ENTANGLEMENT AS AN OBSERVER-DEPENDENT CONCEPT:
AN APPLICATION TO QUANTUM PHASE TRANSITIONS

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1. INTRODUCTION

In 1935 Einstein, Podolsky, and Rosen [1] published a Gedanken-experiment that still surprises us today [2]. Their main observation highlighted a key feature of quantum phenomena in which a pure state of a composite quantum system may cease to be determined by the states of its constituent subsystems. It is this non-classical behavior that Schrödinger later termed entanglement [3]. Entangled states are joint states of two or more distinguishable quantum systems that cannot be expressed as a mixture of products of states of each system. To establish whether or not a state is entangled requires knowledge of the correlations between subsystems, and measurement of one subsystem may affect the results of measurements on another even when the system is in a pure state, something which is against classical intuition. In Schrödinger’s words [3]: “[The] Best possible knowledge of a whole does not necessarily include the same for its parts.” Today we know that entanglement is the defining resource for quantum communications (e.g., allowing quantum teleportation protocols), it provides non-local correlations between subsystems that admit no local classical interpretation (quantum non-locality), and it is believed to be an essential ingredient to understanding and unlocking the power of quantum computation [4].

Until very recently, most studies concentrated on how the entanglement of a number of distinguishable quantum subsystems differs from classical correlations. Typically these studies focus on two subsystems (the bipartite or “Alice and Bob” setting). A notion of entanglement emerged from these investigations that we will term conventional entanglement. This paper addresses the following question: Do we have a theoretical understanding of entanglement applicable to a full variety of physical settings? It is clear that not only the assumption of distinguishability (which is natural and useful in standard quantum information processing (QIP) situations), but also the few-subsystem scenario, are too narrow to embrace all possible physical settings. In particular, the need to go beyond this subsystem-based framework becomes manifest when one tries to apply the conventional concept of entanglement to the physics of matter, since the constituents of a quantum many-body system are indistinguishable particles. We shall discuss here a notion of generalized entanglement.
(GE), which can be applied to any operator language (fermions, bosons, spins, etc.) used to describe a physical system and which includes the conventional entanglement settings introduced to date in a unified fashion. This is realized by noticing that entanglement is an observer-dependent concept, whose properties are determined by the expectations of a distinguished set of observables without reference to a preferred subsystem decomposition, i.e., it depends on the physically relevant point of view. This viewpoint depends in turn upon the relationship between different sets of observables that determine our ability to control the system of interest. Indeed, the extent to which entanglement is present depends on the observables used to measure a system and describe its states. It is worthwhile noting that the role of observables and control in distinguishing subsystems such as qubits and determining the emergence of a preferred tensor product structure has been stressed before by various authors (see e.g., [5, 6]). However, our present approach is conceptually new and more general in that the observable sets need not be associative algebras, thus the resulting entanglement notion is capable of going beyond the identification of factorizations of a system into subsystems. This represents a most conspicuous advantage as will be highlighted by the condensed-matter application we will discuss.

On the other hand, novel materials display complex phase diagrams as a result of competing interactions with complex orderings (states of matter) which are believed to be influenced by being in proximity to quantum critical points (high-$T_c$, heavy fermion compounds, frustrated magnets, etc.). And the truth is that our understanding of the nature of these transitions is very poor. Since entanglement is a property inherent in quantum states and is strongly related to quantum correlations, one would expect that the entanglement present in the ground state of the system changes substantially at a quantum phase transition (QPT), i.e., when there is a qualitative change in the behavior of the correlations between constituents. Can we use concepts and tools borrowed from QIP to better understand QPTs? What is the nature and role of entanglement in a QPT? Can one quantify entanglement in a physically meaningful way? What properties distinguish between broken- and non-broken-symmetry QPTs? These are some of the questions we are going to address by example. (See also Refs. [7-10] for recent work in similar directions.)

