Derived categories and syzygies

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Abstract

We introduce syzygies for derived categories and study their properties. Using these, we prove the derived invariance of the following classes of artin algebras: (1) syzygy-finite algebras, (2) Igusa-Todorov algebras, (3) AC algebras, (4) algebras satisfying the finitistic Auslander conjecture, and (5) algebras satisfying the generalized Auslander-Reiten conjecture. In particular, Gorenstein CM-finite algebras are derived invariants.

MSC2000: Primary 18E30 16E05 Secondary 18G35 16G10 16E65
Keywords: syzygy; derived category; derived equivalence; Igusa-Todorov algebra; CM-finite algebra
Auslander condition

1 Introduction

Syzygies were introduced by Hilbert \( \S \) in 1890. Nowadays, syzygies are of general importance in algebraic geometry, homology algebra and representation theory of groups and algebras etc.. The advantage of syzygies is that they contain important information of modules and they can test some homological dimensions. For instance, Zimmermann-Huisgen and her coauthors showed that the structure of syzygies is important and powerful in understanding and compute various finitistic dimension of some algebras such as string algebras, monomial algebras and serial algebras \[1\] \[20\] \[21\] \[22\].

Monomial algebras and serial algebras are all syzygy-finite. By definition, syzygy-finite algebras means that there is an integer \( s \) such that the class of all \( n \)-th syzygies, where \( n > s \), is representation finite, or equivalently, the number of non-isomorphic indecomposable modules in the class is finite. Syzygy-finite algebras have nice homological properties. They satisfy a group of homological conjectures related to the finitistic dimension conjecture, for instance, Auslander conjecture (c.f. \[5\]) and Auslander-Reiten conjecture \[1\] etc.. Recently, algebras of finite Cohen-Macaulay type, or CM-finite algebras, are more attractive, see for instance \[11\] and references therein. Note that syzygy-finite algebras

*Supported by the National Science Foundation of China (No.10971099)
are clearly CM-finite. We don’t know if CM-finite algebras are also syzygy-finite, but for an important class of algebras, that is, Gorenstein algebras, they are the same.

In this paper, we extend the notion of syzygy to derived categories and study their properties. As is well known, derived categories are very important in modern study of algebraic geometry and representation theory of groups and algebras, and derived equivalences play an important role in the study. In particular, there are remarkable conjectures concerning derived equivalences. For instance, one famous conjecture in algebraic geometry, first made by Bondal and Orlov, asserts that if $X_1$ and $X_2$ are birational smooth projective Calabi-Yau varieties of dimension $n$, then there is an equivalence between their derived categories [2]. While in representation theory of groups, the famous Abelian Defect Group Conjecture of Broué claims that a block algebra $A$ of a finite group algebra and its Brauer correspondent $B$ is derived equivalent provided that their common defect group is abelian, see for instance [14].

Syzygies in derived category is powerful concerning derived equivalences, as we show in this paper. For example, using syzygies in derived category, we prove that derived equivalences preserve the syzygy-finiteness of algebras. This provides an important way to obtain syzygy-finite algebras, in particular, CM-finite algebras. The reader is referred to [19] for other ways to obtain syzygy-finite algebras. Moreover, using syzygies in derived category, we also prove that derived equivalences preserve the following interesting classes of algebras: Igusa-Todorov algebras, AC-algebras, algebras satisfying the finitistic Auslander conjecture and algebras satisfying the generalized Auslander-Reiten conjecture.

Igusa-Todorov algebras were introduced in [18] in connection with the study of the finitistic dimension conjecture using Igusa-Todorov functor. Such algebras have finite finitistic dimension. The class of Igusa-Todorov algebras is large, including syzygy-finite algebras, algebras with representation dimension at most three, algebras with radical cube zero and most algebras which were recently proved to have finite finitistic dimension, see [18] for details. We refer to [18] [19] for other methods to judge when an algebra is Igusa-Todorov.

AC-algebras are algebras satisfying Auslander’s condition. They are studied in detail in [3] recently. Auslander’s condition (AC) for an algebra $R$ asserts that for every finitely generated left $R$-module $M$ there is an integer $n = n_M$, called Auslander bound of $M$, such that $\text{Ext}^i_R(M, N) = 0$ for all $i > n$, whenever $N$ is a finitely generated left $R$-module satisfying $\text{Ext}^i_R(M, N) = 0$ for all but finitely many $i$. Auslander conjectured all finite dimensional algebras satisfy Auslander’s condition (c.f.[3]). However, the conjecture fails by counterexamples firstly given in [10]. A revisited version of Auslander conjecture, named the finitistic Auslander conjecture, asserts that the finitistic Auslander bound of every algebra is finite [17]. Note that the finitistic Auslander conjecture implies the finitistic dimension conjecture. In [17], the generalized Auslander-Reiten conjecture is also formulated, which asserts that for an algebra $R$, if $M$ is a finitely generated left $R$-module such that $\text{Ext}^i_R(M, M \oplus R) = 0$ for all $i \geq n$, then projective dimension of $M$ is at most $n$. In the special case $n = 1$, it is just Auslander-Reiten conjecture [1].

Though we work on artin algebras and finitely generated left modules throughout
This paper, syzygies (resp., cosyzygies) defined here for derived categories make sense for the derived category of any abelian categories with enough projective (resp., injective) objects. It is expected that these notions can have nice applications in more areas.

The paper is organized as follows. In Section 2 we introduce notations used in the paper and recall some basic facts on derived categories. We introduce syzygies for derived categories and study their basic properties in Section 3. In Sections 4 and 5, we prove that derived equivalences preserve syzygy-finite algebras and Igusa-Todorov algebras respectively. In Section 6, we prove that AC-algebras, algebras satisfying the finitistic dimension Auslander conjecture and algebras satisfying the generalized Auslander-Reiten conjecture also have derived invariance.

2 Preliminaries

In this section, we recall some basic definitions and facts which are necessary for our proofs.

Let $R$ be an artin algebra, which means $R$ is a finitely generated $A$-module over a commutative artin ring $A$. We denote by mod$R$ the category of finitely generated left $R$-modules. The subcategory of mod$R$ consisting of projective (resp., injective) modules is denoted by $P_R$ (resp., $I_R$). We denote $fg$ the composition of homomorphisms $f : L \to M$ and $g : M \to N$.

We strengthen that we work on chain complexes. Let $C$ be a class of $R$-modules. A (chain) complex $X$ over $C$ is a set $\{X_i \in C, i \in \mathbb{Z}\}$ equipped with a set of homomorphisms $\{d_X : X_i \to X_{i-1}, i \in \mathbb{Z}|d_X^{i+1}d_X^i = 0\}$. We usually write $X = \{X_i, d_X^i\}$. A chain map $f$ between complexes, say from $X = \{X_i, d_X^i\}$ to $Y = \{Y_i, d_Y^i\}$, is a set of maps $f = \{f_i : X_i \to Y_i\}$ such that $f_id_Y^i = d_X^if_{i-1}$. A complex $X = \{X_i, d_X^i\}$ is right (resp., left) bounded if $X_i = 0$ for all but finitely many negative (resp., positive) integers $i$. A complex $X$ is bounded if it is both left and right bounded, equivalently, $X_i = 0$ for all but finitely many $i$. A complex $X$ is homologically bounded if all but finitely many homologies of $X$ are zero.

Let $I \subseteq \mathbb{Z}$ be an interval. We say that a complex $X = \{X_i, d_X^i\}$ has a homological (resp., representation) interval $I$, if the homologies $H^i(X) = 0$ (resp., terms $X_i = 0$) for all $i \notin I$. We identify an $R$-module with a complex centred on the 0-th term, i.e., a complex with representation interval $[0, 0]$.

The category of all complexes over $C$ with chain maps is denoted by $\mathcal{C}(C)$. The homotopical category of complexes over $C$ is denoted by $\mathcal{K}(C)$. When $C$ is an abelian category, then the derived category of complexes over $C$ is well defined and is denoted by $\mathcal{D}(C)$. The subcategories of $\mathcal{K}(C)$ and $\mathcal{D}(C)$ consisting of bounded (resp., right bounded, left bounded) complexes are denoted by $\mathcal{K}^b(C)$ (resp., $\mathcal{K}^{-b}(C)$, $\mathcal{K}^{+b}(C)$) and $\mathcal{D}^b(C)$ respectively. Let $I$ be an interval. We denote by $\mathcal{K}^{I}(C)$ (resp., $\mathcal{K}^{\geq b}(C)$) and $\mathcal{D}^{I}(C)$ the subcategory of $\mathcal{K}(C)$ and $\mathcal{D}(C)$ consisting of complexes with representation (resp., homological) interval $I$, respectively. Similarly, we denote by $\mathcal{D}^{I}(C)$ the subcategory of $\mathcal{D}(C)$ consisting of complexes with homological interval $I$. For instance, $\mathcal{K}^{I[0,0]}(\mathcal{P}_R)$ is...
just the subcategory \( \mathcal{P}_R \) and \( \mathcal{D}^{[0,0]}(R) \) is just the same as \( \text{mod}R \).

