Discrete Dubrovin Equations and Separation of Variables for Discrete Systems

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Abstract

A universal system of difference equations associated with a hyperelliptic curve is derived constituting the discrete analogue of the Dubrovin equations arising in the theory of finite-gap integration. The parametrisation of the solutions in terms of Abelian functions of Kleinian type (i.e. the higher-genus analogues of the Weierstrass elliptic functions) is discussed as well as the connections with the method of separation of variables.

1 Introduction

The method of finite-gap integration has been developed in the late seventies to deal with the periodic solutions of integrable nonlinear evolution equations, cf. e.g. [1, 2, 3], and the recent monograph [4]. The Dubrovin equations arise in the theory of finite-gap integration, namely as the equations governing the dynamics of the so-called auxiliary spectrum or equivalently of the poles of the Baker-Akhiezer function. In the case of integrable models associated with a hyperelliptic curve they are typically given by a first-order system of coupled ordinary differential equations the form,

\[ \dot{\mu}_i = \frac{\sqrt{R(\mu_i)}}{\prod_{j \neq i} (\mu_i - \mu_j)}, \]  

(1.1)

in which \( R(\mu) \) denotes the discriminant of the hyperelliptic curve in question. Variations on eqs. [1,4] might occur depending on details of the model under consideration, but nonetheless the equations are to a great extent universal among the various integrable

0
systems. They can generally be resolved by using the Lagrange interpolation formula, thus leading to the formulation of a Jacobi inversion problem from which the dynamics of the model can be solved in terms of the zeroes of theta-functions associated with the curve.

In the last few years a gradual shift of emphasis has taken place from continuous to discrete integrable systems. One particular class of such discrete systems represent dynamical systems evolving with discrete time, namely integrable mappings, cf. e.g. [6] for a review. Such mappings can arise as periodic solutions of partial difference analogues of equations of KdV type, cf. [7, 8]. Since these are the discrete analogues of the finite-gap potentials in the continuous theory, it is natural to expect that analogous methods can be applied to these discrete models as the ones used to deal with periodic problems in the continuous case. In a recent paper [9] the finite-gap integration of such mappings was considered and the interpretation of the maps was given as constituting special addition formulae for hyperelliptic Abelian functions involving special winding vectors on the spectral curve. As a byproduct difference analogues of the Dubrovin equations (1.1) were derived which form the equations of the discrete motion of the auxiliary spectrum under the KdV mappings. The term discrete Dubrovin equations stems from the paper [1] where related equations were discussed but no explicit formulae were given.

It is well-known that in problems associated with hyperelliptic curves, such as the periodic problems for equations of KdV type, the poles of the Baker-Akhiezer function play the role of separation variables, cf. e.g. [10, 11]. Recently, this observation was cast into a wider framework by Sklyanin in [12] using the Lax pair approach to obtain separation of variables for a wide class of integrable models. In the last section of this note we will comment on the fine interplay between the discrete-time integrable systems which are the mappings of KdV type and the separation mechanism. We believe that the problem of finding the structure behind the discrete Dubrovin equations might form one of the keys to understand the separation mechanism in the new approach.
2 Discrete Dubrovin Equations

In this note we investigate a class of discrete integrable systems that admit the following Lax description. Let us introduce the elementary matrices depending on a discrete variable $n$ labelling the sites along a chain of length $N$, given by

$$V_n(\lambda) = \begin{pmatrix} v_n & 1 \\ \lambda_n & 0 \end{pmatrix}, \quad (2.1)$$

in which $\lambda_n = \lambda + \alpha_n$, $\lambda$ being the spectral parameter and the $\alpha_n$ being some site-dependent shifts. All variables can be taken to be real or complex valued as we wish. The most important object for us is the monodromy matrix

$$T(\lambda) = \prod_{n=1}^{N} V_n(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \quad (2.2)$$

which we assume to evolve under a discrete-time map $v_n \mapsto \tilde{v}_n$ according to

$$\tilde{T}(\lambda) = M(\lambda)T(\lambda)M(\lambda)^{-1}, \quad M(\lambda) = \begin{pmatrix} w & 1 \\ \lambda & 0 \end{pmatrix}. \quad (2.3)$$

