THE $\mathfrak{gl}_2$ BETHE ALGEBRA ASSOCIATED WITH A NILPOTENT ELEMENT

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Abstract. To any $2 \times 2$-matrix $K$ one assigns a commutative subalgebra $B^K \subset U(\mathfrak{gl}_2[t])$ called a Bethe algebra. We describe relations between the Bethe algebras, associated with the zero matrix and a nilpotent matrix.

1. Introduction

To any $N \times N$-matrix $K$ one assigns a commutative subalgebra $B^K \subset U(\mathfrak{gl}_N[t])$ called a Bethe algebra [T], [MTV1], [CT]. The Bethe algebra acts on any $U(\mathfrak{gl}_N[t])$-module giving an example of a quantum integrable system. In particular, it acts on any evaluation $U(\mathfrak{gl}_N[t])$-module $L_\lambda(0)$, where $L_\lambda$ is the irreducible finite-dimensional $\mathfrak{gl}_N$-module with some highest dominant integral weight $\lambda$.

The most interesting of the Bethe algebras is the Bethe algebra $B^0$ associated with the zero matrix $K$. The Bethe algebra $B^0$ is closely connected with Schubert calculus in Grassmannians of $N$-dimensional subspaces. The eigenvectors of the $B^0$-action on suitable $U(\mathfrak{gl}_N[t])$-modules are in a bijective correspondence with intersection points of suitable Schubert cycles [MTV3], [MTV4]. The most important of those $U(\mathfrak{gl}_N[t])$-modules is the infinite-dimensional module $V_S = (V \otimes \mathbb{C}[z_1, \ldots, z_n])^S$ introduced in [MTV3]. Here $V \otimes \mathbb{C}$ is the $n$-fold tensor power of the vector representation of $\mathfrak{gl}_N$ and the upper index $S$ denotes the subspace of invariants with respect to a natural action of the symmetric group $S_n$. The other $U(\mathfrak{gl}_N[t])$-modules related to Schubert calculus are subquotients of $V_S$.

The Bethe algebra $B^0$ commutes with the subalgebra $U(\mathfrak{gl}_N) \subset U(\mathfrak{gl}_N[t])$. Let $V^S = \bigoplus_\lambda V_{\lambda,0}^S$ be the $\mathfrak{gl}_N$-isotypical decomposition, where $\lambda$ runs through $\mathfrak{gl}_N$-highest weights. The Bethe algebra $B^0$ preserves this decomposition and $B^0_{\nu^S} = \bigoplus_\lambda B^0_{\lambda, \nu^S}$, where $B^0_{\nu^S} \subset \text{End}(V^S)$ and $B^0_{\lambda, \nu^S} \subset \text{End}(V_{\lambda,0}^S)$ are the images of $B^0$. It is shown in [MTV3] that the Bethe algebra $B^0_{\lambda, \nu^S}$ is isomorphic to the algebra $O^\lambda_{\nu^S}$ of functions on a suitable Schubert cell $\Omega_{\lambda, \nu^S}$ in a Grassmannian. It is also shown that the $B^0_{\lambda, \nu^S}$-module $V_{\lambda,0}^S$ is isomorphic to the regular representation of $O^\lambda_{\nu^S}$. These statements give a geometric interpretation of the $B^0_{\lambda, \nu^S}$-module $V_{\lambda,0}^S$ (or representational interpretation of $O^\lambda_{\nu^S}$) and they are key facts for applications of Bethe algebras to Schubert calculus.

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This paper has two goals. The first is to extend these results to the Bethe algebras $B^K$ associated with nonzero matrices $K$. Note that this goal was accomplished in [MTV5] for diagonal matrices $K$ with distinct diagonal entries.

The second goal is to express the $B^K$-action on the infinite-dimensional module $V^S$ in terms of the $B^0$-action on $V^S$ and the $B^K$-actions on finite-dimensional modules $L_\lambda(0)$.

In this paper we achieve these two goals for one example: $N = 2$ and $K = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$.

We denote $B$ the Bethe algebra $B^K$ associated with that nilpotent matrix $K$. We define a decomposition $V^S = \bigoplus \lambda V^S_\lambda$ into suitable $B$-modules called the deformed isotypical components of $V^S$. For any $\lambda$, $V^S_\lambda$ is a suitable deformation of the isotypical component $V^{S,0}_\lambda$. In particular, $V^S_\lambda$ and $V^{S,0}_\lambda$ have equal ranks as $\mathbb{C}[z_1, \ldots, z_n]$-modules. We have $B_{\lambda S} = B_\lambda$, where $B_{\lambda S} \subset \text{End}(V^S_\lambda)$ and $B_\lambda \subset \text{End}(V^S_\lambda)$ are the images of $B$.

For any $\lambda = (n - k, k)$, the image of $B$ in $L_\lambda(0)$ is isomorphic to $A_{n-2k} = \mathbb{C}[b]/(b^{n-2k+1})$. The algebra $A_{n-2k}$ acts on $L_\lambda(0)$ by the formula $b \mapsto e_{21}$ where $e_{21}$ is one of the four standard generators of $gl_2$. We show that the $B_\lambda$-module $V^S_\lambda$ is isomorphic to the regular representation of $A_{n-2k} \otimes O^0_\lambda$. Theorems 5.3 and 5.6 in [MTV3] give a geometric interpretation of the $B_\lambda$-module $V^S_\lambda$ as the regular representation of the algebra of functions with nilpotents on the Schubert cell $\Omega_\lambda$, where the nilpotents are determined by the algebra $A_{n-2k}$. This statement is our achievement of the first goal of this paper.

We define an action of $A_{n-2k} \otimes B^0_\lambda$ on $V^{S,0}_\lambda$ by the formula $b^j \otimes B : v \mapsto (e_{21})^j B v$. The $A_{n-2k} \otimes B^0_\lambda$-module $V^{S,0}_\lambda$ is isomorphic to the regular representation of $A_{n-2k} \otimes O^0_\lambda$ due to Theorems 5.3 and 5.6 in [MTV3].

As a result of these descriptions of the $B_\lambda$-module $V^S_\lambda$ and $A_{n-2k} \otimes B^0_\lambda$-module $V^{S,0}_\lambda$, we construct an algebra isomorphism $\nu_\lambda : A_{n-2k} \otimes B^0_\lambda \rightarrow B_\lambda$ and a linear isomorphism $\eta_\lambda : V^{S,0}_\lambda \rightarrow V^S_\lambda$ which establish an isomorphism of the $B_\lambda$-module $V^S_\lambda$ and $A_{n-2k} \otimes B^0_\lambda$-module $V^{S,0}_\lambda$, see Theorem 9.6. This statement is our achievement of the second goal.

The paper is organized as follows. In Section 2 we discuss representations of $U(gl_2[t])$ and introduce the $U(gl_2[t])$-module $V^S$. We introduce the Bethe algebra $B^K$ in Section 3. We define decompositions $V^S = \bigoplus \lambda V^S_\lambda$ and $B^0_\lambda = B_\lambda$ in Section 4. We study deformed isotypical components in Section 5. Section 6 is on the algebra $O_\lambda \simeq A_{n-2k} \otimes O^0_\lambda$. The first connections between the algebras $B_\lambda$ and $O_\lambda$ are discussed in Section 7. In Section 8 we show that the $B_\lambda$-module $V^S_\lambda$ is isomorphic to the regular representation of $A_{n-2k} \otimes O^0_\lambda$. In Section 9 we show that the $B_\lambda$-module $V^S_\lambda$ and $A_{n-2k} \otimes B^0_\lambda$-module $V^{S,0}_\lambda$ are isomorphic.

In [FFR], the authors study the Bethe algebra associated with a principal nilpotent element. One of our motivations was to relate the picture in [FFR] with our description of Bethe algebras in [MTV3], [MTV5].

2. Representations of current algebra $gl_2[t]$

2.1. Lie algebra $gl_2$. Let $e_{ij}, i, j = 1, 2$, be the standard generators of the complex Lie algebra $gl_2$ satisfying the relations $[e_{ij}, e_{kl}] = \delta_{jk}e_{ik} - \delta_{ik}e_{kj}$. We identify the Lie algebra $sl_2$ with the subalgebra in $gl_2$ generated by the elements $e_{11} - e_{22}, e_{12}, e_{21}$.

The elements $e_{11} + e_{22}$ and $(e_{11} + 1)e_{22} - e_{21}e_{12}$ are free generators of the center of $U(gl_2)$. 
Let $M$ be a $\mathfrak{gl}_2$-module. A vector $v \in M$ has weight $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ if $e_{ij}v = \lambda_i v$ for $i = 1, 2$. A vector $v$ is called singular if $e_{22}v = 0$.

We denote by $M[\lambda]$ the subspace of $M$ of weight $\lambda$, by Sing $M$ the subspace of $M$ of all singular vectors and by Sing $M[\lambda]$ the subspace of $M$ of all singular vectors of weight $\lambda$.

Denote $L_\lambda$ the irreducible finite-dimensional $\mathfrak{gl}_2$-module with highest weight $\lambda$. Any finite-dimensional $\mathfrak{gl}_2$ weight module $M$ is isomorphic to the direct sum $\bigoplus_\lambda L_\lambda \otimes \text{Sing } M[\lambda]$, where the spaces Sing $M[\lambda]$ are considered as trivial $\mathfrak{gl}_2$-modules.

The $\mathfrak{gl}_2$-module $L_{(1,0)}$ is the standard 2-dimensional vector representation of $\mathfrak{gl}_2$. We denote it $V$. We choose a highest weight vector of $V$ and denote it $v_+$. A $\mathfrak{gl}_2$-module $M$ is called polynomial if it is isomorphic to a submodule of $V^\otimes n$ for some $n$.

A sequence of integers $\lambda = (\lambda_1, \lambda_2)$, $\lambda_1 \geq \lambda_2 \geq 0$, is called a partition with at most 2 parts. Denote $|\lambda| = \lambda_1 + \lambda_2$. We say that $\lambda$ is a partition of $|\lambda|$.

The $\mathfrak{gl}_2$-module $V^\otimes n$ contains the module $L_\lambda$ if and only if $\lambda$ is a partition of $n$ with at most 2 parts.

For a Lie algebra $\mathfrak{g}$, we denote $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

2.2. Current algebra $\mathfrak{gl}_2[t]$. Let $\mathfrak{gl}_2[t] = \mathfrak{gl}_2 \otimes \mathbb{C}[t]$ be the complex Lie algebra of $\mathfrak{gl}_2$-valued polynomials with the pointwise commutator. We identify $\mathfrak{gl}_2[t]$ with the subalgebra $\mathfrak{gl}_2 \otimes 1$ of constant polynomials in $\mathfrak{gl}_2[t]$. Hence, any $\mathfrak{gl}_2[t]$-module has a canonical structure of a $\mathfrak{gl}_2$-module.

The standard generators of $\mathfrak{gl}_2[t]$ are $e_{ij} \otimes t^r$, $i, j = 1, 2$, $r \in \mathbb{Z}_{\geq 0}$. They satisfy the relations $[e_{ij} \otimes t^r, e_{sk} \otimes t^p] = \delta_{js}e_{ik} \otimes t^{r+p} - \delta_{ik}e_{sj} \otimes t^{r+p}$.

The subalgebra $\mathfrak{sl}_2[t] \subset \mathfrak{gl}_2[t]$ generated by the elements $(e_{11} + e_{22}) \otimes t^r$, $r \in \mathbb{Z}_{\geq 0}$, is central. The Lie algebra $\mathfrak{gl}_2[t]$ is canonically isomorphic to the direct sum $\mathfrak{sl}_2[t] \oplus \mathfrak{z}_2[t]$.

It is convenient to collect elements of $\mathfrak{gl}_2[t]$ in generating series of a variable $u$. For $g \in \mathfrak{gl}_2$, set

$$g(u) = \sum_{s=0}^{\infty} (g \otimes t^s)u^{-s-1}.$$ 

We have $(u - v)[e_{ij}(u), e_{sk}(v)] = \delta_{js}(e_{ik}(u) - e_{ik}(v)) - \delta_{ik}(e_{sj}(u) - e_{sj}(v))$.

For each $a \in \mathbb{C}$, there is an automorphism $\rho_a$ of $\mathfrak{gl}_2[t]$, $\rho_a : g(u) \mapsto g(u - a)$. Given a $\mathfrak{gl}_2[t]$-module $M$, we denote by $M(a)$ the pull-back of $M$ through the automorphism $\rho_a$. As $\mathfrak{gl}_2$-modules, $M$ and $M(a)$ are isomorphic by the identity map.

For any $\mathfrak{gl}_2[t]$-modules $L$, $M$ and any $a \in \mathbb{C}$, the identity map $(L \otimes M)(a) \to L(a) \otimes M(a)$ is an isomorphism of $\mathfrak{gl}_2[t]$-modules.

We have the evaluation homomorphism, $ev : \mathfrak{gl}_2[t] \to \mathfrak{gl}_2$, $ev : g(u) \mapsto gu^{-1}$. Its restriction to the subalgebra $\mathfrak{gl}_2 \subset \mathfrak{gl}_2[t]$ is the identity map. For any $\mathfrak{gl}_2$-module $M$, we denote by the same letter the $\mathfrak{gl}_2[t]$-module, obtained by pulling $M$ back through the evaluation homomorphism. Then for each $a \in \mathbb{C}$, the $\mathfrak{gl}_2[t]$-module $M(a)$ is called an evaluation module.

