On higher-order Codazzi tensors on complete Riemannian manifolds

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Abstract
We prove several Liouville-type nonexistence theorems for higher-order Codazzi tensors and classical Codazzi tensors on complete and compact Riemannian manifolds, in particular. These results will be obtained by using theorems of the connections between the geometry of a complete smooth manifold and the global behavior of its subharmonic functions. In conclusion, we show applications of this method for global geometry of a complete locally conformally flat Riemannian manifold with constant scalar curvature because its Ricci tensor is a Codazzi tensor and for global geometry of a complete hypersurface in a standard sphere because its second fundamental form is also a Codazzi tensor.

Keywords Complete Riemannian manifold · Higher-order Codazzi tensor · Subharmonic function

Mathematics Subject Classification 53C20 · 53C25 · 53C40

1 Introduction

Let \((M, g)\) be a Riemannian manifold of dimension \(n \geq 2\) with the Levi-Civita connection \(\nabla\). Everywhere in the following we denote by \(\Lambda^q M\) and \(S^p M\) the vector bundles of exterior differential \(q\)-forms (\(1 \leq q \leq n - 1\)) and symmetric differential \(p\)-forms (\(p \geq 2\)) on \(M\).

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Throughout this paper, we will consider the vector spaces of their $C^\infty$-sections denoted by $C^\infty\Lambda^p M$ and $C^\infty S^p M$, respectively. The Riemannian metric $g$ induces a point-wise inner product on the fibers of each of these spaces.

A symmetric bilinear form $T \in C^\infty S^2 M$ on a Riemannian manifold $(M, g)$ is said to be a Codazzi tensor if it satisfies the equation \cite{5, p. 435; 15; 30, p. 382; 36, pp. 24; 56; 68}

\[ (\nabla_X T) (Y, Z) = (\nabla_Y T) (X, Z) \tag{1} \]

for any tangent vector fields $X, Y, Z$ on $M$. The Codazzi tensor $T \in C^\infty S^2 M$ is called non-trivial if it is not a constant multiple of metric \cite{15, p. 15}. Alongside with it we know from \cite{15, p. 17} that every smooth manifold $M$ carries a $C^\infty$-metric $g$ such that $(M, g)$ admits a non-trivial Codazzi tensor $T \in C^\infty S^2 M$. We remark here that this result is essentially local.

Codazzi tensors appear in a natural way in many geometric situations. Therefore, the research on Codazzi tensors is vast and has found many applications, and it would require a very long read to cover all aspects, even superficially. Some of these results can be found in the monograph \cite{5} which was published in 1987. On the other hand, there are many papers on the geometry of Codazzi tensors \cite{1,2,9,13,14,29} which were published in subsequent years.

In turn, we introduced in \cite{37} the notion of Codazzi $p$-tensors ($p \geq 2$) which extends the well-known concept for $p = 2$. Let us recall that a Codazzi $p$-tensor or, in other words, a higher-order Codazzi tensor is a $C^\infty$-section $T$ of the vector bundle $S^p M$ on $M$ satisfying the following equation:

\[ (\nabla_{X_0} T) (X_1, X_2, \ldots, X_p) = (\nabla_{X_1} T) (X_0, X_2, \ldots, X_p) \]

for any tangent vector fields $X_0, X_1, \ldots, X_p$ on $M$. The theory of higher-order Codazzi tensors was developed in the papers from the following list \cite{25,27,28}.

In the present paper, we study the question of nonexistence of higher-order Codazzi tensors and classical Codazzi tensors on complete and compact Riemannian manifolds, in particular. The classical Bochner technique \cite{3,48,49; 34, pp. 333–363} and its generalized version \cite{35} will help us to accomplish this task. We must recall here that the classic Bochner technique is an analytical method to obtain vanishing theorems for some topological or geometrical invariants on a compact (without boundary) Riemannian manifold, under some curvature assumption. The proofs of such theorems apply Bochner maximum principle and Green Theorem \cite{49, pp. 30–31}. We will also use a generalized version of the Bochner technique in the present paper. Therefore, we will use the Hopf maximum principle \cite{11, p. 47}, Yau, Li and Schoen results on the connections between the geometry of a complete smooth manifold and the global behavior of its subharmonic functions \cite{19,26,50}. We have demonstrated in our papers \cite{39,42} and \cite{43} that this generalized version of the Bochner technique is effective for the differential geometry “in the large”. In addition, the theorems and corollaries of the present paper will illustrate the effectiveness of this method for studying the global geometry of higher-order Codazzi tensors and, in particular, Codazzi tensor of order 2 on complete and compact Riemannian manifolds (see our Theorems 1, 3 and Corollary 2).

It is well known that the Ricci tensor $\text{Ric}$ is a Codazzi 2-tensor on an $n$-dimensional ($n \geq 3$) locally conformally flat manifold $(M, g)$ with constant (not necessarily zero) scalar curvature $s = \text{trace}_g \text{Ric}$ \cite{45}. By using this fact, we will give an application of the Bochner technique to the global geometry of complete locally conformally flat Riemannian manifolds (see our Corollary 5 and Theorem 4). On the other hand, if $(M, g)$ is a minimal hypersurface in the standard sphere $S^{n+1}$, then its second fundamental form $S$ is a traceless Codazzi 2-tensor \cite{4, p. 388}. By using this theorem, we will give an application of the Bochner technique to the global geometry of complete minimal hypersurfaces in a sphere (see Theorem 5).
The theorems and corollaries of this paper supplement our results from [37] and the results of other authors from [5, Theorem 16.9]; [18,28] and [45].

2 Main results

Everywhere in this paper we consider a higher-order Codazzi tensor $T$ as a smooth section of the subbundle $S^p_0 M$ of the bundle $S^p M$ on a Riemannian manifold $(M, g)$ defined by the condition

$$\text{trace}_g T = \sum_{i=1,...,n} T(e_i, e_i, X_3, \ldots, X_p) = 0$$

for $T \in C^\infty S^p_0 M$ and orthonormal basis $\{e_i\}$ of $T_x M$ at an arbitrary point $x \in M$. In this case, $T$ is called traceless. It can be proved that an arbitrary traceless Codazzi $p$-tensor $T$ is a divergence-free tensor field, i.e., $\delta T = 0$ for the formal adjoint operator $\delta: C^\infty (\otimes^{p+1} T^* M) \rightarrow C^\infty (\otimes^p T^* M)$ of the covariant derivative $\nabla: C^\infty (\otimes^p T^* M) \rightarrow C^\infty (\otimes^{p+1} T^* M)$ where $T^* M$ is the cotangent bundle on $M$ (see [4] and [5, p. 54]).

