On Coulomb drag in double layer systems

Bruno Amorim$^1$ and N M R Peres$^2$

$^1$ Instituto de Ciencia de Materiales de Madrid, CSIC, Cantoblanco, E-28049 Madrid, Spain
$^2$ Physics Department and CFUM, University of Minho, P-4710-057, Braga, Portugal

E-mail: amorim.bac@icmm.csic.es

Received 30 March 2012, in final form 10 July 2012
Published 27 July 2012
Online at stacks.iop.org/JPhysCM/24/335602

Abstract

We argue, for a wide class of systems including graphene, that in the low temperature, high density, large separation and strong screening limits the drag resistivity behaves as $d^{-4}$, where $d$ is the separation between the two layers. The results are independent of the energy dispersion relation, the dependence on momentum of the transport time, and the electronic wave function structure. We discuss how a correct treatment of the electron–electron interactions in an inhomogeneous dielectric background changes the theoretical analysis of the experimental drag results of Kim et al (2011 Phys. Rev. B 83 161401). We find that a quantitative understanding of the available experimental data (Kim et al 2011 Phys. Rev. B 83 161401) for drag in graphene is lacking.

(Some figures may appear in colour only in the online journal)

1. Introduction

Coulomb drag [1, 2] occurs when a current driven through a metallic layer, referred to as the active layer (which will be denoted by 2), induces a current in another metallic layer, separated by a distance $d$, referred to as the passive layer (which will be denoted by 1). This phenomenon is caused by the transfer of momentum between electrons in different layers due to the interlayer electron–electron interaction. In experimental situations no current is allowed to flow in the passive layer, so that an electrical field, $E_1$, builds up in that layer. In this situation, the drag resistivity, $\rho_D$, is obtained from the ratio

$$\rho_D = \frac{E_1}{j_2},$$

where $j_2$ is the current driven through the active layer.

Recently there has been great deal of interest in the phenomenon of Coulomb drag in graphene double layers. Although there are still only a few experimental works on drag in graphene [3–5], there are already plenty of theoretical works on the topic [6–12]. However, among the theoretical literature we find contradictory statements, particularly in the limit of low temperature, high density, large layer separation and strong screening. In this limit, it is a well established result that the drag resistivity between two 2DEGs (two-dimensional electron gases), with parabolic dispersion relations and a constant transport time, depends on temperature as $T^2$ and on the interlayer separation as $d^{-4}$ [13] (a result previously given in [14] without derivation). Graphene is distinct from a 2DEG in three ways: (i) the low energy dispersion relation is linear, instead of parabolic; (ii) the electron wave function has a spinorial structure, instead of being a scalar; (iii) the transport time, for the dominant kind of impurity is proportional to the momentum, $\tau(k) \propto k$ [15]. There is agreement [7–12] that in the low temperature/high density limit, the $T^2$ dependence should still hold in graphene. However, there are a few contradictory results on the dependence of the drag resistivity on the layer separation, and how considering a constant or a momentum dependent transport time changes the result. In this paper we will give special attention to the limit of low temperature, high density, large interlayer separation and strong screening. Therefore, we will make a brief overview of the available results in the literature in this limit.

(i) Tse et al [7] assume a constant transport time, and a dependence of $d^{-4}$ is obtained.

(ii) Peres et al [8] consider a momentum dependent transport time, $\tau(k) \propto k$, obtaining a $d^{-6}$ dependence (as we explain later this result is due to an algebraic error at the end of the asymptotic calculation; correcting this gives a $d^{-4}$ dependence).

(iii) Katsnelson [9] considers a constant transport time and obtains a $d^{-4}$ dependence.

(iv) Hwang et al [10] consider the cases of a constant transport time and a transport time proportional to the momentum, $\tau(k) \propto k$. For a constant transport time a
the drag resistivity should always depend on temperature high density, large interlayer separation and strong screening presenting a clear proof that in the limit of low temperature, $\sigma$-dependence is obtained in this same limit.

(v) Narozhny et al [11] consider the cases of a constant and a linearly momentum dependent transport time. For the case of a constant transport time a $d^{-4}$ dependence is obtained. It is argued that in the low temperature, high density limit, this result still holds, regardless of the momentum dependence of the transport time.

(vi) Carrega et al [12], in a recent independent work, studied drag between massless Dirac electrons. It is proved that in the low temperature/high density limit, the dependence on momentum of the transport time is irrelevant and a $d^{-4}$ dependence is obtained for large interlayer separations.

In this paper, we attempt to clarify this situation by presenting a clear proof that in the limit of low temperature, high density, large interlayer separation and strong screening the drag resistivity should always depend on temperature as $T^2$ and on distance separation as $d^{-4}$. Our analysis is independent of the energy dispersion relation, electron wave function structure and dependence on momentum of the transport time. The structure of the paper is as follows: in section 2, we present the general theory of Coulomb drag. In section 3, we present a general argument proving that the drag resistivity in the limit of low temperature, high density, large separation between layers and strong screening should be linearly momentum dependent. For the interlayer interaction one usually uses the random phase approximation (RPA) dynamically screened Coulomb interaction [16],

$$U_{12}(\bar{q}, \omega) = \frac{V_{12}(\bar{q})}{\epsilon_{\text{RPA}}(\bar{q}, \omega)},$$

where $V_{ab}(\bar{q})$ is the bare Coulomb interaction between electrons in layer $a$ and $b$, $\epsilon_{\text{RPA}}(\bar{q}, \omega)$ is the RPA dielectric function for the double layer system, which is given by [16, 19]

$$\epsilon_{\text{RPA}}(\bar{q}, \omega) = \left[1 - V_{11}(\bar{q})\chi_1(\bar{q}, \omega)\right]\left[1 - V_{22}(\bar{q})\chi_2(\bar{q}, \omega)\right] - V_{12}(\bar{q})V_{21}(\bar{q})\chi_1(\bar{q}, \omega)\chi_2(\bar{q}, \omega),$$

and $\chi_\alpha(\bar{q}, \omega)$ is the polarizability of layer $\alpha$. The bare Coulomb interactions can in general be written as (see section A)

$$V_{ab}(\bar{q}) = \frac{1}{\epsilon_{ab}(\bar{q})} e^2 2\pi q \delta(d - |\bar{q}|),$$

where $\epsilon_0$ is the vacuum permittivity and $\epsilon_{ab}(\bar{q})$ are effective dielectric functions. If the metallic layers are immersed in a homogeneous dielectric with constant $\epsilon_r$, then $\epsilon_{ab}(\bar{q}) = \epsilon_r$.

