Non-renormalization Theorem
Originating in a New Fixed Point of the Vector Manifestation

Chihiro Sasaki\textsuperscript{a}

\textsuperscript{a}Department of Physics, Nagoya University, Nagoya, 464-8602, Japan.

We study the pion velocity at the critical temperature of chiral symmetry restoration in QCD. Starting from the premise that the bare effective field theory is to be defined from the underlying QCD, we incorporate the effects of Lorentz non-invariance into the bare theory by matching an effective field theory to QCD at a suitable matching scale, and investigate how the Lorentz non-invariance existing in the bare theory influences physical quantities. Using the hidden local symmetry model as the effective field theory, where the chiral symmetry restoration is realized as the vector manifestation (VM), we find that the pion velocity at the critical temperature receives neither quantum nor (thermal) hadronic corrections at the critical temperature even when we start from the bare theory with Lorentz symmetry breaking. This is likely the manifestation of a new fixed point in the Lorentz non-invariant formulation of the VM.

1. Introduction

Chiral symmetry in QCD is expected to be restored under some extreme conditions such as large number of flavor \( N_f \) and high temperature and/or density. In hadronic sector, the chiral symmetry restoration is described by various effective field theories (EFTs) based on the chiral symmetry \([1]\). By using the hidden local symmetry (HLS) model \([2]\) as an EFT and performing the Wilsonian matching which is one of the methods that determine the bare theory from the underlying QCD \([3]\), the vector manifestation (VM) in hot or dense matter was formulated in Refs. \([4,5]\). In the VM, the massless vector meson becomes the chiral partner of pion at the critical point \([6]\) \#1. There, the intrinsic temperature or density dependences of the parameters of the HLS Lagrangian, which are obtained by integrating out the high energy modes (i.e., the quarks and gluons above the matching scale) in hot and/or dense matter, play the essential roles to realize the chiral symmetry restoration consistently.

In the analysis done in Ref. \([7]\), it was shown that the effect of Lorentz symmetry breaking to the bare parameters caused by the intrinsic temperature dependence through the Wilsonian matching are small \([7,8]\). Starting from the bare Lagrangian with Lorentz invariance, it was presented that the pion velocity approaches the speed of light at the critical temperature \([7]\), although in low temperature region \((T \ll T_c)\) the pion velocity deviates from the speed of light due to hadronic corrections \([5]\).

However there do exist the Lorentz non-invariant effects in bare EFT anyway due to the intrinsic temperature and/or density effects. Further the Lorentz non-invariance might be enhanced through the renormalization group equations (RGEs), even if effects of Lorentz symmetry breaking at the bare level are small. Thus it is important to investigate how the Lorentz non-invariance at bare level influences physical quantities.

In this paper, we pick up the pion velocity at the critical temperature and study the quantum and hadronic thermal effects based on the VM. The pion velocity is one of the important quantities since it controls the pion propagation in medium through the dispersion relation. Our main result is that the pion velocity does not receive either quantum or hadronic corrections in the limit \( T \to T_c \) :

\begin{equation}
  v_\pi(T) = V_{\pi,\text{bare}}(T),
\end{equation}

independently of the value of the bare pion velocity \( V_{\pi,\text{bare}} \). This non-renormalization property on the pion velocity is protected by the VM. Equation \((1.1)\) implies that the Lorentz non-invariance at bare level is directly reflected on the physical pion velocity at the critical temperature.

This paper is organized as follows: In section 2 we show the HLS Lagrangian with Lorentz non-invariance and the conditions satisfied at the critical temperature for the bare parameters. In section 3 we show that the conditions for the bare parameters at the critical temperature are satisfied in any energy scale and that this is protected as a fixed point of the relevant RGEs. In section 4 we show the quantum and hadronic corrections to the pion velocity and derive our result \((1.1)\). In section 5 we give a summary and discussions.
2. Hidden Local Symmetry

Since Lorentz symmetry breaking effects are included in the bare theory through the Wilsonian matching, the HLS Lagrangian in hot and/or dense matter is generically Lorentz non-invariant. Its explicit form was presented in Ref. [5]. In this section, we start from this Lagrangian with Lorentz non-invariance, and requiring that the axial-vector current correlator be equal to the vector current correlator at the critical point, we obtain the conditions for the bare parameters.

2.1. Lorentz Non-invariant HLS Lagrangian

In this subsection, we show the HLS Lagrangian at leading order including the effects of Lorentz non-invariance.

The HLS model is based on the $G_{\text{global}} \times H_{\text{local}}$ symmetry, where $G = SU(N_f)_L \times SU(N_f)_R$ is the chiral symmetry and $H = SU(N_f)_V$ is the HLS. The basic quantities are the HLS gauge boson $V_\mu$ and two matrix valued variables $\xi_L(x)$ and $\xi_R(x)$ which transform as $\xi_{L,R}(x) \rightarrow \xi_{L,R}(x) = h(x) e^{i \pi a T_a} g_{L,R}$, where $h(x) \in H_{\text{local}}$ and $g_{L,R} \in [SU(N_f)_L, SU(N_f)_R]_{\text{global}}$. These variables are parameterized as $^2$

$$\xi_{L,R}(x) = e^{i \pi a T_a} / F_\pi \epsilon_{\mu} \xi_{L,R} / F_\pi$$

where $\pi = \pi a T_a$ denotes the pseudoscalar Nambu-Goldstone bosons associated with the spontaneous symmetry breaking of $G_{\text{global}}$ and $\sigma = \sigma a T_a$ denotes the Nambu-Goldstone bosons associated with the spontaneous breaking of $H_{\text{local}}$. This $\sigma$ is absorbed into the HLS gauge boson through the Higgs mechanism, and then the vector meson acquires its mass. $F_\pi$ and $F_\sigma$ denote the temporal components of the decay constant of $\pi$ and $\sigma$, respectively. The covariant derivative of $\xi_L$ is given by

$$D_\mu \xi_L = \partial_\mu \xi_L - i V_\mu \xi_L + i \xi_L \mathcal{L}_\mu,$$

and the covariant derivative of $\xi_R$ is obtained by the replacement of $\mathcal{L}_\mu$ with $\mathcal{R}_\mu$ in the above where $V_\mu$ is the gauge field of $H_{\text{local}}$ and $\mathcal{L}_\mu$ and $\mathcal{R}_\mu$ are the external gauge fields introduced by gauging $G_{\text{global}}$ symmetry. In terms of $\mathcal{L}_\mu$ and $\mathcal{R}_\mu$, we define the external axial-vector and vector fields as $\mathcal{A}_\mu = (\mathcal{R}_\mu - \mathcal{L}_\mu) / 2$ and $V_\mu = (\mathcal{R}_\mu + \mathcal{L}_\mu) / 2$.

