Kerr-like Phantom Wormhole

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In this work we study a Kerr-like wormhole with phantom matter as source. It has three parameters: mass, angular momentum and scalar field charge. This wormhole has a naked ring singularity, other wise it is regular everywhere. The mean feature of this wormhole is that the mouth of the throat lie on a sphere of the same radius as the ring singularity an avoids any observer to see or to reach the singularity, it behaves like an anti-horizon. We analyse the geodesics of the wormhole and find that an observer can go through the geodesics without troubles, but the equator presents an infinity potential barrier which avoids to reach the throat. From an analysis of the Riemann tensor we obtain that the tidal forces permits the wormhole to be traversable for an observer like a human being.

I. INTRODUCTION

The research on wormhole started with Einstein trying to give a field representation of particles [1]. The idea was further developed by Ellis, [2], where instead of particles they try to model them as bridges between two regions of the space-time. Later on, in the seminal Morris and Thorne’s work [3], the idea of considering such solutions as connections between two separated regions of the Universe, has attracted a lot of attention.

In the 80’s, the astronomer Carl Sagan asked for help to physicist and turned out that the solution proposed by Ellis [2], actually could be interpreted as the identification, the union, of two different regions. The idea of Ellis is that no matter how far the two regions are, or even if those two regions were in different space-time, by means of this gluing one could just identify them, and obtain a solution which allows to go from one region to another. Morris and Thorne [3] showed that these wormhole solutions need to violate the energy conditions, such type of matter is called exotic (see [4] for a detailed review on this subject). The solutions can exist but they need to be generated by matter which apparently does not exist. This is the reason why this solutions remained in the realm of fiction.

However, more and more evidence was building towards the presence in our Universe of unknown types of matter and energy which do not necessarily obey the energy conditions. Now we know that the Universe is formed by 73% of dark energy. This new type of matter composes the majority in the Universe and happens to be everywhere [5]. There is now an agreement among the scientific community that matter which violates some of the energy conditions is very plausible to exit. Thus, the issue that the wormholes can be rejected due to the type of matter that they need is, at least, diminished. The existence of wormholes in the universe is very interesting because they could be highways to visit stars and galaxies, otherwise it will be impossible to go enough far away to visit other worlds. Of course this fact is very speculative to this stage, but no observation can discard the existence of phantom energy. Nowadays we know that may be a combination of phantom and quintessence field could be the dominant component in the universe [6]. The fact that phantom energy can be the source of wormholes [1]-[5] is more than exciting, and give rise to investigate about the existence of stars made of this kind of matter.

Maybe the mayor problem faced by the wormhole solutions is their stability. The stability problem of the bridges has been studied since the 60’s by Penrose [7], in connection to the stability of the Cauchy horizons. The stability of the throat of a wormhole was studied numerically by Shinkay and Hayward [8], where they show that the wormhole proposed by Thorne [3] collapses possibly towards a black hole and the throat closes, when it is perturbed by a scalar field with stress-energy tensor defined with the usual sign. They also find that the throat of the wormhole grows exponentially, when the perturbation is due to a scalar field of the same type as that making the wormhole, thus showing that the solution is highly unstable (see also [9]). In [10] it was conjectured that rotation or the magnetic field could stabilize a phantom star. The idea is that a rotating solution would have more possibilities of being stable, as well as more general static spherically symmetric solutions than the one proposed by Thorne.

We look for wormhole solutions which are traversable, that is, a test particle can go from one side of the throat to the other without facing large tidal forces and in a finite time. We follow the conjecture that the rotation can stabilize the wormhole, some rotating solutions were

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studied in the past, as an approximation \([11]\) or as an exact solution of the Einstein equations \([10]\). Nevertheless, this last solution has two main problems, it is not asymptotically flat and must be matched with a static one and it has a spherical singularity surrounding the wormhole. In this work we study the solution obtained in \([12]\), it has three parameters: mass, angular momentum and scalar charge. It has also a gauge parameter which determine the radius of the throat and a naked ring singularity, exactly the same one as the Kerr solution. We will show that it is possible to go though the throat going by the geodesics.

