Scattering state and bound state of scalar field in Schwarzschild spacetime: Exact solution

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ABSTRACT: The main aim of this paper is twofold. (1) Exact solutions of a scalar field in the Schwarzschild spacetime are presented. The exact wave functions of scattering states and bound-states are presented. Besides the exact solution, we also provide explicit approximate expressions for bound-state eigenvalues and scattering phase shifts. (2) By virtue of the exact solutions, we give a direct calculation for the discontinuous jump on the horizon for massive scalar fields, while in literature such a jump is obtained from an asymptotic solution by an analytic extension treatment.

KEYWORDS: Exact solution, Schwarzschild spacetime, Scalar field, Scattering state, Bound state

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1 Introduction

A massive scalar field $\Phi$ with mass $\mu$ in the background of the Schwarzschild spacetime,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2,$$

is described by the scalar equation [1]

$$\left(\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^\nu} - \mu^2\right)\Phi = 0.$$  \hspace{1cm} (1.2)

In this paper, we present the exact solutions of bound states and scattering states. For bound states, we solve the exact bound-state wave function and present an exact implicit expression and an asymptotic expression of the bound-state eigenvalue. For scattering states, using the Eddington-Finkelstein coordinates, we solve the exact solutions of the scattering wave function; this allows us to calculate the discontinuous jump of the wave function on the horizon exactly.

On the horizon, there is a discontinuous jump of the wave function. The calculation of the Hawking radiation relies on the magnitude of the jump. The calculations given by Hawking [2] and Damour and Ruffini [3] are based on an analytic extension of asymptotic wave functions. Concretely, they construct an asymptotic inner-horizon wave function by an analytic extension of an asymptotic outer-horizon wave function. In this paper, instead of the analytic extension treatment, starting from an exact wave function obtained under the Eddington-Finkelstein coordinate rather than an approximate asymptotic one, we calculate the discontinuous jump directly.

In black hole theory, the study of scattering plays an important role [1, 4]. There are many studies on scattering, such as the asymptotic tail [5] and complex angular momenta of scalar scattering [6]. The absorption cross section of regular black holes which have event horizons but not singularities is discussed [7]. Many approximate methods are developed, such as the phase-integral method for scalar scattering [8], the propagation of a massive vector field [9], massive Dirac field scattering [10], the absorption cross section for scalar scattering [11], massive spin-half scattering [12], the WKB approximation for massive Dirac field scattering [13] in the Schwarzschild spacetime, a massive scalar scattering in the Reissner-Nordström spacetime [14], massless planar scalar waves scattered by a charged nonrotating black hole [15], scattering by a deformed non-rotating black hole [16], and massless scalar scattering by a Kerr black hole [17]. Scattering of spin fields and vector fields in curved spacetime is also studied: the analogue of the Mott formula for scattering in...
a Coulomb background and in the Dirac scattering by a black hole [18], massive spin-2 fluctuations of Schwarzschild and slowly rotating Kerr black holes [19], the internal stationary state of a black hole for massless Dirac fields [20], the quasinormal modes of electromagnetic and gravitational perturbations of a Schwarzschild black hole in an asymptotically anti-de Sitter spacetime [21], and the quasinormal mode frequencies for the massless Dirac field in Schwarzschild-AdS spacetime [22]. Scattering on arbitrary dimensional black holes and on black holes with a cosmological constant is considered [23]. Scattering between two black holes is numerically studied [23]. The scalar field perturbations of the 4 + 1-dimensional Schwarzschild black hole immersed in a Godel universe by the Gimon-Hashimoto solution is described [24]. Scattering method can be used in the calculation of the Hawking radiation. A systematic scattering method for the Hawking radiation is developed by Damour and Ruffini [3, 25]. The Hawking radiation of a Reissner-Nordström-de Sitter black hole [26], the scalar particle Hawking radiation of a BTZ black hole [27], the charged Dirac particle Hawking radiation of the Kerr-Newman black hole [28], and the distribution for particles emitted by a black hole [29] are discussed by the Damour-Ruffini method. The Dirac particle Hawking radiation of the Kerr black hole [30] and of the BTZ black hole [31] are calculated by the WKB approximation. The Hawking radiation of acoustic black holes is discussed [32]. The renormalized expectation values $\langle T_{ab} \rangle$ of the relevant energy-momentum tensor operator of a massless scalar field in the Schwarzschild spacetime is calculated [33]. Some exact solutions are also obtained. The analytical solution of the Regge-Wheeler equation and the Teukolsky radial equation is obtained in Ref. [34]. In Ref. [35], using the truncation condition of the confluent Heun function, the authors calculate resonant frequencies for a charged scalar field in a dyonic black hole background, and the asymptotic form of the scattering wave function and Hawking radiation are presented. An exact solution of the Klein-Gordon equation in Kerr-Newman spacetime is calculated, but in which the scattering and bound-state boundary condition are not taken into account [36].

The scattering of a scalar field on the Schwarzschild spacetime can also be dealt with by the integral equation method in which the scattering phase shift can be given explicitly [37]. The method used in the present paper also applies to scalar fields in the Reissner-Nordström spacetime [38].

In section 2, the equation of a scalar field in the Schwarzschild spacetime and the boundary condition are given. In section 3, as a key step, we convert the scalar field equation in the Schwarzschild spacetime to a confluent Heun equation. In section 4, we provide an exact solution of the bound-state wave function and an explicit asymptotic expression for the bound-state eigenvalue; the exact solution of the bound-state eigenvalue will be given in section 6. In section 5, an exact solution of the scattering wave function is given. Moreover, we also give an explicit expression of phase shift under the weak-field approximation. In section 6, we give an exact implicit expression of the bound-state eigenvalue. In section 7, we consider the jump condition of the wave function on the horizon and compare our result of the discontinuous jump on the horizon with the Hawking and Damour-Ruffini treatments. The conclusions are summarized in section 8.
2 Scalar field in Schwarzschild spacetime

2.1 Field equation

The Schwarzschild spacetime is spherically symmetric, so we can perform a partial-wave expansion

$$\Phi(x^\mu) = \sum_{l=0}^{\infty} (2l + 1) e^{-i\omega t} P_l(cos \theta) \phi_l(r),$$

where \(\phi_l(r)\) is the radial wave function satisfying the radial equation

$$\frac{1}{r^2} \left(1 - \frac{2M}{r} \right) \frac{d}{dr} \left(1 - \frac{2M}{r} \right) \frac{d}{dr} + \omega^2 - \left(1 - \frac{2M}{r} \right) l(l + 1) \phi_l(r) = 0, \quad r \geq 2M. \tag{2.2}$$

There are two singularities at \(r = 2M\) and \(r \to \infty\) in the radial equation for \(r \in [2M, \infty)\). \(r = 2M\) is the horizon of the Schwarzschild spacetime and \(r \to \infty\) is the natural boundary of space.

