PERIOD INTEGRAL OF OPEN FERMAT SURFACES
AND SPECIAL VALUES OF HYPERGEOMETRIC FUNCTIONS

TOMOHIDE TERASOMA

Abstract. In the paper [AOT], we prove that the special values $3F_2(1, 1; a; b, c; 1)$ of the hypergeometric function $3F_2$ is a $\mathbb{Q}$-linear combination of $\log(\lambda)$ for $\lambda \in \mathbb{Q}$ and 1, if $a, b, c$ are rational numbers satisfying a certain condition. In [A], [S], the triples $(a, b, c)$ with this condition are completely classified in relation with Hodge cycles on Fermat surfaces. In this paper, we give an explicit expression of $3F_2(1, 1; a; b, c; 1)$ which does not belong to the finite exceptional characters in the list of [S].

Contents

1. Introduction 1
2. Algebraic cycles and one forms with constant residues 10
3. Hypergeometric identities for $\Gamma_1$. 13
4. Hypergeometric identities for $\Gamma_2$ 16
5. Hypergeometric identities for $\Gamma_3$ 18
Appendix A. List of orbits for exceptional characters 20
References 21

1. Introduction

1.1. Introduction and result of [AOT]. Let $p_1, \ldots, p_5$ be real numbers such that $p_1, p_4 \notin -1, -2, \ldots$ and $x$ be a complex number with $|x| < 1$. We define the hypergeometric function $3F_2(p_1, p_2, p_3; p_4, p_5; x) = F(p_1, p_2, p_3; p_4, p_5; x)$ by the series (12)

$$F(p_1, p_2, p_3; p_4, p_5; x) = \sum_{k=0}^{\infty} \frac{(p_1)_k(p_2)_k(p_3)_k}{(p_4)_k(p_5)_k k!} x^k,$$

where $(p)_k (k = 0, 1, \cdots)$ is the Pochhammer symbol defined by

$$(p)_k = p(p+1)\cdots(p+k-1) = \frac{\Gamma(p+k)}{\Gamma(p)}.$$

For a positive integer $m$ and rational numbers $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \frac{1}{m}\mathbb{Z} - \mathbb{Z}$, the limit

$$(1.1.1) \quad F(1, 1, \alpha_1+\alpha_2+\alpha_3; \alpha_1+\alpha_2; \alpha_1+\alpha_3; 1) = \lim_{x \to 1} F(1, 1, \alpha_1+\alpha_2+\alpha_3; \alpha_1+\alpha_2; \alpha_1+\alpha_3; x)$$

exists. The fractional part $\langle \alpha \rangle$ of a rational number $\alpha$ is a rational number characterized by $\alpha - \langle \alpha \rangle \in \mathbb{Z}$ and $0 \leq \langle \alpha \rangle < 1$. 
Theorem 1.1 (see [AOT]). If the parameters $\alpha_0, \ldots, \alpha_3$ satisfy the condition
\[(1.1.2) \quad \langle t\alpha_0 \rangle + \langle t\alpha_1 \rangle + \langle t\alpha_2 \rangle + \langle t\alpha_3 \rangle = 2, \text{ for all } t \in (\mathbf{Z}/m\mathbf{Z})^\times,\]
then
\[(1.1.3) \quad F(1, 1, \alpha_1 + \alpha_2; \alpha_1, \alpha_2; 1) \in \overline{\mathbf{Q}} + \overline{\mathbf{Q}} \log(\mathbf{Q}^\times).\]
Here, $\overline{\mathbf{Q}} + \overline{\mathbf{Q}} \log(\mathbf{Q}^\times)$ denotes the $\overline{\mathbf{Q}}$ linear hull of $\log(\lambda)$ with $\lambda \in \mathbf{Q}^\times$ and 1.

As is explained in the next subsection, the special value of hypergeometric function of the above type is related to the extension class of mixed $K$-mixed Hodge structures
\[(1.1.4) \quad 0 \to H^1_{Hg}(B, K)(\chi_\alpha) \to H^2_{Hg}(X_m, B, K)(\chi_\alpha) \to H^2_{Hg}(X_m, K)(\chi_\alpha) \to 0\]
arising from the relative cohomologies of Fermat surface $X_m$ and its divisor $B$. Theorem 1.1 is a consequence of the fact that the extension class of (1.1.4) is actually a Hodge realization of a mixed Tate motives if the condition (1.1.2) is satisfied. In the proof of Theorem 1.1 in [AOT], we use Lefschetz-Hodge theorem and the fact that an algebraic cycles on algebraic variety defined over $\overline{\mathbf{Q}}$ are defined on $\overline{\mathbf{Q}}$. Thus the method in [AOT] does not give an closed formulas for the special value (1.1.1). In this paper, we give an explicit formula for the above theorem for non-exceptional cases (see Theorem 1.2 for the definition of exceptional cases).

Let us explain the outline of the paper. In Section 1, we recall the result of [AOT] by introducing period integral of an open Fermat surfaces, or dually period for the relative cohomology Fermat surfaces with their divisors. Let $\chi$ be a character of a group $G_m$ acting on $X_m$. If the $\chi$-part of the cohomology of Fermat surface is generated by algebraic cycles, the corresponding periods can be written using the logarithmic function evaluated at algebraic numbers. We recall the result by Aoki and Shioda on algebraic cycles on Fermat surfaces. We also recall that except for finitely many characters, they are obtained by four types of characters. We give an explicit formula belonging to four types.

In Section 2, we explain that the existence of algebraic cycles implies the exactness for certain rational differential forms. The equation of algebraic cycles are key to find differential forms which bound the given differential forms. Actually, if one find such differential form, the story is independent of the existence of algebraic cycles.

From Section 3 to Section 5, we compute the integral and show that the integral expressing the extension class actually are expressed by simpler integration, which gives the explicit expression by logarithmic functions. Here we use Stokes formula for the variety obtained by blowing up $\mathbf{C}^2$.

In the first proof of [AOT], we used regulator maps for a symbols. If the symbol can be written explicitly, we also have a closed formula of the special value (1.1.1). In [AY], they obtained explicit formulas for the special values for some cases using this method. The one form obtained in Section 2 gives a key to find an explicit expression of related $K$-group via symbols. This topics will be left to a future research.

**Acknowledgement** This paper was first considered as a continuation of the paper [AOT]. The author express his acknowledgement to M. Asakura and N. Otsubo for discussion and private communication. He also thank them to let him know the relationship with regulator maps in $K$-theory and hypergeometric functions, which is a driving force to let him go into this subject.
1.2. Algebraic cycles on Fermat surfaces. A proof of the above theorem is based on the fact that the value (1.1.3) is considered as a period integral of certain relative cohomology of Fermat surface. We recall some properties of cohomologies of Fermat surfaces. Let \( X_m \) be an affine Fermat surfaces defined by

\[
X_m = \text{Spec}(\mathbb{Q}[u,v,w]/(u^m + v^m - 1 - w^m))
\]

and \( \mu_m \) be the group of \( m \)-th roots of unities in \( \mathbb{C}^\times \) and set \( K = \mathbb{Q}(\mu_m) \). Then the group \( G_m = (\mu_m)^3 \) acts on \( X_m \) by

\[
\rho(g_1, g_2, g_3) : (u, v, w) \mapsto (g_1 u, g_3 v, g_3^2 w)
\]

for \((g_1, g_2, g_3) \in G_m\). For \( a = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \frac{1}{m} \mathbb{Z}^4 \) with \( 0 < \alpha_i < 1, \sum_i \alpha_i \in \mathbb{Z} \), we define a character \( \chi = \chi_a \) by

\[
\chi_a(g_1, g_2, g_3) = g_1^{a_1} g_2^{a_2} g_3^{a_3}
\]

where \( a_i = m \alpha_i \). The \( \chi_a \)-part of the singular cohomology \( H^2_B(X_m, K) \) and algebraic de Rham cohomology \( H^2_{dR}(X_m/\mathbb{Q}) \) over \( \mathbb{Q} \) of \( X_m \) are denoted by \( H^2_B(X_m, K)(\chi_a) \) and \( H^2_{dR}(X_m/\mathbb{Q})(\chi_a) \), respectively. Then we have

\[
\dim(H^2_{dR}(X_m/\mathbb{Q})(\chi_a)) = \dim(H^2_B(X_m, K)(\chi_a)) = 1.
\]

The condition (1.1.2) is equivalent to the following condition (see [S]):

\[
\text{The space } H^2_B(X_m, K)(\chi_a) \text{ is generated by Hodge cycles.}
\]

The complete classification of the set of indices \((a_0, a_1, a_2, a_3)\) satisfying the condition (1.2.7) is conjectured in [S] and proved by [A].

**Theorem 1.2** ([S], [A]). Let \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) be elements in \( \mathbb{Q} \cap (0, 1) \) with \( \sum_{i=0}^3 \alpha_i \in \mathbb{Z} \) satisfying the condition (1.1.2) and \( m \) be the common denominator. Then one of the following holds.

1. There exists and element \( \alpha, \beta \in \mathbb{Q} \) such that \( (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \) is equal to \( (\alpha, -\alpha, \beta, -\beta) \) up to a permutation.
2. There exists and element \( \alpha \in \mathbb{Q} \) such that \( (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \) is equal to one of the following up to a permutation.
   - (a) \( (2\alpha, 1 - \alpha, -\alpha + \frac{1}{2}, \frac{1}{2}) \)
   - (b) \( (3\alpha, 1 - \alpha, -\alpha + \frac{1}{3}, -\alpha + \frac{2}{3}) \)
   - (c) \( (4\alpha, 1 - 2\alpha, -\alpha + \frac{1}{4}, -\alpha + \frac{3}{4}) \)
3. \( m \leq 180 \) and does not satisfies (1) and (2).

In particular there exist only finitely many vectors \((\alpha_0, \ldots, \alpha_3)\) belonging to the case (3).