3. Low temperature behaviour

We now study the behaviour of the drag conductivity in the limit of low temperature, $\epsilon_{\text{F}1(2)} \gg k_B T$, where $\epsilon_{\text{F}1(2)}$ is the Fermi energy of layer $1(2)$. Unless specified otherwise, we will keep the energy dispersion relation, $\epsilon_{k\lambda}$, the transport time, $\tau_{k\lambda}$, and wave function overlap factors, $f_{k\lambda}^{\alpha\beta}$, general. We assume isotropy and that there is only one band at the Fermi level. Central to the analysis is the realization that the energy dispersion relation close to the Fermi energy is

\[d^{d^{-6}}\text{ dependence is obtained (in contradiction with the result from [7]). For the case of a momentum dependent transport time a } d^{d^{-4}}\text{ behaviour is obtained. The case of drag between two bilayer graphene layers is also studied. Using a constant transport time a } d^{d^{-4}}\text{ behaviour is obtained in this same limit.}\]
always linear in momentum, that is, the dispersion can be approximated by:

\[ \epsilon_{k,c} - \epsilon_F \simeq \hbar v_F (k - k_F), \]  

(8)

where \( v_F \) is the slope of the band at the Fermi energy, termed the Fermi velocity, and the label \( c \) refers to the conduction band. For graphene (8) is exact. We also assume that the two metallic layers are in a vacuum, such that

\[ \epsilon_{RPA}(q, \omega) = 1 + \chi_1(\vec{q}, \omega) \chi_2(\vec{q}, \omega) \left( \frac{e^2}{2\varepsilon_0 q} \right)^2 \frac{2 \sinh(qd)e^{-qd}}{q} \]

(9)

\[ - \frac{e^2}{2\varepsilon_0 q} \left[ \chi_1(\vec{q}, \omega) + \chi_2(\vec{q}, \omega) \right]. \]

Due to the factor \( \sinh^{-2}(\beta \hbar \omega) \) in (3), the main contribution to the integral in \( \omega \) comes from \( \hbar \omega \lesssim k_B T \). Since \( \epsilon_F(\vec{q}, \omega) \gg k_B T \), we can therefore expand the remaining integration kernel to lowest order in \( \omega \) and set \( T = 0 \). Therefore we replace the dynamically screened dielectric function, \( \epsilon_{RPA}(\vec{q}, \omega) \), by its static value, \( \epsilon_{RPA}(\vec{q}, 0) \), and expand the non-linear susceptibility of each layer, (4), to lowest order in \( \omega \). Using the energy conserving \( \delta \)-function in (4), \( \delta(\epsilon_{k,c} - \epsilon_{k,c} + \hbar \omega) \), we expand to lowest order in \( \omega \):

\[ n_F(\epsilon_{k,c}) - n_F(\epsilon_{k,c} + \hbar \omega) = n_F(\epsilon_{k,c}) - n_F(\epsilon_{k,c} + \hbar \omega) \]

\[ \simeq - \hbar \omega \frac{\partial n_F(\epsilon_{k,c})}{\partial \epsilon} \simeq \hbar \omega \delta(\epsilon_F - \epsilon_{k,c}). \]  

(10)

Therefore \( \tilde{\Gamma}(\vec{q}, \omega) \) has a linear contribution in \( \omega \). Since we want \( \tilde{\Gamma}(\vec{q}, \omega) \) to lowest order in \( \omega \) we can now set \( \omega = 0 \) in \( \delta(\epsilon_{k,c} - \epsilon_{k,c} + \hbar \omega) \), obtaining

\[ \tilde{\Gamma}(\vec{q}, \omega) = -\frac{\hbar \omega}{2\pi} \int d^2 k f_{k,c} \delta(\epsilon_F - \epsilon_{k,c}) \delta(\epsilon_F - \epsilon_{k+\vec{q},c}) \]

\[ \times (\vec{v}_{k,c} \cdot \vec{v}_{k,c} - \vec{v}_{k+\vec{q},c} \cdot \vec{v}_{k+\vec{q},c}). \]  

(11)

Since we have isotropy we can write \( \vec{v}_{k,c} \cdot \vec{v}_{k,c} = \vec{v}(k) \), where \( g(k) \) is a general function that satisfies \( k g(k_F) = v_F \tau_F \), \( \tau_F \) being the transport time at the Fermi level. The \( \delta \)-functions set \( [\vec{k}] = [\vec{k} + \vec{q}] = \vec{k}_F \), and therefore we can take \( g(k) \) outside the integral, obtaining to lowest order in \( \omega \)

\[ \tilde{\Gamma}(\vec{q}, \omega) = \frac{\hbar \omega v_F}{2\pi k_F} \vec{q} \int d^2 k f_{k,c} \delta(\epsilon_F - \epsilon_{k,c}) \delta(\epsilon_F - \epsilon_{k+\vec{q},c}) \]

\[ \times (\vec{v}_{k,c} \cdot \vec{v}_{k,c} - \vec{v}_{k+\vec{q},c} \cdot \vec{v}_{k+\vec{q},c}). \]  

