INFEERENCE AND FDR CONTROL FOR SIMULATED ISING MODELS IN HIGH-DIMENSION

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This paper studies the consistency and statistical inference of simulated Ising models in the high dimensional background. Our estimators are based on the Markov chain Monte Carlo maximum likelihood estimation (MCMC-MLE) method penalized by the Elastic-net. Under mild conditions that ensure a specific convergence rate of MCMC method, the $\ell_1$ consistency of Elastic-net-penalized MCMC-MLE is proved. We further propose a decorrelated score test based on the decorrelated score function and prove the asymptotic normality of the score function without the influence of many nuisance parameters under the assumption that accelerates the convergence of the MCMC method. The one-step estimator for a single parameter of interest is purposed by linearizing the decorrelated score function to solve its root, as well as its normality and confidence interval for the true value, therefore, be established. Finally, we use different algorithms to control the false discovery rate (FDR) via traditional p-values and novel e-values.

1. Introduction.

1.1. Backgrounds. The (probabilistic) graphical model consists of a collection of probability distributions that factorize according to the structure of an underlying graph \cite{52}. The graphical model captures the complex dependencies among random variables and build large-scale multivariate statistical models, which has been used in many research areas such as hierarchical Bayesian models \cite{27}, contingency table analysis \cite{20,53} in categorical data analysis \cite{1,23,37}, constraint satisfaction \cite{16,15}, language and speech processing \cite{11,31}, image processing \cite{17,24,28} and spatial statistics more generally \cite{8}.

In our work, we focus on the undirected graphical models, where the probability distribution factorizes according to the function defined on the cliques of the graph. The undirected graphical models have a variety of applications, including statistical physics \cite{32}, natural language processing \cite{38}, image analysis \cite{54} and spatial statistics \cite{43}. Specifically, we pay attention to the undirected graphical models which can be described as exponential families, a broad class of probability distributions elaborately studied in many statistical literature \cite{4,21,13}. The properties of the exponential families provide some connections between the inference methods and the convex analysis \cite{12,29}. There are many well-known examples that are undirected graphical models viewed as exponential families, such as Ising model \cite{32,5}, Gaussian MRF \cite{46} and latent Dirichlet allocation \cite{11}.

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Due to the importance of the graphical models, the problem of structure learning and parameter estimation has been the most important part of the study of the graphical models [52, 10, 33, 39, 30]. The main obstacle in learning the structure of the graphical models is the intractable normalizing constant. When a graphical model has \( m \) vertices and each variable attached to a vertex takes values in a discrete state space consisting of \( r \) different elements, the normalizing constant becomes a summation of \( m^r \) different terms, which is computationally infeasible. There have been many methods proposed to overcome this obstacle. One widely applied method named pseudo-likelihood approach [8] is replacing the likelihood containing the normalizing constant by the product of conditionals, which do not contain the normalizing constant. This idea has been widely used in many works studying the model selection properties of high-dimensional Ising model [55, 33, 30]. Unfortunately, this method only performs well when the pseudo-likelihood serves as a good approximation to the likelihood, which means the structure of the graph is simple. Another well-known way to overcome the intractable normalizing constant involves the Markov chain Monte Carlo (MCMC) method [26, 44], where the normalizing constant is approximated by a path integral of a Markov chain. The computational cost of MCMC method is independent of the complexity of the estimated graph, which means we only need sufficiently large samplings to approximate the normalizing constant.

In this paper, we focus on the Markov chain Monte Carlo maximum likelihood estimation (MCMC-MLE)[25] for the undirected graphical models in an exponential form, where the normalizing constant in the original likelihood is replaced by the approximation of sampling technique using MCMC method.

1.2. The High Dimensional Background. In the high dimensional background, when the number of parameters \( p \) is larger than the number of independent samples \( n \), the maximum likelihood estimation is an ill-posed problem. To handle this, a typical way is to apply the Lasso method [47], where the parameters remain close to the extreme of the likelihood and not too significant. The properties of Lasso-penalized estimators have been thoroughly studied in much literature based on linear models or generalized linear models [9, 14, 49], and more complex statistical models [3, 14, 30, 42, 55].

Lasso method can filter out some parameters, which means some parameters can be 0. However, the ridge regression can find the estimators where the parameters can not be estimated to 0. If we combine the Lasso method and the ridge regression, it yields the Elastic-net estimation [61], which is adopted in this paper. This method inherits the multicollinearity characteristic of the independent variables processed by ridge regression. In many numerical results, the Elastic-net performs better than the Lasso method [61] by showing a group effect. Specifically, the Lasso method can only choose one variable in a set of highly correlated variables, while the Elastic-net can choose more than one.

On the other hand, there is another critical problem that is selecting the most influential factors since scientists want to know which predictors have non-zero influence and which ones do not. For each covariate, we hope to determine whether it is important or not at the same time, i.e. this is a multiple testing issue. When the dimensionality is relatively low, we can use the family-wise error rate (FWER) as a solution. For high-dimensional data sets, for example, among millions of genes, scientists want to study which of them are related to a specific disease. In this case, the family-wise criterion would be somewhat conservative. In contrast, the false discovery rate (FDR) proposed by [6] is a criterion more suitable for the large-scale multiple hypothesis testing problem. After doing some penalized estimation for regressions, FDR control methods can typically be applied via p-value thresholds, such as BH procedure ([6]). BH procedure was proved to be able to control FDR in independent or positive dependent cases, see [6, 7]. It is worthy to note that FDR control is based on regular
estimators which are asymptotic normally distributed. Fortunately, in the high dimensional setting, debiased estimators make the estimator as regular, see \cite{58, 35, 48, 45, 57}. These methods begin from the regularized estimator, use different techniques to construct the debiased estimator, and then draw statistical inferences on the asymptotic normality nature of it.

Except the FDR control, there is another novel tool dealing with multiple testing, i.e. \textit{e-value} purposed by \cite{51}. It has been shown that e-value often are more mathematically convenient than p-values and lead to simpler results, such as they are easier to combine. E-values have attracted extensive attention and been used since it was purposed, see \cite{18, 50} for instance. As a consequence, it is so valuable to study many statistical testing under e-value framework.

1.3. Motivation and Contribution. In \cite{56}, the generalized Lasso-type convex penalty (GLCP), which includes Lasso penalty, Elastic-net estimation and other convex penalties, was considered and an upper bound for the symmetric Bregman divergence between the true parameters and the estimated parameters was provided. Based on this result, we obtain the oracle inequality for MCMC-MLE under Elastic-net penalty and prove the $\ell_1$ consistency of the estimation after we introduce the compatibility factor, which is closely related to the compatibility condition \cite{49} or the restricted eigenvalue condition \cite{9}. We notice to obtain the $\ell_1$ consistency, we need further assume $\sqrt{\log(p)/m} = o(1)$ besides the conventional assumption $\sqrt{\log(p)/n} = o(1)$ without introducing Monte Carlo method.

Although a lot of effort has been put in developing the statistical inferential theory of the high dimensional statistical models, to the best of our knowledge, how to construct statistics for the test hypothesis of graphic models is still an open problem. The main challenge is the existence of high dimensional nuisance parameters, making the existing partial-likelihood-based inference infeasible.

In this paper, to overcome the high dimensional nuisance parameters, we construct a decorrelated score function by projecting the score function of the parameter of interest to the space generated by the nuisance parameters, which immediately implies a decorrelated score test statistic. In the high dimensional background, the classical profile partial score function for the low dimensional setting does not yield a tractable limiting distribution due to the existence of a large number of nuisance parameters. However, the new constructed decorrelated score function is asymptotically normal even in high dimensions, overcoming the problem by handling the effect of nuisance parameters. Based on the decorrelated score function, we propose an asymptotically unbiased one-step estimator inspired by Newton’s method for solving an equation. We prove the asymptotic normality of the one-step estimator and construct the confidence interval. And based on the one-step estimator, we also purpose algorithms to control the FDR under either p-value framework and e-value framework. We notice that our work agrees with the existing work for high dimensional inferential statistics where a stronger assumption $\frac{\log(p)}{\sqrt{n}} = o(1)$ is needed. We emphasize to get the desired asymptotic normality when the Monte Carlo method is introduced, we need to accelerate the convergence of Monte Carlo method and assume $\frac{m}{n} \asymp \log(p)$, which is a remarkable difference from existing work.

1.4. Outline. The paper is outlined as follows. In Section 2, we provide preliminaries on the graphical models, the MCMC-MLE method, the Elastic-net-penalized estimation in the high dimensional background and the concentration inequality for Markov chain. In Section 3, we obtain the oracle inequality and show the $\ell_1$ consistency of the Elastic-net-penalized MCMC-MLE. In Section 4, we construct the decorrelated score function and the corresponding test statistic and show the asymptotic normality. In Section 5, we propose an asymptotically unbiased one-step estimator, based on which we construct an confidence interval for a
single parameter of interest. Finally, we will purpose two different FDR controlling procedures via both p-values and e-values in section 6.

2. Preliminaries. For the simplicity of the following statement and proof, we need some notations. For a series of random variables \( X_n \) and a series of constants \( a_n \), we let \( X_n = a_n p(a_n) \) mean \( \frac{X_n}{a_n} \) converges to zero in probability as \( n \) approaches infinity and let \( X_n = O_p(a_n) \) mean the set of random variables \( \{ \frac{X_n}{a_n} \} \) is stochastically bounded. For two series of constants \( a_n \) and \( b_n \), we write \( a_n \leq b_n \) when there exists a constant \( c > 0 \) such that \( a_n \leq cb_n \). We define \( a_n \geq b_n \) in the same way. And we write \( a_n \asymp b_n \) when \( a_n \asymp b_n \) and \( a_n \asymp b_n \).

Let \( d \) mean convergence in distribution.

2.1. The Graphical Models as Exponential Families. A graph \( G = (V, E) \) is composed by a set of vertices \( V = \{1, 2, \cdots, m\} \) and a set of edges \( E \subset V \times V \). Each edge consists of a pair of different vertices \( s, t \in E \) and may either be directed or undirected. To define a probability distribution associated with the graph, we attach each vertex \( s \in V \) with a random variable \( X_s \) taking values in the state space \( \mathcal{X}_s \). In our work, we focus on the finite discrete state space \( \mathcal{X}_s = \{1, 2, \cdots, r\} \) and denote the Cartesian product of the state spaces \( \mathcal{X}_s (s \in V) \) where the random vector \( X = (X_s, s \in V) \) takes values by \( \prod_{s \in V} \mathcal{X}_s \). We use the lowercase letter \( x_s \) to denote the particular element in \( \mathcal{X}_s \). And \( x = (x_1, \cdots, x_m) \) means a particular value in the space \( \prod_{s \in V} \mathcal{X}_s \). For a subset \( A \) of the vertex set \( V \), define \( X_A := (X_s, s \in A) \) to be the sub-vector of the random vector and \( x_A := (x_s, s \in A) \) to be a particular element of the random sub-vector.

In our work, we focus on the undirected graphical models, where the probability distribution factorizes according to the function defined on the cliques of the graph. A clique \( C \) means a fully connected subset of the vertex set where \( (s, t) \in E \) for each \( s, t \in C \). For each clique \( C \), we define a compatibility function \( \psi_C : \prod_{s \in V} \mathcal{X}_s \rightarrow \mathbb{R}_+ \). Let \( C \) denote a set of cliques of the graph and the undirected graphical model, also known as the Markov random field (MRF), can be written as

\[
p(x) := \frac{1}{Z} \prod_{C \in C} \psi_C(x_C), \quad \text{where} \quad Z := \sum_{x \in \prod_{s \in V} \mathcal{X}_s} \prod_{C \in C} \psi_C(x_C).
\]

\( Z \) is a constant chosen to normalize the distribution.

We in this paper pay our attention to the undirected graphical models in exponential form. Specifically, the probability mass function of \( (\mathcal{X}_s, s \in V) \) is

\[
p(x|\theta) := \frac{1}{C(\theta)} e^{\theta^T \varphi(x)} \frac{1}{C(\theta)} \sum_{\theta = (\theta_1, \cdots, \theta_p)^T} \theta^T \varphi(x), \quad \text{where} \quad C(\theta) := \sum_{x \in \prod_{s \in V} \mathcal{X}_s} e^{\theta^T \varphi(x)}.
\]

\( \theta = (\theta_1, \cdots, \theta_p)^T \) is the unknown parameters of interest. For simplicity, we replace \( \prod_{s \in V} \mathcal{X}_s \) by \( \mathcal{X} \). According to the property of exponential families, we can use \( \nabla \log C(\theta) \) and \( \nabla^2 \log C(\theta) \) to represent the expectation and covariance of random vector \( \varphi(X) \)

\[
E_\theta(\varphi(X)) = \nabla \log(C(\theta)) \quad \text{and} \quad \text{Cov}_\theta(\varphi(X)) := E_\theta(\varphi(X) - E_\theta(\varphi(X))) \otimes 2 = \nabla^2 \log(C(\theta)),
\]

where for a column vector \( r \), \( r \otimes 2 = rr^T \). We denote \( \text{Cov}_\theta(\varphi(X)) \) by \( \Sigma(\theta) \) in the following.
2.2. Markov Chain Monte Carlo Maximum Likelihood Estimation. Suppose \( x_1, \ldots, x_n \) are independent and identically distributed random variables according to the same probability distribution \( p(x|\theta^*) \), where \( \theta^* \) is the true parameter we need to learn from the observed values \( x_1, \ldots, x_n \). We construct the negative log-likelihood function

\[
\mathcal{L}_n(\theta; x_1, \ldots, x_n) = -\frac{1}{n} \sum_{i=1}^{n} \log(p(x_i|\theta)) = -\frac{1}{n} \sum_{i=1}^{n} \theta^T \varphi(x_i) + \log(C(\theta)).
\]

Computing \( C(\theta) \) directly is computationally infeasible and instead we use the MCMC method to approximate it. Suppose \( Y_1, \ldots, Y_m \) is a Markov chain whose stationary distribution on the product space \( \mathcal{X} \) is \( h(y) \). Using the ergodic property of the Markov chain, we can approximate \( C(\theta) \) by a path integral

\[
C(\theta) = \mathbb{E}_{Y \sim h(y)} \frac{e^{\theta^T \varphi(y)}}{h(y)} \approx \frac{1}{m} \sum_{i=1}^{m} e^{\theta^T \varphi(Y_i)}.
\]

Thus we use the expectation in (3) to replace the \( C(\theta) \) in (2) and obtain the new negative log-likelihood function, which we denote by \( \mathcal{L}_n^m \)

\[
\mathcal{L}_n^m(\theta; x_1, \ldots, x_n) = -\frac{1}{n} \sum_{i=1}^{n} \theta^T \varphi(x_i) + \log \left( \frac{1}{m} \sum_{i=1}^{m} \frac{e^{\theta^T \varphi(Y_i)}}{h(Y_i)} \right).
\]

In the low dimensional setting, the Markov chain Monte Carlo maximum likelihood estimation \( \hat{\theta}_n^m \) is the parameter which minimizes the surrogate negative log-likelihood function \( \mathcal{L}_n^m(\theta) \)

\[
\hat{\theta}_n^m = \arg\min_{\theta} \mathcal{L}_n^m(\theta).
\]

In many cases, we use the MCMC-MLE to construct the confidence intervals and the hypothesis test statistics.

2.3. The Elastic-net Penalty in the High Dimensional Background. In the high dimensional cases where the number of parameters is larger than the number of samplings (\( p \gg n \)), the minimum of \( \mathcal{L}_n^m \) is not unique, thus the direct MCMC-MLE is ill-posed. To overcome this problem, we add an extra penalty to the likelihood function. In this article, we use the Elastic-net penalty, which inherits the advantages of both the Lasso method and the ridge regression. And the Elastic-net-penalized MCMC-MLE can be written as

\[
\hat{\theta}_n^m(\lambda_1, \lambda_2) = \arg\min_{\theta} \mathcal{L}_n^m(\theta) + \lambda_1 \|\theta\|_1 + \lambda_2 \|\theta\|_2^2,
\]

where \( \lambda_1, \lambda_2 \) are two tuning parameters, and \( \|\theta\|_1 := \sum_{i=1}^{p} \|\theta_i\| \) and \( \|\theta\|_2 := (\sum_{i=1}^{p} \theta_i^2)^{\frac{1}{2}} \) are \( \ell_1 \)-norm and \( \ell_2 \)-norm of vector \( \theta \) respectively. In the following, we replace \( \hat{\theta}_n^m(\lambda_1, \lambda_2) \) by \( \hat{\theta}_n^m \).

To study the Elastic-net penalty, we consider the generalized Lasso-type convex penalty (GLCP) added to the likelihood function

\[
\hat{\theta}(\lambda_1, \lambda_2) = \arg\min_{\theta} \mathcal{L}(\theta) + \lambda_1 \|\theta\| + \lambda_2 g(\theta),
\]

where \( g(\theta) \) is a nonnegative convex function with \( g(0) = 0 \). It obviously contains the situation for the Elastic-net penalty when we set \( g(\theta) = \|\theta\|_2^2 \). To measure the distance between two vectors, we define the symmetric Bregman divergence

\[
D_\theta^\phi(\hat{\theta}, \theta) := (\hat{\theta} - \theta)^T (\nabla \mathcal{L}(\hat{\theta}) - \nabla \mathcal{L}(\theta) + \lambda_2 (\nabla g(\hat{\theta}) - \nabla g(\theta))).
\]
If $g(\theta) = 0$, we denote the symmetric Bregman divergence by $D^s(\hat{\theta}, \theta) := (\hat{\theta} - \theta)^T(\nabla \mathcal{L}(\hat{\theta}) - \nabla \mathcal{L}(\theta))$, which is a symmetric extension of Kullback-Leibler divergence. Because $g(\theta)$ is a nonnegative convex function, we have $D^s(\hat{\theta}, \theta) \geq D^s(\hat{\theta}, \theta)$. Readers can refer to [40, 56, 60] for more discussion and applications about the symmetric Bregman divergence.

For convenience, we need more notations. Let $z^* := \|\nabla \mathcal{L}(\theta^*) + \lambda_2 \nabla g(\theta^*)\|_\infty$, where $\theta^*$ is the true parameter and $\|\theta\|_\infty := \max_{1 \leq i \leq p} |\theta_i|$ is the $\ell_\infty$-norm of the vector $\theta$, and $\hat{\theta} = \hat{\theta} - \theta^*$, the distance between the likelihood estimation and the true parameter. For the true parameter, let $S_1 = \{i : \theta^*_i \neq 0, 1 \leq i \leq q\}$ be the nonzero coordinates of $\theta^*$ and $T^c$ be the completion of $T$. Let $s := |S_1|$ be the number of nonzero coordinates in $\theta^*$. In [56], it provided an upper bound for the symmetric Bregman divergence using $z^*$ and $\hat{\theta}$:

**Lemma 2.1.** For GLCP maximum likelihood estimation, we have

$$D^s(\hat{\theta}, \theta^*) \leq (\lambda_1 + z^*)\|\hat{\theta}_T\|_1 - (\lambda_1 - z^*)\|\hat{\theta}_{T^c}\|_1.$$

### 2.4. The Concentration Inequality for Markov Chain

Concentration inequalities can derive the explicit non-asymptotic error bound for a summation of a series of random variables, which has become the fundamental tool in theories of high dimensional statistical inference. Concentration inequalities show how closely a summation of random variables distributes around its expectation. For example, it is well-known that concentration inequalities of sub-Gaussian distribution are used to derive oracle inequality of linear models in [9] and to perform testing for high dimensional regression models in [41].

