A bosonic bright soliton in a mixture of repulsive Bose–Einstein condensate and polarized ultracold fermions under the influence of pressure evolution

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Abstract

Repulsive Bose–Einstein condensates, where short-range interaction is included up to the third order by the interaction radius, demonstrate the existence of bright solitons in a narrow interval of parameters. These solitons are studied here for the boson–fermion mixture, where spin-1/2 fermions are considered in the regime of full-spin polarization. The influence of fermions on bosonic bright solitons via the boson–fermion interaction is considered up to the third order by the interaction radius. Fermions themselves are considered within the hydrodynamic model, which includes the kinetic pressure-evolution equation. The interactions between fermions are also considered. The first order by the interaction radius makes a zero contribution to the Euler equation and the kinetic pressure evolution equation for fermions, but the third order by the interaction radius provides nonzero contributions to both equations. The repulsive (attractive) boson–fermion interaction leads to bright (dark) fermionic solitons.

Keywords: bright solitons, hydrodynamics, nonlocal interaction, boson–fermion mixtures, pressure evolution equation

(Some figures may appear in colour only in the online journal)

1. Introduction

Solitons [1–3], vorticities [4–6], and skyrmions [7] are the fundamental non-linear excitations of quantum gases. Quantum droplet formation is also detected in Bose–Einstein condensates (BECs) of rare-earth atoms, due to the dipole–dipole interaction and quantum fluctuations [8–10]. This work is focused on solitons in boson–fermion mixtures [11], which are studied in terms of the quantum hydrodynamic model. Boson–fermion mixtures are experimentally obtained in different combinations, such as $^7$Li–$^6$Li [12, 13], $^{23}$Na–$^6$Li [14], and $^{87}$Rb–$^{40}$K [15]. A comprehensive review of boson–fermion mixtures is given in [16].

The minimal coupling model for BECs is the Gross–Pitaevskii equation obtained in the mean-field limit or its hydrodynamic representation consisting of the continuity and Euler equations [17]. There are also beyond-mean-field models of BECs, as mentioned below. Moreover, in this paper,
we use the beyond-mean-field approximation obtained from quantum hydrodynamics.

Fermions can show different behaviors, depending on the spin polarization and the interaction of fermions with different spin projections. BEC–BCS (Bardeen-Cooper-Schrieffer) crossover in two-component fermion systems is an actively studied branch in the field of ultracold fermions. Sanner et al [18] considered two-component fermions with a strong repulsive interaction between fermions with different spin projections. It was demonstrated that pairing instability was faster than ferromagnetic instability in this regime. However, the average and weak repulsions showed the stable coexistence of fermions with different spin projections. The interactions of fermions of the same spin projection was not discussed in their paper. The metastable Stoner-like ferromagnetic phase, supported by the strong repulsion of fermions with opposite spin projections in excited scattering states was studied in [19].

A mass-imbalanced Fermi–Fermi mixture of $^{161}$Dy and $^{40}$K was created to study the strongly interacting regime [20]. The existence of a second sound in the systems of one-dimensional fermions with repulsive interactions was reported in [21], where the hydrodynamic equations for four conserved macroscopic characteristics of the fluid were presented for a Luttinger liquid with a linear excitation spectrum [22–24]. This theoretical result was in agreement with the earlier experimental observation of the first and second sounds in $^6$Li atoms [25].

Single-state ultracold fermions are also being studied at the moment. Single-spin state-degenerate fermions of $^6$Li are created when confined by heavy bosonic atoms of $^{133}$Cs at the attractive interspecies interaction [26]. Rakshit et al [27] demonstrated that higher-order corrections to the standard mean-field energy were able to lead to the formation of Bose–Fermi liquid droplets for attractive BECs and spin-polarized fermions. A hydrodynamic approach, including a set of two non-linear Schrödinger equations, was used in [27]. A weakly interacting single-component two-dimensional dipolar Fermi gas was considered in [28] to study the zero sound.

If the Bose–Einstein condensate (BEC) is considered in terms of the Gross–Pitaevskii equation [17], it reveals a bright soliton where there is an attraction between bosons or a dark soliton in repulsive BECs. Noninteracting degenerate fermions also demonstrate the existence of a soliton [29], where a pair of bright and dark solitons is obtained in the case of trapped fermions. However, the extended model, with a nonlocal short-range interaction considered up to the third order by the interaction radius, leads to the formation of a bright soliton (previously called the ‘bright-like’ soliton) in repulsive BECs [30]. The bright soliton in repulsive BECs is studied for anisotropic short-range interactions [31], and also considered up to the third order by the interaction radius. Fermionic bright solitons have been found at the attractive boson–fermion interaction, as described in [32, 33]. Fermionic dark solitons were obtained in [12]. Bright solitons in a Fermi superfluid described by the Bogoliubov–de Gennes theory of BEC-to-BCS crossover using a random-phase approximation were considered in [34], where soliton decay via the snake instability was studied. Mixtures of two BECs have also been studied. Bright–bright, dark–dark and dark–bright types of soliton have theoretically been obtained in two-component BECs [35, 36]. Soliton dynamics beyond the mean-field approximation were studied in [37–40]. Dark solitons are associated with Bose–Einstein condensates [38] and bright-dark solitons with two-component quantum gases [37]. Many-body quantum dynamics, including correlation dynamics were also considered in that paper.

The possibility that the boson–fermion interaction in the third order by the interaction radius approximation leads to the formation of a new soliton is discussed in [41], where it corresponds to a step beyond the mean-field approximation. However, here we report the advanced version of the quantum hydrodynamics of fermions. Therefore, the results of [41] can be readdressed in terms of a novel model, where fewer assumptions are made for the force field in the Euler equation. Moreover, the kinetic pressure evolution equation is included here. The presence of the pressure evolution equation becomes especially important, since the force field contains a pressure tensor. Furthermore, the second-interaction constants for each type of interatomic potential are not represented via the scattering length, but they are considered as independent constants, following from their definitions.

In this paper, we present a microscopic derivation of the quantum hydrodynamic model for boson–fermion mixtures, where fermions are located in a single spin state. The general structure of the equations for balancing the particle number, the momentum and the momentum flux is obtained. The two-particle short-range boson–boson, boson–fermion, and fermion–fermion interactions are included in the model. General forms are found for the interactions in the momentum balance equations and the momentum flux-evolution equations. The weakly interacting limit of the interaction terms for bosons in the BEC state and degenerate fermions is derived up to the third order by the interaction radius. In this regime, the mixture is characterized by five interaction constants. A small-amplitude non-linear evolution of the collective excitations is considered to find the bright–bright and bright–dark solitons in the boson–fermion mixtures. These solitons exist purely due to the interaction constants present in the third order by the interaction radius. These solitons are the solitons of concentrations of bosons and fermions. The velocity field of the bosons and fermions together with the diagonal elements of kinetic pressure reveal the soliton structure.

This paper is organized as follows. In section 2, the major steps of the derivation of the hydrodynamic equations from the Schrödinger equation are demonstrated. In section 3, the quantum hydrodynamic equations for the BEC and ultracold spin-polarized fermions are obtained. Both species are described up to the third order by the interaction radius. In section 4, the method for the approximate non-linear solution of the hydrodynamic equations is presented. In section 5, a numerical analysis of the obtained Korteweg–de Vries equation is shown. In section 6, the obtained results are summarized.
2. On the derivation of hydrodynamic equations from microscopic quantum dynamics

2.1. The general form of the quantum hydrodynamic equations for boson–fermion mixtures

A boson–fermion mixture consisting of $N$ particles, which is the superposition of a number of bosons, $N_b$, and a number of fermions, $N_f$, is described by the Schrödinger equation with the following Hamiltonian:

$$
\hat{H} = \sum_{i=1}^{N} \left( \frac{\mathbf{p}_i^2}{2m_i} + V_{\text{ext}}(\mathbf{r}_i, t) \right) + \frac{1}{2} \sum_{i,j \neq i} U(\mathbf{r}_i - \mathbf{r}_j),
$$

