Discrete Space Formulation of Quantum Geometry on $R \times S^{d-1}$

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Abstract. A numerical formulation of $d$-dimensional quantum gravity on $R \times S^{d-1}$ is developed based on the $d$-dimensional dynamical triangulation method. It provides tools to analyze space-time properties of the $(d-1)$-dimensional space with sphere topology along the direction $R$, in which time coordinate can be defined naturally. As examples, we show numerical results of the 2-dimensional ($R \times S$) model, and compare with the matrix model. Furthermore, we measure the 2-space point correlation function on the last scattering surface of 3-dimensional space $S^3$, and compare it to the CMB anisotropy observation results of COBE and WMAP. The numerical simulation exhibits the inflation without any additional fields, and the quantized space-time possesses properties required from observations.

1. Introduction

Recently the observations of the cosmic microwave background (CMB) anisotropy [1, 2], suggest influences of the quantum gravity at the early universe. In order to analyze the anisotropy of CMB in terms of quantum gravity, we construct a numerical method for the quantum geometry with the topology $R \times S^{d-1}$. Although there are several theoretical as well as numerical solutions of the equation of motions of the classical general relativity, it is not possible to explain the correlation properties of the anisotropy of the CMB especially at large angular scale. We show that the numerical method based on the discretized space quantization is a candidate of approaches which can provide a picture of the early universe. From the numerical simulation, it is possible to discover the information comparing with the observation data directly.

For the quantization of the space-time geometry through the numerical approach, we need to discretize the manifold with a suitable method[3]. We introduce the dynamically balanced triangulation method which is a direct extension of the method known as the dynamical triangulation. We construct the $d$-dimensional discretized manifold with boundary of $(d-1)$-dimensional sphere, and compute the partition function as a set of discretized manifolds weighted properly. Numerical simulations based in a 2-dimensional space on this method show results which correspond to the 2-dimensional conformal field theory and Matrix model[4, 5].

To obtain the partition function, we construct a Markov chain by defining moves which satisfy the manifold conditions and the detailed balance. Moves are defined in the similar manner as the $(p, q)$-moves of the DT: adding or taking off $d$-simplices from $S^{(d-1)}$ boundary surface. In 2-dimensional case, we compare numerical results with theoretical analysis, i.e., the 2-dimensional conformal field theory and the matrix model. Our numerical formulation which build discretized surface with boundary, is related to the model discussed in Ref.[6, 7], as the 2-dimensional conformal field theory with boundary. The boundary condition with conformal
invariance is realized by adding a boundary cosmological constant term to the Liouville action. Thus properties of the discretized surface in 2-dimensional case is controlled by the lattice cosmological constant $\mu$, the lattice boundary cosmological constant $\mu_B$. The critical values which separate the topology and the geometry of the discretized surfaces, can be obtained from the $\phi^4$-matrix model. Moreover, it is also possible to compare numerical results and theoretical predictions of the Liouville theory on the disk\cite{6,8} for various disk amplitudes.

Furthermore, we measure the two point correlation function of the scalar curvature fluctuation on the discretized section universe $S^{d-1}$, in order to discuss the anisotropy on the CMB in our numerical formulation. We develop to 4-dimensional case and measure the correlation function on the last scattering surface (lss): the section universe $S^2$ extracted from the section universe $S^3$.

In the next section, we show the algorithm to build the discretized manifold with the topology $R \times S^{d-1}$ and introduce the modified dynamical triangulation rules which satisfy the manifold conditions. In section 3, we show the numerical results in 2, 3 and 4-dimensional cases and discuss in comparison with the theory in 2-dimension.

2. Numerical Formulation on $R \times S^{d-1}$

First, we introduce the dynamically balanced triangulation method for the computation of the partition function on the discretized manifold. We build up the discretized $d$-dimensional manifold with boundary by $d$-simplices. Our numerical formulation is similar to the dynamical triangulation method which has been developed in Euclidean simplicial quantum gravity. However, we need to modify the rules of triangulations in order to generate the discretized surface with topology $R \times S^{d-1}$, instead of the standard dynamical triangulation rules for the closed surface $S^d$. The modified triangulation rule, called $\{\delta S, \delta V\}$-move, generates a new configuration with changing the size of the section area $S$ and the volume $V$. The $\{\delta S, \delta V\}$-moves in 2, 3, and 4 dimensional case are illustrated in Fig. 1, 2 and 3. These rules satisfy the manifold condition for the whole universe $R \times S^{d-1}$ as well as the spatially closed universe $S^{d-1}$. The Euler relations on $R \times S^{d-1}$ and $S^{d-1}$ are fulfilled in the $\{\delta S, \delta V\}$-move, while the section universe satisfies the Dane-Sommerville relations in addition to the Euler relation.

