Nine lectures on quark structure of light hadrons

B.V. Martemyanov

Institute of Theoretical and Experimental Physics
117218, B. Cheremushkinskaya 25, Moscow, Russia

March 27, 2022

Abstract

The following lectures concern with the quark structure of hadrons made up from light quarks. The symmetries of strong interaction were the only instruments used for the description of this structure.

The models of hadrons (there are a lot of them) were not considered. There was not also considered the structure of hadrons with heavy quarks. The first round of problems (models of hadrons) is not considered because of its unexhaustability. The second round of problems (hadrons with heavy quarks) is not considered because of the physics of heavy flavors tends mainly to electroweak interaction and will be considered in the course of electroweak interactions.

Therefore, the presented here course, being the self-bounded part of the theory of elementary particles, is considered separately.

These lectures were formed from 1989 and are the part of the set of courses on the theory of fundamental interactions for the students of Moscow Physical Technical Institute. The basic source of the lectures was the unpublished manuscript "Hadrons and quarks" by L.B. Okun. Another source was the "Lectures on the theory of unitary symmetry in elementary particle physics" by Nguen Van Hieu (Atomizdat, Moscow 1967). The list of references (due to its smallness) is absent.
1 Lecture 1. Particles and interactions

1.1 The classes of particles and the types of interactions

At present there exists a hard belief that all in the Nature is constructed from elementary particles and that all the Natural processes are caused by the interactions of these particles. Elementary particles are assumed today to be gauge bosons, leptons, quarks and Higgs scalar particles. The fundamental interactions are assumed today to be strong, electroweak and gravitational interactions. So, conventionally we can indicate four classes of elementary particles and three types of fundamental interactions.

The first class contains photon ($\gamma$), $W^+$, $W^-$, $Z$-bosons, eight gluons ($g^a, a = 1, \ldots, 8$) and assumes the existence of graviton. All these particles transfer the interactions: $\gamma, W^+, W^-, Z$-bosons transfer electroweak interaction, gluons transfer strong interaction and gravitons are hypothetical quanta of gravitational field.

The second class contains leptons. At present we have six of them: electron ($e$), muon ($\mu$), tau-lepton ($\tau$) and corresponding neutrino ($\nu_e, \nu_\mu, \nu_\tau$). It is convenient to put six leptons into three families

\[
\begin{pmatrix}
\nu_e(<5.1eV) \\
e(0.511MeV)
\end{pmatrix}
\begin{pmatrix}
\nu_\mu(<0.27MeV) \\
\mu(105.6MeV)
\end{pmatrix}
\begin{pmatrix}
\nu_\tau(<31MeV) \\
\tau(1777MeV)
\end{pmatrix}
\]

Neutrinos are electrically neutral; electron, muon and tau-lepton possess electric charge. Leptons take part in electroweak and gravitational interactions.

The third class is represented by quarks. Today we know six quarks- $u, d, s, c, b, t$ and each quark is colored in one of three colors. Like leptons quarks are represented by three families

\[
\begin{pmatrix}
u_c(2\sim8MeV) \\
d(5\sim15MeV)
\end{pmatrix}
\begin{pmatrix}
u_d(1\sim1.6GeV) \\
s(100\sim300MeV)
\end{pmatrix}
\begin{pmatrix}
u_t(175GeV) \\
b(4.1\sim4.5GeV)
\end{pmatrix}
\]

Quarks are not seen as isolated particles. Together with gluons they are the constituents of hadrons and there are some hundreds of hadrons. Hadrons like quarks from which they are formed take part in all types of interactions.

The forth class of particles are Higgs scalar particles not observed yet experimentally. In minimal scheme of interactions there is one Higgs scalar.
The role of Higgs particles in the Nature is mainly the theoretical one and is to make the electroweak interaction renormalizable. In particular, the masses of all elementary particles are due to the condensate of Higgs field. Probably, the existence of Higgs fields is necessary for solving the fundamental problems of cosmology: the homogeneity and causality of the Universe.

The following lectures on quark structure of hadrons will consider light $u,d,s$-quarks and hadrons constructed from these quarks. The main attention will be payed to classification of particles, symmetries and conservation laws.

### 1.2 Particles and antiparticles

**Fermions and bosons**

All particles have the partners-antiparticles that have the same values of mass, spin, lifetime but have opposite in sign electric charge, other charges, for instance, leptonic, baryonic, hypercharge, strangeness etc. Antiparticle for electron $e^-$ is positron $e^+$, for proton $p$ - antiproton $\bar{p}$, for neutron $n$ - antineutron $\bar{n}$ etc. If the particle has no charges its antiparticle coincides with the particle itself and the particle is called truly neutral. The examples of truly neutral particles are $\gamma$-quantum, $Z^0$-boson, $\pi^0$-meson etc. The part of the Universe surrounding us and probably all the Universe is asymmetric: it consists of $e^-, n, p$ and almost does not contain $e^+, \bar{p}, \bar{n}$. The reasons of such asymmetry are proposed in the theories of Grand Unification of particle interactions.

All particles have either integer or half-integer spin. The particles with half-integer spin are called fermions and follow the Fermi-statistics according to which the given state can be occupied by no more than one fermion. The wave function of the system of fermions is antisymmetric under the interchange of fermion variables. The particles with integer spin are called bosons and follow the Bose-statistics according to which the given state can be occupied by any number of bosons. The wave function of the system of bosons is symmetric under the interchange of boson variables. In what follows we will encounter the manifestations of Fermi and Bose principles in the physics of hadrons many times.
1.3 The characteristic of interactions in short

Strong interaction is manifested on two levels: first, it is the interaction of quarks inside hadrons. It is described by the gluon exchange at small distances and by the confinement mechanism at large distances. Second, it is the interaction of hadrons—of protons and neutrons in the nuclei, for example. It is assumed that the second interaction is the result of the first interaction although the exact mechanism of the realization of the interaction between hadrons on the quark level is unclear. Hadrons are divided on baryons (odd number of quarks and antiquarks, fermions) and mesons (even number of quarks and antiquarks, bosons), on particles (stable with respect to strong interaction, long lived) and resonances (decaying due to strong interaction, short lived). The division on particles and resonances becomes now more and more conventional one when short lived decaying due to weak interaction hadrons are being discovered.

Electroweak interaction is mediated by the exchange of $\gamma$, $W$, $Z$-particles. It splits into electromagnetic and weak interactions at the energies lower than the masses of $W$ and $Z$-bosons. The massless photon induces the long range interaction, the massive $W$ and $Z$-bosons induce contact four-fermion interaction with dimensional Fermi constant.

Gravitational interaction of elementary particles is extremely small up to the energies of the order $10^{19} GeV$ where this interaction could unify with other interactions.

1.4 System of units

Normally, in physics of elementary particles the velocities of the order of light velocity $c$, the actions and angular momenta of the order of Plank constant $\hbar$ are encountered. It is natural therefore to use in physics of elementary particles the system of units where $\hbar = c = 1$. Then, angular momentum, action and velocity are dimensionless: $[J] = [s] = [v] = 1$. The energy and momentum have the dimensionality of mass $[E] = [p] = [m]$, and time and length have the dimensionality of inverse mass $[t] = [l] = [m^{-1}]$. As the unit of energy it is convenient to choose $1 GeV$. 
1.5 Problem 1

a) express $1\text{fm} = 10^{-13}\text{cm}$ through $1\text{GeV}^{-1}$,
b) evaluate the gravitational constant and find its value in the system of units $\hbar = c = 1$. 
2 Ground states of light hadrons

2.1 Arithmetic of charges

On the previous lecture we addressed to quarks. There are six quarks
\[
\begin{pmatrix}
u(5\text{MeV}) \\
d(7\text{MeV}) \\
c(1.5\text{GeV}) \\
\end{pmatrix}
\begin{pmatrix}
s(150\text{MeV}) \\
t(175\text{GeV}) \\
b(4.5\text{GeV}) \\
\end{pmatrix}
\]

If one compares the quark masses with the characteristic hadronic scale $1\text{GeV}$ it would be reasonable to split all quarks on light ones ($m_q << 1\text{GeV}$) and heavy ones ($m_q > 1\text{GeV}$). Below we will discuss the properties of hadrons consisting of light quarks.

In strong and electromagnetic interactions the numbers of $u$-quarks, $d$-quarks and $s$-quarks are conserved separately. Therefore we can introduce the charges $U, D, S$ conserved in these interactions. Then

\[
\begin{array}{c|ccc}
\text{charge quark} & U & D & S \\
\hline
u & 1 & 0 & 0 \\
d & 0 & 1 & 0 \\
s & 0 & 0 & 1 \\
\end{array}
\]

Instead of $U, D, S$-charges it is convenient to introduce more familiar charges that are linear combinations of $U, D, S$

\[
B = \frac{U + D + S}{3}
\]

(baryon number),

\[
T_3 = \frac{U - D}{2}
\]

(3-d component of isotopic spin),

\[
Y = \frac{U + D - 2S}{3}
\]

(hypercharge). Then

\[
\begin{array}{c|cccc}
\text{charge quark} & B & T_3 & Y & Q \\
\hline
u & 1/3 & 1/2 & 1/3 & 2/3 \\
d & 1/3 & -1/2 & 1/3 & -1/3 \\
s & 1/3 & 0 & -2/3 & -1/3 \\
\end{array}
\]
Baryonic charge is defined in such a way that the proton (or any other baryon) consisting of three quarks has baryonic charge equal to one. Baryonic charge is conserved in all interactions. 3-d component of isotopic spin is defined in such a way as $u$ and $d$ quarks form the isotopic doublet ($T = 1/2$) and strange quark is isosinglet. Finally, hypercharge is defined in such a way that there holds the relation of Gell-Mann - Nishigima

$$ Q = T_3 + \frac{Y}{2}. $$

In strong and electromagnetic interactions both $T_3$ and $Y$ are conserved. In weak interactions, where the transitions $d \rightarrow u(\Delta T_3 = 1, \Delta Y = 0)$ and $s \rightarrow u(\Delta T_3 = 1/2, \Delta Y = 1)$ are possible $T_3$ and $Y$ are not conserved but their violation has very definite character.

Strong interactions of quarks are due to their color charges and these charges are identical for $u,d,s$- quarks. Therefore strong interactions of $u,d,s$- quarks are identical. If the quark masses are equal $m_u = m_d = m_s$ then $u,d,s$- quarks will be indistinguishable by strong interactions i.e. it could be possible to change $u$-quark by $d$-quark or by $s$-quark in hadrons and neither masses nor strong interaction of these hadrons would be changed. In reality $m_u \neq m_d \neq m_s$ but the inequalities look small compared to the characteristic hadronic scale $1GeV$, especially small in the case of $u$ and $d$-quarks. Therefore, it is possible to expect an almost exact $SU(2)$-symmetry (isotopic spin)

$$ \begin{pmatrix} u \\ d \end{pmatrix} \rightarrow U(2) \begin{pmatrix} u \\ d \end{pmatrix} $$

and approximate $SU(3)$-symmetry

$$ \begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow U(3) \begin{pmatrix} u \\ d \\ s \end{pmatrix} $$

The results of the symmetries are the conserved charges-the generators of these symmetries. In the case of $SU(2)$-symmetry the conserved charges are presented by isotopic spin $T$. Note that $T_3$ is the diagonal generator of $SU(2)$-group and $T_3$ and $Y$ are the diagonal generators of $SU(3)$-group. The states of particles (hadrons) are classified by irreducible representations of these groups, in particular, for $SU(2)$-group the irreducible representations
are labeled by the square of isotopic spin $T^2 = T(T + 1)$ where $T = T_{3\text{max}}$ in the multiplet.

Let us classify the ground states of mesons and baryons by $T, T_3$ and $Y$.

