Physics and Derivatives: Effective-Potential Path-Integral Approximations of Arrow-Debreu Densities

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ABSTRACT: The authors show how effective-potential path-integrals methods, stemming from a simple and nice idea originally due to Feynman and successfully employed in physics for a variety of quantum thermodynamics applications, can be used to develop an accurate and easy-to-compute semianalytical approximation of transition probabilities and Arrow-Debreu densities for arbitrary diffusions. The authors illustrate the accuracy of the method by presenting results for the Black-Karasinski and the GARCH linear models, for which the proposed approximation provides remarkably accurate results, even in regimes of high volatility, and for multyear time horizons. The accuracy and the computational efficiency of the proposed approximation make it a viable alternative to fully numerical schemes for a variety of derivatives pricing applications.

KEY FINDINGS

- The connection between Feynman’s path-integrals and the formalism of derivatives pricing provides powerful computational tools for financial applications.
- An “effective potential” path-integral formalism of quantum statistical mechanics, employed over the years for the study of a number quantum systems, can be employed to develop semianalytical approximations of transition probabilities and Arrow-Debreu prices for nonlinear diffusion.
- The accuracy and the computational efficiency of the proposed approximation make it a viable alternative to fully numerical schemes for a variety of derivatives pricing applications.

TOPICS: Derivatives, options, credit default swaps*

Path integrals (Feynman, Hibbs, and Styer 2010), also known as Wiener integrals in stochastic calculus (Kac 1966; Wiener 1921a, 1921b), are a well-established mathematical formalism that has been used for a long time in physics to develop accurate approximations and efficient computational techniques (Kleinert 2009).

Among these techniques, so-called semiclassical methods (Kleinert 2009) play a central role. These approximations can be developed in several ways, which, although sharing the same limiting behavior, lead to genuinely different results. The renowned Wentzel-Kramers-Brillouin approximation (Brillouin 1926; Kramers 1926; Wentzel 1926), which is equivalent to to a saddle-point approximation of the path integral (Kakushadze 2015; Kleinert 2009; Rajaraman 1975), and the Wigner-Kirkwood expansion (Fujiwara, Osborn, and Wilk 1982;...
Hillery et al. 1984; Kirkwood 1933; Wigner 1932) are well-known theoretical devices in this context.

A prominent role among semiclassical approximations is played by so-called effective potential methods (Feynman 1998; Feynman, Hibbs, and Styer 2010), which is based, borrowing from renormalization group ideas, on “integrating out” the fluctuations around a “classical” trajectory. Although exact in principle, the calculation can be performed only at some level of approximation, using a perturbation scheme in which the choice of the unperturbed system plays a crucial role in the quality of the approximation.

A particularly successful effective potential approximation is the one stemming from a simple and nice idea originally conceived by Feynman (Feynman, Hibbs, and Styer 2010) and independently developed by Giachetti and Tognetti (1985) and Feynman and Kleinert (1986) (GTFK), which is based on a self-consistent (nonlocal) harmonic approximation of the effective potential in a sense that will become clear in the following sections.

Basically, the GTFK effective potential is employed within the usual classical formalism but accounts for the quantum nature of a system through the suitable renormalization parameters it contains; hence, the approximation does not immediately lead to final results but reduces a quantum-mechanical problem to a classical one that can be treated by any known method. Physicists know that this amounts to an enormous simplification.

The most appealing aspect is that the classical behavior is fully accounted for by the GTFK potential, so it opened the way to face challenging quantum systems whose classical analogues were known to be characterized by peculiar nonlinear excitations, for example, those dubbed solitons in one dimension (1D) or vortices in two dimensions (2D). The latter are the “engine” of a topological phase transition, for the study (Kosterlitz and Thouless, 1973) of which Michael Kosterlitz and David Thouless (KT) earned the Nobel Prize in 2016. By the GTFK method, it has been possible to establish that some real magnetic compounds do show a KT transition.

Other quantum systems that were successfully treated by (suitable generalizations of) the same method, are frustrated antiferromagnets, for example, the so-called 2D $J_1$-$J_2$ model (Capriotti et al. 2004), and 2D Josephson-junction arrays, which can be artificially fabricated, also with the inclusion of resistors; in the latter case, the effective potential could naturally be extended to account for the related dissipative coupling with the environment (Cuccoli et al. 1997).

The connection between the so-called Euclidean path integrals (Feynman, Hibbs, and Styer 2010; Kleinert 2009), namely, those employed to describe the thermodynamics of quantum systems, and the formalism of derivatives pricing has also been known since the seminal papers of (Linetsky 1997) and (Bennati, Rosa-Clot, and Taddei 1999) (see also the recent review by Kakushadze 2015). In particular, it is a known fact that a variable following a nonlinear diffusion process can be described by the same formalism used to model the finite-temperature properties of a quantum particle in a potential that is linked to the drift of the diffusion, where the role of the mass is played by the inverse of the volatility squared, that of the temperature by the inverse of time, and that of quantum fluctuations by the Brownian noise (Bennati, Rosa-Clot, and Taddei 1999). The interest in financial engineering for path-integral formalism mainly stems from the possibility of developing accurate approximation schemes that are not otherwise available, or known, in traditional formulations of stochastic calculus (Bennati, Rosa-Clot, and Taddei 1999; Capriotti 2006; Kakushadze 2015).

In this article, we consider the application of the GTFK method to generalized short-rate models of the form $r = r(Y)$ with $Y$ following the nonlinear diffusion process specified by the following stochastic differential equation (SDE)

$$dY_t = \mu(Y_t) dt + \sigma(Y_t) dW_t,$$

for $t > 0$, where $\mu(Y)$ and $\sigma(Y)$ are the drift and volatility functions, respectively, $Y_0 = y_0$, and $W_t$ is a standard Brownian motion.

Short-rate models are of paramount importance in financial modeling, providing the foundation of many approaches used for the pricing of both interest rate and credit derivatives (Andersen and Piterbarg 2010; O’Kane 2010). In particular, celebrated affine models (Duffie, Pan, and Singleton 2000) like those of Vasicek (1977), Hull and White (1990), and Cox et al. (1985), play a prominent role. This is mainly due to their analytical tractability, which allows one to derive closed-form expressions for fundamental building blocks like zero-coupon bonds or, in the context of default intensity models (O’Kane 2010), survival probabilities.

Unfortunately, the availability of closed-form solutions often comes at the price of less than realistic
properties of the underlying rates. For instance, Gaussian models such as those of Vasicek (1977) and Hull and White (1990), when calibrated to financial data, typically imply that rates can assume negative values with sizable probabilities. While this may not be a problem for interest-rate models, especially in a low-interest-rate environment, it is not consistent with the absence of arbitrage in the context of default intensity models (O’Kane 2010). On the other hand, square-root diffusions such as that of Cox, Ingersoll, and Ross (1985)—although guaranteed to be non-negative—may give rise to distributions of the par swap rate (see Andersen and Piterbarg 2010; Li, Mercurio, and Resnick 2018), which do not admit values below a finite threshold and may therefore be considered unrealistic.