The general 'action' on the discretized manifold is assumed as,

$$ S = \sum_{i=0}^{d} u_i N_i, \quad (1) $$

where $N_i$ denotes the number of $i$-simplices and $\{u_i\}$ are 'coupling constant' to control the global properties of the universe. Among $(d+1)$ parameters the Euler relation reduce one parameter for manifolds with boundary. In 2-dimensional case, the action $S$ is written as,

$$ S = \mu N_2 + \mu_B \tilde{N}_1, \quad (2) $$

where $N_2$ gives the universe volume $V$ and $\tilde{N}_1$ denotes the size of the boundary universe, i.e. the section area $S$. This action contains two parameters, the cosmological constant $\mu$ and the boundary cosmological constant $\mu_B$. From the analogy of the Euclidean simplicial quantum gravity, the action in $d$-dimensional case is written as,

$$ S = -\sum_{i}^{d-2} \kappa_i N_i + \mu N_d + \mu_B \tilde{N}_{d-1} \quad (3) $$

where $\kappa_{d-2}$ correspond to the Newton constant in 4-dimension.
Figure 1. Schematic image of $\{\delta S, \delta V\}$-moves in 2 dimensions. The target objects are colored with blue.

We need to compute the partition function for the quantization of the discretized geometry,

$$Z = \sum_T e^{-S(T)},$$

where $\sum_T$ is the sum over all distinct triangulations, $T$, with disk topology. We construct a Markov chain by the Monte-Carlo method to count all possible surfaces of distinct triangulations. We regard the quantization to be a set of distinct configurations with appropriate weight $e^{-S(T)}$. For this end we need to count number of states possible to make from a starting state, carefully. We employ the Metropolis Monte Carlo method for the probability of transition between states A and B as,

$$P_{A \rightarrow B} = \min \left( 1, \frac{n_A}{n_B} \exp (-S(B) + S(A)) \right),$$

where $n_A$ denotes the number of state possible to make transition from a starting state $A$.

3. Numerical results

Let us firstly look at results of the two-dimensional simulation. In Fig. 4 we show the volume changing along the Monte-Carlo computational time step, $\tau$, which indicates the volume $V$ and the sectional area $S$ grow linearly in $\tau$. When we define the physical time, $t$, through

$$V(t) = c \int_0^t S(t') dt',$$

within a scale factor $c$. These two time scales, $t$ and $\tau$, are related as

$$ct = \int_0^t d\tau' \frac{1}{S\tau'} \frac{dV(\tau')}{d\tau'}.$$
Figure 2. Schematic image of $\{\delta S, \delta V\}$-moves in 3 dimensions. The target objects are colored with blue.

From the numerical simulation, the relation between $t$ and $\tau$ is given as,

$$\tau \sim \exp\left(\frac{S_0}{V_0} ct\right),$$

which means the exponential growth of $S(t)$ (i.e. the inflation) with the velocity $S_0/V_0$. We show the volume growing along the physical time $t$ in Fig. 5.

The cosmological constant $\mu$ and the boundary cosmological constant $\mu_B$ control the growth rates of the volume $V$ and the section area $S$. In the parameter space of $\mu$ and $\mu_B$, we find three type universes, i.e., open, closed and collapsed. We show the image of the phase diagram in Fig. 6.

In the case of open universe, both the volume $V$ and the section area $S$ are inflating as shown in Fig. 5. From the analysis of the 2-dimensional Liouville theory[9, 8], the inflation velocity
Figure 3. Schematic image of $\{\delta S, \delta V\}$-moves in 4 dimensions. The target objects are colored with blue.
Figure 4. The volume $V$ and the surface area $S$ changing along the Monte-Carlo computation time $\tau$.

Figure 5. The volume $V$ and the surface area $S$ changing along the physical time $\tau$.

