Relativistic theory of the above-threshold multiphoton ionization of hydrogen-like atoms in the ultrastrong laser fields.

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The relativistic theory of above-threshold ionization (ATI) of hydrogen-like atoms in ultrastrong radiation fields, taking into account the photoelectron induced rescattering in the continuum spectrum is developed. It is shown that the contribution of the latter in the multiphoton ionization probability even in the Born approximation by Coulomb field is of the order of ATI probability in the scope of Keldysh-Faisal-Reiss ansatz.

I. INTRODUCTION

The increasing interest to the process of multiphoton above threshold ionization (ATI) of atoms in superintense laser fields particularly is conditioned by the problem of high harmonic generation and short wave coherent radiation implementation via multiphoton bound-free transitions through free continuum spectrum (one of the possible version of a free electron X-ray laser). During the last two decades numerous investigations have been carried out to study ATI of atoms both theoretically and experimentally and many review papers (see, e.g., [1]-[10] and monographs [11]-[15] are devoted to this problem.

The main ansatz in the nonrelativistic theory of multiphoton ionization of atoms in strong electromagnetic (EM) radiation fields is the Keldysh one [16] (further called Keldysh-Faisal-Reiss ansatz [17],[18]). The advantage of this approach is that it leads in a very simple way to reveal some of the main qualitative features of the photoelectron energy spectrum in ATI experiments ([19],[20] and [21]). Within the scope of this ansatz the photoelectron rescattering in the field of atomic remainder is neglected. In order to cover this gap attempts have been made to describe the photoelectron final state by "Coulomb-Volkov" wave function that is a product of the Coulomb wave function of elastic scattering and a wave function of electron in the EM wave field [22]-[28]. This wave function results in the factorization of the probability of multiphoton ionization and restricts both the frequency (low frequency approximation) and intensity of the wave. The use of another ansatz for definition of multiphoton ionization probabilities [29] should also be noted.

The description of the photoelectron final state taking into account the stimulated bremsstrahlung (SB) at the photoelectron scattering on the electrostatic potential of the ionized atom in the presence of strong EM radiation field (induced free-free transitions) still remains as one of the main problems for the ATI process. Moreover, the definition of dynamic wave function of an electron in SB process already is problematic, so that the main results concerning to multiphoton SB probabilities have been found through S-matrix formalism for "free-free" (over electrostatic potential) transitions in the Born approximation between Volkov states in EM wave [30]. Although, in a lot of cases when the condition of the Born approximation is broken, the scattering process is described in low-frequency [31],[32] or eikonal [33] approximations. Though the Born and low-frequency approximations in SB process are applicable for describing free-free transitions in high intensity radiation fields, but they don’t take into account the mutual influence of the scattering and the radiation fields [i.e. the probability of SB is factorized by elastic scattering and photon emission or absorption processes]. What concerns to the eikonal approximation in SB process it is not applicable beyond the interaction region \( z \ll pa^2/\hbar \) (where \( z \) is the coordinate along the direction of initial momentum \( \vec{p} \) of the particle, \( a \) is the range of the interaction region, and \( \hbar \) is the Plank constant). The description of the electron eigenstates in SB process behind of the scope of these approximations has been made in [34], developing a generalized eikonal approximation (GEA). The obtained GEA wave function enables to leave the framework of ordinary eikonal approximation and abandons from the restriction \( z \ll pa^2/\hbar \). Besides, such wave function simultaneously takes into account the influence of both the scattering and radiation fields on the particle state. So, to determine the multiphoton probabilities of above threshold ionization of an atom it should be known the wave function of the ejected photoelectron in SB process with more accuracy. On the other hand, in the current superintense laser fields the state of electron becomes relativistic already at the distances \( l \ll \lambda \) (\( \lambda \) is the wavelength of a laser radiation) independent on its initial state. Hence, the problem of ATI of atoms with the photodetached electron SB process...
demands a relativistic consideration. The relativistic generalization of multiphoton SB in the first Born and eikonal approximations have been made in the papers [35], [36] and [37] respectively. In [38] on the base of the solution of the Dirac equation the GEA approximation has been developed for relativistic scattering theory in the arbitrary electrostatic and plane EM wave fields, including both the Born and eikonal approximations in corresponding limits and describing the spin interaction as well. Such a wave function allows us to describe the final state of the photoelectron with more accuracy in the ATI process of atoms.

The relativistic description of multiphoton ATI of hydrogen-like atoms for high-intensity laser fields taking into account the spin interaction has been developed analytically in the papers [39]-[41] with an approximation where the stimulated bremsstrahlung of the emergent electron is neglected. The relativistic consideration of ATI is important as it is generally assumed that the problem of stabilization of atoms in ultraintense laser fields must be solved within the framework of relativistic theory [42]. From this point of view some attempts have been made to solve analytically the Klein-Gordon equation or numerically the Dirac equation in fields of a static potential and monochromatic EM wave [using various model potentials of one or two dimensions and various approximations]. In the papers [46]-[48] the relativistic corrections to the nonrelativistic results have been given.

Note, that at the present time an analytic formulas for these probabilities, taking into account the photoelectron rescattering, are unknown even in the first Born approximation for the Coulomb scattering field. So, in the present paper the relativistic probabilities of multiphoton ATI in the limit of the Born approximation for the photoelectron rescattering, are calculated. Moreover, it is shown that the neglect of the photoelectron rescattering in the relativistic domain specially [39]-[41] is invalid, since the contribution of the electron rescattering process in the matrix elements of transitions has the same order by a scattering potential in the Born approximation as the matrix elements of bound-free transitions for the ATI process.

The organization of the paper is as follows. In Sec. II we present the multiphoton cross sections of the above threshold ionization of hydrogen-like atom in ultraintense laser field (with the help of GEA wave function), taking into account the induced free-free transitions of the ejected photoelectron in the continuum spectrum. Because of much complicated expressions in GEA the spin interaction is neglected and the ultimate analytic results for the multiphoton probabilities are performed in the limit of the first Born approximation by ion (atomic remainder) potential which we present in Sec. III. In Sec. IV we treat the dependence of ATI probability on polarization of an EM wave and consider the differences between circular and linear polarizations of electromagnetic wave.