(1)

where $m_i$ is the mass of the $i$th particle, and $\mathbf{p}_i = -i\hbar \nabla_i$ is the momentum of the $i$th particle. The last term in the Hamiltonian (1) is a short representation of boson–boson $(1/2) \sum_{i=1}^{N_b} \sum_{j \neq i} U_{bb}(\mathbf{r}_i - \mathbf{r}_j)$ interparticle interaction, fermion–fermion $(1/2) \sum_{i=1}^{N_f} \sum_{j \neq i} U_{ff}(\mathbf{r}_i - \mathbf{r}_j)$ interparticle interaction, and boson–fermion $(1/2) \sum_{i=1}^{N_b} \sum_{j \neq i} U_{bf}(\mathbf{r}_i - \mathbf{r}_j)$ interparticle interaction. The Schrödinger equation $i\hbar \partial_t \Psi(R, t) = H \Psi(R, t)$ with Hamiltonian (1) describes the evolution of the wave function of the full boson–fermion mixture $\Psi(R, t)$, where the full configuration space $R = \{R_b, R_f\}$ is a combination of the configurational space of the bosons, $R_b$, and the configurational space of the fermions, $R_f$. Following quantum mechanics, all nonrelativistic quantum systems are described by the Schrödinger equation, where the wave function is required to be symmetric for permutations of bosons, and antisymmetric for permutations of fermions. It is correct for arbitrary temperatures, including the zero-temperature limit.

However, there are formulations of the many-body problem for fermions with no interaction between fermions in the same spin state, see, for instance, equation (23) in [42]. This is due to the argument that interactions are strongly inhibited by the Pauli exclusion principle. The antisymmetry of the many-particle wave function is a manifestation of the Pauli exclusion principle. However, we do not use the properties of the wave function in this step. Mentioning some of our results here, we point out that the first order by the interaction radius (an analog of the s-wave) contribution for the fermion–fermion interaction is equal to zero, due to the antisymmetry of the many-particle wave function for permutations of fermions. However, the third order of the interaction radius (an analog of the p-wave) terms is non-zero.

At this stage, bosons and fermions have arbitrary distributions in quantum states. A transition to near-equilibrium states with zero temperatures is made at a later stage of derivation, where the chain of equations is truncated.

The concentration (number density) of bosons is defined as the quantum-mechanical average of the concentration operator, which is a superposition of the delta functions [43, 44]:

$$
n_b(\mathbf{r}, t) = \int d\mathbf{r} \sum_{i=1}^{N_b} \delta(\mathbf{r} - \mathbf{r}_i) \Psi^\ast(\mathbf{r}, t) \Psi(\mathbf{r}, t),
$$

(2)

where $d\mathbf{r} = dR_d dR_b$, $dR_b = \prod_{i=1}^{N_b} d\mathbf{r}_i$ are elements of volume in 3$N_b$-dimensional configuration space, where $N_b$ is the number of bosons, $dR_f = \prod_{i=1}^{N_f} d\mathbf{r}_i$ is the element of volume in 3$N_f$-dimensional configuration space, and $N_f$ is the number of fermions. We need to integrate over the coordinates of all particles, since the wave function describes both species.

The definition of the fermion concentration has a similar structure

$$
n_f(\mathbf{r}, t) = \int d\mathbf{r} \sum_{i=N_b+1}^{N_b+N_f} \delta(\mathbf{r} - \mathbf{r}_i) \Psi^\ast(\mathbf{r}, t) \Psi(\mathbf{r}, t),
$$

(3)

but the fermion concentration operator contains the coordinates of a different set of particles.

Considering the time evolution of each concentration, $n_a$, via the evolution of the wave function $\Psi(R, t)$, we find the continuity equation [43, 44]:

$$
\partial_t n_a + \nabla \cdot \mathbf{j}_a = 0,
$$

(4)

where the subindex $a$ stands for $b$ or $f$, and the current, $\mathbf{j}_a$, is defined via the many-particle wave function of the system:

$$
\mathbf{j}_a(\mathbf{r}, t) = \int d\mathbf{r} \sum_{i=N_a}^{N_b} \delta(\mathbf{r} - \mathbf{r}_i) \times \frac{1}{m_a} (\Psi^\ast(\mathbf{r}, t) \mathbf{p}_i \Psi(\mathbf{r}, t) + c.c.),
$$

(5)

where $c.c.$ is the complex conjugation.

Next, we derive the equation for the current evolution. We consider the time derivative of the current (5) using the Schrödinger equation and some straightforward calculations. As a result, we find the current-evolution equation (it can be also called the momentum-balance equation)

$$
\partial_t \mathbf{j}_a + \mathbf{J}_a \Pi_a^\beta = -\frac{1}{m_a} n_a \partial_a V_{\text{ext}} + \frac{1}{m_a} \mathbf{F}_a^\alpha \Pi_a^\beta,
$$

(6)

where

$$
\Pi_a^\beta = \int d\mathbf{r} \sum_{i=N_a}^{N_b} \delta(\mathbf{r} - \mathbf{r}_i) \frac{1}{4m_i^2} [\Psi^\ast(\mathbf{r}, t) \mathbf{p}_i^\ast \mathbf{p}_i^\beta \Psi(\mathbf{r}, t) + \mathbf{p}_i^\ast \Psi^\ast(\mathbf{r}, t) \mathbf{p}_i^\beta \Psi(\mathbf{r}, t) + c.c.]
$$

(7)

is the momentum flux (containing the kinetic pressure tensor), and the force field

$$
\mathbf{F}_a^\alpha = -\sum_{a'=b,f} \int (\partial^\alpha U_{aa'}(\mathbf{r} - \mathbf{r}')) n_{a',a'}(\mathbf{r}, \mathbf{r}', t) d\mathbf{r}',
$$

(8)

with the following expression for the two-particle concentration
The suggested model includes the kinetic pressure evolution equation for fermions. It is not required for bosons, since bosons are considered to be in the Bose–Einstein state. Therefore, the kinetic pressure of the bosons equals zero and the bosons are completely described by the concentration and the velocity field. Therefore, it is necessary to derive the equation for the momentum-flux evolution, since the momentum flux has a clear relation to the wave function \((7)\), similar to the concentrations and the currents \((5)\). The pressure evolution will be extracted from the momentum-flux evolution below. Similarly to the derivation of the current-evolution equation, we consider the time derivative of the momentum flux \((7)\) using the Schrödinger equation \([45]\):

\[
\frac{\partial \Pi_{\gamma}^{\alpha}}{\partial t} + \partial_\gamma M_{\gamma}^{\alpha \beta} = \frac{1}{m_j} \partial_\gamma \psi_{\text{ext}} - \frac{1}{m_j} \partial_\beta \psi_{\text{ext}}
\]

\[
-\frac{1}{m_j} \int [\partial^\gamma U(\mathbf{r} - \mathbf{r}')] \partial_\gamma \psi_{\text{ext}} d\mathbf{r}'
\]

\[
-\frac{1}{m_j} \int [\partial^\beta U(\mathbf{r} - \mathbf{r}')] \partial_\beta \psi_{\text{ext}} d\mathbf{r}',
\]

\[\text{where}\]

\[
M_{\gamma}^{\alpha \beta} = \int d\mathbf{R} \sum_{\mathbf{r}_i \in N_i} \delta(\mathbf{r} - \mathbf{r}_i) \frac{1}{8m_i^2} \left[ \psi^*(\mathbf{R}, t) \hat{p}_i^\alpha \hat{p}_i^\beta \psi(\mathbf{R}, t) \right.

\[+ \hat{p}_i^\alpha \psi^*(\mathbf{R}, t) \hat{p}_i^\beta \psi(\mathbf{R}, t) + \hat{p}_i^\beta \psi^*(\mathbf{R}, t) \hat{p}_i^\alpha \psi(\mathbf{R}, t) \right] + c.c.
\]

\[\psi(\mathbf{R}, t) \quad \text{and} \quad \hat{p}_i^\alpha \psi^*(\mathbf{R}, t) \hat{p}_i^\beta \psi(\mathbf{R}, t) + c.c. \] (11)

is the flux of tensor \(\Pi^{\alpha \beta}\), and

\[
\mathbf{j}_2(\mathbf{r}, \mathbf{r}', t) = \int d\mathbf{R} \sum_{\mathbf{r}_i \in N_i \cap N_j \neq i} \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{r}' - \mathbf{r}_j)
\]

\[
\times \frac{1}{2m_i} \left[ \psi^*(\mathbf{R}, t) \hat{p}_i \psi(\mathbf{R}, t) + c.c. \right]. \] (12)

If the quantum correlations are dropped, function \(j_2^\alpha(\mathbf{r}, \mathbf{r}', t)\) splits into the product of the current, \(j_2^\alpha(\mathbf{r}, t)\), and the concentration, \(n_t(\mathbf{r}', t)\). In general, one cannot neglect quantum correlations in quantum gases. We do not neglect them in our calculations. However, we provide a simplified form of function \(j_2\) (12) to understand its structure.