**Remark 1** In [27], the dimension of the vector space of traceless Codazzi $p$-tensors ($p \geq 2$) on compact Riemannian surfaces of genus $\gamma$ was determined. It depends only on $p$ and $\gamma$. Additionally the result was extended to genus zero.

The following Bochner–Weitzenböck formula for an arbitrary Codazzi $p$-tensor $T \in C^\infty S^p_0 M$ holds

$$\frac{1}{2} \Delta_B \| T \|^2 = Q_p (T, T) + \| \nabla T \|^2,$$

(2)

where $\Delta_B = \text{div} \circ \text{grad}$ is the Laplace–Beltrami operator, $\| T \|^2 = g(T, T)$, $\| \nabla T \|^2 = g(\nabla T, \nabla T)$ and $Q_p$ is a quadratic form $Q_p: S^p_0 M \otimes S^p_0 M \rightarrow \mathbb{R}$ which can be algebraically expressed through the curvature tensor $R$ and the Ricci tensor $\text{Ric}$ of $(M, g)$.

The curvature tensor $R$ of $(M, g)$ induces an algebraic curvature operator $\bar{\bar{\circ}} R: S^2_0 M \rightarrow S^2_0 M$. The symmetries of the curvature tensor $R$ imply that $\bar{\bar{\circ}} R$ is a selfadjoint operator, with respect to the point-wise inner product on $S^2_0 M$. That is why, the eigenvalues of $\bar{\bar{\circ}} R$ are real numbers at each point $x \in M$. Thus, we say $\bar{\bar{\circ}} R$ is positive semidefinite (resp. strictly positive), or simply $\bar{\bar{\circ}} R \geq 0$ (resp. $\bar{\bar{\circ}} R > 0$), if all the eigenvalues of $\bar{\bar{\circ}} R$ are nonnegative (resp. strictly positive). In the next paragraph we will prove that $Q_p (T, T) \geq 0$ for an arbitrary $T \in C^\infty S^p_0 M$ if the curvature operator $\bar{\bar{\circ}} R$ is positive semidefinite on $S^2_0 M$ and $Q_p (T, T) > 0$ for an arbitrary nonzero $T \in C^\infty S^p_0 M$ if the curvature operator $\bar{\bar{\circ}} R$ is positive definite on $S^2_0 M$.

**Remark 2** The curvature operator $\bar{\bar{\circ}} R$ has been studied in many papers and monographs. It is famous for its numerous applications [5, pp. 51–52; 346–347]; [10,22,40,44]. Beside the curvature operator $\bar{\bar{\circ}} R$ there is a curvature operator $\bar{\bar{\circ}} R: \Lambda^2 M \rightarrow \Lambda^2 M$ (see [34, pp. 83]). It is also widely used in Riemannian geometry. Examples are given by the well-known Gallot–Meyer theorem [34, p. 351] on harmonic $p$-forms $\omega \in C^\infty \Lambda^p M$ on a compact manifold $(M, g)$ and its $p$-th Betti number $\beta_p (M)$, and theorems from [42] and [43]. Therefore, $\bar{\bar{\circ}} R$ is also referred to as the curvature operator of the second kind [22].
Theorem 1 Let \( \hat{R} \geq 0 \) at any point of a connected open domain \( U \) of \((M, g)\) and \( T \) be a traceless Codazzi \( p \)-tensor \((p \geq 2)\) defined at any point of \( U \). If the scalar function \( \| T \|_2^2 \) has a local maximum at some point of \( U \), then \( \| T \|_2^2 \) is a constant function and \( T \) is invariant under parallel translations in \( U \). Moreover, if \( \hat{R} > 0 \) at some point \( x \in U \), then \( T \equiv 0 \).

Let \((M, g)\) be a compact Riemannian manifold with the curvature operator \( \hat{R} \geq 0 \). Then there exists a point \( x \in M \) at which the function \( \| T \|_2^2 \) attains the maximum. At the same time, \( \| T \|_2^2 \) satisfies the condition \( \Delta B \| T \|_2^2 \geq 0 \) everywhere in \((M, g)\). If, moreover, \( \hat{R} > 0 \) at some point \( x \in M \), then \( T \equiv 0 \) everywhere on \((M, g)\). As a result, we obtain the corollary that was proved in our paper [37]. Moreover, it is a generalization of the Berger–Ebin theorem on an arbitrary Codazzi \( 2 \)-tensor with constant trace on a compact Riemannian manifold [4]; [5, p. 436].

Corollary 2 Every traceless Codazzi \( p \)-tensor \((p \geq 2)\) on a compact Riemannian manifold \((M, g)\) with nonnegative curvature operator is invariant under parallel translations. Moreover, if \( \hat{R} > 0 \) at some point \( x \in M \), then there is no nonzero traceless Codazzi \( p \)-tensor \((p \geq 2)\).

Theorem 1 Let \((M, g)\) be a complete noncompact Riemannian manifold with nonnegative curvature operator \( \hat{R} \). Then there is no nonzero traceless Codazzi \( p \)-tensor \((p \geq 2)\) on \((M, g)\) such that \( \int_M \| T \|_2^q \, dV_g \leq +\infty \) for some \( q \geq 1 \).

Remark 3 In [6] was proved that \( g(\hat{R}(T), T) \geq 0 \) for any \( T \in C^\infty S^p_0 M \) and \( p \geq 2 \) if \( \sec \geq 0 \) for the sectional curvature \( \sec \) of \((M, g)\). Therefore, we can reformulate our Lemma 1, Corollary 1 and Theorem 1 by using this statement. In particular, we can state that every traceless higher-order Codazzi tensor on a compact Riemannian manifold with nonnegative sectional curvature is invariant under parallel translations.

Let us formulate a theorem that supplements Theorem 2 from our paper [37] which was proved for a compact Riemannian manifold.

