Copositivity Detection of Tensors: Theory and Algorithm

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Abstract A symmetric tensor is called copositive if it generates a multivariate form taking nonnegative values over the nonnegative orthant. Copositive tensors have found important applications in polynomial optimization, tensor complementarity problems and vacuum stability of a general scalar potential. In this paper, we consider copositivity detection of tensors from both theoretical and computational points of view. After giving several necessary conditions for copositive tensors, we propose several new criteria for copositive tensors based on the representation of the multivariate form in barycentric coordinates with respect to the standard simplex and simplicial partitions. It is verified that, as the partition gets finer and finer, the concerned conditions eventually capture all strictly copositive tensors. Based on the obtained theoretical results with the help of simplicial partitions, we propose a numerical method to judge whether a tensor is copositive or not. The preliminary numerical results confirm our theoretical findings.
Keywords Symmetric tensor · Strictly copositive tensor · Positive semi-definiteness · Simplicial partition

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1 Introduction

A symmetric tensor is called copositive if it generates a multivariate form taking nonnegative values over the nonnegative orthant [1]. Copositive tensors constitute a large class of tensors that contain nonnegative tensors and several kinds of structured tensors in the even order symmetric case such as $M$-tensors [2], diagonally dominant tensors [3], $B$ ($B_0$) tensors [3], $M B$-($MB_0$-)tensors [4] and double (quasi-double) $B$ tensors [5]. Copositive tensors are also related to semi-positive tensors [6] and even order symmetric Cauchy tensors [7]. For example, Song and Qi [6] proved that a real symmetric tensor is a (strictly) semi-positive if and only if it is (strictly) copositive.

It has been found that copositive tensors have important applications in polynomial optimization [8] and vacuum stability of a general scalar potential [9]. Pena et al. [8] provided a general characterization for a class of polynomial optimization problems that can be formulated as a conic program over the cone of completely positive tensors or copositive tensors. As a consequence of this characterization, it follows that recent related results for quadratic problems can be further strengthened and generalized to higher-order polynomial optimization problems. Kannike [9] studied the vacuum stability of a general scalar potential of a few fields. With the help of copositive tensors and its relationship to orbit space variables, Kannike showed that how to find positivity conditions for more complicated potentials. Then, he discussed the vacuum stability conditions of the general potential of two real scalars, without and with the Higgs boson included in the potential.

Recently, Fan et al. [10] formulated a tensor eigenvalue complementarity problem as a constrained polynomial optimization problem, which can be solved by Lasserre’s hierarchy of semi-definite relaxations. In Algorithm 4.1 of [10], the authors assumed the objective tensor is strictly copositive, which can not only guarantee the existence of the eigenvalue, but also can guarantee that the eigenvalues can be normalized. Unfortunately, in practical applications, it is generally difficult to know whether the given tensor is strictly copositive or not. In addition, Che et al. [11] showed that the tensor complementarity problem (TCP) has a nonempty and compact solution set if the involved tensor with even order is strictly copositive, which provides an important theoretical basis for designing effective algorithm to solve the TCP with the involved tensor being strictly copositive (see [12–14] for some related references). Thus, from the view of theoretical and computational points, it is of great importance to check directly whether a tensor is copositive or not. The challenging problem is how to check the copositivity of a given symmetric tensor efficiently.

Several sufficient conditions or necessary and sufficient conditions for copositive tensors were presented in [1,15], but they are difficult to achieve numerically. Song and Qi [16] introduced the concepts of Pareto H(or Z)-eigenvalue and proved that a symmetric tensor is copositive (strictly copositive) if and only if every Pareto H(or...
Z)-eigenvalue of the tensor is nonnegative (positive). Unfortunately, it is NP-hard to compute the minimum Pareto H\((or\) Z)-eigenvalue of a given symmetric tensor. In fact, the copositivity detection of tensors is NP-complete, even for the matrix case \([17,18]\). To the best of our knowledge, there is no any numerical detection method for high-order copositive tensors. In this paper, we further give some theoretical studies on various conditions for (strictly) copositive tensors, and based on some of our theoretical findings, we propose a numerical method to judge whether a tensor is copositive or not, which can be viewed as an extension of some branch-and-bound-type algorithms for testing copositivity of symmetric matrices \([19–21]\).

The rest of this paper is organized as follows. In Sect. 2, we recall some notions about tensors. Three necessary conditions for copositive tensors are given in Sect. 3. In Sect. 4, we give several criteria for (strictly) copositive tensors based on the simplicial subdivision, and an equivalent condition for a symmetric tensor that is not copositive. In Sect. 5, we propose a numerical detection algorithm for copositive tensors based on the results obtained in Sect. 4 and show that the algorithm can always capture strictly copositive tensors in finitely many iterations. The preliminary numerical results are reported in Sect. 6, and conclusions are given in Sect. 7.

2 Preliminaries

Throughout this paper, we denote the set consisting of all positive integers by \(\mathbb{N}\), and always assume that \(m, n \in \mathbb{N}\). Let \(\mathbb{R}^n\) be the \(n\)-dimensional real Euclidean space and the set of all nonnegative vectors in \(\mathbb{R}^n\) be denoted by \(\mathbb{R}^n_+\). Let \(\mathbb{R}^n_{++}\) denote the set of vectors with positive entries. Vectors are denoted by bold lowercase letters, i.e., \(\mathbf{x}, \mathbf{y}, \ldots\), matrices are denoted by capital letters, i.e., \(A, B, \ldots\), and tensors are written as calligraphic capitals such as \(\mathcal{A}, \mathcal{F}, \ldots\). We denote \([n] = \{1, 2, \ldots, n\}\). The \(i\)th unit coordinate vector in \(\mathbb{R}^n\) is denoted by \(e_i\) for any \(i \in [n]\).