In the HLS model it is possible to perform the derivative expansion systematically. In the chiral perturbation theory (ChPT) with HLS, the vector meson mass is to be considered as small compared with the chiral symmetry breaking scale $\Lambda$, by assigning $O(p)$ to the HLS gauge coupling, $g \sim O(p)$ [10,11]. (For details of the ChPT with HLS, see Ref. [10].) The leading order Lagrangian with Lorentz non-invariance can be written as

$$\mathcal{L} = \left[ (F^2_\pi)^2 u_\mu u_\nu + F^2_\pi F^2_\sigma (g_{\mu\nu} - u_\mu u_\nu) \right] \times \text{tr} [\hat{\alpha}^\mu \hat{\alpha}^\nu]$$

$$+ \left[ (F^2_\sigma)^2 u_\mu u_\nu + F^2_\sigma F^2_\pi (g_{\mu\nu} - u_\mu u_\nu) \right] \times \text{tr} [\hat{\alpha}_L^\mu \hat{\alpha}_R^\nu],$$

$$+ \left[ - \frac{1}{g_L^2} u_\mu u_\alpha g_{\nu\beta} - \frac{1}{2 g_f^2} (g_{\mu\alpha} g_{\nu\beta} - 2 u_\mu u_\alpha g_{\nu\beta}) \right] \times \text{tr} [v^{\mu\nu} v^{\alpha\beta}],$$

where

$$\hat{\alpha}^\mu_L = \frac{1}{2 \mu} \left[ D^\mu \xi_R \cdot \xi^t_L \mp D^\mu \xi_L \cdot \xi^t_R \right].$$

Here $F^2_\pi$ denote the spatial pion decay constant and similarly $F^2_\sigma$ for the $\sigma$. The rest frame of the medium is specified by $u^\mu = (1, 0)$ and $V_{\mu\nu}$ is the field strength of $V_\mu$. $g_L$ and $g_R$ correspond to the HLS gauge coupling $g$. The parametric $\pi$ and $\sigma$ velocities are defined by

$$V^2_\pi = F^2_\pi / F^2_\pi, \quad V^2_\sigma = F^2_\sigma / F^2_\sigma.$$

2.2. Conditions for the Bare Parameters

In this subsection following Ref. [5] where the condition for the current correlators with the bare parameters in dense matter were presented, we show the Lorentz non-invariant version of the conditions satisfied at the critical temperature for the bare parameters.

Concept of the matching in the Wilsonian sense is based on the following assumptions: The bare Lagrangian of the effective field theory (EFT) $\mathcal{L}_{\text{bare}}$ is defined at a suitable matching scale $\Lambda$. Generating functional derived from $\mathcal{L}_{\text{bare}}$ leads to the same Green’s function as that derived from the generating functional of QCD at $\Lambda$. In other words, the bare parameters are obtained after integrating out the high energy modes, i.e., the quarks and gluons above $\Lambda$. When we integrate out the high energy modes in hot matter, the bare parameters have a certain temperature dependence, intrinsic temperature dependence, converted from QCD to the EFT. The intrinsic temperature dependence is nothing but the signature that hadrons have an internal structure constructed from quarks and gluons. In the following, we describe the chiral symmetry restoration based on the point of view that the bare HLS theory is defined...
from the underlying QCD. We note that the Lorentz non-invariance appears in the bare HLS theory as a result of including the intrinsic temperature dependence. Once the temperature dependence of the bare parameters is determined through the matching with QCD mentioned above, the parameters appearing in the hadronic corrections pick up the intrinsic thermal effects through the RGEs.

The axial-vector and vector current correlators at bare level are constructed in terms of bare parameters and are divided into the longitudinal and transverse components:

$$G_{A,V}^{\mu
u} = P_{L}^{\mu
u}G_{A,V}^{L} + P_{T}^{\mu
u}G_{A,V}^{T},$$  \hspace{1cm} (2.6)

where $P_{L,T}^{\mu
u}$ are the longitudinal and transverse projection operators, respectively. The axial-vector current correlator in the HLS around the matching scale $\Lambda$ is well described by the following forms with the bare parameters [5,7]:

$$G_{A}^{L}(p_0,\bar{p}) = \frac{\bar{p}^2 F_{\pi,bare}^{L} F_{s,bare}^{s}}{-[\bar{p}_0^2 - V_{\pi,bare}^{\mu\gamma\nu} F_\pi^2]} - 2\bar{p}_0 z_{L}^{\pi,bare},$$

$$G_{A}^{T}(p_0,\bar{p}) = -F_{\pi,bare}^{L} F_{\pi,bare}^{s} - 2\left(\bar{p}_0 z_{L}^{\pi,bare} - \bar{p}_0 z_{T}^{\pi,bare}\right),$$  \hspace{1cm} (2.7)

where $z_{L,T}^{\pi,bare}$ are the coefficients of the higher order terms, and $V_{\pi,bare}$ is the bare pion velocity related to $F_{\pi,bare}^{L}$ and $F_{\pi,bare}^{s}$ as

$$V_{\pi,bare}^{\mu\nu} = \frac{F_{\pi,bare}^{L} F_{s,bare}^{s}}{F_{\pi,bare}^{L}}.$$  \hspace{1cm} (2.8)

Similarly, two components of the vector current correlator have the following forms:

$$G_{V}^{L}(p_0,\bar{p}) = \frac{\bar{p}^2 F_{\sigma,bare}^{L} F_{s,bare}^{s}}{-[\bar{p}_0^2 - V_{\sigma,bare}^{\mu\gamma\nu} F_\pi^2 - 2 M_{\rho,bare}^2]} - 2\bar{p}_0 z_{L}^{\sigma,bare} + O(p_0^4),$$

$$G_{V}^{T}(p_0,\bar{p}) = \frac{F_{\sigma,bare}^{L} F_{s,bare}^{s}}{-[\bar{p}_0^2 - V_{\sigma,bare}^{\mu\gamma\nu} F_\pi^2 - 2 M_{\rho,bare}^2]} \times \left[\bar{p}_0^2 \left(1 - 2g_{L,bare}^{2}\right) - V_{\sigma,bare}^{\mu\gamma\nu} F_\pi^2 \left(1 - 2g_{T,bare}^{2}\right)\right] - 2\left(\bar{p}_0^2 z_{L}^{\sigma,bare} - \bar{p}_0^2 z_{T}^{\sigma,bare}\right) + O(p_0^4),$$  \hspace{1cm} (2.9)

where $z_{L,T}^{\sigma,bare}$ and $z_{L,T}^{\sigma,bare}$ denote the coefficients of the higher order terms. In the above expressions, the bare vector meson mass in the rest frame, $M_{\rho,bare}$, is

$$M_{\rho,bare}^{2} = g_{L,bare}^{2} F_{\sigma,bare}^{L} F_{s,bare}^{s}.$$  \hspace{1cm} (2.10)

