On gravitational radiation in Brans-Dicke gravity

Bertrand Chauvineau

Université Côte d’Azur, OCA, CNRS, Lagrange, France
(e-mail : Bertrand.Chauvineau@oca.eu)

Abstract

Brans-Dicke gravity admits spherical solutions describing naked singularities rather than black holes. Depending on some parameters entering such a solution, stable circular orbits exist for all radius. One argues that, despite the fact that the naked singularity is an infinite redshift location, the far observed orbital motion frequency is unbounded for an adiabatically decreasing radius. This is a salient difference with General Relativity, and the incidence on the gravitational radiation by EMRI systems is stressed. Since this behaviour survives the $\omega \to \infty$ limit, the possibility of such solutions is of utmost interest in the new gravitational wave astronomy context, despite the current constraints on scalar-tensor gravity.

KEY WORDS : scalar-tensor gravity, naked singularity, gravitational radiation

I - Introduction

The recently born gravitational wave (GW) astronomy opens unprecedented ways to observe our Universe. The observation of black hole (BH) merging events, like GW150914 and GW151226 [1][2], can only be achieved by the means of GW observatories, as long as no electromagnetic counterpart is expected. On the other hand, the templates that serve to modelise the time evolution of the signal depend on the gravity theory that is supposed to govern the evolution of the source, especially in the strong field case, like BH merging. As a consequence, the GW astronomy is a very specific way for testing gravity theories in the strong field regime.

So far, the observed GW events are well interpreted in the usual General Relativity (GR) framework. Indeed, GR allows to interpret both GW150914 and GW151226 as the final inspiraling phases and merging of BH pairs, including the ringdown emission by the resulting BH. So far, solar system and binary pulsar tests strongly constrain alternatives to GR in the weak (and intermediate) field regime(s) [3]. The GW events analysis then strengthens our confidence in GR, grounded in long standing tests. Nevertheless, many attempts to quantize gravity (in a unified sheme of interactions or not) return a scalar as a partner of the metric in the effective gravitational sector of the theory [4][5][6]. This promotes Brans-Dicke (BD) and scalar-tensor (ST) theories as serious alternatives to GR, despite the fact that there is no experimental evidence of such a scalar so far. The credibility of such theories is reinforced by the fact that BD/ST gravity includes GR in some limit cases [3], and also by the fact that the Universe’s expansion makes a large family of ST theories to asymptotically behave like GR [7][8].
In any case, it is worth trying other astrophysical scenarios to interpret the observed GW events, in both usual GR and alternative gravity frameworks. The interest of such attempts is strengthened by the fact that GWs are expected to reveal the existence of objects that could escape traditional (electromagnetic) observational means. This justifies a systematic study of the GW emission properties of speculative objects. In these lines, the link between the ringdown phase and the existence of an event horizon is not fully obvious. Some authors argue that the ringdown phase is not the signature of a BH horizon, but rather of a “photonic sphere”, that are associated to wormhole structures as well, for instance [9][10]. Alternative GW sources, like the collision of ultracompact boson stars [10], or the production of a gravastar as the results of the merging of two ultracompact objects [11], have also been considered. Some authors have pointed out the lack of understanding of merging phenomena in alternative theories [12].

The aim of this paper is to spot a salient difference between GR and BD gravity. Unlike GR, BD admits spherical solutions that exhibit a naked singularity (NakS), rather than BH, structure. Depending on a specific parameter related to the NakS structure, stable circular orbits exist for all radius, with the specific property that the corresponding far observed orbital frequency is unbounded. This last points drastically departs from GR, in which (1) the orbital frequencies are bounded by the frequency at the innermost stable circular orbit (ISCO), and (2) frequencies are frozen at the horizon. Besides, one shows that these features exist for all \( \omega \), and survive the \( \omega \to \infty \) limit. The possibility of such a scenario is then not ruled out by current constraints on gravity. Thence, if NakS turn out to exist in our Universe, these features result in a specific signature that carries information of the utmost relevance on the very nature of gravity.

Notations

Unless otherwise stated, the spherical metrics hereafter considered will be written with the help of spherical like coordinates

\[
\text{ds}^2 = g_{00} dt^2 + g_{11} dr^2 + g_{22} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right).
\]  

(0.1)

The geodesic equations of the metric (0.1) read (the prime representing \( r \) derivation)

\[
g_{00} \frac{dt}{dr} = -E
\]  

\[
2 \frac{d}{dr} \left( g_{11} \frac{dr}{dr} \right) = \left( \frac{dt}{dr} \right)^2 g'_{00} + \left( \frac{dr}{dr} \right)^2 g'_{11} + \left( \frac{d\theta}{dr} \right)^2 g'_{22}
\]  

\[
g_{22} \frac{d\theta}{dr} = C
\]

where \( E \) and \( C \) are integration constants, and \( p \) is an affine parameter. These equations admit the first integral

\[
g_{00} \left( \frac{dt}{dr} \right)^2 + g_{11} \left( \frac{dr}{dr} \right)^2 + g_{22} \left( \frac{d\theta}{dr} \right)^2 = -\sigma^2
\]  

(0.3)

where \( \sigma^2 \) is a (positive) constant. In the massless particle (usually referred to as the photonic) orbit case, one has \( \sigma = 0 \). In the timelike orbit case, corresponding to massive particles, \( \sigma^2 \) is positive and will be normalized to unity by choosing the proper time as the affine parameter (and renormalizing \( E \) and \( C \) accordingly).
In the case where isotropic coordinates are chosen, the metric functions satisfy
\[ g_{22} = r^2 g_{11}. \] (0.4)

II - ST & BD gravity

In this section, we first remind the general setting of ST gravity. We then specify to the Brans’s Class I spherical BD solution, presented in a form adapted to our purpose. The detailed properties of circular orbits, that will be relevant for the discussion on far observed frequencies discussed in the next section, are then pointed out.

