THE AVERAGE SIZE OF RAMANUJAN SUMS OVER QUADRATIC NUMBER FIELDS(II)

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Abstract: In this paper we study Ramanujan sums $c_m(n)$, where $m$ and $n$ are integral ideals in an arbitrary quadratic number field. We give some new results about the asymptotic behavior of sums of $c_m(n)$ over both $m$ and $n$.

1. Introduction and statements of results

1.1. Ramanujan sums over rationals. For any positive integers $m$ and $n$, the classical Ramanujan sum $c_m(n)$ is defined by (see, for example, Krätzel [7])

$$c_m(n) := \sum_{1 \leq j \leq m, \gcd(j,m) = 1} e\left(\frac{jn}{m}\right) = \sum_{d | \gcd(m,n)} d \mu\left(\frac{m}{d}\right),$$

where $e(z) := e^{2\pi iz}$ and $\mu(\cdot)$ is the Möbius function. The Ramanujan sum is an interesting and important object in number theory and there are lots of papers in this area.

In 2012, Chan and Kumchev [1] dealt with the question of the average order of $c_m(n)$ with respect to both variables $m$ and $n$. Let $Y \geq X \geq 3$ be two large real numbers and $k \geq 1$ be a fixed integer. Define

$$C_k(X, Y) : = \sum_{1 \leq n \leq Y} \left( \sum_{1 \leq m \leq X} c_m(n) \right)^k.$$

When $k = 1$, they proved that the asymptotic formula

$$C_1(X, Y) = Y - \frac{3}{2\pi^2} X^2 + O(XY^{1/3} \log X) + O(X^3Y^{-1})$$

holds, which implies that if $Y \asymp X^\delta$ then

$$S_1(X, Y) \sim \begin{cases} 
Y, & \text{if } \delta > 2, \\
\frac{3}{2\pi^2} X^2, & \text{if } 1 < \delta < 2.
\end{cases}$$

When $k = 2$, Chan and Kumchev proved that if $Y \geq x^2(\log B)^B$ for some fixed $B > 0$, then

$$C_2(X, Y) = \frac{YX^2}{2\zeta(2)} + O(X^4 + XY \log X).$$
1.2. Ramanujan sums over quadratic number fields. Suppose $\mathbb{F}/\mathbb{Q}$ is a number field of degree $d \geq 2$ and its ring of algebraic integers is denoted by $\mathcal{O}_F$. For any nonzero integral ideal $I$ in $\mathcal{O}_F$, the Möbius function is defined as follows (see, for example [3], Page 100):

$$\mu(I) = 0 \text{ if there exists a prime ideal } P \text{ such that } P^2 \text{ divides } I,$$

and

$$\mu(I) = (-1)^r \text{ if } I \text{ is a product of } r \text{ distinct prime ideals.}$$

For any ideal $I$, the norm of $I$ is denoted by $N(I)$. For two nonzero integral ideals $m$ and $n$, the Ramanujan sum is defined by

$$(1.6) \quad c_m(n) := \sum_{d \in \mathcal{O}_F} N(d)\mu\left(\frac{m}{d}\right),$$

which is an analogue of (1.1). The definition (1.6) of Ramanujan sums can be considered in the much more general context of arbitrary arithmetical semigroups. See, for example, Grytczuk [2] and the monograph by Knopfmacher [6].

For each $n \geq 1$, let $a_F(n)$ denote the number of integral ideals $I$ in $\mathcal{O}_F$ such that $N(I) = n$. Then we have

$$(1.7) \quad \sum_{n \leq X} a_F(n) = \rho_F x + P_F(x), \quad P_F(x) = O(x^{\delta+1}),$$

where $\rho_F$ is a constant depending only on $\mathbb{F}$. The asymptotic formula (1.7) is a classical result of Landau (see [8]), which was improved by many authors (see, for example, Müller [9], Nowak [10]).

Let $X \geq 3$ and $Y \geq 3$ be two large real numbers and $k \geq 1$ be a fixed integer. Define

$$(1.8) \quad C_{F,k}(X, Y) := \sum_{1 \leq N(n) \leq Y} \left( \sum_{1 \leq N(m) \leq X} c_m(n) \right)^k,$$

which is an analogue of the sum $C_k(X, Y)$ defined by (1.2).

In [11], W. G. Nowak proved that if $\mathbb{F}$ is a fixed quadratic number field, then the asymptotic formula

$$(1.9) \quad C_{F,1}(X, Y) \sim \rho_F Y$$

holds provided that $Y > X^\delta$ for some $\delta > 1973/820 = 2.40609 \cdots$. In [12], W. G. Nowak had considered the Gaussian field case $\mathbb{F} = \mathbb{Q}(i)$, where he proved that the asymptotic formula (1.9) holds provided that $Y > X^\delta$ for $\delta > 29/12 = 2.41\bar{6}$.

In [16], the author proved that the asymptotic formula (1.9) holds provided that $Y > X^\delta$ for $\delta > 79/34 = 2.3235 \cdots$. It was also proved that (1.9) holds on average for $2 < \delta \leq 79/34$.

In this paper, we shall prove that (1.9) holds provided that $Y > X^\delta$ for any $\delta > 2$. More precisely, we have the following Theorem 1.

**Theorem 1.** Let $\mathbb{F}$ be a fixed quadratic number field and $3 \leq X < Y$ be two large real numbers. Then we have

$$(1.10) \quad C_{F,1}(X, Y) = \rho_F Y + O\left(XY^{1/2}(\log Y)^7 + X^2\right).$$

**Remark 1.** Let $\lambda = \lambda(t)$ be an increasing function such that $\lim_{t \to \infty} \lambda(t) = \infty$ and $\lambda(t) = o(\log t)$ as $t \to \infty$. If $Y \geq X^2(\log X)^{14}\lambda(X)$, then (1.9) holds.
We can also study the sum for $k = 2$. In this case we have the following Theorem 2, which is an analogue and generalization of (1.5).

**Theorem 2.** Let $F$ be a fixed quadratic number field and $3 \leq X < Y$ be two large real numbers. If $Y > X^2$, then

\[
C_{F,2}(X,Y) = \frac{\rho_F^2}{2\zeta_F(2)}X^2Y + \frac{\zeta_\mathbb{R}(0)\rho_F^2}{4\zeta_F(2)}X^4
\]

\[
+ O\left(X^{\frac{21}{5}}Y^{-\frac{2}{5}} + X^2Y^{\frac{2}{5}}\log^5 Y + X^{\frac{2}{5}}Y\log^3 Y\right)
\]

**Remark 2.** In [16], the method of exponential sums was applied. However, in this paper, we apply the method of complex integration.

**Notation.** Throughout this paper, we use the following notations. $\mathbb{N}$, $\mathbb{Q}$ and $F$ denote the set of positive integers, the set of rational numbers and a number field of degree $d \geq 2$, respectively. We say $n$ is a half integer if $n - 1/2 \in \mathbb{N}$. For each $n \in \mathbb{N}$, $a_F(n)$ denotes the number of integral ideals $\mathcal{I}$ such that $N(\mathcal{I}) = n$, $\tau_\ell(n)$ denotes the number of ways $n$ can be written as a product of $\ell$ positive integers, $\tau(n) = \tau_2(n)$ is the well-known Dirichlet divisor function. $\zeta(s)$ is the Riemann zeta-function, and $\zeta_F(s)$ is the Dedekind zeta-function of the field $F$. $\varepsilon$ always denotes a small positive constant, which may be different at different places.

2. **Preliminary Lemmas**

We suppose that $F$ is a fixed number field of degree $d \geq 2$. The Dedekind zeta function of $F$ is defined by

\[
\zeta_F(s) := \sum_{\substack{\mathcal{I} \in \mathcal{O}_F \\ \mathcal{I} \neq 0}} \frac{1}{N^s(\mathcal{I})} \quad (\text{Re}(s) > 1).
\]

Then

\[
\zeta_F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} \quad (\text{Re}(s) > 1),
\]

where $a_F(n)$ denotes the number of integral ideals $\mathcal{I}$ such that $N(\mathcal{I}) = n$.

We have

\[
\frac{1}{\zeta_F(s)} = \sum_{\substack{n \in \mathcal{O}_F \\ n \neq 0}} \frac{\mu(n)}{N^s(n)} \quad (\text{Re}(s) > 1).
\]

**Lemma 2.1.** Suppose $F$ is a fixed number field of degree $d \geq 2$. Then we have the functional equation

\[
\zeta_F(s) = \chi_F(s)\zeta_F(1 - s)
\]

such that the estimate

\[
\chi_F(s) \ll (|t| + 1)^{d(\frac{1}{2} - \sigma)} \quad (|t| \to \infty)
\]

holds in any fixed critical strip.

**Proof.** See, for example, Iwaniec and Kowalski [5].
Lemma 2.2. Suppose $\mathbb{F}$ is a fixed number field of degree $d \geq 2$.

