Dye’s Theorem and Gleason’s Theorem for $AW^*$-algebras.

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Abstract: We prove that any map between projection lattices of $AW^*$-algebras $A$ and $B$, where $A$ has no Type $I_2$ direct summand, that preserves orthocomplementation and suprema of arbitrary elements, is a restriction of a normal Jordan $*$-homomorphism between $A$ and $B$. This allows us to generalize Dye’s Theorem from von Neumann algebras to $AW^*$-algebras. We show that Mackey-Gleason-Bunce-Wright Theorem can be extended to homogeneous $AW^*$-algebras of Type I. The interplay between Dye’s Theorem and Gleason’s Theorem is shown. As an application we prove that Jordan $*$-homomorphims are commutatively determined. Another corollary says that Jordan parts of $AW^*$-algebras can be reconstructed from posets of their abelian subalgebras.

1. INTRODUCTION

The main goal of the present paper is to show that any map between projection lattices of $AW^*$-algebras that preserves orthocomplementation and arbitrary suprema is a restriction of a normal Jordan $*$-homomorphism. This generalizes famous Dye’s Theorem in a few directions. Moreover, we contribute to the Mackey-Gleason problem by showing that any bounded vector measure on the projection lattice of $AW^*$-algebra of finite Type $I_n$, $(n \geq 3)$, extends to a bounded linear map. Besides its own mathematical interest, this line of the research stems also from a long discussion on mathematical understanding of quantum theory. There are two basic principles of mathematical foundations of quantum mechanics - Gleason’s Theorem and Wigner’s Theorem. These results are very nontrivial, even in the the most special context of matrix algebras (quantum systems with finitely many levels). In this setting they read as follows. Gleason’s Theorem states that any probability measure on the projection structure, $P(M_n(\mathbb{C}))$, of the matrix algebra $M_n(\mathbb{C})$, $n \geq 3$, of all complex $n$ by $n$ matrices, extends to a positive linear functional on $M_n(\mathbb{C})$ [11]. Loosely speaking, it says that any quantum probability measure has its expectation value (integral). On the other hand, Wigner’s Theorem [26] (in its Ulhron’s
version [25]) says that any bijection $\varphi$ acting on $P(M_n(\mathbb{C}))$, $n \geq 3$, that preserves orthogonality in both directions is implemented by a unitary or anti-unitary operator $U$ in the sense

$$\varphi(P) = U^{-1}PU, \quad P \in P(M_n(\mathbb{C})).$$

It means that any transformation that preserves logical structure of the system of quantum propositions is in fact a geometric transformation of the underlying inner product space. Gleason’s Theorem and Wigner’s Theorem have one thing in common. They show that a particular mathematical object attached to projections can be viewed globally as attached to the linear structure of the whole space or algebra. Responding to a well known Mackey’s problem in axiomatic of quantum theory [21], Gleason showed in [11] that any probability measure on the structure $P(H)$ of projections that act on a separable Hilbert space $H$, extends to a positive functional on the algebra $B(H)$ of all bounded operators acting on $H$. After considerable effort of many outstanding mathematicians [1, 5, 22, 27, 28], Gleason’s Theorem has been extended to positive (finitely additive) measures on projection lattices of general von Neumann algebras (see also survey [14, 20] and references therein). However, in order to obtain Gleason’s Theorem for vector measures it was necessary to relax positivity assumption and to prove the result for general complex-valued measures. This progress required further difficult ideas and techniques. It was achieved in a remarkable series of works by Bunce and Wright [4, 5, 7]. This development has led to the Mackey-Gleason-Bunce-Wright Theorem: Let $M$ be a von Neumann algebra without Type $I_2$ direct summand with projection lattice $P(M)$. Any bounded map $\varrho : P(M) \to X$ with values in a Banach space $X$ such that

$$\varrho(p + q) = \varrho(p) + \varrho(q),$$

whenever $p$ and $q$ are orthogonal projections, extends to a bounded operator $T : M \to X$.

Despite the progress in establishing linear extensions of measures for von Neumann algebras, not much is known about solution of Mackey-Gleason problem for general $C^*$-algebras. One of the most known results in this direction is Haagerup’s theorem saying that any quasi trace on exact $C^*$-algebra is linear [12].

Wigner’s Theorem has an intriguing history as well. The main role has been played by the following reformulation of this result: Any orthoisomorphism of the orthomodular lattice $P(H)$, $\dim H \geq 3$, extends to a Jordan $*$-isomorphism of the algebra $B(H)$. Remarkable Dye’s Theorem [10] extends this to a very general context of von Neumann algebras: Let $M$ and $N$ be von Neumann algebras with projection lattices $P(M)$ and $P(N)$, respectively. If $M$ has no Type $I_2$ direct summand, then any orthoisomorphism $\varphi : P(M) \to P(N)$ extends to
a unique Jordan ∗-isomorphism \( \Phi : M \to N \). Dye’s Theorem is one of the deepest results on the geometry of projections in von Neumann algebras. The arguments in the proof of Dye’s Theorem rely on geometry of matricidal structures over von Neumann algebras and on applying special lattice polynomials that, surprisingly, have the power to capture linear structure. Some of these ideas have their origin in von Neumann work on projective geometry [23]. In the proof of Dye’s Theorem the bijectivity of orthoisomorphism is used in an essential way.

