Magnetic Convection in a Nonuniformly Rotating Electroconducting Medium

M. I. Kopp, A. V. Tour, V. V. Yanovsky

Institute of Single Crystals, National Academy of Sciences of Ukraine, Kharkov, 61001 Ukraine
Karazin Kharkov National University, Kharkov, 61000 Ukraine
Université de Toulouse [USP], CNRS, Institut de Recherche en Astrophysique et Planétologie, BP 44346, Toulouse Cedex 4, 31028 France

*–e-mail: Anatoly.Tour@irap.omp.eu
**–e-mail: yanovsky@isc.kharkov.ua
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1. INTRODUCTION

Convective flows induced by thermal processes in a gravitational field are important for explaining many phenomena occurring in the interior of planets, stars, and other space objects. It is generally accepted that convection is the source of generation of large-scale magnetic fields as well as large-scale vortex structures irrespective of the choice of the model (laminar [1–4] or turbulent dynamo [5–7]).

Rotation and magnetic fields undoubtedly strongly affect the convective flows of electroconducting media. The theory of such processes (Rayleigh–Bénard problem) for 1D rotation and constant magnetic field was described in detail in monographs [8, 9]. However, the rotation of most cosmic objects consisting of high-density gases or liquid (Jupiter, Saturn, Sun, galaxies, etc.), as well as electroconducting media in the planetary interior, is nonuniform. In many hydrodynamic problems, the differential rotation of the medium is simulated by the Couette flow confined between two cylinders rotating with different angular velocities (Fig. 1a), which is found to be convenient for performing laboratory experiments [10]. The stability of such a flow for a perfectly conducting medium in a magnetic field was considered for the first time in [11, 12]. It was shown that a weak axial magnetic field destabilizes the azimuthal differential rotation of the plasma; when condition $d\Omega^2/dR < 0$ is satisfied, the magnetorotational instability (MRI) of the standard magnetorotational instability (SMRI) appears in a nondissipative plasma (see Fig. 1a). Since this condition also holds for Kepler flows with $\Omega \sim R^{-3/2}$, the MRI is the most probable source of turbulence in accretion disks.

The MRI discovery served as an impetus for numerous theoretical investigations [13] as well as laboratory experiments on rotation of liquid metals (sodium, gallium) and low-temperature plasmas [14, 15]. The first theoretical studies devoted to the problem of accretion flows were performed in the nondissipative plasma approximation taking into account the radial thermal stratification [16] as well as magnetization of heat flows [17]. In [18–23], the effect of viscosity and magnetic diffusion on the stability of a nonuniformly rotating plasma in a magnetic field was investigated. In [24], the stability of the differentially rotating plasma in an axial magnetic field was considered with simultaneous allowance for dissipative effects (viscosity and Ohmic dissipation) as well as thermal radial stratification of the plasma. The MTI effect in a helical magnetic field (i.e., with nontrivial topology $B_0 \cdot \text{curl}B_0 \neq 0$) was studied in [25, 26]. The MRI effect in an azimuthal magnetic field is referred to in the literature as the azimuthal magnetorotational instability.
As a result of the $\beta$ effect, a more general type of the Rossby waves (so-called Rossby magnetothermal waves) appears in rotational magnetoconvection [33].

In contrast to the Busse model [29–32], we analyze in this study the stability of a nonuniformly rotating plasma in an axial magnetic field and in the presence of a vertical temperature gradient. In other words, we consider here the problem of stability of an electroconducting liquid between two rotating liquids (Couette flow) together with the Rayleigh–Bernard problem in an external constant magnetic field (Fig. 2). In such a formulation of the problem, angular velocity $\Omega(R)$ of rotation of a viscous electroconducting liquid in the cylindrical geometry $(R, \phi, z)$ is described by the relation

$$\Omega(R) = \frac{\Omega_1 R_2^2 - \Omega_2 R_1^2}{R_2^2 - R_1^2} + \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R^2(R_2^2 - R_1^2)} ,$$

where $R_1$ and $R_2$ are the radii of the inner and outer cylinders.

This article has the following structure. In Section 2, we derive the evolution equations for small perturbations (in the Boussinesq approximation) in a rotating viscous incompressible liquid in the gravitational field with a constant temperature gradient. In Section 3, we solve the Rayleigh–Benard problem for an electroconducting liquid layer confined between two rotating cylinders and heated from below. Using the dispersion equation derived in Section 3, we analyze in Section 4 the monotonic and oscillating regimes of convection. In the same section, we obtain the critical values of the Rayleigh numbers for the stationary and oscillating convective instabilities for various angular velocity profiles $\Omega(R)$. In Section 5, we study the weakly nonlinear stage of rotational magnetoconvection for axisymmetric perturbations, at which a chaotic regime leading to random variations of the magnetic field sets in. In recent years, the chaotic behavior of convection has been studied intensely for rotating layers of liquids [34, 35], in conducting media with a uniform magnetic field [36–39], as well as in conducting media rotating with the magnetic field [40]. However, the magnetic field dynamics as such was not considered, which corresponds to the inertialess approximation. Such problems are important for technological applications such as crystal growth, chemical processes of solidification and centrifugal casting of metals.

Applying the Galerkin method to the nonlinear system of equations for a nonuniformly rotating magnetoconvection, we have obtained a dynamic system of Lorentz-type equations, but for the 6D phase space. Analytic and numerical analyses of this system of equations have revealed chaotic behavior of the magnetic field and its inversion. The theory developed in this study can be applied to various astrophysical and geophysical problems in which chaotic behavior of the magnetic field in convective layers on the cores of the Earth, the Sun, hot galactic clusters, accretion disks, and other objects is considered.

Fig. 1. (a) Geometry of the problem for standard MRI: two concentric cylinders of radii $R_{in} = R_1$ and $R_{out} = R_2$ rotate with angular velocities $\Omega_{in} = \Omega_1$ and $\Omega_{out} = \Omega_2$; $B_{loc}$ is the axial magnetic field directed vertically upwards. (b) Busse model (convective dynamo) for the layer of an electroconducting liquid in rotating magnetoconvection.

(AMRI) [22, 23]. The criteria for MRI evolution for a rarefied plasma taking into account the Hall effects and dissipation were formulated in recent publications [27, 28].

The model of rotating cylinders was used in the convective dynamo theory developed in [29–32]. In this theory [29], the outer cylinder rotates with angular velocity $\Omega_2$, while the inner cylinder remains at rest ($\Omega_1 = 0$; Fig. 1b). Convective flows (Bernard cells) appear in the layer of the liquid between the cylinders due to the difference of temperatures of the inner ($T_1$) and outer cylinders ($T_2$), $T_2 > T_1$.

The difference in the heights of the inner ($h_1$) and outer cylinders ($h_2$) leads to an analogous effect of the Coriolis force acting on the $\beta$ plane. The Busse model corresponds to convection in liquid layers located in the equatorial region of a rotating object, in which the role of azimuthal magnetic field is significant. This model was also actively employed in studying the behavior of hydrodynamic waves in rotational magnetoconvection.
2. BASIC EVOLUTION EQUATIONS FOR SMALL PERTURBATIONS

Let us consider a nonuniformly rotating flow of an incompressible viscous electroconducting liquid, which will be simulated by the Couette–Taylor flow between two rotating cylinders with outer radius \( R_{\text{out}} \) and inner radius \( R_{\text{in}} \ll R_{\text{out}} \) [10].

It is obviously convenient to analyze this type of the flow using the cylindrical system of coordinates \((R, \phi, z)\) (Fig. 3); such a choice is substantiated by the possibility of practical application of the theory developed here. Let us suppose that the rotating conducting medium (plasma) in constant gravitational and magnetic fields at a constant vertical temperature gradient \( \nabla T_0 = \text{const} = -Ae \), where \( A > 0 \) is a constant gradient and \( e \) is a unit vector directed vertically upwards along the \( z \) axis. Convective flows induced by the temperature gradient can be described by the equations of motion for a viscous incompressible electroconducting liquid in the Boussinesq approximation [9]:

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho_0} \left( \mathbf{P} + \mathbf{B}^2 \right) + \frac{1}{4\pi\rho_0} \mathbf{B} \cdot \nabla \mathbf{B} + g\beta T \mathbf{e} + \nu \nabla^2 \mathbf{v},
\]

\[
\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} = \eta \nabla^2 \mathbf{B},
\]

\[
\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \chi \nabla^2 T,
\]

\[
\text{div} \mathbf{B} = 0, \quad \text{div} \mathbf{v} = 0,
\]

where \( \beta \) is the thermal expansion coefficient, \( \rho_0 = \text{const} \) is the density of the medium, \( \nu \) is the kinematic viscosity, \( \eta = c^2/4\pi\sigma \) is the magnetic viscosity, \( \sigma \) is the electrical conductivity, and \( \chi \) is the thermal conductivity coefficient of the medium.

