Décalage and Kan’s simplicial loop group functor

Danny Stevenson*
School of Mathematics and Statistics
University of Glasgow
15 University Gardens
Glasgow G12 8QW
United Kingdom
email: Danny.Stevenson@glasgow.ac.uk

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Abstract

Given a bisimplicial set, there are two ways to extract from it a simplicial set: the diagonal simplicial set and the less well known total simplicial set of Artin and Mazur. There is a natural comparison map between these simplicial sets, and it is a theorem due to Cegarra and Remedios and independently Joyal and Tierney, that this comparison map is a weak equivalence for any bisimplicial set. In this paper we will give a new, elementary proof of this result. As an application, we will revisit Kan’s simplicial loop group functor $G$. We will give a simple formula for this functor, which is based on a factorization, due to Duskin, of Eilenberg and Mac Lane’s classifying complex functor $W$. We will give a new, short, proof of Kan’s result that the unit map for the adjunction $G \dashv W$ is a weak equivalence for reduced simplicial sets.

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Contents

1 Introduction
2 The décalage comonad
   2.1 The décalage or shift functor
   2.2 Contractibility of the décalage functor

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1 Introduction

The aim of this paper is to give new and hopefully simpler proofs of two theorems in the theory of simplicial sets, the first being a generalization to simplicial sets of Dold-Puppe’s version \([6]\) of the Eilenberg-Zilber theorem from homological algebra, the second being an old result of Kan’s on simplicial loop groups. This first result is due to Cegarra and Remedios and independently to Joyal and Tierney.

Recall that if \(C\) is a double complex of abelian groups concentrated in the first quadrant then there are two ways in which one can associate to it an ordinary complex. One can form the total complex \(\text{Tot } C\) which in degree \(n\) is equal to
\[
\langle \text{Tot } C \rangle_n = \bigoplus_{p+q=n} C_{p,q},
\]
or one can form the diagonal complex \(dC\) which in degree \(n\) is equal to
\[
\langle dC \rangle_n = C_{n,n}.
\]
There is a natural comparison map \(dC \to \text{Tot } C\) and the generalized Eilenberg-Zilber theorem \([6, 11]\) says that this comparison map is a chain homotopy equivalence.

There is a generalization of this comparison with chain complexes, or equivalently simplicial abelian groups, replaced by simplicial sets. Just as we can form the diagonal of a double complex we can also form the diagonal \(dX\) of a bisimplicial set \(X\). This is the simplicial set obtained by precomposing the functor \(X: \Delta^{\text{op}} \times \Delta^{\text{op}} \to \text{Set}\) with the opposite of the functor \(\delta: \Delta \to \Delta \times \Delta\) given by \(\delta([n]) = ([n], [n])\), i.e. \(dX = X\delta\), so that the set of \(n\)-simplices of \(dX\) is \(\langle dX \rangle_n = X_{n,n}\). There is another, less well known, way to form a simplicial set from \(X\). Namely one can form what is variously known as the total simplicial set or Artin-Mazur codiagonal \(TX\) of \(X\) (see \([1]\)). This construction extends to define a functor \(T: \text{SS} \to \text{S}\) from the category \(\text{SS}\) of bisimplicial sets to the category \(\text{S}\) of simplicial sets.

The construction \(TX\) is the analog for simplicial sets of the process of forming the total complex \(\text{Tot } C\) of a double complex \(C\). In fact, if
\[
N: s\text{Ab} \rightleftarrows \text{Ch}_{\geq 0}: \Gamma
\]
denotes the Dold-Kan correspondence, then \(NTA\) is isomorphic to \(\text{Tot } NA\) (see \([3]\)).

As mentioned above, the Eilenberg-Zilber theorem in homological algebra has a generalization for simplicial sets. For any bisimplicial set \(X\), there is a natural comparison map
\[
dX \to TX
\]
between the diagonal simplicial set of $X$ and the total simplicial set of $X$. We have the following result.

**Theorem 1** ([3], [17]). Let $X$ be a bisimplicial set. Then the comparison map $dX \to TX$ is a weak equivalence.

The first published proof of this result was given in [3], with the authors noting that this fact is stated without proof in [5] where it is attributed to Zisman (unpublished). When $X$ is a bisimplicial group, a closely related result was proven by Quillen [22]. The proof of Theorem 1 given in [3] is unfortunately somewhat complicated, and so it is of interest to have a simpler proof. One such proof, incorporating some ideas of Cisinski, is given by Joyal and Tierney in their forthcoming book [17]. We shall give here a new proof, which we think is fairly elementary — in particular it uses nothing more than the fact that the diagonal functor $d$ sends level-wise weak equivalences to weak equivalences.

In the second part of the paper we present a simple construction of Kan’s simplicial loop group functor as an application of Theorem 1. Recall that in [13], Kan defined a functor $G : S \to sGp$ which is left adjoint to the classifying complex functor $W : sGp \to S$ of Eilenberg and Mac Lane [10]. He was able to show that, when $X$ is reduced (i.e. when $X$ is a simplicial set with only one vertex), the principal $GX$ bundle $X_\eta$ on $X$ induced by the unit map $\eta : X \to WGX$ has weakly contractible total space. Kan’s proof of this last fact involves showing firstly that $X_\eta$ is simply connected, and secondly that $X_\eta$ is acyclic in the sense that it has vanishing reduced homology in all degrees.

We will show that both the construction of the functor $G$ and the proof that the unit map is a weak equivalence can be greatly illuminated and simplified by considering a factorization (first noticed by Duskin) of $W$ involving the functor $T$. In fact we hasten to point out that this last section of the paper makes no great claim to originality, we find it hard to believe that some of the results of this section were not known to Duskin, although we cannot find any evidence for this in his published papers. We also point out that in their forthcoming book [17] and their paper [18] Joyal and Tierney prove more general statements in the context of simplicially enriched groupoids. Using Duskin’s factorization we will give a simple formula for the left adjoint to $W$ (see Proposition 16). In Theorem 21 we will apply this formula to give a simple and direct proof of Kan’s theorem that the unit map of the adjunction $G \dashv W$ is a weak equivalence whenever $X$ is reduced. To the best of our knowledge this proof is new (we note that an essential ingredient for the proof is Theorem 1). We point out that in [24] Waldhausen described another approach to the construction of $G$, nevertheless we feel our approach (which proceeds along different lines) is still of some interest.

# 2 The décalage comonad

We begin by recalling the definition and main properties of the décalage and total décalage functors of Illusie [14].
2.1 The décalage or shift functor

Let $\Delta_a$ denote the augmented simplex category, in other words the simplex category $\Delta$ together with the additional object $[-1]$, the empty set (the initial object of $\Delta_a$). We will write $as\mathcal{C}$ for the category $[\Delta^a_0, \mathcal{C}]$ of augmented simplicial objects in a category $\mathcal{C}$, which we will assume to be complete and cocomplete. Recall (see for example VII.5 of [19]) that $\Delta_a$ is a monoidal category with unit $[-1]$ under the operation of ordinal sum, which operation we will denote by $\sigma$ (following Joyal and Tierney). If $[m], [n] \in \Delta_a$ then $\sigma([m],[n]) = [m+n+1]$, and the operation $\sigma$ gives rise to a bifunctor $\sigma: \Delta_a \times \Delta_a \to \Delta_a$ which sends a morphism

$$(\alpha, \beta): ([m], [n]) \to ([m'], [n'])$$

in $\Delta_a \times \Delta_a$ to the morphism $\sigma(\alpha, \beta): [m+n+1] \to [m'+n'+1]$ in $\Delta_a$ defined by

$$\sigma(\alpha, \beta)(i) = \begin{cases} 
\alpha(i) & \text{if } 0 \leq i \leq m \\
\beta(i-m-1) + m' + 1 & \text{if } m + 1 \leq i \leq m + n + 1.
\end{cases}$$

$(\Delta_a, \sigma)$ is not a symmetric monoidal category — while $\sigma([m], [n]) = \sigma([n], [m])$, it need not be the case that $\sigma(\alpha, \beta) = \sigma(\beta, \alpha)$. The monoidal structure on $\Delta_a$ allows us to define a functor $\sigma(-, [0]): \Delta_a \to \Delta$ which sends $[n] \in \Delta_a$ to $\sigma([n], [0]) = [n + 1]$ in $\Delta$. We have the following definition which we believe is originally due to Illusie.

