ON THE COHOMOLOGY RING OF INFINITESIMAL GROUP SCHEMES

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Abstract. For each \( r \geq 1 \), let \( G_r \) be the \( r \)-th Frobenius kernels of a simple classical group scheme \( G \) defined over an algebraically closed field \( k \) of characteristic \( p > 0 \). These subgroup schemes play an important role in the study of representation theory of finite group schemes. We study in this paper the structure of the cohomology ring \( \text{H}^\ast(G_r, k) \). In particular, we investigate the Cohen-Macaulay property and Krull dimension of this ring as well as the cohomology ring of related subgroups of \( G_r \). Our calculations give a lower bound for the complexity of certain modules over \( G_r \). In the special cases when \( p = 2 \) or \( \text{rank}(G) = 2 \), we explicitly compute the complexity of \( G_r \) and related subgroups.

1. Introduction

Infinitesimal groups are the group schemes such that their coordinate algebras are finite dimensional and local. These groups play a fundamental role in relating cohomology theory of finite groups to that of reductive group schemes. Given a connected algebraic group \( G \) defined over the field of characteristic \( p \), important examples of such groups are Frobenius kernel subgroups \( G_r \) for all \( r \geq 1 \) of \( G \). Then the rational representation theory of \( G \) is equivalent to those of infinitesimal subgroups \( G_r \) for all \( r \geq 1 \). However, cohomology theory for these objects are not well understood except few special cases. In the case \( r = 1 \), the first Frobenius kernel \( G_1 \) is a familiar object and received considerable interest from representation theorists due to the equivalence between the category of \( G_1 \)-modules and that of the restricted Lie algebra \((\mathfrak{g} = \text{Lie}(G), [p])\)-modules. For sufficiently large \( p \), a good deal has been made for cohomology of \( B_1 \) and \( G_1 \) from work of Andersen and Jantzen, Friedlander and Parshall, Kumar, Lauritzen, and Tomson. In joint work with Drupieski and Nakano, we completely described the cohomology ring structure for the first Frobenius kernel of a maximal unipotent subgroup of \( G \) \([\text{DNN}]\). This result combined with earlier results give a complete answer for cohomology of the first Frobenius kernels for large enough \( p \). The problem remains unsolved in the case of small \( p \) (except some small rank cases). However, the geometry of \( G_1 \)-cohomology, under the light of the groundbreaking paper of Suslin, Friedlander, and Bendel \([\text{SFB}]\), is clear from the explicit description of \( N_1 \) of Carlson, Lin, Nakano, and Parshall in \([\text{CLNP}]\). Moreover, under certain assumptions involving Grauert-Riemenschneider vanishing and normality of Richardson orbits’ closure, Bendel, Nakano, Parshall, and Pillen are able to compute exactly the cohomology ring of \( G_1 \) \([\text{BNPP}], \text{Chapter 7}\)\). The state of affairs for higher Frobenius kernels is rather complicated. In the very special case when \( G = SL_2 \), I computed the cohomology of \( B_r \) and \( G_r \) as a \( G \)-module for each \( r \geq 1 \) \([\text{Ngo1}]\). On the other hand, in the paper \([\text{KSTY}]\) 20 years ago, Kaneda et. al. described the module structure of the cohomology of \( B_2 \) when \( G = SL_3 \) and obtained some bounds for the Krull dimension of \( B_r \)-cohomology ring. Calculating cohomology of \( G_r \) in general seems to be a highly difficult task. Hence it requires a new approach to tackle this problem.

Following footsteps of Quillen, Benson, Carlson, Duflot and others, we use commutative algebra to study the ring structure of the \( G_r \)-cohomology. In particular, we are concerned about the question when the cohomology ring of Frobenius kernels is Cohen-Macaulay, which is one of the interests suggested in \([\text{Ben2}], \text{Section 6}\)\). The history of this topic traces back to the paper of Quillen in 1972 where he showed that the cohomology ring of \( GL_n(\mathbb{F}_q) \) is Cohen-Macaulay in characteristic not dividing \( q \). Having this property gives us more understanding on the structure of the cohomology
ring. For instance, it is then represented as a free module over some polynomial ring. Moreover, in [BC] Benson and Carlson showed that a Cohen-Macaulay cohomology ring of a finite group satisfies Poincaré duality, which deduces more information on minimal projective resolutions in the category of modules over that group. This result was later generalized to the category of finite dimensional cocommutative Hopf algebras with finiteness condition of cohomology by Evens and Siegel [ES], hence it is applied for finite group schemes (thanks to the famous result of Friedlander and Suslin in [FS]). For these reasons, it would be nice if one could classify all the finite group schemes having Cohen-Macaulay cohomology. Up to now, one has verified this property for the cohomology ring of many classes of finite groups, see [Ben2], however the author has not seen much progress on group schemes. Recently, Carlson and Nakano [CN] have studied the Cohen-Macaulay property and Poincaré duality for the first Frobenius kernel $U_1$ of the unipotent radical subgroup $U$ for sufficiently large $p$. This stimulates our present work in which we affirm an observation of Carlson and Nakano in a more general setting.

Let $G$ be a simple algebraic group defined over an algebraically closed field $k$ of characteristic $p$. As pointed out in the first paragraph the property of Cohen-Macaulay cohomology ring is well known for the first Frobenius kernels. Explicitly, when $p$ is sufficiently large, i.e. $p \geq h$ the Coxeter number, the cohomology ring $H^i(G_1, k)$ is Cohen-Macaulay since it is isomorphic to the coordinate ring of the nilpotent cone $N$ of $g$ [EP]. The cohomology ring of $B_1$ is even nicer as it is isomorphic to $S^g(u)$, which is a regular polynomial ring. Recently, the ring $H^i(U_1, k)$ has been shown to be Cohen-Macaulay by Carlson and Nakano in [CN]. These results completely answer the question on Cohen-Macaulayness of the first Frobenius kernels for large $p$. Our investigations in this paper appeal that the cohomology ring of $G_r$ for each $r \geq 2$ is rarely Cohen-Macaulay.

For any subgroup scheme $H$ of $G$, one application of support varieties $V_H(M)$ is the complexity $c_H(M)$ of $H$-module $M$, which is the rate of growth for a minimal projective resolution of $M$. The dimension of $V_H(M)$ is shown to be equalled to $c_H(M)$. When $H$ is infinitesimal of height $r$, e.g. $H = G_r$, one can identify $V_{G_r}(k)$ with the variety of restricted commuting nilpotent $r$-tuples in $g$ [SPB]. Note that for $r = 1$ the complexity of $B_1$ and $G_1$ has been completely determined from the study on the restricted nullcones and orbital varieties. For $p \geq h$ (the Coxeter number of $G$), the result of Premet on nilpotent commuting varieties implies the complexity for $G_2$. The problem is still wild open in remaining cases. Using techniques in commuting varieties, we compute the complexity of $B_r$ and $G_r$ (for certain modules) for classical groups $G$ of rank 2, and for $p = 2, G$ of type $A$. Note that the latter is a generalization of a result of Levy in [L] where he computed the dimension of the restricted nilpotent commuting variety $C_2(N_1)$ in the same condition. Our calculations also provide upper bounds for the complexity of any $G_r$-module $M$. For higher ranks, we obtain a lower bound for this amount.

The paper structure is twofold. Sections 2–5 deal with the Cohen-Macaulay property of the cohomology ring of Frobenius kernel subgroups of $G$ while the last two sections 6–7 are devoted to study the complexity of those groups. It is summarized as follows.

Section 2 includes necessary notation and materials for later uses. Then we introduce the concepts of Poincaré series, complexity and Cohen-Macaulay for a graded ring in Section 3. We state here the fact that for any finite group scheme $G$ whose cohomology ring is Cohen-Macaulay then it satisfies the Poincaré duality. This is a generalized statement of Theorem 3.2 in [CN].

In Section 4 we prove that the Frobenius kernels $(SL_2)_r$ for all $r \geq 1$ have Cohen-Macaulay cohomology. Note that these results are the generalization of the ones in [Ngo1] Section 7] in the sense that we remove the reduceness condition on the cohomology rings. Our proofs are more straightforward and based on elementary properties of invariant theory. Consequently, for each $r \geq 1$ the group $(SL_2)_r$ satisfies Poincaré duality.

