Green’s Functions of Partial Differential Equations with Involutions

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Abstract

In this paper we develop a way of obtaining Green’s functions for partial differential equations with linear involutions by reducing the equation to a higher-order PDE without involutions. The developed theory is applied to a model of heat transfer in a conducting plate which is bent in half.

Keywords: Green’s functions, PDEs, linear involution, heat equation.

1 Introduction

The study of differential equations with involutions dates back to the work of Silberstein [10] who, in 1940, obtained the solution of the equation \( f(x) = f(1/x) \). In the field of differential equations there has been quite a number of publications (see for instance the monograph on the subject of reducible differential equations of Wiener [11]) but most of them relate to ordinary differential equations (ODEs). There has also been some work in partial differential equations (PDEs), for instance [11] or [2], where they study a PDE with reflection.

In what Green’s functions for equations with involutions is concerned, we find in [3] the first Green’s function for ODEs with reflection and in [4] we have a framework that allows the reduction of any differential equation with reflection and constant coefficients. This setting is established in a general way, so it can be used as well for other operators (the Hilbert transform, for instance) or in other yet unexplored problems, like PDEs [8]. In this work we take this last approach and find a way of reducing general linear PDEs with linear involutions to usual PDEs.

The paper is structured as follows. In Section 2 we develop an abstract framework, with definitions and adequate notation in order to treat linear PDEs as elements of a vector space.

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consisting of symmetric tensors. This will allow us to systematize the algebraic transformations necessary in order to obtain the desired reduction of the problem. In Section 3 we start providing a simple example that shows how the general process works and then prove the main result of the paper, Theorem 3.3, that permits a general reduction in the case of order two involutions. We end the Section with a problem with an order 3 involution (Example 3.4), illustrating that the same principles could be applied to higher order involutions. Finally, in Section 4, we describe a way to obtain Green’s functions for PDEs with linear involutions and apply it to a model of the process of heat transfer in a conducting plate which is bent in half with the two halves separated by some insulating material. We study the problem for different kinds of boundary conditions and a general heat source.

2 Definitions and notation

2.1 Derivatives

Let \( \mathbb{F} \) be \( \mathbb{R} \) or \( \mathbb{C} \), \( n \in \mathbb{N} \) and \( \Omega \subset V := \mathbb{F}^n \) a connected open subset. For \( p \geq 2 \), note by \( V^{\otimes p} \) the space of symmetric tensors or order \( p \), that is, the space of tensors of order \( p \) modulus the permutations of their components. We note \( V^{\otimes 1} = V \) and \( V^{\otimes 0} = \mathbb{F} \). For the convenience of the reader, we summarize now the properties and operations of the symmetric tensors:

- \( V^{\otimes p} := \{ v_1^1 \otimes \cdots \otimes v_1^k + \cdots + v_r^1 \otimes \cdots \otimes v_r^k : v_j^i \in \mathbb{F}^n; \ j = 1, \ldots, r; \ s = 1, \ldots, k; \ r, k \in \mathbb{N} \} \).
- \( (v_1^1 \otimes \cdots \otimes v_1^k) \otimes (v_1^{k+1} \otimes \cdots \otimes v_1^p) = v_1^1 \otimes \cdots \otimes v_1^p; \ v_1^s \in \mathbb{F}^n, \ s = 1, \ldots, p; \ p \in \mathbb{N} \).
- \( \nabla \) denotes the gradient vector of \( y \).
- \( \lambda (v_1 \otimes v_2) = (\lambda v_1) \otimes v_2; \ v_1, v_2 \in \mathbb{F}^n \).
- \( (v_1 + v_2) \otimes v_3 = v_1 \otimes v_3 + v_2 \otimes v_3; \ v_1, v_2, v_3 \in \mathbb{F}^n \).
- \( 0 \otimes v_1 = 0; \ v_1 \in \mathbb{F}^n \).

With these properties, \( V^{\otimes p} \) is an \( \mathbb{F} \)-vector space of dimension \( \binom{n+p-1}{p} \).

