A canonical form for nonderogatory matrices under unitary similarity

Vyacheslav Futorny† Roger A. Horn‡ Vladimir V. Sergeichuk§

Abstract

A square matrix is nonderogatory if its Jordan blocks have distinct eigenvalues. We give canonical forms for

• nonderogatory complex matrices up to unitary similarity, and
• pairs of complex matrices up to similarity, in which one matrix has distinct eigenvalues.

The types of these canonical forms are given by undirected and, respectively, directed graphs with no undirected cycles.

AMS classification: 15A21

Keywords: Belitskii’s algorithm; Littlewood’s algorithm; Unitary similarity; Classification; Canonical matrices

1 Introduction

A square matrix is nonderogatory if its Jordan blocks have distinct eigenvalues; that is, if its characteristic and minimal polynomials coincide.
We give canonical forms for

• nonderogatory matrices up to unitary similarity, and

• pairs of matrices up to similarity, in which one matrix has distinct
eigenvalues.

All matrices that we consider are complex matrices.

Our canonical matrices are special cases of the canonical matrices that
were algorithmically constructed by Littlewood and Belitskii:

• Littlewood [10] developed an algorithm that reduces each square matrix
  $M$ by unitary similarity transformations

  $$M \mapsto U^{-1}MU, \quad U \text{ is a unitary matrix,}$$

  to a matrix $M_{\text{can}}$ in such a way that $M$ and $N$ are unitarily similar if
  and only if they are reduced to the same matrix $M_{\text{can}} = N_{\text{can}}$. Thus,
  the matrices that are not changed by Littlewood’s algorithm are canonical with respect to unitary similarity. Other versions of Littlewood’s
  algorithm were given in [4] and [12, 14].

• Belitskii [1, 2] developed an algorithm that reduces each pair of $n \times n$
  matrices $(M, N)$ by similarity transformations

  $$\begin{align*}
  (M, N) &\mapsto (S^{-1}MS, S^{-1}NS), \\
  S &\text{ is nonsingular,}
  \end{align*}$$

  to a matrix pair $(M, N)_{\text{can}}$ in such a way that $(M, N)$ and $(M', N')$
  are similar if and only if they are reduced to the same matrix pair
  $(M, N)_{\text{can}} = (M', N')_{\text{can}}$. Thus, the matrix pairs that are not changed
  by Belitskii’s algorithm are canonical with respect to similarity. Belitskii’s algorithm was extended in [15] to the problem of classifying
  arbitrary systems of linear mappings and the problem of classifying
  representations of finite dimensional algebras.

Lists of Littlewood’s canonical 5 × 5 matrices and Belitskii’s canonical
pairs for 4 × 4 matrices are in [8] and [5]. Without restrictions on the size
of matrices, we cannot expect to have explicit descriptions of Littlewood’s
canonical matrices and Belitskii’s canonical matrix pairs since

• The problem of classifying matrices up to unitary similarity contains
  the problem of classifying arbitrary systems of linear mappings on unitary spaces [9, 14]; and
The problem of classifying matrix pairs up to similarity contains the problem of classifying arbitrary systems of linear mappings on vector spaces \([6, 3]\).

When it is applied to nonderogatory matrices, Littlewood’s algorithm can be greatly simplified. Mitchell \([11]\) presented an algorithm intended to reduce nonderogatory matrices to canonical form, but his algorithm is incorrect\(^1\). In Sections 2–5 we give a version of Littlewood’s algorithm for nonderogatory matrices and describe a set of canonical nonderogatory matrices for unitary similarity. Each type of canonical nonderogatory matrices with \(t\) distinct eigenvalues is given by an undirected graph with \(t\) vertices and no cycles.

When it is applied to pairs of \(n \times n\) matrices in which one matrix has distinct eigenvalues, Belitskii’s algorithm can also be greatly simplified. In Section 6 we describe a set of canonical forms for pairs of matrices under similarity. It is analogous to the set of canonical nonderogatory matrices in Section 4, but it involves directed graphs instead of undirected graphs. This description was used in \([5]\) to classify pairs of \(4 \times 4\) matrices up to similarity.

2 Schur’s triangular form for nonderogatory matrices

Schur’s unitary triangularization theorem \([7,\) Theorem 2.3.1\)] ensures that each square matrix \(M\) is unitarily similar to an upper triangular matrix

\[
A = \begin{bmatrix}
\lambda_1 & a_{12} & \cdots & a_{1n} \\
0 & \lambda_2 & \cdots & \vdots \\
& \ddots & \ddots & \vdots \\
& & \ddots & a_{n-1,n} \\
& & & \lambda_n
\end{bmatrix}, \quad \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n, \quad (2)
\]

\(^1\)The following reasoning on page 71 of \([11]\) is incorrect: “Let us agree to go from left to right down the successive diagonals below the main diagonal and pick out each non-zero element as we come to it until we obtain either a total of \(n - 1\) non-zero elements or all non-zero elements off the main diagonal, where \(n\) is the order of the matrix. These chosen non-zero elements can then be made positive by transforming by a diagonal unitary matrix.” Unfortunately, it is impossible to make positive in this way “each non-zero element as we come to it”. In Section 3 we choose a set of nonzero elements that can be made positive.
whose diagonal entries are complex numbers in any prescribed order; for definiteness, we use the lexicographic order:

\[ a + bi \preceq c + di \quad \text{if either } a < c, \text{ or } a = c \text{ and } b \preceq d. \]  

(3)

A unitary matrix \( U \) that transforms \( M \) to an upper triangular matrix \( A = U^{-1}MU \) of the form (2) can be constructed as follows: first find a nonsingular matrix \( S \) such that \( J = S^{-1}MS \) is the Jordan form of \( M \) that has diagonal entries in the prescribed order, then apply the Gram-Schmidt orthogonalization to the columns of \( S \) and obtain a unitary matrix \( U = ST \), in which \( T \) is upper triangular. Alternatively, a unitary \( U \) with the desired property can be constructed directly, without first obtaining the Jordan form [7, Theorem 2.3.1].

