Quantum search of matching on signed graphs

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Abstract
We construct a quantum searching model of a signed edge driven by a quantum walk. The time evolution operator of this quantum walk provides a weighted adjacency matrix induced by the assignment of a sign to each edge. This sign can be regarded as so-called the edge coloring. Then as an application, under an arbitrary edge coloring which gives a matching on a complete graph on \( n + 1 \) vertices we consider a quantum search of a colored edge from the edge set of the complete graph. We show that this quantum walk finds a colored edge within the time complexity of \( O(n^{2-\alpha}) \) with probability \( 1 - o(1) \), while the corresponding random walk on the line graph finds them within the time complexity of \( O(n^{2-\alpha}) \) if we set the number of the edges of the matching by \( t = O(n^{\alpha}) \) for \( 0 \leq \alpha \leq 1 \) red with \( t \leq \frac{n}{2} \).

Keywords Quantum search · Edge signed graphs · Matching

Mathematics Subject Classification 05C50 · 05C81 · 81P68 · 81Q99

1 Introduction

A quantum walk is introduced as a quantum analogue of a classical random walk [14] and the fundamental idea can be seen in [12]. As Y. Aharonov [3] et al. formulated quantum random walks as an antecedent of the current quantum walk and Meyer [20] introduced it as a quantum cell automaton, quantum walks have been in the limelight for the last two decades. One of the reason why quantum walks have been

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intensively studied for a long time is that the quantum walks often exhibit a specific characteristic which cannot be seen in classical random walks in both of infinite and finite graphs. As is seen in [6] and [21], by a combinatorial and the Schrödinger’s approach, the asymptotic probability distribution of a quantum walk in an infinite graph was analyzed, which shows a definitely different behavior from that of a classical random walk [17].

In finite graphs, D. Aharonov [4] et al. formulated quantum mixing, filling and dispersion time and showed that these are quadratically faster than classical ones. Constructing an efficient system to find targets in a graph has been one of the main research topic of quantum walks. In other words, quantum walks enable us to propose an algorithm finding marked elements in a graph efficiently. For example, the original Grover’s search algorithm [13] can be regarded as a quantum search on the complete graph on \( N \) vertices with self-loops driven by a quantum walk and it finds marked items within the time complexity of \( O(\sqrt{N}) \) while that of a classical searching algorithm is \( O(N) \). In that case, a quantum walk gives quadratic speedup to the searching algorithm. Besides this, quantum searching is considered in some classes of graphs, e.g., a finite \( d \)-dimensional grid [8], hypercubes [24], triangular lattices [2], simplicial complexes [19] and complete graphs [23]. Element distinctness problem proposed by Ambainis [7] is regarded as a quantum search on a kind of Johnson graph. See [22] and references therein. A principal idea to construct a quantum search algorithm is driving a quantum walk with a perturbation. In other words, we observe marked elements with sufficiently high probability by proper times applications of a perturbed quantum walk. Details are seen in, e.g., [1,18] and [25].

In this paper, we give a perturbation to the edge set of a complete graph by a sign. This sign is a map from the set of edges to \( \{\pm 1\} \) which corresponds to an edge coloring. We call such a graph a signed graph. Harary [15] introduced signed graphs as a model for a social network. Brown [11] et al. proposed perfect state transfer, which is a specific property of a continuous-time quantum walk in a signed graph and studied how negatively signed edges effect to the perfect state transfer. Here, we call the negatively signed edges the marked edges and regard them as targets of our searching model. Through the research, we aim to see how the existence of negatively signed edges affects to our searching model. The idea of this paper is constructing a perturbed quantum operator \( U_\sigma \) and finding marked edges in a signed complete graph as fast as possible by driving \( U_\sigma \). To this end, the spectral analysis of \( U_\sigma \) is useful. It is known that the spectrum, i.e., the set of the eigenvalues, of the time evolution operator is expressed in terms of that of the operator on vertices called the discriminant. We set our model so that the discriminant becomes a weight adjacency matrix on a signed graph studied by Akbari [5]. In particular, if the marked edges show matching which is a set of disjoint edges, the explicit expression of the spectrum of the weighted adjacency matrix is obtained by Akbari et al. Then we treat the case where the set of marked edges becomes a matching. We call a matching having \( t \) edges a \( t \)-matching. If the matching covers all the vertices, it is called a perfect matching. More precisely, we search a signed \( t \)-matching on the complete graph on \( n + 1 \) vertices. Note that the number of edges of the perfect matching is \( \left\lfloor \frac{n+1}{2} \right\rfloor \). Then we set the number of edges in the match-
ing by $t = O(n^\alpha)$ for $0 \leq \alpha \leq 1$. In addition, we will show that the probability observing marked edges becomes sufficiently high after driving the perturbed quantum walk within the time complexity of $O(n^{2-\alpha})$ while a classical one requires the time complexity of $O(n^{2-\alpha})$. Thus, the quantum algorithm proposed in this paper gives quadratic speedup. The following statement is our main result in this paper.

**Theorem 1.1** For a sufficiently large $n$, quantum search driven by $U_\sigma$ in a signed complete graph on $n + 1$ vertices enables us to find a marked $t$-matching with the time complexity of $O(n^{2-\alpha})$, where $t = O(n^\alpha)$ for $0 \leq \alpha \leq 1$ with $t \leq n^2$.

This paper is organized as follows. In Sect. 2, we define a sign on graphs and give definition of marked edges. In addition, we construct a time evolution operator of the perturbed quantum walk by the sign. In Sect. 3, we estimate searching time finding marked edges by a classical searching algorithm by spectral analysis. In Sect. 4, we consider quantum searching on a signed complete graph and prove our main result; that is, we show that the searching time on the graph given by our quantum walk is quadratically faster than that of a classical searching algorithm. In Sect. 5, we show some numerical simulations which visually show the time complexity of our model. Section 6 is devoted to summarizing our results and making discussion for our future direction.

2 Preliminaries

2.1 Graph and sign

Let $G = (V, E)$ be a connected and simple graph. For $e = uv \in E$, the vertices $u$ and $v$ are called the endpoints of $e$. If two edges $e$ and $f$ with $e \neq f$ share a vertex as their endpoints, we write $e \sim f$. In addition, if $u \in V$ is an endpoint of $e \in E$, we denote by $u \in e$. The oriented edge from $u \in V$ to $v \in V$ is called an arc and denoted by $(u, v)$. Define $\mathcal{A} = \{(u, v), (v, u) \mid uv \in E\}$ which is the set of the symmetric arcs of $G$. For $a \in \mathcal{A}$, $t(a)$ and $o(a)$ denote the terminus and origin of $a$, respectively. In addition, $a^{-1}$ denotes the inverse arc of $a$. Moreover, we write the adjacency matrix of a graph $G$ as $A(G)$. Furthermore, for $r \in \mathbb{N}$, the zero vector and the all-one vector in $\mathbb{C}^r$ are denoted by $\mathbf{0}_r$ and $\mathbf{j}_r$, respectively. Throughout the paper, $\text{Spec}(X)$ denotes the set of eigenvalues of a matrix $X$.

