EQUATION OF STATE FOR DARK ENERGY IN MODIFIED GRAVITY THEORIES

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We explore the equation of state (EoS) for dark energy \( w_{DE} \) in modified gravitational theories to explain the current accelerated expansion of the universe. We explicitly demonstrate that the future crossings of the phantom divide line of \( w_{DE} = -1 \) are the generic feature in the existing viable \( f(R) \) gravity models. Furthermore, we show that the crossing of the phantom divide can be realized in the combined \( f(T) \) theory constructed with the exponential and logarithmic terms. In addition, we investigate the effective EoS for the universe when the finite-time future singularities occur in non-local gravity.

Keywords: Modified theories of gravity; Dark energy; Cosmology.

1. Introduction

It has been suggested that the current expansion of the universe is accelerating by recent cosmological observations such as Supernovae Ia (SNe Ia),1 cosmic microwave background (CMB) radiation,2,3 large scale structure (LSS),4 baryon acoustic oscillations (BAO),5 and weak lensing.6 There are two representative approaches to account for the late time cosmic acceleration. One is the introduction of cosmological constant dark energy in the framework of general relativity. The other is the modification of gravity, for example, \( f(R) \) gravity, where \( f(R) \) is an arbitrary function of the scalar curvature \( R \) (for recent reviews on \( f(R) \) gravity, see, e.g., Refs. 7,8). One of the most important parameter in this issue is the equation of state (EoS) for dark energy \( w_{DE} \equiv P_{DE}/\rho_{DE} \), which is the ratio of the pressure \( P_{DE} \) of dark energy to the energy density \( \rho_{DE} \) of it. Recent cosmological observational data9 also seems to indicate the crossing of the phantom divide line of \( w_{DE} = -1 \) of the EoS for dark energy in the near “past”. In this paper, we concentrate on the evolution of \( w_{DE} \). In particular, we review our
main results on it in $f(R)$ gravity, $f(T)$ theory and non-local gravity. The paper is organized as follows. In Sec. 2, we explicitly show that the future crossings of the phantom divide line $w_{\text{DE}} = -1$ are the generic feature in the existing viable $f(R)$ gravity models. In Sec. 3, we demonstrate that the crossing of the phantom divide can be realized in the combined $f(T)$ theory having the exponential and logarithmic terms. In Sec. 4, we evaluate the effective EoS for the universe when a finite-time future singularity occurs in non-local gravity. The effective EoS corresponds to the ratio of the total pressure to the total energy density of the universe. We use units of $k_B = c = \hbar = 1$ and denote the gravitational constant $8\pi G$ by $\kappa^2 \equiv 8\pi/M_{\text{Pl}}^2$ with the Planck mass of $M_{\text{Pl}} = G^{-1/2} = 1.2 \times 10^{19}\text{GeV}$.

2. Future crossing of the phantom divide in $f(R)$ gravity

The action of $f(R)$ gravity is given by $I_{f(R)} = \int d^4x \sqrt{-g} f(R)/(2\kappa^2)$, where $g$ is the determinant of the metric tensor $g_{\mu\nu}$. Here, we use the standard metric formalism. It is known that viability conditions for $f(R)$ gravity are (a) positivity of the effective gravitational coupling, (b) stability of cosmological perturbations, (c) asymptotic behavior to the standard $\Lambda$-Cold-Dark-Matter ($\Lambda$CDM) model in the large curvature regime, (d) stability of the late-time de Sitter point, (e) constraints from the equivalence principle, and (f) solar-system constraints. We consider the following four viable models: (i) Hu-Sawicki, $f_{\text{HS}} \equiv R - [c_1 R_{\text{HS}} (R/R_{\text{HS}})^p]/[c_2 (R/R_{\text{HS}})^p + 1]$, where $c_1$, $c_2$, $p(>0)$, $R_{\text{HS}}(>0)$ are constant parameters (for an extended model, see Ref. 18). (ii) Starobinsky, $f_S \equiv R + \lambda R_S \left((1 + R^2/R_S^2)^{-n} - 1\right)$, where $\lambda(>0)$, $n(>0)$, $R_S$ are constant parameters. (iii) Tsujikawa, $f_T \equiv R - \mu R_T \tanh(R/R_T)$, where $\mu(>0)$, $R_T(>0)$ are constant parameters. (iv) the exponential gravity, $f_E \equiv R - \beta R_E [1 - \exp(-R/R_E)]$, where $\beta$ and $R_E$ are constant parameters. It has been examined that the crossing of the phantom divide can be realized in the above viable $f(R)$ models on the past. We therefore explore the future evolution of $w_{\text{DE}}$.

