Analysis of the ellipticity induced PMD and general scaling perturbation in a transmitting fiber.

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Abstract: Presented is an analysis of general scaling perturbations in a transmitting fiber. For elliptical perturbations, under some conditions an intermode dispersion parameter characterizing modal PMD is shown to be directly proportional to the mode dispersion.

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In the following paper we derive a generalized Hermitian Hamiltonian approach for the treatment of Maxwell equations in waveguides as well as develop a perturbation theory for the general class of scaling perturbations that include ellipticity and a uniform scaling of an arbitrary index profile. Because of the Hermitian nature of the formulation most of the results from the well developed perturbation theory of quantum mechanical systems can be directly related to the light propagation in the waveguides. Such formulation provides for an intuitive way of understanding PMD and birefringence in the elliptically perturbed fiber profiles. Region of validity of our theory extends to the case of large variations of the dielectric constant across the fiber crosssection and is limited only by an amount of re-scaling. Finally, we establish that if in some range of frequencies a particular mode behaves like a mode of pure polarization $TE, TM$ (where polarization is judged by the relative amounts of the electric and magnetic longitudinal energies in a modal crosssection) its inter-mode dispersion parameter $\tau = \frac{\Delta \beta_e}{\beta}$ is related to its dispersion $D$ as $\tau = \lambda \delta |D|$, where $\delta$ is a measure of the fiber ellipticity and $\Delta \beta_e$ is a split in a wavevector of a linearly polarized doubly degenerate mode of interest due to an elliptical perturbation.

While there has been a wide amount of work done on estimating such quantities as local birefringence induced by perturbations in the fiber profile most of the treatments were geared toward understanding the low contrast, weakly guiding systems such as ubiquitous silica waveguides and are not directly applicable to the high contrast systems such as Bragg fibers, photonic crystal fibers and integrated optics waveguides which are steadily emerging as an integral part of the state of the art transmission systems.

In deriving Hamiltonian formulation for the eigen fields of a generic waveguide exhibiting translational symmetry in longitudinal $\hat{z}$ direction we start with a well known set of Maxwell equations written in terms of transverse and longitudinal fields [1]. Assuming the form of the field

$$
\begin{bmatrix}
E(x, y, z, t) \\
H(x, y, z, t)
\end{bmatrix} = \begin{bmatrix}
E(x, y) \\
H(x, y)
\end{bmatrix} \exp(i\beta z - i\omega t)
$$

and introducing the transverse and longitudinal components of the fields as $F = F_t + F_z$, $F_z = \hat{z}F_z$, $F_t = (\hat{z} \times F) \times \hat{z}$ Maxwell equations can be rewritten as a generalized Hermitian eigen problem [2]

$$
\begin{bmatrix}
E_t \\
H_t
\end{bmatrix} = \begin{bmatrix}
\omega e - \frac{1}{\varepsilon} \nabla_t \times [\hat{z}(\hat{z} \cdot (\nabla_t \times E_t))] \\
0
\end{bmatrix} \begin{bmatrix}
\frac{1}{\varepsilon} \nabla_t \times [\hat{z}(\hat{z} \cdot (\nabla_t \times E_t))] \\
\omega - \frac{1}{\varepsilon} \nabla_t \times [\hat{z}(\hat{z} \cdot (\nabla_t \times E_t))]
\end{bmatrix} \begin{bmatrix}
E_t \\
H_t
\end{bmatrix}.
$$

In this form operators on the left and on the right are Hermitian thus defining a generalized Hermitian eigen problem and allowing for all the convenient properties pertaining to such a form, including real eigenvalues $\beta$ as well as orthogonality of the modes corresponding to the different $\beta$'s (for more discussion see [2]).

Defining hermitian operator on the left of (2) as $\hat{B}$ and the one on the right $\hat{A}$ and introducing Dirac notation $|\psi\rangle = \begin{bmatrix} E_t \\
H_t \end{bmatrix}$ we rewrite a generalized eigen problem as

$$\beta \hat{B} |\psi_{\beta}\rangle = \hat{A} |\psi_{\beta}\rangle.$$

\[ \text{(3)} \]
with a condition of orthogonality between modes of $\beta$ and $\beta'$ in the form

$$<\psi_\beta|\hat{B}|\psi_{\beta'}> = \delta_{\beta,\beta'}.$$  \hfill (4)

In the following we analyze uniform along $\hat{z}$ axis perturbations. If a perturbation such as general re-scaling of coordinates $x_{scaled} = x(1+\delta_x), y_{scaled} = y(1+\delta_y)$ is introduced into the system it will modify an operator $\hat{A}$. A particular case of general re-scaling when $\delta_x = \delta_y$ correspond to the uniform scaling of a structure, while the case of $\delta_x = -\delta_y$ corresponds to a uniform ellipticity. Denoting a correction to an original operator on the left of (2) $\delta \hat{A}$, the new eigen values $\tilde{\beta}$ of the split doubly degenerate eigen mode are found by solving a secular equation $|3|$ and gives

$$\beta^\pm = \beta \pm \frac{<\psi_{\beta,m} | \delta \hat{A} | \psi_{\beta,m} >}{<\psi_{\beta,m} | \hat{B} | \psi_{\beta,m} >} \pm \frac{<\psi_{\beta,m} | \delta \hat{A} | \psi_{\beta,-m} >}{<\psi_{\beta,m} | \hat{B} | \psi_{\beta,m} >}.$$  \hfill (5)

The inter-mode dispersion parameter being proportional to PMD $|4|$ is defined to be equal to the mismatch of the inverse group velocities of the split due to the perturbation modes $\tau = \frac{1}{v_g} - \frac{1}{v_g}$ which is, in turn, can be expressed in terms of the frequency derivative $\tau = \frac{\partial (\beta^+ - \beta^-)}{\partial \omega}$. Now we derive a form of the perturbation operator for the cases of uniform scaling and uniform ellipticity. We start with an elliptical waveguide and a generalized Hermitian formulation (2) where the derivatives in operator $\hat{A}$ should be understood as the derivatives over the coordinates $x_{scaled}$ and $y_{scaled}$. We then transform into the coordinate system in which an elliptical waveguide becomes cylindrical. Assuming normalization (4) after some cumbersome algebra $|5|$ we arrive at the following expressions.

