THE COMBINATORICS OF THE VERLINDE FORMULAS

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1. Introduction

In this short note we discuss the origin and properties of the Verlinde formulas and their connection with the intersection numbers of moduli spaces. Given a simple, simply connected Lie group $G$, the Verlinde formula is an expression $V^G_k(g)$ associated to this group depending on two integers $k$ and $g$. For $G = \text{SL}_2$ the formula is

$$V^\text{SL}_2_k(g) = \sum_{j=1}^{k-1} \left( \frac{k}{2 \sin^2 \frac{j\pi}{k}} \right)^{g-1}.$$  

(1.1)

We describe $V^G_k$ for general groups in §2. These formulas were first written down by E. Verlinde [23] in the context of Conformal Field theory. The interest towards them in Algebraic Geometry stems from the fact that they give the Hilbert function of moduli spaces of principle bundles over projective curves. More precisely, let $C$ be a smooth projective curve of genus $g$, and let $\mathcal{M}_C^G$ be the moduli space of principal $G$-bundles over $C$ (cf. e.g. [16] and references therein). Then there is an ample line bundle $L$ over $\mathcal{M}_C^G$ such that

$$\dim H^0(\mathcal{M}_C^G, L^k) = V^G_{k+h}(g),$$

(1.2)

where $h$ is the dual Coxeter number of $G$. This statement requires some modifications for a general simple $G$, but it holds for $\text{SL}_n$ ([3, 7, 6, 16]).

Proving (1.2) is important, but in this paper we will address a different question: what can be said about the moduli spaces knowing (1.2)? Accordingly, first we concentrate on understanding the formula.

Two rather trivial aspects of (1.2) are that

- $V^G_k(g)$ is integer valued,
- $V^G_k(g)$ is a polynomial in $k$.

Note that looking at the formula itself, none of this is obvious. Our goal is to explain these properties and connect them to the intersection theory of $\mathcal{M}_C^G$.

The paper is structured as follows: in §2 we discuss some of the ideas of Topological Field Theory, which explain the structure of the formula for general $G$ and show its integrality (cf. [13, 21, 8, 5]). In §3 we give our main result, a residue formula for

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$V^G_k$ for $G = \text{SL}_n$. Such a formula gives an explicit way of calculating the coefficients of $V^G_k$ as a polynomial in $k$. Finally, in §4 we give an application of our formulas: a “one-line proof of” (1.2).

This paper is intended as an announcement and overview. As a result, few proofs will be given, and even most of those will be sketchy. A more complete treatment will appear separately.

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2. Topological Field Theory and Fusion Algebras

This section is independent from the rest of the paper. It contains a quick and rather formal overview of the structure of Topological Field Theories [20, 1] and Verlinde’s calculus [23].

Consider a finite dimensional vector space $F$ (the space of fields) with a marked element $1 \in F$ (the vacuum). Assume that a number $F(g)_{v_1,v_2,...,v_n}$ (correlation functions) is associated to every topological Riemann surface of genus $g$, with elements of the algebra $v_1, v_2, \ldots v_n \in F$ inserted at $n$ punctures, which satisfies the following axioms:

- **Normalization**: $F(0)_{1,1,1} = 1$.
- **Invariance**: $F(g)_{v_1,...} = F(g)_{1,v_1,...}$.
- **Linearity**: $F(g)_{v_1,...}$ is linear in $v_i$.

Introduce the symmetric linear 3-form $\omega : F \otimes F \otimes F \to \mathbb{C}$ by $\omega(u, v, w) = F(0)_{u,v,w}$, the bilinear form $(u, v) = F(0)_{u,v}$ and the trace $f u = F(0)_{u}$. Assume that $(,)$ is non-degenerate, and fix a pair of bases $\{u_i, u^j\}$ of $F$, dual with respect to this form, that is $(u_i, u^j) = \delta_{ij}$.

**Verlinde’s fusion rule**: $F(g)_{v_1,...} = \sum_i F(g - 1)_{u_i,v^i}v_1\ldots$

One can extend $F$ to disconnected surfaces by the axiom:

- **Multiplicativity**: $F$ is multiplicative under disjoint union.