### 2.2 Mesons

Basic states of mesons are presented by $9 \ q\bar{q}$ mesons with spin 0 (1). They are:

- **isotopic triplet of** $\pi(140)(\rho(770))$ – mesons
  
  $T = 1; T_3 = +1, 0, -1; Y = 0$
  
  \[
  \begin{pmatrix}
  \frac{1}{\sqrt{2}}(u\bar{d} - \bar{d}d) \\
  du
  \end{pmatrix} = \begin{pmatrix}
  \pi^+ \\
  \pi^0 \\
  \pi^-
  \end{pmatrix}, \begin{pmatrix}
  \rho^+ \\
  \rho^0 \\
  \rho^-
  \end{pmatrix};
  \]

- **isotopic doublets of** $K(490)(K^*(890))$ – mesons
  
  $T = 1/2; T_3 = +1/2, -1/2; Y = +1, -1$
  
  \[
  \begin{pmatrix}
  u\bar{s} \\
  d\bar{s}
  \end{pmatrix} = \begin{pmatrix}
  K^+ \\
  K^0
  \end{pmatrix}, \begin{pmatrix}
  K^{*+} \\
  K^{0*}
  \end{pmatrix};
  \]

  \[
  \begin{pmatrix}
  s\bar{d} \\
  s\bar{u}
  \end{pmatrix} = \begin{pmatrix}
  \bar{K}^0 \\
  K^-
  \end{pmatrix}, \begin{pmatrix}
  K^{0*} \\
  K^{*-}
  \end{pmatrix};
  \]

- **isotopic singlets** $\eta, \eta'$ ($T = 0, T_3 = 0, Y = 0$)
  
  \[
  \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) = \eta(550)
  \]

  \[
  \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) = \eta'(960)
  \]

  for scalar particles

- and isotopic singlets $\omega, \phi$ ($T = 0, T_3 = 0, Y = 0$)
  
  \[
  \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) = \omega(780)
  \]

  \[s\bar{s} = \phi(1020)\]

  for vector particles.
2.3 Baryons

Out of 27=3*3*3 flavor states of $qqq$-baryons with spin 1/2 and 3/2 the Fermi statistics admits only the octet of baryons with spin 1/2 ((8,1/2)) and decuplet of baryons with spin 3/2 ((10,3/2)). The octet of baryons is presented by: isotopic triplet of $\Sigma(1190)$ - hyperons

$$T = 1; T_3 = +1,0,-1; Y = 0$$

\[
\begin{pmatrix}
    uus \\
    uds \\
    dds
\end{pmatrix} = \begin{pmatrix}
    \Sigma^+ \\
    \Sigma^0 \\
    \Sigma^-
\end{pmatrix};
\]

isotopic doublets of nucleons ($N(939)$) and $\Xi(1320)$ - hyperons

$$T = 1/2; T_3 = +1/2,-1/2; Y = +1,-1$$

\[
\begin{pmatrix}
    uud \\
    ddu
\end{pmatrix} = \begin{pmatrix}
    p \\
    n
\end{pmatrix};
\]

\[
\begin{pmatrix}
    ssu \\
    ssd
\end{pmatrix} = \begin{pmatrix}
    \Xi^0 \\
    \Xi^-
\end{pmatrix};
\]

isotopic singlet ($T_3 = 0, Y = 0$)- $\Lambda(1116)$ - hyperon consisting of $uds$-quarks.

The decuplet of baryons with spin 3/2 is presented by: quartet of isobars $\Delta(1232)$

$$T = 3/2; T_3 = +3/2,+1/2,-1/2,-3/2; Y = 1$$

\[
\begin{pmatrix}
    uuu \\
    uud \\
    ddu \\
    ddd
\end{pmatrix} = \begin{pmatrix}
    \Delta^{++} \\
    \Delta^+ \\
    \Delta^0 \\
    \Delta^-
\end{pmatrix};
\]

isotopic triplet of $\Sigma^*(1385)$ - hyperons

$$T = 1; T_3 = +1,0,-1; Y = 0$$

\[
\begin{pmatrix}
    uus \\
    uds \\
    dds
\end{pmatrix} = \begin{pmatrix}
    \Sigma^{*+} \\
    \Sigma^{*0} \\
    \Sigma^{*-}
\end{pmatrix};
\]
isotopic doublet of $\Xi^*(1530)$ - hyperons

$$T = 1/2; T_3 = +1/2, -1/2; Y = -1$$

$$\begin{pmatrix} ssu \\ ssd \end{pmatrix} = \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix}$$

and isotopic singlet ($T_3 = 0, Y = -2$)- $\Omega^-(1672)$ - hyperon consisting of $sss$-quarks.

### 2.4 Decays

The lightest out of all hadrons $\pi^0$ - meson is decaying due to electromagnetic interaction to two $\gamma$ - quanta

$$\pi^0 \rightarrow 2\gamma$$

and lives about $10^{-16}$ seconds.

Its charged partners $\pi^\pm$ are decaying due to weak interaction

$$\pi^\pm \rightarrow \mu^\pm \nu_\mu (\bar{\nu}_\mu)$$

and live about $10^{-8}$ seconds.

$K$ -mesons are decaying due to weak interaction to $2\pi, 3\pi, ...$ and have lifetimes from $10^{-8}$ to $10^{-10}$ seconds.

$\eta$ - meson is decaying to $3\pi, 2\gamma, ...$ violating the isotopic symmetry. Its decay to $2\pi$ - mesons is forbidden by $CP$ - symmetry (see lecture 3 below).

Meson $\eta'$ is decaying to $\eta\pi\pi$ due to strong interaction.

Proton is stable. At present the lower bound of its lifetime is $10^{32}$ years.

Neutron has lifetime 887 seconds and decays to proton, electron and electron antineutrino. $\Lambda, \Sigma^\pm, \Xi^0, \Xi^-, \Omega^- -$ hyperons are decaying due to weak interaction, $\Sigma^0 -$ hyperon is decaying electromagnetically to $\Lambda\gamma$ and $\Delta, \Sigma^*, \Xi^* -$ particles are resonances i.e. are decaying due to strong interactions.

### 2.5 Problem 2

a) Why the decay $\Xi \rightarrow \Sigma\pi$ is not seen experimentally?

b) How has the hypercharge $Y$ to look like in order to formula $Q = T_3 + \frac{Y}{2}$ be valid for all hadrons including the hadrons with heavy $(c, b, t)$ - quarks?
3 Lecture 3. Discrete symmetries

In the present lecture we will turn to the discrete symmetries—space inversion, charge conjugation and G—transformation. These symmetries will help us to understand the peculiarities of the decays of $\eta, \eta', \rho, \omega, \phi$—mesons.

3.1 Space inversion

The transformation of space inversion is realized by the change of the reference frame from $t, x, y, z$ to $t, -x, -y, -z$. Various objects transform under this change of the reference frame differently. So, the polar vectors of space position and momentum ($\mathbf{r}, \mathbf{p}$) change their signs, the axial vector of angular momentum ($\mathbf{l}, l_i = \epsilon_{ijk} r_j p_k$) is unchanged. The scalar product of two polar vectors forms the scalar that is also unchanged under the transformation of space inversion but the scalar product of polar and axial vectors forms the pseudoscalar that change the sign under space inversion. The spatial parity ($P$) of some object is determined by its property to change or not to change the sign under space inversion. Scalar and axial vector are $P$—even but pseudoscalar and vector are $P$—odd.

Now we know that the Lagrangians of strong and electromagnetic interactions are not changed under the transformation of space inversion i.e. are the scalars but the Lagrangian of weak interaction is the sum of scalar and pseudoscalar. Invariance with respect to space inversion on the quantum-mechanical level means that the amplitude of the process and the amplitude of the space inverted process coincide

$$< b^P | V | a^P > = < b | P^{-1} V P | a > = < b | V | a > .$$

If the initial and final states have definite $P$—parity,

$$P | a > = p_a | a >$$

$$P | b > = p_b | b > ,$$

then $p_a \cdot p_b = 1$ or $p_a = p_b$ i.e. $P$—parity is conserved.

P—parities of the particles

Strong and electromagnetic interactions that conserve the $P$—parity conserve also $U, D, S$—charges i.e. the numbers of $u, d, s$—quarks.

Therefore, the quark parities $p_u, p_d, p_s$ can be chosen arbitrarily. It is accepted
to take them equal to 1: $p_u = p_d = p_s = 1$. One of the theorem of the theory of fermions based on the Dirac equation is the statement that the product of the internal parities of fermion and antifermion is equal to -1: $p_u p_{\bar{u}} = p_d p_{\bar{d}} = p_s p_{\bar{s}} = -1$. This means that $p_{\bar{u}} = p_{\bar{d}} = p_{\bar{s}} = -1$ for our convention $p_u = p_d = p_s = 1$. The internal parity of the system of two particles (of the meson consisting from quark and antiquark, for example) is equal to the product of the internal parities of the constituent particles and of the orbital parity of their relative motion

$$\Psi(r_q - r_{\bar{q}}) \xrightarrow{P} p_q p_{\bar{q}} \Psi(r_q - r_{\bar{q}}) = p_q p_{\bar{q}} (-1)^l \Psi(r_q - r_{\bar{q}}),$$

i.e.

$$P(M(q\bar{q})) = (-1)^{l+1},$$

where $l$ is the orbital momentum of relative motion of quarks. For ground states of mesons $l = 0$ and $P(M(q\bar{q})) = -1$ (pseudoscalar and vector mesons).

The internal parity of the system of three particles (of the baryon consisting from three quarks, for example) is equal to the product of the internal parities of the constituents, the parity of relative orbital motion of any pair of constituents and the parity of relative orbital motion of chosen pair and third constituent

$$p(qqq) = p_q p_{\bar{q}} p_q (-1)^l (-1)^L,$$

where $l$ is the orbital momentum of the pair, $L$ is the relative orbital momentum of the pair and third particle. For ground states of baryons $l = L = 0$ and $P(B(qqq)) = 1$.

### 3.2 Charge conjugation

Charge conjugation, by definition, transforms particle to antiparticle, all the charges of which have different sign with respect to the charges of particle. Truly neutral particles (not only electrically neutral) under $C$ - conjugation transform to themselves what allows to introduce the notion of $C$ - parity. For example, wave function (or the state) of some particles ($\gamma, \rho^0, \omega, \phi$) changes the sign under charge conjugation

$$C \gamma = -\gamma, \quad C \gamma = -1, ...,$$
wave function of other particles \((\eta, \eta', \pi^0)\) does not change the sign under charge conjugation

\[ C \eta = \eta, C_{\eta'} = 1, \ldots. \]

\(C\) - parity of mesons constructed from quark and antiquark is determined by the relative orbital momentum of quark and antiquark and their total spin \(s\)

\[
\begin{align*}
C \psi(r_q - r_{\bar{q}}) \psi(s_q, s_{\bar{q}}) a_q^+ (r_q, s_q) a_{\bar{q}}^+ (r_{\bar{q}}, s_{\bar{q}}) |0> &= -C \psi(r_q - r_{\bar{q}}) \psi(s_q, s_{\bar{q}}) a_q^+ (r_q, s_q) a_{\bar{q}}^+ (r_{\bar{q}}, s_{\bar{q}}) |0> \\
&= (-1)^{1+l+s+1} \psi(r_q - r_{\bar{q}}) \psi(s_q, s_{\bar{q}}) a_q^+ (r_q, s_q) a_{\bar{q}}^+ (r_{\bar{q}}, s_{\bar{q}}) |0>
\end{align*}
\]

(the summation over the variables \(r_q, r_{\bar{q}}, s_q, s_{\bar{q}}\) is here assumed). In the above transformations we have taken into account that fermion operators anticommute, that the interchange of quark and antiquark coordinates results in the multiplier \((-1)^l\) and that the interchange of quark and antiquark spins gives multiplier \((-1)^{s+1}\). So, \(C\) - parity of meson \(M = q\bar{q}\) is equal to

\[ C(M(q\bar{q})) = (-1)^{l+s}, \]

i.e. is negative for vector mesons \((l = 0, s = 1; \rho_0, \omega, \phi)\) and positive for pseudoscalar mesons \((l = 0, s = 0; \pi^0, \eta, \eta')\).

