Analytical validation of the helicity conservation for the compressible Euler equations

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Abstract

In [25], Moffatt introduced the concept of helicity in an inviscid fluid and examined the helicity preservation of smooth solution to barotropic compressible flow. In this paper, it is shown that the weak solutions of the above system in Onsager type spaces \( \dot{B}^{1/3}_{p,c(N)} \) guarantee the conservation of the helicity. The parallel results of homogeneous incompressible Euler equations and the surface quasi-geostrophic equation are also obtained.

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1 Introduction

The Euler equations describing incompressible inviscid fluids with constant density in \( \Omega \times (0, T) \) read

\[
\begin{align*}
\frac{\partial v}{\partial t} + v \cdot \nabla v + \nabla \Pi &= 0, \\
\text{div} \, v &= 0, \\
v|_{t=0} &= v_0(x),
\end{align*}
\]

(1.1)

where the unknown vector \( v \) represents the flow velocity field, and the scalar function \( \Pi = -\Delta^{-1} \partial_i \partial_j (v_i v_j) \) stands for the pressure. The initial velocity \( v_0 \) satisfies \( \text{div} \, v_0 = 0 \). Let \( \Omega \) be the whole space \( \mathbb{R}^d \) or the periodic domain \( \mathbb{T}^d \), where \( d \geq 2 \) is the spatial dimension. We denote \( \omega = \text{curl} \, v \) by the vorticity of the flow velocity in (1.1), whose equations read

\[
\omega_t + v \cdot \nabla \omega - \omega \cdot \nabla v = 0, \quad \text{div} \, \omega = 0.
\]

(1.2)

It is well known that there exist two physically conserved quantities for the regular solutions of the Euler system (1.1):

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• energy conservation
\[ \int_{\Omega} |v(x, t)|^2 dx = \int_{\Omega} |v(x, 0)|^2 dx. \] (1.3)

• helicity preservation
\[ \int_{\Omega} \omega(x, t) \cdot v(x, t) dx = \int_{\Omega} \omega(x, 0) \cdot v(x, 0) dx. \] (1.4)

Besides the above two conserved laws of the Euler equations (1.1), one can find others in [24, p.24]. In a seminal paper [27], Onsager conjectured that weak solutions of the Euler equation with Hölder continuity exponent \( \alpha > \frac{1}{3} \) do conserve energy and that turbulent or anomalous dissipation occurs when \( \alpha \leq \frac{1}{3} \). The first attempt to the energy conservation issue for weak solutions of the Euler equations (1.1) was given by Eyink [18] in a stronger Hölder space \( C^{\alpha}_v \) with \( \alpha > \frac{1}{3} \), which is included in \( C^\alpha \). Subsequently Constantin-E-Titi [11] successfully solved the positive part of the Onsager’s conjecture and proved that the energy is conserved if a weak solution \( v \) is in the Besov space \( L^3(0; T; B^{\alpha}_{3,\infty}(\mathbb{T}^3)) \) with \( \alpha > 1/3 \).

Later on, by deriving a local energy equation which contains a term \( D(v) \) representing the dissipation or production of energy caused by the lack of smoothness of solution, Duchon and Robert gave a weaker condition on the solutions to conserve energy in [17], that is, if \( v \) satisfies \( \int |v(t, x + \xi) - v(t, x)|^3 dx \leq C(t)|\xi|\sigma(|\xi|) \), where \( C(t) \in L^1(0, T) \) and \( \sigma(a) \to 0 \) as \( a \to 0 \), then energy is conserved. Recently, Cheskidov-Constantin-Friedlander-Shvydkoy [3] refined the above spaces to the critical space \( L^3(0, T; B^{\alpha/3}_{3,c(\mathbb{N})}) \), where \( B^{\alpha/3}_{3,c(\mathbb{N})} = \{ v \in B^{\alpha/3}_{3,\infty}: \lim_{q \to \infty} 2^q \| \Delta_q v \|^3_{L^3} = 0 \} \) and \( \Delta_q \) represents a smooth restriction of \( v \) into Fourier modes of order \( 2^q \). In the follow-up paper, Shvydkoy [28, 29] stated that this condition of energy conservation can be equivalent to
\[ \lim_{z \to 0} \frac{1}{T} \int_0^T \int_{\mathbb{R}^d} \frac{|v(x + z, t) - v(x, t)|^3}{|z|} dx dt = 0. \]

It is worth remarking that \( B^{\alpha/3}_{3,q} \) for \( 1 \leq q < \infty \) are included in \( B^{\alpha/3}_{3,c(\mathbb{N})} \), which is called as the Onsager-critical space. In this direction, the recent progress can be found in [20]. The proof of the other part of Onsager’s conjecture in three-dimensional space is given by Isett [21], which adopted the approach developed by De Lellis and Szekelyhidi [12–15].

Next, we turn our attention to the helicity conservation (1.4) in the Euler equations. The concept of the helicity in an inviscid fluid was introduced by Moffatt in [25]. As pointed in [23, 26], helicity is important at a fundamental level in relation to flow kinematics because it admits topological interpretation in relation to the linkage or linkages of vortex lines of the flow. Indeed, Moffatt [25] examined that the helicity of smooth solutions to the following compressible Euler equations (1.5) is invariant in time,
\[ \begin{cases} 
  p_t + \nabla \cdot (\rho v) = 0, \\
  (\rho v)_t + \text{div} (\rho v \otimes v) + \nabla \pi = 0.
\end{cases} \] (1.5)

The proof in [25] relies on the transport formula involving moving region. To the best of our knowledge, there has been little literature concerning the helicity preservation of weak solutions to the compressible Euler equations (1.5). In this direction, most previous works focus on the homogeneous incompressible Euler equations (1.1). In particular, the study of
helicity conservation for the incompressible Euler equations (1.1) was originated from Chae’s interesting works [4, 5]. In [4], Chae considered the preservation of the helicity of weak solutions to the Euler equations via $\omega$ in spaces $L^{3}(0, T; B_{\pi,\infty}^{2})$ with $\alpha > \frac{1}{3}$. Subsequently, it is shown that $v \in L^{r_{1}}(0, T; B_{\pi,q}^{2})$ and $\omega \in L^{r_{2}}(0, T; \dot{B}_{\pi,q}^{2})$, with $\alpha > \frac{1}{3}$, $q \in [2, \infty]$, $r_{1} \in [2, \infty]$, $r_{2} \in [1, \infty]$ and $\frac{2}{r_{1}} + \frac{1}{r_{2}} = 1$, ensure the helicity conservation of weak solutions in [5]. After that, Cheskidov-Constantin-Friedlander-Shvydkoy [8] established the helicity conservation class based on the velocity $v$ in Onsager critical space $L^{3}(0, T; B_{\beta,\infty}^{2,\beta})$. 

Very recently, De Rosa [10] proved that the helicity is a constant provided that $v \in L^{2\beta}(0, T; W^{\beta,2\beta})$ and $\omega \in L^{\alpha}(0, T; W^{\alpha,q})$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{\kappa}$ and $2\theta + \alpha \geq 1$. Notice that there holds the embedding relation $W^{\theta,p} \hookrightarrow B_{p,c(\Omega)}^{\theta}$, which can be found in [8, 10]. The first objective of this paper is to address the question how much regularity is needed for a weak solution of the compressible Euler equations (1.5) to conserve the helicity. To this end, we shall need to give the equation for the vorticity $\omega$ from (1.5). Our observation is that for any smooth solutions $(\rho, v)$ of compressible Euler equations (1.5) with $0 < c_{1} \leq \rho \leq c_{2} < \infty$, dividing the both sides of the momentum equation (1.5) by $\rho$, we can reformulate the system (1.5) as

$$
\begin{align*}
\left\{ 
\begin{array}{l}
\rho_{t} + \nabla \cdot (\rho v) = 0, \\
v_{t} + v \cdot \nabla v + \nabla \Pi(\rho) = 0,
\end{array}
\right.
\end{align*}
$$

(1.7)

where $\Pi(\rho) = \int_{1}^{\rho} \frac{\pi'(s)}{s} \, ds$. Then the momentum equation in (1.7) is the same as that in the incompressible case. However, this is not valid for weak solutions $(\rho, v)$ even with $0 < c_{1} \leq \rho \leq c_{2} < \infty$. To use the equations (1.7), we shall apply the techniques due to Feireisl, Gwiazda, Swierczewska-Gwiazda and Wiedemann in [19] to show that the weak solutions of compressible Euler equations (1.5) in Onsager type spaces $B_{p,c(\Omega)}^{1/3}$ are also the weak solutions of equations (1.7). Now we state our first main result on the helicity conservation of weak solutions for the compressible Euler equations (1.7) as follows.

**Theorem 1.1.** Let $(\rho, v)$ be a solution of (1.5) in the sense of distributions and $\text{div } v, \text{curl } v \in C([0, T]; L^{\frac{2d}{d+1}}(\Omega))$. Assume that there exist two positive constants $c_{1}$ and $c_{2}$ such that

$$
\begin{align*}
0 < c_{1} \leq \rho \leq c_{2} < \infty, \quad \rho \in L^{3}(0, T; \dot{B}_{3,\infty}^{\frac{1}{3}}), \quad \rho v \in L^{3}(0, T; \dot{B}_{3,\infty}^{\frac{1}{3}}), \\
v \in L^{3}(0, T; B_{3,c(\Omega)}^{\frac{1}{3}}), \quad \pi \in C^{2}[c_{1}, c_{2}],
\end{align*}
$$

(1.8)

and one of the following four conditions is satisfied

1. $\omega \in L^{3}(0, T; \dot{B}_{3,\infty}^{\frac{1}{3}})$;
2. $\omega \in L^{3}(0, T; L^{3}(\Omega))$;
3. $v \in L^{\frac{p}{p-2}}(0, T; L^{\frac{q}{q-2}}(\Omega)), \omega \in L^{p}(0, T; L^{q}(\Omega))$, with $2 < p, q < \infty$;
4. $\text{div } v, \omega \in L^{3}(0, T; L^{\frac{2}{3}}(\Omega))$, with $d = 3$. 

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Then the helicity of the compressible Euler equation \((1.5)\) is conserved, that is, for any \(0 < t < T\),
\[
\int_{\Omega} \omega(x, t) \cdot v(x, t) dx = \int_{\Omega} \omega(x, 0) \cdot v(x, 0) dx.
\]

**Remark 1.1.** The helicity conservation of weak solutions for the compressible Euler equations \((1.5)\) in Onsager’s critical space \(B^0_{p,c,(\mathbb{N})}\) and \(B^0_{p,\infty}\) in terms of the velocity and the vorticity is studied in Theorem 1.2 and Corollary 1.3. This is consistent with Moffatt’s work \([23]\). The regularity condition that \(\text{div} \ v, \text{curl} \ v \in C([0,T]; L^{\frac{2}{\gamma+1}}(\Omega))\) is necessary to ensure that the helicity is well defined.

