Greedy Rectilinear Drawings*

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Abstract. A drawing of a graph is greedy if for each ordered pair of vertices u and v, there is a path from u to v such that the Euclidean distance to v decreases monotonically at every vertex of the path. The existence of greedy drawings has been widely studied under different topological and geometric constraints, such as planarity, face convexity, and drawing succinctness. We introduce greedy rectilinear drawings, in which each edge is either a horizontal or a vertical segment. These drawings have several properties that improve human readability and support network routing. We address the problem of testing whether a planar rectilinear representation, i.e., a plane graph with specified vertex angles, admits vertex coordinates that define a greedy drawing. We provide a characterization, a linear-time testing algorithm, and a full generative scheme for universal greedy rectilinear representations, i.e., those for which every drawing is greedy. For general greedy rectilinear representations, we give a combinatorial characterization and, based on it, a polynomial-time testing and drawing algorithm for a meaningful subset of instances.

1 Introduction

In a greedy drawing of a graph in the plane every vertex is mapped to a distinct point and, for each ordered pair of vertices u and v, there is a distance-decreasing path from u to v, i.e., a path such that the Euclidean distance to v decreases monotonically at every vertex of the path. Greedy drawings have been

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originally proposed to support *greedy routing schemes* for ad hoc wireless networks \[21,22,23\]. In such schemes, a node that has to send a packet to a destination \(v\) just forwards the packet to one of its neighbors that is closer to \(v\) than itself. In their seminal work, Papadimitriou and Ratajczak \[21,22\] showed that 3-connected planar graphs form the largest class of graphs for which every instance may admit a greedy drawing, and they formulated two conjectures. 

**Weak conjecture:** Every 3-connected planar graph admits a greedy drawing.  

**Strong conjecture:** Every 3-connected planar graph admits a *convex* greedy drawing, i.e., a planar greedy drawing with convex faces. Concerning the weak conjecture, Dhandapani \[7\] provided an existential proof of greedy drawings for maximal planar graphs. Later on, Leighton and Moitra \[17\] and Angelini et al. \[3\] independently settled the weak conjecture positively, by also describing constructive algorithms. Da Lozzo et al. \[6\] strengthened these results, showing that in fact every 3-connected planar graph admits a *planar* greedy drawing. However, the strong conjecture is still open. For graphs that are not 3-connected, Nölenburg and Prutkin \[19\] characterized the trees that admit a greedy drawing.

Greedy drawings have also been investigated in terms of *succinctness*, an important property that helps to make greedy routing schemes work in practice. A drawing is succinct if the vertex coordinates are represented by a polylogarithmic number of bits. Since there exist greedy-drawable graphs in the Euclidean sense that do not admit a succinct greedy drawing \[2\], several papers also studied succinct greedy drawings in spaces different from the Euclidean one or according to a metric different from the Euclidean distance \[10,12,13,14,18,26\].

We finally mention another model, called *self-approaching drawing*, that reinforces the properties of greedy drawings \[1,20\]. A straight-line drawing is self-approaching if for any ordered pair of vertices \(u\) and \(v\), there exists a path \(P\) from \(u\) to \(v\) such that, for any point \(p\) on \(P\), the distance from \(u\) to \(p\) always decreases while continuously moving along \(P\) in the drawing. Clearly, every self-approaching drawing is greedy, but not vice versa. In particular, the *dilation* of self-approaching drawings is bounded by a constant \[15\], while for greedy drawings it may be unbounded \[1\]. The dilation (or "stretch-factor") of a straight-line drawing is the maximum value of the ratio between the length of the shortest path between two vertices in the drawing and their Euclidean distance.

**Motivation and Contribution.** Our work is motivated by the rich literature on greedy drawings that satisfy some interesting topological or geometric requirements, such as planarity \[6\] and face convexity \[13,14,21,26\]. We study greedy drawings in the popular *orthogonal drawing* convention \[8,9,24\]: Vertices are mapped to points and edges are sequences of horizontal and vertical segments (each vertex has degree at most 4). More precisely, we introduce planar *greedy rectilinear drawings*, that is, crossing-free greedy drawings where each edge is either a horizontal or a vertical segment. We address the following question: “Let \(H\) be a planar rectilinear representation, i.e., a plane graph with given values (90, 180, 270 degrees) for the geometric angles around each vertex; is it possible to assign coordinates to the vertices of \(H\) so that the resulting drawing is greedy rectilinear?” (see Appendix \[A\] for a more formal definition of planar rec-
Fig. 1: (a) A rectilinear drawing that is not greedy; (b) A greedy rectilinear drawing of the same representation (the distance-decreasing paths between $u$ and $v$ are dashed). (c) Drawing of a universal greedy rectilinear representation. (d)–(e) $H$ is not greedy realizable if an internal face is not a rectangle or the external face is not orthoconvex.

tilnear representations). Fig. 1a shows a rectilinear drawing that is not greedy; however, the corresponding rectilinear representation has a greedy drawing, as shown in Fig. 1b. Our question fits into the effective topology-shape-metrics approach [4,23], which first computes a planar embedding of the graph, then finds an embedding-preserving orthogonal representation, and finally assigns coordinates to vertices and bends to complete the drawing; we address this last step, but our representations have no bend. Our contribution is as follows.

Section 2 discusses basic properties of greedy rectilinear drawings. We prove that the faces are convex and the dilation is bounded by a small constant, and we show convex (non-rectilinear) greedy drawings in which every distance-decreasing path between two nodes is arbitrarily longer than the Euclidean distance.

Section 3 focuses on planar universal greedy rectilinear representations for which every drawing is greedy (see Fig. 1c). We give a linear-time recognition algorithm that, in the positive case, computes a greedy drawing of minimum area on an integer grid. We also describe a generative scheme for constructing any possible universal greedy rectilinear representation starting from a rectangle.

Section 4 extends the study to general rectilinear greedy representations. We give a non-geometric characterization of this class, which leads to a linear-time testing algorithm for a meaningful subset of instances. If the condition of the characterization is satisfied, a greedy drawing of minimum area within that condition can be computed in quadratic time. However, we show that in general greedy rectilinear representations may require exponential area.

We assume familiarity with basic concepts of graph drawing and planarity [8]; for the reader’s convenience, the terminology used in the paper is reported in Appendix A. For reasons of space, some proofs are moved to the appendix.

2 Basic Properties of Greedy Rectilinear Representations

We denote by $x(v)$ and $y(v)$ the $x$- and the $y$-coordinate of a vertex $v$ in a drawing $\Gamma$ of a graph. For two vertices $u$ and $v$ of $\Gamma$, $d(u,v)$ is the Euclidean distance between $u$ and $v$ and a path from $u$ to $v$ in $\Gamma$ is a $u$-$v$-path. The degree of $v$ is denoted as $\deg(v)$. If $v$ has neighbors $u_1, u_2, \ldots, u_h$, the cell of $v$ in $\Gamma$, denoted as $\text{cell}(v)$, is the (possibly unbounded) region of all points of the plane that are closer to $v$ than to any $u_i$. Fig. 2 shows all types of cells of a vertex $v$ in
a rectilinear drawing, depending on \( \deg(v) \) and on the angles at \( v \); if \( \deg(v) \leq 3 \), \( \text{cell}(v) \) is unbounded. The following geometric characterization is proven in [22].

**Theorem 1 (Papadimitriou and Ratajczak [22]).** A drawing of a graph is greedy if and only if for every vertex \( v \), \( \text{cell}(v) \) contains no vertex other than \( v \).

If a rectilinear representation \( H \) admits a greedy rectilinear drawing, \( H \) is greedy realizable or, equivalently, it is a greedy rectilinear representation. W.l.o.g., we shall assume that \( H \) comes with a fixed “rotation”, i.e., for any edge \((u,v)\), it is fixed whether \( u \) is to the left, to the right, above, or below \( v \) in every rectilinear drawing \( \Gamma \) of \( H \). A flat vertex of \( H \) (or of \( \Gamma \)) is a vertex with a flat angle (180 degrees). A flat angle formed by two horizontal segments is north-oriented (south-oriented) if it is above (below) the two segments. A flat angle between two vertical segments is either east-oriented or west-oriented.

We restrict our study to biconnected graphs, as otherwise the set of greedy rectilinear drawings may be very limited (see, e.g., Theorem 10 in Appendix B.1). Lemma 1 (proved in Appendix B.2) allows us to further restrict to convex rectilinear representations, i.e., those having rectangular internal faces and an orthoconvex polygon as external boundary. Indeed, if \( H \) is not convex, there exist two vertices \( u, v \) such that \( u \in \text{cell}(v) \) in any drawing of \( H \) (see also Figs. 1d-1e).

**Lemma 1.** \( H \) is greedy realizable only if it is convex.

For a rectilinear drawing of a convex rectilinear representation \( H \), let \( R(u,v) \) denote the minimum bounding box (rectangle or segment) including \( u \) and \( v \). The next property immediately follows from the convexity of \( H \).

**Property 1.** Let \( f \) be a face of \( H \) and \( w \) be any vertex of \( H \) with an angle of 90 degrees inside \( f \). Denote by \( u \) and \( v \) the two neighbors of \( w \) along the boundary of \( f \). In any rectilinear drawing of \( H \), there is no vertex properly inside \( R(u,v) \).

We exploit Property 1 to prove that rectilinear greedy drawings have bounded dilation, where the paths that determine the dilation are distance-decreasing. An analogous statement does not hold for general convex greedy drawings; an example of this fact is in Appendix B.3 together with the proof of Theorem 2.

**Theorem 2.** In a rectilinear greedy drawing on an integer grid, for every two vertices \( s, t \) there is a distance-decreasing \( s-t \)-path of length at most \( 3\sqrt{2} \cdot d(s,t) \).
Greedy Rectilinear Drawings

Conflicts in rectilinear representations. We now define two directed acyclic graphs (DAGs) $D_x$ and $D_y$ associated with $H$, already used for orthogonal compaction \[5\]. They are fundamental tools for the rest of the paper. $D_x$ is obtained from $H$ by orienting the horizontal edges from left to right and by contracting each maximal path of vertical edges into a node. $D_y$ is defined symmetrically on the maximal paths of horizontal edges; see Fig. 3. $D_x$ and $D_y$ may have multiple edges and they are st-digraphs (they have a single source and a single sink), since the external face of $H$ is orthoconvex. For any vertex $u$ of $H$, we denote by $c_x(u)$ ($c_y(u)$) the node of $D_x$ ($D_y$) corresponding to the maximal vertical (horizontal) path containing $u$ in $H$. If $c_x(u) \neq c_y(u)$, the notation $u \prec_x v$ ($u \not\prec_x v$) denotes the existence (absence) of a directed path from $c_x(u)$ to $c_y(v)$ in $D_x$. The notation $u \sim_x v$ means that either $u \prec_x v$ or $v \prec_x u$ holds, while $u \not\sim_x v$ means that none of them holds. The notations $u \prec_y v$, $u \not\prec_y v$, $u \sim_y v$, and $u \not\sim_y v$ are symmetric for $D_y$. Clearly, $\prec_x$ and $\prec_y$ are transitive relations. The next lemma (proved in Appendix \[B.2\]) states that there is a directed path between any two vertices of $H$ in at least one of $D_x$ and $D_y$.

**Lemma 2.** For any two vertices $u$ and $v$ of a convex rectilinear representation $H$, at least one of the following holds: (i) $u \sim_x v$ or (ii) $u \sim_y v$.

Let $u$ and $v$ be two vertices of $H$ such that $c_x(u) \neq c_x(v)$ and $c_y(u) \neq c_y(v)$. By Lemma 2, at least one of $u \sim_x v$ and $u \sim_y v$ holds, say the latter. If $u \sim_x v$ also holds, the relative positions (left/right/top/bottom) of $u$ and $v$ are fixed (they are the same in any drawing of $H$); in this case, we prove that none of the two vertices lies in the cell of the other in any drawing of $H$ (Lemma 3). Conversely, this is not guaranteed if $u \not\prec_x v$ (Theorem 3), and we say that $u$ and $v$ are in a conflict, denoted by $\{u, v\}$. In this case, suppose that $u \prec_y v$ (the case $v \prec_y u$ is symmetric) and consider the topmost (flat) vertex $u'$ of the vertical path corresponding to $c_x(u)$ and the bottommost (flat) vertex $v'$ of the vertical path corresponding to $c_x(v)$. We say that $u'$ and $v'$ are responsible for the conflict $\{u, v\}$. A conflict $\{u, v\}$ is an $x$-conflict if $u \not\prec_x v$ and a $y$-conflict if $u \not\prec_y v$. In Fig. 3a $\{u, v\}$ is an $x$-conflict, with $u' = u$ and $v' = v$. A conflict is resolved in a drawing $\Gamma$ of $H$ if none of the two vertices that are responsible for it lies in the cell of the other. The proof of Lemma 3 is in Appendix \[B.2\].
Lemma 3. Let \( H \) be a convex rectilinear representation of a biconnected graph. A rectilinear drawing \( \Gamma \) of \( H \) is greedy if and only if every conflict is resolved in \( \Gamma \).

3 Universal Greedy Rectilinear Representations

A convex rectilinear representation \( H \) is conflict-free if it has no conflict. The following concise characterization holds for universal rectilinear representations.

Theorem 3. Let \( H \) be a convex rectilinear representation of a biconnected plane graph. \( H \) is universal greedy if and only if it is conflict-free.

Proof. By Lemma 3, if \( H \) is conflict-free, every rectilinear drawing of \( H \) is greedy (note that a rectilinear representation may be conflict-free without being convex, which would imply that it is not universal greedy; see Fig. 3d).