Unfortunately, more realistic models lack the same degree of analytical tractability as that shown by affine models. As a result, although widely used in practice, their implementations rely on computationally intensive partial differential equations (PDEs) or Monte Carlo methods for the calculation of bond prices or survival probabilities. This is particularly onerous in the context of multifactor problems, notably the ones involving the calculation of valuation adjustments (Gregory 2010), that are currently very prominent in financial engineering. Indeed, these applications require Monte Carlo simulations and, for example, the valuation of conditional bond prices or survival probabilities at different points on the simulated paths, which are expensive to compute for models that lack closed-form solutions for these quantities. In this context, reliable analytical approximations are particularly important for reducing the numerical burden associated with these computations.

More specifically, in this article we will focus on developing approximations of the so-called (generalized) Arrow-Debreu (AD) densities, see (Andersen and Piterbarg 2010; Karatzas and Shreve 1991), also known as Green’s functions, which are the fundamental building blocks for pricing contingent claims. These are defined, in this setting, as

\[ \int_A dy_T \psi^y_\lambda(y_T, y_0, T) \equiv \mathbb{P}[Y_T \in A | Y_0 = y_0]. \] (3)

The price at time \( t = 0 \) of a European option with expiry \( T \) and payout of the form \( P(t) \),

\[ V(0) = \mathbb{E}[e^{-\int_0^{t_T} \lambda(r_u) du} P(t)], \] (4)
can be obtained by integrating the product of the payout function and the \( \lambda = 1 \) AD density over all the possible values of the short rate at time \( T \), namely,

\[ V(0) = \int dy_T \psi^y_1(y_T, y_0, T)P(y_T), \] (5)

where the integration is performed over the range of the function \( y_0 = r^*(t_0) \). In particular, the moment generating function for the random process \( \int_0^{t_T} du r_u \) can be obtained for \( P \equiv 1 \),

\[ Z_\lambda(t_0, T) = \int dy_T \psi^y_\lambda(y_T, y_0, T), \] (6)

which, for \( \lambda = 1 \), gives the value at time \( t = 0 \) of a zero-coupon bond with maturity \( T \) (Andersen and Piterbarg 2010). In the context of default intensity models, where the default of a firm is modeled by the first arrival of a Poisson process with time-dependent intensity \( \lambda \), (O’Kane 2010), Equation (6) for \( \lambda = 1 \) represents the survival probability up to time \( T \), conditional on survival up to time \( t = 0 \). This is the fundamental building block for the evaluation of cash flows that are contingent on survival or default, see O’Kane (2010).

The structure of the article is as follows. We start by reviewing the formalism of the GTFK effective potential method in the context of the path-integral formulation of quantum statistical mechanics. We then make the connection between the formalism used in quantum physics and the one used in finance by reviewing the path-integral formulation of AD densities for nonlinear diffusion, and we show how the GTFK approximation can be used in the mathematical setting of stochastic calculus to develop a semianalytical approximation for the generalized AD densities (2), and zero-coupon bonds (6) for nonlinear diffusion of the form (1). Remarkably, the GTFK method, yielding exact results in the limit of zero volatility and time to maturity as any semiclas-sical approximation, is also exact whenever the drift potential is quadratic, which means it is exact, as we
will recall, for the Vasicek (1977) and quadratic models (Kakushadze 2015). We finally illustrate the remarkable accuracy of the GTFK method for models for which an analytical solution is not available via the application to the so-called Black-Karasinski (BK) model (Black and Karasinski 1991) and the so-called GARCH linear SDE (Capriotti, Jiang, and Shaimerdenova 2019; Li, Mercurio, and Resnick 2018), both of particular relevance for the valuation of credit derivatives.

**EFFECTIVE POTENTIAL APPROXIMATION IN QUANTUM STATISTICAL MECHANICS**

We start by recalling the path-integral formalism of quantum thermodynamics for a nonrelativistic particle of mass \( m \) described by the standard Hamiltonian

\[
\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}),
\]

where \( \hat{x} \) and \( \hat{p} \) are the canonical coordinate and momentum operators such that \([\hat{x}, \hat{p}] = i\hbar\), with \( \hbar \) the reduced Planck’s constant, and where \( V(\hat{x}) \) is the potential the particle is subject to.

The quantum thermodynamical properties of the particle at temperature \( T \) can be described by the density matrix (Feynman, Hibbs, and Styer 2010),

\[
\hat{\rho} = e^{-\beta \hat{H}},
\]

where \( \beta = 1/k_B T \), with \( k_B \) being Boltzmann’s constant. The elements of the density matrix, in the coordinate representation, can be expressed in terms of Feynman’s path integral (Feynman, Hibbs, and Styer 2010) as

\[
\rho(x_T, x_0, T) = \langle x_T | \hat{\rho} | x_0 \rangle = \int_{x(0)=x_0}^{x(T)=x_T} D[x(t)] e^{\beta S[x(t)]},
\]

where the path integration, \( \int_{x(0)=x_0}^{x(T)=x_T} D[x(t)] \ldots \), is defined over all paths \( x(t) \) such that \( x(0) = x_0 \) and \( x(T) = x_T \), with \( T = \beta \hbar \) the so-called Euclidean time and the functional

\[
S[x(t)] = -\frac{1}{\hbar} \int_0^T dt \left[ \frac{m}{2} \dot{x}^2(t) + V(x(t)) \right],
\]

is the Euclidean action. The functional integration in Equation (9) is formally defined as the limit for \( N \to \infty \) of the expression

\[
\left( \frac{m}{2\pi \hbar \Delta t} \right)^{N/2} \int \ldots \prod_{i=1}^{N-1} dx_i \prod_{i=1}^{N} \exp[S(x_i, x_{i-1})],
\]

with \( \Delta t = T/N \), \( x_N \equiv x_T \) and

\[
S(x_i, x_{i-1}) = -\frac{\Delta t}{\hbar} \left[ \frac{m}{2} \left( x_i - x_{i-1} \right)^2 + V(\left( x_i + x_{i-1} \right) / 2) \right].
\]

Although the evaluation of the path integral in Equation (9) is possible in just a few cases for simple potentials, the formalism allows for new kinds of approximations. In particular, here we pursue an approximation stemming on an idea originally due to Feynman, which consists in classifying the paths according to an equivalence relation, and consequently decompose the integral into a first sum over all paths belonging to the same class, and a second one over all the equivalence classes. In particular, equivalent paths are those who share the average point, defined as the functional

\[
\bar{x}[x(t)] = \frac{1}{T} \int_0^T dx(t),
\]

so that each equivalence class is labeled by a real number \( \bar{x} \) representing the common average point and we can factor out in Equation (9) an ordinary integral over \( \bar{x} \), namely,

\[
\rho(x_T, x_0, T) = \int d\bar{x} \rho_{\bar{x}}(x_T, x_0, T),
\]

where the reduced density matrix

\[
\rho_{\bar{x}}(x_T, x_0, T) = \int_{\bar{x}(0)=\bar{x}_0}^{\bar{x}(T)=\bar{x}_T} D[x(t)] \delta \left[ \bar{x} - \frac{1}{T} \int_0^T dx(t) \right] e^{\beta S[\bar{x}(t)]},
\]

represents the contribution to the path integral in Equation (9) that comes from those paths that have \( \bar{x} \) as average point.