2.2 Boundary condition

In the problem, we need to impose two boundary conditions at \(r = 2M\) and \(r \to \infty\), respectively. The boundary condition at the singular point \(r = 2M\) of the Schwarzschild spacetime is

$$|\phi_l(2M)| < \infty, \tag{2.3}$$

i.e., \(|\phi_l(r)|\) must be finite at the singular point [39].

Furthermore, the boundary condition at \(r = 2M\) is the asymptotic solution of the radial equation (2.2) at \(r \to 2M\) [3, 39–41]. That is, \(\phi_l(r)|_{r=2M} = \phi_l^{2M}(r)\), where \(\phi_l^{2M}(r)\) is the asymptotic solution of the radial equation at \(r \to 2M\). The asymptotics of the radial equation (2.2) at \(r \to 2M\) is

$$\left[\left(1 - \frac{2M}{r} \right) \frac{d}{dr} \left(1 - \frac{2M}{r} \right) \frac{d}{dr} + \omega^2 \right] \phi_l(r) \to ^{r \to 2M} 0. \tag{2.4}$$

The solution of this asymptotic equation reads

$$\phi_l(r) \to ^{r \to 2M} e^{\pm i\omega r_*}, \tag{2.5}$$

where \(r_* = r + 2M \ln \left|\frac{r}{2M} - 1\right|\) is the tortoise coordinate, i.e., \(\phi_l^{2M}(r) = e^{\pm i\omega r_*}\). This gives the boundary condition at \(r = 2M\). The modulus \(|\phi_l(2M)|\) is finite just as required by the boundary condition (2.3). It should be emphasized that though the modulus of the wave function is finite, the wave function is still singular, because, as will be seen later, there is a jump on the phase of the wave function.

The boundary condition at \(r \to \infty\) determines the solution whether a bound state or a scattering state:

$$\phi_l(r)|_{r \to \infty} = 0, \quad \text{bound state,} \tag{2.6}$$

$$\phi_l(r)|_{r \to \infty} = \phi_l^{\infty}(r), \quad \text{scattering state,} \tag{2.7}$$
where $\phi_l^\infty (r)$ is the large-distance asymptotics of the solution of the radial equation.

Furthermore, the boundary condition at $r \to \infty$ is the asymptotic solution of the radial equation (2.2) at $r \to \infty$ [3, 39–41]. The asymptotics of the radial equation (2.2) at $r \to \infty$ is

$$
\frac{1}{r^2} \left( 1 - \frac{2M}{r} \right) \frac{d}{dr} \left[ r^2 \left( 1 - \frac{2M}{r} \right) \frac{d}{dr} + \eta^2 \right] \phi_l (r) \sim 0.
$$

(2.8)

The solution of this asymptotic equation reads

$$
\phi_l (r) \sim \frac{1}{r} e^{\pm i \eta r},
$$

(2.9)

where $\eta = \sqrt{\omega^2 - \mu^2}$.

Scattering by a Schwarzschild spacetime is essentially a kind of long-range scattering [4]. Recall that for potential scattering, the large-distance asymptotics of the solution of the radial equation, $\phi_l^\infty (r)$, is the same for all short-range potential scattering, but for long-range potential scattering [41, 43], like that in our case, $\phi_l^\infty (r)$ is determined by the potential and different potentials have different asymptotic solutions [44, 45].

3 Converting Scalar field equation to confluent Heun equation

3.1 Confluent Heun equation

The key step in solving the scalar field equation in the Schwarzschild spacetime is to convert the radial equation (2.2) to a confluent Heun equation, also called the generalized spheroidal equation [46].

By the variable substitution $z = r/M - 1$, the radial equation (2.2) can be converted to a confluent Heun equation,

$$
\frac{d}{dz} (z^2 - 1) \frac{d}{dz} y(z) + \left[ -p^2 (z^2 - 1) + 2p\beta z - \lambda - \frac{m^2 + s^2 + 2msz}{z^2 - 1} \right] y(z) = 0,
$$

(3.1)

where the parameters

$$
m = s = 2M \sqrt{-\eta^2 - \mu^2},
$$

(3.2)

$$
\beta = iM \eta (2\eta^2 + \mu^2),
$$

(3.3)

$$
p = -iM \eta,
$$

(3.4)

$$
\lambda = l (l + 1) - 8\eta^2 M^2 - 6\mu^2 M^2.
$$

(3.5)

The relation between the radial wave function $\phi_l (r)$ and the confluent Heun function $y(z)$ is

$$
\phi_l (r) = y (z) \big|_{z = r/M - 1}.
$$

(3.6)

Corresponding to the region outside the horizon, i.e., $r \in [2M, \infty)$, the range of the variable $z$ in the confluent Heun equation is $z \in [1, \infty)$. The confluent Heun equation (3.1) has two singular points, $z = 1$ and $z \to \infty$, in the region $z \in [1, \infty)$ [46]. These two singular points just correspond to the two singular points of the Schwarzschild spacetime, $r = 2M$ and $r \to \infty$. 

- 5 -
3.2 Boundary condition

The boundary condition of \( \phi_l (r) \) is then converted to a boundary condition of \( y(z) \).

The boundary condition at singular point \( r = 2M \), Eq. (2.3), becomes

\[
|y(1)| < \infty. \tag{3.7}
\]

The boundary conditions for bound states, Eq. (2.6), and scattering states at \( r \to \infty \), Eq. (2.7), then become

\[
y(z) |_{z=\infty} = 0, \quad \text{bound state}, \tag{3.8}
\]

\[
y(z) |_{z=\infty} = y^\infty (z), \quad \text{scattering state}, \tag{3.9}
\]

where \( y^\infty (z) \) is the large-distance asymptotics of the solution of the confluent Heun equation, Eq. (3.1).

4 Bound state

In this section, we solve the bound-state solution. We solve the exact bound-state wave function and present an exact implicit expression and an asymptotic expression of the bound-state eigenvalue.

4.1 Bound-state wave function

For bound states, let

\[
\eta = ik, \tag{4.1}
\]

so that for bound states \( \eta = -k^2 < 0 \) with \( k \) a real number. Then

\[
m = s = 2M \sqrt{k^2 - \mu^2}, \tag{4.2}
\]

\[
\beta = \frac{M}{k} (2k^2 - \mu^2), \tag{4.3}
\]

\[
p = kM, \tag{4.4}
\]

\[
\lambda = l (l + 1) + 8M^2 k^2 - 6M^2 \mu^2. \tag{4.5}
\]

The bound-state boundary condition is \( \phi_l (\infty) = 0 \), or, \( y(\infty) = 0 \), given by Eqs. (2.6) and (3.8).

