Correlated random walks with a finite memory range

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Abstract. We study a family of correlated one-dimensional random walks with a finite memory range $M$. These walks are extensions of the Taylor’s walk as investigated by Goldstein, which has a memory range equal to one. At each step, with a probability $p$, the random walker moves either to the right or to the left with equal probabilities, or with a probability $q = 1 - p$ performs a move, which is a stochastic Boolean function of the $M$ previous steps. We first derive the most general form of this stochastic Boolean function, and study some typical cases which ensure that the average value $< R_n >$ of the walker’s location after $n$ steps is zero for all values of $n$. In each case, using a matrix technique, we provide a general method for constructing the generating function of the probability distribution of $R_n$; we also establish directly an exact analytic expression for the step-step correlations and the variance $< R_n^2 >$ of the walk. From the expression of $< R_n^2 >$, which is not straightforward to derive from the probability distribution, we show that, for $n$ going to infinity, the variance of any of these walks behaves as $n$, provided $p > 0$. Moreover, in many cases, for a very small fixed value of $p$, the variance exhibits a crossover phenomenon as $n$ increases from a not too large value. The crossover takes place for values of $n$ around $1/p$. This feature may mimic the existence of a non-trivial Hurst exponent, and induce a misleading analysis of numerical data issued from mathematical or natural sciences experiments.
1 Introduction.

Soon after its early developments in probability theory [1], the study of random walks and their applications has gradually carved its way into the concerns of many topics in exact or applied science. In various recent developments, the notion of “walk” in the proper sense of the term has gradually blurred, giving way to a more general concept where a “step” of the walk is not necessarily an algebraic distance between two locations but may become a time interval between two consecutive events of a chronological series, or even a sequence of real numbers.

Before mentioning a few significant fields of application, let us quote more specifically the notion of “correlated random walk,” which underlies the possibility for a step to depend on the values taken by some (or all) of the previous steps. We will address the step correlations and their consequences at length in the course of this paper, and will consider therefore the “direct” problem, in contradistinction with the usual concern prevailing in applied science which asks for answers to the “inverse” problem, i.e., can a given set of data be suitably described by a (correlated or uncorrelated) random walk formalism? Examples of fields resorting to the inverse problem are neuromuscular [2] and cardiovascular medicine [3], structural genetics [4-6], physiology [7], regulation of biological rhythms [8]; to these fields pertaining essentially to the medical domain can be added the soaring irruption of financial analysis [9-11] in view of market predictions.

In this paper we will focus on the formalism of correlated random walks on one-dimensional lattices and discuss the role of memory in some typical models which can be solved exactly. With the help of a direct evaluation of the variance of the walks, we will be able to prove that all walks with short-range memory have a Hurst exponent equal to $\frac{1}{2}$ in the strict limit of an infinite number of steps, although the apparent Hurst exponent may look different from $\frac{1}{2}$. In particular, we will endeavor to produce a simple argument for a necessary condition yielding a Hurst exponent larger than $\frac{1}{2}$.

2 Correlated walks and memory

The transition probabilities in a simple random walk on a one-dimensional infinite lattice are constant and independent of the course of the walk. If these transition probabilities depend upon the $M$ previous steps, we have a correlated random walk whose memory range is $M$. Correlated random walks were introduced by Taylor [12] in an analysis of diffusion by continuous motions. The Taylor’s model is a one-dimensional random walk on a lattice, in which steps can be made to nearest neighbors only. The walker has a probability $\alpha$ to repeat his previous move, and a probability $1 - \alpha$ to move in the opposite direction. The resulting correlated random walk has a memory range equal to 1. If $\alpha > \frac{1}{2}$ (resp. $\alpha < \frac{1}{2}$) the walk is said to be persistent (resp. antipersistent). A complete analysis of the Taylor’s model has been given by Goldstein [13]. More examples
of persistent random walks have been studied by many other authors [14-16]. A detailed bibliography may be found in Weiss [17].

In this paper we study a class of correlated random walks with memory range \( M > 1 \). Our model may be described as follows. Let \((S_k)\) be an infinite sequence of random variables such that, for \( k > M \),

\[
S_k = \begin{cases} 
1 & \text{with probability } \frac{1}{2} p \\
-1 & \text{with probability } \frac{1}{2} p \\
f_M(S_{k-1}, S_{k-2}, \ldots, S_{k-M}) & \text{with probability } q = 1 - p,
\end{cases}
\]

where \( f_M \), called the memory function, is a Boolean function from \( \{-1,1\}^M \) onto \( \{-1,1\} \). For \( k \leq M \), the \( S_k \)'s are independent, identically distributed (iid) Bernoulli random variables, each \( S_k \) taking the values \( \pm 1 \) with equal probabilities. The values of these \( M \) random variables define the initial conditions for the walk. The position of the random walker after \( n \) steps is

\[ R_n = S_1 + S_2 + \cdots + S_n. \]

The set \( \mathcal{F}_M \) of all bounded functions defined on \( \{-1,1\}^M \) is a Euclidean space of dimension \( 2^M \) for the dot product defined by

\[
< f^{(1)}_M | f^{(2)}_M > = \left( \frac{1}{2} \right)^M \text{tr} f^{(1)}_M (S_{k-1}, S_{k-2}, \ldots, S_{k-M}) f^{(2)}_M (S_{k-1}, S_{k-2}, \ldots, S_{k-M}),
\]

where the trace operator \( \text{tr} \) is a sum over all the possible values of the \( M \) variables \( S_{k-1}, S_{k-2}, \ldots, S_{k-M} \). The set of the following \( 2^M \) functions

\[
\begin{align*}
1, \\
S_{k-1}, S_{k-2}, \ldots, S_{k-M}, \\
S_{k-1}S_{k-2}, S_{k-1}S_{k-3}, \ldots, S_{k-M}S_{k-M}, \\
S_{k-1}S_{k-2}S_{k-3}, \ldots, S_{k-M}S_{k-M}S_{k-M}, \\
\cdots, \\
S_{k-1}S_{k-2} \cdots S_{k-M},
\end{align*}
\]

is a complete orthonormal system, and any function in \( \mathcal{F}_M \) can be written as a linear combination of these \( 2^M \) functions. The most general correlated random walk with a finite memory of range \( M \) is, therefore, defined by a sequence \((S_k)\) of random variables such that, for \( k > M \),

\[
S_k = \begin{cases} 
1 & \text{with probability } \frac{1}{2} p \\
-1 & \text{with probability } \frac{1}{2} p \\
\epsilon_{j_1j_2 \cdots j_r} S_{k-j_1} S_{k-j_2} \cdots S_{k-j_r} & \text{with probability } qa_{j_1j_2 \cdots j_r},
\end{cases}
\]

where \( p + q = 1 \), \( 1 \leq j_1 < j_2 < \cdots < j_r \leq M \), and \( \epsilon_{j_1j_2 \cdots j_r} \), which are non-random, are equal either to 1 or to -1. The \( a \)'s are conditional probabilities.
(some of them possibly equal to zero) satisfying the completeness relation:

$$
\sum_{1 \leq j_1 \leq M} a_{j_1} + \sum_{1 \leq j_1 < j_2 \leq M} a_{j_1,j_2} + \cdots \\
+ \sum_{1 \leq j_1 < j_2 < \cdots < j_r \leq M} a_{j_1,j_2-\cdots-j_r} + \cdots + a_{12\cdots M} = 1.
$$

As above, for \( k \leq M \), \( S_k \) is a simple Bernoulli random variable which takes the values \( \pm 1 \) with equal probabilities.