We take the flat Friedmann-Lemaitre-Robertson-Walker (FLRW) metric $ds^2 = -dt^2 + a^2(t) \sum_{i=1,2,3} (dx^i)^2$. In this background, $w_{\text{DE}}$ is given by

$$w_{\text{DE}} = \frac{-(1/2)(FR - f) + \dot{F} + 2H\dot{F} - (1 - F)(2\dot{H} + 3H^2)}{(1/2)(FR - f) - 3H^2 - 3(1 - F)\dot{H}}$$

where $F(R) \equiv df(R)/dR$, and the dot denotes the time derivative of $\partial/\partial t$, and $\dot{H} \equiv \dot{a}/a$ is the Hubble parameter. As a result, it has explicitly been demonstrated that in the future the crossings of the phantom divide are
the generic feature in the viable \( f(R) \) gravity models (i)–(iv) shown above. We mention that \( f(R) \) gravity models with realizing the crossings of the phantom divide have been reconstructed analytically\(^{22}\) and numerically.\(^{23}\) The new cosmological ingredient is that in the future the sign of \( \dot{H} \) changes from negative to positive due to the dominance of dark energy over non-relativistic matter. This is a common physical phenomena to the existing viable \( f(R) \) models and thus it is one of the peculiar properties of \( f(R) \) gravity models characterizing the deviation from the \( \Lambda \)CDM model.

3. Equation of state for dark energy in \( f(T) \) theory

There is another procedure to study gravity beyond general relativity by using the Weitzenböck connection, which has no curvature but torsion, rather than the curvature defined by the Levi-Civita connection. This approach is referred to “teleparallelism”.\(^{24}\) The teleparallel Lagrangian density described by the torsion scalar \( T \) has been extended to a function of \( T \)\(^{25,26}\) to account for the late-time cosmic acceleration as well as inflation in the early universe.\(^{27}\) This idea is equivalent to the concept of \( f(R) \) gravity, in which the Ricci scalar \( R \) in the Einstein-Hilbert action is promoted to a function of \( R \). The modified teleparallel action for \( f(T) \) theory is given by

\[
I_{f(T)} = \frac{1}{2\kappa^2} \int d^4x |e| (T + f(T)),
\]

where \( |e| = \det (e^A_\mu) = \sqrt{-g} \). In the teleparallelism, orthonormal tetrad components \( e_A(x^\mu) \) are used, where an index \( A \) runs over 0, 1, 2, 3 for the tangent space at each point \( x^\mu \) of the manifold. In the flat FLRW background, by using the analysis method in Ref. 17, we explicitly illustrate the cosmological evolution of \( w_{\text{DE}} \) in \( f(T) \) gravity, expressed as

\[
w_{\text{DE}} = -1 + \frac{T' f_T + 2T f_{TT}}{3T f/T - 2f_T} - \frac{f/T - f_T + 2T f_{TT}}{(1 + f_T + 2T f_{TT})(f/T - 2f_T)},
\]

where a prime denotes a derivative with respect to \( \ln a \), \( f_T \equiv df(T)/dT \) and \( f_{TT} \equiv d^2f(T)/dT^2 \). Since we are interested in the late time universe, we consider only non-relativistic matter (cold dark matter and baryon), whose pressure is approximately zero. We have constructed an \( f(T) \) theory by combining the logarithmic and exponential terms in order to realize the crossing of the phantom divide:

\[
f(T) = \gamma \left\{ T_0 \left( u T_0/T \right)^{-1/2} \ln \left( u T_0/T \right) - T \left[ 1 - \exp \left( u T_0/T \right) \right] \right\}
\]

with \( \gamma \equiv \left\{ 1 - \Omega_{m}^{(0)} \right\} / \left\{ 2u^{-1/2} + \left[ 1 - (1 - 2u) \exp (u) \right] \right\} \), where \( T_0 \) is the current torsion and \( u \) is a constant. Moreover, \( \Omega_{m}^{(0)} \equiv \rho_{m}^{(0)}/\rho_{\text{crit}}^{(0)} \), where \( \rho_{m}^{(0)} \) is the energy density of non-relativistic matter at the present time and \( \rho_{\text{crit}}^{(0)} = 3H_0^2/\kappa^2 \) is
the critical density with \( H_0 \) being the current Hubble parameter. We have shown that the crossing in the combined \( f(T) \) theory is from \( w_{\text{DE}} > -1 \) to \( w_{\text{DE}} < -1 \), which is opposite to the typical manner in \( f(R) \) gravity models.

