Inflationary cosmology with nonlinear dispersion relations

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(Dated: May 11, 2014)

We present a technique, the uniform asymptotic approximation, to construct accurate analytical solutions of the linear perturbations of inflation after quantum effects of the early universe are taken into account, for which the dispersion relations generically become nonlinear. We construct explicitly the error bounds associated with the approximations and then study them in detail. With the understanding of the errors and the proper choice of the Liouville transformations of the differential equations of the perturbations, we show that the analytical solutions describe the exact evolution of the linear perturbations extremely well even only in the first-order approximations. As an application of the approximate analytical solutions, we calculate the power spectra and indices of scalar and tensor perturbations in the slow-roll inflation, and find that the amplitudes of the power spectra get modified due to the quantum effects, while the power spectrum indices remain the same as in the linear case.

I. INTRODUCTION

The inflationary cosmology [1], aside from solving several fundamental and conceptual problems, such as flatness, horizon, and exotic relics, of the standard big bang cosmology, also provides a causal mechanism for generating structures in the universe and the spectrum of cosmic microwave background (CMB) anisotropies [2]. The large scale structure and the CMB anisotropies in the universe arose from the gravitational collapse of adiabatic, Gaussian, and nearly scale-invariant primordial fluctuations of space-time, a key prediction of inflation models. These are now matched to observations with unprecedented precision [3], especially after the recent release of the more precise cosmological results from the Planck satellite [4].

Inflationary scenario is based on the assumption that there is a period in the very early universe during which the scale of space is expanding at an exponentially growing rate. Consequently, the wavelengths corresponding to the present large scale structure in the Universe and to the measured CMB anisotropies, were exponentially stretched during inflation. However, it was found that, if the inflationary period is sufficiently long to consistent with observations, the physical wavelength of fluctuations observed at the present time may well originate with a wavelength smaller than the Planck length at the beginning of the inflation — the trans-Planckian issues [5]. Thus, questions immediately arise as to whether the usual predictions of the scenario still remain robust, due to the ignorance of physics in such a small scale, and more interestingly, whether they could leave imprints for future observations, even the usual predictions are indeed robust. Such considerations have attracted a great deal of attention in understanding the quantum effects of the early universe on the inflation and cosmological perturbations [6, 7].

So far, various approaches have been proposed to study the aforementioned effects [8]. One is to choose a different initial state rather than that of the Bunch-Davies vacuum [9]. This corresponds to introducing a time-like new physics hypersurface on which an ad hoc initial state is set. In this perspective, the quantum effects have simply been shifted to modifying the initial states. Another approach is the boundary effective field theory [7]. Its main idea was to integrate out the high energy physics and then derive an effective theory which only involves observable scales. However, the resulting effects from these two approaches are controlled by the initial time of inflation rather than the redshift where Planck physics is relevant. This implies a priori assumption that the quantum effectors of the early universe do not change the existing predictions, except for some suppressed corrections.

Instead of specifying ad hoc initial states, another natural option is to replace the conventional linear dispersion relation by a nonlinear one. This approach was initially applied to inflationary cosmology as a toy model [6], motivated from the studies of the dependence of black hole radiation on Planck scale physics [10]. Later, it was found [11] that it could be naturally realized in the framework of the Horava-Lifshitz gravity [12, 13], a candidate of the ultraviolet complete theory of quantum gravity. The perturbations (of scalar, vector or tensor fields) produced during the inflationary epoch are described by [8],

$$\mu_k''(\eta) + \left( \frac{\omega_k'({\eta}) - \frac{z''}{z}}{z} \right) \mu_k(\eta) = 0, \quad (1.1)$$

where $\mu_k(\eta)$ denotes the mode function, a prime the dif-
where $y$ depends on the background and the types of perturbations, scalar, vector or tensor. The modified dispersion relation $\omega^2_k(\eta)$ takes the form,

$$\omega^2_k(\eta) = k^2 \left[ 1 - b_1 \left( \frac{k}{aM_*} \right)^2 + b_2 \left( \frac{k}{aM_*} \right)^4 \right], \quad (1.2)$$

where $M_*$ is the relevant energy scale of the trans-Planckian physics, $k$ is the comoving wavenumber of the mode, $b_1$ and $b_2$ are dimensionless constants, and in order to get a healthy ultraviolet limit, one requires $b_2 > 0$. When $b_i = 0$, it reduces to that of general relativity (GR).

In the slow-roll inflation, we have $a(\eta) \simeq - (1 - \varepsilon) / (\eta H)$, with $H$ and $\varepsilon \equiv - H^2 / H^2$ being, respectively, the Hubble and slow-roll parameters, and $H \equiv dH / dt = H'/a$. Then, we find that,

$$\frac{d^2 \mu_k(y)}{dy^2} + \left[ \omega^2_k(\eta) - \frac{\nu^2(\eta) - 1/4}{y^2} \right] \mu_k(y) = 0, \quad (1.3)$$

where $y \equiv -k\eta$ and

$$\omega^2_k(\eta) \equiv \frac{\omega^2_k(\eta)}{k^2} = 1 - b_1\varepsilon^2 y^2 + b_2\varepsilon^4 y^4, \quad (1.4)$$

with $\epsilon_* \equiv H / M_*$, $b_1 \equiv (1 + 2\varepsilon)b_1$, $b_2 \equiv (1 + 4\varepsilon)b_2$, and

$$\frac{z''}{z} \equiv \frac{\nu^2(\eta) - 1/4}{\eta^2}. \quad (1.5)$$

To the first-order approximations of the slow-roll inflation, one can treat $\nu(\eta)$ and the parameters $b_1$ and $b_2$ as constants [8].

Obtaining analytically approximate solutions of Eq. (1.3) with proper initial conditions is one of the crucial steps in the understanding of quantum effects on inflation and cosmological perturbations, and has been intensively investigated in the past decade [8]. These studies were carried out mainly by using the Brandenberger-Martin method, in which the evolution of $\mu_k(\eta)$ is divided into several epochs, and in each of them the solution can be obtained either by the WKB approximations when the adiabatic condition

$$\frac{3\omega_k^2}{4\omega_k^4} - \frac{\omega_k''}{2\omega_k^3} \ll 1, \quad (1.6)$$

is satisfied, or by the linear combination of the exponentially decaying and growing modes, otherwise. Then, the individual solutions were matched together at the boundaries. While this often yields reasonable analytical approximations, its validity in various physical situations has been questioned recently [14, 15], and shown that it is valid only when the comoving wavenumber $k \gg aH$. In addition, in this method the error bounds are not known, and can be obtained only by comparing it with the numerical (exact) evolution of the mode function in the case-by-case basis. However, with the arrival of the precision era of cosmological measurements, accurate calculations of cosmological variables are highly demanded [10].

In this paper, we propose another method — the uniform asymptotic approximation, to construct accurate analytical solutions of the mode function $\mu_k(y)$, and present explicitly the error bounds for the error terms associated with the approximations. It is precisely because of the understanding and control of the errors that such constructed approximate solutions describe the exact evolutions of the perturbations extremely well, even only to the first-order approximations. As an application of the approximate analytical solutions, we calculate the power spectra and indices of scalar and tensor perturbations in the slow-roll inflation, and find that the amplitudes of the power spectra get modified due to the quantum effects, while the power spectrum indices remain the same as in the linear case. Specifically, the paper is organized as follows: In Section II, using the Liouville transformations we first write Eq. (1.3) in a proper form. Then, applying the uniform asymptotic approximations we construct approximate solutions and the corresponding error bounds, and study them in details. Thanks to the understanding of the error bounds, the Liouville transformations are chosen so that the errors are minimized, and the analytical solutions describe the exact evolution of the perturbations extremely well, as one can see from Figs. 2-4. In Section III, the particular case $|\xi_0|^2 \gg 1$ is considered. In Section IV, as a simple application of the approximate analytical solutions constructed in Section III and IV, we calculate the power spectra and indices of scalar and tensor perturbations, by paying particular attention on the modifications of them due to the quantum effects. In Section V, we present our main conclusions. Five appendices are also included, in which some useful mathematical expressions and detailed calculations of quantities involved in the context are given.

It should be noted that a relevant consideration was previously carried out by Habib et al [17] for the inflationary universe in GR, in which the dispersion relation is linear, given by Eq. (1.2) with $b_i = 0$. Clearly, their treatment is applicable only to the case where the function $g(\eta)$ defined in Eq. (2.33) has only one single turning point (or a single root of the equation $g(\eta) = 0$). It cannot be applied to the more interesting cases with several turning points, in particular, to those where some turning points may be double, triple or even higher-multiple roots. The method to be proposed in this paper is to treat all the above cases in a unified way, which is mathematically quite different from that of [17], and reduces to it when $b_i = 0$. Part of the results was reported in [18].
II. LIOUVILLE TRANSFORMATIONS AND APPROXIMATE ANALYTICAL SOLUTIONS

Following [19][21], we first write Eq.(1.3) in the form
\[ \frac{d^2\mu_k(y)}{dy^2} = [g(y) + q(y)]\mu_k(y), \quad (2.1) \]
where
\[ g(y) + q(y) = \frac{\nu^2 - 1/4}{y^2} - 1 + b_1\epsilon_2^a y^2 - b_2\epsilon_2^a y^4, \quad (2.2) \]
and the functions \( g(y) \) and \( q(y) \) will be determined by the analysis of the error bounds given below, so that the associated errors will be minimized. Clearly, \( g(y) \) and \( q(y) \) in general have two poles, one is at \( y = 0^+ \) and the other is at \( y = +\infty \). In addition, the function \( q(y) \) can vanish at various points, which are called the turning points or zeros, according to the terminology used in [19][21]. From the theory of the second-order differential equations, the asymptotic solutions of the mode function \( \mu_k(y) \) depend on the behavior of functions \( g(y) \) and \( q(y) \) around the poles and turning points. In the following, we present a technique for determining the analytic approximate solutions around them in a unified way.

To such goals, let us first introduce the Liouville transformations with two new variables \( U(\xi) \) and \( \xi \),
\[ U(\xi) = \chi^{1/4}\mu_k(y), \quad \xi^2 = \left| g(y) \right| \frac{f^{(1)}(\xi)^2}{\psi(\xi)^2}, \quad (2.3) \]
where \( \chi \equiv \xi^2 \), \( \xi' = d\xi/dy \), and
\[ f^{(1)}(\xi) = \int y \sqrt{|g(y)|}dy, \quad f^{(1)}(\xi) = \frac{d^2f(\xi)}{d\xi^2}. \quad (2.4) \]
Note that \( \chi \) must be regular and not vanish in the intervals of interest. Consequently, \( f(\xi) \) must be chosen so that \( f^{(1)}(\xi) \) has zeros and singularities of the same type as that of \( g(y) \). As shown below, such requirements play an essential role in determining the approximate solutions. In terms of \( U \) and \( \xi \), Eq. (2.1) takes the form
\[ \frac{d^2U}{d\xi^2} = \left( \pm f^{(1)}(\xi)^2 + \psi(\xi) \right) U, \quad (2.5) \]
where
\[ \psi(\xi) = \frac{g(y)}{\chi^2} - \chi^{-3/4} \frac{d^2(\chi^{-1/4})}{dy^2}, \quad (2.6) \]
and the signs “±” correspond to \( g(y) > 0 \) and \( g(y) < 0 \), respectively. Considering \( \psi(\xi) = 0 \) as the first-order approximation, one can choose \( f^{(1)}(\xi) \) so that the first-order approximation can be as close to the exact solutions as possible with the guidelines of the error functions constructed below, and then solve it in terms of known functions. Clearly, such a choice sensitively depends on the behaviors of the functions \( g(y) \) and \( q(y) \) near the poles and turning points.