II. THE IONIZATION PROBABILITY BY RELATIVISTIC GEA SOLUTION OF A WAVE EQUATION OF AN ELECTRON

The problem has been reduced to the investigation of the relativistic exploration of the transition S-matrix formalism utilizing the relativistic GEA wave function [38] as a wave function of the final state of a photodetached electron (it has been neglected with the spin interaction in relativistic GEA wave function). Following the relativistic S-matrix formalism the bound-free transition amplitude can be written in this integral form (in natural units \( \hbar = c = 1 \))

\[
T_{i \rightarrow f} = -i \int_{-\infty}^{\infty} \Psi^{(-)}(x) \hat{V} \Phi(x) dx,
\]

where \( x = (t, \mathbf{r}) \) is the four-component radius-vector \( x^\mu \), \( \Phi(x) \) is the initial unperturbed bound state of the atomic system and \( \Psi^{(-)}(x) \) is the final out-state of an electron in the potential of atomic remainder and in the field of a plane EM wave (\( K^\dagger \) is denotes the complex conjugation of \( K \)). We assume the EM wave to be quasimonochromatic and of an arbitrary polarization with the vector potential

\[
\mathbf{A}(\varphi) = A_0(\varphi) (\mathbf{e}_1 \cos \varphi + \mathbf{e}_2 \sin \varphi) \quad \varphi = k \cdot x = \omega t - \mathbf{k} \cdot \mathbf{r},
\]

where \( k = (\omega, \mathbf{k}) \) is the four-wave vector, \( A_0(\varphi) \) is the slow varying amplitude of the vector potential of a plane wave, \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) are unit vectors: \( \mathbf{e}_1 \perp \mathbf{e}_2 \perp \mathbf{k} \), and \( \arctan \zeta \) is the polarization angle.

According to the Klein-Gordon equation the interaction operator is

\[
\hat{V} = -2e \mathbf{A}(\varphi)(-i \hat{\mathbf{r}}) + e^2 \mathbf{A}^2(\varphi),
\]

where \( e \) is the electron charge.

The wave function of the final state of the photodetached electron in the relativistic GEA approximation has the following form ( [38])
\[ \Psi^{(-\dagger)}(x) = \frac{1}{\sqrt{2\Pi_0}} F^{\dagger}(x) \exp[-iS_V(x)], \] (2.4)

The \( S_V(x) \) is the action of photoelectron in the field \( \overline{\Pi} \)

\[ S_V(x) = \overline{\Pi} \cdot \overrightarrow{r} - \Pi_0 t + \alpha \left( \frac{\overrightarrow{p}}{k \cdot p} \right) \sin[\varphi - \theta(\overrightarrow{p})] - \frac{Z}{2}(1 - \zeta^2) \sin 2\varphi. \] (2.5)

Here \( \overline{\Pi} = (\overline{\Pi}_0, \overline{\Pi}) \) is the average four- kinetic momentum or “quasimomentum” of the electron in the plane EM wave field, which is defining via free electron four-momentum \( p = (\varepsilon_0, \overrightarrow{p}) \) and relative parameter of the wave intensity \( Z \) by the following equation

\[ \overline{\Pi} = p + kZ(1 + \zeta^2); \quad Z = \frac{e^2 \Delta_0^2}{4k \cdot p}, \] (2.6)

where \( \Delta_0 \) is the averaged value of the amplitude \( A_0(\varphi) \). The wave function \( F^{\dagger}(x) \) is normalized for the one particle in the unit volume \( V = 1 \).

Including in (2.5) quantity \( \alpha \left( \frac{\overrightarrow{p}}{k \cdot p} \right) \) is the intensity-dependent amplitude of the electron-wave interaction and as a function on any three-vector \( \overrightarrow{b} \) has the following definition

\[ \alpha \left( \overrightarrow{b}_1 \right) = e\Delta_0 \sqrt{\left( \overrightarrow{b} \cdot \overrightarrow{c}_1 \right)^2 + \zeta^2 \left( \overrightarrow{b} \cdot \overrightarrow{c}_2 \right)^2}, \] (2.7)

with the phase angle

\[ \theta(\overrightarrow{p}) = \arctan \left( \frac{\overrightarrow{b} \cdot \overrightarrow{c}_2}{\overrightarrow{b} \cdot \overrightarrow{c}_1} \right). \] (2.8)

The function \( F^{\dagger}(x) \) in Eq. (2.4), which describes the impact of both the scattering and EM radiation fields on the photoelectron state simultaneously, has the following form \( \overline{U} \)

\[ F^{\dagger}(x) = \exp \left[ \frac{1}{4\pi^3} \sum_{n=-\infty}^{\infty} e^{i\varphi} \int \frac{\omega \left[ \alpha \left( \frac{\overrightarrow{p}}{k \cdot p} \right) D^{\dagger}_{1,n}(\theta(\overrightarrow{q}) - \theta(\overrightarrow{p})) - Z(1 - \zeta^2)D^{\dagger}_{2,n} \right]}{\overrightarrow{q}^2 + 2\overline{\Pi} \cdot \overrightarrow{q} - 2n(k \cdot p - \overrightarrow{k} \cdot \overrightarrow{q}) + i0} \right] \times \overline{U}(\overrightarrow{q}) \exp \left[ -i \left\{ \overrightarrow{q} \cdot \overrightarrow{p} + \alpha_1(\overrightarrow{q}) \sin[\varphi - \theta_1(\overrightarrow{q})] - \alpha_2(\overrightarrow{q}) \sin 2\varphi + \theta_{1}(\overrightarrow{q}) \alpha(\overrightarrow{p}) \right\} \right] d\overrightarrow{q}, \] (2.9)

where

\[ \overline{U}(\overrightarrow{q}) = \int U(\overrightarrow{p}) \exp(-i \overrightarrow{q} \cdot \overrightarrow{p}) d\overrightarrow{p} \] (2.10)

is the Fourier transform of the potential of the atomic remainder and \( \alpha_1(\overrightarrow{q}) \), \( \alpha_2(\overrightarrow{q}) \) are dynamic parameters of the interaction defining by expression

\[ \alpha_1(\overrightarrow{q}) = \alpha \left( \frac{\overrightarrow{k} \cdot \overrightarrow{q}}{k \cdot p} \right) / k \cdot p + \overrightarrow{q}, \quad \alpha_2(\overrightarrow{q}) = \frac{\overrightarrow{k} \cdot \overrightarrow{q}}{2(k \cdot p - k \cdot \overrightarrow{q})} Z(1 - \zeta^2), \] (2.11)

and \( \theta_1(\overrightarrow{q}) \) is the phase angle

\[ \theta_1(\overrightarrow{q}) = \theta \left( \frac{\overrightarrow{k} \cdot \overrightarrow{q}}{k \cdot p} \right) / k \cdot p + \overrightarrow{q}. \] (2.12)