3.3 \(G\) - transformation

Under the charge conjugation the components of isotopic triplet of \(\pi\) - mesons transform as follows

\[
\begin{align*}
\pi^0 & \rightarrow \pi^0 & \pi^\pm & \rightarrow \pi^\mp.
\end{align*}
\]

It can be seen that charged \(\pi\) - mesons are not the eigenstates of charge conjugation operator, so, no \(C\) - parity can be ascribed to them. It is possible however to join charge conjugation with some rotation in isotopic space in order to both neutral and charged \(\pi\) - mesons be eigenstates with respect to the combined transformation \((G\) - transformation). The resulting \(G\) - parity is conserved in strong interaction in the same degree as the charge
conjugation and isotopic rotations are the symmetries of strong interaction. The latter (the symmetry of isotopic rotations) is violated only slightly by the difference of \(u\) and \(d\) - quark masses, so, \(G\) - transformation is almost an exact symmetry of strong interaction. The desired rotation in isotopic space is the rotation around the 2-nd axis on the angle \(180^\circ\)

\[
\begin{align*}
\pi^0 & \xrightarrow{T_2^{180}} -\pi^0 & \pi^\pm & \xrightarrow{T_2^{180}} -\pi^\mp
\end{align*}
\]

Then

\[ G : \pi = CT_2^{180} : \pi = -\pi, \]

i.e. \(G_\pi = -1\). For other particles \(G\) -parity can be easily determined

\[ G_\eta = +1, \quad G_\rho = +1, \quad G_\omega = -1, \quad G_\phi = -1. \]

### 3.4 Decays

Let us apply the discussed above discrete symmetries to the decays of \(\eta, \rho, \omega\) - mesons. The following table of forbidden by appropriate symmetry decays is obvious

| \(\eta\) | \(\pi\pi\) | \(P, CP\) |
|---|---|---|
| \(\eta\) | \(\pi\pi\pi\) | \(G\) |
| \(\eta\) | \(\pi^0\gamma\) | \(C\) |
| \(\eta\) | \(\pi^0\pi^0\gamma\) | \(C\) |
| \(\rho\) | \(\pi^0\pi^0\) | \(C, Bose\) |
| \(\rho\) | \(\pi\pi\pi\) | \(G\) |
| \(\omega\) | \(\pi\pi\) | \(G\) |
| \(\omega\) | \(\pi^0\pi^0\pi^0\) | \(C\) |

### 3.5 Problem 3

Write allowed and forbidden quantum numbers \(J^{PC}\) for \(q\bar{q}\) - system.
4 Lecture 4. Isotopic symmetry

4.1 Quarks and antiquarks

Isotopic symmetry of strong interaction is the result of the approximate equality of $u$ and $d$ - quark masses ($m_u = 4MeV, m_d = 7MeV, m_u \approx m_d$) from the point of view of characteristic hadronic scale ($1GeV$)

$$\Delta m_{ud} << 1GeV$$

and is manifested in the invariance of strong interaction for $m_u = m_d$ under isotopic $SU(2)$ - transformations

$$\left( \begin{array}{c} u \\ d \end{array} \right) \rightarrow U(2) \left( \begin{array}{c} u \\ d \end{array} \right).$$

The lowest (fundamental) representation of $SU(2)$ - group is the spinor $\Psi^\alpha; \alpha = 1, 2$, the doublet of quarks, for example. The $SU(2)$ - group

$$U^+ U = I \quad detU = 1$$

has three generators $\frac{\tau}{2}$

$$U = exp(i \omega \frac{\tau}{2}),$$

one of which ($\frac{\tau_3}{2}$) can be made diagonal. The matrices $\tau_i$ are hermitian (unitarity of matrix $U$), traceless (the consequence of the constraint $detU = 1$) and have the form

$$\tau_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \tau_2 = \left( \begin{array}{cc} 0 -i \\ i & 0 \end{array} \right) \tau_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$$

They have the following commutation, anticommutation and normalization relations

$$\left[ \frac{\tau_i}{2}, \frac{\tau_j}{2} \right] = i \epsilon_{ijk} \frac{\tau_k}{2},$$

$$\{\tau_i, \tau_j\} = 2 \delta_{ij} I,$$

$$Tr(\tau_i \tau_j) = 2 \delta_{ij}.$$  

For other representations of $SU(2)$ - group the $SU(2)$ - transformations have similar structure

$$exp(i \omega T),$$
\[ [T_i, T_j] = i\epsilon_{ijk} T_k. \]

The quark doublet is transformed like contravariant spinor

\[ q^\alpha = U^\alpha_\beta q^\beta. \]

Antiquarks are transformed by complex conjugate matrix \( U^* \)

\[ q^{\alpha*} = U^{\alpha*}_\beta q^{\beta*} \]

or like covariant spinor

\[ \tilde{q}'_\alpha = \tilde{q}_\beta U^{-1\beta}_\alpha = \tilde{q}_\beta U^{+\beta}_\alpha = \tilde{q}_\beta U^{*\alpha}_\beta. \]

The \( SU(2) \) - group has two invariant tensors

\[ \delta^{\alpha}_{\beta} \rightarrow U^\alpha_{\alpha'} \delta^{\alpha'}_{\beta} U^{-1\beta'}_\beta = \delta^{\alpha}_{\beta}, \]

\[ \epsilon^{\alpha\beta} \rightarrow U^\alpha_{\alpha'} U^\beta_{\beta'} \epsilon^{\alpha'}_{\beta'} = \epsilon^{\alpha\beta} (\det U) = \epsilon^{\alpha\beta}. \]

Using the second invariant tensor \( \epsilon^{\alpha\beta} \) the covariant spinor can be transformed to the contravariant one i.e. quarks and antiquarks are transformed as unitary equivalent representations

\[ \tilde{q}_\alpha = (\tilde{u}, \tilde{d}) \rightarrow \begin{pmatrix} \tilde{d} \\ -\tilde{u} \end{pmatrix} = \tilde{q}^\beta = \epsilon^{\beta\alpha} \tilde{q}_\alpha. \]

### 4.2 Diquarks and mesons

The state space (isotopic) of two quarks has four basis elements \( q^\alpha q^\beta \). Isotopic wave functions of diquarks, the tensors \( Q^{\alpha\beta} \), form the state space that is reducible with respect to transformations of \( SU(2) \) - group. It can be divided on two invariant (irreducible) subspaces: \( S^{\alpha\beta} \) - subspace of symmetric tensors and \( A^{\alpha\beta} = \epsilon^{\alpha\beta} A \) - subspace of antisymmetric tensors. The subspace \( S^{\alpha\beta} \) has three basis elements: \( uu, \frac{1}{\sqrt{2}}(uu + dd), dd \). They form triplet \( (T = 1) \) where \( T_3 = 1, 0, -1 \), correspondingly. The subspace \( A^{\alpha\beta} \) is one dimensional with one basis element \( \frac{1}{\sqrt{2}}(uu - dd) \). It represent an isotopic singlet \( (T = 0) \).

Analogously, in the isotopic space of states of quark and antiquark (mesons) we have the basis elements \( q^\alpha \tilde{q}_\beta \) and isotopic wave functions \( Q^{\alpha}_{\beta} = Q^{\alpha}_{0\beta} + \).
\[ \delta^{\alpha\beta} Q(Q^\alpha_\alpha = 0) \]. Three basis elements of \( Q^\alpha_{0\beta} \) - subspace \((u\bar{d}, \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}), d\bar{u})\) form isotopic triplet and one basis element of \( \delta^{\alpha\beta} Q \) - subspace \((\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}))\) forms isotopic singlet. Let us write (having in mind pseudoscalar mesons) the triplet \( Q^\alpha_{0\beta} \) as follows

\[
Q^\alpha_{0\beta} = \left( \frac{\pi_0}{\sqrt{2}}, \frac{\pi^+}{\sqrt{2}}, \pi^- \right)
\]

and compare this presentation of tensor \( Q^\alpha_{0\beta} \) with another one where the isotopic vector \( \pi \) is explicitly introduced

\[
Q^\alpha_{0\beta} = \frac{1}{\sqrt{2}} \pi \tau^\alpha_{\beta} = \left( \frac{\pi_3}{\sqrt{2}} \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2), -\frac{\pi_3}{\sqrt{2}} \right).
\]

As a result we have the following one to one correspondence between the two forms of the representation of isotopic triplet

\[
\pi^\pm = \frac{1}{\sqrt{2}}(\pi_1 \pm i\pi_2), \pi_0 = \pi_3,
\]

that was already use in the preceding lecture.

### 4.3 Isotopic states in the systems of two and three pions

Two pions \( \pi_1, \pi_2 \) can have the total isospin equal to 0, 1, 2 \((3 \ast 3 = 1 + 3 + 5)\). An explicit construction of the isotopic wave function of the system of two pions

\[
T = 0 \quad \pi_1 \cdot \pi_2 = \delta_{ij}\pi_{1i}\pi_{2j} \quad S
\]

\[
T = 1 \quad (\pi_1 \times \pi_2)_k = \epsilon_{ijk}\pi_{1i}\pi_{2j} \quad A
\]

\[
T = 2 \quad \pi_{1i}\pi_{2j} + \pi_{1j}\pi_{2i} - \frac{2}{3}\delta_{ij}\pi_{1k}\pi_{2k} \quad S
\]

shows that the isotopic wave function of the system of two pions is symmetric \((S)\) for the total isospin \( T = 0, 2 \) and antisymmetric \((A)\) for \( T = 1 \). Therefore, in accordance with Bose principle the orbital angular momentum of two pions \((l)\) can be even for \( T = 0, 2 \) and odd for \( T = 1 \). So, \( \omega \) - meson \((T = 0, S = 1)\) cannot decay to two pions due to the isospin conserving part
of strong interaction. In the previous lecture we have seen that the decay \( \omega \to 2\pi \) is forbidden by the conservation of \( G \) - parity, now we have looked at this problem from another side.

For the system of three pions \( \pi_1, \pi_2, \pi_3 \) where \( 3 \ast 3 \ast 3 = (1 + 3 + 5) \ast 3 = 3 + (1 + 3 + 5) + (3 + 5 + 7) \) we have one singlet

\[
\pi_1 \cdot \pi_2 \times \pi_3 = \epsilon_{ijk} \pi_i \pi_j \pi_k
\]

three triplets one of which can be chosen with symmetric wave function

\[
\pi_1 (\pi_2 \cdot \pi_3) + \pi_2 (\pi_3 \cdot \pi_1) + \pi_3 (\pi_1 \cdot \pi_2)
\]

etc. If one assumes the conservation of isotopic spin in the decay \( \eta \to 3\pi \) \( (T = 0) \) then in accordance with Bose principle the coordinate/momentum wave function of three pions should be antisymmetric. This leads to the negative \( C \) - parity of the system of three pions and to the violation of \( C \) - parity in the decay. If one assumes that the coordinate/momentum wave function of three pions in the considered decay is symmetric (there is no relative orbital motion of pions due to small phase space, for example) then \( C \) - parity is conserved but the total isospin of pions cannot equal to zero i.e. isospin is not conserved. Before (see lecture 3) we have seen that \( G = CT_2^{180} \) - parity is violated in \( \eta \to 3\pi \) decay, the presented here consideration so to say disentangle this violation.