As the path integration has been reduced to paths belonging to the same class, we can develop a specialized approximation for each class. In particular, the GTFK method approximates the potential in the action Equation (10) with a quadratic potential in the displacement from the average point \( \bar{x} \),

\[
V_{\bar{x}}(x) = w(\bar{x}) + \frac{m}{2} \omega^2(\bar{x})(x - \bar{x})^2,
\]
where the parameters $\omega(x)$ and $\omega^2(x)$ are to be optimized so that the trial reduced density matrix

$$
\bar{\rho}_x(x_T, x_0, T) = \int_{x(0)=x_0}^{x(T)=x_T} D[x(t)] \delta \left( x - \frac{1}{T} \int_0^T dt \, x(t) \right) e^{S_x[x(t)]}, 
$$

(17)

with the action given by

$$
S_x[x(t)] = -\frac{1}{\hbar} \int_0^T dt \left[ \frac{m}{2} \dot{x}^2(t) + V_x(x(t)) \right],
$$

(18)

best approximates the reduced density matrix in Equation (15). Note that one does not need to include a linear term in the trial potential (16), as it would give a vanishing contribution to the trial action (18), because of the very definition of $\bar{x}$.

The path integral in Equation (15), corresponding to the harmonic action (18) can be worked out analytically (Cuccoli, Giachetti, et al. 1995), giving

$$
\bar{\rho}_x(x_T, x_0, T) = \frac{m}{\sqrt{2\pi\hbar^2}} e^{-\beta f(x)} \sinh f 
	imes \frac{1}{\sqrt{2\pi\alpha}} \exp \left[ -\frac{\xi^2}{4\hbar} - m \omega \coth f (x_T - x_0)^2 \right],
$$

(19)

where $\xi = (x_T + x_0) / 2 - \bar{x}$, $f(x) = \beta \hbar \omega / 2$ and

$$
\alpha(x) = \frac{\hbar}{2m\omega(x)} \left( \coth f(x) - \frac{1}{f(x)} \right);
$$

(20)

for the sake of conciseness, above and in the following the argument ($\bar{x}$) of the quantities $\alpha$, $f$, and $\omega$ is often omitted. The diagonal elements of the reduced density matrix read in particular

$$
\bar{\rho}_x(x_0, x_0, T) = \frac{m}{\sqrt{2\pi\hbar^2}} e^{-\beta \omega(x)} \frac{f}{\sinh f} 
	imes \frac{1}{\sqrt{2\pi\alpha}} \exp \left[ -\frac{\xi^2}{4\hbar} \right],
$$

(21)

taking a suggestive form in terms of a Gaussian distribution with mean $\bar{x}$ and variance $\alpha(x)$, describing the fluctuations around the average point. In particular, the so-called partition function, $Z$ (Feynman 1998), assumes the classical form

$$
Z \equiv \int d\bar{x} \int dx_0 \rho_x(x_0, x_0, T) = \frac{m}{\sqrt{2\pi\hbar^2}} \int d\bar{x} e^{-\beta \omega(x)},
$$

(22)

where the GTFK effective potential reads

$$
V_{\text{eff}}(\bar{x}) = \omega(\bar{x}) + \frac{1}{\beta} \ln \sinh f(\bar{x}) / f(\bar{x}).
$$

(23)

To close the approximation, we still need to devise an optimization scheme for the parameters $\omega(x)$ and $\omega^2(x)$ in Equation (16). For example, we could simply identify the trial potential (16) with the expansion of $V(\bar{x})$ up to second order by setting $\omega(x) = V(\bar{x})$ and $\omega^2(x) = V''(\bar{x})$ for any $\bar{x}$. However, this approximation has limitations. For instance, it can happen that $V''(\bar{x})$ is negative: in this case, writing $f = \beta \hbar / 2$ as $f = \beta \phi / \alpha$, can be analytically continued as $\omega = \beta \hbar / 4m (1 / \phi^2 - \cot \phi / \phi)$, which diverges to $+\infty$ for $\phi \rightarrow \pi^-$ (or $f^2 \rightarrow -\pi$) and is negative for $\phi > \pi$ ($f^2 < -\pi$). As a consequence, if $\omega^2(\bar{x})$ is negative, for sufficiently large time horizons $T$ we have $f^2 < -\pi^2$ and $\alpha(x) < 0$. In this situation, the reduced density matrix (19) is not well defined and the approximation breaks down.

A more robust approximation can be devised by observing that the Gaussian density $\rho_x(x_0, x_0, T)$ has to be close to $\rho_x(x_0, x_0, T)$, so that $V_x(x)$ must approximate $V(\bar{x})$ not only at $\bar{x}$: this is accomplished by requiring the equality of the Gaussian averages of the true and the trial potentials, and of their derivatives up to the second one

$$
\langle \langle V(\bar{x} + \xi) \rangle \rangle = \langle \langle V_x(\bar{x} + \xi) \rangle \rangle
$$

(24)

$$
\langle \langle V''(\bar{x} + \xi) \rangle \rangle = \langle \langle V''_x(\bar{x} + \xi) \rangle \rangle = m \omega^2(\bar{x}),
$$

(25)

with the short-hand notation

$$
\langle \langle F(\bar{x} + \xi) \rangle \rangle = \frac{1}{\sqrt{2\pi\alpha(\bar{x})}} \int_{-\infty}^{\infty} d\xi e^{-\xi^2 / 2\alpha(\bar{x})} F(\bar{x} + \xi)
$$

(26)

and $\alpha(x)$ given by Equation (20). The preceding equations impose that the expectation value according to the Gaussian probability distribution in Equation (21) of the potential and of its second order expansion are in
agreement with each other, for every value of $\bar{\pi}$. Under the GTFK approximation, the quantum effects are embedded in the notion of the effective potential (23), which is a renormalized version of the potential $V(\pi)$, where $a(\bar{\pi}) \equiv \langle \langle \dot{\pi}^2 \rangle \rangle$—representing the average quadratic fluctuations around $\bar{\pi}$ because of the quantum effects—is the renormalization parameter. Note that Equation (25) is self-consistent, meaning that its solution $\omega^2(\bar{\pi})$ in turn determines the variance (20).

It can be shown that the preceding determination of the parameters $w(\bar{\pi})$ and $\omega(\bar{\pi})$ satisfies a variational principle based on the so-called Jensen-Feynman inequality, $Z \geq Z_0 e^{b(S_0 - S_1)}$, where the functional average is taken with whatever trial action $S_0$, $Z_0$ being the corresponding partition function. Indeed, taking $S_0 = S_2$ and maximizing the right-hand side of the inequality, one just finds Equations (24) and (25).

