Improving Bayesian posterior correlation analysis on Type Ia supernova luminosity evolution

Keto D. Zhang,1★ Yukei S. Murakami,1,2,3 Benjamin E. Stahl,1,2,4 Kishore C. Patra,1,5 and Alexei V. Filippenko1,6,7

1Department of Astronomy, University of California, Berkeley, CA 94720-3411, USA
2Department of Physics, University of California, Berkeley, CA 94720-7300, USA
3Google Lick Predoctoral Fellow
4Marc J. Staley Graduate Fellow
5Nagaraj-Noll Graduate Fellow
6Miller Institute for Basic Research in Science, University of California, Berkeley, CA 94720, USA
7Miller Senior Fellow

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ABSTRACT

Much of the cosmological utility thus far extracted from Type Ia supernovae (SNe Ia) relies on the assumption that SN Ia peak luminosities do not evolve significantly with the age (local or global) of their stellar environments. Two recent studies have provided conflicting results in evaluating the validity of this assumption, with one finding no correlation between Hubble residuals (HR) and stellar environment age, while the other claims a significant correlation. In this Letter we perform an independent reanalysis that rectifies issues with the statistical methods employed by both of the aforementioned studies. Our analysis follows a principled approach that properly accounts for regression dilution and critically (and unlike both prior studies) utilises the Bayesian-model-produced SN environment age estimates (posterior samples) instead of point estimates. Moreover, the posterior is used as an informative prior in the regression. We find the Pearson correlation between the HR and local (global) age to be in excess of \(3\sigma\). Assuming there exists a linear relationship between HR and local (global) age, we find a corresponding slope of \(0.035 \pm 0.007\) mag Gyr\(^{-1}\) (\(0.036 \pm 0.007\) mag Gyr\(^{-1}\)). We encourage further use of our approach to examine HR and host environment correlations, as well as experiments in correcting for luminosity evolution in SN Ia standardisation.

Key words: distance scale – cosmology: observations – supernovae: general – methods: data analysis – methods: statistical

1 INTRODUCTION

The standardisable property (e.g., Phillips 1993) of Type Ia supernovae (SNe Ia) played a pivotal role in the discovery of the accelerating expansion of the Universe (Riess et al. 1998; Perlmutter et al. 1999). Subsequently, various improvements have been made to reduce biases induced by environmental effects during standardisation. These improvements all serve to reduce the Hubble residuals (HR) — the difference between observed and best-fit-cosmology predicted distance moduli on the Hubble–Lemaître diagram (Lampeitl et al. 2010; Sullivan et al. 2010; Childress et al. 2013), and are typically manifested through correlation statistics between the HR and various SN Ia environment observables. For example, a correction for host-galaxy stellar mass is now routinely included in cosmological analyses (e.g., Betoule et al. 2014; Scolnic et al. 2018). Another candidate HR correlate is the age of the host’s stellar population (Childress et al. 2014), the significance (and even presence) of which has been vigorously debated in recent studies (Rose et al. 2019; Kang et al. 2020; Rose et al. 2020; Lee et al. 2020). The presence of such correlation (and a negative trend) indicates that SN Ia luminosities are fainter as their age (and cosmological redshift) increases. With a sufficiently strong trend, the resulting systematic biases may indicate that the apparent excess decrease in distant SN Ia luminosity is not due to the accelerating expansion of the Universe but merely an artefact of the age-luminosity relationship.

In this Letter, we follow the argument of Lee et al. (2020, hereafter L20) against the methodologies of Rose et al. (2019, hereafter R19) when analysing the relation between HR and two measures of the host stellar population age: (1) the local age of the stellar population near each SN, and (2) the global age of the stellar population of the entire host galaxy. We describe the datasets used by both of the aforementioned studies in Section 2, suggest a principled approach to using the datasets for inference in Section 3, and reanalyse the
correlation (Sec. 4) and slope (Sec. 5), rectifying issues with the statistical methods found in both studies.

2 Data

We collect HR and SN global and local environment age estimates (collectively called “Age”; separately “global Age” and “local Age”) from Table 1 and the data repository of R19. The R19 HR dataset is a modification of the set provided by Campbell et al. (2013), which uses SNe Ia from the SDSS-II supernova survey (Sako et al. 2008). The modification corrects the HR value for those that are significantly correlated with the SN Ia stretch parameter. The HR values in the dataset were determined by a Monte Carlo Markov Chain (MCMC) method used by Campbell et al. (2013). The HR dataset includes only the mean HR and 1σ standard deviations of the MCMC posterior. Without the MCMC posterior samples, we can only assume that the HR values have Gaussian uncertainties.