We define the bare parameters $a_{\pi,bare}$ and $a_{\rho,bare}$ as

$$a_{\pi,bare}^{L} = \left(\frac{F_{\pi,bare}^{L}}{F_{\pi,bare}^{L}}\right)^2, \quad a_{\rho,bare}^{s} = \left(\frac{F_{\pi,bare}^{s}}{F_{\pi,bare}^{L}}\right)^2,$$  \hspace{1cm} (2.11)

and the bare $\sigma$ and transverse $\rho$ velocities as

$$V_{\sigma,bare}^{\mu\nu} = \frac{F_{\sigma,bare}^{L}}{F_{\sigma,bare}^{L}}, \quad V_{\rho,bare}^{\mu\nu} = \frac{g_{L,bare}}{g_{T,bare}}.$$  \hspace{1cm} (2.12)

Now we consider the matching near the critical temperature. At the chiral phase transition point, the axial-vector and vector current correlators must agree with each other: $G_{A}^{L}(p_0,\bar{p}) = G_{V}^{L}(p_0,\bar{p})$. These equalities are satisfied only if the following conditions are met:

$$a_{\pi,bare}^{L} \rightarrow 1, \quad a_{\rho,bare}^{s} \rightarrow 1,$$

$$g_{L,bare} \rightarrow 0, \quad g_{T,bare} \rightarrow 0 \quad \text{for} \quad T \rightarrow T_c.$$  \hspace{1cm} (2.13)

This implies that at bare level the longitudinal mode of the vector meson becomes the real NG boson and couples to the vector current correlator, while the transverse mode decouples.

3. Vector Manifestation Condition

In this section, we show that the conditions for the bare parameters for $T \rightarrow T_c$ are satisfied in any energy scale and that this is protected by the fixed point of the RGEs.

It was shown that the HLS gauge coupling $g = 0$ is a fixed point of the RGE for $g$ at one-loop level [14,15]. The existence of the fixed point $g = 0$ is guaranteed by the gauge invariance. This is easily understood from the fact that the gauge field is normalized as $V_{\mu} = g \rho_{\mu}$. In the present case without Lorentz symmetry, the gauge field is normalized by $V_{\mu} = g_{L} \rho_{\mu}$ and thus $g_{L} = 0$ becomes a fixed point of the RGE for $g_{L}$.

Provided that $g_{L} = 0$ is a fixed point, we can show that $a^{L} = a^{s} = 1$ is also a fixed point of the coupled RGEs for $a^{L}$ and $a^{s}$ as follows: We start from the bare theory defined at a scale $\Lambda$ with $a_{\pi,bare}^{L} = a_{\rho,bare}^{s} = 1$ (and $g_{L} = 0$). The parameters $a^{L}$ and $a^{s}$ at $(\Lambda - \delta \Lambda, \Lambda)$ are calculated by integrating out the modes in $[\Lambda - \delta \Lambda, \Lambda]$. They are obtained from the two-point functions of $A_{\mu}$ and $V_{\mu}$, denoted by $\Pi_{\mu\nu}$ and $\Pi_{\mu\nu}$. We decompose these functions into

$$\Pi_{\mu\nu} = u^{\mu} u^{\nu} \Pi_{\perp\perp} + (g^{\mu\nu} - u^{\mu} u^{\nu}) \Pi_{\parallel\parallel} + P_{L}^{\mu\nu} \Pi_{\perp\parallel} + P_{T}^{\mu\nu} \Pi_{\perp\parallel},$$  \hspace{1cm} (3.1)

where $u^{\mu} u^{\nu}, (g^{\mu\nu} - u^{\mu} u^{\nu}), P_{L}^{\mu\nu}$ and $P_{T}^{\mu\nu}$ denote the temporal, spatial, longitudinal and transverse projection operators, respectively. The parameters $a^{L}$
and $a^s$ are defined by $a^t = \Pi^t_{\perp}/\Pi^t_{\perp} \; \; , \; \; a^s = \Pi^s_{\parallel}/\Pi^s_{\parallel}$. These expressions are further reduced to

$$a^t(\Lambda - \delta \Lambda) = a^t_{\text{bare}} + \frac{1}{(F_{\pi,\text{bare}}^t)^2} \times \left[ \Pi^t_{\parallel}(\Lambda; \Lambda - \delta \Lambda) - a^t_{\text{bare}} \Pi^t_{\perp}(\Lambda; \Lambda - \delta \Lambda) \right],$$

$$a^s(\Lambda - \delta \Lambda) = a^s_{\text{bare}} + \frac{1}{F_{\pi,\text{bare}}^s F_{\pi,\text{bare}}^t} \times \left[ \Pi^s_{\parallel}(\Lambda; \Lambda - \delta \Lambda) - a^s_{\text{bare}} \Pi^s_{\perp}(\Lambda; \Lambda - \delta \Lambda) \right],$$

where $\Pi^t_{\parallel,s}(\Lambda; \Lambda - \delta \Lambda)$ denotes the quantum correction obtained by integrating the modes out between $[\Lambda - \delta \Lambda, \Lambda]$. We show the diagrams for contributions to $\Pi^t_{\perp}$ and $\Pi^s_{\parallel}$ at one-loop level in Figs. 1 and 2.

The contributions (a) in Fig. 1 and (a) in Fig. 2 are proportional to $g_L^2$. The contributions (c) in Fig. 1 and (d) in Fig. 2 are proportional to $(a^t_{\text{bare}} - 1)$. Taking $g_L_{\text{bare}} = 0$ and $a^t_{\text{bare}} = a^s_{\text{bare}} = 1$, these contributions vanish. We note that $\sigma$ (i.e., longitudinal vector meson) is massless and the chiral partner of pion at the critical temperature. Then the contributions (b) and (c) in Fig. 2 have a symmetry factor 1/2 respectively and are obviously equal to the contribution (b) in Fig. 1, i.e., $\Pi^{(b)}_{\perp} = \Pi^{(b)+(c)}_{\parallel}$. Thus from Eq. (3.2), we obtain

$$a^t(\Lambda - \delta \Lambda) = a^t_{\text{bare}} = 1, \quad a^s(\Lambda - \delta \Lambda) = a^s_{\text{bare}} = 1.$$  

This implies that $a^t$ and $a^s$ are not renormalized at the scale $(\Lambda - \delta \Lambda)$. Similarly, we include the corrections below the scale $(\Lambda - \delta \Lambda)$ in turn, and find that $a^t$ and $a^s$ do not receive the quantum corrections. Eventually we conclude that $a^t = a^s = 1$ is a fixed point of the RGEs for $a^t$ and $a^s$.

