Quantitative explanation of circuit experiments and real traffic using the optimal velocity model

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Abstract
We have experimentally confirmed that the occurrence of a traffic jam is a dynamical phase transition (Tadaki et al 2013 New J. Phys. 15 103034, Sugiyama et al 2008 New J. Phys. 10 033001). In this study, we investigate whether the optimal velocity (OV) model can quantitatively explain the results of experiments. The occurrence and non-occurrence of jammed flow in our experiments agree with the predictions of the OV model. We also propose a scaling rule for the parameters of the model. Using this rule, we obtain critical density as a function of a single parameter. The obtained critical density is consistent with the observed values for highway traffic.

1. Introduction
In our recent works, we have experimentally established that a transition from free flow to jammed flow is a dynamical phase transition using circuit experiments [1–3]. In the first experiment, we observed the instability of free flow on a circuit road, and the emergence of a traffic jam without bottleneck. In the second experiment, we showed that density is a control parameter in the transition, and we determined its critical value. When the density of cars exceeds a critical value, a jammed flow occurs.

Since the 1990s, a number of physics-based theoretical models have been proposed to explain the properties of traffic flow, in particular the physical mechanisms underlying the formation of traffic jams [4–9]. The optimal velocity (OV) model is one such model [10]. It is constructed from a microscopic viewpoint, i.e., the motion of each car is described by Newtonian-like equations of motion. In the OV model, the homogeneous flow becomes unstable and transits to jammed flow if the density exceeds a critical value. This model was applied to real traffic and succeeded in reproducing the fundamental diagram [11].

In the present paper, we investigate the applicability of the OV model to both a circuit experiment and real traffic based on the common physical mechanism. The OV model is chosen because of its simple structure and a small number of parameters. We first determine suitable values for the parameters of the OV model in the experiment [3]. The experimental values must be different from those for real traffic because the maximum velocity in a circuit experiment is smaller than that in real traffic. Setting these parameters appropriately allows us to explain the results of the experiment.

Next, we propose a method for predicting the critical density in real traffic without additional estimation of parameters. To achieve this, it is necessary to find the relation between the parameters in the circuit experiment and real traffic. Using this relation, we define a scaling rule for the parameters and obtain the critical density as the function of a single parameter. This is a different type of scaling rule from that introduced in [25]. The predicted critical density is then tested against observations of real traffic.
This paper is organized as follows. In sections 2 and 3, we describe the experiment and the OV model, respectively. The estimation of the model parameters is shown in section 4, and the application to real traffic is shown in section 5. A summary and conclusions are given in section 6.

2. Experiment summary

The experiment in [3] consisted of 19 sessions. Table 1 lists all the sessions. The experiment was conducted on an indoor circuit with a radius of 50 m.7 The resolution of the positions of cars was about 0.2 m. Velocity and acceleration were calculated from the data on their positions. We used a standard car type 3885 mm long.

Figure 1 shows three sample spacetime plots of the positions of cars. Velocities are shown by the colors of the dots. It can be seen that free flow was realized in session 2328, and stop-and-go flow was realized in session 2030. In session 1625, the flow was almost free, but a jam cluster where velocity slightly decreased emerged for some time. We consider this as jammed flow. Stop-and-go flow is identified by stopped cars inside jam clusters.

Applying these criteria, the sessions are classified as follows: free flow in sessions 1010–1520, 1940, 2125, and 2328, and jammed flow, including stop-and-go flow, in all the other sessions. This classification is shown in Table 1. In a previous study [3], we used the data from sessions 2030 to 2934 to analyze the critical density in the

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Table 1. List of session IDs. Four-digit numbers are assigned to the sessions. The first two digits represent the serial number of the session, and last two digits are the number of cars run in the session.

| Session ID | Number of cars | Classification | Session ID | Number of cars | Classification |
|------------|----------------|----------------|------------|----------------|----------------|
| 1010       | 10             | Free           | 2030       | 30             | Stop-and-go    |
| 1210       | 10             | Free           | 2125       | 25             | Free           |
| 1310       | 10             | Free           | 2228       | 28             | Jammed         |
| 1412       | 12             | Free           | 2328       | 28             | Free           |
| 1520       | 20             | Free           | 2432       | 32             | Jammed         |
| 1625       | 25             | Jammed         | 2534       | 34             | Jammed         |
| 1730       | 30             | Jammed         | 2634       | 34             | Jammed         |
| 1835       | 35             | Jammed         | 2732       | 32             | Jammed         |
| 1940       | 40             | Free           | 2830       | 30             | Jammed         |
|            |                |                | 2934       | 34             | Stop-and-go    |

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Figure 1. Space–time plots for sessions 2328, 2030, and 1625. Dots and their colors represent the positions and velocities of cars, respectively.