The GTFK method becomes exact in both limits of high-temperature $\beta \rightarrow 0$ and vanishing quantum effects $\hbar/m \rightarrow 0$, for which the parameter $\alpha$ vanishes as $\beta \hbar^2/12m$ and the effective potential (23) coincides with the exact classical potential:

$$V_{\text{eff}}(\bar{\pi}) = V(\bar{\pi}) + \frac{\beta \hbar^2}{24m} V''(\bar{\pi}) + O(\beta^2 \hbar^4/m^2),$$  \hspace{1cm} (27)

so that the partition function in Equation (22) coincides with the well-known exact classical result (Feynman 1998).

The effective potential can be compared with the semiclassical effective potential introduced by Wigner and Kirkwood (Fujiiwara, Osborn, and Wilk 1982; Hillery et al. 1984; Kirkwood 1933; Wigner 1932) (WK), that was substantially found as an expansion in $\beta$ and $\hbar$ of the exact classical effective potential $V_{\text{eff}}$, defined such that the quantum density bears the classical form

$$\rho(\pi_0, x_0, T) \equiv \frac{1}{Z} e^{-\beta V_{\text{eff}}(\pi_0)}. \hspace{1cm} (28)$$

The WK expansion is exact in principle, but only the first few terms are practically affordable, and while lowering the temperature, all terms soon diverge. One has indeed (Jizba and Zatloukal 2014)

$$V_{\text{eff}}(\pi_0) = V(\pi_0) + \frac{\beta \hbar^2}{12m} V''(\pi_0) - \frac{\beta^2 \hbar^2}{24m} V''(\pi_0) + \ldots$$ \hspace{1cm} (29)

This apparently disagrees with the expansion in Equation (27) because the comparison is a little subtle: $V_{\text{eff}}$ should not be compared directly with $V_{\text{eff}}^0$, because, to obtain $\rho(\pi_0, x_0, T)$, one cannot integrate over $x_0$ as done in Equation (22) but rather over $\bar{\pi}$. Accounting for this (Kleinert 1986), the WK and the GTFK effective potentials do agree (Cuccoli et al. 1992; Vaia and Tognetti 1990). Similarly, GTFK is distinct from the exponential power series expansion of Makri and Miller (1989), previously applied successfully in the financial context (Capriotti 2006; Capriotti, Jiang, and Shaimerdenova 2019; Stelhlikova and Capriotti 2014), and which we use as one of the benchmarks when discussing our numerical results.

With respect to these approaches, the GTFK method has a strong advantage: it still gives a meaningful representation of the thermodynamics down to zero temperature, where it is equivalent to the so-called self-consistent harmonic approximation (Koehler 1966a, 1966b), that was initially applied to quantum crystal lattices. Therefore, increasing the temperature from zero, the accuracy increases more and more, because the renormalization parameter $a(\bar{\pi})$ decreases. The price to pay is that one still has to solve the classical problem with the effective potential, but this is nevertheless a huge simplification, especially in view of the plethora of methods that have been developed to treat classical systems. In particular, because the nonlinear character of the potential is kept, the GTFK approach allows for the study of quantum systems whose classical counterpart is characterized by nonlinear excitations (solitons, vortices) and constitutes a much simpler and clearly interpretable alternative to heavy numerical approaches, such as Quantum Monte Carlo.

The GTFK approach is also distinct from other semiclassical path-integral approximations, like the Wentzel-Kramers-Brillouin (Brillouin 1926; Kramers 1926; Wentzel 1926) or the equivalent saddle-point approximations (Kukushadze 2015; Kleinert 2009; Rajaraman 1975), which are based on a power-series expansion of the action around the classical trajectory $x(t)$ rather than around the average point, that is, the density matrix, Equations (9) and (10), is expressed as

$$\rho(x_T, x_0, T) = e^{S[x_0(t)]} \int_{[\bar{x}(t)=0]} D[\bar{x}(t)] e^{S[\bar{x}(t)]}, \hspace{1cm} (30)$$
where \( x_i(t) \) obeys the classical equation of motion \( \frac{\delta S}{\delta x_i(t)} = 0 \) and satisfies the boundary conditions \( x_i(0) = x_i \) and \( x_i(T) = x_f \), whereas the path summation is over closed paths \( \tilde{x}(t) = x(t) - x_i(t) \) with the expanded action

\[
\tilde{S}[\tilde{x}(t)] = -\frac{1}{\hbar} \int_0^T dt \left[ \frac{m}{2} \tilde{x}^2(t) + \frac{V^*(\tilde{x}(t))}{2} \tilde{x}^2(t) + \cdots \right]. \quad (31)
\]

The Wentzel–Kramers–Brillouin approximation is exact for a quadratic potential, and, the first term being of order \( \hbar^{-1} \), it can include the effect of tunneling (e.g., in a double-well potential) at variance with the GTFK. However, one has to consider that it is not crucial to account for tunneling effects, as they are soon overwhelmed by quantum thermal fluctuations and are practically absent in many-body systems. Moreover, beyond a few relatively simple cases, the evaluation of the path integral (31) is generally hard, mainly due to the dependence of \( \tilde{S} \) upon the classical path. On the other hand, the nonlocal nature of the GTFK approximation yields the possibility of tuning two families of parameters, \( w(\tilde{x}) \) and \( \phi(\tilde{x}) \), allowing one to look for the best approximation of the true action in a richer space while preserving the property of being exact in the classical limit and for harmonic actions. By “richer space,” we mean that the trial action, thanks to its dependence on the average-point functional, is much more general than the local actions corresponding to physical potentials. The GTFK can also be systematically improved, at least in principle, without suffering from the divergencies that appear instead in most perturbative approaches (Kleinert 2009).

The generalizations of the GTFK approach to many degrees of freedom, as well as to Hamiltonian systems (Cuccoli, Giachetti, et al. 1995; Cuccoli et al. 1992), have found numerous applications in physics and physical chemistry. Besides the tests on simple models with one degree of freedom (Feynman and Kleinert 1986; Janke and Kleinert 1986, 1987; Vaia and Tognetti 1990), it is noteworthy that the very first paper regarded the 1D sine–Gordon model (Giachetti and Tognetti 1985; Giachetti, Tognetti, and Vaia 1988a), whose classical version is characterized by the existence of topological nonlinear excitations, the solitons, that determine an anomaly of thermodynamic quantities like the specific heat: the GTFK method allowed for the first time to quantify the same anomaly for the quantum system, and was shown to agree with the outcomes of hard Quantum Monte Carlo calculations (Giachetti, Tognetti, and Vaia 1988a) and to admit a renormalized continuum limit in agreement with exact “Bethe Ansatz” calculations (Giachetti, Tognetti, and Vaia 1988b).

