Polar Varieties in Cayley-Klein Spaces

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ABSTRACT
In this paper, we introduce the notion of a total polar for an arbitrary subspace of a Cayley-Klein space in an analytical framework. We show that the set of all total polars of a subspace is a Schubert variety. The notion of total polar gives a definition for a subspace to be tangent to the absolute figure of the space. By specifying tangent lines, tangent cones and then spheres are defined. This definition of the sphere does not depend on the metric of the space. It is proved that every reflection of a Cayley-Klein space, defined by two subspaces which are total polar to each other, is a motion of the space. On the other hand, each motion in a Cayley-Klein space of dimension $n$ is a product of at most $n + 1$ reflections in point-hyperplane pairs.

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Cayley-Klein space; total polar; Schubert variety; tangent cone; reflection

1. Introduction

Projective geometry is all geometry. This is the famous assertion of the great English geometer Arthur Cayley appeared in his sixth Memoir upon Quanitics \cite{1} in 1859. Whereas up to this time, the projective geometry had been regarded as a part of metric geometry, Cayley considered the Euclidean plane within the projective plane and obtained the Euclidean metric only by using projective notions. In Cayley’s words: ‘A chief object of the present memoir is the establishment, upon purely descriptive principles, of the notion of the distance’. It seems a paradoxical statement, because we have no distances and no angles in projective geometry. In fact, Cayley was able to construct the distance between two points and the angle between two lines by considering a quadric, namely two imaginary points on the line at infinity, in the projective plane. Ten years later, Klein in \cite{3} took up this idea of Cayley and derived distances between points and angles between lines of the hyperbolic and elliptic planes by fixing quadrics in projective plane. These two non-Euclidean spaces along with the Euclidean space and some other ones such as the Minkowskian and Galilean spaces are nowadays called Cayley-Klein spaces. A Cayley-Klein space of dimension $n$ is an $n$-dimensional real projective space with a sequence of quadrics, called the absolute figure of the space, in which each quadric is defined in the vertex of the previous
one and the last quadric is non-degenerate. As the literature of this field, we refer the reader to the report [7] by I. M. Yaglom et al, as well as the textbook [2] of O. Giering.

For a non-degenerate quadric $Q$ in an $n$-dimensional real projective space, that yields a Cayley-Klein space with the absolute figure $Q$, there exists a polarity. This polarity associates to each $k$-dimensional subspace $K$ the polar of it with respect to $Q$ which is a subspace of dimension $n - k - 1$. Giering has used the term total polar to refer to the polar of a subspace with respect to a quadric. In general Cayley-Klein spaces, he has defined this notion only for some special subspaces which are called regular by him. For each regular subspace of dimension $k$ there corresponds a unique total polar which is a subspace of dimension $n - k - 1$. This incomplete generalization of polarity in an arbitrary Cayley-Klein space gives rise to an incomplete definition of the notions of orthogonality or reflection in these spaces. The notion of orthogonality is defined in [2] only for regular subspaces. Every reflection of a Cayley-Klein space is defined in [7] by means of two subspaces that one of them is a polar of the other. Although both of these subspaces may be considered non-regular, a clear definition of polars for non-regular subspaces does not exist there. Until now few researchers have addressed the problem of associating a polar (or total polar) to an arbitrary subspace in a Cayley-Klein space. Richter-Gebert in his book [4] has focused on Cayley-Klein planes. His approach is completely different from Giering. He has introduced a primal-dual pair of conics for defining the absolute figure of a Cayley-Klein plane. For each point and each line of the plane there corresponds a line and a point as a polar respectively which is not necessarily unique. He has proved that each reflection of a Cayley-Klein plane is a motion of the plane. The notion of orthogonality is also defined for any two given lines by using the dual conic. H. Struve and R. Struve in [5,6] have provided a purely synthetic framework for defining Cayley-Klein spaces. They have presented a criterion in this framework for determining when two arbitrary subspaces of a Cayley-Klein space of dimension $n$ are polar to each other. It is proved that all polars of a $k$-dimensional subspace have dimension $n - k - 1$. They have shown that every reflection of a Cayley-Klein space leaves invariant the absolute figure of the space. Also the notion of orthogonality has been defined for any two given subspaces by them. Although both definitions for polars agree with Giering’s definition for regular subspaces, there are some disagreements between them which are expressed in the next section.

