Orthogonal black di-ring solution

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We construct a five dimensional exact solution of the orthogonal black di-ring which has two black rings whose $S^1$-rotating planes are orthogonal. This solution has four free parameters which represent radii of and speeds of $S^1$-rotation of the black rings. We use the inverse scattering method. This method needs the seed metric. We also present a systematic method how to construct a seed metric. Using this method, we can probably construct other solutions having many black rings on the two orthogonal planes with or without a black hole at the center.

I. INTRODUCTION

In recent years, higher dimensional black objects have been actively studied. This is partly because string theory tells us that the spacetime we live in is higher dimensional. In order for us to feel that the number of spacetime dimensions is four on large scales, the extra dimensions must be compactified. However, when we observe a phenomenon on a small scale, the effect of higher dimensions may appear. If the compactification scale is sufficiently large, this effect can be detected in future collider experiments [1].

The only asymptotically flat static solution of the vacuum Einstein equations in higher dimensions is Schwarzschild-Tangherlini solution [2, 3], which is stable against perturbation [4]. These properties are the same as in the four dimensional case. However, in the asymptotically flat, stationary and axisymmetric case, the uniqueness theorem of the black hole does not exist in higher dimensions unlike the four dimensional case. Myers and Perry discovered the higher dimensional black hole [5] whose topology is $S^{D-1}$, which is an extension of the Kerr black hole to higher dimensions. This solution was obtained also by the solitonic solution-generating methods [6] and the inverse scattering method [7]. A five dimensional Myers-Perry black hole is unique [8] if black hole topology is $S^3$ in the asymptotically flat spacetime and if the spacetime has three commuting Killing vectors. The black ring solution with horizon topology $S^1 \times S^2$ was discovered by Emparan and Reall [9]. The black ring rotating on the $S^1$ plane, which helps the balance against its attractive self-gravity force. The $S^2$-rotating black ring solution was discovered by Mishima and Iguchi [10, 11, 12], in which a plane supporting black ring from falling due to the self-gravity is needed. The solution of black ring with $S^1$ and $S^2$ rotations was also found by Pomeransky and Sen’kov [13], which first had been found by numerical method [14]. It was proved that the black ring solution with horizon topology $S^2 \times S^2$ is only Pomeransky and Sen’kov solution [13, 14]. In the case with Maxwell fields, a generalized analysis has done [17]. Moreover, the solution which has a number of black objects, such as black di-ring [15, 16] and black saturn [20] were also discovered. Rotating dipole black ring solution [21, 22, 23] and black saturn with dipole black ring solution [24] have been generated in five-dimensional Einstein-Maxwell-dilaton gravity.

In this paper, we construct the solution having two black rings which are orthogonal to each other. We call this solution “orthogonal black di-ring”. We use the inverse scattering method [25]. In inverse scattering method, we need a seed solution. We present a method of constructing a diagonal seed metric.

This paper is organized as follows. In Sec. II we will review the inverse scattering method. In Sec. III we will present a method of constructing a seed metric, giving a seed metric for the orthogonal black di-ring solution. In Sec. IV we will show how to obtain the orthogonal black di-ring solution using the inverse scattering method and we will write the obtained metric explicitly. In Sec. V we will analyse the regularity of the obtained solution. The solution with several parameters generally has singularities. However, we can remove all of these singularities if we choose the parameters appropriately, leaving four free parameters. We show the conditions that the parameters must satisfy for regularity. In Sec VI we will summarize our results.

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II. INVERSE SCATTERING METHOD

In this section, we briefly explain the inverse scattering method \(^2\), by which a new metric can be obtained from a
known seed metric. This method can be used when the metric which we want has \(D - 2\) commuting killing vector
fields, one of which is timelike, in \(D\)-dimensional spacetime. In this paper, we only present the procedure for generating a
new solution without giving its deviation. For detailed deviation, see Belinsky’s Paper \(^25\).

Thanks to the symmetries, the metric can be written as

\[
d s^2 = f (d \rho^2 + dz^2) + g_{ab} dx^a dx^b,
\]

where \(f\) and \(g_{ab}\) depend only on \(\rho\) and \(z\). Here without loss of generality, we can set the determinant
of \(g_{ab}\) as

\[
det g_{ab} = -\rho^2.
\]

Then the Einstein equations become

\[
\begin{align*}
\partial_\rho U + \partial_z V &= 0, \\
U &= \rho (\partial_\rho g) g^{-1}, \\
V &= \rho (\partial_z g) g^{-1}, \\
\partial_\rho \ln f &= -\frac{1}{\rho} + \frac{1}{4\rho} \text{Tr}(U^2 - V^2), \\
\partial_z \ln f &= \frac{1}{2\rho} \text{Tr}(UV),
\end{align*}
\]

where \(g\) is \(g_{ab}\) in matrix notation. These equations can be classified. The first three are the differential equations
of \(g\). The others are the equations from which \(f\) is obtained for a given \(g\). From eq.\((2.3)\), we find that integrability
condition for \(\ln f, \partial_\rho \partial_z \ln f = 0\), is satisfied.

Suppose that a seed metric \(g_0\) which satisfies the Einstein equations \((2.3)\) is prepared. Then we consider linear
differential equations

\[
\begin{align*}
\left(\partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2} \partial_\lambda\right) \Psi &= \frac{\rho V_0 - \lambda U_0}{\lambda^2 + \rho^2} \Psi, \\
\left(\partial_\rho + \frac{2\lambda \rho}{\lambda^2 + \rho^2} \partial_\lambda\right) \Psi &= \frac{\rho U_0 - \lambda V_0}{\lambda^2 + \rho^2} \Psi,
\end{align*}
\]

where \(U_0\) and \(V_0\) are \(U\) and \(V\) with \(g = g_0\). \(\lambda\) is the complex spectral parameter independent of \(\rho\) and \(z\), and
\(\Psi = \Psi(\lambda, \rho, z)\) is a \((D - 2)\times(D - 2)\) matrix. Solving these equations, we get the matrix \(\Psi_{01}\).

