Lattice Construction $C^*$ from Self-Dual Codes

Maiara F. Bollauf *, Sueli I. R. Costa*, and Ram Zamir ‡

* Institute of Mathematics, Statistic and Scientific Computing
University of Campinas, São Paulo
13083-859, Brazil
Email: bollauf@ieee.org, sueli@ime.unicamp.br

‡ Deptartment Electrical Engineering-Systems
Tel Aviv University, Tel Aviv, Israel
Email: zamir@eng.tau.ac.il

Abstract—Construction $C^*$ was recently introduced as a generalization of the multilevel Construction C (or Forney’s code-formula), such that the coded levels may be dependent. Both constructions do not produce a lattice in general, hence the central idea of this paper is to present a 3-level lattice Construction $C^*$ scheme that admits an efficient nearest-neighborhood decoding. In order to achieve this objective, we choose coupled codes for levels 1 and 3, and set the second level code $C_2$ as an independent linear binary self-dual code, which is known to have a rich mathematical structure among families of linear codes. Our main result states a necessary and sufficient condition for this construction to generate a lattice. We then present examples of efficient lattices and also non-lattice constellations with good packing properties.

Index terms—Multilevel construction, Construction C, Construction $C^*$, self-dual codes, sphere packing.

I. INTRODUCTION

A lattice is a well studied mathematical structure due to an extensive list of applications, including its efficient packing properties. The sphere packing problem has known solutions only for dimensions 2, 3, 8 and 24, [13, 6, 21] and all of them can be reached by lattices. For other dimensions, there are strong beliefs that the best possible packing density can be achieved by lattices.

One way of producing lattice constellations is to use linear codes in the so called Constructions A, B, and D [8]. There are also other interesting constructions that generate more general constellations (lattices and non-lattices) with prominent applications in quantization and coded modulation, such as Constructions C [11] and $C^*$ [4]. The advantage of working with such constructions is mainly the translation of characteristics from the linear code over a finite field to an infinite constellation in the $n$-dimensional real space.

While the condition for Construction C to be a lattice is elegant and directly related to Construction D [13], the lattice condition for its generalization, i.e. Construction $C^*$, cannot be related to any other previous lattice construction [4]. Thus, one proposal of this work is to investigate families of codes which make Construction $C^*$ always a lattice and the result points out to the role of self-dual codes.

In coding theory, self-dual codes are of a peculiar importance as they represent the best known error correcting codes for transmission or data storage [14], when one is interested in transmitting a large number of messages with a large minimum weight, in order to correct maximum number of errors. Their properties and relations with results from group theory, combinatorics and lattices are well known. Self-dual codes underlying Construction A are explored in several works [2, 17, 19] regarding the association of these codes to unimodular lattices.

We are inspired by the 3-level Construction $C^*$ of the Leech lattice presented in [4], which considered coupled codes for levels 1 and 3, while the second level was the $[24,12,8]$—GoI code. We generalize this idea for any even dimension by fixing the choice of the second level code $C_2$ to be a self-dual code and our main result states a necessary and sufficient condition for such construction to produce a lattice. This theory also arises as a promising approach for the open problem of decoding Construction $C^*$, by using an extension of the works from Forney [12] and Amrani et al. [11] to any 3-level lattice Construction $C^*$.

We present alternative constructions for the $E_8$ lattice and known packings in dimensions 32 and 40. Interesting non-lattice constellations (with a code $C_2$ which is not self-dual), including a special one in dimension 4 that achieves the same packing density of the lattice $D_4$, are presented.

This paper is organized as follows: Section II introduces some relevant notions about lattices, Construction $C^*$, and codes. Section III presents a general way of producing lattices via a 3-level Construction $C^*$ by using self-dual codes in the second level. Section IV is devoted to examples of lattice packings. Section V describes non-lattice constellations which have good packing properties, including one that presents the same packing density as the densest known lattice in $\mathbb{R}^4$. Finally, in Section VI conclusions and perspectives of future work are drawn.