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7 The positions of cars were measured from the center of the circuit, and the average distance measured from the center to the running cars was 51 m. In this study, we take this value as the radius of the circuit.
transition from free flow to jammed flow. The data from sessions 1520 to 1940 were discarded because the behavior of drivers in those sessions seemed to be different from that in sessions 2030 to 2934. In the present study, we analyze the data from all sessions and discuss differences of flow on the basis of differences in the parameters of the OV model.

3. Brief review of the OV model

The OV model is described by the equations of motion

\[
\frac{d^2x_i}{dt^2} = a \left[ V(x_{i+1} - x_i) - \frac{dx_i}{dt} \right],
\]

where \(x_i\) is the position of the \(i\)th car. The parameter \(a\) is called sensitivity. The function \(V(h)\) expresses the OV as a function of headway \(h\). A hyperbolic tangent function is typically adopted as the OV function. Sensitivity \(a\) and the OV function \(V\) are assumed to be common to all cars.

The OV model predicts that a homogeneous flow becomes unstable and transits to a jammed flow if

\[
\frac{dV(h)}{dh} \Big|_{h=h_{\text{mean}}} > \frac{1}{2} a,
\]

where \(h_{\text{mean}}\) is the mean headway. When the jammed flow becomes stationary, the trajectories of all cars in the headway-velocity space are expressed by a loop, similar to the hysteresis loop shown in figure 2 [12]. In most periods, cars in motion stay in the states represented by the two cusps of the loop. The lower cusp represents the state of cars inside jam clusters. In stop-and-go flow, the lower cusp indicates the minimum headway at which cars stop. The upper cusp represents the state in which cars are running almost freely in the regions outside the jam clusters. The backward velocity of a jam cluster is given by the velocity-axis intercept of the line connecting the upper and lower cusps (figure 2). Note that the inflection point of the hyperbolic tangent function also falls on this line.

4. Estimation of parameters for the OV model

To explain the experimental results, we must first determine the parameters of the OV model. Sensitivity \(a\) and parameters in the function \(V\) take different values for each car. However, we assume that all cars in a session have the same OV function, and that sensitivity \(a\) is a common constant for all sessions.

4.1. Sensitivity

We first estimate the value of sensitivity \(a\), which represents the strength of reaction of a car to the motion of the preceding car. This sensitivity depends on the reactions of the driver, the output power of the engine, the mass of the car, and other factors. However, it is not necessary to take full account of these dependencies. A relation exists between sensitivity \(a\) and the time delay of the motion, \(T\), of successive cars in the framework of the OV models [13–16]. In the case of stop-and-go flow, the time delay \(T\) is equal to the time interval at which cars depart from a jam cluster one after another. The value of \(aT\) is insensitive to changes in the model parameters. For
example, the values of $aT$ are $aT \sim 1.6$ for the step function, $aT \sim 1.7$ or 1.8 for piecewise linear functions (single or double slope function), and $aT \sim 1.8$ for a hyperbolic tangent function. Here, we set

$$aT = 1.8.$$  

(3)

In the case of stop-and-go flow, the time delay $T$ is given by

$$T = \frac{h_{\text{min}}}{v_{\text{back}}},$$  

(4)

where $h_{\text{min}}$ is the minimum headway and $v_{\text{back}}$ is the backward velocity of jam clusters. Both $h_{\text{min}}$ and $v_{\text{back}}$ can be measured experimentally or by observation of real traffic. The experimental data show that $h_{\text{min}}$ and $v_{\text{back}}$ are approximately 6 m and 6 m s$^{-1}$, respectively. The backward velocity on real highways is approximately 20 km h$^{-1}$ ($\sim$6 m s$^{-1}$). Hence, we can suppose that $v_{\text{back}}$ values are common to the experiment and real traffic, and represent the constants for standard Japanese cars. However, observations suggest that the minimum headway on real highways is greater than 6 m. Estimation of the minimum headway will be discussed in section 5.2. To compare the results of the experiment and the predictions of the OV model, we adopt $aT = 1.8$.