In order to simplify this model, we require that, as in the case of a simple symmetric random walk, the average value \( < R_n > \) of the position of the random walker after \( n \) steps should be equal to zero. While this condition is automatically satisfied if \( n \leq M \), a tedious inspection shows that, for \( n > M \), the above requirement is satisfied if, and only if, the memory function \( f_M \) has the following simple forms:

- \( f_M(S_{k-1}, S_{k-2}, \ldots, S_{k-M}) \) is a linear combination of multilinear terms of odd degree in the \( S_k \)'s;
- \( f_M(S_{k-1}, S_{k-2}, \ldots, S_{k-M}) \) reduces to a single multilinear term containing an even number of \( S_k \)'s.

For example, if \( M = 3 \), the most general form for the memory function \( f_3 \), satisfying the condition \( < R_n > = 0 \), is either

$$
f_3(S_{k-1}, S_{k-2}, S_{k-3}) = a_1\varepsilon_1 S_{k-1} + a_2\varepsilon_2 S_{k-2} + a_3\varepsilon_3 S_{k-3} \\
+ a_{123}\varepsilon_{123} S_{k-1} S_{k-2} S_{k-3},
$$

or

$$
f_3(S_{k-1}, S_{k-2}, S_{k-3}) = a_{ij}\varepsilon_{ij} S_{k-i} S_{k-j},
$$

with \( 1 \leq i \leq 3 \) and \( 1 \leq j \leq 3 \).

In the sequel we will only consider two cases: either \( f_M \) is a linear function of the \( M \) previous steps, or the product of the \( M \) previous steps.

Let us briefly examine these two cases.

In what follows \( (X_k) \) denotes an infinite sequence of \( iid \) Bernoulli random variables such that, for all positive integers \( k \), we have

$$
P(X_k = 1) = P(X_k = -1) = \frac{1}{2},
$$

1. \( f_M \) is linear. If \( (S_k) \) is an infinite sequence of random variables defined by

$$
S_k = \begin{cases} 
X_k & \text{if } k \leq M, \\
X_k & \text{with probability } p, \text{ if } k > M, \\
\varepsilon_j S_{k-j} & \text{with probability } qa_j, \text{ if } k > M,
\end{cases}
$$

(2)
where, for \( j = 1, 2, \ldots, M \), the \( \varepsilon_j \) have fixed values chosen \textit{a priori} equal either to 1 or \(-1\), and the \( a_j \) are \( M \) nonnegative real numbers such that

\[
a_1 + a_2 + \cdots + a_M = 1.
\]

It is clear that, for \( k \leq M \), \( < S_k > = 0 \). Hence

\[
< S_{M+1} > = p < X_{M+1} > + qa_1 \varepsilon_1 < S_M > + \cdots + qa_M \varepsilon_M < S_1 > = 0
\]

\[
< S_{M+2} > = p < X_{M+2} > + qa_1 \varepsilon_1 < S_{M+1} > + \cdots + qa_M \varepsilon_M < S_2 > = 0
\]

\[
\ldots
\]

\[
< S_{M+\ell} > = p < X_{M+\ell} > + qa_1 \varepsilon_1 < S_{M+\ell-1} > + \cdots + qa_M \varepsilon_M < S_\ell > = 0,
\]

and, therefore, \( < R_n > = 0 \), for all positive integers \( n \).

2. \( f_M \) is a single product of \( r \) terms. If \( (S_k) \) is an infinite sequence of random variables defined by

\[
S_k = \begin{cases} 
X_k & \text{if } k \leq M, \\
X_k & \text{with probability } p \text{ if } k > M, \\
\varepsilon_{j_1,j_2,\ldots,j_r}S_{k-j_1}S_{k-j_2}\cdots S_{k-j_r} & \text{with probability } q \text{ if } k > M,
\end{cases}
\]

where \( q = 1 - p \), \( 1 \leq j_1 < j_2 < \cdots < j_r \leq M \), and \( \varepsilon_{j_1,j_2,\ldots,j_r} \) is given and equal to either 1 or \(-1\). Here again, for \( k \leq M \), \( < S_k > = 0 \). Hence, assuming first that \( r = M \), we have

\[
< S_{M+1} > = p < X_{M+1} > + q \varepsilon_{12\ldots M} < S_1S_2\cdots S_M >
\]

\[
= p < X_{M+1} > + q \varepsilon_{12\ldots M} < S_1 > < S_2 > \cdots < S_M > = 0
\]

\[
< S_{M+2} > = p < X_{M+2} > + q \varepsilon_{12\ldots M} < S_2S_3\cdots S_MS_{M+1} >
\]

\[
= pq \varepsilon_{12\ldots M} < S_2S_3\cdots S_MX_{M+1} > + q^2 \varepsilon_{12\ldots M} < S_1S_2^2S_3^2\cdots S_M^2 > = q^2 \varepsilon_{12\ldots M} < S_1 > = 0
\]

where in the expression of \( S_{M+2} \) we have replaced \( S_{M+1} \) by its expression. More generally,

\[
< S_{M+\ell} > = p < X_{M+\ell} > + q < S_{\ell}S_{\ell+1}\cdots S_{M+\ell-1} >
\]

\[
= pq < S_\ell S_{\ell+1}\cdots X_{M+\ell-1} > + q^2 < S_{\ell-1}S_\ell^2S_{\ell+1}\cdots S_{M+\ell-2}^2 > = q^2 < S_{\ell-1} >,
\]

which shows that for all positive integers \( k \), \( < S_k >= 0 \), and, consequently, \( < R_n >= 0 \).
Along similar lines, it can be shown that the result $< R_n > = 0$ remains valid for $r < M$, although its proof requires a larger number of substitutions.

In the following sections we will show that, for any finite value of $M$, we can determine exactly the probability distribution of $< R_n >$, using a transfer matrix method. It will turn out that the rank of the matrix is equal to $2^M$. Due to the increasing complexity of the eigensystem problem with increasing $M$, this method becomes rapidly cumbersome, although it rather easily provides a numerical solution if the number of steps $n$ is not too large. In order to illustrate the method, we will first briefly review the Goldstein’s solution of the Taylor’s model.

Since the asymptotic behavior of the variance $< R_n^2 >$ is of particular interest to natural scientists, we will directly derive its exact expression through a recursive method that applies at any finite space dimensionality. From this expression we will be able to show that the asymptotic behavior $< R_n^2 >$, when $n$ goes to infinity, exhibits a crossover phenomenon.

3 Taylor’s model.

The Taylor’s model corresponds to Model 2 for $M = 1$, that is,

$$S_1 = X_1,$$

and, for all $k > 1$,

$$S_k = \begin{cases} X_k, & \text{with probability } p \\ \varepsilon S_{k-1}, & \text{with probability } q = 1 - p, \end{cases}$$

where $(X_k)$ is a sequence of iid Bernoulli variables such that, for all positive integers $k$,

$$P(X_k = -1) = P(X_k = 1) = \frac{1}{2},$$

and $\varepsilon$ is a non-random number equal to $\pm 1$.