Remark 2.1. These axioms serve as an algebraic model of certain relations among the Hilbert functions of various moduli spaces. The number $F(g)_{v_1,v_2,...,v_n}$ represents the dimension of the space of sections of a certain line bundle over a moduli space of parabolic bundles with weights depending on the insertions $v_1, v_2, \ldots v_n$. The fusion axiom describes how the space of sections of a line bundle decomposes over a family of curves degenerating to a nodal curve (cf. [7, 22]).
Lemma 2.1. The axioms above define the structure of an associative and commutative algebra on $F$, by the formula $vw = \sum_i \omega(v, w, u^i)u_i$, compatible with $(,)$ and $\int$. Then if we denote the invariantly defined element $\sum_i u_iw^i \in F$ by $\alpha$, we have

\[ F(g)_{v_1, v_2, \ldots, v_n} = \int \alpha^g v_1 v_2 \ldots v_n. \tag{2.1} \]

Now assume in addition that the algebra $F$ is *semisimple*. Then it has the form $F \cong L^2(S, \mu)$, where $S = \text{Spec } F$ is a finite set and the complex measure $\mu$ can be given via a function $\mu : S \to \mathbb{C}$.

The elements of $F$ become functions on $S$ and the trace $\int$ turns out to be the actual integral with respect to $\mu$. Now take the following pair of dual bases: $\{\delta_s, \delta_s/\mu(s) | s \in S\}$, where $\delta_s(x) = \delta_{sx}$. We call this the spectral basis. Using this basis and (2.1), we obtain the following formula:

\[ F(g) = \sum_{s \in S} \mu(s)^{1-g}. \tag{2.2} \]

This formula resembles (1.1), but what is the appropriate algebra?

2.1. Fusion algebras. Here we construct the fusion algebras for arbitrary simple, simply connected Lie groups. First we need to introduce some standard notation.

Notation. In this paragraph we will use the compact form of simple Lie groups, still denoting them by the same letter. Thus let $G$ be a compact, simply connected, simple Lie group, $\mathfrak{g}$ its complexified Lie algebra, $T$ a fixed maximal torus, and $\mathfrak{t}$ the complexified Lie algebra of $T$. Denote by $\Lambda$ the unit lattice in $\mathfrak{t}$ and by $\mathcal{W} \subset \mathfrak{t}^*$ its dual over $\mathbb{Z}$, the weight lattice. Let $\Delta \subset \mathcal{W}$ be the set of roots and $W$ the Weyl group of $G$. A fundamental domain for the natural action of the Weyl group on $T$ is called an alcove; a fundamental domain for the associated action on $\mathfrak{t}^*$ is called a chamber. We will use the multiplicative notation for weights and roots, and think of them as characters of $T$. The element of $\mathfrak{t}^*$ corresponding to a weight $\lambda$ under the exponential map will be denoted by $L_\lambda$.

Fix a dominant chamber $\mathfrak{c}$ in $\mathfrak{t}^*$ or a corresponding alcove $a$ in $T$. This choice induces a splitting of the roots into positive ($\Delta^+$) and negative ($\Delta^-$) ones. For a weight $\lambda$, denote its Weyl antisymmetrization by $A\cdot \lambda = \sum_{w \in W} \sigma(w) w \cdot \lambda$, where $\sigma : W \to \pm 1$ is the standard character of $W$. According to the Weyl character formula, for a dominant weight $\lambda$, the character of the corresponding irreducible highest weight representation is $\chi_\lambda = A\cdot \lambda \rho / A\cdot \rho$, where $\rho$ is the square root of the product of the positive roots.

The ring $R(G)$, the representation ring of $G$, can be identified with $R(T)^W$, the ring of Weyl invariant linear combinations of the weights. Denote by $d\mu_T$ be the normalized Haar measure on $T$. If we endow $T/W$ with the Weyl measure

\[ d\mu_W = A\cdot \rho \ A\cdot \bar{\rho} \ d\mu_T, \]
then $R(G)$ becomes a pre-Hilbert space with orthonormal basis $\{\chi_\lambda\}$, i.e., one has

$$\int_{WV} \chi_\lambda \chi_\mu d\mu = \delta_{\lambda,\mu}. \quad \square$$

We need to introduce an integer parameter denoted by $k$ called the level, which can be thought of as an element of $H^3(G, \mathbb{Z}) \cong H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$, and in turn can be identified with a Weyl-invariant integral inner product on $t$.

The basic invariant inner product on $t$ corresponding to $k = 1$ is specified by the condition $(H_{\theta}, H_{\theta}) = 2$, where $H_{\theta} \in t$ is the coroot of the highest root $L_{\theta}$. It has the following properties (see [13, §6],[19, Ch.4]):

- For the induced inner product on $t^*$, we have $(L_{\theta}, L_{\theta}) = 2$.
- For $\lambda \in W$, the inner product $(L_{\theta}, L_{\lambda})$ is an integer, and $(,)$ is the smallest inner product with this property.
- The Killing form is equal to $-2h(,)$, where $h = (L_{\theta}, L_{\rho}) + 1$ is the dual Coxeter number of $G$.

The basic inner product also gives an identification $\nu : t^* \rightarrow t$ between $t^*$ and $t$, by the formula $\beta(x) = (\nu(\beta), x)$.