Equations (4), (6) and (10) are the fundamental equations for a collection of bosons and fermions. These equations contain a number of new functions that should be expressed via the basic hydrodynamic functions. The truncation is to be made for bosons in the BEC state and fermions at a temperature of zero but collected in a single spin state.

Number density (the concentration) applies to stable particles. Moreover, it applies to mixtures where the processes of ionization–recombination or creation–annihilation are in progress. The velocity field refers to the change in the number of particles in some area of space. In quantum systems, it also refers to a change of probability of finding a particle in some area of space. The type of statistic (bosons or fermions) does not affect the physical picture presented.

Moreover, it is necessary to present the hydrodynamic equations (4), (6) and (10) in a more traditional form. To this end, we introduce the velocity field \(\mathbf{v}_a = \mathbf{J}_2/n_a\). This definition allows us to represent the continuity equation in a traditional form. However, the other equations require more detailed descriptions, which can be found in [44, 45]. The method of introduction of the velocity field includes an analysis of the of the velocity deviations of quantum particles introduced as gradients \((\hbar \nabla S/m)\) of the wave function phase \(\Psi(\mathbf{R}, t) = a(\mathbf{R}, t) \exp(iS(\mathbf{R}, t))\) from the velocity field. These deviations also include thermal effects and other mechanisms (e.g., the Fermi surface caused by Pauli blocking) of the distribution of particles in quantum states with velocities shifted from the average velocity, \(v_a\). This method provides the structure of the momentum flux tensor:

\[
\Pi_{\gamma}^{\alpha \beta} = n_a \Psi^* \left( \frac{\partial_{\gamma} n_a}{n_a} + \frac{\partial_{\beta} \mathbf{v}_a}{n_a} \right). \] (13)

The first and second terms on the right-hand side of equation (13) have classical meaning and include the kinetic pressure tensor, \(\Psi^{\alpha \beta}\). The last term in equation (13) has a quantum nature and can be presented in the following approximate form:

\[
T^{\alpha \beta}_a = -\frac{\hbar^2}{4m_i^2} \left[ \partial_{\alpha} \partial_{\beta} n_a - \frac{\partial_{\gamma} n_a \cdot \partial_{\beta} n_a}{n_a} \right]. \] (14)

It is related to the quantum Bohm potential.

Equation (14) appears for the noninteracting bosons in the BEC state. Its linear part (the first term) is straightforward for interacting bosons or interacting fermions, while the second term in equation (14) has no proper justification for fermions, even if their interactions are neglected. Hence, for fermions, the second term in equation (14) is an approximate equation of state.

A representation for the tensor \(M_{\gamma}^{\alpha \beta}\) (11), similar to (14), can also be found. It is given in [45] as a set of four rather large equations (see equations (25)–(28)).

Equations are obtained for particles with arbitrary spin, but a further analysis is performed for spin-0 bosons and spin-1/2 fermions in the single spin-projection state.

Equations (4)–(10) are derived for the arbitrary potential \(U_{ij} = U(\mathbf{r}_i - \mathbf{r}_j)\). It is necessary to specify that neutral particles interact via their short-range potential. To stress the small radius of interaction, we represent the coordinates of interacting particles \(\mathbf{r}_i\) and \(\mathbf{r}_j\) via their relative distance and the coordinates of their centers of mass. Next, we can expand the delta functions and wave function in the interaction-related terms (8), (9), (12) and the last two terms in equation (10) for
the small interparticle distance $r_{ij} = r_i - r_j$, since the potential $U_{ij}$ is zero at large interparticle distances. The straightforward calculations for weakly interacting particles include the symmetry between fermions, between bosons, and the absence of symmetry between bosons and fermions (for more details see [44, 45]). The terms in the zeroth order for the interparticle distance cancel each other. In the first order, there are nonzero terms for boson–boson and boson–fermion interactions, which correspond to the Gross–Pitaevskii approximation. The integral over the interparticle distance contains the interaction potential and gives the interaction constant for each interaction. The fermion–fermion interaction term in the first order equals zero due to the antisymmetry of the wave function. Formally, we have the first interaction constant for fermions, but it is multiplied by the function, which is equal to zero. The second-order terms vanish due to integration over the interparticle distance (its angular dependence). The third order of expansion gives nonzero results for all three interactions. The second interaction constant appears for each interaction.

The derivation of the fundamental hydrodynamic equations is performed by the many-particle quantum hydrodynamic method [31, 43, 44]. Regarding the development of hydrodynamic methods, we should mention that generalized hydrodynamics has been actively developing in recent years [46, 47].

2.2. Hydrodynamics in the third order by the interaction radius

Equations (2)–(12) were obtained for the general case, with no assumptions made about the strength and distance of interactions. Below, these equations are simplified on account of two assumptions.

First, it is assumed that we are considering neutral atoms that interact via short-range interaction. Here, for dilute gases, the range of interaction is assumed to be small in comparison with the average interparticle distance or in comparison with the characteristic length of the collective excitations.

This assumption yields a small dimensionless expansion parameter. Therefore, the terms containing interaction in the hydrodynamic equations, such as the force field (8) and the flux of the force field on the right-hand side of equation (10), can be expanded using the small parameter.

We consider four terms of the expansion. The zeroth and second orders of the expansion yield a value of zero. We obtain nonzero contributions from the first and third terms of the expansion. The terms appearing in the first (third) order of the expansion are called the first (third) order by the interaction radius approximation. We should mention that the first-order expansion gives a zero value for the fermion–fermion interaction of fermions in quantum states with the same spin projection.

No assumptions about the temperature of the system were made in the derivation of equations (2)–(12). Hence, they are obtained for arbitrary temperatures. The expansion on the small radius of interaction is also made at arbitrary temperatures. Nevertheless, the final hydrodynamic equations presented below are obtained in the limit of zero temperature, $T = 0$.

There are different methods for modeling temperature. Many theoretical approaches to the statistical physics of near-equilibrium states are based on the Gibbs statistics for isothermal processes. Hence, the system is located in the thermostat. However, the basic definition of temperature is the average kinetic energy of the chaotic motion of particles. In the general case, we have quantum systems that are not in a state of equilibrium. We introduce the velocity of local ordered motion (the velocity field); deviation from this motion is the chaotic motion of particles. The energy of the chaotic motion is the temperature. Therefore, the temperature is not a parameter of the medium. The temperature is a scalar field, like the concentration, which changes in space and time under influence of processes in the system. In particular, the kinetic pressure of bosons is equal to zero at a temperature of zero, while the kinetic pressure of fermions reduces to the Fermi pressure in the limit of zero temperature. However, arbitrary many-particle wave functions lead to different values of pressure for bosons versus fermions.

The basic Schrödinger equation describes the evolution of quantum particles which evolve via different quantum states. This is the microscopic (mechanical) level of description, where the notion of temperature does not exist. Temperature is introduced on the macroscopic scale, to describe a quantum medium in terms of its collective variables. In general, quantum systems exist at nonzero temperatures.

In particularly, the classical regime (i.e. classical mechanics or classical hydrodynamics) is the high-temperature limit of quantum theory (quantum mechanics or quantum hydrodynamics), where thermal effects dominate over the quantum effects. The quantum hydrodynamic equations (17)–(19) presented below have no classical limit since they are obtained in the zero temperature limit, which is the opposite regime to classical hydrodynamics.