**Definition 2** ([14]). Define $Dec_0: s\mathcal{C} \to as\mathcal{C}$ to be the functor given by restriction along $\sigma(-, [0]): \Delta_a \to \Delta$, so that if $X$ is a simplicial object in $\mathcal{C}$ then $Dec_0 X$ is the augmented simplicial object given by

$$Dec_0 X([n]) = X([n+1]),$$

whose face maps $d_i: (Dec_0 X)_n \to (Dec_0 X)_{n-1}$ are given by $d_i: X_{n+1} \to X_n$ for $i = 0, 1, \ldots, n$, and whose degeneracy maps $s_i: (Dec_0 X)_n \to (Dec_0 X)_{n+1}$ are given by $s_i: X_{n+1} \to X_{n+2}$ for $i = 0, 1, \ldots, n$. The augmentation $(Dec_0 X)_0 \to X_0$ is given by $d_0: X_1 \to X_0$.

$Dec_0 X$ is obtained from $X$ by forgetting the top face and degeneracy map at each level and re-indexing by shifting degrees up by one. Thus the augmented simplicial object $Dec_0 X$ can be pictured as

$$X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} X_2 \xrightarrow{d_0} X_3 \xrightarrow{d_1} \cdots$$

Note that the simplicial identity $d_0 d_1 = d_0 d_0$ shows that $d_0: X_1 \to X_0$ is an augmentation.

There is an analogous functor $Dec^0: s\mathcal{C} \to as\mathcal{C}$ given by restriction along the functor $\sigma([0], -): \Delta_a \to \Delta$ — thus $Dec^0$ is the functor which forgets the bottom face and degeneracy map at each level. The functors $Dec_0$ and $Dec^0$ are usually called the décalage or shifting functors. More generally we can define functors $Dec_n: s\mathcal{C} \to as\mathcal{C}$ and $Dec^0: s\mathcal{C} \to as\mathcal{C}$ induced by restriction along $\sigma(-, [n]): \Delta_a \to \Delta$ and $\sigma([n], -): \Delta_a \to \Delta$ respectively.
The relation between $\text{Dec}_n X$ and $\text{Dec}^n X$ can be easily understood through the notion of the opposite simplicial object. Let $\tau : \Delta \to \Delta$ denote the automorphism of $\Delta$ which reverses the order of each ordinal $[n]$, or equivalently sends the category $[n]$ to its opposite category. Note that $\tau(\sigma([n],[m])) = \sigma(\tau([m]), \tau([n]))$ for any $[n],[m] \in \Delta$. If $X$ is a simplicial object then we write $X^o$ for the simplicial object obtained by precomposing $X$ with the functor $\tau^\text{op}$. The simplicial object $X^o$ is called the opposite simplicial object of $X$ in [15]. Note that $\text{Dec}_0^o X = \text{Dec}^0 (X^o)$ by the following calculation:

$$\text{Dec}_0^o ([n]) = \text{Dec}_0 X(\tau([n]) = X(\tau([0]), \tau([n]))) = X(\sigma(\tau([n]), [0])), $$

since $\tau[0] = [0]$. It follows that $\text{Dec}_n^o X = \text{Dec}^n (X^o)$ for any $n \geq 0$.

There are canonical comonads underlying the functors $\text{Dec}_0$ and $\text{Dec}^0$, when these functors are thought of as endofunctors on $s\mathcal{C}$ by forgetting augmentations. As is well known, $[0]$ determines a monoid in $\Delta$ whose multiplication is given by the canonical map $[1] \to [0]$. This monoid is universal in a certain precise sense (see Proposition 5.1 in Chapter VII of [19]).

The monoid $[0]$ determines a corresponding comonoid in $\Delta^\text{op}$ which in turn induces by composition the two comonads $\text{Dec}_0$ and $\text{Dec}^0$ in $s\mathcal{C}$. The counit of the comonad $\text{Dec}_0$ is induced by the natural transformation $[n] \to \sigma([0],[n])$ and hence is given on a simplicial object $X$ by the simplicial map $\text{Dec}_0 X \to X$ which in degree $n$ is the last face map $d_{n+1} : X_{n+1} \to X_n$. We will write $d_{\text{last}} : \text{Dec}_0 X \to X$ for this map.

Likewise, the counit of the comonad $\text{Dec}^0$ is induced by the natural transformation $[n] \to \sigma([n],[0])$ and hence is given on a simplicial object $X$ by the simplicial map $\text{Dec}^0 X \to X$ which in degree $n$ is the first face map $d_0 : X_{n+1} \to X_n$. We will write $d_{\text{first}} : \text{Dec}^0 X \to X$ for this map.

When $\text{Dec}_0$ and $\text{Dec}^0$ are regarded as endofunctors on $s\mathcal{C}$, we see that the functors $\text{Dec}_n$ and $\text{Dec}^n$ (also thought of as endofunctors on $s\mathcal{C}$) are given by $\text{Dec}_n = (\text{Dec}_0)^n$ and $\text{Dec}^n = (\text{Dec}^0)^n$ respectively.

### 2.2 Contractibility of the décalage functor

It is an important fact that $\text{Dec}_0 X$ and $\text{Dec}^0 X$ are not just augmented simplicial objects, they are actually contractible augmented simplicial objects in the following sense.

Recall that the augmentation map of an augmented simplicial object $\epsilon : X \to X_{-1}$ is a a deformation retraction if there exists a simplicial map $s : X_{-1} \to X$ (with $X_{-1}$ is regarded as a constant simplicial object) which is a section of the projection $\epsilon$ and is such that $s \epsilon$ is simplicially homotopic to the identity map on $X$.

A sufficient condition for $s \epsilon$ to be simplicially homotopic to the identity map on $X$ is that there exist for each $n \geq -1$, maps $s_{n+1} : X_n \to X_{n+1}$ with $s_0 = s$, which act as ‘extra degeneracies on the right’ in the sense that the following identities hold:

$$s_{n+1} \circ s_n = s_{n+1} \circ s_{n+1} = s_{n+2} \circ s_n,$$

where $s_{-1} = \text{id}$.
\[d_i s_n = s_{n-1} d_i \text{ for } 0 \leq i < n,
\]
\[d_n s_n = \text{id},
\]
\[s_i s_n = s_{n+1} s_i \text{ for } 0 \leq i \leq n,
\]

The following definition is standard.

**Definition 3.** Let \( \epsilon : X \to X_{-1} \) be an augmented simplicial object in \( \mathcal{C} \). By a contraction of \( X \) we will mean the data of the section \( s : X_{-1} \to X \) together with the extra degeneracies \( s_{n+1} \) as described above. We will say that \( X \) is contractible if it has such a contraction.

A map of contractible augmented simplicial objects is a map of the underlying augmented simplicial objects which preserves the corresponding sections \( s \) and the extra degeneracies (as in \([7]\) we will sometimes say that such a map is coherent). We will write \( a_c s\mathcal{C} \) for the category of contractible augmented simplicial objects and coherent maps.