This paper aims to give a new definition for polars in a Cayley-Klein space in an analytical framework. The definition is in agreement with [2] for regular subspaces. In section two, we give an overview of the existing definitions for polars and in the third section, we introduce our definition for this notion. The definition associates to each $k$-dimensional subspace of a Cayley-Klein space of dimension $n$, a family of $(n - k - 1)$-dimensional subspaces as total polars. It is proved that this family of subspaces is a Schubert variety ensuring the existence of at least one total polar for each subspace. Also we give the necessary and sufficient conditions for the uniqueness of a total polar and get some important properties of total polars. It is shown that if $K$ is a subspace of $K'$, then every total polar of $K$ contains a total polar of $K'$ and every total polar of $K'$ can be extended to a total polar of $K$. In section four, we give the definition for a subspace to be tangent to the absolute figure of a Cayley-Klein space. Achieving a complete description of tangent lines, enables us to define tangent cones and present their equations. Using these tangent cones, we will obtain the equations of all spheres of the space without applying its metric. In section five, we will give the definition of a reflection in a Cayley-Klein space. We show that every reflection is a motion of the space and each motion of a Cayley-Klein space of dimension $n$ is a product of at most
2. A Survey of Definitions of Polars

In this section, we give a brief overview of the existing definitions of total polars (or polars) in Cayley-Klein spaces. Consider a projective space $P(V)$ for a vector space $V$ over a field $\mathbb{F}$. That is, the quotient of $V \setminus \{0\}$ by the equivalence relation that identifies a nonzero vector $\alpha x$ with any other vector $\alpha x$ for $\alpha \neq 0$. We denote the point $X = [x]$ of $P(V)$ by $X(x)$ if needed. We write $P^n(\mathbb{R})$ and $P^n(\mathbb{C})$ rather than $P(\mathbb{R}^{n+1})$ and $P(\mathbb{C}^{n+1})$ respectively. For a point $X((x_0, \ldots, x_n))$ of $P^n(\mathbb{R})$ the column matrix $(x_0 \ldots x_n)^T$, called the coordinate matrix of $X$, is denoted by $X$. Also the coordinate matrix of a hyperplane of $P^n(\mathbb{R})$ with the equation $a_0 x_0 + \ldots + a_n x_n = 0$ is the row marix $(a_0 \ldots a_n)$. A quadric $Q$ in $P(V)$ is equal to the set $\{X(x) \mid f(x, x) = 0\}$ for a symmetric bilinear form $f : V \times V \to \mathbb{F}$ which is not identical to the zero form. We say that the point $X(x)$ is conjugate to a point $Y(y)$ with respect to $Q$ if $f(x, y) = 0$. The polar of a subspace $K$ with respect to $Q$, denoted by $K^P$, is defined to be the set of all points of $P(V)$ that are conjugate to every point of $K$. For the simplicity of notation, for each point $X$ of $P(V)$ we write $X^p$ instead of $\{X\}^p$. The vertex of $Q$ is defined to be the polar of $P(V)$. The quadric $Q$ is called non-degenerate if its vertex is empty. If $X$ is a point which is not in the vertex, $X^p$ is a hyperplane of $P(V)$; otherwise $X^p = P(V)$. In the first case, $X^p$ is called the polar hyperplane of $X$. If $V^C$ is the complex extension of a real vector space $V$, $P(V)$ is embeddable into its complex extension $P(V^C)$ and every quadric in $P(V)$ can be extended to a quadric in $P(V^C)$. Now we give the formal definition of a Cayley-Klein space according to [2].

**Definition 2.1.** Set $A_0 = P^n(\mathbb{R})$ and let $Q_i$ be a quadric in $A_i$ with the vertex $A_{i+1}$ for $0 \leq i \leq r$. The real projective space $P^n(\mathbb{R})$ with the following sequence

$$Q_0 \supseteq \hat{A}_1 \supseteq Q_1 \supseteq \hat{A}_2 \supseteq \cdots \supseteq \hat{A}_r \supseteq Q_r \supseteq \hat{A}_{r+1} = \emptyset$$

is a Cayley-Klein space of dimension $n$ where for $0 \leq i \leq r$, $\hat{Q}_i$ and $\hat{A}_{i+1}$ are the complex extensions of $Q_i$ and $A_{i+1}$ respectively. The above sequence is called the absolute figure of the space.