The unitary similarity class of \( M \) can contain more than one upper triangular matrix \( A \) of the form (2). For example, the argument of any nonzero entry in the first superdiagonal may be chosen arbitrarily. The following diagonal unitary similarity permits us to standardize the choice of these arguments by replacing every nonzero entry \( a_{i,i+1} \) in the first superdiagonal by the nonnegative real number \( r_i := |a_{i,i+1}| \):

\[ A \mapsto UAU^{-1}, \quad U := \text{diag}(1, u_1, u_1u_2, u_1u_2u_3, \ldots), \]

in which \( u_i := a_{i,i+1}/r_i \) if \( a_{i,i+1} \neq 0 \) and \( u_i := 1 \) if \( a_{i,i+1} = 0 \). This unitary similarity is used in the following example.

**Example 2.1.** Every square matrix \( M \) that is unitarily similar to a matrix of the form (2), in which all entries of the first superdiagonal of \( A \) are nonzero, is unitarily similar to a matrix of the form

\[
B = \begin{bmatrix}
\lambda_1 & b_{12} & \ldots & b_{1n} \\
& \lambda_2 & \ddots & \vdots \\
& & \ddots & b_{n-1,n} \\
& & & \lambda_n
\end{bmatrix}, \quad \text{all } b_{i,i+1} > 0, \quad \lambda_1 \preceq \lambda_2 \preceq \cdots \preceq \lambda_n.
\]

(4)

Such a matrix can be used as a canonical form for \( M \) under unitary similarity since if two matrices of the form (4) are unitarily similar, then they are identical. This canonical form is a special case of a canonical form for nonderogatory matrices that we construct in Section 4.
The number of Jordan blocks with eigenvalue \( \lambda \) in the Jordan form of an \( n \times n \) matrix \( A \) is equal to \( n - \text{rank}(A - \lambda I_n) \). Thus, a matrix of the form \( (2) \) is nonderogatory if and only if \( \lambda_i = \lambda_{i+1} \) implies that \( a_{i,i+1} \neq 0 \). We formalize this observation in the following lemma.

**Lemma 2.1.** A matrix is nonderogatory if and only if it is unitarily similar to a block matrix of the form

\[
A = \begin{bmatrix}
\Lambda_1 & A_{12} & \cdots & A_{1t} \\
A_{21} & \Lambda_2 & \cdots & \vdots \\
& \ddots & \ddots & \ddots \\
0 & \cdots & A_{t-1,t} & \Lambda_t
\end{bmatrix},
\]

in which each diagonal block \( \Lambda_i \) is \( m_i \times m_i \) and has the form

\[
\Lambda_i = \begin{bmatrix}
\lambda_i & * & \cdots & * \\
\lambda_i & \ddots & \ddots & \\
& \ddots & \ddots & *
\end{bmatrix}, \quad \text{all entries of the first superdiagonal of } \Lambda_i \text{ are positive real numbers,}
\]

and the diagonal entries are lexicographically ordered: \( \lambda_1 < \lambda_2 < \cdots < \lambda_t \).

### 3 An algorithm for nonderogatory matrices

Let \( M \) be a nonderogatory matrix. We first reduce it by unitary similarity transformations to a matrix \( A \) of the form described in Lemma 2.1. Then we reduce \( A \) by transformations \( A \mapsto A' := U^{-1}AU \) (\( U \) is unitary) that preserve this form.

We prove in Lemma 5.1 that \( A' \) has the form described in Lemma 2.1 if and only if

\[
U = u_1I_{m_1} \oplus \cdots \oplus u_tI_{m_t}, \quad |u_1| = \cdots = |u_t| = 1.
\]

Thus, we reduce \( A \) to canonical form by transformations

\[
A \mapsto \begin{bmatrix}
\Lambda_1 & u_1^{-1}u_2A_{12} & \cdots & u_1^{-1}u_tA_{1t} \\
A_{21} & \Lambda_2 & \cdots & \vdots \\
& \ddots & \ddots & \ddots \\
0 & \cdots & u_{t-1}^{-1}u_tA_{t-1,t} & \Lambda_t
\end{bmatrix}.
\]
Notice that the blocks $A_{ij}$ are multiplied by complex numbers of modulus 1.

We construct a set of canonical nonderogatory matrices that includes the canonical matrices from Example 2.1. For this purpose, if $A_{12} \neq 0$, then we reduce it to the following form.

**Lemma 3.1.** Let $C = [c_{ij}]$ be a nonzero $p \times q$ matrix. Let $c$ be the first nonzero entry in the sequence formed by the diagonals of $C$ starting from the lower left:

$$
c_{p1}; c_{p-1,1}; c_{p-2,1}; c_{p-1,2}; c_{p3}; \ldots; c_{1q}.
$$

We can replace $c$ by the positive real number $r = |c|$ by multiplying $C$ by a complex number of modulus 1. The resulting matrix is canonical with respect to multiplication by complex numbers of modulus 1.