Given the data of such a collection of maps \( s_{n+1} \) as above, we define maps \( h_i : X_n \to X_{n+1} \) by the formula
\[h_i = s^{n-i} s_{n+1} d_0^{n-i}.
\]

It is easy to check that the maps \( h_i \) satisfy the conditions (i)–(iii) in Definition 5.1 of \([20]\). The \( h_i \) then piece together to define a simplicial homotopy \( h : X \otimes \Delta[1] \to X \) from \( s\epsilon \) to the identity on \( X \), analogous to Proposition 6.2 in \([20]\). Here, if \( K \) is a simplicial set, \( X \otimes K \) denotes the tensor for the usual structure of \( s\mathcal{C} \) as a simplicially enriched category, so that \( X \otimes K \) has \( n \)-simplices given by
\[(X \otimes K)_n = \bigsqcup_{k \in K_n} X_n. \tag{1}\]

In degree \( n \), the map \( h : X \otimes \Delta[1] \to X \) is given by \( d_{i+1} h_i : (X_n)_\alpha \to X_n \) on the summand \((X_n)_\alpha \) of \((X \otimes \Delta[1])_n \) corresponding to the map \( \alpha : [n] \to [1] \) determined by \( \alpha^{-1}(0) = [i] \). We summarize this discussion in the following lemma.

**Lemma 4.** Let \( \epsilon : X \to X_{-1} \) be a contractible augmented simplicial object in \( \mathcal{C} \). Then there is a simplicial homotopy \( h : X \otimes \Delta[1] \to X \) in \( s\mathcal{C} \) between \( s\epsilon \) and \( 1_X \).

Clearly, the degeneracy \( s_{n+1} : X_{n+1} \to X_{n+2} \) for \( n \geq 0 \) equips \( \text{Dec} \epsilon X \) with an extra degeneracy in the above sense. Therefore we have the following well known result.

**Lemma 5.** For any simplicial object \( X \) in \( \mathcal{C} \), the augmentation \( d_0 : \text{Dec} \epsilon X \to X_0 \) is a deformation retract. An analogous statement is true for \( \text{Dec} \epsilon X \).

A prime example where simplicial objects with extra degeneracies appear is in the construction of simplicial comonadic resolutions. Suppose that \( L \) is a comonad on a category \( \mathcal{C} \), and \( X \) is an object of \( \mathcal{C} \). Then, as is well known, \( L \) determines an augmented simplicial
object \( L_*X \) whose object of \( n \)-simplices is \( L^nX \) and whose face and degeneracy maps are defined by
\[
d_i = L^i \epsilon L^{n-i}, \quad s_i = L^i \delta L^{n-i-1}
\]
respectively, where \( \epsilon : L \to 1 \) denotes the counit and \( \delta : L \to L^2 \) denotes the comultiplication of the comonad. Suppose that there exists a section \( \sigma : X \to LX \) of the counit \( \epsilon_X : LX \to X \). Then \( \sigma \) determines extra degeneracies \( s_{n+1} : L^nX \to L^{n+1}X \) given by \( s_{n+1} = L^n \sigma \) (see for example [23]). It follows from the discussion above that there is a simplicial homotopy \( h : L_*X \otimes \Delta[1] \to L_*X \) in \( s\mathcal{C} \) between \( \sigma \epsilon \) and the identity on \( L_*X \).

3 The total décalage and the total simplicial set functors

In this section we will recall some of the main properties of Illusie’s total décalage functor \( \text{Dec} \) [14] and its right adjoint, the Artin-Mazur total simplicial set functor \([1]\). For more details the reader should refer to the excellent discussion of these functors and their properties in the papers [3, 4]. In this section we will mainly be interested in the case where \( \mathcal{C} = \text{Set} \). We begin therefore by explaining our notations and conventions for bisimplicial sets (which follows closely the presentation in [16]).

If \( X \in \text{SS} \) is a bisimplicial set then we will say that \( X_{m,n} = X([m],[n]) \) has horizontal degree \( m \) and vertical degree \( n \). We write \( \text{SS} \) for the category of bisimplicial sets. We say a simplicial space is a simplicial object in \( \text{S} \). There are two ways in which we can regard a bisimplicial set \( X \) as a simplicial space. On the one hand, we can define \( X_m \) to be the simplicial set with \( n \)-simplices \( (X_m)_n = X_{m,n} \). Thus we regard \( X \) as a horizontal simplicial object with vertical simplicial sets. On the other hand we can define \( X_n \) to be the simplicial set with \( m \)-simplices \( (X_n)_m = X_{m,n} \). Thus we regard \( X \) as a vertical simplicial object with horizontal simplicial sets.

Each of these two ways of viewing a bisimplicial set as a simplicial space leads to a simplicial enrichment of \( \text{SS} \), using the canonical simplicial enrichment of \( s\mathcal{S} \) mentioned earlier. If we view bisimplicial sets as a horizontal simplicial objects in \( \text{S} \), then \( \text{SS} = s\mathcal{S} \) is equipped with the structure of a simplicial enriched category for which the tensor \( X \otimes_1 K \), for \( X \) a bisimplicial set and \( K \in \text{S} \), has vertical simplicial set of \( m \)-simplices given by (see [11])
\[
(X \otimes_1 K)_m = \prod_{k \in K_m} X_m = X_m \times K_m,
\]
so that the set of \( (m,n) \)-bisimplices of \( X \otimes_1 K \) is \( (X \otimes_1 K)_{m,n} = X_{m,n} \times K_m \). In other words,
\[
X \otimes_1 K = X \times p_1^* K,
\]
where \( p_1 : \Delta \times \Delta \to \Delta \) denotes projection onto the first factor. The simplicial enrichment is then defined by the formula
\[
\text{Hom}_1(X,Y) = i_1^*(Y^X),
\]
where \( i_1 : \Delta \to \Delta \times \Delta \) denotes the right adjoint to \( p_1 \), so that \( i_1([n]) = ([n], [0]) \).

Similarly, if we view \( X \in \text{SS} \) as a vertical simplicial object, then the tensor \( X \otimes_2 K \) is given by

\[
X \otimes_2 K = X \times p_2^* K
\]

where \( p_2 : \Delta \times \Delta \to \Delta \) denotes projection onto the second factor. The simplicial enrichment is defined by the formula

\[
\text{Hom}_2(X, Y) = i_2^*(Y^X),
\]

where \( i_2 : \Delta \to \Delta \times \Delta \) denotes right adjoint to \( p_2 \), defined by \( i_2([n]) = ([0], [n]) \).

Following Joyal we will say that a bisimplicial set \( X \) is row augmented if there is a map \( X \to p_1^* K \) in \( \text{SS} \) for some simplicial set \( K \), and we will say that \( X \) is column augmented if there is a map \( X \to p_2^* K \) in \( \text{SS} \) for some simplicial set \( K \).

With these conventions understood we can describe Illusie’s total décalage functor \([14]\).

The simplicial comonadic resolution of \( \text{Dec}_0 \) gives rise to a functor \( \text{Dec} : S \to \text{SS} \) which sends a simplicial set \( X \) to the simplicial space \( \text{Dec}X \) which in degree \( n \) is the simplicial set

\[
\text{Dec}_n X = (\text{Dec}_0)^n X.
\]

Here we are thinking of \( \text{Dec}X \) as a vertical simplicial object in \( S \) with horizontal simplicial sets. The set of \((m, n)\)-bisimplices of the bisimplicial set \( \text{Dec}X \) is \( (\text{Dec}X)_{m, n} = X_{m+n+1} \). The horizontal and vertical face operators \( d_i^h : (\text{Dec}X)_{m+1, n} \to (\text{Dec}X)_{m, n} \) and \( d_i^v : (\text{Dec}X)_{m, n+1} \to (\text{Dec}X)_{m, n} \) are given by \( d_i^h = d_i : X_{m+n+2} \to X_{m+n+1} \) and \( d_i^v = d_m+i+1 : X_{m+n+2} \to X_{m+n+1} \) respectively. There are similar formulas for the horizontal and vertical degeneracy operators. Note that if we regard the bisimplicial set \( \text{Dec}X \) as a horizontal simplicial object with vertical simplicial sets then \( \text{Dec}X \) is the simplicial comonadic resolution of \( X \) by \( \text{Dec}_0 \). The following Lemma is straightforward.