Before defining a sign, let us introduce some classes of graphs. For $r \in \mathbb{N}$, the **complete graph** on $r$ vertices, denoted by $K_r$, is a graph in which every pair of distinct two vertices are adjacent. For $r \in \mathbb{N}$, the **cocktail party graph**, denoted by $\text{CP}(r)$, is the graph in which the set of the vertices is decomposed into $r$ distinct subsets $V_1, V_2, \ldots, V_r$ with $|V_i| = 2$ for $1 \leq i \leq r$ and every pair of two vertices are connected unless these are belong to the same partite set. In other words, the adjacency matrix of $K_r$ and $\text{CP}(r)$ is given by
Fig. 1 Complete graph $K_4$

$$A(K_r) = \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 & 0 \end{pmatrix},$$

and

$$A(CP(r)) = \begin{pmatrix} 0 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 0 & 0 \ldots & 11 \\ 1 & 1 & 0 & 0 \ldots & 11 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 & 0 \\ 1 & 1 & 1 & 1 & \ldots & 0 & 0 \end{pmatrix} = A(K_r) \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

respectively (See Figs. 1, 2).

We call the set of disconnected edges a matching. In other words, a matching is the set of edges in which every pair of two edges $(e, f)$ satisfies $e \sim f$. If the number of edges in a matching is $t$, it is called a $t$-matching. See Fig. 3 as an example of a matching (the set of thick edges is a 2-matching).

We give $\sigma : \mathcal{A} \rightarrow \{\pm 1\}$ as a map satisfying that $\sigma(a^{-1}) = 1$ whenever $\sigma(a) = -1$ for $a \in \mathcal{A}$. Define $\mathcal{M} = \{a \in \mathcal{A} \mid \sigma(a) = -1\}$ and $\mathcal{M}^{-1} = \{a \in \mathcal{A} \mid a^{-1} \in \mathcal{M}\}$. 
From the above $\sigma$, we define a sign $\tau : E \to \{\pm 1\}$ by

$$
\tau(uv) = \begin{cases} 
1, & \sigma((u, v)) \cdot \sigma((v, u)) = 1, \\
-1, & \sigma((u, v)) \cdot \sigma((v, u)) = -1.
\end{cases}
$$

In other words, $\tau(uv) = \sigma((u, v)) \cdot \sigma((v, u))$. If $e \in E$ satisfies $\tau(e) = -1$, we call it a marked edge. Define $M = \{e \in E \mid \tau(e) = -1\}$ which is the set of marked edges and $\partial M = \{t(a), o(a) \in V \mid a \in M\}$ which is the set of the endpoints of the marked edges.

For example, let $G = K_5$ with $V(G) = \{v_1, v_2, \ldots, v_5\}$. Set $\{(v_1, v_2), (v_3, v_4)\}$ to be $M$, so we define $\sigma : A(G) \to \{\pm 1\}$ as

$$
\sigma(a) = \begin{cases} 
-1, & a \in \{(v_1, v_2), (v_3, v_4)\}, \\
1, & \text{otherwise}.
\end{cases}
$$

Then it holds that $M = \{v_1v_2, v_3v_4\}$ and $\partial M = \{v_1, v_2, v_3, v_4\}$ (See Fig. 4). In addition, it enables us to define $\tau : E(G) \to \{\pm 1\}$ as

$$
\tau(e) = \begin{cases} 
-1, & e \in \{v_1v_2, v_3v_4\}, \\
1, & \text{otherwise}.
\end{cases}
$$

### 2.2 Time evolution operator

In this part, we construct a time evolution operator of the perturbed quantum walk from the above $\sigma$. First, let us define an operator $S$ on $\mathbb{C}^{|A|}$ by

$$
S_{a, b} = \begin{cases} 
1, & a = b^{-1}, \\
0, & \text{otherwise}.
\end{cases}
$$
Fig. 4 An example of our model: The dashed arcs are negatively signed by $\sigma$

Note that $S^2 = I_A$. It is so-called the flip-flop shift operator. In addition, we define a boundary operator $d : \mathbb{C}^{|A|} \to \mathbb{C}^{|V|}$ by

$$d_{v,a} = \begin{cases} \frac{1}{\sqrt{\deg(t(a))}}, & t(a) = v, \\ 0, & \text{otherwise}. \end{cases}$$

Then it immediately follows that

$$d_{a,v}^* = \begin{cases} \frac{1}{\sqrt{\deg(t(a))}}, & t(a) = v, \\ 0, & \text{otherwise}. \end{cases}$$

Put $C = 2d^*d - I_A$. Remark that

$$C_{a,b} = \begin{cases} \frac{2}{\sqrt{\deg o(a)\deg t(b)}} - \delta_{a,b}, & t(a) = t(b), \\ 0, & \text{otherwise}, \end{cases}$$

where $\delta_{a,b}$ is the Kronecker delta symbol. The quantum walk whose time evolution operator is given by

$$U = SC$$

is the Grover walk. Then it holds that

$$U_{a,b} = \begin{cases} \frac{2}{\sqrt{\deg o(a)\deg t(b)}} - \delta_{a^{-1},b}, & t(b) = o(a), \\ 0, & \text{otherwise}. \end{cases}$$
The above value is regarded as the transmitting or reflecting rate from an arc \( b \) to an
arc \( a \) with \( t(b) = o(a) \).

In this paper, we give a modified boundary operator \( d_\sigma : \mathbb{C}^{|A|} \to \mathbb{C}^{|V|} \) by

\[
(d_\sigma)_{v,a} = \begin{cases} 
\frac{\sigma(a)}{\sqrt{\deg t(a)}}, & t(a) = v, \\
0, & \text{otherwise.}
\end{cases}
\]

Then it similarly follows that

\[
(d_\sigma^*)_{a,v} = \begin{cases} 
\frac{\sigma(a)}{\sqrt{\deg t(a)}}, & t(a) = v, \\
0, & \text{otherwise.}
\end{cases}
\]

Then it is easily checked that \( d_\sigma d_\sigma^* = I_V \) and \( d_\sigma^* d_\sigma \) is a projection operator. Put \( C_\sigma = 2d_\sigma^* d_\sigma - I_A \), that is,

\[
(C_\sigma)_{a,b} = \begin{cases} 
\frac{2\sigma(a)\sigma(b)}{\sqrt{\deg t(a)}\sqrt{\deg t(b)}} - \delta_{a,b}, & t(a) = t(b), \\
0, & \text{otherwise.}
\end{cases}
\]

We define the time evolution operator of our quantum walk model by

\[
U_\sigma := SC_\sigma.
\]

