Large subgraphs without short cycles

F. Foucaud∗ M. Krivelevich† G. Perarnau‡

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Abstract

We study two extremal problems about subgraphs excluding a family \( F \) of fixed graphs. i) Among all graphs with \( m \) edges, what is the smallest size \( f(m, F) \) of a largest \( F \)-free subgraph? ii) Among all graphs with minimum degree \( \delta \) and maximum degree \( \Delta \), what is the smallest minimum degree \( h(\delta, \Delta, F) \) of a spanning \( F \)-free subgraph with largest minimum degree? These questions are easy to answer for families not containing any bipartite graph. We study the case where \( F \) is composed of all even cycles of length at most \( 2r \), \( r \geq 2 \). In this case, we give bounds on \( f(m, F) \) and \( h(\delta, \Delta, F) \) that are essentially asymptotically tight up to a logarithmic factor. In particular for every graph \( G \), we show the existence of subgraphs with either many edges or large minimum degree, and arbitrarily high girth. These subgraphs are created using probabilistic embeddings of a graph into extremal graphs.

1 Introduction

Let \( G = (V, E) \) be a simple undirected graph with \( |V| = n \) vertices. If \( H \) is a given graph, then we say that \( G \) is \( H \)-free if there is no subgraph of \( G \) isomorphic to \( H \). The problem of determining the Turán number with respect to \( n \) and \( H \), i.e. the largest size of an \( H \)-free graph on \( n \) vertices, has been extensively studied in the literature. This is the same as determining the size of a largest \( H \)-free spanning subgraph of \( K_n \), the complete graph on \( n \) vertices, the latter quantity is denoted by \( \text{ex}(K_n, H) \). We can extend this notion in a natural way: for every graph \( G \), let \( \text{ex}(G, H) \) denote the largest size of an \( H \)-free subgraph of \( G \). Also, for a family \( F \) of graphs, we say that \( G \) is \( F \)-free if \( G \) does not contain any graph from \( F \), and denote by \( \text{ex}(G, F) \) the largest size of an \( F \)-free subgraph of \( G \).

In this paper, we provide lower bounds for \( \text{ex}(G, F) \) in terms of different graph parameters of \( G \). If \( F \) does not contain any bipartite graph, it is easy to provide tight bounds for \( \text{ex}(G, F) \). Therefore, it is more interesting to study the behavior of \( \text{ex}(G, F) \) when \( F \) contains bipartite graphs. We mostly address the case of even cycles \( F = \{C_4, C_6, \ldots, C_{2r}\} \), with \( r \geq 2 \).

In the first part of the paper, we derive a lower bound for \( \text{ex}(G, \{C_4, C_6, \ldots, C_{2r}\}) \) in terms of the size of \( G \). In the second part of the paper, we study the largest minimum degree of a \( \{C_4, C_6, \ldots, C_{2r}\} \)-free spanning subgraph of \( G \) in terms of the maximum and minimum degrees of \( G \). In both cases, for every graph \( G \), we show the existence of subgraphs with either many edges or large minimum degree, and arbitrary high girth.

As far as we know, this is the first study of these extremal problems.

∗Department of Mathematics, University of Johannesburg, Auckland Park 2006, South Africa LAMSAD - CNRS UMR 7243, PSL, Université Paris-Dauphine, 75775 Paris, France. florent.foucaud@gmail.com
†School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 69978, Israel. krivelev@post.tau.ac.il. Research supported in part by USA-Israel BSF Grant 2010115 and by grant 912/12 from the Israel Science Foundation.
‡School of Computer Science, McGill University, 845 Sherbrooke Street West, Montreal, Quebec, Canada H3A 0G4. guillem.perarnaullobet@mcgill.ca.
**Key definitions.** Let $H$ be a fixed graph. We define $f(m, H)$ as the smallest possible size of a largest $H$–free subgraph of a graph with $m$ edges, that is,

$$f(m, H) := \min_{|E(G)|=m} \text{ex}(G, H).$$

More generally, for a family $\mathcal{F}$ of fixed graphs, we define

$$f(m, \mathcal{F}) := \min_{|E(G)|=m} \text{ex}(G, \mathcal{F}).$$

Our second key definition is $h(\delta, \Delta, H)$, the smallest possible minimum degree of an $H$–free spanning subgraph $G_0$ of a graph $G$ with minimum degree $\delta$ and maximum degree $\Delta$ such that $\delta(G_0)$ is maximized, that is,

$$h(\delta, \Delta, H) := \min_{\Delta(G)=\Delta, \delta(G)=\delta} \max \{\delta(G_0) : G_0 \subseteq G \text{~is~} H\text{–free}\}.$$

Also, for a family $\mathcal{F}$ define

$$h(\delta, \Delta, \mathcal{F}) := \min_{\Delta(G)=\Delta, \delta(G)=\delta} \max \{\delta(G_0) : G_0 \subseteq G \text{ is } \mathcal{F}\text{–free}\}.$$

**Thomassen’s conjecture.** A related problem for the girth and the average degree was stated by Thomassen [19]. Similarly to our definitions of $f(m, \mathcal{F})$ and $h(\delta, \Delta, \mathcal{F})$, we can define the smallest possible average degree of an $\mathcal{F}$–free subgraph $G_0$ of a graph $G$ with average degree $d$ such that $\overline{d}(G_0)$ is maximized. Formally, denote:

$$f'(d, \mathcal{F}) := \min_{\overline{d}(G) \geq d} \max \{\overline{d}(G_0) : G_0 \subseteq G \text{ is } \mathcal{F}\text{–free}\}.$$

A reformulation of Thomassen’s conjecture is to say that for every fixed $g$, $f'(d, \{C_3, \ldots, C_{g-1}\}) \to \infty$ when $d \to \infty$. If $g = 4$, it is easy to check that $f'(d, \{C_3\}) = d/2$. For $g = 6$, Kühn and Osthus [13] showed that $f'(d, \{C_3, C_4, C_5\}) = \Omega(\sqrt{\log \log d})$, thus confirming the conjecture for this case.

