Algebraic Geometry Approach in Theories with Extra Dimensions
II. Tensor Length Scale, Compactification and Rescaling
in Low-Energy Type I String Theory

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Abstract

In this second part of the paper, dedicated to theories with extra dimensions, a new physical
notion about the "tensor length scale" is introduced, based on the gravitational theories with covariant
and contravariant metric tensor components. Then the notion of "compactification" in low energy
type I string theory is supplemented by the operation of "rescaling" of the contravariant metric
components. For both the cases of "rescaling+compactification" and "compactification+rescaling",
quasilinear differential equations in partial derivatives have been obtained and the corresponding
solutions have been found for the scale (length) function and for the case of a flat 4D Minkowski
space, embedded into a 5D space with an exponential warp factor. A differential equation has been
obtained and investigated also from the equality of the "rescaled" scalar curvature with the usual
one.

1 INTRODUCTION

In [1, 2], the algebraic geometry approach in gravity was developed, based on the important distinction
between covariant and contravariant metric tensor components in the framework of the affine geometry
approach [3, 4].

Also in a previous paper [5], a cubic algebraic equation was proposed, based on the equivalence
between the gravitational Lagrangian with the more generally defined contravariant tensor and the usual
Lagrangian.

In this paper, the same idea shall be exploited. The main difference will be that instead of choosing the
contravariant tensor in the form of the factorized product \( \tilde{g}^{ij} = dX^i dX^j \) and solving the corresponding
algebraic equation with respect to \( dX^i \), now this tensor shall be chosen as \( \tilde{g}^{ij} = lg^{ij} \). Another difference
in comparison with the previous approach will be that instead of solving an algebraic equation, this
time the obtained equation will be considered as a differential equation in partial derivatives, and it will
be solved by means of the known method of characteristics with respect to the function \( l(x) \). This
function is called the "length function" and as explained in the previous part [6], it is a partial case of
the newly introduced notion of a "tensor length scale \( l^k_i(x) \)" which satisfies the relation \( g_{ij}\tilde{g}^{jk} = l^k_i(x) \)
in gravitational theories with separately determined covariant \( g_{ij} \) and contravariant \( \tilde{g}^{jk} \) metric tensor
components.

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One of the main purposes of this paper is to find implementation of the tensor length scale and function in the type I low-energy string theory. For the purpose, the defined in [1,5] "tilda" connection
\[
\bar{\Gamma}^k_{ij} \equiv \frac{1}{2} \hat{g}^{ks} (g_{js,i} + g_{is,j} - g_{ij,s}) = l_m \Gamma^m_{ij},
\]
which turns out to be a linear combination of the Christoffel's connection components \( \Gamma^m_{ij} \) (and therefore is not independent from them), is substituted in the known formule for the "tilda" Riemann scalar curvature
\[
\bar{R} = \bar{g}^{ij} \bar{R}_{ij} = \tilde{g}^{ij} \left( \partial_k \bar{\Gamma}^k_{ij} - \partial_i \bar{\Gamma}^k_{kj} + \bar{\Gamma}^k_{kj} \bar{\Gamma}^l_{ij} - \bar{\Gamma}^m_{kl} \bar{\Gamma}^k_{jm} \right).
\]
which constitutes the gravitational part of the string action. The correctness of such a substitution is ensured by a known theorem from affine geometry [3], the formulation of which and the proof for the simple algebraic relations
\[
M_{ij} \equiv \frac{1}{2} \hat{g}^{ks} (g_{js,i} + g_{is,j} - g_{ij,s}) = l_m \Gamma^m_{ij}
\]
is given, related to the low-energy action of type I string theory in ten dimensions [7, 8, 9, 10].

Further, the coefficients in the type I low-energy string theory action are compared before and after the compactification. Due to the newly introduced operation of "rescaling", in this case not the standard simple algebraic relations
\[
M^2_{(4)} = \frac{(2\pi)^7}{e^{24} m_4 g_4^2} \text{ and } \lambda = \frac{2 \pi V_4 m_4^6}{e^{24}}
\]
and the electromagnetic coupling constant \( g_4 \) are obtained, but a quasilinear differential equation in partial derivatives for the length function \( l(x) \).

Moreover, since the compactification can be performed from the "unrescaled" or from the "rescaled" gravitational part of the string action, two such equations are obtained and the corresponding solutions by the methods of characteristics have been derived. If one assumes that it is irrelevant whether compactification or rescaling is performed at first, then from the two differential equations a simple cubic algebraic equation can be derived. From this equation, important inequalities for the parameters in the low-energy type I string action can be obtained. Since the length function \( l(x) \) does not participate in these inequalities, they might be relevant also for the present theories with extra dimensions.

Another case, not related to type I string theory and to compactification and for which the corresponding solutions have been obtained is the modified gravitational action with the "tilda" connection (1.1) and again under the choice \( \tilde{g}^{ij} = l^2 \hat{g}^{ij} = l \hat{g}^{ij} \) for the contravariant components. For a flat 4D Minkowski space-time metric, embedded in a 5D space of constant (or even non-constant) negative curvature, the following expression for the length function \( l(x) \) as a solution of the quasilinear equation has been found:
\[
l^2 = \frac{1}{1 - \text{const. } e^{24} k \xi y}, \quad \xi = \pm 1
\]
From a physical point of view it is interesting to note that the "scale function" will indeed be equal to one (i.e. we have the usual gravitational theory with \( \tilde{g}^{ij} = g^{ij} \)) for \( \xi = -1 \) and \( y \to \infty \) (the s.c. infinite extra dimensions). However, for \( \xi = +1 \) there will be even a decrease of the "length function" due to the exponential factor in the denominator.

2 TENSOR LENGTH SCALE, RESCALING AND COMPACTIFICATION IN THE LOW ENERGY ACTION OF TYPE I TEN-DIMENSIONAL STRING THEORY

Now an example of the possible application of theories with covariant and contravariant metrics shall be given, related to the low-energy action of type I string theory in ten dimensions [7, 8, 9, 10].
\[
S = \int d^{10}x \left( \frac{m_s^8}{(2\pi)^7 \lambda^2} R + \frac{1}{4} \frac{m_e^6}{(2\pi)^7 \lambda} F^2 + ... \right) = \int d^4x V_6(......)
\]
where \( \lambda \sim \exp(\Phi) \) is the string coupling (as remarked in [10], in the first term the coupling is \( \lambda^2 \), because it is generated by an world-sheet path integral on an sphere and the coupling \( \lambda \) in the second term -
by an world - sheet path integral on the disc), $m_s$ is the string scale, which we can identify with $m_{grav}$. Compactifying to 4 dimensions on a manifold of volume $V_6$, one can identify the resulting coefficients in front of the $R$ and $\frac{1}{4}F^2$ terms with $M_{(4)}^2$ and $\frac{1}{g_s^4}$, from where one obtains [7]

$$ M_{(4)}^2 = \frac{(2\pi)^7}{V_6 m_s^6 g_s^4}; \quad \lambda = \frac{g_s^2 V_6 m_s^8}{(2\pi)^2}. \quad (2.2) $$

The physical meaning of the performed identification is that since the length scale $\sqrt{\alpha}$ of string theory, the volume $V$ of the (Calabi - Yau) manifold and the expectation value of the dilaton field cannot be determined experimentally, they can be adjusted in such a way so that to give the desired values of the Newton’s constant, the GUT (Grand Unified Theory) scale $M_{GUT}$ and the GUT coupling constant $[10]$. It should be stressed that in the weakly coupled heterotic string theory (when there are no different string couplings $\lambda \sim \exp(2\Phi)$ and $\lambda \sim \exp(\Phi)$, but just one), the obtained bound on the Newton’s constant $[10] \frac{G_N}{M_{GUT}^4}$ is too large, but in the same paper $[10]$ it was remarked that “the problem might be ameliorated by considering an anisotropic Calabi - Yau with a scale $\sqrt{\alpha}$ in $d$ directions and $\frac{1}{M_{GUT}}$ in $(6 - d)$ directions”.

Now we shall propose, in the spirit of the affine geometry approach, how such a different metric scale on the given manifold can be introduced by defining more general contravariant tensors. The key idea is that the contraction of the covariant metric tensor $g_{ij}$ with the contravariant one $\bar{g}^{ij} = dX^i dX^j$ gives exactly (when $i = k$) the length interval [5]

$$ l = ds^2 = g_{ij} dX^j dX^i. \quad (2.3) $$

Naturally, for $i \neq k$ the contraction will give a tensor function $l_i^k = g_{ij} dX^j dX^k$, which can be interpreted as a ”tensor” length scale for the different directions. In the spirit of the remark in [10], one can take for example

$$ l_i^k = g_{ij} dX^j dX^k = L_i^k \quad \text{for} \quad i, j, k = 1, ..., d, \quad \text{for} \quad a, b, c = 1, ..., 6 - d. \quad (2.4) $$

For simplicity and as a starting point, further we shall assume that for all indices $i, j, k,...$

$$ l_i^k = l_i \delta_i^k. \quad (2.6) $$

In fact, this will be fulfilled if we assume that the contravariant metric tensor components $\bar{g}^{ij}$ are proportional to the usual inverse contravariant metric tensor $g^{ij}$ with a function of proportionality $l(x)$, i.e. $\bar{g}^{ij} = l(x)g^{ij}$ (it will be called a ”conformal” rescaling). Further we shall call the function $l(x)$ ”a length scale function”.

