On the \( q \)-deformed oscillator algebras: \( su_q(1, 1) \) and \( su_q(2) \)

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Abstract

We study the relations between \( q \)-deformations and \( q \)-coherent states of the single oscillator representations for \( su_q(1, 1) \) and \( su_q(2) \) algebras; Dyson and Holstein-Primakoff type in terms of Biedenharn, Macfarlane and anyonic oscillators. We also discuss the related Fock-Bargmann \( q \)-derivative and integration.

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I. INTRODUCTION

$q$-deformation of oscillator algebra was first introduced by Biedenharn (B) \cite{Biedenharn} and Macfarlane (M) \cite{Macfarlane} in the context of oscillator realization of quantum algebra $su_q(2)$ \cite{Curtright}. They obtained their $q$-deformation of $su(2)$ algebra by using the double oscillators realization of Jordan-Schwinger type. Later, Kulish and Damaskinsky \cite{Kulish} used a single oscillator realization of $su(1,1)$ to obtain the $q$-deformed algebra for a special value of Casimir constant equal to $-\frac{3}{16}$. Another oscillator realization of $su(1,1)$ and its $q$-deformation was obtained for Calogero-type oscillator \cite{Calogero}. This oscillator does not satisfy the Heisenberg algebra but a modified one due to exchange operator, i.e., Dunkle operator. $q$-deformation of this oscillator was also achieved and gave a different commutation relation from those of B and M.

In this paper, we consider the $q$-deformation of a single oscillator representation of $su(1,1)$ and $su(2)$ algebras and their coherent states in terms of Holstein-Primakoff (HP) \cite{Holstein} and Dyson (D) \cite{Dyson} realizations which appear in many applications from spin density wave in condensed matter physics \cite{Giamarchi} to nuclear physics \cite{Poves}. In spite of the existing literature \cite{Sahatci} on the $q$-deformation and $q$-coherent states \cite{Ginoux} of HP and D realizations of those algebras, we find that a comprehensive study on the relations among the various realizations is still lacking, especially the relations among the measures of the $q$-coherent state in the resolution of unity. We are going to resolve the issues by studying other types of $q$-deformations suited to HP and D realizations, starting with the "symmetric" $q$-deformation of the Lie algebras developed by Curtright and Zachos \cite{Curtright2}. We find that this procedure is instructive since we can obtain the B, M, $q$-deformed anyonic oscillators, and their $q$-coherent states naturally. Also, $q$-deformation of Fock-Bargmann (FB) type can be realized in $q$-derivatives.

This paper is organized as follows. In section II, we obtain $su_q(1,1)$ in HP and D realizations in terms of the B, M, and anyonic type oscillators. We construct the $q$-coherent states and compare the measure in each case. We also present the FB representation in $q$-derivative. In Section III, the same analysis is performed for $su_q(2)$ case. Section IV
contains the conclusion and discussion.

Before going into details, we briefly explain our notation. In the oscillator representation of the Heisenberg-Weyl algebra,

\[ [a_-, a_+] = 1, \quad (1.1) \]

we introduce a number eigenstate \( |n> \) of number operator \( N = a_+ a_- \). We require \( |0> \) to be annihilated by \( a_- \), \( a_- |0> = 0 \). Explicitly, the creation and annihilation operators act on the ket,

\[ a_- |n> = n |n-1>, \quad a_+ |n> = |n+1>. \quad (1.2) \]

It should be noted that the normalization of the ket is not fixed yet, which will result from Hermitian property of the generators of \( su_q(1,1) \) and \( su_q(2) \). We use FB holomorphic representation of the oscillator algebra, in which \( \xi |n> = \xi^n \) and

\[ \frac{d}{d\xi} \xi |n> = \xi |a_- |n> \quad \xi |a_+ |n> = \xi |a_+ |n>. \quad (1.3) \]

II. \( SU_q(1,1) \) AND COHERENT STATE

\( su(1,1) \) satisfies the algebra,

\[ [K_0, K_{\pm}] = \pm K_{\pm}, \quad [K_+, K_-] = -2K_0, \quad (2.1) \]

and Casimir invariant is expressed as \( C = K_0(K_0 - 1) - K_+ K_- \). To have a connection with the oscillator algebra, we require the number eigenstate \( |n> \) to be an eigenstate of \( K_0 \),

\[ K_0 |n> = (k_0 + n) |n>. \quad (2.2) \]

Here, \( k_0 \) is assumed to be a positive integer or half odd integer and \( |0> \) is also annihilated by \( K_- \). This representation gives the Casimir number \( k_0(k_0 - 1) \). For definiteness, we will assume that the \( su(1,1) \) algebra act on the ket \( |n> \) as
\[ K_-|n> = n|n-1> \quad K_+|n> = (n + 2k_0)|n+1>. \] (2.3)

This convention is consistent with the holomorphic first-order differential operator representation for \( su(1,1) \) given in [13]

\[
\hat{K}_+(\xi) = \xi^2 \frac{d}{d\xi} + 2k_0\xi, \quad \hat{K}_-(\xi) = \frac{d}{d\xi}, \quad \hat{K}_0(\xi) = \xi \frac{d}{d\xi} + k_0. \quad (2.4)
\]

Since \( \hat{K}_i(\xi) < \xi|n> = <\xi|K_i|n> \), we can check the relation in Eq. (2.3) holds.