From the above, we find that $(g_L, a^t, a^s) = (0, 1, 1)$ is a fixed point of the combined RGEs for $g_L, a^t$ and $a^s$. Thus the VM condition is given by

$$g_L \rightarrow 0, \quad a^t \rightarrow 1, \; a^s \rightarrow 1 \quad \text{for} \; T \rightarrow T_c.$$  

The vector meson mass is never generated at the critical temperature since the quantum correction to $M_\rho^2$ is proportional to $g_L^2$. Because of $g_L \rightarrow 0$, the transverse vector meson at the critical point, in any energy scale, decouples from the vector current correlator. The VM condition for $a^t$ and $a^s$ leads to the equality between the $\pi$ and $\sigma$ (i.e., longitudinal vector meson) velocities:

$$(V_\pi/V_\sigma)^4 = \left( F_{\pi,\text{bare}}^t / F_{\pi,\text{bare}}^s \right)^2 = \frac{a^t}{a^s} \frac{T_c - T}{T_c - T} = 1.$$  

This is easily understood from a point of view of the VM since the longitudinal vector meson becomes the chiral partner of pion. We note that this condition $V_\sigma = V_\pi$ holds independently of the value of the bare pion velocity which is to be determined through the Wilsonian matching.

4. Pion Velocity at Critical Temperature

As we mentioned in Introduction, the intrinsic temperature dependence generates the effect of Lorentz symmetry breaking at bare level. Then how does the Lorentz non-invariance at bare level influence physical quantities? In order to make it clear, in this section we study the pion decay constants and the pion velocity near the critical temperature.

Following Ref. [7], we define the on-shell of the pion from the pole of the longitudinal component $G_\pi^L$ of the axial-vector current correlator. This pole structure is expressed by temporal and spatial components of the two-point function $\Pi^\mu_{\nu}$. The temporal and spatial pion decay constants are expressed as follows [7,8]:

$$\left( f^t_\pi(p; T) \right)^2 = \Pi^t_{\parallel}(V_\pi \bar{p}, \bar{p}; T), \quad f^s_\pi(p; T) f^s_\pi(\bar{p}; T) = \Pi^s_{\parallel}(V_\pi \bar{p}, \bar{p}; T),$$

where the on-shell condition $p_0 \rightarrow V_\pi \bar{p}$ was taken. We divide the two-point function $\Pi^\mu_{\nu}$ into two parts, zero temperature (vacuum) and non-zero temperature parts, as $\Pi^\mu_{\nu} = \Pi^\mu_{\parallel;\text{vac}} + \Pi^\mu_{\parallel;\text{non}}$. The quantum correction is included in the vacuum part $\Pi^\mu_{\parallel;\text{vac}}$, and the hadronic thermal correction is in $\Pi^\mu_{\parallel;\text{non}}$. In the present perturbative analysis, we obtain the pion velocity as [8]:

$$v^t_\pi(\bar{p}; T) = \frac{f^t_\pi(\bar{p}; T)}{f^t_\pi(p; T)} = V_\pi^2 + \Pi_{\parallel}(V_\pi \bar{p}, \bar{p})$$
\[ + \bar{\Pi}_\perp(V_\pi \bar{p}, \bar{p}; T) - V_\perp^2 \bar{\Pi}_\perp(V_\pi \bar{p}, \bar{p}; T), \]

(4.2)

where \( \bar{\Pi}_\perp(V_\pi \bar{p}, \bar{p}) \) denotes a possible finite renormalization effect. Note that the renormalization condition on \( V_\pi \) is determined as \( \bar{\Pi}_\perp(V_\pi \bar{p}, \bar{p}) |_{\bar{p}=0} = 0 \).

In the following, we study the quantum and hadronic corrections to the pion velocity for \( T \to T_c \), on the assumption of the VM conditions [254].

As we defined above, the two-point function associated with the pion velocity \( v_\pi(\bar{p}; T) \) is \( \Pi_\mu^\nu(p_0, \bar{p}; T) \). The diagrams contributing to \( \Pi_\mu^\nu \) are shown in Fig. 1. As mentioned in section 3, diagram (a) is proportional to \( g_l \) and diagram (c) has the factor \( (\alpha_l - 1) \). Then these contributions vanish at the critical point.

We consider the contribution from diagram (b) only.

First we evaluate the quantum correction to the vacuum part \( \Pi_\perp^{(\text{vac})\mu\nu} \). This is expressed as

\[
\Pi_\perp^{(\text{vac})\mu\nu}(p_0, \bar{p}) = N_f \int \frac{d^n k}{i(2\pi)^n} \frac{\Gamma^\mu(k; p) \Gamma^\nu(-k; -p)}{[-k_0^2 + V_\perp^2 k^2][M_\rho^2 - (k_0 - p_0)^2 + V_\perp^2 |k - \bar{p}|^2]},
\]

(4.3)

where \( \Gamma^\mu \) denotes the \( A_\pi \sigma \) vertex as

\[
\Gamma^\mu(k; p) = \frac{i}{2} \sqrt{\alpha_s} \frac{g_\mu}{\sqrt{2}} [u^\mu u^\rho] (2k - p)_\rho + V_\perp^2 (\bar{u}^\rho u^\rho)(2k - p)_\rho.
\]

(4.4)

We note that the spatial component of this vertex \( \Gamma^i \) has an extra-factor \( V_\perp^2 \) as compared with the temporal one. In the present analysis it is important to include the quadratic divergences to obtain the RGEs in the Wilsonian sense. In Refs. [253,12], the dimensional regularization was adopted and the quadratic divergences were identified with the presence of poles of ultraviolet origin at \( n = 2 \) [10]. In this paper when we evaluate four dimensional integral, we first integrate over \( k_0 \) from \(-\infty \) to \( \infty \). Then we carry out the integral over three-dimensional momentum \( k \) with three-dimensional cutoff \( \Lambda_3 \). In order to be consistent with ordinary regularization in four dimension [253,12], we use the following replacement associated with quadratic divergence:

\[
\Lambda_3 \to \frac{1}{\sqrt{2}} \Lambda_4 = \frac{1}{\sqrt{2}} \Lambda,
\]

\[
\int \frac{d^n-1 k}{(2\pi)^{n-1}} \frac{1}{k^i k^j} \to \frac{\Lambda^2}{8\pi^2},
\]

\[
\int \frac{d^n-1 k}{(2\pi)^{n-1}} \frac{k^i k^j}{k^4} \to -\delta^{ij} \frac{\Lambda^2}{8\pi^2}.
\]

(4.6)

When we make these replacements, the present method of integral preserves chiral symmetry.