If $\sigma \geq 1$, then
\begin{equation}
\zeta_{\mathbb{F}}(\sigma + it) \ll_{\mathbb{F}} (|t| + 2) \log(|t| + 2).
\end{equation}

If $\sigma \geq 1$, then
\begin{equation}
\frac{1}{\zeta_{\mathbb{F}}(\sigma + it)} \ll_{\mathbb{F}} (|t| + 2) \log(|t| + 2).
\end{equation}

If $0 \leq \sigma \leq 1$, then
\begin{equation}
\zeta_{\mathbb{F}}(\sigma + it) \ll_{\mathbb{F}} (|t| + 2)^{\frac{d(d-1)}{2}} \log(|t| + 2).
\end{equation}

If $-2 \leq \sigma \leq 0$, then
\begin{equation}
\zeta_{\mathbb{F}}(\sigma + it) \ll_{\mathbb{F}} (|t| + 2)^{d(\frac{1}{2} - \sigma)} \log(|t| + 2).
\end{equation}

Proof. Let $x = (|t| + 2)^{d+1}$. Suppose $\sigma > 1$. We write
\begin{equation}
\zeta_{\mathbb{F}}(\sigma + it) = \Sigma_1 + \Sigma_2,
\end{equation}
where
\begin{equation}
\Sigma_1 := \sum_{n \leq x} \frac{a_{\mathbb{F}}(n)}{n^{\sigma+it}}, \quad \Sigma_2 := \sum_{n > x} \frac{a_{\mathbb{F}}(n)}{n^{\sigma+it}}.
\end{equation}

Let $A_{\mathbb{F}}(u) = \sum_{n \leq u} a_{\mathbb{F}}(n)$. By partial summation we have
\begin{equation}
\Sigma_2 = \int_{x}^{\infty} \frac{dA_{\mathbb{F}}(u)}{u^{\sigma+it}} = \rho_{\mathbb{F}} \int_{x}^{\infty} \frac{du}{u^{\sigma+it}} + \int_{x}^{\infty} \frac{dP_{\mathbb{F}}(u)}{u^{\sigma+it}}
\end{equation}
\begin{equation}
= \rho_{\mathbb{F}} \frac{x^{1-\sigma-it}}{\sigma-1+it} - \frac{P_{\mathbb{F}}(x)}{x^{\sigma+it}} + (\sigma + it) \int_{x}^{\infty} \frac{P_{\mathbb{F}}(u)}{u^{\sigma+1+it}} du.
\end{equation}

By the estimate $P_{\mathbb{F}}(u) \ll u^{(d-1)/(d+1)}$ we see that (2.8) is valid for $\sigma > (d-1)/(d+1)$. So for $\sigma \geq 1$ we have (recalling the definition of $x$)
\begin{equation}
\Sigma_2 \ll 1 + (|t| + 2)x^{-\frac{1}{d+1}} \ll 1.
\end{equation}

For $\Sigma_1$ we have by partial summation that
\begin{equation}
\Sigma_1 \ll \sum_{n \leq x} \frac{a_{\mathbb{F}}(n)}{n} \ll \log x \ll \log(|t| + 2) \quad (\sigma \geq 1).
\end{equation}

Now the estimate (2.3) follows from (2.7)-(2.10). The proof of (2.4) is similar and easier.

By (2.3) and Lemma 2.1 we get
\begin{equation}
\zeta_{\mathbb{F}}(it) \ll_{\mathbb{F}} (|t| + 2)^{\frac{d}{2}} \log(|t| + 2).
\end{equation}

So (2.5) follows from (2.11) and $\zeta_{\mathbb{F}}(1+it) \ll_{\mathbb{F}} \log(|t| + 2)$ with the help of Phragmen-Lindelöf principle. The estimate (2.6) follows from (2.3) and Lemma 2.1. \hfill \Box

Lemma 2.3. Suppose $\mathbb{F}$ is a quadratic number field. Then the estimate
\begin{equation}
\int_{-U}^{U} |\zeta_{\mathbb{F}}(\sigma + it)|^2 dt \ll U (\log U)^4 \quad (U \geq 2)
\end{equation}
holds uniformly for $1/2 \leq \sigma \leq 1$. 
Proof. Suppose $\mathbb{F}$ is a quadratic number field, then there exists a real primitive Dirichlet Character $\chi_D$ of modulo $|D|$ such that $\zeta_{\mathbb{F}}(s) = \zeta(s)L(s, \chi_D)$, where $L(s, \chi_D)$ is the Dirichlet L-function corresponding to $\chi_D$. Now Lemma 2.3 follows from the fourth power moment of $\zeta(s)$ and the fourth power moment of $L(s, \chi_D)$.

**Lemma 2.4.** Suppose $\mathbb{F}$ is a quadratic number field. Then the estimate

$$\int_{-U}^{U} |\zeta_{\mathbb{F}}(\sigma + it)|^4 dt \ll U \quad (U \geq 2)$$

holds uniformly for $2/3 \leq \sigma \leq 1$.

Proof. We have

$$\int_{1}^{U} \left| \zeta\left(\frac{2}{3} + it\right) \right|^8 dt \ll U \quad (U \geq 2).$$

In [4] we can find even much stronger results. Similarly, we have

$$\int_{1}^{U} \left| L\left(\frac{2}{3} + it, \chi_D\right) \right|^8 dt \ll U \quad (U \geq 2).$$

So Lemma 2.4 follows from the above two estimates and Cauchy’s inequality.

**Lemma 2.5.** Suppose $\mathbb{F}$ is a quadratic number field. Then for $1/2 \leq \sigma \leq 1$ we have the estimate

$$\zeta_{\mathbb{F}}(\sigma + it) \ll_{\mathbb{F}} (|t| + 2)^{\frac{2}{3}(1-\sigma)} \log(|t| + 2).$$

Proof. This estimate follows from (2.3) with $d = 2$, the well-known bound

$$\zeta_{\mathbb{F}}(1/2 + it) \ll_{\mathbb{F}} (|t| + 2)^{\frac{1}{3}}$$

and the Phragmen-Lindelöf principle.

**Lemma 2.6.** Let $k \geq 2$ be a fixed integer, and $f(n_1, \ldots, n_k)$ is a multivariable arithmetic function such that its Dirichlet series

$$F(s_1, \ldots, s_k) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{f(n_1, \ldots, n_k)}{n_1^{s_1} \cdots n_k^{s_k}}$$

is absolutely convergent for $\text{Re}(s_j) > \sigma_j$ ($j = 1, \ldots, k$), where $\sigma_1 > 0, \ldots, \sigma_k > 0$. Suppose $x_1, \ldots, x_k, T_1, \ldots, T_k \geq 5$ are parameters such that $x_j \notin \mathbb{N}$ ($j = 1, \ldots, k$), and define

$$b_j = \sigma_j + \frac{1}{\log x_j}, \quad (j = 1, \ldots, k).$$

Then we have

$$\sum_{n_1 \leq x_1} \cdots \sum_{n_k \leq x_k} f(n_1, \ldots, n_k)$$

is absolutely convergent for $\text{Re}(s_j) > \sigma_j$ ($j = 1, \ldots, k$).
where
\begin{equation}
E := \sum_{j=1}^{k} \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{|f(n_1, \ldots, n_k)|}{n_1^{b_1} \cdots n_k^{b_k}} \times \frac{1}{T_j |\log \frac{x_j}{n_j}| + 1}.
\end{equation}

Proof. For $b > 0, a > 0, T > 1$, we have
\begin{equation}
\frac{1}{2\pi i} \int_{b-iT}^{b+iT} a^s \frac{ds}{s} = \delta(a) + O \left( \frac{a^b}{T |\log a| + 1} \right),
\end{equation}
where $\delta(a) = 1$ when $a > 1$, and $\delta(a) = 0$ when $0 < a < 1$. See for example, Chapter 2 in the second part of Tenenbaum [13].

The Dirichlet series (2.12) is absolutely convergent for Re $s_j > \sigma_j$ ($j = 1, \ldots, k$). So we have by (2.15) that
\begin{equation}
\frac{1}{(2\pi i)^k} \int_{b_1-iT_1}^{b_1+iT_1} \cdots \int_{b_k-iT_k}^{b_k+iT_k} F(s_1, \ldots, s_k) \prod_{j=1}^{k} \frac{x_j^{s_j}}{s_1 \cdots s_k} ds_k \cdots ds_1
\end{equation}
\begin{align*}
&= \frac{1}{(2\pi i)^k} \int_{b_1-iT_1}^{b_1+iT_1} \cdots \int_{b_k-iT_k}^{b_k+iT_k} \prod_{j=1}^{k} \frac{x_j^{s_j}}{s_1 \cdots s_k} \left( \frac{1}{\prod_{j=1}^{k} \frac{x_j^{s_j}}{s_j}} \right) \frac{1}{\prod_{j=1}^{k} \frac{x_j^{s_j}}{s_j}} ds_k \cdots ds_1
\end{align*}
\begin{equation}
= \prod_{j=1}^{k} \left( \frac{x_j}{n_j} \right) \delta \left( \frac{x_j}{n_j} \right) + E_j \left( \frac{x_j}{n_j} \right),
\end{equation}
where
\begin{equation}
E_j \left( \frac{x_j}{n_j} \right) = O \left( \frac{(x_j/n_j)^{b_j}}{T_j |\log \frac{x_j}{n_j}| + 1} \right) (j = 1, \ldots, k).
\end{equation}

For any integer $n_j > 0$ we have
\begin{equation}
\delta \left( \frac{x_j}{n_j} \right) \leq \left( \frac{x_j}{n_j} \right)^{b_j} \leq \left( \frac{x_j}{n_j} \right)^{\sigma_j}.
\end{equation}
\begin{equation}
E_j \left( \frac{x_j}{n_j} \right) \leq \left( \frac{x_j}{n_j} \right)^{\sigma_j} \times \frac{1}{T_j |\log \frac{x_j}{n_j}| + 1} (j = 1, \ldots, k).
\end{equation}

Thus we have
\begin{equation}
\prod_{j=1}^{k} \left( \delta \left( \frac{x_j}{n_j} \right) + E_j \left( \frac{x_j}{n_j} \right) \right)
\end{equation}
\begin{align*}
&= \delta \left( \frac{x_1}{n_1} \right) \cdots \delta \left( \frac{x_k}{n_k} \right) + \sum_{j=1}^{k} O \left( \frac{x_1^{\sigma_j} \cdots x_k^{\sigma_j}}{n_1^{b_1} \cdots n_k^{b_k}} \times \frac{1}{T_j |\log \frac{x_j}{n_j}| + 1} \right).
\end{align*}
Now Lemma 2.4 follows from (2.16) and (2.17) by noting that
\[
\sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} f(n_1, \ldots, n_k) \delta \left( \frac{x_1}{n_1} \right) \cdots \delta \left( \frac{x_k}{n_k} \right) = \sum_{n_1 \leq x_1} \cdots \sum_{n_k \leq x_k} f(n_1, \ldots, n_k).
\]
\[\Box\]

3. Some special Dirichlet series

3.1. Dirichlet series involving \( c_m(n) \). For fixed \( k \geq 1 \), we define a multivariate arithmetic function \( f(m_1, \ldots, m_k, n) \) over the number field \( \mathbb{F} \) by
\[
f(m_1, \ldots, m_k, n) := c_{m_1}(n) \cdots c_{m_k}(n).
\]
Note that when \( k = 1 \), \( f(m_1, n) = c_{m_1}(n) \).