The application of the short-range interaction limit gives the intermediate form of the hydrodynamic equations. To get the final form, the second assumption is used, which is the assumption of the weak interaction. All the assumptions described are related to the form of the quantum hydrodynamic equations presented below in section 3.

The solution of the hydrodynamic equations given in section 4 applies an additional assumption. The limit of the small amplitude of the non-linear excitations is considered there. This regime is similar to the assumption that allows the Bogoliubov spectrum of BECs to be obtained [17], but here, the non-linear corrections are included as well.

3. Hydrodynamic equations for a boson–fermion mixture

Here, we present the approximate form of the quantum hydrodynamic equations derived above. Equations (15)–(19) are applicable at a temperature of zero for neutral particles with weak short-range interactions.

In this regime, we have two continuity equations:

$$\partial_t n_b + \nabla \cdot (n_b v_b) = 0,$$

$$\partial_t n_f + \nabla \cdot (n_f v_f) = 0.$$
The Euler equation for bosons:

\[ \partial_t n_f + \nabla \cdot (n_f v_f) = 0. \]  

We also have two Euler (momentum balance) equations:

\[ m_b n_b (\partial_t + v_b \cdot \nabla) v_b^b = \frac{\hbar^2}{2 m_b} \nabla^2 n_b + \frac{g_b n_b \partial^{\alpha} n_b}{2} + \frac{1}{2} g_{2b} \partial^{\alpha} \Delta n_b^b = -n_b \partial^{\alpha} v_{\text{ext}}^b \]

\[ -g_{bf} n_b \partial^{\alpha} n_f - \frac{g_{2bf}}{2} n_b \partial^{\alpha} \Delta n_f \]  

contains the boson–boson interaction in the first and third order by the interaction radius, corresponding to terms proportional to the \( g_b \) and \( g_{2b} \) constants. It includes the boson–fermion interaction terms, which are proportional to \( g_{bf} \) in the first order and \( g_{2bf} \) in the third order.

The Euler equation for fermions is

\[ m_f n_f (\partial_t + v_f \cdot \nabla) v_f^f = \frac{\hbar^2}{2 m_f} \nabla^2 n_f + \partial^{\alpha} p_f^\alpha = -g_{bf} n_f \partial^{\alpha} n_b - \frac{g_{2bf}}{2} n_f \partial^{\alpha} \Delta n_b + g_{2bf} \frac{m_f^2}{2 \hbar^2} b_0^{\alpha \beta \gamma \delta} \partial^\beta (n_f p_f^\gamma), \]  

where all fermions are in quantum states with the same spin projection. Euler equation (18) has the contribution of the boson–fermion interaction. Similarly to the Euler equation for bosons (17), they are proportional to \( g_{2bf} \) in the first order by the interaction radius and to \( g_{2bf} \) in the third order. The fermion–fermion interaction requires a single term in the Euler equation (18). It appears in the third order in the form of the interaction radius being proportional to the \( g_{2bf} \) constant.

The third order by the interaction radius approximation has a similarity to the p-wave interaction. The p-wave fermion–fermion interaction is studied in boson–fermion mixtures as part of the study of solitons in mixtures [48]. The traditional p-wave approximation assumes an equation of state for kinetic pressure in terms of concentration. However, our model gives a more accurate analysis of pressure via the pressure-evolution equation.

The boson–boson interaction in the third order by the interaction radius is presented by the nonlocal interaction term, showing similarity with the models presented in \[ \text{Laser Phys.} \, 31 \, (2021) \, 015501 \, P \, A \, Andreev \].

The pressure-evolution equation (19) appears from equation (10) after extraction of the thermal components or other mechanisms for the distribution of particles in momentum space, such as Pauli blocking for degenerate fermions. Equation (19) has no trace of the external potential and the boson–fermion interaction. The fermion–fermion interaction gives a nonzero contribution in the third order by the interaction radius. It consists of a structure of two terms, which is repeated to give it a symmetric form on free indexes, since the pressure tensor \( p_f^{\alpha \beta} \) is symmetric. Let us have a closer look at each of the two terms. One is highly non-linear and includes the product of three velocities: \( n_f^2 v_f^\alpha v_f^\beta \). Another term is proportional to the pressure tensor. Moreover, it is proportional to the classical hydrodynamic vorticity \( \varepsilon^{\gamma \delta \alpha} \Omega^\gamma = \partial^\gamma v_f^\delta - \partial^\delta v_f^\gamma \), where \( \Omega^\gamma = \varepsilon^{\alpha \beta \gamma} \partial_\beta v_f^\gamma \) is the vorticity of a classic uncharged fluid.

The left-hand side of equation (19) contains the divergence of a third-rank tensor \( Q_f^{\alpha \beta \gamma} \) which is the average of the product of three thermal velocities (the thermal part of the tensor \( M_f^{\alpha \beta \gamma} \) (11)), while \( p_f^{\alpha \beta} \) is the average of the product of two thermal velocities (the velocities relative to the local center of mass). It is assumed to be equal to zero. It is the equation of state obtained as an extension of the equilibrium value of the tensor, \( Q_f^{\alpha \beta \gamma} \).

The boson part of the model is developed in [31, 44]. The fermion part of the hydrodynamic model is derived in [45, 51]. Here, as in [51], the kinetic pressure-evolution equation is considered in the long-wavelength limit, so high-order derivatives are neglected. The interspecies interaction is addressed in terms of the many-particle quantum hydrodynamic method in [44, 45].

### 3.1. Comparison with other models

The presented equations (15)–(19) contain two generalizations. One accounts for the third order by the interaction radius. The second generalization is made for fermions. It is a consideration of the kinetic pressure tensor as an independent function, while it is mostly expressed via concentration, using the Fermi pressure as the equation of state to truncate the set of hydrodynamic equations.

Consider equations (15)–(19) in the first order by the interaction radius. To make this approximation, we need to drop all terms containing the second-interaction constants \( g_{2b}, g_{bf}, \) and \( g_{2bf} \). As a result, we have

\[ \partial_t n_b + \nabla \cdot (n_b v_b) = 0, \]  

\[ m_b n_b (\partial_t + v_b \cdot \nabla) v_b^b = \frac{\hbar^2}{2 m_b} \nabla^2 n_b + \frac{1}{2} g_{2b} \partial^{\alpha} \Delta n_b^b = -n_b \partial^{\alpha} v_{\text{ext}}^b + \frac{g_{2bf}}{2} n_b \partial^{\alpha} \Delta n_f \]  

for bosons, and

\[ \partial_t n_f + \nabla \cdot (n_f v_f) = 0, \]  

where

\[ g_{2bf} = \int r^2 U_{bf} \, d\mathbf{r}, \]  

and \( g_{2bf} = \int r^2 U_{bf} \, d\mathbf{r}, \) and \( g_{2bf} = (1/24) \int r^2 U_{bf} \, d\mathbf{r}, \) the scattering lengths are defined via atom–atom interactions. They give the relation between the interaction constant \( g \) and the scattering length \( a: \) \( g = 4\pi \hbar^2 a / m. \)

The pressure-evolution equation (19) appears from equation (10) after extraction of the thermal components or other mechanisms for the distribution of particles in momentum space, such as Pauli blocking for degenerate fermions. Equation (19) has no trace of the external potential and the boson–fermion interaction. The fermion–fermion interaction gives a nonzero contribution in the third order by the interaction radius. It consists of a structure of two terms, which is repeated to give it a symmetric form on free indexes, since the pressure tensor \( p_f^{\alpha \beta} \) is symmetric. Let us have a closer look at each of the two terms. One is highly non-linear and includes the product of three velocities: \( n_f^2 v_f^\alpha v_f^\beta \). Another term is proportional to the pressure tensor. Moreover, it is proportional to the classical hydrodynamic vorticity \( \varepsilon^{\gamma \delta \alpha} \Omega^\gamma = \partial^\gamma v_f^\delta - \partial^\delta v_f^\gamma \), where \( \Omega^\gamma = \varepsilon^{\alpha \beta \gamma} \partial_\beta v_f^\gamma \) is the vorticity of a classic uncharged fluid.
\((\partial_t + v_f \cdot \nabla) v_f^a - \frac{\hbar^2}{2m_f^2} \partial^a \sqrt{\frac{m_f}{\hbar}} + \frac{\partial^b_p p_f^b}{m_f} = - \frac{g_{sf}}{m_f} \partial^a n_b, \) 
\[ (23) \]

\[ \partial_t p_f^{a\beta} + v_f^\gamma \partial_\gamma p_f^{a\beta} + p_f^{\alpha\gamma} \partial_\gamma v_f^\beta + p_f^{\alpha\beta} \partial_\gamma v_f^\gamma = 0 \]
\[ (24) \]

\( \) for fermions. 

