MARKOV RANDOM FIELDS AND ITERATED TORIC FIBRE PRODUCTS

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Abstract. We prove that iterated toric fibre products from a finite collection of toric varieties are defined by binomials of uniformly bounded degree. This implies that Markov random fields built up from a finite collection of finite graphs have uniformly bounded Markov degree.

1. Introduction and main results

The notion of toric fibre product of two toric varieties goes back to [Sul07]. It is of relevance in algebraic statistics since it captures algebraically the Markov random field on a graph obtained by glueing two graphs along a common subgraph; see [RS16] and also below. In [Sul07, RS16, RS14] it is proved that under certain conditions, one can explicitly construct a Markov basis for the large Markov random field from bases for the components. For related results see [Shi12, EKS14, KR14].

However, these conditions are not always satisfied. Nevertheless, in [RS16, Conjecture 56] the hope was raised that when building larger graphs by glueing copies from a finite collection of graphs along a common subgraph, there might be a uniform upper bound on the Markov degree of the models thus constructed, independent of how many copies of each graph are used. A special case of this conjecture was proved in the same paper [RS16, Theorem 54]. We prove the conjecture in general, and along the way we link it to recent work [SS17] in representation stability. Indeed, an important point we would like to make, apart from proving said conjecture, is that algebraic statistics is a natural source of problems in asymptotic algebra, to which ideas from representation stability apply. Our main theorems are reminiscent of Sam’s recent stabilisation theorems on equations and higher syzygies for secant varieties of Veronese embeddings [Sam17a, Sam17b].

Markov random fields. Let $G = (N, E)$ be a finite, undirected, simple graph and for each node $j \in N$ let $X_j$ be a random variable taking values in the finite set $[d_j] := \{1, \ldots, d_j\}$. A joint probability distribution on $(X_j)_{j \in N}$ is said to satisfy the local Markov properties imposed by the graph if for any two non-neighbours $j, k \in N$ the variables $X_j$ and $X_k$ are conditionally independent given $\{X_l \mid \{j, l\} \in E\}$.

On the other hand, a joint probability distribution $f$ on the $X_j$ is said to factorise according to $G$ if for each maximal clique $C$ of $G$ and configuration $\alpha \in \prod_{j \in C}[d_j]$ of the random variables labelled by $C$ there exists an interaction parameter $\theta_C^\alpha$ such that for each configuration $\beta \in \prod_{j \in N}[d_j]$ of all random variables of $G$:

$$f(\beta) = \prod_{C \in \text{mcl}(G)} \theta_C^{\beta|_C}$$

where $\text{mcl}(G)$ is the set of maximal cliques of $G$, and $\beta|_C$ is the restriction of $\beta$ to $C$. 

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These two notions are connected by the Hammersley-Clifford theorem, which says that a positive joint probability distribution on $G$ factorises according to $G$ if and only if it satisfies the Markov properties; see [HC71] or [Lau98, Theorem 3.9].

The set of all positive joint probability distributions on $G$ that satisfy the Markov properties is therefore a subset of the image of the following map:

$$
\varphi_G : \mathbb{C}[\prod_{C \in \text{mcl}(G)} \prod_{j \in C}[d_j]] \rightarrow \mathbb{C}[\prod_{j \in N}[d_j]], \quad (\theta_C^G)_{C,\alpha} \mapsto \prod_{C \in \text{mcl}(G)} \theta_{\overset{\rightarrow}{\beta}}^C
$$

It is the ideal $I_G$ of polynomials vanishing on $\text{im} \varphi_G$ that is of interest in algebraic statistics. Since the components of $\varphi_G$ are monomials, $I_G$ is generated by finitely many binomials (differences of two monomials) in the standard coordinates on $\mathbb{C}[\prod_{j \in N}[d_j]]$, and any finite generating set of binomials can be used to set up a Markov chain for testing whether given observations of the variables $X_j$ are compatible with the assumption that their joint distribution factorises according to the graph $G$ [DS98]. The zero locus of $I_G$ is often called the graphical model of $G$.

Now suppose we have graphs $G_1, \ldots, G_s$ with node sets $N_1, \ldots, N_s$, that $N_i \cap N_k$ equals a fixed set $N_0$ for all $i \neq k$ in $[s]$, and that the graph induced on $N_0$ by each $G_i$ is equal to a fixed graph $H$. Moreover, for each $j \in \bigcup_i N_i$ fix a number $d_j$ of states. We can then glue copies of the $G_i$ along their common subgraph $H$, by which we mean first taking disjoint copies of the $G_i$ and then identifying the nodes labelled by a fixed $j \in N_0$ across all copies. For $a_1, \ldots, a_s \in \mathbb{Z}_{\geq 0}$, we denote the graph obtained by glueing $a_i$ copies of graph $G_i, i \in [s]$ by $\Sigma_{i=1}^s G_i +_H \cdots +_H \Sigma_{i=s}^s G_i$.

**Theorem 1.** Let $G_1, \ldots, G_s$ be graphs with a common subgraph $H$ and a number of states associated to each node. Then there exists a uniform bound $D \in \mathbb{Z}_{\geq 0}$ such that for all multiplicities $a_1, \ldots, a_s$, the ideal $I_G$ of $G = \Sigma_{i=1}^s G_i +_H \cdots +_H \Sigma_{i=s}^s G_i$ is generated by binomials of degree at most $D$.

Our proof shows that one needs only finitely many combinatorial types of binomials, independent of $a_1, \ldots, a_s$, to generate $I_G$. This result is similar in flavour to the Independent Set Theorem from [HS12], where the graph is fixed but the $d_j$ vary. Interestingly, the underlying categories responsible for these two stabilisation phenomena are opposite to each other; see Remark 23.

**Example 2.** In [RS14] it is proved that the ideal $I_G$ for the complete bipartite graph $G = K_{3,N}$, with two states for each of the random variables, is generated in degree at most 12 for all $N$. The graph $G$ is obtained by gluing $N$ copies of $K_{3,1}$ along their common subgraph consisting of 3 nodes without any edges.