Suppose \( s_1, \ldots, s_k, w \in \mathbb{C} \) with \( \text{Re}(s_j) > 1(j = 1, \ldots, k) \), \( \text{Re}(w) > 2 \). Define the Dirichlet series
\[
\mathcal{F}(s_1, \ldots, s_k, w) := \sum_{m_1, \ldots, m_k, n} \frac{f(m_1, \ldots, m_k, n)}{N^{s_1}(m_1) \cdots N^{s_k}(m_k) N^w(n)}.
\]
For any \( \theta \in \mathbb{C} \) and any non-zero integral ideal \( n \), we define the weighted divisor function
\[
\sigma_\theta(n) := \sum_{d | n} N^\theta(d).
\]
We have the following Lemma 3.1.

**Lemma 3.1.** Suppose \( \theta_1, \theta_2, w \in \mathbb{C} \).

If \( \text{Re} w > \max(1, 1 + \text{Re}(\theta_1)) \), then we have the identity
\[
\sum_n \frac{\sigma_{\theta_1}(n)}{N^w(n)} = \zeta_\mathbb{F}(w) \zeta_\mathbb{F}(w - \theta_1).
\]

If \( \text{Re} w > \max(1, 1 + \text{Re}(\theta_1), 1 + \text{Re}(\theta_1), 1 + \text{Re}(\theta_1 + \theta_2)) \), then we have the Ramanujan’s identity
\[
\sum_n \frac{\sigma_{\theta_1}(n) \sigma_{\theta_2}(n)}{N^w(n)} = \frac{\zeta_\mathbb{F}(w) \zeta_\mathbb{F}(w - \theta_1) \zeta_\mathbb{F}(w - \theta_2) \zeta_\mathbb{F}(w - \theta_1 - \theta_2)}{\zeta_\mathbb{F}(2w - \theta_1 - \theta_2)}.
\]

**Proof.** The formula (3.4) follows from the definitions of \( \sigma_\theta(\cdot) \) and the Dedekind zeta-function (2.1). The formula (3.5) can be proved in the same way as the proof of the formula (1.3.3) in [14]. We omit the details. \(\Box\)

For the function \( \mathcal{F}(s_1, \ldots, s_k, w) \), we then have the following

**Proposition 3.1.** Suppose \( w, s_1, s_2 \in \mathbb{C} \).

If \( \text{Re}(w) > 1, \text{Re}(w + s_1) > 2 \), then
\[
\mathcal{F}(s_1, w) = \frac{\zeta_\mathbb{F}(w) \zeta_\mathbb{F}(w + s_1 - 1)}{\zeta_\mathbb{F}(s_1)}.
\]

If \( \text{Re}(w) > 1, \text{Re}(w + s_1) > 2, \text{Re}(w + s_2) > 2, \text{Re}(w + s_1 + s_2) > 3 \), then
\[
\mathcal{F}(s_1, s_2, w) = \frac{\zeta_\mathbb{F}(w) \zeta_\mathbb{F}(w + s_1 - 1) \zeta_\mathbb{F}(w + s_2 - 1) \zeta_\mathbb{F}(w + s_1 + s_2 - 2)}{\zeta_\mathbb{F}(s_1) \zeta_\mathbb{F}(s_2) \zeta_\mathbb{F}(2w + s_1 + s_2 - 2)}.
\]
Proof. Obviously we can rewrite the formula (3.2) in the form

\[ \mathcal{F}(s_1, \cdots, s_k, w) = \sum_n \frac{1}{N^w(n)} \prod_{j=1}^{k} \left( \sum_{m_j} \frac{c_{m_j}(n)}{N^{s_j}(m_j)} \right). \]  

Suppose \( s \in \mathbb{C} \) such that \( \text{Re}(s) > 1 \). By (1.6) we have

\[ \sum_m \frac{c_{m}(n)}{N^s(m)} = \sum_m \frac{1}{N^s(m)} \sum_{d|m,d|n} N(d) \mu \left( \frac{m}{d} \right) \]

\[ = \sum_{d|n} N^{1-s}(d) \sum_{m} \mu(m^*) \frac{\mu(m)}{N^s(m^*)} \]

\[ = \frac{\sigma_{1-s}(n)}{\zeta_F(s)}. \]

From (3.8) and (3.9) we get

\[ \mathcal{F}(s_1, \cdots, s_k, w) = \frac{1}{\zeta_F(s_1) \cdots \zeta_F(s_k)} \sum_n \frac{\sigma_{1-s_1}(n) \cdots \sigma_{1-s_k}(n)}{N^w(n)}. \]  

Now Proposition 3.1 follows from (3.10) and Lemma 3.1. \( \square \)

3.2. Dirichlet series involving \( c_m^*(n) \). For non-zero integral ideals \( m \) and \( n \), define

\[ c_m^*(n) := \sum_{d|n} N(d) \left| \mu \left( \frac{m}{d} \right) \right|. \]

It is easily seen that

\[ |c_m(n)| \leq c_m^*(n). \]

Suppose \( s \in \mathbb{C} \) such that \( \text{Re}(s) > 1 \). It is easy to see that

\[ \sum_{m} \frac{\mu(m_1)}{N^{s}(m_1)} = \frac{\zeta_F(s)}{\zeta_F(2s)}. \]

So for any non-zero integral ideal \( n \), we have for \( \text{Re}(s) > 1 \) that

\[ \sum_{m} \frac{c_m^*(n)}{N^{s}(m)} = \sum_m \frac{1}{N^s(m)} \sum_{d|m,d|n} N(d) \left| \mu \left( \frac{m}{d} \right) \right| \]

\[ = \sum_{d|n} N^{1-s}(d) \sum_{m_1} \frac{\mu(m_1)}{N^s(m_1)} \]

\[ = \frac{\zeta_F(s)}{\zeta_F(2s)} \sigma_{1-s}(n). \]
4. Estimates of some sums

Suppose $X, Y, T \geq 3$ are large real numbers such that both $X$ and $Y$ are half integers and $X \leq Y$. Let $\sigma_0 = 1 + 1/\log X$ and $k \geq 1$ be a fixed integer. In this section we shall estimate the sums $E_{j,k}(X, T)$ ($j = 1, 2, \cdots, k$) and $\mathcal{E}_k(Y, T)$, which are defined by

$$E_{j,k}(X, T) := \sum_{m_1, \cdots, m_k \in \mathbb{Z}^k} \frac{c^*_m(n) \cdots c^*_m(n)}{N^{\sigma_0}(m_1) \cdots N^{\sigma_0}(m_k) N^{\sigma_0}(n)} \times \frac{1}{T \left| \log \frac{X}{N(m_j)} \right| + 1}$$

and

$$\mathcal{E}_k(Y, T) := \sum_{m_1, \cdots, m_k \in \mathbb{Z}^k} \frac{c^*_m(n) \cdots c^*_m(n)}{N^{\sigma_0}(m_1) \cdots N^{\sigma_0}(m_k) N^{\sigma_0}(n)} \times \frac{1}{T \left| \log \frac{Y}{N(n)} \right| + 1}$$

respectively. It is easy to see that $E_{1,k}(X, T) = E_{2,k}(X, T) = \cdots = E_{k,k}(X, T)$. Thus it suffices to bound $E_{1,k}(X, T)$ and $\mathcal{E}_k(Y, T)$.

4.1. **An auxiliary estimate.** Suppose $T$ and $U$ are large real numbers such that $U$ is a half integer, $g(n)$ is a non-negative arithmetic function such that $g(n) \ll n^\varepsilon$ holds for any $\varepsilon > 0$. Define

$$G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \quad (\text{Re}(s) > 1),$$

which is obviously absolutely convergent for $\text{Re}(s) > 1$. Suppose $1 < \sigma_1 < 11/10$. Define

$$E(U, T; \sigma_1) := \sum_{n=1}^{\infty} \frac{g(n)}{n^{\sigma_1}} \times \frac{1}{T \left| \log \frac{U}{n} \right| + 1}.$$

We write

$$E(U, T; \sigma_1) = E_1(U, T; \sigma_1) + E_2(U, T; \sigma_1) + E_3(U, T; \sigma_1),$$

where

$$E_1(U, T; \sigma_1) := \sum_{n \leq U/2} \frac{g(n)}{n^{\sigma_1}} \times \frac{1}{T \left| \log \frac{U}{n} \right| + 1},$$

$$E_2(U, T; \sigma_1) := \sum_{U/2 < n \leq 2U} \frac{g(n)}{n^{\sigma_1}} \times \frac{1}{T \left| \log \frac{U}{n} \right| + 1},$$

$$E_3(U, T; \sigma_1) := \sum_{n > 2U} \frac{g(n)}{n^{\sigma_1}} \times \frac{1}{T \left| \log \frac{U}{n} \right| + 1}.$$

Trivially we have

$$E_1(U, T; \sigma_1) + E_3(U, T; \sigma_1) \ll \frac{1}{T} \sum_{n=1}^{\infty} \frac{g(n)}{n^{\sigma_1}} = \frac{G(\sigma_1)}{T}. \quad (4.2)$$

For $E_2(U, T; \sigma_1)$, we have

$$E_2(U, T; \sigma_1) \ll U^{\varepsilon-1} \sum_{U/2 < n \leq 2U} \frac{1}{T \left| \log \frac{U}{n} \right| + 1} \ll U^{\varepsilon} T^{-1}, \quad (4.3)$$
where we used the estimate
\[
\sum_{U/2 < n \leq U} \frac{1}{T |\log \frac{U}{n}| + 1} < \frac{U \log U}{T},
\]
which is well-known in analytic number theory.