(12)

Note that in this limit \( q \) is restricted to \( q < 2k_F \). To perform the integration in \( \vec{k} \), we choose, without loss of generality, \( \vec{q} = (q, 0) \) and write

\[ u \equiv \cos \theta = \frac{\vec{k} \cdot \vec{q}}{k_F q}, \]

\[ \int d^2 k = \frac{2k_F}{\pi} \int_0^{\pi/2} du \int_{-1}^{1} du \int_{-1}^{1} \frac{da}{u^2}, \]

\[ \delta(\epsilon_F - \epsilon_{k,c}) = \frac{1}{\hbar v_F} \delta(k - k_F), \]

\[ \delta(\epsilon_F - \epsilon_{k+\vec{q},c}) = \frac{1}{\hbar v_F q} \delta \left( u + \frac{q}{2k_F} \right). \]

Therefore the following result is obtained

\[ \tilde{\Gamma}(\vec{q}, \omega) = \frac{2 \omega v_F}{\pi \hbar v_F q} \left( \frac{\sqrt{\epsilon_{k,c} + \hbar \omega}}{\sqrt{1 - u^2}} \right). \]  

(13)

This result is central to this paper. It shows that in the limit of low temperature and high density, the non-linear susceptibility is independent of both the energy dispersion relation and the dependence of the transport time on momentum, depending only in the particular form of the overlap factor. Therefore, this result can be readily applied for the case of a 2DEG, graphene, bilayer graphene and other systems. Although it was already pointed out in [11, 12] that the non-linear susceptibility is independent of the momentum dependence of the transport time, in those works this result was obtained for the particular case of massless Dirac electrons. Here, we show that this is a general result which is also independent of the energy dispersion relation. Note that this result contradicts [10], where different results for the non-linear susceptibility are obtained for different transport times in the low temperature limit.

Since in the low temperature limit we have \( \tilde{\Gamma}(\vec{q}, \omega) \propto \omega \), the integration in \( \omega \) in (3) reads

\[ \int_0^\infty da \omega^2 \sinh^{2}((\beta \hbar \omega)/2) = 2^{3} \left( \frac{k_B T}{\hbar} \right)^{3} \pi^{2} \frac{n_{\text{re}}}{6}. \]  

(14)

which gives the \( T^2 \) dependence of the drag conductivity and resistivity in the low temperature limit. The \( T^2 \) behaviour is independent of the details of the energy dispersion relation, the transport time and the wave function overlap factors. Notice, however, that the \( T^2 \) behaviour might be modified if one includes corrections to the drag conductivity due to higher order terms in the interlayer interaction [20].

3.1. General system at large interlayer distance and strong screening

We now assume that the interlayer separation is large, \( k_F d \gg 1 \). The interlayer Coulomb interaction decays exponentially with \( d \), thus the integration kernel of (3) is dominated by values of \( q \) such that \( q \lesssim d^{-1} \). Therefore the condition \( k_F d \gg 1 \) allow us to expand the remaining integration kernel to lowest order in \( q \). To lowest order, the overlap factor \( f_{k,c} \) is 1. Therefore at low temperature and for small \( q \) and \( \omega \), with \( \omega < \nu F q \), we have

\[ \tilde{\Gamma}(\vec{q}, \omega) = \frac{2 \omega v_F}{\pi \hbar v_F q} \]  

(15)

a universal result that is independent of all the details of the system. Note that although it is clear that in this limit \( \tilde{\Gamma} \) should only depend on quantities defined at the Fermi level (\( k_F, \tau_F \)), it is not obvious at first that changing the momentum dependence of \( \epsilon_{k,c} \) or \( \tau_{k,c} \) will not change the momentum dependence of \( \Gamma \). For small \( q \) we approximate \( x_{a}(q, 0) \simeq -\rho_a(\epsilon_{F_a}) \), where \( \rho_d(\epsilon) \) is the density of states of layer \( a \), and
the RPA dielectric function \( \varepsilon_{\text{RPA}}(q, 0) \) becomes
\[
\varepsilon_{\text{RPA}}(q, 0) = 1 + \frac{\varepsilon_{\text{TF}}^2}{q^2} 2 \sinh(qd)e^{-qd} + \frac{q_{\text{TF1}} + q_{\text{TF2}}}{q},
\]
with \( q_{\text{TFa}} = \rho_a(E_F) e^2/(2\varepsilon_0) \), the Thomas–Fermi screening momentum in two dimensions of layer \( a \). If we assume that we have strong screening, \( q_{\text{TFa}} \gg 1 \), we further approximate [13]
\[
\varepsilon_{\text{RPA}}(q, 0) = 2 \frac{q_{\text{TF1}} + q_{\text{TF2}}}{q^2} \sinh(qd)e^{-qd}.
\] (16)

If the dispersion relation of layer \( a \) is given by a power law, \( \varepsilon_{k,c}^a = C_a k_b^\beta_a \), then we have \( q_{\text{TFa}} \propto k_{\text{Fea}}^{2-\beta_a} \). Therefore, for a linear dispersion relation the condition \( q_{\text{TFa}}d \gg 1 \) is equivalent to \( k_{\text{Fea}}d \gg 1 \); while for a parabolic dispersion relation \( q_{\text{TFa}} \) is independent of \( k_{\text{Fea}} \), and therefore \( q_{\text{TFa}}d \gg 1 \) becomes an extra assumption. Assuming \( q_{\text{TFa}}d \gg 1 \), and using (15) and (17) in (3), we obtain the following expression for the drag conductivity:
\[
\sigma_D = \frac{e^2}{\hbar} \frac{\zeta(3)g_1 g_2}{2^4 4\pi e_0|\chi_{11}|^2} \frac{q_{\text{TF1}} + q_{\text{TF2}}}{q^2} \frac{q_{\text{TF1}}^2}{q_{\text{TF2}}^2} \frac{\varepsilon_{\text{RPA}}}{\hbar^2 (q_{\text{TF1}}d)^2 (q_{\text{TF2}}d)^2}.
\] (18)

