APPENDIX A: DERIVATION OF EQ. (12)

In this Appendix, we relate $\phi$ and $\psi$ by deriving Eq. (12) used in the main text. At the same time, we verify that this expansion is indeed consistent at all orders. Taylor expanding the right-hand side of Eq. (11),

$$
\psi(s + ks_0) - \psi(s) = \phi(s) + 2 \left( \sum_{j=1}^{k-1} \phi(s + js_0) \right) + \phi(s + ks_0) = 2k\phi(s) + \sum_{n=1}^{\infty} \frac{s^n}{n!} \phi^{(n)}(s) \left\{ k^n + 2 \sum_{j=1}^{k-1} j^n \right\}
$$

where we have used Faulhaber’s formula [46] to expand the sum of powers of integers, and where $B_0 = 1, B_1 = -\frac{1}{2}, \ldots$ denote the Bernoulli numbers (of the first kind) [32]. Expanding $(k-1)^{n+1-i}$ using the binomial theorem and simplifying the binomial coefficients,

$$
\psi(s + ks_0) - \psi(s) = 2k\phi(s) + \sum_{n=1}^{\infty} \frac{s^n}{n!} \phi^{(n)}(s) \left\{ k^n + 2 \sum_{j=0}^{n} (-1)^j \binom{n+1}{j} B_j (k-1)^{n+1-j} \right\},
$$

wherein $a_0, a_1, \ldots, a_{n+1}$ depend on $n$. In particular,

$$
a_0 = 0, \quad a_n = 0, \quad a_{n+1} = \frac{2}{n+1},
$$

of which the last two are obtained by direct computation, and the first one follows using an identity of Bernoulli numbers [32],

$$
\sum_{j=0}^{n} \binom{n+1}{j} B_j = 0 \quad \text{for } n = 1, 2, \ldots.
$$

Moreover, for $i = 1, 2, \ldots, n-1$,

$$
a_i(n) = 2(-1)^{n+1-i} \frac{n!}{i!} \sum_{j=0}^{n+1-i} \frac{B_j}{j!(n+1-i-j)!}.
$$

where we have used Faulhaber’s formula [46] to expand the sum of powers of integers, and where $B_0 = 1, B_1 = -\frac{1}{2}, \ldots$ denote the Bernoulli numbers (of the first kind) [32]. Expanding $(k-1)^{n+1-i}$ using the binomial theorem and simplifying the binomial coefficients,
Accordingly,
\[ A(k, n) = \frac{2k^{n+1}}{n+1} + 2(-1)^{n+1}n! \sum_{i=1}^{n-1} (-1)^i \frac{k^i}{i!} \left\{ \sum_{j=0}^{n-i} \frac{B_j}{j!(n+1-i-j)!} \right\}. \]  
\hspace{1cm} (A7)

Upon inverting the order of summation, Eq. (A2) becomes
\[ \psi(s + ks_0) - \psi(s) = 2k\phi(s) + \sum_{n=1}^{\infty} \frac{\phi^{(n)}(s)}{n!\ell_0^n} \left( \frac{2k^{n+1}}{n+1} \right) + \sum_{n=1}^{\infty} \sum_{m=n+2}^{\infty} 2(-1)^{n+1-i} \frac{\phi^{(n)}(s) k^i}{i!} \left\{ \sum_{j=0}^{n-i} \frac{B_j}{j!(n+1-i-j)!} \right\}. \]  
\hspace{1cm} (A8)

Finally, upon relabelling indices in the first summation and changing variables \( n \rightarrow m = n + 1 - i \) in the last summation,
\[ \psi(s + ks_0) - \psi(s) = \sum_{i=1}^{\infty} \frac{k^i}{i!\ell_0^i} \left\{ \sum_{m=0}^{\infty} \frac{2B_{2m}}{(2m)!} \phi^{(i-1+2m)}(s) \right\}. \]  
\hspace{1cm} (A9)

But, rearranging Eq. (A5),
\[ \sum_{j=0}^{m} \frac{B_j}{j!(m-j)!} = \frac{B_m}{m!} \text{ for } m = 2, 3, \ldots \]  
\hspace{1cm} (A10)