We will not discuss here the details of the explicit map on the level of the local variables $v_n$, (see the comments in section 5), but only work on the level of the monodromy matrix. What is essential is that the discrete-time evolution implies the invariance under the map of the spectral curve

$$\Gamma : \quad \mathcal{R}(\eta, \lambda) = \det(T(\lambda) - \eta) = 0, \quad n = 1, \ldots, P, \quad (2.4)$$

which defines an hyperelliptic curve of genus $g = P - 1$, the branch points of which being defined by the formula

$$\overline{\eta}^2 = R(\lambda) = \sum_{j=0}^{2g+1} r_j \lambda^j = r_{2g+1} \prod_{j=1}^{2g+1} (\lambda - e_j), \quad (2.5)$$

with $\overline{\eta} = \eta - \frac{1}{2}(A(\lambda) + D(\lambda))$. $R(\lambda)$ is the discriminant of the hyperelliptic curve, and its branch points $e_i$, $i = 1, \ldots, 2g + 1$ as well as their symmetric functions $r_i$ of order $i$ are invariant under the discrete-time map (2.3). In principle the branch points, which correspond to the eigenvalues of the associated $N \times N$ tridiagonal matrix, can be complex. To
establish under what conditions they are real-valued and simple requires detailed analysis which is beyond the scope of this note.

As is well-known from the continuous theory, the roots of the polynomial $B(\lambda)$, which incidentally correspond to the poles of the Baker-Akhiezer function, define the so-called *auxiliary spectrum*, and they contain the relevant information on the dynamics of the system under consideration. Let us, therefore, derive the discrete equations for the auxiliary spectrum which then form the natural analogue of Dubrovin equations (1.1). This will, in fact, lead to a universal system of coupled first-order difference equations associated with the hyperelliptic curve and to derive them we only need some global properties of the monodromy matrix.

For genus $g$ the monodromy matrices $T(\lambda)$ takes on either one of two distinct possible forms depending on whether the length $N$ of the chain in (2.2) is even or odd. Writing eq. (2.2) as (in which we have taken $\alpha_N = 0$ in accordance with (2.3)),

$$T(\lambda) = \begin{pmatrix} \lambda^{g+1}A_{g+1} + \lambda^g A_g + \cdots + A_0 & \lambda^g B_g + \lambda^{g-1} B_{g-1} + \cdots + B_0 \\ \lambda \left( \lambda^g C_g + \lambda^{g-1} C_{g-1} + \cdots + C_0 \right) & \lambda \left( \lambda^g D_g + \lambda^{g-1} D_{g-1} + \cdots + D_0 \right) \end{pmatrix}$$

(2.6)

the two cases associated with genus $g$ are either of the following two:

*Odd Case* :

$$\begin{cases} A_{g+1} = 0 , & B_g = 1 \\
C_g = 1 , & D_g = 0 \end{cases} \quad , \quad \text{Even Case} : \quad \begin{cases} A_{g+1} = 1 , & B_g \neq 0 \\
C_g \neq 0 , & D_g = 1 \end{cases}$$

(2.7)

The even/odd distinction is related to the variance in the choice of periodic initial data in the two-dimensional lattice described by lattice equations of KdV type, cf. [7], where we can impose periodicity on chains (“staircases”) with period $2P - 1$ respectively period $2P$ both corresponding to a curve of genus $g = P - 1$. We note in passing that as a function of the invariants $I_j$, $j = 0, \ldots, g$ of the map which are the coefficients of the trace of the monodromy matrix

$$\text{tr} T(\lambda) = I_0 + \sum_{j=1}^{g} I_j \lambda^j \quad , \quad I_j = A_j + D_{j-1}$$

(2.8)

the top coefficient $I_g = A_g + D_{g-1}$ being a Casimir with respect to the natural Poisson algebra associated with the dynamical map, the discriminant of the curve takes on the
form

\[ R(\lambda) = \frac{1}{4} \left( I_0 + \sum_{j=1}^{g} I_j \lambda^j \right)^2 - \prod_{n=1}^{N} (-\lambda_n) = B_g C_g \lambda^{2g+1} + \ldots , \quad (2.9) \]

thus depending on the shift variables \( \alpha_n \) entering via \( \lambda_n = \lambda + \alpha_n \).