A. Liouville-Green Solutions Near Two Poles

From Eq. (2.2), we can see that, except exactly at the two poles, located at \( y = 0^+ \) and \( y = +\infty \), respectively, the functions \( g(y) \) and \( q(y) \) are well-defined in their neighborhoods. With this property, we can choose
\[ f^{(1)}(\xi)^2 = \text{const.} \quad (2.7) \]
Without loss of the generality, we can always set this constant to one. Then, from Eq. (2.3) we find
\[ \xi = \int y \sqrt{\pm g(y)}dy, \quad (2.8) \]
here “+” (“-”) corresponds to the pole \( y = 0^+ \) (\( y = +\infty \)), and the equation of motion (2.1) takes the form
\[ \frac{d^2U}{d\xi^2} = \pm 1 + \psi(\xi) U. \quad (2.9) \]

Let us first consider the approximate solution near the pole \( y = 0^+ \). As the first-order approximation, neglecting the \( \psi(\xi) \) term, from Eq. (2.9) we find
\[ U^+ = c_+ e^{\xi}(1 + \epsilon_1^+), \quad d_+ e^{-\xi}(1 + \epsilon_2^+), \quad (2.10) \]
where \( \epsilon_1^+ \) and \( \epsilon_2^+ \) represent the errors of the approximate solution. Accordingly, the mode function \( \mu_k(y) \) is given by the Liouville-Green (LG) solution
\[ \mu_k^+(y) = \frac{c_+}{g(y)^{1/4}} e^{\int y \sqrt{g(y)}dy(1 + \epsilon_1^+)} + \frac{d_+}{g(y)^{1/4}} e^{-\int y \sqrt{g(y)}dy(1 + \epsilon_2^+)}. \quad (2.11) \]

Similarly, near the pole \( y = +\infty \), Eq. (2.9) has the solution
\[ U^- = c_- e^{i\xi}(1 + \epsilon_1^-) + d_- e^{-i\xi}(1 + \epsilon_2^-), \quad (2.12) \]
where \( \epsilon_1^- \) and \( \epsilon_2^- \) represent the errors of the approximations, and the mode function \( \mu_k(y) \) takes the form
\[ \mu_k^-(y) = \frac{c_-}{(-g(y))^{1/4}} e^{i\int y \sqrt{-g(y)}dy(1 + \epsilon_1^-)} + \frac{d_-}{(-g(y))^{1/4}} e^{-i\int y \sqrt{-g(y)}dy(1 + \epsilon_2^-)}. \quad (2.13) \]

1. Error bounds of the LG solutions

Obviously, the accuracy of the approximate solutions (2.10) and (2.12) depends on the magnitude of \( \psi(\xi) \), which was neglected when solving Eq. (2.9) to the first-order approximations. With the choice of Eq. (2.7), we find
\[ \psi(\xi) = \frac{q(y)}{|g(y)|} - \frac{1}{|g(y)|^{3/4}} \frac{d^2}{dy^2} \left( \frac{1}{|g(y)|^{1/4}} \right). \quad (2.14) \]
Roughly speaking, the resulting approximations are meaningful when $\psi(\xi)$ is much smaller than one, which shall be true, provided that

$$
\left| \frac{q(y)}{g(y)} \right| < 1, \quad \left| q(g^{-3/4} \frac{d^2(g^{-1/4})}{dy^2}) \right| < 1. \quad (2.15)
$$

These two conditions give rise to strong constraints on the choice of the functions $g(y)$ and $q(y)$. To see these, we first consider the error $\epsilon_1^+$. Substituting the first branch of $U^+ = \epsilon^2(1 + \epsilon_1^+)$ into Eq. (2.14), we obtain a second-order differential equation for $\epsilon_1^+$,

$$
d^2\epsilon_1^+ + 2 d\epsilon_1^+ - \psi(\xi)(1 + \epsilon_1^+) = 0. \quad (2.16)
$$

By using the variations of parameters we find that the error term $\epsilon_1^+$ has the solution

$$
\epsilon_1^+ = \frac{1}{2} \int_0^\xi \left( 1 - e^{2(v-\xi)} \right) \psi(v)(1 + \epsilon_1^+) dv, \quad (2.17)
$$

where $\xi \in (0, a_1)$, $v \in (0, \xi)$, and $y \in (0^{+}, \bar{a}_1)$ with $x_1, \bar{a}_1$ are the upper bounds of variable $\xi$ and $y$, respectively, and $\xi(0^+) = 0, \xi(\bar{a}_1) = a_1$. Comparing Eq. (2.16) with Eq. (A.1), we obtain

$$
K(\xi, v) = \frac{1}{2} \left( 1 - e^{2(v-\xi)} \right), \quad J(v) = 1,
$$

$$
\phi(v) = \phi_0(v) = \psi(v), \quad \psi_1(v) = 0. \quad (2.18)
$$

Considering $0 < v \leq \xi$, we find

$$
\left| K(\xi, v) \right| \leq \frac{1}{2} \left| \frac{\partial K(\xi, v)}{\partial \xi} \right| \leq 1. \quad (2.19)
$$

Hence, we obtain

$$
P_0(\xi) = \frac{1}{2}, \quad Q(v) = 1, \quad P_1(\xi) = 1, \quad (2.20)
$$

where $P_i$ and $Q$ are all defined in Appendix A. Thus, applying the theorems presented in Appendix A, the error bounds for $\epsilon_1^+$ read

$$
|\epsilon_1^+|, \quad \frac{|d\epsilon_1^+|}{2|g|^{1/2}} \leq \exp \left( \frac{1}{2} \psi_0(y,F) \right) - 1, \quad (2.21)
$$

where $F(y)$ is the error control function, defined as

$$
F(y) = \int |\psi(v)| dv = \int \left[ \frac{1}{|g|^{1/2}} \frac{d^2}{dy^2} \left( \frac{1}{|g|^{1/2}} \right) - \frac{q}{|g|^{1/2}} \right] dy, \quad (2.22)
$$

and $\psi_{x_1,x_2}(F)$ is defined as

$$
\psi_{x_1,x_2}(F) = \int_{x_1}^{x_2} \left| \frac{dF(y)}{dy} \right| dy, \quad (2.23)
$$

representing the total variation of the function $F(y)$.

Similarly, it can be shown that the error bounds for the error $\epsilon_2^+$ are given by

$$
|\epsilon_2^+|, \quad \frac{|d\epsilon_2^+|}{2|g|^{1/2}} \leq \exp \left( \frac{1}{2} \psi_0(y,F) \right) - 1. \quad (2.24)
$$

Repeating the above analysis for the error bounds near the pole $y = +\infty$, we find that the error bounds for $\epsilon_1, \epsilon_2$ take the forms

$$
|\epsilon_1|, \quad \frac{|d\epsilon_1|}{2|g|^{1/2}} \leq \exp (\psi_{y_0, +\infty}(F) - 1, \quad (2.25)
$$

$$
|\epsilon_2|, \quad \frac{|d\epsilon_2|}{2|g|^{1/2}} \leq \exp (\psi_{y_0, +\infty}(F) - 1, \quad (2.25)
$$

where $\bar{a}_2$ is the lower bound of $y$, i.e., $y \in (\bar{a}_2, +\infty)$.

2. Choice of Functions $g(y)$ and $q(y)$

At the pole $y = 0^+$, as pointed out in [19], the LG approximations are valid only when the function $g(y)$ has a pole of order $m \geq 2$. Assuming that $q(y)$ also has a pole at this point, but of order $n$, then we can expand them in the form,

$$
g(y) = \frac{1}{y^m} \sum_{s=0}^{\infty} y_s y^s, \quad q(y) = \frac{1}{y^n} \sum_{s=0}^{\infty} y_s y^s. \quad (2.26)
$$

When $m > 2$, from Eq. (2.22) we can see that we must have $n = m$, and consequently $q_0 = -q_0$. Thus, near the pole $y = 0^+$, the terms $g_0/y^m$ and $q_0/y^m$ dominate, and the condition $|q(y)| \ll |g(y)|$ [cf. Eq. (2.15)] is violated. Therefore, we must set $m = 2$, for which the condition $|q(y)| \ll |g(y)|$ requires $n \leq 2$. Then, substituting Eq. (2.20) into the integrand of the error control function $F(y)$, we obtain

$$
\frac{1}{|g|^{1/2}} \frac{d^2}{dy^2} \left( \frac{1}{|g|^{1/2}} \right) - \frac{q}{|g|^{1/2}} \approx \frac{1}{y} \left[ 4 |g_0|^{1/2} \right]
$$

$$
+ \frac{49}{8} \frac{q_0}{|g_0|^{3/2}} y + O(y^2) \right] - \frac{1}{y^{n-1}} \left[ \frac{q_0}{|g_0|^{1/2}} \right]
$$

$$
+ \left( q_1 y_0^{1/2} - \frac{|q_1| y_0^{3/2}}{2|g_0|^{3/2}} \right) y + O(y^2). \quad (2.27)
$$

Clearly, the convergence of the error control function $F(y)$ leads to

$$
n = 2, \quad q_0 = -\frac{1}{4}. \quad (2.28)
$$

On the other hand, the terms $q_1/y$ and $g_1/y$ violate the condition $|q(y)| \ll |g(y)|$, unless they vanish identically,

$$
q_1 = 0 = -g_1. \quad (2.29)
$$

Let us now turn to the pole $y = +\infty$. Assuming that $g(y)$ and $q(y)$ have a pole of order $\tilde{m}$ and $\tilde{n}$, respectively, we find that they can be expanded into the forms,

$$
g(y) = y^{\bar{m}} \sum_{s=0}^{\infty} y s y^{s}, \quad q(y) = y^{\bar{n}} \sum_{s=0}^{\infty} y_s y^{-s}. \quad (2.30)
$$
where $\bar{g}_s$ and $\bar{s}_s$ are other sets of constants. Substituting the above expansions into the integrand of the error control function $F(y)$, we find that

$$
\frac{1}{|g|^{1/4}} \frac{d^2}{dy^2} \left( \frac{1}{|g|^{1/4}} - \frac{q}{|g|^{1/2}} \right) = y^{-\hat{m}/2} \sum_{s=0}^{+\infty} c_s^{(1)} y^{-s} + y^{\hat{n}-\hat{m}/2} \sum_{s=0}^{+\infty} c_s^{(2)} y^{-s},
$$

(2.31)

where the coefficients $c_s^{(1)}$ and $c_s^{(2)}$ are functions of $\bar{q}_s$ and $\bar{s}_s$. Then, the convergence of $F(y)$ requires

$$
\hat{m} > -2, \quad \hat{n} < \frac{\hat{m}}{2} - 1.
$$

When $\hat{m} > 4$ or $\hat{m} < 4$, from Eq. (2.29) one finds that $\hat{n}$ must be taken either $\hat{n} = \hat{m}$ or $\hat{n} = -4$. But, these shall violate the above conditions just mentioned. Therefore, we must choose $\hat{m} = 4$, for which $\hat{n}$ must be less than 1, that is,

$$
\hat{m} = 4, \quad \hat{n} < 1.
$$

(2.32)

Combining the conditions given by Eqs. (2.28), (2.29) and (2.32), we find that the function $g(y)$ must take the form, $g(y) = -1/(4y^2) + q_2$ with $q_2 + q_2 = -1$ and $|q_2| \ll |g_2|$. Without loss of the generality, we can always set $q_2 = 0$, so finally we obtain

$$
g(y) = \frac{\nu^2}{y^2} - 1 + b_1\epsilon^2 y^2 - b_2\epsilon^4 y^4,
$$

$$
q(y) = -\frac{1}{4y^2}.
$$

(2.33)

Clearly, depending on values of $\nu$ and $b_1$, the function $g(y)$ has different behavior. Fig. 1 shows these possibilities.