II. BACKGROUND ON LATTICES AND CODES

In this section, we recall the definition of Construction $C^*$ and the condition for it to be a lattice. We also point out some properties of self-orthogonal and self-dual codes.
Definition 1. (Lattice) A lattice $\Lambda \subset \mathbb{R}^n$ is a set of integer linear combinations of independent vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$. We say that a lattice is full rank if $N = n$, which is the case of lattices explored through this paper. The volume $\text{vol}(\Lambda)$ of a full rank lattice is the absolute value of the determinant of a matrix which has its columns as the generator vectors $v_1, v_2, \ldots, v_n$.

Definition 2. (Packing radius and packing density) The packing radius $r_{\text{pack}}(\Lambda)$ of a lattice $\Lambda \subset \mathbb{R}^n$ is half of the minimum distance between lattice points and the packing density $\Delta(\Lambda)$ of the space that is covered by balls $B(\mathbf{x}, r_{\text{pack}}(\Lambda))$ of radius $r_{\text{pack}}(\Lambda)$, centered at a lattice point $\mathbf{x} \in \Lambda$, i.e.,

$$\Delta(\Lambda) = \frac{\text{vol}(B(0, r_{\text{pack}}(\Lambda)))}{\text{vol}(\Lambda)} = \frac{V_n}{V_n} r_{\text{pack}}^n \frac{\text{vol}(\Lambda)}{\text{vol}(\Lambda)}$$

where $V_n$ refers to the volume of the unit ball in $\mathbb{R}^n$.

The packing density is an important measure to compare lattices. However, for increasing dimensions, this value tends to zero and analogies are hard to perform. In that case, instead of analyzing packing densities it is common to compare Hermite constants.

Definition 3. (Hermite constant) The Hermite constant of a lattice $\Lambda \subset \mathbb{R}^n$ is given by

$$\gamma_n(\Lambda) = 4 \left( \frac{\Delta(\Lambda)}{V_n} \right)^{\frac{1}{2n}} = \frac{4r_{\text{pack}}^n(\Lambda)}{\text{vol}(\Lambda)^{\frac{2}{n}}} = \frac{d_{\text{min}}^2(\Lambda)}{\text{vol}(\Lambda)^{\frac{2}{n}}},$$

where $V_n$ refers to the volume of the unit ball in $\mathbb{R}^n$.

The Hermite constant $\gamma_n$ measures the highest attainable coding gain of an $n$-dimensional lattice.

Besides the well known Constructions A and D, that produce lattice constellations from linear codes, another interesting construction is the so called Construction C or construction by code-formula [11].

Definition 4. (Construction C) Consider $L$ binary codes $C_1, \ldots, C_L \subseteq \mathbb{F}_2^n$, not necessarily nested or linear. Then we define an infinite constellation $C_\infty$ in $\mathbb{R}^n$ that is called Construction C as:

$$C_\infty := C_1 + 2C_2 + \cdots + 2^{L-1}C_L + 2^L \mathbb{Z}^n. \quad (3)$$

A generalization of Construction C was introduced in [3, 4] and denoted by Construction $C^*$. It was inspired by bit-interleaved coded modulation (BICM) and asymptotically, it was demonstrated its superior packing efficiency when compared to Construction C.

The main feature of Construction $C^*$ that differs from Construction C is the fact that the levels are inter-coded, i.e., they are dependent.

Definition 5. (Construction $C^*$) Let $C \subseteq \mathbb{F}_2^L$ be a binary code. Then Construction $C^*$ is defined as

$$C^* := \{c_1 + 2c_2 + \cdots + 2^{L-1}c_L + 2^Lz : (c_1, c_2, \ldots, c_L) \in C, c_i \in \mathbb{F}_2, i = 1, \ldots, L, z \in \mathbb{Z}^n \}. \quad (4)$$

Note that Construction C coincides with Construction $C^*$ when $C = C_1 \times \cdots \times C_n$ and we observe that both constructions in general do not produce a lattice. A condition that will assure the latticeness of Construction $C^*$ will be presented next.