Suppose that \( H \) is universal greedy but not conflict-free. Let \( \Gamma \) be any rectilinear drawing of \( H \). Consider two vertices \( u \) and \( v \) that are responsible for a conflict in \( H \); assume w.l.o.g. that \( u \not\sim_x v \). We can further assume that there is no vertex \( w \) such that \( x(u) < x(w) < x(v) \) in \( \Gamma \). Indeed, if such a vertex \( w \) exists (which implies \( w \not\sim_x u \) and \( v \not\sim_x w \)), at least one of \( u \not\sim_x w \) and \( w \not\sim_x v \) holds, as otherwise \( u \prec_x v \). Hence, we could have selected either \( u \) and \( w \) or \( w \) and \( v \) instead of \( u \) and \( v \). If \( x(u) = x(v) \), \( \Gamma \) is not greedy, because \( u \in \text{cell}(v) \) and \( v \in \text{cell}(u) \). If \( x(u) < x(v) \), we can transform \( \Gamma \) into a drawing \( \Gamma' \) by moving \( u \) and all the vertices in its vertical path to the right until \( x(u) = x(v) \). Since \( u \) and \( v \) are consecutive along the \( x \)-axis in \( \Gamma \) and since \( H \) is convex, \( \Gamma' \) is still planar but not greedy, which contradicts the fact that \( H \) is universal greedy. \( \square \)

Theorem 4. Let \( H \) be a rectilinear representation of an \( n \)-vertex biconnected plane graph. There exists an \( O(n) \)-time algorithm to test if \( H \) is universal greedy.

Proof. The algorithm first checks in linear time if \( H \) is convex. If not, the instance is rejected. Otherwise, it checks whether both \( D_x \) and \( D_y \) contain a (directed) Hamiltonian path, which can be done in linear time in the size of \( D_x \) and \( D_y \), which is \( O(n) \). Namely, since each of \( D_x \) and \( D_y \) is an \( st \)-digraph, computing a longest path from \( s \) to \( t \) is done in \( O(n) \) time from a topological sorting. We claim that \( H \) is universal greedy if and only if this test succeeds. By Theorem 3 to prove this claim, it is enough to show that a DAG \( D \) contains a Hamiltonian path if and only if for any two vertices \( u \) and \( v \) of \( D \), there is a directed path either from \( u \) to \( v \) or from \( v \) to \( u \). If \( D \) has a Hamiltonian path \( \pi \), a directed path between any two vertices of \( D \) is a subpath of \( \pi \). Conversely, if there is a directed path between any two vertices of \( D \), then a linear extension of a topological sorting of the vertices corresponds to a Hamiltonian path. \( \square \)

Since conflict-free rectilinear representations are a subclass of the turn-regular orthogonal representations, for which a minimum-area drawing can be found in linear time, we can also state the following.

Corollary 1. Let \( H \) be a universal greedy rectilinear representation. There is a linear-time algorithm to compute a (greedy) drawing of \( H \) with minimum area.
Generative Scheme. Let $H$ be a biconnected universal greedy rectilinear representation. Each of the following operations on $H$ produces a new biconnected universal greedy rectilinear representation (see Lemma 12 in Appendix C), which gives a generative scheme for universal greedy rectilinear representations. The proof is in Appendix C. See Fig. 14 for an example.

- $k$-reflex vertex addition. Attach to the external face of $H$ a path of $1 \leq k \leq 4$ reflex vertices (corners) that forms a new rectangular internal face, provided that the resulting representation is convex (Fig. 13 in Appendix C).

- flat vertex addition. Subdivide an external edge $(u,v)$ of $H$ with a flat vertex of degree two, provided that the strip of the plane between the two lines orthogonal to $(u,v)$ and passing through $u$ and $v$, respectively, has no vertices in its interior.

Theorem 5. Let $H$ be a universal greedy rectilinear representation of a biconnected planar graph. $H$ can be obtained by a suitable sequence of $k$-reflex vertex and flat vertex additions, starting from a rectangle.

4 General Greedy Rectilinear Representations

We now consider convex rectilinear representations $H$ of biconnected plane graphs that may contain conflicts, and investigate conditions under which they are greedy realizable. We present a characterization (Theorem 6), which yields a polynomial-time testing algorithm for a meaningful subclass of instances, namely when $D_x$ and $D_y$ are series-parallel (Theorem 9). Proofs are in Appendix D.

Let $D$ be one of the two DAGs $D_x$ and $D_y$ associated with $H$. Since $D$ is an st-digraph, it has an st-ordering $S = v_1, \ldots, v_m$. For two indices $i,j$, with $1 \leq i < j \leq m$, $D(i,j)$ denotes the subgraph of $D$ induced by $v_i, \ldots, v_j$. We say that $S$ is good if:

(S.1) For any two indices $i,j$, with $1 \leq i < j \leq m$, $D(i,j)$ has at most two connected components, and

(S.2) if $D(i,j)$ has exactly two components, then all nodes of one component precede those of the other in $S$.

Further, we say that a drawing of $H$ respects an st-ordering $S_x$ of $D_x$ ($S_y$ of $D_y$) if for any two vertices $v, w \in H$, we have that $v$ lies to the left of $w$ (below $w$) in the drawing if and only if $c_x(v)$ precedes $c_x(w)$ in $S_x$ ($c_y(v)$ precedes $c_y(w)$ in $S_y$). Finally, when we refer to the $x$-coordinate ($y$-coordinate) of a node $v_i$ of $D_x$ ($D_y$), we mean the one of all the vertices $w \in H$ with $c_x(w) = v_i$ (with $c_y(w) = v_i$), as these vertices belong to the same vertical (horizontal) path.

Theorem 6. A convex rectilinear representation $H$ of a biconnected plane graph is greedy realizable if and only if both DAGs $D_x$ and $D_y$ admit good st-orderings.

We start by proving the necessity of Theorem 6.

Lemma 4. If $D_x$ or $D_y$ admits no good st-ordering, $H$ is not greedy realizable.

Proof sketch. If $H$ admits a greedy drawing $\Gamma$ respecting an st-ordering $S_x$ of $D_x$ that is not good, there exist $i,j$, with $1 \leq i < j \leq m$, such that $D_x(i,j)$ has at least two connected components $C_1$ and $C_2$. First, note that the vertices of $C_1$ and $C_2$ are vertically separated by a horizontal line in $\Gamma$ (say, $C_1$ lies above $C_2$;
see Fig. 4, as otherwise there would be at least a pair of vertical paths whose corresponding nodes in \( D_x \) are joined by a directed path.

Since every internal face of \( H \) is rectangular, the vertices of the bottom (top) boundary of \( C_1 \) (\( C_2 \)) are part of a horizontal path, spanning all \( x \)-coordinates between the ones of \( v_i \) and \( v_j \). Thus, all vertices on the bottom (top) boundary of \( C_1 \) (\( C_2 \)) are south-oriented (north-oriented) flat vertices, and the union of their cells is a connected region spanning all the \( x \)-coordinates between their leftmost and rightmost vertices; see Fig. 4a. So, if a vertex of \( C_1 \) appears between two vertices of \( C_2 \) in \( S_x \), then it lies inside the cell of a vertex of \( C_2 \) in \( \Gamma \), and vice versa. Thus, the vertices of each component are consecutive in \( S_x \). Since \( S_x \) is not good, there is at least another component \( C_3 \) in \( D_x(i,j) \); see Fig. 4b. Consider the vertical line \( \ell \) that is horizontally equidistant to \( v_i \) and \( v_j \) in \( \Gamma \). Any two components of \( D_x(i,j) \) must be separated by \( \ell \) in order for the cells of the vertices of these components to be empty, which is not possible for three components. \( \square \)

To prove the sufficiency of Theorem 6, we assign the \( x \)- and \( y \)-coordinates in two steps, which can be performed independently due to the following lemma.

**Lemma 5.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be two drawings of \( H \) such that all \( x \)-conflicts are resolved in \( \Gamma_1 \) and all \( y \)-conflicts are resolved in \( \Gamma_2 \). Then, the drawing \( \Gamma_3 \) of \( H \) in which the \( x \)-coordinate of each vertex is the same as in \( \Gamma_1 \) and the \( y \)-coordinate of each vertex is the same as in \( \Gamma_2 \) is greedy.

We describe the assignment of the \( x \)-coordinates based on the good st-ordering \( S_x = v_1, \ldots, v_m \) of \( D_x \). The assignment of the \( y \)-coordinates based on the good st-ordering of \( D_y \) works symmetrically. We first prove in Lemma 6 that, to guarantee that every \( x \)-conflict is resolved, it suffices to resolve a specific subset of them, which is called minimal. Namely, an \( x \)-conflict \( \{u, v\} \) dominates an \( x \)-conflict \( \{w, z\} \), with \( c_x(u) = v_i \), \( c_x(v) = v_j \), \( c_x(w) = v_k \), and \( c_x(z) = v_\ell \), if \( k \leq i < j \leq \ell \). A minimal \( x \)-conflict is not dominated by any \( x \)-conflict.

By Lemmas 3 and 6 we conclude that a greedy rectilinear drawing can be obtained by resolving all the minimal conflicts. We finally give a constructive proof that this can always be done since \( S_x \) is good. In particular, we encode that a minimal \( x \)-conflict is resolved with a single inequality on the horizontal distances between the vertices in the \( x \)-conflict. Then, in Lemma 7 we prove that for a minimal \( x \)-conflict \( \{u, v\} \) the nodes \( c_x(u) \) and \( c_x(v) \) of \( D_x \) are consecutive in \( S_x \). We use this property to show that the system of inequalities describing the conditions for the minimal \( x \)-conflicts to be resolved always admits a solution.
Lemma 6. Let $G$ be a rectilinear drawing of $H$ respecting $S_x$. If every minimal $x$-conflict dominating an $x$-conflict $\{u, w\}$ is resolved in $G$, $\{u, w\}$ is resolved.

Proof sketch. We may assume that $u$ and $w$ are responsible for $\{u, w\}$. Let $v_i = c_x(u)$ and $v_j = c_x(w)$, with $i < j$. Since $S_x$ is good, graph $D_x(i, j)$ has at most two connected components $C_1$ and $C_2$. Assume that $v_i \in C_1$.

Suppose first that $v_j \in C_1$. Let $u'$ be the right neighbor of $u$ in $H$; see Fig. 5a. Since $v_i, v_j \in C_1$, node $c_x(u')$ precedes $c_x(w)$ in $S_x$ and $c_x(u') \in C_1$. So, $u'$ lies to the left of $w$ in any drawing of $H$ respecting $S_x$. Hence, the mid-point of edge $(u, u')$, which defines the right boundary of cell($u$), lies to the left of $w$, and thus $w \notin$ cell($u$). Symmetrically, $u \notin$ cell($w$).

Suppose now that $v_j \in C_2$. By symmetry, we assume that $C_1$ lies above and to the left of $C_2$; see Fig. 5b. Let $z$ be the bottommost (topmost) vertex of the vertical path corresponding to the last node $c_x(z)$ of $C_1$ (first node $c_x(r)$ of $C_2$) in $S_x$, i.e., $z$ and $r$ are responsible for a minimal (and thus resolved) $x$-conflict.

We show that $w \notin$ cell($u$) (by symmetry, $u \notin$ cell($w$)). If $u' \in C_1$, the previous case applies. Otherwise, $u$ lies on the right boundary of $C_1$. If $c_x(u)$ is a sink of $C_1$, then $C_1$ contains only $c_x(u)$; see Fig. 5c. Thus, either $c_x(u)$ is not a sink of $C_1$, or $c_x(u) = c_x(z)$. In the latter case, $r \notin$ cell($u$), since the minimal $x$-conflict $\{z, r\}$ is resolved, which implies $w \notin$ cell($u$). Hence, $c_x(u) \neq c_x(z)$ and $c_x(u)$ is not a sink of $C_1$; see Fig. 5d. Since $u' \notin C_1$ and $u$ is south-oriented, $u$ lies below $z$. Let $z'$ be the right neighbor of $z$, with $c_x(z') = v_k$. If $z'$ is to the left of $u'$, $D_x(i, k)$ has two connected components; one contains $v_i$ and $v_k$, the other contains $v_j$, as $(u, u')$ cannot be crossed. This contradicts (S.2) as $i < j < k$. So, $z'$ is to the right of $u'$. Since $z$ is to the right of $u$, the right boundary of cell($z$) is to the right of the one of cell($u$). Since $r \notin$ cell($z$), also $r \notin$ cell($u$), and thus $w \notin$ cell($u$).

Lemma 7. For any two vertices $u$ and $w$ of $H$ such that $\{u, w\}$ is a minimal $x$-conflict, we have that $c_x(u)$ and $c_x(w)$ are consecutive in a good $st$-ordering $S_x$.

Proof. Suppose that there is a vertex $z \in H$ such that $c_x(z) = v_j$ lies between $c_x(u) = v_i$ and $c_x(w) = v_k$ in $S_x$, i.e., $i < j < k$. If $c_x(u)$ and $c_x(w)$ belong to the same connected component $C$ of $D_x(i, k)$, then by definition, $v_i$ and $v_k$ are a source and a sink of $C$, respectively. Since $\{u, w\}$ is an $x$-conflict, we have...
We now present our algorithm to assign $x$-coordinates so that all minimal $x$-conflicts are resolved. We extend some definitions from vertices of $H$ to nodes of $D_x$. Namely, we say $v_i \prec_x v_j$ if there is a directed path in $D_x$ from $v_i$ to $v_j$. Also, we say that there is a (minimal) $x$-conflict $\{v_i, v_j\}$ in $D_x$ if there is a (minimal) $x$-conflict $\{u, w\}$ in $H$ such that $c_x(u) = v_i$ and $c_x(w) = v_j$.