**Families of non-bipartite graphs.** It is rather easy to determine both functions $f(m, \mathcal{F})$ and $h(\delta, \Delta, \mathcal{F})$ asymptotically when $\chi(\mathcal{F}) > 2$ (there are no bipartite graphs in $\mathcal{F}$), as shown by the following proposition (a proof is provided in Section 2.1).

**Proposition 1.** Let $G$ be a graph with $m$ edges, minimum degree $\delta$ and maximum degree $\Delta$. Then, for every $k \geq 3$ there exists a $(k-1)$–partite subgraph $G_0$ of $G$ such that for every $v \in V(G)$,

$$d_{G_0}(v) \geq \left(1 - \frac{1}{k-1}\right) d_G(v).$$

In particular, for every $\mathcal{F}$ with $\chi(\mathcal{F}) = k$, we have

$$f(m, \mathcal{F}) = \left(1 - \frac{1}{k-1} + o_m(1)\right) m,$$

and

$$h(\delta, \Delta, \mathcal{F}) = \left(1 - \frac{1}{k-1} + o(1)\right) \delta.$$
**Even cycles.** It remains to study \( f(m, F) \) and \( h(\delta, \Delta, F) \) when \( F \) contains a bipartite graph. In this paper, we focus on the case \( F = F_r = \{ C_3, C_4, \ldots, C_{2r+1} \} \) and \( F = F_{r, \text{even}} = \{ C_4, C_6, \ldots, C_{2r} \} \) for some \( r \geq 2 \). We define

\[
f(m, r) = f(m, F_{r, \text{even}}),
\]

and

\[
h(\delta, \Delta, r) = h(\delta, \Delta, F_{r, \text{even}}).
\]

Using Proposition 1 one can easily show that for every family \( F \supseteq F_{r, \text{even}} \) such that for every bipartite graph \( H \in F \) we have \( H \in F_{r, \text{even}} \),

\[
f(m, F) = \Theta(f(m, r)) \quad \text{and} \quad h(\delta, \Delta, F) = \Theta(h(\delta, \Delta, r)).
\]

Denoting by \( K_{s,t} \) the complete bipartite graph with parts of size \( s \) and \( t \), our first result is:

**Theorem 2.** For every \( r \geq 2 \) there exists \( c = c(r) > 0 \) such that for every large enough \( m \),

\[
f(m, r) \geq \frac{c}{\log m} \min_{1 \leq k \leq m} \text{ex}(K_{k, m/k}, F_{r, \text{even}}).
\]

Observe that for every \( k \) dividing \( m \), we have

\[
f(m, r) \leq \text{ex}(K_{k, m/k}, F_{r, \text{even}}),
\]

since \( K_{k, m/k} \) has \( m \) edges. Thus, Theorem 2 is asymptotically tight up to a logarithmic factor. This theorem is proved in Section 3.

The value of \( \text{ex}(K_{k, m/k}, F_{r, \text{even}}) \) is not known for general \( k \) and \( r \). Some results for small values of \( r \) can be found in [5, 10]. The results in [17] imply the following explicit upper bound

\[
f(m, r) = O(m(r+1)/2r).
\]

The case \( r = 2 \) of the above problem appears to be more accessible. In particular, by Kovári, Sós and Turán [12] and by Reiman [18], \( \text{ex}(K_n, C_4) = \Theta(n^{3/2}) \) (see Chapter 6.2 of [2] for a discussion). Here we derive the following corollary (proved in Section 3):

**Corollary 3.** There exists a constant \( c > 0 \) such that for every large enough \( m \),

\[
f(m, 2) \geq \frac{cm^{2/3}}{\log m}.
\]

We remark that by applying a standard double counting argument of [12], one can show that \( \text{ex}(K_{m^{1/3}, m^{2/3}}, C_4) = O(m^{2/3}) \). Hence we obtain \( f(m, 2) = \Theta(m^{2/3}) \) (where the \( \Theta \) notation neglects polylogarithmic terms). Then, by [11]

\[
f(m, \{ C_3, C_4, C_5 \}) = \tilde{\Theta}(m^{2/3}).
\]

By using [11] with \( F = F_r \), Theorem 2 provides a lower bound on \( f(m, F_r) \), which implies the existence of large subgraphs with high girth.

We also provide a general lower bound for \( h(\delta, \Delta, r) \) (proved in Section 4):
Theorem 4. For every $r \geq 2$ and every large enough $\Delta$, let $G$ be a graph with $c' \Delta$ vertices, $c' > 2e^4$, girth $g(G) \geq 2r + 2$ and minimum degree $q$. Then there exists $c > c'(r) > 0$ such that for every graph $G$ with minimum degree $\delta$ and maximum degree $\Delta$ such that $\delta^2 \geq \Delta$, there exists a spanning subgraph $G_0$ of $G$ with $g(G_0) \geq 2r + 2$ and $\delta(G_0) \geq \frac{c\delta}{\Delta \log \Delta}$. In particular,

$$h(\delta, \Delta, r) \geq h(\delta, \Delta, \mathcal{F}_r) \geq \frac{c\delta}{\Delta \log \Delta}.$$ 