Before we describe the concept of GE, let us start by recalling the standard framework for conventional entanglement. Given a quantum system with state space $\mathcal{H}$, it can a priori support inequivalent tensor product structures; for instance, $\mathcal{H} \simeq \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, or ($N$ even) $\mathcal{H} \simeq \mathcal{H}_{12} \otimes \mathcal{H}_{34} \otimes \cdots \otimes \mathcal{H}_{(N-1)N}$, where $\mathcal{H}_{ij} \simeq \mathcal{H}_i \otimes \mathcal{H}_j$. Entanglement of states in $\mathcal{H}$ is unambiguously defined only once a preferred subsystem decomposition is chosen: $\mathcal{H} \simeq \bigotimes_j \mathcal{H}_j$. Relative to the selected multipartite structure, a pure state $|\Psi\rangle$ is entangled iff $|\Psi\rangle$ induces at least some mixed subsystem states. For example: if $\mathcal{H} \simeq \mathcal{H}_A \otimes \mathcal{H}_B$, where the Hilbert spaces for Alice and Bob, $\mathcal{H}_A, \mathcal{H}_B$, are two-dimensional ($|\psi\rangle_{A,B} = a_{0}^{A,B}|0\rangle_{A,B} + a_{1}^{A,B}|1\rangle_{A,B}$), and

$$|\Psi\rangle = \frac{|0\rangle_{A} \otimes |1\rangle_{B} + |1\rangle_{A} \otimes |0\rangle_{B}}{\sqrt{2}},$$

then

$$\begin{cases}
\rho_A = \text{tr}_B |\Psi\rangle\langle\Psi| \\
\rho_B = \text{tr}_A |\Psi\rangle\langle\Psi|
\end{cases}$$

are mixed, (1)

whereas if

$$|\Psi\rangle = |0\rangle_{A} \otimes |1\rangle_{B},$$

then $\rho_A$ and $\rho_B$ are pure. (2)

Thus, in the conventional approach, entangled pure states look mixed to at least some local observers, while unentangled pure states still look pure. In standard QIP
settings the choice of preferred subsystem is usually not problematic, as there is a notion of locality in *real space*. Indeed, spatially separated local parties are often taken as synonymous with distinguishable quantum subsystems.

However, as mentioned above, the distinguishability assumption is too narrow for a generic physical setting, and yet many questions remain open:
i) What does entanglement mean for indistinguishable particles?

It has been known since the early days of quantum mechanics that particles in Nature are either bosons or fermions in the three-dimensional space. Indistinguishability places an additional constraint on the space of admissible states which manifests itself in their symmetry properties. The physically accessible state space is the symmetric(bosons)/antisymmetric(fermions) subspace of the full tensor product state space. For example, consider the free electron gas (Fermi liquid) in the real-space $x$-representation ($k_j$ is a label of momentum), described by a many-body wavefunction of the form

$$
\langle x_1, x_2, \cdots | \Psi \rangle \sim \text{Det} \begin{pmatrix}
e^{ik_1 x_1} & e^{ik_1 x_2} & \cdots \\
e^{ik_2 x_1} & e^{ik_2 x_2} & \cdots \\
\vdots & \vdots & \cdots
\end{pmatrix}, \langle x_1, x_2, \cdots | \Psi \rangle = -\langle x_2, x_1, \cdots | \Psi \rangle. \quad (3)
$$

Exchange correlations are present but are not, in principle, a usable resource in the usual QIP sense. (Other recent work on generalizing entanglement to indistinguishable fermions may be found in [11] and references therein.)

ii) Particle or mode entanglement?

The state $\langle x_1, x_2, \cdots | \Psi \rangle$ given above exhibits mode entanglement with respect to the set of momentum modes; yet, the electrons are non-interacting. A set of modes provides a factorization into a distinguishable-subsystem structure, but in principle inequivalent factorizations (e.g., position modes and momentum modes) may occur on the same footing for a given system.

iii) What is the most appropriate operator language?

The algebraic language (fermions, bosons, anyons, gauge fields, etc.) used to describe an interacting quantum system may be intentionally changed by mappings such as the Jordan-Wigner transformation and its generalizations [12].

The bottom line is that nontrivial quantum statistics, or other physical restrictions, can make the choice of preferred subsystem problematic. Our generalized approach can overcome such difficulties, by removing the need for a subsystem decomposition from the start. Even when a subsystem partition exists, we would like to emphasize that from the viewpoint of GE, ordinary separable pure states (i.e., states that can be written as a tensor product of normalized subsystem states, $|\Psi\rangle_s = |\psi\rangle_1 \otimes |\phi\rangle_2 \otimes \cdots \otimes |\varphi\rangle_N$) are not necessarily equivalent to generalized unentangled states. We will illustrate this issue by example in section 3.2.