The characters listed in (3) of Theorem 1.2 which is not appeared in (1), (2) is called the exceptional characters. There are 101 Galois orbits in the exceptional characters and the list of orbits is given in Appendix (Compare for the list in [S]).

1.3. Extension of mixed Tate motives.
1.3.1. Tate Hodge structure. Let $m \geq 2$ be an integer and $K$ be the field generated by $\mu_m = \langle e(1/m) \rangle$ over $\mathbb{Q}$. A mixed $K$-Hodge structure $V_{Hg}$ consists of triple $V_{Hg} = ((V_{dR}, F, T), (V_B, W), (C, K)$ of filtered vector space over $C, K$ and a comparison isomorphism $c : V_B \otimes_K C \xrightarrow{\cong} V_{dR}$ compatible with $W$ such that $(F, T, W)$ is a filtration defined in [D]. For mixed $K$-Hodge structures $V_{1,Hg}, V_{2,Hg}$, a homomorphisms from $\varphi : V_{1,Hg} \to V_{2,Hg}$ is a pair of homomorphisms $\varphi_B : V_{1,B} \to V_{2,B}$ and $\varphi_{dR} : V_{1,dR} \to V_{2,dR}$ preserving comparison isomorphisms $c$ and filtrations $W, F$. By proposition in [D], the category of mixed $K$-Hodge structures whose morphisms are morphism of mixed $K$-Hodge structures becomes an abelian category. For a mixed Hodge structure $V_{Hg} = (V_B, V_{dR}, c)$ over $\mathbb{Q}$, the tensor product $V_{Hg} \otimes_{\mathbb{Q}} K = (V_B \otimes_{\mathbb{Q}} K, V_{dR}, c)$ becomes a mixed $K$-Hodge structure in a natural way.

Tate Hodge structure $\mathbb{Q}(1)_{Hg}$ is defined by the triple $(\mathbb{Q}(1)_{dR}, \mathbb{Q}(1)_B, c)$, where $\mathbb{Q}(1)_{dR}$ and $\mathbb{Q}(1)_B$ are one dimensional vector spaces over $C$ and $Q$ generated by $\xi_{dR}$ and $\xi_B$ with $c(\xi_{dR}) = (2\pi i)\xi_B$. We define $\mathbb{Q}(i)_{Hg} = (\mathbb{Q}(i)_{dR}, \mathbb{Q}(i)_B, c)$, where $\mathbb{Q}(i)_{Hg} = (\mathbb{Q}(i)_{dR}, \mathbb{Q}(i)_B, c)$ is a pair of $\mathbb{Q}(i)_{dR}$ and $\mathbb{Q}(i)_B$.

1.3.2. Extension class. Let $e_{Hg} : 0 \to K \to E \xrightarrow{p} K(-1) \to 0$ be an exact sequence of mixed $K$-Hodge structures. Let $c : K(-1)_B \otimes_K C \to K(-1)_{dR}$ be the comparison isomorphism and $s_B$ and $s_{dR}$ be elements in $E_B$ such that

1. $p(s_B)$ is an element in $E_B$ such that $p(s_B) = \xi_B^{-1}$,
2. $s_{dR}$ is the inverse image of $\xi_{dR}^{-1}$ under the isomorphism $F^1E_{dR} \to F^1K(-1)_{dR} = K(-1)_{dR}$.

Then we have an isomorphism $cl : Ext^1_{MHS(K)}(K(-1), K) = C/2\pi iK$ by setting $cl(e_{Hg}) = 2\pi i(c(s_B) - s_{dR}) \in 2\pi iK_{dR} = C\xi_{dR}^0 \mod 2\pi iK_B = 2\pi iK\xi_B^0$.

1.3.3. Extensions arising from relative cohomologies of Fermat surfaces. Let $(X, B)$ a pair of varieties such that $B \subset X$. Let $H^i_B(X, B, K)$ and $H^i_{dR}(X, B, C)$ be the singular and de Rham cohomologies of the pair $(X, B)$ with the coefficient in $K$ and de Rham cohomology with the coefficient in $C$. By the natural comparison map $c$, we have a mixed $K$-Hodge structure $(H_B(X, B, K), H_{dR}(X, B, C))$. Let $G \to Aut(X, B)$ be a group action of $G$ on $(X, B)$ and $\chi$ be a character of $G$ with the value in $K^\times$. Then $H^i_{Hg}(X, B, K)(\chi) = (H_B(X, B, K)(\chi), H_{dR}(X, B)(\chi), c)$ becomes a mixed $K$-Hodge structure.
We define a subvariety $B$ of Fermat surface $X_m$ by

$$B := \{ u^m = 1, v^m = w^m \} \cup \{ v^m = 1, u^m = w^m \}.$$ 

The the variety $B$ is a union of affine lines.

The $\chi_a$-part of the relative singular cohomology $H_B^r(X, B, K)$ and relative de Rham cohomology $H_{dR}^r(X, B/Q)$ of $(X, B)$ are denoted by $H_B^r(X, B, K)(\chi_a)$ and $H_{dR}^r(X, B/Q)(\chi_a)$, respectively. By the long exact sequence of relative cohomology, we have an exact sequences:

$$0 \to H_B^r(B, K)(\chi_a) \to H_B^r(B, X, B, K)(\chi_a) \to H_B^r(X, K)(\chi_a) \to 0,$$
$$0 \to H_{dR}^r(B/Q)(\chi_a) \to H_{dR}^r(B/X, B/Q)(\chi_a) \to H_{dR}^r(X/Q)(\chi_a) \to 0.$$

These sequences are compatible with the comparison maps

$$c : H_B^r(X, K)(\chi_a) \otimes_K C \xrightarrow{c} H_{dR}^r(X/Q)(\chi_a) \otimes_K C, \quad \text{etc.}$$

We set

$$H_H^r(X, K)(\chi_a) = (H_B^r(X, K)(\chi_a), H_{dR}^r(X, K)(\chi_a) \otimes_K C, c)$$

Therefore the sequences (1.3.8) together with the comparison maps defines a Yoneda extension class $e$ of mixed $K$-Hodge structures

$$e_{H^r}(X, \chi_a) \in Ext^1_{HMS(K)}(H_H^r(X, K)(\chi_a), H_H^r(B, K)(\chi_a)).$$

Under the condition (1.2.7), we have isomorphisms of Hodge structures

$$(1.3.9) \qquad H_H^r(B, K)(\chi_a) \xrightarrow{\iota} K_{H^r}, \quad H_H^r(X, K)(\chi_a) \xrightarrow{\iota_2} K(-1)_{H^r}.$$

To prove Theorem 1.1 we use theory of mixed motives and its Hodge realization. By the Lefschetz-Hodge theorem, these isomorphisms arise from algebraic correspondences. Since the pair of varieties $(X, B)$ are defined over $\overline{Q}$, the above algebraic correspondence is defined over $\overline{Q}$. Let $MM(K)/L$ be the derived category of mixed motives over $L$ with the coefficients in $M$. Therefore we have the following commutative diagram where the horizontal arrow are induced by algebraic correspondence over $\overline{Q}$.

$$\begin{array}{ccc}
Ext^1_{MM(K)/Q}(h(X) K(\chi_a), h(E) K(\chi_a)) & \to & Ext^1_{MM(K)/Q}(K(-1), K) \\
\downarrow & & \downarrow \\
Ext^1_{MM(K)/C}(h(X) K(\chi_a), h(E) K(\chi_a)) & \to & Ext^1_{MM(K)/C}(K(-1), K) \\
\downarrow & & \downarrow \\
Ext^1_{HMS(K)}(h(X) K(\chi_a), h(E) K(\chi_a)) & \to & Ext^1_{HMS(K)}(K(-1), K)
\end{array}$$

Since the Hodge realization map

$$Ext^1_{MM(K)/Q}(K(-1), K) \simeq \overline{Q}^x \otimes K \xrightarrow{\rho} Ext^1_{HMS(K)}(K(-1)_{H^r}, K_{H^r}) = (C/2\pi i K)$$

is given by $e_M \mapsto \log(e_M)$, we have

$$cl(e_{H^r}(X, \chi_a)) \in K \log(\overline{Q}^x) = \text{Im}(\rho),$$
1.3.4. Extension class and relative periods. The extension class $e_{Hg}$ can be computed by the period integral as follows: Let $s_{B}, s_{dR}$ be liftings of a common bases of $H_{B}^{2}(X, K)(\chi_{a}) H_{dR}^{2}(X, \boldsymbol{Q})(\chi_{a})$ satisfying the conditions (1)-(2) in the last subsection, and $b_{dR}$ be a base of $H_{dR}^{1}(B/\boldsymbol{Q})(\chi_{a})$. Then we have $2\pi ic(s_{B}) = s_{dR} + cl(e_{Hg})b_{dR}$, where $e_{Hg} = e_{Hg}(X, \chi_{a})$.

We consider the Betti part of the dual of the exact sequences (1.3.8).

$$0 \to H_{B}^{2}(X, K)(\chi_{-a}) \to H_{B}^{2}(X, B, K)(\chi_{-a}) \xrightarrow{\partial} H_{dR}^{1}(B, K)(\chi_{-a}) \to 0$$

Let $\gamma$ be a $K$-base of $H_{B}^{1}(B, K)(\chi_{-a})$ and $\Gamma$ be an element in $H_{dR}^{1}(X, B, K)(\chi_{-a})$ such that $\partial \Gamma = \gamma$. The cup product induces a map

$$(\ast, \ast): H_{B}^{2}(X, B, K)(\chi_{-a}) \times H_{B}^{2}(X, B, K)(\chi_{a}) \to K.$$ 

Then

$$2\pi i(\Gamma, s_{B}) = 2\pi i(c(\Gamma), c(s_{B})) = (c(\Gamma), s_{dR}) + cl(e_{Hg})(c(\Gamma), b_{dR})$$

$$= (c(\Gamma), s_{dR}) + cl(e_{Hg})(c(\gamma), b_{dR}).$$

Since $(\Gamma, s_{B}) \in K$, $(c(\gamma), b_{dR}) \in \overline{\boldsymbol{Q}}$, we have

$$(c(\Gamma), s_{dR}) \in 2\pi iK + \overline{\boldsymbol{Q}} \log(\overline{\boldsymbol{Q}}^{x}) = \overline{\boldsymbol{Q}} \log(\overline{\boldsymbol{Q}}^{x}).$$

Let $s'_{dR}$ be an arbitrary lifting. Then $s'_{dR}$ is written as $s'_{dR} = s_{dR} + ab_{dR}$ with $a \in \overline{\boldsymbol{Q}}$. Thus we have

$$(c(\Gamma), s'_{dR}) = (c(\Gamma), s_{dR}) + (c(\Gamma), ab_{dR}) = (c(\Gamma), s_{dR}) + a(c(\gamma), b_{dR}) \in \overline{\boldsymbol{Q}} \log(\overline{\boldsymbol{Q}}^{x}) + \overline{\boldsymbol{Q}}.$$

As for the precise argument, see [AOT].