The existence of a graph with a given number of vertices, many edges and large girth is one of the most interesting open problems in extremal graph theory (see Section 4 in [9]). For every large enough $n$ we can prove the existence of a graph on $n$ vertices, girth at least $2r + 2$ and minimum degree $\Theta(n^{\frac{1}{2r - 1}})$ by the following quite standard argument (merely sketched here, as we do not believe the so obtained bound is tight – see below) as follows: consider a random graph $G \sim G(n, p)$ with $p = \varepsilon n^{\frac{1}{2r - 1} - 1}$, for some $\varepsilon > 0$. Then with high probability the number of cycles of length at most $2r$ going through any vertex is at most half of the minimum degree, if $\varepsilon$ is small enough. By deleting one edge for each such cycle we get a graph with girth at least $2r + 1$ and minimum degree $\Omega(n^{\frac{1}{2r - 1}})$. Taking now a largest bipartite subgraph of the so obtained graph eliminates all cycles of length at most $2r + 1$, yet at most halving the degrees. Thus we can prove the following:

Corollary 5. There exists a constant $c > 0$ such that for every large enough $\Delta$ and $\delta$ such that $\delta^2 \geq \Delta$,

$$h(\delta, \Delta, r) \geq h(\delta, \Delta, \mathcal{F}_r) \geq \frac{c\delta}{\Delta^{1 - \frac{1}{2r - 1}} \log \Delta}.$$ 

Using the upper bound for the extremal Turán number of even cycles $\text{ex}(K_d, C_{2r}) = O(d^{\frac{1}{1/r}})$ shown by Bondy and Simonovits [8], we have

$$h(d, d, r) \leq \frac{2}{d} \text{ex}(K_d, C_{2r}) = O(d^2).$$

Since the above cited upper bound for $\text{ex}(K_d, C_{2r})$ is conjectured to be of the right order, Corollary 5 is probably not tight.

For $r = 2$ we can derive a better bound. Erdős, Rényi and Sós [6] and Brown [4] showed that for every prime $p$ there exists a $C_4$–free graph with $p^2 - 1$ vertices and minimum degree $p$. By the density of primes, for every $n$ there exists a graph of order at most $2n$ satisfying the former properties. Thus, we can obtain the following corollary of Theorem 4.

Corollary 6. There exists a constant $c > 0$ such that for every large enough $\Delta$ and $\delta$ such that $\delta^2 \geq \Delta$,

$$h(\delta, \Delta, 2) \geq h(\delta, \Delta, \{C_3, C_4, C_5\}) \geq \frac{c\delta}{\sqrt{\Delta} \log \Delta}.$$ 

This corollary is tight up to a logarithmic factor, as will be shown in Proposition 14 (Section 4). The condition on $\delta$ and $\Delta$ is tight since, if $\Delta \geq \delta^2$, any spanning $C_4$–free subgraph $G_0$ of $K_{\delta, \Delta}$ satisfies $\delta(G_0) \leq 1$. Similar results can be derived for $r = 3, 5$, since there exist graphs with girth at least 8 and 12 respectively and large minimum degree [13] [14].

Outline. We begin with some preliminary considerations in Section 2. We then prove Theorem 4 and Corollary 3 in Section 3. Section 4 is devoted to the proof of Theorem 4 and Proposition 13. We conclude by some remarks and open questions in Section 5.
2 Preliminaries

For every vertex $v \in V$ let $N_G(v)$ denote the set of neighbors of $v$ in $G$ and $d_G(v) = |N_G(v)|$. If the graph $G$ is clear from the context we will denote the above quantities by $N(v)$ and $d(v)$, respectively.

2.1 Proof of Proposition 1

We start by proving Proposition 1. We will use the well-known Erdős-Stone-Simonovits theorem [3, 7]: for every graph $H$ with $\chi(H) = k$,

$$ex(K_k,H) = \left(1 - \frac{1}{k-1} + o(1)\right) \binom{n}{2}.$$ (3)

Proof of Proposition 1 Let $G$ be a graph with $m$ edges and consider a $(k-1)$-partition $\mathcal{P} = \{P_1, P_2, \ldots, P_{k-1}\}$ of $V(G)$ that maximizes the number of edges in the subgraph $G_{\mathcal{P}} = G \setminus (G[P_1] \cup \cdots \cup G[P_{k-1}])$. Then, we claim that for every $v \in V(G)$, $\delta_{G_{\mathcal{P}}}(v) \geq \left(1 - \frac{1}{k-1}\right) \delta_G(v)$.

For the sake of contradiction, suppose that there is a vertex $v \in P_i$, $i \in [k-1]$, with degree in $G_{\mathcal{P}}$ less than $\left(1 - \frac{1}{k-1}\right) \delta_G(v)$. Then, there are more than $\frac{\delta_G(v)}{k-1}$ neighbors of $v$ in $P_i$ and there is a part $P_j$, $j \neq i$, with at most $\frac{\delta_G(v)}{k-1}$ neighbors of $v$. Moving $v$ from $P_i$ to $P_j$ increases the number of edges in $G_{\mathcal{P}}$ by at least one, which gives a contradiction by the choice of $\mathcal{P}$.

Clearly, $G_{\mathcal{P}}$ does not contain any copy of $H$ with $\chi(H) \geq k$, and thus it does not contain any graph in $\mathcal{F}$. Observe also that $G_{\mathcal{P}}$ has at least $\left(1 - \frac{1}{k-1}\right) m$ edges and minimum degree $\delta(G_{\mathcal{P}}) \geq \left(1 - \frac{1}{k-1}\right) \delta(G)$.