### B. Classification of Turning Points

The LG approximate solutions only apply to the region that $g(y)$ does not vanish. Once $g(y)$ is zero, both $\psi(\xi)$ and $\mu_\xi(y)$ becomes divergent near these points, and the LG approximations fail to hold. In order to get the asymptotic solutions around these points, one must take different choices of $f^{(1)}(\xi, \xi^2)$ in the Liouville transformations (2.3). However, such choices depend on the natures of the turning points. Therefore, we first investigate the natures of them. In general, the equation $g(y) = 0$ is a cubic equation, and can be always cast in the form,

$$
b_2\epsilon^6 - b_1\epsilon^4 + x^2 - \nu^2 = 0,
$$

(2.34)

where $x = \epsilon_s y > 0$. Setting $\nu \equiv 3b_2/b_1^2$, the three roots of Eq. (2.34) are given by

$$
x_0 = \left\{ \begin{array}{l}
\frac{b_1}{3b_2} \left[ 1 - 2\sqrt{1 - \nu} \cos \left( \frac{\theta}{3} \right) \right]^{1/2}, \\
\frac{b_1}{3b_2} \left[ 1 - 2\sqrt{1 - \nu} \cos \left( \frac{\theta + 2\pi}{3} \right) \right]^{1/2}, \\
\frac{b_1}{3b_2} \left[ 1 - 2\sqrt{1 - \nu} \cos \left( \frac{\theta + 4\pi}{3} \right) \right]^{1/2},
\end{array} \right.
$$

(2.35)

where

$$
\cos \theta \equiv -\left( 1 - \frac{3}{2} \nu + \frac{3}{2} b_1 \nu^2 \epsilon^2 \right) (1 - \nu)^{-3/2}.
$$

(2.36)

Depending on the signs of $\Delta$, the natures of the three roots are different, where $\Delta$ is defined as

$$
\Delta \equiv (\nu - 1)^3 + \left( 1 - \frac{3}{2} \nu + \frac{3}{2} b_1 \nu^2 \epsilon^2 \right)^2.
$$

(2.37)

In particular, when $\Delta < 0$, the three roots $x_i$ are real and different [Fig. 1(a)]. When $\Delta = 0$, it has one single real approximate solutions near these turning points that are different from the LG solutions, but must go over to them near these two poles. For such approximate solutions, the convergence of the errors requires that $g(y) = 0$ and $q(y)$ must satisfy all the conditions of Eqs. (2.28), (2.29), and (2.32). As a result, the function $g(y)$ is necessarily to take the form $q(y) = -1/(4y^2) + q_2$. Then, Eq. (2.36) follows.
root and one double real root [Fig. 1(b)]. When \( \Delta > 0 \), it has one single real root and two complex roots [Fig. 1(c)].

In terms of \( y \), the three roots will be denoted by \( y_i \) \( (i = 0, 1, 2) \). Without lose of the generality, we further assume \( y_0 < y_1 \leq y_2 \) in the case \( \Delta < 0 \); \( y_0 < y_1 \) and \( y_2 = y_1 \) in the case \( \Delta = 0 \); and \( y_0 \) is real and \( y_{1,2} \) are complex with \( y_1 = y_2^* \) in the case \( \Delta > 0 \). Note that with such identifications, in general we do not have the relations \( y_i = x_i/\epsilon \), where \( x_i \) are the roots given by Eq. (2.35).

Then, in all the three cases, we have \( y_i \sim \mathcal{O}(1) \), while the magnitude of roots \( y_2 \) and \( y_1 \) (for complex roots, Re\((y_1)\) and Re\((y_2)\)) are dependent on the values of \( \epsilon \). When \( \epsilon \ll 1 \), we have \( y_{1,2} \gg 1 \) (Re\((y_{1,2})\) \( \gg 1 \) for complex roots). However, the difference between \( y_1 \) and \( y_2 \) generally depends on the choice of the free parameters appearing in the dispersion relations.

To process further, let us specify the three conditions [19] [22].

(a) \( |q(y)| < |g(y)| \) holds everywhere, except in the neighborhoods of the turning points \( y_i \);

(b) \( |q(y)| < \left| \frac{g(y)}{(y-y_1)} \right| \) holds in the neighborhoods of the turning points \( y_i \);

(c) \( |q(y)| < \left| \frac{g(y)}{(y-y_1)(y-y_2)} \right| \) holds in the neighborhoods of the turning points \( y_1 \) and \( y_2 \).

As we shall show below, these three conditions play essential roles in determining the approximate solutions around the turning points.

### C. Asymptotic Solution Around \( y = y_0 \)

In the neighborhood of \( y_0 \), Conditions (a) and (b) are satisfied, and \( y_0 \) is a single turning point. Then, one can introduce a monotone increasing or decreasing function \( \xi \) via the relations,

\[
    f^{(1)}(\xi)^2 = \pm \xi,
\]

where \( \xi(y_0) = 0 \). Without loss of the generality, we can choose \( \xi \) to have the same sign as \( g(y) \), and thus \( \xi \) is a monotone decreasing function around \( y = y_0 \). Combining Eqs. (2.3) and (2.38), we find

\[
    \xi = \begin{cases} 
        - \left( \frac{2}{3} \int_{y_0}^{y} \sqrt{-g(y)} \, dy \right)^{2/3}, & y \geq y_0, \\
        \left( -\frac{2}{3} \int_{y_0}^{y} \sqrt{g(y)} \, dy \right)^{2/3}, & y \leq y_0,
    \end{cases}
\]

and that Eq. (2.25) now reads,

\[
    \frac{d^2U}{d\xi^2} = \left( \xi + \psi(\xi) \right) U.
\]

Neglecting the \( \psi(\xi) \) term as the first-order approximation, we find that the above equation has the approximate analytical solution

\[
    U(\xi) = \alpha_0 \left( \text{Ai}(\xi) + \epsilon_3 \right) + \beta_0 \left( \text{Bi}(\xi) + \epsilon_4 \right),
\]

where \( \text{Ai}(\xi) \) and \( \text{Bi}(\xi) \) are the Airy functions, \( \alpha_0 \) and \( \beta_0 \) are integration constants, and \( \epsilon_3 \) and \( \epsilon_4 \) denote the errors of the approximations. Then, the mode function is given by,

\[
    \mu_k(y) = \alpha_0 \left( \frac{\xi}{g(y)} \right)^{1/4} \left( \text{Ai}(\xi) + \epsilon_3 \right) + \beta_0 \left( \frac{\xi}{g(y)} \right)^{1/4} \left( \text{Bi}(\xi) + \epsilon_4 \right).
\]

To estimate the corresponding errors, let us first consider \( \epsilon_3 \). Inserting \( U(\xi) \sim \text{Ai}(\xi) + \epsilon_3 \) into Eq. (2.40), we have

\[
    \frac{d^2\epsilon_3}{d\xi^2} = \xi \epsilon_3 + \psi(\xi) \left( \text{Ai}(\xi) + \epsilon_3 \right).
\]

Treating the second term in the right-hand side of the above equation as corrections, and then using the method of variation of parameters, we obtain

\[
    \epsilon_3(\xi) = \int_{\xi}^{a_3} K(\xi, v) |v|^{-1/2} \psi(v) \left( \text{Ai}(v) + \epsilon_1 \right),
\]

where \( a_3 \) is the upper bound of \( \xi \) and the corresponding upper bound of \( y \) is \( \hat{a}_3 \), i.e., \( \xi(\hat{a}_3) = a_3 \), and

\[
    K(\xi, v) = \pi |v|^{1/2} \left( \text{Bi}(\xi) \text{Ai}(v) - \text{Ai}(\xi) \text{Bi}(v) \right).
\]

Bounds of \( K(\xi, v) \) and \( \partial K(\xi, v)/\partial \xi \) are expressible in terms of the auxiliary functions of the Airy functions (See Appendix B for details). Thus, we find

\[
    \left| K(\xi, v) \right| = \pi \sqrt{|v| M(\xi) M(v)} \left| \frac{E(\xi)}{E(v)} \cos \theta(\xi) \sin \theta(v) \right| 
\]

\[
    \leq \pi \sqrt{|v| M(\xi) M(v) E(v)} \left| \frac{E(\xi)}{E(v)} \right| \left| \frac{E(v)}{E(\xi)} \right| \cos \theta(\xi) \sin \theta(v).
\]

for \( \xi \leq v \leq a_3 \), where \( E(x) \) is a non-decreasing function of \( x \). We also have

\[
    \left| \frac{\partial K(\xi, v)}{\partial \xi} \right| \leq \pi \sqrt{|v| N(\xi) M(v) E(v) E^{-1}(\xi)}.
\]

Then, from the above we obtain

\[
    Q(v) = \pi |v|^{1/2} E(v) M(v),
\]

\[
    P_0(\xi) = E^{-1}(\xi) M(\xi),
\]

\[
    P_1(\xi) = E^{-1}(\xi) N(\xi).
\]
Comparing Eq. (2.44) with Eq. (A.1), we find
\[ \phi(v) = \psi_0(v) = |v|^{-1/2} \psi(v), \]
\[ \psi_1(v) = 0, \quad J(v) = A_i(v). \]  
(2.49)

From Appendix A, the quantities \( \kappa, \kappa_0 \) now can be calculated and are given by
\[ \kappa = \sup \{ \pi |v|^{1/2} E(\xi) M(\xi) |A_i(\xi)| \}, \]
\[ \kappa_0 = \sup \{ \pi |v|^{1/2} M^2(\xi) \}. \]  
(2.50)

Numerically, we find that these yield \( \kappa = 1, \kappa_0 = 1.03952 \).

With the above results, the error bounds for \( \epsilon_3 \) read
\[ \frac{|\epsilon_3|}{M(\xi)}, \frac{|\partial \epsilon_3 / \partial \xi|}{N(\xi)} \leq \frac{E^{-1}(\xi)}{\lambda} \left\{ \exp(\lambda \nu_{\xi, a_1}(H)) - 1 \right\}, \]  
(2.51)

where \( \lambda \equiv \kappa_0 \) and \( H(\xi) \) is the corresponding error control function, give by
\[ H(\xi) = \int_{\xi}^{a_3} |v|^{-1/2} \psi(v) dv. \]  
(2.52)

Similarly, the error bound for \( \epsilon_4 \) can be expressed as,
\[ \frac{|\epsilon_4|}{M(\xi)}, \frac{|\partial \epsilon_4 / \partial \xi|}{N(\xi)} \leq \frac{E(\xi)}{\lambda} \left\{ \exp(\lambda \nu_{a_4, \xi}(H)) - 1 \right\}, \]  
(2.53)

where \( a_4 \) is the lower bound of the variable \( \xi \), and the corresponding upper bound for \( y \) is \( \hat{a}_4 \), i.e., \( \xi(\hat{a}_4) = a_4 \). Note that for \( \epsilon_4 \), we have
\[ Q(v) = \pi |v|^{1/2} E^{-1}(v) M(v), \quad P_0(\xi) = E(\xi) M(\xi), \quad P_1(\xi) = E(\xi) N(\xi), \quad J(v) = B_i(v), \]
\[ \kappa = \sup \{ \pi |v|^{1/2} E^{-1}(\xi) M(\xi) |B_i(\xi)| \}, \]  
(2.54)

which are different from those for \( \epsilon_3 \). Numerical calculations also yield \( \kappa = 1 \).