From (4.1)-(4.3) we get the estimate
\[(4.4)\quad E(U, T; \sigma_1) \ll \frac{G(\sigma_1)}{T} + \frac{U^\varepsilon}{T}.
\]

4.2. Estimate of $E_k(Y, T)$. We write
\[
E_k(Y, T) = \sum_{n \in \mathcal{O}_F} \frac{1}{N^{\sigma_0}(n)} \times \frac{1}{T |\log \frac{Y}{N(n)}| + 1} \left( \sum_{m \in \mathcal{O}_F} \frac{c_m^*(n)}{N^{\sigma_0}(m)} \right)^k,
\]
which combining (3.14) gives
\[(4.5)\quad E_k(Y, T) = \sum_{n \in \mathcal{O}_F} \frac{1}{N^{\sigma_0}(n)} \times \frac{1}{T |\log \frac{Y}{N(n)}| + 1} \left( \frac{\zeta_F(\sigma_0)}{\zeta_F(2\sigma_0)} \sigma_{1-\sigma_0}(n) \right)^k.
\]

by noting that
\[
\sigma_{1-\sigma_0}(n) = \sum_{d|n} (N(d))^{-\frac{1}{\log X}} \leq \sum_{d|n} 1 = \sigma_0(n).
\]

Suppose $s$ such that $\text{Re}(s) > 1$. Then we have
\[
\sum_n \frac{\sigma_0(n)}{N^s(n)} = \zeta_F^2(s) = \sum_{n=1}^\infty \frac{a_F * a_F(n)}{n^s},
\]
where
\[
a_F * a_F(n) = \sum_{n=n_1n_2} a_F(n_1)a_F(n_2).
\]

So for $\text{Re}(s) > 1$ we can write
\[(4.6)\quad G_1(s) := \sum_{n \in \mathcal{O}_F} \frac{\sigma_0^k(n)}{N^s(n)} = \sum_{n=1}^\infty \frac{g_1(n)}{n^s},
\]
where
\[
g_1(n) := (a_F * a_F(n))^k a_F(n).
\]
By the well-known bound $a_F(n) \ll n^\varepsilon$ we get that $g_1(n) \ll n^\varepsilon$. So from (4.4)-(4.6) we get
\[(4.7)\quad E_k(Y, T) \ll \frac{\zeta_F^k(\sigma_0)}{\zeta_F^k(2\sigma_0)} \times \frac{1}{T} (G_1(\sigma_0) + Y^\varepsilon).
\]
THE AVERAGE SIZE OF RAMANUJAN SUMS OVER QUADRATIC NUMBER FIELDS (II)

By Euler’s product we have

$$\sum_{n \in \mathcal{O}_F} \frac{\sigma_0^k(n)}{N^s(n)} = \prod_{p \in \mathcal{O}_F} \left(1 + \sum_{\alpha=1}^{\infty} \frac{\sigma_0^k(p^\alpha)}{N^s(p^\alpha)}\right)$$

$$= \prod_{p \in \mathcal{O}_F} \left(1 - \frac{1}{N^s(p)}\right)^{-2k} \prod_{p \in \mathcal{O}_F} \left(1 + \sum_{\alpha=1}^{\infty} \frac{\sigma_0^k(p^\alpha)}{N^s(p^\alpha)}\right) \left(1 - \frac{1}{N^s(p)}\right)^{2k}$$

$$= (\zeta_F(s))^{2k} H(s),$$

where

$$H(s) = \prod_{p \in \mathcal{O}_F} \left(1 + \sum_{\alpha=1}^{\infty} \frac{\sigma_0^k(p^\alpha)}{N^s(p^\alpha)}\right) \left(1 - \frac{1}{N^s(p)}\right)^{2k}.$$

It is easy to see that $H(s)$ is absolutely convergent for Re($s$) > 1/2. So We have

$$(4.8) \quad G_1(\sigma_0) \ll (\zeta_F(\sigma_0))^{2k}.$$

From (2.3), (4.7) and (4.8) we get

$$(4.9) \quad \mathcal{E}_k(Y, T) \ll \frac{(\log X)^{2k+k}}{T} + \frac{Y^\varepsilon}{T} \ll \frac{Y^\varepsilon}{T}$$

by noting that $X \ll Y$.

4.3. Estimate of $E_{1,k}(X, T)$. By (3.14) and the definition of $c_m^*(n)$ in last section we have

$$(4.10) \quad E_{1,k}(X, T)$$

$$= \sum_{m \in \mathcal{O}_F} \frac{c_{m_1}^*(n)}{N^s(m_1)} \sum_{m_1 \in \mathcal{O}_F} \frac{c_m^*(n)}{N^s(m)} \frac{1}{\sigma_{1-s_0}^0(m_1) T \log \frac{X}{N(m_1)}} + 1 \left(\sum_{m \in \mathcal{O}_F} \frac{c_m^*(n)}{N^s(m)}\right)^{k-1}$$

$$= \frac{\zeta_F^{k-1}(\sigma_0)}{\zeta_F^{k-1}(2\sigma_0)} \sum_{n \in \mathcal{O}_F} \frac{\sigma_0^{k-1}(n)\sigma_0^{k-1}(n\mu(m))}{N^s(n)N^s(m_1)N^{2s_0-1}(d)} \frac{1}{T \log \frac{X}{N(m_1)}} + 1$$

$$= \frac{\zeta_F^{k-1}(\sigma_0)}{\zeta_F^{k-1}(2\sigma_0)} \sum_{n, m \in \mathcal{O}_F} \frac{\sigma_0^{k-1}(n)\sigma_0^{k-1}(d)\mu(m)}{N^s(n)N^s(m)N^{2s_0-1}(d)} \frac{1}{T \log \frac{X}{N(m)N(d)}} + 1$$

$$= \frac{\zeta_F^{k-1}(\sigma_0)}{\zeta_F^{k-1}(2\sigma_0)} \sum_{n \in \mathcal{O}_F} \frac{\sigma_0^{k-1}(n)}{N^s(n)} \sum_{d \in \mathcal{O}_F} \frac{\sigma_0^{k-1}(d)\mu(m)}{N^s(m)N^s(d)} \frac{1}{T \log \frac{X}{N(m)N(d)}} + 1$$

$$\ll \zeta_F^{k+1-2k+1}(\sigma_0) \sum_{d, m \in \mathcal{O}_F} \frac{\sigma_0^{k-1}(d)\mu(m)}{N^s(m)N^s(d)} \frac{1}{T \log \frac{X}{N(m)N(d)}} + 1.$$
where in the fifth line we used the bound $\sigma_0(dn) \leq \sigma_0(d)\sigma_0(n)$ and in the final line we used (4.8), which holds for any $k \geq 0$.

Define
\[ g(n) := \sum_{n=md} \sigma_0^{-1}(d)|\mu(m)|, \quad G_2(s) = \sum_{n \in \mathcal{O}_E} \frac{g(n)}{N^s(n)} \quad (\text{Re } s > 1). \]

Then we have
\[
(4.11) \quad \sum_{d,m \in \mathcal{O}_E} \frac{\sigma_0^{-1}(d)|\mu(m)|}{N^s(m)N^s(d)} T \left| \frac{1}{\log \frac{X}{N^s(m)N^s(d)}} + 1 \right| = \sum_{n \in \mathcal{O}_E} \frac{g(n)}{N^s(n)} \left| \frac{1}{\log \frac{x}{N(n)}} + 1 \right|.
\]

It is easily seen that $g(n)$ is multiplicative. For any prime ideal $p$, we have
\[ g(p) = \sigma^{-1}(p) + |\mu(p)| = 2^{k-1} + 1. \]

Thus for $\text{Re } s > 1$ we have
\[ G_2(s) = \sum_{n \in \mathcal{O}_E} \frac{g(n)}{N^s(n)} = \prod_{p \in \mathcal{O}_E} \left( 1 + \sum_{\alpha=1}^{\infty} \frac{g(p^\alpha)}{N^s(p^\alpha)} \right) = (\zeta_F(s))^{2^{k-1}+1}H(s), \]

where $H(s)$ is absolutely convergent for $\text{Re } s > 1/2$. So we have
\[ (4.12) \quad G_2(\sigma_0) \ll (\zeta_F(\sigma_0))^{2^{k-1}+1}. \]

If we write
\[ G_2(s) = \sum_{n=1} g_2(n) n^{-s}, \]
then it is easy to see that $g_2(n) \ll n^\varepsilon$. So from (4.4), (4.10), (4.11)(4.12) and (2.3) we get
\[ (4.13) \quad E_{1,k}(X,T) \ll \frac{X^\varepsilon}{T}. \]

5. Proof of Theorem 1

Without loss of generality, we suppose that both $X$ and $Y$ are half integers with $3 \leq X < Y$. Let $T \geq 3$ be a parameter to be determined later. Define
\[ b := 1 + \frac{1}{\log X}, \quad T_1 := T, \quad T_2 := 2T. \]

By the definition of $C_{\mathbb{F}_1}(X,Y)$ and Lemma 2.6 we have
\[ (5.1) \quad C_{\mathbb{F}_1}(X,Y) = I_{\mathbb{F}_1}(X,Y,T) + O(XYE_{\mathbb{F}_1}(X,T) + XY\mathcal{E}_1(Y,T)), \]

where
\[
I_{\mathbb{F}_1}(X,Y,T) := \frac{1}{(2\pi i)^2} \int_{b-iT_1}^{b+iT_1} ds \int_{b-iT_2}^{b+iT_2} \frac{\zeta_F(w)\zeta_F(w+s-1) X^s Y^w}{sw} dw,
\]
\[
E_{\mathbb{F}_1}(X,T) := \sum_{m} \sum_{n} \frac{|c_m(n)|}{N^b(m)N^b(n)} \times \frac{1}{T \left| \frac{X}{N(n)} \right| + 1},
\]
\[
\mathcal{E}_1(X,T) := \sum_{m} \sum_{n} \frac{|c_m(n)|}{N^b(m)N^b(n)} \times \frac{1}{T \left| \frac{Y}{N(n)} \right| + 1}.
\]
From (4.9) and (4.13) with $k = 1$, we have

$$\tag{5.2} E_{\mathcal{F},1}(X, T) \ll \frac{X^\varepsilon}{T}, \quad \mathcal{E}_1(X, T) \ll \frac{Y^\varepsilon}{T}. $$