This work is organized as follows. In section II we present the metric and give some of its limits. We also write the energy density and specialize it to the north pole, showing that from the north pole an observer measures a finite energy density, corroborating the fact that from the north pole this wormhole is traversable. In section III we study the geodesics. We find that the geodesics are refuse by the ring singularity (see also \([13]\)), the easiest way to go through the throat is by the polar geodesics which can go from one part of the throat to the other side. Here also we calculate the accelerations of a vector going through the throat by the polar geodesics and find that this acceleration is very weak. In section IV we analyze the tidal forces. Of course the tidal forces are infinity on the study the energy conditions. In section V we analyze the geodesics.

II. THE LINE ELEMENT IN DIFFERENT COORDINATES

We consider a stationary and axially symmetric spacetime possessing a time-like Killing vector field \((\xi)^\alpha = (\partial/\partial t)^\alpha\), generating invariant time translations and a space-like Killing vector field \(\mu^\alpha = (\partial/\partial \varphi)^\alpha\) generating invariant rotations with respect to \(\varphi\). The corresponding line element can be expressed in Boyer-Lindquist coordinates,

\[
ds^2 = -f \, dt^2 + \omega \, d\varphi^2 + \frac{1}{f} \left( \Delta \left( \frac{dl^2}{\Delta_l} + d\theta^2 \right) + \Delta_1 \sin^2 \theta \, d\varphi^2 \right),
\]

where

\[
\Delta = (l - l_1)^2 + (l_0^2 - \Delta_1^2) \cos^2 \theta,
\]

\[
\Delta_1 = (l - l_1)^2 + (l_0^2 - \Delta_1^2),
\]

The line element components \(\omega\) and \(f\) read

\[
\omega = \frac{a}{2\Delta} \frac{(l - l_1)}{\Delta} \sin^2 \theta,
\]

\[
f = \frac{(a^2 + k_1^2)}{a^2 + k_1^2} e^{\frac{\lambda}{2}},
\]

and \(\Delta, \Delta_1 \to \Delta^2\), so that the line element

\[
ds^2 \to -dt^2 + dl^2 + l^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right),
\]

is a flat space in spherical coordinates, i.e., metric \([1]\) is asymptotically flat.

The scalar curvature (and the other invariants of the metric) are of the form

\[
\text{Invariants} = \frac{F}{8k_1^2 \Delta r^3 \Delta_1^2 (a^2 + k_1^2 e^{2\lambda})^2},
\]

FIG. 1: The function \(f\) with \(l_1 = 1.0, l_0 = 1.1, a = 0.1, k = 0.11\) for different values of \(\theta\) (\(\theta = 0\) dotted line, \(\theta = \pi/2\) solid line and \(\theta = \pi/4\) dashed line). Observe that there is a minimum in 0.859 when \(\theta = 0\),

where the parameter \(\lambda\) is given by

\[
\lambda = \frac{a^2 + k_1^2}{2k_1 \Delta} \cos \theta,
\]

being \(l_1, l_0\) parameters with units of distance, such that \(g_{rr} > 0\), that is

\[
l_0^2 > l_1^2 > 0,
\]
FIG. 2: The Ricci scalar for the metric (1).

where \( F \) is a complicated function, which is different for the different invariants of the metric but without singularities and \( r_1, r_2 \) and \( r_3 \) are different coefficients depending of the invariant. In Fig. 2 we show the behaviour of the Ricci scalar of metric (1). The restriction (7) for the parameters \( l_0 \) and \( l_1 \) avoids that \( \Delta_1 \) in (3) can be null, but \( \Delta \) given in (2) is zero for \( \theta = \pi/2 \). Thus metric (1) has a naked singularity at \( \Delta = 0 \), which represents a ring on the equatorial plane centred in the centre of the wormhole.