Define a grading on $\mathfrak{gl}_2[t]$ such that the degree of $e_{ij} \otimes t^r$ equals $r + j - i$ for all $i, j, r$. We set the degree of $u$ to be 1. Then the series $g(u)$ is homogeneous of degree $j - i - 1$.

A $\mathfrak{gl}_2[t]$-module is called graded if it has a bounded from below $\mathbb{Z}$-grading compatible with the grading on $\mathfrak{gl}_2[t]$. Any irreducible graded $\mathfrak{gl}_2[t]$-module is isomorphic to an evaluation module $L(0)$ for some irreducible $\mathfrak{gl}_2$-module $L$, see [CG].
Let $M$ be a $\mathbb{Z}$-graded space with finite-dimensional homogeneous components. Let $M_j \subset M$ be the homogeneous component of degree $j$. We call the Laurent series in a variable $q$, 
\[ \text{ch}_M(q) = \sum_j (\dim M_j) q^j, \]
the graded character of $M$.

2.3. Weyl modules. Let $W_m$ be the $\mathfrak{gl}_2[t]$-module generated by a vector $v_m$ with the defining relations:
\[ e_{11}(u)v_m = \frac{m}{u} v_m, \quad e_{22}(u)v_m = 0, \]
\[ e_{12}(u)v_m = 0, \quad (e_{21} \otimes 1)^{m+1} v_m = 0. \]
As an $\mathfrak{sl}_2[t]$-module, we call the module $W_m$ isomorphic to the Weyl module from [CL], [CP], corresponding to the weight $m\omega$, where $\omega$ is the fundamental weight of $\mathfrak{sl}_2$. Note that $W_1 = V(0)$.

**Lemma 2.1** (cf. [MTV3]). The module $W_m$ has the following properties.

(i) The module $W_m$ has a unique grading such that $W_m$ is a graded $\mathfrak{gl}_2[t]$-module and the degree of $v_m$ equals 0.

(ii) As a $\mathfrak{gl}_2[t]$-module, $W_m$ is isomorphic to $V^m$.

(iii) A $\mathfrak{gl}_2[t]$-module $M$ is an irreducible subquotient of $W_m$ if and only if $M$ has the form $L_\lambda(0)$, where $\lambda$ is a partition of $m$ with at most 2 parts.

(iv) Consider the decomposition of $W_m$ into isotypical components of the $\mathfrak{gl}_2$-action, $W_m = \oplus_\lambda (W_m)_\lambda$, where $(W_m)_\lambda$ is the isotypical component corresponding to the irreducible polynomial $\mathfrak{gl}_2$-module with highest weight $\lambda = (m-k,k)$. Then for any $\lambda$, the graded character of $(W_m)_\lambda$ is given by
\[ \text{ch}_{(W_m)_\lambda}(q) = \frac{(1-q^{m-2k+1})^2}{1-q} \frac{(q)_m}{(q)_{m-k+1}(q)_k} q^{2k-m}, \]
where $(q)_a = \prod_{j=1}^a (1-q^j)$.

**Proof.** A proof follows from Lemma 2.2 in [MTV3].

Given sequences $n = (n_1, \ldots, n_k)$ of natural numbers and $b = (b_1, \ldots, b_k)$ of distinct complex numbers, we call the $\mathfrak{gl}_2[t]$-module $\otimes_{s=1}^k W_{n_s}(b_s)$ the Weyl module associated with $n$ and $b$.

2.4. $\mathfrak{gl}_2[t]$-module $\mathcal{V}^S$. Let $\mathcal{V}$ be the space of polynomials in $z_1, \ldots, z_n$ with coefficients in $V^\otimes_n$,
\[ \mathcal{V} = V^\otimes_n \otimes_{\mathbb{C}} \mathbb{C}[z_1, \ldots, z_n]. \]
The space $V^\otimes_n$ is embedded in $\mathcal{V}$ as the subspace of constant polynomials.

For $v \in V^\otimes_n$ and $p(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$, we write $p(z_1, \ldots, z_n)v$ to denote $v \otimes p(z_1, \ldots, z_n)$.

The symmetric group $S_n$ acts on $\mathcal{V}$ by permuting the factors of $V^\otimes_n$ and the variables $z_1, \ldots, z_n$ simultaneously,
\[ \sigma(p(z_1, \ldots, z_n)v_1 \otimes \cdots \otimes v_n) = p(z_{\sigma(1)}, \ldots, z_{\sigma(n)}) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}, \quad \sigma \in S_n. \]
We denote $\mathcal{V}^S$ the subspace of $S_n$-invariants of $\mathcal{V}$.
Lemma 2.2 ([MTV3]). The space $V^S$ is a free $\mathbb{C}[z_1, \ldots, z_n]^S$-module of rank $2^n$.

We consider the space $V$ as a $\mathfrak{gl}_2[t]$-module with a series $g(u), \ g \in \mathfrak{gl}_2$, acting by

\begin{equation}
(2.1) \quad g(u) \left(p(z_1, \ldots, z_n) v_1 \otimes \cdots \otimes v_n\right) = p(z_1, \ldots, z_n) \sum_{s=1}^n \frac{v_1 \otimes \cdots \otimes gv_s \otimes \cdots \otimes v_n}{u - z_s}.
\end{equation}

The $\mathfrak{gl}_2[t]$-action on $V$ commutes with the $S_n$-action. Hence, $V^S$ is a $\mathfrak{gl}_2[t]$-submodule of $V$.

2.5. Weyl modules as quotients of $V^S$. Let $\sigma_s(z), \ s = 1, \ldots, n,$ be the $s$-th elementary symmetric polynomial in $z_1, \ldots, z_n$. For $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, denote $I_a \subset \mathbb{C}[z_1, \ldots, z_n]$ the ideal generated by the polynomials $\sigma_s(z) - a_s, \ s = 1, \ldots, n$. Define

\begin{equation}
(2.2) \quad I_a^V = (V^\otimes n \otimes I_a) \cap V^S.
\end{equation}

Clearly, $I_a^V$ is a $\mathfrak{gl}_2[t]$-submodule of $V^S$ and a free $\mathbb{C}[z_1, \ldots, z_n]^S$-module.

Define distinct complex numbers $b_1, \ldots, b_k$ and natural numbers $n_1, \ldots, n_k$ by the relation

\begin{equation}
(2.3) \quad \prod_{s=1}^k (u - b_s)^{n_s} = u^n + \sum_{j=1}^n (-1)^j a_j u^{n-j}.
\end{equation}

Clearly, $\sum_{s=1}^k n_s = n$.

Lemma 2.3 ([MTV3]). The $\mathfrak{gl}_2[t]$-modules $V^S/I_a^V$ and $\otimes_{s=1}^k W_n(b_s)$ are isomorphic.

2.6. Grading on $V^S$. Let $V^\otimes n = \oplus_{k=0}^n V^\otimes [n-k, k]$ be the $\mathfrak{gl}_2$-weight decomposition. Define a grading on $V^\otimes n$ by setting $\deg v = -k$ for any $v \in V^\otimes [n-k, k]$. Define a grading on $\mathbb{C}[z_1, \ldots, z_n]$ by setting $\deg z_i = 1$ for all $i = 1, \ldots, n$. Define a grading on $V$ by setting $\deg (v \otimes p) = \deg v + \deg p$ for any $v \in V^\otimes n$ and $p \in \mathbb{C}[z_1, \ldots, z_n]$. The grading on $V$ induces a grading on $V^S$ and $\text{End}(V^S)$.

Lemma 2.4. The $\mathfrak{gl}_2[t]$-action on $V^S$ is graded. $\square$

3. Bethe algebra

3.1. Definition. Let $K = (K_{ij})$ be a $2 \times 2$-matrix with complex coefficients. Consider the series

\[ B^K_i(u) = \sum_{j=0}^\infty B_{ij}^K u^{-j}, \quad i = 1, 2, \]

where $B_{ij}^K \in U(\mathfrak{gl}_2[t])$, defined by the formulae

\[ B^K_1(u) = K_{11} + K_{22} - e_{11}(u) - e_{22}(u), \]
\[ B^K_2(u) = (K_{11} + e_{11}(u))(K_{22} + e_{22}(u)) - (K_{12} + e_{21}(u))(K_{21} + e_{12}(u)) - e'_{22}(u), \]

where $'$ stands for the derivative $d/du$. We call the unital subalgebra of $U(\mathfrak{gl}_2[t])$ generated by $B_{ij}^K, \ i = 1, 2, \ j \in \mathbb{Z}_{\geq 0}$, the Bethe algebra associated with the matrix $K$ and denote it $B^K$. The elements $B_{ij}^K$ will be called the standard generators of $B^K$. 

Theorem 3.1. For any matrix $K$, the algebra $B^K$ is commutative. If $K$ is the zero matrix, then $B^K$ commutes with the subalgebra $U(gl_2) \subset U(gl_2[t])$.

Proof. Straightforward. □

Let $\partial$ be the operator of differentiation with respect to a variable $u$. An important object associated with the Bethe algebra is the universal differential operator

$$D^K = \partial^2 + B_1^K(u)\partial + B_2^K(u),$$

see [1], [CT], [MTV1]. It is a differential operator with respect to the variable $u$.

If $M$ is a $B^K$-module, we call the image of $B^K$ in $\text{End}(M)$ the Bethe algebra of $M$. The universal differential operator of a $B^K$-module $M$ is the differential operator

$$\mathcal{D} = \partial^2 + \bar{B}_1(u)\partial + \bar{B}_2(u), \quad \bar{B}_i(u) = \sum_{j=0}^{\infty} (B^K_{ij})|_M u^{-j}. $$

It is an interesting problem to describe the algebra $B^K$. In this paper we will consider the cases

$$(3.1) \quad K = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

and will compare the corresponding objects $D^K, B^K, B^K_{ij}$, etc. The objects associated with the zero matrix $K$ will be denoted $D^0, B^0, B^0_{ij}$, etc., while the objects associated with the nonzero matrix $K$ in (3.1) will be denoted $D, B, B_{ij}$, etc.

We have

$$B^0_1(u) = B_1(u) = -e_{11}(u) - e_{12}(u),$$

$$B^0_2(u) = e_{11}(u)e_{22}(u) - e_{21}(u)e_{12}(u) - e_{22}'(u),$$

$$B_2(u) = B^0_2(u) + e_{21}(u)$$

Writing $B^0_i(u) = \sum_j B^0_{ij}u^{-j}$ and $B_i(u) = \sum_j B_{ij}u^{-j}$, we have

$$(3.2) \quad B_{1,j} = B^0_{1,j}, \quad B_{2,j} = B^0_{2,j} + e_{21} \otimes t^{j-1},$$

for all $j$. Note that the elements

$$(3.3) \quad B^0_{11} = -e_{11} - e_{22} \quad \text{and} \quad B^0_{22} = (e_{11} + 1)e_{22} - e_{21}e_{12}$$

belong to the center of the subalgebra $U(gl_2)$.

3.2. Actions of $B$ and $B^0$ on $\mathcal{L}(\lambda)$. For $b \in \mathbb{C}$ and $\lambda = (n - k, k)$, consider the action of the Bethe algebras $B$ and $B^0$ on the evaluation module $\mathcal{L}(\lambda)$. 

Lemma 3.2.

(i) The image of $B^0$ in $\text{End}(\mathcal{L}(\lambda))$ is the subalgebra of scalar operators.

(ii) The image of $B$ in $\text{End}(\mathcal{L}(\lambda))$ is the unital subalgebra generated by the element $e_{21}|_{\mathcal{L}(\lambda)}$.

Proof. Part (i) follows from Schur’s lemma and the fact that $B^0$ commutes with $U(gl_2)$. Part (ii) follows from commutativity of $B$ and the fact that the image of $B_{21}$ in $\text{End}(\mathcal{L}(\lambda))$ equals the image of $e_{21}$. □
Corollary 3.3. The map $B_{21} |_{L_{\lambda}} \mapsto b$ defines an isomorphism of the image of $B$ in $\text{End}(L_{\lambda})$ and the algebra $\mathbb{C}[b]/\langle t^{n-2k+1} \rangle$.

4. ACTIONS OF $B^0$ AND $B$ ON $V^S$

4.1. Gradings on $B$ and $B^0$. In Section 2.2 we introduced a grading on $\mathfrak{gl}_2[t]$ such that $\deg e_{ij} \otimes t^r = r + j - i$ for all $i, j, r$.

Lemma 4.1. For any $(i, j)$, the elements $B_{ij}, B_{ij} \in U(\mathfrak{gl}_2[t])$ are homogeneous of degree $j - i$.

By Lemma 4.1, the grading on $\mathfrak{gl}_2[t]$ induces a grading on $B^0$ and $B$.

As subalgebras of $U(\mathfrak{gl}_2[t])$, the algebras $B^0$ and $B$ act on any $\mathfrak{gl}_2[t]$-module $M$. Consider the $\mathfrak{gl}_2[t]$-module $V^S$ graded as in Section 2.6.

Lemma 4.2. The actions of $B^0$ and $B$ on $V^S$ are graded.

Denote $B_{V^S}$ (resp. $B^0_{V^S}$) the image of the Bethe algebra $B$ (resp. $B^0$) in $\text{End}(V^S)$.

Lemma 4.3. Each of the Bethe algebras $B_{V^S}$ and $B^0_{V^S}$ contains the algebra of operators of multiplication by elements of $\mathbb{C}[z_1, \ldots, z_n]^S$.