II-1 - General Relativity and scalar-tensor equations

The GR field equations are derived from the action \[ S_{GR}[g, \Psi] = \frac{1}{16\pi G} \int \sqrt{-g} (R - 2\Lambda) d^4x + S_{\text{mat}}[g, \Psi], \] (0.5)
where \( S_{\text{mat}} \) represents the matter sector of the theory, \( \Psi \) the matter fields, and \( \Lambda \) the cosmological constant. The gravitational sector, ie \( \frac{1}{16\pi G} \int \sqrt{-g} (R - 2\Lambda) d^4x \) (the Einstein-Hilbert action, here completed with a cosmological term), depends on the spacetime metric only. Varying (0.5) w.r.t. the metric yields the GR Einstein’s equation
\[ R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}, \] (0.6)
The variation w.r.t. \( \Psi \) yields the matter equations.

The lagrangian of the ST theory reads, in Jordan’s representation \[ S_{ST}[g, \Psi] = \frac{1}{16\pi} \int \left( \Phi R - \frac{\omega(\Phi)}{\Phi} g^{ab} \partial_a \Phi \partial_b \Phi - 2\Phi U(\Phi) \right) \sqrt{-g} d^4x + S_{\text{m}}[g, \Psi]. \] (0.7)
In this representation, the gravitational sector (the r.h.s. discarding the \( S_{\text{m}} \) term) involves both the spacetime metric and a scalar field \( \Phi \). On the other hand, the scalar does not enter the matter sector. This justifies our representation choice: test particles then describe geodesics, even in the vacuum case\(^1\). Each specific choice of the scalar functions \( \omega(\Phi) \) and \( U(\Phi) \) defines a specific ST theory. Varying (0.7) w.r.t. the metric yields the ST Einstein’s equation \[ \Phi \left( R_{ab} - \frac{1}{2} R g_{ab} + U g_{ab} \right) = 8\pi T_{ab} + \frac{\omega}{\Phi} \left( \partial_a \Phi \partial_b \Phi - \frac{1}{2} g_{ab} g^{cd} \partial_c \Phi \partial_d \Phi \right) + \nabla_a \partial_b \Phi - g_{ab} \Box \Phi. \] (0.8)
Varying (0.7) w.r.t. the scalar field, and eliminating the Ricci scalar \( R \) thanks to the contracted version of (0.8), one gets the scalar equation
\[ \Box \Phi = \frac{8\pi T - \omega \partial_c \Phi \partial^c \Phi - 2\Phi (U - \Phi U')}{2\omega + 3}. \] (0.9)

\(^1\)One could split \( S_{\text{m}} \) in two parts: \( S_{\text{m}} = S_{\text{m(active)}} + S_{\text{m(test)}} \), where \( S_{\text{m(test)}} \) describes the physics of some (the ones the behaviour of which we are interested in) test particles, and \( S_{\text{m(active)}} \) includes all the matter that effectively curves spacetime. Setting \( S_{\text{m(active)}} = 0 \) then leads to the vacuum equations of the theory. But the test particles’ behaviour remain fully determined by \( S_{\text{m(test)}} \). Supposing that \( S_{\text{m}} \) is globally scalar independent unambiguously results in the fact that test particles describe geodesics, even if vacuum solutions are considered.
where \( \omega' \equiv \frac{d\omega}{d\Phi} \) and \( U' \equiv \frac{dU}{d\Phi} \). The special case \( U' = 0 \), but with \( U \neq 0 \), corresponds to ST theory with a cosmological constant. The special case \( \omega' = 0 \) corresponds to the BD theory (with a potential \( U \)). The original version of the BD theory was formulated in the potential free case \([15]\). Let us point out that requiring \( \omega(\Phi) > -3/2 \) ensures the Ostrogradskian stability of the theory \([16]\). In fact, we will even be a bit more restrictive in this paper, just considering theories with \( \omega > 0 \). This choice will simplify some aspects of the coming discussions, and will prove to be adapted to the purpose of the paper.

If the scalar field \( \Phi \) is replaced by a constant \( \Phi_0 \), the equation (0.8) reduces to the GR’s equation (0.6) with \( G = 1/\Phi_0 \) and \( \Lambda = U(\Phi_0) \). But of course, to get an ST solution, the equation (0.9) has to be satisfied too. For finite values of \( \omega \) and \( \omega' \), the existence of a solution with a constant scalar then requires \[ \Phi_0 \left[ U(\Phi_0) - \Phi_0 U'(\Phi_0) \right] = 4\pi T. \] (0.10)

Thence, unless the relation (0.10) occurs, the ST theory does not admit spacetimes with constant scalar \(^2\).