We prepare functions \(\mu_k(\rho, z)\) and \(\mu_k(\rho, z)\), which we call solitons and antisolitons, respectively, defined by

\[
\begin{align*}
\mu_k &= \sqrt{\rho^2 + (z - a_k)^2} - (z - a_k), \\
\tilde{\mu}_k &= -\sqrt{\rho^2 + (z - a_k)^2} - (z - a_k),
\end{align*}
\]

where \(a_k\) is a real constant. We choose \(\mu_i'(i = 1, \ldots, n)\) from either \(\mu_k\) or \(\tilde{\mu}_k\). We introduce \(n\) 3-vectors \(m^{(k)}\)
associated with \(\mu_k\). \(m^{(k)}\) is called BZ vector. We make a \(n \times n\) matrix as

\[
\Gamma_{kl} = \frac{m^{(k)}(\Psi_0^{-1}(\mu_k', \rho, z))^{ab} g_{bc}(\Psi_0^{-1}(\mu_k', \rho, z))^{cd} m^{(l)}_d}{\rho^2 + \mu_k' \mu_l'}. \tag{2.10}
\]

Then a metric

\[
\begin{align*}
g_{1ab} &= \left( g_{0ab} - \sum_{kl} (\Gamma^{-1})_{kl} \mu_k' \mu_l' N_{a}^{(k)} N_{b}^{(l)} \right), \\
n_{a}^{(k)} &= m_{b}^{(k)} (\Psi_0^{-1}(\mu_k', \rho, z))^{bc} g_{0ca}, \tag{2.11}
\end{align*}
\]

satisfies the Einstein equations. In general, \(g_1\) does not satisfy \(\det g = -\rho^2\). However, \(g_1'\) multiplied by \(\rho\) and \(\mu_i'\)
also satisfies the Einstein equations. With the help of this property, we can get the metric satisfying both the Einstein
equations and \(\det g_1' = -\rho^2\) as

\[
g_{1ab} = \rho^{-\frac{2g}{g}} \prod_{k=1}^{n} \mu_k' \tilde{g}_{1ab}'. \tag{2.13}
\]
Moreover, if we choose $f$ as

\[ f = C f_0 \rho^{-\frac{n(n+1)}{2}+\frac{2n-2-D_0 n}{D}} \left( \prod_{k=1}^{n} \mu_k^{-\frac{2(n+D-1)}{D}} \right) \left( \prod_{k=1}^{n} (\mu_k^2 \rho^2)^{\frac{2-n}{D}} \right) \left( \prod_{k=1}^{n} (\mu_k - \mu_k')^{\frac{1}{D}} \right)^{-1} \det \Gamma_{kl}, \]  

(2.14)

with a constant $C$, $f$ satisfies eq. (2.5) and eq. (2.6). The above set of operations is called soliton transformation with \{ $\mu'_k$ \}.

In this paper we use the strategy taken in Ref. [26]. Using this method, a new metric $g$ automatically satisfies $\det g = -\rho^2$ without the operation of eq. (2.13). We give a diagonal seed metric $g_0$ and $f_0$ which is constructed using $\mu_k$ and $\rho$ as explained in the next section. First, we apply the soliton transformation with \{ $\mu'_k$ \} and simple BZ vectors to the seed metric $g_0$, to obtain the metric $g'_1$ (see eq. (2.11)). If we operate the soliton transformation to the metric $g'_1$ with \{ $\mu'_l$ \} and the same BZ vectors, the metric is transformed back into the seed metric $g_0$ where we refer to $\mu_i$ \((\mu_i)\) as $\mu_i'$ when $\mu_i'$ is $\mu_k$ \((\mu_k)\). If we operate the soliton transformation to the metric $g_0$ with \{ $\mu'_l$ \} and general BZ vectors, we get a new metric $g$. Since the change of $\det g$ is independent of BZ vectors, change of $\det g$ by soliton transformation from $g'_1$ to $g_0$ is the same as that from $g'_1$ to $g$. This means $\det g = \det g_0$ and we don’t need to rescale the new metric.

Moreover, we don’t care the complicated factors composed of $\rho$ and $\mu'_k$ in eq. (2.11). As a result, the metric $g$ and $f$ take a simple form as

\[ g_{ab} = g'_{1ab} - \sum_{k,l} (\Gamma'^{-1})_{kl} \mu_k^{-1} \mu_l^{-1} N^{(k)}_a N^{(l)}_b, \]

\[ f = C f_0 \frac{\det \Gamma'_{kl}}{\det \Gamma'^{(0)}_{kl}}, \]

(2.15)

(2.16)

where $\Gamma'_{kl}$ and $\Gamma'^{(0)}_{kl}$ are constructed from the metric $g'_1$, and $\Psi_{ij}$ with general BZ vectors the simple ones, respectively.

In the inverse scattering method regular points where $\det g \neq 0$ are transformed to regular points. Some points where $\det g = 0$ become physical singularities. However, we don’t need to care about physical singularities at the level of the seed metric because physical singularities can be transformed to coordinate singularities. In many cases, the seed metric from which a new metric without a physical singularity is obtained has physical singularities on the axis.

### III. SEED METRIC

In this section, we will give a seed solution which is one of the Weyl solutions [27]. We have introduced solitons $\mu_k$ in eq. (2.9). Using these solitons, we can construct a solution of the Einstein equations as follows. We prepare the $3 \times 3$ diagonal metric like

\[ g = \text{diag} \left\{ -\mu_1 \mu_2 \cdots \cdot \rho^2 \mu_1', \cdots, \cdot \mu_1', \cdots, \cdot \mu_1' \right\}, \]

(3.1)

where the total number of each $\mu_k$ in the numerator of all components is equal to that in the denominator. Though in the above example $\rho^2$ appears in the second component, it can be in any component but it appears only once. You can check that the metric $g$ satisfies Einstein equations (2.3) and $\det g = -\rho^2$ (See Appendix A). Next, we give $f$ as

\[ f = \kappa^2 \left( \prod_{k} \mu_k^2 \right) \left( \prod_{k} \mu_k^{-1} \right) \left( \prod_{a=1}^{3} \rho^2 + \mu_k \mu_k' \right) \left( \prod_{k,l} \rho^2 + \mu_k \mu_l \right)^{-1} \left( \prod_{k,l} \mu_k \mu_l \right)^{-1} \left( \prod_{k,l} \mu_k' \mu_l' \right)^{-1} \left( \prod_{k} \rho^2 + \mu_k \mu_k' \right)^{-1} \left( \prod_{k} \mu_k \mu_k' \right)^{-1} \left( \prod_{k} \mu_k' \mu_k \right)^{-1} \left( \prod_{k} \mu_k' \mu_k \right)^{-1} \left( \prod_{k} \mu_k' \mu_k \right)^{-1} \left( \prod_{k} \mu_k' \mu_k \right)^{-1}, \]

(3.2)

where $\kappa$ is a constant and $\mu_k$ \((\mu_k')\) are the solitons that appear in the numerator (denominator) of the $a$-th component. Although, in the above example, $f$ has the product $\prod_{k} \mu_k \mu_k^{-1} \mu_k' \mu_k'$, we must replace it with the product of $\mu_k'$ corresponding to the component having $\rho^2$ in the general case. Then, $f$ satisfies Einstein equations (2.5) and (2.6) (See Appendix A). Since $\mu_i > 0$ except for the rod, the seed metric is regular in the region satisfying $\rho > 0$. Although this seed metric generally has singularities on the rod, it is not a problem at all as we explained above.
Using this prescription, we will show a method for constructing a seed metric corresponding to a given rod structure. Before that, we give a brief explanation about a rod structure. Since $\det g = -\rho^2$, at least one eigenvalue of $g$ becomes zero at $\rho = 0$. An eigenvector with a zero eigenvalue is not always the same but depend on $z$. A rod structure represents how the eigenvector changes depending on $z$. The positive density rod indicates the direction of the eigenvector with a zero eigenvalue. In this paper, we consider only the case in which the eigenvalues corresponding to positive density rods become $O(\rho^2)$ in the limit $\rho \to 0$. Then, if there are two positive density rods, one of the other eigenvalue of $g$ must become infinite as $O(\rho^{-2})$ in the limit $\rho \to 0$. We call it a negative density rod.