**Remark 1.2.** The constraint that \(\rho v \in L^3(0,T; B^\frac{1}{2}_{3,\infty})\) can be removed, if the condition that \(\rho \in L^3(0,T; B^\frac{1}{2}_{3,\infty})\) is replaced by \(\rho \in L^\infty(0,T; B^\frac{3}{2}_{\infty,c,(\mathbb{N})})\) due to Lemma 2.5.

**Remark 1.3.** This theorem is valid for the compressible isentropic Euler system. Indeed, for the isentropic pressure law \(\pi(\rho) = \kappa\rho^\gamma\) with \(\gamma > 1\) and \(\kappa = \frac{(\gamma-1)^2}{4\gamma}\), the pressure condition that \(\pi \in C^2[c_1,c_2]\) is obviously satisfied if \(0 < c_1 \leq \rho \leq c_2 < \infty\).

**Remark 1.4.** The sufficient conditions (3) in this theorem and (4)-(5) in Theorem 1.2 below are partially motivated by the recent works \([31]\) on energy balance of the Navier-Stokes equations based on the combination of the velocity and its gradient.

**Remark 1.5.** Since the following embedding relations in \(\mathbb{R}^3\) is valid
\[
W^{1, \frac{3}{p}} \hookrightarrow H^\frac{7}{p} \hookrightarrow B^\frac{2}{3,c,(\mathbb{N})},
\]
the fourth criterion of this theorem is also in Onsager’s critical space. It is an interesting question to show the velocity \(v\) in Onsager’s critical space \(L^3(0,T; B^\frac{3}{2}_{3,\infty,c,(\mathbb{N})})\) keeping the helicity conservation for the compressible Euler equations.

In what follows, we formulate our second main result involving helicity conservation for the incompressible Euler equations \((1.1)\).

**Theorem 1.2.** Let \(v\) be a weak solution of incompressible Euler equations \((1.1)\) in the sense of Definition 2.2 and \(\omega \in C([0,T]; L^{\frac{2}{\gamma+1}}(\Omega))\). Then the helicity conservation \((1.4)\) is valid provided that one of the following five conditions is satisfied

1. \(v \in L^k(0,T; \dot{B}^{\alpha}_{p,c,(\mathbb{N})}), \omega \in L^{\ell}(0,T; \dot{B}^{\beta}_{q,c,(\mathbb{N})})\), with \(\frac{2}{k} + \frac{1}{\ell} = 1, \frac{2}{p} + \frac{1}{q} = 1, 2\alpha + \beta \geq 1\);
2. \(v \in L^k(0,T; \dot{B}^{\alpha}_{p,\infty}), \omega \in L^{\ell}(0,T; \dot{B}^{\beta}_{q,\infty,c,(\mathbb{N})})\), with \(\frac{2}{k} + \frac{1}{\ell} = 1, \frac{2}{p} + \frac{1}{q} = 1, 2\alpha + \beta \geq 1\);
3. \(\omega \in L^3(0,T; B^\frac{1}{2}_{3,\infty,c,(\mathbb{N})})\);
4. \(v \in L^{\frac{p}{\gamma-2}}(0,T; L^{\frac{q}{\gamma-2}}(\Omega)), \omega \in L^p(0,T; L^q(\Omega))\), with \(2 < p, q < \infty\);
5. \(v \in L^{\frac{2p}{\gamma+1}}(0,T; L^{\frac{2q}{\gamma+1}}(\Omega)), \text{curl} \omega \in L^p(0,T; L^q(\Omega))\), with \(1 \leq p \leq \infty, 1 \leq q < \infty\).

**Remark 1.6.** Theorem 1.2 is an improvement of corresponding results in \([4, 5, 16]\).

**Remark 1.7.** An approach to \((1.6)\) in homogeneous Besov spaces is presented in the proof of Theorem 1.2.
Recently, when dimension $d = 3$, the authors in [23] provide a proof of energy conservation criteria for the Euler equations in terms of the vorticity

$$\omega \in L^3(0, T; L^2(\Omega)),$$

which is also deduced from Cheskidov-Constantin-Friedlander-Shvydkoy’s classical condition that $L^3(0, T; B^{1/3}_{3\infty}(\Omega))$ (see also related work [9]). In the spirit of [23], we have

**Corollary 1.3.** Let $v$ be a weak solution of the 3D incompressible Euler equations (1.1) in the sense of Definition 2.2 and $\omega \in C([0, T]; L^2(\Omega))$. Then the helicity conservation (1.4) is valid provided that one of the following two conditions is satisfied

1. $\omega \in L^3(0, T; L^9(\Omega));$
2. $\text{curl}\omega \in L^3(0, T; L^9(\Omega))$.

**Remark 1.8.** From (1.9) and the condition (1) in Corollary 1.3, we see that there may exist a weak solution of the Euler equations that keep the energy rather than the helicity. This is also previously pointed out by Chae in [4].

**Remark 1.9.** In dimension two, the criterion (1) in Corollary 1.3 can be replaced by a new condition that $\omega \in L^3(0, T; L^2(\Omega))$.

The starting point of (1) in Theorem 1.1 and (1)-(2) in Theorem 1.2 is the study of the cross-helicity in the ideal MHD equations. Precisely, for any vector functions $A, B$, there hold the identities below

$$\nabla(A \cdot B) = A \cdot \nabla B + B \cdot \nabla A + A \times \text{curl}B + B \times \text{curl}A,$$

$$\text{curl}(A \times B) = A \text{div}B - B \text{div}A + B \cdot \nabla A - A \cdot \nabla B,$$

which together with the condition that $\text{div}\omega = 0$ lead to

$$v \cdot \nabla v = \frac{1}{2} \nabla |v|^2 + \omega \times v,$$

$$\text{curl}(\omega \times v) = \omega \text{div}v + v \cdot \nabla \omega - \omega \cdot \nabla v.$$  

Then we can use (1.10) to reformulate the compressible Euler system (1.7) and its vorticity equations as

$$\begin{cases}
v_t + \omega \times v + \nabla(\Pi(\rho) + \frac{1}{2}|v|^2) = 0, \\
\omega_t - \text{curl}(v \times \omega) = 0, \\
\text{div}\omega = 0,
\end{cases}$$

which is very closed to the inviscid ideal Magnetohydrodynamics (MHD) equations

$$\begin{cases}
v_t + \omega \times v - b \cdot \nabla b + \nabla(p + \frac{1}{2}|v|^2 + \frac{1}{2}|b|^2) = 0, \\
b_t - \text{curl}(v \times b) = 0, \\
\text{div}v = \text{div}b = 0,
\end{cases}$$

where $b$ describes the magnetic field. For this ideal MHD system, the cross-helicity $\int_\Omega v(x, t) \cdot h(x, t)dx$ is conserved for the smooth solutions. To the knowledge of the authors, Yu [34]
obtained cross-helicity conservation criterion of weak solutions for ideal MHD equations based on the condition that \( v \in L^3(0, T; B^{\alpha_1}_{3, \infty}) \) and \( b \in L^3(0, T; B^{\alpha_2}_{3, \infty}) \) with \( \alpha_1 + 2\alpha_2 \geq 1 \) and \( \alpha_2 \geq \frac{1}{3} \) (see also \([3, 22, 30]\)). Inspired by Yu’s work \([34]\), we consider the helicity conservation of weak solutions to the incompressible Euler equations \([11]\) in spaces \( B^\theta_{p,c}(\Omega) \) and \( B^\theta_{p,\infty} \). It should be pointed out that this coincides with the appearance of helicity conservation of weak solutions to the compressible Euler equations \([15]\) motivated by the cross-helicity conservation in the MHD system (see \([25, 26]\)). The proof of \((1)\) in Theorem 1.4 is very close to that in \([8, 34]\). The key point is our new observation that functions in the Onsager type spaces \( \dot{B}^{1/3}_{p,c}(\Omega) \) can satisfy the corresponding Constantin-E-Titi type commutator estimates in physical spaces (see Lemma 2.6). It seems that the argument presented here is quite general and can be applied in other fluid equations. Particularly, we shall show that our approach here can be applied in the 2D surface quasi-geostrophic (SQG) equation below,

\[
\begin{align*}
\theta_t + v \cdot \nabla \theta &= 0, & \text{in } (0, T) \times \Omega, \\
v &= \nabla^\perp \Psi = \left(-\frac{\partial \Psi}{\partial x_2}, \frac{\partial \Psi}{\partial x_1}\right), & \theta = -(-\Delta)^{1/4} \Psi, & \text{in } [0, T) \times \Omega, \\
\theta|_{t=0} &= \theta_0, & \text{in } \Omega, \\
where the unknown scalar function } \theta(x, t): \mathbb{R}^2 \times [0, \infty) \to \mathbb{R} \text{ stands for the temperature, } v \text{ is the velocity field, and } \Psi = -\int_{\Omega} \frac{\theta(x, t)}{|x-y|} dy \text{ is the stream function. Here we assume that } \Omega \text{ is the whole space } \mathbb{R}^2 \text{ or torus } \mathbb{T}^2 \text{ with periodic boundary conditions. The surface quasi-geostrophic equation arises in geophysical fluids and shares many striking similarities with the 3D Euler equations (see } [10] \text{ and references therein). Zhou } [35] \text{ studied the following general helicity}
\int_{\Omega} \theta \partial_i \theta dx, i = 1, 2,
\text{of weak solutions for the 2D surface quasi-geostrophic equation } (1.12). \text{ It is shown that if } \nabla \theta \in C([0, T]; L^\frac{3}{2}(\Omega)) \cap L^3(0, T; B^{\alpha}_3(\Omega)) \text{ for } \alpha > 1/3, \text{ then the general helicity of weak solutions for 2-D surface quasi-geostrophic equation is conserved in } [33]. \text{ In the present paper, we are going to give some new sufficient conditions to guarantee the conservation of the helicity for weak solutions to the 2-D surface quasi-geostrophic equation in Onsager-critical spaces. Our third main result can be stated as follows: }

\textbf{Theorem 1.4.} \text{ Let } \theta \text{ be a weak solution to the 2D surface quasi-geostrophic equation } (1.12) \text{ in the sense of Definition } 2.3 \text{ and } \nabla \theta \in C([0, T]; L^\frac{3}{2}(\Omega)). \text{ Then the helicity conservation}
\int_{\Omega} \theta(x, t) \partial_i \theta(x, t) dx = \int_{\Omega} \theta_0(x) \partial_i \theta_0(x) dx, i = 1, 2, \tag{1.13}
\text{is valid provided that } \nabla \theta \in L^3(0, T; B^{\frac{1}{2}}_3, c(\Omega)) \cap C([0, T]; L^\frac{3}{2}(\Omega)).
\textbf{Remark 1.10.} \text{ According to the Bernstein inequality, one may replace the condition that } \nabla \theta \in L^3(0, T; B^{\frac{1}{2}}_3, c(\Omega)) \text{ by } \theta \in L^3(0, T; B^{\frac{1}{2}}_3, c(\Omega)) \text{ in this theorem. }

\text{The rest of this paper is organized as follows. In Section 2, we present some notations and auxiliary lemmas which will be used in the present paper. Particularly, in the spirit of}
we shall show that the functions in Onsager type spaces $B^{\frac{1}{2}}_{p,c}(\Omega)$ mean the Constantin-E-Titi type commutator estimates in physical spaces, which plays an important role in the follow-up study. Section 3 and Section 4 are devoted to the proof of helicity conservation of weak solutions to the compressible Euler equations and the homogeneous incompressible Euler equations, respectively. The helicity conservation of weak solutions to the 2-D surface quasi-geostrophic equation is considered in Section 5.