The proofs of Gleason’s Theorem and Dye’s Theorem were independent for a long time. However, a clever argument was given by Bunce and Wright [6] to the effect that Gleason’s Theorem for positive measures on von Neumann algebras implies quickly Dye’s Theorem. Moreover, it was shown in [6] that any map \( \varphi : P(M) \to P(N) \) between projection lattices of von Neumann algebras \( M \) and \( N \), where \( M \) has no Type \( I_2 \) direct summand, is a restriction of Jordan ∗-homomorphism between \( M \) and \( N \) if and only if \( \varphi(p + q) = \varphi(p) + \varphi(q) \) for any pair of orthogonal projections \( p \) and \( q \) in \( M \). Recently, the problem of linear extensions of projection lattice morphisms has been investigated in the context of \( AW^\ast \)-algebras by Heunen and Reyes [17]. They succeeded in proving the following deep result. Any map \( \varphi : P(A) \to P(B) \) between projection lattices of \( AW^\ast \)-algebras \( A \) and \( B \), where \( A \) has no Type \( I_2 \) direct summand, that preserves arbitrary suprema and orthocomplements, extends to a normal Jordan ∗-homomorphism between \( A \) and \( B \) if and only if the following equivariance condition holds:

\[
(1) \quad \varphi((1 - 2p)q(1 - 2p)) = (1 - 2\varphi(p))\varphi(q)(1 - 2\varphi(q)).
\]

The authors posed the following open problem in [17]: Does any morphism \( \varphi \) specified above satisfy condition (1) automatically? We answer this problem in the positive.

In order to obtain Dye’s Theorem for \( AW^\ast \)-algebras, one might be tempted to follow ideas of Bunce and Wright and to try to establish the Gleason’s Theorem for \( AW^\ast \)-algebras first. But this way seems to be very hard. For example, if the Mackey-Gleason problem has positive solution for \( AW^\ast \)-algebras, then any dimension function on Type \( II_1 \) \( AW^\ast \)-factor extends to a trace. But the existence of such a trace would imply that any factorial \( AW^\ast \)-algebra of Type \( II_1 \) is a von Neumann factor. This would solve difficult Kaplansky’s problem [19]. However, fortunately, condition (1) involves only two projections. We shall carefully examine the structure of \( AW^\ast \)-subalgebras generated by two projections, and establish Mackey-Gleason-Bunce-Wright Theorem for this case. This implies that (1) holds for any map between projection lattices of \( AW^\ast \)-algebras (not having Type \( I_2 \) direct
summand) that preserves arbitrary suprema of projections and orthocomplements. Our main theorem then reads as follows. *Let $A$ be an AW$^*$-algebra without Type $I_2$ direct summand, $B$ be an AW$^*$-algebra, and let $\varphi : P(A) \to P(B)$ be a map between projection lattices that preserves arbitrary suprema and orthocomplements. Then $\varphi$ is the restriction of a normal Jordan $*$-homomorphism $\Phi : A \to B$. Moreover, this result allows us to show that normal Jordan $*$-homomorphisms between AW$^*$-algebras are commutatively determined, that is, a map (linear or not) is a normal Jordan $*$-homomorphism if it is a normal Jordan $*$-homomorphism when restricted to any abelian subalgebra. Besides, we establish Mackey-Gleason-Bunce-Wright Theorem for AW$^*$-algebras of Type $I_n$, $3 \leq n < \infty$, as well. (The case of properly infinite algebras will be treated in a subsequent paper.)*

Let us remark that AW$^*$-algebras seem to be more natural for Mackey-Gleason program as well as for logical considerations on quantum theory than von Neumann algebras. For example, any quasitrace on a C$^*$-algebra is a composition of a $*$-homomorphism and a quasitrace on a finite AW$^*$-algebra [2]. Therefore, AW$^*$-algebras play a key role in linearity problem for quasi traces. On the other hand, only in the category of AW$^*$-algebras we have a perfect bijective correspondence between commutative AW$^*$-algebras and complete Boolean algebras. In this correspondence AW$^*$-algebra is sent to its projection lattice. This underlines crucial role of AW$^*$-algebras for logical structures. Finally, AW$^*$-algebras are more suitable for recent topos theoretic approach to quantum theory (see e.g. [13]). This approach is based on the structure $\text{Abel}(A)$ of commutative C$^*$-subalgebras of a given C$^*$-algebra $A$, ordered by the set inclusion. It was shown in [9, 15] that $\text{Abel}(A)$ determines the Jordan structure of a von Neumann algebra $A$. As an application of Dye’s Theorem for AW$^*$-algebras, we show that the same holds for AW$^*$-algebras: *Let $A$ be an AW$^*$-algebra without Type $I_2$ direct summand and $B$ be any AW$^*$-algebra. Suppose that $\varphi : \text{Abel}(A) \to \text{Abel}(B)$ is an order isomorphism. Then there is a unique Jordan $*$-isomorphism $\Phi : A \to B$ such that $\varphi(C) = \Phi(C)$, $C \in \text{Abel}(A)$.*

This generalizes hitherto known results in [15, 16].

The paper is organized as follows. In the second section we deal with the the geometry of the structure of projections in AW$^*$-algebras, especially we describe AW$^*$-algebras generated by two projections and analyze isoclinicity of projections. Inclusions of two by two matricidal substructures into AW$^*$-algebras is examinated. In Section 3 we establish Gleason’s Theorem for finite homogeneous AW$^*$-algebras. This enables us to show linearity of quasi linear functionals on subalgebras...
generated by two projections. In the concluding section main results described above are presented.