This expression is valid for \( q_{\text{TF1}}, q_{\text{TF2}}, q_{\text{TFa}}d \gg 1 \) and is universal in the sense that does not depend on the particular forms of the energy dispersion relations, transport time dependence on momentum or wave function structure. We obtain the familiar 2DEG \( T^2 \) and \( d^{-4} \) behaviour for the drag conductivity, proving that it is indeed a much more general result. If the metallic layers are immersed in a homogeneous dielectric medium, with dielectric constant \( \varepsilon_r \), one should multiply (18) by \( \varepsilon_r^2 \). Now, we notice that in the low temperature limit the intralayer conductivity for isotropic systems is given by the Boltzmann result
\[
\sigma_{aa} = \frac{e^2}{2} \rho(E_F) \chi_{11} q_{\text{TF}},
\] (19)
where the factor of 1/2 comes from the fact that we are in two dimensions, the density of states at the Fermi energy is given by
\[
\rho(E_F) = \frac{g_a}{2\pi} k_{\text{Fea}}/\hbar v_F a,
\] (20)
and that the carrier density is related to the Fermi momentum in two dimensions by
\[
k_{\text{Fea}} = \sqrt{4\pi n_a / g_a}.
\] (21)

This allows us to express the drag resistivity in terms of the carrier densities as
\[
\rho_D = -\frac{\hbar}{e^2} \frac{\zeta(3)}{2^{9/2} \sqrt{8} g_2} \left( \frac{4\pi \varepsilon_0}{e_1^2} \right)^2 \frac{(k_B T)^2}{n_1^{3/2} n_2^{3/2} d^4}.
\] (22)

It is also usual to express the drag resistivity in this limit in terms of the Fermi energy, momentum and Thomas–Fermi screening momentum. To do this we assume a power law energy dispersion relation, \( \varepsilon_{k,c}^a(k) = C_a k_b^\beta_a \), obtaining
\[
\rho_D = -\frac{\hbar}{e^2} \frac{\zeta(3)^2}{2^{9/2} \sqrt{8} g_2} \left( \frac{k_B T}{\varepsilon_0} \right)^2 \frac{(k_B T)^2}{(q_{\text{TF1}}d)(q_{\text{TF2}}d) (q_{\text{TF1}}d)^2 (q_{\text{TF2}}d)^2}.
\] (23)

For drag between two 2DEGs, \( \beta_1(2) = 2, \beta_2(2) = 2 \) (spin degeneracy), we re-obtain the known formula from [13]. For graphene we obtain exactly the same result, since \( \beta_1(2) = 1, \beta_2(2) = 4 \) (spin and valley degeneracy). Finally, for the case where each of the two layers is formed by graphene bilayers, \( \beta_1(2) = 2, \beta_2(2) = 4 \) (spin and valley degeneracy), we have an extra factor of \( 1/\beta \).

### 3.2. The case of graphene at small interlayer distance

Now we specialize to the case where both metallic layers are formed by single layer graphene (SLG) and analyse the behaviour of the drag conductivity when the layer separation is small, \( q_{\text{TFa}}d \ll 1 \). In this situation we can no longer expand the non-linear susceptibility for small \( q \) and need to consider its full dependence on \( q \). In graphene the wave function overlap factor is
\[
f_{k,k+\vec{q}} = \frac{1}{2} \left( 1 + \lambda \lambda' \frac{k}{|k||k+\vec{q}|} \right),
\] (24)
with \( \lambda = +, - \) for the conduction and valence band, respectively. Therefore, the non-linear susceptibility in the low temperature limit (13) reads
\[
\chi_{\text{SLG}}(q, \omega) = \frac{4\sigma_F q}{\pi \hbar v_F q} \left( 1 - \frac{q}{2k_F} \right)^2,
\] (25)
where the factor of 4 comes from the spin and valley degeneracies. Equation (25) is in disagreement with the expressions obtained in [10] both for the momentum independent and for the linearly momentum dependent transport time cases. However, we emphasize that in the low temperature limit (25) holds for an arbitrary transport time. Now we notice that for \( q < 2k_F \), the static polarizability for graphene is given by [21]
\[
\chi_{\text{SLG}}(q < 2k_F, 0) = -\frac{2k_F}{\pi \hbar v_F} = -\frac{2e_0}{q}. \] (26)

Therefore the dielectric function \( \varepsilon_{\text{RPA}}(q, 0) \) still has the form given by (16), even if we do not assume that \( q \) is small. Since we have \( k_{\text{TFa}}d \ll 1 \), we expand to first order in \( d \)
\[
|U_{12}(q, 0)|^2 = \frac{|V_{12}(q, 0)|^2}{|\varepsilon_{\text{RPA}}(q, 0)|^2} \approx \frac{1}{d^2 (q + q_{\text{TF1}} + q_{\text{TF2}})^2} \left( q + q_{\text{TF1}} + q_{\text{TF2}} \right) \left( q + q_{\text{TF1}} + q_{\text{TF2}} \right)^3.
\] (27)
and the drag conductivity becomes
\[
\sigma_D = \frac{e^2}{h} \frac{2}{3} \alpha_g^2 \frac{t_p^2 (k_B T)^2}{h^2} \left[ \mathcal{T}^{(0)}(k_{F1}, k_{F2}) - d \mathcal{T}^{(1)}(k_{F1}, k_{F2}) \right],
\]
(28)
where \( \alpha_g = e^2/(4\pi \varepsilon_0 v_F h) \) is the fine structure constant of graphene and we have defined the functions
\[
\mathcal{T}^{(0)}(k_{F1}, k_{F2}) = \int_0^K \frac{dq q}{(q + q_{TF1} + q_{TF2})^2} \times \sqrt{1 - \frac{q^2}{4k_{F1}^2} \left( 1 - \frac{q^2}{4k_{F2}^2} \right)}
\]
\[
\mathcal{T}^{(1)}(k_{F1}, k_{F2}) = \int_0^K \frac{dq}{q^2} \frac{2(2q_{TF1}q_{TF2} + q(q + q_{TF1} + q_{TF2}))}{(q + q_{TF1} + q_{TF2})^3} \times \sqrt{1 - \frac{q^2}{4k_{F1}^2} \left( 1 - \frac{q^2}{4k_{F2}^2} \right)}.
\]
(29)
where \( K = 2 \min(k_{F1}, k_{F2}) \), and for graphene \( q_{TF} = 4\alpha_g k_F \).