Then it is easily checked that

\[
(U_\sigma)_{a,b} = \begin{cases} 
\frac{2\sigma(a^{-1})\sigma(b)}{\sqrt{\deg t(b)}\deg o(a)} - \delta_{a^{-1},b}, & t(b) = o(a), \\
0, & \text{otherwise.}
\end{cases}
\]

In other words, the transmitting or the reflecting rate is modified by \( \sigma \). Indeed, this
operator is expressed by an oracle operator. Let us define an operator \( O_\sigma \) on \( \mathbb{C}^{|A|} \) by

\[
(O_\sigma)_{a,b} = \begin{cases} 
\sigma(a), & a = b, \\
0, & \text{otherwise.}
\end{cases}
\]

This is so-called the oracle. Then it is easily checked that

\[
U_\sigma = SO_\sigma CO_\sigma.
\]

Remark that \( U_\sigma \) is a unitary operator on \( \mathbb{C}^{|A|} \) because of \( S^2 = I_A \) and \( d_\sigma d_\sigma^* = I_V \). For an initial state \( u \in \mathbb{C}^{|A|} \), the time evolution of the quantum walk is given by \( \varphi_k = U_\sigma^k u \) and the finding probability on an edge \( uv \) at time \( k \) is given by

\[
|\varphi_k((u, v))|^2 + |\varphi_k((v, u))|^2.
\]
Furthermore, we define \( T_\sigma = d_\sigma Sd_\sigma^T \). It is checked that \( T_\sigma \) is an operator on \( \mathbb{C}^{[V]} \) whose entry is

\[
(T_\sigma)_{u,v} = \begin{cases} \frac{\tau(uv)}{\deg u \deg v}, & u \sim v, \\ 0, & \text{otherwise}. \end{cases}
\]

(1)

For example, we consider the graph as in Fig. 4. Then \( \mathcal{M} = \{(v_1, v_2), (v_3, v_4)\} \). Label the arcs and vertices by \((v_1, v_2), \ldots, (v_1, v_5), (v_2, v_1), \ldots, (v_2, v_5), (v_3, v_1), \ldots, (v_3, v_5), (v_4, v_1), \ldots, (v_4, v_5), (v_5, v_1), \ldots, (v_5, v_4)\) and \(v_1, v_2, \ldots, v_5\). In this case, the operator \( d_\sigma \) is given by

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
\end{pmatrix}.
\]

In addition, the operator \( U_\sigma \) is given by

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
\end{pmatrix}.
\]
Moreover, $T_\sigma$ is given by

$$
\begin{pmatrix}
0 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
-\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0
\end{pmatrix}.
$$

Here, it is known that a part of the spectrum of $U_\sigma$ is expressed in terms of that of $T_\sigma$.

**Theorem 2.1** (Higuchi–Konno–Sato–Segawa [16]) Let $U_\sigma$ and $T_\sigma$ be defined as in the above. Then it holds that

$$
\{ e^{\pm i\theta_\lambda} | \lambda \in \text{Spec}(T_\sigma) \backslash \{\pm 1\} \} \subset \text{Spec}(U_\sigma),
$$

where $\theta_\lambda = \cos^{-1} \lambda$. In addition, the normalized eigenvector of $U_\sigma$ associated with $e^{\pm i\theta_\lambda}$ is given by

$$
\varphi_{\pm \lambda} = \frac{1}{\sqrt{2|\sin \theta_\lambda|}} (d^\tau f - e^{\pm i\theta_\lambda} Sd^\tau f),
$$

where $f$ is the normalized eigenvector of $T_\sigma$ associated with $\lambda$. That is,

$$
\varphi_{\pm \lambda}(a) = \frac{1}{\sqrt{2|\sin \theta_\lambda|}} \left( \frac{\sigma(a)}{\sqrt{\deg t(a)}} f(t(a)) - e^{\pm i\theta_\lambda} \frac{\sigma(a^{-1})}{\sqrt{\deg o(a)}} f(o(a)) \right).
$$

The above vector plays an important role. In this paper, we treat the case where $G$ is the complete graph on $n+1$ vertices and $M$ becomes a $t$-matching with $t = O(n^\alpha)$ for $0 \leq \alpha \leq 1$. We set $c > 0$ to be the coefficient of $n^\alpha$ of $t$, that is, $t = [cn^\alpha]$. Note that $c$ is small enough to satisfy $cn^\alpha \leq \frac{n}{2}$.

**3 Classical search algorithm**

In this part, we estimate a classical searching time finding an edge in $M$ on a signed complete graph $G = K_{n+1}$. To this end, we introduce a simple random walk on edges of $G$ by the following operator $P$ on $\mathbb{C}^{|E|}$:

$$
P_{e,f} = \begin{cases} 
\frac{1}{2(n-1)}, & e \sim f, \\
0, & \text{otherwise}.
\end{cases}
$$

In other words, it holds that

$$
P = \frac{1}{2(n-1)} A(L(G)),$$
where $L(G)$ is so-called the line graph of $G$ whose definition is given by

$$V(L(G)) = E(G),$$
$$E(L(G)) = \{ef \mid e \sim f \text{ and } e \neq f \text{ in } G\}.$$  

Let us denote by $P_M$ the $(|E| - t) \times (|E| - t)$-matrix obtained by removing all the rows and columns corresponding to the edges in $M$ from $P$, that is,

$$P_M = \frac{1}{2(n - 1)} A(L(G_M)),$$

where $G_M$ is the graph obtained by removing all the edges of $M$. Let $\mu_m$ be the maximum eigenvalue of $P_M$. According to Lemma 5 of [25], the classical searching time finding edges in $M$, which is the expectation of the first hitting time to the matching is $O\left(\frac{1}{1 - \mu_m}\right)$ if $P_M$ is a symmetric matrix and the associated eigenvector for $\mu_m$ is sufficiently close to the all-one vector. Clearly, $P_M$ is a symmetric matrix since so is $A(L(G_M))$. Thus, we will evaluate the maximum eigenvalue of $P_M$ and its eigenvector to estimate the classical searching time in the following.

Define a boundary operator $B_\sigma : \mathbb{C}^{(|E| - M)} \rightarrow \mathbb{C}^{(|V|)}$ by

$$(B_\sigma)_{u,e} = \begin{cases} 1, & u \in e, \\ 0, & \text{otherwise}. \end{cases}$$

**Lemma 3.1** Let $P_M$ and $B_\sigma$ be defined as in the above. Then we have

$$P_M = \frac{1}{2(n - 1)} (B_\sigma^\top B_\sigma - 2I_{|E|-t}).$$

**Proof** Here, $B_\sigma^\top B_\sigma$ is a matrix indexed by $E \setminus M$ whose entry is

$$(B_\sigma^\top B_\sigma)_{e,f} = \sum_{u \in V} (B_\sigma)_{u,e} \cdot (B_\sigma)_{u,f}$$

$$= \sum_{\substack{u \in V \\text{and } e,u \in f}} 1$$

for $e, f \in E \setminus M$. The right-hand side of the above equation is nothing but the number of the vertices which is an endpoint of both of $e$ and $f$. Thus, we have

$$(B_\sigma^\top B_\sigma)_{e,f} = \begin{cases} 2, & e = f, \\ 1, & e \neq f, \ e \sim f, \\ 0, & \text{otherwise}. \end{cases}$$
Then it holds that \((B_\sigma^\top B_\sigma) - 2I_{|E|-t} = A(L(G_M)))\), which completes the proof. \(\square\)

In order to estimate the maximum eigenvalue of \(P_M\), it is useful to obtain that of \(B_\sigma^\top B_\sigma\) by Lemma 3.1. As spectra of \(B_\sigma^\top B_\sigma\) and \(B_\sigma B_\sigma^\top\) are in coincidence except for 0, we analyze the spectrum of the latter one instead of the former one.