We derive Theorem 1 from a general stabilisation result on toric fibre products, which we introduce next.
Toric fibre products. Fix a ground field $K$, let $r$ be a natural number, and let $U_1, \ldots, U_r, V_1, \ldots, V_r$ be finite-dimensional vector spaces over $K$.

Define a bilinear operation

$$\prod_j U_j \times \prod_j V_j \to \prod_j (U_j \otimes V_j), \quad (u, v) \mapsto u \ast v := (u_1 \otimes v_1, \ldots, u_r \otimes v_r).$$

**Definition 3.** The toric fibre product $X \ast Y$ of Zariski-closed subsets $X \subseteq \prod_j U_j$ and $Y \subseteq \prod_j V_j$ equals the Zariski-closure of the set $\{u \ast v \mid u \in X, v \in Y\}$. ▲

**Remark 4.** In [Sul07], the toric fibre product is defined at the level of ideals: if $(x'_i)_i$ are coordinate functions on $U_i$ and $(y'_j)_j$ are coordinate functions on $V_j$, then $(z'_{i,k} := x'_i \otimes y'_j)_k$ are coordinate functions on $U_i \otimes V_j$. The ring homomorphism of coordinate rings

$$K[(z'_{i,k})_{i,k}] = K\left[\prod_j (U_j \otimes V_j)\right] \to K\left[\prod_j U_j \times \prod_j V_j\right] = K[(x'_i)_{i}, (y'_j)_{j}]$$

dual to $[1]$ sends $z'_{i,k}$ to $x'_i \cdot y'_j$. If we compose this homomorphism with the projection modulo the ideal of $X \times Y$, then the kernel of the composition is precisely the toric fibre product of the ideals of $X$ and $Y$ as introduced in [Sul07]. In that paper, multigradings play a crucial role for computing toric fibre products of ideals, but do not affect the definition of toric fibre products. ▲

The product $\ast$ is associative and commutative up to reordering tensor factors. We can iterate this construction and form products like $X^2 \ast Y^2 \ast Z^3$, where $Z$ also lives in a product of $r$ vector spaces $W_i$. This variety lives in $\prod_j (U_j^{r^2}) \otimes V_j \otimes W_j^{r^2}$. We will not be taking toric fibre products of general varieties, but rather Hadamard-stable ones. For this, we have to choose coordinates on each $U_j$, so that $U_j = K^{d_j}$.

**Definition 5.** On $K^d$ the Hadamard product is defined as $(a\ldots a, a\ldots a) \circ (b\ldots b, b\ldots b) = (a_1b_1, \ldots, a_rb_r)$. On $U := U_1 \times \cdots \times U_r$, where $U_j = K^{d_j}$, it is defined component-wise. A set $X \subseteq \prod_j U_j$ is called Hadamard-stable if $X$ contains the all-one vector $1_U$ (the unit element of $\circ$) and if moreover $x \circ z \in X$ for all $x, z \in X$. ▲

**Remark 6.** By [ES96], Remark after Proposition 2.3), $X$ is Hadamard-stable if and only if its ideal is generated by differences of two monomials.

In particular, the Zariski-closure in $U$ of a subtorus of the $\sum_j d_j$-dimensional torus $\prod_i (K \setminus \{0\})^{d_i}$ is Hadamard-stable. These are the toric varieties from the abstract.

Suppose that we also fix identifications $V_j = K^{d_j}$ and a corresponding Hadamard multiplication $\prod_j V_j \times \prod_j V_j \to \prod_j V_j$. Equipping the spaces $U_j \otimes V_j$ with the natural coordinates and corresponding Hadamard multiplication, the two operations just defined satisfy $(u \circ u') \ast (v \circ v') = (u \ast v) \circ (u' \ast v')$ as well as $1_U \otimes V_j = 1_U \otimes 1_V$. Consequently, if both $X \subseteq \prod_j U_j$ and $Y \subseteq \prod_j V_j$ are Zariski-closed and Hadamard-stable, then so is their toric fibre product $X \ast Y$. We can now formulate our second main result.
Theorem 7. Let $s, r \in \mathbb{Z}_{\geq 0}$. For each $i \in [s]$ and $j \in [r]$ let $d_{ij} \in \mathbb{Z}_{\geq 0}$ and set $V_{ij} := K^{d_{ij}}$. For each $i \in [s]$, let $X_i \subseteq \prod_j V_{ij}$ be a Hadamard-stable Zariski-closed subset. Then there exists a uniform bound $D \in \mathbb{Z}_{\geq 0}$ such that for any exponents $a_1, \ldots, a_s$ the ideal of $X_i^{a_i}$ is generated by polynomials of degree at most $D$.

Remark 8. A straightforward generalisation of this theorem also holds, where each $X_i$ is a closed sub-scheme given by some ideal $J_i$ in the coordinate ring of $\prod_j V_{ij}$. Hadamard-stable then says that the pull-back of $J_i$ under the Hadamard product lies in the ideal of $X_i \times X_i$, and the toric fibre product is as in Remark 4. Since this generality would slightly obscure our arguments, we have decided to present explicitly the version with Zariski-closed subsets—see also Remark 22.

Also, the theorem remains valid if we remove the condition that the $X_i$ contain the all-one vector, but require only that they be closed under Hadamard-multiplication; see Remark 21.

Organisation of this paper. The remainder of this paper is organised as follows. In Section 2 we introduce the categories of (affine) Fin-varieties and, dually, Fin$^{\text{op}}$-algebras. The point is that, as we will see in Section 4, the iterated toric fibre products together form such a Fin-variety (or rather a Fin$^s$-variety, where Fin$^s$ is the product category of $s$ copies of Fin).

Indeed, they sit naturally in a Cartesian product of copies of the Fin-variety of rank-one tensors, which, as we prove in Section 3, is Noetherian (Theorem 12). This Noetherianity result is of a similar flavour as the recent result from [SS17] (see also [DK14, Proposition 7.5] which follows the same proof strategy) that any finitely generated Fin$^{\text{op}}$-module is Noetherian; this result played a crucial role in a proof of the Artinian conjecture. However, our Noetherianity result concerns certain Fin$^{\text{op}}$-algebras rather than modules, and is more complicated. Finally, in Section 4 we first prove Theorem 7 and then derive Theorem 1 as a corollary.