The ADM mass and angular momentum \([14]\) are given by

\[
M = \frac{1}{8\pi} \lim_{S_t \to \infty} \oint_{S_t} (k - k_0) \sqrt{\sigma} d^2 \theta \tag{11}
\]

\[
J = -\frac{1}{8\pi} \lim_{S_t \to \infty} \oint_{S_t} (K_{ij} - K \gamma_{ij}) \phi^i r^i \sqrt{\sigma} d^2 \theta \tag{12}
\]

In our case we obtain

\[
M = -l_1 \tag{13}
\]

\[
J = a \tag{14}
\]

Thus the wormhole has a negative mass given by the size of the throat and an angular momentum given by the parameter \( a \).

If we drop out the acceleration \( a = 0 \), the rotation \( \omega = 0 \), the metric (1) transforms into

\[
ds^2 = -f \, dt^2 + \frac{1}{f} \left( \frac{dl^2}{\Delta_1} + \Delta_1 \sin^2 \theta \, d\varphi^2 \right) \tag{15}
\]

where the line element component \( f \) now reads

\[
f = e^{-\lambda} \tag{16}
\]

and the parameter \( \lambda \) is give by

\[
\lambda = \frac{k_1}{2 \Delta} \cos \theta, \tag{17}
\]

Other interesting case is when the mass parameter \( l_1 = 0 \). The metric transforms into

\[
ds^2 = -f \, (dt + \omega \, d\varphi)^2 + \frac{1}{f} \left( \frac{l_1^2 + l_0^2 \cos^2 \theta}{l^2 + l_0^2} \, dt^2 + (l^2 + l_0^2) \left[ (l^2 + l_0^2 \cos^2 \theta \, \theta^2 + \sin^2 \theta \, d\varphi^2 \right] \right), \tag{18}
\]

Observe that the term inside of the bracket \([\ ]\) is the solid angle element, but with a factor multiplying \( d\theta \). This is the modification of the solid angle due to the axial symmetry.

To see the throat of the wormhole, it is convenient to write metric (1) as

\[
ds^2 = -f \, (dt + \omega \, d\varphi)^2 + \frac{K}{f} \, dt^2 + \frac{\Delta_1}{f} \left[ K \, dt^2 + \sin^2 \theta \, d\varphi^2 \right], \tag{19}
\]

where \( K = \Delta/\Delta_1 \). Again, the quantity between the brackets \([\ ]\) can be interpreted as the solid angle, but now with a modification given by the \( K \) function multiplying \( d\theta^2 \), see the plot of \( K \) for different values of \( \theta \) in Fig. 3. The modification of the solid angle is due the fact that the space-time is now axially symmetric and not spherically symmetric. Nevertheless, in this case, this modification is small everywhere, except near a sphere of radius \( l_1 \), the radius of the ring singularity, indicating that the space-time is almost spherically symmetric outside this sphere, (see figure 3). Amazingly, the space-time is just spherically symmetric at the equator \( \theta = \pi/2 \) of the phantom star.

FIG. 3: The function \( K \) with \( l_1 = 1, l_0 = 1.1, a = 0.1, k = 0.11 \) for different values of \( \theta \). Observe that \( K = 1 \) everywhere except close to the ring singularity.
The function $\Delta_1/f$ can be also interpreted as the radial coordinate to the square. In this coordinates this function never reaches the value zero, see Fig. 4 indicating that the area of this region is never zero and thus that there is a throat.

![Diagram](image)

**FIG. 4:** The function $R = \sqrt{\Delta_1/f}$ with $l_1 = 1, l_0 = 1.1, a = 1, k = 1$, for different values of $\theta$. Amazingly, the function $R$ is well behave everywhere, and is smooth on the equator ($\theta = \frac{\pi}{2}$ solid line).

Now we write metric (1) in cylindrical coordinates ($\hat{\rho}, z$). These coordinates are defined by the transformation,

$$l = \sqrt{\hat{\rho}^2 + z^2} = \frac{l_0^2 - l_1^2}{4\sqrt{\hat{\rho}^2 + z^2}} + l_1$$

$$\theta = \arctan\left(\frac{\hat{\rho}}{z}\right)$$

with inverse given by

$$\hat{\rho} = \frac{1}{2} \left( l - l_1 + \sqrt{\Delta_1} \right) \sin \theta$$