2. ENTANGLEMENT AS AN OBSERVER-DEPENDENT CONCEPT

Our approach to tackle these issues will be to reformulate entanglement as a *subsystem-independent* property that depends on the physically relevant point of view. The formal development of GE may be found in Refs. [13,14]. Here we will simply recall the key concepts and further illustrate them by example.
The notions of Hilbert space and linear maps or operators are central to the formulation of quantum mechanics. Pure states are elements of a Hilbert space and physical observables are associated to Hermitian (self-adjoint) operators which act on that space, and whose eigenvalues are selected by a measuring apparatus. The role of linear operators in quantum mechanics is not restricted to the representation of physical observables. Non-Hermitian operators are very often used as well, for instance in the context of describing non-unitary quantum dynamics. Linear operators form a complex vector space under the sum and the multiplication by a scalar over the field of complex numbers. If we augment this vector space with a bilinear operation, the set forms an algebra. Quantum mechanics also requires that this operation be non-commutative and associative. A Lie algebra \( \mathfrak{h} \) is a vector space endowed with a bilinear map \([\cdot, \cdot]\) satisfying antisymmetry and the Jacobi identity

\[
[x, x] = 0, \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad \forall \, x, y, z \in \mathfrak{h}.
\] (4)

An ideal \( I \) in a Lie algebra is a subalgebra fixed by the commutator map: For \( x \in I \) and arbitrary \( h \in \mathfrak{h} \), \([h, x] \in I \). A simple Lie algebra is one with no proper ideals. \( \mathfrak{h} \) is said to be semisimple iff it is the direct sum of simple Lie algebras. A Cartan subalgebra (CSA) \( c \) of a semisimple Lie algebra \( \mathfrak{h} \) is a maximal commutative subalgebra. A vector space carrying a representation of \( \mathfrak{h} \) decomposes into orthogonal joint eigenspaces \( V_\lambda \) of the operators in \( c \). That is, each \( V_\lambda \) consists of the set of states \( |\psi\rangle \) such that for \( x \in c \), \( x|\psi\rangle = \lambda(x)|\psi\rangle \). The label \( \lambda \) is therefore a linear functional on \( c \), called the weight of \( V_\lambda \). The subspace of operators in \( \mathfrak{h} \) orthogonal in trace inner product to \( c \) can be organized into orthogonal raising (\( e_\mu \)) and lowering (\( e_{-\mu} \)) operators, which connect different weight spaces. For a fixed CSA, the weights form a convex polytope; a lowest (or highest) weight is an extremal point of such a polytope, and the one-dimensional weight-spaces having those weights are known as lowest-weight states. The set of lowest-weight states for all CSAs is the orbit of any one such state under the Lie group generated by \( \mathfrak{h} \). These are the group-theoretic generalized coherent states (GCSs) [15], which may be formally represented as

\[
|\text{GCS}\rangle = e^{\sum_\mu (\eta_\mu e_\mu - \bar{\eta}_\mu e_{-\mu})}|\text{ext}\rangle, \quad \text{where } |\text{ext}\rangle \text{ is an extremal state}.
\] (5)

The GCSs attain minimum uncertainty in an appropriate invariant sense [16], thus providing a natural generalization of the familiar harmonic-oscillator coherent states.

Our notion of GE is relative to a distinguished subspace of observables \( \Omega \) of the quantum system. The basic idea is to generalize the observation that standard entangled pure states are those that look mixed to local observers. Given a pure state of a system, we remind the reader that the associated reduced state, or \( \Omega \)-state, is obtained by only considering the expectation values of the distinguished observables \( \Omega \). In other words, an \( \Omega \)-state is a linear and positive functional \( \omega \) on the operators of \( \Omega \) induced by a density matrix \( \rho \) according to \( \omega(x) = \text{tr}(\rho x) \). The set of reduced states is convex. We then have the following

**Definition** :

A pure state \( |\Psi\rangle \) is generalized unentangled relative to the distinguished set of observables \( \Omega \), if its reduced state is pure (extremal), and generalized entangled otherwise. Similarly, a mixed state is generalized unentangled relative to \( \Omega \), if it can be written as a convex combination of generalized unentangled pure states.
When Ω is an irreducibly represented Lie algebra \( \Omega \simeq h \), the set of unentangled pure states is identical to the set of GCSs. Another, more physically transparent, characterization of generalized-unentangled states is as the set of states that are unique ground states of a distinguished observable (e.g., a Hamiltonian).

### 2.1 Measure of generalized entanglement

One would like to have a way to quantify how entangled a given state is. In our theory of GE (as well as in conventional entanglement theory), no single measure is able to uniquely characterize the entanglement properties of a state. Indeed, a hierarchy of characterizations of quantum correlations is necessary. In order to connect the notions of generalized unentanglement and coherence, a natural way is to introduce the quadratic purity \( P_h \) relative to the distinguished subspace of observables \( h \).