For example, if the first nonzero diagonal (starting from the lower left) of $C$ is below the main diagonal, then its canonical matrix from Lemma 3.1 has the form

$$
\begin{bmatrix}
* \\
0 \\
0 * \\
0 \cdots 0 * \\
0 \cdots \cdots 0 * \\
0 \cdots \cdots \cdots \cdots 0 \\
\end{bmatrix}
\quad r \in \mathbb{R}, \ r > 0,
$$

*’s are complex numbers.

We sequentially reduce the blocks $A_{ij}$ of the matrix (5) to canonical form in the following order (i.e., arranging them along the block superdiagonals of $A$):

$$
A_{12}, A_{23}, \ldots, A_{t-1,t}; A_{13}, A_{23}, \ldots, A_{t-2,t}; \ldots; A_{1t}.
$$

We begin with the block $A_{12}$. If $A_{12} = 0$, then it is not changed by transformations of the form (7), and so it is already canonical. If $A_{12} \neq 0$, then we reduce it as in Lemma 3.1 to preserve the block $A_{12}$ obtained, we must impose the condition $u_1 = u_2$ on the transformations (7).

Then we reduce $A_{23}$ in the same way and so on, until all blocks in the first superdiagonal have been reduced. We obtain a matrix $A$ in which all nonzero blocks in the first superdiagonal have the form described in Lemma 3.1.
This matrix is uniquely determined by the unitary similarity class of $A$, up to transformations of the form (7) that satisfy the conditions $u_i = u_{i+1}$ if $A_{i,i+1} \neq 0$; we say that such transformations are admissible. It is convenient to describe these conditions by a graph $G^{(1)}$ with vertices $1, \ldots, t$ and with edges $i \rightarrow (i + 1)$ that correspond to all $A_{i,i+1} \neq 0$.

Next we reduce the blocks of the second superdiagonal to canonical form. If $A_{13} = 0$ or if $u_1 = u_2 = u_3$ (i.e., if $G^{(1)}$ contains the path $1 \rightarrow 2 \rightarrow 3$), then $A_{13}$ is not changed by admissible transformations of the form (7); it is already canonical. If $A_{13} \neq 0$ and $G^{(1)}$ does not contain the path $1 \rightarrow 2 \rightarrow 3$, then we reduce $A_{13}$ as in Lemma 3.1 and add the edge $1 \rightarrow 3$ to the graph. Then we reduce $A_{24}$ and so on until we have reduced all blocks in the sequence (8).

This algorithm can be formalized as follows. For each graph $G$ with vertices $1, \ldots, t$, we say that (7) is a $G$-transformation if $u_i = u_j$ for all edges $i \rightarrow j$ in $G$.

**Algorithm 3.1.** Let $M$ be a nonderogatory matrix, let $A$ be its upper triangular form (5) for unitary similarity described in Lemma 2.1, and let $G_0$ be the graph with vertices $1, \ldots, t$ and without edges.

*The first step:* We construct a pair $(A_1, G_1)$ as follows. Let $A_{p_1q_1}$ be the first nonzero block of $A$ in the sequence (8). Reduce $A_{p_1q_1}$ as in Lemma 3.1 by transformations of the form (7) and denote the resulting matrix by $A_1$. Add the edge $p_1 \rightarrow q_1$ to $G_0$ and denote the resulting graph by $G_1$.

*The $\alpha$th step ($\alpha \geq 2$):* Using the pair $(A_{\alpha-1}, G_{\alpha-1})$ constructed at the $(\alpha - 1)$st step, we construct $(A_\alpha, G_\alpha)$. Let $A_{p_\alpha q_\alpha}$ be the first block of $A_{\alpha-1}$ that is to the right of $A_{p_{\alpha-1}q_{\alpha-1}}$ in (8) and is changed by $G_{\alpha-1}$-transformations (this means that $A_{p_\alpha q_\alpha} \neq 0$ and $G_{\alpha-1}$ does not contain a path from $p_\alpha$ to $q_\alpha$). We reduce $A_{p_\alpha q_\alpha}$ as in Lemma 3.1 and denote the resulting matrix by $A_\alpha$. Add the edge $p_\alpha \rightarrow q_\alpha$ to $G_{\alpha-1}$ and denote the resulting graph by $G_\alpha$.

*The result:* The process stops at a pair $(A_r, G_r)$ such that all blocks of $A_r$ to the right of $A_{p_rq_r}$ in (8) are not changed by $G_r$-transformations. The number $r$ of steps is less than $t$ since the graph $G_r$ has $t$ vertices, $r$ edges, and no cycles. Write $M_{\text{can}} := A_r$ and $G := G_r$.

In the proof of Theorem 4.1 we show that the pair $(M_{\text{can}}, G)$ is uniquely determined by the unitary similarity class of $M$; that is, $M_{\text{can}}$ is a canonical form for $M$ with respect to unitary similarity.
4 Canonical nonderogatory matrices and the classification theorem

Algorithm 3.1 constructs a pair \((M_\text{can}, G)\) for each nonderogatory matrix \(M\). The structure of \(M_\text{can}\) is determined by the graph \(G\) as follows:

- The blocks
  \[ A_{p_1q_1}, A_{p_2q_2}, \ldots, A_{p_rq_r} \]  
  of \(M_\text{can}\) have the form described in Lemma 3.1; they correspond to the edges of \(G\).

