A New Approach to the Spectral Theory
of the Fourth Moment of the Riemann Zeta-Function

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The aim of the present article is to exhibit a method to embed the fourth power moment of the Riemann zeta-function

$$Z_2(g) = \int_{-\infty}^{\infty} |\zeta(\frac{1}{2} + it)|^4 g(t) dt$$

into the structure of $L^2(\Gamma \backslash G)$, with $\Gamma = \text{PSL}_2(\mathbb{Z})$ and $G = \text{PSL}_2(\mathbb{R})$. It is shown that there exists a $\Gamma$-automorphic function on $G$, whose value at the unit element is closely related to $Z_2(g)$, and whose spectral decomposition in $L^2(\Gamma \backslash G)$ gives rise to that of $Z_2(g)$. This amounts to an alternative and direct proof of the explicit formula for $Z_2(g)$ that was established in Chapter 4 of [7]. Especially, we are now able to dispense with the spectral theory of sums of Kloosterman sums that played an essential rôle in [7]. Our argument seems to provide a new insight into the nature of the zeta-function, particularly in its relation with linear Lie groups.

Convention. Notations are introduced where they are needed first time, and will continue to be effective thereafter. In particular, $\varepsilon$ and $B$ are positive parameters for which one may set the values, respectively, as small and large as to be appropriate at each occurrence. All implicit constants are possibly dependent on them. We stress that our choice of the pair $G$ and $\Gamma$ is made for the sake of convenience. We could work instead with the pair $\text{PGL}_2(\mathbb{R})$ and $\text{PGL}_2(\mathbb{Z})$, which is perhaps more suitable to our present purpose. We have, however, taken into account that most of recent applications of the spectral theory to the Riemann zeta and allied functions are done with the same choice of the groups as ours.

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1. Introduction. The discussion in [7] on $Z_2(g)$ begins with the expression

$$I(w; g) = \int_{-\infty}^{\infty} \zeta(w_1 - it)\zeta(w_2 + it)\zeta(w_3 + it)\zeta(w_4 - it) g(t) dt$$

$$= \sum_{a,b,c,d \geq 1} a^{-w_1} b^{-w_2} c^{-w_3} d^{-w_4} \hat{g} \left( \frac{1}{2\pi} \log(ad/(bc)) \right), \quad (1.1)$$

where $w = (w_1, w_2, w_3, w_4)$ is in the region of absolute convergence. Here, the weight function $g$ is assumed to be even, entire, and of rapid decay in any fixed horizontal strip; and for $x \in \mathbb{R}$

$$\hat{g}(x) = \int_{-\infty}^{\infty} g(t) \exp(-2\pi i xt) dt. \quad (1.2)$$

Note that $\hat{g} \in C^\infty(\mathbb{R})$, and

$$\hat{g}(x) = \hat{g}(|x|) \ll e^{-B|x|}. \quad (1.3)$$
The sum (1.1) is divided into three parts according as \( ad = bc, ad < bc \) and \( ad > bc \), which is called the Atkinson dissection:

\[
I(w; g) = \frac{\zeta(w_1 + w_2)\zeta(w_1 + w_3)\zeta(w_2 + w_4)\zeta(w_3 + w_4)}{\zeta(w_1 + w_2 + w_3 + w_4)} \hat{g}(0) + J(w; g) + J(w'; g),
\]

where \( w' = (w_2, w_1, w_4, w_3) \). A spectral argument is applied to \( J(w; g) \), with the aim to have its meromorphic continuation to \( \mathbb{C}^4 \). There the arithmetic expression

\[
J(w; g) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sigma_{w_1-w_4}(m)\sigma_{w_2-w_3}(m+n)}{m^{w_1}(m+n)^{w_2}} \hat{g}
\left( \frac{1}{2\pi} \log(1 + n/m) \right)
\]

is exploited, where \( \sigma_a(n) \) is the sum of \( a \)-th powers of divisors of \( n \). The Ramanujan expansion is applied to \( \sigma_{w_2-w_3}(m+n) \), and the Voronoi scheme is employed; namely, the functional equation for the Estermann zeta-function is invoked. This transforms (1.5) into sums of Kloosterman sums, save for a residual term. The Kloosterman–Spectral sum formula is now applied; and a spectral decomposition of (1.5) emerges, but initially only in a quite limited domain of \( w \). Certain involved technicalities have to be utilized to establish the existence of \( J(w; g) \) as a meromorphic function over \( \mathbb{C}^4 \). The success of the argument is much due to the fact that each cuspidal contribution is expressed in terms of Hecke series, which are entire and of polynomial growth. The expression (1.4) holds throughout \( \mathbb{C}^4 \) as a relation of the four meromorphic functions. The rest of the argument is to make the specialization \( w \rightarrow p_+ = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) in the decomposition.

In this way it is proved that

\[
\zeta_2(g) = M(g) + \sum_{\nu} \alpha_{\nu} H_{\nu}(\frac{1}{2})^3 \Theta(g, \nu) + \int_{0}^{1} \frac{(\zeta(\frac{1}{2} + \nu) \zeta(\frac{1}{2} - \nu))^3}{\zeta(1 + 2\nu) \zeta(1 - 2\nu)} \Theta(g, \nu) \frac{d\nu}{\pi},
\]

with \( \omega \) the vertical line passing \( \omega \). Here \( V \), with \( \nu \) the spectral data, runs over all irreducible cuspidal \( \Gamma \)-automorphic representations of \( G \); all \( V \) are assumed to be Hecke invariant. The \( H_{\nu} \) is the Hecke \( L \)-function attached to \( V \), and \( \alpha_{\nu} \) is a metric normalization factor. These concepts are to be made precise in Section 3. The \( M \) and \( \Theta \) are integral transforms; the kernel of \( M \) is given in terms of logarithmic derivatives of the Gamma function, and that of \( \Theta(g, \cdot) \) involves the Bessel function of representation. This is Theorem 4.2 of [7], with a reformulation in terms of unitary representations of \( G \).

The end result (1.6) of the above procedure does not contain any trace of the use of Kloosterman sums. The right side has a characteristic pertinent to the structure of \( L^2(\Gamma \backslash \mathbb{G}) \). From this observation, a problem comes out: Find a way to reach (1.6) as directly as possible, especially without recourse to the reduction to sums of Kloosterman sums. In what follows we shall show an answer to this basic problem in the theory of the Riemann zeta-function. It is a realization of the programme given in Section 4.2 of [7].

2. Poincaré series. That programme concerns, in hindsight, a spectral decomposition of certain Poincaré series over \( \Gamma \backslash \mathbb{G} \) whose value at the identity has the same structure as (1.5). In this section we shall fix the Poincaré series on which our discussion is to be developed.

The right side of (1.1) suggests us to use the seed

\[
G \ni g = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \rightarrow |a|^{-w_1} |b|^{-w_2} |c|^{-w_3} |d|^{-w_4} \psi(ad/(bc)),
\]

where the matrix is in the projective sense, and \( w \) is in an appropriate domain of \( \mathbb{C}^4 \). In view of (1.3), we may assume that

\[
\psi^{(l)}(x) \ll \min(|x|^B, |x|^{-B}),
\]
for any $l \geq 0$. We should remark here that the above condition on $g$ is imposed for the sake of simplicity; as is done in [7], one may suppose more generally that $g$ is regular and decays sufficiently rapidly in a fixed horizontal strip of certain width. This means in terms $\psi$ that the $B$ in (2.2) is possibly large but fixed, with which our argument below in fact works well.

To implement the Atkinson dissection, we introduce the factor $\tau(ad)$, where

$$\tau(x) = 0, \; x > 0; \quad \tau^{(l)}(x) \ll \min(|x|^B, |x|^{-B})$$

for each fixed $l$. We put

$$f_{\psi\tau}(g) = |a|^{-w_1}|b|^{-w_2}|c|^{-w_3}|d|^{-w_4} \psi(ad/(bc)) \tau(ad).$$

Then, let us consider the Poincaré series

$$\mathcal{P}f_{\psi\tau}(g) = \sum_{\gamma \in \Gamma} f_{\psi\tau}(\gamma g),$$

ignoring temporarily the convergence issue. We apply to this the operator

$$\mathcal{T}_w = \sum_{n=1}^{\infty} n^{-z_1} T_n, \quad z_1 = \frac{1}{2}(w_1 + w_2 + w_3 + w_4) - 1,$$

where $T_n$ is the Hecke operator

$$T_n = \frac{1}{\sqrt n} \sum_{d|n} \sum_{b \mod d} L_n[b/d] a[n/d^2],$$

with $L$ the left translation and $n[x] = \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right], \; a[y] = \left[\begin{smallmatrix} \sqrt n & 1 \\ 0 & \sqrt n \end{smallmatrix}\right]$. We have

$$\mathcal{T}_w \mathcal{P}f_{\psi\tau}(1) = \frac{1}{2} \sum_{g \in M_2(\mathbb{Z}) \atop \det g > 0} \left(\det g\right)^{-z_1-1} f_{\psi\tau} \left((\det g)^{-z_1} g\right)$$

$$= 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{w_1-w_4} \frac{\sigma_{w_1-w_4}(m) \sigma_{w_2-w_3}(m+n)}{w_2(m+n)w_2} \psi(m/(m+n)) \tau(-m/n).$$

We may take the limit as $\tau$ tends to the characteristic function of the negative reals. The result is comparable with (1.5).

Thus, (2.5) can be dealt with. However, in general the series does not converge for all $g$. To see this, let $\gamma_0 \in \Gamma$ be a hyperbolic element, and $g_0 \in G$ be such that $g_0^{-1} \gamma_0 g_0 = a[\lambda]$ with $\lambda > 1$. Then for any integer $n$

$$f_{\psi\tau}(\gamma_0^n g_0) = f_{\psi\tau}(g_0) \lambda^{\frac{1}{2}(w_2+w_4-w_1-w_3)}.$$

which obviously implies the divergence of $\mathcal{P}f_{\psi\tau}$ at $g_0$, provided $f_{\psi\tau}(g_0) \neq 0$. Hence $\mathcal{T}_w \mathcal{P}f_{\psi\tau}$ is not well-defined. To overcome this difficulty, we introduce the modification

$$f_{\psi\tau\eta} = |a|^{-w_1}|b|^{-w_2}|c|^{-w_3}|d|^{-w_4} \psi(ad/(bc)) \tau(ad) \eta(d/c),$$

with an $\eta$ satisfying

$$\eta(-x) = \eta(x); \quad \eta^{(l)} \ll \min(|x|^B, |x|^{-B})$$

for each fixed $l$. 

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Lemma 1. Let $f = f_{\psi_{\tau}}$, with bounded $w$ in the domain

$$\Re (w_3 + w_4) > 2 + \Re (w_1 + w_2) > 4. \quad (2.12)$$

Then the Poincaré series $T \! f$ converges absolutely and uniformly to a smooth $\Gamma$-automorphic function on $G$. The same holds for $T_w f$, and in particular

$$T_w T f(1) = 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi(m/(m+n)) \tau(-m/n) \sum_{ad=m} b \sum_{bc=m+n} \sum_{a,b,c,d} \eta(d/c) \frac{\eta(d/c)}{a^{w_1} b^{w_2}} e^{w_3 d^{w_4}}, \quad (2.13)$$

with positive integers $a, b, c, d$.