2.2. The simply-laced subgroup. Let $\Delta_l \in \Delta$ be the set of long roots of $G$. Denote by $W_l$ the lattice in $t^*$ generated by $\Delta$, and by $W_l$ the lattice generated by $\Delta_l$. By definition $\Lambda$ is the dual of $W$ over $\mathbb{Z}$ with respect to the canonical pairing $(,)$ between $t^*$ and $t$. The dual of $W_r$ is the center lattice in $t$. Denote the dual of $W_l$ by $\Lambda_l$.

The root system $\Delta_l$ corresponds to a subgroup $G_l$ of $G$ with maximal torus $T$ and Weyl group $W_l \subset W$, which is generated by reflections corresponding to the elements of $\Delta_l$. Denote the center of $G_l$ by $Z_l$. Then $Z_l$ can be described as the set of elements of $T$ invariant under $W_l$, and we have $\exp^{-1} \Lambda_l = Z_l$. It is important to note that in view of the second property of $(,)$ above, $\nu : W_l \hookrightarrow \Lambda$ is an isomorphism. Since $W$ is paired to $\Lambda$ and $W_l$ is paired to $\Lambda_l$ over $\mathbb{Z}$, it follows that $\nu : W \hookrightarrow \Lambda_l$ is also an isomorphism. Then the map $\exp : \nu : \{\alpha \in \mathfrak{C} | (L_{\theta}, \alpha) \leq 1\} \hookrightarrow \mathfrak{a}$ is a bijection.

Naturally, if $G$ is simply laced, then $G_l = G$. For the non-simply laced groups one has the following subgroups:

- Spin$_{2n} \subset$ Spin$_{2n+1}$
- SU$_2 \subset$ Sp$_n$
- SU$_3 \subset$ G$_2$
- Spin$_8 \subset$ F$_4$

2.3. The definition of the fusion algebra. We give a different definition from the standard one via co-invariants of infinite dimensional Lie algebras [22], but one which is very natural from the point of view of representation theory.

To motivate the construction, recall the procedure of holomorphic induction [4]: the flag variety $F = G/T$ has a complex structure and every character $\lambda$ of $T$ induces a holomorphic equivariant line bundle $\mathcal{L}_\lambda = G \times_\lambda \mathbb{C}$ over $F$. Then one can define
the induction map $\mathcal{I} : \mathcal{R}(T) \to \mathcal{R}(G)$ as a homomorphism of additive groups by the formula $\lambda \mapsto \sum (-1)^i H^i(F, \mathcal{L}_\lambda)$, where the cohomology groups in the latter expression are thought of as $G$-modules. The Borel-Weil-Bott theorem then says that

\begin{equation}
\text{for } \lambda \text{ dominant } \mathcal{I}(\lambda) = \chi_\lambda
\end{equation}

and

\begin{equation}
\mathcal{I}(\lambda') = \sigma(w)\mathcal{I}(\lambda), \text{ whenever } \lambda' \rho = w(\lambda \rho) \text{ for some } w \in W.
\end{equation}

Note that $\mathcal{I}$ is not expected to be a ring homomorphism.

This procedure applies to the loop group $\hat{\mathcal{L}}G$ as well ([19, 15, 17]). Once the action of the central elements is fixed as $c \mapsto c^k$, where $k \in \mathbb{N}$ is the level, again, we have a map $\mathcal{I} : \mathcal{R}(T) \to \mathcal{R}_k(\hat{\mathcal{L}}G)$. This last object $\mathcal{R}_k(\hat{\mathcal{L}}G)$ has only additive structure, since the tensor product of two level $k$ representations has level $2k$. The role of the Weyl group is played by the affine Weyl group $W_k$ obtained by adjoining to $W$ the translation by $(k + h)L_\theta$. Again the Borel-Weil-Bott theorem applies, and (2.4), with $W$ replaced with $W_k$, gives a description of the kernel of $\mathcal{I}$.

Since $W \subset W_k$, the map $\mathcal{I}$ factors through $\mathcal{I}$, and as a result we have a map $\mathcal{J} : \mathcal{R}(G) \to \mathcal{R}_k(\hat{\mathcal{L}}G)$. It is easy to see from the (2.4) that the set of characters: $\Xi_k = \{\chi_\lambda | (L_\theta, L_\lambda) \leq k\}$ forms a basis of $\mathcal{R}_k(\hat{\mathcal{L}}G)$ if we identify $\mathcal{J}(\chi_\lambda)$ with $\chi_\lambda$.

**Lemma 2.2.** The additive group $\mathcal{R}_k(\hat{\mathcal{L}}G)$ can be endowed with a ring structure $\mathcal{F}_G^k$ so that the map $\mathcal{J}$ becomes a homomorphism of rings.