In the case of MCMC-MLE, we use the average of a trajectory of a Markov chain to approximate the normalizing constant $C(\theta)$. Thus we need a concentration inequality of Markov chain to describe how the average distributes around $C(\theta)$. Before we introduce the desired inequality, we need more preparation. As what we construct in the MCMC method, we consider a Markov chain $\{Y_i, i \geq 1\}$ on the state space $\mathcal{X}$ with the transition kernel $P(y, x)$, the initial distribution $q(y)$ and the stationary distribution $h(y)$. We assume the Markov chain is irreducible and aperiodic, which is satisfied in the implementation of MCMC method. Therefore, the Markov chain has an unique stationary distribution and is ergodic from any initial distribution. We define two crucial quantities

$$M_1 := E_{Y \sim h}\left(\frac{p(Y|\theta^*)}{h(Y)}\right)^2 = \frac{1}{C^2(\theta^*)} E_{Y \sim h}\left(\frac{e^{(\theta^*)^T \phi(Y)}}{h(Y)}\right)^2 \quad \text{and} \quad M_2 := \max_{y \in \mathcal{X}} \frac{p(y|\theta^*)}{h(y)},$$

where $M_1$ and $M_2$ describe the difference between the stationary distribution $h(y)$ and the true distribution $p(y|\theta^*)$.

For the stationary distribution $h(y)$, let $L^2(h)$ denote the Hilbert space with inner product $(f, g) = \sum_{y \in \mathcal{X}} f(y) g(y) h(y)$. Define a linear operator $P$ related to the transition kernel $P(y, x)$ by

$$(P f)(y) := \sum_{x \in \mathcal{X}} f(x) P(y, x).$$

We define the Markov chain has a spectral gap $1 - \kappa$ when

$$\kappa = \sup\{|\rho| : \rho \in \text{Spec}(P)\},$$

where $\text{Spec}(P)$ denotes the spectrum of linear operator $P$. Define

$$\beta_1 := \frac{1 + \kappa}{1 - \kappa} \quad \text{and} \quad \beta_2 := \begin{cases} 1, & \text{if } \kappa = 0 \\ \frac{\kappa}{1 - \kappa}, & \text{if } \kappa \in (0, 1) \end{cases}.$$
which describe how fast the average of a trajectory convergences to the expectation. Now we can state the concentration inequality for Markov chains, which we will frequently use in the proof of this article.

**Lemma 2.2 (Theorem 1 in [36]).** Let \( Y_1, \cdots, Y_m \) be a stationary Markov chain with invariant distribution \( h(y) \) and non-zero absolute spectral gap \( 1 - \kappa > 0 \). Suppose \( g_i : X \rightarrow [-M, +M] \) is a sequence of functions with \( E_{Y \sim h} g_i(Y) = 0 \). Let \( \sigma^2 = \frac{1}{m} \sum_{i=1}^{m} E_{Y \sim h} g_i^2(Y) \). Then for any \( t > 0 \), we have

\[
P \left( \frac{1}{m} \sum_{i=1}^{m} g_i(Y_i) \geq t \right) \leq \exp \left( - \frac{mt^2}{2(\beta_1 \sigma^2 + \beta_2 Mt)} \right),
\]

where \( \beta_1 \) and \( \beta_2 \) are defined in (6).

2.5. More Notations. For the simplicity when we analysis the likelihood \( L_n^m \), let

\[
w_i(\theta) := \frac{e^{\theta^T \varphi(Y_i)}}{h(Y_i)} \quad \text{and} \quad \bar{\varphi}(\theta) := \frac{\sum_{i=1}^{m} w_i(\theta) \varphi(Y_i)}{\sum_{i=1}^{m} w_i(\theta)}.
\]

Then the gradient \( \nabla L_n^m \) and the Hessian \( \nabla^2 L_n^m \) can be reformulated to

\[
\nabla L_n^m(\theta) = -\frac{1}{n} \sum_{i=1}^{n} \varphi(X_i) + \frac{\sum_{i=1}^{m} w_i(\theta) \varphi(Y_i)}{\sum_{i=1}^{m} w_i(\theta)} = -\frac{1}{n} \sum_{i=1}^{n} \varphi(X_i) + \bar{\varphi}(\theta)
\]

and

\[
\nabla^2 L_n^m(\theta) = \frac{\sum_{i=1}^{m} w_i(\theta) \varphi(Y_i) \varphi(Y_i)}{\sum_{i=1}^{m} w_i(\theta)} - \left( \frac{\sum_{i=1}^{m} w_i(\theta) \varphi(Y_i)}{\sum_{i=1}^{m} w_i(\theta)} \right)^2 \frac{\sum_{i=1}^{m} w_i(\theta) (\varphi(Y_i) - \bar{\varphi}(\theta))^2}{\sum_{i=1}^{m} w_i(\theta)}.
\]

From (8), we can see \( \nabla^2 L_n^m(\theta) \) is convex.

To deal with hypotheses testing for elements in \( \theta^* \), we need notations to represent the partitions of vectors \( \theta, \nabla L_n^m \) and matrix \( \nabla^2 L_n^m \). Let \( \theta = (\alpha, \beta^T) \), where \( \alpha \) is the first scalar of \( \theta \). We define

\[
\nabla L_n^m = (\nabla_\alpha L_n^m, (\nabla_\beta L_n^m)^T)^T \quad \text{and} \quad \nabla^2 L_n^m = \left( \nabla^2_{\alpha \alpha} L_n^m, \nabla^2_{\alpha \beta} L_n^m, \nabla^2_{\beta \beta} L_n^m \right)^T.
\]

Let \( H^* := \text{Cov}_\theta^*(\varphi(X)) \) and

\[
H^* = \left( H_{\alpha \alpha}^*, H_{\alpha \beta}^*, H_{\beta \beta}^* \right).
\]

3. \( \ell_1 \) Consistency via Compatibility Factor. In this section, we state the oracle inequality and show the \( \ell_1 \) consistency of the MCMC-MLE penalized by Elastic-net under some regularity conditions.

Define the cone set

\[
C(\zeta, T) := \{ \theta : \|\theta^*\|_1 \leq \zeta \|\theta_T\|_1 \}.
\]
If there exists a constant $\zeta > 1$ such that $z^* \leq \frac{\zeta - 1}{\zeta + 1} \lambda_1$. Therefore, $\lambda_1 - z^* \geq \frac{2}{\zeta + 1} \lambda_1$ and $\lambda_1 + z^* \leq \frac{2\zeta}{\zeta + 1} \lambda_1$. We have

$$
\frac{2}{\zeta + 1} \lambda_1 \|\hat{\theta}_T\|_1 \leq D^*_g(\hat{\theta}, \theta^*) + \frac{2}{\zeta + 1} \lambda_1 \|\hat{\theta}_T\|_1 \leq \frac{2\zeta}{\zeta + 1} \lambda_1 \|\hat{\theta}\|_1,
$$

which means $\hat{\theta}$ belongs to the cone $C(\zeta, T)$ on the event $\Omega_1 := \{z^* \leq \frac{\zeta - 1}{\zeta + 1} \lambda_1\}$.

Another very important value for the deriving of $\ell_1$ consistency is the compatibility factor, which is defined as

$$
F(\zeta, T, \Sigma) := \inf_{0 \neq \theta \in C(\zeta, T)} \frac{s(\theta^T \Sigma \theta)}{\|\theta\|_1^2},
$$

where $\Sigma$ is a $p \times p$ nonnegative-definite matrix.

To show the desired consistency, we need some regularity conditions:

**Assumption 1.** $\varphi$ is bounded: $\|\varphi\|_\infty \leq K$ for some $K > 0$. In particular, we can deduce the $\ell_\infty$-norm of covariance matrix $H^*$ is bounded: $\|H^*\|_\infty \leq C_{uni}$ for a constant $C_{uni} > 0$.

**Assumption 2.** $F(\zeta, T, H^*)$ have an uniform positive lower bound: $F(\zeta, T, H^*) \geq C_{\min}$ for some $C_{\min} > 0$.

**Assumption 3.** The $\ell_\infty$-norm of $\theta^*$ is bounded: $\|\theta^*\|_\infty \leq B$ for some $B > 0$.

For simplicity, we define

$$
F(\zeta, T) := F(\zeta, T, \nabla^2 L_n^m(\theta^*)) \quad \text{and} \quad F(\zeta, T) := F(\zeta, T, H^*).
$$

Now we present an important lemma which controls $||\hat{\theta} - \theta^*||$ by $F(\zeta, T)$.

**Lemma 3.1 (Appendix A.1).** Suppose Assumption 1 is satisfied. Define the event $\mathcal{P} = \{(\zeta + 1) s \lambda_1 \leq \frac{1}{2F(\zeta, T)}\}$. On the event $\mathcal{P} \cap \Omega_1$, we have

$$
||\hat{\theta} - \theta^*|| \leq \frac{c(\zeta + 1) s \lambda_1}{2F(\zeta, T)}.
$$

Before we provide the main theorem about consistency, we need to prove two concentration inequalities of $\nabla L_n^m(\theta^*)$ and $\nabla^2 L_n^m(\theta^*)$, which show $\nabla L_n^m(\theta^*)$ and $\nabla^2 L_n^m(\theta^*)$ respectively concentrate around 0 and $H^*$.

**Lemma 3.2 (Appendix A.2).** Under the Assumption 1, we have

$$
P(\|\nabla L_n^m(\theta^*)\|_\infty \leq t) \geq 1 - 2pe^{-\frac{m^2t^2}{2n}} - 2pe^{-\frac{m^2t^2}{4(n_1 M_1 + n_2 M_2)}} - e^{-\frac{m^2t^2}{4(n_1 M_1 + n_2 M_2)}}.
$$

**Lemma 3.3 (Appendix A.3).** Under the Assumption 1, we have

$$
P(\|\nabla^2 L_n^m(\theta^*) - H^*\|_\infty \leq t) \geq 1 - 2e^{-\frac{m^2t^2}{4(n_1 M_1 + n_2 M_2)}} - 2p^2e^{-\frac{m^2t^2}{16K^2(8K\beta_1^2 M_1 + 1)M_2}} - 2pe^{-\frac{m^2t^2}{64K^2(8K\beta_1^2 M_1 + 1)M_2}}.
$$

**Remark 3.1.** From the above two lemma, we can see $\|\nabla L_n^m(\theta^*)\|_\infty$ and $\|\nabla^2 L_n^m(\theta^*) - H^*\|_\infty$ are stochastically bounded,

$$
\|\nabla L_n^m(\theta^*)\|_\infty = O_p\left(\frac{\log(p)}{n} + \sqrt{\frac{\log(p)}{n}} \right) \quad \text{and} \quad \|\nabla^2 L_n^m(\theta^*) - H^*\|_\infty = O_p\left(\frac{\log(p)}{m}\right).
$$
Another tricky obstacle is the right hand of (12) contains a random variable \( F(\zeta, T) \). We want to use the deterministic \( F(\zeta, T) \) to approximate the stochastic \( F(\zeta, T) \), and thus we have the following lemma.

**Lemma 3.4** (Appendix A.4). With \( F(\zeta, T) \) and \( F(\zeta, T) \) defined in (11), we have
\[
F(\zeta, T) \geq F(\zeta, T) - s(\zeta + 1)^2 \| \nabla^2 \mathcal{L}_m - H^* \|_\infty.
\]

Now we state the main result of this section.

**Theorem 3.1** (Appendix A.5). Assume the Assumptions 1, 2 and 3 are satisfied. Assume there exists a constant \( \zeta > 1 \) such that \( \frac{\zeta - 1}{\zeta + 1} \lambda_1 - 2\lambda_2 B > 0 \) and \( \frac{(\zeta + 1)s\lambda_1}{2(\zeta + 1)^2} \leq \frac{1}{4K} \). Define \( \tau_1 := \frac{\zeta - 1}{\zeta + 1} \lambda_1 - 2\lambda_2 B \) and \( \tau_2 := \frac{C_{\min}}{2s(\zeta + 1)^2} \). Then we have
\[
\| \hat{\theta} - \theta^* \|_1 \leq \frac{e(\zeta + 1)s\lambda_1}{C_{\min}},
\]
with probability at least
\[
1 - \delta_1 - \delta_2,
\]
where we define
\[
\delta_1 = 2pe^{-\frac{m\tau_1^2}{8K\tau_2}} + 2pe^{-\frac{m\tau_1^2}{16K(8K\beta \lambda_1 M_1 + \tau_1\beta_2 M_2)}} + e^{-\frac{4(2(\zeta + 1)^2\beta_1 M_1 + \beta_2 M_2)^2}{2(\zeta + 1)^2}} \quad \text{and}
\]
\[
\delta_2 = 2e^{-\frac{m\tau_1^2}{4(2(\zeta + 1)^2\beta_1 M_1 + \beta_2 M_2)}} + 2pe^{-\frac{m\tau_1^2}{16K^2(8K\beta \lambda_1 M_1 + \tau_2\beta_2 M_2)}} + 2pe^{-\frac{m\tau_1^2}{64K^3(8K\beta \lambda_1 M_1 + \tau_2\beta_2 M_2)}}.
\]

Notice when we choose \( \lambda_1 = \frac{1}{4B(\zeta + 1)} \lambda_1 \), we have \( \tau_1 = \frac{\zeta - 1}{2(\zeta + 1)} \lambda_1 \). Fix a constant \( r_1 > 1 \) and set
\[
\lambda_1 > \frac{4K(\zeta + 1)}{\zeta - 1} \sqrt{\frac{2r_1 \log(p)}{n}},
\]
we have \( e^{\frac{m\tau_1^2}{8K\tau_2}} > e^{r_1} \), from which we have \( 2pe^{-\frac{m\tau_1^2}{8K\tau_2}} \to 0 \). And when we choose
\[
\frac{32K(\zeta + 1)}{\zeta - 1} \sqrt{\frac{\beta_1 M_1 r_1 \log(p)}{m}} < \lambda_1 < \frac{16K(\zeta + 1)}{\zeta - 1} \beta_2 M_2
\]
we have \( 8K\beta \lambda_1 M_1 > \tau_1\beta_2 M_2 \) and therefore,
\[
e^{-\frac{m\tau_1^2}{16K(8K\beta \lambda_1 M_1 + \tau_1\beta_2 M_2)}} > e^{-\frac{m\tau_1^2}{256K^2\beta \lambda_1 M_1}} > e^{-\frac{m\tau_1^2}{64K^3(8K\beta \lambda_1 M_1 + \tau_2\beta_2 M_2)}} \to 0.
\]

Therefore other components of \( \delta_1 \) and \( \delta_2 \) converge to 0. Noticing \( \| \hat{\theta} - \theta^* \|_1 \leq s\lambda_1 \) from the Theorem 3.1, we need \( \lambda_1 = o(1) \) to get the \( \ell_1 \) consistency, which means \( \sqrt{\frac{\log(p)}{n}} = o(1) \) and \( \sqrt{\frac{\log(p)}{m}} = o(1) \) according to (14) and (15). And under the condition \( \sqrt{\frac{\log(p)}{m}} = o(1) \), (16) is automatically satisfied, from which we have \( \delta_1, \delta_2 \to 0 \). From which we have the following important corollary on the \( \ell_1 \) consistency of the penalized MCMC-MLE.
Assume the Assumptions 1, 2 and 3 are satisfied. Then $\|\hat{\theta} - \theta^*\|_1$ is stochastically bounded with respect to $s(\sqrt{\log(p) n} + \sqrt{\log(p) m})$,

$$
\|\hat{\theta} - \theta^*\|_1 = O_p\left(s\left(\sqrt{\log(p) n} + \sqrt{\log(p) m}\right)\right),
$$

moreover we have the asymptotic consistency $\|\hat{\theta} - \theta^*\|_1 = o_p(1)$.

Remark 3.2. From the Corollary 3.1 and our assumption $\lambda_1 \sim \lambda_2 = o(1)$, we can see $\|\hat{\theta}\|_1 = o_p(1)$. We notice besides the conventional assumption $\sqrt{\log(p) n} = o(1)$ and the sparsity of $\theta^*$ for the consistency of high dimensional estimators, we need to further assume $\sqrt{\log(p) m} = o(1)$ to ensure a necessary convergence rate of MCMC method to obtain the consistency when Monte Carlo method is introduced.

4. Decorrelated Score Test. In this section, we propose a decorrelated score test for the testing of hypothesis $H_0 : \alpha^* = \alpha_0$ versus $H_1 : \alpha^* \neq \alpha_0$, which can be seen as the analogue of the conventional score test in the low dimensional background. The direct extension of profile partial score test is infeasible because the limiting distribution is intractable because of the large number of nuisance parameters. To overcome the difficulty, we propose a decorrelated method to eliminate the influence of nuisance parameters inspired by a projection method.

To test the hypothesis $H_0 : \alpha^* = \alpha_0$, we first estimate the true nuisance parameters $\beta$ by $\hat{\beta}$ using the Elastic-net-penalized MCMC-MLE defined in (4). Next, we use the combination of partial score function $\nabla_\beta L_n(\theta^*)$ to approximate $\nabla_\alpha L_n(\theta^*)$ in the sense of expectation

$$
w^* = \arg\min_{w \in \mathbb{R}^{p-1}} E(\nabla_\alpha L_n(\theta^*) - w^T \nabla_\beta L_n(\theta^*))^2
$$

$$
= (H^*_{\beta\beta})^{-1} H^*_{\alpha\beta},
$$

where $w^T \nabla_\beta L_n(\theta^*)$ can be viewed as the projection of $\nabla_\alpha L_n(\theta^*)$ onto the linear space spanned by the elements of $\nabla_\beta L_n(\theta^*)$. In the high dimensional background, the direct computation of $w^*$ by the sample version is ill posed, thus we use a lasso-type estimator $\hat{w}$ defined below to estimate $w^*$

$$
\hat{w} = \arg\min_{w \in \mathbb{R}^{p-1}} \frac{1}{2} w^T \nabla^2_{\beta\beta} L_n(\hat{\theta}) w - w^T \nabla^2_{\alpha\beta} L_n(\hat{\theta}) w + \lambda^* \|w\|_1,
$$

where $\lambda^*$ is a tuning parameter. Then we propose a decorrelated score function

$$
\hat{U}(\alpha, \hat{\beta}) := \nabla_\alpha L_n^m(\alpha, \hat{\beta}) - \hat{w}^T \nabla_\beta L_n^m(\alpha, \hat{\beta}).
$$

To derive the limiting distribution of the decorrelated score function under the null hypothesis $\alpha^* = \alpha_0$, we need the following assumptions.

Assumption 4. The eigenvalues of covariance matrix $H^*$ are bounded from both sides: $\lambda_{\min} \leq \lambda_{\min}(H^*) \leq \lambda_{\max}(H^*) \leq \lambda_{\max}$ for some $0 < \lambda_{\min} < \lambda_{\max}$.

Assumption 5. The $\ell_\infty$-norm of $w^*$ is bounded: $\|w^*\|_\infty \leq D$ for some $D > 0$.

