Conflict-Free Coloring on Open Neighborhoods

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Abstract. In an undirected graph, a conflict-free coloring (with respect to open neighborhoods) is an assignment of colors to the vertices of the graph $G$ such that every vertex in $G$ has a uniquely colored vertex in its open neighborhood. The conflict-free coloring problem asks to find the smallest number of colors required for a conflict-free coloring.

The conflict-free coloring problem is NP-complete. From results in Abel et. al. [SODA 2017], it can be inferred that every planar graph has a conflict-free coloring with at most nine colors. As the best known lower bound for planar graphs is four colors, it was asked in the same paper if fewer colors would suffice. We make progress in answering this question, by showing that every planar graph can be colored using at most six colors. The same proof idea is used to show that every outerplanar graph can be colored using at most five colors. Using a different approach, we further show that every outerplanar graph can be colored using at most four colors.

Finally, we study the problem on Kneser graphs. We show that $k + 2$ colors are necessary and sufficient to color the Kneser graph $K(n, k)$ when $n \geq k(k+1)^2 + 1$.

1 Introduction

A proper coloring of a graph is an assignment of a color to every vertex of the graph such that adjacent vertices are of distinct colors. Conflict-free coloring is a variant of the graph coloring problem. A conflict-free coloring of a graph $G$ is a coloring such that for every vertex in $G$, there exists a uniquely colored vertex in its neighborhood. This problem was first introduced in 2002 by Even, Lotker, Ron and Smorodinsky [1]. This problem was originally motivated by wireless communication systems consisting of base stations and clients. The clients and base stations have to send each other information and hence they communicate with each other. Each base station is assigned a frequency and if two base stations with the same frequency try to communicate with a client, it leads to interference. So for each client, there has to be a base station with a unique frequency. Since each frequency band is expensive, there is a need to minimize the number of frequencies used by the base stations. Over the past two decades, this problem has been very well studied, see for instance the survey by Smorodinsky [2].

The conflict-free coloring problem has been studied with respect to open neighborhoods as well as closed neighborhoods. In this paper, we study the conflict-free coloring problem with respect to open neighborhoods.
Definition 1 (Conflict-Free Coloring). A complete conflict-free (CF) coloring of a graph $G = (V,E)$ using $k$ colors is an assignment $C : V(G) \rightarrow \{1,2,\ldots,k\}$ such that for every $v \in V(G)$, there exists an $i \in \{1,2,\ldots,k\}$ such that $|N(v) \cap C^{-1}(i)| = 1$. The smallest number of colors required for a complete conflict-free coloring of $G$ is called the conflict-free chromatic number of $G$, denoted by $\chi_{CF}(G)$.

The conflict-free coloring problem and many of its variants are known to be NP-complete [3,4]. It was further shown in [4] that the CF coloring problem on open neighborhoods is hard to approximate within a factor of $n^{1/2-\varepsilon}$, unless P = NP. Since the problem is NP-hard, the parameterized aspects of the problem have been studied. The problems are fixed parameter tractable when parameterized by vertex cover number, neighborhood diversity [4], distance to cluster, distance to threshold graphs [5], and more recently, tree-width [6,7].

In this paper, we look at the conflict-free open neighborhood problem, which is considered as the harder of the open and closed neighborhood variants, see for instance, remarks in [8,9]. It is easy to construct examples of bipartite graphs $G$, for which $\chi_{CF}(G)$ is $\Theta(\sqrt{n})$. Since any proper coloring is also a valid conflict-free closed neighborhood coloring, these examples have a CF closed neighborhood coloring using two colors. Further, Cheilaris [10] showed that for every graph $G$, we have $\chi_{CF}(G) \leq 2\sqrt{n}$. On the contrary, a graph with maximum degree $\Delta$ has a conflict-free closed neighborhood coloring with at most $O(\log^{2+\varepsilon} \Delta)$ colors [8].

Restrictions of the conflict-free coloring problem to special classes of graphs have been studied extensively. Of these, graphs arising out of intersection of geometric objects have attracted special interest, see for instance, [9,11,12]. The problem has also been studied for structural classes of graphs such as bipartite graphs and split graphs [5].

In [3], Abel et. al. considered the partial coloring variant of the problem where not all vertices need to be assigned a color.

Definition 2 (Partial Conflict-Free Coloring). A partial conflict-free (CF) coloring of a graph $G = (V,E)$ using $k$ colors is an assignment $C : V(G) \rightarrow \{0,1,2,\ldots,k\}$ such that for every $v \in V(G)$, there exists an $i \in \{1,2,\ldots,k\}$ such that $|N(v) \cap C^{-1}(i)| = 1$.

Let us refer to the variant of the problem that we originally stated in Definition 1 where all the vertices have to be colored, as the complete coloring variant. The key difference between partial CF coloring and complete CF coloring is that in the partial variant, we allow some vertices to be assigned the color 0. It is convenient to think of the vertices 0 as uncolored vertices. However, the uniquely colored neighbor is not allowed to be of color 0. If a graph can be colored using $k$ colors in the partial coloring variant, then all the uncolored vertices can be assigned the color $k+1$, and thus a $k+1$ complete coloring for the same graph can be obtained.

For the partial coloring variant, eight colors suffice to color a planar graph [3]. It is easy to construct a planar graph that requires four colors. Starting with $K_4$, each original edge is subdivided by introducing a degree-two vertex on this
In addition, a pendant vertex is attached to every original vertex of the $K_4$. This graph (see Figure 1) is planar and requires four colors. One of the open questions asked in [3] was to close the gap between the upper bound of eight and lower bound of four for the partial coloring variant of the conflict-free chromatic number of a planar graph. In this paper we reduce this gap, by showing that five colors suffice for the partial CF coloring of a planar graph. Using the same proof idea, we show that four colors are enough for the partial coloring of an outerplanar graph. The lower bound for the partial CF coloring of an outerplanar graph is three, see Figure 2. As noted before, a partial coloring of an outerplanar graph using four colors implies a complete coloring using five colors. Using a different approach, we show that four colors are sufficient for a complete CF coloring of an outerplanar graph.

The last section in this paper studies the conflict-free coloring on Kneser graphs. The Kneser graph $K(n, k)$ is the graph whose vertices are $k$-subsets of $[n]$, and two such vertices are adjacent if and only if the corresponding sets are disjoint. Several properties of Kneser graphs have been subject to study. The chromatic number of the Kneser graph $K(n, k)$ was conjectured by Kneser [13] in 1955 to be $n - 2k + 2$. This remained open till Lovász proved [14] the conjecture in 1978. When $n \geq k(k+1)^2 + 1$, we determine the exact conflict-free chromatic number of the Kneser graph $K(n, k)$.