The confluent Heun equation (3.1) with the boundary conditions (3.7) and (3.8) at \( z = 1 \) and \( z = \infty \), which corresponds to the radial equation (2.2) with the boundary conditions (2.3) and (2.6) at \( r = 2M \) and \( r \to \infty \), has a solution \( \Pi(p, \beta, z) \) satisfying \( |\Pi(p, \beta, 1)| < \infty \) and \( |\Pi(p, \beta, \infty)| = 0 \), called the radial generalized spheroidal function of \( p \)-type (RGSF) [46]:

\[
\Pi(p, \beta, z) = N (z - 1)^{(m+s)/2} (z + 1)^{(m-s)/2} e^{-p(1+z)}
\]

\[
\times \text{He}^{(a)} \left( p, -\beta + m + 1, m + s + 1, m - s + 1, \sigma: \frac{z + 1}{2} \right), \tag{4.6}
\]
where \( \text{He}^{(a)}(p, \alpha, \gamma, \delta, \sigma; z) \) is the angular confluent Heun function, \( N \) is the normalization constant, and \( \sigma = \lambda + 2p(-2\beta + m + s + 1) - m(\mu + 1) \) with the restriction
\[
m + s \geq 0, \quad m - s \geq 0, \quad p \geq 0, \quad \beta \in \mathbb{R}.
\] (4.7)

The eigenvalue of the Heun equation (3.1) is denoted as \( \lambda^{(r)}(p, \beta) \), where the superscript \( r \) stands for the radial generalized spheroidal function of \( p \)-type (RGSF). It should be noted here that the eigenvalue \( \lambda^{(r)}(p, \beta) \) is the eigenvalue of the Heun equation (3.1) rather than the eigenvalue of the radial equation (2.2).

The restrictions (4.4) and (4.7) require that \( k > 0 \). (4.8)

The bound-state wave function can be then expressed as
\[
\Pi (k, r) = N \left( \frac{r}{M} - 2 \right)^{2M\sqrt{k^2 - \mu^2}} e^{-kr} \times \text{He}^{(a)} \left( Mk, \frac{M}{k} \left( 2k^2 - \mu^2 \right) + 2M \sqrt{k^2 - \mu^2} + 1, 4M \sqrt{k^2 - \mu^2} + 1, 1, \sigma; \frac{r}{2M} \right)
\]
\[
= 2Ne^{2M\sqrt{k^2 - \mu^2} \ln(\frac{2M}{r} - 1)} e^{-kr} \times \text{He}^{(a)} \left( Mk, 1 + \frac{M}{k} \left( k + \sqrt{k^2 - \mu^2} \right)^2, 1 + 4M \sqrt{k^2 - \mu^2}, 1, \sigma; \frac{r}{2M} \right),
\] (4.9)

where \( N \) is the normalization constant and
\[
\sigma = l(l + 1) + 2M^2 (k^2 + \mu^2) + 8M^2 \left( \sqrt{k^2 - \mu^2} + \frac{k}{2} \right)^2 + 2M \left( k - \sqrt{k^2 - \mu^2} \right).
\] (4.10)

An asymptotic explicit expression of the eigenvalue \(-k^2\) will be given later in section (4.2); an exact implicit expression of the eigenvalue \(-k^2\) is given by Eq. (6.2) in section 6. After substituting the expression of bound-state eigenvalue \(-k^2\) into the wave function (4.9), we arrive at the bound-state wave function.

4.2 Bound-state eigenvalue

For bound states, the eigenvalue of the Heun equation (3.1) takes the discrete values [46]:
\[
\lambda^{(r)}(p, \beta) = \lambda_n(p, \beta)
\] (4.11)

with \( n \) an integer. Together with Eqs. (4.3), (4.4), and (4.5), we have
\[
l(l + 1) + 8M^2 k^2 - 6M^2 \mu^2 = \lambda_n \left( Mk, -\frac{M}{k} (2k^2 - \mu^2) \right).
\] (4.12)

The energy eigenvalue \(-k^2\) can be solved from Eq. (4.12). Eq. (4.12) is an implicit expression of the eigenvalue of the radical equation.

Before solving the exact result of the eigenvalue which will be given by analyzing the analytic property of the \( S \)-matrix of scattering in section 6, we now give an asymptotic expression of the eigenvalue of bound states.

Note that substituting the eigenvalue given by Eq. (4.12) into Eq. (4.9) gives the bound-state wave function and, then, the radial generalized spheroidal function of \( p \)-type (RGSF) in the bound-state wave function (4.9) reduces to the Heun polynomials [46].
4.2.1 Eigenvalue with large $Mk$

*Eigenvalue.* For a large $p = Mk$, the eigenvalue $\lambda_n$ has the following asymptotics [46]:

\[
\lambda_n = 2Mk \left[ 2\chi + \frac{M}{k} \left( 2k^2 - \mu^2 \right) \right] + \left\{ -2\chi \left[ \chi + \frac{M}{k} \left( 2k^2 - \mu^2 \right) \right] + \frac{1}{2} \left( 8M^2 \left( k^2 - \mu^2 \right) - 1 \right) \right\} \\
+ \frac{1}{2Mk} \left\{ -\chi \left[ \chi + \frac{M}{k} \left( 2k^2 - \mu^2 \right) \right] - \left( \chi^2 - \frac{1}{4} \right) \left[ \chi + \frac{M}{k} \left( 2k^2 - \mu^2 \right) \right] - \frac{1}{4} \chi \left[ 1 - 16M^2 \left( k^2 - \mu^2 \right) \right] \right\} \\
+ O\left( \frac{1}{(Mk)^2} \right),
\]

(4.13)

where

\[
\chi = n + \frac{1}{2}
\]

(4.14)

with $n$ an integer. The eigenvalue $\eta^2 = -k^2$ can be solved from Eqs. (4.12) and (4.13) directly.

For instance, up to the first order of $1/Mk$ for simplicity, by Eqs. (4.12), (4.13), and (4.14), we can obtain the eigenvalue. Solving

\[
l (l + 1) + 2M^2 \mu^2 + 8M^2 \left( k^2 - \mu^2 \right) \\
= 2Mk \left[ 2 \left( n + \frac{1}{2} \right) + \frac{M}{k} \left( 2k^2 - \mu^2 \right) \right] + \left\{ -2 \left( n + \frac{1}{2} \right) \left[ n + \frac{1}{2} + \frac{M}{k} \left( 2k^2 - \mu^2 \right) \right] \\
+ \frac{1}{2} \left( 8M^2 \left( k^2 - \mu^2 \right) - 1 \right) \right\}
\]

(4.15)

gives

\[
k = \frac{2n + 1}{2n(n + 1) + l(l + 1) + 1/4} \mu^2 M.
\]

(4.16)

Then the eigenvalue reads

\[
\eta^2 = -k^2 = -\left( \frac{2n + 1}{2n(n + 1) + l(l + 1) + 1} \right)^2 \mu^4 M^2.
\]

(4.17)