We simply write \( \mathcal{D}(R) \) and \( \mathcal{D}^b(R) \) for \( \mathcal{D}(\text{mod}R) \) and \( \mathcal{D}^b(\text{mod}R) \) respectively. It is well known that \( \mathcal{D}^b(R) \subseteq \mathcal{D}(R) \), \( \mathcal{K}^b(\mathcal{P}_R) \subseteq \mathcal{K}^{b,-}(\mathcal{P}_R) \), \( \mathcal{K}^b(\mathcal{I}_R) \subseteq \mathcal{K}^{b,+}(\mathcal{I}_R) \) are all triangulated categories. For basic results in triangulated categories and derived categories, we refer to [7] and [16]. Moreover, \( \mathcal{D}^b(R) \) is equivalent to both \( \mathcal{K}^{b,-}(\mathcal{P}_R) \) and \( \mathcal{K}^{b,+}(\mathcal{I}_R) \) as triangulated categories.

We denote by \([-] \) the shift functor on complexes. In fact, for a complex \( X \), the complex \( X[1] \) is obtained from \( X \) by shifting \( X \) to the left one degree. The notation add\( X \) denotes the class of all direct summands of finite sums of a complex \( X \).

Two algebras \( R \) and \( S \) are said to be derived-equivalent if \( \mathcal{D}^b(R) \) and \( \mathcal{D}^b(S) \) are equivalent as triangulated categories. Rickard [13] proved that two algebras \( R \) and \( S \) are derived-equivalent if and only if there is a complex \( T \in \mathcal{K}^b(\mathcal{P}_R) \) with \( S \simeq \text{Hom}_{\mathcal{D}(R)}(T, T) \) such that \( \text{Hom}_{\mathcal{D}(R)}(T, T[i]) = 0 \) for all \( i \neq 0 \) and \( \text{add} T \) generates \( \mathcal{K}^b(\mathcal{P}_R) \) as a triangle category. Such complex is called a tilting complex. In fact, if \( F : \mathcal{D}^b(R) \rightleftharpoons \mathcal{D}^b(S) : G \) define an equivalence, then \( T := G(S) \) is just a tilting complex. In particular, if \( T \) is a tilting \( R \)-module, then \( T \) induces a derived equivalence between \( \mathcal{D}^b(R) \) and \( \mathcal{D}^b(\text{End}_R T) \) [7]. We refer to [9] for recent development on constructing derived equivalences.

For an interval \( I \subseteq \mathbb{Z} \), we denote by \( \sigma_I(X) \) the brutal truncated complex which is obtained from the complex \( X \) by replacing each \( X_i \), where \( i \notin I \), with 0. For instance, if \( M \) is a complex in \( \mathcal{K}^{[n,\infty)}(\mathcal{P}_R) \), then \( \sigma_{[n,\infty)}(M) = M \).

Throughout the paper, \( R \) stands for an artin algebra. Homomorphisms and isomorphisms between complexes always means in \( \mathcal{D}(R) \). In particular, restricting to module category, they are just the usual homomorphisms and isomorphisms in module category.

## 3 Syzygies in derived categories

Recall that \( \mathcal{D}^b(R) \) is equivalent to \( \mathcal{K}^{b,-}(\mathcal{P}_R) \) as triangulated categories. For a complex \( M \in \mathcal{D}^b(R) \), a projective resolution of \( M \) is a complex \( P \in \mathcal{K}^{b,-}(\mathcal{P}_R) \) such that \( P \simeq M \) in \( \mathcal{D}(R) \). In case \( M \) is an \( R \)-module, \( P \) is just a usual projective resolution of \( M \).

### Definition 3.1

Let \( M \in \mathcal{D}^b(R) \) and \( n \in \mathbb{Z} \). Let \( P \) be a projective resolution of \( M \). We say that a complex in \( \mathcal{D}^b(R) \) is an \( n \)-th syzygy of \( M \), if it is isomorphic to \( (\sigma_{[n,\infty)}(P))[\neg n] \) in \( \mathcal{D}(R) \). In the case, the \( n \)-th syzygy of \( M \) is denoted by \( \Omega^n_d(M_P) \), or simply by \( \Omega^n_d(M) \) if there is no danger of confusion.

Thus, the \( n \)-th syzygy of a complex \( M \in \mathcal{D}^b(R) \) depends on the choice of projective resolution of \( M \). In this case that \( M \) is an \( R \)-module and \( n > 0 \), the brutal truncated complex \( (\sigma_{[n,\infty)}(P))[\neg n] \) is just the projective resolution of the \( n \)-th syzygy of \( M \) defined by \( P \). Hence, syzygies of \( M \) defined here coincide with the usual syzygies in module category.

We leave to the reader the entire proof of the following lemma.
Lemma 3.2 Let $M, N \in \mathcal{D}^b(R)$ and $n, m, a, k, i$ be integers. Let $P$ be a projective resolution of $M$.

1. $\Omega^n_\mathcal{D}(M) \in \mathcal{D}^{[0,\infty)}(R)$ and $\text{Hom}_\mathcal{D}(Q, \Omega^n_\mathcal{D}(M)[i]) = 0$ for any projective module $Q$ and any integer $i > 0$.

2. If $M \in \mathcal{D}^{[a,k]}(R)$ for some $a \leq k$, then $\Omega^n_\mathcal{D}(M)$ is contained in $\mathcal{D}^{[0,0]}(R)$, $\mathcal{D}^{[0,k-n]}(R)$, $\mathcal{D}^{[a-n,k-n]}(R)$ for cases $n \geq k$, $a \leq n \leq k$, $n \leq a$, respectively. In particular, $\Omega^n_\mathcal{D}(M)$ is isomorphic to an $R$-module and $M \simeq \Omega^n_\mathcal{D}(M)[a]$.

3. $\Omega^{n+m}_\mathcal{D}(M[m]) \simeq \Omega^n_\mathcal{D}(M)$.

4. $\Omega^n_\mathcal{D}(M) \simeq \Omega^n_\mathcal{D}(\Omega^n_\mathcal{D}(M))$, for $m \geq 0$.

5. $\Omega^n_\mathcal{D}(M) \oplus Q$ is also an $n$-th syzygy of $M$, where $Q$ is a projective module.

6. $M \in \mathcal{K}^b(\mathcal{P}_R)$ if and only if any/some syzygy of $M$ is also in $\mathcal{K}^b(\mathcal{P}_R)$.

7. $\Omega^n_\mathcal{D}(M \oplus N) \simeq \Omega^n_\mathcal{D}(M) \oplus \Omega^n_\mathcal{D}(N))$.

Let $f$ be a chain map between complexes, say from $X = \{X_i, d_X^i\}$ to $Y = \{Y_i, d_Y^i\}$. Recall that the cone of $f$, denoted by cone($f$), is a complex such that, for each $i$, $(\text{cone}(f))_i = Y_i \oplus X_{i-1}$, $d_{\text{cone}(f)}^i = (d_Y^i, f_{i-1} - d_X^i)$. It is well known that there is a canonical triangle $X \rightarrow f Y \rightarrow \text{cone}(f) \rightarrow$ in $\mathcal{D}(R)$. Using the construction of cones, one obtains the following canonical triangles in $\mathcal{D}(R)$, for any complex $X$ and any $n \geq m$:

$$
\begin{align*}
(\sigma 1) \quad (\sigma_{[n+1,\infty]}(X))[-1] \rightarrow \sigma_{[m,n]}(X) \rightarrow \sigma_{[m,\infty]}(X) \rightarrow, \\
(\sigma 2) \quad (\sigma_{[m,n]}(X))[-1] \rightarrow \sigma_{[-\infty,m-1]}(X) \rightarrow \sigma_{(-\infty,n]}(X) \rightarrow, \\
(\sigma 3) \quad (\sigma_{[n,\infty]}(X))[-1] \rightarrow X_{(-\infty,n-1]} \rightarrow X. 
\end{align*}
$$

Lemma 3.3 Let $M \in \mathcal{D}^b(R)$. Then there is a triangle $(\Omega^{n+1}_\mathcal{D}(M))[n - m] \rightarrow B \rightarrow \Omega^n_\mathcal{D}(M) \rightarrow$, where $B \in \mathcal{K}^{[0,n-m]}(\mathcal{P}_R)$ and $n \geq m$. In particular, for any $n$, there is a triangle $\Omega^{n+1}_\mathcal{D}(M) \rightarrow Q \rightarrow \Omega^n_\mathcal{D}(M) \rightarrow$ with $Q$ projective.