A Riemannian manifold \((M, g)\) is locally conformally flat if, for any \( x \in M \), there exists a neighborhood \( U \) of \( x \) and \( C^\infty \)-function \( f \) on \( U \) such that \( (U, e^2 f g) \) is flat [5, p. 60]. We remind that a manifold \((M, g)\) of dimension \( n \) \((n \geq 4)\) is locally conformally flat if and only if its Weyl tensor \( W \) is identically zero [5, p. 60]. The Schouten tensor \( Sch \) plays an important role in the description of such manifolds [20]. This tensor has the form \( Sch = (n - 2)^{-1} (Ric - s (2n - 2)^{-1} g) \) for the Ricci tensor \( Ric \) and the scalar curvature \( s = \text{trace}_g Ric \) of \((M, g)\). Let us formulate the following statement.

Corollary 2 Let \((M, g)\) be a complete noncompact locally conformally flat Riemannian manifold of dimension \( n \) \((n \geq 4)\) with the nonnegative definite Schouten tensor. Then there is no nonzero traceless Codazzi \( p \)-tensor \((p \geq 2)\) on \((M, g)\) such that \( \int_M \| T \|_2^q \, dV_g \leq +\infty \) for some \( q \geq 1 \).

Remark 4 Corollary 2 supplements to some results on compact locally conformally flat Riemannian manifolds from [28]. We note that the condition of the nonnegative definiteness of the Schouten tensor in the case of constant positive scalar curvature means for a complete manifold that it is compact [34, p. 251].
Let us now consider a nonzero Codazzi 2-tensor on a Riemannian manifold, i.e., a symmetric bilinear form \( T \in C^\infty S^2 M \) satisfying the Codazzi equation (1).

First, we consider a symmetric bilinear form \( T \in C^\infty S^2 M \) as a 1-form with values in the cotangent bundle \( T^* M \). This bundle is equipped with the Levi-Civita covariant derivative \( \nabla \), thus there is an induced exterior differential \( d^\nabla : C^\infty S^2 M \to C^\infty (A^2 M \otimes T^* M) \) on \( T^* M \)-valued differential one-forms such as \( d^\nabla T (X, Y, Z) = (\nabla_X T) (Y, Z) - (\nabla_Y T) (X, Z) \) for any tangent vector fields \( X, Y, Z \) on \( M \) and an arbitrary \( T \in C^\infty S^2 M \) [5, p. 355]; [7]; [34, pp. 349–350]. In this case, a symmetric bilinear form \( T \in C^\infty S^2 M \) is a Codazzi 2-tensor if and only if \( d^\nabla T \) vanishes [7]; [34, p. 350].

On the other hand, Petersen called [34, p. 350] a symmetric bilinear form \( T \in C^\infty S^2 M \) harmonic if \( T \in \text{Ker} \ d^\nabla \bigcap \text{Ker} \delta \). In this case, \( T \) is a divergence-free Codazzi 2-tensor. In addition, from (1) we obtain the equation \( \delta T = -d \left( \text{trace}_g T \right) \) for an arbitrary Codazzi 2-tensor \( T \in C^\infty S^2 M \). Therefore, we can conclude that a bilinear form \( T \in C^\infty S^2 M \) is harmonic if and only if it is a Codazzi 2-tensor with constant trace [34, p. 350]. Moreover, the following statement holds.

**Theorem 2** The vector space of harmonic symmetric bilinear forms on a compact Riemannian manifold is finite dimensional.

**Remark 5** Bourguignon proved [8, p. 281] that a compact orientable Riemannian four-manifold admitting a non-trivial Codazzi tensor of order 2 with constant trace (harmonic symmetric bilinear form) must have signature zero.

Everywhere in the following we will consider a harmonic symmetric bilinear form or, in other words, we will consider a Codazzi 2-tensor with constant trace.

We have the Bochner–Weitzenböck formula

\[
\frac{1}{2} \Delta_B \| T \|^2 = Q_2 (T, T) + \| \nabla T \|^2, \tag{3}
\]

for an arbitrary harmonic symmetric bilinear form. Here the sign of the quadratic form \( Q_2 (T, T) \) depends on the sign of the sectional curvature \( \text{sec} \) of \((M, g)\) [4]. We remind here that an arbitrary Codazzi 2-tensor \( T \) on \((M, g)\) commutes with the Ricci tensor \( Ric \) at each point \( x \in M \) [5, p. 439]; [15]. Therefore, the eigenvectors of an arbitrary Codazzi tensor \( T \) determine the principal directions of the Ricci tensor at each point \( x \in M \) [17, pp. 113–114]. The converse is also true. Then taking into account of (3) and using the “Hopf maximum principle”, we will prove in the next paragraph that the following lemma holds.

**Lemma 2** Let \( U \) be a connected open domain \( U \) of a Riemannian manifold \((M, g)\) and \( T \) be a harmonic symmetric bilinear form defined at any point of \( U \). If the sectional curvature \( \text{sec} \ (e_i \wedge e_j) \geq 0 \) for all vectors of the orthonormal basis \( \{e_i\} \) of \( T_x M \) which is determined by the principal directions of the Ricci tensor \( Ric \) at an arbitrary point \( x \in U \) and \( \| T \|^2 \) has a local maximum in the domain \( U \), then \( \| T \|^2 \) is a constant function and \( T \) is invariant under parallel translations in \( U \). Moreover, if \( \text{sec} \ (e_i \wedge e_j) > 0 \) at some point \( x \in U \), then \( T \) is trivial.

If \((M, g)\) is a compact manifold and a harmonic symmetric bilinear form \( T \) is given in a global way on \((M, g)\) then due to the “Bochner maximum principle” for compact manifold it follows the classical Berger–Ebin theorem [4] and [5, p. 436] which is a corollary of our Lemma 2 (and see also Remark 3).

**Corollary 3** Every harmonic symmetric bilinear form \( T \in C^\infty S^2 M \) on a compact Riemannian manifold \((M, g)\) with nonnegative sectional curvature is invariant under parallel translations.
translations. If at the same time, the sectional curvature $\text{sec} > 0$ at some point, then $T \in C^\infty S^2M$ is trivial.

**Remark 6** It is well known that every parallel symmetric tensor field $T \in C^\infty S^2M$ on a connected locally irreducible Riemannian manifold $(M, g)$ is proportional to $g$, i.e., $T = \lambda g$ for some constant $\lambda$. Due to this, the second parts of Corollary 3 can be reformulated in the following form: Moreover, if $(M, g)$ a connected locally irreducible Riemannian then $T$ is trivial.