Figure 6. Three phases diagram and critical line for $\mu$ and $\mu_B$. 
$S/V$ is related to the renormalized cosmological constant $\Lambda$, 

$$\frac{S}{V} \propto \Lambda^{1/2}. \quad (9)$$

While in the numerical calculation the volume $V$ and the section $S$ grow as,

$$\begin{align*}
V & \propto \frac{1}{\Lambda^{1/2}} e^{\Lambda^{1/2} t}, \\
S & \propto e^{\Lambda^{1/2} t}. \quad (10)
\end{align*}$$

The boundary cosmological constant term control the growth rate of the section area $S$, and the space-time manifold turns to be closed for those $\mu_B$’s above a critical value. We can regard that the boundary cosmological constant $\mu_B$ corresponds to the one discussed in the 2-dimensional conformal field theory\[6\]. In the case of collapse universe, both the volume $V$ and the section $S$ shrink to a points. The tri-critical point is found at $\mu^c = 1.12(3), \mu^c_B = 0.84(3)$, which is close to those expected by the matrix model\[10, 11\] as $\mu^c = 1.12467, \mu^c_B = 0.83698$.

We extend the 2-dimensional algorithm to the higher-dimensional case. In the 3 and 4-dimensional cases, we observe four types of universes in the parameter space $\kappa_{d-2}$, $\mu$ and $\mu_B$. Similar to the 2-dimensional case, the open, closed and collapsed universe also appears. Besides, the open universe is separated into two phases, which we call dimple phase and crumpled phase in analogy to the Euclidean calculation of the 3- and 4-dimensional DT simulations, according to different expansion rates.

As one of the important indication of quantum space-time, we apply our method to the CMB anisotropy observed in COBE and WMAP. We measure the two point correlation function on the section universe $S^{d-1}$ for the analysis of the anisotropy of the CMB.

We measure the curvature-curvature correlation between two points separated by angle $\theta$ defined by the ratio of the geodesic distance and the total length of the great circle of the section universe $S^{d-1}$, 

$$\left\langle \frac{(R(0) - < R >)(R(\theta) - < R >)}{< R >^2} \right\rangle, \quad (11)$$

where the curvature $R$ is obtained by the deficit angle which is associated to the number of triangles sheared at the vertex in the 2-dimensional case. We show the correlation function in 2-dimensional case in Fig. 7. This result suggests that the initial quantum correlation which is dominated by nearest neighbors develops to extend into the large angular scale. Regarding the cross section $S^2$ extracted from the section $S^3$ as the last scattering surface, we also measure the curvature two point function on the section $S^2$ and show the numerical result in Fig. 8.

Moreover, we compute the power spectrum of the correlation function and compare with the observation data in Fig. 9. From these numerical results, we expect that the correlation function has similar trend with the observation data at the large angular scale.

4. Summary

In this talk, we discuss about a numerical construction of the quantum universe with topology $R \times S^{d-1}$. In order to satisfy the manifold condition, we introduce the triangulation rules, called $\{\delta S, \delta V\}$-moves, in an analogy to the rules known in the dynamical triangulation method. In principle we can generate all possible configurations for the discretized manifold with a boundary by the $\{\delta S, \delta V\}$-moves. Using the same algorithm we can obtain the section universe $S^{d-1}$ as an inverse process. By this method we are enabled to define the physical time $t$ along the growing of the universe and the last scattering surface by reversing the process. From the numerical results, the volume $V$ and the section are $S$ are growing exponentially and it is consistent to a classical solution of 2-dimensional Liouville theory. In the parameter space of $\mu$ and $\mu_B$, 

\begin{align*}
\mu & = 1.12(3), \\
\mu_B & = 0.84(3).
\end{align*}
Two Point Function \( S = 200 \) Average

**Figure 7.** Correlation function for the curvature \( R \) in 2-dimensional case.

Power Spectrum for Two Point Function at 4D Inflation phase

**Figure 8.** Correlation function for the curvature \( R \) in 4-dimensional case.

**Figure 9.** The power spectrum of the correlation function and compare with the observation data (WMAP2003[2]).
three distinct types of universes are expected and the numerical estimation of the tri-critical point which divides those phase agrees to the result of the matrix model in the 2-dimensional simulation. We also perform Monte-Carlo simulation to 3 and 4-dimensional case. In the 4-dimensional case, the space also expands exponentially. We measure the curvature correlation function on the section $S^2$ extracted in $S^3$ as the last scattering surface. From the comparison with the WMAP observation data, the correlation function on the section universe has similar trend with the observation data at large angular distances. We expect the effects of the quantum gravity at early stage of the universe can be achieved by the numerical simulations. We expect that the numerical formulation on simplicial space will give a basic tool to analyze the quantum universe and compare with the experimental observation data.

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