The functions \( J_n(u, v, \Delta), D_n, D_{1,n}(\theta_1(\overrightarrow{q}) - \theta(\overrightarrow{p})) \), and \( D_{2,n} \) are defined by the expressions (also see Ref. \( \overline{U} \))

\[ D_n = J_n(\alpha_1(\overrightarrow{q}), -\alpha_2(\overrightarrow{q}), \theta_1(\overrightarrow{q})), \] (2.13)

\[ D_{1,n}(\theta_1(\overrightarrow{q}) - \theta(\overrightarrow{p})) = \frac{1}{2} \left[ J_{n-1}(\alpha_1(\overrightarrow{q}), -\alpha_2(\overrightarrow{q}), \theta_1(\overrightarrow{q}) \alpha(\overrightarrow{p})) e^{-i\theta_1(\overrightarrow{q}) - \theta(\overrightarrow{p})} \right] \]
+J_{n+1}(\alpha_1(\vec{q}), -\alpha_2(\vec{q}), \theta_1(\vec{q}))e^{i(\theta_1(\vec{q})-\theta(\vec{p}))}].

(2.14)

and

\[ D_{2,n} = \frac{1}{2} \left[ J_{n-2}(\alpha_1(\vec{q}), -\alpha_2(\vec{q}), \theta_1(\vec{q}))e^{-i2\theta_1(\vec{q})} 
+ J_{n+2}(\alpha_1(\vec{q}), -\alpha_2(\vec{q}), \theta_1(\vec{q}))e^{i2\theta_1(\vec{q})} \right]. \]

(2.15)

In the denominator of the integral in expression \(2.9\), \(+0i\) is an imaginary infinitesimal, which shows how the path around the pole in the integrand should be chosen to obtain a certain asymptotic behavior of the wave function, i.e., the ingoing spherical wave [to determine that one must be passed to the limit of the Born approximation at \(\vec{A}(\varphi) = \vec{0}\)].

Since we consider the ATI problem for hydrogen-like atoms (\(Z_a \ll 137\)), the initial velocities of atomic electrons are nonrelativistic and as a initial-state wave function \(\Phi\) in the transition amplitude \((2.1)\) will be taken a stationary wave function of hydrogen-like atom bound state in nonrelativistic limit

\[ \Phi(\vec{r}, t) = \frac{1}{\sqrt{2m}} \Phi_0(\vec{r}) \exp(-i\varepsilon_0 t), \quad \varepsilon_0 = m - E_B, \]

(2.16)

where \(E_B > 0\) is the binding energy of the valence electron in the atom

\[ 2mE_B = a^{-2}. \]

(2.17)

Concerning the relativism of the photoelectron final state in strong EM field it is followed to mention that at the wave intensities already \(\xi \sim 10^{-1}\), where

\[ \xi = \frac{\varepsilon \lambda_0}{m} \]

(2.18)

is the relativistic invariant parameter of the wave intensity, relativistic effects become observable and the final state of the photoelectron should be described in the scope of relativistic theory. Moreover, at the currently available laser intensities \(\xi > 1\) (even \(\xi \gg 1\)) a free electron becomes essentially relativistic already at the distances smaller than one wavelength. On the other hand, in such fields becomes actual the production of electron-positron pairs from intense photon field on the electrostatic potential of atomic remainder through multiphoton channels. However, we can calculate separately the ATI probability in superstrong laser fields without restricting intensities by the threshold value of multiphoton pairs production (\(\xi \sim 2\); see [49] and [50]) since those are independent processes.

Since \(\hat{V}\) is a Hermitian operator, the transition amplitude \((2.1)\) can be written in the form

\[ T_{i \to f} = -i \int_{-\infty}^{\infty} \Phi(x) \hat{V} \Psi(-) \uparrow(x) dx. \]

(2.19)

To integrate this expression it is convenient to turn from variables \(t, \vec{r}\) to \(\varphi, \vec{\eta}\)[see \(2.2\)]

\[ T_{i \to f} = -i \omega \int_{-\infty}^{\infty} \Phi(\varphi, \vec{\eta}) \hat{V} \Psi(-) \uparrow(\varphi, \vec{\eta}) d\varphi d\vec{\eta}. \]

(2.20)

and make a Fourier transformation of the function \(F^\dagger(x)\) over variable \(\varphi\)

\[ F^\dagger(\varphi, \vec{\eta}) = \sum_{l=-\infty}^{\infty} \tilde{F}_l(\vec{\eta}) \exp(-il\varphi), \]

(2.21)

\[ \tilde{F}_l(\vec{\eta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\varphi, \vec{\eta}) \exp(il\varphi) d\varphi. \]

(2.22)