#### Definition:

Let \( \{x_i\} \) be a Hermitian \((x_i^\dagger = x_i)\), and commonly-normalized orthogonal basis for \( h \) \((\text{tr}(x_i x_j) \propto \delta_{ij})\). For any \(|\Psi\rangle \in \mathcal{H}\), the purity of \(|\Psi\rangle\) relative to \( h \), \( h \)-purity, is

\[
P_h(|\Psi\rangle) = \sum_i |\langle \Psi | x_i | \Psi \rangle|^2.
\]

This measure has the following geometric meaning: \( P_h(|\Psi\rangle) \) is the square-distance from 0 of the projection of \(|\Psi\rangle\langle \Psi|\) onto \( h \) according to the usual trace-inner-product norm. If \( h \) is a Lie algebra, it is invariant under group transformations, \( \tilde{x}_i = g^\dagger x_i g \), where \( g \in e^{ih} \).

So far, \( h \) has been assumed to be a real Lie algebra of Hermitian operators. These may be thought of as a preferred family of Hamiltonians, which generate (via \( h \mapsto e^{it h} \)) a Lie group of unitary operators, or as a preferred family of observables.

The following characterizations of unentangled states (proven in Ref. [9]) demonstrate the power of the Lie-algebraic setting (Fig. 1 summarizes equivalent statements for generalized unentangled states).

#### Theorem:

The following statements are equivalent for an irreducible representation of a Lie algebra \( h \) on \( \mathcal{H} \) and state \( \rho \)

1. \( \rho \) is generalized unentangled relative to \( h \).
2. \( \rho = |\Psi\rangle\langle \Psi| \) with \(|\Psi\rangle\) the unique ground state of some Hamiltonian \( H \) in \( h \).
3. \( \rho = |\Psi\rangle\langle \Psi| \) with \(|\Psi\rangle\) a lowest-weight vector of \( h \).
4. \( \rho \) has maximum \( h \)-purity.
3. A FEW EXAMPLES

3.1 Alice and Bob revisited

As mentioned above, this example corresponds to a bipartite situation with preferred distinguishable subsystems $A$ and $B$. The total Hilbert space $\mathcal{H} \simeq \mathcal{H}_A \otimes \mathcal{H}_B$, such that $\dim \mathcal{H}_A = m$, and $\dim \mathcal{H}_B = n$. Physical access is restricted to local observables acting on one subsystem i.e., observables in the set

$$\Omega_{\text{loc}} = \{ \hat{A} \otimes \mathbb{I} \oplus \mathbb{I} \otimes \hat{B} \} ,$$

with $\hat{A}$ and $\hat{B}$ Hermitian and traceless. For each pure state $|\Psi\rangle \in \mathcal{H}$, the reduced state $\rho_{\text{red}}$ is then determined by the pair of reduced density operators $\rho_A$ and $\rho_B$.

Theorem:

An $\Omega_{\text{loc}}$-state is pure iff it is induced by a pure product state, meaning that conventional bipartite entanglement is equivalent to GE relative to $\Omega_{\text{loc}}$.

For the particular case of two $S=1/2$ parties ($m = n = 2$), and $\Omega = \mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) = \{ S_j^x, S_j^y, S_j^z | j = 1, 2 \}$, where the three generators

$$S_j^x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_j, \quad S_j^y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_j, \quad S_j^z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_j$$

satisfy the $\mathfrak{su}(2)$ algebra $[S_j^\alpha, S_k^\beta] = i\delta_{jk}\varepsilon_{\alpha\beta\delta}S_k^\delta$ ($\varepsilon_{\alpha\beta\delta}$ being the totally antisymmetric tensor, with $\alpha, \beta, \delta = x, y, z$) and a generic normalized state $|\Psi\rangle = a_1|\uparrow\uparrow\rangle + a_2|\uparrow\downarrow\rangle + a_3|\downarrow\uparrow\rangle + a_4|\downarrow\downarrow\rangle$, the purity relative to $\mathfrak{h}$ is

$$P_\mathfrak{h}(|\Psi\rangle) = 2 \sum_{j, \alpha} \langle \Psi | S_j^\alpha | \Psi \rangle^2 ,$$

with the result

$$P_\mathfrak{h} = \begin{cases} 1 & \text{product states} \\ \vdots & \vdots \\ 0 & \text{Bell states} \end{cases} .$$

However, when physical access is unrestricted (such as $\Omega = \mathfrak{h}' = \mathfrak{su}(4)$), all pure states have maximal purity meaning that all of them are generalized unentangled.