Then the homogeneous flow becomes unstable, if

$$\frac{1}{2} a = \frac{1.8}{2T} \sim 0.9 \ (1 \text{ s}^{-1}).$$  

(5)

4.2. OV function in stop-and-go flows

We now discuss the estimation of parameters in the OV functions for all sessions in the experiment. OV functions express the relation between headway and velocity. We estimate them from the experimental data. Figure 3(a) shows the original data on headway and velocity for session 2030, which showed a typical stop-and-go flow. In the headway-velocity space, each car shows a loop-like trajectory with various shape, size, and location, as shown in our previous paper [2]. Because figure 3 is a superposition of all trajectories, the loop structure shown in figure 2 is not clear. We assume that there is a loop which characterizes the motion of all cars. Then we consider that there is an OV function which corresponds to the loop. The OV function is determined by choosing several representative points. Details of the method are shown in appendix A.

In the case of stop-and-go flow (2030 and 2934), we can find five representative points sufficient to determine the OV functions. The result from session 2030 is shown in figure 3, as an example. Figure 3(a) shows the original data, and figure 3(b) shows the distribution of data points by color, and five representative points: two peaks and the saddle point of the distribution and two points corresponding to two cusps. Figure 3(b) also shows the OV function fitted to these points. In the analysis, we supposed that the function takes the form

$$V(h) = \alpha \tanh[\beta(h - h_0)] + v_0,$$  

(7)

and regarded the saddle point as the inflection point $(h_0, v_0)$. Table 2 shows the estimated parameters of the OV functions for sessions 2030 and 2934.

The OV model predicts that a jammed flow appears if the gradient of the OV function at a mean headway $h_{\text{mean}}$ is greater than $\frac{1}{2} a \approx 0.9$. The mean headway in each session is a constant value defined as the length of the
circuit divided by the number of cars. The mean headways in sessions 2030 and 2934 were 10.7 m and 9.4 m, respectively. The estimated OV functions and the mean headways for these sessions are shown in Figure 4. The gradients of the OV functions at the mean headways are clearly greater than 0.9 in both cases, consistent with the experimental results.

4.3. OV function in other cases
In cases where stop-and-go flow does not appear, we are unable to obtain the number of representative points needed to find the hyperbolic tangent function fitted to the data. As can be seen in Figure 4, however, the hyperbolic tangent functions in this experiment can be well approximated by piecewise linear functions with two segments.

In order to find these piecewise linear functions, we suppose that the maximum velocity is the 11 m s\(^{-1}\) mean velocity in warming-up sessions 1010–1412, and the minimum headway is the 6 m value in the stop-and-go flow of session 2030. An OV function for each session is then derived from two lines: the straight line connecting the point of minimum headway to the point of the peak of distribution, and the horizontal line corresponding to the maximum velocity (see appendix B). This estimation of the OV function is consistent with the acceleration data (see appendix C).

Figure 5 gives two examples of piecewise linear OV functions and the position of mean headways. The gradient of the OV function at the mean headway is less than \(0.91\) in session 1520 and 0 in session 2125. The OV model predicts that the homogeneous flow will be stable. Free flow is realized in these sessions, and the result agrees with the prediction. Free flow is also realized in all the other sessions where the model predicts that the homogeneous flow will be stable (table 3).

Figure 6 shows the results for two cases where the OV model predicts that the homogeneous flow will be unstable. Jammed flow is realized in session 2432, but not in session 2328. This discrepancy will be discussed in section 6. Jammed flow is realized in all the other sessions where the homogeneous flow is predicted to be unstable.

The agreement between the predictions of the OV model and the results of experiment can be summarized as follows (see appendix B and table 3). In most sessions, the predictions agree with the experiment. In sessions 1940 and 2328, the predictions do not agree with the experiment. In session 2228, the agreement is ambiguous, as the mean headway lies exactly at the junction of the two lines.

