Derivation of crack propagation velocity formula for investigation dynamic problems

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Abstract. The following thesis will illustrate derivations of crack propagation’s formula for investigating brittle and elastic bodies. This work will look at the static and dynamic cases for crack initiation and propagation. Some approaches will be outlined in this work, which are the technics widely used in the field of mechanics for investigating the crack propagation and initiation related problems. At the end of the work crack propagation velocity formula will be derived.

1. Introduction
Currently, we live in the world of gadgets and software packages. Without a doubt, gadgets integrated our life thoroughly. The applications of such are everywhere. They can be found in the various activities of daily life routines, for example, mobile phones, vacuum cleaners, powerbanks, irons, TVs, etc. It is clear that every device/gadget is run by software. The complexity of software will depend on the basis of it use. One can assume and outline that iron will use simple software to do its functions. Whereas mobile phones will have more complex software embedded to perform complex tasks. Therefore, software plays a vital role in modern-day devices. Similarly, software packages also are greatly influencing our life and pushing the boundaries to the next level. Alike to gadgets they can be found everywhere. We use apps on our mobile phones, computers and etc. They are well used in the industry to perform some complex tasks that can not be done in real life. Also, they are utilized to fasten the analytical process and to minimize the cost. The following software packages can be seen in the industry as the standards, such as Solidworks, Inventor, Abaqus, and others. Each software package has its own unique technique when it comes to the analysis, but the basis is the same. The software packages use engineering formulas to analysis the problem. Hence the software packages are itself a combination of formulas (brought up to a certain degree of complexity) that are aimed to tackle dawn the given problem. As the work will look up on the fracture and particularly on crack propagation the formulas used in the analysis will be outlined. The crack propagation velocity formula will be derived by using different methods and formulas.

Fracture can be traced back to when human-made structures start occurring. And the problem getting worse each year as structures become more complex in the technological world. There is much to be learned, however understanding of the failure of material and the ability to prevent such failure is
increased since World War II. Dramatic failure related to fracture happened during World War II with Liberty ships, these ships were fabricated during the war in a short time and sustained serious fractures as an outcome of shortage of time and money to build with consideration of standards, in this case mainly using the riveted method, but instead, large cylinders were welded together. This ominous event gave sufficient knowledge on how to avoid the cracks occurrence and their further spreading in the structure of ships. Therefore, transformed the way of perception and the attitude toward how the it have to be treated to avoid failures in general.

2. Different approaches to deal with the fracture
In this paragraph, different approaches will be outlined for both static and dynamic. To work crack propagation velocity formula the below presented approaches will be utilized. These approaches can be utilized in different scenarios. The problems (tasks) can be simple, complex, require some amendments such as idealizing the structure for the utilization of formulas and etc.

2.1 LEFM
Fracture mechanics (FM) is the part of mechanics which is mainly aimed at dealing with the formation and propagation of cracks in the different structures. LEFM is a tool of representing qualitative measure in the brittle material (BM), which resists the formation of erratic crack growth. Griffith was the first to propose the theory that the main contributors for the fracture occurrence are flows and existing cracks [1]. The tool can be defined utilizing Griffiths approach (Energy Balance). Similarly, it can be described by the Stress intensity Factor, which is used in this particular time. If one is about applying one of two before mentioned approaches, the correct mode should be selected for the valid outcome. The following three modes are chosen when dealing with a such problem: opening, sliding and tearing. Fracture in highly BMs occurs under mode I (opening), and brittle crack propagates with displacement surfaces on its planes in a manner to minimize the result of tensile stresses, which is shear loading. Earlier the the concept of FM was mostly valid to materials that obey Hooke’s law [2]. Simply saying LEFM is a simple approach, yet sophisticated, theory of fracture, which capable of dealing exclusively with a sharp crack in elastic bodies [3]. Crack propagation comes in two forms: transgranular and intergranular. In the first case as the name suggests it goes through the material grains and the second one goes along the boundaries due to its weakness. So, the tool (LEFM) can be used if the following steps are taken into account, for instance, the presence of an ideal condition in which all materials should be elastic, except for a small area close the crack tip (SACC). The area occurs to be in an inelastic state due to the high stress in the region. However, the plastic zone formed by the high stress near the crack tip is relatively small to the linear dimension of the structure and even the size of the crack, then inelastic region and its effect to the analysis can be neglected. Therefore, LEFM theory can be utilized to analyse the fundamental behaviour of any material, which tends to crack in the manner mentioned above.

