ITERATED CROSSED PRODUCTS FOR GROUPOID FIBRATIONS

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Abstract. We define and study fibrations of topological groupoids. We interpret a groupoid fibration \( L \to H \) with fibre \( G \) as an action of \( H \) on \( G \) by groupoid equivalences. Our main result shows that a crossed product for an action of \( L \) is isomorphic to an iterated crossed product first by \( G \) and then by \( H \). Here “groupoid action” means a Fell bundle over the groupoid, and “crossed product” means the section \( C^* \)-algebra.

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1. Introduction

What does it mean for a topological groupoid \( H \) to act on another topological groupoid \( G \)? The definition of an action by groupoid isomorphisms is straightforward if complicated (see [12]). Some examples, however, require arrows in \( H \) to act by equivalences, not isomorphisms. If \( H \) is an étale groupoid, such actions by equivalences are defined in [12] using the inverse semigroup of bisections of \( H \). Here we extend this notion of action to non-étale topological groupoids through the notion of a fibration of topological groupoids, briefly groupoid fibration.

The idea is the following. An action of \( H \) on \( G \) should give a transformation groupoid \( L := G \rtimes H \) that contains \( G \) and comes with a continuous functor \( L \to H \). Thus defining actions of topological groupoids on topological groupoids amounts to characterising which chains of continuous functors \( G \leftarrow L \to H \) correspond to actions. We require \( L \to H \) to be a “groupoid fibration” with “fibre” \( G \subseteq L \) (see Definition 27). This gives the same notion of action as in [12] if \( H \) is étale.

Groupoid fibrations are inspired by higher category theory. The thesis of Li Du [30] describes actions of \( \infty \)-groupoids on \( \infty \)-groupoids through Kan fibrations. By definition, a groupoid fibration between two topological groupoids is a Kan fibration between the associated topological \( \infty \)-groupoids.

Let \( F : L \to H \) be a groupoid fibration with fibre \( G \subseteq L \). Then we describe an induced action of \( H \) on the \( C^* \)-algebra of \( G \), such that the crossed product is \( C^*(L) \). This generalises the well known decomposition \( C^*(X \rtimes H) \cong C_0(X) \rtimes H \).

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for an action of a groupoid $H$ on a space $X$. In general, an “action” of a locally compact groupoid on a C$^*$-algebra is a (saturated) Fell bundle over the groupoid, and its “crossed product” is the section C$^*$-algebra of the Fell bundle. Saturated Fell bundles are interpreted as actions by Morita–Rieffel equivalences in [13]. We tacitly assume all Fell bundles over groupoids to be saturated.

More generally, we decompose the “crossed product” for any “action” of $L$ in such a way. That is, for a Fell bundle over $L$, we construct a Fell bundle over $H$ with the restricted section algebra over $G$ as unit fibre and show that the section algebra of this Fell bundle over $H$ is the section algebra of the original Fell bundle over $L$. In brief, $(A \rtimes G) \rtimes H \cong A \rtimes L$ for an action of $L$ on a C$^*$-algebra $A$.

If $G$ and $H$ are groups, then a groupoid fibration $L \to H$ with fibre $G$ is nothing but an extension of topological groups $G \hookrightarrow L \twoheadrightarrow H$. Our result on iterated crossed products is known in this case in the language of Green twisted actions.

As in [12], a motivating example for our theory is to construct an action of a locally Hausdorff, but non-Hausdorff groupoid $H$ on a C$^*$-algebra that represents the action of $H$ on its arrow space $H^1$ by left multiplication. This is the simplest example of a free and proper action. Since $H^1$ is non-Hausdorff, the C$^*$-algebra that best describes $H^1$ is the groupoid C$^*$-algebra of the Čech groupoid of a Hausdorff, open covering of $H^1$. There is usually no classical action of $H$ by automorphisms on this Čech groupoid. There is, however, a groupoid fibration describing this action, and an associated Fell bundle over $H$ describing the action in C$^*$-algebraic terms.

Groupoid fibrations of plain groupoids without topology or other extra structure are defined already by Ronald Brown [6]. A definition for topological groupoids is given in [15]; but the definition in [15] is not used in [15], works only in the étale case, and contains an unnecessary extra assumption. Our construction is similar to the one in [15], except that we insist on getting saturated Fell bundles and allow non-étale, locally Hausdorff, locally compact groupoids (with a Hausdorff object space and a Haar system). We do not require amenability assumptions as in [15] since we work with full crossed products throughout.

Reduced crossed products for non-Hausdorff groupoids do not always behave well for iterated crossed products. Counterexamples in [10] in the étale case show this.

Section 2 defines groupoid fibrations and illustrates the notion by some examples and basic properties. In particular, classical groupoid actions by automorphisms and groupoid extensions give examples of groupoid fibrations. Most of the general theory works for arbitrary topological groupoids, even for groupoids in a category with pretopology as in [32].

Section 3 compares groupoid fibrations with étale $H$ to the actions of $H$ defined in [12] using inverse semigroups. Section 4 shows that the transformation groupoid $L$ inherits the properties of being (locally) Hausdorff and locally compact from $G$ and $H$. Section 5 describes how Haar systems on $G$ and $H$ induce a Haar system on $L$. Section 6 contains our main result on iterated crossed products. Section 7 constructs the Fell bundle describing the translation action on the arrow space of a locally Hausdorff groupoid and explains how certain results in [5,11,22,24,32,35] are contained in our main theorem.

2. Groupoid fibrations

A topological groupoid consists of two topological spaces $G^1$ and $G^0$ with open, continuous range and source maps $r, s : G^1 \to G^0$ and continuous multiplication, inversion and unit maps satisfying the usual algebraic conditions. The range and source maps are automatically open if the groupoid has a Haar system (see [36]).
Until we consider Haar systems and groupoid $C^*$-algebras, we allow arbitrary topological spaces $G^0$ and $G^1$ as in [12]. We need neither Hausdorffness nor local compactness, but we do need open range and source maps.

**Definition 2.1.** Let $L$ and $H$ be topological groupoids. A groupoid fibration is a continuous functor $F: L \to H$ (continuous maps $F^i: L^i \to H^i$ for $i = 0, 1$ that intertwine $r, s$ and the multiplication maps), such that the map

\[(F^1, s): L^1 \to H^1 \times_{s,H^0,F^0} L^0 := \{(h, x) \in H^1 \times L^0 \mid s(h) = F^0(x)\}\]

is an open surjection. Its fibre is the subgroupoid $G$ of $L$ defined by $G^0 = L^0$ and $G^1 = \{g \in L^1 \mid F^1(g) = 1_{F^0(s(g))}\}$, equipped with the subspace topology on $G^1 \subseteq L^1$.

A groupoid covering is a functor $F: L \to H$ for which (2.2) is a homeomorphism.

**Lemma 2.3.** The fibre of a groupoid fibration is a topological groupoid.

**Proof.** If $g \in G^1$, then $F^1(g) = 1_{F^0(s(g))}$, so $F^0(r(g)) = r(F^1(g)) = s(F^1(g)) = F^0(s(g))$. Thus $g^{-1} \in G^1$ as well. We also get $g_1 \cdot g_2 \in G^1$ if $g_1, g_2 \in G^1$, so that $G$ is a subgroupoid of $L$. It remains to prove that the source map (and hence the range map) in $G$ is open. We use that the pull-back of an open surjection is again an open surjection (see [32]). There is a fibre product diagram

\[
\begin{array}{ccc}
G^1 & \xrightarrow{s} & G^0 = L^0 \\
\downarrow & & \downarrow \quad (u \circ F^0, \text{Id}) \\
L^1 & \xrightarrow{(F^1, s)} & H^1 \times_{s,H^0,F^0} L^0,
\end{array}
\]

where $u: H^0 \hookrightarrow H^1$ denotes the unit map. Since $(F^1, s)$ is an open surjection by assumption, so is $s: G^1 \to G^0$. \hfill $\Box$

**Remark 2.4.** The inversion map turns (2.2) into the map

\[(F^1, r): L^1 \to H^1 \times r,H^0,F^0 L^0.
\]

Hence $F: L \to H$ is a fibration if and only if $(F^1, r)$ is an open surjection.

**Proposition 2.5.** Let $F: L \to H$ be a groupoid fibration with fibre $G$. The left multiplication action of $G$ on $L^1$ with the map (2.2) as bundle projection is a principal $G$-bundle, that is, the bundle projection is an open surjection and the following map is a homeomorphism:

\[(2.6) \quad G^1 \times_{s,L^0,F} L^1 \xrightarrow{\sim} L^1 \times (F^1,s)H^1 \times_{s,H^0,F,L^0,(F^1,s)} L^1, \quad (g, l) \mapsto (g \cdot l, l).
\]

Actions that are part of principal bundles are called basic in [12,32].

**Proof.** That $(F^1, s)$ is an open surjection is exactly the assumption of $F$ being a groupoid fibration. The map (2.2) is well defined because $F^1(gl) = F^1(l)$ and $s(gl) = s(l)$ for all $g \in G^1$, $l \in L^1$ with $s(g) = r(l)$. We claim that $(l_1, l_2) \mapsto (l_1 \cdot l_2^{-1}, l_2)$ is a well defined map $L^1 \times_{(F^1,s),H^1 \times_{s,H^0,F,L^0,(F^1,s)}} L^1 \to G^1 \times_{s,L^0,F} L^1$.

Let $(l_1, l_2) \in L^1 \times L^1$ satisfy $(F^1, s)(l_1) = (F^1, s)(l_2)$. Since $s(l_1) = s(l_2)$, $g := l_1^{-1}l_2$ is well defined. Since $F^1(l_1) = F^1(l_2)$ and $F^1$ is multiplicative, $F^1(g) = 1_{F^0(s(g))}$, so $g \in G^1$. And $s(g) = r(l_2)$ by construction, so $(g, l_2) \in G^1 \times_{s,L^0,F} L^1$. The map $(l_1, l_2) \mapsto (l_1l_2^{-1}, l_2)$ is continuous and a two-sided inverse for (2.6). \hfill $\Box$
Remark 2.7. The definitions of a groupoid fibration and covering carry over to the setting of groupoids in categories with pretopology studied in [32]. A pretopology specifies a class of special morphisms in the category called “covers,” subject to some axioms. In particular, pull-backs of covers are again covers. In this article, we use the category of topological spaces with open surjections as covers.

For groupoids in a category with pretopology, a groupoid fibration is defined as a functor where (2.2) is a cover, and a groupoid covering as a functor where (2.2) is an isomorphism. The proof of Lemma 2.3 still works because it only uses that pull-backs of covers remain covers. Similarly, the proof of Proposition 2.5 still works for groupoids in any category with pretopology. Most definitions, results and examples in this section generalise to groupoids in categories with pretopology. Namely, this is the case for Example 2.9, Proposition 2.10, Definition 2.12, Lemma 2.14, Proposition 2.21, Example 2.22, Lemma 2.23, Definition 2.24 and Lemma 2.25. Lemma 2.31 carries over if [32, Assumption 2.9] holds for our pretopology.

Remark 2.8. Lie groupoids are groupoids in the category of smooth manifolds with surjective submersions as covers. The definition of a Lie groupoid fibration in Remark 2.7 is weaker than the usual one in [20, 31], which also asks for the map \( F^0 : L^0 \to H^0 \) on objects to be a cover. This is reasonable if one wants \( H \) to be determined by \( L \) and \( G \) as a generalised quotient (see [31, Theorems 2.4.6 and 2.4.8]). But it rules out important examples. For a groupoid covering, requiring \( F^0 \) to be a cover restricts to actions with a cover as anchor map. An important counterexample is the action of \( H \) by right translations on \( H^x := \{ h \in H \mid r(h) = x \} \) for \( x \in H^0 \).

2.1. Groupoid coverings and actions on spaces.

Example 2.9. Let \( X \) be a topological space with an action of a topological groupoid \( H \). View \( X \) as a groupoid with only identity arrows. The functor from the transformation groupoid \( H \ltimes X \) to \( H \) which is the anchor map on objects and the map \( (h, x) \mapsto h \) on arrows is a groupoid covering with fibre \( X \). The map (2.2) is an isomorphism by the definition of \( H \ltimes X : (H \ltimes X)^1 := \{(h, x) \in H^1 \times X \mid s(h) = r(x)\} \).

Proposition 2.10. Any groupoid covering is isomorphic to \( H \ltimes X \to H \) for some \( H \)-action \( X \). A groupoid fibration is a groupoid covering if and only if its fibre is a groupoid with only identity arrows, that is, a space viewed as a groupoid.

This is the main result in [17]. We sketch the proof to show how the argument carries over to groupoids in categories with pretopology as in Remark 2.7.

Proof. First let \( F : L \to H \) be a groupoid covering. Its fibre is \( L^0 \) viewed as a groupoid with only identity arrows. We construct an \( H \)-action on \( X := L^0 \). Its anchor map is \( F^0 \). Let \( \tau : H^1 \times_{s, H^0, F^0} L^0 \simto L^1 \) be the inverse of the isomorphism in (2.2). We define the \( H \)-action to be \( \tau \circ \tau : H^1 \times_{s, H^0, F^0} L^0 \to L^0 \); this is indeed a groupoid action. The identity on objects and \( \tau \) on arrows give a groupoid isomorphism \( H \ltimes L^0 \simto L \). This is the only isomorphism for which \( F : L \to H \) becomes the canonical functor \( H \ltimes L^0 \to H \).

Now let \( F : L \to H \) be a groupoid fibration whose fibre \( G \) is the space \( X = L^0 \) of unit arrows. The map (2.2) is a bundle projection for a principal \( G \)-bundle by Proposition 2.5 and \( G \) has only identity arrows by assumption. Hence (2.2) must be a homeomorphism by [32, Proposition 5.9]; that is, \( F \) is a groupoid covering. □

Thus a groupoid fibration \( L \to H \) where the fibre \( G \) is a space is equivalent to an \( H \)-action on \( G \) with transformation groupoid \( L \). This suggests to view a groupoid fibration \( F : L \to H \) with a groupoid \( G \) as its fibre as a generalised action of \( H \) on \( G \) with transformation groupoid \( L \).
2.2. Groupoid equivalences defined by a fibration. Let \( F : L \to H \) be a groupoid fibration. We are going to construct an action of the groupoid \( H \) on \( G \) by equivalences. For \( h \in H^1 \), let \( L_h = (F^0)^{-1}(h) \subseteq L^1 \). If \( y \in H^0 \), then \( L_y \) is contained in \( G^0 \) and consists of those \( g \in G^1 \) with \( F^0(r(g)) = y \). Since \( F^0(r(g)) = F^0(s(g)) \) for \( g \in G \), the subset \( (F^0)^{-1}(y) \subseteq G^0 \) is \( G \)-invariant, and \( G_y := L_y \) is the restriction of \( G \) to this \( G \)-invariant subset. This is the subgroupoid of \( G \) with \( G_y^0 = (F^0)^{-1}(y) \) and \( G_y^1 = s^{-1}_G((G_y)') = r^{-1}_G(G_y) \).

**Lemma 2.11.** The subgroupoids \( G_{r(h)} \) and \( G_{s(h)} \) act on \( L_h \) by left and right multiplication, respectively. With these actions, \( L_h \) is an equivalence between \( G_{r(h)} \) and \( G_{s(h)} \). The inversion and multiplication in \( L \) restrict to isomorphisms of groupoid equivalences \( L_h \cong L_{h^{-1}}^* \) for \( h \in H^1 \) and \( L_{h_1} \times_{G_y} L_{h_2} \cong L_{h_1h_2} \) for composable \( h_1, h_2 \in H^1 \) and \( y := s(h_1) = r(h_2) \); here the star denotes the inverse equivalence with left and right actions exchanged, and \( \times_{G_y} \) denotes the composition of equivalences.

**Proof.** We have \( L_{h_1} \times L_{h_2} \subseteq L_{h_1h_2} \) for all composable \( h_1, h_2 \in H^1 \). In particular, \( L_{1_{H^1}} \subseteq L_h \) and \( L_{h_{H^1}} \subseteq L_h \). Since \( G^0_y = L_y \) for all \( y \in H^0 \), this gives commuting left and right actions of \( G_{r(h)} \) and \( G_{s(h)} \) on \( L_h \), respectively. Their anchor maps are the restrictions of \( r \) and \( s \) to \( L_h \), respectively. The left action of \( G_{r(h)} \) gives a principal bundle with bundle projection \( s : L_h \to G_0^0 \) because of (2.6): we have simply restricted the principal \( G \)-bundle \( L^1 \to H^1 \times_{s,H^0,F} L^0 \to \{h\} \times_{G^0_0} G^0_0 \subseteq H^1 \times_{s,H^0,F} L^0 \). Taking inverses everywhere gives the same statement for the right action of \( G_{s(h)} \) on \( L_h \).

The isomorphism \( L_h \cong L^*_{h^{-1}} \) is trivial. The multiplication in \( L^1 \) restricts to a continuous map \( L_{h_1} \times_{s,H^1} L_{h_2} \to L_{h_1h_2} \) that equals \((l_1, g, l_2) \) and \((l_1 g, l_2) \) for all \( l_1 \in L_{h_1}, g \in G_y \) with \( s(l_1) = r(g), s(g) = r(l_2) \). Hence it induces a continuous map \( L_{h_1} \times_{G_y} L_{h_2} \to L_{h_1h_2} \). This map is equivariant with respect to the left action of \( G_{r(h_1)} \) and the right action of \( G_{s(h_2)} \). An equivariant, continuous map between two groupoid equivalences such as \( L_{h_1} \times_{G_y} L_{h_2} \) and \( L_{h_1h_2} \) is automatically a homeomorphism; this follows from the statement in [12, Theorem 7.15] that all 2-arrows in the bicategory of bibundle functors are invertible.

If we equip \( H \) with the discrete topology, then the equivalences \( L_h \) for \( h \in H \) with the isomorphisms of equivalences \( L_{h_1} \times_{G_y} L_{h_2} \to L_{h_1h_2} \) for \( y := s(h_1) = r(h_2) \) form an action of \( H \) on the groupoid \( \bigsqcup_{y \in H^0} G_y^0 \) by equivalences, compare [12]. The continuity of this action is expressed by putting a topology on \( L = \bigsqcup_{y \in H} L_y \) such that the involution and multiplication are continuous and the maps \( r, s : L \to L^0 \) and \( 2.3 \) are open and surjective.

2.3. Classical groupoid actions. We define actions of one topological groupoid on another by automorphisms and construct a transformation groupoid and a groupoid fibration from it. This corroborates our interpretation of groupoid fibrations as generalised groupoid actions.

**Definition 2.12.** Let \( H \) and \( G \) be topological groupoids. A classical action of \( H \) on \( G \) consists of \( H \)-actions on \( G^0 \) and \( G^1 \) such that

1. \( r, s : G^1 \to G^0 \) are \( H \)-equivariant;
2. the multiplication map \( G^1 \times_{G_0^0} G^1 \to G^1 \) is \( H \)-equivariant. Here we use the diagonal action of \( H \) on \( G^1 \times_{G_0^0} G^1 \), which exists because \( s \) and \( r \) are equivariant.

The equivariance of the multiplication in \( G \) implies that the unit and inversion maps are \( H \)-equivariant as well. Let \( r_H : G^0 \to H^0 \) and \( r_H : G^1 \to H^0 \) denote the
anchor maps. The map \( r_H : G^0 \to H^0 \) is \( G \)-invariant because \( r_H(r(g)) = r_H(g) = r_H(g_1) \) for all \( g \in G^1 \).