Then with the help of Eqs. \((2.9), (2.15), (2.16)\) [using as well Eq. \(A3\)] and taking into account the Lorentz condition for the plane wave field \(\vec{F}\) : \(\vec{A}(\varphi) = 0\), we can accomplish the integration over the variable \(\varphi\) in Eq. \((2.20)\). After a simple transformation with the help of the formula \(A5\) we obtain the next expression for the transition amplitude
\[
T_{i \rightarrow f} = \frac{i2\pi(k \cdot p)}{\omega \sqrt{\mu m_0}} \sum_{N,l=-\infty}^{\infty} \left\{ (N + l - Z(1 + \zeta^2)) \tilde{\Phi}_l(\vec{q}) J_L \left( \alpha \left( \frac{\vec{p}}{k \cdot p} \right), -\frac{Z}{2}(1 - \zeta^2), \theta(\vec{q}) \right) \right. \\
\times \delta \left( \frac{\Pi_0 - \varepsilon_0}{\omega} - L - l \right) \\
+ 2 \sum_{n=-\infty}^{\infty} \int \frac{d\vec{q}}{(2\pi)^3} \tilde{\Phi}_l(\vec{q} + \vec{q}') \tilde{U}(\vec{q}') \\
\times \alpha \left( \frac{\vec{q}}{k \cdot p} \right) C_{1,L}^l \left( \theta(\vec{p} + \vec{q}) - \theta(\vec{q}') \right) e^{-in\theta(\vec{q}') + iL\theta(\vec{p} + \vec{q}')}
\]

\[
\times \left\{ \omega \left[ \alpha \left( \frac{\vec{p}}{k \cdot p} \right) D_{1,n}^l \left( \theta(\vec{q}) - \theta(\vec{q}') \right) - Z(1 - \zeta^2) D_{2,n}^l - \Pi_0 D_{0,n}^l \right] \right\} \\
\times \left[ \frac{1}{\vec{q}^2 + 2\Pi \cdot \vec{q} - 2n(k \cdot p - \vec{k} \cdot \vec{q}) + i0} \right]
\]

\[
\times \delta \left( \frac{\Pi_0 - \varepsilon_0}{\omega} - L - l + n \right) \right\},
\]

where \( \vec{q} \) is the three-vector

\[
\vec{q} = \vec{p} - \frac{(\varepsilon - \varepsilon_0) \vec{k}}{\omega}.
\]

and the function \( \tilde{\Phi}_l(\vec{q}) \) is the Fourier transform of \( \Phi_l(\vec{q}) \equiv \Phi(\vec{q}) \tilde{F}_l(\vec{q}) \) and as a function on any three-vector \( \vec{q} \) is defined by the formula (2.10), and

\[
C_{1,n} \left( \theta(\vec{p} + \vec{q}) - \theta(\vec{q}') \right)
\]

\[
= \frac{1}{2} \left[ J_{n-1} \left( \alpha(\vec{p} + \vec{q}), -\frac{Z}{2}(1 - \zeta^2), \theta(\vec{p} + \vec{q}) \right) e^{-i(\theta(\vec{p} + \vec{q}) - \theta(\vec{q}'))} \\
+ J_{n+1} \left( \alpha(\vec{q}), -\frac{Z}{2}(1 - \zeta^2), \theta(\vec{p} + \vec{q}) \right) e^{i(\theta(\vec{p} + \vec{q}) - \theta(\vec{q}'))} \right],
\]

where the parameters \( \alpha \left( \frac{\vec{p} + \vec{q}}{k \cdot p - \vec{k} \cdot \vec{q}} \right), \theta(\vec{p} + \vec{q}) \) are determined by the expressions (2.7) and (2.8), and

\[
Z_1 = \frac{e^2 \epsilon_0}{4(k \cdot p - \vec{k} \cdot \vec{q})}.
\]

Using the general conservation law of considering process the probability amplitude of the above-threshold ionization in concluding form can be presented in this ultimate form

\[
T_{i \rightarrow f} = \frac{i2\pi(k \cdot p)}{\sqrt{\mu m_0}} \sum_{N,l=-\infty}^{\infty} \left\{ (N + l - Z(1 + \zeta^2)) \tilde{\Phi}_l(\vec{q}) \right. \\
\times J_{N-l} \left( \alpha \left( \frac{\vec{p}}{k \cdot p} \right), -\frac{Z}{2}(1 - \zeta^2), \theta(\vec{p}) \right) e^{i(N-l)\theta(\vec{p})}
\]

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where the scattering potential. The latter takes place if the corresponding part of the action function (2.4), i.e., the exponent of the function \( F \) on quasimomenta in the interval \( -\rightarrow \) \( \rightarrow \) in accordance with normalization of electron wave function) taking into account the all final states of photoelectron with the "effective mass" of the relativistic electron in the EM wave field.

Expanding Eq. (2.27) into the series and keeping only the terms to the first order over \( \Pi_0 \), \( \Pi \) and \( d\Pi \) is

\[
dW_{i\rightarrow f} = w_{i\rightarrow f} \frac{d\Pi}{(2\pi)^3}
\]

where \( d\Omega \) is the differential solid angle and

\[
m_* = \sqrt{\Pi_0^2 - \Pi^2} = \sqrt{m^2 + e^2A_0^2(1 + \zeta^2)/2}
\]

is the "effective mass" of the relativistic electron in the EM wave field.

Including in Eq. (2.28) the transition probability per unit time \( w_{i\rightarrow f} \) has the general definition

\[
w_{i\rightarrow f} = \lim_{t\rightarrow \infty} \frac{1}{t} |T_{i\rightarrow f}|^2
\]

III. THE RELATIVISTIC BORN APPROXIMATION BY THE POTENTIAL OF ATOMIC REMAINDER FOR HYDROGEN-LIKE ATOM IONIZATION

The impact of rescattering effect on the ATI process is more transparent in the limit of the Born approximation by the scattering potential. The latter takes place if the corresponding part of the action \( S_1(\vec{p},t) \) in the GEA wave function (2.4), i.e., the exponent of the function \( F^1(x) \) \( (F^1(x) = \exp \{iS_1(\vec{p},t)\}) \) is enough small (see Ref. [38]):

\[
|S_1(\vec{p},t)| \ll 1 .
\]

Expanding Eq. (2.27) into the series and keeping only the terms to the first order over \( U(\vec{p}) \), after a simple transformation, utilizing Eqs. (A2), (A3) and (A5), we obtain

\[
T_{i\rightarrow f} = \frac{i2\pi}{\sqrt{m\Pi_0}} \sum_{N=-\infty}^{\infty} \left\{ \left[ N - Z(1 + \zeta^2) \right] (k \cdot p) \tilde{\Phi}(\vec{q}) e^{iN\theta(\vec{q})} J_N \left( \alpha \left( \frac{\vec{q}}{k \cdot p} \right) , -\frac{Z}{2} (1 - \zeta^2), \theta(\vec{q}) \right) \right.
\]

\[
+ 2 \sum_{n=-\infty}^{\infty} \int \frac{d\vec{q}'}{(2\pi)^3} \left[ N + n - Z_1(1 + \zeta^2) \right] (k \cdot p - \vec{k} \cdot \vec{q}') \tilde{\Phi}(\vec{q}' + \vec{q}) \tilde{U}(\vec{q}')
\]

\[\Rightarrow \]