From the discrete-time map (2.3) we have the following discrete relations for its entries

\[ \tilde{A}(\lambda) = wB(\lambda) + D(\lambda) , \quad \tilde{C}(\lambda) = \lambda B(\lambda) \]

\[ \lambda \tilde{B}(\lambda) = C(\lambda) + w(A(\lambda) - D(\lambda)) - w^2 B(\lambda) , \quad \tilde{D}(\lambda) = A(\lambda) - wB(\lambda) . \quad (2.10) \]

Expanding eq. (2.10) in powers of \( \lambda \) we are lead to the following set of equations:

\[ \tilde{A}_0 = wB_0 = I_0 , \quad \tilde{A}_j = wB_j + D_{j-1} , \quad j = 1, \ldots, g . \quad (2.11) \]

The entry \( B(\lambda) \) has the following factorisation:

\[ B(\lambda) = B_g \prod_{j=1}^{g} (\lambda - \mu_j) , \quad (2.12) \]

leading to the expressions for \( B_0/B_g, \ldots, B_{g-1}/B_g \) as elementary symmetric functions of the zeroes \( \mu_1, \ldots, \mu_g \). Similarly, the coefficients of \( C(\lambda) \) are symmetric functions of its zeroes \( \mu_1, \ldots, \mu_g \), where the undertilde denotes the backward time-shift, and from (2.10) we establish that the top coefficients \( B_g \) and \( C_g \) can be taken to be equal and constant.

From the fact that

\[ \frac{1}{2}(A - D)(\lambda) = \kappa \sqrt{R(\lambda) - B(\lambda)C(\lambda)} , \quad (2.13) \]

where the \( \kappa \) denotes the sign \( \kappa = \pm \) corresponding to the choice of sheet of the Riemann surface, subject to the condition \( \kappa = -\kappa \) (as follows from (2.13) and the relations (2.10)), and taking \( \lambda = \mu_i \) in (2.13) we obtain the system

\[ \begin{pmatrix} \frac{1}{2}I_1 - D_0 \\ \vdots \\ \frac{1}{2}I_g - D_{g-1} \end{pmatrix} = \begin{pmatrix} \mu_1 & \cdots & \mu_1^g \\ \vdots & \ddots & \vdots \\ \mu_g & \cdots & \mu_g^g \end{pmatrix}^{-1} \begin{pmatrix} \kappa \sqrt{R(\mu_1)} - \frac{1}{2}I_0 \\ \vdots \\ \kappa \sqrt{R(\mu_g)} - \frac{1}{2}I_0 \end{pmatrix} . \quad (2.14) \]

On the one hand, (2.14) provides us with the values of \( I_j - D_{j-1} - \tilde{D}_{j-1} = wB_j, \quad (j = 1, \ldots, g) \), whereas on the other hand from the first of eq. (2.11) we note that \( w = I_0/B_0, \)
hence we need the expressions for the factors $B_j/B_0$ which can be expressed in terms of symmetric functions of the $\mu_j$ as a consequence of (2.12). Alternatively, we can take $\lambda = \mu_i$ in (2.11) and thus obtain set of first-order difference equations for the $\mu_i$, namely

\[
\mathcal{M}^{-1} \cdot \left( \kappa \sqrt{\mathcal{R}(\mu)} + \frac{1}{2} I_0 e \right) + \tilde{\mathcal{M}}^{-1} \cdot \left( \tilde{\kappa} \sqrt{\mathcal{R}(\tilde{\mu})} - \frac{1}{2} I_0 e \right) = 0.
\]  

(2.15)

In eq. (2.15) $\mu = (\mu_1, \ldots, \mu_g)^t$ denotes the vector with entries $\mu_j$ and $e = (1, 1, \ldots, 1)^t$, whereas $\mathcal{M}$ denotes the VanderMonde matrix

\[
\mathcal{M} = \begin{pmatrix}
\mu_1 & \cdots & \mu_1^g \\
\vdots & \ddots & \vdots \\
\mu_g & \cdots & \mu_g^g
\end{pmatrix}.
\]

We note that the terms with $I_0$ can be expressed in terms of the symmetric functions of the $\mu$’s and $\tilde{\mu}$’s, namely by

\[
\mathcal{M}^{-1} \cdot e = -S \left( -\mu_1^{-1}, \ldots, -\mu_g^{-1} \right),
\]

where $S(x_1, \ldots, x_n)$ is the vector of symmetric products $S_k$ of its arguments, i.e.

\[
S_k(x_1, \ldots, x_g) \equiv \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.
\]

Furthermore, in the spirit of [9] we can obtain from (2.10) and (2.13) by using an asymptotic expansion as $\lambda \to \infty$ a reconstruction formulae for the variable $w$, namely

\[
w = \kappa \left( \sqrt{A} - \sqrt{\tilde{A}} \right), \quad A \equiv \sum_{j=1}^{g} (\mu_j + \tilde{\mu}_j) - \sum_{j=1}^{2g+1} e_j
\]

(2.16)

in terms of symmetric functions of the auxiliary spectrum. The ambiguity in the choice of sign reflects the reversibility of the map in the forward/backward discrete-time direction, and can be fixed at our choice subject to the condition that $\tilde{\kappa} = -\kappa$.