1.4. Period integrals for relative cohomologies and hypergeometric function. We consider a sequence of morphisms:

$$X_{m} \quad \xrightarrow{\pi''} \quad \mathbb{A}^{2}$$

$$(u, v, w) \quad \mapsto \quad (\xi, \eta) = (u^{m}, v^{m})$$

$$\bigcup \Gamma''_{1} \quad \mapsto \quad \bigcup \Gamma_{1}.$$

Let $\Gamma_{1}$ be chains of $\mathbb{A}^{2}(\mathbb{C})$ defined by

$$\Gamma_{1} = \{(\xi, \eta) \in \mathbb{R}^{2} \mid 0 \leq \xi \leq 1, 0 \leq \eta \leq 1, 1 \leq \xi + \eta\},$$

with the standard orientations and $\Gamma''_{1}$ be a topological cycle on $X_{m}$ defined by

$$\Gamma''_{1} = \left\{(u, v, w) \in X_{pm} \mid \pi' \circ \pi(u, v, w) \in \Gamma_{1}, u, v, w \in \mathbb{R}_{+}\right\}.$$ 

Let $\gamma_{0}$ be a one chain defined by

$$\{t: [0, 1] \to (1, t) \in X_{m}\} + \{t: [0, 1] \to (1 - t, 1) \in X_{m}\}.$$ 

For a chain $\beta$ in $X_{m}$, we set

$$pr_{x}(\beta) = \frac{1}{|G|} \sum_{g \in G} \chi(g)^{-1}g(\beta).$$
Proposition 1.3. Under the above notation, we have \( \text{pr}_\chi(\Gamma_1) \in H^2_B(X, B, K)(\chi) \), \( \text{pr}_\chi(\gamma_0) \in H^1_B(B, K)(\chi) \) and
\[
\partial \text{pr}_\chi(\Gamma_1) = \text{pr}_\chi(\gamma_0).
\]
Moreover, \( \text{pr}_\chi(\gamma_0) \) is a base of \( H^1_B(B, K)(\chi) \).

We set
\[
\omega = (\xi + \eta - 1)^{\alpha_1 - 1} \xi^{\alpha_2 - 1} \eta^{\alpha_3 - 1}.
\]
Since the pairing is given by integrals, we have
\[
(c(\text{pr}_\chi(\Gamma_1)), \omega) = \int_{\text{pr}_\chi(\Gamma_1)} \omega = \frac{1}{|G|} \sum_g \chi^{-1}(g) \int_{\Gamma_1} \omega
\]
\[
= \frac{1}{|G|} \sum_g \chi^{-1}(g) \int_{\Gamma_1} (g^{-1})^* \omega = \int_{\Gamma_1} \omega
\]

In the following, we give an explicit formula for \( (c(\text{pr}_\chi(\Gamma_1)), \omega) \) for the classification (1)–(2) of Theorem 1.2. The following proposition give a relation between the period integrals for relative cycles and special values of the hypergeometric function.

Proposition 1.4. Let \( \alpha_1, \alpha_2, \alpha_3 > 0 \) be real numbers. Then we have
\[
\int_{\Gamma_1} (\xi + \eta - 1)^{\alpha_1 - 1} \xi^{\alpha_2 - 1} \eta^{\alpha_3 - 1} d\xi d\eta
\]
\[
= \frac{1}{(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_3)} F(1, 1, \alpha_1 + \alpha_2 + \alpha_3; \alpha_1 + \alpha_2 + 1, \alpha_1 + \alpha_3 + 1; 1)
\]

Proof. By changing the variables
\[
\xi = \frac{1 - s}{1 - st}, \eta = \frac{1 - t}{1 - st},
\]
the domain \( \Gamma_1 \) of the integral are transformed into the domain
\[
\Gamma'_1 = \{0 < t_1 < 1, 0 < t_2 < 1\}.
\]

By changing the variable, we have
\[
\int_{\Gamma_1} (\xi + \eta - 1)^{\alpha_1 - 1} \xi^{\alpha_2 - 1} \eta^{\alpha_3 - 1} d\xi d\eta
\]
\[
= \int_{\Gamma'_1} (1 - s)^{\alpha_1 + \alpha_2 - 1} (1 - t)^{\alpha_1 + \alpha_3 - 1} (1 - st)^{-\alpha_1 - \alpha_2 - \alpha_3} ds dt
\]
\[
= B(1, \alpha_1 + \alpha_2) B(1, \alpha_1 + \alpha_3) F(1, 1, \alpha_1 + \alpha_2 + \alpha_3; \alpha_1 + \alpha_2 + 1, \alpha_1 + \alpha_3 + 1; 1)
\]

1.5. First example. We consider the case (1) in Theorem 1.2 \( (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\alpha, 1 - \alpha, \beta, 1 - \beta) \), where \( \alpha, \beta \in \mathbb{Q} \) and \( 0 < \alpha, \beta < 1 \). We compute the integral
\[
\int_{\Gamma_1} (\xi + \eta - 1)^{-\alpha} \xi^{\alpha - 1} \eta^{\beta - 1} d\xi d\eta
\]
with $\Gamma_1 = \{\xi + \eta > 1, \xi < 1, \eta < 1\}$. By changing variable by $\eta' = \frac{\xi + \eta - 1}{\xi}$, $\xi' = \eta$ (i.e. $\xi = 1 - \xi'$, $\eta = \xi'$), we have

$$
\int_{\Gamma_1} (\xi + \eta - 1)^{-\alpha} \xi^{\alpha-1} \eta^{\beta-1} d\xi d\eta = \int_{\Gamma_1^*} \eta'^{-\alpha} \xi'^{\beta-1} \frac{1}{1 - \eta'} d\xi' d\eta'
$$

where $\Gamma_1^* = \{\eta' < \xi' < 1, 0 < \eta' < 1\}$.

We extend the above equality using analytic continuation using Pochhammer integral. Let $\alpha \not\in \mathbb{Z}$ and $f(x)$ be a rational function of $C$ without pole on $[0,1] \subset \mathbb{R}$. We define Pochhammer integral by

\begin{equation}
(1.5.11) \quad \int_{C_{\epsilon}} x^{\alpha} f(x) dx = \int_{\epsilon}^{1} x^{\alpha} f(x) dx + \frac{1}{e^{\alpha} - 1} \int_{C_{\epsilon}} x^{\alpha} f(x) dx
\end{equation}

Here the path $C_{\epsilon}$ is defined by

$$
C_{\epsilon} : [0,1] \rightarrow C : t \mapsto \epsilon e(t).
$$

Here we choose the branch of $x^\alpha$ on $C_{\epsilon}$ to be $\arg(x^\alpha) = 2\pi \alpha t$ for the above parameter $t$. Then the integral (1.5.11) is analytic function of $\alpha$ for $\alpha \not\in \mathbb{Z}$ and

$$
\int_{0}^{1} x^{\alpha} f(x) dx = \int_{P(0,1)} x^{\alpha} f(x) dx
$$

for $-1 < \alpha$. Thus we have the following theorem.

**Theorem 1.5.** Let $\alpha, \beta$ be real numbers such that $\alpha, \beta, \alpha - \beta \not\in \mathbb{Z}$. We have the following identity:

$$
\frac{1}{(-\alpha + 1 + \beta)} F(1,1,1 + \beta;2,\alpha + 2 + \beta;1) = \int_{P(0,1)} \eta^{-\alpha} \frac{1 - \eta^\beta}{\beta(1 - \eta)} d\eta.
$$

By the following proposition, the above integral expression gives an explicit formula as an element in $\mathbb{Q} + \mathbb{Q} \log(\mathbb{Q}^\times)$.