The drag resistivity becomes
\[
\rho_D = -\frac{h}{e^2} \frac{2^2 \pi^2 (k_B T)^2}{\varepsilon_{F1F2}} \times \alpha_g^2 \left[ \mathcal{T}^{(0)}(k_{F1}, k_{F2}) - d \mathcal{T}^{(1)}(k_{F1}, k_{F2}) \right].
\]
(30)
An identical result for \( d = 0 \) has recently been derived in an independent work [12]. For the case where both layers are at the same carrier density \( k_{F1} = k_{F2} = k_F \) the functions \( \mathcal{T}^{(0)} \) and \( \mathcal{T}^{(1)} \) simplify considerably and we obtain
\[
\mathcal{T}^{(0)}(k_F, k_F) = 12\alpha_g - \frac{3}{2} + (1 - 48\alpha_g^2) \log \left( 1 + \frac{1}{4\alpha_g} \right),
\]
\[
\mathcal{T}^{(1)}(k_F, k_F)/k_F = \frac{2}{3} + 44(1 - 8\alpha_g)\alpha_g
\]
\[+ \frac{2}{1 + 4\alpha_g} + 32\alpha_g (4\alpha_g^2 - 1) \log \left( 1 + \frac{1}{4\alpha_g} \right),
\]
(31)
Therefore, for drag between two SLG layers as the interlayer separation is increased, the behaviour of the drag resistivity changes from a linear dependence on \( d \) for \( q_{TFd} \ll 1 \) to a \( d^{-4} \) dependence at large \( q_{TFd} \gg 1 \). In figure 1, we can see that the small separation expression (30) is only reliable for \( q_{TFd} \lesssim 0.2 \) and the large separation expression (22) for \( q_{TFd} \gtrsim 20 \). If the graphene layers are immersed in a homogeneous dielectric with constant \( \varepsilon_r \), this would correspond to \( k_Fd \lesssim 0.02\varepsilon_r \) and \( k_Fd \gtrsim 2\varepsilon_r \), respectively. We therefore see, that these limits are not easy to achieve experimentally.

4. Drag in graphene: general formula and asymptotic limit

As we have just seen, the dependence on momentum of the transport time is irrelevant in the low temperature and high density limits, but it will be important in general, however. The dependence of the transport time on momentum in graphene depends on the dominant scattering mechanism. Both for strong short-range impurities (resonant scatterers) and Coulomb impurities the transport time depends linearly on the momentum, \( \tau_{\lambda,\lambda'} = \tau_0 |k| \), where \( \tau_0 \) is a constant with units of length \( \times \) time [15]. The linear dependence of the transport time is assumed in [8] and we make the same assumption here. In this case, the non-linear susceptibility for graphene becomes
\[
\tilde{\chi}(q, \omega) = -8\pi v_F \tau_0 \sum_{\lambda,\lambda'} \int \frac{d^3k}{(2\pi)^2} \lambda' \delta(k_{\lambda} - k_{\lambda'})
\times \left. \left( n_F(\epsilon_{k,\lambda}^+) - n_F(\epsilon_{k,\lambda}^-) \right) \right| \delta (\epsilon_{\lambda} - \epsilon_{\lambda'} + \hbar \omega).
\]
(32)
We will follow the steps of [8] and assume that both layers have high electron doping, so that the existence of the valence band can be ignored. In this case, we take only the \( \lambda, \lambda' = +, + \) contribution into account. Taking the non-linear susceptibilities at zero temperature, the drag resistivity can be written as
\[
\rho_D = -\frac{h}{2e^2} \frac{\sqrt{\varepsilon_{F1F2}}}{k_B T} \alpha_g^2 \mathcal{F}(k_{F1}, k_{F2}, d).
\]
(33)
The function \( \mathcal{F} \) is defined as
\[
\mathcal{F}(k_{F1}, k_{F2}, d) = \int_0^\infty dy \times \int_0^\infty dx \times \frac{dy}{\sinh^2 \left( y \sqrt{\varepsilon_{F1F2} k_F} \right)} \times \frac{e^{-2d/\sqrt{\varepsilon_{F1F2}}} \Phi_1(x, y) \Phi_2(x, y)}{[\varepsilon_{RPA}(x, y)]^2 \left( 1 - \left( \frac{x}{\ell} \right)^2 \right)},
\]
(34)
where the functions \( \Phi_{a}(x, y) \) are introduced in appendix B and \( x = q/\sqrt{k_{F1}k_{F2}} \) and \( y = q/(\sqrt{\varepsilon_{F1F2} k_F}) \). This is exactly the same expression as derived in [8] if one notices that the function \( \epsilon(x, y) \) used there is related to \( \varepsilon_{RPA}(x, y) \) by \( \epsilon(x, y) = x^2(y^2 - \ell^2)\varepsilon_{RPA}(x, y) \). So far no approximation has been made in the sense that no asymptotic limit has been considered.
clarify that in [8], as in this paper, the group velocity used in [8] depends linearly on momentum. We dependent on the momentum, implying that the carrier group for a constant group velocity and a transport time linearly separation limit should behave as $d^{-6}$ in this same limit a dependence of $d^{-6}$ in all plots. No attempt to fit the data was made.