We summarize our results in this paper below:

1. Five colors are sufficient for the partial conflict-free coloring of a planar graph. This improves the previous best known bound of [3] that required eight colors.

2. Four colors suffice for the complete conflict-free coloring of an outerplanar graph. Moreover, three colors are sufficient and sometimes necessary for a complete conflict-free coloring of cactus graphs. These results are shown in Section 4.

3. In Section 5 we compute bounds on the conflict-free coloring of Kneser graphs. We also determine that the $\chi_{CF}(K(n, k)) = k + 2$ when $n \geq k(k + 1)^2 + 1$. 

Fig. 1. Graph that requires 4 colors for a partial CF coloring.

Fig. 2. Graph that requires 3 colors for a partial CF coloring.
2 Preliminaries

For any two vertices \( u, v \in G \), we denote the shortest distance between them in \( G \) by \( \text{dist}(u, v) \). The open neighborhood of \( v \), denoted by \( N(v) \), is the set of vertices adjacent to \( v \). We denote the graph induced by a set of vertices \( V' \) in \( G \) as \( G[V'] \).

In this paper, we consider only connected graphs with at least two vertices because the colorings of the connected components can combine to give a coloring of the graph. Also, an isolated vertex does not have a conflict-free coloring (in the open neighborhood setting) since there are no neighbors.

A planar graph is a graph that can be drawn in \( \mathbb{R}^2 \) (a plane) such that the edges do not cross each other in the drawing. Each such drawing divides the plane into regions and each region is called a face. A planar drawing of a graph has one face that is unbounded. This face is called the outer face. All the other faces are referred to as inner faces. An outerplanar graph is a planar graph that has a drawing in a plane such that all the vertices of the graph belong to the outer face. Throughout the paper, we use terminology from the textbook “Graph Theory” by Diestel [15].

3 Partial CF Coloring of Planar Graphs

In [3] Abel et. al., showed that eight colors are sufficient for the partial CF coloring of a planar graph. In this section, we improve the bound to five colors.

We need the following definition:

**Definition 3 (Maximal Distance-3 Set).** For a graph \( G = (V, E) \), a maximal distance-3 set is a set \( S \subseteq V(G) \) that satisfies the following:

1. For every pair of vertices \( w, w' \in S \), we have \( \text{dist}(w, w') \geq 3 \).
2. For every vertex \( w \in S \), there exists a vertex \( w' \in S \) such that \( \text{dist}(w, w') = 3 \).
3. For every vertex \( x \notin S \), there exists a vertex \( x' \in S \) such that \( \text{dist}(x, x') < 3 \).

The set \( S \) is constructed by initializing \( S = \{v\} \) where \( v \) is an arbitrary vertex. We proceed in iterations. In each iteration, we add a vertex \( w \) to \( S \) if (1) for every \( v \) already in \( S \), \( \text{dist}(v, w) \geq 3 \), and (2) there exists a vertex \( w' \in S \) such that \( \text{dist}(w, w') = 3 \). We repeat this until no more vertices can be added.

The main component of the proof is the construction of an auxiliary graph \( G' \) from the given graph \( G \).

**Construction of \( G' \):** The first step is to pick a maximal distance-3 set \( V_0 \). Notice that any distance-3 set is an independent set by definition. We let \( V_1 \) denote the neighborhood of \( V_0 \). More formally, \( V_1 = \{w \mid \{w, w'\} \in E(G), w' \in V_0\} \). Let \( V_2 \) denote the remaining vertices i.e., \( V_2 = V \setminus (V_0 \cup V_1) \).

We note the following properties satisfied by the above partitioning of \( V(G) \).

1. The set \( V_0 \) is an independent set.
2. For every vertex \( w \in V_1 \), there exists a unique vertex \( w' \in V_0 \) such that \( \{w, w'\} \in E(G) \). This is because if there are two such vertices, this will violate the distance-3 property of \( V_0 \).

3. Every vertex in \( V_0 \) has a neighbor in \( V_1 \). If there exists \( v \in V_0 \) without a neighbor in \( V_1 \), then \( v \) is an isolated vertex. By assumption, \( G \) does not have isolated vertices.

4. There are no edges from \( V_0 \) to \( V_2 \).

5. Every vertex in \( V_2 \) has a neighbor in \( V_1 \), and is hence at distance 2 from some vertex in \( V_0 \). This is due to the maximality of the distance-3 set \( V_0 \).

Now we define \( A = V_0 \cup V_2 \). We first remove all the edges of \( G[V_2] \) making \( A \) an independent set. We then contract every vertex \( v \in A \) to a neighbor \( f(v) \in N(v) \subseteq V_1 \). The contraction process is as follows: we first identify vertex \( v \) with \( f(v) \). Then for every edge \( \{v, v'\} \), we add an edge \( \{f(v), v'\} \). The resulting graph is \( G' \).

**Theorem 4.** Five colors are sufficient to partial CF color a planar graph.

**Proof.** Let \( G \) be a planar graph. We first construct the graph \( G' \) as above. Since the steps for constructing \( G' \) involve only edge deletion, and edge contraction, \( G' \) is also a planar graph. By the planar four-color theorem [16], every planar graph has a proper four coloring. That is, there is an assignment \( C : V(G') \rightarrow \{2, 3, 4, 5\} \) such that no two adjacent vertices of \( G' \) are assigned the same color.

Notice that we have colored all the vertices of \( G' \), that is the entire set \( V_1 \).

Now, we extend \( C \) to get a CF coloring for \( G \). For all vertices \( v \in V_0 \), we assign \( C(v) = 1 \). The vertices in \( V_2 \) are assigned the color 0.

We will show that \( C \) is indeed a partial CF coloring of \( G \). Consider a vertex \( v \in A \) which is contracted to a neighbor \( f(v) = w \in V_1 \). The color assigned to \( w \) is distinct from all \( w \)'s neighbors in \( G' \). Hence the color assigned to \( w \) is the unique color among the neighbors of \( v \) in \( G \).

For each vertex \( w \in V_1 \), \( w \) is a neighbor of exactly one vertex \( v \in V_0 \). Every vertex \( v \in V_0 \) is colored 1, which is different from all the colors assigned to the neighbors of \( w \) in \( G' \). \( \square \)

**Algorithmic Note:** The steps in the proof of Theorem 4 lead to an algorithm. The steps involved are construction of maximal distance-3 set, contraction of vertices in \( A \) and the planar 4 coloring [16]. All these steps can be performed in \( O(|V(G)|^2) \) time. Thus we have an \( O(|V|^2) \) time algorithm, that given a planar graph \( G \), determines a partial CF coloring for \( G \) that uses five colors.