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\[ \times e^{-in\theta_1(|\vec{q}|)+i(\Delta(N+n)|\vec{p}+\vec{q}|) \left[ \omega \left\{ \alpha \left( \frac{\vec{p}}{k \cdot p} \right) D_{1,n}^1(\theta_1(\vec{q}) - \theta(\vec{p})) - Z(1 - \zeta^2)D_{2,n}^1 \right\} - \Pi_0 D_{n}^1 \right] } \]

\[ \times J_{(N+n)}(\alpha \left( \frac{\vec{p}+\vec{q}}{k \cdot p - \vec{q}} \right), -\frac{Z_0}{2}(1 - \zeta^2), \theta(\vec{p}+\vec{q})) \] \[ \frac{\vec{q}^2 + 2\Pi \cdot \vec{q} - 2n(k \cdot p - \vec{k} \cdot \vec{q}) + i0}{\left( \frac{\vec{q}^2 + 2\Pi \cdot \vec{q} - 2n(k \cdot p - \vec{k} \cdot \vec{q}) + i0}{\right) } \]

\[ \times \delta (\Pi_0 - \varepsilon_0 - \omega N) . \] (3.2)

For hydrogen-like atoms with the charge number \( Z_a \) the condition of the Born approximation for the photoelectron scattering (in Coulomb field)

\[ \frac{Z_a e^2}{\hbar v} \ll 1 \] (3.3)

requires electron velocities \( v \gg Z_a \alpha \), where \( \alpha = e^2/\hbar c = 1/137 \) is the fine-structure constant (it is assumed that \( Z_a \ll 137 \), \( \hbar \) and \( c \) are restored for clarity). The photoelectron acquires such velocities in the EM wave field at the intensities

\[ \xi \gg \frac{Z_a}{137} \] (3.4)

As will be shown below the Eq. (3.4) is the condition of the Born approximation in ATI process of hydrogen-like atoms taking into account the photoelectron rescattering.

The initial bound state enters into Eq. (2.27) through its momentum space wave function \( \Phi(\vec{b}) \). For hydrogen-like atom the bound state wave function has the following form:

\[ \Phi(\vec{b}) = \frac{\exp(-\eta/a)}{\sqrt{\pi a^3}} \] (3.5)

where \( a = a_0/Z_a \) (\( a_0 = 1/mc^2 \) is the Bohr radius) (3.1) and the corresponding momentum space wave function has the following form:

\[ \tilde{\Phi}(\vec{b}) = \frac{2^3(\pi a^3)^{1/2}}{b^4 a^4} \] (3.6)

Note, that in Eq. (3.6) it has been taken into account that \( |\vec{b}| a \gg 1 \) in accordance with the Born approximation.

Then the function \( \tilde{\Phi}(\vec{q} + \vec{q}) \) in the second term in figured brackets can be replaced by the quantity \( \delta(\vec{q} + \vec{q})/\sqrt{\pi a^3} \) because of the small contributions of the other terms in expansion of \( T_{i \rightarrow f} \) over parameter \( \gamma^2 a^2 \) [see, e.g., (54)] which will be shown below. Such a delta function can be used to accomplish the integration over \( \vec{q} \) in the second term of the sum in the figured brackets of Eq. (3.2).

For the scattering of a charged particle in the Coulomb field for which the Fourier transform is

\[ \tilde{U}(\vec{q}) = \frac{4\pi}{am \gamma^2} \] (3.7)

we have the following expression for the transition amplitude in the field of arbitrary polarization of EM wave

\[ T_{i \rightarrow f} = \frac{i2^4(\pi a)^{3/2}}{\sqrt{mH_0 \gamma^2 a^4}} \sum_{N=-\infty}^{\infty} \left\{ (N - Z(1 + \zeta^2)) e^{i\lambda_0(\vec{q})} J_N \left( \alpha \left( \frac{\vec{q}}{k \cdot p} \right), -\frac{Z_0}{2}(1 - \zeta^2), \theta(\vec{p}) \right) \right\} \]

\[ - \frac{\omega \varepsilon_0 \gamma \gamma^2}{m(k \cdot p)} \sum_{n=-\infty}^{\infty} (2n - \alpha' (1 + \zeta^2)) e^{-i(2n - N) \theta(\vec{p})} \]

\[ 7 \]
\[
\times \left\{ (2\omega^2 N + 2\omega^2 \zeta^2 + \omega^2 (1 - \zeta^2) \zeta^2) J_n \left( \frac{-\alpha'(1 - \zeta^2)}{2} \right) \right\}
\]

\[
\times \delta (\Pi_0 - \omega N), \quad (3.8)
\]

where \( \alpha' \) is defined by Eq. (2.26) at \( \tilde{q} = -\tilde{q} \) and \( \alpha' = e^2 A_0^2 / 4\omega \varepsilon_0 \), then \( J_n \left( \frac{-\alpha'(1 - \zeta^2)}{2} \right) \) is the ordinary Bessel function [ \( J_{2n}(0, x, 0) = J_n(x) \) [15]], \( C_s \) and \( C_{2,s} \) are defined by the expressions

\[
C_s = J_s(\alpha(\tilde{p} / \kappa \varepsilon)), (Z - \alpha')(1 - \zeta^2)/2, \theta(\tilde{p}),
\]

and

\[
C_{2,s} = \frac{1}{2} \left[ J_{s-2}(\alpha(\tilde{p} / \kappa \varepsilon)), (Z - \alpha')(1 - \zeta^2)/2, \theta(\tilde{p})e^{-i2\phi(\tilde{p})} \right.
\]

\[
+ J_{s+2}(\alpha(\tilde{p} / \kappa \varepsilon)), (Z - \alpha')(1 - \zeta^2)/2, \theta(\tilde{p})e^{i2\phi(\tilde{p})} \left. \right] . \quad (3.10)
\]

