SMC and MPC based composite control for a constrained second-order nonlinear system with external disturbances

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Abstract—The paper proposes a novel structure of composite control consisting of sliding mode control (SMC) and model predictive control (MPC) for a constrained second-order nonlinear system with external disturbances. Based on linear sliding mode surface, the control law is first generated by MPC problem to drive the system state to a certain region, and then is switched to SMC which can promise the input-to state stable (ISS). The recursive feasibility of MPC and closed-loop stability is guaranteed, while the whole process behaves robustness and constraints satisfaction. Finally, a simulation case is provided to demonstrate the effectiveness of the composite control.

Index Terms—Composite control, sliding mode control (SMC), model predictive control (MPC), constrained second-order nonlinear system.

I. INTRODUCTION

Sliding mode control (SMC) [1] has been widely employed in industrial applications such as power systems [2], unmanned vehicle [3] and spacecraft [4]. The main idea of SMC is to construct sliding mode surface to decompose the order of controlled system, and then drive each decomposed system to desired properties. The whole process of SMC consists of two stage: reaching mode and sliding mode. In reaching mode, the reaching law is implemented to drive sliding variable to sliding mode surface ensuring the attainment of the sliding motion. Next, the system state converges to equilibrium point along sliding mode surface at sliding mode. Therefore, SMC behaves simplicity and high robustness to external disturbances [5–8]. However, the drawback of SMC is incapable of dealing with system constraints on both input and state [9]. In particular, when the system state is far away from sliding mode surface, the input and state can violate system constraints under SMC, which may lead to instability of system.

Recently, model predictive control (MPC) [10] has attracted more attention since it can be able to manage both input and state constraints. By constructing performance index and constraints, the optimal control is solved to stabilize controlled system. Nevertheless, MPC suffers from low robustness, i.e some properties or even stability of system under MPC may not be ensured when existing external disturbance. To cope with the problem, the sevral robust MPC approach has been investigated. In [11–13], a min-max MPC is proposed by considering the worst case caused by disturbance. However, the solution provided by min-max MPC is too conservative and it is hard to realize due to its computational complexity. To overcome the disadvantage of min-max MPC, the inherent robustness MPC is introduced in [14], [15]. Corresponding, the optimization problem is solved under tighted input and state constraints to promise the system robustness, while the precondition is the bound of disturbance is relative small. To weaken the effect of disturbance, the disturbance observer is introduced to MPC in [16]. However, the robustness of MPC itself is not improved. Meanwhile, with the increase of system order, the computational complexity of MPC grows. It is worth noting that the advantages and disadvantages of MPC and SMC are complementary. Therefore, an intuitive idea to combine the two control approaches.

A few techniques has been proposed to merge SMC and MPC, combining the robustness of SMC with the constraints satisfaction of MPC [17–21]. In above articles, an inner-outer loop structure is adopted, where SMC rejects disturbances at inner loop by keeping initial state on the sliding mode surface all the time, and then MPC solves the optimization problem based on nominal system to stabilized closed-loop system at outer loop. However, SMC only serves eliminating disturbance in this structure which functions as disturbance observer. What’s more, the implementation of discontinuous control signal in SMC can only guarantee the state on sliding mode surface theoretically. In fact, discontinuous control signal can lead to chattering problem [22], which can further break the theoretical properties of MPC. In [23], a discrete-time sliding mode control law is proposed for nonlinear systems by formulating a nonlinear predictive control problem. The resulting control law has all the properties of discrete-time SMC while satisfying input and state constraints. Inspiring by this work, a composite control based on SMC and MPC is developed.

In this paper, we proposed a novel structure to merge SMC and MPC based on a consider a constrained second-order continuous system with external disturbances. The optimization problem is constructed based on the linear sliding mode surface. By solving the optimization problem, the control law drives the sliding variable to the neighbouring region of sliding mode surface and then is switched to SMC to realize input-to-state stability. The contributions of this paper can be summarized as follows:

1. A novel structure is proposed to realize a composite control consisting of SMC and MPC for a constrained second-order nonlinear system with external disturbances. The composite control takes the advantages of SMC and MPC, guaranteeing the robustness and constraints satisfaction during the whole process. The sufficient conditions and rigorous proof are also presented.
(2) The inherent robustness of MPC is improved by selecting proper parameters of sliding mode surface. Therefore, the MPC in this paper shows better robustness compared with existing papers.

(3) Utilizing the features of decomposing system of SMC, a reduced-order system is optimized by MPC, which reduces the computational complexity of optimization theoretically compared with using MPC directly.