2.2 Griffith Energy Balance
Griffiths applied the idea of the first law of thermodynamics to understand the crack formation, which states when a system changes from a non-equilibrium state to an equilibrium state, there is a net decrease in energy [1]. Around the 1920s he established study of fracture in the glass. The approach worked with the sharp cracked specimen with the existence of the idealized condition in which all materials (bodies) should be elastic and under static loading condition. Also, the following material should have to have brittle manner but for the small area close the crack tip. The idea of his work was focused on taking account of the global energy balance of the system rather than on crack tip stresses directly. As well to work out the energy’s amount required to propagate the initial crack in a thermodynamically permanent way. He was aware of Inglis studies on stress concentration near the crack tip when he started his studies of fracture in the glass. However, Inglis work introduces mathematical complexity: stresses at the crack tip approaches infinity with a perfectly sharp crack and utilisation of such solution would predict near-zero strength in the material, which will lead to the rupture of bounds near the crack tip with a very small load applied. Stress at the crack tip of the stress plate was defined by Inglis as illustrated below in
formula 1. Griffith illustrated the total energy of the system can be written as in formula 2. Where \( W \) is the system’s total energy or (work done), \( dU_f \) is the energy available for the formation of new crack surfaces and \( dU_E \) is the internal potential energy. Or it can be written as the following formula 3. Where the right side represents the fracture, resistance associated with \( G_{\text{critical}} \), which is the crack growth resistance of the material \( R \), and the left side shows the elastic strain release rate \( G \). A crack will propagate with the energy release rate criterion \( G \geq R \). Griffith was capable to illustrate that internal strain energy is as shown in formula 4 by utilising Inglis description for the stress at the tip of the crack. Where the potential energy of an uncracked plate is represented by \( U_{E_0} \) with the plate thickness \( B \). \( U_r = 4aBy \) as a crack requires a formation of two new surfaces, therefore factor 2 is required and where \( y \) is the surface energy or energy required to create the new cracked surface. The surface energy value per unit area is constant, hence it is a linear function of \( a \), but the stored strain energy released in the growth of crack is a function of \( a^2 \), and is, therefore, parabolic [4]. Using formula 3 it can be written as shown below in formula 5 by setting the derivative of total energy \( U_E + U_f = 0 \). Finally, by solving it to get formula 6 for fracture stress. Where \( E \) is young’s modulus.

\[
\begin{align*}
\sigma_m & = \sigma(1 + \frac{2a}{b}) & (1) \\
\frac{dW}{dA} = & \frac{dU_f}{dA} + \frac{dU_E}{dA} = 0 & (2) \\
\frac{dU_f}{dA} = & \frac{dU_E}{dA} & (3) \\
U_E = & U_{E_0} - \frac{\sigma^2 a^2 \pi b}{E} & (4) \\
\frac{dU_E}{dA} = & - \frac{\sigma^2 \pi a}{E} \text{ or } \frac{dU_f}{dA} = 2\gamma & (5) \\
\sigma_f = & \left( \frac{2\gamma}{\pi a} \right)^\frac{1}{2} & (6)
\end{align*}
\]

2.3 Stress intensity factor (SIF)

Stress intensity factor (SIF) is a friendly user tool when dealing with the prediction of stress near the tip of the crack as a result of remote load. It also defines whether there will be propagation of the crack and measures the real forces, which are applied at the tip of the crack. There is three type of modes of loading and each of them produces \( \frac{1}{\sqrt{r}} \) singularity at the tip of the crack. Hence the proportionality constants scaling factor \( k \) and function of \( \theta \) depends on the mode and it is better to introduce stress intensity factor \( \mathcal{K} \), which equals to \( k\sqrt{2\pi} \) or \( \mathcal{K} = k\sqrt{2\pi} \). Mode I loading is considered as loading stress is acting perpendicular to the crack plane and hence stress field ahead of the tip of the crack can be written as illustrated below formula 7. It can be observed that formula 7 is on the left side with the assumption of usage of an isotropic leaner elastic material and at the right side rearranged making the scaling factor \( \mathcal{K} \) object [1]. The assumption is made for the non-linear behaviour of the material at the tip of the crack that the effect of the region on the solution can be negligible due to the small area compared to the crack length. Otherwise different methods are applicable such as: J-integral and crack opening displacement [4]. Once, \( \theta = 0 \), then shear stress equals zero and function \( f = 1 \) and the following formula 8 is assumed to be, which describes stress in the close vicinity of the crack tip. Formula 8 only valid close to the crack tip where the \( \frac{1}{\sqrt{r}} \) singularity dictates the stress field as mentioned before [3]. Hence stresses near the tip of the crack grows in proportion to \( \mathcal{K} \).