2 Notations and some auxiliary lemmas

First, we introduce some notations used in this paper. For $p \in [1, \infty]$, the notation $L^p(0, T; X)$ stands for the set of measurable functions $f$ on the interval $(0, T)$ with values in $X$ and $\|f\|_X$ belonging to $L^p(0, T)$. The classical Sobolev space $W^{k,p}(\Omega)$ is equipped with the norm $\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| = 0}^k \|D^\alpha f\|_{L^p(\Omega)}$. $S$ represents the Schwartz class of rapidly decreasing functions, $S'$ the space of tempered distributions, $S'/P$ the quotient space of tempered distributions which modulo polynomials. We use $Ff$ or $\hat{f}$ to denote the Fourier transform of a tempered distribution $f$. $a \approx b$ means that $C^{-1}b \leq a \leq Cb$ for some constant $C > 1$. For simplicity, we write

$$\int_0^t \int_\Omega f(t, x)dxds = \int_0^t f$$

and $\|f\|_{L^p(0,T;X(\Omega))} = \|f\|_{L^p(0,T;X)}$.

To define Besov spaces, we need the following dyadic unity partition (see e.g. [1]). Choose two nonnegative radial functions $\varrho, \varphi \in C^\infty(\mathbb{R}^d)$ supported respectively in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq \frac{3}{4}\}$ and the shell $\{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{5}\}$ such that

$$\varrho(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^d; \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \neq 0.$$

Then for every $\xi \in \mathbb{R}^d$, $\varphi(\xi) = \varrho(\xi/2) - \varrho(\xi)$. Denote $h = F^{-1}\varphi$ and $\tilde{h} = F^{-1}\varrho$, then nonhomogeneous dyadic blocks $\Delta_j$ are defined by

$$\Delta_j u := 0 \quad \text{if} \quad j \leq -2, \quad \Delta_{-1} u := \varrho(D)u = \int_{\mathbb{R}^d} \tilde{h}(y)u(x - y)dy,$$

and $\Delta_j u := \varphi(2^{-j}D)u = 2^{jd} \int_{\mathbb{R}^d} h(2^jy)u(x - y)dy$ if $j \geq 0$.

The nonhomogeneous low-frequency cut-off operator $S_j$ is defined by

$$S_j u := \sum_{k \leq j - 1} \Delta_k u.$$

The homogeneous dyadic blocks $\tilde{\Delta}_j$ and homogeneous low-frequency cut-off operators $\hat{S}_j$ are defined for every $j \in \mathbb{Z}$ by

$$\tilde{\Delta}_j u := \varphi(2^{-j}D)u = 2^{jd} \int_{\mathbb{R}^d} h(2^jy)u(x - y)dy,$$

and $\hat{S}_j u := \varrho(2^{-j}D)u = 2^{jd} \int_{\mathbb{R}^d} \tilde{h}(2^jy)u(x - y)dy$. 

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Lemma 2.1 (Bernstein inequality). For any $b \geq a \geq 1$, there holds that
\[
\left\| \Delta_j f \right\|_{L^b} \leq 2^{jd(b^{-1} - a^{-1})} \left\| \Delta_j f \right\|_{L^a}.
\] (2.2)

The following lemma gives a characterization of Besov spaces in terms of finite differences in the same spirit of [17, 28, 34].

Lemma 2.2. Let $\alpha \in (0, 1)$ and $p \in [1, \infty]$, then for any $f \in S'_h$, there holds
\[
f \in \dot{B}^\alpha_{p,\infty} \iff \text{ess sup}_{|y| > 0} \frac{\|f(\cdot - y) - f(\cdot)\|_{L^p}}{|y|^{\alpha}} < \infty; \] (2.3)
\[
\lim_{j \to \infty} 2^{j\alpha} \left\| \Delta_j f \right\|_{L^p} = 0 \iff \lim_{|y| \to 0} \frac{\|f(\cdot - y) - f(\cdot)\|_{L^p}}{|y|^{\alpha}} = 0. \] (2.4)

Proof. (1) “LHS $\Rightarrow$ RHS” in (2.3). First, by the mean value theorem, the Bernstein inequality and the Minkowski inequality, we have for any tempered distribution $f$ in $\mathbb{R}^d$,
\[
\|f(\cdot - y) - f(\cdot)\|_{L^p} \leq C \left( \sum_{j \leq N} 2^j |y| \|\Delta_j f\|_{L^p} + \sum_{j > N} \|\Delta_j f\|_{L^p} \right). \] (2.5)
Before going further, just as \[8\], we set the following localized kernel

\[
K(j) = \begin{cases} 
2^{j\alpha}, & \text{if } j \leq 0, \\
2^{-(1-\alpha)j}, & \text{if } j > 0,
\end{cases}
\]

and denote \(d_j = 2^{j\alpha}\|\hat{\Delta}_j f\|_{L^p}\).

As a consequence, using the mean value theorem, the Bernstein inequality and the Minkowski inequality again, we rewrite (2.5) as

\[
\|f(\cdot - y) - f(\cdot)\|_{L^p} \leq C \left(2^{N(1-\alpha)}|y|\sum_{j \leq N} 2^{-(N-j)(1-\alpha)}2^{j\alpha}\|\hat{\Delta}_j f\|_{L^p} + 2^{-\alpha N} \sum_{j > N} 2^{(N-j)\alpha}2^{j\alpha}\|\hat{\Delta}_j f\|_{L^p}\right)
\]

\[
\leq C \left(2^{N(1-\alpha)}|y| + 2^{-\alpha N}\right) (K * d_j)(N).
\]

Thanks to the localized kernel \(K(\cdot) \in l^1\), by choosing \(N\) such that \(2^{N(1-\alpha)}|y| \approx 2^{-N\alpha}\), we arrive at

\[
\|f(\cdot - y) - f(\cdot)\|_{L^p} \leq C|y|^\alpha (K * d_j)(N) \leq C|y|^\alpha \sup_{j \in \mathbb{Z}} d_j.
\]

This implies that \(\text{ess sup}_{|y| > 0} \frac{\|f(\cdot - y) - f(\cdot)\|_{L^p}}{|y|^\alpha} \leq C\|f\|_{\dot{B}^\alpha_{p,\infty}}\).

Now we will prove the reverse inequality “LHS \(\leq\) RHS” in (2.3). As the mean value of the function \(h\) is 0, we can write

\[
\hat{\Delta}_j f(x) = 2^{jd} \int h(2^j y) f(x - y)dy
\]

\[
= 2^{jd} \int h(2^j y) (f(x - y) - f(x)) dy,
\]

which together with Minkowski inequality yields that

\[
2^{j\alpha}\|\hat{\Delta}_j f\|_{L^p} \leq 2^{jd} \int 2^{j\alpha}|h(2^j y)||f(\cdot - y) - f(\cdot)|_{L^p}dy
\]

\[
\leq \text{ess sup}_{y \in \mathbb{R}^d} \frac{\|f(\cdot - y) - f(\cdot)\|_{L^p}}{|y|^\alpha} \int 2^{j(\alpha + d)}|y|^\alpha|h(2^j y)|dy
\]

\[
\leq C \text{ess sup}_{y \in \mathbb{R}^d} \frac{\|f(\cdot - y) - f(\cdot)\|_{L^p}}{|y|^\alpha}.
\]

Then we conclude the result (2.3).

(2) The proof of (2.4) is similar to (2.3). Due to (2.6) and (2.8), we immediately obtain the result (2.4) after taking the limits. We complete the proof of this lemma.

As a corollary of Lemma 2.2, we have the following properties.
Corollary 2.3. Let $0 < \alpha, \beta < 1$ and $1 \leq q_1, q_2 \leq \infty$. Assume that $f \in \dot{B}^\alpha_{q_1,\infty}$ and $g \in \dot{B}^\beta_{q_2,c(\mathbb{N})}$, then we have

\[
f \in \dot{B}^\alpha_{q_1,\infty} \iff \text{ess sup}_{|y|>0} \frac{||f(\cdot) - f(\cdot)||_{L^{q_1}}}{|y|^\alpha} < \infty, \]

\[
g \in \dot{B}^\beta_{q_2,c(\mathbb{N})} \iff \text{ess sup}_{|y|>0} \frac{||g(\cdot) - g(\cdot)||_{L^{q_2}}}{|y|^\beta} < \infty \quad \text{and} \quad \lim_{|y| \to 0} \frac{||g(\cdot) - g(\cdot)||_{L^{q_2}}}{|y|^\beta} = 0. \tag{2.9}
\]

Moreover, there hold

\[
||f(\cdot) - f(\cdot)||_{L^{q_1}} = O(|y|^\alpha) ||f||_{\dot{B}^\alpha_{q_1,\infty}},
\]

\[
||g(\cdot) - g(\cdot)||_{L^{q_2}} = o(|y|^\beta) ||g||_{\dot{B}^\beta_{q_2,c(\mathbb{N})}}. \tag{2.10}
\]

Combining the properties of mollifier with Corollary 2.3, we derive the following Lemma.

Lemma 2.4. Let $\alpha, \beta \in (0,1)$, $p, q \in [1, \infty]$, and $k \in \mathbb{N}^+$. Assume that $f \in L^p(0, T; \dot{B}^\alpha_{q,\infty})$, $g \in L^p(0, T; \dot{B}^\beta_{q,c(\mathbb{N})})$, then there hold

1. $||f^\varepsilon - f||_{L^p(0, T; L^q)} \leq O(\varepsilon^\alpha) ||f||_{L^p(0, T; \dot{B}^\alpha_{q,\infty})}$;
2. $||\nabla^k f^\varepsilon||_{L^p(0, T; L^q)} \leq O(\varepsilon^{\alpha-k}) ||f||_{L^p(0, T; \dot{B}^\alpha_{q,\infty})}$;
3. $||g^\varepsilon - g||_{L^p(0, T; L^q)} \leq o(\varepsilon^\beta) ||g||_{L^p(0, T; \dot{B}^\beta_{q,c(\mathbb{N})})}$;
4. $||\nabla^k g^\varepsilon||_{L^p(0, T; L^q)} \leq o(\varepsilon^{\beta-k}) ||g||_{L^p(0, T; \dot{B}^\beta_{q,c(\mathbb{N})})}$.