For every \( v = (v_1, \ldots, v_n) \in V \), we define the directional derivative operator as

\[
\mathcal{C}^1(\Omega, \mathbb{F}) \xrightarrow{D_v} \mathcal{C}(\Omega, \mathbb{F})
\]

\[
y \longmapsto v_1 \frac{\partial y}{\partial x^1} + \cdots + v_n \frac{\partial y}{\partial x^n}
\]

If \( \nabla y \) denotes the gradient vector of \( y \), then \( D_v(y) = v^T \nabla y \). Observe that \( D_{\lambda u + v} = \lambda D_u + D_v \) for every \( u, v \in \mathbb{F}^n \) and \( \lambda \in \mathbb{F} \), that is, \( D_v \) is linear in \( v \). Also, for \( u, v \in \mathbb{F}^n \), if \( y \in \mathcal{C}^2(\Omega, \mathbb{F}) \), then \( D_u(D_v(y)) = D_v(D_u(y)) \). Furthermore, \( D_u \circ D_v \) is bilinear –that is, linear in both \( u \) and \( v \), so we can write the identification \( D_v \circ D_u \equiv D_{v \otimes u} \) whenever \( \otimes \) denotes de symmetric tensor product of \( u \) and \( v \). In the same way, we define the composition of higher order derivatives by \( D_{\omega_2} \circ D_{\omega_1} = D_{\omega_2 \otimes \omega_1} \) where \( \omega_1 \in V^{\otimes q} \) and \( \omega_2 \in V^{\otimes p} \), \( p, q \in \mathbb{N} \).

In this way, a linear partial differential equation is given by

\[
Ly := \sum_{k=0}^m D_{\omega_k}^k y = 0,
\]

(2.1)
where \( \omega_k \in V^\otimes k \) for \( k = 1, \ldots, m \) and \( D^0_{\omega^0} u \equiv \omega_0 u \) where \( \omega_0 \in \mathbb{F} \) (that is, \( V^\otimes 0 := \mathbb{F} \)). Now, the operator \( L \) can be identified with \( \omega_0 + \omega_1 + \cdots + \omega_m \), which is an element of the symmetric tensor algebra

\[
S^*V := \bigoplus_{k=0}^{\infty} V^\otimes n = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots
\]

It is interesting to point out the the Hilbert space completion of \( S^*V \), that is, \( F_+(V) := S^*V \), is called the symmetric or bosonic Fock space, which is widely used in quantum mechanics \([5]\).

### 2.2 Involution

**dfn 2.1.** Let \( \Omega \) be a set and \( A: \Omega \to \Omega, p \in \mathbb{N}, p \geq 2 \). We say that \( A \) is an order \( p \) involution if

1. \( A^p \equiv \Lambda^0 \cdots \Lambda A = \text{Id} \),
2. \( A^j \neq \text{Id}, j = 1, \ldots, p-1 \).

We will consider linear involutions in \( \mathbb{F}^n \). They are characterized by the following theorem.

**thm 2.2** ([1]). A necessary and sufficient condition for a linear transformation \( A \) on a finite dimensional complex vector space \( V \) to be an involution of order \( p \) is that \( A = \alpha_1 P_1 + \cdots + \alpha_k P_k \) where \( \alpha_j \) is a \( p \)-th root of the unity, and \( P_1, \ldots, P_k \) are projections such that \( P_i P_j = 0, i \neq j \) and \( P_1 + \cdots + P_k = \text{Id} \).

**rem 2.3.** As an straightforward consequence of this result we have that there are only order two linear involutions in \( \mathbb{R}^n \). This is because the only real \( p \)-th roots of the unity are contained in \( \{\pm 1\} \).

The characterization provided in Theorem 2.2 can be rewritten in the following way.

**cor 2.4.** A necessary and sufficient condition for a linear transformation \( A \) on \( V \) to be an involution of order \( p \) is that \( A = U^{-1} \Lambda U \) where \( \Lambda, U \in \mathcal{M}_n(\mathbb{F}) \), \( U \) is invertible and \( \Lambda \) is a diagonal matrix where the elements of the diagonal are \( p \)-th roots of the unity.

**Proof.** Consider the characterization of involutions given by Theorem 2.2. Take the vector subspaces \( H_j := P_j V, j = 1, \ldots, k \). Then, \( V = H_1 \oplus \cdots \oplus H_k \). Take \( U^{-1} \) to be the matrix of which its columns are, consecutively, a basis of \( H_k \). Hence, \( A = U^{-1} \Lambda U \) where \( \Lambda \) is a diagonal matrix of diagonal

\[
(\alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_2, \ldots, \alpha_k, \ldots, \alpha_k),
\]

where every \( \alpha_j \) is repeated according to the dimension of \( H_k \). \( \blacksquare \)