If $p_n(k)$ denotes the probability that the walker will be at site $k$ after the $n$th step has been completed, we have

$$p_n(k) = p_n^+(k) + p_n^-(k),$$

where the superscript $\pm$ refers to the sign of the $n$-th step, that is

$$p^+(k, n) = P((R_n = k) \cap (S_n = 1)).$$

Taking into account the conditional probabilities:

\begin{align*}
P(S_n = 1 \mid S_{n-1} = 1) &= P(S_n = -1 \mid S_{n-1} = -1) = \frac{1}{2} (p + (1 + \varepsilon)q) \\
P(S_n = 1 \mid S_{n-1} = -1) &= P(S_n = -1 \mid S_{n-1} = 1) = \frac{1}{2} (p + (1 - \varepsilon)q),
\end{align*}
we obtain the following recursion relations

\[ p^+_n(k) = \frac{1}{2} (p + (1 + \varepsilon)q) p^+_{n-1}(k - 1) + \frac{1}{2} (p + (1 - \varepsilon)q) p^-_{n-1}(k - 1) \]

\[ = \frac{1}{2} (1 + \varepsilon) p^+_{n-1}(k - 1) + \frac{1}{2} (1 - \varepsilon) q p^-_{n-1}(k - 1) \]

(4)

\[ p^+_n(k) = \frac{1}{2} (p + (1 - \varepsilon)q) p^+_{n-1}(k + 1) + \frac{1}{2} (p + (1 + \varepsilon)q) p^-_{n-1}(k + 1) \]

\[ = \frac{1}{2} (1 - \varepsilon) p^+_{n-1}(k + 1) + \frac{1}{2} (1 + \varepsilon) q p^-_{n-1}(k + 1) \]

(5)

or, in a more condensed form which will be useful when we consider the case \( M > 1 \),

\[ p^\sigma_n(k) = \frac{1}{2} (1 - \varepsilon q) p^+_{n-1}(k - \sigma) + \frac{1}{2} (1 + \varepsilon q) p^-_{n-1}(k - \sigma), \]

(6)

with \( \sigma = \pm 1 \). The probability \( \alpha \) in Goldstein’s notations coincides here with \( \frac{1}{2} (1 + \varepsilon q) \).

The generating function of the probability distribution of the random walk is defined by

\[ f_n(x) = f^+_n(x) + f^-_n(x), \]

with

\[ f^+_n(x) = \sum_k p^+_n(k) x^k \]

\[ f^-_n(x) = \sum_k p^-_n(k) x^k, \]

where the summation index \( k \) runs from \(-n\) to \( n\) with step 2. Let

\[ f_n(x) = \begin{pmatrix} f^+_n(x) \\ f^-_n(x) \end{pmatrix} \]

and

\[ M(x) = \begin{pmatrix} \frac{1}{2} (1 + \varepsilon q) x & \frac{1}{2} (1 - \varepsilon q) x \\ \frac{1}{2} (1 - \varepsilon q) \frac{1}{x} & \frac{1}{2} (1 + \varepsilon q) \frac{1}{x} \end{pmatrix}. \]

Then

\[ f_n(x) = M(x) f_{n-1}(x). \]

(7)

Iterating this recursion relation, we obtain

\[ f_n(x) = M^{n-1}(x) f_1(x), \]

(8)

where

\[ f_1(x) = \begin{pmatrix} f^+_1(x) \\ f^-_1(x) \end{pmatrix} = \begin{pmatrix} \frac{x}{2} \\ \frac{1}{2x} \end{pmatrix}. \]
Let $\mathbf{R}(x)$ be the transformation matrix such that

$$
\mathbf{D}(x) = \mathbf{R}^{-1}(x) \mathbf{M}(x) \mathbf{R}(x) = \begin{pmatrix}
\lambda_1(x) & 0 \\
0 & \lambda_2(x)
\end{pmatrix},
$$

where $\lambda_1(x)$ and $\lambda_2(x)$ are the eigenvalues of $\mathbf{M}(x)$. Then

$$
f_n(x) = \mathbf{R}(x) \mathbf{D}^{n-1}(x) \mathbf{R}^{-1}(x) f_1(x)
$$

The above relation involves the eigenvalues and eigenvectors of $\mathbf{M}(x)$ which can be readily obtained. However, the exact expression for the generating function of $p_n(k)$ is cumbersome to manipulate, and does not easily yield the variance. Therefore, we resort to an alternative and more direct method, which allows for a recursive evaluation of $< R_n^2 >$

$$
<R_n^2> = < (S_1 + S_2 + \cdots + S_n)^2 >
$$

$$
= < (S_1 + S_2 + \cdots + S_{n-1})^2 > + 2 < S_n (S_1 + S_2 + \cdots + S_{n-1}) > + < S_n^2 >
$$

$$
= < R_{n-1}^2 > + 2 < S_n (S_1 + S_2 + \cdots + S_{n-1}) > + < S_n^2 >.
$$

As will appear below, the correlation coefficient $< S_n S_{n-k} >$ does not depend explicitly upon $n$, thus we use the notation

$$
c_k = < S_n S_{n-k} >,
$$

and obtain

$$
<R_n^2> = < R_{n-1}^2 > + 2(c_1 + c_2 + \cdots + c_{n-1}) + 1,
$$

where we have taken into account that $< S_n^2 > = 1$. Evaluating the $c_k$’s, we get

$$
c_1 = < S_n S_{n-1} > = p < X_n S_{n-1} > + \varepsilon q < S_{n-1}^2 > = \varepsilon q
$$

$$
c_2 = < S_n S_{n-2} > = p < X_n S_{n-2} > + \varepsilon q < S_{n-1} S_{n-2} > = \varepsilon q c_1 = q^2
$$

$$
\cdots
$$

$$
c_k = < S_n S_{n-k} > = p < X_n S_{n-k} > + \varepsilon q < S_{n-1} S_{n-k} > = \varepsilon q c_{k-1} = (\varepsilon q)^k.
$$

Substituting the $c_k$’s in the expression of $< R_n^2 >$ we obtain

$$
<R_n^2> = < R_{n-1}^2 > + 2 (\varepsilon q + q^2 + \cdots + (\varepsilon q)^{n-1}) + 1
$$

$$
= < R_{n-1}^2 > + 2 \frac{\varepsilon q - (\varepsilon q)^n}{1 - \varepsilon q} + 1,
$$

and iteration of this recursion relation yields

$$
<R_n^2> = n + \frac{2}{(1 - \varepsilon q)^2} \left( (n - 1)\varepsilon q - n q^2 + (\varepsilon q)^{n+1} \right)
$$

$$
= n + \frac{2}{(1 - \varepsilon (1-p))^2} \times \left( (n - 1)\varepsilon (1-p) - n (1-p)^2 + (\varepsilon)^{n+1} (1-p)^{n+1} \right).
$$

(9)
It is quite remarkable that for a similar random walk on a $d$-dimensional simple cubic lattice, the above result is preserved. Of course, in this case, the sequence of one-dimensional $X_k$’s is replaced by a sequence of $d$-dimensional random vectors $X_k$’s, and each elementary step can be performed along each of the $d$ axes, either in the positive or the negative direction. For any fixed nonzero value of $p$, when $n$ goes to infinity, the term $(1 - p)^{n+1}$ goes to zero, and $< R_n^2 >$ behaves as

$$\frac{1 + \varepsilon q}{1 - \varepsilon q} n,$$

and the random walk is eventually Gaussian at very large $n$. However, when $q$ goes to 1, according to whether $\varepsilon$ is equal to $+1$ or $-1$, the prefactor of $n$ goes, respectively, either to infinity or to zero. Consequently, if $p$ is very small, the asymptotic behavior of $< R_n^2 >$ should be carefully analyzed, distinguishing the two cases $\varepsilon = 1$ and $\varepsilon = -1$.

