COMPLEX CONNECTIONS WITH TRIVIAL HOLOMONY

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Abstract. Given an almost complex manifold \((M, J)\), we study complex connections with trivial holonomy and such that the corresponding torsion is either of type \((2,0)\) or of type \((1,1)\) with respect to \(J\). Such connections arise naturally when considering Lie groups, and quotients by discrete subgroups, equipped with bi-invariant and abelian complex structures.

1. Introduction

Let \(M\) be an \(n\)-dimensional connected manifold together with an affine connection with trivial holonomy, hence flat. This amounts to having an absolute parallelism on \(M\), which in turn is equivalent to a smooth trivialization of the frame bundle \(B\) (see [29, Proposition 2.2]). Given a pseudo-Riemannian metric \(g\) on \(M\), J. Wolf studied in [29,30] the problem of existence of metric connections with trivial holonomy and having the same geodesics as the Levi-Civita connection. Equivalently, he considered the class of pseudo-Riemannian manifolds \((M, g)\) which carry connections \(\nabla\) such that \(\nabla g = 0\), \(\text{Hol}(\nabla) = 1\) and whose torsion is totally skew-symmetric (see [2] for a different approach using geometries with torsion in the Riemannian case). When the connection is required to be complete with parallel torsion, the resulting manifolds are of the form \(\Gamma \backslash G\) with \(G\) a simply connected Lie group carrying a bi-invariant pseudo-Riemannian metric and \(\Gamma\) a discrete subgroup of \(G\). Moreover, \(\nabla\) is induced by the affine connection corresponding to the parallelism of left translation on \(G\) and the pseudo-Riemannian metric \(g\) is induced from a bi-invariant metric on \(G\) [29, Theorem 3.8]. He also provided a complete classification of all complete pseudo-Riemannian manifolds admitting such connections in the reductive case [30, Theorem 8.16].

It is our aim to investigate an analogue of the previous problem in the case of almost complex manifolds instead of pseudo-Riemannian manifolds. More precisely, given an almost complex manifold \((M, J)\), we will be interested in studying complex connections \(\nabla\) on \(M\) with trivial holonomy and such that the corresponding torsion \(T\) is either of type \((2,0)\) or of type \((1,1)\) with respect to \(J\).

Our first observation (see Proposition 2.2) is that when \(\nabla J = 0\) and the torsion \(T\) of \(\nabla\) is of type \((2,0)\) or \((1,1)\) then \(J\) is necessarily integrable, that is, \((M, J)\) is a complex manifold. We prove a general result for affine complex manifolds with trivial holonomy (Theorem 4.1, analogous to [28 Theorem 1] (see also [18 Theorem 3.6]))

We show that on a complex parallelizable manifold the existence of \(n\) holomorphic vector fields \(Z_1,\ldots,Z_n\) which are linearly independent at every point of \(M\) is equivalent to the existence of a complex connection \(\nabla\) on \(M\) with trivial holonomy whose torsion tensor field

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is of type (2,0) (Proposition 3.4). In particular the Chern connection of any metric compatible with $J$ and having constant coefficients on the given trivialization has trivial holonomy, hence it is flat.

On the other hand, the existence of $n$ commuting vector fields $Z_1, \ldots, Z_n$ which are linearly independent sections of $T^{1,0}M$ at every point of $M$ is equivalent to the existence of a complex connection $\nabla$ on $M$ with trivial holonomy whose torsion tensor field $T$ is of type $(1,1)$ (Proposition 3.8). Such a connection will be called an abelian connection. The class of abelian connections is the $(1,1)$-counterpart of the class of Chern-type connections on complex parallelizable manifolds (Proposition 3.4 and Definition 3.6).

Our motivation for studying abelian connections arises from the fact that quotients by a discrete subgroup of Lie groups carrying abelian complex structures, which have been studied by several authors (see, for instance [3, 6, 8, 9, 12, 21, 27]), are natural examples of such manifolds. One of our results (Corollary 4.4), which is analogous to [29, Theorem 3.8], asserts that when $\nabla$ is complete with parallel torsion then $M = \Gamma \backslash G$ with $G$ a simply connected Lie group, $\Gamma$ a discrete subgroup of $G$, $\nabla$ is induced by the connection corresponding to the parallelism of left translation on $G$ and the almost complex structure $J$ comes from a left invariant abelian complex structure on $G$.

We point out that the class of complex connections we consider in this work is not the same as the class of complex-flat connections introduced by D. Joyce in [17]. According to [17], an affine connection $\nabla$ on a complex manifold $(M, J)$ is called complex-flat when $\nabla J = 0$, $\nabla$ is torsion-free and the curvature tensor of $\nabla$ satisfies certain condition which is always fulfilled by the curvature tensor of a Kähler metric. The tangent bundle $TM$ of $M$ is naturally a complex manifold, with a complex structure $J$ induced by $J$. It was shown in [17, Theorem 6.2] that given a complex-flat connection on $(M, J)$ it is possible to endow $TM$ with a complex structure $K$ commuting with $J$. In this case, both $J$ and $K$ induce complex structures on the cotangent bundle $T^*M$ and it turns out that the natural symplectic structure on $T^*M$ gives rise to a pseudo-Riemannian metric on $T^*M$ which is pseudo-Kähler with respect to $K$.