- Let \(A_{ij}\) \((i < j)\) be a block of \(M_\text{can}\) that is not a member of the list in (9). Let \(A_{p_\alpha q_\alpha}\) be the nearest block in the list (9) that is to the left of \(A_{ij}\) in (8). If there is no such block (i.e., if \(A_{ij}\) is to the left of \(A_{p_1q_1}\)), we put \(\alpha := 0\). Then
  
  (i) \(A_{ij} = 0\) if \(G_\alpha\) does not contain a path from \(i\) to \(j\), and
  
  (ii) \(A_{ij}\) is arbitrary if \(G_\alpha\) contains a path from \(i\) to \(j\).

The graph \(G_\alpha\) in (i) and (ii) can be obtained from \(G\) by removing the edges \(u - v\) that correspond to those \(A_{uv}\) in the list (9) that are reduced after \(A_{p_\alpha q_\alpha}\) if \(\alpha \neq 0\), and by removing all the edges of \(G\) if \(\alpha = 0\). Thus, \(A_{uv}\) is to the right of \(A_{ij}\) in (8); i.e., either \(v - u > j - i\), or \(v - u = j - i\) and \(u > i\). Hence, \(M_\text{can}\) is a \(G\)-canonical matrix in the sense of the following definition.

**Definition 4.1.** Let \(G\) be an undirected graph with vertices \(1, 2, \ldots, t\) and no cycles. By a \(G\)-canonical matrix, we mean a block matrix of the form (5) in which every diagonal block has the form (6), \(\lambda_1 < \lambda_2 < \cdots < \lambda_t\), and each block \(A_{ij}\) \((i < j)\) satisfies the following conditions:

- \(A_{ij}\) has the form described in Lemma 3.1 if \(G\) contains the edge \(i - j\);

- \(A_{ij} = 0\) if either \(G\) contains no path from \(i\) to \(j\), or the path from \(i\) to \(j\) (which is unique since \(G\) without cycles) contains an edge \(u - v\) \((u < v)\) such that
  
  - either \(v - u > j - i\),
  
  or \(v - u = j - i\) and \(u > i\);
• $A_{ij}$ is arbitrary, otherwise.

**Example 4.1.** (a) Each matrix of the form \[ \begin{bmatrix} 1 & 2 & \cdots & t \end{bmatrix} \] is $G$-canonical with

\[ G : \quad 1 \rightarrow 2 \rightarrow \cdots \rightarrow t \]

(b) Each $G$-canonical matrix with

\[ G : \quad 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1 \]

has the form

\[
\begin{bmatrix}
\Lambda_1 & 0 & 0 & C_4 & \ast \\
\Lambda_2 & C_1 & \ast & \ast & \\
\Lambda_3 & C_2 & C_3 & \\
0 & \Lambda_4 & 0 & \\
& & & \Lambda_5
\end{bmatrix}
\]

in which

- each block $\Lambda_i$ has the form \[ \begin{bmatrix} \lambda & 0 & \cdots & 0 \end{bmatrix} \] and $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5$,
- each block $C_i$ has the form described in Lemma 3.1,
- the stars denote arbitrary blocks.

A $G$-canonical matrix is a **canonical nonderogatory block** if $G$ is a tree. It follows from the uniqueness in (b) of the next theorem that canonical nonderogatory blocks are indecomposable under unitary similarity, i.e., they are not unitarily similar to a direct sum of square matrices of smaller sizes. Their role is analogous to the role of Jordan blocks in the Jordan canonical form.

**Theorem 4.1.** (a) For each nonderogatory matrix $M$, there is a unique undirected graph $G$ and a unique $G$-canonical matrix $M_{\text{can}}$ such that $M$ is unitarily similar to $M_{\text{can}}$. Thus, $M$ is unitarily similar to $N$ if and only if $M_{\text{can}} = N_{\text{can}}$.

(b) Each nonderogatory matrix $M$ is unitarily similar to a direct sum of canonical nonderogatory blocks. This direct sum is uniquely determined by $M$, up to permutation of summands.
Remark 4.1. The direct sum in Theorem 4.1(b) is permutationally similar to $M_{\text{can}}$ and can be obtained from it as follows: The graph $G$ is a disjoint union of trees; denote them by $G_1, \ldots, G_s$. Let $u_{i_1} < u_{i_2} < \cdots < u_{i_{t_i}}$ be the vertices of $G_i$. Let $A_i$ be the $t_i \times t_i$ submatrix of $M_{\text{can}}$ formed by rows $u_{i_1}, \ldots, u_{i_{t_i}}$ and columns $u_{i_1}, \ldots, u_{i_{t_i}}$. Definition 4.1 ensures that the $u_{i_1},u_{j_k}$ block of $M_{\text{can}}$ is zero if $i \neq j$. Therefore, $M_{\text{can}}$ is permutationally similar to

$$A_1 \oplus A_2 \oplus \cdots \oplus A_s,$$

which is the desired direct sum. Each $A_i$ is a $G_i'$-canonical matrix, in which $G_i'$ is the tree obtained from $G_i$ by relabeling the vertices $u_{i_1}, \ldots, u_{i_{t_i}}$ with $1, \ldots, t_i$.

5 Proof of Theorem 4.1

Our proof is based on the following lemma about unitary similarity of matrices of the form described in Lemma 2.1. This lemma was proved in greater generality in [10] and in [11]; we offer a proof for the reader’s convenience.