### 2.3 Pullbacks and equations

Let \( \mathcal{F}(\Omega, \mathbb{F}) \) be the set of functions from \( \Omega \subset \mathbb{F}^n \) to \( \mathbb{F} \). We define the pullback operator by a function \( \varphi \in \mathcal{F}(\Omega, \Omega) \) as

\[
\mathcal{F}^1(\Omega, \mathbb{F}) \xrightarrow{\varphi^*} \mathcal{F}(\Omega, \mathbb{F})
\]

\[
y \longmapsto y \circ \varphi
\]
Assume $A$ is a linear order $p$ involution on $\Omega$ ($\Omega$ has to be such that $\Omega = A(\Omega)$). From now on, we will omit the composition signs. Observe that, for $v \in V$, $x \in \Omega$ and $y \in \mathcal{C}^1(\Omega, \mathbb{F})$,

$$((D_v A^*) y)(x) = D_v (y(Ax)) = v^T \nabla (y(Ax)) = v^T A^T \nabla y(Ax) = (Av)^T \nabla y(Ax) = D_{Av} \nabla y(Ax) = (A^* D_{Av}) y(x),$$

or, written briefly, $D_v A^* = A^* D_{Av}$. All the same, for $v_1, \ldots, v_j \in V$,

$$D_{v_1 \oplus \cdots \oplus v_j} A^* = A^* D_{Av_1 \oplus \cdots \oplus Av_j}.$$ 

If $\omega_k = v_1 \cdots v_k \in V^\otimes k$, we denote $A \omega_k \equiv Av_1 \cdots Av_k$. This way, $D_{\omega_k} A^* = A^* D_{\omega_k}$. 

We can consider now linear partial differential equations with linear involutions of the form

$$L y := \sum_{j=0}^{p-1} \sum_{k=0}^{m} (A^*)^j D^k_{\omega_k} y = 0,$$

where $\omega_k^j \in V^\otimes k$ for $k = 0, \ldots, m$; $j = 0, \ldots, p-1$. This time we can identify $L$ with

$$\left( \omega_1^0 + \cdots + \omega_k^0, \omega_1^1 + \cdots + \omega_k^1, \ldots, \omega_1^{p-1} + \cdots + \omega_k^{p-1} \right) \in (S^* V)^p.$$

The interest in these equations appears when they can be reduced to usual partial differential equations.

**dfn 2.5 ([4]).** If $\mathbb{F}[D]$ is the ring of polynomials on the usual differential operator $D$ and $\mathcal{A}$ is any operator algebra containing $\mathbb{F}[D]$, then an equation $Lx = 0$, where $L \in \mathcal{A}$, is said to be a reducible differential equation if there exits $R \in \mathcal{A}$ such that $RL \in \mathbb{F}[D]$.

In our present case, the first projection of the algebra $(S^* V)^p$ is precisely the algebra of partial differential operators on $n$ variables $\text{PD}_n[\mathbb{F}]$, so we want to find elements $R \in (S^* V)^p$ such that they nullify the last $p-1$ components of $L$.

### 3 Reducing the operators

We start with an illustrative example.

**EXA 3.1.** Let $V = \mathbb{R}^2$, $v = (v_1, v_2) \in V$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

$A$ is an order 2 involution. Consider the equation

$$v_1 \frac{\partial y}{\partial x_1}(x) + v_2 \frac{\partial y}{\partial x_2}(x) + y(Ax) = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2. \quad (3.1)$$

Here we work with the operator $L = D_v + A^*$. Take then $R = D_{-Av} + A^*$ and consider the identity operator $\text{Id}$. We have that

$$RL = (D_{-Av} + A^*)(D_v + A^*) = D_{-Av} D_v + A^* D_v + D_{-Av} A^* + (A^*)^2 = D_{-Av \oplus v} A^* D_v + A^* D_{Av} + \text{Id} = D_{-Av \oplus v} A^* D_v + A^* D_{-Av} + \text{Id} = D_{-Av \oplus v} A^* D_v - \text{Id} + \text{Id} = \text{Id} = D_{-Av \oplus v} + \text{Id}.$$
Hence, every two-times differentiable solution of equation (3.1) has to be a solution of the partial differential equation

$$-v_2^2 \frac{\partial^2 y}{\partial x_1^2}(x) + v_2^2 \frac{\partial^2 y}{\partial x_2^2}(x) + y = 0, \ x = (x_1, x_2) \in \mathbb{R}^2.$$  

**rem 3.2.** With the notation we have introduced, it is extremely important the use of parentheses. Observe that every \( \omega \in (\mathbb{P}^n)^{\mathbb{N}} \) can be expressed as \( \omega = v_1 \otimes \cdots \otimes v_k + \cdots + v_r \otimes \cdots \otimes v_r \) for some \( v_j \in \mathbb{P}^n, \ j = 1, \ldots, r, \ s = 1, \ldots, k; \ r, k \in \mathbb{N}. \) Hence, for \( c \in \mathbb{F}, \)

\[
(cA)\omega = cAv_1 \otimes \cdots \otimes cAv_k + \cdots + cAv_r \otimes \cdots \otimes cAv_r
= c^k(Av_1 \otimes \cdots \otimes Av_k) + \cdots + c^k(Av_r \otimes \cdots \otimes Av_r) = c^k(A\omega) \equiv c^kA\omega.
\]