3.0.1 Persistent random walk ($\varepsilon = 1$).

Replacing $\varepsilon$ by $+1$ in (9), and expanding the resulting expression in powers of $p$, we obtain

$$< R_n^2 > = n + \frac{2}{p^2} \left( \frac{n^2 p^2}{2} - \frac{np^2}{2} - \frac{n(n^2 - 1)p^3}{6} + O(p^4) \right),$$

which shows that, as expected, when $p$ tends to zero, $< R_n^2 >$ tends to $n^2$. Therefore, for a small fixed value of $p$, when $n$ increases, we expect, for the variance, a crossover from a $n^2$ behavior to an $n$ behavior. This crossover is controlled by the term $(1 - p)^{n+1}$ in (9) and the above expansion of $< R_n^2 >$ shows that the relevant parameter is the variable $a = np$. To characterize this behavior, let define the exponent

$$E(n, p) = \frac{\partial \log < R_n^2 >}{\partial \log n},$$

which is twice the Hurst exponent. A simple calculation yields

$$\lim_{p \to 0} E\left( \frac{a}{p}, p \right) = \frac{a(1 - e^{-a})}{a + e^{-a} - 1}.$$

This expression shows that, if $n$ is large but small compared to $1/p$ (i.e., $a$ is small), the exponent $E$ tends to 2, whereas if $n$ is large compared to $1/p$ (a is large), $E$ tends to 1. This result is illustrated in Figure 1 for $p = 0.00005$. Note that the crossover takes place around $n = e^{11} \approx 6 \times 10^4$, $np$ being of the order of unity.

The crossover from $E(n, p) = 2$ to $E(n, p) = 1$, clearly observed in the above figure, where $p$ has a small fixed value and $n$ varies, can also be evidenced from the figures representing the probability distribution for a fixed value of $p$:

- Figure 2: at small $p$ ($np \ll 1$) most of the weight of the distribution is concentrated at the edges of the distribution range.

9
1.2. Exponent $E(n,p)$ of the persistent walk as a function of $\log n$, for $p = 5 \times 10^{-5}$ and $n_{\text{max}} = 1.6 \times 10^7$.

- Figure 2 at large $p$ ($np \gg 1$) the significant weight of the distribution spreads out in a Gaussian-like profile, with a width roughly proportional to $\sqrt{n}$.

Figure 2: $p_n(k)$ for $M = 1$, $\varepsilon = 1$, $p = 0.002$ and $n = 100$.

Figure 3: $p_n(k)$ for $M = 1$, $\varepsilon = 1$, $p = 0.2$ and $n = 100$.

The fulfillment of both conditions (with perhaps the possibility that, in the case corresponding to $np \ll 1$, the localization takes place at various points of the distribution range, and not necessarily at the edges, as, for example, for the antipersistent walk discussed below) will be from now on considered as a sufficient condition for the occurrence of a crossover.

3.0.2 Antipersistent random walk ($\varepsilon = -1$).

Clearly, if $p = 0$ the walk will be a zigzag walk, and $< R_n^2 > = 0$ (resp. $< R_n^2 > = 1$) if $n$ is even (resp. odd). If $p$ is small, a calculation similar to the previous one shows that:
• If \( np \gg 1 \), \( \langle R_n^2 \rangle \) behaves as \( \frac{1}{2} np \), and the oscillatory terms related to the parity of \( n \) are negligible.
• If \( np \ll 1 \), we find that \( \langle R_n^2 \rangle \) behaves as \( np \) when \( n \) is even, and as 1 when \( n \) is odd. Both expressions are the order of unity or less, so that \( \langle R_n^2 \rangle \) behaves as \( n^0 \).

Figure 4 represents the exponent characterizing the behavior of the variance. In order to avoid, for values of \( n \) such that \( np \ll 1 \), the spurious behavior of \( \langle R_n^2 \rangle \) in \( n \) for even values of \( n \), we have only considered the odd values of \( n \).

Figure 4: Exponent \( E(n, p) \) of the antipersistent walk as a function of \( \log n \) (\( n \) odd), for \( p = 5 \times 10^{-5} \) and \( n_{\text{max}} = 1.6 \times 10^7 \).

4 Linear memory function.

We first address explicitly the case \( M = 2 \), as a significant illustration of the complexity which manifests as soon as \( M > 1 \). Even in this case, a closed form for the probability distribution is already difficult to write down. However, we provide a formalism that can be fruitfully applied, at least for numerical purposes, to larger values of \( M \).

4.1 Probability distribution for \( M = 2 \).

Let \( p_{n}^{\sigma_1,\sigma_2}(k, n) \) denote the probability that, after \( n \) steps, the random walker is at site \( k \) and the steps \( S_{n-1} \) and \( S_n \) are, respectively, equal to \( \sigma_1 \) and \( \sigma_2 \), that is

\[
p_{n}^{\sigma_1,\sigma_2}(k) = P((R_n = k) \cap ((S_{n-1} = \sigma_1) \cap (S_n = \sigma_2))).
\]
If \( S_{n-1} = \sigma_1 \) and \( S_n = \sigma_2 \), then \( R_{n-1} = k - \sigma_2 \), and \( p_n^{\sigma_1 \sigma_2}(k) \) is a linear combination of two probabilities of the form \( p_n^{\sigma_1}(k - \sigma_2) \) where \( \sigma = \pm 1 \):

\[
p_n^{\sigma_1 \sigma_2}(k) = \frac{1}{2} (1 + q(a_1 \sigma_1 \sigma_2 \varepsilon_1 + a_2 \sigma_2 \varepsilon_2)) p_n^{\sigma_1}(k - \sigma_2) \\
+ \frac{1}{2} (1 + q(a_1 \sigma_1 \sigma_2 \varepsilon_1 - a_2 \sigma_2 \varepsilon_2)) p_n^{\sigma_1}(k - \sigma_2).
\]

(10)

To this equation we associate the equation expressing \( p_n^{\sigma_1 \sigma_2}(k) \), where \( \sigma_2 = -\sigma_2 \), since this probability is a linear combination of the two probabilities \( p_n^{\sigma_1}(k - \sigma_2) = p_n^{\sigma_1}(k + \sigma_2) \) where \( \sigma = 1 \) or \(-1\). We have

\[
p_n^{\sigma_1 \sigma_2}(k) = \frac{1}{2} (1 + q(a_1 \sigma_1 \sigma_2 \varepsilon_1 + a_2 \sigma_2 \varepsilon_2)) p_n^{\sigma_1}(k - \sigma_2) \\
+ \frac{1}{2} (1 + q(a_1 \sigma_1 \sigma_2 \varepsilon_1 - a_2 \sigma_2 \varepsilon_2)) p_n^{\sigma_1}(k - \sigma_2).
\]

(11)

The generating function of the probability distribution of \(< R_n >\) is defined by

\[
f_n(x) = f_n^{++}(x) + f_n^{+-}(x) + f_n^{--}(x) + f_n^{--}(x),
\]

with

\[
f_n^{\sigma_1 \sigma_2}(x) = \sum_k p_n^{\sigma_1 \sigma_2}(k) x^k,
\]

where the summation index \( k \) runs from \(-n\) to \( n \) with step 2. From Equations \[10\] and \[11\], replacing \( \sigma_1 \) and \( \sigma_2 \) by all their possible respective values, we obtain

\[
\begin{pmatrix}
  f_n^{++}(x) \\
  f_n^{+-}(x)
\end{pmatrix} = M^+(x) \begin{pmatrix}
  f_{n-1}^{++}(x) \\
  f_{n-1}^{+-}(x)
\end{pmatrix},
\]