To determine the differential probability of ATI it should be integrate the expression (3.8) over \( \Pi_0 \) according to (2.28). In result we have

\[
\frac{dW_{i-f}}{d\Omega} = \frac{2^4}{\pi m^2 \alpha^2} \sum_{N=N_0}^\infty \frac{(N - Z(1 + \zeta^2))^2 (k \cdot \Pi)^2 |\tilde{p}|}{\tilde{q}^4}
\]

\[
\times \left\{ \left| e^{iN\theta(\tilde{p})} J_N \left( \alpha \left( \frac{\tilde{p} \cdot \Pi}{k \cdot \Pi} \right), - \frac{Z}{2} (1 - \zeta^2), \theta(\tilde{p}) \right) \right|^2 + \frac{\tilde{q}^2}{2m(N - Z(1 + \zeta^2))(k \cdot \Pi)} \sum_{n=-\infty}^\infty e^{-i(2n - N)\phi(\tilde{p})} J_n \left( \frac{-\alpha'(1 - \zeta^2)}{2} \right)
\]

\[
\times \left| \left( \varepsilon_0 + 2n\omega \right) C_{N-2n}^0 + \omega \alpha'(1 - \zeta^2) C_{2,N-2n}^0 \right| \right\}^2 \right; \quad (3.11)
\]

where \( \tilde{q} \) is the three-vector

\[
\tilde{q} = \tilde{p} - N \tilde{k}, \quad |\tilde{p}| = \sqrt{(\varepsilon_0 + \omega N)^2 - m^2}. \quad (3.12)
\]

The number \( N_0 \) over which is carried out summation in Eq. (3.11) is defined from the energy conservation law of ATI process: \( N_0 = \langle (m_s - \varepsilon_0) / \omega \rangle \).

The first term in the figured brackets of Eq. (3.11) corresponds to result of the KFR approximation. And the second term shows the dependance of ATI probability on the ejected photoelectron stimulated bremsstrahlung (SB) probability, i.e., it takes into account the rescattering process.

**IV. PROBABILITY OF ATI PROCESS FOR THE CIRCULAR AND LINEAR POLARIZATION OF EM WAVE**

The state of photoelectron in the field of a strong EM wave and consequently the ionization probability essentially depends on the polarization of the wave [nonlinear effect of intensity conditioned by the impact of strong magnetic field]. Thus, for the circular polarization the relativistic parameter of the wave intensity \( \zeta^2 = \text{const} = \zeta_0^2 \) and the longitudinal velocity of the electron in the wave \( v_{LL} = \text{const} \) (eliminating this inertial motion - in the framework connecting with the electron - we have the uniform rotation in the polarization plane with the wave frequency
\( \omega \), meanwhile for the linear one \( \xi^2 = \xi_0^2 \cos^2 \varphi \) and \( v_{II} \) oscillates with the frequencies of all wave harmonics \( n \omega \) corresponding to strongly unharmonic oscillatory motion of photoelectron. The later leads to principally different behavior of ionization process and corresponding formulas depending on the polarization of strong wave. Therefore, we shall consider the cases of circular and linear polarizations of EM wave field separately.

From the Eq. (3.11) for the circularly polarized wave (\( \zeta = 1 \)) in the first Born approximation by ionized atom potential we obtain the next formula for differential probability of ATI process

\[
\frac{dW_{i \rightarrow f}}{d\Omega} = \frac{2^4}{\pi m a^5} \sum_{N=N_0}^{\infty} \frac{(N-2Z)^2 (k \cdot \Pi)^2}{g^8} \left| \Pi \right| J_N^2 \left( \alpha \left( \frac{\Pi}{k \cdot \Pi} \right) \right) \times \left\{ 1 + \frac{\chi^2}{2(N-2Z)(k \cdot \Pi)} \right\}^2.
\] (4.1)

As is seen from this formula in contrast to the case of another polarizations the differential probability of ATI process is defined by ordinary Bessel function instead of the function \( J_n(u,v,\Delta) \) and the sum over \( n \) vanishes. The latter corresponds the above mentioned fact that for the circular polarization the parameter of intensity of the wave \( \xi^2 = \text{const} \) and effect of intensity of strong wave is appeared in the form of constant renormalization of the characteristic parameters of the interacting system.

Let us estimate the contribution of photoelectron rescattering in the probability of ATI process that is the second term in the figured brackets in Eq. (4.1). The latter is

\[
\frac{\chi^2}{2(N-2Z)(k \cdot \Pi)} \approx 1
\] (4.2)

for the most probable number of absorbed photons at which the Bessel function has the maximum value. So, the rescattering effect has the same order as the probability of the direct transition in SFA. It is followed to note that the derivations rely upon the SFA (e.g., [1]) are expected to become more accurate at high intensity EM field. However, the prediction of SFA regarding to rescattering effect in high intensity EM field, i.e., for relativistic photoelectron-according to which the rescattering will be negligible small in relativistic domain with the increasing field. However, the prediction of SFA regarding to rescattering effect in high intensity EM field, i.e., for relativistic photoelectron-according to which the rescattering will be negligible small in relativistic domain with the increasing field.

In the scope of current approximation \( \xi \gg Z_a/137 \) the explicit analytic formulas for total ionization rate can be obtained utilizing properties of Bessel function. At the condition (3.4) the argument of Bessel function \( X(N) \gg 1 \) and always \( X < N \). Therefore the terms with \( N \gg 1 \) and \( N \sim X \) give the main contribution in the sum (4.1). Besides, in this limit one can replace the summation over \( N \) with integration and approximate the Bessel function by Airy one

\[
J_N(x) \approx \left( \frac{2}{N} \right)^{1/3} Ai \left( \left( \frac{N}{2} \right)^{2/3} \left( 1 - \frac{x^2}{N^2} \right) \right)
\] (4.3)