D. Asymptotic Solution Near \( y_1 \) and \( y_2 \)

Depending on \( \Delta < 0, \Delta = 0 \) and \( \Delta > 0 \), the roots \( y_1 \) and \( y_2 \) have different properties, as shown above. Although these three different cases can be treated in a unfired way, let us first consider them, separately.

1. When \( y_1 \) and \( y_2 \) Are Real

In this case, near the turning points \( y_1 \) and \( y_2 \), Conditions (a) and (c) are always satisfied. When Condition (b) is also satisfied, one can treat \( y_1 \) and \( y_2 \) as simple turning points, and similar to that near the turning point \( y_0 \), one can get the asymptotic solutions near these points. However, when \( |y_2 - y_1| < 1 \), Condition (b) is not satisfied, and then the method used for the turning point \( y_0 \) is no longer valid. Following Olver[20], we shall adopt a method to treat these cases together. The crucial point is to choose \( f^{(1)}(\xi)^2 \) in the Liouville transformations (2.3) as
\[ f^{(1)}(\xi)^2 = |\xi^2 - \xi_0^2|, \]  
(2.55)

where \( \xi \) is an increasing function of \( y \), and with the choices \( \xi(y_1) = -\xi_0 \) and \( \xi(y_2) = \xi_0 \). When \( \xi_0 = 0 \), it corresponds to the case \( y_1 = y_2 \). Then, we find that in the region \( y \geq y_2 \) we have \( g(y) < 0 \) and \( \xi \geq \xi_0 \), and
\[ \int_{y_2}^{y} \sqrt{-g(y)} dy = \frac{1}{2} \xi \sqrt{\xi^2 - \xi_0^2} - \frac{1}{2} \xi_0^2 \arccosh \left( \frac{\xi}{\xi_0} \right), \quad (y \geq y_2). \]  
(2.56)

Similarly, in the region \( y \leq y_1 \), we have \( g(y) > 0 \) and \( \xi \leq \xi_0 \), while in the region \( y \in (y_1, y_2) \), we have \( g(y) > 0 \) and \( -\xi_0 \leq \xi \leq \xi_0 \). Thus, we find that
\[ \int_{y_1}^{y} \sqrt{g(y)} dy = \frac{1}{2} \xi_0^2 \arccosh \left( \frac{\xi}{\xi_0} \right) - \frac{1}{2} \xi \left\{ \sqrt{\xi^2 - \xi_0^2}, \quad y \leq y_1, \right\}  \]  
(2.57)

\[ \xi_0^2 = \frac{2}{\pi} \int_{y_1}^{y_2} \sqrt{g(y)} dy. \]  
(2.58)

Then, Eq. (2.5) reduces to
\[ \frac{d^2 U}{d \xi^2} = \left[ (\xi_0^2 - \xi^2) + \psi(\xi) \right] U. \]  
(2.59)

Neglecting the \( \psi(\xi) \) term, we find that the approximate solutions can be expressed in terms of the parabolic cylinder functions \( W(\frac{1}{2} \xi_0^2, \pm \sqrt{2}\xi) \), and are given by
\[ U(\xi) = \alpha_1 \left\{ W \left( \frac{1}{2} \xi_0^2, \sqrt{2}\xi \right) + \epsilon_5 \right\} + \beta_1 \left\{ W \left( \frac{1}{2} \xi_0^2, -\sqrt{2}\xi \right) + \epsilon_6 \right\}, \]  
(2.60)

for which we have
\[ \mu_k(y) = \alpha_1 \left( \frac{\xi_0^2 - \xi_0^2}{-g(y)} \right)^{1/4} W \left( \frac{1}{2} \xi_0^2, \sqrt{2}\xi \right) + \beta_1 \left( \frac{\xi_0^2 - \xi_0^2}{-g(y)} \right)^{1/4} W \left( \frac{1}{2} \xi_0^2, -\sqrt{2}\xi \right), \]  
(2.61)

where \( \epsilon_5 \) and \( \epsilon_6 \) are the errors of the corresponding approximates.
2. When $y_1$ and $y_2$ Are Complex

In this case, $y_1$ and $y_2$ are complex conjugate, $y_1 = y_2^*$. To have a unified treatment, here we still choose $f^{(1)}(\xi)^2$ in the Liouville transformations (2.3) as [20]

$$f^{(1)}(\xi)^2 = \xi^2 - \xi_0^2, \quad (2.62)$$

where $\xi$ is still an increasing variable and is chosen so that $\xi(y_1) = -\xi_0$ and $\xi(y_2) = \xi_0$, respectively, but now with $\xi_0^2 < 0$, that is $\xi_0$ becomes imaginary. When $\xi_0 = 0$, it reduces to the case $y_1 = y_2$. Since now there are no real turning points, the function $f^{(1)}(\xi)^2$ does not need to have any real zeros or singularities in the neighborhoods of $y = \text{Re}(y_1)$. Then, we can choose

$$\xi_0^2 = -\frac{2}{\pi} \int_{y_1}^{y_2} \sqrt{-y} dy. \quad (2.63)$$

Note that in above integral the path lies along the imaginary axis, and the branch of $\sqrt{-y}$ is real and positive. Since now $\xi_0$ is purely imaginary, without loss of the generality, we set $\text{Im}(\xi_0) \geq 0$. Note that we also have $g(y) = g(-y)$ for both real and purely imaginary values of $y$, we find

$$\int_{y_1}^{y_2} \sqrt{-g(y)} dy = \int_{0}^{\xi} \sqrt{\xi^2 - \xi_0^2} d\xi. \quad (2.64)$$

Then, $\xi$ can be obtained from the integral,

$$\int_{y_1}^{y_2} \sqrt{-g(y)} dy = \frac{1}{2} \xi \sqrt{\xi^2 - \xi_0^2} - \frac{1}{2} \xi_0^2 \left( \frac{\xi + \sqrt{\xi^2 - \xi_0^2}}{|\xi_0|} \right), \quad (2.65)$$

and now the integral path lies along the real axis. With such chosen $\xi$, it can be shown that Eq. (2.5) takes the same form as that of Eq. (2.59). As a result, the approximate solution is also given by Eq. (2.61), but now $\xi_0$ is given by Eq. (2.6). The above shows clearly that the three cases ($\Delta > 0$, $\Delta = 0$, $\Delta < 0$) can be treated in a unified way, although initially we considered them separately.

3. Error Bounds

Since the approximate solutions are given by Eq. (2.61) in all the three cases considered above, we do not need to distinguish them, when we consider the corresponding errors $\epsilon_5$ and $\epsilon_6$. In particular, for $U$ given by

$$U = W \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) + \epsilon_5(\xi), \quad (2.66)$$

from Eq. (2.59), we find that

$$\frac{d^2 \epsilon_5}{d \xi^2} = -\left( \xi^2 - \xi_0^2 \right) \epsilon_5$$

$$+ \psi(\xi) \left\{ W \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) + \epsilon_5 \right\}. \quad (2.67)$$

Considering the second term in the right-hand side of the above equation as corrections, we find that the error can be written in the form,

$$\epsilon_5(\xi) = \int_{a_5}^{a_5} K(\xi, v) \psi(v) \left\{ W \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) + \epsilon_5 \right\} dv, \quad (2.68)$$

where $a_5$ is the upper bound of $\xi$ and

$$K(\xi, v) = v \left\{ W \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) W \left( \frac{1}{2} \xi_0^2, -\sqrt{2} v \right) - W \left( \frac{1}{2} \xi_0^2, \sqrt{2} v \right) W \left( \frac{1}{2} \xi_0^2, -\sqrt{2} v \right) \right\}. \quad (2.69)$$

Then, in terms of the weight function $E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)$, the modulus function $M \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)$ for the function $W \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)$ (See Appendix C for definitions of these auxiliary functions), we find that

$$|K(\xi, v)| = |v| M \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) M \left( \frac{1}{2} \xi_0^2, \sqrt{2} v \right)$$

$$\times \left| \frac{E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)}{E \left( \frac{1}{2} \xi_0^2, \sqrt{2} v \right)} \sin \theta_1 \cos \theta_2 - \frac{E \left( \frac{1}{2} \xi_0^2, \sqrt{2} v \right)}{E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)} \sin \theta_2 \cos \theta_1 \right|, \quad (2.70)$$

for $v \leq \xi \leq 0$, where $\theta_1 = \theta \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)$, $\theta_2 = \theta \left( \frac{1}{2} \xi_0^2, \sqrt{2} v \right)$. When the function $E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)$ is non-decreasing for $\xi \geq 0$, we also have

$$|K(\xi, v)| \leq |v| M \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) M \left( \frac{1}{2} \xi_0^2, \sqrt{2} v \right)$$

$$\times \left| \frac{E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)}{E \left( \frac{1}{2} \xi_0^2, \sqrt{2} v \right)} \right|. \quad (2.71)$$

Similarly, it can be shown that

$$\left| \frac{\partial K(\xi, v)}{\partial \xi} \right| \leq \sqrt{2} |v| N \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) M \left( \frac{1}{2} \xi_0^2, \sqrt{2} v \right)$$

$$\times \left| \frac{E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)}{E \left( \frac{1}{2} \xi_0^2, \sqrt{2} v \right)} \right|, \quad (2.72)$$

where $N \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)$ is the modulus function of $W^* \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)$ and defined in Appendix C. Then, we find that

$$Q(v) = |v| M \left( \frac{1}{2} \xi_0^2, \sqrt{2} v \right) E \left( \frac{1}{2} \xi_0^2, \sqrt{2} v \right),$$

$$P_0(\xi) = \frac{M \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)}{E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)},$$

$$P_1(\xi) = \frac{N \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)}{E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)}. \quad (2.73)$$
Comparing Eq. (2.68) with Eq. (A.1), we obtain
\[ \phi(v) = \psi_0(v) = |v|^{-1}\psi(v), \]
\[ \psi_1(v) = 0, \quad J(v) = W\left(\frac{1}{2}\xi_0^2, \sqrt{2v}\right). \quad (2.74) \]

With the above expressions, \( \kappa \) and \( \kappa_0 \) are given by
\[ \kappa = \sup \left\{ |\xi| M\left(\frac{1}{2}\xi_0^2, \sqrt{2\xi}\right) E\left(\frac{1}{2}\xi_0^2, \sqrt{2\xi}\right) \right\}, \]
\[ \kappa_0 = \sup \left\{ |\xi| M^2\left(\frac{1}{2}\xi_0^2, \sqrt{2\xi}\right) \right\}. \quad (2.75) \]

Therefore, for \( \xi \leq 0 \) the error bounds for \( \epsilon_5 \) can be expressed as
\[ \frac{|\epsilon_5|}{M\left(\frac{1}{2}\xi_0^2, \sqrt{2\xi}\right)} \leq \frac{\kappa}{\lambda E\left(\frac{1}{2}\xi_0^2, \sqrt{2\xi}\right)} \left\{ \exp\left(\lambda \gamma_{\xi_0\dot{\alpha}_5}(H)\right) - 1 \right\}. \quad (2.76) \]

Similarly, it can be shown that the error bound for \( \epsilon_6 \) are given by
\[ \frac{|\epsilon_6|}{M\left(\frac{1}{2}\xi_0^2, \sqrt{2\xi}\right)} \leq \frac{\kappa E\left(\frac{1}{2}\xi_0^2, \sqrt{2\xi}\right)}{\lambda} \left\{ \exp\left(\lambda \gamma_{\xi_0\dot{\alpha}}(I)\right) - 1 \right\}, \quad (2.77) \]

where \( \lambda \equiv \kappa_0 \), and \( I(\xi) \) is the corresponding error control function, given by
\[ I(\xi) = \int_{\xi}^{\infty} |v|^{-1}\psi(v)dv. \quad (2.78) \]

It is easy to get the error bounds for \( \xi < 0 \) by replacing \( \xi \) by \( -\xi \) in the above expressions.