More explicitly, the \( H \)-equivariance of the multiplication map means that
\[
h \cdot (g_1 \cdot g_2) = (h \cdot g_1) \cdot (h \cdot g_2)
\]
for \( h \in H^1, g_1, g_2 \in G^1 \) with \( s(h) = r_H(g_1), s(g_1) = r(g_2) \). We check that \( h \cdot (g_1 \cdot g_2) \) and \( (h \cdot g_1) \cdot (h \cdot g_2) \) are defined. The product \( g_1 \cdot g_2 \) is defined because \( s(g_1) = r(g_2) \), and \( h \cdot (g_1 \cdot g_2) \) is defined because \( s(h) = r_H(g_1) \). The product \( h \cdot g_1 \) is defined because \( s(h) = r_H(g_1) \), and \( h \cdot g_2 \) is defined because \( s(h) = r_H(g_1) \). Since \( s(h) = g \cdot s(g_1) = h \cdot r(g_2) = r(h \cdot g_2) \), the product \((h \cdot g_1) \cdot (h \cdot g_2)\) is defined.

**Remark 2.13.** Since \( r_H : G^0 \to H^0 \) is \( G \)-invariant, the subspaces \( r_H^{-1}(y) \subseteq H^0 \) are \( G \)-invariant, so that we may restrict \( G \) to topological subgroupoids \( G_y := G|_{r_H^{-1}(y)} \) for \( y \in H^0 \). Then \( G = \bigsqcup_{y \in H^0} G_y \) as a set. An arrow \( h \in H^1 \) acts on \( G^3 \) by an isomorphism of topological groupoids \( \alpha_h : G_{s(h)}^3 \to G_{r(h)}^3 \), such that \( \alpha_{h_1 h_2} = \alpha_h \alpha_{h_2} \) for all \( h_1, h_2 \in H^1 \) with \( s(h_1) = r(h_2) \). Moreover, the map \( H^1 \times_{s, r} G^0 \to G^1 \) is \( s \)-invariant, the subspaces \( r_H^{-1}(y) \subseteq H^0 \) are defined. And
\[
\alpha_{h_1} \alpha_{h_2} = \alpha_h
\]
for composable \( h_1, h_2 \in H^1 \) and such that \( H^1 \times_{s, r} H^0 \to G^1 \) is \( (h, g) \mapsto \alpha_h(g) \), is continuous. Conversely, we assume that we have a decomposition \( G = \bigsqcup G_y \) through a \( G \)-invariant continuous map \( G^0 \to H^0 \) and groupoid isomorphisms \( \alpha_h : G_{s(h)} \to G_{r(h)} \) for \( h \in H^1 \), such that \( \alpha_{h_1} \alpha_{h_2} = \alpha_{h_1 h_2} \). Then the map \( H^1 \times_{s, r} H^0 \to G^0 \) is \( s \)-invariant, the subspaces \( r_H^{-1}(y) \subseteq H^0 \) are defined. And
\[
\alpha_{h_1} \alpha_{h_2} = \alpha_h
\]
for composable \( h_1, h_2 \in H^1 \) and such that \( H^1 \times_{s, r} H^0 \to G^1 \) is \( (h, x) \mapsto \alpha_h(x) \), is continuous as well, and these actions of \( H \) on \( G^0 \) and \( G^1 \) form a classical action as in Definition 2.12. Thus a classical action of \( H \) on \( G \) is the same as a decomposition of \( G \) into a bundle of topological groupoids over \( H^0 \) (that is, a continuous, \( G \)-equivariant map \( r_H : G^0 \to H^0 \) and an action of \( H \) by topological groupoid isomorphisms between the fibres of this bundle, such that the induced action on \( G^1 \) is continuous.

In particular, if \( H \) is a topological group, then a classical action is a group homomorphism from \( H \) to the group of groupoid automorphisms of \( G \) that is continuous in the sense that the action map \( H^1 \times G^0 \to G^1 \) is continuous (this implies the continuity of \( H^1 \times G^0 \to G^0 \)). Actions of this type are used in [24–27].

Now we build a transformation groupoid \( L \) for a classical action of \( H \) on \( G \); this may also be called a semidirect product groupoid because that is what it is for an action of a topological group on another topological group. Let \( L^0 := G^0 \) and \( L^1 := H^1 \times_{s, r} G^0 \). Define \( s, r : L^1 \to L^0 \) by \( s(h, g) = s(g), r(h, g) = h \cdot r(g) \) for \( h \in H^1, g \in G^1 \) with \( s(h) = r_H(g) \). Define the multiplication by
\[
(h_1, g_1) \cdot (h_2, g_2) := (h_1 \cdot h_2, (h_2^{-1} \cdot g_1) \cdot g_2)
\]
for \( h_1, h_2 \in H^1, g_1, g_2 \in G^1 \) with \( s(h_1) = r_H(g_1), s(h_2) = r_H(g_2), s(g_1) = h_2 \cdot r(g_2) \).

**Lemma 2.14.** The data above defines a topological groupoid.

**Proof.** The range map is well defined because \( r : G^1 \to G^0 \) is \( H \)-equivariant. We check that the multiplication is well defined. Let \( h_1, h_2 \in H^1, g_1, g_2 \in G^1 \) satisfy \( s(h_1) = r_H(g_1), s(h_2) = r_H(g_2), s(g_1) = h_2 \cdot r(g_2) \). Then \( s(h_1) = r_H(g_1) = r_H(s(g_1)) = r_H(h_2 \cdot r(g_2)) = r(h_2) = s(h_2^{-1}), s(g_1) = r(g_2), s(h_2^{-1} \cdot g_1) \) is defined. And \( s((h_2^{-1} \cdot g_1) \cdot g_2) = r_H((h_2^{-1} \cdot g_1) \cdot g_2) = r(h_2^{-1}) \cdot s(g_1) = s(h_2 \cdot h_2^{-1}), s(g_1) = s(h_2 \cdot g_2) \).

Direct computations give the following:

- \( s((h_1, g_1) \cdot (h_2, g_2)) = s(g_2) = s(h_2, g_2) \)
- \( r((h_1, g_1) \cdot (h_2, g_2)) = h_1 \cdot r(g_1) = r(h_1, g_1) \)

- the arrow \((1_{r_H(x)}, x)\) for \( x \in G^0 \) is the unit arrow on \( x \);
the arrow \((h^{-1}, h \cdot g^{-1})\) is inverse to \((h, g)\) for \(h \in H^1, g \in G^1\) with \(s(h) = r_H(g)\), and the inversion map is a homeomorphism;

- the multiplication is associative.

The source map of \(L\) is an open surjection as the product of two open surjections, namely, \(s: G^1 \to G^0 = L^0\) and the coordinate projection \(L^1 = H^1 \times_{s,H^0,r_H} G^1 \to G^1\), which is the pull-back of the open surjection \(s: H^1 \to H^0\) along \(r_H: G^1 \to H^0\).

**Proposition 2.15.** The map \(r_H: L^0 = G^0 \to H^0\) on objects and the coordinate projection \(pr_1: L^1 = H^1 \times_{s,H^0,r_H} G^1 \to H^1\) on arrows define a continuous functor \(F: L \to H\), which is a groupoid fibration.

**Proof.** Direct computations show that \(F\) is a functor. The map \((F^1, s): L^1 \to H^1 \times_{s,H^0,r_H} G^1\) in (2.2) is equivalent to the map \(H^1 \times_{s,H^0,r_H} G^1 \to H^1 \times_{s,H^0,r_H} G^0\), \((h, g) \mapsto (h, s(g))\). This is equivalent to the pull-back of the open surjection \(s: G^1 \to G^0\) along the map \(pr_2: H^1 \times_{s,H^0,r_H} G^0 \to G^0\). Hence \((F^1, s)\) is an open surjection.

**Example 2.16.** Let \(X\) be a topological space with commuting left and right actions of topological groupoids \(G\) and \(H\). Then \(H\) acts on the transformation groupoid \(G \times X\) on the right by the given action on objects \(x\) in \(X\) and by \((g, x) \cdot h := (g, x \cdot h)\) for \((g, x) \in (G \times X)^1\), \(h \in H^1\) with \(s(x) = s(g, x) = r(h)\). The transformation groupoid for this classical action of \(H\) on \(G \times X\) is the bi-transformation groupoid \((G \times X) \times H\), which has unit space \(X\), arrow space \(G \times_{s,r} X \times_{s,r} H\), \(r(g, x, h) := g \cdot x, s(g, x, h) = x \cdot h^{-1}\), and \((g_1, x_1, h_1) \cdot (g_2, x_2, h_2) := (g_1 \cdot g_2, x_1 \cdot h_1 x_2, h_1 h_2)\) for \(g_1, g_2 \in G^1\), \(x_1, x_2 \in X\), \(h_1, h_2 \in H\) with \(s(g_1) = r(x_i), s(x_i) = r(h_i)\) for \(i = 1, 2\) and \(x_1 h_1^{-1} = g_2 x_2\). The canonical functor \(F^1: G \times X \times H \to H\) is the anchor map \(s: X \to H^0\) on objects and the coordinate projection \(pr_3: G \times_{s,r} X \times_{s,r} H \to H\) on arrows. This is a groupoid fibration by Proposition 2.15.

We may characterise exactly which groupoid fibrations come from classical actions as above. The transformation groupoid \(L = H \times G\) for a classical action comes with a canonical action of \(H\) on \(L^1 := H^1 \times_{s,H^0,r_H} G^1\) by \(h_1 \cdot (h_2, g) := (h_1 \cdot h_2, g)\), with anchor map \(r_H: L^1 \to H^0\), \((h, g) \mapsto r(h)\). This action commutes with the \(\\Box\) action on \(L^1\) itself by right multiplication. Thus it is an actor from \(H\) to \(L\) as in [32, Definition 4.20]. The action of \(H\) on \(L^1\) makes \(F^1\) equivariant: \(F^1(h \cdot l) = h \cdot F^1(l)\) for all \(h \in H^1\), \(l \in L^1\) with \(s(h) = r_H(l)\). There is, however, no canonical functor \(H \to H \times G\) unless \(F^0\) is a homeomorphism because we lack a canonical map \(H^0 \to L^0\).

**Proposition 2.17.** A groupoid fibration \(F: L \to H\) with fibre \(G\) comes from a classical action of \(H\) on \(G\) if and only if there is an actor from \(H\) to \(L\) that makes \(F^1\) \(H\)-equivariant.

**Proof.** We have already described the actor from \(H\) to \(L\) for a classical action. Conversely, assume that an actor from \(H\) to \(L\) is given that makes \(F^1\) \(H\)-equivariant. Hence \(r_H(l) = r(F^1(l)) = F^0(r(l))\). Write \(\bullet\) for the left \(H\)-action on \(L^1\) to distinguish it from the multiplication in \(L\). We claim that the maps \(H^1 \times_{s,H^0,F^0 \circ r} G^1 \to L^1, (h, g) \mapsto h \bullet g\), and \(L^1 \to H^1 \times_{s,H^0,F^0 \circ r} G^1, l \mapsto (F^1(l), F^1(l)^{-1}\bullet l)\), are well-defined, continuous and inverse to each other. Hence they are both homeomorphisms. The first map is clearly well defined, and the second map is well defined because the anchor map \(L^1 \to H^0\) is \(F^0 \circ r\) and \(F^1(F^1(l)^{-1}\bullet l) = F^1(l)^{-1}\cdot F^1(l) = 1_{F^0(s(l))}\), that is, \(F^1(l)^{-1}\bullet l\) belongs to \(G^1\). Both maps are clearly continuous. They are inverse to each other because \(F^1(h \cdot g) = h \cdot F^1(g) = h\) for all \((h, g) \in H^1 \times_{s,H^0,F^0 \circ r} G^1\).
Identify \( L^1 \cong H^1 \times_{s,H^0, p_{0\circ p}} G^1 \) as above. The multiplication in \( L \) must satisfy

\[
(h_1, g_1) \cdot (h_2, g_2) = (h_1 \cdot g_1) \cdot (h_2 \cdot g_2) = (h_1 h_2) \cdot (h_2^{-1} \cdot g_1) \cdot (h_2 \cdot 1_{s(g_1)} \cdot g_2)
\]

\[
= (h_1 h_2) \cdot (h_2^{-1} \cdot 1_{r(g_1)}) \cdot g_1 \cdot (h_2 \cdot 1_{s(g_1)} \cdot g_2).
\]

Since \( F^1 \) is \( H \)-equivariant, \( F^1(h_2^{-1} \cdot 1_{r(g_1)}) = F^1(h_2)^{-1} \) and \( F^1(h_2 \cdot 1_{s(g_1)}) = F^1(h_2) \), so that the product \((h_2^{-1} \cdot 1_{r(g_1)}) \cdot g_1 \cdot (h_2 \cdot 1_{s(g_1)}) \) belongs to \( G^1 \). Since \( F^1 \) is \( H \)-equivariant, the multiplication in \( L \) becomes \((h_1, g_1) \cdot (h_2, g_2) = (h_1 h_2, (h_2^{-1} \cdot g_1) \cdot g_2)\) with

\[
h \cdot g := (h \cdot 1_{r(g)}) \cdot g \cdot (h^{-1} \cdot 1_{s(g)}).
\]

This is exactly as in the transformation groupoid for a classical action. We also define an action of \( H \) on \( G^0 \) by \( h \cdot x = r(h) \cdot l_x \) as in the proof of [32] Proposition 4.21. Reversing the computations in the proof of Lemma 2.14 we see that the formulas above must define a classical action of \( H \) on \( G \) by automorphisms because \( L \) is a groupoid.

\[\square\]

### 2.4. Translation action on the arrow space

A motivating example in [12] is to associate an action on a \( C^* \)-algebra to the translation action of a locally Hausdorff, locally compact groupoid \( H \) on its arrow space \( H^1 \). Since \( H^1 \) is non-Hausdorff, we cannot use the commutative \( C^* \)-algebra of \( C_{0}\)-functions on \( H^1 \). Instead, we cover \( H^1 \) by Hausdorff, open subsets and form the resulting Čech groupoid \( G \). It should carry an action of \( H \), which then induces an action of \( H \) on the groupoid \( C^* \)-algebra \( C^*(G) \). This is accomplished in [12] for étale groupoids. Groupoid fibrations allow to do the same for arbitrary locally Hausdorff groupoids (the \( C^* \)-algebraic assertions also need a Hausdorff, locally compact object space, a locally compact arrow space, and a Haar system). We construct the relevant example at the end of this section, based on some simpler examples and a proposition on the composition of groupoid fibrations.

#### Example 2.18

Let \( H \) be a topological groupoid and \( p: X \to H^0 \) an open, continuous surjection. The pull-back \( p^*(H) \) of \( H \) along \( p \) is the topological groupoid with object space \( p^*(H)^0 := X \), arrow space

\[
p^*(H)^1 := X \times_{p,H^0} H^1 \times_{s,H^0,p} X = \{(x,h,y) \mid p(x) = r(h), p(y) = s(h)\},
\]

\( s(x,h,y) = y, r(x,h,y) = x \), and \( (x,h,y) \cdot (y,h',z) = (x, hh', z) \) (see [32] Example 3.13]). If \( H \) is a space viewed as a groupoid with only identity arrows, this gives the Čech groupoid \( p^*(Y) \) of \( p \), which has unit space \( X \) and arrow space \( X \times_{p,Y,p} X = \{(x,x') \mid p(x) = p(y)\} \).

The maps \( F^0 := p \) on objects and \( F^1(x,h,y) := h \) on arrows give a functor \( F: p^*(H) \to H \) (see [32] Example 3.18]). It is always a groupoid fibration: identifying \( H^1 \times_{s,H^0,p} p^*(H)^0 = H^1 \times_{s,H^0,p} \), the map \((F^1, s)\) in (2.2) becomes the coordinate projection \( X \times_{p,H^0,p} H^1 \times_{s,H^0,p} X \to H^1 \times_{s,H^0,p} X \), which is an open surjection because it is the pull-back of the open surjection \( p \) along \( H^1 \times_{s,H^0,p} X \to H^0, (h, x) \mapsto r(h) \). The fibre of \( F \) is the Čech groupoid \( p^*(H^0) \) of \( p \).

#### Lemma 2.19

Any functor from a topological groupoid \( L \) to a space \( Y \) is a groupoid fibration with fibre \( L \). It is a groupoid covering if and only if \( L \) is a space viewed as a groupoid.

**Proof.** A functor \( F: L \to Y \) is equivalent to a continuous map \( F: L^0 \to Y \) that is \( L \)-invariant, that is, \( f(s(t)) = f(r(t)) \) for all \( t \in L^1 \). There is a homeomorphism \( Y \times_{id,Y} L^0 \to Y \), \( (y,x) \mapsto x \), which identifies the map in (2.2) with the source map \( s: L^1 \to L^0 \). Since we assume \( s \) to be an open surjection, \( F \) is a groupoid fibration automatically. It is a groupoid covering if and only if \( s: L^1 \to L^0 \) is a
Remark 2.20. A functor from a space $X$ to a topological groupoid $H$ is a fibration only if the relevant part of $H$ is a space viewed as a groupoid. More precisely, such a functor is the same as a continuous map $f : X \to H^0$, and this is a fibration if and only if any arrow in $H$ with source or range in the image of $f$ is an identity arrow.

Proposition 2.21. Let $F_i : L_i \to H_i$ be groupoid fibrations with fibre $G_i$ for $i = 1, 2$ with $H_1 = L_2$. Let $F := F_2 \circ F_1 : L_1 \to H_1 = L_2 \to H_2$. Then $F$ is a groupoid fibration as well; let $G$ be its fibre. The functor $F_1$ restricts to a functor $G \to G_2$, which is a groupoid fibration with fibre $G_1$. If $F_1$ is a groupoid covering, then $F|_G : G \to G_2$ is a groupoid covering. If $F_2$ is a groupoid covering, then $G_1 = G$.

Proof. We have assumed that the maps $(F^1_i, s_{L_i}) : L^1_i \to H^1_i \times_{s_{H_i}, H^0_i} L^0_i$ for $i = 1, 2$ are open surjections. The pull-back of the open surjection $(F^1_2, s_{L_2})$ along the map $F^0_1 : L^0_1 \to H^0_1 = L^0_2$ remains an open surjection. The homeomorphism

$$(H^1_2 \times_{s_{H_2}, H^0_2} L^0_2) \times_{pr_2, L^0_1} F^0_1 \to L^0_1 \simeq H^1_2 \times_{s_{H_2}, H^0_2} F^0_1 \times_{pr_1} L^0_1$$

identifies this pull-back with the map

$$(H^1_2 \times_{s_{H_2}, H^0_2} F^0_1 \times_{pr_1} L^0_1) = L^2_2 \times_{s_{H_2}, F^0_2} L^0_2 \to H^1_2 \times_{s_{H_2}, F^0_2} L^0_1, \quad (g, x) \mapsto (F^1_2(g), x).$$

Composing with $(F^1_1, s_{L_1})$ gives $(F, s) : L^1_1 \to H^1_2 \times_{s_{H_2}, F^0_2} L^0_1$. So $F$ is a groupoid fibration.

Since $F^2_2 \circ F^1_1(g)$ is a unit if and only if $F^1_1(g)$ belongs to the fibre $G_2$ of $F_2$, the preimage of $G_2 \times_{s_{F^2_2}, F^0_2} L^0_1 \subseteq H^1_1 \times_{s_{F^0_1}, F^0_1} L^0_1$ under the map $(F^1_1, s)$ is naturally isomorphic to the fibre of $F$. Restricting an open surjection to the preimage of a subspace is also a case of pull-back, so the restriction remains an open surjection. This says exactly that the restriction $F_1|_G : G \to G_2$ is a groupoid fibration. Since $G_1 \subseteq G \subseteq L_1$, the fibre of $F_1|_G$ is the same as for $F_1$. So the groupoid fibration $F_1|_G : G \to G_2$ has fibre $G_1$ as asserted.

