VISCOSITY CHARACTERIZATION OF THE ARBITRAGE FUNCTION UNDER MODEL UNCERTAINTY

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Abstract

We show that in an equity market model with Knightian uncertainty regarding the relative risk and covariance structure of its assets, the arbitrage function – defined as the reciprocal of the highest return on investment that can be achieved relative to the market using nonanticipative strategies, and under any admissible market model configuration – is a viscosity solution of an associated HAMILTON-JACOBI-BELLMAN (HJB) equation under appropriate boundedness, continuity and Markovian assumptions on the uncertainty structure. This result generalizes that of Fernholz & Karatzas (2011), who characterized this arbitrage function as a classical solution of a CAUCHY problem for this HJB equation under much stronger conditions than those needed here.

1. INTRODUCTION

We consider an equity market with asset capitalizations \( \mathbf{x}(t) = (X_1(t), \ldots, X_n(t))^\intercal \in (0, \infty)^n \) at time \( t \in [0, \infty) \), and with local covariation rates \( \alpha(t, \mathbf{x}) = (\alpha_{ij}(t, \mathbf{x}))_{1 \leq i, j \leq n} \) and local relative risk rates \( \vartheta(t, \mathbf{x}) = (\vartheta_1(t, \mathbf{x}), \ldots, \vartheta_n(t, \mathbf{x}))^\intercal \), which are nonanticipative functionals of (i.e., are determined by) the past and present capitalizations for any given time \( t \). We denote by \( \mathbb{S}_+(n) \) the space of real, symmetric and positive-definite \( n \times n \) matrices, fix a collection \( \{\mathcal{K}(y)\}_{y \in (0, \infty)} \) of nonempty, compact and convex subsets on \( \mathbb{R}^n \times \mathbb{S}_+(n) \), and pose the following question:

If the pair \( (\vartheta(t, \mathbf{x}), \alpha(t, \mathbf{x})) \) is restricted to take values in a given nonempty subset \( \mathcal{K}(\mathbf{x}(t)) \) of \( \mathbb{R}^n \times \mathbb{S}_+(n) \), what is the highest return on investment relative to the market portfolio over the given time horizon \([0, T]\), that can be achieved using nonanticipative investment rules, when starting with initial capitalizations \( \mathbf{x} = (x_1, \ldots, x_n)^\intercal \in (0, \infty)^n \), and with probability one under all possible market model configurations with the above covariance and relative risk structure?

Equivalently, if the initial configuration of asset capitalizations is \( \mathbf{x} = (x_1, \ldots, x_n) \), what is the smallest proportion of the initial total market capitalization \( x_1 + \cdots + x_n \), starting with which one can match or outperform the market capitalization over a given time horizon \([0, T]\), by using nonanticipative investment rules, and with probability one under all possible market model configurations with the above covariance and relative risk structure?

Our main result offers the following answers to these two questions: \( 1/\mathbf{u}(T, x) \) and \( \mathbf{u}(T, x) \), respectively. Here the function \( \mathbf{u} : [0, \infty) \times \mathbb{R}_+^n \to (0, 1] \) is, subject to appropriate conditions that will be specified as we progress, a viscosity solution to the CAUCHY problem for the HAMILTON-JACOBI-BELLMAN (HJB) fully nonlinear partial differential equation

\[
(u_t - \mathcal{L}u)(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}_+^n
\]

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of parabolic type, subject to the initial condition
\[ u(0, \cdot) = 1, \quad x \in \mathbb{R}^n. \]

Here we are using the notation
\[
\tilde{L}u(t, x) := \sup_{a \in A(x)} \mathcal{L}_a u(t, x), \quad \mathcal{L}_a u(t, x) := \sum_{i,j} x_i x_j a_{ij} \left( \frac{D_{ij}^2}{2} + \frac{D_i}{||x||_1} \right) u(t, x),
\]
for \((t, x) \in (0, \infty) \times \mathbb{R}^n_+\) with \(a = (a_{ij})_{1 \leq i, j \leq n}\); we are also using the \(\ell^1\)-norm \(||x||_1 := \sum_i x_i\),
\[
A(x) := \{ a \in S_+(n) : \exists \theta \in \mathbb{R}^n \text{ s.t. } (\theta, a) \in K(x) \}
\]
and employ the notation \(D_i u = u_{x_i}, D_{ij}^2 u = u_{x_i(x_j)}\), and \(\mathbb{R}^n_+ := (0, \infty)^n\). Furthermore, the above function \(u\) is dominated by any nonnegative classical supersolution of this Cauchy problem; thus, it is the smallest nonnegative classical supersolution of this Cauchy problem, whenever it is of class \(C \left([0, \infty) \times \mathbb{R}^n_+ \right) \cap C^{1,2} \left((0, \infty) \times \mathbb{R}^n_+ \right)\).

The function \(u\) is called the arbitrage function for a model with uncertainty, in the terminology of [12, Sections 1 and 4]; this extends the arbitrage function \(u_M\) for a specified model \(M\) in the terminology of [11, Section 6]. In [12] the authors characterized the arbitrage function \(u\) as a classical solution of the HJB equation (1.1), subject to the initial condition of (1.2), but under much stronger assumptions on the uncertainty structure; see Theorem 3.3 below.

Under much weaker conditions than in [12], we develop here a different characterization of the arbitrage function \(u\), as a viscosity solution to the Cauchy problem of (1.1), (1.2). We first prove in Theorems 4.5 and 4.6 that the function \(\tilde{\Phi}\) – defined as the supremum of \(u_M\) over all possible market models \(M\) that satisfies certain strong Markov property (strongly Markovian admissible systems in Definition 2.2) – and the function \(\Phi\) – defined as the supremum of \(u_M\) over all possible market models \(M\) – are viscosity subsolution and viscosity supersolution of this Cauchy problem, respectively.

Moreover, we show in Theorem 7.2 that the function \(u\) coincides with \(\Phi\), if this latter function is continuous. As a consequence, the function \(u\) is shown to be a viscosity supersolution of (1.1), and further, a viscosity solution of (1.1) if \(\Phi \equiv \tilde{\Phi}^*\) (the upper-semicontinuous envelope of \(\Phi\); see (4.4)).

1.1. Preview. Section 2 sets up the model for an equity market with model uncertainty regarding its covariance and relative risk characteristics, and Section 3 interprets the variables in this model, introduces the concepts of investment rules and portfolios as well as the notion of arbitrage function, and reviews the results of [12].

Section 4 recalls the definition of viscosity solutions, states our main results and discusses related work. Section 5 characterizes the function \(\tilde{\Phi}\) as a viscosity subsolution – and further, in Section 6 the function \(\Phi\) as a viscosity supersolution – to the Cauchy problem of (1.1), (1.2).

Section 7 provides conditions, under which the arbitrage function \(u\) coincides with the function \(\Phi\) (Theorems 7.1 and 7.2), and thus becomes a viscosity solution to the Cauchy problem (1.1), (1.2). Furthermore, these conditions imply that, if \(u\) is of class \(C \left([0, \infty) \times \mathbb{R}^n_+ \right) \cap C^{1,2} \left((0, \infty) \times \mathbb{R}^n_+ \right)\), it is a classical solution and in fact the smallest nonnegative (super)solution of this Cauchy problem (Corollary 7.3). Additional results, namely, Propositions 7.6 and 7.11 provide conditions on the covariance and relative risk structure, under which \(u \equiv \Phi \equiv \tilde{\Phi}\) and it is indeed the smallest nonnegative (super)solution of this Cauchy problem.

Section 8 develops the proof of Theorem 7.1. Section 9 concludes with examples from the (generalized) volatility-stabilized model of [13], [37]. Finally, Appendix B presents an alternative proof for the viscosity characterizations of the functions \(\tilde{\Phi}\) and \(\Phi\).
2. Notation and Terminology

We shall fix the dimension $n$, let $\Omega := C([0, \infty); \mathbb{R}^+_n)$ be the canonical space of continuous paths $\omega : [0, \infty) \to \mathbb{R}^+_n$ equipped with the topology of locally uniform convergence. We shall also denote by $\mathcal{F}$ the Borel $\sigma$-field of $\Omega$, and $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$ the raw filtration generated by the canonical process $\mathbb{B}(t, \omega) := \omega(t)$.

We shall let $0 = (0, \ldots, 0)'$ denote the origin in $\mathbb{R}^n$, and
\begin{equation}
\mathbb{K} = \{\mathcal{K}(y)\}_{y \in [0, \infty)^n \setminus \{0\}}
\end{equation}
be a collection of nonempty, compact and convex subsets on $\mathbb{R}^n \times S_+(n)$ (recall that $S_+(n)$ is the space of real, symmetric, positive-definite matrices). We denote by $\mathbb{R}$ the collection of pairs $(\sigma, \vartheta)$ consisting of progressively measurable functionals $\sigma = (\sigma_{ik})_{n \times n} : [0, \infty) \times \Omega \to \text{GL}(n)$ and $\vartheta = (\vartheta_1, \ldots, \vartheta_n)' : [0, \infty) \times \Omega \to \mathbb{R}^n$, such that
\begin{equation}
(\vartheta(T, \omega), \alpha(T, \omega)) \in \mathcal{K}(\omega(T)) \quad \text{and} \quad \int_0^T \left(||\vartheta(t, \omega)||^2 + \text{Tr}(\alpha(t, \omega))\right) dt < \infty
\end{equation}
hold for all $\omega \in \Omega$, $T \in (0, \infty)$, where
\begin{equation}
\alpha := \sigma \vartheta'.
\end{equation}

Here and throughout the paper, $\cdot'$ denotes transposition and GL($n$) the space of $n \times n$ invertible real matrices.

**Definition 2.1. Admissible Systems** \[12\] Sections 1 and 2: For a given $x = (x_1, \ldots, x_n)' \in \mathbb{R}^+_n$, we shall call **admissible system**, subject to the Knightian uncertainty $\mathbb{K}$ with initial configuration $x$, a quintuple $\mathcal{M} = (\sigma, \vartheta, \mathbb{P}, W, \mathcal{X})$ consisting of

(i) a pair $(\sigma, \vartheta) \in \mathbb{R}$; of

(ii) a probability measure $\mathbb{P}$ on the measurable space $(\Omega, \mathcal{F})$; of

(iii) an $n$-dimensional $\mathbb{P}$-Brownian motion $W(\cdot) = (W_1(\cdot), \ldots, W_n(\cdot))'$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{F}$; and of

(iv) a continuous, $\mathbb{F}$-adapted process $\mathcal{X}(\cdot) = (X_1(\cdot), \ldots, X_n(\cdot))'$ with values in $\mathbb{R}^+_n$ and
\begin{equation}
\text{d}X_i(t) = X_i(t) \sum_k \sigma_{ik}(t, \mathcal{X})(\vartheta_k(t, \mathcal{X})) \text{d}t + \text{d}W_k(t), \quad i = 1, \ldots, n, \quad \mathcal{X}(0) = x.
\end{equation}

The integrability condition (2.2) guarantees that the process $\mathcal{X}(\cdot)$ indeed takes values in $\mathbb{R}^+_n$, $\mathbb{P}$-a.s.

We shall write $\sigma^\mathcal{M}, \vartheta^\mathcal{M}, \mathbb{P}^\mathcal{M}, W^\mathcal{M}$ and $\mathcal{X}^\mathcal{M}$ for the elements $\sigma, \vartheta, \mathbb{P}, W$ and $\mathcal{X}$ of the quintuple $\mathcal{M}$, respectively, and $\mathfrak{M}(x)$ for the collection of admissible systems with initial configuration $x \in \mathbb{R}^+_n$. \[\square\]

In Definition 2.1 and throughout this paper, all vectors are assumed to be column vectors, and summations to extend from 1 to $n$.

**Definition 2.2. Strongly Markovian Admissible Systems**: For a given initial configuration $x \in \mathbb{R}^+_n$, we shall call **strongly Markovian admissible system**, subject to the Knightian uncertainty $\mathbb{K}$ with initial configuration $x$, an admissible system $\mathcal{M} = (\sigma, \vartheta, \mathbb{P}, W, \mathcal{X}) \in \mathfrak{M}(x)$ satisfying:

(i) the functionals $\sigma$ and $\vartheta$ are Markovian and time-homogeneous, i.e.,
\begin{equation}
\sigma(t, \omega) = s(\omega(t)) = (s_{ij}(\omega(t)))_{1 \leq i,j \leq n} \quad \text{and} \quad \vartheta(t, \omega) = \theta(\omega(t)) = (\theta_1(\omega(t)), \ldots, \theta_n(\omega(t)))'
\end{equation}
for some measurable functions $s : \mathbb{R}^+_n \to \text{GL}(n)$ and $\theta : \mathbb{R}^+_n \to \mathbb{R}^n$; and

(ii) for every $y \in \mathbb{R}^+_n$, there exists an admissible system $\mathcal{M}^y \in \mathfrak{M}(y)$ with the same $s(\cdot)$ and $\theta(\cdot)$ as in $\mathcal{M}$, and a strongly Markovian state process $\mathcal{X}(\cdot)$.

We shall denote by $\hat{\mathfrak{M}}(x)$ the subcollection of $\mathfrak{M}(x)$ consisting of all strongly Markovian admissible systems with initial configuration $x$. \[\square\]
Remark 2.3. It follows from the Markovian selection results of Krylov (see \[25\], \[12\] Chapter 12 and \[10\] Theorem 5.4) that, if the collection of subsets $\mathcal{K}$ satisfies the linear growth condition

\begin{equation}
(2.6) \sup_{(\theta,a)\in\mathcal{K}(y),b=c,T} \left[ \sum_{i,j} y_i y_j a_{ij} + \sum_i (y_i b_i)^2 \right] \leq C(1 + ||y||)^2, \quad \forall \ y \in [0, \infty)^n \setminus \{0\},
\end{equation}

for some constant $C > 0$, then the state process $\mathcal{X}(\cdot)$ can be chosen to be strongly Markovian under $\mathbb{P}^\mathcal{M}$ for any admissible system $\mathcal{M}$ with Markovian and time-homogeneous $\sigma$ and $\vartheta$ as in (2.5). \hfill $\Box$

Remark 2.4. (i) A sufficient condition for $\mathcal{M}(x) \neq \emptyset$ to hold for all $x \in \mathbb{R}^n_+$, is that there exist locally Lipshitz functions $s(\cdot)$ and $\theta(\cdot)$ satisfying Condition (i) of Definition 2.2, that $\left( (\theta(y), s(y)s'(y)) \right) \in \mathcal{K}(y)$ for all $y \in \mathbb{R}^n_+$, and that $s(\cdot)$ and $b(\cdot) := s(\cdot)\vartheta(\cdot)$ are linearly growing, i.e.,

\begin{equation}
(2.7) \quad ||s(y)|| + ||b(y)|| \leq C(1 + ||y||) \quad \text{for all} \ y \in \mathbb{R}^n_+,
\end{equation}

for some real constant $C > 0$. Under this condition and for any $x \in \mathbb{R}^n_+$, the SDE (2.4) with $\sigma$ and $\vartheta$ as in (2.5), always has a pathwise unique, strong solution starting at $x$ (\[17\] Theorem 5.2.2; \[41\] p. 8]).

