TURAEV-VIRO MODULES OF SATELLITE KNOTS

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Abstract. Given an oriented knot $K$ in $S^3$ and a TQFT, Turaev and Viro defined modules somewhat analogous to the Alexander module. We work with the $(V_p, Z_p)$ theories of Blanchet, Habegger, Masbaum and Vogel [BHMV] for $p \geq 3$, and consider the associated modules. In [G], we defined modules which also depend on the extra data of a color $c$ which is assigned to a meridian of the knot in the construction of the module. These modules can be used to calculate the quantum invariants of cyclic branched covers of knots and have other uses.

Suppose now that $S$ is a satellite knot with companion $C$, and pattern $P$. We give formulas for the Turaev-Viro modules for $S$ in terms of the Turaev-Viro modules of $C$ and similar data coming from the pattern $P$. We compute these invariants explicitly in several examples.

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§1 Turaev-Viro modules

Let $(V, Z)$ be a Topological Quantum Field Theory over a field $f$ defined on a cobordism category whose morphisms are oriented 3-manifolds perhaps with extra structure. Let $(M, \chi)$ be a closed oriented 3-manifold $M$ with this extra structure together with $\chi \in H^1(M)$ where $\chi : H_1(M) \to Z$ is onto. Let $M_\infty$ denote the infinite cyclic cover of $M$ given by $\chi$. Consider a fundamental domain $E$ for the action of the integers on $M_\infty$ bounded by lifts of a surface $\Sigma$ dual to $\chi$, and in general position. $E$ can be viewed as a cobordism from $\Sigma$ to itself. $Z(E)$ can be viewed as an endomorphism of $V(\Sigma)$.

Let $\mathcal{K}(V)$ be the generalized 0-eigenspace for the action of $Z(E)$ on $V(\Sigma)$, i.e. $\mathcal{K}(V) = \bigcup_{k \geq 1} \text{Kernel}(Z^k)$. $Z(E)$ induces an automorphism $Z^b(E)$ of $V^b(\Sigma) = V(\Sigma)/\mathcal{K}(V)$. Alternatively $V^b$ can be defined as $\bigcap_{k \geq 1} \text{Image}(Z^k)$. The Turaev-Viro module $(M, \chi)$ associated to $(V, Z)$ is simply is $V(\Sigma)^b$ viewed as a $f[t, t^{-1}]$-module where $t$ acts by $Z^b(E)$.

Theorem 1.1 (Turaev-Viro). This module does not depend on the choice of $E$.

Sketch of Proof. A detailed exposition of Turaev-Viro’s proof [TV] is given in [G,§1]. Here we give the main idea. Suppose $E'$ is another choice of fundamental domain. Without loss of generality we may assume that $E'$ has been shifted by the covering
transformation so that $E$ and $E'$ are disjoint. Let $W$ denote the cobordism indicated by the following schematic diagram for the infinite cyclic cover.

\[ W \]

\[ T(W) \]

As $E \cup T(W) = W \cup E'$, we have $Z(W) \circ Z(E) = Z(E') \circ Z(W)$ after identifying $V(\Sigma)$ with $V(T(\Sigma))$, and $V(\Sigma')$ with $V(T(\Sigma'))$. After dividing out by $\mathcal{K}(V)$, and $\mathcal{K}(V)'$, $Z_W$ becomes invertible and so provides a similarity between $Z^p(E)$ and $Z^p(E')$. □

We will now specialize to the case that $(V, Z)$ is the $(V_p, Z_p)$ theory of Blanchet, Habegger, Masbaum and Vogel [BHMV] for $p \geq 3$. These are combinatorial versions of the Witten-Reshetikhin-Turaev TQFTs associated to $SU(2)$ and $SO(3)$ [W,RT]. We will work over the field of fractions of $k_p$, which we denote $f_p$. A “color” for this theory is an integer from zero to $p/2 - 2$ if $p$ is even. If $p$ is odd, a color is an even positive integer less than or equal to $p - 3$. Here we depart from the usage in [BHMV,G], by assuming that colors are always even, when $p$ is odd. One advantage is that the tensor product axiom will always hold. We let $\mathcal{C}$ denote the set of colors. A triple of colors $\{i, j, k\}$ is called admissible if $i + j + k \equiv 0 \pmod{2}$, and $i \leq j + k$, $j \leq i + k$, and $k \leq i + k$. Moreover their sum must be small. In particular $i + j + k \leq p - 4$, if $p$ is even. Also $i + j + k \leq 2p - 4$, if $p$ is odd. Let $\mathcal{A}$ denote the set of admissible triples. Let $\mathcal{A}(i, j)$ denote the set of colors $k$ such that $\{i, j, k\} \in \mathcal{A}$. Let $\mathcal{A}(i)$ denote the set of ordered pairs of colors $(j, k)$ such that $\{i, j, k\} \in \mathcal{A}$. We also let $\mathcal{A}(i)$ denote the set of colors $j$ such that $\{i, j, j\} \in \mathcal{A}$.

The objects of our cobordism theory are oriented surfaces with colored banded points and $p_1$-structure. A banded point is simply a point with an oriented arc through it. The empty set $\emptyset$ is also an object. $V(\emptyset)$, according to the axioms of for a TQFT, is the scalar field $f_p$. A morphism is an oriented 3-manifold with $p_1$-structure with admissibly colored trivalent banded graphs. A trivalent banded graph is an oriented surface which deformation retracts to a trivalent graph. A coloring is an assignment of colors to the edges of the core graph so that the colors at each vertex are admissible. By closed 3-manifold, from now on we mean a morphism from $\emptyset$ to $\emptyset$. If $M$ is a closed 3-manifold, then $Z_p(M)$ is multiplication by a scalar which is denoted $<M>$. 

Figure 1
We let $\omega$ denote the linear combination of links in solid torus $\eta \sum_{e \in E} \Delta_c e_c$, where $\Delta_c$ denotes the evaluation of an unknot diagram colored $c$ and $e_c$ denotes the closure of the Jones-Wenzl idempotent of the c-strand Temperley-Lieb algebra. $\omega$ is need for the surgery formula [BHMV, §1.C]. See the discussion at the end of the proof of [G, 8.2]. If we have a linear combination admissibly colored trivalent banded graphs in $\mathbb{R}^3$, by an evaluation, we mean the scalar obtained as in [KL] with $A$ a primitive $4p$th root of unity. Thus the empty graph evaluates to one. In general, if $M$ denotes the same graph in $S^3$, the one point compactification of $\mathbb{R}^3$ with $p_1$-structure which extends over the 4-ball, then $< M >_p$ is $\eta$ times the evaluation of the graph in $\mathbb{R}^3$.

Let $K$ be an oriented knot in $S^3$. Let $M(K)$ denote 0-framed surgery to $S^3$ along $K$ equipped with a $p_1$-structure with zero $\sigma$-invariant. Let $M(K,c)$ denote $M(K)$ with a meridian colored by a color $c$. Let $\chi$ evaluate to one on a positive meridian of $K$. A surface dual to $\chi$ will be called a splitting surface for $M(K)$. Let $Z_p(K,c)$ denote the Turaev-Viro module of $(M(K,c), \chi)$, thought of as a similarity class of automorphisms for a finite dimensional vector space over $f_p$. Thus it may be described by a matrix $M$ with entries in $f_p$. In this case, we will write $Z_p(K,c) = M$. Sometimes we may wish to describe $Z_p(K,c)$ with a matrix $M$ or automorphism $Z$ which may have a non trivial generalized 0-eigenspace. Then we write $Z_p(K,c) \equiv M$, or $Z$ as the case may be, and it is understood that $Z_p(K,c)$ is given by the induced map on the quotient after we divide out by this generalized 0-eigenspace. We let $Z_p(K)$ denote $Z_p(K,0)$. It is easy to see that $Z_p(K,c)$ is zero if $c$ is odd, and $p$ is even. $Z_p(K,c)$ is undefined if $c$ is odd, and $p$ is odd. It is shown in [G], that $Z_p(K)$ is unchanged if we change the string orientation on $K$.

As motivation for studying $Z_p(K,c)$, we mention some results of [G].

**Theorem 1.2.** If $K$ is a fibered knot in a homology sphere which is a homotopy ribbon knot, then one is an eigenvalue of $Z_p(K)$.

Let $M(K)_d$ denote the d-fold cyclic cover of $M(K)_d$ associated to $\chi$ with the $p_1$-structure induced from $M(K)$ by the projection.

**Theorem 1.3.** $< M(K)_d >_p$ is the trace of $Z_p(K)^d$. Thus $< M(K)_d >_p$ can be computed by a linear recursion formula given by the characteristic polynomial of $Z_p(K)$.

Let $\sigma_\lambda(K) = \text{Sign}((1-\lambda)V + (1-\bar{\lambda})V^\dagger)$, where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $V$ is a Seifert matrix for $K$. Following [KM], let $\sigma_d(K) = \sum_{i=1}^{d-1} \sigma_{\lambda_d^i}(K)$, where $\lambda_d = e^{2\pi i/d}$. These are called the total $d$-signatures of $K$. The $\sigma$ invariant of the $p_1$-structure on $M(K)_d$ is $3\sigma_d(K)$. Let $K_d$ denote the branched cyclic d-fold cover of $S^3$ along $K$ with a $p_1$-structure with $\sigma$-invariant $3\sigma_d(K)$.