**Lemma 3.2** Let \(B_\sigma\) be defined as in the above. Then it holds that

\[
(B_\sigma B_\sigma^\top)_{u,v} = \begin{cases} 
  n, & u = v, \ u \notin \partial M, \\
  n - 1, & u = v, \ u \in \partial M, \\
  1, & u \neq v, \ uv \notin M, \\
  0, & u \neq v, \ uv \in M.
\end{cases}
\]

**Proof** It holds that

\[
(B_\sigma B_\sigma^\top)_{u,v} = \sum_{e \in \setminus M} (B_\sigma)_{u,e} \cdot (B_\sigma)_{v,e} 
= \sum_{e \in E \setminus M} 1 
\]

for \(u, v \in V\). The right-hand side is the number of edges in \(E \setminus M\) whose endpoints are \(u\) and \(v\). We first consider the case of \(u = v\). If \(u = v \notin \partial M\), then the number of edges in \(E \setminus M\) whose endpoint is \(u\) is \(n\) since \(u\) is adjacent to all the vertices in \(G\). If \(u = v \in \partial M\), there is only one edge in \(M\) whose endpoint is \(u\) since \(M\) is a matching. Thus, (4) is \(n - 1\) in this case.

We next consider the case of \(u \neq v\). If \(uv \notin M\), there is only one edge in \(E \setminus M\) connecting \(u\) and \(v\). Thus, (4) is 1 in this case. If \(uv \in M\), there is no edge in \(E \setminus M\) whose endpoints are \(u\) and \(v\), which implies that (4) is 0 in this case. Therefore, we conclude that

\[
(B_\sigma B_\sigma^\top)_{u,v} = \begin{cases} 
  n, & u = v, \ u \notin \partial M, \\
  n - 1, & u = v, \ u \in \partial M, \\
  1, & u \neq v, \ uv \notin M, \\
  0, & u \neq v, \ uv \in M.
\end{cases}
\]

\(\square\)

Thus, by taking a proper labeling of vertices, we express \(B_\sigma B_\sigma^\top\) as

\[
B_\sigma B_\sigma^\top = \begin{pmatrix} 
  A(CP(t)) + (n - 1)I_{2t} & J_{2t,n+1-2t} \\
  J_{n+1-2t,2t} & A(K_{n+1-2t}) + nI_{n+1-2t}
\end{pmatrix},
\]

where \(J_{r,s}\) is the \((r \times s)\)-all-one matrix.
Lemma 3.3 Let \( B_\sigma \) be defined as in the above. Then it holds that

\[
\text{Spec}(B_\sigma B_\sigma^\top) = \{n - 3\}^{t-1} \cup \{n - 1\}^{n-t} \cup \{\mu_\pm\},
\]

where

\[
\mu_\pm = \frac{3n - 3 \pm \sqrt{n^2 + 6n + 9 - 16t}}{2}.
\]

Proof Put \( \hat{B} = B_\sigma B_\sigma^\top \). As is seen in (5), \( \hat{B} \) is expressed in terms of the adjacency matrices of \( \text{CP}(t) \) and \( K_{n+1-2t} \). The spectrum of a complete graph is known to be

\[
\text{Spec}(A(K_r)) = \{r - 1\}^1 \cup \{-1\}^{r-1},
\]

for example, see [10]. Then we have

\[
\text{Spec}(A(K_{n+1-2t})) = \{n - 2t\}^1 \cup \{-1\}^{n-2t}.
\]

In addition, it is also known that the eigenvector associated with \( (n - 2t) \) is \( \mathbf{j}_{n+1-2t} \). We next analyze the spectrum of \( A(\text{CP}(t)) \). Since \( A(\text{CP}(t)) = A(K_t) \otimes J_{2,2} \), an eigenvalue of \( A(\text{CP}(t)) \) is expressed by the product of those of \( A(K_t) \) and \( J_{2,2} \). Thus, it follows from \( \text{Spec}(J_{2,2}) = \{0\}^1 \cup \{2\}^1 \) and (6) that

\[
\text{Spec}(A(\text{CP}(t))) = \{-2\}^{t-1} \cup \{0\}^t \cup \{2(t - 1)\}^1.
\]

Similarly, the eigenvector associated with \( 2(t - 1) \) is \( \mathbf{j}_{2t} \). Let \( \Psi_1 = (\mathbf{j}_{2t}, \mathbf{0}_{n+1-2t})^\top \in \mathbb{C}^{|V|} \) and \( \Psi_2 = (\mathbf{0}_{2t}, \mathbf{j}_{n+1-2t})^\top \). Using the above facts, we have

\[
\hat{B}\Psi_1 = \begin{pmatrix} A(\text{CP}(t)) + (n - 1)I_{2t} & J_{2t,n+1-2t} \\ J_{n+1-2t,2t} & A(K_{n+1-2t}) + nI_{n+1-2t} \end{pmatrix} \begin{pmatrix} \mathbf{j}_{2t} \\ \mathbf{0}_{n+1-2t} \end{pmatrix}
\]

\[
= \begin{pmatrix} A(\text{CP}(t)) + (n - 1)I_{2t} \\ (J_{n+1-2t,2t})^\top \end{pmatrix} \mathbf{j}_{2t}
\]

\[
= \left( \begin{array}{c} (2(t - 1) + n - 1)\mathbf{j}_{2t} \\ 2t\mathbf{j}_{n+1-2t} \end{array} \right)
\]

\[
= (n + 2t - 3)\Psi_1 + 2t\Psi_2
\]

and

\[
\hat{B}\Psi_2 = \begin{pmatrix} A(\text{CP}(t)) + (n - 1)I_{2t} & J_{2t,n+1-2t} \\ J_{n+1-2t,2t} & A(K_{n+1-2t}) + nI_{n+1-2t} \end{pmatrix} \begin{pmatrix} \mathbf{0}_{2t} \\ \mathbf{j}_{n+1-2t} \end{pmatrix}
\]

\[
= \begin{pmatrix} (J_{2t,n+1-2t})^\top \mathbf{j}_{n+1-2t} \\ (A(K_{n+1-2t}) + nI_{n+1-2t})^\top \mathbf{j}_{n+1-2t} \end{pmatrix}
\]

\[
= \left( \begin{array}{c} (n + 1 - 2t)\mathbf{j}_{2t} \\ (n - 2t + n)\mathbf{j}_{n+1-2t} \end{array} \right).
\]
Thus, $\hat{B}$ is reduced to the following $2 \times 2$-matrix on $\text{Span}\{\Psi_1, \Psi_2\}$:

$$Q = \begin{pmatrix} n + 2t - 3 & n + 1 - 2t \\ 2t & 2n - 2t \end{pmatrix}. \quad (7)$$