We assume that uniform magnetic field \( \mathbf{B}_0 \), which will henceforth be referred to as the axial field, is directed along the \( z \) axis. The magnetic field direction coincides with rotational axis \( \Omega \) of the medium and with the \( z \) axis. The plasma rotates in the azimuthal direction with velocity \( v_\phi = R\Omega(R) \), where \( \Omega(R) \) is the angular velocity of rotation.

System of equations (1)–(4) has stationary solutions of the form

\[
P = \rho_0(R, z), \quad B_\phi(R, z) = 0, \quad B_\rho(R, z) = 0,
\]

\[
B_z(R) = B_0 = \text{const}, \quad v_\phi(R) = 0, \quad v_\rho(R) = R\Omega(R),
\]

\[
v_z(R) = 0, \quad T_0(z) = \text{const} + AZ.
\]

For such a flow, the centrifugal equilibrium sets in the radial direction:

\[
\frac{1}{\rho_0} \frac{dp_0}{dz} = -g\beta T_0,
\]

Our main task is the determination of stability of small perturbations of physical quantities \((\mathbf{u}, \mathbf{b}, \rho, \theta)\) against the background of the stationary state described by Eqs. (5), (6). Representing all quantities in Eqs. (1)–(4) as the sum of the stationary and perturbed components,

\[
\mathbf{v} = \mathbf{v}_0 + \mathbf{u}, \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \quad P = \rho_0 + \rho, \quad T = T_0 + \theta,
\]

we obtain the evolution equations for small perturbations in the linear approximation:

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{v}_0 \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{v}_0 = -\frac{1}{\rho_0} \nabla \rho
\]

\[
+ \frac{1}{4\pi\rho_0} \left[ (\mathbf{B}_0 \cdot \nabla) \mathbf{b} - \nabla (\mathbf{B}_0 \cdot \mathbf{b}) \right] + g\beta \theta \mathbf{e} + \nu \nabla^2 \mathbf{u},
\]
where  \( v_0 = (0, R\Omega(R), 0) \), and \( B_0 = (0, 0, B_0) \). System of equations (7) will be used for testing the stability of small perturbations. For this purpose, we write Eqs. (7) in the cylindrical system of coordinates, using the following relations:

\[
\nabla^2 \rightarrow \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} + \frac{1}{R^2} \frac{\partial^2}{\partial \phi^2},
\]

\[
\left( \nabla^2 \begin{pmatrix} u \\ b \end{pmatrix} \right) = \nabla^2 \begin{pmatrix} u_R \\ b_R \end{pmatrix} - \frac{2}{R^2} \frac{\partial}{\partial \phi} \begin{pmatrix} u_b \\ b_b \end{pmatrix} - \frac{1}{R^2} \begin{pmatrix} u_R \\ b_R \end{pmatrix},
\]

\[
\left( \nabla^2 \begin{pmatrix} u \\ b \end{pmatrix} \right)_\phi = \nabla^2 \begin{pmatrix} u_\phi \\ b_\phi \end{pmatrix} + \frac{2}{R^2} \frac{\partial}{\partial \phi} \begin{pmatrix} u_b \\ b_b \end{pmatrix} - \frac{1}{R^2} \begin{pmatrix} u_\phi \\ b_\phi \end{pmatrix}.
\]

As a result, we obtain basic equations for analyzing stability of small perturbations:

\[
\frac{\partial u_\phi}{\partial t} + \Omega \frac{\partial u_\phi}{\partial \phi} - 2\Omega u_\phi - \frac{1}{4\pi \rho_0} B_0 \frac{\partial b_\phi}{\partial z} = -\frac{1}{\rho_0} \frac{\partial \tilde{p}}{\partial R},
\]

\[+} \nabla \left( \nabla^2 u_R - \frac{2}{R^2} \frac{\partial}{\partial \phi} \frac{u_R}{R} \right),
\]

\[
\frac{\partial u_\phi}{\partial t} - \frac{2}{R^2} \frac{\partial}{\partial \phi} \frac{u_\phi}{R} + \frac{\kappa^2}{2\Omega} u_R - \frac{1}{4\pi \rho_0} B_0 \frac{\partial b_\phi}{\partial z} = -\frac{1}{\rho_0} \frac{\partial \tilde{p}}{\partial \phi},
\]

\[+} \nabla \left( \nabla^2 u_R + \frac{2}{R^2} \frac{\partial}{\partial \phi} \frac{u_R}{R} \right),
\]

\[
\frac{\partial u_z}{\partial t} + \Omega \frac{\partial u_z}{\partial \phi} - \frac{1}{4\pi \rho_0} B_0 \frac{\partial b_z}{\partial z} = -\frac{1}{\rho_0} \frac{\partial \tilde{p}}{\partial z} + \nu \nabla^2 u_z,
\]

\[
\frac{\partial b_R}{\partial t} + \Omega \frac{\partial b_R}{\partial \phi} - R \frac{\partial \Omega}{\partial R} b_R - B_0 \frac{\partial u_R}{\partial z} = \eta \left( \nabla^2 b_R - \frac{2}{R^2} \frac{\partial b_R}{\partial \phi} - \frac{b_R}{R} \right),
\]

\[
\frac{\partial b_b}{\partial t} + \Omega \frac{\partial b_b}{\partial \phi} - R \frac{\partial \Omega}{\partial R} b_b - B_0 \frac{\partial u_b}{\partial z} = \eta \left( \nabla^2 b_b + \frac{2}{R^2} \frac{\partial b_b}{\partial \phi} + \frac{b_b}{R} \right),
\]

\[
\frac{\partial b_\phi}{\partial t} + \Omega \frac{\partial b_\phi}{\partial \phi} - \frac{1}{4\pi \rho_0} B_0 \frac{\partial u_\phi}{\partial z} = \eta \left( \nabla^2 b_\phi - \frac{2}{R^2} \frac{\partial b_\phi}{\partial \phi} - \frac{b_\phi}{R} \right),
\]

\[
\frac{\partial b_\phi}{\partial t} + \Omega \frac{\partial b_\phi}{\partial \phi} - \frac{1}{4\pi \rho_0} B_0 \frac{\partial u_\phi}{\partial z} = \eta \left( \nabla^2 b_\phi + \frac{2}{R^2} \frac{\partial b_\phi}{\partial \phi} + \frac{b_\phi}{R} \right),
\]

\[
\frac{\partial \theta}{\partial t} + \Omega \frac{\partial \theta}{\partial \phi} - u_\cdot A = \chi \nabla^2 \theta.
\]
3. RAYLEIGH–BERNARD PROBLEM
FOR A THIN LAYER OF A NONUNIFORM ROTATING MAGNETIZED PLASMA

The system of equations (8)–(15) obtained in the previous section will be used for describing convective phenomena in a thin layer of a nonuniformly rotating conducting medium (plasma) of thickness \(h = R_{out} - R_{in}\). We denote by \(T_1\) and \(T_2\) the temperatures of the lower and upper parts of the layer, respectively, where \(T_1 > T_2\) (heating from below; see Fig. 3). Such a formulation of the problem generalizes the classical Rayleigh–Bernard problem for free convection.

The characteristic scale of inhomogeneity of the medium in the horizontal plane for the given problem is larger than in the vertical direction \((L_R \gg L_h)\). Therefore, we can employ the local WKB approximation for perturbations depending on horizontal coordinates \((R, \phi)\). We expand all quantities into a Taylor series in the vicinity of fixed points \((R_0, \phi_0)\), retaining zero-order terms in local coordinates \(R = R - R_0, \phi = \phi - \phi_0\). This gives system of differential equations (8)–(15) with constant coefficients. Here, we take into account the following relations:

\[\Omega_0 = \Omega(R_0), \quad \nabla^2 \to \hat{D}^2 + \frac{\partial^2}{\partial R^2} + \frac{1}{R_0} \frac{\partial}{\partial R} + \frac{1}{R_0^2} \frac{\partial}{\partial \phi},\]

\[\hat{D} \equiv \frac{d}{dz},\]

\[\left(\nabla^2 u \right)_R = \nabla^2 \left( u_R \right) - \frac{2}{R_0^2} \frac{\partial}{\partial \phi} \left( u_\phi \right) - \frac{1}{R_0^2} \left( u_R \right),\]

\[\left(\nabla^2 u \right)_\phi = \nabla^2 \left( u_\phi \right) + \frac{2}{R_0^2} \frac{\partial}{\partial \phi} \left( u_R \right) - \frac{1}{R_0^2} \left( u_\phi \right).\]