$$z = \frac{1}{2} \left( l - l_1 + \sqrt{\Delta_1} \right) \cos \theta$$

leaving ($t, \varphi$) as before. We can rewrite the line element as

$$ds^2 = -f (dt + \omega d\varphi)^2 + \frac{1}{(\hat{\rho}^2 + z^2)^2} f \left( K (d\hat{\rho}^2 + dz^2) + \hat{\rho}^2 \Delta_+ d\varphi^2 \right),$$

being

$$\Delta_+ = \hat{\rho}^2 + z^2 \pm \frac{l_0^2 - l_1^2}{4},$$

$$K = \Delta_+ \pm \left( l_0^2 - l_1^2 \right) z^2,$$

$$\omega = a \frac{\hat{\rho}^2 \Delta_+}{2 \sqrt{\hat{\rho}^2 + z^2} K},$$

$$\lambda = \left( \frac{a^2 + k_1^2}{2 k_1} \right)^2 \frac{\sqrt{\hat{\rho}^2 + z^2}}{K}.$$

where $f$ has the same form as before, (5). Using the Weyl coordinates (14) ($\rho, \zeta$) defined by

$$\rho = \sqrt{l_1^2 - \rho^2 + l_0^2 \sin(\theta)}$$

$$\zeta = (l - l_1) \cos(\theta)$$

we can write the metric in terms of the prolate coordinates defined as

$$\rho = \sigma (x^2 - 1)(1 - y^2)$$

$$\zeta = \sigma x y$$

being $\sigma^2 = l_0^2 - l_1^2$. The inverse of this coordinates is given by

$$\rho = \sigma \frac{x + \sqrt{x^2 - 1}}{1 - y^2}$$

$$\zeta = \sigma \frac{y}{x^2 - 1}$$

The relationship between the prolate and the Boyer-Lindquist coordinates is

$$\sigma x = l_1 - l_0$$

$$y = \cos(\theta).$$

In prolate coordinates, metric (1) is

$$ds^2 = -f (dt + \omega d\varphi)^2 + \frac{\sigma^2(x^2 - 1)(1 - y^2)^2}{f} \left( \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) + \frac{\sigma^2(x^2 - 1)(1 - y^2)}{f} d\varphi^2,$$

where the functions $\lambda$ and $\omega$ written in these coordinates read

$$\lambda = \frac{a^2 + k_1^2}{2 k_1 \Delta} y$$

$$\omega = \frac{a \sigma x}{2 \Delta} (1 - y^2)$$

In these coordinates function $\Delta$ is

$$\Delta = \sigma^2 (x^2 + y^2).$$

The wormhole geometry may be described by an embedding diagram in three dimensional Euclidean space at fixed moment in time and for a fixed value of $\theta$. Accordingly, we will write the line element as follows

$$ds^2 = f^{-1} K \left( \frac{dl}{d\rho} \right)^2 d\rho^2 + \rho^2 \left( \sin^2(\theta_0) - \frac{\omega^2 f^2}{\Delta_1} \right) d\varphi^2,$$

where $\rho^2 = f^{-1} \Delta_1$. The resulting surface of revolution has the parametric form $b(\rho, \varphi) = (\rho \cos(\varphi), \rho \sin(\varphi), z(\rho, \theta_0))$. The requirements are met by $z = z(\rho, \theta_0)$ for any fixed $\theta$ such that:

$$\frac{dz}{d\rho}^2 = f^{-1} K \left( \frac{dl}{d\rho} \right)^2 - 1.$$
Furthermore, the flare-out condition \( \frac{d^2 \rho}{d \tau^2} > 0 \) (for the upper universe) at or near the throat, are

\[
\frac{d^2 \rho}{d \tau^2} = (f^{-1} K - (\frac{d \rho}{d \tau})^2)^{-1} \frac{d^2 \rho}{d l^2} + \frac{1}{2} \frac{d \rho}{d l} (f^{-1} K - (\frac{d \rho}{d \tau})^2)^{-1} > 0.
\]  

(33)

The embedding surface give by (32) and the flare-out condition (33) are plotted using the values: \( l_1 = 1, l_2 = 1.5, a = 0.1 \) and \( k = 0.11 \), see Fig 5 and Fig 6. It is important to mention that given this values the flare out condition holds everywhere except in a vicinity of the singularity \((\theta = \pi/2 \text{ and } l = 1)\).