Proof. (1) Since $\int_{\mathbb{R}^d} \eta_\varepsilon(y) dy = 1$, we deduce from direct calculations that

\[
f^\varepsilon(x) - f(x) = \int f(x-y) \eta_\varepsilon(y) dy - f(x) \int \eta_\varepsilon(y) dy = \int [f(x-y) - f(x)] \eta_\varepsilon(y) dy.
\]

According to the Minkowski inequality and Corollary 2.3, we see that

\[
||f^\varepsilon - f||_{L^p(0, T; L^q)} \leq C \int_{B(0, \varepsilon)} ||f(x-y) - f(x)||_{L^p(0, T; L^q)} \eta_\varepsilon(y) dy
\]

\[
\leq \int_{B(0, \varepsilon)} O(|y|^\alpha) ||f||_{L^p(0, T; \dot{B}^\alpha_{q,\infty})} \eta_\varepsilon(y) dy
\]

\[
\leq \int_{B(0, \varepsilon)} O(\varepsilon^\alpha) ||f||_{L^p(0, T; \dot{B}^\alpha_{q,\infty})} \eta_\varepsilon(y) dy
\]

\[
\leq O(\varepsilon^\alpha) ||f||_{L^p(0, T; \dot{B}^\alpha_{q,\infty})},
\]

which concludes (1).
(2) Some straightforward computations yield that

\[
\nabla^k f^\varepsilon(x) = \nabla^k (f * \eta^\varepsilon)(x) = \int_{B(0,\varepsilon)} \nabla^k f(x - y)\eta^\varepsilon(y)dy \\
= (-1)^k \int_{B(0,\varepsilon)} \nabla_y f(x - y)\eta^\varepsilon(y)dy \\
= \varepsilon^{-k} \int_{B(0,\varepsilon)} f(x - y)\nabla^k \eta^\varepsilon(y)dy.
\]

Using the fact \(\int_{B(0,\varepsilon)} \nabla^k \eta^\varepsilon(y)dy = 0\), we infer that

\[
\varepsilon^{-k} \int_{B(0,\varepsilon)} \nabla^k \eta^\varepsilon(y)dy = 0. \tag{2.11}
\]

Hence, we arrive at

\[
\nabla f^\varepsilon(x) = \varepsilon^{-k} \int_{B(0,\varepsilon)} [f(x - y) - f(x)]\nabla^k \eta^\varepsilon(y)dy.
\]

Then making use of the Minkowski inequality and Corollary 2.8 once again, we know that

\[
\|\nabla^k f^\varepsilon\|_{L^p(0,T;L^q)} \leq C\varepsilon^{-k} \int_{B(0,\varepsilon)} \|f(x - y) - f(x)\|_{L^p(0,T;L^q)}\|\nabla^k \eta^\varepsilon(y)\|dy \\
\leq \varepsilon^{-k} \int_{B(0,\varepsilon)} O(|y^\alpha\|_{L^p(0,T;\dot{B}^\alpha_{q,\infty})})\|\nabla^k \eta^\varepsilon(y)\|dy \\
\leq \varepsilon^{-k} \int_{B(0,\varepsilon)} O(\varepsilon^\alpha)\|f\|_{L^p(0,T;\dot{B}^\alpha_{q,\infty})}\|\nabla^k \eta^\varepsilon(y)\|dy \\
\leq O(\varepsilon^{\alpha-k})\|f\|_{L^p(0,T;\dot{B}^\alpha_{q,\infty})}.
\]

This verifies the second part of this lemma. Exactly as the above derivation, we can finish the proof of the rest part of this lemma. \(\square\)

**Lemma 2.5.** Let \(\alpha \in (0,1)\) and \(p \in [1,\infty]\), then there hold

\[
\|fg\|_{\dot{B}^\alpha_{p,\infty}} \leq C\left(\|f\|_{L^\infty}\|g\|_{\dot{B}^\alpha_{p,\infty}} + \|f\|_{\dot{B}^\alpha_{\infty,\infty}}\|g\|_{L^p}\right), \tag{2.12}
\]

and

\[
\|fg\|_{\dot{B}^\alpha_{p,c(\mathbb{N})}} \leq C\left(\|f\|_{L^\infty}\|g\|_{\dot{B}^\alpha_{p,c(\mathbb{N})}} + \|f\|_{\dot{B}^\alpha_{\infty,c(\mathbb{N})}}\|g\|_{L^p}\right), \tag{2.13}
\]

**Proof.** Using the triangle inequality, we have

\[
|f(x - y)g(x - y) - f(x)g(x)| \\
\leq |f(x - y)(g(x - y) - g(x))| + \left|\left(f(x - y) - f(x)\right)g(x)\right|, \tag{2.14}
\]

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which together with the Minkowski inequality implies that
\[
\frac{\|f(\cdot - y)g(\cdot - y) - f g(\cdot)\|_{L^p}}{|y|^\alpha} \leq C \left( \frac{\|f(\cdot - y)g(\cdot - y) - g(\cdot)\|_{L^p}}{|y|^\alpha} + \frac{\|f(\cdot - y) - f(\cdot)\|_{L^\infty}}{|y|^\alpha} \|g\|_{L^p} \right)
\]
\[
\leq C \left( \frac{\|f\|_{L^\infty}}{|y|^\alpha} \|g(\cdot - y) - g(\cdot)\|_{L^p} + \frac{\|f(\cdot - y) - f(\cdot)\|_{L^\infty}}{|y|^\alpha} \|g\|_{L^p} \right)
\]
\[
\leq C \left( \|f\|_{L^\infty} \|g\|_{\dot{B}^{\alpha}_{p,\infty}} + \|f\|_{\dot{B}^{\alpha}_{\infty,c}(\mathbb{R}^n)} \|g\|_{L^p} \right),
\]
where we have used the Corollary 2.3. Moreover, taking the limits on (2.15) and using the Corollary 2.3 again, we have
\[
\lim_{|y| \to 0} \frac{\|f(\cdot - y)g(\cdot - y) - f g(\cdot)\|_{L^p}}{|y|^\alpha} \leq C \left( \|f\|_{L^\infty} \lim_{|y| \to 0} \frac{\|g(\cdot - y) - g(\cdot)\|_{L^p}}{|y|^\alpha} + \lim_{|y| \to 0} \frac{\|f(\cdot - y) - f(\cdot)\|_{L^\infty}}{|y|^\alpha} \|g\|_{L^p} \right)
\]
\[
\leq C \left( \|f\|_{L^\infty} \|g\|_{\dot{B}^{\alpha}_{p,\infty}(\mathbb{R}^n)} + \|f\|_{\dot{B}^{\alpha}_{\infty,c}(\mathbb{R}^n)} \|g\|_{L^p} \right).
\]

Next, we will state a new Constantin-E-Titi type commutator estimate as follows.

**Lemma 2.6.** Assume that $0 < \alpha, \beta < 1$, $1 \leq p, q, p_1, p_2 \leq \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, there holds
\[
\|fg^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q)} \leq o(\varepsilon^{\alpha+\beta}),
\]
provided that one of the following three conditions is satisfied,

1. $f \in L^{p_1}(0,T;\dot{B}^{\alpha}_{q_1,c}(\mathbb{R}^n)), \ g \in L^{p_2}(0,T;\dot{B}^{\beta}_{q_2,\infty}), \ 1 \leq q_1, q_2 \leq \infty, \ \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$；
2. $\nabla f \in L^{p_1}(0,T;\dot{B}^{\alpha}_{q_1,c}(\mathbb{R}^n)), \ \nabla g \in L^{p_2}(0,T;\dot{B}^{\beta}_{q_2,\infty}), \ \frac{2}{q} + \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \ 1 \leq q_1, q_2 < d$；
3. $f \in L^{p_1}(0,T;\dot{B}^{\alpha}_{q_1,c}(\mathbb{R}^n)), \ \nabla g \in L^{p_2}(0,T;\dot{B}^{\beta}_{q_2,\infty}), \ \frac{2}{q} + \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \ 1 \leq q_2 < d, \ 1 \leq q_1 \leq \infty$.

**Remark 2.1.** The estimate that $\int_0^T \int (fg)^\varepsilon - f^\varepsilon g^\varepsilon \|\nabla h\|dxdt$ frequently appears in the study of energy (helicity) conservation of fluid equations (see [7, 11, 16, 31, 33]). Therefore, it seems that this lemma is very helpful in this research of weak solutions in critical Besov spaces.

**Remark 2.2.** We would like to point out that $o(\cdot)$ should be replaced by $O(\cdot)$ if the space $\dot{B}^{\alpha}_{q_1,c}(\mathbb{R}^n)$ in the condition of this lemma is replaced by the space $\dot{B}^{\alpha}_{q_1,\infty}$.

**Remark 2.3.** It is worth remarking that $\nabla$ may be replaced by curl for incompressible flow if $1 < q_1, q_2 < d$.

**Remark 2.4.** The results of Lemma 2.2-2.6 still hold for the nonhomogeneous Besov spaces.
Proof. First, we recall the following identity observed by Constantin-E-Titi in [11] that
\[(fg)^{\varepsilon}(x) - f^{\varepsilon}g^{\varepsilon}(x)\]
\[= \int_{\Omega} \eta_{\varepsilon}(y)[f(x - y) - f(x)][g(x - y) - g(x)]dy - (f - f^{\varepsilon})(g - g^{\varepsilon})(x).\]

(1) Using the Hölder’s inequality and Minkowski inequality, we obtain
\[
\|f^{\varepsilon}g^{\varepsilon} - f^{\varepsilon}g^{\varepsilon}\|_{L^p(0,T;L^q)}
\leq C \int_{|y| \leq \varepsilon} \eta_{\varepsilon}(y)\|f(\cdot - y) - f(\cdot)\|_{L^p(0,T;L^{q_1})}\|g(\cdot - y) - g(\cdot)\|_{L^p(0,T;L^{q_2})}dy
\leq C \|f - f^{\varepsilon}\|_{L^p(0,T;L^{q_1})}\|g - g^{\varepsilon}\|_{L^p(0,T;L^{q_2})}
\leq o(\varepsilon^{\alpha + \beta})\|f\|_{L^p(0,T;B^{\alpha}_{q_1,c(\Omega)})}\|g\|_{L^p(0,T;B^{\beta}_{q_2,\infty})},
\]
where Corollary 2.3 and Lemma 2.4 are used.