**thm 3.3.** Let \( A \) be an order 2 linear involution on \( \mathbb{P}^n. \) Let \( L \in (S^*V)^p \) be defined as in (2.1). Then there exists \( R \in (S^*V)^p \) defined as

\[
Ry := \sum_{j=0}^{p-1} \sum_{k=0}^m (A^j)^iD_{\xi_k}^j y = 0,
\]

where \( \xi_k^0 = -A\omega_k^0, \xi_k^1 = \omega_k^1, \) for \( k = 0, 1, \ldots, \) such that \( RL \in \text{PD}_n[\mathbb{F}]. \) Furthermore, \( L \) and \( R \) commute.

**Proof.** For convenience, define \( \xi_k^j \) and \( \omega_k^j \) outside the index range \( j = 0, \ldots, p-1, \ k = 0, \ldots, m \) to be zero. In general,

\[
RL = \sum_{l=0}^{p-1} \sum_{k=0}^m (A^j)^iD_{\xi_k}^j \left( \sum_{j=0}^{p-1} \sum_{k=0}^m (A^j)^iD_{\omega_k}^j \right) = \sum_{l,j=0}^{p-1} \sum_{r,k=0}^m (A^j)^iD_{\xi_k}^j (A^j)^iD_{\omega_k}^j
= \sum_{l,j=0}^{p-1} \sum_{r,k=0}^m (A^j)^{i+j}D_{A^j\xi_k}^r (A^j)^iD_{\omega_k}^r
= \sum_{l,j=0}^{p-1} (A^j)^{i+j} \left( \sum_{s=0}^{2m} \sum_{k=0}^s D_{A^s\xi_{s-k}^0 \otimes \omega_k^0}^j \right) = \sum_{l,j=0}^{p-1} (A^j)^{i+j} \left( \sum_{s=0}^{2m} D_{A^s\xi_{s-k}^0 \otimes \omega_k^0}^j \right).
\]

In the particular case \( p = 2, \) we have that

\[
RL = \sum_{s=0}^{2m} D_{A^s\xi_{s-k}^0 \otimes \omega_k^0}^0 + \sum_{s=0}^{2m} D_{A^s\xi_{s-k}^1 \otimes \omega_k^0}^1 + A^2 \left( \sum_{s=0}^{2m} D_{A^s\xi_{s-k}^1 \otimes \omega_k^0}^0 + \sum_{s=0}^{2m} D_{A^s\xi_{s-k}^1 \otimes \omega_k^1}^1 \right)
= \sum_{s=0}^{2m} D_{A^s\xi_{s-k}^0 \otimes \omega_k^0}^0 + A^2 \sum_{s=0}^{2m} D_{A^s\xi_{s-k}^0 \otimes \omega_k^1}^1.
\]

So it is enough to check that, for \( s = 0, \ldots, 2m, \)

\[
\sum_{k=0}^s (\xi_{s-k}^1 \otimes \omega_k^0 + A\xi_{s-k}^0 \otimes \omega_k^1) = 0.
\]
Substituting the $\xi_{s_k}^j$ by their given values,

$$\sum_{k=0}^{s} \left( \xi_{s_k}^1 \otimes \omega_{s_k}^0 + A \xi_{s_k}^1 \otimes \omega_{s_k}^0 \right) = \sum_{k=0}^{s} \left( \omega_{s_k}^1 \otimes \omega_{s_k}^0 - A^2 \omega_{s_k}^0 \otimes \omega_{s_k}^1 \right)$$

$$= \sum_{k=0}^{s} \left( \omega_{s_k}^1 \otimes \omega_{s_k}^0 - \omega_{s_k}^0 \otimes \omega_{s_k}^1 \right) = \sum_{k=0}^{s} \omega_{s_k}^1 \otimes \omega_{s_k}^0 - \sum_{k=0}^{s} \omega_{s_k}^0 \otimes \omega_{s_k}^1$$

$$= \sum_{k=0}^{s} \omega_{s_k}^1 \otimes \omega_{s_k}^0 - \sum_{k=0}^{s} \omega_{s_k}^0 \otimes \omega_{s_k}^1 = 0.$$ 

Let us see that $L$ and $R$ commute.

$$LR = \sum_{s=0}^{2m} D^s_{\omega_0 \otimes \xi_{s_k}^0 + A \omega_0 \otimes \xi_{s_k}^0} + A^* \left( \sum_{s=0}^{2m} D^s_{\omega_0 \otimes \xi_{s_k}^0 + A \omega_0 \otimes \xi_{s_k}^0} \right).$$