Proof. An element $B_{1j} = B^0_{1j} = e_{11} \otimes t^{j-1} + e_{22} \otimes t^{j-1}$ acts on $V^S$ as the operator of multiplication by $\sum_{s=1}^{n} z_s^{j-1}$.

For $i = 1, \ldots, n$, let $\sigma_i$ denote the $i$-th elementary symmetric function of $z_1, \ldots, z_n$. We have $\mathbb{C}[\sigma_1, \ldots, \sigma_n] = \mathbb{C}[z_1, \ldots, z_n]^S$. The embeddings in Lemma 4.3 of $\mathbb{C}[\sigma_1, \ldots, \sigma_n]$ to $B_{V^S}$ and $B^0_{V^S}$ provide $B_{V^S}$ and $B^0_{V^S}$ with structures of $\mathbb{C}[\sigma_1, \ldots, \sigma_n]$-modules.

4.2. Weight, isotypical and graded decompositions of $V^S$. As a $\mathbb{C}[z_1, \ldots, z_n]^S$-module, $V^S$ has the form

$$V^S \cong V^\otimes n \otimes \mathbb{C}[z_1, \ldots, z_n]^S.$$ 

This is an isomorphism of $\mathfrak{gl}_2$-modules, if $\mathfrak{gl}_2$ acts on $\mathbb{C}[z_1, \ldots, z_n]^S$ trivially and acts on $V^\otimes n$ in the standard way.

The $\mathfrak{gl}_2$-weight decomposition of $V^S$ has the form

$$V^S = \bigoplus_{m=0}^{n} V^S[n - m, m] \cong \bigoplus_{m=0}^{n} V^\otimes n[n - m, m] \otimes \mathbb{C}[z_1, \ldots, z_n]^S.$$ 

We say that a weight $(n - m, m)$ is lower than a weight $(n - m', m')$ if $n - m < n - m'$.

Consider the decomposition of $V^S$ into isotypical components of the $\mathfrak{gl}_2$-action,

$$V^S = \bigoplus_{\lambda} V^S_{\lambda} \cong \bigoplus_{\lambda} (V^\otimes n)_{\lambda} \otimes \mathbb{C}[z_1, \ldots, z_n]^S,$$

where $V^S_{\lambda}$, $(V^\otimes n)_{\lambda}$ are the isotypical components corresponding to the irreducible polynomial $\mathfrak{gl}_2$-module with highest weight $\lambda = (n - k, k)$.

The graded decomposition of $V^S$ has the form

$$V^S = \bigoplus_{j=-n}^{\infty} (V^S)_j.$$
Decompositions (4.2), (4.3) and (4.4) are compatible. Namely, we can choose a graded basis \( v_i, i \in I \), of the \( \mathbb{C}[z_1, \ldots, z_n]^S \)-module \( V^S \) which agrees with decompositions (4.2), (4.3), (4.4). That means that each basis vector \( v_i \) lies in one summand of each of decompositions (4.2), (4.3), (4.4).

**Lemma 4.4.** For any \( \lambda = (n-k, k) \), the graded character of \( V^{S,0}_\lambda \) is given by the formula
\[
\chi_{V^{S,0}_\lambda}(q) = \frac{(1 - q^{n-2k+1})^2}{1 - q} \frac{1}{(q)_{n-k+1}(q)_k} q^{2k-n}.
\]

The lemma follows from Lemma 2.1.

Decomposition (4.3) of \( V^S \) into \( \mathfrak{gl}_2 \)-isotypical components is preserved by the action of \( B^0 \). By formula (3.3), for any \( \lambda = (n-k, k) \), the summand \( V^{S,0}_\lambda \) is the eigenspace of the operator \( B^0_{22} \) with the eigenvalue \( k(n-k+1) \). Hence
\[
B^0_{V^S} = \bigoplus \lambda B^0_\lambda,
\]
where \( B^0_\lambda \) is the image of \( B^0 \) in \( \text{End}(V^{S,0}_\lambda) \).

**Lemma 4.5.** The image \( B^0_\lambda \) of \( B^0 \) in \( \text{End}(V^{S,0}_\lambda) \) is canonically isomorphic to the image of \( B^0 \) in \( \text{End}(\text{Sing } V^{S,0}_\lambda) \), where \( \text{Sing } V^{S,0}_\lambda \subset V^S \) is the subspace of singular vectors of weight \( \lambda \).

The lemma follows from Schur’s lemma.

By [MTV3] the graded character of \( B^0_\lambda \) is given by the formula
\[
\chi_{B^0_\lambda}(q) = \frac{1 - q^{n-2k+1}}{(q)_{n-k+1}(q)_k} q^{2k-n}.
\]

### 4.3. Algebra \( A_{n-2k} \otimes B^0_\lambda \) and its module \( V^{S,0}_\lambda \).

Given an integer \( d \), let \( A_d = \mathbb{C}[b]/(b^{d+1}) \).

The algebra \( A_{n-2k} \otimes B^0_\lambda \) acts on \( V^{S,0}_\lambda \) by the rule,
\[
b^j \otimes B \mapsto e^j_{21}B
\]
for any \( j \) and \( B \in B^0_\lambda \). Define a grading on \( A_{n-2k} \otimes B^0_\lambda \) by setting \( \deg (b^j \otimes B) = -j + \deg B \).

The action of \( A_{n-2k} \otimes B^0_\lambda \) on \( V^{S,0}_\lambda \) is graded.

### 4.4. Deformed isotypical components of \( V^S \).

In this section we obtain a decomposition of the algebra \( B^0_\lambda \) similar to decomposition (4.4) of the algebra \( B^0_{V^S} \).

For \( \lambda = (n-k, k) \), denote \( V^S_\lambda \subset V^S \) the generalized eigenspace of the operator \( B_{22} \in B \) with the eigenvalue \( k(n-k+1) \). Clearly, \( V^S_\lambda \) is a \( \mathbb{C}[z_1, \ldots, z_n]^S \)-submodule.

**Lemma 4.6.** We have the following three properties.

(i) Consider a graded basis \( v_i, i \in I \), of the free \( \mathbb{C}[z_1, \ldots, z_n]^S \)-module \( V^S \) which agrees with decompositions (4.2), (4.3), (4.4), see Section 2.4. Let a subset \( I_\lambda \subset I \) be such that the vectors \( v_i, i \in I_\lambda \), form a basis of \( V^{S,0}_\lambda \). Then the \( \mathbb{C}[z_1, \ldots, z_n]^S \)-module \( V^S_\lambda \) has a basis \( w_i, i \in I_\lambda \), such that for all \( i \), we have \( \deg w_i = \deg v_i \) and \( w_i = v_i + v'_i \), where \( v'_i \) lies in the sum of the \( \mathfrak{gl}_2 \)-weight components of \( V^S \) of weight lower than the weight of \( v_i \).

(ii) We have
\[
V^S = \bigoplus \lambda V^S_\lambda.
\]
(iii) $V^S_\lambda$ is a graded free $\mathbb{C}[z_1, \ldots, z_n]^S$-module of rank equal to the rank of the isotypical component $V^S_{\lambda,0}$. The graded character of $V^S_\lambda$ is given by the formula

\begin{equation}
\text{ch}_{V^S_\lambda}(q) = \frac{(1-q^{n-2k+1})^2}{(1-q)q^{2k-n}}.
\end{equation}

Proof. The operator $B_{22} : V^S \rightarrow V^S$ is of degree zero. The matrix $g = (g_{ij})$ of $B_{22}$ in the basis $v_i, i \in I$, has entries in $\mathbb{C}[z_1, \ldots, z_n]^S$. By (3.2), the matrix $g$ is lower triangular with the diagonal entries $g_{ii} = k(n - k + 1)$ for all $i \in I$. Let $B_{ii}, i \in I$, be a basis of the isotypical component $V^S_{\lambda,0}$, which agrees with decompositions (4.2), (4.3), (4.4). Let $I_{\lambda,s} \subset I_\lambda$ be the subset such that the vectors $v_i, i \in I_{\lambda,s},$ form a basis of the $\mathbb{C}[z_1, \ldots, z_n]^S$-module Sing $V^S_{\lambda,0}$, where Sing $V^S_{\lambda,0}$ is the submodule of singular vectors.

Let $w_i, i \in I_\lambda,$ be a basis of the deformed isotypical component $V^S_\lambda$, which has properties described in Lemma 4.6 with respect to the basis $v_i, i \in I_\lambda$.

Define a $\mathbb{C}[z_1, \ldots, z_n]^S$-module epimorphism

\begin{equation}
p^\lambda_\Sigma : V^S_\lambda \twoheadrightarrow \text{Sing } V^S_{\lambda,0}
\end{equation}

by the formula: $w_i \mapsto v_i$ for $i \in I_{\lambda,s}$ and $w_i \mapsto 0$ for $i \in I_\lambda \setminus I_{\lambda,s}$.

Lemma 4.7. We have the following properties.

(i) The kernel of $p^\lambda_\Sigma$ is a $\mathcal{B}$-submodule of the deformed isotypical component $V^S_\lambda$ and, therefore, $p^\lambda_\Sigma$ induces a $\mathcal{B}$-module structure on $\text{Sing } V^S_{\lambda,0} \simeq V^S_\lambda/(\ker p^\lambda_\Sigma)$.

(ii) For this $\mathcal{B}$-module structure on Sing $V^S_{\lambda,0}$, the image of the $\mathcal{B}$ in $\text{End } (\text{Sing } V^S_{\lambda,0})$ is canonically isomorphic to the image of $\mathcal{B}^0$ in $\text{End } (\text{Sing } V^S_{\lambda,0})$. More precisely, for every $(i,j)$, the elements $B_{ij} \in \mathcal{B}$ and $B^0_{ij} \in \mathcal{B}^0$ have the same image.

Proof. Lemma follows from Lemma 4.6, formula (3.2) and Theorem 3.1. □

By Lemmas 4.5 and 4.7, the epimorphism $p^\lambda_\Sigma$ determines an algebra epimorphism

\begin{equation}
p^\lambda_\mathcal{B} : \mathcal{B}_\lambda \twoheadrightarrow \mathcal{B}^0_\lambda.
\end{equation}

It is clear $p^\lambda_\mathcal{B}$ is graded and $p^\lambda_\mathcal{B}$ is a homomorphism of $\mathbb{C}[\sigma_1, \ldots, \sigma_n]$-modules.
5. More on deformed isotypical components

5.1. Deformed isotypical components of $\mathcal{M}_a$. Given a sequence of complex numbers $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, consider the $\mathfrak{gl}_2$-module $V^S/I^V_a$ as in Section 2.5. As a $\mathfrak{gl}_2$-module, $V^S/I^V_a$ is isomorphic to $V^\otimes n$ by Lemma 2.3.

Consider the $\mathfrak{gl}_2$-weight decomposition of $V^S/I^V_a$ and its decomposition into $\mathfrak{gl}_2$-isotypical components, respectively,

\begin{equation}
V^S/I^V_a = \bigoplus_{m=0}^n (V^S/I^V_a)[n-m,m],
\end{equation}

\begin{equation}
V^S/I^V_a = \bigoplus_{\lambda} (V^S/I^V_a)_{\lambda}.
\end{equation}

Consider a graded basis $v_i, i \in I$, of the free $\mathbb{C}[z_1, \ldots, z_n]$-module $V^S$ which agrees with decompositions (4.2), (4.3), (4.4). This basis induces a $\mathbb{C}$-basis $\bar{v}_i, i \in I$, of $V^S/I^V_a$, which agrees with both decompositions in (5.1). For any $\lambda$, the vectors $\bar{v}_i, i \in I_\lambda$, form a weight basis of the isotypical component $(V^S/I^V_a)_{\lambda}$.

Denote

$$\mathcal{M}_a = V^S/I^V_a.$$

For $\lambda = (n-k,k)$, denote

$$\mathcal{M}_{\lambda,a} \subset \mathcal{M}_a$$

the generalized eigenspace of the operator $B_{22} \in \mathcal{B}$ with eigenvalue $k(n-k+1)$.

Lemma 4.6 has the following analog.

**Lemma 5.1.** We have the next three properties.

(i) $\mathcal{M}_{\lambda,a}$ is a $\mathbb{C}$-vector space of the dimension equal to the dimension of $(V^S/I^V_a)_{\lambda}$.

(ii) $\mathcal{M}_{\lambda,a}$ has a basis $w_i, i \in I_\lambda$, such that for all $i$, $w_i = \bar{v}_i + v'_i$ where $v'_i$ lies in the sum of the $\mathfrak{gl}_2$-weight components of $\mathcal{M}_a$ of weight lower than the weight of $\bar{v}_i$.

(iii) We have

\begin{equation}
\mathcal{M}_a = \bigoplus_{\lambda} \mathcal{M}_{\lambda,a}.
\end{equation}

It is clear that the subspaces $\mathcal{M}_{\lambda,a} \subset \mathcal{M}_a$ are $\mathcal{B}$-submodules. We call the $\mathcal{B}$-modules $\mathcal{M}_{\lambda,a}$ the deformed isotypical components of $\mathcal{M}_a$.

5.2. Bethe eigenleaves. Let $\phi: \mathcal{B}^0 \to \mathbb{C}$ be a homomorphism. Let $W_\phi \subset \mathcal{M}_a$ be the generalized eigenspace of the $\mathcal{B}^0$-action with eigenvalue $\phi$. Since the $\mathcal{B}^0$-action commutes with the $\mathfrak{gl}_2$-action, $W_\phi$ is a $\mathfrak{gl}_2$-module. Assume that $W_\phi$ is an irreducible $\mathfrak{gl}_2$-module with highest weight $\lambda = (n-k,k)$. This means, in particular, that $Bw = \phi(B)w$ for all $w \in W_\phi$ and $B \in \mathcal{B}^0$.