We consider the rectangle domain of $w$ formed by the four points $b \pm iT_2$ and $1/2 \pm iT_2$. Let

$$G(w; x, X, Y) := \frac{\zeta_{\mathcal{F}}(w)\zeta_{\mathcal{F}}(w + s - 1)}{\zeta_{\mathcal{F}}(s)} \frac{X^sY^w}{sw}. $$

In this domain, $G(w; x, X, Y)$ has two simple poles, which are $w = 1$ and $w = 2 - s$, respectively. It is easy to see that

$$\text{Res}_{w=1}G(w; s, X, Y) = \rho_{\mathcal{F}}Y \frac{X^s}{s} ,$$
$$\text{Res}_{w=2-s}G(w; s, X, Y) = \rho_{\mathcal{F}} \frac{\zeta_{\mathcal{F}}(2-s)X^sY^{2-s}}{s(2-s)} . $$

By the residue theorem we get

$$\tag{5.3} I_{\mathcal{F},1}(X, Y, T) = \mathcal{J}_1(X, Y, T) + \mathcal{J}_2(X, Y, T) + H_1(X, Y, T) + H_2(X, Y, T) - H_3(X, Y, T), $$

where

$$\tag{5.4} \mathcal{J}_1(X, Y, T) := \rho_{\mathcal{F}}Y \frac{1}{2\pi i} \int_{b-iT_1}^{b+iT_1} \frac{X^s}{s} ds,$$
$$\mathcal{J}_2(X, Y, T) := \rho_{\mathcal{F}} \frac{1}{2\pi i} \int_{b-iT_1}^{b+iT_1} \frac{\zeta_{\mathcal{F}}(2-s)X^sY^{2-s}}{\zeta_{\mathcal{F}}(s)} \frac{1}{s(2-s)} ds,$$
$$H_1(X, Y, T) := \frac{1}{(2\pi i)^2} \int_{b-iT_1}^{b+iT_1} ds \int_{1/2+iT_2}^{1/2+iT_2} \frac{\zeta_{\mathcal{F}}(w)\zeta_{\mathcal{F}}(w + s - 1)X^sY^w}{sw} dw,$$
$$H_2(X, Y, T) := \frac{1}{(2\pi i)^2} \int_{b-iT_1}^{b+iT_1} ds \int_{1/2-iT_2}^{1/2-iT_2} \frac{\zeta_{\mathcal{F}}(w)\zeta_{\mathcal{F}}(w + s - 1)X^sY^w}{sw} dw,$$
$$H_3(X, Y, T) := \frac{1}{(2\pi i)^2} \int_{b-iT_1}^{b+iT_1} ds \int_{1/2-iT_2}^{1/2-iT_2} \frac{\zeta_{\mathcal{F}}(w)\zeta_{\mathcal{F}}(w + s - 1)X^sY^w}{sw} dw.$$

We consider $H_1(X, Y, T)$ first. Suppose

$$s = b + it, \ |t| \leq T, \ w = u + 2iT, \ 1/2 \leq u \leq b.$$  

From Lemma 2.2 we have

$$G(w; x, X, Y) \ll \begin{cases} 
\frac{X^{bY^uT^{1-2u}}}{|t|+1} \log^3 T, & 1/2 \leq u \leq 1, \\
\frac{X^{bY^u}}{|t|+1} \log^3 T, & 1 \leq u \leq b, 
\end{cases} $$

which implies that

$$\tag{5.5} H_1(X, Y, T) \ll \int_{b-iT_1}^{b+iT_1} \frac{X^b}{|t|+1} dt \left( \int_{1/2}^{1} Y^u T^{1-2u} du + \int_{1}^{b} \frac{Y^u}{T} du \right) \log^3 T $$
$$\ll XY^{1/2} \log^4 T + XY^{bT^{-1}} \log^4 T.$$

Similarly we have

$$\tag{5.6} H_2(X, Y, T) \ll XY^{1/2} \log^4 T + \frac{XY^b}{Y T} \log^4 T.$$
Now we consider $H_2(X, Y, T)$. Suppose
\[ s = b + it, \ |t| \leq T, \ w = 1/2 + iv, \ |v| \leq 2T. \]
We have
\[ H_2(X, Y, T) \ll XY^{1/2} \log T \times \mathcal{H}(X, Y, T), \]
where
\[ \mathcal{H}(X, Y, T) := \int_{b-iT_1}^{b+iT_1} dt \int_{1/2-iT_2}^{1/2+iT_2} \frac{\left| \zeta_F\left(\frac{1}{2} + iv\right) \zeta_F\left(\frac{1}{2} + \frac{1}{\log X} + i(t + v)\right) \right|}{(|t| + 1)(|v| + 1)} dv. \]
Write
\[ \mathcal{H}(X, Y, T) = \mathcal{H}_1(X, Y, T) + \mathcal{H}_2(X, Y, T), \]
where
\[ \mathcal{H}_1(X, Y, T) := \int_{|t| \leq |v|} \left| \frac{\zeta_F\left(\frac{1}{2} + iv\right) \zeta_F\left(\frac{1}{2} + \frac{1}{\log X} + i(t + v)\right)}{(|t| + 1)(|v| + 1)} \right| dv dt, \]
\[ \mathcal{H}_2(X, Y, T) := \int_{|v| \leq |t|} \left| \frac{\zeta_F\left(\frac{1}{2} + iv\right) \zeta_F\left(\frac{1}{2} + \frac{1}{\log X} + i(t + v)\right)}{(|t| + 1)(|v| + 1)} \right| dv dt. \]

From Lemma 2.3 and partial integration we have that
\[ \int_{-U}^{U} \frac{\left| \zeta_F(u + iv) \right|^2}{|v| + 1} dv \ll (\log U)^5 \quad (1/2 \leq u \leq 1). \]
and
\[ \int_{-U}^{U} \frac{\left| \zeta_F(u + iv) \right|}{|v| + 1} dv \ll (\log U)^3 \quad (1/2 \leq u \leq 1). \]
If $|t| \leq |v|$, then $|v + t| \leq |v| + |t| \leq 2|v|$, which combining with (5.9) and Cauchy’s inequality implies that
\[ \mathcal{H}_1(X, Y, T) \ll \int_{|t| \leq T} \frac{1}{|t| + 1} dt \int_{|t| \leq |v|} \left| \frac{\zeta_F\left(\frac{1}{2} + iv\right) \zeta_F\left(\frac{1}{2} + \frac{1}{\log X} + i(t + v)\right)}{(|v| + 1)^{1/2}(|v + t| + 1)^{1/2}} \right| dv \]
\[ \ll \int_{|t| \leq T} \frac{dt}{|t| + 1} \left( \int_{|t| \leq |v|} \left| \frac{\zeta_F\left(\frac{1}{2} + iv\right)^2}{|v| + 1} \right| dv \right)^{1/2} \left( \int_{|t| \leq |v|} \left| \frac{\zeta_F\left(\frac{1}{2} + \frac{1}{\log X} + i(v + t)\right)^2}{|v + t| + 1} \right| dv \right)^{1/2} \]
\[ \ll (\log T)^6. \]
If $|v| \leq |t|$, then $|v + t| \leq |v| + |t| \leq 2|t|$, which combining with (5.10) gives
\[ \mathcal{H}_2(X, Y, T) \ll \int_{|v| \leq T} \left| \frac{\zeta_F\left(\frac{1}{2} + iv\right)}{|v| + 1} \right| dv \int_{|v| \leq |t|} \left| \frac{\zeta_F\left(\frac{1}{2} + \frac{1}{\log X} + i(t + v)\right)}{|v + t| + 1} \right| dt \]
\[ \ll (\log T)^6. \]

From (5.7), (5.8), (5.11) and (5.12) we get
\[ H_2(X, Y, T) \ll XY^{1/2}(\log T)^7. \]
From (2.15) we get

\[(5.14) \quad \tilde{J}_1(X, Y, T) = \rho_Y Y + O\left(\frac{XY}{T \log X}\right).\]

Finally we consider \( \tilde{J}_2(X, Y, T) \). Let

- \( IP_1 = \{ s = \sigma - iT_1 : b \leq \sigma \leq 2 \} \), \( IP_2 = \{ s = 2 + it : -T \leq t \leq -\frac{1}{\log X} \} \),
- \( IP_3 = \{ s = \frac{e^{i\theta}}{\log X} : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \} \),
- \( IP_4 = \{ s = 2 + it : \frac{1}{\log X} \leq t \leq T \} \), \( IP_5 = \{ s = \sigma + iT_1 : b \leq \sigma \leq 2 \} \).

By the residue theorem we have

\[(5.15) \quad \tilde{J}_2(X, Y, T) = \frac{\zeta_{\varphi}(0)}{2\zeta_{\varphi}(2)} X^2 + \sum_{j=1}^{4} \frac{\zeta_{\varphi}(2-s)}{\zeta_{\varphi}(s)} \int_{IP_j} X^s Y^{2-s} ds \quad (j = 1, 2, 3, 4).\]

By Lemma 2.2 we have

\[(5.16) \quad \tilde{J}_{21}(X, Y, T) \ll \int_{b}^{2} T^{\sigma-3} X^\sigma Y^{2-\sigma} \log^2 T d\sigma \ll \frac{XY}{T^2} \log^2 T + \frac{X^2}{T} \log^2 T.\]

and

\[(5.17) \quad \tilde{J}_{25}(X, Y, T) \ll \int_{b}^{2} T^{\sigma-3} X^\sigma Y^{2-\sigma} \log^2 T d\sigma \ll \frac{XY}{T^2} \log^2 T + \frac{X^2}{T} \log^2 T.\]

By Lemma 2.2 again we have

\[(5.18) \quad \tilde{J}_{24}(X, Y, T) \ll X^2 \int_{\log X}^{T_1} \frac{\log(t+1)}{t(t+1)} dt = X^2 \left( \int_{\log X}^{1} \frac{\log(t+1)}{t(t+1)} dt + \int_{1}^{T_1} \frac{\log(t+1)}{t(t+1)} dt \right) \ll X^2.\]

by noting that \( \log(1+t) \ll t \) (0 < t < 1). Similarly

\[(5.19) \quad \tilde{J}_{22}(X, Y, T) \ll X^2 \int_{\log X}^{T_1} \frac{\log(t+1)}{t(t+1)} dt \ll X^2.\]

For \( \tilde{J}_{23}(X, Y, T) \) we have

\[(5.20) \quad \tilde{J}_{23}(X, Y, T) \ll X^2.\]

From (5.15)-(5.20) we get

\[(5.21) \quad \tilde{J}_2(X, Y, T) \ll \frac{XY}{T^2} \log^2 T + \frac{X^2}{T} \log^2 T + X^2.\]
From (5.1)-(5.6), (5.13), (5.14) and (5.21) we have

\[ C_{F,1}(X, Y) = \rho_2 Y + O \left( \frac{X Y^{1+\varepsilon}}{T} + X Y^{1/2} \log^7 T \right) + O \left( X^2 + \frac{X^2}{T} \log^4 T \right) \]

\[ = \rho_2 Y + O \left( X Y^{1/2} \log^7 Y + X^2 \right) \]

by choosing \( T = XY \). This completes the proof of Theorem 1.