II-2 - Brans’s Class I spherical solution

The complete stationary solution of the BD equations with no potential \((U = 0)\) is known in the spherical case \([15][17]\). It is made of four families, usually referred to as Brans’s Class I, II, III and IV solutions. We specify here to the Class I family. It can be written

\[
\Phi = \left( \frac{r - k}{r + k} \right)^s \] (0.11)

\[
ds^2 = -\left( \frac{r - k}{r + k} \right)^{-s+2\lambda} \, dt^2 + \left( \frac{r + k}{r} \right)^4 \left( \frac{r - k}{r + k} \right)^{2-s-2\lambda} \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)
\]

where \( \lambda \) is defined by

\[
\lambda = \frac{1}{2} \sqrt{4 - (3 + 2\omega) s^2} \in [0, 1].
\] (0.12)

In (0.11), \( \epsilon \) is a sign, and \( s \) and \( k \) are two arbitrary constants, but with \( s \) satisfying

\[
|s| \leq \frac{2}{\sqrt{3 + 2\omega}}
\] (0.13)

for (0.12) to get sense.

\(^2\)Let us remark that the possibility for GR like solutions with \( G = 1/\Phi_0 \) in the case where (0.10) occurs does not contradict that the effective local gravitational constant reads \( G_{eff} = (2\omega_0 + 4) (3 + 2\omega_0 + 3)^{-1} / \Phi_0 \) (and not simply \( G_{eff} = 1/\Phi_0 \)) where \( \omega_0 = \omega(\Phi_0) \). Indeed, \( G_{eff} \) results from (1) the (00) Einstein’s BD equation (with no potential) in the weak field, slow motion and quasi-stationary approximations, (2) the elimination of \( \Box \Phi \) thanks to the scalar BD equation, (3) the reduction of the perfect fluid stress tensor to \( T_{\mu\nu} = \rho \delta_{\mu0} \delta_{\nu0} \) under the previous hypotheses, leading to the Poisson like equation \( \triangle V = -4\pi [2\omega_0 + 4] (3 + 2\omega_0 + 3)^{-1} \Phi_0^{-1} \rho \), where \( V \) is the Newtonian potential \([20]\). But in the same time, the equation (0.10) reduces to \( \rho = 0 \) (since there is no potential). In this condition, \( \Delta V = 0 \), and defining the effective Newton’s constant is meaningless.

\(^3\)The correspondance between this form and the form given in \([17]\) reads \( (s, \epsilon) = \left( \frac{2}{s}, \frac{C + 2}{s} \right) \) and \( (l, C) = \frac{(2.2a)}{-s+\epsilon\sqrt{4-(3+2\omega)s^2}} \) (notations of ref \([17]\), with \( \lambda \) replaced by \( l \)).
The solutions \((s, k, \epsilon)\) and \((-s, -k, -\epsilon)\) being the same, let us choose \(\epsilon = +\) once for all. We will also consider that \(k \neq 0, k = 0\) being the Minkowski spacetime.

Because of its nice geometrical interpretation, it is worth defining the areal radius, that reads, from (0.11) (with \(\epsilon = +\))

\[
R(r) = r \left(\frac{r + k}{r}ight)^2 \left(\frac{r - k}{r + k}\right)^{1 - 2\lambda}. \tag{0.14}
\]

For large values of \(r\), the \(g_{00}\) component of (0.11) expands as

\[
g_{00} = -1 + \frac{2(2\lambda - s)k}{r} + O\left(\frac{1}{r^2}\right). \tag{0.15}
\]

Thence, if it is positive, the quantity

\[
m = (2\lambda - s)k \tag{0.16}
\]

is the newtonian mass of the gravitational field, as measured by a far observer.

The solution (0.11) is the BD counterpart of the GR Schwarzschild solution. However, while the physical meaning of the Schwarzschild solution (as well as its Kruskal maximal extension) is well understood in terms of BH spacetime, the interpretation of the Brans’s solution (0.11) is more complex, generically describing NakS or wormhole spacetimes depending on the \((s, \epsilon)\) (or on Brans’s \((l, C)\)) parameters [18][19][20]. For any (finite) \(\omega\), the minimally scalarized case \(s = 0\) corresponds to the GR Schwarzschild’s metric in isotropic coordinates. (If \(U = 0\), Schwarzschild is indeed a peculiar BD vacuum solution whatever the finite value of \(\omega\), in accordance with (0.10).)

In the following, we specify to positive mass NakS like solutions. From (0.16), the positive mass condition reads

\[
(2\lambda - s)k > 0. \tag{0.17}
\]

Coming from \(r = +\infty\) regions, the first singularity one meets is located at \(r = |k|\). The condition (0.17) leads to

\[
g_{00}(r = |k|) = 0 \tag{0.18}
\]

which means that this singularity is an infinite redshift location. Discarding the Schwarzschild case, its areal radius is either zero or infinite. The infinite case corresponds to a wormhole like structure (that corresponds to the case discussed in [21], section II-A-3). Let us discard this case and demand the areal radius to be zero. From (0.14), this requires

\[
(2 - s - 2\lambda)k > 0. \tag{0.19}
\]

For \(k > 0\), and thanks to \(\lambda \in [0, 1]\), (0.17) and (0.19) are fulfilled iff

\[
s \in \left[\frac{-2}{\sqrt{3} + 2\omega}, 0\right] \cup \left[\frac{2}{\sqrt{2} + \omega}, \frac{\sqrt{2}}{\sqrt{2} + \omega}\right]. \tag{0.20}
\]
The corresponding $\lambda$ intervals are respectively
\[
\lambda(s < 0) \in [0, 1]
\]
\[
\lambda(s > 0) \in \left[ \frac{1}{\sqrt{2(2+\omega)}}, \frac{1+\omega}{2+\omega} \right].
\]  
(0.21)

As this will prove to be sufficient for the purpose of this paper, we will just consider solutions with $k > 0$. (0.20) is then a necessary and sufficient condition for the solution to represent a positive mass NakS spacetime (given that we have restricted our study to BD with $\omega > 0$). Accordingly, the spacetime region with $r > k$ is considered.