Now,

$$\sum_{k=0}^{s} \left( \omega_{s_k}^0 \otimes \xi_{s_k}^0 + A \omega_{s_k}^0 \otimes \xi_{s_k}^1 \right) = \sum_{k=0}^{s} \omega_{s_k}^0 \otimes \xi_{s_k}^0 + \sum_{k=0}^{s} A \omega_{s_k}^0 \otimes \xi_{s_k}^1.$$

On the other hand,

$$\sum_{k=0}^{s} \left( \omega_{s_k}^1 \otimes \xi_{s_k}^0 + A \omega_{s_k}^0 \otimes \xi_{s_k}^1 \right) = \sum_{k=0}^{s} \left( \omega_{s_k}^1 \otimes (A \omega_{s_k}^0) + A \omega_{s_k}^0 \otimes \omega_{s_k}^1 \right)$$

$$= \sum_{k=0}^{s} \left( - \omega_{s_k}^1 \otimes A \omega_{s_k}^0 + A \omega_{s_k}^0 \otimes \omega_{s_k}^1 \right) = \sum_{k=0}^{s} \left( - A^2 \omega_{s_k}^0 \otimes \omega_{s_k}^1 \right) = 0.$$

Hence, the result is proven.

Similar reductions can be found for higher order involutions, although the coefficients may have a much more complex expression.

**Example 3.4.** Let $A$ be an order 3 linear involution in $\mathbb{C}^n$, $\nu \in \mathbb{C}^n \setminus \{0\}$ and consider the operator $L = D_\nu + A^*$. Define now

$$R := D_{\nu \otimes A^* \nu} - A^* D_{A^* \nu} + (A^*)^2.$$

Observe that second derivatives occur in $R$ but not in $L$. We have that

$$RL = D_{\nu \otimes A^* \nu} D_\nu - A^* D_{A^* \nu} D_\nu + (A^*)^2 D_\nu + D_{\nu \otimes A^* \nu} A^* - A^* D_{A^* \nu} A^* + (A^*)^2 A^*$$

$$= D_{\nu \otimes A^* \nu} - A^* D_{A^* \nu} + (A^*)^2 D_\nu + A^* D_{\nu \otimes A^* \nu} - (A^*)^2 D_\nu + \Id$$

$$= D_{\nu \otimes A^* \nu} + \Id.$$

Unfortunately, we do not have commutativity in general:

$$LR = D_\nu D_{\nu \otimes A^* \nu} - D_\nu A^* D_{A^* \nu} + D_\nu (A^*)^2 + A^* D_{A^* \nu} - (A^*)^2 D_{A^* \nu} + \Id$$

$$= D_{\nu \otimes A^* \nu} - A^* D_{A^* \nu \otimes A^* \nu} + (A^*)^2 D_{A^* \nu} + A^* D_{\nu \otimes A^* \nu} - (A^*)^2 D_{A^* \nu} + \Id$$

$$= D_{\nu \otimes A^* \nu} + A^* D_{(\nu \otimes A^* \nu) \otimes A^* \nu} + \Id.$$

In the particular case $\nu$ is a fixed point of $A$, $RL = LR$.

The obtaining of a general expression for associated operators in the case of order 3 involutions and the conditions under which such operators commute is an interesting open problem.
4 Green's functions

Consider now the following problem
\[ Lu = h; \ B_\lambda u = 0, \ \lambda \in \Lambda, \] (4.1)
where \( L \in (S^*V)^p, \ h \in L^1(\mathbb{F}^n, \mathbb{F}), \) the \( B_\lambda : \mathcal{C}(\mathbb{F}^n, \mathbb{F}) \to \mathbb{F} \) are linear functionals, \( \lambda \in \Lambda \) and \( \Lambda \) is an arbitrary set.

Let \( R \in (S^*V)^p, \ f \in L^1(\mathbb{F}^n, \mathbb{F}) \) and consider the problem
\[ RLv = f; \ B_\lambda v = 0, \ B_\lambda Rv = 0, \ \lambda \in \Lambda. \] (4.2)

Given a function \( G : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}, \) we define the operator \( H_G \) such that \( H_G(h)|_x := \int_{\mathbb{F}^n} G(x,s)h(s)\,ds \) for every \( h \in L^1(\mathbb{F}^n, \mathbb{F}), \) assuming such an integral is well defined. Also, given an operator \( R \) for functions of one variable, define the operator \( R_\lambda, \) as \( R_\lambda G(t,s) := R(G(\cdot,s))|_t \) for every \( s, \) that is, the operator acts on \( G \) as a function of its first variable.

