Non-normal and Stochastic Amplification of Magnetic Energy in the Turbulent Dynamo: Subcritical Case

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Abstract

Our attention focuses on the stochastic dynamo equation with non-normal operator that gives an insight into the role of stochastics and non-normality in galactic magnetic field generation. The main point of this Letter is a discussion of the generation of a large-scale magnetic field that cannot be explained by traditional linear eigenvalue analysis. We present a simple stochastic model for the thin-disk axisymmetric $\alpha\Omega$ dynamo involving three factors: (a) non-normality generated by differential rotation, (b) nonlinearity reflecting how the magnetic field affects the turbulent dynamo coefficients, and (c) stochastic perturbations. We show that even for the subcritical case, there are three possible mechanisms for the generation of magnetic field. The first mechanism is a deterministic one that describes an interplay between transient growth and nonlinear saturation of the turbulent $\alpha$–effect and diffusivity. It turns out that the trivial state is nonlinearly unstable to small but finite initial perturbations. The second and third are stochastic mechanisms that account for the interaction of non-normal effect generated by differential rotation with random additive and multiplicative fluctuations. In particular, we show that in the subcritical case the average magnetic energy can grow exponentially with time due to the multiplicative noise associated with the $\alpha$–effect.
The generation and maintenance of large scale magnetic fields in stars and galaxies has attracted enormous attention in past years \[1\]-\[4\] (see also a recent review \[5\]). The main candidate to explain the process of conversion of the kinetic energy of turbulent flow into magnetic energy is the mean field dynamo theory \[2\]. The standard dynamo equation for the large scale magnetic field $B(t, x)$ reads

$$\frac{\partial B}{\partial t} = \text{curl}(\alpha B) + \beta \Delta B + \text{curl}(u \times B),$$

where $u$ is the mean velocity field, $\alpha$ is the coefficient of the $\alpha$-effect and $\beta$ is the turbulent magnetic diffusivity. This equation has been widely used for analyzing the generation of the large-scale magnetic field. Traditionally the mathematical procedure consists of looking for exponentially growing solutions of the dynamo equation with appropriate boundary conditions. While this approach has been quite successful in the prediction of large scale magnetic field generation, it fails to predict the subcritical onset of a large-scale magnetic field for some turbulent flow. Although the trivial solution $B = 0$ is linearly stable for the subcritical case, the non-normality of the linear operator in the dynamo equation for some turbulent flow configurations leads to the transient growth of initial perturbations \[3\]. It turns out that the non-linear interactions and random fluctuations might amplify this transient growth further. Thus, instead of the generation of the large scale magnetic field being a consequence of the linear instability of trivial state $B = 0$, it results from the interaction of transient amplifications due to the non-normality with nonlinearities and stochastic perturbations. The importance of the transient growth of magnetic field for the induction equation has been discussed recently in \[7, 8\]. Comprehensive reviews of subcritical transition in hydrodynamics due to the non-normality of the linearized Navier-Stokes equation, and the resulting onset of shear flow turbulence, can be found in \[9, 10\].

The main purpose of this Letter is to study the non-normal and stochastic amplification of the magnetic field in galaxies. Our intention is to discuss the generation of the large-scale magnetic field that cannot be explained by traditional linear eigenvalue analysis. It is known that non-normal dynamical systems have an extraordinary sensitivity to stochastic perturbations that leads to great amplifications of the average energy of the dynamical system \[11\]. Although the literature discussing the mean field dynamo equation is massive, the effects of non-normality and random fluctuations are relatively unexplored. Several attempts have been made to understand the role of random fluctuations in magnetic field generation. The motivation was the observation of rich variability of large scale magnetic fields in stars and galaxies. Small scale fluctuations parameterized by stochastic forcing were the subject of recent research by Farrell and Ioannou \[7\]. They examined the mechanism of stochastic field generation due to the transient growths for the induction equation. They did not use the standard closure involving $\alpha$ and $\beta$ parameterization. Hoyon with his colleagues has studied the effect of random alpha-fluctuations on the solution of the kinematic mean-field dynamo \[12\]. However they did not discuss the non-normality of the dynamo equation and the possibility of stochastic transient growth of magnetic energy. Both attempts have involved only the linear stochastic theory. Numerical simulations of magnetoconvection equations with noise and non-normal transient growth have been performed in \[8\].

