The category of supermodules

I Hadi*, M Makmuri and S Sudarwanto
Department of Mathematics, Faculty of Mathematics and Science, Universitas Negeri Jakarta, Jl. Rawamangun Muka, Jakarta, 13220, Indonesia

*ibnu_hadi@unj.ac.id

Abstract. Given a commutative algebra with unit element \( R \) and an \( R \)-module \( U \), in this paper we will construct a category consists as objects all pairs of \( R \)-modules \( (N,M) \) where \( N \) is direct sum of \( U \) and \( M \) contains \( U \). To define the morphisms between any two objects, we will identify an object \( (N,M) \) with an \( R \)-module \( E \) determined by \( N \) and \( M \). By this method, we obtain a category denoted by \( C_U \). Further, if \( W \) is a direct summand of \( U \) then there exists a functor from \( C_U \) to \( C_W \).

1. Introduction
Given a commutative algebra with unit element \( R \) and an \( R \)-module \( U \), our goal is to construct a category \( C \) of supermodules of \( U \) whose objects are all pairs of \( R \)-modules \( (N,M) \) where \( N \) is direct sum of \( U \) and \( M \) contains \( U \). To define the morphisms between any two objects in \( C \), we will identify every object \( (N,M) \) uniquely with an \( R \)-module \( E \) determined from \( N \) and \( M \). We will call such category as “supermodule category of \( U \)”.

The idea of this research was generated from the notion of \( U \)-exact sequence introduced by Davvaz and Parnian-Garamaleky in [1] and by Davvaz and Shabani-Solt in [2]. Mahatma and Muchtadi Alamsyah then developed the \( U \)-extension module in [3] and, further, Mahatma investigated the equivalence between the first \( U \)-extension module and the short \( U \)-exact sequence [4]. The last manuscript gives a method for determining a pair of supermodule with a module.

This research can be continued by further question about the existence of equivalence between two categories of supermodules, either generated using the same algebra or different algebras.

2. Method of research
The category \( C \) we construct in this article whose objects are all pairs of \( R \)-modules \( (N,M) \) where \( N \) is direct sum of \( U \) and \( M \) contains \( U \). To define the morphisms between two objects, say from \( (N,M) \) to \( (N',M') \), we identify the object \( (N,M) \) to \( (N',M') \) uniquely with, respectively, an \( R \)-module \( E \) and \( E' \), determined from \( N \) and \( M \). The morphism from \( (N,M) \) to \( (N',M') \) thus defined as any \( R \)-module homomorphisms from \( E \) to \( E' \).
We note first that when defining the module $E$ of $(N,M)$, we must involve the submodule $U$ in the process. For if not then the category of supermodules of $U$ will be the same with the category of supermodules of $W$ for any submodule $W$ of $U$.

3. Constructing the category
Let $R$ be commutative algebra with unit element and $U$ be projective $R$-module. Let $N=U \oplus V$ and $M$ be projective $R$-module contains $U$. Consider the $R$-module homomorphism $z:U \rightarrow V$. Define the $R$-module $I$ by $I:=\{(u \oplus z(u)) \oplus -u | u \in U\}$. Clearly, $I$ is a $R$-submodule of $N \oplus M$. Let $E=(N \oplus M)/I$. We will show that the module $E$ does not depend on the choice of $z$.

Suppose that $w:U \rightarrow V$ is an $R$-module homomorphism, $J:=\{(u \oplus w(u)) \oplus -u | u \in U\}$, and $F=(N \oplus M)/J$. Note first that every element in $F$ is in the form of $((a \oplus b) \oplus c)+J$ where $a \in U$, $b \in V$, and $c \in M$. Let us denote this element by the class $[a,b,c]_J$. Now define a map $\delta:E \rightarrow F$ by $\delta([u,v,m]) := [u,v+(w-z)(u),m]$ for every $[u,v,m] \in E$. Note that if $[u,v,m]_J = I$ then $m=-u$ and $v=z(u)$. This implies $\delta(I) = \delta([u,z(u),-u])_J = [u,z(u)+(w-z)(u),-u]_J = [u,w(u),-u]_J = J$. Thus, $\delta$ is well-defined. Clearly, $\delta$ is an $R$-module homomorphism and is injective for if $\delta([u,v,m])_J = J$ then $m=-u$ and $v+(w-z)(u)=w(u)$ which implies $v=z(u)$ and so $[u,v,m]_J = [u,z(u),-u]_J = I$. Finally, every $[u,v,m] \in F$ is the image of $[u,v+(z-w)(u),m] \in E$ by $\delta$. Therefore, $\delta$ is an isomorphism. We call this isomorphism as “standard isomorphism”. Notice that the standard isomorphism $\delta:E \rightarrow E$ is actually the identity map $1_E$.