Proof. Let $P$ be a projective resolution of $M$. Then the first triangle is obtained from $(\sigma 1)$ by shifting, letting $B = (\sigma_{[m,n]}(P))[-m] \in \mathcal{K}^{[0,n-m]}(\mathcal{P}_R)$ in the case. Taking $m = n$, we then obtain the second triangle. \hfill \Box

Recall that two modules $N, N'$ are projectively equivalent if $N \oplus P \simeq N' \oplus P'$ for some projective modules $P, P'$. Let $M$ be a module. It is well known that any two $n$-th syzygies of $M$ are projectively equivalent. It is also the case for syzygies in derived category. We say that two complexes $N, N'$ are projectively equivalent if $N \oplus P \simeq N' \oplus P'$ for some projective modules $P, P'$.

Proposition 3.4 Let $M \in \mathcal{D}^b(R)$ and $P, P'$ be two projective resolutions of $M$. Then $\Omega^n_\mathcal{D}(M_P)$ and $\Omega^n_\mathcal{D}(M_{P'})$ are projectively equivalent.
Proof. We prove by induction on $n$. Assume that $M \in \mathcal{D}^{[a,k]}(R)$ for some integers $a < k$. Then $\Omega^n_{\mathcal{D}}(M_P) \simeq M[-n] \simeq \Omega^n_{\mathcal{D}}(M_{P'})$ in case $n \leq a$, by Lemma 3.2 (2).

Now we consider the case $n + 1$. By the induction assumption, there are projective modules $Q, Q'$ such that $\Omega^n_{\mathcal{D}}(M_P) \oplus Q \simeq \Omega^n_{\mathcal{D}}(M_{P'}) \oplus Q'$. Then we have triangles

$$\Omega^{n+1}_{\mathcal{D}}(M_P) \to P_n \oplus Q \to \Omega^n_{\mathcal{D}}(M_P) \oplus Q, \quad \text{and}$$

$$\Omega^{n+1}_{\mathcal{D}}(M_{P'}) \to P'_n \oplus Q' \to \Omega^n_{\mathcal{D}}(M_{P'}) \oplus Q',$$

where $P_n$ and $P'_n$ are projective, by Lemma 3.3. Since we have a homomorphism between the last terms of these two triangles and $\text{Hom}_{\mathcal{D}}(P_n \oplus Q, \Omega^{n+1}_{\mathcal{D}}(M_{P'})[1]) = 0$ by Lemma 3.2 (1), there is the following commutative diagram:

$$\begin{array}{ccc}
\Omega^{n+1}_{\mathcal{D}}(M_P) & \to & P_n \oplus Q \\
& | & | \\
\Omega^{n+1}_{\mathcal{D}}(M_{P'}) & \to & P'_n \oplus Q'
\end{array}$$

Since the homomorphism in the right column is an isomorphism, we have a canonical triangle

$$\Omega^{n+1}_{\mathcal{D}}(M_P) \to P_n \oplus Q \oplus \Omega^{n+1}_{\mathcal{D}}(M_{P'}) \to P'_n \oplus Q' \to.$$

Note again that $P_n \oplus Q'$ are projective and that $\text{Hom}_{\mathcal{D}}(P_n \oplus Q', \Omega^{n+1}_{\mathcal{D}}(M_{P})[1]) = 0$ by Lemma 3.2 (1), so the above triangle splits. Hence we obtain that

$$\Omega^{n+1}_{\mathcal{D}}(M_P) \oplus (P_n \oplus Q') \simeq (P_n \oplus Q) \oplus \Omega^{n+1}_{\mathcal{D}}(M_{P'}).$$

It follows that $\Omega^{n+1}_{\mathcal{D}}(M_P)$ and $\Omega^{n+1}_{\mathcal{D}}(M_{P'})$ are projectively equivalent. \hfill $\square$

By the above result, we see that the $n$-th syzygy of $M$ is unique up to projectively equivalences. By abuse of language, we speak of the $n$-syzygy of a complex in $\mathcal{D}^b(R)$.

We also have the following easy result. The proof is left to the reader.

**Proposition 3.5** (Dimension shifting) Let $M \in \mathcal{D}^b(R)$. Assume that $N \in \mathcal{D}^{[c,d]}(R)$ for some integers $c \leq d$. Then there is an isomorphism

$$\text{Hom}_{\mathcal{D}}(\Omega^{n+m}_{\mathcal{D}}(M), N[j]) \simeq \text{Hom}_{\mathcal{D}}(\Omega^{n}_{\mathcal{D}}(M), N[j + m])$$

for any integers $n, m, j$ such that $m \geq 1$ and $j > -c$.

The following result provides a way to compare syzygies for complexes in some triangles.

**Lemma 3.6** Let $M \to B \to N$ be a triangle in $\mathcal{D}^b(R)$. If $B \in \mathcal{K}^{(-\infty,k]}(\mathcal{P}_R)$ for some integer $k$, then $\Omega^n_{\mathcal{D}}(M) \simeq \Omega^{k+1}_{\mathcal{D}}(N)$. Hence, $\Omega^n_{\mathcal{D}}(M) \simeq \Omega^{k+1}_{\mathcal{D}}(N)$ for all $n \geq k$.

**Proof.** Let $i : M \to B$ be the homomorphism in the triangle. Assume that $P$ is a projective resolution of $M$, then we have a homomorphism $f : P \to B$ in $\mathcal{K}^{b}(-\mathcal{P}_R)$ followed from the equivalence between $\mathcal{D}^b(R)$ and $\mathcal{K}^{-b}(\mathcal{P}_R)$. Note that cone($f$) $\in \mathcal{K}^{-b}(\mathcal{P}_R)$ and $N \simeq \text{cone}(f)$, i.e., cone($f$) is a projective resolution of $N$. Since $B \in \mathcal{K}^{(-\infty,k]}(\mathcal{P}_R)$, it
is easy to see that $\sigma_{[k+1,\infty)}(\text{cone}(f)) \simeq \sigma_{[k+1,\infty)}(P[1])$, by the construction of cone($f$). It follows that

$$
\Omega_{\mathcal{D}}^{k+1}(N) \simeq (\sigma_{[k+1,\infty)}(\text{cone}(f)))[- (k + 1)]
\simeq (\sigma_{[k+1,\infty)}(P[1]))[- (k + 1)]
\simeq \Omega_{\mathcal{D}}^{k+1}(M[1]) \simeq \Omega_{\mathcal{D}}^{k}(M).
$$

by Lemma 3.2 (3). Hence, the conclusion follows. \hfill \square

For general triangles, we have the following result.

**Proposition 3.7** Let $L \to M \to N \to$ be a triangle in $\mathcal{D}^b(R)$. Then, for any integer $n$, there exists a triangle $\Omega_{\mathcal{D}}^n(L) \to \Omega_{\mathcal{D}}^n(M) \to \Omega_{\mathcal{D}}^n(N) \to$.

**Proof.** The proof is given by induction on $n$. We may assume that $L, M, N \in \mathcal{D}^{[a,k]}(R)$ for some integers $a \leq k$. In case $n \leq a$, we obtain that $\Omega_{\mathcal{D}}^n(L) \simeq L[-n]$, $\Omega_{\mathcal{D}}^n(M) \simeq M[-n]$ and $\Omega_{\mathcal{D}}^n(N) \simeq N[-n]$, by Lemma 3.2 (2). Hence we have a triangle $\Omega_{\mathcal{D}}^n(L) \to \Omega_{\mathcal{D}}^n(M) \to \Omega_{\mathcal{D}}^n(N) \to$ by assumption.

Now consider the case $n + 1$. By Lemma 3.3 we have triangles $\Omega_{\mathcal{D}}^{n+1}(L) \to B \to \Omega_{\mathcal{D}}^{n+1}(L) \to$ and $\Omega_{\mathcal{D}}^{n+1}(N) \to C \to \Omega_{\mathcal{D}}^{n+1}(N) \to$, where $B, C$ are projective. By the induction assumption, there is a triangle $\Omega_{\mathcal{D}}^n(L) \to \Omega_{\mathcal{D}}^n(M) \to \Omega_{\mathcal{D}}^n(N) \to$. Note that $\text{Hom}(\mathcal{D}(R), (\Omega_{\mathcal{D}}^n(L))[1]) = 0$ by Lemma 3.2 (1), so we can obtain the following triangle commutative diagram for some $M' \in \mathcal{D}^b(R)$.

\[
\begin{array}{cccccc}
\Omega_{\mathcal{D}}^{n+1}(L) & \to & M' & \to & \Omega_{\mathcal{D}}^{n+1}(N) & \\
 & & | & & | & \\
 B & \to & B \oplus C & \to & C & \\
 & & | & & | & \\
 & \Omega_{\mathcal{D}}^n(L) & \to & \Omega_{\mathcal{D}}^n(M) & \to & \Omega_{\mathcal{D}}^n(N)
\end{array}
\]