Lemma 5.1. Let

$$A = \begin{bmatrix} A_1 & A_{12} & \cdots & A_{1t} \\ \Lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots \\ 0 & \cdots & \Lambda_t \end{bmatrix}, \quad B = \begin{bmatrix} \Lambda'_1 & B_{12} & \cdots & B_{1t'} \\ \Lambda'_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots \\ 0 & \cdots & \Lambda'_{t'} \end{bmatrix}$$

be nonderogatory matrices of the form described in Lemma 2.1. Assume that they are unitarily similar: $U^{-1}AU = B$ with unitary $U$. Then $t = t'$, 

$$\Lambda_1 = \Lambda'_1, \ldots, \Lambda_t = \Lambda'_t, \quad \text{(10)}$$

and $U$ has the form

$$U = u_1I_{m_1} \oplus \cdots \oplus u_tI_{m_t}, \quad \text{(11)}$$

in which $u_1, \ldots, u_t$ are complex numbers of modulus 1 and the size of $\Lambda_i$ is $m_i \times m_i$ for each $i$.

Proof. The matrices $A$ and $B$ have the same main diagonal since they are similar and the entries along their main diagonals are lexicographically ordered. This means that $t = t'$ and for each $i$ the diagonal blocks $\Lambda_i$ and $\Lambda'_i$
are \( m_i \times m_i \) matrices of the form (6) with the same \( \lambda_i \). The proof is divided into three steps.

Step 1: Prove that \( U \) has the form

\[
U = U_1 \oplus U_2 \oplus \cdots \oplus U_t
\]

(12)

in which every block \( U_i \) is \( m_i \times m_i \). If \( t = 1 \) there is nothing to prove, so assume that \( t \geq 2 \). Partition \( U \) into blocks \( U_{ij} \) of size \( m_i \times m_j \). Our strategy is to exploit the equality of corresponding blocks of both sides of the identity \( AU = UB \).

The \( t,1 \) block of \( AU \) is \( \Lambda_t U_{11} \), and the \( t,1 \) block of \( UB \) is \( U_{11} \Lambda'_1 \). Since \( \lambda_t \neq \lambda_1 \), \( U_{11} = 0 \) is the only solution to \( \Lambda_t U_{11} = U_{11} \Lambda'_1 \).

If \( t > 2 \), then the \( t,2 \) block of \( AU \) is \( \Lambda_t U_{12} \), and the \( t,2 \) block of \( UB \) is \( U_{12} \Lambda'_2 \) (since \( U_{11} = 0 \)); we have \( \Lambda_t U_{12} = U_{12} \Lambda'_2 \). Since \( \lambda_t \neq \lambda_2 \), we have \( U_{12} = 0 \). Proceeding in this way across the last block row of \( AU = UB \), we find that \( U_{11}, U_{12}, \ldots, U_{t-1,t-1} \) are all zero.

Now equate the blocks of \( AU = UB \) in positions \( (t-1),k \) for \( k = 1,2,\ldots,t-2 \) and conclude in the same way that \( U_{t-1,1}, U_{t-1,2}, \ldots, U_{t-1,t-2} \) are all zero. Working our way up the block rows of \( AU = UB \), left to right, we conclude that \( U_{ij} = 0 \) for all \( i > j \). Since \( U^{-1} = U^* \), it follows that \( U_{ij} = 0 \) for all \( j > i \) and hence

\[
U = U_{11} \oplus U_{22} \oplus \cdots \oplus U_{tt}.
\]

This proves (12) with \( U_i := U_{ii} \).

Step 2: Prove that \( U \) is diagonal. Since \( AU = UB \), we have \( t \) identities \( \Lambda_i U_i = U_{i} \Lambda'_i \), \( i = 1,\ldots,t \), and all the entries in the first superdiagonal of each \( \Lambda_i \) and \( \Lambda'_i \) are positive real numbers. Thus, it suffices to consider the case \( t = 1 \). In this case

\[
A = \begin{bmatrix}
\lambda & a_{12} & \cdots & a_{1n} \\
0 & \lambda & \ddots & \vdots \\
& \ddots & \ddots & a_{n-1,n} \\
0 & & \cdots & \lambda
\end{bmatrix}, \quad B = \begin{bmatrix}
\lambda & b_{12} & \cdots & b_{1n} \\
0 & \lambda & \ddots & \vdots \\
& \ddots & \ddots & b_{n-1,n} \\
0 & & \cdots & \lambda
\end{bmatrix},
\]

\( a_{i,i+1} \) and \( b_{i,i+1} \) are positive real numbers for all \( i = 1,\ldots,n-1 \), and \( AU = UB \). As in Step 1, we equate corresponding entries of the identity

\[
(A - \lambda I_n)U = U(B - \lambda I_n).
\]

(13)
In position \(n,1\) we have \(0 = 0\). In position \(n,2\) we have \(0 = u_{n1}b_{12}\); since \(b_{12} \neq 0\) it follows that \(u_{n1} = 0\). Proceeding across the last row of (13), we obtain
\[
u_{n1} = u_{n2} = \cdots = u_{n,n-1} = 0.\]

Working our way up the rows of (13) in this fashion, left to right, we find that \(u_{ij} = 0\) for all \(i > j\). Thus, \(U\) is upper triangular. Since \(U\) is unitary, it is diagonal: \(U = \text{diag}(u_1, \ldots, u_n)\).

Step 3: Prove that \(U = \text{diag}(u_1, \ldots, u_n)\) has the form (11). We continue to assume that \(t = 1\). Equating the entries of \(AU = UB\) in position \(i,i\), we have \(a_{i,i+1}u_{i+1} = u_i b_{i,i+1}\), so \(a_{i,i+1}/b_{i,i+1} = u_i/u_{i+1}\), which is positive real and has modulus one. We conclude that \(u_i/u_{i+1} = 1\) for each \(i = 1, \ldots, n-1\), and hence \(u_1 = \cdots = u_n\). This proves (11), which implies (10).