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2. Preliminaries

Let $\nabla$ be an affine connection on a manifold $M$ with torsion tensor field $T$, where $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, for all $X, Y$ vector fields on $M$.

Given an almost complex structure $J$ on $M$, we denote by $N$ the Nijenhuis tensor of $J$, defined by

$$N(X, Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y].$$

Recalling that

$$(\nabla_X J)Y = \nabla_X (JY) - J(\nabla_X Y),$$
we obtain the following identity:
\[
N(X, Y) = (\nabla_J X) J Y - (\nabla_J Y) X + (\nabla_X J) J Y - (\nabla_Y J) J X \\
+ T(X, Y) - T(JX, JY) + J(T(JX, Y) + T(X, JY)),
\]
(3)
for all \(X, Y\) vector fields on \(M\).

The almost complex structure \(J\) is called integrable when \(N \equiv 0\), and in this case \((M, J)\) is a complex manifold [24]. The tensor field \(J\) is called parallel with respect to \(\nabla\) when \(\nabla J = 0\), that is, \(\nabla_X J Y = 0\) for all \(X, Y\) vector fields on \(M\) (see [2]). Also in this case one says that \(\nabla\) is a complex connection (see [19, p. 143]).

The next lemma follows from equation (3).

**Lemma 2.1.** Let \((M, J)\) be an almost complex manifold with a complex connection \(\nabla\). Then \(J\) is integrable if and only if the torsion \(T\) of \(\nabla\) satisfies:
\[
T(X, Y) - T(JX, JY) + J(T(JX, Y) + T(X, JY)) = 0,
\]
for all vector fields \(X, Y\) on \(M\).

The torsion \(T\) of a connection \(\nabla\) on the almost complex manifold \((M, J)\) is said to be:
- of type \((1, 1)\) if \(T(JX, JY) = T(X, Y)\),
- of type \((2, 0)\) if \(T(JX, Y) = JT(X, Y)\),
- of type \((2, 0) + (0, 2)\) if \(T(JX, JY) = -T(X, Y)\),
for all vector fields \(X, Y\) on \(M\).

**Proposition 2.2.** Let \((M, J)\) be an almost complex manifold.

(i) If \(\nabla\) is a complex connection on \(M\) whose torsion is of type \((1, 1)\) with respect to \(J\), then \(J\) is integrable.

(ii) If \(\nabla\) is a complex connection on \(M\) whose torsion is of type \((2, 0)\) with respect to \(J\), then \(J\) is integrable.

(iii) If \(\nabla\) is a complex connection on \(M\) whose torsion is of type \((2, 0) + (0, 2)\) and \(J\) is integrable, then \(T\) is of type \((2, 0)\).

(iv) If \(J\) is integrable, then there exists a complex connection \(\nabla\) whose torsion is of type \((1, 1)\) with respect to \(J\).

(v) If \(J\) is integrable, then there exists a complex connection \(\nabla\) whose torsion is of type \((2, 0)\) with respect to \(J\).

**Proof.** (i), (ii) and (iii) are a straightforward consequence of Lemma 2.1.

To prove (iv) and (v), we introduce a Hermitian metric \(g\) on \(M\), that is, \(g\) is a Riemannian metric on \(M\) satisfying \(g(JX, JY) = g(X, Y)\) for all vector fields \(X, Y\) on \(M\). If \(\nabla^g\) is the Levi-Civita connection of \(g\), then we consider the connections \(\nabla^1\) and \(\nabla^2\) defined by
\[
g(\nabla^1_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{4} (d\omega(X, JY, Z) + d\omega(X, Y, JZ)),
\]
(4)
\[
g(\nabla^2_X Y, Z) = g(\nabla^g_X Y, Z) - \frac{1}{2} d\omega(JX, Y, Z),
\]
where \(\omega(X, Y) := g(JX, Y)\) is the Kähler form corresponding to \(g\) and \(J\). These connections satisfy
\[\nabla^1 g = 0, \quad \nabla^1 J = 0, \quad T^1 \text{ is of type } (1, 1),\]
\[ \nabla^2 g = 0, \quad \nabla^2 J = 0, \quad T^2 \text{ is of type } (2,0), \quad (5) \]

(see [20, 13]), thus proving the claim. \(\square\)

Remark 1. The connections \(\nabla^1\) and \(\nabla^2\) appearing in the proof of Proposition 2.2 are known, respectively, as the first and second canonical connection associated to the Hermitian manifold \((M, J, g)\). The connection \(\nabla^2\) is also known as the Chern connection, and it is the unique connection on \((M, J, g)\) satisfying (5). In the almost Hermitian case, the Chern connection is the unique complex metric connection whose torsion is of type \((2,0) + (0,2)\), equivalently, the \((1,1)\)-component of the torsion vanishes.

Remark 2. If \(\nabla\) is a torsion-free affine connection on \(M\), define

\[ \nabla_X Y := \nabla_X Y + \frac{1}{2} (\nabla_X J) JY = \frac{1}{2} (\nabla_X Y - J\nabla_X JY), \]

for \(X, Y\) vector fields on \(M\). It is easy to see that \(\nabla J = 0\) and using (3), we obtain that \(T(X, Y) = T(JX, JY)\), i.e. \(T\) is of type \((1,1)\) with respect to \(J\).