*Comparison with Coulomb potential.* It is worth comparing the above result, a Klein-Gordon particle in the Schwarzschild spacetime, with the solution of a Klein-Gordon particle in the Coulomb potential. The large-$k$ (large-$Mk$) case is the high-energy case, i.e., the case of large $n$. With a large $n$, Eq. (4.17) becomes

\[
\eta^2 \sim -\frac{M^2 \mu^4}{n^2}.
\]

(4.18)

While the eigenvalue of a Klein-Gordon particle in the Coulomb potential is $\eta_{\text{Coulomb}}^2 = \mu^2 \left[ 1 + \alpha^2 / (n + \beta)^2 \right]^{-1/2} - \mu^2$, where $\alpha$ and $\beta$ are some constants [47]. For a large $n$, $\eta_{\text{Coulomb}}^2 \sim -1/n^2$. These two results are similar to each other.
4.2.2 Eigenvalue with small $Mk$

*Eigenvalue.* For a small $p = Mk$, Eq. (3.1) can be approximately written as

$$
\frac{d}{dz} (z^2 - 1) \frac{d}{dz} y(z) + \left( -\lambda - 2m^2 \frac{1}{z-1} \right) y(z) = 0,
$$

(4.19)

where the relation $m = s$ given by Eq. (4.2) is used. This is a hypergeometric equation [48]. The eigenvalue of this equation reads

$$
\lambda_n = m^2 - (2n + 1) m + n (n + 1).
$$

(4.20)

With Eqs. (4.2) and (4.5), Eq. (4.20) becomes

$$
l(l + 1) + 2M^2 \mu^2 + 8M^2 (k^2 - \mu^2) = \left( -2M \sqrt{k^2 - \mu^2} \right) - (2n + 1) \left( -2M \sqrt{k^2 - \mu^2} \right) + n (n + 1).
$$

(4.21)

Then we have

$$
k = -\left\{ \frac{\mu^2}{2} + \frac{2n + 1}{8M^2} \left[ 8n (n + 1) - 4l (l + 1) + 1 - 8\mu^2M^2 \right] \right\}^{1/2}
$$

$$
+ \frac{3n(n + 1)}{4M^2} - \frac{l (l + 1)}{4M^2} + 1 \right\}^{1/2}.
$$

(4.22)

The eigenvalue then reads

$$
k = -\left\{ \frac{\mu^2}{2} + \frac{2n + 1}{8M^2} \left[ 8n (n + 1) - 4l (l + 1) + 1 - 8\mu^2M^2 \right] \right\}^{1/2}
$$

$$
+ \frac{3n(n + 1)}{4M^2} - \frac{l (l + 1)}{4M^2} + 1 \right\}.
$$

(4.23)

Comparison with Coulomb potential. For small $k$ (small $Mk$), which is the low-energy case, we also compare our result with the solution of a Klein-Gordon particle in the Coulomb potential. For the low-energy case, $n$ is small. With a small $n$, Eq. (4.23) becomes

$$
\eta^2 \sim A - Bn,
$$

(4.24)

where $A$ and $B$ are some constants. While the eigenvalue of a Klein-Gordon particle in the Coulomb potential with small $n$ becomes $\eta^2_{\text{Coulomb}} \sim A' - B'n$. These two results are similar to each other.

Moreover, an exact implicit expression of the eigenvalue of bound states will be given by analyzing the analytic property of the $S$-matrix of scattering in section 6.

5 Scattering state

For scattering states,

$$
p = -ic, \quad \beta = i\zeta,
$$

(5.1)

where $c = M\eta$ and $\zeta = (M/\eta) \left( 2\eta^2 + \mu^2 \right)$.

For scattering states, similarly to bound states, we also solve the radial equation (2.2) with the boundary conditions at $r = 2M$ and at $r \to \infty$, respectively, replacing the bound-state boundary condition (2.6) with the scattering-state boundary condition (2.7) at $r \to \infty$.
5.1 Scattering wave function: Regular solution

The regular solution for scattering in the Schwarzschild spacetime is a solution satisfying the boundary condition at \( r = 2M \). As a comparison, in common quantum-mechanical central potential scattering the regular solution satisfies the boundary condition at the singular point \( r = 0 \) [44, 49].

At \( r = 2M \), the solution of the radial equation (2.2) still needs to satisfy the boundary condition (2.3), but the parameters \( p \) and \( \beta \) given by Eq. (5.1) are now pure imaginary numbers. The solution now is the angular generalized spheroidal function of \( c \)-type, \( \Pi(c, \zeta, z) \), which can be achieved by substituting \( \eta = ik \) into Eq. (4.9):

\[
\Pi(\eta, r) = 2Ne^{2iM\sqrt{\eta^2 + \mu^2}\ln\left(\frac{r}{2M}\right)}\left(1 + 4iM\sqrt{\eta^2 + \mu^2} - 2e^{i\eta r}\right),
\]

(5.2)

where

\[
\sigma = l(l + 1) - 2M^2(\eta^2 - \mu^2) - 8M^2\left(\sqrt{\eta^2 + \mu^2} - \frac{\eta}{2}\right)^2 - 2iM\left(\eta + \sqrt{\eta^2 + \mu^2}\right).
\]

(5.3)

This solution is an analogue of the regular solution in quantum-mechanical scattering theory.

It should be noted that here the bound-state wave function \( \Pi(k, r) \) given by Eq. (4.9) satisfies the boundary condition at \( r = 2M \), but leaves alone the boundary condition at \( r \to \infty \) before substituting the bound-state eigenvalue which is determined by the boundary condition at \( r \to \infty \) into the eigenfunction. Therefore after performing the replacement \( \eta = ik \), \( \Pi(\eta, r) \) can serve as the regular solution in scattering.

5.2 Scattering wave function: Irregular solution

The irregular solution for scattering in the Schwarzschild spacetime is the solution satisfying the boundary condition \( \phi(l)(r)|_{r \to \infty} = \phi^{\infty}(r) \) at \( r \to \infty \). As a comparison, in quantum-mechanical central potential scattering the irregular solution satisfies the boundary condition at \( r \to \infty \) [44].

At \( r \to \infty \), the bound-state boundary condition (2.6) is replaced by the scattering boundary condition (2.7) and the parameters \( p \) and \( \beta \) are taken as imaginary numbers.

5.2.1 Scattering boundary condition: Asymptotic behavior

Before seeking exact solutions, we first investigate the large-distance asymptotic behavior of the scattering wave function. The asymptotic solution of the scattering wave function will serve as boundary conditions for scattering states [41, 43, 44].