Let us now recall basic notions and fix the notation. For all unmentioned details on operator algebras we refer the reader to monographs [3, 18]. For a $C^*$-algebra $A$ we shall denote by $1$ its unit (if it exists). By $A_{sa}$ we shall understand the real subspace of $A$ consisting of all self-adjoint elements. We write $A^+$ and $A_1$ for the positive part and the closed unit ball of $A$, respectively. By $P(A)$ we shall denote the set of all projections in $A$; that is the set of all self-adjoint idempotents. $P(A)$ is ordered by order relation $e \leq f$ if $ef = e$. Suprema and infima of two projections $e$ and $f$ will be denoted by $e \lor f$ and $e \land f$, respectively (if they exist). An orthocomplement $e^\perp$ of a projection $e$ is defined as $e^\perp = 1 - e$. The central cover, $c(e)$, of a projection $e$ is a smallest central projection $z$ for which $z \geq e$. We say that two projections are very orthogonal if their central covers are orthogonal. A projection is called faithful if its central cover is the unit. A projection $e \in A$ is called abelian if the hereditary subalgebra $eAe$ is abelian. Finally, two projections $e$ and $f$ are said to be equivalent (in symbols $e \sim f$) if there is an element $v$ (partial isometry) such that $vv^* = e$ and $v^*v = f$. A symmetry we mean a self-adjoint element $s$ with $s^2 = 1$. Given a $C^*$-algebra $A$, we shall write $M_n(A)$ for the $C^*$-algebra of all $n \times n$ matrices with entries from $A$. If $X$ is a compact Hausdorff space, then by $C(X, A)$ we denote the $C^*$-algebra of all continuous maps from $X$ to $A$. If $A = \mathbb{C}$, we write simply $C(X)$.

A Jordan $*$-homomorphism $\Phi$ is a linear map between two $C^*$-algebras that preserves $*$ operation and squares of self-adjoint elements. Jordan $*$-isomorphism is a Jordan $*$-homomorphism that is bijective and whose inverse is also a Jordan $*$-homomorphism. An AW*-algebra is a $C^*$-algebra $A$ that is a Bear $*$-ring. That is, if the following holds: For any nonempty subset $S \subset A$ there is a projection $e \in A$ such that for the right annihilator $R(S) = \{a \in A : sa = 0 \text{ for all } s \in S\}$ we have that $R(S) = eA$. Throughout the paper $A$ will always represent AW*-algebra. Given an element $a \in A$ there exists a left support projection $LP(a)$ of $a$ which is a smallest projection $g \in A$ such that $ga = a$. Analogously, there is a right support projection $RP(a)$, that is a smallest projection $h$ with $ah = h$. Explicitly, $R(\{a\}) = (1 - RP(a))A$. It is known that $P(A)$ is a complete lattice. Given elements $a_1, \ldots, a_n \in A$ we write $AW^*(a_1, \ldots, a_n)$ for the smallest AW*-subalgebra of $A$ containing elements $a_1, \ldots, a_n$. 
2. Geometry of projections in $AW^*$-algebras

2.1. Definition. Let $e$ and $f$ be projections in a $C^*$-algebra $C$. We say that $e$ and $f$ are isoclinic with angle $\alpha$, $0 < \alpha < \frac{\pi}{2}$, if

$$efe = \cos^2 \alpha e \quad \text{and} \quad fef = \cos^2 \alpha f.$$ 

The following Proposition gathers important facts about isoclinic projections. The proofs can be found in [14, p. 129-130] and [20].

2.2. Proposition. Let $e$ and $f$ be projections in a $C^*$-algebra $C$ that are isoclinic with angle $\alpha$. Then the following statements are true:

(i) $C^*$-algebra $C^*(e, f)$ generated by $e$ and $f$ is $*$-isomorphic to $M_2(\mathbb{C})$.

(ii) $e$ and $f$ are unitarily equivalent in $C^*(e, f)$.

(iii) $\|e - f\| = \sin \alpha$.

We are going to analyze the position of two projections in a general $AW^*$-algebra $A$. Let us recall a few notions (see [3]). Two projections $e$ and $f$ are said to be in position $p'$ if $e \wedge (1 - f) = (1 - e) \wedge f = 0$. Projections $e$ and $f$ that are in position $p'$ are said to be in position $p$ if, moreover, $e \wedge f = (1 - e) \wedge (1 - f) = 0$. Let us remark that $e$ and $f$ are in position $p'$ if, and only if, $LP(ef) = e$ and $RP(ef) = f$. Further, $e$ and $f$ are in position $p$ if, and only if, $RP(ef - fe) = 1$ (see [3]).

2.3. Proposition. Suppose that $e$ and $f$ in $A$ are projections in position $p$. Then

$$RP(e^\perp fe) = e.$$  

Proof. Denote $g = RP(x)$. In other words,

$$R(\{e^\perp fe\}) = g^\perp A.$$ 

As $e^\perp A \subset R\{e^\perp fe\}$ we infer that $e \geq g$. Put $z = e - g$. The proof will be completed if we show that $z = 0$. Suppose, for a contradiction, that $z \neq 0$. Using the fact that $g^\perp \in R(\{e^\perp fe\})$, we have

$$e^\perp fez = e^\perp fe - e^\perp feg = e^\perp fe - e^\perp fe = 0.$$ 

Consequently, $z \in R(\{e^\perp fe\})$. This, together with the inequality $z \leq e$, means that

$$z \in R(\{e^\perp f\}) = (1 - RP(e^\perp f))A = f^\perp A.$$ 

Therefore, $z \leq f^\perp \wedge e = 0$.  

2.4. Proposition. Let $e$ and $f$ be projections in $A$ in position $p$. Then $AW^*(e, f)$ lies in an $AW^*$-subalgebra isomorphic to $M_2(C)$, where $C$
is an abelian $AW^*$-algebra. Moreover, when identifying $M_2(C)$ with $C(X, M_2(C))$, where $X$ is the spectrum of $C$, we can arrange for
\begin{equation}
(2)\ e(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f(x) = \begin{pmatrix} a(x) & \sqrt{a(x) - a^2(x)} \\ \sqrt{a(x) - a^2(x)} & 1 - a(x) \end{pmatrix},
\end{equation}
where $a(x)$ is a continuous function on $X$ with values in $[0, 1]$.

**Proof.** Put $x = e^\perp fe$. Then $x^* x = efe^\perp fe \in eAe$. Consider the polar decomposition $x = uh$, where $h = (x^* x)^{1/2} \in eAe$, and $u$ is a partial isometry with $u^* u = RP(x)$, $uu^* = LP(x)$.