**Remark.** Throughout the sequel, this definition for $f$ will be retained. The condition $\Re (w_3 + w_4) > 2 + \Re (w_1 + w_2) > 3$ is sufficient for the convergence of $T \! f$, but $T_w \! f$ requires $2.8$. The heuristic identity $2.8$ can be understood to be the limit of $2.13$ as $\eta$ tends to the characteristic function of $\Re \{ 0 \}$. Also, the limiting procedure mentioned above with respect to $\tau$ is to be considered. We shall perform this with an explicit choice of $\tau$ and $\eta$, in the final section.

**Proof.** Let $G = AN \sqcup ANwN$ be the Bruhat decomposition of $G$, where $w = \begin{bmatrix} -1 \end{bmatrix}$, $A = \{ a[y] \mid y > 0 \}$, $N = \{ n[x] \mid x \in \mathbb{R} \}$. Also, let $G = NAK$ be the Iwasawa decomposition, with $K = \{ k[\theta] = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \mid \theta \in \mathbb{R}/(\pi\mathbb{Z}) \}$, which we shall read as $g = n ak = n[x]a[y]k[\theta]$. We have, for $\sin \theta \neq 0$, i.e., in the big Bruhat cell,

$$n[x]a[y]k[\theta] = \begin{bmatrix} \sqrt{y}/\sin \theta \\ \sin \theta/\sqrt{y} \end{bmatrix} n[x y^{-1} \sin^2 \theta - \sin \theta \cos \theta] w n[\cot \theta]. \quad (2.14)$$

Note that $f$ is left $A$-equivariant:

$$f(\begin{bmatrix} a[y] \end{bmatrix}) = y^{z_2} f(g), \quad z_2 = 1/2 (w_3 + w_4 - w_1 - w_2). \quad (2.15)$$

Since $f$ vanishes on $AN$, we can restrict ourselves to the case $\sin \theta \neq 0$. We have, by (2.14),

$$f(n[x]a[y]k[\theta]) \ll y^{\Re z_2} |x/y - \cot \theta|^{\Re w_1} |x/y + \tan \theta|^{\Re w_2} \times |\sin \theta|^{\Re (w_1+w_3)} |\cos \theta|^{\Re (w_2+w_4)} \left| \psi \left( \frac{x/y - \cot \theta}{x/y + \tan \theta} \right) \eta(-\cot \theta) \right|. \quad (2.16)$$

We claim that if $\Re w_j$ are all bounded then

$$|\sin \theta|^{-w_1 - w_3} |\cos \theta|^{-w_2 - w_4} \eta(-\cot \theta) \ll 1, \quad (2.17)$$

and if moreover $\Re (w_1 + w_2) \geq 0$ then

$$|x/y - \cot \theta|^{-w_1} |x/y + \tan \theta|^{-w_2} \psi \left( \frac{x/y - \cot \theta}{x/y + \tan \theta} \right) \ll \min(1, |x/y|^{-\Re (w_1+w_2)}). \quad (2.18)$$

To prove (2.17) we may assume that $|\cos \theta| \leq |\sin \theta|$, and consequently $|\cos \theta| \leq 1/\sqrt{2} \leq |\sin \theta|$. Then, by (2.11) the left side is $\ll |\sin \theta|^{B - \Re (w_1+w_3)} |\cos \theta|^{B - \Re (w_2+w_4)} \ll 1$. To prove (2.18) we need to consider the two cases $|x/y| < 2$ and $|x/y| \geq 2$ separately. In the first case we note that $|x/y - \cot \theta| + |x/y + \tan \theta| \geq |\cot \theta + \tan \theta| \geq 2$. Thus we may assume, for instance, that $U = |x/y - \cot \theta| \geq 1$. We have, with $V = |x/y + \tan \theta|$, that either $V \geq U \geq 1$ or $U \geq V \geq 1$ or $U \geq V \geq 1$. The left side of (2.18) is, by (2.2),

$$\ll U^{-\Re w_1} V^{-\Re w_2} \min((U/V)^B, (V/U)^B) \leq 1, \quad (2.19)$$
provided $\Re(w_1 + w_2) \geq 0$. In the remaining case the left side of (2.18) is
\begin{equation}
\ll |x/y|^\Re(w_1 + w_2) U_1^{-\Re w_1} V_1^{-\Re w_2} \min((U_1/V_1)^B, (V_1/U_1)^B),
\end{equation}
where $U_1 = |1 - (y/x) \cot \theta|$, $V_1 = |1 + (y/x) \tan \theta|$. We may assume, for instance, that $|\tan \theta| \leq 1$. Then we have $\frac{1}{2} \leq V_1 \leq \frac{3}{2}$, and thus (2.20) is $\ll |x/y|^\Re(w_1 + w_2)$, which gives (2.18). Summing up, we have
\begin{equation}
f(n[x]a[y]k[\theta]) \ll y^{\Re z_2} \min\left(1, |x/y|^\Re w_1 + w_2\right),
\end{equation}
provided $\Re w_j$ are all bounded and $\Re(w_1 + w_2) \geq 0$. It should be remarked that this is proved without taking into account the effect of the factor $\tau(\alpha \beta d)$. The bound (2.21) gives
\begin{equation}
\sum_{\mu \in \Gamma_\infty} |f(\mu g)| \ll (1 + y) y^{\Re z_2}, \quad \Gamma_\infty = \Gamma \cap N,
\end{equation}
if $\Re(w_1 + w_2) > 1$, and it follows that
\begin{equation}
\mathcal{P}f(g) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \sum_{\mu \in \Gamma_\infty} f(\mu \gamma g)
\end{equation}
is absolutely convergent for any $g$ if (2.12) holds. In fact this is the result of comparing (2.23) with the Eisenstein series
\begin{equation}
E_p(g; \nu) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \phi_p(\gamma g; \nu), \quad \phi_p(g; \nu) = y^{\frac{1}{2} + \nu} \exp(2ip\theta),
\end{equation}
which converges absolutely for $\Re \nu > \frac{1}{2}$. The assertion on the convergence of $\mathcal{P}_w \mathcal{P}f$ is now immediate, and the formula (2.13) follows readily.

It remains to prove that $\mathcal{P}f$ is smooth. We shall restrict ourselves to the case where none of the elements of $g$ is equal to 0, for $\tau(\alpha \beta d) \neq 0$ implies this. The boundary situation can be discussed likewise. Let $\mathfrak{g}$ be the Lie algebra of $G$, and $\mathfrak{g}$ its universal enveloping algebra. Note that $\mathfrak{g}$ is spanned by $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Computing it explicitly, we see that $Xf(g) = (d/dt)_{t=0} f(g \exp(Xt))$ with $g$ as in (2.1) is a linear combination of the five functions
\begin{align}
a[a]^{-w_1} b^{-w_2} c^{-w_3} d^{-w_4} \psi(\alpha d/\beta c))\tau(\alpha d)\eta(d/c), \\
c[a]^{-w_1} b^{-w_2} c^{-w_3} d^{-w_4} \psi(\alpha d/\beta c))\tau(\alpha d)\eta(d/c), \\
a^{-w_1} b^{-w_2} c^{-w_3} d^{-w_4} \psi(\alpha d/\beta c))\tau(\alpha d)\eta(d/c), \\
a^{-w_1} b^{-w_2} c^{-w_3} d^{-w_4} \psi(\alpha d/\beta c))\tau(\alpha d)\eta(d/c), \\
a^{-w_1} b^{-w_2} c^{-w_3} d^{-w_4} \psi(\alpha d/\beta c))\tau(\alpha d)\eta(d/c).
\end{align}
They are majorized by the right side of (2.21), and $\mathcal{P}Xf$ is absolutely and uniformly convergent throughout $G$, provided (2.12). That is, $X\mathcal{P}f = \mathcal{P}Xf$. Similarly, one may show the same for $Y\mathcal{P}f$ and $H\mathcal{P}f$. This procedure can of course be repeated indefinitely. Hence, for any $u \in \mathfrak{g}$ we have proved that $u\mathcal{P}f = \mathcal{P}uf$ in the pointwise sense, if (2.12) holds. We end the proof of the lemma.
More precisely, we have, for any fixed \( u \),

\[
  uPf(g) = \sum_{\mu \in \Gamma_\infty} u_f(\mu g) + \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_\infty} \sum_{\mu \in \Gamma_\infty} u_f(\mu \gamma g)
  = \sum_{\mu \in \Gamma_\infty} u_f(\mu g) + O\left(y^{1-\Re z_2}\right),
\]

as \( y \uparrow \infty \), provided \( w \) is as in Lemma 1. By Poisson’s sum formula

\[
  \sum_{\mu \in \Gamma_\infty} u_f(\mu g) = \int_{-\infty}^{\infty} u_f(n[u]g)\,du + O\left(y^{1-\Re z_2}\right),
\]

as \( y \uparrow \infty \). To see this, we use that

\[
  u_f(n[x]a[y]k[\theta]) = y^z u_f(n[x/y]k[\theta]),
\]

and hence

\[
  \int_{-\infty}^{\infty} u_f(n[x]a[y]k[\theta])\,dx = y^{z+1} \int_{-\infty}^{\infty} u_f(n[z]k[\theta])\,dx,
\]

which is \( \ll (|m| + 1)^{-B} \) via integration by parts. The relations (2.26)–(2.28) show that \( uPf \) is not in \( L^2(\Gamma \setminus \Gamma_\infty) \), in general. Because of this, we subtract a \( \Gamma \)-invariant function from \( uPf \) to have a square integrable function: We put

\[
  P_0f = Pf - P_\infty f,
\]

where

\[
  P_\infty f(g) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_\infty} \int_{-\infty}^{\infty} f(n[u]\gamma g)\,du,
\]

the convergence of which follows from (2.28) with \( m = 0, \ u = 1 \). An examination of the above shows readily that for each \( u \in \mathbb{U} \)

\[
  uP_0f \in L^2(\Gamma \setminus \Gamma_\infty),
\]

provided (2.12).

Observe that

\[
  \int_{-\infty}^{\infty} f(n[u]g)\,du = \sum_p f_p \phi_p(g; z_2 + \frac{1}{2}),
\]

with \( f_p \ll (|p| + 1)^{-B} \). Hence

\[
  P_\infty f(g) = \sum_p f_p E_p(g; z_2 + \frac{1}{2}).
\]

Also, a computation shows that

\[
  \int_{-\infty}^{\infty} f(n[u]g)\,du = \frac{\psi(x/(x+1))\tau(-x)}{x^{w_1}(x+1)^{w_2}}\eta(\psi(x/(x+1)))\int_{0}^{\infty} dx,
\]

with \( g \) as in (2.1). In particular,

\[
  P_\infty f(1) = 2\psi(0) \sum_{c=1}^{\infty} \sum_{d=1}^{\infty} c^{w_2-w_3-1} d^{w_1-w_4-1} \eta(\psi(1)),
\]

(2.35)
orthogonal decomposition into invariant subspaces

Here $0$ $N$ respect to the left action of $d$ short, which are square integrable against the measure $\Gamma$ $E$ by integrals of $\nu$.

we shall perform an initial reduction for the spectral decomposition of $\mathcal{H}$ Haar measures on the groups $d$

space $L$ where $d$ runs over all integers. If it is not trivial, $V$ is the set of all smooth vectors in $\mathcal{H}$.