To understand the relevance of the operator language used to describe our physical system, let us consider a maximally entangled (Bell) state

$$|\text{Bell}\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} = \frac{S_1^- - S_2^-}{\sqrt{2}} |\downarrow\downarrow\rangle,$$

where $S_j^\pm = S_j^x \pm iS_j^y$, and change our spin into a fermionic description. After a Jordan-Wigner isomorphic mapping

$$\begin{cases} S_1^+ = c_1^\dagger \\ S_2^+ = (1 - 2n_1)c_2^\dagger \end{cases}$$

the Bell state turns into $|\text{Bell}\rangle = \frac{c_1^\dagger - c_2^\dagger}{\sqrt{2}} |0\rangle_F$, where $|0\rangle_F$ is the fermionic vacuum. Written in the fermionic language this is a single-particle state which can only display mode entanglement relative to the local spin algebra $\mathfrak{h}$. However, had we asked whether $|\text{Bell}\rangle$ is entangled relative to the
\[ \left( c_1^\dagger c_2 - c_2^\dagger c_1 \right) / \sqrt{2}, n_1 - \frac{1}{2}, n_2 - \frac{1}{2} \right) = \left\{ (S_x^1 S_x^2 + S_y^1 S_y^2) / \sqrt{2}, (S_x^1 S_y^2 - S_y^1 S_x^2) / \sqrt{2}, S_z^1, S_z^2 \right\}, \]

The answer would be No. The message is that the property of entanglement is independent of the operator language used to describe a state, and only dependent on the observer.

### 3.2 A single spin \( S = 1 \)

The extent to which our viewpoint extends the subsystem-based definition may be appreciated in situations where no physically natural subsystem partition exists and conventional entanglement is not directly applicable. Consider a single spin \( S=1 \) system, whose three-dimensional state space \( \mathcal{H} \) carries an irreducible representation of \( su(2) \), with generators \( S_x, S_y, S_z \). Suppose that operational access to the system is restricted to observables in the given representation of \( su(2) \)

\[
S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]  

The reduced states can be identified with vectors of expectation values of these three observables (see Fig. 2(a)) which form a unit ball in \( \mathbb{R}^3 \), \( \langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2 \leq 1 \), and the extremal points are those on the surface, having maximal spin component 1 for some linear combination of \( S_x, S_y, S_z \). These are the spin coherent states \( |\eta\rangle \), or GCSs for SU(2). For any choice of spin direction, \( \mathcal{H} \) is spanned by the \( |1\rangle, |0\rangle, |-1\rangle \) eigenstates of that spin component; the first and last are GCSs (i.e., unentangled), but \( |0\rangle \) is not, characterizing \( |0\rangle \) as a generalized (maximally) entangled state relative to \( \mathfrak{h} = su(2) \). Note, however, that all pure states appear generalized-unentangled if access to the full algebra \( \mathfrak{h}' = su(3) \) is available (that is, \( su(3) \) is distinguished). This is pictorially illustrated in Fig. 2(b).
Figure 3. Measures of entanglement for the Ising model in a transverse magnetic field ($\gamma = 1$). The blue line represents the magnetization in the $x$-direction $M_x$ (order parameter), the black line is the nearest neighbors concurrence $C(1)$, and the red line identifies the purity relative to the $u(N)$ Lie algebra, $P'_b = P_b - 1/(1 + \gamma)$. QCP stands for quantum critical point.

To appreciate the distinction between separability and generalized unentanglement, let us consider the case of two spins $S = 1$. The only product states that are generalized unentangled with respect to $h = su(2) \oplus su(2)$ are those that are products of local SU(2) spin coherent states, e.g., the state $|1\rangle \otimes |1\rangle$. Instead, product states such as $|0\rangle \otimes |0\rangle$ will be entangled (in particular, this state is maximally entangled).

4. QUANTUM PHASE TRANSITIONS AND ENTANGLEMENT

QPTs occur at zero temperature due to changes in the interactions among the constituents of the system. Each quantum phase is characterized by different behaviors of appropriate correlation functions (such as, long-range order, algebraic or exponential decay). Since the notions of entanglement and quantum correlations are intimately related, it becomes quite natural to look for a good measure of entanglement as a way to characterize the QPT.

In this section we study the paramagnetic-to-ferromagnetic QPT present in the one-dimensional ($S = 1/2$) anisotropic XY model in a transverse magnetic field, applying the concepts of GE described in the previous sections (a more expanded discussion is also given in [17]). Fortunately, this model is exactly solvable and the ground state properties are well known, allowing for a better understanding of the entanglement properties at and across the transition point.