For a real tensor \(\mathcal{A} = (a_{ij1i_2 \ldots i_m}), i_j \in [n], j \in [m]\), it is said to be nonnegative if all its entries are nonnegative, and \(\mathcal{A}\) is called a symmetric tensor if the entries \(a_{ij1i_2 \ldots i_m}\) are invariant under any permutation of their indices. In this paper, we always consider real symmetric tensors. The identity tensor \(\mathcal{I}\) with order \(m\) and dimension \(n\) is given by \(\mathcal{I}_{i_1 \ldots i_m} = 1\) if \(i_1 = \cdots = i_m\) and \(\mathcal{I}_{i_1 \ldots i_m} = 0\) otherwise. For any \(J \subseteq [n]\), \(|J|\) denotes the number of elements of \(J\), and \(A_J\) denotes a principle subtensor of \(\mathcal{A}\).

We denote \(S_{m,n}\) the set of all real symmetric tensors with order \(m\) dimension \(n\). For any \(\mathcal{A}, \mathcal{B} \in S_{m,n}\), we define the inner product by \(\langle \mathcal{A}, \mathcal{B} \rangle := \sum_{i_1 \ldots i_m=1}^n a_{i_1 \ldots i_m} b_{i_1 \ldots i_m}\), and the corresponding norm is denoted by \(\|\mathcal{A}\| = (\langle \mathcal{A}, \mathcal{A} \rangle)^{1/2}\).

For any \(\mathbf{x} \in \mathbb{R}^n\), we use \(x_i\) to denote its \(i\)th component and use \(\|\mathbf{x}\|_m\) to denote the \(m\)-norm of \(\mathbf{x}\). For \(m\) vectors \(\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z} \in \mathbb{R}^n\), \(\mathbf{x} \circ \mathbf{y} \circ \cdots \circ \mathbf{z}\) denotes an \(m\)-th order \(n\)-dimensional rank one tensor with \((\mathbf{x} \circ \mathbf{y} \circ \cdots \circ \mathbf{z})_{i_1j_1 \ldots i_mj_m} = x_{i_1} y_{j_1} \cdots z_{i_m}\) for all \(i_1, \ldots, i_m \in [n]\). Particularly, for any \(\mathcal{A} \in S_{m,n}\), we denote \(\mathcal{A} \mathbf{x}^m = \langle \mathcal{A}, \mathbf{x} \circ \cdots \circ \mathbf{x} \rangle_m\) and \(\mathcal{A} \mathbf{x}^k \mathbf{y}^{m-k} = \langle \mathcal{A}, \mathbf{x} \circ \cdots \circ \mathbf{x} \circ \mathbf{y} \circ \cdots \circ \mathbf{y} \rangle_k\) where \(k \in [m]\). Then,
\[
A^m x^m = \sum_{i_1, \ldots, i_m=1}^n a_{i_1 \ldots i_m} x_{i_1} \cdots x_{i_m}, \\
A^k y^{m-k} = \sum_{i_1, \ldots, i_m=1}^n a_{i_1 \ldots i_m} x_{i_1} \cdots x_{i_k} y_{i_{k+1}} \cdots y_{i_m},
\tag{1}
\]

Moreover, \(A^{m-1}\) is a vector such that

\[
(A^{m-1})_i = \sum_{i_2, i_3, \ldots, i_m \in [n]} a_{i_1 i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}, \forall i \in [n].
\]

### 3 Necessary Conditions for Copositivity of Tensors

In this section, we first introduce the definition of copositive tensors and then establish three necessary conditions for copositive tensors, which are given based on binomial expansion, principle subtensor and convex combination of symmetric tensors, respectively.

**Definition 3.1** 
Let \(A \in S_{m,n}\) be a given tensor. If \(A^{m} x^{m} \geq 0\) (or \(A^{m} x^{m} > 0\)) for any \(x \in \mathbb{R}_{+}^{n}\) (or \(x \in \mathbb{R}_{+}^{n} \setminus \{0\}\)), then \(A\) is called a copositive (strictly copositive) tensor.

To move on, we first prove the following result, which looks like the binomial expansion of two real variables.

**Lemma 3.1** Let \(A = (a_{i_1 i_2 \ldots i_m}) \in S_{m,n}\) be given. For any \(x, y \in \mathbb{R}^{n}\), it holds that

\[
A(x + y)^m = A x^m + \binom{m}{1} A^{m-1} y + \binom{m}{2} A^{m-2} y + \cdots + \binom{m}{m} A y^m.
\]

**Proof** For any \(x, y \in \mathbb{R}^{n}\), it follows that

\[
A(x + y)^m = (A, (x + y) \circ (x + y) \circ \cdots \circ (x + y)) = \sum_{i_1, i_2, \ldots, i_m \in [n]} a_{i_1 i_2 \ldots i_m} (x_{i_1} + y_{i_1})(x_{i_2} + y_{i_2}) \cdots (x_{i_m} + y_{i_m}).
\]

By using the symmetry property of \(A\) and (1), we further obtain that

\[
A(x + y)^m = \sum_{i_1, i_2, \ldots, i_m \in [n]} a_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}
\]

\[
+ \binom{m}{1} \sum_{i_1, i_2, \ldots, i_m \in [n]} a_{i_1 i_2 \ldots i_m} x_{i_1} \cdots x_{i_{m-1}} y_{i_m}.
\]
\[ + \cdots + \binom{m}{m} \sum_{i_1, i_2, \ldots, i_m \in [n]} a_{i_1 i_2 \ldots i_m} y_{i_1} y_{i_2} \cdots y_{i_m} \]

\[ = \mathcal{A} x^m + \binom{m}{1} \mathcal{A} x^{m-1} y + \binom{m}{2} \mathcal{A} x^{m-2} y + \cdots + \binom{m}{m} \mathcal{A} y^m, \]

which completes the proof. \[\Box\]

Now, by Lemma 3.1 we establish a necessary condition for the copositive tensor.

**Theorem 3.1** Let \( \mathcal{A} \in S_{m,n} \) be copositive. If there is \( x \in \mathbb{R}_+^n \) such that \( \mathcal{A} x^m = 0 \), then \( \mathcal{A} x^{m-1} \geq 0 \).