A method for constructing a seed metric corresponding to a given rod structure is given as follows. First, we put a minus sign to the $a_i$ to the numerator of the component having rod at the left-end ($z = -\infty$). Next, starting with the left-end, we add $\mu_i$ to the numerator of the corresponding diagonal component if the positive density rod appears at $a_i$ or if the negative density rod disappears at $a_i$. Similarly, we add $\mu_i$ to the denominator of the corresponding diagonal component if the positive density rod disappears at $a_i$ or if the negative density rod appears at $a_i$. Constructing $f$ by eq. (3.2), we complete the construction of a seed metric. The metric $g$ obtained by the above operations has to the rod structure. Suppose a positive (or negative) rod exists on the left of $z$ and cancels the positive density rod on the segments $[a_i, a_j]$. Then we can make the positive (negative) density rod appearing from $z = a_i$ by adding $\mu_i$ to the numerator (denominator) of the metric component. The disappearance of rod can be explained similarly.

Next, we construct the seed metric of the orthogonal black di-ring. The rod structure of the orthogonal black di-ring is shown in fig.1. The thick solid lines correspond to the positive density rods, while the dashed lines correspond to the negative density rods. Naively one may think that we need to prepare the seed metric as the one corresponding to this rod structure directly. However, referring to the construction of the black saturn solution [20], we expect that we cannot add the angular momentum of $S^1$-plane of black ring starting from the seed metric of this rod structure. Following the case of black saturn, we introduce a negative density rod as shown in fig.2. Then the metric corresponding to this rod structure can be written as

$$g_0 = \text{diag} \left\{ -\frac{\mu_1\mu_5}{\mu_3\mu_7}, \frac{\rho^2\mu_3\mu_7}{\mu_2\mu_4\mu_6}, \frac{\mu_2\mu_4\mu_6}{\mu_1\mu_5} \right\}, \quad (3.3)$$

where the constants $a_i$ contained in $\mu_i$ are ordered such that $a_i \geq a_j$ for $i > j$. Then $f_0$ becomes

$$f_0 = \frac{k^2\mu_2\mu_4\mu_6}{\mu_1\mu_5}(\rho^2 + \mu_1\mu_2)(\rho^2 + \mu_1\mu_3)(\rho^2 + \mu_1\mu_4)(\rho^2 + \mu_1\mu_6)(\rho^2 + \mu_1\mu_7)(\rho^2 + \mu_2\mu_3)(\rho^2 + \mu_2\mu_5) \times (\rho^2 + \mu_2\mu_7)(\rho^2 + \mu_3\mu_4)(\rho^2 + \mu_3\mu_5)(\rho^2 + \mu_3\mu_6)(\rho^2 + \mu_4\mu_5)(\rho^2 + \mu_4\mu_7)(\rho^2 + \mu_5\mu_6)(\rho^2 + \mu_5\mu_7) \times (\rho^2 + \mu_6\mu_7)(\rho^2 + \mu_1\mu_5)^{-2}(\rho^2 + \mu_2\mu_4)^{-2}(\rho^2 + \mu_2\mu_6)^{-2}(\rho^2 + \mu_3\mu_7)^{-2}(\rho^2 + \mu_4\mu_6)^{-2} \prod_{i=1}^{7}(\rho^2 + \mu_i^2)^{-1}. \quad (3.4)$$

This solution has singularities on the rods $z \in [a_1, a_2]$ and $z \in [a_6, a_7]$. and is of no physical interest by itself. However, once soliton transformation is applied to this solution appropriately, the negative density rod moves to the $t$-direction and cancels the positive density rod on the segments $z \in [a_1, a_2]$ and $z \in [a_6, a_7]$. Although in general it leaves the singularities at $z = a_1$ and $z = a_7$, we will find that we can remove these singularities by choosing the appropriate BZ vectors.

We must construct the matrix $\Psi_{g_0}$ solving eq. (2.7) and eq. (2.8). Although it seems at first sight difficult to solve these equations, it is easy to obtain one of the solutions in fact. The method is as follows. We eliminate $\rho^2$ using
The construction of the black Saturn solution \[20\]. The orthogonal black di-ring solution can be obtained by performing

\[ g = \Psi / (\mu) \]

FIG. 2: the rod structure of the seed metric of the orthogonal black di-ring the solid (dashed) lines means positive (negative) density rods.

\[ \mu_i \bar{\mu}_i = -\rho^2 \]. Here we can use any \( \mu_i \) other than those used in soliton transformation\(^1\). Then basically we only have to change \( \mu_i \) in \( g_0 \) into \((\mu_i - \lambda)\) to get \( \Psi_{g_0} \). However, if we perform a soliton transformation with \( \mu_i \) when \( g_0 \) has \( \mu_i \), \( \Psi_{g_0}(\mu_i, \rho, z) \) or \( \Psi_{g_0}^{-1}(\mu_i, \rho, z) \) becomes infinity. In order to avoid this pathology, we replace \( \mu_i \) contained in \( g_0 \) with \(-\mu_i, \bar{\mu}_i / \bar{\mu}_i \). Then, we obtain the matrix \( \Psi_{g_0}(\lambda, \rho, z) \) which has no pathology at \( \lambda = \mu_i \).

IV. SOLITON TRANSFORMATION

In this section, we explain the soliton transformation to obtain the orthogonal black di-ring solution following the construction of the black saturn solution \[20\]. The orthogonal black di-ring solution can be obtained by performing the following soliton transformation on the seed metric \([20]\).

We perform a 2-soliton transformation\(^2\) with \( \mu_1' = \{\mu_1, \mu_7\} \) and both of BZ vectors being \((1,0,0)\). The metric obtained by this transformation is

\[ g_1' = \text{diag} \left\{ \frac{\mu_5 \mu_7}{\mu_3 \mu_1}, \frac{\rho^2 \mu_3 \mu_7}{\mu_2 \mu_4 \mu_6}, \frac{\mu_2 \mu_4 \mu_6}{\mu_1 \mu_5} \right\} \]. \quad (4.1)

We rescale this metric as

\[ \tilde{g}_1' = \frac{\mu_1}{\mu_7} g_1' = \text{diag} \left\{ \frac{\mu_5}{\mu_3}, \frac{\rho^2 \mu_1 \mu_3}{\mu_2 \mu_4 \mu_6}, \frac{\mu_2 \mu_4 \mu_6}{\mu_5 \mu_7} \right\} \]. \quad (4.2)

This rescaling makes the calculation easier, although it does not change the result\(^3\). We make \( \Psi \) corresponding to \( \tilde{g}_1' \) as

\[ \Psi_{\tilde{g}_1'} = \text{diag} \left\{ \frac{(\mu_5 - \lambda)}{\mu_3 - \lambda}, \frac{(\mu_1 - \lambda)(\mu_3 - \lambda)(\bar{\mu}_4 - \lambda)}{(\mu_2 - \lambda)(\mu_6 - \lambda)}, \frac{(\mu_2 - \lambda)(\mu_6 - \lambda)(\bar{\mu}_7 - \lambda)}{(\mu_5 - \lambda)(\bar{\mu}_4 - \lambda)} \right\} \]. \quad (4.3)