The solution of Eq. (2.2) satisfying the scattering boundary condition at \( r \to \infty \) is the radial generalized spheroidal function (RGSF) of \( c \)-type, \( \Pi(c, \zeta, z) \) [46]. The large-distance asymptotics of the RGSF of \( c \)-type is [46]

\[
\Pi^{(3)}(\eta, r) \sim \frac{1}{r} \exp\left(i\left(\eta(r-M) + 2\eta M + M\frac{\mu^2}{\eta}\right)\ln\left(\frac{r}{M}-1\right) + \chi(\eta, M)\right),
\]

(5.4)

\[
\Pi^{(4)}(\eta, r) \sim \frac{1}{r} \exp\left(-i\left(\eta(r-M) + 2\eta M + M\frac{\mu^2}{\eta}\right)\ln\left(\frac{r}{M}-1\right) + \chi(\eta, M)\right),
\]

(5.5)
where $\chi(\eta, M)$ is the phase of the RGSF of $c$-type [46]. The asymptotics at $r \to \infty$ of the two solutions, (5.4) and (5.5), satisfies the boundary condition (2.9). For high-energy scattering, we have [46]

$$
\chi(\eta, M) = -2M\eta \ln 2 - \frac{7\pi}{2} - 2M\eta\pi + O\left(\frac{1}{M\eta}\right).
$$

(5.6)

### 5.2.2 Exact solution of outgoing wave function: Outgoing Eddington-Finkelstein coordinate

In order to solve the scattering wave function, we use the Eddington-Finkelstein coordinate.

**Scalar equation with outgoing Eddington-Finkelstein coordinate.** To solve the outgoing wave function, we use the outgoing Eddington-Finkelstein coordinate $(u, r)$ with $u = t - r_*$ [50]. By the outgoing Eddington-Finkelstein coordinate, we have

$$
d\sigma^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2du dr + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.
$$

(5.7)

It is worthy to note that to calculate the Hawking radiation, one needs to know the discontinuous jump of the outgoing wave function on the horizon. The outgoing wave function can be solved under the outgoing Eddington-Finkelstein coordinate and the discontinuous jump can be obtained directly from the outgoing wave function without the analytic extension. This is because the outgoing Eddington-Finkelstein coordinate does not diverge at $r = 2M$ and, then, holds for both regions inner and outer the horizon.

The scalar equation (1.2) then becomes

$$
\left[ -\frac{1}{r^2} \frac{\partial}{\partial u} r^2 \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial u} + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \left(1 - \frac{2M}{r}\right) \frac{\partial}{\partial r} + \frac{L^2}{r^2} - \mu^2 \right] \Phi = 0.
$$

(5.8)

By means of the variable separation

$$
\Phi = \beta(u, r) Y_{lm}(\theta, \phi),
$$

(5.9)

we arrive at

$$
\left[ -\frac{1}{r^2} \frac{\partial}{\partial u} r^2 \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial u} + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \left(1 - \frac{2M}{r}\right) \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} - \mu^2 \right] \beta(u, r) = 0.
$$

(5.10)

This equation can be solved exactly.

**Outer horizon.** Outer the horizon, the outgoing solution of Eq. (5.10) reads [46]

$$
\beta_{\text{outer}}(u, r) = e^{-i\sqrt{\eta^2 + \mu^2}u} e^{-i\sqrt{\eta^2 + \mu^2}r} e^{i2M\eta(1-r/M)}
$$

$$
\times \text{He}^{(a)}\left(-iM\eta, i2M \left(\eta + \sqrt{\eta^2 + \mu^2}\right) + iM \frac{\mu^2}{\eta} + 1, 1, 1 + i4M\sqrt{\eta^2 + \mu^2}, \sigma_{\text{outer}}, 1 - \frac{r}{2M}\right),
$$

(5.11)

where $\text{He}^{(a)}(\alpha, \beta, \gamma, \delta, \sigma, z)$ is the confluent Heun function and

$$
\sigma_{\text{outer}} = l(l+1) + 4M^2 \left(\eta^2 + \frac{\mu^2}{2}\right) - i2M \left(\eta + \sqrt{\eta^2 + \mu^2}\right).
$$

(5.12)
Inner horizon. Inner the horizon, the outgoing solution of Eq. (5.10) reads \[ 46 \]
\[
\beta^{\text{inner}}(u, r) = e^{-i\sqrt{\eta^2 + \mu^2} u} e^{-i\sqrt{\eta^2 + \mu^2} r} e^{i\eta r} (r - 2M)^{-i4\sqrt{\eta^2 + \mu^2} \eta} e^{i2M\eta(1-r/M)} \times Hc^{(a)}(\eta, r) \]
\[
\times \left( -iM\eta, i2M \left( \eta - \sqrt{\eta^2 + \mu^2} \right) + iM\eta \right) + 1, 1, 1 - i4\sqrt{\eta^2 + \mu^2} \eta - i4\sqrt{\eta^2 + \mu^2} \eta, \sigma^{\text{inner}}, 1 - \frac{r}{2M} \right), \tag{5.13}
\]

where
\[
\sigma^{\text{inner}} = l(l + 1) + 4M^2 \left( \eta^2 + \frac{\mu^2}{2} \right) - i2M \left( \eta - \sqrt{\eta^2 + \mu^2} \right). \tag{5.14}
\]

It can be seen that the solutions (5.11) and (5.13) satisfy the boundary conditions given in section 2.2: \( \beta^{\text{outer}}(u, r) \sim e^{-i\omega t} e^{i\omega r} \) and \( \beta^{\text{inner}}(u, r) \sim e^{-i\omega t} e^{i\omega r} \) when \( r \to 2M \).

Scattering wave function. For scattering, we only concern ourselves with the wave function outside the horizon, so the exact irregular solution reads
\[
e^{-i\omega t} \Pi^{(3)}(\eta, r) = \beta^{\text{outer}}(u, r). \tag{5.15}
\]
Here we have directly written out the time-dependent part.

5.2.3 Exact solution of ingoing wave function: Ingoing Eddington-Finkelstein coordinate

Scalar equation with ingoing Eddington-Finkelstein coordinate. For the ingoing wave function, instead of the outgoing Eddington-Finkelstein coordinate, we use the ingoing Eddington-Finkelstein coordinate \((v, r)\) with \( v = t + r \) [50]. By the ingoing Eddington-Finkelstein coordinate, we have
\[
ds^2 = -\left( 1 - \frac{2M}{r} \right) dv^2 + 2dvdr + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{5.16}
\]

Similarly, the ingoing wave function can be solved under the ingoing Eddington-Finkelstein coordinate and the discontinuous jump can be obtained directly from the ingoing wave function without the analytic extension. This is because the ingoing Eddington-Finkelstein coordinate does not diverge at \( r = 2M \) and, then, holds for both regions inner and outer the horizon.