The \( q \)-deformed algebra \( su_q(1,1) \) is given as [3]

\[ [Q_0, Q_\pm] = \pm Q_\pm, \quad [Q_+, Q_] = -[2Q_0]_q, \quad (2.5) \]

where the \( q \)-deformation is defined as

\[ [x]_q = \frac{q^x - q^{-x}}{q-q^{-1}}. \quad (2.6) \]

\( q \)-deformed Casimir invariant is given by \( C_q = [Q_0]_q [Q_0 - 1]_q - Q_+ Q_- \). One can obtain the explicit form of the \( q \)-deformed generators following [12],

\[ Q_0 = K_0, \quad Q_- = K_- f(K_0), \quad Q_+ = f(K_0) K_. \quad (2.7) \]

Noting a useful identity \(-[2K_0]_q = g(K_0) - g(K_0 + 1) \) where \( g(K_0) = [K_0 - k_0]_q [K_0 + k_0 - 1]_q \) and \( g(K_0 = k_0) = 0 \), we may identify \( g(K_0) \) as \( f(K_0)^2(K_0(K_0 - 1) - C)_q \),

\[ f(K_0) = \sqrt{\frac{[K - k_0]_q [K_0 + k_0 - 1]_q}{(K - k_0)(K + k_0 - 1)}} = \sqrt{\frac{[N]_q [N + 2k_0 - 1]_q}{N(N + 2k_0 - 1)}}. \quad (2.8) \]

This realization is not the unique choice. In the following, we give some other examples which will result in the \( q \)-deformed oscillator algebra or the \( q \)-deformed FB representation. In addition, independent of the realization we are requiring the conjugate relations,

\[ Q_-^\dagger = Q_+, \quad Q_0^\dagger = Q_0. \quad (2.9) \]

This will determine the norm of each state for the given realization and the resolution of unity for coherent state.
A. B oscillator realization.

(1) D type.

Let us consider the realization,

\[ Q_0 = K_0 = N + k_0, \quad Q_- = K_- \sqrt{\frac{[N]_q}{N}}, \quad Q_+ = \sqrt{\frac{[N]_q}{N}} \frac{[N + 2k_0 - 1]_q}{(N + 2k_0 - 1)} K_+ . \]  

(2.10)

This is obtained if we re-scale \( Q_+ \) and \( Q_- \) in Eq. (2.7). This ladder operators act on the ket as

\[ Q_- |n >_D = \sqrt{n[n]_q} |n - 1 >_D, \quad Q_+ |n >_D = \sqrt{\frac{n + 1}{n + 1}} [n + 2k_0]_q |n + 1 >_D . \]  

(2.11)

Eq. (2.10) becomes an oscillator realization if we interpret it as

\[ Q_0 = N + k_0, \quad Q_- = (a_q)_-, \quad Q_+ = [N + 2k_0 - 1]_q (a_q)_+, \]  

(2.12)

and identify \((a_q)_\pm\) as

\[ (a_q)_- = a_- \sqrt{\frac{[N]_q}{N}}, \quad (a_q)_+ = \sqrt{\frac{[N]_q}{N}} a_+. \]  

(2.13)

\((a_q)_\pm\) satisfy the \(q\)-deformed oscillator algebra of B type,

\[ (a_q)_-(a_q)_+ - q(a_q)_+(a_q)_- = q^{-N} . \]  

(2.14)

\(q\)-deformed coherent state is defined by ( à la Perelomov’ coherent state [13])

\[ |z >_D = e^{zQ_+}|0 >_D = \sum_{n=0}^{\infty} \tilde{z}^n \frac{1}{[n]_q!n!} \frac{[n + 2k_0 - 1]_q!}{[2k_0 - 1]_q!} |n >_D , \]  

(2.15)

where we use the \(q\)-deformed exponential function. The subscript \(D\) stands for D type. (Note that we do not add a normalization constant in this definition since this will introduce \(z\) in addition to \(\tilde{z}\)). Conjugate relation, Eq. (2.9) gives the normalization of the number eigenstate,

\[ _D < n |n >_D = \frac{n! [2k_0 - 1]_q!}{[n + 2k_0 - 1]_q!} . \]  