Giering has defined total polars for some subspaces which are called regular by him.

**Definition 2.2.** Consider an $n$-dimensional Cayley-Klein space with the absolute figure

$$Q_0 \supseteq \hat{A}_1 \supseteq Q_1 \supseteq \hat{A}_2 \supseteq \cdots \supseteq \hat{A}_r \supseteq Q_r \supseteq \hat{A}_{r+1} = \emptyset.$$

A subspace $K$ of $P^n(\mathbb{R})$ satisfying $K \cap A_l \neq \emptyset$ and $K \cap A_{l+1} = \emptyset$ for an integer $l$, $0 \leq l \leq r$, is called a regular subspace, if $K + A_l = P^n(\mathbb{R})$.

In this case the total polar of $K$ is defined to be the polar of $K \cap A_l$ with respect to $Q_l$. By regularity, the total polar of a $k$-dimensional subspace is of dimension $n-k-1$. Throughout this paper, we denote the polar of $K \cap A_j$ with respect to $Q_j$ by $(K \cap A_j)^P$, for $0 \leq j \leq r$. In particular $K^P = K^{P_0}$.

For instance, the absolute figure of Euclidean plane is as follows. The first conic is a line $L_\infty$ equal to its own vertex and the second one is a non-degenerate conic $Q$. 

$n + 1$ reflections in point-hyperplane pairs.
consisting of two imaginary points. The line \( L_\infty \) which is considered as the line at infinity in Euclidean plane, is the only non-regular line of the plane. The total polar of any other line \( L \) is the polar of the point \( L \cap L_\infty \) with respect to \( Q \). Each point of the plane not at infinity is a regular point whose total polar is \( L_\infty \).

Richter-Gebert in [4] has defined a polar for each point and each line of a Cayley-Klein plane. He describes the absolute figure of the plane by means of a primal-dual pair of conics. The dual conic represents a set of lines consisting of all tangent lines to the primal conic. This pair of conics is given by a pair \((A, B)\) of real symmetric nonzero \( 3 \times 3 \) matrices that satisfy \( AB = \lambda I \) for a real number \( \lambda \). Let \( (x_0 \ x_1 \ x_n)^T \) and \( (l_0 \ l_1 \ l_2 \ ) \) be the coordinate matrices of a point \( X \) and line \( L \) respectively. \( X \) and \( L \) are said to be polar to each other if there exist constants \( \lambda, \mu \in \mathbb{R} \) such that \( AX = \lambda L^T \) and \( LB = \mu X^T \). For example, matrices \( A \) and \( B \) used for constructing the absolute figure of Euclidean plane are

\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

The matrix \( A \) gives the point conic \( x_0^2 = 0 \) which is denoting the line \( L_\infty \) and \( B \) describes the line conic \( l_1^2 + l_2^2 = 0 \). This conic consists of all lines passing through the point with coordinate matrices \((0 \ 1 \ i)^T\) or \((0 \ 1 \ -i)^T\) denoting the two imaginary points at infinity. In this way to each regular point and line there corresponds a unique polar which is equal to the one given by Giering. All points of the plane are polars of the line \( L_\infty \). For each point \( X \) on this line there has corresponded the pencil of lines through the polar of \( X \) with respect to \( Q \).

H. Struve and R. Struve in [5,6] establish a correspondence between subspaces of an arbitrary Cayley-Klein space associating at least one polar to each subspace. They have defined Cayley-Klein spaces in a synthetic framework, in which the ambient structure is a projective lattice. Let \( (L, \wedge, \vee) \) be a projective lattice with the least and the greatest elements 0 and 1 respectively. Polarities of a projective lattice are anti-automorphisms of order 2. The synthetic definition of a Cayley-Klein space introduced by them is as follows. \( (L, ((\varepsilon_0, \varepsilon_1), \pi_0), \ldots, ((\varepsilon_r, \varepsilon_{r+1}), \pi_r)) \) is a Cayley-Klein lattice of dimension \( n \geq 0 \), if the following conditions are satisfied:

1. \( L \) is a projective lattice of finite length \( n + 1 \).
2. \( 1 = \varepsilon_0 > \varepsilon_1 > \cdots > \varepsilon_{r+1} = 0 \) is a chain of \( L \).
3. \( \pi_k, 0 \leq k \leq r \), is a polarity on the interval \([\varepsilon_k, \varepsilon_{k+1}] = \{ \alpha \in L | \varepsilon_k \geq \alpha \geq \varepsilon_{k+1} \} \).