Case of uniform scaling $\delta_x = \delta_y = \delta$

$$\Delta \beta_s = <\psi_{\beta,m} | \delta \hat{A} | \psi_{\beta,m} > = 2\delta \int_S ds \begin{vmatrix} E_r & \omega \epsilon & 0 & 0 & -\beta & E_r \\ E_\theta & 0 & \omega \epsilon & \beta & 0 \\ H_r & 0 & \beta & \omega & 0 \\ H_\theta & -\beta & 0 & 0 & \omega \end{vmatrix}_{\beta,m} = 2\eta \omega \int_S ds (\epsilon |E_r|^2 + |H_z|^2)$$  \hfill (6)

Another important result about change in the propagation constant of a mode under uniform scaling is that its frequency derivative is proportional to the dispersion of a mode. To derive this result we consider a dispersion relation for some mode of a waveguide $\beta = f(\omega)$. From the form of Maxwell equations it is clear that if we uniformly re-scale all the transverse dimensions in a system by a factor of $(1 + \delta)$ then the new $\tilde{\beta} = \beta + \Delta \beta_s$ for the same $\omega$ will satisfy $\tilde{\beta} = \frac{f(\omega(1+\delta))}{1+\delta}$. Decomposing the last expression in Taylor series and collecting terms of the same order in $\delta$ we derive expressions for $\Delta \beta_s$ and its derivative $\Delta \beta_s = \delta (\omega \frac{\partial \beta}{\partial \omega} - \beta)$, $\frac{\partial \Delta \beta_s}{\partial \omega} = \delta \omega \frac{\partial^2 \beta}{\partial \omega^2} = -\lambda \delta D(\omega)$ where $D(\omega)$ is a dispersion of the mode.

Case of uniform ellipticity $\delta_x = -\delta_y = \delta$. A first order correction to the split in the values of propagation constants of the modes $(\beta, 1)$ and $(\beta, -1)$ due to the uniform re-scaling becomes (5)

$$\Delta \beta_e = 2|<\psi_{\beta,1} | \delta \hat{A} | \psi_{\beta,-1} |> = 2\eta \omega \int_S ds \begin{vmatrix} E_r & -\epsilon & -i\epsilon & 0 & 0 & E_r \\ E_\theta & -i\epsilon & \epsilon & 0 & 0 \\ H_r & 0 & 0 & -1 & -i \\ H_\theta & 0 & 0 & -i & 1 \end{vmatrix}_{\beta,1} = 2\eta \omega \int_S ds [(-\epsilon |E_z|^2 + |H_z|^2) + 2Im(\epsilon E_r^* E_\theta - H_r^* H_\theta)]$$  \hfill (7)

where $E$’s and $H$’s are those of the $(\beta, 1)$ mode.

From expression (7) we find that the split between the degenerate modes due to the ellipticity is proportional to the difference in the longitudinal magnetic and electric energies in the crosssection of a fiber. The rest of the crossterms in expression (7) usually do not contribute substantially to the split, unless special structures are considered when longitudinal magnetic and longitudinal electric energies are of the same order.
Fig. 1. Effective birefringence due to the elliptical perturbation in a double core high dielectric contrast fiber. Data is presented for the fundamental linearly polarized doubly degenerate mode. While the width of a ring was kept at a constant 1 µm corresponding inner radii \( R \) was varied in the interval (3,10) µm. Split in a wavevector of an originally degenerate mode due to the uniform elliptical perturbation of magnitude \( \delta \) as predicted by the perturbation theory (circles) is compared to the results of the Finite Difference numerical simulations (crosses). Excellent correspondence over the whole range of inner radii is observed.

An important conclusion about PMD of a structure can be drawn when electric or magnetic longitudinal energy dominates substantially over the other (for a longer discussion see [5]). In the case of pure-like TE \( (E_z \sim 0) \) or TM \( (H_z \sim 0) \) modes split due to the uniform scaling (6) becomes almost identical to the split in the degeneracy of the modes due to the uniform ellipticity perturbation (7). Thus, in the case when the mode is predominantly TE or TM as judged by the amounts of the corresponding longitudinal energies in the cross section we expect \( \Delta \beta_e = \Delta \beta_c \). As PMD is proportional to the intermode dispersion parameter \( \tau = \frac{\partial \Delta \beta_e}{\partial \omega} \) and taking into account expressions for the frequency derivatives of \( \Delta \beta_e \) we arrive to the conclusion that for such a mode PMD is proportional to the dispersion of a mode

\[
\tau = \left| \frac{\partial \Delta \beta_e}{\partial \omega} \right| = \left| \frac{\partial \Delta \beta_c}{\partial \omega} \right| = \lambda \delta |D(\omega)| \tag{8}
\]

We conclude by presenting the results of calculations of the normalized birefringence due to the uniform elliptical perturbation in the case of a double core high dielectric contrast fiber Fig.1. Fundamental doubly degenerate mode of \( m = 1 \) was studied in a ring-like fiber of 1.0 to 1.5 index contrast. Excellent correspondence between the predictions of the perturbation theory and, in principle, exact Finite Difference numerical simulations is observed.

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