(12)

where

\[
M^+(x) = \begin{pmatrix}
  \frac{1}{2} (1 + q(a_1 \varepsilon_1 + a_2 \varepsilon_2)) x & \frac{1}{2} (1 + q(a_1 \varepsilon_1 - a_2 \varepsilon_2)) x \\
  \frac{1}{2} (1 - q(a_1 \varepsilon_1 + a_2 \varepsilon_2)) x & \frac{1}{2} (1 - q(a_1 \varepsilon_1 - a_2 \varepsilon_2)) x
\end{pmatrix},
\]

(13)

and

\[
\begin{pmatrix}
  f_n^{+-}(x) \\
  f_n^{--}(x)
\end{pmatrix} = M^-(x) \begin{pmatrix}
  f_{n-1}^{+-}(x) \\
  f_{n-1}^{--}(x)
\end{pmatrix},
\]

(14)

where

\[
M^-(x) = \begin{pmatrix}
  \frac{1}{2} (1 - q(a_1 \varepsilon_1 + a_2 \varepsilon_2)) x & \frac{1}{2} (1 - q(a_1 \varepsilon_1 - a_2 \varepsilon_2)) x \\
  \frac{1}{2} (1 + q(a_1 \varepsilon_1 - a_2 \varepsilon_2)) x & \frac{1}{2} (1 + q(a_1 \varepsilon_1 + a_2 \varepsilon_2)) x
\end{pmatrix}.
\]

(15)

Equations \[12\] and \[14\] can be grouped together in a unique equation that reads

\[
\begin{pmatrix}
  f_n^{++}(x) \\
  f_n^{+-}(x) \\
  f_n^{+-}(x) \\
  f_n^{--}(x)
\end{pmatrix} = \begin{pmatrix}
  M^+(x) & 0 \\
  0 & M^-(x)
\end{pmatrix} \begin{pmatrix}
  f_{n-1}^{++}(x) \\
  f_{n-1}^{+-}(x) \\
  f_{n-1}^{+-}(x) \\
  f_{n-1}^{--}(x)
\end{pmatrix}.
\]
Since the column vector on the right hand side of the above equation is not correctly ordered, we have to introduce a $4 \times 4$ permutation matrix, and write this equation under the final form

$$f_n(x) = M(x) P_4 f_{n-1}(x)$$

(16)

where

$$f_n(x) = \begin{pmatrix} f_{n}^{++}(x) \\ f_{n}^{-+}(x) \\ f_{n}^{+-}(x) \\ f_{n}^{--}(x) \end{pmatrix},$$

$$M(x) = \begin{pmatrix} M^+(x) & 0 \\ 0 & M^-(x) \end{pmatrix},$$

and

$$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Iterating Relation (16), we obtain

$$f_n(x) = (M(x) P_4)^{n-2} f_2(x),$$

(17)

where

$$f_2(x) = \begin{pmatrix} f_{2}^{++}(x) \\ f_{2}^{-+}(x) \\ f_{2}^{+-}(x) \\ f_{2}^{--}(x) \end{pmatrix} = \begin{pmatrix} \frac{x^2}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4x^2} \end{pmatrix}.$$

Further progress toward the expression of the generating function $f_n(x)$ requires the explicit diagonalization of the $4 \times 4$ matrix $M(x)P_4$, followed by the procedure outlined in the $M = 1$ case. The formal derivation of $f_n(x)$ is extremely complicated, and would reveal untractable for $M > 2$ as will be seen below. However, the formalism is well suited to numerically evaluate the probability distribution of $R_n$ recursively using (17). Figures 5 and 6 show the probability distribution $p_n(k)$ when $a_1 = a_2 = 0.5$ and $\varepsilon_1 = \varepsilon_2 = 1$ for two values of $p$ to illustrate cases $np \ll 1$ and $np \gg 1$.

Clearly, at the light of the comment at the end of subsection 3.0.1, the set of these two figures is a sure sign of the existence of a crossover.
4.2 Variance for \( M = 2 \).

Correlations are more tedious to evaluate than in the case \( M = 1 \). Let

\[ c_{1, \ell} = < S_\ell S_{\ell+1} >; \]

then

\[
\begin{align*}
  c_{1,1} &= < S_1 S_2 > = 0 \\
  c_{1,2} &= < S_2 S_3 > = < S_2(pX_3 + q(a_1 \varepsilon_1 S_2 + a_2 \varepsilon_2 S_1)) > = qa_1 \varepsilon_1
\end{align*}
\]

and, for \( \ell \geq 3 \),

\[
\begin{align*}
  c_{1, \ell} &= < S_\ell S_{\ell+1} > = < S_\ell(pX_{\ell+1} + qa_1 \varepsilon_1 S_\ell + qa_2 \varepsilon_2 S_{\ell-1}) > = qa_1 \varepsilon_1 + qa_2 \varepsilon_2 c_{1, \ell-1}.
\end{align*}
\]

Hence

\[ c_{1, \ell} = qa_1 \varepsilon_1 \frac{1 - (qa_2 \varepsilon_2)^\ell}{1 - qa_2 \varepsilon_2} \quad (\ell \geq 1) \]  

(18)

Similarly, if

\[ c_{2, \ell} = < S_\ell S_{\ell+2} >; \]

then, for all positive integers \( \ell \),

\[
\begin{align*}
  c_{2, \ell} &= < S_\ell S_{\ell+2} > = < S_\ell(pX_{\ell+2} + q(a_1 \varepsilon_1 S_{\ell+1} + a_2 \varepsilon_2 S_\ell)) > = qa_1 \varepsilon_1 c_{1, \ell} + qa_2 \varepsilon_2
\end{align*}
\]

so that

\[ c_{2, \ell} = \frac{(qa_1 \varepsilon_1)^2}{1 - qa_2 \varepsilon_2}((1 - qa_2 \varepsilon_2)^\ell - 1) + qa_2 \varepsilon_2 \quad (\ell \geq 1). \]  

(19)

More generally, for \( r > 2 \),

\[
\begin{align*}
  c_{r, \ell} &= < S_\ell S_{\ell+r} > = < S_\ell(pX_{\ell+r} + q(a_1 \varepsilon_1 S_{k+r-1} + a_2 \varepsilon_2 S_{k+r-2})) > = q(a_1 \varepsilon_1 c_{r-1, \ell} + a_2 \varepsilon_2 c_{r-2, \ell}).
\end{align*}
\]  

(20)
From this recursion relation, we obtain the following expression of $c_{r,\ell}$

$$c_{r,\ell} = \frac{c_{1,\ell}\lambda_2 - c_{2,\ell}}{\lambda_2 - \lambda_1} \lambda_1^{r-1} + \frac{c_{2,\ell} - c_{1,\ell}}{\lambda_2 - \lambda_1} \lambda_2^{r-1},$$

(21)

where $\lambda_1$ and $\lambda_2$ are the roots of

$$\lambda^2 - qa_1\varepsilon_1\lambda - qa_2\varepsilon_2 = 0.$$

It is easy to verify that $|\lambda_1| \leq 1$ and $|\lambda_2| \leq 1$, the equal sign being possible if $p = 0$. For $p > 0$, these conditions imply that the correlations $c_{r,\ell}$ decrease exponentially in $r$, and this, in turn, will ensure a linear dependence in $n$ for the variance.

