omega^s_{\Omega|\Omega}(\phi_0, \phi_1, \ldots, \phi_{n-2}) - \sum_{i=1}^{n} (-1)^{i-1} \mathcal{F}^{-1}(K^s(\phi_0, \ldots, \phi_{i-1})^* \omega^s_{\Omega|\Omega}(\phi_i, \ldots, \phi_{n-1})),
\end{equation}

(28)

\begin{equation}
\omega^s_{\Omega|\Omega}(\phi_0, \phi_1, \ldots, \phi_{n-1}) = 0 \text{ for } n \geq k,
\end{equation}

(29)

\begin{equation}
\omega^s_{\Omega|\Omega}(\phi_0, \phi_1, \ldots, \phi_{n-1}) = 0 \text{ if some } \phi_i = id.
\end{equation}

(30)

Proof. Let $q^*_n(\phi_0, \ldots, \phi_{n-1}): K^s_n(W_0|U_0) \times (I^*_n) \to K_{*|\Omega}^b(W_n|U_n)$ be the simplicial map given by the composition of the snake $\mathcal{S}_n^{1}$, so that

\begin{equation}
\omega^s_{\Omega|\Omega}(\phi_0, \phi_1, \ldots, \phi_{n-1}) = \mathcal{F}^n(q^*_n(\phi_0, \phi_1, \ldots, \phi_{n-1})^* \omega^b_{\Omega|\Omega}).
\end{equation}

For convenience, let us use the notation

\begin{equation}
q^*_n(\phi_0, t_1, \phi_1, t_2, \ldots, t_{n-1}, \phi_{n-1}, t_n): K^s_n(W_0|U_0) \to K_{*|\Omega}^b(W_n|U_n)
\end{equation}

with $t_i \in I^*_n$. Similarly to (28) we have the identities

\begin{equation}
q^*_n(\phi_0, t_1, \ldots, t_{n-1}, \phi_{n-1}, t_n) =
\begin{cases}
q^*_n(\phi_0, t_1, \ldots, t_{n-1}, \phi_{n-1}, t_n) & \text{if } \phi = id \text{ if } \phi = id, 1 \leq i \leq n-1 \\
q^*_n(\phi_0, t_1, \ldots, t_{i-1}, \phi_i \circ \phi_{i-1}, \ldots, \phi_{n-1}, t_n) & \text{if } t_i = 0 \text{ identically, } 1 \leq i \leq n-1 \\
(\phi_0, t_1, \ldots, \phi_{n-1}, t_n) & \text{if } t_n = 0 \text{ identically} \\
q^*_n(\phi_0, t_1, \ldots, t_{i-1}, \phi_i, t_n) & \text{if } t_i = 1 \text{ identically, } 1 \leq i \leq n.
\end{cases}
\end{equation}

(31)

Since $\delta \omega^b_{\Omega|\Omega} = 0$, we have

\begin{equation}
\delta \omega^s_{\Omega|\Omega}(\phi_0, \phi_1, \ldots, \phi_{n-1}) = (\delta \mathcal{F}^{-1} - (-1)^n \mathcal{F}^n)(q^*_n(\phi_0, \phi_1, \ldots, \phi_{n-1})^* \omega^b_{\Omega|\Omega})
\end{equation}

and Proposition 11.2 and the last three cases of Equation (31) give us Equation (28). Equation (30) follows from the first two cases of Equation (31) and finally Equation (29) is obvious. \hfill \Box

Remark (Courant algebroids and Dirac structures). When $k = 1$ then, by (29), $\omega^s_{\Omega|\Omega}(\phi_0) = 0$, and Equation (28) with $n = 1$ becomes

\begin{equation}
\omega^s_{\Omega|\Omega} = K^s(\phi_0)^* \omega^s_{\Omega|\Omega}.
\end{equation}

The first non-trivial case is thus $k = 2$. Equation (28) for $n = 1$ is

\begin{equation}
\delta \omega^s_{\Omega|\Omega}(\phi_0) = \omega^s_{\Omega|\Omega} - K^s(\phi_0)^* \omega^s_{\Omega|\Omega},
\end{equation}

and for $n = 2$ (using $\omega^s_{\Omega|\Omega}(\phi_0, \phi_1) = 0$)

\begin{equation}
\omega^s_{\Omega|\Omega}(\phi_0) = \omega^s_{\Omega|\Omega}(\phi_0 \circ \phi_0) + K^s(\phi_0)^* \omega^s_{\Omega|\Omega}(\phi_1) = \mathcal{F} K^s(\phi_0, \phi_0)^* \omega^s_{\Omega|\Omega}.
\end{equation}

(32a)

(32b)

In particular if we have a Courant algebroid over a manifold $M$, if $Z$ is the corresponding 2-symplectic NQ-manifold (and thus $0 \cdot Z = M$) and if $W_i|U_i$ are (isomorphic to) local pieces of $Z$ and $\phi_i$’s are their identifications on the overlaps, Equations (32) show us in what sense the symplectic forms $\omega^s_{\Omega|\Omega}$ agree up to a coherent homotopy.