The discrete Dubrovin eqs. (2.15) is a set of $g$ first-order difference equations for the $\mu_j$ and similarly as in the continuous case (1.1) depending on the invariants $I_j$ only. Discrete Dubrovin equations were mentioned first in [1], but not given in explicit form. In [9] they were interpreted as special addition formulae for Abelian functions. From the rather general and elementary derivation given above we conclude that these equations
ar quite universal and linked to the Jacobi inversion problem on the hyperelliptic curve in a fundamental way. It would be of interest, therefore, to find a mechanism similar to the Lagrange interpolation trick that works in the continuous case to connect the discrete Dubrovin equations to linear motion on the Jacobian of the curve. So far, we have not been able to find such a “direct” integration, although some of the original works by Abel, e.g. [13], suggest that such a mechanism might exist. We have for the time being to resort to an indirect approach to linearise eqs (2.15) which is by exploiting what is effectively the interpolating flows of the corresponding discrete-time maps. This will be discussed in the next section.

We mention at this point also an alternative form of the discrete Dubrovin equations, namely as a second-order implicit system of equations of the form

$$\tilde{B}(\mu_i) - B(\mu_i) = 2\kappa \frac{w_i}{\mu_i} \sqrt{R(\mu_i)}, \quad i = 1, \ldots, g.$$  

This form of the discrete Dubrovin equations, which is closer to the forms proposed in [1], was the one discussed in [9].

3 Explicit Examples: \( g = 1 \) and \( g = 2 \)

Let us investigate the form of the discrete Dubrovin equations (2.15) in the special cases of genus \( g = 1 \) and \( g = 2 \). In the case of \( g = 1 \) the discrete Dubrovin equations (2.15) reduce to one single equation, namely

$$\frac{1}{2} I_0 \left( \frac{1}{\mu} - \frac{1}{\bar{\mu}} \right) = \frac{1}{\mu} \sqrt{R(\mu)} + \frac{1}{\bar{\mu}} \sqrt{R(\bar{\mu})},$$  

where for simplicity we have omitted the sign designation \( \kappa \). It is not hard to see that if we properly normalise the corresponding elliptic curve, i.e. (2.4) for \( g = 1 \), eq. (3.1) is actually nothing else than the addition formula for the Weierstrass elliptic \( \wp \)-function, namely by identifying

$$\mu = \wp(\delta) - \wp(t), \quad \bar{\mu} = \wp(\delta) - \wp(t + \delta),$$  

together with

$$\bar{\eta} = \sqrt{R(\mu)} = \wp'(t), \quad -\frac{1}{2} I_0 = \wp'(\delta),$$  

(3.2a)
the parameter $\delta$ playing the role of the discrete time-step. Thus, in this case the discrete Dubrovin equation forms a natural difference equation for the Weierstrass elliptic $\wp$-function. Generic difference equations for elliptic functions were proposed by Potts \[14\] a decade ago.

In the case of genus $g = 2$ the discrete Dubrovin equations are given by the following set of two coupled first-order difference equations:

\[
\frac{1}{2} I_0 \left( \frac{1}{\mu_1 \mu_2} - \frac{1}{\tilde{\mu}_1 \tilde{\mu}_2} \right) = \frac{1}{\mu_1} \sqrt{R(\mu_1)} - \frac{1}{\mu_2} \sqrt{R(\mu_2)} \frac{1}{\mu_1 - \mu_2} \tilde{\mu}_1 - \tilde{\mu}_2 \sqrt{R(\tilde{\mu}_1)} - \frac{1}{\mu_2} \sqrt{R(\mu_2)} \frac{1}{\mu_1 - \mu_2} \tilde{\mu}_1 - \tilde{\mu}_2 \sqrt{R(\tilde{\mu}_2)},
\]

\[
(3.3a)
\]

\[
\frac{1}{2} I_0 \left( \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} - \frac{\tilde{\mu}_1 + \tilde{\mu}_2}{\tilde{\mu}_1 \tilde{\mu}_2} \right) = \frac{\mu_2}{\mu_1} \sqrt{R(\mu_1)} - \frac{\mu_1}{\mu_2} \sqrt{R(\mu_2)} \frac{\mu_2}{\mu_1 - \mu_2} \tilde{\mu}_1 - \tilde{\mu}_2 \sqrt{R(\tilde{\mu}_1)} + \frac{\mu_1}{\mu_2} \sqrt{R(\mu_2)} \frac{\mu_2}{\mu_1 - \mu_2} \tilde{\mu}_1 - \tilde{\mu}_2 \sqrt{R(\tilde{\mu}_2)}.
\]

\[
(3.3b)
\]