(2.16)

This normalization provides us the resolution of unity as
\[ I = \sum_{n=0}^{\infty} \frac{[n + 2k_0 - 1]_q!}{n! [2k_0 - 1]_q!} |n > D \quad D < n| = \int d_q^2 z G(z) |z > D \quad D < z|, \quad (2.17) \]

and the measure \( G(z) \) is given as
\[
G(z) = \begin{cases} 
\frac{[2k_0-1]_q}{\pi} (1 - |z|^2)_q^{2k_0-2} & \text{for } 2k_0 = \text{integer} > 1 \\
\frac{|z|^2}{\pi} & \text{for } k_0 = 1
\end{cases}
\quad (2.18)
\]

where the \( q \)-deformed function is defined as \( (1-x)_q^n = \sum_{m=0}^{n} [m]_q [n-m]_q (-x)_q^m \). (For \( k_0 = \frac{1}{2} \), see below Eq. (2.29)). Here, the two dimensional integration is defined as
\[
d_q^2 z \equiv \frac{1}{2} d\theta \ d_q |z|^2.
\quad (2.19)
\]

The angular integration is an ordinary integration, \( 0 \leq \theta \leq 2\pi \). The radial part is a \( q \)-integration, which is the inverse operation of \( q \)-derivative defined as
\[
\frac{d}{d_q z} f(z) = \frac{f(qz) - f(q^{-1}z)}{z(q - q^{-1})}.
\quad (2.20)
\]

One can check that \( I \) commutes with the \( su_q(1,1) \) generators.

We note that in this Hilbert space, \( (a_q)_+ \) is not an adjoint of \( (a_q)_- \). To have the conjugation property between \( (a_q)_- \) and \( (a_q)_+ \) as well as between \( Q_- \) and \( Q_+ \), we may resort to HP realization. Therefore, we need to compare the quantities in different realization. It turns out to be useful to express the quantities in unit normalized eigenstate basis \( |n \rangle \) instead of \( |n > \). For later comparison, we give the explicit expression;
\[
Q_- |n \rangle_D = \sqrt{[n]_q [n + 2k_0 - 1]_q} |n - 1 \rangle_D, \quad Q_+ |n \rangle_D = \sqrt{[n+1]_q [n + 2k_0]_q} |n + 1 \rangle_D.
\quad (2.21)
\]

And the coherent state in Eq. (2.13) becomes
\[
|z > D = e^z Q_+ |0 > D = \sum_{n=0}^{\infty} z^n \frac{[n + 2k_0 - 1]_q!}{[n]_q! [2k_0 - 1]_q!} |n \rangle_D.
\quad (2.22)
\]

(2) HP type.

Let us consider the realization in terms of \( q \)-deformed oscillator in Eq. (2.13) to have the HP realization,
\[ Q_0 = N + k_0, \]
\[ Q_- = K_- \sqrt{\frac{[N]_q [N + 2k_0 - 1]_q}{N}} = (a_q)_- \sqrt{[N + 2k_0 - 1]_q}, \]
\[ Q_+ = \sqrt{\frac{[N]_q [N + 2k_0 - 1]_q}{N}} \frac{1}{(N + 2k_0 - 1)} K_+ = \sqrt{[N + 2k_0 - 1]_q (a_q)_+}. \] (2.23)

The ladder operators act on the ket as
\[ Q_-|n >_H = \sqrt{n [n]_q [n + 2k_0 - 1]_q} |n - 1 >_H \]
\[ Q_+|n >_H = \sqrt{\frac{[n + 1]_q [n + 2k_0]_q}{n + 1}} |n + 1 >_H. \] (2.24)

The subscript \( H \) stands for HP.

Conjugate relation between \( Q_\pm \)'s requires the normalization in this Hilbert space, \( H < n|n >_H = n! \), which is different from the previous one, Eq. (2.16). This is not surprising since two Hilbert spaces are different. If we use the unit normalized ket \( |n >_H \), then the relation given in Eq. (2.24) becomes exactly the same form given in Eq. (2.21) with subscript \( D \) replaced by subscript \( H \), and the \( q \)-deformed coherent state corresponding to Eq. (2.15) is given as
\[ |z >_H = e^{\bar{z}Q_+}|0 >_H = \sum_{n=0}^{\infty} \bar{z}^n \sqrt{\frac{[n + 2k_0 - 1]_q!}{[n]_q! [2k_0 - 1]_q!}} |n >_H \] (2.25)
in terms of the normalized ket \( |n >_H \). This is again exactly the same form given in Eq. (2.22). Therefore, the resolution of unity for the coherent state is expressed in terms of the same measure \( G(z) \) in Eq. (2.18), even though two realizations look so different at first sight.