The algebra $\mathcal{F}_G^k$ is called the *fusion algebra* of $G$ of level $k$. As noted above, we can consider $\Xi_k$ to be a basis of $\mathcal{F}_G^k$. Endow $\mathcal{F}_G^k$ with the trace function $f$ by the formula $f(\chi_\lambda) = 0$ except for the trivial character $\chi_1$, which has trace equal to 1. Also note that since $\text{Spec}(\mathcal{R}(G)) = T/W$ and $\mathcal{F}_G^k$ is a quotient of $\mathcal{R}(G)$, we expect $\text{Spec}(\mathcal{F}_G^k) \subset T/W$.

**Lemma 2.3.** $\mathcal{F}_G^k$ can be identified with “$L^2$” of the finite normalized measure space $Z_k = \{t \in T| t^{k+h} \in \mathbb{Z}_l, \text{t is regular}\}/W$, with measure $d\mu_k$ given by the function

\begin{equation}
\frac{A \cdot \rho(t) \cdot \bar{\rho}(t)}{|Z_l|(k + h)^r}.
\end{equation}

Note the surprising fact that the discrete measure remains unchanged up to a normalization factor as $k$ varies.

Now we can define the quantity $V_{G}^k(g)$ which appeared in (1.2) as the number associated to a Riemann surface of genus $g$ and the fusion algebra $\mathcal{F}_G^{k-h}$. Combining (2.2) and (2.5) we obtain

\[ V_{G}^k(g) = \sum_{t \in T/W; t^{k} \in \mathbb{Z}_l} \left( \frac{|Z_l|^r}{A \cdot \rho(t) \cdot \bar{\rho}(t)} \right)^{g-1}, \]
where $T_r$ is the set of regular elements of $T$. This can be easily seen to give (1.1) for the case of $G = \mathrm{SU}_2$. Indeed, embedding the maximal torus of $\mathrm{SU}_2$ into $\mathbb{C}$ as the unit circle, we have: $\rho(z) = z$, $z_1 = \pm 1$, $h = 2$ and $A(z) = 1/z$.

Finally, note that since the relations in the fusion algebras given in (2.4) have integer coefficients, and the $\{\chi_\lambda\}$ form an orthonormal basis of $F$, we see that $f_\alpha^g$ from (2.1) has to be an integer. This proves that $V^G_k(g)$ is an integer for all groups and values of $k$ and $g$.

Remark 2.2. That this definition of the fusion algebras is equivalent to the standard one via coinvariants of current algebras [22] can be shown to be equivalent to Verlinde’s conjecture on the diagonalization of the fusion rules, which gives a formula for the product in $F^G_k$ using the $S$-matrix. The definition given above is simpler to use for calculations and it gives the correct prescription for non simply connected groups (see also [8]).

3. Residue formulas

In this section we study $V^G_k(g)$ as a function of $k$. We show that $V^G_k(g)$ is a polynomial in $k$ for $G = \mathrm{SL}_3$, and give a simple formula for the coefficients of this polynomial. The generalization of these results to $\mathrm{SL}_n$ is straightforward.

Consider the case $G = \mathrm{SL}_2$ first. Again, as at the end of the previous section embed the maximal torus of $\mathrm{SL}_2$ into $\mathbb{C} \subset \mathbb{P}^1$.

Consider the differential form

$$\mu = \frac{dz}{z} + \frac{z^{-1}}{z - z^{-1}}$$

on $\mathbb{P}^1$. This form has simple poles: at $z = \pm 1$ with residue 1, and at $z = 0$, $\infty$ with residue $-1$. Thus if we pull back $\mu$ by the $k$-th power map we obtain a differential form $\mu_k$ with poles at the $2k$-th roots of unity and residues $+1$, and simple poles at $z = 0$, $\infty$ with residue $-k$. It is given by the following formula:

$$\mu_k = k \frac{dz}{z} \frac{z^k + z^{-k}}{z^k - z^{-k}}.$$ 

Note that $\mu_k$ is invariant under multiplication by a $2k$th root of unity and under the Weyl reflection $z \rightarrow 1/z$.