They bring the notion of a polar for an arbitrary element of \( L \), by considering for each \( k, 0 \leq k \leq r \), the projection \( \varphi_k \) which is a mapping from \( L \) onto \([\varepsilon_k, \varepsilon_{k+1}] \) with \( \varphi_k(\alpha) = (\alpha \wedge \varepsilon_k) \vee \varepsilon_{k+1} \). For a given \( \alpha \in L \), \( \beta \in L \) is called a polar of \( \alpha \) if \( \pi_k(\varphi_k(\alpha)) = \varphi_k(\beta) \) for \( 0 \leq k \leq r \). Since the set of all subspaces of a projective space is a projective lattice, it has shown that every \( n \)-dimensional Cayley-Klein space associates to a Cayley-Klein lattice of dimension \( n \) [5]. The lattice associated to the Cayley-Klein space with the absolute figure

\[
\hat{Q}_0 \supseteq \hat{A}_1 \supseteq \hat{Q}_1 \supseteq \hat{A}_2 \supseteq \cdots \supseteq \hat{A}_r \supseteq \hat{Q}_r \supseteq \hat{A}_{r+1} = \emptyset.
\]

is \( (L(P^0(\mathbb{R})), ([A_0, A_1], \pi_0), \ldots, ([A_r, A_{r+1}], \pi_r)) \), where \( L(P^0(\mathbb{R})) \) is the set of all subspaces of \( P^n(\mathbb{R}) \) and \( \pi_i \) is a polarity of \([A_i, A_{i+1}] \) mapping \( K, A_{i+1} \subseteq K \subseteq A_i \), to \( K^{p_i} \) for \( 0 \leq i \leq r \). In this way, all polars of a subspace in a Cayley-Klein space are
determined. It has proved that all polars of a $k$-dimensional subspace have dimension $n-k-1$. Consider the absolute figure of Euclidean plane again in which the first conic is the line $L_\infty$ and the second one is the conic $Q$ consisting of two imaginary points. Assume that $\pi_1$ is the polarity of $[L_\infty, \emptyset]$ induced by $Q$ and $\pi_0$ is that of $[P^2(\mathbb{R}), L_\infty]$ interchanging the only two elements $L_\infty$ and $P^2(\mathbb{R})$ of it. With these assumptions $(L(P^2(\mathbb{R})), ([P^2(\mathbb{R}), L_\infty], \pi_0), ([L_\infty, \emptyset], \pi_1))$ is the Cayley-Klein lattice associated to Euclidean plane. It is easy to see that the polar of each regular point or each regular line is identical with the one given by Giering. To a point $X$ at infinity there have corresponded all lines through the polar of $X$ with respect to $Q$ with the exception of $L_\infty$, and every point of the plane is a polar of $L_\infty$ except the points on it. It is in disagreement with the previous result obtained from [4] in which, all lines of the pencil are polars of $X$ and each point of the plane is a polar of the line $L_\infty$.

3. Polar Variety

In this section, we give our definition for total polars in a Cayley-Klein space. We will prove that the set of all total polars of a subspace is a Schubert variety. Finally, some properties of total polars are provided.

Definition 3.1. Let $X$ be a point in a Cayley-Klein space of dimension $n$ with the absolute figure

$$\hat{Q}_0 \supseteq \hat{A}_1 \supseteq \hat{Q}_1 \supseteq \hat{A}_2 \supseteq \cdots \supseteq \hat{A}_r \supseteq \hat{Q}_r \supseteq \hat{A}_{r+1} = \emptyset.$$  

Assume that $X \in A_l \setminus A_{l+1}$ for some $l$, $0 \leq l \leq r$. A total polar of $X$ with respect to the absolute figure is a hyperplane of $P^n(\mathbb{R})$ that contains $X$.