**Proof** Suppose \( u \in \mathbb{R}_+^n \) and \( \varepsilon \in \mathbb{R} \) with \( \varepsilon > 0 \). Since \( \mathcal{A} \) is copositive, it follows that \( \mathcal{A} z^m \geq 0 \), where \( z = x + \varepsilon u \). So, by Lemma 3.1, we have

\[ \mathcal{A} z^m = \mathcal{A} (x + \varepsilon u)^m = \mathcal{A} x^m + \binom{m}{1} \varepsilon \mathcal{A} x^{m-1} u + \binom{m}{2} \varepsilon^2 \mathcal{A} x^{m-2} u^2 + \cdots + \binom{m}{m} \varepsilon^m \mathcal{A} u^m. \]  

(2)

Now, assume that there is an index \( i \in [n] \) such that \( \mathcal{A} x^{m-1} i < 0 \). Take \( u = e_i \) in (2), we obtain that

\[ \mathcal{A} z^m = \mathcal{A} (x + \varepsilon e_i)^m = \mathcal{A} x^m + \binom{m}{1} \varepsilon \mathcal{A} x^{m-1} e_i + \binom{m}{2} \varepsilon^2 \mathcal{A} x^{m-2} e_i^2 + \cdots + \binom{m}{m} \varepsilon^m \mathcal{A} e_i^m < 0 \]

holds for any sufficiently small \( \varepsilon > 0 \), which contradicts the fact that \( \mathcal{A} \) is copositive.

\[\Box\]

Next, we establish a necessary condition for the copositive tensor by using the concept of principle subtensor.

**Theorem 3.2** Let \( \mathcal{A} \in S_{m,n} \) be a copositive tensor. Then, for any \( J \subseteq [n] \), the system \( \mathcal{A} J x^{m-1} \geq 0 \) admits a nonzero solution \( x \in \mathbb{R}_{+}^{|J|} \).

**Proof** We prove it by contradiction. If there is \( J \subseteq [n] \) such that \( \mathcal{A} J x^{m-1} < 0 \) for all nonzero \( x \in \mathbb{R}_{+}^{|J|} \), then we may define a vector \( y \in \mathbb{R}^n \) with \( y_i = \frac{x_i}{\|x\|_m} \) when \( i \in J \), and \( y_i = 0 \) otherwise. By a direct computation, we obtain \( \mathcal{A} y^m = \frac{\mathcal{A} J x^m}{\|x\|_m} < 0 \), which contradicts the fact that \( \mathcal{A} \) is copositive.

\[\Box\]

To end this section, we establish a necessary condition for the copositivity of a convex combination of two symmetric tensors.

**Theorem 3.3** Given \( \mathcal{A}, \mathcal{B} \in S_{m,n} \). If there exists \( t \in [0, 1] \) such that \((1-t) \mathcal{A} + t \mathcal{B} \) is copositive, then \( \max\{\mathcal{A} u^m + \mathcal{A} y^m, \mathcal{B} u^m + \mathcal{B} y^m\} \geq 0 \) for all \( u, v \in \mathbb{R}_{+}^n \).

\[\Box\] Springer
Proof Since \((1 - t)\mathcal{A} + t\mathcal{B}\) is copositive, it follows that
\((1 - t)\mathcal{A}u^m + t\mathcal{B}u^m \geq 0\) and \((1 - t)\mathcal{A}v^m + t\mathcal{B}v^m \geq 0\) for all \(u, v \in \mathbb{R}^n_+\). Adding these two inequalities, one has that
\[
\max\{\mathcal{A}u^m + \mathcal{A}v^m, \mathcal{B}u^m + \mathcal{B}v^m\} \geq (1 - t)(\mathcal{A}u^m + \mathcal{A}v^m) + t(\mathcal{B}u^m + \mathcal{B}v^m) \geq 0
\]
holds for all \(u, v \in \mathbb{R}^n_+\). This completes the proof. \(\square\)

When \(m = 2\) (i.e., in the matrix case), Crouzeix et al. [22] proved the conclusions of Theorem 3.3 are sufficient and necessary. However, it is difficult to prove the sufficiency for the case with high-order tensors, which may be an interesting topic in the future.

4 Detection Criteria Based on Simplicial Partition

In this section, several sufficient conditions or necessary conditions of copositive tensors are characterized based on some simplices. First of all, we show a useful result obtained by Song and Qi [15].

Lemma 4.1 Let \(\mathcal{A} \in S_{m,n}\) be given and \(\|\cdot\|\) denote any norm on \(\mathbb{R}^n\). Then, we have

(i) \(\mathcal{A}\) is copositive iff \(\mathcal{A}x^m \geq 0\) for all \(x \in \mathbb{R}^n_+\) with \(\|x\| = 1\);

(ii) \(\mathcal{A}\) is strictly copositive iff \(\mathcal{A}x^m > 0\) for all \(x \in \mathbb{R}^n_+\) with \(\|x\| = 1\).

It is well known that the set \(S_0 = \{x \in \mathbb{R}^n_+ : \|x\|_1 = 1\}\) is the so-called standard simplex with vertices \(e_1, e_2, \ldots, e_n\). So, it follows from Lemma 4.1 that the copositivity of tensor \(\mathcal{A} \in S_{m,n}\) can be translated to check \(\mathcal{A}x^m \geq 0\) for all \(x \in S_0\). Thus, our main goal in this section is to search for conditions that can guarantee the homogeneous polynomial \(\mathcal{A}x^m\) to be nonnegative on a simplex. A simple way to describe a polynomial with respect to a simplex is to use barycentric coordinates, which gives a convenient verifiable sufficient condition for a tensor to be copositive on a simplex. This approach has been much used for convex surface fitting in computer-aided geometric design [23] and the copositivity detection of matrices [19–21].

Lemma 4.2 Let \(S_1 = \text{conv}\{u_1, u_2, \ldots, u_n\}\) be a simplex. If

\[
\langle \mathcal{A}, u_{i_1} \circ u_{i_2} \cdots \circ u_{i_m} \rangle \geq 0 \quad (\langle \mathcal{A}, u_{i_1} \circ u_{i_2} \cdots \circ u_{i_m} \rangle > 0)
\]

for all \(i_1, i_2, \ldots, i_m \in [n]\), then \(\mathcal{A}x^m \geq 0\) (\(\mathcal{A}x^m > 0\) respectively) for all \(x \in S_1\).