All perturbations in system (8)–(15) can be represented as plane waves

\[
\begin{pmatrix}
    u \\
    b \\
    \theta \\
    \bar{\rho}
\end{pmatrix} =
\begin{pmatrix}
    U(z) \\
    H(z) \\
    \Theta(z) \\
    \bar{\rho}(z)
\end{pmatrix}
\exp(yt + im\bar{\phi} + ik\bar{R}),
\]

after which we obtain the following equations in the short-wavelength approximation \(k \gg 1/R_0\) disregarding terms \(ik/R_0 - 1/R_0^2\):

\[
\hat{L}_u U_R + \frac{2\Omega_0}{v} U_0 - \frac{2im}{v R_0^2} U_R + \frac{B_{0z}}{4\pi\rho_0 v} \hat{D} H_R - \frac{ik}{\nu \rho_0} \bar{\rho} = 0, \quad (17)
\]

\[
\hat{L}_u U_0 - \frac{2\Omega_0}{v} (1 + Ro) U_R + \frac{2im}{v R_0^2} U_R + \frac{B_{0z}}{4\pi\rho_0 v} \hat{D} H_0 - \frac{im}{\nu \rho_0 R_0} \bar{\rho} = 0, \quad (18)
\]

\[
\hat{L}_u U_z + \frac{B_{0z}}{4\pi\rho_0 v} \hat{D} H_z + \frac{g\Theta}{v} \frac{\partial}{\partial \rho} \bar{\rho} = 0, \quad (19)
\]

\[
\hat{L}_\eta H_R + \frac{B_{0z}}{\eta} \hat{D} U_R = 0, \quad (20)
\]

\[
\hat{L}_\eta H_0 + \frac{B_{0z}}{\eta} \hat{D} U_0 + \frac{2\Omega_0}{\eta} Ro H_R = 0, \quad (21)
\]

\[
\hat{L}_\eta H_z + \frac{B_{0z}}{\eta} \hat{D} U_z = 0, \quad (22)
\]

\[
\hat{L}_z \Theta + \frac{A}{\chi} U_z = 0, \quad \hat{D} U_z + ik U_R + \frac{im}{R_0} U_0 = 0, \quad (23)
\]

\[
\hat{D} H_z + ik H_R + \frac{im}{R_0} H_0 = 0.
\]

Here, \(Ro\) is the Rossby number, and the following notation for operators has been introduced:

\[
\hat{L}_\nu \equiv \hat{D}^2 - \left( \frac{\gamma + m\Omega_0}{\nu} + k^2 + \frac{m^2}{R_0^2} \right),
\]

\[
\hat{L}_\eta \equiv \hat{D}^2 - \left( \frac{\gamma + m\Omega_0}{\eta} + k^2 + \frac{m^2}{R_0^2} \right),
\]

\[
\hat{L}_z \equiv \hat{D}^2 - \left( \frac{\gamma + m\Omega_0}{\chi} + k^2 + \frac{m^2}{R_0^2} \right).
\]

It is convenient to reduce system of equations (17)–(23) to dimensionless form by introducing the following quantities marked by asterisk:

\[
(R^*_0, z^*) = \hat{h}^{-1}(R_0, z),
\]

\[
(U^*_R, U^*_\phi, U^*_z) = \chi \hat{h}^{-1}(U_R, U_\phi, U_z),
\]

\[
(H^*_R, H^*_\phi, H^*_z) = B_0^*(H_R, H_\phi, H_z),
\]

\[
\phi^* = \phi, \quad \Theta^* = \Theta(Ah)^{-1}, \quad \bar{\rho}^* = \bar{\rho} \left( \frac{h^2}{R_0 \nu \chi} \right),
\]

\[
t^* = \frac{t}{h^2}, \quad \frac{\partial}{\partial t^*} = \frac{h^2}{\nu} \frac{\partial}{\partial t}.
\]

Omitting the asterisk symbol, we obtain the following system of dimensionless equations:

\[
\hat{L}_u U_R + \sqrt{T}a U_\phi - \frac{2im}{R_0^2} U_R + \frac{\eta}{\nu} U_0 = 0, \quad (24)
\]

\[
\hat{L}_u U_0 - \sqrt{T}a (1 + Ro) U_R + \frac{2im}{R_0^2} U_R + \frac{\eta}{\nu R_0} \bar{\rho} = 0, \quad (25)
\]

\[
\hat{L}_u U_z + \frac{PRm^{-1} H^2}{R_0^2} \hat{D} H_z + \frac{Ra}{\nu} \Theta - \hat{D} \bar{\rho} = 0, \quad (26)
\]
Thus, we have obtained in this section Eq. (31) describing the convective phenomena in a thin layer of a nonuniformly rotating magnetized liquid.

4. MONOTONIC AND OSCILLATING CONVECTION REGIMES

In many problems in the linear theory of convective instability [9], only the unimodal approximation is conventionally used. We will seek the solution to Eq. (31) with boundary condition (32) in the form

\[ U_z = W_0 \sin(\pi z), \]

where \( W_0 \) is a constant amplitude. Substituting solution (33) into Eq. (31) and integrating over layer thickness \( z = (0, 1) \), we obtain the dispersion equation

\[ Ra = \frac{\Gamma_\eta \Gamma_A}{k^2 \Gamma_\nu} \left[ a^2 \Gamma_A + \pi^2 Ta(1 + Ro) \Gamma_\eta^2 \right. \]

\[ + \left. \pi^4 Ha^2 Ta Ro Pm \right], \]

where the following notation has been introduced:

\[ \Gamma_A = (\gamma + a^2)(\gamma Pm + a^2) + \pi^2 Ha^2, \quad \Gamma_\chi = \gamma Pr + a^2, \]

\[ \Gamma_\eta = \gamma Pm + a^2, \quad a^2 = \pi^2 + k^2. \]

Disregarding thermal processes, i.e., in the absence of heating \( (Ra = 0) \), Eq. (34) coincides with the dispersion equation in the standard MRI model taking into account dissipation processes (see, for example, [26]). The threshold value of the hydrodynamic Rossby number \( Ro \) can be determined from the condition \( \gamma = 0 \) and has the form

\[ Ro_{cr} = -\frac{a^2(\gamma^2 + \pi^2 Ha^2)^2 + \pi^2 \pi^4 Ta}{\pi^2 Ta a^4 + \pi^4 Ha^2 Pm}. \]

Passing to dimensional variables

\[ \frac{\pi^2}{a^4} \rightarrow \frac{\omega_\nu}{\omega_\nu \omega_\eta}, \quad \frac{\pi^2}{a^4} \rightarrow \frac{\omega_\chi}{\omega_\eta}, \quad \frac{\pi^2}{a^2} \rightarrow \frac{\zeta^2}{\omega_\xi}, \]

\[ \frac{Ta}{a^4} \rightarrow \frac{4 \Omega^2}{\omega_\nu}, \quad \frac{\pi^2}{a^2} \rightarrow \frac{\xi^2}{\omega_\xi}, \]

we obtain the expression for \( Ro_{cr} \) [26]:

\[ Ro_{cr} = -\frac{(\omega_\nu + \omega_\eta)(\omega_\eta + \omega_\xi)^2 + 4 \xi^2 \Omega^2 \omega_\nu^2}{4 \Omega^2 \xi^2 (\omega_\xi^2 + \omega_\nu^2)}, \]

where the following notation has been introduced: \( \omega_\nu = kv^2 \) and \( \omega_\eta = \eta k^2 \) are the viscous and Ohmic frequencies, respectively, and \( \omega_\xi = \alpha_0^2 \) is the Alfven frequency.

\[ \omega_\xi = k_0^2 c_A^2 = k_0^2 B_0^2 / 4 \pi \rho. \]

Therefore, in the limiting case of \( Ra = 0 \), magnetorotational instability appears in a nonuniformly rotating electroconducting liquid in a magnetic field. The criterion for its appearance is the condition imposed
on the angular velocity profile $\Omega(R)$ of the rotating liquid, i.e., Rossby number $Ro > Ro_{cr}$.

Let us now analyze a more general case when heating of the liquid layer ($Ra \neq 0$) and its nonuniform rotation ($Ro \neq 0$) take place. We consider here a convective flow in a planar nonuniformly rotating layer in the form of bores (cells). Perturbation growth rate $\gamma$ in the general case is complex-valued ($\gamma = \gamma_r + i\omega$). Clearly, the system is stable if $\gamma_r < 0$ and unstable if $\gamma_r > 0$. Let us now determine the stability boundary for monotonic ($\omega_i = 0$) and oscillating ($\omega_i \neq 0$) perturbations. At the stability boundary (neutral states), we have $\gamma_r = 0$; therefore, performing the substitution $\gamma = i\omega_r$ in Eq. (34), we obtain

$$Ra = Ra_r + i\omega Ra_r,$$

where the following notation has been introduced:

$$Ra_r = \frac{a^2}{k^2 \zeta}[\zeta(a^4 + \omega_3^2 PmPr) - \omega_2^2 a^4(l + Pm)(Pr - Pm)] + \pi^2 Ta(l + Ro)(a^4 - \omega_3^2 PmPr) + \omega_2^2 a^4(l + Pm)(Pr + Pr) k^2[\zeta^2 + \omega_3^2 a^4(l + Pm)^2]$$

$$+ \pi^4 Q Ta Ro Pm(a^4 + \omega_3^2 PmPr) + \omega_2^2 a^4(l + Pm)(Pr - Pm),$$

$$Ra_r = \frac{a^2}{k^2 \zeta}[(1 + Pr)(a^4 + \omega_3^2 PmPr) + (Pr - Pm)\zeta] + \pi^2 Ta(l + Ro)a^2[(Pr + Pr)\zeta - (1 + Pr)(a^4 - \omega_3^2 PmPr)] k^2[\zeta^2 + \omega_3^2 a^4(l + Pm)^2]$$