### 4.4 Pion-nucleon scattering

Let us apply the isotopic symmetry to the pion-nucleon scattering. There are 10 processes

\[
\begin{align*}
\pi^+ p &\to \pi^+ p & \pi^- n &\to \pi^- n \\
\pi^0 p &\to \pi^0 p & T_2^{180} &\pi^0 n &\to \pi^0 n \\
\pi^- p &\to \pi^- p & \pi^+ n &\to \pi^+ n \\
\pi^0 p &\to \pi^+ n & \pi^0 n &\to \pi^- p \\
\pi^- p &\to \pi^0 n & \pi^+ n &\to \pi^0 p ,
\end{align*}
\]

where the processes in the right column are isotopically connected to the processes in the left column by the 180° rotation around the 2-nd axis in isotopic space. So, we have to consider only the processes in the left column. The fourth process of the left column is time reverse of the fifth process of
the right column, so, its amplitude is equal to the amplitude of the fifth process (no matter of which column). The second process is not observed experimentally because of the lack of $\pi^0$ - beams. The remaining three processes $\pi^+ p \rightarrow \pi^+ p, \pi^- p \rightarrow \pi^- p, \pi^- p \rightarrow \pi^0 n$ have the amplitudes $M_+, M_-, M_0$ connected by isotopic symmetry. Considering the states of pion and nucleon with definite total isospin one gets

$$\pi^+ p = \left(\frac{3}{2}, \frac{3}{2}\right)$$

$$\pi^- p = \sqrt{\frac{1}{3}}(\frac{3}{2}, \frac{1}{2}) - \sqrt{\frac{2}{3}}(\frac{1}{2}, -\frac{1}{2})$$

$$\pi^0 n = \sqrt{\frac{2}{3}}(\frac{3}{2}, -\frac{1}{2}) + \sqrt{\frac{1}{3}}(\frac{1}{2}, -\frac{1}{2})$$

Due to isotopic symmetry the amplitude of pion-nucleon scattering does not depend on the value of third component of isospin i.e. we have only two independent amplitudes $M_{3/2}$ and $M_{1/2}$. Thus $M_+, M_-, M_0$ are expressed through $M_{3/2}$ and $M_{1/2}$

$$M_+ = M_{3/2}$$

$$M_- = \frac{1}{3}M_{3/2} + \frac{2}{3}M_{12}$$

$$M_0 = \frac{\sqrt{2}}{3}(M_{3/2} - M_{1/2})$$

and

$$\sqrt{2}M_0 + M_- = M_+.$$ 

The last relation is just what we wanted to obtain. For the energy of incoming pion corresponding to the $\Delta$ - resonance region one amplitude $M_{3/2}$ is much larger than the other $M_{1/2}$. In that case

$$M_+ : M_- : M_0 \approx 1 : \frac{1}{3} : \frac{\sqrt{2}}{3},$$

and the cross sections follow the ratios

$$\sigma_+ : \sigma_- : \sigma_0 \approx 9 : 1 : 2.$$ 

These ratios are in good agreement with experiment.
4.5 Problem 4

Find the ratio of the widths of $\eta \to \pi^0\pi^0\pi^0$ and $\eta \to \pi^+\pi^-\pi^0$ decays assuming that the total isospin of three pions is equal to $T = 1$ and that the coordinate/momentum wave function of pions is symmetric (see lecture).
5 $SU(3)$ - symmetry

5.1 Fundamental representation

The comparison of $u, d, s$ - quark masses with characteristic hadronic scale ($m_u = 4MeV, m_d = 7MeV, m_s = 150MeV << 1GeV$) tells us that $SU(3)$ - symmetry

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow U(3) \begin{pmatrix} u \\ d \\ s \end{pmatrix},$$

can be considered as an approximate symmetry of strong interaction. Mainly it is violated by the mass difference of strange ($s$) and nonstrange ($u$ and $d$) quarks. The consequences of just this reason of violation of $SU(3)$ - symmetry are the mass formulae for ground states of mesons and baryons that will be considered in the next lecture. In the present lecture we will classify the $SU(3)$ - multiplets of mesons and baryons.

The fundamental representation of $SU(3)$ - group is the triplet of quarks

$$q^\alpha = \begin{pmatrix} u \\ d \\ s \end{pmatrix}.$$ 

The $SU(3)$ - transformations of the triplet are produced by the unitary matrices $3\times3$

$$U = exp(i\omega^a\lambda^a),$$

where $\lambda^a$ are Gell-Mann matrices

$$\lambda^{1,2,3} = \begin{pmatrix} \tau^{1,2,3} & 0 \\ 0 & 0 \end{pmatrix}; \lambda^{4,5} = \begin{pmatrix} 0 & 0 & 1(-i) \\ 0 & 0 & 0 \\ 1(+i) & 0 & 0 \end{pmatrix},$$

$$\lambda^{6,7} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1(-i) \\ 0 & 1(+i) & 0 \end{pmatrix}; \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$ 

The $SU(3)$ - group has two diagonal generators: the third component of isotopic spin and hypercharge,

$$T_3 = \frac{\lambda^3}{2} \quad Y = \frac{1}{\sqrt{3}} \lambda^8.$$
that are conserved in strong and electromagnetic interactions (see lecture 2).

Antiquarks are transformed as complex conjugate of the fundamental (contravariant spinor) representation and these transformations coincide with the transformations of the covariant spinor due to the unitarity of the group. Unlike the case of $SU(2)$ - group the covariant spinor representation is not now equivalent to the contravariant representation (quarks and antiquarks transform nonequivalently).

The $SU(3)$ - group has two invariant tensors

$$\delta^{\alpha \beta} \rightarrow U^{\alpha \alpha'} \delta^{\alpha' \beta'} U^{-1 \beta} = \delta^{\alpha \beta},$$

$$\epsilon^{\alpha \beta \gamma} \rightarrow U^{\alpha \alpha'} U^{\beta \beta'} U^{\gamma \gamma'} \epsilon^{\alpha' \beta' \gamma'} = \epsilon^{\alpha \beta \gamma} (\det U) = \epsilon^{\alpha \beta \gamma}.$$

5.2 Mesons

Mesons consist of quark and antiquark and are described by the tensor (flavor wave function)

$$Q_{\alpha \beta}^\star = Q_{\alpha \beta}^0 + \frac{1}{3} \delta_{\alpha \beta} Q \quad (Q_{\alpha 0}^0 = 0),$$

two parts of which represent the octet and singlet states. The basis states of the octet are (we indicate pseudoscalar mesons as an example):

isotopic triplet of $\pi$ - mesons

$$u\bar{d}, \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}), d\bar{u};$$

isotopic doublets of $K$ - mesons

$$u\bar{s}, d\bar{s}; \quad s\bar{d}, s\bar{u}$$

and isotopic singlet $\eta_8$

$$\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}).$$

The basis state of the singlet is $\eta_0$

$$\eta_0 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}).$$
Using the above octet basis states the octet part of tensor $Q^\alpha_\beta (Q^0_{0,\beta})$ can be represented in the following form

$$Q^0_{0,\beta} = \begin{pmatrix}
\frac{\pi^0}{\sqrt{2}} + \frac{\eta^s}{\sqrt{6}} & \frac{\pi^+}{\sqrt{2}} & K^+ \\
\pi^- & -\frac{2\eta^s}{\sqrt{6}} & K^0 \\
K^- & \frac{\eta^s}{\sqrt{6}} & -\frac{2\eta^s}{\sqrt{6}}
\end{pmatrix}. $$

5.3 Baryons

Baryons consist of three quarks. Their wave function presented by the tensor $B^{\alpha\beta\gamma}$ is reducible. Let us briefly describe the division of tensor $B^{\alpha\beta\gamma}$ in irreducible parts. First, let us divide tensor $B^{\alpha\beta\gamma}$ in two parts one of which is symmetric and another antisymmetric in indexes $\alpha\beta$

$$B^{\alpha\beta\gamma} = B^{\{\alpha\beta\}}\gamma + B^{[\alpha\beta]\gamma}. $$

With the help of invariant tensor $\epsilon_{\alpha\beta\gamma}$ the antisymmetric part can be presented in the equivalent form

$$B^{[\alpha\beta]\gamma} \to B^{\gamma\gamma'} = \epsilon_{\alpha\beta\gamma'} B^{[\alpha\beta]\gamma}. $$

Like tensor $Q^\alpha_\beta$ for mesons tensor $B^{\gamma\gamma'}$ is divided in octet and singlet parts

$$B^{\gamma\gamma'} = B^{0}_{0,\gamma'} + \frac{1}{3} \delta^{\gamma\gamma'}\epsilon_{\alpha\beta\delta} B^{[\alpha\beta]\delta}. $$

Coming back to tensor

$$B^{[\alpha\beta]\gamma} = \frac{1}{2} \epsilon^{\alpha\beta\gamma'} B^{\gamma\gamma'},$$

we see that the singlet is described by tensor

$$\frac{1}{6} \epsilon^{\alpha\beta\gamma} B,$$

where

$$B = \epsilon_{\alpha\beta\gamma} B^{[\alpha\beta]\gamma}.$$ 

So, the singlet is presented by totally antisymmetric part of the initial tensor $B^{\alpha\beta\gamma}$. 

23
The symmetric part $B^{(\alpha\beta)\gamma}$ can be further symmetrized or antisymmetrized in indexes $\beta\gamma$

$$B^{(\alpha\beta)\gamma} = B^{(\alpha\beta\gamma)} + B^{(\alpha[\beta)\gamma]}.$$  

Symmetrization in indexes $\beta\gamma$ gives totally symmetric tensor $B^{(\alpha\beta\gamma)}$ ten independent components of which form the irreducible representation of baryon-decuplet. Antisymmetrization in indexes $\beta\gamma$ gives tensor $B^{(\alpha[\beta)\gamma]}$ that can be transformed to the equivalent tensor $B'^{\alpha\alpha'}$ as follows

$$B^{(\alpha[\beta)\gamma]} \to B'^{\alpha\alpha'} = \epsilon_{\alpha'\beta\gamma} B^{(\alpha[\beta)\gamma]}.$$  

Because the trace of tensor $B'^{\alpha\alpha'}$ is equal to zero ($B'^{\alpha\alpha} = \epsilon_{\alpha\beta\gamma} B^{(\alpha[\beta)\gamma]} = 0$) it represents a pure octet.

Finally, we have divided the wave function $B^{\alpha\beta\gamma}$ in symmetric decuplet, antisymmetric singlet and two octets with mixed symmetry. Let us write the result in the following form

$$3 \times 3 \times 3 = 10 + 8' + 8 + 1.$$  

If one compares this result with the experimentally observed multiplets of baryon ground states (decuplet of baryons with spin 3/2 and octet of baryons with spin 1/2) it can be noted that the octet of baryons is singly presented and the singlet baryon is absent at all. As we will see below the reason of such incompleteness is the Pauli principle for constituent quarks. Let us write the baryon wave function in the following form

$$\Psi = \Psi(x_1, x_2, x_3) \times \Psi(c_1, c_2, c_3) \times B^{\alpha\beta\gamma} \times \Psi(s_1, s_2, s_3),$$  

where the first multiplier is the coordinate wave function that is assumed to be symmetric for the ground state of the baryon;

the second multiplier is the color wave function that is assumed to be antisymmetric in $SU(3)$ color indexes $c_1, c_2, c_3$ (baryons like all hadrons are assumed to be singlets of $SU(3)$ - color group; compare the antisymmetry of color wave function for $SU(3)$ color singlet with the antisymmetry of flavor wave function for $SU(3)$ flavor singlet).
the third multiplier $B^{\alpha \beta \gamma}$ is flavor wave function the symmetry of which was just described above;
the fourth multiplier $\Psi(s_1, s_2, s_3)$ is the spin wave function the symmetry of which can be described in the same way as the symmetry of flavor wave function

$$2 \times 2 \times 2 = 4 + 2^r + 2.$$  

The antisymmetry of the total wave function (Pauli principle) will be reached if the product of flavor and spin wave functions is symmetric. One way to make this product symmetric is obvious: the decuplet of spin 3/2. Another way, the octet of spin 1/2, is not so obvious. Let us clarify it by the following simple argument. If one considers the flavor and spin variables simultaneously the common index will take 6 values: $u, d, s$ - quarks with up and down spins. The symmetric in common flavor-spin indexes tensor (flavor-spin wave function) contains $6 \cdot 7 \cdot 8 = 56$ independent components. Forty of them correspond to the decuplet of spin 3/2 ($10 \cdot 4$). There remain 16 components that cannot to represent the octet of spin 1/2 ($16 = 8 \cdot 2$).
So, the Pauli principle selects the possible flavor multiplets and spins of ground state baryons. Note once more that there is no place for flavor singlet among the ground state baryons.

### 5.4 Two presentations of baryon octet

As we have seen in the example of mesons the octet of particles can be described by the traceless matrix $3 \times 3$. Let us describe in such a way the octet of baryons. The problem what particle should be putted in one or another place in the matrix can be solved uniquely by the comparison with meson matrix and by the observation that the third component of isotopic spin and the electric charge of compared particles should be equal ($T_3$ and $Q = T_3 + Y/2$ are both the generators of $SU(3)$ - group) Hence we have the correspondence

$$
\begin{pmatrix}
\frac{\pi_0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\
\pi^- & \frac{\pi_0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\
K^- & -\frac{\pi_0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \bar{K}^0 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\frac{\Sigma_0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Lambda^+ & \Lambda \\
\Sigma^- & \frac{\Sigma_0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Xi^- \\
\Xi^- & -\frac{\Sigma_0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\
\end{pmatrix}
$$

25
This description of baryon octet will be used in the following lecture for the
derivation of mass formulae for baryon octet.