In the next section, we show that the cohomology ring $H^*(B_r, k)$ is Cohen-Macaulay if $H^*(U_r, k)$ is so. However, our observation later in the section on commuting varieties of type $G = GL_n$ indicates that this is rarely the case. In fact, for $p \geq n - 1$ our result implies that the cohomology
ring of $U_r$ might be Cohen-Macaulay only when $r \leq 2$ or $n \leq 3$. Our strategy, same as in [BC, Proposition 1.2], is showing that the corresponding support variety is not equidimensional. It is encouraged, from the last two sections, to study tools in invariant theory applied to the cohomology ring of Frobenius kernels.

Our results in previous sections reveal that the cohomology ring of Frobenius kernels in general could be complicated. In particular, non-equidimensionality creates difficulties in studying the structure of these rings. So finding the Krull dimension of $H^*(G_r, k)$ becomes the first task in exploring further structure of the cohomology ring. In Section 6, we develop the techniques in the previous section to establish lower bounds for the Krull dimension of $H^*(B_r, k)$ and $H^*(G_r, k)$ (as well as the complexity of $B_r$ and $G_r$). Our strategy is to employ the existence of a commutative square zero algebra in $C_r(N_1)$ to get lower bounds for $\dim C_r(N_1)$ in terms of $r$ and $\ell := \text{rank}(G)$. This also suggests a lower bound for the complexity of simple modules of restricted dominant weights. In the very special case when $G = A_\ell$ and $p = 2$, these bounds become exactly the complexity for $B_r$ and $G_r$.

One of the reasons for the theory of support varieties for infinitesimal groups is considerably complicated to study is the lack of explicit examples. This motivates us to compute in the final section the complexity for certain modules over rank 2 classical groups. Hopefully, our explicit results provide some insights for the cohomology theory of infinitesimal groups. Note that our results significantly strengthen the computations of Kaneda et. al. on the Krull dimension of $B_r$ cohomology ring for $SL_3$ [KSTY, 4.7].

2. Notation

2.1. Representation theory. Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $G$ be a simple classical algebraic group over $k$. Fix a maximal torus $T$ of $G$, let $B \subset G$ be the Borel subgroup of $G$ containing $T$ and corresponding to the set of positive roots, and let $U \subset B$ be the unipotent radical of $B$. Set $\mathfrak{g} = \text{Lie}(G)$, the Lie algebra of $G$, $\mathfrak{b} = \text{Lie}(B)$, $\mathfrak{u} = \text{Lie}(U)$. Denote by $S^*(\mathfrak{u}^*)$ and $\Lambda^*(\mathfrak{u}^*)$ respectively the symmetric algebra and exterior algebra over $\mathfrak{u}^* = \text{Hom}(\mathfrak{u}, k)$. Throughout this paper, the symbol $\otimes$ means the tensor product over the field $k$, unless otherwise stated. Suppose $H$ is an algebraic group over $k$ and $M$ is a (rational) module of $H$. Denote by $M^H$ the submodule consisting of all the fixed points of $M$ under the $H$-action.

Let $Y$ be a group scheme. Then for every positive integer $r$, the scheme $Y^{(r)}$ is defined by for each $k$-algebra $A$,

$$Y^{(r)}(A) = Y(A^{(r)})$$

where $A^{(r)}$ is identified with $A$ as a ring but the action of $b \in k$ on $A^{(r)}$ is the same as $b^{mp_r}$ acting on $A$ (see [Jan1, I.9.2]). Now consider $G$ as a group scheme. Then, for each $r$ let $F_r : G \to G^{(r)}$ be the $r$-th Frobenius morphism. The scheme-theoretic kernel $G_r = \ker(F_r)$ is called the $r$-th Frobenius kernel of $G$. Given a closed subgroup (scheme) $H$ of $G$, write $H_r$ for the scheme-theoretic kernel of the restriction $F_r : H \to H^{(r)}$. In other words, we have

$$H_r = H \cap G_r.$$  

Given a rational $G$-module $M$, write $M^{(r)}$ for the module obtained by twisting the structure map for $M$ by $F_r$. Note that $G_r$ acts trivially on $M^{(r)}$. Conversely, if $N$ is a $G$-module on which $G_r$ acts trivially, then there is a unique $G$-module $M$ with $N = M^{(r)}$. We denote the module $M$ by $N^{(-r)}$.

Let $M$ be a $B$-module. Then the induced $G$-module can be defined as

$$\text{ind}^G_B M = (k[G] \otimes M)^B.$$  

The higher derived functor of $\text{ind}^G_B (-)$ is denoted by $R^i \text{ind}^G_B (-)$.
2.2. **Geometry.** Let $R$ be a commutative Noetherian ring with identity. We use $R_{\text{red}}$ to denote the reduced ring $R/\text{Nilrad } R$ where $\text{Nilrad } R$ is the radical ideal of $0$ in $R$, which consists of all nilpotent elements of $R$. Let $\text{Spec}(R)$ be the spectrum of all prime ideals of $R$. This set is a topological space under the Zariski topology. Let $X$ be a variety. We denote by $k[X]$ the algebra of regular functions defined on $X$. Note that when $X$ is an affine variety, $k[X]$ coincides with the coordinate algebra of $X$. The notation $\dim(-)$ will be interchangeably used as the dimension of a variety or the Krull dimension of a ring.

Denote by $\mathcal{N}$ the nilpotent cone of $\mathfrak{g}$. The adjoint action of $G$ on $\mathfrak{g}$ stabilizes $\mathcal{N}$ and is denoted by “.”. Let $x_{\text{reg}}$ be a fixed regular nilpotent element and $z_{\text{reg}}$ be its centralizer in $\mathfrak{g}$. It is well known that $z_{\text{reg}} \subset \mathfrak{u}$, $\dim z_{\text{reg}} = \text{rank } \mathfrak{g} =: \ell$, and the regular orbit $G \cdot x_{\text{reg}}$ is dense in $\mathcal{N}$. For $H$ a subgroup of $G$, suppose $X, Y$ are $H$-varieties. Then the morphism $f : X \to Y$ is called $H$-equivariant if it is compatible with $H$-action. The restricted nullcone $\mathcal{N}_1$ of $\mathfrak{g}$ is defined as a subvariety of $\mathcal{N}$ such that

$$x \in \mathcal{N}_1 \iff x^p = 0.$$  

Complete description of $\mathcal{N}_1$ is referred to the paper of Carlson, Lin, Nakano, and Parshall [CLNP]. Set $\mathfrak{u}_1 = \mathcal{N}_1 \cap \mathfrak{u}$. Note that $\mathcal{N}_1 = \mathcal{N}$ and $\mathfrak{u}_1 = \mathfrak{u}$ whenever $p \geq h$.

2.3. **Commuting varieties.** Suppose $V$ is closed affine subvarieties of $\mathfrak{g}$. We define the commuting variety of $r$-tuples as follows:

$$C_r(V) = \{(x_1, \ldots, x_r) \in V^r \mid [x_i, x_j] = 0, \ 1 \leq i \leq j \leq r\}.$$  

We will just call it the commuting variety over $V$ for short. In case $V = \mathcal{N}$ (or $\mathcal{N}_1$), we call $C_r(V)$ the (restricted) nilpotent commuting variety of $\mathfrak{g}$. For more details of such varieties, one can refer to [Pr] [Ngo2]. The structure of the variety $C_r(\mathcal{N})$ is complicated when $r > 2$, however one can always describe an irreducible component associated to regular nilpotent elements as follows.

**Proposition 2.3.1.** For each $r \geq 1$, the subvariety $V_{\text{reg}} := \overline{G \cdot (x_{\text{reg}}, z_{\text{reg}}, \ldots, z_{\text{reg}})}$ is an irreducible component of $C_r(\mathcal{N})$ whose dimension is $\dim \mathcal{N} + (r - 1)\ell$.

**Proof.** First note that $V_{\text{reg}}$ is irreducible as there is a surjective morphism $m : G \times z_{\text{reg}}^{r-1} \to V_{\text{reg}}$ defined as $m(g, x_1, \ldots, x_{r-1}) = g \cdot (x_{\text{reg}}, x_1, \ldots, x_{r-1})$ for all $g \in G$ and $x_i \in z_{\text{reg}}$. Now consider the projection from $C_r(\mathcal{N})$ to its first component

$$p : C_r(\mathcal{N}) \to \mathcal{N}.$$  

Since $G \cdot x_{\text{reg}}$ is open in $\mathcal{N}$, so is its preimage $p^{-1}(G \cdot x_{\text{reg}}) = G \cdot (x_{\text{reg}}, z_{\text{reg}}, \ldots, z_{\text{reg}})$ (here we use the fact that $z_{\text{reg}}$ is commutative). So the closure $\overline{p^{-1}(G \cdot x_{\text{reg}})} = V_{\text{reg}}$ is an irreducible component of $C_r(\mathcal{N})$.