It is called a positive odd integer. According to the right action of $K$, the space $V$ is decomposed into $K$-irreducible subspaces

$$V = \bigoplus_p V_p, \quad \dim V_p \leq 1,$$

which is implicit in (1.6). The Casimir operator becomes a constant multiplication in each $V$; that is,

$$\Omega|_{V^\infty} = (\nu_p^2 - \frac{1}{4}) \cdot 1, \quad \Omega = y^2 (\partial_x^2 + \partial_y^2) - iy\partial_x\partial_y,$$

where $V^\infty$ is the set of all smooth vectors in $V$. Under our present supposition that $\Gamma = \text{PSL}_2(\mathbb{Z})$, we can restrict our attention to two cases: either $\nu_\Gamma < 0$ or $\nu_\Gamma$ is equal to half a positive odd integer. According to the right action of $K$, the space $V$ is decomposed into $K$-irreducible subspaces

$$0L^2(\Gamma\backslash G) = \bigoplus V,$$

where $p$ runs over all integers. If it is not trivial, $V_p$ is spanned by a $\Gamma$-automorphic function on which the right translation by $k[\theta]$ becomes the multiplication by the factor $\exp(2ip\theta)$. It is called a $\Gamma$-automorphic form of spectral parameter $\nu_\Gamma$ and weight $2p$.

Let us assume temporarily that $V$ belongs to the unitary principal series, i.e., $\nu_\Gamma < 0$. Then one can show that $\dim V_p = 1$ for all $p \in \mathbb{Z}$ and that there exists a complete orthonormal system $\{\varphi_p \in V_p : p \in \mathbb{Z}\}$ of $V$ such that

$$\varphi_p(g) = \sum_{n \neq 0} \frac{\varphi_v(n)}{\sqrt{|n|}} A^{q_\text{sgn}(n)} \delta_p n |g; \nu_\Gamma|,$$

with

$$A^0 \phi_p(g; \nu) = \int_{-\infty}^{\infty} \exp(-2\pi i\delta x) \phi_p(wn[x]g; \nu) dx, \quad \delta = \pm.$$

The $A^0$ is a specialization of the Jacquet operator. This follows from a study of the Fourier coefficients of $\varphi_p$. We note that

$$A^0 \phi_p(g; \nu) = y^{\frac{3}{2} - \nu} \exp(2\pi i\delta x) \int_{-\infty}^{\infty} \exp\left(\frac{2\pi iy\xi}{\sqrt{2}}\right) \delta_p n \left(\frac{\xi + i}{\xi - i}\right)^{\delta} d\xi \cdot \exp(2pi\theta)$$

$$= (-1)^p \pi^{\frac{1}{2} + \nu} \exp(2\pi i\delta x) \frac{W_{\delta p, \nu}(4\pi y)}{\Gamma(\delta p + \frac{1}{2} + \nu)} \exp(2pi\theta), \quad (3.7)$$
where $W_{\lambda,\mu}(y)$ is the Whittaker function. The first line is valid for $\text{Re}\nu > 0$, while the second defines $A^\delta \phi_p$ for all $\nu \in \mathbb{C}$. It should be observed that the coefficients $\varrho_V(n)$ in (3.5) do not depend on the weight, a fact that can be shown by using the Maass operators. In particular, we have the expansion

$$
\varphi_0(g) = \frac{2\pi^{\frac{1}{2}+\nu}}{\Gamma(\frac{1}{2}+\nu)} \sqrt{y} \sum_{n \neq 0} \varrho_V(n) \exp(2\pi inx) K_{\nu}(2\pi|n|y),
$$

(3.8)

where $K_{\nu}$ is the $K$-Bessel function of order $\nu$. This corresponds to the Fourier expansion of cuspidal Maass forms of weight zero on the upper half plane, but with a normalization different from that in (1.1.33) of [7].

We may assume that each $V$ is Hecke invariant; that is, for all $n \geq 1$,

$$
T_n|_V = t_V(n) \cdot 1
$$

(3.9)

with a $t_V(n) \in \mathbb{R}$. Also, the invariance

$$
\varphi_0(n^{-1}a) = \epsilon_V \varphi_0(na), \quad \epsilon_V = \pm 1,
$$

(3.10)

can be assumed. Thus we have

$$
\varrho_V(n) = \varrho_V(1) \epsilon_V^{(1-sgn(n))} t_V(|n|).
$$

(3.11)

Next, let us consider a $V$ in the discrete series; that is, $\nu_V = \ell - \frac{1}{2}$, $1 \leq \ell \in \mathbb{Z}$. We have either

$$
V = \bigoplus_{p \geq \ell} V_p, \quad \dim V_p = 1,
$$

(3.12)

or

$$
V = \bigoplus_{p \leq -\ell} V_p, \quad \dim V_p = 1.
$$

(3.13)

The involution $g = nak \mapsto n^{-1}ak^{-1}$ interchanges the rôle of these two. As a counterpart of (3.5), we have, in the complete orthonormal system $\{\varphi_p : p \geq \ell\}$ in $V$, such that

$$
\varphi_p(g) = \pi^{\frac{1}{2} - \ell} \left( \frac{\Gamma(p + \ell)}{\Gamma(p - \ell + 1)} \right)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\varrho_V(n)}{\sqrt{n}} \phi_p(a[n]g; \ell - \frac{1}{2}).
$$

(3.14)

The same as (3.9) can be assumed, and thus $\varrho_V(n) = \varrho_V(1) t_V(n)$. In particular, we have

$$
\varphi_\ell(g) = (-1)^{\ell} \frac{2^2 \pi^{\ell+\frac{1}{2}}}{\sqrt{\pi}(2\ell)\Gamma(2\ell)} \exp(2i\ell\theta) \sum_{n=1}^{\infty} \varrho_V(n) n^{\ell-\frac{1}{2}} \exp(2\pi in(x + iy)),
$$

(3.15)

which corresponds to (2.2.3) of [7].

In passing, we note that as $M \uparrow \infty$

$$
\sum_{|\nu_V| \leq M} |\varrho_V(1)|^2 \ll M^2,
$$

(3.16)

with the implied constant being absolute. See Lemmas 2.3 and 2.4 of [7], while noting our present normalization (3.8) and (3.15), or more precisely, the second line of (6.23) as well as (6.27) below.
Remark. The proof of this in [7] may appear to come rather close to the spectral theory of sums of Kloosterman sums. A closer examination will, however, reveal that the proof depends only on a non-trivial bound for individual Kloosterman sums; Estermann’s elementary bound works fine.

To each $V$, in both types of cuspidal representations, we associate the Hecke series

$$H_V(s) = \sum_{n=1}^{\infty} t_V(n)n^{-s}. \quad (3.17)$$

This converges absolutely for $\text{Re} \ s > \frac{1}{2}$, for we have

$$t_V(n) \ll n^{\frac{1}{2} + \epsilon}, \quad (3.18)$$

with the implicit constant depending only on $\epsilon$. It is known, however, that (3.17) in fact converges absolutely for $\text{Re} \ s > 1$, and there an Euler product representation holds also. Further, $H_V$ continues to an entire function, satisfying the functional equation

$$H_V(s) = 2^{2s-1} \pi^{2s-1} \Gamma(1-s+\nu_V)\Gamma(1-s-\nu_V) \times \{\epsilon_V \cos \pi \nu_V - \cos \pi s\} H_V(1-s), \quad (3.19)$$

where $\epsilon_V \cos \pi \nu_V = 0$ for $V$ in the discrete series. In particular, the Phragmén–Lindelöf convexity principle implies that $H_V(s)$ is of polynomial order with respect to both $\nu_V$ and $\text{Im} \ s$, with the exponent as well as the implied constant depending only on $\text{Re} \ s$. See Chapter 3 of [7].

As to the Eisenstein series, we have the Fourier expansion

$$E_p(g; \nu) = \phi_p(g; \nu) + c_p(\nu)\phi_p(g; -\nu) + \frac{1}{\zeta(1+2\nu)} \sum_{n \neq 0} |n|^{-\frac{2}{2} - \nu} \sigma_{2\nu}(|n|) A^{\text{sgn}}(n) \phi_p(a||n||g; \nu), \quad (3.20)$$

provided the right side is finite, where

$$c_p(\nu) = \frac{(-1)^p \pi \Gamma(2\nu) \zeta(2\nu)}{2^{2\nu-1} \pi \Gamma(1+\frac{3}{2} + \nu + p) \Gamma(\frac{3}{2} + \nu - p)}. \quad (3.21)$$

We have the functional equation

$$E_p(g; \nu) = c_p(\nu)E_p(g; -\nu), \quad c_p(\nu)c_p(-\nu) = 1. \quad (3.22)$$

The proof of (3.20)–(3.22) is the same as that of Lemma 1.2 of [7].

With this, we state:

**Lemma 2.** Let $\varpi_V$ be the orthogonal projection to $V$, and $\varpi_E$ to $^0L^2(\Gamma \setminus G)$. Let $\varphi$ be a vector such that $u\varphi \in L^2(\Gamma \setminus G)$ for any $u \in U$. Then the spectral decomposition

$$\varphi(g) = \frac{3}{\pi} \langle \varphi, 1 \rangle r_{\Gamma \setminus G} + \sum_V \varpi_V \varphi(g) + \varpi_E \varphi(g) \quad (3.23)$$

converges absolutely for each $g$. Similarly

$$\varpi_V \varphi = \sum_p \langle \varphi, \varphi_p \rangle r_{\Gamma \setminus G} \varphi_p, \quad (3.24)$$
where \( \varphi_p \) are as above together with an obvious convention for \( V \) in the discrete series. Also

\[
\varphi E \varphi(g) = \sum_p \int_{(0)} e_p(\varphi; \nu) E_p(g; \nu) \frac{d\nu}{4\pi i},
\]

(3.25)

with

\[
e_p(\varphi; \nu) = \int_{\Gamma \setminus G} \varphi(g) \overline{E_p(g; \nu)} \, dg.
\]

(3.26)

**Proof.** This assertion is taken from Section 1.2 of [3], which is based on [4]. Other approaches are possible. See the remark at the end of this section.

Hence, we have, by (2.31), a pointwise spectral decomposition of \( \mathcal{P}_0 f \). We may put the result as

\[
\mathcal{P} f(g) = \mathcal{P}_\infty f(g) + \sum_V \varphi V \mathcal{P}_0 f(g) + \varphi E \mathcal{P}_0 f(g),
\]

(3.27)

with

\[
\varphi E \mathcal{P}_0 f(g) = \sum_p \int_{(0)} e_p^{(1)}(\mathcal{P}_0 f; \nu) E_p(g; \nu) \frac{d\nu}{4\pi i}.
\]

(3.28)

Here, \( e_p^{(1)} \) is the part of \( e_p \) corresponding to the third term on the right of (3.20). The identity (3.27) depends on the fact that \( \langle \mathcal{P}_0 f, 1 \rangle_{\Gamma \setminus G} = 0 \) and \( e_p(\mathcal{P}_0 f; \nu) = e_p^{(1)}(\mathcal{P}_0 f; \nu) \) for all \( p \). Both follow readily from the definition (2.29) of \( \mathcal{P}_0 f \).