Notice that $f(m, \mathcal{F}) \leq f(m, \mathcal{H})$ and $h(\delta, \Delta, \mathcal{F}) \leq h(\delta, \Delta, \mathcal{H})$ for every $H \in \mathcal{F}$. Let $H \in \mathcal{F}$ be such that $\chi(H) = k$. The upper bound for $f(m, H)$ follows from (3) by choosing $n$ for which $\frac{(n-1)}{2} < m \leq \binom{n}{2}$, and then by taking $G$ to be any subgraph of $K_n$ with exactly $m$ edges. For the upper bound on $h(\delta, \Delta, \mathcal{H})$ (assuming $\delta \geq 2$) take $\Delta$ disjoint copies of $K_{\delta+1}$, add a new vertex $v$ and connect it to one vertex from each of the cliques $K_{\delta+1}$. If a subgraph $G_0$ of the so obtained graph $G$ is $H$-free, then the subgraph of $G_0$ spanned by the vertex set of each of the cliques $K_{\delta+1}$ is $H$-free as well, thus implying by (3) that $\delta(G_0) \leq \left(1 - \frac{1}{k-1} + o(1)\right) \delta$. \hfill $\square$

2.2 Useful definitions

The following definitions will be useful in our proofs.

Definition 7. For every graph $G$, every $G$ with $V(G) = [\ell]$ and every vertex labeling $\chi : V(G) \to [\ell]$ we define the spanning subgraph $G'_{(\chi, G)} \subseteq G$ as the subgraph with vertex set $V(G)$ where an edge $e = uv$ is present if and only if $uv \in E(G)$ and $\chi(u) \chi(v) \in E(\mathcal{G})$.

Definition 8. For every graph $G$, every $G$ with $V(G) = [\ell]$ and every vertex labeling $\chi : V(G) \to [\ell]$ we define the spanning subgraph $G^*_{(\chi, G)} \subseteq G$ as the subgraph with vertex set $V(G)$ such that an edge $e = uv$ is present in $G^*$ if all the following properties are satisfied:

i) $uv \in E(G)$ and $\chi(u) \chi(v) \in E(\mathcal{G})$, that is $e \in E(G')$,

ii) for every $w \neq v$, $w \in N_G(u)$, we have $\chi(w) \neq \chi(v)$, and

iii) for every $w \neq u$, $w \in N_G(v)$, we have $\chi(w) \neq \chi(u)$.

The concept of frugal coloring was introduced by Hind, Molloy and Reed in [11]. We say that a proper coloring $\chi : V(G) \to [\ell]$ is $t$-frugal if for every vertex $v$ and every color $c \in [\ell]$,

$$|N_G(v) \cap \chi^{-1}(c)| \leq t,$$
that is, there are at most $t$ vertices of the same color in the neighborhood of each vertex. For instance, a 1–frugal coloring of $G$ is equivalent to a proper coloring of $G^2$.

### 2.3 Probabilistic tools

Here we state some (standard) lemmas we will use in the proofs.

**Lemma 9** (Chernoff inequality for binomial distributions [1]). Let $X \sim \text{Bin}(N, p)$ be a Binomial random variable, then for all $0 < \varepsilon < 1$,

1. $\Pr(X \leq (1 - \varepsilon)Np) < \exp \left( -\frac{\varepsilon^2}{2} Np \right)$.
2. $\Pr(X \geq (1 + \varepsilon)Np) < \exp \left( -\frac{\varepsilon^2}{2} Np \right)$.

Let $L : S^T \to \mathbb{R}$ be a functional. We say that $L$ satisfies the Lipschitz condition if for every $g$ and $g'$ differing in just one coordinate from the product space $S^T$, we have

$$|L(g) - L(g')| \leq 1.$$

**Lemma 10** (Azuma inequality, Theorem 7.4.2 in [1]). Let $L$ satisfy the Lipschitz condition relative to a gradation of length $l$ ($|T| = l$). Then for all $\lambda > 0$

1. $\Pr(X \leq E(X) - \lambda \sqrt{\lambda}) < e^{-\frac{\lambda^2}{2}}$,
2. $\Pr(X \geq E(X) + \lambda \sqrt{\lambda}) < e^{-\frac{\lambda^2}{2}}$.

**Lemma 11** (Weighted Lovász Local Lemma [16]). Let $A = \{A_1, \ldots, A_N\}$ be a set of events and let $H$ be a dependency graph for $A$.

If there exist weights $w_1, \ldots, w_N \geq 1$ and a real $p \leq \frac{1}{4}$ such that for each $i \in [N]$:

1. $\Pr(A_i) \leq p^{w_i}$, and
2. $\sum_{j : ij \in E(H)} (2p)^{w_j} \leq \frac{w_i}{p}$,

then

$$\Pr \left( \bigcap_{i=1}^{N} \overline{A_i} \right) > 0.$$

### 3 Subgraphs with large girth and many edges

This section is devoted to the proofs of Theorem 2 and Corollary 3.

**Proof of Theorem 2** Let $G$ be a graph with $m$ edges. Define $V_1$ to the set of vertices of $G$ with degree at least $2\sqrt{m}$, and $V_2 = V \setminus V_1$. Observe that

$$|V_1| \leq 2m/2\sqrt{m} = \sqrt{m}$$

and thus $V_1$ spans at most $m/2$ edges. Recall that $F^\text{even}_r = \{C_4, C_6, \ldots, C_{2r}\}$ is the family of all even cycles of length at most $2r$. In order to find an $F^\text{even}_r$–free subgraph with many edges, we will remove all the edges inside $V_1$ and only look at the edges between $V_1$ and $V_2$, and the edges inside $V_2$. We will split the proof into two cases in terms of the number of edges between $V_1$ and $V_2$. 


Case 1: $e(V_1, V_2) \geq m/4$. Consider the following partition of $V_1$ into sets $U_1, \ldots, U_s$, where $U_p$ contains all vertices of $V_1$ whose degree into $V_2$ is between $2^p$ and $2^{p+1}$.