Note that in this HP realization, the conjugate relation between \( a_+^\dagger = a_- \) is satisfied automatically, since
\[ (a_q)_-|n >_H = \sqrt{[n]_q} |n - 1 >_H \quad (a_q)_+|n >_H = \sqrt{[n + 1]_q} |n + 1 >_H. \] (2.26)

Therefore, one can equally use a new coherent state, \( q \)-deformed version of Glauber coherent state [14],
\[ |z >_H = e^{\bar{z}a_+}|0 >_H. \] (2.27)
Explicit form of this coherent state is given as \(|z\rangle_H = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle_H\). The resolution of unity is given as

\[
I = \sum_{n=0}^{\infty} |n\rangle_H H(n) = \int d^2z \, g(z) |z\rangle_H H\langle z|
\]

where \(g(z)\) is given by

\[
g(z) = \frac{1}{\pi} e^{-|z|^2},
\]

and the domain is over an infinite plane. Unlike the D case, this holds for any value of \(k_0\). Therefore, for \(k_0 = \frac{1}{2}\), one can use the \(q\)-analogue of Glauber coherent state. For other value of \(k_0\), one can use the Bargmann measure defined in Eq. (2.29) or Liouville measure in Eq. (2.18) depending on the definition of coherent state.

### B. M oscillator realization.

(1) D type.

We consider a different realization from the B type oscillator realization,

\[
Q_0 = K_0 = N + k_0,
\]

\[
Q_- = K_\sqrt{\frac{|N\rangle_q}{N}} q^{-\frac{N-2}{2}} = (b_q)_-, \quad Q_+ = q^{-\frac{N-1}{2}} \sqrt{\frac{|N\rangle_q}{N}} \frac{[N + 2k_0 - 1]_q}{(N + 2k_0 - 1)} K_+ = q^{-(N-1)}[N + 2k_0 - 1]_q (b_q)_+, \quad (2.30)
\]

with new oscillator given as

\[
(b_q)_- = (a_q)_- q^{-\frac{N-1}{2}} = a_- \sqrt{\frac{|N\rangle_q}{N}}, \quad (b_q)_+ = q^{-\frac{N-1}{2}} (a_q)_+ = \sqrt{\frac{|N\rangle_q}{N}} a_+,
\]

(2.31)

where we introduce a new definition of \(q\)-number,

\[
\{x\}_q = \frac{q^{2x} - 1}{q^2 - 1} = [x]_q q^{x-1}. \quad (2.32)
\]

This oscillator realization gives the \(q\)-deformed oscillator algebra of M type,

\[
(b_q)_- (b_q)_+ - q^2 (b_q)_+ (b_q)_- = 1. \quad (2.33)
\]
In terms of this realization, the ladder operators act on the ket as

\[ Q_- |n \rangle = \sqrt{n} \{ n \}_q |n - 1 \rangle, \quad Q_+ |n \rangle = q^{-n} \sqrt{\frac{\{ n + 1 \}_q}{n + 1}} [n + 2k_0]_q |n + 1 \rangle. \quad (2.34) \]

In the following, for notational simplicity, we will delete the subscript on the ket which distinguishes the Hilbert space, since there is no possibility of confusion. The conjugate relation between \( Q_\pm \)'s gives the normalization,

\[ < n | n > = n! [2k_0 - 1]_q ! q^{-\frac{n(n - 1)}{2}}. \quad (2.35) \]

In terms of unit normalized ket \( |n\rangle \), we have the canonical operator relations for \( Q_\pm \) as in Eq. (2.21). In addition, \( q \)-deformed coherent state for \( su_q(1, 1) \) has the same form as in Eq. (2.22) and therefore, the measure \( G(z) \) in Eq. (2.18) is used for the resolution of unity for the coherent state.

(2) HP type.

Another HP type realization is given as

\[ Q_0 = N + k_0, \]
\[ Q_- = (b_q)_- \sqrt{q^{-(N - 1)} [N + 2k_0 - 1]_q}, \]
\[ Q_+ = \sqrt{q^{-(N - 1)} [N + 2k_0 - 1]_q (b_q)_+}, \quad (2.36) \]

with \( b_q \)'s defined in Eq. (2.31). However, these generators coincide with the one given in HP type of the B oscillator realization, Eq. (2.23). That is, the Hilbert space is exactly same for both cases as far as \( su_q(1, 1) \) is concerned. Therefore, the \( su_q(1, 1) \) \( q \)-deformed coherent state in terms of the normalized ket \( |n\rangle \) is exactly the same form given in Eq. (2.22) and the same measure \( G(z) \) in Eq. (2.18) is used for the resolution of unity.