It is the purpose of this Letter to present a simple stochastic dynamo model for the thin-disk axisymmetric $\alpha\Omega$ dynamo involving three factors: non-normality, non-linearity and stochastic perturbations. Recently it has been found \[13\] that the interactions of these factors leads to noise-induced phase transitions in a “toy” model mimicking a laminar-to-turbulent transition. In this Letter we discuss three possible mechanisms for the generation of a magnetic field that are not based on standard linear eigenvalue analysis of the dynamo equation. The first mechanism is a deterministic one that describes an interplay between linear transient growth and nonlinear saturation of both turbulent parameters: $\alpha$ and $\beta$. The second and third are stochastic mechanisms that account for the interaction of the non-normal effect generated by differential rotation.
with random additive and multiplicative fluctuations.

Here we study the nonnormality and stochastic perturbation effects on the growth of galactic magnetic field by using a Moss’s “no-z” model for galaxies [14]. Despite its simplicity the “no-z” model proves to be very robust and gives reasonable results compared with real observations. We consider a thin turbulent disk of conducting fluid of uniform thickness 2\(h\) and radius \(R\) (\(R \gg h\)), which rotates with angular velocity \(\Omega(r)\) [3, 4]. We consider the case of \(\alpha\Omega\)-dynamo for which the differential rotation dominates over the \(\alpha\)-effect. Neglecting the radial derivatives one can write the stochastic equations for the azimuthal, \(B_\varphi(t)\), and radial, \(B_r(t)\), components of the axisymmetric magnetic field

\[
\frac{dB_r}{dt} = -\frac{\alpha(|B|, \xi_\alpha(t))}{h} B_\varphi - \frac{\pi^2\beta(|B|)}{4h^2} B_r + \xi_f(t),
\]

\[
\frac{dB_\varphi}{dt} = g B_r - \frac{\pi^2\beta(|B|)}{4h^2} B_\varphi, \tag{1}
\]

where \(\alpha(|B|, \xi_\alpha(t))\) is the random non-linear function describing the \(\alpha\)-effect, \(\beta(|B|)\) is the turbulent magnetic diffusivity, \(g = rd\Omega/dr\) is the measure of differential rotation (usually \(rd\Omega/dr < 0\)).

Nonlinearity of the functions \(\alpha(|B|, \xi_\alpha(t))\) and \(\beta(|B|)\) reflects how the growing magnetic field \(B\) affects the turbulent dynamo coefficients. This nonlinear stage of dynamo theory is a topic of great current interest, and, numerical simulations of the non-linear magneto-hydrodynamic equations are necessary to understand it. There is an uncertainty about how the dynamo coefficients are suppressed by the mean field and current theories seem to disagree about the exact form of this suppression [19]. Here we describe the dynamo saturation by using the simplified forms [5]

\[
\alpha(|B|, \xi_\alpha(t)) = (\alpha_0 + \xi_\alpha(t))\varphi_\alpha(|B|), \quad \beta(|B|) = \beta_0\varphi_\beta(|B|), \tag{2}
\]

where \(\varphi_{\alpha,\beta}(|B|)\) is a decaying function such that \(\varphi_{\alpha,\beta}(0) = 1\). In what follows we use [5]

\[
\varphi_\alpha(|B|) = \left(1 + k_\alpha(B_\varphi/B_{eq})^2\right)^{-1}, \quad \varphi_\beta(|B|) = \left(1 + \frac{k_\beta}{1 + (B_{eq}/B_{\varphi})^2}\right)^{-1}, \tag{3}
\]

where \(k_\alpha\) and \(k_\beta\) are constants of order one, and \(B_{eq}\) is the equipartition strength. It should be noted that for the \(\alpha\Omega\)-dynamo the azimuthal component \(B_\varphi(t)\) is much larger than the radial field \(B_r(t)\), therefore, \(B^2 \simeq B_{\varphi}^2\). We did not include the strong dependence of \(\alpha\) and \(\beta\) on the magnetic Reynolds number \(R_m\). The back reaction of the magnetic field on the differential rotation is also ignored.