Since the variance is given by

$$< R_n^2 > = < (S_1 + S_2 + \cdots + S_n)^2 >$$

$$= n + 2 \sum_{\ell=1}^{n-1} \sum_{r=1}^{n-\ell} c_{r,\ell},$$

to find its explicit expression, we need to evaluate the double summation. Summing first over $r$, we obtain

$$\sum_{r=1}^{n-\ell} c_{r,\ell} = \frac{1 - \lambda_1^{n-\ell}}{(\lambda_2 - \lambda_1)(1 - \lambda_1)} \left( \frac{qa_1\varepsilon_1}{1 - qa_2\varepsilon_2} (1 - (qa_2\varepsilon_2)^{\ell-1}) (\lambda_2 - qa_1\varepsilon_1) - qa_2\varepsilon_2 \right)$$

$$+ \frac{1 - \lambda_2^{n-\ell}}{(\lambda_2 - \lambda_1)(1 - \lambda_2)} \left( \frac{qa_1\varepsilon_1}{1 - qa_2\varepsilon_2} (1 - (qa_2\varepsilon_2)^{\ell-1}) (qa_1\varepsilon_1 - \lambda_1) + qa_2\varepsilon_2 \right),$$

where we have replaced $c_{2,\ell}$ by its expression given by (21). Summing now over $\ell$ finally yields

$$< R_n^2 > = n + \frac{2(n-1)}{(\lambda_2 - \lambda_1)(1 - \lambda_1)} \left( \frac{qa_1\varepsilon_1 (\lambda_2 - qa_1\varepsilon_1)}{1 - qa_2\varepsilon_2} - qa_2\varepsilon_2 \right)$$

$$- \frac{2}{(\lambda_2 - \lambda_1)(1 - \lambda_1)} \times \frac{qa_1\varepsilon_1 (\lambda_2 - qa_1\varepsilon_1)}{1 - qa_2\varepsilon_2} \times \frac{1 - (qa_2\varepsilon_2)^{n-1}}{1 - qa_2\varepsilon_2}$$

$$- \frac{2}{(\lambda_2 - \lambda_1)(1 - \lambda_1)} \times \frac{qa_1\varepsilon_1 (\lambda_2 - qa_1\varepsilon_1)}{1 - qa_2\varepsilon_2} \times \frac{\lambda_1 (1 - \lambda_1^{n-1})}{1 - \lambda_1}$$

$$+ \frac{2}{(\lambda_2 - \lambda_1)(1 - \lambda_1)} \times \frac{qa_1\varepsilon_1 (\lambda_2 - qa_1\varepsilon_1)}{1 - qa_2\varepsilon_2} \times \frac{\lambda_1 (1 - \lambda_1^{n-1} - (qa_2\varepsilon_2)^{n-1})}{1 - qa_2\varepsilon_2}$$

$$+ \frac{2(n-1)}{(\lambda_2 - \lambda_1)(1 - \lambda_2)} \left( \frac{qa_1\varepsilon_1 (qa_1\varepsilon_1 - \lambda_1)}{1 - qa_2\varepsilon_2} + qa_2\varepsilon_2 \right)$$

$$- \frac{2}{(\lambda_2 - \lambda_1)(1 - \lambda_2)} \times \frac{qa_1\varepsilon_1 (qa_1\varepsilon_1 - \lambda_1)}{1 - qa_2\varepsilon_2} \times \frac{1 - (qa_2\varepsilon_2)^{n-1}}{1 - qa_2\varepsilon_2}$$

$$- \frac{2}{(\lambda_2 - \lambda_1)(1 - \lambda_2)} \times \frac{qa_1\varepsilon_1 (qa_1\varepsilon_1 - \lambda_1)}{1 - qa_2\varepsilon_2} \times \frac{\lambda_2 (1 - \lambda_2^{n-1})}{1 - \lambda_2}$$

$$+ \frac{2}{(\lambda_2 - \lambda_1)(1 - \lambda_2)} \times \frac{qa_1\varepsilon_1 (qa_1\varepsilon_1 - \lambda_1)}{1 - qa_2\varepsilon_2} \times \frac{\lambda_2 (\lambda_2^{n-1} - (qa_2\varepsilon_2)^{n-1})}{1 - qa_2\varepsilon_2}.$$
For our purpose, a complete analytic discussion of this expression would take us too far. Instead, we will illustrate the behavior of the variance with four typical choices for the set of the parameters $\varepsilon_1, \varepsilon_2, a_1, a_2$ in Figures 7, 8, 9, and 10. Note that we obtain crossovers, respectively, similar to those obtained for the persistent and antipersistent walks of the Taylor’s model when $\varepsilon_2 = 1$ and $\varepsilon_1$ is either equal to 1 or $-1$. On the contrary when $\varepsilon_2 = -1$, there is no crossover, the variance behaves as $n$ for both signs of $\varepsilon_1$.

Figure 7: Log-log plot of $< R_n^2 >$ vs. $n$ for $M = 2$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $a_1 = a_2 = 0.5$, $p = 5 \times 10^{-5}$, $n_{\text{max}} = 1.6 \times 10^8$.

Figure 8: Log-log plot of $< R_n^2 >$ vs. $n$ for $M = 2$, $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, $a_1 = a_2 = 0.5$, $p = 5 \times 10^{-5}$, $n_{\text{max}} = 1.6 \times 10^8$.

Figure 9: Log-log plot of $< R_n^2 >$ vs. $n$ for $M = 2$, $\varepsilon_1 = -1$, $\varepsilon_2 = 1$, $a_1 = a_2 = 0.5$, $p = 5 \times 10^{-5}$, $n_{\text{max}} = 1.6 \times 10^8$.

Figure 10: Log-log plot of $< R_n^2 >$ vs. $n$ for $M = 2$, $\varepsilon_1 = -1$, $\varepsilon_2 = -1$, $a_1 = a_2 = 0.5$, $p = 5 \times 10^{-5}$, $n_{\text{max}} = 1.6 \times 10^8$.

4.3 Linear memory for $M > 2$.

The matrix method used for the case $M = 2$ is formally easy to generalize. It yields, for the generating function, a recursion relation similar to (16) where $f_n$ is a $2^M$-component vector, $M$ a $2^M \times 2^M$ matrix and $P_4$ is replaced by
a \(2^M \times 2^M\) permutation matrix \(P_{2^M}\). With increasing values of \(M\) it becomes rapidly untractable. Unfortunately, due to the large number of parameters when \(M > 2\), even the variance \(\langle R_n^2 \rangle\) is unmanageable. This is not the case for the single product memory, as will be seen in the next section. However, there is, at least, one case for which we can predict that the variance exhibits the crossover observed in the persistent walk of the Taylor’s model. This occurs when all the \(\varepsilon_j (j = 1, 2, \ldots, M)\) are equal to 1, since in this case, it is easy to verify that for \(p = 0\), the variance behaves as \(n^2\).

5 Single product memory function.

Here again we will first consider the case \(M = 2\) as an illustration of the complexity which manifests as soon as \(M > 1\). Since we will only consider the case of a single product, the notation \(\varepsilon_{j_1, j_2, \ldots, j_r}\) will be replaced by \(\varepsilon\).