This system can be resolved along the following lines. Without giving any details of the analysis we mention that the $\mu_i$ arise as the poles, and the $\tilde{\mu}_i$ as the zeroes of a “transition factor” $\tilde{\varphi}/\varphi$ where $\varphi$ is the relevant Baker-Akhiezer function. In this case the independent variable $t$ lives on the Jacobian $\text{Jac}(\Gamma)$ of the curve and we have the Jacobi inversion problem in the form:

\[
t = \int_{e_1}^{\mu_1} d\mu + \int_{e_2}^{\mu_2} d\mu , \quad d\mu = a + \mu b \sqrt{R(\mu)} d\mu ,
\]

\[
(3.4)
\]

$d\mu$ being a properly chosen vector (i.e. with appropriately chosen vectors $a$ and $b$) of holomorphic differentials on the curve, and $e_1,e_2$ being two of the branch points. Using Abel’s theorem the map $\mu_i \rightarrow \tilde{\mu}_i$ is resolved via the shifts on the Jacobian of the form

\[
\delta = \sum_{i=1,2} \int_{\mu_i}^{\tilde{\mu}_i} d\mu = - \int_0^0 d\mu \in \text{Jac}(\Gamma)
\]

\[
(3.5)
\]

in terms of the special winding vector (on the right hand side) on the Jacobian.

In \[9\] we have used Kleinian functions to parametrise the solution in terms of the genus $g = 2$ analogue of the Weierstrass $\wp$-function, namely in terms of the Kleinian functions $\wp_{ij}$, $i,j = 1,2$. The definitions of the generalised Weierstrass functions go back to Klein, cf. \[13\], and have also been discussed at great length in the monographs by Baker, \[16, 17\]. The general theory has been revived in the papers \[18\]. (For an outline of the relevant
definitions, cf. the Appendix). We restrict ourselves here to list the relevant formulae for the case \( g = 2 \) to parametrise the solutions \( \mu = (\mu_1, \mu_2) \) of the discrete Dubrovin equations (3.3), are the following, cf. [17]:

\[
\begin{align*}
\varphi_{22}(u) &= \mu_1 + \mu_2, & \varphi_{12}(u) &= -\mu_1 \mu_2, & \varphi_{11}(u) &= \frac{F(\mu_1, \mu_2) - 2\overline{\eta}_1 \overline{\eta}_2}{4(\mu_1 - \mu_2)^2}, \\
\end{align*}
\] (3.6a)

where

\[
F(\mu_1, \mu_2) = \sum_{k=0}^{2} \mu_1^k \mu_2^k (2r_{2k} + r_{2k+1}(\mu_1 + \mu_2)) ,
\] (3.6b)

in terms of the coefficients of the \( g = 2 \) curve, cf. eq. (2.4). We note that only two of the functions \( \varphi_{ij}(u) \) are independent. The three functions \( \varphi_{22}(u), \varphi_{21}(u) \) and \( \varphi_{11}(u) \) are connected by a quartic relation which is the remarkable Kummer surface \( \mathcal{K} \) in \( \mathbb{C}^3 \), the equation of which is given by

\[
\mathcal{K} : \quad \det K = \det \begin{pmatrix}
-r_0 & \frac{1}{2}r_1 & 2\varphi_{11} & -2\varphi_{12} \\
\frac{1}{2}r_1 & -r_2 + 4\varphi_{11} & \frac{1}{2}r_3 + 2\varphi_{12} & 2\varphi_{22} \\
2\varphi_{11} & \frac{1}{2}r_3 + 2\varphi_{12} & -r_4 - 4\varphi_{22} & 2 \\
-2\varphi_{12} & 2\varphi_{22} & 2 & 0
\end{pmatrix} = 0 .
\] (3.7)