Now suppose we have a function $f(z)$, with poles only at $z = \pm 1$, vanishing at 0 and $\infty$, and invariant under the substitution $z \rightarrow z^{-1}$. Then by applying the Residue Theorem to the differential form $f\mu_k$ and using the Weyl symmetry at hand, we have

$$\sum_{j=1}^{k-1} f(\exp(\pi i j/k)) = - \text{Res}_{z=1} \mu_k f(z),$$
where $I^2 = -1$. Applying this argument to the function

$$f(z) = \left(\frac{2k}{-(z-z^{-1})^2}\right)^{g-1}$$

we obtain the formula

$$V_{k}^{SL_2}(g) = (-1)^g (2k)^{g-1} \text{Res}_{z=1} \frac{k}{z} \frac{dz}{z^k - z^{-k}} \left(\frac{1}{(z-z^{-1})^2}\right)^{g-1}.$$  

Now using the invariance of the residue under substitutions we can obtain different formulas for $V_{k}^{SL_2}(g)$. For example, the polynomial nature of $V_{k}^{SL_2}(g)$ becomes transparent if we perform the substitution $z \to \exp(ix)$:

$$V_{k}^{SL_2}(g) = -(2k)^{g-1} \text{Res}_{x=0} \frac{k \cot(kx)}{(2\sin x)^2(g-1)}.$$  

It is easy to check using this formula that the degree of $V_{k}^{SL_2}(g)$ as a polynomial in $k$ is $3(g-1)$, which, as expected, coincides with the dimension of $\mathfrak{M}_{C}^{SL_2}$.

Before we proceed, we need an understanding of higher dimensional residues. The notion that a top dimensional differential form has an invariantly defined number assigned to it, does not carry over to the higher dimensions. The correct object in $\mathbb{C}^n$ is $\text{Res} : H^n_{\text{loc}}(\Omega^n, \mathbb{C}^n) \mapsto \mathbb{C}$ mapping from the $n$th local Čech cohomology group in a neighborhood of 0 with values in holomorphic $n$-forms to complex numbers. To define this map let $\omega$ be a meromorphic $n$-form defined in a neighborhood of 0 in $\mathbb{C}^n$. Then $\omega$ can be represented in the form $dz_1 dz_2 \ldots dz_n h(z)/f(z)$ where $f$ and $h$ are holomorphic functions. The additional data necessary to represent an element of $H^n_{\text{loc}}(\Omega^n, \mathbb{C}^n)$ is a splitting of $f$ into the product of $n$ functions $f = a_1 a_2 \ldots a_n$. Such a splitting defines $n$ open sets in the a neighborhood of 0: $A_i = \{a_i \neq 0\}$. These define a local Čech cocycle. A detailed explanation of this and an algorithm to calculate the residue can be found in [9, 10].

We will call a differential $n$-form with such a splitting a residue form.

**Definition 3.1.** A non-trivial residue form $\omega$ is called flaglike if $a_i$ only depends on $z_1, \ldots z_i$. This notion depends on choice and the order of the coordinates $z_1, \ldots, z_n$.

**Lemma 3.1.** Let $\omega$ be a flaglike residue form. Then

$$\text{Res}(\omega) = \text{Res}_{z_n} \ldots \text{Res}_{z_1} \omega.$$

Here $\text{Res}_{z_i}$ is the ordinary 1-dimensional residue, taken assuming all the other variables to be constants.

The proof is straightforward. Note that the order of the variables is important, while there is some freedom in the way the denominator is split up.
For simplicity we restrict ourselves to the case of $\text{SL}_3$. According to Lemma 2.3 and (2.2) the Verlinde formula can be written as

$$V_{\text{SL}_3}^{k}(g) = (3k^2)^{g-1} \sum_{i,j,k-i-j>0} (8 \sin(i\pi/k) \sin(j\pi/k) \sin((i+j)\pi/k))^{-2(g-1)}$$

Now we can write down the main result of the paper:

**Theorem 3.2.**

$$V_{\text{SL}_3}^{k}(g) = (3k^2)^{g-1} \sum_{Y=1}^{\infty} \sum_{X=1}^{\infty} \frac{X^k + X^{-k}}{X^k - X^{-k}} \frac{Y^k + Y^{-k}}{Y^k - Y^{-k}} \times$$

$$\frac{(-1)^{g-1} k^2 dX dY}{((X - X^{-1})(Y - Y^{-1})(XY - (XY)^{-1}))^{2(g-1)} XY}.$$

The proof is analogous to the case of $\text{SL}_2$. Denote the residue form in (3.2) by $\omega_k(g)$. Again at the points $p_{ij} = (e^{i\pi n/k}, e^{i\pi n/k})$, with $i, j, k-i-j \geq 1$ the residues of $\omega_k(g)$ reproduce the sum (3.1). However, now it is not immediately obvious that the residue theorem can localize this sum at the point $(1,1)$, since the residue form in (3.2) has non-trivial residues at other points as well. To illustrate the situation consider the matrix $M_k$ whose $(i,j)$th entry is the residue of $\omega_k(g)$ taken at the point $p_{ij}$ instead of $(1,1)$, where $i, j = 0, \ldots, k-1$.