Choose a weight basis $u_i, i = 0, \ldots, n-2k$, of $W_\phi$. Choose a finite set $B^0_{ij}, (i, j) \in J$, of the standard generators of $\mathcal{B}^0$, such that $W_\phi$ is the common generalized eigenspace of the operators $B_{ij}^0 \in \mathcal{B}^0, (i, j) \in J$, with eigenvalues $\phi(B_{ij}^0)$, respectively.

Under these assumptions, denote $\mathcal{M}_{\phi, \lambda,a} \subset \mathcal{M}_a$ the generalized eigenspace of the operators $B_{ij} \in \mathcal{B}, (i, j) \in J$, with eigenvalues $\phi(B_{ij})$, respectively.

Lemmas 4.6 and 5.1 have the following analog.

**Lemma 5.2.** Under these assumptions, we have the next two properties.
(i) $M_{\phi, \lambda, a}$ is a $C$-vector subspace of $M_{\lambda, a}$ of the dimension equal to the dimension of $W_\phi$.

(ii) $M_{\phi, \lambda, a}$ has a basis $w_i, i \in \{0, \ldots, n-2k\}$, such that for all $i$, $w_i = u_i + u'_i$ where $u'_i$ lies in the sum of the $\mathfrak{gl}_2$-weight components of $M_{\lambda, a}$ of weight lower than the weight of $u_i$.

It is clear that $M_{\phi, \lambda, a} \subset M_{\lambda, a}$ is a $B$-submodule. We call the $B$-module $M_{\phi, \lambda, a}$ a Bethe eigenleaf of $M_{\lambda, a}$.

Lemma 5.3. Let $a \in \mathbb{R}^n$ be such that all roots of the polynomial $u^n + \sum_j (-1)^j a_j u^{n-j}$ are distinct and real. Then the $B$-module $M_a$ is the direct sum of its Bethe eigenleaves,

\begin{equation}
M_a = \sum_{\phi, \lambda} M_{\phi, \lambda, a}.
\end{equation}

Proof. Denote $\text{Sing } M_a = \{ v \in M_a \mid e_{21}v = 0 \}$ the subspace of singular vectors. By [MTV3], the action of $B^0$ on $\text{Sing } M_a$ has simple spectrum if all roots of the polynomial $u^n + \sum_j (-1)^j a_j u^{n-j}$ are distinct and real. This fact and property (3.2) imply the lemma. \qed

5.3. The universal differential operator of $V^S$.

Lemma 5.4 (cf. Lemma 5.9 in [MTV3]). Denote $D_{V^S}$ the universal differential operator of the $B$-module $V^S$. Then $D_{V^S}$ has the form

\begin{equation}
D_{V^S} = \partial^2 - \bar{B}_1(u) \partial + \bar{B}_2(u),
\end{equation}

where

\begin{align*}
\bar{B}_1(u) &= \frac{W'(u)}{W(u)}, & \bar{B}_2(u) &= \frac{U(u)}{W(u)}, \\
W(u) &= \prod_{i=1}^n (u - z_i), & U(u) &= \sum_{i=1}^n U_i u^{n-i},
\end{align*}

with $U_i \in \text{End}_{C[z_1, \ldots, z_n]}(V^S)$ and

\begin{equation}
U_1 = B_{21} = \sum_{s=1}^n e_{21}^{(s)}.
\end{equation}

5.4. The universal differential operator of $M_a$.

Lemma 5.5. Let $D_{M_a}$ be the universal differential operator of the $B$-module $M_a$ and $y(u)$ an $M_a$-valued function of $u$. Then all solutions to the differential equation $D_{M_a}y(u) = 0$ are $M_a$-valued polynomials.

Proof. By Theorem 8.4 in [MTV2], every solution is a linear combination of the functions of the form $e^{cu}p(u)$, where $p(u)$ is an $M_a$-valued polynomial and $c \in C$. Writing $D_{M_a}e^{cu}p(u) = 0$ and computing the leading term, we conclude that $c = 0$. \qed
5.5. The universal differential operator of a Bethe eigenleaf.

**Lemma 5.6.** Let \( \lambda = (n - k, k) \). Let \( M_{\phi, \lambda, a} \) be a Bethe eigenleaf. Then the universal differential operator \( D_{M_{\phi, \lambda, a}} \) of the \( B \)-module \( M_{\phi, \lambda, a} \) has the form

\[
D_{M_{\phi, \lambda, a}} = \partial^2 - B_1(u) \partial + B_2(u),
\]

where

\[
B_1(u) = \frac{W'(u)}{W(u)}, \quad B_2(u) = \frac{U(u)}{W(u)},
\]

\[
W(u) = u^n + \sum_{i=1}^{n} (-1)^i a_i u^{n-i}, \quad U(u) = \sum_{i=1}^{n} U_i u^{n-i},
\]

with \( U_i \in \text{End}_C(M_{\phi, \lambda, a}) \). Moreover,

\[
U_1 = B_{21}|_{M_{\phi, \lambda, a}} = e_{21}|_{M_{\phi, \lambda, a}}
\]

and for any \( i > 1 \), we have

\[
U_i = \sum_{j=0}^{n-2k} c_{ij} (B_{21}|_{M_{\phi, \lambda, a}})^j
\]

where \( c_{ij} \in \mathbb{C} \) and \( c_{20} = k(n - k + 1) \).

**Proof.** We need to prove (5.7) and formula \( c_{20} = k(n - k + 1) \). Everything else follows from Lemma 5.4.

The operators \( U_i \) are elements of the Bethe algebra of \( M_{\phi, \lambda, a} \). The Bethe algebra of \( M_{\phi, \lambda, a} \) contains the scalar operators and the nilpotent operator \( B_{21}|_{M_{\phi, \lambda, a}} \). On the complex \( n - 2k + 1 \)-dimensional vector space \( M_{\phi, \lambda, a} \), we have \( (B_{21}|_{M_{\phi, \lambda, a}})^{n-2k} \neq 0 \) and \( (B_{21}|_{M_{\phi, \lambda, a}})^{n-2k+1} = 0 \). Hence, every element of that algebra is a polynomial in \( B_{21}|_{M_{\phi, \lambda, a}} \) with complex coefficients. Formula (5.7) is proved.

Formula \( c_{20} = k(n - k + 1) \) follows from (3.2) and properties of the universal differential operator of the algebra \( B^0 \) associated with the isotypical component \( (V^s/I^V_a)_{\lambda} \), see [MTV3].

**Lemma 5.7.** Let \( M_{\phi, \lambda, a} \) be a Bethe eigenleaf and \( D_{M_{\phi, \lambda, a}} \) the universal differential operator of the \( B \)-module \( M_{\phi, \lambda, a} \), see Lemma 5.6. Then all solutions to the \( M_{\phi, \lambda, a} \)-valued differential equation \( D_{M_{\phi, \lambda, a}} y(u) = 0 \) are \( M_{\phi, \lambda, a} \)-valued polynomials.

The lemma follows from Lemma 5.5.

To a Bethe eigenleaf \( M_{\phi, \lambda, a} \), we assign a scalar differential operator

\[
D_{M_{\phi, \lambda, a}, 0} = \partial^2 - \frac{W'(u)}{W(u)} \partial + \sum_{i=2}^{n} c_{i0} u^{n-i} \frac{W(u)}{W(u)},
\]

see notation in Lemma 5.6. It is clear, that any solution to the differential equation \( D_{M_{\phi, \lambda, a}, 0} y(u) = 0 \) is a polynomial of degree \( k \) or \( n - k + 1 \).
Let \( w_i, i \in 0, \ldots, n - 2k \) be a basis of \( \mathcal{M}_{\phi, \lambda, a} \) indicated in Lemma 5.2. Let \( y(u) = \sum_i y_i(u)w_i \) be a solution to the differential equation \( D_{\mathcal{M}_{\phi, \lambda, a}}y(u) = 0 \), then \( y_0(u) \) is a solution to the differential equation \( D_{\mathcal{M}_{\phi, \lambda, a, 0}}y(u) = 0 \).

Let \( F_0(u), G_0(u) \in \mathbb{C}[u] \) be polynomials of degrees \( k \) and \( n - k + 1 \), respectively. Then the kernel of the differential operator

\[
(5.9) \quad D_{F_0,G_0} = \partial^2 - \frac{\text{Wr}'(F_0,G_0)}{\text{Wr}(F_0,G_0)} \partial + \frac{\text{Wr}(F'_0,G'_0)}{\text{Wr}(F_0,G_0)}
\]

is the two-dimensional subspace of \( \mathbb{C}[u] \) generated by \( F_0(u), G_0(u) \).

**Lemma 5.8.** For any generic pair of polynomials \( F_0(u), G_0(u) \in \mathbb{C}[u] \) with \( \deg F_0(u) = k \), \( \deg G_0(u) = n - k + 1 \), there exists a unique Bethe eigenleaf \( \mathcal{M}_{\phi, \lambda, a} \) such that \( D_{\mathcal{M}_{\phi, \lambda, a, 0}} = D_{F_0,G_0} \).

**Proof.** For \( F_0(u), G_0(u) \in \mathbb{C}[u] \) with \( \deg F_0(u) = k \), \( \deg G_0(u) = n - k + 1 \), define \( a = (a_1, \ldots, a_n) \) by the formula

\[
\text{Wr}(F_0(u), G_0(u)) = (n - 2k)(u^n + \sum_{j=1}^n (-1)^j a_j u^{n-j}).
\]

By [MTV3], for every generic pair \( F_0(u), G_0(u) \in \mathbb{C}[u] \), there exists a unique eigenvector \( v \in \text{Sing} \mathcal{M}_a \) of the Bethe algebra \( \mathcal{B}^0 \) with

\[
\mathcal{B}^0_{ij}v = c_{ij}v
\]

for some \( c_{ij} \in \mathbb{C} \) and all \((i,j)\), such that

\[
\mathcal{D}_{F_0,G_0} = \partial^2 - \sum_j c_{1j}u^{-j} \partial + \sum_j c_{2j}u^{-j}.
\]

This fact and property (3.2) imply the lemma. \qed

6. **Algebra \( \mathcal{O}_\lambda \)**

6.1. **Wronskian conditions.** Fix nonnegative integers \( k \) and \( d \). Define an algebra \( \mathcal{A}_d = \mathbb{C}[b]/(b^{d+1}) \), with \( b \) a generator of \( \mathcal{A}_d \). Consider the expressions:

\[
(6.1) \quad f(u) = \sum_{i=0}^{k-1} f_i u^i + u^k + \sum_{i=1}^d \tilde{f}_{k+i} b^i u^{k+i},
\]

\[
g(u) = \sum_{i=0}^{k-1} g_i u^i + \sum_{i=k+1}^{k+d} \tilde{g}_{k+i} u^i + u^{k+d+1} + \sum_{i=1}^d \tilde{g}_{k+d+1+i} b^i u^{k+d+1+i}.
\]

These are polynomials in \( u, f_i, g_i, \tilde{f}_{k+i}, \tilde{g}_{k+d+1+i} \) with coefficients in \( \mathcal{A}_d \).

Consider the polynomials

\[
(6.2) \quad \text{Wr}(f(u), g(u)) = \sum_{j=0}^{2k+3d} U_j u^j, \quad \text{Wr}(f'(u), g'(u)) = \sum_{j=0}^{2k+3d-2} V_j u^j,
\]

where \( U_j, V_j \) are suitable polynomials in \( f_i, g_i, \tilde{f}_{k+i} b^i, \tilde{g}_{k+d+1+i} b^i \) with integer coefficients.
It is easy to see that

\[ U_i = 0 , \quad V_{i-2} = 0 , \quad \text{for} \ i > 2k + 2d , \]

**Theorem 6.1.** Consider the system of 2d equations

\[
\begin{align*}
U_{2k+d+1} &= 0 , \quad V_{2k+d-2+1} - U_{2k+d}b = 0 , \\
U_{2k+d+i} &= 0 , \quad V_{2k+d-2+i} = 0 , \quad \text{for} \ i = 2, \ldots, d ,
\end{align*}
\]

with respect to \( \tilde{f}_{k+ib^i}, \tilde{g}_{k+d+1+ib^i}, i = 1, \ldots, d \). Then there exist 2d polynomials \( \tilde{\phi}_{k+i}, \tilde{\psi}_{k+d+1+i} \) in \( 2k + d \) variables

\[
(6.4) \quad f_i = 0, \ldots, k - 1, \quad \text{and} \quad g_i = 0, \ldots, k - 1, k + 1, \ldots, k + d,
\]

with coefficients in \( A_d \), such that system \( (6.3) \) is equivalent to the system of 2d equations:

\[
(6.5) \quad \tilde{f}_{k+ib^i} = \tilde{\phi}_{k+i} , \quad \tilde{g}_{k+d+1+ib^i} = \tilde{\psi}_{k+d+1+i} , \quad i = 1, \ldots, d .
\]

Let \( \mathcal{E} \) be a \( \mathbb{C} \)-algebra. Abusing notation, we will write \( b^i y \) instead of \( b^i \otimes y \in A_d \otimes \mathcal{E} \) for any \( 0 \leq j \leq d \) and \( y \in \mathcal{E} \).