The multiplicative noise \(\xi_\alpha(t)\) describes the effect of rapid random fluctuations of \(\alpha\). We assume that they are more important than the random fluctuations of the turbulent magnetic diffusivity \(\beta\) [12]. The additive noise \(\xi_f(t)\) represents the stochastic forcing of unresolved scales [7]. Both noises are independent Gaussian random processes with zero means \(<\xi_\alpha(t)>=0\), \(<\xi_f(t)>=0\) and correlations:

\[
<\xi_\alpha(t)\xi_\alpha(s)> = 2D_\alpha \delta(t-s), \quad <\xi_f(t)\xi_f(s)> = 2D_f \delta(t-s). \tag{4}
\]

The intensity of the noises is measured by the parameters \(D_\alpha\) and \(D_f\). One can show [13] that the additive noise in the second equation in (1) is less important.
The governing equations (1) can be nondimensionalized by using an equipartition field strength $B_{eq}$, a length $h$, and a time $\Omega_0^{-1}$, where $\Omega_0$ is the typical value of angular velocity. By using the dimensionless parameters

$$g \to -\Omega_0 |g|, \quad \delta = \frac{R_{\omega}}{R_\omega}, \quad \varepsilon = \frac{\alpha_\omega h}{4R_\omega}, \quad R_\alpha = \frac{\alpha_\omega h^2}{\beta}, \quad R_\omega = \frac{\Omega_0 h^2}{\beta},$$  \tag{5}$$

we can write the stochastic dynamo equations in the form of SDE’s

$$dB_r = -\left(\delta \varphi_\alpha(B_\varphi)B_\varphi + \varepsilon \varphi_\beta(B_\varphi)B_\varphi\right) dt - \sqrt{2\sigma_1 \varphi_\alpha(B_\varphi)B_\varphi}dW_1 + \sqrt{2\sigma_2}dW_2,$$

$$dB_\varphi = -\left(|g|B_r + \varepsilon \varphi_\beta(B_\varphi)B_\varphi\right) dt,$$ \tag{6}

where $W_1$ and $W_2$ are independent standard Wiener processes. The dynamical system (5) is subjected to the multiplicative and additive noises with the corresponding intensities:

$$\sigma_1 = \frac{D_\alpha}{h^4 \Omega_0}, \quad \sigma_2 = \frac{D_f}{B_{eq}^4 \Omega_0}.$$ \tag{7}

It is well-known that the presence of noise can dramatically change the properties of a dynamical system [18]. Since the differential rotation dominates over the $\alpha$-effect ($R_\alpha \ll |R_\omega|$), the system (5) involves two small parameters $\delta = R_\alpha/R_\omega$ and $\varepsilon = 1/R_\omega$ whose typical values are $0.01 - 0.1$ ($R_\omega = 10 - 100, \quad R_\alpha = 0.1 - 1$). These parameters play very important roles in what follows. For small values $\delta$ and $\varepsilon$, the linear operator in (5) is a highly non-normal one ($|g| \sim 1$). This can lead to a large transient growth of the azimuthal component $B_\varphi(t)$ in a subcritical case. We then expect a high sensitivity to stochastic perturbations. Similar deterministic low-dimensional models have been proposed to explain the subcritical transition in the Navier-Stokes equations (see, for example, [15, 16]). The main difference is that the nonlinear terms in (1) are not energy conserving.

The probability density function $p(t, B_r, B_\varphi)$ obeys the Fokker-Planck equation associated with (5) \[17\]

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial B_r} [(\delta \varphi_\alpha(B_\varphi)B_\varphi + \varepsilon \varphi_\beta(B_\varphi)B_\varphi)p] - \frac{\partial}{\partial B_\varphi} [(|g|B_r + \varepsilon \varphi_\beta(B_\varphi)B_\varphi)p] +$$

$$(\sigma_1 \varphi^2_\alpha(B_\varphi)B_{\varphi}^2 + \sigma_2) \frac{\partial^2 p}{\partial B_\varphi^2}.$$ \tag{8}

Using this equation in the linear case one can find a closed system of ordinary differential equations for the moments $<B_r^2>$, $<B_r B_\varphi>$, and $<B_\varphi^2>$

$$\frac{d}{dt} \left( \begin{array}{c} <B_r^2> \\ <B_r B_\varphi> \\ <B_\varphi^2> \end{array} \right) = \left( \begin{array}{ccc} -2\delta & -2\varepsilon & \sigma_1 \\ -|g| & -2\varepsilon & -\delta \\ 0 & -2|g| & -2\varepsilon \end{array} \right) \left( \begin{array}{c} <B_r^2> \\ <B_r B_\varphi> \\ <B_\varphi^2> \end{array} \right) + \left( \begin{array}{c} \sigma_2 \\ 0 \\ 0 \end{array} \right).$$ \tag{9}