Here, we refer to the analysis of the fundamental diagram in a previous paper [3]. The critical density was obtained from a single fundamental diagram based on many sessions. The analysis was implicitly based on the assumption that all cars ran in the same manner, and that only the car density varied among sessions. In this study, we assume that the OV function may change for each session. However, the OV functions from the sessions analyzed in the previous study (2125–2934) vary relatively little, as shown in appendix B, justifying the

| Table 2. List of parameters of OV functions for sessions 2030 and 2934. |
|-------------------|---|---|---|---|
| ID    | α  | β  | h₀ | v₀ |
| 2030  | 5.5 | 0.37 | 9.1 | 4.9 |
| 2934  | 6.4 | 0.34 | 8.0 | 4.4 |

Figure 4. Red and green curves represent fitted OV functions for sessions 2030 and 2934, respectively. Blue line shows a reference line with a gradient of 0.9. Each black dot on an OV function corresponds to a point \((h_{\text{mean}}, V(h_{\text{mean}}))\), where \(h_{\text{mean}}\) is the mean headway of the session.
use of the single fundamental diagram constructed from those sessions. The data from sessions 1010 to 1412 were also used in the previous study. Though the OV functions in these sessions are obviously different from those in sessions 2125–2934, all cars run at the maximum velocity and only the horizontal part of the piecewise linear OV functions is meaningful. Thus, the OV functions from all sessions reported in the previous study are essentially the same.

**Figure 5.** Piecewise linear OV functions for sessions (a) 1520 and (b) 2125. Blue and black lines represent the OV function and the reference line with a gradient of 0.9, respectively. Red cross represents the peak position. Black cross indicates the position of the mean headway on the OV function. Colored contours represent the logarithm of the histogram.

**Table 3.** Agreements between the results of experiment and the predictions of the OV model.

| Session ID | Experiment | Prediction | Session ID | Experiment | Prediction |
|------------|------------|------------|------------|------------|------------|
| 1010       | Free       | Free       | 2030       | Jammed*    | Jammed     |
| 1210       | Free       | Free       | 2125       | Free       | Free       |
| 1310       | Free       | Free       | 2228       | Jammed     | ambiguous  |
| 1412       | Free       | Free       | 2328       | Free       | Jammed     |
| 1520       | Free       | Free       | 2432       | Jammed     | Jammed     |
| 1625       | Jammed     | Jammed     | 2534       | Jammed     | Jammed     |
| 1730       | Jammed     | Jammed     | 2634       | Jammed     | Jammed     |
| 1835       | Jammed     | Jammed     | 2732       | Jammed     | Jammed     |
| 1940       | Free       | Jammed     | 2830       | Jammed     | Jammed     |
|            |            |            | 2934       | Jammed     | Jammed     |

* Stop-and-go flow in sessions 2030 and 2934 is classified as jammed flow.

**Figure 6.** Piecewise linear OV functions for sessions (a) 2432 and (b) 2328. Blue and black lines represent the OV function and the reference line with a gradient of 0.9, respectively. Red cross represents the peak position. Black cross indicates the position of the mean headway on the OV function. Colored contours represent the logarithm of the histogram.
5. Application to real traffic

5.1. Backward velocity of jam cluster
In previous study \cite{1,3}, we estimated the backward velocity of jam clusters by drawing a line on the spacetime plot. In contrast, when stop-and-go flow appears, the backward velocity is determined by the two cusps of the hysteresis-like loop (figure 2). We apply the latter analysis to the data from the earlier experiment \cite{2}, as shown in appendix D. Table 4 shows the backward velocities in the experiments. Though small differences are found in the backward velocities and minimum headways, the approximate estimations (6 m s\(^{-1}\) and 6 m, respectively) used in this study are also valid for the earlier experiment. We therefore use the data from both sets of experiments when we consider a scaling method from experiment to real traffic.

5.2. Scaling of the OV function
We investigate the relation between the experiments and real traffic through scaling of the OV functions. We assume that the shape of the OV function is equation (7). To find a scaling rule from the OV functions obtained experimentally to those for real traffic on highways, we suppose that the backward velocities of jam clusters have a common value of 6 m s\(^{-1}\). However, the minimum headways (∼9 m) in real traffic turn out to be different from those in our experiments based on the relation between occupancy and car density \cite{17,18}.