To derive the asymptotic property of the decorrelated score function, we need some crucial lemmas. The Lemma 4.1 shows the asymptotic normality of $\nabla L_n^m$. 
LEMMA 4.1 (Appendix B.1). Assume $\frac{n}{m} \asymp \log(p)$. Assume the Assumptions 1 and 4 are satisfied. Then for vector $v \in \mathbb{R}^p$ with $\|v\|_0 = s' = O(1)$, we have

$$\frac{\sqrt{n}v^T \nabla L_n^m(\theta^*)}{\sqrt{v^T H^* v}} \xrightarrow{d} N(0, 1).$$

REMARK 4.1. We notice from the Lemma 4.1, to ensure the asymptotic normality of $\nabla L_n^m(\theta^*)$, we need to further assume $\frac{n}{m} \asymp \log(p)$, which means we need more samples to accelerate the convergence of MCMC method.

Another lemma below shows the consistency of the estimator $\hat{w}$, which is essential in the later proof.

LEMMA 4.2 (Appendix B.2). Assume the Assumptions 1, 2, 3, 4 and 5 are satisfied. Suppose $\lambda_1 \asymp (\sqrt{\frac{\log(p)}{m}} + \sqrt{\frac{\log(p)}{m}})$, $\lambda_1 \asymp \lambda_2 \asymp \lambda' = o(1)$ and $s = s' = O(1)$. We have

$$\|\hat{w} - w^*\|_1 = O_p\left((s + s')(\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}})\right),$$

where $s' := \|w^*\|_0$.

Notice the sparsity of $w^*$ and true parameter $\theta^*$ is necessary for the asymptotic normality of $\nabla L_n^m(\theta^*)$ and the consistency of $\hat{w}$. Besides, we need $\lambda' \asymp \lambda_1$ to ensure the consistency of $\hat{w}$.

Now we state the main result of this section, which shows the asymptotic normality of the decorrelated score function $\hat{U}(\alpha, \hat{\beta})$ under the null hypothesis.

THEOREM 4.1 (Appendix B.4). Assume the Assumptions 1, 2, 3, 4 and 5 are satisfied. Suppose $\lambda_1 \asymp (\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}})$, $\lambda_1 \asymp \lambda_2 \asymp \lambda' = o(1)$ and $s = s' = O(1)$. For the relations among $m$, $n$ and $p$, we further assume $(\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}}) = o(1)$ and $\frac{m}{n} \asymp \log(p)$. Then under the null hypothesis $\alpha^* = \alpha_0$, the decorrelated score test $\hat{U}(\alpha, \hat{\beta})$ defined in (17) satisfies

$$\sqrt{n}\hat{U}(\alpha_0, \hat{\beta}) \xrightarrow{d} N(0, H^*_{\alpha|\beta}),$$

where the variance $H^*_{\alpha|\beta}$ of the limiting distribution is defined by

$$H^*_{\alpha|\beta} := H_{\alpha\alpha} - w^*H^*_{\alpha|\beta} = H_{\alpha\alpha} - (H_{\alpha\beta}^*)^T H_{\beta\beta}^{-1} H_{\alpha\beta}^*.$$

REMARK 4.2. From the Theorem 4.1, we notice to achieve the asymptotic normality of the decorrelated score function, we should further assume $(\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}}) = o(1)$, which is stronger than the assumption $(\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}}) = o(1)$ needed for the consistency of $\|\hat{\theta}\|_1$. And the stronger assumption here is consistent with the existing work for the proportional hazards model [22] and more general statistical models [41].

We estimate the variance of the limiting normal distribution by $\hat{H}_{\alpha|\beta}$ defined below

$$\hat{H}_{\alpha|\beta} := \nabla^2_{\alpha\alpha} L_n^m(\hat{\theta}) - \hat{w}^T \nabla^2_{\alpha\beta} L_n^m(\hat{\theta}).$$

Next lemma shows the consistency of $\hat{H}_{\alpha|\beta}$. 
LEMMA 4.3 (Appendix B.5). Assume the Assumptions 1, 2, 3, 4 and 5 are satisfied. Suppose \( \lambda_1 \asymp (\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}}) \), \( \lambda_1 \asymp \lambda_2 \asymp \lambda' = o(1) \) and \( s = s' = O(1) \). We have
\[
|\hat{H}_{\alpha|\beta} - H^*_{\alpha|\beta}| = O_p \left( (s + s') \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \right).
\]

Therefore, we define the decorrelated score test statistic by
\[
\hat{S}_n := \sqrt{n \hat{H}_{\alpha|\beta}} \hat{U}(\alpha_0, \hat{\beta}).
\]

Because of the Theorem 4.1 and the Lemma 4.3, we can see the limiting distribution of \( \hat{S}_n \) is standard normal distribution
\[
\hat{S}_n \xrightarrow{d} N(0, 1).
\]

Thus given a significance level \( \eta \in (0, 1) \), the decorrelated score test \( \psi(\eta) \) is defined by
\[
\psi(\eta) = \begin{cases} 
0 & \text{if } |\hat{S}_n| \leq \Psi^{-1}(1 - \frac{\eta}{2}), \\
1 & \text{others,}
\end{cases}
\]

where \( \Psi^{-1} \) is the inverse of the cumulative distribution function of the standard normal distribution \( \Psi \) and \( \Psi^{-1}(1 - \frac{\eta}{2}) \) is also the \( (1 - \frac{\eta}{2}) \)th quantile of the standard normal distribution. According to the construction of \( \psi(\eta) \), we should reject the null hypothesis \( \alpha^* = \alpha_0 \) if and only if \( \psi(\eta) = 1 \) and the type I error of the decorrelated score test \( \psi(\eta) \) asymptotically converges to \( \eta \).

5. Confidence Interval via One-step Estimator. In this section, we propose a confidence interval for the true parameter \( \alpha^* \), overcoming the shortcoming that the decorrelated score function in last section does not directly imply a confidence interval for \( \alpha^* \).

The key idea to derive a confidence interval for \( \alpha^* \) is based on the decorrelated score function \( \tilde{U}(\alpha, \tilde{\beta}) \). From the Theorem 4.1, the root of the equation \( \tilde{U}(\alpha, \tilde{\beta}) = 0 \) serves as a good approximation to \( \alpha^* \). However, direct computation to solve the equation is computationally infeasible. An alternative way we apply here is to linearize \( \tilde{U}(\alpha, \tilde{\beta}) \) at the penalized estimator \( \hat{\alpha} \) and use the following one-step estimator \( \tilde{\alpha} \)

\[
\tilde{\alpha} := \hat{\alpha} - \left( \frac{\partial \tilde{U}(\hat{\alpha}, \tilde{\beta})}{\partial \alpha} \right)^{-1} \tilde{U}(\hat{\alpha}, \tilde{\beta}),
\]

where \( \tilde{\theta} = (\hat{\alpha}, \hat{\beta}^T)^T \) is the Elastic-net-penalized MCMC-MLE in (4) and \( \tilde{U}(\alpha, \tilde{\beta}) \) is the decorrelated score test defined in (17). Notice the construction of \( \tilde{\alpha} \) is similar to the one step iteration at the initial point \( \alpha = \hat{\alpha} \) using Newton’s method to solve the equation \( \tilde{U}(\alpha, \tilde{\beta}) = 0 \).

For the asymptotic normality of \( \tilde{\alpha} \), we have the following theorem.

THEOREM 5.1 (Appendix C). Assume the Assumptions 1, 2, 3, 4 and 5 are satisfied. Suppose \( \lambda_1 \asymp (\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}}) \), \( \lambda_1 \asymp \lambda_2 \asymp \lambda' = o(1) \) and \( s = s' = O(1) \). For the relations among \( m, n \) and \( p \), we further assume \( (\frac{\log(p)}{n} + \frac{\log(p)}{m}) = o(1) \) and \( \frac{m}{n} \asymp \log(p) \). Then the decorrelated one-step estimator \( \tilde{\alpha} \) defined in (18) is asymptotic normality satisfying
\[
\sqrt{n}(\tilde{\alpha} - \alpha^*) \xrightarrow{d} N(0, H_{\alpha|\beta}^{-1}).
\]
From the Theorem 5.1 and the Lemma 4.3, we have

\[
\sqrt{n\hat{H}_{\alpha|\beta}}(\tilde{\alpha} - \alpha^*) \overset{d}{\to} N(0, 1),
\]

from which we can easily construct a 100(1 - \eta)\% confidence interval for \alpha^*

\[
\left[ \tilde{\alpha} - \Psi^{-1}(1 - \frac{\eta}{2}) \sqrt{n\hat{H}_{\alpha|\beta}} , \tilde{\alpha} + \Psi^{-1}(1 - \frac{\eta}{2}) \sqrt{n\hat{H}_{\alpha|\beta}} \right].
\]

6. False Discovery Rate Control via both p-values and e-values.

In this section, we propose a false discovery rate (FDR) control procedure with both p-values and e-values, which controls the FDR via slightly different ways. The p-value FDR control provides the classic results, which can be compared with the existent results, while FDR via e-values is able to control the FDR at level \epsilon under any unknown dependence between e-values.

6.1. General Setting for FDR Control. For \(k \in \mathcal{P} := \{1, 2, \cdots, p\}\), let the \(k\)-th hypothesis be \(H_k : \theta_k = \theta_k^*\), where \(\theta_k\) is the \(k\)-th scalar of the parameter \(\theta = (\theta_1, \cdots, \theta_p)^T\) in the model and \(\theta_k^*\) is the parameter of interest. Notice \(\theta_k^*\) may not be the true parameter of the model. Let \(\mathcal{N} \subset \mathcal{P}\) be the set of unknown true null hypotheses.

A FDR control procedure \(\mathcal{G}\) accepts or rejects each hypothesis \(H_k\) based on the observation \(e_k\). Define the set \(\mathcal{D} \subseteq \mathcal{P}\) to be the hypotheses rejected by the procedure \(\mathcal{G}\), which is called discoveries by the literature. Thus the set \(\mathcal{N} \cap \mathcal{D}\) contains the true null hypotheses rejected by the procedure \(\mathcal{G}\), which is called false discoveries. Define the false discovery proportion (FDP) to be the ratio of the number of false discoveries to discoveries \(\text{FDP}_\mathcal{G} = |\mathcal{N} \cap \mathcal{D}|/|\mathcal{D}|\), which is a crucial value reflecting the performance of FDR control procedure. Since the FDP\(_\mathcal{G}\) is a random variable, the expectation of FDP\(_\mathcal{G}\) under the distribution generating the data is called the false discovery rate (FDR) of the procedure \(\mathcal{G}\)

\[
\text{FDR}_\mathcal{G} = E(\text{FDP}_\mathcal{G}) = E\left[ \frac{|\mathcal{N} \cap \mathcal{D}|}{|\mathcal{D}|} \right].
\]

Designing an appropriate procedure to control the FDR is a central task in multiple hypothesis testing. We will provide two different FDR procedures based on traditional p-values and the novel e-values.

6.2. FDR Control via classic p-values. It is worthy to bear in mind that for classic p-values, the correlation between them makes we cannot simply summarize them to construct the statistics, extra technique to tackle the correlation are required.

Fortunately, in spirit of [19], we can use data-splitting and mirror statistic to control it. The mirror statistic is defined as follows

\[
M_j = \text{sgn} \left( T_j^{(1)} T_j^{(2)} \right) f(|T_j^{(1)}|, |T_j^{(2)}|),
\]

where \(f(u, v)\) is a non-negative, symmetric, and monotonically increasing in \(u\) and \(v\), and \(T_j^{(\ell)}\) is the normalized estimates for \(\theta_j\) in \((\ell)\)-th part of the whole data. Here we use

\[
T_j^{(\ell)} = \hat{\theta}_j \sqrt{n\hat{H}_{j|-j}/2}
\]

in which \(\hat{\theta}_j\) is the one-step estimator for \(\theta_j\) in Section 5. Then we can apply the following algorithm similar to [19] and then asymptotically control the FDR.
Algorithm 1 FDR control via single data split

1: For each \( j \), calculate the mirror statistic \( M_j \).
2: Given a designated level \( q \in (0, 1) \), the cutoff is
   \[
   \tau_q = \inf \left\{ t > 0 : \frac{\# \{ j : M_j < -t \}}{\# \{ j : M_j > t \}} \leq q \right\}.
   \]
3: Set \( \hat{S} = \{ j : M_j > \tau_q \} \).

We require the following extra assumptions to theoretically justify our method to control FDR.

**Assumption 6.** \( \{ j \in S_0 \} \) are exchangeable, that is for population version \( X \), we have \( X_j \overset{d}{=} X_k \) and \( X_j | X_{-j} \overset{d}{=} X_k | X_{-k} \).

**Assumption 7.** The number of null signal \( p_0 = |S_0| = |\{ 1 \leq j \leq p : \theta_j^* = 0 \}| \) satisfies
\[
p_0 = O\left( \sqrt{n \log p} \right).
\]

**Assumption 8.** The sparsity \( v^* \) on \( \Theta = H^{* \perp} \) satisfies
\[
v^* = \max_{j=1, \cdots, p} \left| \{ 1 \leq k \leq p, \Theta_{jk} \neq 0 \} \right| = o\left( \sqrt{\frac{n}{\log p}} \right).
\]

In contrast to the setting in previous sections, here we require stronger sparsity on the true coefficient vector \( \theta^* \) (\( p_0 \gg \log p \) and \( p_0 \geq \sqrt{n} / \log p \)) as well as extra conditions on the precision \( \Theta \). Assumption 6 can let the elastic net estimator enjoys a fast convergence rate, see [9], and Assumption 7 also appears in [34], [48], and [19], which is natural and gives a stronger structure for \( H^* \).

**Lemma 6.1 (Appendix D.1).** Suppose the condition is the same as that in Theorem 5.1, then for the elastic-net estimator \( \hat{\theta} \) in Section 2.3, if Assumption 6 and Assumption 7 holds, then we have
\[
\text{cov}(\hat{\theta}_j, \hat{\theta}_k) = O(n^{-1}), \quad j, k \in S_0.
\]

**Theorem 6.1 (Appendix D.2).** Assume the condition is the same as that in Theorem 5.1. Furthermore, we also suppose \( \hat{H}_{|j|} \) is uniformly integrable for any \( j \in S_0 \). Then if Assumption 6, 7, and 8 also holds, we have
\[
\limsup_{n,p \to \infty} \text{FDR}_G := \limsup_{n,p \to \infty} \text{E} \left[ \frac{\# \{ j : j \in S_0, j \in \hat{S}_{\tau_q} \}}{\# \{ j \in \hat{S}_{\tau_q} \}} \right] \leq q.
\]

Algorithm 1, i.e. single data split, can theoretically achieve a good performance. However, single data split may have some disadvantages, like potential power loss or non-stability, see [19]. We can apply the following multiple data splits similar to that in [19]. Denote
\[
I_j = \text{E} \left[ \frac{\mathbb{I}(j \in \hat{S})}{|\hat{S}|} \right], \quad \hat{I}_j = \frac{1}{m} \sum_{k=1}^{m} \frac{\mathbb{I}(j \in \hat{S}^{(k)})}{|\hat{S}^{(k)}|},
\]
where \( \hat{S}^{(k)} \) represents the selected features in \( k \)-th data split by using Algorithm 1. We call \( I_j \) as the inclusion rate, and assume \( I_j = 0 \) if \( |\hat{S}| = 0 \).
Algorithm 2 FDR control via multiple data split

1: Sort $I_j$ by $0 \leq I_{(1)} \leq I_{(2)} \leq \cdots \leq I_{(p)}$.
2: Find the largest $\ell$ such that $I_{(1)} + \cdots + I_{(\ell)} \leq q$.
3: Select the features $\hat{S}_{\text{mul}} = \{ j : I_j > I_{(\ell)} \}$.

Theorem 6.2 (Appendix D.3). Suppose the condition is the same as that in Theorem 6.1, then the $\hat{S}_{\text{mul}}$ selected by Algorithm 2 satisfies

$$\limsup_{n,p \to \infty} \text{FDP}_G := \limsup_{n,p \to \infty} \frac{\# \{ j : j \in S_0, j \in \hat{S}_{\text{mul}} \}}{\# \{ j : j \in \hat{S}_{\text{mul}} \}} \leq q.$$  

Theorem 6.2 guarantees Algorithm 2 can also control the FDR efficiently. Furthermore, since

$$\text{var} \left( \frac{\mathbb{1}(j \in \hat{S})}{|\hat{S}|} \right) \geq \text{var} \left( \mathbb{E} \left[ \frac{\mathbb{1}(j \in \hat{S})}{|\hat{S}|} \right] \bigg| \text{data} \right),$$

multiple data split is essentially a Rao-Blackwell improvement of single data split under our algorithms.

6.3. E-BH Procedure and Theoretical Guarantee. In this subsection, we will provide a procedure controlling the FDR based on the e-BH procedure in [50, 51] and then analyze the asymptotic performance of the procedure we propose. As said in the Introduction, e-values share additivity regardless of correlation between covariates. This property will allow our FDR controlling procedure via e-values to be much easier and more direct than the FDR procedure via p-values.

For $k \in \mathcal{P}$, let $\theta_{-k}$ be the parameter $\theta$ excluding the $k$-th component $\theta_k$

$$\theta_{-k} := (\theta_1, \cdots, \theta_{k-1}, \theta_{k+1}, \cdots, \theta_p)^T.$$  

The FDR control procedure is given as follows.

Algorithm 3 FDR control via e-values

1: For each $k \in \mathcal{P}$, compute the e-values $e_k$ corresponding to each hypothesis $H_k$ by

$$e_k = \sqrt{\frac{\pi}{2}} \cdot \sqrt{n H_{\theta_k | \theta_{-k}}} |\hat{\theta}_k - \theta_k^*|.$$  

2: For each $k \in \mathcal{P}$, let $e_{(k)}$ be the $k$-th order statistics of $e_1, \cdots, e_p$ from the largest to the smallest.

3: Define $k^*$ to be

$$k^* := \max \left\{ k \in \mathcal{K} : \frac{ke_{(k)}}{p} \geq \frac{1}{\epsilon} \bigg\{ \max \{ \emptyset \} = 0 \right\},$$

reject the largest $k^*$ e-values (hypotheses).

When the hypothesis $H_k$ is true, by (19) we see

$$\sqrt{\frac{2}{\pi}} e_k \overset{d}{\to} N(0,1),$$

which means we are more likely to reject the hypothesis when $e_k$ is large. The coefficient $\sqrt{\pi/2}$ in Step 2 in the algorithm above is used to normalize the expectation value of e-values, which we will see more clearly in the following proof.
When analyze the asymptotic performance of the FDR control procedure, we take e-values as random variables and denote

\[ E_k = \sqrt{\frac{\pi}{2}} \cdot \sqrt{n \tilde{H}_{\theta_k|\theta_k}} |\hat{\theta}_k - \theta_k^*| . \]

To obtain an appropriate theoretical guarantee for the procedure above, we need an assumption on the limiting behavior of e-values \( E_k \).

**ASSUMPTION 9.** Let \( \theta^* \) be the parameter which generates the data and \( E_k \) be the random variables of e-values in (20). For any sequence of subsets \( S_n \subseteq \mathcal{P} \), the average of the expectation values of \( E_k \) over the subset \( S_n \) has limit superior less than 1, which means

\[
\limsup_{n,p \to \infty} \frac{1}{|S_n|} \sum_{k \in S_n} E(E_k) \leq 1.
\]

**REMARK 6.1.** Notice in the Assumption 9, \( S_n \) and \( \mathcal{P} \) both depend on different \( n \). The rationality of the Assumption 9 comes from the following observation. By (19), we have

\[
\lim_{n \to \infty} E(E_k) = \sqrt{\frac{\pi}{2}} \lim_{n \to \infty} E \left( \sqrt{n \tilde{H}_{\theta_k|\theta_k}} |\hat{\theta}_k - \theta_k^*| \right) = \sqrt{\frac{\pi}{2}} \cdot E|Z| = 1. \quad (Z \sim N(0,1))
\]

Thus we assume the average of \( E(E_k) \) over any subset of \( \mathcal{P} \) has limit superior when \( n \) tends to infinity.