The linear system of equations (9) allows us to determine the initial evolution of the average magnetic energy $E(t) = <B_r^2> + <B_r B_\varphi> + <B_\varphi^2>$. Similar equations emerge in a variety of physical situations, such as models of stochastic parametric instability that explain why the linear oscillator subjected to multiplicative noise can be unstable [20].

Now we are in a position to discuss three possible scenarios for the subcritical generation of galactic magnetic field.
Figure 1: Linear case: the azimuthal component $B_\varphi$ as a function of time ($B_\varphi(0) = 0$). for $|g| = 1$, $\delta = 10^{-4}$ and $\varepsilon = 2 \cdot 10^{-2}$ and different initial values of $B_r$: $-0.017, -0.021, -0.03$.

**Deterministic subcritical generation.** Let us examine the deterministic transient growth of the magnetic field in the subcritical case. To illustrate the non-normality effect consider first the linear case without noise terms. The dynamical system (6) takes the form

$$\frac{d}{dt} \begin{pmatrix} B_r \\ B_\varphi \end{pmatrix} = \begin{pmatrix} -\varepsilon & -\delta \\ -|g| & -\varepsilon \end{pmatrix} \begin{pmatrix} B_r \\ B_\varphi \end{pmatrix}. \tag{10}$$

Since $\delta \ll 1$, $\varepsilon \ll 1$ and $|g| \sim 1$, this system involves a highly non-normal matrix. Even in the subcritical case ($0 < \delta < \varepsilon^2/|g|$ see below) when all eigenvalues are negative, $B_\varphi$ exhibits a large degree of transient growth before the exponential decay. Assuming that $B_r(t) = e^{\gamma t}$ and $B_\varphi(0) = b e^{\gamma t}$ we find two eigenvalues $\gamma_{1,2} = -\varepsilon \pm \sqrt{\delta|g|}$ (the corresponding eigenvectors are almost parallel). The supercritical excitation condition $\gamma_1 > 0$ can be written as $\sqrt{\delta|g|} > \varepsilon$ or $\sqrt{R_\alpha R_\omega |g|} > \pi^2/4 [4]$. Consider the subcritical case when $0 < \delta < \varepsilon^2/|g|$. The solution of the system (10) with the initial conditions $B_r(0) = -2c\sqrt{\delta/|g|}$, $B_\varphi(0) = 0$ is

$$B_r(t) = -c\sqrt{\frac{\delta}{|g|}}(e^{\gamma_1 t} + e^{\gamma_2 t}), \quad B_\varphi(t) = c(e^{\gamma_1 t} - e^{\gamma_2 t}). \tag{11}$$

Thus $B_\varphi(t)$ exhibits large transient growth over a timescale of order $1/\varepsilon$ before decaying exponentially. In Fig. 1 we plot the azimuthal component $B_\varphi$ as a function of time for $|g| = 1$, $\delta = 10^{-4}$ and $\varepsilon = 2 \cdot 10^{-2}$ and different initial values of $B_r$ ($B_\varphi(0) = 0$).

Of course without nonlinear terms any initial perturbation decays. However if we take into account the back reaction suppressing the effective dissipation ($\varphi_\beta(|B|)$ is a decaying function), one can expect an entirely different global behaviour. In the deterministic case there can be three stationary solutions to (4). In Fig. 2 we illustrate the role of transient growth and nonlinearity in the transition to a non-trivial state using (3) with $k_\alpha = 0.5$ and $k_\beta = 3$. We plot the azimuthal component $B_\varphi$ as a function of time with the initial condition $B_\varphi(0) = 0$. We use the same values of parameters $|g|$, $\delta$ and $\varepsilon$ and three initial values of $B_r(0)$ as in Fig. 1. One can see from
Figure 2: Nonlinear case: the azimuthal component $B_\phi$ as a function of time ($B_\phi(0) = 0$). for $|g| = 1$, $\delta = 10^{-4}$ and $\varepsilon = 2 \cdot 10^{-2}$ and different initial values of $B_r: -0.017, -0.021, -0.03$.