Next, we apply the 2-soliton transformation to \( \tilde{g}_1' \) with \( \mu_1' \). The BZ vectors associated with \( \bar{\mu}_1 \) and \( \mu_7 \) are chosen to be \((1,0,c)\) and \((1,b,0)\), respectively. We denote the resulting metric by \( \tilde{g} \). Finally, to undo the rescaling in eq.(4.2) we rescale \( \tilde{g} \) as

\[ g = \frac{\mu_7}{\mu_1} \tilde{g} \]. \quad (4.4)

We must also compute \( f \) given in eq.(2.17). \( \Gamma_{kl} \), which is necessary to compute eq.(2.18), is obtained in the process of constructing \( \tilde{g} \). Moreover, \( \Gamma_{kl}^{(0)} \) is given by \( \Gamma_{kl}^{(0)} = \Gamma_{kl}|_{b=0,c=0} \).

\(^1\) You might worry that, if we replace \(-\rho^2\) with \(\mu_i \bar{\mu}_i\) of the different \(\mu_i\), the different \(\Psi\) is obtained. Then the difference of the component in \(\Psi^{-1}(\lambda = \mu_k)\) is any factor. In the new metric, \(\Psi\) appears in the only form of \(m_a^{(k)} \Psi_{a\bar{b}}^{-1}(\mu_k)\). These facts mean the difference of \(\Psi\) can be finally absorbed in the parameters in BZ vectors.

\(^2\) The metric obtained by 2-soliton transformation is the same as that obtained by two 1-soliton transformations with the same solitons and the same BZ vectors. The calculation of two 1-soliton transformations is easier than that of 2-soliton transformation.

\(^3\) Due to the rescaling, \(\Psi\) is also rescaled as \(\Psi_{\tilde{g}}' = (\mu_1 - \lambda)/(\mu_7 - \lambda) \Psi_{\tilde{g}}\). This affect \(\Gamma_{kl}\) of eq.(2.10) and \(N^{(k)}\) of eq.(2.18) as \(\Gamma_{kl} \rightarrow (\mu_1 - \mu_k)(\mu_1 - \mu_i)/(\mu_7 - \mu_k)(\mu_7 - \mu_i)\Gamma_{kl}\) and \(N^{(k)} \rightarrow (\mu_1 - \mu_k)/(\mu_7 - \mu_k)N^{(k)}\). We can find from eq.(2.17) that in the new metric the effects cancel each other.
The obtained metric is given by Eq.2.1 with $a, b = t, \phi$ and

$$
f = C_f f_0 H / F, \quad H = F + b^2 F_{(b)} + c^2 F_{(c)} + b^2 c^2 F_{(bc)},
$$

$$
F = - \frac{\mu_1^2 \mu_2^2 (\mu_3 - \mu_5)^2 (\mu_1 \mu_3 + \rho^2)^2 (\mu_1 \mu_5 + \rho^2)^2}{\mu_3^2 (\mu_1 - \mu_5)^2 (\mu_1 \mu_3 + \rho^2)^2 (\mu_1 \mu_5 + \rho^2)^2 (\mu_3^2 + \rho^2)^2}
$$

$$
F_{(b)} = - \frac{\mu_1^2 \mu_2^2 \mu_3^2 (\mu_2 - \mu_5)^2 (\mu_6 - \mu_7)^2 (\mu_1 \mu_3 + \rho^2)^2}{\mu_2 \mu_3 \mu_6 (\mu_3 - \mu_7)^2 (\mu_1 \mu_5 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2},
$$

$$
F_{(c)} = - \frac{\mu_1^2 \mu_2^2 \mu_3^2 (\mu_1 - \mu_4)^2 (\mu_3 - \mu_5)^2 (\mu_1 \mu_5 + \rho^2)^2}{\mu_3 \mu_4 (\mu_5 - \mu_7)^2 (\mu_1 \mu_2 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2},
$$

$$
F_{(bc)} = - \frac{\mu_1^2 \mu_2^2 \mu_3^2 (\mu_1 - \mu_4)^2 (\mu_2 - \mu_5)^2 (\mu_1 \mu_5 + \rho^2)^2}{\mu_5 \mu_6 (\mu_1 - \mu_7)^2 (\mu_1 \mu_2 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2},
$$

$$
g_{tt} = H^{-1} (A + b^2 A_{(b)} + c^2 A_{(c)} + b^2 c^2 A_{(bc)}),
$$

$$
A = \frac{\mu_1^2 \mu_3^2 (\mu_3 - \mu_5)^2 (\mu_1 \mu_3 + \rho^2)^2 (\mu_1 \mu_5 + \rho^2)^2}{\mu_3^2 \mu_7 (\mu_5 - \mu_7)^2 (\mu_1 \mu_5 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2 (\mu_3^2 + \rho^2)^2},
$$

$$
A_{(b)} = - \frac{\mu_1^2 \mu_3^2 \mu_7 (\mu_2 - \mu_5)^2 (\mu_6 - \mu_7)^2 (\mu_1 \mu_5 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2}{\mu_2 \mu_3 \mu_6 (\mu_3 - \mu_7)^2 (\mu_1 \mu_5 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2 (\mu_3^2 + \rho^2)^2},
$$

$$
A_{(c)} = - \frac{\mu_1^2 \mu_3^2 \mu_6 (\mu_1 - \mu_4)^2 (\mu_3 - \mu_5)^2 (\mu_1 \mu_5 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2}{\mu_3 \mu_4 \mu_5 (\mu_5 - \mu_7)^2 (\mu_1 \mu_2 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2 (\mu_3^2 + \rho^2)^2},
$$

$$
A_{(bc)} = - \frac{\mu_1^2 \mu_3^2 \mu_6 (\mu_1 - \mu_7)^2 (\mu_3 - \mu_5)^2 (\mu_1 \mu_2 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2}{(\mu_1 - \mu_7)^2 (\mu_3 - \mu_7)^2 (\mu_1 \mu_5 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2 (\mu_3^2 + \rho^2)^2},
$$

$$
g_{\psi \psi} = B + H^{-1} (b^2 B_{(b)} + b^2 c^2 B_{(bc)}),
$$

$$
B = \frac{\mu_3 \mu_7 \rho^2}{\mu_2 \mu_4 \mu_6},
$$

$$
B_{(b)} = \frac{\mu_1 \mu_3 \mu_5 (\mu_2 - \mu_5)^2 (\mu_6 - \mu_7)^2 (\mu_1 \mu_3 + \rho^2)^2}{\mu_2 \mu_3 \mu_5 \mu_7 (\mu_3 - \mu_7)^2 (\mu_1 \mu_5 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2 (\mu_3^2 + \rho^2)^2},
$$