For example, let $(M, g)$ be a *Riemannian symmetric space of compact type* that is a compact Riemannian manifold with nonnegative sectional curvature and positive-definite Ricci tensor (see [23, p. 256]). Moreover, if a Riemann symmetric space of compact type is a locally irreducible Riemannian manifold $(M, g)$ then it is a compact Riemannian manifold with positive sectional curvature [16]. Therefore, we can formulate the following corollary.

**Corollary 4** Every harmonic symmetric bilinear form on a Riemannian symmetric manifold of compact type is invariant under parallel translations. If in addition to the above mentioned the manifold is locally irreducible then harmonic symmetric bilinear forms are trivial.

The following theorem supplements the Berger–Ebin theorem [4] and [5, p. 436] for the case of a complete noncompact Riemannian manifold.

**Theorem 3** Let $(M, g)$ be a connected complete noncompact Riemannian manifold with nonnegative sectional curvature. Then there is no a nonzero harmonic symmetric bilinear form $T$ which satisfies the condition $\int_M \| T \|^q \, d \text{Vol}_g < +\infty$ at least for one $q \geq 1$.

**Remark 7** In the case of a locally conformally flat $n$-dimensional $(n \geq 4)$ Riemannian manifold $(M, g)$ the following equalities hold [45,46]

$$\text{sec} \left( e_i \wedge e_j \right) = \frac{1}{n-2} \left( r_i + r_j - \frac{r_1 + \cdots + r_n}{n-1} \right); \quad \text{sec} \left( e_i \wedge e_j \right) = \frac{\lambda_i + \lambda_j}{n-2},$$

where $\{ e_i \}$ is an orthonormal basis of $T_x M$ at an arbitrary point $x \in M$ such that $\text{Ric}(e_i, e_j) = r_i \delta_{ij}$, $\text{Sch}(e_i, e_j) = \lambda_i \delta_{ij}$ and $\lambda_i = (n-2)^{-1} \left( r_i - \frac{s}{2n-2} \right)$. Due to these equations, we can formulate analogues of Lemma 2 and Corollary 3 for a locally conformally flat Riemannian manifold where the inequality $\text{sec} \left( e_i \wedge e_j \right) \geq 0$ can be replaced by $\lambda_i + \lambda_j \geq 0$ or $r_i + r_j \geq (n-1)^{-1} \left( r_1 + \cdots + r_n \right)$ for any $i \neq j$.

For an $n$-dimensional $(n \geq 3)$ locally conformally flat manifold $(M, g)$ with constant (not necessarily zero) scalar curvature its Ricci tensor $\text{Ric}$ is a Codazzi 2-tensor with constant trace or, in other words, a harmonic symmetric bilinear form [5, p. 444]; [45]. Therefore, from Theorem 3 we conclude that the following corollary holds.

**Corollary 5** Let $(M, g)$ be an $n$-dimensional $(n \geq 3)$ connected complete noncompact locally conformally flat Riemannian manifold with nonnegative sectional curvature and constant scalar curvature. If $(M, g)$ is not locally flat then $\int_M \| \text{Ric} \|^q \, d \text{Vol}_g = +\infty$ for the Ricci tensor $\text{Ric}$ and an arbitrary $q \geq 1$.

**Remark 8** Our Corollary 5 is a supplement to the theorem of Tani [45].

We strengthen Corollary 5 by proving the validity of the following

**Theorem 4** Let $(M, g)$ be an $n$-dimensional $(n \geq 3)$ connected complete locally conformally flat Riemannian manifold such that $\| \text{Ric} \|^2 < (n-1)^{-1} s^2$ for the Ricci tensor $\text{Ric}$ and the positive constant scalar curvature $s$. If one of the following conditions is satisfied:

$$\sum_{i,j} r_i r_j < \frac{s^2}{n}.$$
(i) \( \| \text{Ric} \|_2^2 \) has a global maximum point;
(ii) \( \int_M \| \text{Ric} \|_q^q \ d\text{Vol}_g < +\infty \) at least for one \( q \geq 2 \);
(iii) \((M, g)\) is a parabolic manifold, then \((M, g)\) is a spherical space form.

Remark 9 Theorem 4 is a supplement to the following Goldberg theorem [18].

Let \((M, g)\) be an \( n \)-dimensional connected complete minimal hypersurface in the standard sphere \((\mathbb{S}^{n+1}, g_0)\) where \( \mathbb{S}^{n+1} \subset \mathbb{R}^m \) for \( M > n + 1 \). In this case, we know that the second fundamental form \( S \) of the hypersurface \((M, g)\) is a traceless Codazzi tensor \([4, \text{p. 388}]\). Therefore, (3) can be rewritten in the form \([33]\)

\[
\frac{1}{2} \Delta_B \| S \|^2 = \| S \|^2 (n - \| S \|^2) + \| \nabla S \|^2
\] (4)

The following theorem holds.

Theorem 5 Let \((M, g)\) be a connected complete minimal hypersurface in the standard sphere \((\mathbb{S}^{n+1}, g_0)\) such that its second fundamental form \( S \) satisfies the inequality \( \| S \|^2 \leq n \). If one of the following conditions is satisfied:

(i) \( \| S \|^2 \) has a global maximum point;
(ii) \( \int_M \| S \|^q \ d\text{Vol}_g < +\infty \) at least for one \( q \geq 2 \);
(iii) \((M, g)\) is a parabolic manifold

then one of the following occurs:

(i) \((M, g)\) is an equator of \((\mathbb{S}^{n+1}, g_0)\);
(ii) \((M, g)\) is locally isometric to a generalized Clifford torus.

Remark 10 If \((M, g)\) is a complete Riemannian manifold with finite volume then the conditions (ii) of Theorems 4 and 5 are satisfied. For example, from \( \| S \|^2 \leq n \) we obtain \( \int_M \| S \|^2 \ d\text{Vol}_g \leq n \int_M \text{dVol}_g \leq n \text{Vol}_g (M) < +\infty \).