One more condition is needed to determine a scaled OV function, and we propose to use the inflection point of the function. As discussed in section 4, the OV function cannot be definitively determined unless stop-and-go traffic is realized. However, the inflection point of the OV function can be found, even when this condition is not satisfied by assuming that the inflection point is at the middle of two local maxima of the distribution in the headway-velocity space. Headway-velocity relations in real traffic were obtained by measurements of car-following behavior on Japanese highways \cite{19,20}. Some of these data are shown in appendix F. Figure 7 shows the inflection points obtained from those data and our experiments. All data are well fitted by the straight line

\[
v = 0.7(h - 2).
\]

Consequently, the rule for scaling of the OV functions is defined by two properties: the common backward velocity (6 m s\(^{-1}\)) of jam clusters, and the linear relation (8) among inflection points. In the OV model, the point of backward velocity, the point of minimum headway, and the inflection point of the OV function all lie on a straight line in the headway-velocity space. The inflection point and the backward velocity determine the minimum headway and thereby the scaled OV function. This scaling rule is illustrated in figure 8.

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Table 4. List of backward velocities for the two experiments.

| ID   | Lower cusp h | Lower cusp v | Upper cusp h | Upper cusp v | Backward Velocity |
|------|--------------|--------------|--------------|--------------|------------------|
| 2030 | 6.0          | 0.0          | 15.9         | 10.2         | −6.3             |
| 2934 | 5.6          | 0.0          | 16.6         | 10.7         | −5.5             |
| Run(I)| 5.6          | 0.0          | 14.2         | 10.2         | −6.7             |
| Run(II)| 5.4      | 0.0          | 14.0         | 7.7          | −4.9             |

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Figure 7. Filled squares represent inflection points from our experiments. Circles and filled circles represent inflection points from Japanese highways reported in \cite{19,20}, respectively. Red line represents the line fitted to the data.

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8 We assume that occupancy is proportional to car length, which is roughly estimated at 7 m. The minimum headway is given by the car length plus a minimum clearance of 2 m.
Suppose that parameters $\alpha$, $\beta$, $h_0$, and $v_0$ of the OV function (7) are obtained by experiment and that an inflection point $(h', v')$ is found by observation of real traffic. A scaled OV function $V'(h)$ for real traffic can then be obtained as follows

$$ V'(h) = \alpha' \tanh[\beta'(h - h')] + v'_0, $$

$$ \alpha' = \frac{v'_0}{v_0}, $$

$$ \beta' = \frac{h_0 - h_{\min}}{h'_0 - h_{\min}}, $$

where $h_{\min}$ and $h'_{\min}$ are the minimum headways for the OV functions (7) and (9), respectively. Hereafter, all scaled quantities are represented by primed symbols. We note that the scaling is determined by a single parameter, because while there are four parameters $\alpha$, $\beta$, $h_0$, and $v_0$, only three independent relations (8), (10), and (11) exist among them. Details of the derivation are shown in appendix E.

### 5.3. Critical density

When sensitivity $a$ and the OV function is obtained, the critical density $\rho_c$ at which the transition from free flow to jammed flow occurs can be predicted. In the OV model, the critical density is calculated from the OV function (7) and the stability condition (2) as

$$ \rho_c = \left[ \frac{1}{\beta} \cosh^{-1} \left( \frac{2 \alpha \beta}{a} + h_0 \right) \right]^{-1}. $$

In the scaled OV function (9), the time delay of motion changes to $\tau' = h'_{\min}/v_{\text{back}}$ instead of equation (4). By setting $\alpha' \tau' = 1.8$ as in section 4.1, we find

$$ \rho_{c} = \left[ \frac{1}{\beta'} \cosh^{-1} \left( \frac{\alpha' \beta' T'}{0.9} + h'_0 \right) \right]^{-1}. $$

As mentioned in the previous section, relations (8), (10), and (11) exist among the parameters $\alpha$, $\beta$, $h'_0$, and $v'_0$. $\tau'$ is also expressed by these parameters and $v_{\text{back}}$ as shown in equation (E.17). The scaling of the critical density is therefore expressed by a single parameter. Here, we use the maximum velocity $\alpha' + v'_0$ as the scaling parameter for convenience.

Figure 9 shows the critical density given by equation (13), as a function of the maximum velocity. Each of the four curves represents the critical density using the parameters for the four cases 2030, 2934, run (I) and run (II) in tables 2 and D2. We first verify the consistency of the critical density obtained by mathematical analysis based on the OV model with that estimated from the fundamental diagram for the same experiment [3]. The black solid bar in figure 9 represents the range of estimated critical density. The two results are in good agreement. In order to compare the critical densities observed on real highways with those obtained by our scaling method, we plot the observed values (figure 9). The critical densities and maximum velocities on real highways can be estimated from the fundamental diagrams that have been investigated by many researchers [17, 18, 21–23].