On the other hand, from the oscillator point of view, one can define a new coherent state since the conjugate relation between \( (b_q)_- \) is satisfied automatically;

\[ (b_q)_- |n\rangle = \sqrt{\{ n \}_q |n - 1\rangle} \quad (b_q)_+ |n\rangle = \sqrt{\{ n + 1 \}_q |n + 1\rangle}. \quad (2.37) \]

Let us define another version of \( q \)-deformed Glauber coherent state as
\[ |z\rangle = E_q^{z(b_q)}|0\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\{n\}_q} |n\rangle, \quad (2.38) \]

where \( q \)-deformed exponential function \( E_q^x \) differs from \( e_q^x \) in that \( q \)-number \( [n]_q \) is replaced by \( \{n\}_q \),

\[ E_q^x = \sum_{n=0}^{\infty} \frac{x^n}{\{n\}_q} \quad (2.39) \]

The resolution of unity is given as

\[ I = \sum_{n=0}^{\infty} |n\rangle \langle n| = \int d_q^2 z h(z)|z\rangle \{z| . \quad (2.40) \]

where \( h(z) \) is given by

\[ h(z) = \frac{1}{\pi} E_q^{-|z|^2}, \quad (2.41) \]

and the domain is over an infinite plane.

**C. \( q \)-anyonic oscillator realization**

Let us consider the HP type realization again. As seen in the previous section, one can have the B oscillator Eq. (2.23) or M oscillator Eq. (2.36) from the same \( q \)-deformed form of the \( Q_i \)’s. We give another useful form of oscillator realization, \( q \)-anyonic oscillator. Since D and HP type realizations are now trivially connected, we present only HP realization which maintains the conjugate condition for oscillator algebra also. Let us put \( Q_i \)’s as

\[ Q_0 = N + k_0, \]
\[ Q_- = K_- \sqrt{\frac{[N]_q[N + 2k_0 - 1]_q}{N}} = (A_q)- \sqrt{(A_q)_+(A_q)_+ + 2[k_0 - \frac{1}{2}]_q}, \]
\[ Q_+ = \sqrt{\frac{[N]_q[N + 2k_0 - 1]_q}{N}} \frac{1}{(N + 2k_0 - 1)} K_+ = \sqrt{(A_q)_+(A_q)_+ + 2[k_0 - \frac{1}{2}]_q (A_q)_+}, \quad (2.42) \]

Then we have the \( q \)-deformed oscillator as

\[ (A_q)_- = a_- \sqrt{\frac{[N + k_0 - \frac{1}{2}]_q - [k_0 - \frac{1}{2}]_q}{N}}, \quad (A_q)_+ = \sqrt{\frac{[N + k_0 - \frac{1}{2}]_q - [k_0 - \frac{1}{2}]_q}{N}} a_+. \quad (2.43) \]
Its commutation relation looks complicated,
\[(A_q^-)(A_q^+) + [k_0 - \frac{1}{2}]_q - q((A_q^+)(A_q^-) + [k_0 - \frac{1}{2}]_q) = q^{-(N + k_0 + \frac{1}{2})}. \tag{2.44} \]

However, the meaning of this commutation relation becomes clear if we rewrite the relation in M’s form,
\[(B_q^-)(B_q^+) - q^2(B_q^+)(B_q^-) = 1, \tag{2.45} \]

by identifying
\[(B_q^-)(B_q^+) = q^{N + k_0 - \frac{1}{2}}[N + k_0 + \frac{1}{2}]_q = q^{N + k_0 - \frac{1}{2}} ((A_q^-)(A_q^+) + [k_0 - \frac{1}{2}]_q), \]
\[(B_q^+)(B_q^-) = q^{N + k_0 - \frac{3}{2}}[N + k_0 - \frac{1}{2}]_q = q^{N + k_0 - \frac{3}{2}} ((A_q^+)(A_q^-) + [k_0 - \frac{1}{2}]_q). \tag{2.46} \]

In this realization, the vacuum \(|0\rangle\) is not annihilated by \((B_q^-)\) unless \(k_0 = \frac{1}{2}\), since
\[(B_q^+) (B_q^-) |0\rangle = \{k_0 - \frac{1}{2}\}_q |0\rangle. \tag{2.47} \]

This feature reflects the fact that this realization corresponds to the \(q\)-deformed non-trivial one dimensional analogue of anyon which appears in two dimensional oscillator representation with \(k_0\) being related with statistical parameter in anyon physics \cite{15}.