Since \( B_{2n} = 0 \) for odd \( n > 1 \), and using \( B_0 = 1 \), we finally obtain
\[ \psi(s + ks_0) - \psi(s) = \sum_{i=1}^{\infty} \frac{k^i}{i!\ell_0^i} \left\{ \sum_{m=0}^{\infty} \frac{2B_{2m}}{(2m)!} \phi^{(i-1+2m)}(s) \right\}. \]  
\hspace{1cm} (A11)

Comparing this to the Taylor expansion of the left-hand side,
\[ \psi(s + ks_0) - \psi(s) = \sum_{i=1}^{\infty} \frac{k^i}{i!\ell_0^i} \psi^{(i)}(s), \]  
\hspace{1cm} (A12)

we deduce that the expansion is consistent at all orders, with, in particular,
\[ \psi'(s) = \sum_{m=0}^{\infty} \psi_m \phi^{(2m+1)}(s) \]  
where \( \psi_m = \frac{2B_{2m}}{(2m)!}. \]  
\hspace{1cm} (A13)

which is Eq. (12). As noted in the main text, we are not aware of a closed form for the inverted series that expresses \( \phi \) as a function of the derivatives of \( \psi \). Formally, inverting Eq. (12) gives
\[ \phi(s) = \sum_{m=0}^{\infty} \psi_m \phi^{(2m+1)}(s), \]  
\hspace{1cm} (A14)

where the coefficients \( \phi_0, \phi_1, \ldots \) are determined recursively by \( \phi_0 \psi_0 = 1 \) and
\[ \sum_{j=0}^{m} \phi_j \psi_{m-j} = 0 \text{ for } m = 1, 2, \ldots. \]  
\hspace{1cm} (A15)

In agreement with Eq. (13), we find
\[ \phi_0 = \frac{1}{2}, \quad \phi_1 = -\frac{1}{24}, \quad \phi_2 = \frac{1}{384}, \ldots \]  
\hspace{1cm} (A16)

\section{Appendix B: Eigenmodes of the Buckled Epithelium}

Eigenmodes of the epithelium are nonzero solutions of the governing Eq. (21) with \( \mu = 0 \). They thus obey
\[ \ddot{\psi} = 6\Xi^2 \dot{\psi} - 3\Delta \Xi \dot{\psi} \dot{\psi} + \frac{15}{4} \psi^2 \ddot{\psi}, \]  
\hspace{1cm} (B1)

subject to
\[ \psi(0) = \psi(1) = 0, \quad \dot{\psi}(0) = \dot{\psi}(1) = 0. \]  
\hspace{1cm} (B2)

To find eigenmodes numerically, we remove the trivial, zero solution by imposing a nonzero compression \( D \) and varying this compression until a solution with \( \mu = 0 \) is found.

Numerically, we obtain eigenmodes if \( \Delta \geq \Delta_* \), but find no solutions if \( \Delta < \Delta_* \), for some value \( \Delta_* \) depending on \( \Xi \) (Fig. 5). Plotting \( \Delta_* \) against \( \Xi \) (Fig. 5, inset) suggests that \( \Delta_* \) approaches a constant value as \( \Xi \) grows large. We observe that the numerical data are well approximated by a
power-law $\Delta_\alpha = c_1 + c_2 \Xi^{-5/4}$, where $c_1 \approx 3.96$, $c_2 \approx 19.5$ (Fig. 5, inset).

Some of the solutions in Fig. 5 have energy $\mathcal{E} < 2$, lower than the energy of the uncompressed, flat solution; these are spontaneous buckled modes that arise in the absence of external forces, but, as is apparent from the corresponding values $D > 1$ (Fig. 5), these solutions are unphysical. In Ref. [10], the flat configuration of the epithelium becomes unstable at large enough differential tension. This instability, absent in the present description, arises because the analysis of Ref. [10] does not impose the condition that the cells match up exactly along their lateral sides [35].

The “large” values of $\Delta$ and hence $\delta$ for these eigenmodes beckon a comment on the formal range of validity of the continuum model: stability of the underlying discrete model requires $\alpha, \beta \geq 0$ [7], and hence $\delta \leq \ell_0^2$. While the asymptotic expansion leading to the geometric relation (13) was an expansion in the large parameter $\ell_0$, it did not involve $\delta$. By contrast, the expansion of the Lagrangian (20), which did involve $\delta$, was an expansion in a different large parameter, $\Xi$. Hence, large values of $\delta \lesssim \ell_0^2$ are indeed in the formal range of validity of the continuum model provided that $\Xi$ is large enough.