If $F_1$ is a groupoid covering, then its fibre is just a space. Hence the fibration $F_1|_G$ also has a space as its fibre and thus is a groupoid covering by Proposition 2.10. If $F_2$ is a groupoid covering, then $G_2$ is just a space. Hence the fibre $G_1$ of the groupoid fibration $F_1|_G : G \to G_2$ is $G$ by Lemma 2.19.

Example 2.22. Let $H$ be a topological groupoid and let $p : X \to H$ be an open surjection. We let $H$ act on $H^1$ by the left translation action and form its transformation groupoid $H \ltimes H^1$. Example 2.13 applied to the map $p$ and the groupoid $H \ltimes H^1$ gives a groupoid fibration $F_1 : p^*(H \ltimes H^1) \to H \ltimes H^1$ with the Čech groupoid of $p$ as its fibre. Example 2.13 gives a groupoid covering $F_2 : H \ltimes H^1 \to H$. Proposition 2.21 shows that the composite functor $F := F_2 \circ F_1 : p^*(H \ltimes H^1) \to H \ltimes H^1 \to H$ is a groupoid fibration with the same fibre $p^*(H^1)$ as $F^1$. The groupoid fibration $F : p^*(H \ltimes H^1) \to H$ describes an action of $H$ on the Čech groupoid $p^*(H^1)$ of $p$ with transformation groupoid $p^*(H \ltimes H^1)$.

Lemma 2.23. The groupoid $p^*(H \ltimes H^1)$ is isomorphic to the Čech groupoid of the open surjection $s \circ p : X \to H^1 \to H^0$.

Proof. There is a canonical homeomorphism $(r, s) : (H \ltimes H^1)^1 \simeq H^1 \times_{s, H^0} H^1, \quad (h_1, h_2) \mapsto (h_1 \cdot h_2, h_2)$. This together with the identity on objects identifies $H \ltimes H^1$ with the Čech groupoid $s^*(H^0)$ of $s : H^1 \to H^0$. Thus $p^*(H \ltimes H^1) \cong p^*s^*(H^0) \cong (s \circ p)^*(H^0)$.

\[ \square \]
Now let $H$ be a topological groupoid with locally Hausdorff arrow space $H^1$. Choose a cover $H^1 = \bigcup_{U \in \mathcal{U}} U$ by Hausdorff, open subsets and let $X := \bigcup_{U \in \mathcal{U}} U$ with the canonical map $p: X \to H^1$ that is the inclusion map on each component $U \subseteq Y$. This is a local homeomorphism and a fortiori an open surjection. The construction above gives a groupoid fibration from the Čech groupoid $(s \circ p)^*(H^0)$ of $s \circ p: X \to H^0$ to $H$ with the Čech groupoid $p^*(H^1)$ as its fibre. Both $(s \circ p)^*(H^0)$ and $p^*(H^1)$ have object space $X$, which is Hausdorff, and arrow spaces contained in $X \times X$, which forces their arrow spaces to be Hausdorff as well. The Čech groupoid $p^*(H^1)$ is étale because $p$ is a surjective, open local homeomorphism.

2.5. Groupoid extensions.

**Definition 2.24.** A (topological) groupoid extension is a diagram $G \hookrightarrow L \to H$ with the functor $F: L \to H$ is a groupoid fibration of topological groupoids with fibre $G \subseteq L$, such that $F^0: L^0 \xrightarrow{\sim} H^0$ is a homeomorphism.

**Lemma 2.25.** A functor $F: L \to H$ that is a homeomorphism on objects is a groupoid fibration and only if $F^1: L^1 \to H^1$ is an open surjection, and a groupoid covering if and only if $F^1$ is a homeomorphism as well, that is, $F$ is an isomorphism of topological groupoids.

**Proof.** Simplify (2.2) using the homeomorphism $H^1 \times_{s,H^0, F^0} L^0 \xrightarrow{\sim} H^1$, $(h, x) \mapsto h$. □

**Proposition 2.26.** Let $G$ and $H$ be topological groups. A groupoid fibration $L \to H$ with fibre $G$ is the same as an extension of topological groups $G \hookrightarrow L \to H$.

**Proof.** We have $L^0 = G^0 = \ast$, so $L$ is a group as well. A group fibration is the same as a continuous, open, surjective group homomorphism by Lemma 2.25. The fibre is the kernel of this homomorphism. □

A topological group extension $G \to L \to H$ comes from a classical action of $H$ on $G$ by group automorphisms if and only if it splits by a continuous group homomorphism $H \to L$: this is well known, and also follows from Proposition 2.17. We consider any topological group extension as an “action” of $H$ on $G$. For Polish groups, such extensions may be classified by Borel measurable 2-cocycles, see [5].

**Remark 2.27.** Proposition 2.26 works in a category with pretopology if we assume that our category has a final object $\ast$. Then we may define a group as a groupoid with $G^0 = \ast$. (Any map from a non-empty space to the one-point space is an open surjection. Some basic features of topological groups and their actions only work in the abstract setting if we assume that any map to the final object is a cover, except possibly for the map from the initial object, if that exists, compare [5] Assumption 2.10 and Examples 3.14 and 4.9.1.)

**Lemma 2.28.** Let $G \hookrightarrow L \to H$ be a groupoid extension. Then $G \subseteq L$ is a normal subgroup bundle on which the range map is open, and $H = L/G$. Conversely, any normal subgroup bundle $G \subseteq L$ on which the range map is open appears in a groupoid extension that is unique up to isomorphism.

**Proof.** All three groupoids in a groupoid extension have the same or homeomorphic object spaces: $G^0 = L^0 \cong H^0$. If $g \in G^1$, then $F^0(s(g)) = s(F^1(g)) = r(F^1(g)) = F^0(r(g))$, so $G$ is a bundle of groups contained in $L$. If $g \in G^1$, $t \in L^1$ with $s(g) = r(t)$, then $l_{g^{-1}}l^g \in G^1$ as well, that is, the subgroup bundle $G$ is normal in $L$. The range and source maps of $L$ restrict to open mappings on $G$ by Lemma 2.23. Conversely, let $G \subseteq L$ be a normal subgroup bundle. This is a topological groupoid with the subspace topology if and only if the range map on $L$ restricts to an open map on $G$. Assume this. Since $G$ is a normal subgroup bundle, there is a unique
multiplication on \( H^1 := L^1/G \) so as to give a groupoid \( H \) with object set \( H^0 \) and such that the quotient map \( L \to H \) is a functor. We equip \( H^1 \) with the quotient topology. Then \( H \) is a topological groupoid. The quotient map \( L^1 \to L^1/G \cong H^1 \) is automatically an open surjection by [32 Proposition 9.34]. Hence \( G \to L \to H \) is an extension of topological groupoids by Lemma 2.25. If \( G \to L \to H \) is a groupoid extension, then \( H \) is canonically homeomorphic to the quotient \( L/G \) described above by Proposition 2.25, simplified using \( H^1 \times_s, H^0, F^0 \) \( L^0 \cong H^1 \) as in the proof of Lemma 2.25. Thus any groupoid extension comes from a unique normal subgroup bundle \( G \subseteq L \) on which the range map is open.

Example 2.29. Let \( L \) be an étale groupoid and consider the interior \( G := \text{Iso}^0(L) \) of its isotropy bundle \( \{ g \in L \mid s(g) = r(g) \} \). This is an open, normal group bundle. So we may form a groupoid extension \( G \hookrightarrow L \to H \) with \( H^1 := L^1/G \) as in Lemma 2.25.

Example 2.30. A central groupoid extension \( G \hookrightarrow L \to H \) is an extension where \( G = L^0 \times \Gamma \) is a trivial group bundle such that the conjugation action of arrows in \( L \) induces the trivial map on \( \Gamma \). More precisely, let \( \iota : L^0 \times \Gamma \to L \) be the embedding, then we require \( \iota(r(l), \gamma) \cdot l = l \cdot \iota(s(l), \gamma) \) for all \( \gamma \in \Gamma \) and \( l \in L \). This forces \( \Gamma \) to be Abelian. Such extensions have been extensively studied, especially for \( \Gamma = \mathbb{T} \), which leads to twisted groupoid \( C^* \)-algebras, see [33, 40]. These extensions also appear in the study of groupoid cohomology (see, for instance, [37]) and are closely related to gerbes and thus to twisted \( K \)-theory, see [29, Remark 2.14).

The class of groupoid fibrations where \( F^0 \) is not a homeomorphism but only an open surjection also deserves special attention (compare Remark 2.8). They should behave like groupoid extensions where the “kernel” is no longer a group bundle.

Lemma 2.31. Let \( F : L \to H \) be a groupoid fibration. The map \( F^1 : L^1 \to H^1 \) is surjective or open if and only if \( F^0 : L^0 \to H^0 \) is surjective or open, respectively.

Proof. Since \( s : H^1 \to H^0 \) is an open surjection, so is the coordinate projection \( H^1 \times_{s, F^0} L^0 \to L^0 \). Therefore, the coordinate projection \( H^1 \times_{s, F^0} L^0 \to H^1 \) is an open surjection if and only if \( F^0 : L^0 \to H^0 \) is an open surjection (this is the locality of covers for the pretopology of open surjections in the notation of [32]). Even more, it can be checked by hand that \( F^0 \) is open or surjective, respectively, if and only if the coordinate projection \( H^1 \times_{s, F^0} L^0 \to H^1 \) is. If \( F \) is a groupoid fibration, then the map \((F^1, s) : L^1 \to H^1 \times_{s, F^0} L^0\) is an open surjection as well. The two-out-of-three property for the pretopology of open surjections says that the composite map \( F^1 : L^1 \to H^1 \times_{s, F^0} L^0 \to H^1 \) is an open surjection if and only if the projection \( H^1 \times_{s, F^0} L^0 \to H^1 \) is one; once again, it can be checked that the statement remains true for open maps and surjective maps separately. Since the coordinate projection \( H^1 \times_{s, F^0} L^0 \to H^1 \) is open or surjective if and only if the map \( F^0 : L^0 \to H^0 \) is so, \( F^1 \) is open or surjective if and only if \( F^0 \) is so.

Example 2.32. A (continuous) open, surjective functor \( F : L \to H \) need not be a fibration. For instance, let \( K \) be the pair groupoid on the 2-element set. This is a finite, discrete groupoid with two objects and four arrows \( K^1 := \{a, b, \gamma, \gamma^{-1}\} \), where \( a, b \) are the two unit arrows and \( \gamma \) is the non-trivial arrow with \( s(\gamma) = a, r(\gamma) = b \). Let \( K_n := \{a_n, b_n, \gamma_n, \gamma_n^{-1}\} \) be a copy of \( K \) for each \( n \in \mathbb{N} \) and \( L := \bigsqcup_{n \in \mathbb{N}} K_n \). Let \( F : L \to \mathbb{Z} \) be the unique functor that sends \( \gamma_n \mapsto n \). This functor between discrete groupoids is surjective, but not a fibration.

Remark 2.33. The notion of an “extension” of Borel groupoids in [31 Definition 5.2.7] is closely related to our definition of a groupoid fibration with an open surjection \( F^0 \).

In the world of Borel structures, the condition of being an open surjection is replaced...
by the condition of being surjective, so the requirement is that the maps $F^0$ and $(F^1,s)$ be surjective.

Under mild extra conditions, amenability of Borel groupoids is preserved under extensions by $[13]$ Theorem 5.2.14]. Moreover, Renault $[41]$ proves that Borel amenability is equivalent to topological amenability for locally compact and locally Hausdorff topological groupoids with Haar systems and Hausdorff unit space. Therefore, if $L \to H$ is a groupoid fibration with fibre $G$ and the map $F^0 : L^0 \to H^0$ is surjective, then $L$ is amenable if $G$ and $H$ are. We expect this to remain true without the surjectivity assumption on $F^0$, but have not examined the matter closely.

3. Étale Groupoid Fibrations and Inverse Semigroup Gradings

Let $H$ be an étale groupoid. We are going to show that groupoid fibrations $L \to H$ are essentially equivalent to the gradings by inverse semigroups used in $[12]$ to model groupoid actions on other groupoids. Since this section will not be needed in the rest of the paper, we assume that the reader is familiar with the relevant notions from $[12]$. Recall that every étale groupoid $H$ is isomorphic to a groupoid of germs $S \ltimes Z$ for some action of a unital inverse semigroup $S$ on a space $Z$ (see $[17]$). We assume $H$ to be of this form. 

**Theorem 3.1.** A groupoid fibration $F : L \to S \ltimes Z$ with fibre $G$ is equivalent to an $S$-grading on $L$ with $L_1 = G$—so that $L$ is the transformation groupoid $S \ltimes G$ for an action of $S$ on $G$ by partial equivalences—together with a $G$-invariant continuous map $G^0 = L^0 \to Z$ that is $S$-equivariant for the induced action of $S$ on $G^0 / G$.

**Proof.** First let $F$ be a groupoid fibration. We define an $S$-grading on $L$ by $L_t := (F^1)^{-1}(t)$ for $t \in S$, viewed as an open subset of $(S \ltimes Z)^1$. More precisely, each $t \in S$ is viewed as the set of germs $[t,x] \in S \ltimes Z$ with $x$ in the domain $\text{dom}(t) = D_{1,t} \subseteq Z$ of the $S$-action. The subspaces $L_t$ are open because $F^1$ is continuous. The unit fibre $L_1$ of the grading is equal to the fibre of $G$ of $F$ because the unit arrows of $S \ltimes Z$ are exactly the germs of the form $[1,x]$. The following properties required for an $S$-grading are trivial:

$$L_t \cdot L_u \subseteq L_{tu}, \quad L_t^{-1} = L^{-t}, \quad \bigcup_{t \in S} L_t = L^1.$$

If $l \in L_t \cap L_u$, then $F^1(l) \in t \cap u$; by the definition of $S \ltimes Z$, this means that $F^1(l) \in v$ for some $v \in S$ with $v \leq t,u$. Thus

$$L_t \cap L_u = \bigcup_{v \in S, v \leq t,u} L_v.$$

The only property of an $S$-grading that requires the fibration condition is

$$L_t \cdot L_u \supseteq L_{tu}$$

for all $t,u \in S$. Let $l \in L_{tu}$. Then $F^1(l) \in tu$, so we may factor $F^1(l) = h_1 h_2$ with $h_1 \in t$, $h_2 \in u$. Since $s(h_2) = s(F^1(l)) = F^0(s(l))$ and $(2.2)$ is surjective, there is $l_2 \in L^1$ with $s(l_2) = s(l)$ and $F^1(l_2) = h_2$. Then $l_2 \in L_u$ because $h_2 \in u$, and $l_1 := l \cdot l_2^{-1} \in L_t$ because $F^1(l_1) = h_1 \in t$. Thus $l \in L_t L_u$ as desired.

Since $F^0(s(g)) = s(F^1(g)) = r(F^1(g)) = F^0(r(g))$ for all $g \in G^1$, the continuous map $F^0 : G^0 = L^0 \to Z$ is $G$-invariant and hence descends to a continuous map $G^0 / G \to Z$. We must show that this map is $S$-equivariant. We recall how the action of $S$ on $G^0 / G$ by partial homeomorphisms is defined (see the comments before Remark 2.14 in $[12]$). For $t \in S$, let $U_{tt^*} := r(L_t) = s(L_{t^*})$; these are $G$-invariant open subsets of $L^0$, which we view as open subsets of $G^0 / G$. If $x \in U_{tt^*} = s(L_t)$, then pick $l \in L_t$ with $s(l) = x$ and define $t : [x] := [r(l)]$, where the brackets mean
that we take the $G$-orbit. This does not depend on the choice of $l$ because all choices of $L$ are of the form $g \cdot l$ with $g \in G^1$ and we divided out the $G$-action. This is indeed a homeomorphism from $[U_{1,t}]$ onto $[U_{1r}]$, and these partial homeomorphisms form an action of $S$.

Since $F$ is a functor, $F^1(l) \in t$ has range $F^0(r(l))$ and source $F^0(s(l))$. Since $t \cdot F^0(s(l)) = F^0(r(l))$, the map $G^0/G \to Z$ induced by $F^0$ is $S$-equivariant. We have built an $S$-grading and an $S$-equivariant map from a groupoid fibration.

Conversely, take an $S$-grading on $L$ with $L_1 = G$ and an $S$-equivariant continuous map $\rho: G^0/G \to Z$. We are going to define a groupoid fibration $L \to S \times Z$. We let $F^0$ be the composite of the orbit space projection $L^0 \to G^0/G$.

For $l \in L$, there is $t \in S$ with $l \in L_t$. We want to define $F^1(l) \in (S \times Z)^1$ as the germ $[t, \rho(s(l))]$ of $t$ at $\rho(s(l))$. Since $\rho$ is $S$-equivariant, $\rho(s(l))$ belongs to the domain $D_{1,t}$ of the partial homeomorphism on $Z$ given by $t$, so $[t, \rho(s(l))]$ is a well-defined arrow in $S \times Z$. We must also check that $F^1(l)$ does not depend on the choice of $t$. Let $l \in L_t \cap L_u$. The assumptions for an $S$-grading give $v \in S$ with $v \leq t, u$ and $l \in L_v$. Then $\rho(s(l)) \in D_{1,v}$, so $[t, \rho(s(l))] = [v, \rho(s(l))]$. The map $F^1$ is continuous because its restriction to $L_t$ is continuous for each $t \in S$. It is compatible with range maps by the equivariance condition $t \cdot \rho(s(l)) = \rho(r(l))$.

Multiplicativity follows from $L_1 L_u \subseteq L_{1u}$, so $F$ is a continuous functor.

If $(x, [t, z]) \in L^0 \times_{F^0, Z,S} (S \times Z)^1$, then $\rho(x) = z \in D_{1,t}$. Since $\rho$ is $S$-equivariant, this implies $x \in \rho^{-1}(D_{l,t}) = U_{1,t} = s(L_{1,t}) = s(L_t)$. Thus there is $l \in L_t$ with $s(l) = x$. That is, the map

$$L^1 \xrightarrow{(s,F^1)} L^0 \times_{F^0, Z,S} (S \times Z)^1$$

is surjective.

This map is open because $S \times Z$ is étale and $s: L^1 \to L^0$ is open. We now prove this claim in detail. It suffices to check that the restriction of $(s, F^1)$ to $L_t$ is open because the subsets $L_t \subseteq L^1$ form an open cover of $L$. The $F^1$-image of $L_t$ is contained in a bisection of $S \times Z$ associated to $t$. Since the source map of the étale groupoid $S \times Z$ restricts to a homeomorphism on any bisection, the map in (2.2) restricted to $L_t$ is open if and only if $s: L_t \to L^0$ is open; this is assumed for all topological groupoids.

Next, we check that the $S$-grading on $L$ associated to the functor $F$ is the given one; that is, $F^1(l) \in t$ if and only if $l \in L_t$. By construction, if $l \in L_t$ then $F^1(l) \in t$. Conversely, let $l \in L_t$ satisfy $F^1(l) \in t$. There is $u \in S$ with $l \in L_u$, so $F^1(l) = [u, \rho(s(l))] \in t$. Hence there is an idempotent $e \in S$ with $\rho(s(l)) \in D_e$ and $te = ue$. Then $s(l) \in U_e = s(L_e)$ and hence $l = l \cdot 1_s(l) \in L_u \cdot U_e = L_{ue} = L_{te} = L_t \cdot U_e \subseteq L_t$ as desired. In particular, $F^1(l)$ is a unit, that is, belongs to the bisection $1 \in S$, if and only if $l \in L_1 = G$. Thus $G$ is the fibre of the groupoid fibration $F$.