Under the assumption above, we have the following theorem controlling the FDR of the procedure in the asymptotic sense.

**THEOREM 6.3 (Appendix D.4).** Under the Assumptions 1 to 9, let \( \mathcal{G} \) be the FDR control procedure given in the above algorithm. Then for any \( \epsilon \in (0, 1) \), the FDR has limit superior at most \( \epsilon \)

\[
\limsup_{n,p \to \infty} \text{FDR}_\mathcal{G} \leq \epsilon.
\]

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**REFERENCES**

[1] *Agresti, A.* (2003). *Categorical Data Analysis* 482. John Wiley & Sons.
[2] *Azriel, D.* and *Schwartzman, A.* (2017). The empirical distribution of a large number of correlated normal variables. *Journal of the American Statistical Association* 110 1217-1228.
[3] *Banerjee, O.*, *Ghoudi, L. E.* and *D’Aspremont, A.* (2008). Model Selection through Sparse Maximum Likelihood Estimation for Multivariate Gaussian or Binary Data. *Journal of Machine Learning Research* 9 485–516.
[4] *Barndorff-Nielsen, O. E.* (2014). *Information and Exponential Families: in Statistical Theory*. John Wiley & Sons.
[5] *Baxter, R. J.* (2016). *Exactly Solved Models in Statistical Mechanics*. Elsevier.
[6] *Benjamini, Y.* and *Hochberg, Y.* (1995). Controlling the false discovery rate: a practical and powerful approach to multiple testing. *Journal of the Royal statistical society: series B (Methodological)* 57 289–300.
[7] *Benjamini, Y.* and *Yekutieli, D.* (2001). The control of the false discovery rate in multiple testing under dependency. *Annals of statistics* 1165–1188.
[8] *Besag, J.* (1974). Spatial Interaction and the Statistical Analysis of Lattice Systems. *Journal of the Royal Statistical Society: Series B (Methodological)* 36 192–225.
[9] BICKEL, P. J., RITOV, Y. and TSYBAKOV, A. B. (2009). Simultaneous Analysis of Lasso and Dantzig Selector. The Annals of Statistics 37 1705–1732.
[10] BISHOP, C. M. (2006). Pattern Recognition and Machine Learning, springer.
[11] BLEI, D. M., NG, A. Y. and JORDAN, M. I. (2003). Latent Dirichlet Allocation. Journal of Machine Learning Research 3 993–1022.
[12] BORWEIN, J. and LEWIS, A. S. (2010). Convex Analysis and Nonlinear Optimization: Theory and Examples. Springer Science & Business Media.
[13] BROWN, L. D. (1986). Fundamentals of Statistical Exponential Families: With Applications in Statistical Decision Theory. Institute of Mathematical Statistics, USA.
[14] BUHLMANN, P. and VAN DE GEER, S. (2011). Statistics for High-Dimensional Data: Methods, Theory and Applications. Springer Science & Business Media.
[15] CHANDRASEKARAN, V., SREBRO, N. and HARSHA, P. (2012). Complexity of Inference in Graphical Models. arXiv preprint arXiv:1206.3240.
[16] COOK, S. A. (1971). The Complexity of Theorem-Proving Procedures. In Proceedings of the third annual ACM symposium on Theory of computing 151–158.
[17] CROSS, G. R. and JAIN, A. K. (1983). Markov Random Field Texture Models. IEEE Transactions on Pattern Analysis and Machine Intelligence PAMI-5 25-39.
[18] CUI, C., JIA, J., XIAO, Y. and ZHANG, H. (2021). Directional FDR Control for Sub-Gaussian Sparse GLMs. arXiv preprint arXiv:2105.00393.
[19] DAI, C., LIN, B., XING, X. and LIU, J. S. (2020). A scale-free approach for false discovery rate control in generalized linear models. arXiv preprint arXiv:2007.01237.
[20] EDWARDS, D. and KREINER, S. (1983). The Analysis of Contingency Tables by Graphical Models. Biometrika 70 553–565.
[21] EFRON, B. (1978). The Geometry of Exponential Families. The Annals of Statistics 6 362–376.
[22] FANG, E. X., NING, Y. and LIU, H. (2017). Testing and Confidence Intervals for High Dimensional Proportional Hazards Models. Journal of the Royal Statistical Society. Series B: (Methodological) 79 1415–1437.
[23] FIENBERG, S. E. (2000). Contingency Tables and Log-Linear Models: Basic Results and New Developments. Journal of the American Statistical Association 95 643–647.
[24] GEMAN, S. and GEMAN, D. (1984). Stochastic Relaxation, Gibbs Distributions, and the Bayesian Restoration of Images. IEEE Transactions on Pattern Analysis and Machine Intelligence PAMI-6 721-741.
[25] GEYER, C. J. (1994). On the Convergence of Monte Carlo Maximum Likelihood Calculations. Journal of the Royal Statistical Society: Series B (Methodological) 56 261–274.
[26] GILKS, W. R. (2005). Markov Chain Monte Carlo. Encyclopedia of Biostatistics 4.
[27] GYFTODIMOS, E. and FLACH, P. A. (2002). Hierarchical Bayesian Networks: A Probabilistic Reasoning Model for Structured Domains. In Proceedings of the ICML-2002 Workshop on Development of Representations 23–30. Citeseer.
[28] HASSNER, M. and SKLANSKY, J. (1981). The Use of Markov Random Fields as Models of Texture. In Image Modeling 185–198. Elsevier.
[29] HIRIART-URRUTY, J.-B. and LEMARÉCHAL, C. (2013). Convex Analysis and Minimization Algorithms I: Fundamentals 305. Springer science & business media.
[30] HÖFLING, H. and TIBSHIRANI, R. (2009). Estimation of Sparse Binary Pairwise Markov Networks using Pseudo-likelihoods. Journal of Machine Learning Research 10 883–906.
[31] HUANG, X., ACERO, A., HON, H.-W. and REDDY, R. (2001). Spoken Language Processing: A Guide to Theory, Algorithm, and System Development, 1st ed. Prentice Hall PTR, USA.
[32] ISING, E. (1925). Beitrag zur theorie des ferromagnetismus. Zeitschrift für Physik 31 253–258.
[33] JALALI, A., JOHNSON, C. C. and RAVIKUMAR, P. K. (2011). On Learning Discrete Graphical Models using Greedy Methods. In Advances in Neural Information Processing Systems 1935–1943.
[34] JAVANMARD, A. and MONTANARI, A. (2013). Nearly optimal sample size in hypothesis testing for high-dimensional regression. In 2013 51st Annual Allerton Conference on Communication, Control, and Computing (Allerton) 1427–1434. IEEE.
[35] JAVANMARD, A. and MONTANARI, A. (2014). Confidence intervals and hypothesis testing for high-dimensional regression. The Journal of Machine Learning Research 15 2869–2909.
[36] JIANG, B., SUN, Q. and PAN, J. (2018). Bernstein’s inequality for general Markov chains. arXiv preprint arXiv:1805.10721.
[37] LAURITZEN, S. L. (1979). Lectures on Contingency Tables. University of Copenhagen.
[38] MANNING, C. D., MANNING, C. D. and SCHÜTZE, H. (1999). Foundations of Statistical Natural Language Processing. MIT press.
[39] MIASOFEDOW, B. and REICHEL, W. (2018). Sparse Estimation in Ising Model via Penalized Monte Carlo Methods. The Journal of Machine Learning Research 19 2979–3004.
[40] Nielsen, F. and Nock, R. (2009). Sided and Symmetrized Bregman Centroids. *IEEE Transactions on Information Theory* 55: 2882–2904.

[41] Ning, Y. and Liu, H. (2017). A General Theory of Hypothesis Tests and Confidence Regions for Sparse High Dimensional Models. *The Annals of Statistics* 45: 158–195.

[42] Ravikumar, P., Wainwright, M. J. and Lafferty, J. D. (2010). High-dimensional Ising Model Selection using L1-Regularized Logistic Regression. *The Annals of Statistics* 38: 1287–1319.

[43] Ripley, B. D. (1984). Spatial Statistics: Developments 1980-3, Correspondent Paper. *International Statistical Review/Revue Internationale de Statistique* 141–150.

[44] Robert, C. and Casella, G. (2013). *Monte Carlo Statistical Methods*. Springer Science & Business Media.

[45] Robertson, D. S., Wildenhain, J., Javanmard, A. and Karp, N. A. (2019). onlineFDR: an R package to control the false discovery rate for growing data repositories. *Bioinformatics* 35: 4196–4199.

[46] Speed, T. P. and Kivéri, H. T. (1986). Gaussian Markov Distributions over Finite Graphs. *The Annals of Statistics* 138–150.

[47] Tibshirani, R. (1996). Regression Shrinkage and Selection via the Lasso. *Journal of the Royal Statistical Society: Series B (Methodological)* 58: 267–288.

[48] Van de Geer, S., Bühlmann, P., Ritov, Y., Dezeure, R. et al. (2014). On asymptotically optimal confidence regions and tests for high-dimensional models. *The Annals of Statistics* 42: 1166–1202.

[49] Van de Geer, S. A. (2008). High-dimensional Generalized Linear Models and the Lasso. *The Annals of Statistics* 36: 614–645.

[50] Vovk, V. and Wang, R. (2019). True and false discoveries with e-values. *arXiv preprint arXiv:1912.13292* 54.

[51] Vovk, V. and Wang, R. (2021). E-values: Calibration, combination and applications. *The Annals of Statistics* 49: 1736–1754.

[52] Wainwright, M. J. and Jordan, M. I. (2008). *Graphical Models, Exponential Families, and Variational Inference*. Now Publishers Inc.

[53] Wermuth, N. and Lauritzen, S. L. (1982). *Graphical and Recursive Models for Contingency Tables*. Institut for Elektroniske Systemer, Aalborg Universitetssenter.

[54] Woods, J. (1978). Markov Image Modeling. *IEEE Transactions on Automatic Control* 23: 846–850.

[55] Xue, L., Zou, H. and Cai, T. (2012). Nonconcave Penalized Composite Conditional Likelihood Estimation of Sparse Ising Models. *The Annals of Statistics* 40: 1403–1429.

[56] Yu, Y. (2010). High-dimensional Variable Selection in Cox Model with Generalized Lasso-type Convex Penalty.

[57] Yu, Y., Bradic, J. and Samworth, R. J. (2021). Confidence intervals for high-dimensional Cox models. *Statistica Sinica* 31: 243–267.

[58] Zhang, C.-H. and Zhang, S. S. (2014). Confidence intervals for low dimensional parameters in high dimensional linear models. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 76: 217–242.

[59] Zhang, H. and Chen, S. X. (2021). Concentration inequalities for statistical inference. *Communications in Mathematical Research* 37: 1–85.

[60] Zhang, H. and Jia, J. (2022). Elastic-net Regularized High-dimensional Negative Binomial Regression: Consistency and Weak Signals Detection. *Statistica Sinica* 32: 181–207.

[61] Zou, H. and Hastie, T. (2005). Regularization and Variable Selection via the Elastic Net. *Journal of the Royal Statistical Society: Series B (Methodological)* 67: 301–320.
APPENDIX A: PROOFS OF THEOREM AND LEMMAS IN SECTION 3

A.1. Proof of Lemma 3.1. Define $\theta^\dagger = \frac{\hat{\theta}}{\|\hat{\theta}\|_1}$, we notice $\theta^\dagger \in \mathcal{C}(\zeta, T)$ and $\|\theta^\dagger\|_1 = 1$.

Define the function

$$g(t) := (\theta^\dagger)^T (\nabla L^m_n(\theta^* + t\theta^\dagger) - \nabla L^m_n(\theta^*))$$

which is nondecreasing because $\nabla^2 L^m_n$ is convex.

For $t \in (0, \|\hat{\theta}\|_1)$, according to Lemma 2.1 and (9), we have

$$g(t) \leq g(\|\hat{\theta}\|_1) = (\theta^\dagger)^T (\nabla L^m_n(\theta^* + \|\hat{\theta}\|_1\theta^\dagger) - \nabla L^m_n(\theta^*))$$

$$= \|\hat{\theta}\|_1^(-1) (\hat{\theta} - \theta^*)^T (\nabla L^m_n(\hat{\theta}) - \nabla L^m_n(\theta^*))$$

$$= \|\hat{\theta}\|_1^(-1) D^g(\hat{\theta}, \theta^*) \leq \|\hat{\theta}\|_1^(-1) D^d(\hat{\theta}, \theta^*)$$

$$\leq \|\hat{\theta}\|_1^(-1) \left\{ \frac{2\zeta}{1 + \zeta} |\sqrt{1 + \zeta}| \right\}$$

Let $\tilde{t}$ be the maximal value which satisfies

$$g(t) \leq \frac{2\zeta}{1 + \zeta} |\sqrt{1 + \zeta}|$$

Notice

$$w_i(\theta^* + t\theta^\dagger) = \frac{e^{(\theta^* + t\theta^\dagger)^T \varphi(Y_i)}}{h(Y_i)} = e^{t(\theta^\dagger)^T \varphi(Y_i)} w_i(\theta^*)$$

According to (7), we have

$$\nabla L^m_n(\theta^* + t\theta^\dagger) - \nabla L^m_n(\theta^*) = \frac{\sum_{i=1}^m w_i(\theta^* + t\theta^\dagger) \varphi(Y_i)}{\sum_{i=1}^m w_i(\theta^* + t\theta^\dagger)} - \frac{\sum_{i=1}^m w_i(\theta^*) \varphi(Y_i)}{\sum_{i=1}^m w_i(\theta^*)}$$

$$= \sum_{i=1}^m \frac{e^{(\theta^\dagger)^T \varphi(Y_i)} w_i(\theta^*) \varphi(Y_i)}{\sum_{i=1}^m e^{(\theta^\dagger)^T \varphi(Y_i)} w_i(\theta^*)} - \sum_{i=1}^m \frac{w_i(\theta^*) \varphi(Y_i)}{\sum_{i=1}^m w_i(\theta^*)}$$

Let $a_i := t(\theta^\dagger)^T (\varphi(Y_i) - \varphi^\dagger(Y_i))$ and $w_i = w_i(\theta^*)$, we have

$$t\theta^\dagger (\nabla L^m_n(\theta^* + t\theta^\dagger) - \nabla L^m_n(\theta^*)) = \frac{\sum_{i=1}^m w_i(\theta^*) e^{t(\theta^\dagger)^T \varphi(Y_i)} t(\theta^\dagger)^T \varphi(Y_i)}{\sum_{i=1}^m w_i(\theta^*) e^{t(\theta^\dagger)^T \varphi(Y_i)}} - \frac{\sum_{i=1}^m w_i(\theta^*) (a_i + t(\theta^\dagger)^T \varphi^\dagger(\theta^*))}{\sum_{i=1}^m w_i}$$

$$= \frac{\sum_{i=1}^m w_i e^{a_i} (a_i + t(\theta^\dagger)^T \varphi^\dagger(\theta^*))}{\sum_{i=1}^m w_i} - \frac{\sum_{i=1}^m w_i a_i}{\sum_{i=1}^m w_i}$$

$$= \sum_{i=1}^m w_i e^{a_i} a_i - \sum_{i=1}^m w_i a_i$$

$$= \sum_{1 \leq i,j \leq m} w_i w_j a_i e^{a_i} - \sum_{1 \leq i,j \leq m} w_i w_j e^{a_i}$$
\[
\sum_{1 \leq i, j \leq m} w_i w_j (a_i - a_j)(e^{a_i} - e^{a_j}) = 2 \sum_{1 \leq i, j \leq m} w_i w_j (a_i - a_j)^2
\]

Notice \(|a_i| \leq t\|\theta^\dagger\|_1\|\varphi(Y_i) - \varphi(\theta^*)\|_\infty \leq 2Kt\) and
\[
\sum_{1 \leq i, j \leq m} w_i w_j (a_i - a_j)^2 = 2 \sum_{1 \leq i \leq m} w_i \sum_{1 \leq j \leq m} w_j a_j^2
\]
due to \(\sum_{1 \leq i \leq m} w_i a_i = 0\). Using the inequality
\[
\frac{e^y - e^x}{y - x} \geq e^{-(|y|\|x|)},
\]
we have
\[
\sum_{1 \leq i, j \leq m} w_i w_j (a_i - a_j)(e^{a_i} - e^{a_j}) \geq e^{-4Kt} \sum_{1 \leq i, j \leq m} w_i w_j (a_i - a_j)^2 \\
= e^{-4Kt} \sum_{1 \leq i \leq m} w_i \sum_{1 \leq j \leq m} w_j a_j^2 \\
= e^{-4Kt} \sum_{1 \leq i \leq m} w_i a_i^2 = e^{-4Kt} t^2 (\theta^\dagger)^T \nabla^2 L^m_n(\theta^*) \theta^\dagger,
\]
where the last step is from the reformulation of \(L^m_n(\theta^*)\) in (8). Therefore, we have
\[
t\theta^\dagger (\nabla L^m_n(\theta^* + t\theta^\dagger) - \nabla L^m_n(\theta^*)) \geq e^{-4Kt} t^2 (\theta^\dagger)^T \nabla^2 L^m_n(\theta^*) \theta^\dagger.
\]

According to the definition of \(\hat{F}(\zeta, T)\) and (21), we have
\[
t e^{-4Kt} \hat{F}(\zeta, T) \|\theta^\dagger\|_1^2 \leq t e^{-4Kt} s(\theta^\dagger)^T \nabla^2 L^m_n(\theta^*) \theta^\dagger \\
\leq s\theta^\dagger (\nabla L^m_n(\theta^* + t\theta^\dagger) - \nabla L^m_n(\theta^*)) \\
\leq \frac{2\zeta}{\zeta + 1} s\lambda_1 \|\theta^\dagger\|_1 - \frac{2}{\zeta + 1} s\lambda_1 \|\theta^\dagger\|_1 \\
\leq \frac{\zeta + 1}{2} s\lambda_1 \|\theta^\dagger\|_1^2,
\]
which means
\[
t e^{-4Kt} \leq \frac{(\zeta + 1)s\lambda_1}{2F(\zeta, T)} \quad \text{for} \quad t \in (0, \hat{t}).
\]

Consider the function \(y(t) = t e^{-4Kt}\), \(y(t)\) gets its unique maximum \(y_{\max} = \frac{1}{4K} e\) at \(t_{\max} = \frac{1}{4K}\).
On the event
\[ P \cap \Omega_1 = \left\{ \frac{(\zeta + 1)s\lambda_1}{2F(\zeta, T)} \leq \frac{1}{4Ke} \right\} \cap \Omega_1, \]
we have \( \|\hat{\theta}\|_1 \leq \tilde{t} \leq \frac{1}{4K} \), by which we have \( e^{-1} \leq e^{-4K\tilde{t}} \), and therefore
\[ \|\hat{\theta}\|_1 e^{-1} \leq \tilde{t} e^{-4K\tilde{t}} \leq \frac{(\zeta + 1)s\lambda_1}{2F(\zeta, T)}. \]
\[ \square \]