(2) Let \(\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}\) and \(\frac{1}{q_i} = \frac{1}{q_i} - \frac{1}{q}, i = 1, 2\). A slight variant of the above proof implies that
\[
\|f^{\varepsilon}g^{\varepsilon} - f^{\varepsilon}g^{\varepsilon}\|_{L^p(0,T;L^q)}
\leq C \int_{|y| \leq \varepsilon} \eta_{\varepsilon}(y)\|f(\cdot - y) - f(\cdot)\|_{L^p(0,T;L^{q_1})}\|g(\cdot - y) - g(\cdot)\|_{L^p(0,T;L^{q_2})}dy
\leq C \|f - f^{\varepsilon}\|_{L^p(0,T;L^{q_1})}\|g - g^{\varepsilon}\|_{L^p(0,T;L^{q_2})}
\leq o(\varepsilon^{\alpha + \beta})\|\nabla f\|_{L^p(0,T;B^{\alpha}_{q_1,c(\Omega)})}\|\nabla g\|_{L^p(0,T;B^{\beta}_{q_2,\infty})},
\]
where we have used the Hölder inequality and Sobolev embedding theorem.

(3) Arguing in the same manner as above, we know that
\[
\|f^{\varepsilon}g^{\varepsilon} - f^{\varepsilon}g^{\varepsilon}\|_{L^p(0,T;L^q)}
\leq C \int_{|y| \leq \varepsilon} \eta_{\varepsilon}(y)\|f(\cdot - y) - f(\cdot)\|_{L^p(0,T;L^{q_1})}\|g(\cdot - y) - g(\cdot)\|_{L^p(0,T;L^{q_2})}dy
\leq C \|f - f^{\varepsilon}\|_{L^p(0,T;L^{q_1})}\|g - g^{\varepsilon}\|_{L^p(0,T;L^{q_2})}
\leq o(\varepsilon^{\alpha + \beta})\|\nabla f\|_{L^p(0,T;B^{\alpha}_{q_1,c(\Omega)})}\|\nabla g\|_{L^p(0,T;B^{\beta}_{q_2,\infty})},
\]
where the indices are required to satisfy \(\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}\) and \(\frac{1}{q_i} = \frac{1}{q_i} - \frac{1}{d}\) with \(1 \leq q_2 < d\).

Then the proof of this lemma is completed.
Lemma 2.7. Let \( p, q, p_1, q_1, p_2, q_2 \in [1, \infty) \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \). Assume that \( f \in L^{p_1}(0, T; L^{q_1}(\Omega)) \) and \( g \in L^{p_2}(0, T; L^{q_2}(\Omega)) \). Then, as \( \varepsilon \to 0 \), there hold

\[
\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q(\Omega))} \to 0, \quad (2.18)
\]

and

\[
\|(f \times g)^\varepsilon - f^\varepsilon \times g^\varepsilon\|_{L^p(0,T;L^q(\Omega))} \to 0. \quad (2.19)
\]

Proof. First, it follows from the triangle inequality that

\[
\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q)} \\
\leq C \left( \|(fg)^\varepsilon - (fg)\|_{L^p(0,T;L^q)} + \|f - f^\varepsilon\|_{L^p(0,T;L^q)} + \|g - g^\varepsilon\|_{L^p(0,T;L^q)} \right) \\
\leq C \left( \|(fg)^\varepsilon - fg\|_{L^p(0,T;L^q)} + \|f - f^\varepsilon\|_{L^p(0,T;L^q)} + \|g - g^\varepsilon\|_{L^p(0,T;L^q)} \right) \\
+ \|f^\varepsilon\|_{L^p(0,T;L^q)} \|g - g^\varepsilon\|_{L^p(0,T;L^q)},
\]

which together with the properties of the standard mollifiers yields (2.18).

Furthermore, to conclude (2.19), we only consider the case that the spatial dimension \( d = 3 \), since the proof for the case \( d = 2 \) is similar. Without loss of generality, we assume that \( f = (f_1, f_2, f_3) \) and \( g = (g_1, g_2, g_3) \). Then by a direct computation, we have

\[
\|(f \times g)^\varepsilon - f^\varepsilon \times g^\varepsilon\|_{L^p(0,T;L^q)} = \left\| \begin{bmatrix} f_1 \varepsilon & f_2 \varepsilon & f_3 \varepsilon \\ g_1 \varepsilon & g_2 \varepsilon & g_3 \varepsilon \end{bmatrix} \right\|_{L^p(0,T;L^q)} \\
\leq \left\| \begin{bmatrix} f_1 \varepsilon & f_2 \varepsilon & f_3 \varepsilon \\ g_1 \varepsilon & g_2 \varepsilon & g_3 \varepsilon \end{bmatrix} \right\|_{L^p(0,T;L^q)} \\
= \left\| \begin{bmatrix} (f_2 g_3 - f_3 g_2) + (f_1 g_3 - f_3 g_1) \varepsilon \\ (f_1 g_2 - f_2 g_1) \varepsilon \\ f_1 \varepsilon & f_2 \varepsilon & f_3 \varepsilon \end{bmatrix} \right\|_{L^p(0,T;L^q)} \\
\leq \left\| \begin{bmatrix} (f_2 g_3 - f_3 g_2) \varepsilon \\ (f_1 g_2 - f_2 g_1) \varepsilon \\ f_1 \varepsilon & f_2 \varepsilon & f_3 \varepsilon \end{bmatrix} \right\|_{L^p(0,T;L^q)} \\
= \left\| \begin{bmatrix} ((f_2 g_3 - f_3 g_2) \varepsilon) - ((f_3 g_2 - f_2 g_3) \varepsilon) \\ ((f_1 g_2 - f_2 g_1) \varepsilon) - ((f_2 g_1 - f_1 g_2) \varepsilon) \\ f_1 \varepsilon & f_2 \varepsilon & f_3 \varepsilon \end{bmatrix} \right\|_{L^p(0,T;L^q)},
\]

which together with the triangle inequality and (2.18) leads to (2.19). \( \square \)

For the convenience of readers, we end this section by presenting the definition of weak solutions to the compressible Euler equations (1.5), incompressible Euler equations (1.1) and the surface quasi-geostrophic equation (1.12), respectively.

Definition 2.1. A pair \((\rho, v)\) is called a weak solution to the compressible Euler equations (1.5), if \((\rho, v)\) satisfies

(i) for any test function \(\phi \in C^\infty_0((0, T) \times \Omega)\), there holds

\[
\int_0^T \int_\Omega \rho(x, t) \partial_t \phi(x, t) + \rho(x, t)v(x, t) \nabla \phi(x, t) dx dt = 0.
\]
(ii) for any test vector field $\varphi \in C^\infty_0((0,T) \times \Omega)$, there holds
\[
\int_0^T \int_\Omega \rho(x,t)v(x,t)\partial_t \varphi(x,t) + (\rho(x,t)v(x,t) \otimes v(x,t)) \nabla \varphi(x,t) + \pi(\rho)(x,t) \text{div} \varphi(x,t) \, dx \, dt = 0.
\]

(iii) the energy inequality holds
\[
\mathcal{E}(t) \leq \mathcal{E}(0),
\]
where $\mathcal{E}(t) = \int_\Omega \left( \frac{1}{2} \rho |v|^2 + \kappa \frac{\gamma}{\gamma-1} \right) \, dx$.

**Definition 2.2.** A vector field $v \in C^\text{weak}([0,T]; L^2(\Omega))$ is called a weak solution of the Euler equations (1.1) with initial data $v_0 \in L^2(\Omega)$, if $v$ satisfies

(i) for any divergence-free test function $\varphi \in C^\infty_0((0,T) \times \Omega)$, there holds
\[
\int_0^T \int_\Omega v(x,t)\partial_t \varphi(x,t) + v(x,t) \otimes v(x,t) \cdot \nabla \varphi(x,t) \, dx \, dt = 0.
\]

(ii) $v$ is weakly divergence-free, namely, for every scalar test function $\psi \in C^\infty_0((0,T) \times \Omega)$,
\[
\int_0^T \int_\Omega \text{div} v \cdot \psi \, dx \, dt = 0.
\]

**Definition 2.3.** A vector field $\theta \in C([0,T]; L^2(\Omega))$ is called a weak solution of the 2-D quasi-geostrophic equation (1.12) with initial data $\theta_0 \in L^2(\Omega)$, if there hold
\[
\int_\Omega [\theta(x,T)\varphi(x,T) - \theta(x,0)\varphi(x,0)] \, dx = \int_0^T \int_\Omega \theta(x,t)(\partial_t \varphi(x,t) + v \cdot \nabla \varphi) \, dx \, dt,
\]
and
\[
v(x,t) = -\nabla^\perp \int_\Omega \frac{\theta(y,t)}{|x-y|} \, dy
\]
for any test function $\varphi \in C^\infty_0([0,T]; C^\infty(\Omega))$, where $\nabla^\perp$ in (2.23) is in the sense of distributions.

### 3 Helicity conservation for the compressible Euler equations

This section contains the proof of helicity conservation of weak solutions in Onsager type spaces $\dot{B}^{\frac{3}{p}}_{p,c(N)}$ for the compressible Euler equations.

**Proposition 3.1.** Let $(\rho, v)$ be a weak solution of compressible Euler equations (1.5) in the sense of Definition 2.1 and $\text{div} v, \text{curl} v \in C([0,T]; L^{\frac{2d}{d+1}}(\Omega))$. If $(\rho, v)$ additionally satisfies
\[
0 < c_1 \leq \rho \leq c_2 < \infty, \quad \rho \in L^3(0,T; \dot{B}^{\frac{3}{2}}_{3,c(N)}), \quad \rho v \in L^3(0,T; \dot{B}^{\frac{3}{2}}_{3,\infty}),
\]
\[
v \in L^3(0,T; \dot{B}^{\frac{3}{3}}_{3,c(N)}),
\]
then $(\rho, v)$ is also a weak solution of system (1.7) in the sense of distributions.
Proof. First, we mollify the momentum equation (3.5) in space (with the kernel and notation as in Section 2):

\[(\rho v)_t^\varepsilon + \text{div}(\rho v \otimes v)_t^\varepsilon + \nabla \pi(\rho)_t^\varepsilon = 0.\]  
(3.1)

Then, in virtue of appropriate commutators, the above equation can be rewritten as

\[(\rho^\varepsilon v^\varepsilon)_t + \text{div}(\rho v^\varepsilon \otimes v^\varepsilon) + \nabla \pi(\rho^\varepsilon) = -[\{(\rho v)^\varepsilon - (\rho v^\varepsilon)^\varepsilon\}]_t - \text{div}\{(\rho v \otimes v)^\varepsilon - (\rho v)^\varepsilon \otimes v^\varepsilon\} - \nabla \pi(\rho^\varepsilon) - \pi(\rho)^\varepsilon].\]  
(3.2)

Taking into account of mollified version of the continuity equation

\[\rho_t^\varepsilon + \text{div}(\rho v)^\varepsilon = 0,\]  
(3.3)

a direct computation shows

\[\rho_t^\varepsilon v^\varepsilon + \rho^\varepsilon v^\varepsilon \cdot \nabla v^\varepsilon + \nabla \pi(\rho^\varepsilon) = -[\{(\rho v)^\varepsilon - \rho^\varepsilon v^\varepsilon\} \cdot \nabla v^\varepsilon - \{(\rho v)^\varepsilon - (\rho^\varepsilon v^\varepsilon)\}]_t - \text{div}\{(\rho v \otimes v)^\varepsilon - (\rho v)^\varepsilon \otimes v^\varepsilon\} - \nabla \pi(\rho^\varepsilon) - \pi(\rho)^\varepsilon].\]  
(3.4)