The current form of the hydrodynamic equations for bosons \((20), \) \((21)\), given the additional conditions of the curl-free velocity field \( v_b = \nabla \phi_b\), can be represented as the Gross–Pitaevskii equation \([17]\)

\[ \hbar \delta_t \phi_b = \left( -\frac{\hbar^2}{2m_f} + V_{\text{ext}} + g_b \left| \phi_b \right|^2 + g_{bf} n_f \right) \phi_b \]
\[ (25) \]

where the macroscopic wave function has the following form:

\( \phi_b = \sqrt{n_b} \exp(\im \epsilon \phi_b / \hbar) \).

If we consider the first order by the interaction radius for fermions, we have the pressure-evolution equation \((24)\). We can make an additional assumption that the pressure corresponds to the equilibrium expression via the concentration \( p_f^{a\beta} = p_{0f} \delta^{a\beta} \), where \( p_{0f} = \left( \frac{6\pi^2}{\hbar} \right)^{1/3} n_f^0 \hbar^2 / 5m \) is the Fermi pressure and \( \delta^{a\beta} \) is the Kronecker symbol. Hence, the Euler equation \((23)\) simplifies to

\[ \left( \partial_t + v_f \cdot \nabla \right) v_f^a - \frac{\hbar^2}{2m_f^2} \partial^a \sqrt{\frac{m_f}{\hbar}} + \frac{\partial^b p_f^b}{m_f} + \frac{6\pi^2}{2m_f^2} \nabla n_f^{a/3} + \frac{g_{sf}}{m_f} \partial^a n_b = 0. \]
\[ (26) \]

This Euler equation \((26)\) corresponds to the models used in the literature \([52–55], \) \([56–59]\), since equation \((26)\) can be represented as the non-linear Schrödinger equation under the additional conditions of the curl-free velocity field, \( v_f = \nabla \phi_f \).

4. Perturbation method

Here we deal with a specific soliton solution, where there is a bright bosonic soliton at the repulsive boson–boson interaction \([30] \). It appears due to the interaction in the third order by the interaction radius. Similar phenomena are found in \([60] \) for bosons, but we consider this phenomenon for a boson–fermion mixture. Wang et al \([60] \) used a three-particle interaction for the interpretation of their experimental data. Following papers \([30] \) and \([31] \), we use the reductive perturbation method \([61, 62] \) to study solitons in boson–fermion mixtures. According to this method, all hydrodynamic values may be represented as:

\[ n_b = n_{0b} + \epsilon n_{1b} + \epsilon^2 n_{2b} + \ldots, \]
\[ (27) \]

\[ n_f = n_{0f} + \epsilon n_{1f} + \epsilon^2 n_{2f} + \ldots, \]
\[ (28) \]

\[ v_f^a = \epsilon v_{1f} + \epsilon^2 v_{2f} + \ldots, \]
\[ (29) \]

\[ v_f^a = \epsilon v_{1f} + \epsilon^2 v_{2f} + \ldots, \]
\[ (30) \]

\[ p_f^{a\beta} = p_{0f}^{a\beta} + \epsilon p_{1f}^{a\beta} + \epsilon^2 p_{2f}^{a\beta} + \ldots, \]
\[ (31) \]

where \( i \) stands for \( xx, yy \) and \( zz \), since all diagonal elements of the pressure tensor are involved in the dynamics of longitudinal perturbations. It is assumed that the equilibrium concentrations and fermion pressure are nonzero and constant. The velocity fields are equal to zero in equilibrium. We also performed the following ‘scaling’ of variables:

\[ \xi = \epsilon^{1/2}(x - V_t), \]
\[ (32) \]

\[ \tau = \epsilon^{3/2} V_t. \]
\[ (33) \]

The latter expression introduces so-called ‘slow’ time. The parameter epsilon is an indicator of scale.

4.1. First-order perturbations

We substitute the scaling of hydrodynamic functions \((27)–(31)\) and space-time variables \((32), \) \((33)\) into the basic equations \((15)–(19)\). Separate contributions appear in different orders on the parameter \( \epsilon \). We extract equations in the lowest order for the parameter \( \epsilon \) and find the continuity equation for bosons

\[ n_{0b}\partial_t v_{1b} = V\partial_{\xi} n_{1b} = 0, \]
\[ (34) \]

the Euler equation for bosons

\[ m_b V\partial_{\xi} v_{1b} - g_{bf}\partial_{\xi} n_{1f} - g_{bf}\partial_{\xi} n_{1f} = 0, \]
\[ (35) \]

the continuity equation for fermions

\[ n_{0f}\partial_t v_{1f} = V\partial_{\xi} n_{1f} = 0, \]
\[ (36) \]

the Euler equation for fermions

\[ m_{nf} V\partial_{\xi} v_{1f} + \partial_{\xi} p_{1f}^{\alpha\beta} = - g_{bf} n_{0b}\partial_{\xi} n_{1b} + \frac{m^2}{2\hbar^2} \partial_{\xi} [n_{0f}(3p_{1f}^{\alpha\beta} + p_{1f}^{\alpha\beta} + p_{1f}^{\alpha\beta}) + (3p_{1f}^{\alpha\beta} + p_{1f}^{\alpha\beta} + p_{1f}^{\alpha\beta}) n_{1f}], \]
\[ (37) \]

and equations for the evolution of the elements of the kinetic pressure tensor

\[ V\partial_{\xi} p_{1f}^{\alpha\beta} - 3p_{1f}^{\alpha\beta}\partial_{\xi} v_{1f} = 0, \]
\[ (38) \]

and

\[ V\partial_{\xi} p_{1f}^{\alpha\beta} - p_{1f}^{\alpha\beta}\partial_{\xi} v_{1f} = 0, \]
\[ (39) \]

where the \( zz \) element is the same as the \( yy \) element. The Fermi surface in the equilibrium regime is assumed to be a sphere. Therefore, we have \( p_{1f}^{\alpha\beta} = p_{1f}^{\alpha\beta} = p_{1f}^{\alpha\beta} \equiv p_{1f}. \)
After expansion, assuming that the partial velocity of bosons–fermion interaction is small, we can expand the square root. The influence of fermions on the boson solution is important for the repulsive interaction between bosons. Consider the limit of the Bogoliubov spectrum. The velocity squared in equation (30) has the following form:

\[ V^2 = \frac{n_{0b} \delta_b}{m_b} + \frac{3p_0f}{m_f n_f} + \frac{8p_0 m_f}{\hbar^2} \]

\[ \pm \sqrt{\left( \frac{n_{0b} \delta_b}{m_b} - \frac{3p_0f}{m_f n_f} - \frac{8p_0 m_f}{\hbar^2} \right)^2 + \frac{4n_{0b} n_0 g_0^2}{m_b m_f}}. \]  

where ‘+’ corresponds to perturbations in the system of bosons affected by fermions and ‘-’ corresponds to perturbations in the system of fermions affected by bosons.

Choosing the ‘+’ sign in front of the square root and dropping the fermion contribution, we get

\[ V_b^2 = \frac{n_{0b} \delta_b}{m_b}. \]  

(41) dominates over the partial velocity of fermions \( V_f^2 = 3p_0f/m_f n_f + 8g_0 p_0 m_f / \hbar^2 \), we find the following expression

\[ V^2 = \frac{n_{0b} \delta_b}{m_b} + \frac{n_{0b} n_0 g_0^2}{m_b m_f} \]  

The general behavior of velocity (40) shows that the chosen solution should have a positive second term. The sign of the boson–fermion interaction does not affect the velocity of non-linear perturbations. It corresponds to the general solution (40).