Proof For any \(x \in S_1\), we have that

\[
x = \lambda_1 u_1 + \cdots + \lambda_n u_n, \quad \sum_{i=1}^n \lambda_i = 1, \quad \lambda_i \geq 0, \quad \forall i \in [n].
\]
So, it holds that
\[
\mathcal{A} x^m = \langle \mathcal{A}, (\lambda_1 u_1 + \cdots + \lambda_n u_n)^m \rangle \\
= \sum_{j_1, j_2, \ldots, j_m \in [n]} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_m} \langle \mathcal{A}, u_{j_1} \circ u_{j_2} \cdots \circ u_{j_m} \rangle.
\]
By (3) and (4), we further obtain that \( \mathcal{A} x^m \geq 0 \) (or \( \mathcal{A} x^m > 0 \)). \( \square \)

Suppose \( \mathcal{A} \) is a nonnegative tensor, if we apply Lemma 4.2 to the standard simplex \( S_0 = \{ x \in \mathbb{R}^n_+ : \|x\|_1 = 1 \} \), it shows that \( \langle \mathcal{A}, e_{i_1} \circ e_{i_2} \circ \cdots \circ e_{i_m} \rangle = a_{i_1, i_2, \ldots, i_m} \geq 0 \) for all \( i_1, i_2, \ldots, i_m \in [n] \). This means that all nonnegative tensors are copositive tensors.

Let \( S_1, S_2, \ldots, S_r \) be finite simplices in \( \mathbb{R}^n \). The set \( \tilde{S} = \{ S_1, S_2, \ldots, S_r \} \) is called a simplicial partition of \( S \) if it satisfies that \( S = \bigcup_{i=1}^{r} S_i \) and \( \text{int} S_i \cap \text{int} S_j = \emptyset \) for any \( i, j \in [r] \) with \( i \neq j \), where \( \text{int} S_i \) denotes the relative interior of \( S_i \) for any \( i \in [r] \).

**Theorem 4.1.** Given \( \mathcal{A} \in S_{m,n} \). Suppose \( \tilde{S} = \{ S_1, S_2, \ldots, S_r \} \) is a simplicial partition of simplex \( S_0 = \{ x \in \mathbb{R}^n_+ : \|x\|_1 = 1 \} \), and the vertices of simplex \( S_k \) are denoted by \( u^k_1, u^k_2, \ldots, u^k_n \) for any \( k \in [r] \). Then, \( \mathcal{A} \) is copositive iff
\[
\langle \mathcal{A}, u^k_{i_1} \circ u^k_{i_2} \circ \cdots \circ u^k_{i_m} \rangle \geq 0 \quad \text{for all } k \in [r], i_j \in [n], j \in [m], \text{ and } \mathcal{A} \text{ is strictly copositive iff } \langle \mathcal{A}, u^k_{i_1} \circ u^k_{i_2} \circ \cdots \circ u^k_{i_m} \rangle > 0 \quad \text{for all } k \in [r], i_j \in [n], j \in [m].
\]
**Proof.** By Lemma 4.1, it suffices to prove that \( \mathcal{A} x^m \geq 0 \) for all \( x \in S_0 \). For any \( x \in S_0 \), since \( \tilde{S} \) is a simplicial partition of \( S_0 \), it follows that there is an index \( k \in [r] \) such that \( x \in S_k \subseteq \tilde{S} \). By assumptions and Lemma 4.2, the desired results follow. \( \square \)

It is easy to represent that a simplex \( S \) is determined by its vertices, which can be further represented by a matrix \( V_S \) whose columns are vertices of the simplex. It is obvious that \( V_S \) is nonsingular and unique up to a permutation of its columns. So, using the product of a tensor and a matrix defined in [24], we have the following results, which is analog to Theorem 4.1.

**Theorem 4.2.** Let \( \mathcal{A} \in S_{m,n} \) be given. Suppose \( \tilde{S} = \{ S_1, S_2, \ldots, S_r \} \) is a simplicial partition of simplex \( S_0 = \{ x \in \mathbb{R}^n_+ : \|x\|_1 = 1 \} \), and the vertices of simplex \( S_k \) are denoted by \( u^k_1, u^k_2, \ldots, u^k_n \) for any \( k \in [r] \). Let \( V_{S_k} = (u^k_1, u^k_2, \ldots, u^k_n) \) be the matrix corresponding to simplex \( S_k \) for any \( k \in [r] \). Then, the following results hold:
(i) if \( V_{S_k}^T \mathcal{A} V_{S_k} \) is copositive for all \( k \in [r] \), then \( \mathcal{A} \) is copositive;
(ii) if \( V_{S_k}^T \mathcal{A} V_{S_k} \) is strictly copositive for all \( k \in [r] \), then \( \mathcal{A} \) is strictly copositive.
**Proof.** It is sufficient to prove (i), since (ii) can be verified similarly. For any \( k \in [r] \) and \( i_1, i_2, \ldots, i_m \in [n] \), we have that
\[
(V_{S_k}^T \mathcal{A} V_{S_k})_{i_1 i_2 \ldots i_m} = \sum_{j_1, j_2, \ldots, j_m \in [n]} (V_{S_k}^T)_{i_1 j_1} a_{j_1 j_2 \ldots j_m} (V_{S_k})_{j_1 j_2 \ldots j_m} (V_{S_k})_{j_m i_m}
= \sum_{j_1, j_2, \ldots, j_m \in [n]} a_{j_1 j_2 \ldots j_m} (u^k_{j_1})_{i_1} (u^k_{j_2})_{i_2} \cdots (u^k_{j_m})_{i_m}
= \langle \mathcal{A}, u^k_{i_1} \circ u^k_{i_2} \circ \cdots \circ u^k_{i_m} \rangle.
\]
For any $x \in S_0$, it follows that $x \in S_k$ for some $k \in [r]$ such that

$$x = \lambda_1 u_1^k + \lambda_2 u_2^k + \cdots + \lambda_n u_n^k, \quad \sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \geq 0, \quad \forall \, i \in [n].$$