Definition 6. (Projection codes) Let $c = (c_1, \ldots, c_L)$ be a partition of a codeword $c = (c_1, \ldots, c_{n_1}, \ldots, c_{L_1}, \ldots, c_{L_n}) \in C \subseteq \mathbb{F}_2^L$ into length $-n$ subvectors $c_i = (c_{i1}, \ldots, c_{in_i})$, $i = 1, \ldots, L$. Then, a projection code $C_i$ consists of all subvectors $c_i$ that appear as we scan through all possible codewords $c \in C$.

In what follows, we denote by $+$ the real addition and by $\oplus$ the sum in $\mathbb{F}_2$, i.e., $x + y = (x + y) \mod 2$.

Definition 7. (Antiprojection) The antiprojection $S_i(c_1, \ldots, c_{i-1}, 1, c_{i+1}, \ldots, c_L) \subseteq \mathbb{F}_2^L$ of a vector $c_i \in C_i$, $i = 1, \ldots, L$ is the set of all subvectors $c_i$ that appear in $c$.

In this paper we set $L = 3$ for Construction $C^*$ and analyze the case where the second level code $C_2$ is a self-orthogonal linear code in $\mathbb{F}_2^3$, independent of the other two levels. In $\mathbb{F}_2$, the standard inner product of $c = (c_1, c_2, \ldots, c_n)$ and $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n)$ is defined as $\langle c, \tilde{c} \rangle = \sum_{i=1}^{n} c_i \tilde{c}_i \mod 2$ and the orthogonal set $C^\perp$ of a code $C \subseteq \mathbb{F}_2^n$ is also defined as the set $C^\perp = \{ c \in \mathbb{F}_2^n : \langle c, \tilde{c} \rangle = 0, \forall \tilde{c} \in C \}$.

Definition 9. (Self-orthogonal and self-dual codes) A code $C$ is self-orthogonal if $C \subseteq C^\perp$ and it is self-dual if $C = C^\perp$. 


A code $C$ is self-orthogonal if and only if $\langle c, c \rangle = 0$, for all $c \in C$. Each codeword in a self-orthogonal code has even Hamming weight and $(1, \ldots, 1) \in C^\perp$. Indeed, let $c \in C$, which is a self-orthogonal code, then $\langle c, c \rangle = 0$, and it means that the Hamming weight of $c$, i.e., $\omega(c)$, is always even for all $c \in C$. Also, $(1, \ldots, 1) \in C^\perp$ due to the fact that $\langle c, (1, \ldots, 1) \rangle = 0$, for all $c \in C$ and $\omega(c)$ is even.

A characterization of self-dual codes is given by [10] p. 81[16]: a $[n, k, d]$—linear code $C$ is self-dual if and only if $C \subset C^\perp$ and $k = \frac{n}{2}$.

Example 1. The Reed-Muller code $RM(1, 4)$, which is a $[16, 5, 8]$—binary linear code is self-orthogonal, while the $[8, 4, 4]$—extended Hamming code and the $[24, 12, 8]$—extended Golay code are both examples of self-dual codes.

III. GENERAL LATTICES VIA 3-LEVEL CONSTRUCTION $C^*$

Inspired by the Leech lattice construction via $C^*$ presented in [9], we aim to describe a more general 3-level lattice Construction $C^*$ by fixing the level (projection) codes as

- $C_1 = \{(0, \ldots, 0), (1, \ldots, 1)\} \subset \mathbb{F}_2^n$, which is the repetition code;
- $C_2 \subset \mathbb{F}_2^n$ as a convenient code we are going to explore later;
- $C_3 = \hat{C}_3 \cup \bar{C}_3 = \mathbb{F}_2^n$, and we require that if $c_1 = (0, \ldots, 0)$ then $c_3 \in \hat{C}_3 = \{(x_1, \ldots, x_n) \in \mathbb{F}_2^n : \sum_{i=1}^{n} x_i \equiv 0 \mod 2\}$ and if $c_1 = (1, \ldots, 1)$ then $c_3 \in \bar{C}_3 = \{(y_1, \ldots, y_n) \in \mathbb{F}_2^n : \sum_{i=1}^{n} y_i \equiv 1 \mod 2\}$.