Here, it holds that $\text{Spec}(Q) \subset \text{Spec}(\hat{B})$. By direct computation, the eigenvalues of $Q$ are

$$\mu_{\pm} = \frac{3n - 3 \pm \sqrt{n^2 + 6n + 9 - 16t}}{2}.$$

Now, we analyze the remaining eigenvalues of $\hat{B}$. Let $g$ be an eigenvector of $A(K_{n+1-2t})$ associated with $-1$. Then $g$ is orthogonal to $j_{n+1-2t}$, see [10]. Put $\hat{g} = (0_{2t}, g)^T \in \mathbb{C}^{|\mathcal{V}|}$. Since $(J_{2t,n+1-2t})g = 0_{2t}$ and $A(K_{n+1-2t})g = -g$, it is easily checked that $\hat{g}$ is an eigenvector of $\hat{B}$ associated with $-1$ by similar computation. Taking $g$ as an eigenvector of $A(K_{n+1-2t})$ associated with $-1$, we have thus found $n - 2t$ linearly independent eigenvectors of $\hat{B}$ associated with $-1$. We next consider the remaining eigenvalues given by those of $A(\text{CP}(t))$. Since $A(\text{CP}(t))$ is a symmetric matrix, eigenvectors associated with $-2$ and $0$ are orthogonal to the one associated with the maximum eigenvalue $2(t - 1)$, that is, $j_{2t}$. Let $f$ be an eigenvector of $A(\text{CP}(t))$ associated with $\eta \in \{-2, 0\}$ and $\hat{f} = (f, 0_{n+1-2t})^T$. Similarly, it is checked that $\hat{f}$ is an eigenvector of $\hat{B}$ associated with the eigenvalue $\eta + 1$ for $\eta \in \{-2, 0\}$. Recall that the multiplicity of the eigenvalues $-2$ and $0$ of $A(\text{CP}(t))$ are $t - 1$ and $t$, respectively. Then the multiplicities of eigenvalues $n - 3$ and $n - 1$ of $\hat{B}$ are $t - 1$ and $t$, respectively. Therefore, eigenvalues of $\hat{B}$ that we have found are $\mu_{\pm}, n - 1$ with multiplicity $n - 2t + t = n - t$, and $n - 3$ with multiplicity $t - 1$. Since $2 + (n - t) + (t - 1) = n + 1 = |\mathcal{V}|$, these are all the eigenvalues of $\hat{B}$ and we conclude that

$$\text{Spec}(\hat{B}) = \{n - 3\}^{t-1} \cup \{n - 1\}^{n-t} \cup \left\{\frac{3n - 3 \pm \sqrt{n^2 + 6n + 9 - 16t}}{2}\right\}.$$

Clearly, $n - 1 < \frac{3n - 3 + \sqrt{n^2 + 6n + 9 - 16t}}{2} = \mu_+$. Thus, $\mu_+$ is the maximum eigenvalue of $\hat{B}$. \hfill \Box

Note that if $M$ achieves a perfect matching, that is, $t = \frac{n+1}{2}$ for an odd $n$, we have $V \setminus \partial M = \phi$ and

$$B_\sigma B_\sigma^\top = A(\text{CP}(t)) + (n - 1)I_{2t}.$$ 

Then the maximum eigenvalue of $B_\sigma B_\sigma^\top$ is $2(t - 1) + (n - 1) = 2(n - 1)$ which coincides with the above $\mu_+$ with $t = \frac{n+1}{2}$. Hence, we employ $\mu_+$ in the following Theorem for every case.
Theorem 3.4 Let $\alpha$, $M$ and $P_M$ be defined as in the above. Then the classical searching time finding an edge in $M$ is of the leading order of $n^{2-\alpha}$.

Proof By Lemma 3.3 and (3), the maximum eigenvalue of $P_M$ is

$$\mu_m = \frac{1}{2(n-1)}(\mu + 2)$$

$$= \frac{3n - 7 + \sqrt{n^2 + 6n + 9 - 16t}}{4(n-1)}$$

Since $t = [cn^\alpha] = O(n^\alpha)$ with $0 \leq \alpha \leq 1$, we have

$$\sqrt{n^2 + 6n + 9 - 16t} = n + 3 - 8cn^{\alpha-1} + O(n^{\alpha-1}),$$

and

$$\mu_m = \frac{3n - 7 + n + 3 - 8cn^{\alpha-1} + O(n^{\alpha-1})}{4(n-1)}$$

$$= 1 - 2cn^{\alpha-2} + O(n^{\alpha-2}).$$

By computation, the eigenvector of $P_M$ associated with the above $\mu_m$ is obtained as

$$(X_n, \ldots, X_n, Y_n, \ldots, Y_n, 1, \ldots, 1)$$

where

$$X_n = \frac{-n - 3 + \sqrt{n^2 + 6n + 9 - 16t} + 8t}{8t},
Y_n = \frac{-n - 3 + \sqrt{n^2 + 6n + 9 - 16t} + 4t}{4t},
\alpha_n = 2t(n - 2t + 1),
\beta_n = 2t(t - 1)$$

and $\gamma_n = \binom{n + 1}{2} - t - 2t(n - 2t + 1) - 2t(t - 1)$. It is easily checked that the above vector is close to the all-one vector as $n$ tends to $\infty$. Therefore, the leading order of $\frac{1}{1-\mu_m}$ is $n^{2-\alpha}$, which completes the proof.

$\square$

4 Quantum search algorithm

4.1 Spectrum of $T_\sigma$

As $G$ is an $n$-regular graph, it follows from (1) that

$$(T_\sigma)_{u,v} = \begin{cases} 
-\frac{1}{n}, & \tau(uv) = -1, \\
\frac{1}{n}, & \tau(uv) = 1, \\
0, & \text{otherwise},
\end{cases}$$

(8)
which is a weighted adjacency matrix of a signed complete graph \([5]\). According to \([5]\), the spectrum of the adjacency matrix \(A\) of a signed complete graph \(K_{n+1}\) in which the set of negatively signed edges is a \(t\)-matching is obtained as follows:

(i) If \(t < \left\lfloor \frac{n+1}{2} \right\rfloor\),

\[
\text{Spec}(A) = \{-3\}^{t-1} \cup \{a_t\}^1 \cup \{-1\}^{n-2t} \cup \{1\}^t \cup \{b_t\}^1,
\]

where \(a_t = \frac{n-3-\sqrt{(n+1)^2+4s}}{2}\), \(b_t = \frac{n-3+\sqrt{(n+1)^2+4s}}{2}\) and \(s = n - 4t + 2\).