II-3 - Circular orbits in a Brans’s NakS spacetime

Massless ($\sigma = 0$) circular orbital radii solve, from (0.3) and the second of (0.2)
\[
\left( \frac{g_{00}}{g_{22}} \right)' = 0.
\]  
(0.22)

Replacing by the metric functions entering (0.11) with $\epsilon = +$, one gets the circular radius as a function of $s$. Since $r > k > 0$, it reads, using the mass (0.16)
\[
r_0(s) = \frac{2\lambda + \sqrt{4\lambda^2 - 1}}{2\lambda - s} m.
\]  
(0.23)

If the areal radius (0.14) is prefered to $r$, it reads
\[
R_0(s) = \frac{m}{2\lambda - s} \left( \frac{2\lambda + \sqrt{4\lambda^2 - 1}}{2\lambda - s} \right)^2 \left( \frac{2\lambda + \sqrt{4\lambda^2 - 1} - 1}{2\lambda + \sqrt{4\lambda^2 - 1} + 1} \right)^{1 - \frac{s}{2} - \lambda}.
\]  
(0.24)

Let us now consider massive circular orbits ($\sigma = 1$). Eliminating $(dt/dp)^2$ from (0.3) and the second of (0.2), and eliminating proper time derivatives thanks to the third of (0.2), one gets the Binet equation
\[
2 \frac{d^2 r}{dt^2} + \left[ \ln \left| \frac{g_{00}}{r^2 g_{22}} \right] ' \right] \left( \frac{dr}{dt} \right)^2 = r^2 \left[ \left( \frac{g_{22}}{C_2^2 + 1} \right) \left( \ln |g_{00}| \right) ' - \left( \frac{g_{22}}{C_2^2 + 1} \right) \left( \ln |g_{00}| \right) \right] (r = r_{circ}) = 0.
\]  
(0.25)

where one has used (0.4). The radius $r_{circ}$ of a circular orbit with areal constant $C$ then solves
\[
\left[ \left( \frac{g_{22}}{C_2^2 + 1} \right) \left( \ln |g_{00}| \right) ' - \left( \frac{g_{22}}{C_2^2 + 1} \right) \left( \ln |g_{00}| \right) \right]_{(r = r_{circ})} = 0.
\]  
(0.26)

Let us now consider an orbit close to the circular one
\[
r = r_{circ} + \rho \quad \text{with} \quad |\rho| << r_{circ}.
\]  
(0.27)

Linearizing (0.25) w.r.t. $\rho$, one gets
\[
\frac{d^2 \rho}{dt^2} = \frac{r_{circ}^2}{2} \left[ \left( \frac{g_{22}}{C_2^2 + 1} \right) \left( \ln |g_{00}| \right) ' \right]_{(r = r_{circ})} \rho + \frac{C'' - C}{C^3} r_{circ}^2 \left[ g_{22} \left( \ln |g_{00}| \right) ' \right]_{(r = r_{circ})}.
\]  
(0.28)
\( C' \) being the areal constant on the perturbed orbit. Thence, a radius that solves
\[
\left[ (\ln g_{22}') - \left( \frac{g_{22}}{C'^2} + 1 \right) (\ln |g_{00}|)' \right]' = 0
\] (0.29)
separates stable circular orbits from unstable ones (thence generalizes the GR ISCO). Eliminating \( C^2 \) thanks to (0.26), one gets
\[
\left( \frac{\ln |g_{22}|'}{g_{22} (\ln |g_{00}|)'} \right)' = 0.
\] (0.30)
Replacing by the metric functions entering (0.11), one gets a quartic equation on \( r \), that achieves the form
\[
\left( \frac{r}{k} - H_{\varepsilon_r} \right)^2 = H^2_{\varepsilon_r} - 1
\] (0.32)
with
\[
H_{\varepsilon_r} = 3\lambda + \frac{s}{2} + \varepsilon_r \frac{\sqrt{5}}{\sqrt{4\lambda^2 - 1}} \sqrt{\left( 5\lambda^2 - \frac{s}{2} \right)^2 - 5 + \frac{1 - \lambda^2}{3 + 2\omega}}.
\] (0.33)
A first condition of existence of solutions requires the r.h.s. of (0.31), or equivalently the argument of the square root entering (0.33), to be positive. A second condition reads \( H^2_{\varepsilon_r} \geq 1 \), from (0.32). Finally, for a given \( H_{\varepsilon_r} \) satisfying \( H^2_{\varepsilon_r} \geq 1 \), only one of the two solutions of (0.32) is the product of the two roots being \( k^2 \).

II-4 - The large \( \omega \) case
Motivated by the current experimental context, let us now examine more specifically the large \( \omega \) case. From (0.13), \( s \to 0 \) when \( \omega \to \infty \). On the other hand, from (0.12), \( \lambda \) can achieve any value in \([0, 1]\), even in the limit \( \omega \to \infty \). Let us remark that, for both signs of \( s \), and discarding Schwarzschild (NakS solutions being considered), the positive mass NakS condition (0.21) becomes \([0, 1]\) in the limit, i.e. the full set of possible \( \lambda \) with non-zero mass. This means that the Brans’s Class I solution generically describes a positive mass (remind \( k > 0 \)) NakS spacetime for large \( \omega \), and that wormhole like solutions no longer exist in the limit.