**Lemma 6.** The functor \( \text{Dec} : S \to \text{SS} \) is given by restriction along the ordinal sum map \( \sigma : \Delta \times \Delta \to \Delta \) so that

\[
\text{Dec} X([m], [n]) = X(\sigma([m], [n])) = X_{n+m+1}
\]

for \( X \in S \).

We see from this Lemma that in fact \( \text{Dec}X \) is the restriction of a bi-augmented simplicial object in the sense that the functor \( \text{Dec} : \Delta^{op} \times \Delta^{op} \to \mathcal{C} \) extends canonically to a functor \((\Delta^{op}_a \times \Delta^{op}) \cup (\Delta^{op} \times \Delta^{op}) \to \mathcal{C}\). Therefore \( \text{Dec}X \) is both row and column augmented. The row augmentation \( \epsilon_r : \text{Dec}X \to p_1^* X \) is given by the map \( d_{\text{last}} : \text{Dec}_0 X \to X \), while the column augmentation \( \epsilon_c : \text{Dec}X \to p_2^* X \) is given by the map \( d_{\text{first}} : \text{Dec}_0 X \to X \).

Suppose that \( X \) is a simplicial set, and regard \( \text{Dec}X \) as a (vertical) simplicial space whose rows are the simplicial sets \( \text{Dec}_n X \) for \( n \geq 0 \). Then the functor \( \text{disc} : S \to \text{SS} \) which sends a simplicial set \( K \) to the constant simplicial space whose rows are \( K \), has a left adjoint \( \pi_0 : \text{SS} \to S \). Note that the functor \( \text{disc} \) is nothing other than the functor \( p_1^* : S \to \text{SS} \). It is easy to compute the value of \( \pi_0 \text{Dec}X \) as the next lemma shows.
Lemma 7. For any simplicial set $X$, we have $\pi_0 \text{Dec} X = X$.

Proof. The value of the functor $\pi_0 : \text{SS} \to \text{S}$ on a bisimplicial set $Y$ is given by the coequalizer

$$\pi_0 Y = \text{coeq}(Y_1 \xrightarrow{d_0} Y_0),$$

where we have written $Y_n$ for the $n$-th row of $Y$. Since colimits in $\text{S}$ are computed pointwise we see that the set of $n$-simplices $\pi_0(Y)_n$ is the set of path components of the $n$-th column of $Y$. Therefore the set of $n$-simplices of $\pi_0 \text{Dec} X$ is the set of path components of $\text{Dec}^n X$, which we have seen is equal to $X_n$. One can further check that this isomorphism is compatible with face and degeneracy maps so that we get the identification $\pi_0 \text{Dec} X = X$. 

Thus when $X = \Delta[n]$ we have an isomorphism $\pi_0 \text{Dec} \Delta[n] = \Delta[n]$. In fact, we will show that more is true: we will see that the row and column augmentation maps $\text{Dec} \Delta[n] \to p_1^* \Delta[n]$ and $\text{Dec} \Delta[n] \to p_2^* \Delta[n]$ are simplicial homotopy equivalences. To see this we will need the following result.

Lemma 8. For any $n \geq 0$ there are sections of the maps $d_{\text{last}} : \text{Dec} \Delta[n] \to \Delta[n]$ and $d_{\text{first}} : \text{Dec}^0 \Delta[n] \to \Delta[n]$.

Proof. Consider the map $\sigma_r : \Delta[n] \to \text{Dec} \Delta[n]$ given in degree $m$ by

$$\sigma_r(x) = s^{n+1} \sigma(x, [0])$$

for $x : [m] \to [n]$. Clearly this is natural in $x$. It is also a section of $d_{\text{last}}$ by the following calculation. Since $d_{\text{last}} \sigma_r(x) = d_{m+1} s^{n+1} \sigma(x, [0]) = s^{n+1} \sigma(x, [0]) d^{m+1}$, then for any $i \in [m]$ we have

$$s^{n+1} \sigma(x, [0]) d^{m+1}(i) = s^{n+1} \sigma(x, [0])(i) = s^{n+1}(x(i)) = x(i),$$

so that $d_{\text{last}} \sigma_r = \sigma_r$. In a completely analogous way one can define a section of $d_{\text{first}}$. 

The section $\sigma_r : \Delta[n] \to \text{Dec} \Delta[n]$ is a section of the counit $d_{\text{last}} : \text{Dec} \Delta[n] \to \Delta[n]$ and hence defines an extra degeneracy of the simplicial comonadic resolution $\text{Dec} \Delta[n]$ of $\Delta[n]$. As discussed in Section 2.2 above, this means that the map $\sigma_r : \Delta[n] \to \text{Dec} \Delta[n]$ exhibits $\epsilon_r : \text{Dec} \Delta[n] \to p_1^* \Delta[n]$ as a deformation retraction: thus $\epsilon_r \sigma_r$ is the identity on $p_1^* \Delta[n]$ and there is a simplicial homotopy between $\sigma_r \epsilon_r$ and the identity on $\text{Dec} \Delta[n]$. Here we are viewing $\text{Dec} \Delta[n]$ as a vertical simplicial object with horizontal simplicial sets and thus the simplicial homotopy is a map $h : \text{Dec} \Delta[n] \otimes_{2} \Delta[1] \to \text{Dec} \Delta[n]$. Completely analogous statements apply for the section $\sigma_c$. We summarize this discussion in the following lemma.

Lemma 9. There are maps $\sigma_r : p_1^* \Delta[n] \to \text{Dec} \Delta[n]$ and $\sigma_c : p_2^* \Delta[n] \to \text{Dec} \Delta[n]$ in $\text{SS}$ such that the following are true:
1. σ is a section of \( \epsilon_r : \text{Dec} \Delta[n] \to p_1^* \Delta[n] \) and \( \sigma_c \) is a section of \( \epsilon_c : \text{Dec} \Delta[n] \to p_2^* \Delta[n] \),
2. there is a simplicial homotopy \( h : \text{Dec} \Delta[n] \otimes_2 \Delta[1] \to \text{Dec} \Delta[n] \) from \( \sigma_r \epsilon_r \) to the identity,
3. there is a simplicial homotopy \( k : \text{Dec} \Delta[n] \otimes_1 \Delta[1] \to \text{Dec} \Delta[n] \) from \( \sigma_c \epsilon_c \) to the identity.

From the description of \( \text{Dec} \) in Lemma 6 above it is clear that \( \text{Dec} \) has both a left and right adjoint. The left adjoint of \( \text{Dec} \) is related to the notion of the join of simplicial sets. The right adjoint to \( \text{Dec} \) is denoted \( T : \mathbf{SS} \to \mathbf{S} \), it was introduced in [1] where it was called the total simplicial set functor. It is also known as the Artin-Mazur codiagonal. It has the following explicit description: if \( X \) is a bisimplicial set then the set \( (T X)_n \) of \( n \)-simplices of the simplicial set \( T X \) is given by the equalizer of the diagram

\[
(TX)_n \to \prod_{i=0}^{n} X_{i,n-i} \rightrightarrows \prod_{i=0}^{n-1} X_{i,n-i-1}
\]

where the components of the two maps are defined by the composites

\[
\prod_{i=0}^{n} X_{i,n-i} \xrightarrow{\rho_i} X_{i,n-i} \xrightarrow{d_i^h} X_{i,n-i-1}
\]

and

\[
\prod_{i=0}^{n} X_{i,n-i} \xrightarrow{p_{i+1}} X_{i+1,n-i-1} \xrightarrow{d_{i+1}^h} X_{i,n-i-1}.
\]

The face maps \( d_i : (TX)_n \to (TX)_{n-1} \) are given by

\[
d_i = (d_i^np_0, d_i^np_1, \ldots, d_i^np_{i-1}, d_i^hp_{i+1}, d_i^hp_{i+2}, \ldots, d_i^hp_n)
\]

while the degeneracy maps \( s_i : (TX)_n \to (TX)_{n+1} \) are given by

\[
s_i = (s_i^np_0, s_i^np_1, \ldots, s_i^np_{i-1}, s_i^hp_{i+1}, s_i^hp_{i+2}, \ldots, s_i^hp_n).
\]

The unit map \( \eta : X \to T \text{Dec} X \) of the adjunction \( \text{Dec} \dashv T \) is given by the map

\[
x \mapsto (s_0(x), s_1(x), \ldots, s_n(x))
\]

in degree \( n \) (see [3]). In general it is rather difficult to give a simple description of the simplicial set \( T X \) for an arbitrary bisimplicial set \( X \). When \( X \) is constant however, we have the following well-known result.