Outerplanar graphs have a proper coloring using three colors. By argument analogous to Theorem 4, we infer the following.

**Corollary 5.** Four colors are sufficient to partial CF color an outerplanar graph.

The famous Hadwiger’s conjecture states that if a graph \( G \) does not contain \( K_{k+1} \) as a minor, then \( G \) is \( k \)-colorable (in the sense of proper coloring). By an analogous argument again, we obtain the following:

**Corollary 6.** Suppose the Hadwiger’s conjecture is true and that \( G \) has no \( K_{k+1} \) minor. Then \( G \) admits a partial CF coloring using \( k + 1 \) colors.
4 Complete CF Coloring of Outerplanar Graphs

We saw in Corollary 5 that outerplanar graphs can be partially CF colored using 4 colors. This implies a complete CF coloring using 5 colors. In this section, we show an improved bound.

Theorem 7. Four colors are sufficient for a complete CF coloring of an outerplanar graph $G$.

Note that whenever we refer to an outerplanar graph $G$, we will also be implicitly referring to a planar drawing of $G$ with all the vertices appearing in the outer face. We will abuse language and say “faces of $G$” when we want to refer to faces of the above planar drawing.

Theorem 7 is proved using a two-level induction process. The first level is using a block decomposition of the graph. Any connected graph can be viewed as a tree of its constituent blocks. We color the blocks in order so that when we color a block, at most one of its vertices is previously colored. Each block is colored without affecting the color of the already colored vertex. The second level of the induction is required for coloring each of the blocks. We use ear decomposition on each block and color the faces of the block in sequence. However, the proof is quite technical and involves several cases of analysis at each step.

We summarize the relevant aspects of block decomposition below. The reader is referred to a standard textbook in graph theory [15] for more details on this.

- A block is a maximal connected subgraph without a cut vertex.
- Blocks of a connected graph are either maximal 2-connected subgraphs, or edges (the edges which form a block will be bridges).
- Two distinct blocks overlap in at most one vertex, which is a cut vertex.
- Any connected graph can be viewed as tree of its constituent blocks.

In the following discussion, we explain how to construct a coloring $C : V(G) \rightarrow \{1, 2, 3, 4\}$ for an outerplanar graph $G$. At any intermediate stage, the coloring $C$ will satisfy the following invariants:

**Invariants of $C$**

- Every vertex $v$ that has already been assigned a color $C(v)$ has a neighbor $w$, such that $C(w) \neq C(x)$, where $x \in N(v) \setminus \{w\}$. For $v$, the function $U : V(G) \rightarrow \{1, 2, 3, 4\}$ denotes the color of $w$, its uniquely colored neighbor.
- $\forall v \in V(G), C(v) \neq U(v)$.
- $\forall \{v, w\} \in E(G), C(v) \neq C(w)$ and $|\{C(v), U(v), C(w), U(w)\}| = 3. (\star)$

Theorem 7 is proved by using an induction on the block decomposition of the graph $G$ and the below results.

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1 The condition marked $\star$ is violated in a few cases. In the exceptional cases where it is violated, we shall explain how the cases are handled.
Lemma 8. If $G$ is a 2-connected outerplanar graph such that all its inner faces contain exactly 5 vertices, then $G$ has a complete CF coloring using 3 colors.

Theorem 9. Let $G$ be an outerplanar graph.

1. If $B$ is a block of $G$ that is either a bridge, or contains an inner face $F$ with $|V(F)| \neq 5$, then $B$ has a complete CF coloring using at most 4 colors.
2. If $B$ is a block of $G$, with exactly one vertex $v$ precolored with color $C(v)$ and unique color $U(v)$, then the rest of $B$ has a complete CF coloring using at most 4 colors, while retaining $C(v)$ and $U(v)$.

Proof (Proof of Theorem 9). Let $G$ be an outerplanar graph. We apply block decomposition on $G$ which results in blocks that are either maximal 2-connected subgraphs or single edges.

If $G$ is 2-connected and all its inner faces have exactly 5 vertices, then by Lemma 8, $G$ has a complete CF coloring using 3 colors.

If $G$ does not fit the above description, then $G$ has a block $B$ such that either $B$ is an edge, or $B$ has an inner face $F$ with $|V(F)| \neq 5$. In this case, by Theorem 9.1, $B$ has a complete CF coloring using at most 4 colors.

Viewing $G$ as a tree of its blocks, we can start coloring blocks that are adjacent to blocks that are already colored. Suppose the block $B$ is already colored, and let $B'$ be a block adjacent to $B$. Let $x$ be the cut-vertex between the blocks $B$ and $B'$. We use Theorem 9.2 to obtain a complete CF coloring of $B'$ using at most 4 colors, while retaining $C(x)$ and $U(x)$.

We now proceed towards proving Lemma 8 and Theorem 9. Lemma 8 and Theorem 9 discusses the coloring of blocks, which is accomplished by means of induction on the faces of the blocks. Towards this end, we use the following fact about ear decomposition of 2-connected outerplanar graphs. For a proof of the below lemma, we refer the reader to [17] where this is stated as Observation 2.

Lemma 10 (Ear Decomposition). Let $B$ be a 2-connected block in an outerplanar graph. Then $B$ has an ear decomposition $F_0, P_1, P_2, \ldots, P_q$ satisfying the following:

- $F_0$ is an arbitrarily chosen inner face of $B$.
- Every $P_i$ is a path with end points $v, w$ such that $\{v, w\}$ is an edge in $F_0 \cup \bigcup_{1 \leq j < i} P_j$. Thus $P_i$ together with the edge $\{v, w\}$ forms a face of $B$.

We first prove Lemma 8

Proof. (Proof of Lemma 8) Since $G$ is 2-connected, the entire graph forms a single block. Let $F_0, P_1, \ldots, P_q$ be an ear decomposition of $G$. Recall that all the faces have exactly five vertices. Let $F_0 = v_1 - v_2 - v_3 - v_4 - v_5 - v_1$. We assign the following colors to the vertices in $F_0$: $C(v_1) = 1, C(v_2) = 1, C(v_3) = 2$.

The coloring assigned in this proof does not satisfy the condition marked $\star$. However, this is not an issue since we are coloring the whole of $G$ in this lemma.
2, \( C(v_4) = 2 \), \( C(v_5) = 3 \). We also have \( U(v_1) = 3, U(v_2) = 2, U(v_3) = 1, U(v_4) = 3, U(v_5) = 1 \).