The relevant limit equations are: (1) the metric, coming from (0.11), that reads (with \( \varepsilon = + \))
\[
ds^2 = - \left( \frac{r - k}{r + k} \right)^{2\lambda} dt^2 + \left( \frac{r + k}{r - k} \right)^4 \left( \frac{r - k}{r + k} \right)^{2 - 2\lambda} \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)
\] (0.34)
(2) the massless circular radius, coming from (0.24), that reads
\[
R_0(\lambda) = \frac{m}{2\lambda} \left( \frac{2\lambda + \sqrt{4\lambda^2 - 1} + 1}{2\lambda + \sqrt{4\lambda^2 - 1} - 1} \right)^{\frac{1}{2\lambda}} \left( \frac{2\lambda + \sqrt{4\lambda^2 - 1} - 1}{2\lambda + \sqrt{4\lambda^2 - 1} + 1} \right)^{1-\lambda}
\] (0.35)
(3) the "ISCO" radius equation, coming from (0.32) and (0.33), that reads
\[ \left( \frac{r}{k} - 3\lambda - \epsilon_r \sqrt{5\lambda^2 - 1} \right)^2 = \left( 3\lambda + \epsilon_r \sqrt{5\lambda^2 - 1} \right)^2 - 1. \] (0.36)

The metric (0.34) is a GR solution, but filled by a massless scalar, as expected in the \( \omega \rightarrow \infty \) limit of vacuum BD context. Indeed, a massless scalar, originating from the vanishing gradient of the BD scalar, has to be added to the original stress tensor \( T_{ab} \) for the full richness of the large \( \omega \) BD (filled by \( T_{ab} \)) to be restored \[22][23\]. Explicitly, the metric (0.34) solves
\[ R_{ab} = \partial_a \varphi \partial_b \varphi, \]
with
\[ \varphi = \sqrt{2} \left( 1 - \lambda^2 \right) \ln \left| \frac{r - k}{r + k} \right|. \] (0.37)

The mass of the field, coming from (0.16), reads
\[ m = 2\lambda k. \] (0.38)

From (0.35), massless circular orbits exist for \( \lambda < 1/2 \) only.

The scalar (0.37) vanishes for \( \lambda = 1 \), as expected since (0.34) reduces to Schwarzschild then. On the other hand, the metric of the maximally scalarized case \( \lambda = 0 \), that yields \( m = 0 \) reads
\[ ds^2 = -dt^2 + \left( 1 - \frac{k^2}{r^2} \right)^2 \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right). \] (0.39)

This is the metric (2.7) of [21], which is then just one of the full set of all possible limits. Let us remark that the coordinate change
\[ u = r + \frac{k^2}{r} \] (0.40)
puts (0.39) in the following "close to Minkowski" form
\[ ds^2 = -dt^2 + du^2 + \left( u^2 - 4k^2 \right) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \] (0.41)

In some sense, the Schwarzschild solution and (0.39) are the two extreme cases that can be encountered in the \( \omega \rightarrow \infty \) limit of Brans’s solution.

It appears that (0.34)-(0.37) is nothing but the Janis-Newman-Winicour (JNW) solution [24], that precisely solves the GR equation filled by a massless scalar, up to a radial coordinate transform. Circular orbits in this metric have been studied in [25], and the BD limit equations (0.35) and (0.36) are in agreement with their results. In our notations and coordinates, massive circular orbits exist for all \( r > r_0 \) if \( 1/2 < \lambda \leq 1 \), and for all \( r \) up to the singularity if \( 0 < \lambda < 1/2 \). All the circular orbits are stable if \( \lambda < 1/\sqrt{5} \). There is an ISCO orbit for \( \lambda > 1/2 \). For \( 1/\sqrt{5} < \lambda < 1/2 \), there are two branches of orbits that separate stable from unstable orbits, that join together at the point \( \left( \lambda, \frac{\Delta}{m} \right) = \left( \frac{1}{\sqrt{5}}, 2 \left( \frac{2 + \sqrt{5}}{2} \right) ^{1/3} \right) \simeq (0.44721, 3.0758). \)

\footnote{The link between the two forms is got making first the shift \( u = R + \frac{\Delta}{m} \) (JNW notations), and then the coordinate change (0.40).}
All these results strengthen that, in agreement with [22][23], the GR filled by a massless scalar (instead of Schwarzschild) equations constitutes the relevant framework to treat the large $\omega$ vacuum BD problem, with no loss of the Full richness of large $\omega$ BD solutions.

III - Observed frequencies measured by a far observer

The fact that stable circular orbits exist in BD gravity for all areal radius is a salient difference with GR, for which such orbits exist for $R > 6m$ only ($R > 3m$ if the stability requirement is removed). Determining the radius dependence of the corresponding orbital frequency that measures a far observer is of utmost relevance. Indeed, this quantity masters a lot of observable physical phenomena originating in such orbital motions.

Since experimental data favour ST gravity with $\omega \gtrsim 4.10^4$ [3], we will take advantage of the conclusions got in the previous section to make the calculations using the metric (0.34). We will then check that the main conclusions remain valid for large, but finite, values of $\omega$, with the help of the formulae got in II-3. We then discuss the incidences of the results on the gravitational radiation by systems that can reasonably be modeled by the Brans’s solution.