The notation adopted in this paper are as follows. \( \mathbb{R} \) and \( \mathbb{R}^2 \) are the real space and its 2-dimensional Euclidean space respectively. \( \mathbb{N} \) represents the collection of all natural number. \( \| x \| = \sqrt{x^T x} \) denotes the Euclidean norm for a vector \( x \), and \( | y | \) denotes absolute value for a number \( y \). The bold 0 is a zero vector of two dimensions. Given two sets \( A \in \mathbb{R}^2, B \in \mathbb{R}^2 \), \( A \sim B \) stands for the Pontryagin set difference defined by \( A \sim B \triangleq \{ z \in \mathbb{R}^2 | z + b \in A, \forall b \in B \} \). \( \tau(t) \) indicates the value of a variable at time predicted from time \( t \). Moreover, we mark feasible variables satisfying all constraints as \( \gamma \) and mark optimal variables attained by solving optimization problem as \( \bar{\gamma} \).

**II. PROBLEM FORMULATION**

**A. Controlled plant**

Consider the following second-order nonlinear system

\[
\begin{align*}
    \dot{x}_1(t) &= x_2(t), \\
    \dot{x}_2(t) &= f(x(t), u(t)) + w(t),
\end{align*}
\]

where \( x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2, u(t) \in \mathbb{R} \) and \( w(t) \in \mathbb{R} \) denote the states, control input and external disturbance, respectively. Both the states and control input are constrained by \( x(t) \in \mathcal{X}, u(t) \in \mathcal{U} \), where \( \mathcal{X} \) and \( \mathcal{U} \) are compact and convex sets containing the origin in their interior. The disturbance is assumed as bounded by \( |w(t)| \leq \mu \) and.

For the sake of convenience, represent the system (1) as

\[
\dot{x}(t) = F(x(t), u(t)) + w(t),
\]

where \( F(x(t), u(t)) = [x_2(t), f(x(t), u(t))]^T \) and \( w(t) = [0, w(t)]^T \). The nominal dynamics of (2) is defined by

\[
\dot{\bar{x}}(t) = \bar{F}(\bar{x}(t), \bar{u}(t))
\]

**Assumption 1.** The function \( F(x(t), u(t)) \) with \( F(0, 0) = 0 \) is Lipschitz continuous for its first argument \( x(t) \in \mathcal{X} \), and its Lipschitz constant is \( \nu \), such that

\[
\| F(x_1(t), u(t)) - F(x_2(t), u(t)) \| \leq \nu \| x_1(t) - x_2(t) \|
\]

**B. Sliding Mode Surface**

To stabilize the system (2) by SMC, the following first-order linear sliding variable \( s \) is designed as

\[
s(t) = c_1 x_1(t) + c_2 x_2(t),
\]

where \( c_1, c_2 \) is a positive constant. Then, by designing the SMC, the system states reaches to sliding mode surface \( s = 0 \) and then is driven to origin along sliding mode surface. Here, we mark the SMC law based on (3) as \( u_s(x(t)) \).

**Assumption 2.** For \( x(t) \in \mathcal{X} \), \( u_s(x(t)) \) is Lipschitz continuous with the Lipschitz constant \( \eta \), such that

\[
|u_s(x_1(t)) - u_s(x_2(t))| \leq \eta \| x_1(t) - x_2(t) \|
\]

**C. Optimization problem for reaching mode**

Define the series \( \{ t_k \}, k \in \mathbb{N} \) as the time instant, where \( \delta = t_{k+1} - t_k \) is the sampling interval, and the optimization problem is solved at each sampling instant. The optimization problem is formulated as follows:

**Problem 1**

\[
\min_{\bar{u}(\cdot)} J(s(t_k), \bar{u}(\cdot)),
\]

subject to

\[
\begin{align*}
    \bar{x}(t_k|t_k) &= x(t_k), \\
    \bar{x}(\tau|t_k) &= F(\bar{x}(\tau|t_k), \bar{u}(\tau|t_k)), \\
    \bar{x}(\tau|t_k) &\in \bar{X}(\tau - t_k), \\
    \bar{u}(\tau|t_k) &\in \mathbb{U}, \\
    \bar{s}(\tau|t_k) &= c_1 \bar{x}_1(\tau|t_k) + c_2 \bar{x}_2(\tau|t_k), \\
    \bar{s}(t_k + T|t_k) &\in \mathcal{O},
\end{align*}
\]

with \( \tau \in [t_k, t_k + T], \bar{X}(\tau - t_k) = \mathcal{X} \sim \mathcal{Y}(\tau - t_k) \) is the tightened state constraint sets with \( \mathcal{Y}(\tau) = \{ x \in \mathbb{R}^2 | \| x \| \leq \nu (e^{\gamma \tau} - 1) \}, \mathcal{O} = \{ s \in \mathbb{R} | |s| \leq \varepsilon \} \) is the terminal constraint set, and \( T \) is the prediction horizon. The cost function \( J(s(t_k), \bar{u}(\cdot)) \) in Problem 1 is defined as

\[
J(s(t_k), \bar{u}(\cdot)) = \delta \int_{t_k}^{t_k + T} L(\bar{s}(\tau|t_k), \bar{u}(\tau|t_k)) d\tau + V_f(\bar{s}(t_k + T|t_k))
\]

in which \( L(\bar{s}(t_k), \bar{u}(t_k)) = q \bar{s}^2(\tau|t_k) + \tau |\bar{u}(\tau|t_k) - u_s(\bar{x}(\tau|t_k))| \) is stage cost with \( q > 0, r > 0, \) and \( V_f(\bar{s}(t_k + T|t_k)) = p \bar{s}^2(t_k + T|t_k) \) is terminal cost with \( p > 0 \).