We have now turned an $S$-grading with a compatible map $L^0 \to Z$ into a fibration $L \to S \times Z$ and vice versa. And we have checked that when we turn an $S$-grading into a groupoid fibration and back, this gives the same $S$-grading we started with. Conversely, let us start with a groupoid fibration $F: L \to S \times Z$, turn it into an $S$-grading $L_t = (F^1)^{-1}(t)$ on $L$ with compatible map $G^0/G \to Z$, and then construct a groupoid fibration $\tilde{F}$ from this. The new groupoid fibration has the same map $F^0$ on objects, and it has $(\tilde{F}^1)^{-1}(t) = (F^1)^{-1}(t)$ for all $t \in S$. This implies $\tilde{F}^1 = F^1$ because elements of $t$ are distinguished by their source or range. □

Remark 3.2. The proof of Theorem 3.1 shows that the map in (2.2) is open for any functor $F: L \to H$ if $H$ is an étale groupoid. So $F$ is a groupoid fibration if and only if the map in (2.2) is surjective.
We have already seen that \( F^1 \) are the same as
groupoid fibrations in our sense with the extra property that \( F^1 \) is an open surjection.
This is equivalent to \( F^0 \) being an open surjection by Lemma [2.41].

**Example 3.3.** Let \( X \) be a groupoid equivalence between two topological groupoids \( G \) and \( H \). Its linking
groupoid \( L \) is a topological groupoid with unit space \( G^0 \sqcup H^0 \) and
arrow space \( G^1 \sqcup X \sqcup X^* \sqcup H^1 \), where \( X^* \) denotes the dual (or inverse)
equivalence of \( X \). Here \( G \) acts on the left and \( H \) on the right of \( X \). The groupoid structure is
the canonical one involving the groupoid structures of \( G \) and \( H \) and the structure
of the equivalence bibundle \( X \). Let \( K := \{ a, b, \gamma, \gamma^{-1} \} \) be the pair groupoid on the
2-element set as in Example [2.32]. There is an obvious functor \( X \)
maps \( G \) and \( G \) open surjections.

[12, Proposition 2.15] and [32, Proposition 9.18]).

4. **Locally Hausdorff and locally compact groupoids**

The main result in this section says that \( L \) inherits certain topological properties
from \( G \) and \( H \). Furthermore, we relate some properties of \( H \) to the property of \( G \)
being open or closed in \( L \).

**Theorem 4.1.** Let \( F : L \to H \) be a groupoid fibration with fibre \( G \). If both \( H 
\text{and} \ G \) are Hausdorff or locally Hausdorff, respectively, then so is \( L \). If \( G \) and \( H 
\text{are} \text{locally} \text{Hausdorff and locally compact}, \text{then} \text{so is} \ L \).

**Proof.** The proofs for the Hausdorff and locally Hausdorff case are the same, merely
adding or removing the word “locally” where needed. The main tool is the following.

Let \( f : X \to Y \) be a continuous open surjection. Then \( Y \) is (locally) Hausdorff if
and only if \( \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\} \) is (locally) closed in \( X \times X \) (see
[12], Proposition 2.15) and [32, Proposition 9.18]).

Assume first that \( G^1 \) and \( H^l \) are (locally) Hausdorff for \( i = 0, 1 \). Hence \( L^0 = G^0 \)
is (locally) Hausdorff. The space \( L^1 \) is (locally) Hausdorff if and only if the diagonal
in \( L^1 \times L^1 \) is (locally) closed; this is the above criterion applied to \( \text{Id}_{L^1} \).

Since \( G^1 \) is (locally) Hausdorff, the diagonal in \( G^1 \times G^1 \) is (locally) closed. Its
preimage under the continuous map \( G^1 \to G^1 \times G^1 \), \( g \mapsto (g, \text{ls}(g)) \), is \( G^0 \subseteq G^1 \).

Since preimages of (locally) closed subsets are (locally) closed, \( G^0 \subseteq G^1 \) is (locally)
closed. Then the preimage of \( G^0 \) under the coordinate projection \( G^1 \times s_{L^0,1} L^1 \to G^1 \) is also
(locally) closed in \( G^1 \times s_{L^0,1} L^1 \). The homeomorphism (2.6) identifies this with the diagonal in \( L^1 \) as a subspace of the fibre product
\( L^1 \times_{H^1 \times_{H^0} L^0} L^1 \).

The criterion above also applies to the open surjection in (2.2). The space
\( H^1 \times s_{L^0,1} L^0 \) is (locally) Hausdorff because it is a subspace of the (locally)
Hausdorff space \( H^1 \times L^0 \). Hence \( L^1 \times_{H^1 \times_{H^0} L^0} L^1 \) is (locally) closed in \( L^1 \times L^1 \).

If \( A \) is (locally) closed in \( B \) and \( \bar{B} \) is (locally) closed in \( C \), then \( A \) is (locally)
closed in \( C \). Thus the results of the previous two paragraphs together say that the
diagonal is (locally) closed in \( L^1 \times L^1 \) as desired.

Now assume that \( G^1 \) and \( H^1 \) are locally Hausdorff and locally compact for \( i = 0, 1 \).
We have already seen that \( L^0 \) and \( L^1 \) are locally Hausdorff. And \( L^0 = G^0 \) is locally
compact as well. We must show that each \( l \in L^1 \) has a compact, Hausdorff neigh-
bourhood; then any neighbourhood of \( l \) contains a compact, Hausdorff neighbour-
hood. Let \( A \subseteq H^1 \times_{H^0} L^0 \) be a compact, Hausdorff neighbourhood of \( (F^1(l), s(l)) \).
Then \( \tilde{A} := (F^1(s^{-1}(A))) \) is a neighbourhood of \( l \), so we may restrict attention
to this subspace of \( L^1 \). Choose a compact, Hausdorff neighbourhood \( B \) of \( 1_{s(l)} \)
in \( G^1 \). The homeomorphism (2.6) maps \( B \times s_{L^0,1} \tilde{A} \) onto a neighbourhood of \( (l, l) \)
in $L^1 \times_{H^1 \times_{H^0} L^0} L^1$. Hence there is an open neighbourhood $C$ of $l$ in $\hat{A}$ such that all $(l_1, l_2) \in C \times C$ with $(F^1, s)(l_1) = (F^1, s)(l_2)$ have $l_1l_2^{-1} \in B$. We may assume $C$ Hausdorff because we already know that $L^1$ is locally Hausdorff. Since $(F^1, s)$ is open, the subset $(F^1, s)(C)$ is open in $A \subseteq H^1 \times_{H^0} L^0$, so it contains a compact neighbourhood $D'$ of $(F^1(l), s(l))$. Since $A$ is Hausdorff, compact subsets are closed. So $D := \{ k \in C \mid (F^1, s)(k) \in D' \}$ is relatively closed in $C$.

Let $(k_i)_{i \in I}$ be a net in $D$. We claim that some subnet converges in $C$. First, since $D'$ is compact, we can choose a subnet $(k'_j)_{j \in J}$ such that the net $(F^1, s)(k'_j)$ converges in $D'$. To simplify notation, we assume that already $(F^1, s)(k_i)$ converges. Since the map in (22) is open, we may lift $(F^1, s)(k_i)$ to a convergent net $(k'_j)_{j \in J}$ in $C$ (see [17, Proposition 1.15]). Lifting means that the index set $J$ maps to $I$ by a cofinal map and $(F^1, s)(k(\iota_j)) = (F^1, s)(k'_j)$. Once again, we simplify notation by assuming that the net $k'$ is indexed by the same directed set $I$, so we have $(F^1, s)(k) = (F^1, s)(k'_j)$. Hence (2.6) gives $g_i \in G^1$ with $g_i \cdot k'_j = k_i$. Since $k_i, k'_j \in C$, even $g_i \in B$. Since $B$ is compact, we may choose a convergent subnet of $(g_i)$. As before, we simplify notation by assuming that $(g_i)$ itself converges. Since the multiplication is continuous and $(g_i)$ and $(k'_j)$ converge, it follows that $(k_i)$ converges towards some limit point in $C$.

Since $D$ is relatively closed in $C$, the limits of nets in $D$ that converge in $C$ belong to $D$. Thus every net in $D$ has a convergent subnet, that is, $D$ is compact. It is Hausdorff as well by construction.

The following proposition describes when $H$ is Hausdorff. This is unrelated to $L$ or $G$ being Hausdorff because in the example after Lemma 2.23 $L$ and $G$ are always Hausdorff, but $H$ is only locally Hausdorff.

**Proposition 4.2.** Let $F: L \to H$ be a fibration of topological groupoids with Hausdorff object spaces. If $H$ is Hausdorff, then $G^1$ is closed in $L^1$. Conversely, if $F^0$ is open and surjective and $G^1$ is closed in $L^1$, then $H$ is Hausdorff.

**Proof.** The topological groupoid $H$ is Hausdorff if and only if its unit space $H^0$ is Hausdorff and closed in $H^1$ by [10, Lemma 5.2]. By definition, $G^1$ is the inverse image of the units of $H$ under $F^1$. So $G^1$ is closed in $L^1$ if $H$ is Hausdorff. Conversely, if $F^1$ is open and surjective and $G^1$ is closed, then $F^1(L^1 \setminus G^1) = H^1 \setminus H^0$ is open in $H^1$ because $F^1$ is open. Since we assume $H^0$ to be Hausdorff, [10, Lemma 5.2] shows that $H^1$ is Hausdorff. Lemma 2.31 shows that $F^0$ is an open surjection if and only if $F^1$ is.

**Example 4.3.** Let $G \hookrightarrow L \twoheadrightarrow H = L/G$ be the extension (hence fibration) associated to an étale groupoid $L$ and its open isotropy subgroupoid $G$ as in (2.3) by Proposition 2.5. $H$ is Hausdorff if and only if $G^1$ is closed in $L^1$ (compare [10, Proposition 2.5]). The space $L^1$ need not be Hausdorff for this to hold.

**Remark 4.4.** A groupoid $H$ is étale if and only if $H^0$ is open in $H^1$ (see [10] Theorem 1.4) and recall our standing assumption that $s$ and $r$ be open). As in the proof of Proposition 4.2 this implies that $G^1$ is open in $L^1$ if $G$ is the fibre of a fibration $F: L \to H$. Conversely, if $F^1$ is an open surjection and $G^1$ is open in $L^1$, then $F^1(G^1) = \{ x \mid x \in H^0 \}$ is open in $H^1$, so that $H$ is étale.

5. HAAR SYSTEMS AND GROUPOID FIBRATIONS

Let $F: L \to H$ be a groupoid fibration with fibre $G$. We assume that $G$ and $H$ are locally Hausdorff, locally compact groupoids with Hausdorff object spaces. Then $L$ is also locally Hausdorff and locally compact by Theorem 4.1.
Theorem 5.1. Let \((\lambda^x)_{x \in G^0}\) and \((\mu^y)_{y \in H^0}\) be Haar systems on \(G\) and \(H\). These induce a Haar system \((\nu^z)_{z \in L^0}\) on \(L\), given by

\[
\int_{L^1} f(l) \, d\nu^z(l) = \int_{H^1} \int_{L^1} f(l) \, d\lambda^{(h,x)}(l) \, d\mu^{\nu^z(h)}(h)
\]

for the continuous family of measures \(\lambda\) along the fibres of the open surjection \((F^1, r) : L^1 \to H^1 \times_{r,H^0,F^0} L^0\) given by \(\int_{L^1} f(l) \, d\lambda^{(h,x)}(l) = \int_{H} f(ky) \, d\lambda^{\nu^z(k)}(y)\) for any \(k \in L^1\) with \(F^1(k) = h\) and \(r(k) = x\).

Proof. The Haar systems for \(G\) and \(H\) are continuous families of measures along the fibres of the maps \(r : G^1 \to G^0\) and \(r : H^1 \to H^0\). We need a continuous family of measures along the fibres of \(r : L^1 \to L^0\). To use our data, we are going to factorise \(r : L^1 \to L^0\) into two maps related to \(G\) and \(H\).

We apply the inversion in \(L\) to the principal bundle in Proposition 2.5 to see that the right action of \(G\) on \(L^1\) also gives a principal bundle, with bundle projection

\[
(F^1, r) : L^1 \to H^1 \times_{r,H^0,F^0} L^0.
\]

The Haar system for \(G\) induces a continuous family of measures on the fibres of any principal \(G\)-bundle, see [37]. In our case, this gives the following family of measures \(\lambda\). For \((h, x) \in H^1 \times_{r,H^0,F^0} L^0\), choose some \(k \in L^1\) with \(F^1(k) = h\) and \(r(k) = x\). Then the map \(G^0(k) \to (F^1, r)^{-1}(h, x)\), \(g \mapsto k \cdot g\), is a homeomorphism. Thus we may transfer the measure \(\lambda^{(k)}\) to a measure \(\lambda^{(k)}(x)\) on this fibre. This measure does not depend on the choice of \(k\) because \((\lambda^x)_{x \in G^0}\) is left invariant and \(k\) is unique up to right multiplication by some \(g_0 \in G^1\). We are going to use [37, Lemmas 1.2 and 1.3] to check that the family \(\lambda\) is continuous.

First we pull back \((\lambda^x)\) to the family of measures \(l \mapsto \lambda^{(l)}\) on the fibres of the map \(m : L^1 \times_{s,G^0} G^1 \to L^1\), \((l, g) \mapsto l \cdot g\). This family is continuous by [37, Lemma 1.2]. The map \(m\) is a \(G\)-equivariant map if the \(G\)-action on \(L^1 \times_{s,G^0} G^1\) is defined by \((l, g_1) \cdot g_2 := (l, g_1 g_2)\) and on \(L^1\) as usual. Both \(G\)-actions are parts of principal bundles, and the induced map on orbit spaces is the orbit space projection of \(L^1\). [37, Lemma 1.3] says that the induced family of measures for the orbit space projection \(L^1 \to L^1/G\) is also continuous. More precisely, Renault defines principal bundles using free and proper actions; we use basic actions instead, that is, we only require a homeomorphism \(X \times_{s,G^0} G^1 \to X \times_{X/G} X\), \((x, g) \mapsto (x, x \cdot g)\). What we call a principal bundle is one in Renault’s notation if and only if the orbit space is Hausdorff by [32, Corollary 9.35]. Since \(H^1 \times_{s,H^0,F^0} L^0\) is locally Hausdorff, we may apply Renault’s result to the restrictions of our principal bundle to all Hausdorff open subsets of \(H^1 \times_{s,H^0,F^0} L^0\). This is exactly the meaning of “continuity” for a family of measures over a locally Hausdorff base space.

The groupoid \(L\) acts on \(H^1 \times_{s,H^0,F^0} L^0\) with anchor map \((h, x) \mapsto x\) and multiplication \(l \cdot (h, s(l)) = (F^1(l) \cdot h, r(l))\). The map \((F^1, r)\) in (5.3) is \(L\)-invariant. The family of measures \(\lambda\) on the fibres of this map is \(L\)-invariant because \(\lambda\) is \(G\)-invariant.

Next we construct a continuous family of measures along the fibres of

\[
H^1 \times_{r,H^0,F^0} L^0 \to L^0, \quad (h, x) \mapsto x.
\]

We simply take the measure \(\mu^{\nu^z(x)}\) on the fibre of \(x\). This family is the pull-back of \((\mu^y)_{y \in H^0}\) along \(F^0 : L^0 \to H^0\), so it is continuous by [37, Lemma 1.2]. The map in (5.4) is also \(L\)-invariant. The family of measures \((\mu^{\nu^z(x)})_{z \in L^0}\) is \(L\)-invariant because \((\mu^y)_{y \in H^0}\) is \(H\)-invariant.

Now we combine the continuous families of measures in (5.3) and (5.4) to a continuous family of measures \((\nu^z)_{z \in L^0}\) along the fibres of the composite map \(r : L^1 \to L^0\). The integral of a Borel function \(f : L^1 \to \mathbb{C}\) with quasi-compact support against
this family of measures is given by (5.2). The integration over \( \lambda \) in (5.2) maps quasi-continuous functions on \( L^1 \) to quasi-continuous functions on \( H^1 \times_{\nu,H^0,\tau,L^0} L^0 \), and the second integration over \( \mu \) maps quasi-continuous functions on \( H^1 \times_{\tau,H^0,\nu,L^0} L^0 \) to (quasi)continuous functions on \( H^0 \). Hence \( \nu \) is a continuous family of measures. The \( L \)-invariance of \( \lambda \) and \( \mu \) implies that \( \nu \) is \( L \)-invariant. It is also clear that the support of \( \nu^* \) is all of \( L^2 \).

\[ \square \]

**Example 5.5.** Our theorem applies, in particular, to extensions of locally Hausdorff and locally compact groupoids \( G \hookrightarrow L \twoheadrightarrow H \) with the same unit space \( G^0 = L^0 = H^0 \). For an extension of locally compact groups \( G \hookrightarrow L \twoheadrightarrow H = L/G \), Theorem 5.1 gives the usual formula for the Haar measures on \( L \) in terms of the Haar measures on \( G \) and \( H \). For a twist, that is, \( G = L^0 \times \tau \) with trivial conjugation action of \( L \) on \( \tau \), it is already observed in [33] that a Haar system on \( H \) induces one on \( L \). Implicitly, this uses the normalised Haar measure on the compact group \( \tau \).

For the transformation groupoid (or semidirect product) \( L = H \ltimes G \) of a classical action of a locally compact group \( H \) on a Hausdorff, locally compact groupoid \( G \), the existence of the Haar system on \( L \) is proved in [24, Proposition 6.4] under an extra "invariance" condition for the \( H \)-action on \( G \). This condition rules out some basic examples such as the \(.ax+b\)-group \( \mathbb{R} \ltimes \mathbb{R}_{>0} \). It is not necessary by Theorem 5.1, which needs no condition on the Haar systems of \( G \) and \( H \) and even allows \( G \) and \( H \) to be locally Hausdorff.

6. Crossed Products

As before, let \( F : L \to H \) be a groupoid fibration with fibre \( G \). We assume that \( G \) and \( H \) are locally Hausdorff, locally compact groupoids with Hausdorff object spaces and with Haar systems \( \lambda \) and \( \mu \), respectively. Then \( L \) is locally Hausdorff and locally compact by Theorem 4.4. Let \( \nu \) be the canonical Haar syston \( L \) constructed in Theorem 5.1. The available disintegration theory of representations also requires that all our groupoids are second countable and all Fell bundles separable, but these assumptions may probably be removed with better technology, on which we are working at the moment.

Let \( \mathcal{B} \) be a Fell bundle over \( L \). By convention, all Fell bundles are separable, saturated and upper semicontinuous. As in [12], we denote the space of quasi-continuous sections of \( \mathcal{B} \) on \( L \) by \( \mathcal{S}(L,\mathcal{B}) \); these are finite linear combinations of compactly supported continuous sections on Hausdorff open subsets of \( L \), extended by 0 outside. With standard formulas for convolution and involution, \( \mathcal{S}(L,\mathcal{B}) \) becomes a \( * \)-algebra. It carries a canonical bornology, that is, a collection of bounded subsets such that the convolution and involution are bounded, see [12, Appendix B]. We call a seminorm or representation on \( \mathcal{S}(L,\mathcal{B}) \) bounded if it is bounded on all bounded subsets. The so-called inductive limit topology is the locally convex topology generated by the bounded seminorms.

The \textit{section} \( C^* \)-algebra \( C^*(L,\mathcal{B}) \) for the Fell bundle \( \mathcal{B} \to L \) is the completion of \( \mathcal{S}(L,\mathcal{B}) \) for the maximal bounded \( C^* \)-seminorm on \( \mathcal{S}(L,\mathcal{B}) \) or, equivalently, for the maximal \( C^* \)-seminorm that is continuous in the inductive limit topology on \( \mathcal{S}(L,\mathcal{B}) \).

We assume that the Disintegration Theorem holds for \( \mathcal{B} \). It says that any bounded representation of \( \mathcal{S}(L,\mathcal{B}) \) comes from a representation of the Fell bundle \( \mathcal{B} \). This implies that any bounded \( C^* \)-seminorm is already dominated by a certain norm called the \( I \)-norm; hence the maximum of all bounded \( C^* \)-norms on \( \mathcal{S}(L,\mathcal{B}) \) exists.