$$
B_{(bc)} = - \frac{\mu_1 \mu_3 \mu_5 \mu_6 (\mu_1 - \mu_7)^2 (\mu_3 - \mu_7)^2 (\mu_1 \mu_2 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2}{\mu_2 \mu_3 \mu_5 \mu_6 (\mu_1 - \mu_7)^2 (\mu_3 - \mu_7)^2 (\mu_1 \mu_2 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2 (\mu_3^2 + \rho^2)^2},
$$

$$
g_{\phi \phi} = C + H^{-1} (c^2 C_{(c)} + b^2 c^2 C_{(bc)}),
$$

$$
C = \frac{\mu_3 \mu_4 \mu_6}{\mu_1 \mu_5},
$$

$$
C_{(c)} = \frac{\mu_1 \mu_3 \mu_5 (\mu_1 - \mu_4)^2 (\mu_3 - \mu_5)^2 (\mu_1 \mu_5 + \rho^2)^2}{\mu_3 \mu_5 (\mu_5 - \mu_7)^2 (\mu_1 \mu_2 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2},
$$

$$
C_{(bc)} = - \frac{\mu_1 \mu_3 \mu_5 \mu_6 (\mu_1 - \mu_7)^2 (\mu_3 - \mu_7)^2 (\mu_1 \mu_2 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2}{\mu_5 \mu_6 (\mu_1 - \mu_7)^2 (\mu_3 - \mu_7)^2 (\mu_1 \mu_2 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1^2 + \rho^2)^2 (\mu_3^2 + \rho^2)^2},
$$

$$
g_{t \psi} = H^{-1} b (D + c^2 D_{(c)}),
$$

$$
D = \frac{\mu_1 \mu_3 \mu_5 (\mu_2 - \mu_5)^2 (\mu_6 - \mu_7)^2 (\mu_1 \mu_3 + \rho^2)^2 (\mu_1 \mu_5 + \rho^2)^2}{\mu_2 \mu_3 \mu_6 \mu_7 (\mu_5 - \mu_7)^2 (\mu_1 \mu_5 + \rho^2)^2 (\mu_1 \mu_7 + \rho^2)^2 (\mu_1^2 + \rho^2)^2},
$$

$$
D_{(c)} = - \frac{\mu_1 \mu_3 \mu_5 (\mu_1 - \mu_4)^2 (\mu_2 - \mu_5)^2 (\mu_6 - \mu_7)^2 (\mu_1 \mu_5 + \rho^2)^2}{\mu_4 \mu_5 (\mu_1 - \mu_7)^2 (\mu_5 - \mu_7)^2 (\mu_1 \mu_2 + \rho^2)^2 (\mu_1 \mu_6 + \rho^2)^2 (\mu_1 \mu_7 + \rho^2)^2 (\mu_1^2 + \rho^2)^2},
$$
FIG. 3: General rod structure after the transformation. Although this solution has the singularities at \( z = a_1 \) and \( z = a_7 \), we can remove these singularities if we choose the appropriate parameters.

\[
g_{\psi\phi} = H^{-1} c(E + b^2 E(b)),
\]
\[
E = \frac{\mu_2 \mu_5 \mu_6 (\mu_1 - \mu_4)(\mu_3 - \mu_7)^2 (\mu_1 \mu_3 + \rho^2)(\mu_1 \mu_7 + \rho^2)}{\mu_3^2 (\mu_5 - \mu_2)^2 (\mu_1 \mu_2 + \rho^2)(\mu_1 \mu_6 + \rho^2)(\mu_1^2 + \rho^2)^2},
\]
\[
E(b) = -\frac{\mu_1 \mu_4 \mu_7 (\mu_1 - \mu_4)(\mu_2 - \mu_7)^2 (\mu_6 - \mu_7)^2 (\mu_1 \mu_3 + \rho^2)}{(\mu_1 - \mu_7)(\mu_3 - \mu_7)^2 (\mu_1 \mu_2 + \rho^2)(\mu_1 \mu_6 + \rho^2)(\mu_1 \mu_7 + \rho^2)^2 (\mu_1^2 + \rho^2)^2},
\]
\[
g_{\psi\phi} = H^{-1} b c\frac{\mu_1 \mu_4 - \mu_4 \mu_7 (\mu_6 - \mu_7)(\mu_1 \mu_3 + \rho^2)}{(\mu_1 - \mu_7)(\mu_3 - \mu_7)(\mu_1 \mu_2 + \rho^2)(\mu_1 \mu_6 + \rho^2)(\mu_1 \mu_7 + \rho^2)^2 (\mu_1^2 + \rho^2)^2}.
\]

This solution has generally point singularities at \((\rho, z) = (0, a_1)\) and \((0, a_7)\) and conical singularities on the rods \( z \in [-\infty, a_2] \), \( z \in [a_3, a_4] \), \( z \in [a_4, a_5] \) and \( z \in [a_6, \infty] \). However, in the next section, we show these singularities can be removed if we choose the parameters appropriately. When we remove the singularities, this solution becomes the orthonormal black di-ring solution.

**V. ANALYSIS**

First, we analyse the rod structure of the orthogonal black di-ring solution. Although generally singularities appear at \( z = a_1 \) and \( a_7 \), we can remove these singularities by choosing parameters \( c \) and \( b \) appropriately. After setting the parameters \( b \) and \( c \) to the particular values that eliminate the singularities at \( z = a_1 \) and \( a_7 \), we next analyse the asymptotic behavior of this solution. If we choose \( C \kappa^2 = 1 \), we will find that this solution becomes asymptotically flat. Finally, we analyse the conical structure around the axis. We will find that all conical singularities can be removed by choosing the parameters appropriately. As a result, four parameters are left in the end.

**A. rod structure**

The rod structure of the obtained metric with general parameters is illustrated in fig.3. We calculate the direction of the rod which is defined in Ref. [28]. The direction of the rod represents the vector whose norm becomes zero at \( \rho = 0 \). The semi-infinite rod \( z \in [-\infty, a_2] \) and the finite rod \( z \in [a_3, a_4] \) have directions \((0, 1, 0)\) which correspond to the \( \psi \)-axis. The semi-infinite rod \( z \in [a_6, \infty] \) and the finite rod \( z \in [a_4, a_5] \) have directions \((0, 0, 1)\) which correspond to the \( \phi \)-axis. The finite rods \( z \in [a_2, a_3] \) and \( z \in [a_5, a_6] \), which correspond to the location of the black ring horizons, have directions

\[
v = (1, \Omega^{(1)}_\psi, \Omega^{(1)}_\phi) \quad \text{and} \quad u = (1, \Omega^{(2)}_\psi, \Omega^{(2)}_\phi),
\]