**Theorem 1.4.**

$$< K_d >_p = \eta \sum_{c \in \mathbb{C}} \Delta_c \text{Trace}(Z_p(K,c)^k)$$

Note that $\text{Trace}(Z_p(K,c)^k$ can be computed recursively from the characteristic polynomial of $Z_p(K,c)$.

Let $d_g(p,c)$ denote the dimension of $V_p$ of a surface of genus $g$ with a single point colored $c$. We have the following theorem of Walker’s [Wa1] who proved that the rank of $Z(E)$ is an invariant of the pair $(M, \chi)$. His work stimulated Turaev and Viro to refine his theorem and prove (1.1). Theorem (1.5) may be used to estimate the genus of a knot.
Theorem 1.5 (Walker). If \( K \) has genus \( g \),

\[
\text{rank}(Z_p(K, c)) \leq d_g(p, c).
\]

Perhaps Walker did not consider the case \( c \neq 0 \). I do not know. From now on, to cut down on the clutter of subscripts, we will omit the subscript \( p \) from \( Z \) and \( < > \).

§2 Satellite Knots

Let \( C \) be an oriented knot in \( S^3 \), equipped with its standard framing. The pushoff with this framing is a longitude which bounds in the complement. A pattern consists of a oriented link of two components in \( S^3 \). One component \( A \) is called the axis and must be unknotted. The other component \( E \) is called the embellishment. Given \( C \) and a pattern \( E \), a satellite knot \( S \) is formed with \( C \) as its companion. Because \( C \) is framed, its tubular neighborhood comes equipped with an identification to a standard solid torus. We give \( A \) the standard framing. The exterior of \( A \) is also a standard solid torus with a knot \( E \) in it. \( S \) is the image of \( E \) if we replace the tubular neighborhood of \( C \) by the exterior of \( A \). More precisely, we recover the 3-sphere if we glue the exterior of \( C \) to the exterior of \( A \) such that the oriented meridian of \( A \) goes to the oriented longitude of \( C \), and the oriented longitude of \( A \) goes to the oriented meridian of \( C \). Note that this gluing map is orientation reversing. We sometimes denote \( S \) by \( C \ast P \).

![Diagram of satellite knots](image)

Figure 2

The above example is the (2,1) cable of the figure eight knot. A cable knot is where \( E \) is a torus knot on the boundary of the exterior of \( A \). The winding number of the pattern is the linking number of \( A \) and \( E \). In the above example it is two. Since the invariant we are calculating is actually insensitive to the string orientation of a knot, we assume from now on that the winding number is nonnegative.
There is a long tradition in knot theory for expressing invariants of satellite knots in terms of invariants of the companion and the pattern. See the following papers and the references therein: [ML] for abelian invariants, [Li] for signatures and Casson-Gordon Invariants, and [MS1,MS2] for the Jones polynomial. We mention now some precursor results from [G] on Turaev-Viro modules of satellite knots.

A connected sum of $K_1$ and $K_2$ can be viewed as a satellite with companion $K_1$ and the pattern of winding number one obtained by taking $E$ to be $K_2$ and $A$ to be a meridian of $K_2$. In [G,(7.4)], we showed

$$Z(K_1 \# K_2) = \bigoplus_{c \in \mathcal{E}} Z(K_1, c) \otimes Z(K_2, c).$$

(2.1)

More generally:

$$Z(K_1 \# K_2, c) = \bigoplus_{(i,j) \in \mathcal{A}(c)} Z(K_1, i) \otimes Z(K_2, j).$$

(2.2)

Below we will develop a formula which generalizes these. Another important satellite construction is that of the k-twisted double. The winding number is zero. Here is the pattern $D(k)$ (with $k=-1$):

![Figure 3]

In [G], we derived formulas for $Z(C \ast D(k), c)$. The formulas were rather complicated. Let $C_s$ denote the evaluation of a diagram of $C$ with zero writhe colored by $s$. We will show in §8 how using the methods of [G], one can obtain the following formula under the hypothesis that $C_s$ is nonzero for all colors $s$:

$$Z(C \ast D(k)) \equiv \eta \kappa^{-3} \left( \sum_{s \in \mathcal{A}(i,j)} \mu_s^k C_s \right)_{i,j \in \mathcal{E}}$$

(2.3)

where $\mu_i = (-1)^i A^{i^2+2i}$ is the contribution of a positive curl colored $i$. Here we also use the notation that $(a_{ij})_{i,j \in \mathcal{E}}$ denotes the square matrix with entries $a_{ij}$ as $i$ and $j$ vary.
$j$ range over $I$. More generally, if $C_i$ is nonzero for all $i \in A(c)$, we will obtain,

\begin{equation}
Z(C \ast D(k), c) \equiv \eta_{k}^{-3} \left( \frac{\Delta_i \mu_i}{\theta(i, i, c)} \sum_{t \in A(i, j)} \frac{\mu_t^k C_t}{\theta(i, j, t)} \right) \operatorname{Tet} \left( \begin{array}{ccc} t & i & i \\ c & j & j \end{array} \right)_{i, j \in A(c)}.
\end{equation}

Here we adopt some notation from [KL]. $\theta(i, j, k)$ denotes the evaluation of a planar theta curve with edges colored $i, j,$ and $k$. $\operatorname{Tet} \left[ \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{array} \right]$ denotes the evaluation of a planar tetrahedron with edges around some face colored $a_1, a_2,$ and $a_3$, and for each $i$, the edge opposite the edge colored $a_i$ colored $b_i$. Note that $\operatorname{Tet} \left[ \begin{array}{ccc} t & i & i \\ c & j & j \end{array} \right]$ becomes $\theta(i, j, t)$ when we let $c = 0$. Also $\theta(i, i, c)$ becomes $\Delta_i$. So (2.4) becomes (2.3).

Below we will also give a different formula for $Z(C \ast D(k))$ coming from the satellite description. In §8, we also use these other formulas to give a new derivation of (2.4) without the requirement that any $C_i$ be nonzero.

We need to give a slightly different description of the satellite $S$ which will be more suitable for glueing formulas. Let $m_C$ denote a meridian of $C$ in $M(C)$. Now $m_C$ is isotopic in $M(C)$ to core of the solid torus added to the exterior of $C$ in constructing $M(C)$. So the exterior of $m_C$ in $M(C)$ is just the exterior of $C$ in $S^3$. But the meridian of $m_C$ is the longitude of $C$ with the opposite orientation and the longitude of $m_C$ using the obvious framing for $m_C$ is a meridian of $C$. Thus we have that:

**Lemma 2.5.** $S$ is the image of $E$ in the union of the exterior of $A$ with the exterior of $m_C$ in $M(C)$ by an orientation reversing diffeomorphism which preserves the longitudes but reverses the meridians. So $M(S)$ is the union of the exterior of $A$ in $M(E)$ with the exterior of $m_C$ in $M(C)$ by an orientation reversing diffeomorphism which preserves the longitudes but reverses the meridians.

**§3 Gluing formulas**

Let $T$ denote a solid torus $S^1 \times D^2$ with a fixed $p_1$-structure. The meridian $m(T)$ is given by $\{1\} \times S^1$. The longitude $l(T)$ is given by $S^1 \times \{1\}$. Let $e_i$ denote the element of $V(T)$ obtained by coloring $S^1 \times \{0\}$ with $i$.

Let $K$ a framed knot in a closed 3-manifold $M$. By this we mean a framed knot disjoint from the colored graph. We may then isotope the $p_1$-structure on $M$ so that there is an orientation reversing diffeomorphism from a tubular neighborhood of $K$ to $T$ which preserves $p_1$-structure, and meridian, and sends the longitude to minus the longitude. Here the longitude is specified by the framing and the orientation on the knot. The meridian is oriented so that $K$ intersects the meridional disk with intersection number plus one. We assume this has been done. Call a knot $K$ equipped with such a diffeomorphism $\phi_K$, a framed knot. In this situation, let $(M, K, a)$ denote the closed 3-manifold given by $M$ after adjoining $K$ colored $a$ to the graph. Also let $\operatorname{Ext}(K)$, the exterior of $K$, be the complement the interior of the domain of $\phi_K$, with the boundary identified with $\partial T$ by $\phi_K$, which is now orientation preserving. We need the following lemmas.