It is proved in [1, p. 21] that when \(\nabla\) is the Levi-Civita connection of a Hermitian metric on \(M\), then the connection thus obtained is the first canonical connection \(\nabla^1\) defined in (4).

### 3. Complex connections with trivial holonomy

Let \(M\) be an \(n\)-dimensional connected manifold and \(\nabla\) an affine connection on \(M\) with trivial holonomy. Then the space \(P^\nabla\) of parallel vector fields on \(M\) is an \(n\)-dimensional real vector space (see for instance [29, Proposition 2.2]). If \(T\) denotes the torsion tensor field corresponding to \(\nabla\), then

\[ T(X, Y) = -[X, Y], \quad \text{for all } X, Y \in P^\nabla. \quad (6) \]

We point out that, in general, the space \(P^\nabla\) of parallel vector fields is not closed under the Lie bracket. More precisely, there is the following well-known result (see for instance [29, p. 323]):

**Lemma 3.1.** The space \(P^\nabla\) of parallel vector fields is a Lie subalgebra of \(\mathfrak{X}(M)\) if and only if the torsion \(T\) of \(\nabla\) is parallel.

In the next result we give equivalent conditions for an affine connection with trivial holonomy on an almost complex manifold to be complex.

**Lemma 3.2.** Let \(M, \dim M = 2n\), be a connected manifold with an almost complex structure \(J\). Assume that there exists an affine connection \(\nabla\) on \(M\) with trivial holonomy. Then the following conditions are equivalent:

(i) \(\nabla J = 0\);

(ii) the space \(P^\nabla\) of parallel vector fields is \(J\)-stable;

(iii) there exist parallel vector fields \(X_1, \ldots, X_n, JX_1, \ldots, JX_n\), linearly independent at every point of \(M\).
3.1. Complex parallelizable manifolds. We recall from [28] that a complex manifold $(M, J)$ is called complex parallelizable when there exist $n$ holomorphic vector fields $Z_1, \ldots, Z_n$, linearly independent at every point of $M$ [19, 23, 28]. The following classical result, due to Wang, characterizes the compact complex parallelizable manifolds.

**Theorem 3.3** ([28]). Every compact complex parallelizable manifold may be written as a quotient space $\Gamma \backslash G$ of a complex Lie group by a discrete subgroup $\Gamma$.

We prove next a result which relates the notion of complex parallelizability with the existence of a flat complex connection with torsion of type $(2,0)$.

**Proposition 3.4.** Let $M$ be a connected $2n$-dimensional manifold with a complex structure $J$. Then the following conditions are equivalent:

(i) there exist vector fields $X_1, \ldots, X_n, JX_1, \ldots, JX_n$, linearly independent at every point of $M$, such that

$$[X_k, X_l] = -[JX_k, JX_l], \quad k < l, \quad [JX_k, X_l] = J[X_k, X_l], \quad k \leq l,$$

(ii) there exist $n$ holomorphic vector fields $Z_1, \ldots, Z_n$ which are linearly independent at every point of $M$ (in other words, $(M, J)$ is complex parallelizable);

(iii) there exist $n$ linearly independent holomorphic $(1,0)$-forms $\alpha_1, \ldots, \alpha_n$ on $M$ such that $d\alpha_i$ is a section of $\Lambda^2 M$ for every $i$;

(iv) there exists a complex connection $\nabla$ on $M$ with trivial holonomy whose torsion tensor field $T$ is of type $(2,0)$.

**Proof.** We recall that a vector field $X$ on $M$ is an infinitesimal automorphism of $J$ if $[X, JY] = J[X, Y]$ for every vector field $Y$ on $M$.

We note first that (i) is equivalent to (i)', where

(i)' there exist vector fields $X_1, \ldots, X_n, JX_1, \ldots, JX_n$, linearly independent at every point of $M$, such that $X_j, JX_j$ is an infinitesimal automorphism of $J$ for each $j$.

The proof of this equivalence is straightforward. Moreover, it follows from [19, Proposition IX.2.11] that (i)' is equivalent to (ii).

Let $\alpha_1, \ldots, \alpha_n$ be the holomorphic 1-forms dual to the holomorphic vector fields $Z_1, \ldots, Z_n$. It is well known that these holomorphic 1-forms satisfy (iii), and the converse also holds.

If (iv) holds, then there exist parallel vector fields $X_1, \ldots, X_n, JX_1, \ldots, JX_n$, linearly independent at every point of $M$ (see Lemma 3.2). Using (6) and the fact that $T$ is of type $(2, 0)$, relations (7) hold and therefore (i) follows. Conversely, given vector fields $X_1, \ldots, X_n, JX_1, \ldots, JX_n$ as in (i), let $\nabla$ be the affine connection such that the space $P^\nabla$ of parallel vector fields is spanned by these vector fields. It follows that $\nabla$ has trivial holonomy and Lemma 3.2 implies that $\nabla J = 0$. Moreover, using equations (6) and (7), we have that $T$ is of type $(2, 0)$ with respect to $J$, and this proves (iv). \qed

**Corollary 3.5.** Let $(M, J)$ be a complex manifold. The following conditions are equivalent:

(i) $(M, J)$ is complex parallelizable;

(ii) there exists a Hermitian metric $g$ on $M$ such that the Chern connection associated to $(M, J, g)$ has trivial holonomy.
Proof. (ii) implies (i) follows from Proposition 3.4.