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2. Affine Fin-varieties and Fin$^{\text{op}}$-algebras

The category Fin has as objects all finite sets and as morphisms all maps. Its opposite category is denoted Fin$^{\text{op}}$. When C is another category whose objects are called somethings, then a Fin-something is a covariant functor from Fin to C and a Fin$^{\text{op}}$-something is a contravariant functor from Fin to C. The C-homomorphism associated to a Fin-morphism $\pi$ is denoted $\pi_*$ (the push-forward of $\pi$) in the covariant case and $\pi^*$ (the pull-back of $\pi$) in the contravariant case.

More generally, we can replace Fin by any category D. The D-somethings themselves form a category, in which morphisms are natural transformations. In our paper, D is always closely related to Fin or to Fin$^s$, the $s$-fold product of Fin.

Here are three instances of Fin or Fin$^{\text{op}}$-somethings crucial to our paper.
Example 9. Fix $n \in \mathbb{Z}_{\geq 0}$. Then the functor $S \mapsto [n]^S$ is a $\text{Fin}^{\text{op}}$-set, which to $\pi \in \text{Hom}_{\text{Fin}}(S, T)$ associates the map $\pi^* : [n]^T \to [n]^S$, $\alpha \mapsto \alpha \pi$, the composition of $\alpha$ and $\pi$.

Building on this example, the functor $A_n : S \mapsto K[x_\alpha \mid \alpha \in [n]^S]$, a polynomial ring in variables labelled by $[n]^S$, is a $\text{Fin}^{\text{op}}$-$K$-algebra, which associates to $\pi$ the $K$-algebra homomorphism $\pi^* : A(T) \to A(S)$, $x_\alpha \mapsto x_{\alpha \pi}$.

Third, we define an affine $\text{Fin}$-$K$-vector space $Q_n$ by $Q_n : S \mapsto (K^n)^{\otimes S}$, the space of $n \times \cdots \times n$-tensors with factors labelled by $S$, which sends $\pi \in \text{Hom}_{\text{Fin}}(S, T)$ to the linear morphism $\pi_* : (K^n)^{\otimes S} \to (K^n)^{\otimes T}$ determined by

$$\pi_* : \bigotimes_{i \in S} v_i \mapsto \bigotimes_{j \in T} \left( \bigotimes_{i \in \pi^{-1}(j)} v_i \right),$$

where $\otimes$ is the Hadamard product in $K^n$. We follow the natural convention that an empty Hadamard product equals the all-one vector $1 \in K^n$; in particular, this holds in the previous formula for all $j \in T$ that are not in the image of $\pi$.

The ring $A_n$ and the space $Q_n$ are related as follows: $Q_n(S)$ has a basis consisting of vectors $e_\alpha := \bigotimes_{i \in S} e_{\alpha_i}$, $\alpha \in [n]^S$, where $e_1, \ldots, e_n$ is the standard basis of $K^n$; $A_n(S)$ is the coordinate ring of $Q_n(S)$ generated by the dual basis $(x_\alpha)_{\alpha \in [n]^S}$; and for $\pi \in \text{Hom}_{\text{Fin}}(S, T)$ the pullback $\pi^* : A_n(T) \to A_n(S)$ is the homomorphism of $K$-algebras dual to the linear map $\pi_* : Q_n(S) \to Q_n(T)$. Indeed, this is verified by the following computation for $\alpha \in [n]^T$:

$$x_\alpha(\pi_* \bigotimes_{i \in S} v_i) = x_\alpha\left(\bigotimes_{j \in T} \left( \bigotimes_{i \in \pi^{-1}(j)} v_i \right)\right) = \prod_{j \in T} \left( \bigotimes_{i \in \pi^{-1}(j)} v_i \right)_{\pi_j} = \prod_{i \in S} (v_i)^{\pi_j} = \pi^* (x_\alpha) \left( \bigotimes_{i \in S} v_i \right).$$

This is used in Section 3.

In general, by algebra we shall mean an associative, commutative $K$-algebra with 1, and homomorphisms are required to preserve 1. So a $\text{Fin}^{\text{op}}$-algebra $B$ assigns to each finite set $S$ an algebra and to each map $\pi : S \to T$ an algebra homomorphism $\pi^* : B(T) \to B(S)$. An ideal in $B$ is a $\text{Fin}^{\text{op}}$-subset $I$ of $B$ (i.e., $I(S)$ is a subset of $B(S)$ for each finite set $S$ and $\pi^*$ maps $I(T)$ into $I(S)$) such that each $I(S)$ is an ideal in $B(S)$; then $S \mapsto B(S)/I(S)$ is again a $\text{Fin}^{\text{op}}$-algebra, the quotient $B/I$ of $B$ by $I$.

Given a $\text{Fin}^{\text{op}}$-algebra $B$, finite sets $S_j$ for $j$ in some index set $J$, and an element $b_j \in B(S_j)$ for each $j$, there is a unique smallest ideal $I$ in $B$ such that each $I(S_j)$ contains $b_j$. This ideal is constructed as:

$$I(S) = \left\{ \pi^*(b_j) \mid j \in J, \pi \in \text{Hom}(S, S_j) \right\}$$

This is the ideal generated by the $b_j$. A $\text{Fin}^{\text{op}}$-algebra is called Noetherian if each ideal $I$ in it is generated by finitely many elements in various $I(S_j)$, i.e., $I$ can be taken finite.

Example 10. The $\text{Fin}^{\text{op}}$-algebra $A_1$ is Noetherian. Indeed, $A_1(S)$ is the polynomial ring $K[t]$ in a single variable $t$ for all $S$, and the homomorphism $A_1(T) \to A_1(S)$ is the identity $K[t] \to K[t]$. So Noetherianity follows from Noetherianity of the algebra $K[t]$.