Consider the triangle $M' \to B \oplus C \to \Omega_{\mathcal{D}}^n(M) \to$ from the middle column in the diagram. Since $B \oplus C$ is a projective $R$-module, $B \oplus C \in \mathcal{K}^{[0,0]}(\mathcal{P}_R)$. By Lemmas 3.6 and 3.2 we obtain that $\Omega_{\mathcal{D}}^n(M') \simeq \Omega_{\mathcal{D}}^1(\Omega_{\mathcal{D}}^n(M)) \simeq \Omega_{\mathcal{D}}^{n+1}(M)$. Note that $M' \in \mathcal{D}^{[0,\infty)}(R)$ followed from the triangle $\Omega_{\mathcal{D}}^{n+1}(L) \to M' \to \Omega_{\mathcal{D}}^{n+1}(N) \to$ from the up row in the diagram, so $\Omega_{\mathcal{D}}^n(M') \simeq M'$ by Lemma 3.2 (2). It follows that $M' \simeq \Omega_{\mathcal{D}}^{n+1}(M)$ and hence we have a triangle $\Omega_{\mathcal{D}}^{n+1}(L) \to \Omega_{\mathcal{D}}^{n+1}(M) \to \Omega_{\mathcal{D}}^{n+1}(N) \to$. \hfill \square

There is also the notion dual to syzygies. Recall that $\mathcal{D}^b(R)$ is equivalent to $\mathcal{K}^{+,b}(\mathcal{I}_R)$ as triangulated categories. For a complex $M \in \mathcal{D}^b(R)$, an injective resolution of $M$ is a complex $I \in \mathcal{K}^{+,b}(\mathcal{I}_R)$ such that $I \simeq M$. In case $M$ is an $R$-module, $I$ is just a usual injective resolution of $M$. Using the injective resolution of a complex in $\mathcal{D}^b(R)$, we can define the notion of cosyzygies.

**Definition 3.1** Let $M \in \mathcal{D}^b(R)$ and $n \in \mathbb{Z}$. Let $I$ be an injective resolution of $M$. We say that a complex in $\mathcal{D}^b(R)$ is an $n$-th cosyzygy of $M$, if it is isomorphic to
there is an isomorphism $\text{Hom}_{\mathcal{D}}(\Omega_n^R(M), Q[i]) = 0$ for any injective module $Q$ and any integer $i > 0$.

(2) If $M \in \mathcal{D}^{[a,k]}(R)$, for some $a \leq k$, then $\Omega_n^R(M)$ is contained in $\mathcal{D}^{[0,0]}(R)$, $\mathcal{D}^{[a-n,0]}(R)$, $\mathcal{D}^{[a-n,k-n]}(R)$ for corresponding cases $n \leq a$, $a \leq n \leq k$, $n \geq b$ respectively. In particular, $\Omega_n^R(M)$ is isomorphic to an $R$-module and $M \simeq \Omega_n^R(M)[k]$. 

(3) $\Omega_{n+m}^R(M[m]) \simeq \Omega_n^R(M)$.

(4) $\Omega_{n+m}^R(M) \simeq \Omega_n^R(\Omega_m^R(M))$ for $m \leq 0$.

(5) $\Omega_n^R(M) \oplus Q$ is also an $n$-th cosyzygy of $M$, where $Q$ is any injective module.

(6) $M \in \mathcal{K}^b(I_R)$ if and only if any/some cosyzygy of $M$ is also in $\mathcal{K}^b(I_R)$.

(7) $\Omega_n^R(M \oplus N) \simeq \Omega_n^R(M) \oplus \Omega_n^R(N)$.

Lemma 3.3 Let $M \in \mathcal{D}^b(R)$. Then there is a triangle $\Omega_n^R(M) \rightarrow B \rightarrow (\Omega_{m-1}^R(M))[m-n] \rightarrow$ for some $B \in \mathcal{K}^b[m-n,0](I_R)$, where $n \geq m$. In particular, for any $n$, there is a triangle $\Omega_n^R(M) \rightarrow I \rightarrow \Omega_{n-1}^R(M) \rightarrow$ with $I$ injective.

Proposition 3.4 Let $M \in \mathcal{D}^b(R)$ and $I, I'$ be two injective resolutions of $M$. Then $\Omega_n^R(M_I)$ and $\Omega_n^R(M_{I'})$ are injectively equivalent.

Proposition 3.5 (Dimension shifting) Assume that $N \in \mathcal{D}^{[c,d]}(R)$, where $c \leq d$. Then there is an isomorphism $\text{Hom}_{\mathcal{D}}(N, (\Omega_n^R(M)))[j] \simeq \text{Hom}_{\mathcal{D}}(N, (\Omega_{m+n}^R(M))[j+m])$ for any integers $n, m, j$ such that $m \geq 1$ and $j > -d$.

Lemma 3.6 Let $M \rightarrow B \rightarrow N \rightarrow$ be a triangle in $\mathcal{D}^b(R)$. If $B \in \mathcal{K}^b[a,\infty](I_R)$ for some integer $a$, then $\Omega_a^R(M) \simeq \Omega_{a+1}^R(N)$. Hence, $\Omega_n^R(M) \simeq \Omega_{n+1}^R(N)$ for all $n \leq a$.

Proposition 3.7 Let $L \rightarrow M \rightarrow N \rightarrow$ be a triangle in $\mathcal{D}^b(R)$. Then there is a triangle $\Omega_n^R(L) \rightarrow \Omega_n^R(M) \rightarrow \Omega_n^R(N) \rightarrow$, for any integer $n$.

Let us remark that one can define syzygies (resp., cosyzygies) in the derived category of any abelian category with enough projective (resp., injective) objects.

4 Syzygy-finite algebras

Let $\mathcal{C} \subseteq \mathcal{D}^b(R)$. For an integer $n$, we denote by $\Omega_n^R(\mathcal{C})$ the class of all $n$-th syzygies of complexes in $\mathcal{C}$. The class $\mathcal{C}$ is representation-finite provided that $\mathcal{C} \subseteq \text{add}M$ for some $M \in \mathcal{D}^b(R)$, or equivalently, the number of non-isomorphic indecomposable direct summands of objects in $\mathcal{C}$ is finite. It is easy to see that if $\mathcal{E} \subseteq \mathcal{C}$ and $\mathcal{C}$ is representation-
finite then \( \mathcal{E} \) is also representation-finite. We say that \( \mathcal{C} \) is \( n \)-syzygy-finite provided that \( \bigcup_{i \geq n} \Omega_i^r(\mathcal{C}) \) is representation-finite. By that \( \mathcal{C} \) is syzygy-finite, we mean that \( \mathcal{C} \) is \( n \)-syzygy-finite for some \( n \). It follows from Lemma 3.2 (3) that \( \mathcal{C} \) is syzygy-finite if and only if \( \mathcal{C}[m] \) is syzygy-finite for any/some integer \( m \).

An artin algebra \( R \) is called syzygy-finite if the class \( \text{mod} R \) is syzygy-finite. The following algebras are known to be syzygy-finite.

- Algebras of finite representation type
- Algebras of finite global dimension.
- Monomial algebras [20].
- Left serial algebras [21].

- Torsionless-finite algebras, c.f. [15], including:
  - Algebras \( R \) with \( \text{rad}^n R = 0 \) and \( A/\text{rad}^{n-1} A \) representation-finite.
  - Algebras \( R \) with \( \text{rad}^2 R = 0 \).
  - Minimal representation-infinite algebras.
  - Algebras stably equivalent to hereditary algebras.
  - Right glued algebras and left glued algebras.
  - Algebras of the form \( R/\text{soc} R \) with \( R \) a local algebra of quaternion type.
  - Special biserial algebras.

- Algebras possessing a left idealized extension which is torsionless-finite (indeed 2-syzygy-finite) [18].
- Algebras possessing an ideal \( I \) of finite projective dimension such that \( I\text{rad} R = 0 \) and \( R/I \) is syzygy-finite [19].

The following result gives a characterization of syzygy-finite algebras in term of derived category.

**Theorem 4.1** An algebra \( R \) is syzygy-finite if and only if \( \mathcal{D}^{[a,k]}(R) \) is syzygy-finite for any/some integers \( a \leq k \).

**Proof.** Assume that \( \mathcal{D}^{[a,k]}(R) \) is syzygy-finite for some integers \( a \leq k \). Since \( (\text{mod} R)[a] \subseteq \mathcal{D}^{[a,k]}(R) \), \( (\text{mod} R)[a] \) is syzygy-finite and hence \( \text{mod} R \) is syzygy-finite, i.e., \( R \) is a syzygy-finite algebra.

On the other hand, assume that \( R \) is syzygy-finite. For any integers \( a \leq k \), we have \( \Omega^b(\mathcal{D}^{[a,k]}(R)) \subseteq \text{mod} R \) by Lemma 3.2 (2). Since \( \text{mod} R \) is syzygy-finite, we see that \( \Omega^k(\mathcal{D}^{[a,k]}(R)) \) is syzygy-finite. It follows that \( \mathcal{D}^{[a,k]}(R) \) is syzygy-finite from Lemma 3.2 (4). \( \square \)

**Proposition 4.2** Let \( \mathcal{C}, \mathcal{E} \subseteq \mathcal{D}^b(R) \) and \( \mathcal{B} \subseteq \mathcal{K}(-\infty,k)(\mathcal{P}_R) \) for some \( b \). Assume that, for any \( E \in \mathcal{E} \), there is a triangle \( C_E \to B_E \to E \to \) in \( \mathcal{D}^b(R) \) with \( C_E \in \mathcal{C} \) and \( B_E \in \mathcal{B} \). If \( \mathcal{C} \) is syzygy-finite, then \( \mathcal{E} \) is also syzygy-finite.