Assume now that (i) holds. Consider the linearly independent vector fields \{X_k, JX_k\} and the complex connection \(\nabla\) with trivial holonomy given in (i) and (iv) of Proposition 3.4, respectively. Let \(g\) be the Hermitian metric on \(M\) such that the basis above is orthonormal. Then it follows that \(\nabla g = 0\), hence by uniqueness, \(\nabla\) is the Chern connection associated to \((M, J, g)\). \(\square\)

Remark 3. In the compact case, a result similar to Corollary 3.5 was obtained in [11]. We notice that in [10, 11], a Hermitian metric \(g\) on \((M, J)\) whose associated Chern connection has trivial holonomy is called Chern-flat.

**Definition 3.6.** An affine connection \(\nabla\) on a connected complex manifold \((M, J)\) will be called a Chern-type connection if it satisfies condition (iv) of Proposition 3.4.

**Corollary 3.7.** Let \((M, J)\) be a connected complex manifold and \(\nabla\) an affine connection with trivial holonomy. Then \(\nabla\) is a Chern-type connection on \((M, J)\) if and only if the space \(P^\nabla\) of parallel vector fields is \(J\)-stable and \(J\) satisfies

\[
[\nabla J X, J Y] = -[\nabla X, Y] \quad \text{for any} \quad X, Y \in P^\nabla.
\]  

Proof. We just have to observe that (8) is equivalent to condition (i) of Proposition 3.4. \(\square\)

Remark 4. When \(J\) is an almost complex structure on \(M\), we have the following equivalences:

(i) there exist vector fields \(X_1, \ldots, X_n, JX_1, \ldots, JX_n\), linearly independent at every point of \(M\), such that

\[
[X_k, X_l] = -[JX_k, JX_l], \quad k < l, \quad [JX_k, X_l] = [X_k, JX_l], \quad k \leq l,
\]

(ii) there exists a complex connection \(\nabla\) on \(M\) with trivial holonomy whose torsion tensor field \(T\) is of type \((2,0) + (0,2)\).

Moreover, analogously to Corollary 3.7, we have that the torsion \(T\) of a complex connection \(\nabla\) with trivial holonomy is of type \((2,0) + (0,2)\) if and only if \(J\) satisfies

\[
[JX, JY] = -[X, Y] \quad \text{for any} \quad X, Y \in P^\nabla.
\]  

3.2. **Flat complex connections with \((1,1)\)-torsion.** Given an almost complex structure \(J\) on \(M\), we study complex connections \(\nabla\) on \(M\) with trivial holonomy such that the corresponding torsion \(T\) is of type \((1,1)\) with respect to \(J\). It follows from Corollary 2.2 that when the almost complex structure \(J\) admits such a connection, then \(J\) satisfies the integrability condition \(\bar{N} \equiv 0\).

The following proposition is the analogue of Proposition 3.4 in the case when the torsion of the flat complex connection is of type \((1,1)\).

**Proposition 3.8.** Let \(M\) be a connected \(2n\)-dimensional manifold with an almost complex structure \(J\). Then the following conditions are equivalent:

\[
\text{\textit{(i)}} \quad \text{there exist vector fields} \quad X_1, \ldots, X_n, JX_1, \ldots, JX_n, \quad \text{linearly independent at every point of} \quad M, \quad \text{such that}
\]

\[
[X_k, X_l] = -[JX_k, JX_l], \quad k < l, \quad [JX_k, X_l] = [X_k, JX_l], \quad k \leq l,
\]

\[
\text{\textit{(ii)}} \quad \text{there exists a complex connection} \quad \nabla \quad \text{on} \quad M, \quad \text{with trivial holonomy whose torsion tensor field} \quad T \quad \text{is of type} \quad (2,0) + (0,2).
\]

Moreover, analogously to Corollary 3.7, we have that the torsion \(T\) of a complex connection \(\nabla\) with trivial holonomy is of type \((1,1)\) if and only if \(J\) satisfies

\[
[JX, JY] = -[X, Y] \quad \text{for any} \quad X, Y \in P^\nabla.
\]
(i) there exist vector fields $X_1, \ldots, X_n, JX_1, \ldots, JX_n$, linearly independent at every point of $M$, such that

$$[X_k, X_l] = [JX_k, JX_l], \quad [JX_k, X_l] = -[X_k, JX_l], \quad k < l; \quad (10)$$

(ii) there exist $n$ commuting vector fields $Z_1, \ldots, Z_n$ which are linearly independent sections of $T^{1,0}M$ at every point of $M$;

(iii) there exist $n$ linearly independent $(1,0)$-forms $\alpha_1, \ldots, \alpha_n$ on $M$ such that $d\alpha_i$ is a section of $\Lambda^{1,1}M$ for every $i$;

(iv) there exists a complex connection $\nabla$ on $M$ with trivial holonomy whose torsion tensor field $T$ is of type $(1,1)$.