Proof of Theorem 4.1. (a) Let \(M\) be a nonderogatory matrix. Algorithm 3.1 constructs the graph \(G\) and the matrix \(M_{\text{can}}\), which is unitarily similar to \(M\). As shown at the beginning of Section 4, \(M_{\text{can}}\) is a \(G\)-canonical matrix.

Let \(M\) and \(N\) be nonderogatory matrices that are unitarily similar. Our goal is to prove that Algorithm 3.1 reduces them to the same matrix \(M_{\text{can}} = N_{\text{can}}\). Following the algorithm, we first reduce \(M\) and \(N\) to matrices \(A\) and \(B\) of the form described in Lemma 2.1. They are unitarily similar; that is, \(U^{-1}AU = UB\) for a unitary matrix \(U\). Lemma 5.1 ensures that \(t = t', \Lambda_i = \Lambda_i'\) for all \(i\), and \(U\) has the form (11). This means that \(B\) is obtained from \(A\) by a transformation of the form (7):
\[
B_{ij} = u_i^{-1}u_j A_{ij}, \quad |u_i| = 1, \quad i, j = 1, \ldots, t. \tag{14}
\]

We arrange the superdiagonal blocks \(A_{ij}\) in \(A\) and \(B_{ij}\) in \(B\) along the block superdiagonals, as in (8). By (14), the first nonzero superdiagonal block of \(A\) and the first nonzero superdiagonal block of \(B\) occur in the same position \(p_1,q_1\). In Step 1 of Algorithm 3.1 we reduce them to the same form described in Lemma 3.1 and obtain the matrices \(A_1\) and \(B_1\), in which the \(p_1,q_1\) blocks are equal.

In Step \(\alpha\), we reduce the first superdiagonal block of \(A_{\alpha-1}\) that is changed by \(G_{\alpha-1}\)-transformations, and the first superdiagonal block of \(B_{\alpha-1}\) that is changed by \(G_{\alpha-1}\)-transformations. They occur in the same position \(p_\alpha,q_\alpha\) and are reduced to the same form described in Lemma 4.1. We obtain the matrices \(A_\alpha\) and \(B_\alpha\), in which the blocks in position \(p_\alpha,q_\alpha\) coincide; the superdiagonal blocks that precede them coincide as well. The matrix \(B_\alpha\) can be obtained from \(A_\alpha\) by a \(G_\alpha\)-transformation, which preserves these blocks.
The process stops at a matrix $A_r$ such that none of its blocks are changed by $G_r$-transformations. Then $A_r = B_r$ and so $M_{\text{can}} = N_{\text{can}}$.  
(b) This statement follows from Remark 4.1. 

6 Canonical matrix pairs for similarity

Let $(M, N)$ be a pair of $n \times n$ matrices, and let $M$ have $n$ distinct eigenvalues. In this section, we give a canonical form for $(M, N)$ with respect to the similarity transformations (1).

The pair $(M, N)$ is similar to some pair $(A, B)$ such that

$$
(A, B) = \begin{pmatrix}
\lambda_1 & 0 \\
\vdots & \ddots \\
0 & \lambda_n
\end{pmatrix},
\begin{pmatrix}
b_{11} & \ldots & b_{1n} \\
\vdots & \ddots & \vdots \\
b_{n1} & \ldots & b_{nn}
\end{pmatrix}, \quad \lambda_1 < \cdots < \lambda_n, \quad (15)
$$

in which $<$ is the strict lexicographic order on $\mathbb{C}$; see (3).

Let

$$(A', B') = \begin{pmatrix}
\lambda'_1 & 0 \\
\vdots & \ddots \\
0 & \lambda'_n
\end{pmatrix},
\begin{pmatrix}
b'_{11} & \ldots & b'_{1n} \\
\vdots & \ddots & \vdots \\
b'_{n1} & \ldots & b'_{nn}
\end{pmatrix}, \quad \lambda_1' < \cdots < \lambda_n', \quad (16)
$$

be another pair of this form, and let it be similar to $(A, B)$; that is, $(S^{-1}AS, S^{-1}BS) = (A', B')$ for some nonsingular $S$. Then $A = A'$, $AS = SA$, and so $S = \text{diag}(s_1, \ldots, s_n)$ in which $s_1, \ldots, s_n \in \mathbb{C}$. Thus, the pair (15) is uniquely determined by $(M, N)$, up to transformations

$$B \mapsto B' = \begin{pmatrix}
b_{11} & s_1^{-1}s_2b_{12} & \ldots & s_1^{-1}s_nb_{1n} \\
\frac{s_2^{-1}s_1b_{21}}{s_2} & b_{22} & \ldots & s_2^{-1}s_nb_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
s_1^{-1}s_nb_{n1} & s_2^{-1}s_nb_{n2} & \ldots & b_{nn}
\end{pmatrix}, \quad (16)$$

in which $s_1, \ldots, s_n$ are arbitrary nonzero complex numbers.

Example 6.1. Suppose that a pair $(M, N)$ of $n \times n$ matrices is similar to a pair of the form (15) in which $b_{12}, b_{13}, \ldots, b_{1n}$ are all nonzero. Taking $s_1 = 1,
s_2 = b_{12}^{-1}, \ldots, s_n = b_{1n}^{-1} in (16), we reduce (M, N) to the form

\[
\begin{pmatrix}
\lambda_1 & 0 \\
\lambda_2 & \ddots \\
0 & \ddots & \lambda_n
\end{pmatrix}
\begin{pmatrix}
\ast & 1 & \ldots & 1 \\
\ast & \ast & \ldots & \ast \\
\vdots & \vdots & \ddots & \vdots \\
\ast & \ast & \ldots & \ast
\end{pmatrix}, \quad \lambda_1 < \cdots < \lambda_n, \quad (17)
\]

in which the stars denote arbitrary complex numbers. We can use (17) as a canonical form for (M, N) for similarity since if B and B' in (16) have 1 in positions 1, k, k = 2, 3, \ldots, n, then s_1 = \cdots = s_n, and so B = B'. Thus, if pairs of the form (17) are similar, then they are equal.