The conjugate relation between \((B_q^-)\) and \((B_q^+)\) can be seen formally at the operator level in Eq. (2.46) since the conjugate relation between \((A_q^-)\) and \((A_q^+)\) does hold. However, the fact that \((B_q^-)\) does not annihilate the vacuum \(|0\rangle\) implies that one cannot define a proper Hilbert space. Therefore, the measure of the coherent state of the Glauber type for the \(B_q\) oscillator system cannot be defined. On the other hand, the measure of the \(q\)-deformed coherent state of Perelomov type is given in Eq. (2.18). One may also define the coherent state of the Glauber type in terms of the \(A_q\) oscillator, whose explicit form of the measure turns out to be very complicated and will not be reproduced here.

As far as \(B_q\) oscillator is concerned, we may construct a Hilbert space from a new vacuum which is annihilated by \((B_q^-)\). Then, since the commutation relation Eq. (2.43) is the same form as in Eq. (2.33), the generators act on the new Hilbert space as in Eq. (2.37).
this case, one can contruct the $q$-deformed Glauber type coherent state and the measure is given in Eq. (2.41). However, this representation has nothing to do with the anyonic representation mentioned above.

We comment in passing that there is another well-known one dimensional oscillator representation for anyon type; Calogero oscillator system, which turns out to be the realization of parabose system [17]. Its $q$-deformed realization does not satisfy the commutation relation of M type Eq. (2.45). The explicit measure for the $q$-deformed coherent state of the Glauber type in this case is already known [3].

**D. FB realization with symmetric $q$-derivative**

Let us consider a realization,

$$
Q_0 = K_0 = N + k_0, \quad Q_- = K - \frac{[N]_q}{N}, \quad Q_+ = \frac{[N + 2k_0 - 1]_q}{(N + 2k_0 - 1)} K_+.
$$

(2.48)

These generators act on the ket as

$$
Q_-|n> = [n]_q |n - 1>, \quad Q_+|n> = [n + 2k_0]_q |n + 1>.
$$

(2.49)

Conjugate relation between $Q_\pm$’s requires the normalization of the number eigenstate,

$$
<n|n> = \frac{[n]_q ![2k_0 - 1]_q!}{[n + 2k_0 - 1]_q!}.
$$

(2.50)

In terms of the unit normalized ket $|n\rangle$, we reproduce the same form of $su_q(1,1)$ coherent state and resolution of unity as seen in the previous subsections, A and B.

What makes this realization different from the previous ones is that it gives a natural $q$-deformation of the FB representation of $su(1,1)$. By using $<\xi|n> = \xi^n$, we have

$$
\hat{Q}_+ (\xi) = \xi \frac{d}{d\xi} + 2k_0]_q, \quad \hat{Q}_-(\xi) = \frac{d}{d_q\xi}, \quad \hat{Q}_0(\xi) = \xi \frac{d}{d\xi} + k_0.
$$

(2.51)

The $q$-derivative in $\hat{Q}_-$ is defined in Eq. (2.20). This implies that the oscillator realization is given as

$$
(a_q)_-(\xi) = \frac{d}{d_q\xi}, \quad (a_q)_+(\xi) = \xi,
$$

(2.52)

which satisfies the $q$-deformed oscillator algebra of B type, Eq. (2.14).
E. FB realization with a-symmetric $q$-derivative

We may consider a little modified version of Eq. (2.48),
\[
Q_0 = K_0 = N + k_0 , \quad Q_- = K_- \frac{[N]_q}{N} q^{N-1}, \quad Q_+ = q^{-(N-1)} \frac{[N + 2k_0 - 1]_q}{(N + 2k_0 - 1)} K_+ .
\] (2.53)

Then the generators act on the ket as
\[
Q_- |n > = \{ n \} q |n - 1 > , \quad Q_+ |n > = q^n [n + 2k_0]_q |n + 1 > .
\] (2.54)

Conjugate relation between $Q_\pm$’s requires the normalization of the ket as,
\[
< n |n > = q^{n(n+2k_0-2)} \frac{\{ n \} q ! \{ 2k_0 - 1 \} q !}{\{ n + 2k_0 - 1 \} q !}.
\] (2.55)

One can easily check that in this Hilbert space, the same form of $su_q(1,1)$ coherent state and resolution of unity are reproduced as in the previous sections if we use the unit normalized ket $| n >$.