III-1 - Massless circular orbits

The (proper) period on the massless circular orbit, measured by an observer at rest at a point of the orbit, reads $2 \pi R_0 (\lambda)$, since the orbit is travelled at the unit speed. The frequency measured by a far observer is affected by the Einstein effect, and reads

$$\nu_\infty (\lambda) = \frac{1}{2 \pi R_0 (\lambda)} \sqrt{-g_{00} (r = \tau_0)} - g_{00} (r = \infty)$$

$$= \frac{\lambda}{\pi m} \frac{2 \lambda + \sqrt{4 \lambda^2 - 1} - 1}{(2 \lambda + \sqrt{4 \lambda^2 - 1} + 1)^2 \left( 2 \lambda + \sqrt{4 \lambda^2 - 1} - 1 \right) 2^{\lambda-1}} \left( \frac{2 \lambda + \sqrt{4 \lambda^2 - 1} - 1}{2 \lambda + \sqrt{4 \lambda^2 - 1} + 1} \right)^{2^{\lambda-1} - 1}$$

since $g_{00} (r = \infty) = -1$. For $\lambda = 1$, one recovers the GR value $(6 \pi \sqrt{3} m)^{-1}$, as expected. Let us define $f$ by normalizing (0.42) by this GR value

$$f (\lambda) \equiv \frac{\nu_\infty (\lambda)}{\nu_\infty (\lambda = 1)}$$

$$= 6 \sqrt{3} \lambda \frac{2 \lambda + \sqrt{4 \lambda^2 - 1} - 1}{(2 \lambda + \sqrt{4 \lambda^2 - 1} + 1)^2} \left( \frac{2 \lambda + \sqrt{4 \lambda^2 - 1} - 1}{2 \lambda + \sqrt{4 \lambda^2 - 1} + 1} \right)^{2^{\lambda-1} - 1}.$$  

For $\lambda \rightarrow 1/2$, $f$ gets the value

$$f \left( \lambda \rightarrow \frac{1}{2} \right) \rightarrow \frac{3\sqrt{3}}{4} \simeq 1.299.$$  

Interestingly, this value is not infinite, despite the fact that the massless orbital circumference is $2 \pi R (\lambda = 1/2) = 0$. The local frequency is then infinite, but it is also infinitely redshifted since the massless orbits meets the NakS, that is an infinite redshift location, for $\lambda = 1/2$. The two effects compete and finally compensate each other, resulting in the finite far observed frequency given by (0.42).
For $\lambda > 1/2$, $f$ decreases until the GR value $f(\lambda = 1) = 1$.

### III-2 - Massive circular orbits

On any circular orbit, the time needed to complete a revolution, as measured by an observer at rest at a point of the orbit, reads

$$\tau = \sqrt{-g_{00}(r)} \frac{2\pi}{d\theta/dt}.$$  \hspace{1cm} (0.45)

The frequency, as measured by a far observer, is then

$$\nu_\infty = \frac{1}{\tau} \sqrt{-g_{00}(r)} = \frac{1}{2\pi} \frac{d\theta}{dt}.$$ \hspace{1cm} (0.46)

Using the second equation of (0.2) and the metric functions entering (0.34), one gets

$$\nu_\infty(\lambda, r) = \frac{1}{2\pi} \left( \frac{2\lambda r}{m^2} \right)^{3/2} \left( \frac{2\lambda r}{m^2} - 1 \right)^{2\lambda - 1} \left( \frac{2\lambda r}{m^2} + 1 \right)^{-2\lambda - 1} \sqrt{\left( \frac{2\lambda r}{m^2} \right)^2 - 2\lambda^2 \lambda^2 + 1}.$$ \hspace{1cm} (0.47)

For $\lambda = 1$, one recovers the GR value that reads, in terms of the areal radius $R$, $\nu_\infty(\lambda = 1, r) = \left[ \frac{2\pi}{(R/m)^2} \right]^{-1}$.

In the GR case, the maximum frequency on stable orbits corresponds to the ISCO ($R = 6m$) and reads $(12\pi\sqrt{6}m)^{-1}$. For $1/\sqrt{5} < \lambda < 1$, the frequency of the $\lambda$ dependent ISCO is got inserting the relevant solution of (0.36) in (0.47). It increases when $\lambda$ decreases, and reaches the value $(7 - 3\sqrt{5})^{1/\sqrt{5}} (4\pi\sqrt{2}m)^{-1} \approx 2.197 \times (12\pi\sqrt{6}m)^{-1}$ for $\lambda = 1/\sqrt{5}$. For $\lambda = 1/2$, the frequency of the ISCO is $(12\pi\sqrt{3}m)^{-1} \approx 1.414 \times (12\pi\sqrt{6}m)^{-1}$.

For $\lambda < 1/\sqrt{5}$, (0.36) has no solution and $\lambda$-ISCO(s) no longer exist. In this case, circular orbits exist until the naked singularity (zero areal radius), and are all stable. For $1/\sqrt{5} < \lambda < 1/2$, there are unstable circular orbits, but stable circular orbits still exist close to the NakS. For $\lambda < 1/2$, (0.47) behaves, when approaching the singularity, as

$$\nu_\infty(\lambda, r \to m) \sim \frac{m}{2\lambda} \left( \frac{2\lambda r}{m} \right)^{3/2} \left( \frac{2\lambda r}{m} - 1 \right)^{2\lambda - 1} \left( \frac{2\lambda r}{m} + 1 \right)^{-2\lambda - 1} \sqrt{\left( \frac{2\lambda r}{m} \right)^2 - 2\lambda^2 \lambda^2 + 1}.$$ \hspace{1cm} (0.48)

This quantity is unbounded for all $\lambda < 1/2$. As in the massless case, the infinite local proper frequency competes with the infinite redshifting, but unlike the massless case, the local proper frequency now dominates the behaviour. As it should be, the massless case is nevertheless back using (0.48), since the massless circular orbit and the singularity meet for $\lambda = 1/2$. Indeed, for $\lambda = 1/2$, the divergent term is killed and (0.48) yields $\nu_\infty = (8\pi m)^{-1}$, in accordance with (0.44).