Let \( P_i \) be any subsequent face \( P_i = w_1 - w_2 - w_3 - w_4 - w_5 - w_1 \) with \( \{w_1, w_2\} \) being the pre-existing edge in \( F \cap \bigcup_{1 \leq j < i} P_j \). Depending on the values already assigned to \( C(w_1), U(w_1), C(w_2), U(w_2) \), we assign the colors to \( w_3, w_4 \) and \( w_5 \). We always ensure that \( C(v) \neq U(v) \) for all vertices \( v \). We note that the values \( C(w_1), U(w_1), C(w_2), U(w_2) \) can take only the four below combinations, w.l.o.g.

We explain the coloring for the rest of \( P_i \) in each of these cases.

1. \( C(w_1) = C(w_2) \) and \( |\{C(w_1), U(w_1), U(w_2)\}| = 3 \). W.l.o.g., let \( C(w_1) = 1, U(w_1) = 2, C(w_2) = 1, U(w_2) = 3 \). Assign \( C(w_3) = 2, C(w_4) = 2, C(w_5) = 3 \) and \( U(w_3) = 1, U(w_4) = 3, U(w_5) = 1 \).

2. \( C(w_1) \neq C(w_2), U(w_1) \neq U(w_2) \), and \( |\{C(w_1), C(w_2), U(w_1), U(w_2)\}| = 3 \). Either \( w_1 \) serves as the uniquely colored neighbor of \( w_2 \) or vice versa. W.l.o.g., let \( C(w_1) = 1, U(w_1) = 2, C(w_2) = 2, U(w_2) = 3 \). Assign \( C(w_3) = 1, C(w_4) = 3, C(w_5) = 3 \) and \( U(w_3) = 2, U(w_4) = 1, U(w_5) = 1 \).

3. \( C(w_1) = U(w_2) \) and \( C(w_2) = U(w_1) \). W.l.o.g., let \( C(w_1) = 1, U(w_1) = 2, C(w_2) = 2, U(w_2) = 1 \). Assign \( C(w_3) = 2, C(w_4) = 3, C(w_5) = 1 \) and \( U(w_3) = 3, U(w_4) = 2, U(w_5) = 3 \).

4. \( C(w_1) = C(w_2) \) and \( U(w_1) = U(w_2) \). W.l.o.g., let \( C(w_1) = C(w_2) = 1, U(w_1) = U(w_2) = 2 \). Assign \( C(w_3) = 1, C(w_4) = 2, C(w_5) = 3 \) and \( U(w_3) = 2, U(w_4) = 3, U(w_5) = 1 \).

5. The case \( U(w_1) = U(w_2) \) and \( |\{U(w_1), C(w_1), C(w_2)\}| = 3 \) does not arise in the above colorings.

\[ \Box \]

At this point, to complete the proof of Theorem 7, we need to prove Theorem 9. We now state a few results that would help us towards this end.

**Lemma 11.** An uncolored face \( F \), such that \( |V(F)| \neq 5 \), can be CF colored using 4 colors satisfying the invariants.

**Proof.** Let \( F = v_1 - v_2 - v_3 - \cdots - v_{k-1} - v_k - v_1 \) be a face with \( |V(F)| = k \), \( k \neq 5 \). We assign \( C(v_1) = 1, C(v_2) = 2, C(v_3) = 3 \) and for the remaining vertices (if any), we set \( C(v_i) = C(v_{i-3}) \). In order to satisfy the invariants, we need to make the following changes:

- \( k \equiv 0 \pmod{3} \). No change is necessary.
- \( k \equiv 1 \pmod{3} \). Reassign \( C(v_{3k}) = 4 \).
- \( k \equiv 2 \pmod{3} \). Reassign \( C(v_{3k-3}) = 4, C(v_{3k-2}) = 2, C(v_{3k-1}) = 3, C(v_{3k}) = 4 \). Notice that this coloring does not satisfy the invariants if \( k = 5 \). However, the smallest \( k \) that we consider in this case is \( k = 8 \).

In each of the above cases the unique color for each vertex \( v_i \) is provided by its cyclical successor i.e., \( U(v_i) = C(v_{i+1}) \). □

**Lemma 12.** Let \( F \) be a face (cycle) in \( G \) with one vertex \( v \) such that \( C(v) \) and \( U(v) \) are already assigned, with \( C(v) \neq U(v) \). Then the rest of \( F \) can be complete CF colored using at most 4 colors, while retaining \( C(v) \) and \( U(v) \), and satisfying the invariants.
Proof. Let \( v_1 \) be the colored vertex in the cycle \( F \). We may assume w.l.o.g. that \( C(v_1) = 1 \) and \( U(v_1) = 2 \). Now, we extend \( C \) to the remainder of \( F \).

1. \(|V(F)| = 3\) with \( F = v_1 - v_2 - v_3 - v_1 \).
   We assign: \( C(v_2) = 3 \), \( C(v_3) = 4 \) and \( U(v_2) = 1 \), \( U(v_3) = 1 \).
2. \(|V(F)| \geq 4\) with \( F = v_1 - v_2 - v_3 - \cdots - v_{k-1} - v_k - v_1 \).
   We first assign: \( C(v_2) = 3 \) and \( C(v_3) = 2 \). For the remaining vertices \( v_i \), we set \( C(v_i) = C(v_{i-3}) \) for \( 4 \leq i \leq k \). However, we need to make some changes to this in order to satisfy the invariants. We have the following subcases:
   - \( 4 \leq k \equiv 0 \text{ or } 1 \pmod{3} \). Reassign \( C(v_k) = 4 \).
   - \( 4 \leq k \equiv 2 \pmod{3} \). Reassign \( C(v_{k-1}) = 4 \).

In each of the above cases the unique color for each vertex \( v_i \) is provided by its cyclical successor i.e., \( U(v_i) = C(v_{i+1}) \). Observe that \( U(v_1) \) is left unchanged, by ensuring \( v_2 \) and \( v_k \), the neighbors of \( v_1 \), are not assigned the color \( U(v_1) \).

\( \square \)

**Lemma 13.** Let \( P \) be a path in \( G \) whose endpoints are \( v_1, v_2 \). Suppose \( \{v_1, v_2\} \in E(G) \) and that \( v_1, v_2 \) are already assigned the functions \( C \) and \( U \) satisfying the invariants. Then the rest of \( P \) can be CF colored using at most 4 colors, while retaining \( C \) and \( U \) values of the endpoints, and satisfying the invariants.

Since the proof of the above lemma is a bit long and involved, we first prove Theorem 9 using Lemmas 11, 12 and 13.

**Proof (Proof of Theorem 9).**

1. If the block is a bridge, say \( \{v, w\} \), then we color it \( C(v) = 1, C(w) = 2 \) with \( U(v) = 2, U(w) = 1 \). Note that the invariant marked \( \star \) is violated in this case. However, this does not cause an issue since this edge is a bridge, and it does not appear in any inner face.