**Proposition 1.6.** Let $\alpha = \frac{n}{m}, \beta = \frac{n'}{m} \in (0,1)$ is a rational number. We choose a suitable branch of $\log$ and a suitable path $[0,1]$. Then we have

1. Let $c \in \mathbb{C}^\times$, $|c| \neq 1$. We choose $\gamma \in C$ such that $\gamma^m = c$. then

$$
\int_{0}^{1} \frac{x^{\alpha} dx}{c - x} = -\sum_{i=0}^{m-1} e(-ni/m) \log(1 - \frac{e(i/m)}{\gamma})(= \frac{1}{c\alpha} {}_2F_1(1,\alpha;\alpha + 1;1/c))
$$

2. (Gauss’s digamma theorem)

$$
\int_{0}^{1} \frac{x^{\beta} - x^{\alpha} dx}{1 - x} = -\sum_{i=1}^{m-1} (e(-ni'/m) - e(-ni/m)) \log(1 - e(i/m))
$$
Proof. We have
\[
\int_0^1 \frac{x^n}{x - x} \, dx = \int_0^1 \frac{m\xi^{n-1}}{c - \xi^m} \, d\xi = \int_0^1 \sum_{i=0}^{m-1} e(-i(n-1)/m) \, d\xi \\
= -\sum_{i=0}^{m-1} e(-ni/m) \log(1 - \frac{e(i/m)}{\gamma})
\]

The second statement follows from the first. □

1.6. **Permutations of exponent indices.** Let \(\Gamma_i (i = 1, 2, 3)\) be chains of \(A^2(C)\) defined by

(1.6.12) \[\begin{align*}
\Gamma_1 &= \{ (\xi, \eta) \in \mathbb{R}^2 \mid 0 \leq \xi \leq 1, 0 \leq \eta \leq 1, \xi + \eta \leq 1 \}, \\
\Gamma_2 &= \{ (\xi, \eta) \in \mathbb{R}^2 \mid 0 \leq \xi \leq 1, -1 \leq \eta \leq 0, 0 \leq \xi + \eta \leq 1 \}, \\
\Gamma_3 &= \{ (\xi, \eta) \in \mathbb{R}^2 \mid -1 \leq \xi \leq 0, 0 \leq \eta \leq 1, 0 \leq \xi + \eta \leq 1 \},
\end{align*}\]

with the standard orientations. We set \(\xi = \eta', \eta = 1 - \xi' - \eta'\) and \(\xi = 1 - \xi'' - \eta'', \eta = \xi''\). Then we have

\[
\int_{\Gamma_2} (\xi + \eta - 1)^{\alpha_1 - 1} \xi^{\alpha_2 - 1} \eta^{\alpha_3 - 1} \, d\xi d\eta \\
= (-1)^{\alpha_1 + \alpha_3} \int_{\Gamma_1} (\xi')^{\alpha_1 - 1} (\eta')^{\alpha_2 - 1} (\xi' + \eta' - 1)^{\alpha_3 - 1} \, d\xi d\eta
\]

\[
\int_{\Gamma_3} (\xi + \eta - 1)^{\alpha_1 - 1} \xi^{\alpha_2 - 1} \eta^{\alpha_3 - 1} \, d\xi d\eta \\
= (-1)^{\alpha_1 + \alpha_2} \int_{\Gamma_1} (\eta'')^{\alpha_1 - 1} (\xi'' + \eta'' - 1)^{\alpha_2 - 1} (\xi'')^{\alpha_3 - 1} \, d\xi d\eta
\]

Thus the integral for \((\alpha_3, \alpha_1, \alpha_2)\) and \((\alpha_2, \alpha_3, \alpha_1)\) on \(\Gamma_1 \subset X_p\) is reduced to the integral for \((\alpha_1, \alpha_2, \alpha_3)\) on \(\Gamma_2\) and \(\Gamma_3\).

| \(X\) | \(\Gamma_1\) | reduced to \(\Gamma_2\) | reduced to \(\Gamma_3\) |
|---|---|---|---|
| \(X_m\) | \((\beta, -\alpha, \alpha)\) | \((\alpha, \beta, -\alpha)\) | \((-\alpha, \alpha, \beta)\) |
| \(X_{2m}\) | \((2\alpha, -\alpha, -\alpha + 1/2)\) | \((-\alpha + 1/2, 2\alpha, -\alpha)\) | \((-\alpha, -\alpha + 1/2, 2\alpha)\) |
| \(X_{3m}\) | \((3\alpha, -\alpha + 1/3, -\alpha + 2/3)\) | \((-\alpha + 2/3, 3\alpha, -\alpha + 1/3)\) | \((-\alpha + 1/3, -\alpha + 2/3, 3\alpha)\) |
| \(X_{4m}\) | \((4\alpha, -\alpha + 1/4, -\alpha + 3/4)\) | \((-\alpha + 3/4, 4\alpha, -\alpha + 1/4)\) | \((-\alpha + 1/4, -\alpha + 3/4, 4\alpha)\) |

We have a symmetry on \((\alpha_1, \alpha_2, \alpha_3)\) and \((\alpha_1, \alpha_3, \alpha_2)\). Therefore \((X_{2m}, \Gamma_3)\)-case is reduced to \((X_{2m}, \Gamma_2)\)-case by the shifting \(\alpha \mapsto \alpha - 1/2\). Similarly, \((X_{m}, \Gamma_2)\)-case is reduced to \((X_{m}, \Gamma_3)\)-case. The cases \((X_{m}, \Gamma_3)\) is computed in the previous example. As for the relation with Watson’s formula, see [AOT]. We set \(\alpha_0 = (\alpha_1 \alpha_2 \alpha_3)^{-1}\). Let \((\alpha_{i_0}, \alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3})\) be a quadruple obtained by a permutation of \((\alpha_0, \alpha_1, \alpha_2, \alpha_3)\). Then \(\alpha' = (\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3})\)-part is also generated by algebraic cycles if \(\alpha = (\alpha_1, \alpha_2, \alpha_3)\)-part is. The corresponding period integral is an easy linear combination of periods corresponding to permutations of \((\alpha_1, \alpha_2, \alpha_3)\).
2. Algebraic cycles and one forms with constant residues

2.1. Algebraic cycles on Fermat surfaces. As for the case (1)–(2) in Theorem 1.2, algebraic cycles generating the Hodge classes are given as follows (see [AS]). We define varieties $Z^{(1)}_{i,j}, Z^{(2)}_{i,\pm}, Z^{(3)}_{i,j,k} \text{ and } Z^{(4)}_{i,k,\pm}$ by

\[
\begin{align*}
Z^{(1)}_{i,j} : w &= e(i/m + 1/2m), \quad u = e(j/m + 1/2m)v, \quad (0 \leq i, j \leq m - 1) \\
Z^{(2)}_{i,\pm} : w^2 &= (\mp 1)^{1/m} e(i/m)2^{1/m}uv, \quad u^m \pm v^m = 1, \quad (0 \leq i \leq m - 1) \\
Z^{(3)}_{i,j,k} : w^3 &= 3^{1/m} e((i + j)/3m + k/m + 1/2m)uv, \quad \omega^i u^m + \omega^j v^m = 1 \\
&\quad (0 \leq i, j \leq 2, 0 \leq k \leq m - 1) \\
Z^{(4)}_{i,k,\pm} : w^4 &= e((k + 4i)/4m)(2\sqrt{2})^{1/m}uv, \\
&\quad u^{2m} + (e(k/2)v^{2m} \mp e(k/4)\sqrt{2}u^mv^m = \mp 1, \quad (0 \leq i \leq m, 0 \leq k \leq 3)
\end{align*}
\]

**Proposition 2.1.** We have the following inclusions

\[
Z^{(1)}_{i,j} \subset X_m, \quad Z^{(2)}_{i,\pm} \subset X_{2m}, \quad Z^{(3)}_{i,j,k} \subset X_{3m}, \quad Z^{(4)}_{i,k,\pm} \subset X_{4m}
\]

**Proof.** We set $x = u^m, y = v^m, z = w^m$. Using relations

\[
\begin{align*}
z &= -1, \quad y + y = 0 \quad (\text{on } Z^{(1)}_{i,j}) \\
z^2 &= \mp 2xy, \quad x \mp y = 1 \quad (\text{on } Z^{(2)}_{i,\pm}) \\
z^3 &= -3\omega^{i+j}xy, \quad \omega^i x + \omega^j y = 1 \quad (\text{on } Z^{(3)}_{i,j,k}) \\
z^4 &= 2\sqrt{2}i^kxy, \quad x + (-1)^ky \mp i^kxy \mp 1 = 0 \quad (\text{on } Z^{(4)}_{i,k,\pm})
\end{align*}
\]

we have

\[
\begin{align*}
x^2 + y^2 - z &= 1 \quad (\text{on } Z^{(1)}_{i,j}) \\
x^2 + y^2 - z^2 - 1 &= x^2 + y^2 \mp 2xy - 1 = (x \pm y - 1)(x \pm y + 1) \quad (\text{on } Z^{(2)}_{i,\pm}) \\
x^3 + y^3 - z^3 - 1 &= x^3 + y^3 + 3\omega^{i+j}xy - 1 \\
&= (\omega^i x + \omega^j y - 1)(\omega^{i+1}x + \omega^{j+1}y - 1)(\omega^{i-1}x + \omega^{j+1}y - 1) = 0, \quad (\text{on } Z^{(3)}_{i,j,k}) \\
x^4 + y^4 - z^4 - 1 &= x^4 + y^4 - i^k 2\sqrt{2}xy - 1 \\
&= (x^2 + (-1)^ky^2 - i^k \sqrt{2}xy - 1)(x^2 + (-1)^ky^2 + i^k \sqrt{2}xy + 1) = 0, \quad (\text{on } Z^{(4)}_{i,k,\pm})
\end{align*}
\]

Thus we have the required inclusions. \qed

By the above proposition, we have

\[
Z_1 = \bigcup_{i,j} Z^{(1)}_{i,j} \subset X_m, \quad Z_2 = \bigcup_i Z^{(2)}_{i,\pm} \subset X_{2m}, Z_3 = \bigcup_{i,j,k} Z^{(3)}_{i,j,k} \subset X_{3m}, \quad Z_4 = \bigcup_{i,k} Z^{(4)}_{i,k,\pm} \subset X_{4m}
\]

Then the varieties $Z$ is stable under the action of $G_{pm} = \mu^3_{pm}$ given as (1.2.5). We set $x = u^m, y = v^m, z = w^m$ and for an element of $\alpha \in \frac{1}{pm} \mathbb{Z}$ and use a notation $x^\alpha = u^{m\alpha}$, etc.
We define characters $\phi_p$ of $G_{pm}$ by setting $\phi_p = \chi_a$, where

$$a = \begin{cases} 
(\alpha, -\alpha, \beta, -\beta) & (p = 1) \\
(2\alpha, 1 - \alpha, -\alpha + \frac{1}{2}, \frac{1}{2}) & (p = 2) \\
(3\alpha, 1 - \alpha, -\alpha + \frac{1}{3}, -\alpha + \frac{2}{3}) & (p = 3) \\
(4\alpha, 1 - 2\alpha, -\alpha + \frac{1}{4}, -\alpha + \frac{3}{4}) & (p = 4)
\end{cases}$$

Here the Kummer character $\chi_a$ is defined by \[\text{The following proposition is proved in } [\text{AS}].\]

**Proposition 2.2 ([AS]).** The $\phi_p$ parts of the cohomology classes of the components in $Z_p$ generates $H^2(X_{pm})(\phi_p)$. In other words, the map $H^2_{Z_p}(X_{pm})(\phi_p) \to H^2(X_{pm})(\phi_p)$ is surjective. As a consequence, the map $H^2(X_{pm})(\phi_p) \to H^2(X_{pm} - Z_p)(\phi_p)$ is the zero map by the localization exact sequence.