Another presentation of baryon octet refers directly to the quark structure
of baryons. Let us construct the proton state, for example. The proton
consists of two \( u \)-quarks and one \( d \)-quark i.e. its flavor composition is \( uud \).
According to Pauli principle the total spin of two \( u \)-quarks should be equal
to 1 (see the symmetry of flavor-spin wave function stated above). Let us
sum the unit spin of two \( u \)-quarks with 1/2 spin of \( d \)-quark to obtain 1/2
spin of proton

\[
\hat{p} = \sqrt{\frac{2}{3}} \hat{u} \hat{u} \hat{d} - \sqrt{\frac{1}{3}} \sqrt{\frac{1}{2}} (\hat{u} \hat{n} + \hat{n} \hat{u}) \hat{d}
\]

(here the point above (below) the letter means spin up (down) for quark)
and symmetrize the obtained state by permutations of \( d \)-quark

\[
\hat{p} = \sqrt{\frac{1}{18}} (2(\hat{u} \hat{d} \hat{d} + \hat{u} \hat{d} \hat{u} + \hat{d} \hat{u} \hat{u}) - (\hat{u} \hat{d} \hat{u} + \hat{u} \hat{d} \hat{u} + \hat{d} \hat{u} \hat{u} + \hat{u} \hat{d} \hat{u} + \hat{d} \hat{u} \hat{u} + \hat{u} \hat{d} \hat{u}))
\]

The obtained description of baryon octet will be used in the lecture on mag-
netic moments of baryons.

5.5 Problem 5

Classify the baryons with one \( c \)-quark (\( cqq \)) in possible \( SU(3) \) - multiplets
and spin.
6 Lecture 6. Mass formulae. Mixing

Last lecture we were classifying the $SU(3)$ - multiplets of ground states of mesons $q\overline{q}$ and baryons $qqq$. These were the octets and singlets of pseudoscalar and vector mesons and the octet and decuplet of baryons. If $SU(3)$ - symmetry were exact symmetry the masses of particles in each multiplet would be equal to each other. In the real world the $SU(3)$ - symmetry is an approximate symmetry and the masses of particles in the multiplets are essentially different:

$$P^{\alpha\beta}(\begin{array}{cccc} \pi & K & \eta & \eta' \\ 140 & 490 & 550 & 960 \end{array})$$

$$V^{\alpha\beta}(\begin{array}{cccc} \rho & K^* & \omega & \phi \\ 770 & 890 & 780 & 1020 \end{array})$$

$$B^{\alpha\beta}(\begin{array}{cccc} N & \Lambda & \Sigma & \Xi \\ 940 & 1116 & 1190 & 1320 \end{array})$$

$$D^{\alpha\beta\gamma}(\begin{array}{cccc} \Delta & \Sigma^* & \Xi^* & \Omega \\ 1230 & 1380 & 1530 & 1670 \end{array})$$

The violation of $SU(3)$ - symmetry initiates on the quark level and is connected to the mass difference of nonstrange $u,d$ - quarks (4,7MeV) and strange $s$ - quark (150MeV). The consequences of just this mechanism of symmetry violation are the mass formulae that can be deduced by considerations of symmetry properties of quark Lagrangian only.

On the quark level the Lagrangian violating the $SU(3)$ - symmetry is mass Lagrangian of quarks

$$L^q_m = m_q(\overline{u}u + \overline{d}d) + m_s(\overline{s}s) + (m_s - m_q)\overline{s}s.$$ 

Here we have neglected the violation of isotopic symmetry connected to the mass difference of nonstrange $u,d$ - quarks (4,7MeV). Its account (see the problem in the end of this lecture) without an account of electromagnetic
interaction violating both $SU(3)$ and isotopic symmetries would be unjustified.

The mass Lagrangian of quarks is presented by the sum of two terms: $SU(3)$ symmetrical term (singlet under $SU(3)$ - group transformation) and $SU(3)$ violating term (transforming like $(3,3)$ component of the tensor under $SU(3)$ - group transformation)

$$L^q_0 = m_q(\bar{u}u + \bar{d}d + \bar{s}s)$$
$$L^q_{m3} = (m_u - m_q)\bar{s}s.$$ 

It is reasonable to expect that on the hadronic level the mass Lagrangians of hadrons will also contain two terms obeying the same transformation properties under $SU(3)$ - group transformations.

### 6.1 Octet of baryons

First, let us consider the octet of baryons with spin 1/2

$$B^\alpha_\beta = \begin{pmatrix} \Sigma^+ & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \Sigma^- & -\frac{\sqrt{2}}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \Xi^- & -\frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{pmatrix}. $$

The singlet under $SU(3)$ - group transformations mass Lagrangian of baryon octet is unique - the masses of all baryons of the octet ($m^0_B$) are equal in this Lagrangian

$$L^B_{m0} = m^0_B(\bar{p}p + \bar{n}n + \cdots + \bar{\Lambda}\Lambda).$$

The $SU(3)$ violating part of mass Lagrangian of baryon octet, $(3,3)$ component of the tensor, can be presented in two forms, hence, there are two independent mass parameters ($m^8_B$ and $m^8_B'$)

$$L^B_{m3} = m^8_B\bar{B}^3_\beta B^3_\alpha = m^8_B(\bar{p}p + \bar{n}n + 2\frac{2}{3}\bar{\Lambda}\Lambda) + m^8_B'(\bar{\Xi}^-\Xi^- + \bar{\Xi}^0\Xi^0 + \frac{2}{3}\bar{\Lambda}\Lambda).$$

The masses of baryons are contributed by both $L^B_{m0}$ and $L^B_{m3}$

$$m_p = m_n = m^0_B + m^8_B.$$
\[ m_{\Xi^-} = m_{\Xi^0} = m_B^0 + m_B^{8'} \]
\[ m_{\Sigma^+} = m_{\Sigma^0} = m_{\Sigma^-} = m_B^0 \]
\[ m_A = m_B^0 + \frac{2}{3}(m_B^8 + m_B^{8'}) \].

So, four baryon masses (the violation of isotopic symmetry is not considered) are expressed through three parameters. Thus, there exits the constraint

\[ 3m_A + m_{\Sigma} = 2 (m_N + m_{\Xi}) \],

the so called Gell-Mann - Okubo mass formula. The left handed part of this formula is equal to

\[ 3 \cdot 1116 + 1193 = 3348 + 1193 = 4541, \]

whereas the right handed part of this formula is equal to

\[ 2 \cdot (939 + 1318) = 2 \cdot 2257 = 4517. \]

If we have used the Gell-Mann - Okubo mass formula for the prediction of the mass of \( \Lambda \)-hyperon we would obtain the result \( m_A = 1107 \text{MeV} \) that is only 9\( \text{MeV} \) deviated from the experimental value. The difference is comparable with electromagnetic/isotopic mass differences. Thus we can conclude that the accuracy of Gell-Mann - Okubo mass formula is very large.

### 6.2 Decuplet of baryons

Let us now consider the decuplet of baryons \( D^{\alpha\beta\gamma} \). The invariant part of mass Lagrangian for decuplet particles can be written as follows

\[
L_{m_D}^{D0} = m_D^0 \bar{D}_{\alpha\beta\gamma} D^{\alpha\beta\gamma} = m_D^0 \{(111) + (222) + (333) \\
+ 3[(112) + (113) + (221) + (223) + (331) + (332)] + 6(123)\},
\]

where the notation \((\alpha\beta\gamma)\) means \( \bar{D}_{\alpha\beta\gamma} D^{\alpha\beta\gamma} \) with no summation over indexes. Taking into account the equality of the masses of decuplet particles in this invariant part of mass Lagrangian we obtain the following correspondence
between the components of tensor $D^{\alpha\beta\gamma}$ and the fields of the decuplet particles

$$
\begin{align*}
\Delta^{++} & \leftrightarrow D^{111} \\
\Delta^{+} & \leftrightarrow D^{112} \\
\Delta^{0} & \leftrightarrow D^{221} \\
\Delta^{-} & \leftrightarrow D^{222}
\end{align*}
\begin{align*}
\Sigma^{+} & \leftrightarrow D^{113} \\
\Sigma^{0} & \leftrightarrow D^{123} \\
\Xi^{0} & \leftrightarrow D^{223}
\end{align*}
\begin{align*}
\Omega^{-} & \leftrightarrow D^{331} \\
\Xi^{-} & \leftrightarrow D^{332} \\
\Xi^{-} & \leftrightarrow D^{333}.
\end{align*}
$$

$SU(3)$ violating part of mass Lagrangian for decuplet particles is described by the unique structure

$$L_{m3}^{D} = m_{D}^{8} \bar{D}_{3\alpha\beta}D^{3\alpha\beta} = m_{D}^{8}\{(311) + (322) + (333) + 2[(312) + (313) + (323)]\}.$$  

Taking into account the above stated correspondence between the components of tensor $D^{\alpha\beta\gamma}$ and the fields of the decuplet particles we get for the masses of decuplet particles the following results

$$m_{\Delta} = m_{D}^{0} \quad (1230)$$

$$m_{\Sigma^{*}} = m_{D}^{0} + m_{D}^{8}/3 \quad (1380)$$

$$m_{\Xi^{*}} = m_{D}^{0} + 2m_{D}^{8}/3 \quad (1520)$$

$$m_{\Omega} = m_{D}^{0} + m_{D}^{8} \quad (1670).$$

The masses of particles are equidistant in a good agreement with the experiment.

### 6.3 Mesons. Mixing

Turning now to mesons let us note two peculiarities of this case. First, the mass Lagrangian of mesons contains as parameters the masses of mesons squared in contrast with baryon case where the mass Lagrangian was linear in mass parameters. Therefore the mass formulae for mesons will relate the masses of mesons squared. So, the Gell-Mann - Okubo formula transforms in the case of pseudoscalar and vector mesons to the following formulae

$$3m_{\eta_{s}}^{2} + m_{\pi}^{2} = 4m_{K}^{2}$$

$$3m_{\omega_{s}}^{2} + m_{\rho}^{2} = 4m_{K}^{2}.$$
Here $\eta_8$ and $\omega_8$ are isotopic singlet components of unitary octets

$$P_{3}^{3} = \frac{-2\eta_8}{\sqrt{6}} \quad V_{3}^{3} = \frac{-2\omega_8}{\sqrt{6}}. $$

This particles are not observed experimentally like are not observed also isotopic and unitary singlets $\eta_0$ and $\omega_0$. The reason of their inobservability (the second peculiarity of mesons in comparison with baryons) is that $SU(3)$ singlet and octet mesons can mix due to $SU(3)$ - symmetry violation. The mixing term in the mass Lagrangian of mesons has the transformation property of (3,3)-component of tensor i.e. is allowed by symmetry reasons

$$L_{m}^{mix} = m_{mix}^2 M_{3}^{3} M_0, $$

where $M^{a\beta}$ is the tensor of the meson octet and $M_0$ is the singlet meson. The result of mixing of mesons $\eta_8$ and $\eta_0$ ($\omega_8$ and $\omega_0$) are the mesons with definite mass $\eta$ and $\eta'$ ($\phi$ and $\omega$). Let us describe the mixing process.

Let us picture the axis where the square of particle masses will be indicated. First, consider the pseudoscalar mesons. Due to the mass formula we know the mass squared of $\eta_8$ - meson: $m_{\eta_8}^2 = (4m_K^2 - m_\pi^2)/3 = (566 MeV)^2$.