Notice that $s \leq \log m$. Hence, there is a subset $U_q$ incident to at least $m/4 \log m$ edges leading to $V_2$. Let $|U_q| = k$, then $k = \Omega(m/(2^q \log m))$ and $k = O(m/2^q)$ since all the degrees in $U_q$ are between $2^q$ and $2^{q+1}$. Let $u_1, \ldots, u_k$ be the vertices of $U_q$.

Fix an $\mathcal{F}_r^{\text{even}}$-free bipartite graph $G$ with parts $A = [k]$ and $B = [m/k]$, and $\text{ex}(K_k, m/k, \mathcal{F}_r^{\text{even}})$ edges, and let $\chi : V_2 \to B$ be a random $m/k$-coloring of $V_2$, where each vertex is colored uniformly at random. We say that $j \in [m/k]$ is good for $u_i \in U_q$ if:

1. $(i, j) \in E(G)$, and
2. exactly one neighbor of $u_i$ in $G$ is colored $j$ by $\chi$.

Let $G_1$ be the subgraph of $G$ that only contains the edges $e = (u_i, v) \in E(G)$, where $u_i \in U_q$, $v \in V_2$, and $\chi(v) = j$ is good for $u_i$. We claim that $G_1$ is an $\mathcal{F}_r^{\text{even}}$-free graph. Assume that there is a cycle of length $l$ of even length in $G_1$ with $l \leq 2r$ and let $u_1, \ldots, u_l$ be its vertices. Observe that there should be at least one repeated color in the vertices of the cycle, otherwise $G$ would contain a $C_2r$. Let $a, b \in [l]$ such that $\chi(u_a) = \chi(u_b)$ and for every $a < j < b$, $\chi(u_a) \neq \chi(u_j)$. Besides, since $\chi$ is 1-frugal in $G_1$ (in the sense that for each $u_i \in U_q$, all neighbors of $u_i$ in $G_1$ are colored differently by $\chi$), $b - a \geq 3$. Thus, there exists a cycle of length $b - a$ in $G$, a contradiction since $G$ is bipartite and, since it is $\mathcal{F}_r^{\text{even}}$-free, it has no even cycles of length at most $2r$.

Let us compute the expected size of $G_1$. Fix $u_i \in U_q$. Then, the probability that a given edge $e = (u_i, v) \in E(G)$ exists in $G_1$ is

$$\Pr(e \in E(G_1)) = \frac{d_G(i)}{m/k} \left(1 - \frac{1}{m/k}\right)^{2^{q+1}} = \Omega \left(\frac{d_G(i) k}{m \cdot \frac{m}{k}}\right) = \Omega \left(\frac{d_G(i) k}{m}\right),$$

since $2^q k = O(m)$.

Thus, the expected degree of $u_i$ in $G_1$ is of order

$$\mathbb{E}(d_{G_1}(u_i)) = \Omega \left(2^q \frac{d_G(i) k}{m}\right).$$

Recall that $\sum d_G(i) = \text{ex}(K_k, m/k, \mathcal{F}_r^{\text{even}})$, hence the expected number of edges of $G_1$ is of order $2^q \text{ex}(K_k, m/k, \mathcal{F}_r^{\text{even}}) k/m$. Since $2^q k \geq cm/\log m$, the expected number of edges is of order at least

$$\min_k 2^q \frac{\text{ex}(K_k, m/k, \mathcal{F}_r^{\text{even}}) k}{m} \geq \frac{c}{\log m} \min_k \text{ex}(K_k, m/k, \mathcal{F}_r^{\text{even}}).$$

Case 2: $e(V_1, V_2) < m/4$. Then $e(V_2) > m/4$, and all the degrees in $G[V_2]$ are less than $2\sqrt{m}$. Fix a coloring $\chi$ of $G[V_2]$ with $2\sqrt{m}$ colors.

We will find a large $\mathcal{F}_r^{\text{even}}$-free graph inside $V_2$. For this, let $G$ be an $\mathcal{F}_r^{\text{even}}$-free graph on $2\sqrt{m}$ vertices with the largest possible number of edges. Assume $V(G) = \lfloor 2\sqrt{m} \rfloor$ and let $\chi : V_2 \to V(G)$ be a random labeling of the vertices of $V_2$.

Consider the graph $G^* = G^*(\chi, G)$ from Definition 1 applied to the induced subgraph $G[V_2]$. For each edge
e = uv of \( G \) spanned by \( V_2 \), its probability to belong to \( G^* \) is at least

\[
\frac{1}{2\sqrt{m}} \sum_{i \in V(G)} \frac{d_G(i)}{2\sqrt{m}} \left( 1 - \frac{1}{2\sqrt{m} - 2} \right)^{d_G(u)+d_G(v)-2} = \Omega \left( \frac{\text{ex}(K_{2\sqrt{m}, F^*_r})}{m} \right),
\]

since \( d_G(u), d_G(v) \leq 2\sqrt{m} \). To justify the above estimate, first choose a label \( i \) for \( u \), then require a label \( j \) of \( v \) to fall between the neighbors of \( i \) in \( G \), and finally for each of the neighbors of \( u \) and \( v \) in \( G \) choose a label different from \( i \) and \( j \).

Thus, we expect

\[
\Omega \left( \text{ex}(K_{2\sqrt{m}, F^*_r}) \right) = \Omega \left( \text{ex}(K_{\sqrt{m}, \sqrt{m}, F^*_r}) \right) = \omega \left( \text{ex}(K_{\sqrt{m}, \sqrt{m}, F^*_r}) \right) = \omega \left( \text{ex}(K_{\sqrt{m}, \sqrt{m}, F^*_r}) \right) = \omega \left( \text{ex}(K_{\sqrt{m}, \sqrt{m}, F^*_r}) \right).
\]

We now prove Corollary 3.