2.2. **Algebraic cycles and differential forms.** Let $p = 1, 2, 3$. We express the de Rham cohomology $H^2(X_{pm})(\phi_p)$ by that of an open set $X^0_{pm}$ defined by

$$X^0_{pm} = \{(u, v, w) \in X_{pm} \mid uvw \neq 0\}$$

By branch condition, we have

$$H^2(X_{pm})(\phi_p) \simeq H^2(X^0_{pm})(\phi_p), \quad H^2(X_{pm} - Z_p)(\phi_p) \simeq H^2(X^0_{pm} - Z_p)(\phi_p).$$

Let $\Omega = \Omega_p \in \Gamma(X^0_{pm}, \Omega^2)(\phi_p)$ be a closed form whose cohomology class generates $H^2(X_{pm})(\phi_p)$. We set $U_p = X^0_{pm} - Z_p$ and the inclusion $U_p \to X^0_{pm}$ is denoted by $j$. We consider the following diagram:

$$
\begin{array}{ccc}
\Gamma(X^0_{pm}, \Omega^1_{X_{pm}})(\phi_p) & \xrightarrow{d} & \Gamma(X^0_{pm}, \Omega^2_{X_{pm}})(\phi_p) \\
\downarrow j^* & & \downarrow j^* \\
\Gamma(X^0_{pm}, \Omega^1_{X_{pm}}(\log Z_p))(\phi_p) & \xrightarrow{\text{res}} & \Gamma(X^0_{pm}, \Omega^2_{X_{pm}}(\log Z_p))(\phi_p) \\
\oplus_{Z_p^{(i)} \subset Z_p} \Gamma(X^0_{pm} \cap Z_p^{(i)}, \mathcal{O})(\phi_p) & \xrightarrow{d} & \oplus_{Z_p^{(i)} \subset Z_p} \Gamma(X^0_{pm} \cap Z_p^{(i)}, \Omega^1)(\phi_p)
\end{array}
$$

By Proposition 2.2 there exists an element $\psi \in \Gamma(X^0_{pm}, \Omega^1_{X_{pm}}(\log Z_p))(\phi_p)$ such that

$$d\psi = j^* \Omega.$$

By the above diagram, $\text{res}(\psi)$ is a locally constant function. Vice versa, if $\text{res}(\psi)$ of a one form $\psi$ is a constant function on each component, then $\Omega = d\psi$ is a closed form and an image under the map $j^*: \Gamma(X^0_{pm}, \Omega^2_{X_{pm}})(\phi_p) \to \Gamma(X^0_{pm}, \Omega^2_{X_{pm}}(\log Z_p))(\phi_p)$

Moreover, if the class $[\Omega] \in H^2(X^0_{pm})(\phi_p) \simeq H^2_{Z_p}(X^0_{pm})(\phi_p)$ is not zero, it becomes a base. In the next subsection, we construct $\psi$ satisfying the above equation.
2.3. One forms \( \psi \) with constant residues along \( Z_p \). For \( p = 1, 2, 3, 4 \), we consider varieties

\[
X_p = \text{Spec}(\mathbb{C}[x, y]/(x^p + y^p - 1 - z^p))
\]

and consider a sequence of morphisms:

\[
(2.3.1) \quad \begin{align*}
X_{pm} & \xrightarrow{\pi} (u, v, w) \quad (x, y, z) = (u^m, v^m, w^m) \\
\xrightarrow{\pi'} & \quad A^2
\end{align*}
\]

We define one form \( \psi_0 \) on \( X_{pm} \) by

\[
\psi_0 = \begin{cases}
(x + y - 1)^\beta \frac{dx + dy}{x + y}, & \text{on } X_m \\
\frac{dx + dy}{x + y - 1} - \frac{dx - dy}{x - y - 1}, & \text{on } X_{2m} \\
\sum_{1 \leq i,j \leq 3} \omega^{2i+j} \frac{\omega^i dx + \omega^j dy}{\omega^x + \omega^y y - 1}, & \text{on } X_{3m} \\
\sum_{0 \leq k \leq 3} i^k d \log (x^2 + (iky)^2 + \sqrt{2}x(iky) + 1) \\
- \sum_{0 \leq k \leq 3} i^k d \log (x^2 + (iky)^2 - \sqrt{2}x(iky) - 1), & \text{on } X_{4m}
\end{cases}
\]

and rational functions \( f \) by

\[
f = \begin{cases}
\frac{y}{x}, & \text{on } X_m \\
\frac{x^2 + y^2 - 1}{2x^2 y^2}, & \text{on } X_{2m} \\
\frac{x^3 + y^3 - 1}{x^3 y^3}, & \text{on } X_{3m} \\
\frac{x^4 + y^4 - 1}{x^4 y^4}, & \text{on } X_{4m}
\end{cases}
\]

Since \( f|_{Z_p} \) is a constant function, the residue of a differential form \( \psi = f^\alpha \psi_0 \) is constant on each component of \( Z_p \). Using relations \( df_0 = 0 \) and

\[
df_0 \wedge \psi_0 = \begin{cases}
-(x + y - 1)^\beta \frac{dx \wedge dy}{x^2}, & \text{on } X_m \\
-4(x^2 + y^2 - 1) \frac{dx \wedge dy}{x^3 y^2}, & \text{on } X_{2m} \\
-27(x^3 + y^3 - 1) \frac{dx \wedge dy}{x^3 y^2}, & \text{on } X_{3m} \\
-64 \sqrt{2} \frac{(x^4 + y^4 - 1)^3 dx \wedge dy}{x^4 y^2}, & \text{on } X_{4m}
\end{cases}
\]

we have

\[
d\psi = \alpha f^{\alpha - 1} df \wedge \psi_0 = \begin{cases}
-\alpha y^{\alpha - 1} x^{\alpha - 1} (x + y - 1)^\beta dx \wedge dy, & \text{on } X_m \\
-4\alpha (x^2 + y^2 - 1)^{2\alpha - 1} x^{-2\alpha - 1} y^{-2\alpha} dx \wedge dy, & \text{on } X_{2m} \\
-27\alpha (x^3 + y^3 - 1)^{3\alpha - 1} x^{-3\alpha} y^{-3\alpha + 1} dx \wedge dy, & \text{on } X_{3m} \\
-64 \sqrt{2} \alpha (x^4 + y^4 - 1)^{4\alpha - 1} x^{-4\alpha} y^{-4\alpha + 2} dx \wedge dy, & \text{on } X_{4m}
\end{cases}
\]
By setting \( x^p = \xi, y^p = \eta \), we have the following multi valued differential form on \( \mathbb{A}^2 \):

\[
\Omega = d\psi = \begin{cases} 
-\alpha \eta^{-1} \xi^{-1} (\xi + \eta - 1) \beta d\xi \wedge d\eta, & \text{on } X_m \\
-\alpha (\xi + \eta - 1)^{2\alpha - 1} \xi^{-\alpha - 1} \eta^{-\alpha - 1/2} d\xi \wedge d\eta, & \text{on } X_{2m} \\
-3\alpha (\xi + \eta - 1)^{3\alpha - 1} \xi^{-\alpha - 2/3} \eta^{-\alpha - 1/3} d\xi \wedge d\eta, & \text{on } X_{3m} \\
-4\sqrt{2}\alpha (\xi + \eta - 1)^{4\alpha - 1} \xi^{-\alpha - 3/4} \eta^{-\alpha - 1/4} d\xi \wedge d\eta, & \text{on } X_{4m}.
\end{cases}
\]

Then the integral \( \int_{\Gamma_i} \Omega \) converges, if \( \alpha \) and \( \beta \) are real numbers and satisfy the following conditions.

| \( \Gamma \) | \( \Gamma_1 \) | \( \Gamma_2 \) | \( \Gamma_3 \) |
|---|---|---|---|
| \( X_m \) | \( -1 + |\alpha|^\beta \) | | |
| \( X_{2m} \) | \( 0 < \alpha \) | \( 0 < \alpha < 1/4 \) | |
| \( X_{3m} \) | \( 0 < \alpha \) | \( -1/3 < \alpha < 1/2 \) | \( -1/6 < \alpha < 1/3 \) |
| \( X_{4m} \) | \( 0 < \alpha \) | \( -1/4 < \alpha < 1/2 \) | \( -1/12 < \alpha < 1/4 \) |

**Table 1. Convergent condition**

### 3. Hypergeometric identities for \( \Gamma_1 \)

#### 3.1. Blowing up and Stokes’ formula.

For a real number \( \alpha \), we set \((-1)^\alpha = e(\alpha/2)\). Let \( \Gamma_i \) and be \( (i = 1, 2, 3) \) topological chains defined in [1.6.12]. We define a topological cycle \( \Gamma''_1, \Gamma''_2, \Gamma''_3 \) on \( X_{pm} \) for \( p = 1, 2, 3, 4 \) by

\[
\Gamma''_1 = \left\{(u, v, w) \in X_{pm} \mid \pi' \circ \pi(u, v, w) \in \Gamma_1, u, v, w \in \mathbb{R}_+ \right\},
\]

\[
\Gamma''_2 = \left\{(u, v, w) \in X_{pm} \mid \pi' \circ \pi(u, v, w) \in \Gamma_2, w, v \in (-1)^{1/pm} \mathbb{R}_+, u \in \mathbb{R}_+ \right\},
\]

\[
\Gamma''_3 = \left\{(u, v, w) \in X_{pm} \mid \pi' \circ \pi(u, v, w) \in \Gamma_3, w, u \in (-1)^{1/pm} \mathbb{R}_+, v \in \mathbb{R}_+ \right\}.
\]