By solving Problem 1, the optimal input sequence is obtained over \( \tau \in [t_k, t_k + T] \) as

\[
\bar{u}^*(\tau|t_k) = \arg \min_{\bar{u}(\cdot)} J(s(t_k), \bar{u}(\cdot)),
\]

and applied control law is \( \bar{u}^*(\tau|t_k), \tau \in [t_k, t_{k+1}] \).

Before next section, some standing Assumptions are presented as follows.

**Assumption 3.** For the nominal system (3), there exist a set \( \Omega_\alpha = \{ s \in \mathbb{R} | |s| \leq \alpha \} \) with \( \alpha > \varepsilon \) and a local feedback control law \( \kappa_f(\bar{s}) = k\bar{s} + u_s(\bar{x}(\bar{s})) \) such that for all \( \bar{s} \in \Omega_\alpha \), by implementing the control law \( \kappa_f(\bar{s}) \), it holds that \( \kappa_f(\bar{s}) \in \mathcal{U} \) and

\[
\dot{V}_f(\bar{s}) + L(\bar{s}, \kappa_f(\bar{s})) \leq 0.
\]

**Assumption 4.** For the nominal system (3), if it is true that \( \bar{s}(t_k) \in \Omega_\alpha \) and \( \bar{x}(t_k) \in \bar{X}(T - \delta) \), then \( \bar{x}(\tau + t_k) \in \bar{X}(\tau + T - \delta) \) holds under the feedback control law \( \kappa_f(\bar{s}(\tau + t_k)) \) with \( \tau \in [0, \delta] \).

**Assumption 5.** For \( s \in \Omega_\varepsilon \), it holds that the control law \( u_s(x) \in \mathcal{U} \) and the closed-loop (2) is input-to-state stable (ISS) by implementing \( u_s(x) \).

Based on the discussion above, the procedure of composite control based on SMC and MPC is described in Algorithm 1.
Algorithm 1 Composite control based on SMC and MPC
1: Set \( k = 0 \) and initialize the nominal system states by the actual ones.
2: While \( s(t) \notin \Omega_x \)
3: Solve Problem 1
4: Apply \( u(t_k) = u^*(t_k|t_k) \) to the real system for \( t \in [t_k, t_{k+1}] \)
5: Update \( k = k + 1 \)
6: End while
7: Apply the control law \( u(t) = \kappa_f(s(t)) \)

Remark 1. It is noted that the sliding variable \( s \) is used in the cost function of Problem 1 instead of system state \( x_1, x_2 \). Compared with controlling system (1) directly by MPC, the composite control approach reduces the order of optimized system, which further brings down the computational complexity caused by system order theoretically.

III. THEORETICAL ANALYSIS

In this section, the recursive feasibility of MPC algorithm and the stability of closed-loop system is developed

A. Recursive feasibility analysis

Theorem 1. For system (1), assume that Problem 1 is feasible at the initial time \( t_0 \). Then the Problem 1 is recursive feasible for any \( t_k \), if the following conditions are satisfied: (1) the disturbance is bounded by \( \mu \leq \frac{c_1 e^{-\beta t}}{e^{\beta t} - 1} + \frac{c_2}{\sqrt{\sigma}} (x_1^2 + x_2^2) \leq \varepsilon \).

To prove Theorem 1, the following lemmas are first stated.

Lemma 1. Consider the real system (2) and the nominal system (3) from the same initial state. Applying the same control law, the absolute value \( |s(t) - s(t)| \) satisfies the following inequality

\[ |s(t) - s(t)| \leq \frac{\mu}{e^{\beta t}} (e^{\beta t} - 1), \tag{15} \]

where \( \bar{c} = c_1 + c_2 \).