Since 0 < \(c_1 \leq \rho \leq c_2 < \infty\), setting \(\Pi(\rho^\varepsilon) = \int_0^\varepsilon \frac{\pi'(s)}{s} ds\) and dividing both sides of equation (3.4) by \(\rho^\varepsilon\), we arrive at

\[v_t^\varepsilon + \omega \times v^\varepsilon + \nabla \left(\Pi(\rho^\varepsilon) + \frac{1}{2}|v^\varepsilon|^2\right) = \frac{1}{\rho^\varepsilon}\left[(\rho v)^\varepsilon - \rho^\varepsilon v^\varepsilon\right] \cdot \nabla v^\varepsilon - \frac{1}{\rho^\varepsilon}\left[(\rho v)^\varepsilon - (\rho^\varepsilon v^\varepsilon)\right]_t - \frac{1}{\rho^\varepsilon}\text{div}\{(\rho v \otimes v)^\varepsilon - (\rho v)^\varepsilon \otimes v^\varepsilon\} - \frac{1}{\rho^\varepsilon}\nabla \pi(\rho^\varepsilon) - \pi(\rho)^\varepsilon,\]  
(3.5)

where we have used the identity that \(v^\varepsilon \cdot \nabla v^\varepsilon = \frac{1}{2}\nabla|v^\varepsilon|^2 + \omega \times v^\varepsilon\) with \(\omega = \text{curl} v\).

Let \(\varphi(x, t) \in C_0^\infty((0,T) \times \Omega)\) be a test function. Multiplication with \(\varphi\) and then integration yield that

\[\int_0^t \int_\Omega \left(v_t^\varepsilon + \omega \times v^\varepsilon + \nabla \left(\Pi(\rho^\varepsilon) + \frac{1}{2}|v^\varepsilon|^2\right)\right) \varphi dx d\tau = \int_0^t \int_\Omega \left(-\frac{1}{\rho^\varepsilon}\left[(\rho v)^\varepsilon - \rho^\varepsilon v^\varepsilon\right] \cdot \nabla v^\varepsilon - \frac{1}{\rho^\varepsilon}\left[(\rho v)^\varepsilon - (\rho^\varepsilon v^\varepsilon)\right]_t - \frac{1}{\rho^\varepsilon}\text{div}\{(\rho v \otimes v)^\varepsilon - (\rho v)^\varepsilon \otimes v^\varepsilon\} - \frac{1}{\rho^\varepsilon}\nabla \pi(\rho^\varepsilon) - \pi(\rho)^\varepsilon\right) \varphi dx d\tau = I_1 + I_2 + I_3 + I_4.\]  
(3.6)

Next we will show that the four terms \(I_1 - I_4\) on the RHS of (3.6) tend to zero as \(\varepsilon \to 0\).

Indeed, note that

\[\rho \in L^3(0,T; B^\frac{1}{2}_{3,c(\varepsilon)}), \rho v \in L^3(0,T; B^\frac{1}{2}_{3,c(\varepsilon)}), v \in L^3(0,T; B^\frac{1}{2}_{3,c(\varepsilon)}).\]

Using Corollary 2.3, Lemma 2.4 and the Hölder inequality, the first term \(I_1\) on the RHS of (3.6) can be handled as

\[|I_1| \leq C\|(\rho v)^\varepsilon - \rho^\varepsilon v^\varepsilon\|_{L^3(0,T; L^3(\Omega))}\|\nabla v^\varepsilon\|_{L^3(0,T; L^3(\Omega))}\|\varphi\|_{L^\infty(0,T; L^\infty(\Omega))} \leq o(1)\|\rho\|_{L^3(0,T; B^\frac{1}{2}_{3,c(\varepsilon)})}\|v\|^2_{L^3(0,T; B^\frac{1}{2}_{3,c(\varepsilon)})}\]  
(3.7)

\[\leq o(1).\]
Then it follows from integration by parts, combination the Corollary 2.3 with Lemma 2.4 we have

\[
|I_2| = \left| \int_0^t \int_\Omega \left[ (\rho u^\varepsilon - \rho^\varepsilon v^\varepsilon) \right] \left( -\frac{\partial \rho^\varepsilon}{(\rho^\varepsilon)^2} \varphi + \frac{1}{\rho^\varepsilon} \partial \varphi \right) \, dx \, d\tau \right|
\]

\[
= \left| \int_0^t \int_\Omega \left[ (\rho u^\varepsilon - \rho^\varepsilon v^\varepsilon) \right] \left( \frac{\text{div}(\rho u^\varepsilon)}{(\rho^\varepsilon)^2} \varphi + \frac{1}{\rho^\varepsilon} \partial \varphi \right) \, dx \, d\tau \right|
\]

\[
\leq C \| (\rho u^\varepsilon - \rho^\varepsilon v^\varepsilon) \|_{L^2(0,T;L^2(\Omega))} \left( \| \nabla (\rho u^\varepsilon) \|_{L^3(0,T;L^3(\Omega))} + \| \partial \varphi \|_{L^3(0,T;L^3(\Omega))} \right)
\]

\[
\leq o(\varepsilon^\frac{2}{3}) \| \rho \|_{L^3(0,T;B_{a,b}^{\frac{1}{2}}(\Omega))} \| \rho \|_{L^3(0,T;B_{a,b}^{\frac{1}{2}}(\Omega))} \left( o(\varepsilon^\frac{2}{3}) \| \rho \|_{L^3(0,T;B_{a,b}^{\frac{1}{2}}(\Omega))} + 1 \right)
\]

\[
\leq o(1) + o(\varepsilon^\frac{2}{3}).
\]

The third term \( I_3 \) can be estimated similarly as \( I_2 \):

\[
|I_3| = \left| \int_0^t \int_\Omega \left[ (\rho \otimes v)^\varepsilon - (\rho u^\varepsilon \otimes v^\varepsilon) \right] \left( -\frac{\nabla \rho^\varepsilon}{(\rho^\varepsilon)^2} \varphi + \frac{\nabla \varphi}{\rho^\varepsilon} \right) \, dx \, d\tau \right|
\]

\[
\leq C \| (\rho \otimes v)^\varepsilon - (\rho u^\varepsilon \otimes v^\varepsilon) \|_{L^2(0,T;L^2(\Omega))} \left( \| \nabla \rho^\varepsilon \|_{L^3(0,T;L^3(\Omega))} + \| \nabla \varphi \|_{L^3(0,T;L^3(\Omega))} \right)
\]

\[
\leq o(\varepsilon^\frac{2}{3}) \| \rho \|_{L^3(0,T;B_{a,b}^{\frac{1}{2}}(\Omega))} \| \rho \|_{L^3(0,T;B_{a,b}^{\frac{1}{2}}(\Omega))} \left( o(\varepsilon^\frac{2}{3}) \| \rho \|_{L^3(0,T;B_{a,b}^{\frac{1}{2}}(\Omega))} + 1 \right)
\]

\[
\leq o(1) + o(\varepsilon^\frac{2}{3}).
\]

For the last term \( I_4 \), notice that if \( \pi(s) \in C^2([a,b]) \), then for any \( s, s_0 \in [a,b] \) there holds

\[
|\pi(s) - \pi(s_0) - \pi'(s_0)(s-s_0)| \leq C(s-s_0)^2,
\]

where the constant \( C \) can be independent of \( s \) and \( s_0 \). This yields that

\[
|\pi(\rho^\varepsilon(t,x)) - \pi(\rho(t,x)) - \pi'(\rho(t,x))(\rho^\varepsilon(t,x) - \rho(t,x))| \leq C|\rho^\varepsilon(t,x) - \rho(t,x)|^2. \tag{3.10}
\]

Similarly, we also have

\[
|\pi(\rho(t,y)) - \pi(\rho(t,x)) - \pi'(\rho(t,x))(\rho(t,y) - \rho(t,x))| \leq C|\rho(t,y) - \rho(t,x)|^2. \tag{3.11}
\]

Applying convolution with respect to \( y \) to the last inequality and invoking Jensen’s inequality, we get

\[
|\pi^\varepsilon(\rho(t,x)) - \pi^\varepsilon(\rho(t,x)) - \pi'(\rho(t,x))(\rho^\varepsilon(t,x) - \rho(t,x))| \leq C(\rho(t,\cdot) - \rho(t,x))^2 * \eta_\varepsilon(x), \tag{3.12}
\]

which together with (3.10) gives

\[
|\pi(\rho^\varepsilon(t,x)) - \pi^\varepsilon(\rho(t,x))| \leq C|\rho^\varepsilon(t,x) - \rho(t,x)|^2 + C(\rho(t,\cdot) - \rho(t,x))^2 * \eta_\varepsilon(x). \tag{3.13}
\]

Then it follows from integration by parts that \( I_4 \) can be handled as

\[
|I_4| = \left| \int_0^t \int_\Omega \left[ \pi(\rho^\varepsilon) - \pi^\varepsilon(\rho) \right] \left( \frac{1}{\rho^\varepsilon} \text{div} \varphi - \varphi \cdot \nabla \rho^\varepsilon \right) \, dx \, d\tau \right|
\]

\[
\leq C \| \rho^\varepsilon(t,x) - \rho(t,x) \|_{L^3(0,T;L^3(\Omega))} \| \nabla \varphi \|_{L^3(0,T;L^3(\Omega))} + \| \nabla \rho^\varepsilon \|_{L^3(0,T;L^3(\Omega))}
\]

\[
\leq o(\varepsilon^\frac{2}{3}) \| \rho \|_{L^3(0,T;B_{a,b}^{\frac{1}{2}}(\Omega))} \left( 1 + o(\varepsilon^\frac{2}{3}) \| \rho \|_{L^3(0,T;B_{a,b}^{\frac{1}{2}}(\Omega))} \right)
\]

\[
\leq o(\varepsilon^\frac{2}{3}) + o(1).
\]
Hence, letting \( \varepsilon \to 0 \) and using the Lebesgue’s dominated convergence theorem for LHS of (3.6), we have
\[
\int_0^t \int_\Omega \left( v_\tau + \omega \times v + \nabla \left( \Pi(\rho) + \frac{1}{2} |v|^2 \right) \right) \varphi \, dx \, d\tau = 0,
\]
which means that \((\rho, v)\) is also a weak solution of equations
\[
\begin{align*}
\rho_t + \nabla \cdot (\rho v) &= 0, \\
v_t + \omega \times v + \nabla \left( \Pi(\rho) + \frac{1}{2} |v|^2 \right) &= 0.
\end{align*}
\]
(3.16)
This concludes the proof of this proposition.