Similarly, the R19 Age dataset is derived from an MCMC method. However, the Age dataset includes the entire MCMC posterior sample of size 1,020,000 for each SN. We further discuss this dataset in Section 3. Although L20 also source their dataset from R19, they use summary statistical descriptors of the Age dataset (in contrast to the full MCMC posterior samples we use). In particular, it appears that L20 have retrieved the columns from Table 7 of R19 that correspond to the Age posterior sample’s mean, median, standard deviation (SD), −1 SD quantile, and +1 SD quantile for each SN. We strongly emphasise that assembling a dataset in this manner makes assumptions that are incorrect (e.g., modalities in the underlying Age posteriors for each SN, as can be seen in Fig. 1). Consequently, their subsequent estimates of slope and correlation are inherently flawed; we further discuss this in Section 3.

For our analysis, we have removed two SNe that uniquely exist in the HR dataset (SNID 3256) and the Age dataset (SNID 15459). These SNe are also missing in Tables 1 and 7 of R19, respectively. After removing these two SNe, the resulting dataset comprises a total of 102 SNe. Owing to the computationally prohibitive size of the Age dataset (1,020,000 samples × 102 SNe), we downsample the Age dataset to 50,000 for each SN by uniformly sampling 50,000 rows without replacement for each of the 102 retrieved SNe. We use the Kolmogorov-Smirnoff (KS) test to determine how different the downsample is from the original sample, and we resample until all 102 downsamples have a KS p-value greater than 5% — there are no significant differences between the downsample and original sample above the 2σ level.

3 Age Posterior Inference

The Age posterior can be a useful informant in forecasting future observations of the data it initially describes (e.g., the local Age estimates posterior may inform the possible light-curve parameters a particular SN in that environment can have as these two are significantly correlated in a recent study by Rigault et al. (2020)). More applicable to our problem of making statistical inference between the relationship of HR and Age, we may use the Age posterior as a prior to a Bayesian model that estimates the correlation and slope (see Sec. 4 and Sec. 5, respectively). This technique is very similar to hierarchical Bayesian models where priors are conditioned on other priors (which is also true in our case). When we use the

Age posterior distribution as a prior distribution in our model, it is conditioned on other priors defined in Equation 9 of R19. Unlike hierarchical Bayesian models, however, we do not fit for every prior at once, but instead opt for a stepwise process so that we can take advantage of Bayesian simulation results that others have already computed and published.

3.1 Estimating the Age Posterior Distribution

Although we describe above the predictive power yielded by the posterior distribution, we actually do not have the Age posterior distribution. Instead, we have a posterior sample generated by the MCMC sampler under the assumption of the posterior distribution. Fortunately, with a fairly large posterior sample, we can fit for the posterior distribution using a parametric probability density function (PDF).

Examining the various distributions of each SN in Figure 1, we infer that a multimodal distribution better fits the data. Thus, we choose the Gaussian Mixture Model (GMM) as the posterior distribution that fits the Age posterior sample (\(A_1, A_2, \ldots, A_N\)). We fit the GMM to our Age posterior samples as described in Section 3.1.

We set \(k = 3\) (i.e., three Gaussian components) after observing that all the posterior samples have no more than three significant modes.

![Figure 1. Histograms of the downsampled local Age posterior samples of the first 9 SNe in our selected dataset sorted by highest kurtosis (the set of all 102 Age posterior samples is provided as supplementary data). Strong multimodal features are clearly visible in many of the posterior samples and as a result, many of these samples are very unlikely to be Gaussian. The blue solid line is the three-component Gaussian mixture model fit to the posterior samples as described in Section 3.1.](https://doi.org/10.5281/zenodo.3875481)
4 CORRELATION

As expressed above, the estimation of the correlation between HR and Age is independent and separate from the estimation of the slope. Correlation estimation models cannot initially assume that a linear relationship exists between HR and Age while slope estimation models (i.e., linear models) do. Here, we use the Pearson correlation coefficient, r, to gauge how strongly two variables are linearly related. The correlation coefficient with uncertainties in both variables (denoted as σx and σy) can be biased by a relation that is inversely proportional to these uncertainties,

\[ r_{x,y} = \frac{\text{Cov}[x^*, y^*]}{\sqrt{\text{Var}[x^*] + \sigma_{x^*}^2} \sqrt{\text{Var}[y^*] + \sigma_{y^*}^2}}, \quad (4.1) \]

where the asterisks denote true variables (e.g., \( x = x^* + \text{error} \)) and \( \sigma_{x^*}^2 \), \( \sigma_{y^*}^2 \) is the covariance between the uncertainties (often set to zero by invoking the classical assumption of independent measurement error; we invoke the same assumption hereafter). This relationship reveals that for large errors (when noise dominates the signal), the correlation tends to zero. We can correct for this bias by applying a correction factor which removes the error terms in the above correlation coefficient, \( r \), to get

\[ r'_{x,y} = \left( \frac{\text{Var}[x]}{\text{Var}[x^*]} \right) \left( \frac{\text{Var}[y]}{\text{Var}[y^*]} \right)^{-1/2} r_{x,y}, \quad (4.2) \]