For $n \geq 2$ the $\text{Fin}^{\text{op}}$-algebra $A_n$ is not Noetherian. For instance, consider the monomials

$$u_2 := x_{21}x_{12} \in A_n([2]), u_3 := x_{211}x_{121}x_{112} \in A_n([3]), u_4 := x_{2111}x_{1211}x_{1121}x_{1112} \in A_n([4]),$$
and so on. For any map \( \pi : [k] \to [l] \) with \( k > l \), by the pigeon hole principle there are two indices \( i, j \in [k] \) such that \( \pi(i) = \pi(j) =: m \in [l] \). Then \( \pi^*x_{1...1;2} \), where the 2 is in the \( m \)-th position, is a variable with at least two indices equal to 2. Since \( u_k \) contains no such variable, \( \pi^*u_k \) does not divide \( u_k \). So \( u_2, u_3, \ldots \) generates a non-finitely generated monomial \( \text{Fin}^{op} \)-ideal in \( A_n \). (On the other hand, for each \( d \) the piece of \( A_n \) of homogeneous polynomials of degree at most \( d \) is Noetherian as a \( \text{Fin}^{op} \)-module, see [DK14, Proposition 7.5].)\[\square\]

We shall see in the following section that certain interesting quotients of each \( A_n \) are Noetherian.

3. Rank-one tensors form a Noetherian \( \text{Fin} \)-variety

Let \( Q_{\leq 1}^n(S) \) be the variety of rank-one tensors, i.e., those of the form \( \otimes_{i \in S} v_i \) for vectors \( v_i \in K^n \). We claim that this defines a Zariski-closed \( \text{Fin} \)-subvariety \( Q_{\leq 1}^n \) of \( Q_n \).

For this, we must verify that for a map \( \pi : S \to T \) the map \( Q_n(S) \to Q_n(T) \) dual to the algebra homomorphism \( A_n(T) \to A_n(S) \) sends \( Q_{\leq 1}^n(S) \) into \( Q_{\leq 1}^n(T) \). And indeed, in Example 9 we have seen that this map sends

\[
\otimes_{i \in S} v_i \mapsto \otimes_{j \in T} \left( \bigotimes_{i \in \pi^{-1}(j)} v_i \right).
\]

It is well known that (if \( K \) is infinite) the ideal in \( A_n(S) \) of \( Q_{\leq 1}^n(S) \) equals the ideal \( I_n(S) \) generated by all binomials constructed as follows. Partition \( S \) into two parts \( S_1, S_2 \), let \( \alpha, \beta \in [n]^S \); and write \( \alpha \| \beta \) for the element of \([n]^S \) which equals \( \alpha \) on \( S_1 \). Then we have the binomials \( x_{\alpha \| \beta} - x_{\alpha \| \beta} \in I_n(S) \), and \( I_n(S) \) is the ideal generated by these for all partitions and all \( \alpha, \alpha, \beta_1, \beta_2 \). The functor \( S \mapsto I_n(S) \) is an ideal in the \( \text{Fin}^{op} \)-algebra \( A_n \); for infinite \( K \) this follows from the computation above, and for arbitrary \( K \) it follows since the binomials above are mapped to binomials by pull-backs of maps \( T \to S \) in \( \text{Fin} \). Moreover, \( I_n \) is finitely generated (see also [DK14, Lemma 7.4]):

**Lemma 11.** The ideal \( I_n \) in the \( \text{Fin}^{op} \)-algebra \( A_n \) is finitely generated.

**Proof.** In the determinantal equation (2), if there exist distinct \( j, l \in S_1 \) such that \( \alpha_1(j) = \alpha_2(l) \) and \( \beta_1(j) = \beta_2(l) \), then the equation comes from an equation in \( I_n(S \setminus \{j\}) \) via the map \( S \to S \setminus \{j\} \) that is the identity on \( S \setminus \{j\} \) and maps \( j \to l \). By the pigeon hole principle this happens when \( |S_1| > n^2 \). Similarly for \( |S_2| > n^2 \). Hence \( I_n \) is certainly generated by \( I_n(2n^2 - 1) \). \( \square \)

The main result in this section is the following.

**Theorem 12.** For each \( n \in \mathbb{Z}_{\geq 20} \) the coordinate ring \( A_n/I_n \) of the \( \text{Fin} \)-variety \( Q_{\leq 1}^n \) of rank-one tensors is a Noetherian \( \text{Fin}^{op} \)-algebra.

Our proof follows the general technique from [SS17], namely, to pass to a suitable category close to \( \text{Fin} \) that allows for a Gröbner basis argument. However, the relevant well-partial-orderedness proved below is now and quite subtle. We use the category \( \text{OS} \) from [SS17] (also implicit in [DK14, Section 7]) defined as follows.
Definition 13. The objects of the category OS (“ordered-surjective”) are all finite sets equipped with a linear order and the morphisms \( \pi : S \to T \) are all surjective maps with the additional property that the function \( T \to S, \ j \mapsto \min \pi^{-1}(j) \) is strictly increasing.

Any Fin-algebra is also an OS-algebra, and OS-Noetherianity implies Fin-Noetherianity. So to prove Theorem 12 we set out to prove the stronger statement that \( A_n/I_n \) is, in fact, OS-Noetherian.

We get a more concrete grip on the K-algebra \( A_n/I_n \) through the following construction. Let \( M_n \) denote the (Abelian) Fin\(^{op}\)-monoid defined by

\[
M_n(S) := \left\{ \alpha \in \mathbb{Z}^{[n] \times S} \mid \forall j, l \in S : \sum_{i=1}^{n} \alpha_{ij} = \sum_{i=1}^{n} \alpha_{il} \right\},
\]

in which the multiplication is given by addition, and where the pull-back of a map \( \pi : S \to T \) is the map \( \pi' : M_n(T) \to M_n(S) \) sending \( (\alpha_{ij})_{i \in [n], j \in T} \) to \( (\alpha_{ij})_{i \in [n], j \in S} \).

So elements of \( M_n(S) \) are matrices with nonnegative integral entries and constant column sum. Let \( K M_n \) denote the Fin-algebra sending \( S \) to the monoid \( K \)-algebra \( K M_n(S) \). The following proposition is a reformulation of a well-known fact.

**Proposition 14.** The Fin\(^{op}\)-algebra \( A_n/I_n \) is isomorphic to the Fin\(^{op}\)-algebra \( K M_n \) (and the same is true when both are regarded as OS\(^{op}\)-algebras).