In other words, the main code $C \subset \mathbb{F}_2^{3n}$ is given by

$$C = \{(0, \ldots, 0, a_1, \ldots, a_n, x_1, \ldots, x_n), \quad (1, \ldots, 1, a_1, \ldots, a_n, y_1, \ldots, y_n)\}, \quad (8)$$

One can notice that the dependence between levels is crucial in the definition of the main code $C \subset \mathbb{F}_2^{3n}$, as in Eq. (8). We can then define a constellation $\Gamma_C^*$, as the 3-level Construction $C^*$ given by

$$\Gamma_C^* = \{c_1 + 2c_2 + 4c_3 + 8z : (c_1, c_2, c_3) \in C, z \in \mathbb{Z}^n\}. \quad (9)$$

The choice of $C_2$ in Eq. (8) is directly related to Theorem [1] as we are interested in constructing lattice constellations.

Theorem 2. (Lattice Construction $C^*$ with self-orthogonal codes) Let $C \subset \mathbb{F}_2^{3n}$ be a linear code according to Eq. (8). The resulting constellation $\Gamma_C^*$, Eq. (9) obtained via Construction $C^*$ from the code $C$ is a lattice if and only if $C_2 \subset \mathbb{F}_2^n$ is a self-orthogonal code that contains $(1, \ldots, 1)$.

Proof. ($\Rightarrow$) Suppose that $\Gamma_C^*$. constructed from $C \subset \mathbb{F}_2^{3n}$ is a lattice. Then, given $x, y \in \Gamma_C^*$, it is true that $x + y \in \Gamma_C^*$. We can write

$$x = c_1 + 2c_2 + 4c_3 + 8z \quad y = c'_1 + 2c'_2 + 4c'_3 + 8z'$$

and $x + y \in \Gamma_C^*$ implies that the vector

$$\langle c_1 + c'_1, c_2 + c'_2, c_3 + c'_3, (c_1 + c'_1) * (c_2 + c'_2) * (c_2 + c'_2) \rangle \in C \quad (10)$$

and in particular, $c_2 + c'_2 + (c_1 + c'_1) \in C$. Due to linearity, $c_2 + c'_2 \in C_2$ and for $c_1 = c'_1 = (1, \ldots, 1)$, we must have that $(1, \ldots, 1) \in C_2$.

It remains to demonstrate the $C_2$ is self-orthogonal. There are only four possible choices for $c_1$ and $c'_1$, which we discuss case by case below:

- $c_1 = c'_1 = (0, \ldots, 0)$: from Eq. (10) we have that $(0, \ldots, 0, c_2 + c'_2, c_3 + c'_3, c_2 + c'_2) \in C_2$, where by construction $c_3 + c'_3$ has even weight, so it is straightforward to conclude that the sum of the coordinates of $c_2 + c'_2$ is equal to zero and $c_2, c'_2 = 0$.
- $c_1 = (1, \ldots, 1)$ and $c'_1 = (0, \ldots, 0)$: from Eq. (10) we have that $(1, \ldots, 1, c_2 + c'_2, c_3 + c'_3, c_2 + c'_2) \in C$, where by construction the coordinates of $c_3$ sum one modulo 2 and the coordinates of $c'_3$ sum zero modulo 2, thus the only possibility is that the sum of $c_2 + c'_2$ is equal to zero and $c_2, c'_2 = 0$. An analogous argument applies to the case where $c_1 = (0, \ldots, 0)$ and $c'_1 = (1, \ldots, 1)$.
- $c_1 = c'_1 = (1, \ldots, 1)$: from Eq. (10) we have that $(0, \ldots, 0, c_2 + c'_2 (1, \ldots, 1), c_3 + c'_3 (c_2 = c_2') (c_2 + c'_2) (c_2 + c'_2) \in C$, where in this case both coordinates of $c_3$ and $c'_3$ sum one modulo 2, hence $c_3 + c'_3$ has even weight and consequently also $c_2 + c'_2$ and $c_2 + c'_2$ must have even weight. We need to prove that the coordinates of $c_2 + c'_2$ sum zero modulo 2. Assume that $c_2 + c'_2$ has odd weight, by contradiction (because it will force $c_2 + c'_2$ to have odd weight as well). Due to the linearity of $C_2$, $c_2 + c'_2 = c_2 = c_2 \in C_2$. Then, we consider in Eq. (10), $c_2 = c'_2 = \hat{c}_2$, which yields:

$$(0, \ldots, 0, 1, \ldots, 1, c_3 + c'_3 (c_2 + c'_2) (c_2 + c'_2) \hat{c}_2 \hat{c}_2 \in C, \quad (11)$$

and $(\hat{c}_2 + \hat{c}_2) (c_2 + c'_2) = c_2$, which makes the third coordinate to have odd weight. Thus, the element written in Eq. (11) does not belong to the code $C$ and we have a contradiction. Therefore, both $c_2 + c'_2$ and $c_2 + c'_2$ must have even weight, which implies that $(c_2, c'_2) = 0$.

We can then conclude that $C_2$ is self-orthogonal.

($\Leftarrow$) To assure the latticeness condition from Theorem 1 we hold one needs to first verify that

$$C_1 \subseteq S_2(0, \ldots, 0) \subseteq C_2 \subseteq S_3(0, \ldots, 0) \subseteq C_3, \quad (12)$$

and due to the structure of $C \subset \mathbb{F}_2^{3n}$ in Eq. (8) we have that $S_2(0, \ldots, 0) = C_2$ and $S_3(0, \ldots, 0) = C_3$. By hypothesis, $(1, \ldots, 1) \in C_2$, what allow us to conclude that $C_1 \subseteq S_2(0, \ldots, 0)$ and this nesting is clearly closed under Schur product.

Since $C_2$ is self-orthogonal, all codewords have even weight and $C_2 \subseteq \hat{C}_3$. It remains to show that this nesting is closed under Schur product, i.e., given any $c_2, c'_2 \in C_2$, the sum of all coordinates of the vector defined by $c_2 + c'_2$ should be zero.
modulo 2. Observe that the Schur product is the coordinate-by-coordinate product and the action of summing all components of the resulting Schur product vector is the same as \((c_2, c_2')\). Thus, we want to prove that \((c_2, c_2') = 0 \text{ mod } 2\), which is true since \(C_2\) is self-orthogonal.

One can observe that for self-dual codes, the condition required by Theorem \(\text{[2]}\) is automatically satisfied, because \(C = C^\perp\) and also \((1, \ldots, 1) \in C^\perp\).

IV. CONSTRUCTIONS OF KNOWN LATTICES VIA C*

We can only expect to have interesting lattice constellations via Construction C* following the procedure described in Section [III] for \(n\) even, because we need to assure that \((1, \ldots, 1) \in C_2 \subset S_3(0, \ldots, 0) = \tilde{C}_3\).

This section summarizes some new lattice constructions for even dimensions built from a 3-level Construction C* with the main code \(C \subset F_2^{2n}\) as in Eq. \(\text{[6]}\), whose resulting constellation is \(\Gamma_{C^*}\) as in Eq. \(\text{[9]}\).

Observe that an essential feature to calculate the packing efficiency or Hermite constant of a lattice is the minimum distance. A closed formula for the minimum distance of a constellation generated by Construction C* is still an open problem and in general, what is known is just an upper and lower bound for it \(\text{[4]}\). However, for particular cases, when the codes are established, as it is the case of the examples explored in this section, this calculation can be done by brute force, i.e., by investigating all possible minimum weight codewords and calculating the minimum among them.

**Dimension 8 - E8 lattice:** Define \(C_2\) as the \([8, 4, 4]\)-extended Hamming code, which is self-dual and whose basis vectors are displayed in the rows of the following generator matrix,

\[
G = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 
\end{bmatrix}.
\]