**Lemma 3.1.** $\operatorname{Ext}(K) = \sum_{a \in \mathbb{Z}} \mu_a \{ M, K, a \}$. 
Proof. Ext(\(K\)) paired with \(e_a\) is given by \(< (M, K, a) >\) using the pairing \(Q_2\) of [BHMV]. \(\{e_a\}_{a \in \mathbb{C}}\) is a basis and pairing is nondegenerate. \(\square\)

Lemma 3.2. Suppose \(K_1\) and \(K_2\) are pramed knots in a closed 3-manifolds \(M_1\), and \(M_2\). \(< \text{Ext}(K_1) \cup_s \phi K_2^{-1} \circ \phi K_1 (\text{Ext}(K_2)) > = \sum_{a \in \mathbb{C}} < (M, K_1, a) > < (M, K_2, a) >\)

Proof. \(< \text{Ext}(K_1) \cup_s \phi K_2^{-1} \circ \phi K_1 (\text{Ext}(K_2)) > \geq < \text{Ext}(K_1), (\text{Ext}(K_2)) >\). Moreover \(\{e_a\}_{a \in \mathbb{C}}\) is orthonormal with respect the Hermitian pairing \(Q_2\). \(\square\)

Let \(K^*\) denote \(K\) in \(M\) after reversing the orientation on \(M\), but preserving the string orientation on \(K\). Note that by the above construction starting with \(K^*\), we have \(\text{Ext}(K^*)\) has boundary identified with \(\partial J\) by \(\phi K^*\), which is orientation preserving. Also the longitude of \(K^*\) is the longitude of \(K\), but the meridian of \(K^*\) is oriented opposite to that of \(K\). Consider now \(\text{Ext}(K_1) \cup_s \phi K_2^{-1} \circ \phi K_1 (\text{Ext}(K_2^*))\), which we denote by \(M_1 K_1 \wedge K_2 M_2\). This obtained by gluing the exterior of \(K_1\) with the exterior of \(K_2\) by an orientation reversing diffeomorphism which preserves the longitudes but reverses the meridians just as in Lemma(2.5). (3.2) in this situation becomes:

\[(\text{3.3}) \quad < M_1 K_1 \wedge K_2 M_2 > = \sum_{a \in \mathbb{C}} < (M, K_1, a) > < (M, K_2, a) >\]

Also \(M(\mathcal{E})\), and \(M(\mathcal{C})\) both bound 4-manifolds: \(B(\mathcal{E})\), and \(B(\mathcal{C})\) respectively with zero signature such that the inclusions \(M(\mathcal{C}) \hookrightarrow B(\mathcal{C})\), and \(M(\mathcal{E}) \hookrightarrow B(\mathcal{E})\) induce isomorphisms on first homology. This follows from the fact that \(\Omega_3(S^1) = 0\).

As \(M(\mathcal{E})\), \(M(\mathcal{C})\) both have a \(p_1\)-structure with trivial \(\sigma\)-invariant, the \(p_1\)-structure extends over \(B(\mathcal{E})\), \(B(\mathcal{C})\) respectively. Thus the \(p_1\)-structure on \(M(\mathcal{E}) A \wedge \mathcal{m}_C M(\mathcal{C})\) extends over \(B\) obtained by gluing \(B(\mathcal{E})\) to \(B(\mathcal{C})\) along tubular neighborhoods of \(\mathcal{E}\) and \(\mathcal{C}\) identified by the above identification. Moreover the kernels of the maps on \(H_1\) induced by the inclusions \(\text{Ext}(\mathcal{m}_C) \hookrightarrow B(\mathcal{C})\), and \(\text{Ext}(\mathcal{E}) \hookrightarrow B(\mathcal{E})\) are both generated by the longitudes. Thus the Maslov index of the triple of kernels needed to compute the signature of \(B\) is zero. It follows that the \(\sigma\)-invariant of the induced \(p_1\)-structure on \(\partial B = M(\mathcal{E}) A \wedge \mathcal{m}_C M(\mathcal{C})\) is zero. Thus:

\[(\text{3.4}) \quad M(S) = M(\mathcal{E}) A \wedge \mathcal{m}_C M(\mathcal{C})\]

Suppose \(M_2\) above is not a closed 3-manifold, but a morphism from \(\Sigma\) to \(\Sigma'\), and \(K_2\) is in the interior of \(M_2\). Then one has:

\[(\text{3.5}) \quad Z(M_1 K_1 \wedge K_2 M_2) = \sum_{a \in \mathbb{C}} < (M, K_1, a) > Z(M, K_2, a)\]

We also want to glue morphisms along the exteriors of arcs. Let \(J\) denote the solid tube \(I \times D^2\). We equip \(J\) with a fixed \(p_1\)-structure. The meridian \(m(\mathcal{J})\) is given by \(\{1\} \times S^1\), oriented with the standard orientation on \(S^1\). The parallel \(p(\mathcal{J})\) is given by \(I \times [1]\) oriented with the orientation on \(I\) from zero to one.
Let $M$ be a morphism from $\Sigma$ to $\Sigma'$, and suppose neither of these surfaces are empty. Let $\gamma$ be a smooth framed arc in $M$ from $p \in \Sigma$ to $q \in \Sigma'$. We may then isotope the $p_1$-structure on $M$ so that there is an orientation reversing diffeomorphism $\phi_{\gamma}^+$ from a tubular neighborhood of $\gamma$ to $\mathcal{I}$ which preserves $p_1$-structure, and parallel, and sends the meridian to minus the meridian. Call an arc $\gamma$ equipped with such a diffeomorphism, a $(+)$pramed arc. Here the parallel is an arc in the boundary of the tubular neighborhood and is specified by the framing and the orientation on the knot. The meridian is oriented so that $\gamma$ intersects the meridional disk with intersection number plus one.

Let $-\mathcal{I}$ denote $\mathcal{I}$ with the same $p_1$-structure and with the same parallel but with the opposite ambient orientation and with the oppositely oriented meridian. We may also isotope the $p_1$-structure on $M$ so that there is an orientation reversing diffeomorphism $\phi_{\gamma}^-$ from a tubular neighborhood of $\gamma$ to $-\mathcal{I}$ which preserves $p_1$-structure, and parallel, and sends the meridian to minus the meridian. Call an arc $\gamma$ equipped with such a diffeomorphism, a $(-)$pramed arc.

In this situation, let $(M, \gamma, a)$ denote morphism from say $(\Sigma, p, a)$ to $(\Sigma', q, a)$ given by $M$ after adjoining $\gamma$ colored $a$ to the graph. Also let $\text{Ext}(\gamma)$, the exterior of $\gamma$, be the complement the interior of the domain of $\phi_{\gamma}^\pm$, with the boundary identified with $\pm I \times S^1$ by $\phi_{\gamma}^\pm$, which is now orientation preserving. Suppose $\gamma_1$ is a $(+)$pramed arcs in a morphism $M_1$, and $\gamma_2$ is a $(-)$pramed arc in a morphism $M_2$. Let $M_1\gamma_1 \#_\mathcal{I} M_2$ denote $\text{Ext}(\gamma_1) \cup (\phi_{\gamma_2}^-)^{-1}(\text{Ext}(\gamma_2))$. $M_1\gamma_1 \#_\mathcal{I} M_2$ is a morphism from $\Sigma_1 \#_\mathcal{I} \Sigma_2$ to $\Sigma'_1 \#_\mathcal{I} \Sigma'_2$ where the connect sum has been taken by deleting neighborhoods of the endpoints of $\gamma_1$ and $\gamma_2$. Let $\Sigma_{1,a}$ denote $\Sigma_1$ with the relevant point colored $a$, etc. The colored splitting theorem [BHMV,1.14] describes an isomorphism $V_p(\Sigma_1 \#_\mathcal{I} \Sigma_2) \approx \oplus_{a \in C} \Sigma_{1,a} \otimes \Sigma_{2,a}$. The following gluing formula follows easily from the description of this isomorphism and the definition of the maps induced by a morphism.

**Lemma 3.6.** If $\gamma_1$ is a $(+)$pramed arcs in a morphisms $M_1$, and $\gamma_2$ is a $(-)$pramed arc in a morphisms $M_2$, then

$$Z(M_1\gamma_1 \#_\mathcal{I} M_2) = \oplus_{a \in C} Z(M_1, \gamma_1, a) \otimes Z(M_2, \gamma_2, a)$$

Much more general gluing formulas are described in [Wa2] and the more recent paper [Ge]. The above formulas follow easily from the set-up in [BHMV]. Lemma (3.6) was used implicitly in [G] to prove (2.1) and (2.2) above. Note also that the trace of (3.6) yields a special case of (3.3).

§4 Winding number zero

The contribution of the companion $C$ to the formulas we derive for $Z(S, c)$ is $< (M(C), m_C, a) >$. This may be more convenient than the contribution $C_a$ of $C$ to the formula (2.4) for $Z(C \ast D(k), c)$, as $< (M(C), m_C, a) > \geq \text{Trace}Z(C, a)$. For instance, if $C$ itself is a satellite knot, we may use the methods of this paper to get our hands on $Z(C, a)$, and thus $< (M(C), m_C, a) >$. We will let $C(a)$ denote $< (M(C), m_C, a) >$. The data $(C_a)_{a \in C}$ and $(C(a))_{a \in C}$ is equivalent. In (8.1)and (8.2), we will give a change of basis matrix which relates these two vectors. In this way we will rederive (2.4) from (4.4) but without the additional hypothesis, that any $C_s$ be nonzero.

If $E$ has linking number zero with the axis $A$, then we may pick a Seifert surface $E_s$ for $E$ which misses $A$. Let $\Sigma_{E_s}$ denote $E_s$ capped off in $M(E)$. We may view $A$
as a subset of $M(S)$, and $\Sigma_P$ misses $A$. Let $E_\xi$ be a fundamental domain for the $Z$-action on $M(\mathcal{E})_\infty$ with boundary two lifts of $\Sigma$, and continue to call the lift of $A$ in $E_\xi$, by $A$.

Let $Z(\Sigma_P; a) = Z((E_\xi; a))$ where $(E_\xi; a)$ denotes $E_\xi$ constructed as above with $A$ colored $a$. Note that $Z^b((E_\xi; a))$ represents the Turaev-Viro module of $M(\mathcal{E})$ with $A$ colored $a$, which we denote $Z(P; a)$.