Applying the theorem on the dimension of fibers to the restriction of $p : V_{\text{reg}} \to \mathcal{N}$, we have

$$\dim V_{\text{reg}} = \dim \mathcal{N} + \dim p^{-1}(x_{\text{reg}}) = \dim \mathcal{N} + \dim z_{\text{reg}}^{r-1} = \dim \mathcal{N} + (r - 1)\ell,$$

which completes our proof. □

Replacing $G$ by $B$ in the above argument, we obtain a similar result for $C_r(\mathfrak{u})$ as follows.

**Proposition 2.3.2.** For each $r \geq 1$, the subvariety $B \cdot (x_{\text{reg}}, z_{\text{reg}}, \ldots, z_{\text{reg}})$ is an irreducible component of $C_r(\mathfrak{u})$ whose dimension is $\dim \mathfrak{u} + (r - 1)\ell$.

It follows an easy corollary.

**Corollary 2.3.3.** For each $r \geq 1$, if $C_r(\mathcal{N})$ (or $C_r(\mathfrak{u})$) is irreducible or equidimensional then its dimension is $\dim \mathcal{N} + (r - 1)\ell$ (or $\dim \mathfrak{u} + (r - 1)\ell$).
Later on this paper we show that \( C_r(\mathcal{N}) \) is not equidimensional for large enough \( r \). One of our main tools in this paper is the homeomorphism between \( C_r(\mathcal{N}_1) \) and the spectrum of \( H^*(G_r, k) \) established in [SFB]. (Hence from now on we identify \( \text{Spec}(H^*(G_r, k)_{\text{red}}) \) with \( C_r(\mathcal{N}_1) \).) In particular, one has the following identifications

\[
\dim C_r(u_1) = \dim H^*(U_r, k) = \dim H^*(B_r, k),
\]

and

\[
\dim C_r(\mathcal{N}_1) = \dim H^*(G_r, k).
\]

3. Cohen-Macaulay cohomology ring

We review in this section some definition of a Cohen-Macaulay ring and then state Poincaré duality for cohomology rings of the \( r \)-th Frobenius kernels. Note that this property was stated in a paper of Carlson and Nakano [CN] for the first Frobenius kernels. Most of terminology presented here can be found in [BC], [Ben1], or [CN].

3.1. Poincaré series and duality. Let \( V \) be a graded vector space of finite type over \( k \), i.e.

\[
V = \bigoplus_{i \geq 0} V_i
\]

where each \( V_i \) is a finite dimensional vector space. The Poincaré series is defined as

\[
p_V(t) = \sum_{i \geq 0} (\dim V_i)t^i.
\]

If \( V \) is a finitely generated module over a finitely generated commutative graded algebra of finite type, then one can rewrite \( p_V(t) \) as a rational function of the form

\[
\frac{f(t)}{\prod_{j=1}^s (1 - t^{h_j})}.
\]

Then the rate of growth of \( \{V_i\} \) is defined to be the order of the pole of above rational function at \( t = 1 \). In other words, it is the least integer \( s \geq 0 \) satisfying

\[
\lim_{n \to \infty} \frac{\dim V_n}{n^s} = 0.
\]

If no such \( s \) exists, one says that the rate of growth for \( \{V_i\} \) is \( \infty \).

Now we consider a finite group scheme \( G \) defined over \( k \). It is well-known that the category of \( G \)-modules is equivalent to that of modules over a finite-dimensional cocommutative Hopf algebra. The latter was intensitively studied in [BN], [BS], [Pa], and [W]. Let

\[
H^*(G, k) = \bigoplus_{i \geq 0} H^i(G, k)\text{\footnote{The reader should note that we use \( H^* \) to denote the cohomology ring instead of \( H^* \). This is to make it consistent with the notation in [BC] [Ben2] [CN], but inconsistent with that in [Ngo1].}}, \quad H^{2*}(G, k) = \bigoplus_{i \geq 0} H^{2i}(G, k)
\]

where the latter is sometimes called the even cohomology ring of \( G \) and it is always commutative. It is easy to see that \( H^*(G, k) \) is a graded commutative \( k \)-algebra. In [FS], Friedlander and Suslin proved that this ring is finitely generated. Moreover, they showed in general that any finite group scheme \( G \) satisfies the finiteness condition of cohomology, i.e. for each finitely generated \( G \)-module \( M \), \( H^*(G, M) \) is a finitely generated \( H^*(G, k) \)-module.

The Poincaré series associated to the cohomology ring \( H^*(G, k) \) is denoted by

\[
p_G(t) = \sum_{i \geq 0} \dim H^i(G, k)t^i.
\]
Now let
\[ P : \cdots \to P_1 \to P_0 \to M \]
be a minimal projective resolution of \( M \) in the category of \( G \)-modules. We define the complexity \( c_G(M) \) to be the rate of growth of the complex \( P \). As \( \text{Ext}_G^*(M, M) \) can be considered as a \( \text{H}^*(G, k) \)-module via Yoneda product, set \( J(M) \) the annihilator ideal in \( \text{H}^*(G, k) \) for this action. Then the support variety for \( M \), denoted by \( V_G(M) \), is defined as the spectrum of the reduced quotient ring \((\text{H}^*(G, k)/J(M))_{\text{red}}\). It turns out that \( c_G(M) \) is the same as the Krull dimension of this quotient ring, see \([NPV, \text{Theorem 2.2.2}]\). It follows that \( c_G(M) = \dim V_G(M) \). In particular, one has for each \( r \geq 1 \) and each finite dimensional \( G_r \)-module \( M \)
\[ V_{U_r}(M) = V_{B_r}(M) \subseteq C_r(u_1), \quad V_{G_r}(M) \subseteq C_r(N_1). \]
Specializing \( M = k \), one has the following identities, from the ones in Section 2.3
\[ c_{B_r}(k) = c_{U_r}(k) = \dim C_r(u_1), \]
and
\[ c_{G_r}(k) = \dim C_r(N_1). \]

3.2. Let \( R = \bigoplus_{i \geq 0} R_i \) be a finitely generated graded commutative \( k \)-algebra and \( M = \bigoplus_{i \geq 0} M_i \) be a graded \( R \)-module. A sequence \( r_1, \ldots, r_m \) of homogeneous elements of degree \( n_i > 0 \) is said to be a regular sequence for \( M \) if for every \( i = 1, \ldots, m \), we have that multiplication by \( r_i \) is an injective map from \( M/(r_1, \ldots, r_{i-1})M \) to itself. Some authors call it a homogeneous system of parameters, e.g. \([BC]\). The depth of \( M \) is the length of the longest regular sequence for \( M \). The module \( M \) is called to be Cohen-Macaulay if its depth is equal to its Krull dimension. The algebra \( R \) is Cohen-Macaulay if it is Cohen-Macaulay as a module over itself. In fact, one has the following characterization of a Cohen-Macaulay ring.

**Proposition 3.2.1.** \([Ben2, \text{Proposition 2.5.1}]\) The following are equivalent.
(a) \( R \) is Cohen-Macaulay
(b) There exists a homogeneous polynomial subring \( k[x_1, \ldots, x_r] \) such that \( R \) is finitely generated free module over \( k[x_1, \ldots, x_r] \).
(c) If \( k[x_1, \ldots, x_r] \) is a homogeneous polynomial subring of \( R \) over which \( R \) is a finitely generated module then \( R \) a free module over it.

In other words, the structure of a ring is more rigorous if it is Cohen-Macaulay. Moreover, in the context of cohomology rings, Cohen-Macaulayness implies Poincaré duality. The proof is just a combination of \([Ben1, \text{Theorem 5.18.1}]\) and \([ES, \text{Theorem 12}]\). In particular, we have

**Theorem 3.2.2.** Let \( G \) be a finite group scheme. If the cohomology ring \( \text{H}^*(G, k) \) is Cohen-Macaulay, then the Poincaré series \( p_G(t) \) satisfies the Poincaré duality, i.e.
\[ p_G(1/t) = (-t)^d p_G(t) \]
where \( d \) is the Krull dimension of \( \text{H}^*(G, k) \).