Among many accomplishments, one should mention the quantitative explanation (Cuccoli et al. 1991) of experimental data regarding a quasi-1D magnet, \( \text{CsNiF}_3 \), that behaves similarly to the sine–Gordon model, whereas a major one has been the study of 2D quantum anisotropic magnets (Cuccoli et al. 1995, 1998), whose classical counterpart shows the topological phase transition studied (Kosterlitz and Thouless 1973) by Kosterlitz and Thouless (KT); the GTFK approach allowed also to quantitatively characterize (Cuccoli et al. 2006) earlier experiments, showing that magnetic and calorimetric measurements performed in 1983 were the first known experimental observation of KT behavior in a real magnet; a further success in the magnetic realm was providing a consistent picture of the elusive Ising phase transition in a frustrated model such as the 2D quantum \( J_1-J_2 \) Heisenberg antiferromagnet (Capriotti et al. 2004).

2D Josephson-junction arrays are also typical KT systems: the effective potential was extended to include the dissipative effect of resistive shunts among the junctions used in experiments, getting quantitative accuracy for the phase diagram (Cuccoli et al. 2000). The versatility of the GTFK potential is witnessed also by recent applications in the theoretical interpretation of thermal expansion measurements obtained by x-ray absorption spectroscopy in alloys (Yokoyama and Eguchi 2013; Yokoyama, Koide, and Uemura 2018).

**PATH-INTEGRAL FORMULATION OF STOCHASTIC CALCULUS**

In this section, we briefly review how the formalism of stochastic calculus can be recast in the language of path-integrals in Euclidean time, focusing for simplicity on the case of a single SDE as in Equation (1). As a first step, to simplify the derivation, it is convenient to transform the original process into an auxiliary one, \( X_t \), with constant volatility \( \sigma \). Following Aït Sahalia (1999), this can be achieved in general through the so-called Lamperti’s transform

\[
X_t = \gamma(Y_t) \equiv \sigma \int_0^t dz \frac{\dot{Y}_z}{\sigma_0(z)}.
\]  

(32)
A straightforward application of Ito’s Lemma gives the SDE satisfied by $X_t$ for $t \geq 0$:
\[
dX_t = \mu(X_t)dt + \sigma dW_t, \tag{33}
\]
where
\[
\mu(x) = \sigma \left[ \frac{\mu(\gamma^{-1}(x))}{\sigma(\gamma^{-1}(x))} - \frac{1}{2} \frac{\partial \sigma}{\partial y}(\gamma^{-1}(x)) \right]. \tag{34}
\]

Here, $\gamma = \gamma^{-1}(x)$ is the inverse of the transformation (32). The generalized AD density (2) for the processes $X_t$ and $Y_t$ are related by the Jacobian associated with (32) giving
\[
\psi(y, y_0, T) = \sigma \frac{\psi(x, x_0, T)}{\sigma(\gamma(y))}. \tag{35}
\]

It is well known (see, e.g., Andersen and Piterbarg 2010; Karatzas and Shreve 1991) that the generalized AD density (2) for the process (33) satisfies the following conjugate forward (Fokker-Planck) PDE
\[
\partial_t \psi(x, x_0, t) = \left( -\lambda r(x) - \partial_x \mu(x) + \frac{1}{2} \sigma^2 \partial^2_x \right) \psi(x, x_0, t), \tag{36}
\]
with the initial condition $\psi(x, x_0, 0) = \delta(x_0 - x)$.

A path-integral representation of the AD density can be constructed (Bennati, Rosa-Clot, and Taddei 1999) starting from the Euler approximation, correct up to $O(\Delta t)$, for the solution of the Fokker-Planck PDE (36)
\[
\psi_\lambda(x_\Delta, x_0, \Delta t) = e^{-\lambda r(x_0) \Delta t} \times \frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} \exp \left[ -\frac{(x_\Delta - x_0 - \mu(x_0) \Delta t)^2}{2 \sigma^2 \Delta t} \right]. \tag{37}
\]

Using the Markov property, Equation (37) gives a prescription to write the solution of the Fokker-Planck AD density in the form of a convolution product of short-time AD densities as:
\[
\psi_\lambda(x_T, x_0, T) = \left( \frac{1}{2\pi \sigma^2 \Delta t} \right)^{N/2} \times \int \cdots \int d\xi N \prod_{i=1}^{N-1} d\xi_i N \exp[\tilde{S}(x_i, x_{i-1})], \tag{38}
\]
with $\Delta t = T/N$, $x_N \equiv x_T$ and
\[
\tilde{S}(x_i, x_{i-1}) = -\frac{\Delta t}{2\sigma^2} \left[ \frac{(x_i - x_{i-1}) + \mu((x_{i-1} + x_i)/2)^2}{\Delta t} \right] - \Delta t \left[ \partial_x \mu((x_{i-1} + x_i)/2) + \lambda r((x_{i-1} + x_i)/2) \right], \tag{39}
\]
where the term
\[
\Delta t \partial_x \mu((x_{i-1} + x_i)/2), \tag{40}
\]
arises, at order $O(\Delta t)$, from using the analytically convenient Stratonovich mid-point discretization (for more details, see, e.g., Bennati, Rosa-Clot, and Taddei 1999). As a result, the limit $N \to \infty$ of Equation (38) can be formally written as
\[
\psi_\lambda(x_T, x_0, T) = e^{-W(x_T, x_0)} \rho(x_T, x_0, T), \tag{41}
\]
where
\[
\rho(x_T, x_0, T) = \int_{x(0)=x_0}^{x(T)=x_T} D[x(t)] e^{\tilde{S}(x(t))}, \tag{42}
\]
has the same form of the density matrix in Equation (9), the functional
\[
S[x(t)] = -\int_0^T dt \left[ \frac{1}{2\sigma^2} \tilde{S}(x(t)) + V(x(t)) \right], \tag{43}
\]
has the same form of the Euclidean action in Equation (10),
\[
V(x) = \frac{\mu(x)^2}{2\sigma^2} + \frac{\mu'(x)}{2} + \lambda r(x), \tag{44}
\]
can be called drift potential and we have defined
\[
W(x_T, x_0) = -\frac{1}{\sigma^2} \int_{x_0}^{x_T} dx \mu(x), \tag{45}
\]
to give Equation (43) a suggestive Lagrangian structure, as in Equation (10).

The key observation is that the path integral in Equation (42) is formally equivalent to density matrix in Equation (9) describing the quantum thermodynamics of a particle of mass $m = \hbar / \sigma^2$ in a potential $\hbar V(x)$, at temperature $T = \hbar / \kappa_B T$ (such that $\beta \hbar = T$).