If $Y \subset Z$ is a Lagrangian NQ-submanifold (i.e. a (generalized) Dirac structure of the Courant algebroid), if $W_0|U_0$ is a local piece of $Y$, $W_1|U_1$ a local piece of $Z$, and $\phi_0 : W_0|U_0 \to W_1|U_1$ the inclusion $Y \subset Z$, then we have $\omega^s_{\Omega|\Omega}(\phi_0) = 0$ and Equation (32a) becomes

\begin{equation}
\delta \omega^s_{\Omega|\Omega}(\phi_0) = -K^s(\phi_0)^* \omega^s_{\Omega|\Omega}.
\end{equation}

The closed 2-form $K^s(\phi_0)^* \omega^s_{\Omega|\Omega}$ thus doesn’t have to vanish, i.e. $K^s(\phi_0) : K^s_2(W_0|U_0) \to K^s_1(W_1|U_1)$ is not necessarily a Lagrangian embedding, however $K^s(\phi_0)^* \omega^s_{\Omega|\Omega}$ is homotopic to zero.
Theorem 11.1 can be interpreted as $A_{\infty}$-functoriality. Let $\mathcal{C}$ be the category of the NQ-manifolds of the form $W|_{U}$. Let $F(W|_{U})$ denote the cochain complex
\[ F(W|_{U}) := (\Omega^{2,cl}(K_{s}^{s}(W|_{U})), \delta) \]
and for
\[ f := \left( W_{0}|_{U_{n}} \xrightarrow{\phi_{0}} W_{1}|_{U_{1}} \xrightarrow{\phi_{1}} \cdots \xrightarrow{\phi_{n-1}} W_{n}|_{U_{n}} \right) \]
let
\[ F(f) := F(W|_{U}) \]
Proposition 11.2. $F$ is a (strictly unital) contravariant $A_{\infty}$-functor from $\mathcal{C}$ to the category of cochain complexes, i.e. $\deg F(\phi_{0}, \ldots, \phi_{n-1}) = 1 - n$, $F(id) = id$ and $F(\phi_{0}, \ldots, \phi_{n-1}) = 0$ if $n \geq 2$ and some $\phi_{i} = id$, and finally
\[ \delta \circ F(\phi_{0}, \ldots, \phi_{n-1}) = (-1)^{1-n} F(\phi_{0}, \ldots, \phi_{n-1}) \circ \delta = \sum_{p=1}^{n-2} (-1)^{p} F(\phi_{0}, \ldots, \phi_{p-1}) \circ F(\phi_{p}, \ldots, \phi_{n-1}). \]

Proof. The claim follows from the properties of $K_{s}(f)$ listed in Equation (23) and from the property of $F$ given in Proposition 11.1 \qed

Let now $\mathcal{C}^{\infty}_{x,k}$ be the cocone category of $\mathcal{C}_{x,k}$. By definition, it contains $\mathcal{C}_{x,k}$ as a full subcategory, a unique additional object $\ast$, and a unique additional morphism $X \to \ast$ for any $X \in \mathcal{C}_{x,k}$. Let us extend $F$ from $\mathcal{C}$ to $\mathcal{C}^{\infty}_{x,k}$ as follows: for any $f \in N_{p}^{\ast} \mathcal{C}_{x,k}$ ($p = 0, 1, \ldots$),
\[ f = (X_{0} \to X_{1} \to \cdots \to X_{p}), \]
let
\[ \hat{F}(f) = F(f) \text{ if } X_{p} \neq \ast \]
\[ \hat{F}(\ast) = \mathbb{R}[-k] \]
\[ \hat{F}(f) = \omega_{X_{p}}^{s}(X_{0} \to X_{1} \to \cdots \to X_{p-1}) \text{ if } p > 0 \text{ and } X_{p} = \ast, \]
where $\omega_{X_{p}}^{s}(X_{0} \to \cdots \to X_{p-1}) \in \Omega^{2,cl}(K_{s}^{s}(X_{0}))$ is understood as a map $\mathbb{R}[-k] \to \Omega^{2,cl}(K_{s}^{s}(X_{0}))$ of degree $-p$.

Theorem 11.2. $\hat{F}$ is a (strictly unital) contravariant $A_{\infty}$-functor from $\mathcal{C}^{\infty}_{x,k}$ to the category of cochain complexes.

Proof. The claim is a combination of Theorem 11.1 and Proposition 11.2 \qed

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