Moreover, any of the above conditions implies that $J$ is integrable.

Proof. Let $X$ and $Y$ be vector fields on $M$. A simple calculation shows that

$$[X - iJX, Y - iJY] = 0 \quad \text{if and only if} \quad [X, Y] = [JX, JY] \quad \text{and} \quad [JX, Y] = -[X, JY].$$

Thus, if (i) holds, $Z_l = X_l - iJX_l, l = 1, \ldots, n$, is a commuting family of $(1,0)$ vector fields, linearly independent at every point of $M$. Conversely, given $Z_1, \ldots, Z_n$ as in (ii), setting $X_l = Z_l + Z_l$, it turns out that $X_1, \ldots, X_n$ satisfy (i).

We note first that an almost complex structure satisfying (ii) or (iii) is integrable (see [19, Theorem IX.2.8]). Therefore, given a $(1,0)$ form $\alpha$, it follows that

$$d\alpha$$

is a section of $\Lambda^{1,1}M$ if and only if $d\alpha(Z, W) = 0 \ \forall Z, W \in T^{1,0}M$.

In case $T^{1,0}M$ has a basis $Z_1, \ldots, Z_n$ of commuting vector fields, let $\alpha_1, \ldots, \alpha_n$ be the dual basis of $(1,0)$ forms. We calculate

$$d\alpha_i(Z_j, Z_l) = Z_j(\alpha_i(Z_l)) - Z_l(\alpha_i(Z_j)) - \alpha_i([Z_j, Z_l]), \quad (11)$$

where $\alpha_i(Z_k)$ is constant on $M$ and the last summand is zero since $Z_k$ are commuting vector fields, yielding $d\alpha_i(Z_j, Z_l) = 0$. This clearly implies that $d\alpha_i(Z_j, Z_l) = 0$ for any $Z, W \in T^{0,1}M$, thus $d\alpha_i$ is a section of $\Lambda^{1,1}M$ for every $i$. Conversely, if (iii) holds, let $Z_1, \ldots, Z_n$ be the basis of $T^{1,0}M$ dual to $\alpha_1, \ldots, \alpha_n$. By assumption, $d\alpha_i(Z_j, Z_l) = 0$ for every $1 \leq i, j, l \leq n$, hence (11) implies that $\alpha_i([Z_j, Z_l]) = 0$ for any $i$, therefore, $[Z_j, Z_l] = 0$ and (ii) follows.

If (iv) holds, then there exist parallel vector fields $X_1, \ldots, X_n, JX_1, \ldots, JX_n$, linearly independent at every point of $M$ (see Lemma 3.2). Using (6) and the fact that $T$ is of type $(1,1)$, relations (10) hold and therefore (i) follows. Conversely, given vector fields $X_1, \ldots, X_n, JX_1, \ldots, JX_n$ as in (i), let $\nabla$ be the affine connection such that the space $P^\nabla$ of parallel vector fields is spanned by these vector fields. It follows that $\nabla$ has trivial holonomy and Lemma 3.2 implies that $\nabla J = 0$. Moreover, using equations (6) and (10), we have that $T$ is of type $(1,1)$ with respect to $J$, and this proves (iv). \qed

The following definition is motivated by Proposition 3.8 (ii).

Definition 3.9. An affine connection $\nabla$ on a connected almost complex manifold $(M, J)$ will be called an abelian connection if it satisfies condition (iv) of Proposition 3.8.

The next corollary is a straightforward consequence of Lemma 3.2 and Proposition 3.8.
Corollary 3.10. Let \((M, J)\) be a connected complex manifold and \(\nabla\) an affine connection with trivial holonomy. Then \(\nabla\) is an abelian connection on \((M, J)\) if and only if the space \(P^\nabla\) of parallel vector fields is \(J\)-stable and \(J\) satisfies
\[
[JX, JY] = [X, Y] \quad \text{for any} \quad X, Y \in P^\nabla.
\] (12)

Proof. We just have to observe that (12) is equivalent to condition (i) of Proposition 3.8. \(\square\)

4. Complete complex connections with parallel torsion and trivial holonomy

We begin this section by exhibiting a large class of complex manifolds equipped with complex connections with trivial holonomy whose torsion tensors are of type \((2, 0)\) or \((1, 1)\) (compare with (iv) in Propositions 3.4 and 3.8).

We begin by recalling known facts on invariant complex structures and affine connections on Lie groups.

A complex structure on a real Lie algebra \(\mathfrak{g}\) is an endomorphism \(J\) of \(\mathfrak{g}\) satisfying \(J^2 = -I\) and such that \(N(x, y) = 0\) for all \(x, y \in \mathfrak{g}\), where \(N\) is defined as in (11). It is well known that (11) holds if and only if \(\mathfrak{g}^{1,0}\), the \(i\)-eigenspace of \(J\), is a complex subalgebra of \(\mathfrak{g}^C := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}\).