The critical densities obtained by the scaling are to some extent larger than those observed on real highways. We explain this difference as follows. In the circuit experiments, drivers can watch the other cars and understand

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9 In some studies, only a flow-occupancy relation is shown. The average length of cars is supposed to be 7 m in such cases.
the situation of the whole circuit. Thereby, the drivers can maintain a larger velocity even when the headway is small. This causes the OV function to become steep because the minimum headway is bounded by the length of the cars. The critical density is therefore larger than that on real highways where drivers have more limited information.

6. Summary

In this study, we investigated the applicability of the OV model. The model parameters were estimated using experimental data. When the parameters are obtained, the model predicts which type of flow is realized. We compared the predictions of the model with the results of the experiments and found that the results can be explained by the OV model in most sessions.

In a few cases, however, we found discrepancies between the results and predictions. In sessions 2228 and 2328, it is difficult to predict which type of flow is realized because the density is near the critical value. In session 1940, where the density is very high, the drivers may behave differently. It is difficult to estimate the OV function in this case.

We also investigated the scaling rule from circuit experiments to real highways and proposed a scaling rule for the OV function, assuming a common backward velocity of jam clusters and a linear relation between headway and velocity at the inflection points of the OV functions. Thus, we were able to obtain the critical density as a function of a single parameter, e.g., maximum velocity, using the relation between the sensitivity and the time delay of car motion. The critical densities obtained by this scaling rule are consistent with those observed on real highways.

In conclusion, in both circuit experiments and real traffic, the OV model consistently explains the emergence of traffic jams and predicts the critical density.

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Appendix A. Representative points for determining the OV function

The data of headway and velocity contain noise, which may not be random. To remove the effect of such noise, we adopt the Parzen window density estimation [24]. We assign a Gaussian distribution for each data point and sum them over all data points as follows

\[
K(h, v) = \sum_i \exp \left[ -\left( \frac{h - h_i}{w_h} \right)^2 - \left( \frac{v - v_i}{w_v} \right)^2 \right]. \tag{A.1}
\]
where $h_i$ and $v_i$ are the headway and velocity of the $i$th data point, and $w_h$ and $w_v$ are parameters for headway and velocity, respectively. In this study, we set $w_h = 1.6$ and $w_v = 0.8$ to obtain smooth functions for all sessions.

Figure A1 shows the results for two typical sessions (a) 2432 and (b) 2030. We find a single peak distribution similar to figure A1 (a) in most sessions. The peak positions for all sessions except stop-and-go flows (2030 and 2934) are shown in table A1. For stop-and-go flows, we observe double peaks reflecting the existence of jam clusters.

Five representative points are sufficient to determine the OV function. In sessions 2030 and 2934, we can identify two local maxima of $K(h, v)$ in a large velocity region (upper max.) and in a small velocity region (lower max.). Between the two maxima, there is a saddle point, which corresponds to the inflection point of the hyperbolic tangent function. We also find two cusps of the hysteresis-like loop in the headway-velocity space. The lower cusp is the point of minimum headway at which cars stop. The minimum headway is determined by averaging the headways of all data points that satisfy the condition $v < 0.1 \text{ m s}^{-1}$. To determine the upper cusp, we first select time slices where stopped cars exist. For each selected time slice, we search for the maximum velocity and the corresponding headway. We obtain the upper cusp of the loop in each session by averaging these data. Table A2 shows points $(h, v)$ for the local maxima, saddle point, and two cusps. We can determine the OV function by minimizing the sum of squared distances between the curve of the OV function and these points.

### Appendix B. Piecewise linear OV functions

Piecewise linear OV functions for all sessions are shown in figure B1.

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| ID   | $h$  | $v$  | ID   | $h$  | $v$  | ID   | $h$  | $v$  |
|------|------|------|------|------|------|------|------|------|
| 1010 | 28.6 | 9.9  | 1730 | 9.6  | 3.9  | 2432 | 9.1  | 5.3  |
| 1210 | 14.6 | 11.3 | 1835 | 7.4  | 2.3  | 2534 | 8.9  | 4.7  |
| 1310 | 19.9 | 10.7 | 1940 | 7.0  | 1.8  | 2634 | 8.3  | 4.4  |
| 1412 | 15.9 | 10.3 | 2125 | 10.5 | 8.0  | 2732 | 8.8  | 4.7  |
| 1520 | 14.4 | 6.4  | 2228 | 9.2  | 6.6  | 2830 | 9.9  | 7.5  |
| 1625 | 10.6 | 4.5  | 2328 | 10.7 | 7.6  |      |      |      |