**Proof.** Assume that \( \mathcal{C} \) is \( m \)-syzygy-finite, for some \( m \). For any \( E \in \mathcal{E} \), consider the triangle \( C_E \to B_E \to E \to \) in \( \mathcal{D}^b(R) \) with \( C_E \in \mathcal{C} \) and \( B_E \in \mathcal{B} \). Since \( B_E \in \mathcal{B} \), \( B_E \) is syzygy-finite. Thus, \( E \) is syzygy-finite. Since \( \mathcal{C} \) is syzygy-finite, \( \mathcal{E} \) is also syzygy-finite.
Hence, \( \bigcup_{n \geq n+1} \Omega_{\mathcal{G}}(E) \) is representation-finite, for some \( n \geq k \), if and only if the class \( \{ \Omega_{\mathcal{G}}(C_E) \mid E \in \mathcal{E}, i \geq n \} \) is representation-finite. Since \( \{ \Omega_{\mathcal{G}}(C_E) \mid E \in \mathcal{E}, i \geq n \} \subseteq \bigcup_{i \geq n} \Omega_{\mathcal{G}}(C) \) and the latter is representation-finite whenever \( n \geq m \), we obtain that \( \bigcup_{i \geq n+1} \Omega_{\mathcal{G}}(E) \) is representation-finite for \( n = \max\{m, k\} \). It follows that \( \mathcal{E} \) is syzygy-finite.

\[ \blacksquare \]

We need the following result on basic properties of tilting complexes.

**Lemma 4.3** Assume that there is an equivalence \( \mathcal{F} : \mathcal{D}(R) \cong \mathcal{D}(S) : \mathcal{G} \). Let \( T := \mathcal{G}(S) \). Assume that \( T \in \mathcal{K}^{[a,k]}(P_R) \) for some integers \( a \leq k \). Then

1. \( \mathcal{F}(\mathcal{D}^{[c,d]}(R)) \subseteq \mathcal{D}^{[a-d,k-c]}(S) \), for any \( c \leq d \).
2. \( \mathcal{G}(\mathcal{K}^{[c,d]}(P_S)) \subseteq \mathcal{K}^{[a+c,k+d]}(P_R) \), for any \( c \leq d \).

**Proof.** (1) For any \( M \in \mathcal{D}^{[a]}(R) \) and any \( i \), we have isomorphisms:

\[
\text{H}^i(\mathcal{F}(M)) \simeq \text{Hom}_{\mathcal{D}(S)}(S, \mathcal{F}(M)[i]) \\
\quad \simeq \text{Hom}_{\mathcal{D}(R)}(\mathcal{G}(S), \mathcal{G}(\mathcal{F}(M))[i]) \\
\quad \simeq \text{Hom}_{\mathcal{D}(R)}(T, M[i]).
\]

Since \( T \in \mathcal{K}^{[a,k]}(P_R) \) and \( M \in \mathcal{D}^{[c,d]}(R) \), we obtain that \( \text{Hom}_{\mathcal{D}(R)}(T, M[i]) = 0 \) for \( i \notin [a-d, k-c] \). The conclusion then follows.

(2) Take any \( B \in \mathcal{K}^{[c,d]}(P_S) \). Note that there are triangles \( \Omega_{\mathcal{G}}^{i+1}(B) \to P_i \to \Omega_{\mathcal{G}}^{i}(B) \to \) for all integers \( i \), where each \( P_i \) is a projective \( S \)-module, by Lemma 3.3. Since \( B \simeq \Omega_{\mathcal{G}}^{c}(B)[c] \) by Lemma 3.2 (2). Applying the functor \( \mathcal{G} \) to these triangles, we obtain triangles \( \mathcal{G}(\Omega_{\mathcal{G}}^{i+1}(B)) \to \mathcal{G}(P_i) \to \mathcal{G}(\Omega_{\mathcal{G}}^{i}(B)) \to \). Note that \( \mathcal{G}(\Omega_{\mathcal{G}}^{d}(B)), \mathcal{G}(P_i) \in \text{add}(\mathcal{G}(S)) = \text{add} T \subseteq \mathcal{K}^{[a,k]}(P_R) \), so we obtain that \( \mathcal{G}(\Omega_{\mathcal{G}}^{c}(B)) \in \mathcal{K}^{[a+c,k+d]}(P_R) \) from the above triangles, by using the construction of cones. Consequently, we see that \( \mathcal{G}(B) \simeq \mathcal{G}((\Omega_{\mathcal{G}}^{c}(B))[c]) \simeq (\mathcal{G}(\Omega_{\mathcal{G}}^{c}(B)))[c] \in \mathcal{K}^{[a+c,k+d]}(P_R) \).

The following result shows that derived equivalences preserve syzygy-finite classes.

**Proposition 4.4** Assume that there is an equivalence \( \mathcal{F} : \mathcal{D}(R) \cong \mathcal{D}(S) : \mathcal{G} \). Let \( T := \mathcal{G}(S) \in \mathcal{K}^{[a,k]}(P_R) \) and \( C \subseteq \mathcal{D}(S) \). If \( C \) is syzygy-finite, then \( \mathcal{G}(C) \) is also syzygy-finite.

**Proof.** Since \( \bigcup_{i \geq n} \Omega_{\mathcal{G}}(C) \) is representation finite for some \( n \), we have some \( M \in \mathcal{D}(S) \) such that \( \bigcup_{i \geq n} \Omega_{\mathcal{G}}(C) \subseteq \text{add} M \).

**Claim:** \( \mathcal{G}(\bigcup_{i \geq n} \Omega_{\mathcal{G}}(C)) \) is \( k \)-syzygy-finite.

**Proof.** Take any \( C \in C \). Note that, for any \( i \), there is a triangle \( \Omega_{\mathcal{G}}^{i+1}(C) \to C_i \to \Omega_{\mathcal{G}}^{i}(C) \to \), where \( C_i \) is projective, by Lemma 3.3. So, by applying the functor \( \mathcal{G} \), we obtain a triangle \( \mathcal{G}(\Omega_{\mathcal{G}}^{i+1}(C)) \to \mathcal{G}(C_i) \to \mathcal{G}(\Omega_{\mathcal{G}}^{i}(C)) \to \).
Lemma 3.2 (2), so we have a triangle \((\Omega^j_R(\mathcal{G}(\Omega^{j+1}_S(C))) = \Omega^{j+1}_R(\mathcal{G}(\Omega^{j+1}_S(C))) \) for all \( j \geq k \), by Lemma 3.6. It follows that 

\[ \Omega^j_R(\mathcal{G}(\Omega^j_S(C))) \simeq \Omega^{j-1}_R(\mathcal{G}(\Omega^{j+1}_S(C))) \simeq \cdots \simeq \Omega^0_R(\mathcal{G}(\Omega^1_S(C))) \]

for all \( j \geq k \). Hence, by Proposition 4.2, we have that \( \mathcal{G}(\Omega^{j-i_k}_S(C)) \) is syzygy-finite, by Lemma 3.6. Hence, we obtain that the class \( \bigcup_{j \geq k} \Omega^j_R(\mathcal{G}(\bigcup_{i \geq n} \Omega^i_S(C))) = \bigcup_{j \geq k} \Omega^j_R(\mathcal{G}(\bigcup_{i \geq n} \Omega^{j-k+i}_S(C))) \) is representation finite. Since \( \bigcup_{i \geq n} \Omega^i_S(C) \subseteq \text{add} \mathcal{M} \), we see that the last class is contained in the class \( \bigcup_{i \geq n} \Omega^i_S(\mathcal{G}(\text{add} \mathcal{M})) = \Omega^k_S(\mathcal{G}(\text{add} \mathcal{M})) \) and hence is representation finite. The claim then follows.

Now take any \( C \in \mathcal{C} \). Note that there is some \( m_C \) such that \( C[-m_C] \simeq \Omega^{m_C}_S(C) \) by Lemma 3.3 (2), so we have a triangle \((\Omega^n_S(C)) \simeq \mathcal{G}(\Omega^{n-1}_S(C)) \) for all \( n \geq m_C \). Then we obtain a triangle \( \mathcal{G}((\Omega^n_S(C)) \simeq \mathcal{G}(\Omega^{n-1}_S(C)) \) for all \( n \geq m_C \). Hence, \( \mathcal{G}(\Omega^n_S(C)) \) is syzygy-finite, by Lemma 3.2 (3). Hence, by Proposition 4.4. Thus, \( \mathcal{G}(C) \) is syzygy-finite.

\[ \square \]

Now we prove that derived equivalences preserve syzygy-finite algebras.