4.1 HR and Age Correlation

For the case of HR and Age, only for HR can we separate its uncertainties. This changes Equation 4.1 and Equation 4.2 to become

\[ r'_{HR,A} = \frac{\text{Cov}[HR^*, A^*]}{\sqrt{\text{Var}[HR^*] + \sigma_{HR^*}^2} \text{Var}[A]}, \quad (4.3) \]

\[ r'_{HR,A} = \left( \frac{\text{Var}[HR]}{\text{Var}[HR^*]} \right)^{-1/2} r_{HR,A}, \quad (4.4) \]

respectively, where A denotes Age.

Since we only have a sample of all these variables (HR is the observed sample and A is the posterior sample), we can only estimate the sample correlation (the general calculation of which we defer to Appendix A), We determine the biased sample correlation to be \(-0.330 \pm 0.043\) with significance 3.9σ and the corrected sample correlation to be \(-0.370 \pm 0.047\) with significance 4.0σ, where the uncertainty in the estimate is derived from the variance of bootstrap samples. The bootstrap samples are generated by randomly sampling 102 rows (i.e., SNe) with replacement, estimating the correlation using the same technique in each case, and repeating this to get 100 correlation estimates. The same procedure is applied with the global Age in place of the local Age. We determine the biased sample correlation to be \(-0.320 \pm 0.070\) with significance 3.3σ and the corrected sample correlation to be \(-0.360 \pm 0.078\) with significance 3.4σ.

R19 calculates the Spearman correlation coefficient instead of the Pearson coefficient we have presented herein. Unfortunately, we cannot use the same correction factor to estimate the Spearman correlation coefficient as it requires a nontrivial estimation of the variance of the rank statistics for a sample of independent, nonidentical Gaussian-distributed random variables. However, we do attempt to make a better estimate than R19 using MC simulations on the HR and Age posterior samples. Unlike R19, we account for the variability in HR by sampling under their (assumed) Gaussian distribution. Our simulation results in a Spearman correlation coefficient of \(-0.255 \pm 0.091\) with significance 2.5σ and \(-0.245 \pm 0.084\) with significance 2.5σ using local and global Age, respectively. These values are greater (in an absolute sense) and more significant than the R19 estimates, albeit still insignificant at a 3σ threshold. However, we caution that we cannot confirm if our MC simulation is unbiased as we did for the Pearson coefficient.

5 SLOPE

As previously stated, the estimation of the parameters of a linear relationship between HR and Age (e.g., slope) is independent from the estimation of correlation owing to the necessary assumption that there indeed exists a linear relationship (forcing the correlation coefficient to be 1). Here we make that assumption and examine the resulting slopes.

5.1 Models

We compare the slope estimates from five models: ordinary least squares (OLS), orthogonal distance regression (ODR), LINMIX (model from Kelly 2007; results taken from L20), direct estimation using posterior samples (results taken from R19), and our proposed model. All models share the Bayesian linear regression form,

\[ y_i = \beta x_i + \alpha + \epsilon_{\text{scatter}}, \quad (5.1) \]

with β and α being the slope and intercept (respectively), the asterisks denoting true variables, and the \( \epsilon_{\text{scatter}} \) being the intrinsic scatter term.

Sharing only Equation 5.1, the models differ in their assumption about the errors in the observed values of HR \( y_i \) (no asterisk) and true Age value \( x_i^* \) for each SN: (1) OLS is a naive model that ignores all errors in both variables for each observation, (2) ODR assumes classical Gaussian errors in both variables, (3) direct estimation ignores errors in HR and associates every value of HR with every value in the Age posterior sample, (4) LINMIX assumes Gaussian errors in each observation of HR \( y_i \) but assumes the entire population of SNe Ages \( x_i^* \) has the GMM distribution, and (5) our proposed model assumes Gaussian errors in HR and that each SN’s Age \( x_i^* \) (notice the subscript i) has the GMM distribution (see Sec. 5.2 for more details).