   If the block is not a bridge, then by assumption, it contains a face \( F \) such that \(|V(F)| \neq 5\). By Lemma 11 we have a coloring of \( F \) using 4 colors and satisfying the invariants. By the Lemma 10 (Ear Decomposition), the block has an ear decomposition \( F, P_1, P_2, \ldots \) with \( F \) as the starting inner face. Recall that for every path \( P_i \), the end points form an edge in \( F_0 \cup \bigcup_{1 \leq j < i} P_j \).

   We color the paths \( P_1, P_2, \ldots \) in this order. By Lemma 13 we have a coloring for each of these paths using 4 colors and satisfying the invariants.

2. Let \( v \) be the vertex in the block that is already colored. W.l.o.g., we may assume that \( C(v) = 1 \) and \( U(v) = 2 \).

   If the block is a bridge \( \{v, w\} \), we color \( w \) with \( C(w) = 3 \) and set \( U(w) = 1 \).

   If the block is not a bridge, choose an inner face \( F \) that contains \( v \). Using Lemma 12 we color the remainder of \( F \) using at most 4 colors and satisfying the invariants. The rest of the proof follows from the fact that we have an ear decomposition with \( F \) as the starting face, and Lemma 13. This is very similar to the argument in the proof of part 1 of this theorem and hence the details are omitted.
Case 1: \( C(v_1) \neq C(v_2), U(v_1) \neq U(v_2) \). W.l.o.g. we may assume \( C(v_1) = 1 \), \( C(v_2) = 2 \) and \( U(v_1) = 2, U(v_2) = 3 \).

- \( |V(P)| = 3 \), \( P = v_2 - v_3 - v_1 \). Assign \( C(v_3) = 4 \) with \( U(v_3) = 2 \).
- \( |V(P)| = 4 \), \( P = v_2 - v_3 - v_4 - v_1 \). Assign \( C(v_3) = 4, C(v_4) = 3 \) with \( U(v_3) = 3, U(v_4) = 1 \).
- \( |V(P)| \geq 5 \), \( P = v_2 - v_3 - \cdots - v_{k-1} - v_k - v_1 \). We first assign \( C(v_3) = 1, C(v_4) = 3, C(v_5) = 4 \). For the remaining vertices \( v_i \), we initially assign \( C(v_i) = C(v_{i-3}) \) for \( 6 \leq i \leq k \). However, we need to make some changes to satisfy the invariants. We have the following subcases:
  - \( k \equiv 0 \pmod{3} \). Reassign \( C(v_{k-1}) = 2 \) and \( C(v_k) = 4 \).
  - \( k \equiv 1 \pmod{3} \). Reassign \( C(v_{k-1}) = 2 \).
  - \( k \equiv 2 \pmod{3} \). No change is necessary.

In each of the above cases the unique color for each vertex \( v_i \) is provided by its cyclical successor i.e., \( U(v_i) = C(v_{i+1}) \).

Case 2: \( U(v_1) = U(v_2) \). W.l.o.g., we may assume \( C(v_1) = 1, C(v_2) = 2 \) and \( U(v_1) = U(v_2) = 3 \).

- Case 2(i): \( |V(P)| = 3 \) and \( P = v_2 - v_3 - v_1 \).
  - Case 2(i)(a): Vertices \( v_1 \) and \( v_2 \) are the only neighbors of \( v_3 \). Assign \( C(v_3) = 4 \) and \( U(v_3) = 2 \). The invariant marked \( \star \) is not satisfied, but that does not matter as \( v_3 \) does not participate in any further faces.
  - Case 2(i)(b): One of the edges \( \{v_1, v_3\} \) or \( \{v_2, v_3\} \) does not feature in another face. W.l.o.g., say \( \{v_2, v_3\} \) be that edge. Assign \( C(v_3) = 4 \) with \( U(v_3) = 1 \). The \( \star \) invariant is violated for \( \{v_2, v_3\} \) here but it does not affect the further coloring.
  - Case 2(i)(c): One of the edges \( \{v_1, v_3\} \) or \( \{v_2, v_3\} \) features in an uncolored face \( F \) such that \( |V(F)| \neq 3 \). W.l.o.g., say \( \{v_2, v_3\} \) is that edge. We assign \( C(v_3) = 4 \) with \( U(v_3) = 1 \). Let \( |V(F)| = k \) with \( F = v_3 - w_1 - w_2 - \cdots - w_{k-2} - v_2 - v_3 \). We assign \( C(w_1) = 3, C(w_2) = 1 \) and \( C(w_3) = 4 \) (if \( w_3 \) exists). For all \( 4 \leq i \leq k-2, C(w_i) = C(w_{i-3}) \). If \( k \equiv 0 \pmod{3} \), we reassign \( C(w_{k-4}) = 2, C(w_{k-3}) = 1 \) and \( C(w_{k-2}) = 4 \).

  The unique colors \( U \) for the vertices are assigned as follows:
  * For \( k = 6 \), \( U(w_1) = 4, U(w_2) = 3, U(w_3) = 2 \) and \( U(w_4) = 2 \).
  * For \( k \neq 6 \), we have for \( 1 \leq i \leq k-3, U(w_i) = C(w_{i+1}) \) and \( U(w_{k-2}) = C(w_2) = 2 \).

  - Case 2(i)(d): The only remaining case is when both the edges \( \{v_1, v_3\} \) or \( \{v_2, v_3\} \) feature in uncolored triangular faces. Let \( \{v_1, v_3\} \) form a triangular face with \( x \) and \( \{v_2, v_3\} \) with \( y \). We have two subcases:
The edge \( \{x, v_3\} \) forms a triangular face with another vertex \( z \) (see Figure 3). Assign \( C(v_3) = 1, C(x) = 2, C(y) = 4, C(z) = 3 \) and \( U(v_3) = 4, U(x) = 3, U(y) = 2, U(z) = 1 \). Some edges violate the invariant marked \( \star \), but these edges are already part of two faces, and hence do not feature in the further coloring.

The edge \( \{x, v_3\} \) is not part of a triangular face with another vertex. In this case, we assign \( C(v_3) = 4, C(x) = 4, C(y) = 1 \) and \( U(v_3) = 2, U(x) = 1, U(y) = 2 \). Out of the edges that violate the invariant marked \( \star \), the only one that can participate in the further coloring is the edge \( \{x, v_3\} \). By assumption, \( \{x, v_3\} \) is not part of a triangular face. In Lemma 14, we explain how to color the uncolored face that is \( \{x, v_3\} \) may be a part of.