The last sum can be taken inside the integral as Lemma 2 asserts. Applying the Hecke operator to (3.27) and (3.28) termwise, and invoking (3.18), we get, on (2.12),

\[
\mathcal{T}_w \mathcal{P} f(g) = \mathcal{T}_w \mathcal{P}_\infty f(g) + \sum_V \mathcal{T}_w \varphi V \mathcal{P}_0 f(g) + \mathcal{T}_w (\varphi E + \varphi E) \mathcal{P}_0 f(g),
\]

(3.29)

where

\[
\mathcal{T}_w \mathcal{P}_\infty f(g) = \zeta(w_1 + w_2 - 1) \zeta(w_3 + w_4) \mathcal{P}_\infty f(g),
\]

\[
\mathcal{T}_w \varphi V \mathcal{P}_0 f(g) = H_V(z_1 + \frac{1}{2}) \varphi V \mathcal{P}_0 f(g),
\]

\[
\mathcal{T}_w \varphi E \mathcal{P}_0 f(g) = \int_{(0)} \zeta(z_1 + \frac{1}{2} + \nu) \zeta(z_1 + \frac{1}{2} - \nu) E^{(j)}(\mathcal{P}_0 f; g, \nu) \frac{d\nu}{4\pi i},
\]

(3.30)

with \( z_1 \) as in (2.6). Here we have used \( T_n E_p(g; \nu) = n^{-\nu} \sigma_{2\nu}(n) E_p(g; \nu) \), and put

\[
E^{(j)}(\mathcal{P}_0 f; g, \nu) = \sum_p e_p^{(1)}(\mathcal{P}_0 f; \nu) E_p^{(j)}(g; \nu),
\]

(3.31)

where \( E_p^{(0)} \) is the sum of the first two terms on the right of (3.20) and \( E_p^{(1)} \) the rest. Observe that the three Hecke series in (3.30), i.e., \( H_V(z_1 + \frac{1}{2}) \) and its analogues, are all absolutely bounded under (2.12).

In view of (2.13), we shall use (3.29) when \( g \) is the unit element. Namely, our original problem has been reduced to computing the quantities \( \varphi V \mathcal{P}_0 f(1) \) and \( E^{(j)}(\mathcal{P}_0 f; 1, \nu) \), for we have already (2.35).

**Remark.** The spectral decomposition in \( L^2(\Gamma \setminus G) \) can be derived, via the Fourier expansion with respect to the right action of \( K \), from that in \( L^2(\Gamma \setminus \mathbb{H}) \) with \( \mathbb{H} \) the hyperbolic upper half plane, thus for instance, from a minor extension of Chapter 1 of [7]. Naturally, the pointwise convergence in (3.27) is crucial for our purpose. Thus, it should be stressed that there is a way, based on explicit estimation, to achieve the same without recourse to Lemma 2 but
rather starting with the convergence in \(L^2(\Gamma \backslash G)\). The necessary estimate is in fact provided by (5.7) below for both the unitary principal series representations and the Eisenstein part. As to the discrete series representations, the same follows from a combination of (5.20), (5.24) and (5.30) but with \(R_k f\) in place of \(f\), where \(R_k\) is the right translation by an element in \(K\). Another alternative is to begin similarly and appeal to the Sobolev inequality; see, e.g., p. 393 of [6].

4. Big cell. The unfolding argument reduces our task further to an application of the harmonic analysis in the big cell of the Bruhat decomposition, as will be seen in the next section. Hence we shall collect here fundamentals in this context, which may be termed the Kirillov scheme.

We first extend (3.6) by

\[
A^\delta \phi(g) = \sum_p c_p A^\delta \phi_p, \quad \phi = \sum_p c_p \phi_p, \tag{4.1}
\]

where \(\phi\) is smooth, i.e., \(|c_p| \ll (|p| + 1)^{-B}\). Note that we shall occasionally omit to mention \(\nu\). We shall show that (4.1) exists for any \(\nu\). For this and other purposes, the following estimates will be useful; bounds up to (4.5) are all uniform for \(p\) and \(|\text{Re}\nu| < \frac{1}{2}\).

The first line of (3.7) gives

\[
A^\delta \phi_p(a[y]) = A^\delta \phi_0(a[y]) + y^{\frac{1}{2} - \nu} \int_{-\infty}^{\infty} \frac{\exp(2\pi i y \xi)}{(\xi^2 + 1)^{\frac{3}{2} + \nu}} \left( \frac{\xi + i}{\xi - i} \right)^{\delta p} d\xi
\]

By the power series expansion for \(K_\nu\), we get, as \(y \downarrow 0\),

\[
A^\delta \phi_p(a[y]) \ll (|p| + |\nu| + 1) y^{\frac{1}{2} - |\text{Re}\nu| - \epsilon}. \tag{4.3}
\]

On the other hand, we have, by integration by parts,

\[
A^\delta \phi_p(a[y]) = \frac{y^{\frac{1}{2} - \nu}}{2\pi i} \int_{-\infty}^{\infty} \frac{(1 + 2\nu)\xi + 2\delta p) \exp(2\pi i y \xi)}{\left(\xi^2 + 1\right)^{\frac{3}{2} + \nu}} \left( \frac{\xi + i}{\xi - i} \right)^{\delta p} d\xi. \tag{4.4}
\]

Shifting the contour to \(\text{Im}\ \xi = (|\nu| + 1)^{-1}\), we see that

\[
A^\delta \phi_p(a[y]) \ll (|p| + |\nu| + 1) y^{\frac{1}{2} - |\text{Re}\nu|} \exp \left( -\frac{y}{|\nu| + 1} \right). \tag{4.5}
\]

Repeating integration by parts in (4.4), we find that (4.1) converges in any fixed vertical strip of \(\nu\). Note that

\[
A^\delta \phi = \int_{\mathbb{R}} \exp(-2\pi \delta i x)\phi(wn[x]g)dx, \tag{4.6}
\]

for those \(\nu\) in the domain where the integral converges uniformly. In fact the equality holds at least for \(\text{Re}\nu > 0\), and the assertion follows with analytic continuation.

We then define the Kirillov map \(\mathcal{K}\) by

\[
\mathcal{K}\phi(u) = A^\mathcal{K} \phi(a[|u|]), \quad u \in \mathbb{R}^\times = \mathbb{R} \setminus \{0\}. \tag{4.7}
\]

This concept will play a crucial rôle in our argument, via the following three lemmas:
Lemma 3. Let $\phi$ be smooth as in (4.1). We have, with the right translation $R,$
\begin{equation}
KR_{n(x)}\phi(u) = \exp(2\pi iux)K\phi(u), \quad KR_{a[y]}\phi(u) = K\phi(u).
\end{equation}
Also, if $|\Re \nu| < \frac{1}{2},$ then
\begin{equation}
KR_{\nu}\phi(u) = \int_{\mathbb{R}^\times} j_{\nu}(u\lambda)K\phi(\lambda)d^2\lambda, \quad d^2\lambda = d\lambda/|\lambda|,
\end{equation}
where
\begin{equation}
J_{\nu}(u) = \frac{\sqrt{|u|}}{\sin \pi\nu} \left(J_{-2\nu}^{\text{sgn}(u)}(4\pi \sqrt{|u|}) - J_{2\nu}^{\text{sgn}(u)}(4\pi \sqrt{|u|})\right)
\end{equation}
with $J_{\nu}^+ = J_{\nu}$ and $J_{\nu}^- = I_{\nu}$ in the ordinary notation for Bessel functions.

Proof. This is probably due originally to N.Ja. Vilenkin (see Section 7 of Chapter VII, [9]). A rigorous proof can be found in Theorem 2 of [8], which is developed in the context of automorphy but in fact asserts the above. It is shown there that the function
\begin{equation}
\Gamma_{p}(s) = \int_{0}^{\infty} A^+ \phi_{p}(a[y])y^{s-\frac{1}{2}}dy
\end{equation}
continues meromorphically to $\mathbb{C},$ and satisfies the Jacquet–Langlands local functional equation
\begin{equation}
(-1)^{p}\Gamma_{p}(s) = 2^{-2s-1} \Gamma(s+\nu)\Gamma(s-\nu)
\end{equation}
\begin{equation}
\times \left(\cos \pi s \Gamma_{p}(1-s) + \cos \pi \nu \Gamma_{-p}(1-s)\right).
\end{equation}
The Mellin inversion of this gives (4.9) for $\phi = \phi_{p}.$ A combination of (4.3), (4.5) and
\begin{equation}
j_{\nu}(u) \ll \begin{cases} |u|^{\frac{1}{2}-|\Re \nu|} & \text{if } |u| \leq 1, \\ |u|^\frac{1}{2} & \text{otherwise,} \end{cases}
\end{equation}
for any bounded $\nu$ with $|\Re \nu| < \frac{1}{2},$ yields the necessary analytic continuation in $\nu,$ and the extension to smooth $\phi.$ The first case in (4.13) follows from the series expansions of the relevant Bessel functions, and the second from their well-known asymptotic expansions.

Lemma 4. Let $\nu \in i\mathbb{R},$ and introduce the Hilbert space
\begin{equation}
U_{\nu} = \bigoplus_{p} \mathbb{C}\phi_{p}, \quad \phi_{p}(g) = \phi_{p}(g;\nu),
\end{equation}
equipped with the norm
\begin{equation}
\|\phi\|_{\nu} = \sqrt{\sum_{p}|c_{p}|^2}, \quad \phi(g) = \sum_{p} c_{p}\phi_{p}(g).
\end{equation}
Then $K$ is a unitary map from $U_{\nu}$ onto $L^{2}(\mathbb{R}^{\times}, \pi^{-1}d^{\times}).$

Proof. This seems to stem from A.A. Kirillov [5]. A proof of the unitarity is given in Theorem 1 of [8], though disguised in the context of automorphy. It depends on the following integral formula: For any $\alpha, \beta \in \mathbb{C}$ and $|\Re \nu| < \frac{1}{2}$
\begin{equation}
\int_{0}^{\infty} W_{\alpha,\nu}(u)W_{\beta,\nu}(u)\frac{du}{u} = \frac{\pi}{(\alpha - \beta) \sin(2\pi \nu)}
\end{equation}
\begin{equation}
\times \left[ \frac{1}{\Gamma(\frac{1}{2} - \alpha + \nu)\Gamma(\frac{1}{2} - \beta - \nu)} - \frac{1}{\Gamma(\frac{1}{2} - \alpha - \nu)\Gamma(\frac{1}{2} - \beta + \nu)} \right].
\end{equation}
The proof in [8] of this employs the Whittaker differential equation. Here we shall show the surjectivity of the map. Thus, let us assume that \( \nu \in i\mathbb{R} \), and that a smooth function \( \omega \), compactly supported on \( \mathbb{R}^\times \), is orthogonal to all \( \mathcal{K}\phi_p \). Multiply (4.4) by \( \omega \) and integrate, change the order of integration, and undo the integration by parts with respect to the outer integral. We have

\[
0 = \int_{\mathbb{R}^\times} \omega(u)\overline{\mathcal{K}\phi_p(u)}du
= \int_{-\infty}^{\infty} \frac{1}{(\xi^2 + 1)^{\frac{1}{2}+\nu}} \left( \frac{\xi + i}{\xi - i} \right)^p \int_{-\infty}^{\infty} \omega(u)|u|^{-\frac{1}{2}+\nu} \exp(-2\pi iu\xi)du d\xi. \tag{4.17}
\]

Observe that the system \( \{(\xi + i)/(\xi - i)^p : p \in \mathbb{Z}\} \) is complete orthonormal in the space \( L^2(\mathbb{R}, d\xi/\pi(\xi^2 + 1)) \). Hence the Fourier transform of \( \omega(u)|u|^{-\frac{1}{2}+\nu} \) vanishes identically, whence the assertion.