For $0 < i, j < m$, let $x_{i,j} := x_j - x_i$ be the $x$-distance between $v_i$ and $v_j$. To prove that a good $st$-ordering $S_x$ allows for a greedy realization, we set up a system of inequalities describing the geometric requirements for the $x$-distance of consecutive nodes in $S_x$ in a greedy drawing, and then prove that this system always admits a solution since $S_x$ is good. First note that, for every $0 < i < m$ such that there is no minimal $x$-conflict $\{v_i, v_{i+1}\}$, we only require the $x$-distance to be positive, so we define the trivial inequality $x_{i,i+1} > 0$.

For every $0 < i < m$ such that there is a minimal $x$-conflict $\{v_i, v_{i+1}\}$, we define two inequalities that describe the necessary conditions for the $x$-conflict to be resolved. Let $u$ and $w$, with $c_x(u) = v_i$ and $c_x(w) = v_{i+1}$, be responsible for $\{v_i, v_{i+1}\}$. We assume that $u \prec_w w$; the other case is symmetric.

By assumption, $v_i$ lies to the bottom left of $v_{i+1}$, so we only have to consider the part $\text{cell}_x(w)$ of $\text{cell}(w)$ to the bottom left of $w$ (dark region in Fig. 6). Let $(w', w)$ be the bottommost incoming edge of $v_{i+1}$ with $c_x(w') = v_{i+1}$. Then, the left boundary of $\text{cell}_x(w)$ is delimited by the vertical line through the midpoint of $(w', w)$. Thus, we require $x_{i,i+1} > x_{\ell_{i,i+1},i+1}/2 \iff x_{i,i+1} > x_{\ell_{i,i+1},i}$. Symmetrically, we only consider the part $\text{cell}_x(u)$ of $\text{cell}(u)$ to the top right of $u$ (light region in Fig. 6), which is bounded by the vertical line through the midpoint of the topmost outgoing edge $(u, u')$ of $v_i$ with $c_x(u') = v_r$. Thus, we require $x_{i,i+1} > x_{i,r}/2 \iff x_{i,i+1} > x_{i+1,r}$. Since $v_{\ell_{i+1}}$ and $v_i$ (and $v_{i+1}$ and $v_r$) are not necessarily consecutive in the $st$-ordering, we express the $x$-distance $x_{i,i+1}$ (and $x_{i,r}$) as the sum of the $x$-distances between the consecutive nodes between them in the $st$-ordering. This gives the left and the right inequality.

Fig. 6: Solving the inequalities of $x_{i,i+1}$ implies $u \not\prec w$ and $w \not\prec cell(u)$.

Fig. 7: The relation graph defined by the left and right inequalities.
We express the left and trivial (right and trivial) inequalities as $Ax > 0$ (as $Bx > 0$). Any vector $x > 0$ determines a unique rectilinear drawing: we assign to each vertex the $y$-coordinate defined by $S_y$, we assign to $v_1$ the $x$-coordinate $x_1 = 0$ and to every other $v_i$ the $x$-coordinate $x_i = x_{i-1} + x_{i-1,j}$. Since $x > 0$, the $x$-coordinates preserve the good $st$-ordering and resolve all $x$-conflicts.

**Lemma 8.** A vector $x = (x_{1,2}, \ldots, x_{m-1,m})^T > 0$ solves both $Ax > 0$ and $Bx > 0$ if and only if it determines a drawing where all $x$-conflicts are resolved.

Note that we can always solve $Ax > 0$ and $Bx > 0$ independently by solving the linear equation systems $Ax = 1$ and $Bx = 1$ via forward substitution, since $A$ and $B$ are triangular. We prove that there is always a vector $x > 0$ solving $Ax > 0$ and $Bx > 0$ simultaneously. Let $C = A + B - I_{m-1}$ be the matrix defined by the values of $c_{i,j}$. Any solution to the linear inequality system $Cx > 0$ is also a solution to both $Ax > 0$ and $Bx > 0$. We show that $C$ can be triangulated. For this, we define the relation graph corresponding to the adjacency matrix $I_{m-1} - C$ that contains a vertex $u_i$ for each interval $x_{i,i+1}$, $1 \leq i < m$, and a directed edge from a vertex $u_i$ to a vertex $u_j$ if and only if $c_{i,j} = 1$; see Fig. 7.

**Lemma 9.** The relation graph of a good $st$-ordering is acyclic.

Proof sketch. We show that a shortest cycle $C$ in the relation graph has length 2, finding a “shortcut” for every longer cycle. Then, we analyze the relative order of the $y$-coordinates of the responsible vertices for the two minimal $x$-conflicts in $H$ corresponding to $C$, and find a contradiction for every combination.

**Lemma 10.** The matrix $C$ is triangularizable.

Proof. By Lemma 9, the relation graph described by the matrix $I_{m-1} - C$ is acyclic. Hence, there is a permutation matrix $P$ (corresponding to a topological sort) such that $P(I_{m-1} - C)P^{-1}$ is triangulated with only 0’s on the diagonal. Thus, $PI_{m-1}P^{-1} - PCP^{-1} = I_{m-1} - PCP^{-1}$ is triangulated with only 0’s on the diagonal, so $PCP^{-1}$ is triangulated with only 1’s on the diagonal.
Since $C$ is triangularizable by Lemma 10, the system of linear equations $Cx = 1$ always has a solution, which solves $Ax > 0$ and $Bx > 0$ simultaneously. This concludes the sufficiency proof for Theorem 6.

Note that the given construction ensures that all the coordinates are integer; however, the area of the drawing is in general not minimum. A rectilinear greedy drawing with minimum area respecting the given $st$-orderings can be constructed in polynomial time by solving a linear program that minimizes $\sum_{i=1}^{m-1} x_{i,i+1}$ under the constraints $Ax \geq 1$ and $Bx \geq 1$. We analyze the integrality of the solution and the running time in Appendix D.

**Theorem 7.** Let $H$ be a convex rectilinear representation of a biconnected plane graph and let $S_x$ and $S_y$ be good $st$-orderings of $D_x$ and $D_y$. We can compute a greedy drawing of $H$ that respects $S_x$ and $S_y$ with minimum area in $O(n^2)$ time.

Although minimum, the area of the drawings yielded by our algorithm may be non-polynomial in some cases; Theorem 3 states that there exist convex rectilinear representations (see Fig. 3) whose DAGs admit good $st$-orderings, but there is no combination of them resulting in a succinct greedy drawing, since the solutions of the corresponding system of inequalities are always exponential in the input size. On the contrary, every universal greedy rectilinear representation of an $n$-vertex graph is succinct, since by Corollary 1 it has a (greedy) drawing of minimum area on an integer grid of size $O(n^2)$ [24].

**Theorem 8.** There exist rectilinear representations whose every greedy rectilinear drawing has exponential area, even if $D_x$ and $D_y$ are series-parallel.

When $D_x$ and $D_y$ are series-parallel, the conditions can be tested efficiently.

**Theorem 9.** Let $H$ be a convex rectilinear representation of a biconnected plane graph. If $D_x$ and $D_y$ are series-parallel, we can test in $O(n)$ time if $H$ is greedy realizable. If the test succeeds, a greedy drawing of $H$ is computed in $O(n^2)$ time.

**Proof sketch.** To find a good $st$-ordering for $D_x$, we recursively apply the following procedure. Let $D_x$ be composed of a set of subgraphs $D_1, \ldots, D_k$, forming a parallel or a series composition. If $D_1, \ldots, D_k$ form a parallel composition, then either $k = 2$, or $k = 3$ and $D_3$ is a single edge; otherwise, the graph violates Condition (S.1). If we remove the sink and source from $G$, then, by Condition (S.2) we have a good $st$-ordering if and only if all nodes of $D_1$ precede all nodes of $D_2$, there is exactly one sink in $D_1$, and there is exactly one source in $D_2$. If $D_1, \ldots, D_k$ form a series composition, we construct good $st$-orderings of $D_1, \ldots, D_k$ recursively and merge them in a good $st$-ordering of $D_x$.

\[\square\]

### 5 Open Problems

We introduced rectilinear greedy drawings, i.e., planar greedy drawings in the orthogonal drawing style with no bends. Our work reveals several interesting
open problems. (1) What if we allow bends along the edges? (2) Can we always test in polynomial time whether a planar DAG admits a good st-ordering? (3) Given a biconnected plane graph $G$, what is the complexity of deciding whether $G$ admits a (universal) greedy rectilinear representation? This question pertains the intermediate step of the topology-shape-metrics approach [24].

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Appendix

A Preliminary Notions and Definitions

Planarity. Let $G = (V, E)$ be a graph. A drawing of $G$ is a geometric representation $\Gamma$ of $G$ in the plane such that each vertex $v \in V$ is drawn as a distinct point $p_v$ and each edge $e = (u, v) \in E$ is drawn as a simple curve connecting $p_u$ and $p_v$. $\Gamma$ is planar if no two edges intersect except at their common end-vertices (if they are adjacent). A graph is planar if it admits a planar drawing. A planar drawing $\Gamma$ of $G$ divides the plane into topologically connected regions, called faces. Exactly one face of $\Gamma$ is an infinite region, and it is called the external face of $\Gamma$; the other faces are called internal. Each internal face is described by the counterclockwise sequence of vertices and edges that form its boundary; the external face is described by the clockwise sequence of vertices and edges of its boundary. The description of the set of (internal and external) faces determined by a planar drawing of $G$ is called a planar embedding of $G$. A planar graph $G$ together with one of its planar embeddings is called a plane graph: If $\Gamma$ is a planar drawing of $G$ whose set of faces coincides with that described by the planar embedding of $G$, we say that $\Gamma$ preserves this embedding. We finally recall that a plane graph uniquely determines, for each vertex $v$, a specific clockwise ordering of the edges incident to $v$.

Directed Graphs and Series-Parallel Compositions. A directed acyclic graph, DAG for short, is a directed graph without directed cycles. A node of a DAG with only outgoing (incoming) edges is a source (sink). An st-digraph $D$ is a DAG with a single source and a single sink. An st-ordering of $D$ is a linear order $S = v_1, \ldots, v_m$ of its nodes such that $v_1$ and $v_m$ are the source and the sink of $D$, respectively, and for every directed edge $(v_i, v_j) \in D$, we have that $i < j$. Note that every st-digraph admits an st-ordering, which can be computed in linear time in the size of the graph \[11\]. An st-digraph $D$ is series-parallel if one of the following holds:

(i) $D$ is a single edge $(s, t)$ connecting a source to a sink;
(ii) $D$ is obtained from a set of series-parallel st-digraphs $D_1, \ldots, D_k$ with sources $s_1, \ldots, s_k$ and sinks $t_1, \ldots, t_k$, by identifying $s_1, \ldots, s_k$ into a single node $s$, which becomes the source of $D$, and by identifying $t_1, \ldots, t_k$ into a single node $t$, which becomes the sink of $D$. This operation is a parallel composition;
(iii) $D$ is obtained from a set of series-parallel st-digraphs $D_1, \ldots, D_k$ with sources $s_1, \ldots, s_k$ and sinks $t_1, \ldots, t_k$, by identifying node $t_i$ with $s_{i+1}$, for each $i = 1, \ldots, k - 1$. Here, $s = s_1$ and $t = t_k$ are the source and the sink of the resulting graph $D$. This operation is called series composition.

Orthogonal Drawings and Representations. The concept of (rectilinear) orthogonal drawing has been already defined in the introduction. Here we recall more formally the definition of (rectilinear) orthogonal representation. Suppose
that \( G \) is a plane graph. If \( v \) is a vertex of \( G \) and if \( e_1 \) and \( e_2 \) are two (possibly coincident) edges incident to \( v \) that are consecutive in the clockwise order around \( v \), we say that \( a = (e_1, v, e_2) \) is an **angle at** \( v \) of \( G \) or simply an **angle** of \( G \). Let \( \Gamma \) and \( \Gamma' \) be two rectilinear orthogonal drawings of \( G \) that preserve its planar embedding. We say that \( \Gamma \) and \( \Gamma' \) are **shape equivalent** if for any angle \( a \) of \( G \), the geometric angle corresponding to \( a \) is the same in \( \Gamma \) and \( \Gamma' \). A **rectilinear orthogonal representation** \( H \) of \( G \) is a class of shape equivalent rectilinear orthogonal drawings of \( G \); \( H \) can be described by the embedding of \( G \) together with the geometric value of each angle of \( G \). If \( \Gamma \) is a rectilinear orthogonal drawing within class \( H \), we also say that \( \Gamma \) is a **rectilinear orthogonal drawing of** \( H \). For the sake of simplicity, we will use the term **rectilinear drawing** in place of rectilinear orthogonal drawing and the term **rectilinear representation** in place of rectilinear orthogonal representation.

**Greedy Drawings.** Let \( \Gamma \) be a drawing of \( G \). Given two vertices \( u \) and \( v \) of \( G \), let \( d(u, v) \) denote the Euclidean distance between \( u \) and \( v \) in \( \Gamma \). A path \((v_0, v_1, \ldots, v_k)\) of \( G \) is **distance-decreasing** if \( d(v_{i+1}, v_k) < d(v_i, v_k) \), for \( i = 0, \ldots, k - 1 \). Drawing \( \Gamma \) is **greedy** if for any ordered pair of vertices \( u \) and \( v \), there exists a distance-decreasing path from \( u \) to \( v \).

**B Additional Material for Section 2**

**B.1 Greedy Rectilinear Drawings of non-Biconnected Graphs**

The next result characterizes the trees that have a greedy rectilinear drawing. They are very restricted, thus motivating the study of biconnected graphs only.

**Theorem 10.** A tree \( T \) of vertex degree at most four admits a greedy rectilinear drawing if and only if it has at most four leaves.