3 Proofs of the statements

Let us deduce the Weitzenböck formula (2) for a traceless Codazzi \( p \)-tensor \( T \) for \( p \geq 2 \). For this purpose, we remind that the local components of the Ricci tensor \( \text{Ric} \) of the manifold \((M, g)\) are calculated from the identity \( \text{Ric} (\partial_i, \partial_j) = R_{ij} = R_{iklj} \) for the local components \( R_{iklj} \) of the curvature tensor \( R \) determined from the equality \( R_{iklj} X^l = \nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k \) where \( \nabla_i = \nabla_{\partial_i} \) and \( X = X^i \partial_i \). Let us denote by \( s := g^{ij} R_{ij} \) the scalar curvature of the metric \( g \) for \( (g^{ij}) = (g_{ij})^{-1} \). Then direct calculations give
\[ \frac{1}{2} \Delta_B \parallel T \parallel^2 := \frac{1}{2} g^{kl} \nabla_k \nabla_l \left( T^{k_1k_2\ldots k_p} T_{k_1k_2\ldots k_p} \right) \]

\[ = \left( g^{kl} \nabla_k \nabla_l T_{k_1k_2\ldots k_p} \right) T^{k_1k_2\ldots k_p} + g^{kl} \left( \nabla_k T_{k_1k_2\ldots k_p} \right) \left( \nabla_l T^{k_1k_2\ldots k_p} \right) \]

\[ = R_{ij} T^{ik_2\ldots k_p} T_{k_2\ldots k_p}^j - (p - 1) R_{ijkl} T^{ik_3\ldots k_p} T_{k_3\ldots k_p}^j + g^{kl} \left( \nabla_k T_{k_1k_2\ldots k_p} \right) \left( \nabla_l T^{k_1k_2\ldots k_p} \right) \]

\[ = Q_p \left( T, T \right) + \parallel \nabla T \parallel^2 \]

where the quadratic form \( Q_p \left( T, T \right) \) has the form

\[ Q_p \left( T, T \right) = R_{ij} T^{ik_2\ldots k_p} T_{k_2\ldots k_p}^j - (p - 1) R_{ijkl} T^{ik_3\ldots k_p} T_{k_3\ldots k_p}^j \]  

(5)

for the components \( T_{k_1\ldots k_p} \) of an arbitrary Codazzi \( p \)-tensor \( T \in C^\infty S^p_0 M \) with respect to a local coordinate system \( x^1, \ldots, x^n \). In deriving the formula, we have used the condition of divergence-free of the Codazzi \( p \)-tensor \( T \) and well-known Ricci identity (11.16) from [17].

The curvature operator \( \tilde{\mathcal{R}} : S^2_0 M \to S^2_0 M \) is defined by equations [5, pp. 51–52]

\[ \left( \tilde{\mathcal{R}} T \right)_{ie} = R_{ijkl} T^{ik} = g^{km} g^{jp} R_{ijkl} T_{mp} \]

for the local components \( T_{ij} \) of arbitrary \( T \in S^2_0 M \). Everywhere else we assume that the curvature operator \( \tilde{\mathcal{R}} \) is nonnegative definite on an arbitrary section of the bundle \( S^2_0 M \), i.e., the inequality \( R_{ijkl} T^{ik} T^{jk} \geq 0 \) is true for an arbitrary \( T \in S^2_0 M \) and then \( R_{ijkl} T^{ik_3\ldots k_p} T_{k_3\ldots k_p}^{jk} \geq 0 \) for any \( T \in S^0_0 M \). As a result, the second term in (5) will be nonpositive, i.e., \( (p - 1) R_{ijkl} T^{ik_3\ldots k_p} T_{k_3\ldots k_p}^{jk} \leq 0 \).

At an arbitrary point \( x \in M \), we choose orthogonal unit vectors \( X, Y \in T_x M \) and define the tensor \( \theta \in S^2_0 M \) by the equality

\[ \theta = 2^{-1} \left( X \otimes Y + Y \otimes X \right) \]

then

\[ g \left( \tilde{\mathcal{R}} \left( \theta, \theta \right), \theta \right) = 2 g \left( R \left( X, Y \right), Y, X \right) = 2 \sec \left( X \wedge Y \right). \]

That is why the sectional curvature of a manifold \( (M, g) \) is everywhere nonnegative if the operator \( \tilde{\mathcal{R}} \) is nonnegative definite on any section of the bundle \( S^2_0 M \). Let \( X \in T_x M \) is a unit vector and we complete it to an orthonormal basis \( X, e_2, \ldots, e_n \) for \( T_x M \) at an arbitrary point \( x \in M \), then [34, p. 86]

\[ \text{Ric} \left( X, X \right) = \sum_{a=2}^n \sec \left( X \wedge e_a \right). \]

Therefore, the Ricci curvature is also nonnegative definite. Thus, if the operator \( \tilde{\mathcal{R}} \) is nonnegative definite on sections of the bundle \( S^2_0 M \), then \( Q_p \left( T, T \right) \geq 0 \). As a result of (3) it follows that \( \Delta_B \parallel T \parallel^2 \geq 0 \), i.e., \( \parallel T \parallel^2 \) is a nonnegative subharmonic function. Moreover, we note that \( Q_p \left( T_x, T_x \right) \geq 0 \) for all nonzero \( T_x \in S^p_0 \left( T_x M \right) \) at some point \( x \in M \) if \( \tilde{\mathcal{R}} > 0 \) for all nonzero \( T_x \in S^2_0 \left( T_x M \right) \) at this point.

Let us prove our Lemma 1. Consider a traceless Codazzi \( p \)-tensor \( (p \geq 2) \) in the connected open domain \( U \) of \( (M, g) \) where the curvature operator \( \tilde{\mathcal{R}} \geq 0 \) then \( Q_p \left( T, T \right) \geq 0 \). And
according to (3) we conclude that $\Delta_B \| T \|^2 \geq 0$, i.e., $\| T \|^2$ is a subharmonic function in the domain $U$. Suppose also that $\| T \|^2$ has a local maximum at some point $C \in U$ then according to the “Hopf’s maximum principle” [2, p. 47] we have that $\| T \|^2$ is a constant function in the domain $U \subset M$. In this case, $\Delta \| T \|^2 = 0$ and as a consequence of (3) we obtain $\| \nabla T \|^2 = 0$ which means that the Codazzi $p$-tensor $T$ is parallel.