### F. Matching the Analytical Solutions

So far, we have obtained the analytical approximate solutions near poles \( y = 0^+, +\infty \), given by Eqs. (2.41) and (2.13), respectively, and in the neighborhoods of the turning points \( y_i \), given, respectively, by Eqs. (2.42) and (2.61). Now, we need to determine all the integration constants from the initial conditions by matching them on their boundaries (or common regions) with the requirements that the mode function \( \mu_k(\eta) \) and its first derivative \( d\mu_k(\eta)/d\eta \) be continuous across each of their boundaries (or in their common regions).

We first note that the approximate solutions (2.42) around the turning point \( y_i \) reduce to the LG solutions \( \mu_k^i(y) \), as \( y \to 0^+ \), while the approximate solutions (2.61) around the turning points \( y_1 \) and \( y_2 \) reduce to the LG solution \( \mu_k^i(y) \), as \( y \to +\infty \).

Assume that the universe was initially at the adiabatic vacuum \( \mathcal{N} \),
\[ \lim_{y \to +\infty} \mu_k(y) = \frac{1}{\sqrt{2\omega}} e^{-i \int \omega d\eta} \]
\[ \simeq \sqrt{\frac{k}{2}} \frac{1}{\sqrt{-g}^{1/4}} \exp\left(-i \int_{y_i}^{y} \sqrt{-g} dy\right), \quad (2.80) \]
while the second condition shall be the Wronskian condition
\[ \mu_k(y)\mu_k^i(y)' - \mu_k^i(y)\mu_k(y)' = i. \quad (2.81) \]

Applying the above conditions to the solution \( \mu_k(y) \) given by Eq. (2.13), we find that
\[ c_+ = 0, \quad d_- = \sqrt{\frac{1}{2k}} e^{i\phi}, \quad (2.82) \]
where \( \phi \) is an irrelevant phase factor. Without loss of the generality, we can always set it to zero. Similar considerations will be applied, when other integration constants are concerned.

On the other hand, to consider the matching between the solution \( \mu_k(y) \) and the one given by Eq. (2.61), we note that for a positive and large \( \xi \) the parabolic cylinder
functions take the asymptotical forms [23],
\[
W \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) \simeq \left( \frac{2 j^2(\xi_0)}{\xi^2 - \xi_0^2} \right)^{1/4} \cos \mathcal{D},
\]
\[
W \left( \frac{1}{2} \xi_0^2, -\sqrt{2} \xi \right) \simeq \left( \frac{2 j^{-2}(\xi_0)}{\xi^2 - \xi_0^2} \right)^{1/4} \sin \mathcal{D},
\]
where
\[
j(\xi_0) \equiv \sqrt{1 + e^{\pi \xi_0^2}} - e^{\pi \xi_0^2/2},
\]
and
\[
\mathcal{D} \equiv \frac{1}{2} \xi \sqrt{\xi^2 - \xi_0^2} - \frac{1}{2} \xi_0^2 \ln \left( \frac{\xi + \sqrt{\xi^2 - \xi_0^2}}{|\xi_0|} \right) + \frac{\pi}{4} + \phi \left( \frac{1}{2} \xi_0^2 \right),
\]
with
\[
\phi(x) \equiv \frac{x}{2} - \frac{x}{4} \ln x^2 + \frac{1}{2} \mathrm{ph} \Gamma \left( \frac{1}{2} + ix \right),
\]
where the phase \( \mathrm{ph} \Gamma \left( \frac{1}{2} + ix \right) \) is zero when \( x = 0 \), and determined by continuity, otherwise. Inserting the above into Eq. (2.61), and then comparing the resulting solution with \( \mu_k(y) \), we find that the continuities of the mode function and its first derivatives with respect to \( \xi \) yield,
\[
\alpha_1 = 2^{-3/4} k^{-1/2} j^{-1/2}(\xi_0),
\]
\[
\beta_1 = -i 2^{-3/4} k^{-1/2} j^{1/2}(\xi_0).
\]

To determine the coefficients \( \alpha_0 \) and \( \beta_0 \), let us consider the matching of the solutions (2.61) and (2.42) in the region \( y \in (y_0, y_1) \). In this region, \( |y_0 - y_1| \) is large, as mentioned above, and \( \xi \) is very negative. Thus, using the asymptotical form (2.83) of \( W \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) \), and the asymptotic form of the Airy functions,
\[
\mathrm{Ai}(-x) = \frac{1}{\pi^{1/2} x^{1/4}} \cos \left( \frac{2}{3} x^{3/2} - \frac{\pi}{4} \right),
\]
\[
\mathrm{Bi}(-x) = -\frac{1}{\pi^{1/2} x^{1/4}} \sin \left( \frac{2}{3} x^{3/2} - \frac{\pi}{4} \right),
\]
for \( x \gg 1 \), we find that the coefficients \( \alpha_0 \) and \( \beta_0 \) are given by,
\[
\alpha_0 = \sqrt{\frac{\pi}{2k}} \left[ j^{-1}(\xi_0) \sin \mathfrak{B} - ij(\xi_0) \cos \mathfrak{B} \right],
\]
\[
\beta_0 = \sqrt{\frac{\pi}{2k}} \left[ j^{-1}(\xi_0) \cos \mathfrak{B} + ij(\xi_0) \sin \mathfrak{B} \right],
\]
where
\[
\mathfrak{B} = \int_{y_0}^{y_1} \sqrt{-y} dy + \phi(\xi_0^2/2).
\]

Finally, we consider the matching between \( \mu_k^+(y) \) given by Eq. (2.11) and the one given by Eq. (2.42) in the region \( y \in (0, y_0) \). It can be shown that the continuities of the mode function and its first derivatives with respect to \( \eta \) yield
\[
d_+ = \frac{\alpha_0}{2 \sqrt{\pi}} \exp \left( - \int_{0+}^{y_0} \sqrt{y} dy \right),
\]
\[
c_+ = \frac{\beta_0}{\sqrt{\pi}} \exp \left( \int_{0+}^{y_0} \sqrt{y} dy \right).
\]
The above completed the matching among these individual solutions given in each of the regions specified above. From such a process, it can be seen that all the integration constants are uniquely determined form the initial conditions.

Once we have determined these integration constants, let us now turn to consider some representative cases. In particular, in the case with three different single turning points, the numerical (exact) and our approximate solutions are plotted in Fig. 2(a). The cases with two and one turning point(s) are plotted, respectively, in Figs. 2(b) and 2(c). From these figures, one can see clearly how well the exact solutions are approximated by our analytical ones. Many other cases have been also considered, and found that in all those cases the exact solutions are extremely well approximated by the analytical ones, thanks to the understanding of the error bounds, and the proper choice of the Liouville transformations.

### III. SOLUTIONS FOR LARGE \( |\xi_0| \)

In the last section, the approximate analytical solutions around the turning point \( y_0 \) is expressed in terms of the Airy functions, while around the ones \( y_1 \) and \( y_2 \) they are expressed in terms of the parabolic cylinder functions, which are uniform with respect to both real and purely imaginary values of \( \xi_0 \). This difference comes from the fact that Condition (b) holds at \( y_0 \), while in general only Condition (c) holds at \( y_1 \) and \( y_2 \). In this section, we consider the case where \( |\xi_0| \) is very large, so Condition (b) also holds at both \( y_1 \) and \( y_2 \). In the latter case, we shall show that the approximate solutions around the turning points \( y_1 \) and \( y_2 \) are also expressible in terms of the Airy functions.

#### A. Asymptotic Solutions for Real \( y_{1,2} \)

We first consider the case where \( y_1 \) and \( y_2 \) are real. Similar to what we did around the turning point \( y_0 \), when Condition (b) is satisfied, the function \( f^{(1)}(\xi)^2 \) needs to have the same order of zeros as \( g(y) \) around the turning points \( y_1 \) and \( y_2 \). Thus, we can define a monotone increasing or decreasing function \( \xi \) via the relations
\[
f^{(1)}(\xi)^2 = \pm \xi^i,
\]
where \( \xi_i \) vanishes at the turning points \( y_1 \) and \( y_2 \), i.e., \( \xi_i(y_1) = \xi_i(y_2) = 0 \). Note that, in order to distinguish
Thus, neglecting the $\psi(\xi_i)$ term as the first-order approximation, the above equation has the following general asymptotic solutions in terms of the Airy functions,

$$\mu_k^{(i)}(y) \simeq \hat{\alpha}_i \left( \frac{\xi_i}{g(y)} \right)^{1/4} \text{Ai}(\xi_i)[1 + \epsilon_i^{(i)}(\xi_i)]$$

$$+ \hat{\beta}_i \left( \frac{\xi_i}{g(y)} \right)^{1/4} \text{Bi}(\xi_i)[1 + \epsilon_i^{(i)}(\xi_i)], \quad (3.5)$$

here $\epsilon_i^{(i)}(\xi_i)$ denote the errors of the approximations, which are given by similar expressions of Eqs. (2.51) and (2.53) for the turning point $y_0$, so we shall not rewrite them here.

Now let us turn to determine the integrations constants for the new solutions (3.5) near the turning points $y_1$ and $y_2$, which are valid only for $|\xi_0| \gg 1$, as we emphasized above. First, the coefficients $c_-$ and $d_-$ are also given by Eq. (2.82), determined from the initial conditions (2.80). To consider the matching between the solution $\mu_k^{(i)}(y)$ and $\mu_k^{(2)}(y)$, we note that $\xi_2$ is very large and negative. Thus, using the asymptotical form (2.88) we find

$$\hat{\alpha}_2 = \sqrt{\frac{\pi}{2k}}, \quad \hat{\beta}_2 = i \sqrt{\frac{\pi}{2k}}. \quad (3.6)$$

To determine the coefficients $\hat{\alpha}_1$ and $\hat{\beta}_1$, let us consider the matching of the solutions $\mu_k^{(2)}(y)$ and $\mu_k^{(1)}(y)$ in the region $y \in (y_1, y_2)$. Since $|\xi_0| \gg 1$, we find that both $\xi_1$ and $\xi_2$ are very large in this region. Then, using the asymptotical form of the Airy functions

$$\text{Ai}(x) = \frac{x^{-1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}x^{3/2}},$$

$$\text{Bi}(x) = \frac{x^{-1/4}}{\sqrt{\pi}} e^{-\frac{2}{3}x^{3/2}}, \quad (3.7)$$

we find that the coefficients $\hat{\alpha}_1$ and $\hat{\beta}_1$ are given by

$$\hat{\alpha}_1 \simeq 2\beta_2 \exp \left( \int_{y_1}^{y_2} \sqrt{g(y)} dy \right) = i \sqrt{\frac{2\pi}{k}} e^{-\alpha},$$

$$\hat{\beta}_1 \simeq \frac{1}{2} \sqrt{\alpha_2} \exp \left( -\int_{y_1}^{y_2} \sqrt{g(y)} dy \right) = \sqrt{\frac{\pi}{8k}} e^{-\alpha}, \quad (3.8)$$

where

$$\alpha = \int_{y_1}^{y_2} \sqrt{g(y)} dy. \quad (3.9)$$

In the region $y \in (y_0, y_1)$, we consider the matching of the solutions $\mu_k^{(1)}(y)$ and that given by Eq. (3.5). In this region, $|y_0 - y_1|$ is large, as mentioned above, and $\xi$ near both of the turning points $y_0$ and $y_1$ is very negative.
Thus, using the asymptotical form (2.88), we obtain
\[ \alpha_0 = \tilde{\alpha}_1 \sin \beta \cos \beta, \]
\[ \beta_0 = \tilde{\alpha}_1 \cos \beta - \tilde{\beta}_1 \sin \beta, \]
where we use the subscripts \(\beta\) and \(\tilde{\beta}\) as asymptotic form (2.88), we find that the coefficients \(\alpha_0\) and \(\beta_0\) are given by
\[ \alpha_0 = \sqrt{\frac{\pi}{2k}} \cos \beta, \quad \beta_0 = i \sqrt{\frac{\pi}{2k}} \cos \beta. \] (3.11)

Finally, we note that the above results can be also obtained by directly expand the parabolic cylinder functions in terms of the Airy functions when \(\xi_0\) is large. For details, see Appendix D.