We may restrict \( \mathcal{B} \) to a Fell bundle over \( G \subseteq L \) and get a bornological \( * \)-algebra \( \mathcal{S}(G,\mathcal{B}|_G) = \mathcal{S}(G,\mathcal{B}) \) and a \( C^* \)-algebra \( C^*(G,\mathcal{B}|_G) = C^*(G,\mathcal{B}) \) in the same way.
Similarly, we may restrict to the subgroupoids $G_y \subseteq G$ for $y \in H^0$ (see \cite{ALCIDES BUSS AND RALF MEYER} and construct $C^\ast$-algebras $C^\ast(G_y, \mathcal{B})$.

We also assume that the Equivalence Theorem holds for $C^\ast(G_y, \mathcal{B})$, that is, the groupoid equivalences $L_h$ in Lemma \ref{lemma-equivalence} induce Morita–Rieffel equivalences between $C^\ast(G_r(h), \mathcal{B})$ and $C^\ast(G_s(h), \mathcal{B})$.

The Disintegration Theorem and the Equivalence Theorem have been shown for several classes of Fell bundles over groupoids: for Green twisted actions of non-Hausdorff groupoids on continuous fields of $C^\ast$-algebras over $L^0$ in \cite{Non-Hausdorff Groupoids}; for arbitrary (separable and saturated) upper semicontinuous Fell bundles over Hausdorff groupoids in \cite{Non-Hausdorff Groupoids}; and for ordinary actions of non-Hausdorff groupoids in \cite{Non-Hausdorff Groupoids}.

So far, there seems to be no source that covers Fell bundles and non-Hausdorff groupoids simultaneously. We are working on proving these results in this generality; for now, we must assume these basic technical results to hold for our proofs to work.

We now construct a pre-Fell bundle over $H$, which we will later complete to a Fell bundle using appropriate $C^\ast$-products of the Hausdorff open subsets on which $L$ is closed in the same partition of unity for all functions $f$ of unity to decompose, respectively. The product $h \in L^1$-algebras over $G$ is quasi-continuous sections $L_h \to \mathcal{B}$ is well defined. We define an involution

$$\mathfrak{A}_h \to \mathfrak{A}_{h^{-1}}, \quad f^* := f(\cdot^{-1})$$

using that the inversion map restricts to a homeomorphism from $L_h$ to $L_{h^{-1}}$ by Lemma \ref{lemma-equivalence}. This map is bounded, conjugate-linear, and involutive, that is, $f^{**} = f$. We want to define a convolution $\mathfrak{A}_{h_1} \times \mathfrak{A}_{h_2} \to \mathfrak{A}_{h_1 h_2}$ for $h_1, h_2 \in H$ by

$$(f_1 * f_2)(l) := \int_{L^1} f_1(l_1) \cdot f_2(l^{-1}_1 l) \, d\hat{\lambda}^{(h_1, r(l))}(l_1)$$

for $f_1 \in \mathfrak{A}_{h_1}, f_2 \in \mathfrak{A}_{h_2}$. We explain why this formula works. We integrate over the $L$-invariant family of measures $\hat{\lambda}$ on the fibres of the map

$$(F^1, r) : L^1 \to H^1 \times_{r, H^0, F_0} L^0$$

as in Theorem \ref{theorem-disintegration}. The support of $\hat{\lambda}^{(h_1, r(l))}$ is the set of all $l_1 \in L^1$ with $F^1(l_1) = h_1$ and $r(l_1) = r(l)$ or, equivalently, $l_1 \in L_{h_1}$ with $r(l_1) = r(l)$. Then $F^1(l^{-1}_1 l) = h_1^{-1} h = h_2$, so the integral in (6.1) only sees values of $f_1$ on $L_{h_1}$ and of $f_2$ on $L_{h_2}$, respectively. The product $f_1(l_1) \cdot f_2(l^{-1}_1 l)$ belongs to $\mathfrak{B}_{h_1} \cdot \mathfrak{B}_{l^{-1}_1} \subseteq \mathfrak{B}_1$. Thus $(f_1 * f_2)(l)$ is a well defined element of $\mathfrak{B}_1$.

The same argument as for the convolution in $\mathfrak{S}(L, \mathcal{B})$ shows that $f_1 * f_2$ is quasi-continuous, that is, belongs to $\mathfrak{S}(L_{h_1 h_2}, \mathcal{B})$ (see \cite{Non-Hausdorff Groupoids} Proposition 4.4). It suffices to prove this if both $f_1$ and $f_2$ are compactly supported continuous functions on some Hausdorff open subsets of $L_{h_1}$ and $L_{h_2}$, respectively. We may further use partitions of unity to decompose $f_1$ and $f_2$ into functions with smaller supports, so that the product of the Hausdorff open subsets on which $f_1$ and $f_2$ live is again Hausdorff in $L_{h_1 h_2}$. Then the continuity of $\hat{\lambda}$ implies that $f_1 * f_2$ is continuous with compact support on the relevant Hausdorff open subset of $L_{h_1 h_2}$. Since we may choose the same partition of unity for all functions $f_1, f_2$ with a given support, the convolution is bounded; that is, if $f_i$ for $i = 1, 2$ run through bounded subsets of $\mathfrak{S}(L_{h_i}, \mathcal{B})$, then the set of products $f_1 * f_2$ is bounded.
The identity follows easily from the definitions of the involutions. The quality of \( \eta \) is the restriction of \( \nu \) to \( C \). For a Hausdorff, open subset \( U \subseteq H^1 \), let \( V := (F^1)^{-1}(U) \), which is an open subset of \( L^1 \). To a quasi-continuous section \( \xi \in \mathcal{G}(V, \mathfrak{B}) \), we assign a section \( \xi^* \) of \( \mathfrak{A}_h \) over \( U \) by \( \xi(h) := \xi|_{L_h} \). We take this as the space of continuous sections with compact support of the bundle \( \mathfrak{A}_h \) over the Hausdorff subset \( U \). The space of quasi-continuous sections of \( \mathfrak{A}_h \) over an arbitrary open subset \( U \subseteq H^1 \) is defined as for Banach bundles, as the space of finite linear combinations of continuous sections over Hausdorff open subsets. This is canonically isomorphic to \( \mathcal{G}((F^1)^{-1}(U), \mathfrak{B}) \) by [12, Proposition B.2]. This finishes the construction of the pre-Fell bundle \( \mathfrak{A}_h \) over \( H \).

The assignment \( \xi \mapsto \tilde{\xi} \) used above also preserves the algebraic structure, that is, we have \( \xi \ast \eta = \tilde{\xi} \ast \tilde{\eta} \) and \( (\tilde{\xi})^* = \tilde{\xi}^* \) for all \( \xi, \eta \in \mathcal{G}(L, \mathfrak{B}) \). The equality \( (\tilde{\xi})^* = \tilde{\xi}^* \) follows easily from the definitions of the involutions. The equality \( \xi \ast \eta = \tilde{\xi} \ast \tilde{\eta} \) uses how the Haar system \( \nu \) in Theorem 6.1 is built. Let \( h \in H^1 \). Then \( \xi \ast \eta(h) \) is the restriction of \( \xi \ast \eta \) to \( L_h \subseteq L^1 \). And for \( l \in L_h \), that is, \( l \in L^1 \) with \( F^1(l) = h \), this equals

\[
(\tilde{\xi} \ast \tilde{\eta})(h)(l) = \xi \ast \eta(l) = \int_{L} \xi(l_1)\eta(l_1^{-1}l) \, d\nu(l_1) = \int_{L} \int_{L} \xi(l_1)\eta(l_1^{-1}l) \, d\lambda(l_1) \, d\nu(l_1) = \int_{L} \int_{L} \xi(l_1)\eta(l_1^{-1}l) \, d\lambda(l_1) \, d\nu(l_1).
\]

The assumption \( F^1(l) = h \) gives \( F^0(r(l)) = r(h) \), so that this is equal to the section \( (\tilde{\xi} \ast \tilde{\eta})(h) \in \mathcal{G}(L_h, \mathfrak{B}) \) evaluated at the point \( l \) via (6.1); hence \( \xi \ast \eta = \tilde{\xi} \ast \tilde{\eta} \). Therefore, the map \( \xi \mapsto \xi \) gives a *-algebra isomorphism \( \mathcal{G}(L, \mathfrak{B}) \xrightarrow{\sim} \mathcal{G}(H, \mathfrak{B}) \).

Next we complete \( \mathfrak{A}_h \) to a C*-algebraic Fell bundle \( \mathfrak{A}_h \). The groupoids \( G_y \) inherit Haar systems from \( G \) by restricting \( (\lambda^x)^* \) to \( x \in (F^0)^{-1}(y) \). Our space \( \mathfrak{A}_1 \) is exactly the space \( \mathcal{G}(G_y, \mathfrak{B}) \) of quasi-continuous sections of the restriction of \( \mathfrak{B} \) to \( G_y \). Thus we may complete \( \mathcal{G}(G_y, \mathfrak{B}) \) to a C*-algebra using the maximal bounded C*-norm.

The C*-algebras \( C^*(G_y, \mathfrak{B}) \) for \( y \in H^0 \) are the fibres of an upper semicontinuous field on \( H^0 \) with section algebra \( C^*(G, \mathfrak{B}) \) or, more precisely, \( C^*(G, \mathfrak{B}|_G) \). To construct this field, we use the \( G \)-invariant function \( F^0 : G^0 = L^0 \to H^0 \). The resulting non-degenerate *-homomorphism \( (F^0)^* : C_0(H^0) \to \mathcal{M}(C^*(G, \mathfrak{B})) \) takes
values in the centre because $F^0$ is $G$-invariant. Thus it turns $C^*(G, \mathfrak{B})$ into a $C_0(H^0)$-$C^*$-algebra. The fibre at $y \in H^0$ for this $C_0(H^0)$-$C^*$-algebra structure is $C^*(G_y, \mathfrak{B})$ because $\mathfrak{S}(G, \mathfrak{B})/C_0(H^0 \setminus \{y\}) \cdot \mathfrak{S}(G, \mathfrak{B}) \cong \mathfrak{S}(G_y, \mathfrak{B})$.

Recall that $L_h$ is an equivalence between the groupoids $G_{r(h)}$ and $G_{s(h)}$. The formulas we used to define the pre-Hilbert bimodule $\mathfrak{B}_h = \mathfrak{S}(L_h, \mathfrak{B})$ are the usual ones for an equivalence of Fell bundles. By assumption, Renault’s Equivalence Theorem holds for the groupoids $G_y$ for all $y \in H^0$, see [54, Corollaire 5.4] and [55, Theorem 5.5]. Hence we may complete $\mathfrak{B}_h$ to an imprimitivity bimodule $\mathfrak{B}_h = C^*(L_h, \mathfrak{B})$ between $C^*(G_{r(h)}, \mathfrak{B})$ and $C^*(G_{s(h)}, \mathfrak{B})$. The convolution products $\mathfrak{B}_{h_1} \times \mathfrak{B}_{h_2} \to \mathfrak{B}_{h_1 h_2}$ and the involutions $\mathfrak{B}_h \to \mathfrak{B}_{h^{-1}}$ defined in (6.1) extend to the completions $\mathfrak{B}_h$. The proof uses the $C^*$-identity and that these convolution products are the left or right bimodule actions on $\mathfrak{B}_h$ if restricted to units. Hence the spaces $\mathfrak{B}_h$ for $h \in H$ are the fibres of a (saturated) Fell bundle $\mathfrak{A}$ over $H$ if we specify a suitable space of quasi-continuous sections. This is done via the previously defined map $\xi \mapsto \tilde{\xi}$, which identifies $\mathfrak{S}(L, \mathfrak{B}) \cong \mathfrak{S}(H, \mathfrak{A})$. This determines a unique topology on the bundle $\mathfrak{A}$ with fibres $\mathfrak{B}_h$ and turns it into a Fell bundle (see [5, Propositions 2.4 and 2.7]).

By construction, the map $\xi \mapsto \tilde{\xi}$ identifies $\mathfrak{S}(L, \mathfrak{B})$ with the dense $*$-subalgebra $\mathfrak{S}(H, \mathfrak{A})$ of the section $C^*$-algebra $C^*(H, \mathfrak{A})$. This extends to an isomorphism $C^*(L, \mathfrak{B}) \overset{\sim}{\to} C^*(H, \mathfrak{A})$ of $C^*$-algebras:

**Theorem 6.2** (Iterated crossed-product decomposition). Let $L$ and $H$ be second countable, locally Hausdorff and locally compact groupoids and let $F: L \to H$ be a groupoid fibration with fibre $G$. Assume that $G$ and $H$ carry Haar systems, and endow $L$ with the Haar system constructed in Theorem 5.1. Let $\mathfrak{B}$ be a saturated, separable Fell bundle over $L$. Then $C^*(L, \mathfrak{B})$ is isomorphic to the section $C^*$-algebra of a saturated Fell bundle over $H$ with fibres $C^*(L_h, \mathfrak{B})$ at $h \in H^1$ and unit fibre $C^*(G, \mathfrak{B})$. In particular, $C^*(L)$ is isomorphic to the section $C^*$-algebra of a saturated Fell bundle over $H$ with fibres $C^*(L_h)$ at $h \in H^1$ and unit fibre $C^*(G)$.

Before we prove this theorem, we interpret it. We view a fibration $F: L \to H$ with fibre $G$ as a continuous action by groupoid equivalences of $H$ on $G$ with transformation groupoid $L$, see [22]. As in [11][13], we view a saturated Fell bundle $\mathfrak{B}$ over $L$ as an action (by $C^*$-algebra equivalences) of $L$ on the $C^*$-algebra $A := C^*(L^0, \mathfrak{B})$ of the restriction of $\mathfrak{B}$ to $L^0$, which we called unit fibre above. We view the section $C^*$-algebra $C^*(L, \mathfrak{B})$ as the “crossed product” $A \rtimes L$ of this action. We may restrict the action (that is, the Fell bundle) from $L$ to $G \subseteq L$. Theorem 6.2 says that there is a new action (in the form of a Fell bundle) of $H$ on $A \rtimes G = C^*(G, \mathfrak{B})$ such that $(A \rtimes G) \rtimes H \cong A \rtimes L$.

We begin to prove Theorem 6.2. The proof will be finished after Lemma 6.8.

The $C^*$-algebra $C^*(L, \mathfrak{B})$ is the completion of $\mathfrak{S}(L, \mathfrak{B})$ for the maximal bounded $C^*$-seminorm on $\mathfrak{S}(L, \mathfrak{B})$. Similarly, the section $C^*$-algebra of the Fell bundle $\mathfrak{A}$ over $H$ is the completion of $\mathfrak{S}(H, \mathfrak{A})$ in the maximal bounded $C^*$-seminorm. By construction, $\mathfrak{S}(L, \mathfrak{B})$ is a dense $*$-subalgebra in $\mathfrak{S}(H, \mathfrak{A})$, and the inclusion map is bounded. It remains to prove that this dense inclusion extends to an isomorphism between the $C^*$-completions. Equivalently, any bounded $*$-representation of $\mathfrak{S}(L, \mathfrak{B})$ on a Hilbert space extends to a bounded $*$-representation of $\mathfrak{S}(H, \mathfrak{A})$. This result looks plausible, but the proof is rather technical, and it took us some time to finish this argument.

We prove that any bounded Hilbert space representation of $\mathfrak{S}(L, \mathfrak{B})$ is bounded in norm by a variant of the $I$-norm on $\mathfrak{S}(H, \mathfrak{A})$. This is non-trivial because of the $C^*$-completion $\mathfrak{A}$ of $\mathfrak{A}$ in the direction of $G$. A first step is to construct a morphism
**Lemma 6.3.** Define

\[(\varphi \ast \psi)(l) := \int_G \varphi(g)\psi(g^{-1}l)\,d\lambda^{(l)}(g)\]

for \(\varphi \in \mathcal{S}(\mathfrak{G}, \mathfrak{B}), \psi \in \mathcal{S}(\mathfrak{L}, \mathfrak{B}), l \in L,\) where \(\lambda\) denotes the Haar system on \(G.\) This defines a \(*\)-homomorphism from \(\mathcal{S}(\mathfrak{G}, \mathfrak{B})\) into the multiplier \(*\)-algebra of \(\mathcal{S}(\mathfrak{L}, \mathfrak{B}),\) that is, the convolution is bilinear and satisfies \((\varphi_1 \ast \varphi_2) \ast \psi = \varphi_1 \ast (\varphi_2 \ast \psi),\)

\[\varphi_1 \ast \psi_2 \ast \psi_2 = (\varphi_1 \ast \psi_1) \ast \psi_2 \ast \psi_2 = (\varphi_1 \ast \psi_1)^* \ast \psi_2 \ast \psi_2 \text{ for } \varphi, \varphi_1, \varphi_2 \in \mathcal{S}(\mathfrak{G}, \mathfrak{B}) \text{ and } \psi, \psi_1, \psi_2 \in \mathcal{S}(\mathfrak{L}, \mathfrak{B}).\]

**Proof.** Bilinearity is trivial, and \((\varphi_1 \ast \varphi_2) \ast \psi = \varphi_1 \ast (\varphi_2 \ast \psi)\) and \((\varphi \ast \psi_1) \ast \psi_2 =\)

\[(\varphi \ast \psi_1)^* \ast \psi_2\text{ follow from the left invariance of the Haar systems on } G \text{ and } L.\] We prove \(\psi_1^* \ast (\varphi \ast \psi_2) = (\varphi \ast \psi_1)^* \ast \psi_2\) in detail. Let \(l \in L.\) Let \(\lambda\) and the Haar systems \(\mu, \nu\) on \(G, H \text{ and } L\) be as in Theorem 5.1. By definition,

\[\psi_1^* \ast (\varphi \ast \psi_2)(l) = \int \int \int \psi_1((l_2^{-1})^* \varphi(g)^* \psi_2(g^{-1}l_2^{-1})\,d\lambda^{(l_2)}(g)\,d\mu^{(l_2)}(h),\]

\[= \int \int \psi_1((\tilde{l}_2)^{-1})^* \varphi(g)^* \psi_2((\tilde{l}_2)^{-1})\,d\lambda^{(l_2)}(g)\,d\mu^{(l_2)}(h).\]

The equality of these two triple integrals is proved by three substitutions for fixed \(h.\) First, we use the homeomorphism \(L_h^{(l)} \times_s L_{l_0}^{(l_0)} \xrightarrow{\sim} G \times \lambda),\) \((l_2, g) \mapsto (l_2, l_2g).\) Secondly, we use the homeomorphism \(L_h^{(l)} \times L_h^{(l)} \xrightarrow{\sim} G \times \lambda),\) \((l_2, l_3) \mapsto (l_2^{-1}l_2l_3),\) and finally to \(\psi_1 \ast (\varphi \ast \psi_2)(l) = \int \int \psi_1((\tilde{l}_2)^{-1})^* \varphi(g)^* \psi_2((\tilde{l}_2)^{-1})\,d\lambda^{(l_2)}(g)\,d\mu^{(l_2)}(h).\]

To construct a morphism \(C^*(G, \mathfrak{B}) \to \mathcal{M}(C^*(L, \mathfrak{B}))\) from the above lemma, we must extend multipliers defined only on \(\mathcal{S}(\mathfrak{L}, \mathfrak{B})\) to \(C^*(L, \mathfrak{B}).\) This can be done assuming a stronger form of the Disintegration Theorem for densely defined representations. To reduce the number of technical assumptions, we construct this morphism only for the special case of a trivial Fell bundle, where the relevant technical result is already proved. This special case provides some information on quasi-invariant measures and modular functions which, together with the Disintegration Theorem for representations by globally defined bounded operators, gives the morphism \(C^*(G, \mathfrak{B}) \to \mathcal{M}(C^*(L, \mathfrak{B}))\) also for Fell bundle coefficients.