Let $\| T \|^2 = C$ (where $C$ is a constant), then $T$ does not become zero anywhere in the domain $U$ and at the same time $\Delta_B \| T \|^2 = 0$. If there is a point $x \in U$ where $\hat{R} > 0$ then, as we stated above, $Q_p(\bigtriangledown_x, \bigtriangledown_x) > 0$. In this case, the left side of (3) is equal to zero and its right side is greater than zero, so it follows that $T = 0$. Lemma 1 is proved.

Corollary 1 of our Lemma 1 does not require any proof.

For the proof of our Theorem 1 in the case $q = 1$, we use Theorem 1 from [19] according to which any nonnegative subharmonic function $f \in C^\infty M$ on a connected complete noncompact Riemannian manifold with nonnegative sectional curvature must satisfy the condition $\int_M f \text{ dVol}_{g} = +\infty$.

Let us suppose that $(M, g)$ is a complete noncompact Riemannian manifold with nonnegative curvature operator $\hat{R}$ (and hence with nonnegative sectional curvature) and with a globally defined nonzero Codazzi $p$-tensor $T \in C^\infty S^p_0 M$ for an arbitrary $p \geq 2$.

By direct calculation, we find the following

$$\frac{1}{2} \Delta_B \| T \|^2 = \| T \| \Delta_B \| T \| + \| d \| T \| \| T \|^2.$$

Then the equation (2) can be rewritten in the form

$$\| T \| \Delta_B \| T \| = Q_p(\bigtriangledown, \bigtriangledown) + \| \nabla T \|^2 - \| d \| T \|^2.$$

By using the first Kato inequality $\| \nabla T \|^2 \geq \| d \| T \|^2$ (see [12]), we can write

$$\| T \| \Delta_B \| T \| \geq Q_p(\bigtriangledown, \bigtriangledown).$$

Therefore, if $Q_p(\bigtriangledown, \bigtriangledown) \geq 0$ then we have $\Delta \| T \| \geq 0$ and as a result of this $\| T \|$ is a nonnegative subharmonic function. Now according to Theorem 1 from [19] we come to the conclusion that $\int_M \| T \| \text{ dVol}_{g} = +\infty$ for an arbitrary nonzero Codazzi $p$-tensor $T \in C^\infty S^p_0 M$. Thus, on a complete noncompact Riemannian manifold with nonnegative curvature operator (and that is why with nonnegative sectional curvature) there exists no nonzero Codazzi tensor $T \in C^\infty S^p_0 M$ such that $\int_M \| T \| \text{ dVol}_{g} < +\infty$.

For the proof of our Theorem 1 in the case $q > 1$ we use Theorem 7 from [50]. According to this theorem any nonnegative subharmonic function on a connected complete noncompact Riemannian manifold satisfies the condition $\int_M f^q \text{ dVol}_{g} = +\infty$ for $q \geq 1$ or $f$ is const.

Let us suppose that a nonzero traceless Codazzi $p$-tensor $T$ is globally defined on a complete noncompact Riemannian manifold $(M, g)$ with nonnegative curvature operator $\hat{R}$. In this case $\| T \|$ is a nonnegative subharmonic function. If in addition to the above-mentioned $T$ satisfies the condition $\int_M \| T \|^q \text{ dVol}_{g} < +\infty$ for some $q > 1$, then due to the Yau theorem we conclude that $\| T \| = C$ for some nonnegative constant $C$. Therefore, the condition $\int_M \| T \|^q \text{ dVol}_{g} < +\infty$ can be rewritten in the form $C^q \int_M \text{ dVol}_{g} < +\infty$. On the other hand, every complete noncompact Riemannian manifold with nonnegative Ricci curvature or with nonnegative sectional curvature has infinite volume [19,50]. Therefore, from the inequality $C^q \int_M \text{ dVol}_{g} < +\infty$ we conclude that $\| T \| = C = 0$. This completes our proof.
Let us prove Corollary 2. Suppose that \((M, g)\) is a \(n\)-dimensional \((n \geq 4)\) complete noncompact locally conformally flat Riemannian manifold. In this case, the curvature tensor \(R\) has the following local components [5, pp. 60–61]; [45]:

\[
R_{ijkl} = \frac{1}{n-2} \left( R_{jil} g_{ik} - R_{jkgil} + R_{ikgjl} - R_{ilgjk} \right) - \frac{s}{(n-1)(n-2)} \left( g_{jl} g_{ik} - g_{jkgil} \right). \tag{6}
\]

Then from (6) we obtain the following equalities

\[
R_{ijkl} \theta^{jk} \theta^{il} = \frac{2}{n-2} \left( R_{ij} \theta^{ik} \theta^{j}_k - \frac{s}{2(n-1)} \| \theta \|^2 \right)
= \frac{1}{n-2} \left( \left( R_{jl} - \frac{s}{2(n-1)} g_{jl} \right) g_{ik} + \left( R_{ik} - \frac{s}{2(n-1)} g_{ik} \right) g_{jl} \right) \theta^{il} \theta^{jk}
= (S_{jl} g_{ik} + S_{ik} g_{jl}) \theta^{il} \theta^{jk} = 2 S_{ij} \theta^{ik} \theta^{j}_i.
\]

for the local component \(S_{ik}\) of the Schouten tensor \(Sch \in C^\infty S^2 M\) and the local components \(\theta_{ik}\) of an arbitrary traceless tensor \(\theta \in S^2_0 M\). Then from the condition of nonnegative definiteness of the Schouten tensor \(Sch \in C^\infty S^2 M\) we obtain the inequality \(\tilde{R} \geq 0\). After that we should repeat the proof of Theorem 1.

Let us prove Theorem 2. We define the following differential operator: \(\delta^*: C^\infty T^* M \to C^\infty S^2 M\) by the equality \((\delta^* \xi)(X, Y) = 2^{-1}((\nabla_X \xi)Y + (\nabla_Y \xi)X)\) for any tangent vector fields \(X, Y, Z\) on \(M\) and an arbitrary one-form \(\xi \in C^\infty T^* M\). Moreover, we denote by \(\delta^* V\) the adjoint operator of \(d^V [4, \text{p. 380}; 388]; [5, \text{p. 355}]. \) If \((M, g)\) is a compact manifold then the condition \(T \in \text{Ker} d^V \cap \text{Ker} \delta\) is equivalent to \(T \in \text{Ker} \Psi\) for \(\Psi = \delta^* d^V + 2 \delta^* \delta\) [4, p. 388]. The differential operator \(\Psi\) is an elliptic operator with injective symbol and \(\text{Ker} \Psi = \text{Ker} d^V \cap \text{Ker} \delta\) [4, p. 388]. Therefore, \(\text{Ker} \Psi\) is a finite-dimensional vector space of harmonic symmetric bilinear form on a compact manifold \((M, g)\) [5, p. 464]. This completes our proof.