![FIG. 5: Function z for fixed \( \theta = 0 \). Here we set \( l_1 = 1, l_2 = 1.5, a = 0.1 \) and \( k = 0.11 \). This plot shows embedding diagram for a wormhole with metric given by (1). Since our metric is axially symmetric for different values of \( \theta \) the shape of the throat may change.](image)

![FIG. 6: Function z for fixed \( \theta = \pi \). The throat is located at \( l = 1 \). Here we set \( l_1 = 1, l_2 = 1.5, a = 0.1 \) and \( k = 0.11 \). The flare out condition is satisfied, i.e., it is positive near the throat.](image)

**III. GEODESICS**

We start with the study of the geodesics using the Boyer-Lindquist coordinates. We first are interested in radial geodesics to see whether an observer can penetrate the wormhole or not. Of course the ring singularity avoids any observed to penetrate the wormhole, at least going by the equator. Nevertheless, the easiest possibility for a traveler for going thorough the throat is that the observer travels by the polar geodesic, that means, the line that joints the north pole with the south pole. In the surface of the sphere of radius \( l = l_1 \) the traveler meets the mouth of the throat, where the tidal forces reduce its magnitude due to the symmetry of the singularity as we saw in the latter section. In order to see that, we obtain the polar geodesics in the different coordinate systems.

For doing so, let \( \tau \) be an affine parameter and \( u^\mu = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi}) \), with \( \dot{t} = \frac{dt}{d\tau} \), etc., the vector velocity of an observer, such that the equation \( u^\mu u_\mu = -1 \) holds. In Boyer-Lindquist coordinates this expression reduces to

\[-1 = -f (\dot{t} + \omega \dot{\varphi})^2 + \frac{1}{f} \left( \Delta \left( \frac{i^2}{\Delta_1} + \dot{\theta}^2 \right) + \Delta_1 \sin^2 \theta \dot{\varphi}^2 \right). \]  

(34)

For the polar geodesic \( \theta = 0 \), this implies that \( \Delta_1 = \Delta \) and \( \omega = 0 \), that means that from this geodesic, the observer does not feel the rotation of the wormhole. The equation (34) reduces to

\[-1 = -f \dot{t}^2 + \frac{1}{f} \dot{i}^2, \]  

(35)

In this metric \( \frac{\partial}{\partial t} \) is a Killing vector, thus we have that \( f \dot{t} = \sqrt{2} E \), being \( E \) a constant. Then the geodesic equation (35) reduces to

\[\frac{1}{2} \dot{i}^2 + \frac{1}{2} f = E^2 \]  

(36)

This last is a dynamical equation where we can know the motion of the test particle if we know the potential, in this case given by \( 1/2 f \).

From the geodesic equations

\[\frac{d^2 x^\alpha}{d \tau^2} + \Gamma^\alpha_{\beta \gamma} \frac{dx^\beta}{d \tau} \frac{dx^\gamma}{d \tau} = 0 \]  

(37)

where \( \tau \) parametrize the curve \( x^\alpha (\tau) \), we can obtain the same results. In terms of the prolate coordinates, the non-vanishing Christoffel symbols of this metric are given by

\[\Gamma^t_{\tau t} = -\frac{\dot{t}^2}{2f}, \quad \Gamma^t_{r r} = -\frac{l_1 \dot{r}}{2f}, \quad \Gamma^r_{\tau t} = \frac{\dot{r}}{2f}. \]  

(38)

Thus the geodesic equations reads

\[\ddot{x} + \frac{\Gamma^x_{xx}}{x^2} \dot{x}^2 + \frac{\Gamma^x_{tt}}{t^2} \dot{t}^2 = 0. \]  