One can notice that the minimum distance of \(C_1\) is 8, of \(C_2\) is 4, and of \(\tilde{C}_3\) and \(\overline{C}_3\) is 2. Then, because of the dependence created by the main code \(C\) (Eq. \(\text{[8]}\)), in order to calculate the squared minimum distance of \(\Gamma_{C^*}\), we may consider the combinations of codewords that yields in the minimum, i.e.,

\[
d_{\min}^2(\Gamma_{C^*}) = \min\{d_H(C_1) + 2^2d_H(C_3), \quad 2^2d_H(C_2), \quad 2^4d_H(C_3), \quad 2^4d_H(C_7)\} = \min\{8 + 16, \quad 2^2 \cdot 4, \quad 2^4 \cdot 2, \quad 2^4 \cdot 2\} = 16
\]

and \(d_{\min}(\Gamma_{C^*}) = 4\). Here, \(d_H\) denotes the minimum Hamming weight of the respective code. Hence, the packing density of this construction is calculated by

\[
\Delta(\Gamma_{C^*}) = \frac{|C| \cdot \text{vol}(E_8(0, d_{\min}))}{2^{3n}} = \frac{2 \cdot 4^2 \cdot 7 \cdot \pi^4 \cdot 2^8}{2^{24} \cdot 4^7} = 0.25367.
\]

which coincides with the packing density of the \(E_8\) lattice and \(E_8 = \frac{1}{\sqrt{8}} \Gamma_{C^*}\). This construction is just to illustrate that one can achieve the same packing density as \(E_8\) lattice via Construction \(C^*\), although the most efficient way of representing this lattice is via Construction A.

**Dimension 14:** Consider \(C_2\) as the self-dual code \([14, 7, 4]\). Thus,

\[
d_{\min}^2(\Gamma_{C^*}) = \min\{32 + 14, \quad 2^2 \cdot 4, \quad 2^4 \cdot 2\} = 16,
\]

whose Hermite constant is

\[
\gamma_{14}(\Gamma_{C^*}) = \frac{d_{\min}^2(\Gamma_{C^*})}{\text{vol}(\Gamma_{C^*})^{2/n}} = \frac{16}{(2^{48})^{2/14}} = 2.
\]

The upper bound for the Hermite constant in this dimension is 2.4886, according to \(\text{[5]}\).

In dimension 16, the best known packing density is given by the decoupled version of Eq.\(\text{[5]}\), where \(C_2 = \mathcal{R}\mathcal{M}(2, 4)\), where \(\mathcal{R}\mathcal{M}(r, m)\) denotes the Reed-Muller code of length \(2^m\) and order \(r\). In this particular case, Construction C, D and \(C^*\) coincides.

**Dimension 24:** (Leech lattice) This construction was already presented in \(\text{[3, 4]}\) and it assumes \(C_2\) as the \([24, 12, 8]\)-extended Golay code.

**Dimension 32:** Define \(C_2\) as the \(\mathcal{R}\mathcal{M}(2, 5)\), which is a \([32, 16, 8]\)–self dual code. Then, we have that, following an analogous calculation for the minimum distance as it was done in the \(E_8\) case,

\[
d_{\min}^2(\Gamma_{C^*}) = \min\{32 + 16, \quad 2^2 \cdot 8, \quad 2^4 \cdot 2\} = 32.
\]

Hence, the Hermite constant is

\[
\gamma_{32}(\Gamma_{C^*}) = \frac{d_{\min}^2(\Gamma_{C^*})}{\text{vol}(\Gamma_{C^*})^{2/n}} = \frac{32}{(2^{48})^{2/32}} = 4,
\]

which coincides with Hermite constant of the Barnes-Wall lattice \(BW_{32}\).

**Dimension 40:** Define \(C_2\) as an extremal self-dual \([40, 20, 8]\)-code, i.e., its minimum distance achieves the highest possible value for given \(k\) and \(n\). The squared minimum distance is given by

\[
d_{\min}^2(\Gamma_{C^*}) = \min\{40 + 16, \quad 2^2 \cdot 8, \quad 2^4 \cdot 2\} = 32.
\]

The Hermite constant of this lattice constellation is

\[
\gamma_{40}(\Gamma_{C^*}) = \frac{d_{\min}^2(\Gamma_{C^*})}{\text{vol}(\Gamma_{C^*})^{2/n}} = \frac{32}{(2^{90})^{2/40}} = 4,
\]

which coincides the Hermite constant given by the extremal even unimodular lattice in dimension 40.