Experimentally the masses squared of $\eta$ and $\eta'$ - mesons are known: $m_\eta^2 = (549 MeV)^2$ and $m_{\eta'}^2 = (958 MeV)^2$

| $\eta$ | $\eta_8$ | $\eta_0$ | $\eta'$ |
|-------|----------|----------|--------|
| $(549)^2$ | $(566)^2$ | $(949)^2$ | $(958)^2$ |

\[ \frac{1}{\sqrt{6}} (u\bar{u} + d\bar{d} - 2s\bar{s}) \quad \frac{1}{\sqrt{3}} (u\bar{u} + d\bar{d} + s\bar{s}) \]

The mass squared of $\eta_0$ - meson can be found by the formula of two level mixing theory:

$$m_{\eta'}^2 - m_{\eta_0}^2 = m_{\eta_8}^2 - m_{\eta}^2$$

( two levels are repelling in opposite directions and on equal distances). We see that the nondiagonal $\eta_8$ - meson is lighter than the nondiagonal $\eta_0$ - meson although $\eta_8$ - meson contains more heavy strange quarks than $\eta_0$ -
meson and this is a surprise that will be discussed in the next lecture. The mixing of pseudoscalar mesons is described by the mixing angle $\theta_P$

$$\eta = \cos \theta_P \eta_8 + \sin \theta_P \eta_0$$

$$\eta' = -\sin \theta_P \eta_8 + \cos \theta_P \eta_0,$$

that can be found one inverts the mixing formulae

$$\eta_8 = \cos \theta_P \eta - \sin \theta_P \eta'$$

$$\eta_0 = \sin \theta_P \eta + \cos \theta_P \eta'$$

and one calculates the matrix element of mass squared operator, say, over the state of $\eta_8$ - meson

$$m_{\eta_8}^2 = \cos^2 \theta_P m_\eta^2 + \sin^2 \theta_P m_{\eta'}^2.$$  

Then we obtain

$$\sin^2 \theta_P = \frac{m_{\eta_8}^2 - m^2_\eta}{m^2_{\eta'} - m^2_\eta} \quad \rightarrow |\theta_P| \approx 10^o.$$  

The mass formulae prediction of the mixing angle $\theta_P$ will be discussed in the following lecture where the annihilation of pseudoscalar mesons into two photons will be considered.

Let us turn now to vector mesons. We know the mass squared of $\omega_8$ - meson: $m_{\omega_8}^2 = (4m_K^2 - m_\rho^2)/3 = (929 MeV)^2$. The masses squared of diagonal $\omega$ and $\phi$ - mesons are known experimentally: $m_\omega^2 = (780 MeV)^2$ $m_\phi^2 = (1020 MeV)^2$

$$\omega \quad \omega_0 \quad \omega_8 \quad \phi$$

$$\begin{array}{c}
(780)^2 \\
(900)^2 \\
(929)^2 \\
(1020)^2
\end{array}$$

$$\begin{array}{c}
\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \\
\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})
\end{array}$$

The mass of $\omega_0$ - meson obtained by the use of mixing formulae is less than
the mass of $\omega_8$ - meson and this is not a surprise because $\omega_0$ - meson contains less strange quarks than $\omega_8$ - meson.

The mixing of vector mesons is described by the mixing angle $\theta_V$

$$\omega = \cos \theta_V \omega_0 + \sin \theta_V \omega_8$$

$$\phi = -\sin \theta_V \omega_0 + \cos \theta_V \omega_8.$$ 

According to mass formulae this angle is equal to

$$\sin^2 \theta_V = \frac{m_{\omega_0}^2 - m_{\omega}^2}{m_{\phi}^2 - m_{\omega}^2} \quad \Rightarrow \quad |\theta_V| \approx 40^\circ.$$ 

The angle $\theta_V$ is close to the ideal mixing angle

$$\cos \theta_{V\text{ideal}} = \sqrt{\frac{2}{3}} \quad \sin \theta_{V\text{ideal}} = \sqrt{\frac{1}{3}} \quad \theta_{V\text{ideal}} \approx 35^\circ,$$

for which $\omega$ - meson contains only nonstrange quarks and $\phi$ - meson contains only strange quarks. In more detail the mixing of mesons will be discussed in the following lecture.

6.4 Problem 6

Find the mass formulae for the baryon octet taking into account the violation not only of $SU(3)$ - symmetry but also of $SU(2)$ - symmetry.
7 Lecture 7. Mixing of pseudoscalar and vector mesons

7.1 SU(3) - symmetry limit for the masses of pseudoscalar and vector mesons

In the previous lecture we have considered the effects of SU(3) - symmetry violation in the masses of mesons and baryons. The result was the Gell-Mann - Okubo mass formulae. In the case of vector mesons the mass formula has predicted the mass of nondiagonal $\omega_8$ - meson to be $m_{\omega_8} = 929\,\text{MeV}$. The mixing theory then has predicted the mass of nondiagonal $\omega_0$ - meson to be $m_{\omega_0} = 900\,\text{MeV}$: for this mass of nondiagonal $\omega_0$ - meson the mixing of $\omega_8$ and $\omega_0$ - mesons produces the diagonal $\phi$ and $\omega$ - mesons with observed masses. The result that $\omega_8$ - meson is more heavy than $\omega_0$ - meson was not an unexpected one because $\omega_8$ - meson contains more strange quarks than $\omega_0$ - meson. Then it is natural to expect that the decreasing of strange quark mass to nonstrange quark mass would result in the degeneracy of $\omega_8$ and $\omega_0$ - mesons ($m_{\omega_8} \rightarrow m_{\omega_0}$) and that in the SU(3) - symmetry limit we would have the nonet (octet plus singlet) of degenerate vector mesons.

On the contrary in the case of pseudoscalar mesons $m_{\eta_8} << m_{\eta_0}$ although $\eta_8$ - meson contains more strange quarks than $\eta_0$ - meson. Then it is natural to expect that the decreasing of strange quark mass to nonstrange quark mass would result in the increase of $m_{\eta_0} - m_{\eta_8}$ mass difference and that in the SU(3) - symmetry limit we would have the octet of pseudoscalar mesons with masses $\sim m_{\pi}$ pseudoscalar singlet with mass $m_{\eta_0} >> m_{\pi}$. For this reason it is widely accepted to think that some part of $\eta_0$ - meson (and hence of $\eta'$ - meson) is presented by the gluonic component i.e.

$$\eta_0 = a\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) + \beta G\tilde{G},$$

where the symbol $G\tilde{G}$ denotes the pseudoscalar glueball with quantum numbers $J^{PC} = 0^{-+}$. 

34
7.2 Mixing of pseudoscalar mesons

The information on the pseudoscalar mesons mixing angle $\theta_P$ defined by the formulae

$$\eta = \cos \theta_P \eta_8 + \sin \theta_P \eta_0$$

$$\eta' = -\sin \theta_P \eta_8 + \cos \theta_P \eta_0,$$

can be obtained not only from mass formulae where $|\theta_P| \approx 10^\circ$ but also from the decays of pseudoscalar mesons. There are three pseudoscalar mesons decaying to two photons - $\pi_0, \eta, \eta'$. The amplitude of the transition of $q \bar{q}$ pair to two photons is proportional to the electric charge of $q$-quark squared $e_q^2 \equiv g_{q \bar{q}}$. In the limit of exact $SU(3)$ - symmetry the amplitudes of $\pi_0, \eta_8, \eta_0 \rightarrow 2\gamma$ transitions would differ only by the values of $g_{\pi_0}, g_{\eta_8}, g_{\eta_0}$ constants

$$g_{\pi_0} = \frac{1}{\sqrt{2}}(e_u^2 - e_d^2) = \frac{1}{\sqrt{6}}(\frac{4}{9} - \frac{1}{9}) = \frac{4}{\sqrt{6}} \frac{1}{3}$$

$$g_{\eta_8} = \frac{1}{\sqrt{6}}(e_u^2 + e_d^2 - 2e_s^2) = \frac{1}{\sqrt{6}}(\frac{4}{9} + \frac{2}{9} - 2\frac{1}{9}) = \frac{1}{\sqrt{6}} \frac{2}{3}.$$

The violation of $SU(3)$ - symmetry contributes at least two modifications. First, the decaying particles $\eta$ and $\eta'$ are the mixtures of $\eta_8$ and $\eta_0$ and

$$g_{\eta} = \cos \theta_P g_{\eta_8} + \sin \theta_P g_{\eta_0} = \frac{1}{3}(\frac{1}{\sqrt{6}} \cos \theta_P + \frac{2}{\sqrt{3}} \sin \theta_P)$$

$$g_{\eta'} = -\sin \theta_P g_{\eta_8} + \cos \theta_P g_{\eta_0} = \frac{1}{3}(\frac{1}{\sqrt{6}} \sin \theta_P + \frac{2}{\sqrt{3}} \cos \theta_P).$$

Second, the amplitudes and the widths of the considered decays are different because the decaying particles have different masses, different energies of photons. Let us take into account the dependence on the mass of decaying pseudoscalar meson phenomenologically

$$A(P \rightarrow 2\gamma) \sim g_P P F_{\mu \nu} \bar{F}_{\mu \nu} \sim g_P m_P^2$$

$$\Gamma(P \rightarrow 2\gamma) \sim |A(P \rightarrow 2\gamma)|^2 \frac{1}{2m_P} \sim g_P^2 m_P^3.$$

Here $(\bar{F}_{\mu \nu}) F_{\mu \nu}$ (dual) tensor of electromagnetic field. So, we have the ratios

$$\frac{\Gamma_{\eta}}{\Gamma_{\pi_0}} = \frac{g_{\eta}^2 m_{\eta}^3}{g_{\pi_0}^2 m_{\pi_0}^3} = (\cos \theta_P + 2\sqrt{2} \sin \theta_P)^2 \frac{1}{3} \frac{m_{\eta}^3}{m_{\pi_0}^3},$$

that predicts $\theta_P \approx 11.8^\circ$ and

$$\frac{\Gamma_{\eta'}}{\Gamma_{\pi_0}} = \frac{g_{\eta'}^2 m_{\eta'}^3}{g_{\pi_0}^2 m_{\pi_0}^3} = (-\sin \theta_P + 2\sqrt{2} \cos \theta_P)^2 \frac{1}{3} \frac{m_{\eta'}^3}{m_{\pi_0}^3},$$

35
that predicts $\Gamma_{\eta'}^{th} = 783\Gamma_\pi = 6.07KeV(\Gamma_{\eta'}^{exp} = 4.54KeV)$. If the second ratio was used for the determination of angle $\Theta_P$ the result would be $\theta_P \approx 22.8^o$ in disagreement with the result $|\theta_P| \approx 10^o$ of mass formulae. The possible reason of this disagreement can be the presence of gluonic component in $\eta'$-meson discussed above. Let us assume that in the mixture
\[ \eta_0 = \alpha \frac{1}{\sqrt{3}} (u\bar{u} + d\bar{d} + s\bar{s}) + \beta \tilde{G} \tilde{G} \]
the gluonic component does not annihilates to two photons. Then the width $\Gamma_{\eta'}^{th}$ will be equal to
\[ \Gamma_{\eta'}^{th} = \alpha^2 6.07KeV \]
and $\alpha^2 \approx 75\%$ to explain the experimental width $\Gamma_{\eta'}^{exp} = 4.54KeV$. Thus we have the second argument in favor of gluonic component in $\eta'$-meson. The nature of this gluonic component is a separate interesting problem that we will not discuss here.