Thus,

$$\mathcal{A}x^m = (\mathcal{A}, (\lambda_1 u_1^k + \lambda_2 u_2^k + \cdots + \lambda_n u_n^k)^m) = \sum_{i_1, i_2, \ldots, i_m \in [n]} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_m} (\mathcal{A}, u_{i_1}^k \circ u_{i_2}^k \circ \cdots \circ u_{i_m}^k) = \sum_{i_1, i_2, \ldots, i_m \in [n]} (V_{S_k}^T \mathcal{A} V_{S_k})_{i_1 i_2 \ldots i_m} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_m} = (V_{S_k}^T \mathcal{A} V_{S_k}) \lambda^m,$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n_+$, and the second equality is obtained from (5). By conditions and $\lambda \in \mathbb{R}^n_+$, it holds that $\mathcal{A}x^m \geq 0$ holds for any $x \in S_0$, and hence, the desired results follow. \hfill \square

To show the simplicial partition is fine enough, we will give a necessary condition for strictly copositivity of a tensor. For the standard simplex $S_0$ with a simplicial partition $\bar{S} = \{S_1, S_2, \ldots, S_r\}$, the vertices of simplex $S_k$ are denoted by $u_1^k, u_2^k, \ldots, u_n^k$ for any $k \in [r]$. Let $d(\bar{S})$ denote the maximum diameter of a simplex in $\bar{S}$:

$$d(\bar{S}) = \max_{k \in [r]} \max_{i, j \in [n]} \|u_i^k - u_j^k\|_2.$$

**Theorem 4.3** Let $\mathcal{A} \in S_{m,n}$ be a strictly copositive tensor. Then, there exists $\varepsilon > 0$ such that for all finite simplicial partitions $\bar{S} = \{S_1, S_2, \ldots, S_r\}$ of $S_0$ with $d(\bar{S}) < \varepsilon$, it follows that $(\mathcal{A}, u_{i_1}^k \circ u_{i_2}^k \circ \cdots \circ u_{i_m}^k) > 0$ for all $k \in [r], i_j \in [n], j \in [m]$, where $u_1^k, u_2^k, \ldots, u_n^k$ are vertices of the simplex $S_k$.

**Proof** First of all, we define the following function:

$$f(w) = \langle \mathcal{A}, x \circ y \circ \cdots \circ z \rangle, \quad \forall \, w := (x, y, \ldots, z) \in S_0 \times S_0 \times \cdots \times S_0 \in \mathbb{R}^{mn}.$$

For any $w_x := (x, x, \ldots, x) \in S_0 \times S_0 \times \cdots \times S_0$, the strictly copositivity of $\mathcal{A}$ implies that $f(w_x) > 0$. By continuity, it follows that, for any $x \in S_0$, there exists $\varepsilon_x > 0$ such that $f(w) > 0$ for all $w$ satisfying $\|w - w_x\|_2 \leq \varepsilon_x$. Let $\varepsilon = \min_{x \in S_0} \varepsilon_x > 0$. Then, it follows from uniformly continuity of $f$ that

$$f(w) > 0 \quad \text{for all } w \text{ satisfying } \|w - w_x\|_2 \leq \varepsilon \text{ for all } x \in S_0. \quad (6)$$
For any simplicial partition \( \tilde{S} = \{S_1, S_2, \ldots, S_r\} \) of \( S_0 \) with \( d(\tilde{S}) < \frac{1}{m} \epsilon \), it holds that
\[
\|u^k_i - u^k_j\|_2 \leq \frac{1}{m} \epsilon \quad \text{for all} \ k \in [r] \ 	ext{and} \ i, j \in [n].
\]
Moreover, for any \( x \in S_k \subseteq \tilde{S} \), it follows that, for any \( i \in [n] \),
\[
\|u^k_i - x\|_2 \leq \frac{1}{m} \epsilon
\]
which implies that \( \|(u^k_{i_1}, u^k_{i_2}, \ldots, u^k_{i_m}) - wx\|_2 \leq \epsilon \) for all \( i_j \in [n] \) with \( j \in [m] \). Combining this with (6), we obtain that
\[
f(u^k_{i_1}, u^k_{i_2}, \ldots, u^k_{i_m}) = \langle \mathcal{A}, u^k_{i_1} \circ u^k_{i_2} \circ \cdots \circ u^k_{i_m} \rangle > 0
\]
holds for all \( k \in [r], i_j \in [n], j \in [m] \), and hence, the desired results follow. \( \square \)

The following theorem gives a detection criterion for the case of a tensor being not copositive.

**Theorem 4.4** Given \( \mathcal{A} \in S_{m,n} \). Then, tensor \( \mathcal{A} \) is not copositive iff there exists \( \epsilon > 0 \) such that, for all simplicial partition \( \tilde{S} = \{S_1, S_2, \ldots, S_r\} \) of \( S_0 \) with \( d(\tilde{S}) < \epsilon \), there are at least one \( k \in [r] \) and one \( i \in [n] \) satisfying \( \mathcal{A}(u^k_i)^m < 0 \).

**Proof** Sufficiency is obvious. To prove necessity, we assume that \( \mathcal{A} \) is not copositive. Then, there is \( x \in S_0 \) such that \( \mathcal{A}x^m < 0 \). By continuity, there exists \( \epsilon > 0 \) such that
\[
\mathcal{A}y^m < 0 \quad \text{for all} \ y \text{ satisfying} \|y - x\|_2 < \epsilon.
\]

For any simplicial partition \( \tilde{S} = \{S_1, S_2, \ldots, S_r\} \) of \( S_0 \) with \( d(\tilde{S}) < \epsilon \), there is at least one \( k \in [r] \) such that \( x \in S_k \) with \( \|u^k_x - x\|_2 \leq d(\tilde{S}) < \epsilon \). Thus, it follows from (7) that \( \mathcal{A}(u^k_x)^m < 0 \), which implies that the desired results follow. \( \square \)

## 5 Detection Algorithm Based on Simplicial Partition

Based on the results obtained in the last section, we can develop an algorithm to verify whether a tensor is copositive or not, which is stated as follows.
Algorithm 5.1. Test whether a given symmetric tensor is copositive or not