Eq. (1.2) then becomes
\[
\left[ \frac{1}{r^2} \frac{\partial}{\partial v} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial v} + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \right] \left( 1 - \frac{2M}{r} \right) \frac{\partial}{\partial r} - \frac{L^2}{r^2} - \mu^2 \right] \Phi = 0 \tag{5.17}
\]
By means of the variable separation
\[
\Phi = \alpha(v, r) Y_{lm}(\theta, \phi), \tag{5.18}
\]
we arrive at
\[
\left[ \frac{1}{r^2} \frac{\partial}{\partial v} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial v} + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \right] \left( 1 - \frac{2M}{r} \right) \frac{\partial}{\partial r} - \frac{l(l + 1)}{r^2} - \mu^2 \right] \alpha(v, r) = 0. \tag{5.19}
\]
Here, the coefficients $C_s$ satisfies the boundary condition at $r = T$.

To calculate the scattering phase shift, we first express the regular solution $\Pi(\eta, \varphi)$ for scattering, we only concern ourselves with the wave function outside the horizon, so the exact irregular solution reads

$$e^{-i\omega t} = \alpha^{\text{outer}}(v, r),$$

where $\sigma^{\text{outer}}$ is given by Eq. (5.12).

The regular solution is obtained by Eqs. (5.20) and (5.21) is an analytic extension of $\alpha^{\text{outer}}(v, r)$. It can be seen that the solutions satisfy the boundary conditions given in section 2.2: $\alpha^{\text{outer}}(v, r) \sim e^{-i\omega t} e^{-i\omega r}$, and $\alpha^{\text{inner}}(v, r) \sim e^{-i\omega t} e^{-i\omega r}$ when $r \rightarrow 2M$.

Scattering wave function. For scattering, we only concern ourselves with the wave function outside the horizon, so the exact irregular solution reads

$$e^{-i\omega t} \Pi(\eta, \varphi) = \alpha^{\text{outer}}(v, r).$$

5.3 Scattering phase shift

5.3.1 Formal expression

To calculate the scattering phase shift, we first express the regular solution $\Pi(\eta, \varphi)$ which satisfies the boundary condition at $r = 2M$ as a linear combination of the irregular solutions $\Pi^{(3)}(\eta, \varphi)$ and $\Pi^{(4)}(\eta, \varphi)$ which satisfy the boundary condition at $r \rightarrow \infty$:

$$\Pi(\eta, \varphi) = C_l \Pi^{(4)}(\eta, \varphi) + D_l (-i)^{l+1} \Pi^{(3)}(\eta, \varphi).$$

Here, the coefficients $C_l$ and $D_l$ is determined by [51]

$$C_l = (-i)^{l+1} \frac{W_r [\Pi^{(3)}(\eta, \varphi), \Pi(\eta, \varphi)]}{W_r [\Pi^{(3)}(\eta, \varphi), \Pi^{(4)}(\eta, \varphi)]},$$

$$D_l = (-i)^{l+1} \frac{W_r [\Pi^{(4)}(\eta, \varphi), \Pi(\eta, \varphi)]}{W_r [\Pi^{(3)}(\eta, \varphi), \Pi^{(4)}(\eta, \varphi)]},$$

where the Wronskian determinant $W_x [f(x), g(x)] = f(x) g'(x) - f'(x) g(x)$.

The $S$-matrix can be obtained by Eqs. (5.20) and (5.21):
The scattering phase shift then reads
\[ \delta_l(\eta) = \frac{1}{2} \arg \left(-(-1)^{l+1} \frac{W_r[\Pi^{(4)}(\eta, r), \Pi(\eta, r)]}{W_r[\Pi^{(3)}(\eta, r), \Pi(\eta, r)]}\right). \] (5.27)

The scattering phase shift (5.27) can be rewritten as
\[ \delta_l(\eta) = W_r[\Pi^{(4)}(\eta, r), \Pi(\eta, r)] + \frac{l\pi}{2}, \] (5.28)
since \(W_r[\Pi^{(3)}(\eta, r), \Pi(k, r)]\) and \(W_r[\Pi^{(4)}(\eta, r), \Pi(\eta, r)]\) are complex conjugate to each other.

In principle, one can calculate the phase shift by Eq. (5.28) with the asymmetries of the exact wave function, Eqs. (5.2), (5.11), and (5.20) directly. The expression Eq. (5.27), however, is too complicated, so we provide an explicit approximation under the weak-field in the next section.

### 5.3.2 Weak-field approximation

In this section, we calculate the scattering phase shift under the weak-field approximation.

In section 5, we take \(p = -ic\) and \(\beta = i\zeta\), so \(c\) and \(\zeta\) are real numbers: \(c = M\eta\) and \(\zeta = \frac{M}{\eta} (2\eta^2 + \mu^2)\).

In weak-field approximation, i.e., \(c \ll 1 (M\eta \ll 1)\), the confluent Heun equation (3.1) reduces to
\[ \frac{d}{dz} \left((z^2 - 1) \frac{d}{dz} y(z) + [c^2 (z^2 - 1) + 2c\zeta z - \lambda] y(z)\right) = 0. \] (5.29)

The asymptotic solution at \(z \to \infty (r \to \infty)\) of Eq. (5.29) reads [52]
\[ \Pi(c, \zeta) \sim \frac{1}{z} \sin \left(cz - \zeta \ln 2cz - \frac{l\pi}{2} + \delta_l\right). \] (5.30)

For \(c \ll 1 (M\eta \ll 1)\), the \(S\)-matrix can be approximately expressed as [52]
\[ e^{2i\delta_l} \sim \frac{\Gamma(l + 1 - i\zeta)}{\Gamma(l + 1 + i\zeta)} \exp \left(\frac{2c^2\zeta}{(2l - 1)(2l + 3)}\right) \frac{1 - i\rho e^{il\pi} \Gamma(l+1+i\zeta)\Gamma(l+1-i\zeta)}{1 + i\rho e^{-il\pi} \Gamma(l+1+i\zeta)\Gamma(l+1-i\zeta)} \] (5.31)

where
\[ \rho \sim \left(\frac{c}{4}\right)^{2l+1} \frac{\pi^2}{(l + \frac{1}{2})^2 \Gamma^4(l + \frac{1}{2})} \left[1 - c^2 \frac{4(2l + 1)}{(2l - 1)^2(2l + 3)^2} \left(\zeta^2 + \frac{1}{4}\right)\right]. \] (5.32)