Table A1. List of the peaks for all sessions except 2030 and 2934.

| ID   | Upper max. | Saddle point | Lower max. | Lower cusp | Upper cusp |
|------|------------|--------------|------------|------------|------------|
|      | $h$ | $v$ | $h$ | $v$ | $h$ | $v$ | $h$ | $v$ | $h$ | $v$ |
| 2030 | 10.7 | 7.6 | 9.1 | 4.8 | 7.3 | 1.9 | 6.0 | 0.0 | 15.9 | 10.2 |
| 2934 | 9.6  | 7.4 | 8.0 | 4.4 | 5.8 | 0.1 | 5.6 | 0.0 | 16.6 | 10.7 |

Table A2. List of representative points for sessions 2030 and 2934.
Appendix C. Comparison of piecewise linear OV functions and acceleration

The validity of piecewise linear functions is shown by another analysis. In the OV model, cars accelerate or decelerate depending on whether their velocities are smaller than the OV. When we assign different colors to the data points according to the sign of acceleration of cars, the OV function is expected to be a border between two colors. To determine the color, we create a mesh of size $0.4 \times 0.2$, and the color of each site is decided by the majority rule. Figure C1 shows the distributions of data points for the sessions where the jammed flow is realized. The estimated piecewise linear OV functions agree with the border of the red (accelerating) and green (decelerating) regions. In other words, data points for accelerating cars exist below the estimated OV function, and those for decelerating cars exist above the OV function. This shows the existence of the hysteresis-like loop structure shown in figure 2.

Appendix D. Analysis of the first experiment

In order to apply the same analysis to the data of the first experiment [1], we select time slices where stopped cars exist. The range of time slices are $t = 150–250 \text{ s}$ for run (I), and $t = 330–370 \text{ s}$ and $t = 445–475 \text{ s}$ for run (II). In the same manner as in section 2, we find five representative points: two local maxima of data distribution, a saddle point, and two cusps. Tables D1 and D2 show the representative points and the estimated parameters of the OV functions, respectively.

Figure D1 shows data points and fitted OV functions.
Figure C1. Red and green represent data points at which cars accelerate and decelerate, respectively. Solid line represents the piecewise linear OV function.

Table D1. List of representative points for earlier experiment.

| ID     | Upper max. | Saddle point | Lower max. | Lower cusp | Upper cusp |
|--------|------------|--------------|------------|------------|------------|
|        | h          | v            | h          | v          | h          | v          |
| run(I) | 11.1       | 8.0          | 7.5        | 3.5        | 6.3        | 1.3        | 5.6        | 0.0        | 14.2       | 10.2       |
| run(II)| 11.2       | 6.2          | 8.8        | 3.3        | 6.2        | 0.5        | 5.4        | 0.0        | 14.0       | 7.7        |

Table D2. List of parameters of OV functions for run (I) and run (II).

| ID     | \(\alpha\) | \(\beta\) | \(h_0\) | \(v_0\) |
|--------|-------------|------------|---------|---------|
| Run(I) | 7.2         | 0.24       | 7.5     | 3.5     |
| Run(II)| 5.4         | 0.22       | 8.8     | 3.3     |

Figure D1. Solid curves represent fitted OV functions for run (I) and run (II). Black dots represent the position of the data points.
Appendix E. Scaling of critical density

We rewrite the OV function as

$$V(h) - v_0 = \alpha \tanh[\beta(h - h_0)],$$  \hspace{1cm} (E.1)

and the minimum headway $h_{\text{min}}$ is defined by $\alpha \tanh[\beta(h_{\text{min}} - h_0)] = -v_0$. When we write a scaled OV function as

$$V'(h) - v'_0 = \alpha' \tanh[\beta'(h - h'_0)],$$  \hspace{1cm} (E.2)

the minimum headway $h'_{\text{min}}$ is given by $\alpha' \tanh[\beta'(h'_{\text{min}} - h'_0)] = -v'_0$. The scaling of the OV function is defined as follows.