Fig. 2 that the trivial solution $B_\phi = B_r = 0$ is nonlinearly unstable to small but finite initial perturbations of $B_r$, such as, $B_r(0) = -0.03$. For fixed values of the parameters in nonlinear system (6), there exists a threshold amplitude for the initial perturbation, above which $B_\phi(t)$ grows and below which it eventually decays.

**Stochastic subcritical generation due to additive noise.** This scenario has been already discussed in the literature [7] (see also [11] for hydrodynamics). The physical idea is that the average magnetic energy is maintained by additive Gaussian random forcing representing unresolved scales. It is clear that the non-zero additive noise ($\sigma_2 \neq 0$) ensures the stationary solution to (9). If we assume for simplicity $\sigma_1 = 0$ and $\delta = 0$ then the dominant stationary moment is

$$< B^2_\phi >_{st} = \frac{g^2 \sigma_2}{4 \varepsilon^3}. \quad (12)$$

We can see that due to the non-normality of the system (10) the average stationary magnetic energy $E_{st} \sim < B^2_\phi >_{st}$ exhibits a high degree of sensitivity with respect to the small parameter $\varepsilon : E_{st} \sim \varepsilon^{-3}$. [11, 13].

**Stochastic subcritical generation due to multiplicative noise.** Here we discuss the divergence of the average magnetic energy $E(t) = < B^2_r > + < B_r B_\phi > + < B^2_\phi >$ with time $t$ due to the random fluctuations of the $\alpha-$parameter. Although the first moments tend to zero in the subcritical case, the average energy $E(t)$ grows as $e^{\lambda t}$ when the intensity of noise $\sigma_1$ exceeds a critical value. The growth rate $\lambda$ is the positive real root of the characteristic equation for the system (9)

$$(\lambda + 2\varepsilon)^3 - 4\delta |g| (\lambda + 2\varepsilon) - 2\sigma_1 |g| = 0. \quad (13)$$

For $\delta = 0$, the growth rate is $\lambda_0 = -2\varepsilon + (2\sigma_1 |g|)^{1/3}$ as long as it is positive, and the excitation condition can be written as $\sigma_1 > \sigma_{cr} = 4\varepsilon^3/|g|$. It means that the generation of average magnetic energy occurs for $\alpha_0 = 0$! It is interesting to compare this criterion with the classical supercritical excitation condition: $\delta |g| > \varepsilon^2$. [4]. To assess the significance of this parametric instability it is useful to estimate the magnitude of the critical noise intensity $\sigma_{cr}$. First let us estimate the
parameter $\varepsilon = \pi^2 \beta / (4\Omega_0 h^2)$. The turbulent magnetic diffusivity is given by $\beta \simeq lv/3$, where $v$ is the typical velocity of turbulent eddy $v \simeq 10 \text{ km s}^{-1}$, and $l$ is the turbulent scale, $l \simeq 100 \text{ pc}$. For spiral galaxies, the typical values of the thickness, $h$, and the angular velocity, $\Omega_0$, are $h \simeq 800 \text{ pc}$ and $\Omega_0 \simeq 10^{-15} \text{ s}^{-1}$; $|g| \simeq 1$. It gives an estimate for $\varepsilon \simeq 3.2 \times 10^{-2}$, that is, $\sigma_{cr} \simeq 1.3 \times 10^{-4}$. In general $\lambda(\delta) = \lambda_0 + (4/3|g|(2\sigma_1|g|)^{-1/3})\delta + o(\delta)$. This analysis predicts an amplification of the average magnetic energy in a system where no such amplification is observed in the absence of noise. The value of the critical noise intensity parameter $\sigma_{cr}$, above which the instability occurs, is proportional to $\varepsilon^3$, that is, very small indeed. To some extent, the amplification process exhibits features similar to those observed in the linear oscillator submitted to parametric noise. To avoid the divergence of the average magnetic energy, it is necessary to go beyond the kinematic regime and consider the effect of nonlinear saturations.