This is a generalized statement of \([CN, \text{Theorem 3.2}]\). As Cohen-Macaulay rings are well known in literature, we hope this result stimulates further study for commutative algebra of cohomology ring of finite group schemes.

4. A Revisit to \( SL_2 \) Case

Assume only in this section that \( G = SL_2 \). We show that the cohomology ring of the \( r \)-th Frobenius kernel of \( G \) is Cohen-Macaulay. Then the last theorem in the previous section implies that \( G_r \) satisfies Poincaré duality for each \( r \geq 1 \). Note that the author already verified the Cohen-Macaulayness for \( \text{H}^*(B_r, k)_{\text{red}} \) and \( \text{H}^*(G_r, k)_{\text{red}} \) in \([Ngo1]\). We now remove the reducedness condition, however, most of calculations are similar as in \([Ngo1]\). So we skip unnecessary details.
4.1. $U_r$-cohomology ring. Recall from Corollary 4.1.2 in [Ngo1] that there is a $B$-algebra isomorphism
\[ \text{H}^*(U_r, k) \cong k[x_1, \ldots, x_r] \otimes \Lambda^*(y_0, \ldots, y_{r-1}) \]
where each $x_i$ is of degree 2 and of weight $p^i\alpha$, while each $y_j$ is of degree 1 and weight $p^j\alpha$ and $\Lambda^*(y_0, \ldots, y_{r-1})$ is the exterior algebra over the vector space spanned by $y_0, \ldots, y_{r-1}$. This description implies that the ring $\text{H}^*(U_r, k)$ is Cohen-Macaulay by definition.

4.2. $B_r$-cohomology ring. In this case we can not apply Hochster-Robert’s result for free as the ring $\text{H}^*(U_r, k)$ is not regular. However, we are able to describe the structure of $B_r$-cohomology as a free module over a polynomial ring. In particular, applying $\lambda = 0$ in Proposition 4.2.1 [Ngo1], we obtain the following

**Proposition 4.2.1.** For each $r \geq 1$ and $n \geq 0$, there is a $B$-isomorphism
\[ \text{H}^n(B_r, k)^{(-r)} \cong \bigoplus \left\{ \left( x_1^{a_1} y_0^{b_1} x_2^{a_2} y_1^{b_2} \cdots x_r^{a_r} y_{r-1}^{b_{r-1}} \right) \right\} \]
where the direct sum is taken over all $a_i, b_j$ satisfying the following conditions
\[
\begin{align*}
  a_i &\in \mathbb{N} \text{ and } b_i \in \{0, 1\} \text{ for all } 1 \leq i \leq r \\
  n &\leq 2(a_1 + \cdots + a_r) + b_1 + \cdots + b_r \\
  b_1 + (a_1 + b_2)p + \cdots + a_r p^{r} &\in p^r X^+.
\end{align*}
\]

It follows that $k[x_1^{p^{r-1}}, x_2^{p^{r-2}}, \ldots, x_r]$ is a subring of $\text{H}^*(B_r, k)$. Moreover, we can describe the cohomology ring of $B_r$ as a free module over that polynomial ring with the basis $\mathcal{B}_r^* = \{x_1^{a_1} y_0^{b_1} x_2^{a_2} y_1^{b_2} \cdots x_r^{a_r} y_{r-1}^{b_{r-1}}\}$ where $a_i, b_j$ satisfy (1) plus $0 \leq a_i < p^{r-i}$. In summary, let $R = k[x_1^{p^{r-1}}, x_2^{p^{r-2}}, \ldots, x_r]$, we then have

**Corollary 4.2.2.** For each $r \geq 1$, there is an isomorphism of $R$-modules as well as $B$-modules
\[ \text{H}^*(B_r, k) \cong \bigoplus_{v \in \mathcal{B}_r^*} v \otimes R. \]

In other words, the ring $\text{H}^*(B_r, k)$ is Cohen-Macaulay.

4.3. $G_r$-cohomology ring. Again we follow the strategy in [Ngo1] Theorem 4.3.1 and combine with Corollary 4.2.2 to obtain that
\[ \text{H}^*(G_r, k)^{(-r)} \cong \bigoplus_{v \in \mathcal{B}_r^*} \text{ind}_B^G(v \otimes R)^{(-r)}. \]

Now it is easy to see that each graded $k$-algebra $\text{ind}_B^G(v \otimes R)^{(-r)}$ is the graded $k$-algebra $\text{ind}_B^G(R^{(-r)})$ shifted by certain degree. Hence it suffices to prove that the latter is Cohen-Macaulay. This could be established by the same argument as in [Ngo1] Lemma 7.1.3], which heavily relies on the computation in [Ngo2]. We provide here an alternative proof which is more straightforward. The following lemma is another approach to attack the commutative algebra aspect of cohomological objects.

**Lemma 4.3.1.** Let $S$ be a $B$-algebra on which $U$ acting trivially. Suppose that $S$ is regular as a commutative ring. Then the ring $\text{ind}_B^G S$ is Cohen-Macaulay.

**Proof.** We have
\[ \text{ind}_B^G S = [k[G] \otimes S]^B \]
\[ \cong [(k[G] \otimes S)^U]^B/U \]
\[ \cong [k[G]_U \otimes S]^T. \]
Now it is not hard to compute that \( k[G]^U \) is in fact a polynomial ring over 2 variables, see [Po, 2.1], so that it is a regular ring. As tensoring preserves regularity, we get \( k[G]^U \otimes S \) is regular. Now since \( T \) is linearly reductive, a result of Hochster-Robert implies that the invariant ring \( [k[G]^U \otimes S]^T \) is Cohen-Macaulay; hence completing our proof.

Now we can tackle the Cohen-Macaulayness of the \( G_r \)-cohomology ring.

**Theorem 4.3.2.** For each \( r \geq 1 \), the ring \( H^*(G_r, k) \) is Cohen-Macaulay.

**Proof.** Note that \( R^{(-r)} \cong k[u^r] \) as a \( B \)-algebra. As \( U \) trivially acts on \( u^* \), \( U \) does the same on \( R^{(-r)} \). Applying the lemma above, we get \( \text{ind}^G_B(R^{(-r)}) \) is Cohen-Macaulay. Thus the theorem follows from the fact that it is the direct sum of copies of \( \text{ind}^G_B(R^{(-r)}) \).

Now combining with Theorem 3.2.2, we have

**Corollary 4.3.3.** The Poincaré series \( p_{B_r}(t) \) and \( p_{G_r}(t) \), associated to \( H^*(B_r, k) \) and \( H^*(G_r, k) \), satisfy the Poincaré duality.

**Remark 4.3.4.** The Lemma 4.3.1 would not hold if \( S \) was just Cohen-Macaulay. In fact, Hochster gave an explicit example in [Ho, p. 900] for the more general fact, that is, the invariant subring of a Cohen-Macaulay ring under a torus action is not Cohen-Macaulay. As a consequence, it is not true in general that the ring \( \text{ind}^G_B R \) is Cohen-Macaulay provided that \( R \) is so. In other words, this disproves Conjecture 7.2.1 in an earlier paper of the author [Ngo1]. It remains interesting to know under what conditions the conjecture holds.

5. Cohen-Macaulay of \( U_r \) and \( B_r \)-cohomology

In this section we are back to the assumption that \( G \) is a classical group, however the below result is still true for any reductive algebraic group. Our goal is to generalize some properties in the previous section. Explicitly, our goal is to investigate whether the Cohen-Macaulayness of \( H^*(U_r, k) \) implies the same for \( H^*(B_r, k) \). The first part of this section give a solution to this problem while the second one point out when this possibly happens.

5.1. We first recall a result in Hashimoto’s paper as follows.

**Proposition 5.1.1.** [Ha, pp. 4] Given an exact sequence of \( k \)-group schemes

\[
0 \to H \to G \to N \to 0
\]

where \( H \) is a linearly reductive finite group scheme, \( N \) is an affine algebraic group scheme. Let \( R \) be a \( G \)-algebra and set \( S = R^H \). If \( R \) is Cohen-Macaulay, then so is \( S \).

This proposition suggests the following result on the Cohen-Macaulayness of \( H^*(B_r, k) \).

**Theorem 5.1.2.** If for each \( r \geq 1 \), the algebra \( H^*(U_r, k) \) is a Cohen-Macaulay ring, then so is \( H^*(B_r, k) \).