Similarly let $E_S$ be a fundamental domain for the $Z$-action on $M(S)_\infty$ with boundary two lifts of $\Sigma$. Then we have $E_S = M(C)_{m_c \wedge A}E(\mathcal{E})$. So by (3.5) we have

\begin{equation}
Z(S) \equiv \sum_{a \in \mathcal{E}} C(a)Z(\Sigma_P; a)
\end{equation}

However we cannot replace $Z(\Sigma_P; a)$ by $Z(P; a)$, as the summation above requires that the maps $Z(\Sigma_P; a)$ have the same domain for each $a$. Also of course the sum of singular maps may be nonsingular. At this point we remind the reader that when we write $Z(S) \equiv X$ where $X$ is an endomorphism or a matrix, we mean $Z(S) = X^b$, where $X^b$ denotes the map after dividing out by the generalized 0-eigenspace.

Similarly let $Z(\Sigma_P; a, c) = Z((E_\xi; a, c))$ where $(E_\xi; a, c)$ denotes $E_\xi$ constructed as above with $A$ colored $a$, and and the inverse image of a meridian for $\mathcal{E}$ colored $c$. Note that $Z(\Sigma_P; a, c)^b$ represents the Turaev-Viro module of $M(\mathcal{E})$ with $A$ colored $a$, and a meridian for $\mathcal{E}$ colored $c$. We denote this by $Z(P; a, c)$.

\begin{equation}
Z(S, c) \equiv \sum_{a \in \mathcal{E}} C(a)Z(\Sigma_P; a, c)
\end{equation}

For the pattern $D(k)$ of the $k$-twisted double we have:

\begin{equation}
Z(F_{D(k)}; a) = \eta \kappa^{-3} \left( \mu_i \sum_{t \in A(i, j)} \mu_t^k (-1)^{a+t}[(a+1)(t+1)] \right)
\end{equation}

Here $[n]$ denotes $\frac{A^n - A^{-n}}{A - A^{-1}}$. We also have that $Z(\Sigma_D(2); a, c)$ is given by:

\begin{equation}
\eta \kappa^{-3} \left( \mu_i \Delta_i \sum_{t \in A(i, j)} \frac{\mu_t^k (-1)^{a+t}[(a+1)(t+1)]}{\theta(i, j, t)} \right)
\end{equation}

Note that (4.4) becomes (4.3) if $c = 0$. Let $U$ denote the unknot. By [G,7.2],

$U(a) = \delta_0^a$. Thus (4.2) shows that $Z(U \ast D(k), c) \equiv Z(P, 0, c)$. Similarly $Z(U \ast D(k)) \equiv Z(P, 0)$. Thus

\begin{equation}
Z(U \ast D(k)) \equiv \eta \kappa^{-3} \left( \mu_i \sum_{t \in A(i, j)} \mu_t^k (-1)^{t}(t+1) \right)
\end{equation}
(4.6)

\[ Z(U \ast D(k), c) \equiv \eta \kappa^{-3} \left( \mu_i \frac{\Delta_i}{\theta(i, i, c)} \sum_{t \in A(i, j)} \frac{\mu_t^k(-1)^t[(t + 1)] \ Tet \frac{[t \ i \ i \ j \ j]}{t \ c \ j \ j}}{\theta(i, j, t)} \right)_{i,j \in A(c)} \]

(4.5) becomes especially simple when \( k = \pm 1 \). \( U \ast D(1) \) is the figure eight knot, denoted \( F_8 \). \( U \ast D(-1) \) is the right hand trefoil knot, denoted \( RT \).

(4.7) \[ Z(F_8) \equiv \eta \kappa^{-3} \left( \mu_i^2 \mu_j (-1)^{i+j}[(i + 1)(j + 1)] \right)_{i,j \in \mathcal{E}} \]

(4.8) \[ Z(RT) \equiv \eta \kappa^{-3} \left( \mu_j^{-1} (-1)^{i+j}[(i + 1)(j + 1)] \right)_{i,j \in \mathcal{E}} \]

These last two formulas, in the case \( p \) is even, are very close to the formulas for the maps induced by the monodromies of \( F_8 \) and \( RT \) [G,11.1 & 11.2] obtained from [J] after making certain substitutions and some slight corrections. We haven’t yet seen directly that they are similar.

The rest of this section contains a derivation of (4.3), and an indication of the proof of (4.4). We will also give the derivation of (4.7) and (4.8) from (4.5). As in [G,§5], we obtain the following description for \( E_{\mathcal{E}} \) with the lift of \( A \) colored \( a \) as surgery on \( S^2 \times I \) with a tunnel drilled out from the bottom and a 1-handle added to the top. Here and in later diagrams, the ‘slab’ \( D^2 \times I \) denotes a part of a copy of \( S^2 \times I \), the rest of which is left out of the picture but is included in the manifold depicted.

\[ \omega \]

\[ \kappa^{-3} \]

\[ \kappa \]

\[ a \]

\[ 2k+1 \text{ full twists} \]

\[ k \text{ full twists} \]

Figure 4

Here the scalar \( \kappa^{-3} \) indicates that we must multiply \( Z \) of the manifold pictured by this scalar to correct for a change in \( p \)-structure. We may compute the \( i,j \)
entrie of the matrix for $Z$ with respect to the basis orthonormal basis $\{e_i\}_{i \in C}$ by adding a solid torus to bottom with core colored $j$ and a solid torus to the top with core colored $i$ to get a colored link in $S^1 \times S^2$. The invariant of this manifold is the evaluation of the diagram on the left of Figure 4. We expand out the lower $\omega = \eta \sum_{s \in C} \Delta_s e_s$, and remove the $2k+1$ full twists at the cost of introducing $\mu_s^{2k+1}$. The sum on the right of Figure 5 is over $s \in C$.

$$\eta \kappa^{-3} \omega = \eta 2 \kappa^{-3} \sum \mu_s^{2k+1} \Delta_s$$

Figure 5

Then we use the following simplification where the sum is over $t \in A(i, s)$. The second inequality uses [L1, Lemma 6] which is legitimate when $p$ is odd as we have chosen our colors to be even. The sum is over $t \in C$.

$$\omega = \sum_{t \in A(i, s)} \frac{\Delta t}{\theta(i, s, t)} \omega = \frac{\delta_s}{\Delta_i} \omega = \frac{\delta_s}{\eta \Delta_i} \omega$$

Figure 6

We obtain:

$$\eta \kappa^{-3} \mu_i^{2k+1} \omega$$

Figure 7
We simplify the two strands with \( k \) full twists by repeatedly using:

\[
\sum_{\theta(i,j,t)} \Delta_t \mu_i \mu_j = \sum_{\theta(i,j,t)} \Delta_t \mu_t \mu_j \quad \text{for} \quad i = j = \sum_0^\theta(t,j,t) \Delta_t \mu_i \mu_j \; t_j = \sum_0^\theta(t,j,t) \Delta_t \mu_i \mu_j \; t_j
\]

Figure 8

Here the sum is over \( t \in A(i,j) \). Alternatively, one may use the same trick to take care of all \( k \) twists at once. Then one uses, say, [KL9.10(iii)] to collapse forks. The evaluation is seen to be

\[
(4.9) \quad \eta \kappa^{-3} \mu_i 2^{k+1} \sum_{t \in A(i,j)} \left( \frac{\mu_t}{\mu_i \mu_j} \right)^k H(a,t) = \left( \frac{\mu_i}{\mu_j} \right)^k \eta \kappa^{-3} \mu_i \sum_{t \in A(i,j)} \mu_t^k (-1)^{a+t} [(a+1)(t+1)]
\]

Here \( H(a,t) \) denotes the evaluation on the standard Hopf link with components colored \( a \) and \( t \). Morton and Strickland evaluated \( H(a,t) \) to be as \((-1)^{a+t}[(a+1)(t+1)]\) in the \( p \) even case [MS1,MS2]. However the same argument works in the \( p \) odd case. There is shift by one in the index of corresponding colors between this paper and [MS1,MS2]. The matrix on the right of (4.9) is simplified by removing the factor \( (\frac{\mu_i}{\mu_j}) \) by the change of basis \( \{ e_j \} \rightarrow \{ \mu_j^k e_j \} \). So we obtain the matrix given on the left hand side of (4.3).

\( E_\mathcal{E} \) with the lift of \( A \) colored \( a \) and the inverse image of a meridian of \( \mathcal{E} \) colored \( c \) can be pictured as in Figure 4 except one must add a single vertical line colored \( c \). Consider the basis \( \{ f_i \} \) for the vector space of a boundary of a solid torus with one point colored \( c \), given by the core of the solid torus colored \( i \) and an edge joining the core to the point colored \( c \), where \( i \in \mathcal{A}(c) \). This basis is orthogonal but not orthonormal. \( \mathcal{Z}(P; a, c) \) is given by a matrix whose \( i, j \) entrie is the quotient of the evaluations pictured in Figure 9. Here the indices \( i \) and \( j \) run over \( \mathcal{A}(c) \). The same
methods of evaluation then yield (4.4).

To derive (4.7,) we note that

$$\mu_i^{-1} \mu_j^{-1} \sum_{t \in A(i,j)} \mu_t (-1)^t (t + 1) = H(i, j)$$

using Figure 8. Then we make use of the Morton-Strickland formula for $H(i, j)$. (4.8) follows from the conjugate of the above equation.