A.2. Proof of Lemma 3.2. Notice
\[
\nabla L_{n,j}^m(\theta^*) = -\frac{1}{n} \sum_{i=1}^n \varphi_j(Y_i) + \frac{\sum_{i=1}^m w_i(\theta^*)\varphi_j(Y_i)}{\sum_{i=1}^m w_i(\theta^*)}
\]
\[ = -\frac{1}{n} \sum_{i=1}^n (\varphi_j(Y_i) - E_{\theta^*}\varphi_j(X)) + \frac{\sum_{i=1}^m w_i(\theta^*) (\varphi_j(Y_i) - E_{\theta^*}\varphi_j(X))}{\sum_{i=1}^m w_i(\theta^*)}. \]

Define
\[ A_j = \left\{ \left| \frac{1}{n} \sum_{i=1}^n \varphi_j(X_i) - E_{\theta^*}\varphi_j(X) \right| \geq \frac{t}{2} \right\}, \quad B_j = \left\{ \left| \frac{\sum_{i=1}^m w_i(\theta^*) (\varphi_j(Y_i) - E_{\theta^*}\varphi_j(X))}{\sum_{i=1}^m w_i(\theta^*)} \right| \geq \frac{t}{2} \right\}, \]
\[ C_j = \left\{ \left| \frac{1}{m} \sum_{i=1}^m w_i(\theta^*) (\varphi_j(Y_i) - E_{\theta^*}\varphi_j(X)) \right| \geq \frac{t}{4} C(\theta^*) \right\} \quad \text{and} \quad D = \left\{ \frac{1}{m} \sum_{i=1}^m w_i(\theta^*) \leq \frac{1}{2} C(\theta^*) \right\}. \]

For different events, we have the following relations
\[ (i) \{\| \nabla L_{n,j}^m(\theta^*) \|_\infty \leq t \} = \bigcap_{1 \leq j \leq p} \{ | \nabla L_{n,j}^m(\theta^*) | \leq t \}, \]
\[ (ii) \{ | \nabla L_{n,j}^m(\theta^*) | \geq t \} \subseteq A_j \cup B_j \quad \text{due to the decomposition of } \nabla L_{n,j}^m(\theta^*), \]
\[ (iii) \quad B_j \setminus D \subseteq C_j \rightarrow B_j \subseteq C_j \cup D, \]
from which we have
\[ \bigcup_{1 \leq j \leq p} \{ | \nabla L_{n,j}^m(\theta^*) | \|_\infty \geq t \} \subseteq \bigcup_{1 \leq j \leq p} A_j \cup C_j \cup D. \]

According to the Hoeffding inequality [Corollary 2.1 in [59]] and \( |\varphi_j(Y_i)| \leq K \), we have
\[ P(A_j) \leq 2 \exp \left( -\frac{2 \left( \frac{nt}{2} \right)^2}{\sum_{i=1}^m (2K)^2} \right) = 2e^{-\frac{nt^2}{4K^2}}. \]
\[ \text{(22)} \]

Let
\[ g(Y) = \frac{e^{(\theta^*)^T \varphi(Y)} (\varphi_j(Y) - E_{\theta^*}\varphi_j(X))}{h(Y)}. \]
Because
\[ E_{Y \sim h} g(Y) = \sum_{y \in \mathcal{X}} e^{(\theta^*)^T \varphi(y)} \varphi_j(y) - C(\theta^*) E_{\theta^*} \varphi_j(X) = 0, \]
use Lemma 2.2 and let \( g_i(Y) = g(Y) \). Notice \( \|g\|_{\infty} \leq 2KM_2C(\theta^*) \) and
\[ E_{Y \sim h} g^2(Y) \leq 4K^2 E_{Y \sim h} \left( \frac{e^{(\theta^*)^T \varphi(Y)}}{h(Y)} \right)^2 = (2KC(\theta^*))^2 M_1, \]
because
\[ |\varphi_j(Y) - E_{\theta^*} \varphi_j(X)| \leq |\varphi_j(Y)| + E_{\theta^*} |\varphi_j(X)| \leq 2K. \]
Apply Lemma 2.2 and let \( M = 2KM_2C(\theta^*) \). Noticing \( \sigma^2 \leq (2KC(\theta^*))^2 M_1 \), we have
\[ P(C_j) \leq 2 \exp \left( -\frac{m(\frac{4}{3}C(\theta^*))^2}{2(\beta_1(2KC(\theta^*))^2 M_1 + \beta_22KC(\theta^*)M_2\frac{4}{3}C(\theta^*))} \right) \]
\[ = 2 \exp \left( -\frac{ml^2}{16K(8K\beta_1 M_1 + t\beta_2 M_2)} \right), \]
Now, define a new \( g(Y) \)
\[ g(Y) = \frac{e^{(\theta^*)^T \varphi(Y)}}{h(Y)} - C(\theta^*), \]
noticing \( E_{Y \sim h} g(Y) = 0 \) and \( \|g\|_{\infty} \leq M_2C(\theta^*) \).
Notice
\[ E_{Y \sim h} g^2(Y) \leq E_{Y \sim h} \left( \frac{e^{(\theta^*)^T \varphi(Y)}}{h(Y)} \right)^2 = C^2(\theta^*)M_1 \]
Again using Lemma 2.2, let \( g_i(Y) = g(Y) \) and \( M = C(\theta^*)M_2 \). Noticing \( \sigma^2 \leq C^2(\theta^*)M_1 \), we have
\[ P(D) \leq \exp \left( -\frac{m(C(\theta^*)^2}{2(2\beta_1 C^2(\theta^*)M_1 + \beta_2 C(\theta^*)M_2 C(\theta^*)^2)} \right) \]
\[ = \exp \left( -\frac{m}{4(2\beta_1 M_1 + \beta_2 M_2)} \right). \]
With (22), (23) and (24), we conclude
\[ P(\|\nabla L_n^m(\theta^*)\|_{\infty} \leq t) = 1 - P(\|\nabla L_n^m(\theta^*)\|_{\infty} \geq t) \]
\[ \geq 1 - 2pe^{-\frac{ml^2}{8K^2} - \frac{ml^2}{16K(8K\beta_1 M_1 + t\beta_2 M_2)} + \frac{ml^2}{4(2\beta_1 M_1 + \beta_2 M_2)}}. \]
\[ \square \]

**A.3. Proof of Lemma 3.3.** According to the reformulation of \( \nabla^2 L_n^m(\theta^*) \) in (8) and the definition of \( H^* = \text{Cov}_{\theta^*} \varphi(X) \) in (1), we have
\[ \nabla L_n^m(\theta^*) - H^* = \left( \sum_{i=1}^{m} w_i(\theta) \varphi(Y_i)^2 \right) - \left( \sum_{i=1}^{m} w_i(\theta) \varphi(X)^2 \right) \]
\[ =: F + E. \]
Notice

\[ F_{kl} = \frac{1}{m} \sum_{i=1}^{m} w_i(\theta) (\varphi_k(Y_i)\varphi_l(Y_i) - E_{\theta^*} \varphi_k(X)\varphi_l(X)) \quad (1 \leq k, l \leq m) \]

Define the events

\[ A_{kl} := \left\{ |F_{kl}| \geq \frac{t}{2} \right\}, \quad B_{kl} := \left\{ \left| \frac{1}{m} \sum_{i=1}^{m} w_i(\theta) (\varphi_k(Y_i)\varphi_l(Y_i) - E_{\theta^*} \varphi_k(X)\varphi_l(X)) \right| \geq \frac{t}{4} C(\theta^*) \right\}, \]  

and \[ D := \left\{ \frac{1}{m} \sum_{i=1}^{m} w_i(\theta^*) \leq \frac{1}{2} C(\theta^*) \right\}. \]

We have the following relations for the events defined above \( A_{kl} \setminus D \subseteq B_{kl} \rightarrow \ A_{kl} \subseteq B_{kl} \cup D. \)

Let

\[ g(Y) = \frac{e^{(\theta^*)^T \varphi(Y)} (\varphi_k(Y)\varphi_l(Y) - E_{\theta^*} \varphi_k(X)\varphi_l(X))}{h(Y)}. \]

Notice

\[ E_{Y \sim h} g(Y) = \sum_{y \in \mathcal{Y}} e^{(\theta^*)^T \varphi(y)} \varphi_k(y)\varphi_l(y) - C(\theta^*) E_{\theta^*} \varphi_k(X)\varphi_l(X) = 0, \]

\[ \|g\|_{\infty} \leq 2K^2 C(\theta^*) M_2 \quad \text{and} \]

\[ E_{Y \sim h} g^2(Y) \leq 4K^4 E_{Y \sim h} \left( \frac{e^{(\theta^*)^T \varphi(Y)}}{h(Y)} \right)^2 = (2K^2 C(\theta^*))^2 M_1, \]

where the last inequality is because

\[ |\varphi_k(Y)\varphi_l(Y) - E_{\theta^*} \varphi_k(X)\varphi_l(X)| \leq |\varphi_k(Y)| \cdot |\varphi_l(Y)| + E_{\theta^*} |\varphi_k(Y)| \cdot |\varphi_l(Y)| \leq 2K^2. \]

Applying Lemma 2.2, let \( g_l(Y) = g(Y) \) and \( M = 2K^2 C(\theta^*) M_2. \) Noticing \( \sigma^2 \leq (2K^2 C(\theta^*))^2 M_1, \) we have

\[ P(B_{kl}) \leq 2 \exp \left( -\frac{m(\frac{1}{4} C(\theta^*))^2}{2(\beta_1(2K^2 C(\theta^*))^2 M_1 + \beta_2 2K^2 C(\theta^*) M_2 (\frac{1}{4} C(\theta^*)^2))} \right) \]

\[ = 2 \exp \left( -\frac{ml^2}{16K^2(8K^2 \beta_1 M_1 + t\beta_2 M_2)} \right). \]

Thus for \( P(\|F\|_{\infty} \geq \frac{t}{2}) \), we have

\[ P \left( \|F\|_{\infty} \geq \frac{t}{2} \right) = P \left( \bigcup_{1 \leq k,l \leq r} B_{kl} \cup D \right) \leq 2p^2 e^{- \frac{ml^2}{16K^2(8K^2 \beta_1 M_1 + t\beta_2 M_2)}} + e^{- \frac{ml^2}{4(2\beta_1 M_1 + \beta_2 M_2)}}, \]

where we use the upper bound for \( P(D) \) in the proof of Lemma 3.2 in the last step.
Now we turn to bound \( \|E\|_\infty \), we have
\[
E_{kl} = \left( \sum_{i=1}^{m} w_i(\theta^*) \phi_k(Y_i) \right) \left( \sum_{i=1}^{m} w_i(\theta^*) \phi_l(Y_i) \right) - E_{\theta^*} \phi_k(X) \cdot E_{\theta^*} \phi_l(X)
\]
\[
= \left( \sum_{i=1}^{m} w_i(\theta^*) \phi_k(Y_i) \right) \left( \sum_{i=1}^{m} w_i(\theta^*) \phi_l(Y_i) \right) + \left( \sum_{i=1}^{m} w_i(\theta^*) \phi_l(Y_i) \right) \left( \sum_{i=1}^{m} w_i(\theta^*) \right) - E_{\theta^*} \phi_l(X) E_{\theta^*} \phi_k(X).
\]

Therefore,
\[
|E_{kl}| \leq K \left| \sum_{i=1}^{m} w_i(\theta^*) \phi_k(Y_i) - E_{\theta^*} \phi_k(X) \right| + K \left| \sum_{i=1}^{m} w_i(\theta^*) \phi_l(Y_i) - E_{\theta^*} \phi_l(X) \right| + 2K \left| \sum_{i=1}^{m} w_i(\theta^*) \phi_l(Y_i) - E_{\theta^*} \phi_l(X) \right| \left( \sum_{i=1}^{m} w_i(\theta^*) \right) - E_{\theta^*} \phi_l(X)
\]
\[
\leq 2K \left| \sum_{i=1}^{m} w_i(\theta^*) \phi_l(Y_i) \right| \left( \sum_{i=1}^{m} w_i(\theta^*) \right) - E_{\theta^*} \phi_l(X) \right|_\infty.
\]

Using the proof of Lemma 3.2, we find the upper bound for \( P(\|E\|_\infty \geq \frac{t}{2}) \),
\[
P(\|E\|_\infty \geq \frac{t}{2}) \leq P \left( \left| \sum_{i=1}^{m} w_i(\theta^*) \phi(Y_i) \right| \left( \sum_{i=1}^{m} w_i(\theta^*) \right) - E_{\theta^*} \phi(X) \right| \geq \frac{t}{4K} \right)
\]
\[
\leq 2pe^{-\frac{m^2}{4K^2(8K_\beta^2_1 M_1 + 8K_\beta^2_2 M_2)}} + e^{-\frac{t}{4(2K_1 M_1 + \beta_2 M_2)}}.
\]

Using (25) and (26), we conclude
\[
P(\|\nabla \mathcal{L}_m(\theta^*) - H^*\|_\infty \leq t) \geq 1 - 2pe^{-\frac{m^2}{16K^2(8K_\beta^2_1 M_1 + 8K_\beta^2_2 M_2)}} - e^{-\frac{m}{4(2K_1 M_1 + \beta_2 M_2)}}
\]
\[
- 2pe^{-\frac{m^2}{4K^2(8K_\beta^2_1 M_1 + 8K_\beta^2_2 M_2)}} - e^{-\frac{m}{4(2K_1 M_1 + \beta_2 M_2)}} - 2pe^{-\frac{m^2}{4K^2(8K_\beta^2_1 M_1 + 8K_\beta^2_2 M_2)}} - 2pe^{-\frac{m^2}{4K^2(8K_\beta^2_1 M_1 + 8K_\beta^2_2 M_2)}}.
\]

\[\square\]

**A.4. Proof of Lemma 3.4.** Let \( \bar{b}, b \in \mathcal{C}(\zeta, T) \) be the vectors of cone set respectively satisfying
\[
\bar{F}(\zeta, T) = \frac{s(\bar{b}^T \nabla^2 \mathcal{L}_m(\theta^*) \bar{b})}{\|\bar{b}^T\|^2} \quad \text{and} \quad F(\zeta, T) = \frac{s(b^T H^* b)}{\|b^T\|^2},
\]
By the definition of $F(\zeta, T)$ we have

$$F(\zeta, T) = \frac{s(b^T H^* b)}{\|b_T\|_1^2} \leq \frac{s(b^T H^* b)}{\|b_T\|_1^2},$$

and therefore

$$F(\zeta, T) - \tilde{F}(\zeta, T) = \frac{s(b^T H^* b)}{\|b_T\|_1^2} - \frac{s(b^T \nabla^2 \mathcal{L}^m_n(\theta^*) b)}{\|b_T\|_1^2} \leq \frac{s(b^T H^* b)}{\|b_T\|_1^2} - \frac{s(b^T \nabla^2 \mathcal{L}^m_n(\theta^*) b)}{\|b_T\|_1^2} \leq \frac{s(b^T (H^* - \nabla^2 \mathcal{L}^m_n(\theta^*)) b)}{\|b_T\|_1^2},$$

where we use the fact $a^T A a \leq \|A\|_\infty \|a\|_1^2$ and $\|\tilde{b}\|_1 = \|b_T\|_1 + \|b_{T^c}\|_1 \leq (\zeta + 1) \|\tilde{b}\|_1$. □

### A.5. Proof of Theorem 3.1

Define the event

$$\Omega_2 := \left\{ s(\zeta + 1)^2 \|\nabla^2 \mathcal{L}^m_n - H^*\|_\infty \leq \frac{C_{\min}}{2} \right\} \quad \text{and} \quad \Omega_3 := \left\{ \frac{(\zeta + 1)s\lambda_1}{C_{\min}} \leq \frac{1}{4Ke} \right\}.$$

Notice on the event $\Omega_2$, we have

$$\tilde{F}(\zeta, T) \geq F(\zeta, T) - s(\zeta + 1)^2 \|\nabla^2 \mathcal{L}^m_n - H^*\|_\infty \geq C_{\min} - \frac{C_{\min}}{2} = \frac{C_{\min}}{2}.$$

On the event $\Omega_2 \cap \Omega_3$, we have

$$\frac{(\zeta + 1)s\lambda_1}{2\tilde{F}(\zeta, T)} \leq \frac{(\zeta + 1)s\lambda_1}{C_{\min}} \leq \frac{1}{4Ke}.$$

Therefore, according to Lemma 3.1, on the event $\Omega_1 \cap \Omega_2 \cap \Omega_3$, we have

$$\|\tilde{\theta}\|_1 \leq \frac{e(\zeta + 1)s\lambda_1}{2\tilde{F}(\zeta, T)} \leq \frac{e(\zeta + 1)s\lambda_1}{C_{\min}}.$$

Because we assume $\frac{(\zeta + 1)s\lambda_1}{C_{\min}} \leq \frac{1}{4Ke}$, we have $P(\Omega_3) = 1$. Thus

$$P \left( \|\tilde{\theta}\|_1 \leq \frac{e(\zeta + 1)s\lambda_1}{C_{\min}} \right) \geq P(\Omega_1 \cap \Omega_2 \cap \Omega_3) = P(\Omega_1 \cap \Omega_2) \geq 1 - P(\Omega_1^c) - P(\Omega_2^c).$$

Next, we show $\Omega_1$ occurs in a high probability. With the Assumption 3, $z^* \leq \|\nabla \mathcal{L}(\theta^*)\|_\infty + 2\lambda_2 B$. Noticing we let $\tau_1 = \frac{\zeta - 1}{\zeta + 1}\lambda_1 - 2\lambda_2 B$, by Lemma 3.2 we have

$$P(\Omega_1^c) = P \left( \left\{ z^* \geq \frac{\zeta - 1}{\zeta + 1}\lambda_1 \right\} \right) \leq P \left( \|\nabla \mathcal{L}(\theta^*)\|_\infty \geq \frac{\zeta - 1}{\zeta + 1}\lambda_1 - 2\lambda_2 B \right) \leq 2pe^{-\frac{\zeta^2}{8K^2}} + 2pe^{-\frac{\zeta^2}{16K^2(\beta K^3\lambda_1 + \tau_1/\beta_2 M_2^2 + \tau_2 M_2^2) + e^{-\frac{4(\beta K^3\lambda_1 + \tau_1/\beta_2 M_2^2 + \tau_2 M_2^2)}}} =: \delta_1.$$\)

Finally, we show $\Omega_2$ occurs in a high probability. Because we define $\tau_2 = \frac{C_{\min}}{2s(\zeta + 1)}$, by Lemma 3.3 we have

$$P(\Omega_2^c) = P(s(\zeta + 1)^2 \|\nabla^2 \mathcal{L}^m_n - H^*\|_\infty \geq \frac{C_{\min}}{2}) = P(\|\nabla^2 \mathcal{L}^m_n - H^*\|_\infty \geq \tau_2) \leq 2e^{-\frac{m^2}{16sK^2(\beta K^3\lambda_1 + \tau_1/\beta_2 M_2^2) + e^{-\frac{4(\beta K^3\lambda_1 + \tau_1/\beta_2 M_2^2) + e^{-\frac{4(\beta K^3\lambda_1 + \tau_1/\beta_2 M_2^2)}}}} =: \delta_2.$$
from which we have
\[ P \left( \| \hat{\theta} \|_1 \leq \frac{e(\zeta + 1)s\lambda_1}{C_{\text{min}}} \right) \geq 1 - P(\Omega_1^c) - P(\Omega_2^c) \geq 1 - \delta_1 - \delta_2 \]
and complete the proof. \qed

APPENDIX B: PROOFS OF THEOREM AND LEMMAS IN SECTION 4

B.1. Proof of Lemma 4.1. Notice
\[ \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbf{v}^T \varphi(X_i) \right) = \frac{1}{n} \text{Var}(\mathbf{v}^T \varphi(X)) = \frac{1}{n} \mathbf{v}^T H^* \mathbf{v}. \]
According to the reformulation of \( \nabla \mathcal{L}_p^m(\theta^*) \) in (7), we have
\[
\frac{\sqrt{n} \mathbf{v}^T \nabla \mathcal{L}_p^m(\theta^*)}{\sqrt{\mathbf{v}^T H^* \mathbf{v}}} = -\frac{1}{n} \sum_{i=1}^{n} \mathbf{v}^T (\varphi(X_i) - E_{\theta^*} \varphi(X)) \frac{\sqrt{\text{Var}(\frac{1}{n} \sum_{i=1}^{n} \mathbf{v}^T \varphi(X_i))}}{\sqrt{n} \sum_{i=1}^{n} \mathbf{w}_i(\theta)} + \frac{\sqrt{\mathbf{v}^T H^* \mathbf{v}}}{\sum_{i=1}^{m} \mathbf{w}_i(\theta)}
\]
\[ =: A + B. \]
Using the central limit theorem, we have
\[ A \xrightarrow{d} N(0, 1). \]
If we can show \( B = O_p(1) \), we can get the desired result using Slutsky’s theorem.