We denote by \( \mathbb{C}[\{ f, g, \tilde{f}, \tilde{g} \}] \) the polynomial algebra in all variables \( f_i, g_i, \tilde{f}_{k+i}, \tilde{g}_{k+d+1+i} \) appearing in (6.1), and by \( \mathbb{C}[\{ f, g \}] \) the polynomial algebra of all variables \( f_i, g_i \) described in (6.4).

Let

\[
(6.6) \quad C_{k,d} \subset A_d \otimes \mathbb{C}[\{ f, g, \tilde{f}, \tilde{g} \}]
\]

be the \( \mathbb{C} \)-subalgebra generated by all elements 1, \( f_i, g_i, \tilde{f}_{k+i}, \tilde{g}_{k+d+1+i}, b \).

**Corollary 6.2.** Consider the ideal \( I \) in \( C_{k,d} \) generated by the left hand sides of equations (6.3). Then the quotient algebra \( C_{k,d}/I \) is canonically isomorphic to the algebra \( A_d \otimes \mathbb{C}[\{ f, g \}] \).

**Proof of Theorem 6.1.** The four equation in (6.3) have the following form

\[
(6.7) \quad d \tilde{f}_{k+1b^i} + (d + 2) \tilde{g}_{k+d+2b^i} + Y_{2k+d+1} = 0 , \quad d(k + 1)(k + d + 1) \tilde{f}_{k+1b^i} + (d + 2)k(k + d + 2) \tilde{g}_{k+d+2b^i} - (d + 1)b + Z_{2k+d-2+1} = 0 ,
\]

\[
(6.8) \quad (d + 1 - i) \tilde{f}_{k+ib^i} + (d + 1 + i) \tilde{g}_{k+d+1+i}b^i + \sum_{j=1}^{i} (d + 1 + i - 2j) \tilde{f}_{k+jb^i} \tilde{g}_{k+d+1+i-j}b^{i-j} + Y_{2k+d+i} = 0 , \quad (d + 1 - i)(d + k + 1) \tilde{f}_{k+ib^i} + (d + 1 + i)(d + k + 1) \tilde{g}_{k+d+1+i}b^i + \\
\sum_{j=1}^{i} (d + 1 + i - 2j)(d + k + j)(d + k + 1 + i - j) \tilde{f}_{k+jb^i} \tilde{g}_{k+d+1+i-j}b^{i-j} + Z_{2k+d-2+i} = 0 .
\]

In equations (6.7), \( Y_{2k+d+1} \) and \( Z_{2k+d-1} \) are suitable polynomials in the variables \( f_j, g_j, \tilde{f}_{k+jb^j}, \tilde{g}_{k+d+1+j}b^j \) such that every monomial of \( Y_{2k+d+1} \) and every monomial of \( Z_{2k+d-1} \) has
degree at least two with respect to \( b \). In equations (6.8), \( Y_{2k+d-i} \) and \( Z_{2k+d-2+i} \) are suitable polynomials in the variables \( f_j, g_j, \tilde{f}_{k+j}b^j, \tilde{g}_{k+d+1+j}b^j \) such that every monomial of \( Y_{2k+d+i} \) and every monomial of \( Z_{2k+d-2+i} \) has degree at least \( i + 1 \) with respect to \( b \).

Transforming equations (6.3) to equations (6.7) and (6.8) we distinguished the leading terms (with respect to powers of \( b \)) of the polynomials in (6.3).

The variables \( \tilde{f}_{k+1}b, \tilde{g}_{k+d+2}b \) enter linearly the two equations in (6.7). The determinant of this \( 2 \times 2 \) system is nonzero. Solving this linear system, gives

\[
\begin{align*}
\tilde{f}_{k+1}b &= c_{k+1}b + W_{k+1}, \\
\tilde{g}_{k+d+2}b &= c_{k+d+2}b + W_{k+d+2},
\end{align*}
\]

where \( c_{k+1}, c_{k+d+2} \in \mathbb{C} \) and \( W_{k+1}, W_{k+d+2} \) are suitable polynomials in the variables \( f_j, g_j, \tilde{f}_{k+j}b^j, \tilde{g}_{k+d+1+j}b^j \) such that every monomial of \( W_{k+1} \) and every monomial of \( W_{k+d+2} \) has degree at least two with respect to \( b \).

Consider the two equations of (6.8) corresponding to \( i = 2 \),

\[
\begin{align*}
a_1 \tilde{f}_{k+2}b^2 + a_2 \tilde{g}_{k+d+3}b^2 + a_3 \tilde{f}_{k+1}b \tilde{g}_{k+d+2}b + Y_{2k+d+2} &= 0, \\
b_1 \tilde{f}_{k+2}b^2 + b_2 \tilde{g}_{k+d+3}b^2 + b_3 \tilde{f}_{k+1}b \tilde{g}_{k+d+2}b + Z_{2k+d} &= 0,
\end{align*}
\]

where the numbers \( a_j, b_j \) are determined in (6.8). It is easy to see that the determinant of the matrix \( \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \) is nonzero. Replace in (6.10) the product \( \tilde{f}_{k+1}b \tilde{g}_{k+d+2}b \) with

\[
(c_{k+1}b + W_{k+1})(c_{k+d+2}b + W_{k+d+2}).
\]

Then solving the linear system in (6.10) with respect to \( \tilde{f}_{k+2}b^2, \tilde{g}_{k+d+3}b^2 \) we get

\[
\begin{align*}
\tilde{f}_{k+2}b^2 &= c_{k+2}b^2 + W_{k+2}, \\
\tilde{g}_{k+d+3}b^2 &= c_{k+d+3}b^2 + W_{k+d+3},
\end{align*}
\]

where \( c_{k+2}, c_{k+d+3} \in \mathbb{C} \) and \( W_{k+2}, W_{k+d+3} \) are suitable polynomials in the variables \( f_j, g_j, \tilde{f}_{k+j}b^j, \tilde{g}_{k+d+1+j}b^j \) such that every monomial of \( W_{k+2} \) and every monomial of \( W_{k+d+3} \) has degree at least three with respect to \( b \).

Repeating this procedure we obtain for every \( i = 1, \ldots, d \), equations

\[
\begin{align*}
\tilde{f}_{k+i}b^i &= c_{k+i}b^i + W_{k+i}, \\
\tilde{g}_{k+d+1+i}b^i &= c_{k+d+1+i}b^i + W_{k+d+1+i},
\end{align*}
\]

where \( c_{k+i}, c_{k+d+1+i} \in \mathbb{C} \) and \( W_{k+i}, W_{k+d+1+i} \) are suitable polynomials in the variables \( f_j, g_j, \tilde{f}_{k+j}b^j, \tilde{g}_{k+d+1+j}b^j \) such that every monomial of \( W_{k+i} \) and every monomial of \( W_{k+d+1+i} \) has degree at least \( i + 1 \) with respect to \( b \).

For every \( m \), replace in \( W_m \) every variable \( \tilde{f}_{k+j}b^j \) and \( \tilde{g}_{k+d+1+j}b^j \) with \( c_{k+j}b^j + W_{k+j} \) and \( c_{k+d+1+j}b^j + W_{k+d+1+j} \), respectively. Then for every \( i = 1, \ldots, d \), we have

\[
\begin{align*}
\tilde{f}_{k+i}b^i &= X^1_{k+i} + X^2_{k+i}, \\
\tilde{g}_{k+d+1+i}b^i &= X^1_{k+d+1+i} + X^2_{k+d+1+i},
\end{align*}
\]

where \( X^1_{k+i}, X^2_{k+i} \) are suitable polynomials in the \( k + 2d \) variables \( f_j, g_j \), and \( X^1_{k+i}, X^2_{k+d+1+i} \) are suitable polynomials in the variables \( f_j, g_j, \tilde{f}_{k+j}b^j, \tilde{g}_{k+d+1+j}b^j \) such that every
monomial of $X_{k+i}^2$ and every monomial of $X_{k+d+1+i}^2$ has degree at least $i + 2$ with respect to $b$.

Iterating this procedure we prove the theorem. \hfill \square

6.2. Algebras $O_\lambda$ and $O^0_\lambda$. For given $\lambda = (k + d, k)$, we define an algebra $O_\lambda$ by the formula

\[ O_\lambda = C_{k,d}/I , \]

where $C_{k,d}$ is defined in (6.6). For any $x \in C_{k,d}$, its image in $O_\lambda$ will be denoted $\{x\}$. Let

\[ O^0_\lambda = \mathbb{C}\{f, g\} . \]

By Corollary 6.2, the algebra homomorphism

\[ g_\lambda : A_d \otimes O^0_\lambda \rightarrow O_\lambda ; \quad f_i \mapsto \{f_i\}, \ g_i \mapsto \{g_i\}, \ b \mapsto \{b\} , \]

for all $i$, is an isomorphism.

Introduce the polynomials $\{f\}(u), \{g\}(u) \in O_\lambda[u]$ by the formulae:

\begin{align*}
\{f\}(u) &= \sum_{i=0}^{k-1} \{f_i\} u^i + u^k + \sum_{i=1}^{d} \{\bar{f}_{k+i} b^i\} u^{k+i} , \\
\{g\}(u) &= \sum_{i=0}^{k-1} \{g_i\} u^i + \sum_{i=k+1}^{k+d} \{g_i\} u^i + u^{k+d+1} + \sum_{i=1}^{d} \{\bar{g}_{k+d+i} b^i\} u^{k+d+1+i} .
\end{align*}

The polynomials $\{f\}(u), \{g\}(u)$ lie in the kernel of the differential operator

\[ D_{O_\lambda} = \partial^2 - \frac{\text{Wr}(\{f\}, \{g\})}{\text{Wr}(\{f\}, \{g\})} \partial + \frac{\text{Wr}(\{f\}', \{g\}')}{{\text{Wr}(\{f\}, \{g\})}} . \]

The operator $D_{O_\lambda}$ will be called the universal differential operator associated with $O_\lambda$.

**Corollary 6.3.** In formula (6.13), $\text{Wr}(\{f\}, \{g\})$ is a polynomial in $u$ of degree $2k + d$, $\text{Wr}(\{f\}', \{g\}')$ is a polynomial in $u$ of degree $2k+d-1$ and the residue at $u = \infty$ of the ratio $\text{Wr}(\{f\}', \{g\}')/\text{Wr}(\{f\}, \{g\})$ equals $\{b\}$.

Introduce a notation for the coefficients of the universal differential operator $D_{O_\lambda}$:

\begin{align*}
F_1(u) &= \frac{\text{Wr}(\{f\}, \{g\})}{\text{Wr}(\{f\}, \{g\})} , \quad F_2(u) = \frac{\text{Wr}(\{f\}', \{g\}')}{{\text{Wr}(\{f\}, \{g\})}} .
\end{align*}

Expand the coefficients in Laurent series at $u = \infty$:

\begin{align*}
F_1(u) &= \sum_{j=1}^{\infty} F_{1j} u^{-j} , \quad F_2(u) = \sum_{j=1}^{\infty} F_{2j} u^{-j} ,
\end{align*}

where $F_{sj} \in O_\lambda, F_{11} = 2k + d, F_{21} = \{b\}$.

**Lemma 6.4.** The $\mathbb{C}$-algebra $O_\lambda$ is generated by the elements $F_{sj}, s = 1, 2, j = 1, 2, \ldots$
Proof. By Theorem 6.11 we have an isomorphism $q_\lambda : A_d \otimes O^0_\lambda \rightarrow O_\lambda$. Hence, for all $(s,j)$, we can write $F_{sj} = \sum_{t=0}^d F_{sj}^t \{b\}^t$, where $F_{sj}^t$ are polynomials in the generators $\{f_i\}, \{g_i\}$. The operator

$$\partial^2 - \sum_{j=1}^\infty F_{1j}^0 u^{-j} \partial + \sum_{j=2}^\infty F_{2j}^0 u^{-j}$$

annihilates the polynomials $\{f_0\} + \cdots + \{f_{k-1}\} u^{k-1} + u^k$ and $\{g_0\} + \{g_1\} u + \cdots + \{g_{k-1}\} u^{k-1} + \{g_k\} u^{k+1} + \cdots + \{g_{k+d}\} u^{k+d} + u^{k+d+1}$. By Lemma 3.3 in [MTV3], every $\{f_m\}, \{g_m\}$ can be written as a polynomial in $F_{sj}^0$, $s = 1, 2, j = 2, 3, \ldots$, with coefficients in $\mathbb{C}$:

$$\{f_m\} = \phi_m^0(F_{sj}^0), \quad \{g_m\} = \psi_m^0(F_{sj}^0).$$

We have

$$\{f_m\} = \psi_m(F_{sj}) + (\phi_m(F_{sj}^0) - \phi_m(F_{sj})) = \phi_m(F_{sj}) + \{b\} \phi_m^1,$n

$$\{g_m\} = \psi_m(F_{sj}) + (\psi_m(F_{sj}^0) - \psi_m(F_{sj})) = \psi_m(F_{sj}) + \{b\} \psi_m^1,$n

where $\phi_m^1, \psi_m^1 \in O_\lambda$. These formulae give a presentation of the elements $\{f_m\}, \{g_m\}$ in terms of $F_{sj}$ modulo the ideal $\langle \{b\} \rangle \subset O_\lambda$.