Proof. According to [24], we have

\[ \|x(t) - \bar{x}(t)\| \leq \frac{\mu}{e^{\beta t}} (e^{\beta t} - 1). \tag{16} \]

Then, it can be deduced that

\[ |s(t) - s(t)| = |c_1 x_1(t) - \bar{x}_1(t)| + c_2 |x_2(t) - \bar{x}_2(t)| \leq c_1 |x_1(t) - \bar{x}_1(t)| + c_2 (x_2(t) - \bar{x}_2(t)) \leq \frac{\mu}{e^{\beta t}} (e^{\beta t} - 1). \]

Lemma 2. (Gronwall inequality) \[25\] If

\[ x(t) \leq h(t) + \int_{t_0}^{t} \beta(t)x(t)dt, \quad t \in [t_0, T), \]

with all the functions involved are continuous on \([t_0, T), T \leq +\infty \) and \( \beta(t) \geq 0 \), then \( x(t) \) satisfies the integral inequality

\[ x(t) \leq h(t) + \int_{t_0}^{t} \beta(t)x(t)dt, \quad t \in [t_0, T). \tag{17} \]

Lemma 3. For \( x \in \mathbb{R}(\tau-t_k) \) and \( y \) such that \( \|y - x\| \leq \frac{\mu}{e^{\beta t}} (e^{\beta t} - 1) e^{\beta t} \), then \( y \in \mathbb{R}(\tau-t_k) \).

Proof. Consider \( e \in \mathbb{R}(\tau-t_k) \) and define auxiliary variable \( z = y - x + e \). It is clear that

\[ \|z\| \leq \|y - x\| + \|e\| \leq \frac{\mu}{e^{\beta t}} (e^{\beta t} - 1) e^{\beta t} + \frac{\mu}{e^{\beta t}} (e^{\beta t} - 1) \]

therefore \( \|z\| \in \mathbb{R}(\tau-t_k) \). Since \( y + e = x + z \in \mathbb{R} \), it obtains that \( y \in \mathbb{R}(\tau-t_k) \).

With the aid of lemmas 1, 2 and 3, the proof of Theorem 1 is derived as follows.

Proof. To show the recursive feasibility of Problem 1, the candidate solution at time \( t_{k+1} \) is constructed as follows:

\[ \tilde{u}(\tau|t_{k+1}) = \begin{cases} u^*(\tau|t_k), & \tau \in [t_{k+1}, t_k + T) \\ \kappa_f(\bar{s}(\tau|t_{k+1})), & \tau \in [t_k + T, t_{k+1} + T) \end{cases} \tag{18} \]

and thus the feasible state trajectory is generated by

\[ \tilde{x}(\tau|t_{k+1}) = F(\tilde{x}(\tau|t_{k+1}), \tilde{u}(\tau|t_{k+1})), \tag{19} \]

where \( \tau \in [t_{k+1}, t_{k+1} + T) \), \( \bar{x}(t_{k+1}|t_{k+1}) = x(t_{k+1}) \). Then the proof is divided into three parts.

Part 1: To show the feasible state \( \tilde{s}(t_{k+1} + T|t_{k+1}) \in \Omega_x \) under control law (18).

For \( \tau \in [t_{k+1}, t_k + T) \), the difference between feasible state \( \tilde{x}(\tau|t_{k+1}) \) and optimal state \( x^*(\tau|t_k) \) can be bounded as

\[ \|\tilde{x}(\tau|t_{k+1}) - x^*(\tau|t_k)\| \]

\[ = \left\| x(t_{k+1}) + \int_{t_{k+1}}^{\tau} F(\bar{x}(\tau|t_{k+1}), \tilde{u}(\tau|t_{k+1})) d\tau - x^*(t_{k+1}|t_k) - \int_{t_{k+1}}^{\tau} F(x^*(\tau|t_k), \tilde{u}^*(\tau|t_k)) d\tau \right\| \tag{20} \]

\[ \leq \frac{\mu}{e^{\beta t}} (e^{\beta t} - 1) + \int_{t_{k+1}}^{\tau} \|\tilde{x}(\tau|t_{k+1}) - x^*(\tau|t_k)\| d\tau \]

\[ \leq \frac{\mu}{e^{\beta t}} (e^{\beta t} - 1) e^{\beta t} \tag{17} \]
Then, the boundness of $|\bar{s}(\tau|t_{k+1}) - \bar{s}^*(\tau|t_k)|$ can be deduced that
\[
|\bar{s}(\tau|t_{k+1}) - \bar{s}^*(\tau|t_k)|
= |c_1(\bar{x}_1(\tau|t_{k+1}) - \bar{x}_1^*(\tau|t_k)) + c_2(\bar{x}_2(\tau|t_{k+1}) - \bar{x}_2^*(\tau|t_k))|
\leq c_1 |\bar{x}_1(\tau|t_{k+1}) - \bar{x}_1^*(\tau|t_k)| + c_2 |\bar{x}_2(\tau|t_{k+1}) - \bar{x}_2^*(\tau|t_k)|
\leq \varepsilon \|\bar{\bar{x}}(\tau|t_{k+1}) - \bar{s}^*(\tau|t_k)\|.
\]

Substituting $\tau = t_k + T$ and applying triangle inequality yields
\[
|\bar{s}(t_k + T|t_{k+1})| \\
\leq |\bar{s}^*(t_k + T|t_{k+1})| + \frac{\bar{\mu}}{v}(e^{\delta} - 1)e^{v(T - \delta)} \\
\leq \varepsilon + (\alpha - \varepsilon) \\
= \alpha
\]
which implies the feasible state $\bar{s}(t_k + T|t_{k+1})$ enters the region $\Omega_\alpha$.