§ 5 Winding number one

We may find a splitting surface $\Sigma_1$ in $M(C)$ which meets $m_C$ in a single point $x$. Note $\Sigma_1$ itself may be formed from a Seifert surface for $C$ in $S^3$ capped off in $M^2$. If $\mathcal{E}$ has linking number one with the axis $A$, then we may also pick a Seifert surface $F_2$ for $\mathcal{E}$ which meets $A$ in a single point say $y$. Let $\Sigma_2$ denote $F$ capped off in $M(\mathcal{E})$. $\Sigma_1 x \# y \Sigma_2$ forms a splitting surface $\Sigma$ in $M(S)$ assuming that the meridian around the point $x$ is glued to the meridian around $y$ as in (2.5). Here $x \# y$ indicates that the connect summing takes places at the points $x$ and $y$.

Let $E_1$ be a fundamental domain for the Z-action on $M(C)_\infty$ with boundary two lifts of $\Sigma_1$. Let $\gamma_1$ denote the inverse image of $m_C$ in $E_1$. Let $E_2$ be a fundamental domain for the Z-action on $M(\mathcal{E})_\infty$ with boundary two lifts of $\Sigma_2$. Let $\gamma_2$ denote the inverse image of $A$ in $E_2$. Similarly let $E_S$ be a fundamental domain for the Z-action on $M(S)_\infty$ with boundary two lifts of $\Sigma$. Then we have $E_S = E_1 \gamma_1 \# \gamma_2 E_2$.

So by (3.6)

$$Z(S) = \bigoplus_{a \in C} Z(C, a) \otimes Z(P, a)$$

Similarly

$$Z(S, c) = \bigoplus_{a \in \mathcal{E}} Z(C, a) \otimes Z(P; a, c)$$

Although there is a clear analogy between (5.1) and (5.2) and (4.1) and (4.2), they are really quite different because of the differences between $\sum$ and $\otimes$. In particular the direct sum of invertible automorphisms in still invertible but their sum need not be. Also $\oplus$ and $\otimes$ are well behaved with respect to the operation $X \mapsto X^\flat$. 
The most important example of a winding number one satellite construction is the connect sum of two knots $K_1$ and $K_2$. Here we take $C$ to be $K_1$ and take $E$ to be $K_2$ with the axis $A$, a meridian to $K_2$. Then (5.1) yields (2.1). It is less obvious that (5.2) yields (2.2). Note that the meridian to $E$ and the axis in $P$ are parallel.

Suppose $M$ is a morphism from a surface $\Sigma$ to itself which contains a framed arc $\gamma$ going from point $x$ in one copy of $\Sigma$ to this same point $x$ in the other copy. Suppose $\gamma'$ is pushoff of $\gamma$ going from, say, $y$ in one copy of $\Sigma$ to this same point $y$ in the other copy. Let $\Sigma(b)$ denotes $\Sigma$ with $x$ colored $b$. Let $\Sigma(a,c)$ denotes $\Sigma$ with $x$ and $y$ colored $a$ and $c$. Let $M(b)$ denote $M$ with $\gamma$ colored $b$, $M(a,c)$ denote $M$ with $\gamma$ and $\gamma'$ colored $a$ and $c$. We have an isomorphism from $V(\Sigma(a,c))$ to $\bigoplus_{b \in A(a,c)} V(\Sigma(b))$. The components of this map are given by $Z$ of a $Y$ graph colored $a$, $b$, and $c$ embedded in $\Sigma \times I$. Moreover it is not hard to see that under this isomorphism $Z(M(a,c)) = \bigoplus_{b \in A(a,c)} Z(M(b))$. The above observation shows that $Z(P,a,c) = \bigoplus_{b \in A(a,c)} Z(P,b)$. Thus (5.2) implies (2.2).

We calculate now $Z(P,a)$ for certain pattern $P$. Consider Figure 10a. Here we have drawn $E$ as unknotted and $A$ as tangled. However one can isotope $A$ into a standard unknot and then $E$ becomes tangled. The resulting picture is the pattern $P$ we mean to study. However the link we have drawn is actually symmetric so, in this case, $P$ can be obtained by switching the labels of $A$ and $E$ in Figure 10a.

For a general pattern $E$ might not be isotopic to the unknot. Then one would have to do surgery to $S^3$ in the complement of $P$ to unknot $E$ first. This is Rolfsen’s method for calculating the Alexander module of a knot. The resulting Figure which would play the role of Figure 10a then would have some circles labelled with an $\omega$ and a scalar correction for the $p_1$-structure as in Figure 4. Figure 10b shows $E$ where $E$ is the morphism for which $Z(E) = Z(P,a)$. $Z(E)$ is an endomorphism.
of a vector space with the basis: \( \{ f_i \} \) indexed by \( i \in \mathcal{A}(a) \). In fact using methods already developed in §4 we have that \( Z(E)f_j = \sum_i z_{i,j} f_i \), where \( z_{i,j} \) is the following quotient of two evaluations.

One then calculates that \( z_{i,j} \) is zero if \( i \neq a \), or if \( \{ a, a, a \} \notin \mathcal{A} \). Note that if \( \{ a, a, a \} \in \mathcal{A} \), then \( a \) is even. If \( \{ a, a, a \} \in \mathcal{A} \), we let \( \gamma_a \) denote \( z_{a,a} \). We calculate that
\[
\gamma_a = \frac{(-1)^{\frac{a(a+2)}{2}}}{\theta(a,a,a)} \sum_{t \in \mathcal{A}(a)} \frac{\mu_t \Delta_t}{\theta(a,a,t)} \text{Tet} \begin{bmatrix} t & a & a \\ a & a & a \end{bmatrix}
\]

Let \( \mathcal{T} \) denote the set of colors \( c \) such that \( \{ c, c, c \} \) is admissible. Then \( Z^2(E) \) is zero if \( a \notin \mathcal{T} \). If \( a \in \mathcal{T} \), then \( Z^2(E) \) is multiplication by \( \lambda_a \) on a one dimensional space. Thus by (5.1) we have:
\[
Z(S) = \bigoplus_{a \in \mathcal{T}} \gamma_a Z(C, a).
\]

One could easily work out \( Z(S, c) \) in a similar way, but the answer would be more complicated.

§6 n-Wheels

The type of data that a pattern with winding number greater than one contributes to our formula for the Turaev-Viro module of a satellite is more complicated than an \( f_p[t, t^{-1}] \) module or equivalently, a similarity class of an \( f_p \)-automorphism. In this section, we define and study \( n \)-wheels. We also construct invariants of any ordered link in \( S^3 \) with linking number \( n \) with values in \( n \)-wheels of isomorphisms. The contributions of a pattern and of a companion to the satellite formula will be \( n \)-wheels of isomorphisms.

Let \( n \) be a positive integer. An \( n \)-wheel \( U \) is a sequence of \( n \) vector space homomorphisms \( u_i : U_i \to U_{i+1} \) where \( i \in \mathbb{Z}_n \). A equivalence from an \( n \)-wheel \( v_i : V_i \to V_{i+1} \) to an \( n \)-wheel \( u_i : U_i \to U_{i+1} \) is a sequence of isomorphisms \( T_i : U_i \to V_{i+k} \), for some fixed \( k \) such that \( u_{i+k} \circ T_i = T_{i+1} \circ v_i \) for all \( i \). If \( U \) is equivalent to \( V \), we write \( U \approx V \). These conditions may be visualized clearly as a commutative diagrams on an annulus. Note that a 1-wheel is simply a vector space endomorphism, and equivalence of 1-wheels is similarity. The dimension of an \( n \)-wheel \( u_i : U_i \to U_{i+1} \) is simply \( \dim(U_0) \). We also have zero dimensional \( n \)-wheels, which we denote as 0. The maps of a 1-dimensional \( n \)-wheel will be denoted by scalars.
Let \( \hat{u}_i \) denote the endomorphism of \( U_i \) given by the composition \( u_{i-1} \circ u_{i-2} \cdots u_0 \circ u_{n-1} \cdots u_{i+1} \circ u_i \). Let \( K(U_i) = \cup_{k \geq 1} \text{Kernel}(\hat{u}_i^k) \). Let \( U_i^b = U_i/K(U_i) \). \( u_i \) induces a map \( u_i^b \) from \( U_i^b \) to \( U_{i+1}^b \). In this way we get a new \( n \)-wheel of vector space isomorphisms.

If \( U = (u_i : U_i \to U_{i+1}) \), and \( V = (v_i : v_i \to v_{i+1}) \) are two \( n \)-wheels. We may define the tensor product by \( U \otimes V = (u_i \otimes v_i : U_i \otimes V_i \to U_{i+1} \otimes V_{i+1}) \). One has the easily proved lemma:

**Lemma 6.1.** \( 0 \otimes U = 0 \). \( U \otimes V \approx V \otimes U \). \( U^b \otimes V^b \approx (U \otimes V)^b \). If \( U \approx U' \) and \( V \approx V' \), then \( U \otimes V \approx U' \otimes V' \).

Given an \( n \)-wheel \( U = (u_i : U_i \to U_{i+1})_i \), we may form a new \( n \)-wheel \( U(k) \) with the following replacement by shifting all indices by \( k \) i.e. \( U(k)_i = U_{k+i} \) and \( u_i(k) = u_{k+i} \).