\[
\begin{align*}
\lim_{r \to 0} \sigma & = \frac{k}{\sqrt{2\pi r}} f(\theta) \text{ or } \mathcal{K} = \lim_{r \to 0} \sigma \sqrt{2\pi r} f(\theta) & (7) \\
\sigma & = \frac{k}{\sqrt{2\pi r}} & (8)
\end{align*}
\]
\[ K = \sigma \sqrt{2\pi r} \]  

(9)

2.4 J-Integral

Stress intensity factor was applied to analyse the problem with plastic zone near the crack tip, which is considered too small if to be compared with the size of the specimen (small scale yielding). However, in event of the presence of a plastic zone, but considered as large-scale yielding stress intensity factor is not applicable due to the providence of precise results. J-Integral comes as a significant tool when dealing with elastic-plastic FM. It can be defined analogous to the strain energy release when considered non-linear formula 10 or equal under linear elastic loading. When solving the problem, the J Integral is not dependent on the path around the tip of the crack [5] and it can be easily defined experimentally. J-integral can be defined as depicts formula 11.

\[ J = \frac{dU}{dA} \]  

(10)

\[ \delta_c = \frac{4K_c^2}{\pi \sigma_y} \]  

(11)

2.5 Crack opening displacement (COD)

Wells was a pioneer of COD for the analysis of crack initiation with large-scale yielding. It was said that as the result of strain crack will extend with help of voids growth and coalescence to the tip of the crack. The approach measures the strain at the crack tip and the crack is estimated to extend at a critical \( \delta_c \), which is estimated to be as stated in formula 12 by Irwin. Where \( \sigma_y \) is yield stress and its can be assumed small scale yielding with

\[ \frac{\sigma}{\sigma_y} \]  

(12)

COD can be related to (strain energy release) \( G \) by the following formula 13 which is also related to J-integral as it was mentioned before. However, one needs to consider that those relationships formula 14 valid only under linear elastic fracture mechanics.

\[ G = \sigma_y \delta_c \]  

(13)

\[ G = J = \sigma_y \delta_c \]  

(14)

2.6 Irwin

G. R. Irwin was an outstanding scientist in the fields of FM and he was the one who suggested a fundamentally new amendment of the Griffiths theory. Irwin suggested the concept of effective surface energy and changed the theory of Griffith in terms of singular stresses at the tip of the crack and showed that his approach is identical to Griffith’s energy approach. Normal stress \( \sigma \) \( (\sigma_{yy}) \) can be defined as in formula 8, which is on the crack plane of linear elastic material. It can be said that when \( \sigma_{ys} = \sigma_{yy} \) yielding occurs for plane stress conditions. Replacing the right-hand side with yield strength to before mentioned formula 8 therefore the one can solve for \( r \) gives a first-order estimation of the plastic zone size formula 15. It can be observed that the formula 15 has a stress intensity factor, which was introduced by Irwin to relate microstructure characteristics to a local mechanical state near the tip of the crack, which results in providing a very modest description of fracture processes as an uninterrupted process of steady (stable) crack growth to the inception of instability [6].

\[ r = \frac{1}{2\pi} \left( \frac{K_I}{\sigma_{ys}} \right)^2 \]  

(15)
2.7 Mott model
A lot of work has been done to examine BM under rapid loading, under which the material tends to shatter into small fragments. With the higher magnitude more pieces of the shattered brittle body occur. There is much more to be learned in the field of rate sensitivity of BM behaviour. Griffith was the one who introduced the energy balance formula 2 of the system of BM under static conditions describing crack propagation as mentioned before. Mott was the one who started analysing dynamic crack propagation using the analytical approach. Considering propagation of centrally located through a crack in an infinite plate. Similarly, as Griffith, Mott developed a model, which defines crack propagation with the dynamic energy release as illustrated in formula 16. Where $F$ is the work done by external force and $U$ represents the internal work. To enable the equation function as it was intended the following assumptions were suggested: constant speed for the traveling crack in the body under uniform time dependent stress, which is applied perpendicular to the plane of the crack, crack speed $a \approx \sqrt{E}\frac{\sigma}{\rho}$ is comparably small than shear wave speed and the displacement and stress fields are similar as in the electrostatic problem [7]. Using assumptions before mentioned Mott modified Griffiths formula 2 to formula 16 by including kinetic terms $E_k$, to reach a configuration of keeping the total energy constant. $E_k$ for a cracked plate (assuming a unit thickness) was derived to be as depicted in formula 17 for a crack with repeat propagation and taking account of the stress field. Where $k$ is constant dependent on crack speed and $\sigma$ is stress perpendicular to the crack plane, and $\rho$ is the mass per unit area of the plate. Substituting kinetic energy formula 17 and formula 4 to modified Griffiths energy balance formula 16 then one can get Mott’s dynamic energy release rate formula 18 as illustrated below. Where $y$ is work of fracture or surface energy in ideally brittle material, but with $G(t)$ equal to constant $\gamma$ at crack initiating stage assumed to be constant as well. The equation gets form as illustrated below. Now the right side can be differentiated with respect to $a$ ignoring the kinetic terms and the one can have the following formula 20. The one can rearrange the equation making $a = a_o$ mane subject, which results in providing from fracture stress initial static crack length shown in formula 21 [4].