(ii) If \(t = \frac{n+1}{2}\) for an odd \(n\), that is, \(M\) achieves a perfect matching,

\[
\text{Spec}(A) = \{-3\}^{t-1} \cup \{1\}^t \cup \{b_t\}^1.
\]

Note that \(b_t = n - 2\) in this case.

Hence, it follows from the fact of \(T = \frac{1}{n}A\) that if \(t < \left\lfloor \frac{n+1}{2} \right\rfloor\),

\[
\text{Spec}(T_\sigma) = \left\{-3 \right\}^{t-1} \cup \left\{a_t \right\}^1 \cup \left\{-1 \right\}^{n-2t} \cup \left\{1 \right\}^t \cup \left\{b_t \right\}^1.
\]

Moreover, the normalized eigenvector of \(T_\sigma\) associated with the maximum eigenvalue \(\lambda_m = \frac{b_t}{n}\) is

\[
f_M = \frac{1}{\sqrt{cn}} \left(\rho_n, \ldots, \rho_n, 1, \ldots, 1\right)^\top,
\]

where \(\rho_n = \frac{-(s+1)+\sqrt{\Delta_n}}{4t}\), \(\Delta_n = (n + 1)^2 + 4s\) and \(c_n = 2t \rho_n^2 + n + 1 - 2t\). If \(M\) achieves a perfect matching, it holds

\[
\text{Spec}(T_\sigma) = \left\{-3 \right\}^{t-1} \cup \left\{1 \right\}^t \cup \left\{b_t \right\}^1
\]

and the normalized eigenvector of \(T_\sigma\) associated with \(\lambda_m = \frac{b_t}{n}\) is \(\frac{1}{\sqrt{n+1}} \mathbf{e}_{n+1}\). Remark that \(\rho_n = 1\), \(\sqrt{\Delta_n} = (n - 1)^2\) and \(c_n = n + 1\) in this case.

### 4.2 Proof of theorem 1.1

First, we consider the case where \(M\) does not achieve a perfect matching. Let us estimate the values \(\sqrt{\Delta_n} = \sqrt{(n + 1)^2 + 4s}\), \(\rho_n = \frac{-(s+1)+\sqrt{\Delta_n}}{4t}\) and \(c_n = 2t \rho_n^2 + n + 1 - 2t\). Indeed, \(\sqrt{\Delta_n}\) is expanded as

\[
\sqrt{\Delta_n} = n + 3 - 8cn^{\alpha-1} + O(n^{\alpha-2}).
\]

Hence, we have

\[
\rho_n = \frac{-(s+1)+\sqrt{\Delta_n}}{4t}
\]
\[
= -(n - 4t + 3) + (n + 3 - 8cn^{2} + O(n^{2})) \\
= 1 + O\left(\frac{1}{n}\right)
\]

and
\[
c_n = 2t\rho_n^2 + n + 1 - 2t \\
= n + 1 + 2t(\rho_n^2 - 1). \\
= n + O(1).
\]

Then, for a sufficiently large \(n\), the values \(\rho_n\) and \(c_n\) are approximated as \(1\) and \(n\), respectively.

Next, let us analyze the eigenvector of \(U_\sigma\) associated with the eigenvalue induced by \(\lambda_m\). To this end, we decompose \(A\) into six distinct subsets \(A_1, A_2, A_3, A_4, A_5, A_6\) given by

\[
A_1 = \mathcal{M}, \\
A_2 = \mathcal{M}^{-1}, \\
A_3 = \{a \in A \mid a \notin \mathcal{M} \cup \mathcal{M}^{-1}, \ t(a), o(a) \in \partial \mathcal{M}\}, \\
A_4 = \{a \in A \mid t(a) \notin \partial \mathcal{M}, o(a) \in \partial \mathcal{M}\}, \\
A_5 = \{a \in A \mid t(a) \notin \partial \mathcal{M}, o(a) \notin \partial \mathcal{M}\}, \\
A_6 = \{a \in A \mid t(a) \notin \partial \mathcal{M}, o(a) \notin \partial \mathcal{M}\}.
\]

Put \(\theta_m = \cos^{-1} \lambda_m\). Inserting (12) into (2), we obtain the normalized eigenvector \(\varphi_{\pm \theta_m}\) of \(U_\sigma\) associated with \(e^{\pm i\theta_m}\) as follows:

\[
\varphi_{\pm \theta_m}(a) = \frac{1}{\sqrt{2nc_n \sin \theta_m}} \times \begin{cases} 
-\rho_n(1 + e^{\pm i\theta_m}), & a \in A_1, \\
\rho_n(1 + e^{\pm i\theta_m}), & a \in A_2, \\
\rho_n(1 - e^{\pm i\theta_m}), & a \in A_3, \\
\rho_n - e^{\pm i\theta_m}, & a \in A_4, \\
1 - \rho_n e^{\pm i\theta_m}, & a \in A_5, \\
1 - e^{\pm i\theta_m}, & a \in A_6.
\end{cases}
\]

Here, we count the number of arcs in each set. Clearly, both of the numbers of arcs in \(A_1\) and \(A_2\) are \(t\). For a pair of two distinct arcs \((u, v), (w, x)\) in \(\mathcal{M}\), the arcs \((u, w), (u, x), (v, w), (v, x)\) and their inverse arcs satisfy the condition as in \(A_3\). Thus, the number of arcs in \(A_3\) is \(8 \times (\begin{pmatrix} 1 \end{pmatrix}) = 4t(t - 1)\). For an arc \(a\) in \(\mathcal{M}\), the number of arcs \(b(\neq a)\) satisfying \(t(b) = t(a)\) and \(o(b) \notin \partial \mathcal{M}\) coincides with the number of vertices in \(V \setminus \partial \mathcal{M}\), that is, \(n + 1 - 2t\). Similarly, the number of arcs \(b(\neq a)\) satisfying \(t(b) = o(a)\) and \(o(b) \notin \partial \mathcal{M}\) is \(n + 1 - 2t\). Thus, the number of arcs in \(A_4\) is \((n + 1 - 2t) \times 2 \times t = 2t(n + 1 - 2t)\), which is same as \(|A_5|\). Finally, the number of
the other arcs is \[ |A| - \sum_{i=1}^{5} |A_i| = n(n + 1) - 2t - 4t(t - 1) - 4t(n + 1 - 2t) = n^2 + n + 4t^2 - 4nt - 2t. \] We make a list of the numbers of arcs satisfying the above conditions as follows:

| \(A_1\) | \(A_2\) | \(A_3\) | \(A_4\) | \(A_5\) | \(A_6\) |
|---|---|---|---|---|---|
| \(t\) | \(t\) | \(4t(t - 1)\) | \(2t(n + 1 - 2t)\) | \(2t(n + 1 - 2t)\) | \(n^2 + n + 4t^2 - 4nt - 2t\) |