**Lemma 6.2.** If \( U \) is a \( n \)-wheel of isomorphisms, then \( U \approx U(k) \) for all \( k \).

**Proof.** The equivalence \( U \approx U(1) \) is given by the maps \( u_i \). \( \Box \)

Given an \( n \)-wheel \( U = (u_i : U_i \to U_{i+1})_i \), we may form an endomorphisms \( S(U) \) of the single vector space \( \oplus_{i \in \mathbb{Z}} U_i \) by

\[
S(u_i)(\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n) = (u_n(\alpha_n) \oplus u_1(\alpha_1) \oplus \cdots \oplus u_{n-1}(\alpha_{n-1})).
\]

The similarity class of \( S(U) \) is determined by the equivalence class of \( U \).

Suppose now for each \( i \in \mathbb{Z}_n \) we have an endomorphism \( g_i \) of a vector space \( G_i \), then we may form an \( n \)-wheel denoted by \( \mathcal{W}(\{g_i\}) \), as follows. Let \( \mathcal{W}_i = G_{-i} \otimes G_{-i+1} \otimes \cdots \otimes G_{-i+n-1} \). Define \( w_i : \mathcal{W}_i \to \mathcal{W}_{i+1} \) by \( w_i(\alpha_{-i} \otimes \alpha_{-i+1} \otimes \cdots \otimes \alpha_{-i+n-1}) = g_{-i+n-1}(\alpha_{-i+n-1}) \otimes \alpha_{-i} \otimes \cdots \otimes \alpha_{-i+n-2} \). This choice of indexing may seem complicated but:

\[
\mathcal{W}_0 = G_0 \otimes G_1 \otimes \cdots \otimes G_{n-1}, \quad \mathcal{W}_1 = G_1 \otimes G_2 \otimes \cdots \otimes G_n,
\]

\[
w_0(\alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_{n-1}) = g_{n-1}(\alpha_{n-1}) \otimes \alpha_0 \otimes \cdots \otimes \alpha_{n-2}.
\]

**Lemma 6.3.** \( S(U^b) = S(U)^b \). Also \( \mathcal{W}(\{g_i^b\}) = (\mathcal{W}(\{g_i\}))^b \).

Suppose we have an oriented framed knot \( K \) in a closed 3-manifold \( M \) and an epimorphism \( \chi : H_1(M) \to \mathbb{Z} \). Assume that \( n = \chi([K]) \) is greater than one. Let \( \Sigma \) be surface dual to \( K \) which meets \( K \) in exactly \( n \) points. Choose an arbitrary such point to call \( x_0 \). Now travel along \( K \) in the direction of its orientation to the next point. Call this point \( x_1 \). Continuing in this way, we may name all \( n \) points: \( \{x_0, x_1, \cdots, x_{n-1}\} \). Suppose we are given an ordered \( w \)-tuple of colors \( \bar{a} = (a_0, a_2, \cdots, a_{n-1}) \) in \( \mathbb{C}^n \). Let \( F(\bar{a}) \) denote \( F \) with \( x_i \) colored \( a_i \). Let \( \sigma \) denote the transformation which sends \( \bar{a} = (a_0, a_2, \cdots, a_{n-1}) \) to \( \sigma(\bar{a}) = (a_{n-1}, a_0, \cdots, a_{n-2}) \).

Let \( n(\bar{a}) \) be the least exponent \( e \) such that \( \sigma^e(\bar{a}) = \bar{a} \). Let \( E \) be a fundamental domain for the \( \mathbb{Z} \) action on \( M_\infty \) with boundary a copies of \( -\Sigma \) and \( \Sigma \). The inverse image of \( K \) consists of \( n \) framed arcs. Let \( E(\bar{a}) \) be obtained by coloring the arc which starts at \( x_i \) in \( -\Sigma \) and goes to \( x_{i+1} \) in \( F \) by \( a_i \), for all \( i \). \( E(\bar{a}) \) is a morphism from \( F(\bar{a}) \) to \( F(\sigma^e(\bar{a})) \).

Let \( W_p(L; \bar{a}) \) denote the \( n(\bar{a}) \)-wheel given by \( U^b \) where \( U \) is the wheel given by \( U_i = V_p(F(\sigma^i(\bar{a}))) \), and \( u_i = Z(E(\sigma^i(\bar{a}))) \). We will usually omit the subscript \( p \). By (6.3) \( W_p(L; \bar{a}) \) is equivalent to \( W(L, \sigma^i(\bar{a})) \), so the equivalence class of \( Z(L; \bar{a}) \) only depends on the cyclic ordering of \( \bar{a} \).

The proof of Theorem (1.1) extends to this situation and we have:
Theorem (6.4). The equivalence class of $W(K; a)$ is an invariant of the isotopy class of $K$.

Now suppose $L$ is a link two components $K_1$ and $K_2$ with linking number $w$ greater than one. Let $M$ be zero framed surgery along $K_1$ with $p_1$-structure with zero sigma invariant, and let $K = K_2$. Define $W(L; \vec{a})$ to be the $n(\vec{a})$-wheel $W(K_2; \vec{a})$ defined above. It will also be useful to do all the above with a color $c$ assigned to a meridian of $K_2$ and the inverse images of this meridian in $E$. In this way, we obtain an $n$-wheel $W(L; \vec{a}, c)$ well defined up to equivalence. We will let $U(L; \vec{a}, c)$ denote $U(\vec{a})^c$ in the above construction. Similarly we let $u(L; \vec{a}, c)$ denote the map from $U(\vec{a})$ to $U(\sigma \vec{a})$.

§7 Higher winding numbers

If view a pattern link $P$ as a link of two components where $E$ is taken for $K_1$ and $A$ is taken for $K_2$, and $P$ has winding number $w$, we obtain $n(\vec{a})$-wheels denoted $Z(P; \vec{a})$, and $Z(P; \vec{a}, c)$, for each $\vec{a}$ in $C^w$. These only depend on $\vec{a}$ up to cyclic permutation.

The (2,1) cable pattern. We consider the pattern $P(2,1)$ from Figure 2 with winding number two, and calculate $W(P(2,1); \vec{a}, c)$. $U(P(2,1), (a_1, a_2), c)$ is zero if $c \notin A(a_1, a_2)$. If $c \in A(a_1, a_2)$, then $U(P(2,1), (a_1, a_2), c)$ is one dimensional and $u(P(2,1), (a_1, a_2), c)$ is the map induced by the manifold pictured on the left of Figure 12.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure12.png}
\caption{Figure 12}
\end{figure}

So $u(P(2,1); (a_1, a_2), c)$ is multiplication given by the quotient of evaluations on the left of Figure 12, which we denote by $\nu_{a_1,a_2,c}$. One easily has $\nu_{a_1,a_2,c} = \mu_{a_1}^{-1}(\lambda_c^{a_1,a_2})^{-1}$. We use the $\lambda^{ab}$ notation for the 3-vertex term [KL, 9.9]. Note $(\lambda^{ab})^0 = \delta^b_a \mu_a$. See [MV] for a simple derivation of $\lambda^{ab}$. Of course $u(P(2,1); (a_2, a_1), c)$ is multiplication by $\nu_{a_2,a_1,c}$.

If we specialize to $p=5$, then $C = \{0, 2\}$, and $A = \{0, 0, 0\} \{2, 2, 0\} \{2, 2, 2\}$. We have the following non-zero wheels for this pattern. The one-dimensional 1-wheels $W(P(2,1); (0, 0), 0) = 1$, $W(P(2,1); (2, 2), 0) = -A$, and $W(P(2,1); (2, 2); 2) = A^3$. We also have a one dimensional 2-wheel $W(P; (0, 2), 2)$ with $u(P(2,1); (0, 2), 2) = 1$, and $u(P(2,1); (2, 0), 2) = A^2$.

The (3,1) cable pattern. Consider now the pattern $P(3,1)$ of the (3,1) cable shown on the left of Figure 13. This link is symmetric. However the framed
axis $A$ develops two negative twists in the isotopy. $W(P(3,1);(a,b,c))$ is zero if \{a, b, c\} $\notin A$. If \{a, b, c\} $\in A$, $W(P(3,1);(a,b,c))$ is one dimensional. The maps (with respect to the basis given a ‘T’ diagram with edges labelled $a$, $b$, and $c$) are multiplication by the quotient of the evaluations on the right of Figure 13.

The numerator becomes the denominator after removing the two kinks and two 3-vertex moves. So this quotient is $\mu^{-2}(\lambda^{ab}_c)(\lambda^{ac}_b)^{-1} = \mu^{-1}$. Thus for each $c \in T$ we have a one wheel $W(P(1,3);(c,c,c)) = \mu^{-1}$. If $a \neq b$, for each $c \in A(a,b)$ we have the 3-wheel $W(P(3,1);(a,b,c))$ with $u(P(3,1);(a,b,c)) = \mu^{-1}$.

If we specialize to $p=5$, we have the following non-zero wheels for this pattern.

We have the following one-dimensional 1-wheels or (simply vector space automorphisms).

$W(T(3,1);(0,0,0)) = 1$, and $W(T(3,1);(2,2,2)) = A^2$. We also have two one dimensional 3-wheels: $W(T(3,1);(0,0,2))$ with $u(T(3,1);(0,0,2)) = 1$, $u(T(3,1);(2,0,0)) = A^2$, and $u(T(3,1);(0,2,0)) = 1$. and $W(T(3,1);(2,2,0))$ with $u(T(3,1);(2,2,0)) = A^2$, $u(T(3,1);(0,2,2)) = 1$, and $u(T(3,1);(2,0,2)) = A^2$.