Our next purpose will be to prove that if one imposes the requirement for invariance of the low - energy type I string action (2.1) under the ”conformal” rescaling, i.e.

$$ S = \int d^{10}x \left( \frac{m_s^8}{(2\pi)^7} \bar{R} + \frac{1}{4} \frac{m_s^6}{(2\pi)^2} \bar{F}^2 \right) = \int d^4x V_6 (....) = $$$$ = \int d^4x \left( M_{(4)}^2 R + \frac{1}{4} \frac{1}{g_s^4} \bar{F}^2 \right). \quad (2.7) $$

then the length scale $l(x)$ will be possible to be determined from a differential equation in partial derivatives. In other words, unlike the previously described in [7, 8, 9, 10] case, when the coefficients in front of $R$ and $\bar{F}^2$ before and after the compactification are identified, here we shall propose another approach to the same problem. Concretely, first a rescaling of the contravariant metric components shall be performed, and after that the compactification shall be realized, resulting again in the R.H.S. of the standard $4D$ action (2.1).
However, in principle another approach is also possible. One may start from the "unrescaled" ten-dimensional action (2.1), then perform a compactification to the four-dimensional manifold and afterwards a transition to the usual "rescaled" scalar quantities $\tilde{R}$ and $\tilde{F}^2$. Then it is required that the "unrescaled" ten-dimensional effective action (2.1) (i.e. the L. H. S. of (2.1)) is equivalent to the four-dimensional effective action after compactification, but in terms of the rescaled quantities $\tilde{R}$ and $\tilde{F}^2$ in the R. H. S of (2.1). This can be expressed as follows

$$S = \int d^{10}x \left( \frac{m_s^8}{(2\pi)^7 \lambda^2} R + \frac{1}{4} \frac{m_s^6}{(2\pi)^7 \lambda} F^2 \right) = \int d^4x V_6 (...,)= \int d^4x \left( M_4^2 \tilde{R} + \frac{1}{4} g_4^2 \tilde{F}^2 \right). \quad (2.8)$$

In the next subsections both cases shall be investigated, deriving the corresponding (quasilinear) differential equations in partial derivatives and moreover, finding concrete solutions of these equations for the special case of the metric of a flat 4D Minkowski space, embedded in a five-dimensional $\text{ADS}$ space of constant negative curvature. It will be shown also that for a definite scale factor $h(y) = \beta y^n$ ($\beta$ is a constant) in front of the extra-coordinate $y$ in the metric, the derived differential equations are still solvable, in spite of the fact that the five-dimensional space is no longer of a constant negative curvature. Besides the opportunity to extend the results to such spaces of non-constant curvature, there is one more reason for the necessity to investigate such quasilinear differential equations for concrete cases - in principle, examples can be given, when such equations cannot be explicitly solved. But evidently, some special kinds of chosen metrics will allow the solution of these equations and consequently the determination of the scale length function $l(x)$ in terms of all the important parameters in the low-energy type I string theory. If for certain metrics this is possible, then it will turn out to be possible to test whether there will be deviations from the standardly known gravitational theory with $l = 1$, if the electromagnetic coupling constant $g_4$, the 4D Planck constant $M_4$, the string scale $m_s$ and the string coupling $\lambda$ are known, presumably from future experiments or cosmological data. Even if one assumes that there are no deviations from the standard theory with $l = 1$, the obtained solutions will allow to find new relations between the above mentioned parameters. The obtained differential equations in the limit of $l = 1$ will result in the simple algebraic relations (2.2), already found in the literature.

One may also require the equivalence of the two approaches, expressed mathematically by (2.7) and (2.8), although for the moment it is not known whether there is some physical reason for this equivalence.

### 3 ALGEBRAIC RELATION AND A QUASILINEAR DIFFERENTIAL EQUATION IN PARTIAL DERIVATIVES FROM THE "RESCALED + COMPACTIFIED" LOW-ENERGY TYPE I STRING ACTION

In order to rewrite the "rescaled+compactified" string action (2.7), let us first define the "rescaled" square of the electromagnetic field strength as

$$\tilde{F}^2 = \tilde{F}_{AB}\tilde{F}^{AB} = F_{AB}\tilde{g}^{AM}\tilde{g}^{BN} F_{MN} = l^2 F_{AB}\tilde{g}^{AM}g^{BN} F_{MN} = l^2 F^2. \quad (3.1)$$

Using the formulae for the Riemann tensor and for the rescaled affine connection

$$\tilde{\Gamma}^D_{AC} = \tilde{g}^{DG} g_{GF} \Gamma^F_{AC} = \Gamma^D_{AC}, \quad (3.2)$$
the rescaled scalar gravitational curvature \( \tilde{R} \) can be written as

\[
\tilde{R} = \tilde{g}^{DG} g_{GF} \tilde{R}_{ABCD} = \frac{1}{2} l^2 g^{AC} g^{BD} (g_{AD,BC} + g_{BC,AD} - g_{AC,BD} - g_{BD,AC})
\]

\[
= l^4 R - \frac{1}{2} l^2 (l^2 - 1) g^{AC} g^{BD} (g_{AD,BC} + g_{BC,AD} - g_{AC,BD} - g_{BD,AC})
\] (3.3)

Substituting the above expressions (3.1) and (3.4) for \( \tilde{F}^2 \) and \( \tilde{R} \) into the L. H. S. of the low - energy string action (2.7) and setting up equal the corresponding coefficients in front of the \( \frac{1}{2} F^2 \) term in the L. H. S. and the R. H. S. of (2.7), one can derive

\[
\lambda = \frac{g_4 m_5^6 V_6}{(2\pi)^7} l^2
\] (3.5)

This is almost the same expression as in (2.2), but now corrected with the function of proportionality \( l^2(x) \).

The string coupling \( \lambda \) is thus a non - local physical quantity, depending on the space - time coordinates.

Next, after the elimination of the terms with \( \frac{1}{2} F^2 \) on both sides of (2.7) and substituting the found formulae for \( \lambda \) into the resulting expression on both sides of (2.7), one derives the algebraic relation

\[
\left[ \frac{(2\pi)^7}{V_6 m_5^4 g_4^2} - M^2_{(4)} \right] R = \frac{(2\pi)^7}{2 m_5^4 V_6 l^2 g_4^2} g^{AC} g^{BD} (\ldots)
\] (3.6)

For brevity, the brackets (\( \ldots \)) will denote the term in (3.4) with the second derivatives of the metric tensor. For \( l = 1 \), as expected, we obtain the usual relation for \( M^2_{(4)} \) as in (2.2). Therefore, physically any possible deviations from relation (2.2) can be attributed to the appearence of the new length scale \( l(x) \). Let us introduce the notation

\[
\beta = \left[ \frac{(2\pi)^7}{V_6 m_5^4 g_4^2} - M^2_{(4)} \right] m_5^4 V_6 \frac{2}{(2\pi)^7}
\] (3.7)

and assume that the deviation from the relation \( M^2_{(4)} = \frac{(2\pi)^7}{V_6 m_5^4 g_4^2} \) is small, i.e. \( \beta \ll 1 \). Then the length scale \( l(x) \) can be expressed from the algebraic relation (3.6) as

\[
l^2 = \frac{1}{1 - \beta \frac{R}{g^{AC} g^{BD} (\ldots)}} \approx 1 + \beta \frac{R}{g^{AC} g^{BD} (\ldots)}
\] (3.8)

Consequently the deviation from the "standard" length scale \( l = 1 \) in the case of a gravitational theory with \( l \neq 1 \) in the case of small \( \beta \) shall be proportional to the ratio \( \frac{R}{g^{AC} g^{BD} (\ldots)} \). In the concrete example of an 4D Minkowski space, embedded in a 5D \( \text{ADS} \) space of constant negative curvature, this ratio will be

\[
\frac{R}{g^{AC} g^{BD} (\ldots)} = \left( \frac{-8k^2}{-32k^2} \right) = \frac{1}{4}
\] (3.9)

and therefore, this constant factor will not affect the smallness of the number \( \beta \frac{R}{g^{AC} g^{BD} (\ldots)} \). The above result has also an important physical meaning - the zero value of the number \( \beta \) (which signifies the fulfillment of the relation \( M^2_{(4)} = \frac{(2\pi)^7}{V_6 m_5^4 g_4^2} \)) is directly connected with the usual length scale \( l = 1 \) in gravity theory.