Next, we prove \( B = O_p(1) \). Using the fact \( \| \mathbf{v} \|_1 \leq \sqrt{s} \| \mathbf{v} \|_2 \), we have
\[ |B| \leq \frac{\sqrt{n} \sqrt{s}}{\sqrt{\lambda_{\text{min}}}} \left\| \frac{\sum_{i=1}^{m} \mathbf{w}_i(\theta)(\varphi(X_i) - E_{\theta^*} \varphi(X))}{\sum_{i=1}^{m} \mathbf{w}_i(\theta)} \right\|_{\infty} \]
According to the proof Lemma 3.2, we have
\[ P \left( \left\| \frac{\sum_{i=1}^{m} \mathbf{w}_i(\theta)(\varphi(X_i) - E_{\theta^*} \varphi(X))}{\sum_{i=1}^{m} \mathbf{w}_i(\theta)} \right\|_{\infty} \geq t \right) \leq 2pe^{-\frac{t^2}{8Kn(4K\beta_1 M_1 + \beta_2 M_2)}} + e^{-\frac{m}{4(2\beta_1 M_1 + \beta_2 M_2)}} \]
Therefore,
\[ P(|B| \geq \epsilon) \leq P \left( \left\| \frac{\sum_{i=1}^{m} \mathbf{w}_i(\theta)(\varphi(X_i) - E_{\theta^*} \varphi(X))}{\sum_{i=1}^{m} \mathbf{w}_i(\theta)} \right\|_{\infty} \geq \frac{\epsilon \sqrt{\lambda_{\text{min}}}}{\sqrt{n} \sqrt{s}} \right) \]
\[ \leq 2p \exp \left( -\frac{\lambda_{\text{min}} \epsilon^2 m}{8Ks'n(4K\beta_1 M_1 + \beta_2 M_2)} \right) + \exp \left( -\frac{m}{4(2\beta_1 M_1 + \beta_2 M_2)} \right) \]
\[ = 2pI_1 + I_2. \]
With the assumption \( s' = O(1) \), if there exists a constant \( r_3 > 1 \) such that
\[ \frac{\lambda_{\text{min}} \epsilon^2 m}{8Ks'n(4K\beta_1 M_1 + \beta_2 M_2)} > r_3 \log(p) \]
for sufficient large $n$, then we have for sufficient large $n$, $2pI_1 \leq 2p^{1-r_3} \longrightarrow 0$ and thus $B = o_p(1)$. \hfill \Box

**B.2. Proof of Lemma 4.2.** Define $M(w) := \frac{1}{2}w^T\nabla^2_{\beta\beta}L_n^m(\hat{\theta})w - w^T\nabla^2_{\alpha\alpha}L_n^m(\hat{\theta}) + \lambda'\|w\|_1$ and $\hat{\Delta} := \tilde{w} - w^*$. Notice $\tilde{w}$ minimizes $M(w)$, and thus $M(\tilde{w}) \leq M(w^*)$, from which we have

$$\frac{1}{2}\hat{\Delta}^T\nabla^2_{\beta\beta}L_n^m(\hat{\theta})\hat{\Delta} \leq \hat{\Delta}^T\nabla^2_{\beta\beta}L_n^m(\hat{\theta}) - \hat{\Delta}^T\nabla^2_{\alpha\alpha}L_n^m(\hat{\theta})w^* + \lambda'\|w^*\|_1 - \lambda'\|\tilde{w}\|_1$$

$$= \hat{\Delta}^T(\nabla^2_{\beta\beta}L_n^m(\theta^*) - \nabla^2_{\beta\beta}L_n^m(\theta^*)w^*) + \lambda'\|w^*\|_1 - \lambda'\|\tilde{w}\|_1 +$$

$$+ \hat{\Delta}^T(\nabla^2_{\alpha\alpha}L_n^m(\hat{\theta}) - \nabla^2_{\alpha\alpha}L_n^m(\theta^*)) - \hat{\Delta}^T(\nabla^2_{\beta\beta}L_n^m(\hat{\theta}) - \nabla^2_{\beta\beta}L_n^m(\theta^*))w^*$$

$$= I_1 + I_2 + I_3 - I_4,$$

where we define

$I_1 := \hat{\Delta}^T(\nabla^2_{\beta\beta}L_n^m(\theta^*) - \nabla^2_{\beta\beta}L_n^m(\theta^*)w^*)$, \quad $I_2 := \lambda'\|w^*\|_1 - \lambda'\|\tilde{w}\|_1$,

$I_3 := \hat{\Delta}^T(\nabla^2_{\alpha\alpha}L_n^m(\hat{\theta}) - \nabla^2_{\alpha\alpha}L_n^m(\theta^*))$ and $I_4 := \hat{\Delta}^T(\nabla^2_{\beta\beta}L_n^m(\hat{\theta}) - \nabla^2_{\beta\beta}L_n^m(\theta^*))w^*$.

For $I_1$, we have

$$|I_1| \leq \|\hat{\Delta}\|_1 \cdot \|\nabla^2_{\beta\beta}L_n^m(\theta^*) - \nabla^2_{\beta\beta}L_n^m(\theta^*)w^*\|_\infty$$

$$= \|\hat{\Delta}\|_1 \cdot \|\nabla^2_{\beta\beta}L_n^m(\theta^*) - \nabla^2_{\beta\beta}L_n^m(\theta^*)w^*\|_\infty$$

$$\leq \|\hat{\Delta}\|_1 \cdot \|\nabla^2_{\beta\beta}L_n^m(\theta^*) - \nabla^2_{\beta\beta}L_n^m(\theta^*)w^*\|_\infty + s'B\|\nabla^2_{\beta\beta}L_n^m(\theta^*) - \nabla^2_{\beta\beta}L_n^m(\theta^*)w^*\|_\infty$$

$$\leq C\|\hat{\Delta}\|_1 \cdot \|\nabla^2_{\beta\beta}L_n^m(\theta^*) - \nabla^2_{\beta\beta}L_n^m(\theta^*)w^*\|_\infty + s'B\|\nabla^2_{\beta\beta}L_n^m(\theta^*) - \nabla^2_{\beta\beta}L_n^m(\theta^*)w^*\|_\infty$$

(27)

$$\lesssim \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \|\hat{\Delta}\|_1,$$

where in the last step we use the Remark 3.1.

For $I_2$, let $S$ be the support of $w^*$. We have

$$|I_2| = \lambda'\|w^*\|_1 - \lambda'\|\tilde{w}\|_1$$

$$= \lambda'\|w^*\|_1 - \lambda'\|\tilde{w}\|_1 - \lambda'\|\tilde{w}^c\|_1$$

$$\leq \lambda'\|\tilde{w}^c\|_1 - \lambda'\|\tilde{w}^c\|_1$$

(28)

where we use the fact $\|w^*\|_1 - \|\tilde{w}\|_1 \leq \|\tilde{w}^c\|_1$ and $\tilde{w}^c = \tilde{w}^c$. Now we focus on $I_3$, note that the same treatment can be applied to $I_4$. According to the reformulation of $\nabla^2_{\alpha\beta}L_n^m(\hat{\theta})$ in (8), we rewrite $\nabla^2_{\alpha\beta}L_n^m(\hat{\theta})$ to

$$\nabla^2_{\alpha\beta}L_n^m(\hat{\theta}) = \frac{\sum_{i=1}^m w_i(\hat{\theta})(\varphi_\alpha(Y_i) - \varphi_\alpha)(\varphi_\beta(Y_i) - \varphi_\beta)}{\sum_{i=1}^m w_i(\hat{\theta})},$$

where we use the partition $\varphi(Y_i) = (\varphi_\alpha(Y_i), \varphi_\beta(Y_i)^T)^T$ and $\tilde{\varphi} = (\varphi_\alpha, \varphi_\beta)^T$. Notice

$$w_i(\hat{\theta}) = \frac{e^{\tilde{\theta}^T\varphi(Y_i)}e^{\tilde{\theta}^T\varphi(Y_i)}}{h(Y_i)} = e^{\tilde{\theta}^T\varphi(Y_i)}w_i(\theta^*).$$

For simplicity, we define

$$g_i := \tilde{\theta}^T(\varphi(Y_i) - \varphi), \quad w_i := w_i(\theta^*), \quad b_i := \varphi_\alpha(Y_i) - \varphi_\alpha \quad \text{and} \quad h_i := \hat{\Delta}^T(\varphi_\beta(Y_i) - \varphi_\beta).$$
Therefore, we reformulate $\hat{\Delta} T \nabla^2_{\alpha \beta} \mathcal{L}^m_n(\hat{\theta})$ and $\hat{\Delta} T \nabla^2_{\alpha \beta} \mathcal{L}^m_n(\theta^*)$ respectively to

$$\hat{\Delta} T \nabla^2_{\alpha \beta} \mathcal{L}^m_n(\hat{\theta}) = \frac{\sum_{i=1}^{m} w_i b_i h_i e^{g_i}}{\sum_{i=1}^{m} w_i e^{g_i}} \quad \text{and} \quad \hat{\Delta} T \nabla^2_{\alpha \beta} \mathcal{L}^m_n(\theta^*) = \frac{\sum_{i=1}^{m} w_i b_i h_i}{\sum_{i=1}^{m} w_i}.$$ 

Thus for $I_3$, we have

$$|I_3| \leq \left| \sum_{i=1}^{m} w_i b_i h_i (e^{g_i} - 1) \right| + \left( \sum_{i=1}^{m} w_i b_i h_i e^{g_i} \right) \left( \frac{\sum_{i=1}^{m} 1}{\sum_{i=1}^{m} w_i} - \frac{1}{\sum_{i=1}^{m} w_i} \right) =: I_{31} + I_{32}.$$ 

For $I_{31}$, by Cauchy’s inequality we have

$$|I_{31}| \leq 2K \left| \frac{\sum_{i=1}^{m} w_i h_i (e^{g_i} - 1)}{\sum_{i=1}^{m} w_i} \right| \leq 2K \frac{\sqrt{\sum_{i=1}^{m} w_i h_i^2}}{\sqrt{\sum_{i=1}^{m} w_i}} \cdot \frac{\sqrt{\sum_{i=1}^{m} w_i (e^{g_i} - 1)^2}}{\sqrt{\sum_{i=1}^{m} w_i}},$$

where in the last step, we use the fact that $e^{g_i} - 1 \approx g_i$ when $g_i = o(1)$. Notice

$$\frac{\sum_{i=1}^{m} w_i g_i^2}{\sum_{i=1}^{m} w_i} = \bar{\theta}^T \nabla^2 \mathcal{L}^m_n(\theta^*) \bar{\theta} \leq \bar{\theta}^T \nabla^2 \mathcal{L}^m_n(\theta^*) H^* \bar{\theta} + |\bar{\theta}^T (\nabla^2 \mathcal{L}^m_n(\theta^*) - H^*) \bar{\theta}| \leq \lambda_{\max} \|\bar{\theta}\|_2^2 + \|\nabla^2 \mathcal{L}^m_n(\theta^*) - H^*\|_{\infty} \cdot \|\bar{\theta}\|_2 \lesssim \|\bar{\theta}\|_1^2.$$ 

By Corollary 3.1 and the fact that

$$\frac{\sum_{i=1}^{m} w_i h_i^2}{\sum_{i=1}^{m} w_i} = \hat{\Delta} T \nabla^2_{\alpha \beta} \mathcal{L}^m_n(\theta^*) \hat{\Delta},$$

we have an upper bound for $I_{31}$,

$$|I_{31}| \lesssim s \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \sqrt{\hat{\Delta} T \nabla^2_{\alpha \beta} \mathcal{L}^m_n(\theta^*) \hat{\Delta}}.$$  

For $I_{32}$, we have

$$I_{32} = \left| \frac{\sum_{i=1}^{m} w_i b_i h_i e^{g_i}}{\sum_{i=1}^{m} w_i e^{g_i}} \right| \left| \frac{\sum_{i=1}^{m} w_i (e^{g_i} - 1)}{\sum_{i=1}^{m} w_i} \right| \lesssim \frac{\sum_{i=1}^{m} w_i h_i^2}{\sum_{i=1}^{m} w_i} \cdot \frac{\sum_{i=1}^{m} w_i g_i^2}{\sum_{i=1}^{m} w_i},$$

where

$$\sum_{i=1}^{m} w_i h_i^2 \leq \lambda_{\max} \|\bar{\theta}\|_2^2 \quad \text{and} \quad \sum_{i=1}^{m} w_i g_i^2 \leq \lambda_{\max} \|\bar{\theta}\|_2^2.$$
where we use the fact $e^{g_i} - 1 \asymp g_i$ and $e^{g_i} \asymp 1$ when $g_i = o(1)$. With the same argument for $I_{31}$, we have

\begin{align}
|I_{32}| & \lesssim s \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \sqrt{\Delta^T \nabla^2_{\beta \beta} L^m_n(\hat{\theta}^*) \tilde{\Delta}}.
\end{align}

Therefore, according to (29) and (30), we have

\begin{align}
|I_3| & \lesssim s \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \sqrt{\Delta^T \nabla^2_{\beta \beta} L^m_n(\theta^*) \tilde{\Delta}}.
\end{align}

According to (27), (28) and (31), with the same proof of Lemma 1 in [22], we get the desired conclusion. \hfill \square

B.3. A Technical Lemma.

**Lemma B.1.** Assume the Assumptions 1, 2, 3, 4 and 5 are satisfied. Suppose $\lambda_1 \asymp (\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}})$, $\lambda_1 \asymp \lambda_2 \asymp \lambda' = o(1)$ and $s = s' = O(1)$. We have

\begin{align}
\|\nabla^2_{\alpha, \beta} L^m_n(\hat{\theta}) - w^T \nabla^2_{\beta, \beta} L^m_n(\hat{\theta})\|_\infty &= O_p \left( s(\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}}) \right) \quad \text{and}
\end{align}

\begin{align}
\|\nabla^2_{\alpha, \beta} L^m_n(\theta) - (\hat{w})^T \nabla^2_{\beta, \beta} L^m_n(\hat{\theta})\|_\infty &= O_p \left( (s + s')\left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \right).
\end{align}

**Proof.** We only prove the first claim, since the same treatment can be applied to prove the second claim after we notice

\begin{align}
\|\nabla^2_{\alpha, \beta} L^m_n(\hat{\theta}) - (\hat{w})^T \nabla^2_{\beta, \beta} L^m_n(\hat{\theta})\|_\infty &\leq \|\nabla^2_{\alpha, \beta} L^m_n(\hat{\theta}) - w^T \nabla^2_{\beta, \beta} L^m_n(\hat{\theta})\|_\infty + \|w - w^*\|_1 \|\nabla^2_{\beta, \beta} L^m_n(\hat{\theta})\|_\infty.
\end{align}

Using the fact $H^*_\alpha = w^T H^*_\beta$ and by the triangle inequality, we have

\begin{align}
\|\nabla^2_{\alpha, \beta} L^m_n(\hat{\theta}) - w^T \nabla^2_{\beta, \beta} L^m_n(\hat{\theta})\|_\infty &\leq \|\nabla^2_{\alpha, \beta} L^m_n(\hat{\theta}) - w^T \nabla^2_{\beta, \beta} L^m_n(\hat{\theta})\|_\infty + \|\nabla^2_{\alpha, \beta} L^m_n(\theta^*) - H^*_\alpha\|_\infty + \|w^T(\nabla^2_{\beta, \beta} L^m_n(\theta^*) - H^*_\alpha)\|_\infty + \|w^T(\nabla^2_{\beta, \beta} L^m_n(\theta^*) - H^*_\beta)\|_\infty
\end{align}

\begin{align}
&=: J_1 + J_2 + J_3 + J_4.
\end{align}

Notice $\|\nabla^2_{\alpha, \beta} L^m_n(\theta^*) - H^*_\alpha\|_\infty \leq \|\nabla^2 L^m_n(\theta^*) - H^*\|_\infty$ and $s = O(1)$. By the Remark 3.1, we have

\begin{align}
J_2 &= O_p \left( s\left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \right).
\end{align}

Because $\|w^*\|_\infty \leq D$ and $\|w^*\|_0 = s' = O(1)$, we have

\begin{align}
\|w^T(\nabla^2_{\beta, \beta} L^m_n(\theta^*) - H^*_\beta)\|_\infty &\leq s' D \|\nabla^2 L^m_n(\theta^*) - H^*\|_\infty \lesssim \|\nabla^2 L^m_n(\theta^*) - H^*\|_\infty.
\end{align}

By the Remark 3.1 and $s = O(1)$, we have

\begin{align}
J_4 &= O_p \left( s\left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \right).
\end{align}
Now we prove
\[ J_1 = O_p \left( s \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \right). \]

Let \( \nabla^2_{\alpha \beta,j} L_n^m(\hat{\theta}) \) and \( \varphi_{\beta,j}(Y_i) \) respectively denote the \( j \)-th element of the vector \( \nabla^2_{\alpha \beta} L_n^m(\hat{\theta}) \) and \( \varphi_{\beta}(Y_i) \). Notice
\[
\nabla^2_{\alpha \beta,j} L_n^m(\hat{\theta}) = \frac{\sum_{i=1}^{m} w_i(\hat{\theta})(\varphi_{\alpha}(Y_i) - \bar{\varphi}_{\alpha})(\varphi_{\beta,j}(Y_i) - \bar{\varphi}_{\beta,j})}{\sum_{i=1}^{m} w_i(\hat{\theta})} = \frac{\sum_{i=1}^{m} w_i e^{g_i} a_i b_{ij}}{\sum_{i=1}^{m} w_i e^{g_i}},
\]
and
\[
\nabla^2_{\alpha \beta,j} L_n^m(\theta^*) = \frac{\sum_{i=1}^{m} w_i(\theta^*)(\varphi_{\alpha}(Y_i) - \bar{\varphi}_{\alpha})(\varphi_{\beta,j}(Y_i) - \bar{\varphi}_{\beta,j})}{\sum_{i=1}^{m} w_i(\theta^*)} = \frac{\sum_{i=1}^{m} w_i a_i b_{ij}}{\sum_{i=1}^{m} w_i},
\]
where we define \( a_i := \varphi_{\alpha}(Y_i) - \bar{\varphi}_{\alpha}, \ b_{ij} := \varphi_{\beta,j}(Y_i) - \bar{\varphi}_{\beta,j}, \ w_i := w_i(\theta^*) \) and \( g_i := \bar{\theta}^T \varphi(Y_i) \), and use the fact \( w_i(\hat{\theta}) = e^{\bar{\theta}^T \varphi(Y_i)} w_i(\theta^*) \).