**Input:** $\mathcal{A} \in S_{m,n}$

Set $\tilde{S} := \{S_1\}$ where $S_1 = \text{conv}\{e_1, e_2, \ldots, e_n\}$ is the standard simplex and $k := 1$

while $k \neq 0$

set $S := S_k = \text{conv}\{u_1, u_2, \ldots, u_n\} \in \tilde{S}$

if there exists $i \in [n]$ such that $\mathcal{A}u_i^m < 0$, then

return “$\mathcal{A}$ is not copositive”

else if $\langle \mathcal{A}, u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_m} \rangle \geq 0$ for all $i_1, i_2, \ldots, i_m \in [n]$, then

set $\tilde{S} := \tilde{S}\{S_k\}$ and $k := k - 1$

else

take $v = \frac{u_0 + u_n}{2}$, $[p, q] = \text{arg} \text{max}_{i,j \in [n]} \|u_i - u_j\|_2$ and $p < q$, and set

$S_k := \text{conv}\{u_1, \ldots, u_{p-1}, v, u_{p+1}, \ldots, u_n\}$;

$S_{k+1} := \text{conv}\{u_1, \ldots, u_{q-1}, v, u_{q+1}, \ldots, u_n\}$,

set $\tilde{S} := \tilde{S}\{S\} \cup \{S_k, S_{k+1}\}$ and $k := k + 1$

end if

end while

return “$\mathcal{A}$ is copositive.”

**Output:** “$\mathcal{A}$ is copositive” or “$\mathcal{A}$ is not copositive”.

From Theorems 4.3 and 4.4, it is easy to see that the following result holds.

**Theorem 5.1** In Algorithm 5.1, if the input symmetric tensor $\mathcal{A}$ is strictly copositive or $\mathcal{A}$ is not copositive, then the method will terminate in finitely many iterations.

Thus, it is clear that Algorithm 5.1 can capture all strictly copositive tensors and non-copositive tensors. Unfortunately, when $\mathcal{A}$ is copositive but not strictly copositive, it is possible that the partition procedure of the algorithm leads to $d(\tilde{S}) \rightarrow 0$, and in this case, the algorithm does not stop in general. This case also exists for the matrix detecting process [19, 20]. The reason for this is the following result.

**Proposition 5.1** Suppose $\mathcal{A} \in S_{m,n}$ is copositive. Let $S = \text{conv}\{u_1, u_2, \ldots, u_n\}$ be a simplex with $\mathcal{A}u_i^m > 0$ for all $i \in [n]$. If there exists $x \in S\{u_1, u_2, \ldots, u_n\}$ such that $\mathcal{A}x^m = 0$, then there are $i_1, i_2, \ldots, i_m \in [n]$ such that $\langle \mathcal{A}, u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_m} \rangle < 0$.

**Proof** We assume that $\langle \mathcal{A}, u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_m} \rangle \geq 0$ for all $i_1, i_2, \ldots, i_m \in [n]$. By the given conditions, there is $x \in S\{u_1, u_2, \ldots, u_n\}$ such that $\mathcal{A}x^m = 0$. It follows that there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+$ such that $x = \sum_{i=1}^n \lambda_i u_i$ and $\sum_{i=1}^n \lambda_i = 1$. Thus,

$$\mathcal{A}x^m = \sum_{i_1, i_2, \ldots, i_m \in [n]} \lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_m} \langle \mathcal{A}, u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_m} \rangle \geq \sum_{i=1}^n \lambda_i \mathcal{A}u_i^m > 0,$$

which contradicts the result $\mathcal{A}x^m = 0$. Therefore, the desired results follow.

For the given tensor $\mathcal{A}$ being copositive but not strictly copositive, we can first try to check the copositivity of $\mathcal{A}$ by Algorithm 5.1. If it terminates in finitely iterations, then we get a definitive answer, and if not, we test the tensor $\mathcal{B} = \mathcal{A} + \sigma E$, where $E$ is the tensor of all ones and $\sigma > 0$ is a small tolerance.

**Theorem 5.2** Given $\mathcal{A} \in S_{m,n}$, Then, $\mathcal{A}$ is copositive iff $\mathcal{B} = \mathcal{A} + \sigma E$ is strictly copositive for any $\sigma > 0$. 
Proof It is obvious that the necessary condition holds. For the sufficient statement, since \( \mathcal{B} \mathbf{x}^m = (\mathcal{A} + \sigma \mathcal{E}) \mathbf{x}^m = \mathcal{A} \mathbf{x}^m + \sigma > 0 \) for all \( \mathbf{x} \in S_0 \) and \( \sigma > 0 \), by letting \( \sigma \to 0 \), we can obtain the desired result. \( \square \)

From Theorems 5.1 and 5.2, the following conclusion holds.

Corollary 5.1 If the given tensor \( \mathcal{A} \) is copositive but not strictly copositive, by replacing \( \mathcal{A} \) by \( \mathcal{B} = \mathcal{A} + \sigma \mathcal{E} \) for some \( \sigma > 0 \), then Algorithm 5.1 terminates in finitely many iterations.

For \( \sigma > 0 \), we call the symmetric tensor \( \mathcal{A} \) a \( \sigma \)-copositive tensor with respect to simplex \( S_0 \) if \( \mathcal{A} \mathbf{x}^m \geq -\sigma \) for all \( \mathbf{x} \in S_0 \). And we have the following conclusion.

Proposition 5.2 Let \( \mathcal{A} \in S_{m,n} \), \( S = \text{conv}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} \) be a simplex and \( \sigma > 0 \). If \( \langle \mathcal{A}, \mathbf{u}_{i_1} \circ \mathbf{u}_{i_2} \circ \cdots \circ \mathbf{u}_{i_m} \rangle \geq -\sigma \) for all \( i_1, i_2, \ldots, i_m \in [n] \), then \( \mathcal{A} \mathbf{x}^m \geq -\sigma \) for any \( \mathbf{x} \in S \).