respectively, where

\[
\Omega^{(1)}_\psi = -\frac{b a_{75} a_{76}}{2 a_{73} a_{71}}, \quad \Omega^{(1)}_\phi = -\frac{c a_{51}^2}{2 a_{31} a_{61} a_{71}},
\]
\[
\Omega^{(2)}_\psi = -\frac{b a_{76}}{2 a_{71} a_{74}}, \quad \Omega^{(2)}_\phi = \frac{c a_{41}}{2 a_{61} a_{71}},
\]
\[
a_{ij} \equiv a_i - a_j.
\]
In the limit $r \rightarrow 0$, the other components of $g$ become
\[ g_{\psi} \rightarrow \frac{a_{41} a_{51} a_{61} a_{71} + 2 a_{21} a_{41} a_{61} a_{71} \sin^2(1 - \cos \theta)}{a_{31} a_{41} a_{51} a_{61} a_{71} + 2 a_{21} a_{41} a_{61} a_{71} \sin^2(1 - \cos \theta)} t^{-1}. \]
\[ g_{\phi} \rightarrow \frac{a_{41} a_{51} a_{61} a_{71} + 2 a_{21} a_{41} a_{61} a_{71} \sin^2(1 - \cos \theta)}{a_{31} a_{41} a_{51} a_{61} a_{71} + 2 a_{21} a_{41} a_{61} a_{71} \sin^2(1 - \cos \theta)} t^{-1}. \]

Then the term proportion to $r^{-1}$ in $g_{\phi}$ becomes 0, and the leading term of $g_{\phi}$ becomes
\[ g_{\phi} \rightarrow \frac{2 a_{21} a_{41} a_{61} a_{71}}{a_{31} a_{41} a_{51} a_{61} a_{71}} (5.7) \]

In this limit, the other components of $g$ and $f$ become
\[ g_{tt} = O(r), \quad g_{\psi\psi} = \frac{a_{31} a_{71}}{2 a_{21} a_{41} a_{61}} r^2 \sin^2 \theta, \quad g_{t\phi} = \frac{b a_{31} a_{72} a_{75} a_{76}}{4 a_{21} a_{41} a_{61} a_{71} a_{73}^2} r^2 \sin^2 \theta, \]
\[ g_{\phi\phi} = \frac{c a_{41} a_{51}}{a_{31} a_{71}} r^2 \sin^2 \theta \quad \text{and} \quad f = \frac{a_{31} a_{71}}{2 a_{21} a_{41} a_{61}}. \]

We find that, in this limit, $g_{t\phi}$ and $g_{\phi\phi}$ are constant. Moreover, when we introduce new variable
\[ \psi_1 = \psi - \frac{b a_{72} a_{75} a_{76}}{2 a_{71} a_{73}^2} t + \frac{2 a_{21} a_{41} a_{61} a_{71}}{a_{31} a_{71}} \phi, \]
the metric becomes
\[ ds^2 \propto f (dt^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi_1^2) + 2 g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2. \]

Since this metric is the Minkowski metric, we find that there is no singularity at this point. In a similar way, we can remove the singularity at $z = a_7$ if we choose the parameter as
\[ b^2 = \frac{2 a_{71} a_{72} a_{74}}{a_{72} a_{75} a_{76}}. \]

With these conditions \( 5.7 \) and \( 5.12 \), the metric is smooth across $z = a_1$ and $a_7$. Then the rod structure is as illustrated in Fig. 4.
B. asymptotic structure

To analyse the asymptotic structure, we take other coordinates defined by
\[
\rho = \frac{1}{2} r^2 \sin 2\theta, \quad z = \frac{1}{2} r^2 \cos 2\theta,
\]
where \(0 < \theta < \pi/2\). Then in the asymptotic limit \(r \to \infty\), we have \(g_{t\psi} \to 0\), \(g_{t\phi} \to 0\) and \(g_{\psi\phi} \to 0\). Since
\[
d\rho^2 + dz^2 = r^2 (dr^2 + r^2 d\theta^2),
\]
and the asymptotically value of \(f\) becomes
\[
f = C\kappa^2 \frac{1}{r^2} + \cdots,
\]
we find we must choose \(C\kappa = 1\) in order to get the asymptotic flat solution. Setting \(C\kappa = 1\) the asymptotic metric becomes
\[
ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \cos^2 \theta d\psi^2 + r^2 \sin^2 \theta d\phi^2.
\]
The asymptotic flatness also requires that the angles \(\psi\) and \(\phi\) should have periodicities
\[
\Delta \psi = \Delta \phi = 2\pi.
\]

C. regularities on the axis

In order to avoid conical singularities, the period \(\Delta \eta\) of the spacelike coordinate \(\eta = \psi\) or \(\phi\) corresponding to the angle around each positive density rod must satisfy [28]
\[
\Delta \eta = 2\pi \lim_{\rho \to 0} \sqrt{g_{\eta\eta}}.
\]
When we impose the conditions (5.7) and (5.12), the regularity conditions on the rods \(z \in [-\infty, a_2]\) and \(z \in [a_6, \infty]\) are automatically satisfied for \(\Delta \psi = \Delta \phi = 2\pi\).

Next we consider the regularity condition on the rod \(z \in [a_3, a_4]\). The regularity condition is written as
\[
\Delta \psi = 2\pi \sqrt{\frac{a_{14} a_{16} a_{17} a_{25} a_{27} a_{34} a_{35} a_{36}}{a_{15} a_{24} a_{26} a_{37}}}.
\]
Since \(\Delta \psi\) must be \(2\pi\), this implies
\[
1 = \frac{a_{14} a_{16} a_{17} a_{25} a_{27} a_{34} a_{35} a_{36}}{a_{15} a_{24} a_{26} a_{37}}.
\]
Similarly, the regularity condition on the rod \(z \in [a_4, a_5]\) implies
\[
1 = \frac{a_{74} a_{72} a_{71} a_{63} a_{61} a_{54} a_{53} a_{52}}{a_{73} a_{64} a_{62} a_{51}}.
\]
Under the transformation, \(a_i \to a_{7-i}\), eq. (5.20) becomes eq. (5.21). This is expected because the solution we want has a symmetry corresponding to the exchange of \(\psi\) and \(\phi\).

We must check the existence of the parameters which satisfy eq. (5.20) and eq. (5.21) with \(a_i > a_j\) for \(i > j\). Without loss of generality, we can set \(a_4 = 0\). For the sake of simplicity, we consider the symmetric case where
\[
a_1 = -a_7, \quad a_2 = -a_6, \quad a_3 = -a_5.
\]
In this case, eq. (5.20) and eq. (5.21) become the same equation. Moreover, we set \(a_7 = 1\), which corresponds to fixing the scale. Then, eq. (5.20) and eq. (5.21) are written as
\[
a_5^2 (1 + a_6)^2 (a_5 + a_6)^2 - (1 + a_5)^4 a_6^4 = 0.
\]
Here, we denote the left hand side of this equation by $f(a_6)$ as a function of $a_6$. Then

\[
    f(a_5) = a_5^2(1 + a_5)^2(1 - a_5)(3 + a_5), \quad (5.24)
\]

\[
    f(1) = (1 + a_5)^2(3a_5 + 1)(a_5 - 1). \quad (5.25)
\]

Since $a_5$ satisfies the condition $0 < a_5 < 1$, $f(a_5)$ is always positive. On the other hand, $f(1)$ is always negative. Therefore the parameter $a_6$ which satisfies eq. (5.23) exists between $a_5$ and $a_7(= 1)$. This means that the orthogonal black di-ring solution which has no singularity outside the horizon exists.