$$+ \pi^4 Q Ta R o Pm a^2[(Pr - Pr)\zeta - (1 + Pr)(a^4 + \omega_3^2 PmPr)] k^2[\zeta^2 + \omega_3^2 a^4(l + Pm)^2],$$

$$\zeta = a^4 + \omega_3^2 Pm^2,$$  

$$\zeta = a^4 + \pi^2 Q - \omega_3^2 Pm.$$  

4.1. Stationary Convection Regime

If coefficient $\gamma$ is equal to zero, Eq. (34) gives the critical value of Rayleigh number $Ra_{st}$ for stationary convection:

$$Ra_{st} = \frac{(\pi^2 + k^2)^3}{k^2} + \frac{\pi^2(\pi^2 + k^2)Q}{k^2}$$

$$+ \frac{\pi^2(\pi^2 + k^2)Ta}{k^2(\pi^2 + k^2)^2} + \frac{\pi^2 Ta Ro (\pi^2 + k^2)^2 + \pi^2 Q Pm}{k^2(\pi^2 + k^2)^2 + \pi^2 Q}.$$  

The minimal value of the critical Rayleigh number can be determined from the condition $\partial Ra_{st}/\partial k = 0$ and corresponds to wavenumbers $k = k_c$, which satisfy the following equation:

$$\frac{2k_c^2 - \pi^2}{k_c} - \frac{\pi^4 Q}{k_c a_2^4} + \frac{2\pi^2 k_c T a(l + Ro)}{k_c a_2^4} - \frac{\pi^2 Ta (a_4^2 + \pi^2 Q + 2k_c^2 a_4^2)}{k_c (a_4^2 + 2\pi^2 Q)^2}$$

$$- \frac{\pi^2 Ta Ro (a_4^2 + \pi^2 Q Pm) (a_4^2 + 2\pi^2 Q + 2k_c^2 a_4^2)}{k_c a_2^4 (a_4^2 + \pi^2 Q)^2} = 0,$$

$$a_c^2 = \pi^2 + k_c^2.$$  

Number $Ra_{st}^{\min}$ in Fig. 4 corresponds to a point on the neutral curve separating the regions of stable and unstable perturbations. It can be seen that upon an increase in the positive profile of Rossby number $Ro$, the minimal value of critical Rayleigh number $Ra_{st}^{\min}$ also increases (i.e., the instability evolution threshold becomes higher). On the other hand, for negative rotation profiles, viz., the Kepler profile ($Ro = -3/4$) and the Rayleigh profile ($Ro = -1$), the critical Rayleigh number decreases, i.e., the instability evolution threshold is lower than in the case of uniform ($Ro = 0$) and nonuniform rotation ($Ro = 2$). In the absence of rotation ($Ta = 0$, $Ro = 0$) in zero magnetic field ($B_0 = 0$), expression (36) leads to the familiar result [8, 9]

$$Ra_{st} = \frac{(k^2 + \pi^2)^3}{k^2}.$$  

In this case, the minimal critical Rayleigh number reaches the value $Ra_{st}^{\min} = 27\pi^4/4$ for wavenumbers $k_c = \pi/\sqrt{2}$.  

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In the case of nonconducting ($\sigma = 0$) and uniformly rotating (Ro = 0) medium, expression (36) also leads to the familiar result [8, 9]

$$Ra_{st} = \frac{(k^2 + \pi^2)^3}{k^2} + \frac{\pi^2 Ta}{k^2}.$$  

Here, the values of the critical Rayleigh number were obtained for $k = k_c$ satisfying the relation

$$1 + \frac{Ta}{\pi^4} = 2\left(\frac{k_c}{\pi}\right)^6 + 3\left(\frac{k_c}{\pi}\right)^4.$$  

A generalization of this result to the case of a nonuniformly rotating medium boils down to the substitution of the Taylor number $Ta \rightarrow Ta(1 + Ro)$. In the absence of rotation ($Ta = 0$) of a medium in a uniform axial magnetic field $B_0$, the critical value of the Rayleigh number assumes the form [8, 9]

$$Ra_{st} = \frac{(k^2 + \pi^2)^3}{k^2} \left(1 + \frac{\pi^2 Ha^2}{(k^2 + \pi^2)^2}\right)$$

for wavenumbers $k = k_c$, satisfying the relation

$$1 + \frac{Ha^2}{\pi^2} = 2\left(\frac{k_c}{\pi}\right)^6 + 3\left(\frac{k_c}{\pi}\right)^4.$$  

In the presence of a uniform rotation (Ro = 0) in a constant magnetic field $B_0$, the critical Rayleigh number was also calculated by Chandrasekhar [8]:

$$Ra_{st} = \frac{(k^2 + \pi^2)^3}{k^2} + \frac{\pi^2 (k^2 + \pi^2)^2}{k^2} \left(\frac{Ha^2 + 2Ta(k^2 + \pi^2)}{(k^2 + \pi^2)^2 + \pi^2 Ha^2}\right).$$  

(37)

For analyzing the effects of a nonuniform rotation in a magnetic field, we calculate derivatives $d\tilde{R}/d\tilde{Q}$, $d\tilde{R}/d\tilde{T}$, and $d\tilde{R}/dRo$ in new variables $\tilde{R} = \frac{Ra_{st}}{\pi^4}$, $\tilde{T} = \frac{Ta}{\pi^4}$, $\tilde{Q} = \frac{Q}{\pi^2}$, $x = \frac{k^2}{\pi^2}$.

These derivatives have the form

$$\frac{d\tilde{R}}{d\tilde{Q}} = \frac{1 + x - (1 + x)^2 \tilde{T}(1 + Ro)}{x[1 + (1 + x)^2 + \tilde{Q}^2]} - \frac{\tilde{Ro}Q \tilde{Pm}}{x[1 + (1 + x)^2 + \tilde{Q}^2]} + \frac{\tilde{T}\tilde{Ro} \tilde{Pm}}{x[1 + (1 + x)^2 + \tilde{Q}^2]}.$$  

(38)

$$\frac{d\tilde{R}}{d\tilde{T}} = \frac{(1 + x)^2 + \tilde{Ro}(1 + x)^2 + \tilde{Q} \tilde{Pm}}{x[1 + (1 + x)^2 + \tilde{Q}^2]},$$  

$$\frac{d\tilde{R}}{dRo} = \frac{\tilde{T}(1 + x)^2 + \tilde{Q} \tilde{Pm}}{x[1 + (1 + x)^2 + \tilde{Q}^2]}.$$  

(40)

Expressions (38) and (39) show that in the case of stationary convection, quantities $d\tilde{R}/d\tilde{Q}$ and $d\tilde{R}/d\tilde{T}$ can be either positive or negative. Consequently, the magnetic field and nonuniform rotation can produce a stabilizing or a destabilizing effect. Nonuniform rotation produces the stabilizing effect in the case of positive Rossby numbers, Ro > 0. In the opposite case (Ro < 0), nonuniform rotation can produce a destabilizing effect. Equations (38) and (39) were solved numerically.

The dependences of critical Rayleigh number $\tilde{R}$ on Chandrasekhar numbers $\tilde{Q}$ are plotted in Fig. 5 for rotation parameter $\tilde{T} = 100$ for different Rossby numbers and wavenumbers $x = 0.1, 0.2, 0.5, 1.0$. In the case of uniform rotation (Ro = 0), the magnetic field stabilizes the instability evolution, which is manifested most clearly upon an increase in wavenumbers x. For a positive rotation profile (Ro = 2), the stabilization of instability by the magnetic field is also observed, but for larger values of $\tilde{R}$ as compared to the case of uniform rotation. Such stabilization is most effective for short-wavelength perturbations. For the Kepler (Ro = –3/4) and Rayleigh (Ro = -1) rotation profiles, the magnetic field produces a destabilizing effect, which is most sensitive to short-wavelength (large x) perturbations.