Example for $g = 2$:

$$M_6 = \begin{bmatrix}
166 & -45 & -29 & -18 & -29 & -45 \\
-45 & 36 & 9 & 9 & 36 & -45 \\
-29 & 9 & 4 & -29 & 36 \\
-18 & 9 & 9 & -18 & 9 & 9 \\
-29 & 36 & -29 & 9 & 4 & 9 \\
-45 & -45 & 36 & 9 & 9 & 36
\end{bmatrix}.$$

We can apply the Residue Theorem to “each column” by fixing a value of $X$. By degree count, one can see that $\omega(g)$ has trivial residues at $Y = 0, \infty$ and this implies that the sum of the entries in each column of $M_k(g)$ is 0. Next, note that $M_k(0, i) = M_k(i, 0)$, since these residues are split, i.e. they have the form $dX dY X^{-m} Y^{-n} f(X, Y)$, where $f$ is holomorphic at the point where the residue is taken.

Now to prove the Theorem it is sufficient to show that $M_k(j, 0) = M_k(j, k-j)$ for every $j > 0$. To see this, note that both residues are simple (first order) in $X$ at $\alpha = \exp(j\pi/k)$. This means that after taking the $X$-residue, we are left with the form

$$\omega_\alpha = \text{const} \cdot \frac{dY Y^k + Y^{-k}}{Y Y^k - Y^{-k} ((Y - Y^{-1})(\alpha Y - \alpha^{-1}Y^{-1}))^{2(g-1)}}.$$
The two numbers we need to compare are the residues of this form at 0 and \( \alpha \) respectively. But these two residues clearly coincide since \( \omega_\alpha \) is invariant under the substitution \( Y \rightarrow \alpha^{-1} Y^{-1} \). \( \square \)

The formula for \( G = \text{SL}_n \) reads as follows:

\[
V^G_k(g) = (-1)^{n-1+(g-1)|\Delta^+|} (nk^{n-1})^{g-1} \left( \prod_{X_{n-1}=1}^{n-1} \right) \left( \prod_{X_{1}=1}^{n-1} \right) W^{-2(g-1)} \frac{X^k + 1}{X^k - 1} \frac{k dX}{2X_i}
\]

where \( X_i, \ i = 1, \ldots, n-1 \) are the simple (multiplicative) roots and \( W = \prod_{\alpha \in \Delta^+} (\alpha^{1/2} - \alpha^{-1/2}) \) is the Weyl measure.

As we pointed out after Lemma 3.1, the ordering of the variables matters when taking the subsequent residues. In the special case of \( G = \text{SL}_3 \) this ordering does not matter (i.e. \( M_k(i, j) = M_k(j, i) \)), but for higher rank groups a finer argument is necessary.

Finally, note that similarly to the case of \( \text{SL}_2 \), (3.3) gives a simple prescription for calculating the coefficients of \( V^G_k(g) \) as a polynomial in \( k \), via the exponential substitution. For example, for \( \text{SL}_3 \) we obtain

\[
(3.4) \quad V^\text{SL}_3_k(g) = (3k^2)^{g-1} \left( \prod_{x=0}^{y=0} \frac{k^2 \cot(kx) \cot(ky) dx dy}{8 \sin(x) \sin(y) \sin(x+y)^2} \right).
\]

A different generating function was obtained for the case of \( G = \text{SL}_3 \) by Zagier [26].

4. Multiple \( \zeta \)-values and intersection numbers of moduli spaces

In this final section we show how (1.2) and (3.2) can be related via the Riemann-Roch formula to Witten's conjectures on the intersection numbers of moduli spaces. Our argument below gives a quick proof of (1.2) for \( \text{SL}_n \) assuming Witten's formulas. This is a generalization of the work of Thaddeus who considered the case of \( \text{SL}_2 \) [18].

4.1. Multiple \( \zeta \)-values and intersection numbers of moduli spaces. Consider the case of \( \text{SL}_2 \) first. If we want to find the asymptotic behavior of \( V^\text{SL}_2_k \) for large \( k \), the best way to think about the formula is that it is a discrete approximation to the (divergent) integral \( \int_0^1 \sin(\pi x)^{-2(g-1)} dx \). To find the leading asymptotics, we can replace \( \sin(x) \) by \( x \), and taking the large \( k \) limit we obtain:

\[
(k/2)^{g-1} \sum_{j=1}^{\infty} (k/(j\pi))^{2(g-1)} = k^{3(g-1)} \zeta(2(g-1))/((2^{g-1}) \pi^{2(g-1)}). \]

This can be easily proven, and in fact, a generalization of this formula for arbitrary groups appeared in Witten's work [25].