**Proof.** Given a leaf \( v \) of \( T \) and a rectilinear drawing \( \Gamma \) of \( T \), we say that \( v \) is **north-oriented** (south-oriented) if \( v \) is above (below) its neighbor in \( \Gamma \). Similarly, \( v \) is **east-oriented** (west-oriented) if \( v \) is to the right (left) of its neighbor in \( \Gamma \). If \( T \) has at least five leaves, then there are at least two leaves \( u \) and \( v \) in \( \Gamma \) that are equally oriented, say north-oriented. This implies that \( \text{cell}(u) \) contains \( v \) or \( \text{cell}(v) \) contains \( u \) (or both). By Theorem 1, \( \Gamma \) is not greedy.

Suppose vice versa that \( T \) has at most four leaves \( u, v, w, z \). A greedy drawing \( \Gamma \) of \( T \) is constructed as follows (see Figs. 9a and 9b). All vertices of the path \( \pi \) between \( u \) and \( v \) in \( T \) are horizontally aligned, so that \( u \) is west-oriented and \( v \) is east-oriented. Let \( \pi' \) be the path connecting \( \pi \) to \( w \) and \( \pi'' \) be the path connecting \( \pi \) to \( z \). All vertices of \( \pi' \) are vertically aligned in such a way that \( w \)

\[ \text{Note that every degree-1 vertex has a single angle of 360 degrees, thus one can avoid to specify it.} \]
Fig. 9: (a) A tree $T$ with four leaves; (b) a greedy rectilinear representation of $T$

is north-oriented, while all vertices of $\pi''$ are vertically aligned in such a way that $z$ is south-oriented. It is immediate to see that $\Gamma$ is a greedy drawing. □

With the same argument used in the proof of Theorem 10 any graph with more than four degree-1 vertices does not admit a greedy rectilinear drawing.

B.2 Convexity and Conflicts of Rectilinear Greedy Representations

In this subsection, we provide the full proofs of Lemmas 1, 2, and 3.

**Lemma 1.** $H$ is greedy realizable only if it is convex.

*Proof.* Suppose first that $H$ has an internal face $f$ that is not a rectangle. This means that there is a vertex $v$ with an angle of 270 degrees inside $f$. Let $\Gamma$ be any rectilinear drawing of $H$. Without loss of generality, assume that moving counterclockwise along the boundary of $f$, we enter $v$ horizontally from west and leave $v$ vertically, towards south (this situation is always achievable by a suitable rotation and/or mirroring of the drawing). Since $\Gamma$ has no bend, there exists a vertex $u$ to the right of $v$ and above $v$ (see Fig. 10a). Therefore, cell$(v)$ contains $u$, which means that $\Gamma$ is not greedy by Theorem 1.

Suppose now that the polygon $P$ defined by the external boundary of $H$ is not orthoconvex. Recall that a simple polygon $P$ is orthoconvex if for any vertical or horizontal line $\ell$, the intersection between $P$ and $\ell$ is either empty or

![Fig. 10: (a) $H$ is not greedy realizable if an internal face is not a rectangle. (b) $H$ is not greedy realizable if the external face is not orthoconvex.](image)
a single segment. Let \( \Gamma \) be any rectilinear drawing of \( H \) and let \( P \) be the polygon defined by the boundary of the external face of \( \Gamma \). If \( P \) is not orthoconvex, we can assume, without loss of generality, that there exists a vertical line \( \ell \) whose intersection with \( P \) consists of at least two segments \( s' \) and \( s'' \). Suppose that \( s' \) is above \( s'' \). Clearly, \( \ell \) cuts \( P \) into at least three distinct polygons, two of which lie on the same side of \( \ell \), say to the right of \( \ell \). The polygon having \( s' \) as a leftmost side is denoted by \( P' \) while the polygon having \( s'' \) as a leftmost side is denoted by \( P'' \). Refer to Fig. 10 for an illustration. Let \( r' \) be a rightmost vertical side of \( P' \) and let \( r'' \) be a rightmost vertical side of \( P'' \); also, denote by \( x(r') \) and \( x(r'') \) the \( x \)-coordinate of \( r' \) and the \( x \)-coordinate of \( r'' \), respectively. Assume first that \( x(r') \geq x(r'') \) and let \( p \) be the topmost point on \( r'' \). Clearly, \( \Gamma \) has a vertex \( v \) at point \( p \), and \( v \) is a degree-2 vertex that forms an angle of 270 degrees on the external face of \( \Gamma \). Also, since \( P' \) and \( P'' \) cannot intersect, there must be at least a vertex \( u \) of \( \Gamma \) on \( P' \) that is above \( v \) and not to the left of \( v \). It follows that \( cell(v) \) contains \( u \), and hence \( \Gamma \) is not greedy. The case in which \( x(r') < x(r'') \) is handled symmetrically, choosing \( p \) as the bottommost point on \( r' \). \( \square \)

To present the proof of Lemma 2, we give an auxiliary lemma.

**Lemma 11.** Let \( \Gamma \) be a drawing of a convex rectilinear representation \( H \). For two distinct nodes \( c_x(u) \) and \( c_x(v) \) of \( D_x \) such that there exists a horizontal line crossing both the vertical paths corresponding to \( c_x(u) \) and to \( c_x(v) \) in \( \Gamma \), we have that \( u \sim_x v \). A symmetric property holds for the nodes of \( D_y \).

**Proof.** We give the proof for the first part of the statement; the argument for the second part is symmetric. Consider a horizontal line crossing both the vertical paths corresponding to \( c_x(u) \) and \( c_x(v) \) in \( \Gamma \). Let \( s_x \) be the portion of this line between the two vertical paths. If \( s_x \) does not traverse any face, then it overlaps with a set of horizontal edges in \( \Gamma \). Thus, there is a path in \( D_x \) between \( c_x(u) \) and \( c_x(v) \). Otherwise, since every face of \( H \) is rectangular, there exists a path in \( D_x \) between \( c_x(u) \) and \( c_x(v) \) whose internal vertices are the vertical paths containing the vertical edges of the faces traversed by \( s_x \). \( \square \)

**Lemma 2.** For any two vertices \( u \) and \( v \) of a convex rectilinear representation \( H \), at least one of the following holds: (i) \( u \sim_x v \) or (ii) \( u \sim_y v \).

**Proof.** If \( c_x(u) = c_x(v) \), then \( u \) and \( v \) belong to the same vertical path, and thus \( u \sim_y v \). The case \( c_y(u) = c_y(v) \) is symmetric. Suppose now that \( c_x(u) \neq c_x(v) \) and \( c_y(u) \neq c_y(v) \). Let \( \Gamma \) be any drawing of \( H \), and assume that \( u \) is below and to the left of \( v \); the other cases are symmetric. Consider a maximal path \( \pi = (u_1, \ldots, u_k) \) in \( H \), with \( u_1 = u \), such that for each edge \( (u_i, u_{i+1}) \), \( u_i \) lies either below or to the left of \( u_{i+1} \). Since all internal faces of \( H \) are rectangles, \( u_k \) is a top-right corner of \( H \) on the external face.

Note that, if there is a node \( u_i \) in \( \pi \) such that \( c_x(u_i) = c_x(v) \), then \( u \sim_x v \) and \( u \sim_y v \). Otherwise, \( \pi \) crosses either the horizontal line \( \ell_h \) through \( v \) (to the left of \( v \)) or the vertical line through \( v \) (below \( v \)). In the former case, let \( u_i \) be a node of \( \pi \) such that the vertical path containing \( u_i \) is crossed by \( \ell_h \). Note that,
by the construction of $\pi$, either $c_x(u) = c_x(u_i)$ or $u \prec_x u_i$. Also, by Lemma 11, $u_i \prec_x v$. By transitivity, $u \prec_x v$. In the latter case, a symmetric argument is used to prove that $u \prec_y v$. This concludes the proof of the lemma.

**Lemma 3.** Let $H$ be a convex rectilinear representation of a biconnected graph. A rectilinear drawing $\Gamma$ of $H$ is greedy if and only if every conflict is resolved in $\Gamma$.

**Proof.** By Theorem 1, drawing $\Gamma$ is greedy if and only if for any vertex $v$ of $H$, we have that cell($v$) contains no vertex distinct from $v$. This already proves the necessity, since a conflict that is not resolved implies that a vertex lies in the cell of another vertex, by definition.

We now prove the sufficiency. First note that, if $v$ is a vertex on the external face, then the portion of cell($v$) that belongs to the external face is empty, since the external boundary defines an orthoconvex polygon. Also, since all internal faces of $H$ are rectangles, there is no internal angle of 270 degrees. Further, by Property 1 if two edges incident to a vertex $v$ create an angle of 90 degrees, then the portion of cell($v$) delimited by these two edges is always empty. Thus, the only possible vertices $v$ whose cell cell($v$) may be non-empty in $\Gamma$ are the flat vertices. Let $v$ be a flat vertex, and assume that the flat angle at $v$ is south-oriented (the other cases are symmetric). Consider any other vertex $u$. If $u$ and $v$ are not in conflict, then either $u \prec_x v$ or $v \prec_x u$, say the former. Then, $u$ lies to the left of $v$ in $\Gamma$; also, it does not lie to the right of the left neighbor of $v$, which implies that it lies to the left of cell($v$). Finally, if $u$ and $v$ are in conflict, then this conflict is resolved, by the assumption of the lemma. Hence, $u \notin$ cell($v$) by definition. Hence, $\Gamma$ is greedy.

**B.3 Dilation of Greedy Rectilinear Representations**

Here we report the proof of Theorem 2, which shows that the dilation of greedy rectilinear representations is always bounded by a small constant. We remark that this property is not guaranteed for general convex greedy drawings. For instance, Fig. 11 depicts a greedy (non-orthogonal) convex drawing of a biconnected planar graph in which all distance-decreasing paths from a vertex $s$ to a vertex $t$ have unbounded dilation.

Let $s$, $t$ be two vertices in a drawing $\Gamma$. We recall that $d(s,t)$ denotes the Euclidean distance between $s$ and $t$. Also, in the following $d_M(s,t)$ denotes the Manhattan distance between $s$ and $t$. Finally, a path from $s$ to $t$ in $\Gamma$ will be called an $s$-$t$-path.

**Fig. 11:** The bold zigzag path is the shortest distance-decreasing $s$-$t$-path.
Theorem 2. In a rectilinear greedy drawing on an integer grid, for every two vertices $s, t$ there is a distance-decreasing $s$-$t$-path of length at most $3\sqrt{2} \cdot d(s, t)$.

Proof. We prove that for every two vertices $s$ and $t$ in $\Gamma$ there exists a distance-decreasing $s$-$t$-path of length at most $3d_M(s, t)$. Then, the statement will follow by the fact that $d_M(s, t) \leq \sqrt{2}d(s, t)$. We use induction on $d_M(s, t)$, which is always an integer number as $\Gamma$ has integer vertex coordinates. First, note that in $\Gamma$, every vertex $s$ is connected to every vertex that is closest to $s$ with respect to the Euclidean distance $[22]$.

In the base case $d_M(s, t) = 1$, we have that $x(s) = x(t)$ or $y(s) = y(t)$. Since $t$ is the closest vertex to $s$, we have that $s$ and $t$ are adjacent in $\Gamma$, and the statement trivially holds.

Suppose now that $d_M(s, t) > 1$ and that the statement holds for every pair of vertices whose Manhattan distance is less than $d_M(s, t)$. If $x(s) = x(t)$ or $y(s) = y(t)$, then there must be in $\Gamma$ a distance-decreasing straight $s$-$t$-path (horizontal or vertical), otherwise $\Gamma$ would not be greedy. In this case, the length of this path equals $d_M(s, t) = d(s, t)$. Suppose now that $s$ and $t$ are not horizontally or vertically aligned. Without loss of generality, let $t$ lie to the right of $s$ and below it. Recall that $R(s, t)$ denotes the bounding box of $s$ and $t$. We distinguish between the following cases:

Case 1. There is a vertex $v \neq s$ on the top or left boundary of $R(s, t)$. Then, $s$ is connected to $v$ by a straight path. Since $d_M(v, t) < d_M(s, t)$, by induction there exists a distance-decreasing $v$-$t$-path of length at most $3d_M(v, t)$. Concatenating this $v$-$t$-path and the straight $s$-$v$-path creates a distance-decreasing $s$-$t$-path of length at most $d_M(s, v) + 3d_M(v, t) < 3(d_M(s, v) + d_M(v, t)) = 3d_M(s, t)$.

Case 2. There is no vertex $v \neq s$ on the top or left boundary of $R(s, t)$. Consider the shortest distance-decreasing $s$-$t$-path $\rho$. Let $(s, u)$ be the first edge of this path, and assume, without loss of generality, that this edge is horizontal (the other case is symmetric). Then, we have $x(t) < x(u) < x(s) + 2(x(t) - x(s))$ and hence $d_M(u, t) < d_M(s, t)$. Note that, in this case, $s$ cannot have a neighbor $w$ below it, as in this case it would be $y(w) < y(t)$, thus violating Property 1; see Fig. 12a. Then, due to greediness of $\Gamma$, the region delimited by the vertical lines $x = x(s)$, $x = \frac{1}{4}(x(s) + x(u))$, and by the upper horizontal line $y = y(s)$ is a subset of $\text{cell}(s)$, so it contains no vertex other than $s$; see Fig. 12b. Let $v$ be the latest successor of $u$ along $\rho$, such that the $u$-$v$-subpath of $\rho$ is a staircase. The following subcases are possible:

Case 2.1. $v = t$. Then the length of $\rho$ is $x(u) - x(s) + d_M(u, t) \leq 2d_M(s, t) + d_M(u, t) \leq 3d_M(s, t)$.