Figure 6 shows the variation of $\tilde{R}$ relative to $\tilde{T}$ for parameter $\tilde{Q} = 100$ for different Rossby numbers and wavenumbers $x = 0.1, 0.2, 0.5, 1.0$. It is known [9] that rapid solid-state rotation (Ro = 0) produces a stabilizing effect irrespective of the magnetic field induction. An analogous effect also appears in the case of a nonuniform rotation for positive Rossby numbers (Ro > 0). However, in the case of negative rotation profiled (Ro = –3/4 and Ro = -1), the rotation itself can produce a destabilizing effect, which is enhanced for short-wavelength perturbations. Figure 7 shows the variation of $\tilde{R}$ as a function of Rossby number Ro. It can be seen that the stabilizing effect is permanent irrespective of the magnetic field magnitude for positive Rossby numbers Ro > 0 for parameters $\tilde{T} = 1000$, $\tilde{Q} = 100$, and $x = 0.1, 0.2, 0.5, 1.0$. As can be seen from

\[\frac{d\tilde{R}}{dRo} = \frac{\tilde{T}(1 + x)^2 + \tilde{Q} \tilde{Pm}}{x[1 + (1 + x)^2 + \tilde{Q}^2]}\]
4.2. Vibrational (Oscillating) Convection Regime

It follows from physical considerations that quantity $Ra$ is obviously real-valued; then, the imaginary part of Eq. (35) must vanish. In this case, the following variants are possible: $\omega_i = 0$ and $Ra_i = 0$. In the former variant ($\omega_i = 0$), we obtain the critical value of the Rayleigh number $Ra_c$ for monotonic perturbations, which coincides with expression (36) for the stationary convection regime. In the case of a vibrational perturbation, $\omega_i \neq 0$ ($Ra_i = 0$), formula (35) gives the critical Rayleigh number for oscillatory instability,

$$Ra_{osc} = \frac{a^2}{k^2 \zeta^2} \left[ \zeta (a^4 + \omega_0^2 Pm Pr) - \omega_0^2 a^4 (1 + Pm)(Pr - Pm) \right]$$

$$+ \pi^2 T a(1 + Ro) \frac{(a^4 - \omega_0^2 Pm Pr) \zeta + \omega_0^2 a^4 (1 + Pm)(Pr + Pr)}{k^2 \zeta^2 + \omega_0^2 a^4 (1 + Pm)^2}$$

$$+ \pi^4 Q T a R o P m \frac{(a^4 + \omega_0^2 Pm Pr) \zeta + \omega_0^2 a^4 (1 + Pm)(Pr - Pm)}{k^2 \zeta^2 + \omega_0^2 a^4 (1 + Pm)^2}, \quad (41)$$

and the neutral vibration frequency $\omega = \omega_0$ satisfying the equation

$$K_0(\omega^6) + K_i(\omega^4) + K_2(\omega^2) + K_3 = 0,$$

where the following notation has been introduced:

$$K_0 = Pm^2 (1 + Pr) Pm^2,$$

$$K_i = [a^4 (1 + Pm) + (Pr - Pm)(a^4 + \pi^2 Q)] Pm^2$$

$$+ Pm^2 (1 + Pr) [a^4 (1 + Pm)^2 - 2 Pm (a^4 + \pi^2 Q)]$$

$$+ \frac{\pi^2}{a^2} T a(1 + Ro) P m^3 (Pr - 1),$$

and

$$K_2 = \frac{\pi^2}{a^2} T a(1 + Ro) P m^3 (Pr - 1).$$

$\omega_0$ is the neutral vibration frequency, $\zeta$ is the Chandrasekhar parameter, $Ro$ is the Rossby number, $Pm$ is the Prandtl number, $Pr$ is the Prandtl number, $T$ is the temperature, $\pi$ is the angular frequency, $Q$ is the temperature gradient, $a$ is the characteristic length, $R$ is the Rayleigh number, $K$ is the inequality constant, $k$ is the wave number, $x$ is the wavenumber, and $\zeta$ is the Chandrasekhar parameter.
Formula (41) contains familiar results in some limiting cases. For example, for a uniform rotation ($Ro = 0$) and a nonconducting liquid ($\sigma = 0$), Chandrasekhar [8] obtained the following expression for the critical Rayleigh number of oscillatory instability:

$$Ra_{osc} = \frac{1}{k^2} \left[ a^4 - \omega^2 a^2 Pr + \frac{\pi^2 Ta(a^4 + \omega^2 Pr)}{a^4 + \omega^2} \right].$$

In the absence of rotation ($Ta = 0$, $Ro = 0$) in a magnetized conducting liquid ($Q \neq 0$), critical Rayleigh number $Ra_{osc}$ for oscillatory convection was also obtained by Chandrasekhar [8]:

$$Ra_{osc} = \frac{a^2}{k^2} \left[ a^4 - \omega^2 Pr + \frac{\pi^2 Q(a^4 + \omega^2 Pr Pm)}{a^4 + \omega^2 Pm^2} \right].$$

Figure 8 shows the dependences of critical Rayleigh number $Ra_{osc}$ for oscillatory instability on $\pi/k$ for different nonuniform rotation profiles. It can be seen that for negative Rossby numbers ($Ro < 0$), threshold Rayleigh number $Ra_{osc}^{min}$ decreases.

5. WEAKLY NONLINEAR INSTABILITY EVOLUTION REGIME

For describing nonlinear convective phenomena in a nonuniformly rotating layer of an electroconducting liquid, it is convenient to pass to a rotating frame of reference with local Cartesian coordinates $(x, y, z)$ (see Fig. 3b). This reference system rotates with angular
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The solenoidality equations for axisymmetric velocity and magnetic field perturbations assume the form

\[
\frac{\partial \mathbf{u}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{b}) + \nabla \times (\mathbf{u} \mathbf{V}) = \mathbf{0},
\]

\[
\frac{\partial \mathbf{b}}{\partial t} - \nabla \times (\mathbf{b} \times \mathbf{u}) = \mathbf{0},
\]

\[
\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{b} = 0.
\]

The solenoidal equations for axisymmetric velocity and magnetic field perturbations assume the form

\[
\frac{\partial u}{\partial t} + \nabla \times (u \nabla \times \mathbf{b}) + \nabla \times (u \mathbf{V}) = \mathbf{0},
\]

\[
\frac{\partial b}{\partial t} - \nabla \times (b \times u) = \mathbf{0},
\]

\[
\nabla \cdot u = 0, \quad \nabla \cdot b = 0.
\]

Let us use Eqs. (42) in terms of stream function \( \psi \) and function \( \phi \), transposing nonlinear terms to the right-hand sides of the equations:

\[
\frac{\partial \psi}{\partial t} + \mathbf{V} \cdot \nabla \psi = -J(\psi, \nabla^2 \phi),
\]

\[
\frac{\partial \phi}{\partial t} - \mathbf{V} \cdot \nabla \phi = -J(\phi, \nabla^2 \psi).
\]

The solenoidal equations for axisymmetric velocity and magnetic field perturbations assume the form

\[
\frac{\partial u}{\partial t} + \nabla \times (u \nabla \times \mathbf{b}) + \nabla \times (u \mathbf{V}) = \mathbf{0},
\]

\[
\frac{\partial b}{\partial t} - \nabla \times (b \times u) = \mathbf{0},
\]

\[
\nabla \cdot u = 0, \quad \nabla \cdot b = 0.
\]
where
\[ J(a, b) = \frac{\partial a \partial b}{\partial x \partial z} - \frac{\partial a \partial b}{\partial z \partial x} \]
is the Jacobian operator or the Poisson bracket, \( J(a, b) \equiv \{ a, b \} \). It should be noted that for the absence of thermal effects, system of equations (44)–(48) was derived in [42], where the MRI saturation mechanism was investigated. Since thermal effects are taken into account in our case, it is convenient to pass in Eqs. (44)–(48) to dimensionless variables
\[ (x, z) = h(x^*, z^*), \quad t = \frac{h^2}{v}, \]
\[ \psi = \chi \psi^*, \quad \phi = h B_0 \phi^*, \]
\[ \nu \equiv \frac{h}{v} \phi^*, \quad \theta = 4 \pi h \phi^*. \]
Omitting the asterisk symbol, we can write Eqs. (44)–(48) in dimensionless variables:
\[
\frac{\partial}{\partial t} \nabla^2 \psi + \sqrt{\frac{\nu}{\rho}} \frac{\partial \psi}{\partial t} - \frac{\partial}{\partial z} \left[ \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \frac{\partial}{\partial x} \right] \psi - \frac{\partial}{\partial z} \left[ \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \frac{\partial}{\partial y} \right] \psi = \frac{\partial}{\partial z} \psi + \frac{\partial}{\partial y} \psi - \frac{\partial}{\partial x} \psi + \frac{\partial}{\partial y} \psi - \frac{\partial}{\partial x} \psi
\]
\[ = -4 \pi \psi \]
\[ \frac{\partial \psi}{\partial t} + \nabla \psi = \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} = 0. \]
\[ \frac{\partial \psi}{\partial z} = 0, \quad \theta = 0 \quad \text{for} \quad z = 0, \]
\[ \frac{\partial \phi}{\partial z} = 0, \quad \theta = 0 \quad \text{for} \quad z = 1, \quad \text{and} \quad \frac{\partial \psi}{\partial z} = 0, \quad \theta = 0 \quad \text{for} \quad z = 1, \]
which will be solved using the Galerkin method.