**Proof.** Note that the result of Hashimoto is valid for commutative algebras while our cohomology ring is graded commutative (We are not sure if there is another version applicable for graded commutative algebras). However, we are still able to use it for the even cohomology ring. To do this, we first prove the claim that if \( H^*(U_r, k) \) is Cohen-Macaulay then it is a direct sum of copies of the cohomology ring \( H^{2*}(U_r, k) \) as a \( T \)-algebra. Indeed, suppose \( H^*(U_r, k) \) is a free module over

\[\text{ind}^G_B(R^{(-r)})\]

\[^2\text{It is stated in [Ha] that } R \text{ is a domain but not necessary here. We especially thanks Mitsuyasu Hashimoto for pointing out this fact.}\]
a polynomial ring $R$, then $R$ must be contained in $H^{2s}(U_r, k)$ as all cocycles in odd degree are nilpotent. Now let $\mathcal{B}$ be a basis of $H^*(U_r, k)$ over $R$. It follows that

$$H^{2s}(U_r, k) = \bigoplus_{b \in \mathcal{B}, \deg(b) \text{ even}} b \cup R$$

and so

$$H^*(U_r, k) = \bigoplus_{b \in \mathcal{B}, \deg(b) \text{ odd}} b \cup H^{2s}(U_r, k)$$

which proves our claim.

For each $r \geq 1$, we have the following short exact sequence

$$0 \to T_r \to T \to T^{(r)} \to 0$$

where $T_r$ is linearly reductive. It can be seen that $H^*(U_r, k)$ is a $T$-algebra. Hence we have the following

$$H^*(B_r, k) \cong H^*(U_r, k)^{T_r} \cong \bigoplus_{b \in \mathcal{B}, \deg(b) \text{ odd}} (b \cup H^{2s}(U_r, k))^{T_r}$$

Each direct summand in the last item can be considered as a graded algebra $H^{2s}(U_r, k)^{T_r}$ shifted by $\deg(b)$. On the other hand, $H^{2s}(U_r, k)^{T_r}$ is Cohen-Macaulay by earlier proposition. So $H^*(B_r, k)$ is a direct sum of Cohen-Macaulay rings; hence it is so. \(\square\)

**Remark 5.1.3.** The theorem is not so helpful as the fact that $H^*(U_r, k)$ is Cohen-Macaulay rarely happens when $r > 1$. In particular, our following observations on groups of type $A$, i.e. $G = GL_n$, reveal that this cohomology ring is Cohen-Macaulay only when $n \leq 3$ or possibly some exceptions for small values of $r$.

5.2. **Observation.** Let $G = GL_n$. To be convenient, we assume for this section that $p \geq n - 1$. Our aim is to show that the cohomology ring $H^*(U_r, k)$ is not Cohen-Macaulay for $n \geq 4$ and $r \geq 3$. Thanks to the result of Suslin, Friedlander, and Bendel, this is reduced to showing that the commuting variety $C_r(u)$ is not equidimensional for such values of $n$ and $r$.

Before proving the main result of this section, we set up an additional notation which plays a key role in our later proof. Let $u'$ be the subspace of $u$ consisting of all $n$ by $n$ matrices of the form

$$\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$$

where $A$ is an $\frac{n}{2}$ by $\frac{n}{2}$ matrix if $n$ is even, otherwise it is an $\frac{n+1}{2}$ by $\frac{n-1}{2}$ matrix. Observe that $u'$ is a commutative subalgebra of $g$. Moreover, we have

$$\dim u' = \begin{cases} \frac{n^2}{2} & \text{if } n \text{ even,} \\ \frac{n^2}{2} - 1 & \text{if } n \text{ odd.} \end{cases}$$

**Proposition 5.2.1.** The variety $C_r(u)$ is reducible and not equidimensional for all $n \geq 4$ and $r \geq 3$.

**Proof.** By contradiction, we assume that $C_r(u)$ were irreducible or equidimensional. Then from Corollary 2.3.3 we have

$$\dim C_r(u) = \dim u + (r - 1)\ell = \frac{n^2 - n}{2} + (r - 1)(n - 1).$$
On the other hand, it is easy to see that \( u^r \) is a proper subvariety of \( C_r(u) \). This led us to the following inequalities

\[
\begin{align*}
\frac{n^2-n}{2} + (r-1)(n-1) > \frac{r n^2}{4} & \quad \text{if } n \text{ even}, \\
\frac{n^2-n}{2} + (r-1)(n-1) > \frac{r(n^2-1)}{4} & \quad \text{if } n \text{ odd}.
\end{align*}
\]

It would follow that \( r < 3 \) or \( n < 4 \). Contradiction! \( \square \)

An easy corollary follows immediately.

**Corollary 5.2.2.** For all \( n \geq 4 \) and \( r \geq 3 \), the ring \( H^*(U_r, k) \) is not a domain and not Cohen-Macaulay.

**Remark 5.2.3.** Even in the case \( r = 2 \), the cohomology ring \( H^*(U_2, k) \) is also rare to be Cohen-Macaulay for the same reason. Goodwin and Röhrle showed in their recent preprint [GR] that \( C_2(u) \) is not equidimensional when \( u \) has infinitely many \( B \)-orbits, i.e. \( n \geq 6 \). Although their work is assumed for \( k \) of characteristic 0, it is expected to be true as well for large enough \( p \).

6. Bounding the complexity of Frobenius kernels

Keep assumptions as in the last section. It is easy to see that Propositions 2.3.2 and 2.3.1 give lower bounds for \( \dim C_r(u) \) and \( \dim C_r(N) \), hence giving those for \( c_{U_r}(k), c_{B_r}(k) \), and \( c_{G_r}(k) \) when \( p \) is sufficiently large. Our goal in this section is to improve these bounds and remove the restriction on \( p \). One can see that as \( p \) gets smaller, \( u_1 \) (and \( N_1 \)) gets smaller. So the strategy is to develop the square zero subvariety \( u' \) of \( u_1 \) (in the last section) to make it applicable for other types. Then we induce subvarieties of \( C_r(u_1) \) and \( C_r(N_1) \) from this object. It turns out that for large enough \( p \) such varieties have dimensions greater, for most \( r \) and \( \ell \), than the ones described in the aforementioned propositions. This therefore imply that \( C_r(N) \) is not equidimensional in these cases so that the cohomology ring \( H^*(G_r, k) \) is not Cohen-Macaulay for sufficiently large characteristic \( p \). In the special case when \( p = 2 \) and \( G \) is of type \( A \) (note that 2 is not a good prime for other types), we further obtain the exact complexity for \( U_r, B_r, \) and \( G_r \).

6.1. We first have the following

**Theorem 6.1.1.** For each \( r \geq 1 \), we have

\[
c_{B_r}(k) = c_{U_r}(k) \geq \begin{cases} 
\frac{r (\ell+1)^2}{4} & \text{if } G \text{ of type } A, \\
\frac{r(\ell^2+\ell)}{2} & \text{otherwise}
\end{cases}
\]

and

\[
c_{G_r}(k) \geq \begin{cases} 
\frac{(r+1) (\ell+1)^2}{4} & \text{if } G \text{ of type } A, \\
\frac{(r+1) (\ell^2+\ell)}{2} & \text{otherwise}.
\end{cases}
\]

**Proof.** We consider 2 cases:

- \( G \) is of type \( A_\ell \): From the computation in Section 5.2, we have \( u^r \subseteq C_r(u_1) \). Note that this is true for all \( p > 0 \) since \( x^2 = 0 \) for each \( x \in u' \) so that \( u' \subseteq u_1 \). Hence \( \dim C_r(u_1) \geq \dim u^r = r \dim u' \), and then the first inequality follows from (2), in terms of \( n = \ell + 1 \).

- \( G \) is of other types: We need to modify the commutative algebra \( u' \) to make it a subalgebra of \( g \). Explicitly, we embed \( g \) into the general linear algebra \( gl_n \). Note that

\[
n = \begin{cases} 
2\ell & \text{if } G \text{ of type } C_\ell \text{ or } D_\ell, \\
2\ell + 1 & \text{if } G \text{ of type } B_\ell.
\end{cases}
\]

Now one can consider

\[
v = \left\{ \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \in gl_n : B \text{ be arbitrary symmetric } \ell \text{ by } \ell \text{ matrix} \right\}
\]
As a commutative subalgebra of \( g \). Again it is easy to see that \( \mathfrak{v} \subseteq \mathfrak{u}_1 \) and \( \dim \mathfrak{v} = \frac{G + \ell}{2} \). So \( \dim \mathfrak{c}_{r}(u_1) \geq r \dim \mathfrak{v} \) which implies the second inequality for \( c_{U_r}(k) \) and \( c_{B_r}(k) \).