Next, we shall consider the complementary series. This is included here only for the sake of completeness; such a representation of \( G \) in \( L^2(\Gamma \backslash G) \) does not occur, as indicated above. Obviously, Lemma 2 remains valid. The (4.14) is the same, but (4.15) is replaced by the norm

\[
\| \phi \|_{D_\ell} = \sqrt{\pi^{2\nu} \sum_{p \geq \ell} \frac{\Gamma(p - \ell + 1)}{\Gamma(p + 1)} |c_p|^2}, \quad -\frac{1}{2} < \nu < \frac{1}{2}. \tag{4.18}
\]

With this, the above proof extends readily, and Lemma 4 holds for these \( \nu \) as well.

On the other hand, in dealing with the discrete series, (4.14) needs to be replaced by the Hilbert space

\[
D_\ell = \bigoplus_{p \leq \ell} \mathbb{C}\phi_p, \quad \nu = \ell - \frac{1}{2}, \quad 1 \leq \ell \in \mathbb{Z}, \tag{4.19}
\]

equipped with the norm

\[
\| \phi \|_{D_\ell} = \sqrt{\pi^{2\ell-1} \sum_{p \geq \ell} \frac{\Gamma(p - \ell + 1)}{\Gamma(p + \ell + 1)} |c_p|^2}, \quad \phi = \sum_{p \geq \ell} c_p\phi_p. \tag{4.20}
\]

We need also to treat a similar space with \( p \leq -\ell \), but we skip it because the discussions are identical. Since \( A^- \) annihilates \( D_\ell \), we are concerned with \( A^+ \) only. The expression (3.7), \( \delta = + \), holds without changes. With this, the map \( \mathcal{K} \) is defined as before. The extension of Lemmas 3 and 4 to the discrete series is as follows:

**Lemma 5.** The \( \mathcal{K} \) is a unitary map from \( D_\ell \) onto \( L^2((0, \infty), \pi^{-1}d^\times) \). Also, for any smooth vector \( \phi \in D_\ell \), we have (4.9) with \( j_{\ell-\frac{1}{2}}(u) = 0 \) for \( u < 0 \) and \( = 2\pi(-1)^\ell \sqrt{u}J_{\ell-1}(4\pi\sqrt{u}) \) for \( u > 0 \).

**Proof.** The unitarity of \( \mathcal{K} \) is proved with a minor change of the above argument. The Whittaker function \( W_{p,\ell-\frac{1}{2}}(u) \) \( (p \geq \ell) \) is a product of \( \exp(-u/2)u^\ell \) and a polynomial on \( u \) of degree \( p - \ell \), as (3.7) implies. Thus the proof of (4.16) in [8] can be carried out also for the product \( W_{p,\ell-\frac{1}{2}}(u)W_{q,\ell-\frac{1}{2}}(u) \) with integers \( p, q \), although the condition on \( \text{Re} \nu \) is violated. The result is equal to the limit of (4.16) as \( (\alpha, \beta, \nu) \) tends to \( (p, q, \ell - \frac{1}{2}) \). As to the surjectivity, we argue as follows: Let \( \omega \) be smooth and compactly supported on \((0, \infty)\). If \( \omega \) is orthogonal to all \( \mathcal{K}\phi_p \), \( \ell \leq p \), then we have, by the remark just made on \( W_{p,\ell-\frac{1}{2}}(u) \),

\[
\int_0^\infty \omega(u) \exp(-2\pi u)u^\ell \frac{du}{u} = 0, \quad \ell \leq p. \tag{4.21}
\]
This implies that the Fourier transform of \( \omega(u) \exp(-2\pi u)u^{\ell-1} \) vanishes identically; in fact it suffices to expand the additive character into a power series and integrate termwise. Hence \( \omega \equiv 0 \). The counterpart of (4.9), with \( \phi = \phi_p \), can be proved in much the same way as before. Its extension to smooth vectors \( \phi \) is immediate once the following bounds are noted:

\[
K\varphi_p(u) = A^+ \phi_p(a|u|) \lesssim \min(u, |p| + 1)u^{-\ell}, \quad u \in \mathbb{R}^x,
\]

as well as

\[
j_{-\frac{1}{2}}(u) \lesssim \min(u^4, u^\ell), \quad u > 0.
\]

The implicit constants may depend on \( \ell \) but not on \( p \). The first comes from (3.7) and (4.4), and the second from well-known bounds for \( J_{-\nu} \)-Bessel functions.

**Remark.** The identity (4.9) is crucial for our purpose. In a context related to ours, this is given in Theorem 4.1 of [3], but the proof there lacks an adequate discussion on the convergence issue; the same can be said about the relevant argument in [9]. The first rigorous proof is given in [8], and outlined above; the argument is different from those in [3] and [9]. On the other hand, the recent preprint [1] provides ingredients to handle the convergence issue in the discussion in [3]. In passing, we note that Lemmas 3 and 4 can be extended to \( \text{PSL}_2(\mathbb{C}) \); see [2].

5. **Projections.** We are now ready to compute \( \varpi_V P_0 f(1) \) and \( E^{(j)}(P_0 f; 1, \nu) \). The condition (2.12) is assumed throughout the section.

Let us first consider \( V \) in the unitary principal series, so that \( \nu_V \in i\mathbb{R} \). Let \( \varphi_p \) be as in (3.5). Since Eisenstein series are orthogonal to any cuspidal element, we have

\[
\langle P_0 f, \varphi_p \rangle_{\Gamma \setminus G} = \langle \mathcal{P} f, \varphi_p \rangle_{\Gamma \setminus G} = \int_G f(g)\overline{\varphi_p(g)}dg.
\]

The unfolding procedure in the second line is justified by (2.21) and the exponential decay of \( \varphi_p \) as \( y \uparrow \infty \). The latter follows from (3.18) and (4.5). We have

\[
\langle P_0 f, \varphi_p \rangle_{\Gamma \setminus G} = \sum_{n \neq 0} \frac{\varpi_V(n)}{\sqrt{|n|}} \int_G f(g)A^{\text{sgn}(n)}\phi_p(a|n|g)dg
\]

\[
= \varpi_V(1)H_V(z_2 + \frac{1}{2}) \left( \Phi_p^+ + \epsilon_V \Phi_p^- \right) f(\nu_V),
\]

where (2.15) has been used, and

\[
\Phi_p^0 f(\nu) = \int_G f(g)A^0 \phi_p(g;\nu)dg.
\]

The absolute convergence that is necessary to have the first line of (5.2) follows from that of (5.3), which in turn results from (2.21) and (4.5). By (3.24) or gathering together the projections of \( P_0 f \) to \( V_p \), we have now

\[
\varpi_V P_0 f(g) = \left| \varpi_V(1) \right|^2 H_V(z_2 + \frac{1}{2})
\]

\[
\times \sum_{n=1}^{\infty} \frac{t\nu_V(n)}{\sqrt{|n|}} \left( B^{(+,+)} + B^{(-,-)} + \epsilon_V B^{(+,-)} + \epsilon_V B^{(-,+)} \right) f(a|n|g; \nu_V),
\]

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with
\[ \mathcal{B}^{(\delta_1, \delta_2)} f(g; \nu) = \sum_p \Phi_p^\delta \phi_p(g; \nu) \]
\[ = \exp(2\pi i \delta_2 x) \sum_p \Phi_p^\delta f(\nu) A^{\delta_2} \phi_p(a|y|) \exp(2ip\theta). \]  
(5.5)

We are going to prove that the right side of (5.4) converges absolutely to a continuous function in \( V \).

To this end, we shall show the bound
\[ \Phi_p^\delta f(\nu) \ll (|p| + |\nu| + 1)^{-B}, \quad |\Re \nu| < \frac{1}{2}. \]  
(5.6)

A combination of (4.5) and (5.6) yields
\[ \mathcal{B}^{(\delta_1, \delta_2)} f(g; \nu) \ll y^{\frac{1}{2} - |\Re \nu| - \epsilon}((y + 1)(|\nu| + 1))^{-B}, \]  
(5.7)
in the same region of \( \nu \), whence the above claim on (5.4). To prove (5.6), observe, as in the proof of Lemma 1, that the function \( \Phi^\delta \) is bounded by the right side of (2.21), for any fixed \( u \in \mathfrak{U} \). Thus, the second line of (3.7) and (4.5) give
\[ \Phi_p^\delta u f \ll \int_0^\infty y^{\Re z_2 - 1} |A^\delta \phi_p(a|y|)| \, dy \]
\[ \ll (|p| + |\nu| + 1)(|\nu| + 1)^{\Re z_2 - \Re \nu - \frac{1}{2}}. \]  
(5.8)

Since \( A^\delta \) is an intertwining operator with respect to the action of the elements of \( g \), we have, for any positive integer \( q \),
\[ \Phi_p^\delta f(\nu + i\partial_\theta^2)^q f = (\nu^2 - \frac{1}{4} - 4ip^2)^q \Phi_p^\delta f, \]  
(5.9)

by integration by parts, which can be justified with (5.8). This obviously gives (5.6).

Let us look at \( \mathcal{B}^{(\delta_1, \delta_2)} f(a|y|; \nu) \) closer, with the Kirillov scheme. We assume that \( \nu \in i\mathbb{R} \).

We have
\[ \mathcal{B}^{(\delta_1, \delta_2)} f(a|y|; \nu) = \sum_p \Phi_p^\delta f(\nu) \mathcal{K} \phi_p(\delta_2 y) = \mathcal{K} \mathcal{L}^\delta f(\delta_2 y), \]  
(10.10)

where
\[ \mathcal{L}^\delta f = \sum_p \Phi_p^\delta f(\nu) \phi_p \]  
(11.11)
is a smooth vector in \( U_\nu \). Lemma 4 gives
\[ \Phi_p^\delta f = \langle \mathcal{L}^\delta f, \phi_p \rangle_{U_\nu} = \frac{1}{\pi} \int_{\mathbb{R}^\times} \mathcal{K} \mathcal{L}^\delta f(u) \mathcal{K} \phi_p(u) d^\times u. \]  
(12.12)

This means that if one can transform (5.3) into
\[ \Phi_p^\delta f = \frac{1}{\pi} \int_{\mathbb{R}^\times} \mathcal{Y}^\delta (u) \mathcal{K} \phi_p(u) d^\times u \]  
(13.13)
then it should follow that
\[ \mathcal{B}^{(\delta_1, \delta_2)} f(a|y|) = \mathcal{Y}^\delta_1 (\delta_2 y). \]  
(14.14)

Note that we have used implicitly a simple continuity argument, which will be made explicit in (5.22).
We may write (5.3) as

\[ \Phi_p^\delta f = \frac{1}{\pi} \int_0^\infty u^{z-1} \int_{\mathbb{R}^2} f(n[x_1]wn[x_2]) \exp(-2\pi i\delta x_1 u) \times \int_{\mathbb{R}^2} \exp(-2\pi i\delta x_2 \lambda) j_{\nu}(\delta u \lambda) \overline{\mathcal{X}\phi_p(\lambda)} d^x \lambda d^x x \, du. \] (5.18)