(ii) In particular, if $\mathcal{K}(y) = \{ (\theta(y), s(y)s'(y)) \}$ for all $y \in \mathbb{R}^n_+$ with such $s$ and $\theta$, then we have $\mathcal{M}(x) = \mathcal{M}(x) \neq \emptyset$ for all $x \in \mathbb{R}^n_+$. \hfill $\Box$ 

3. Interpretation and Previous Results

The above variables can be interpreted in a model for an equity market with $n$ assets, say stocks, as follows:

(i) $\mathcal{X}(t)$ as the vector of capitalizations for the various assets $i = 1, \cdots, n$ at time $t$, and

\begin{equation}
X(t) := \sum_i X_i(t)
\end{equation}

as the total capitalization at that time;

(ii) $W(\cdot)$ as the vector of independent factors (sources of randomness) in the resulting model;

(iii) $\sigma_k(t, \mathcal{X})$, $k = 1, \cdots, n$ as the local volatilities for the $i$th asset at time $t$;

(iv) $\alpha_{ij}(t, \mathcal{X})$ as the local covariation rate between assets $i$ and $j$ at time $t$;

(v) $\vartheta(t, \mathcal{X})$ as the vector of local market prices of risk at time $t$; and

(vi) $\beta(t, \mathcal{X}) := (\sigma\vartheta)(t, \mathcal{X})$ as the vector of local rates of return at time $t$.

3.1. Investment Rules and Portfolios. Consider now an investor who is “small”, in the sense that his actions have no effect on market prices. Starting with initial fortune $v > 0$, he uses a rule that invests a proportion $\Pi_i(t, \mathcal{X})$ of current wealth in the $i$-th asset of the equity market at time $t \in [0, \infty)$ ($i = 1, \ldots, n$), and holds the remaining proportion in cash – or equivalently in a zero-interest money market.

We shall call investment rule a progressively measurable functional $\Pi = (\Pi_1, \cdots, \Pi_n)' : [0, \infty) \times \Omega \to \mathbb{R}^n$ satisfying

\begin{equation}
(3.2) \quad \int_0^T \left( ||\Pi'(t, \omega)\sigma(t, \omega)\vartheta(t, \omega)|| + ||\Pi'(t, \omega)\alpha(t, \omega)\Pi(t, \omega)|| \right) dt < \infty \quad \text{for all} \ T \in (0, \infty), \ \omega \in \Omega,
\end{equation}

and denote by $\mathfrak{P}$ the set of all such (nonanticipative) investment rules.

We shall call an investment rule $\Pi$ bounded, if $\Pi$ is bounded uniformly on $[0, \infty) \times \Omega$; for a bounded investment rule, the requirement (3.2) is satisfied automatically, on the strength of (2.2).

We shall call an investment rule $\Pi$ a portfolio, if $\sum_i \Pi_i = 1$ on $[0, \infty) \times \Omega$; in other words, if it never invests, in or borrows from, the money market. We shall call a portfolio $\Pi$ long-only if $\Pi_i \geq 0$, \hfill $\Box$. 


Given an initial wealth \( v \), an investment rule \( \Pi \) and an admissible model \( \mathcal{M} \in \mathcal{M}(x) \), the resulting wealth process \( Z(\cdot) := Z^{v,\Pi}(\cdot) \) satisfies the initial condition \( Z(0) = v \) and

\[
\frac{dZ(t)}{Z(t)} = \sum_{i} \Pi_i(t, \mathcal{X}) \frac{dX_i(t)}{X_i(t)} = \Pi'(t, \mathcal{X}) \sigma(t, \mathcal{X}) [\vartheta(t, \mathcal{X}) \, dt + dW(t)] , \quad \text{by (2.3)}.
\]

3.2. The Market Portfolio. In the special case with

\[
\Pi_i(t, \omega) \equiv \mu_i(t, \omega) := \frac{\omega_i(t)}{\omega_1(t) + \cdots + \omega_n(t)} , \quad \forall \; i = 1, \cdots, n , \quad 0 \leq t < \infty ,
\]

we have \( \mu(\cdot, \mathcal{X}) = X(\cdot) / \mathcal{X}(\cdot) \): the resulting strategy \( \mu \) invests in all stocks in proportion to their relative market weights. We call the resulting strategy \( \Pi \equiv \mu \) the (long-only) market portfolio. It follows from the first equality in the dynamics (3.3) that investing according to the market portfolio amounts to owning the entire market, in proportion of course to the initial wealth: \( Z^{v,\mu}(\cdot) = v X(\cdot) / X(0) \).

3.3. The Arbitrage Function. With these ingredients in place, we define the arbitrage function \( u : [0, \infty) \times \mathbb{R}_+^n \to (0, 1) \), as

\[
u(T, x) := \inf \left\{ r > 0 : \exists \Pi \in \mathcal{P} \text{ s.t. } \mathbb{P}^\mathcal{M} \left[ Z^{r,\mathcal{M}(0),\Pi}(T) \geq X^{\mathcal{M}}(T) \right] = 1 , \quad \forall \; \mathcal{M} \in \mathcal{M}(x) \right\}.
\]

For the strict positivity of this quantity, see (3.11) below. \( \Box \)

We call the function \( u(\cdot, \cdot) \) the arbitrage function because, for the initial configuration \( x = (x_1, \ldots, x_n)' \in \mathbb{R}_+^n \) of asset capitalizations, the quantity \( u(T, x) \) can be thought of as the smallest proportion of the initial total market capitalization \( x_1 + \cdots + x_n \), starting with which one can find a nonanticipative investment rule, whose performance matches or outperforms that of the market portfolio over the time horizon \([0, T]\), with probability one under all admissible systems. Equivalently, \( u(T, x) \) can be thought of as the reciprocal of the highest return on investment relative to the market portfolio over the time horizon \([0, T]\), that can be achieved using nonanticipative investment rules when starting with the vector \( x = (x_1, \ldots, x_n)' \in \mathbb{R}_+^n \) of initial capitalizations, and with probability one under all admissible systems.

Given an admissible system \( \mathcal{M} = (\sigma, \vartheta, \mathcal{P}, \mathcal{W}, \mathcal{X}) \in \mathcal{M}(x) \), we define the stochastic discount factor \( L(\cdot) \) as the associated exponential \( \mathbb{P} \)-local martingale

\[
L(t) := \exp \left( - \int_0^t \vartheta'(s, \mathcal{X}) \, dW(s) - \int_0^t \frac{1}{2} ||\vartheta(s, \mathcal{X})||^2 \, ds \right) , \quad 0 \leq t < \infty .
\]

This process is well-defined and a strictly positive \( \mathbb{P} \)-local martingale (thus a \( \mathbb{P} \)-supermartingale), on the strength of the integrability condition (2.2); but is not necessarily a \( \mathbb{P} \)-martingale. It plays the role of a state-price-density or “deflator” in the present context. We also write \( L^\mathcal{M}(\cdot) \) for this \( L(\cdot) \) under \( \mathcal{M} \) when needed.

Assuming that \( \widehat{\mathcal{M}}(x) \neq \emptyset \) holds for all \( x \in \mathbb{R}_+^n \), we consider the functions

\[
\Phi(T, x) := \sup_{\mathcal{M} \in \mathcal{M}(x)} u_\mathcal{M}(T, x) \quad \text{and} \quad \widehat{\Phi}(T, x) := \sup_{\mathcal{M} \in \widehat{\mathcal{M}}(x)} u_\mathcal{M}(T, x)
\]

for \( (T, x) \in [0, \infty) \times \mathbb{R}_+^n \), where

\[
u_\mathcal{M}(T, x) := \mathbb{E}^\mathcal{M} \left[ L^\mathcal{M}(T) X^{\mathcal{M}}(T) \right] / ||x||_1
\]

\( i = 1, \ldots, n \) also holds on this domain, that is, it never sells any stock short. A long-only portfolio is also bounded, since it satisfies \( 0 \leq \Pi_i \leq 1 , \; i = 1, 2, \ldots, n \).

\( \bullet \) Given an initial wealth \( v \), an investment rule \( \Pi \) and an admissible model \( \mathcal{M} \in \mathcal{M}(x) \), the resulting wealth process \( Z(\cdot) := Z^{v,\Pi}(\cdot) \) satisfies the initial condition \( Z(0) = v \) and

\[
\frac{dZ(t)}{Z(t)} = \sum_{i} \Pi_i(t, \mathcal{X}) \frac{dX_i(t)}{X_i(t)} = \Pi'(t, \mathcal{X}) \sigma(t, \mathcal{X}) [\vartheta(t, \mathcal{X}) \, dt + dW(t)] , \quad \text{by (2.3)}.
\]
(recall the $\ell^1$-norm $||x||_1 = \sum_i x_i$) and the total capitalization
\begin{equation}
X^M(T) := ||X^M(T)||_1 = \sum_i X^M_i(T).
\end{equation}

As was shown in [14] Section 10, pp.127–129], [23], or [39], the quantity $u_M(T, x)$ in (3.8) is obtained by fixing an admissible system $M$ in the definition (3.5) of $u$, namely,
\begin{equation}
u_M(T, x) = \inf \left\{ r > 0 : \exists \Pi \in \mathcal{P} \text{ s.t. } \mathbb{P}^M \left[ Z^r X^M(0) \Pi (T) \geq X^M(T) \right] = 1 \right\} \in (0, 1].
\end{equation}

This can be interpreted as the reciprocal of the highest return on investment over the time horizon $[0, T]$, that can be achieved relative to the market portfolio in the context of the model $M$, by using nonanticipative strategies and starting with the vector $x$ of initial capitalizations. It can also be interpreted as the \textit{arbitrage function} for $M$ in the terminology of [11] Section 6, at least when $(\mathbb{P}, \mathcal{F})$-martingales can be represented as stochastic integrals with respect to the $W(\cdot)$ in (2.4).

Since the processes $L(\cdot)$ and $X(\cdot)$ are strictly positive, so is the function $u_M(\cdot, \cdot)$ for all admissible system $M$. It then follows from the definitions (3.5)–(3.10) that
\begin{equation}1 \geq u(T, x) \geq \Phi(T, x) \geq \Phi(T, x) > 0, \quad \forall (T, x) \in [0, \infty) \times \mathbb{R}_+^n.
\end{equation}

\textbf{Remark 3.1. Strong Arbitrage:} If $u(T, x) < 1$, then a \textit{strong arbitrage} relative to the market portfolio in the terminology of [14] Definition 6.1 exists on $[0, T]$ with the initial capitalizations $x$. Such strong arbitrage is \textit{robust}, that is, holds under every possible admissible system or model that might materialize.

Instances of $u(T, x) < 1$ with $T \in (0, \infty)$ occur when there exists a real constant $C > 0$ such that either
\[
\inf_{a \in \mathcal{A}(y)} \left( \sum_i \frac{y_i a_{ii}}{y_1 + \cdots + y_n} - \sum_{i,j} \frac{y_i y_j a_{ij}}{(y_1 + \cdots + y_n)^2} \right) \geq C
\]
or
\[
\frac{(y_1 \cdots y_n)^{1/n}}{y_1 + \cdots + y_n} \cdot \inf_{a \in \mathcal{A}(y)} \left( \sum_i a_{ii} - \frac{1}{n} \sum_{i,j} a_{ij} \right) \geq C
\]
holds for every $y \in \mathbb{R}_+^n$ (recall $\mathcal{A}(\cdot)$ from (1.4) and see [14] Examples 11.1, 11.2, [13] and [15]).

\textbf{Remark 3.2. No Unbounded Profits with Bounded Risk:} The inequality $u(T, x) > 0$ in (3.11) rules out scalable arbitrage opportunities, also known as \textit{Unbounded Profits with Bounded Risk} (UPBR). We refer the reader to [7] for the origin of the resulting “No Unbounded Profit with Bounded Risk” (NUPBR) concept, and to [22] for an elaboration of this point in a different context, namely, the existence and properties of the so-called “numéraire” portfolio.

\textbf{3.4. Previous Results.} The Knightian uncertainty in the above model shares a lot with the uncertainty regarding the underlying volatility structure of assets in [31]. The approach in [12] is reminiscent of the \textit{Dubins-Savage} ([8]) and \textit{Sudderth} ([20], [34], [36], [33]) approaches to stochastic optimization.

The arbitrage function $u$ of (3.5) was characterized in [12] as a classical solution and in fact, the smallest nonnegative classical (super) solution of the \textit{Cauchy} problem (1.1), (1.2), but under rather strong assumptions on the uncertainty structure (see Theorem 3.3 below), which amount to: $\Phi \equiv u_M$ for some strongly Markovian admissible system $M$, and $u_M$ solves (1.1).

\textbf{Theorem 3.3.} [12] Proposition 3, Remark 2] \textit{We have $u \equiv \Phi$ on $[0, \infty) \times \mathbb{R}_+^n$, and in fact, this function is the smallest nonnegative (super)solution of (1.1), (1.2), if there exists a strongly Markovian admissible system $M$, under which EITHER:}
\begin{enumerate}[(i)]
\item \textit{the functions $s$ and $\theta$ of (2.5) are locally LIPSCITZ, and}
\end{enumerate}
(ii) the function \( u(t, x) := u_{M^y}(t, x) \), which, by [40, Theorem 4.7], is of class \( C^{1.2} \) and solves
\[
(3.12) \quad (u_t - \mathcal{L}(a(x))u)(t, x) = 0 \quad \text{with} \quad a(x) := s(x)s'(x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n
\]
\((M^y := (\mathcal{M}_y)^x);\;
\text{recall Definition 2.2 for \( M^y \) and (1.3) for \( \mathcal{L}_a \)}, \text{is a classical supersolution of (1.1)};\]

OR both of the following conditions hold:

(i)’ the functions \( s \) and \( \theta \) of (2.5) are continuous,

(ii)’ there exists a positive constant \( C \) such that
\[
\sum_{i,k} y_i s_{ik}(y) \theta_k(y) \leq C(1 + ||y||)
\]
holds for all \( y \in \mathbb{R}^n_+ \),

(iii)’ there exists a \( C^2 \)-function \( h : \mathbb{R}^n_+ \rightarrow \mathbb{R} \) such that \( \theta_k(y) = \sum_i y_i s_{ik}(y) D_i h(y), \; k = 1, \ldots, n \),

(iv)’ the function
\[
\mathcal{G}(t, x) := \mathbb{E}^{\mathbb{P} M^y} \left[ \mathcal{F}(\mathcal{X}(t)) \exp \left( \int_0^t \mathcal{K}(\mathcal{X}(t)) \right) \right] \in C \left( [0, \infty) \times \mathbb{R}^n_+ \right) \cap C^{1.2} \left( (0, \infty) \times \mathbb{R}^n_+ \right),
\]
where
\[
\mathcal{F}(y) := \frac{1}{2} \sum_{i,j} a_{ij}(y) \left[ D_i^2 h + D_i h \cdot D_j h \right](y) \quad \text{and} \quad \mathcal{K}(y) := ||y||_1 \exp \left( - h(y) \right),
\]
and

(v)’ the function \( \mathcal{U}(t, x) := \mathcal{G}(t, x) / \mathcal{F}(x) \) is a classical supersolution of (1.1).

A natural question to ask then is whether the arbitrage function \( u \) of (3.3) is still a solution to (1.1), perhaps in some weak or generalized sense, when regularity and other conditions are weakened. The answer turns out to be affirmative, though it is somewhat indirect; it is provided in Theorems 4.5, 4.6 and Corollary 7.3 below.

4. Viscosity Characterizations of the Functions \( \Phi \) and \( \hat{\Phi} \)

We first recall from [4] the definition of viscosity (sub/super)solutions for a second-order parabolic partial differential equation, and then state our main results with a discussion of related results.