Let \( \mathfrak{w} = \mathfrak{u}' \) or \( \mathfrak{v} \) respectively up to the type of \( G \). Then denote \( V = G \cdot \mathfrak{w}^r \). Note that \( V \subseteq C_r(N_1) \) and it is irreducible. Now we consider the projection \( p : V \rightarrow G \cdot \mathfrak{w} \).

Let \( \mathcal{O} = G \cdot \mathfrak{v} \) be the Richardson orbit corresponding to \( \mathfrak{w} \). Then we have \( \mathcal{O} \) is an open dense subset of \( G \cdot \mathfrak{w} \) \cite{Jan2} Lemma 8.8(a). Combining with the formula (2) in \cite{Jan2} 8.8, we obtain that

\[
\dim G \cdot \mathfrak{w} = \dim(\mathcal{O}) = \dim(G/P) + \dim \mathfrak{w} = 2 \dim \mathfrak{w}.
\]

As each fiber \( p^{-1}(g \cdot \mathfrak{v}) = g(\mathfrak{v}, \mathfrak{w}, \ldots, \mathfrak{w}) \cong \mathfrak{w}^{r-1} \) with \( g \in G \) is of dimension \( (r - 1) \dim \mathfrak{w} \), we have

\[
\dim V = \dim G \cdot \mathfrak{w} + \dim p^{-1}(g \cdot \mathfrak{v}) = (r + 1) \dim \mathfrak{w}
\]

Finally the result follows from the dimension of \( \mathfrak{w} \).

Let \( h \) be the Coxeter number of \( G \), that is \( h = \ell + 1, 2\ell \), or \( 2\ell - 2 \) when \( G \) is respectively of type \( A_{\ell}, B_{\ell}, C_{\ell}, \) or \( D_{\ell} \). Note that if \( p \geq h \), one has \( N_1 = N \). We can show further that the dimension of the subvariety \( V \) in the proof above is greater than that of the irreducible component \( V_{\text{reg}} \).

**Corollary 6.1.2.** Suppose \( p \geq h \). If \( G \) is of type \( A \), suppose \( r \geq 6, \ell \geq 3 \). Otherwise, suppose \( r \geq 4 \). Then the variety \( C_r(N) \) is not equidimensional. Therefore, the ring \( H^*(G_r, k) \) is not Cohen-Macaulay.

**Proof.** Assume that \( C_r(N) \) were equidimensional. Then Corollary 2.3.3 would imply that \( \dim C_r(N) = \dim N + (r - 1)\ell \). Hence, we would have

\[
\dim G \cdot \mathfrak{w}^r \leq \dim N + (r - 1)\ell
\]

where the dimension of \( G \cdot \mathfrak{w}^r \) is either \( (r + 1)(\ell + 2) / 4 \) (for type \( A \)) or \( (r + 1)(\ell^2 + \ell) / 2 \) (for other types). Now the values of \( r \) and \( \ell \) in the hypothesis would indicate that the above inequality were impossible. So the proof follows by contradiction.

This result disproves Conjecture 7.2.2 in \cite{Ngo1} on the Cohen-Macaulay property of the \( G_r \) cohomology ring. The cases when this conjecture hopefully hold are \( r = 2 \) (and arbitrary \( \ell \)) or low rank \( \ell \) (and arbitrary \( r \)). We have recently verified that the variety \( C_r(N) \) is Cohen-Macaulay for type \( A_2 \) in \cite{Ngo3} however it remains open whether the ring \( H^*(G_r, k) \) is Cohen-Macaulay. As there is an \( F \)-isomorphism between this cohomology ring and \( k[C_r(N)] \), it would be nice if we could develop some relations on commutative algebra properties between the two rings in this context.

Now let \( X \) be the weight lattice of \( \Phi \). Write \( X^+ \) for the set of dominant weights in \( X \), and \( X_1 \) for the set of \( p \)-restricted dominant weights in \( X^+ \). Let \( c = \left( \frac{\ell + 1}{2} \right)^2 \) (resp. \( \frac{\ell(\ell + 1)}{2}, \frac{\ell^2}{2} \) or \( \frac{\ell(\ell - 1)}{2} \)) if \( G \) is of type \( A_{\ell} \) (resp. \( B_{\ell}, C_{\ell} \) or \( D_{\ell} \)). Given \( \lambda \in X_1 \) and suppose \( p > hc \), our calculations give a lower bound for the complexity of the simple module \( L(\lambda) \).

**Corollary 6.1.3.** Let \( \lambda \in X_1 \). Suppose \( p > hc \). Then one has for each \( r \geq 2 \)

\[
c_{G_r}(L(\lambda)) \geq \begin{cases} r \left( \frac{(\ell + 1)^2}{4} \right) & \text{if } G \text{ of type } A, \\ r\left( \frac{\ell^2 + \ell}{2} \right) & \text{otherwise} \end{cases}
\]

**Proof.** Proposition 3.1 in \cite{So} gives us \( V_{G_r}(L(\lambda)) \supseteq V_{G_{r-1}}(k) \) so that \( c_{G_r}(L(\lambda)) \geq c_{G_{r-1}}(k) \). The last inequality and Theorem 6.1.1 prove our result.
Regarding upper bounds for $c_{G_r}(k)$, it is easy to see that there is always the following
\[ c_{G_r}(k) = \dim C_r(N_1) \leq \dim N_1' = r \dim N_1 = r c_{G_1}(k). \]
However, this upper bound is not useful at all when $r > 1$. So finding sharper bounds for this amount could be an interesting problem. Given a finite dimensional $G_r$-module $M$, what we prove as follows is an upper bound for $c_{G_r}(M)$ in terms of $C_B(M)$.

**Proposition 6.1.4.** For each $r \geq 1$, $c_{G_r}(M) \leq c_{B_r}(M) + \dim u$.

*Proof.* Again it suffices to prove the inequality in terms of support varieties, that is $\dim V_{G_r}(M) \leq \dim V_{B_r}(M) + \dim u$. As the support variety $V_{B_r}(M)$ is a $B$-variety and $V_{G_r}(M) = G \cdot V_{B_r}(M)$, we establish the moment morphism $G \times^B V_{B_r}(M) \to V_{G_r}(M)$, which is surjective. It follows that
\[ \dim V_{G_r}(M) \leq \dim (G \times^B V_{B_r}(M)) = \dim (G/B) + \dim V_{B_r}(M) = \dim u + \dim V_{B_r}(M). \]
This proves our proposition. \hfill \square

**Remark 6.1.5.** When $M = k$, this upper bound is sharp and the equality occurs in the case of $r = 1$ and $p \geq h$ or when $C_r(u_1)$ is irreducible. As a consequence, an upper bound of $c_{B_r}(M)$ gives that of $c_{G_r}(M)$.

### 6.2. The case $p = 2$ and $G$ of type $A_t$

We keep employing the square zero subvariety $V$ of $C_r(N_1)$. In this very special case, we can obtain the exact dimension of $C_r(N_1)$, which is $\dim V$. In particular, one has the following.

**Theorem 6.2.1.** For each $r \geq 1$, the subvarieties $u^r$ and $G \cdot u^r$ are irreducible components of maximal dimensions in $C_r(u_1)$ and $C_r(N_1)$. Consequently,
\[ \dim C_r(u_1) = r \left( \left( \frac{\ell + 1}{2} \right) + 1 \right), \quad \dim C_r(N_1) = (r + 1) \left( \frac{\ell + 1}{2} \right). \]

*Proof.* From [CLNP Theorem 3.1(b)(ii)], we have
\[ \dim N_1 = \frac{(\ell + 1)^2 + s^2 - 1}{2} \]
where $s = \ell \pmod{2}$. Hence, one has
\[ \dim u_1 = \frac{1}{2} \dim N_1 = \frac{(\ell + 1)^2 + s^2 - 1}{4} = \dim u'. \]
It follows that $u'$ is an irreducible component of maximal dimension of $u_1$. Thus, $u^r$ is an irreducible component of maximal dimension of $C_r(u_1)$. Then the dimension of $C_r(u_1)$ follows.