5.1. Galerkin Expansion

We will solve nonlinear system of equations (49)–(53) using the Galerkin expansion in the \( x \) and \( z \) directions for stream functions \( \psi, \phi \), as well as the \( y \) components of perturbations of velocity \( v \), magnetic field \( \nu \), and temperature \( \theta \):
\[
\psi(x, z, t) = A(x) \sin(kx) \sin(\pi z),
\]
\[
\nu = V(t) \sin(kx) \cos(\pi z),
\]
\[
\phi(x, z, t) = B(t) \sin(kx) \cos(\pi z),
\]
\[
\theta(x, y, t) = C(t) \sin(kx) \sin(\pi z) + C_2(t) \sin(2\pi z),
\]
where \( k = \frac{2\pi}{L} \) is the dimensionless wavenumber; \( L \) is the characteristic length of the layer in the horizontal direction, and \( A_1, V_1, B_1, W_1, C_1 \), and \( C_2 \) are the amplitudes of perturbations. Substituting expansion (55) into Eqs. (49)–(53) and integrating in the entire domain \([0, 1] \times [0, L/h]\) with allowance for the orthogonality of the functions,
\[
\int_0^1 \sin(m \pi x) \sin(n \pi x) dx = \begin{cases} 0, & \text{for} \quad m \neq n, \\ 1/2, & \text{for} \quad m = n, \end{cases}
\]
we obtain the evolution equations for the perturbation amplitudes:
\[
\frac{\partial A}{\partial t} = -A + \frac{\pi \sqrt{\nu} \alpha}{a} \frac{V_1}{a} \frac{\partial}{\partial z} B^2 + \frac{\pi \nu A_1}{a^2} C_1, \tag{56}
\]
\[
\frac{\partial V}{\partial t} = -V + \frac{\pi \sqrt{\nu} \alpha}{a^2} \frac{V_1}{a} \frac{\partial}{\partial z} A + \frac{\pi \nu B_1}{a^2} W_1, \tag{57}
\]
\[
\frac{\partial B}{\partial t} = -B + \frac{\pi \nu Pm}{a} \frac{\partial}{\partial z} A, \tag{58}
\]
\[
\frac{\partial W}{\partial t} = -W + \frac{\pi \nu Pm}{a^2} \frac{V_1}{a} \frac{\partial}{\partial z} B + \frac{\nu \sqrt{\nu} \alpha \frac{\partial}{\partial z} B}{a^2} B_1, \tag{59}
\]
\[
\frac{\partial C}{\partial t} = -C + \frac{k}{a} A + \frac{\pi \nu C_2}{a} \frac{\partial}{\partial z} A, \tag{60}
\]
\[
\frac{\partial C_2}{\partial t} = -\frac{4 \pi^2 C_2}{a^2} + \frac{\pi k}{2 a^2} A C_1. \tag{61}
\]
Here, \( a = \sqrt{k^2 + \pi^2} \) is the general wavenumber and \( t = a^2 \tau \) is the reduced time. The resultant system of ordinary differential equations (56)–(61) is a low-dimensional spectral model, which, however, can reproduce qualitatively the convective processes in the complete (self-consistent) nonlinear system of equations (49)–(53). Therefore, dynamic system of equations (56)–(61) is suitable for describing the weakly
nonlinear stage of evolution of convection. For convenience, we introduce the notation

\[ R = \frac{k^2 Ra}{d^3}, \quad T = \frac{\pi^2 \sqrt{Ta}}{a^2}, \quad H = \frac{\pi^2 Q Pr}{a^4 Pm}, \quad \gamma = \frac{4\pi^2}{a^2} \]

and perform rescaling of amplitudes \( A_1, V_1, B_1, W_1, C_1, \) and \( C_2 \) in the form

\[ X(\bar{t}) = k\pi A(\bar{t}), \quad V(\bar{t}) = k \frac{V(\bar{t})}{\sqrt{2}}, \]

\[ U(\bar{t}) = \frac{kB(\bar{t})}{\sqrt{2}}, \quad W(\bar{t}) = \frac{a^2 k}{\pi \sqrt{2}} \frac{W(\bar{t})}{\sqrt{2}}, \]

\[ Y(\bar{t}) = \frac{\pi C(\bar{t})}{\sqrt{2}}, \quad Z(\bar{t}) = -\pi C_2(\bar{t}). \]

Then Eqs. (56)–(61) assume the form of the following nonlinear dynamic system of equations:

\[
\begin{align*}
\dot{X} &= -X + RY - TV - HU, \\
\dot{V} &= -V + HW + \sqrt{Ta}(1 + Ro)X, \\
\dot{U} &= -Pm^{-1}U + Pr^{-1}X, \\
\dot{W} &= -Pm^{-1}W - Pr^{-1}V + Ro\sqrt{Ta}U, \\
\dot{Y} &= Pm^{-1}(-\gamma Y + X - XZ), \\
\dot{Z} &= Pm^{-1}(-\gamma Z + XY),
\end{align*}
\]

(62)

where the dot on the symbol indicates the differentiation with respect to time \( \bar{t} \). The last two nonlinear equations in system (62) resemble analogous equations in the Lorentz system \([43, 44]\). Therefore, we can attribute resultant nonlinear system of equations (62) to the Lorentz-type equations for the 6D phase space.

5.2. Stability Analysis

In this section, we analyze the stability of convective flows in nonuniformly rotating layers of an electroconducting medium under the influence of a uniform magnetic field. Qualitative and numerical analyses of dynamic system of equations (62) make it possible to determine the type of stationary points and the conditions for the emergence of a chaotic regime. It can easily be seen that system of equations (62) is dissipative because the divergence of the vector field is negative:

\[
\text{div} \Phi = \frac{\partial \Phi}{\partial X} + \frac{\partial \Phi}{\partial V} + \frac{\partial \Phi}{\partial U} + \frac{\partial \Phi}{\partial W} + \frac{\partial \Phi}{\partial Y} + \frac{\partial \Phi}{\partial Z} = -2(1 + Pm^{-1}) - Pr^{-1}(1 + \gamma) < 0.
\]

The 6D phase volume decreases exponentially with time during the flow along the trajectories of the phase space,

\[
\Phi(\bar{t}) = \Phi(0) \exp\{-2(1 + Pm^{-1}) - Pr^{-1}(1 + \gamma)\bar{t}\}.
\]

Therefore, the phase volume is compressed as a result of dissipation. This means that attracting sets (attractors) appear in the phase space of dissipative systems. In addition, system of equations (62) is invariant to the substitution

\[ (X, V, U, W, Y, Z) \rightarrow (-X, -V, -U, -W, -Y, Z). \]

Equating the left-hand sides of Eqs. (62) to zero, we obtain three equilibrium states:

\[
\begin{align*}
O_1(X_1, V_1, U_1, W_1, Y_1, Z_1) &= (0, 0, 0), \\
O_2(X_2, V_2, U_2, W_2, Y_2, Z_2) &= \pm \frac{1}{r} \sqrt{\gamma r(R - r)}, \\
O_3(X_3, V_3, U_3, W_3, Y_3, Z_3) &= \pm \frac{1}{r} \sqrt{\gamma r(R - r)},
\end{align*}
\]

(63)

where the coordinates of stationary points \( X_{1,2,3}, V_{1,2,3}, U_{1,2,3}, W_{1,2,3}, Y_{1,2,3}, \) and \( Z_{1,2,3} \) are given, respectively, by

\[
\begin{align*}
(X_1, V_1, U_1, W_1, Y_1, Z_1) &= (0, 0, 0), \\
(X_2, X_3) &= \pm \frac{1}{r} \sqrt{\gamma r(R - r)}, \\
(U_2, U_3) &= \pm \frac{Pm}{r Pr} \sqrt{\gamma r(R - r)}, \quad (W_2, W_3) = \pm \frac{\sqrt{Ta} (Ro Pm - Ro - 1)}{r (H Pm + Pr)} \sqrt{\gamma r(R - r)}, \\
(Y_2, Y_1) &= \pm \frac{1}{R} \sqrt{\gamma r(R - r)}, \quad (Z_2, Z_3) = 1 - \frac{R}{r}.
\end{align*}
\]

where

\[ r = 1 + \frac{Pm}{Pr}(1 + \frac{H}{Pr}) \left( 1 + \frac{Pm^2}{Pr} \right). \]

To determine the type of stationary points, we linearize system of equations (62) in a small neighborhood of stationary points using the standard method. As a result, we write the linearized equations in the form of Jacobi matrix \( ||J|| \) with elements

\[
\begin{align*}
J_{11} &= -1, \quad J_{12} = -1, \quad J_{13} = -1, \quad J_{14} = 0, \\
J_{15} &= R, \quad J_{16} = 0, \quad J_{21} = \sqrt{Ta}(1 + Ro), \quad J_{22} = 1, \quad J_{23} = 0, \quad J_{24} = H, \quad J_{25} = 0, \quad J_{26} = 0, \quad J_{31} = Pr^{-1}, \quad J_{32} = 0, \\
J_{33} &= -Pm^{-1}, \quad J_{34} = 0, \quad J_{35} = 0, \quad J_{36} = 0, \quad J_{41} = 0, \quad J_{42} = -Pr^{-1}, \quad J_{43} = Ro\sqrt{Ta}, \quad J_{44} = -Pm^{-1}, \quad J_{45} = 0, \quad J_{46} = 0.
\end{align*}
\]