Considering $\tau \in [t_k + T, t_{k+1} + T]$, the control law switches to $\kappa_f(\bar{s}(\tau|t_{k+1}))$. According to Assumption 3, we have
\[
\dot{V}_f(\bar{s}(\tau|t_{k+1})) \leq -L(\bar{s}(\tau|t_{k+1}), \kappa_f(\bar{s}(\tau|t_{k+1}))) \\
\leq -\frac{q}{p} V_f(\bar{s}(\tau|t_{k+1})) - \frac{r_k}{p} V_f(\dot{\bar{s}}(\tau|t_{k+1}))
\]
By solving the differential equation, it follows that
\[
V_f(\bar{s}(\tau|t_{k+1})) \leq \left(-\frac{r_k}{q} + \frac{r_k}{q} e^{-\frac{\delta}{T}(\tau - t_{k+1} - T)} + V_f(\bar{s}(\tau|t_{k+1})) e^{-\frac{\delta}{T}(\tau - t_{k+1} - T)}\right)^2
\]
At time $\tau = t_{k+1} + T$, the above inequality can be equivalently written as
\[
|\bar{s}(t_{k+1} + T|t_{k+1})| \\
\leq \frac{1}{\sqrt{p}} \left(-\frac{r_k}{q} + \frac{r_k}{q} e^{-\frac{\delta}{T}} + \sqrt{p} |\bar{s}(t_k + T|t_{k+1})| e^{-\frac{\delta}{T}}\right)
\]
Due to $|\bar{s}(t_k + T|t_{k+1})| \leq \alpha$ and $\frac{1}{\sqrt{p}} \left(-\frac{r_k}{q} + \frac{r_k}{q} e^{-\frac{\delta}{T}} + \sqrt{p} e^{-\frac{\delta}{T}}\right) \leq \varepsilon$, we obtain
\[
|\bar{s}(t_{k+1} + T|t_{k+1})| \leq \varepsilon
\]
which implies the terminal state constraint $\bar{s}(t_{k+1} + T|t_{k+1}) \in \Omega_\varepsilon$ is satisfied. The proof of Part 1 is completed.

Part 2: To show the candidate solution conforms to input constraint (9).

For $\tau \in [t_k, t_k + T]$, the control law is derived from $\bar{u}^*(\tau|t_k)$, and thus it satisfies input constraint (9). Meanwhile, referring to Assumption 3, the terminal control law $\kappa_f(\bar{s}(\tau|t_{k+1}))$ is implemented during $\tau \in [t_k + T, t_{k+1} + T]$, which means the input constraint is also satisfied. Therefore, Part 2 is proven.

Part 3: To show the state $\bar{s}(\tau|t_{k+1})$ under (18) satisfies constraint (8).

For $\tau \in [t_{k+1}, t_k + T]$, let $x = \bar{s}^*(\tau|t_k)$ and $y = \bar{x}(\tau|t_{k+1})$. Because $\|\bar{s}(\tau|t_{k+1}) - \bar{s}^*(\tau|t_k)\| \leq \frac{\mu}{v}(e^{\delta} - 1)e^{v(T - t_{k+1})}$, $\bar{x}(\tau|t_{k+1}) \in \bar{x}(\tau - t_{k+1})$ by Lemma 3.

Then, take $\tau \in [t_k + T, t_{k+1} + T]$ into account. Since $\bar{s}(t_k + T|t_{k+1}) \in \Omega_\alpha$ by (26) and $\bar{x}(t_k + T|t_{k+1}) \in \bar{x}(T - \delta)$, according to Assumption 4, we have $\bar{x}(\tau|t_{k+1}) \in \bar{x}(\tau - t_{k+1})$.

The proof of Theorem 1 is completed.

\[\square\]

Remark 2. Lemma 1 shows that by selecting $\varepsilon < 1$, the difference is compressed in (15) compared with (16). It implies that by mapping the system state $x_1$ and $x_2$ to sliding mode variable $s$, the effect caused by disturbances can be attenuated with respect to optimization problem. Therefore, the inherent robustness of MPC can be improved.

Remark 3. Theorem 1 reveals that the recursive feasibility is affected by the disturbance bound and it can be observed that the maximum disturbance is allowed to be $\mu_0 = \frac{v}{\alpha - \varepsilon} \frac{\mu}{\alpha - \varepsilon}$. Meanwhile, the bound can be uprisen by decreasing $\varepsilon$.