- **Case 2(ii):** \(|V(P)| = 4, P = v_2 - v_3 - v_4 - v_1\).
  - **Case 2(ii)(a):** The edge \( \{v_3, v_4\} \) forms a triangular face with a vertex \( x \). We assign \( C(v_3) = 1, C(v_4) = 4, C(x) = 3 \), with \( U(v_3) = 3, U(v_4) = 1, U(x) = 4 \).
  - **Case 2(ii)(b):** The edge \( \{v_3, v_4\} \) is not part of an uncolored triangular face. We assign \( C(v_3) = C(u_4) = 4 \), with \( U(v_3) = 2, U(v_4) = 1 \). If the edge \( \{v_3, v_4\} \) is part of an uncolored face \( F \), by assumption, we know that \(|V(F)| \geq 4 \) and hence we can use Lemma 14 to color \( F \) satisfying the invariants.

- **Case 2(iii):** \(|V(P)| = 5, P = v_2 - v_3 - v_4 - v_5 - v_1\). We assign \( C(v_3) = 1, C(v_4) = 3, C(v_5) = 2 \), with \( U(v_3) = 3, U(v_4) = 2, U(v_5) = 1 \).

- **Case 2(iv):** \(|V(P)| \geq 6, P = v_2 - v_3 - \cdots - v_{k-2} - v_{k-1} - v_k - v_1\).
  - We first assign \( C(v_k) = 4 \) and \( C(v_4) = 3 \). For \( 5 \leq i \leq k \), assign \( C(v_i) = C(v_{i-3}) \). If \( k \equiv 1 \pmod{3} \), then reassign \( C(v_{k-2}) = 1 \) and \( C(v_k) = 2 \). For each vertex \( v_i \), the unique color is provided by its cyclical successor i.e., \( U(v_i) = C(v_{i+1}) \).

\( \square \)

**Lemma 14.** Let \( F \) be a face with \(|V(F)| \geq 4 \) with such that the edge \( \{v_1, v_2\} \in E(F) \) and \( v_1 \) and \( v_2 \) already colored such that \( C(v_1) = C(v_2) \) and \( U(v_1) \neq U(v_2) \). Then the rest of \( F \) can be CF colored using 4 colors satisfying the invariants.
Proof. W.l.o.g., we may assume $C(v_1) = C(v_2) = 4$, $U(v_1) = 1$ and $U(v_2) = 2$. We have the following cases:

- $|V(F)| = 4$ with $F = v_2 - v_3 - v_4 - v_1 - v_2$. We assign: $C(v_3) = 1$, $C(v_4) = 3$ and $U(v_3) = 4$ and $U(v_4) = 4$.
- If $|V(F)| = 5$ with $F = v_2 - v_3 - v_4 - v_5 - v_1 - v_2$. We assign: $C(v_3) = 1$, $C(v_4) = 2$, $C(v_5) = 3$ and $U(v_3) = 2$, $U(v_4) = 3$ and $U(v_4) = 4$.
- If $|V(F)| \geq 6$ with $F = v_2 - v_3 - \cdots - v_{k-1} - v_k - v_1 - v_2$. We assign: $C(v_3) = 3$ and $C(v_4) = 2$. For all $5 \leq i \leq k$, $C(v_i) = C(v_{i-3})$.
  - $k \equiv 0 \pmod{3}$. Reassign $C(v_{k-1}) = 1$.
  - $k \equiv 1 \pmod{3}$. No change is required.
  - $k \equiv 2 \pmod{3}$. Reassign $C(v_{k-1}) = 1$ and $C(v_k) = 2$.

The unique color of each vertex $v_i$ is provided its cyclical successor i.e., $U(v_i) = C(v_{i+1})$.

\[\Box\]

Algorithmic Note: The steps in the proof of Theorem 16 leads to an algorithm. Block decomposition, outerplanarity testing and embedding outerplanar graphs \[18\] can all be done in linear time, i.e., $O(|V(G)|)$. Thus we have an $O(|V(G)|)$ time algorithm, that given an outerplanar graph $G$, determines a complete CF coloring for $G$ that uses four colors.

4.1 Cactus Graphs

Now, we show that a cactus graph can be complete CF colored using 3 colors. This is a tight bound.

Definition 15. A cactus graph is a connected graph in which any two cycles have at most one vertex in common.

Theorem 16. Three colors are sufficient and sometimes necessary to complete CF color cactus graphs.

Proof. Cactus graphs are outerplanar and by definition \[15\] any two cycles have at most one vertex in common. We apply the block decomposition on the cactus graph $G$. Note that each block is a cycle or a bridge. Throughout the coloring, we maintain the invariant that for each vertex $v$, the unique color seen by $v$, $U(v) \neq C(v)$.

Let $B_0$ be the first block considered. If $B_0$ is a bridge $\{v, w\}$, we assign $C(v) = 1$, $C(w) = 2$ and $U(v) = 2$, $U(w) = 1$. Else $B_0$ is a cycle with $F = v_1 - v_2 - \cdots - v_{k-1} - v_k - v_1$. Initially we assign $C(v_1) = 1$, $C(v_2) = 2$ and $C(v_3) = 3$. For all $4 \leq i \leq k$, $C(v_i) = C(v_{i-3})$. If $k \equiv 2 \pmod{3}$, $C(v_{k-1}) = 3$. In each of the cases, for each vertex $v$, we can identify $U(v)$ such that $U(v) \neq C(v)$.

Now, we choose a block that is adjacent to an already colored block. Such a block has exactly one colored vertex. Let $v_1$ be the that vertex with $C(v_1) = 1$ and $U(v_1) = 2$. If the block is a bridge, say $\{v_1, w\}$, then assign $C(w) = 3$ with $U(w) = 1$. Else the block is an inner face $F = v_1 - v_2 - \cdots - v_{k-1} - v_k - v_1$ with
a colored vertex $v_1$. We initially assign $C(v_2) = 3$, $C(v_3) = 2$. For all $4 \leq i \leq k$, $C(v_i) = C(v_{i-3})$. In the case when $k \equiv 0 \pmod{3}$, we reassign $C(v_k) = 3$, and in the case when $k \equiv 2 \pmod{3}$, we reassign $C(v_{k-1}) = 2$. In this case too, we maintain the invariant that $C(v) \neq U(v)$ for each $v$.

To see that the bound is tight, observe that Figure 2 is a cactus graph that requires three colors. \hfill \Box

5 Kneser graphs

In this section, we study the CF coloring of Kneser graphs. Throughout this section, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. The Kneser graph $K(n, k)$ is formally defined as follows:

Definition 17 (Kneser graph). The Kneser graph $K(n, k)$ is the graph whose vertices are $\binom{[n]}{k}$, the $k$-sized subsets of $[n]$, and the vertices $x$ and $y$ are adjacent if and only if $x \cap y = \emptyset$ (when $x$ and $y$ are viewed as sets).