5.2 Our Proposed Model

We propose our own Bayesian model to apply a principled approach to make statistical inference using a Bayesian posterior as described in Section 3. This approach rectifies two major issues: (1) the underestimation of the slope due to uncertainties (regression dilution), and (2) incorrect probability density distribution assumed for the
Age posterior samples as apparent in L20. Our proposed model is composed of the following linear model, likelihood, and prior components:

\[
\begin{align*}
HR_i & = HR^*_i + \epsilon_{HR,i} & A^*_i & \sim \text{GMM}(w_i, \mu_i, \tau_i; k) \\
HR^*_i & = \beta A^*_i + \alpha + \epsilon_{\text{scatter}} & \alpha & \sim \text{Uniform}(-1, 1) \\
\epsilon_{HR,i} & \sim \text{Normal}(0, \sigma_{HR,i}) & \beta & \sim \text{Uniform}(-1, 0) \\
\epsilon_{\text{scatter}} & \sim \text{Normal}(0, \sigma_{\text{scatter}}) & \sigma_{\text{scatter}} & \sim \text{HalfNormal}(2),
\end{align*}
\]

where the tilde symbol “~” denotes that the left-hand side is distributed as the right-hand side. \(A^*_i\) has an informative prior modeled after its posterior GMM fitted distribution with \(w_i\) being the GMM weight vector, \(\mu_i\) being the GMM mean vector, and \(\tau_i\) being the GMM standard deviation vector for the given \(i\)-th observation of vector size \(k = 3\) (the number of components in our employed GMM). Parameters \(\alpha\) and \(\beta\) are assumed with top-hat priors while \(\sigma_{\text{scatter}}\) is a latent variable with uninformative half-normal prior since the parameter is nonnegative. We have set the scale parameter in the half-normal prior to be 2 such that it is sufficiently large for about 95% of its values to be < 4.

The proposed model estimates the slope using MCMC. We use the pyMC3 NUTS (Salvatier et al. 2016) implementation resulting in a slope and intercept posterior sample of size 100,000, then 76,000 after burn-in as shown in Figure 2. The posterior samples for both parameters are nearly Gaussian with slight differences in the upper and lower uncertainties. We report the point estimate of the intercept is 0.080 ± 0.035 mag and the slope is \(-0.030 \pm 0.010\) mag Gyr\(^{-1}\).

5.3 HR and Age Slope Estimations

Under the assumption that there exists a linear relation between HR and local Age, our proposed model yields a slope of \(-0.035 \pm 0.007\) mag Gyr\(^{-1}\) and an intercept of 0.151 ± 0.04 mag. With the same procedure applied for global Age, our proposed model yields a fitted slope of \(-0.036 \pm 0.007\) mag Gyr\(^{-1}\) and an intercept of 0.16 ± 0.04 mag.

6 DISCUSSION AND CONCLUSION

We have reanalysed the results of R19 and L20 for the correlation and linear-model parameter estimates (respectively) that describe the relationship between HR and SN Ia local and global environment ages. Our estimates properly account for uncertainties in the HR and the Age dataset that are posterior samples produced by a Bayesian model in R19 and Campbell et al. (2013). In stark contrast to L20, we do not assume the uncertainties in the Age dataset to be Gaussian-distributed (given the multimodality observed in the posterior samples for some SNe), and unlike R19, we do not directly use the Age posterior samples as if we observed all values in the Age posterior samples.

We compare our correlation coefficient estimates with those of R19. First, our Spearman correlation coefficient calculated with an MC simulation results in greater (in an absolute sense) and more significant Spearman values than R19. Although we cannot test whether our method fully accounts for biases due to uncertainties in the HR and Age dataset, we agree with R19 that there exists no significant Spearman correlation relation between HR and Age (local and global) above 3\(\sigma\). The Spearman correlation does not paint the full picture of possible linear relation between HR and Age. As R19 mentioned, the Spearman correlation gauges the monotonic relation while the Pearson correlation gauges the linear relation.