**Lemma 10.** Let \( X \) be a simplicial set. Then there are isomorphisms \( T p_1^* X = T p_2^* X = X \), natural in \( X \).

**Proof.** Observe that the functor \( T p_1^* \) is right adjoint to the functor \( \pi_0 \text{Dec} \). Lemma 7 implies that the functor \( \pi_0 \text{Dec} \) is the identity on \( \mathbf{S} \), from which it follows that there is an isomorphism \( T p_1^* X = X \), natural in \( X \). The other statement is proven in an analogous fashion. \( \square \)
4 The generalized Eilenberg-Zilber theorem for simplicial sets

Our goal in this section is to present an elementary proof of Theorem 1. Recall that this theorem states that there is a weak equivalence
\[ dX \rightarrow TX \] (4)
of simplicial sets, natural in X. As mentioned earlier, the proof of Theorem 1 in [3] is rather lengthy, and so it is of interest to have a simpler approach. We will describe here another proof, which we think is fairly elementary (as mentioned in the Introduction, the forthcoming book [17] of Joyal and Tierney contains another proof, which proceeds along different lines).

We begin by describing the map (4). This map is obtained from the map of cosimplicial bisimplicial sets
\[ \text{Dec} \Delta \rightarrow (p_1^* \Delta \times p_2^* \Delta) \delta \] (5)
by applying the functor \( SS(-, X) : eSS \rightarrow S \). Here \( (p_1^* \Delta \times p_2^* \Delta) \delta \) is the cosimplicial bisimplicial set which in degree \( n \) is \( p_1^* \Delta[n] \times p_2^* \Delta[n] \). The map (5) in degree \( n \) is the canonical map induced by the row and column augmentations \( \epsilon_r : \text{Dec} \Delta[n] \rightarrow p_1^* \Delta[n] \) and \( \epsilon_c : \text{Dec} \Delta[n] \rightarrow p_2^* \Delta[n] \) respectively. Note that it is possible to describe the map \( dX \rightarrow TX \) much more explicitly at the level of simplices (see [3]) but we will not need this.

The proof of Theorem 1 that we shall give essentially boils down to the well known fact that the diagonal functor \( d : SS \rightarrow S \) sends level-wise weak equivalences of bisimplicial sets to weak equivalences of simplicial sets. In other words, if \( f : X \rightarrow Y \) is a map in \( SS \) such that the map \( f_n : X_n \rightarrow Y_n \) on \( n \)-th rows is a weak equivalence for all \( n \geq 0 \), then \( df : dX \rightarrow dY \) is also a weak equivalence. Alternatively, if the map \( f_n : X_n \rightarrow Y_n \) on the \( n \)-th columns is a weak equivalence for all \( n \geq 0 \), then \( df : dX \rightarrow dY \) is a weak equivalence.

Recall that \( d \) has a right adjoint \( d_* : S \rightarrow SS \) (see for instance [11] page 222) defined by the formula
\[ (d_* X)_{m,n} = S(\Delta[m] \times \Delta[n], X). \] (6)
Using the fact that the diagonal \( d \) sends level-wise weak equivalences to weak equivalences one can prove (see for instance [21]) that the counit \( \epsilon : dd_* K \rightarrow K \) of this adjunction is a weak equivalence for any simplicial set \( K \), and so in particular \( dd_* TX \rightarrow TX \) is a weak equivalence for any bisimplicial set \( X \). Therefore, since we can factor (4) as
\[ dX \rightarrow dd_* TX \rightarrow TX, \]
we see that to prove Theorem 1 it suffices to prove the following proposition.

**Proposition 11.** The map \( dX \rightarrow dd_* TX \) is a weak equivalence for any bisimplicial set \( X \).

**Proof.** First note that the underlying map \( X \rightarrow d_* TX \) of simplicial sets is induced by the following map in \( eSS \):
\[ \epsilon_r \times \epsilon_c : \text{Dec} \Delta \times \text{Dec} \Delta \rightarrow p_1^* \Delta \times p_2^* \Delta, \] (7)
where Dec $\Delta$ denotes the cosimplicial bisimplicial set $[m] \mapsto \text{Dec} \Delta[m]$. This relies on (6) and the fact that Dec preserves products, together with the observation that the map (5) factors as

$$\text{Dec}\Delta \to \text{Dec}(\Delta \times \Delta)\delta \to (p_1^*\Delta \times p_2^*\Delta)\delta$$

Here the map $\Delta \to (\Delta \times \Delta)\delta$ is the canonical map inducing the counit $dd_* \to 1$ of the adjunction $d \dashv d_*$. The map (7) in turn factors as

$$\text{Dec}\Delta \times \text{Dec}\Delta \overset{1 \times \epsilon_c}{\longrightarrow} \text{Dec}\Delta \otimes \Delta \overset{\epsilon_r \times 1}{\longrightarrow} p_1^*\Delta \times p_2^*\Delta,$$

where $\text{Dec}\Delta \otimes \Delta$ denotes the bicosimplicial bisimplicial set which in bidegree $(m, n)$ is $\text{Dec}\Delta[m] \otimes \Delta[n]$. Applying the functor $\mathbf{SS}(-, X) : \mathbf{ccSS} \to \mathbf{SS}$ we get a pair of maps of bisimplicial sets

$$(1 \times \epsilon_c)^* : \mathbf{SS}(\text{Dec}\Delta \otimes \Delta, X) \to d_*TX \quad (8)$$

and

$$(\epsilon_r \times 1)^* : X \to \mathbf{SS}(\text{Dec}\Delta \otimes \Delta, X). \quad (9)$$

We will prove that the first map is a row-wise weak equivalence and that the second map is a column-wise weak equivalence. Since $d$ sends level-wise weak equivalences to weak equivalences, this is enough to prove that the map $dX \to dd_*TX$ is a weak equivalence.