(64)
Table 1. Eigenvalues \( \lambda_{i, 2, 3, 4, 5, 6} \) (Lyapunov indices) for stationary point \( O_i \), calculated for different values of parameter \( R \) for \( Pm = 1, \text{Pr} = 9, H = 5, T = 1, Ta = 2, \) and \( \gamma = 1 \) and for the Rayleigh rotation profile \( (Ro = –1) \)

| \( R \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) | \( \lambda_4 \) | \( \lambda_5 \) | \( \lambda_6 \) |
|---|---|---|---|---|---|---|
| 1 | -0.1111 | -1.0106 + i1.1861 | -0.0045 | -0.4906 | -1.5944 | -1.0106 - i1.1861 |
| 5 | -0.1111 | 0.2771 | -1.0463 + i1.1316 | -0.6180 | -1.6775 | -1.0463 - i1.1316 |
| 10.3 | 0.5483 | -1.0776 + i1.0686 | -0.1111 | -0.7024 | -1.8017 | -1.0776 - i1.0686 |
| 10.5 | 0.5574 | -1.0784 + i1.0664 | -0.1111 | -0.7048 | -1.8067 | -1.0784 - i1.0664 |
| 22.878 | 1.0323 | -1.1012 + i0.9576 | -0.1111 | -0.8023 | -2.1385 | -1.1012 - i0.9576 |
| 22.998 | 1.0362 | -1.1012 + i0.9568 | -0.1111 | -0.8030 | -2.1418 | -1.1012 - i0.9568 |
| 45 | -0.1111 | 1.6572 | -1.0848 + i0.8625 | -0.8766 | -2.7220 | -1.0848 - i0.8625 |

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The necessary and sufficient conditions must hold for all roots of polynomial \( P(\lambda) \) to have negative real-valued parts:

(i) all coefficients of polynomial \( P(\lambda) \) must be positive \( (a_n > 0, n = 1 \sim 6) \);

(ii) the Hurwitz determinants must satisfy the inequalities \( \Delta_{m-1} > 0, \Delta_{m-2} > 0, \ldots \), where \( \Delta_m \) denotes the \( m \)-th Hurwitz determinant:

\[
\Delta_m = \begin{vmatrix}
  a_1 & a_3 & a_5 & \cdots \\
  a_2 & a_4 & a_6 & \cdots \\
  0 & a_1 & a_3 & \cdots \\
  \vdots & \vdots & \vdots & \ddots \\
  0 & a_0 & a_2 & \cdots \\
\end{vmatrix}
\]

Obviously, when the Routh–Hurwitz criterion holds, stationary points are stable, and their equilibrium position is classified as a stable node.

Let us analyze Eq. (65) numerically for stationary point \( O_i \) in the case of the Rayleigh rotation profile \( (Ro = –1) \). Choosing the values of parameter \( Pm = 1, \text{Pr} = 9, H = 5, T = 1, Ta = 2, \) and \( \gamma = 1 \), we calculate eigenvalues \( \lambda_i \) depending on Rayleigh parameter \( R \). These results are compiled in Table 1; for the Kepler rotation profile \( (Ro = –3/4) \), analogous results are given in Table 2. It can be seen that for negative values of \( Re\lambda < 0 \), the trajectories enter point \( O_i \), i.e., correspond to stable proper directions, while for positive \( Re\lambda > 0 \), the trajectories emerge from point \( O_i \) and, hence, correspond to unstable proper directions. The stationary state of convections \( (\lambda = 0) \) corresponds to the critical value of parameter

\[
R_{ke} = 1 + \frac{Pm^2}{\text{Pr} H + T\sqrt{Ta}} \left[ 1 + \frac{Pm^2}{\text{Pr} H} \right],
\]

or the critical value of the Rayleigh number

\[
Ra_{cr} = \frac{a^6}{k^2} + a^2 + \frac{\pi^2}{k^2} \left[ \frac{\text{Pr}^2}{T\sqrt{Ta} a^4 + \text{Ro}(a^4 + \pi^2 Q Pm)} \right],
\]
which coincides with formula (35) and the expression for \( r \). For the above numerical values of the parameters, critical number \( R_{1cr} \approx 1.05 \). In the case of the Kepler rotation profile \((R_0 = -3/4)\), the critical Rayleigh parameter is slightly larger, \( R_{1cr} \approx 1.4 \). If the Rayleigh parameter is given by

\[
\lambda = 0
\]

\( \lambda \) corresponds to the critical value of parameter \( R_{2cr} \), which turns out to be equal to the first critical value \( R_{2cr} = R_{1cr} \).

### 6. DISCUSSION OF NUMERICAL RESULTS

In this section, we report on the results of numerical analysis of nonlinear system of equations (62) with initial conditions \( X(0) = V(0) = U(0) = W(0) = Y(0) = Z(0) = 1 \) in the time interval \( 0 \leq t \leq 1000 \) for the Rayleigh \((R_0 = -1)\) and Kepler \((R_0 = -3/4)\) rotation profiles. For parameters \( R > R_{1cr} \), stability is lost, and convective flows appear in the system. As can be seen in Fig. 9a, convection does not appear for \( R = 1 \), and the initially perturbed magnetic field decays (Fig. 9b). For parameter \( R = 5 \), helical trajectories appear in phase plane \( UY \) around stationary point \( O_2 \) (Figs. 9c, 9d, and Figs. 10a, 10b), which are wound with increasing parameter \( R \). This is noticeable even for \( R = 10.3 \) (Fig. 9e) for Rossby number \( Ro = -1 \) and for \( R = 14.6 \) (Fig. 10c) for Rossby number \( Ro = -3/4 \). In this case, the perturbed magnetic field performs oscillations with damped amplitude (Figs. 9f and 10d). In this

### Table 2. Eigenvalues \( \lambda_{1,2,3,4,5,6}(\text{Lyapunov indices}) \) for stationary point \( O_1 \), calculated for different values of parameter \( R \) for \( Pm = 1, Pr = 9, H = 5, T = 1, Ta = 2, \) and \( \gamma = 1 \) and for the Kepler rotation profile \((R_0 = -3/4)\)

| \( R \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) | \( \lambda_4 \) | \( \lambda_5 \) | \( \lambda_6 \) |
|---|---|---|---|---|---|---|
| 5  | -0.1111 | 0.2246 | -1.0531 + \(i1.1978 \) | -0.7212 | -1.5082 | -1.0531 - \(i1.1978 \) |
| 14.6 | 0.6708 | -1.1108 + \(i0.10685 \) | -0.1111 | -0.8242 | -1.7360 | -1.1108 - \(i0.10685 \) |
| 14.7 | 0.6748 | -1.1111 + \(i0.10674 \) | -0.1111 | -0.8249 | -1.7360 | -1.1111 - \(i0.10674 \) |
| 28.2369 | 1.1449 | -1.1201 + \(i0.9441 \) | -0.1111 | -0.8863 | -2.1294 | -1.1201 - \(i0.9441 \) |
| 30.8455 | 1.2234 | -1.1173 + \(i0.9279 \) | -0.1111 | -0.8936 | -2.2061 | -1.1173 - \(i0.9279 \) |
| 30.8457 | 1.2234 | -1.1173 + \(i0.9279 \) | -0.1111 | -0.8936 | -2.2061 | -1.1173 - \(i0.9279 \) |

### Table 3. Eigenvalues \( \lambda_{1,2,3,4,5,6}(\text{Lyapunov indices}) \) for stationary points \( O_{2,3} \), calculated for different values of parameter \( R \) for \( Pm = 1, Pr = 9, H = 5, T = 1, Ta = 2, \) and \( \gamma = 1 \) and for the Rayleigh rotation profile \((R_0 = -1)\)

| \( R \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) | \( \lambda_4 \) | \( \lambda_5 \) | \( \lambda_6 \) |
|---|---|---|---|---|---|---|
| 1  | 0.0084 | -1.0111 + \(i1.1855 \) | -0.1206 | -0.4923 | -1.5952 | -1.0111 - \(i1.1855 \) |
| 5  | 0.2966 | -1.0482 + \(i1.1287 \) | -0.1173 | -0.6223 | -1.6826 | -1.0482 - \(i1.1287 \) |
| 10.3 | 0.5760 | -1.0799 + \(i1.0634 \) | -0.1169 | -0.7076 | -1.8137 | -1.0799 - \(i1.0634 \) |
| 10.5 | 0.5854 | -1.0808 + \(i1.0612 \) | -0.1168 | -0.7100 | -1.8190 | -1.0808 - \(i1.0612 \) |
| 22.878 | 1.0740 | -1.1016 + \(i0.9504 \) | -0.1166 | -0.8075 | -2.1687 | -1.1016 - \(i0.9504 \) |
| 22.998 | 1.0780 | -1.1016 + \(i0.9496 \) | -0.1166 | -0.8081 | -2.1722 | -1.1016 - \(i0.9496 \) |
| 45  | 1.7156 | -1.0830 + \(i0.8571 \) | -0.1165 | -0.8807 | -2.7744 | -1.0830 - \(i0.8571 \) |
Table 4. Eigenvalues \( \lambda_{1,2,3,4,5,6} \) (Lyapunov indices) for stationary points \( O_{2,3} \) calculated for different values of parameter \( R \) for \( Pm = 1, Pr = 9, H = 5, T = 1, Ta = 2, \) and \( \gamma = 1 \) and for the Kepler rotation profile \( (Ro = -3/4) \)