The GTFK can be therefore applied straightforwardly and here for convenience we restate the results with the notation of stochastic calculus:
\[ \tilde{\rho}_\psi(x_T, x_0, T) = \sqrt{\frac{1}{2\pi \sigma^2 T}} e^{-\frac{1}{4T} \frac{f}{\sinh f}} \times \frac{1}{\sqrt{2\pi \alpha}} \exp \left[ -\frac{\xi^2}{2\alpha} - \frac{\omega^2}{4} \coth f(x_T - x_0)^2 \right], \tag{46} \]

where \( \xi = (x_T + x_0)/2 - \bar{x}, \) \( f(\bar{x}) = \omega(\bar{x})T/2 \) and

\[ \alpha(\bar{x}) = \frac{\sigma^2}{2\omega(\bar{x})} \left( \coth f(\bar{x}) - \frac{1}{f(\bar{x})} \right), \tag{47} \]

with \( w(\bar{x}) \) and \( \omega(\bar{x}) \) solutions of the self-consistent equations:

\[ \langle \langle V(\bar{x} + \xi) \rangle \rangle = \langle \langle V_\psi(\bar{x} + \xi) \rangle \rangle = w(\bar{x}) + \frac{\omega^2(\bar{x})\alpha(\bar{x})}{2\sigma^2}, \tag{48} \]

\[ \langle \langle V^2(\bar{x} + \xi) \rangle \rangle = \langle \langle V_\psi^2(\bar{x} + \xi) \rangle \rangle = \frac{\omega^2(\bar{x})}{\sigma^2}. \tag{49} \]

From a computational perspective, within the GTFK method, the calculation of the AD prices involves a single integration over the average point \( \bar{x} \). This compares favorably with fully numerical schemes, such as trees and PDEs, as it does not require a discretization (and multiple convolutions) in the time dimension.

The GTFK method becomes exact in the limit of short time to maturity \( T \to 0 \) and vanishing volatility \( \sigma \to 0 \), for which the parameter \( \alpha \) vanishes as \( \sigma^2 T/12 \). Furthermore, given the form of the chosen trial potential, for harmonic actions, the GTFK approximation is, in fact, exact. This is, for instance, the case for the Vasicek model (Vasicek 1977), as is illustrated in the next section.

**NUMERICAL RESULTS**

In this section, we illustrate the effectiveness of the GFTK approach by discussing its application to a few diffusions processes of the form \( (1) \), starting from two cases in which the method gives exact results, namely, the Vasicek and the so-called quadratic short-rate model. We then discuss the BK model (Black and Karasinski 1991) and GARCH linear SDE model (Capriotti, Jiang, and Shaimerdenova 2019; Li, Mercurio, and Resnick 2018)—for which the AD density \( (2) \) or zero-coupon bonds \( (6) \) are not known analytically—by presenting the comparison of the GTFK results with those obtained by solving the relevant PDEs numerically and by employing other approximations.

**Vasicek Model**

The Vasicek model (Vasicek 1977) is a simple example of affine process (Duffie, Pan, and Singleton 2000)

\[ dX_t = (b - X_t)dt + \sigma dW_t \tag{50} \]

where \( a \) is the mean-reversion speed, \( b \) the mean-reversion level, \( \sigma \) the volatility, and \( r(X_t) = X_t \). The drift potential \( (44) \) is given by the quadratic form

\[ V_\psi(x) = \frac{a^2(b - x)^2}{2\sigma^2} - \frac{a}{2} + \lambda x. \tag{51} \]

The path integral for quadratic potentials is known to be analytically tractable and corresponds in quantum physics to the so-called harmonic oscillator (Feynman, Hibbs, and Styer 2010). In this case, the GTFK self-consistent conditions \( (48) \) and \( (49) \) read

\[ w(\bar{x}) = V_\psi(\bar{x}), \quad \omega^2(\bar{x}) = a^2, \tag{52} \]

and the reduced density matrix \( (46) \) reads

\[ \tilde{\rho}_\psi(x_T, x_0, T) = \left( \frac{1}{2\pi \sigma^2 T} e^{-\frac{1}{4T} \frac{f}{\sinh f}} \times \frac{1}{\sqrt{2\pi \alpha}} \exp \left[ -\frac{\xi^2}{2\alpha} - \frac{\omega^2}{4} \coth f(x_T - x_0)^2 \right] \right), \tag{53} \]

with \( \alpha = \sigma^2/2a(\coth f - 1/f), f = aT/2, \) both independent of \( \bar{x} \). The integral over \( x \) in Equation \( (14) \) can then be performed analytically giving, after a somewhat tedious but straightforward calculation,

\[ \psi_\psi(x_T, x_0, T) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{\frac{1}{2} \frac{\lambda^2}{\sigma^2} - \frac{1}{2} \frac{\lambda^2}{\sigma^2} e^{-\frac{1}{2} \frac{\lambda^2}{\sigma^2} / \sigma^2}} \times \exp \left[ -\left( \frac{x_T - b + \frac{\lambda \sigma^2}{a^2}}{2\sigma^2} \right)^2 \right] e^{-\frac{1}{2} \frac{\lambda^2}{\sigma^2} / \sigma^2}, \tag{54} \]
where $\sigma^2 = \sigma^2(1 - \exp(-2aT))/2a$, in agreement with the known result (Jamshidian 1989).

**Quadratic Short Rate Model**

In the quadratic short rate model, the short rate is defined as

$$r(X_t) = 1 + \beta X_t + \gamma X_t^2,$$  \hspace{1cm} (55)

with $X_t$ following the OU diffusion (50), which is positive definite for $\beta > 0$ and $\frac{\gamma}{\beta} < 4\beta$. In this case, the drift potential (44) reads

$$V_Q(x) = \frac{a^2(b-x)^2}{2\sigma^2} - \frac{a}{2} + \lambda(1 + \beta x + \gamma x^2),$$  \hspace{1cm} (56)

whereas the GTFK conditions, (48) and (49), can be determined as

$$w(\bar{x}) = V_{Q}(\bar{x}), \quad \omega^2(\bar{x}) = a^2 + 2\lambda\gamma\sigma^2,$$  \hspace{1cm} (57)

which, as in the Vasicek model just discussed, give a frequency $\omega$ that is not dependent on the average point and a function $w(\bar{x})$, which is quadratic in $\bar{x}$. Also in this case, the Gaussian integration can be performed analytically, leading to the exact result.

**Black-Karasinki Model**

The BK model (Black and Karasinski 1991) is a conspicuous example of a diffusion that is particularly suitable for financial applications because the short rate at any time horizon follows an intuitive lognormal distribution. Unfortunately, it lacks the same degree of analytical tractability as that shown by affine models. As a result, although widely used in practice, BK implementations rely on computationally intensive numerical simulations based on PDE or Monte Carlo (Andersen and Piterbarg 2010).

The short rate in the BK model is defined as

$$r(X_t) = \exp X_t,$$  \hspace{1cm} (58)

with $X_t$ following the OU diffusion (50). In this case, the drift potential (44) reads

$$V_{BK}(x) = \frac{a^2(b-x)^2}{2\sigma^2} - \frac{a}{2} + \lambda e^x,$$  \hspace{1cm} (59)

whereas the GTFK conditions, (48) and (49), can be determined with some straightforward algebra as

$$w(\bar{x}) = V_{BK}(\bar{x}) + \frac{a^2 - \omega^2(\bar{x})}{\sigma^2} \alpha(\bar{x})$$
$$+ \lambda(e^{\alpha(\bar{x})/2} - 1)e^{\bar{x}},$$  \hspace{1cm} (60)

$$\omega^2(\bar{x}) = a^2 + 2\lambda\sigma^2 e^{\alpha(\bar{x})/2} e^{\bar{x}},$$  \hspace{1cm} (61)

with the second to be solved self-consistently with the renormalization parameter in Equation (47).