Definition 3.2. Let $K$ be a $k$-dimensional subspace of a Cayley-Klein space of dimension $n$ with the absolute figure

$$\hat{Q}_0 \supseteq \hat{A}_1 \supseteq \hat{Q}_1 \supseteq \hat{A}_2 \supseteq \cdots \supseteq \hat{A}_r \supseteq \hat{Q}_r \supseteq \hat{A}_{r+1} = \emptyset.$$  

Assume that $B = \{X_i|0 \leq i \leq k\}$ is an independent subset of $K$ satisfying

$$|B \cap A_j| = \dim(K \cap A_j) + 1$$

for $0 \leq j \leq r$. If $H_i$ is a total polar of $X_i$ for $0 \leq i \leq k$ and $\{H_i|0 \leq i \leq k\}$ is independent, the $(n-k-1)$-dimensional subspace \( \bigcap_{i=0}^{k} H_i \) is a total polar of $K$ with respect to the absolute figure of the space.

Remark 1. By Definition 3.2 the subspaces $P^n(\mathbb{R})$ and $\emptyset$ are total polar to each other in all Cayley-Klein spaces of dimension $n$.

The set of all total polars of a point in a Cayley-Klein plane may be a single line, a pencil of lines or all lines of the plane and the set of all total polars of a line may be a single point, a projective range of points or all points of the plane. Each of these sets is a certain subvariety of a Grassmannian called a Schubert variety.
Definition 3.3. Let
\[ \emptyset = W_{-1} \subset W_0 \subset \cdots \subset W_i \subset \cdots \subset W_k \]
be a strictly increasing sequence of subspaces in a projective space. The Schubert
variety associated with this sequence is, by definition, the set of all \( k \)-dimensional
subspaces such as \( K \) that satisfies \( \dim(K \cap W_i) \geq i \) for \( 0 \leq i \leq k \).

For instance, in a projective plane \( \mathbb{P} \) the pencil of lines passing through a point \( X \)
is obtained by assuming that \( W_0 = X \) and \( W_1 = \mathbb{P} \). As an additional example, all
lines of the plane \( \mathbb{P} \) is the Schubert variety associated to a sequence \( \emptyset \subset L \subset \mathbb{P} \), in
which \( L \) is an arbitrary line of the plane. In Theorem \[3.9\] we show that the set of all
total polars of a subspace in an arbitrary Cayley-Klein space is a Schubert variety.

Definition 3.4. Let \( K \) be a subspace of a Cayley-Klein space with the absolute figure
\[ \hat{Q}_0 \supset A_1 \supset \hat{Q}_1 \supset A_2 \supset \cdots \supset A_r \supset \hat{Q}_r \supset \hat{A}_{r+1} = \emptyset. \]
The polar sequence of \( K \) is a subsequence of
\[ K^p = (K \cap A_0)^{p_0} \supset \cdots \supset (K \cap A_j)^{p_j} \supset \cdots \supset (K \cap A_r)^{p_r} \supset \emptyset \]
obtained by deleting each subspace \((K \cap A_j)^{p_j}\) which is equal to \( A_{j+1} \).

In the next theorem, we characterize all total polars of a subspace by using its polar
sequence. To prove the theorem we need some lemmas.

Lemma 3.5. Let \( Q \) be a quadric with vertex \( A \) in \( \mathbb{P}^n(\mathbb{R}) \). If \( H \) is a hyperplane that contains \( A \), then \( H^p = X + A \) for a point \( X \) not in \( A \).

Lemma 3.6. Let \( Q \) be a quadric with vertex \( A \) in \( \mathbb{P}^n(\mathbb{R}) \). For a given subspace \( K \), we have

1. If \( H \) is a hyperplane that contains \( K^p \), there exists a point \( X \in K \setminus A \) such that
   \( H^p = X + A \) and so, \( X^p = H \).
2. Let \( H_1, \ldots, H_m \) be independent hyperplanes which are containing \( K^p \). If \( H_i^p = X_i + A \) for a point \( X_i(x_i) \in K \setminus A \) with \( 1 \leq i \leq m \), then for any independent
   subset \( Y \) of \( A \) the set \( Y \cup \{X_i|1 \leq i \leq m\} \) is independent.