Turning to spherical coordinates, we carry out the integration over the \( \varphi \) since there is azimuthal symmetry with respect to the direction \( \vec{k} \) (the OZ axis) and for ionization rate we have

\[
W_{i \rightarrow f} = \frac{2^5}{ma^5} \int_0^\pi \sin \theta d\theta \int_{N=N_0}^{\infty} \frac{2^{2/3} (N-2Z)^2 (k \cdot \Pi)^2}{g^8} \left| \Pi \right| Ai^2 \left[ y(N,\theta) \right] \times \left\{ 1 + \frac{\chi^2}{2(N-2Z)(k \cdot \Pi)} \right\}^2.
\] (4.4)

where

\[
y(N,\theta) = \left( \frac{N}{2} \right)^{2/3} \left[ 1 - \frac{\alpha^2 (\Pi/k \Pi)}{N^2} \right]
\] (4.5)
The \( y(N, \theta) \) has a minimum as a function of \( N \) and \( \theta \) and since Airy function exponentially decreases with increasing argument one can use Laplace method (method of the steepest descent) in order to carry out the integration as over \( N \) as well as over \( \theta \). The extremum points of the function \( y(N, \theta) \), i.e., the most probable values of \( N \) and \( \theta \) are

\[
N_m = \frac{m^2 - \varepsilon_0^2}{\varepsilon_0 \omega} \simeq \frac{m}{\omega} \xi^2, \quad \cos \theta_m = \frac{\overline{\Pi}(N_m)}{\Pi_0(N_m)} \tag{4.6}
\]

and

\[
y_m = y(N_m, \theta_m) = \frac{2^{1/3} E_B}{N_m^{1/3} \omega} \left( \frac{F_{at}}{2 F_0} \right)^{2/3}, \tag{4.7}
\]

where \( F_0 \) and \( F_{at} = Z_a^2 m^2 e^5 \) are wave and atomic electric field strengths. At \( N = N_m \) and \( \theta = \theta_m \) we have a peak for angular and energetic distribution. Let us note that the contribution of the rescattering effect to the angular distribution of the photoelectrons is nonessential.

For \( y_m << 1 \) when the wave electric field strength much exceeds the atomic one: \( F_0 >> F_{at} \) the main contribution in the integral give the areas

\[
\delta \theta \simeq (N_m/2)^{-1/3} / \sqrt{1 + \xi^2} \text{ and } \delta N \simeq 2(N_m/2)^{2/3} \tag{4.8}
\]

(angular and energetic widths of the peak) and for ionization rate we have an explicit formula which expresses directly the dependence upon the wave intensity

\[
W_{i \to f} = \frac{2^{7/3}}{3^{4/3} \Gamma^2(2/3)} \pi \omega \left( \frac{\omega}{E_B} \right)^3 \left( \frac{F_{at}}{F_0} \right)^{11/3}. \tag{4.9}
\]

For \( y_m >> 1 \) or \( F_0 << F_{at} \) (so called tunneling regime of ionization) we shall use the following asymptotic formula for Airy function

\[
Ai(x) \simeq \frac{1}{2 \sqrt{\pi}} x^{-1/4} \exp \left( -\frac{2x^{3/2}}{3} \right) \tag{4.10}
\]

and applying Laplace method we have

\[
W_{i \to f} = 2\omega \left( \frac{\omega}{E_B} \right)^3 \left( \frac{F_{at}}{F_0} \right)^3 \exp \left\{ -\frac{2 F_{at}}{3 F_0} \right\}. \tag{4.11}
\]

Let’s revert to the Born condition (3.3) to substantiate the condition (3.4). As is shown above we have a peak for angular and energetic distribution (4.1) at \( \theta_m \) and \( N_m \) (4.6), and the electron mean velocity will be defined by these values

\[
v = \frac{\overline{\Pi}(N_m)}{\Pi_0(N_m)} \simeq \frac{\xi}{\sqrt{1 + \xi^2}} \tag{4.12}
\]

Substituting (4.12) into Eq. (3.3) we have the condition of the Born approximation in ATI process of hydrogen-like atoms (3.4).

Using the explicit analytic formulas for total ionization rate we can conclude that at \( N = N_m \) and \( \theta = \theta_m \) we have a peak for angular and energetic distribution which are given by Eq. (4.1) with the angular and energetic widths of the peak \( \delta \theta \) and \( \delta N \), respectively (4.8).

In the case of linear polarization of the wave from Eq. (3.11) we have

\[
\frac{dW_{i \to f}}{d\Omega} = \frac{2^4}{\pi m a^5} \sum_{N=N_0}^{\infty} \frac{(N - Z)^2(k \cdot \Pi)^2}{\overline{\Pi}^4} \times \left\{ J_N \left( \alpha \left( \frac{\overline{\Pi}}{k \cdot \Pi} \right), \frac{Z}{2} \right) + \frac{\overline{\Pi}^2}{2m(N - Z)(k \cdot \Pi)} \sum_{n=-\infty}^{\infty} J_n \left( -\alpha'/2 \right) \right\}
\]

\[
\times \left\{ \frac{2^{1/3} E_B}{N_m^{1/3} \omega} \left( \frac{F_{at}}{2 F_0} \right)^{2/3} \right\} \tag{4.13}
\]
\[ \times \left[ (\varepsilon_0 + 2n\omega) J_{N-2n} \left( \frac{\alpha}{k \cdot \Pi}, \frac{(Z - \alpha')}{2} \right) + \frac{\omega' \alpha'}{2} \right] \]

\[ \times \left( J_{N-2n-2} \left( \frac{\alpha}{k \cdot \Pi}, \frac{(Z - \alpha')}{2} \right) + J_{N-2n+2} \left( \frac{\alpha}{k \cdot \Pi}, \frac{(Z - \alpha')}{2} \right) \right) \right] \right)^2, \tag{4.13} \]

where \( J_n(u, v) \) is the real generalized Bessel function (e.g., see [13]). As is seen from the formula (4.13), in this case the total probability of ATI process includes all intermediate transitions of photoelectron through the virtual vacuum states as well, corresponding the emission and absorption of wave photons of number \(-\infty < n < \infty\) (the sum over \( n \)) in accordance with the above mentioned behavior of wave intensity effect at linear polarization (strongly unharmonic oscillatory motion of photoelectron).