As we know from Proposition 2.3
\begin{equation}
RP(x) = e
\end{equation}

\begin{equation}
LP(x) = RP(x^*) = RP(efe^\perp) = e^\perp.
\end{equation}

Therefore $u$ is a partial isometry with initial projection $e$ and final projection $e^\perp$. By a standard argument $u$ introduces a matrix unit which organizes whole algebra $A$ as $M_2(eAe)$. Identifying $eAe$ with upper left corner of the corresponding matrix we can identify
\begin{equation}
(3)\ e = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \quad e^\perp = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \quad u = \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix}.
\end{equation}

We shall find matrix representation of $f$. Suppose
\begin{equation}
(4)\ f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}.
\end{equation}

Then
\begin{equation}
\begin{pmatrix} 0 & 0 \\ f_{21} & 0 \end{pmatrix} = e^\perp fe = uh = \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix}.
\end{equation}

It gives $f_{21} = f_{12} = h$. Expanding the identity $f = f^2$ we obtain the following conditions:
\begin{align}
(5)\ & f_{11}^2 + h^2 = f_{11} \\
(6)\ & f_{11} h + h f_{22} = h \\
(7)\ & h f_{11} + f_{22} h = h \\
(8)\ & h^2 + f_{22}^2 = f_{22}.
\end{align}

It implies that $f_{11}$ and $f_{22}$ commute with $h$ and so we have
\begin{equation}
(9)\ h(f_{11} + f_{22} - e) = 0.
\end{equation}

We shall show that $f_{11} + f_{22} = e$. Put $y = f_{11} + f_{22} - e$. We can see that
\begin{equation}
y \in R(\{h\}) \subset R(\{x\}) = (1 - e)A.
\end{equation}
So \( y \in (1 - e)A \cap eA = \{0\}\). Therefore \( f_{11} + f_{22} = e \). Set \( C = AW^*(e, f_{11}) \). Employing the previous identities we have that \( f_{22}, h \in C \). Now (3) implies that \( AW^*(e, f) \) is a \(*\)-subalgebra of \( M_2(C) \).

As \( e \) can be identified with identity of \( C \) and \( f_{11} \geq 0 \), (in particular \( f_{11} \) is self-adjoint) identifications (2) follows. \( \square \)

2.5. Proposition. Let \( e \) and \( f \) be projections in \( A \). Then the algebra \( AW^*(e, f) \) is contained in a \( AW^* \)-subalgebra of \( A \) isomorphic to \( B \oplus M_2(C) \), where \( B \) and \( C \) are abelian \( AW^* \)-algebras.

Proof. Passing to hereditary subalgebra we can assume that \( e \lor f = 1 \). We set

\[
\begin{align*}
e_0 &= e - e \wedge f - e \wedge f^\perp \quad e_1 = e \wedge f + e \wedge f^\perp \\
f_0 &= f - e \wedge f - e^\perp \wedge f \quad f_1 = e \wedge f + e^\perp \wedge f.
\end{align*}
\]

Then

\[
1 = e \wedge f + e \wedge f^\perp + e^\perp \wedge f + e^\perp \wedge f^\perp + e_0 \lor f_0.
\]

Let us observe that \( e_0 \) and \( f_0 \) are in position \( p \) in the hereditary subalgebra

\[
(e_0 \lor f_0)A(e_0 \lor f_0).
\]

The proof is completed by application of Proposition 2.4. \( \square \)

2.6. Theorem. Let \( e \) and \( f \) be projections in a \( AW^* \)-algebra \( A \) with \( \|e - f\| < 1 \) and \( e \wedge f = 0 \). Then there is a projection \( g \) in \( A \) isoclinic to both \( e \) and \( f \) with the angle

\[
\alpha = \frac{1}{2} \sin^{-1}\|e - f\|.
\]

Proof. As \( \|e - f\| < 1 \), we have that \( e \wedge f^\perp = e^\perp \wedge f = 0 \). Moreover, without loss of generality we can assume that \( e \lor f = 1 \). In that case \( e \) and \( f \) are in position \( p \) and so they can live in an \( AW^* \)-algebra isomorphic to \( C(X, M_2(\mathbb{C})) \) for some compact Hausdorff space \( X \). Moreover, using Proposition 2.4, we can represent \( e \) and \( f \) as matrix valued functions \( e(x) \) and \( f(x) \) on \( X \) such that

\[
\begin{align*}
e(x) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f(x) = \begin{pmatrix} a(x) & \sqrt{a(x) - a(x)^2} \\ \sqrt{a(x) - a(x)^2} & 1 - a(x) \end{pmatrix},
\end{align*}
\]

where \( a(x) \) is a continuous function on \( X \) with values in \([0, 1]\). Now we can proceed exactly as in the proof of analogous theorem for von Neumann algebras (see surveys [14, Theorem 5.3.5, p.130], [20]). It is shown there that the desired isoclinic projection \( g \in C(X, M_2(\mathbb{C})) \) is given by the following formula

\[
g(x) = \begin{pmatrix} l & \sqrt{l - l^2} \omega(x) \\ \sqrt{l - l^2} \omega(x) & 1 - l \end{pmatrix},
\]

where \( l = \sqrt{\|e - f\|} \).
where \( l \) and \( \omega(x) \) are specified as follows:

\[
l = \cos^2 \alpha;
\]

\[
\omega(x) = e^{i \cos^{-1}\left( \frac{1}{\sqrt{a(x) - 1}} \right)}
\]

if \( a(x) \neq 0 \); and \( \omega(x) = 1 \) otherwise.

2.7. Lemma. Given a projection \( e \) in \( A \) there are projections \( p, q, r \) in \( A \) such that

(i) \( p + q + r = e \)

(ii) \( p \sim q \)

(iii) \( r \) is abelian.