When \(\mathfrak{g}^{1,0}\) is a complex ideal we say that \(J\) is bi-invariant and when \(\mathfrak{g}^{1,0}\) is abelian we say that \(J\) is abelian. In terms of the bracket on \(\mathfrak{g}\), these conditions can be expressed as follows:
\[
J \text{ is bi-invariant if and only if } J[x, y] = [x, Jy],
\] (13)
and
\[
J \text{ is abelian if and only if } [Jx, Jy] = [x, y],
\] (14)
for all \(x, y \in \mathfrak{g}\). We note that a complex structure on a Lie algebra cannot be both abelian and bi-invariant, unless the Lie bracket is trivial.

If \(G\) is a Lie group with Lie algebra \(\mathfrak{g}\), by left translating the endomorphism \(J\) we obtain a complex manifold \((G, J)\) such that left translations are holomorphic maps. A complex structure of this kind is called left invariant. If \(\Gamma \subset G\) is any discrete subgroup of \(G\) with projection \(\pi : G \to \Gamma \backslash G\) then the induced complex structure on \(\Gamma \backslash G\) makes \(\pi\) holomorphic. It will be denoted \(J_0\).

Let \(G\) be a Lie group with Lie algebra \(\mathfrak{g}\) and suppose that \(G\) admits a left invariant affine connection \(\nabla\), i.e., each left translation is an affine transformation of \(G\). In this case, if \(X, Y\) are two left invariant vector fields on \(G\) then \(\nabla_X Y\) is also left invariant. Moreover, there is a one-one correspondence between the set of left invariant connections on \(G\) and the set of \(\mathfrak{g}\)-valued bilinear forms \(\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\) (see [15, p.102]). It is known that the completeness of a left invariant affine connection \(\nabla\) on \(G\) can be studied by considering the corresponding connection on the Lie algebra \(\mathfrak{g}\). Indeed, the left invariant connection \(\nabla\) on \(G\) will be complete if and only if the differential equation on \(\mathfrak{g}\)
\[
\dot{x}(t) = -\nabla_{x(t)} x(t)
\]
admits solutions \(x(t) \in \mathfrak{g}\) defined for all \(t \in \mathbb{R}\) (see for instance [7] or [14]).
The left invariant affine connection $\nabla$ on $G$ defined by $\nabla_X Y = 0$ for all $X, Y$ left invariant vector fields on $G$ is known as the $(-)$-connection. This connection satisfies:

1. Its torsion $T$ is given by $T(X,Y) = -[X,Y]$ for all $X, Y$ left invariant vector fields on $G$;
2. $\nabla T = 0$ and $\mathcal{P}^\nabla = \mathfrak{g} \subset \mathfrak{X}(G)$;
3. The holonomy group of $\nabla$ is trivial, thus, $\nabla$ is flat;
4. The geodesics of $\nabla$ through the identity $e \in G$ are Lie group homomorphisms $\mathbb{R} \to G$, therefore, $\nabla$ is complete;
5. The parallel transport along any curve joining $g \in G$ with $h \in G$ is given by the derivative of the left translation $(dL_{h^{-1}})_g$.

If $\Gamma \subset G$ is any discrete subgroup of $G$ then the $(-)$-connection on $G$ induces a unique connection on $\Gamma \backslash G$ such that the parallel vector fields are $\pi$-related with the left invariant vector fields on $G$, where $\pi : G \to \Gamma \backslash G$ is the projection. This induced connection on $\Gamma \backslash G$ is complete, has trivial holonomy, its torsion is parallel and $\pi$ is affine. It will be denoted $\nabla^0$.

If $J$ is a left invariant complex structure on $G$, then $J$ is parallel with respect to the $(-)$-connection $\nabla$. Moreover, $J_0$ is parallel with respect to $\nabla^0$, therefore, $(\Gamma \backslash G, J_0)$ carries a complete complex connection with trivial holonomy and parallel torsion.

In the next result we prove that the converse also holds.

**Theorem 4.1.** The triple $(M, J, \nabla)$ where $M$ is a connected manifold endowed with a complex structure $J$ and a complex connection $\nabla$ with trivial holonomy is equivalent to a triple $(\Gamma \backslash G, J_0, \nabla^0)$ as above if and only if $\nabla$ is complete and its torsion is parallel.

**Proof.** The “only if” part follows from the previous paragraphs.

For the converse, let $\mathcal{P}^\nabla$ be the space of parallel vector fields on $M$. According to Lemmas 3.1 and 3.2, $\mathcal{P}^\nabla$ is a $J$-stable Lie algebra. Let $G$ be the simply connected Lie group with Lie algebra $\mathcal{P}^\nabla$ with the left invariant complex structure induced by $J$ on $\mathcal{P}^\nabla$. This complex structure on $G$ will also be denoted by $J$. If $\nabla^-$ denotes the $(-)$-connection on $G$, we note that $\nabla^-$ is induced by the connection $\nabla$ on $M$. Moreover, $\nabla^-$ is a complex connection on $(G, J)$.