Below we will concentrate on the case of \( \text{SL}_3 \), however the formulas can be extended to \( \text{SL}_n \) as well.
If we perform the trick above for $SL_3$, up to a constant, the leading behavior of the Verlinde formula appears to be

$$V_g^{SL_3}(k) \sim \text{const} \cdot k^{8(g-1)-6(g-1)} \sum_{i,j=1}^{\infty} (ij(i+j))^{-2(g-1)}.$$ 

One can write down more general sums, e.g.:

$$S(a, b, c) = \sum_{i,j=1}^{\infty} i^{-a} j^{-b} (i+j)^{-c},$$

closely related to the so-called multiple zeta values [27].

It was discovered by Witten that all intersection numbers of moduli spaces are given by combinations of multiple $\zeta$-values [24, 25]. Below we give a couple of useful formulas for them. We restrict ourselves to the case $S(2g, 2g, 2g)$ for simplicity. Similar formulas exist in greater generality.

**Lemma 4.1.**

\[
S(2g, 2g, 2g) = \frac{1}{2} \int_0^1 B_{2g}(x)^3 \, dx,
\]

where $B_n(x)$ is a modified nth Bernoulli polynomial, $B_n(x) = -(2\pi i)^n B_n(x)/n!$.

\[
S(2g, 2g, 2g) = \frac{1}{3} \text{Res}_{(0,0)} \cot(x) \cot(y) (xy(x+y))^{-2g}.
\]

**Sketch of Proof:** The first formula follows from the definition of the Bernoulli polynomials:

$$B_n(x) = \sum_{j \neq 0} e^{2i\pi x} \frac{x^n}{j^n}.$$ 

Indeed, substituting this into $\frac{1}{2} \int_0^1 B_{2g}(x)^3 \, dx$, one obtains $S(2g, 2g, 2g)$ on the nose; the coefficient $1/2$ is a combinatorial factor.

The proof of the second formula is similar to the proof of Theorem 3.2. On has to apply the Residue Theorem in two steps. That the residue at infinity vanishes follows from the expansion $\cot(x)$:

$$\pi \cot(\pi x) = \sum_{n \in \mathbb{Z}} (x-n)^{-1}.$$ 

**4.2. Intersection numbers of the moduli spaces.** First we recall some facts about the cohomology of the moduli spaces. We will ignore that the moduli spaces are not smooth in general, and accordingly, we will assume the existence of a universal bundle, Riemann-Roch formula, etc. However, formulas analogous to (3.2) exist for the smooth moduli spaces as well (e.g. when the degree and rank are coprime for $SL_n$), and all of our statements are rigorous for these cases. Some of the singular moduli spaces (e.g. vector bundles) can be handled using the methods of [3]. We will also
ignore certain difficulties which arise for Spin$_n$, $n > 6$, and the exceptional groups, where the ample line bundle exists only for $k = 0 \mod l$, for some $l$, depending on the type of the group. In these cases the Verlinde formula is a polynomial only when restricted to these values. Thus what follows should be perceived as a scheme of a proof, which works as it is in some cases, but requires modification and more work in greater generality.

There is a universal principal $G$-bundle $U$ over the space $C \times \mathcal{M}^G$, which induces a map $\mathcal{M}^G \to BG$, and consequently a map $s : H^*(BG) \to H^*(\mathcal{M}^G) \otimes H^*(C)$.

Recall that $H^*(BG) \cong \text{Sym}(g^*)^G$, the space of $G$-invariant polynomial functions on $g$. This is a polynomial ring itself in rank($G$) generators and it is isomorphic by restriction to $S^G = \text{Sym}(t^*)^W$. For every $\alpha \in H_i(C)$ and $P \in S^G$ we obtain a cohomology class of $\alpha \cap s(P) \in H^2(C \times \mathcal{M}^G)$, the $\alpha$-component of $s(P)$. In fact, $s$ induces a map $\bar{s} : H^*(C) \otimes H^*(\mathcal{M}^G) \to H^*(\mathcal{M}^G)$, where $H^*(C) \otimes H^*(\mathcal{M}^G)$ is the free commutative differential algebra generated by the ring $S^G$ and the differentials of negative degree modeled on $H^*(C)$. For the case of SL$_n$ and coprime degree and rank it is known that $\bar{s}$ is surjective [2, 14]. In particular, denoting the fundamental class of $C$ by $\eta_C$, and the basic invariant scalar product from §2 by $P_2$ we obtain a class $\omega = \eta_C \cap s(P_2) \in H^2(\mathcal{M}^G)$, which turns out to be the first Chern class of the line bundle from (1.2).

To simplify the notation, below we omit the map $s$ and also $\alpha$ if $\alpha = 1$, when writing down the classes $H^*(\mathcal{M}^G)$. Thus $1 \cap s(P)$ will be denoted simply by $P$.