### D. regularities at end points of rods

In this subsection, we show there is no singularity at the end points of rods. From the symmetry of this solution (see footnote-3), we only have to check the points $z = a_2$, $a_3$ and $a_4$. Near each point $(\rho, z) = (0, a_i)$, we transform $\rho$ and $z$ as

\[
    \rho = \frac{1}{2} r^2 \sin 2\theta, \quad z = \frac{1}{2} r^2 \cos 2\theta - a_i. \quad (5.26)
\]

Near the point $(\rho, z) = (0, a_2)$, we take basis as $v$ (see eq. (5.1)), $q = (0, 1, 0)$ and $s = (0, 0, 1)$. In the limit $r \to 0$, inner products and $f$ become

\[
    g(v, v) \to -4c_2 a_{42} a_{62} a_{51}^2 r^2 \sin^2 \theta, \quad g(q, q) \to \frac{a_{32}}{a_{42}} a_{62} r^2 \cos^2 \theta, \quad g(s, s) \to 4c_2 a_{21} a_{31} a_{61} a_{71} a_{52},
\]

\[
    g(v, q) = O(r^4), \quad g(v, s) = \alpha_2 r^2 \sin^2 \theta, \quad g(q, s) = \beta_2 r^2 \cos^2 \theta, \quad f \to \frac{c_2^2 a_{41} a_{51}^2 a_{62} a_{72}}{2a_{21} a_{31} a_{61} a_{71} a_{42} a_{62} r^{-2}}, \quad (5.27)
\]

where $\alpha_2$ and $\beta_2$ are some constants. We find that $g(s, s)$ is constant. When bases are changed like

\[
    \psi_2 = \psi + \frac{\beta_2}{f} \phi \quad \text{and} \quad \eta_2 = \frac{1}{f} \left( 4 \frac{c_2^2 a_{42} a_{62} a_{51}^2}{a_{31} a_{52} a_{61} a_{71}} \eta - \alpha_2 \phi \right), \quad (5.28)
\]

where $\eta$ is defined as $(\partial/\partial \eta)^a \equiv v^a$, the metric becomes

\[
    ds^2 \simeq f(dr^2 + r^2 d\theta^2 + r^2 \cos^2 \theta d\psi_2^2 - r^2 \sin^2 \theta d\eta_2^2) + 4\frac{a_{21} a_{41} a_{51} a_{61} a_{71} a_{52}}{c_2 a_{31}^2 a_{51}} d\phi^2, \quad (5.29)
\]

where we used eq. (5.27). Since eq. (5.24) is locally the Minkowski metric, there is no singularity at the point $(\rho, z) = (0, a_2)$.

Similarly, we can obtain the metric near the point $(\rho, z) = (0, a_3)$. The metric around this point becomes

\[
    ds^2 \simeq \frac{a_{32}}{a_{43}} \frac{a_{73}}{a_{63}} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi_2^2 - r^2 \cos^2 \theta d\eta_3^2) + 2\frac{a_{31}}{a_{43} a_{63}} d\phi^2, \quad (5.30)
\]

where

\[
    \eta_3 = \frac{2a_{31} a_{43} a_{53} a_{63}}{a_{32} a_{73}} \eta. \quad (5.31)
\]

From this, we find there is no singularity at the point $(\rho, z) = (0, a_3)$.

Finally we check the point at $(\rho, z) = (0, a_4)$. The metric around this point becomes

\[
    ds^2 \simeq \frac{a_{24}}{a_{43}} \frac{a_{74}}{a_{63}} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi_4^2 + r^2 \cos^2 \theta d\phi_4^2) - \frac{a_{34}}{a_{41} a_{74}} dt^2, \quad \psi_4 = \psi + \frac{b a_{41} a_{75} a_{64} t}{2 a_{71} a_{64} a_{74}} \quad \text{and} \quad \phi_4 = \phi + \frac{c a_{51}^2}{2 a_{41} a_{61} a_{71}} t, \quad (5.32)
\]

where we used eq. (5.20) and eq. (5.21). We find that there is no singularity at the point $\rho = 0$ and $z = a_4$. 
VI. SUMMARY AND DISCUSSION

We present a method for constructing a seed metric. Using this method, we can obtain a seed metric corresponding to any rod structure that we want. In this paper, we constructed a solution which has two mutually orthogonal black rings. We call this solution the orthogonal black di-ring. Although the solution that we obtained by using the inverse scattering method generally has singularities, we have shown that we can remove these singularities by choosing the parameters appropriately. This solution has four free parameters which represent radii of black rings, and speeds of $S^1$-rotation of black rings.

If we want to construct many black rings solution, probably we should prepare the seed metric corresponding to the rod structure as shown in Fig. 5, which can be easily obtained by our method. The soliton transformation applied to this seed metric leaves naked singularities with the general parameters. However, in generating the orthogonal black di-ring solution, the naked singularities can be removed when we choose the parameters by imposing the continuity of the periodicities at this point. This seems to tell us that these singularities are related to the discontinuity of the periodicities. The physical reason why the discontinuity of the periodicities exists in the seed metric is that a plane which helps the balance against its attractive self-gravity force is needed. If the rotation of the black ring balances against its attractive self-gravity force, such a plane is not needed. Since the rotation of the black ring is introduce through the BZ parameters, it seems that we can choose the parameters such that no discontinuity of the periodicities appears. Therefore, the naked singularities can be probably removed.

Similarly, it seems that the solution, which has black rings on the two orthogonal planes and a black hole at the center, can be probably obtained. We will report such a solution in the forthcoming paper.

Acknowledgments

The author is grateful to Takahiro Tanaka for careful reading the manuscript and useful comments. He also thanks Takashi Nakamura for his valuable comments and continuous encouragement. This work is supported in part by the 21st Century COE “Center for Diversity and Universality in Physics” at Kyoto university, from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

APPENDIX A

In this section, we show that eq. (3.1) and eq. (3.2) satisfy the Einstein equations \((2.3, 2.6)\) and \(\det g = -\rho^2\). Thanks to the diagonalization, we can divide eq. (2.4) into that for each component of \(g\). The differential equation of \(g_{tt}\) becomes

\[
\partial_\rho \{\rho \partial_\rho (\ln g_{tt})\} + \partial_z \{\rho \partial_z (\ln g_{tt})\} = 0.
\]

Suppose that \(g_{tt} = g_i \cdots / (g_j \cdots)\). Since \(g_{tt}\) appears only in the form of \(\ln g_{tt}\), \(g_{tt}\) becomes the solution of eq. (A1) if each \(g_j\) satisfies

\[
\partial_\rho \{\rho \partial_\rho (\ln g_j)\} + \partial_z \{\rho \partial_z (\ln g_i)\} = 0.
\]