Let $e \in G$ be the identity and fix $m \in M$. We identify the tangent space to $G$ at $e$ with $\mathcal{P}^\nabla$. Let $\phi : \mathcal{P}^\nabla \to T_m M$ be the complex linear isomorphism defined by $\phi(X) = X_m$. Since the left invariant vector fields on $G$ correspond to the parallel vector fields on $M$, the Cartan-Ambrose-Hicks Theorem (see [31 Theorem 1.9.1]) applies and there exists a unique affine covering $f : (G, \nabla^-) \to (M, \nabla)$ such that $(df)_e = \phi$. Moreover, since both $\nabla^-$ and $\nabla$ are complex connections and $f$ is affine, it follows that $f$ is holomorphic.

Setting $\Gamma = f^{-1}(m)$, it follows from [16 Lemmas 2 and 3] that $\Gamma$ acting on $G$ by left translations, coincides with the deck transformation group of the covering. Therefore, $f$ induces a holomorphic affine diffeomorphism between $(\Gamma \backslash G, J_0, \nabla^0)$ and $(M, J, \nabla)$. \(\square\)

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ equipped with a left invariant complex structure $J$ and the $(-)$-connection. Since the torsion $T$ is given by $T(X,Y) = -[X,Y]$ for all left invariant vector fields $X, Y$ on $G$, it follows that

$$T$$ is of type $(2,0)$ with respect to $J$ if and only if $J$ is bi-invariant on $\mathfrak{g}$, \hspace{1cm} (15)
and

\[ T \text{ is of type (1,1) with respect to } J \text{ if and only if } J \text{ is abelian on } g. \]  \hspace{1cm} (16)

These properties are shared also by the induced connection \( \nabla^0 \) on \( \Gamma \setminus G \) with respect to \( J_0 \).

**Corollary 4.2.** Let \((M, J)\) be a complex manifold with a Chern-type connection \( \nabla \). If the torsion tensor field \( T \) is parallel, then:

(i) the space \( P^\nabla \) of parallel vector fields on \( M \) is a complex Lie algebra and \( J \) is a bi-invariant complex structure on \( P^\nabla \);

(ii) if, furthermore, \( \nabla \) is complete, then \((M, J, \nabla)\) is equivalent to \((\Gamma \setminus G, J_0, \nabla^0)\), where \( G \) is a simply connected complex Lie group and \( \Gamma \subset G \) is a discrete subgroup.

**Proof.** The space \( P^\nabla \) is a \( J \)-stable Lie algebra since \( T \) is parallel and \( \nabla J = 0 \). The first assertion now follows from (15), noting that a bi-invariant complex structure on a real Lie algebra gives rise to a complex Lie algebra.

(ii) follows from (i) and Theorem 4.1, since \( G \) is the simply connected Lie group with Lie algebra \( P^\nabla \).

**Corollary 4.3.** Let \((M, J, g)\) a Hermitian manifold such that the associated Chern connection \( \nabla \) is complete, has trivial holonomy and parallel torsion. Then \((M, J, g)\) is equivalent to a triple \((\Gamma \setminus G, J_0, g_0)\), where \( G \) is a simply connected Lie group and \( g_0 \) is induced by a left invariant Hermitian metric on \( G \). Furthermore, the Chern connection on the quotient coincides with \( \nabla^0 \).

**Proof.** According to Corollary 4.2, we have that \((M, J, \nabla)\) is equivalent to \((\Gamma \setminus G, J_0, \nabla^0)\), with \( J_0 \) a complex structure induced by a bi-invariant complex structure on \( G \).

Following the argument in the proof of Theorem 4.1, replacing the complex structure by a Hermitian metric, we obtain a left invariant Hermitian metric on \( G \) such that the affine covering \( f : G \to M \) becomes a local isometry. Moreover, the Chern connection of this Hermitian structure on \( G \) is the \((-\)-connection \( \nabla^- \). The induced Hermitian structure on \( \Gamma \setminus G \) is equivalent to the given one on \( M \).

**Corollary 4.4.** Given an affine connection \( \nabla \) with torsion \( T \), we consider the tensor field \( T^{(2)} \) defined by

\[ T^{(2)}(X,Y,Z,W) = T(T(X,Y),T(Z,W)), \]

where \( X, Y, Z, W \) are vector fields on \( M \) (see [18, p. 389]).

**Corollary 4.4.** Let \( \nabla \) be an abelian connection on a connected complex manifold \((M, J)\) such that the torsion tensor field \( T \) is parallel. Then:

(i) the space \( P^\nabla \) of parallel vector fields on \( M \) is a Lie algebra and \( J \) is an abelian complex structure on \( P^\nabla \);

(ii) the Lie algebra \( P^\nabla \) is 2-step solvable, that is, \( T^{(2)} \equiv 0 \);

(iii) if, furthermore, \( \nabla \) is complete, then \((M, J, \nabla)\) is equivalent to \((\Gamma \setminus G, J_0, \nabla^0)\), where \( G \) is a simply connected 2-step solvable Lie group equipped with a left invariant abelian complex structure and \( \Gamma \subset G \) a discrete subgroup.
Proof. It follows from Lemmas 3.1 and 3.2 that $P^\nabla$ is a $J$-stable Lie algebra. Corollary 3.10 implies that $J$ is an abelian complex structure on $P^\nabla$. Therefore, $P^\nabla$ is 2-step solvable (see [25, 4]). Moreover, it is straightforward to verify that the 2-step solvability of $P^\nabla$ is equivalent to $T^{(2)} \equiv 0$. This proves (i) and (ii).