**Theorem 4.5** Assume that \( R, S \) are derived equivalent algebras. If \( S \) is syzygy-finite, then \( R \) is also syzygy-finite.

**Proof.** By assumption, there is an equivalence \( \mathcal{F} : \mathcal{D}^b(R) \simeq \mathcal{D}^b(S) : \mathcal{G} \). Assume that 

\[ T := \mathcal{G}(S) \in \mathcal{K}^{a[k]}(\mathcal{P}_R) \]. Then \( \mathcal{F}((\Omega^n_S(C)) \simeq \mathcal{G}(\Omega^{n-1}_S(C))) \), by Lemma 1.3 (1). If \( S \) is syzygy-finite, then \( \mathcal{D}^{a[k]}(S) \) is syzygy-finite by Theorem 4.4. It follows that \( \mathcal{F}(\text{mod} R) \) is syzygy-finite. Hence, we obtain that \( \text{mod} R = \mathcal{G}(\mathcal{F}(\text{mod} R)) \) is syzygy-finite, by Proposition 4.4. Thus, \( R \) is syzygy-finite.

\[ \square \]

For example, if \( R \) is derived equivalent to a minimal representation-infinite algebra or a monomial algebra, then \( R \) is syzygy-finite by the above theorem. In particular, \( R \) is CM-finite in the case.

Note that derived equivalences preserve Gorenstein algebras and that syzygy-finiteness coincides with CM-finiteness for Gorenstein algebras, we obtain the following corollary.

**Corollary 4.6** Assume that \( R, S \) are derived equivalent algebras. If \( S \) is Gorenstein CM-finite, then \( R \) is also Gorenstein CM-finite.

### 5 Igusa-Todorov algebras

Recall that an artin algebra \( R \) is called \( n \)-**Igusa-Todorov** provided that there exists a fixed \( R \)-module \( V \) and a nonnegative integer \( n \) such that, for any \( M \in \text{mod} R \), there is
an exact sequence $0 \rightarrow V_1 \rightarrow V_0 \rightarrow \Omega^n_C M \rightarrow 0$ with $V_0, V_1 \in \text{add} V$. A remarkable property of Igusa-Todorov algebras that they satisfy the finitistic dimension conjecture.

The following algebras are Igusa-Todorov.

- Algebras with radical cube zero.
- Algebras with representation dimension at most three.
- Syzygy-finite algebras.
- Algebras which are endomorphism algebras of modules over representation-finite algebras.
- Algebras with an ideal $I$ of finite projective dimension such that $I\text{rad}^2R = 0$ (or $I^2\text{rad}R = 0$) and $R/I$ is syzygy-finite.
- Algebras possessing a left idealized extension which is 2-syzygy-finite.
- Algebras which are endomorphism algebras of projective modules over 2-Igusa-Todorov algebras.
- Algebras with representation dimension at most three.
- Algebras with radical cube zero.

Let $C$ be a subclass of $\mathcal{D}^b(R)$. We say that $C$ is relative hereditary provided that there is a complex $V \in \mathcal{D}^b(R)$ such that, for any $M \in C$, there is a triangle $V_1 \rightarrow V_0 \rightarrow M \rightarrow$ with $V_0, V_1 \in \text{add} V$. We say that $C$ is an $n$-Igusa-Todorov class, for some integer $n$, provided that $\Omega^n_C(C)$ is relative hereditary. It is easy to see that $C$ is an Igusa-Todorov (resp., relative hereditary) class if and only if $C[n]$ is an Igusa-Todorov (resp., relative hereditary) class for any/some $n$. It is also obvious that if $E \subseteq C$ and $C$ is Igusa-Todorov (resp., relative hereditary) then $E$ is also Igusa-Todorov (resp., relative hereditary).

**Lemma 5.1** Let $C \subseteq \mathcal{D}^b(R)$. Then $C$ is an Igusa-Todorov class if and only if $\Omega^n_C(C)$ is an Igusa-Todorov class for any/some $n$.

**Proof.** We first prove that if $\Omega^n_C(C)$ is relative hereditary, then $\Omega^n_C(C)$ is also relative hereditary, for any $n \geq m$. In fact, by definition there is a complex $V \in \mathcal{D}^b(R)$ such that, for any $M \in \Omega^n_C(C)$, there is a triangle $V_1 \rightarrow V_0 \rightarrow M \rightarrow$ with $V_0, V_1 \in \text{add} V$. By Proposition 3.1, we have a triangle $\Omega^{n-m}_C(V_1) \rightarrow \Omega^{n-m}_C(V_0) \rightarrow \Omega^{n-m}_C(M) \rightarrow$. Note that $\Omega^{n-m}_C(V_1), \Omega^{n-m}_C(V_0) \in \Omega^{n-m}_C(\text{add} V)$ is independent of $M$, and $\Omega^n_C(C) = \Omega^{n-m}_C(\Omega^n_C(C))$ for $n \geq m$ by Lemma 3.2, so we obtain that $\Omega^n_C(C)$ is relative hereditary, for any $n \geq m$, by definition.

Now assume that $C$ is an $m$-Igusa-Todorov algebra. Then $\Omega^n_C(C)$ is relative hereditary. Let $n$ be an integer, and take an integer $t > \max\{0, m - n\}$, then we have that $\Omega^n_C(\Omega^{n+t}_C(C)) = \Omega^{n+t}_C(C)$ is relative hereditary, by the above argument. It follows that $\Omega^n_C(C)$ is an Igusa-Todorov algebra by definition. Conversely, assume that $\Omega^n_C(C)$ is a $t$-Igusa-Todorov algebra, for some integers $n, t$, then $\Omega^{n+t}_C(C)$ is relative hereditary. Hence $C$ is an Igusa-Todorov algebra by definition. □

The following result gives a characterization of Igusa-Todorov algebras in term of derived categories.
Theorem 5.2 An algebra $R$ is Igusa-Todorov if and only if $\mathcal{D}^{[a,k]}(R)$ is an Igusa-Todorov class for any/some $a \leq k$.

Proof. If $R$ is an Igusa-Todorov algebra, then $\text{mod}R$ is an Igusa-Todorov class by definition. Note that $\text{mod}R = \Omega^k_\mathcal{D}(\mathcal{D}^{[a,k]}(R))$, so we further have that $\mathcal{D}^{[a,k]}(R)$ is an Igusa-Todorov class, by Lemma 5.1.

Conversely, assume that $\mathcal{D}^{[a,k]}(R)$ is an Igusa-Todorov class, then $\text{mod}R$ is also an Igusa-Todorov class by Lemma 5.1 again. Thus there is a fixed complex $V \in \mathcal{D}^b(R)$ and an integer $v$ such that, for any $M \in \text{mod}R$, there is a triangle $V_1 \to V_0 \to \Omega^v_\mathcal{D}(M) \to$ with $V_1, V_0 \in \text{add}V$. Assume that $V \in \mathcal{D}^{[c,d]}(R)$ for some $c \leq d$, then $\Omega^1_\mathcal{D}(V) \in \text{mod}R$ for all $n \geq \max\{d, 0\}$, by Lemma 3.2 (2). By Proposition 3.7 and Lemma 3.2 (4), there is a triangle $\Omega^a_\mathcal{D}(V_1) \to \Omega^a_\mathcal{D}(V_0) \to \Omega^a_\mathcal{D}(\Omega^v_\mathcal{D}(M)) \to$. Take the integer $n$ such that $n \geq \max\{0, d, -v\}$, all terms in the last triangle are $R$-modules. Hence, we have an exact sequence $0 \to \Omega^a_\mathcal{D}(V_1) \to \Omega^a_\mathcal{D}(V_0) \to \Omega^{a+v}_\mathcal{D}(M) \to$. Note that $\Omega^a_\mathcal{D}(V_1), \Omega^a_\mathcal{D}(V_0) \in \Omega^a_\mathcal{D}(\text{add}V) \subseteq \text{mod}R$ and $n + v$ are independent of $M$, so we obtain that $R$ is an Igusa-Todorov algebra by definition.

The following result shows that derived equivalences preserve Igusa-Todorov classes.

Proposition 5.3 Assume that there is an equivalence $\mathcal{F} : \mathcal{D}^b(R) \cong \mathcal{D}^b(S) : \mathcal{G}$. Let $T := \mathcal{G}(S) \in \mathcal{K}^{[a,k]}(\mathcal{P}_R)$ and $\mathcal{C} \subseteq \mathcal{D}^b(S)$. If $\mathcal{C}$ is an Igusa-Todorov class, then $\mathcal{G}(\mathcal{C})$ is also an Igusa-Todorov class.