The model Hamiltonian for an $N$ ($N$ even) spin system is given by

$$H = -g \sum_{j=1}^{N} \left[ (1 + \gamma)\sigma^z_j \sigma^z_{j+1} + (1 - \gamma)\sigma^x_j \sigma^x_{j+1} \right] + \sum_{j=1}^{N} \sigma^z_j ,$$

(13)

where the Pauli spin operators $\sigma^z_j = 2S^z_j$ (see Eq. (8)), $g \in [0, \infty)$ is the ratio between the spin-spin exchange coupling and the external magnetic field, and $\gamma \in [0, 1]$ is the anisotropy in the $xy$-plane. Periodic boundary conditions will be assumed (i.e., $\sigma^N_{\alpha} = \sigma^1_{\alpha}$). The operator $Z_2 = \prod_{j=1}^{N} j^z$ that maps $(\sigma^x_j, \sigma^y_j, \sigma^z_j) \rightarrow (-\sigma^x_j, -\sigma^y_j, \sigma^z_j), \forall j$, is a global ($\mathbb{Z}_2$) symmetry of the model, leaving the Hamiltonian
invariant \([H, Z_2] = 0\). Therefore, for finite \(N\), the non-degenerate energy eigenstates \(|E\rangle\) satisfy \(Z_2^2|E\rangle = z_2|E\rangle\), where \(z_2 = \pm 1\). The ground state \(|G\rangle\) belongs to the sector \(z_2 = +1\). For \(\gamma = 1\) the model reduces to the Ising model in a transverse magnetic field. In this case and for \(g \to 0\) the spins align in the direction of the external magnetic field (z-direction), so the ground state is \(|G\rangle = |↓↓ \cdots ↓\rangle\). On the other hand, when \(g \to \infty\) the external field becomes irrelevant and the spins align in the x-direction: \(|G\rangle \simeq |\to \cdots \to\rangle + |\leftarrow \cdots \leftarrow\rangle\), which becomes degenerate with \(|G_\perp\rangle \simeq |\to \cdots \to\rangle - |\leftarrow \cdots \leftarrow\rangle\) in the thermodynamic limit \((N \to \infty)\). Therefore, the system undergoes a QPT at some critical value \(g_c\) such that for \(g > g_c\) a long range order in the x-direction develops \((\langle \sigma^x_z \sigma^N_z \rangle = M^2_x \neq 0\), where \(M_x\) is the total magnetization in the x-direction) and the global \(\mathbb{Z}_2\) symmetry breaks spontaneously. In other words, for \(g > g_c\) the quantum phase is ferromagnetic (see Fig. 3).

The situation is similar for other anisotropies \(\gamma\). In principle, one would expect that the critical point \(g_c\) depends on the anisotropy. Remarkably, this is not the case and \(g_c = 1/2\), \(\forall \gamma\) such that \(1 \geq \gamma > 0\). (For \(\gamma = 0\), there is a second order non-broken symmetry QPT, i.e. with no symmetry order parameter, between an insulator and a quantum critical phase [17]. The universality class is different than for \(\gamma > 0\).)

### 4.1 The hydrogen atom of QPT

As mentioned before, the anisotropic XY model in a transverse magnetic field is exactly solvable. Finding its ground state is a simple task if we first map the Pauli spin operators into spinless fermionic operators through the Jordan-Wigner formula

\[
S^j_+ = \frac{\sigma^j_x + i\sigma^j_y}{2} = \left( \prod_{k=1}^{j-1} -\sigma^k_z \right) c^\dagger_j, \quad \sigma^j_z = 2n_j - 1,
\]

where the operators \(c^\dagger_j\) and \(c_j = (c^\dagger_j)^\dagger\), create and destroy a fermion at site \(j\), respectively, and \(n_j = c^\dagger_j c_j\) is the fermionic number operator.