The corresponding evaluations on the curve can be expressed in terms of the Kleinian three-indexed symbols, [17, 18], by the relations

\[
\overline{\eta}_1 = \mu_1 \varphi_{222}(u) + \varphi_{221}(u), \quad \overline{\eta}_2 = \mu_2 \varphi_{222}(u) + \varphi_{221}(u) .
\] (3.8)

where these odd Kleinian functions are related to the even functions via

\[
\begin{align*}
\varphi_{222}^2 &= 4\varphi_{11} + r_3 \varphi_{22} + 4\varphi_{22}^3 + 4\varphi_{12} \varphi_{22} + r_4 \varphi_{22}^2 + r_2 , \\
\varphi_{221}^2 &= r_0 - 4\varphi_{11} \varphi_{12} + r_4 \varphi_{12}^2 + 4\varphi_{22} \varphi_{12}^2 .
\end{align*}
\] (3.9a)

What is needed still is to identify the arguments of the Kleinian functions with the variable \( t \) on the Jacobian. This is rather subtle and will be discussed separately, cf. [19].

### 4 Connection with Separation of Variables

As has been mentioned earlier, the auxiliary spectrum \( \{\mu_j\} \) are on the one hand the poles of the properly normalised Baker-Akhiezer function, on the other hand they can be considered
to be separation variables for the integrable Hamiltonian systems, cf. \cite{10, 11}. In recent papers, \cite{12, 20}, this connection has been put into a bigger framework to understand the general problem of separation of variables for integrable systems going beyond the classical coordinate separation. The general statement is the following, \cite{12}:

*For a finite-dimensional integrable Hamiltonian system* \((\{q_i\}, \{p_i\}, H)\) admitting a Lax representation, i.e. for which a matrix \(L(\lambda)\) exists such that the spectral curve

\[
\Gamma : \det (L(\lambda) - \eta) = 0
\]

*yields a complete set of integrals of the motion in involution, the poles \(\{\mu_i\}\) of the Baker-Akhiezer function, i.e. the eigenfunction of the Lax matrix when properly normalised, together with the corresponding evaluations on the curve, i.e. \(\eta_i = \eta(\mu_i)\) form a set of separation variables for the Hamiltonian system, i.e. there is a canonical transformation*

\[
(q_i, p_i) \rightarrow (\mu_i, \eta_i)
\]

*and the separation equations are given by*

\[
\det (L(\mu_i) - \eta_i) = 0
\]

*expressing the fact that the \((\mu_i, \eta_i)\) are lying on the spectral curve.*

This statement has been exemplified by a large number of examples, cf. \cite{12}, and notably recently by generic systems of Calogero-Moser and Ruijsenaars type even in the full elliptic case, \cite{21}. We point out that the question of the normalisation of the Baker-Akhiezer function is a crucial point and no general prescription exists to date.

The separation mechanism is rather independent of the type of dynamics we impose on the Hamiltonian system. Thus, it applies equally well to situations where we have an integrable discrete-time dynamics described by a Lax pair, in which case the corresponding discrete-time map is an (iterative) canonical transformation which usually can be cast into a discrete Lagrangian form, cf. e.g. \cite{6}. This applies to the mappings of the type studied in the present paper, cf. \cite{8}, and in \cite{22} for mappings of this type a classical \(r\)-matrix
structure was found from which we can extract the following Poisson brackets:

\[
\{ A(\lambda), B(\lambda') \} = \frac{B(\lambda)A(\lambda') - A(\lambda)B(\lambda')}{\lambda - \lambda'} + \frac{1}{\lambda'} B(\lambda) D(\lambda') \tag{4.1a}
\]

\[
\{ B(\lambda), B(\lambda') \} = 0 \tag{4.1b}
\]

\[
\{ D(\lambda), B(\lambda) \} = \frac{1}{\lambda'} \left[ \lambda D(\lambda) B(\lambda') - \lambda B(\lambda) D(\lambda') \right] \tag{4.1c}
\]

Eqs. (4.1) can be used to establish the canonicity of the separation variables, i.e.

\[
\{ \eta_i, \eta_j \} = \{ \mu_i, \mu_j \} = 0 , \quad \{ \eta_i, \mu_j \} = \delta_{ij} \eta_i , \tag{4.2}
\]

using \( \eta(\mu_i) = A(\mu_i) \). The symplecticity of the discrete-time map is guaranteed from the Poisson brackets involving the variable \( w \):

\[
\{ A(\lambda), w \} = \frac{1}{\lambda} \left[ w A(\lambda) - w^2 B(\lambda) - C(\lambda) \right] , \quad \{ C(\lambda), w \} = \frac{w}{\lambda} (C(\lambda) - w D(\lambda)) \\
\{ B(\lambda), w \} = \frac{1}{\lambda} \left[ A(\lambda) - D(\lambda) - w B(\lambda) \right] , \quad \{ D(\lambda), w \} = \frac{1}{\lambda} \left( C(\lambda) - w D(\lambda) \right) \tag{4.3}
\]

Thus, the general separation mechanism is consistent with the discrete-time evolution.