**The 3-strand 1-bight turk’s head pattern.** Consider now the pattern $T(3,1)$ shown on the left of Figure 14. This link is symmetric. As the write of $\mathcal{E}$ is zero, no twists develop in the framed axis $A$ during the isotopy. $W(T(3,1);(a,b,c))$ is zero if \{a, b, c\} $\notin A$. If \{a, b, c\} $\in A$, $W(T(3,1);(a,b,c))$ is one dimensional. The maps (with respect to the basis given by a ‘T’ diagram with edges labelled $a$, $b$, and $c$) are multiplication by the quotient of the evaluations on the right of Figure 14.

The numerator becomes the denominator after two 3-vertex moves. So this quotient is $(\lambda^{ab}_c)(\lambda^{ac}_b)^{-1} = \mu_b\mu_c^{-1}$. Thus for each $c \in T$, we have a one wheel $W(T(1,3);(c,c,c)) = 1$. If $a \neq b$, for each $c \in A(a,b)$, we have the 3-wheel $W(T(1,3);(a,b,c))$ with $u(T(1,3);(a,b,c)) = \mu_b\mu_c^{-1}$.
$W(T(3,1); (2, 2, 2)) = 1$. We also have two one dimensional 3-wheels: $W(T(3,1); (0, 0, 2))$ with $u(T(3,1); (0, 0, 2)) = A^2$, $u(T(3,1); (2, 0, 0)) = 1$, $u(T(3,1); (0, 2, 0)) = A^8$, and $W(T(3,1); (2, 2, 0))$ with $u(T(3,1); (2, 2, 0)) = A^8$, $u(T(3,1); (0, 2, 2)) = 1$, and $u(T(3,1); (2, 0, 2)) = A^2$.

**Wheels associated to the companion.** Again let $\vec{a} = (a_0, a_2, \ldots, a_{w-1})$ in $C^w$. Then we obtain a sequence of $n(\vec{a})$ isomorphisms $Z_p(C,a_i)$. We may define the $n(\vec{a})$-wheel of the companion associated to a sequence of colors $\vec{a}$ by $W_p(C,\vec{a}) = W(\{Z_p(C,a_i)\})$. We will usually omit the subscript $p$. Note that the isomorphism class of this $n(\vec{a})$-wheel also only depends on the cyclic ordering of $\vec{a}$. We let $\mathcal{U}(C, (a_1, a_2, \cdots n(\vec{a})))$ denote the vector space $Z(C, a_1) \otimes Z(C, a_2) \otimes \cdots \otimes Z(C, a_{n(\vec{a})})$. Let $u(C, (a_1, a_2, \cdots n(\vec{a})))$ denote associated isomorphism from $\mathcal{U}(C, (a_1, a_2, \cdots n(\vec{a})))$ to $\mathcal{U}(C, (a_{n(\vec{a})}, a_2, \cdots a_{n(\vec{a})-1}))$.

**Wheels associated to figure eight companion at $p=5$.** As an example, suppose $C$ is the figure eight knot $F_8$. This is the 1-twisted double of the unknot. In [G], we calculated that has $Z_5(F_8)$ two eigenvectors: $e_1$ with eigenvalue $A$, and $e_2$ with eigenvalue $\bar{A}$. Also $Z_5(F_8, 2)$ is the identity map on a one dimensional space. Let $f$ denote a vector in this space. Of course these calculations also follow from (4.6) and (4.7).

We calculate the nonzero wheels $W(F_8, \vec{a})$ associated to a color vector $\vec{a}$ of length two. So $W(F_8, (0, 0))$ is a 4-dimensional 1-wheel. $\mathcal{U}(F_8, 0, 0)$ has a basis of elements of the form $e_1 \otimes e_2$ ordered lexicographically. With respect to this basis $u(F_8, 0, 0)$ is given by $\mathfrak{S}_1$, the direct sum of the three matrices: $(A), \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ and $(\bar{A})$. $\mathfrak{S}_1$ has eigenvalues 1, $-1$, $A$, and $\bar{A}$.

$W(F_8, (2, 2))$ is a 1-dimensional 1-wheel given by the identity. $\mathcal{U}(F_8, 0, 2)$ has $e_1 \otimes f$, $e_2 \otimes f$ as an ordered basis. $\mathcal{U}(F_8, 2, 0)$ has $f \otimes e_1$, $f \otimes e_2$ as an ordered basis. With respect to these bases $W(F_8, (0, 2))$ is a 2-dimensional 2-wheel with $\mathcal{U}(F_8, 0, 2)$ the identity and $\mathcal{U}(F_8, 2, 0)$ given by $\mathfrak{S}_2$, the direct sum of the two matrices: $(A)$, $(\bar{A})$.

We calculate now the nonzero wheels $W(F_8, \vec{a})$ associated to a color vector $\vec{a}$ of length three. $W(F_8, (0, 0, 0))$ is a 8-dimensional 1-wheel. $\mathcal{U}(F_8, 0, 0, 0)$ has $e_1 \otimes e_1 \otimes e_1$, $e_2 \otimes e_2 \otimes e_2$, $e_1 \otimes e_2 \otimes e_2$, $e_2 \otimes e_1 \otimes e_2$, $e_2 \otimes e_2 \otimes e_1$, $e_1 \otimes e_1 \otimes e_2$, $e_2 \otimes e_1 \otimes e_1$, $e_1 \otimes e_2 \otimes e_1$ as ordered basis. With respect to this basis $u(F_8, 0, 0, 0)$ is given by $\mathfrak{S}_3$ the direct sum of the four matrices: $(A)$, $(\bar{A})$, $\begin{pmatrix} 0 & 0 & A \\ \bar{A} & 0 & 0 \\ 0 & \bar{A} & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & A \\ \bar{A} & 0 & 0 \\ 0 & \bar{A} & 0 \end{pmatrix}$.

$\mathfrak{S}_3$ has eigenvalues $A$, and $\bar{A}$, the three cube roots of $A$, and the three cube roots of $\bar{A}$.

$W(F_8, (2, 2, 2))$ is a 1-dimensional 1-wheel given by the identity.

$W(F_8, (0, 0, 2))$ is a 4-dimensional 3-wheel. $\mathcal{U}(F_8, (0, 0, 2))$ has a basis of elements of form $e_i \otimes e_j \otimes f$ ordered lexicographically. $\mathcal{U}(F_8, (2, 0, 0))$ has a basis of elements of form $f \otimes e_i \otimes e_j$ ordered lexicographically. $\mathcal{U}(F_8, (0, 2, 0))$ has a basis of elements of form $e_i \otimes f \otimes e_j$ ordered lexicographically. With respect to these basis $u(F_8, (0, 0, 2))$ is by the the identity matrix. $u(F_8, (2, 0, 0))$ and $u(F_8, (0, 2, 0))$ are both given by $\mathfrak{S}_1$.

$W(F_8, (2, 2, 0))$ is a 2-dimensional 3-wheel. $\mathcal{U}(F_8, (2, 2, 0))$ has a basis of elements of form $f \otimes f \otimes e_i$ ordered lexicographically. $\mathcal{U}(F_8, (0, 2, 2))$ has a basis of elements of form $e_i \otimes e_j \otimes e_j$ ordered lexicographically.
Proof. Let \( \Sigma \) be a splitting surface for \( M(C) \) which meets \( m_C \) in a single point \( x \). Let \( \Sigma_P \) be a splitting surface for in \( M(\mathcal{E}) \), which meets the axis in consecutive (along the axis) points \( x_0, x_2, \ldots, x_{w-1} \). Let \( \Sigma_S \) be the connected sum of \( \Sigma_P \) at the points \( x_0, x_2, \ldots, x_{w-1} \) with \( w \) copies of \( \Sigma_C \) at the point \( x \). \( \Sigma_P \) serves as a splitting surface for \( M(S) \) in a natural way.

Let \( E_C \) be the fundamental domain for the \( \mathbb{Z} \)-action on \( M(C)_{\infty} \) with boundary two lifts of \( \Sigma_C \). Let \( \gamma \) denote the inverse image of \( m_C \) in \( E_C \). Let \( E_P \) be the fundamental domain for the \( \mathbb{Z} \)-action on \( M(\mathcal{E})_{\infty} \) with boundary two lifts of \( \Sigma_P \). The inverse image of \( A \) in \( E_P \) consists of \( w \) arcs \( \gamma_0, \gamma_1 \cdots \gamma_{w-1} \) where \( \gamma_i \) goes from \( x_i \) in \( -\Sigma_P \) to \( x_{i+1} \) in \( -\Sigma_P \). Let \( T \) be \( \Sigma_C \times I, \tau \) be the arc \( \{ x \} \times I \) in \( T \). Let \( E_S \) be the fundamental domain for the \( \mathbb{Z} \)-action on \( M(S)_{\infty} \) with boundary two lifts of \( \Sigma_P \). Then we have:

\[
E_S = (\cdots ((E_P \gamma_{w-1} \land \gamma E_C)_{\gamma_0} \land \tau (T)_{\gamma_1} \land \cdots)_{\gamma_{w-2}} \land \tau T)
\]

The result then follows from (3.6). \( \square \)

The \((2,1)\) cable of the figure eight. We consider first the case \( p=5 \). \( \mathcal{O} \) has three elements \((0,0), (2,2) \) and \((2,0) \). We have \( Z_5(S) \) is the direct sum of contributions from each element of \( \mathcal{O} \). \((2,0) \) contributes zero, as \( W(P(2,1); (0,2)) = 0 \). \((2,2) \) contributes \( \mathbb{W}(F8, (2,2)) \otimes W(P(2,1); (2,2)) \). This is a one dimensional vector space with eigenvalue \(-A \). \((0,0) \) contributes \( \mathbb{W}(F8, (0,0)) \otimes W(P(2,1); (0,0)) \). This is a 4-dimensional vector space with eigenvalues: 1 \(-1, A, \) and \(-\tilde{A} \). So the eigenvalues of \( Z_5(S) \) are \( 1, -1, A, \tilde{A}, \) and \(-\tilde{A} \).