Proof. Assume that $\mathcal{C}$ is an $n$-Igusa-Todorov class for some $n$, then there is a fixed complex $V \in \mathcal{D}^b(R)$ such that, for any $C \in \mathcal{C}$, there is a triangle $V_1 \to V_0 \to \Omega^n_\mathcal{D}(C) \to$ with $V_1, V_0 \in \text{add}V$. Hence, we also have a triangle $\mathcal{G}(V_1) \to \mathcal{G}(V_0) \to \mathcal{G}(\Omega^n_\mathcal{D}(C)) \to$, by applying the functor $\mathcal{G}$. Similarly as in the proof of Proposition 4.3, there is a triangle $\mathcal{G}((\Omega^n_\mathcal{D}(C))[n-1]) \to \mathcal{G}(B) \to \mathcal{G}(C) \to$ with $\mathcal{G}(B) \in \mathcal{K}^{(-\infty,k+n-1)}(\mathcal{P}_R)$. So, we have that

$$
\Omega^{k+n}_\mathcal{D}(\mathcal{G}(C)) \simeq \Omega^{k+n-1}_\mathcal{D}(\mathcal{G}((\Omega^n_\mathcal{D}(C))[n-1])) \\
\simeq \Omega^{k+n-1}_\mathcal{D}(\mathcal{G}((\Omega^n_\mathcal{D}(C))[n-1])) \simeq \Omega^{k}_\mathcal{D}(\mathcal{G}((\Omega^n_\mathcal{D}(C))),
$$

by Proposition 3.6 and Lemma 3.2 (3). Note that we have a triangle $\Omega^k_\mathcal{D}(\mathcal{G}(V_1)) \to \Omega^k_\mathcal{D}(\mathcal{G}(V_0)) \to \Omega^{k+n}_\mathcal{D}(\mathcal{G}(C)) \to$, by Proposition 3.7 and the above argument. Since $b + n$ and $\Omega^b_\mathcal{D}(\mathcal{G}(V_1)), \Omega^b_\mathcal{D}(\mathcal{G}(V_0)) \in \Omega^b_\mathcal{D}(\text{add}V)$ are independent of $C$, so we obtain that $\Omega^{k+n}_\mathcal{D}(\mathcal{G}(C))$ is relative hereditary, i.e., $\mathcal{G}(C)$ is a $(k+n)$-Igusa-Todorov class.

Now we obtain the following important result on Igusa-Todorov algebras.

Theorem 5.4 Assume that $R, S$ are derived equivalent algebras. If $S$ is an Igusa-Todorov algebra, then $R$ is also an Igusa-Todorov algebra.

Proof. The proof is similar to that of Theorem 4.5. Namely, if the equivalence is given by $\mathcal{F} : \mathcal{D}^b(R) \cong \mathcal{D}^b(S) : \mathcal{G}$, then we can obtain that $\mathcal{F}(\text{mod}R)$ is an Igusa-Todorov class provided that $S$ is an Igusa-Todorov algebra. Hence, we obtain that $\text{mod}R = \mathcal{D}^{[a,k]}(R)$ is an Igusa-Todorov algebra for some $a \leq k$. 

$$
\Omega^1_\mathcal{D}(R) \subseteq \mathcal{D}^{[a,k]}(R) \subseteq \mathcal{D}^{[b,n]}(R).
$$

Therefore, $\mathcal{D}^{[b,n]}(R)$ is also a derived equivalent class.
$\mathcal{G}(\mathcal{F}(\text{mod} R))$ is an Igusa-Todorov class, by Proposition 5.3. Thus, $R$ is an Igusa-Todorov algebra by Theorem 5.2. □

For instance, if $R$ is derived equivalent to an algebra with radical cube zero, or an algebra with representation dimension at most three, then $R$ must be an Igusa-Todorov algebra.

6 Auslander’s condition

Let $M \in D^b(R)$ and $C \subseteq D^b(R)$. We define the $C$-Auslander bound of $M$ to be the minimal integer $m$, or $\infty$ if such minimal integer doesn’t exist, such that $\text{Hom}_{D^b(R)}(M, N[i]) = 0$ for all $i > m$, whenever $N \in C$ satisfies that $\text{Hom}_{D^b(R)}(M, N[i]) = 0$ for all but finitely many $i$. Let $E \subseteq D^b(R)$. The global $C$-Auslander bound of $E$ is the supremum of all $C$-Auslander bounds of objects in $E$. The finitistic $C$-Auslander bound of $E$ is the supremum of all $C$-Auslander bounds of objects in $E$ whose $C$-Auslander bound is finite.

In case that $M$ is an $R$-module and $C = E = \text{mod} R$, the notions given here coincide with the usual ones in module categories [3, 17], i.e., Auslander bound of $M$, global Auslander bound of the algebra $R$ and the finitistic Auslander bound of the algebra $R$, respectively.

We have the following easy observation.

Lemma 6.1 Let $M \in D^b(R)$ and $C \subseteq D^b(R)$. Then $M$ has finite $C$-Auslander bound if and only if $M[m]$ has finite $(C[n])$-Auslander bound for any/some integers $m, n$.

It is also easy to see that if a complex $M$ has finite $C$-Auslander bound, then $M$ also has finite $E$-Auslander bound for any $E \subseteq C$.

An algebra $R$ is called an AC-algebra provided that every $R$-module has finite $(\text{mod} R)$-Auslander bound [3]. Auslander conjecture asserts that all algebras are AC-algebras. However, the conjecture fails in general [10,12]. We refer to [3] for the list of AC-algebras. In [17], the author suggests a revisited version of Auslander conjecture, named the finitistic Auslander conjecture, which asserts that the finitistic Auslander bound of every algebra is finite. Note that the finitistic Auslander conjecture implies the finitistic dimension conjecture.

We have the following characterization of AC-algebras in term of derived categories.

Theorem 6.2 (1) $R$ is an AC-algebra if and only if every complex in $D^b(R)$ has finite $(D^{[c,d]}(R))$-Auslander bound for any/some integers $c \leq d$.

(2) The global Auslander bound of $R$ is finite if and only if, for any/some integers $a \leq k$ and $c \leq d$, the global $(D^{[c,d]}(R))$-Auslander bound of $D^{[a,k]}(R)$ is finite.

(3) The finitistic Auslander bound of $R$ is finite if and only if, for any/some integers $a \leq k$ and $c \leq d$, the finitistic $(D^{[c,d]}(R))$-Auslander bound of $D^{[a,k]}(R)$ is finite.
Proof. (1) The if-part. Assume that every complex in $\mathcal{D}^b(R)$ has finite $(\mathcal{D}^{[c,d]}(R))$-Auslander bound for some integers $c \leq d$. Then every complex has finite $((\text{mod}R)[c])$-Auslander bound since $(\text{mod}R)[c] \subseteq \mathcal{D}^{[c,d]}(R)$. It follows that every complex, in particular every $R$-module, has finite $(\text{mod}R)$-Auslander bound by Lemma 6.1. Hence, $R$ is an AC-algebra.

The only-if part. Take any $M \in \mathcal{D}^b(R)$ and any integers $c \leq d$. Assume that $M \in \mathcal{D}^{[a,k]}(R)$ for some integers $a \leq k$, then we have that $\Omega^b_{\mathcal{D}}(M)$ is an $R$-module and $M \simeq \Omega^a_{\mathcal{D}}(M)[a]$, by Lemma 3.2 (2). Now take any $N \in \mathcal{D}^{[c,d]}(R)$, then we obtain that
\[
\text{Hom}_{\mathcal{D}(R)}(M, N[j]) \simeq \text{Hom}_{\mathcal{D}(R)}(\Omega^b_{\mathcal{D}}(M)[a], N[j]) \\
\simeq \text{Hom}_{\mathcal{D}(R)}(\Omega^a_{\mathcal{D}}(M), N[j - a]) \simeq \text{Hom}_{\mathcal{D}(R)}(\Omega^b_{\mathcal{D}}(M), N[j - k])
\]
for all $j > k - c$, by dimension shifting in Proposition 3.5. Now note that $\Omega^b_{\mathcal{D}}(M)$ is an $R$-module and $N = \Omega^a_{\mathcal{D}}(N)[d]$ by Lemma 3.2 (2), so we further obtain that
\[
\text{Hom}_{\mathcal{D}(R)}(\Omega^k_{\mathcal{D}}(M), N[j - k]) \simeq \text{Hom}_{\mathcal{D}(R)}(\Omega^k_{\mathcal{D}}(M), (\Omega^b_{\mathcal{D}}(N))[j - k + d])
\]
for all $j > k - c$, by dimension shifting in Proposition 3.5. It follows that
\[
\text{Hom}_{\mathcal{D}(R)}(M, N[j]) \simeq \text{Hom}_{\mathcal{D}(R)}(\Omega^b_{\mathcal{D}}(M), (\Omega^b_{\mathcal{D}}(N))[j - k + c]),
\]
for all $j > k - c$.