Note that $C_r(N_1) = G \cdot C_r(u_1)$. As $G \cdot u' = N_1$ and $u^r$ is an irreducible component of maximal dimension in $C_r(u_1)$, the saturation $G \cdot u^r$ is an irreducible component of maximal dimension in $C_r(N_1)$; hence proving our theorem. \hfill \square

**Corollary 6.2.2.** For each $r \geq 1$, one has
\[ c_{U_r}(k) = c_{B_r}(k) = r \left( \left( \frac{(\ell + 1)^2}{4} \right), \quad c_{G_r}(k) = \dim H^r(G_r, k) = (r + 1) \left( \frac{(\ell + 1)^2}{4} \right). \]

This result implies that the lower bounds in Theorem 6.1.1 are actually sharpened. Moreover, it generalizes the computation in [L Theorem 2.3] where $r = 2$. The problem for larger prime $p$ is difficult with $r \geq 3$ as one does not know any irreducible component of maximal dimension of $C_r(u_1)$ then. We end this section by leaving this as an open question.

**Question 6.2.3.** What are irreducible components of maximal dimension in $C_r(u_1)$ and $C_r(N_1)$?
7. Rank 2 cases

We show here our explicit results for the complexity of $G_\tau$ when $G$ is a simply classical group of rank 2. Our calculations are performed under the assumption that characteristic $k$ is a good prime, i.e. all primes for type $A$ and $p \neq 2$ for the other types. The strategy is computing the dimension of the corresponding restricted nilpotent commuting variety.

7.1. Type $A_2$. We first consider the case when $p \geq 3$. Then the restricted nullcone $N_1$ coincides with the nilpotent cone $N$ and $u_1 = u$. Hence we immediately obtain the following

**Proposition 7.1.1.** For each $r \geq 1$, we have

$$c_{U_r}(k) = c_{B_r}(k) = \dim C_r(u) = 2r + 1,$$

and $c_{G_r}(k) = \dim C_r(N) = 2r + 4$.

**Proof.** Follows from Proposition 7.1.1 and Theorem 7.1.2 in [Ngo2]. □

Now suppose $p = 2$. Then the restricted nullcone $N_1 = \overline{O_{sub}}$ and $u_1 = u \cap \overline{O_{sub}}$ which implies that

$$u_1 = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : x, y \in k \right\}.$$

Moreover, $C_r(u_1) = u_1^r$. It follows the first dimension formula for $\dim C_r(u_1)$ below.

**Proposition 7.1.2.** For each $r \geq 1$, we have

$$c_{U_r}(k) = c_{B_r}(k) = \dim C_r(u_1) = 2r,$$

and

$$c_{G_r}(k) = \dim C_r(N_1) = 2r + 2.$$

**Proof.** We have

$$c_{G_r}(k) = \dim C_r(N_1) = \dim C_r(\overline{O_{sub}}) = 2r + 2$$

by [Ngo2, Proof of Theorem 7.2.3]. □

Finally we combine our earlier results in the complexity formula (as well as the Krull dimension formula of the cohomology ring) as follows.

**Theorem 7.1.3.** For each $r \geq 1$, one has

$$c_{U_r}(k) = c_{B_r}(k) = \dim H^*(U_r, k) = \dim H^*(B_r, k) = \begin{cases} 2r + 1 & \text{if } p > 2, \\ 2r & \text{if } p = 2, \end{cases}$$

and

$$c_{G_r}(k) = \dim H^*(G_r, k) = \begin{cases} 2r + 4 & \text{if } p > 2, \\ 2r + 2 & \text{if } p = 2. \end{cases}$$

**Remark 7.1.4.** This result generalizes the one of Kaneda et. al. in [KSTY] where they established the bounds for the Krull dimension of $B_r$-cohomology ring in the case $p > 2$, $2r \leq \dim H^*(B_r, k) \leq 2r + 1$.

Our result is much more powerful as it not only gives an exact formula but also include the case $p = 2$. 
7.2. Type $B_2 \cong C_2$. We first assume $p > 3$. This means $u_1 = u$ and so we have

**Proposition 7.2.1.** For each $r \geq 2$, $C_r(u)$ has two irreducible components of dimensions $3r$ and $2r + 2$. Hence, $c_{U_r}(k) = c_{B_r}(k) = 3r$.

**Proof.** We first have

$$ u = \begin{cases} 
0 & x & z & t \\
0 & 0 & y & z \\
0 & 0 & 0 & -x \\
0 & 0 & 0 & 0 
\end{cases} \mid x, y, z, t \in k. $$

Analyzing commutators of $u$, one can obtain that $C_r(u)$ is defined by polynomials $\{x_iy_j - x_jy_i, x_iz_j - x_jz_i \mid 1 \leq i \leq j \leq r\}$. So it splits up into 2 components where the additional equations for the first are $x_1 = \cdots = x_r = 0$, and those for the second are $y_i z_j - y_j z_i$ for $1 \leq i \leq j \leq r$. It is now not hard to see that the first component is the affine space of dimension $3r$ and the second one is the product of a determinantal variety generated by $2 \times 2$ minors of a $3 \times r$ matrix of indeterminants and an affine space of dimension $r$, so that of dimension $r + r + 2 = 2r + 2$. $\square$

It now follows the complexity of $G_r$.

**Proposition 7.2.2.** For each $r \geq 2$, $c_{G_r}(k) = \dim C_r(\mathcal{N}) = \max\{2r + 6, 3r + 3\}$. In particular, $c_{G_r}(k) = 3r + 3$ for $r \geq 3$.

**Proof.** Fix $r \geq 2$, from the previous proposition, let $V_1$ and $V_2$ be the two irreducible components of $C_r(u)$. As the variety $C_r(u)$ is $B$-stable, so is each $V_i$ for $i = 1, 2$. One has

$$ C_r(\mathcal{N}) = G \cdot C_r(u) = G \cdot V_1 \cup G \cdot V_2. $$

The dimension of $G \cdot V_1$ is $(r + 1) \dim V_1 = 3r + 3$ as computed in Theorem 6.1.1. For the second one, we consider the following birational morphism

$$ G \times^B V_2 \to G \cdot V_2. $$

So $\dim G \cdot V_2 = \dim G/B + \dim V_2 = 2r + 6$. This completes our proof. $\square$

Note that for $r = 2$ we have $G \cdot V_1$ belongs to the other so that $C_2(\mathcal{N})$ is irreducible. For $r \geq 3$, the two components are different and then $C_2(\mathcal{N})$ is reducible.

Before moving on to the case when $p = 3$, we need some extra notation. Suppose $R$ is a polynomial ring and $J$ is a subset of $R$. Then denote by $V(J)$ the variety generated by polynomials in $J$.

**Lemma 7.2.3.** Suppose $p = 3$. We have $u_1 = (u \cap V(x)) \cup (u \cap V(y))$.

**Proof.** It follows from the observation that the cube of a generic element in $u$ is 0 if and only if $x^2y = 0$. $\square$

Identify $u_1^r$ with the affine space with indeterminates $x_i, y_i, z_i, t_i$ for $1 \leq i \leq r$. Then easy computations on commutators of $u_1$ give us the following

**Proposition 7.2.4.** For $r \geq 2$, $\dim C_r(u_1) = 3r$ and the irreducible component of maximal dimension is $V(x_1, \ldots, x_r)$. Hence $\dim C_r(\mathcal{N}_1) = 3r + 3$.

**Corollary 7.2.5.** For $r \geq 1$, one has

$$ c_{U_r}(k) = c_{B_r}(k) = \dim H^r(U_r, k) = \dim H^r(B_r, k) = \begin{cases} 
4 & \text{if } r = 1, p > 3, \\
3r & \text{else},
\end{cases} $$

and

$$ c_{G_r}(k) = \dim H^r(G_r, k) = \begin{cases} 
2r + 6 & \text{if } r \leq 2, p \neq 3, \\
3r + 3 & \text{else}.
\end{cases} $$
7.3. **Type** $D_2$. It is well known that $D_2$ is isomorphic to $A_1 \oplus A_1$, i.e. $\mathfrak{so}_4 = \mathfrak{sl}_2 \times \mathfrak{sl}_2$. Hence, one has the restricted nullcone of $\mathfrak{so}_4$

$$\mathcal{N}_1 \cong \mathcal{N}_1(\mathfrak{sl}_2) \times \mathcal{N}_1(\mathfrak{sl}_2)$$

where $\mathcal{N}_1(\mathfrak{sl}_2)$ is the restricted nullcone of $\mathfrak{sl}_2$. It follows that $C_r(\mathcal{N}_1) = C_r(\mathcal{N}_1(\mathfrak{sl}_2)) \times C_r(\mathcal{N}_1(\mathfrak{sl}_2))$.