Let us prove Lemma 2. First, we rewrite the Bochner–Weitzenböck formula (3) in the form

\[
\frac{1}{2} \Delta_B \| \tilde{T} \|^2 = Q_2 (\tilde{T}, \tilde{T}) + \| \nabla \tilde{T} \|^2 \tag{7}
\]

where \(Q_2 (\tilde{T}, \tilde{T}) = R_{ij} \tilde{T}^{ik} \tilde{T}^j_k - R_{ijkl} \tilde{T}^{ik} \tilde{T}^{jl}_k\) for a traceless Codazzi 2-tensor \(\tilde{T}\).

Second, if \(T\) is a Codazzi tensor with constant trace, then \(\tilde{T} = T - 1/n (\text{trace}_g T) g\) is a traceless Codazzi tensor such that \(\Delta_B \| \tilde{T} \|^2 = \Delta_B \| T \|^2\) and \(\nabla \tilde{T} = \nabla T\). In addition, one can prove that \(Q_2 (\tilde{T}, \tilde{T}) = Q_2 (T, T) = R_{ij} T^{ik} T^j_k - R_{ijkl} T^{ik} T^{jl}_k\). Then (7) can be rewritten in the form

\[
\frac{1}{2} \Delta_B \| T \|^2 = Q_2 (T, T) + \| \nabla T \|^2 \tag{8}
\]

According to [4, p. 388] and [5, p. 436] we conclude that

\[
Q_2 (T, T) = \sum_{i \neq j} \sec (e_i \wedge e_j) (\lambda_i - \lambda_j)^2, \tag{9}
\]

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where \( \{e_i\} \) is a such orthonormal basis of the tangent space \( T_xM \) at an arbitrary point \( x \in M \) that the Codazzi tensor \( T_x(e_i, e_j) = \lambda_i(x) \delta_{ij} \) where \( \delta_{ij} \) is the Kronecker symbol and \( \sec(e_i \wedge e_j) \) is the sectional curvature in the direction of subspace \( \pi(x) \subset T_xM \) such that \( \pi(x) = \text{span} \{e_i, e_j\} \). It is known from [5, p. 439] and [15] that an arbitrary Codazzi tensor \( T \) on a manifold \((M, g)\) commutes with its Ricci tensor \( Ric \), and therefore, the eigenvectors \( \{e_i\} \) of the Codazzi tensor \( T \) determine the principal directions of the Ricci tensor at each point \( x \in U \) [11, pp.113–114]. The converse is also true. Next, let us suppose that \( \sec(e_i \wedge e_j) \geq 0 \) in some connected open domain \( U \subset M \) then \( Q_2(T) \geq 0 \). Moreover, if there is a nonzero Codazzi tensor \( T \) given in \( U \subset M \) then from (8) we conclude that \( \Delta_B \|T\|^2 \geq 0 \), i.e., \( \|T\|^2 \) is a nonnegative subharmonic function in \( U \). Let us suppose \( \|T\|^2 \) has a local maximum at some point \( x \in U \) then \( \|T\|^2 \) is a constant function in \( U \subset M \) according to the “Hopf’s maximum principle” [11, p. 47]. In this case, \( \Delta_B \|T\|^2 = 0 \) and \( \nabla \|T\|^2 = 0 \). Then the latter equation means that the Codazzi tensor \( T \) is a parallel tensor field.

Let \( \|T\|^2 = C \) (\( C \) is a constant) then from (8) we obtain that \( Q_2(T, T) + \|\nabla T\|^2 = 0 \). Since \( \sec(e_i \wedge e_j) \geq 0 \) it means that \( Q_2(T, T) = 0 \) and \( \nabla T = 0 \). If there is a point \( x \in U \) such that \( \sec(e_i \wedge e_j) > 0 \) then from (9) we come to the conclusion that \( \lambda_1 = \cdots = \lambda_n = \lambda \) which is equivalent to \( T = 1/n \) (trace \( T \)) \( g \) (see [5, p. 436]). Lemma 2 is proved.

Assume that the manifold \((M, g)\) is compact and the Codazzi tensor is globally defined on \((M, g)\) then the “Bochner maximum principle” comes into force [49, p. 30] according to which a subharmonic function on a compact manifold is a constant. As a result, from our Lemma 2 we obtain Corollary 3 which is essentially the Berger–Ebin theorem [4], [5, p. 436].

Let us prove our Theorem 3. Let \((M, g)\) be a complete noncompact Riemannian manifold with nonnegative sectional curvature and \( T \) be a globally defined nonzero Codazzi 2-tensor with a constant trace. In this case \( Q_2(T, T) \geq 0 \) and according to Theorem 1 the norm \( \|T\|^2 \) is a subharmonic function. Therefore, due to Theorem 1 from [19] we come to the conclusion that there is no nonzero Codazzi tensor \( T \) on \((M, g)\) such that \( \int_M \|T\| \, dVol_g < +\infty \).

Let us turn to the Yau theorem [50, p. 663]. Yau’s theorem states the following: If \( \int_M \|T\|^q \, dVol_g < +\infty \) for some \( q > 1 \) on a complete \((M, g)\) then \( \|T\| = C \) (\( C \) is a constant). In this case, the condition \( \int_M \|T\|^q \, dVol_g < +\infty \) is equivalent to \( C^q \int_M dVol_g < +\infty \). The latter inequality is not feasible on a complete noncompact Riemannian manifold \((M, g)\) with nonnegative sectional curvature [19]. So \( \|T\| = C = 0 \) that contradicts the existence of a nonzero Codazzi tensor. Theorem 2 is proved.

Corollary 5 does not require any proof.