**Proof.** For each finite set \( S \), the \( K \)-algebra homomorphism \( \Phi_S : A_n(S) \to K M_n(S) \) that sends \( x_\alpha, \alpha \in [n]^S \) to the \([n] \times S\) matrix in \( M_n(S) \) with a 1 at the positions \((\alpha, j), \ j \in S \) and zeroes elsewhere is surjective and has as kernel the ideal \( I_n(S) \). Moreover, if \( \pi : S \to T \) is a morphism in Fin, then we have \( \Phi_T \pi' = \pi' \Phi_S \), i.e., the \((\Phi_S)_S\) define a natural transformation. \( \square \)

Choose any monomial order \( >_0 \) on \( \mathbb{Z}_{\geq 0}^n \), i.e., a well-order such that \( a > b \) implies \( a + c > b + c \) for every \( a, b, c \in \mathbb{Z}_{\geq 0}^n \). Then for each object \( S \) in OS we define a linear order \( >_0 \) on \( M_n(S) \), as follows: \( \alpha >_0 \beta \) if \( \alpha \neq \beta \) and the smallest \( j \in S \) with \( \alpha_{\cdot, j} \neq \beta_{\cdot, j} \) (i.e., the \( j \)-th column of \( \alpha \) is not equal to that of \( \beta \)) satisfies \( \alpha_{\cdot, j} >_0 \beta_{\cdot, j} \) in the chosen monomial order on \( \mathbb{Z}_{\geq 0}^n \). A straightforward verification shows that this is a monomial order on \( M_n(S) \) (we call the elements of \( M_n(S) \) monomials, even though \( K M_n(S) \) is not a polynomial ring). Moreover, for various \( S \), these orders are interrelated as follows.

**Lemma 15.** For any \( \pi \in \text{Hom}_{OS}(S, T) \) and \( \alpha, \beta \in M_n(T) \), we have \( \alpha >_0 \beta \Rightarrow \pi' \alpha >_0 \pi' \beta \).

**Proof.** If \( j \in T \) is the smallest column index where \( \alpha \) and \( \beta \) differ, then \( \alpha' := \pi' \alpha \) and \( \beta' := \pi' \beta \) differ in column \( l := \min \pi^{-1}(j) \), where they equal \( \alpha_{\cdot, j} \) and \( \beta_{\cdot, j} \), respectively, and the former is larger than the latter. Furthermore, if \( l' \) is the smallest position where \( \alpha'_{\cdot, j} \) differs, then \( \alpha_{\cdot, \pi(l')} \neq \beta_{\cdot, \pi(l')} \) and hence \( \pi(l') \geq j \) and hence \( l' = \min \pi^{-1}(\pi(l')) \geq \min \pi^{-1}(j) = l \). Hence in fact \( l = l' \) and \( \pi' \alpha >_0 \pi' \beta \). \( \square \)

In addition to the well-order \( \leq \) on each individual \( M_n(S) \), we also need the following partial order \( | \) on the union of all of them.
Definition 16. Let $S, T$ be objects in $\text{OS}$. We say that $\alpha \in M_n(T)$ divides $\beta \in M_n(S)$ if there exist a $\pi \in \text{Hom}_{\text{OS}}(S, T)$ and a $\gamma \in M_n(S)$ such that $\beta = \gamma + \pi^*\alpha$. In this case, we write $\alpha \mid \beta$.

The key combinatorial property of the relation just defined is the following.

**Proposition 17.** The relation $\mid$ is a well-quasi-order, that is, for any sequence $\alpha^{(1)} \in M_n(S_1), \alpha^{(2)} \in M_n(S_2), \ldots$ there exist $i < j$ such that $\alpha^{(i)} \mid \alpha^{(j)}$.

*Proof.* First, to each $\alpha \in M_n(S)$ we associate the monomial ideal $P(\alpha)$ in the polynomial ring $R := K[z_1, \ldots, z_n]$ (here $K$ is but a place holder) generated by the monomials $z^{\alpha(j)}, j \in S$. The crucial fact that we will use about monomial ideals in $R$ is that in any sequence $P_1, P_2, \ldots$ of such ideals there exist $i < j$ such that $P_i \supseteq P_j$—in other words, monomial ideals are well-quasi-ordered with respect to reverse inclusion [Mac01].

To prove the proposition, suppose, on the contrary, that there exists a sequence as above with $\alpha^{(i)} \mid \alpha^{(j)}$ for all $i < j$. Such a sequence is called bad. Then by basic properties of well-quasi-orders, some bad sequence exists in which moreover

(3) $P(\alpha^{(1)}) \supseteq P(\alpha^{(2)}) \supseteq \ldots$

Among all bad sequences with this additional property choose one in which, for each $j = 1, 2, \ldots$, the cardinality $|S_j|$ is minimal among all bad sequences starting with $\alpha^{(1)}, \ldots, \alpha^{(j-1)}$.

Write $\alpha^{(j)} = (\gamma^{(j)}|\beta^{(j)})$, where $\beta^{(j)} \in \mathbb{Z}_{\geq 0}^n$ is the last column (the one labelled by the largest element of $S_j$), and $\gamma^{(j)}$ is the remainder. By Dickson’s lemma, there exists a subsequence $j_1 < j_2 < \ldots$ such that $\beta^{(j_1)}, \beta^{(j_2)}, \ldots$ increase weakly in the coordinate-wise ordering on $\mathbb{Z}_{\geq 0}^n$. By restricting to a further subsequence, we may moreover assume that also

(4) $P(\gamma^{(j_1)}) \supseteq P(\gamma^{(j_2)}) \supseteq \ldots$

Then consider the new sequence

$\alpha^{(1)}, \ldots, \alpha^{(j_1)}, \gamma^{(j_1)}, \gamma^{(j_2)}, \ldots$

By (1) and (3), and since $P(\alpha^{(1)}) \supseteq P(\gamma^{(j)}))$, this sequence also satisfies (3). We claim that, furthermore, it is bad.