Let \(u \in \mathbb{C}^{|A|}\) be the uniform state, that is,

\[
u(a) = \frac{1}{\sqrt{n(n + 1)}}, \quad a \in A(G).
\]

Now, we employ \(u\) as the initial state. In this paper, we estimate the total of the time complexity of our quantum searching model based on [22]. In other words, we obtain the leading order of the time at which the finding probability on the marked edges is sufficiently high. We give the outline of the proof as follows:

1. We define two vectors \(\beta_+\) and \(\beta_-\) and see that \(\beta_-\) is sufficiently close to \(u\).
2. We estimate the leading order of the time complexity \(k_f\) such that \(U_\sigma^{k_f} \beta_-\) is close to \(\beta_+\).
3. We estimate the leading order of the finding probability \(FP_n\) on the marked edges of \(\beta_+\).

It is known that the total of the time complexity is obtained as the product of the leading orders of \(k_f\) and \(\sqrt{FP_n}\) by using the amplitude amplification [9]. Multiplying the leading orders of \(k_f\) and \(FP_n\), we get the leading order of the total of the time complexity. In order to complete the proof, we get their leading orders.

Define

\[ \beta_\pm := \frac{1}{\sqrt{2}} (\varphi_+ \theta_m \pm \varphi_- \theta_m). \]

Then we have

\[
\beta_+(a) = \frac{1}{\sqrt{n c_n \sin \theta_m}} \times \begin{cases} -\rho_n(1 + \cos \theta_m), & a \in A_1, \\ \rho_n(1 + \cos \theta_m), & a \in A_2, \\ \rho_n(1 - \cos \theta_m), & a \in A_3, \\ \rho_n - \cos \theta_m, & a \in A_4, \\ 1 - \rho_n \cos \theta_m, & a \in A_5, \\ 1 - \cos \theta_m, & a \in A_6, \end{cases}
\]
and
\[ \beta_-(a) = \frac{1}{\sqrt{nc_n}} \times \begin{cases} -i \rho_n, & a \in A_1 \cup A_3 \cup A_5, \\ i \rho_n, & a \in A_2, \\ -i, & a \in A_4 \cup A_6, \end{cases} \]
by (14). Let us analyze the overlap between \( \beta_- \) and \( u \). Now, it holds that
\[ |\langle u, \beta_- \rangle| = \frac{1}{n\sqrt{c_n(n+1)}} \times \left\{ \right. \]
\[ \left. \begin{array}{c}
(-i \rho_n) + |A_2| \times (i \rho_n) + (|A_4| + |A_6|) \times (-i) \\
\end{array} \right) \\
\]
\[ = \frac{1}{n\sqrt{c_n(n+1)}} \times \left\{ (2nt - t) \times (-i \rho_n) + t \times (i \rho_n) + n(n - 2t + 1) \times (-i) \right\} \\
\]
\[ = \frac{1}{n\sqrt{c_n(n+1)}} \times \left\{ (2t - 2nt) \rho_n - (n^2 - 2nt + n) \right\}. \]

As \( \rho_n \approx 1 \) and \( c_n \approx n \), we have
\[ |\langle u, \beta_- \rangle| = 1 - O \left( \frac{1}{n} \right). \]

Thus, \( \beta_- \) is so close to \( u \) and we regard \( \beta_- \) as the initial state. Let
\[ \psi_k = U_0^k \beta_- = \frac{1}{\sqrt{2}} \left( e^{i \theta_m k} \varphi_{+\theta_m} - e^{-i \theta_m k} \varphi_{-\theta_m} \right) \]
and \( k_f = \left\lfloor \frac{n}{2\theta_m} \right\rfloor \). Then \( \psi_{k_f} \) is regarded as \( i \beta_+ \) since \( e^{\pm i \theta_m k_f} \) is close to \( \pm i \). Now, we compute the time complexity converting \( \beta_- \) to its orthogonal vector \( i \beta_+ \), that is, the leading order of \( \frac{1}{\theta_m} \). To this end, let us estimate \( \theta_m \) for a sufficiently large \( n \). Since \( \lambda_m = \cos \theta_m \) tends to 1 as \( n \to \infty \), \( \theta_m \) is approximated as \( \sin \theta_m \). Hence, we have
\[ \theta_m \approx \sin \theta_m = \sqrt{1 - \cos^2 \theta_m} \]
\[ = \sqrt{1 - \left( \frac{n - 3 + \Delta n}{2n} \right)^2} \]
\[ = \sqrt{1 - \left( \frac{2n - 8cn^{\alpha-1} + O(n^{\alpha-2})}{2n} \right)^2} \]
\[ = \sqrt{8cn^{\frac{\alpha-2}{2}}} + o \left( n^{\frac{\alpha-2}{2}} \right). \]

Then the leading order of \( k_f \) is \( n^{\frac{2-\alpha}{2}} \).

At this time, the finding probabilities on arcs \( a \in M \) and \( a^{-1} \in M^{-1} \) are close to
\[ |\beta_+(a)|^2 = |\beta_+(a^{-1})|^2 = \left( \frac{\rho_n}{\sqrt{nc_n \sin \theta_m}} (1 + \cos \theta_m) \right)^2 \]
(15)
\[ \odot \] Springer
by (14). As \( c_n \approx n \), \( \rho_n \approx 1 \), \( \cos \theta_m \approx 1 \) and \( \sin \theta_m \approx \sqrt{8cn^{-\alpha^2}} \), (15) is of the leading order of \( n^{-\alpha} \). Hence, the finding probability on the edges in \( M \) is

\[
FP_n = \sum_{a \in M} \left( |\beta_+(a)|^2 + |\beta_+(a^{-1})|^2 \right)
\]

\[
= 2t \cdot \left| \frac{\rho_n}{\sqrt{ncn \sin \theta_m}} (1 + \cos \theta_m) \right|^2
\]

\[
= 2cn^\alpha \cdot \left| \frac{1}{\sqrt{n \cdot n \sqrt{\frac{\alpha - 2}{2}}} (1 + 1)} \right|^2 - o(1)
\]

\[
= 1 - o(1).
\]

By the amplitude amplification [9,22], the total of the time complexity \( k_{total} \) is given by

\[
k_{total} = k_f \times \sqrt{\frac{1}{FP_n}}.
\]

Recall that the leading orders of \( k_f \) and \( FP_n \) are \( n^{\frac{2-\alpha}{2}} \) and 1, respectively. Therefore, the leading order of \( k_{total} \) becomes \( n^{\frac{2-\alpha}{2}} \).