Elements $\phi_m^1, \psi_m^1$ can be written as polynomials in the generators $\{f_i\}, \{g_i\}$ with coefficients in $\mathbb{C}[\{b\}]$:

$$\phi_m^1 = \phi_m^1(\{f_i\}, \{g_i\}), \quad \psi_m^1 = \psi_m^1(\{f_i\}, \{g_i\}).$$

Then

$$\phi_m^1 = \phi_m^1(\{f_i\}, \{g_i\}) = \phi_m^1(\phi_i(F_{sj}), \psi_i(F_{sj}))$$

and

$$\psi_m^1 = \psi_m(\{f_i\}, \{g_i\}) = \psi_m(\phi_i(F_{sj}), \psi_i(F_{sj})).$$

These formulae give a presentation of elements $\{f_m\}, \{g_m\}$ in terms of $F_{sj}$ modulo the ideal $\langle \{b\} \rangle \subset O_\lambda$. Continuing this procedure we prove the lemma.

Define an algebra epimorphism

$$\nu_\lambda : O_\lambda \rightarrow O^0_\lambda$$

by the formulae $\{b\} \mapsto 0$, $\{f_i\} \mapsto f_i, \{g_i\} \mapsto g_i$ for all $i$. Define an algebra monomorphism

$$i_\lambda : A_d \rightarrow O_\lambda$$

by the formula $b \mapsto \{b\}$.
6.3. Grading on $\mathcal{O}_\lambda$ and $\mathcal{O}_\lambda^0$. Define the degrees of the elements $u, b, f_i, g_i, \tilde{f}_{k+i}, \tilde{g}_{k+i+1}, \tilde{b}^i$ to be $1, -1, k - i, k + d + 1 - i, -i, -i$, respectively. Then the polynomials $f(u), g(u)$, defined in (6.1), are homogeneous of degree $k, k + d + 1$, respectively.

Equations of system (6.3) are homogeneous. Hence $\mathcal{O}_\lambda$ has an induced grading. The same rule defines a grading on $\mathcal{O}_\lambda^0$. The isomorphism $g_\lambda : A_d \otimes \mathcal{O}_\lambda^0 \rightarrow \mathcal{O}_\lambda$ and epimorphism $p_\lambda : \mathcal{O}_\lambda \rightarrow \mathcal{O}_\lambda^0$ are graded.

**Lemma 6.5.** The graded character of $\mathcal{O}_\lambda$ and $\mathcal{O}_\lambda^0$ are given by the following formulae:

\[
\text{ch}_{\mathcal{O}_\lambda}(q) = \frac{(1 - q^{d+1})^2}{1 - q} \frac{q^{-d}}{(q)_{k+d+1}(q)_k} = \frac{(1 - q^{n-2k+1})^2}{1 - q} \frac{q^{2k-n}}{(q)_{n-k+1}(q)_k},
\]

\[
\text{ch}_{\mathcal{O}_\lambda^0}(q) = \frac{1 - q^{d+1}}{(q)_{k+d+1}(q)_k} = \frac{1 - q^{n-2k+1}}{(q)_{n-k+1}(q)_k}.
\]

\[\square\]

Let $F_{ij} \in \mathcal{O}_\lambda$ be the elements defined in (6.15).

**Lemma 6.6.** For any $(i, j)$, the element $F_{ij}$ is homogeneous of degree $j - i$. \[\square\]

7. Special homomorphism of $\mathcal{O}_\lambda$ and Bethe eigenleaves

We keep notations of Section 6.

7.1. Differential operators with polynomial kernel. Let $W(u) \in \mathbb{C}[u]$ be a monic polynomial of degree $2k + d$. Let $U(u) \in A_d[u]$ be a polynomial of the form

\[(7.1) \quad U(u) = bu^{2k+d-1} + \sum_{i=0}^{2k+d-2} \sum_{j=0}^{d} v_{ij} b^i u^j\]

with $v_{ij} \in \mathbb{C}$. Denote

\[(7.2) \quad \mathcal{D} = \partial^2 - \frac{W'}{W} \partial + \frac{U}{W},\]

\[(7.3) \quad \chi(\alpha) = \alpha(\alpha - 1) - (2k + d)\alpha + v_{2k+d-2,0},\]

where $\alpha$ is a variable.

Consider the differential equation $\mathcal{D} y(u) = 0$ on an $A_d$-valued function $y(u)$.

**Lemma 7.1.** Assume that all solutions to the differential equation $\mathcal{D} y(u) = 0$ are polynomials and $\chi(\alpha) = (\alpha - k)(\alpha - k - d - 1)$. Then the differential equation $\mathcal{D} y(u) = 0$ has unique solutions $F(u), G(u)$ such that

\[
F(u) = \sum_{i=0}^{k-1} \sum_{j=0}^{d} F_{ij} b^i u^j + u^k + \sum_{i=1}^{d} \sum_{j=i}^{d} F_{k+i,j} b^i u^{k+i},
\]

\[
G(u) = \sum_{i=0}^{k-1} \sum_{j=0}^{d} G_{ij} u^i + \sum_{i=k+1}^{k+d} \sum_{j=0}^{d} G_{ij} b^i u^i + u^{k+d+1} + \sum_{i=1}^{d} \sum_{j=i}^{d} G_{k+d+1+i,j} b^i u^{k+d+1+i},
\]

where $F_{ij}, G_{ij} \in \mathbb{C}$.
Proof. Write
\[ U(u) = bu^{2k+d-1} + \sum_{j=0}^{d} b^j U_j(u) \]
with \( U_j(u) \in \mathbb{C}[u] \) and \( \deg U_j \leq 2k + d - 2 \) for all \( j \).

Let \( y(u) = y_0(u) + by_1(u) + \cdots + b^d y_d(u) \) be a solution with \( y_i(u) \in \mathbb{C}[u] \). Assume that \( y_0(u) \neq 0 \). Then \( y_0(u) \) is of degree \( k \) or \( k + d + 1 \) and \( y_0(u) \) satisfies the equation \( D_0 y_0(u) = 0 \), where
\[ D_0 = \partial^2 - \frac{W'}{W} \partial + \frac{U_0}{W}. \]

Assume that \( y_0 \) is of degree \( k \) and monic. The polynomial \( y_1(u) \) is a solution of the inhomogeneous differential equation
\[ (7.4) \quad D_0 y_1(u) + \frac{u^{2k+d-1} + U_1(u)}{W(u)} y_0(u) = 0. \]

The term \( D_0 y_1(u) \) is of order \( u^{k-1} \) as \( u \to \infty \). The polynomial \( y_1(u) \) is defined up to addition of a solution of the homogeneous equation. Therefore, \( y_1(u) \) does exist and unique if it has the form
\[ (7.5) \quad y_1(u) = \frac{-1}{\chi(k+1)} u^{k+1} + \sum_{i=0}^{k-1} y_{i1} u^i \]
with \( y_{i1} \in \mathbb{C} \). Continuing this reasoning, we can show that a solution \( y(u) = y_0(u) + by_1(u) + \cdots + b^d y_d(u) \) with \( y_i(u) \in \mathbb{C}[u] \) does exist and unique if \( y_0(u) \) is a monic polynomial of degree \( k \) and for \( j = 1, \ldots, d \), the polynomial \( y_j(u) \) has the form
\[ y_j(u) = \frac{(-1)^j}{\prod_{m=1}^{j} \chi(k+m)} u^{k+j} + \sum_{i=0}^{k-1} y_{ij} u^i + \sum_{i=k+1}^{k+j-1} y_{ij} u^i \]
with \( y_{ij} \in \mathbb{C} \). We take this \( y(u) \) to be \( F(u) \) in the lemma. Similarly, we can construct the polynomial \( G(u) \) in the lemma. \( \square \)

7.2. Special homomorphisms \( \mathcal{O}_\lambda \to \mathcal{A}_d \). Let \( \{f\}(u), \{g\}(u) \) be the polynomials defined in (6.12). Let \( \mathcal{D}_{\mathcal{O}_\lambda} \) be the universal differential operator defined in (6.13).

Let \( W(u) \in \mathbb{C}[u] \) be a monic polynomial of degree \( 2k + d \). Let \( U(u) \in \mathcal{A}_d[u] \) be a polynomial of the form described in (7.1). Let \( \mathcal{D} \) and \( \chi(\alpha) \) be defined by (7.2) and (7.3), respectively. Assume that all solutions to the differential equation \( D y(u) = 0 \) are polynomials and \( \chi(\alpha) = (\alpha - k)(\alpha - k - d - 1) \). Consider the two polynomials \( F(u), G(u) \), described in Lemma 7.1. Write them in the form:
\[ F(u) = \sum_{i=0}^{k-1} F_i u^i + u^k + \sum_{i=1}^{d} F_{k+i} u^{k+i}, \]
\[ G(u) = \sum_{i=0}^{k-1} G_i u^i + \sum_{i=k+1}^{k+d} G_{i} u^i + u^{k+d+1} + \sum_{i=1}^{d} G_{k+d+1+i} u^{k+d+1+i}, \]
where \( F_i = \sum_{j=0}^{d} F_{ij} b^j, \) \( G_i = \sum_{j=0}^{d} G_{ij} b^j, \) \( \tilde{F}_{k+i} = \sum_{j=i}^{d} F_{k+i,j} b^{j-i}, \) \( \tilde{G}_{k+d+1+i} = \sum_{j=i}^{d} G_{k+d+1+i,j} b^{j-i}. \)

**Lemma 7.2.** A map

\[
\{f_i\} \mapsto F_i, \quad \{g_i\} \mapsto G_i, \quad \{\tilde{f}_{k+i} b^i\} \mapsto \tilde{F}_{k+i} b^i,
\]

\[
\{\tilde{g}_{k+d+1+i} b^i\} \mapsto \tilde{G}_{k+d+1+i} b^i, \quad \{b\} \mapsto b
\]
defines an algebra homomorphism \( \eta : \mathcal{O}_\lambda \to \mathcal{A}_d. \) Under this homomorphism,

\[
\eta(\{f\}(u)) = F(u), \quad \eta(\{g\}(u)) = G(u), \quad \eta(D_{\mathcal{O}_\lambda}) = D.
\]

Here \( \eta(\{f\}(u)) \) is the polynomial in \( u \) obtained from \( \{f\}(u) \) by replacing the coefficients with their images in \( \mathcal{A}_d. \) Similarly, \( \eta(\{g\}(u)) \) and \( \eta(D_{\mathcal{O}_\lambda}) \) are defined.

**Proof.** It is enough to prove that \( \eta(\{f\}(u)) = F(u), \) \( \eta(\{g\}(u)) = G(u) \) and this follows from the definition of \( \mathcal{O}_\lambda. \)



Lemma 7.2 assigns a homomorphism \( \eta : \mathcal{O}_\lambda \to \mathcal{A}_d \) to every differential operator \( D \) satisfying the assumptions of Lemma 7.1.

The homomorphism \( \eta \) of Lemma 7.2 is such that

\[
\eta(\text{Wr}(\{f\}(u), \{g\}(u))) \in \mathbb{C}[u].
\]

We call an arbitrary homomorphism \( \eta : \mathcal{O}_\lambda \to \mathcal{A}_d \) a *special homomorphism* if \( \eta : \{b\} \mapsto b \) and \( \eta \) has property (7.8).

**7.3. Special homomorphisms and Bethe eigenleaves.** Under notations of Section 7.2 define \( n \) by the formula \( n = 2k + d. \) Then \( d = n - 2k. \) Define \( \lambda = (k + d, k) = (n - k, k). \)

For \( a \in \mathbb{C}^n, \) consider the \( \mathcal{B} \)-module \( \mathcal{M}_a \) and its submodule \( \mathcal{M}_{\lambda,a}, \) see definitions in Section 5.1. Assume that \( \mathcal{M}_{\lambda,a} \) has a Bethe eigenleaf \( \mathcal{M}_{\phi,\lambda,a}. \) Consider the universal differential operator \( D_{\mathcal{M}_{\phi,\lambda,a}} \) of the Bethe eigenleaf \( \mathcal{M}_{\phi,\lambda,a}. \) By Lemmas 5.6 and 5.7 the differential operator \( D_{\mathcal{M}_{\phi,\lambda,a}} \) satisfies the assumptions of Lemma 7.1 if we identify the operator \( B_{21} : \mathcal{M}_{\phi,\lambda,a} \to \mathcal{M}_{\phi,\lambda,a} \) in Lemmas 5.6 and 5.7 with the element \( b \in \mathcal{A}_d \) in Lemma 7.1.

By Lemma 7.2 the differential operator \( D_{\mathcal{M}_{\phi,\lambda,a}} \) determines a special homomorphism \( \eta : \mathcal{O}_\lambda \to \mathcal{A}_d, \) which will be called the *special homomorphism associated with a Bethe eigenleaf*. We have \( \eta(D_{\mathcal{O}_\lambda}) = D_{\mathcal{M}_{\phi,\lambda,a}} \) by Lemma 7.2.

**7.4. Wronski homomorphisms.** Set again \( n = 2k + d. \) The Wronskian \( \text{Wr}(\{f\}(u), \{g\}(u)) \in \mathcal{O}_\lambda[u] \) has the form

\[
\text{Wr}(\{f\}(u), \{g\}(u)) = \sum_{j=0}^{n} (-1)^j W_j u^{n-j},
\]

with \( W_j \in \mathcal{O}_\lambda \) for all \( j \) and \( W_0 = d + 1 + w_0, \) where \( w_0 \) is an element of the ideal \( \langle \{b\} \rangle \subset \mathcal{O}_\lambda. \) Thus, the coefficient \( W_0 \) is invertible in \( \mathcal{O}_\lambda. \)
Let $\sigma_s$, $s = 1, \ldots, n$, be indeterminates. Define a grading on $\mathbb{C}[\sigma_1, \ldots, \sigma_n]$ by setting $\deg \sigma_s = s$ for all $s$. The algebra homomorphism,

$$\pi_\lambda : \mathbb{C}[\sigma_1, \ldots, \sigma_n] \to \mathcal{O}_\lambda, \quad \sigma_s \mapsto \frac{W_s}{W_0}, \quad s = 1, \ldots, n,$$

will be called the Wronski homomorphism for $\mathcal{O}_\lambda$. The composition $\pi_\lambda = p_\lambda \pi : \mathbb{C}[\sigma_1, \ldots, \sigma_n] \to \mathcal{O}_\lambda$ will be called the Wronski homomorphism for $\mathcal{O}_\lambda$. Both Wronski homomorphism $\pi_\lambda$ are graded.