Proof. Let us observe that using orthogonal additivity of \( AW^* \)-algebras [3, Corollary 1, p.80] and the fact that sum of very orthogonal family of abelian projections is an abelian projection again [3, Proposition 8, p. 91] the following holds: If there is a central partition of unity \( \sum z_\alpha = 1 \) such that the present lemma holds for every \( z_\alpha A \), then it holds for whole of \( A \). Employing decomposition of \( A \) into Types [3], we can reduce the proof to the case where \( A \) is properly infinite, finite Type II or homogeneous finite Type I. If \( A \) is properly infinite or of finite Type II, then each projection \( e \) can be halved: \( e = p + q \), where \( p \sim q \), and so the Lemma 2.7 holds with \( r = 0 \). It remains to prove the statement under condition that \( A \) is of finite Type \( I_n \), \( n \geq 2 \).

The hereditary subalgebra \( eAe \) is finite again [3, Proposition 1, p. 89]. Moreover it remains to be of Type I for the following reason: Suppose, on the contrary that \( eAe \) is not of Type I. Then there is a nonzero central projection \( z \) in \( eAe \) such that \( z \) majorizes no nonzero abelian projection. But in Type I algebra every nonzero projection majorizes some nonzero abelian projection by [3, Lemma 1, p. 113] Therefore \( eAe \) decomposes into finite direct sum of subalgebras of Type \( I_k \), where \( k \leq n \). (This is due to the fact that Type \( I_n \) algebra cannot contain more than \( n \) equivalent nonzero abelian projections.) Working in the hereditary subalgebra \( eAe \) we can again pass to its homogeneous direct summands. Consequently, we can assume without loss of generality that \( eAe \) is of Type \( I_l \), where \( l \in \mathbb{N} \). There are equivalent orthogonal abelian projections \( g_1, \ldots, g_l \) with sum \( e \). If \( l \) is even, then we can halve \( g \) into \( g_1 + \cdots + g_{l/2} \) and \( g_{l/2+1} + \cdots + g_l \). If \( l \) is odd, then we can decompose \( e \) into sum of abelian projection \( g_1 \) and two equivalent orthogonal projections \( g_2 + \cdots + g_{l-1} \) and \( g_{l-1+1} + \cdots + g_l \). This completes the proof. \( \square \)

2.8. Lemma. Let \( A \) has no Type II direct summand. Suppose that \( B \) is a \( C^* \)-subalgebra of \( A \) \( * \)-isomorphic to \( M_2(\mathbb{C}) \). Then \( B \) is a subalgebra of the direct sum \( C \oplus D \), where

(i) \( C \) is either zero or a copy of \( M_4(\mathbb{C}) \),
(ii) \( D \) is either zero or a subalgebra of another algebra that is isomorphic to \( M_3(\mathbb{C}) \).

\[ \text{Proof.} \]

Suppose that \( B \) is an \( AW^\ast \)-subalgebra of \( A \) \( \ast \)-isomorphic to \( M_2(\mathbb{C}) \). Then \( B \) is generated by matrix units corresponding to two equivalent nonzero orthogonal projections (atoms in \( B \)) \( e \) and \( f \). By Lemma 2.7 we can find projections \( e_1, e_2, e_3 \) and \( f_1, f_2, f_3 \) such that \( e = e_1 + e_2 + e_3 \), \( f = f_1 + f_2 + f_3 \), \( e_1 \sim e_2 \), \( f_1 \sim f_2 \), and \( e_3 \) and \( f_3 \) are abelian. Then \( e_1 \sim e_2 \sim f_1 \sim f_2 \) are orthogonal. If they are nonzero then the corresponding matrix unit allows us to embed these projections into a subalgebra \( C \) that is a copy of \( M_4(\mathbb{C}) \). Let \( D \) be a subalgebra generated by \( e_3 \) and \( f_3 \). Suppose that \( D \) is nonzero. We shall prove that there is a projection \( h \leq 1 - e_3 - f_3 \), that is equivalent to \( e_3 \). This will complete the proof. Let us put \( z = c(e_3) = c(f_3) \). As \( A \) has no direct summand of Type \( I_2 \), we see that \( z - e_3 - f_3 \) is nonzero, for otherwise \( z \) would be a sum of two equivalent orthogonal abelian projections and this would induce Type \( I_2 \) direct summand of \( A \). For this reason we can work in subalgebra \( zA \) in which \( e_3 \) is a faithful projection. In particular, there is no loss of generality in assuming that \( A \) is of Type \( I \). By considerations analogous to that in the beginning of the proof of Lemma 2.7 we are able to reduce the argument to the case when \( A \) is homogeneous; that is, of Type \( I_n \), \( n \geq 3 \). Therefore \( A \) contains three equivalent orthogonal faithful abelian projections \( h_1 \sim h_2 \sim h_3 \). All are equivalent to \( e_3 \). According to [3, Proposition 5, p.106] if two finite projections in a \( AW^\ast \)-algebra are equivalent, then the same holds for their complements Having this in mind and using the fact that \( h_1 + h_2 \) and \( e_3 + f_3 \) are equivalent finite projections, we see that \( 1 - e_3 - f_3 \) contains an abelian projection equivalent to \( h_3 \) and therefore to \( e_3 \) as well. \( \square \)

3. QUASI LINEAR FUNCTIONALS ON \( AW^\ast \)-ALGEBRAS

3.1. Definition. Let \( A \) be an \( AW^\ast \)-algebra. A mapping \( \mu : A \to \mathbb{C} \) is called quasi-linear functional if the following holds

(i) \( \mu \) is linear on any abelian \( AW^\ast \)-subalgebra of \( A \).

(ii) \( \mu(x + iy) = \mu(x) + i\mu(y) \) for all self-adjoint elements \( x, y \in A \).

(iii) \( \mu \) is bounded on the unit ball of \( A \).