In Exhibit 1, we plot the GTFK self-consistent parameters $\omega^2(\bar{x})$, and $\alpha(\bar{x})$ and the diagonal trial reduced density matrix $\bar{\rho}_x(x_0, x_0, T)$ in Equation (46) as a function of the average point $\bar{x}$ for different strength of the diffusive effects, namely, of the time to maturity and volatility. For weak diffusive effects, the parameter $\alpha(\bar{x})$ is relatively small and the trial reduced density matrix has a sharp peak around $x_0$. In this region, both $\alpha(\bar{x})$ and $\omega^2(\bar{x})$ display a weak dependence on $\bar{x}$, which signals the adequacy of a local harmonic approximation to capture the purely diffusive effects in the problem. However, as the diffusive effects increase, with larger volatility and/or time to maturity, the renormalization parameter $\alpha(\bar{x})$ increases, the trial density broadens and both $\alpha(\bar{x})$ and $\omega^2(\bar{x})$ display a more marked dependency on the average point $\bar{x}$, signaling that a nonlocal approximation is needed to best capture the diffusive effects, given an harmonic ansatz of the effective potential.

An illustration of the accuracy of the BK AD densities (2) obtained with the GTFK approximation is displayed for a high volatility case in Exhibit 2 for different values of time to maturity, by comparing with a numerical solution of the Fokker–Planck Equation (36). Here we observe that the GTFK approximation is hardly distinguishable from the PDE result up to $T = 5$, and remains very accurate even for large time horizons.

This is also confirmed by the results for zero-coupon bonds (6) reported in Exhibit 3, illustrating how the GTFK method compares favorably with the results obtained with recently proposed semianalytical approximations, namely, the Exponent Expansion (EE) (Stehlíková and Capriotti 2014), and the Karhunen–Loéve (KL) expansions (Daniluk and Muchorski 2016) when benchmarked against a numerical solution of the associated PDE. In particular, for short time horizons, the GTFK approximation has comparable accuracy with the EE. For larger time horizons, the GTFK compares better and better and...
remains very accurate even when the EE, which has a finite convergence ratio in $T$, eventually breaks down. Similarly, the GTFK method has better accuracy than the first-order KL expansion, and comparable accuracy with the second-order KL expansion for short time horizons, while it has significantly better accuracy for large time horizons. Even for time horizons as large as 20 years, the GTFK approximation produces zero-coupon bond prices within 50 basis points from the exact result, as also illustrated in Exhibit 4. Similar conclusions can also be drawn when comparing with other recently proposed approaches as those in Antonov and Spector (2011) and Tourrucôø, Hagan, and Schleiniger (2007).

**GARCH Linear SDE**

As an example of a more challenging application, we consider the GARCH linear SDE or Inhomogeneous Geometric Brownian Motion (Capriotti, Jiang, and Shaimerdenova 2019; Li, Mercurio, and Resnick 2020).
The 2018) model, which is a special case of the so-called Continuous Elasticity of Variance (CEV) diffusion (Cox and Ross 1976), namely

$$dY_t = a(b - Y_t)dt + \sigma Y_t dW_t,$$  \hspace{1cm} (62)

with $r(Y_t) = Y_t$.

The process defined by the SDE in Equation (62) can be shown to be strictly positive (Kloeden and Platen 1992). As a result, like the BK model, it is well suited to represent default intensities. It can be also shown to have probability density profiles, which are more intuitive than those generated by the widely used square-root processes (Cox, Ingersoll, and Ross 1985; Li, Mercurio, and Resnick 2018). Unfortunately, even if it can be solved exactly (Kloeden and Platen 1992), it does not admit a closed form for the (generalized) AD prices (2).
**EXHIBIT 3**

BK T Maturity Zero-Coupon Bonds

| T   | EE       | KL(1)    | KL(2)    | GTFK     | PDE     |
|-----|----------|----------|----------|----------|---------|
| 0.1 | 0.9939 (0.00%) | 0.9939 (0.00%) | 0.9939 (0.00%) | 0.9939 (0.00%) | 0.9939 |
| 0.5 | 0.9681 (0.00%) | 0.9681 (0.00%) | 0.9681 (0.00%) | 0.9681 (0.00%) | 0.9681 |
| 1.0 | 0.9331 (0.00%) | 0.9331 (0.00%) | 0.9331 (0.00%) | 0.9331 (0.00%) | 0.9331 |
| 2.0 | 0.8581 (0.01%) | 0.8580 (0.02%) | 0.8581 (0.01%) | 0.8582 (0.00%) | 0.8582 |
| 3.0 | 0.7845 (0.01%) | 0.7842 (0.05%) | 0.7844 (0.02%) | 0.7847 (0.01%) | 0.7846 |
| 5.0 | 0.6595 (0.04%) | 0.6582 (0.24%) | 0.6593 (0.08%) | 0.6602 (0.06%) | 0.6598 |
| 10.0| –        | 0.4545 (1.69%) | 0.4601 (0.48%) | 0.4628 (0.10%) | 0.4623 |
| 20.0| –        | 0.2440 (9.06%) | 0.2592 (3.38%) | 0.2672 (0.41%) | 0.2683 |

Notes: Obtained with the GTFK approximation, the Exponent Expansion (EE) of Stehlíková and Capriotti (2014), the Karhunen-Loéve (KL) expansion of Daniluk and Muchorski (2016) to the first and second orders, and by solving the associated PDE numerically. The parameters of the BK process are mean-reversion speed $a = 0.1$, level $b = \ln 0.04$, volatility $\sigma = 0.85$, and initial rate $r_0 = 0.06$. BK = Black–Karasinski; GTFK = Giachetti and Tognetti (1985) and Feynman and Kleinert (1986); PDE = partial differential equation.