Let us now derive the differential equation in partial derivatives, starting from the second representation of the "rescaled" scalar gravitational curvature \( \tilde{R} \) by means of the "rescaled" Ricci tensor \( \tilde{R}_{ij} \)

\[
\tilde{R} = \tilde{g}^{AB} \tilde{R}_{AB} = lg^{AB} \left[ \frac{\partial \tilde{\Gamma}^{C}_{AB}}{\partial x^C} - \frac{\partial \tilde{\Gamma}^{C}_{AC}}{\partial x^B} + \tilde{\Gamma}^{C}_{AB} \tilde{\Gamma}^{D}_{CD} - \tilde{\Gamma}^{D}_{AC} \tilde{\Gamma}^{C}_{BD} \right]
\] (3.10)
It can easily be found that the rescaled gravitational curvature is expressed through the usual one as
\[
\tilde{R} = \tilde{g}^{AB} \tilde{R}_{AB} = l^2 R + l^2 (l - 1) g^{AB} (\Gamma^C_{AB} \Gamma^D_{CD} - \Gamma^D_{AC} \Gamma^C_{BD}) + \\
+ l \frac{\partial}{\partial x^C} g^{AB} \Gamma^C_{AB} - l \frac{\partial}{\partial x^E} g^{AB} \Gamma^C_{AC} .
\] (3.11)

Again, this expression and also (2.9) for \( \tilde{F}^2 \) are substituted into the L. H. S. of the action (2.7) and the corresponding coefficients in front of the term \( \frac{1}{4} F^2 \) in the L. H. S. and the R. H. S. of (2.2) are set up equal. Thus one obtains
\[
\lambda^2 = \frac{g_4^4 m_s^{12} V_6 l^4}{(2\pi)^{14}} .
\] (3.12)

Substituting this expression into the resulting one on both sides of (2.2), we receive the following equation in partial derivatives with respect to the scale function \( l(x) \):
\[
\left[ \frac{(2\pi)^7}{m_s^2 V_6 g_4^2 l^2} - M_4^2 \right] R + \frac{(2\pi)^7}{m_s^2 V_6 g_4^2 l^2} g^{AB} (\Gamma^C_{AB} \Gamma^D_{CD} - \Gamma^D_{AC} \Gamma^C_{BD}) + \\
+ \frac{(2\pi)^7}{m_s^4 V_6 g_4^4 l^4} \left[ \frac{\partial}{\partial x^C} g^{AB} \Gamma^C_{AB} - \frac{\partial}{\partial x^E} g^{AB} \Gamma^C_{AC} \right] = 0 .
\] (3.13)

Note that for \( l = 1 \) (the known gravitational theory) we obtain again expression (2.2) for \( M_4^2 \). It turns out that for \( l(x) \) we have both the algebraic relation (3.6) and the above differential equation (3.13).

4 \textbf{(ANOTHER) ALGEBRAIC RELATION AND A QUASILINEAR DIFFERENTIAL EQUATION FROM THE "COMPACTIFIED+RESCALED" LOW ENERGY TYPE I STRING THEORY ACTION}

This time we start from the action (2.8) and substitute the expressions for the "unrescaled" scalar quantities \( F^2 \) and \( \tilde{R} \)
\[
F^2 = \frac{1}{l^2} \tilde{F}^2 ; \quad R = \frac{1}{l^4} \tilde{R} + \frac{(l^2 - 1)}{2l^4} g^{AC} g^{BD} (\ldots)
\] (4.1)
into the L. H. S. of (2.8).

Following the method, described in the previous section, we find for \( \lambda \)
\[
\lambda = \frac{g_4 m_s^6 V_6}{(2\pi)^2 l^2} ,
\] (4.2)
which with respect to the function \( l(x) \) can be considered as the "dual" one, if compared with (3.5). However, the obtained algebraic relation will be different from (3.6)
\[
\left[ \frac{(2\pi)^7}{V_6 m_s^2 g_4^2} - M_4^2 \right] \tilde{R} + \frac{(2\pi)^7 l^2 (l^2 - 1)}{2 m_s^4 V_6 g_4^4} g^{AC} g^{BD} (\ldots) = 0 .
\] (4.3)

If again expression (3.4) for \( \tilde{R} \) is used, the algebraic relation (4.3) can be rewritten as
\[
\frac{1}{2} l^2 (l^2 - 1) g^{AC} g^{BD} (\ldots) = \frac{P^2 R}{(P - NR)^2} + l^4 R ,
\] (4.4)
where \( P \) and \( N \) denote the expressions

\[
P = \frac{(2\pi)^7}{2m_4^7 V_6^3 g_4} g^{AC} g^{BD}(...) \quad ; \quad N = \frac{(2\pi)^7}{m_4^7 V_6 g_4^3} - M_4^2 .
\]  

(4.5)

In the same way, starting from the second representation (3.11) of the gravitational Lagrangian for \( R \) in terms of \( \tilde{R} \) and again making use of formulae (3.11) for \( \tilde{R} \), one can obtain the second quasilinear equation in partial derivatives

\[
\left[ \frac{(2\pi)^7}{m_4^7 V_6 g_4^3} \right] R - \left[ \frac{(2\pi)^7 l^2}{m_4^7 V_6 g_4^3} - M_4^2 \right] \frac{l^2(l-1)}{2} g^{AC} g^{BD}(...) - \\
\frac{(2\pi)^7 l^4 l(l-1)}{m_4^7 V_6 g_4^3} g^{AB} \left( \Gamma^{C}_{AB} \Gamma^{D}_{CD} - \Gamma^{D}_{AC} \Gamma^{C}_{BD} \right) - \\
\frac{(2\pi)^7 l^3}{m_4^7 V_6 g_4^3} \left( \frac{\partial l}{\partial x^C} g^{AB} \Gamma^{C}_{AB} - \frac{\partial l}{\partial x^B} g^{AB} \Gamma^{C}_{AC} \right) = 0 .
\]  

(4.6)

Substituting the algebraic relation (4.4) into the second term of (4.6), the differential equation is obtained in a simpler form

\[
\frac{(2\pi)^7 l^3}{m_4^7 V_6 g_4^3} \left( \frac{\partial l}{\partial x^C} g^{AB} \Gamma^{C}_{AB} - \frac{\partial l}{\partial x^B} g^{AB} \Gamma^{C}_{AC} \right) + \frac{RP^2}{l^3(P - NR)^2} \left[ \frac{(2\pi)^7 l^2}{m_4^7 V_6 g_4^3} - M_4^2 \right] + \\
\frac{(2\pi)^7 l^4 l(l-1)}{m_4^7 V_6 g_4^3} g^{AB} \left( \Gamma^{C}_{AB} \Gamma^{D}_{CD} - \Gamma^{D}_{AC} \Gamma^{C}_{BD} \right) = 0 .
\]  

(4.7)

This (second) differential equation evidently is different from the first one (3.13) and in this aspect an interesting conclusion can be made. Suppose that the two differential equations (3.13) and (4.7) simultaneously hold, which means that it does not matter in what sequence we perform "rescaling + compactification" or "compactification + rescaling" in the low energy type I string theory action. If the initial term with the derivatives in (4.7) is expressed and substituted into the first differential equation (3.13), then the square of the length scale function \( l^2 \) can be found as a solution of the following cubic algebraic equation

\[
M_4^2 l^6 - \frac{(2\pi)^7}{m_4^7 V_6 g_4^3} l^4 + \frac{(2\pi)^7}{m_4^7 V_6 g_4^3} \frac{P^2}{(P - NR)^2} l^2 - M_4^2 \frac{P^2}{(P - NR)^2} = 0 .
\]  

(4.8)

Both the above mentioned approaches of "rescaling + compactification" and "compactification + rescaling" would be consistent in the case of a non-imaginary Lobachevsky space, if the function \( l(x) \) is a real one and not a complex one, i.e. the roots of the above equation should not be imaginary functions and there should be at least one root, which is a real function. This may lead additionally to some restrictions on the parameters in the initial string action. Also, for \( l = 1 \) the above equation can be written as

\[
\left[ M_4^2 - \frac{(2\pi)^7}{m_4^7 V_6 g_4^3} \right] \frac{P^2}{(P - NR)^2} + 1 = 0 .
\]  

(4.9)

There is no other relation from this equation besides the known one \( M_4^2 - \frac{(2\pi)^7}{m_4^7 V_6 g_4^3} = 0 \), since the nominator of the second term can be written as

\[
-2 \left[ \left( \frac{P}{N} \right)^2 - \left( \frac{P}{N} \right) R + \frac{1}{2} R^2 \right] = \frac{1}{2} \left[ \left( \frac{P}{N} - R \frac{1}{2} \right)^2 + R^2 \right] = 0 .
\]  

(4.10)

and evidently this term is positive and different from zero.