(iii) follows from (i), (ii) and Theorem 4.1, since $G$ is the simply connected Lie group with Lie algebra $P^\nabla$. □

Remark 5. We note that a complete classification of the Lie algebras admitting abelian complex structures is known up to dimension 6 (see [3]), and there are structure results for arbitrary dimensions ([6]).

4.1. Examples. We show next a compact complex manifold $M$ which is not complex parallelizable but admits abelian connections.

Example 1. Let $N$ be the Heisenberg Lie group, given by

$$N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$  

The subgroup $\Gamma$ of matrices in $N$ with integer entries is discrete and cocompact. The 4-dimensional compact manifold $M = (\Gamma \backslash N) \times S^1 = (\Gamma \times \mathbb{Z}) \backslash (N \times \mathbb{R})$ is known as the Kodaira-Thurston manifold. The Lie group $N \times \mathbb{R}$ admits a left invariant abelian complex structure (see for instance [5, 3]), and therefore $M$ inherits a complex structure $J$ admitting an abelian connection. On the other hand, $M$ is not complex parallelizable. Indeed, if it were, it would follow by Lemma 3.3 that $M$ is a quotient $\Lambda \backslash G$, where $G$ is a 2-dimensional complex Lie group and $\Lambda$ is a discrete cocompact subgroup of $G$. There are only two 2-dimensional simply connected complex Lie groups, namely $\mathbb{C}^2$ and $\tilde{\text{Aff}}(\mathbb{C})$, where $\tilde{\text{Aff}}(\mathbb{C})$ is the universal cover of the group

$$\text{Aff}(\mathbb{C}) = \left\{ \begin{pmatrix} z \\ w \\ 1 \end{pmatrix} : z \in \mathbb{C}^*, w \in \mathbb{C} \right\}. \quad (17)$$

The group $\tilde{\text{Aff}}(\mathbb{C})$ does not admit any discrete cocompact subgroup, since it is not unimodular ([22]). Thus, $G = \mathbb{C}^2$ and $M$ is biholomorphic to a complex torus. Hence, $M$ would admit a Kähler structure, which is impossible since the Kodaira-Thurston manifold does not admit any such structure (see [26]).

Example 2. Consider the complex Lie group $\text{Aff}(\mathbb{C})$ given in (17), and the discrete subgroup

$$\Gamma = \left\{ \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} : w \in \mathbb{Z}[i] \right\}.$$  

The quotient $M := \Gamma \backslash \text{Aff}(\mathbb{C})$ is topologically $\mathbb{C}^* \times \mathbb{T}^2$. Considering any left invariant Hermitian metric on $\text{Aff}(\mathbb{C})$, we obtain a Hermitian metric on the quotient whose associated Chern connection is induced by the $(−)$-connection on the group, hence, it has trivial holonomy.
The next example shows that a complex manifold can admit an abelian connection with non-parallel torsion.

Example 3. Let $M = \mathbb{R}^4$ with canonical coordinates $(x_1, x_2, x_3, x_4)$ and corresponding vector fields $\partial_1, \ldots, \partial_4$. Let $f, g \in C^\infty(\mathbb{R}^4)$, such that $\partial_k(f) = \partial_k(g) = 0$, $k = 1, 2$, and $\partial_3(f)$ or $\partial_3(g)$ is not constant. We define an affine connection $\nabla$ so that the space $\mathcal{P}^\nabla$ of parallel vector fields is

$$\mathcal{P}^\nabla = \text{span}_\mathbb{R}\{\partial_1, \partial_2, \partial_3, -f\partial_1 + g\partial_2 + \partial_4\}.$$

We define an almost complex structure on $\mathbb{R}^4$ as follows:

$$J\partial_1 = \partial_2, \quad J\partial_3 = -f\partial_1 + g\partial_2 + \partial_4, \quad J\partial_2 = -\partial_1, \quad J\partial_4 = g\partial_1 + f\partial_2 - \partial_3.$$

It turns out that $\nabla$ is an abelian connection on $(\mathbb{R}^4, J)$. In fact, we can check that condition (i) of Proposition 3.8 is satisfied. To do this, we compute

$$[J\partial_1, J\partial_3] = [\partial_2, -f\partial_1 + g\partial_2 + \partial_4] = -\partial_2(f)\partial_1 + \partial_2(g)\partial_2 = 0,$$

$$[\partial_1, J\partial_3] = [\partial_1, -f\partial_1 + g\partial_2 + \partial_4] = -\partial_1(f)\partial_1 + \partial_1(g)\partial_2 = 0,$$

and the assertion follows. The complex affine manifold $(\mathbb{R}^4, J, \nabla)$ cannot be obtained as in Corollary 4.4, since

$$[\partial_3, J\partial_3] = -\partial_3(f)\partial_1 + \partial_3(g)\partial_2,$$

therefore $\mathcal{P}^\nabla$ is not a Lie algebra, that is, the torsion tensor field $T$ of $\nabla$ is not parallel, since $f$ and $g$ have been chosen so that $\partial_3(f)$ or $\partial_3(g)$ is not constant.

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