4.1. Viscosity (Super/sub)solution of a Second-order Parabolic PDE. Let \( O \) be an open subset of \( \mathbb{R}^n \), let \( S(n) \) be the set of \( n \times n \) real symmetric matrices, and consider a continuous, real-valued mapping \( (t, x, r, p, q) \mapsto F(t, x, r, p, q) \) defined on \( (0, \infty) \times O \times \mathbb{R} \times \mathbb{R}^n \times S(n) \) and satisfying the ellipticity condition
\[
(4.1) \quad F(t, x, r, p, q_1) \leq F(t, x, r, p, q_2) \quad \text{whenever} \quad q_1 \geq q_2, \quad \text{for all} \quad (t, x, r, p) \in (0, \infty) \times O \times \mathbb{R} \times \mathbb{R}^n.
\]

Consider the second-order parabolic partial differential equation
\[
(4.2) \quad u_t + F \left( t, x, u(t, x), Du(t, x), D^2 u(t, x) \right) = 0, \quad (t, x) \in (0, \infty) \times O
\]
with the gradient \( Du = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})' \) and the Hessian \( D^2 u = (u_{x_ix_j})_{n \times n} \).

Definition 4.1. Viscosity Solution: (i) We say that a function \( u : (0, \infty) \times O \rightarrow \mathbb{R} \) is a viscosity subsolution of the equation (1.2), if
\[
(4.3) \quad \varphi_t + F \left( t_0, x_0, u^*(t_0, x_0), D\varphi(t_0, x_0), D^2\varphi(t_0, x_0) \right) \leq 0
\]
holds for all \( (t_0, x_0) \in (0, \infty) \times O \) and test functions \( \varphi \in C^{1.2}((0, \infty) \times O) \) such that \( (t_0, x_0) \) is a (strict) (local) maximum of \( u^* - \varphi \) on \( (0, \infty) \times O \). We have denoted here by
\[
(4.4) \quad u^*(t, x) := \limsup_{(s,y) \rightarrow (t,x)} u(s, y), \quad (t, x) \in (0, \infty) \times O
\]
the upper-semicontinuous envelope of \( u \), i.e., the smallest upper-semicontinuous function that dominates pointwise the function \( u \).

(ii) Similarly, we say that \( u : (0, \infty) \times \mathcal{O} \to \mathbb{R} \) is a viscosity supersolution of \((1.2)\), if

\[
\varphi_t + F (t_0, u, \partial_x (t_0, x_0), D \varphi (t_0, x_0), D^2 \varphi (t_0, x_0) ) \geq 0
\]

holds for all \((t_0, x_0) \in (0, \infty) \times \mathcal{O}\) and test functions \( \varphi \in C^{1,2} ((0, \infty) \times \mathcal{O}) \) such that \((t_0, x_0)\) is a (strict) (local) minimum of \( u - \varphi \) on \((0, \infty) \times \mathcal{O}\). We have denoted here by

\[
u (t, x) := \lim \inf_{(s, y) \to (t, x)} u (s, y), \quad (t, x) \in (0, \infty) \times \mathcal{O}
\]

the lower-semicontinuous envelope of \( u \), i.e., the largest lower-semicontinuous function dominated pointwise by the function \( u \).

(iii) Finally, we say that \( u : (0, \infty) \times \mathcal{O} \to \mathbb{R} \) is a viscosity solution of \((1.2)\), if it is both a viscosity subsolution and a viscosity supersolution of this equation.

\begin{remark}
The above definition implies that \( u \) is a viscosity subsolution (supersolution) of \((1.2)\) if and only if \( u^* \) (\( u_* \)) is a viscosity subsolution (supersolution) of this equation. \hfill \square
\end{remark}

4.2. Main Results. In our setting we have \( \mathcal{O} = \mathbb{R}^n_+ \) and

\[
F (t, x, r, p, q) = - \sup_{a \in \mathcal{A} (x)} \left( \sum_{i, j} x_i x_j a_{ij} \left( \frac{q_{ij}}{2} + \frac{p_i}{||x||_1} \right) \right)
\]

for \( q = (q_{ij})_{1 \leq i, j \leq n} \), \( p = (p_1, \ldots, p_n)' \), and thus the left-hand sides of \((4.3)\) and \((4.5)\) simplify to \((\varphi_t - \hat{L} \varphi) (t_0, x_0) \) in the notation of \((1.3)\).

Since each matrix \( a \) in the collection \( \mathcal{A} (x) \) of \((1.4)\) is positive-definite, we deduce that the matrix \((x_i x_j a_{ij})_{1 \leq i, j \leq n} = x' a x \) is always positive-definite, and hence \( F \) satisfies the ellipticity condition \((1.1)\).

In the results that follows, we shall also need \( F \) to be a continuous mapping, as well as the following conditions:

\begin{assumption}
Local Boundedness: The collection \( \mathbb{K} \) of \((2.1)\) is locally bounded on \( \mathbb{R}^n_+ \); that is, for any \( x \in \mathbb{R}^n_+ \), there exists a neighborhood \( \mathcal{D} (x) \subset \mathbb{R}^n_+ \) of \( x \) such that \( \bigcup_{y \in \mathcal{D} (x)} \mathcal{K} (y) \) is bounded.
\end{assumption}

\begin{assumption}
Continuity: For any \( \iota > 0 \), \( x \in \mathbb{R}^n_+ \) and \( a = (a_{ij})_{1 \leq i, j \leq n} \in \mathcal{A} (x) \), there exist a positive number \( \delta < \iota \) and locally Lipschitz functions \( s : \mathbb{R}^n_+ \to \mathrm{GL} (n) \) and \( \theta : \mathbb{R}^n_+ \to \mathbb{R}^n \) such that \( s (\cdot) \) and \( b (\cdot) := s (\cdot) \theta (\cdot) \) are linearly growing (i.e., satisfy the condition \((2.7)\)), and that for \( a (\cdot) = (a_{ij} (\cdot))_{1 \leq i, j \leq n} \) we have \((\theta (y), a (y)) \in \mathcal{K} (y) \) for all \( y \in \mathbb{R}^n_+ \) and

\[
|a_{ij} (y) - a_{ij}| < \iota, \quad 1 \leq i, j \leq n, \quad \text{for all } y \in B_{\delta} (x).
\]

\begin{remark}
All of the conditions in Assumption \( \mathbf{4.3} \) except for \((1.8)\), are inspired by Remark \( \mathbf{2.4} \). The aim is to guarantee the existence of an admissible system with the functional \( \sigma (t, \omega) = s (\omega (t)) \), as in \((2.5)\).
\end{remark}

We have then the following results.

\begin{theorem}
Viscosity Subsolution: Suppose that the real-valued function \( F \) of \((1.4)\) is continuous on \((0, \infty) \times \mathbb{R}^n_+ \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S} (n) \), and that Assumption \( \mathbf{4.3} \) holds.

The function \( \hat{\Phi} \) of \((3.7)\) is a viscosity subsolution of the HJB equation \((1.1)\), and thus a viscosity subsolution of the Cauchy problem \((1.1)\), \((1.2)\), since it satisfies \( \hat{\Phi} (0, \cdot) = 1 \).
\end{theorem}

\begin{theorem}
Viscosity Supersolution: Suppose that the real-valued function \( F \) of \((1.7)\) is continuous on \((0, \infty) \times \mathbb{R}^n_+ \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S} (n) \), and that Assumptions \( \mathbf{4.3} \) and \( \mathbf{4.4} \) hold.

The function \( \Phi \) of \((3.7)\) is a viscosity supersolution of the HJB equation \((1.1)\), and thus a viscosity supersolution of the Cauchy problem \((1.1)\), \((1.2)\), since it satisfies \( \Phi (0, \cdot) = 1 \).
\end{theorem}
4.3. Discussion of Related Work. These results echo similar themes from the literature on models with an analogous type of uncertainty, under which the functionals $\sigma$ and $\vartheta$ are fixed; instead, the uncertainty comes from a control process $C(\cdot)$. At any time $t$, the values of $\sigma$ and $\vartheta$ are determined not only by the present capitalizations $X(t)$, but also by the present value $C(t)$ of the control process $C$, i.e., the local volatility matrix and the relative risk vector at time $t$ are $\sigma(X(t), C(t))$ and $\vartheta(X(t), C(t))$, respectively. A control process is a progressively measurable process that takes values in a given subset $\Gamma$ of some Euclidean space and satisfies certain integrability condition.

Among those papers in the literature are the ground-breaking works [27]–[30] by P.L. Lions, specifically, [30, Theorem III.1] (or [28, Theorem I.1]). These impose much stronger assumptions on the volatility and drift structure: namely, $\sup_{\gamma \in \Gamma} ||h(\cdot, \gamma)||_{W^{2,\infty}(\mathbb{R}^n)} < \infty$, and continuity of $h(x, \cdot)$ for all $x$, where $h = \sigma_{ik}, \beta_i, 1 \leq i, k \leq n$.

A similar result was proved in [41, Theorem 4.1], but under the stronger assumptions that both functions $\sigma$ and $\vartheta$ be bounded and Lipschitz, that the analogue in their formulation of the function $F$ of (4.7) be locally Lipschitz, and that the set $\Gamma$ be compact.

If the functions $\alpha_{ij}(\cdot, \gamma)$ ($\gamma \in \Gamma, 1 \leq i, j \leq n$) are all of class $C_{loc}^{1,\eta}(\mathbb{R}^n)$ for some constant $\eta \in (0, 1]$, then in [2, Theorem 3.3], and more generally, in [24, Theorem 2.1], the asymptotic-growth-optimal trading strategy is characterized in terms of a generalized version of the principal eigenvalue of the following fully nonlinear elliptic operator and its associated eigenfunction:

$$\tilde{\mathcal{L}}u(t, x) := \frac{1}{2} \sum_{i,j} a_{ij} D^2_{ij} u(t, x).$$

For a model with no uncertainty and with local volatility matrix $\sigma(X(t))$ and relative risk vector $\vartheta(X(t))$ at time $t$, the viscosity characterization was obtained in [3, Proposition 4.5] but with additional local Lipschitz condition on $\sigma$ and $\vartheta$. This (local) Lipschitz condition is also a typical assumption in previous literature on stochastic control and dynamic programming, e.g., [4], [19] and [44] (it is even assumed in [16] that $\sigma(y, \gamma)$ and $\vartheta(y, \gamma)$ are continuous and twice differentiable in $y$).

In the one-dimensional case ($n = 1$) with zero drift ($\beta \equiv 0$) but no uncertainty, the authors of [5] removed the local Lipschitz condition and hence chose not to pursue a viscosity characterization; instead, provided that the function $\vartheta$ is continuous and satisfies $\int_{t}^{\infty} x^{-2}(x) \, dx = \infty$, they approximated the arbitrage function by classical solutions to Cauchy problems [5, Theorem 5.3].

5. The Proof of Theorem 4.5: Viscosity Subsolution

We first highlight the main idea without many of the technicalities. We argue by contradiction, assuming the negation of (4.3) in Definition 4.1 with the function $F$ as in (4.7); namely, that there exist $\varphi \in C^{1,2}((0, \infty) \times \mathbb{R}^n_+)$ and $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^n_+$, such that $(t_0, x_0)$ is a strict maximum of $\hat{\Phi} - \varphi$; that the maximal value is equal to zero; and that

$$\varphi_t - \tilde{\mathcal{L}} \varphi(t_0, x_0) > 0.$$  

It follows from the definition (4.4) of $\hat{\Phi}$ that we can take a pair $(t^*, x^*)$ close to $(t_0, x_0)$ such that the nonnegative difference $(\varphi - \hat{\Phi})(t^*, x^*)$ is sufficiently small, say less than a small positive constant $C_3$; further, by the definition (3.7) of $\hat{\Phi}$, we can take an admissible system $M^x \in \hat{\mathcal{M}}(x^*)$ such that $0 \leq (\hat{\Phi} - u_{M^x})(t^*, x^*) < C_3$. Therefore $0 \leq (\varphi - u_{M^x})(t^*, x^*) < 2C_3$. Under this system, we have

$$||x^*||_1 \varphi(t^*, x^*) - \mathbb{E} \left[ L(\rho) X(\rho) \varphi(t^* - \rho, X(\rho)) \right] = \mathbb{E} \left[ \int_0^\rho L(s) X(s) g(t^* - s, s, X) \, ds \right] > 0,$$

for any sufficiently small positive stopping time $\rho$, where

$$g(t, s, X) := (\varphi_t - \mathcal{L}_{a(s, X)} \varphi)(t, X(s)) \geq (\varphi_t - \tilde{\mathcal{L}} \varphi)(t, X(s)).$$
for \((t, s) \in (0, \infty) \times [0, \infty)\) and with \(\mathcal{L}_a\) and \(\hat{\mathcal{L}}\) as in (1.3). This displayed quantity is positive for any sufficiently small \(s\), and \(t\) sufficiently close to \(t^*\), by virtue of (5.1) and the continuity of the function \(F\) in (1.7).

On the other hand, on the left-hand side of (5.2) we can estimate \(\varphi(t^*, x^*)\) from above by \(u_{\mathcal{M}^x}(t^*, x^*) + 2C_3\), and \(\varphi(t^* - \rho, \mathcal{X}(\rho))\) from below by \(\Phi(t^* - \rho, \mathcal{X}(\rho)) + C_2\) (for some \(\omega\)'s in \(\Omega\)) or by \(\Phi(t^* - \rho, \mathcal{X}(\rho))\) (for other \(\omega\)'s in \(\Omega\)) with \(C_2\) a small positive constant; this allows us to deduce

\[
\|\varphi\|_1 u_{\mathcal{M}^x}(t^*, x^*) > \mathbb{E}\left[ L(\rho)\mathcal{X}(\rho) \Phi(t^* - \rho, \mathcal{X}(\rho)) \right].
\]

But the inequality (5.3) turns out to contradict the martingale property of the process \(L(\cdot)X(\cdot)\mathcal{M}^x(\cdot)\), see Proposition 5.3 and recall \(\mathcal{M}^x\), \(x \in \mathbb{R}_+^n\) from Definition 2.2.

When implementing this program, the stopping time \(\rho\) needs to be not only small, but also such that on \([0, \rho]\) the processes \(L(\cdot)\) and \(\mathcal{X}(\cdot)\) are bounded, and \(\mathcal{X}(\cdot)\) is close to \(x^*\); however, \(\rho\) cannot be too small, in order to ensure that \(\varphi(t^* - \rho, \mathcal{X}(\rho)) \geq \Phi(t^* - \rho, \mathcal{X}(\rho)) + C_2\) holds with a probability greater than some positive constant independent of \(C_2\) (1/2 in the following proof, see Lemma 5.2). These considerations inspire us to construct \(\rho\) as in (5.10) - (5.11) below.

**Proof of Theorem 4.3:** According to Definition 4.1 (i) of viscosity subsolution with the function \(F\) as in (4.7), it suffices to show that for any test function \(\varphi \in C^{1,2}((0, \infty) \times \mathbb{R}_+^n)\) and \((t_0, x_0) \in (0, \infty) \times \mathbb{R}_+^n\) with

\[
(\Phi^* - \varphi)(t_0, x_0) > 0,
\]

(i.e., such that \((t_0, x_0)\) is a strict maximum of \(\Phi^* - \varphi\)), we have

\[
(\varphi_t - \hat{\mathcal{L}}\varphi)(t_0, x_0) \leq 0.
\]

Here \(\hat{\mathcal{L}}\) is defined in (1.3), and \(\Phi^*\) is the upper-semicontinuous envelope of \(\Phi\) as in the definition (4.4). We shall argue this by contradiction, assuming that

\[
\hat{G}(t_0, x_0) > 0\]

holds for the function \(\hat{G}(t, x) := (\varphi_t - \hat{\mathcal{L}}\varphi)(t, x)\).

Since the function \(F\) of (4.7) is continuous, so is the function \(\hat{G}\) just introduced in (5.3). There will exist then, under this hypothesis and Assumption 4.3, a neighborhood \(D_\delta := (t_0 - \delta, t_0 + \delta) \times B_\delta(x_0)\) of \((t_0, x_0)\) in \((0, \infty) \times \mathbb{R}_+^n\) with \(0 < \delta < \|x_0\|_1/n\), on which \(K(\cdot)\) is bounded and \(\hat{G}(\cdot, \cdot) > 0\) holds.