**Lemma 12.** The map (8) is a row-wise weak equivalence.

**Proof.** The induced map on the $n$-th row induced by (8) is obtained by applying the functor $\mathbf{SS}(-, X) : \mathbf{ccSS} \to \mathbf{SS}$ to the map of cosimplicial bisimplicial sets

$$\text{Dec}\Delta \times \text{Dec}\Delta[n] \to \text{Dec}\Delta \times p_2^*\Delta[n].$$

Since $\mathbf{SS}$ is cartesian closed (it is a presheaf category), we see that the induced map on rows is given by the map of simplicial sets

$$\epsilon_c^* : \mathbf{SS}(p_2^*\Delta[n], X^{\text{Dec}\Delta}) \to \mathbf{SS}(\text{Dec}\Delta[n], X^{\text{Dec}\Delta}), \quad (10)$$

where $X^{\text{Dec}\Delta}$ denotes the simplicial object in $\mathbf{SS}$ whose bisimplicial set of $p$-simplices is given by the exponential

$$X^{\text{Dec}\Delta[p]}$$

in $\mathbf{SS}$. Note that for any simplicial set $K$, there is a bijection

$$\mathbf{S}(K, \mathbf{SS}(\text{Dec}\Delta[n], X^{\text{Dec}\Delta})) = \mathbf{SS}(\text{Dec}\Delta[n], X^{\text{Dec}K}),$$

which is natural in $K$. Thus there is an isomorphism

$$\mathbf{SS}(\text{Dec}\Delta[n], X^{\text{Dec}\Delta})^{\Delta[1]} = \mathbf{SS}(\text{Dec}\Delta[n] \times \text{Dec}\Delta[1], X^{\text{Dec}\Delta}),$$

where $\mathbf{SS}(\text{Dec}\Delta[n], X^{\text{Dec}\Delta})^{\Delta[1]}$ denotes the simplicial path space of $\mathbf{SS}(\text{Dec}\Delta[n], X^{\text{Dec}\Delta})$. We will use these remarks to prove that the map (10) is a simplicial homotopy equivalence.
Note that the map \( \sigma_c: p_2^* \Delta[n] \to \text{Dec} \Delta[n] \) induces a left inverse \( \sigma_c^* \) of the map \( \epsilon^*_c \). Therefore, we need to show that there is a simplicial homotopy

\[
(\sigma_c \epsilon_c)^* \simeq \text{id}.
\]

By the previous description of the simplicial path space of \( \text{SS}(\text{Dec} \Delta[n], X^{\text{Dec} \Delta}) \), to find such a simplicial homotopy it suffices to find a map

\[
\text{Dec} \Delta[n] \times \text{Dec} \Delta[1] \to \text{Dec} \Delta[n], \tag{11}
\]

in \( \text{SS} \) making the obvious diagram commute. A map as in (11) is given by the following composite:

\[
\text{Dec} \Delta[n] \times \text{Dec} \Delta[1] \xrightarrow{1 \times \epsilon_r} \text{Dec} \Delta[n] \otimes_1 \Delta[1] \xrightarrow{k} \text{Dec} \Delta[n], \tag{12}
\]

where \( k: \text{Dec} \Delta[n] \otimes_1 \Delta[1] \to \text{Dec} \Delta[n] \) is the simplicial homotopy between \( \sigma_c \epsilon_c \) and \( \text{id} \) of Lemma 9. It is easy to check that this map satisfies the required commutativity condition.

**Lemma 13.** The map (9) is a column-wise weak equivalence.

**Proof.** The map induced on the \( m \)-th column by (9) is obtained by applying the functor \( \text{SS}(-, X) \) to the map of cosimplicial bisimplicial sets

\[
\epsilon_r \times 1: \text{Dec} \Delta[m] \times p_2^* \Delta \to p_1^* \Delta[m] \times p_2^* \Delta.
\]

Again, since \( \text{SS} \) is cartesian closed, we see that the induced map on columns is given by

\[
\text{SS}(p_1^* \Delta[m], X^{p_2^* \Delta}) \to \text{SS}(\text{Dec} \Delta[m], X^{p_2^* \Delta}).
\]

Our strategy again is to show that this map is a simplicial homotopy equivalence. Since \( \sigma_r \) is a right inverse of \( \epsilon_r \), the map \( (\sigma_r \times 1)^* \) is a left inverse of \( (\epsilon_r \times 1)^* \). Again, a little calculation shows that the simplicial path space of \( \text{SS}(\text{Dec} \Delta[m], X^{p_2^* \Delta}) \) is given by

\[
\text{SS}(\text{Dec} \Delta[m] \times p_2^* \Delta[1], X^{p_2^* \Delta}).
\]

Therefore, to find a simplicial homotopy \( (\sigma_r \epsilon_r \times 1)^* \simeq \text{id} \), it suffices to find a bisimplicial map

\[
\text{Dec} \Delta[m] \otimes_2 \Delta[1] \to \Delta[m].
\]

making the obvious diagram commute. Such a map is given for example by the simplicial homotopy

\[
h: \text{Dec} \Delta[m] \otimes_2 \Delta[1] \to \text{Dec} \Delta[m]
\]

between \( \sigma_r \epsilon_r \) and \( \text{id} \) of Lemma 9. This completes the proof of the lemma.

The proof of Proposition 11 is now complete, by the remark above.
In analogy with the fact that the map \( dC \rightarrow \text{Tot} C \) of the generalized Eilenberg-Zilber Theorem for a chain complex \( C \) is a chain homotopy equivalence \([6, 11]\), one may wonder whether the analogous map \( dX \rightarrow TX \) of simplicial sets is a simplicial homotopy equivalence. In some cases this is known, for instance when \( X \) is the degree-wise nerve \( NG \) of a simplicial group \( G \), as we will discuss in the next section. We suspect that the map \( dX \rightarrow TX \) is not a simplicial homotopy equivalence for arbitrary \( X \), however it seems a little difficult to construct a counter-example to support this. In this direction we can say the following however: there is no map \( TX \rightarrow dX \) which is natural in \( X \). For such a map would be induced in degree \( n \) by a map \( \Delta[n, n] \rightarrow \text{Dec} \Delta[n] \), natural in \( [n] \), which by adjointness would in turn be induced by a map \( \Delta[2n + 1] \rightarrow \Delta[n] \), natural in \( [n] \). However it is not hard to see that no such map can exist.

5 Kan’s simplicial loop group construction revisited

The classifying complex \( \overline{W}G \) of a simplicial group \( G \) was introduced in \([10]\) (see Section 17 of that paper). We recall the definition.

**Definition 14** (Eilenberg-Mac Lane \([10]\)). Let \( G \) be a simplicial group. Then \( \overline{W}G \) is the simplicial set with a single vertex, and whose set of \( n \)-simplices, \( n \geq 1 \), is given by

\[
(\overline{W}G)_n = G_{n-1} \times G_{n-2} \times \cdots \times G_0.
\]

The face and degeneracy maps of \( \overline{W}G \) are given by the following formulas:

\[
d_i(g_{n-1}, \ldots, g_0) = \begin{cases} 
(g_{n-2}, \ldots, g_0) & \text{if } i = 0, \\
(d_i(g_{n-1}), \ldots, d_1(g_{n-i+1}), g_{n-i-1}, g_{n-i-2}, \ldots, g_0) & \text{if } 1 \leq i \leq n
\end{cases}
\]

and

\[
s_i(g_{n-1}, \ldots, g_0) = \begin{cases} 
(1, g_{n-1}, \ldots, g_0) & \text{if } i = 0, \\
(s_i-1(g_{n-1}), \ldots, s_0(g_{n-i}), 1, g_{n-i-1}, \ldots, g_0) & \text{if } 1 \leq i \leq n.
\end{cases}
\]

The motivation for the above formula for \( \overline{W}G \) is perhaps not so clear. We will show that there is a very natural ‘explanation’ for the above formula in terms of the décalage functors. For this, we first need some background on principal twisted cartesian products.

Recall that a principal twisted cartesian product (PTCP) with structure group \( G \) consists of a simplicial set \( P \) (the total space) and a simplicial set \( M \) (the base space) together with a map \( \pi: P \rightarrow M \) and an action of \( G \) on \( P \) which is principal in the sense that the diagram

\[
\begin{array}{ccc}
P \times G & \longrightarrow & P \\
p_1 \downarrow & & \downarrow \pi \\
P & \longrightarrow & M
\end{array}
\]
is a pullback, where $p_1$ denotes projection onto the first factor, the top arrow is the action of $G$ on $P$, and $\pi$ denotes the projection to the base. Moreover, $\pi: P \to M$ is required to have a pseudo-cross section (on the left), i.e. a family of sections $\sigma_n$ of the maps $\pi_n: P_n \to M_n$ for all $n \geq 0$ such that $\sigma_{n+1}s_i = s_i\sigma_n$ for all $0 \leq i \leq n$ and $d_i\sigma_n = \sigma_{n-1}d_i$ for all $0 < i < n$.