We have a similar $q$-deformation of the FB representation of $su(1,1)$ as in the previous section,
\[
\hat{Q}_0(\xi) = \xi \frac{d}{d\xi} + k_0 , \quad \hat{Q}_-(\xi) = \frac{D}{D_q \xi} , \quad \hat{Q}_+(\xi) = \xi q^{-\left( \frac{2\xi}{q} + 2k_0 - 1 \right)} \left( \xi \frac{d}{d\xi} + 2k_0 \right) .
\] (2.56)

The derivative in $\hat{Q}_-$ is replaced by a new $q$-derivative which is given by
\[
\frac{D}{D_q z} f(z) = \frac{f(q^2 z) - f(z)}{z(q^2 - 1)} ,
\] (2.57)

This implies that the oscillator realization is given by
\[
(b_q)_-(\xi) = \frac{D}{D_q \xi} , \quad (b_q)_+(\xi) = \xi ,
\] (2.58)

which satisfies the $q$-deformed oscillator algebra of M type, Eq. (2.33).

III. $SU_Q(2)$ AND COHERENT STATE

$su_q(2)$ and its coherent state can be studied in close analogy with the previous section and therefore, we will describe briefly about B oscillator realization only. $su(2)$ satisfies the algebra,
\[ [K_3, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = 2K_3, \quad (3.1) \]

and Casimir operator is given as \( C = K_3(K_3 + 1) + K_+K_- \). In addition,
\[
K_-|n \rangle = n|n - 1 \rangle, \quad K_+|n \rangle = (J-n)|n + 1 \rangle. \quad (3.2)
\]

The Hilbert space is finite dimensional with dimension \( J+1 \) where
\[
K_-|n \rangle = 0, \quad K_+|n \rangle = J|n \rangle, \quad (3.3)
\]

where \( J \) is an integer. Since \(|n \rangle \) is an eigenstate of \( K_3 \),
\[
K_3|n \rangle = (n - J/2)|n \rangle, \quad (3.4)
\]

we have the Casimir constant, \( C = \frac{J}{2} \left( \frac{J}{2} + 1 \right) \).

\( q \)-deformed \( su(2) \) algebra is given as [3]
\[
[Q_3, Q_\pm] = \pm Q_\pm, \quad [Q_+, Q_-] = [2Q_3]_q^2. \quad (3.5)
\]

We require conjugate relation \( Q_-^\dagger = Q_+, \ Q_3^\dagger = Q_3 \) independent of the realization. Repeating the same procedure as \( su(1,1) \) case, we find
\[
Q_3 = K_3, \quad Q_- = K_-F(K_3), \quad Q_+ = F(K_3)K_+. \quad (3.6)
\]

where
\[
F(K_3) = \frac{[\frac{J}{2} + K_3]_q [\frac{J}{2} + 1 - K_3]_q}{(\frac{J}{2} + K_3)(\frac{J}{2} + 1 - K_3)} = \sqrt{\frac{[N]_q [J + 1 - N]_q}{N(J + 1 - N)}}. \quad (3.7)
\]

A. \( D \) type of \( B \) oscillator representation.

\[
Q_3 = K_3 = \frac{J}{2} - N, \quad Q_- = K_-\sqrt{\frac{[N]_q}{N}}, \quad Q_+ = \sqrt{\frac{[N]_q [J + 1 - N]_q}{N(J + 1 - N)}}K_+. \quad (3.8)
\]

These generators act on the ket as
\[
Q_-|n \rangle = \sqrt{n[n]_q}|n - 1 \rangle, \quad Q_+|n \rangle = \sqrt{\frac{n + 1}{n + 1}} [J - n]_q |n + 1 \rangle. \quad (3.9)
\]
We get an oscillator realization if

\[ Q_0 = \frac{J}{2} - N, \quad Q_- = (a_q)_-, \quad Q_+ = [J - N + 1]_q (a_q)_+. \quad (3.10) \]

\((a_q)_\pm\) is as defined in Eq. (2.13).

Introducing unit normalized ket, \(|n\rangle = \sqrt{n/J-n_n!} |n\rangle\), we have the canonical operator relations of \(su_q(2)\).

\[ Q_- |n\rangle = \sqrt{|n|_q[J + 1 - n]_q} |n - 1\rangle, \quad Q_+ |n\rangle = \sqrt{|n + 1|_q[J - n]_q} |n + 1\rangle. \quad (3.11) \]

\(q\)-deformed coherent state is given as

\[ |z\rangle = e^{zQ_-} |J\rangle = \sum_{n=0}^{\infty} \sqrt{n!_q \sqrt{|n|_q[J + 1 - n]_q}} |n\rangle. \quad (3.12) \]

Resolution of unity is expressed as

\[ I = \sum_{n=0}^{J} |n\rangle \langle n| = \int d^2_q z H(z) |z\rangle < z |, \quad (3.13) \]

and the measure is given as

\[ H(z) = \frac{[J + 1]_q}{\pi} \frac{1}{(1 + |z|^2)^{2+J}}. \quad (3.14) \]

One can check that \(I\) commutes with the \(su_q(2)\) generators.