Then we arrive at
\[ \Pi(\eta, r) \sim \frac{1}{M-1} \sin \left(M\eta \left(\frac{r}{M} - 1\right) - \frac{M}{\eta} (2\eta^2 + \mu^2) \ln 2M\eta \left(\frac{r}{M} - 1\right) - \frac{l\pi}{2} + \delta_l\right) \]
\[ \sim \frac{1}{r} \sin \left(\eta (r - M) - \frac{M}{\eta} (2\eta^2 + \mu^2) \ln \left(\frac{r}{M} - 1\right) - \frac{l\pi}{2} + \delta_l\right). \] (5.33)
\[ e^{2i\delta_l} \sim \frac{\Gamma(l + 1 - i \left[ \frac{M}{\eta} (2\eta^2 + \mu^2) \right])}{\Gamma(l + 1 + i \left[ \frac{M}{\eta} (2\eta^2 + \mu^2) \right])} \exp \left( 2M^2\eta^2 \left[ \frac{M}{\eta} (2\eta^2 + \mu^2) \right] \frac{1}{(2l - 1)(2l + 3)} \right) \times \frac{1 - i\rho e^{il\pi} \Gamma\left(l + 1 + i \left[ \frac{M}{\eta} (2\eta^2 + \mu^2) \right] \right)}{1 + i\rho e^{-il\pi} \Gamma\left(l + 1 - i \left[ \frac{M}{\eta} (2\eta^2 + \mu^2) \right] \right)}. \]  

(5.34)

When the mass \( M \) is very small, we have
\[ e^{2i\delta_l} \sim \frac{\Gamma(l + 1 - 2\eta M)}{\Gamma(l + 1 + 2\eta M)}. \]  

(5.35)

This is just the scattering phase shift of the Newtonian inverse-square potential.

### 5.4 Asymptotic behavior of scattering wave function

In scattering, we often concern the large-distance asymptotic behavior of the scattering wave function. The asymptotics at \( r \to \infty \) of the scattering wave function (5.23) can be achieved in virtue of Eqs. (5.4) and (5.5):
\[ \Pi(\eta, r) \xrightarrow{r \to \infty} \frac{1}{r} \sin \left( \eta r + \left(2\eta M + M \frac{\mu^2}{\eta} \right) \ln \left( \frac{R}{M} - 1 \right) + \delta_l + \chi(\eta, M) - \frac{l\pi}{2} - \eta M \right). \]  

(5.36)

The asymptotic expression of \( \chi(\eta, M) \) for high-energy scattering is given by Eq. (5.6).

### 6 Bound-state eigenvalue: Exact solution

Now we return to the problem of bound states. Based on the solution of scattering obtained above, we present the eigenvalue of the bound state.

In section 4.2, we provide an explicit asymptotic expression for the eigenvalue of bound states. Here we provide an exact implicit expression for the eigenvalue of bound states.

The eigenvalue spectrum of bound states, according to the \( S \)-matrix theory, is the singularity of the \( S \)-matrix on the positive real axis [51]. Taking \( \eta = ik \), we have
\[ S_l(k) = -(-1)^{l+1} \frac{W_r[\Pi^{(4)}(k, r) \Pi(k, r)]}{W_r[\Pi^{(3)}(k, r) \Pi(k, r)]}. \]  

(6.1)

The singularity of the \( S \)-matrix (6.1) is just the zero of the denominator [51] \( W_r[\Pi^{(3)}(k, r) \Pi(k, r)] \) with \( k > 0 \), i.e.,
\[ W_r[\Pi^{(3)}(k, r) \Pi(k, r)] = 0, \ k > 0. \]  

(6.2)

This is an exact implicit expression of the eigenvalue of bound states. The bound-state eigenvalue \(-k^2\) can be solved from Eq. (6.2).
7 Discontinuous jump on horizon: Jump condition

The scattering wave function on the horizon \( r = 2M \) has a discontinuous jump. In order to calculate the scattering wave function, we need to first determine the jump of the wave function on the horizon, i.e., the jump condition.

In literature, the discontinuous jump of the wave function is obtained by analytically extending an asymptotic solution [2, 3].

In this paper, without analytic extension, we calculate the discontinuous jump from the exact result obtained in the present paper directly. The reason why we need not to perform the analytic extension is that the coordinate adopted here is the Eddington-Finkelstein coordinate which has no singularity on the horizon and holds for both regions inner and outer the horizon.

7.1 Jump condition: Outgoing wave function

In section 5.2.2, we obtain the exact solution of the outgoing wave function outside the horizon, Eq. (5.13), and obtain the outgoing wave function inside the horizon, Eq. (5.11), under the Eddington-Finkelstein coordinate. Comparing the wave functions inside and outside the horizon, we can directly obtain the jump of the wave function on the horizon.

By the outgoing solutions (5.11) and (5.13), the jump of the outgoing wave function on the horizon \( r = 2M \) is

\[
\frac{\beta^{\text{outer}}(u, r)}{\beta^{\text{inner}}(u, r)} \bigg|_{r\to 2M} = \lim_{r\to 2M^-} \frac{1}{(r - 2M)^{-i4\sqrt{\eta^2 + \mu^2}M}} = e^{-4M\sqrt{\eta^2 + \mu^2}\pi}, \tag{7.1}
\]

where \( H_c^{(a)}(\alpha, \beta, \gamma, \delta, \sigma) = 1 \) is used in the calculation. Note that the limit here is a left limit since the factor \((r - 2M)^{-i4\sqrt{\eta^2 + \mu^2}M}\) comes from \( \beta^{\text{inner}}(u, r) \).

It can be directly seen that there exists a jump on the phase of the scattering wave function.

In literature, e.g., Refs. [2, 3], the jump is calculated from an asymptotic solution. Our result agrees the result in literature.

7.2 Jump condition: Ingoing wave function

Similarly, by the exact ingoing scattering wave functions inside and outside the horizon obtained in section 5.2.2, we can directly calculate the jump on the horizon of the ingoing wave function.

By the ingoing solutions (5.20) and (5.21), the jump of the ingoing wave function on the horizon \( r = 2M \) is

\[
\frac{\alpha^{\text{outer}}(v, r)}{\alpha^{\text{inner}}(v, r)} \bigg|_{r\to 2M} = \lim_{r\to 2M^+} (2M - r)^{4i\sqrt{\eta^2 + \mu^2}M} = e^{-4M\sqrt{\eta^2 + \mu^2}\pi}. \tag{7.2}
\]

Comparing Eqs. (7.1) and (7.2) shows that the jump of the outgoing wave and the jump of the ingoing wave are the same. Note that the limit here is a right limit since the factor \((2M - r)^{4i\sqrt{\eta^2 + \mu^2}M}\) comes from \( \alpha^{\text{outer}}(v, r) \).
7.3 Discontinuous jump on horizon: Comparison with Hawking and Damour-Ruffini methods

In order to calculate the Hawking radiation, one needs to know the discontinuous jump of the wave function on the horizon.

Hawking and Ruffini dealt with this problem from different viewpoints respectively. They, mathematically speaking, did almost the same thing. The methods they used are both based on asymptotic solutions.

In the above, we have obtained the exact wave functions inner and outer the horizon, so we can calculate the jump exactly rather than asymptotically.

It should be emphasized that the starting point of all the methods is the solution of the outgoing wave of the radial equation (2.2).

In the following, we compare our method with the Hawking treatment and the Damour and Ruffini treatment.