5. Comparison with experiments

In the experimental setup of [3] we have two graphene layers, which we denote by $t$ (top) and $b$ (bottom). Between the two graphene sheets we have a layer of Al$_2$O$_3$ of thickness $d = d_t = 7$ nm (in [8] it a value of 14 nm was used). The bottom layer is on top of a thickness $d_b = 280$ nm of SiO$_2$. Finally, these layers are on top of a silicon wafer. The carrier density in the graphene sheets is controlled using a back gate voltage between the silicon wafer and the bottom layer. For the relation between the gate voltage and the carrier densities the reader is referred to [3, 8]. Given the carrier densities, one can compute the drag resistivity using (33) and (34). For the fine structure constant of graphene we will use the accepted value of $\alpha = 1.27 \times 10^{-3}$. However, as we have argued in section 3, the momentum dependence of the vertex function $\Phi(x,y)$ was taken to behave as $\Phi(x,y) \sim \frac{y}{x}$, which changes the integration kernel obtained in [8] in the asymptotic limit from $q^2 \sinh^2(qd)$ to $q^2 \sinh^2(qd)$, which changes the dependence from $d^{-6}$ to the correct behaviour of $d^{-4}$. Therefore, for $\epsilon_{F1(2)}\beta$, $k_{F1(2)}d \gg 1$, the exact (33) and (34) give

$$\rho_D = -\frac{h}{e^2} \frac{\zeta(3) \pi^2}{24} \frac{(k_B T)^2}{\epsilon_{F1} \epsilon_{F2}(k_{F1} d)(k_{F2} d)(q_{TF1} d)(q_{TF2} d)},$$

(35)

in agreement with the general result discussed in section 3. Expressing (35) in terms of the carrier density we get

$$\rho_D = -\frac{h}{e^2} \frac{\zeta(3) \pi^2}{24} \frac{(k_B T)^2}{\epsilon_{F1} \epsilon_{F2}(k_{F1} d)(k_{F2} d)(q_{TF1} d)(q_{TF2} d)},$$

(36)

as obtained before, but here we have started from the general expression for the drag, that is, (33). Note that in this limit the drag resistivity decreases as $\alpha g$ is increased. We can understand this as follows: as $\alpha g$ increases the screening becomes more effective making the momentum transfer between layers less effective.
exact form of the Coulomb interactions in a layered dielectric medium one must solve Poisson’s equation (see section A). For a three layered dielectric structure, for the effective dielectric constants $\varepsilon_{ab}(q)$ in (7) we would have to use the functions $\varepsilon_{ab}^{(3)}(q)$ defined in section A, with $\varepsilon_1 = \varepsilon_{air} = 1$, $\varepsilon_2 = \varepsilon_{AlO_3} = 5.6$ and $\varepsilon_3 = \varepsilon_{SiO_2} = 3.8$. In figure 2(b), we can see the comparison between the computed resistivity using these interactions and the experimental results. Although this is a more rigorous treatment for the interactions than the naive one, (37), the computed drag curves deviate even further from the experimental results. One could proceed as in [8] and use $\alpha_g$ as a fitting parameter. We found, however, that it would be necessary to use values of the order $\alpha_g \sim 5$ and $\alpha_g \sim 16$ for the interactions with $\varepsilon_{ab}^{(naive)}$ and $\varepsilon_{ab}^{(3)}$, respectively, to fit the data; both values are unrealistic. Taking into account that the SiO$_2$ layer is finite when solving Poisson’s equation has virtually no effect on the computed drag resistivity. This situation is puzzling at least. However, more attention must be paid to how the devices from [3] are constructed. The Al$_2$O$_3$ dielectric layer is deposited in two steps: (i) First a 2 nm aluminum layer is deposited on top of the bottom graphene layer, and is later oxidized. We will refer to this as AlO$_x$ layer. (ii) On top of the oxidized aluminum layer, a 5 nm Al$_2$O$_3$ layer is directly deposited through atomic layer deposition (ALD). We will refer to this as the ALD layer. When the procedure is completed a 7 nm Al$_2$O$_3$ layer is obtained. The reason why the Al$_2$O$_3$ is not directly deposited on top of the graphene is to due to graphene being chemically inert. In [22], the dielectric constant of the AlO$_x$ alumina was determined by studying how the capacitance of devices similar to those from [3] scales with the thickness of the AlO$_x$ layer keeping the AlO$_x$ layer thickness fixed, obtaining the already noted value of $\varepsilon_{AlO_3} = 5.6$. The AlO$_x$ layer contributed with a finite capacitance, which corresponds to a dielectric constant of $\varepsilon_{AlO_3} = 2.7$. The reason why $\varepsilon_{AlO_3}$ differs from $\varepsilon_{AlO_3}$ is not clear, but it is most likely due to interface effects between the graphene layer and the dielectric or between the ALD and the AlO$_x$ layers. In this case, it is not clear if we can attribute a bulk value for $\varepsilon_{AlO_3}$. Nevertheless, we still considered this situation. For this situation of a four layered dielectric we use the functions $\varepsilon_{ab}^{(4)}(q)$ from section A, now with $\varepsilon' = \varepsilon_{AlO_3} = 2.7$ and $w = 2$ nm. The comparison between the drag curves computed this way and the experimental data can be seen in figure 2(c). We can see that there is a better approximation with the experimental results, but this should be regarded with caution.