Here \( \pi' \circ \pi \) is a morphism defined in [2.3.1]. The image \( \pi(\Gamma''_1) \) of \( \Gamma''_1 \) under the map \( \pi \) defined in [2.3.1] is denoted by \( \Gamma'_1 \). Then we have the following diagram:

\[
X_{pm} \xrightarrow{\pi'} X_p \xrightarrow{\pi'} \mathbb{A}^2
\]

We set \( B_y = \{x = (-1)^{1/p} y\}, B_x = \{y = (-1)^{1/p} x\}, B_2 = \{x = 1\} \) and \( B_3 = \{y = 1\} \). Then we have

\[
\partial \Gamma'_1 \subset \{x^p + y^p = 1\} \cup B_2 \cup B_3, \\
\partial \Gamma'_2 \subset B_x \cup B_2 \cup \{y = 0\}, \\
\partial \Gamma'_3 \subset B_y \cup \{x = 0\} \cup B_3,
\]

Tomohide Terasoma 13
In the last section, we have a relation \( d\psi = j^*\Omega \) on \( X_p \). Since the pole of the differential form \( \psi_p \) passes through the singular points

\[
p_2 = \{(x, y) = (1, 0)\}, \quad p_3 = \{(x, y) = (0, 1)\}
\]

in \( \Gamma'_2, \Gamma'_3 \), we can not apply Stokes’ formula for \( \Gamma'_i \) in \( X_p \). Thus we consider the blowing up \( b : \widetilde{X}_p \to X_p \) of the variety \( X_p \) with the center \( B' \cap Z' \). Let \( E_2 \) and \( E_3 \) be the exceptional divisors over the points \( p_2 \) and \( p_3 \), respectively.

Let \( \widetilde{\Gamma}_i \) be the closure of \( b^{-1}(\Gamma'_i) \) where \( \Gamma'_i^0 \) is the relative interior of \( \Gamma'_i \). The chain \( \widetilde{\Gamma}_i \) is called the proper transform of \( \Gamma'_i \) for short. From now on we assume that \( \alpha \) and \( \beta \) satisfy the convergent condition Table I. Via the birational map \( b \), we have an equality

\[
\int_{\Gamma'_i} \Omega = \int_{\widetilde{\Gamma}_i} \widetilde{\Omega}.
\]

Let \( \widetilde{B}_i \) and \( \widetilde{R} \) be the proper transform of \( B_i \) (\( i = 2, 3, x, y \)) and \( R = \{x^p + y^p = 1\} \cup \{x = 0\} \cup \{y = 0\} \), respectively. Then we have

\[
\begin{align*}
\partial \widetilde{\Gamma}_1 & \subset \widetilde{B}_3 \cup \widetilde{B}_2 \cup \widetilde{R}, \\
\partial \widetilde{\Gamma}_2 & \subset \widetilde{B}_x \cup \widetilde{B}_2 \cup \widetilde{R} \cup E_2, \\
\partial \widetilde{\Gamma}_3 & \subset \widetilde{B}_y \cup \widetilde{B}_3 \cup \widetilde{R} \cup E_3.
\end{align*}
\]

Since \( d\widetilde{\psi}_p = \widetilde{\Omega} \) on \( \widetilde{X}_p \) and the pole of the rational differential form \( \widetilde{\psi}_p \) does not intersect with the proper transforms \( \widetilde{\Gamma}_i \) for \( i = 2, 3 \), we have

\[
(3.1.2) \quad \int_{\widetilde{\Gamma}_i} \widetilde{\Omega} = \int_{\partial \widetilde{\Gamma}_i} \widetilde{\psi}_p
\]

by Stokes’ formula.

3.2. Computation of restrictions for \( \Gamma_1 \). The restriction of the rational one form \( \psi \) to \( \widetilde{B}_i \) (\( i = x, y, 2, 3 \)) and \( E_j \) (\( j = 1, 2 \)) are denoted by \( \psi_{B_i} = \psi|_{B_i} \) and \( \psi_{E_j} = \psi|_{E_j} \), respectively. We have \( \psi \big|_{x^p + y^p = 1} = 0 \) on \( X_p \) for \( p = 1, 2, 3 \). The restrictions \( \psi_{B_2}, \psi_{B_3} \) are computed as
follows.

\[ \begin{align*}
on X_m : \psi_{B2} &= y^{\beta + \alpha} \frac{dy}{1 + y}, & \psi_{B3} &= x^{\beta - \alpha} \frac{dx}{1 + x}, \\
on X_{2m} : \psi_{B2} &= 0, & \psi_{B3} &= 2x^{2\alpha - 1} \frac{dx}{2 - x}, \\
on X_{3m} : \psi_{B2} &= -y^{6\alpha} \frac{9y(3 + y^3)dy}{27 + y^6}, & \psi_{B3} &= x^{6\alpha} \frac{9(9 + x^3)dx}{27 + x^6}, \\
on X_{4m} : & \left\{ \begin{array}{l}
\psi_{B2} = -y^{12\alpha} \frac{8\sqrt{2}y^2(-8 + 4y^4 + y^8)}{-64 + y^{12}} dy, \\
\psi_{B3} = x^{12\alpha} \frac{8\sqrt{2}(-8x^4 - 32 + x^8)}{-64 + x^{12}} dx.
\end{array} \right.
\end{align*} \]

3.3. Stokes’ theorem for \( \Gamma_1 \). By Stokes’ theorem 3.1.2 and the computations for the restriction of \( \eta \), we have the following theorem.

**Theorem 3.1.** Suppose that \( \alpha, \beta \) be real number satisfying the condition of Table 1. Then we have the following equalities.

1. For \( -1 + |\alpha| < \beta \), we have

\[
\alpha \int_{\Gamma_1} (x + y - 1)^\beta x^{-\alpha - 1} y^{\alpha - 1} dx \wedge dy = -\int_0^1 y^{\beta + \alpha} \frac{dy}{1 + y} + \int_0^1 x^{\beta - \alpha} \frac{dx}{1 + x}
\]

2. For \( \alpha > 0 \), we have

\[
\alpha \int_{\Gamma_1} (\xi + \eta - 1)^{2\alpha - 1} \xi^{-\alpha - 1} \eta^{-\alpha - 1/2} d\xi \wedge d\eta = 2 \int_0^1 x^{2\alpha - 1} \frac{dx}{2 - x}
\]

3. For \( \alpha > 0 \), we have

\[
\alpha \int_{\Gamma_1} (\xi + \eta - 1)^{3\alpha - 1} \xi^{-\alpha - 2/3} \eta^{-\alpha - 1/3} d\xi \wedge d\eta
= 3 \int_0^1 y^{6\alpha} \frac{9(3 + y^3)dy}{27 + y^6} + 3 \int_0^1 x^{6\alpha} \frac{(9 + x^3)dx}{27 + x^6}
\]

4. For \( \alpha > 0 \), we have

\[
\alpha \int_{\Gamma_1} (\xi + \eta - 1)^{4\alpha - 1} \xi^{-\alpha - 3/4} \eta^{-\alpha - 1/4} d\xi \wedge d\eta
= 2 \int_0^1 y^{12\alpha} \frac{y^2(-8 + 4y^4 + y^8)}{64 - y^{12}} dy + 2 \int_0^1 x^{12\alpha} \frac{-8x^4 - 32 + x^8}{64 - x^{12}} dx
\]

By analytic continuation, we have the following hypergeometric identities.

**Theorem 3.2.**

1. For \( \alpha \notin \mathbb{Z} \), we have the following equalities.

\[
\frac{\alpha}{(\beta - \alpha + 1)(\beta + 1 + \alpha)} F(1, 1, \beta + 1; \beta - \alpha + 2, \beta + 2 + \alpha; 1)
= \int_{\Gamma_{(0,1)}} x^{\beta - \alpha} \frac{dx}{1 + x}
\]
(2) For $2\alpha \not\in \mathbb{Z}$, we have the following equalities.

\[ \frac{1}{\alpha + 1/2} F(1, 1, 1/2; \alpha + 1, \alpha + 3/2; 1) = 2 \int_{P(0,1)} \frac{x^{2\alpha-1} dx}{2 - x} \]

(3) For $3\alpha \not\in \mathbb{Z}$, we have the following equalities.

\[ \frac{\alpha}{(6\alpha + 1)(2\alpha + 2/3)} F(1, 1, \alpha + 1; 2\alpha + 4/3, 2\alpha + 5/3; 1) = \int_{P(0,1)} y^{6\alpha + 1} \frac{(3 + y^3) dy}{27 + y^6} + \int_{P(0,1)} x^{6\alpha} \frac{(9 + x^3) dx}{27 + x^6} \]

(4) For $4\alpha \not\in \mathbb{Z}$, we have the following equalities.

\[ \frac{4\alpha}{(12\alpha + 1)(3\alpha + 3/4)} F(1, 1, 2\alpha + 1; 3\alpha + 5/4, 3\alpha + 7/4; 1) = \int_{P(0,1)} \frac{y^{12\alpha} 2y^2(8 - 4y^4 - y^8)}{64 - y^{12}} dy + \int_{P(0,1)} x^{12\alpha} \frac{2(8x^4 + 32 - x^8)}{64 - x^{12}} dx \]

4. Hypergeometric identities for $\Gamma_2$

4.1. **Restriction of $\psi$ to the boundary for $\Gamma_2$.** In this section, we give a hypergeometric identities arising from the integration on $\Gamma_2$