### 7.3 Mixing of vector mesons

Let us now consider the mixing of vector mesons. According to mass formulae the mixing
\[ \omega = \cos \theta_V \omega_0 + \sin \theta_V \omega_8 \]
\[ \phi = -\sin \theta_V \omega_0 + \cos \theta_V \omega_8 \]
with $|\theta_V| \approx 40^o$ is close to the ideal mixing with $\theta_V^{ideal} \approx 35^o$ for which $\omega$-meson does not contain strange quarks and $\phi$-meson does not contain non-strange quarks. Like the case of pseudoscalar mesons the mixing of vector mesons can be studied in electromagnetic decays of vector mesons - annihilation to electron-positron pair. The amplitude of $q\bar{q}$ - pair annihilation to electron-positron pair is proportional to the quark charge $e_q \equiv h_{q\bar{q}}$. In the case of exact $SU(3)$ - symmetry we would have diagonal $\rho_0, \omega_8, \omega_0$ - mesons and their couplings
\[ h_{\rho_0} = \frac{1}{\sqrt{2}} \left( \frac{2}{3} - \left( -\frac{1}{3} \right) \right) = \frac{1}{\sqrt{2}} \]
\[ h_{\omega_8} = \frac{1}{\sqrt{6}} \left( \frac{2}{3} + \left( -\frac{1}{3} \right) - 2\left( -\frac{1}{3} \right) \right) = \frac{1}{\sqrt{6}} \]
\[ h_{\omega_0} = \frac{1}{\sqrt{3}} \left( \frac{2}{3} + \left( -\frac{1}{3} \right) + \left( -\frac{1}{3} \right) \right) = 0. \]
Like the case of pseudoscalar mesons the violation of $SU(3)$ - symmetry also contributes two modifications here. First, we have the mixing of vector mesons and

\[
\begin{align*}
    h_\omega &= \cos\theta_V h_{\omega_0} + \sin\theta_V h_{\omega_8} = \sin\theta_V \frac{1}{\sqrt{6}} \\
    h_\phi &= -\sin\theta_V h_{\omega_0} + \cos\theta_V h_{\omega_8} = \cos\theta_V \frac{1}{\sqrt{6}}.
\end{align*}
\]

Second, the decay width $\Gamma(V \to e^+ e^-)$ depend on the mass of vector meson. This dependence can be found by the following arguments. The widths of the annihilation of vector meson is proportional to the probability for quark and antiquark to meet - the square of wave function of quark and antiquark at origin. Making the desired dimensionality for the width by the use of the proper power of the vector meson mass we get

\[
\Gamma(V \to e^+ e^-) \sim \frac{|\psi_V(0)|^2}{m_V^2} h_V^2.
\]

Finally we have the ratio

\[
\frac{\Gamma_\omega}{\Gamma_\rho} = \frac{\sin^2\theta_V m_\rho^2}{3 m_\omega^2},
\]

from which $\theta_V \approx 32^\circ$ close to $\theta_{V\text{ideal}} \approx 35^\circ$ can be obtained and the ratio

\[
\frac{\Gamma_\phi}{\Gamma_\rho} = \frac{\cos^2\theta_V m_\rho^2}{3 m_\phi^2},
\]

from which one follows that $\Gamma_{\phi}^{th} \approx 1 KeV$. The 30% difference with the experimental width $\Gamma_{\phi}^{exp} \approx 1.3 KeV$ can be explained by larger value of the wave function at the origin in the case of strange (heavy) quarks compared to the case of nonstrange (light) quarks.

### 7.4 Problem 7

Make all the numerical calculations omitted in the lecture.
8 Lecture 8. Magnetic moments of baryons

8.1 SU(3) symmetric limit for magnetic moments of baryons

SU(3) - symmetry of strong interaction predicts the relations for various physical characteristics of hadrons. One of the most impressive example is the magnetic moments of baryons. Like the case of mass formulae let us start from the quark level - from electromagnetic current of quarks. In the limit of SU(3) - symmetry $u, d, s$ - quarks differ only by the values of their electric charges. The electromagnetic current of quarks has the following SU(3) - structure

$$j_\mu = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d - \frac{1}{3} \bar{s} \gamma_\mu s =$$

$$= \bar{u} \gamma_\mu u - \frac{1}{3} (\bar{u} \gamma_\mu u + \bar{d} \gamma_\mu d + \bar{s} \gamma_\mu s)$$

or

$$j_\mu = J^1_{\mu 1} - \frac{1}{3} (J^a_{\mu a}).$$

Considering the hadrons we will construct the electromagnetic current of hadrons that possesses the same transformation properties with respect to SU(3) - transformations as electromagnetic current of quarks. In particular, for the octet of baryons that is described by the matrix

$$B^\alpha_\beta = \begin{pmatrix} \Sigma^+ \sqrt{2} & \Sigma^0 & p \\ \Sigma^- & -\Sigma^+ \sqrt{2} & n \\ \Xi^- & 0 & -2\Lambda \sqrt{6} \end{pmatrix},$$

the electromagnetic current can be presented by two independent flavor structures

$$< j_\mu >_B = \tilde{B}^\alpha \Gamma^a_\beta B^\beta_\alpha + \frac{1}{3} \tilde{B}^\alpha \beta \Gamma^a_\mu B^\beta_\mu + \frac{1}{3} \tilde{B}^\alpha \beta \Gamma^\prime_\mu B^\beta_\mu,$$

where the quantities $\Gamma^a_\mu$ and $\Gamma^\prime_\mu$ are the combinations of electric and magnetic formfactors

$$\Gamma^a_\mu = \gamma_\mu f_1(q^2) + \sigma_\mu q^2 f_2(q^2)$$

$$\Gamma^\prime_\mu = \gamma_\mu f'_1(q^2) + \sigma_\mu q^2 f'_2(q^2).$$

Here $q$ is the four-momentum transferred to the baryon. Electric and magnetic formfactors of baryons are the definite combinations of two electric ($f_1(q^2)$)
and \( f'_1(q^2) \) and two magnetic (\( f_2(q^2) \) and \( f'_2(q^2) \) formfactors. In accordance to this the electric charges and magnetic moments of baryons are the definite combinations of constants \( Q, Q' \) and \( \mu, \mu' \) defined as follows

\[
Q = f_1(0) \quad Q' = f'_1(0) \\
\mu_A = f_2(0) \quad \mu'_A = f'_2(0) \\
\mu = \frac{Q}{2m} + \mu_A \quad \mu' = \frac{Q'}{2m} + \mu'_A.
\]

So, magnetic moments of baryons being the combinations of two independent parameters \( \mu, \mu' \) can be expressed through two independent baryon magnetic moments, say, the magnetic moments of proton and neutron. Thus we get the following table

| Baryon | \( \mu - \frac{1}{3}(\mu + \mu') \) | Input | \( \mu - \frac{1}{3}(\mu + \mu') \) | Input |
|--------|---------------------------------|-------|---------------------------------|-------|
| \( \mu_p \) | \( \mu - \frac{1}{3}(\mu + \mu') \) | 2.79  | \( \mu - \frac{1}{3}(\mu + \mu') \) | 2.79  |
| \( \mu_n \) | \( -\frac{1}{3}(\mu + \mu') \) | -1.91 | \( -\frac{1}{3}(\mu + \mu') \) | -1.91 |
| \( \mu_{\Sigma^+} \) | \( \mu - \frac{1}{3}(\mu + \mu') = \mu_p \) | 2.79  | \( \mu - \frac{1}{3}(\mu + \mu') = \mu_p \) | 2.79  |
| \( \mu_{\Sigma^0} \) | \( \frac{1}{2} - \frac{1}{3}(\mu + \mu') = \mu_n \) | 0.96  | \( \frac{1}{2} - \frac{1}{3}(\mu + \mu') = \mu_n \) | 0.96  |
| \( \mu_{\Sigma^-} \) | \( \mu - \frac{1}{3}(\mu + \mu') = -(\mu_p + \mu_n) \) | -0.88 | \( \mu - \frac{1}{3}(\mu + \mu') = -(\mu_p + \mu_n) \) | -0.88 |
| \( \mu_{\Xi^-} \) | \( \mu - \frac{1}{3}(\mu + \mu') = -(\mu_p + \mu_n) \) | -0.88 | \( \mu - \frac{1}{3}(\mu + \mu') = -(\mu_p + \mu_n) \) | -0.88 |
| \( \mu_{\Xi^0} \) | \( -\frac{1}{3}(\mu + \mu') = \mu_n \) | -1.91 | \( -\frac{1}{3}(\mu + \mu') = \mu_n \) | -1.91 |
| \( \mu_{\Lambda} \) | \( \frac{1}{6} - \frac{1}{3}(\mu + \mu') = \frac{1}{3}\mu_n \) | -0.96 | \( \frac{1}{6} - \frac{1}{3}(\mu + \mu') = \frac{1}{3}\mu_n \) | -0.96 |
| \( \mu_{\Sigma\Lambda} \) | \( \frac{1}{\sqrt{12}}(\mu + \mu') = -\frac{\sqrt{3}}{\sqrt{12}}\mu_n \) | 1.65  | \( \frac{1}{\sqrt{12}}(\mu + \mu') = -\frac{\sqrt{3}}{\sqrt{12}}\mu_n \) | 1.65  |

(all numerical values here are expressed in units \( \frac{1}{2m} \) where \( m \) is the nucleon mass)

The difference among the \( SU(3) \) - predictions for magnetic moments of baryons (left numbers) and experimental data (right numbers) are rather large. It means that \( SU(3) \) - symmetry in magnetic moments of baryons is noticeably violated . It is quite educative to look for the relations among the magnetic moments of baryons in another way that will let both to take into account the effects of \( SU(3) \) - symmetry violation and relate the magnetic moments of proton and neutron (they were not related in the above considerations). The assumption of additivity of quark magnetic moments in the magnetic moment of baryon will be the key in this another way.
8.2 Magnetic moment of baryon in additive quark model

The magnetic moment ($\mu$) of spin 1/2 particle is defined by the relation

$$\mu = \mu \sigma,$$

where $\sigma$ are Pauli matrices acting on spin variables of particle. Assuming that baryons consist of three quarks and that the quark magnetic moments are simply added in the magnetic moment of baryon we get

$$\mu = \Sigma_i \mu_i \sigma_i \quad (i = 1, 2, 3)$$

The matrices $\sigma_i$ act on spin variables of constituent quarks. The baryon magnetic moment ($\mu$) is, by definition, the matrix element of operator $\mu_z$ over the baryon state with spin projection on z-axis equal to 1/2

$$\mu = <s_z = 1/2 | \mu_z | s_z = 1/2 >.$$  

For the calculation of magnetic moment of baryons we will use the explicit constructions of baryon states found in lecture 5. So, for the proton we have

$$\dot{p} = \sqrt{\frac{2}{3}} \dot{u} \dot{u} \dot{d} - \sqrt{\frac{1}{3}} \sqrt{\frac{1}{2}} (\dot{u} \dot{u} + \dot{u} \dot{u}) \dot{d},$$

where the exact symmetrization with respect to the third quark is omitted (this symmetrization does not change the result). Calculating the corresponding matrix element of operator $\mu_z$ we get the proton magnetic moment

$$\mu_p = \frac{2}{3}(\mu_u + \mu_u - \mu_d) + \frac{2}{6}(\mu_u - \mu_u + \mu_d) = \frac{1}{3}(4\mu_u - \mu_d).$$

The magnetic moments of six other components of baryon octet follow immediately from the comparison with the result for the proton magnetic moment

$$\mu_n = \frac{1}{3}(4\mu_d - \mu_u) \quad \mu_{\Sigma^+} = \frac{1}{3}(4\mu_u - \mu_s)$$

$$\mu_{\Sigma^-} = \frac{1}{3}(4\mu_d - \mu_s) \quad \mu_{\Xi^-} = \frac{1}{3}(4\mu_s - \mu_d)$$
\[ \mu_{\Xi^0} = \frac{1}{3}(4\mu_s - \mu_u) \quad \mu_{\Sigma^0} = \frac{1}{3}(2(\mu_u + \mu_d) - \mu_s) \]

The separate consideration is needed for \( \Lambda \) - hyperon. Its wave function (without simmetrization with respect to quark interchanges that does not change the result) has the form

\[ \hat{\Lambda} = \frac{1}{\sqrt{2}}(\hat{u}\sigma \hat{d} - \hat{u}\sigma \hat{d}) \hat{s} \]

Therefore, the magnetic moment of \( \Lambda \) - hyperon is equal to

\[ \mu_{\Lambda} = \frac{1}{2}(\mu_u - \mu_d + \mu_s - \mu_u + \mu_d + \mu_s) = \mu_s \]

The separate consideration is also needed for the transition matrix element \( \mu_{\Sigma \Lambda} \) that determines the radiative decay \( \Sigma^0 \to \Lambda + \gamma \) (see the problem in the end of this lecture). In close analogy to the proton wave function the wave function of \( \Sigma^0 \) - hyperon is equal to

\[ \hat{\Sigma^0} = \sqrt{\frac{2}{3}} \hat{u}\hat{d}\hat{s} - \sqrt{\frac{1}{3}} \sqrt{\frac{1}{2}}(\hat{u}\hat{d} + \hat{u}\hat{d}) \hat{s} \]

Then \( \mu_{\Sigma \Lambda} \) is easily calculated

\[ \mu_{\Sigma \Lambda} = -\sqrt{\frac{1}{3}} \left( \frac{1}{2}(\mu_u - \mu_d + \mu_s + \mu_u - \mu_d - \mu_s) \right) = -\sqrt{\frac{1}{3}}(\mu_u - \mu_d) \]

The quark magnetic moments \( (\mu_u, \mu_d, \mu_s) \) can be considered as parameters and defined, say, by magnetic moments of proton, neutron and \( \Lambda \) - hyperon

\[ \mu_u = \frac{4\mu_p + \mu_n}{5} = 1.85 \quad \mu_d = \frac{4\mu_n + \mu_p}{5} = -0.97 \quad \mu_s = \mu_{\Lambda} = -0.61 \]

If \( SU(3) \) - symmetry was exact symmetry the quark magnetic moments would be proportional to their electric charges

\[ \mu_u : \mu_d : \mu_s = \frac{2}{3} : \frac{2}{3} : \frac{1}{3} : \frac{2}{3} : \frac{1}{3} = 2 : -1 : -1 \]

As it can be seen the ratio \( \mu_u : \mu_d = 1.85 : -0.97 \approx 2 : -1 \) is satisfied rather well i.e. the \( SU(2) \) - symmetry is a good symmetry while the ratio
$\mu_d : \mu_s = -0.97 : -0.61 \neq -1 : -1$ is largely violated. If $\mu_u : \mu_d = 2 : -1$ (SU(2)-symmetry) then $\mu_p : \mu_n = 3 : -2$ that is very close to the experimental result $\mu_p : \mu_n = 2.79 : -1.91$. In such a way the additive quark model relates the magnetic moments of proton and neutron.