Let \(C\) be a constant such that \(\|\theta\| < C\) and \(|a_{ij}| < C\) \((1 \leq i, j \leq n)\) hold for all pairs \((\theta, a = (a_{ij})_{n \times n}) \in K(x)\) and all \(x \in B_\delta(x_0)\). We notice that \(|x_i - (x_0)_i| \leq |x - x_0| < \delta\) holds for any \(x = (x_1, \ldots, x_n) \in D_\delta\), thus

\[
0 < \|x_0\|_1 - n\delta < \|x\|_1 < \|x_0\|_1 + n\delta,
\]

and introduce the strictly positive constants

\[
C_1 := \sqrt{32/\delta C^2 + 4\delta^2 C^4},\quad C_2 := -\max_{D_\delta} (\Phi^* - \varphi)(t, x),\quad C_3 := \frac{C_2 e^{-C_1(\|x_0\|_1 - n\delta)}}{4(\|x_0\|_1 + n\delta)}
\]

(the positivity of \(C_2\) and \(C_3\) follows from 5.4 and 5.6, respectively). We observe that

\[
\limsup_{(t, x) \to (t_0, x_0)} (\Phi - \varphi)(t, x) = (\Phi^* - \varphi)(t_0, x_0) = 0,
\]

hence there exists \((t^*, x^*) \in D_\delta\) such that

\[
(\Phi - \varphi)(t^*, x^*) > -C_3;
\]
and by the definition (3.7) of $\hat{\Phi}$, there exists an admissible system $\mathcal{M}^{x^*} \in \hat{\mathcal{M}}(x^*)$ such that

$$u_{\mathcal{M}^{x^*}}(t^*, x^*) > \hat{\Phi}(t^*, x^*) - C_3 > \varphi(t^*, x^*) - 2C_3,$$

by (5.8).

The remaining discussion in this section (with the exception of Proposition 5.3) will be carried out under this admissible system.

- Let us start by recalling the definitions of $\mathcal{D}_\delta$ and $t^*$, and by constructing the positive stopping times

$$\nu(=\nu(\omega)) := \inf \{s \in (0, t^*): (t^* - s, \mathcal{X}(s)) \notin \mathcal{D}_\delta\} \leq t^* - (t_0 - \delta) = (t^* - t_0) + \delta < t^* \wedge 2\delta,$$

$$\lambda(=\lambda(\omega)) := \inf \{s > 0: |\log L(s)| > C_1\}, \quad \rho(=\rho(\omega)) := \nu \wedge \lambda$$

with the usual convention $\inf \emptyset = \infty.$ From the definitions (5.3) and (1.3), we see that

$$g(t, s, \mathcal{X}) := (\varphi_t - L_{\alpha(s, \mathcal{X})}\varphi)(t, \mathcal{X}(s)) \geq \hat{G}(t, \mathcal{X}(s)), \quad \forall (t, s) \in (0, \infty) \times (0, \infty).$$

Recall that $\hat{G}(\cdot, \cdot) > 0$ holds on $\mathcal{D}_\delta$, from the discussion right below (5.5). Combining with (5.12), this observation leads to

$$g(t^* - s, s, \mathcal{X}) > 0, \quad \forall s \in [0, \rho).$$

Thanks to the assumption $\varphi \in C^{1,2}((0, \infty) \times \mathbb{R}_+^n)$, we can apply Itô’s change of variable rule to $X(t)L(t)\varphi(T - t, \mathcal{X}(t))$, $0 \leq t \leq T$ and derive the following decomposition (see Appendix A for a detailed proof).

**Lemma 5.1.** For any $0 \leq t < T < \infty$, $x \in \mathbb{R}^n$, $\varphi \in C^{1,2}((0, \infty) \times \mathbb{R}^n_+)$, and diffusion $\mathcal{X}(\cdot)$ satisfying (2.4), we have

$$dL(t)X(t)\varphi(T - t, \mathcal{X}(t))) = -L(t)X(t)g(T - t, t, \mathcal{X})\,dt$$

$$-X(t)\varphi(T - t, \mathcal{X}(t))L(t)\varphi'(t, \mathcal{X})\,dW(t)$$

$$+L(t)\sum_{i,k}X_{i}X_{k}[\varphi + X(t)D_{i}\varphi](T - t, \mathcal{X}(t))\sigma_{ik}(t, \mathcal{X})\,dW_{i}(t).$$

Let us apply now Lemma 5.1 with $T = t^*$, integrating (5.14) with respect to $t$ over $[0, \rho]$ and taking the expectation under $\mathbb{P}$, to obtain

$$||x||_1\varphi(t^*, x^*) - \mathbb{E}[L(\rho)X(\rho)\varphi(t^* - \rho, \mathcal{X}(\rho))] = \mathbb{E}\left[\int_0^\rho L(s)X(s)\,g(t^* - s, s, \mathcal{X})\,ds\right] > 0.$$ 

Here, the strict inequality comes from (5.13) and the positivity of $\rho$; whereas, in the inequality, the expectations of the integrals with respect to $dW(t)$ or $dW_{k}(t)$ have all vanished. This is due to the the boundedness of the processes $\mathcal{X}(\cdot)$ and $L(\cdot)$ on $[0, \rho]$, of the functions $\varphi$ and $D_{i}\varphi$ on $\overline{\mathcal{D}}_\delta$, and of the functionals $\varphi(\cdot, \mathcal{X})$, $\alpha_{ij}(\cdot, \mathcal{X})$ (by Assumption 1.3) and thus $\sigma_{ik}(\cdot, \mathcal{X})$ on $[0, \rho]$.

(We have made use here of the following facts. The eigenvalues $e_i$ of $\alpha$ are the nonnegative roots of the characteristic polynomial of $\alpha$, which is determined by the entries $\alpha_{ij}$; since the $\alpha_{ij}(\cdot, \mathcal{X})$’s are bounded on $[0, \rho]$, so are the $e_i$’s. Thus $\sigma$, which can be written as $Q\mathbf{D}$ for some $n \times n$ orthonormal matrix $Q$ and diagonal matrix $\mathbf{D}$ with diagonal entries $\sqrt{e_i}$, is also bounded.)

Notice that $(t^* - \nu, \mathcal{X}(\nu)) \in \partial \mathcal{D}_\delta$ holds by the definition (5.10) of $\nu$, so we have

$$\varphi(t^* - \nu, \mathcal{X}(\nu)) \geq \hat{\Phi}(t^* - \nu, \mathcal{X}(\nu)) + C_2 \geq \hat{\Phi}(t^* - \nu, \mathcal{X}(\nu)) + C_2.$$
Plugging (5.4), (5.9) and (5.16) into (5.15) yields
\[
0 < \|x^*\|_1 \left[ u_{M^*}(t^*, x^*) + 2C_3 \right] - \mathbb{E} \left[ 1_{\rho=\nu} L(\rho) X(\rho) \left( \hat{\Phi}(t^* - \rho, \pi(\rho)) + C_2 \right) \right]
\]
\begin{align}
&+ 1_{\rho \neq \nu} L(\rho) X(\rho) \hat{\Phi}(t^* - \rho, \pi(\rho)) \\
&= \|x^*\|_1 u_{M^*}(t^*, x^*) - \mathbb{E} \left[ L(\rho) X(\rho) \hat{\Phi}(t^* - \rho, \pi(\rho)) \right] + C_3 \|x^*\|_1 - C_2 \mathbb{E} \left[ 1_{\rho=\nu} L(\rho) X(\rho) \right].
\end{align}

We start by estimating the last term on the right-hand side of (5.17). Recalling the definition (5.11) of \( \rho \) and the second inequality in (5.6), we see that
\[
L(\rho) \geq e^{-C_1} \quad \text{and} \quad X(\rho) > \|x_0\|_1 - n\delta > 0,
\]
hence
\[
\mathbb{E} \left[ 1_{\rho=\nu} L(\rho) X(\rho) \right] \geq e^{-C_1} (\|x_0\|_1 - n\delta) \ P(\rho = \nu).
\]

**Lemma 5.2.** We have
\[
\mathbb{P}(\rho = \nu) = \mathbb{P}(\lambda \geq \nu) \geq \frac{1}{2}.
\]

**Proof.** For any \( t \in (0, \nu) \), we have
\[
(\log L(t))^2 = \left| - \int_0^t \vartheta'(s, \pi) \, dW(s) - \int_0^t \frac{1}{2} \| \vartheta(s, \pi) \|^2 \, ds \right|^2
\]
\[
\leq 2 \left| \int_0^t \vartheta'(s, \pi) \, dW(s) \right|^2 + 2 \left| \int_0^t \frac{1}{2} \| \vartheta(s, \pi) \|^2 \, ds \right|^2.
\]
It follows from \( t \leq \nu < 2\delta \) that
\[
\int_0^t \frac{1}{2} \| \vartheta(s, \pi) \|^2 \, ds \leq \frac{t}{2} C^2 \leq \delta C^2,
\]
and therefore
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq \nu} (\log L(t))^2 \right] \leq 2 \mathbb{E} \left[ \sup_{0 \leq t \leq \nu} \left| \int_0^t \vartheta'(s, \pi) \, dW(s) \right|^2 \right] + 2 \delta^2 C^4.
\]
Further, the **Burkholder-Davis-Gundy Inequality** gives
\[
2 \mathbb{E} \left[ \sup_{0 \leq t \leq \nu} \left| \int_0^t \vartheta'(s, \pi) \, dW(s) \right|^2 \right] \leq 8 \mathbb{E} \left[ \int_0^\nu \| \vartheta'(s, \pi) \|^2 \, ds \right] \leq 8 \mathbb{E} \left[ \nu C^2 \right] \leq 16 \delta C^2,
\]
thus
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq \nu} (\log L(t))^2 \right] \leq 16 \delta C^2 + 2 \delta^2 C^4.
\]
Finally, appealing to Markov’s Inequality yields
\[
\mathbb{P}(\lambda < \nu) = \mathbb{P} \left[ \sup_{0 \leq t \leq \nu} |\log L(t)| > C_1 \right] \leq \frac{16 \delta C^2 + 2 \delta^2 C^4}{C_1^2} = \frac{1}{2}
\]
(this is why we defined \( C_1 \) as in (5.7); in fact, setting \( C_1 \) to be any value greater than the right-hand side of the first equation in (5.7) would also work), and the claim (5.20) follows. \( \Box \)
Substituting the estimate of Lemma 5.2 into (5.19), we obtain
\[
C_2 \mathbb{E}\left[1_{\{\rho=t\}} L(\rho) X(\rho)\right] \geq \frac{1}{2} C_2 e^{-C_1} \left(||x_0||_1 - n\delta\right) = 2 C_3 \left(||x_0||_1 + n\delta\right) > 2 C_3 ||x^*||_1,
\]
where we used the definition (5.7) of \(C_3\) and the last inequality in (5.6). Plugging into (5.17) yields the inequality (5.3); however, this inequality contradicts Proposition 5.3 (ii) right below with \(T = t^*\) and \(\tau = \rho\). (This explains why we constructed \(C_3\) as we did in (5.7); in fact, setting \(C_3\) to be any value less than the right-hand side of (5.7) would also work.)

The proof of Theorem 4.5 is complete. \(\Box\)

**Proposition 5.3.** Martingale Property: Recall the strongly Markovian admissible systems \(\mathcal{M}^y \in \hat{\mathcal{M}}(y) \in \mathbb{R}_+^n\) from Definition 2.2.

(i) For \(0 \leq t \leq T < \infty\), we have
\[
L(t)X(t)u_{M^x(t)}(t - T, \mathcal{F}(t)) = \mathbb{E}\left[L(T)X(T) \mid \mathcal{F}(t)\right], \quad \mathbb{P}\text{-a.s.}
\]
In particular, the process on the left-hand side is a martingale.

(ii) For any stopping time \(\tau \leq T < \infty\), we have
\[
\mathbb{E}\left[L(\tau)X(\tau)u_{M^x(\tau)}(T - \tau, \mathcal{F}(\tau))\right] = ||x^*||_1 u_{M^x*}(T, x^*).
\]

**Proof.** (i) To alleviate notation somewhat, we write \(\mathbb{P}^y, W^y(\cdot), X^y(\cdot)\) and \(L^y(\cdot)\) for \(\mathbb{P}^\mathcal{M}^y, W^{\mathcal{M}^y}(\cdot)\), \(X^{\mathcal{M}^y}(\cdot)\) and \(L^{\mathcal{M}^y}(\cdot)\) \((y \in \mathbb{R}_+^n)\), respectively. The definitions (3.3) of \(u_M\) and (3.6) of \(L^M\) give
\[
\text{LHS} = L(t) \mathbb{E}^{\mathcal{P}^X(t)} \left[ X^{T}(T - t) L^{X}(T - t) \right] \]
\[
= L(t) \mathbb{E}^{\mathcal{P}^X(t)} \left[ X^{T}(T - t) \exp \left( - \int_0^{T-t} \vartheta(s, \mathcal{X}(t)) \, dW^{X}(s) - \int_0^{T-t} \frac{1}{2} ||\vartheta(s, \mathcal{X}(t))||^2 \, ds \right) \right]
\]
\[
= L(t) \mathbb{E} \left[ X(T) \exp \left( - \int_t^{T} \vartheta(s, \mathcal{X}) \, dW(s) - \int_t^{T} \frac{1}{2} ||\vartheta(s, \mathcal{X})||^2 \, ds \right) \right] \left| \mathcal{F}(t) \right|
\]
\[
= L(t) \mathbb{E} \left[ X(T) L(T) / L(t) \mid \mathcal{F}(t) \right] = \text{RHS}, \quad \mathbb{P}\text{-a.s.}
\]
We note that in the third equality we took advantage of (2.5) and of the strong Markov property for the process \(\mathcal{X}(\cdot)\).

(ii) On the strength of the martingale property from (i), the Optional Sampling Theorem gives
\[
\text{LHS} = L(0)X(0)u_{M^x*}(T, \mathcal{X}(0)) = \text{RHS}. \quad \Box
\]

**Remark 5.4.** In the above proof of Theorem 4.5, the special structure of strongly Markovian admissible systems that we selected in Definition 2.2 is indispensable in the context of Proposition 5.3. On the other hand, the Assumption 4.3 is important for the existence of the neighborhood \(D_\delta\) with the stated properties; see the discussion right below (5.5).

6. **Proof of Theorem 4.6** Viscosity Supersolution

The proof that follows shares many similarities with that in Section 5 for Theorem 4.5, the counterpart of Theorem 4.6, but also requires the additional Assumption 4.4 and a much stronger result – the Dynamic Programming Principle (or DPP, Proposition 6.1 below) – than the martingale property of Proposition 5.3. Before outlining and presenting the proof, we explain the reasons for such differences.

- We begin with an idea similar to that in Section 5 (with corresponding inequalities in opposite directions, and with \(\hat{\Phi}\) replaced by \(\Phi\)); however, we cannot proceed in the same way for two reasons:
(i) The reverse inequality to (5.12), namely, $g(t, s, x) \leq \hat{G}(t, x(s))$ does not hold in general, by the definition (1.3) of $\hat{L}$ (recall $g$ from (5.12) and $\hat{G}$ from (5.5)). Therefore, we cannot obtain
\begin{equation}
(6.1) \quad g(t^* - s, s, x) < 0 \quad \text{for all } s \in [0, \rho], \quad \text{with } g \text{ as in (5.12) and } \rho \text{ as in (5.11)},
\end{equation}
the reverse inequality to (5.13), as we did in Section 5. Instead, we need to find an admissible system to the definitions (1.3) of $L$.