Proof. (1) The hyperplane \( H \) contains \( A \). So \( H^p = Z + A \) for some point \( Z \notin A \).
Suppose that \( Z \notin K \). Since \( K^p \subseteq H \), \( H^p \subseteq (K^p)^p = K + A \). Hence, \( Z \in X + W \) for
some \( X \in K \setminus A \) and \( W \in A \). This gives
\[ H^p = Z + A \subseteq (X + W) + A = X + A. \]
It follows that \( H^p = X + A \) and consequently, \( X^p = H \).

(2) Let \( Y = \{X_{m+1}(x_{m+1}), \ldots, X_{m+r}(x_{m+r})\} \) and \( M \) be the matrix associated to \( Q \).
Suppose that \( \sum_{i=1}^{m+r} \alpha_i X_i = 0 \) for given scalars \( \alpha_i \)s. So \( \sum_{i=1}^{m+r} \alpha_i (MX_i) = 0 \), implying

1 This sequence is called the system of total polar subspaces of \( K \) by Hermann Vogel in [3]. He considers it as
a generalization of the total polar of a regular subspace.
\[ \sum_{i=1}^{m} \alpha_i (MX_i) = 0. \] The hyperplanes \( H_i \)'s are independent. This gives \( \alpha_i = 0 \) for 

\[ 1 \leq i \leq m, \] and then \[ \sum_{i=m+1}^{m+r} \alpha_i X_i = 0. \] Since \( Y \) is independent, we get \( \alpha_i = 0 \) for 

\[ 1 \leq i \leq m + r. \]

\[ \square \]

**Lemma 3.7.** Let \( S \) and \( T \) be subspaces of \( P^n(\mathbb{R}) \) satisfying \( S \subseteq T \). If \( \dim S = n - m - 1 \) and \( \dim T = n - r - 1 \), there exist hyperplanes \( H_0, \ldots, H_m \) such that \( T = \bigcap_{i=0}^{r} H_i \) and 

\[ S = \bigcap_{i=0}^{m} H_i. \] Moreover, \( \{H_i \cap T | r < i \leq m\} \) is an independent set of hyperplanes of \( T \).

**Proof.** Suppose that \( T = S^+ < X_1, \ldots, X_{m-r} > \) for an independent subset \( \{X_1, \ldots, X_{m-r}\} \subseteq T \setminus S \). For each \( i, 1 \leq i \leq m - r \), let \( H_i \) be a hyperplane of \( P^n(\mathbb{R}) \) that contains \( S^+ < X_1, \ldots, X_i, X_{m-r} > \) and does not contain \( X_i \). If \( T = \bigcap_{i=0}^{r} H_i \) where the hyperplanes \( H_0, \ldots, H_r \) are independent, then \( S = \bigcap_{i=0}^{m} H_i \) and \( \{H_i \cap T | r < i \leq m\} \) is independent. \[ \square \]

**Theorem 3.8.** In a Cayley-Klein space of dimension \( n \) with the absolute figure 

\[ \hat{Q}_0 \supseteq \hat{A}_1 \supseteq \hat{Q}_1 \supseteq \hat{A}_2 \supseteq \cdots \supseteq \hat{A}_r \supseteq \hat{Q}_r \supseteq \hat{A}_{r+1} = \emptyset, \]

let 

\[ (K \cap A_{j_0})^{p_0} \supseteq \cdots \supseteq (K \cap A_{j_i})^{p_i} \supseteq \cdots \supseteq (K \cap A_{j_s})^{p_s} \supseteq \emptyset \]

be the polar sequence of a \( k \)-dimensional subspace \( K \) of \( P^n(\mathbb{R}) \). An \( (n - k - 1) \)-dimensional subspace \( Y \) is a total polar of \( K \) if and only if for each \( i, 0 \leq i \leq s \), 

\[ \dim(Y \cap (K \cap A_{j_i})^{p_i}) \geq m_j - k_j - 1 \] where \( m_j = \dim A_j \) and \( k_j = \dim(K \cap A_j) \) for 

\[ 0 \leq j \leq r. \]