Therefore, we have
\[
|\nabla^2_{\alpha \beta,j} L_n^m(\hat{\theta}) - \nabla^2_{\alpha \beta,j} L_n^m(\theta^*)| = \left| \frac{\sum_{i=1}^{m} w_i e^{g_i} a_i b_{ij}}{\sum_{i=1}^{m} w_i e^{g_i}} - \frac{\sum_{i=1}^{m} w_i a_i b_{ij}}{\sum_{i=1}^{m} w_i} \right| \leq \frac{\sum_{i=1}^{m} w_i (e^{g_i} - 1) a_i b_{ij}}{\sum_{i=1}^{m} w_i e^{g_i}} + \left| \frac{\sum_{i=1}^{m} w_i a_i b_{ij}}{\sum_{i=1}^{m} w_i} \right| \left| \frac{1}{\sum_{i=1}^{m} w_i e^{g_i}} - \frac{1}{\sum_{i=1}^{m} w_i} \right| \frac{\sum_{i=1}^{m} w_i e^{g_i}}{\sum_{i=1}^{m} w_i e^{g_i}} =: J_{11} + J_{12}.
\]

Notice \( |a_i| \leq 2K, |b_{ij}| \leq 2K \) and \( |g_i| \leq K \| \bar{\theta} \|_1 \). Using the fact \( e^{g_i} - 1 \approx g_i \) and \( e^{g_i} \approx 1 \) when \( g_i = o(1) \), we have
\[ J_{11} \lesssim \| \bar{\theta} \|_1 \quad \text{and} \quad J_{12} \lesssim \| \bar{\theta} \|_1 \]

By the Corollary 3.1, we have \( J_1 = O_p \left( s \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \right) \).

For the term \( J_3 \), using the same method for \( J_1 \), we have \( J_3 = O_p \left( s \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \right) \), thus we get the first claim.
B.4. Proof of the Theorem 4.1. We begin the proof by decomposing the \( \hat{U}(\alpha, \beta) \) into several terms

\[
\hat{U}(\alpha, \beta) = \nabla_\alpha \mathcal{L}_n^m(\alpha, \beta) - \hat{w}^T \nabla_\beta \mathcal{L}_n^m(\alpha, \beta)
\]

\[
= \nabla_\alpha \mathcal{L}_n^m(\alpha, \beta) + (\nabla_{\alpha \beta} \mathcal{L}_n^m(\alpha, \beta_1)) - \hat{w}^T \nabla_\beta \mathcal{L}_n^m(\alpha, \beta_1) - \hat{w}^T \nabla_{\beta \beta} \mathcal{L}_n^m(\alpha, \beta_2)(\hat{\beta} - \beta^*)
\]

\[
= \left[ \nabla_\alpha \mathcal{L}_n^m(\alpha, \beta) - \hat{w}^* \nabla_\beta \mathcal{L}_n^m(\alpha, \beta^*) \right] + \left[ (\hat{\beta} - \beta^*)^T (\nabla_{\alpha \beta} \mathcal{L}_n^m(\alpha, \beta_1) - \nabla_{\beta \beta} \mathcal{L}_n^m(\alpha, \beta_2)w^*) \right] + \left[ (w^* - \hat{w})^T \nabla_{\beta \beta} \mathcal{L}_n^m(\alpha, \beta_2)(\hat{\beta} - \beta^*) \right]
\]

\[
= E_1 + E_2 + E_3 + E_4,
\]

where by using Taylor’s expansion \( \beta_1 = u_1(\hat{\beta} - \beta^*) + \beta^* \) and \( \beta_2 = u_2(\hat{\beta} - \beta^*) + \beta^* \) for \( u_1, u_2 \in [0,1] \).

For the term \( E_1 \), let \( v := (1, -w^T) \) and notice \( v^T H^* v = H_{\alpha \beta}^* \). By the Lemma 4.1, we have

\[
\sqrt{n}E_1 \xrightarrow{d} N(0, H_{\alpha \beta}^*).
\]

According to our assumption about the relations among \( m, n \) and \( p \), we have

\[
\sqrt{n}(\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}}) = \sqrt{\log(p)} + \sqrt{\frac{n \log(p)}{m}} \lesssim \sqrt{\log(p)},
\]

where we use the assumption \( \frac{n}{m} \asymp \log(p) \).

For the term \( E_2 \), by the Lemma 4.2 and the Remark 3.1, we have

\[
\sqrt{n}|E_2| \leq \sqrt{n} \| w^* - \hat{w} \|_1 \| \nabla_\beta \mathcal{L}_n^m(\alpha, \beta^*) \|_\infty
\]

\[
= O_p \left( (s + s') \sqrt{n} \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right)^2 \right),
\]

where we use the fact that if \( X_n = O_p(a_n) \) and \( Y_n = O_p(b_n) \), then \( c_nX_nY_n = O_p(c_n a_n b_n) \) in the last step. According to (32), notice

\[
(s + s') \sqrt{n} \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right)^2 \lesssim \frac{\log(p)}{\sqrt{n}} + \frac{\log(p)}{\sqrt{m}}.
\]

Therefore, we bound \( E_2 \) by

\[
\sqrt{n}|E_2| = O_p \left( \frac{\log(p)}{\sqrt{n}} + \frac{\log(p)}{\sqrt{m}} \right) \xrightarrow{\sqrt{n}} |E_2| = o_p(1),
\]

where we use our assumption \( \frac{\log(p)}{\sqrt{n}} + \frac{\log(p)}{\sqrt{m}} = o(1) \).

For the term \( E_3 \), notice

\[
|E_3| \leq \| \hat{\beta} - \beta^* \|_1 \| \nabla_{\alpha \beta} \mathcal{L}_n^m(\alpha, \beta_1) - \nabla_{\beta \beta} \mathcal{L}_n^m(\alpha, \beta_2)w^* \|_\infty.
\]

Use the same method in the proof of the technical Lemma B.1, we have

\[
\| \nabla_{\alpha \beta} \mathcal{L}_n^m(\alpha, \beta_1) - \nabla_{\beta \beta} \mathcal{L}_n^m(\alpha, \beta_2)w^* \|_\infty = O_p \left( \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \right).
\]
By the Corollary 3.1, we have $\sqrt{n}|E_3| = O_p \left( \sqrt{n}s^2 \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right)^2 \right)$. Use the same treatment for $E_2$, we can show

\begin{equation}
\sqrt{n}|E_3| = o_p(1).
\end{equation}

For the term $E_4$, notice

$$\| \nabla_{\beta_\beta}^2 L^m_n (\alpha_0, \beta_2) \|_\infty \leq \| H_{\beta_\beta}^* \|_\infty + \| \nabla_{\beta_\beta}^2 L^m_n (\alpha_0, \beta_2) - H_{\beta_\beta}^* \|_\infty \leq C_{uni} + \| \nabla_{\beta_\beta}^2 L^m_n (\alpha_0, \beta_2) - H_{\beta_\beta}^* \|_\infty \leq C_{uni} + \| \nabla_{\beta_\beta}^2 L^m_n (\alpha_0, \beta_2) - \nabla_{\beta_\beta}^2 L^m_n (\theta^*) \|_\infty + \| \nabla_{\beta_\beta}^2 L^m_n (\theta^*) - H_{\beta_\beta}^* \|_\infty = O_p(1),$$

where we use the same the treatment to the term $J_1$ in the proof of Lemma B.1 and show

$$\| \nabla_{\beta_\beta}^2 L^m_n (\alpha_0, \beta_2) - \nabla_{\beta_\beta}^2 L^m_n (\theta^*) \|_\infty \leq \| \hat{\beta} - \beta^* \|_1 = o_p(1),$$

and by the same treatment of the term $J_2$ in the proof of Lemma B.1, we show

$$\| \nabla_{\beta_\beta}^2 L^m_n (\theta^*) - H_{\beta_\beta}^* \|_\infty = O_p \left( s\left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \right) = o_p(1).$$

Thus we have $|E_4| \leq \| \nabla_{\beta_\beta}^2 L^m_n (\alpha_0, \beta_2) \|_\infty \| w^* - \hat{\omega} \|_1 \| \hat{\beta} - \beta^* \|_1$ and

$$\sqrt{n}|E_4| \leq \sqrt{n}\| \nabla_{\beta_\beta}^2 L^m_n (\alpha_0, \beta_2) \|_\infty \| w^* - \hat{\omega} \|_1 \| \hat{\beta} - \beta^* \|_1 \leq \sqrt{n}\| w^* - \hat{\omega} \|_1 \| \hat{\beta} - \beta^* \|_1 = o_p(1),$$

where in the last step we use the same treatment to the term $E_2$.

Combine (33), (34), (35) and the fact $\sqrt{n}E_1 \xrightarrow{d} N(0, H_{\alpha|\beta}^*)$ we proved before, we get the desired asymptotic normality for the score test

$$\sqrt{n}U (\alpha, \hat{\beta}) \xrightarrow{d} N(0, H_{\alpha|\beta}^*).$$

\[\square\]

**B.5. Proof of Lemma 4.3.** According to the definition of $\hat{H}_{\alpha|\beta}$ and $H_{\alpha|\beta}^*$, we have

$$|\hat{H}_{\alpha|\beta} - H_{\alpha|\beta}^*| \leq |\nabla_{\alpha \alpha}^2 L^m_n (\hat{\theta}) - H_{\alpha|\beta}^*| + |(\hat{\omega} - w^*)^T H_{\alpha|\beta}^*| + |\hat{\omega}^T (\nabla_{\alpha \alpha}^2 L^m_n (\hat{\theta}) - H_{\alpha|\beta}^*)|$$

$$=: F_1 + F_2 + F_3.$$

For $F_1$, we have

$$F_1 \leq |\nabla_{\alpha \alpha}^2 L^m_n (\hat{\theta}) - \nabla_{\alpha \alpha}^2 L^m_n (\theta^*)| + |\nabla_{\alpha \alpha}^2 L^m_n (\theta^*) - H_{\alpha|\beta}^*| =: F_{11} + F_{12}.$$

Using the same treatment for the term $J_1$ in the proof of Lemma B.1, we have $F_{11} \leq \| \hat{\theta} \|_1$, and thus $F_{11} = O_p \left( s\left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \right)$. By the Remark 3.1, we have $F_{12} = O_p \left( s\left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \right)$. Therefore, $F_1 = O_p \left( s\left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \right)$.

For $F_2$, notice under the Assumption 4, $\| H^* \|_\infty \leq C_{uni}$, and thus $F_2 \leq \| \hat{\omega} - w^* \|_1 \| H^* \|_\infty$. Then by the Lemma 4.2, we have $F_2 = O_p \left( (s + s') \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right) \right)$.
At last, for $F_3$, notice $\| \hat{w} \|_1 \leq \| \hat{w} - w^* \|_1 + \| w^* \|_1$, $\| w^* \|_1 \leq s'D$ and $\| \hat{w} - w^* \|_1 = o_p(1)$ according to the Lemma 4.2. We have

$$F_3 \leq \| \hat{w} \|_1 \| \nabla^2_{\alpha\beta} L_n^m(\hat{\theta}) - H^*_{\alpha\beta} \|_\infty \leq \| \nabla^2_{\alpha\beta} L_n^m(\hat{\theta}) - \nabla^2_{\alpha\beta} L_n^m(\theta^*) \|_\infty + \| \nabla^2_{\alpha\beta} L_n^m(\theta^*) - H^*_{\alpha\beta} \|_\infty.$$

By the Remark 3.1, we have $\| \nabla^2_{\alpha\beta} L_n^m(\theta^*) - H^*_{\alpha\beta} \|_\infty = O_p\left(s\left(\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}}\right)\right)$.

By the same argument for the treatment of $J_1$ in the proof of Lemma B.1, we have $\| \nabla^2_{\alpha\beta} L_n^m(\hat{\theta}) - \nabla^2_{\alpha\beta} L_n^m(\theta^*) \|_\infty = O_p\left(s\left(\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}}\right)\right)$, and therefore, we have $E_3 = O_p\left(s\left(\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}}\right)\right)$. Combine the results for $E_1$, $E_2$ and $E_3$, we get the desired conclusion.

**APPENDIX C: PROOF OF THE THEOREM 5.1**

According to the definition of $\hat{\alpha}$ in (18), we have

$$\tilde{\alpha} - \alpha^* = \alpha - \alpha^* - H^{-1}_{\alpha|\beta} \hat{U}(\alpha, \hat{\beta}) + \hat{U}(\alpha, \hat{\beta}) \left(H^{-1}_{\alpha|\beta} - \left(\frac{\partial \hat{U}(\alpha, \hat{\beta})}{\partial \alpha}\right)^{-1}\right)$$

$$= \alpha - \alpha^* - H^{-1}_{\alpha|\beta} \left(\hat{U}(\alpha^*, \hat{\beta}) + (\alpha - \alpha^*) \frac{\partial \hat{U}(\alpha, \hat{\beta})}{\partial \alpha}\right) + \hat{U}(\alpha, \hat{\beta}) \left(H^{-1}_{\alpha|\beta} - \left(\frac{\partial \hat{U}(\alpha, \hat{\beta})}{\partial \alpha}\right)^{-1}\right)$$

$$= -H^{-1}_{\alpha|\beta} \hat{U}(\alpha^*, \hat{\beta}) + (\alpha - \alpha^*)H^{-1}_{\alpha|\beta} \left(H^{-1}_{\alpha|\beta} - \left(\frac{\partial \hat{U}(\alpha, \hat{\beta})}{\partial \alpha}\right)^{-1}\right) + \hat{U}(\alpha^*, \hat{\beta}) \left(H^{-1}_{\alpha|\beta} - \left(\frac{\partial \hat{U}(\alpha, \hat{\beta})}{\partial \alpha}\right)^{-1}\right)$$

$$= E_1 + E_2 + E_3 + E_4,$$

where $\tilde{\alpha} = u(\hat{\alpha} - \alpha^*) + \alpha^*$ for $u \in [0, 1]$.

For the term $E_1$, by the Theorem 4.1 we have $\sqrt{n}E_1 \overset{d}{\to} N(0, H^{-1}_{\alpha|\beta})$.

Notice $\frac{\partial \hat{U}(\alpha, \hat{\beta})}{\partial \alpha} = \hat{H}_{\alpha|\beta}$, then by the Lemma 4.3, we have

$$H^*_{\alpha|\beta} = \frac{\partial \hat{U}(\alpha, \hat{\beta})}{\partial \alpha} = O_p\left(s + s'\right)\left(\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}}\right)$$

and thus,

$$H^{-1}_{\alpha|\beta} - \left(\frac{\partial \hat{U}(\alpha, \hat{\beta})}{\partial \alpha}\right)^{-1} = O_p\left(s + s'\right)\left(\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}}\right).$$

Under the assumption $\left(\sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}}\right) = o(1)$, we have

$$H^{-1}_{\alpha|\beta} - \left(\frac{\partial \hat{U}(\alpha, \hat{\beta})}{\partial \alpha}\right)^{-1} = o_p(1).$$
For the term \( E_3 \), notice \( \sqrt{n} \hat{U}(\alpha^*, \beta) \xrightarrow{d} N(0, H_{\alpha^\beta}^*) \), then we have \( \sqrt{n}E_3 = o_p(1) \) because of \((38)\) and the Slutsky’s theorem.

For the term \( E_2 \), notice

\[
|E_2| \lesssim |\hat{\alpha} - \alpha^*| \left| H_{\alpha^\beta}^{* -1} - \left( \frac{\partial \hat{U}(\hat{\alpha}, \hat{\beta})}{\partial \alpha} \right)^{-1} \right| \| \hat{\theta} - \theta^* \|_1 \leq O_p \left( s + s' \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right)^2 \right),
\]

where we use the Corollary 3.1 and \((36)\) in the last step. Under the assumptions \( \frac{m}{n} \propto \log(p) \) and \( \left( \frac{\log(p)}{\sqrt{n}} + \frac{\log(p)}{\sqrt{m}} \right) = o(1) \), using \((32)\) we have

\[
\sqrt{n} \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right)^2 = o(1),
\]

from which we have

\[
\sqrt{n}|E_2| = o_p(1).
\]

For the term \( E_4 \), with the same proof of the Lemma 4.3, we have \( \frac{\partial \hat{U}^\alpha, \hat{\beta}}{\partial \alpha} = O(1) \) after we notice

\[
\frac{\partial \hat{U}^\alpha, \hat{\beta}}{\partial \alpha} - \hat{H}_{\alpha^\beta} = O_p((1 - u)(\hat{\alpha} - \alpha^*)) = o_p(1) \quad \text{and} \quad \hat{H}_{\alpha^\beta} = O_p(1).
\]

By \((37)\) we have

\[
|E_4| \lesssim |\hat{\alpha} - \alpha^*| \left| H_{\alpha^\beta}^{* -1} - \left( \frac{\partial \hat{U}(\hat{\alpha}, \hat{\beta})}{\partial \alpha} \right)^{-1} \right| \| \hat{\theta} - \theta^* \|_1 \leq O_p \left( s + s' \left( \sqrt{\frac{\log(p)}{n}} + \sqrt{\frac{\log(p)}{m}} \right)^2 \right).
\]

We have \( \sqrt{n}|E_4| = o_p(1) \) by the same treatment for \( E_2 \).