**Proof of Theorem 1.1.** Let \((\rho, v)\) be a weak solution to the compressible Euler equations (1.5). Combining (1.8) and Proposition 3.1, we know that \((\rho, v)\) is also a weak solution to the system (3.16). Then one can mollify the equation (3.16) in space direction to deduce that
\[
\begin{align*}
v_\varepsilon + (\omega \times v)_\varepsilon + \nabla (\Pi(\rho) + \frac{1}{2} |v|^2)_\varepsilon &= 0, \\
\omega_\varepsilon - \text{curl}(v \times \omega)_\varepsilon &= 0, \\
\text{div} \omega_\varepsilon &= 0.
\end{align*}
\]
Consequently, by a straightforward computation, we have
\[
\frac{d}{dt} \int_\Omega (v_\varepsilon \cdot \omega_\varepsilon) dx = \int_\Omega v_\varepsilon \cdot \omega_\varepsilon + v_\varepsilon \cdot v_\varepsilon dx
\]
\[
= \int_\Omega - \left( (\omega \times v)_\varepsilon + \nabla (\Pi(\rho) + \frac{1}{2} |v|^2)_\varepsilon \right) \cdot \omega_\varepsilon + (\text{curl}(v \times \omega))_\varepsilon \cdot v_\varepsilon dx
\]
\[
= \int_\Omega - (\omega \times v)_\varepsilon \cdot \omega_\varepsilon + (v \times \omega)_\varepsilon \cdot \omega_\varepsilon dx
\]
\[
= \int_\Omega - 2 ((\omega \times v)_\varepsilon - \omega_\varepsilon \times v_\varepsilon) \cdot \omega_\varepsilon - 2 (\omega_\varepsilon \times v_\varepsilon) \cdot \omega_\varepsilon dx
\]
\[
= \int_\Omega - 2 ((\omega \times v)_\varepsilon - \omega_\varepsilon \times v_\varepsilon) \cdot \omega_\varepsilon dx
\]
\[
= I,
\]
where we have used the identity that \( \omega^\varepsilon \times v^\varepsilon \cdot v^\varepsilon = \omega^\varepsilon \times \omega^\varepsilon \cdot v^\varepsilon = 0. \)

Next we will show that \( I \) tends to zero as \( \varepsilon \to 0. \)

1. The Hölder inequality ensures that
\[
|I| \leq C \| (\omega \times v)_\varepsilon - \omega_\varepsilon \times v_\varepsilon \|_{L^\frac{3}{2}(0,T;L^\frac{3}{2}(\Omega))} \| \omega_\varepsilon \|_{L^3(0,T;L^3(\Omega))}
\]
\ [
\leq C \| (\omega \times v)_\varepsilon - \omega_\varepsilon \times v_\varepsilon \|_{L^\frac{3}{2}(0,T;L^\frac{3}{2}(\Omega))} \| \nabla v_\varepsilon \|_{L^3(0,T;L^3(\Omega))}.
\]
(3.18)
By \( v \in L^3(0,T;B^\frac{1}{2}_{3,\infty}(\Omega)), \omega \in L^3(0,T;B^\frac{1}{2}_{3,\infty}) \) and Lemma 2.6, we deduce that
\[
\| (\omega \times v)_\varepsilon - \omega_\varepsilon \times v_\varepsilon \|_{L^\frac{3}{2}(0,T;L^\frac{3}{2}(\Omega))} \leq o(\varepsilon^\frac{3}{2}).
\]
Combining $v \in L^3(0, T; B^{\frac{1}{3}}_{3,\infty}(\Omega))$ and (2) in Lemma 2.4 we observe that
\[ \|\nabla v^\varepsilon\|_{L^3(0, T; L^3(\Omega))} \leq o(\varepsilon^{-2}). \]

Hence there holds
\[ |I| \leq o(1). \]

(2) By the Hölder inequality and Lemma 2.7 we get
\[ |I| \leq C\|\omega \times v\|_{L^9(0, T; L^9(\Omega))}^2\|\omega\|_{L^3(0, T; L^3(\Omega))} \to 0, \tag{3.19} \]
due to the condition that $v \in L^3(0, T; B^{\frac{1}{3},\infty}_{3,\infty}(\Omega))$ and $\omega \in L^3(0, T; L^3(\Omega))$.

(3) It follows from the Hölder inequality that
\[ |I| \leq C\|\omega \times v\|_{L^{p'}(0, T; L^{q'}(\Omega))}^2\|\omega\|_{L^q(0, T; L^q(\Omega))} \tag{3.20} \]
and
\[ \|\omega \times v\|_{L^{p'}(0, T; L^{q'}(\Omega))} \leq C\|v\|_{L^{p'}(0, T; L^{q'}(\Omega))}^2\|\omega\|_{L^q(0, T; L^q(\Omega))}. \]
Thus, by using the properties of the standard mollifiers, we know that
\[ \limsup_{\varepsilon \to 0} |I| = 0. \]

(4) Recall the identical equation
\[ -\Delta A = \text{curl} \, \text{curl} \, A - \nabla \text{div} \, A. \]
The classical elliptic estimate yields that
\[ \|\nabla A\|_{L^p(\Omega)} \leq C(\|\text{curl} \, A\|_{L^p(\Omega)} + \|\text{div} \, A\|_{L^p(\Omega)}). \]
Therefore, there holds
\[ \|\nabla v\|_{L^9(\Omega)} \leq C(\|\text{curl} \, v\|_{L^9(\Omega)} + \|\text{div} \, v\|_{L^9(\Omega)}). \]
When dimension $d = 3$, the Sobolev inequality implies that
\[ \|v\|_{L^p(\Omega)} \leq C\|\nabla v\|_{L^q(\Omega)}^2. \]
Integrating this estimate in time direction, we derive
\[ \|v\|_{L^3(0, T; L^3(\Omega))} \leq C\|\nabla v\|_{L^3(0, T; L^3(\Omega))}. \]
Finally, we get $v \in L^3(0, T; L^9(\Omega))$ and $\text{div} \, u, \omega \in L^3(0, T; L^9(\Omega))$. From the previous case, we complete the proof of this case. This concludes Theorem 1.1. \qed
4 Helicity conservation for the homogeneous incompressible Euler equations

This section is devoted to the proof of helicity conservation of weak solutions in Onsager type spaces $B^{1}_{p,c(N)}$ for the homogeneous incompressible Euler equations.

Proof of Theorem 1.2. Let $v$ be a weak solution to the homogeneous incompressible Euler equations $(1.1)$. Then one can mollify the equation $(1.1)$ in space direction to deduce that

$$
\begin{cases}
  v_t^\varepsilon + (\omega \times v)^\varepsilon + \nabla (\Pi + \frac{1}{2}|v|^2)^\varepsilon = 0, \\
  \omega_t^\varepsilon - \text{curl}(v \times \omega)^\varepsilon = 0, \\
  \text{div}\omega^\varepsilon = 0.
\end{cases}
$$

Consequently, by a straightforward computation, we have

$$
\frac{d}{dt} \int_{\Omega} (v^\varepsilon \cdot \omega^\varepsilon) dx = \int_{\Omega} v_t^\varepsilon \cdot \omega^\varepsilon + \omega_t^\varepsilon \cdot v^\varepsilon dx
$$

$$
= \int_{\Omega} - (\omega \times v)^\varepsilon \cdot \omega^\varepsilon + (\text{curl}(v \times \omega))^\varepsilon \cdot v^\varepsilon dx
$$

$$
= \int_{\Omega} - (\omega \times v)^\varepsilon \cdot \omega^\varepsilon + (v \times \omega)^\varepsilon \cdot \omega^\varepsilon dx
$$

$$
= \int_{\Omega} -2 (\omega \times v)^\varepsilon \cdot \omega^\varepsilon - 2 (\omega^\varepsilon \times v^\varepsilon) \cdot \omega^\varepsilon dx
$$

$$
= \int_{\Omega} -2 (\omega \times v)^\varepsilon \cdot \omega^\varepsilon + (\omega \times v)^\varepsilon \cdot \omega^\varepsilon dx,
$$

where integration by parts and the fact that $(v^\varepsilon \times \omega^\varepsilon) \cdot \omega^\varepsilon = v^\varepsilon \cdot (\omega^\varepsilon \times \omega^\varepsilon) = 0$ were utilized.

From $(1.10)_1$ and the divergence-free condition that $\text{div} v = 0$, we infer that

$$
\omega \times v = v \cdot \nabla v - \frac{1}{2} \nabla|v|^2 = \text{div}(v \otimes v) - \frac{1}{2} \nabla|v|^2,
$$

which entails that

$$
(\omega \times v)^\varepsilon = (v \cdot \nabla v)^\varepsilon - \frac{1}{2} \nabla(|v|^2)^\varepsilon = \text{div}(v \otimes v)^\varepsilon - \frac{1}{2} \nabla(|v|^2)^\varepsilon
$$

and

$$
\omega^\varepsilon \times v^\varepsilon = v^\varepsilon \cdot \nabla v^\varepsilon - \frac{1}{2} \nabla|v^\varepsilon|^2 = \text{div}(v^\varepsilon \otimes v^\varepsilon) - \frac{1}{2} \nabla|v^\varepsilon|^2.
$$

Plugging the latter two equations into $(4.1)$, we use integration by parts and observe that

$$
\frac{d}{dt} \int_{\Omega} v^\varepsilon \cdot \omega^\varepsilon dx = -2 \int_{\Omega} \text{div}((v \otimes v)^\varepsilon - (v^\varepsilon \otimes v^\varepsilon)) \cdot \omega^\varepsilon + \frac{1}{2} \nabla(|v^\varepsilon|^2 - (|v|^2)^\varepsilon) \cdot \omega^\varepsilon dx
$$

$$
= 2 \int_{\Omega} ((v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon) \nabla \omega^\varepsilon dx.
$$

Then integrating the latter relation in time over $(0,t)$, we see that

$$
\int_{\Omega} v^\varepsilon(x,t) \cdot \omega^\varepsilon(x,t) dx - \int_{\Omega} v^\varepsilon(x,0) \cdot \omega^\varepsilon(x,0) dx = 2 \int_{0}^{t} \int_{\Omega} (v^\varepsilon \otimes v^\varepsilon - (v \otimes v)^\varepsilon) \nabla \omega^\varepsilon dx ds. \quad (4.2)
$$
Now we are in a position to apply Lemma 2.6 to complete the proof of Theorem 1.2.