**Proof.** We first show that each total polar of \( K \) satisfies the above inequalities. Let 

\[ K^\perp = \bigcap_{t=0}^{k} H_t \] be a total polar of \( K \), where the hyperplanes \( H_t \) for \( 0 \leq t \leq k \) are total polars of points of an independent subset \( B \) of \( K \) satisfying \( |B \cap A_j| = k_j + 1 \) for 

\[ 0 \leq j \leq r. \] Moreover, we may assume that the hyperplanes \( H_t \) for \( 0 \leq t \leq k_j \) are total polars of points of \( B \cap A_j \). We have 

\[ \dim(\bigcap_{t=0}^{k} H_t \cap (K \cap A_{j_i})^{p_i}) = \dim(\bigcap_{t=0}^{k_j+1} H_t \cap (K \cap A_{j_i})^{p_i}) \]

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m_j - k_j - 1

Now, let \( Y \) be an \((n-k-1)\)-dimensional subspace satisfying \( \dim(Y \cap (K \cap A_j)^{p_i}) \geq m_j - k_j - 1 \) for \( 0 \leq i \leq s \). We show that \( Y \) is a total polar of \( K \). Since the polar sequence of \( \emptyset \) is the sequence of vertices of the absolute figure of the space and \( \dim(P^n(\mathbb{R}) \cap A_j) = m_j \) for \( 0 \leq j \leq r \), the theorem holds for \( K = \emptyset \). If \( K \neq \emptyset \), it is proved that there exists an independent subset \( B = \{X_i | 0 \leq i \leq k \} \) of \( K \) that satisfies \( |B \cap A_j| = k_j + 1 \) for \( 0 \leq j \leq r \) such that for each \( i \), \( 0 \leq i \leq k \), there exists a total polar \( H_i \) for \( X_i \) satisfying \( Y = \bigcap_{i=0}^{k} H_i \). Assume that
\[
K \cap A_t \neq \emptyset, \quad K \cap A_{t+1} = \emptyset,
\]
for some integers \( 0 \leq d \leq l \leq r \). Notice that \( (K \cap A_j)^{p_i} = A_j \neq A_{j+1} \) for \( 0 \leq j < d \) and \( l < j \leq r \). It is easily seen that the polar sequence of \( K \) includes \((K \cap A_t)^{p_i}\) and \((K \cap A_l)^{p_i} \subseteq Y \). Consider the following sequence of subspaces
\[
Y = Y + (K \cap A_l)^{p_i} \subseteq \cdots \subseteq Y + (K \cap A_j)^{p_i} \subseteq Y + A_j \subseteq \cdots \subseteq Y + A_d.
\]
Applying Lemma 3.7 for any two subsequent subspaces of the sequence gives
\[
Y = \bigcap_{i=0}^{r_1} H_i \cap \bigcap_{i=r_1+1}^{n_1-1} H_i \cap \ldots \cap \bigcap_{i=r_d+1}^{k} H_i
\]
for hyperplanes \( H_0, \ldots, H_k \) where (i) for each \( j \), \( d \leq j \leq l \), \( Y + A_j = \bigcap_{i=r_j+1}^{k} H_i \) and
\[
Y + (K \cap A_j)^{p_i} = \bigcap_{i=r_{j+1}}^{k} H_i \text{ by assuming } n_l = -1; \text{ and (ii) } \{H_i \cap (Y + A_j) \mid n_j < i \leq r_j\}
\]
for \( d \leq j \leq l \) and \( \{H_i \cap (Y + (K \cap A_j)^{p_i}) \mid r_{j+1} < i \leq n_j\} \) for \( d \leq j < l \) are independent sets of hyperplanes of \( Y + A_j \) and \( Y + (K \cap A_j)^{p_i} \) respectively.

Fix some integer \( j \), \( d \leq j \leq l \). We show that \( \{H_i \cap A_j \mid n_j < i \leq r_j\} \) is an independent set of hyperplanes of \( A_j \). For each \( i \), \( n_j < i \leq r_j \), since \( Y \subseteq H_i \) and \( Y + A_j \nsubseteq H_i \), \( A_j \nsubseteq H_i \). It is easy to see that \( (H_i \cap A_j) + Y = H_i \cap (Y + A_j) \) and
\[
\bigcap_{t=n_{j+1}}^{r_j} (H_t \cap A_j) + Y = \bigcap_{t=n_{j+1}}^{r_j} (H_t \cap (Y + A_j)).
\]
Lemma 3.6 gives an independent subset \{X_i\} satisfying \( d \) not yet been determined any total polar for them. The hyperplanes \( H_t \) in independent set of hyperplanes of \( (K \cap A_j)^{p_i} \) for \( d \leq j < l \). From the independency of these sets, it is concluded that \( \{H_t \cap (K \cap A_j)^{p_i} | 0 \leq t \leq n_j \} \) is also an independent set of hyperplanes of \( (K \cap A_j)^{p_i} \) for \( d \leq i < s - (r - l - 1) \) provided that \( j_i \neq l \). From \( \dim(Y \cap (K \cap A_j)^{p_i}) \geq m_j, -k_j, -1 \), it is followed that