During the discussion, we shall use the words $k$-set or $k$-subset to refer to a set of size $k$. We shall sometimes refer to the $k$-subsets of $[n]$ and the vertices of $K(n, k)$ in an interchangeable manner.

Lemma 18. $k + 2$ colors are sufficient to complete CF color a $K(n, k)$ when $n \geq 3k - 1$.

Proof. We first assign a coloring$^3$ to the vertices of $K(n, k)$ and then argue that this coloring is a complete CF coloring.

- For any vertex ($k$-set) $v$ that is a subset of $\{1, 2, \ldots, 2k - 1\}$, we assign $C(v) = \max_{\ell \in v} \ell - (k - 1)$.
- The set $\{2k, 2k + 1, \ldots, 3k - 1\}$ is assigned the color $k + 1$.
- All the remaining vertices are assigned the color $k + 2$.

For example, for the Kneser graph $K(n, 3)$, we assign the color 1 to the vertex $\{1, 2, 3\}$, color 2 to the vertices $\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$, color 3 to the vertices $\{1, 2, 5\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}$, color 4 to the vertex $\{6, 7, 8\}$ and color 5 to all the remaining vertices.

Now, we prove that in the above coloring, every vertex has a uniquely colored neighbor. Let $C_1$ be the set of all vertices assigned the color $i$, i.e., the color class of the color $i$. Notice that $C_1 \cup C_2 \cup \cdots \cup C_k = \binom{[2k-1]}{k}$. Let $w_{k+1}$ denote the $k$-set $\{2k, 2k + 1, \ldots, 3k - 1\}$. Any vertex $v \in C_1 \cup C_2 \cup \cdots \cup C_k$ is a neighbor of $w_{k+1}$. Since $w_{k+1}$ is the lone vertex colored $k + 1$, it serves as the uniquely colored neighbor for any $v \in C_1 \cup C_2 \cup \cdots \cup C_k$.

Now we have to show the presence of uniquely colored neighbors for vertices that have some elements from outside $\{1, 2, \ldots, 2k - 1\}$. Let $v$ be the vertex

$^3$ In this coloring, the uniquely colored neighbor is not colored $k + 2$ for any of the vertices. Thus, by recoloring the color class $k + 2$ with the color 0, we get a partial coloring that uses $k + 1$ colors.
such that it has some elements from outside \{1, 2, \ldots, 2k - 1\}. That is, \( v \cap \{1, 2, \ldots, 2k - 1\} \neq v \). Let \( t = t(v) \) be the smallest nonnegative integer such that \(|\{1, 2, \ldots, k + t\} \setminus v| = k\). Since \( v \) has at least one element from outside \{1, 2, \ldots, 2k - 1\}, \( t \) is at most \( k - 1 \).

We claim that \( v \) has a lone neighbor colored \( t + 1 \), and this neighbor is given by the set \( \{1, 2, \ldots, k + t\} \setminus v \). By the choice of \( t \), this is the only neighbor of \( v \) that is colored \( t + 1 \). It can be further observed that there are no neighbors of \( v \) that are assigned a color smaller than \( t + 1 \).

Now we show that \( k + 2 \) colors are necessary, when \( n \) is large enough.

**Theorem 19.** \( k + 2 \) colors are necessary to complete CF color \( K(n, k) \) when \( n \geq k(k + 1)^2 + 1 \).

**Proof.** We prove this by contradiction. Suppose that \( K(n, k) \) can be colored using the \( k + 1 \) colors \( 1, 2, 3, \ldots, k + 1 \). For each \( 1 \leq i \leq k + 1 \), let \( C_i \) denote the color class corresponding to the color \( i \), i.e., the set of all vertices colored with the color \( i \). Let \( q = \binom{n}{k} / (k + 1) \).

If for all \( i \), \( |C_i| < q \), this implies that the total number of vertices is strictly less than \( q(k + 1) = \binom{n}{k} \). This is a contradiction. Hence there is at least one \( i \), such that \( |C_i| \geq q \). For any vertex \( v \), let \( d_i(v) \) denote the number of neighbors of \( v \) in \( C_i \).

Our strategy is to find one vertex, say \( x \), which does not have a uniquely colored neighbor. More formally, we want \( x \) to satisfy \( d_i(x) \neq 1 \), for all \( 1 \leq i \leq k + 1 \). We construct the vertex \( (k-set) \) \( x \), by choosing elements in it as follows. Suppose there are \( C_i \)'s that are singleton, i.e., \( |C_i| = 1 \). For all the singleton \( C_i \)'s we choose a hitting set. In other words, we choose entries in \( x \) so as to ensure that \( x \) intersects with the vertices in all the singleton \( C_i \)'s. This partially constructed \( x \) may also intersect with vertices in other \( C_i \)'s. Some of the other \( C_i \)'s might become “effectively singleton”, that is \( x \) may intersect with all the vertices in those \( C_i \)'s except one. We now choose further entries in \( x \) so as to hit these effectively singleton \( C_i \)'s too. Finally, we terminate this process when all the remaining \( C_i \)'s are not singleton.

At this stage, \( x \) can have potentially \( k + 1 \) entries, one each to hit the \( k + 1 \) color classes. However, the below claim shows that not all the color classes need to be hit.

**Claim:** There exists an \( i \) for which \( C_i \) does not become singleton/effectively singleton.

**Proof of claim.** We have already seen that there is at least one \( C_i \) for which \( |C_i| \geq q = \binom{n}{k} / (k + 1) \). We show that this \( C_i \) does not become effectively singleton.

Let \( t \) be the number of entries in \( x \) when the above process terminates. Notice that each entry in \( x \) can cause \( x \) to intersect with at most \( \binom{n-1}{k-1} \) other vertices. We have \( t \leq k + 1 \) entries in \( x \), so \( x \) can intersect with at most \( (k + 1) \binom{n-1}{k-1} \) vertices. When \( n \geq k(k + 1)^2 + 1 \), it can be verified that \( (k + 1) \binom{n-1}{k-1} < q - 2 \), leaving at least two vertices in \( C_i \) that do not intersect with \( x \). \( \square \)
Due to the above claim, the number of entries in \( x \) is \( t \leq k \). To fill up the remaining entries of \( x \) (if any), we consider the set(s) \( C_j \) that have not become effectively singleton. For each of these sets \( C_j \), we choose two distinct vertices, say \( y_j, y'_j \in C_j \). We choose the remaining entries of \( x \) so that \( x \cap y_j = \emptyset \) and \( x \cap y'_j = \emptyset \). The number of such sets \( C_j \) is at most \( k + 1 \). So for choosing the remaining entries of \( x \), we have at least \( n - t - 2k(k + 1) \) choices. Because \( n > k^3 \), we can choose such entries.

\[ \square \]

It is worth noting that the above proof technique cannot be applied for showing a lower bound of \( k + 3 \). For such a proof, we would start with a \( k + 2 \) coloring, and try for a contradiction. In this case, we could have \( k + 1 \) singletons and effective singletons, which could require \( k + 1 \) elements of \([n]\) to hit. However, \( x \) can hold at most \( k \) elements. This is where the proof breaks down.