B. Stability analysis

To guarantee stability of closed-loop system under composite control in the whole process, the following theorem is given.

Theorem 2. For system (1) under the control input generated by Problem 1, if the conditions in Theorem 1 and the following condition
\[
\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 < 0
\]
are satisfied, where $\zeta_1 = (\varepsilon^2(T - \delta) + 2\varepsilon^2\mu(\frac{1}{v}(e^{v(T - \delta)} - 1) - T + \delta))e^{(e^{2\varepsilon(T - \delta)} - 1) - 2\varepsilon^2\mu(\frac{1}{v}(e^{v(T - \delta)} - 1) + T - \delta)}$, $\zeta_2 = \frac{\varepsilon^2\mu}{(e^{2\varepsilon(T - \delta)} - 1)T(1 - \delta)}$, $\zeta_3 = \frac{2\varepsilon^2\mu}{(e^{2\varepsilon(T - \delta)} - 1)T(1 - \delta)}$, and $\zeta_4 = \frac{\varepsilon^2\mu}{(e^{2\varepsilon(T - \delta)} - 1)T(1 - \delta)}$, then

(1) the sliding variable $s$ converges asymptotically to the set $\Omega_c$ under the control law (13),

(2) the closed-loop system of (1) is ISS under $u_s(x)$ after the sliding variable $s$ enters the set $\Omega_c$.

Proof. Define $\Delta J \triangleq J(\bar{s}(\tau|t_{k+1}), \bar{u}(\tau|t_{k+1})) - J(\bar{s}^*(\tau|t_k), \bar{u}^*(\tau|t_k))$. Then expanding the term $\Delta J$ yields that
\[
\Delta J = \int_{t_{k+1}}^{t_{k+1} + T} L(\bar{s}(\tau|t_{k+1}), \bar{u}(\tau|t_{k+1}))d\tau \\
+ V_f(\bar{s}(t_{k+1} + T|t_{k+1})) - \int_{t_k}^{t_{k+1} + T} L(\bar{s}^*(\tau|t_k), \bar{u}^*(\tau|t_k))d\tau - V_f(\bar{s}^*(t_k + T|t_k))
= \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4,$
where
\[
\Delta_1 = \int_{t_{k+1}}^{t_{k+1}+T} q \left( \tilde{s}^2(\tau|t_{k+1}) - \tilde{s}^*(\tau|t_k) \right) d\tau,
\]
\[
\Delta_2 = \int_{t_{k+1}}^{t_{k+1}+T} r \left( \|	ilde{u}(\tau|t_{k+1}) - u_s(\tilde{x}(\tau|t_{k+1}))\| \right)\left| \tilde{u}^*(\tau|t_{k+1}) - u \tilde{u}(\tau|t_{k+1}) \right| d\tau,
\]
\[
\Delta_3 = \int_{t_{k+1}}^{t_{k+1}+T} \left( L(\tilde{s}(\tau|t_{k+1}), \bar{u}(\tau|t_{k+1})) d\tau + V_f(\tilde{s}(t_{k+1} + T|t_{k+1})) - V_f(\tilde{s}(t_{k+1} + T|t_{k+1})) \right),
\]
\[
\Delta_4 = - \int_{t_{k+1}}^{t_{k+1}+T} L(\tilde{s}^*(\tau|t_{k+1}), \bar{u}^*(\tau|t_{k+1})) d\tau.
\]

For \(\Delta_1\), it holds that
\[
\Delta_1 \leq q \int_{t_{k+1}}^{t_{k+1}+T} |\tilde{s}(\tau|t_{k+1}) - \tilde{s}^*(\tau|t_k)| \times |\tilde{s}(\tau|t_{k+1}) + \tilde{s}^*(\tau|t_k)| d\tau 
\leq q \int_{t_{k+1}}^{t_{k+1}+T} |\tilde{s}(\tau|t_{k+1}) - \tilde{s}^*(\tau|t_k)| \times (|\tilde{s}(\tau|t_{k+1}) - \tilde{s}^*(\tau|t_k)| + 2|\tilde{s}^*(\tau|t_k)|) d\tau 
\leq q \int_{t_{k+1}}^{t_{k+1}+T} \left( \frac{2q_2^2 c_2^2}{v^2} (e^{v\delta_1} - 1)^2 (e^{2v(T - \delta_2)} - 1) \right) d\tau.
\]