If we still want to argue by contradiction, assuming the reverse inequality to (5.3), then according to the definitions (1.3) of $L$ and (5.5) of $\hat{G}$, there exists $a_0 \in A(x_0)$ such that $(\varphi_t - L_{a_0}) (t_0, x_0) < 0$. Plugging in the definition (1.3) of $L_{a_0}$ and comparing the left-hand side of this inequality with the $g(t^* - s, s, x)$ of (5.12), we see that (6.1) holds if the $\alpha$ in (5.12) is very close to $a_0$ when $s$ is sufficiently small. This accounts for the requirement (4.8) of Assumption 4.4. Other conditions in Assumption 4.4 are inspired by Remark 2.4 aimed for the existence of an admissible system with such $\alpha$.

(ii) The reverse inequality of (5.3) with $\hat{G}$ replaced by $\Phi$, namely
\begin{equation}
(6.2) \quad ||x^*||_1 u_{M^{a*}}(t^*, x^*) < \mathbb{E} \left[ L(\rho) X(\rho) \Phi(t^* - \rho, x(\rho)) \right],
\end{equation}
actually holds in general, on the strength of Proposition 5.3 and the definition (3.7) of $\Phi$. Therefore we need to estimate more accurately the value of $\varphi$ on the left-hand side of the counterpart of (5.2), by using $\Phi$ instead of $u_{M^{a*}}$, so that we arrive at
\begin{equation}
(6.3) \quad ||x^*||_1 \Phi(t^*, x^*) < \mathbb{E} \left[ L(\rho) X(\rho) \Phi(t^* - \rho, x(\rho)) \right],
\end{equation}
instead of (6.2). We then need the DPP of Proposition 6.2 to obtain a contradiction to (6.3).

6.0.1. Informal Outline. Now we outline the main steps of the proof. We prove by contradiction, assuming the negation of (4.5) in Definition 4.1 with the function $F$ as in (4.7), that there exist $\varphi \in C^{1,2}((0, \infty) \times \mathbb{R}_+^n)$ and $(t_0, x_0) \in (0, \infty) \times \mathbb{R}_+^n$ such that: $(t_0, x_0)$ is a strict minimum of $\Phi - \varphi$; the minimal value is equal to zero; and $(\varphi_t - \hat{L}) (t_0, x_0) < 0$.

Since $\hat{L} \varphi = \sup_{a \in A(x)} L_a \varphi$ (definition (1.3)), there exists $a_0 \in A(x_0)$ such that
\begin{equation}
(6.4) \quad (\varphi_t - L_{a_0}) (t_0, x_0) < 0.
\end{equation}
We take $(x, a) = (x_0, a_0)$ and a sufficiently small $\delta$ in Assumption 4.4, and let $\delta$, $s$ and $\theta$ be the corresponding elements. Further, by the definition (4.6) of $\Phi_*$, we can take a pair $(t^*, x^*)$ close to $(t_0, x_0)$ such that the nonnegative difference $(\Phi - \varphi)(t^*, x^*)$ is sufficiently small, say less than a small positive constant $C_3^*$ (depending on $\delta$; defined similarly to the $C_3$ of (5.7)).

Thanks to Assumption 4.4 there exists an admissible system $M^{a_*} \in \mathcal{M}(x^*)$ with the functionals $\sigma$ and $\bar{\sigma}$ defined by (2.5). Under this admissible system, we derive (6.1) from (6.4), and thus
\begin{equation}
(6.5) \quad ||x^*||_1 \varphi(t^*, x^*) - \mathbb{E} \left[ L(\rho) X(\rho) \varphi(t^* - \rho, x(\rho)) \right] = \mathbb{E} \left[ \int_0^\rho L(s) X(s) g(t^* - s, s, x) \, ds \right] < 0.
\end{equation}

On the other hand, on the left-hand side of (6.5) we estimate the real number $\varphi(t^*, x^*)$ from below by $\Phi(t^*, x^*) - C_3^*$, and the random quantity $\varphi(t^* - \rho, x(\rho))$ from above by $\Phi(t^* - \rho, x(\rho)) - C_2^*$ (for some $\omega$’s in $\Omega$) or $\Phi(t^* - \rho, x(\rho))$ (for other $\omega$’s in $\Omega$) with $C_2^*$ a small positive constant similar to the $C_2$ of (5.7), and then deduce (6.3), which contradicts the Dynamic Programming Principle of Proposition 6.2.

6.1. The Supersolution Property. We are ready now to present the argument proper.

Proof of Theorem 4.6: According to Definition 4.1(ii) of viscosity supersolution with the function $F$ as in (4.7), it suffices to show that for any test function $\varphi \in C^{1,2}((0, \infty) \times \mathbb{R}_+^n)$ and $(t_0, x_0) \in (0, \infty) \times \mathbb{R}_+^n$ with
\begin{equation}
(6.6) \quad (\Phi_* - \varphi)(t_0, x_0) = 0 < (\Phi_* - \varphi)(t, x), \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}_+^n
\end{equation}

Recalling the number $g_\ell$ from (1.3), it suffices to establish $(\varphi_t - \mathcal{L}_a \varphi)(t_0, x_0) < g_0$ for every fixed $a \in A(x)$. We shall argue this by contradiction, assuming that for some $a_0 \in A(x)$ we have

$$g_0 := -\mathcal{G}_{a_0}(t_0, x_0) > 0, \quad \text{where } \mathcal{G}_a(t, x) := (\varphi_t - \mathcal{L}_a \varphi)(t, x), \quad (a, t, x) \in A(x) \times (0, \infty) \times \mathbb{R}^n_+.$$ 

Under Assumption 1.3 there exists a positive number $\delta_1 < t_0 \wedge (||x_0||/n)$ such that $K(\cdot)$ is bounded on $D_{\delta_1} := (t_0 - \delta_1, t_0 + \delta_1) \times B_{\delta_1}(x_0)$. Let $C > 1$ be a constant such that $||\theta||, |a_{ij}| < C$ ($1 \leq i, j \leq n$) hold for all pairs $(\theta, a = (a_{ij})_{n \times n}) \in K(x)$ and all $x \in B_{\delta_1}(x_0)$.

Since the functions $\mathcal{G}_{a_0}(\cdot, \cdot), \varphi_t(\cdot, \cdot)$ and

$$H_{ij}(s, y) := D_i \varphi(s, y) / ||y||_1 + y_i y_j D^2_{ij} \varphi(s, y) / 2, \quad (s, y) \in (0, \infty) \times \mathbb{R}^n_+, \quad 1 \leq i, j \leq n$$

are continuous, there exists under the hypothesis (6.7), a positive number $\delta_2 < \delta_1$ such that for all $H \in \{\varphi_t, H_{ij} (1 \leq i, j \leq n)\}$, we have

$$|H(t, x) - H(t_0, x_0)| < g_0 / 3n^2 C < g_0 / 3, \quad \forall (t, x) \in D_{\delta_2} := (t_0 - \delta_2, t_0 + \delta_2) \times B_{\delta_2}(x_0).$$

**Lemma 6.1.** With $\mathcal{G}_a(\cdot, \cdot)$ defined in (6.7), the inequality

$$|\mathcal{G}_a(t, x) - \mathcal{G}_{a_0}(t_0, x_0)| < g_0$$

holds for all $(t, x) \in D_{\delta_2}, \; a \in A(x)$ with

$$\max_{1 \leq i, j \leq n} |a_{ij} - (a_0)_{ij}| < \iota := \delta_2 \wedge g_0 \left(1 + 3n^2 \max_{i,j} |H_{ij}(t_0, x_0)|\right)^{-1}.$$ 

Recalling the number $g_0$ from the definition (6.7), we have also

$$\mathcal{G}_a(t, x) < 0 \quad \text{for all } (a, t, x) \in (6.11).$$

**Proof.** Plugging the definition (6.7) of $\mathcal{G}_a$ into the left-hand side of (6.10) yields

$$\text{LHS of (6.10)} = ||(\varphi_t - \mathcal{L}_a \varphi)(t, x) - (\varphi_t - \mathcal{L}_{a_0} \varphi)(t_0, x_0)||$$

$$\leq |\varphi_t(t, x) - \varphi_t(t_0, x_0)| + |\mathcal{L}_a \varphi(t_0, x_0) - \mathcal{L}_{a_0} \varphi(t_0, x_0)| + |\mathcal{L}_{a_0} \varphi(t_0, x_0) - \mathcal{L}_a \varphi(t_0, x_0)|$$

$$=: \Lambda_1 + \Lambda_2 + \Lambda_3,$$

i.e., $\Lambda_j (j = 1, 2, 3)$ denotes the $j$-th term in (6.13). It suffices to show that $\Lambda_j < g_0 / 3$ for all $j$.

Since $(t, x) \in D_{\delta_2}$, we can take advantage of the property (6.9) and get $\Lambda_1 < g_0 / 3$. Moreover, we notice that $\mathcal{L}_a \varphi(t, x) = \sum_{i,j} a_{ij} H_{ij}(t, x)$ (from the definitions (1.3) of $\mathcal{L}_a$ and (6.8) of $H_{ij}$) and obtain

$$\Lambda_2 = \left|\sum_{i,j} a_{ij} [H_{ij}(t_0, x_0) - H_{ij}(t, x)]\right| < n^2 \cdot C \cdot g_0 / (3n^2 C) = g_0 / 3 \quad \text{by (6.9)},$$

$$\Lambda_3 = \left|\sum_{i,j} [(a_0)_{ij} - a_{ij}] H_{ij}(t_0, x_0)\right| < n^2 \cdot \iota \cdot \left(\max_{i,j} |H_{ij}(t_0, x_0)|\right) < g_0 / 3 \quad \text{by (6.11)}.$$ 

This completes the proof. \hfill \Box

Take $x = x_0$ and $a = a_0$ in Assumption 4.4 with $\iota$ defined in (6.11). Let $\delta, s, \theta$ and $a$ be the corresponding elements described in Assumption 4.4. We shall now adopt the definitions of $C_1$ from (5.7) and introduce the strictly positive constants

$$C_2^* := \min_{\partial D_3} (\Phi^*_\theta - \varphi)(t, x) > 0 \quad \text{by (6.6)}, \quad C_3^* := \frac{C_2^* e^{-C_1(||x_0||_1 - n\delta)}}{2(||x_0||_1 + n\delta)} > 0 \quad \text{by (5.6)}.$$
by analogy with $C_2$ and $C_3$ in (5.7). We observe from the definition (4.6) of $\Phi_*$ that
\[
\liminf_{(t, x) \to (t_0, x_0)} (\Phi - \varphi)(t, x) = (\Phi_* - \varphi)(t_0, x_0) = 0,
\]
hence there exists $(t^*, x^*) \in \mathcal{D}_\delta$ such that
\[
(\Phi - \varphi)(t^*, x^*) < C_2^*;
\]
and thanks to Assumption 4.4 there exists an admissible system $\mathcal{M}^{x^*} \in \mathcal{M}(x^*)$ with the functionals $\sigma$ and $\vartheta$ defined by (2.5). The remaining discussion in this section (with the exception of Proposition 6.2) will be carried out under this admissible system.

Now we shall adopt the definitions of $\nu$, $\lambda$ and $\rho$ from (5.10) and (5.11). For any $0 \leq s \leq \rho$, we have $(t^* - s, \mathcal{X}(s)) \in \overline{\mathcal{D}}_\delta \subset \overline{\mathcal{D}}_{\delta_2}$ and therefore (6.11) holds for $(a, t, x) = (\alpha(s, \mathcal{X}), t^* - s, \mathcal{X}(s))$ by virtue of (4.8) (recall from (2.5) that $\alpha(s, \mathcal{X}) = a(\mathcal{X}(s))$). Therefore, we can apply (6.12) and obtain
\[
(6.16) \quad \mathcal{G}_{\alpha(s, x)}(t^* - s, \mathcal{X}(s)) < 0.
\]

Let us apply now Lemma 5.1 with $T = t^*$, integrating (5.14) with respect to $t$ over $[0, \rho]$ and taking the expectation under $\mathbb{P}$, to obtain
\[
(6.17) \quad ||x^*||_1 \varphi(t^*, x^*) - \mathbb{E} \left[ L(\rho)X(\rho)\varphi(t^* - \rho, \mathcal{X}(\rho)) \right] = \mathbb{E} \left[ \int_0^\rho L(s)X(s)g(t^* - s, s, \mathcal{X}) ds \right] < 0,
\]
by (6.16) and the same reasoning as right below (5.15). Here $g$ is defined in (5.3), and thus the quantity $g(t^* - s, s, \mathcal{X})$ is the left-hand side of (6.16) with $\ell = \ell^*$.

Notice that $(t^* - \nu, \mathcal{X}(\nu)) \in \partial \mathcal{D}_\delta$ holds by the definition (5.10) of $\nu$, thus
\[
(6.18) \quad \varphi(t^* - \nu, \mathcal{X}(\nu)) \leq \Phi_*(t^* - \nu, \mathcal{X}(\nu)) - C_2^*.
\]
Plugging (6.6), (5.6) and (6.18) into (6.17) yields
\[
0 > ||x^*||_1 \left[ - C_3^* + \Phi(t^*, x^*) \right] - \mathbb{E} \left[ 1_{\{\rho = \nu\}} L(\rho)X(\rho) \left( \Phi_* (t^* - \rho, \mathcal{X}(\rho)) - C_2^* \right) \right] + 1_{\{\rho \neq \nu\}} L(\rho)X(\rho) \Phi_*(t^* - \rho, \mathcal{X}(\rho)) \right]
\]
\[
= -C_3^* ||x^*||_1 + ||x^*||_1 \Phi(t^*, x^*) - \mathbb{E} \left[ L(\rho)X(\rho) \Phi_*(t^* - \rho, \mathcal{X}(\rho)) \right] + C_2^* \mathbb{E} \left[ 1_{\{\rho = \nu\}} L(\rho)X(\rho) \right].
\]
On the strength of (5.18), Lemma 5.2, the definition (6.14) of $C_3^*$, and (5.6), we obtain now
\[
C_2^* \mathbb{E} \left[ 1_{\{\rho = \nu\}} L(\rho)X(\rho) \right] \geq C_2^* e^{C_1} \left( ||x_0||_1 + n\delta \right) \mathbb{P}(\rho = \nu) \geq C_3^* e^{C_1} \left( ||x_0||_1 + n\delta \right) / 2 = C_3^* ||x^*||_1.
\]
(This explains why we constructed $C_3^*$ as we did in (6.14); in fact, setting $C_3^*$ to be any value less than the right-hand side of (6.14) would also work.)

Substituting this inequality into (6.19), and recalling $\Phi_*(\cdot, \cdot) \leq \Phi(\cdot, \cdot)$ from the definition (4.6) of $\Phi_*$, leads now to the inequality
\[
||x^*||_1 \Phi(t^*, x^*) < \mathbb{E} \left[ L(\rho)X(\rho) \Phi(t^* - \rho, \mathcal{X}(\rho)) \right]
\]
of (6.3). However, this inequality contradicts the Dynamic Programming Principle of Proposition 6.2 right below, so the proof of Theorem 4.9 is complete.