In the case where \( M \) achieves a perfect matching, it is similarly shown that the total of the time complexity is \( O(n^{\frac{1}{2}}) \) by repeating the same argument with \( \rho_n = 1 \), \( c_n = n + 1 \) and \( \sqrt{\Delta_n} = (n - 1)^2 \). Therefore, we complete the proof.

5 Numerical simulation

In this section, we compare the probability distribution of our model and its asymptotic distribution by a numerical simulation. As is seen in the previous section, the finding probability on the marked edges at time \( k \) is obtained by

\[
F(k) = \sum_{a \in M} \left( |(U^k_\sigma u)(a)|^2 + |(U^k_\sigma u)(a^{-1})|^2 \right).
\]

(16)

On the other hand, the asymptotic probability is obtained by

\[
Asy(k) = \sum_{a \in M} \left( |(U^k_\sigma \beta_+)(a)|^2 + |(U^k_\sigma \beta_+)(a^{-1})|^2 \right).
\]

Then it holds that

\[
U^k_\sigma \beta_- = U^k \left( \frac{1}{\sqrt{2}} (\varphi_{+\theta_m} - \varphi_{-\theta_m}) \right)
\]
\[
\frac{1}{\sqrt{2}} \left( e^{ik\theta_m} \varphi_{+\theta_m} - e^{-ik\theta_m} \varphi_{-\theta_m} \right).
\]

It follows from (14) that

\[
\varphi_{\pm\theta_m}(a) = -\frac{1}{\sqrt{2n c_n \sin \theta_m}} \left( \rho_n (1 + e^{\pm i\theta_m}) \right)
\]

for \(a \in \mathcal{M}\). Then we have

\[
(U^k_{\sigma \beta} -)(a) = -\frac{\rho_n}{2\sqrt{n c_n \sin \theta_m}} \left( e^{ik\theta_m} (1 + e^{i\theta_m}) - e^{-ik\theta_m} (1 + e^{-i\theta_m}) \right)
\]

\[
= -\frac{i \rho_n}{\sqrt{n c_n \sin \theta_m}} (\sin (k+1)\theta_m + \sin \theta_m)
\]

\[
= -\frac{2i \rho_n}{\sqrt{n c_n \sin \theta_m}} \frac{(2k+1)\theta_m}{2} \cos \frac{\theta_m}{2}
\]

for \(a \in \mathcal{M}\). Similarly, we have

\[
(U^k_{\sigma \beta} -)(a^{-1}) = \frac{2i \rho_n}{\sqrt{n c_n \sin \theta_m}} \frac{(2k+1)\theta_m}{2} \cos \frac{\theta_m}{2}
\]

for \(a \in \mathcal{M}\). Hence, it holds that

\[
|U^k_{\sigma \beta} -(a)|^2 = |U^k_{\sigma \beta} -(a^{-1})|^2 = \frac{4\rho_n^2}{n c_n \sin^2 \theta_m} \frac{\sin^2 (2k+1)\theta_m}{2} \cos^2 \frac{\theta_m}{2}
\]

\[
= \frac{\rho_n^2}{n c_n \sin^2 \theta_m} (1 - \cos (2k+1)\theta_m)(1 + \cos \theta_m)
\]

and

\[
\text{Asy}(k) = \sum_{a \in \mathcal{M}} \left( |(U^k_{\sigma \beta} -(a)|^2 + |(U^k_{\sigma \beta} -(a^{-1})|^2 \right)
\]

\[
= 2t |U^k_{\sigma \beta} -(a)|^2
\]

\[
= \frac{2t \rho_n^2}{n c_n \sin^2 \theta_m} (1 - \cos (2k+1)\theta_m)(1 + \cos \theta_m).
\]

Here, we plot (16) and (17) in the following two cases: (i) \(n = 9, t = 3\), (ii) \(n = 19, t = 4\). The results are illustrated in Figs. 5, 6. (The horizontal and vertical axes in the figures imply the time and the finding probability on the marked edges, respectively.) Let \(V(K_{n+1}) = \{v_1, v_2, \ldots, v_{n+1}\}\). We set the marked arcs as \(\{(v_1, v_2), (v_3, v_4), (v_5, v_6)\}\) in the case of (i). (Similarly, the marked arcs in the case of (ii) are \(\{(v_1, v_2), (v_3, v_4), (v_5, v_6), (v_7, v_8)\}\). As is seen in the previous section, the time at which the marked edges are found with high probability is \(k_f = \lfloor \frac{\pi}{2\theta_m} \rfloor \approx \frac{\pi}{2\sqrt{8}} n^{\frac{2-a}{2}}\). For example, in the case of \(n = 19\) and \(t = 4\), we have
\[
\alpha = \log_{19} 4 \approx 0.470818 \quad \text{and} \quad k_f \approx \frac{\pi}{2\sqrt{8}} n^{\frac{2-n}{2}} \approx 5.33891,
\]

which is close to the first time achieving the peak of the probability distribution as in Fig. 6. In that case, the finding probability on the marked edges at time 0 is

\[
2t \times \left\{ \sqrt{\frac{1}{n(n+1)}} \right\}^2 \approx 0.0210526.
\]

Furthermore, we have \(\rho_n \approx 0.905869\), \(c_n \approx 18.5648\), \(\cos \theta_m \approx 0.960366\) and \(\sin \theta \approx 0.278743\) by inserting \(n = 19\) and \(t = 4\) to the values stated in subsection 4.1. Then
the value as in (15) becomes
\[
|\beta_+(a)|^2 = |\beta_+^{-1}(a)|^2 = \left| \frac{\rho_n}{\sqrt{nc_n} \sin \theta_m} (1 + \cos \theta_m) \right|^2 
\]
\[\approx 0.115068\]
for a marked arc \(a\). Thus, the finding probability at \(k_f \approx 5.55324\) is
\[
FP_n = 2t \times |\beta_+(a)|^2 \approx 0.920544,
\]
which agrees the result as in Fig. 6.

6 Summary and discussion

In this paper, we introduced a signed graph whose edges are negatively signed. In particular, we proposed a quantum searching algorithm in a signed complete graph \(K_{n+1}\) and it enabled us to find negatively signed edges, say, marked edges, quadratically faster than a classical algorithm, where the number of the marked edges is \(t = O(n^\alpha)\) for \(0 \leq \alpha \leq 1\) with \(t \leq \frac{n}{2}\). In addition, the set of marked ones becomes a \(t\)-matching which is a combination of disjoint \(t\) edges. In other words, we found a quantum search algorithm detecting the marked edges within the time complexity of \(O(n^{\frac{2}{2-a}})\) while an algorithm based on a classical random walk requires the time complexity of \(O(n^{2-a})\). The algorithm proposed in this paper only reveals quadratic speedup to find one marked edge from a matching. Hence, considering an algorithm to know how marked edges are located is one of our future problems. An extension of our proposed algorithm to the other graphs is also still open.

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