Since the model exhibits translational symmetry, it is useful to map the fermionic operators in the site representation to the Fourier-transformed operators; i.e., to

\[
c^\dagger_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-ikj} c^\dagger_j,
\]

where the operator \(c^\dagger_k\) \((c_k = (c^\dagger_k)^\dagger)\) creates (destroys) a fermion in the mode \(k\). In the fermionic language the sector \(z_2 = +1\) corresponds to fermionic anti-periodic boundary conditions and thus \(k \in K = K_+ + K_- = \left[ \pm \frac{\pi}{N}, \pm \frac{3\pi}{N}, \ldots, \pm \frac{(N-1)\pi}{N} \right]\). Since the ground state \(|G\rangle\) belongs to this sector we will only consider these \(2^{N-1}\) states. In this new operator algebra, the Hamiltonian possess terms that correlate the \(k\) and \(-k\) modes (Cooper pairs). At this point, the Hamiltonian can be diagonalized by performing a Bogoliubov transformation; that is, by defining a new set of fermionic (quasiparticle) operators \(\gamma^\dagger_k = u_k c^\dagger_k + iv_k c_{-k}\), with \(u^2_k + v^2_k = 1\), \(u_k = u_{-k}\), and \(v_k = -v_{-k}\). The transformed Hamiltonian reads

\[
H = \sum_{k,\gamma \in K} \epsilon_k (\gamma^\dagger_k \gamma_k - 1/2),
\]

where \(\gamma^\dagger_k \gamma_k\) is the quasiparticle number operator. Since the energy per mode \(\epsilon_k = 2\sqrt{(-1 + 2g \cos k)^2 + 4g^2 \gamma^2 \sin^2 k} > 0 \forall k \in K\), the ground state of the system con-
contains no quasiparticles: \( \gamma_k |G\rangle = 0 \). Thus, \(|G\rangle = |\text{BCS}\rangle = \prod_{k \in K} (u_k + iv_k c_k^\dagger c_{-k}^\dagger)|0\rangle_F \),
where \(|0\rangle_F\) is the fermionic vacuum state (i.e., \(c_k^\dagger |0\rangle_F = 0\)), and the name \(|\text{BCS}\rangle\) refers to the Bardeen-Cooper-Schrieffer superconducting state.

### 4.2 h-Purity as an indicator of the QPT

Figure 3 displays the behavior of the concurrence \( C(1) \) (a measure of bipartite entanglement) between nearest neighbors for the case \( \gamma = 1 \) (Ising model in a transverse magnetic field) as a function of \( g \) (see also [8]). Although it is possible to show that its derivative, \( \partial_g C(1) \), diverges at the critical point \((g_c = 1/2)\), the function itself has a smooth behavior across \( g_c \), thus it does not directly characterize the QPT.

The relevant question is whether GE may be able to characterize the QPT, and to what extent. In order to answer this question we first have to identify the relevant algebra \( \mathfrak{h} \) of observables that should be considered. Since in terms of fermionic operators (after the Jordan-Wigner transformation) the Hamiltonian of the system is a linear combination of quadratic operators belonging to the Lie algebra \( \text{so}(2N) = \{c_k^\dagger c_{k'}, c_k c_{k'} \mid k, k' \in K\} \), the ground state (if not degenerate) must be a GCS of this algebra. In other words, the \(|\text{BCS}\rangle\) state can be obtained by applying \( \text{so}(2N) \) group operations to the fully polarized state (which is also a GCS of \( \text{so}(2N) \)):

\[ |\text{BCS}\rangle = \prod_{k \in K} e^{i\phi_k c_k^\dagger c_{-k}^\dagger}|0\rangle_F, \]
with \( u_k = \cos \phi_k \) and \( v_k = \sin \phi_k \). In this way, the \( \mathfrak{h} \)-purity relative to this algebra is always maximum \( (P_h = 1, \forall g, \gamma) \) and does not give information about the QPT.

The \(|\text{BCS}\rangle\) state has no longer the property of being GCS and becomes generalized entangled \( (P_h \leq 1) \) when we choose a Lie subalgebra of \( \text{so}(2N) \) that does not contain operators of the form \( c_k^\dagger c_{-k}^\dagger \). This is the case if we consider the subalgebra that conserves the number of fermions; that is \( u(N) = \{c_k^\dagger c_{k'} \mid k, k' \in K\} \).

In particular, in the \(|\text{BCS}\rangle\) state the expectation values of the operators in \( u(N) \) are \( \langle c_k^\dagger c_{k'} \rangle = d_{kk'} v_k^2 \). The \( \mathfrak{h} \)-purity becomes

\[ P_h(|\text{BCS}\rangle) = \frac{4}{N} \sum_{k \in K} \langle c_k^\dagger c_k - 1/2 \rangle^2 = \frac{4}{N} \sum_{k \in K} (v_k^2 - 1/2)^2, \quad (17) \]

where the constant 1/2 has been subtracted for normalization reasons. Then, any fermionic product state of the form \( |\psi\rangle = c_{k_1}^\dagger c_{k_2}^\dagger \cdots c_{k_n}^\dagger |0\rangle_F \) is a GCS of this algebra, thus, it has maximum purity \( (P_h = 1) \).