Combine the result \( \sqrt{n}E_1 \xrightarrow{d} N(0, H_{\alpha^\beta}^{* -1}) \) and \( \sqrt{n}|E_i| = o_p(1) \) \( (i = 2, 3, 4) \), we get the Theorem 5.1.

\[
\square
\]

APPENDIX D: PROOFS OF THEOREM AND LEMMAS IN SECTION 6

**D.1. Proof of the Lemma 6.1.** By Cauchy-Schwartz inequality, it suffices to show that \( \text{var}(\hat{\theta})_j = O(n^{-1}) \) for any \( j \in S_0 \). Let the event \( \mathcal{A} \) be

\[
\mathcal{A} := \left\{ \| \hat{\theta} - \theta^* \|_1 \leq e(\zeta + 1)s\lambda_1/C_{\min} \right\}.
\]

Then since \( \theta^*_j = 0 \) and note that the exchangeable property in Assumption 6, we have

\[
E \left| E[\hat{\theta}_j | \mathcal{A}] \right| \leq E[|\hat{\theta}_j - \theta^*_j| | \mathcal{A}] = \frac{1}{p_0} E[|\hat{\theta}_{S_0} - \theta_{S_0}|_1 | \mathcal{A}] \leq \frac{1}{p_0} E[|\hat{\theta} - \theta|_1 | \mathcal{A}] \leq \frac{e(\zeta + 1)s\lambda_1}{p_0 C_{\min}},
\]

Similarly,
\[
\var(\hat{\theta}_j | \mathcal{A}) \leq E[\hat{\theta}_j^2 | \mathcal{A}] = E[(\hat{\theta}_j - \theta_j^*)^2 | \mathcal{A}]
\]
\[
\leq \frac{1}{p_0} E[\|\hat{\theta} - \theta^*\|_2^2 | \mathcal{A}] \leq \frac{1}{p_0} E[\|\hat{\theta} - \theta^*\|_2^2 | \mathcal{A}] \leq \frac{e^2(\zeta + 1)^2s^2\lambda_1}{p_0C_{\min}^2} \mathbb{I}_{\mathcal{A}}.
\]
By Theorem 3.1, \( P(\mathcal{A}) \rightarrow 1 \). Hence, by Assumption 7
\[
\var(\hat{\theta}_j) = E[\var(\hat{\theta}_j | \mathcal{A})] + \var[E(\hat{\theta}_j | \mathcal{A})]
\]
\[
\leq \frac{e(\zeta + 1)s\lambda_1}{p_0C_{\min}} P(\mathcal{A}) + \frac{e^2(\zeta + 1)^2s^2\lambda_1^2}{p_0C_{\min}^2} P(\mathcal{A})(1 - P(\mathcal{A}))
\]
\[
= O(s\lambda_1/p_0) = O\left(\frac{1}{p_0} \sqrt{\frac{\log p}{n}} + \sqrt{\frac{\log p}{n \log p}}\right) = O\left(\frac{1}{\sqrt{n \log p}} \frac{\log p}{n}\right) = O(n^{-1}),
\]
where we use the fact that \( s = O(1) \) and \( m/n \sim \log p \). \( \square \)

**D.2. Proof of the Theorem 6.1.** Since \( M_j \) is naturally symmetric about 0 for any \( j \in S_0 \), from Proposition 1 in [19], we only need to verify that there exists some constants \( C > 0 \) and \( \alpha \in (0, 2) \) such that
\[
\var\left(\sum_{j \in S_0} \mathbb{I}(M_j > t)\right) \leq Cp_0^\alpha, \quad \forall \ t \in \mathbb{R}.
\]

It is suffices to show
\[
\sup_{t \in \mathbb{R}} \var\left(\frac{1}{p_0} \sum_{j \in S_0} \mathbb{I}(M_j > t)\right) \rightarrow 0.
\]

Denote the correlated set as \( \mathcal{C} := \{(j, k) : j, k \in S_0, \Theta_{j,k} \neq 0\} \) and the uncorrelated set as \( \mathcal{C}^c := \{(j, k) : j, k \in S_0, \Theta_{j,k} = 0\} \), then
\[
\var\left(\frac{1}{p_0} \sum_{j \in S_0} \mathbb{I}(M_j > t)\right) = \frac{1}{p_0^2} \sum_{(j,k) \in \mathcal{C}} \cov(\mathbb{I}(M_j > t), \mathbb{I}(M_k > t))
\]
\[
+ \frac{1}{p_0^2} \sum_{(j,k) \in \mathcal{C}^c} \cov(\mathbb{I}(M_j > t), \mathbb{I}(M_k > t))
\]
\[
\leq \frac{\lvert \mathcal{C} \rvert}{p_0^2} + \max_{(j,k) \in \mathcal{C}^c} \left| P(M_j > t)P(M_k > t) - H^2(t) \right|
\]
\[
+ \frac{1}{p_0^2} \sum_{j,k \in \mathcal{C}^c} \left| P(M_j > t, M_k > t) - H^2(t) \right|.
\]

where \( H(t) = P(\sgn(Z_1Z_2)f(Z_1, Z_2) > t) \) with \( Z_1 \) and \( Z_2 \) are independent standard normal distribution. First, by Assumption 8,
\[
\frac{\lvert \mathcal{C} \rvert}{p_0^2} \leq \frac{v^* p_0}{p_0^2} = o\left(\frac{\sqrt{n}}{p_0 \log p}\right) = \frac{1}{\log^{3/2} p} \rightarrow 0.
\]
From Theorem 5.1, for whole-data based statistic $T_j$, denote $	ilde{T}_j := \hat{\theta}_j \sqrt{nH^*_j-i}$

$$\sup_{t \in \mathbb{R}} |P(T_j \leq t) - \Phi(t)| \leq \sup_{t \in \mathbb{R}} |P(T_j \leq t) - P(\tilde{T}_j \leq t)| + \sup_{t \in \mathbb{R}} |P(\tilde{T}_j \leq t) - \Phi(t)|$$

$$= \sup_{t \in \mathbb{R}} E[P(T_j \leq t | \tilde{H}_{j-i}, \tilde{H}_{j-ij}) - P(\tilde{T}_j \leq t | \tilde{H}_{j-i}, \tilde{H}_{j-ij})] + o(1)$$

$$= \sup_{t \in \mathbb{R}} E \left| \int_{t}^{H^*_j-i/\tilde{H}_{j-ij}} \phi(x) \, dx \right| + o(1)$$

$$\leq \sup_{t \in \mathbb{R}} E \left| t \left( \sqrt{\frac{H^*_j-i}{\tilde{H}_{j-ij}}} - 1 \right) \right| + o(1)$$

$$= \lim_{n \to \infty} \sup_{t \in [-n, n]} |t| E \left( \sqrt{\frac{H^*_j-i}{\tilde{H}_{j-ij}}} - 1 \right) + o(1) = 0,$$

where the last step is by the uniform integrability of $\tilde{H}_{j-i}$. By Lemma A.5 in [19], we have $\sup_{t \in \mathbb{R}, j \in \mathbb{R}} |P(M_j > t) - H(t)| \to 0$, which implies

$$\sup_{t \in \mathbb{R}, (j,k) \in \mathcal{E}^*} |P(M_j > t)P(M_k > t) - H^2(t)| \to 0.$$

For the last term in (41), we have the following decomposition

$$P(M_j > t, M_k > t) = P \left( T_j^{(2)} > I_t(T_j^{(1)}), T_k^{(2)} > I_t(T_k^{(1)}), T_j^{(1)} > 0, T_k^{(1)} > 0 \right)$$

$$+ P \left( T_j^{(2)} > I_t(T_j^{(1)}), T_k^{(1)} < -I_t(T_j^{(1)}), T_j^{(1)} > 0, T_k^{(1)} < 0 \right)$$

$$+ P \left( T_j^{(2)} < -I_t(T_j^{(1)}), T_k^{(2)} > I_t(T_k^{(1)}), T_j^{(1)} < 0, T_k^{(1)} > 0 \right)$$

$$+ P \left( T_j^{(2)} < -I_t(T_k^{(1)}), T_j^{(2)} < -I_t(T_k^{(1)}), T_j^{(1)} < 0, T_k^{(1)} < 0 \right)$$

$$:= D_1 + D_2 + D_3 + D_4$$

where $I_t = \inf\{u \geq 0 : f(u, v) > t\}$. For $D_1$,

$$D_1 = E \left[ P(T_j^{(2)} > I_t(x), T_j^{(2)} > I_t(y)) \mid T_j^{(1)} = x, T_k^{(1)} = y \right]$$

$$= E \left[ P(\tilde{T}_j^{(2)} > I_t(x), \tilde{T}_k^{(2)} > I_t(y)) \mid T_j^{(1)} = x, T_k^{(1)} = y \right] + o(1)$$

$$\leq E \left[ Q(I_t(x))Q(I_t(y)) \mid T_j^{(1)} = x, T_k^{(1)} = y \right] + c_1 |\text{cov}(\tilde{T}_j^{(2)}, \tilde{T}_k^{(2)})| + o(1)$$

where $Q(t) = 1 - \Phi(t)$, $c_1$ is some positive number, and the last "≤" follows from Mehler’s identity and Lemma 1 in [2]. Indeed, for the zero mean bivariate normal distribution function $\Phi_\rho(t_1, t_2)$ with covariance matrix $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, Mehler’s identity guarantees that

$$\Phi_\rho(t_1, t_2) = \Phi(t_1)\Phi(t_2) + \sum_{n=1}^{\infty} \frac{\rho^n}{n!} \phi^{(n-1)}(t_1)\phi^{(n-1)}(t_2)$$

$$= \Phi(t_1)\Phi(t_2) + \rho\phi(t_1)\phi(t_2) + \sum_{n=2}^{\infty} \frac{\rho^n}{n!} \phi^{(n-1)}(t_1)\phi^{(n-1)}(t_2)$$

$$\leq \Phi(t_1)\Phi(t_2) + \rho\phi(t_1)\phi(t_2) + c_2\rho \leq \Phi(t_1)\Phi(t_2) + (1/(2\pi)) + c_2\rho.$$
where the first "$\leq"$ is due to
\[ \sum_{n=2}^{\infty} \frac{\sup_{t \in \mathbb{R}} \phi(n^{-1})}{n!} < \infty, \]
by Lemma 1 in [2]. In the next step, we will prove that for any \((j, k) \in \mathcal{E}_c\), \(\text{cov}(\tilde{\theta}_j, \tilde{\theta}_k) = O(n^{-1})\), which implies \(\text{cov}(\tilde{T}_j^{(2)}, \tilde{T}_k^{(2)}) = O\left(\sqrt{H_{j|j-1}^* H_{k|k-1}^*}\right)\) in turn.

By the definition of one-step estimator (41)
\[
\text{cov}(\tilde{\theta}_j, \tilde{\theta}_k) = \text{cov}\left(\theta - \left(\nabla_j \hat{U}(\theta_j, \hat{\theta}_j)\right)^{-1}\hat{U}(\hat{\theta}), \theta - \left(\nabla_k \hat{U}(\theta_k, \hat{\theta}_k)\right)^{-1}\hat{U}(\hat{\theta})\right)
= (\text{var}(\hat{\theta}))_{jk} - \text{cov}\left(\nabla_j \hat{U}(\theta_j, \hat{\theta}_j)\right)^{-1}\hat{U}(\hat{\theta}) - \text{cov}\left(\nabla_k \hat{U}(\theta_k, \hat{\theta}_k)\right)^{-1}\hat{U}(\hat{\theta}) + \text{cov}\left(\nabla_j \hat{U}(\theta_j, \hat{\theta}_j)\right)^{-1}\hat{U}(\hat{\theta}), \nabla_k \hat{U}(\theta_k, \hat{\theta}_k)\right)^{-1}\hat{U}(\hat{\theta})\).
\]
The \(U\) function above is not equal for each term, but it does not matter. The first term is \(O(n^{-1})\) by Lemma 6.1. Due to Cauchy-Schwartz inequality, it is sufficient to show the last term above is \(O(n^{-1})\). Indeed, note that
\[
\hat{U}(\hat{\theta}) = \hat{U}(\theta_j^*, \hat{\theta}_j) + \sqrt{\sum_j \hat{U}(\hat{\theta}_j, \hat{\theta}_j)(\theta_j - \theta_j^*)},
\]
where \(\hat{\theta}_j\) lies between \(\theta_j\) and \(\hat{\theta}_j\). By the proof of Theorem 5.1, we know that
\[
\nabla_j \hat{U}(\hat{\theta}_j, \hat{\theta}_j) = H_{j|j-1}^* + o_p(1),
\]
and
\[
\nabla_j \hat{U}(\theta_j^*, \hat{\theta}_j) = \nabla_j \hat{U}(\hat{\theta}_j, \hat{\theta}_j) + O_p(|\theta_j^* - \theta_j^*|) = H_{j|j-1}^* + o_p(1).
\]
Since Assumption 4 confirms that \(H_{j|j-1}^*\) must be in an interval away from zero point. By the continuity of \(\nabla_j \hat{U}(\cdot, \hat{\theta}_j)\), we know that
\[
\nabla_j \hat{U}(v, \hat{\theta}_j) \in [c_2, c_3]
\]
for \(v = \theta^*_j, \hat{\theta}_j\), and \(\hat{\theta}_j\), in which \(c_2\) and \(c_3\) are some positive number independent with \(n, m,\) and \(p\). Hence,
\[
\text{cov}\left(\nabla_j \hat{U}(\theta_j^*, \hat{\theta}_j)\right)^{-1}\hat{U}(\hat{\theta}), \nabla_k \hat{U}(\theta_k^*, \hat{\theta}_k)\right)^{-1}\hat{U}(\hat{\theta})\right) \leq c_2^{-2} \text{cov}(\hat{U}(\theta_j^*, \hat{\theta}_j), \hat{U}(\theta_k^*, \hat{\theta}_k)) = c_2^{-2} \text{cov}(\hat{U}(\theta_j^*, \hat{\theta}_j) + \nabla_j \hat{U}(\theta_j^*, \hat{\theta}_j)(\theta_j - \theta_j^*), \hat{U}(\theta_k^*, \hat{\theta}_k)) + \nabla_k \hat{U}(\theta_k^*, \hat{\theta}_k)(\theta_k - \theta_k^*))
\leq \text{cov}(\hat{U}(\theta_j^*, \hat{\theta}_j), \hat{U}(\theta_k^*, \hat{\theta}_k)) + \text{cov}(\hat{U}(\theta_j^*, \hat{\theta}_j)) + \text{cov}(\hat{U}(\theta_k^*, \hat{\theta}_k)).
\]
For the first term above, a slight generalization of Theorem 4.1 guarantees it should be \(n^{-1} \sqrt{H_{j|j-1}^* H_{k|k-1}^*}\): Cauchy-Schwartz inequality that \(\text{cov}(\xi_1, \xi_2) \leq \sqrt{\text{var}(\xi_1) \text{var}(\xi_2)}\) for any random variables \(\xi_1\) and \(\xi_2\), together with Lemma 6.1 gives the second term above is also \(O(n^{-1})\); The third term above is naturally \(O(n^{-1})\) by Cauchy-Schwartz inequality again. As a result, for \(D_1\), we have
\[
D_1 \leq E \left[Q(I(x))Q(I(y)) \mid T_j^{(1)} = x, T_k^{(1)} = y\right] + c_4 \sqrt{H_{j|j-1}^* H_{k|k-1}^*} + o(1),
\]
where \(c_4\) is some positive number free of \(n, m, p\). Similarly, we can upper bound \(D_2, D_3, D_4,\) and combine them together to get an upper bound on \(P(M_i > t, M_j > t)\) specified as below
\[
P(\text{sgn}(Z_j^{(2)}T_j^{(1)}) f(Z_j^{(2)}T_j^{(1)} > t, \text{sgn}(Z_k^{(2)}T_k^{(1)}) f(Z_k^{(2)}T_k^{(1)} > t) + c_5 \sqrt{H_{j|j-1}^* H_{k|k-1}^*} + o(1).
\]
with some positive $c_5$, in which $Z^{(2)}_j$ and $Z^{(2)}_k$ are two independent random variables following the standard normal distribution. By conditioning on the signs of $Z^{(2)}_j$ and $Z^{(2)}_k$ and decomposing the above display as previous, it can be shown

$$P(M_j > t, M_k > t) \leq H^2(t) + c_6 \sqrt{H^*_j H^*_k} + o(1)$$

with $c_6 > 0$. Similarly, we can establish the corresponding lower bound. Therefore,

$$\frac{1}{p_0} \sum_{j,k \in \mathcal{S}_1} |P(M_j > t, M_k > t) - H^2(t)| \leq \frac{c_6}{p_0} \sum_{j,k \in \mathcal{S}_0} \sqrt{H^*_j H^*_k} + o(1)$$

$$\leq \frac{c_6}{p_0} p_0 \lambda_{\max}(H^*) + o(1) \to 0$$

uniformly on $t \in \mathbb{R}$ by Assumption 6. Then (39) is valid, we complete the proof. \hfill \Box

D.3. Proof of the Theorem 6.2. From Theorem 6.1, we know procedure faithfulness holds, i.e.

$$\limsup_{n,p \to \infty} \sum_{j \in \mathcal{S}_0} I_j \leq q.$$ 

Therefore, $\limsup_{n,p \to \infty} \max_{k \in \mathcal{S}_1} I_k \geq 1 - q$, which implies

$$\limsup_{n,p \to \infty} \max_{k \in \mathcal{S}_1} I_k \geq \frac{1 - q}{s}.$$ 

And for any $\alpha \in (0, 1)$, note that $p_0 \to \infty$, we have $(1 - q)/s > \alpha/p_0$. Then

$$\limsup_{n,p \to \infty} \frac{1}{s} \sum_{k \in \mathcal{S}_1} \mathbb{I}(I_k \leq \frac{\alpha}{p_0}) \leq \limsup_{n,p \to \infty} \frac{1}{s} \mathbb{I}(\max_{k \in \mathcal{S}_1} I_k \leq \frac{\alpha}{p_0}) \leq \limsup_{n,p \to \infty} \frac{1}{s} \mathbb{I}(\max_{k \in \mathcal{S}_1} I_k < \frac{1 - q}{s}) = 0.$$ 

So rank faithfulness holds. Finally, since for any $j \in \mathcal{S}_0$, $M_j$ has the same limit distribution, then

$$j \in \mathcal{S} \iff M_j > \tau_q$$

shares with the same law by Assumption 6, the null exchangeability are also satisfied. By Proposition 2 in [19], we obtain this theorem. \hfill \Box

D.4. Proof of the Theorem 6.3. Let $\mathcal{N}_1(t)$ be the number of true null hypotheses with e-values larger than or equal to $t$ and $\mathcal{N}_2(t)$ be the number of all hypotheses with e-values larger than or equal to $t$. Define

$$t_e := \inf \left\{ t \geq 0 : \frac{\mathcal{N}_2(t) t}{p} \geq \frac{1}{e} \right\}.$$ 

Notice

$$\frac{k e_{(k)}}{p} = \frac{N_2(e_{(k)}) e_{(k)}}{p},$$
therefore in Step 2 of the algorithm, we actually reject all hypotheses with e-values larger than or equal to \( t_\epsilon \), from which the FDR can be written as

\[
\text{FDR}_G = E \left( \frac{N_1(t_\epsilon)}{N_2(t_\epsilon)} \right) = \frac{\epsilon}{p} \cdot E(t_\epsilon N_1(t_\epsilon)),
\]

where we use \( N_2(t_\epsilon) = \frac{p}{ct_\epsilon} \) in the last step.

Notice \( N_1(t_\epsilon) \) can be written to

\[
N_1(t_\epsilon) = \sum_{k \in \mathcal{N}} 1(E_k \geq t_\epsilon).
\]

Therefore we have

\[
E(t_\epsilon N_1(t_\epsilon)) = \sum_{k \in \mathcal{N}} E(t_\epsilon 1(E_k \geq t_\epsilon)) \leq \sum_{k \in \mathcal{N}} E(E_k),
\]

where the last step is because \( t_\epsilon 1(E_k \geq t_\epsilon) \leq E_k \) (a.s.).

Thus we bound the FDR by

\[
\text{FDR}_G \leq \epsilon \cdot \frac{|\mathcal{N}|}{p} \cdot \frac{1}{|\mathcal{N}|} \sum_{k \in \mathcal{N}} E(E_k) \leq \epsilon \cdot \frac{1}{|\mathcal{N}|} \sum_{k \in \mathcal{N}} E(E_k).
\]

Taking limit superior on both side of (42), by the Assumption 9, we have

\[
\lim_{n \to \infty} \text{FDR}_G \leq \epsilon \lim_{n \to \infty} \frac{1}{|\mathcal{N}|} \sum_{k \in \mathcal{N}} E(E_k) \leq \epsilon.
\]

\[\square\]