6. Proof of Theorem 2

Without loss of generality, we suppose that both \( X \) and \( Y \) are half integers and \( X < Y \). Let \( T \geq 3 \) be a parameter to be determined later. Define

\[ b_1 := 1 + \frac{1}{\log X}, \quad b_2 := 1 + \frac{2}{\log X}, \quad b_3 := 1 + \frac{3}{\log X} \]

\[ T_1 = T, \quad T_2 = 2T, \quad T_3 := 4T. \]

By the definition of \( C_{F,2}(X, Y) \) and Lemma 2.6 we have

\[ C_{F,2}(X, Y) = I_{F,2}(X, Y, T) + O(X^2 Y^b E_{F,1}(X, T) + X^2 Y^b \mathcal{E}_2(Y, T)), \]

where

\[ I_{F,2}(X, Y, T) := \frac{1}{(2\pi i)^3} \int_{b_1 - iT_1}^{b_1 + iT_1} ds_1 \int_{b_2 - iT_2}^{b_2 + iT_2} ds_2 \int_{b_3 - iT_3}^{b_3 + iT_3} \mathcal{G}(w; s_1, s_2)dw, \]

\[ E_{F,1}(X, T) := \sum_{m_1} \sum_{m_2} \sum_n \frac{|c_{m_1}(n)c_{m_2}(n)|}{N^{b_1}(m_1)N^{b_1}(m_2)N^{b_1}(n)} \times \frac{1}{T \left\lfloor \frac{X}{N(m)} \right\rfloor + 1}, \]

\[ \mathcal{E}_2(X, T) := \sum_{m_1} \sum_{m_2} \sum_n \frac{|c_{m_1}(n)c_{m_2}(n)|}{N^{b_1}(m_1)N^{b_1}(m_2)N^{b_1}(n)} \times \frac{1}{T \left\lfloor \frac{Y}{N(n)} \right\rfloor + 1} \]

and

\[ \mathcal{G}(w; s_1, s_2) := \frac{\zeta_F(w)\zeta_F(w + s_1 - 1)\zeta_F(w + s_2 - 1)\zeta_F(w + s_1 + s_2 - 2)}{\zeta_F(s_1)\zeta_F(s_2)\zeta_F(2w + s_1 + s_2 - 2)} \frac{X^{s_1 + s_2} Y^w}{s_1 s_2 w}. \]

From (4.9) and (4.13) with \( k = 2 \), we have

\[ E_{F,1}(X, T) \ll \frac{X^\varepsilon}{T}, \quad \mathcal{E}_2(X, T) \ll \frac{Y^\varepsilon}{T}. \]

We consider the rectangle domain of \( w \) formed by the four points \( b_3 \pm iT_3 \) and \( 2/3 \pm iT_3 \). In this domain, \( \mathcal{G}(w; s_1, s_2) \) has four simple poles, which are \( w_1 = 1 \), \( w_2 = 2 - s_1 \), \( w_3 = 2 - s_2 \) and \( w_4 = 3 - s_1 - s_2 \) respectively. By the residue theorem we get

\[ I_{F,2}(X, Y, T) = \mathcal{L}_1(X, Y, T) + \mathcal{L}_2(X, Y, T) + \mathcal{L}_3(X, Y, T) + \mathcal{L}_4(X, Y, T) + K_1(X, Y, T) + K_2(X, Y, T) - K_3(X, Y, T). \]
where
\[
\mathcal{L}_j(X, Y, T) := \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{b_2-iT_2}^{b_2+iT_2} Res_{w=w_j} G(w; s_1, s_2) ds_2 \quad (j = 1, 2, 3, 4),
\]
\[
K_1(X, Y, T) := \frac{1}{(2\pi i)^3} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{b_2-iT_2}^{b_2+iT_2} ds_2 \int_{b_3-iT_3}^{b_3+iT_3} G(w; s_1, s_2) dw,
\]
\[
K_2(X, Y, T) := \frac{1}{(2\pi i)^3} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{b_2-iT_2}^{b_2+iT_2} ds_2 \int_{2/3+iT_3}^{2/3+iT_3} G(w; s_1, s_2) dw,
\]
\[
K_3(X, Y, T) := \frac{1}{(2\pi i)^3} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{b_2-iT_2}^{b_2+iT_2} ds_2 \int_{2/3-iT_3}^{2/3-iT_3} G(w; s_1, s_2) dw.
\]

6.1. Upper bound of $K_j(X, Y, T)$ ($j=1,2,3$). We first consider $K_1(X, Y, T)$. Suppose
\[
s_1 = b + it_1, \quad |t_1| \leq T_1, \quad s_2 = b + it_2, \quad |t_2| \leq T_2, \quad w = u + iT_3, \quad 2/3 \leq u \leq b.
\]
By Lemma 2.5 we have
\[
G(w; s_1, s_2) \ll \frac{X^{2b}}{(|t_1| + 1)(|t_2| + 1)} T^{\frac{2}{3} - \frac{s_1}{Y}} Y^u \log^6 T, \quad 2/3 \leq u \leq b,
\]
which implies that
\[
K_1(X, Y, T) \ll X^2 Y^{\frac{2}{3}} T^{-\frac{1}{3}} \log^8 T + X^2 Y^b T^{-1} \log^8 T.
\]
Similarly we have
\[
K_3(X, Y, T) \ll X^2 Y^{\frac{2}{3}} T^{-\frac{1}{3}} \log^8 T + X^2 Y^b T^{-1} \log^8 T.
\]
We now consider $K_2(X, Y, T)$. By (2.4) of Lemma 2.2 we write
\[
K_2(X, Y, T) \ll J X^2 Y^{\frac{2}{3}} \log^2 T,
\]
where
\[
J := \int_{b_1-iT_1}^{b_1+iT_1} dt_1 \int_{b_2-iT_2}^{b_2+iT_2} dt_2 \int_{2/3-iT_3}^{2/3+iT_3} \frac{|g(t_1, t_2, v)|}{(|t_1| + 1)(|t_2| + 1)(|v| + 1)} dv
\]
with
\[
g(t_1, t_2, v) := \zeta_F \left( \frac{2}{3} + iv \right) \zeta_F \left( \frac{2}{3} + \frac{1}{\log X} + i(v + t_1) \right) \zeta_F \left( \frac{2}{3} + \frac{1}{\log X} + i(v + t_2) \right) \zeta_F \left( \frac{2}{3} + \frac{2}{\log X} + i(v + t_1 + t_2) \right)
\]
With the help of Lemma 2.4 we can show that
\[
J \ll \log^3 T.
\]
The proof of (6.8) is similar to the arguments of $H_2(T)$ in Tóth and Zhai [15]. So we omit its details.
From (6.6)-(6.8) we get
\[
K_2(X, Y, T) \ll X^2 Y^{\frac{2}{3}} \log^5 T.
\]
We now consider the first integral in (6.10). We have

\[
\mathcal{L}_1(X, Y, T) = \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{b_2-iT_2}^{b_2+iT_2} ds_2 \frac{\zeta(s) - 1}{\zeta(s + 1)} \frac{X^{s_1+s_2}Y}{s_1s_2} ds_2.
\]

By the residue theorem, we have

\[
\mathcal{L}_1(X, Y, T) = \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} \frac{\rho_{\pi}^2}{\zeta(2)} \frac{X^{s_1+s_2}}{s_1s_2} ds_1.
\]

By Lemma 2.2 it is easy to see that

\[
\mathcal{L}_1(X, Y, T) \ll \frac{X^{2Y}}{T} \log T + \frac{X^{2Y}}{T^{1/2}} \log T.
\]

For \(\mathcal{L}_2(X, Y, T)\), we have

\[
\mathcal{L}_2(X, Y, T) \ll \frac{YX^{2Y}}{T} \log T + \frac{X^{2Y}}{T^{1/2}} \log T.
\]
where \( \int_{(b)} \) means that \( \int_{b-i\infty}^{b+i\infty} \). Moving the integral line from \( b \) to \( \text{Re}(s_1) = 1 \), we get

\[
(6.14) \quad \frac{1}{2\pi i} \int_{b-iT_1}^{b+iT_1} \frac{\rho^2_F}{\zeta_F(2)} \frac{X^2Y}{s_1(2 - s_1)} ds_1 = c_F X^2Y + O \left( \frac{X^2Y \log T}{T} \right),
\]

with

\[
c_F = \frac{\rho^2_F}{\zeta_F(2)} \frac{1}{2\pi i} \int_{1} \frac{1}{s_1(2 - s_1)} ds_1 = \frac{\rho^2_F}{2\zeta_F(2)}.
\]

From (6.10)-(6.14) we get

\[
(6.15) \quad \mathcal{L}_1(X, Y, T) = \frac{\rho^2_F}{2\zeta_F(2)} X^2Y + O \left( \frac{X^2Y \log T}{T} + Y X^{3/2} \log T \right).
\]