Let us finish with some general remarks. It seems that the discrete-time systems and integrable mappings play a more profound role in the general problem of separation, cf. \[23\]. In principle, the problem of finding the separation of variables in terms of discrete Dubrovin equations of the type (2.15), which for a \( N \)-dimensional Hamiltonian system generically take the form

\[
G_i(\{ \tilde{\mu}_i \}, \{ \mu_i \}; I_1, \ldots, I_N) = 0 , \quad i = 1, \ldots, N \tag{4.4}
\]

can be formulated as a problem of finding \textit{commuting canonical transformations}. Thus, we might conjecture that in the relevant cases the following diagram actually represents a commuting diagram of integrable canonical maps as follows:

\[
(q_i, p_i) \quad \xrightarrow{S(q_i, \tilde{q}_i)} \quad (\tilde{q}_i, \tilde{p}_i) \\
\downarrow F(q_i, \mu_i) \quad \downarrow F(\tilde{q}_i, \tilde{\mu}_i) \\
(\mu_i, \eta_i) \quad \xrightarrow{W(\mu_i, \tilde{\mu}_i)} \quad (\tilde{\mu}_i, \tilde{\eta}_i)
\]
In this diagram the $S$ is the action functional describing the canonical transformation which is the discrete mapping (as can be derived from a discrete Lagrangian), and $F$ denotes the generating function of the canonical transformation from the original variables to the separating variables. The canonical transformation, with generating function $W$, realising the dynamical mapping in terms of the separation variables, is obviously, by construction, an integrable map itself. Unfortunately, however, it does not seem easy in general to obtain explicit expressions for the generating function $W$, since this requires the elimination of the invariants entering as coefficients of the spectral curve. Noting that the general issue of superposition of canonical transformations and the connected problem of finding commuting canonical maps has to our knowledge not been addressed in full generality in classical mechanics, it seems that the advent of integrable discrete dynamical systems will make the study of these questions imminent.

5 Conclusions

In [7] we constructed a family of exactly integrable finite-dimensional mappings from a lattice KdV equation by considering ‘local’ initial value problems on so-called ‘staircases’ in the lattice. These mappings are symplectic as consequence of the Lagrange structure of the original lattice KdV equation, and their complete integrability in the sense of Liouville, [6], was established in [8, 22]. The resulting mappings can be written in the form of multidimensional rational mappings $\mathbb{R}^N \rightarrow \mathbb{R}^N : \{v_n\} \mapsto \{\tilde{v}_n\}, n = 1, \ldots, N$, where the variables $v_n$ coincide with the ones entering in the monodromy matrix (2.2). In fact, their Lax description is based on the matrices $V_n$ used in section 2. In the recent paper, [9], V. Enolskii and the author investigated the finite-gap integration of the resulting mappings and their parametrisation in terms of the Kleinian functions. It was pointed out there that these integrable mappings have, in fact, the interpretation of being addition formulae for hyperelliptic Abelian functions for special winding vectors on the Riemann surface. In that paper we concentrated on the local description of the mappings, taking into account the dependence of the variables $\mu_i$ on the variable $n$ labelling the sites along the periodic chain as encoded in the Lax matrices $V_n$. Interestingly, the shift variables $\alpha_n$ play an
essential role in this description since they enter as distinct singularities on the curve
defining special winding vectors corresponding to the shifts $n \mapsto n + 1$. In the present
paper we have not at all addressed that issue, nor the actual reconstruction formulae for
the potential $v_n$. We should also mention the work by Bobenko and Pinkall, [24], on the
geometric aspects of lattice systems associated with equations of KdV type, where from a
different perspective finite-gap formulae have also been presented.