Case 2.2. $v \neq t$ and $y(v) > y(t)$. Then $x(v) > \frac{1}{2}(x(s) + x(u))$. Furthermore, $x(v) < x(t)$, as otherwise the edge following $v$ would not be distance decreasing; see Fig. 12b. The $s$-$v$-subpath of $\rho$ has length $x(u) - x(s) + x(u) - x(v) + y(s) - y(v) < 3(x(v) - x(s)) + y(s) - y(v) < 3d_M(s, v)$. Since $d_M(v, t) < d_M(s, t)$ by induction, the $v$-$t$-subpath of $\rho$ has length at most $3d_M(v, t)$. Therefore, the length of $\rho$ is at most $3d_M(s, v) + 3d_M(v, t) = 3d_M(s, t)$.

Case 2.3. $v \neq t$ and $y(v) \leq y(t)$. If $x(v) < x(t)$, then for the predecessor $v'$ of $v$ on $\rho$, we have $y(v') > y(t)$; see Fig. 12c. Then, Case 2.2 can be applied to $v'$.
Fig. 12: Illustration for the proof of Theorem 2. (a) There is no edge \((s, w)\) with \(y(w) < y(s)\), and (b)–(d) the path from \(u\) to \(v\) is the longest staircase on \(\rho\).

Instead of \(v\). Conversely, assume that \(x(v) \geq x(t)\); see Fig. 12d. In this case, starting from \(t\), we repeatedly go upwards or to the right in \(\Gamma\). We cannot get stuck, since otherwise we have a vertex with no edge to the right and no edge upwards, a contradiction to face convexity. This implies that at some point we reach (intersect) the \(s-v\)-subpath of \(\rho\) by going upwards or to the right, and thus we construct a distance-decreasing \(s-t\)-path shorter than \(\rho\), a contradiction. ☐

C Additional Material for Section 3

In this section, we prove Theorem 5. To this aim we need a corollary and two auxiliary lemmas. Suppose that \(H\) is a rectilinear representation of a biconnected plane graph and suppose that there exists a staircase path from any two vertices \(u\) and \(v\). This immediately implies that \(u\) and \(v\) belong to the same horizontal or vertical path in \(H\) (i.e., \(c_x(u) = c_x(v)\lor c_y(u) = c_y(v)\)) or there exist two directed paths connecting \(c_x(u)\) and \(c_x(v)\) in \(D_x\) and \(c_y(u)\) and \(c_y(v)\) in \(D_y\). Thus, in this case \(H\) is conflict-free. It is not difficult to prove that the reverse is also true, which implies the following alternative characterization as for universal greedy representations, as a corollary of Theorem 3.

**Corollary 2.** \(H\) is universal greedy if and only if there exists a staircase path between any two vertices of \(H\).

**Lemma 12.** Let \(H\) be a universal greedy rectilinear representation of a biconnected plane graph. Let \(H'\) be the rectilinear representation obtained from \(H\) by

Fig. 13: Schematic illustration of \(k\)-reflex vertex additions. The new face introduced by the operation is shaded; the new \(k\)-reflex vertices are in black. (a) \(k = 1\); (b)-(c) \(k = 2\); (d)-(e) \(k = 3\); (f) \(k = 4\).
applying either a \( k \)-reflex vertex addition or a flat vertex addition. \( H' \) is biconnected and it is a universal greedy rectilinear representation.

Proof. \( H' \) is biconnected because subdividing an edge or attaching a simple path between two vertices of a biconnected graph cannot create cutvertices. By definition, a \( k \)-reflex vertex addition creates a new rectangular face and keeps \( H' \) convex. Also, a flat vertex addition does not change the shape of any face of \( H \). Therefore \( H' \) remains convex.

Concerning the property of being universal greedy, consider first a flat vertex addition that subdivides an edge \((u,v)\) into two edges \((u,w)\) and \((w,v)\). W.l.o.g., assume that \((u,v)\) is horizontal and that \( u \) is to the left of \( v \). Since the strip \( S \) between the two lines orthogonal to \((u,v)\) and passing through \( u \) and \( v \), respectively, does not contain vertices in its interior, every vertex \( z \neq w \) is either to the left or to the right of the interior of \( S \). In the former case, a staircase path from \( z \) to \( w \) is obtained by merging a staircase path from \( z \) to \( u \) with edge \((u,w)\). In the latter case, a staircase path from \( z \) to \( w \) is obtained by merging a staircase path from \( z \) to \( v \) with edge \((w,v)\). Thus, if we apply a flat vertex addition, \( H' \) is universal greedy by Corollary 2.

Assume now that \( H' \) is obtained by applying a \( k \)-reflex vertex addition, and suppose that \( k = 4 \); the cases \( k = 1, 2, 3 \) can be treated similarly. This implies that \( u \) and \( v \) are horizontally or vertically aligned. W.l.o.g., assume that \( u \) and \( v \) are horizontally aligned, \( u \) is to the left of \( v \), and the new four reflex vertices are not below the horizontal line \( \ell_{uv} \) through \( u \) and \( v \). Denote by \( \ell_u \) and \( \ell_v \) the vertical lines through \( u \) and \( v \), respectively. Let \( z \) be any vertex of \( H \) and let \( w \) be one of the new four reflex vertices (i.e., \( w \in H' \setminus H \)). Note that \( z \) may coincide with \( u \) or \( v \). Since \( H' \) is convex, \( z \) is in the closed infinite region delimited by \( \ell_{uv}, \ell_u, \ell_v \), whose interior is below \( \ell_{uv} \). Also, observe that the relative left/right position between \( w \) and \( z \) is the same in every rectilinear drawing of \( H' \) (because \( H \) is conflict-free and because the internal face added by the 4-reflex vertex addition is a rectangle). If \( z \) is to the left of \( w \), a staircase path from \( z \) to \( w \) in \( H' \) is obtained by merging a staircase path from \( z \) to \( v \) in \( H \) with the staircase path from \( v \) to \( w \) (which is along the boundary of the new rectangular face). Symmetrically, if \( z \) is to the right of \( w \), a staircase path from \( z \) to \( w \) in \( H' \) is obtained by merging a staircase path from \( z \) to \( u \) in \( H \) with the staircase path from \( u \) to \( w \). Hence, \( H' \) is universal greedy by Corollary 2.

\[\square\]

**Lemma 13.** Let \( H \) be a rectilinear representation of a biconnected plane graph \( G \), such that all internal faces of \( H \) are rectangles. If \( G \) is not a simple cycle then there exists an internal face \( f \) of \( G \) such that:

i. \( f \) is adjacent to the external face of \( G \);

ii. \( f \) contains a degree-2 vertex that is a reflex vertex in the external face of \( H \);

iii. removing from \( G \) all the external degree-2 vertices of \( f \) and their incident edges, \( G \) remains biconnected.

**Proof.** Let \( G^* \) be the weak dual of \( G \), i.e. the node set of \( G^* \) is the set of the internal faces of \( G \), and for each edge \( e \) of \( G \) shared by two internal faces \( f \)
and \( g \), there is a dual edge of \( e \) in \( G^* \) that connects the two nodes corresponding to \( f \) and \( g \). Since \( G \) is biconnected its dual graph is biconnected (see, e.g., [24]); therefore \( G^* \) is (at least) connected, as it is obtained from the dual graph of \( G \) by simply deleting the node corresponding to the external face of \( G \). Also, as soon as \( G \) becomes non-biconnected due to the removal of some edges of the external face, then \( G^* \) becomes disconnected. Indeed, in this case, \( G \) would have a cutvertex \( c \) on the external face, which means that in \( G^* \) there would be no path from any two nodes corresponding to faces that belong to different biconnected components of \( G \) with respect to \( c \). Therefore, it is sufficient to prove that there exists a face \( f \) in \( G \) that verifies properties (i) and (ii), and such that \( G^* \) remains connected after the removal from \( G \) of all the external degree-2 vertices of \( f \). To this aim, we distinguish between two cases, based on whether \( G^* \) is biconnected or simply connected.

**Case 1.** \( G^* \) is biconnected. Since the external boundary of \( H \) is a rectilinear polygon, the external face of \( H \) has at least four reflex vertices. Let \( v \) be one of them and \( f \) be the internal face containing \( v \). Removing from \( G \) all the external degree-2 vertices of \( f \) (included \( v \)) causes the removal of the node corresponding to \( f \) in \( G^* \). Since \( G^* \) was biconnected, it remains connected after such a removal.

**Case 2.** \( G^* \) is connected but not biconnected. Let \( T \) be the block-cutvertex tree of \( G^* \), and let \( C^* \) be the block (i.e., biconnected component) of \( G^* \) corresponding to a leaf node of \( T \). By definition of \( T \), \( C^* \) contains only one cutvertex of \( G^* \), which corresponds to an internal face \( f_c \) of \( G \). Denote by \( F_c \) the set of internal faces of \( G \) distinct from \( f_c \) and whose corresponding nodes of \( G^* \) are in \( C^* \).

It can be seen that there is a face \( f \in F_c \) that contains a reflex vertex in the external face of \( H \). More precisely, let \( s \) be the number of sides of \( f_c \) in \( H \) that are incident to some face of \( F_c \), and let \( r \) be the number of reflex vertices in the external face of \( H \) that belong to some faces of \( F_c \). Since the boundary of the rectilinear representation \( H \) restricted to \( f_c \) is a rectilinear polygon, we have that: (a) if \( s = 1 \), then \( r \geq 2 \); (b) if \( s = 2 \), then \( r \geq 3 \); (c) if \( s \in \{3,4\} \), then \( r \geq 4 \). Hence, removing from \( G \) all the external degree-2 vertices of \( f \) causes the removal of the node corresponding to \( f \) in \( C^* \). Since \( C^* \) was biconnected, it remains connected after such a removal, and \( G^* \) remains connected as well.

We are now ready to prove Theorem 5.

**Theorem 5.** Let \( H \) be a universal greedy rectilinear representation of a biconnected planar graph. \( H \) can be obtained by a suitable sequence of \( k \)-reflex vertex and flat vertex additions, starting from a rectangle.

**Proof.** We prove that there exists a sequence \( H_0, H_1, \ldots, H_r \) (\( r \in \mathbb{N} \)) of universal greedy rectilinear representations such that \( H_0 \) is a rectangle, \( H_r = H \), and \( H_{i+1} \) is obtained by applying either a \( k \)-reflex vertex addition or a flat vertex addition on \( H_i \) (\( i = 0, \ldots, r-1 \)). To this aim, it suffices to show that from each \( H_{i+1} \) we can derive \( H_i \) by applying a reverse operation of either a \( k \)-reflex vertex addition or a flat vertex addition. We distinguish between two cases:

**Case 1.** \( H_{i+1} \) has a flat degree-2 vertex \( w \) on the external face. Let \( u \) and \( v \) be the neighbors of \( w \). Let \( H_i \) be the rectilinear representation obtained from \( H_{i+1} \)
Fig. 14: A sequence of primitive operations that generates a universal greedy rectilinear representation. (a) A single rectangular face; (b) flat vertex addition; (c) 2-reflex vertex addition; (d) 1-reflex vertex addition; (e) 3-reflex vertex additions; (f) 2-reflex vertex addition; (g) 4-reflex vertex addition.

by deleting the edges \((u, w)\) and \((w, v)\), and by adding the edge \((u, v)\) (as a single segment). Clearly, \(H_i\) remains biconnected, convex, and greedy universal. \(H_{i+1}\) is obtained from \(H_i\) by applying a flat vertex addition that subdivides \((u, v)\).

**Case 2.** Every degree-2 vertex on the external face of \(H_{i+1}\) is a reflex vertex. Note that the external face contains at least four reflex vertices. Let \(f\) be an internal face having the properties (i)–(iii) in the statement of Lemma 13 (this lemma guarantees that such a face exists). By the proof of Lemma 13 the external degree-2 vertices of \(f\) form a path \(\pi\), and their removal preserves biconnectivity. Since by hypothesis there is no external flat vertex of degree two in \(H_{i+1}\), all vertices of \(\pi\) are reflex vertices in the external face of \(H_{i+1}\). Also, since \(f\) is rectangular, \(\pi\) is formed by at most \(k\) vertices, with \(k \in \{1, 2, 3, 4\}\).

Now, let \(u\) and \(v\) be the two vertices of \(f\) to which \(\pi\) is attached, and let \(\pi'\) be the path from \(u\) to \(v\) containing all the internal edges of \(f\) (the boundary of \(f\) is the union of \(\pi\) and \(\pi'\)). Since \(H_{i+1}\) is universal greedy, \(\pi'\) cannot contain two vertices with an angle of 90 degrees inside \(f\) (i.e., \(\pi'\) is either a straight-line path or it is an \(L\)-shaped path). Indeed, in such a case, \(u\) and \(v\) would be two flat vertices on opposite sides of \(f\), which, as already observed, contradicts the fact that \(H_{i+1}\) is universal greedy. Let \(H_i\) be the rectilinear representation derived from \(H_{i+1}\) by removing \(\pi\). For the above properties, \(H_i\) remains convex. Also, \(H_{i+1}\) is universal greedy, because \(\pi'\) is a staircase path from \(u\) to \(v\) and thus every staircase path of \(H_{i+1}\) that contains \(\pi\) can be replaced with a staircase path in which \(\pi\) is substituted with \(\pi'\). This proves that \(H_{i+1}\) is obtained from \(H_i\) by applying a \(k\)-reflex vertex addition. \(\square\)
D Additional Material for Section 4

We present in this section the full version of the proofs that have been sketched or omitted in Section 4.

Lemma 4. If $D_x$ or $D_y$ admits no good st-ordering, $H$ is not greedy realizable.