Remark. The map $\pi_\lambda^0 : \mathbb{C}[\sigma_1, \ldots, \sigma_n] \to \mathcal{O}_\lambda^0$ is the standard Wronski map, see for example [EG].

7.5. Fibers of Wronski map. Let $A$ be a commutative $\mathbb{C}$-algebra. The algebra $A$ considered as an $A$-module is called the regular representation of $A$. The dual space $A^*$ is naturally an $A$-module, which is called the coregular representation.

A bilinear form $(,): A \otimes A \to \mathbb{C}$ is called invariant if $(ab, c) = (a, bc)$ for all $a, b, c \in A$. A finite-dimensional commutative algebra $A$ with an invariant nondegenerate symmetric bilinear form $(,): A \otimes A \to \mathbb{C}$ is called a Frobenius algebra.

For $a \in \mathbb{C}^n$, let $I_{\lambda,a}^0$ be the ideal in $\mathcal{O}_\lambda$ generated by the elements $\pi(\sigma_s) - a_s$, $s = 1, \ldots, n$. Let

$$\mathcal{O}_{\lambda,a} = \mathcal{O}_\lambda/I_{\lambda,a}^0$$

be the quotient algebra. The algebra $\mathcal{O}_{\lambda,a}$ is a scheme-theoretic fiber of the Wronski homomorphism.

Lemma 7.3. If the algebra $\mathcal{O}_{\lambda,a}$ is finite-dimensional, then it is a Frobenius algebra.

Proof. We have a natural isomorphism

$$\mathcal{O}_\lambda \simeq A_d \otimes \mathbb{C}[\{f, g\}] = \mathbb{C}[f_i, g_i, b]/(b^{d+1}) \, .$$

The ideal $I_{\lambda,a}^0 \subset \mathcal{O}_\lambda$ is generated by $n$ elements $\pi(\sigma_s) - a_s$, $s = 1, \ldots, n$. Hence, $\mathcal{O}_{\lambda,a}$ is the quotient of the polynomial algebra $\mathbb{C}[f_i, g_i, b]$ with $n + 1$ generators by an ideal with $n + 1$ generators. Any such a finite-dimensional quotient is a Frobenius algebra, see for instance, Lemma 3.9 in [MTV3].

8. ISOMORPHISMS

8.1. Isomorphism $\tau_\lambda : \mathcal{O}_\lambda \to B_\lambda$. Let $\mathcal{V}^S_\lambda$ be a deformed isotypical component of $\mathcal{V}^S$, see Section 4.4. Let $B_\lambda$ be the image of $B$ in $\text{End}(\mathcal{V}^S_\lambda)$. Denote $\hat{B}_{ij} \in B_\lambda$ the image of the standard generators $B_{ij} \in B$.

Consider a map

$$\tau_\lambda : \mathcal{O}_\lambda \to B_\lambda, \quad F_{ij} \mapsto \hat{B}_{ij},$$

where the generators $F_{ij}$ of the algebra $\mathcal{O}_\lambda$ are defined in (6.15). In particular,

$$\tau_\lambda : \mathcal{F}_{21} = \{b\} \to \hat{B}_{21} = e_{21}|_{\mathcal{V}^S_\lambda}.$$
**Theorem 8.1.** The map \( \tau_\lambda \) is a well-defined isomorphism of graded algebras.

**Proof.** Let \( R(\hat{F}_{ij}) \) be a polynomial in generators \( F_{ij} \in \mathcal{O}_\lambda \) with complex coefficients. Assume that \( R(\hat{F}_{ij}) \) is equal to zero in \( \mathcal{O}_\lambda \). We will prove that the corresponding polynomial \( R(\hat{B}_{ij}) \) is equal to zero in \( \mathcal{B}_\lambda \). This will prove that \( \tau_\lambda \) is a well-defined algebra homomorphism.

Consider the vector bundle over \( \mathbb{C}^n \) with fiber \( \mathcal{M}_{\lambda,a} \) over a point \( a \). The polynomialal \( R(\hat{B}_{ij}) \) defines a section of the associated bundle with fiber \( \text{End}(\mathcal{M}_{\lambda,a}) \). If \( R(\hat{B}_{ij}) \) is not equal to zero identically, then there exist a fiber \( \mathcal{M}_{\lambda,a} \) and a Bethe eigenleaf \( \mathcal{M}_{\phi,\lambda,a} \subset \mathcal{M}_{\lambda,a} \), such that \( R(\hat{B}_{ij}|_{\mathcal{M}_{\phi,\lambda,a}}) \in \text{End}(\mathcal{M}_{\phi,\lambda,a}) \) is not equal to zero. Let

\[
(8.2) \quad \mathcal{D}_{\mathcal{M}_{\phi,\lambda,a}} = \partial^2 - \hat{B}_1(u)\partial + \hat{B}_2(u) ,
\]

be the universal differential operator of the Bethe eigenleaf \( \mathcal{M}_{\phi,\lambda,a} \), see (5.6). Write

\[
(8.3) \quad \hat{B}_1(u) = \sum_{j=1}^{\infty} \hat{B}_{1j}u^{-j} , \quad \hat{B}_2(u) = \sum_{j=1}^{\infty} \hat{B}_{2j}u^{-j} .
\]

Then \( \hat{B}_{ij} = \hat{B}_{ij}|_{\mathcal{M}_{\phi,\lambda,a}} \) for all \((i,j)\). Consider the special homomorphism \( \eta : \mathcal{O}_\lambda \to \mathcal{A}_d \) associated with the Bethe eigenleaf \( \mathcal{M}_{\phi,\lambda,a} \), see Sections 7.2 and 7.3. By Lemma 7.2 \( \eta(\mathcal{D}_{\mathcal{O}_\lambda}) = \mathcal{D}_{\mathcal{M}_{\phi,\lambda,a}} \). This equality contradicts to the fact that \( R(\hat{F}_{ij}) \) is zero in \( \mathcal{O}_\lambda \) and \( R(\hat{B}_{ij}) \) is nonzero in \( \text{End}(\mathcal{M}_{\phi,\lambda,a}) \). Thus, \( R(\hat{B}_{ij}) \) is zero in \( \mathcal{B}_\lambda \).

By Lemmas 4.1 and 6.6 the elements \( \hat{F}_{ij} \) and \( \hat{B}_{ij} \) are of the same degree. Therefore, the homomorphism \( \tau_\lambda \) is graded.

Since the elements \( \hat{B}_{ij} \) generate the algebra \( \mathcal{B}_\lambda \), the map \( \tau_\lambda \) is surjective.

Let \( R(\hat{F}_{ij}) \) be a polynomial in generators \( F_{ij} \in \mathcal{O}_\lambda \) with complex coefficients. Assume that \( R(\hat{F}_{ij}) \) is a nonzero element of \( \mathcal{O}_\lambda \). We will prove that the corresponding polynomial \( R(\hat{B}_{ij}) \) is not equal to zero in \( \mathcal{B}_\lambda \). This will prove that the homomorphism \( \tau_\lambda \) is injective.

Since \( \mathcal{O}_\lambda \simeq \mathbb{C}[\{f, g\}] \otimes \mathcal{A}_d \). Any nonzero element \( R(\hat{F}_{ij}) \in \mathcal{O}_\lambda \) can be written in the form

\[
R(\hat{F}_{ij}) = \sum_{j=0}^{d} R_j(\{f_i\}, \{g_i\}) \{b\}^j ,
\]

where \( R_j(\{f_i\}, \{g_i\}) \in \mathbb{C}[\{f_i\}, \{g_i\}] \) and \( R_{0j}(\{f_i\}, \{g_i\}) \) is a nonzero polynomial.

For generic numbers \( F_i^0, G_i^0 \in \mathbb{C} \), we have \( R_{0j}(F_i^0, G_i^0) \neq 0 \). Consider two polynomials \( F_0(u) = u^k + \sum_i F_i^0 u^i \) and \( G_0(u) = u^{k+d+1} + \sum_i G_i^0 u^i \). By Lemma 5.8 there exists a Bethe eigenleaf such that \( \mathcal{D}_{\mathcal{M}_{\sigma,\lambda,a},0} = \mathcal{D}_{F_0, G_0} \). Let \( \hat{B}_{ij} \) be the coefficients of \( \mathcal{D}_{\mathcal{M}_{\sigma,\lambda,a}} \), see (8.2) and (8.3). Then \( R(\hat{B}_{ij}) \neq 0 \). Hence, \( R(\hat{B}_{ij}) \) is not equal to zero in \( \mathcal{B}_\lambda \). \( \square \)

8.2. Algebras \( \mathcal{O}_\lambda \) and \( \mathcal{B}_\lambda \) as \( \mathbb{C}[\sigma_1, \ldots, \sigma_n] \)-modules. The algebra \( \mathbb{C}[z_1, \ldots, z_n]^S = \mathbb{C}[\sigma_1, \ldots, \sigma_n] \) is embedded into the algebra \( \mathcal{B}_\lambda \) as the subalgebra of operators of multiplication by symmetric polynomials, see Lemma 4.3. This embedding makes \( \mathcal{B}_\lambda \) a \( \mathbb{C}[\sigma_1, \ldots, \sigma_n] \)-module.

The Wronski homomorphism \( \pi_\lambda : \mathbb{C}[\sigma_1, \ldots, \sigma_n] \to \mathcal{O}_\lambda \) makes \( \mathcal{O}_\lambda \) a \( \mathbb{C}[\sigma_1, \ldots, \sigma_n] \)-module.

**Lemma 8.2.** The map \( \tau_\lambda : \mathcal{O}_\lambda \to \mathcal{B}_\lambda \) is an isomorphism of \( \mathbb{C}[\sigma_1, \ldots, \sigma_n] \)-modules, that is, for any \( s = 1, \ldots, n \), \( \tau_\lambda(\pi_\lambda(\sigma_s)) \) is the operator of multiplication by \( \sigma_s \).
Proof. The proof follows from the two formulae:

\[ B_1(u) = e_{11}(u) + e_{22}(u), \quad F_1(u) = -\frac{\text{W}_1'(\{f\}(u), \{g\}(u))}{\text{W}_1(\{f\}(u), \{g\}(u))}. \]

\[ \square \]

Corollary 8.3. The Wronski homomorphism \( \pi_\lambda : \mathbb{C}[\sigma_1, \ldots, \sigma_n] \to \mathcal{O}_\lambda \) is an embedding.

Consider the projection \( p_\lambda^\sigma : \mathcal{O}_\lambda \to \mathcal{O}_\lambda^0 \) defined in (6.16). The composition

\[ \pi_\lambda^0 = p_\lambda^\sigma \pi_\lambda : \mathbb{C}[\sigma_1, \ldots, \sigma_n] \to \mathcal{O}_\lambda^0 \]

is the standard Wronski map. Its degree \( d_\lambda^0 \) is given by the Schubert calculus. In particular, we have

\[ (d + 1) d_\lambda^0 = \dim (V^\otimes n)_\lambda, \]

where \( (V^\otimes n)_\lambda \subset V^\otimes n \) is the \( \mathfrak{gl}_2 \)-isotypical component corresponding to the irreducible polynomial \( \mathfrak{gl}_2 \)-representation with highest weight \( \lambda = (n - k, k) \) and \( d = n - 2k \).

Proposition 8.4. For \( a \in \mathbb{C}^n \), let \( I_{\lambda, a}^\sigma \) be the ideal in \( \mathcal{O}_\lambda \) generated by the elements \( \pi(\sigma_s) - a_s \), \( s = 1, \ldots, n \). Let \( \mathcal{O}_{\lambda, a} = \mathcal{O}_\lambda/I_{\lambda, a}^\sigma \) be the quotient algebra. Then

\[ \dim \mathcal{O}_{\lambda, a} = \dim (V^\otimes n)_\lambda. \]

Proof. The proposition follows from Lemma 8.5. \( \square \)

Let \( H_s(x_1, \ldots, x_m, b) \), \( s = 1, \ldots, m \), be \( m \) polynomials in \( \mathbb{C}[x_1, \ldots, x_m, b] \) such that

\[ H_s(x_1, \ldots, x_m) = \sum_{j=0}^d H_{sj}(x_1, \ldots, x_m) b^j. \]

Let \( I \subset \mathbb{C}[x_1, \ldots, x_m, b] \) be the ideal generated by \( m + 1 \) polynomials: \( b^{d+1} \) and \( H_s(x_1, \ldots, x_m, b) \), \( s = 1, \ldots, m \). Let \( I_0 \subset \mathbb{C}[x_1, \ldots, x_m] \) be the ideal generated by the polynomials \( H_{s0}(x_1, \ldots, x_m) \), \( s = 1, \ldots, m \).