However, if solutions of the two quasilinear differential equations are found, then some new relations may be written. It should be kept in mind that these solutions are found by means of the characteristic system of equations, and the general solutions depend on the solutions of the characteristic system.
5 ALGEBRAIC INEQUALITIES FOR THE PARAMETERS IN THE LOW-ENERGY TYPE I STRING THEORY ACTION

Taking into account expressions (4.5) for \( P \) and \( N \), eq.(4.8) after dividing by \( Q^2 M_4^2 m_4^2 V_6 g_4^2 \) can be written in the form of the following cubic algebraic equation with respect to the variable \( l_1 = l^2 \)

\[
l_1^3 + a_1 l_1^2 + a_2 l_1 + a_3 = 0 ,
\]

where \( Q, a_1, a_2 \) and \( a_3 \) denote the expressions

\[
Q = \frac{g^{AC} g_{BD} (2\pi)^7 g_4^4 (\ldots)}{(g^{AC} g_{BD} (2\pi)^7 g_4^4 - 2R((2\pi)^7 - M_4^2 m_4^2 g_4^2))} ,
\]

\[
a_1 = -\frac{(2\pi)^7}{M_4^2 m_4^2 V_6 g_4^2} ; \quad a_2 = \frac{(2\pi)^7}{M_4^2 m_4^2 V_6 g_4^2 Q^2} ; \quad a_3 = -\frac{g_4^2}{Q^2} .
\]

After the variable change \( l_1 = x - \frac{a_1}{3} \) equation (5.1) is brought to the form

\[
x^3 + ax + b = 0 ,
\]

where \( a \) and \( b \) are the expressions

\[
a = a_2 - \frac{a_1^2}{3} ; \quad b = \frac{2a_1^3}{27} - \frac{a_1 a_2}{3} + a_3 .
\]

The roots of the cubic equation (5.4) are given by the formulae

\[
x = \sqrt[3]{p} - \frac{a}{3\sqrt[3]{p}} ,
\]

where \( p \) denotes the expression

\[
p = -\frac{b}{2} \pm \sqrt{\left(\frac{b^2}{4} + \frac{a^3}{27}\right)} .
\]

The roots of the cubic equation will not depend on the + or − sign in front of the square in the above expression.

It may be noted that if the expression for \( \frac{b^2}{4} + \frac{a^3}{27} \) is negative, then the corresponding roots \( x_1, x_2, x_3 \) and the length function \( l(x) \) will be imaginary. From a geometrical point of view, this would be unacceptable, but with one exception - in the imaginary Lobachevsky space [12], which is realized by all the straight lines outside the absolute cone (on which the scalar product is zero, i.e. \( [x,x] = 0 \)), the length may may take imaginary values in the interval \([0, \frac{\pi}{2k}]\) (\( k \) is the Lobachevsky constant). Further we shall assume that \( l(x) \) is a real function, but in principle it is interesting that the sign of the inequalities, relating the parameters in the string action, may change, if the spacetime is an imaginary Lobachevsky one.

The expression for \( \frac{b^2}{4} + \frac{a^3}{27} \) can be written as

\[
\frac{b^2}{4} + \frac{a^3}{27} = \frac{1}{Q^n d^n} g_4^6 d^6 - \frac{1}{4.27} g_4^8 d^2 Q^2 - \frac{2}{27} g_4^{12} Q^6 + \frac{1}{4} g_4^2 Q^2 - \frac{1}{6} g_4^6 Q^2 d^4 + \frac{1}{27} g_4^8 Q^4 d^4} ,
\]
where \( d \) is the introduced notation for

\[
d \equiv \frac{M_s^6 V_0 m_4^2 g_4^4}{(2\pi)^7} .
\]

(5.9)

It is difficult to check when expression (5.8) will be non-negative, since \( Q \) depends also on \( d \) and a higher-degree polynomial with respect to \( d \) will be obtained. However, it may be noted that since

\[
l^2 = l_1 = x - \frac{a_1}{3} > 0
\]

(5.10)

and since \( a_1 \) is a non-complex quantity, then all the roots \( x_1, x_2, x_3 \) are real. Therefore from the Wiet formulae

\[
-a = x_1 + x_2 + x_3 > a_1
\]

(5.11)

and with account of the expressions for \( a \) and \( a_1 \), an equality can be obtained with respect to \( d \) and the electromagnetic constang \( g_4 \) in the type I low energy string theory action

\[
\frac{1}{3} g_4^2 > W(W + 2)d^4 - 2W(W + 1)d^3 + W^2 d^2 ,
\]

(5.12)

where \( W \) is the notation for

\[
W \equiv \frac{2R}{g^{AB} g^{CD} (...) g_4^4} .
\]

(5.13)

The last (third) inequality with the parameters of the low-energy type I string theory action can be obtained from the restriction (5.10) \( x > \frac{a_1}{3} \) for the roots of the cubic equation and expression (5.6) for \( x \)

\[
\frac{3 \sqrt{p^2} - a}{3 \sqrt{p}} > \frac{a_1}{3} .
\]

(5.14)

Denoting

\[
q_1 = \sqrt{p^2} ,
\]

(5.15)

the above inequality can be rewritten as

\[
9q_1^2 - (a_1^2 + 6a)q_1 + a^2 > 0
\]

(5.16)

which is satisfied for

\[
p^2 = \frac{b^2}{2} + \frac{a^3}{27} - b \sqrt{\frac{b^2}{2} + \frac{a^3}{27}} > \left[ \frac{a_1 + 6a}{18} + \frac{a_1}{18} \sqrt{a_1^2 + 12a} \right]^3
\]

(5.17)

or for

\[
p^2 = \frac{b^2}{2} + \frac{a^3}{27} - b \sqrt{\frac{b^2}{2} + \frac{a^3}{27}} < \left[ \frac{a_1 + 6a}{18} - \frac{a_1}{18} \sqrt{a_1^2 + 12a} \right]^3 .
\]

(5.18)

The last two inequalities are the new inequalities between the parameters in the low-energy type I string theory action, which cannot be obtained in the framework of the usual gravity theory. Note that in these inequalities the length function \( l(x) \) no longer appears, so the obtained result should be valid also for the case of standard gravitational theory.

6 SOLUTIONS OF THE FIRST QUASILINEAR DIFFER-
ENTHALY EQUATION (3.13) FOR THE CASE OF A FLAT 4D MINKOWSKI METRIC, EMBEDDED IN A FIVE - DIMENSIONAL SPACE (OF CONSTANT NEGATIVE OR NON - CONSTANT CURVATURE)

The purpose will be to show that the differential equation in partial derivatives (3.13) will be solvable for the previously considered case of the metric (2.4), written now as

$$ds^2 = e^{-2k\varepsilon} \eta_{\mu\nu} dx^\mu dx^\nu + h(y) dy^2$$, (6.1)

$$\eta_{\mu\nu} = (+, -, -, -)$$ with $$h(y) = 1$$ and $$\varepsilon = \pm 1$$. Moreover, the equation will turn out to be solvable also for the case of a power - like dependence of the scale factor $$h(y) = \gamma y^n (\gamma = \text{const})$$, when the five - dimensional scalar curvature is no longer a constant one. In principle, it is necessary to know for what kind of metrics quasilinear differential equations of the type (3.13) admit exact analytical solutions. It may be shown that for more complicated metrics (for example, when the embedded four - dimensional spacetime is a Schwarzschild Black hole with an warp factor) such analytical solutions cannot be found. This will be shown elsewhere.

As usual, the Greek indices $$\mu, \nu, \alpha, \beta$$ will run from 1 to 4 and the extra - dimensional metric tensor component is $$h(y) \equiv g_{55}$$. The big letters $$A, B, C...$$ will denote the coordinates of the five - dimensional spacetime.

The corresponding affine connection components are

$$\Gamma_{\mu\nu}^\alpha = \Gamma_{5\nu}^\alpha = 0 ; \quad \Gamma_{\mu5}^5 = \frac{k\varepsilon}{h} \eta_{\mu\nu} e^{-2k\varepsilon y}$$, (6.2)

$$\Gamma_{5\mu}^5 = 0 ; \quad \Gamma_{55}^5 = \frac{1}{2} h'(y) ; \quad \Gamma_{\alpha\alpha}^5 = \frac{1}{2} e^{2k\varepsilon y} \eta_{\alpha\alpha} h'(y)$$. (6.3)

The expressions for the scalar curvature $$R$$ and for $$g^{AB}(\Gamma^{C}_{AB,C} - \Gamma^{C}_{AC,B})$$ are

$$R = -\frac{8k^2}{h} - 4k\varepsilon \frac{h'}{h^2} = g^{AB}(\Gamma^{C}_{AB,C} - \Gamma^{C}_{AC,B})$$, (6.4)

from where, taking the difference of the two expressions, it is found that $$g^{AB}(\Gamma^{C}_{AB,C} \Gamma^{D}_{CD} - \Gamma^{D}_{AC} \Gamma^{C}_{BD}) = 0$$ and the differential equation (3.13) is written as

$$\frac{(2\pi)^7}{m^4 V_5 g_5^2} \frac{e^{2k\varepsilon y} h'}{2h} \left[ \frac{\partial l}{\partial x^1} \frac{\partial l}{\partial x^2} - \frac{\partial l}{\partial x^3} \frac{\partial l}{\partial x^4} \right] +$$

$$+ \frac{(2\pi)^7 4k\varepsilon \partial l}{m^4 V_5 g_5^2 h} \frac{dy}{dy} = C l - D i^3$$, (6.5)

where $$C$$ and $$D$$ denote the expressions

$$C = \frac{(2\pi)^7 4k (2kh + e'h')}{m^4 V_5 g_5^2 h^2} ; \quad D = \frac{M^2 4k (2kh + e'h')}{h^2}$$. (6.6)