**Proof of Corollary 3** By Theorem 2 we have

\[
f(m, 2) \geq c \log m \min_{1 \leq k \leq m} \text{ex}(K_{k,m/k}, C_4),
\]

for some small constant \( c > 0 \). By the symmetry of \( K_{s,t} \) and \( K_{t,s} \), we may assume without loss of generality that the minimum is attained when \( k \leq \sqrt{m} \).

We provide two constructions. First, the disjoint union of \( k \) stars of degree \( m/k \), which is a \( C_4 \)-free graph, shows that \( \text{ex}(K_{k,m/k}, C_4) \geq m/k \).

On the other hand, one can construct a \( C_4 \)-free graph by selecting a subgraph of a larger \( C_4 \)-free graph. Let \( G_1 \) be a largest \( C_4 \)-free subgraph of \( K_{m/m/k} \). We construct a \( C_4 \)-free bipartite graph \( G_2 \) by keeping the \( k \) vertices with highest degrees in one of the parts of \( G_1 \). Then \( G_2 \) is a subgraph of \( K_{k,m/k} \) and has at least \( \frac{m}{k} \text{ex}(K_{m/k,m/k}, C_4) = \Omega(\sqrt{mk}) \) edges.

Hence, for every \( k \),

\[
\text{ex}(K_{k,m/k}, C_4) \geq \max\{m/k, \Omega(\sqrt{mk})\} \geq \Omega \left( m^{2/3} \right).
\]

4 Subgraphs with large girth and large minimum degree

We devote this section to the proof of Theorem 4. Before proving the theorem, let us state an auxiliary lemma.

In a vertex-colored graph, a cycle is called **rainbow** if all its vertices have distinct colors. A path is called **maximal inner-rainbow** if its endpoints have the same color \( i \), but all other vertices of the path are colored with distinct colors (other than \( i \)). We will use the following lemma:

**Lemma 12.** Let \( G \) be a graph with maximum degree \( \Delta \) and minimum degree \( \delta \), that admits a \( t \)-frugal coloring \( \chi \) without rainbow cycles of length at most \( 2r+1 \) and maximal inner-rainbow paths of length \( l \) for every \( 3 \leq l \leq 2r \). If \( \delta > 129t^3 \log \Delta \), then there exists a subgraph \( G_0 \subseteq G \) such that

1. \( \forall v \in V(G), d_{G_0}(v) \geq \frac{d_G(v)}{4t}, \) and
2. \( g(G_0) \geq 2r + 2. \)
Proof. Let the color classes of \( \chi \) be \( S_1, S_2, \ldots \). Assign to each edge \( e \in E(G) \) a random variable \( f(e) \) uniformly distributed in \( (0,1) \). We construct the following subgraph \( G_0 \): for every pair of color classes \( (S_i, S_j) \) of \( \chi \), an edge \( e \) between \( S_i \) and \( S_j \) is retained in \( G_0 \) if \( f(e) \) is less than \( f(e') \) for every \( e' \) between \( S_i \) and \( S_j \) incident with \( e \). Observe that, by construction, \( \chi \) is a 1-frugal coloring of \( G_0 \), that is, the vertices of any pair of color classes of \( \chi \) induce a matching.

We claim that deterministically (i.e., with probability 1) \( g(G_0) \geq 2r + 2 \). For the sake of contradiction, suppose that \( g(G_0) < 2r + 2 \) and let \( C = (u_1,\ldots,u_l) \) be a cycle in \( G_0 \) with \( l < 2r + 2 \). Since \( \chi \) does not induce any rainbow cycle of length at most \( l \) in \( G \), there exist at least two vertices of \( C \) with the same color. Let \( a,b \in [l] \) be such that, \( a < b \), \( \chi(u_a) = \chi(u_b) \) and for every \( a < j < b \), \( \chi(u_j) \neq \chi(u_j) \). Then \( 3 \leq b - a \leq 2r \) since \( \chi \) is a 1-frugal coloring in \( G_0 \) and \( C \) is not rainbow. Then \( u_a,\ldots,u_b \) is a maximal inner-rainbow path with a forbidden length, a contradiction.

Now, it remains to show that with positive probability the obtained subgraph \( G_0 \) has the desired minimum degree.

Observe first that a given edge \( e = (u,v) \), with \( u \in S_i \) and \( v \in S_j \), is preserved in \( G_0 \) with probability
\[
\frac{1}{d_G(u,S_j) + d_G(v,S_i) - 1} \geq \frac{1}{2t - 1},
\]
where \( d_G(w,S) = |N_G(w) \cap S| \).

For every \( v \in V \) consider the random variable \( L(v) \) equal to the degree of \( v \) in \( G_0 \). We have: \( \mathbb{E}(L(v)) \geq d_G(v)/2t \). By applying Azuma’s inequality (Lemma 10) we will now show that with probability exponentially close to 1, \( L(v) \) is large enough.

First of all, observe that \( L(v) \) only depends on the edges that connect a neighbor of \( v \) to a vertex of color \( \chi(v) \). Let \( T_v \) be the set of these edges:
\[
T_v = \bigcup_{u \in N_G(v)} \{vu\} \cup \{uw : w \in N_G(u) \text{ and } \chi(v) = \chi(w)\}.
\]
Since \( \chi \) is \( t \)-frugal, we have \( |T_v| \leq td_G(v) \). Let \( S = (0,1) \). Then \( L(v) \) depends only on a vector in \( ST_v \).