In the general case, we reduce B by transformations of the form (16) using the following algorithm. We arrange the entries of B along the rows starting from the first; that is, b_{ij} precedes b_{pq} if (i, j) < (p, q) with respect to the lexicographic order. For each directed graph G with vertices 1, \ldots, n, we say that (16) is a G-transformation if s_i = s_j for all directed edges i \to j in G.

Algorithm 6.1. Let B = [b_{ij}] be an n \times n matrix. Denote by G_0 the graph with vertices 1, \ldots, n and without edges.

The first step. The entry b_{11} is not changed by transformations of the form (16); we mark it as reduced and write (B_1, G_1) := (B, G_0).

The second step. If b_{12} = 0 then it is not changed by $G_1$-transformations, we mark $b_{12}$ as reduced and write $(B_2, G_2) := (B_1, G_1)$. If $b_{12} \neq 0$ then we make $b_{12} = 1$ by $G_1$-transformations, add the directed edge 1 \to 2 to $G_1$, and denote by $B_2$ and $G_2$ the resulting matrix and directed graph.

The kth step. Let $b_{pq}$ be the kth entry; that is, (p - 1)n + q = k. If p = q, or $b_{pq} = 0$, or $G_{k-1}$ has an undirected path from p to q, then $b_{pq}$ is not changed by $G_{k-1}$-transformations; we mark $b_{pq}$ as reduced and write $(B_k, G_k) := (B_{k-1}, G_{k-1})$. Otherwise, we make $b_{pq} = 1$ by $G_{k-1}$-transformations, add the directed edge p \to q to $G_{k-1}$, and denote by $B_k$ and $G_k$ the resulting matrix and directed graph.

The result. After $n^2$ steps, we obtain a matrix $B_{n^2}$, in which all entries have been marked as reduced. Write $(B_{\text{can}}, G) := (B_{n^2}, G_{n^2})$. 

14
Let us show that $B_{\text{can}}$ is a canonical form for $B$ with respect to transformations of the form (16); that is, if $B$ and $C$ are $n \times n$ matrices such that $B$ can be reduced to $C$ by transformations of the form (16) then $B_{\text{can}} = C_{\text{can}}$. Indeed, after $k$ steps of Algorithm 6.1 applied to $B$ and $C$, we obtain the matrices $B_k$ and $C_k$ and the same directed graph $G_k$. One can prove by induction on $k$ that $B_k$ reduces to $C_k$ by $G_k$-transformations, and so the first $k$ entries of $B_k$ and $C_k$ coincide. Taking $k = n^2$, we obtain $B_{\text{can}} = C_{\text{can}}$.

Let $(M, N)$ be a pair of $n \times n$ matrices, and let $M$ have $n$ distinct eigenvalues. Then $(M, N)$ is similar to a pair $(A, B)$ of the form (15), which is uniquely determined by $(M, N)$, up to transformations of the form (16). Taking $B = B_{\text{can}}$, we obtain the pair $(M, N)_{\text{can}} := (A, B_{\text{can}})$, which is similar to $(M, N)$ and is uniquely determined by $(M, N)$. Thus,

$$(M, N)_{\text{can}} \text{ is a canonical form for } (M, N) \text{ for similarity.} \quad (18)$$

In the $k$th step of Algorithm 6.1, we reduce the $k$th entry $b_{pq}$ and construct the directed graph $G_k$. The graph $G_k$ can be also obtained from $G = G_{n^2}$ by removing the directed edges $i \to j$ that correspond to those entries $b_{ij}$ that were reduced to 1 after $b_{pq}$; this means that $(i, j) > (p, q)$. Thus,

the pair $(M, N)_{\text{can}}$ is $G$-canonical \quad (19)

in the sense of the following definition.

**Definition 6.1.** Let $G$ be a directed graph with vertices $1, 2, \ldots, n$ and no undirected cycles. By a $G$-canonical matrix pair we mean a matrix pair of the form

$$\begin{pmatrix} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & \lambda_r \end{pmatrix}, \begin{bmatrix} b_{11} & \ldots & b_{1r} \\ \vdots & \ddots & \vdots \\ b_{r1} & \ldots & b_{rr} \end{bmatrix}, \quad \lambda_1 < \cdots < \lambda_r,$$

in which every entry $b_{pq}$ satisfies the following conditions:

(i) $b_{pq} = 1$ if $G$ has the directed edge $p \to q$;

(ii) $b_{pq} = 0$ if either $G$ has no undirected path from $p$ to $q$, or the undirected path from $p$ to $q$ contains a directed edge $i \to j$ such that $(i, j) > (p, q)$ with respect to the lexicographic order;

(iii) $b_{pq}$ is arbitrary, otherwise.
Example 6.2. (a) Each pair of the form (17) is $G$-canonical with

$$G: \begin{array}{cccc}
 & 3 & 4 \\
2 & \downarrow & \downarrow \\
1 & \rightarrow & \cdot & \cdot \\
n &
\end{array}$$

(b) Each $G$-canonical matrix pair with

$$G: \begin{array}{cccc}
 & 5 \\
2 & \rightarrow & \rightarrow & \rightarrow \\
1 & \rightarrow & \rightarrow & \rightarrow \\
3 & \rightarrow & \rightarrow & \rightarrow \\
4 &
\end{array}$$

has the form

$$\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & 0 \\
0 & 0 & 0 & 0 & \lambda_5
\end{pmatrix}, \begin{pmatrix}
* & 0 & 1 & 0 & 1 \\
1 & * & * & 0 & * \\
* & * & * & 0 & * \\
0 & 0 & 1 & * & * \\
* & * & * & * & *
\end{pmatrix}$$

in which $\lambda_1 < \cdots < \lambda_5$ and the stars denote arbitrary complex numbers.