Here \( g = n[x_1]wn[x_2] \) and \( dg = dx_1 dx_2 / \pi \); the formula (2.14) gives the Jacobian for this change of variables. We observe

\[ R_w A^\delta \phi_p(a[u]) = R_{wn[x_2]} A^\delta \phi_p(n[x_1]u[a][u]) = \exp(2\pi i\delta x_1 u) R_{wn[x_2]} A^\delta \phi_p(a[u]) = \exp(2\pi i\delta x_1 u) A^\delta R_w R_{n[x_2]} \phi_p(a[u]). \] (5.16)

By Lemma 3

\[ A^\delta R_w R_{n[x_2]} \phi_p(a[u]) = \mathcal{K} R_w R_{n[x_2]} \phi_p(\delta u) = \int_{\mathbb{R}^2} j_{\nu}(\delta u \lambda) \mathcal{K} R_{n[x_2]} \phi_p(\lambda) d^x \lambda \]

Thus

\[ \Phi_p^\delta f = \frac{1}{\pi} \int_0^\infty u^{z-1} \int_{\mathbb{R}^2} f(n[x_1]wn[x_2]) \exp(-2\pi i\delta x_1 u) \times \int_{\mathbb{R}^2} \exp(-2\pi i\delta x_2 \lambda) j_{\nu}(\delta u \lambda) \overline{\mathcal{X}\phi_p(\lambda)} d^x \lambda d^x x \, du, \] (5.18)

where we have used that \( j_{\nu} = j_{-\nu} \). Applying change of variables \( x_1 \to x_1/x_2, x_2 \to -x_2 \), we have

\[ \Phi_p^\delta f = \frac{1}{\pi} \int_0^\infty u^{z_2-1} \int_{x_2 \| \omega_1 - \omega_4} \hat{\psi}_\tau(\delta u/x_2) \eta(x_2) \times \int_{\mathbb{R}^2} \exp(-2\pi i\delta x_2 \lambda) j_{\nu}(\delta u \lambda) \overline{\mathcal{X}\phi_p(\lambda)} d^x \lambda d^x x_2 \, du, \] (5.19)

with

\[ \hat{\psi}_\tau(u) = \int_0^\infty \frac{\psi(x_1/(x_1 + 1))\tau(-x_1)}{x_1^\omega_1 (x_1 + 1)^\omega_2} \exp(-2\pi i u x_1) d x_1. \] (5.20)

Here (2.3) and (2.11) have been used. The triple integral in (5.19) converges absolutely. In fact, a multiple application of integration by parts gives, for any fixed \( l \),

\[ \left( \frac{d}{du} \right)^l \hat{\psi}_\tau(u) \ll (|u| + 1)^{-B}, \] (5.21)

because of (2.3). A combination of (4.3), (4.5), (4.13) and (5.21) yields that the integral whose integrand is the absolute value of that in (5.19) is \( \ll |p| + 1 \), with the implied constant depending on \( \nu \). Hence we have, for any smooth \( \phi \in U_\nu \),

\[ \langle L^\delta f, \phi \rangle_{U_\nu} = \frac{1}{\pi} \int_{\mathbb{R}^2} \int_0^\infty u^{z_2-1} \int_{x_2 \| \omega_1 - \omega_4} \hat{\psi}_\tau(\delta u/x_2) \eta(x_2) \exp(2\pi i\delta x_2 \lambda) d^x \lambda d^x x_2 \, du \int_{\mathbb{R}^2} \overline{\mathcal{X}\phi(\lambda)} d^x \lambda. \] (5.22)

Via (5.10), Lemma 4 now gives rise to

\[ \mathcal{B}^{(\delta_1, \delta_2)} f(a[y]; \nu) = \mathcal{B}^{\delta_1, \delta_2} f(a[y]; \nu), \] (5.23)
with

$$B^\delta f(a[y]; \nu) = \int_0^\infty t^{\nu-1} j_\nu(\delta uy)$$

$$\times \int_{\mathbb{R}^+} |x|^{w_1-w_2+1} \dot{\psi}_r(\delta u/x) \eta(x) \exp(2\pi ixy) dx \, du. \quad (5.24)$$

Inserting this into (5.4), we find that

$$\varpi_V P_0 f(1) = 2|\varphi_V(1)|^2 H_V(z_2 + \frac{1}{2}) \sum_{n=1}^\infty \frac{t_V(n)}{\sqrt{n}} (B^+ + \epsilon V B^-) f(a[n]; \nu_V). \quad (5.25)$$

Next, we shall treat the discrete series. We assume that $V$ is as in (3.12), having the complete orthonormal system \{\varphi_p : p \geq \ell\} with \varphi_p given in (3.14). The relation (5.1) extends as it is; the unfolding procedure depends on the observation on the Whittaker function $W_{p, \ell - \frac{1}{2}}$ made in the proof of Lemma 5. Then, (5.2), with an obvious interpretation of (5.3), is replaced by

$$\langle \mathcal{P} f, \varphi_p \rangle_{\Gamma \setminus G} = \pi^{1-2\ell} |\varphi_V(1)|^2 H_V(z_2 + \frac{1}{2}) \left( \frac{\Gamma(p+\ell)}{\Gamma(p-\ell+1)} \right)^{\frac{1}{2}} \Phi_p^+(\ell - \frac{1}{2}), \quad (5.26)$$

and (5.4) by

$$\varpi_V P_0 f(g) = |\varphi_V(1)|^2 H_V(z_2 + \frac{1}{2}) \sum_{n=1}^\infty \frac{t_V(n)}{\sqrt{n}} B f(a[n]g; \ell - \frac{1}{2}). \quad (5.27)$$

Here

$$B f(g; \ell - \frac{1}{2}) = \pi^{1-2\ell} \sum_{p \geq \ell} \frac{\Gamma(p+\ell)}{\Gamma(p-\ell+1)} \Phi_p^+(\ell - \frac{1}{2}) A^+ \varphi_p(g; \ell - \frac{1}{2}), \quad (5.28)$$

which replaces (5.5). The $B f$ exists as a continuous function in $V$. On noting (4.22), this follows from $\Phi_p^+(\ell - \frac{1}{2}) \ll (|p|+1)^{-B}$, with implicit constant depending on $\ell$. To get the latter, we observe that for any $u \in U$

$$\langle \mathcal{P} u f, \varphi_p \rangle_{\Gamma \setminus G} = \langle u \mathcal{P} f, \varphi_p \rangle_{\Gamma \setminus G} = \pm \langle \mathcal{P} f, u \varphi_p \rangle_{\Gamma \setminus G}. \quad (5.29)$$

Set $u = \partial_y^q$, with a positive integer $q$; and use (5.25) on the right side and $\|\langle \mathcal{P} u f, \varphi_p \rangle_{\Gamma \setminus G} \leq \|\mathcal{P} u f\|_{\Gamma \setminus G}$ on the left, which confirms the claim. We have actually proved that $\varpi_V P_0 f$ exists as a continuous function in $V$.

We shall prove an extension of (5.24). This is now easy: We put, in place of $\mathcal{L}^\delta f$,

$$\mathcal{L} f = \pi^{1-2\ell} \sum_{p \geq \ell} \frac{\Gamma(p+\ell)}{\Gamma(p-\ell+1)} \Phi_p^+(\ell - \frac{1}{2}) \varphi_p, \quad (5.30)$$

which is a smooth vector in $D_\ell$. Then, we can proceed much like (5.10)–(5.22), relying on Lemma 5 and (4.22)–(4.23). Thus, we have

$$B f(a[y]) = B^+ f(a[y]; \ell - \frac{1}{2}) \quad (5.31)$$

with an obvious extended use of notation. We have, as a counterpart of (5.25),

$$\varpi_V P_0 f(1) = |\varphi_V(1)|^2 H_V(z_2 + \frac{1}{2}) \sum_{n=1}^\infty \frac{t_V(n)}{\sqrt{n}} B^+ f(a[n]; \ell - \frac{1}{2}). \quad (5.32)$$
We now turn to the contribution of Eisenstein series. We see readily that
\[ c_p^{(1)}(P_0 f; \nu) = \frac{\zeta(z_2 + \frac{1}{2} + \nu) \zeta(z_2 + \frac{1}{2} - \nu)}{\zeta(1-2\nu)} \left( \Phi_+^+ + \Phi_-^- \right) f(\nu). \] (5.33)

This and (5.6) confirm our claim on the convergence of (3.28) that is made prior to (3.29). The discussion of \( \mathcal{E}^{(1)}(P_0 f; a[y], \nu) \) is obviously analogous to that of \( \varpi V P_0 f(a[y]) \) with \( V \) in the unitary principal series. Hence, it suffices to state only the end result:
\[ \mathcal{E}^{(1)}(P_0 f; 1, \nu) = \frac{2\zeta(z_2 + \frac{1}{2} + \nu) \zeta(z_2 + \frac{1}{2} - \nu)}{\zeta(1-2\nu)} \sum_{n=1}^{\infty} \frac{\sigma_{2\nu}(n)}{n^{\frac{1}{2} + \nu}} \left( \mathcal{B}^+ + \mathcal{B}^- \right) f(a[n]; \nu). \] (5.34)

As to \( \mathcal{E}^{(0)}(P_0 f; a[y], \nu) \), we observe that the functional equation (3.22) implies the relation \( c_p(\nu)c_p(\nu) = c_p(-\nu) \). Thus
\[ \mathcal{E}^{(0)}(P_0 f; 1, \nu) = \mathcal{D}(P_0 f; \nu) + \mathcal{D}(P_0 f; -\nu), \] (5.35)
where
\[ \mathcal{D}(P_0 f; \nu) = \frac{\zeta(z_2 + \frac{1}{2} + \nu) \zeta(z_2 + \frac{1}{2} - \nu)}{\zeta(1-2\nu)} \left( \mathcal{C}^+ + \mathcal{C}^- \right) f(\nu), \] (5.36)
with
\[ \mathcal{C}^\delta f(\nu) = \sum_p \Phi_\delta^p f(\nu). \] (5.37)

The computation of this sum requires certain technicalities. We put
\[ \mathcal{C}_\delta^\nu f(\nu) = \sum_p \int_G f(g) A^{-\delta} \phi_{-p}(g; -\nu) dg, \] (5.38)
which is regular for \( |\text{Re}\nu| < \frac{1}{2} \), since the integral satisfies the same bound as (5.6). We have \( \mathcal{C}_\delta^\nu f(\nu) = \mathcal{C}^\delta f(\nu) \) on the imaginary axis. Let us suppose \( -\frac{1}{2} < \text{Re}\nu < 0 \). In (5.38), use the first line of (3.7), but with the contour \( \text{Im}\xi = \frac{1}{2} \), so that the quadruple integral converges absolutely; here we need (2.21). Take the integral over \( K \) innermost, and apply integration by parts many times, while noting that \( \partial_\nu^\delta f \) with any fixed \( q \) still satisfies the bound (2.21). We see now that the sum over \( p \) can be taken inside the first triple integral. Then we may shift the \( \xi \)-contour back to \( \mathbb{R} \). Undoing integration by parts, we get
\[ \mathcal{C}_\delta f(\nu) = \int_0^\infty y^{-\frac{1}{2} + \nu} \int_0^\infty \int_0^\infty \frac{\exp(2\pi i(y\xi - \delta x))}{(\xi^2 + 1)^{-\nu + \frac{1}{2}}} \]
\[ \times \sum_p \left( \frac{\xi + i}{\xi - i} \right)^{3p} \int_0^\pi f(n[x]a[y]k|\theta) \exp(-2pi\theta) \frac{d\theta}{\pi} d\xi dx dy, \] (5.39)
and thus
\[ \mathcal{C}_\delta f(\nu) = \int_0^\infty y^{\frac{3}{2} - \frac{1}{2} + \nu} \int_{-\infty}^{\infty} \int_0^\infty \frac{\exp(2\pi i y(\xi - \delta x))}{(\xi^2 + 1)^{-\nu + \frac{1}{2}}} f(n[x]k|\xi) d\xi dx dy, \] (5.40)
with
\[ k_\xi = \frac{1}{\sqrt{\xi^2 + 1}} \begin{bmatrix} \xi & \delta \\ -\delta & \xi \end{bmatrix} \in K. \] (5.41)
The last double integral over \((\xi, x)\) converges absolutely. We take the \(x\)-integral innermost, and perform the change of variable \(x \mapsto \delta \xi + \delta x(\xi + 1/\xi)\), getting

\[
\mathcal{E}_0 f(\nu) = \int_0^\infty y^{z_2 - \frac{1}{2} + \nu} \int_{-\infty}^{\infty} |\xi|^{w_1 - w_4 - 1} (\xi^2 + 1)^{\frac{1}{2} + \nu} \hat{\psi}_r((\xi + 1/\xi) y) \eta(\xi) d\xi dy
\]

\[
= \left( (\hat{\psi}_r)^+ + (\hat{\psi}_r)^- \right) (z_2 + \frac{1}{2} + \nu) \eta^*(z_3 + \frac{1}{2} + \nu),
\]