However, it is possible that the Pearson correlation is stronger than the Spearman correlation and vice versa. Fortunately, for estimating the Pearson correlation coefficient, we can confirm Equation 4.1 and correct Equation 4.2 for the bias due to uncertainties in the datasets. After applying this bias correction, we find significant Pearson correlations of 4.0\(\sigma\) and 3.4\(\sigma\) for HR with the respective local and global Age sample.
Table 1. Linear regression parameter estimates for HR vs. local and global Age

|                  | Local |                  |                | Global |                  |                |
|------------------|-------|------------------|----------------|--------|------------------|----------------|
|                  | slope (mag Gyr$^{-1}$) | $\sigma$ | intercept (mag) | $\sigma$ | slope (mag Gyr$^{-1}$) | $\sigma$ | intercept (mag) | $\sigma$ |
| OLS              | −0.053 | 0.233 | 0.080 | −0.040 | 0.178 |
| ODR              | −0.055 | 0.250 | 0.080 | −0.051 | 0.122 | 0.260 | 0.070 |
| Rose et al. (2019)$^b$ | −0.012 | 0.050 | 0.040 | −0.011 | 0.050 |
| Lee et al. (2020)$^b$ | −0.057 | 0.220 | 0.040 | −0.047 | 0.011 | 0.200 |
| Proposed Model   | −0.035 | 0.151 | 0.040 | −0.036 | 0.007 | 0.162 | 0.039 |

$^a$ Estimated from Rose et al. (2019, Fig. 5 and Fig. 6); $^b$ Slope taken and intercept estimated from Lee et al. (2020, Fig. 1); Not all models give a standard deviation ($\sigma$) of the estimate.

While the monotonic relation may be insignificant, the linear relation is indeed significant, motivating a linear model. We estimate linear model parameters (slope and intercept) assuming that there exists a linear relationship between Hubble residuals and Age. Classical methods of regression (e.g., ordinary least squares and orthogonal distance regression) do not fully account for the nature of the Bayesian model posteriors, especially non-Gaussian-distributed posteriors that are the Age posterior samples. We feed our estimator (MCMC) not the posterior samples but instead a posterior distribution from fitting a three-component Gaussian mixture model on the Age posterior samples. This posterior distribution in our MCMC is treated as an informative prior — a stepwise approach motivated from hierarchical Bayesian models. Our proposed model’s estimated slopes are in strong disagreement with — and in between — those reported by both R19 (our values are higher) and L20 (our values are lower), as summarised in Table 1.

The correlation between HR and Age is significant within our dataset, suggesting that current standardisation procedures do not fully account for the effects of host environment age. In this Letter, we do not attempt to correct standardisation, and we discourage direct usage of our linear parameters (e.g., Age step) without accounting for correlations between age and the light-curve-shape parameters. Other models can be inferred from Figure 2; for example, instead of fitting a single slope to all of the data, the bulk of SNe Ia at relatively young ages ($\leq 5$ Gyr) may suggest a constant HR while the relation for older SNe Ia still has an appreciable slope. Alternatively, neither mass nor age can fully describe the relationship themselves and other host properties could be at play. In our separate work, Murakami et al. (2021) found the separations in HR values, between host-galaxy morphology separated into early- and late-types, are also consistent with both the HR–age slope (from R19 and this work) as well as the HR–mass slope (from in Uddin et al. (2020)). We encourage further experimentation with standardisation to correct for luminosity evolution.

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DATA AVAILABILITY
The raw data used in our analysis will be shared upon request to an author of this paper.

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APPENDIX A: COMPONENTS FOR PEARSON CORRELATION COEFFICIENT

Here we define all three terms that were undefined in Equations 4.3 and 4.4. All expected values (\(E[\cdot]\)) mentioned below are estimated with the sample mean. First, the sample variance for HR is

\[
\text{Var}[\text{HR}] = \frac{1}{N-1} \sum_{i=1}^{N} (E[\text{HR}] - \text{HR}_i)^2, \quad \text{as } N \to \infty. \quad (A1)
\]

The two remaining terms (the sample variance of \(A\) and the sample covariance of HR and \(A\)) are nontrivial. The sample variance of \(A\) is calculated by relying on the law of total variance,

\[
\text{Var}[A] = E[\text{Var}[A | \text{HR}]] + \text{Var}[E[A | \text{HR}]], \quad (A3)
\]

\[
\text{Var}[A] = \sum_{i=1}^{N} \left( \frac{\text{Var}[A_i]}{N} + \frac{[E[A] - E[A_i]]^2}{N-1} \right), \quad \text{as } N \to \infty. \quad (A4)
\]

Finally, the sample covariance of HR and \(A\),

\[
\text{Cov}[\text{HR}, A] = E[\text{HR} \cdot A] + E[\text{HR}] \cdot E[A], \quad (A5)
\]

\[
\text{Cov}[\text{HR}, A] = \left( \frac{1}{N} \sum_{i=1}^{N} E[A_i] \cdot \text{HR}_i \right) + E[\text{HR}] \cdot E[A], \quad \text{as } N \to \infty. \quad (A6)
\]

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