Applying H"older inequality yields that
\[
\Delta_1 \leq \left( \int_{t_{k+1}}^{t_{k+1}+T} (|\tilde{s}(\tau|t_k)|^2 d\tau \right)^{\frac{1}{2}} \times \left( \int_{t_{k+1}}^{t_{k+1}+T} \left( \frac{2q_2^2 c_2^2}{v^2} (e^{v\delta_1} - 1)^2 (e^{2v(T - \delta_2)} - 1) \right)^{\frac{1}{2}} d\tau \right)^{\frac{1}{2}}
\]
\[
\leq \left( \epsilon^2 (T - \delta_1) + \frac{2q_2^2 c_2^2}{v^2} \left( \frac{1}{v^2} (e^{v\delta_1} - 1) - \frac{2}{v^2} (e^{v\delta_2} - 1) + T - \delta_1 \right) \left( \frac{1}{v^2} (e^{v\delta_1} - 1) - \frac{2}{v^2} (e^{v\delta_2} - 1) + T - \delta_1 \right) \right)^{\frac{1}{2}}
\]
\[
+ \left( \frac{2q_2^2 c_2^2}{v^2} \left( \frac{1}{v^2} (e^{v\delta_1} - 1) - \frac{2}{v^2} (e^{v\delta_2} - 1) + T - \delta_1 \right) \left( \frac{1}{v^2} (e^{v\delta_1} - 1) - \frac{2}{v^2} (e^{v\delta_2} - 1) + T - \delta_1 \right) \right)^{\frac{1}{2}}
\]
\[
\leq \left( \epsilon^2 (T - \delta_1) + \frac{2q_2^2 c_2^2}{v^2} \left( \frac{1}{v^2} (e^{v\delta_1} - 1) - \frac{2}{v^2} (e^{v\delta_2} - 1) + T - \delta_1 \right) \right)^{\frac{1}{2}}
\]
\[
\leq \left( \epsilon^2 (T - \delta_1) + \frac{2q_2^2 c_2^2}{v^2} \left( \frac{1}{v^2} (e^{v\delta_1} - 1) - \frac{2}{v^2} (e^{v\delta_2} - 1) + T - \delta_1 \right) \right)^{\frac{1}{2}}
\]
\[
\leq \left( \epsilon^2 (T - \delta_1) + \frac{2q_2^2 c_2^2}{v^2} \left( \frac{1}{v^2} (e^{v\delta_1} - 1) - \frac{2}{v^2} (e^{v\delta_2} - 1) + T - \delta_1 \right) \right)^{\frac{1}{2}}
\]
\[
+ \frac{q_2^2 c_2^2}{v^2} \left( e^{v\delta_1} - 1 \right)^2 (e^{2v(T - \delta_1)} - 1).
\]

Considering \(\Delta_2\), we have
\[
\Delta_2 \leq r \int_{t_{k+1}}^{t_{k+1}+T} |u_s(\tilde{x}(\tau|t_{k+1})) - u_s(\tilde{x}^*(\tau|t_k))| d\tau.
\]

By referring to Assumption 2, \(\Delta_2\) is bounded by
\[
\Delta_2 \leq \frac{r \eta \mu}{\nu} (e^{\nu \delta_1} - 1) \int_{t_{k+1}}^{t_{k+1}+T} e^{v|\tau - t_{k+1}|} d\tau 
\leq \frac{r \eta \mu}{\nu^2} (e^{\nu |\tau - t_{k+1}|} - 1).
\]

The upper bound of \(\Delta_3\) can be obtained as
\[
\Delta_3 = \int_{t_{k+1}}^{t_{k+1}+T} L(\tilde{s}(\tau|t_{k+1}), \bar{u}(\tau|t_{k+1})) d\tau + V_f(\tilde{s}(t_{k+1} + T|t_{k+1})) - V_f(\tilde{s}(t_{k+1} + T|t_{k+1})) 
\leq V_f(\tilde{s}(t_{k+1} + T|t_{k+1})) - V_f(\tilde{s}(t_{k+1} + T|t_{k+1})) 
\leq \frac{p \mu \bar{c}}{\nu} (e^{\nu |\tau - t_{k+1}|} - 1) (e^{v(T - \delta_1)} - 1).
\]

For \(\Delta_4\), it is clear that
\[
\Delta_4 \leq - q \int_{t_{k+1}}^{t_{k+1}+T} (\tilde{s}^*|^2(\tau|t_k) d\tau.
\]

Then, we first consider the condition \(s(t_k|t_k) = s(t_k) \notin \Omega_\varepsilon\) and \(s(\tau)\) cannot enter into \(\Omega_\varepsilon\). For \(\tau \in [t_k, t_{k+1}]\). According to (16), the following inequality holds
\[
|\tilde{s}^*(\tau|t_k)| \geq |s(\tau)| - \frac{\bar{c} \mu}{v} (e^{v(T - t_{k+1})} - 1)\]
\[
(29)
\]