### III-3 - Far frequency for large, but finite, $\omega$

The fact that the far measured frequency (0.48) on stable circular orbits is unbounded for decreasing radius in the $\lambda < 1/2$ case is an appealing difference with GR, where this quantity

10
is bounded by the ISCO value $(12\pi/\sqrt{6\alpha})^{-1}$. For that reason, it is worth to check that this behaviour is not an artefact of the limit, but still occurs in BD theory, for which $\omega$ is actually finite, even if very high.

The far measured frequency is still given by (0.48), but with $d\theta/dt$ calculated from the metric functions entering (0.11) (with $\epsilon = +$). From the second of (0.2), one gets

$$
\nu_\infty (\omega, s, r) = \frac{1}{2\pi} \sqrt{-g_{00} / g_{22}},
$$

(0.49)

$$
= \frac{1}{2\pi} (r - k)^{2\lambda - 1} (r + k)^{-2\lambda - 1} \sqrt{\frac{(2\lambda - s) kr^3}{r^2 - (s + 2\lambda) kr + k^2}}.
$$

For $r \to k$, one gets the behaviour

$$
\nu_\infty (\omega, s, r \to k) \to \frac{1}{2^{2\lambda + 2\pi k} \sqrt{2\lambda - s}} \frac{2\lambda - s}{2 - s - 2\lambda} \left( \frac{r}{k} - 1 \right)^{2\lambda - 1},
$$

(0.50)

(which leads back to (0.48) for $\omega \to \infty$). The square root in front of (0.50) is defined thanks to (0.17) & (0.19). Obviously, (0.50) diverges for $\lambda < 1/2$, as (0.48) did. To be complete, let us remark that, for all positive $\omega$, the quantity

$$
8 - (7 + 5\omega) s^2 + s \sqrt{4 - (3 + 2\omega) s^2}
$$

is negative for values of $s^2$ sufficiently close to $4/(3 + 2\omega)$. This means that the square root entering (0.33) is negative then, in such a way that (0.31) has no solution. The stability of the circular orbits is then ensured whatever their radii for these values of $s$.

III-4 - Physical impacts and gravitational radiation by EMRI binaries in Brans-Dicke gravity

The main objection usually raised against NakSs is that they generate unpredictability in the parts of the Universe belonging to their causal future. Thence the requirement to hide singularities behind horizons, to preserve the part of the Universe we are belonging to from such an unwelcome behaviour (Penrose’s conjecture). However, the unpredictable behaviour attached to a (naked or hidden) singularity is deeply related to the classical character of the considered gravity theories. It is commonly believed that singular states should be removed when the quantum nature of the spacetime will be properly taken into account. In this sense, the presence of a (naked or hidden) singularity is nothing but the hallmark of the fact that one enters a region where the classical approximation of the "true" (presumably quantum) gravity theory is no longer valid.

The possibility of NakSs being accepted along these lines, discussing the potential astronomical applications of the solutions previously described makes sense. The peculiar features with respect to GR got in the previous sections result in potential differences in the expected behaviour of physical observables that originate in strong gravitational fields. Some of them could turn out as being of the utmost importance in the present astrophysical context.

For instance, the existence or not of an ISCO is expected to drastically impact the accretion disk behaviour. This point has been discussed in details in the JNW metric framework [25]. From
the results here highlighted on the large $\omega$ BD case, the conclusions got by these authors apply to ST theories that are admissible in the current experimental context. Let us also mention that similar phenomena have been observed in the framework of the q-metric (also named $\gamma$-metric), which is another kind of spacetime exhibiting NakS (while not related to ST gravity)\cite{25}\cite{26}.

The existence or not of a massless circular orbit obviously impacts geometrical optics, at least for light rays penetrating strong field regions. Indeed, in the GR framework, there is an infinite number of massless geodesics circling the ”photonic sphere”, resulting in an infinite multiplication of images (both of background and foreground objects) stuck on this sphere. This feature is lost if there is no massless circular orbit. Specifying to a large $\omega$ BD solution of mass $m$, the bending is weaker than in GR. This means that the lensing object gravitational mass would be underestimated if calculated on the ground of a given bending angle interpreted in the framework of GR instead of BD. However, let us stress that the differences occur only for orbits entering spacetime regions with $r/m$ of the order of some units. Indeed, the $\gamma_{Edd}$ Eddington parameter of $0.34$ is $=1$, meaning that geometrical optics does not differ from GR in the far regions of the metric. Therefore, it seems unlikely that this effect could significantly participate to the "missing mass" problem.