Solution (40) with a negative sign corresponds to the acoustic wave in fully spin-polarized fermions. Separation into the bosonic and fermionic branches is partially conventional. If the partial velocity of fermions, \( V_f \), dominates over the partial velocity of bosons, \( V_b \), (41) we have a negative sign for the bosonic branch (found in solution (41)) for the small boson–fermion interaction and a positive sign for the fermionic branch. However, a study of the mixture for the intermediate boson–fermion interaction does not allow such a straightforward separation into the bosonic and fermionic branches. So, we keep studying the non-linear solution corresponding to the ‘+’ sign in (40) and call it the bosonic branch by convention. The second branch, conventionally called the fermionic branch, will be studied elsewhere.

Fermions in a partially polarized regime demonstrate two acoustic waves and a spin wave with \( \omega(k = 0) \neq 0 \) [63]. The Hamiltonian of the non-linear Pauli equation in [63] contains the interaction term corresponding to the total energy of a two-component Fermi gas presented in [64].
4.2. Second-order perturbations

The next order of the $\varepsilon$ contribution in the hydrodynamic equations leads to the following set of non-linear differential equations: the continuity equation for bosons

$$V\partial_t n_{1b} - V\partial_k n_{2b} + n_{0b} \partial_k v_{2b} + \partial_k (n_{1b} v_{1b}) = 0; \quad (43)$$

equation for fermions

$$m_{f} n_{0f} V\partial_{t} \nu_{1f} - m_{f} n_{0f} V\partial_{k} \nu_{2f} - m_{f} n_{1b} V\partial_{k} \nu_{1b}$$

$$- \frac{\hbar^2}{4m_{f}} \partial_{k} n_{1b} + m_{f} n_{0f} V\partial_{k} \nu_{1b} = -g_{0b} n_{0b} \partial_{k} n_{2b}$$

$$-g_{0b} n_{1b} \partial_{k} n_{1b} - g_{0f} n_{0f} \partial_{k} n_{1f} - g_{0b} n_{0b} \partial_{k} n_{2f}$$

$$-g_{0f} n_{1f} \partial_{k} n_{1f} - \frac{1}{2} g_{2f} n_{0f} \partial_{k} n_{1f}; \quad (44)$$

equation for fermions

$$V\partial_{t} n_{1f} - V\partial_{k} n_{2f} + n_{0f} \partial_{k} v_{2f} + \partial_{k} (n_{1f} v_{1f}) = 0; \quad (45)$$

equation for bosons

$$m_{f} n_{0f} V\partial_{t} \nu_{1f} - m_{f} n_{0f} V\partial_{k} \nu_{2f} - m_{f} n_{1b} V\partial_{k} \nu_{1b}$$

$$+ m_{f} n_{0f} V\partial_{k} \nu_{1f} = \frac{\hbar^2}{4m_{f}} \partial_{k} n_{1f} + \partial_{k} p_{2}^{\nu_{2f}}$$

$$= \frac{\hbar^2}{2m_{f}} \partial_{k} n_{1f} [n_{0f}(3p_{2f}^{\nu_{2f}} + p_{2f}^{\nu_{2f}} + p_{2f}^{\nu_{2f}})]$$

$$+ (3p_{2f}^{\nu_{2f}} + p_{2f}^{\nu_{2f}} + p_{2f}^{\nu_{2f}} + p_{2f}^{\nu_{2f}} + p_{2f}^{\nu_{2f}}) n_{1f}]$$

$$-g_{0f} n_{0f} \partial_{k} n_{2f} - g_{0f} n_{1f} \partial_{k} n_{1f} - \frac{1}{2} g_{2f} n_{0f} \partial_{k} n_{1f}; \quad (46)$$

equations for the evolution of the elements of the pressure tensor $F_{ij}$

$$V\partial_{t} p_{1f}^{\nu_{1f}} - V\partial_{k} p_{2f}^{\nu_{1f}} + \partial_{k} (n_{1f} v_{1f}) = 0,$$  

$$(47)$$

$$V\partial_{t} p_{1f}^{\nu_{1f}} - V\partial_{k} p_{2f}^{\nu_{1f}} + \partial_{k} (n_{1f} v_{1f}) = 0,$$  

$$(48)$$

and

$$V\partial_{t} p_{1f}^{\nu_{1f}} - V\partial_{k} p_{2f}^{\nu_{1f}} + \partial_{k} (n_{1f} v_{1f}) = 0.$$  

$$(49)$$

All the functions of the first order can be represented via the first-order perturbations for the concentrations of bosons, $n_{1b}$, and fermions, $n_{1f}$. Presenting the second-order hydrodynamic perturbations via the derivative of the second-order concentration of fermions, $\partial_{k} n_{2f}$, and the first-order perturbations for the concentrations of bosons, $n_{1b}$, and fermions, $n_{1f}$, we find an equation for three variables, where the coefficient in front of $\partial_{k} n_{2f}$ equals zero if expression (40) for $V^2$ is included.

After the described manipulations, we obtain the equation for concentrations $n_{2f}$, $n_{1f}$, $n_{1b}$:

$$g_{bf} n_{bf} n_{bf} \partial_{k} n_{2f} + \partial_{k} n_{2f} \left[ \frac{8g_{bf} p_{bf} m_{bf}^2}{n_{bf}^2} \right]$$

$$\times \left[ \frac{3p_{bf}}{n_{bf}^2} \right]$$

$$\times \left[ 2m_{bf} V^2 \left( \partial_{k} n_{1f} + \frac{n_{1f}}{n_{bf}} \partial_{k} n_{1f} \right) \right]$$

$$\times \left[ \frac{6p_{bf} + 20g_{bf} p_{bf} m_{bf}^2}{n_{bf}^2} \right] + g_{bf} n_{bf} \partial_{k} n_{1b} + \frac{1}{2} g_{bf} m_{bf} \partial_{k} n_{1bf}$$

$$+ g_{bf} n_{bf} \partial_{k} n_{1bf} + g_{bf} m_{bf} \partial_{k} n_{1bf} + g_{bf} m_{bf} \partial_{k} n_{1bf}$$

$$+ \frac{1}{2} g_{bf} m_{bf} \partial_{k} n_{1bf} = 0.$$  

$$(50)$$
The first and second terms in equation (50) contain all the contributions of the second-order functions (in this case, they are expressed via the second-order concentration of fermions, \(n_2\)). The coefficient in front of \(n_2\) goes to zero if the explicit form of velocity \(V\) (40) is used. Therefore, equation (50) reduces to the equation involving two functions: \(n_{1b}\) and \(n_{1f}\).

The first term in equation (50) and the combination inside the square bracketed group of the last seven terms present the contribution of bosons. The other terms present the contribution of fermions.

Equation (50) appears as the Euler equation for fermions. Therefore, the contribution of bosons vanishes when the interspecies interaction constant, \(g_{bf}\), becomes zero.

The lowest-order analysis in \(\varepsilon\) gives the relation between the concentrations of bosons and fermions

\[
n_{1f} = \frac{m_b}{n_{1b} S_{bf}} \left( V^2 - \frac{n_{1b} G_b}{m_b} \right) n_{1b}.
\]

(51)

This relation can be used to obtain an equation for a single function (\(n_{1b}\) for instance) from (50). Next, when \(n_{1b}\) is found, we obtain the structure of the soliton solution for \(n_{1f}\) using relation (51).