The characteristic system of equations for the equation (6.5) is

$$\frac{dl}{Cl - Di^3} = \frac{\varepsilon m^4 V_5 g_5^4 h}{(2\pi)^7 4k} dy = d\sigma$$, (6.7)

$$\frac{2m^4 V_5 g_5^4 e^{-2k\varepsilon y}}{(2\pi)^7 (\ln h)} dx^1 = - \frac{2m^4 V_5 g_5^4 e^{-2k\varepsilon y}}{(2\pi)^7 (\ln h)} dx^i = d\sigma$$, (6.8)
where the indice \( i = 2, 3, 4 \) and \( \sigma \) is some parameter. The solution of the first characteristic equation for the \( y \) and \( l \) variables is

\[
\frac{\varepsilon_1 l}{[\varepsilon_2(l^2 - \alpha_1^2)]^{\frac{1}{2}}} = C_1(x_1, x_i) e^{2k\varepsilon_3 y} h, \tag{6.9}
\]

where

\[
\alpha_1 = \sqrt{\frac{C}{D}} \tag{6.10}
\]

and \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) take values \( \pm 1 \) independently one from another.

In order to find the function \( C_1(x_1, x_i) \), the obtained solution (6.9) should be differentiated by \( x_1 \) and the characteristic equations for \( \frac{\partial l}{\partial y} \) and \( \frac{\partial y}{\partial x_1} \) have to be used. As a result, the function \( C_1(x_1, x_i) \) is found as a solution of the following simple differential equation

\[
M = S \frac{\partial C_1(x_1, x_i)}{\partial x_1} + TC_1(x_1, x_i), \tag{6.11}
\]

where the functions \( M, S \) and \( T \) are defined as follows

\[
M \equiv -\frac{\varepsilon_1 (2l^2 - \alpha_1^2)l8kM_2^2(2kh + \varepsilon_3 h')m_1^4V_5g^4}{[\varepsilon_2(l^2 - \alpha_1^2)]^{\frac{1}{2}}} (2\pi)^7 h' e^{2k\varepsilon_3 y} \quad \text{for} \quad \varepsilon_1 \varepsilon_2 = -1 , \tag{6.12}
\]

\[
S \equiv h e^{2k\varepsilon_3 y} ; \quad T \equiv \frac{8k(2k + \varepsilon_3 h')}{h'} . \tag{6.14}
\]

The solution of the differential equation (6.11) can be written as

\[
C_1(x_1, x_i) = \frac{M}{T} - \varepsilon_1 \frac{C_2(x_i)}{T} e^{-\int \frac{S}{T} dx_1} . \tag{6.15}
\]

Again, the obtained solution (6.9) can be differentiated with respect to \( x_i \) and taking into account from the characteristic equations that

\[
\frac{\partial l}{\partial x_i} = -\frac{\partial l}{\partial x_1} ; \quad \frac{\partial y}{\partial x_i} = -\frac{\partial y}{\partial x_1}, \tag{6.16}
\]

the solution of the corresponding equation (6.11) with \( \tilde{M} = -M, \tilde{T} = -T, \tilde{S} = S \) can be represented as

\[
C_1(x_1, x_i) = \frac{M}{T} + \varepsilon_4 \frac{C_3(x_1)}{T} e^{\int \frac{S}{T} dx_1} . \tag{6.17}
\]

Substracting the two relations (6.15) and (6.17), the following relation between the functions \( C_2(x_1) \) and \( C_3(x_1) \) can be derived

\[
C_2(x_1) = e^{\int \frac{S}{T} dx_1} C_3(x_1) e^{\int \frac{S}{T} dx_1} . \tag{6.18}
\]

Now let us differentiate relation (6.17) with respect to \( x_1 \) and make use of (6.18) and its derivative also with respect to \( x_1 \). Then the following differential equation is derived with respect to the function \( C_3(x_1) \)

\[
\frac{\partial C_3(x_1)}{\partial x_1} - C_3(x_1) \frac{T}{S} = 0 , \tag{6.19}
\]

from where

\[
C_3(x_1) = \text{const.} e^{\int \frac{S}{T} dx_1} . \tag{6.20}
\]
Therefore from (6.17)

\[ C_1(x_1, x_i) = \frac{M + \varepsilon_4}{T} \]

(6.21)

and substituting into (6.9), a final expression for \( l \) can be found (for the case \( \varepsilon_1 \varepsilon_2 = +1 \))

\[ l^2 = \frac{\varepsilon_2 \alpha_4 e^{\varepsilon_3 k y} I_2^2}{\varepsilon_2 k^2 e^{\varepsilon_3 k y} - \frac{T_2}{(M + \varepsilon_4)^2}} \]

(6.22)

The solution for the other case \( \varepsilon_1 \varepsilon_2 = -1 \) can be found analogously.

The general solution of the quasilinear differential equation in partial derivatives will be not only expression (6.22), but also any function \( V \), depending on the first integrals \( K_1, K_2, K_3, ..., K_6 \) of the characteristic system of equations [13]

\[ V = V(K_1, K_2, K_3, K_4, K_5, K_6) \]

(6.23)

Now let us find a solution of the characteristic equation for the \( x_i \) and \( y \) variables for the case of the function \( h(y) = \gamma y^n \). The equation can be written as

\[ e^{2k\varepsilon_4 y} y^{n-1} = -\varepsilon_4 \frac{8k}{n\gamma} dx^i \]

(6.24)

from where \( x^i \) can be expressed as

\[ x^i = -\varepsilon_4 \frac{n\gamma}{8k} I_{n-1}(k, y) \]

(6.25)

and \( I_{n-1}(k, y) \) denotes the integral

\[ I_{n-1}(k, y) = \int e^{2k\varepsilon_4 y} y^{n-1} dy \]

(6.26)

This integral can be exactly calculated and the result can be found in the monograph by Timofeev [14]

\[ I_{n-1} = \frac{e^{2k\varepsilon_4 y}}{(2k\varepsilon_4)^n} \sum_{p=0}^{n-1} (-1)^p \binom{n-1}{p} p! (2k\varepsilon_4)^{n-1-p} y^{n-1-p} \]

(6.27)

Note that because of the complicated expression for the integral \( I_{n-1}(k, y) \), \( y \) cannot be expressed as a function of the \( x_i \) and the \( x_1 \) coordinate. Also, the solvability of the quasilinear differential equation is determined mostly by the presence of the embedded flat 4D Minkowski spacetime. Therefore, it may be expected that there might be another functions \( h(y) \), for which exact analytical solution may be found.

7 SOLUTIONS OF THE SECOND QUASILINEAR DIFFERENTIAL EQUATION (4.6) FOR THE CASE OF A FLAT 4D MINKOWSKI METRIC, EMBEDDED IN A FIVE - DIMENSIONAL SPACE

The same approach, developed in the previous section, shall be applied with respect to the second quasilinear differential equation in partial derivatives (4.6). The aim will be to show that the analytical solution will be different, compared to the first one for the differential equation (3.13).

The differential equation (4.6) for the case of the metric (6.1) can be written as

\[ -D \frac{\partial l}{\partial x^1} + D \left[ \frac{\partial l}{\partial x^2} + \frac{\partial l}{\partial x^3} + \frac{\partial l}{\partial x^4} \right] + E \frac{\partial l}{\partial y} + A l^2 + B l^2 + C = 0 \]

(7.1)
where $A, B, C, D, E$ denote the expressions

\[ A \equiv \frac{(2\pi)^7 4k(2kh - \varepsilon h')}{h^2 m_5^2 V_5 g_5^4} ; \quad C \equiv \frac{16k^2 M_5^2}{h}, \quad (7.2) \]

\[ B \equiv \frac{16k^2 (2\pi)^7}{h m_5^2 g_5^4 V_5} + \frac{4k M_5^2}{h^2} (\varepsilon h' - 2kh), \quad (7.3) \]

\[ D \equiv -\frac{(2\pi)^7 l e^{2kh} h'}{m_5^2 V_5 g_5^4 2h} ; \quad E \equiv -\frac{4k \varepsilon (2\pi)^7 l}{m_5^2 V_5 g_5^4 h}. \quad (7.4) \]

The characteristic system of equations is

\[ \frac{dx_1}{D} = - \frac{dx_i}{D} = - \frac{dy}{E} = \frac{dl}{Al^4 + Bl^2 + C}. \quad (7.5) \]