5.1 Probability distribution for \(M = 2\).

As in the linear memory function case, \(p^{\sigma_1, \sigma_2}_n(k)\) denotes the probability \(P((R_n = k) \cap ((S_{n-1} = \sigma_1) \cap (S_n = \sigma_2)))\), and as above, if \(S_{n-1} = \sigma_1\) and \(S_n = \sigma_2\), then \(R_{n-1} = k - \sigma_2\), which implies that \(p^{\sigma_1, \sigma_2}_n(k)\) is a linear combination of the two probabilities \(p^{\sigma_1, \sigma_2}_n(k - \sigma_2)\), where \(\sigma = \pm 1\). Since, with a probability \(q\), \(S_n = \varepsilon S_{n-1} S_{n-2}\), we have \(\sigma_2 = \varepsilon \sigma_1\sigma_1\) or \(\sigma = \varepsilon \sigma_1\sigma_2\). Therefore, using notations similar to those of the linear case, we have

\[
\begin{pmatrix}
  f_{n+}^{++}(x) \\
  f_{n+}^{-}(x)
\end{pmatrix} = M^+ \begin{pmatrix}
  f_{n-}^{++}(x) \\
  f_{n-}^{-}(x)
\end{pmatrix},
\]

where

\[
M^+(x) = \begin{pmatrix}
  \frac{1}{2}(1 + q\varepsilon) x & \frac{1}{2}(1 - q\varepsilon) x \\
  \frac{1}{2}(1 - q\varepsilon) \frac{1}{x} & \frac{1}{2}(1 + q\varepsilon) \frac{1}{x}
\end{pmatrix},
\]

(22)

and

\[
\begin{pmatrix}
  f_{n-}^{++}(x) \\
  f_{n-}^{-}(x)
\end{pmatrix} = M^- \begin{pmatrix}
  f_{n+}^{++}(x) \\
  f_{n+}^{-}(x)
\end{pmatrix},
\]

where

\[
M^-(x) = \begin{pmatrix}
  \frac{1}{2}(1 - q\varepsilon) x & \frac{1}{2}(1 + q\varepsilon) x \\
  \frac{1}{2}(1 + q\varepsilon) \frac{1}{x} & \frac{1}{2}(1 - q\varepsilon) \frac{1}{x}
\end{pmatrix}.
\]

(23)

The above equations can be grouped together in a unique equation that reads

\[
\begin{pmatrix}
  f_{n+}^{++}(x) \\
  f_{n+}^{-}(x) \\
  f_{n-}^{++}(x) \\
  f_{n-}^{-}(x)
\end{pmatrix} = \begin{pmatrix} M^+(x) & 0 \\
  0 & M^-(x) \end{pmatrix} \begin{pmatrix}
  f_{n-}^{++}(x) \\
  f_{n-}^{-}(x)
\end{pmatrix}.
\]

17
Since the column vector on the right hand side of the above equation is not correctly ordered, we have to introduce a $4 \times 4$ permutation matrix, and write this equation under the final form

$$f_n(x) = M(x) \, P_4 \, f_{n-1}(x)$$

where

$$f_n(x) = \begin{pmatrix} f_{n+}^+(x) \\ f_{n-}^-(x) \\ f_{n+}^-(x) \\ f_{n-}^+(x) \end{pmatrix},$$

$$M(x) = \begin{pmatrix} M^+(x) & 0 \\ 0 & M^-(x) \end{pmatrix}$$

and

$$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Iterating Relation 24, we obtain

$$f_n(x) = (M(x) \, P_4)^{n-2} \, f_2(x),$$

where

$$f_2(x) = \begin{pmatrix} f_2^+(x) \\ f_2^+(x) \\ f_2^+(x) \\ f_2^-(x) \end{pmatrix} = \begin{pmatrix} \frac{x^2}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4x^2} \end{pmatrix}.$$

As in the linear case, further progress toward the expression of the generating function $f_n(x)$ is extremely complicated, and would reveal untractable for $M > 2$. However the formalism is well suited to numerically evaluate the probability distribution of $R_n$ recursively using 25. Figures 11, 12, 13, 14 show the probability distribution $p_n(k)$ for two values of $p$ to illustrate cases $np \ll 1$ and $np \gg 1$ when $\varepsilon = \pm 1$.

At the light of the comment at the end of subsection 3.0.1, Figures 11, 12, 13, 14 reveal the existence of crossovers. This property will be confirmed directly in the next subsection.
5.2 Variance for $M = 2$.

In order to evaluate recursively the variance, we need to determine the correlations. We have

\[
< S_{\ell} S_{\ell+1} > = \varepsilon q < S_\ell (S_\ell S_{\ell-1}) > = \varepsilon q < S_{\ell-1} >= 0;
\]

\[
< S_{\ell} S_{\ell+2} > = \varepsilon q < S_\ell (S_{\ell+1} S_\ell) > = \varepsilon q < S_{\ell+1} >= 0;
\]

\[
< S_{\ell} S_{\ell+3} > = \varepsilon q < S_\ell (S_{\ell+2} S_{\ell+1}) > = \varepsilon^2 q^2 < S_\ell ((S_{\ell+1} S_\ell) S_{\ell+1}) > = q^2;
\]

and, for $m > \ell + 3$,\[
< S_{\ell} S_m > = \varepsilon q < S_\ell (S_{m-1} S_{m-2}) > = \varepsilon^2 q^2 < S_\ell ((S_{m-2} S_{m-3}) S_{m-2}) > = q^2 < S_\ell S_{m-3} > .
\]

These results provide all the correlations:

\[
< S_\ell S_m > = \begin{cases} 
0 & \text{if } m - \ell \text{ is not a multiple of 3,} \\
q^{2(m-\ell)/3} & \text{if } m - \ell \text{ is a multiple of 3.}
\end{cases}
\]

(26)

Note that the correlations do not depend upon $\varepsilon$. On the other hand, the variance satisfies the recursion relation

\[
< R^2_n >=< R^2_{n-1} > + 1 + 2 < S_1 S_n + S_2 S_n + \cdots + S_{n-1} S_n > .
\]

Figure 11: $p_n(k)$ for $M = 2$, $\varepsilon = 1$, $p = 0.005$, and $n = 50$.  
Figure 12: $p_n(k)$ for $M = 2$, $\varepsilon = 1$, $p = 0.5$, and $n = 50$.  

19
Iterating this relation and taking into account $26$, we finally obtain
\[
<R_n^2> = n + (n-3)q^2 + (n-6)q^4 + \cdots + (n-3\lfloor n/3 \rfloor)q^{2\lfloor n/3 \rfloor} \\
= n + nq^2 \frac{1 - q^{2\lfloor n/3 \rfloor}}{1 - q^2} \\
-3q^2 \frac{\lfloor n/3 \rfloor q^{2\lfloor n/3 \rfloor+2} - (\lfloor n/3 \rfloor + 1)q^{2\lfloor n/3 \rfloor} + 1}{(1-q^2)^2},
\]
(27)
where $\lfloor x \rfloor$ denotes the largest integer less or equal to $x$. On this expression of $<R_n^2>$ we readily verify that for $q = 0$ the variance is equal to $n$, while for $q = 1$ it takes the value
\[
n + (2n-3)\lfloor n/3 \rfloor - 3(\lfloor n/3 \rfloor)^2 = \lceil n^2/3 \rceil,
\]
where $\lceil x \rceil$ denotes the smallest integer greater or equal to $x$. Since the functions floor and ceiling are not differentiable, we define the crossover exponent $E(n, p)$ by
\[
E(n, p) = \frac{n}{2} \frac{<R_{n+1}^2> (p) - <R_{n-1}^2> (p)}{<R_n^2> (p)}.
\]
Figure 15 represents the variation of $E(n, p)$ as a function of $\log n$ for $p = 5 \times 10^{-5}$.