Consider now just the contribution of \( \tilde{O} = (0,0) \) to \( Z_p(S) W_p(P(2,1); \tilde{O}) \) is the identity map on a one dimensional space. So the contribution is just the 1-wheel \( \mathbb{W}_p(F8, (0,0)) \). In general, we know that \( Z(F8) \) is a unitary matrix, and so is diagonalizable with eigenvalues all of norm one as F8 is fibered and has genus one. Since \( F8 \) is amphichiral, the non-real eigenvalues come in conjugate pairs. Consider a pair \( e_1 \) and \( e_2 \) of eigenvectors with conjugate and therefore inverse eigenvalues. Then the automorphism restricted to the subspace spanned by \( e_1 \otimes e_2 \) and \( e_2 \otimes e_1 \) is direct summand with eigenvalues \( 1 \) and \(-1 \). Thus in general \( Z_p(F8 \ast P(2,1)) \) has one and minus one among its eigenvalues.

In his thesis, Miyazaki showed \( F8 \ast P(2,1) \) was not a ribbon knot [M]. It is an algebraically slice fibered knot. He showed that the monodromy does not extend over any handlebody. If the knot were ribbon, a theorem of Casson and Gordon shows that the fundamental group of the complement of a 4-dimensional neighborhood would have a non-abelian free factor. This is impossible for a fibered knot.
asserts that the closed off monodromy of a fibered homotopy ribbon knot must extend over a handlebody [CG]. This same theorem was used to prove Theorem (1.2). We had hoped to recover Miyazaki’s result using Theorem (1.2). So far we have not been able to do this. We working on some refinements of Theorem (1.2). Perhaps using another TQFT would also help.

Two winding number three satellites of the figure eight knot at $p=5$. These examples will illustrate the operation $\mathcal{S}$ in a nontrivial way. We consider first $Z_5(F8 \star P(3,1))$. The contribution of $(0,0,0)$ is just the 1-wheel $\mathcal{W}(F8,(2,2,2))$, given by $\mathfrak{G}_3$. The contribution of $(2,2,2)$ is just $W(P(3,1),(2,2,2))$, given by $(A^2)$. The contribution of orbit of $(0,0,2)$ is given by the block matrix

$$
\begin{pmatrix}
    0 & 0 & \mathfrak{G}_1 \\
    I_{4 \times 4} & 0 & 0 \\
    0 & A^2 \mathfrak{G}_1 & 0
\end{pmatrix}
$$

The contribution of orbit of $(2,2,0)$ is given by the block matrix

$$
\begin{pmatrix}
    0 & 0 & A^2 I_{2 \times 2} \\
    A^2 \mathfrak{G}_2 & 0 & 0 \\
    0 & I_{2 \times 2} & 0
\end{pmatrix}
$$

In particular, the characteristic polynomial of $Z_3(F8 \star P(3,1))$ is: $(x - A) (x - \bar{A}) (x^3 - A) (x^3 - \bar{A}) (x - A^2) (x^6 + (1 - A^3)x^3 - A^3) (x^{12} - (A^3 + A^2 + A)x^9 + 2(A^3 - 1)x^6 + (A^2 + A + 1)x^3 - A^3)$

$Z_5(F8 \star T(3,1))$ may be worked out in the same way as $Z_5(F8 \star P(3,1))$. We will just give its characteristic polynomial: $(x - A) (x - \bar{A}) (x^3 - A) (x^3 - \bar{A}) (x - 1) (x^{12} - (A^3 + A^2 + A)x^9 + 2(A^3 - 1)x^6 + (A^2 + A + 1)x^3 - A^3) (x^6 + (A^3 - A^2 - 1)x^3 + 1)$

§8 Derivations of (2.4)

Derivation along the lines of [G]. We need the hypothesis that $C_s$ are nonzero for $s \in \mathcal{A}(c)$ for this approach. Consider the exterior of the loop labelled $i$ in Figure 8a of that paper. Instead of completing it to a diagram in $S^3$, we should instead complete it to a diagram in $S^1 \times S^2$. In this way one may make use of the orthogonality of the bases described in [BHMV,4.11]. The matrix one then needs to invert is then already diagonal. In fact we have the following variant of Theorem (7.7) of [G]. $Z(C \star D(k),c)$ is given by matrix whose $i,j$ entrie is quotient of the evaluations in Figure 15 below. Here $i,j$ range over $\mathcal{A}(c)$. 
Let \( B(c)_{i,j} \) be the evaluation of the numerator and \( L(c)_i \) be the evaluation of the denominator. The grey disk labelled \( C \) indicates that a string diagram for \( C \) with zero writhe should be inserted. The grey disk labelled \( C(k) \) stands for two parallel strands along a string diagram for \( C \) with zero writhe with \( k \) full additional twists between the strands added. \( \kappa^{-3} \) simply means multiply that scalar times the evaluation of the rest of diagram. To evaluate expand the lower loop labelled \( \omega \) using \( \omega = \eta \sum_{s \in C} \Delta_s e_s \). Then we use the simplification of Figure 6.

Now one has a strand colored \( i \) with \( 2k + 1 \) full twists and \( C \) with zero writhe tied into it. The twists contribute a factor of \( \mu_i^{2k+1} \), and \( C \) contributes \( \frac{C_i}{\Delta_i} \). Thus \( B(c)_{i,j} = \kappa^{-3} \mu_i^{2k+1} \frac{C_i}{\Delta_i} \) times the evaluation of:

\[
\sum_{t \in A(i,j)} C_t \theta(i,j,t) \mu_i^{-k} \mu_j^{-k} \mu_t \mu_t \mu_i \mu_j.
\]

Thus

\[
B(c)_{i,j} = \kappa^{-3} \mu_i^{2k+1} \frac{C_i}{\Delta_i} \sum_{t \in A(i,j)} \frac{C_t}{\theta(i,j,t)} (\mu_t \mu_i \mu_j)^k \text{Tet} \left[ \frac{t \ i \ i}{c \ j \ j} \right].
\]
Similarly

\[ L(c)_i = \frac{C_i \theta(i, i, c)}{\eta \Delta^2_i}. \]

Note \( L(c)_i \) is invertible if and only if \( C_i \) is invertible. Thus \( L(c)^{-1}B(c) \) is given by \((\frac{\mu_j}{\mu_i})^k\) times the left hand side of (2.4). Under the change of basis \( \{f_j\} \rightarrow \{\mu_j f_j\} \), we obtain the matrix given on the left hand side of (2.4).

**A change of basis for the vector space of a torus.** Let \( T^2 \) be the boundary of a solid torus. Above we have made use of the basis \( \{e_i\} \in C \) for \( V(T^2) \) where \( e_i \) is given by coloring a framed core for the solid torus \( i \). Consider the basis \( \{g_i\} \in C \) for \( V(T^2) \) where \( g_i \) is given by labelling a framed core for the solid torus with \( \omega \) and coloring the meridian \( j \). Then \( <g_i, e_j>_T \) is given by the evaluation of Figure 18. Using Figure 6, this evaluates to \( \eta H(i, j) = \eta(-1)^{i+j}[(i+1)(j+1)] \). Thus

\[ g_i = \eta \sum_{j \in C} (-1)^{i+j}[(i+1)(j+1)]e_j. \]

Applying this in a tubular neighborhood of a knot \( C \), and recalling that \( \eta^2 = \frac{-(A^2-A^{-2})^2}{p} \), we obtain

(8.1) \[ C(i) = \frac{-(A^2-A^{-2})^2}{p} \sum_{j \in C} (-1)^{i+j}[(i+1)(j+1)]C_j \]

The extra factor of \( \eta \) comes from the fact that the invariant of \( S^3 \) with standard \( p_1 \)-structure with \( C \) colored \( j \) is \( \eta C_j \). Now it is shown that \( \eta((-1)^{i+j}[(i+1)(j+1)]) \) is equal to its own inverse in \[MS2\] in the case that \( p \) is even. One can check that this is true in general. Actually the symmetry of the above picture shows that \( e_j = \eta \sum_{i \in C} (-1)^{i+j}[(i+1)(j+1)]g_i \). This shows that \( \eta((-1)^{i+j}[(i+1)(j+1)]) i, j \in C \) is its own inverse! Inverting the above equation, then yields

(8.2) \[ C_t = \sum_{a \in C} (-1)^{a+t}[(a+1)(t+1)]C(a). \]

**Derivation of (2.4) from (4.2)& (4.4).** Substitute (4.4) into (4.2). Interchange the order of summation. Then making use of (8.2), we obtain (2.4) without any hypothesis on \( C_t \).

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