V. SPECIAL NON-LATTICE CONSTELLATIONS

One can notice that the scheme proposed for Construction \(C^*\) may be also used to get non-lattice constellations when the code \(C_2\) is not self-orthogonal or it is but does not contain the codeword \((1, \ldots, 1)\).

**Dimension 4:** It is believed that the best known packing density for any constellation in dimension \(n = 4\) is given by
the lattice $D_4$ [7, 8], which is, up to congruence, the unique lattice that achieves this density. In the sequel, we present a non-lattice constellation that achieves the same packing density as $D_4$.

We consider $C_1$ and $C_2$ as the coupled codes according to Section III and $C_2$ is the $\mathcal{R}(1, 2) = [4, 3, 2]$—code, i.e.,

$$\mathcal{R}(1, 2) = \{ (0, 0, 0, 0), (1, 0, 0, 1), (1, 1, 1, 1),$$

$$(0, 0, 1, 1), (0, 1, 1, 0), (1, 0, 1, 0), (1, 1, 0, 0) \}.$$

we can see that this code is not self-orthogonal. Moreover, if we apply a Construction $C^*$ as proposed in Eq. (9), it does not give a lattice. Indeed, consider $(4, 6, 0, 2), (4, 4, 2, 2) \in \Gamma_C$. Their real sum is $(8, 10, 2, 4) = (0, 0, 0, 0) + 2(0, 1, 1, 0) + 4(0, 0, 0, 1) + 8(1, 1, 0, 0)$ and $(0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1) \notin C \subseteq \mathbb{F}_2^{12}$.

When we calculate the squared minimum distance of this constellation, we have that $d_{\min}^2(\Gamma_{C^*}) = \min\{4 + 16, 2^2 \cdot 2, 2^4 \cdot 2 \} = \{20, 8, 32 \} = 8$ and $d_{\min}(\Gamma_{C^*}) = 2\sqrt{2}$. The packing density of this construction is then

$$\Delta(\Gamma_{C^*}) = \frac{|C| \text{ vol}(B_2(0, d_{\min}))}{2^{3n}} = 2 \cdot 2^3 \cdot 2^3 \pi^2 \frac{\pi}{2^3} \sqrt{2}^4$$

$$= \frac{\pi^2}{24} \approx 0.6168...$$

which is the same packing density as the $D_4$ lattice.

Other interesting non-lattice cases obtained by an analogous construction are the following:

**Dimension 18:** Considering $C_2$ to be the $[18, 9, 6]$—binary linear code [18], the resulting constellation achieves the best known Hermite constant in this dimension [5].

**Dimension 20:** The best sphere packing in dimension 20 is presented in the work of Vardy [20] and it can be seen as a Construction $C^*$, where the three levels are coupled.

**Dimension 40:** By assuming $C_2$ as the $[40, 23, 8]$—binary linear code [8, p. 146], we can slightly improve the Hermite constant of the lattice presented in Section IV in dimension 40, which in this case reaches $\gamma_{40} = 4.287$.

VI. CONCLUSION AND FUTURE WORK

We detailed some lattice constructions under the perspective of a special scheme of Construction $C^*$, using coupled first and third levels and admitting as second level self-dual codes. This construction is only interesting for low dimensions, because the choice of the most significant bit code (third level) forces an upper bound for the squared minimum distance equal to 32, which does not depend on the dimension. This drawback may be solved by applying Construction $C^*$ to other families of coupled codes or by increasing the number of levels.

We also presented non-lattice constructions, including a four dimensional Construction $C^*$ that achieves the same packing density as the $D_4$ lattice and interesting potentially interesting results for dimensions 18, 20, and 40. We aim in a future work to apply other self-dual codes to Construction $C^*$, also with different alphabet sizes, and compare it with known results for Construction A [17].

In terms of efficient decoding, the idea is to generalize the bounded-distance decoding scheme for the Leech lattice proposed by Forney [12] to any 3—level lattice Construction $C^*$ built according the structure proposed by this paper.