Hence, if $\text{Hom}_{\mathcal{D}(R)}(M, N[j]) = 0$ for all but finitely many $j$, then we have that $\text{Hom}_{\mathcal{D}(R)}(\Omega^k_{\mathcal{D}}(M), (\Omega^b_{\mathcal{D}}(N))[i]) = 0$ for all but finitely many $i$. Since $R$ is an AC-algebra and $\Omega^k_{\mathcal{D}}(M), \Omega^b_{\mathcal{D}}(N) \subseteq \text{mod}R$, there is an integer $l \geq 0$, independent of $\Omega^b_{\mathcal{D}}(N)$, such that $\text{Hom}_{\mathcal{D}(R)}(\Omega^k_{\mathcal{D}}(M), (\Omega^b_{\mathcal{D}}(N))[i]) = 0$ for all $i > l$. It follows from the above isomorphism that $\text{Hom}_{\mathcal{D}(R)}(M, N[j]) = 0$ for all $j$ such that $j - k + c > l$ and $j > k - c$, i.e., for all $j > k - c + l$. Note that $k - c + l$ is independent of $N$, so we obtain that $M$ has finite $(\mathcal{D}^{[c,d]}(R))$-Auslander bound.

(2) and (3). The proofs are similar to (1), just note that Auslander bounds are unique in both cases. □

Then, we obtain an important property of AC-algebras and algebras satisfying the finitistic Auslander conjecture.

Theorem 6.3 Assume that $R, S$ are derived equivalent algebras.

(1) If $S$ is an AC algebra, then $R$ is also an AC algebra.

(2) If the global Auslander bound of $S$ is finite, then the global Auslander bound of $R$ is also finite.

(3) If the finitistic Auslander bound of $S$ is finite, then the finitistic Auslander bound of $R$ is also finite.

Proof. (1) Take any $M, N \in \text{mod}R$. If $\text{Hom}_{\mathcal{D}(R)}(M, N[i]) = 0$ for all but finitely many $i$, then $\text{Hom}_{\mathcal{D}(S)}(\mathcal{F}(M), \mathcal{F}(N)[i]) = 0$ for all but finitely many $i$. Note that $\mathcal{F}(N) \in \mathcal{F}(\text{mod}S) \subseteq \mathcal{D}^{[a,k]}(S)$ for some fixed integer $a \leq k$, by Lemma 4.3. Since $S$ is an AC
algebra, we have that \( F(M) \in \mathcal{D}^b(S) \) has finite \( (\mathcal{D}^{a,k}(S)) \)-Auslander bound by Theorem 6.3. Then there is an integer \( m \), independent of \( N \), such that \( \text{Hom}_{\mathcal{D}(S)}(F(M), F(N)[i]) = 0 \) for all \( i > m \). It follows that \( \text{Hom}_{\mathcal{D}(R)}(M, N[i]) = \text{Hom}_{\mathcal{D}(R)}(GF(M), GF(N)[i]) = 0 \) for all \( i > m \), i.e., \( M \) has finite \((\text{mod}R)\)-Auslander bound. Hence, \( R \) is an AC-algebra.

(2) and (3). The proofs are similar. \( \square \)

Now we turn to algebras satisfying the generalizedAuslander-Reiten conjecture. We note that an equivalent statement of the generalized Auslander-Reiten conjecture is that if \( M \) is an \( R \)-module such that \( \text{Hom}_{\mathcal{D}(R)}(M, (M \oplus R)[i]) = 0 \) for all but finitely many \( i \), then \( M \) is of finite projective dimension, i.e., \( M \) is (isomorphic to) a complex in \( \mathcal{K}^b(\mathcal{P}_R) \).

**Lemma 6.4** The following conditions are equivalent for a complex \( M \in \mathcal{D}^b(R) \).

1. \( \text{Hom}_{\mathcal{D}(R)}(M, (M \oplus R)[i]) = 0 \) for all but finitely many \( i \).
2. \( \text{Hom}_{\mathcal{D}(R)}(\Omega^m_{\mathcal{D}}(M), (\Omega^m_{\mathcal{D}}(M) \oplus R)[i]) = 0 \) for any/some integer \( n \) and all but finitely many \( i \).
3. \( \text{Hom}_{\mathcal{D}(R)}(M, (M \oplus T)[i]) = 0 \) for any/some tilting complex \( T \) and all but finitely many \( i \).

**Proof.** (1) \( \Leftrightarrow \) (2) Note that \( M = \Omega^m_{\mathcal{D}}(M)[m] \) for some integer \( m \), so we obtain that \( \text{Hom}_{\mathcal{D}(R)}(\Omega^m_{\mathcal{D}}(M), (\Omega^m_{\mathcal{D}}(M) \oplus R)[i]) = 0 \) for all but finitely many \( i \) if and only if \( \text{Hom}_{\mathcal{D}(R)}(M, (M \oplus R)[i]) = 0 \) for all but finitely many \( i \). By Proposition 3.5, there is a triangle \( \Omega^j_{\mathcal{D}}(M) \to P_j \to \Omega^j_{\mathcal{D}}(M) \to \) with \( P_j \) projective, for any \( j \). So, one can easily check that \( \text{Hom}_{\mathcal{D}(R)}(\Omega^j_{\mathcal{D}}(M), (\Omega^j_{\mathcal{D}}(M) \oplus R)[i]) = 0 \) for all but finitely many \( i \) if and only if \( \text{Hom}_{\mathcal{D}(R)}(\Omega^{j+1}_{\mathcal{D}}(M), (\Omega^{j+1}_{\mathcal{D}}(M) \oplus R)[i]) = 0 \) for all but finitely many \( i \). The conclusion then follows.

(2) \( \Leftrightarrow \) (3) If \( T \) is a tilting complex, then \( R \) generates \( T \) and \( T \) generates \( R \) in \( \mathcal{K}^b(\mathcal{P}_R) \), both via finitely many steps. Also note that \( \text{Hom}_{\mathcal{D}(R)}(B, M[i]) = 0 \) for all but finitely many \( i \), whenever \( B \in \mathcal{K}^b(\mathcal{P}_R) \). It follows that \( \text{Hom}_{\mathcal{D}(R)}(M, (M \oplus T)[i]) = 0 \) for all but finitely many \( i \) if and only if \( \text{Hom}_{\mathcal{D}(R)}(M, (M \oplus R)[i]) = 0 \) for all but finitely many \( i \). \( \square \)

Now we can provide a derived version of the generalized Auslander-Reiten conjecture.

**Theorem 6.5** An algebra \( R \) satisfies the generalized Auslander-Reiten conjecture if and only if, for any \( M \in \mathcal{D}^b(R) \) such that \( \text{Hom}_{\mathcal{D}(R)}(M, (M \oplus T)[i]) = 0 \) for any/some tilting complex \( T \) and all but finitely many \( i \), it holds that \( M \in \mathcal{K}^b(\mathcal{P}_R) \).

**Proof.** The sufficient part follows from Lemma 6.4.

The necessary part. Note that there is some \( n \) such that \( \Omega^n_{\mathcal{D}}(M) \) is an \( R \)-module, by Lemma 3.2 (2). If \( M \) satisfies that \( \text{Hom}_{\mathcal{D}(R)}(M, (M \oplus R)[i]) = 0 \) for some tilting complex and all but finitely many \( i \), then we have that \( \text{Hom}_{\mathcal{D}(R)}(\Omega^n_{\mathcal{D}}(M), (\Omega^n_{\mathcal{D}}(M) \oplus R)[i]) = 0 \) for all but finitely many \( i \), by Lemma 6.4. Since \( R \) satisfies the generalized Auslander-Reiten conjecture, we obtain the \( \Omega^n_{\mathcal{D}}(M) \in \mathcal{K}^b(\mathcal{P}_R) \). It follows that \( M \in \mathcal{K}^b(\mathcal{P}_R) \), by Lemma 3.2 (6). \( \square \)
Then, we can show that derived equivalences preserve generalized Auslander-Reiten conjecture.

**Theorem 6.6** Assume that $R, S$ are derived equivalent algebras. If $S$ satisfies the generalized Auslander-Reiten conjecture, then $R$ also satisfies the generalized Auslander-Reiten conjecture.

**Proof.** Take any $M \in \mathcal{D}^b(R)$ such that $\text{Hom}_{\mathcal{D}^b(R)}(M, (M \oplus R)[i]) = 0$ for all but finitely many $i$. Then $\text{Hom}_{\mathcal{D}^b(S)}(\mathcal{F}(M), (\mathcal{F}(M) \oplus \mathcal{F}(R))[i]) = 0$ for all but finitely many $i$. Note that $\mathcal{F}(R)$ is a tilting complex in $\mathcal{D}^b(S)$ and that $S$ satisfies the generalized Auslander-Reiten conjecture, so we obtain that $\mathcal{F}(M) \in \mathcal{K}^b(\mathcal{P}_S)$ by Theorem 6.5. It follows that $M \simeq \mathcal{G}\mathcal{F}(M) \in \mathcal{K}^b(\mathcal{P}_R)$. Hence $R$ satisfies the generalized Auslander-Reiten conjecture by Theorem 6.5 again. 

\[\square\]

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