In fact, if \(g_i = \mu_i\) or \(-\rho^2\), eq. (A2) holds. Therefore, \(g_{tt}\) which is a product of \(\rho^2\) and \(\mu_i\) satisfies eq. (A1). We can construct \(g_{\psi\psi}\) and \(g_{\phi\phi}\) similarly. Moreover, \(\det g = -\rho^2\) is achieved provided that the total number of each \(\mu_k\) in the numerator in all components is equal to that in the denominator and that \(\rho^2\) appears once in the numerator among all components. This means that \(g\) given in (3.1) satisfies eq. (2.3) and \(\det g = -\rho^2\).
We must also solve eq. (2.5) and eq. (2.6). Thanks to the diagonal form of the metric $g$, the traces $\text{Tr}(U^2 - V^2)$ and $\text{Tr}(UV)$ in these equations become the summations of the contributions from each component of $g$. In addition, since $f$ appears only in the form of $\ln f$ in the equations 2.5 and 2.6, the left hand sides of these equations can be written as a summation of $\ln f_m$ with $f = f_1 f_2 \cdots f_n$. By this fact, we have only to solve for $f_m$ corresponding to each term of the right hand side of eq. (2.5) and eq. (2.6). First, we consider a component without $\rho^2$, which is written as

$$\frac{\mu_{i_1} \cdots \mu_{i_m}}{\mu_{i_1'} \cdots \mu_{i_n'}} \tag{A3}$$

Then $(1/4\rho)\text{Tr}(U^2 - V^2)$ and $(1/2\rho)\text{Tr}(UV)$ becomes

$$\frac{1}{4\rho} \text{Tr}(U^2 - V^2) = \frac{\rho}{4} \sum_k \left( \frac{(\partial_{\rho} \mu_{i_k})^2 - (\partial_{\rho} \mu_{i_k})^2}{\mu_{i_k}^2} \right) + \frac{\rho}{4} \sum_k \left( \frac{(\partial_{\rho} \mu_{i_k'})^2 - (\partial_{\rho} \mu_{i_k'})^2}{\mu_{i_k'}^2} \right)$$

$$+ \frac{\rho}{2} \sum_{k \neq l} \left( \frac{(\partial_{\rho} \mu_{i_k})(\partial_{\rho} \mu_{i_l})}{\mu_{i_k} \mu_{i_l}} - \frac{(\partial_{\rho} \mu_{i_k})(\partial_{\rho} \mu_{i_l})}{\mu_{i_k} \mu_{i_l}} \right) + \frac{\rho}{2} \sum_{k \neq l} \left( \frac{(\partial_{\rho} \mu_{i_k'})(\partial_{\rho} \mu_{i_l'})}{\mu_{i_k'} \mu_{i_l'}} - \frac{(\partial_{\rho} \mu_{i_k'})(\partial_{\rho} \mu_{i_l'})}{\mu_{i_k'} \mu_{i_l'}} \right)$$

$$- \frac{\rho}{2} \sum_{k, l} \left( \frac{(\partial_{\rho} \mu_{i_k})(\partial_{\rho} \mu_{i_l'})}{\mu_{i_k} \mu_{i_l'}} + \frac{(\partial_{\rho} \mu_{i_k'})}{\mu_{i_k} \mu_{i_l'}} \right) \tag{A4}$$

and

$$\frac{1}{2\rho} \text{Tr}(UV) \geq \frac{\rho}{2} \sum_k \left( \frac{\partial_{\rho} \mu_{i_k} \partial_{\rho} \mu_{i_k'}}{\mu_{i_k}^2 \mu_{i_k'}} \right) + \frac{\rho}{2} \sum_k \left( \frac{\partial_{\rho} \mu_{i_k} \partial_{\rho} \mu_{i_k'}}{\mu_{i_k}^2 \mu_{i_k'}} \right)$$

$$+ \frac{\rho}{2} \sum_{k \neq l} \left( \frac{\partial_{\rho} \mu_{i_k} \partial_{\rho} \mu_{i_l}}{\mu_{i_k} \mu_{i_l}} + \frac{\partial_{\rho} \mu_{i_k} \partial_{\rho} \mu_{i_l}}{\mu_{i_k} \mu_{i_l}} \right) + \frac{\rho}{2} \sum_{k \neq l} \left( \frac{\partial_{\rho} \mu_{i_k'} \partial_{\rho} \mu_{i_l'}}{\mu_{i_k'} \mu_{i_l'}} + \frac{\partial_{\rho} \mu_{i_k'} \partial_{\rho} \mu_{i_l'}}{\mu_{i_k'} \mu_{i_l'}} \right)$$

$$- \frac{\rho}{2} \sum_{k, l} \left( \frac{\partial_{\rho} \mu_{i_k} \partial_{\rho} \mu_{i_l'}}{\mu_{i_k} \mu_{i_l'}} + \frac{\partial_{\rho} \mu_{i_k} \partial_{\rho} \mu_{i_l'}}{\mu_{i_k} \mu_{i_l'}} \right) \tag{A5}$$

As we explained above, we only have to solve the equations

$$\partial_{\rho} \ln f_m = \frac{1}{2\rho} \left( \frac{(\partial_{\rho} \mu_{i_k})(\partial_{\rho} \mu_{i_l})}{\mu_{i_k} \mu_{i_l}} - \frac{(\partial_{\rho} \mu_{i_k})(\partial_{\rho} \mu_{i_l})}{\mu_{i_k} \mu_{i_l}} \right) \tag{A6}$$

$$\partial_{z} \ln f_m = \frac{1}{2\rho} \left( \frac{(\partial_{\rho} \mu_{i_k})(\partial_{\rho} \mu_{i_l})}{\mu_{i_k} \mu_{i_l}} + \frac{(\partial_{\rho} \mu_{i_k})(\partial_{\rho} \mu_{i_l})}{\mu_{i_k} \mu_{i_l}} \right) \tag{A7}$$

A solution of these equations is given by

$$f_m = \frac{\mu_{i_k} \mu_{i_l}}{\rho^2 + \mu_{i_k} \mu_{i_l}} \tag{A8}$$

Next, we consider the component with $\rho^2$. The difference from the case of the component without $\rho^2$ is that $(1/4\rho)\text{Tr}(U^2 - V^2)$ and $(1/2\rho)\text{Tr}(UV)$, respectively, have extra terms

$$\frac{1}{\rho} + \sum_k \frac{\partial_{\rho} \mu_{i_k}}{\mu_{i_k}} - \sum_k \frac{\partial_{\rho} \mu_{i_k'}}{\mu_{i_k'}} \tag{A9}$$

and

$$\sum_k \frac{\partial_{z} \mu_{i_k}}{\mu_{i_k}} - \sum_k \frac{\partial_{z} \mu_{i_k'}}{\mu_{i_k'}} \tag{A10}$$

The first term of (A9) cancels $-1/\rho$ on the right hand side of eq. (2.5). $f_m$ corresponding to the the second and the third of (A9) and (A10) is $\mu_k$. As a result, remembering that $f$ can be written with the product of $f_m$, $f$ is obtained as eq. (2.2).

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