5.3 Probability distribution for $M > 2$.

The matrix method used for the case $M = 2$ is formally easy to generalize. It yields, for the generating function, a recursion relation similar to (24), where $f_n$ is a $2^M$-component vector, $M$ a $2^M \times 2^M$ matrix and $P_4$ is replaced by a $2^M \times 2^M$ permutation matrix $P_{2^M}$. With increasing values of $M$ it becomes rapidly untractable. However, the variance $<R_n^2>$ can be exactly evaluated as we will show in next subsection.
5.4 Variance for $M > 2$.

In order to evaluate recursively the variance, we need to determine the correlations. By definition, $< S_\ell S_m > = 0$ if $\ell < m \leq M$. This is also true if $m = M + 1$:

$$< S_\ell S_{M+1} > = q \varepsilon < S_\ell (S_1 S_2 \cdots S_M) >$$
$$= q \varepsilon < S_1 > < S_2 > \cdots < S_M > = 0,$$

due the fact that the random variables $S_1, S_2, \ldots, S_M$ are independent, and $< S_\ell^2 > = 1$.

Considering now the case: $< S_\ell S_m >$ (any $\ell$ and $m > \ell$), we first note that

$$< S_\ell S_{\ell+M+1} > = q \varepsilon < S_\ell (S_{\ell+1} S_{\ell+2} \cdots S_{\ell+M}) >$$
$$= (q \varepsilon)^2 < (S_\ell S_{\ell+1} \cdots S_{\ell+M-1})^2 >$$
$$= q^2,$$

whereas, in the general case,

$$< S_\ell S_{\ell+r} > = q \varepsilon < S_\ell (S_{\ell+r-M} S_{\ell+r-M+1} \cdots S_{\ell+r-1}) >$$
$$= q^2 < S_\ell S_{\ell+r-(M+1)}>,$$  \hspace{1cm} (28)

where it is assumed that $\ell + r - (M + 1)$ is positive but may be smaller than $\ell$.

Iterated application of (28) in the presence of the initial conditions shows that, for $m - \ell = \left\lfloor \frac{m-\ell}{M+1} \right\rfloor + b$ ($0 \leq b < M + 1$),

$$< S_\ell S_m > = \begin{cases} 
0 & \text{if } b > 0 \\
q^2 \left\lfloor \frac{b}{M+1} \right\rfloor & \text{if } b = 0
\end{cases}$$  \hspace{1cm} (29)
Substitution of 29 in the recursion relation satisfied by the variance:

\[ < R_n^2 > = < R_{n-1}^2 > + 1 + 2 < S_1 S_n + S_2 S_n + \cdots + S_{n-1} S_n > . \]

yields

\[
< R_n^2 > = n + (n - (M + 1))q^2 + (n - 2(M + 1))q^4 + \cdots
\]
\[
+ \left( n - \left\lfloor \frac{n}{M+1} \right\rfloor (M + 1) \right) q^{2\left\lfloor \frac{n}{M+1} \right\rfloor}
\]
\[
= n + nq^2 \frac{1 - q^{2\left\lfloor \frac{n}{M+1} \right\rfloor}}{1 - q^2} - (M + 1)q^2
\]
\[
\times 1 - \left( \left\lfloor \frac{n}{M+1} \right\rfloor + 1 \right) q^{2\left\lfloor \frac{n}{M+1} \right\rfloor} + \left\lfloor \frac{n}{M+1} \right\rfloor q^{2\left\lfloor \frac{n}{M+1} \right\rfloor + 2} \right),
\]

which is the generalization for any \( M \) of the result 27 obtained for \( M = 2 \). Note that \( M = 1 \) is not a particular case of 30.

When \( q = 0 \), \( < R_n^2 > = n \), whereas, when \( p \) goes to zero \( (q \to 1) \), \( < R_n^2 > \) tends to

\[
= n + (2n - (M + 1)) \left\lfloor \frac{n}{M+1} \right\rfloor,
\]

which is of the order of \( \frac{n^2}{M+1} \) for large \( n \). These two limiting behaviors are the signal of a crossover, which is evidenced in Figure 16. This figure shows that the variance is a decreasing function of \( M \) at a given \( n \).

Figure 16: Log-log plot of the variance as a function of \( n \), for \( M = 2, 8, 32, 128 \) (to be read downwards), \( p = 5 \times 10^{-5} \) and \( n_{\max} = 1.6 \times 10^8 \).

The general expression of the variance is an even function of \( q \), and, therefore, does not depend upon \( \varepsilon \). Such is not the case for the probability distribution \( p_n(k) \) of \( R_n \) as in Figures 17 and 18.

Here the probability distribution are symmetric. This is not always the case as can be observed in Figures 11, 12, 13, and 14.
6 Conclusion and perspectives

We have studied a family of correlated random walks with a finite memory range. These walks are a generalization of the Taylor’s walk. We have established a matrix formalism for the generating functions of the probability distributions of these walks. We have also derived in many cases an analytic expression for the variance. As a function of the number of steps \( n \), for each of these walks, the variance becomes ultimately linear in \( n \) in the limit \( n \to \infty \). However, at comparatively small \( n \), a different behavior of the variance can be observed leading to the existence of a crossover phenomenon. This feature shows that, when analyzing experimental results, one finds a Hurst exponent different from \( \frac{1}{2} \), this does not necessarily imply an infinite-range memory. If, experimentally feasible, one should increase significantly the number of steps to cross-check the stability of the exponent before concluding that the system under consideration exhibits a non-Gaussian behavior.

The ultimate Gaussian behavior of the variance at very large \( n \) is a direct consequence of the exponential decay of the step-step correlations as a function of the absolute value of the difference between step indices. Therefore, in order to find a different behavior, one has to consider step-step correlations decreasing more slowly than exponentially. This is what may happen in the case of a power law decay as can be shown as follows. For a symmetric random walk, the variance is given by

\[
< R_n^2 > = n + 2 \sum_{\ell=1}^{n-1} \sum_{m=\ell+1}^{n} < S_\ell S_m >
\]

\[
= n + 2 \sum_{\ell=1}^{n-1} \sum_{r=1}^{n} < S_\ell S_{\ell+r} >
\]

\[
= n + 2 \sum_{\ell=1}^{n-1} \sum_{r=1}^{n} c_{r,\ell},
\]
where as above $c_{r,\ell} = \langle S_{r} S_{r+\ell} \rangle$. If we now assume that, for some reason,

$$c_{r,\ell} = \frac{1}{r^\alpha}$$

independently of $\ell$, then

$$\langle R_n^2 \rangle = n + 2 \sum_{r=1}^{n-1} \frac{n-r}{r^\alpha}.$$

The possible non-Gaussian behavior of $\langle R_n^2 \rangle$ can only result from the asymptotic behavior at large $n$ of the summation in the right hand side of the above relation. According to the value of $\alpha$, the asymptotic behaviors of

$$S(n, \alpha) = \sum_{r=1}^{n-1} \frac{n-r}{r^\alpha}$$

depends on $\alpha$ as follows:

- if $\alpha > 1$, $S(n, \alpha) \sim n \zeta(\alpha)$, where $\zeta$ is the Riemann function.
- if $\alpha = 1$, $S(n, 1) \sim n \log n$.
- if $\alpha < 1$, $S(n, \alpha) \sim \frac{1}{(1-\alpha)(2-\alpha)} n^{2-\alpha}$.

Note that the power law decay for the step-step correlations is only a sufficient condition to obtain a non-trivial Hurst exponent equal to $1 - \alpha/2$.

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