We have shown that every $(N,M) \in C$ can be identified with the module $E$. Now we define the morphism from $(N,M)$ to $(N',M') \in C$ as any $R$-module homomorphisms from $E$ to $E'$ for all possible choices of $E$ and $E'$. Now suppose that $\varphi:E \rightarrow E'$ and $\phi:F \rightarrow F'$ both are morphisms from $(N,M)$ to $(N',M')$. We consider $\varphi$ and $\phi$ to be the same if there exist standard isomorphisms $f:E \rightarrow F$ and $f':E' \rightarrow F'$ such that the diagram

$$\begin{array}{ccc}
E & \rightarrow & E' \\
\downarrow f & & \downarrow f' \\
F & \rightarrow & F'
\end{array}$$

commutes, that is $f'\varphi=\phi f$. We define the composition of two morphisms in $C$ as follows: Suppose that $\varphi:E \rightarrow F$ is a morphism from $(N,M)$ to $(N',M')$ and $\phi:G \rightarrow H$ is a morphism from $(N',M')$ to $(N'',M'')$. Let $\delta:F \rightarrow G$ be standard isomorphism. Define $\phi \circ \varphi = \delta \delta \varphi$. Clearly, $\phi \circ \varphi$ is a morphism from $(N,M)$ to $(N'',M'')$.

**Theorem 1** $C$ is a category.

**Proof:** We only need to show the associativity of the composition and the existence of the identity morphism for every object in $C$. First suppose that $f:E \rightarrow E_i$ is any morphism from $(N,M)$ to $(N_i,M_i)$. $g:F_i \rightarrow F_2$ is any morphism from $(N_i,M_i)$ to $(N_2,M_2)$, and $h:G_2 \rightarrow G_3$ is any
morphism from \((N_2, M_2)\) to \((N_3, M_3)\). Let \(\delta_1 : E_1 \to F_1\) and \(\delta_2 : F_2 \to G_2\) both be standard isomorphisms. Then

\[
h \circ (g \circ f) = h \circ g \circ \delta_1 f = h \delta_2 (g \delta_1 f) = (h \delta_2 g) \delta_1 f = (h \circ g) \circ f
\]

Hence the composition is associative.

Now consider the identity map \(1_E : E \to E\) which is a morphism from \((N, M)\) to \((N, M)\). Suppose that \(s : F \to F'\) is any morphism from \((N, M)\) to \((N', M')\) and \(t : G' \to G\) is any morphism from \((N', M')\) to \((N, M)\). Let \(\mu : E \to F\) and \(\lambda : G \to E\) both be standard isomorphisms. Then \(s \circ 1_E = s \mu\) which is the same with \(s\) by the diagram

\[
\begin{array}{c}
E \xrightarrow{\mu} F' \\
\downarrow \mu \quad \downarrow \lambda' \\
F \xrightarrow{t} F'
\end{array}
\]

and \(1_E \circ t = \lambda t\) which is the same with \(t\) by the diagram

\[
\begin{array}{c}
G' \xrightarrow{t} G \\
\downarrow \lambda' \quad \downarrow \lambda \\
G' \xrightarrow{\mu} E
\end{array}
\]

Hence \(1_E\) acts as identity morphism from \((N, M)\) to \((N, M)\). To show that such morphism is unique, suppose that \(z : H \to K\) is an identity morphism from \((N, M)\) to \((N, M)\). Let \(\alpha : E \to H\) be standard isomorphism. Now, since \(z\) is an identity morphism then \(z \circ 1_E = z \alpha\) must be the same with \(1_E\). This equity is described by the diagram

\[
\begin{array}{c}
E \xrightarrow{z} E \\
\downarrow 1_E \quad \downarrow \beta \\
E \xrightarrow{z\alpha} K
\end{array}
\]

where \(\beta : E \to K\) is standard isomorphism. Hence \(\beta 1_E = z \alpha\) which says that \(z\) and \(1_E\) are considered to be the same morphisms. □

4. Functor between two categories
Consider the category \(C\) of supermodules of an \(R\)-module \(U\). We shall denote this category by \(C_U\). Suppose that \(W\) is a direct summand of \(U\). Now we have the category \(C_w\) of all pairs \((A, B)\) where \(A\) is direct sum of \(W\) and \(B\) is \(R\)-module contains \(W\). Consider the injection \(i : W \to U\) and the natural projection \(\pi : U \to W\) whose composition \(\pi i = 1_w\), the identity map on \(W\). Define the map \(F : C_U \to C_w\) as follows:
For every object \((N, M) \in C_v\), if \(N = U \oplus V\) then \(F((N, M)) = (W \oplus V, M)\). Now suppose that \((N, M) = (U \oplus V, M)\) and \((N', M') = (U' \oplus V', M')\) both are objects in \(C_v\) and let \(f : (N, M) \to (N', M')\) be a morphism in \(C_v\) given by the \(R\)-module homomorphism \(f : (N \oplus M)/I \to (N' \oplus M')/I'\). We will give method for constructing the morphism \(F(f) : F((N, M)) \to F((N', M'))\) in \(C_w\).