So we have a result.

**Proposition 7.3.1.** For $p > 2$ and $r \geq 1$, one has $\dim C_r(\mathcal{N}_1) = 2r + 2$. Therefore, $c_{G_r}(k) = 2r + 2$.

**Proof.** Follow from the fact that for each $r \geq 1$, $\dim C_r(\mathcal{N}_1(\mathfrak{sl}_2)) = \dim C_r(\mathcal{N}(\mathfrak{sl}_2)) = r + 1$ in [Ngo2, Proposition 5.1.1].

7.4. As a consequence, we establish an upper bound for the complexity of arbitrary $G_r$-module. Suppose $M$ is a finite dimensional module over $G_r$. It is well known that the support variety for $M$ is a closed subvariety of $V_{G_r}(k) = C_r(\mathcal{N}_1)$; hence $c_{G_r}(M) = \dim V_{G_r}(M) \leq \dim C_r(\mathcal{N}_1)$. In summary, combining all previous results in this section we have the following

**Corollary 7.4.1.** For $r \geq 1$ and $M$ a finite dimensional $G_r$-module, one has

$$c_{G_r}(M) \leq \begin{cases} 2r + 4 & \text{if } G \text{ of } A_2, \\ \max\{2r + 6, 3r + 3\} & \text{if } G \text{ of } B_2 \text{ or } C_2, \\ 2r + 2 & \text{if } G \text{ of } D_2. \end{cases}$$

7.5. **Complexity for simple modules over** $G_r$. We assume further that $p > h$ here. The goal of this section is to generalize the calculations in previous section to compute the dimension of support varieties for a simple module of a restricted dominant weight over $G_r$, hence obtain the complexity of that module. Our tool is a mix of results of Sobaje and Drupieski-Nakano-Parshall which identifies the aforementioned support varieties with commuting varieties of the form

$$C_r(V, \mathcal{N}) = C_r(\mathcal{N}) \cap (V \times \mathcal{N}^{r-1})$$

with $V$ a closure of some Richardson orbit in $\mathcal{N}$. To be more precise, we set up some preliminary notation as follows.

Let $J \subseteq \Pi$ be a subset of the set of simple roots of $G$. Then set $\Phi_J = \Phi \cap \mathbb{Z}J$, a closed subroot system of $\Phi$ corresponding to $J$. Hence, $\Phi_J^+$ and $\Phi_J^-$ are respectively the sets of positive and negative roots of $\Phi_J$. Now, for any $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ be the $\alpha$-root space of $\mathfrak{g}$. We define the parabolic Lie algebra associated to $J$ as

$$\mathfrak{p}_J = \mathfrak{l}_J \oplus \mathfrak{u}_J$$

where

$$\mathfrak{l}_J = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_J} \mathfrak{g}_\alpha , \quad \mathfrak{u}_J = \bigoplus_{\alpha \in \Phi^- \setminus \Phi_J^-} \mathfrak{g}_\alpha.$$ 

In particular, if $J = \emptyset$ then $\mathfrak{p}_J = \mathfrak{b}$ the Borel Lie subalgebra of $\mathfrak{g}$. For each $J \subseteq \Pi$, the set $G \cdot \mathfrak{u}_J$ is an irreducible closed subvariety of the nilpotent cone $\mathcal{N}$, (and coincides with $\mathcal{N}$ when $J = \emptyset$). In such a variety, there exists a dense orbit of $G$ which is called Richardson orbit and denoted by $O_J$. Recall that $X$ is the weight lattice of $\Phi$. Write $X^+$ for the set of dominant weights in $X$, and $X_1$ for the set of $p$-restricted dominant weights in $X^+$. Given $\lambda \in X^+$, set

$$\Phi_{\lambda,p} = \{ \beta \in \Phi \mid (\lambda + \rho, \beta^\vee) \in p\mathbb{Z} \}.$$ 

Particularly, $\lambda$ is called $p$-regular if $\Phi_{\lambda,p} = \emptyset$, otherwise it’s called $p$-singular. There is $w \in W$ such that $w(\Phi_{\lambda,p}) = \Phi_J$ for some $J \subseteq \Pi$. Then combining results of Drupieski-Nakano-Parshall and Sobaje, we have a description of the support variety for $L(\lambda)$, the simple $G$-module of highest weight $\lambda$, over $G_r$. 

...
Proposition 7.5.1. \[\text{Suppose } p > hc. \text{ For each } \lambda \in X^+ \text{ and } J \subseteq \Pi \text{ such that } w(\Phi_{\lambda,p}) = \Phi_J, \text{ then one has}\]

\[V_{G_1}(L(\lambda)) = G \cdot u_J.\]

For \( r \geq 2 \) and \( \lambda \in X_1 \), then

\[V_{G_r}(L(\lambda)) = C_r(G \cdot u_J, N^{r-1})\]

with \( J \subseteq \Pi \) such that \( w(\Phi_{\lambda,p}) = \Phi_J \) for some \( w \in W \).

Assume from now on \( p > hc \). Note from earlier that \( c = \frac{9}{4} \) (resp. 3 or 1) if \( G \) is of type \( A_2 \) (resp. \( B_2 = C_2 \) or \( D_2 \)). Then for each \( p \)-restricted dominant weight \( \lambda \), the above result reduces the complexity of \( L(\lambda) \) over \( G_r \) to the dimension of the variety \( C_r(V_J, N^{r-1}) \) for \( V_J = G \cdot u_J \) with suitable \( J \subseteq \Pi \). We will not explicitly show our computation here but the following strategy:

- List all the nilpotent orbits \( O_i \) for each \( 1 \leq i \leq n_J \) in \( V_J \), i.e. \( V_J = \bigcup_{i=1}^{n_J} O_i \). Then for each \( i \) choose \( x_i \) a representative of \( O_i \). We can set \( x_i \) the corresponding canonical Jordan form of \( O_i \).
- For each \( 1 \leq i \leq n_J \), compute \( m_i = \dim G \cdot (x_i, C_{r-1}(z(x_i))) = \dim O_i + \dim C_{r-1}(z(x_i)) \).
- The dimension of \( C_r(V_J, N^{r-1}) \) is \( \max_{1 \leq i \leq n_J} m_i \).

As \( G \) is of rank 2, set \( \Pi = \{ \alpha, \beta \} \) and \( J \) is a subset of \( \Pi \). The tables below show explicit results for the complexity of \( L(\lambda) \) in terms of \( J \), associated to \( \lambda \). Note that the information in the first table (for type \( A_2 \)) is a part of [Ngo4, Theorem 5.3.2].

**Table 1. Type \( A_2 \)**

| \( J \) | \( V_J \) | \( c_{G_r}(L(\lambda)) = \dim C_r(V_J, N^{r-1}) \) |
|---|---|---|
| \( \emptyset \) | \( N \) | \( 2r + 4 \) |
| \( \alpha \text{ or } \beta \) | \( O_{\text{sub}} \) | \( 2r + 3 \) |
| \( \Pi \) | \{0\} | \( 2r + 2 \) |

**Table 2. Type \( B_2 = C_2 \)**

| \( J \) | \( V_J \) | \( c_{G_r}(L(\lambda)) = \dim C_r(V_J, N^{r-1}) \) |
|---|---|---|
| \( \emptyset \) | \( N \) | \( \max(2r + 6, 3r + 3) \) |
| \( \alpha \) | \( O_{\text{sub}} \) | \( \max(2r + 4, 3r + 1) \) |
| \( \beta \) | \( O_{\text{min}} \) | \( 3r + 1 \) |
| \( \Pi \) | \{0\} | \( \max(2r + 4, 3r) \) |

**Table 3. Type \( D_2 \)**

| \( J \) | \( V_J \) | \( c_{G_r}(L(\lambda)) = \dim C_r(V_J, N^{r-1}) \) |
|---|---|---|
| \( \emptyset \) | \( N(sl_2)^2 \) | \( 2r + 2 \) |
| \( \alpha \text{ or } \beta \) | \( N(sl_2) \) | \( 2r + 1 \) |
| \( \Pi \) | \{0\} | \( 2r \) |

\(^4\text{The hypothesis of Lusztig Character formula in the original theorem is omitted as it is satisfied for the classical groups of rank 2 with } p > h.\)
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