The characteristic equation for the $y$ and $l$ variables can be written as

\[ d \left[ \ln \left( \varepsilon_1 \left( \frac{s - G}{s + G} \right) \right) \right] = 2AC \frac{\varepsilon_4 h m_5^2 V_5 g_5^4}{4k (2\pi)^7} dy, \quad (7.6) \]

where

\[ G \equiv \frac{B^2}{4A} - \frac{C}{A} = \frac{M_5 \sqrt{(F - \frac{1}{2})^2 + k^2 (2(2\pi)^7 - \frac{1}{4})}}{\sqrt{(2\pi)^3} \left| 2k - \varepsilon_4 h' h \right|}. \quad (7.7) \]

is a non-negative function and

\[ s \equiv l^2 + \frac{B}{2A} ; \quad F \equiv m_5^4 V_5 g_5^4 (2k - \varepsilon_4 h' h). \quad (7.8) \]

The integration of the characteristic equation (7.6) results in the expression

\[ l^2 = \frac{B}{2A} + G \left( \frac{1 + \varepsilon_1 D_1(x_1, x_i) e^Z \tilde{J}(y)}{1 - \varepsilon_1 D_1(x_1, x_i) e^Z \tilde{J}(y)} \right), \quad (7.9) \]

where $Z$ is given by

\[ Z := \frac{2M_5 \varepsilon_4 m_5^2 g_5^2 \sqrt{V_5}}{(2\pi)^3} \quad (7.10) \]

and $\tilde{J}(y)$ is the integral

\[ \tilde{J}(y) := \int \frac{\sqrt{K_1 y^2 + K_2 y + 1}}{y} dy. \quad (7.11) \]

The functions $K_1$ and $K_2$ are the following

\[ K_1 \equiv \frac{k^2 \left[ (2(2\pi)^7 - \frac{1}{4}) \right]}{m_5^8 V_5^2 g_5^8} + k^2 \left( 2 - \frac{1}{2m_5^4 V_5 g_5^4} \right)^2, \quad (7.12) \]

\[ K_2 \equiv -2\varepsilon_4 \left( 2k - \frac{k}{2m_5^4 V_5 g_5^4} \right). \quad (7.13) \]

It is important to note that the free term under the square in the integral $\tilde{J}(y)$ (7.11) is positive (it is +1) and the function $K_1$ is also positive. The analytical solution of integrals of the type (7.11) depends on the sign of the function $K_1$ and of the free term, which in the present case are both positive. The explicit solution can be found in the book of Timofeev [14]:

\[ \tilde{J}(y) \equiv \sqrt{K_1 y^2 + K_2 y + 1} + \frac{K_2}{2\sqrt{K_1}} \ln(K_1 y + \frac{K_2}{2}). \]
Now it remains to determine the function $D_1(x_1, x_i)$ in expression (7.9) for $l^2$. For the purpose, let us denote the under-integral expression in (7.11) by $J(y)$, differentiate both sides of (7.9) by $x_1$ and take into account the expressions for $\frac{\partial l}{\partial x_1}$ and $\frac{\partial y}{\partial x_1}$ from the characteristic system of equations. After rearranging the terms and denoting by $V$

$$V \equiv \varepsilon_1 - \frac{2l(At^4 + Bt^2 + C)}{E} \frac{\partial}{\partial y} \left( \frac{B}{F} \right) - \frac{\partial Q}{\partial y} \left( \frac{I^2 + I^4}{G} \right),$$

(7.15)

the following differential equation can be obtained for the function $D_1(y)$

$$\frac{\partial D_1}{\partial y} + D_1(ZJ(y) + V) - Ve^{-Z\hat{J}(y)} = 0 .$$

(7.16)

It may seem strange at first glance that the function $D_1$ depends on the $y$ coordinate, while in (7.9) it was assumed that $D_1 = D_1(x_1, x_i)$. In fact, from the characteristic equations (7.5) for the $y$ and $x_1$ variables it follows

$$\frac{dy}{e^{2k\varepsilon_4 y^2}} = -\frac{\varepsilon_4}{8k} dx_1 .$$

(7.17)

For the concrete expression for the function $h(y) = \gamma y^n$, the coordinates $x_1$ and $x_i$ can be expressed as

$$x_1 = -\varepsilon_4 \frac{8k}{n\gamma} f\left( -k, 1 - n \right) + \text{const.} ; \quad x_i = -x_1$$

(7.18)

and therefore it is reasonable to consider that $D_1 = D_1(x_1(y), x_i(y)) = D_1(y)$. Note however that due to the complicated structure of the integral $I(-k, 1 - n)$, it is impossible to express $y$ as a function of $x_1$ (or $x_i$).

As in the previous case, the general solution of the equation depends on all the first integrals of the characteristic system of equations.

8 LENGTH FUNCTION $l(x)$ FROM THE CONSTANCY

OF THE SCALAR CURVATURE $R$ UNDER "RESCALINGS" OF THE CONTRAVARIANT METRIC TENSOR FOR THE CASE OF A FLAT 4D MINKOWSKI METRIC, EMBEDDED IN A 5D SPACETIME.

Now we shall find solutions of the corresponding differential equation in partial derivatives, when the second representation of the scalar curvature $\tilde{R}$ (3.11) is equal to the initial scalar curvature $R$ (i.e. $\tilde{R} = R$). The obtained differential equation under this identification is

$$I^2 \left[ R - \frac{1}{2} g^{AC} g^{BD} (...) \right] + I^2 \left[ -R + g^{AB} (\Gamma^C_{AB,C} - \Gamma^C_{AC,B}) \right] +$$

$$+ l \left[ \frac{1}{2} g^{AC} g^{BD} (...) - g^{AB} (\Gamma^C_{AB,C} - \Gamma^C_{AC,B}) \right] +$$

$$+ \frac{\partial l}{\partial x^B} g^{AB} \Gamma^C_{AC} - \frac{\partial l}{\partial x^C} g^{AB} \Gamma^C_{AB} = 0 .$$

(8.1)
The expression in the small brackets is the same as in (3.4), i.e. \((g_{AD,BC} + g_{BC,AD} - g_{AC,BD} - g_{BD,AC})\). The equation (8.1) for the case of the metric (6.1) with the affine connection components (6.2) - (6.3) acquires the form
\[
\frac{\partial l}{\partial y} + \frac{h'}{8k}e^{2kxy} \left( \frac{\partial l}{\partial x_1} - \frac{\partial l}{\partial x_2} - \frac{\partial l}{\partial x_3} - \frac{\partial l}{\partial x_4} \right) =
\]
\[
= (2k - \frac{\epsilon h'}{h})(t^3 - l) \quad .
\]
(8.2)

The characteristic system of equations is
\[
\frac{dl}{(2k - \frac{\epsilon h'}{h})l(l^2 - 1)} = \varepsilon dy =
\]
\[
= \frac{dx_1}{h}8ke^{-2kxy} = -\frac{dx_1}{h}8ke^{-2kxy} \quad .
\]
(8.4)

The solutions of the characteristic system for the \(x_1\) and \(y\) variables are correspondingly
\[
x_1 = C_4(x_1, l) + \varepsilon_2 \frac{e^{2k_1y}h}{8k} - \frac{1}{4} \int he^{2k_1y}dy \quad ,
\]
(8.5)

\[
l^2 = \frac{h^2 - D_2(x_1, x_i)e^{4k_1y}}{h^2} \quad .
\]
(8.6)

Unfortunately, (8.6) cannot be considered as an expression for \(l\), since from (8.5) it is obvious that \(D_2(x_1, x_i)\) also depends on the function \(l\). Also, it should be understood that \(D_2\) depends on all the variables \(x_i, i = 2, 3, 4\).

Let us differentiate both sides of (8.6) by \(x_1\)
\[
2l \frac{\partial l}{\partial x_1} = 2\frac{h'}{h} \frac{\partial y}{\partial x_1} - \frac{l^4}{h^2} \frac{2h'h' \frac{\partial y}{\partial x_1}}{\partial x_1} - 2D_2(\frac{\partial D_2}{\partial x_1} e^{4k_1y} - D_2 4k_1e^{4k_1y} \frac{\partial y}{\partial x_1}) \quad .
\]
(8.7)

If the same operation is applied also with respect to the \(x_i\) coordinate and the derived equation is summed up with (8.7) with account also of \(\frac{\partial l}{\partial x_1} = -\frac{\partial l}{\partial x_1}, \frac{\partial y}{\partial x_1} = -\frac{\partial y}{\partial x_1},\) then it can be obtained
\[
2l^2 e^{4k_1y}D_2 (\frac{\partial D_2}{\partial x_1} + \frac{\partial D_2}{\partial x_1}) = 0 \quad .
\]
(8.8)

The equation is satisfied also for \(D_2 = 0\), which evidently corresponds to the standard case \(l = 1\) in gravity theory. The other case, when the equation is fulfilled, is \(\frac{\partial D_2}{\partial x_1} = -\frac{\partial D_2}{\partial x_1}\).