(39)
\[ \ddot{t} + 2\Gamma^i_{jk}\dot{x}^j = 0 \]  
Explicitly, the geodesic equation for the variable \( t \) on the north pole is given by

\[ f\dot{t} = \sqrt{2E} \]  
being \( E \) the integrations constant and the corresponding one for the variable \( x \)

\[ \frac{\sigma^2}{f}\dot{x}^2 - f\dot{t}^2 = -1 \]  
Again we can write both equations (41) and (42) together to obtain

\[ \frac{1}{2}\dot{x}^2 + \frac{f}{2\sigma^2} = E^2. \]  
In order to have a better interpretation of equation (43), we can transform it into

\[ \frac{1}{2}\dot{x}^2 - \frac{f}{2\sigma^2} + \frac{1}{2\sigma^2} = E^2 + \frac{1}{2\sigma^2} \]  
where now the potential and the total energy are given by

\[ V \rightarrow -\frac{f}{2\sigma^2} + \frac{1}{2\sigma^2}, \quad E^2 \rightarrow E^2 + \frac{1}{2\sigma^2} \]  
Interpreting the dynamical equation \( \frac{\dot{x}^2}{\sigma^2} + V = E^2 \), we see that the potential \( V \) is a potential well, the particle can fall into the well and remain there or, if the total energy of the test particle is bigger than \( E^2 \), the test particle continues through the throat. It is important to mention that this analysis is only valid for null geodesics with \( \theta = 0 \), since other values of \( \theta \) = const. lead us to non-physical geodesics.

We can integrate equation (43) in terms of the variables \( \tau \) and \( x \). This equation transform into

\[ \frac{d\tau}{dx} = \pm \left( 2E^2 + \frac{f}{\sigma^2} \right)^{-1/2} \]  
which can be reduced into quadratures as

\[ \tau = \pm \int \left( 2E^2 - \frac{(a^2 + k_1^2)e^{\frac{a^2+k_1^2}{2\sigma^2}(x-x_0^2)+\frac{a^2+k_1^2}{2\sigma^2}(x_0-x_1^2)}}{\sigma\left((a^2+k_1^2)e^{\frac{a^2+k_1^2}{2\sigma^2}(x-x_0^2)+\frac{a^2+k_1^2}{2\sigma^2}(x_0-x_1^2)}\right)} \right)^{-1/2} dx \]  
This integration can be performed numerically, the results are shown in Fig. 7, where we see that an observer can travel through the throat in finite proper time \( \tau \). The plot show different possibilities for the journey.

For the general case, with \( \theta \neq \text{const.} \), in terms of momenta

\[ p_t = \frac{\Delta f}{\Delta_t}, \quad p_\theta = \frac{\Delta}{f} \]  
and the conserved quantities

\[ E = f(\dot{t} + \omega \dot{\varphi}), \quad L + \omega E = \frac{f}{\Delta_1 \sin^2 \theta} \]  
we get the Euler-Lagrange equations for the geodesics of (1) are,

\[ \dot{p}_t = \frac{(a^2 + k_1^2)(l - l_1)\cos \theta}{2k_1 f_+ \Delta^2} E^2 + \frac{a(l - l_1)f \Delta_1 - E(L + \omega E)}{\Delta_1 \Delta^2} \]  
\[ + \frac{(l_1 - l) f^2 \Delta_1}{\Delta^2} \left( \frac{1}{f} + \frac{(a^2 - k_1^2)(l - l_1)\cos \theta}{2k_1 f_+ \Delta^2} \right) \left( p_t^2 + p_\theta^2 \right) \]  
\[ + \frac{2f^2}{\Delta^2 \sin^2 \theta} \left( \frac{1}{f} + \frac{(a^2 - k_1^2)(l - l_1)\cos \theta}{2k_1 \Delta^2 f_+} \right)(L + \omega E)^2 + \frac{1}{(l - l_1)}f_L \right) \]  
\[ \frac{\Delta_1}{\Delta} - \frac{p_t^2}{\Delta} \]  
\[ \dot{p}_\theta = \frac{(a^2 - k_1^2) \sin \theta \Delta_1 - E^2}{4k_1 \Delta^2 f_+} E(L + \omega E) \]  
\[ + \frac{f^2}{\Delta_1 \sin \theta} \left( \frac{\Delta_1 \sin \theta}{2f \sin \theta} + \frac{(a^2 - k_1^2) \Delta_1}{4k_1 f_+ \Delta} \right)(L + \omega E)^2 \]  
\[ + \frac{f^2}{\Delta_1 \sin \theta} \left( \frac{(a^2 - k_1^2) \Delta_1}{4k_1 f_+ \Delta} - \frac{l_1^2 - l_1^2}{2f} \right) \]  
\[ \times \left( p_t^2 + p_\theta^2 \right) \]  
being \( \Delta_+ = (l - l_1)^2 - (l_0^2 - l_1^2) \) and