| \( R \)   | \( \lambda_1 \)         | \( \lambda_2 \)         | \( \lambda_3 \)         | \( \lambda_4 \)         | \( \lambda_5 \)         | \( \lambda_6 \)         |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 5     | \(-0.0366 + i0.2328\) | \(-1.0182 + i1.2451\) | \(-0.6631\) | \(-1.4493\) | \(-1.0182 - i1.2451\) | \(-0.6631 - i0.2328\) |
| 14.6  | \(0.557 + i0.2743\)   | \(-1.0437 + i1.1942\) | \(-0.7346\) | \(-1.5114\) | \(-1.0437 - i1.1942\) | \(-0.7346 - i0.2743\) |
| 14.7  | \(0.0565 + i0.2745\)   | \(-1.0439 + i1.1937\) | \(-0.7352\) | \(-1.5121\) | \(-1.0439 - i1.1937\) | \(-0.7352 - i0.2745\) |
| 28.2369 | \(0.1626 + i0.2849\)  | \(-1.07167 + i1.1298\) | \(-0.7941\) | \(-1.6100\) | \(-1.07167 - i1.1298\) | \(-0.7941 - i0.2849\) |
| 30.8455 | \(0.1811 + i0.2829\)  | \(-1.0758 + i1.1184\) | \(-0.8026\) | \(-1.6301\) | \(-1.0758 - i1.1184\) | \(-0.8026 - i0.2829\) |
| 30.8457 | \(0.1811 + i0.2829\)  | \(-1.0758 + i1.1184\) | \(-0.8026\) | \(-1.6301\) | \(-1.0758 - i1.1184\) | \(-0.8026 - i0.2829\) |
| 105   | \(0.2571\)         | \(0.9028\)         | \(-1.0952 + i0.9125\) | \(-0.9106\) | \(-2.2810\) | \(-1.0952 - i0.9125\) |

In case, eigenvalues \( \lambda_i \) are complex-valued with a negative real part (see Tables 3 and 4) and, hence, we classify the stationary point as a stable focus. The significant increase of the Rayleigh parameter from \( R = 10.3 \) to \( R = 10.5 \) (Fig. 11a) and, analogously, from \( R = 14.6 \) to \( R = 14.7 \) (Figs. 10c, 10d, 10e, 10f) leads to sign (direction) reversal of the oscillating perturbed magnetic field, which also decays (Figs. 10f and 11b). Here, phase trajectories are wound in the helix around stationary point \( O_3 \) located in the negative sector of the \( UY \) plane; this point is also classified as a stable focus.

Figures 11c and 11d show the emergence of a homoclinic trajectory in the phase space for parameters \( R = 22.878 \) and \( Ro = -1 \), which contains an equilibrium state loop of the saddle–focus type (see Tables 1 and 3). An analogous pattern is observed for parameters \( R = 28.2369455 \) and \( Ro = -3/4 \) (Figs. 12a and 12b). For \( R = 22.998 \) (\( Ro = -1 \)), a transition from the homoclinic trajectory to chaotic motion is observed (see Figs. 11e and 13a). Figures 12c and 12d show metastable chaos in which the chaotic phase lasts approximately \( t \approx 300 \), and then a transition to damped oscillations occurs. For the Kepler rotation profile \( (Ro = -1) \), the transition to chaotic motion is observed for Rayleigh parameters \( R = 30.8457 \) (see Figs. 12e and 13c). Figures 11f and 12f show irregular oscillations with aperiodic variation of the amplitude and direction (inversion) of the perturbed magnetic field.

A further increase in parameter \( R \) facilitates the evolution of chaotic behavior of convection in which chaotic fractal structures (strange attractors) are formed (see Fig. 13). It should be noted that the convection regimes considered above also appear for larger values of \( H \) and \( T \); accordingly, parameter \( R \) must also increase.

Figure 14 shows bifurcation diagrams demonstrating the emergence of a chaotic regime via a number of period doubling bifurcations. The results are shown for the \( Z \) amplitudes as a function of quantity \( \gamma \). Comparison of bifurcation diagrams shows that the chaotic regime sets in the system for smaller values of the Rayleigh parameter \( (R = 45) \) for the Rayleigh rotation profile \( (Ro = -1) \). Both bifurcation diagrams show that with increasing parameter \( \gamma \), a large number of complex cycles ultimately leading to a chaotic (turbulent) state appear in the system.

With the help of spectral analysis of the system of equations (62) performed numerically, we have obtained the aperiodic behavior of magnetic field perturbations with a noiselike frequency spectrum. Figures 15a and 15b show the dependences of the Fourier components of magnetic field perturbations \( F|U| \) on frequency \( f \) for Rayleigh parameter \( R = 45.105 \) and
Fig. 9. (a, c, e) Projections of phase trajectories in the $UY$ plane upon the variation of parameter $R$ for $\gamma = 1$, $P\text{m} = 1$, $Pr = 9$, $H = 5$, $T = 1$, $Ta = 2$, and $Ro = -1$. (b, d, f) Time dependences of magnetic component amplitude variations $U(t)$. 
Fig. 10. Same as in Fig. 9 for Ro = –3/4.
Fig. 11. Same as in Fig. 9 for Ro = -1.
Fig. 12. Same as in Fig. 10, but for other values of parameter R.
Rossby numbers $\text{Ro} = -1$ and $-3/4$. It can be seen that the spectrum does not decrease with increasing frequency but, on the contrary, appears after a certain dip region.

Consequently, the solution obtained for the above parameters is chaotic indeed. More intense spikes of the Fourier energy spectrum $E_{uu}$ of the perturbed magnetic field are observed for the Kepler rotation profile ($\text{Ro} = -3/4$) and Rayleigh parameter $R = 105$ as compared to the Rayleigh profile ($\text{Ro} = -1$) for $R = 45$ (see Figs. 15c and 15d).

The chaotic behavior of convection for different rotation profiles with Rossby numbers $\text{Ro} = -1$ and $\text{Ro} = -3/4$ is confirmed by the results of numerical calculation of the autocorrelation function $K_{uu}(\tau)$ for a perturbed magnetic field, which are shown in Fig. 16. The chaotic motion corresponds to the parts of trajectories with an exponential decrease of function $K_{uu}(\tau)$. Obviously, the region of the exponential decrease on logarithmic scale of autocorrelation function $K_{uu}(\tau)$ is approximated by straight lines (Fig. 17).

7. CONCLUSIONS

We have analyzed the stability of a nonuniformly rotating electroconducting liquid in an axial magnetic field.
field in the presence of a constant temperature gradient. In the linear approximation, we have derived the dispersion equation which leads to the well-known criterion for the emergence of MRI instability in the absence of a temperature gradient \( Ra = 0 \) [25]. With allowance for the temperature gradient (i.e., convective flows), we have considered the stationary and oscillatory regimes of convection. Analysis of these convection regimes depending on the nonuniform rotation profile (Rossby number \( Ro \)) has revealed that negative values of \( Ro < 0 \) produce a destabilizing

**Fig. 14.** Bifurcation diagrams for the \( Z \) amplitudes upon the variation of \( \gamma \). Calculations were performed for parameters \( Pm = 1, Pr = 9, H = 5, T = 1 \), and \( Ta = 2 \) for two different rotation profiles.

**Fig. 15.** Fourier spectra of magnetic field perturbations \( U(f) \) and Fourier energy spectra \( E_{UU} \) of the perturbed magnetic field on the logarithmic scale for parameters \( \gamma = 1, Pm = 1, Pr = 9, H = 5, T = 1 \), and \( Ta = 2 \) for the Rayleigh (a, c) and Kepler (b, d) rotation profiles.
effect. In this case, critical Rayleigh number $R_{\text{a}_{\text{min}}}$ becomes smaller than in the case of uniform rotation ($R_\text{o} = 0$) or rotation with positive number $R_\text{o} > 0$ (see Fig. 4). In addition, we have studied the chaotic behavior of 3D magnetoconvection in nonuniformly rotating layers of an electroconducting liquid on the basis on the nonlinear dynamics equations of the 6D phase space. These equations have been obtained in our study using the Galerkin approximation of the minimal order. Qualitative analysis of the nonlinear system of dynamic equations, which has been performed analytically and numerically, demonstrates the existence of a complex chaotic structure, viz., a strange attractor. Therefore, we have determined the convection regime in which chaotic changes in the direction (inversion) and amplitude of the perturbed magnetic field appear on account of the nonuniform convection of the medium. The theory developed in this study can be used as a scenario of the emergence of turbulence in hot accretion disks.

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