Moreover, we shall call \( \mu \) self-adjoint if it takes real values on self-adjoint elements. We shall define a norm of a quasi-linear functional \( \mu \) by

\[ \|\mu\| = \sup\{|\mu(x)| : x \in A, \|x\| \leq 1\}. \]

3.2. Definition. A measure \( \varrho \) on \( A \) with values in a normed space \( X \) is a bounded map \( \mu : P(A) \to X \) satisfying the following condition:

\[ \varrho(e + f) = \varrho(e) + \varrho(f) \]

whenever \( e \) and \( f \) are orthogonal projections in \( A \).
3.3. Proposition. Every complex measure \( \nu \) on \( A \) extends uniquely to a quasi-linear functional \( \mu \) on \( A \). Moreover, if \( \nu \) is real then \( \mu \) is self-adjoint.

Proof. The proof is the same as in [14, Proposition 5.2.6, p.125]. □

3.4. Proposition. Suppose that \( A \) is an AW*-algebra for which every quasi-linear functional is linear. Then any bounded measure \( \nu \) on \( P(A) \) with values in Banach space \( X \) extends to a bounded linear operator \( T \) from \( A \) to \( X \).

Proof. The proof is the same as the proof of [14, Theorem 5.2.4, p.123] □

We shall often use the following fact:

3.5. Proposition. Let \( A \) be an AW*-algebra. For any \( 0 \leq x \leq 1 \) in \( A \) there are projections \((e_n)\) lying in the commutative AW*-algebra generated by \( x \) such that

\[
x = \sum_{n=1}^{\infty} \frac{1}{2^n} e_n.
\]

Proof. It holds for any \( C^* \)-algebra enjoying the spectral axiom (see [24, p.367]). □

3.6. Proposition. Let \( A \) be an AW*-algebra without Type I_2 direct summand. Then every quasi-linear functional is linear on every subalgebra of \( A \) isomorphic to \( M_2(\mathbb{C}) \).

Proof. Any copy \( B \) of \( M_2(\mathbb{C}) \) is embedded into matricial subalgebra of \( A \) for which the classical Gleason’s Theorem holds [11]. Therefore \( \mu \) is linear on \( B \). □

3.7. Proposition. Any quasi-linear functional \( \mu \) on an AW*-algebra \( A \), that has no Type I_2 direct summand, is Lipschitz on \( P(A) \).

Proof. The proof is the same as in Theorem 5.3.8 in [14], we present it here for the sake of completeness.

As \( \|e - f\| \leq 1 \) for all projections \( e \) and \( f \), we can verify Lipschitz condition for a pair of projections with \( \|e - f\| < 1 \). Moreover, we can (by discarding \( e \wedge f \)) suppose that \( e \wedge f = 0 \). Employing Theorem 2.6 we can find a projection \( h \) isoclinic to both \( e \) and \( f \) with angle \( \alpha = \frac{1}{2} \sin^{-1} \|e - f\| < \frac{\pi}{4} \). As we know, \( C^*(e, h) \) and \( C^*(f, h) \) are isomorphic to \( M_2(\mathbb{C}) \). Therefore \( \mu \) is linear on these algebras (Proposition 3.6). In particular,

\[
|\mu(e) - \mu(h)| \leq \|\mu\| \cdot \|e - h\| = \|\mu\| \sin \left( \frac{1}{2} \sin^{-1} \|e - f\| \right) \leq \|\mu\| \cdot \|e - f\|.
\]

Similarly,

\[
|\mu(f) - \mu(h)| \leq \|\mu\| \cdot \|e - f\|.
\]
This gives

$$|\mu(e) - \mu(f)| \leq 2\|\mu\|\|e - f\|.$$  

□

We shall need some notation. Suppose that $X$ is a Stonean space and consider $AW^*$-algebra $A = C(X, M_n(\mathbb{C})) \simeq C(X) \otimes M_n(\mathbb{C})$. An element $f \in A$ is called locally constant if it attains finitely many values. In other words, $f$ is locally constant if, and only if, there is a partition $O_1, \ldots, O_k$ of $X$ consisting of clopen sets such that $f$ is constant on each $O_i$. It is clear that the set $B$ of all locally constant functions is a $*$-subalgebra of $A$. Moreover, $B$ is dense in $A$. It follows directly from the fact that in any $AW^*$-algebra the set of self-adjoint elements with finite spectrum is dense in self-adjoint part of the algebra. (Alternatively, it can be established by topological considerations.)

3.8. Theorem. Let $A$ be of Type $I_n$, $n \geq 3$. Then any quasi linear functional on $A$ is linear.

Proof. Let $\mu$ be a quasi linear functional on $A$. It is enough to assume that $\mu$ is self-adjoint. By the structure theory of $AW^*$-algebras $A$ can be identified with $C(X, M_n(\mathbb{C}))$, where $X$ is a Stone space. Let $B$ be a subalgebra of $A$ consisting of locally constant functions. First we show that $\mu$ is linear on $B$. For this take $a, b \in B$. We can find a partition of $X$, consisting of clopen sets $O_1, \ldots, O_k$, such that both $a$ and $b$ are constant on each $O_i$. The set of all locally constant functions with this property forms a $*$-subalgebra $C$ of $B$ that can be identified with $k$-fold direct sum of the matrix algebras $M_n(\mathbb{C})$. By the classical Gleason’s Theorem, $\mu$ is linear on $C$. Since $a, b \in C$ we can see that $\mu(a + b) = \mu(a) + \mu(b)$. This way we have established linearity of $\mu$ on $B$. Therefore, there is a unique bounded linear extension, $\varrho$, of $\mu|B$ to $A$ since $B$ is dense in $A$. It is clear that $\varrho$ coincides with $\mu$ on $P(B)$. However, $\mu$ is uniformly continuous on $P(A)$ by Proposition 3.7 and $P(B)$ is dense in $P(A)$. Therefore, $\varrho$ and $\mu$ coincide on $P(A)$. However every quasi linear functional on $A$ is uniquely determined by its values on projections. Therefore $\mu = \varrho$ and the proof is completed. □

3.9. Corollary. Let $A$ be an $AW^*$-algebra without Type $I_2$ direct summand and let $\mu$ be a quasi linear functional on $A$. Let $e, f$ be projections in $A$. Then $\mu$ is linear on subalgebra $AW^*(1, e, f)$.