**EXHIBIT 4**

GTFK Zero-Coupon Bond Prices as a Function of Time to Maturity for the BK Model, with Mean-Reversion Speed $a = 0.1$, Level $b = \ln 0.04$, Initial Rate $r_0 = 0.06$, and Different Values of the Volatility

Under the Lamperti’s transformation (32) for this process, namely $X_t = \log Y_t$, Equation (62) reads

$$dX_t = \mu_c(X_t)dt + \sigma dW_t,$$

with

$$\mu_c(x) = ab e^{-x} - a - \sigma^2/2. \quad (63)$$

The drift potential (44) associated with the SDE (63) reads therefore

$$V_c(x) = \frac{a^2b^2}{2\sigma^2} e^{-2x} - \frac{ab}{\sigma^2} e^{-x} (a + \sigma^2)$$

$$+ \frac{1}{2\sigma^2} (a^2 + \sigma^2/2)^2 + \lambda e^x, \quad (65)$$

which is related to the so-called Morse potential (Bentaiba, Chetouani, and Hammann 1994). The GTFK conditions, (48) and (49), can be determined with some straightforward algebra as

$$\psi(\bar{x}) = \frac{a^2b^2}{2\sigma^2} e^{-2x} e^{2\alpha} - \frac{ab}{\sigma^2} e^{-x} (a + \sigma^2)e^{\alpha/2}$$

$$+ \frac{1}{2\sigma^2} (a^2 + \sigma^2/2)^2 + \lambda e^x e^{\alpha/2} - \frac{\omega^2(\bar{x})\alpha(\bar{x})}{2\sigma^2} \quad (66)$$

$$\omega^2(\bar{x}) = 2a^2b^2 e^{-2x} e^{2\alpha} - ab e^{-x} (a + \sigma^2)e^{\alpha/2}$$

$$+ \lambda \sigma^2 e^x e^{\alpha/2}. \quad (67)$$

Notes: Crosses indicate the partial differential equation results. The inset is an enlargement for short times to maturity. BK = Black–Karasinski; GTFK = Giachetti and Tognetti (1985) and Feynman and Kleinert (1986).
Examples of AD densities (2) obtained with the GTFK approximation for the GARCH linear SDE are displayed in Exhibit 5 for different values of the diffusion parameters, with a comparison with a numerical solution of the Fokker-Planck Equation (36). Here we observe that the GTFK approximation, as in the BK case, is difficult to distinguish from the PDE result up to several years’ maturity, and for large enough volatilities. As in the BK case, the accuracy of the approximations depends on the chosen model parameters, and the maturity being considered. The approximation becomes less accurate for larger maturities $T$ and volatility. The behavior with respect to the mean-reversion speed $a$ is instead less clear-cut, as this parameter affects both the variance of the process and the nonlinearity of the drift potential (65).

The accuracy of the GTFK method for the GARCH linear SDE is also illustrated for zero-coupon bonds (6) in Exhibit 6 and Exhibit 7 for two sets of model parameters,
showing how the GTFK method compares favorably with the results obtained with recently proposed semi-analytical approximations, namely the EE (Capriotti, Jiang, and Shaimerdenova 2019), when benchmarked against a numerical solution of the associated PDE. In general, although less accurate than in the BK case, because of the more complex form of the drift potential (65), the approximation produces satisfactory results for maturities up to several years, even in regimes of high volatility.

**CONCLUSION**

An effective-potential path-integral formalism of quantum statistical mechanics—dubbed GTFK after the authors (Feynman and Kleinert 1986; Giachetti and Tognetti 1985) who originally introduced it—has been widely used in physics for the study of the quantum thermodynamics of condensed matter systems. The method is based on a self-consistent harmonic approximation of the pure-quantum contributions to the thermodynamics, while fully accounting for the classical behavior of the system (Cuccoli, Giachetti, et al. 1995). As a semiclassical approach, it is exact in the high-temperature and zero-quantum fluctuations limits, but remarkably it also gives a meaningful representation in the zero-temperature limit, where it is equivalent to a self-consistent harmonic approximation of the potential.

By exploiting the path-integral formulation of stochastic calculus, we have shown how the GTFK approach can be used to develop an accurate semi-analytical approximation of (generalized) AD densities, and zero-coupon bonds for nonlinear diffusions. The method is exact in the limit of zero volatility, of zero time to maturity, and for Ornstein-Ulhenbeck diffusions.

The GTFK provides remarkably accurate results for the BK and GARCH linear SDE for interest rates or default intensities, even for high volatilities and long time horizons, with results that compare favorably with previously presented approximation schemes (Antonov and Spector 2011; Capriotti, Jiang, and Shaimerdenova 2019; Daniluk and Muchorski 2016; Stehlíková and Capriotti 2014; Tourrucôo, Hagan, and Schleininger 2007), with expressions that are more compact and easier to compute, and with less severe limitations that arise from a finite convergence radius in the time to maturity or volatility. Similar to the approach in Capriotti (2006), the range of application of the expansion can be further extended to even larger time horizons by means of a fast numerical convolution (Bennati, Rosa-Clot, and Taddei 1999).

The GTFK approximation can potentially be improved in one of two ways: by pursuing higher-order corrections as in the so-called variational perturbation theory (Kleinert 2009) or by its generalization to Hamiltonian systems (Cuccoli, Giachetti, et al. 1995; Cuccoli et al. 1992) that would allow avoiding the nonlinearities in the potential introduced (e.g., as for the GARCH linear SDE) via the Lamperti’s transformation (32). The generalization to explicitly time-dependent coefficients in the SDE (1) is also left as the topic of future research.
The accuracy and ease of computation of the GTFK method makes it a computationally efficient alternative to fully numerical schemes such as binomial trees, PDE, or Monte Carlo for the calculation of transition densities—whether for the maximization of classical likelihoods or the computation of posterior distributions—and for the evaluation of European-style derivatives. This is of practical utility for econometric applications, for example, Aït-Sahalia (1999), for speeding up pricing or calibration routines for valuation of derivatives (Andersen and Piterbarg 2010), or in the context of time-consuming multifactor simulations that are commonplace in financial engineering in a variety of applications (Hull 2017).

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**ADDITIONAL READING**

**Physics and Derivatives: On Three Important Problems in Mathematical Finance**

**ALEXANDER LIPTON AND VADIM KAUSHANSKY**

The Journal of Derivatives

**ABSTRACT:** In this article, we use recently developed extension of the classical heat potential method in order to solve three important but seemingly unrelated problems of financial engineering: (A) American put pricing; (B) default boundary determination for the structural default problem; and (C) evaluation of the hitting time probability distribution for the general time-dependent Ornstein–Uhlenbeck process. We show that all three problems boil down to analyzing behavior of a standard Wiener process in a semi-infinite domain with a quasi-square-root boundary.

**An Efficient Convergent Willow Tree Method for American and Exotic Option Pricing under Stochastic Volatility Models**

**JUNMEI MA, SIHUAN HUANG, AND WEIXU**

The Journal of Derivatives

https://jod.pm-research.com/content/27/3/75

**ABSTRACT:** Stochastic volatility models can describe the evolution of financial assets, such as stocks, currencies, and commodities, better than the classic Black–Scholes model. Some strategic decision-making problems also involve path-dependent and American-style options. In this article, the authors propose a novel, efficient, accurate, and unified two-factor willow tree method to price exotic and American options under the stochastic volatility models, such as the Heston, 3/2, 4/2, Hull–White, Stein–Stein, and a-Hypergeometric models. They also present the convergence analysis of their proposed tree method. They then apply the tree method to price European and American options, and the expected present value and survival rate in a dividend-and-ruin problem. Numerical results demonstrate the efficiency, accuracy, and convergence of their method.