We have now the following theorem relating problems (4.1) and (4.2). The proof for the case of ordinary differential equations can be found in [4]. The case of PDEs is analogous.

**thm 4.1.** Let \( L, R \in (S^*V)^p, \ h \in L^1(\mathbb{F}^n, \mathbb{F}). \) Assume \( L \) commutes with \( R \) and that there exists \( G \) such that \( H_G \) is well defined satisfying

\[ (I) \ (RL)_\lambda G = 0, \]
\[ (II) \ B_\lambda G = 0, \ \lambda \in \Lambda, \]
\[ (III) \ (B_\lambda R)_\lambda G = 0, \ \lambda \in \Lambda, \]
\[ (IV) \ RLH_G h = H_{(RL)_\lambda G} h + h, \]
\[ (V) \ LH_{R_\lambda G} h = H_{L_{R_\lambda G} h + h}, \]
\[ (VI) \ B_\lambda H_G = H_{B_\lambda G}, \ \lambda \in \Lambda, \]
\[ (VII) \ B_\lambda RH_G = B_\lambda H_{R_\lambda G} = H_{(B_\lambda R)_\lambda G}, \ \lambda \in \Lambda. \]

Then, \( v := H_G f \) is a solution of problem (4.2) and \( u := H_{R_\lambda G} h \) is a solution of problem (4.1).

4.1 A model of stationary heat transfer in a bent plate

We now consider a circular plate which is bent in half, with each of the two distinct halves separated by a very small distance which may be filled with some kind of (imperfect) heat insulating material (see Figure 4.1).

The heat equation which determines the temperature \( u \) on the plate for this situation is given by
\[ \frac{\partial u}{\partial t}(t,x,y) = \alpha \left[ \frac{\partial^2 u}{\partial x^2}(t,x,y) + \frac{\partial^2 u}{\partial y^2}(t,x,y) \right] + \beta [u(t,x,−y)−u(t,x,y)], \]
where
\[ \frac{\partial u}{\partial t}(t,x,y) = \alpha \left[ \frac{\partial^2 u}{\partial x^2}(t,x,y) + \frac{\partial^2 u}{\partial y^2}(t,x,y) \right], \]
is the usual heat equation with heat transfer coefficient \( \alpha > 0 \) and the term that goes with \( \beta > 0 \) relates to the heat transfer from the corollaryresponding point in the other half of the plate due to Newton’s law of cooling.

If we consider the associated stationary problem
\[ \alpha \left[ \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) \right] + \beta [u(x,−y)−u(x,y)] = 0, \]
it can be rewritten in a convenient way as

\[ Lu := \alpha \Delta u + \beta (A^* - \text{Id})u = 0, \]

where

\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

If we think of a circular plate in which the boundary is constantly cooled and the surface has a constant heat source given by a function \( h \), we are imposing Dirichlet boundary conditions in the ball \( B \) of radius \( \rho \in \mathbb{R}^+ \) and considering the problem

\[ Lu = h, \quad u|_{\partial B} = 0. \quad (4.3) \]

Observe that, \( \Delta \), expressed in tensor notation, is \( \Delta = D_{\omega_2^0} \) where

\[ \omega_2^0 = \frac{1}{2} [(1,1) \odot (1,1) + (1,-1) \odot (1,-1)]. \]

Besides, \( A \omega_2^0 = \omega_2^0 \) and, thus, \( \Delta A^* = \Delta A^* \). Hence, using Theorem 3.3, we have to take \( R = -\alpha \Delta + \beta A^* + \beta \text{Id} \) and thus

\[ RL = -\alpha^2 \Delta^2 - \alpha \beta A^* \Delta + \alpha \beta \Delta - \alpha \beta A^* \Delta + \alpha \beta \Delta + \beta^2 \text{Id} - \beta^2 A^* + \alpha \beta \Delta + \beta^2 A^* - \beta^2 \text{Id} \]

\[ = -\alpha^2 \Delta^2 + 2 \alpha \beta \Delta = (-\alpha^2 \Delta + 2 \alpha \beta \text{Id}) \Delta. \]

Now, the boundary conditions transformed by \( R \) are

\[ 0 = Ru = -\alpha \Delta u + \beta A^* u + \beta u = -\alpha \Delta u, \]

that is, the reduced problem becomes

\[ RLu = Rh =: f, \quad u|_{\partial B} = 0, \quad \Delta u|_{\partial B} = 0, \quad (4.4) \]
which is equivalent to the sequence of problems

\[ \Delta u = v, \quad u|_{\partial B} = 0, \quad (4.5) \]
\[ (-\alpha^2 \Delta + 2\alpha \beta I) v = f, \quad v|_{\partial B} = 0. \quad (4.6) \]