Proof. First we show that $\mu$ is linear on $AW^*(e, f)$. By Proposition 2.5 there is no loss in assuming that $AW^*(e, f)$ is a direct sum of abelian algebra and an algebra $D$ that can be identified with $C(X, M_2(\mathbb{C}))$, where $X$ is a Stonean space. It is enough to establish linearity of $\mu$ on the latter direct summand. For this we can proceed verbatim like in the proof of Theorem 3.8. Consider a subalgebra $B \subset D$ of locally constant functions in $C(X, M_2(\mathbb{C}))$. Given two elements $a$ and $b$ of $B$ we can embed them into the direct sum of copies of $M_2(\mathbb{C})$. By Proposition 3.6
\[ \mu \text{ is linear on this subalgebra. It implies that } \mu(a + b) = \mu(a) + \mu(b). \]

Then rest is the same as in the proof of Theorem \ref{thm3.8}.

When \( \mu \) is linear on \( AW^*(e, f) \), then it must be linear on \( AW^*(1, e, f) \) since \( AW^*(1, e, f) = C1 + AW^*(e, f) \).

\[ \square \]

4. Dye’s Theorem for \( AW^* \)-algebras

The next proposition can be proved in the same way as in [14, Theorem 8.1.1, p. 255] for von Neumann algebras. In fact it holds not only for \( AW^* \)-algebras but for all \( C^* \)-algebras of real rank zero.

4.1. Proposition. Any bounded linear map \( \Phi : A \to B \) between \( AW^* \)-algebras that preserves projections (that is, \( \Phi(P(A)) \subseteq P(B) \)) is a Jordan \( * \)-homomorphism.

Following terminology in [17] we say that a map between projection lattices of \( AW^* \)-algebras is a \textbf{COrtho}-morphism if it preserves orthocomplementations and suprema of arbitrary projections. Then it preserves order, unit, and orthogonality of projections.

We now proceed to the main theorem.

4.2. Theorem. Let \( A \) be an \( AW^* \)-algebra without Type I\(_2\) direct summand and \( B \) be an \( AW^* \)-algebra. Let \( \varphi : P(A) \to P(B) \) be a \textbf{COrtho}-morphism. Then \( \varphi \) extends to a Jordan \( * \)-homomorphism \( \Phi : A \to B \).

Proof. First we show that \( \varphi \) induces a bounded measure on \( P(A) \) with values in \( P(B) \). Indeed, let us take two orthogonal projections \( e \) and \( f \) in \( A \). Then \( \varphi(e) \) and \( \varphi(f) \) are orthogonal. As \( \varphi \) preserves suprema we have

\[ \varphi(e + f) = \varphi(e \lor f) = \varphi(e) \lor \varphi(f) = \varphi(e) + \varphi(f). \]

Therefore \( \varphi \) is a bounded measure on \( P(A) \). Let us take two projections \( e \) and \( f \) in \( P(A) \). By Corollary \ref{cor3.9} every quasi linear functional on \( G = AW^*(1, e, f) \) is linear and so, by Proposition \ref{prop3.4}, the restriction of \( \varphi \) to this algebra extends to a bounded operator, say \( T \), from \( G \) into \( B \). By Proposition \ref{prop4.1} \( T \) is a Jordan \( * \)-homomorphism. As any Jordan \( * \)-homomorphism preserves triple products we have

\[ \varphi((1 - 2e)f(1 - 2e)) = T((1 - 2e)f(1 - 2e)) = (1 - 2\varphi(e))T(f)(1 - 2\varphi(e)) = (1 - 2\varphi(e))\varphi(f)(1 - 2\varphi(e)). \]

But, according to deep result of Heunen and Reyes [17, Theorem 4.6] the above equality is equivalent to the fact that \( \varphi \) extends to a Jordan \( * \)-homomorphism between \( A \) and \( B \).

\[ \square \]

Next we extend Dye’s Theorem to \( AW^* \)-algebras. A map \( \varphi : P(A) \to P(B) \) between projection lattices of \( AW^* \)-algebras \( A \) and \( B \) is called orthoisomorphism if it is a bijection preserving orthogonality in both directions, in the sense that \( ef = 0 \) if, and only if, \( \varphi(e)\varphi(f) = 0 \). Let us remark that every orthoisomorphism is a \textbf{COrtho}-morphisms.
The following theorem generalizes the Dye’s Theorem.

4.3. Theorem. Let \( \varphi : P(A) \to P(B) \) be an orthoisomorphism between projections lattices of \( AW^* \)-algebras \( A \) and \( B \), where \( A \) has no Type \( I_2 \) direct summand. Then there is a Jordan \( * \)-isomorphism \( \Phi : A \to B \) that extends \( \varphi \).

Proof. As \( \varphi \) is a \textbf{COrtho}-morphism, we have by Theorem \ref{4.2} that there is a Jordan \( * \)-homomorphism \( \Phi \) from \( A \) to \( B \) extending \( \varphi \). It remains to show that \( \Phi \) is an isomorphism. Every Jordan \( * \)-homomorphism has a closed range. This, together with the fact that image of \( \Phi \) contains \( P(B) \) that generates \( B \) as a Banach space, we infer that \( \Phi \) is surjective. Now we establish the injectivity of \( \Phi \). Using the fact that kernel of \( \Phi \) is a Jordan \( * \)-ideal generated linearly by positive elements, we can see that for proving injectivity it suffices to show that \( \Phi \) is nonzero on every element \( 0 \leq x \leq 1 \). It follows from Proposition \ref{3.5} that any such \( x \) dominates a nonzero positive multiple \( le \) of a projection \( e \). Then \( \Phi(le) = l\Phi(e) \) is nonzero by the hypothesis. As \( \Phi \) preserves the order, we obtain that \( \Phi(x) \) must be nonzero. \( \square \)

4.4. Definition. By a quasi Jordan \( * \)-homomorphism between \( AW^* \)-algebras \( A \) and \( B \) we mean a map \( \Phi : A \to B \) that satisfies the following conditions:

(i) \( \Phi \) preserves the \( * \) operation; that is, \( \Phi(a^*) = \Phi(a)^* \) for all \( a \in A \).