First, the inflection point $(h_0, v_0)$ of equation (E.1) is translated to a new inflection point $(h'_0, v'_0)$. Then $(h_{\text{min}}, 0)$ shifts to $(h_2, v_2)$ where

$$h_2 = h_{\text{min}} + (h'_0 - h_0),$$  \hspace{1cm} (E.3)

$$v_2 = (v'_0 - v_0).$$  \hspace{1cm} (E.4)

The backward velocity $v_{\text{back}}$ of jam clusters is unchanged under this scaling rule. The OV model requires that the point $(h'_0, 0)$ lies on the line

$$V'(h) + v_{\text{back}} = \frac{v'_0 + v_{\text{back}}}{h'_0} h,$$

which connects the point $(0, -v_{\text{back}})$ and the inflection point $(h'_0, v'_0)$ of the new OV function (see figure 2). Then, we obtain

$$h'_{\text{min}} = \frac{v_{\text{back}}}{v'_0 + v_{\text{back}}} h'_0.$$  \hspace{1cm} (E.6)

Next, the translated function is scaled such that $(h_2, v_2)$ coincides with $(h'_0, 0)$ (see figure 8). This condition determines $\alpha'$ and $\beta'$

$$\alpha' = \frac{v'_0}{v_0} \alpha = \frac{v'_0}{v_0} \alpha,$$  \hspace{1cm} (E.7)

$$\beta' = \frac{h'_0 - h_2}{h'_0 - h'_{\text{min}}} \beta = \frac{h_0 - h_{\text{min}}}{h'_0 - h'_{\text{min}}} \beta = \left(\frac{h_0 - h_{\text{min}}}{h'_0 - h'_{\text{min}}} \right) \left(\frac{v'_0 + v_{\text{back}}}{v'_0 + v_{\text{back}}}ight) \beta.$$  \hspace{1cm} (E.8)

The scaled OV function still has two free parameters $h'_0$ and $v'_0$.

However, inflection points are supposed to satisfy the relation

$$v'_0 = 0.7(h'_0 - 2)$$  \hspace{1cm} (E.9)

obtained from our experiments and observations on Japanese highways (figure 7). By use of equation (E.9), we can express the scaling rule by a single parameter. We choose the maximum velocity $v_{\text{max}}$ as the scale parameter for convenience. The maximum velocity of the OV function is given by

$$v_{\text{max}} - v'_0 = \alpha' = \left(\frac{v'_0}{v_0}\right) \alpha,$$  \hspace{1cm} (E.10)

As a result, the scaled OV function $V'(h)$ is given by $h_0, v_0, \alpha, \beta, h_{\text{min}}, v_{\text{back}},$ and $v_{\text{max}}$ as follows

$$V'(h) = \alpha' \tanh[\beta'(h - h'_0)] + v'_0,$$  \hspace{1cm} (E.11)

$$\alpha' = \left(\frac{v'_0}{v_0}\right) \alpha,$$  \hspace{1cm} (E.12)

$$\beta' = \left(\frac{h_0 - h_{\text{min}}}{h'_0 - h'_{\text{min}}} \right) \left(\frac{v'_0 + v_{\text{back}}}{v'_0 + v_{\text{back}}}ight) \beta,$$  \hspace{1cm} (E.13)

$$h'_0 = \frac{v'_0}{0.7} + 2,$$  \hspace{1cm} (E.14)

$$v'_0 = \left(\frac{v_0}{v_0 + \alpha}\right) v_{\text{max}}.$$  \hspace{1cm} (E.15)

Finally, the critical density is given by the scaled parameters. Note that equations (3) and (4) are replaced by

$$\alpha'T' = 1.8,$$  \hspace{1cm} (E.16)
Then the critical density $\rho_c$ is given by the condition

$$v' = \frac{h'_\text{min}}{v_{\text{back}}} = \frac{h'_0}{v_0 + v_{\text{back}}}.$$  \hspace{1cm} (E.17)

and the result is

$$\rho_c = \left[ \frac{1}{\beta} \cosh^{-1} \sqrt{\frac{\alpha' \beta' T'}{0.9} + h'_0} \right]^{-1}. \hspace{1cm} (E.19)$$

**Appendix F. Inflection points of observed data**

We show some observed data of headway and velocity obtained using a test car running on Japanese highways [19]. Figure F1 shows the data taken on (a) Chuo highway, (b) Tomei highway, and (c) Tokyo metropolitan highway10. Inflection points were estimated visually.

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