As shown in Appendix A, \( \Pi_\perp^{(\text{vac})t} \) and \( \Pi_\perp^{(\text{vac})s} \) are independent of the external momentum. Accordingly, the finite renormalization effect \( \bar{\Pi}_\perp \) is also independent of the external momentum and then vanishes:

\[ \bar{\Pi}_\perp(V_\pi \bar{p}, \bar{p}) = 0. \]

(4.7)

Thus in the following, we take the external momentum as zero. In that case, the temporal and spatial components of \( \Pi_\perp^{(\text{vac})\mu\nu} \) are expressed as \( \Pi_\perp^{(\text{vac})t} = \Pi_\perp^{(\text{vac})00} \) and \( \Pi_\perp^{(\text{vac})s} = -(\delta^{ij}/3)\Pi_\perp^{(\text{vac})ij} \).

Taking the VM limit (\( M_\rho \to 0 \) and \( V_\sigma \to 0 \)), these components become

\[
\lim_{\text{VM}} \Pi_\perp^{(\text{vac})00}(p_0 = \bar{p} = 0) = \frac{N_f}{4} \int \frac{dk_0 d^{n-1} \bar{k}}{(2\pi)^n} \frac{4k_0^2}{-k_0^2 + V_\perp^2 \bar{k}^2},
\]

\[
\Pi_\perp^{(\text{vac})ij}(p_0 = \bar{p} = 0) = -\frac{N_f}{4} \int \frac{d^{n-1} \bar{k}}{(2\pi)^{n-1}} \frac{\bar{k}^i \bar{k}^j}{V_\perp^4 \bar{k}^4} \Lambda^2.
\]

(4.8)

Thus we obtain the temporal and spatial parts as

\[
\lim_{\text{VM}} \Pi_\perp^{(\text{vac})t}(p_0 = \bar{p} = 0) = -\frac{N_f}{4} \frac{1}{V_\pi} \frac{\Lambda^2}{8\pi^2},
\]

\[
\lim_{\text{VM}} \Pi_\perp^{(\text{vac})s}(p_0 = \bar{p} = 0) = -\frac{N_f}{4} \frac{\Lambda^2}{V_\pi} \frac{8\pi^2}{8\pi^2}.
\]

(4.9)

These quadratic divergences are renormalized by \( (F_\pi^{t,\text{bare}})^2 \) and \( F_\pi^{t,\text{bare}} F_\pi^{s,\text{bare}} \), respectively. Then RGEs for the parameters \( (F_\pi^t)^2 \) and \( F_\pi^t F_\pi^s \) are expressed as

\[
\frac{d(F_\pi^t)^2}{d\mu} = \frac{N_f}{(4\pi)^2} \frac{\Lambda^2}{V_\pi} \mu^2,
\]

\[
\frac{d(F_\pi^t F_\pi^s)}{d\mu} = \frac{N_f}{(4\pi)^2} \frac{\Lambda^2}{V_\pi} \mu^2.
\]

(4.10)

(4.11)

Both \( (F_\pi^t)^2 \) and \( F_\pi^t F_\pi^s \) scale following the quadratic running \( \mu^2 \). However, the coefficient of \( \mu^2 \) in the RGE for \( (F_\pi^t)^2 \) is different from that for \( F_\pi^t F_\pi^s \).

When we use these RGEs, the scale dependence of the parametric pion velocity is

\[
\frac{dV_\pi^2}{d\mu} = \mu \frac{d[(F_\pi^t F_\pi^s/)(F_\pi^t)^2]}{d\mu}.
\]
This implies that the parametric pion velocity at the critical temperature does not scale. In other words, the Lorentz non-invariance at bare level is not enhanced through the RGEs as long as we consider the pion velocity. As we noted below Eq. (4.11), the factor $V_\sigma^2$ is in the spatial component of the vertex $\Gamma^\mu$. If $V_\sigma$ were not equal to $V_\pi$, the coefficients of running in the right-hand-side of Eqs. (4.10) and (4.11) would change. However, since the VM conditions do guarantee $V_\sigma = V_\pi$, the quadratic running caused from $\Lambda^2$ in $(F^2_\pi)$ and $F^t_F^s_\pi$ are exactly canceled in the second line of Eq. (4.12).

Next we study the hadronic thermal correction to the pion velocity at the critical temperature. The temporal and spatial parts of the hadronic thermal correction $\tilde{\Pi}_\perp^{\mu\nu}$ contribute to the pion velocity, which have the same structure as those of the quantum correction $\Pi_\perp^{(\text{vac})\mu\nu}$, except for a Bose-Einstein distribution function. Thus by the replacement of $\Lambda^2/(4\pi)^2$ with $T^2/12$ in $\Pi_\perp^{(\text{vac})\mu\nu}$, hadronic corrections to the temporal and spatial parts of $\tilde{\Pi}_\perp^{\mu\nu}$ are obtained as follows:

$$
\begin{align*}
\lim_{VM} \tilde{\Pi}_\perp^t(p_0, \vec{p}; T) &= -\frac{N_f}{24} \frac{1}{V_\pi} T_c^2, \\
\lim_{VM} \tilde{\Pi}_\perp^s(p_0, \vec{p}; T) &= -\frac{N_f}{24} \frac{1}{V_\pi} T_c^2.
\end{align*}
$$

(4.13)

Substituting Eq. (4.13) into Eq. (4.12) with $\tilde{\Pi}_\perp = 0$ as shown in Eq. (4.7), we obtain the physical pion velocity in the VM as

$$
\begin{align*}
v_\pi^2(\vec{p}; T) &\rightarrow T \rightarrow \frac{\tilde{\Pi}^t_\perp(V_\pi \vec{p}, \vec{p}; T_c) - V_\pi^2 \tilde{\Pi}^s_\perp(V_\pi \vec{p}, \vec{p}; T_c)}{(F_\pi^2)^2} \\
&= V_\pi^2.
\end{align*}
$$

(4.14)

Since the parametric pion velocity in the VM does not scale with energy [see Eq. (4.12)], $V_\pi$ in the above expression is equivalent to the bare pion velocity:

$$
v_\pi(\vec{p}; T) = V_{\pi, \text{bare}}(T) \quad \text{for } T \rightarrow T_c.
$$

(4.15)

This implies that the pion velocity in the limit $T \rightarrow T_c$ receives neither hadronic nor quantum corrections due to the protection by the VM. This is our main result.