**B. HP type of B oscillator representation.**

\[ Q_3 = K_3 = \frac{J}{2} - N, \]
\[ Q_- = K_- \sqrt{\frac{|N|_q[J + 1 - N]_q}{N}} = (a_q)_- \sqrt{|J + 1 - N]_q}, \]
\[ Q_+ = \sqrt{\frac{|N|_q[J + 1 - N]_q}{N}} \frac{1}{(J + 1 - N)} K_+ = \sqrt{|J + 1 - N]_q(a_q)_+. \quad (3.15) \]

\((a_q)_\pm\) is defined in Eq. (2.13). The ladder operators act on the ket as

\[ Q_- |n\rangle = \sqrt{n|n|_q[J + 1 - n]_q} |n - 1\rangle, \quad Q_+ |n\rangle = \sqrt{\frac{|n + 1|_q[J - n]_q}{n + 1}} |n + 1\rangle. \quad (3.16) \]
Using the normalized ket, $|n\rangle = \sqrt{\frac{1}{n!}} |n\rangle$, we have the canonical ladder operator realization as in Eq. (3.11) and $q$-deformed coherent state is given as

$$|z\rangle = e^{z Q} |J\rangle = \sum_{n=0}^{\infty} \frac{[J]_q!}{[n]_q! [J-n]_q!} |n\rangle.$$  (3.17)

Therefore, the measure $H(z)$ given in Eq. (3.14) is used for the resolution of unity.

Because of the conjugate relation between $(a_q)_+$ and $(a_q)_-$, we can equally consider the $q$-coherent state of finite Glauber coherent state. However, the Hilbert space is finite dimensional, so one has to modify the definition of the coherent state from the $su(1,1)$ case, Eq. (2.38);

$$|z\rangle = e^{z (a_q)_-} |J\rangle = \sum_{n=0}^{\infty} \frac{[J]_q!}{[n]_q! [J-n]_q!} |n\rangle.$$  (3.18)

It is interesting to note that the oscillator coherent state reproduces the same form of $su_q(2)$ coherent state given in Eq. (3.12). This is because the Hilbert space is finite dimensional in contrast with $su_q(1,1)$ case.

IV. CONCLUSION

We have presented and compared various type of oscillator algebra realizations of $su_q(1,1)$ and $su_q(2)$ algebras, and their coherent states. For $su_q(1,1)$, if we impose the conjugate condition for the generators, the Perelomov $q$-coherent states has a common measure in the resolution of unity independently of the explicit forms of realization. Another type of $q$-coherent state, the Glauber type is considered in the HP realization since $a_-$ and $a_+$ are automatically conjugate to each other. The explicit measure for this type of $q$-coherent state depends on the oscillator realization such as B or M type; the Liouville type measure defined in Eq. (2.18) or the Bargmann type in Eq. (2.29), or the other Bargmann type in Eq. (2.41). In addition, it is shown that the explicit forms of the generators of $su_q(1,1)$ can be modified such that $q$-anyonic oscillator and various definition of $q$-derivatives can be accommodated in the realizations.
\( su_q(2) \) shares much of the same results with \( su_q(1,1) \). However, in HP realization, the finite Glauber \( q \)-coherent state does not have the Bargmann measure, but has the Liouville measure. The difference comes from the finiteness of the dimension of the Hilbert space. Therefore, the measure of coherent state in \( su_q(2) \) is distinguishable from that of the oscillator coherent state on a plane.

We also presented two different types of FB realization which provide two different definitions of \( q \)-derivative and \( q \)-integration such that we can describe their \( q \)-deformed oscillators algebra in a natural and simple fashion.

We conclude with a couple of remarks. The D representations can be extended to the \( SU(N) \) case [18]. The HP version in the \( SU(N) \) case can also be constructed [19]. In addition, its \( q \)-deformation was considered in [20]. It would be interesting to go through the same analysis in this higher case, especially in connection with FB realization.

\( q \)-deformed FB representation will be useful for evaluating the \( q \)-deformed version of the path integral [21]. In our approach, \( q \)-deformation of FB representation is understood in terms of the oscillator representation and the role of the \( q \)-derivatives are illustrated. However, \( q \)-integration is performed essentially for one dimensional direction, radial part. Angular part is treated as an ordinary integration Eq. (2.19). So our resolution of unity cannot be used directly in evaluating the \( q \)-deformed version of path integral at this stage. To overcome the shortcomings, one has to fully develop \( q \)-deformed higher dimensional integral in terms of non-commuting numbers. We expect that this direction of research should accommodate \( q \)-calculus on plane and sphere [22].

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