Problem (4.5) is the well-known Poisson equation with Dirichlet conditions on the circle of radius \( \rho \). The Green’s function can be written in polar coordinates as

\[
G_1(r, \varphi, \tilde{r}, \tilde{\varphi}) = -\frac{1}{4\pi} \ln \left[ \frac{r^2 \tilde{r}^2 - 2\rho^2 r \tilde{r} \cos(\varphi - \tilde{\varphi}) + \rho^4}{\rho^2 r^2 - 2\rho^2 \tilde{r} \tilde{r} \cos(\varphi - \tilde{\varphi}) + \tilde{\rho}^2 \tilde{r}^2} \right].
\]

See [9, Section 7.2.3]. On the other hand, problem (4.6) is a Helmholtz equation, and the Green’s function can be described in terms of the eigenfunctions of the associated homogeneous problem (see [9, Section 7.3.3]). More concretely, the associated Green’s function in polar coordinates is written as

\[
G_2(r, \varphi, \tilde{r}, \tilde{\varphi}) = \frac{1}{\alpha^2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\mu_{nm}^2 + \frac{2\beta}{\alpha}} \|w_{nm}^{(1)}(\tilde{r}, \tilde{\varphi})\|^{2} \left[ w_{nm}^{(1)}(r, \varphi)w_{nm}^{(1)}(\tilde{r}, \tilde{\varphi}) + w_{nm}^{(2)}(r, \varphi)w_{nm}^{(2)}(\tilde{r}, \tilde{\varphi}) \right],
\]

where \( \mu_{nm} \) are the positive zeroes of the Bessel functions \( J_n \), the eigenfunctions are given by

\[
w_{nm}^{(1)} = J_n \left( \frac{\mu_{nm}}{\rho} r \right) \cos n \varphi, \quad w_{nm}^{(2)} = J_n \left( \frac{\mu_{nm}}{\rho} r \right) \sin n \varphi,
\]

and

\[
\|w_{nm}^{(1)}\|^2 = \frac{1}{2} \pi \rho^2 (1 + \delta_{nm}) \left[ J_n'(\mu_{nm}) \right]^2,
\]

where \( \delta_{ij} = 1 \) if \( i = j \) and 0 if \( i \neq j \).

Now, the Green’s function associated to problem (4.4) is given by

\[
G_3(r, \varphi, \tilde{r}, \tilde{\varphi}) = \int_0^\rho \int_0^{2\pi} G_2(r, \varphi, \tilde{r}, \tilde{\varphi}) G_1(\tilde{r}, \tilde{\varphi}) \, d\tilde{\varphi} \, d\tilde{r}.
\]

In conclusion, the Green’s function related to problem (4.3) is

\[
G_4(\eta, \xi) = R_\eta G_3(\eta, \xi) = \int_0^\rho \int_0^{2\pi} R_\eta G_2(r, \varphi, \tilde{r}, \tilde{\varphi}) G_1(\tilde{r}, \tilde{\varphi}) \, d\tilde{\varphi} \, d\tilde{r},
\]

where \( R_\eta \) has to be expressed in polar coordinates in order to act in the first two variables of \( G_3 \):

\[
R = -\alpha \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right] + \beta A^* + \beta I d.
\]

Also, it is known that \( J_n'(z) = (n/z)J_n(z) - J_{n+1}(z) \), so

\[
R_\eta G_2(r, \varphi, \tilde{r}, \tilde{\varphi}) \]

\[
= \frac{1}{\alpha^2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\mu_{nm}^2 + \frac{2\beta}{\alpha}} \|w_{nm}^{(1)}\|^2 \left[ \tilde{w}_{nm}^{(1)}(r, \varphi)w_{nm}^{(1)}(\tilde{r}, \tilde{\varphi}) + \tilde{w}_{nm}^{(2)}(r, \varphi)w_{nm}^{(2)}(\tilde{r}, \tilde{\varphi}) \right],
\]

\[ 9 \]
where

\[
\tilde{\psi}^{(1)}_{nm} = \left( \frac{\mu_{nm}}{\rho} \right)^2 \left[ \left( \frac{n\rho}{\mu_{nm}r} \right)^2 J_n - \left( 1 + \frac{(n+1)\rho}{\mu_{nm}r} \right) J_{n+1} + J_{n+2} \right] \\
+ \frac{n}{r} \left[ \frac{\rho}{\mu_{nm}r} \right] - n^2 \left( \frac{\mu_{nm}r}{\rho} \right)^2 J_n \right) \left( \frac{\rho}{\mu_{nm}r} \right)^2 \cos n\varphi,
\]

\[
\tilde{\psi}^{(2)}_{nm} = \left( \frac{\mu_{nm}}{\rho} \right)^2 \left[ \left( \frac{n\rho}{\mu_{nm}r} \right)^2 J_n - \left( 1 + \frac{(n+1)\rho}{\mu_{nm}r} \right) J_{n+1} + J_{n+2} \right] \\
+ \frac{n}{r} \left[ \frac{\rho}{\mu_{nm}r} \right] - n^2 \left( \frac{\mu_{nm}r}{\rho} \right)^2 J_n \right) \left( \frac{\rho}{\mu_{nm}r} \right) \sin n\varphi.
\]