Proposition 6.2. Dynamic Programming Principle (32–33): For any given $(T, x) \in (0, \infty) \times \mathbb{R}^n_+$ and any stopping time $\tau \leq T < \infty$, we have
\[
||x||_1 \Phi(T, x) = \sup_{\mathcal{M} \in \mathcal{M}(x)} \mathbb{E}^\mathcal{M} \left[ L^\mathcal{M}(\tau)X^\mathcal{M}(\tau) \Phi(T - \tau, X^\mathcal{M}(\tau)) \right].
\]

Proof. We refer to [32, Proposition 2.2, Theorem 2.4, Remark 2.7] and [33, Theorem 2.3].
7. Viscosity Characterization of the Arbitrage Function

Let us go back to the arbitrage function $u$ of (3.5). As a consequence of the minimality result Theorem 7.1 below, if $\Phi$ of (3.7) is a classical supersolution of (1.1), then the function $u$ coincides with $\Phi$ and hence is the smallest nonnegative classical supersolution of the Cauchy problem of (1.1), (1.2); in fact, we have $u \equiv \Phi$ if $\Phi$ is only continuous (see Theorem 7.2 below).

**Theorem 7.1.** ([3.11] and [12, Proposition 2]) For any nonnegative classical supersolution $U$ of the Cauchy problem (1.1), (1.2), we have

$$U(T, x) \geq u(T, x) \geq \Phi(T, x) \geq \hat{\Phi}(T, x) > 0, \quad \forall (T, x) \in [0, \infty) \times \mathbb{R}^n_+.$$ 

**Proof.** We adopt the idea from the proof in [12, Proposition 2, (5.3)–(5.15)]; the detailed proof is provided in section 8. □

**Theorem 7.2.** The arbitrage function $u$ coincides with the function $\Phi$ of (3.7) if $\Phi$ is continuous.

This theorem is proved right below. Combining it with Theorems 4.5, 4.6 and 7.1, and recalling Remark 7.4. If a robust strong arbitrage relative to the market exists on some time horizon $[0, T]$, then $u(T, x) < 1$. This amounts to a failure of uniqueness of classical/viscosity solutions for the Cauchy problem of (1.1), (1.2), since the constant $u \equiv 1$ is always a (trivial) solution to this problem.

We refer the reader to [12, p. 2205] or to [31], for an interpretation of Theorem 7.2. □

**Proof of Theorem 7.2.** Let $U$ be the collection of positive classical supersolutions of the Cauchy problem (1.1), (1.2), and $\hat{U}$ the collection of continuous functions $\hat{U} : [0, \infty) \times \mathbb{R}^n_+ \to \mathbb{R}_+$ that satisfies (1.2) and that the process $L(t)X(t)\Phi(T-t,X(t))$ is a supermartingale under every admissible system. Note that $\Phi \in \hat{U}$ by virtue of [33, Theorem 2.3].

Following the idea in [12, Theorem 1], we have for $T = 0$ the identities $u(0, x) = 1 = \Phi(0, x)$ for all $x \in \mathbb{R}^n_+$ by the initial condition (1.2). Now we fix an arbitrary pair $(T, x) \in (0, \infty) \times \mathbb{R}^n_+$. For every $\varepsilon > 0$, there exists a mollification $U_\varepsilon \in U$ of the function $\Phi$ with $0 < U_\varepsilon(T, x) \leq \Phi(T, x) + \varepsilon$. Combining with Theorem 7.1 gives

$$u(T, x) \leq U_\varepsilon(T, x) \leq \Phi(T, x) + \varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrary, this leads to $u(T, x) \leq \Phi(T, x)$. On the other hand, the reverse inequality $u(T, x) \geq \Phi(T, x)$ holds on the strength of (3.11). Hence, $u(T, x) = \Phi(T, x)$ on $[0, \infty) \times \mathbb{R}^n_+$. □

**Remark 7.5.** With slight modifications our approach can also show that, under appropriate conditions described in Theorems 4.5, 4.6 and Corollary 7.3 but now with

$$F(t, x, r, p, q) = -\frac{1}{2} \sum_{i,j} a_{ij} q_{ij} \bigg| x_i x_j a_{ij} q_{ij} \bigg|, \quad p = (p_1, \ldots, p_n)'$$

and $a = (a_{ij})_{1 \leq i, j \leq n}$, the functions

$$\Psi(T, x) = ||x||_1 \Phi(T, x), \quad \hat{\Psi}(T, x) = ||x||_1 \hat{\Phi}(T, x)$$

where $\Phi$ is subject to the

and
\[ v(T, x) := ||x||_1 u(T, x), \quad (T, x) \in [0, \infty) \times \mathbb{R}^n_+ \]
are classical/viscosity (super/sub)solutions of an HJB equation simpler than (1.1) – namely, the Pucci-maximal type equation
\[ u_t(t, x) - \frac{1}{2} \sup_{a \in A(x)} \left( \sum_{i,j} x_i x_j a_{ij} D^2_i u(t, x) \right) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n_+ \]
subject to the initial condition \( u(0, x) = ||x||_1 \), and are dominated by any nonnegative classical supersolution of the Cauchy problem (7.1), (1.2). □

7.1. Sufficient Conditions for \( u \equiv \Phi \equiv \hat{\Phi} \) to be a classical supersolution of (1.1). Now let us provide some sufficient conditions under which we have \( u \equiv \Phi \equiv \hat{\Phi} \), and this function is a classical solution of (1.1) – thus also the smallest nonnegative classical (super)solution of the Cauchy problem (1.1), (1.2) by virtue of Theorem 7.1.

In particular, via the discussions below, we will see that one sufficient condition is the following specific requirements on the Knightian uncertainty \( K \).

**Proposition 7.6.** Suppose that there exist locally Lipschitz functions \( s : \mathbb{R}^n_+ \to \text{GL}(n) \) and \( \theta : \mathbb{R}^n_+ \to \mathbb{R}^n \), and subsets \( \mathcal{R}(y) \ (y \in \mathbb{R}^n_+) \) of \( \mathbb{R} \) such that the functions \( s(\cdot) \) and \( b(\cdot) := s(\cdot)\theta(\cdot) \) are linearly growing \( (i.e., \text{satisfy} \ (2.7)) \) and with \( a(y) := s(y)s'(y) \ (y \in \mathbb{R}^n_+) \) we have
\[ (\theta(y), a(y)) \in K(y), \quad \mathcal{A}(y) = \{ r \cdot a(y) : r \in \mathcal{R}(y) \}, \quad \min \mathcal{R}(y) = 1 \quad \text{for all} \ y \in \mathbb{R}^n_+ . \]
Then \( u \equiv \Phi \equiv \hat{\Phi} \) is the smallest nonnegative classical (super)solution of the Cauchy problem (1.1), (1.2), and the smallest nonnegative classical (super)solution of the Cauchy problem (3.12), (1.2).

**Proof.** This result follows directly from Remark 7.9 and Theorem 7.11 below. □

We start with the following observation.

**Proposition 7.7.** If there exist admissible systems \( \mathcal{M}^y \in \mathcal{M}(y) \ (y \in \mathbb{R}^n_+) \) such that
\[ V(t, y) := u_{\mathcal{M}^y}(t, y), \quad (t, y) \in [0, \infty) \times \mathbb{R}^n_+ \]
is a classical supersolution of (1.1), then \( u \equiv \Phi \equiv \hat{\Phi} \equiv V \) is the smallest nonnegative classical (super)solution of the Cauchy problem (1.1), (1.2).

**Proof.** Theorem 7.1 and the definition (3.7) of \( \hat{\Phi} \) give \( V(t, y) \geq u(t, y) \geq \Phi(t, y) \geq \hat{\Phi}(t, y) \geq u_{\mathcal{M}^y}(t, y) = V(t, y) \), hence \( u \equiv \Phi \equiv \hat{\Phi} \equiv V \) is a classical supersolution of (1.1). □

To proceed further, we need the following assumption.

**Assumption 7.8.** There exist admissible systems \( \mathcal{M}^x \in \mathcal{M}(x) \), \( x \in \mathbb{R}^n_+ \) such that
(i) they share the same functionals \( \sigma(t, \mathcal{X}) = s(\mathcal{X}(t)) \) and \( \vartheta(t, \mathcal{X}) = \theta(\mathcal{X}(t)) \) as in (2.5); and
(ii) for every \( x \in \mathbb{R}^n_+ \), the process \( \mathcal{X} \) in \( \mathcal{M}^x \) is unique in distribution in the following sense (and thus strongly Markovian): for any admissible system \( \tilde{\mathcal{M}} \in \mathcal{M}(x) \) with the same functionals \( \sigma \) and \( \vartheta \) as in \( \mathcal{M}^x \), the two processes \( \mathcal{X}^{\tilde{\mathcal{M}}} \) and \( \tilde{\mathcal{X}}^{\tilde{\mathcal{M}}} \) have the same law. □

**Remark 7.9.** Assumption 7.8 holds when the conditions in Remark 2.4(i) are satisfied. □

**Proposition 7.10.** Under Assumption 7.8, the function \( V \) of (7.3) is
(i) dominated by any nonnegative classical supersolution of the Cauchy problem (3.12), (1.2);
(ii) a viscosity solution of (3.12), if \( \theta(\cdot) \) and \( a(\cdot) \) are locally bounded and \( a(\cdot) \) is continuous; and
(iii) a classical solution of the Cauchy problem (3.12), (1.2) (and thus its smallest nonnegative (super)solution), if \( s(\cdot) \) and \( \theta(\cdot) \) are locally Lipschitz.
Proof. We will see that (i) and (ii) are special cases of Theorems 7.11 and Theorems 4.5–4.6, respectively, with \( \mathcal{K}(y) = \{(\theta(y), a(y))\} \) \((y \in \mathbb{R}^n_+)) via the following observations. First, in this case we have \( \hat{L}(t, y) = L_{a(y)}(t, y) \) (recall the definition \( \ref{def:La} \) for \( \hat{L} \) and \( L_a \)). Moreover, by virtue of Assumption 7.8 and definition \( \ref{def:La} \), we have \( u_{M^y}(t, y) = u_M(t, y) \) for all \( M \in \mathcal{M}(y) \), and by the definition \( \ref{def:Phi} \) of \( \Phi \) and \( \hat{\Phi} \) gives
\[
\Phi(t, y) = \hat{\Phi}(t, y) = u_{M^y}(t, y) = V(t, y).
\]

(iii) Under these conditions, we have \( V(\cdot, \cdot) \in C^{1,2}(0, \infty) \times \mathbb{R}^n_+ \) (see \[10, \text{Theorem 4.7}\] for a proof that uses results from the theory of stochastic flows \([20, 38]\) and from parabolic partial differential equations \([9, 21]\)), and conclude by invoking (ii) since the local Lipschitz condition on \( s \) and \( \theta \) implies the condition in (ii).

\[\text{Proposition 7.11.} \text{ If Assumption 7.8 holds with locally Lipschitz functions } s(\cdot) \text{ and } \theta(\cdot), \text{ and there exist subsets } R(y) (y \in \mathbb{R}^n_+) \text{ of } \mathbb{R} \text{ such that } \ref{cond:R(y)} \text{ holds, then, with } M^y \in \mathcal{M}(y) \text{ as in Assumption 7.8, the function}
\]
\[
V(t, y) = \Phi(t, y) \equiv \hat{\Phi}(t, y) \equiv u_{M^y}(t, y)
\]
is the smallest nonnegative classical (super)solution of the Cauchy problem \ref{eq:cauchy}, \ref{eq:cauchy2}, as well as the smallest nonnegative classical (super)solution of the Cauchy problem \ref{eq:cauchy}, \ref{eq:cauchy2}.

Proof. By Proposition 7.10(iii), the right-hand side of \ref{eq:cauchy}, i.e., the function \( V \) of \ref{eq:cauchy} solves \ref{eq:cauchy2}:
\[
V_t(t, y) = L_{a(y)}V(t, y), \quad (t, y) \in (0, \infty) \times \mathbb{R}^n_+.
\]

Thus
\[
L_{r-a(y)}V(t, y) = r \cdot L_{a(y)}V(t, y) = r \cdot V_t(t, y), \quad (t, y) \in (0, \infty) \times \mathbb{R}^n_+.
\]

Once we have shown that \( V_t(t, y) \leq 0 \), i.e., that \( V \) is nonincreasing in \( t \) on \((0, \infty)\), for all \( y \in \mathbb{R}^n_+ \), then \( V \) is a classical supersolution of \ref{eq:cauchy} on the strength of \ref{cond:R(y)}, and the proof will be complete by Proposition 7.7. In fact, under any given admissible system, the positive process \( L(\cdot)X(\cdot) \) is a local martingale, hence a supermartingale (one can derive the formula \( d(L(t)X(t)) = L(t)X(t) (\pi' \sigma - \sigma') (t, X) \) \(dW(t) \) with \( \pi \) the market portfolio, via Itô’s Rule; see \ref{eq:ito} for details). Therefore \( V(t, y) = \mathbb{E}^P[M^y][L^M(t)X^M(t)] / ||y||_1 \) is indeed nonincreasing in \( t \). \qed

Remark 7.12. This result is in agreement with general regularity theory for fully nonlinear parabolic equations, as in \[29, \text{Theorem II.4}\].

Remark 7.13. We have tried to find weaker conditions for Theorem 7.2 to hold, or for the function \( \Phi \) to be continuous, but did not succeed. Even if all the functions \( u_M \) are of class \( C^{1,2} \), their supremum \( \Phi \) might still fail to be continuous. \qed

8. THE PROOF OF THEOREM 7.11

Minimality

The proof consists of two parts, Theorems 8.1 and 8.2. Theorem 8.1 shows that any nonnegative classical supersolution \( U \) of the Cauchy problem \ref{eq:cauchy}, \ref{eq:cauchy2} is strictly positive, by proving that \( U(T, x) > \Phi(T, x) \) and then applying the fact \( \Phi(T, x) > 0 \) from \ref{cond:Phi}.\ref{cond:Phi2}.

In Theorem 8.2 the positivity of \( U \) from Theorem 8.1 enables us to construct an investment rule from \( U \) (see \ref{def:investment} below) that matches or outperforms the market portfolio over the time horizon \([0, T]\), with probability one under all admissible systems. We then conclude that \( U(T, x) \geq u(T, x) \) from the definition \ref{def:u} of \( u(T, x) \).

The following proofs of Theorems 8.1 and 8.2 adopt the idea from \[12, \text{Proposition 2, (5.3)–(5.15)}\] and provide details for completeness.
Theorem 8.1. For any nonnegative classical supersolution $U$ of the Cauchy problem (1.1), (1.2), we have
\begin{align}
U(T, x) &\geq \Phi(T, x) > 0, \quad \forall (T, x) \in [0, \infty) \times \mathbb{R}^n. 
\end{align}

Proof. The second inequality was shown in (3.11). For the first inequality, let us fix an admissible system $\mathcal{M} \in \mathcal{M}(x)$; the remaining discussion in this proof will be carried out under this system. The key point, is to show that the process
\begin{align}
\Xi(t) := X(t)L(t)U(T-t, \mathcal{X}(t))
\end{align}
is a supermartingale. Once this is proved, with the initial condition $U(0, \cdot) \geq 1$, we obtain
\begin{align}
\left\|x\right\|_1 U(T, x) &= \mathbb{E}[\Xi(0)] \geq \mathbb{E}[\Xi(T)] = \mathbb{E}[X(T)L(T)U(0, \mathcal{X}(T))] \\
&\geq \mathbb{E}[X(T)L(T)] = \left\|x\right\|_1 u_M(T, x), \quad \text{by the definition } (3.8).
\end{align}
Since $\left\|x\right\|_1 > 0$, we deduce $U(T, x) \geq u_M(T, x)$, which leads to (8.1) by the definition (3.7).