The simplicial set $\hat{W}G$ is a classifying space for PTCPs with structure group $G$ in the sense that there is a universal PTCP $\hat{W}G$ with base space $\hat{W}G$ with the property that every PTCP $P$ on $M$ with structure group $G$ is induced by pullback from $WG \to \hat{WG}$ along a map $M \to \hat{WG}$, the classifying map of $P$.

In [7] Duskin explained how this classical notion of pseudo-cross section has a convenient reformulation in terms of $\text{Dec}^0$. In this reformulation, $\sigma$ is required to be a section of the induced map $\text{Dec}^0\pi: \text{Dec}^0P \to \text{Dec}^0M$ in the category $acS$ of contractible augmented simplicial sets and coherent maps (see Section 2.2).

Since $G$ acts principally on $P$, there is a canonical map of bisimplicial sets

$$\text{cosk}_0P \to NG,$$

where $NG$ denotes the bisimplicial set which, when viewed as a (vertical) simplicial object in $S$, has as its object of $n$-simplices the (horizontal) simplicial set $NG_n$, i.e. the nerve of the group $G_n$. Also here $\text{cosk}_0P$ denotes the 0-coskeleton (or Čech nerve) of $P$, viewed as an object in $S/M$. Therefore, $\text{cosk}_0P$ has as its object of $n$-simplices the (horizontal) simplicial set $\check{C}(P_n)$ which is the Čech nerve of the map $\pi_n: P_n \to M_n$. In degree $n$ the canonical map $\text{cosk}_0 \to NG$ is just the canonical map $\check{C}(P_n) \to NG_n$ arising from the principal action of $G_n$ on $P_n$.

One of the advantages of this reformulation of the notion of PTCP is that it allows for a very simple and conceptual description of the classifying map of $P$ (we find it hard to believe that this description was not known to Duskin). We have a commutative diagram

$$\begin{array}{ccc}
\text{Dec}^0P & \rightarrow & P \\
\downarrow & & \downarrow \\
\text{Dec}^0M & \rightarrow & M.
\end{array}$$

Composing the pseudo-cross section $s: \text{Dec}^0M \to \text{Dec}^0P$ with the map $\text{Dec}^0P \to P$ gives rise to a map $\text{Dec}^0M \to P$ over $M$ which extends canonically to a simplicial map

$$\text{Dec}M \to \text{cosk}_0P$$

between simplicial objects in $S/M$. Here $\text{Dec}^0M$ is thought of as the vertical simplicial set of 0-simplices of the bisimplicial set $\text{Dec}M$. We can compose this with the canonical map $\text{cosk}_0P \to NG$ to obtain a map $\text{Dec}M \to NG$. The adjoint of the map $\text{Dec}M \to NG$ is a map

$$M \to TNG$$

which serves as a classifying map for $P$. One can go further and show that there is a canonical PTCP with base space $TNG$ from which $P$ arises via pullback along the above map. The next result shows that $TNG$ is precisely the classifying complex $\hat{WG}$.
Lemma 15 (Duskin). The classifying complex functor $\overline{W}$ factors as $\overline{W} = TN$, so that $\overline{W}G = TNG$ for any simplicial group $G$.

This factorization of $\overline{W}$ is due as far as we know to Duskin, who observed that this factorization persists when simplicial groups are replaced by simplicially enriched groupoids, i.e. the functor $\overline{W}: SGpd \to S$ introduced by Dwyer and Kan in [8] also factors as $\overline{W} = TN$ (this last observation also appears in the MSc thesis of Ehlers [9]).

Proof. This is an essentially straightforward computation, so we will just give a sketch of the details. To an $n$-simplex of $\overline{W}G$ consisting of a tuple $(g_{n-1}, g_{n-2}, \ldots, g_0)$ as above, we associate the element $(x_0, x_1, \ldots, x_n)$ of $TNG$, where $x_0 = 1$ and

$$x_i = (d_i^{n-1}(g_{n-1}), d_i^{n-2}(g_{n-2}), \ldots, d_0(g_{n-i+1}), g_{n-i}) \in (NG_{n-i}),$$

for $i \geq 1$. This sets up a bijection $(\overline{W}G)_n = (TNG)_n$ which respects face and degeneracy maps. \hfill \Box

It is well known that $\overline{W}G$ is weakly equivalent to the simplicial set $dNG$, obtained by applying the diagonal functor to the degree-wise nerve $NG$ of the simplicial group $G$. Of course this can be seen as an instance of Theorem 1 in light of the identification $\overline{W}G = TNG$, but there are easier proofs, see for example [12]. In fact, $\overline{W}G$ is simplicially homotopy equivalent to $dNG$, the point being that both $\overline{W}G$ and $dNG$ are fibrant (a proof of the latter fact can be found in [18]). In [23] it is shown via explicit calculation that the map $f: dNG \to \overline{W}G$ defined by

$$f(h_1, \ldots, h_n) = (d_0(h_1), \ldots, d_0^n(h_n))$$

for $h_i \in G_n$ exhibits $dNG$ as a deformation retract of $\overline{W}G$. There is a further relationship between $dNG$ and $\overline{W}G$ (see [2]): after passing to geometric realizations there is an isomorphism of spaces $|\overline{W}G| = |dNG|$. It is not clear that this isomorphism is induced by a simplicial map however. It would be interesting to give a more conceptual proof of the isomorphism from [2].

There are several advantages of the description of $\overline{W}$ in Lemma 15 over the traditional description. One such advantage of the present description is that it becomes manifestly clear that $\overline{W}$ has a left adjoint since both of the functors $N$ and $T$ do.

Proposition 16. A left adjoint for the functor $\overline{W} = TN$ is given by the functor

$$G = \pi_1R Dec: S \to sGp,$$

where $R: SS \to sS_0$ is the left adjoint of the inclusion $sS_0 \subset SS$. If $X$ is a simplicial set, then the value of $G$ on $X$ is the simplicial group $GX$ defined by

$$[n] \mapsto \pi_1(Dec_nX/X_{n+1}).$$

16
Proof. Observe that the functor $R$ is induced by the left adjoint of the inclusion $S_0 \subset S$, i.e. the functor which sends a simplicial set $X$ to the reduced simplicial set $X/\sk_0 X$. To describe $RX$ for $X$ a bisimplicial set whose $n$-th row is $X_n$, we let $\sk_0 X$ denote the bisimplicial set whose $n$-th row is $\sk_0 X_n$, i.e. the constant simplicial set $[m] \mapsto X_{0,n}$. Then $RX = X/\sk_0 X$ so that the $n$-th row of $RX$ is $RX_n = X_n/X_{0,n}$. The proposition then follows from the fact that $\sk_0 \Dec_n X$ is the constant simplicial set $X_{n+1}$.

Recall that a simplicial group $G$ is said to be a loop group for a simplicial set $X$ if there is a PTCP $P$ on $X$ with structure group $G$ such that $P$ is weakly contractible. In [13] Kan showed that the left adjoint $G: S \to s\text{Gp}$ of the classifying complex functor $W$ had the property that $G(X)$ was a loop group for any reduced simplicial set $X$. We will shortly give a simplified proof of his theorem by exploiting the description of $G$ given in Proposition [16] above. Before we do this however we need the following lemmas.

**Lemma 17.** Suppose that $X$ is a bisimplicial set whose first column is weakly contractible, i.e. the simplicial set $[n] \mapsto X_{0,n}$ is weakly contractible. Then $X \to RX$ is a column-wise weak equivalence.

**Proof.** For every $m \geq 0$, the vertical simplicial set $(\sk_0 X)_m$ is weakly contractible and so $X_m \to X_m/(\sk_0 X)_m$ is a weak equivalence of vertical simplicial sets for every $m \geq 0$.