Finally, let us compare the predictions of SU(3) - symmetry, the additive quark model and the experimental data

|       | Input | Input | 2.79 | 1.91 | 2.42 | 2.67 | 1.16 | 0.68 | 1.43 | 1.25 | 0.61 | 1.61 |
|-------|-------|-------|------|------|------|------|------|------|------|------|------|------|
| $\mu_p$ |       |       |      |      |      |      |      |      |      |      |      |      |
| $\mu_n$ |       |       |      |      |      |      |      |      |      |      |      |      |
| $\mu_{\Sigma^+}$ | 2.79 |      |      |      |      |      |      |      |      |      |      |      |
| $\mu_{\Sigma^-}$ | -0.88 |      |      |      |      |      |      |      |      |      |      |      |
| $\mu_{\Xi^-}$ | -0.88 |      |      |      |      |      |      |      |      |      |      |      |
| $\mu_{\Xi^0}$ | -1.91 |      |      |      |      |      |      |      |      |      |      |      |
| $\mu_{\Lambda}$ | -0.96 |      |      |      |      |      |      |      |      |      |      |      |
| $\mu_{\Sigma\Lambda}$ | 1.65 |      |      |      |      |      |      |      |      |      |      |      |

It can be seen from the table that the additive quark model gives better description of experimental data. Note that the hypothesis of additivity is a natural but not an exact one. Gluon corrections of quark interactions gives, for example, non additive contributions. These contributions make the description of experimental data better.

8.3 Problem 8

Calculate the width of $\Sigma^0 \to \Lambda + \gamma$ decay and find $\mu_{\Sigma\Lambda}$ from the comparison with the experiment.
9 Lecture 9. Chiral symmetry and quark masses

This lecture closes the round of lectures on the quark structure of hadrons constructed from light quarks. We began with the introduction of the masses of light $u,d,s$-quarks ($4,7,150\,\text{MeV}$) and said that the difference of quark masses is small compared to the characteristic hadronic scale $1\,\text{GeV}$. From this followed the isotopic ($SU(2)$) and unitary ($SU(3)$) symmetries of strong interaction. We considered the violation of $SU(3)$-symmetry caused by the difference of strange and nonstrange quark masses. Now it is quite a time to say where are the quark masses used overall the lectures came from. For this we will consider the so-called chiral symmetry of strong interaction corresponding to the case of massless quarks. We have all the reasons to expect that the chiral symmetry is violated no stronger than $SU(3)$-symmetry based on the neglect of strange and nonstrange quark mass difference. Analyzing the violation of the chiral symmetry we will find the quark masses.

9.1 Left and right quarks

If to put the quark masses to zero two nontrivial things happen: the Lagrangian of strong interaction becomes invariant under the separate $SU(3)$-transformations of left and right (in the sense of chirality) quark fields; left and right (in the sense of chirality) quark fields goes in one to one correspondence with left and right (in the sense of spirality) states of massless quarks. The symmetry for which left and right quark fields transform separately is called the chiral symmetry. In our case it is the chiral $SU(3)_L \times SU(3)_R$-symmetry. This symmetry means that the left and right quark currents are conserved separately

$$ j^a_{\mu L} = \bar{q}_L \gamma_\mu \frac{\lambda^a}{2} q_L \quad j^a_{\mu R} = \bar{q}_R \gamma_\mu \frac{\lambda^a}{2} q_R, $$

where $q_{L,R} = \frac{1}{2}(1 \pm \gamma_5)q$. The conservation of quark current that is the sum of the left and right quark currents (vector current) corresponds to the $SU(3)$-symmetry considered in lecture 5. The conservation of quark current that is the difference of the left and right quark currents (axial current) corresponds to the axial $SU(3)$-symmetry. These currents are

$$ j^a_{\mu V} = \bar{q} \gamma_\mu \frac{\lambda^a}{2} q \quad j^a_{\mu A} = \bar{q} \gamma_\mu \gamma_5 \frac{\lambda^a}{2} q. $$
Vector and axial currents generates conserved charges $Q^a, Q^a_A$. The charges $Q^a$ are the charges that are conserved due to the $SU(3)$ - symmetry. In the limit of massless quarks the symmetry is enlarged and new conserved charges - axial charges $Q^a_A$ have appeared.

9.2 Nonlinear realization of chiral symmetry

Let us consider the matrix elements of conserved vector and axial currents. Let the vector current be, for example, an electromagnetic one

\[ j^{em}_\mu = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d - \frac{1}{3} \bar{s} \gamma_\mu s = \bar{q} \gamma_\mu \lambda^{em}_2 q, \]

where $\frac{\lambda^{em}_2}{2} = \frac{\lambda^3}{2} + \frac{\lambda^8}{2\sqrt{3}}$, and matrix element be taken over the proton states. In the limit of zero momentum transfer $q \to 0$ the considered matrix element is equal to

\[ <p(k_2) | j^{em}_\mu | p(k_1) > = \bar{\Psi}_2 \gamma_\mu \Psi_1. \]

The conservation of electromagnetic current means that

\[ (k_2 - k_1) \bar{\Psi}_2 \gamma_\mu \Psi_1 = 0. \]

This equation is valid due to the Dirac equations for proton spinors $\Psi_2$ and $\Psi_1$.

Let now the axial current be the current corresponding to $\beta$ - decay of neutron

\[ \gamma_\mu \to \gamma_\mu \gamma_5 \quad \lambda^{em} \to \tau^+. \]

If in analogy to the case of electromagnetic current we assume that the matrix element of the axial current over the states of neutron and proton and in the limit of zero momentum transfer is equal to

\[ < p(k_2) | j^+_A | n(k_1) > = g_A \bar{\Psi}_2 \gamma_\mu \gamma_5 \Psi_1, \]

then the conservation of the axial current will require the equation

\[ (k_2 - k_1) \bar{\Psi}_2 \gamma_\mu \gamma_5 \Psi_1 = g_A 2m_p \bar{\Psi}_2 \gamma_5 \Psi_1 = 0 \]

and hence will require the mass of nucleon be equal to zero $m_p = 0$. With experimental mass of nucleon of the order $1GeV$ we will be forced to conclude
that the chiral symmetry is strongly violated. We will assume that the matrix element of the axial current over the states of neutron and proton and in the limit of zero momentum transfer is equal to

\[ < p(k_2) | j_{\mu A}^+ | n(k_1) > = g_A \bar{\Psi}_2 \gamma_\nu \gamma_5 \Psi_1 (\delta_{\mu \nu} - \frac{q_\mu q_\nu}{q^2}) , \]

i.e. is conserved automatically. For \( q^2 = 0 \) the matrix element has a pole that can be interpreted as follows: neutron emits massless pseudoscalar particle (\( \pi^- \)) and transforms to proton but \( \pi^- \) is annihilated by axial current. Eight massless pseudoscalar mesons correspond to eight conserved axial currents. The situation when the masses of pseudoscalar mesons are equal to zero in the chiral limit of massless quarks seems to be more interesting from the physical point of view because experimentally all eight pseudoscalar mesons are substantially lighter than other hadrons. Especially it is true for the case of \( \pi^- \) mesons. So, the chiral symmetry can be realized by making massless only pseudoscalar mesons.

Vector charges \( Q^a \) acting on the states of baryon octet transform these states into themselves. Axial charge \( Q^3_A \), for example, acting, say, on the proton state transforms the proton state to another state that has the same spin, isospin, hypercharge but has opposite parity. In the octet of baryons there is no such states, it is natural to think that we get the two particle state "\( \pi^- \) nucleon". For zero energy of \( \pi^- \) meson the states "nucleon" and "\( \pi^- \) nucleon" are degenerate as it should be in the case when there is a symmetry. The described realization of chiral symmetry is called the nonlinear realization.

## 9.3 Partial conservation of axial currents and quark masses

The arguments in favor of zero masses of pseudoscalar mesons for the limit of massless r\( \Phi \) can be obtained also by the considerations of matrix elements of axial currents over the pseudoscalar meson states and vacuum. For example, for \( \pi^+ \) - meson the matrix element

\[ < 0 | \bar{u} \gamma_\mu \gamma_5 d | \pi^+ > = f_\pi p^+_\mu \]

45
enters the amplitude of weak decay $\pi^+ \rightarrow \mu^+ \nu_\mu$. Calculating the divergence of axial current we obtain

$$<0|(m_u + m_d)\bar{u}\gamma_5 d|\pi^+> = f_\pi m_\pi^2,$$

from where it can be seen that the mass of $\pi$ - meson goes to zero in the limit of massless quarks $m_{u,d} \rightarrow 0$. Similarly

$$<0|(m_u + m_s)\bar{u}\gamma_5 s|K^+> = f_K m_K^2.$$ 

Assuming that $<0|\bar{u}\gamma_5 d|\pi^+> \approx <0|\bar{u}\gamma_5 s|K^+>$ and $f_\pi \approx f_K$ (experimentally) we get the following quark mass ratio

$$\frac{m_u + m_d}{m_u + m_s} \approx \frac{m_\pi^2}{m_K^2} \approx \frac{1}{13}.$$

If one takes the strange quark mass to be $m_s \approx 150 MeV$ (the mass differences of baryons in the decuplet that differ by the numbers of strange quarks are approximately of this value) then

$$m_u + m_d \approx 11 MeV.$$ 

Now we have to evaluate the quark masses $m_u$ and $m_d$ separately. Up to now we did not take into account the electromagnetic interaction that can give the electromagnetic corrections to the masses of mesons. The size of these corrections is comparable to the masses of $u$ and $d$ - quarks. Therefore, our wishes to evaluate the quark masses $m_u$ and $m_d$ separately encounter the necessity to consider the electromagnetic corrections. There is no rigorous way to do it and we will bound ourselves by the following speculation. In the spirit of above relations among the masses of mesons squared and quark masses let us write the modified relations that will take into account the difference of charged particles from ones

$$m_{\pi^+}^2 \sim m_u + m_d + \gamma$$

$$m_{\pi^0}^2 \sim m_u + m_d$$

$$m_{K^+}^2 \sim m_u + m_s + \gamma$$

$$m_{K^0}^2 \sim m_d + m_s$$

The unknown electromagnetic correction $\gamma$ can be eliminated and we get as a result

$$\frac{m_d - m_u}{m_d + m_u} = \frac{(m_{\pi^+}^2 - m_{\pi^0}^2) - (m_{K^+}^2 - m_{K^0}^2)}{m_{\pi^0}^2} \approx 0.29.$$ 

46
So, for $m_u + m_d \approx 11\text{MeV}$ one obtains $m_u \approx 4\text{MeV}$ and $m_d \approx 7\text{MeV}$.
We see that $m_d - m_u \sim m_d \sim m_u$, hence, the chiral $SU(2)_L \times SU(2)_R$ - symmetry should be as good as usual isotopic $SU(2)$ - symmetry and be better than $SU(3)$ - symmetry in any case.

9.4 Problem 9

Find the relation between the ratio $\frac{m_d^2}{m_{\eta_S}}$ and the ratio of quark masses and verify its accuracy for the values of quark masses obtained on the lecture.