ACKNOWLEDGMENT

The authors would like to thank Joseph J. Boutros for fruitful discussions and also the reviewers for meaningful suggestions. SIRC was supported by CNPq (313326/2017-7) and FAPESP (2013/25977-7) Foundations, and RZ was supported by Israel Science Foundation (676/15).

REFERENCES

[1] O. Amrani, Y. Beery, A. Vardy, F-W. Sun, and H. C. A. van Tilborg, “The Leech lattice and the Golay code: bounded-distance decoding and multilevel constructions”, IEEE Trans. on Inf. Th., vol. 40, no. 4, pp. 1030-1043, Jul. 1994.

[2] C. Bachoc, “Applications of coding theory to construction of unimodular lattices”, Jour. of Comb. Th., vol. 78, n. 1, pp. 92-119, Apr. 1997.

[3] M. F. Bollauf, R. Zamir and Sueli I. R. Costa, “Construction $C^*$ : an inter-level coded version of Construction C”, 2018 Int. Zav. Sem. on Inf. and Comm, Zurich, pp. 118-122, Feb. 2018.

[4] M. F. Bollauf, R. Zamir and S.I.R. Costa, “Multilevel constructions: coding, packing and geometric uniformity”, IEEE Trans. on Inf. Th., vol. 65, n. 12, pp. 7669-7681, Dec. 2019.

[5] H. Cohn and N. Elkies, “New upper bounds on sphere packings I”, Ann. of Math., vol. 157, no. 2, pp. 689-714, 2003.

[6] H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, and M. Viazovska, “The sphere packing problem in dimension 24”, Ann. Math., vol. 185, no. 3, pp. 1017-1033, Apr. 2017.

[7] J. H. Conway and N. J. A. Sloane, “What are all the best sphere packings in low dimensions?”, Discr. Comput. Geom, vol. 13, pp. 382 - 403, 1995.

[8] J. H. Conway and N.J.A. Sloane, Sphere Packings, Lattices and Groups, 3rd ed. New York, USA: Springer, 1999.

[9] R. de Buda, “Fast FSK signals and their demodulation”. Can. Electron. Eng. Journal, vol. 1, pp. 2834, Jan. 1976.

[10] W. Ebeling and F. Hirzebruch, Lattices and codes: a course partially based on lectures by F. Hirzebruch. Wiesbaden: Vieweg, 2002.

[11] G. D. Forney, “Coset codes-part I: introduction and geometrical classification”. IEEE Trans. Inf. Theory, vol. 34, no. 5, pp. 1123-1151, Sep. 1988.

[12] G. D. Forney, “A bounded-distance decoding algorithm for the Leech lattice with generalizations”. IEEE Trans. Inf. Theory, vol. 35, no. 4, pp. 906-909, Jul. 1989.

[13] T. Hales, “A proof of the Kepler conjecture”, Ann. Math., vol. 162, pp. 1065-1185, 2005.

[14] W. C. Huffman, “On the classification and enumeration of self-dual codes”, Fin. E. Their Appl., vol. 11, pp. 451-490, 2005.

[15] W. Kositwattanarerk and F. Oggier, “Connections between Construction D and related constructions of lattices”. Designs, Codes and Cryptography, v. 73, pp. 441-455, Nov. 2014.

[16] F. J. MacWilliams and N. J. A. Sloane, The theory of error-correcting codes. New York, NY: North Holland Publishing Co., 1977.

[17] G. Nebe, E. M. Rains, and N. J. A. Sloane, Self-dual codes and invariant theory. Netherlands: Springer-Verlag Berlin, 2006.

[18] J. Simonis, “The [18,9,6] code is unique”, Discr. Math., vol. 106-107, pp. 439-448, 1992.

[19] N. J. A. Sloane, “Self-dual codes and lattices”, Proc. Symp. Pure Math., vol. 34, pp.273-308.

[20] A. Vardy, “A New Sphere Packing in 20-Dimensions”, Inven. Math., vol. 121, no. 1, pp. 119-134, 1995.

[21] M. Viazovska, “The sphere packing problem in dimension 8”, Ann. Math., vol. 185, no. 3, pp. 991-1015, Apr. 2017.