Proof For any \( \mathbf{x} \in S \), there is \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}_+ \) such that \( \mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{u}_i \) and \( \sum_{i=1}^n \lambda_i = 1 \). Thus,

\[
\mathcal{A} \mathbf{x}^m = \langle \mathcal{A}, (\lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n)^m \rangle = \sum_{i_1, i_2, \ldots, i_m \in [n]} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_m} \langle \mathcal{A}, \mathbf{u}_{i_1} \circ \mathbf{u}_{i_2} \circ \cdots \circ \mathbf{u}_{i_m} \rangle \geq -\sigma,
\]

which implies that the desired result holds. \( \square \)

6 Numerical Examples

In this section, we use Algorithm 5.1 to detect whether a tensor is copositive or not. All experiments are finished in Matlab2014b on a Philips desktop computer with 4 GB of RAM. We detect several classes of tensors from four aspects, which are given in the following four parts, respectively.

Part I Suppose that \( \mathcal{B} \in S_{m,n} \) is a nonnegative tensor and \( \rho(\mathcal{B}) \) denotes its spectral radius. Let \( \mathcal{I} \in S_{m,n} \) denote the identity tensor. Then, by Definition 3.1 and Theorem 3.12 of [2], it follows that the tensor \( \eta \mathcal{I} - \mathcal{B} \) is copositive iff \( \eta \geq \rho(\mathcal{B}) \), and the tensor \( \eta \mathcal{I} - \mathcal{B} \) is strictly copositive iff \( \eta > \rho(\mathcal{B}) \). Based on these results, we construct several tensors for testing.

Example 6.1 Suppose \( \mathcal{A} \in S_{3,3} \) (or \( \mathcal{A} \in S_{4,4} \)) is given by \( \mathcal{A} = \eta \mathcal{I} - \mathcal{B} \), where \( \mathcal{B} \in S_{3,3} \) (or \( \mathcal{B} \in S_{4,4} \)) is a tensor of all ones and \( \eta \) is specified in Table 1.

The numerical results are given in Table 1, where “\( \rho \)” denotes the spectral radius of the tensor \( \mathcal{B} \), “IT” denotes the number of iterations, “CPU(s)” denotes the CPU time in seconds, and “Result” denotes the output result in which “No” denotes the output result that the tested tensor is not copositive and “Yes” denotes the output result that the tested tensor is copositive.

Example 6.2 Suppose that \( \mathcal{A} \in S_{m,n} \) is given by \( \mathcal{A} = \eta \mathcal{I} - \mathcal{B} \), where \( \mathcal{B} \in S_{m,n} \) is randomly generated with all its elements in the interval \((0, 1)\).
In our experiments, we use the higher-order power method to compute the spectral radius $\rho$ of every tensor $\mathbf{B}$. For the same $m$ and $n$, we generate randomly every tested problem 10 times, and the numerical results are shown in Table 2, where for every tested problem, “MinIT” and “MaxIT” denote the minimal number and the maximal number of iterations among ten times experiments, respectively; “MinCPU(s)” and “MaxCPU(s)” denote the smallest and the largest CPU times in second among ten times experiments, respectively. “Nyes” (“Nno”) denotes the number of the output results that the tested tensors are copositive (not copositive).

Part 2 It is obvious that any nonnegative tensor is copositive. By Corollary 6.1 of [15], we also know that for any $\mathbf{A} \in S_{m,n}$, if $\mathbf{A}$ is (strictly) copositive, then $(a_{ii\ldots i} > 0)$ $a_{ii\ldots i} \geq 0$ for all $i \in [n]$. Based on these results, we detect the following example.

Example 6.3 (i) Consider the tensor $\mathbf{A} \in S_{m,n}$ which is randomly generated with all its elements are in the interval $(0, 1)$; (ii) we set $\mathbf{B} := \mathbf{A}$ and $b_{11\ldots 1} = -1$.

In our experiments, for the same $m$ and $n$, we generate randomly every tested problem 10 times, the numerical results are shown in Table 3, in which we use the same notations as those used in Table 2.

Part 3 It is well known that there is a one-to-one relationship between the homogeneous polynomial and the symmetric tensor. In this part, we consider several tensors.
which come from several famous homogeneous polynomials. For convenience, we use the following notation: for any $i_1, i_2, \ldots, i_m \in [n]$, we use $\pi(i_1i_2\ldots i_m)$ to denote a permutation of $i_1i_2\ldots i_m$, and $S_{\pi(i_1i_2\ldots i_m)}$ to denote the set of all these permutations.

**Example 6.4** Suppose that $A \in S_{6,3}$ is given by

$$
\sum_{i_1i_2i_3i_4i_5i_6 \in S_{\pi(111122)}} a_{i_1i_2i_3i_4i_5i_6} = 1, \quad \sum_{i_1i_2i_3i_4i_5i_6 \in S_{\pi(122222)}} a_{i_1i_2i_3i_4i_5i_6} = 1, \quad a_{333333} = 1, \quad \sum_{i_1i_2i_3i_4i_5i_6 \in S_{\pi(112223)}} a_{i_1i_2i_3i_4i_5i_6} = -3,
$$

The homogeneous polynomial defined by the tensor $A$ is the famous Motzkin polynomial, which is nonnegative but not a sum of squares, and hence, the tensor $A$ is copositive. It is easy to see that the tensor $A$ is not strictly copositive. We use Algorithm 5.1 to test this tensor, and the algorithm does not terminate within 100 iterations. We use Algorithm 5.1 to test the tensor $A + \sigma B$ with $\sigma > 0$; however, the algorithm can correctly detect the copositivity of the tensor. For example, when $\sigma = 0.001$, the algorithm can correctly detect the copositivity of the tensor with 27 iterations in 0.874 s, and when $\sigma = 0.0001$, the algorithm can correctly detect the copositivity of the tensor with 71 iterations in 2.25 s.