**Lemma 18.** Let $X$ be a CW complex whose path components are all contractible. Then $X/X^0$ is a $K(\pi,1)$, where $X^0$ denotes the set of vertices of $X$.

**Proof.** $X/X^0$ can be written as a wedge

$$\bigvee_{\alpha \in \pi_0(X)} X_{\alpha}/X^0_{\alpha},$$

where $X_\alpha$ denote the path components of $X$. Therefore without loss of generality we can assume that $X$ is a path connected, pointed CW complex. We then have to show that $X/X^0$ is a $K(\pi,1)$. Choose a strong deformation retraction of $X$ onto a maximal tree $T$ in the 1-skeleton $X^1$ of $X$ (see for example I Theorem 5.9 of [26]). Then $T/X^0$ is a deformation retract of $X/X^0$ and so $X/X^0$ is a wedge of circles, from which the result follows.

**Corollary 19.** For any simplicial set $X$, $\Dec_n X/X_{n+1}$ has the weak homotopy type of a $K(\pi,1)$.

**Proof.** Since $\Dec_n X = \Dec_0 \Dec_{n-1} X$, it is enough to prove this for $\Dec_0 X/X_1$. However this follows immediately from the Lemma since $\Dec_0 X$ deformation retracts onto $X_0$ (see Lemma [5]).

With a little extra effort one can use this corollary to construct an explicit isomorphism between $GX$ and the simplicial group described by Kan in [13], however we will not do this here.

17
We can now give a simple proof of Kan’s result from [13] that $X \to \Omega GX$ is a weak equivalence when $X$ is reduced. We will need the following property of the total simplicial set functor $T$: as observed in [3], since $d$ sends level-wise weak equivalences of bisimplicial sets to weak equivalences of simplicial sets, Theorem 1 implies that this property is inherited by $T$. As an immediate consequence of this observation, Cegarra and Remedios prove the following:

**Lemma 20 ([3]).** For any simplicial set $X$, the unit map $X \to T \text{Dec} X$ is a weak equivalence.

We briefly review the proof of this result from [3].

**Proof.** Cegarra and Remedios observe that the composite of the unit $X \to T \text{Dec} X$ with the map $T \text{Dec} X \to T \pi_1 X$ is the identity on $X$, in light of the identification $T \pi_1 X = X$ of Lemma 10. Since $T$ sends level-wise weak equivalences to weak equivalences it follows that $T \text{Dec} X \to X$ is a weak equivalence and hence the unit map is a weak equivalence.

We are now ready to prove that $GX$ is a loop group for $X$ whenever $X$ is reduced.

**Theorem 21 ([13]).** Let $X$ be a reduced simplicial set. Then the unit map

$$\eta: X \to \Omega GX$$

is a weak equivalence. Hence $GX$ is a loop group for $X$.

**Proof.** The units of the adjunctions $\text{Dec} \dashv T$, $R \dashv U$, and $N_0 \dashv \pi_1$ give a factorization of $\eta$

$$X \to T \text{Dec} X \to TR \text{Dec} X \to TN\pi_1 R \text{Dec} X$$

in $S$. The map $X \to T \text{Dec} X$ is a weak equivalence by Lemma 20. The maps $T \text{Dec} X \to TR \text{Dec} X$ and $TR \text{Dec} X \to T\pi_1 R \text{Dec} X$ are induced by the maps

$$\text{Dec} X \to R \text{Dec} X$$

and $R \text{Dec} X \to N\pi_1 R \text{Dec} X$

in $SS$. We will show that both of these maps are level-wise weak equivalences. The first map is a level-wise weak equivalence by Lemma 17 since $sk_0 \text{Dec} X = \text{Dec}^0 X$ and $X$ is reduced. Corollary 19 shows that $R \text{Dec}_n X$ has the weak homotopy type of a $K(\pi, 1)$ and so the second map is also a level-wise weak equivalence.

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References

[1] M. Artin and B. Mazur, On the van Kampen theorem, Topology 5 (1966) 179–189.

[2] C. Berger and J. Huebschmann, Comparison of the geometric bar and $W$ constructions, J. Pure Appl. Algebra 131 (1998) no. 2, 109–123.

[3] A. Cegarra and J. Remedios, The relationship between the diagonal and the bar constructions on a bisimplicial set, Topology Appl. 153 (2005), no. 1, 21–51.

[4] A. Cegarra and J. Remedios, The behaviour of the $\overline{W}$-construction on the homotopy theory of bisimplicial sets, Manuscripta Math. 124 (2007), no. 4, 427–457.

[5] J.-M. Cordier, Sur les limites homotopiques de diagrammes homotopiquement cohérents, Compositio Math. 62 (1987) 367–388.

[6] A. Dold and D. Puppe, Homologie nicht-additiver Funktoren Anwendungen, Ann. Inst. Fourier Grenoble 11 (1961), 201–312.

[7] J. Duskin, Simplicial methods and the interpretation of “triple” cohomology, Mem. Amer. Math. Soc. 3 (1975), issue 2, no. 163.

[8] W. Dwyer and D. Kan, Homotopy theory and simplicial groupoids, Nederl. Akad. Wetensch. Indag. Math. 46 (1984), no. 4, 379–385.

[9] P. Ehlers, Simplicial groupoids as models for homotopy types, MSc thesis, 1991, University of Wales, Bangor.

[10] S. Eilenberg and S. Mac Lane, On the groups $H(\Pi, n)$, I, Ann. of Math. (2) 58, (1953). 55–106.

[11] P. Goerss and J. F. Jardine, Simplicial homotopy theory, Progress in Mathematics, 174. Birkhäuser Verlag, Basel, 1999

[12] J. F. Jardine and Z. Luo, Higher principal bundles, Math. Proc. Cambridge Philos. Soc., 140 (2006) no. 2 221–243.

[13] D. Kan, A combinatorial definition of homotopy groups, Ann. of Math. (2) 67 1958, 282–312.

[14] L. Illusie, Complexe cotangent et déformations, II Lecture Notes in Mathematics, Vol. 283. Springer-Verlag, Berlin-New York, 1972.

[15] A. Joyal, The theory of quasi-categories and its applications, Advanced course on simplicial methods in higher categories, vol. 2, Centre de Recerca Matemàtica, 2008.
[16] A. Joyal and M. Tierney, Quasi-categories vs Segal spaces, *Categories in algebra, geometry and mathematical physics*, 277–326, Contemp. Math., 431, Amer. Math. Soc., Providence, RI, 2007.

[17] A. Joyal and M. Tierney, *Elements of simplicial homotopy theory*, book in preparation.

[18] A. Joyal and M. Tierney, On the homotopy theory of sheaves of simplicial groupoids, *Math. Proc. Cambridge Philos. Soc.* **120** (1996), no. 2, 263–290.

[19] S. Mac Lane, *Categories for the working mathematician*, Second Edition, Graduate Texts in Mathematics, Vol. 5, Springer-Verlag, New York, 1998.

[20] J. P. May, *Simplicial objects in algebraic topology*, Van Nostrand Mathematical Studies No. 11, D. Van Nostrand Co., Inc., Princeton, N.-J.-Toronto, Ont.-London 1967.

[21] I. Moerdijk, Bisimplicial sets and the group completion theorem, in *Algebraic K-theory: connections with geometry and topology*, Kluwer Academic, Dordrecht, 1989, 225–240.

[22] D. G. Quillen, Spectral sequences of a double semi-simplicial group, *Topology* **5** (1966), 155–157.

[23] S. Thomas, The functors Wbar and Diag Nerve are simplicially homotopy equivalent, *Journal of Homotopy and Related Structures* **3** (1) (2008), pp. 359-378.

[24] F. Waldhausen, On the construction of the Kan loop group, *Doc. Math.* 1 (1996), No. 05, 121–126.

[25] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, Cambridge 1994.

[26] G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, **61**. Springer-Verlag, New York-Berlin, 1978.