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Appendix: Kleinian Functions

In this appendix we collect a number of relevant formulae on Kleinian functions, [15, 16,
17]. We refer the reader to the excellent review papers [18], (from which this material is
taken), in which this theory has been cast in a modern context and where it was shown
that these functions arise naturally within the KdV theory.

Let the hyperelliptic curve be given by $\Gamma : y^2 = R(x)$, as in (2.4), of genus $g$ having
$2g + 2$ branch points $e_1, \ldots, e_{2g+1}, \infty$, cf. also (2.5), and as is well-known we can equip it
with a canonical homology basis $(a_1, \ldots, a_g; b_1, \ldots, b_g)$, and define on $\Gamma$ a canonical set
of holomorphic differentials,
\[
du = (du_1, \ldots, du_g), \quad du_k = \frac{x^{k-1}dx}{y}, \quad (A.1)
\]
and differentials of the second kind with a pole at infinity
\[
d\Omega = (d\Omega_1, \ldots, d\Omega_g), \quad d\Omega_j = \sum_{k=j}^{2g-j} \frac{(k+1-j)r_{k+1+j}x^kdx}{4y}. \quad (A.2)
\]
Introducing $g \times 2g$ period matrices $(2\omega, 2\omega')$ and $(2\eta, 2\eta')$ of their respective integrals
over the $a$- and $b$ cycles, we note that $\det \omega \neq 0$ and the matrix $\tau = \omega^{-1}\omega'$ is symmetric
and its imaginary part is positive definite. Introducing also the matrix \( \kappa = \eta (2\omega)^{-1} \), we can now define the fundamental hyperelliptic \( \sigma \)-function by the formula

\[
\sigma(u) = \sqrt[\det 2\omega]{\prod_{i \neq j} (e_i - e_j)} \exp\{u' \kappa u\} \theta[\varepsilon][(2\omega)^{-1}u|\tau],
\]

where \([\varepsilon] = \begin{bmatrix} \varepsilon' \\ \varepsilon \end{bmatrix}\), is the characteristic of the vector of Riemann constants, and \(\theta[\varepsilon](v|\tau)\) is the Riemann theta function with characteristic,

\[
\theta[\varepsilon](v|\tau) = \sum_{m \in \mathbb{Z}^g} \exp\{\pi i (m + \varepsilon')^t \tau (m + \varepsilon') + 2\pi i (m + \varepsilon')^t (v + \varepsilon)\}. \tag{A.4}
\]

The Kleinian \( \wp \)-functions are defined by the second- and third order logarithmic derivatives of the Kleinian \( \sigma \)-function, namely

\[
\varphi_{ij}(u) = -\frac{\partial^2 \ln \sigma(u)}{\partial u_i \partial u_j}, \quad \varphi_{ijk}(u) = -\frac{\partial^3 \ln \sigma(u)}{\partial u_i \partial u_j \partial u_k}, \quad i, j, k = 1, \ldots, g. \tag{A.5}
\]

The Abelian map \( \mathfrak{A} : (\Gamma)^g \to \text{Jac}(\Gamma) \) of the symmetrised product \( \Gamma \times \cdots \times \Gamma \) to the Jacobian variety \( \text{Jac}(\Gamma) = \mathbb{C}/2\omega \oplus 2\omega' \) of the curve \( \Gamma \) is defined by the vector relation

\[
\sum_{i=1}^{g} \int_{(0,e_i)}^{(y_i,x_i)} du = u, \tag{A.6}
\]

and where the \( e_i, i = 1, \ldots, g \) are a properly chosen selection of branch points among the \( 2g + 2 \) branch points (except \( \infty \)) on the curve.

The Jacobi inversion problem of inverting the Abelian map, in the formulation by Weierstrass, can be solved in terms of the Kleinian functions as follows: The Abelian preimage of the point \( u \in \text{Jac}(\Gamma) \) is given by the set \( \{(y_1,x_1), \ldots, (y_g,x_g)\} \in (\Gamma)^g \), where \( \{x_1, \ldots, x_g\} \) are the zeros of the polynomial

\[
\mathcal{P}(x; u) = x^g - x^{g-1} \varphi_{g,g}(u) - x^{g-2} \varphi_{g,g-1}(u) - \ldots - \varphi_{g,1}(u), \tag{A.7}
\]

and \( \{y_1, \ldots, y_g\} \) are given by

\[
y_k = -\frac{\partial \mathcal{P}(x; u)}{\partial x_k} \bigg|_{x=x_k}. \tag{A.8}
\]

More details can be found in [18].
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