In order to clarify the reason of this non-renormalization property, let us recall the fact that only the diagram (b) in Fig. 1 contributes to the pion velocity at the critical temperature. Away from the critical temperature, the contribution of the massive $\sigma$ (i.e., the longitudinal mode of massive vector meson) is suppressed owing to the Boltzmann factor $\exp[-M_\sigma/T]$, and then only the pion loop contributes to the pion velocity. Then there exists the $O(T^4)$ correction to the pion velocity [8]. Near the critical temperature, on the other hand, $\sigma$ becomes massless due to the VM since $\sigma$ (i.e., the longitudinal vector meson) becomes the chiral partner of the pion. Thus the absence of the hadronic corrections in the pion velocity at the critical temperature is due to the exact cancellation between the contribution of pion and that of its chiral partner $\sigma$. Similarly the quantum correction generated from the pion loop is exactly cancelled by that from the $\sigma$ loop.

5. Summary and Discussions

In this paper, we started from the Lorentz non-invariant HLS Lagrangian at bare level and studied the pion velocity at the critical temperature based on the VM. From the analysis of the quantum and hadronic thermal corrections to the pion velocity, we obtained the result that the pion velocity in the limit $T \rightarrow T_c$ is equal to the bare pion velocity. In other words, the pion velocity does not receive either quantum or hadronic corrections at the critical temperature. This occurs due to the exact cancellation between the contribution of pion and that of the longitudinal vector meson (i.e., the chiral partner of pion).

Now we consider the meaning of our result (4.15). Based on the point of view that the bare HLS theory is defined from QCD, we presented the VM conditions realizing the chiral symmetry in QCD consistently, i.e., $(g_L, a^t, a^s) \rightarrow (0, 1, 1)$ for $T \rightarrow T_c$. This is the fixed point of the RGEs for the parameters $g_L, a^t$ and $a^s$. As we showed in section 4 although both pion decay constants $(F_\pi^t)^2$ and $F^t_F^s_\pi$ scale following the quadratic running, $(F_\pi^t)^2$ and $F^t_F^s_\pi$ show a different running since the coefficient of $\mu^2$ in Eq. (4.10) is different from that in Eq. (4.11). Nevertheless in the pion velocity at the critical temperature, the quadratic running in $(F_\pi^t)^2$ is exactly cancelled by that in $F^t_F^s_\pi$ [see second line of Eq. (4.12)]. There it was crucial for intricate cancellation of the quadratic running that the velocity of $\sigma$ (i.e., longitudinal vector meson) is equal to its chiral partner, i.e., $V_\sigma \rightarrow V_\pi$ for $T \rightarrow T_c$. Note that this is not an extra condition but a consequence from the VM conditions for $a^t$ and $a^s$: we started simply from the VM conditions alone and found that $V_\pi$ does not receive quantum corrections at the restoration point. As we showed in Eq. (4.13), the hadronic correction to $(F_\pi^t)^2$ is different from that to $F^t_F^s_\pi$.
In the pion velocity, however, the hadronic correction from \((F^\perp_\pi)^2\) is exactly cancelled by that from \(F_\pi^a F_\pi^a\) [see second line of Eq. (4.14)]. The VM conditions guarantee these exact cancellations of the quantum and hadronic corrections. This implies that \((g_L, a^t, a^s, V_\pi) = (0, 1, 1, \text{any})\) forms a fixed line for four RGEs of \(g_L, a^t, a^s\) and \(V_\pi\). When one point on this fixed line is selected through the matching procedure as done in Ref. [17], namely the value of \(V_{\pi, \text{bare}}\) is fixed, the present result implies that the point does not move in a subspace of the parameters. This is likely the manifestation of a new fixed point in the Lorentz non-invariant formulation of the VM. Approaching the restoration point of chiral symmetry, the physical pion velocity itself would flow into the fixed point.

Several comments are in order:

We should distinguish the consequences within HLS/VM from those beyond HLS/VM. Clearly the determination of the definite value of the bare pion velocity is done outside HLS/VM. On the other hand, our main result (4.15) holds independently of the value of the bare pion velocity itself. Applying this result to the case where one starts from the bare HLS theory with Lorentz invariance, i.e., \(V_{\pi, \text{bare}} = 1\), one finds that the pion velocity at \(T_c\) becomes the speed of light since \(v_\pi = V_{\pi, \text{bare}} = 1\), as obtained in Ref. [7].

As a consequence of the relation (4.15), we can determine the temporal and spatial pion decay constants at the critical temperature when we take the bare pion velocity as finite. In the following, we study these decay constants and discuss their determination based on Eq. (4.15). Using Eq. (4.14), we solve the RGEs (4.10) and (4.11) and obtain a relation between two parametric pion decay constants as \(F^t_\pi(0; T_c) F^s_\pi(0; T_c) = V^2_\pi (F^t_\pi(0; T))^2\). From this and (4.13), the temporal and spatial pion decay constants with the quantum and hadronic corrections are obtained as

\[
(f^t_\pi)^2 = \left( F^t_\pi(0; T_c) \right)^2 - \frac{N_f}{24} \frac{1}{V_\pi} T_c^2;
\]

\[
f^t_\pi f^s_\pi = F^t_\pi(0; T_c) F^s_\pi(0; T_c) - \frac{N_f}{24} V_\pi T_c^2
= V^2_\pi (f^t_\pi)^2.
\]

Since the order parameter \((f^t_\pi f^s_\pi)\) vanishes as expected at the critical temperature, we find that \(f^t_\pi f^s_\pi = V^2_\pi (f^t_\pi)^2 = 0\). Multiplying both side by \(\frac{3}{2} v_\pi^2 = V_\pi^2\), the above expression is reduced to

\[
(f^t_\pi)^2 = V^2_\pi (f^t_\pi)^2 = 0.
\]

Now, the spatial pion decay constant vanishes at the critical temperature, \(f^s_\pi(T_c) = 0\). In the case of a vanishing pion velocity, \(f^t_\pi\) can be finite at the restoration point. On the other hand, when \(V_\pi\) is finite, Eq. (5.2) leads to \(f^s_\pi(T_c) = 0\). Thus we find that both temporal and spatial pion decay constants vanish simultaneously at the critical temperature when the bare pion velocity is determined as finite.