Proof. Let $S_x$ be any st-ordering of $D_x$ that is not good. We prove that $H$ does not admit any greedy drawing respecting $S_x$. Suppose, for a contradiction, that there exists such a greedy drawing $\Gamma$ of $H$. Since $S_x$ is not good, there exist two indices $i$ and $j$, with $1 \leq i < j \leq m$, such that $D_x(i,j)$ consists of at least two connected components.

Let $\ell_A$ be a vertical line with $x$-coordinate between $x(v_{i-1})$ and $x(v_i)$, and let $\ell_B$ be a vertical line with $x$-coordinate between $x(v_j)$ and $v_{j+1}$ in $\Gamma$. Observe that, for a connected component $C$ of $D_x(i,j)$, the following property holds. Consider the smallest rectangle $R(C)$ having its vertical sides along $\ell_A$ and $\ell_B$ and containing all the vertices of $H$ corresponding to nodes of $C$ in its interior; then, every horizontal segment connecting two points on the two vertical sides of $R(C)$ intersects at least a vertical edge between two vertices $u$ and $w$ of $H$ such that $c_x(u) = c_x(w) \in C$.

Let $C_1$ and $C_2$ be two components of $D_x(i,j)$, and consider two vertices $u_1$ and $u_2$ of $H$ such that $c_x(u_1) \in C_1$ and $c_x(u_2) \in C_2$; see Fig. 15a. This implies that $u_1 \lessdot_x u_2$. Thus, by Lemma 2 either $u_1 \lessdot_y u_2$ or $u_2 \lessdot_y u_1$ holds; assume the latter. Consider another pair of vertices $u'_1$ and $u'_2$ of $H$ such that $c_x(u'_1) \in C_1$ and $c_x(u'_2) \in C_2$. By the same argument, either $u'_1 \lessdot_y u'_2$ or $u'_2 \lessdot_y u'_1$ holds; we claim that $u'_2 \lessdot_y u'_1$. Suppose for a contradiction that $u'_1 \lessdot_y u'_2$. Consider the two rectangles $R(C_1)$ and $R(C_2)$ as defined above. Note that, since $u_2 \lessdot_y u_1$ and $u'_1 \lessdot_y u'_2$, we have $R(C_1) \cap R(C_2) \neq \emptyset$, as the rectangle $R(C_1)$ contains $u_1$ and $u'_1$, and thus it contains also $u_2$ and $u'_2$. Therefore, there exist a vertical path corresponding to a node of $C_1$ and a vertical path corresponding to a node of $C_2$ that are crossed by the same horizontal line. By Lemma 1 there exists a directed path in $D_x$ between the two nodes of $C_1$ and $C_2$, contradicting the fact that $C_1$ and $C_2$ are different connected components. Repeating this argument for any pair of vertices, we conclude that there exists a horizontal line-segment $h$ from $\ell_A$ to $\ell_B$ such that all the vertices of $H$ corresponding to nodes of $C_1$ lie above $h$ and all those corresponding to nodes of $C_2$ lie below $h$ in $\Gamma$.

Further, since for each node of $C_1$ there is a flat vertex of $H$ that is south-oriented (the bottommost vertex of the vertical path corresponding to the node of $C_1$), we have that the union of the cells of these flat vertices, restricted to the region below $h$, consists of a rectangle of infinite height spanning at least all the $x$-coordinates between those of the leftmost and of the rightmost node of $C_1$ (see the tiled region in Fig. 15a). Since the same holds for the cells of the north-oriented flat vertices that are the topmost points of the vertical paths corresponding to nodes of $C_2$, we have that all the nodes of $C_1$ are to the left of all the nodes of $C_2$ in $\Gamma$, or vice versa. Therefore, $D_x(i,j)$ contains at least another connected component $C_3$, as otherwise the st-ordering $S_x$ would be good. By the
same argument as before, we can claim that $C_1$, $C_2$, and $C_3$ are separated by horizontal line-segments; we assume that these components appear in this order from top to bottom in $\Gamma$. Also, either all the nodes of $C_3$ lie to the left of all the nodes of $C_2$ in $\Gamma$, or vice versa, and the same holds for the nodes of $C_3$ and of $C_1$; see Fig. 15(b).

Assume that all the nodes of $C_1$ are to the left of all those of $C_2$, which are to the left of those of $C_3$; the other cases are analogous. Let $\ell_C$ be the vertical line that is equidistant from $\ell_A$ and $\ell_B$. We claim that all the nodes of $C_2$ are required to lie to the right of $\ell_C$. Namely, if there is at least a node of $C_1$ to the right of $\ell_C$, this is trivially true since the nodes of $C_2$ are to the right of those of $C_1$, by assumption. Further, if all the nodes of $C_1$ lie to the left of $\ell_C$, let $v_r$ be the bottommost vertex of the vertical path corresponding to the rightmost node of $C_1$. Let $x_r$, $x_A$, $x_B$, and $x_C$ be the $x$-coordinates of $v_r$, $\ell_A$, $\ell_B$, and $\ell_C$, respectively. Assuming all positive $x$-coordinates, we have that $x_C - x_r < x_C - x_A = x_B - x_C$; thus, $x_r + (x_B - x_r)/2 > x_C$. This implies that the right boundary of cell($v_r$) lies to the right of $\ell_C$, since the neighbor of $v_r$ in $H$ with its same $y$-coordinate and with larger $x$-coordinate lies to the right of $\ell_B$, as otherwise $c_x(v_r)$ would not be the rightmost node of $C_1$. Hence, the claim follows, since the nodes of $C_2$ must lie to the right of cell($v_r$). With analogous arguments we can prove that the topmost vertex of the vertical path corresponding to the leftmost node of $C_3$ enforces all the nodes of $C_2$ to lie to the left of $\ell_C$. This results in a contradiction and concludes the proof. □

**Lemma 5.** Let $\Gamma_1$ and $\Gamma_2$ be two drawings of $H$ such that all $x$-conflicts are resolved in $\Gamma_1$ and all $y$-conflicts are resolved in $\Gamma_2$. Then, the drawing $\Gamma_3$ of $H$ in which the $x$-coordinate of each vertex is the same as in $\Gamma_1$ and the $y$-coordinate of each vertex is the same as in $\Gamma_2$ is greedy.

**Proof.** By Theorem 1, in order to prove that $\Gamma_3$ is greedy, it is enough to prove that for any vertex $v$ of $H$, we have that cell($v$) contains no vertex distinct from $v$. Since $H$ is convex, there is no internal angle of 270 degrees. Also, by
Property [4] if two edges incident to a vertex $v$ create an angle of 90 degrees, then the portion of cell($v$) delimited by these two edges is always empty. Thus, the only possible vertices $v$ whose cell cell($v$) may be non-empty in $\Gamma_3$ are those forming one or two flat angles. However, the fact that a vertex lies inside a cell determined by a north-oriented or by a south-oriented flat angle only depends on the $x$-coordinates of the vertices in the drawing; thus, all these cells are empty in $\Gamma_3$ since they are empty in $\Gamma_1$. Analogously, all the cells determined by east-oriented or by west-oriented flat angles are empty in $\Gamma_3$ since they are empty in $\Gamma_2$. This concludes the proof of the lemma.

**Lemma 6.** Let $\Gamma$ be a rectilinear drawing of $H$ respecting $S_x$. If every minimal $x$-conflict dominating an $x$-conflict $\{u, w\}$ is resolved in $\Gamma$, $\{u, w\}$ is resolved.

**Proof.** We may assume without loss of generality that $u$ and $w$ are responsible for $\{u, w\}$. Let $v_i = c_x(u)$ and $v_j = c_x(w)$, with $i < j$. Consider the graph $D_x(i, j)$. Since $S_x$ is good, this graph has at most two connected components $C_1$ and $C_2$. Assume that $v_i \in C_1$.

Suppose first that also $v_j \in C_1$. Consider the right neighbor $u'$ of $u$ in $H$, which exists since $u$ is a flat vertex: see Fig. 16a. Note that node $c_x(u')$ precedes $c_x(w)$ in $S_x$, that is, $c_x(u') \in D_x(i, j)$; in fact, if this were not the case, then $v_j$ would not belong to $C_1$. Thus, $u'$ lies to the left of $w$ in any rectilinear drawing of $H$ respecting $S_x$. Hence, the mid-point of edge $(u, u')$, which defines the right boundary of cell($u$), lies to the left of $w$, which implies that $w \notin$ cell($u$). Symmetrically, we can show that $u \notin$ cell($w$).

Suppose now that $v_j \in C_2$. As in the proof of Lemma 4, we can assume that all the vertices corresponding to nodes of $C_1$ lie above those corresponding to nodes of $C_2$. Further, we can assume that all the nodes of $C_2$ follow all those of $C_1$ in the good $st$-ordering $S_x$. The other combinations of cases are symmetric. Let $z$ be the bottommost vertex of the vertical path corresponding to the last node $c_x(z)$ of $C_1$ in $S_x$; see Fig. 16b also, let $r$ be the topmost vertex of the vertical path corresponding to the first node $c_x(r)$ of $C_2$ in $S_x$. Note that vertices $z$ and $r$ are responsible for a minimal $x$-conflict $\{z, r\}$, which is resolved by assumption. We now show that also $\{u, w\}$ is resolved. In particular, we show that $w \notin$ cell($u$); the argument for $u \notin$ cell($w$) is symmetric.

---

**Fig. 16:** Illustration for the proof of Lemma 6. (a) $w \notin$ cell($u$) and $u \notin$ cell($w$); (b) the $x$-conflict $\{z, r\}$ is resolved; (c) $c_x(u)$ and $c_x(z)$ are sinks in $D_x(i, j)$; and (d) $c_x(u)$ is not a sink in $C_1$. 

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First observe that, if the right neighbor \( u' \) of \( u \) in \( H \) belongs to \( C_1 \), \( \text{cell}(u) \) does not extend beyond \( c_x(z) \); since every node of \( C_2 \) is completely to the right of \( c_x(z) \), we have \( w \notin \text{cell}(u) \); see Fig. 16d. Thus, we assume that \( u \) lies on the right boundary of \( C_1 \), i.e., its right neighbor \( u' \) does not belong to \( C_1 \). Note that, if \( c_x(u) \) is also a sink of \( C_1 \), then \( C_1 \) does not contain any other node other than \( c_x(u) \), since \( c_x(u) \) is the first node of \( C_1 \); see Fig. 16c. Thus, either \( v_i \) is not a sink of \( C_1 \), or \( c_x(u) = c_x(z) \). In the latter case, \( r \notin \text{cell}(u) \), since the minimal \( x \)-conflict \( \{ z, r \} \) is resolved, which implies \( w \notin \text{cell}(u) \). Hence, it remains to consider the case that \( c_x(u) \neq c_x(z) \) and \( c_x(u) \) is not a sink of \( C_1 \); see Fig. 16d. This implies that there is a directed path from \( c_x(u) \) to \( c_x(z) \) in \( C_1 \).

Since \( u' \notin C_1 \) and since \( u \) is a south-oriented flat vertex, \( u \) lies below \( z \). Consider the right neighbor \( z' \) of \( z \). Recall that \( u' \) lies to the right of \( w \). Assume first that \( z' \) lies to the left of \( u' \), and let \( c_x(z') = v_k \). Consider now the graph \( D_x(i, k) \) from \( v_i \) to \( v_k \). This graph contains two connected components, one containing \( c_x(u) = v_i \) and \( c_x(z') = v_k \), and another one containing \( c_x(w) = v_j \), due to the presence of the edge \(( u, u') \), which cannot be crossed. However, this implies a contradiction to Condition \( [S.2] \) of a good \( st \)-ordering, since \( i < j < k \). Thus, \( z' \) must lie to the right of \( u' \); since \( z \) is to the right of \( u \), the right boundary of \( \text{cell}(z) \) is to the right of the right boundary of \( \text{cell}(u) \). Hence, the fact that \( r \notin \text{cell}(z) \) implies that \( r \notin \text{cell}(u) \), and thus \( w \notin \text{cell}(u) \).

\[ \square \]

**Lemma 8.** A vector \( x = (x_{1,2}, \ldots, x_{m-1,m})^\top > 0 \) solves both \( Ax > 0 \) and \( Bx > 0 \) if and only if it determines a drawing where all \( x \)-conflicts are resolved.

**Proof.** First, suppose that \( x \) solves both \( Ax > 0 \) and \( Bx > 0 \). Let \( \{v_i, v_j\} \) be a minimal \( x \)-conflict. By Lemma 7, we have \( j = i + 1 \). Consider the \( i \)-th row \( a_i \) in matrix \( A \). By definition of \( A \), we have that \( a_i \cdot x > 0 \) is equivalent to the left inequality of \( \{v_i, v_j\} \). If the left inequality of \( \{v_i, v_j\} \) is resolved, then \( v_i \) lies outside the cell of \( v_j \); see Fig. 6. Analogously, the \( i \)-th row \( b_i \) in matrix \( B \) gives the right inequality \( b_i \cdot x > 0 \) of \( \{v_i, v_j\} \), which implies that \( v_j \) lies outside the cell of \( v_i \). Hence, the minimal \( x \)-conflict \( \{v_i, v_j\} \) is resolved and, by Lemmas 3 and 6, all \( x \)-conflicts are resolved in \( \Gamma' \).