Now let us rewrite equation (8.7) with account of the expressions for \(\frac{\partial l}{\partial x_1}\) and \(\frac{\partial y}{\partial x_1}\) from the characteristic system of equations (8.3 - 8.4). Then the following quasilinear differential equation with respect to the function \(D_2\) is derived
\[
\frac{\partial D_2}{\partial y} - 4k_1e_1(1 + \frac{4}{h}D_2) - 2e_1e_5 \frac{h'}{h} (2k - \frac{\epsilon h'}{h})e^{-4k_1y}(h^2 - D_2^2 e^{4k_1y}) \frac{h}{h} = 0 \quad .
\]
(8.9)

After finding the function \(D_2 = D_2(y)\) as a solution of this equation, from (8.6) \(l^2\) can also be found as a function of the extra - coordinate \(y\), i.e. \(l = l(y)\). Then after differentiating the solution for \(x_1\) (8.5) by \(y\), the function \(C_4(x_1, l)\) can be derived as a solution of the following differential equation
\[
\frac{\partial C_4(x_1, l)}{\partial y} = F_1 \quad ,
\]
(8.10)
where the function $F_1$ is determined as

$$F_1 \equiv e^{2k\varepsilon_1 y} \left[ \frac{\varepsilon_2 (l^2 - 1) h'}{8k} - \frac{\varepsilon_1 \varepsilon_2 h}{4} - \frac{\varepsilon_2 h'}{8k} + \frac{h}{4} \right] \quad (8.11)$$

and $l$ has to be substituted with expression (8.6), in which $D_2$ is determined as a solution of the differential equation (8.9). The representation in the form (8.10) is particularly convenient for the case $h(y) = \gamma y^n$, when the difficulty will be only in calculating the integral along $y$ in the first term of (8.11). The advantage of the representation (8.10) will become evident if we differentiate the solution (8.5) by $l$, obtaining thus the differential equation

$$E_1(y) = \frac{\partial C_4(x_i, l)}{\partial l} + \frac{E_2(y)}{l(l^2 - 1)} , \quad (8.12)$$

where

$$E_1(y) \equiv \frac{\varepsilon_2 h e^{2k\varepsilon_1 y}}{8k(2kh - \varepsilon_2 h')} , \quad (8.13)$$

$$E_2(y) \equiv \varepsilon_2 h e^{2k\varepsilon_1 y} \left[ (\varepsilon_1 \varepsilon_2 - 1) + \frac{\varepsilon_2 h'}{8k} \right] . \quad (8.14)$$

Now it is important to stress that the second representation (8.12) is inconvenient to use for the case $h(y) = \gamma y^n$. The reason is that the integration is along the $l$ coordinate, which means that the $y$ coordinate in $E_1(y)$ and $E_2(y)$ has to be expressed from expression (8.6) as a function of $l$. However, in view of the extremely complicated expression, this is not possible. Instead, the differential equation (8.12) will be very helpful for the case $h(y) \equiv 1$, which is frequently encountered in most of the papers on theories with extra dimensions. Indeed, then the nonlinear differential equation (8.9) is of a particularly simple form:

$$\frac{\partial D_2^2}{\partial y} - 20k\varepsilon D_2^2 = 0 , \quad (8.15)$$

from where with the help of (8.6)

$$l^2 = \frac{1}{1 - \text{const.} e^{2k\varepsilon_1 y}} . \quad (8.16)$$

Note one interesting property of the obtained solution, already mentioned in the Introduction - when $\varepsilon_1 = -1$ and $y$ tends to infinity, the known case in gravity theory $l^2 = 1$ is recovered.

Now $y$ can be expressed easily and the resulting differential equation (8.12) with $E_1(y) = 0$ can be rewritten as

$$\frac{\partial C_4(x_i, l)}{\partial l} + \frac{\varepsilon_2 (\varepsilon_1 \varepsilon_2 - 1)e^{\frac{h}{y}}}{\text{const.} 8kl^3} = 0 . \quad (8.17)$$

The solution of the equation can be represented as

$$C_4(x_i, l) = -\frac{\varepsilon_2 e^{\frac{h}{y}} (1 - \text{const.} e^{2k\varepsilon_1 y})}{8k \text{ const.}} \tilde{C}_4(x_i) . \quad (8.18)$$

The unknown function $\tilde{C}_4(x_i)$ can be found if expression (8.5) for $x_1$ is differentiated with respect to $x_i$. Unfortunately, the resulting formulae will contain the expressions for $\frac{\partial y}{\partial x_i}$ and $\frac{\partial l}{\partial x_i}$, which are singular when $h'(y) = 0$. But if we multiply by $\frac{\partial x_i}{\partial y}$, the following differential equation will be obtained

$$\frac{\partial x_i}{\partial y} = \frac{\varepsilon_1 (\varepsilon_2 - \varepsilon_1)}{4} e^{2k\varepsilon_1 y} + \frac{\varepsilon_2 e^{\frac{h}{y}}}{8k \text{ const.} l^3} \tilde{C}_4(x_i) \frac{\partial l}{\partial y} - \frac{\varepsilon_2 e^{\frac{h}{y}}}{8k \text{ const.} l^2} \frac{\partial \tilde{C}_4(x_i)}{\partial y} . \quad (8.19)$$
which is no longer singular in the limit \( h'(y) = 0 \), because the expressions for \( \frac{\partial x_1}{\partial y} \) and \( \frac{\partial l}{\partial y} \) are

\[
\frac{\partial x_1}{\partial y} = 0 \quad ; \quad \frac{\partial l}{\partial y} = \varepsilon_2 2kl(l^2 - 1) .
\] (8.20)

Taking into account these formulae, the solution of the differential equation (8.19) with respect to the function \( \tilde{C}_4(x_i) \) can be found in the form

\[
\tilde{C}_4(x_i) = \text{const} \left| 1 - \text{const} \, e^{24k\varepsilon_1 y} \right|^{\frac{1}{12}} .
\] (8.21)

Substituting into (8.18), the final expressions for the function \( C_4(x_i, l) \) can be found and also for the coordinate \( x_1 \), which for \( \varepsilon_1 = \varepsilon_2 \) and \( h(y) = 1 \) is simply

\[
x_1 = C_4(x_i(y), l(y)) = -\varepsilon_2 \text{const} \left| 1 - \text{const} \, e^{24k\varepsilon_1 y} \right|^{\frac{1}{12}} .
\] (8.22)

\section{Discussion}

It is perhaps surprising that the quasilinear differential equation in the preceeding section and its solution for the length function is simpler than the other two differential equations in sections 6 and 7, when the two cases of "compactification+rescaling" and "rescaling+compactification" have been investigated. Nevertheless, it is recommendable to find more solutions for the length function \( l(x) \), from where it can be seen in which other cases the transition to \( l = 1 \) can be performed. Also, solutions of these equations can be found not only by the method of characteristics, but also in terms of complicated functions of the Weierstrass elliptic function and its derivative, following with slight modifications the algorithm in [2, 5]. As a matter of fact, quite a broader class of equations, concerning the Ginzburg-Landau model and \( \lambda \Phi^4 \) scalar field models from quantum field theory admit such solutions, representing uniformization functions, depending on elliptic functions.

In this paper only one length scale has been taken into account. However, there is a motivation for taking into consideration (at least) two different length scales - these are the models with intersecting \( D5 \)-branes on \( 4D \) (orientifold) compactifications of type II B string theories [15,16]. These models have two transverse to the \( D5 \)-branes directions and thus the string scale is lowered to the \( TeV \) region.

Now let us try to implement the introduced notion of length scale with respect to type II A string theory compactifications on a compact variety of the form \( T^2 \times B_4 \) [17], where several sets of \( D \)-branes with one worldvolume dimention are wrapped on different cycles within a two torus. The torus is obtained by quotienting the two dimensional flat space \( R^2 \) by the lattice of translations, generated by the vectors \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \). The length of the cycle \((n, m)\) with the two compact dimensions of size \( R_1 \) and \( R_2 \) is

\[
| (n, m) | = (g_{ab} v^a v^b)^{\frac{1}{2}} = 2\pi \sqrt{n^2 R_1^2 + m^2 R_2^2 + 2nm R_1 R_2 \cos \Theta} ,
\] (9.1)

where \( \Theta \) is the angle between the two vectors and \( g_{ab} \) is the symmetric matrix

\[
g_{ab} = \begin{pmatrix}
R_1n & R_1 R_{2nm} \cos \Theta \\
R_1 R_{2nm} \cos \Theta & R_2 m
\end{pmatrix} .
\] (9.2)

With respect to this two-dimensional model, let us define the contravariant metric \( \tilde{g}^{ab} \), which here shall be identified with the factorized product

\[
\tilde{g}^{ab} := v^a v^b .
\] (9.3)
This metric has the (constant) components
\[ \tilde{g}^{11} := n^2; \quad \tilde{g}^{12} := nm; \quad \tilde{g}^{22} := m^2. \] (9.4)

In Appendix B the inverse contravariant metric components \( g^{11}, g^{12}, \) and \( g^{22} \) and also the Christoffel components have been calculated, from where it can be seen that they have a singularity at
\[ 1 - R_1 R_2 n m \cos^2 \Theta = 0. \] (9.5)

The scalar curvature \( R \) will be singular too at the values for \( \Theta \), satisfying (9.5). However, if a new scalar curvature \( \tilde{R} \) with the contravariant components (9.4) is defined and the equality of the two curvatures \( R = \tilde{R} \) is required, i.e.
\[ R = \tilde{R}, \] (9.6)

then from (9.6) and the resulting algebraic equation with respect to \( \cos \Theta \) the angle \( \Theta \) can be determined.