4.1.1. **Restriction to the component $B_x$.** On $X_{pm}$, we have

\[
\psi_{B_x} = \begin{cases} 
-\frac{1}{2}x^{4\alpha-1} dx, & p = 2 \\
-1 \cdot \frac{27 \omega x^{6\alpha+2}(3x^3 + 1)}{1 + 27x^6} dx, & p = 3 \\
-1 \cdot \frac{(32 - 32i)(1 + 8x^4)x^3}{64x^8 + 1} dx, & p = 4 
\end{cases}
\]

4.1.2. **Restriction to the component $B_2$.** We restrict $\psi$ to $B_2$ and change parameter of integrals on $X_{pm}$ by $y = (-1)^{1/p} x$. Then we have

\[
\psi_{B_2} = \begin{cases} 
0, & p = 2 \\
-\frac{9\omega x(3 - x^3)}{27 + x^6} dx, & p = 3 \\
-\frac{(8 - 8i)x^2(8 - 4x^4 + 8x^8)}{64 + x^{12}} dx, & p = 4 
\end{cases}
\]

4.1.3. **Contribution from an exceptional divisor on $\Gamma_2$.** By setting $x - 1 = uy$, we compute the contribution of $\psi$ from the exceptional divisor $E_2$ on $X_{2m}$ at $x - 1 = y = 0$ as follows:

\[
\psi = 2 \left( \frac{x^2 + y^2 - 1}{(x^2y^2)^\alpha(x - 1)^2 - y^2} \right) (-ydx + (x - 1)dy) = 2 \left( \frac{u^2y^2 + 2uy + y^2}{(uy + 1)^2y^2} \right)^{2\alpha} (-ydy - y^2du + udy) \\
\psi_{E_2} = 2 \left( \frac{2u}{1 - u^2} \right)^{2\alpha} du = 2 \left( \frac{4u^2}{1 - u^2} \right)^{\alpha} du.
\]
Changing the coordinates by \( u = -e(-1/2p)v \), \((v \in \mathbb{R}_+)\), we have
\[
\psi_{E_2} = (-1)^{\alpha} \frac{2i(4v^2)^{\alpha} dv}{v^2 + 1}.
\]
Similarly on \( X_{3m} \) and \( X_{4m} \), using parameters \( x - 1 = uy \), \( u = -e(-1/2p)v \) \((v \in \mathbb{R}_+)\), we have
\[
\psi_{E_2} = \begin{cases} 
(-1)^{2\alpha} \omega^{-1} \frac{3(27v^3)^{\alpha} dv}{v^3 + 1}, & \text{on } X_{3m} \\
(-1)^{3\alpha} (4v)^{4\alpha} \frac{(4 - 4i)}{4v^4 + 1} dv, & \text{on } X_{4m}.
\end{cases}
\]
Therefore, we have
\[
\int_0^\infty \psi_{E_2} = \begin{cases} 
(-1)^{\alpha} 4 \alpha \frac{\pi i}{\cos(\pi \alpha)}, & \text{on } X_{2m} \\
(-1)^{2\alpha} \omega \frac{\pi i}{\sin(\pi \alpha + \pi/3)}, & \text{on } X_{3m} \\
(-1)^{3\alpha} 4 \alpha \frac{(1/2 - 1/2i) \sqrt{2\pi}}{\sin(\pi \alpha + \pi/4)}, & \text{on } X_{3m}.
\end{cases}
\]

4.2. Hypergeometric identities for \( \Gamma_2 \).

4.2.1. Hypergeometric identities for \( X_{2m} \). By Stokes’s theorem on \( X_{2m} \), we have the following equality:
\[
\alpha \int_{\Gamma_2} (\xi + \eta - 1)^{2\alpha - 1} \xi^{-\alpha - 1} \eta^{\alpha - 1/2} d\xi \wedge d\eta
\]
\[
= (-1)^{\alpha - 1/2} \alpha \int_{\Gamma_1} \xi^{2\alpha - 1} \eta^{\alpha - 1} (\xi + \eta - 1)^{-\alpha - 1/2} d\xi \wedge d\eta
\]
\[
= \int_0^1 (-1)^{\alpha} \frac{-2i x^{-4\alpha} dx}{2x^2 - 2x + 1} - 2 \int_0^{i\infty} \frac{(4u^2)^{\alpha} du}{1 - u^2}
\]
\[
= \int_0^1 (-1)^{\alpha} \frac{-2i x^{-4\alpha} dx}{2x^2 - 2x + 1} - (-1)^{\alpha} 4 \alpha \frac{\pi i}{\cos(\pi \alpha)}
\]
By analytic continuation, we have the following theorem.

**Theorem 4.1.** Let \( 2\alpha \notin \mathbb{Z} \). Then we have the following equalities.

(1)
\[
\frac{\alpha}{(\alpha + 1/2)(-2\alpha + 1/2)} F(1, 1, 1/2; \alpha + 3/2, -2\alpha + 3/2; 1)
= \int_{\Gamma_1} \frac{2x^{-4\alpha} dx}{2x^2 - 2x + 1} + \frac{4 \alpha \pi}{\cos(\pi \alpha)}
\]

(2)
\[
\frac{\alpha - 1/2}{(-2\alpha + 3/2)\alpha} F(1, 1, 1/2; \alpha + 1, -2\alpha + 5/2; 1) = \int_{\Gamma_1} \frac{2x^{-4\alpha+2} dx}{2x^2 - 2x + 1} + \frac{2^{2\alpha-1} \pi}{\cos(\pi \alpha - \pi/2)}
\]
By analytic continuation, we have the following theorem.
4.2.2. Hypergeometric identities for $X_{3m}, X_{4m}$. Using Stokes’ theorem, the integral is computed using the boundary integral on $\Gamma_2$. By Proposition 1.3, we have the following theorem.

**Theorem 4.2.** (1) Let $\alpha$ be a real number such that $-1/3 < \alpha < 1/2$.

\[
\frac{3\alpha}{(2\alpha + 2/3)(-2\alpha + 1)} F(1, 1, \alpha + 1; 2\alpha + 5/3, -2\alpha + 2; 1)
\]

\[
= 3\alpha \int_{\Gamma_1} \xi^{3\alpha - 1} \eta^{-\alpha - 2/3} (\xi + \eta - 1)^{-\alpha - 1/3} d\xi \wedge d\eta
\]

\[
= \int_{P(0, 1)} \frac{27x^{-6\alpha + 2}(3x^3 + 1)}{1 + 27x^6} dx - \int_{P(0, 1)} \frac{9x^{6\alpha + 1}(3 - x^3)}{27 + x^6} dx - \frac{3\alpha \pi}{\sin(\pi\alpha + \pi/3)}.
\]

Moreover the first line is equal to the third line if $3\alpha \notin \mathbb{Z}$.

(2) Let $\alpha$ be a real number such that $-1/4 < \alpha < 1/2$.

\[
\frac{-\alpha}{(3\alpha + 3/4)(-2\alpha + 1)} F(1, 1; 2\alpha + 1; 3\alpha + 7/4, -2\alpha + 2; 1)
\]

\[
= -\alpha \int_{\Gamma_1} (\xi + \eta - 1)^{-\alpha - 1/4} \xi^{4\alpha - 1} \eta^{-\alpha - 3/4} d\xi \wedge d\eta
\]

\[
= - \int_{P(0, 1)} x^{-8\alpha} \frac{8(1 + 8x^4)x^3}{64x^8 + 1} dx - \int_{P(0, 1)} x^{12\alpha} \frac{2x^2(-8 - 4x^4 + x^8)}{64 + x^{12}} dx
\]

\[
+ 4^{3\alpha - 1} \frac{\pi}{\sqrt{2}\sin(\pi\alpha + \pi/4)}.
\]

Moreover the first line is equal to the third line if $4\alpha \notin \mathbb{Z}$.

**Proof.** (1) We apply Skokes’ formula for $X_{3m}$ and we have

\[
- 3\alpha \int_{\Gamma_2} (\xi + \eta - 1)^{3\alpha - 1} \xi^{-\alpha - 2/3} \eta^{-\alpha - 1/3} d\xi \wedge d\eta
\]

\[
= -3(-1)^{2\alpha + 2/3} \alpha \int_{\Gamma_1} \xi^{3\alpha - 1} \eta^{-\alpha - 2/3} (\xi + \eta - 1)^{-\alpha - 1/3} d\xi \wedge d\eta
\]

\[
= -(-1)^{2\alpha} \omega \int_0^1 \frac{27x^{-6\alpha + 2}(3x^3 + 1)}{1 + 27x^6} dx + (-1)^{2\alpha} \omega \int_0^1 x^{6\alpha} \frac{9x(3 - x^3)}{27 + x^6} dx
\]

\[
- (-1)^{2\alpha} \omega \frac{3\alpha \pi}{\sin(\pi\alpha + \pi/3)}.
\]

(2) is similar by applying Stokes’ formula to $X_{4m}$.

\[ \square \]

5. Hypergeometric identities for $\Gamma_3$

5.1. **Restriction of $\psi$ to the boundary for $\Gamma_3$.** In this section, we give a hypergeometric identities arising from the integration on $\Gamma_3$ for $p = 3, 4$.