\[ f = \frac{(a^2 - k_1^2)e^x}{a^2 - k_1^2 e^{2x}}. \]
This system can be integrated analytically for $\theta = 0$. For the general case with a non constant $\theta$ it can be solved numerically. In fig. 8 we can see the geodesics for different initial values of $\theta$. As we can see no null-geodesic is able to reach the singularity, even when the geodesic falls into the wormhole (see [13]).

An important condition to build wormholes is the violation of the Null Energy Condition (NEC) at or near the throat, i.e., $T_{\alpha\beta}^{\mu\nu} \mu^{\hat{\beta}} < 0$, for all null vector $\mu$. For simplicity we use a radial outgoing null vector, $\mu = e_t + e_r$, thus

$$T_{\alpha\beta}^{\mu\nu} \mu^{\hat{\beta}} = T_{ii} + T_{rr} = \frac{1}{8\alpha}(R_{ii} + R_{rr}).$$

The expression associated to (51) is very large and it is not interesting to write it down. Instead of that we plot the expression at and near the throat, see Fig. 9. We see that, as expected, the NEC is violated in this region.

IV. NEC VIOLATION

To make the analysis tractable, we choose an orthonormal basis $e_i, e_r, e_{\theta}, e_{\phi}$,

$$e_i = f^{-\frac{1}{2}}e_t,$$
$$e_r = (f^{-1}K)^{\frac{1}{2}}e_r,$$
$$e_{\theta} = f^{-1}K\Delta_1^{-\frac{1}{2}}e_{\theta},$$
$$e_{\phi} = \frac{\omega e_t + e_{\phi}}{\sqrt{f^{-1}K\Delta_1 \sin(\theta)}}.$$

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V. THE GRAVITATIONAL TIDAL FORCES

From the physical point of view it is important to know whether the wormhole is traversable or not. In this section we study the tidal forces constraints [3, 11] of metric [1]. For simplicity we consider a traveller going radially through the wormhole, beginning at rest in space station in the lower universe and ending at rest in space station in the upper universe. We introduce the orthonormal basis of its own reference frame,

$$e_0 = \gamma e_i \mp \gamma (v/c)e_r,$$
$$e_1 = \mp \gamma e_r + \gamma (v/c)e_i,$$
$$e_2 = e_\theta, \quad e_3 = e_\phi.$$

being $\gamma = [1 - (\frac{v}{c})^2]^{-\frac{1}{2}}$. For simplicity we assume here that the traveller do not feel an acceleration larger than about 1 Earth’s gravity. Following the same idea as in the work of Morris and Thorne [3] and [11], the radial tidal constraint is given by

$$|R_{1010}| \leq g_0/c^2 \times 2m \approx 1/(10^5 km)^2,$$

where the height of our traveller is 2m. We have $|R_{1010}| = |R_{ii}|$. Then the first tidal constraint reduces to

$$|R_{ii}| = \frac{\Delta_1}{4f\sin^2 \theta} [2 \sin^2 \theta \Delta f \frac{\partial^2 f}{\partial l^2} - \Delta_1 f \sin^2 \theta \frac{\partial f}{\partial l} \frac{\partial K}{\partial l} - K \sin^2 \theta \left(\frac{\partial f}{\partial \theta}\right)^2 + f \sin^2 \theta \frac{\partial f}{\partial \theta} \frac{\partial K}{\partial \theta} + f^4 K \left(\frac{\partial \omega}{\partial \theta}\right)^2] \leq (10^5 km)^{-2}.$$