If two functions \( f,f' : ST_v \to \mathbb{R} \) differ only on one edge of \( T_v \), \( |L(v)(f(T_v)) - L(v)(f'(T_v))| \leq 1 \). Thus \( L = L(v) \) satisfies the 1-Lipschitz condition. Let \( A_v \) be the event that \( L(v) \leq d_G(v)/4t \). Since the martingale length is at most \( td_G(v) \), by setting \( \lambda = \frac{\sqrt{d_G(v)}}{4t^{3/2}} \) in Lemma 11,
\[
\Pr(A_v) = \Pr \left( L(v) \leq \frac{d_G(v)}{4t} \right) = \Pr \left( L(v) \leq \mathbb{E}(L(v)) - \frac{d_G(v)}{4t} \right) < e^{-\frac{d_G(v)}{32t^2}} < e^{-\frac{1}{32t^2}}.
\]
Since \( A_v \) is influenced only by the edges in \( T_v \), it is thus independent of all \( A_u \) but those for which \( u \) is at distance at most 4 from \( v \), and there are at most \( \Delta^4 \) such events.

Notice that, if \( \Delta \) is large enough,
\[
2 \Pr(A_v) \Delta^4 < 2e^{-\frac{1}{32t^2}} \Delta^4 < \frac{1}{2},
\]
since \( \delta > 129t^3 \log \Delta \). Thus, by the Lovász Local Lemma (Lemma 11) with \( p = \Pr(A_v) \) and \( w_i = 1 \) we have
\[
\Pr(\bigcap_{v \in V} A_v) > 0,
\]
and thus, there is a way to assign values to \( f(e) \) such that \( \delta(G_0) \geq d_G(v)/4t \). \qed
Now we are ready to prove Theorem 4.

**Proof of Theorem 4.** The idea in this proof is to randomly color the vertices from $G$ with $\ell$ colors, where $\ell = |V(G)|$ for a graph $G$ satisfying the conditions in the statement. We then to consider the subgraph $G' = G'_{(\chi, G)}$ from Definition 7 induced by the coloring and the graph $G$. We will show that with positive probability, such a coloring is $t$-frugal and induces a graph $H$ with neither rainbow cycles of length at most $2r$ nor maximal inner-rainbow paths of length $3 \leq l \leq 2r$. The value of $t$ will be set later in the proof. Then, we will use Lemma 12 to obtain the desired subgraph.

Let $\chi$ be a random coloring of $V(G)$ with $\ell$ colors. Consider the spanning subgraph $G' = G'_{(\chi, G)}$ of $G$. Since $g(G) \geq 2r + 2$, $\chi$ does not induce any rainbow cycle of length at most $2r + 1$ in $G'$, nor any maximal inner-rainbow path of length at least $3$ and at most $2r$. Moreover, $\chi$ is a proper coloring on $G'$, since $G$ has no loops.

We will use the Lovász Local Lemma to show that there is a positive probability that the random $\ell$-coloring of $G$ satisfies the following properties:

1. for every $v \in V(G)$, $d_{G'}(v) = \Omega\left(\frac{q\delta}{\ell}\right)$ and
2. $\chi$ is $t$-frugal in $G'$.

For this purpose we will define the following events:

1. **Type A**: for each $v \in V(G)$, $A_v$ is the event $d_{G'}(v) \leq \frac{q{d}(v)}{8\ell}$.
2. **Type B**: for each vertex $v \in V(G)$ and each set $X = \{x_1, \ldots, x_{t+1}\} \subseteq N_G(v)$, $B_{v,X}$ is the event $\chi(x_i) = \chi(x_j)$, for every $i, j \in [t+1]$.

Since $d_G(v) \geq \delta$, using Lemma 12 Part 1 with $\varepsilon = 1/2$ one can check that

$$\Pr(A_v) = \Pr\left(d_{G'}(v) \leq \frac{q{d(G)}(v)}{8\ell}\right) \leq \Pr\left(d_{G'}(v) \leq \frac{1}{4}E(d_{G'}(v))\right) \leq e^{-\Omega(\varepsilon d_{G}(v))} \leq e^{-\Omega\left(\frac{q\delta}{\ell}\right)}.$$

We also have

$$\Pr(B_{v,X}) = \ell^{-t}.$$

Here, it is convenient to define the auxiliary event $D_e$ as the event $e \in E(G')$. Observe that any event $A_v$ or $B_{v,X}$ can be expressed in terms of the events $D_e$.

**Claim.** Let $v \in V(G)$ and let $F \subseteq E(G)$ a set of edges not incident to $v$. Then, for each $i \in [\ell]$,

$$\Pr(\chi(v) = i \mid \cup_{e \in F} D_e) = \frac{1}{\ell}.$$

To prove the claim, observe that all the unveiled information is about non-incident edges and thus no information about the color of $v$ has been provided. While information on the existence of the edges in $F$ may affect the degree of $v$, it cannot affect its color.

Since the existence of an edge $e = uv$ in $G'$ depends only on the colors of $u$ and $v$, we have at most the number of dependencies given in Table 1.

Then, by applying the weighted version of the Local Lemma (Lemma 11) with $p = \ell^{-1}$, $w_A = \Omega\left(\frac{q\delta}{\ell \log \ell}\right)$ and $w_B = t$, we have that a subgraph $G'$ avoiding all events exists if:

$$\Delta^2e^{-\Omega\left(\frac{q\delta}{\ell \log \ell}\right)} + \Delta^2\left(\frac{\Delta}{t}\right)(2p)^t = o\left(\frac{q\delta}{\ell \log \ell}\right),$$

and
| Type          | Type A | Type B |
|--------------|--------|--------|
| Type A       | $\Delta^2$ | $\Delta^2(t/\ell)$ |
| Type B       | $(t + 2)\Delta$ | $(t + 2)\Delta(t/\ell)$ |

Table 1: Table of dependencies

$$(t + 2)\Delta e^{-O(\frac{1}{t^2})} + (t + 2)\Delta\left(\frac{\Delta}{t}\right)(2p)^t \leq \frac{t}{2}.$$  

Since $\delta^2 \geq \Delta$, this is satisfied for $p^{-1} = \ell \geq 2\Delta^{1+\frac{1}{4}}$, if $\Delta$ is large enough.