**Example 6.5** Suppose that $A \in S_{6,3}$ is given by

$$
a_{111111} = 1, \quad a_{222222} = 1, \quad a_{333333} = 1, \quad \sum_{i_1i_2i_3i_4i_5i_6 \in S_{\pi(111122)}} a_{i_1i_2i_3i_4i_5i_6} = -1, \quad \sum_{i_1i_2i_3i_4i_5i_6 \in S_{\pi(122222)}} a_{i_1i_2i_3i_4i_5i_6} = -1, \quad \sum_{i_1i_2i_3i_4i_5i_6 \in S_{\pi(111133)}} a_{i_1i_2i_3i_4i_5i_6} = -1, \quad \sum_{i_1i_2i_3i_4i_5i_6 \in S_{\pi(222223)}} a_{i_1i_2i_3i_4i_5i_6} = -1, \quad \sum_{i_1i_2i_3i_4i_5i_6 \in S_{\pi(222233)}} a_{i_1i_2i_3i_4i_5i_6} = -1, \quad \sum_{i_1i_2i_3i_4i_5i_6 \in S_{\pi(222233)}} a_{i_1i_2i_3i_4i_5i_6} = 3,
$$

The homogeneous polynomial defined by the tensor $A$ is the famous Robinson polynomial, which is nonnegative but not a sum of squares, and hence, the tensor $A$ is copositive. It is easy to see that the tensor $A$ is not strictly copositive. We use Algorithm 5.1 to test this tensor; the algorithm does not terminate within 100 iterations. We use Algorithm 5.1 to test the tensor $A + \sigma B$ with $\sigma > 0$; however, the algorithm can correctly detect the copositivity of the tensor. For example, when $\sigma = 0.001$, the algorithm can correctly detect the copositivity of the tensor with 27 iterations in 0.842 s, and when $\sigma = 0.0001$, the algorithm can correctly detect the copositivity of the tensor with 83 iterations in 2.5 s.
Example 6.6 Suppose that $\mathcal{A} \in S_{6,3}$ is given by
\[
\begin{align*}
\sum_{i_1i_2i_3i_4i_5i_6 \in S_{(111222)}} a_{i_1i_2i_3i_4i_5i_6} &= 1, \\
\sum_{i_1i_2i_3i_4i_5i_6 \in S_{(222233)}} a_{i_1i_2i_3i_4i_5i_6} &= 1, \\
\sum_{i_1i_2i_3i_4i_5i_6 \in S_{(333311)}} a_{i_1i_2i_3i_4i_5i_6} &= 1, \\
\sum_{i_1i_2i_3i_4i_5i_6 \in S_{(112233)}} a_{i_1i_2i_3i_4i_5i_6} &= -3,
\end{align*}
\]

The homogeneous polynomial defined by the tensor $\mathcal{A}$ is the famous Choi-Lam polynomial, which is nonnegative but not a sum of squares, and hence, the tensor $\mathcal{A}$ is copositive. It is easy to see that the tensor $\mathcal{A}$ is not strictly copositive. We use Algorithm 5.1 to test this tensor; the algorithm does not terminate within 100 iterations. We use Algorithm 5.1 to test the tensor $\mathcal{A} + \sigma \mathcal{E}$ with $\sigma > 0$; however, the algorithm can correctly detect the copositivity of the tensor. For example, when $\sigma = 0.001$, the algorithm can correctly detect the copositivity of the tensor in 0.858 s, and when $\sigma = 0.0001$, the algorithm can correctly detect the copositivity of the tensor with 41 iterations in 1.29 s.

Part 4 Given positive integers $m, n \geq 2$ and a vector $v = (v_0, v_1, \ldots, v_{m(n-1)})^\top$. An $m$th-order $n$-dimensional tensor $\mathcal{A}$ is called a Hankel tensor if its entries satisfy
\[
a_{i_1i_2\ldots i_m} = v_{i_1+i_2+\ldots+i_m-m} \quad \text{for all } i_j \in [n] \text{ and } j \in [m].
\]
The vector $v$ is called the generating vector of the Hankel tensor $\mathcal{A}$. It is known that Hankel tensors appear in many engineering problems such as signal processing, automatic control and geophysics, which have been studied in recent years (see, for example, [25, 26] and references therein). In this part, we detect the copositivity of Hankel tensors with their generating vectors being given in the following way:
\[
v = (1, \alpha, \alpha^2, \ldots, \alpha^{m(n-1)})^\top + \beta \ast \text{ones}(m(n-1), 1) \quad \text{where } \alpha, \beta \in \mathbb{R}.
\]
It is easy to see that the Hankel tensor $\mathcal{A}$ is strictly copositive if (a) $m$ is an even number, $\alpha \neq 0$ and $\beta \geq 0$, or (b) $\alpha > 0$ and $\beta \geq 0$, and the Hankel tensor $\mathcal{A}$ is not copositive when $m$ is an odd number, $\alpha < 0$ and $\beta \leq 0$. It is possible that the Hankel tensor $\mathcal{A}$ is not copositive when $m$ is an even number, $\alpha < 0$ and $\beta < 0$.

In our numerical experiments, we use the fast Fourier transform to reduce the computational complexity of the algorithm. The numerical results are listed in Table 4.

| $\alpha$ | $\beta$ | $m$ | $n$ | Num | IT  | CPU(s) | Result |
|---------|--------|----|----|-----|-----|--------|--------|
| −0.999  | 0.5    | 10 | 9  | 3486784401 | 16755 | 3940 | Yes |
|         | 8      | 1073741824 | 2915 | 303 | Yes |
|         | 7      | 282475249 | 669 | 281 | Yes |
| 0.999   | 0.0    | 4  | 300 | 8100000000 | 1 | 3940 | Yes |
| −0.999  | −0.5   | 4  | 500 | 625000000000 | 1 | 0.0312 | No |
| 0.0     | 3      | 1000 | 10000000000 | 1 | 0.0 | No |

Table 4 The numerical results of the problem in Part 4
where “Num” denotes the number of nonzero entries of the tested tensor $\mathcal{A}$, and the symbols “IT”, “CPU(s)” and “Result” are the same as those in Table 1.

From the numerical results given in Part 1–Part 4, we can see that Algorithm 6.1 is effective for the problems we tested.

## 7 Conclusions

In this paper, we proposed new criteria to judge whether a tensor is (strictly) copositive or not, and several necessary conditions or sufficient conditions which are investigated by taking advantage of the simplicial partition. Moreover, by the obtained criteria based on the simplicial partition, we proposed a detection algorithm for the copositive tensor. The preliminary numerical results demonstrate that the proposed algorithm is effective. However, there are some more questions need to be answered in the future. For Theorem 3.3, is this condition still sufficient in the higher-order tensor case? For the proposed algorithm, is it possible to upper bound the iterations in theory, if the algorithm really stops in finitely many iterations?

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