4. Effective equation of state in non-local gravity

Non-local gravity produced by quantum effects has been proposed in Ref. 28. It is known that matter instability occurs in \( f(R) \) gravity and the curvature inside matter sphere becomes very large and hence the curvature singularity could appear.\(^{29,30}\) It is important to examine whether there exists the curvature singularity, called “the finite-time future singularities”, in non-local gravity. The starting action of non-local gravity is given by

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[ R \left( 1 + f^{-1}(\Box R) \right) - 2\Lambda \right] + \mathcal{L}_{\text{matter}} \right\}. \tag{3}
\]

Here, \( g \) is the determinant of the metric tensor \( g_{\mu\nu} \), \( f \) is some function, \( \Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu \) with \( \nabla_\mu \) being the covariant derivative is the covariant d’Almbertian for a scalar field, \( \Lambda \) is a cosmological constant, and \( \mathcal{L}_{\text{matter}} \) is the matter Lagrangian. The above action in Eq. (3) can be rewritten by introducing two scalar fields \( \eta \) and \( \xi \) in the following form:

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[ R \left( 1 + f(\eta) \right) - \partial_\mu \xi \partial^\mu \eta - \xi R - 2\Lambda \right] + \mathcal{L}_{\text{matter}} \right\}. \tag{4}
\]

We take the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric. We consider the case in which the scalar fields \( \eta \) and \( \xi \) only depend on time. In this background, by deriving the gravitational field equations and the equations of motion for the scalar fields \( \eta \) and \( \xi \) and using these equations, we examine whether there exists the finite-time future singularities in non-local gravity. We analyze an asymptotic solution of the gravitational field equations in the limit of the time when the finite-time future singularities appear. We consider the case in which the Hubble parameter \( H \) is expressed as

\[
H \sim h_s (t_s - t)^{-q},
\]

where \( h_s \) is a positive constant, \( q \) is a non-zero constant larger than \(-1\) \((q > -1, q \neq 0)\), and \( t_s \) is the time when the finite-time future singularity appears. We only consider the period \( 0 < t < t_s \) because \( H \) should be real number. When \( t \to t_s \), for \( q > 1 \), \( H \sim h_s (t_s - t)^{-q} \) as well as \( \dot{H} \sim q h_s (t_s - t)^{-(q+1)} \) become infinity and hence the scalar curvature \( R \) diverges. For \(-1 < q < 0 \) and \( 0 < q < 1 \), \( H \) is finite, but \( \dot{H} \) becomes infinity and therefore \( R \) also diverges. From \( H \sim h_s (t_s - t)^{-q} \), we obtain

\[
a \sim a_s \exp \left\{ \frac{[h_s/(q-1)](t_s - t)^{-(q-1)}}{1-q} \right\},
\]