To show the supermartingale property of $\Xi(t)$, we apply Lemma 5.1 with $\varphi = U$ and get
\begin{align}
\begin{aligned}
d(\Xi(t)) &= -L(t)X(t)(U_t - L_{\alpha(t, x)}) (T-t, \mathcal{X}(t)) dt - X(t)U(T-t, \mathcal{X}(t)) L(t) \vartheta'(t, \mathcal{X}) dW(t) \\
&\quad + L(t) \sum_{i,k} X_i(t) \left[ U(T-t, \mathcal{X}(t)) + X(t)D_iU(T-t, \mathcal{X}(t)) \right] \sigma_{ik}(t, \mathcal{X}) dW_k(t).
\end{aligned}
\end{align}

Thanks to the supersolution property of $U$ that
\begin{align}
(U_t - \hat{L}_{\alpha(t, x)})(s, y) \geq (U_t - \hat{L}U)(s, y) \geq 0, \quad \forall (s, y) \in [0, \infty) \times \mathbb{R}^n
\end{align}
(recall $L_{\alpha}$ and $\hat{L}$ from (1.3)) and the nonnegativity of the processes $L(\cdot)$, $X(\cdot)$ and the function $U(\cdot, \cdot)$, we conclude that $\Xi(t)$ is a nonnegative local martingale, hence a supermartingale. \hfill \Box

Theorem 8.2. For any nonnegative classical supersolution $U$ of the Cauchy problem (1.1), (1.2), the investment rule $\pi^U \in \mathcal{P}$ generated by this function $U$ through
\begin{align}
\pi^U_i(t, \omega) := \omega_i(t)D_i \log U(T-t, \omega(t)) + \frac{\omega_i(t)}{||\omega(t)||_1}, \quad i = 1, \ldots, n, \quad t \in [0, T]
\end{align}
for continuous function $\omega : [0, \infty) \to \mathbb{R}^n_+$, satisfies the inequality
\begin{align}
Z^{U(T, x)X^{\mathcal{M}(0)}, \pi^U}(T) \geq X^{\mathcal{M}}(T), \quad \mathbb{P}-\text{a.s., } \forall \mathcal{M} \in \mathcal{M}(x).
\end{align}
It then follows from the definition (3.5) of $u(T, x)$ that
\begin{align}
U(T, x) \geq u(T, x), \quad \forall (T, x) \in [0, \infty) \times \mathbb{R}^n_+.
\end{align}

Proof. The investment rule $\pi^U$ is well-defined since $U$ is positive by Theorem 8.1. Let us fix $\mathcal{M} \in \mathcal{M}(x)$; the remaining discussion in this proof will be carried out under this system.

We shall set $v := U(T, x)X(0)$ and $\pi := \pi^U$. The main goal is to show that the growth rate of the process $\log (L(t)Z^{\nu, \pi}(t))$ is no less than that of $\log \Xi(t)$ with $\Xi(t)$ defined in (8.2). Once this is proved, noticing that these two processes start at the same initial value $v$, we obtain
\begin{align}
L(T)Z^{\nu, \pi}(T) \geq \Xi(T) = X(T)L(T)U(0, \mathcal{X}(T)) \geq X(T)L(T),
\end{align}
as $U(0, \cdot) \geq 1$ by the initial condition. This leads to (8.6) as $L(T) > 0$.

To start, we observe from (3.3) with $\pi = \Pi$ that the wealth process $Z^{\nu, \pi}(\cdot)$ satisfies the dynamics
\begin{align}
dZ^{\nu, \pi}(t) = Z^{\nu, \pi}(t)\pi'(t, \mathcal{X})\sigma(t, \mathcal{X})[\vartheta(t, \mathcal{X}) dt + dW(t)] \quad \text{with } Z^{\nu, \pi}(0) = v.
\end{align}
We apply Itô’s Rule for the product function $f_1(r_1, r_2) := r_1 r_2$ with (A.4) and (8.7) yields
\begin{align}
(8.8) \quad d \left( L(t) Z^{v, \pi}(t) \right) &= L(t) dZ^{v, \pi}(t) + Z^{v, \pi}(t) dL(t) + d\langle L, Z^{v, \pi} \rangle(t) \\
&= L(t) Z^{v, \pi}(t) [\pi' \sigma \vartheta dt + \pi' \sigma dW(t) - \vartheta' dW(t) - \pi' \sigma \vartheta dt](t, \mathfrak{X}) \\
&= L(t) Z^{v, \pi}(t) \mathcal{H}(t, \mathfrak{X}) dW(t),
\end{align}
where
\begin{align}
(8.9) \quad \mathcal{H}(t, \mathfrak{X}) := (\pi' \sigma - \vartheta')(t, \mathfrak{X}),
\end{align}
whose $k$-th component
\begin{align}
(8.10) \quad \mathcal{H}_k(t, \mathfrak{X}) = \sum_i \left[ X_i(t) D_i \log U(T - t, \mathfrak{X}(t)) + \frac{X_i(t)}{X(t)} \right] \sigma_{ik}(t, \mathfrak{X}) - \vartheta_k(t, \mathfrak{X}), \quad \text{by (8.5)}.
\end{align}

Applying Itô’s Rule to the logarithm function for $L(\cdot) Z^{v, \pi}(\cdot)$, we obtain
\begin{align}
(8.11) \quad d \log (L(t) Z^{v, \pi}(t)) = \mathcal{H}(t, \mathfrak{X}) dW(t) - \frac{1}{2} (\mathcal{H}'(t, \mathfrak{X}) dt.
\end{align}

To determine the growth rate for $\log \Xi(\cdot)$, we recast (8.3) into
\begin{align*}
\frac{d}{dt} (\log \Xi(t)) = \Xi(t) [I(T - t, \mathfrak{X}(t))] dt + \mathcal{H}(t, \mathfrak{X}) dW(t),
\end{align*}
by virtue of
\begin{align*}
\left( \frac{D_i U}{U} \right)(s, y) = D_i \log U(s, y), \quad (s, y) \in [0, \infty) \times \mathbb{R}_+^n
\end{align*}
and (8.10), where
\begin{align*}
I(s, y) := - \left( \frac{U_t - L_v(t, x)}{U} \right)(s, y) \leq 0, \quad (s, y) \in [0, \infty) \times \mathbb{R}_+^n, \quad \text{by (8.4) and } U > 0.
\end{align*}
Applying Itô’s Rule again to the logarithm function for $\Xi(\cdot)$ and juxtaposing with (8.11) leads to
\begin{align*}
\frac{d}{dt} \log \Xi(t) = I(T - t, \mathfrak{X}(t)) dt + \mathcal{H}(t, \mathfrak{X}) dW(t) - \frac{1}{2} (\mathcal{H}'(t, \mathfrak{X}) dt \leq \frac{d}{dt} \log (L(t) Z^{v, \pi}(t)) , \quad \text{as desired.}
\end{align*}

**Remark 8.3.** In the special case of a model without uncertainty, the HJB equation (1.1) reduces to a linear PDE. If additionally, the functions $\sigma$ and $\vartheta$ have the form of (2.5) and are locally Lipschitz continuous, then the arbitrage function $u$ is also shown to be dominated by every nonnegative and lower-semicontinuous viscosity supersolution of the Cauchy problem for the linear PDE (1.1) and (1.2) [3 Proposition 4.7], that satisfies certain convexity and continuity conditions.

This local Lipschitz condition on $\sigma$ and $\vartheta$ is indispensable in the proof of [3]. It is the subject of future research, to determine whether this result still holds with weaker assumptions and in the presence of model uncertainty.

\section{9. Examples}

The volatility-stabilized model was introduced in [13] and further generalized in [37], but now we add some uncertainty regarding its local volatility and relative risk structure.

**Example 9.1. Volatility-Stabilized Model:** Take constants $c_1 \geq c_2 \geq 1/2$ and $c_2 \geq 1$, and set
\begin{align*}
\mathcal{K}(y) = \{ (\gamma_2 a(y), \gamma_1 \gamma_2 \theta(y)) : \gamma_1 \in [c_1, c_1^*], \gamma_2 \in [1, c_2] \},
\end{align*}
where
\begin{align}
(9.1) \quad a(y) = s(y)s'(y) \quad \text{with} \quad s_{ij}(y) = 1_{i=j}(|y|/y_i)^{1/2}, \quad \theta_i(y) = (|y|/y_i)^{1/2}, \quad 1 \leq i, j \leq n.
\end{align}
Then the system of Stochastic Differential Equations (2.4) becomes
\[ dX_i(t) = \gamma_1 \gamma_2^2 \left( X_1(t) + \cdots + X_n(t) \right) dt + \gamma_2 \sqrt{X_i(t)(X_1(t) + \cdots + X_n(t))} \, dW_i(t), \quad i = 1, \ldots, n, \]
or equivalently, and a bit more succinctly,
\[ d \log(X_i(t)) = \left( \gamma_1 - \frac{1}{2} \right) \frac{\gamma_2^2}{\mu_i(t, X)} \, dt + \frac{\gamma_2}{\mu_i(t, X)} \, dW_i(t), \quad i = 1, \ldots, n, \]
with \( \mu(t, X) \) the market portfolio defined in (3.4).

For every \( x \in \mathbb{R}_+^n \), \( \gamma_1 \in [c_1, c_1^*] \) and \( \gamma_2 \in [1, c_2] \), this system of SDEs has a unique-in-distribution solution \( X(\cdot) \) starting at \( X(0) = x \) whose \( X_i(\cdot) \)'s are time-changed versions of independent squared-Bessel processes (see [1], [13] and [18] for more details). In particular, we have \( X(\cdot) \in \mathbb{R}_+^n \).

Moreover, this uncertainty structure satisfies the conditions in Remark 3.1 and Proposition 7.11 with the \( s, \theta \) as in (9.1) and \( \mathcal{R}(y) = [1, c_2] \). Hence
\[
\Phi(t, y) \equiv \Phi(t, y) \equiv u_{\mathcal{M}^y}(t, y) \begin{cases} < 1, & \text{if } t > 0 \\ = 1, & \text{if } t = 0 \end{cases}
\]
is the smallest nonnegative classical (super)solution of the Cauchy problem (1.1), (1.2), as well as the smallest nonnegative classical (super)solution of the Cauchy problem (3.12), (1.2) (recall \( \mathcal{M}^y \) from Assumption 7.8) see [18] and [35] for a computation of the joint density of \( X_1(\cdot), \ldots, X_n(\cdot) \), which leads to an explicit formula for
\[ u_{\mathcal{M}^y}(t, y) = \frac{\prod_{i=1}^n y_i}{\prod_{i=1}^n \mu_i^{\mu_i}(t)} \cdot \mathbb{E}^{\mathcal{M}^y} \left[ \left[ \prod_{i=1}^n X_i^{\mathcal{M}^y}(t) \right] \right] \]
and shows that this function is indeed of class \( C^{1,2} \).

Example 9.2. Generalized Volatility-Stabilized Model: Take constants \( c_i^* \geq c_i \geq 0, i = 1, 2, \ldots, n \) and \( c_{n+1} \geq 1 \), and set
\[
\mathcal{K}(y) = \left\{ (a, \theta) : a = \gamma_{n+1} a(y), \; \theta_i = \gamma_i + \frac{\gamma_{n+1}^2}{2\gamma_{n+1}} \theta_i(y), \; \gamma_i \in [c_i, c_i^*], \; \gamma_{n+1} \in [1, c_{n+1}], \; 1 \leq i \leq n \right\},
\]
where \( a(y) = s(y)s'(y) \) with
\[
s_{ij}(y) = \mathbf{1}_{\{i=j\}} \left( \frac{\|y\|_1}{y_i} \right)^\kappa G(y), \quad \theta_i(y) = \left( \frac{\|y\|_1}{y_i} \right)^\kappa G(y), \quad 1 \leq i, j \leq n
\]
where \( \kappa \) is a positive constant and \( G : \mathbb{R}_+^n \to \mathbb{R}_+ \) is a bounded and locally Lipschitz function (Example 9.1 is a special case of this model with \( \kappa = 1/2 \) and \( G \equiv 1 \)).

Then the system of Stochastic Differential Equations (2.4) becomes
\[ dX_i(t) = X_i(t) \left[ \frac{\gamma_i + \gamma_{n+1}^2}{2(\mu_i(t, X))^{2\kappa}} G^2(X(t)) \right] dt + \frac{\gamma_{n+1}}{(\mu_i(t, X))^{\kappa}} G(X(t)) \, dW_i(t), \quad i = 1, \ldots, n \]
with \( \mu(t, X) \) the market portfolio defined in (3.4), or equivalently,
\[ d \log(X_i(t)) = \left( \frac{\gamma_i}{2(\mu_i(t, X))^{2\kappa}} G^2(X(t)) \right) dt + \frac{\gamma_{n+1}}{(\mu_i(t, X))^{\kappa}} G(X(t)) \, dW_i(t), \quad i = 1, \ldots, n, \]
For every \( x \in \mathbb{R}_+^n \), \( \gamma_i \in [c_i, c_i^*], i = 1, 2, \ldots, n \) and \( \gamma_{n+1} \in [1, c_{n+1}] \), this system of SDEs has a unique-in-distribution solution \( X(\cdot) \) starting at \( X(0) = x \), the components \( X_i(\cdot) \) of this solution are time-changed versions of independent squared-Bessel processes (see [1] and [37], Sections 2 and 4 for more details). In particular, we have \( X(\cdot) \in \mathbb{R}_+^n \).
This uncertainty structure also satisfies the conditions in Proposition 7.11 with the $s, \theta$ as in (9.3) and $R(y) = [1, c_{n+1}]$, therefore
\[
u(t, y) \equiv \Phi(t, y) \equiv \tilde{\Phi}(t, y) \equiv u_{M^v}(t, y)
\]
is the smallest nonnegative classical (super)solution of the Cauchy problem (1.1), (1.2), as well as the smallest nonnegative classical (super)solution of the Cauchy problem (3.12), (1.2) (recall $M^v$ from Assumption 7.8). If in addition $G(\cdot)$ is bounded away from zero, then the condition in Remark 3.1 is satisfied as well and (9.2) follows.

**Appendix A. The Proof of Lemma 5.1**

*Proof.* Let $\phi(t) := \varphi(T-t, X(t))$,
\[
s_{ik}(t, x) := X_i(t)\sigma_{ik}(t, x),
\]
and $b(t, x) = (b_1, \ldots, b_n)(t, x) := (s \vartheta)(t, x)$. Then the SDE (2.4) can be rewritten as
\[
dX(t) = b_i(t, x) \, dt + \sum_{k} s_{ik}(t, x) \, dW_k(t), \quad i = 1, 2, \ldots, n, \quad X(0) = x.
\]

Applying Itô’s Rule to $f_2(x, y_1, \ldots, y_n) := \varphi(T-x, y_1, \ldots, y_n)$ with (A.1):
\[
d\phi(t) = \left[-\varphi_t \, dt + \sum_i D_i\varphi \left(b_i \, dt + \sum_k s_{ik} \, dW_k(t)\right) + \frac{1}{2} \sum_{i,j} D_{ij}^2 \varphi \sum_{k} s_{ik} s_{jk} \, dt\right](T-t, x),
\]
where for convenience, throughout the paper the values of $L, \phi, b_i, s_{ik}, \vartheta, \varphi_t, D_i\varphi$ and $D_{ij}^2\varphi$ at $(T-t, x)$ stand for $L(t), \phi(t), b_i(t, x), s_{ik}(t, x), \vartheta(t, x), \varphi_t(T-t, x(t)), D_i\varphi(T-t, x(t))$ and $D_{ij}^2\varphi(T-t, x(t))$, respectively.