Then, there is a subgraph $G'$ such that for every $v \in V$,

$$d_{G'}(v) \geq \frac{q\delta}{8\ell},$$

and $G'$ admits a $t$–frugal coloring with $\ell$ colors, no rainbow cycle of length at most $2r + 1$, and no maximal inner-rainbow path of length at least 3 and at most $2r$.

Now, set $t = \log \Delta$, which implies $\ell \geq 2e^4\Delta \geq 2\Delta^{1+\frac{1}{4}}$. The existence of a graph $G$ of order at least $2e^4\Delta$ and $g(G) \geq 2r + 2$ is provided by the hypothesis of the theorem. Since $\delta \geq \sqrt{\Delta} \geq 129\log^4\Delta = 129\ell^4\log\Delta$ if $\Delta$ is large enough, we can apply Lemma 12 to obtain a subgraph $G_0$ with $g(G_0) \geq 2r + 2$ and for every $v \in V$,

$$d_{G_0}(v) \geq \frac{q\delta}{32t\ell} = \frac{cq\delta}{\ell \log \Delta}.$$

for some small constant $c > 0$.

The following proposition shows that Corollary 6 is tight up to a logarithmic factor.

**Proposition 13.** For every $\delta, \Delta$ satisfying $\Delta \leq \delta^2 \leq \Delta^2$, there exists a graph $G$ with minimum degree $\delta$ and maximum degree $\Delta$ such that for every spanning $C_4$–free subgraph $G_0$ of $G$,

$$\delta(G_0) = O\left(\frac{\delta}{\sqrt{\Delta}}\right).$$

**Proof.** Let $G$ be the complete bipartite graph with parts $A, B$ of sizes $\Delta$ and $\delta$, respectively. For the sake of contradiction, suppose that there exists a $C_4$–free subgraph $G_0$ such that $\delta(G_0) > 2\delta/\sqrt{\Delta}$. We call a pair of edges incident to a common vertex $v$, a cherry of $v$. We will get a contradiction by double counting the number of cherries of vertices of $A$. On the one hand, $\delta(G_0) > 2\delta/\sqrt{\Delta}$, there are at least $\Delta \cdot \left(\frac{2\delta}{\sqrt{\Delta}}\right)$ cherries of vertices of $A$.

On the other hand, since $G_0$ is $C_4$–free, each pair of vertices in $B$ has at most one common neighbor in $A$ having a cherry, thus there are at most $\binom{\delta}{2}$ cherries of vertices of $A$, and we have:

$$2\delta^2 - \delta\sqrt{\Delta} = \Delta\left(\frac{2\delta}{\sqrt{\Delta}}\right) \leq \binom{\delta}{2} = \frac{\delta^2 - \delta}{2},$$

providing a contradiction. $\square$
5 Remarks and open questions

1. There are still logarithmic gaps between the lower bounds (Theorem 2 and Corollary 6) and the upper bounds for $f$ and $h$. We conjecture that the upper bounds are asymptotically tight.

2. In order to give a more explicit result in Theorem 2 it is interesting to determine the value $k^*(m, F_r)$ that minimizes $ex(K_k, F_r)$. It is clear that for every $r \geq 2$, $k^* = \Omega(m^{1/3})$ and $k^* = O(m^{2/3})$. Indeed, any extremal bipartite $F$-free graph has at least as many edges as the size of the largest stable set which is $\Omega(m^{2/3})$ in both previous cases.

In the proof of Corollary 3 we showed that $k^*(m, C_4) = \Theta(m^{1/3})$.

Observe that

$$\lim_{r \to \infty} k^*(m, F_r) = m^{1/2},$$

When $r$ tends to infinity, $F_r$ is composed of all cycles of length up to $2r + 1$, and thus, the extremal graph tends to a tree. In this case, the number of edges is of the order of the number of vertices in the graph, which is minimized when both stable sets are of the same size approximately. Thus, we get that for every $r \geq 2$,

$$f(m, r) = \Theta(f(m, F_r)) = \Omega \left( \frac{m^{1/2}}{\log m} \right).$$

However this is meaningless since it is clear that any graph $G$ with $m$ edges has a spanning forest with at least $\Omega(m^{1/2})$ edges: such a $G$ contains a star with $\sqrt{m}$ edges, or a matching of size $\Omega(m^{1/4})$.

3. Regarding function $h(\delta, \Delta, F)$, we conjecture that the following holds:

Conjecture 14. For every $\delta$, $\Delta$ and every family $F$, we have

$$h(\delta, \Delta, F) = \Omega \left( \frac{ex(K_\Delta, F)}{\Delta^2 \delta} \right).$$

Proposition 4 shows that this conjecture is true for $F$ not containing any bipartite graph. The bipartite case remains widely open. Our results imply that the conjecture is true up to a logarithmic factor for $F$ containing only the bipartite graphs in $F_2^{even}$, $F_3^{even}$ or $F_5^{even}$. Observe that in order to set the conjecture for a given $F$, one does not need to establish $ex(K_\Delta, F)$. The only important feature would be to show that there exists an extremal graph with (close to) $ex(K_\Delta, F)$ edges and with large minimum degree.

In this direction we ask the following question,

Question 15. For every $r \geq 2$ and large $n$, does there exist a graph $G$ with $n$ vertices, $g(G) \geq 2r + 2$ and minimum degree

$$\delta(G) = \Omega \left( \frac{ex(K_n, F_r^{even})}{n} \right)?$$

The existence of such a graph would allow to improve Corollary 5.

4. It would be interesting to study the functions $f$ and $h$ for other families of bipartite graphs, such as complete bipartite graphs.

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