Suppose that the submodule \(I\) of \(N \oplus M\) and the submodule \(I'\) of \(N' \oplus M'\) in the preceding paragraph were determined by the homomorphisms \(z : U \to V\) and \(z' : U \to V'\), respectively. Let us identify the objects \(F((N, M)) = (W \oplus V, M)\) and \(F((N', M')) = (W \oplus V', M')\) in \(C_w\) by, respectively, the module \(((W \oplus V) \oplus M)/J\) where \(J = \{(w \oplus z(w)) \oplus w | w \in W\}\) and the module \(((W \oplus V') \oplus M')/J'\) where \(J' = \{(w \oplus z'(w)) \oplus -w | w \in W\}\). Define the map \(\varphi : ((W \oplus V) \oplus M)/J \to ((W \oplus V') \oplus M')/J'\) by \(\varphi([w, v, m]) = [w, v, m]_{J'}\) for every \([w, v, m] \in ((W \oplus V) \oplus M)/J\). Since \(W \subseteq U\) and \(J \subseteq I\), it is clear that the map \(\varphi\) is well-defined.

Now define the map \(\psi : ((U \oplus V') \oplus M')/I' \to ((W \oplus V') \oplus M')/J'\) by \(\psi([u, v, m]) = [\pi(u), z'(u) + z'(\pi(u) - u), m + u - \pi(u)]_{J'}\) for every \([u, v, m] \in ((U \oplus V') \oplus M')/I'\). Notice that if \([u, v, m]_{I'} = I'\) then \(v = z'(u)\) and \(m = -u\) and thus

\[
\psi(I') = \psi([u, z'(u), -u]_{J'}) = [\pi(u), z'(u) + z'(\pi(u) - u), -u + u - \pi(u)]_{J'} = [\pi(u), z'(u), -\pi(u)]_{J'} = J'
\]

Hence \(\psi\) is well-defined.

Next, define \(F(f) = \psi \circ f \circ \varphi\), which can be illustrated by the diagram below

\[
\begin{array}{ccc}
(U \oplus V) \oplus M)/I & \xrightarrow{\varphi} & (U \oplus V') \oplus M')/I' \\
\uparrow \psi & & \downarrow \psi \\
(W \oplus V) \oplus M)/J & \xrightarrow{\psi \circ f \circ \varphi} & (W \oplus V') \oplus M')/J'
\end{array}
\]

Note that the top row in the diagram above was morphism in \(C_v\), while the bottom row was morphism in \(C_w\).

Now suppose that \(g : (N', M') \to (N'', M'')\) is a morphism in \(C_v\). Without loss of generality, we may assume that \(g\) is given by the \(R\)-module homomorphism \(g : (N' \oplus M')/I' \to (N'' \oplus M'')/I''\). By the construction we have just made, \(F(g) = \psi \circ g \circ \varphi\), for some \(R\)-module homomorphisms \(\varphi\) and \(\psi\) defined similarly with \(\varphi\) and \(\psi\), respectively. Notice that \(\varphi : ((W \oplus V') \oplus M')/J' \to ((U \oplus V') \oplus M')/I'\) with \(\varphi([w, v, m]) = [w, v, m]_{J'}\) for every \([w, v, m] \in ((W \oplus V') \oplus M')/J'\). Hence, for every \([u, v, m] \in ((U \oplus V') \oplus M')/I', \varphi \psi([u, v, m]) = [\pi(u), v + z'(\pi(u) - u), m + u - \pi(u)]_{J'} = [u, v, m]_{J'}\). We see that \(\varphi \psi\) is an identity function in \(((U \oplus V') \oplus M')/I'\). As a consequence, we have that \(F(g) \circ F(f) = (\psi \circ g \circ \varphi) \circ (\psi \circ f \circ \varphi) = \psi \circ g \circ f \circ \varphi = F(g \circ f)\).
Finally, notice that \( F(1_{(N,M)}) = \psi \varphi \) since \( \pi(w) = w \) for every \( w \in W \). We have just shown that \( F : \mathcal{C}_U \to \mathcal{C}_W \) is a functor.

5. Result and discussion

We have given a method for constructing a category of supermodules of an \( R \)-module \( U \), called \( \mathcal{C}_U \), consists of all pairs of \( R \)-modules \( (N = U \oplus V, M \supseteq U) \), all morphisms between any pairs \( (N, M) \) and \( (N', M') \), and the composition rule. Further, if \( W \) is a direct summand of \( U \) then there exists a functor \( F : \mathcal{C}_U \to \mathcal{C}_W \). It is not hard to verify that if \( U \) and \( W \) are isomorphic then \( F \) become a category equivalence. There is still a question about what the necessary condition is for the existence of equivalence between two categories. It will be interesting to investigate further whether there is an equivalence between two categories of supermodules over different algebras.

6. Conclusion

Given commutative algebra with unit element \( R \) and an \( R \)-module \( U \) there exists a category of supermodules of \( U \) consists of all pairs of \( R \)-modules \( (N = U \oplus V, M \supseteq U) \) as objects. Further, if \( W \) is direct sum of \( U \) then there exists a functor from the category of supermodules of \( U \) to the category of supermodules of \( W \).

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