In summary, we have discussed galactic magnetic field generation that cannot be explained by traditional linear eigenvalue analysis of dynamo equation. We have presented a simple stochastic model for the $\alpha\Omega$ dynamo involving three factors: (a) non-normality due to differential rotation, (b) nonlinearity of the turbulent dynamo $\alpha$ effect and diffusivity $\beta$, and (c) additive and multiplicative noises. We have shown that even for the subcritical case, there are three possible scenarios for the generation of large scale magnetic field. The first mechanism is a deterministic one that describes an interplay between transient growth and nonlinear saturation of the turbulent $\alpha$-effect and diffusivity. We have shown that the trivial state $B = 0$ can be nonlinearly unstable with respect to small but finite initial perturbations. The second and third are stochastic mechanisms that account for the interaction of non-normal effect generated by differential rotation with random additive and multiplicative fluctuations. We have shown that multiplicative noise associated with the $\alpha$-effect leads to exponential growth of the average magnetic energy even in the subcritical case.

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References

[1] H. K. Moffatt, Magnetic Field Generation in Electrically Conducting Fluids (Cambridge University Press, New York, 1978).

[2] F. Krause and K. H. Radler, Mean-Field Magnetohydrodynamics and Dynamo Theory (Academic-Verlag, Berlin, 1980).

[3] Ya. B. Zeldovich, A.A. Ruzmaikin and D. D. Sokoloff, Magnetic Fields in Astrophysics (Gordon and Breach Science Publishers, New York, 1983).

[4] A.A. Ruzmaikin, A. M. Shukurov and D. D. Sokoloff, Magnetic Fields in Galaxies (Kluwer Academic Publishers, Dordrecht, 1988).

[5] L. Widrow, Rew. Mod. Phys. 74, 775 (2002).

[6] An operator is said to be non-normal if it does not commute with its adjoint in the corresponding scalar product. Typical examples of transient growth can be represented by the functions: $f_1(t) = t \exp(-t)$ and $f_2(t) = \exp(-t) - \exp(-2t)$ for $t \geq 0$.

[7] B. Farrel, and P. Ioannou, 1999, ApJ, 522, 1088.

[8] J. R. Gog, I. Opera, M. R. E. Proctor, and A. M. Rucklidge, Proc. R. Soc. Lond. A, 455, 4205 (1999).
[9] S. Grossmann, Rev. Mod. Physics, 72, 603 (2000).

[10] P. J. Schmid, D. S. Henningson, Stability and Transition in Shear Flows, (Springer, Berlin, 2001).

[11] B. F. Farrell, and P. J. Ioannou, Phys. Fluids, 5, 2600 (1993); Phys. Rev. Lett. 72, 1188 (1994); J. Atmos. Sci., 53, 2025 (1996).

[12] P. Hoyng, D. Schmitt, and L. J. W. Teuben, Astron. Astrophys. 289, 265 (1994).

[13] S. Fedotov, I. Bashkirtseva, and L. Ryashko, Phys. Rev. E., 66, 066310 (2002).

[14] R. Beck, A. Brandenburg, D. Moss, A. Shukurov, and D. Sokoloff, Ann. Rev. Astron. Astrophys. 34, 155 (1996).

[15] L. N. Trefethen, A. E. Trefethen, S. C. Reddy, and T. A. Driscoll, Science 261, 578 (1993).

[16] T. Gebhardt and S. Grossmann, Phys. Rev. E 50, 3705 (1994); J. S. Baggett, T. A. Driscoll, and L. N. Trefethen, Phys. Fluids 7, 833 (1995); J. S. Baggett and L. N. Trefethen, Phys. Fluids 9, 1043 (1997).

[17] C. W. Gardiner, Handbook of Stochastic Methods, 2nd ed. (Springer, New York, 1996).

[18] W. Horsthemke and R. Lefever, Noise-Induced Transitions (Springer, Berlin, 1984).

[19] S. I. Vainshtein and F. Cattaneo, ApJ 393, 165 (1992); F. Cattaneo, and D. W. Hughes, Phys. Rev. E 54, R4532 (1996); A. V. Gruzinov and P. H. Diamond, Phys. Rev. Lett. 72, 1651 (1994); A. Brandenburg, ApJ 550, 824 (2001); I. Rogachevskii and N. Klecorin Phys. Rev. E 64, 056307 (2001); E. G. Blackman, and G. B. Field, Phys. Rev. Lett. 89, 265007 (2002).

[20] R. Bourret, Physica 54, 623 (1971); U. Frisch, and A. Pouquet, ibid, 65, 303 (1973).