where \( a_s \) is a constant. \( \eta \) is described as \( \eta \) in the limit \( t \to t_s \), for \( q > 1 \),
\[ H \ll H^2 \] and hence \( R \sim 12H^2 \), whereas for \(-1 < q < 0 \) and \( 0 < q < 1 \), \( H \gg H^2 \) and hence \( R \sim 6H \). By taking the leading term in terms of \( (t_s - t) \), we obtain \( \eta \sim -\frac{4h_s}{(q - 1)}(t_s - t)^{-(q - 1)} + \eta_c \) \((q > 1)\), \( \eta \sim -\frac{6h_s}{(q - 1)}(t_s - t)^{-(q - 1)} + \eta_c \) \((-1 < q < 0, 0 < q < 1)\), where \( \eta_c \) is an integration constant. We take a form of \( f(\eta) \) as \( f(\eta) = f_s \eta^\sigma \), where \( f_s(\neq 0) \) and \( \sigma(\neq 0) \) are non-zero constants. From the expression of \( a \), we see that when \( t \to t_s \), for \( q > 1 \), \( a \to \infty \), whereas for \(-1 < q < 0 \) and \( 0 < q < 1 \), \( a \to a_s \). Moreover, it follows from \( H \sim h_s(t_s - t)^{-q} \) and \( \rho_{\text{eff}} = 3H^2/\kappa^2 \) that for \( q > 0 \), \( H \to \infty \) and therefore \( \rho_{\text{eff}} = 3H^2/\kappa^2 \to \infty \), whereas for \(-1 < q < 0 \), \( H \) asymptotically becomes finite and also \( \rho_{\text{eff}} \) asymptotically approaches a finite constant value \( \rho_s \). On the other hand, from \( H \sim qh_s(t_s - t)^{-(q + 1)} \) and \( P_{\text{eff}} = -\left(2\dot{H} + 3H^2\right)/\kappa^2 \) we find that for \( q > -1 \), \( \dot{H} \to \infty \) and hence \( P_{\text{eff}} \to -\left(2\dot{H} + 3H^2\right)/\kappa^2 \to \infty \). Here, \( \rho_{\text{eff}} \) and \( P_{\text{eff}} \) are the effective energy density and pressure of the universe, respectively. It is known that the finite-time future singularities can be classified in the following manner:\(^{31}\) Type I ("Big Rip")\(^{32}\): In the limit \( t \to t_s \), \( a \to \infty \), \( \rho_{\text{eff}} \to \infty \) and \( |P_{\text{eff}}| \to \infty \). The case in which \( \rho_{\text{eff}} \) and \( P_{\text{eff}} \) becomes finite values at \( t = t_s \) is also included. Type II ("sudden")\(^{33}\): In the limit \( t \to t_s \), \( a \to a_s \), \( \rho_{\text{eff}} \to \rho_s \) and \( |P_{\text{eff}}| \to \infty \). Type III: In the limit \( t \to t_s \), \( a \to a_s \), \( \rho_{\text{eff}} \to \infty \) and \( |P_{\text{eff}}| \to \infty \). Type IV: In the limit \( t \to t_s \), \( a \to a_s \), \( \rho_{\text{eff}} \to 0 \), \( |P_{\text{eff}}| \to 0 \), and higher derivatives of \( H \) diverge. The case in which \( \rho_{\text{eff}} \) and/or \( |P_{\text{eff}}| \) asymptotically approach finite values is also included. For \( q > 1 \), the Type I ("Big Rip") singularity, for \( 0 < q < 1 \), the Type III singularity, and for \(-1 < q < 0 \), the Type II ("sudden") singularity. If \( \eta_c \neq 0 \) and \( \xi_c = 1 \), in a model with \( \sigma < 0 \), there can exist the finite-time future singularities with the property of the Type I ("Big Rip") singularity for \( q > 1 \). If \( \eta_c \neq 0 \), in a model with satisfying the condition \( f_s \eta_c^{\sigma - 1} (6\sigma - \eta_c) + \xi_c - 1 = 0 \), there can exist the finite-time future singularities with the property of the Type III singularity for \( 0 < q < 1 \) and that of the Type II ("sudden") singularity for \(-1 < q < 0 \). In the special case of \( \eta_c = 0 \), the finite-time future singularities described by \( H \sim h_s(t_s - t)^{-q} \) cannot occur. The limit on a constant EoS for dark energy in a flat universe has been estimated as \( w_{\text{DE}} = -1.10 \pm 0.14 \) (68% CL) by combining the data of Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations\(^3\) with the latest distance measurements from the BAO in the distribution of galaxies and the Hubble constant measurement. We estimate the present value of \( w_{\text{eff}} \). Here, we regard \( w_{\text{eff}} \) as being approximately equal to \( w_{\text{DE}} \) at the present time \( (w_{\text{eff}} \approx w_{\text{DE}}) \) because the energy density of dark energy is dominant.
over that of non-relativistic matter at the present time. For \( q > 1 \) with \( \sigma < 0 \), we take \( \sigma = -1 \), \( q = 2 \), \( h_s = 1 \) [GeV]\(^{-1} \) and \( t_s = 2t_p \), where \( t_p \) is the present time. The current value of the Hubble parameter is given by \( H_p = 2.1h \times 10^{-42} \text{GeV}^3 \) with \( h = 0.7 \).\(^{3,35} \) In this case, if \( f_s = -3.0 \times 10^{-43} \), \( w_{\text{eff}} = -1.10 \), and if \( f_s = -2.1 \times 10^{-43} \), \( w_{\text{eff}} = -0.93 \). For \( 0 < q < 1 \), we take \( \sigma = 1 \), \( q = 1/2 \), \( h_s = 1 \) [GeV]\(^{1/2} \), \( \eta_c = 1 \) and \( t_s = 2t_p \). In this case, if \( f_s = 7.9 \times 10^{-2} \), \( w_{\text{eff}} = -1.10 \), and if \( f_s = 6.6 \times 10^{-2} \), \( w_{\text{eff}} = -0.93 \). For \( -1 < q < 0 \), we have \( w_{\text{eff}} > 0 \). Thus, the present observed value of \( w_{\text{DE}} \) can be realized in our models.

5. Summary

We have investigated the EoS for dark energy \( w_{\text{DE}} \) in \( f(R) \) gravity as well as \( f(T) \) theory. We have shown that the future crossings of the phantom divide line are the generic feature in the existing viable \( f(R) \) gravity models. It has also been demonstrated that the crossing of the phantom divide line can be realized in an \( f(T) \) theory constructed by combining the exponential and logarithmic terms. Furthermore, we have explored the effective EoS for the universe when the finite-time future singularities occur in non-local gravity.

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