Now, suppose that \( x \) does not solve both \( Ax > 0 \) and \( Bx > 0 \); w.l.o.g., assume that some row \( a_i \) of \( A \) is not resolved, that is, we have \( x_{i+1} \leq \sum_{j=i+1}^{i+1} x_{j,j+1} = x_{i+1,i+1} = x_{i,i+1,i+1} - x_{i,i+1} \). Then, \( v_i \) lies in cell(\( v_{i+1} \)), so the drawing determined by \( x \) is not greedy.

\[ \square \]

**Lemma 9.** The relation graph of a good \( st \)-ordering is acyclic.

**Proof.** Let \( S_x = v_1, \ldots, v_m \) be a good \( st \)-ordering of the DAG \( D_x \), let \( A, B, C \) be the matrices as defined above, and let \( u_1, \ldots, u_m \) be the vertices of its relation graph. We call a directed edge \((u_i, u_j)\) a left edge if \( i < j \) and a right edge otherwise. Note that a left (right) edge corresponds to a part of a left (right) inequality. We first have to prove the following property for the values \( c_{i,j} \) of the matrices \( A, B, C \).

**Property 2.** For any \( 0 < i < j < k < m \), we have \( c_k,j \leq c_{k,i} \) and \( c_{i,j} \leq c_{i,k} \).
Assume that $0 = c_{k,j} > c_{k,i} = -1$. By definition of $c_{k,i}$, we have that $i \geq \ell_{k+1}$. But then $j > i \geq \ell_{k+1}$ implies $c_{k,j} = -1$. Furthermore, assume that $0 = c_{i,k} > c_{j,-1} = -1$. By definition of $c_{i,j}$, we have that $k \leq r_i$. But then $j < k \leq r_i$ implies $c_{i,k} = -1$. 

Consider the shortest cycle $u_{\lambda_1}, \ldots, u_{\lambda_k}, u_{\lambda_i}$ in the relation graph. Obviously, there is at least one left edge and at least one right edge in the cycle. We will first show that this shortest cycle has length 2. W.l.o.g. assume that $(u_{\lambda_1}, u_{\lambda_2})$ is a right edge; the other case is symmetric. Let $i$ be the smallest number such that $(u_{\lambda_i}, u_{\lambda_i+1})$ is a left edge. Then $\lambda_1 < \ldots < \lambda_i$, and we consider three cases.

**Case I.** $\lambda_{i+1} = \lambda_{i-1}$. Then there is a cycle $u_{\lambda_{i-1}}, u_{\lambda_i}, u_{\lambda_{i+1}} = u_{\lambda_{i-1}}$ of length 2. 

**Case II.** $\lambda_{i+1} < \lambda_{i-1} < \lambda_i$. The edge $(u_{\lambda_i}, u_{\lambda_{i+1}})$ is a left edge, so $c_{\lambda_i, \lambda_{i+1}} = -1$. However, by Property 3 $c_{\lambda_i, \lambda_{i-1}} \leq c_{\lambda_i, \lambda_{i+1}} = -1$, so there must be a left edge $(u_{\lambda_i}, u_{\lambda_{i-1}})$. Then there is a cycle $u_{\lambda_{i-1}}, u_{\lambda_i}, u_{\lambda_{i-1}}$ of length 2.

**Case III.** $\lambda_{i-1} < \lambda_{i+1} < \lambda_i$. The edge $(u_{\lambda_{i-1}}, u_{\lambda_i})$ is a right edge, so $c_{\lambda_{i-1}, \lambda_i} = -1$. However, by Property 2 $c_{\lambda_{i-1}, \lambda_{i+1}} \leq c_{\lambda_{i-1}, \lambda_i} = -1$, so there must be a right edge $(u_{\lambda_{i-1}}, u_{\lambda_{i+1}})$. Hence, there is a shorter cycle $u_{\lambda_1}, \ldots, u_{\lambda_{i-1}}, u_{\lambda_{i+1}}, \ldots, u_{\lambda_k}$.

From our case analysis, it follows that $k = 2$. Let $\alpha = \lambda_1$ and $\beta = \lambda_2$. Then there are two minimal $x$-conflicts $\{\alpha, \alpha + 1\}$ and $\{\beta, \beta + 1\}$ with $\alpha < \beta$, $\alpha \geq \ell_{\beta+1}$, and $\beta + 1 \leq r_{\beta+1}$. Let $w_\alpha, w_{\alpha+1}, w_\beta, w_{\beta+1}$ be the responsible vertices for these two $x$-conflicts with $c_x(w_\alpha) = v_\alpha$, $c_x(w_{\alpha+1}) = v_{\alpha+1}$, $c_x(w_\beta) = v_\beta$, and $c_x(w_{\beta+1}) = v_{\beta+1}$. Assume that $w_\alpha \prec_\beta w_{\alpha+1}$; the other case is symmetric. By the definition of $x$-conflicts, we have $v_\alpha \not< x v_\beta$ and $v_\beta \not< x v_{\beta+1}$. We first show that we cannot have $\beta = \alpha + 1$.

**Property 3.** $\beta > \alpha + 1$.

**Proof.** Assume that $\beta = \alpha + 1$; see Fig. 17. If there is no directed path between $v_\alpha$ and $v_{\beta+1}$, then the graph $D_x(\alpha, \beta + 1)$ contains three connected components, which contradicts Condition [S.1]. On the other hand, if there is a directed path between $v_\alpha$ and $v_{\beta+1}$, then the graph $D_x(\alpha, \beta + 1)$ contains two connected components: one component that contains exactly $v_\alpha$ and $v_{\beta+1}$, and one component that contains only $v_\beta$; however, since $\alpha < \beta < \beta + 1$, this contradicts Condition [S.2].

Fig. 17: Illustration for the proof of Property [3] $\beta > \alpha + 1$ by assuming that $\beta = \alpha + 1$. (a) $v_\alpha \not< x v_{\beta+1}$; (b) $v_\alpha \prec_\beta v_{\beta+1}$; (c) $v_\alpha \not< x v_{\beta+1}$.
We now show some properties on the existence of directed paths between vertices \( v_\alpha, v_{\alpha+1}, v_\beta, v_{\beta+1} \).

**Property 4.** \( v_\alpha \prec_x v_\beta \).

**Proof.** Assume that \( v_\alpha \not\prec_x v_\beta \). Consider the graph \( D_x(\alpha, \beta + 1) \). If \( v_\alpha \not\prec_x v_{\beta+1} \), then the graph has three connected components, which contradicts Condition [S.1]. Otherwise, there is a connected component that contains \( v_\alpha \) and a connected component that contains \( v_{\beta+1} \), which contradicts Condition [S.2]. \( \square \)

**Property 5.** \( v_{\alpha+1} \prec_x v_{\beta+1} \).

**Proof.** Assume that \( v_{\alpha+1} \not\prec_x v_{\beta+1} \). Consider the graph \( D_x(\alpha, \beta + 1) \). If \( v_{\alpha+1} \not\prec_x v_{\beta+1} \), then the graph has three connected components, which contradicts Condition [S.1]. Otherwise, there is a connected component that contains \( v_{\alpha+1} \) and one that contains \( v_\alpha \) and \( v_{\beta+1} \), which contradicts Condition [S.2]. \( \square \)

**Property 6.** \( v_{\alpha+1} \prec_x v_\beta \).

**Proof.** Assume that \( v_{\alpha+1} \prec_x v_\beta \). Since \( v_{\alpha+1} \prec_x v_\beta \), by Property 4 the graph \( D_x(\alpha, \beta) \) has a connected component that contains \( v_{\alpha+1} \) and one that contains \( v_\alpha \) and \( v_\beta \); a contradiction to Condition [S.2]. \( \square \)

Assume that we know the exact \( y \)-coordinates of every vertex. We now have to analyze the relative positions of the vertices \( w_\alpha, w_{\alpha+1}, w_\beta, w_{\beta+1} \) with \( y \)-coordinates \( \psi_\alpha, \psi_{\alpha+1}, \psi_\beta, \psi_{\beta+1} \). We will show that any choice of \( y \)-coordinates gives a contradiction. Note that \( \psi_\alpha < \psi_{\alpha+1} \) by assumption and \( \psi_\beta \neq \psi_{\beta+1} \) by the \( x \)-conflict \( \{\beta, \beta + 1\} \). Recall that \( \alpha \geq \ell_{\beta+1} \) and \( \beta + 1 \leq r_\alpha \). Further, let \( (w_\alpha, w_{\alpha+1}) \) with \( c_x(w_{\alpha+1}) = v_{\alpha+1} \) be the right horizontal edge of \( w_\alpha \), let \( (w_\beta, w_{\beta+1}) \) with \( c_x(w_{\beta+1}) = v_{\beta+1} \) be the left horizontal edge of \( w_\beta \), let \( (w_\beta, w_\beta) \) with \( c_x(w_\beta) = c_x(w_{\beta+1}) = v_{\beta+1} \) be the left horizontal edge of \( w_\beta \), and let \( (w_{\ell_{\beta+1}}, w_{\beta+1}) \) with \( c_x(w_{\beta+1}) = v_{\beta+1} \) be the left horizontal edge of \( w_{\beta+1} \); refer to the definition of the left and right inequalities. We distinguish between the following cases.

**Case 1.** \( \psi_\beta < \psi_{\beta+1} \). This implies that \( w_\beta \prec_y w_{\beta+1} \) and thus \( w_\beta \) has a north-oriented flat angle and \( w_{\beta+1} \) has a south-oriented flat angle; see Fig. [18]

**Case 1.1.** \( \psi_{\beta+1} > \psi_\alpha+1 \). By Property 4 \( v_{\alpha+1} \prec_x v_{\beta+1} \). If the corresponding path starts in \( v_{\alpha+1} \) at a \( y \)-coordinate \( \geq \psi_{\beta+1} \), then we have that \( \alpha \geq \ell_{\beta+1} \geq \alpha + 1 \), a contradiction; see Fig. [18a]. Otherwise, since \( v_{\beta+1} \) has a south-oriented flat angle, this path has to end at a \( y \)-coordinate \( \geq \psi_{\beta+1} \) and its last segment is a horizontal segment. Hence, the path has to traverse some point with a \( y \)-coordinate \( \psi_{\beta+1} \) and with a \( x \)-coordinate between \( \alpha + 1 \) and \( \beta + 1 \); see Fig. [18b]. However, all of these points lie on the edge \( (w_{\ell_{\beta+1}}, w_{\beta+1}) \), due to \( \ell_{\beta+1} \leq \alpha \), which contradicts planarity.

**Case 1.2.** \( \psi_{\beta+1} = \psi_{\alpha+1} \). Then we have \( \ell_{\beta+1} = \alpha + 1 > \ell_{\beta+1} \); a contradiction.

**Case 1.3.** \( \psi_\beta < \psi_{\beta+1} \prec \psi_{\alpha+1} \); see Fig. [18c]. By Property 6 \( v_{\alpha+1} \prec_x v_\beta \). Since \( w_{\alpha+1} \) has a south-oriented flat angle and \( w_\beta \) has a north-oriented flat angle,
the corresponding path has to traverse some point with \( y \)-coordinate \( \psi_{\beta+1} \) and with \( x \)-coordinate between \( v_{\alpha+1} \) and \( v_\beta \). However, all of these points lie on the edge \((w_\ell_{\beta+1}, w_{\beta+1})\), due to \( \ell_{\beta+1} \leq \alpha \), which contradicts planarity.

**Case 2.** \( \psi_{\beta+1} < \psi_\beta \). This implies that \( w_{\beta+1} \prec_y w_\beta \) and thus \( w_{\beta+1} \) has a north-oriented flat angle and \( v_\beta \) has a south-oriented flat angle; see Fig. 19.

**Case 2.1.** \( \psi_\beta > \psi_{\beta+1} > \psi_{\alpha+1} \). By Property 6 \( v_{\alpha+1} \prec_x v_\beta \). If the corresponding path starts in \( v_{\alpha+1} \) at a \( y \)-coordinate \( > \psi_{\beta+1} \), then we have that \( \alpha \geq \ell_{\beta+1} \geq \alpha+1 \), a contradiction; see Fig. 19a. Otherwise, since \( w_\beta \) has a south-oriented flat angle, this path has to end at a \( y \)-coordinate \( \geq \psi_\beta > \psi_{\beta+1} \) and its last segment is a horizontal segment. Hence, the path has to traverse some point with \( y \)-coordinate \( \psi_{\beta+1} \) and with \( x \)-coordinate between \( \alpha+1 \) and \( \beta \); see Fig. 19b. However, all of these points lie on the edge \((w_\ell_{\beta+1}, w_{\beta+1})\), due to \( \ell_{\beta+1} \leq \alpha \), which contradicts planarity.

**Case 2.2.** \( \psi_{\beta+1} = \psi_{\alpha+1} \). Then we have \( \ell_{\beta+1} = \alpha + 1 > \ell_{\beta+1} \); a contradiction.

**Case 2.3.** \( \psi_{\beta+1} < \psi_{\alpha+1} \); see Fig. 19c. By Property 5 \( v_{\alpha+1} \prec_x v_{\beta+1} \). Since \( v_{\alpha+1} \) has a south-oriented flat angle and \( w_{\beta+1} \) has a north-oriented flat angle, the corresponding path has to traverse some point with \( y \)-coordinate \( \psi_{\beta+1} \) and with \( x \)-coordinate between \( v_{\alpha+1} \) and \( v_{\beta+1} \). However, all of these points lie on the edge \((w_\ell_{\beta+1}, w_{\beta+1})\), due to \( \ell_{\beta+1} \leq \alpha \), which contradicts planarity.

From the above case analysis, it follows that there is no valid \( y \)-coordinate \( \psi_{\beta+1} \) in any rectilinear drawing. Thus, there cannot be any cycle in the relation graph and the proof of the lemma follows.

\[ \square \]

**Theorem 7.** Let \( H \) be a convex rectilinear representation of a biconnected plane graph and let \( S_x \) and \( S_y \) be good st-orderings of \( D_x \) and \( D_y \). We can compute a greedy drawing of \( H \) that respects \( S_x \) and \( S_y \) with minimum area in \( O(n^2) \) time.