While the lateral constraints are reduced to the study of $|R_{2020}| \leq (10^5 km)^{-2}$, and $|R_{3030}| \leq (10^5 km)^{-2}$ since
our metric is axially symmetric, we have that
\[ |R_{2020}| = \gamma^2 |R_{\theta\theta\theta\theta}| + \gamma^2 (v^2/c) |R_{\theta\theta t t}|. \tag{55} \]

Now we assume that, for example, the spaceship is at rest at the throat \([3]\), this implies \(v \to 0\) and \(\gamma \to 1\). Then \(|R_{2020}| = |R_{\theta\theta\theta\theta}|\). Then second tidal constrain is given by
\[ |R_{\theta\theta\theta\theta}| = \frac{1}{4\Delta^2 f \sin^2 \theta} | - \Delta \Delta_1 \sin^2 \theta \left( \frac{\partial f}{\partial l} \right)^2 + f \Delta_1 \sin^2 \theta \frac{\partial f}{\partial l} \frac{\partial \Delta}{\partial l} + 2 f \Delta \sin^2 \theta \frac{\partial^2 f}{\partial \theta^2} - f \sin^2 \theta \frac{\partial f}{\partial \theta} \frac{\partial \Delta}{\partial \theta} + f^4 K \left( \frac{\partial \omega}{\partial \theta} \right)^2 | \leq (10^5 km)^{-2} \tag{56} \]

Finally our last constraint can be written as
\[ |R_{\phi\phi t t}| = \frac{1}{4f \Delta \sin^2 \theta} | - \Delta \sin^2 \theta \left( \frac{\partial f}{\partial l} \right)^2 + f \sin^2 \theta \frac{\partial f}{\partial l} \frac{d \Delta}{d l} + f^4 \left( \frac{\partial \omega}{\partial l} \right)^2 + 2 f \sin \theta \cos \theta \frac{\partial f}{\partial \theta} - \sin^2 \theta \left( \frac{\partial f}{\partial \theta} \right)^2 + f^4 \Delta_1 \left( \frac{\partial \omega}{\partial \theta} \right)^2 | \leq (10^5 km)^{-2}. \tag{57} \]

We plot the tidal constraints (55), (56) and (57) in the Fig. 10.

We can also force the traveller to approach the throat using the geodesic with \(\theta = 0\), in that case we have that \(\partial w/\partial \theta = 0\) and \(\partial f/\partial \theta = 0\). We can see that trough this geodesic the Riemann tensors simplifies hardly. We plot this result in Fig. 11.

\[ \text{FIG. 10: Tidal forces for metric (1), given by equations (55), (56) and (57). Here } l_1 = 1, l_2 = 1.5, a = 0.1, k = 0.11. \text{ This plot shows the tidal constraints with } \theta = 0. \]

\[ \text{FIG. 11: Where } l_1 = 1, l_2 = 1.5, a = 0.1, k = 0.11. \text{ This plot shows the tidal constraints with } \theta = 0. \]

VI. CONCLUSIONS

We have analysed metric (1) and shown that this metric represents a wormhole, which throat is shown in Fig. 5. This metric contains a ring singularity very similar to the Kerr solution, all the invariants of the metric are regular everywhere except on this ring. The throat is found outside of the ring singularity, the mouth of the throat is on a sphere of radius \(l = l_1\), around the wormhole. We have shown that the null polar geodesics, that means, geodesics going through the polar line are regular, an observer can go through the throat if the observer trajectory remains on the polar geodesics and contains an energy bigger than \(1/2f\), for any values of the free parameters. On this trajectory the tidal forces are very small, therefore this wormhole is traversable. For any other angle the mouth of the wormhole lie on the sphere, but close to the equator, the effect of the wormhole is to repel the test particles. On the equator, the repulsion is infinity and nothing can reach the singularity, even the light is repealed by the wormhole (see also [13]). Thus, the sphere of radius \(l = l_1\) has an effect contrary to the horizon of a black hole, namely, an observer can reach the sphere, goes thought the throat, but this sphere avoids the traveller to observe or to reach the singularity. Of course, the traveller can come back to its original world without much troubles.

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