And now we prove Theorem 4. Let, as before, \((M, g)\) be an \( n \)-dimensional \( (n \geq 3) \) locally conformally flat Riemannian manifold with positive constant scalar curvature \( s = \text{trace}_g Ric \). In this case, for the traceless Ricci tensor \( \overline{Ric} = Ric - n^{-1}s \, g \) we have

\[
Q_2(Ric, Ric) = Q_2(\overline{Ric}, \overline{Ric}) = R_{ij} \overline{R}^{ik} \overline{R}^j_k - R_{ijkl} \overline{R}^{ik} \overline{R}^j_l = \frac{1}{n-1} s \, \| \overline{Ric} \| ^2 + \frac{n}{n-2} R_{ij} \overline{R}^{ik} \overline{R}^j_k
\geq \frac{1}{n-1} \| \overline{Ric} \| ^2 \left( s - \sqrt{n(n-1)} \, \| \overline{Ric} \| \right)
\]

where, due to Lemma 2.1 from [32], we used the inequality

\[
\overline{R}_{ij} \overline{R}^{ik} \overline{R}^j_k \geq -\frac{n-2}{\sqrt{n(n-1)}} \| \overline{Ric} \| ^3.
\]

Then, using the formula (7), we obtain the inequality

\[
\frac{1}{2} \Delta_B \| \overline{Ric} \| ^2 \geq \frac{1}{n-1} \| \overline{Ric} \| ^2 \left( s - \sqrt{n(n-1)} \, \| \overline{Ric} \| \right) + \| \nabla \overline{Ric} \| ^2. \tag{10}
\]
One can prove that $\Delta_B \| Ric \|^2 = \Delta_B \| \widetilde{Ric} \|^2$. Therefore, we can rewrite (10) in the following form

$$\frac{1}{2} \Delta_B \| Ric \|^2 \geq \frac{1}{n-1} \| \widetilde{Ric} \|^2 \left( s - \sqrt{n(n-1)} \| \widetilde{Ric} \| \right) \quad (11)$$

If we suppose that $\| Ric \|^2 < (n-1)^{-1} s^2$ then from (11) we conclude that $\Delta_B \| Ric \|^2 \geq 0$, i.e., $\| Ric \|^2$ is a nonnegative subharmonic function.

In the first case, if $(M, g)$ is a connected complete manifold and $\| Ric \|^2$ has a global maximum point, then due the “Hopf maximum principle” we obtain $\| Ric \|^2 = C$ where $C$ is a constant.

In the second case, we rewrite (10) in the following form

$$\| Ric \| \Delta_B \| Ric \| \geq \frac{1}{n-1} \| \widetilde{Ric} \|^2 \left( s - \sqrt{n(n-1)} \| \widetilde{Ric} \| \right)$$

$$+ \| \nabla Ric \| \| Ric \| \geq \frac{1}{n-1} \| \widetilde{Ric} \|^2 \left( s - \sqrt{n(n-1)} \| \widetilde{Ric} \| \right) \quad (12)$$

where we used the first Kato inequality $\| \nabla Ric \|^2 \geq \| d \| Ric \|^2$ (see [12]).

Let $\| Ric \|^2 < (n-1)^{-1} s^2$ then from (12) we conclude that $\| Ric \|$ is a nonnegative subharmonic function. Then for an arbitrary $q \geq 2$, either $\int_M \| Ric \|^q d \text{Vol}_g = +\infty$ or $\| Ric \| = C$ [50, p. 664]. Therefore, if we suppose that $\int_M \| Ric \|^q d \text{Vol}_g < +\infty$ at least for one $q \geq 2$, then $\| Ric \| = C$.

In the third case, we remind that a complete manifold $(M, g)$ is said to be parabolic if it does not admit a positive Green’s function. In addition, if $(M, g)$ is a parabolic manifold then every subharmonic and bounded-above function on $M$ is constant [35, p. 147]. In our case, $\| Ric \|^2$ is a subharmonic function on $(M, g)$ such as $\| Ric \|^2 < (n-1)^{-1} s^2$ for the constant scalar curvature $s > 0$. Therefore, for the case of parabolic manifold $(M, g)$ we have $\| Ric \|^2 = C$.

Finally, if $\| Ric \|^2 = C$ and $\| Ric \|^2 < (n-1)^{-1} s^2$ then from (11) we obtain $\| \widetilde{Ric} \|^2 = 0$. In this case $g$ is an Einstein metric. Therefore, $(M, g)$ becomes a complete Riemannian manifold with positive constant curvature, that means $(M, g)$ is a spherical space form [47, p. 69].

Let us prove our Theorem 5. In the first case, if $\| S \|^2 \leq n$ then from (4) we conclude that $\| S \|^2$ is a nonnegative subharmonic function. If, moreover, $(M, g)$ is a connected complete manifold and $\| S \|^2$ has a global maximum point, then due the “Hopf maximum principle” we obtain $\| S \|^2 = C$ where $C$ is a constant.

In the second case, we rewrite (4) in the following form

$$\| S \| \Delta_B \| S \| \geq \| S \|^2 \left( n - \| S \|^2 \right) \quad (13)$$

Let $\| S \|^2 \leq n$ then from (13) we conclude that $\| S \|$ is a nonnegative subharmonic function. Then for an arbitrary $q \geq 1$, either $\int_M \| S \|^q d \text{Vol}_g = +\infty$ or $\| S \| = C$ [50, p. 664]. Therefore, if we suppose $\int_M \| S \|^q d \text{Vol}_g < +\infty$ for some $q \geq 2$, then $\| S \| = C$.

In the third case, if $(M, g)$ is a parabolic manifold then $\| S \|^2 = C$ because $\| S \|^2$ is a subharmonic function on $(M, g)$ such that $\| S \|^2 \leq n$.

Finally, if $\| S \|^2 = C$ and $\| S \|^2 \leq n$ then, using the formula (4), we obtain either $\| S \|^2 = 0$ or $\| S \|^2 = n$. In the first case, $(M, g)$ must be totally geodesic. At the same time, we know that an arbitrary $n$-dimensional complete totally geodesic submanifold of the sphere $S^{n+p}$ is a sphere $S^n$ [31]. Therefore, $(M, g)$ has to be an equator $S^n \subset S^{n+1}$. In the second
case, \((M, g)\) is locally isometric to a generalized Clifford torus \(S^k(r_1) \times S^{n-k}(r_2)\), which is the standard product embedding of the product of two spheres of radius \(R_1 = \sqrt{k} \frac{n-1}{n}\) and \(r_2 = \sqrt{(n-k) \frac{n-1}{n}}\), respectively [24].

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