(5.42)

where \((\hat{\psi}_r)^\pm\) and \(\eta^*\) are Mellin transforms on \((0, \infty)\) of \(\hat{\psi}_r(\pm \cdot)\) and \(\eta\), respectively; and

\[
z_3 = \frac{1}{2}(w_1 + w_3 - w_2 - w_4).
\]

(5.43)

In deducing (5.42), we have used (2.11) and (5.21). The second line of (5.42) is a regular function of \(\nu\) in a neighbourhood of the imaginary axis; thus, it is equal to \(\mathcal{E}_0 f(\nu)\) if \(\nu \in i\mathbb{R}\).

Summing up, we find that

\[
\int_0^\infty \zeta(z_1 + \frac{1}{2} + \nu)\zeta(z_1 + \frac{1}{2} - \nu)\mathcal{E}_0^{(0)}(\mathcal{P}_0 f; 1, \nu) \frac{d\nu}{4\pi i} \int_0^\infty Z(\nu) \left( (\hat{\psi}_r)^+ + (\hat{\psi}_r)^- \right) (z_2 + \frac{1}{2} + \nu) \eta^*(z_3 + \frac{1}{2} + \nu) \frac{d\nu}{\pi i},
\]

(5.44)

with

\[
Z(\nu) = \zeta(z_1 + \frac{1}{2} + \nu)\zeta(z_1 + \frac{1}{2} - \nu)\zeta(z_2 + \frac{1}{2} + \nu)\zeta(z_2 + \frac{1}{2} - \nu)/(\zeta(1 - 2\nu)).
\]

(5.45)

Remark. Our use of the Kirillov scheme should be compared with that in Chapter 5 of [3].

6. Explicit formula. Collecting (2.35), (3.29), (5.25), (5.32), (5.34) and (5.44), we obtain a spectral decomposition of \(\mathcal{T}_w \mathcal{P} f(1)\). Here we shall discuss the behaviour of this decomposition as \(\eta\) tends to the characteristic function of \(\mathbb{R}^\times\), and subsequently \(\tau\) to that of negative reals.

To facilitate the convergence issue, we shall work with bounded \(w\) satisfying

\[
\text{Re } w_3 > \text{Re } w_2 + 3 > 4; \quad \frac{3}{2} + \text{Re } w_1 > \text{Re } w_4 > \text{Re } w_1 + 1 > 2,
\]

(6.1)

which is obviously contained in (2.12). We set

\[
\eta(x) = \exp \left( -\frac{1}{2} \kappa_1 (|x| + 1/|x|) \right), \quad x \in \mathbb{R}^\times,
\]

(6.2)

\[
\tau(x) = 0, \quad x \geq 0; \quad \tau(x) = \exp \left( \frac{1}{2} \kappa_2 (x + 1/x) \right), \quad x < 0,
\]

(6.3)

where \(\kappa_1, \kappa_2 > 0\) are supposed to tend to 0. It is immediate that (2.13) implies

\[
\frac{1}{2} \lim_{\kappa_2 \to 0} \lim_{\kappa_1 \to 0} \int_0^\infty \mathcal{T}_w \mathcal{P} f(1) = \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\sigma_{w_1 - w_4}(m)\sigma_{w_2 - w_3}(m + n)}{m^{w_1}(m + n)^{w_2}} \psi(m/(m + n)).
\]

(6.4)

We are going to make explicit the spectral decomposition of the right side that results via this relation, as is given in (6.22)–(6.28). The specialization (6.2)–(6.3) allows us to prove that \(\mathcal{B}^0 f(a[y]; \nu)\) tends, in a uniform manner, to an object essentially the same as \(\Phi_{\pm}\) in Section 4.4 of [7]. Once this has been achieved, the subsequent discussion is the same as Sections 4.5–4.7 there, and can largely be skipped.
We shall consider first the contribution of the unitary principal series representations. Returning to (5.20), we put
\[ \psi^*_z(s) = \int_0^\infty x^{s-1} \frac{\psi(1/(x+1))\tau(-x)}{(x+1)^{w_1}} dx, \] 
(6.5)
which is regular and of rapid decay in any fixed vertical strip. We have
\[ \frac{\psi(x/(x+1))\tau(-x)}{(x+1)^{w_2}} = \frac{1}{2\pi i} \int_{(\alpha)} \psi^*_x(s)x^{s-w_2} ds, \]
(6.6)
with an arbitrary \( \alpha \). Here we have used \( \tau(x) = \tau(1/x) \). Multiply both sides by the factor \( x^{-w_1} \exp(-ax - 2\pi iux) \), \( a > 0 \), and integrate over \( (0, \infty) \). The left side converges uniformly for \( a \geq 0 \). Moving the contour \( (\alpha) \) to the right if necessary, the double integral on the right side converges absolutely, provided \( a > 0 \). Exchange the order of integration, compute the inner integral, and observe that the resulting integral is uniformly convergent for \( a \geq 0 \), because of the rapid decay of \( \psi^*_x \). Thus, for \( u \neq 0 \),
\[ \hat{\psi}_\tau(u) = \frac{1}{2\pi i} \int_{(\alpha)} \psi^*_x(s)(2\pi |u|)^{w_1+w_2-1-s} \Gamma(s+1-w_1-w_2) \]
\[ \times \exp \left( -\frac{1}{2} \pi i \text{sgn}(u)(s+1-w_1-w_2) \right) ds, \]
(6.7)
with any \( \alpha > \text{Re}(w_1 + w_2) - 1 \).

To the inner integral of (5.24) we apply a similar procedure: multiply the integrand by the factor \( \exp(-a|x|) \), \( a > 0 \), replace \( \psi^*_x \) by (6.7), and exchange the order of integration. We get the expression
\[ \frac{1}{2\pi i} \int_{(\alpha)} \psi^*_x(s)(2\pi |u|)^{w_1+w_2-1-s} \Gamma(s+1-w_1-w_2) \sum_{\pm} \exp \left( \pm \frac{1}{2} \pi i \delta(s_1+1-w_1-w_2) \right) \]
\[ \times \int_0^\infty x^{s_1-w_2-w_4} \exp(-(a \pm 2\pi iy)x)\eta(x)dx ds_1. \]
(6.8)

On noting that \( \eta(x) = \eta(1/x) \), use the Mellin inverse of \( \eta^* \). Because of the uniform convergence for \( a \geq 0 \), we see that the integral in question is equal to
\[ \frac{-1}{2\pi} \int_{(\alpha)} \eta^*_x(s) \int_{(\alpha)} \psi^*_x(s)(2\pi |u|)^{w_1+w_2-1-s_1} (2\pi |u|)^{w_2+w_4-1-s_2} \]
\[ \times \Gamma(s_1+1-w_1-w_2) \Gamma(s_2+1-w_2-w_4) \]
\[ \times \cos \left( \frac{1}{2} \pi (s_1 + s_2 + 1 - w_2 - w_4 - \delta(s_1+1-w_1-w_2)) \right) ds_1 ds_2, \]
(6.9)
provided (6.1). Let us assume temporarily that \( \alpha \) is such that \(-\frac{1}{2} < \text{Re} \, \alpha < -\frac{1}{4} \), which does not conflict with (6.1). We insert (6.9) into (5.24). The resulting triple integral converges absolutely, because of (4.13). We take the \( u \)-integral innermost, and invoke that for \(-\frac{1}{2} < \text{Re} \, s < -\frac{1}{4} \)
\[ \int_0^\infty j_\nu(\delta u)u^{s-1}du = 2^{-2s} \pi^{-2s-1} \cos \left( \frac{1}{2} \pi (1+\delta)(s+\frac{1}{2}) + (1-\delta)\nu \right) \]
\[ \times \Gamma \left( s + \frac{1}{2} + \nu \right) \Gamma \left( s + \frac{1}{2} - \nu \right), \]
(6.10)
which is a consequence of Mellin transforms of \( J \) and \( K \)-Bessel functions. Note that we now have \( \nu \in i\mathbb{R} \). Thus, after some rearrangement, we get
\[ B^\delta f(a[y]; \nu) = y^{-s_2} \int_{(0)} \eta^*_x(s_2)(2\pi y)^{-s_2} \Theta^\delta_x(s_2; \nu) ds_2, \]
(6.11)
with
\[
\Theta^\delta_\tau(s_2; \nu) = -4(2\pi)^{w_2-w_3-3} \int_\alpha \psi^*_\tau(s_1) \cos \left( \frac{\pi}{2}(1+\delta)(z_1-s_1 + \frac{1}{2}) + (1-\delta)\nu \right) \\
\times \cos \left( \frac{\pi}{2}(s_1 + 1 - w_2 - w_4 - \delta(s_1+1 - w_1-w_2)) \right) \\
\times \Gamma(z_1-s_1 + \frac{1}{2} + \nu) \Gamma(z_1-s_1 + \frac{1}{2} - \nu) \\
\times \Gamma(s_1+1 - w_1-w_2) \Gamma(s_1+s_2+1 - w_2-w_4) ds_1.
\] (6.12)

Here \( \alpha \) is to satisfy \( \text{Re} \ z_1 + \frac{1}{2} > \alpha > \text{Re} \ (w_1+w_2) - 1 \). Such an \( \alpha \) exists, if (6.1) holds. Observe that a shift to the far right of the contour in (6.12) gives
\[
\Theta^\delta_\tau(s_2; \nu) \ll ((|s_2| + 1)/(|\nu| + 1))^\delta,
\] (6.13)
uniformly for \( |\text{Re} \ \nu|, |\text{Re} \ s_2| < \varepsilon \), which is a consequence of the rapid decay of \( \psi^*_\tau \). The formula (6.12) corresponds to (4.4.16)–(4.4.17) of [7] (see the remark following (6.22)).