A $G$-canonical matrix pair is an indecomposable canonical matrix pair if $G$ is a tree. It is not similar to a direct sum of square matrices of smaller sizes. This is a consequence of the uniqueness assertion in (b) of the following theorem.

Theorem 6.1. (a) For each pair $(M, N)$ of $n \times n$ matrices in which $M$ has $n$ distinct eigenvalues, there exist a unique directed graph $G$ and a unique $G$-canonical matrix pair $(M, N)_{\text{can}}$ such that $(M, N)$ is similar to $(M, N)_{\text{can}}$. Thus, $(M, N)$ is similar to $(M', N')$ if and only if $(M, N)_{\text{can}} = (M', N')_{\text{can}}$.

(b) Each pair $(M, N)$ of $n \times n$ matrices in which $M$ has $n$ distinct eigenvalues is similar to a direct sum of indecomposable canonical matrix pairs. This direct sum is uniquely determined by $(M, N)$, up to permutation of summands.

The statement (a) of Theorem 6.1 follows from (18) and (19). The statement (b) is a consequence of the following remark.
Remark 6.1. The direct sum in Theorem 6.1(b) is permutationally similar to \((M, N)_{\text{can}}\) and can be obtained from it as follows: The directed graph \(G\) is a disjoint union of trees; denote them by \(G_1, \ldots, G_s\). Let \(u_{i1} < u_{i2} < \cdots < u_{it_i}\) be the vertices of \(G_i\). Let \((A_i, B_i)\) be the pair of \(t_i \times t_i\) submatrices of the matrices in \((M, N)_{\text{can}}\) formed by rows \(u_{i1}, \ldots, u_{it_i}\) and columns \(u_{i1}, \ldots, u_{it_i}\). Definition 6.1 ensures that the \(u_{il}, u_{jk}\) entries of the matrices in \((M, N)_{\text{can}}\) are zero if \(i \neq j\). Therefore, \((M, N)_{\text{can}}\) is permutationally similar to

\[
(A_1, B_1) \oplus (A_2, B_2) \oplus \cdots \oplus (A_s, B_s),
\]

which is the desired direct sum. Each \((A_i, B_i)\) is a \(G_i'\)-canonical matrix pair, in which \(G_i'\) is the tree obtained from \(G_i\) by relabeling its vertices \(u_{i1}, \ldots, u_{it_i}\) as \(1, \ldots, t_i\).

References

[1] G.R. Belitskiï, Normal forms in a space of matrices, in V.A. Marchenko (Ed.), Analysis in Infinite-Dimensional Spaces and Operator Theory, Naukova Dumka, Kiev, 1983, pp. 3–15 (in Russian).

[2] G. Belitskii, Normal forms in matrix spaces, Integral Equations Operator Theory, 38 (2000) 251–283.

[3] G.R. Belitskii, V.V. Sergeichuk, Complexity of matrix problems, Linear Algebra Appl. 361 (2003) 203–222.

[4] R. Benedetti, P. Cragolini, Versal families of matrices with respect to unitary conjugation, Adv. Math. 54 (1984) 314–335.

[5] D.V. Galinskiï, V.V. Sergeichuk, Classification of pairs of linear operators in a four-dimensional vector space, in N.S. Chernikov (Ed.), Infinite Groups and Related Algebraic Structures, Akad. Nauk Ukraine, Inst. Mat., Kiev, 1993, 413–430 (in Russian).

[6] I.M. Gelfand, V.A. Ponomarev, Remarks on the classification of a pair of commuting linear transformations in a finite dimensional vector space, Functional Anal. Appl. 3 (1969) 325–326.

[7] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.
[8] E.N. Klimenko, Classification of linear operators on a 5-dimensional unitary space, Master Thesis, Kiev State University, Kiev, 1993 (in Russian).

[9] S.A. Kruglyak, Yu.S. Samoilenko, Unitary equivalence of sets of self-adjoint operators, Functional Anal. Appl. 14 (no. 1) (1980) 48–50.

[10] D.E. Littlewood, On unitary equivalence, J. London Math. Soc. 28 (1953) 314–322.

[11] B.E. Mitchell, Unitary transformations, Canad. J. Math. 6 (1954) 69–72.

[12] V.V. Sergeichuk, Classification of linear operators in a finite-dimensional unitary space, Functional Anal. Appl. 18 (no. 3) (1984) 224–230.

[13] V.V. Sergeichuk, Classification problems for systems of forms and linear mappings, Math. USSR-Izv. 31 (no. 3) (1988) 481–501.

[14] V.V. Sergeichuk, Unitary and Euclidean representations of a quiver, Linear Algebra Appl. 278 (1998) 37–62.

[15] V.V. Sergeichuk, Canonical matrices for linear matrix problems, Linear Algebra Appl. 317 (2000) 53–102.