**IV. POWER SPECTRA AND INDICES OF SCALAR AND TENSOR PERTURBATIONS**

In the above sections, we constructed the approximate analytical solutions of the linear perturbations (1.1) for scalar, vector or tensor fields in the framework of slow-roll inflation. In GR and Hořava-Lifshitz gravity, vector perturbations decay rapidly with the expansion of the universe and have negligible effects on observations [2] [24]. So, in this section we consider only the scalar and tensor perturbations, and pay particular attention on the modifications of them due to the quantum effects.

As mentioned previously, although the linear perturbations of all the types can be written in the form (1.1), the constants \(b_i\) and the function \(z\) are different for different types of perturbations. from which it can be shown that \(z\) for the scalar and tensor perturbations is given, respectively, by [2],
\[ z_s = \sqrt{2} \epsilon a, \quad z_T = a, \] (4.1)
where we use the subscripts \(s\) and \(T\) to denote the scalar and tensor perturbations, respectively. Recall that the slow-roll parameter \(\epsilon\) is defined as \(\epsilon = -\dot{H}/H^2\). Then, the corresponding \(\nu_s\) and \(\nu_T\) are given by Eq.(1.5). To the first-order of the slow-roll parameters, we have [17]
\[ \nu_s \approx 3 + 2\epsilon + \delta_1, \quad \nu_T = \frac{3}{2} + \epsilon, \] (4.2)
with \(\delta_1 \equiv \frac{\ddot{\phi}}{H \dot{\phi}}\), and \(\phi\) denotes the inflaton. Similarly, we shall use \(b_i^s\) and \(b_i^T\) to denote, respectively, the scalar and tensor perturbations, \(\mu_s^y(y, b_i^s, \nu_s)\) and \(\mu_T^y(y, b_i^T, \nu_T)\) the corresponding mode functions. With the above in mind, in the following we shall ignore the subscripts whenever it is possible.

**A. Power Spectra**

To calculate the power spectra, we only need to consider the limit \(y \to 0^+\). Then, from Eq.(3.7) we can see that in this limit only the growing mode is relevant, so we have
\[ \mu_k(y) \approx \beta_0 \left( \frac{1}{\pi^2 g(y)} \right)^{1/4} \exp \left( \int_y^{y_0} dy \sqrt{g(y)} \right), \] (4.3)
where \(\beta_0\) is given by Eq. (2.89). Then, the corresponding power spectrum is given by,
\[ \Delta^2(k) \equiv \frac{k^3}{2\pi^2} \left| \frac{\mu_k(y)}{z} \right|^2 = \left( \frac{k^2 y}{4\pi^2 z^2 \nu} \right) A \exp \left( 2 \int_y^{y_0} \sqrt{g(y)} dy \right), \] (4.4)
where
\[ A \equiv \frac{2k|\beta_0|^2}{\pi}, \]
\[ = 1 + 2e^{\pi\xi_0^2} + 2e^{\pi\xi_0^2/2} \sqrt{1 + e^{\pi\xi_0^2}} \cos 2\beta, \] (4.5)
with \(\xi_0^2\) being given by
\[ \xi_0^2 = \pm \frac{1}{\pi} \int_{y_1}^{y_2} \sqrt{\pm g(y)} dy, \] (4.6)
where \(\pm\) corresponds to the case where \(y_1\) and \(y_2\) are real, and “-” to the case where \(y_1\) and \(y_2\) are complex conjugate \(y_1 = y_2^*\). The factor \(\beta\) is given by Eq.(2.90). The integrals in \(\xi_0^2\) and \(\beta\) can be carried out explicitly. Let us first consider the integral (4.6). For the case where the equation \(g(y) = 0\) has three real and different roots,
we find

\[
\xi_0^2 = \frac{2\sqrt{b_2}c^2}{3\pi \sqrt{y^2-y_0^2}} \int_{y_1}^{y_2} \sqrt{\frac{(y^2-y_0^2)(y^2-y_1^2)(y^2-y_2^2)}{y^2-y_0^2}} dy
\]

\[
= \frac{2\sqrt{b_2}c^2}{3\pi \sqrt{y^2-y_0^2}} \times \left\{ (y_0^2-y_1^2)(y_0^2+y_1^2+y_2^2)E \left( \frac{y_3^2-y_4^2}{y_3^2-y_0^2} \right) \right. \\
+\left. (-y_0^4+2y_0^2y_2^2+y_0^2y_2^2+y_0^2y_2^2) \right\},
\]

where \( E(x), K(x), \Pi(x, y) \) denote the Elliptic integrals \([23]\), and are given explicitly in Appendix E. It is easy to show that the above expression is also valid for the case where \( y_1 \) and \( y_2 \) are complex conjugate, for which \( \xi_0^2 \) is negative. In addition, in the case where the equation \( g(y) = 0 \) has one single real root \( y_0 \) and one double real root \( y_1 = y_2 \), we have \( \xi_0^2 = 0 \). On the other hand, we also find

\[
\mathfrak{B} = \int_{y_0}^{y_2} \sqrt{-g(y)dy} + \phi \left( \frac{1}{2} \xi_0^2 \right)
\]

\[
= \frac{\sqrt{b_2}c^2}{3 \sqrt{y_2-y_0^2}} \times \left\{ (y_0^2-y_2^2)(y_0^2+y_1^2+y_2^2) \right. \\
\times \left. \left[ \frac{1}{y_0^2-y_2^2} \right] \right. \\
+\left. (y_0^4-2y_0^2y_1^2-y_0^2y_2^2-y_0^2y_2^2) \right\} + \phi \left( \frac{1}{2} \xi_0^2 \right),
\]

where \( y_R = \text{Re}(y_1) \) when \( y_1 \) is complex, and \( y_R = y_1 \) when \( y_1 \) is real. The functions \( E(x, y), F(x, y), \Pi(x, y) \) denote the other types of the Elliptic integrals \([23]\), and are given explicitly in Appendix E. The phase \( \phi \left( \frac{1}{2} \xi_0^2 \right) \) is given by Eq. \([2.86]\). When \( y_1 \) and \( y_2 \) are real, the above expression can be simplified into the form

\[
\mathfrak{B} = \sqrt{b_2}c^2 \left\{ \frac{\sqrt{y_1^2-y_0^2}}{3} (y_0^2+2y_1^2) - \frac{\pi}{2} y_0y_1 \right. \\
+\left. \arctan \left( \frac{y_0}{\sqrt{y_1^2-y_0^2}} \right) \right\}, \quad (y_1 = y_2).
\]

To process further, let us first write \( g(y) \) in the form,

\[
g(y) = \frac{y_0^2-y_1^2}{y_0^2} (h_0 + h_1y^2 + h_2y^4),
\]

where

\[
h_0 = b_2c_4y_0^4 - b_1c_2y_0^2 + 1,
\]

\[
h_1 = b_2c_4y_0^4 - b_1c_2y_0^2,
\]

and

\[
\sqrt{h_0} + h_1y^2 + h_2y^4 = 1 - \frac{b_1}{2} (y^2 + y_0^2)c_2^2 \\
- \frac{b_2}{8} (y^2 + y_0^2) - \frac{b_2}{2} (y^4 + y_2^2y_0^2 + y_0^4) c_4^2 \\
+ \mathcal{O}(c_6).
\]
Then, we obtain

\[
\int_{y_0}^{y_0} \sqrt{g(y)} \, dy = -y_0 y_0 \ln \frac{y}{y_0 + \sqrt{y_0^2 - y^2}} + \left( \frac{y_0^2 - y^2}{} \right)^{3/2} \left( 5z_1 + 3z_2 y^2 + 2z_2 y_0^2 \right)
\]

\[-z_0 \sqrt{y_0^2 - y^2} + O \left( e^2 \right),
\]

(4.14)

where

\[
z_0 = 1 - \frac{b_1}{2} y_0^2 c_2 + \frac{4b_2 - b_1^2}{8} y_0 c_4,
\]

\[
z_1 = -\frac{b_1}{2} c_2 + \frac{4b_2 - 2b_1}{8} y_0^2 c_4,
\]

\[
z_2 = \frac{4b_2 - b_1^2}{8} c_4.
\]

(4.15)

On the other hand, from Eq. (2.35) we find

\[
y_0 \approx \nu + \delta_1 c_2 + \delta_2 c_4 + O \left( e^2 \right),
\]

(4.16)

where

\[
\delta_1 = \frac{b_1 \nu^3}{2}, \quad \delta_2 = \frac{1}{8} \left( 7b_2^2 - 4b_2 \right) \nu^5.
\]

(4.17)

Inserting the above expressions into Eq. (4.4), we obtain

\[
\Delta^2(k) = \Delta_0^2(k) Q,
\]

(4.18)

where

\[
\Delta_0^2 = \left( \frac{k^2}{4\pi^2 \nu^2} \right) y^{1-2\nu} \left( \frac{2\nu}{e} \right)^{2\nu},
\]

(4.19)

represents the power spectra of GR (for \( b_1 = 0 = b_2 \)), obtained in \[17\]. The function \( Q \) is defined as

\[
Q = A \left\{ 1 + \frac{5}{6} b_1 \nu^3 c_2 + \frac{\nu^5}{360} \left( -30b_1 + 444b_1^2 - 275b_2 + 125b_2^2 \nu \right) c_4 + O \left( e^2 \right) \right\}
\]

(4.20)

B. The Spectral Indices

The corresponding scale and tensor spectral indices can be calculated directly from the power spectra \( 4.4 \), which are given, respectively, by

\[
n_s = 1 + \frac{d \ln \Delta^2}{d \ln k} = 4 - 2 \int_{y_0}^{y_0} \frac{1 - 2b_1 \nu^2 y^2 + 3b_2 \nu^4 y^4 \, dy}{\sqrt{g(y)}}
\]

(4.21)

\[
n_T = \frac{d \ln \Delta_T^2}{d \ln k} = 3 - 2 \int_{y_0}^{y_0} \frac{1 - 2b_1 \nu^2 y^2 + 3b_2 \nu^4 y^4 \, dy}{\sqrt{g_T(y)}}
\]

(4.22)

with \( g_T(y) \) \equiv g(y, b_1, \nu) \) \( (I = s, T) \). That is, to the first-order approximations of the slow-roll parameters, the power spectrum indices of scalar and tensor perturbations are the same as those given in GR. This is an important conclusion and different from the results obtained initially in the first reference of \[6\], but similar to the ones obtained later in \[15\], in which only the term \( b_1 \) was considered, and then the corresponding mode function can be obtained analytically.