6. Conclusions

In this paper we have shown that in the limit of low temperature, high density, large interlayer distance and strong screening the drag resistivity should always behave as $T^2$ and $d^{-4}$. This result was obtained for a general dispersion relation, momentum dependence of the transport time and electronic wave function structure. It is therefore a more general result than previous ones. Central to this fact is the general expression derived for the non-linear susceptibility, (13), in the low temperature limit to lowest order in frequency. Being general, this result also applies to graphene, and should close the ongoing debate regarding the behaviour of drag for this system in the aforementioned limit. We also derived an asymptotic expression for drag in graphene in the limit of low temperature, high density and small layer separation. Finally, we compared the available experimental data on drag in graphene from [3] with our theoretical model and found out that if one insists on the conventional value for the graphene fine structure constant, $\alpha_g = 2$, a quantitative understanding of the data is still lacking. This is most likely due to the complexity of the dielectric substrate used, and one hopes that the theoretical model will be quantitatively more successful in devices built using simpler substrates, such as boron nitride [23]. Another possibility to account for this discrepancy could be spatial inhomogeneities in the chemical potential of the graphene layers, which are not taken into account with the present formalism. Although it is not clear if these should be important in the experimental range investigated, spatial inhomogeneities in graphene’s carrier density will certainly play an important role in the electron–hole puddle regime and should be the object of future work.

Acknowledgments

We acknowledge useful discussions with F Guinea, J M B Lopes dos Santos and A H Castro Neto. Special thanks to E Tutuc for helpful discussions regarding the construction of graphene double layer structures. BA was supported by Fundação para a Ciência e a Tecnologia (FCT) through grant no. SFRH/BD/78987/2011.

Appendix A. Electron–electron Coulomb interaction

To obtain the correct form of the electron–electron interaction in a dielectric material one must solve Poisson’s equation, $-\nabla (\varepsilon(x,y,z)\nabla \phi) = \rho_{free}/\varepsilon_0$, where $\rho_{free}$ is the free charge density. We are interested in situations where the dielectric constant is a piecewise constant function of the $z$ coordinate, with discontinuities at positions $z_i$. Since we want the interaction matrix element between states with well defined momentum in the $x, y$ directions and well defined position in the $z$ direction, it is useful to introduce $\phi(q,z) = \int d^2x \phi(x,z)e^{-i\vec{q}\cdot\vec{x}}$, where $\vec{q} = (q_x, q_y)$ and $\vec{x} = (x, y)$. For a point charge located at $(\bar{x}, z) = (0, z_p)$, $\rho_{free} = e\delta(0)\delta(z_p)$, we obtain the following equation

$$-\frac{\partial}{\partial z} \left( \varepsilon(z) \frac{\partial \phi}{\partial z}(z, z_p) \right) + q^2 \phi(q,z) = \frac{e}{\varepsilon_0} \delta(z_p). \quad (A.1)$$

The potential $\phi(q,z)$ is continuous everywhere, and the function $\varepsilon(z)\phi(q,z)z$ is continuous except at the position of the point charge. Therefore we have the boundary conditions

$$\phi(q, z_{i-}^+) = \phi(q, z_{i-}^-), \quad \varepsilon(z_{i-}^+) \frac{\partial \phi(q, z_{i-}^+)}{\partial z} - \varepsilon(z_{i-}^-) \frac{\partial \phi(q, z_{i-}^-)}{\partial z} = \frac{e}{\varepsilon_0} \delta_{z=z_p}. \quad (A.2)$$
Solving equation (A.1) together with the boundary conditions (2.2) and imposing that $\phi(q, z)$ decays at infinity, we can obtain the potential created by the point charge and from that the bare electron–electron Coulomb interaction. Let us consider that the metallic plates are located at $z = 0$ and $z = d$. The electron–electron interaction can be cast in the form given in (7) with $a, b = t, b$, where $t$ refers to the top layer and $b$ refers to the bottom layer. For a three layered dielectric

$$
\epsilon^{(3)}(z) = \begin{cases} 
\epsilon_1, & z > d \\
\epsilon_2, & d > z > 0 \\
\epsilon_3, & 0 > z,
\end{cases}
$$

and solving Poisson’s equation gives

$$
\begin{align*}
\epsilon^{(3)}_tt(q) &= \frac{e^{2qd}(\epsilon_3 + \epsilon_2)(\epsilon_1 + \epsilon_2) - (\epsilon_3 - \epsilon_2)(\epsilon_1 - \epsilon_2)}{2(1 + e^{2qd})\epsilon_2 - 2(1 - e^{2qd})\epsilon_3}, \\
\epsilon^{(3)}_bb(q) &= \frac{e^{2qd}(\epsilon_1 + \epsilon_2)(\epsilon_3 + \epsilon_2) - (\epsilon_1 - \epsilon_2)(\epsilon_3 - \epsilon_2)}{2(1 + e^{2qd})\epsilon_2 - 2(1 - e^{2qd})\epsilon_3}, \\
\epsilon^{(3)}_bb(q) &= \frac{e^{2qd}(\epsilon_1 + \epsilon_2)(\epsilon_3 + \epsilon_2) - (\epsilon_1 - \epsilon_2)(\epsilon_3 - \epsilon_2)}{2(1 + e^{2qd})\epsilon_2 - 2(1 - e^{2qd})\epsilon_3}.
\end{align*}
$$