Summing (A.1) over $i$ from 1 to $n$ yields
\[
dX(t) = \left(\sum_i b_i \, dt + \sum_{i,k} s_{ik} \, dW_k(t)\right)(t, x).
\]

Finally, apply Itô’s Rule to the exponential function for $L(\cdot)$:
\[
dL(t) = -L(t) \vartheta'(t, x) \, dW(t).
\]

Plugging (A.2) − (A.4) into Itô’s Rule for $f_3(r_1, r_2, r_3) := r_1 r_2 r_3$ gives
\[
d(XL\varphi)(t) = [L\phi \, dX(t) + X\phi \, dL(t) + XL \, d\phi(t) + X \, d\langle L, \phi \rangle_t + L \, d\langle X, \phi \rangle_t + \phi \, d\langle X, L \rangle_t](t)
\]
\[
= \left[L\phi \sum_i b_i \, dt + \sum_{i,k} s_{ik} \, dW_k(t)\right] - X\phi L \vartheta' \, dW(t) - XL\varphi_t \, dt
\]
\[
+ XL \sum_i D_i \varphi \left[b_i \, dt + \sum_k s_{ik} \, dW_k(t)\right] + \frac{1}{2} XL \sum_{i,j} D_{ij}^2 \varphi \sum_k s_{ik} s_{jk} \, dt
\]
\[
- XL \sum_k \vartheta_k \sum_i D_i \varphi s_{ik} \, dt + L \sum_{k,j} s_{jk} \sum_i D_i \varphi s_{ik} \, dt - \phi L \sum_{k,i} s_{ik} \vartheta_k \, dt\bigg|_{(T-t,t,x)}.
\]
Rearranging (A.5), we obtain
\[
\begin{align*}
d(XL\phi)(t) &= -XL \left( \phi_t - \frac{1}{2} \sum_{i,j} D_{ij}^2 \phi \sum_k s_i s_j k - \frac{1}{X} \sum_{j,k} s_j s_k \sum_i D_i \phi s_i k \right) dt + L\phi \sum_{i,k} s_i k dW_k(t) \\
n &\quad - X\phi L \partial_t^i dW(t) + XL \sum_i D_i \phi \sum_k s_i k dW_k(t) + L\phi \left( \sum_i b_i - \sum_{k,i} \delta_k \right) dt \\
n &\quad + XL \left( \sum_i D_i \phi b_i - \sum_k \delta_k \sum_i D_i \phi s_i k \right) dt \\
n &\quad = -XLg dt + L\phi \sum_{i,k} s_i k dW_k(t) - X\phi L \partial_t^i dW(t) + XL \sum_i D_i \phi \sum_k s_i k dW_k(t) \\
n &\quad = -XLg dt - X\phi L \partial_t^i dW(t) + L \sum_{i,k} s_i k dW_k(t) (\phi + XD_i \phi) \\
\end{align*}
\]
where we used the definition (5.3) of $g$ and the fact that $b = s \bar{\vartheta}$.

**Appendix B. An Alternative Proof for Theorem 4.5**

We present here an alternative proof for Theorem 4.5. We still argue by contradiction, but avoid introducing the stopping time $\lambda$ of (5.11) and thus also the stopping time $\rho$ and the constant $C_1$. We also avoid using Lemma 5.2, instead, we provide a lower bound for $E[L(\nu)]$ in (B.15) below. The goal is to prove (5.15) for $\nu$ instead of $\rho$. We shall approximate $\nu$ by a sequence of stopping times $\nu_t$ for which (5.15) holds, then apply Fatou’s Lemma. This approach can also be applied to the proof in Section 6 for the supersolution property.

**Proof.** According to Definition 4.1(i) of viscosity subsolution with the $F$ in (4.7), it suffices to show that for any test function $\varphi \in C^{1,2}((0, \infty) \times \mathbb{R}^n)$ and $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^n_{+}$ with
\[
(\widehat{\varphi}^* - \varphi)(t, x_0) = 0 > (\widehat{\varphi}^* - \varphi)(t, x), \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}^n_{+},
\]
(i.e., such that $(t_0, x_0)$ is a strict maximum of $\widehat{\varphi}^* - \varphi$), we have
\[
(\varphi - \widehat{\varphi})(t_0, x_0) \leq 0.
\]
Here $\widehat{\varphi}$ is defined in (1.3), and $\widehat{\varphi}^*$ is the upper-semicontinuous envelope of $\widehat{\varphi}$ as in the definition (4.1). We shall argue this by contradiction, assuming that
\[
\widehat{G}(t_0, x_0) > 0 \quad \text{for the function} \quad \widehat{G}(t, x) := (\varphi - \widehat{\varphi})(t, x).
\]
Since the function $F$ of (4.2) is continuous, so is the function $\widehat{G}$ just introduced in (B.2). There will exist then, under this hypothesis and Assumption 4.3, a neighborhood $D_\delta := (t_0 - \delta, t_0 + \delta) \times B_\delta(x_0)$ of $(t_0, x_0)$ in $(0, \infty) \times \mathbb{R}^n_{+}$ with $0 < \delta < ||x_0||1/n$, on which $\mathcal{K}(\cdot)$ is bounded and $\widehat{G}(\cdot, \cdot) > 0$ holds.

Let $C$ be a constant such that $\varphi(t, x), ||\theta||, |a_{ij}| < C$ ($1 \leq i, j \leq n$) hold for all pairs $(\theta, a = (a_{ij})_{n \times n}) \in \mathcal{K}(x)$ and all $(t, x) \in D_\delta$. We can assume that
\[
16\delta^2 C^2 + 2\delta^2 C^4 < 1/2
\]
by selecting a sufficiently small $\delta > 0$. We notice that $|x_i - (x_0)_i| \leq |x - x_0| < \delta$ holds for any $x = (x_1, \ldots, x_n) \in D_\delta$, thus
\[
0 < ||x_0||_1 - n\delta < ||x||_1 < ||x_0||_1 + n\delta.
\]
and introduce the constants

\begin{equation}
C_2 := - \max_{\partial D_\delta} \left( \tilde{\Phi}^* - \varphi \right)(t, x) \quad \text{and} \quad C_3^* := \frac{C_2(||x_0||_1 - n\delta)}{2(||x_0||_1 + n\delta) \left( \frac{1}{2} - 16\delta C^2 - 2\delta^2 C^4 \right)},
\end{equation}

which are strictly positive by \((B.1)\) and \((B.3)\), respectively. We observe that

\[
\limsup_{(t, x) \to (t_0, x_0)} (\tilde{\Phi} - \varphi)(t, x) = (\tilde{\Phi}^* - \varphi)(t_0, x_0) = 0,
\]

hence there exists \((t^*, x^*) \in D_\delta\) such that

\[
(\tilde{\Phi} - \varphi)(t^*, x^*) > -C_3^*;
\]

and by the definition \((3.7)\) of \(\tilde{\Phi}\), there exists an admissible system \(\mathcal{M}^* \in \widehat{\mathcal{M}}(x^*)\) such that

\[
u_{\mathcal{M}^*} > \Phi(t^*, x^*) - C_3^* > \varphi(t^*, x^*) - 2C_3^*, \quad \text{by (B.6).}
\]

The remaining discussion in this section will be carried out under this admissible system, unless otherwise specified.

- Let us start by constructing stopping times

\begin{equation}
\nu(\nu(\omega)) := \inf \{ s \in (0, t^*): (t^* - s, \mathfrak{X}(s)) \notin D_\delta \} \leq t^* - (t_0 - \delta) = (t^* - t_0) + \delta < t^* \wedge 2 \delta
\end{equation}

(by the definitions of \(D_\delta\) and \(t^*\)), and for \(\ell = 1, 2, \ldots\),

\[
\lambda_\ell(\lambda_\ell(\omega)) := \inf \{ s > 0: |\log L(s)| > \ell \} \uparrow \infty, \quad \nu_\ell(\nu_\ell(\omega)) := \nu \wedge \lambda_\ell \uparrow \nu, \quad \mathbb{P}\text{-a.s. as } \ell \uparrow \infty
\]

with the usual convention \(\inf \emptyset = \infty\).

From definitions \((B.2)\) and \((1.3)\), we see that

\[
g(t, s, \mathfrak{X}) := (\varphi - \mathcal{L}_x(t^*, x^*))(t, \mathfrak{X}(s)) \geq \hat{g}(t, \mathfrak{X}(s)), \quad \forall (t, s) \in (0, \infty) \times [0, \infty).
\]

Recall that \(\hat{g}(\cdot, \cdot) > 0\) on \(D_\delta\) from the discussion right below \((B.2)\). Combining with \((B.10)\) leads to

\[
g(t^* - s, s, \mathfrak{X}) > 0, \quad \forall s \in [0, \nu_\ell).
\]

Let us apply now Lemma \(5.1\) with \(T = t^*\), integrating \((5.14)\) with respect to \(t\) over \([0, \nu_\ell]\) and taking the expectation under \(\mathbb{P}\), to obtain

\[
||x^*||_1 \varphi(t^*, x^*) - \mathbb{E} \left[ L(\nu_\ell) X(\nu_\ell) \varphi(t^* - \nu_\ell, \mathfrak{X}(\nu_\ell)) \right] = \mathbb{E} \left[ \int_0^{\nu_\ell} L(s) X(s) g(t^* - s, s, \mathfrak{X}) ds \right] > 0.
\]

Here, the strict inequality comes from \((B.11)\) and the positivity of \(\nu_\ell\); whereas, in the equality, the expectations of the integrals with respect to \(dW(t)\) or \(dW_k(t)\) have all vanished – due to the boundedness of the processes \(\mathfrak{X}(\cdot)\) and \(L(\cdot)\) on \([0, \nu_\ell]\), of the functions \(\varphi\) and \(D_i \varphi\) on \(D_\delta\), and of the functionals \(\vartheta(\cdot, \mathfrak{X}), \alpha_{ij}(\cdot, \mathfrak{X})\) (by Assumption \((1.3)\) and thus \(\sigma_{ik}(\cdot, \mathfrak{X})\) on \([0, \nu_\ell]\).

(We have made use here of the following facts. The eigenvalues \(e_i\) of \(\alpha\) are the nonnegative roots of the characteristic polynomial of \(\alpha\), which is determined by the entries \(\alpha_{ij}\); since the \(\alpha_{ij}(\cdot, \mathfrak{X})\)'s are bounded on \([0, \nu_\ell]\), so are the \(e_i\)'s. Thus \(\sigma\), which can be written as \(Q D\), for some \(n \times n\) orthonormal matrix \(Q\) and diagonal matrix \(D\) with diagonal entries \(\sqrt{e_i}\), is also bounded.)

Since almost surely \(\nu_\ell \uparrow \nu\) \((B.9)\) and \(L(\nu_\ell) X(\nu_\ell) \varphi(t^* - \nu_\ell, \mathfrak{X}(\nu_\ell)) > 0\) for all \(\ell\) (the positivity of \(\varphi\) follows from \((B.1)\) and \((3.11)\)), FATOU’s Lemma gives

\[
\mathbb{E} \left[ L(\nu) X(\nu) \varphi(t^* - \nu, \mathfrak{X}(\nu)) \right] = \mathbb{E} \left[ \liminf_{\ell \to \infty} L(\nu_\ell) X(\nu_\ell) \varphi(t^* - \nu_\ell, \mathfrak{X}(\nu_\ell)) \right] \leq \liminf_{\ell \to \infty} \mathbb{E} \left[ L(\nu_\ell) X(\nu_\ell) \varphi(t^* - \nu_\ell, \mathfrak{X}(\nu_\ell)) \right] \leq ||x^*||_1 \varphi(t^*, x^*), \quad \text{by (B.12)}.
\]
Notice that \((t^* - \nu, \mathcal{X}(\nu)) \in \partial \mathcal{D}_\delta\) holds by the definition \((B.8)\) of \(\nu\), so we have
\[(B.14) \quad \varphi(t^* - \nu, \mathcal{X}(\nu)) \geq \bar{\Phi}(t^* - \nu, \mathcal{X}(\nu)) + C_2 \geq \bar{\Phi}(t^* - \nu, \mathcal{X}(\nu)) + C_2.\]

Plugging \((B.7)\) and \((B.14)\) into \((B.13)\) yields
\[0 < ||x^*||_1 \left[ 2C^*_3 + u_{M^*}(t^*, x^*) \right] - \mathbb{E} \left[ L(\nu)X(\nu) \left( \bar{\Phi}(t^* - \nu, \mathcal{X}(\nu)) + C_2 \right) \right] = ||x^*||_1 u_{M^*}(t^*, x^*) - \mathbb{E} \left[ L(\nu)X(\nu) \bar{\Phi}(t^* - \nu, \mathcal{X}(\nu)) \right] + 2C^*_3 ||x^*||_1 - C_2 \mathbb{E} \left[ L(\nu)X(\nu) \right] \leq 2C^*_3 ||x^*||_1 - C_2 \mathbb{E} \left[ L(\nu)X(\nu) \right] < 2C^*_3(||x_0||_1 + n\delta) - C_2(||x_0||_1 - n\delta) \mathbb{E} \left[ L(\nu) \right],\]

we have used Proposition 5.3 in the third step and the last inequality of \((B.4)\) at last.

Recall the definition \((B.5)\) of \(C^*_3\). We will arrive at a contradiction and hence complete the argument, as soon as we have shown the following inequality:
\[(B.15) \quad \mathbb{E} \left[ L(\nu) \right] \geq \frac{1}{2} - 16 \delta C^2 - 2 \delta^2 C^4.\]

This explains why we constructed \(C^*_3\) as we did in \((B.5)\); in fact, setting \(C^*_3\) to be any value less than the right-hand side of \((B.5)\) would also work. First, we observe the following double inequality
\[(B.16) \quad \varepsilon = \frac{3}{2e} - \frac{r^2}{2e} > \frac{1}{2} - r^2, \quad \forall \ r \in \mathbb{R}.\]

The second inequality is obvious since \(2 < e < 3\). For the first inequality, we set
\[f(r) := \varepsilon - \frac{3}{2e} + \frac{r^2}{2e}\]

and find that \(f(-1) = 0, f'(1) = 0\) and \(f''(r) > 0\). Hence \(f(r)\) achieves its minimum 0 at \(r = -1\).

Applying \((B.16)\) to \(\log L(\nu)\) yields
\[\mathbb{E} \left[ L(\nu) \right] \geq \frac{1}{2} - \mathbb{E} \left[ (\log L(\nu))^2 \right].\]

Therefore, it suffices to show that
\[(B.17) \quad \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \left( \log L(t) \right)^2 \right] \leq 16 \delta C^2 + 2 \delta^2 C^4.\]

For any \(t \in (0, \tau)\), we have
\[(\log L(t))^2 = -\int_0^t \vartheta'(s, \mathcal{X}) \, dW(s) - \int_0^t \frac{1}{2} \left| \vartheta(s, \mathcal{X}) \right|^2 \, ds \leq 2 \int_0^t \left| \vartheta'(s, \mathcal{X}) \right|^2 \, ds + 2 \int_0^t \frac{1}{2} \left| \vartheta(s, \mathcal{X}) \right|^2 \, ds \]

It follows from \(t \leq \tau < 2\delta\) that
\[\int_0^t \frac{1}{2} \left| \vartheta(s, \mathcal{X}) \right|^2 \, ds \leq \frac{t}{2} C^2 \leq \delta C^2,\]

and therefore
\[\mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \left( \log L(t) \right)^2 \right] \leq 2 \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \left| \int_0^t \vartheta'(s, \mathcal{X}) \, dW(s) \right|^2 \right] + 2 \delta^2 C^4.\]

Finally, the Burkholder-Davis-Gundy Inequality gives
\[\mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \left| \int_0^t \vartheta'(s, \mathcal{X}) \, dW(s) \right|^2 \right] \leq 4 \mathbb{E} \left[ \int_0^\tau \left| \vartheta'(s, \mathcal{X}) \right|^2 \, ds \right] \leq 4 \mathbb{E} [\nu C^2] \leq 8 \delta C^2,\]
and (B.17) follows.

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