It should be noted that in the above expressions we have assumed that the quantities \( \nu, b_1 \) and \( b_2 \) are all constants. This assumption is correct if one only considers the first-order slow-roll approximations. In this case, the quantity \( A \) does not depend on \( k \), and, as a result, the spectral indices \( n_{s,T} \) do not receive any corrections from \( A \). To consider the quantum effects of the high-order derivative terms on \( n_{s,T} \), we need to go beyond the first-order approximations or consider other backgrounds of the universe. However, the factor \( A \) does affect the amplitudes of the power spectra \( \Delta_{s,T}^2(k) \), as one can see from Eqs. \( 4.18 \) and \( 4.20 \).

C. Error Analysis

In the last subsections, we have obtained the explicit expressions for the power spectra and their spectral indices. In order to show the corresponding errors, in this subsection, we compare them with the numerical (exact) results. The numerical solutions \( \mu_{k, \mathrm{num}}(y) \) of Eq. \( 2.1 \) with the initial conditions \( 2.30 \), can be easily obtained by the numerical integrations. Then, we find that
FIG. 3: The numerical (exact) (red dotted curves) solution of $\eta^2 z^2 \Delta^2\text{num}(k)$, the analytical (blue solid curves) one $\eta^2 z^2 \Delta^2\text{GR}(k)\mathcal{Q}$, and their relative errors defined by Eq. (4.23) vs the free parameter $b_2$ for $\nu = 3/2$, $\epsilon_* = 0.01$. Top panel: $b_1 = 2$. Middle Panel: $b_1 = 3$. Low Panel: $b_1 = 6$.

\[ \left| \frac{\Delta^2(k) - \Delta^2\text{num}(k)}{\Delta^2\text{num}} \right| = \left| \frac{\eta^2 z^2 \Delta^2(k) - |\mu^\text{num}_k(y)|/(2\pi^2)}{|\mu^\text{num}_k(y)|/(2\pi^2)} \right|, \quad (4.23) \]

where

\[ \Delta^2\text{num}(k) \equiv \frac{k^3}{2\pi^2} \left| \frac{\mu^\text{num}_k(y)}{z(\eta)} \right|^2. \quad (4.24) \]

In Fig. 3 we plot out the analytical and numerical values of $\eta^2 z^2 \Delta^2(k)$ vs $b_2$, and their relative errors, from which it can be seen that for most of the cases, the relative errors are about 10\%, which is approximately in the same level as that obtained in GR by using the uniform approximation in the first-order approximations [17].

When $b_1 = 0 = b_2$, the power spectra in Eq. (4.19) reduce to the usual results $\Delta^2_0(k)$, whose relative error is well understood by comparing it with the exact analytical result $\Delta^2\text{GR}(k)$, i.e.,

\[ \Delta^2\text{GR}(k) = \Delta^2_0(k)(1 + \delta_e), \quad (4.25) \]

where $\delta_e$ is in the level of 10\%. Thus, the relative errors of the power spectra (4.19) can be divided into two parts, one comes from $\Delta^2_0(k)$ and the other comes from $\mathcal{Q}$,

\[ \Delta^2\text{exact}(k) = \Delta^2_0(k)(1 + \delta_e)\mathcal{Q}(1 + \delta_\mathcal{Q}), \quad (4.26) \]

where $\Delta^2\text{exact}$ represents the exact power spectra. Since $\delta_e$ is well understood in GR, to improve the uniform approximations, one can replace the $\Delta^2_0(k)$ in Eq. (4.19) by $\Delta^2\text{GR}(k)$, i.e.,

\[ \Delta^2\text{exact}(k) = \Delta^2\text{GR}(k)(1 + \delta_\mathcal{Q}) = \Delta^2(k)(1 + \delta_\mathcal{Q}), \quad (4.27) \]

with $\Delta^2(k) = \Delta^2\text{GR}(k)\mathcal{Q}$ and

\[ \Delta^2\text{GR}(k) \equiv \frac{k^2}{8\pi^3 z^2} \Gamma^2(\nu)2^{2\nu}y^{1-2\nu}. \quad (4.28) \]

Then, in Fig. 4 we show that the relative errors of the power spectra can be dropped from $\sim 10\%$ to roughly 0.5\% for most of cases. However, from Fig. 4 we can also see that in some very narrow regions of the parameter $b_2$, the relative errors become large. This usually
happens when \( g(y) \) has three real turning points and \( \xi_0^2 \) is very large. When we derive the expression of \( Q \), the errors come from two parts. The first one is from the uniform approximation itself, while the other is from the asymptotic expressions (2.83) and (2.88) that were used to determine the coefficients \( a_0 \) and \( b_0 \). Both of them will lead to errors in the phase of \( \cos 2B \). We find that even with a very small such error, it could lead to a large error in the power spectra. To see this more explicitly, let us assume that \( B \) has a small relative error \( \delta \). When \( \xi_0 \) is large, the quantity \( \mathcal{A} \) can be written as

\[
\mathcal{A} \simeq 2e^{\pi \xi_0^2} \left[ 1 + \cos \left( 2B + \delta \right) \right]. \tag{4.29}
\]

Now considering the region in which \( \cos 2B \approx -1 \), while \( \mathcal{A} \) is still very small. Then, the resulted relative error for power spectra is about

\[
\left| \frac{8 \cos 2B}{1 + \cos 2B} \right| \delta \gg 1. \tag{4.30}
\]

The improvement of such errors requires the considerations of the more accurate expansions of the parabolic cylinder and Airy functions, and meanwhile the high-order uniform approximations. These are out of the scope of this paper and will be considered in our future work.

V. DISCUSSIONS AND CONCLUSIONS

In this paper, we have presented a technique, the uniform asymptotic approximation, to construct accurate analytical solutions of the linear perturbations of inflation with a nonlinear dispersion relation in order to account for the quantum effects of the early universe. Such a relation can be naturally realized in the framework of the Hořava-Lifshitz gravity [12, 13], a candidate of the ultraviolet complete theory of quantum gravity.

In particular, using the Liouville transformations of Eq. (2.3), we have first written the equation of motion of the mode function that describes the linear perturbations of scalar, vector or tensor fields in the form (2.5) in terms of the two new variables \( U(\xi) \) and \( \xi \). According to the theory of the second-order differential equations, the accuracy of the approximate solutions of \( U(\xi) \) sensitively depends on the behavior of the functions \( g(y) \) and \( q(y) \).
defined in Eq. (2.2) around their poles and zeros (turning points). To obtain the best approximate analytical solutions, we have written them down formally in term of $g(y)$ and $q(y)$, and then constructed the corresponding error bounds associated with the approximations. By minimizing the error bounds and error control functions, we have uniquely determined the two functions $g(y)$ and $q(y)$, given by Eq. (2.55). After obtaining all the individual solutions, using the initial conditions, we have then uniquely determined the integrations constants, by matching them in their common regions. Because of the understanding of the error bounds, and the proper choice of the Liouville transformations near each pole or turning point, our approximate analytical solutions describe the exact evolution of the linear perturbations extremely well even only in the first-order approximations, as one can see from Figs. 2–4.

In Section IV, we have considered a particular case where the parameter $|\xi_0|$ defined in Eqs. (2.55) and (2.62) is very large. When there is only one real root, this corresponds to the case considered in [11]. As shown explicitly in Appendix D, the solutions constructed in this section can be also obtained from the ones constructed in Section III with the large $|\xi_0|$ limit, as it is expected.

As an application of the approximate analytical solutions constructed in Section III and IV, in Section V we have calculated the power spectra and indices of scalar and tensor perturbations in the slow-roll inflation, and found that the amplitudes of the power spectra get modified due to the quantum effects, while the power spectrum indices remain the same as in the linear case, as far as the first-order approximations of the slow-roll conditions are concerned.

Certainly, the method developed in this paper can be easily applied to other cosmological backgrounds, and it would be very interesting to study the quantum effects of the early universe on inflationary cosmology, including the power spectra and indices of the linear (scalar, vector and tensor) perturbations, non-Gaussianities, primordial gravitational waves, temperature and polarization of CMB [26].

In addition, high-order approximations can be also constructed [19, 21].

**Acknowledgements**

We thank Yongqing Huang, Jiro Soda, Qiang Wu, and Wen Zhao for valuable discussions and suggestions. This work is supported in part by DOE, DE-FG02-10ER41692 (AW), Ciência Sem Fronteiras, No. 004/2013 - DRI/CAPES (AW), NSFC No. 11375153 (AW), No. 11173021 (AW), No. 11047008 (TZ), No. 11105120 (TZ), and No. 11205133 (TZ).

**Appendix A: Theorems of Singular Integral Equations**

In this Appendix, we shall present a basic introduction to the theorems of the error bounds studied in this paper. For more details, we refer readers to [19].

Assume that $h(\xi)$ represents the errors of the approximate solutions of a second-order differential equation. One can substitute the corresponding solution with this error term to the corresponding differential equation, and using the method of variation of parameters, one can cast $h(\xi)$ in the form,

$$h(\xi) = \int_{a}^{\xi} K(\xi, v) \left[ \phi(v) J(v) + \psi_0(\xi) h(v) + \psi_1(v) h'(v) \right] dv,$$

(A.1)

where $J(v)$, $\phi(v)$, $\psi_0(v)$, and $\psi_1(v)$ are continuous functions, and the kernel $K(\xi, v)$ and its first two partial $\xi$ derivatives are continuous.

Then, if the kernel $K(\xi, v)$ and its first two $\xi$ derivatives are bounds, that is, if

$$|K(\xi, v)| \leq P_0(\xi) Q(v),$$

$$\left| \frac{\partial K(\xi, v)}{\partial \xi} \right| \leq P_1(\xi) Q(v),$$

$$\left| \frac{\partial^2 K(\xi, v)}{\partial \xi^2} \right| \leq P_2(\xi) Q(v),$$

(A.2)

where the $P_j(\xi)$ and $Q(v)$ are continuous real functions and $P_j(\xi)$ are positive, we find

$$\frac{|h(\xi)|}{P_0(\xi)}, \frac{|h'(\xi)|}{P_1(\xi)} \leq \kappa \Phi(\xi) \exp \left\{ \kappa_0 \Psi_0(\xi) + \kappa_1 \Psi_1(\xi) \right\},$$

(A.3)

where

$$\Phi(\xi) \equiv \int_{a}^{\xi} |\phi(v)| dv, \quad \Psi_0(\xi) \equiv \int_{a}^{\xi} |\psi_0(v)| dv,$$

$$\Psi_1(\xi) \equiv \int_{a}^{\xi} |\psi_1(v)| dv,$$

(A.4)

are convergent, and

$$\kappa \equiv \sup \{Q(\xi) J(\xi)\}, \quad \kappa_0 \equiv \sup \{Q(\xi) P_0(\xi)\},$$

$$\kappa_1 \equiv \sup \{Q(\xi) P_1(\xi)\},$$

(A.5)

are finite. Then, for the special case where $\psi_0(v) = \phi(v)$ and $\psi_1(v) = 0$, the error bounds read [19]

$$\frac{|h(\xi)|}{P_0(\xi)}, \frac{|h'(\xi)|}{P_1(\xi)} \leq \frac{\kappa}{\kappa_0} \left[ \exp \{\kappa_0 \Psi_0(\xi)\} - 1 \right].$$