Suppose, for instance, that $\gamma^{(j)} \mid \gamma^{(j_i)}$. Set $a_i := \max S_{j_i}$ for $i = 1, 2$. Then there exists a $\pi \in \text{Hom}_{\text{OS}}(S_1 \setminus \{a_2\}, S_2 \setminus \{a_1\})$ such that $\gamma^{(j_i)} - \pi^* \gamma^{(j_i)} \in M_n(S_1 \setminus \{a_2\})$. But then extend $\pi$ to an element $\pi'$ of $\text{Hom}_{\text{OS}}(S_2, S_1)$ by setting $\pi(a_2) := a_i$; since $\beta^{(j)}$ is coordinate-wise smaller than $\beta^{(j_i)}$ we find that $\alpha^{(j_i)} - \pi' \alpha^{(j_i)} \in M_n(S_1)$, so $\alpha^{(j)} \mid \alpha^{(j_i)}$, in contradiction to the badness of the original sequence.

On the other hand, suppose for instance that $\alpha^{(1)} \mid \gamma^{(j)}$ and write $a_2 := \max S_{j_2}$. Then there exists a $\pi \in \text{Hom}_{\text{OS}}(S_1 \setminus \{a_2\}, S_1)$ such that $\gamma^{(j)} - \pi^* \alpha^{(1)} \in M_n(S_1 \setminus \{a_2\})$. Now, and this is why we required that (3) holds, since $P(\alpha^{(1)}) \supseteq P(\alpha^{(j)})$, there exists an element $s \in S_1$ such that the column $\beta^{(j)}$ is coordinate-wise at least as large as the $s$-th column of $\alpha^{(1)}$. Extend $\pi$ to an element of $\text{Hom}_{\text{OS}}(S_1, S_1)$ by setting $\pi(a_2) = s$. Since $a_2$ is the maximal element of $S_{j_2}$, this does not destroy the property that the
function  \( \pi^{-1}(\cdot) \) be increasing in its argument. Moreover, this \( \pi \) has the property that \( \alpha^{(i)} - \pi^{*}\alpha^{(1)} \in M_n(S_i) \), again a contradiction.

Since we have found a bad sequence satisfying \([3]\) but with strictly smaller underlying set at the \( j_{i} \)-st position, we have arrived at a contradiction. \( \square \)

Next we use a Gröbner basis argument.

**Proof of Theorem**\(^{12} \) We prove the stronger statement that \( K_{\eta} \) is Noetherian as an \( \text{OS}-\text{algebra} \). Let \( P \) be any ideal in \( A_{\eta}/I_{\eta} = K_{\eta} \). For each object \( S \) in \( \text{OS} \), let \( L(S) \subseteq M_{\eta}(S) \) denote the set of leading terms of nonzero elements of \( P(S) \) relative to the ordering \( \triangleright \). Proposition\(^{17} \) implies that there exists a finite collection \( S_1, \ldots, S_N \) and \( \alpha^{(1)} \in L(S_j) \) such that each element of each \( L(S) \) is divisible by some \( \alpha^{(i)} \). Correspondingly, there exist elements \( f_j \in P(S_j) \) with leading monomial \( \alpha^{(i)} \) and leading coefficient 1. To see that the \( f_j \) generate \( P \), suppose that there exists an \( S \) such that \( P(S) \) is not contained in the ideal generated by the \( f_j \), and let \( g \in P(S) \) have minimal leading term \( \beta \) among all elements of \( P(S) \) not in the ideal generated by the \( f_j \); without loss of generality \( g \) has leading coefficient 1. By construction, there exists some \( j \) and some \( \pi \in \text{Hom}_{\text{OS}}(S, S_j) \) such that \( \beta - \pi^{*}\alpha^{(i)} \in M_{\eta}(S) \). But now, by Lemma\(^{15} \) we find that the leading monomial of \( \pi^{*}f_j \) equals \( \pi^{*}\alpha^{(i)} \), hence subtracting a monomial times \( \pi^{*}f_j \) from \( g \) we obtain an element of \( P(S) \) with smaller leading monomial that is not in the ideal generated by the \( f_j \)—a contradiction. \( \square \)

Below, we need the following generalisation of Theorem\(^{12} \)

**Theorem 18.** For any \( n_1, \ldots, n_r \in (\mathbb{Z})_{\geq 0} \) the \( \text{Fin}-\text{algebra} \) (or \( \text{OS}-\text{algebra} \) \( (A_{n_i}/I_{n_i}) \otimes \cdots \otimes (A_{n_r}/I_{n_r}) \) is Noetherian.

**Proof.** This algebra is isomorphic to \( B := K(M_{n_1} \times \cdots \times M_{n_r}) \). There is a natural embedding \( \iota : M_{n_1} \times \cdots \times M_{n_r} \to M_{n_1 + \cdots + n_r} =: M_n \) by forming a block matrix; its image consists of block matrices with constant partial column sums. And while a subalgebra of a Noetherian algebra is not necessarily Noetherian, this is true in the current setting.

The crucial point is that if \( \alpha_i \in (M_{n_1} \times \cdots \times M_{n_r})(S_i) \) for \( i = 1, 2 \), then *a priori* \( \iota(\alpha_1) \iota(\alpha_2) \) only means that \( \iota(\alpha_2) - \pi^{*}\iota(\alpha_1) \in M_n(S_2) \); but since both summands have constant partial column sums, so does their difference, so in fact, the difference lies in the image of \( \iota \). With this observation, the proof above for the case where \( r = 1 \) goes through unaltered for arbitrary \( r \). \( \square \)

**Remark 19.** Similar arguments for passing to sub-algebras are also used in \([13,12]\) and \([10]\).

## 4. Proofs of the main results

In this section we prove Theorems\(^{11} \) and \(^{7} \)

**Toric fibre products.** To prove Theorem\(^{7} \) we work with a product of \( s \) copies of the category \( \text{Fin} \); one for each of the varieties \( X_i \) whose iterated fibre products are under consideration. Let \( s, r \in \mathbb{Z}_{\geq 0} \). For each \( i \in [s] \) and \( j \in [r] \) let \( d_{ij} \in \mathbb{Z}_{\geq 0} \) and set
Consider the \textbf{Fin}$^s$-variety $\mathcal{V}$ that assigns to an $s$-tuple $S = (S_1, \ldots, S_s)$ the product
\[
\prod_{j=1}^r \bigotimes_{i=1}^s V_{ij}^{s_i}
\]
and to a morphism $\pi = (\pi_1, \ldots, \pi_s) : S \rightarrow T := (T_1, \ldots, T_s)$ in \textbf{Fin}$^s$ the linear map $\mathcal{V}(S) \rightarrow \mathcal{V}(T)$ determined by
\[
(\otimes_i \otimes_{k \in S_i} v_{ijk})_{j \in [r]} \mapsto (\otimes_i \otimes_{l \in T_i} \left( \bigcirc_{k \in S_i} (\otimes_{j \in [r]} v_{ijk}) \right)_{j \in [r]},
\]
where the Hadamard product $\bigcirc$ is the one on $V_{ij}$. Let $Q^{\leq 1}(S)$ be the Zariski-closed subset of $\mathcal{V}(S)$ consisting of $r$-tuples of tensors of rank at most one; thus $Q^{\leq 1}$ is a \textbf{Fin}$^s$-subvariety of $\mathcal{V}$.