From (6.11) follows
\[
B^\delta f(a[y]; \nu) = 2\pi i y^{-z_1} \Theta^\delta_\tau(0; \nu) + O(y^{-\text{Re} \ z_2}(\kappa_1 y)^\varepsilon (|\nu| + 1)^{-\delta}),
\] (6.14)
uniformly as \( \kappa_1 \downarrow 0 \). To confirm this, we shift the contour in (6.11) to \( (\varepsilon) \), and note that, since (6.2) implies \( \eta^*(s_2) = 2K_{s_2}(\kappa_1) \),
\[
\eta^*(s_2) = \frac{\pi}{\sin \pi s_2} (I_{-s_2}(\kappa_1) - I_{s_2}(\kappa_1)) \\
= \frac{\pi}{\sin \pi s_2} \frac{(\kappa_1/2)^{-s_2}}{\Gamma(1-s_2)} + O\left(\kappa_1^{\text{Re} \ s_2} \exp(-|s_2|)\right),
\] (6.15)
with the implied constant being absolute. The bound (6.13) yields that the error-term contributes negligibly; and as to the main term it suffices to shift the contour to \( (-\varepsilon) \).

We insert (6.14) into (5.25), and get
\[
\varpi_V p_0 f(1) = 4\pi i |q_V(1)|^2 H_V (z_2 + \frac{1}{2}) H_V (z_3 + \frac{1}{2}) \left( (\Theta^\delta_\tau + \epsilon_V \Theta^-_\tau)(0; \nu_V) \\
+ O(|q_V(1)|^2 \kappa_1^\varepsilon (|\nu_V| + 1)^{-\delta}) \right),
\] (6.16)
in which we have used the fact that (6.1) implies \( \text{Re} \ z_3 > \frac{3}{4} \). Because of (3.16), we find that
\[
\lim_{\kappa_1 \downarrow 0} \sum_V \varpi_V p_0 f(1) = 4\pi i \sum_V |q_V(1)|^2 H_V (z_1 + \frac{1}{2}) H_V (z_3 + \frac{1}{2}) \left( (\Theta^\delta_\tau + \epsilon_V \Theta^-_\tau)(0; \nu_V) \right),
\] (6.17)
with \( V \) running over all irreducible representations in the unitary principal series.

Next, we observe that for \( \text{Re} \ s > 0 \)
\[
\psi^*_\tau(s) = \frac{1}{\pi i} \int_{(\alpha)} K_{\mu}(\kappa_2) \psi^*(s-\mu) d\mu.
\] (6.18)

Here \( \psi^* \) is defined to be the right side of (6.5) without the factor \( \tau(-x) \). It is regular and of rapid decay for \( \text{Re} \ s > 0 \). We have
\[
\psi^*_\tau(s) = I_0(\kappa_2) \psi^*(s) + \psi^{**}_\tau(s),
\] (6.19)
with
\[ \psi^*_s(s) = -\frac{1}{2i} \int_{(c)} I(s) \left( \psi^*(s + \alpha) + \psi^*(s - \alpha) \right) \frac{d\mu}{\sin \pi \mu}. \]  
(6.20)

Obviously, for \( \text{Re } s > 0 \) the \( \psi^*_s(s) \) is regular and \( \ll \kappa_2^2(|s| + 1)^{-B} \). This gives readily
\[ \Theta^\delta(0; \nu) = \Theta^\delta(\nu) + O(\kappa_2^2(|\nu| + 1)^{-B}), \]  
(6.21)
where \( \Theta^\delta(\nu) \) is defined by (6.12) with \( s_2 = 0 \) and \( \psi^* \) in place of \( \psi^*_s \).

Hence, we have now proved that on (6.1)
\[ \frac{1}{2} \sum_{V} |\varphi_V(1)|^2 H_V(z_1 + \frac{1}{2}) H_V(z_2 + \frac{1}{2}) H_V(z_3 + \frac{1}{2}) \Theta^+(\nu_V), \]  
(6.22)
with \( V \) running over all irreducible representations in the unitary principal series. We compare this with (4.5.5) and (4.5.9) of [7]. Taking into account our current normalization (3.8) and the remark on (6.12) made after (6.13), we find that the agreement is perfect. We end the treatment of the unitary principal series.

**Remark.** To expedite the comparison, we give the table:

\[
\begin{align*}
(w_1, w_2, w_3, w_4) & \mapsto (u, w, z, v), \\
|\varphi_V(1)|^2 & \mapsto \frac{1}{4} \alpha_j, \\
\Theta^\delta & \mapsto \frac{2}{\pi i} \Phi_s, \\
\psi^* & \mapsto \tilde{g},
\end{align*}
\]  
(6.23)

where on the left are our present objects and on the right those corresponding in Sections 4.3–4.4 of [7]. Likewise \( \epsilon_V \mapsto \epsilon_j, H_V \mapsto H_j \). Note that (4.4.16) in [7] is to be corrected: the second \( \xi \) on the right side should have the opposite sign.

In the Eisenstein part in (3.29), \( g = 1 \), the term \( \mathcal{T}_w \varphi_v^{(1)} \mathcal{P}_0 f(1) \) is treated via (5.34), and completely analogous to the above. After the limiting procedure, we obtain a result equivalent to the sum of (4.5.4) and (4.5.8) of [7].

As to the contribution of the discrete series, we need to return to (5.32). We stress that we have (3.12)–(3.13); thus one may count only those \( V \) as in (3.12) but each of their contributions has to be doubled. Also, the formula (6.10) is replaced by
\[ \int_0^\infty j_{\ell - \frac{1}{2}}(u) u^{s-1} du = (-1)^{\ell} (2\pi)^{-2s} \frac{\Gamma(s+\ell)}{\Gamma(\ell-s)} \quad 1 \leq \ell \in \mathbb{Z}. \]  
(6.24)

This and the argument leading to (6.22) yield
\[ \frac{1}{2} \sum_{V} \mathcal{T}_w \varphi_v \mathcal{P}_0 f(1) = 2\pi i \sum_{V,+} |\varphi_V(1)|^2 H_V(z_1 + \frac{1}{2}) H_V(z_2 + \frac{1}{2}) H_V(z_3 + \frac{1}{2}) \Theta^+(\ell - \frac{1}{2}). \]  
(6.25)

The sum on the left side is over all irreducible representations in the discrete series, whereas on the right \( V, + \) indicates that \( V \) are such as in (3.12). Note that we have
\[ \Theta^+(\ell - \frac{1}{2}) = 2i (-1)^{\ell-1} (2\pi)^{w_2-w_3-1} \cos \left( \frac{1}{2} \pi (w_1 - w_4) \right) \Xi(\ell - \frac{1}{2}; w_3, w_1, w_4, w_2). \]  
(6.26)
with the identification $\psi^* \mapsto \tilde{g}$, where $\Xi$ is defined by (4.4.18) of [7]. Also, the normalization (3.15) means that in (6.25)

$$|g_v(1)|^2 = \frac{1}{2} \alpha_{j, \ell},$$

with $\alpha_{j, \ell}$ as in (4.4.3) of [7]. Inserting these two translations into (6.25), we get a result equivalent to (5.6.6) of [7].

This amounts to an alternative proof of Lemma 4.5 of [7]; note that the domain (6.1) is not disjoint with (4.3.10) of [7].

Hence, it remains to discuss the residual terms (2.35) and (5.44). As to the former, we note that (6.5) gives $\tilde{\psi}(0) = \psi^*(w_1 + w_2 - 1)$, and have

$$\frac{1}{2} \lim_{\kappa_1 \to 0} \lim_{\nu \to 0} \mathcal{T}_w \mathcal{P}_\infty f(1) = \psi^*(w_1 + w_2 - 1)$$

$$\times \zeta(w_1 + w_2 - 1)\zeta(w_3 + w_4)\zeta(w_4 - w_1 + 1)/\zeta(2z_2 + 2).$$

Dealing with (5.44), we apply, on the right side, the change of variable $\nu \mapsto \nu - (z_3 + \frac{1}{2})$, and shift the contour to ($\varepsilon$). We do not encounter any singularity, because of (6.1). Following the argument for (6.14), we have

$$\lim_{\kappa_1 \to 0} \lim_{\nu \to 0} \mathcal{T}_w \mathcal{P}_E \mathcal{P}_0 f(1) = 2Z(-z_3 - \frac{1}{2}) \left( (\tilde{\psi})^+ + (\tilde{\psi})^- \right)(w_4 - w_1).$$

The formula (6.7) gives

$$(\tilde{\psi})^+ + (\tilde{\psi})^-)(w_4 - w_1)$$

$$= 2(2\pi)^w_1 - w_4 \cos \left( \frac{\pi}{2}(w_1 - w_4) \right) \Gamma(w_4 - w_1)\psi^*(w_2 + w_4 - 1).$$

Hence we have

$$\frac{1}{2} \lim_{\kappa_1 \to 0} \lim_{\nu \to 0} \mathcal{T}_w \mathcal{P}_E \mathcal{P}_0 f(1) = \psi^*(w_2 + w_4 - 1)$$

$$\times \zeta(w_1 + w_3)\zeta(w_2 + w_4 - 1)\zeta(w_3 - w_2 - 1)\zeta(w_4 - w_1 + 1)/\zeta(2z_2 + 2),$$

in which we have used the functional equation for the zeta-function. The sum of this and (6.28) is equivalent to (4.3.16) of [7]. That is, the residual contribution in the present context has turned out to be equivalent to that in Section 4.3 of [7]. With this, we conclude our discussion.

**Correction:** The definition of $e_p^{(1)}(\mathcal{P}_0 f; \nu)$ following (3.28) is incorrect. The correct definition is

$$e_p^{(1)}(\mathcal{P}_0 f; \nu) = \int_G f(g)E_p^{(1)}(g; \nu)dg,$$

where $E_p^{(1)}(g; \nu)$ is as in (3.31). Also, prior to (4.5) the contour should be shifted to $\text{Im} \xi = (|\nu| + |p| + 1)^{-1}$; accordingly the exponentiated factor in (4.5) needs to be replaced by $\exp(-y/(|\nu| + |p| + 1))$. This causes minor changes in Section 5, although the overall argument stays the same.

**References**

[1] E.M. Baruch and Z. Mao: Bessel identities in Waldspurger correspondence, the archimedean theory. Preprint, 2002.
[2] R.W. Bruggeman and Y. Motohashi: A note on the mean value of the zeta and $L$-functions. XIII. Proc. Japan Acad., 78A, 87–91 (2002).

[3] J.W. Cogdell and I. Piatetski-Shapiro: The Arithmetic and Spectral Analysis of Poincaré Series. Perspectives in Math., 13, Academic Press, San Diego, 1990.

[4] J. Dixmier and P. Malliavin: Factorisations de fonction et de vecteurs indéfiniment différentiables. Bull. Soc. Math. France, 102, 305–330 (1978).

[5] A.A. Kirillov. On $\infty$-dimensional unitary representations of the group of second-order matrices with elements from a locally compact field. Soviet Math. Dokl., 4, 748–752 (1963).

[6] S. Lang: $\text{SL}_2(\mathbb{R})$. Addison-Wesley, Reading, 1975.

[7] Y. Motohashi: Spectral Theory of the Riemann Zeta-Function. Cambridge Tracts in Math., 127, Cambridge Univ. Press, Cambridge, 1997.

[8] —: A note on the mean value of the zeta and $L$-functions. XII. Proc. Japan Acad., 78A, 36–41 (2002).

[9] N.Ja. Vilenkin. Special Functions and the Theory of Group Representations. Amer. Math. Soc., Providence, 1968.

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