(A.6)
Appendix B: Auxiliary Functions of Airy Functions

In this Appendix, we introduce some auxiliary functions of the Airy functions $\text{Ai}(x)$ and $\text{Bi}(x)$. In particular, the modulus function $M(x)$ and $N(x)$ are defined as

\[
M(x) = \begin{cases} 
\sqrt{2\text{Ai}(x)\text{Bi}(x)}, & \text{for } x \leq c, \\
\sqrt{\text{Ai}^2(x) + \text{Bi}^2(x)}, & \text{for } x \geq c
\end{cases}, \tag{B.1}
\]

\[
N(x) = \begin{cases} 
\sqrt{\text{Ai}^2(x) + \text{Bi}^2(x)}, & \text{for } x \leq c, \\
\sqrt{\text{Ai}^2(x)\text{Bi}^2(x) + \text{Bi}^2(x)\text{Ai}^2(x)}, & \text{for } x \geq c,
\end{cases} \tag{B.2}
\]

where $c = -0.366046$ is the smallest root (in the sense of the absolute value) of the equation $\text{Ai}(x) = \text{Bi}(x)$. Then, the weight function $E(x)$ is defined as

\[
E(x) = \begin{cases} 
\frac{\text{Bi}(x)}{\text{Ai}(x)}, & \text{for } x \geq c, \\
1, & \text{for } x \leq c.
\end{cases} \tag{B.3}
\]

With the phase functions $\theta(x)$ and $\omega(x)$, the Airy functions can be expressed in terms of the modulus functions $M(x)$, $N(x)$ and weight function $E(x)$ as

\[
\text{Ai}(x) = \frac{M(x)}{E(x)} \sin \theta(x), \quad \text{Bi}(x) = E(x)M(x) \cos \theta(x),
\]

\[
\text{Ai}'(x) = \frac{N(x)}{E(x)} \sin \omega(x), \quad \text{Bi}'(x) = N(x)E(x) \cos \omega(x). \tag{B.4}
\]

Appendix C: Auxiliary Functions of Parabolic Cylinder Functions

In this Appendix, we present all the definitions of the auxiliary functions for the parabolic cylinder functions. First, we introduce the modulus functions $M\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right)$, $N\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right)$, and the weight function $E\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right)$. When $\sqrt{2}\xi \leq c(\xi_0)$, we have

\[
M\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right) = \sqrt{2W\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right)W\left(\frac{1}{2}\xi_0^2, -\sqrt{2}\xi\right)},
\]

\[
N\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right) = \left\{ \frac{W'\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right)W\left(\frac{1}{2}\xi_0^2, -\sqrt{2}\xi\right)}{W\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right)} - \frac{W'\left(\frac{1}{2}\xi_0^2, -\sqrt{2}\xi\right)W\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right)}{W\left(\frac{1}{2}\xi_0^2, -\sqrt{2}\xi\right)} \right\}^{1/2}, \tag{C.1}
\]

and

\[
E\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right) = \sqrt{\frac{j(\xi_0)W\left(\frac{1}{2}\xi_0^2, -\sqrt{2}\xi\right)}{W\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right)}}, \tag{C.2}
\]

where $E\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right)$ is of non-decreasing in the region $\xi \in [0, +\infty)$, and $c(\xi_0)$ is the smallest root, in the sense of the absolute value, of the equation

\[
j(\xi_0)^{-1/2}W'^2\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right) = j(\xi_0)^{1/2}W'^2\left(\frac{1}{2}\xi_0^2, -\sqrt{2}\xi\right).
\]

Then, when $\sqrt{2}x \geq c(\xi_0)$, we have

\[
M\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right) = \left\{ j(\xi_0)^{-1}W'^2\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right) + j(\xi_0)W'^2\left(\frac{1}{2}\xi_0^2, -\sqrt{2}\xi\right) \right\}^{1/2},
\]

\[
N\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right) = \left\{ j(\xi_0)^{-1}W'^2\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right) + j(\xi_0)W'^2\left(\frac{1}{2}\xi_0^2, -\sqrt{2}\xi\right) \right\}^{1/2}, \tag{C.3}
\]

and

\[
E\left(\frac{1}{2}\xi_0^2, \sqrt{2}\xi\right) = 1. \tag{C.4}
\]
Thus, with the phase functions \( \theta \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) \) and \( \omega \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) \), the Parabolic cylinder functions can be expressed in terms of the modulus functions \( M \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) \), \( N \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) \) and weight function \( E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) \) as

\[
\begin{aligned}
j(\xi_0)^{-1/2}W \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) & = \frac{M \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)}{E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)} \sin \theta \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right), \\
j(\xi_0)^{1/2}W \left( \frac{1}{2} \xi_0^2, -\sqrt{2} \xi \right) & = M \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) \sin \theta \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right), \\
j(\xi_0)^{-1/2}W' \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) & = \frac{N \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)}{E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)} \sin \omega \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right), \\
j(\xi_0)^{1/2}W' \left( \frac{1}{2} \xi_0^2, -\sqrt{2} \xi \right) & = -N \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) \cos \omega \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right),
\end{aligned}
\]

for \( \xi \geq 0 \), and

\[
\begin{aligned}
j(\xi_0)^{1/2}W \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) & = \frac{M \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)}{E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)} \cos \theta \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right), \\
j(\xi_0)^{-1/2}W \left( \frac{1}{2} \xi_0^2, -\sqrt{2} \xi \right) & = M \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) \sin \theta \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right), \\
j(\xi_0)^{1/2}W' \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) & = \frac{N \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)}{E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right)} \sin \omega \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right), \\
j(\xi_0)^{-1/2}W' \left( \frac{1}{2} \xi_0^2, -\sqrt{2} \xi \right) & = -N \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) E \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) \cos \omega \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right),
\end{aligned}
\]

for \( \xi \leq 0 \).

**Appendix D: Parabolic Cylinder Functions in Terms of Airy Functions**

When \( |\xi_0| \) is large, one can also express the parabolic cylinder functions in terms of the Airy functions.

1. When \( y_{1,2} \) Are Real

Near the real turning points \( y_1 \) and \( y_2 \), the parabolic cylinder functions for \( |\xi_0| \gg 1 \) and \( \xi > -\xi_0 \) are given by

\[
\begin{aligned}
W \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) & \approx 2^{-1/4} \pi^{1/2} e^{-\pi \xi_0^2/4} \\
& \times \left( -\frac{\xi}{\xi_0^2 - \xi_0^2} \right)^{1/4} \text{Bi}(\xi), \\
W \left( \frac{1}{2} \xi_0^2, -\sqrt{2} \xi \right) & \approx 2^{3/4} \pi^{1/2} e^{\pi \xi_0^2/4} \\
& \times \left( -\frac{\xi}{\xi_0^2 - \xi_0^2} \right)^{1/4} \text{Ai}(\xi),
\end{aligned}
\]

where in writing down the above expressions, we have neglected all the high-order terms and token

\[
-\frac{2}{3} \xi_0^{3/2} = \frac{1}{2} \xi_0 \sqrt{\xi_0^2 - \xi^2} - \frac{1}{2} \xi_0^2 \arccos \left( -\frac{\xi}{\xi_0} \right).
\]

for \( \xi \in (-\xi_0, \xi_0) \), and

\[
\frac{2}{3} \left( -\xi \right)^{3/2} = \frac{1}{2} \xi_0 \sqrt{\xi_0^2 - \xi^2} - \frac{1}{2} \xi_0^2 \arccos \left( \frac{\xi}{\xi_0} \right).
\]

for \( \xi \geq \xi_0 \).

In the neighborhoods of \( y_2 \), we have \( \hat{\xi} = \xi_2 \). Thus, the solution (2.61) now takes the form

\[
\mu_{\xi}(y) = \alpha_1 2^{-1/4} \pi^{1/2} e^{-\pi \xi_0^2/4} \left( \frac{\xi_2}{g(y)} \right)^{1/4} \text{Bi}(\xi_2) + \beta_1 2^{3/4} \pi^{1/2} e^{\pi \xi_0^2/4} \left( \frac{\xi_2}{g(y)} \right)^{1/4} \text{Ai}(\xi_2),
\]

which is exactly the asymptotic solution \( \mu_{\xi}^{(2)}(y) \) given in Eq. (3.5) around the turning point \( y_2 \) with

\[
\hat{\beta}_2 = \alpha_1 2^{-1/4} \pi^{1/2} e^{-\pi \xi_0^2/4}, \\
\hat{\alpha}_2 = \beta_1 2^{3/4} \pi^{1/2} e^{\pi \xi_0^2/4}.
\]

Considering the solutions in the neighborhoods of \( y_1 \), similarly one can find now \( \hat{\xi} = \xi_1 \). Then, taking the asymptotic expansions (D.1) of the parabolic functions into account, we find that the solution (2.61) now takes
the form

\[
\mu_k(y) = \alpha_1 2^{3/4} \pi^{1/4} e^{\pi \xi_0^2 / 4} \left( \frac{\xi_1}{g(y)} \right)^{1/4} \text{Ai}(\xi_1) + \beta_1 2^{-1/4} \pi^{1/4} e^{-\pi \xi_0^2 / 4} \left( \frac{\xi_1}{g(y)} \right)^{1/4} \text{Bi}(\xi_1),
\]

(D.6)

which is the same as the asymptotic solution \( \mu_k^{(1)}(y) \) given in Eq. (3.5) around the turning point \( y_1 \), with

\[
\hat{\alpha}_1 = \alpha_1 2^{3/4} \pi^{1/4} e^{\pi \xi_0^2 / 4},
\]

\[
\hat{\beta}_1 = \beta_1 2^{-1/4} \pi^{1/4} e^{-\pi \xi_0^2 / 4}.
\]

(D.7)

With \( \alpha_1 \) and \( \beta_1 \) given by Eq. (1.5), it can be shown that \( \hat{\alpha}_2, \hat{\beta}_2 \) given in Eq. (D.5) and \( \hat{\alpha}_1, \hat{\beta}_1 \) given in Eq. (D.7) are consistent with Eqs. (3.6) and (3.8), except an irrelevant phase factor.

2. When \( y_1 \) and \( y_2 \) Are Complex

For complex \( y_1 \) and \( y_2 \), when \( |\xi_0| \) is large, we find that \( \frac{1}{2} \xi_0^2 \) is very negative. Then, as mentioned above, the solutions given in terms of the Airy functions in Eq. (2.42) are valid in the whole region \( y \in (0^+, \infty) \), so the solutions in this case are already written in terms of the Airy functions, and no further considerations are needed.

Appendix E: The Elliptic Integrals

In this section, we present various definitions of the Elliptic integrals used in this paper. For detail, see, for example, [25]. First, \( F(\phi, m) \) denotes the Elliptic integral of the first kind, defined as

\[
F(\phi, m) = \int_0^\phi \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}},
\]

(E.1)

for \(-\pi < \phi \leq \pi / 2\). \( K(m) \) denotes the complete Elliptic integral of the first kind, defined as

\[
K(m) = F\left( \frac{\pi}{2}, m \right).
\]

(E.2)

The Elliptic integral \( E(\phi, m) \) of the second kind is defined as

\[
E(\phi, m) = \int_0^\phi \sqrt{1 - m \sin^2 \theta} d\theta,
\]

(E.3)

for \(-\pi < \phi \leq \pi / 2\), while the corresponding complete Elliptic integral \( E(m) \) of the second kind is defined as

\[
E(m) = E\left( \frac{\pi}{2}, m \right).
\]

(E.4)

\( \Pi(n, \phi, m) \) denotes the Elliptic integral of the third kind, defined as

\[
\Pi(n, \phi, m) = \int_0^\phi \frac{d\theta}{(1 - n \sin^2 \theta) \sqrt{1 - m \sin^2 \theta}},
\]

(E.5)

and the corresponding complete Elliptic integral \( \Pi(n, m) \) is defined as

\[
\Pi(n, m) = \Pi\left( n, \frac{\pi}{2}, m \right).
\]

(E.6)
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