The \( \mathfrak{h} \)-purity relative to the \( u(N) \) algebra for the \(|\text{BCS}\rangle\) state can be exactly calculated in the thermodynamic limit (where the sum becomes an integral), yielding

\[ P_h(|\text{BCS}\rangle) = \begin{cases} \frac{1}{1-\gamma} \left[ 1 - \frac{\gamma^2}{\sqrt{1-4\gamma^2(1-\gamma^2)}} \right] & \text{for } g \leq g_c, \\ \frac{1}{1+\gamma} & \text{for } g > g_c \end{cases}, \quad (18) \]

Obviously, \( P_h = 1 \) for \( g = 0 \), where the state \(|\text{BCS}\rangle\) reduces to \(|0\rangle_F \equiv |\uparrow\uparrow \cdots \uparrow\rangle \). On the other hand, the \( \mathfrak{h} \)-purity can be related with the fluctuations in the total number of fermions in this model,

\[ P_h(|\text{BCS}\rangle) = 1 - \frac{2}{N} \langle (\hat{N}^2) - \langle \hat{N} \rangle^2 \rangle, \quad \text{where } \hat{N} = \sum_{k \in K} c_k^\dagger c_k. \quad (19) \]
Figure 4. (a) Purity (shifted) relative to the $u(N)$ algebra in the ground state $|\text{BCS}\rangle$ of the anisotropic XY model in a transverse magnetic field. (b) Scaling behavior of the purity relative to the $u(N)$ algebra.

Figure 4(a) shows the (shifted) $\mathfrak{h}$-purity vs $g$, for different anisotropies $\gamma$. Remarkably, this function behaves as a disorder parameter, being 0 in the ferromagnetic (ordered) phase and different from 0 in the paramagnetic (disordered) phase, sharply identifying the QPT. (For $\gamma = 0$, the $\mathfrak{h}$-purity with respect to $\bigoplus_j su(2)_j$ is a good indicator of the non-broken symmetry QPT [17].)

The critical behavior of the $\mathfrak{h}$-purity can also be extracted by performing a Taylor-expansion near the critical point (in the disordered phase, $\gamma > 0$). One finds $P'_h \sim (g_c - g)^\nu$ (Fig. 4(b)), with $\nu = 1$ being the critical exponent that characterizes the Ising universality class; that is, the correlation length $\ell$ in this model diverges with the same exponent (Fig. 4(b)): $\ell \sim (g_c - g)^{-\nu}$.

All these interesting features make the quadratic purity measure of GE a physically appealing measure that characterizes the QPT. In principle, for other kinds of systems one could think in terms of different algebras of observables. However, identifying the appropriate algebra remains in general an open problem.

4. CONCLUSIONS

Entanglement is of one of the key features of quantum mechanics, and consequently a deep understanding of its properties and physical implications is essential not only to fundamentally advance our understanding of how Nature works, but also to design more powerful technologies. In this work we have described a generalization of conventional entanglement which is applicable to arbitrary physical settings, going beyond the standard distinguishable-subsystem approach. It is clear from our formulation that entanglement is a (relative) context-dependent notion, whose properties are determined by the expectations of a distinguished set of observables $\Omega$ of the system of interest, thus being observer-dependent. The concept is independent of the operator language used to describe the system but depends upon the way we can physically access information. (In the QIP setting locality is usually involved.)
Most importantly, we have tied together the theory of entanglement and the theory of
generalized coherent states in those cases where the set \( \Omega \) forms a Lie algebra \( \mathfrak{h} \), thus
establishing that generalized unentangled states are the (highest-weight) generalized
coherent states of \( \mathfrak{h} \). We have also introduced measures of generalized entanglement.
In particular, we introduced the purity relative to a subalgebra which can represent
a global measure of entanglement. Different choices of observable subalgebras may
provide different information on the hierarchical structure of quantum correlations.

In the second part of the paper we discussed an application of our theory of
generalized entanglement to the study of quantum phase transitions. We showed
that the purity relative to a subalgebra can be used as a quantum phase transition indicator. Indeed, it is a useful diagnostic tool playing the role of a disorder (or order
[7]) parameter for broken-symmetry quantum phase transitions. Several important open questions remain. In particular: Given the model Hamiltonian describing a
physical system, what is the minimum number of observable algebras \( \mathfrak{h} \) one needs
to fully characterize a quantum critical point in arbitrary space dimensions? How
would one choose them?

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