**Proof.** By Theorem 6 there is always a rectilinear greedy drawing of \( H \), and we can construct one by solving the linear equality system \( Cx = 1 \) as described above for both \( D_x \) and \( D_y \). Since all inequalities are necessary and sufficient, by Lemma 8 a solution of minimum area will have the form \( x_{i,i+1} = 1 \), if there is no minimal \( x \)-conflict \( \{v_i, v_{i+1}\} \), and \( x_{i,i+1} = \max\{\sum_{j=i+1}^{r_i-1} x_{i,j+1}, \sum_{j=\ell_i+1}^{i-1} x_{j,j+1}\} + \)

---

**Fig. 18:** Illustration for Case 1: (a) \( \psi_\beta < \psi_{\beta+1} \); (a)–(b) The two cases for the path from \( v_{\alpha+1} \) to \( v_{\beta+1} \) in Case 1.1 \( \psi_{\beta+1} > \psi_{\alpha+1} \), in (a) the path starts above \( \psi_{\beta+1} \), while in (b) the path starts below \( \psi_{\beta+1} \). (c) Case 1.3 \( \psi_\beta < \psi_{\beta+1} < \psi_{\alpha+1} \). The path from \( v_{\alpha+1} \) to \( v_\beta \).
1 otherwise. We can find such a solution in quadratic time by solving the following linear program.

\[
\text{minimize } \sum_{i=1}^{m-1} x_{i,i+1} \quad \text{subject to } Ax \geq 1 \text{ and } Bx \geq 1.
\]

Note that the inequalities already imply \( x \geq 1 \). By the acyclicity of the constraints, there is always a solution that satisfies \( a_i \cdot x_i = 1 \) or \( b_i \cdot x_i = 1 \) for each \( 1 \leq i < m \), where \( a_i \) and \( b_i \) correspond to the \( i \)-th row of the matrices \( A \) and \( B \), respectively; hence, the linear program will assign to each \( x_{i,i+1} \) the value \( x_{i,i+1} = \max\{\sum_{j=i+1}^{r-1} x_{j,j+1}, \sum_{j=l_i+1}^{m-1} x_{j,j+1}\} + 1 \), which is an integer.

For the running time, we first have to find all minimal \( x \)-conflicts. To this end, we only have to check whether two consecutive nodes in the \( st \)-orderings have an \( x \)-conflict; this can clearly be done in linear time per node pair, so in \( O(n^2) \) time in total. Then, we have to create the matrices \( A \), \( B \), and \( C \), which have at most \( m - 1 \) rows and columns each (since \( D_x \) and \( D_y \) might have fewer nodes than \( H \)). This takes \( O(n^2) \) time each. In order to triangularize \( C \), we have to compute a topological order on the DAG defined by the adjacency matrix \( I_m - C \); this can be done in \( O(n^2) \) time using, e.g., depth-first search. Finally, we have to solve the linear program, which can be done in polynomial time.

In general, it is not known whether the linear program can be solved in \( O(n^2) \) time; the best-known bound is \( O(n^3.5L^2) \) where \( L \) is the number of bits in the input. However, we can reduce the required runtime for finding a rectilinear greedy drawing of \( H \) with minimum area by solving the inequalities “by hand”.

Let \( A^* = (a_1^*, \ldots, a_m^*)^\top = P A P^{-1} \) and \( B^* = (b_1^*, \ldots, b_m^*)^\top = P B P^{-1} \). Let \( C^* = (a_1^*, b_1^*, \ldots, a_m^*, b_m^*)^\top \) and \( x^* = x P^{-1} \). Obviously, the following linear program is equivalent to the one above and since both \( A^* \) and \( B^* \) are upper triangulated.

Formally, one would have to prove that the constraint matrix is totally unimodular, from which we refrain here since the fact that we obtain an integral solution should be clear.
we can solve it bottom-up two rows at a time in \(O(n^2)\) time.

\[
\text{minimize } \sum_{i=1}^{m-1} x_{i,i+1} \quad \text{subject to } C^*x^* \geq 1
\]

We can also use a more algorithmical approach. We can assign the values to each \(x_{i,i+1}\) already while using the topological sort to triangulate \(I_m - C\); according to this topological sort, we can assign \(x_{i,i+1} = 1\) to all sources of the DAG \(I_m - C\) and the maximum of \(\sum_{j=i+1}^{n-1} x_{j,j+1} + 1\) and \(\sum_{j=i+2}^{n} x_{j,j+1} + 1\) to all non-sources. By this, all linear inequalities are resolved and the minimality follows by the necessity of the constraints. Since the DAG \(I_m - C\) has at most \(O(m^2)\) edges, this algorithm works in \(O(n^2)\) time.

**Theorem 8.** There exist rectilinear representations whose every greedy rectilinear drawing has exponential area, even if \(D_x\) and \(D_y\) are series-parallel.

**Proof.** We first describe a rectilinear representation \(H\), and then we show that it satisfies the properties of the statement; see Fig. 20a. The vertex set of \(H\) consists of four sets \(v_1, \ldots, v_q, w_1, \ldots, w_{q-1}, z_1, \ldots, z_q\), and \(u_2, \ldots, u_q\), connected as follows. Vertices \(v_1, \ldots, v_q\) belong to a vertical path \(\pi_x(v)\), so that they appear in this order from bottom to top. Then, for each \(i = 2, \ldots, q - 1\), we add a horizontal path \(\pi^1_y = u_i, w_i, z_i, v_i\) such that these vertices appear in this left-to-right order. Also, we add a horizontal path \(\pi^q_y = u_q, z_q, v_q\) such that these vertices appear in this left-to-right order. Finally, for each \(i = 1, \ldots, q - 1\), we add a vertical path \(\pi_x^i\) composed of a single edge \((w_i, u_{i+1})\). Observe that \(H\) is convex.

We now consider DAGs \(D_x\) and \(D_y\), and their possible good \(st\)-orderings. The first observation is that \(D_y\) is a directed path from the vertex \(c_y(v_1)\) corresponding to \(\pi_x^1\) to the vertex \(c_y(v_q)\) corresponding to \(\pi_x^q\). Thus, \(D_y\) admits a unique \(st\)-ordering, which is trivially good. As for \(D_x\), it consists of the series-parallel graph depicted in Fig. 20b, whose unique source is the vertex \(c_x(w_1)\) corresponding to \(\pi_x^1\) and whose unique sink is the vertex \(c_x(v_1)\) corresponding to \(\pi_x^q\).
to $\pi_x(v)$. Although $D_x$ admits several st-orderings, we claim that only two of them are good.

Observe that $D_x$ contains a directed path $c_x(w_1), \ldots, c_x(w_{q-1}), c_x(z_q), c_x(v_1)$, and thus these vertices appear in this order in any st-ordering. Thus, the only possible st-orderings differ by the placement of vertices $c_x(z_1), \ldots, c_x(z_{q-1})$. Note that $c_x(z_{q-1})$ must appear after $c_x(w_{q-1})$ in any st-ordering. Thus, if we consider the subgraph of $D_x$ induced by the vertices from the one following $c_x(w_{q-1})$ to the one preceding $c_x(v_1)$ in any st-ordering, we always have a connected component consisting only of $c_x(z_{q-1})$, and another connected component consisting only of $c_x(z_q)$. This implies that no other vertex can be placed after $c_x(w_{q-1})$, as otherwise the resulting st-ordering would not be good. Consider now the subgraph of $D_x$ induced by the vertices from the one following $c_x(w_{q-2})$ to the one preceding $c_x(v_1)$ in any st-ordering. Again, we have already two connected components, namely one consisting only of $c_x(z_{q-2})$ and one consisting of $c_x(w_{q-1}), c_x(z_{q-1})$, and $c_x(z_q)$. This implies that no other vertex can be placed after $c_x(w_{q-2})$, as otherwise the resulting st-ordering would not be good. In particular, this implies that $c_x(w_{q-2}), c_x(z_{q-2})$, and $c_x(w_{q-1})$ are consecutive in any good st-ordering of $D_x$. Repeating this argument for every $i \leq q - 2$, we obtain that $c_x(w_1), c_x(z_1), c_x(w_2), c_x(z_2), \ldots, c_x(w_{q-2}), c_x(z_{q-2}), c_x(w_{q-1})$ are consecutive in any st-ordering of $D_x$. Thus, there exist only two good st-orderings, which only differ for the position of $c_x(z_{q-1})$ with respect to the positions of $c_x(w_q)$ and of $c_x(z_q)$; recall that, in a good st-ordering, $c_x(z_{q-1})$ must appear either before or after both of $c_x(w_q)$ and $c_x(z_q)$.

Assume that $c_x(z_{q-1})$ appears before $c_x(z_q)$ in the good st-ordering; see Fig. 20c. The other case is analogous. For ease of notation, we say $w_q := z_q$.

By the good st-ordering, we have:

$$x(z_i) > x(w_i)$$

for every $1 \leq i < q$. Recall the definitions of minimal $x$-conflict and right inequality from Section 4. For each $1 \leq i < q$, we have a minimal $x$-conflict between $c_x(z_i)$ and $c_x(w_{i+1})$ with $r_{c_x(z_i)} = c_x(v_1)$. This gives us the right inequality:

$$x(w_{i+1}) - x(z_i) > x(v_1) - x(w_{i+1})$$

$$\Leftrightarrow x(v_1) - x(z_i) > 2(x(v_1) - x(w_{i+1})) > 2(x(v_1) - x(z_{i+1}))$$

Hence, we obtain:

$$x(v_1) - x(z_1) > 2(x(v_1) - x(z_2)) > \ldots > 2^{q-1}(x(v_1) - x(z_q)).$$

Since a solution to the right inequalities is necessary for a rectilinear greedy drawing by Lemma 8, any greedy drawing of $H$ must satisfy Equation 3. However, by $q \in \Omega(n)$, this implies a lower bound on the area of $2^{\Omega(n)}$.

**Theorem 9.** Let $H$ be a convex rectilinear representation of a biconnected plane graph. If $D_x$ and $D_y$ are series-parallel, we can test in $O(n)$ time if $H$ is greedy realizable. If the test succeeds, a greedy drawing of $H$ is computed in $O(n^2)$ time.
Proof. By Theorem [6], we need to check whether both \( D_x \) and \( D_y \) admit a good \( st \)-ordering. We show how to check this for \( D_x \) in linear time, the algorithm for \( D_y \) is the same.

Consider the recursive construction of \( D_x \) through series and parallel compositions. For the base case, notice that a graph consisting of a single edge trivially has a good \( st \)-ordering. Let \( D_x \) be composed of a set of subgraphs \( D_1, \ldots, D_k \), forming a parallel or a series composition. If we assume that if \( D_x \) is composed by a parallel (resp. series) composition, then each of \( D_i \) was composed by series (resp. parallel) composition. A construction with this property can be obtained by considering each composition to be maximal.

First note that, if \( D_1, \ldots, D_k \) form a parallel composition, then either \( k = 2 \) or \( k = 3 \) and one of \( D_1, D_2, D_3 \) is a single edge. In fact, let \( s \) and \( t \) be the source and sink of \( D_1, \ldots, D_k \). Thus, for any \( st \)-ordering \( S_x = v_1, \ldots, v_m \) of \( D_x \), it holds that \( s = v_1, t = v_m \), and for each internal vertex \( u \) of a component in \( D_1, \ldots, D_k \), we have \( u = v_q \), for some \( 1 < q < m \). This implies that, for each component in \( D_1, \ldots, D_k \) that is not a single edge, there exists a connected component in \( D_x(x_1, m) \). Hence, both \( k > 3 \) and \( k = 3 \) where none of \( D_1, D_2, D_3 \) is a single edge would violate Condition [S.1] of a good \( st \)-ordering.

Consider now a parallel composition between two vertices \( s \) and \( t \) consisting of exactly two components \( D_1 \) and \( D_2 \) that are not a single edge. Let \( D^o \) denote the subgraph of \( D_x \) induced by the nodes \( V(D_x) \setminus \{s, t\} \). Recall that, by Condition [S.2] all nodes of \( D^o \) must precede all nodes of \( D^o \) in a good \( st \)-ordering, or vice versa. Consider the case in which all nodes of \( D^o \) precede those of \( D^o \), the other one is analogous. We claim that this results in a good \( st \)-ordering only if \( D^o \) has a single sink and \( D^o \) has a single source. This follows from the observation that, for any set of sinks of \( D^o \) and sources of \( D^o \), it is possible to find a pair of nodes \( v_p \) and \( v_q \) in any \( st \)-ordering \( S_x = v_1, \ldots, v_m \) such that each of these sources/sinks define a connected component in \( D_x(p, q) \); thus, if there exist more than two sources/sinks, then there exists no good \( st \)-ordering. On the other hand, if \( D^o \) has only one sink and \( D^o \) only one source, none of the conditions for a good \( st \)-ordering are violated. From the above discussion, it follows that the only two checks to perform are whether either \( D^o \) has only one sink and \( D^o \) only one source, and vice versa. If one of the checks succeeds, we compute a good \( st \)-ordering of \( D^o \) and of \( D^o \), and we merge them according to the result of the check; otherwise, we reject the instance.

When \( D_1, \ldots, D_k \) form a series composition, the number of components \( D_1, \ldots, D_k \) and their structure can be arbitrary. We construct good \( st \)-orderings of \( D_1, \ldots, D_k \) recursively and merge them in a good \( st \)-ordering of \( D_x \).

To conclude, the necessary and sufficient condition for \( D_x \) to have a good \( st \)-ordering is that at every parallel composition either exactly two components are merged or exactly three components are merged and additionally one of them is a single edge; one has a single source and one has a single sink. This condition can be checked in time linear to the number of nodes of \( D_x \). The time complexity for the construction of a rectilinear greedy drawing follows from Theorem [7]. \( \square \)