This result was previously given in[9, 24]. If we consider a four layered dielectric,

$$
\epsilon^{(4)}(z) = \begin{cases} 
\epsilon_1, & z > d \\
\epsilon_2, & d > z > w \\
\epsilon_2', & w > z > 0 \\
\epsilon_3, & 0 > z,
\end{cases}
$$

we obtain

$$
\begin{align*}
\epsilon^{(4)}_tt(q) &= \frac{D^{(4)}(q)}{2} e^{2q(d+w)}(\epsilon_2 + \epsilon_2')(\epsilon_3 + \epsilon_2') + e^{2qd}(\epsilon_2 - \epsilon_2')(\epsilon_3 - \epsilon_2) + e^{4qw}(\epsilon_2 - \epsilon_2')(\epsilon_1 + \epsilon_2') + e^{2q\omega}(\epsilon_2 + \epsilon_2')(\epsilon_3 - \epsilon_2')]^{-1}, \\
\epsilon^{(4)}_bb(q) &= \frac{D^{(4)}(q)}{8e^{2q(d+w)}\epsilon_2}, \\
\epsilon^{(4)}_bb(q) &= \frac{D^{(4)}(q)}{2} e^{2q(d+w)}(\epsilon_1 + \epsilon_2)(\epsilon_2 + \epsilon_2') - e^{2qd}(\epsilon_1 + \epsilon_2)(\epsilon_2 - \epsilon_2') + e^{4qw}(\epsilon_2 - \epsilon_2')(\epsilon_1 + \epsilon_2') - e^{2q\omega}(\epsilon_1 + \epsilon_2)(\epsilon_2 - \epsilon_2')]^{-1},
\end{align*}
$$

where $D^{(4)}(q)$ is defined as

$$
D^{(4)}(q) = e^{2q(d+w)}(\epsilon_1 + \epsilon_2)(\epsilon_2 + \epsilon_2')(\epsilon_3 + \epsilon_3') + e^{2qd}(\epsilon_1 + \epsilon_2)(\epsilon_2 - \epsilon_2')(\epsilon_3 - \epsilon_3') + e^{4qw}(\epsilon_2 + \epsilon_2')(\epsilon_3 - \epsilon_3') + e^{2q\omega}(\epsilon_1 + \epsilon_2)(\epsilon_2 - \epsilon_2')(\epsilon_3 - \epsilon_3') + e^{2q\omega}(\epsilon_1 - \epsilon_2)(\epsilon_2 + \epsilon_2')(\epsilon_3 - \epsilon_3').
$$

Appendix B. Details of the computation of the drag resistivity in graphene

The contribution to the non-linear susceptibility of graphene (32) coming only from the conductance band, $\lambda, \lambda' = +, +$, is

$$
\tilde{\chi}^{zz}(q, \omega) = 8\pi v_F t^0 \tilde{q} \int \frac{d^2 k}{(2\pi)^2} f^{+} f^{-} \left[ (n_0(\epsilon_{k}^{+} - \epsilon_{\tilde{q} + k}^{+}) - n_0(\epsilon_{\tilde{q} + k}^{+} - \epsilon_{k}^{+})) \delta(\epsilon_{k}^{+} - \epsilon_{\tilde{q} + k}^{+} + \hbar \omega) ight] = -2\pi v_F t^0 \bar{q} \text{Im} \chi^{zz}(q, \omega),
$$

where $\chi^{zz}(q, \omega)$ is the contribution to the graphene polarizability coming only from the conductance band,

$$
\chi^{zz}(q, \omega) = 4\int \frac{d^2 k}{(2\pi)^2} f^{+} f^{-} \left[ n_0(\epsilon_{k}^{+} - \epsilon_{\tilde{q} + k}^{+}) - n_0(\epsilon_{\tilde{q} + k}^{+} - \epsilon_{k}^{+}) \right] (\epsilon_{k}^{+} - \epsilon_{\tilde{q} + k}^{+} + \hbar \omega + i0^+).
$$

At zero temperature, this can be computed analytically and the result can be written as

$$
\text{Im} \chi^{zz}(q, \omega) = \frac{1}{4\pi v_F} \sqrt{1 - \frac{(\omega/v_F)^2}{q^2}},
$$

where the function $\Phi(q, \omega)$ is defined as

$$
\Phi(q, \omega > 0) = \Phi^+(q, \omega) \Theta \left( \frac{(\omega/v_F) - q - 2k_F}{\sqrt{2k_F/q}} \right) - \Phi^-(q, \omega) \Theta \left( \frac{(\omega/v_F) - |k_F - q|}{|\sqrt{2k_F/q}|} \right),
$$

where

$$
\Phi^\pm(q, \omega) = \pm \cosh^{-1} \left( \frac{2k_F \pm q/v_F}{q} \right) \times \pm \frac{2k_F \pm q/v_F}{q} \sqrt{2k_F \pm q/v_F} - 1.
$$

The drag conductivity therefore becomes

$$
\sigma_D = \frac{e^2 (10)^2}{2\pi^3 h^3 k_B T} \int_0^\infty dq \int_0^\infty d\omega |U_{12}^{\tilde{q}}(q, \omega)|^2 \phi_1(q, \omega) \phi_2(q, \omega) \frac{1}{1 - \frac{(\omega/v_F)^2}{q^2}},
$$

where $U_{12}^{\tilde{q}}(q, \omega)$ are the interlayer interaction as $U_{12} = e^2 \exp(-qd)/(2\pi e \hbar k_F)$. Adding the dimensional quantities, $x = q/\sqrt{k_F k_B}$ and $y = \omega(v_F \sqrt{k_F k_B})$, the drag resistivity becomes

$$
\rho_D = \frac{1}{2\pi^3} \frac{\hbar}{e^2} \sqrt{\frac{1}{k_B T}} \int_0^\infty dx \int_0^\infty dy \frac{dy}{\sin^2 \left( \frac{\omega v_F \sqrt{k_F k_B}}{2\hbar} \right)} e^{-2d/\sqrt{k_F k_B} x} \left( \frac{\phi_1(x, y) \phi_2(x, y)}{\sigma_0} \right)^2 \frac{1}{1 - \frac{(\omega v_F \sqrt{k_F k_B})^2}{2\hbar} \left( \frac{\phi_1(x, y) \phi_2(x, y)}{\sigma_0} \right)^2}.
$$
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