Lemma 8.5. Assume that \( \mathbb{C}[x_1, \ldots, x_m]/I_0 \) is finite-dimensional. Then

\[ \dim \mathbb{C}[x_1, \ldots, x_m, b]/I = (d + 1) (\dim \mathbb{C}[x_1, \ldots, x_m]/I_0). \]

\[ \square \]

8.3. Isomorphism \( \mu_\lambda : \mathcal{O}_\lambda \to \mathcal{V}_\lambda^S \). By Lemma 4.6, the space \( \mathcal{V}_\lambda^S \) is a graded free \( \mathbb{C}[\sigma_1, \ldots, \sigma_n] \)-module. It has a unique (up to proportionality) vector of degree \( 2k - n \). Fix such a vector \( v_\lambda \in \mathcal{V}_\lambda^S \) and consider a linear map

\[ \mu_\lambda : \mathcal{O}_\lambda \to \mathcal{V}_\lambda^S, \quad F \mapsto \tau_\lambda(F) v_\lambda. \]

Theorem 8.6. The map \( \mu_\lambda : \mathcal{O}_\lambda \to \mathcal{V}_\lambda^S \) is an isomorphism of graded vector spaces. The maps \( \tau_\lambda \) and \( \mu_\lambda \) intertwin the action of multiplication operators on \( \mathcal{O}_\lambda \) and the action of the Bethe algebra \( \mathcal{B}_\lambda \) on \( \mathcal{V}_\lambda^S \), that is, for any \( F, G \in \mathcal{O}_\lambda \), we have

\[ \mu_\lambda(FG) = \tau_\lambda(F) \mu_\lambda(G). \]

\[ (8.6) \]
In other words, the maps $\tau_\lambda$ and $\mu_\lambda$ give an isomorphism of the regular representation of $O_\lambda$ and the $B_\lambda$-module $V_\lambda^S$.

**Proof.** For any nonzero $H \in \mathbb{C}[\sigma_1, \ldots, \sigma_n]$, the vector $(B_21)^dHv_\lambda$ is a nonzero vector. Thus, the kernel of $\mu_\lambda$ is an ideal $I$ in $B_\lambda$, which does not contain elements of the form $(B_21)^dH$. Hence, $\tau_\lambda^{-1}(I)$ is an ideal in $O_\lambda$, which does not contain elements of the form $\{b\}^d\hat{H}$, where $\hat{H} \in \pi(\mathbb{C}[\sigma_1, \ldots, \sigma_n])$. It is easy to see that any ideal in $O_\lambda$, which does not contain elements of the form $\{b\}^d\hat{H}$, is the zero ideal. This reasoning proves that $\mu_\lambda$ is injective.

The map $\mu_\lambda$ is a graded linear map. We have the equality of characters, $\text{ch}_{\lambda}(q) = \text{ch}_{O_\lambda}(q)$, due to formulae (4.9) and (6.18). Hence, the map $\mu_\lambda$ is surjective. Formula (8.6) follows from Theorem 8.1. □

### 8.4. Isomorphism of algebras $O_{\lambda,a}$ and $B_{\lambda,a}$

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$. Consider the $B$-module $M_{\lambda,a}$. Denote $\tilde{B}_{\lambda,a}$ the image of $B$ in $\text{End}(M_{\lambda,a})$.

Let $I^B_{\lambda,a} \subset B_\lambda$ be the ideal generated by the elements $\sigma_s(z) - a_s$, $s = 1, \ldots, n$. Consider the subspace $I^V_{\lambda,a} = I^B_{\lambda,a} V_\lambda^S$.

**Lemma 8.7.** We have

$$\tau_\lambda(I^O_{\lambda,a}) = I^B_{\lambda,a}; \quad \mu_\lambda(I^O_{\lambda,a}) = I^V_{\lambda,a}; \quad B_{\lambda,a} = B_\lambda/I^B_{\lambda,a}; \quad M_{\lambda,a} = V_\lambda^S/I^V_{\lambda,a}.$$  

**Proof.** The lemma follows from Theorems 8.1 8.6 and Lemmas 8.2 2.3. □

By Lemma 8.7, the maps $\tau_\lambda$ and $\mu_\lambda$ induce the maps

$$\tau_{\lambda,a} : O_{\lambda,a} \rightarrow B_{\lambda,a}; \quad \mu_{\lambda,a} : O_{\lambda,a} \rightarrow M_{\lambda,a}.$$

**Theorem 8.8.** The map $\tau_{\lambda,a}$ is an isomorphism of algebras. The map $\mu_{\lambda,a}$ is an isomorphism of vector spaces. The maps $\tau_{\lambda,a}$ and $\mu_{\lambda,a}$ intertwine the action of multiplication operators on $O_{\lambda,a}$ and the action of the Bethe algebra $B_{\lambda,a}$ on $M_{\lambda,a}$, that is, for any $F, G \in O_{\lambda,a}$, we have

$$\mu_{\lambda,a}(FG) = \tau_{\lambda,a}(F) \mu_{\lambda,a}(G).$$

In other words, the maps $\tau_{\lambda,a}$ and $\mu_{\lambda,a}$ give an isomorphism of the regular representation of $O_{\lambda,a}$ and the $B_{\lambda,a}$-module $M_{\lambda,a}$.

**Proof.** The theorem follows from Theorems 8.1 8.6 and Lemma 8.7. □

**Remark.** By Lemma 7.3, the algebra $O_{\lambda,a}$ is Frobenius. Therefore, its regular and coregular representations are isomorphic.

### 9. Comparison of actions of $B$ and $B^0$ on $V^S$

#### 9.1. Isomorphism $\nu_\lambda : A_d \otimes B^0_\lambda \rightarrow B_\lambda$

**Lemma 9.1.** Consider the principal ideal $\langle \hat{B}_{21} \rangle \subset B_\lambda$ and the graded algebra epimorphism $p^B_\lambda : B_\lambda \rightarrow B^0_\lambda$, defined in (4.12). Then $\langle \hat{B}_{21} \rangle = \ker p^B_\lambda$. 

Proof. Clearly, we have \( \langle \hat{B}_{21} \rangle \subset \ker p^R_\lambda \). Consider the commutative diagram of algebra homomorphisms,

\[
\begin{array}{ccc}
\mathcal{A}_d & \xrightarrow{i^R_\lambda} & \mathcal{O}_\lambda \\
\downarrow & & \downarrow \tau_\lambda \\
\mathcal{A}_d & \xrightarrow{i^R} & \mathcal{B}_\lambda
\end{array}
\]

(9.1)

We have \( \ker p^Q_\lambda = \langle \{ b \} \rangle \). The graded characters of \( \mathcal{O}_\lambda^0 \) and \( \mathcal{B}_\lambda^0 \) are equal due to (4.7), (6.18). Hence \( \langle \hat{B}_{21} \rangle = \ker p^R_\lambda \). \( \square \)

Corollary 9.2. The isomorphism \( \tau_\lambda \) induces an isomorphism

\[
\tau^0_\lambda : \mathcal{O}_\lambda^0 \rightarrow \mathcal{B}_\lambda^0 .
\]

Remark. The isomorphism \( \tau^0_\lambda : \mathcal{O}_\lambda^0 \rightarrow \mathcal{B}_\lambda^0 \) is the isomorphism denoted \( \tau_\lambda \) in Theorem 5.3 of [MTV3].

Denote

\[
r^O_\lambda : \mathcal{A}_d \otimes \mathcal{O}_\lambda^0 \rightarrow \mathcal{O}_\lambda^0
\]

the algebra epimorphism such that \( b \otimes x \mapsto 0 \), \( 1 \otimes x \mapsto x \) for any \( x \in \mathcal{O}_\lambda^0 \). Denote

\[
r^B_\lambda : \mathcal{A}_d \otimes \mathcal{B}_\lambda^0 \rightarrow \mathcal{B}_\lambda^0
\]

the algebra epimorphism such that \( b \otimes x \mapsto 0 \), \( 1 \otimes x \mapsto x \) for any \( x \in \mathcal{B}_\lambda^0 \).

Theorem 9.3. The following diagram is commutative,

\[
\begin{array}{ccc}
\mathcal{A}_d & \xrightarrow{id \otimes 1} & \mathcal{A}_d \otimes \mathcal{B}_\lambda^0 \\
\downarrow & & \downarrow \nu_\lambda \\
\mathcal{A}_d & \xrightarrow{id \otimes \lambda} & \mathcal{B}_\lambda
\end{array}
\]

(9.2)

where \( \nu_\lambda \) is the isomorphism defined by the formula \( \nu_\lambda = \tau_\lambda q_\lambda (id \otimes (\tau^0_\lambda)^{-1}) \).

Proof. The theorem follows from the commutativity of the following diagram:
9.2. $A_d \otimes \mathcal{B}^0_\lambda$-module $\mathcal{V}^S_{\lambda,0}$. By Lemma 4.3 the space $\text{Sing} \mathcal{V}^S_{\lambda,0}$ is a graded free $\mathbb{C}[\sigma_1, \ldots, \sigma_n]$-module. It has a unique (up to proportionality) vector of degree $2k - n$. Fix such a vector $\nu^0_\lambda \in \text{Sing} \mathcal{V}^S_{\lambda,0}$ and consider a linear map

$$\mu^0_\lambda : \mathcal{O}^0_\lambda \rightarrow \text{Sing} \mathcal{V}^S_{\lambda,0}, \quad F \mapsto \tau^0_\lambda(F) \nu^0_\lambda.$$ 

**Theorem 9.4** (Theorem 5.6 of [MTV3]). The map $\mu^0_\lambda$ is an isomorphism of graded vector spaces. The maps $\tau^0_\lambda$ and $\mu^0_\lambda$ intertwine the action of multiplication operators on $\mathcal{O}^0_\lambda$ and the action of the Bethe algebra $\mathcal{B}^0_\lambda$ on $\text{Sing} \mathcal{V}^S_{\lambda,0}$, that is, for any $F, G \in \mathcal{O}^0_\lambda$, we have

$$\mu^0_\lambda(FG) = \tau^0_\lambda(F) \mu^0_\lambda(G).$$

In other words, the maps $\tau^0_\lambda$ and $\mu^0_\lambda$ give an isomorphism of the regular representation of $\mathcal{O}^0_\lambda$ and the $\mathcal{B}^0_\lambda$-module $\text{Sing} \mathcal{V}^S_{\lambda,0}$.

Consider the linear map

$$\bar{\mu}^0_\lambda : A_d \otimes \mathcal{O}^0_\lambda \rightarrow \mathcal{V}^S_{\lambda,0}, \quad b^j \otimes F \mapsto (e_{21})^j \tau^0_\lambda(F) \nu^0_\lambda,$$

and the algebra isomorphism

$$\text{id} \otimes \tau^0_\lambda : A_d \otimes \mathcal{O}^0_\lambda \rightarrow A_d \otimes \mathcal{B}^0_\lambda.$$ 

**Corollary 9.5.** The map $\bar{\mu}^0_\lambda$ is an isomorphism of graded vector spaces. The maps $\text{id} \otimes \tau^0_\lambda$ and $\bar{\mu}^0_\lambda$ intertwine the action of multiplication operators on $A_d \otimes \mathcal{O}^0_\lambda$ and the action of the algebra $A_d \otimes \mathcal{B}^0_\lambda$ on $\mathcal{V}^S_{\lambda,0}$, that is, for any $F, G \in \mathcal{O}^0_\lambda$ and $i, j \geq 0$, we have

$$\bar{\mu}^0_\lambda(b^i \otimes FG) = (\text{id} \otimes \tau^0_\lambda)(b^i \otimes F) \bar{\mu}^0_\lambda(b^j \otimes G).$$

In other words, the maps $\text{id} \otimes \tau^0_\lambda$ and $\bar{\mu}^0_\lambda$ give an isomorphism of the regular representation of $A_d \otimes \mathcal{O}^0_\lambda$ and the $A_d \otimes \mathcal{B}^0_\lambda$-module $\mathcal{V}^S_{\lambda,0}$ defined in Section 4.3.

9.3. **Comparison of $A_d \otimes \mathcal{B}^0_\lambda$-module $\mathcal{V}^S_{\lambda,0}$ and $\mathcal{B}_\lambda$-module $\mathcal{V}^S_{\lambda}$.** Define a linear map

$$\eta_\lambda : \mathcal{V}^S_{\lambda,0} \rightarrow \mathcal{V}^S_{\lambda}$$

by the formula

$$(e_{21})^j B \nu^0_\lambda \mapsto \nu_\lambda(b^j \otimes B)\nu_\lambda$$

for any $j \geq 0$ and $B \in \mathcal{B}^0_\lambda$.

**Theorem 9.6.** The map $\eta_\lambda$ is an isomorphism of graded vector spaces. The maps $\nu_\lambda : \mathcal{A}_d \otimes \mathcal{B}^0_\lambda \rightarrow \mathcal{B}_\lambda$ and $\eta_\lambda$ intertwine the action of $\mathcal{A}_d \otimes \mathcal{B}^0_\lambda$ on $\mathcal{V}^S_{\lambda,0}$ and the action of $\mathcal{B}_\lambda$ on $\mathcal{V}^S_{\lambda}$. In other words, the maps $\nu_\lambda$ and $\eta_\lambda$ give an isomorphism of the $\mathcal{A}_d \otimes \mathcal{B}^0_\lambda$-module $\mathcal{V}^S_{\lambda,0}$ and $\mathcal{B}_\lambda$-module $\mathcal{V}^S_{\lambda}$.

The theorem is a direct corollary of Theorems 8.6, 9.3 and Corollary 9.5.
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