Moreover, we can use the solution for \(V^2\) given by equation (40) to analyze relation (51).

\[
n_{1f} = \frac{1}{2} \frac{m_b}{n_{1b} S_{bf}} \left[ \frac{3p_0}{m_{1f}} + \frac{8p_0 m_f}{h^2} - \frac{n_{1b} G_b}{m_b} \right] + \left( \frac{3p_0}{m_{1f}} + \frac{8p_0 m_f}{h^2} - \frac{n_{1b} G_b}{m_b} \right)^2 + 4 \frac{n_{1b} S_{bf} G_b^2}{m_p m_f}.
\]

(52)

The right-hand side of equation (52) is the product of two functions: the interaction constant \(g_{bf}\) and the combination of parameters located in brackets. The structure of the parameters in brackets can be expressed as follows: \(\Xi + \sqrt{\Xi^2 + \Lambda^2}\). The sign of this structure does not depend on the signs and values of the parameters, since \(\Xi\) and \(\Lambda\) are always positive. Therefore, the perturbations for bosons and fermions have the same sign if the boson–fermion interaction is repulsive, \(g_{bf} > 0\), and they have opposite signs if the boson–fermion interaction is attractive, i.e. \(g_{bf} < 0\).

Expression (52) can be rewritten in the different equivalent form:

\[
n_{1f} = \frac{n_{1b} G_{bf}}{m_f} \left( V^2 - \frac{3p_0}{m_{1f}} - \frac{8p_0 m_f}{h^2} \right) n_{1b}.
\]

(53)

4.3. Korteweg–de Vries equation for perturbations of bosons

The Korteweg–de Vries (KdV) equation for the concentration of bosons has the following structure:

\[
\tilde{a} \partial_t n_{1b} + \tilde{b} m_{1b} \partial_x n_{1b} + \tilde{c} \partial_x^3 n_{1b} = 0,
\]

(54)

where we find the coefficients

\[
\tilde{a} = 2 m_b V^2 \left[ 1 + \frac{m_{1b} G_{bf}}{m_p} \left( V^2 - \frac{3p_0}{m_{1f}} - \frac{8p_0 m_f}{h^2} \right) \right],
\]

(55)

The coefficient \(\tilde{a}\) is always positive, since \(V^2\) should be positive for a solution to exist. However, the condition \(V^2 > 0\) places a restriction on the parameters. For instance, if we drop the contribution of fermions, \(V^2 = g_{bf} n_{1b} / m_b\). Hence, the interaction between bosons should be repulsive, i.e. \(g_{bf} > 0\).

Equation (54) can be reduced to a single variable after the introduction of the new variable \(\zeta = \tau - U \xi\). Following this, the KdV equation can be integrated. As a result of the integration, we find a non-linear perturbation of boson concentration in the first order:

\[
n_{1b} = \frac{3U \tilde{a}}{b} \times \cosh^2 \left( \frac{1}{b} \sqrt{U \frac{\tilde{a}}{b}} \right).
\]

(58)

Since the coefficient \(\tilde{a}\) is positive, solution (58) can exist if the coefficient \(\tilde{c}\) is positive. The sign of coefficient \(\tilde{b}\) defines the type of soliton: a bright soliton for \(\tilde{b} > 0\), or a dark soliton for \(\tilde{b} < 0\).

For bosons, the coefficient \(\tilde{c}\) consists of two terms: \(\tilde{c} = g_{2b} n_{1b} - \tilde{h}^2 / 4m_b^2\). Hence, it is positive for a nonzero interaction between bosons in the third order by the interaction radius. Moreover, it requires the repulsive boson–boson interaction, \(g_{2b} > 0\). This conclusion is in agreement with the condition \(V^2 > 0\) requiring \(g_{bf} > 0\). For bosons, the coefficient \(\tilde{a}\) simplifies to \(\tilde{a} = 2 m_b V^2 > 0\). The coefficient \(\tilde{b}\) also appears in a simple form in this limit: \(\tilde{b} = g_{bf} + 2 m_b V^2 / n_{1b} > 0\). The transition to bosons is made by the limit \(g_{bf} \rightarrow 0\). It shows the existence of a specific soliton solution with a positive amplitude in the system of bosons studied up to the third order by the interaction radius. The terms proportional to \(g_{bf}\) give the contribution of fermions, which is discussed numerically below.

This bright-soliton solution purely for bosons is obtained in [30, 31]. The physical picture behind the bosonic bright-soliton solution for repulsive bosons demonstrates a deep relation to the soliton solution experimentally obtained in [60].
that originally exists in the boson subsystem: and the concentration of bosons, since we consider the soliton existence, we represent the KdV equation (5. Numerical analysis of the bright soliton

Figure 5. The dimensionless amplitude of soliton (54) \( A = 3 \mu \text{m} \) as a function of \( F \) for two values of the boson–boson interaction constant, \( g = 5 \) and \( g = 10 \), as demonstrated in the figure. The other parameters have the following values: \( M_0 = 4, N_0 = 2, F = 0.01, G = 1, L = 0.01 \).

Figure 6. Dimensionless coefficient \( c \) is shown as a function of the dimensionless boson–fermion interaction \( l \) for fixed \( L \) for two values of the boson–boson interaction constant, \( g = 5 \) and \( g = 10 \), as demonstrated in the figure. The other parameters have the following values: \( M_0 = 4, N_0 = 2, F = 0.01, G = 1, L = 0.01 \).

Figure 7. Dimensionless coefficient \( c \) is presented as a function of \( l \) for a simultaneous change of \( L \) as \( L = \alpha_L l \) for two values of \( \alpha_L \): \( \alpha_L = 0.01 \) and \( \alpha_L = 0.001 \). Each value of \( \alpha_L \) is shown for two values of the boson–boson interaction constant, \( g = 2 \) and \( g = 10 \). The other parameters have the following values: \( M_0 = 4, N_0 = 2, F = 0.01, G = 1 \).

5. Numerical analysis of the bright soliton

To perform an analysis of the soliton’s properties and its area existence, we represent the KdV equation (54) and the velocity of perturbation (40) in dimensionless form.

The dimensionless velocity is given via the mass of bosons and the concentration of bosons, since we consider the soliton that originally exists in the boson subsystem:

\[
W^2 = \frac{m_b^2 v^2}{\hbar^2 n_{0b}^3} = \frac{1}{2} \left( g + \frac{3}{5} C_0 \frac{N_0^2}{M_0^2} \left( 1 + \frac{8}{3} F \right) \right) + \left( g - \frac{3}{5} C_0 \frac{N_0^2}{M_0^2} \left( 1 + \frac{8}{3} F \right) \right) + \frac{4 N_0}{M_0^2} \xi^2, \tag{59}
\]

where \( N_0 = n_{0f}/n_{0b}, \ M_0 = m_f/m_b, \ g \equiv m_b g_{2b} n_{0b}^{1/3}/\hbar^2, \ G \equiv m_b g_{2b} n_{0b}^{1/3}/\hbar^2, \ l \equiv m_b g_{2b} n_{0b}^{1/3}/\hbar^2, \ L \equiv m_b g_{2b} n_{0b}^{1/3}/\hbar^2, \) and \( F \equiv \frac{m_b g_{2b} n_{0b}^{1/3}}{\hbar^2} \). The explicit form of the equilibrium pressure for degenerate fermions with full-spin polarization is used in the form of Fermi pressure: \( p_f = \left( 6 \pi^2 \right)^{2/3} \hbar^2 n_{0f}^{5/3}/5m_f \). The parameters \( l \) and \( L \), which are related, introduce the following relation: \( L = \alpha_L l \), where \( \alpha_L < 1 \) is a parameter which does not depend on \( l \) or the other parameters, and represents the independent variation of the interaction constant, \( L \). The coefficient \( \alpha_L \) is an independent parameter. Hence, if \( L \) is fixed for a changing \( l \), it means that the parameter \( \alpha_L \) changes to compensate for the contribution of \( l \) in \( L \).

The dimensionless KdV can be written as follows:

\[
a \partial_t N + b N \partial_x \tilde{\tau} + c \partial_x^3 \xi N = 0, \tag{60}
\]

where \( N = n_{1b}/n_{0b}, \ \tilde{\tau} = \sqrt{n_{0b}} \tau, \ \xi = \sqrt{n_{0b}} \xi, \)

\[
a = \frac{m_b \tilde{\alpha}}{\hbar^2 n_{0b}^2} = 2 \frac{W^2}{\left( 1 + \frac{N_0}{M_0} \xi^2 \right)}, \tag{61}
\]

\[\]
However, if the parameter $g$ changes from 1 to 10 at a fixed $G = 1$ leads to a decrease in the amplitude and width of the soliton. A further increase of $g$ at a fixed $G = 1$ leads to a decrease of the width, with no change of amplitude. Figure (1) shows a relatively small fermion influence.