The "singular" value \( \cos^2 \Theta = \frac{1}{R_1 R_2 n m} \) has to be excluded, because then (9.6) cannot be satisfied. This example clearly shows that although the length scale is properly defined, the singularity in the scalar curvature might not be removed by the choice of another contravariant metric. However, it may be interesting to solve equation (9.6) (now treated as a nonlinear differential equation), assuming that the contravariant "tilda" components \( \tilde{g}^{ij} \) are not constant values, but some arbitrary functions \( f^a(R_1, R_2) \), and see which are the singular and non-singular solutions.

APPENDIX A: AFFINE GEOMETRY THEOREM ABOUT THE EQUIAFFINE CONNECTION

The formulation of the theorem is the following: if \( \tilde{\Gamma}^k_{ij} \) is another connection, not compatible with the initial one \( \Gamma^k_{ij} \), then the Ricci tensor \( \tilde{R}_{ij} \) is again symmetric and is equal to
\[ \tilde{R}_{ij} = \partial_h \tilde{\Gamma}^k_{ij} - \partial_i \tilde{\Gamma}^k_{kj} + \tilde{\Gamma}^k_{kl} \tilde{\Gamma}^l_{ij} - \tilde{\Gamma}^m_{ki} \tilde{\Gamma}^k_{jm} \] (A1)
if and only if the connection \( \tilde{\Gamma}^k_{ij} \) is an equiaffine one. The last means that this connection for \( j = k \) should be possible to be represented as a gradient of a scalar quantity \( \tilde{\epsilon} \):
\[ \tilde{\Gamma}^k_{ij} = \partial_i \ln \tilde{\epsilon}. \] (A2)

The equiaffine properties of the connection \( \tilde{\Gamma}^k_{ij} \) in the general case had been proved in a previous paper [1].

Now we shall give a very simple proof for the concrete case \( \tilde{g}^{ij} = l \delta^i_j g^{ri} \), investigated in this paper. Namely, provided that the equiaffine property (A2) is valid for the usual Christoffel connection, after resolution of the differential equation with respect to \( \epsilon \) one obtains
\[ \epsilon = \exp \left[ C(x^0, x^1, \ldots x^{i-1}, x^{i+1}, \ldots, x^n) \int \Gamma^k_{ik} dx^i \right]. \] (A3)

Then, making use of the defining equality (1.1), the "tilda" connection \( \tilde{\Gamma}^k_{ij} \) will be an equiaffine one if
\[ \tilde{\epsilon} = \exp \left[ \tilde{C}(x^0, x^1, \ldots x^{i-1}, x^{i+1}, \ldots, x^n) \int l \partial_i \ln \tilde{\epsilon} dx^i \right]. \] (A4)

Substituting above the expression (A3) for \( l \epsilon \) and choosing for convenience \( \tilde{C} = \frac{1}{C} \) (we presume that \( C \) is known), one obtains the simple expression
\[ \tilde{\epsilon} = \exp \left[ \int l \Gamma^k_{ik} dx^i \right]. \] (A5)

Thus, if the scalar density \( \tilde{\epsilon} \) is determined in this way, (A2) is fulfilled, which proves that \( \tilde{\Gamma}^k_{ij} \) is an equiaffine connection. Consequently, the use of formulae (A1) for the "modified" Ricci tensor is fully justified and legitimate.
APPENDIX B: CONNECTION AND CURVATURE COMPONENTS FOR THE TWO-DIMENSIONAL METRIC

The components $g^{11}$, $g^{22}$ and $g^{12}$, obtained after the solution of the linear algebraic system $g_{ij}g^{ik} = \delta^k_i$ for values of $(i,k) = (1,1), (1,2), (2,1)$ and $(2,2)$ are

$$g^{11} = \frac{g_{22}}{(g_{12} - g_{11}g_{22})} = \frac{1}{R_1n(1 - R_1R_2n\cos^2\Theta)} , \quad (B1)$$
$$g^{12} = \frac{g_{12}}{(g_{12} - g_{11}g_{22})} = \frac{\cos\Theta}{(1 - R_1R_2n\cos^2\Theta)} , \quad (B2)$$
$$g^{22} = \frac{g_{11}}{(g_{12} - g_{11}g_{22})} = \frac{1}{R_2m(1 - R_1R_2n\cos^2\Theta)} . \quad (B3)$$

The "usual" Christoffel connection components $\Gamma^k_{ij}$, calculated by means of the inverse contravariant metric and the derivatives in respect to the variables $(R_1, R_2)$ are

$$\Gamma^1_{11} = \frac{1 - 2R_1R_2n\cos^2\Theta}{2R_1(1 - R_1R_2n\cos^2\Theta)} = g^{12}g_{12,1} + \frac{1}{2}(g^{11}g_{11,1} - g^{12}g_{11,2}) , \quad (B4)$$
$$\Gamma^1_{22} = \frac{m\cos\Theta}{2(1 - R_1R_2n\cos^2\Theta)} = g^{11}g_{12,2} + \frac{1}{2}(g^{12}g_{22,2} - g^{11}g_{22,1}) , \quad (B5)$$
$$\Gamma^1_{12} = \frac{m\cos\Theta}{2(1 - R_1R_2n\cos^2\Theta)} = g^{22}g_{12,1} + \frac{1}{2}(g^{12}g_{11,1} - g^{22}g_{11,2}) , \quad (B6)$$
$$\Gamma^2_{22} = \frac{1 - 2R_1R_2n\cos^2\Theta}{2R_2(1 - R_1R_2n\cos^2\Theta)} = g^{12}g_{12,2} + \frac{1}{2}(g^{22}g_{22,2} - g^{12}g_{22,1}) , \quad (B7)$$
$$\Gamma^1_{12} = \Gamma^2_{12} = 0 \quad (B8)$$

The components of the "tilda" connection $\tilde{\Gamma}^k_{ij}$, calculated as a linear combination of the above connection components (B4) - (B8) are

$$\tilde{\Gamma}^1_{11} = (\tilde{g}^{11}g_{11} + \tilde{g}^{12}g_{12})\Gamma^1_{11} + (\tilde{g}^{11}g_{12} + \tilde{g}^{12}g_{22})\Gamma^2_{11} , \quad (B9)$$
$$\tilde{\Gamma}^1_{22} = (\tilde{g}^{11}g_{11} + \tilde{g}^{12}g_{12})\Gamma^1_{22} + (\tilde{g}^{11}g_{12} + \tilde{g}^{12}g_{22})\Gamma^2_{22} , \quad (B10)$$
$$\tilde{\Gamma}^2_{11} = (\tilde{g}^{12}g_{11} + \tilde{g}^{22}g_{12})\Gamma^1_{11} + (\tilde{g}^{12}g_{12} + \tilde{g}^{22}g_{22})\Gamma^2_{11} , \quad (B11)$$
$$\tilde{\Gamma}^2_{22} = (\tilde{g}^{12}g_{11} + \tilde{g}^{22}g_{12})\Gamma^1_{22} + (\tilde{g}^{12}g_{12} + \tilde{g}^{22}g_{22})\Gamma^2_{22} , \quad (B12)$$
$$\tilde{\Gamma}^1_{12} = \tilde{\Gamma}^2_{12} = 0 \quad (B13)$$

The "non - tilda" scalar curvature $R$, calculated by means of the formulae (B1) - (B8) is

$$R = \frac{\cos\Theta(2 - R_1R_2n\cos^2\Theta)}{4R_1R_2(1 - R_1R_2n\cos^2\Theta)^2} . \quad (B14)$$

The "tilda" scalar curvature, calculated by means of the connections (B9) - (B13) is

$$\tilde{R} = \frac{m^3n^3\cos\Theta(1 - 2R_1R_2n\cos^2\Theta)A}{4(1 - R_1R_2n\cos^2\Theta)^2} + \frac{m^3n^3(m + n^2R_1\cos\Theta)(n + m^2R_2\cos\Theta)B}{4R_2(1 - R_1R_2n\cos^2\Theta)^2} . \quad (B15)$$
where \( A \) and \( B \) denote the expressions

\[
A := mR_2(m + n^2R_1\cos\Theta)^2 + nR_1(n + m^2R_2\cos\Theta)^2, \quad (B16)
\]

\[
B := -1 + 5mnR_1R_2\cos^2\Theta - 4R_1^2R_2^2m^2n^2\cos^4\Theta. \quad (B17)
\]

Now it is seen that the equality \( R = \tilde{R} \) is not satisfied at the singular value \( \cos^2\Theta = \frac{1}{R_1R_2mn} \), although in the formal sense it is well defined. But yet the tilda contravariant components \( \tilde{g}^{ij} \) are well-defined, unlike \( g^{ij} \).

The other values of \( \Theta \) can be determined from the resulting algebraic equation with respect to \( \Theta \).

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