5.1.1. **Restriction to the boundary $B_y$.** On $X_{pm}$, we have

\[
\psi_{B_y} = \begin{cases} 
(-1)^{2\alpha} \frac{27\omega^2 y^{-6\alpha + 2}(-3y^3 + 1)}{1 + 27y^6} dy & p = 3 \\
(-1)^{3\alpha} \frac{y^{-8\alpha}(32 + 32\xi)(8y^4 - 1)y^3}{64y^8 + 1} dy & p = 4
\end{cases}
\]
5.1.2. Restriction to the boundary $B_3$. We restrict $\psi$ to $B_3$ and change parameters of integrals on $X_{pm}$,

$$\psi_{B_3} = \begin{cases} \frac{-(-1)^2 y^6 (9-y^3)dy}{27+y^6} & p = 3 \\ \frac{(-1)^3 y^{12} (8+8i)(8y^4-32+y^8)dy}{64+y^{12}} & p = 4 \end{cases}$$

5.1.3. Contribution from an exceptional divisor. By setting $y - 1 = ux, u = -e(-1/2p)v$ on $X_{pm}, (v \in \mathbb{R}_+)$, the restriction of $\psi$ to the exceptional divisor $E_3 \subset \Gamma_3$ is equal to

$$\psi_{E_3} = \begin{cases} \frac{-(-1)^2 \omega^2 3(27v^3)^\alpha v dv}{v^3 + 1} & \text{on } X_{3m} \\ \frac{(-1)^3 (4v)^4 (8-8i)v^2}{4v^4+1} dv & \text{on } X_{4m} \end{cases}$$

Then the integrals are equal to the following.

$$\int_0^\infty \psi_{E_3} = \begin{cases} \frac{3^{3\alpha} \pi}{\sin(\pi \alpha + 2\pi/3)} & \text{on } X_{3m} \\ \frac{-(-1)^3 \alpha^4 3^{3\alpha} \pi}{\cos(\pi \alpha + \pi/4)} & \text{on } X_{4m} \end{cases}$$

5.2. Hypergeometric identities for $\Gamma_3$. We similarly compute the integral on $\Gamma_3$ and get the following theorem.

**Theorem 5.1.**

1. Let $\alpha$ be a real number such that $-1/6 < \alpha < 1/3$.

$$\frac{3\alpha}{(-2\alpha + 1)(2\alpha + 1/3)} F(1,1,\alpha+1;-2\alpha + 2, 2\alpha + 4/3;1)$$

$$= 3\alpha \int_{\Gamma_1} \eta^{3\alpha-1}(\xi + \eta - 1)^{-\alpha-2/3} \xi^{\alpha-1/3} d\xi \wedge d\eta$$

$$= - \int_{P(0,1)} \frac{27y^{-6\alpha+2}(-3y^3 + 1)}{1+27y^6} dy - \int_{P(0,1)} \frac{9y^{6\alpha}(9-y^3)}{27+y^6} dy + \frac{3^{3\alpha} \pi}{\sin(\pi \alpha + 2\pi/3)}$$

Moreover the first line is equal to the third line if $3\alpha \notin \mathbb{Z}$.

2. Let $\alpha$ be a real number such that $-1/12 < \alpha < 1/4$.

$$- \frac{\alpha}{(-2\alpha + 1)(3\alpha + 4/3)} F(1,1,2\alpha + 1;-2\alpha + 2, 3\alpha + 5/4;1)$$

$$= - \alpha \int_{\Gamma_1} (\xi + \eta - 1)^{-\alpha-3/4} \xi^{-\alpha-1/4} \eta^{4\alpha-1} d\xi \wedge d\eta$$

$$= \int_{P(0,1)} \frac{y^{-8\alpha} 8(1-8y^4)y^3}{64y^6+1} dy - \int_{P(0,1)} \frac{y^{12\alpha} 2(8y^4-32+y^8)}{64+y^{12}} dy$$

$$- \frac{4^{3\alpha-1}}{\sqrt{2} \cos(\pi \alpha + \pi/4)}$$

Moreover the first line is equal to the third line if $4\alpha \notin \mathbb{Z}$.
APPENDIX A. LIST OF ORBITS FOR EXCEPTIONAL CHARACTERS

In this section, we give a list of orbits of \((\alpha_0, \alpha_1, \alpha_2, \alpha_3)\) under the action of multiplicative group \((\mathbb{Z}/m\mathbb{Z})^\times\) satisfying the condition \((1, 1, 2)\). The number of exceptional characters are written as \(\epsilon_m\), the number of orbits are written as \(\omega_m\).

\[
m = 12, \quad \epsilon_m = 8, \quad \omega_m = 2 \\
\{ (1, 4, 9, 10), (1, 6, 8, 9) \}
\]

\[
m = 14, \quad \epsilon_m = 2, \quad \omega_m = 1 \\
\{ (1, 7, 9, 11) \}
\]

\[
m = 15, \quad \epsilon_m = 8, \quad \omega_m = 1 \\
\{ (1, 6, 10, 13) \}
\]

\[
m = 18, \quad \epsilon_m = 18, \quad \omega_m = 3 \\
\{ (1, 6, 14, 15), (1, 7, 12, 16), (1, 9, 12, 14) \}
\]

\[
m = 20, \quad \epsilon_m = 26, \quad \omega_m = 4 \\
\{ (1, 4, 17, 18), (1, 6, 16, 17), (1, 9, 13, 17), (1, 10, 12, 17) \}
\]

\[
m = 21, \quad \epsilon_m = 12, \quad \omega_m = 1 \\
\{ (1, 4, 18, 19) \}
\]

\[
m = 24, \quad \epsilon_m = 38, \quad \omega_m = 8 \\
\{ (1, 6, 19, 22), (1, 8, 17, 22), (1, 8, 19, 20), (1, 11, 17, 19), (1, 12, 16, 19), (1, 12, 17, 18), (1, 13, 16, 18), (2, 9, 16, 21) \}
\]

\[
m = 28, \quad \epsilon_m = 10, \quad \omega_m = 2 \\
\{ (1, 9, 21, 25), (1, 15, 18, 22) \}
\]

\[
m = 30, \quad \epsilon_m = 98, \quad \omega_m = 15 \\
\{ (1, 4, 27, 28), (1, 7, 25, 27), (1, 8, 25, 26), (1, 10, 24, 25), (1, 11, 24, 24), (1, 12, 20, 27), (1, 12, 23, 24), (1, 15, 17, 27), (1, 15, 19, 25), (1, 15, 20, 24), (1, 16, 21, 22), (1, 17, 19, 23), (1, 18, 20, 21), (1, 19, 20, 20), (2, 15, 21, 22) \}
\]

\[
m = 36, \quad \epsilon_m = 18, \quad \omega_m = 2 \\
\{ (1, 19, 24, 28), (2, 9, 28, 33) \}
\]

\[
m = 40, \quad \epsilon_m = 16, \quad \omega_m = 2 \\
\{ (1, 21, 24, 34), (1, 21, 26, 32) \}
\]

\[
m = 42, \quad \epsilon_m = 166, \quad \omega_m = 16 \\
\{ (1, 6, 37, 40), (1, 6, 38, 39), (1, 8, 37, 38), (1, 12, 33, 38), (1, 13, 32, 38), (1, 15, 31, 37), (1, 15, 32, 36), (1, 16, 30, 37), (1, 16, 33, 34), (1, 18, 32, 33), (1, 19, 24, 40), (1, 21, 24, 38), (1, 21, 25, 37), (1, 21, 29, 33), (1, 21, 30, 32), (1, 24, 29, 30) \}
\]

\[
m = 48, \quad \epsilon_m = 16, \quad \omega_m = 2 \\
\{ (1, 25, 32, 38), (1, 25, 34, 36) \}
\]

\[
m = 60, \quad \epsilon_m = 204, \quad \omega_m = 23 \\
\{ (1, 12, 49, 58), (1, 15, 49, 55), (1, 17, 49, 53), (1, 20, 41, 58), (1, 20, 49, 50), (1, 23, 47, 49), (1, 24, 41, 54), (1, 24, 46, 49), (1, 25, 45, 49), (1, 27, 41, 51), ...
\]
\( (1, 29, 41, 49), (1, 30, 40, 49), (1, 30, 41, 48), (1, 31, 34, 54), (1, 31, 38, 50), \)
\( (1, 31, 40, 48), (1, 31, 42, 46), (1, 36, 41, 42), (2, 15, 48, 55), (2, 21, 40, 57), \)
\( (2, 21, 46, 51), (2, 25, 38, 55), (3, 32, 33, 52) \}

\( m = 66, \ e_m = 30, \ o_m = 2 \)

\( \{ (1, 25, 44, 62), (2, 39, 45, 46) \} \)

\( m = 72, \ e_m = 12, \ o_m = 1 \)

\( \{ (3, 16, 57, 68) \} \)

\( m = 78, \ e_m = 32, \ o_m = 2 \)

\( \{ (1, 32, 61, 62), (1, 39, 55, 61) \} \)

\( m = 84, \ e_m = 66, \ o_m = 6 \)

\( \{ (1, 29, 63, 75), (1, 43, 48, 76), (1, 43, 50, 74), (1, 43, 58, 66), (1, 43, 60, 64), (2, 33, 58, 75) \} \)

\( m = 90, \ e_m = 24, \ o_m = 1 \)

\( \{ (3, 20, 72, 85) \} \)

\( m = 120, \ e_m = 72, \ o_m = 5 \)

\( \{ (1, 49, 83, 107), (1, 61, 80, 98), (1, 61, 82, 96), (2, 25, 98, 115), (4, 25, 96, 115) \} \)

\( m = 156, \ e_m = 24, \ o_m = 1 \)

\( \{ (1, 79, 110, 122) \} \)

\( m = 180, \ e_m = 24, \ o_m = 1 \)

\( \{ (3, 40, 147, 170) \} \)

References

[A] Aoki, N., On some arithmetic problems related to the Hodge cycles on the Fermat varieties. Math. Ann. 266 (1983), no. 1, 23–54.

[AOT] Asakura, M., Otsubo, N., Terasoma, T. An algebro-geometric study of the unit arguments \( 3 \mathcal{F}_2(1, 1, q; a, b; 1) \) I, preprint. [arXiv:1603.04558]

[AS] Aoki, N., Shioda, T., Generators of the Neron-Severi group of a Fermat surface. Arithmetic and geometry, Vol. I, 1–12, Progr. Math., 35, Birkhauser Boston, Boston, MA, 1983.

[AY] Asakura, M., Yabu, To appear.

[S] Shioda, T., On the Picard number of a Fermat surface. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1981), no. 3, 725–734

[D] Deligne, P., Theorie de Hodge II, Publications Mathematiques de l’IHES (1971) Volume: 40, page 5-57

[E] A. Erdélyi (Editor), Higher Transcendental Functions, vol. I, McGraw-Hill, New York, 1953.