6.3. Upper bound of \( \mathcal{L}_2(X, Y, T) \). It is easy to see that

\[
\text{Re} s_{w=2-s_1} G(w; s_1, s_2) = \frac{\zeta_F(2 - s_1) \zeta_F(1 - s_1 + s_2)}{\zeta_F(s_1) \zeta_F(2 - s_1 + s_2)} \frac{X^{s_1+s_2} Y^{2-s_1}}{s_1 s_2 (2 - s_1)}
\]

So we have

\[
\mathcal{L}_2(X, Y, T) = \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{b_2-iT_2}^{b_2+iT_2} \rho_F \frac{\zeta_F(2 - s_1) \zeta_F(1 - s_1 + s_2)}{\zeta_F(s_1) \zeta_F(2 - s_1 + s_2)} \frac{X^{s_1+s_2} Y^{2-s_1}}{s_1 s_2 (2 - s_1)} ds_2.
\]

We consider the rectangle domain of \( s_2 \) formed by the four points \( 1/2 \pm iT_2 \) and \( b_2 \pm iT_2 \). In this domain, the integral function in the above integral is \( s_2 = s_1 \) with residue

\[
\frac{\rho^2_F}{\zeta_F(2)} \frac{\zeta_F(2 - s_1)}{\zeta_F(s_1)} \frac{X^{2s_1} Y^{2-s_1}}{s_1^2 (2 - s_1)}.
\]

By the residue theorem we get

\[
(6.16) \quad \mathcal{L}_2(X, Y, T) = \mathcal{L}_{20}(X, Y, T) + \mathcal{L}_{21}(X, Y, T) + \mathcal{L}_{22}(X, Y, T) - \mathcal{L}_{23}(X, Y, T)
\]

where

\[
\begin{align*}
\mathcal{L}_{20}(X, Y, T) &:= \frac{\rho^2_F}{\zeta_F(2)} \frac{1}{2\pi i} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \frac{\zeta_F(2 - s_1)}{\zeta_F(s_1)} \frac{X^{2s_1} Y^{2-s_1}}{s_1^2 (2 - s_1)} ds_1, \\
\mathcal{L}_{21}(X, Y, T) &:= \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{1/2+iT_2}^{1/2-iT_2} \rho_F \frac{\zeta_F(2 - s_1) \zeta_F(1 - s_1 + s_2)}{\zeta_F(s_1) \zeta_F(2 - s_1 + s_2)} \frac{X^{s_1+s_2} Y^{2-s_1}}{s_1 s_2 (2 - s_1)} ds_2, \\
\mathcal{L}_{22}(X, Y, T) &:= \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{1/2-iT_2}^{1/2+iT_2} \rho_F \frac{\zeta_F(2 - s_1) \zeta_F(1 - s_1 + s_2)}{\zeta_F(s_1) \zeta_F(2 - s_1 + s_2)} \frac{X^{s_1+s_2} Y^{2-s_1}}{s_1 s_2 (2 - s_1)} ds_2, \\
\mathcal{L}_{23}(X, Y, T) &:= \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{1/2-iT_2}^{1/2+iT_2} \rho_F \frac{\zeta_F(2 - s_1) \zeta_F(1 - s_1 + s_2)}{\zeta_F(s_1) \zeta_F(2 - s_1 + s_2)} \frac{X^{s_1+s_2} Y^{2-s_1}}{s_1 s_2 (2 - s_1)} ds_2.
\end{align*}
\]

By Lemma 2.2 we have

\[
(6.17) \quad \mathcal{L}_{21}(X, Y, T) \ll Y \log T \int_{-T_1}^{T_1} \frac{|\zeta(2 - b_1 - it_1)|}{(|t_1| + 1)^2} dt_1 \int_{1/2}^{b_2} T^{-\sigma_2} X^{b_1+\sigma_2} d\sigma_2
\]

\[
\ll \frac{X^2Y \log T}{T} + Y X^{3/2} \log T
\]
and

\begin{equation}
(6.18) \quad \mathcal{L}_{23}(X, Y, T) \ll \frac{X^2 Y \log T}{T} + \frac{Y X^{3/2} \log T}{T^{1/2}}.
\end{equation}

By (2.4) of Lemma 2.2 we have

\begin{equation}
(6.19) \quad \mathcal{L}_{22}(X, Y, T) \ll X^{3/2} Y \log^2 T \int_{-T_1}^{T_1} dt_1 \int_{-T_2}^{T_2} \frac{|\zeta_F(\frac{1}{2} - \frac{1}{\log X} + i(t_2 - t_1))|}{(|t_1| + 1)^2(|t_2| + 1)} dt_2 \ll X^{3/2} Y \log^5 T,
\end{equation}

where we used the bound

\[ \int_{-T_1}^{T_1} dt_1 \int_{-T_2}^{T_2} \frac{|\zeta_F(\frac{1}{2} - \frac{1}{\log X} + i(t_2 - t_1))|}{(|t_1| + 1)^2(|t_2| + 1)} dt_2 \ll \log^3 T, \]

whose proof is similar to (5.11) and (5.12).

Finally we consider \( \mathcal{L}_{20}(X, Y, T) \). We write

\[ \mathcal{L}_{20}(X, Y, T) = \frac{\rho_F^2}{\zeta_F(2)} \cdot \frac{1}{2\pi i} \int_{(b_1)} \frac{\zeta_F(2 - s)}{\zeta_F(s)} \cdot \frac{X^{2s_1} Y^{2 - s_1}}{s_1^2(2 - s)} ds_1 + O \left( \frac{X^2 Y \log^2 T}{T^2} \right). \]

Moving the integral line to \( \text{Re} s_1 = 12/5 \), we encounter a simple pole \( s_1 = 2 \). We have

\[ \frac{\rho_F^2}{\zeta_F(2)} \cdot \frac{1}{2\pi i} \int_{(b_1)} \frac{\zeta_F(2 - s)}{\zeta_F(s)} \cdot \frac{X^{2s_1} Y^{2 - s_1}}{s_1^2(2 - s)} ds_1 = \frac{\zeta_F(0)\rho_F^2}{4\zeta_F^2(2)} X^4 + \frac{\rho_F^2}{2\pi i} \int_{(12/5)} \frac{\zeta_F(2 - s)}{\zeta_F(s)} \cdot \frac{X^{2s_1} Y^{2 - s_1}}{s_1^2(2 - s)} ds_1. \]

By Lemma 2.1 we see that if \( s_1 = 12/5 + it \), then

\[ \frac{\zeta_F(2 - s_1)}{\zeta_F(s_1)s_1^2(2 - s_1)} \ll \frac{1}{(|t| + 1)^{6/5}}, \]

which implies that

\[ \int_{(12/5)} \frac{\zeta_F(2 - s)}{\zeta_F(s)} \cdot \frac{X^{2s_1} Y^{2 - s_1}}{s_1^2(2 - s)} ds_1 \ll X^{\frac{24}{5}} Y^{-\frac{2}{5}}. \]

From the above estimates we have

\begin{equation}
(6.20) \quad \mathcal{L}_{20}(X, Y, T) = \frac{\zeta_F(0)\rho_F^2}{4\zeta_F^2(2)} X^4 + O \left( \frac{X^2 Y \log^2 T}{T^2} + X \frac{24}{5} Y^{-\frac{2}{5}} \right).
\end{equation}

From (6.16) to (6.20) we get

\begin{equation}
(6.21) \quad \mathcal{L}_2(X, Y, T) = \frac{\zeta_F(0)\rho_F^2}{4\zeta_F^2(2)} X^4 + O \left( \frac{X^2 Y \log T}{T} + X^\frac{3}{2} Y \log^5 T + X \frac{24}{5} Y^{-\frac{2}{5}} \right).
\end{equation}
6.4. Upper bound of $\mathcal{L}_3(X, Y, T)$. It is easy to see that
\[
\text{Res}_{w=2-s_2} G(w; s_1, s_2) = \rho_F \frac{\zeta_F(2-s_2)\zeta_F(1+s_1-s_2)}{\zeta_F(2-s_1+s_2)} X^{s_1+s_2} Y^{2-s_2}.
\]
So we have
\[
\mathcal{L}_2(X, Y, T) = \frac{1}{(2\pi i)^2} \int_{b_1+iT_1}^{b_1+iT_1} ds_1 \int_{b_2+iT_2}^{b_2+iT_2} \rho_F \frac{\zeta_F(2-s_2)\zeta_F(1+s_1-s_2)}{\zeta_F(2-s_1+s_2)} X^{s_1+s_2} Y^{2-s_2} ds_2.
\]
We consider the rectangle domain of $s_2$ formed by the four points $b_2 \pm iT_2$ and $7/4 \pm iT_2$. By the residue theorem we get
\[
(6.22) \quad \mathcal{L}_3(X, Y, T) = -\mathcal{L}_{31}(X, Y, T) + \mathcal{L}_{32}(X, Y, T) + \mathcal{L}_{33}(X, Y, T)
\]
where
\[
\mathcal{L}_{31}(X, Y, T) := \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{b_2+iT_2}^{b_2+iT_2} \rho_F \frac{\zeta_F(2-s_2)\zeta_F(1+s_1-s_2)}{\zeta_F(2-s_1+s_2)} X^{s_1+s_2} Y^{2-s_2} ds_2,
\]
\[
\mathcal{L}_{32}(X, Y, T) := \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{7/4-iT_2}^{7/4+iT_2} \rho_F \frac{\zeta_F(2-s_2)\zeta_F(1+s_1-s_2)}{\zeta_F(2-s_1+s_2)} X^{s_1+s_2} Y^{2-s_2} ds_2,
\]
\[
\mathcal{L}_{33}(X, Y, T) := \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{b_2-iT_2}^{b_2+iT_2} \rho_F \frac{\zeta_F(2-s_2)\zeta_F(1+s_1-s_2)}{\zeta_F(2-s_1+s_2)} X^{s_1+s_2} Y^{2-s_2} ds_2.
\]
From Lemma 2.2 it is easy to see that
\[
(6.23) \quad \mathcal{L}_{31}(X, Y, T) \ll \frac{X^2 Y}{T^2} \log^3 T + \frac{X^{\frac{11}{2}} Y^{\frac{1}{2}}}{T^2} \log^3 T
\]
and
\[
(6.24) \quad \mathcal{L}_{33}(X, Y, T) \ll \frac{X^2 Y}{T^2} \log^3 T + \frac{X^{\frac{11}{2}} Y^{\frac{1}{2}}}{T^2} \log^3 T.
\]
By (2.4) of Lemma 2.2 we can write
\[
\mathcal{L}_{32}(X, Y, T) \ll X^{\frac{11}{2}} Y^{\frac{1}{2}} \log T \times \mathcal{L}_{32}^*(X, Y, T),
\]
where
\[
\mathcal{L}_{32}^*(X, Y, T) := \int_{-T_1}^{T_1} dt_1 \int_{-T_2}^{T_2} \frac{|\zeta_F\left(\frac{1}{4} - it_2\right)\zeta_F\left(\frac{1}{4} + \frac{1}{\log X} + i(t_1 - t_2)\right)|}{(|t_1| + 1)(|t_2| + 1)^2} dt_2.
\]
By Lemma 2.1 we have
\[
\mathcal{L}_{32}^*(X, Y, T) \ll \int_{-T_1}^{T_1} dt_1 \int_{-T_2}^{T_2} \frac{|\zeta_F\left(\frac{3}{4} + it_2\right)\zeta_F\left(\frac{3}{4} - \frac{1}{\log X} - i(t_1 - t_2)\right)|}{(|t_1| + 1)(|t_2| + 1)^{\frac{3}{2}}} \times (|t_1 - t_2| + 1)^{\frac{1}{2}} dt_2.
\]
Similar to (5.11) and (5.12) we have
\[
\mathcal{L}_{32}^*(X, Y, T) \ll \log^2 T.
\]
Thus
\[
(6.25) \quad \mathcal{L}_{32}(X, Y, T) \ll X^{\frac{11}{2}} Y^{\frac{1}{2}} \log^3 T.
\]
From (6.22)-(6.25) we get
\[
(6.26) \quad \mathcal{L}_3(X, Y, T) \ll X^{\frac{11}{2}} Y^{\frac{1}{2}} \log^3 T + \frac{X^2 Y}{T^2} \log^3 T.
\]
6.5. **Upper bound of \( \mathcal{L}_4(X, Y, T) \).** It is easy to see that