5.1 CF Closed Neighborhood Coloring of Kneser Graphs

In this section, we see some results on the CF closed neighborhood coloring of Kneser graphs. We abbreviate CF closed neighborhood coloring as CF-CN coloring and denote by \( \chi_{CF-CN}(G) \), the conflict-free closed neighborhood chromatic number of \( G \). It is easy to see that a proper coloring of a graph \( G \) is also a CF-CN coloring. That is, \( \chi_{CF-CN}(G) \leq \chi(G) \) for all graphs \( G \). Since \( \chi(K(n, k)) \leq n - 2k + 2 \) \[14\], we have that \( \chi_{CF-CN}(K(n, k)) \leq n - 2k + 2 \).

**Lemma 20.** When \( n \geq 2k + 1 \), we have \( \chi_{CF-CN}(K(n, k)) \leq k + 1 \).

**Proof.** We assign the following coloring to the vertices of \( K(n, k) \):

- For any vertex \((k\text{-set}) v \) that is a subset of \( \{1, 2, \ldots, 2k - 1\} \), we assign \( C(v) = \max_{\ell \in v} \ell - (k - 1) \).
- All the uncolored vertices are assigned color \( k + 1 \).

For \( 1 \leq i \leq k + 1 \), let \( C_i \) be the color class of the color \( i \). Notice that \( C_1 \cup C_2 \cup \cdots \cup C_k = \binom{2k - 1}{k} \). Since any two \( k \)-subsets of \( \{1, 2, \ldots, 2k - 1\} \) intersect, it follows that \( \binom{2k - 1}{k} \) is an independent set. Hence each of the color classes \( C_1, C_2, \ldots, C_k \) are independent sets. So if \( v \) is colored with color \( i \), where \( 1 \leq i \leq k \), it has no neighbors of its own color. Hence, it serves as its own uniquely colored neighbor.

If \( v \) is colored \( k + 1 \), then \( v \notin [2k - 1] \). That is, \( v \) has some elements from outside \([2k - 1] = \{1, 2, \ldots, 2k - 1\} \). Let \( t = t(v) \) be the smallest nonnegative integer such that \( |\{1, 2, \ldots, k + t\} \setminus v| = k \). Since \( v \) has at least one element from outside \( \{1, 2, \ldots, 2k - 1\} \), \( t \) is at most \( k - 1 \). It is easy to verify that the vertex corresponding to the set \( \{1, 2, \ldots, k + t\} \setminus v \) is the lone neighbor of \( v \) that is colored \( t + 1 \), and thus serves as the uniquely colored neighbor of \( v \).

\[ \square \]

**Lemma 21.** \( \chi_{CF-CN}(K(2k + 1, k)) = 2 \), for all \( k \geq 1 \).
Proof. Consider a vertex \( v \in V(K(2k + 1, k)) \). If \( v \cap \{1, 2\} \neq \emptyset \), we assign color 1 to \( v \). Else, we assign color 2 to \( v \).

Let \( C_1 \) and \( C_2 \) be the sets of vertices colored 1 and 2 respectively. Below, we discuss the unique colors for every vertex of \( K(n, k) \).

1. If \( v \in C_1 \) and \( \{v \cap \{1, 2\}\} \subseteq v \), then \( v \) is the uniquely colored neighbor of itself. This is because all the vertices in \( C_1 \) contain either 1 or 2 and hence \( v \) has no neighbors in \( C_1 \).
2. Let \( v \in C_1 \) and \( |v \cap \{1, 2\}| = 1 \). W.l.o.g., let \( 1 \in v \) and \( 2 \notin v \). In this case, \( v \) has a uniquely colored neighbor \( w \in C_2 \). The vertex \( w \) is the k-set \( w = [2k + 1] \setminus (v \cup \{2\}) \).
3. If \( w \in C_2 \), \( w \) is the unique color neighbor of itself. This is because \( C_2 \) is an independent set. For two vertices \( w, w' \in C_2 \) to be adjacent, we need \( |w \cup w'| = 2k \), but vertices in \( C_2 \) are subsets of \( \{3, 4, 5, \ldots, 2k + 1\} \), which has cardinality \( 2k - 1 \).

Lemma 22. \( \chi_{CF-CN}(K(2k + d, k)) \leq d + 1 \), for all \( k \geq 1 \).

Proof. We prove this by induction on \( d \). The base case is when \( d = 1 \) which is true from Lemma 21. Suppose \( K(2k + d, k) \) has a CF-CN coloring with \( d + 1 \) colors. Let us consider \( K(2k + d + 1, k) \). For all the vertices of \( K(2k + d + 1, k) \) that appear in \( K(2k + d, k) \) we use the same coloring as in \( K(2k + d, k) \). The new vertices (the vertices that contain \( 2k + d + 1 \)) are assigned the new color \( d + 2 \). As all the new vertices contain \( 2k + d + 1 \), they form an independent set. Hence each of the new vertices serve as their own uniquely colored neighbor.

The vertices of \( K(2k + d + 1, k) \) already present in \( K(2k + d, k) \) get new neighbors, but all the new neighbors are colored with the new color \( d + 2 \). Hence the unique color of the existing vertices are retained.

So, from Lemma 20 and Lemma 22 we get the following.

\[
\chi_{CF-CN}[K(n, k)] \leq \begin{cases} 
  n - 2k + 1, & \text{for } 2k + 1 \leq n \leq 3k \\
  k + 1, & \text{for } n \geq 3k + 1
\end{cases}
\]

6 Discussion

We note a few directions that are left open by this paper:

- We showed that a planar graph has a partial CF coloring that uses at most 5 colors. The best known lower bound is 4 colors.
- Along similar lines, an outerplanar graph can be partial CF colored using 4 colors, while the lower bound is 3.
- We showed that the complete CF chromatic number of \( K(n, k) \) is \( k + 2 \) when \( n \geq k(k + 1)^2 + 1 \). We believe this requirement on \( n \) can be relaxed.

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