In order to know the value of the (bare) pion velocity, we need to specify a method that determines the bare parameters of the effective field theory. As we stressed in subsection 2.2, the bare parameters of the HLS Lagrangian are determined by the underlying QCD. One possible way to determine them is the Wilsonian matching proposed in Ref. [8] which is done by matching the axial-vector and vector current correlators derived from the HLS with those by the operator product expansion (OPE) in QCD at the matching scale \(\Lambda\). From the analysis performed on the basis of a Wilsonian matching, the bare pion velocity at the critical temperature is found to be finite, i.e., \(V_{\pi, \text{bare}} \neq 0\) [17]. Thus, by combining Eq. (4.15) with estimation of \(V_{\pi, \text{bare}}\), the value of the physical pion velocity \(v_\pi(T)\) at the critical temperature is obtained to be finite [17]. This is in contrast to the result obtained from the chiral theory [18,19], where the relevant degree of freedom near \(T_c\) is only the pion. Their result is that the pion velocity becomes zero for \(T \to T_c\). Therefore from the experimental data, we may be able to distinguish which picture is correct, \(v_\pi \sim 1\) or \(v_\pi \to 0\).

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Appendix

A. Two-point Function

In this appendix, we show that the temporal and spatial components of the two-point function \(\Pi^{(\text{vac})\mu\nu} \not\perp\) are independent of the external momentum in the VM limit.

From Eq. (3.9), \(\Pi^{(\text{vac})\mu\nu} \not\perp\) is rewritten as

\[
\Pi^{(\text{vac})\mu\nu} \not\perp(p_0, \vec{p})
= N_f \frac{q^t}{4} X^\mu_{\phantom{\mu} \tilde{\mu}} X^\nu_{\phantom{\nu} \tilde{\nu}} B^{(\text{vac})\tilde{\mu}\tilde{\nu}}(p_0, \vec{p}; M_\rho, 0)
\equiv N_f \frac{q^t}{4} \bar{B}^{(\text{vac})\mu\nu}(p_0, \vec{p}; M_\rho, 0),
\]

where we define

\[
X^\mu_{\phantom{\mu} \tilde{\mu}} = u^\mu u_{\tilde{\mu}} + V^2_{\pi} (g^\mu_{\tilde{\mu}} - u^\mu u_{\tilde{\mu}}).
\]
In the above expression, we define the function \( B^{(\text{vac})\mu\nu} \) contributed to the diagram (b) in Fig. [1] as
\[
B^{(\text{vac})\mu\nu}(p_0, \bar{p}; M_\rho, 0) = \int \frac{dk_0}{i(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} (2k - p)^\mu (2k - p)^\nu \frac{1}{|k_0^2 - \omega_\pi^2|[(k_0 - p_0)^2 - (\omega_\rho^2)^2]},
\]
with
\[
\omega_\pi^2 = V_\pi^2 k^2, \\
(\omega_\rho^2)^2 = V_\rho^2 |\bar{k} - \bar{p}|^2 + M_\rho^2.
\]
In terms of each component of \( B^{(\text{vac})\mu\nu} \), the temporal and spatial parts of \( B^{\mu\nu} \) are given by
\[
\begin{align*}
\tilde{B}^{(\text{vac})t}(p_0, \bar{p}; M_\rho, 0) &= \left[ B^{(\text{vac})t}(p_0, \bar{p}; M_\rho, 0) \\
&\quad + (1 - V_\sigma^2) \frac{\bar{p}^2}{\vec{p}^2} B^{(\text{vac})L}(p_0, \bar{p}; M_\rho, 0) \right], \\
\tilde{B}^{(\text{vac})s}(p_0, \bar{p}; M_\rho, 0) &= V_\sigma^2 \left[ B^{(\text{vac})s}(p_0, \bar{p}; M_\rho, 0) \\
&\quad + \frac{1 - V_\sigma^2}{V_\sigma^2} \frac{\bar{p}^2}{\vec{p}^2} B^{(\text{vac})L}(p_0, \bar{p}; M_\rho, 0) \right].
\end{align*}
\]

By using the expressions in Ref. [8], the components \( B^{(\text{vac})t} \), \( B^{(\text{vac})s} \) and \( B^{(\text{vac})L} \) take the following forms:
\[
\begin{align*}
B^{(\text{vac})t}(p_0, \bar{p}; M_\rho, 0) &= \int \frac{d^3k}{(2\pi)^3} \left[ \frac{-1}{2\omega_\pi} \frac{(2\omega_\pi - p_0)^2}{(\omega_\pi - p_0)^2 - (\omega_\rho^2)^2} \\
&\quad + \frac{1 - (2\omega_\rho^2 + p_0)^2}{2\omega_\rho^2 (\omega_\rho^2 + p_0)^2 - \omega_\pi^2} \right] - \frac{\bar{p} \cdot (2\vec{k} - \bar{p})}{p_0} \left[ \frac{1}{2\omega_\pi} \frac{(2\omega_\pi - p_0)}{(\omega_\pi - p_0)^2 - (\omega_\rho^2)^2} \\
&\quad + \frac{1}{2\omega_\rho^2} \frac{2\omega_\pi - p_0}{(\omega_\rho^2 + p_0)^2 - \omega_\pi^2} \right], \\
B^{(\text{vac})s}(p_0, \bar{p}; M_\rho, 0) &= \int \frac{d^3k}{(2\pi)^3} \left[ \frac{(2\vec{k} \cdot \bar{p} - \bar{p}^2)^2}{\bar{p}^2} \\
&\quad + \frac{1}{2\omega_\pi} \frac{1}{(\omega_\pi - p_0)^2 - (\omega_\rho^2)^2} \\
&\quad + \frac{1}{2\omega_\rho^2} \frac{1}{(\omega_\rho^2 + p_0)^2 - \omega_\pi^2} \right] + \frac{p_0 \bar{p} \cdot (2\vec{k} - \bar{p})}{\bar{p}^2} \left[ \frac{1}{2\omega_\pi} \frac{2\omega_\pi - p_0}{(\omega_\pi - p_0)^2 - (\omega_\rho^2)^2} \\
&\quad + \frac{1}{2\omega_\rho^2} \frac{2\omega_\pi - p_0}{(\omega_\rho^2 + p_0)^2 - \omega_\pi^2} \right], \\
B^{(\text{vac})L}(p_0, \bar{p}; M_\rho, 0) &= \int \frac{d^3k}{(2\pi)^3} \left[ \frac{-1}{(2\omega_\pi)^2} \frac{(2\omega_\pi - p_0)^2}{(\omega_\pi - p_0)^2 - (\omega_\rho^2)^2} \\
&\quad + \frac{1 - (2\omega_\rho^2 + p_0)^2}{2\omega_\rho^2 (\omega_\rho^2 + p_0)^2 - \omega_\pi^2} \right] \left[ \frac{2\omega_\pi}{(\omega_\rho^2 + p_0)^2 - \omega_\pi^2} \right].
\end{align*}
\]