*First: a focus on the velocity properties.* The dependence of the dimensionless velocity (59) on the boson–boson interaction, $g$, for different boson–boson ($l$) and fermion–fermion ($F$) interactions is presented in Figures 2, 3, and 4. Each figure is made for different mass ratios, $M_0$, and concentrations, $N_0$.

The main change of the dependence of the velocity square $W^2(g)$ (59) happens for small boson–boson interactions. Figures 2, 3, and 4 show relatively strong boson–fermion interactions, $l \sim 1$. The fermion–fermion interaction is considered in an interval from average ($F = 0.1$) to strong ($F = 1$) values. A mass (concentration) increase of each species decreases (increases) the velocity $V$ (40). The relatively small influence of fermions is demonstrated in figure (2), where the mass of fermions is relatively large ($M_0 = 9$), while the variation of $V$ is noticeable at small values of $g$. An increase of $l$ increases the velocity $W$, as can be seen from the analytical dependence (59). An increase of the fermion–fermion repulsion causes a small increase in the velocity $W$ for small values of $g$ and a fixed $l$. The role of fermion–fermion interaction increases if the mass and concentration ratios are approaching one, as is demonstrated by the transitions in figures (3) and (4).

On the coefficients $b$ and $c$. Coefficient $c$ is a symmetric function of $l$ for a fixed $\alpha_1$. However, coefficient $b$ shows a nonsymmetric dependence on the boson–fermion interaction, $l$. The third term in (62) is positive since $D_w > 0$. However, the fourth and last terms can be negative for attractive boson–fermion interaction. Hence, boson–fermion repulsion increases the amplitude of the soliton. The increase can be nonmonotonic, since $W^2(l^2)$ are located in the denominator of the amplitude.

There is a competition between different terms defining the amplitude $A = 3Ua/bm_0$ for the attraction between bosons and fermions. The fourth and last terms become negative in this regime, while the third term is positive. Hence, the sign of the amplitude change depends on the parameters of the system (see figure 5).

The area of the soliton’s existence is restricted by the condition that the width of the soliton $D \sim \sqrt{\alpha}/\sqrt{\sigma}$ is real. This means that the coefficient $c$ should be positive since the coefficient $\alpha > 0$ is positive for all parameters. The third term in $c$ contains a dependence on $L$. This is a positive term for a positive product $IL$. Let us point out that $D_w$ is positive for all parameters.

If we consider the dependence of $c$ on $l$ at fixed $L$, the third term in (63) plays a crucial role (for instance at $m = 4, n = 2, g = 10, F = 0.01, G = 1, L = 0.01$). The dependence appears numerically as an almost parabolic dependence (see figure 6), despite a more complex analytical dependence via $W^2(l)$. This parabola has branches going below the maximum located at a positive value of $l$. However, $l$ and $L$ are moments of the same potential of boson–fermion interaction. Therefore, we
use the representation \( L = \alpha_{L} l \) introduced above. This changes the dependence of \( c \) on \( l \). In this case, the parameter \( c \) is a function of \( \tilde{L} \). The value \( G = 1 \) is chosen, so that the boson–fermion interaction shifts the coefficient \( c \) from the value \( c_{0} = 0.75 \). For small positive values of \( l \) at a fixed \( L = 0.01 \), the shift of \( c \) is positive (see figure 6). There is a value of \( l = l_{0}(g) \), where the shift becomes equal to zero. The value of \( l_{0} \) becomes larger for a larger boson–boson interaction, \( g \). For a further increase of \( l \) above \( l_{0} \), the shift becomes negative, i.e. \( c < 0.75 \). However, the parameter \( c \) shows a small deviation from \( G = 1/4 \) and has a positive value. Therefore, the presence of fermions does not destroy the soliton solution.

Figures 6 and 7 show that deviations of \( c \) from the value \( G = 0.25 \) are small. Therefore, small values of \( G \) can be chosen, down to \( G_{\text{min}} = 0.26 \). Consider the behavior of \( c \) at a fixed \( \alpha_{L} \). A monotonic increase of \( c \) as a function of \( \tilde{L} \) is found at a relatively large \( \alpha_{L} = 0.01 \) and a relatively large \( g = 10 \). A small \( g = 2 \) with a large \( \alpha_{L} = 0.01 \) and different values of \( g \) at a smaller \( \alpha_{L} = 0.001 \) lead to a decrease of \( c \) at small \( l \), which is replaced by an increase of \( c \) at larger \( l \). The area of decrease of \( c \) from \( c_{0} = 0.75 \) becomes wider and \( c_{\text{min}} \) becomes smaller at smaller values of \( \alpha_{L} \) and smaller values of \( g \), as presented in figure 7. All of these are obtained for the small fermion–fermion interaction, \( F = 0.01 \). An area of larger \( F \) is presented for the width of soliton \( D \) in figure 8.

The second constant of the boson–boson interaction \( G > 0.25 \) plays a crucial role in soliton existence. The first interaction constants for the boson–boson and boson–fermion interactions define the properties of the solution. The constants of the boson–fermion and fermion–fermion interaction that exist in the third order of the interaction radius have a small influence if the boson–boson repulsive interaction is strong (\( g \geq 1 \)). The condition \( G > 0.25 \) also corresponds to this criterion.

The solution presented in this manuscript is obtained in the limit of the strong boson–boson interaction. Hence, the second interaction constant for the boson–boson interaction in the third order by the interaction radius is larger than \( 1/4 \). Otherwise, this solution does not exist. Consequently, the terms in the third order by the interaction radius make a small contribution to the evolution of the boson–fermion mixture. Therefore, all the physical phenomena are caused by the terms in the first order by the interaction radius. In particular, the Bose–Fermi mixture supports bright solitons (both in bosonic and fermionic components), provided that the bosons and fermions attract each other strongly enough, even though bosons alone are repulsive [11, 65]. However, this result is not captured by the reductive perturbation method in the chosen form of expansion. The goal of this paper is to demonstrate the novel phenomenon caused by the third-order terms. Hence, a soliton solution is found in a regime where the third-order terms qualitatively change the response of the system.

6. Conclusion

Boson–fermion mixtures have been studied in terms of a hydrodynamic model. The boson–boson, fermion–fermion and boson–fermion interactions have been considered up to the third order by the interaction radius. Stress has been placed on the fermion models, where the kinetic-pressure tensor has been considered as an independent function. Hence, no equation of state has been used for perturbations of pressure, but an additional hydrodynamic equation for pressure evolution is derived from the microscopic quantum model. An equation of state can be used for the equilibrium pressure.

The developed model has been used to study a bright soliton in a repulsive Bose–Einstein condensate fraction. It exists because the repulsive boson–boson interaction causes a positive interaction constant in the third order by the interaction radius. The formation of a soliton in the fermion fraction has been found. It has been seen that the type of soliton in a fermion concentration depends on the sign of the boson–fermion interaction constant in the first order by the interaction radius. Hence, the boson–fermion repulsion (attraction) leads to a bright (dark) soliton in the fermion fraction. The influence of fermions on the properties of a soliton in the boson fraction was analyzed. The model obtained contains the first order on the interaction radius, including the boson–boson interaction corresponding to the Gross–Pitaevskii equation, and the boson–fermion interaction (existing in the well-known works cited above on boson–fermion mixtures), which are three-dimensional zeroth moments of the interaction potential. However, the consideration of the interaction terms in the third order by the interaction radius introduces three additional interaction constants which are the second moments of the interaction potential for boson–boson, boson–fermion, and fermion–fermion interactions.

It is possible to estimate new constants via well-known constants (for boson–boson and boson–fermion interactions), as presented in [44]. However, all constants are independent and introduce additional information about the interaction potential. Moreover, the solitons found are themselves of great interest, since they are examples of new non-linear phenomena in ultracold mixtures.

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