For each $i \in [s]$ let $X_i \subseteq \prod_{j=1}^r V_{ij}$ be a Hadamard-stable Zariski-closed subset. Then for any tuple $S = (S_1, \ldots, S_s)$ in \textbf{Fin}$^s$ the variety $X(S) := X_1^{s_1} \cdots X_s^{s_s}$ is a Zariski-closed subset of $\mathcal{V}(S)$.

\textbf{Lemma 20.} The association $S \mapsto X(S)$ defines a \textbf{Fin}$^s$-closed subvariety of $Q^{\leq 1}$.

\textit{Proof.} From the definition of $\star$ in \textbf{1}, it is clear that the elements of $X(S)$ are $r$-tuples of tensors of rank at most 1. Furthermore, for a morphism $S \rightarrow T$ in \textbf{Fin}$^s$ the linear map $\mathcal{V}(S) \rightarrow \mathcal{V}(T)$ from \textbf{5} sends $X(S)$ into $X(T)$—here we use that if $(v_{i,j,k})_{j \in [r]} \in X_i$ for each $k \in \pi^{-1}(l)$, then also $(\bigcirc_{k \in \pi^{-1}(l)} v_{ijk})_{j \in [r]} \in X_i$ since $X_i$ is Hadamard-stable. \hfill $\Box$

Now Theorem \textbf{7} follows once we know that the coordinate ring of $Q^{\leq 1}$ is a Noetherian $(\textbf{Fin})^{\oplus}$-algebra. For $s = 1$ and $r = 1$ this is Theorem \textbf{12} with $n$ equal to $d_{11}$. For $s = 1$ and general $r$, this is Theorem \textbf{18} with $n_j$ equal to $d_{1j}$.

For $r = 1$ and general $s$, Theorem \textbf{7} follows from a \textbf{Fin}$^s$-analogue of Theorem \textbf{12} which is proved as follows. The coordinate ring of $Q^{\leq 1}(S_1, \ldots, S_s)$ is the subring of $K(M_{d_{11}}(S_1) \times \cdots \times M_{d_{1s}}(S_s))$ spanned by the monomials corresponding to $s$-tuples of matrices with the \textit{same} constant column sum. Using Proposition \textbf{17} and the fact that a finite product of well-quasi-ordered sets is well-quasi-ordered one finds that the natural \textbf{Fin}$^s$-analogue on $M_{d_{11}} \times \cdots \times M_{d_{1s}}$ of the division relation $|$ is a well-quasi-order; and this implies, once again, that the coordinate ring of $Q^{\leq 1}$ is a Noetherian $(\textbf{OS})^{\oplus}$-algebra.

Finally, for general $s$ and general $r$, the result follows as in the proof of Theorem \textbf{18}. This proves the Theorem \textbf{7} in full generality. \hfill $\Box$

\textbf{Remark 21.} The only place where we used that the $X_i$ contain the all-one vector is in the proof of Lemma \textbf{20} when $\pi^{-1}(l)$ happens to be empty. If we do not require this, then the conclusion of Theorem \textbf{7} still holds, since one can work directly with the category $\textbf{OS}$ in which morphisms $\pi$ are surjective.

\textbf{Remark 22.} If we replace the $X_i$ by Hadamard-stable closed subschemes rather than subvarieties, then $S \mapsto X(S)$ is still a \textbf{Fin}$^s$-closed subscheme of $Q^{\leq 1}$, and since the coordinate ring of the latter is Noetherian, the proof goes through unaltered.
Remark 23. In the Independent Set Theorem from [HST12], the graph $G = (N, E)$ is fixed but the state space sizes $d_j$ grow unboundedly for $j$ in an independent set $T \subseteq N$ and are fixed for $j \notin T$. In this case, given a $T$-tuple of maps $(\pi_j : S_j \to P_j)_{j \in T}$ of finite sets, where $S_j$ is thought of as the state space of $j \in T$ in the smaller model and $P_j$ as the state space in the larger model, we obtain a natural map from the larger model into the smaller model. Hence then the graphical model is naturally a $(\text{Fin}^{op})^T$-variety and its coordinate ring is a $\text{Fin}^T$-algebra. Note the reversal of the roles of these two categories compared to Lemma 20.

Markov random fields. Given a finite (undirected, simple) graph $G = (N, E)$ with a number $d_j$ of states attached to each node $j \in N$, the graphical model is $X_G := \overline{\text{im}} \varphi_G$, where $\varphi_G$ is the parameterisation

$$\varphi_G : C^{\prod_{\text{cl}(\mathcal{G})} \prod_{\alpha \in [d]} C} \to C^{\prod_{\text{cl}(\mathcal{O})} C}, \quad (\theta^C_{\alpha})_{\mathcal{C}, \alpha} \mapsto \left( \prod_{\text{cl}(\mathcal{G})} \theta^C_{\beta} \right)_\beta.$$ 

Lemma 24. For any finite graph $G$, the graphical model $X_G$ is Hadamard-closed.