**EXA 4.2.** Inspired by the previous problem, we now change the term due to Newton’s law of cooling by a diffusion term in the following way.

\[
\frac{\partial K}{\partial t}(t,x,y) = \alpha \left[ \frac{\partial^2 K}{\partial x^2}(t,x,y) + \frac{\partial^2 K}{\partial y^2}(t,x,y) \right] + \beta \left[ \frac{\partial^2 K}{\partial x^2}(t,x,-y) + \frac{\partial^2 K}{\partial y^2}(t,x,-y) \right],
\]

where \( \alpha, \beta > 0, \beta \neq \alpha. \)

If we consider the associated stationary problem

\[
\alpha \left[ \frac{\partial^2 K}{\partial x^2}(x,y) + \frac{\partial^2 K}{\partial y^2}(x,y) \right] + \beta \left[ \frac{\partial^2 K}{\partial x^2}(x,-y) + \frac{\partial^2 K}{\partial y^2}(x,-y) \right] = 0,
\]

it can be rewritten as

\[
LK := \alpha \Delta K + \beta A^* \Delta K = 0,
\]

Using Theorem 3.3 we take \( R = -\alpha \Delta + \beta A^* \Delta \) and then

\[
RL = -\alpha^2 \Delta^2 - \alpha \beta \Delta A^* \Delta + \beta \alpha A^* \Delta^2 + \beta^2 (A^* \Delta)^2 = \beta^2 \Delta^2 - \alpha^2 \Delta^2 = (\beta^2 - \alpha^2) \Delta^2.
\]

Now, if we consider the fundamental solution of the bi-Laplacian \( \Delta^2 \) \[6\] equation (2.61)] we obtain a Green’s function given by

\[
G_1(\eta, \xi) = \frac{1}{8\pi} \|\eta - \xi\|^2 \ln \|\eta - \xi\|, \quad \eta, \xi \in \mathbb{R}^2.
\]

Hence, in that case, the Green’s function associated to \( L \) is given by

\[
G_2(\eta, \xi) = R \cdot G_1(\eta, \xi) = (\beta - \alpha) \frac{\ln \|\eta - \xi\| + 1}{2\pi}, \quad \eta, \xi \in \mathbb{R}^2.
\]

If we consider the problem

\[
LK = h, \quad u|\partial B = 0,
\]

the reduced problem becomes

\[
(\beta^2 - \alpha^2) \Delta^2 K = h, \quad u|\partial B = 0, \quad Ru|\partial B = 0. \tag{4.7}
\]

Now, the condition \( Ru = -\alpha \Delta u + \beta A^* \Delta u = 0 \) is satisfied if we can guarantee that \( \Delta u = 0 \), so we can consider the problem

\[
(\beta^2 - \alpha^2) \Delta^2 K = h, \quad u|\partial B = 0, \quad \Delta u|\partial B = 0. \tag{4.8}
\]
For problem (4.8) we have that the Green’s function is given by
\[ G_3(\eta, \xi) = \frac{1}{8\pi} \left| \eta - \xi \right|^2 (\ln \rho - 1 + \ln \| \eta - \xi \|) + \frac{\rho^2}{8\pi}, \eta, \xi \in \mathbb{R}^2. \]

Hence, the Green’s function related to problem (4.7) is
\[ G_4(\eta, \xi) = \frac{\ln \rho + \ln \| \eta - \xi \|}{2\pi}. \]

In general, the functions
\[ G_5(\eta, \xi) = \frac{1}{8\pi} \left| \eta - \xi \right|^2 (\mu + \ln \| \eta - \xi \|) + \frac{\nu}{8\pi}, \eta, \xi \in \mathbb{R}^2, \]
with \( \mu, \nu \in \mathbb{R} \), are Green’s functions related to the operator \( \Delta^2 \) with different boundary conditions. The associated function for the operator \( L \) is given by
\[ G_6(\eta, \xi) = \frac{1 + \mu + \log \| \eta - \xi \|}{2\pi}. \]

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