(ii) \( \Phi \) is a Jordan \( * \)-homomorphism on every abelian \( AW^* \)-subalgebra \( C \) of \( A \). That is, \( \Phi \) is linear on \( C \) and \( \Phi(a^2) = \Phi(a)^2 \) for every \( a \in C \).

(iii) \( \Phi(a + ib) = \Phi(a) + i\Phi(b) \) for all self-adjoint \( a, b \in A \).

Moreover we shall call a quasi Jordan \( * \)-homomorphism normal if it preserves increasing nets of projections, that is if \( \Phi(e_\alpha) \rightharpoonup \Phi(e) \) in \( P(B) \) whenever \( e_\alpha \rightharpoonup e \) in \( P(A) \). Finally, by a quasi Jordan \( * \)-isomorphism we understand a quasi Jordan \( * \)-homomorphism that is a bijection and whose inverse is again a quasi Jordan \( * \)-homomorphism.

4.5. Theorem. Any normal quasi Jordan \( * \)-homomorphism \( \Phi : A \to B \), where \( A \) and \( B \) are \( AW^* \)-algebras, \( A \) not having Type \( I_2 \) direct summand, is a Jordan \( * \)-isomorphism. Moreover, any quasi Jordan \( * \)-isomorphism \( \Phi : A \to B \) is a Jordan \( * \)-isomorphism.

Proof. First we show that \( \Phi \) restricts to a \textbf{COrtho}-morphism \( \varphi \) between projection lattices. For this it is enough to establish that \( \Phi \) preservers suprema of two elements (see e.g. [17, Lemma 3.2]). Let us take projections \( e \) and \( f \) in \( A \). As \( \Phi \) preserves order we have that \( \Phi(e) \vee \Phi(f) \leq \Phi(e \vee f) \). The sum \( e + f \) is a self-adjoint element and so \( AW^*(e + f) \) is abelian algebra isomorphic to some \( C(X) \), where \( X \) is Stonean. From
this one can deduce that there is an increasing sequence \((h_n)\) of projections in \(AW^*(e + f)\) such that \(h_n \not\preceq RP(e + f) = e \lor f\) and \(e + f \geq \frac{1}{n} h_n\) for each \(n\). Then \(\Phi(h_n) \not\preceq \Phi(e \lor f)\). Observe further that working in \(AW^*\)-algebra generated by two projections \(e\) and \(f\) we have linearity of \(\Phi\) on this subalgebra (see the proof of Theorem 4.2) and so we have that \(\Phi(e + f) = \Phi(e) + \Phi(f)\). Therefore, \(\Phi(e + f) = \Phi(e) + \Phi(f) \geq \frac{1}{n} \Phi(h_n)\). It implies that \(\Phi(h_n) \leq RP(\Phi(e) + \Phi(f)) = \Phi(e) \lor \Phi(f)\). Therefore \(\Phi(e \lor f) \leq \Phi(e) \lor \Phi(f)\), giving the reverse inequality. By Theorem 4.2 \(\varphi\) extends to a bounded linear map \(\Psi\) from \(A\) to \(B\). Since \(\Phi\) and \(\Psi\) coincide on \(P(A)\), they have to be equal.

To prove the second statement, let us observe first that \(\varphi\) is an orthoisomorphism. As \(\varphi\) is injective it will suffice to show that \(\Phi(P(A)) = P(B)\). To this end consider a projection \(p\) in \(P(B)\). There is a self-adjoint element \(x \in A\) such that \(\Phi(x) = p\). Then \(\Phi(x) = \Phi(x)^2 = \Phi(x^2)\), giving by injectivity of \(\Phi\), that \(x = x^2\). Therefore \(x\) is a projection. By Theorem 4.3 \(\varphi\) extends to a Jordan \(*\)-isomorphism \(\Psi\) between \(A\) and \(B\). As above \(\Phi = \Psi\).

In conclusion of this paper, we apply our main result to show that structure of abelian \(C^*\)-subalgebras of a \(AW^*\)-algebra determines Jordan structure of \(A\).

Let \(Abel(A)\) be a set of all abelian \(C^*\)-subalgebra of a unital \(C^*\)-algebra \(A\) that contain the unit of \(A\). When ordered by inclusion, we obtain the posets that play an important role in foundations of physics [13]. The following application of our main results allows one to identify order isomorphisms of the structure of abelian subalgebras with Jordan \(*\)-isomorphisms. Let us recall that an order isomorphism between two posets is a bijection preserving the order in both directions.

4.6. Theorem. Let \(A\) be an \(AW^*\)-algebra without Type \(I_2\) direct summand and \(B\) be another \(AW^*\)-algebra. Then for any order isomorphism \(\varphi : Abel(A) \to Abel(B)\) there is a unique Jordan \(*\)-isomorphism \(\Phi : A \to B\) such that

\[
\Phi(C) = \varphi(C)
\]

for all \(C \in Abel(A)\).

Proof. It was shown in [15] that any order isomorphism between \(Abel(A)\) and \(Abel(B)\) is implemented in the above sense by a quasi Jordan \(*\)-isomorphism between \(A\) and \(B\). The result then follows from Theorem 4.5.

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15
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