Any power series in the variables $\alpha \cap P$ can be integrated over $\mathcal{M}^G$ and these numbers are called the intersection numbers of the moduli space. Naturally, only the terms of degree $\dim \mathcal{M}^G = \dim(G)(\text{genus}(C) - 1)$ will contribute.

Witten, using non-rigorous methods, gave a complete description of these intersection numbers in the most general case [25]. His formulas are combinations of multiple $\zeta$-values, and are rather difficult to calculate. In this paper, we will focus only on a subset of these intersection numbers, which are of the form $\int_{\mathcal{M}} \omega l P$, where $l \in \mathbb{N}$ and $P$ is a not necessarily homogeneous Weyl-symmetric function on $t$.

**Conjecture 4.2.** For every group $G$, there exists a residue form $\Omega^G$ depending on $g$, defined in a neighborhood of $0 \in t$, the Cartan subalgebra of $G$, such that

$$\int_{2\pi} e^\omega P = \text{Res}_{0 \in t} \Omega^G P,$$

For $G = \text{SL}_n$,

$$\Omega = n^{g-1} \prod_{x_{n-1}=0}^{x_2=0} \ldots \prod_{x_1=0}^{x_1=0} \prod_{\alpha \in \Delta^+} L_\alpha^{2(g-1)} \prod_{i=1}^{n-1} \cot(x_i) dx_i,$$

where the $x_i$-s are halves of the simple (additive) roots of SL$_n$, ordered according to the Dynkin diagram.
Let us write down the formula for $\text{SL}_3$ more explicitly and inserting the “grading”:

$$
\int_{\mathfrak{m}} e^{k\omega} P = (3k^2)^{g-1} \text{Res}_{x=0} \text{Res}_{y=0} \frac{k^2 \cot(kx) \cot(ky) \, dx \, dy}{(8xy(x+y))^{2(g-1)}} P
$$

(4.5)

**Remark 4.1.** It can be shown that (4.4) is consistent with Witten’s formulas. We will not give the proof here, but note that the link between the two types of formulas is given by equalities like (4.2).

At the moment we do not know $\Omega^G$ for general $G$.

Our formulas seem to be related to those given in the works of Jeffrey and Kirwan [11, 12].

Finally, we present another evidence for (4.4): the consistency with the Verlinde formula. First we need a few facts about the moduli spaces. Fix a curve $C$ of genus $g$ and a group $G$. They will be omitted from the notation.

**Lemma 4.3.**

1. $c_1(T\mathfrak{m}) = h\omega$,
2. $p(T\mathfrak{m}) = c(\text{Ad} \, U_z)^{(2g-1)} = \prod_{\alpha \in \Delta} (1 + \alpha)^{(2g-1)}$,
3. $\hat{A}(T\mathfrak{m}) = \prod_{\alpha \in \Delta^+} \left( \frac{\alpha/2}{\sinh(\alpha/2)} \right)^{2(g-1)}$.

Here $h$ is the dual Coxeter number of $G$, $p$ denotes the total Pontryagin class, $c$ the total Chern class, $U_z$ is the bundle over $\mathfrak{m}$ obtained by restricting the universal principal bundle $U$ to a slice $z \times \mathfrak{m}$ for some $z \in C$ and $\text{Ad} \, U_z$ is the vector bundle associated to $U_z$ via the adjoint representation of $G$.

For the proof of the first two statements in some partial cases see [2]. The second statement statement follows from the Kodaira-Spencer construction. From the second statement we find that the Pontryagin roots of $T\mathfrak{m}$ are the roots of the Lie algebra $\mathfrak{g}$, and this in turn implies the third statement.

Finally, we can put everything together. We will calculate $\dim H^0(\mathfrak{m}^G, \mathcal{L}^k)$. Again, consider $G = \text{SL}_3$ for simplicity. First, the Kodaira vanishing theorem applies to $\mathcal{L}^k$, because the canonical bundle of $\mathfrak{m}$ is negative, (this follows from the Lemma 4.3(1), see also [3]). Thus we can replace the dimension of $H^0$ by the Euler characteristic, and apply the Riemann-Roch theorem:

$$
\dim H^0(\mathfrak{m}^G, \mathcal{L}^k) = \chi(\mathfrak{m}^G, \mathcal{L}^k) = \int_{\mathfrak{m}} e^{k\omega} \text{Todd}(\mathfrak{m}).
$$

According to Lemma 4.3(1), and using the standard shifting trick we can rewrite this integral as

$$
\int_{\mathfrak{m}} e^{(k+h)\omega} \hat{A}(\mathfrak{m}).
$$

We can calculate this integral using (4.5) and Lemma 4.3(3), and the result is exactly $V^G_k(g)$ according to (3.4). This proves (1.2).
VERLINDE FORMULAS

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