Proof. The parameterisation $\varphi$ sends the all-one vector in the domain to the all-one vector $1$ in the target space, so $1 \in \overline{\text{im}} \varphi$. Moreover, if $\theta, \theta'$ are two parameter vectors, then $\varphi(\theta \odot \theta') = \varphi(\theta) \odot \varphi(\theta')$, so $\overline{\text{im}} \varphi$ is Hadamard-closed. Then so is its closure. \hfill $\square$

Following [Sul07], we relate graph glueing to toric fibre products. We are given finite graphs $G_1, \ldots, G_s$ with node sets $N_1, \ldots, N_s$ such that $N_i \cap N_k = N_0$ for all $i \neq k$ in $[s]$ and such that each $G_i$ induces the same graph $H$ on $N_0$. Moreover, for each $j \in \bigcup_i N_i$ we fix a number $d_j$ of states.

For each $\beta_0 \in \prod_{j \in N_0} [d_j]$ and each $i \in [s]$ set $V_{i, \beta_0} := C^{\prod_{j \in N_i \setminus N_0} [d_j]}$, which we interpret as the ambient space of the part of the probability table of the variables $X_j, j \in N_i$ where we have fixed the states of the variables in $N_0$ to $\beta_0$—up to scaling, these are the conditional joint probabilities for the $X_j, j \in N_i \setminus N_0$ given that the $X_j, j \in N_0$ are in joint state $\beta_0$. For $\beta \in \prod_{j \in N_0} [d_j]$ write $\beta = \beta_0 \| \beta'$ where $\beta_0, \beta'$ are the restrictions of $\beta$ to $N_0$ and $N_i \setminus N_0$, respectively. For each maximal clique $C$ in $G_i$ define $C_0 = C \cap N_0$ and $C' = C \setminus N_0$. Correspondingly, decompose $\alpha \in \prod_{j \in C} [d_j]$ as $\alpha = \alpha_0 \| \alpha'$, where $\alpha_0, \alpha'$ are the restrictions of $\alpha$ to $C_0$ and $C'$, respectively.

Then the graphical model $X_{G_i} := \overline{\text{im}} \varphi_{G_i}$ is the closure of the image of the parameterisation

$$\varphi_1 : C^{\prod_{\text{cl}(\mathcal{G})} \prod_{\alpha \in [d]} C} \to \bigotimes_{\beta_0 \in \prod_{j \in N_0} [d_j]} V_{i, \beta_0}, \quad (\theta^C_{\alpha})_{\mathcal{C}, \alpha} \mapsto \left( \prod_{\text{cl}(\mathcal{G})} \theta^C_{\beta} \right)_{\beta_0}.$$ 

Setting $r := \prod_{j \in N_0} d_j$, we are exactly in the setting of the previous sections: for each $i, k \in [s]$, we have the bilinear map

$$\star : \bigotimes_{\beta_0} V_{i, \beta_0} \times \bigotimes_{\beta_0} V_{k, \beta_0} \to \bigotimes_{\beta_0} (V_{i, \beta_0} \otimes V_{k, \beta_0}), \quad ((v_{i, \beta_0})_{\beta_0}, (v_{k, \beta_0})_{\beta_0}) \mapsto (v_{i, \beta_0} \otimes v_{k, \beta_0})_{\beta_0},$$

and we can take iterated products of this type. The space on the right is naturally isomorphic to $C^{\prod_{j \in N \setminus N_0} [d_j]}$, the space of probability tables for the joint distribution...
of the variables labelled by the vertices in the glued graph $G_1 +_H G_k$. Under this identification we have the following.

**Proposition 25.** For $G := \sum^a_H G_1 + \cdots + \sum^a_H G_k$ we have $X_G = X^{a_1}_{G_1} \ast \cdots \ast X^{a_k}_{G_k}$.

**Proof.** It suffices to prove this for the gluing of two graphs. Note that a clique in $G := G_1 +_H G_2$ is contained entirely in either $G_1$ or $G_2$, or in both but then already in $H$. Let $\theta, \eta$ be a parameter vectors in the domains of $\phi_{G_1}, \phi_{G_2}$, respectively. Then

$$\phi_{G_1}(\theta) \ast \phi_{G_2}(\eta)$$

$$= \left( \prod_{C \in \mathrm{cl}(G_1)} \theta^C_\alpha \mid (\theta^C_\alpha) \right) \ast \left( \prod_{C \in \mathrm{cl}(G_2)} \theta^C_\beta \mid (\theta^C_\beta) \right)$$

$$= \left( \prod_{C \in \mathrm{cl}(G_1)} \theta^C_\alpha \mid (\theta^C_\alpha) \right) \ast \left( \prod_{C \in \mathrm{cl}(G_2)} \theta^C_\beta \mid (\theta^C_\beta) \right)$$

$$\ast \left( \prod_{C \in \mathrm{cl}(G_2)} \eta^C_\beta \mid (\eta^C_\beta) \right)$$

$$= \left( \prod_{C \in \mathrm{cl}(G)} \mu^C_\alpha \mid (\mu^C_\alpha) \right)$$

where, for $C \in \mathrm{mcl}(G)$ and $\alpha \in \prod_{j \in C} [d_j]$, the parameter $\mu^C_\alpha$ is defined as

$$\mu^C_\alpha := \begin{cases} 
\theta^C_\alpha & \text{if } C \subseteq N_1 \text{ and } C \nsubseteq N_0,
\eta^C_\beta & \text{if } C \subseteq N_2 \text{ and } C \nsubseteq N_0, \\
\theta^C_\alpha & \text{if } C \subseteq N_0. 
\end{cases}$$

This computation proves that $X_G \ast X_{G_2} \subseteq X_G$. Conversely, given any parameter vector $\mu$ for $G$, we can let $\theta$ be the restriction of $\mu$ to maximal cliques of the first and third type above, and set $\eta^C_\alpha$ equal to $\mu^C_\alpha$ if $C$ is of the second type above and equal to 1 if it is of the third type. This yields the opposite inclusion. \qed

**Proof of Theorem 24.** By Proposition 25, the ideal $I_G$ is the ideal of the iterated toric fibre product $X^{a_1}_{G_1} \ast \cdots \ast X^{a_k}_{G_k}$. By Lemma 24, each of the varieties $X_{G_i}$ is Hadamard closed. Hence Theorem 22 applies, and $I_G$ is generated by polynomials of degree less than some $D$, which is independent of $a_1, \ldots, a_k$. Then it is also generated by the binomials of at most degree $D$. \qed

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