\[
Rc s_{w=3-s_1-s_2} G(w; s_1, s_2) = \rho \frac{\zeta_F(3-s_1-s_2)\zeta_F(2-s_2)}{\zeta_F(s_1)\zeta_F(s_2)\zeta_F(4-s_1-s_2)} \frac{X^{s_1+s_2}Y^{3-s_1-s_2}}{s_1s_2(3-s_1-s_2)} = m(s_1, s_2),
\]
say. By the residue theorem we can write

\[
(6.27) \quad \mathcal{L}_4(X, Y, T) = -\mathcal{L}_{41}(X, Y, T) + \mathcal{L}_{42}(X, Y, T) + \mathcal{L}_{43}(X, Y, T),
\]

where

\[
\mathcal{L}_{41}(X, Y, T) = \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{b_2+iT_2}^{\frac{3}{2}+iT_2} m(s_1, s_2) ds_2,
\]

\[
\mathcal{L}_{42}(X, Y, T) = \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{\frac{3}{2}-iT_2}^{\frac{3}{2}+iT_2} m(s_1, s_2) ds_2,
\]

\[
\mathcal{L}_{43}(X, Y, T) = \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} ds_1 \int_{b_2-iT_2}^{\frac{3}{2}+iT_2} m(s_1, s_2) ds_2.
\]

Suppose \( b_1 \leq \sigma_1 \leq 3/2, |t_1| \leq T \) and \( b_2 \leq \sigma_2 \leq 3/2, |t_2| \leq 2T \). By Lemma 2.2 it is easy to check that

\[
(6.28) \quad m(s_1, s_2) \ll \frac{X^{\sigma_1+\sigma_2}Y^{3-\sigma_1-\sigma_2} \log^2(|t_1|+1) \log^2(|t_2|+1) \log(|t_1+t_2|+1)}{(|t_1|+1)^{2-\sigma_1}(|t_2|+1)^{2-\sigma_2}(|t_1+t_2|+1)^{3-\sigma_1-\sigma_2}}.
\]

From (6.28) we get \((t_2 = T_2 = 2T)\)

\[
(6.29) \quad \mathcal{L}_{41}(X, Y, T) \ll \frac{X^2Y \log^5 T}{T^2} + \frac{X^{\frac{5}{2}}Y^{\frac{1}{2}} \log^5 T}{T}
\]

and \((t_2 = -T_2 = -2T)\)

\[
(6.30) \quad \mathcal{L}_{43}(X, Y, T) \ll \frac{X^2Y \log^5 T}{T^2} + \frac{X^{\frac{5}{2}}Y^{\frac{1}{2}} \log^5 T}{T}.
\]

Now we consider \( \mathcal{L}_{42}(X, Y, T) \). Change the order of \( s_1 \) and \( s_2 \) and then use the residue theorem to \( s_1 \) we get

\[
(6.31) \quad \mathcal{L}_{42}(X, Y, T) = -\mathcal{L}_{421}(X, Y, T) + \mathcal{L}_{422}(X, Y, T) + \mathcal{L}_{423}(X, Y, T),
\]

where

\[
\mathcal{L}_{421}(X, Y, T) = \frac{1}{(2\pi i)^2} \int_{\frac{3}{2}-iT_2}^{\frac{3}{2}+iT_2} ds_2 \int_{b_1+iT_1}^{\frac{3}{2}+iT_1} m(s_1, s_2) ds_1,
\]

\[
\mathcal{L}_{422}(X, Y, T) = \frac{1}{(2\pi i)^2} \int_{\frac{3}{2}-iT_2}^{\frac{3}{2}+iT_2} ds_2 \int_{\frac{3}{2}-iT_1}^{\frac{3}{2}+iT_1} m(s_1, s_2) ds_1,
\]

\[
\mathcal{L}_{423}(X, Y, T) = \frac{1}{(2\pi i)^2} \int_{\frac{3}{2}-iT_2}^{\frac{3}{2}+iT_2} ds_2 \int_{b_1-iT_1}^{\frac{3}{2}+iT_1} m(s_1, s_2) ds_1.
\]
From (6.28) we get

\[(6.32) \quad \mathcal{L}_{421}(X, Y, T) \ll \int_{-T_2}^{T_2} \frac{1}{(|t_2| + 1)^{1/2}} dt_2 \int_{b_1}^{\frac{T}{2}} \frac{X^{\sigma_1 + \frac{3}{2} Y^{\frac{3}{2} - \sigma_1} \log^5 T}}{T^{2 - \sigma_1} (|T + t_2| + 1)^{\frac{3}{2} - \sigma_1}} d\sigma_1
\]

\[\ll \frac{X^{\frac{5}{2}} Y^{\frac{1}{2}} \log^5 T}{T} \int_{-T_2}^{T_2} \frac{1}{(|t_2| + 1)^{1/2} (|T + t_2| + 1)^{1/2}} dt_2
\]

\[+ \frac{X^{3} \log^5 T}{T^{1/2}} \int_{-T_2}^{T_2} \frac{1}{(|t_2| + 1)^{1/2}} dt_2
\]

\[\ll \frac{X^{\frac{5}{2}} Y^{\frac{1}{2}} \log^5 T}{T} + X^{3} \log^5 T
\]

by noting that

\[\int_{-T_2}^{T_2} \frac{1}{(|t_2| + 1)^{1/2} (|T + t_2| + 1)^{1/2}} dt_2 \ll 1.
\]

Similarly we have

\[(6.33) \quad \mathcal{L}_{423}(X, Y, T) \ll \frac{X^{\frac{5}{2}} Y^{\frac{1}{2}} \log^5 T}{T} + X^{3} \log^5 T.
\]

Finally we consider \(\mathcal{L}_{422}(X, Y, T)\). Suppose \(s_1 = 3/2 + it_1, s_2 = 3/2 + it_2\). By Lemma 2.2 we have

\[m(s_1, s_2) \ll X^{3} \log^{3} T \times \frac{|\zeta_{\mathbb{F}}(\frac{1}{2} + it_1)\zeta_{\mathbb{F}}(\frac{1}{2} + it_2)|}{(|t_1| + 1)(|t_2| + 1)},
\]

which combining with (5.10) implies that

\[(6.34) \quad \mathcal{L}_{422}(X, Y, T) \ll X^{3} \log^{9} T.
\]

From (6.27) and (6.29)-(6.34) we have

\[(6.35) \quad \mathcal{L}_{4}(X, Y, T) \ll X^{3} \log^{9} T + \frac{X^{\frac{5}{2}} Y^{\frac{1}{2}} \log^5 T}{T} + \frac{X^{2} Y \log^5 T}{T^2}.
\]

6.6. Proof of Theorem 2: completion.
Choose $T = Y^2$. From (6.1)-(6.5), (6.9), (6.15), (6.21), (6.26), (6.35) we get

$$C_{p,2}(X, Y) = \frac{\rho_p^2}{2 \zeta_p(2)} X^2 Y + \frac{\zeta_{2p}(0) \rho_p^2}{4 \zeta_p^2(2)} X^4 + O \left( \frac{X^2 Y^{1+\varepsilon} + X^{\frac{5}{2}} Y Y \log^7 T}{T} \right)$$

$$+ O \left( X^3 \log^9 T + X^{\frac{11}{4}} Y^{\frac{1}{4}} \log^3 T + X^{\frac{5}{2}} Y \log^3 T \right)$$

$$+ O \left( X^{\frac{24}{5}} Y^{\frac{3}{5}} + X^2 Y^{\frac{3}{4}} \log^5 T \right)$$

by noting that $Y \geq X^2$. This completes the proof of Theorem 2.

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THE AVERAGE SIZE OF RAMANUJAN SUMS OVER QUADRATIC NUMBER FIELDS (II)

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