Let us now consider the ATI process with the rescattering effect in nonrelativistic limit since the theoretical treatments of this problem- the main of those are Keldysh-Faisal-Reiss ansatz (13-18) - in general have been carried out for nonrelativistic photoelectron when the rescattering effect is neglected. In the pioneer result of Keldysh (13) the rescattering of photoelectron on the potential of atomic remainder has been approximately estimated and putted in the form of coefficient in the ultimate formula for the ionization probability (for the wave fields much smaller than atomic ones). Further the same approach has been made in [13] for relatively large wave fields up to the atomic ones. Besides, in the existing nonrelativistic theory of ATI the gauge problem for description of interaction with the wave field and different views concerning the role of wave intensity in the dipole approximation have arose. For discussion of these problems the special paper has been devoted (132). Moreover, in the scope of the same Keldysh-Faisal-Reiss ansatz the existence of stabilization effect depends on the gauge of the wave field (13). So that, we shall consider the results of the present paper in the nonrelativistic limit taking into account the photoelectron rescattering.

From the formula (1.1) for the differential probability of ATI transition rate in the case of circular polarization of EM wave in the nonrelativistic limit we have

\[ \frac{dW_{\text{rel}}^{\text{rel}}}{d\Omega} = \frac{8\omega}{\pi} \left( \frac{E_B}{\omega} \right)^{5/2} \sum_{N=N_0}^{\infty} \frac{(N-2z-E_B/\omega)^{1/2}}{(N-2z)^2} J_N^2(\vartheta) \]

\[ \times \left[ 1 + \frac{N-2z-E_B/\omega}{N-2z} \right]^2, \tag{4.14} \]

where

\[ \vartheta = \frac{eA_0}{\omega m} \sqrt{\left( \frac{\vec{p} \cdot \vec{e}_1}{z} \right)^2 + \left( \frac{\vec{p} \cdot \vec{e}_2}{z} \right)^2}, \tag{4.15} \]

\[ z = Z = Z_1 = \frac{e^2A_0^2}{4m\omega}, \text{ and } N_0 = \langle (\vec{p}^2/2m - E_B)/\omega + z \rangle. \]

The corresponding condition of the Born approximation (Eq. (3.4)) in nonrelativistic limit is

\[ 1 \gg \xi \gg \frac{Z_0}{137}. \tag{4.16} \]

The first term in the quadratic brackets of Eq. (4.13) coincides with the above-threshold ionization differential probability obtained in the SFA for the nonrelativistic photoelectron [13] without the rescattering effect. According to the [13] SFA is expected to become valid when the ponderomotive potential \( U_p = e^2A_0^2/2m \) due to EM radiation field larger than the ionization potential of the atom: \( U_p \gg E_B \) and consequently \( p^2/2m \gg E_B \) which is the condition of the Born approximation. Then taking into account the scattering potential by perturbation theory we obtain (4.13) that the contribution of the photoelectron rescattering in the ATI probability (in the first order of the Born approximation over the Coulomb potential) is of the order of the main results of KFR ansatz. So, the neglecting of SB process for the photoelectron in the Coulomb field of an atomic remainder is incorrect.

In the case of linear polarized EM wave from the formula (4.13) we have the differential probability of the ATI process in the nonrelativistic domain

\[ \frac{dW_{\text{rel}}^{\text{rel}}}{d\Omega} = \frac{8\omega}{\pi} \left( \frac{E_B}{\omega} \right)^{5/2} \sum_{N=N_0}^{\infty} \frac{(N - z - E_B/\omega)^{1/2}}{(N-2z)^2} J_N^2(\vartheta, \frac{z}{2}) \]
\begin{equation}
\times \left\{ 1 + \frac{(N - z - E_B/\omega)}{(N - z)} \right\}
\end{equation}

where

\begin{equation}
\begin{aligned}
&u = z^{1/2} \chi, \\
&\chi = 8^{1/2} \left( N - z - \frac{E_B}{\omega} \right)^{1/2} \cos \theta
\end{aligned}
\end{equation}

in the polar coordinate system has been chosen for this case, \( \theta \) is the angle between the velocity vector of the emitted photoelectron and the wave polarization vector.

**APPENDIX A: DEFINITION OF THE FUNCTION** \( J_N(U, V, \Delta) \)

A function \( J_n(u, v, \Delta) \) may be defined by

\begin{equation}
J_n(u, v, \Delta) = (2\pi)^{-1} \int_{-\pi}^{\pi} d\theta \exp \left\{ i \left( u \sin(\theta + \Delta) + v \sin 2\theta - n(\theta + \Delta) \right) \right\}
\end{equation}

or by an infinite series representation

\begin{equation}
J_n(u, v, \Delta) = \sum_{k=-\infty}^{\infty} e^{-i2k\Delta} J_{n-2k}(u) J_k(v).
\end{equation}

We are performing two important theorems, which can be proved from Eq. (A1):

\begin{equation}
\sum_{n=-\infty}^{\infty} e^{in(\varphi+\Delta)} J_n(u, v, \Delta) = \exp \{ i [u \sin(\varphi + \Delta) + v \sin 2\varphi] \}
\end{equation}

and

\begin{equation}
\sum_{k=-\infty}^{\infty} J_{n+k}(u, v, \Delta) J_k(u', v', \pm \Delta) = J_n(u \pm u', v \pm v', \Delta).
\end{equation}

An integration by parts in Eq. (A1) yields the relation

\begin{equation}
2nJ_n(u, v, \Delta) = u [J_{n-1}(u, v, \Delta) + J_{n+1}(u, v, \Delta)]
\end{equation}

\begin{equation}
+ 2v \left[ e^{-i2\Delta} J_{n-2}(u, v, \Delta) + e^{i2\Delta} J_{n+2}(u, v, \Delta) \right].
\end{equation}

From either Eq. (A1) or (A2) follows that

\begin{equation}
J_n(u, 0, \Delta) = J_n(u),
\end{equation}

and

\begin{equation}
J_n(0, v, \Delta) = \begin{cases} 
e^{-i\Delta n} J_{2}(v), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}
\end{equation}

References:

[1] L. Rosenberg, Adv. At. Mol. Phys. **18**, 1 (1982).
[2] N. B. Delone and V.P. Krainov, *Atom in Strong Light Field* (Nauka, Energoatomizdat, 1984).
[3] N.B. Delone and M.V. Fedorov, Usp. Fiz. Nauk **158**, 215 (1989).
[4] G. Manfray and C. Manus, Rep. Prog. Phys. **54**, 1333 (1991).