(1) Note that \( v \in L^k(0, T; \dot{B}_p^{\alpha}(\Omega)) \), we apply Lemma 2.6 to obtain

\[
\|(v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon\|_{L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Omega))} \leq o(\varepsilon^{2\alpha}).
\]

Since \( \omega \in L^\ell(0, T; \dot{B}_q^{\beta}(\Omega)) \), we adopt Lemma 2.4 to get

\[
\|
abla \omega^\varepsilon\|_{L^\ell(0, T; L^q(\Omega))} \leq O(\varepsilon^{\beta-1}).
\]

In light of the Hölder inequality, we see that

\[
\left| \int_0^t \int_{\Omega} ((v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon) \nabla \omega^\varepsilon dx ds \right|
\leq C \|(v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon\|_{L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Omega))} \|
abla \omega^\varepsilon\|_{L^\ell(0, T; L^q(\Omega))}
\leq o(\varepsilon^{2\alpha})O(\varepsilon^{\beta-1})
\leq o(\varepsilon^{\beta+2\alpha-1}).
\]

Then letting \( \varepsilon \to 0 \) in (4.2), this completes the proof of the first part.

(2) In the same manner as above, making use of Remark 2.2, we have

\[
\left| \int_0^t \int_{\Omega} ((v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon) \nabla \omega^\varepsilon dx ds \right|
\leq C \|(v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon\|_{L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Omega))} \|
abla \omega^\varepsilon\|_{L^\ell(0, T; L^q(\Omega))}
\leq O(\varepsilon^{2\alpha})o(\varepsilon^{\beta-1})
\leq o(\varepsilon^{\beta+2\alpha-1}).
\]

Then letting \( \varepsilon \to 0 \), this together with (4.2) yields the second part of Theorem 1.2.

(3) Choose \( q_1 = q_2 = \frac{3d}{d+2} \). Then there holds \( \frac{3}{2} + \frac{2d-2}{3d} = \frac{d+2}{3d} + \frac{d+2}{3d} \). Hence, we apply (2) in Lemma 2.6 and Remark 2.3 to obtain

\[
\|(v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon\|_{L^{\frac{4}{3}}(0, T; L^{\frac{4d}{3d+2}}(\Omega))} \leq o(\varepsilon^{\frac{2}{3}}),
\]

where we have used the condition that \( \omega \in L^\ell(0, T; \dot{B}_{\frac{3d}{d+2}}^{\frac{1}{2}}(\Omega)) \). Following the same path as above, we get

\[
\left| \int_0^t \int_{\Omega} ((v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon) \nabla \omega^\varepsilon dx ds \right|
\leq C \|(v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon\|_{L^{\frac{4}{3}}(0, T; L^{\frac{4d}{3d+2}}(\Omega))} \|
abla \omega^\varepsilon\|_{L^\ell(0, T; L^\frac{4d}{3d+2}(\Omega))}
\leq O(\varepsilon^{\frac{2}{3}})o(\varepsilon^{-\frac{2}{3}})
\leq o(1),
\]

which together with (4.2) concludes the third part of this theorem.

(An alternative approach to (3) and (1.6)):

Next we provide an alternative approach to proving this part. According to the boundedness of Riesz Transform in homogeneous Besov spaces and the condition that \( \omega \in \)
we have \( \nabla v \in L^3(0, T; \dot{B}^\frac{3}{
abla^2}\text{c}(\mathbb{N})) \). The Bernstein inequality further helps us to get \( v \in L^3(0, T; \dot{B}^\frac{3}{
abla^2}\text{c}(\mathbb{N})) \). Recall that \( \dot{B}^\frac{3}{
abla^2}\text{c}(\mathbb{N}) \hookrightarrow \dot{B}^\frac{3}{2}\text{c}(\mathbb{N}) \), we may achieve the desired proof of (3) from (1.6).

Subsequently, to make the paper more readable and more self-contained, we also present the proof of (1.6). Notice that

\[
\|\nabla \omega \|^q_{L^q(\Omega)} \leq C \|\nabla^2 v\|^q_{L^q(\Omega)}
\]

for \( 1 < q < \infty \), which together with the fact that \( v \in L^3(0, T; B^\frac{3}{2}\text{c}(\mathbb{N})) \) and Lemma 2.4 means that

\[
\|\nabla \omega \|^q_{L^q(\Omega)} \leq C \|\nabla^2 v\|^q_{L^q(\Omega)} \leq o(\varepsilon^{-\frac{4}{3}}).
\]

Using the Hölder inequality and (1) in Lemma 2.6 we arrive at

\[
\left| \int_0^t \int_\Omega \left( (v \otimes v) - v \otimes v \right) \nabla \omega dx ds \right| \\
\leq C \| (v \otimes v) - v \otimes v \|^q_{L^q(0, T; L^\frac{q}{2}(\Omega))} \| \nabla \omega \|^q_{L^q(0, T; L^q(\Omega))} \\
\leq o(\varepsilon^{-\frac{4}{3}}) o(\varepsilon^{-\frac{4}{3}}) \\
\leq o(1).
\]

This together with (4.2) gives (1.6).

(4) Following the same path of derivation of (2) in Theorem 1.2, Lemma 2.7 gives the proof of this part.

(5) With (4.2) and \( \text{div} \omega = 0 \) in hand, a slight variant of the proof of (2) in Theorem 1.2 provides the proof of this part. \( \square \)

5 Helicity conservation for 2-D surface quasi-geostrophic equations

In this section, we are concerned with the helicity conservation of weak solutions to 2-D surface quasi-geostrophic equations.

Proof of Theorem 1.4. First, due to the divergence-free condition of velocity \( v(x, t) \), we rewrite the equation (1.12) as

\[
\begin{align*}
\theta_t + \text{div} (v \otimes \theta) &= 0, \\
v(x, t) &= (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta), \\
\theta|_{t=0} &= \theta_0,
\end{align*}
\]

where \( \mathcal{R}_i \) represents the Riesz transform for \( i = 1, 2 \). Next, we regularize the equation (5.1) to obtain

\[
\theta_t^\varepsilon + \partial_j (v_j \theta)^\varepsilon = 0,
\]

and

\[
\partial_t \theta_t^\varepsilon + \partial_j (\partial_k v_j \theta)^\varepsilon + \partial_j (v_j \partial_k \theta)^\varepsilon = 0.
\]
Thus, it follows from a direct computation that for $i = 1, 2$,
\[
\frac{d}{dt} \int_\Omega \theta^\varepsilon \partial_i \theta^\varepsilon \, dx = \int_\Omega \theta^\varepsilon \partial_i \partial_i \theta^\varepsilon \, dx + \int_\Omega \partial_i \theta^\varepsilon \partial_i \theta^\varepsilon \, dx = -\int_\Omega \theta^\varepsilon [\partial_j (v_j \partial_i \theta^\varepsilon) + \partial_j (v_j \partial_i \theta^\varepsilon)] \, dx - \int_\Omega \partial_j (v_j \partial_i \theta^\varepsilon) \partial_i \theta^\varepsilon \, dx. \tag{5.2}
\]

Using the divergence-free condition of $q$, we deduce from Sobolev’s inequality and Lemma 2.4, we deduce from a direct computation that for $i = 1, 2$,
\[
\frac{d}{dt} \int_\Omega \theta^\varepsilon \partial_i \theta^\varepsilon \, dx = -\int_\Omega \theta^\varepsilon [\partial_j (v_j \partial_i \theta^\varepsilon) + \partial_j (v_j \partial_i \theta^\varepsilon)] \, dx - \int_\Omega \partial_j (v_j \partial_i \theta^\varepsilon) \partial_i \theta^\varepsilon \, dx.
\]

which leads to
\[
\int_\Omega \theta^\varepsilon (x,t) \partial_i \theta^\varepsilon (x,t) \, dx - \int_\Omega \theta^\varepsilon (x,0) \partial_i \theta^\varepsilon (x,0) \, dx
\]
\[
= \int_0^t \int_\Omega \partial_j \theta^\varepsilon [(\partial_i v_j \theta^\varepsilon) - (\partial_i v_j \theta^\varepsilon)] \, dx \, ds + \int_0^t \int_\Omega [(v_j \partial_i \theta^\varepsilon) - (v_j \partial_i \theta^\varepsilon)] \partial_i \theta^\varepsilon \, dx \, ds
\]
\[
+ \int_0^t \int_\Omega [(v_j \theta^\varepsilon) - v_j \theta^\varepsilon] \partial_i \partial_j \theta^\varepsilon \, dx \, ds
\]
\[
=: I + II + III.
\]

For the first term $I$, we conclude by the Hölder inequality that
\[
|I| \leq \|(\partial_i v_j \theta^\varepsilon) - (\partial_i v_j \theta^\varepsilon)\|_{L^3(0,T;L^\frac{6}{5}(\Omega))} \|\partial_j \theta^\varepsilon\|_{L^3(0,T;L^\frac{6}{5}(\Omega))}.
\]

Take $q = \frac{6}{5}$, $d = 2$, $q_1 = \frac{3}{2}$, $q_2 = \frac{3}{2} < 2$. Note that $\frac{1}{q} + \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\nabla \theta \in L^3(0,T;B^{\frac{3}{2}}_{2,c(N)})$, we can invoke (3) in Lemma 2.26 to get
\[
\|(\partial_i v_j \theta^\varepsilon) - (\partial_i v_j \theta^\varepsilon)\|_{L^3(0,T;L^\frac{6}{5}(\Omega))} \leq o(\varepsilon^{\frac{2}{5}}).
\]

Combining Sobolev’s inequality and Lemma 2.4, we deduce from $\nabla \theta \in L^3(0,T;B^{\frac{3}{2}}_{2,c(N)})$ that
\[
\|\partial_j \theta^\varepsilon\|_{L^3(0,T;L^\frac{6}{5}(\Omega))} \leq C \|\nabla \partial_j \theta^\varepsilon\|_{L^3(0,T;L^\frac{6}{5}(\Omega))} \leq o(\varepsilon^{\frac{2}{5}}).
\]

Collecting the above estimates, we get
\[
|I| \leq o(1).
\]
Likewise, we have

$$|II| \leq o(1).$$

It remains to estimate the third term $III$. Using the Hölder inequality, we find

$$|III| \leq \|(v_j \theta^\varepsilon - \theta^\varepsilon v_j^\varepsilon)\|_{L^3_2(0,T;L^3(\Omega))}\|\partial_i \partial_j \theta^\varepsilon\|_{L^3(0,T;L^3(\Omega))}.$$  

Choosing $q = 3, d = 2, q_1 = \frac{3}{2} = q_2 = \frac{3}{2} < 2$, (2) in Lemma 2.6 enables us to get

$$\|(v_j \theta^\varepsilon - \theta^\varepsilon v_j^\varepsilon)\|_{L^4_2(0,T;L^4(\Omega))} \leq o(\varepsilon^{\frac{2}{3}}),$$

where we have used the condition that $\nabla \theta \in L^3(0,T;\dot{B}^{\frac{1}{2}}_{\infty,1}(\mathbb{R}^d))$. Besides, Lemma 2.4 allows us to conclude that

$$\|\partial_i \partial_j \theta^\varepsilon\|_{L^3(0,T;L^3(\Omega))} \leq o(\varepsilon^{-\frac{2}{3}}).$$

In summary, we know that

$$|III| \leq o(1).$$

We achieve the proof of this theorem. \qed

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