7.3.1 Hawking method: Brief review

Hawking [2] and Damour and Ruffini [3] both start from the $r \to \infty (r_* \to \infty)$ asymptotics of the outgoing wave function

\[ \beta_{\text{outer}} r^{-\infty} \frac{1}{r} e^{-i \omega u} = \frac{1}{r} e^{2i \omega r_*} e^{-i \omega v}, \]  
(7.3)

where $u$ and $v$ are the outgoing and ingoing Eddington-Finkelstein coordinates, respectively.

Hawking in Ref. [2] uses quantum field theory to deal with the radiation of a black hole.

In Hawking’s treatment, the outgoing wave function $\Phi_{\text{outer}}^\omega = \beta_{\text{outer}} Y_{lm}(\theta, \phi)$ is expanded as

\[ \Phi_{\text{outer}}^\omega = \int d\omega' (a_{\omega\omega'} f_{\omega'} + b_{\omega\omega'} \bar{f}_{\omega'}), \]  
(7.4)

with the expansion coefficients

\[ a_{\omega\omega'} \sim \frac{1}{2\pi} \left( \frac{\omega'}{\omega} \right) (-\omega')^{-i \frac{\omega'}{\omega} - 1}, \]  
(7.5)

\[ b_{\omega\omega'} = -i \alpha_{\omega(-\omega')}, \]  
(7.6)

where $f_{\omega}$ and $\bar{f}_{\omega}$ are the solutions on past null infinity $f^- (t = -\infty, r = +\infty)$ containing only positive frequencies and only negative frequencies, respectively. The number of particles created by the gravitational field and emitted to infinity then reads [2]

\[ \langle 0_- | N_{\omega} | 0_- \rangle = \int d\omega' |b_{\omega\omega'}|^2. \]  
(7.7)

Notice that the subscript of $\alpha_{\omega(-\omega')}$ in Eq. (7.6) is $-\omega'$. In order to calculate the integral in Eq. (7.7), we need to analytic extend $\omega'$. In Eq. (7.5) $\omega' = 0$ is a singular point. Hawking in Ref. [2] performs the analytic extension by anticlockwisely rounding this singularity

\[ \omega' \to \omega' e^{-i \pi}. \]  
(7.8)
By virtue of such an analytic extension, Eq. (7.6) becomes

$$a_{\omega(-\omega')} = ie^{-\omega/\kappa}a_{\omega\omega'}.$$  

(7.9)

That is to say, $a_{\omega\omega'}$ has a discontinuous jump at $\omega = 0$

$$\frac{a_{\omega(-\omega')}}{a_{\omega\omega'}} = ie^{-\omega/\kappa}.$$  

(7.10)

Accordingly, Hawking arrives at

$$\langle 0_- | N_{\omega} | 0_- \rangle = \frac{1}{e^{\frac{\omega}{\kappa}} - 1}, \quad T = \frac{1}{8\pi M k}.$$  

(7.11)

### 7.3.2 Damour-Ruffini method: Brief review

Damour and Ruffini, also starting from the asymptotic outgoing wave function (7.3), different from Hawking, use scattering method to deal with the radiation of a black hole.

Damour and Ruffini rewrite the asymptotic outgoing wave outside the horizon (7.3) as

$$\beta_{\text{outer}} \sim \frac{1}{r} e^{2i\omega r^*} e^{-i\omega r} = \frac{1}{r} e^{2i\omega r} e^{-i\omega r} \left( \frac{r - 2M}{2M} \right)^{4i\omega M}$$

to expose the singularity at the horizon $r = 2M$. In order to analytically extend the outgoing wave to the inner horizon, they analytically extend $r - 2M$ as [3]

$$r - 2M \rightarrow |r - 2M| e^{-i\pi}$$

$$= (2M - r) e^{-i\pi}.$$  

(7.12)

The outgoing wave in the horizon then reads

$$\beta_{\text{inner}} = \frac{1}{r} e^{4\pi\omega M} e^{2i\omega r^*} e^{-i\omega r}.$$  

(7.13)

There is also a discontinuous jump

$$\frac{\beta_{\text{outer}}}{\beta_{\text{inner}}} = e^{-4\pi\omega M}.$$  

(7.14)

Then they obtain a relative scattering probability [3]

$$P_{\omega} = e^{-8\pi\omega M}$$  

(7.15)

and then obtain the Hawking’s result:

$$N_{\omega} = \frac{1}{e^{8\pi\omega M} - 1} = \frac{1}{e^{\frac{\omega}{\kappa}} - 1}, \quad T = \frac{1}{8\pi M k}.$$  

(7.16)
7.3.3 Direct calculation through exact solution: Comparison

The starting points of Hawking method and Damour-Ruffini method are both the $r \to 2M$ asymptotic outgoing wave function (7.3), though their treatments are somewhat different.

In this paper, using the outgoing Eddington-Finkelstein coordinate, we obtain the exact solution of the outgoing wave function outside the horizon, Eq. (5.11), and we also obtain the exact result inside the horizon, Eq. (5.13) by analytic extension. This allows us to calculate the jump exactly.

From the exact results, we can calculate the jump (7.1) exactly:

$$\frac{\beta_{\text{outer}}}{\beta_{\text{inner}}} \bigg|_{r \to 2M} = e^{-4M \sqrt{\eta^2 + \mu^2}} = e^{-4M \omega \pi}.$$  

(7.17)

The result obtained in the present is based on the exact solution rather than the asymptotic solution.

8 Conclusion

An exact solution for a scalar field in the Schwarzschild spacetime is provided. For bound states, we obtain an exact bound-state wave function and an exact implicit expression of eigenvalues. For scattering, we obtain an exact scattering wave function.

Moreover, besides the exact solutions, we also provide some approximate solutions. For bound states, we provide an explicit asymptotic expression for bound-state eigenvalues. For scattering, an approximate expression under the weak-field approximation for phase shift.

It is worthy to note that the solution of the scalar equation inner the horizon can be obtained by analytically extending the outer solution. One may also discuss the solution inner the horizon further based on interior metrics of a Schwarzschild spacetime [53, 54].

In literature, the discontinuous jump on horizon of the wave function is obtained by analytically extending an asymptotic solution [2, 3]. In this paper, since we have obtained the exact solution of wave functions, we can calculate this discontinuous jump straightforwardly from the exact result without analytic extensions.

Scattering by a Schwarzschild spacetime is a long-range potential scattering. In further researches, we can attempt the heat-kernel method to calculate the scattering phase shift using the approach developed in Refs. [55–61]. Moreover, starting from the exact solution, we can investigate the large-distance asymptotic behavior which is important in the scattering problem [5, 41, 43, 45]. Starting from the exact solution given in the present paper, we can also calculate the corresponding heat kernel and the partition function [62]. Using the heat kernel, we can calculate various quantum field theory quantities, such as one-loop effective actions, vacuum energies, etc. [63, 64].

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