In the current astrophysical context, the most obvious impact probably concerns the new GW astronomy. The unboundedless character of the orbital, and then of the gravitational, frequency for some values of the parameter $\lambda$ has no equivalent in the GR framework, where the far measured gravitational frequency is bounded by its ISCO value $(6\pi\sqrt{6m})^{-1}$ (twice the orbital frequency). Several works have already been achieved on GW emission in the ST framework, but it is generally presupposed that the central object is a (supermassive) BH \cite{27}, or the study deals with Post-Newtonian approximation at some order \cite{28}. Considering the strong field region of a NakS like solution is clearly out of the scope of these studies, and then requires a specific approach. Taking advantage of the current experimental constraints on ST gravity, the solution \cite{34} defines the natural framework to get the frequency and amplitude of the GW emitted by a body of mass $\mu$ inspiraling a ”non rotating” BD like NakS of mass $m$, if $\mu \ll m$ (extreme mass ratio inspiraling –EMRI– approximation). The Brans’s Class I metric serving as the background solution, the orbital velocity, as measured by a local observer, reads

$$V(\lambda, r) = \sqrt{\frac{m}{r - m + \frac{m^2}{4\lambda r^2}}}.$$ \hspace{1cm} (0.51)

For a given value or the areal radius $R$, this velocity decreases when $\lambda$ decreases. This suggests that, for a given areal radius $R$, the amplitude of the GW should decrease for an increasing amplitude of the scalar \cite{37}. For $\lambda < 1/2$, (0.51) is bounded by the limit of its value when approaching the singularity, that reads $\sqrt{\lambda/(1-\lambda)}$. The EMRI scheme should also return the energy lost per revolution resulting from the GW radiation, giving the adiabatic evolution of the orbit and of the frequency. If $\lambda > 1/2$, one could expect a behaviour qualitatively similar to GR, with and inspiraling phase followed by a plunge, the differences just entering the numerical

\footnote{Let us remind that, in ST gravity, for the mass $\mu$ body to describe a geodesics of the mass $m$ spacetime not only requires $\mu \ll m$, but also the mass $\mu$ body self gravitational effects to be negligible.}

\footnote{Indeed, $\left(\frac{\partial (v^2)}{\partial \lambda} \right)_R = \left(\frac{\partial (v^2)}{\partial r} \right)_{\lambda} + \left(\frac{\partial (v^2)}{\partial \lambda} \right)_r > 0$ since $\left(\frac{\partial (v^2)}{\partial r} \right)_{\lambda} < 0$ and $\left(\frac{\partial (v^2)}{\partial \lambda} \right)_r > 0$ from (0.51), and $\left(\frac{\partial v}{\partial r} \right)_R < 0$ from the $s = 0$ version of (0.13).}
details. If $\lambda < 1/\sqrt{5}$, one could expect an inspiraling phase with unbounded frequency (as long as only classical gravity, and no non gravitational phenomena, enter the game). If $1/\sqrt{5} < \lambda < 1/2$, a plunge phase should occur after an inspiraling one, as in GR, but unlike GR, a new inspiraling phase could occur at higher frequency and with higher amplitude when the small body enters the inner stable circular orbits region. In any cases, the process ends up with the "NakS hitting".

Since "NakS hitting" actually means that the small body enters a quantum gravity region, it sounds reasonable to expect that this phase should be encoded, in some way, in the corresponding part of the GW signal, offering an opportunity to observe quantum gravity at work. However, as long as the falling body remains far enough from the quantum region, classical considerations should still be relevant. A rough estimate suggests that the characteristic dimension of the quantum gravity region should be $\delta \sim l_P (m/m_{Pl})^{1/3} \sim (3.3 \times 10^{-28} \text{ m}) (m/M_{Sun})^{1/3}$, where $l_P$ and $m_{Pl}$ are the Planck length and mass, and $M_{Sun}$ the mass of the Sun. Comparing to the size $R$ of the critical orbits entering the figure 1, all of the order of $m$, ie of $R \sim (1.5 \times 10^3 \text{ m}) (m/M_{Sun})$, one gets

$$\frac{\delta}{R} \sim (2 \times 10^{-31}) \left( \frac{m}{M_{Sun}} \right)^{-2/3}.$$  \hspace{1cm} (0.52)

Thence, for astrophysical expected masses, the quantum phenomena should enter the game long after the falling body enters the internal stable orbits region for $1/\sqrt{5} < \lambda < 1/2$.

IV - Conclusion

The ability of solar system and classical tests to separate ST from GR decreases for increasing $\omega$ [3]. In the case where the classical sector of gravity in our Universe is of ST nature but with $\omega >$ (say) $10^{10}$, this makes such tests unlikely to unveil the ST nature of gravity in a foreseeable future.

Much could change thanks to the new GW astronomy. Indeed, one has argued in the present paper that if BD like NakS solutions do exist in our Universe (as primordial objects, for instance), the GW signal generated by a small but massive inspiraling body could drastically differ from GR models, in a way not depending on how large $\omega$ could be. This could result in very specific ways to learn on the deep nature of gravity.

This should stimulate specific studies of GW emission by a massive body orbiting the Brans like NakS solution. In the EMRI case, the Brans's solution can serve as the background metric and scalar, in which the small but massive body moves. Better: as argued in [22][23], a generic way to tackle a $T_{ab}$ filled BD (and, to some extent, ST, see [22]) problem in the large $\omega$ case is to consider GR instead, but filled by $T_{ab}$ plus an arbitrary massless scalar field. The inclusion of this scalar, of finite amplitude, is a necessary condition to restore the full $T_{ab}$ filled large $\omega$ BD richness. This point has been more specifically illustrated in the vacuum case in the present paper. Because of the current experimental constraints on gravity theories, this approach is a natural way to tackle the binaries evolution and radiation problem, especially in the EMRI case, since an exact analytical BD solution can be used as the starting point then. This is an alternative approach to existing works in ST gravity [27][28], which in any case are not adapted to the strong field NakS case.

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