\[
G(t) = \frac{dF}{dA} - \frac{dU}{dA} - \frac{dE_k}{dA} 
\]

(16)

\[
E_k = \frac{1}{2} k^2 a^2 v^2 \rho \left( \frac{\sigma}{E} \right)^2 
\]

(17)

\[
G(t) = \frac{1}{2} \frac{d}{da} \left[ \frac{\pi \sigma^2 a^2}{E} \right] - \frac{1}{2} \frac{d}{da} \left[ \frac{k}{2} \rho a^2 v^2 \left( \frac{\sigma}{E} \right) \right] = 2\gamma 
\]

(18)

\[
2\gamma = \frac{1}{2} \frac{d}{da} \left[ \frac{\pi \sigma^2 a^2}{E} \right] - \frac{1}{2} \frac{d}{da} \left[ \frac{k}{2} \rho a^2 v^2 \left( \frac{\sigma}{E} \right) \right] 
\]

(19)

\[
2\gamma = \frac{\pi \sigma^2 a^2}{E} 
\]

(20)

\[
a_o = \frac{2E\gamma}{\pi \sigma^2} 
\]

(21)

2.8 Crack velocity
It is important in the dynamic investigation propagation of crack to be able to determine the resulting effect of the kinetic term to formula 21, which can be observed by the speed of the crack Equation. The equation of speed of the crack was first derived by Mott making wrong assumption that $\frac{dv}{dA} = 0$, but later Dulaney, Brace and Berry modified Mott’s analysis and derived Equation 22 shown below. Where $c_o = \sqrt{\frac{E}{\rho}}$, which is the speed of sound for 1D wave propagation through the material. Wells and Roberts obtained an approximation for $k$ and after few more assumptions, they illustrated that $\sqrt{\frac{2\pi}{k}} \approx 0.38$. When $a > a_o$ the crack speed reaches its value of limit and formula 22 becomes unstable, which is
0.38c_o by Wells and Roberts. Freund was the one who done more in-depth numerical analysis and come up with the following relationship formula 23. Where c_r is Rayleigh (surface) wave speed and it has been observed performing experimental analysis that speed of the crack does not generally reach c_r and (maximum is 0.6c_o) [4], which typically range from 0.2-0.4c_o. For the distinct event when γ = 0, a propagating crack is simply a disturbance on a free surface which have to move at the Raleigh wave velocity. In formula 22 and formula 23 the limiting velocity is independent of the fracture energy, hence the speed of the crack that can be reached should be equal to c_r for all γ [2]. A more detailed dynamic analysis of Freund assumed that crack length or crack speed does not rely on the fracture energy and the resistance of the material grows with the crack speed [8].

\[ v = \sqrt{\frac{2\pi}{k}} c_o \left(1 - \frac{a_o}{a}\right) \quad (22) \]

\[ v = c_r \left(1 - \frac{a_o}{a}\right) \quad (23) \]

3. Conclusion
To conclude, the following work was done to derive crack propagation velocity formula. Describing the methods used in the field of fracture mechanics some formulas were derived. Using the formulas and rearranging them in the right manner the crack propagation velocity formula was derived. The work illustrated that to derive the formula some conditions should be met, which are necessary to correctly utilize the method and derive the formula. It is obvious that to derive such a formula many researchers have contributed by performing various studies. With the right formulas and software packages run on powerful computing machines and yet it is not enough to analysis fracture. There is much more to be learned and improved in the field of mechanics.

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