Taking (29) into (28) yields
\[
\Delta_4 \leq - q \int_{t_k}^{t_{k+1}+T} ((s(\tau)| - \frac{\bar{c} \mu}{v} (e^{v(T - t_{k+1})} - 1))^2 d\tau 
\leq - q \int_{t_k}^{t_{k+1}+T} ((\tau - \frac{\bar{c} \mu}{v} (e^{v(T - t_{k+1})} - 1)^2 d\tau 
\leq - q \delta^2 z + \frac{2q_2^2 c_2^2}{v^2} \left( \frac{1}{v^2} (e^{v\delta_1} - 1) - \frac{2}{v^2} (e^{v\delta_2} - 1) + T - \delta_1 \right) 
\leq - q \delta^2 z + \frac{2q_2^2 c_2^2}{v^2} \left( \frac{1}{v^2} (e^{v\delta_1} - 1) - \frac{2}{v^2} (e^{v\delta_2} - 1) + T - \delta_1 \right) 
\leq - q \delta^2 z + \frac{2q_2^2 c_2^2}{v^2} \left( \frac{1}{v^2} (e^{v\delta_1} - 1) - \frac{2}{v^2} (e^{v\delta_2} - 1) + T - \delta_1 \right) \}
\]

By (27), it can be induced that \(\Delta J \leq 0\), which implies \(J(\tilde{s}^*(\tau|t_k), \bar{u}^*(\tau|t_k)) \rightarrow -\infty\) for \(k \rightarrow \infty\). However, it contradicts the fact that \(J(\tilde{s}^*(\tau|t_k), \bar{u}^*(\tau|t_k)) \geq 0\). Thus there exists a time instant \(t\) such that \(s(t) \in \Omega_\varepsilon\). The proof of (1) is proven.

After \(s(t)\) enters \(\Omega_\varepsilon\), the controller law is switched to \(u_s(x)\). According to Assumption 5, the closed-loop system of (1) is ISS.

The proof of Theorem 2 is completed \(\Box\)
IV. Simulation

Consider a nonlinear cart-damper-spring system shown in Fig. 1 with the following dynamics

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= -\frac{\gamma}{M_c}e^{-x_1(t)}x_1(t) - \frac{h_d}{M_c}x_2(t) + \frac{u(t)}{M_c} + \omega(t),
\end{align*}
\]

where \(x_1(t)\) and \(x_2(t)\) denote the displacement of the cart and its velocity respectively, \(M_c = 1.25\) kg represents the mass of the cart, \(\gamma = 0.90\) N/m is the stiffness factor of spring, and \(h_d = 0.42\) N.s/m is the damper factor. The control input \(u(t)\) is constrained by \(|u(t)| \leq 8\), and the bound of external disturbances \(\frac{|\omega(t)|}{M_c} \leq \mu = 0.1\). For the optimization problem in Problem 1, the weight coefficients of cost function are set as \(q = 0.35\), \(r = 0.1\) and \(p = 0.85\) respectively. The local feedback coefficient \(k\) is determined as 1.65, the Lipschitz constant \(\nu\) is computed as \(\nu = 1.15\), the sampling interval \(\delta\) and prediction horizon are chosen as 0.1s and 2s respectively, the terminal region parameters are designed as \(\alpha = 0.0105\) and \(\varepsilon = 0.01\), and the sliding mode parameters are taken as \(c_1 = c_2 = 0.18\). Referring to [26], the control law generated by SMC is designed as

\[
u_s(x) = -\frac{1}{c_2M_c}(c_2F_x + c_1x_1 + ks) \quad (30)
\]

where \(F_x = -\frac{\gamma}{M_c}e^{-x_1(t)}x_1(t) - \frac{h_d}{M_c}x_2(t)\). The initial state of system is set as [2, 3].

Fig. 1. Schematic illustration of a cart–damper–spring system

Fig. 2. Comparison of displacements of the closed-loop system

The simulation example is conducted following Algorithm 1 by MATLAB package. In order to compare the performance of the composite control, the MPC method without disturbance observer in [16] and the single SMC method (30) are executed. The displacement and velocity of closed-loop are illustrated in Fig. 2 and 3, respectively. The control input signal is shown in Fig 4. It can be seen that the steady error of displacement and velocity under composite control is reduced comparable with that under MPC in [16], which owes to high robustness of SMC. Meanwhile, the displacement, velocity and control input are always subject to state constraints under the composite control, while the displacement and control input of SMC (30) violate the state constraint, which implies that SMC behaves the function of handling constraints in the composite control with the assistance of MPC.

V. Conclusions

In this paper, a structure of composite control consisting of SMC and MPC has been developed for a constrained second-order nonlinear system with external disturbances. Based on
linear sliding mode surface, MPCs first exploited to generate control law for sliding mode and then the control law is switched at the neighboring region of sliding mode surface. The recursive feasibility of MPC and stability of closed-loop has been shown to be satisfied. The whole process guarantees robustness and constraints satisfaction. The simulation has verified the performance of the proposed composite control.

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