Now we take the VM limit. Note that the VM condition for \( a^t \) and \( a^s \) implies that \( \sigma \) velocity becomes equal to the pion velocity, \( V_\pi \rightarrow V_\pi^c \) for \( T \rightarrow T_c \) as shown in Eq. (3.5). The components \( B^{(\text{vac})t} \), \( B^{(\text{vac})s} \) and \( B^{(\text{vac})L} \) are calculated as follows:
\[
\begin{align*}
\lim_{V_\pi \rightarrow V_\pi^c} B^{(\text{vac})t}(p_0, \bar{p}; M_\rho, 0) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_\pi} \\
&\quad \times \frac{\bar{p} \cdot (2\vec{k} - \bar{p}) \vec{J}(\vec{k}; p_0, \bar{p})}{[(\omega_\pi - p_0)^2 - (\omega_\rho^2)^2][\omega_\pi (p_0^2 + 2\omega_\pi^2 - (\omega_\rho^2)^2)]},
\end{align*}
\]
\[
\begin{align*}
\lim_{V_\pi \rightarrow V_\pi^c} B^{(\text{vac})s}(p_0, \bar{p}; M_\rho, 0) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_\pi} \\
&\quad \times \frac{\bar{p} \cdot (2\vec{k} - \bar{p}) \vec{J}(\vec{k}; p_0, \bar{p})}{[(\omega_\pi - p_0)^2 - (\omega_\rho^2)^2][\omega_\pi (p_0^2 + 2\omega_\pi^2 - (\omega_\rho^2)^2)]},
\end{align*}
\]
where we define the functions as
\[
\begin{align*}
\vec{I}(\vec{k}; p_0, \bar{p}) &= \omega_\pi^2 [4\omega_\pi^2 - 4(\omega_\pi^2)^2 - 3p_0^2] + p_0^2 |p_0 - (\omega_\rho^2)^2], \\
\vec{J}(\vec{k}; p_0, \bar{p}) &= -p_0 [-3\omega_\pi^2 + p_0^2 - (\omega_\rho^2)^2], \\
\vec{K}(\vec{k}; p_0, \bar{p}) &= \omega_\pi^2 p_0^2 - (\omega_\rho^2)^2.
\end{align*}
\]

Using Eqs. (A.7) - (A.9) with these functions, we obtain \( B^{(\text{vac})t,s} \) as
\[
\begin{align*}
\lim_{V_\pi \rightarrow V_\pi^c} \tilde{B}^{(\text{vac})t}(p_0, \bar{p}; M_\rho, 0) &= \int \frac{d^3k}{(2\pi)^3} \frac{\bar{p} \cdot (2\vec{k} - \bar{p}) \vec{J}(\vec{k}; p_0, \bar{p})}{[(\omega_\pi - p_0)^2 - (\omega_\rho^2)^2][\omega_\pi (p_0^2 + 2\omega_\pi^2 - (\omega_\rho^2)^2)]}, \\
\lim_{V_\pi \rightarrow V_\pi^c} \tilde{B}^{(\text{vac})s}(p_0, \bar{p}; M_\rho, 0) &= \int \frac{d^3k}{(2\pi)^3} \frac{\bar{p} \cdot (2\vec{k} - \bar{p}) \vec{J}(\vec{k}; p_0, \bar{p})}{[(\omega_\pi - p_0)^2 - (\omega_\rho^2)^2][\omega_\pi (p_0^2 + 2\omega_\pi^2 - (\omega_\rho^2)^2)]}.
\end{align*}
\]
\begin{align}
\times \frac{1}{\left[ \left( \omega_\pi - p_0 \right)^2 - \left( \omega_\rho^p \right)^2 \right] \left[ \left( \omega_\pi + p_0 \right)^2 - \left( \omega_\rho^p \right)^2 \right]} \\
\times \left[ \left( V_\pi^2 \bar{p} \cdot \left( 2 \vec{k} - \vec{p} \right) \right)^2 \mathcal{K}(\vec{k}; p_0, \bar{p}) \right. \\
\left. + p_0^2 V_\pi^2 \bar{p} \cdot \left( 2 \vec{k} - \vec{p} \right) \mathcal{J}(\vec{k}; p_0, \bar{p}) \right].
\end{align}

(A.12)

The integrand of the functions \( \bar{B}^{(\text{vac})_t} \) and \( \bar{B}^{(\text{vac})_s} \) in the VM limit are same as those of \( B^{(\text{vac})_i} \) and \( B^{(\text{vac})_s} \) with \( V_\pi = 1 \) when we make the following replacement in \( \bar{B}^{(\text{vac})_t} \): \( V_\pi \bar{k} \rightarrow \vec{k}, \quad V_\pi |\vec{k} - \vec{p}| \rightarrow |\vec{k} - \bar{p}|. \) (A.13)

The functions \( B^{(\text{vac})_t} \)'s with \( M_\rho \rightarrow 0 \) are independent of the external momentum \( p_0 \) and \( \bar{p} \). Thus we find that the functions \( \bar{B}^{(\text{vac})_t} \) and \( B^{(\text{vac})_s} \) in the VM limit are obtained independently of the external momentum \( p_0 \) and \( \bar{p} \):

\begin{align}
\lim_{\text{VM}} \bar{B}^{(\text{vac})_t}_i(p_0, \bar{p}; M_\rho, 0) &= -\frac{1}{V_\pi} \frac{\Lambda^2}{8\pi^2}, \\
\lim_{\text{VM}} \bar{B}^{(\text{vac})_s}_i(p_0, \bar{p}; M_\rho, 0) &= -V_\pi \frac{\Lambda^2}{8\pi^2}.
\end{align}

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#3 In Ref. 2 where \( V_\pi = 1 \) was taken, it was shown that the hadronic corrections \( \Pi^{\pi^0}_{i\pi}(p_0, \bar{p}; T) \) at the VM limit are independent of the external momentum \( p_0 \) and \( \bar{p} \). The structure of the integrand in the vacuum part is the same as that in the hadronic part except for the absence of the Bose-Einstein distribution function. Thus the vacuum part is also independent of \( p_0 \) and \( \bar{p} \).