**Lemma 6.4.** There is a unique non-degenerate \(*\)-homomorphism \(\iota: C^*(G) \to \mathcal{M}(C^*(L))\) such that \(\iota(\phi)\psi = \phi \ast \psi\) for \(\phi \in \mathcal{S}(\mathfrak{G}), \psi \in \mathcal{S}(\mathfrak{L}).\)

**Proof.** Since \(C^*(L)\) is separable, it has a faithful representation \(\pi: C^*(G) \to \mathfrak{B}(\mathcal{H})\) on a separable Hilbert space \(\mathcal{H}.\) The extension of \(\pi\) to \(\mathcal{M}(C^*(L))\) remains faithful.
We want to define a densely defined representation $\pi_G$ of $\mathfrak{S}(G)$ on $\mathcal{H}$ by $\pi_G(\phi)\xi = \sum_{i=1}^n \pi(\phi * \psi_i)\xi_i$ if $\phi \in \mathfrak{S}(G)$ and $\xi = \sum_{i=1}^n \pi(\psi_i)\xi_i$ with $\psi_i \in \mathfrak{S}(L)$, $\xi_i \in \mathcal{H}$ for $i = 1, \ldots, n$. This is well defined, that is, $\pi_G(\phi)\xi$ does not depend on how we decompose $\xi$ because

$$\left\langle \sum_{j=1}^m \pi(\tau_j)\eta_j \mid \sum_{i=1}^n \pi(\phi * \psi_i)\xi_i \right\rangle = \left\langle \sum_{j=1}^m \pi(\phi_* * \tau_j)\eta_j \mid \sum_{i=1}^n \pi(\psi_i)\xi_i \right\rangle$$

for any $\tau_j \in \mathfrak{S}(L)$, $\eta_j \in \mathcal{H}$. The right hand side no longer depends on the decomposition of $\xi$ and determines $\sum_{j=1}^m \pi(\tau_j)\eta_j$ because vectors of the form $\sum_{j=1}^m \pi(\tau_j)\eta_j$ are dense in $\mathcal{H}$.

The densely defined representation $\pi_G$ of $\mathfrak{S}(G)$ on $\mathcal{H}$ satisfies the assumptions in Proposition 4.2. Hence the operators $\pi_G(\phi)$ extend to bounded operators on $\mathcal{H}$ that define a $\ast$-homomorphism $\pi_G : C^*(G) \to \mathfrak{B}(\mathcal{H})$. Since $\pi_G(\phi)\pi(\psi) = \pi(\phi * \psi)$ for all $\phi \in \mathfrak{S}(G)$, $\psi \in \mathfrak{S}(L)$ and $\pi_G(C^*(G))$ is a $\ast$-algebra, it is contained in $\mathcal{M}(C^*(G)) \subseteq \mathfrak{B}(\mathcal{H})$.

Let $\pi : C^*(L, \mathfrak{B}) \to \mathfrak{B}(\mathcal{H})$ be a faithful representation on a separable Hilbert space $\mathcal{H}$. The Disintegration Theorem gives an $L$-quasi-invariant measure $\alpha$ on $L^0$, an $\alpha$-measurable field of Hilbert spaces $(\mathcal{H}_x)_{x \in L^0}$ over $L^0$, and a measurable representation $\Pi_1 : \mathfrak{B}_1 \to \mathfrak{B}(\mathcal{H}(0), \mathcal{H}(l))$, $l \in L^1$, of the Fell bundle, such that $\mathcal{H} \cong \mathcal{L}^2(L, \mathcal{H}_x, \alpha)$ is the space of $\mathcal{L}^2$-sections of the field $(\mathcal{H}_x)$ over $L^0$, and

$$\pi(\psi)\xi(x) = \int_{L^0} \Pi_1(\psi(l))\xi(s(l)) \frac{d(\alpha \circ \tilde{\nu})}{d(\alpha \circ \nu)}(l) \, d\nu^x(l)$$

for all $\psi \in \mathfrak{S}(L, \mathfrak{B})$, $\xi \in \mathcal{H}$, and $\alpha$-almost all $x \in L^0$. Here $\tilde{\nu}(l) = \nu(l^{-1})$, and the quasi-invariance of $\alpha$ says that the measures $\alpha \circ \tilde{\nu}$ and $\alpha \circ \nu$ on $L^1$ are equivalent, so that the Radon–Nikodym derivative $\frac{d(\alpha \circ \tilde{\nu})}{d(\alpha \circ \nu)}$ is defined $\alpha \circ \nu$-almost everywhere.

This theorem is proved in [34,35,37] for many cases, but no proof for arbitrary Fell bundles over locally Hausdorff groupoids seems to be published yet. More precisely, there is an $\alpha$-nullset $E \subseteq L^0$ such that the representation $\Pi_1$ is defined for all $l \in L^1$ with $s(l), r(l) \notin E$, and has the properties required of a representation for all such $l$; most notably, $\Pi_{1,1}^2(b_1b_2) = \Pi_1(b_1)\Pi_1(b_2)$ for composite $l_1, l_2 \in L^1$ and $b_i \in \mathfrak{B}_1$ for $i = 1, 2$ and $\Pi_{1,-1}(b^*) = \Pi_1(b)^*$ for $l \in L^1$, $b \in \mathfrak{B}_1$ provided the source and range objects of $l, l_1, l_2$ do not belong to $E$. It is important that the nullsets where things do not work are of this particular form.

**Theorem 6.6.** A quasi-invariant measure on $L$ is also quasi-invariant for $G$, and

$$\frac{d(\alpha \circ \lambda)}{d(\alpha \circ \lambda)}(g) \frac{d(\alpha \circ \tilde{\nu})}{d(\alpha \circ \nu)}(l) = \frac{d(\alpha \circ \tilde{\nu})}{d(\alpha \circ \nu)}(g)$$

holds for almost all $(g, l) \in G^1 \times_{s,c^0,F} L^1$ with respect to the measure defined by $\mathfrak{S}(G^1 \times_{s,c^0,F} L^1) \ni f \mapsto \int f(g, l) \, d\nu^x(g) \, d\nu^y(l) \frac{d(\alpha \circ \lambda)}{d(\alpha \circ \nu)}(g)$.

Both Radon–Nikodym derivatives in [6.7] are characters almost everywhere. If they are continuous, then (6.7) is equivalent to $\frac{d(\alpha \circ \lambda)}{d(\alpha \circ \nu)}(g) = \frac{d(\alpha \circ \lambda)}{d(\alpha \circ \nu)}(g)$ for $g \in G^1$.

Without continuity, this may be meaningless because $G^1$ may be an $\alpha \circ \nu$-null set.

**Proof.** Any quasi-invariant measure $\alpha$ on $L^0$ appears in some representation of the trivial Fell bundle $\mathbb{C}$ and thus comes from some representation of $C^*(L)$: we may take the “regular representation” associated to $\alpha$. Lemma 6.3 shows that this representation of $C^*(L)$ induces a representation of $C^*(G)$. The representations of $C_0(\mathbb{C}) = C_0(L^0)$ associated to the representations of $C^*(G)$ and $C^*(L)$ coincide. Hence the Disintegration of the representation of $C^*(G)$ gives the same measure
The construction of the morphism $C^*(G) \to \mathcal{M}(C^*(L))$ in Lemma 6.3 and one obvious substitution give

$$\pi_G(\varphi)\pi(\psi)\xi(x) = \int \int \varphi(g)\psi(l)\xi(s(l))\frac{d(\alpha \circ \tilde{\lambda})}{d(\alpha \circ \tilde{\lambda})}(g)\frac{d(\alpha \circ \tilde{\nu})}{d(\alpha \circ \tilde{\nu})}(l)\,d\nu^\phi(g)\,d\lambda^\psi(g).$$

Both formulas coincide for $\alpha$-almost all $x \in L^0$ if and only if (6.7) holds. □

The measure $\alpha$ on $G^0 = L^0$ is quasi-invariant for $G$ by Theorem 6.6. The representation $\Pi_\alpha$ restricts to a representation of $\mathcal{B}|G$. This works even if $G \subseteq L$ is an $\alpha \circ \nu$-nullset because of the special form of the set where $\Pi$ is not defined. The measure $\alpha$ on $G^0$, the measurable field of Hilbert spaces $|H_x\rangle \in G^0$, and $\Pi|G$ form a representation of $\mathcal{B}|G$. It integrates to a $^*$-representation $\pi_G$ of $\mathbb{S}(G, \mathcal{B}|G)$, given by

$$\pi_G(\varphi)\xi(x) := \int_{G^x} \Pi_\nu(\varphi(g))\xi(s(g))\sqrt{\frac{d(\alpha \circ \tilde{\nu})}{d(\alpha \circ \tilde{\nu})}(g)}\,d\lambda^\nu(g)$$

for all $\varphi \in \mathbb{S}(G, \mathcal{B})$, $\xi \in H$, and $x \in G^0 \setminus E$ for the nullset $E$ above.

Let $\varphi \in \mathbb{S}(G, \mathcal{B})$, $\psi \in \mathbb{S}(L, \mathcal{B})$, $\xi \in H$, and $x \in G^0 \setminus E$. Then

$$\pi_G(\varphi)\pi(\psi)\xi(x) = \int_{G^x} \int_{L^*(\lambda)} \Pi_\nu(\varphi(g))\Pi_\nu(\psi(l))\xi(s(l))\sqrt{\frac{d(\alpha \circ \tilde{\nu})}{d(\alpha \circ \tilde{\nu})}(g)}\frac{d(\alpha \circ \tilde{\nu})}{d(\alpha \circ \tilde{\nu})}(l)\,d\nu(\varphi(g))\,d\lambda^\psi(l) = \pi(\varphi \ast \psi)\xi(x).$$

We compute as in the proof of Theorem 6.6 but now we know (6.7) and deduce backwards that $\pi_G(\varphi)\pi(\psi) = \pi(\varphi \ast \psi)$. Thus $\pi_G$ gives a $^*$-homomorphism $C^*(G, \mathcal{B}) \to \mathcal{M}(C^*(L, \mathcal{B}))$ as in the proof of Lemma 6.3. It is nondegenerate because the convolution maps $\mathbb{S}(G, \mathcal{B}) \otimes \mathbb{S}(L, \mathcal{B})$ to a dense subspace of $\mathcal{S}(L, \mathcal{B})$.

**Lemma 6.8.** Let $\psi \in \mathbb{S}(L, \mathcal{B})$. Then $\|\pi(\psi)\|_{\mathcal{B}(H)} \leq \|\psi\|_{H^\alpha}^{1/2} \|\psi^\ast\|_{H^\alpha}^{1/2}$ with

$$\|\psi\|_{H^\alpha} := \sup_{\nu \in H^\alpha} \int_{H^\alpha} \|\psi|_{L^\alpha}\|_{\mathcal{C}^*(L^\alpha)} \,d\mu^\nu(h).$$

**Proof of Lemma 6.8.** Let $\psi = \psi_1 \cdot \psi_2$ with pointwise multiplication, where $\psi_1$ is a measurable function $H^1 \times_{r,F^0} L^0 \to [0, \infty)$, viewed as a function on $L^1$ through the map $(F^1, r)$, and $\psi_2$ is an $\alpha \circ \nu$-measurable section of $\mathcal{B}$. We choose $\psi_1(h, x) := \|\psi|_{L^\alpha}\|_{\mathcal{C}^*(L^\alpha)}^{1/2}$ and $\psi_2(l) := \psi(l)/\psi_2(F^1(l), l)$). The function $\psi_1$ is upper semicontinuous and hence Borel, and hence $\psi_2$ is a Borel section.

We are going to decompose $\pi(\psi) = T_{\psi_1} \circ T_{\psi_2}$ for two operators

$$T_{\psi_1}, T_{\psi_2} : L^2(L^0, (H_x), \alpha) \to L^2(H^1 \times_{r,F^0} L^0, \|\psi_1\|_{\mathcal{C}^*(L^\alpha)} \,d\mu^\nu, (H_x), \alpha \circ \mu).$$

Here

$$\alpha \circ \mu(f) = \int_L \int_{H^\alpha} f(h, x) \,d\mu^\nu(h) \,d\alpha(x)$$
and \( \text{pr}^2(\mathcal{H}_x) \) means the \( \alpha \circ \mu \)-measurable field of Hilbert spaces over \( H^1 \times_{r,H^0,F^0} L^0 \) with fibre \( \mathcal{H}_x \) at \( (h,x) \).

We choose \( \psi \) and \( \xi \). This allows to compute

\[
(T_{\psi_1})(h,x) := \psi_1(h,x) \cdot \xi(x)
\]

for \( \xi \in \mathcal{L}^2(L^0,(\mathcal{H}_x),\alpha) \) and \( (h,x) \in H^1 \times_{r,H^0,F^0} L^0 \). There is a canonical isomorphism between \( \mathcal{L}^2(H^1 \times_{r,H^0,F^0} L^0,\text{pr}^2_2(\mathcal{H}_x),\alpha \circ \mu) \) and the tensor product \( \mathcal{L}^2(H^1,\tau,\mu) \otimes_{L^\infty(H^0,(\mathcal{H}_x),\alpha)} \mathcal{L}^2(L^0,(\mathcal{H}_x),\alpha) \), where we view the measurable field of Hilbert spaces \( \mathcal{L}^2(H^2,\mu^2) \) as a Hilbert module \( \mathcal{L}^2(H^1,\tau,\mu) \) over \( L^\infty(H^0,F^0,\alpha) \), which acts on \( \mathcal{L}^2(L^0,(\mathcal{H}_x),\alpha) \) by pointwise multiplication through \( F^0 \). With this identification, \( T_{\psi_1}(\xi) = \psi_1 \otimes \xi \).

This allows to compute \( T_{\psi_1}^* \) and \( T_{\psi_1} \). First,

\[
T_{\psi_1}(\omega)(x) = \int_{H^0(x)} \frac{\psi_1(h,x)}{\omega(h,x)} d\mu^0(x)(h)
\]

for \( \omega \in \mathcal{L}^2(H^1 \times_{r,H^0,F^0} L^0,\text{pr}^2_2(\mathcal{H}_x),\alpha \circ \mu) \) and \( x \in L^0 \). Secondly, \( T_{\psi_1}^* T_{\psi_1} \) multiplies a section in \( \mathcal{L}^2(L^0,(\mathcal{H}_x),\alpha) \) pointwise with the function

\[
x \mapsto \int_{H^0(x)} |\psi_1(h,x)|^2 d\mu^0(x)(h).
\]

So

\[
\|T_{\psi_1}\|^2 = \sup_{x \in L^0} \int_{H^0(x)} |\psi_1(h,x)|^2 d\mu^0(x)(h).
\]

When we choose \( \psi_1 \) as above, we get \( \|T_{\psi_1}\| = \|\psi\|_{H^0}^{1/2} \).

Let \( \Delta(l) := \frac{d(\alpha \circ \mu)}{d(\alpha \circ \mu)}(l) \) for \( l \in L^1 \). We define

\[
\tilde{T}_{\psi_2}(\xi)(h,x) := \int_{L^*} \Pi_l(\psi_2(l)) \xi(s(l)) \Delta(l) d\lambda(h,x)(l)
\]

for \( \xi \in \mathcal{L}^2(L^0,(\mathcal{H}_x),\alpha) \) and \( (h,x) \in H^1 \times_{r,H^0,F^0} L^0 \). The function \( \Delta \) is defined \( \alpha\mu\)-almost everywhere on \( L^1 \). Therefore, there is an \( \alpha\mu \)-nullset in \( H^1 \times_{r,H^0,F^0} L^0 \) so that \( \Delta(l) \) is defined for \( \lambda(\cdot,x) \)-almost all \( l \in L^*_x \) for \( (h,x) \) outside this nullset. Hence the integrand above exists on sufficiently many points to be meaningful.

We assume that \( \psi_2 \) has quasi-compact support until we have proved estimates that allow to extend to more general functions. Then the integral defining \( \tilde{T}_{\psi_2}(\xi)(h,x) \) is over a quasi-compact subset and hence finite, and the resulting function on \( H^1 \times_{r,H^0,F^0} L^0 \) has quasi-compact support and hence belongs to the Hilbert space \( \mathcal{L}^2(H^1 \times_{r,H^0,F^0} L^0,\text{pr}^2_2(\mathcal{H}_x),\alpha \circ \mu) \). By definition,

\[
T_{\psi_1}^* \tilde{T}_{\psi_2}(\xi)(x) = \int_{H^0(x)} \int_{L^*_x} \overline{\psi_1(h,x)} \Pi_l(\psi_2(l)) \xi(s(l)) \Delta(l) d\lambda(h,x)(l) d\mu^0(x)(h)
\]

\[
= \pi(\psi_1 \cdot \psi_2)\xi(x).
\]

Hence

\[
(6.9) \quad \|\pi(\psi)\| \leq \|T_{\psi_1}\|\|\tilde{T}_{\psi_2}\|.
\]

It remains to estimate \( \|\tilde{T}_{\psi_2}\| \). For this purpose, we compute \( \tilde{T}_{\psi_2}^* \) and \( \tilde{T}_{\psi_2}^* \tilde{T}_{\psi_2} \).
Let $\omega \in L^2(L^1 \times_r \mathbb{R}, \mu \circ \alpha \circ \mu)$ and $\xi \in L^2(L^0, (\mathcal{H}, \alpha))$ with quasi-compact support. Then

\[
\langle \omega | \hat{T}_{\psi_2} \xi \rangle = \iint \iint \langle \omega(h, x) | \Pi_l(\psi_2(l)) \xi(s(l)) \Delta(l) \rangle \ d\hat{\lambda}(h, x)(l) \ d\mu^F(x)(h) \ d\alpha(x)
\]

\[
= \iint \langle \omega(F^1(l), r(l)) | \Pi_l(\psi_2(l)) \xi(s(l)) \Delta(l) \rangle \ d\nu(l)(l)
\]

\[
= \iint \langle \omega(F^1(l^{-1}), s(l)) | \Pi_{l^{-1}}(\psi_2(l^{-1})) \xi(r(l)) \rangle \Delta(l) \ d\nu(l)(l)
\]

\[
= \iint \langle \Pi_l(\psi_2(l^{-1})), \omega(F^1(l^{-1}), s(l)) | \xi(r(l)) \rangle \Delta(l) \ d\nu^F(l) \ d\alpha(x).
\]

Here the first step uses the definition of $\hat{T}_{\psi_2}$; the second step uses the construction of the Haar system on $L$ and the definition of the composite measure $\nu \circ \alpha$ on $L^1$; the third step uses the substitution $l \mapsto l^{-1}$, which multiplies with the Radon–Nikodym derivative $\Delta(l)^{-1}$, and uses that $\Delta$ is a character almost everywhere to simplify $\Delta(l^{-1}) \Delta(l)^2 = \Delta(l)$; the last step expands $d(\nu(l))(l) = d\nu^F(l) \ d\alpha(x)$ and uses that $\Pi$ is a $\ast$-representation. Thus $\langle \omega | \hat{T}_{\psi_2} \xi \rangle = \langle \hat{T}_{\psi_2}^* \omega | \xi \rangle$ with

\[
\hat{T}_{\psi_2}^* \omega(x) = \int \Pi_l(\psi_2^*(l)) \omega(F^1(l^{-1}), s(l)) \Delta(l) \ d\nu^F(l).
\]

Let $\Delta_C(g) := \frac{d\alpha \circ \lambda}{d\alpha \circ \alpha}(g)$ for $g \in G^1$. Then

\[
\hat{T}_{\psi_2}^* \hat{T}_{\psi_2} \xi(x)
\]

\[
= \iint \Pi_l(\psi_2(l_1)) \Pi_{l_2}(\psi_2(l_2)) \xi(s(l_2)) \Delta(l_1) \Delta(l_2) \ d\hat{\lambda}(F^1(l_1^{-1}), s(l_1))(l_2) \ d\nu^F(l_1)
\]

\[
= \iint \Pi_{l_1 l_2}(\psi_2(l_1) \psi_2(l_2)) \xi(s(l_1 l_2)) \Delta(l_1) \Delta(l_2) \ d\hat{\lambda}(F^1(l_1^{-1}), s(l_1))(l_2) \ d\nu^F(l_1)
\]

\[
= \int \Pi_g(\psi_2^*(l_1) \psi_2(l_1^{-1} g)) \xi(s(g)) \Delta(l_1) \Delta(l_1^{-1} g) \ d\lambda^F(g) \ d\nu^F(l_1).
\]

The first step uses the definitions. The second step uses that $\Pi$ is multiplicative and $s(l_1 l_2) = s(l_2)$. The third step substitutes $g := l_1 l_2$ for $l_2$, using the left $L$-invariance of $\hat{\lambda}$. Since $F^1(l_2) = F^1(l_1^{-1})$, this gives elements of $G$. Since $\Delta(l^{-1}) = \Delta(l)^{-1} \alpha \circ \nu$-almost everywhere and $\Delta_C(g^{-1}) = \Delta_C(g)^{-1} \alpha \circ \lambda$-almost everywhere, we may rewire $\Delta(l_1) \Delta(l_1^{-1} g) = \Delta(l_1) \Delta(g^{-1} l_1^{-1}) = \Delta(l_1) \Delta(l_1^{-1} \Delta_C(g^{-1})^{-1}) = \Delta_C(g)$ by (6.7); this holds almost everywhere with respect to the relevant measure. Thus $\hat{T}_{\psi_2}^* \hat{T}_{\psi_2} = \pi_C(\varphi)$ with

\[
\varphi(g) := \int \psi_2^*(l) \psi_2(l^{-1} g) \ d\nu^F(g)(l)
\]

\[
= \int_{H^p} \int_{L^*} \psi_2^*(l) \psi_2(l^{-1} g) \ d\lambda^F(h, r(g))(l) \ d\mu^F(r(g))(h).
\]

The $C^*$-norm of this element of $\mathfrak{S}(G, \mathfrak{B})$ is a supremum over the $C^*$-norms of its restrictions in $\mathfrak{S}(G_y, \mathfrak{B})$ for $y \in H^0$. For $g \in G_y$,

\[
\varphi(g) = \int_H \langle \psi_2 | \nu_{l_{-1}} | \psi_2 | \nu_{l_{-1}^{-1}} \rangle_{C^*(L_h)}(g) \ d\mu^h(h).
\]

Hence

\[
\|\varphi\|_{C^*(G, \mathfrak{B})} = \sup_{y \in H^0} \|\varphi|_{G_y}\|_{C^*(G_y, \mathfrak{B})} = \sup_{y \in H^0} \int_{H^y} \|\psi_2 | \nu_{l_{-1}} \|_{C^*(L_{-1}, \mathfrak{B})}^2 \ d\mu^h(h)
\]

\[
= \sup_{y \in H^0} \int_{H^y} \|\psi_2^2 | L_{h} \|_{C^*(L_{h}, \mathfrak{B})}^2 \ d\mu^h(h).
\]
The last step uses that the involution is an isometry between $C^*(L_{h,^*}, \mathcal{B})$. With the choice of $\psi_2$ above, this gives
\[
\|\tilde{T}_{\psi_2}\| = \|\tilde{T}_{\psi_1}^*\tilde{T}_{\psi_2}\|^{1/2} = \|\psi\|^{1/2}_{C^*(G, \mathcal{B})} = \|\psi\|^{1/2}_{H^0}.
\]
In particular, $\tilde{T}_{\psi_1}$ is bounded and well defined on $L^2(L^0, (H_x), \alpha)$ although the relevant $\psi_2$ need not have quasi-compact support. Putting the norm estimates for $T_{\psi_1}$ and $\tilde{T}_{\psi_2}$ into Equation (6.9) gives the desired estimate for $\pi(\psi)$. \hfill \Box

Since $\mathcal{G}(L, \mathcal{B}) \cong \mathcal{G}(H, \mathcal{A})$ is dense in $\mathcal{G}(H, \mathcal{A})$, Lemma 6.5 implies that the faithful representation $\pi$ of $C^*(L, \mathcal{B})$ extends uniquely to a representation of $\mathcal{G}(H, \mathcal{A})$ that is bounded for the $L$-norm. Hence we may extend further to the $C^*$-algebra $C^*(H, \mathcal{A})$.

The resulting representation of $C^*(H, \mathcal{A})$ maps the dense subalgebra $\mathcal{S}(H, \mathcal{A}) = \mathcal{S}(L, \mathcal{B})$ into $\pi(C^*(L, \mathcal{B}))$. Hence it maps $C^*(H, \mathcal{A})$ to $\pi(C^*(L, \mathcal{B})) \cong C^*(L, \mathcal{B})$ as well. We already know that $C^*(L, \mathcal{B})$ maps into $C^*(H, \mathcal{A})$ via the assignment $\xi \mapsto \bar{\xi}$. The map backwards is its inverse and shows that $C^*(L, \mathcal{B}) \cong C^*(H, \mathcal{A})$.

This finishes the proof of Theorem 6.2.

\textbf{Remark 6.10.} Theorem 6.2 has a large overlap with [12, Theorem 5.5], which asserts a similar crossed product decomposition for actions of inverse semigroups on groupoids. The result in [12] implies Theorem 6.2 in case the groupoid $H$ is étale. Here we use Theorem 3.1 to replace the fibration by an action of the inverse semigroup $\mathcal{B}$.

Without the groupoid fibration condition, $L_h$ need not be an equivalence between $G_{r(h)}$ and $G_{s(h)}$ and so the Fell bundle over $H$ constructed above need not be saturated. The Fell bundle constructed in [15, Theorem 3.4] under weaker assumptions is not saturated. Actually, the assumptions that are made in [13, Theorem 3.4] are not used in the proof: any continuous functor $L \to H$ between étale groupoids yields a (possibly non-saturated) Fell bundle over $H$ with fibres $L_h$ as above. Theorem 6.2 should also carry over to this situation, but we did not check details carefully.

\textbf{Remark 6.12.} Some results about crossed products and Fell bundles are only proved if $\mathcal{B}$ restricts to a continuous field on the unit space $L^0$. The Fell bundle $\mathcal{A}$ over $H$ has this property if $\mathcal{B}$ has it and $F^0 : L^0 \to H^0$ is open.

\textbf{Remark 6.13.} An action of $H$ on a $C^*$-algebra, even in the form of a Fell bundle, induces a continuous action of $H$ on the primitive ideal space of the unit fibre (see [23]). We briefly explain this construction. We first assume $H$ to be Hausdorff.

The $C_0(H^0)$-$C^*$-algebra structure on $C^*(G, \mathcal{B})$ with fibres $C^*(G_y, \mathcal{B})$ induces a continuous map $\psi$ from the primitive ideal space $\text{Prim } C^*(G, \mathcal{B})$ to $H^0$. This is the anchor map of the action. When we pull back $C^*(G, \mathcal{B})$ along $r, s : H^1 \to H^0$ to $C_0(H^1)$-$C^*$-algebras $r^*(C^*(G, \mathcal{B}))$ and $s^*(C^*(G, \mathcal{B}))$, these have the primitive ideal spaces $H^1 \times_{r,H^0,\psi} \text{Prim } C^*(G, \mathcal{B})$ and $H^1 \times_{s,H^0,\psi} \text{Prim } C^*(G, \mathcal{B})$, respectively. The space of $C_0$-sections of the Fell bundle $(C^*(L_h, \mathcal{B}))_{h \in H^1}$ is a $C_0(H^1)$-linear imprimitivity bimodule between these two pull-backs of $C^*(G, \mathcal{B})$. By the Rieffel correspondence, this imprimitivity bimodule induces a homeomorphism over $H^1$ between the primitive ideal spaces
\[
H^1 \times_{s,H^0,\psi} \text{Prim } C^*(G, \mathcal{B}) \cong H^1 \times_{r,H^0,\psi} \text{Prim } C^*(G, \mathcal{B}).
\]
This homeomorphism is of the form \((h, p) \mapsto (h, h \cdot p)\) for a continuous action of \(H\) on \(\text{Prim } \mathcal{C}^*(G, \mathfrak{B})\).

If \(H\) is not Hausdorff, then we should replace \(H^1\) by a disjoint union \(\tilde{H} := \bigcup_{U \in \mathcal{U}} U\) for a cover by Hausdorff, open subsets. Then \(C_0(\tilde{H})\) makes sense, and we get a map

\[
\tilde{H} \times_{s, H^0, \psi} \text{Prim } \mathcal{C}^*(G, \mathfrak{B}) \xrightarrow{\sim} \tilde{H} \times_{r, H^0, \psi} \text{Prim } \mathcal{C}^*(G, \mathfrak{B})
\]

as above. Furthermore, the maps coming from \(U_1, U_2 \in \mathcal{U}\) coincide on the intersection \(U_1 \cap U_2\), so that we get a well defined, continuous action of \(H\) on \(\text{Prim } \mathcal{C}^*(G, \mathfrak{B})\) even if \(H\) is only locally Hausdorff.

7. Special cases and applications

We now exhibit some special cases of Theorem 6.2. In this section, we tacitly assume all groupoids to be second countable, locally Hausdorff, locally compact, with Hausdorff unit space and with a Haar system, and we tacitly assume all \(\mathcal{C}^*\)-algebras and Fell bundles to be separable.

7.1. Action on the arrow space of a non-Hausdorff groupoid. Let \(H\) be a groupoid. To make the problem non-trivial, assume that the arrow space \(H^1\) is non-Hausdorff. We continue the construction in §2.4. Let \(H^1 = \bigcup_{U \in \mathcal{U}} U\) be a cover by Hausdorff, open subsets and let \(X := \bigcup_{U \in \mathcal{U}} U\) with the canonical map \(p: X \to H^1\). We have constructed a groupoid fibration \((sp)^*(H^0) \to H\) with fibre \(p^*(H^1)\). Theorem 6.2 gives a Fell bundle over \(H\) with unit fibre \(\mathcal{C}^*(p^*(H^1))\) and section \(\mathcal{C}^*-algebra \mathcal{C}^*((sp)^*(H^0))\). The \(\mathcal{C}^*\)-algebra of the Čech groupoid \(p^*(H^1)\) is the standard way to turn the locally Hausdorff space \(H^1\) into a \(\mathcal{C}^*\)-algebra. It is a Fell algebra with spectrum \(H^1\) and trivial Dixmier–Douady invariant (see [21]). As in [12], the action of \(H\) on \(\text{Prim } \mathcal{C}^*(G)\) is the translation action of \(H\) on \(H^1\); this follows from the description of the action in Remark 6.13. All this justifies viewing the Fell bundle over \(H\) constructed above as a good \(\mathcal{C}^*\)-algebraic description of the action of \(H\) on \(H^1\).

Similarly, \(\mathcal{C}^*((sp)^*(H^0))\) is a Fell algebra with spectrum \(H^0\) and trivial Dixmier–Douady invariant. Since \(H^0\) is Hausdorff, it is even a continuous trace \(\mathcal{C}^*\)-algebra, and Morita–Rieffel equivalence to \(\mathcal{C}^*(H^0) = C_0(H^0)\). This generalises the well known Morita–Rieffel equivalence \(\mathcal{C}^*(H \times H^1) \sim C_0(H^0)\) for a Hausdorff groupoid.

More generally, let \(X\) be a basic \(H\)-space, that is, \(X\) is a locally Hausdorff, locally compact space with a basic \(H\)-action (see [12],[22]). Then we may cover \(X\) by Hausdorff open subsets, producing a Čech groupoid \(G\) that is equivalent to \(X\).

The action of \(H\) on \(X\) transfers to an “action” on \(G\) in the form of a groupoid fibration \(L \to H\) with fibre \(G\). Here \(L := p^*(H \times X)\), where \(p: G^0 \to X\) is obtained from the open covering of \(X\). Theorem 6.2 gives a Fell bundle over \(H\) with unit fibre \(\mathcal{C}^*(G)\) and section \(\mathcal{C}^*-algebra \mathcal{C}^*(p^*(H \times X))\). This Fell bundle describes the \(H\)-action on \(X\) in \(\mathcal{C}^*-algebraic\) terms.

The \(\mathcal{C}^*\)-algebras \(\mathcal{C}^*(G)\) and \(\mathcal{C}^*(p^*(H \times X))\) are Fell algebras with trivial Dixmier–Douady invariants and spectrum \(X\) and \(H \backslash X\), respectively. For \(\mathcal{C}^*(p^*(H \times X))\), the proof uses that \(H \times X\) is isomorphic to the Čech groupoid of the open surjection \(X \to H \backslash X\) because the \(H\)-action on \(X\) is basic.

The following proposition says that the action of \(H\) on its arrow space \(H^1\) cannot be modelled by a classical action by automorphisms:

**Proposition 7.1.** Let \(H\) be a non-Hausdorff, locally Hausdorff groupoid. There is no classical action of \(H\) on a groupoid \(G\) such that \(G\) has Hausdorff object space, is equivalent to the space \(H^1\), and has an open surjective anchor map \(r_H: G^1 \to H^0\), and such that the transformation groupoid is equivalent to \(H^0\).
Proof. Let $L \to H$ be a groupoid fibration with fibre $G$, such that $G$ is equivalent to $H^1$ and $L$ is equivalent to $H^0$. Since $G$ is equivalent to a space, it is a basic groupoid. So $(r,s): G^1 \to G^0 \times G^0$ is injective, forcing $G^1$ to be Hausdorff because $G^0$ is Hausdorff. The same argument shows that $L^1$ is Hausdorff.

The arrow space for a classical action of $H$ on $G$ is $H^1 \times_{s,H^0,r_H} G^1$. We claim that this is non-Hausdorff if $H^1$ is non-Hausdorff and $r_H: G^0 \to H^0$ is an open surjection. Hence the fibration $L \to H$ cannot come from a classical action.

Since $H^1$ is non-Hausdorff, there is a sequence $(h_n)$ in $H^1$ with two limits $h \neq h'$. Since $r_H(g) = r_H(r(g))$ for $g \in G^1$ and both $r: G^1 \to G^0$ and $r_H: G^0 \to H^0$ are open surjections, so is $r_H: G^1 \to H^0$. Hence we may lift the convergent sequence $s(h_n)_{n \in \mathbb{N}}$ in $H^0$ to a convergent sequence $(g_n)$ in $G^1$ with limit, say, $g$ (see [17, Proposition 1.15]). Then $(h_n, g_n)$ is a sequence in $H^1 \times_{s,H^0,r_H} G^1$ that converges both to $(h,g)$ and $(h',g)$. □

If the $H$-action on $G$ models the translation action on $H^1$, then $G$ should be equivalent to $H^1$, the anchor map $G^0 \to H^0$ should correspond to the anchor map $r: H^1 \to H^0$ of the translation action and hence be an open surjection, and the transformation groupoid should be equivalent to $X \rtimes H \sim H^0$. Thus the assumptions in Proposition 7.1 should hold for any model of this action.

Proposition 7.1 is related to [12, Theorem 7.1], which forbids the existence of a classical action (by isomorphisms) of $H$ on a $C^*$-algebra $A$ with $\text{Prim}(A) \cong H^1$ so that the induced action on $\text{Prim}(A)$ is the translation action on $H^1$. The results in [12] are written only for étale groupoids. The idea of the proof may, however, be extended to general locally compact groupoids.

7.2. Groupoid fibration from a bibundle. Consider the groupoid fibration associated to a $G,H$-bibundle $X$ in Example 2.16. We assume $X$ to be second countable, Hausdorff and locally compact, so that the groupoid $G \rtimes X \rtimes H$ satisfies the standing assumptions for this section.

The canonical functor $G \times X \times H \to H$ is a fibration with fibre $G \times X$, see Example 2.16. The groupoid $G \times X$ inherits a Haar system from $G$, and $G \ltimes X \rtimes H$ inherits a Haar system from $G \times X$ and $H$ by Theorem 5.1. This is equal to the Haar system induced in the most obvious way from the Haar systems on $G$ and $H$. Given a (separable, saturated) Fell bundle $\mathcal{B}$ over $G \times X \rtimes H$, Theorem 6.2 provides a Fell bundle over $H$ whose section $C^*$-algebra is canonically isomorphic to $C^*(G \times X \rtimes H, \mathcal{B})$ and with unit fibre $C^*(G \times X, \mathcal{B})$.

Brown, Goehle and Williams [3] have recently proved this under the extra assumptions that $X$ be an equivalence bibundle and $\mathcal{B}$ be the Fell bundle associated to an ordinary action of $G \times X \rtimes H$ on some $C^*$-algebra $A$ by automorphisms. Our analysis works for any $G,H$-space $X$. If $\mathcal{B}$ comes from an action by automorphisms, then the Fell bundle over $X \rtimes H$ constructed above is also associated to an action of $H$ on $A \rtimes (G \times X) = A \times G$ by automorphisms; this $H$-action is defined as in [3, Proposition 3.7]. As explained in [3], the crossed product decomposition $A \rtimes (G \times X \rtimes H) \cong (A \times G) \rtimes H$ is useful to study Brauer semigroups and symmetric imprimitivity theorems for groupoid actions as in [2].

7.3. Crossed products for groupoid extensions. The statement of Theorem 6.2 was already proved for several classes of groupoid extensions $G \to L \to H$. If $F^0$ is a homeomorphism as in 2.5 then we replace $L$ by an isomorphic groupoid so that $F^0$ is the identity map. We assume this in this subsection.

7.3.1. Group extensions. First assume that we are dealing with a group extension, that is, the object spaces are a single point. This is the most classical case of an iterated crossed product decomposition. The assertion of Theorem 6.2 in this
The special case of this result where with fibres $C^*$-ideals of the coefficient $C^*$ and essentially free. Counterexamples to Renault’s theorem for non-Hausdorff is amenable and essentially free; see [38] for the precise definitions. Renault’s theorem is described in [18].

7.3.2. The open isotropy group bundle. Let $L$ be an étale groupoid. Let $G := \text{Iso}^0(L)$ be the interior of the isotropy subgroupoid of $L$ as in Example [22] that is, $G^0 = L^0$ and $G^1$ is the interior of the subset \( \{ g \in L^1 \mid s(g) = r(g) \} \) of $L^1$. This subset is open and hence also locally compact, and it gives an étale, normal subgroup bundle in $L$. The quotient $H = L/G$ is an étale groupoid with $H^0 \cong G^0 = L^0$ which fits into an extension $G \hookrightarrow L \twoheadrightarrow H$ (Remark [1] explains why $H$ is étale).

Theorem 6.2 says, in particular, that $C^*(L)$ is the $C^*$-algebra of a saturated Fell bundle over $H$ with fibres $C^*(L_0)$ and with unit fibre $C^*(G)$. More generally, if $\mathcal{B}$ is a Fell bundle over $L$, then $C^*(L, \mathcal{B})$ is the $C^*$-algebra of a Fell bundle over $H$ with fibres $C^*(L_h, \mathcal{B})$ and with unit fibre $C^*(G, \mathcal{B})$.

A special case of this result is proved in [22] Corollary 3.12, under several extra assumptions. Namely, in [22] $L$ is assumed amenable and Hausdorff, $G$ is assumed closed and of the special form $G^0 \times \Gamma$ for some amenable group $\Gamma$, and $\mathcal{B}$ is assumed to be the Fell line bundle associated to a continuous 2-cocycle on $L$ (see also 7.3.3 below). Theorem 6.2 shows that all these assumptions are unnecessary.

The crossed product decomposition theorem for the open isotropy group bundle is particularly useful to describe the ideal structure of a crossed product. By Renault’s pioneering result [35] Corollary 4.9, the ideals of a groupoid crossed product $A \rtimes L$ for a Green twisted action of a groupoid $L$ are equivalent to $L$-invariant ideals of the coefficient $C^*$-algebra $A$ provided the induced action of $L$ on $\text{Prim}(A)$ is amenable and essentially free; see [35] for the precise definitions. Renault’s theorem applies, in particular, if the groupoid that is acting is Hausdorff, amenable, and essentially free. Counterexamples to Renault’s theorem for non-Hausdorff $L$ are described in [18].
Renault assumes the coefficient algebra to be a continuous field of \( \mathbb{C}^* \)-algebras over \( L^0 \). Presumably this was because the larger class of \( C_0(L^0) \)-\( \mathbb{C}^* \)-algebras that we use was not yet so widely used at the time of \cite{35}. Similarly, Renault only studies Green twisted actions because Fell bundles were not so widely used then. His result carries over to saturated Fell bundles for the following reason.

Every Fell bundle over a Hausdorff groupoid \( L \) is Morita–Rieffel equivalent to an ordinary action of \( L \). This is proved in \cite[Corollary 3.9]{22}, and it also follows from \cite[Theorem 5.3]{13} because Fell bundles over \( L \) are equivariantly equivalent to weak actions of \( L \) in the sense of \cite{13}; these are exactly the actions by \( \mathbb{C}^* \)-equivalences that we use here and in \cite{11}. Hence Renault’s result carries over to (saturated) continuous Fell bundles over Hausdorff groupoids. Continuity here means that the underlying field of \( \mathbb{C}^* \)-algebras over \( L^0 \) is continuous. In this case, the corresponding ordinary action is also continuous and we may use Renault’s original theorem in \cite{35}; this is exactly what is done in \cite[Corollary 3.9]{22}. The continuity assumption is probably not necessary here, but Hausdorffness is crucial: for non-Hausdorff \( L \), there are Fell bundles that are not equivariantly equivalent to ordinary actions, see \cite[§7]{11}.

The quotient \( H := L/\text{Iso}(L)^0 \) for an étale groupoid \( L \) is the largest quotient that is essentially free. Thus any action of \( H \) is essentially free. If \( H \) is Hausdorff and amenable, then the ideal structure of section \( \mathbb{C}^* \)-algebras of Fell bundles over \( H \) is completely understood in terms of the action of \( H \) on the ideal lattice of the unit fibre. This is determined by the action of \( H \) on the primitive ideal space, which is described in Remark \cite[6.13]{6}. The quotient \( L/\text{Iso}(L)^0 \) is Hausdorff if and only if \( \text{Iso}(L)^0 \) is closed in \( L \) by Proposition \cite{12}.

Theorem \cite[6.2]{6} decomposes a crossed product for a general étale groupoid \( L \) into one by the étale, normal group bundle \( \text{Iso}(L)^0 \) and one by the essentially free quotient \( H := L/\text{Iso}(L)^0 \). Renault’s description of the ideal structure applies to the crossed product by \( H \) provided \( H \) is Hausdorff and amenable. This idea is used \cite[Theorem 4.5]{22} to describe the ideal structure of \( \mathbb{C}^* \)-algebras of twisted higher-rank graphs without singular vertices. Such algebras are section \( \mathbb{C}^* \)-algebras of Fell bundles over étale groupoids \( L \) as above, and the assumptions imply that the isotropy group bundle \( \text{Iso}(L)^0 \) is of the form \( L^0 \times \Gamma \) for some abelian group \( \Gamma \).

7.3.3. Twists over groupoids. A twist of a groupoid \( H \) is another groupoid \( L \) satisfying our standing assumptions, which fits into a central extension of the form \( X \times \mathbb{T} \to L \to H \), where \( X = L^0 = H^0 \), see Example \cite[30]{2}. Theorem \cite{5} gives a canonical Haar system on \( L \), which allows to form the groupoid \( \mathbb{C}^* \)-algebra \( C^*(L) \).

A Fell line bundle over \( H \) is a Fell bundle \( \mathfrak{B} \) where all fibres \( \mathfrak{B}_l \) have dimension 1. Twists over \( H \) correspond to Fell line bundles over \( H \) by going back and forth between principal \( \mathbb{T} \)-bundles and (complex) Hermitian line bundles (see \cite{15}). If \( L \) is a twist as above, then \( \mathbb{T} \) acts on \( L \) by pointwise multiplication, and we get a Hermitian line bundle \( \mathfrak{B} := \mathbb{C} \times \mathbb{T} \); this is the orbit space of \( \mathbb{C} \times L \) by the diagonal \( \mathbb{T} \)-action \( z \cdot (\lambda, \sigma) = (\lambda z, z \cdot \sigma) \). The groupoid structure of \( L \) gives \( \mathfrak{B} \) the structure of a Fell line bundle over \( G \). Conversely, a Fell line bundle \( \mathfrak{B} \) gives a twist \( L := \{ u \in \mathfrak{B} | u^*u = 1 \} \) or, equivalently, the subspace of unitary elements of \( \mathfrak{B} \). These two constructions are inverse to each other.

The twisted groupoid \( \mathbb{C}^* \)-algebra \( C^*(G, L) \) is, by definition, the section \( \mathbb{C}^* \)-algebra of the corresponding Fell line bundle \( \mathfrak{B} \). We are going to identify this with a certain direct summand in \( C^*(L) \).

**Corollary 7.2.** Let \( L \) be a twist over \( H \) as above and let \( \mathfrak{B} \) be the associated Fell line bundle. Then \( C^*(L) \) is isomorphic to the section \( \mathbb{C}^* \)-algebra of a Fell bundle \( \mathfrak{A} \).
over $H$ with unit fibre

$$C^*(H^0, \mathfrak{A}) \cong C^*(X \times T) \cong \bigoplus_{n \in \mathbb{Z}} C_0(X).$$

The Fell bundle $\mathfrak{A}$ is a direct sum of Fell line bundles, where the $n$th summand is the $n$-fold tensor product $\mathfrak{B}^{\otimes n} = \mathfrak{B} \otimes \mathfrak{B} \otimes \cdots \otimes \mathfrak{B}$. Hence

$$C^*(L) \cong \bigoplus_{n \in \mathbb{Z}} C^*(\mathfrak{A}_n) \cong \bigoplus_{n \in \mathbb{Z}} C^*(\mathfrak{B}^{\otimes n}).$$

If $L$ is the extension associated to a Borel 2-cocycle $\omega: H^1 \times_{s,r} H^1 \to T$, then $C^*(\mathfrak{A}_n) \cong C^*(H, \omega^n)$ and $C^*(L) \cong \bigoplus_{n \in \mathbb{Z}} C^*(H, \omega^n)$.

This is proved in [4] for twists coming from continuous 2-cocycles over Hausdorff groupoids.

Proof. Since $X \times T \hookrightarrow L$ is central, the resulting $^*$-homomorphism $\varphi: C^*(T) \to \mathcal{M}(C^*(L))$ takes values in the centre. Since $C^*(T) \cong C_0(\mathbb{Z})$, $C^*(L)$ is a $C^*$-algebra over the discrete space $\mathbb{Z}$. That is, $C^*(L) \cong \bigoplus_{n \in \mathbb{Z}} C^*(L_n)$, where $C^*(L_n) \subseteq C^*(L)$ is the ideal generated by the central projection $p_n = \varphi(z^n)$. The issue is to identify these summands.

This is fairly easy for the dense $^*$-subalgebra $\mathcal{G}(L)$: the $n$th spectral subspace $\mathcal{G}(L)_n$ consists of all quasi-continuous functions $L \to \mathcal{C}$ that satisfy $f(z \cdot \sigma) = z^n f(\sigma)$ for all $z \in T$, $\sigma \in L$. For $n = 1$, this is the space of quasi-continuous sections of $\mathfrak{B}$ (see, for instance, [10]). For $n \in \mathbb{Z}$, $\mathcal{G}(L)_n$ is the space of quasi-continuous sections of $\mathfrak{B}^{\otimes n}$. The direct sum decomposition $\mathcal{G}(L) \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{G}(H, \mathfrak{B}^{\otimes n})$ remains the same after taking $C^*$-completions by Theorem 6.2. This is easier than the general situation considered in Theorem 6.2 because the fibre has the simple form $X \times T$, but it is still non-trivial.

A Borel 2-cocycle $\omega$ gives rise to a twist and thus to a Fell line bundle $\mathfrak{B}$ with $C^*(\mathfrak{B}, \omega) = C^*(H, \mathfrak{B})$. The multiplication of cocycles corresponds to the tensor product of Fell line bundles. Hence the claim for Fell line bundles contains the statement for Borel 2-cocycles as a special case. \qed

7.4. Linking groupoids. Now we return to the fibration involving the linking groupoid of an equivalence of groupoids in Example 5.3. Let $G$ and $H$ be groupoids that satisfy our standing assumptions, and let $X$ be an equivalence $G, H$-bibiundle. Let $L$ be the linking groupoid of $X$ as in Example 5.3, and let $F: L \to K$ be the fibration to the finite groupoid with four arrows $\gamma, \gamma^{-1}, 1_{s(\gamma)}, 1_{r(\gamma)}$. The fibre of $F$ is the disjoint union $G \sqcup H$ of the groupoids $G$ and $H$. The groupoid $K$ is discrete and hence étale, so it has a canonical Haar system. Theorem 5.1 gives a canonical Haar system on the linking groupoid $L$ which is induced by the Haar systems on $G$ and $H$ (see also [14]).

A (saturated) Fell bundle over $K$ consists of $C^*$-algebras $A$ and $B$, the fibres over $r(\gamma)$ and $s(\gamma)$, and an equivalence $A, B$-bimodule $\mathcal{H}$, namely, the fibre over $\gamma$. The section $C^*$-algebra of such a Fell bundle over $K$ is simply the direct sum of the fibres with the canonical $^*$-algebra structure: this is already complete in the unique $C^*$-norm. Hence it is the linking $C^*$-algebra of $\mathcal{H}$. In particular, the Fell bundle $(C^*(L_h))_{h \in K}$ over $K$ constructed in [14] has the fibres $C^*(G)$ and $C^*(H)$ at the unit arrows and $C^*(X)$ and $C^*(X^*)$ at the non-identity arrows in $K$.

Thus the assertion of Theorem 5.2 in this case is that the $C^*$-algebra of the linking groupoid of an equivalence is the linking $C^*$-algebra of the induced equivalence of groupoid $C^*$-algebras. This result and its analogue for reduced norms are known for Hausdorff groupoids, see [14] Remark 2.4 and [23] Example 3.5(ii).

More generally, Theorem 6.2 yields a similar result with Fell bundle coefficients. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Fell bundles over $G$ and $H$, respectively, and let $\mathcal{H}$ be an equivalence
over \( X \) between them. That is, \( \mathcal{H} \) is an upper-semicontinuous Banach bundle over \( X \) with commuting actions of \( \mathfrak{A} \) on the left and \( \mathfrak{B} \) on the right and “inner products” satisfying the usual algebraic conditions (see [34]). We may combine this data into a Fell bundle \( L(\mathcal{H}) \) over \( L \) whose restrictions to \( G, H \) and \( X \) are the original bundles \( \mathfrak{A}, \mathfrak{B} \) and \( \mathcal{H} \), respectively; this is the linking Fell bundle as constructed in [12]. Theorem 6.2 says that the linking \( C^* \)-algebra of the equivalence \( C^*(G, \mathfrak{A}) \), \( C^*(H, \mathfrak{B}) \)-bimodule \( C^*(\mathcal{H}) \) associated to the \( \mathfrak{A}, \mathfrak{B} \)-equivalence \( \mathcal{H} \) is the section \( C^* \)-algebra of the linking Fell bundle \( L(\mathcal{H}) \). In brief, \( C^*(L(\mathcal{H})) \cong L(C^*(\mathcal{H})) \). For Hausdorff groupoids, this result is proved in [13] together with the analogous statement for reduced groupoid \( C^* \)-algebras.

### 7.5. Strongly surjective cocycles

Let \( L \) be a groupoid and \( H \) a discrete group. A functor \( F : L \to H \) is also called a cocycle. It is called strongly surjective if the map \( (F, r) : L^1 \to H \times L^0 \) is surjective. Equivalently, the map \( (F, s) \) is surjective. This map is automatically open by Remark 6.2. Hence a groupoid fibration to a discrete group \( H \) is the same as a strongly surjective cocycle \( L \to H \). We interpret such a cocycle as an \( H \)-action on the fibre \( G := F^{-1}(\{1\}) \), which may also be called the kernel of the cocycle. Theorem 6.2 describes \( C^*(L) \) as the section \( C^* \)-algebra of a Fell bundle over \( H \) with unit fibre \( C^*(G) \).

The case \( H = \mathbb{Z} \) is particularly simple. The section \( C^* \)-algebra of a saturated Fell bundle \( (\mathfrak{B}_n)_{n \in \mathbb{Z}} \) over \( \mathbb{Z} \) is the same as the Cuntz–Pimsner algebra of the Hilbert bimodule \( \mathfrak{B}_1 \) over the \( C^* \)-algebra \( \mathfrak{B}_0 \) because both have the same universal property. Thus Theorem 6.2 specialises to the main result of [12] in this case. The theorem in [12] also covers cocycles that are only “unperforated,” which is weaker than being strongly surjective. In this case, the construction of the Fell bundle over \( \mathbb{Z} \) still goes through, but it is no longer saturated. The unperforation assumption ensures that the Fell bundle is “generated” by \( \mathfrak{B}_1 \), which is good enough to identify its section \( C^* \)-algebra with the Cuntz–Pimsner algebra of \( \mathfrak{B}_1 \).

If \( H \) is a locally compact group such as \( \mathbb{R} \), then the correct generalisation of a strongly surjective cocycle \( L \to H \) is a groupoid fibration, that is, a functor \( F^1 : L \to H \) such that \( (F^1, s) \) or, equivalently, \( (F^1, r) \) is surjective and open.

The map \( F^0 \) to the one-point space \( H^0 \) is automatically an open surjection. Hence \( F^1 : L^1 \to H^1 \) is an open surjection for any groupoid fibration to a topological group by Lemma 2.31. But \( F \) need not be a groupoid fibration if \( F^1 \) is just an open surjection, see Example 2.32.

### References

1. Claire Anantharaman-Delaroche and Jean Renault, *Amenable groupoids*, Monographies de L’Enseignement Mathématique, vol. 36, L’Enseignement Mathématique, Geneva, 2000. [MR 1799683](http://www.ams.org/mathscinet-getitem?mr=1799683)

2. Jonathan Henry Brown and Geoff Goehle, *The Brauer semigroup of a groupoid and a symmetric imprimitivity theorem*, Trans. Amer. Math. Soc. **366** (2014), no. 4, 1943–1972, doi: 10.1090/S0002-9947-2013-05953-1 [MR 3152718](http://www.ams.org/mathscinet-getitem?mr=3152718)

3. Jonathan Henry Brown, Geoff Goehle, and Dana P. Williams, *Groupoid equivalence and the associated iterated crossed product*, Houston J. Math. **41** (2015), no. 1, 153–175, available at [http://www.math.uh.edu/~hjm/restricted/pdf41(1)/09brown.pdf](http://www.math.uh.edu/~hjm/restricted/pdf41(1)/09brown.pdf) [MR 3347942](http://www.ams.org/mathscinet-getitem?mr=3347942)

4. Jonathan Henry Brown and Astrid an Huef, *Decomposing the \( C^* \)-algebras of groupoid extensions*, Proc. Amer. Math. Soc. **142** (2014), no. 4, 1261–1274, doi: 10.1090/S0002-9939-2014-11492-4 [MR 3162248](http://www.ams.org/mathscinet-getitem?mr=3162248)

5. Lawrence G. Brown, *Extensions of topological groups*, Pacific J. Math. **39** (1971), 71–78, available at [http://projecteuclid.org/euclid.pjm/1102969870](http://projecteuclid.org/euclid.pjm/1102969870) [MR 0307264](http://www.ams.org/mathscinet-getitem?mr=0307264)

6. Ronald Brown, *Fibrations of groupoids*, J. Algebra **15** (1970), 103–132, doi: 10.1016/0021-8693(70)90089-X [MR 0271194](http://www.ams.org/mathscinet-getitem?mr=0271194)

7. Ronald Brown, G. Danesh-Naruie, and J. P. L. Hardy, *Topological groupoids. II. Covering morphisms and \( G \)-spaces*, Math. Nachr. **74** (1976), 143–156, doi: 10.1002/mana.3210740110 [MR 0442937](http://www.ams.org/mathscinet-getitem?mr=0442937)
Paul S. Muhly and Dana P. Williams, *Continuous trace groupoid C*-algebras, II*, Math. Scand. 70 (1992), no. 1, 127–145, available at [http://www.mscand.dk/article/view/12390](http://www.mscand.dk/article/view/12390). MR 1174207

Paul S. Muhly and Dana P. Williams, *Equivalence and disintegration theorems for Fell bundles and their C*-algebras*, Dissertationes Math. (Rozprawy Mat.) 456 (2008), 1–57, doi: 10.4064/dm456-0-1 MR 2446021

Jean Renault, *A groupoid approach to C*-algebras*, Lecture Notes in Mathematics, vol. 793, Springer, Berlin, 1980, doi: 10.1007/BFb0091072 MR 584266

Jean Renault, *Représentation des produits croisés d'algèbres de groupoides*, J. Operator Theory 18 (1987), no. 1, 67–97, available at [http://www.theta.ro/jot/archive/1987-018-001/1987-018-001-005.html](http://www.theta.ro/jot/archive/1987-018-001/1987-018-001-005.html). MR 912813

Jean Renault, *The ideal structure of groupoid crossed product C*-algebras*, J. Operator Theory 25 (1991), no. 1, 3–36, available at [http://www.theta.ro/jot/archive/1991-025-001/1991-025-001-001.html](http://www.theta.ro/jot/archive/1991-025-001/1991-025-001-001.html). With an appendix by Georges Skandalis. MR 1191252

Jean Renault, *Transverse properties of dynamical systems*, Representation theory, dynamical systems, and asymptotic combinatorics, Amer. Math. Soc. Transl. Ser. 2, vol. 217, Amer. Math. Soc., Providence, RI, 2006, pp. 185–199. MR 2276108

Adam Rennie, David Robertson, and Aidan Sims, *Groupoid algebras as Cuntz–Pimsner algebras*, Math. Scand. (2014), accepted. arXiv: 1402.7126

Pedro Resende, *Étale groupoids and their quantales*, Adv. Math. 208 (2007), no. 1, 147–209, doi: 10.1016/j.aim.2006.02.004 MR 2304314

Aidan Sims and Dana P. Williams, *Renault’s equivalence theorem for reduced groupoid C*-algebras*, J. Operator Theory 68 (2012), no. 1, 223–239, available at [http://www.theta.ro/jot/archive/2012-068-001/2012-068-001-012.html](http://www.theta.ro/jot/archive/2012-068-001/2012-068-001-012.html). MR 2966043

Aidan Sims and Dana P. Williams, *An equivalence theorem for reduced Fell bundle C*-algebras*, New York J. Math. 19 (2013), 159–178, available at [http://nyjm.albany.edu/j/2013/19-11.html](http://nyjm.albany.edu/j/2013/19-11.html). MR 3084702

Dana P. Williams, *Crossed products of C*-algebras*, Mathematical Surveys and Monographs, vol. 134, Amer. Math. Soc., Providence, RI, 2007. doi: 10.1090/surv/134 MR 2288954

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