\[
\dim(\bigcap_{j=0}^{n_j} H_j) \cap (\bigcap_{j=n_j+1}^{k} H_j) \cap (K \cap A_j)^{p_i} \geq m_j, -k_j, -1.
\]

Hence \( (m_j, -k_j, +k_{j+1}) - (n_j, +1) \geq m_j, -k_j, -1 \), implying \( k_{j+1} \geq n_j \). Now, let \( B_{l+1} = \emptyset \) and fix some integer \( j \) with \( d \leq j \leq l \). Suppose that \( B_{j+1} \) is an independent subset of \( K \cap A_{j+1} \) satisfying \( |B_{j+1} \cap A_t| = k_{t+1} \) for \( j+1 \leq t \leq l+1 \). \( \{H_t \cap A_j | n_j \leq i \leq r_j \} \) is an independent set of hyperplanes of \( A_j \) that each of them contains \( (K \cap A_j)^{p_i} \). Applying Lemma 3.6 gives an independent subset \{\( X_{i_j+1}, \ldots, X_{i_{j+1}+r_j-n_j} \)\} \( \cap (K \cap A_j)^{p_i} \) for \( n_j \leq i < r_j \). We consider \( H_i, n_j \leq i \leq r_j \), as a total polar for \( X_{i_j+1}, \ldots, X_{i_{j+1}+r_j-n_j} \). The set \( B_{j+1} \cup \{X_{i_j+1}, \ldots, X_{i_{j+1}+r_j-n_j} \} \) is an independent subset of \( K \cap A_j \). Extend this set to a maximal independent subset \( B_j = \{X_0, \ldots, X_{k_j} \} \) of \( K \cap A_j \). In this way, we get an independent subset \( B_d \) of \( K \cap A_j \) for \( 0 \leq t \leq r \) such that we have determined individual total polars for some of its elements. For each \( j, d < j < l \), if \( r_j \neq m_{j-1}, (K \cap A_j)^{p_{j-1}} \neq A_j \). So \( j - 1 = j_i \) for some \( i, d \leq i < s - (r - l - 1) \). Regarding the inequality \( n_j, k_j \leq k_{j+1} \), we consider the hyperplanes \( H_t \) for \( r_j < t \leq n_{j-1} \), as total polars for those points of \( B_j \) that have not yet been determined any total polar for them. The hyperplanes \( H_j \) for \( j > r_d \), are considered as total polars for the remaining points of \( B_d \).

\[ \square \]

**Theorem 3.9.** In a Cayley-Klein space, the set of all total polars of a subspace is a Schubert variety.

**Proof.** Consider a Cayley-Klein space of dimension \( n \) with the absolute figure

\[ \mathcal{Q}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{Q}_1 \supseteq \mathcal{A}_2 \supseteq \cdots \supseteq \mathcal{A}_r \supseteq \mathcal{Q}_r \supseteq \mathcal{A}_{r+1} = \emptyset \]

and let \( K \) be a \( k \)-dimensional subspace of \( P^n(\mathbb{R}) \). We show that all total polars of \( K \) is a Schubert variety. Set \( m_j = \dim A_j \) and \( k_j = \dim(K \cap A_j) \) for \( 0 \leq j \leq r \). We know that \( (K \cap A_j)^{p_i} \supseteq A_{j+1} \). This gives \( \dim(K \cap A_j)^{p_i} \geq \dim A_{j+1} \) and then \( m_j - k_j - 1 \geq m_{j+1} - k_{j+1} - 1 \) consequently. Moreover, two sides are equal if and only if \( (K \cap A_j)^{p_i} = A_{j+1} \). Let

\[ (K \cap A_j)^{p_{j_0}} \supseteq \cdots \supseteq (K \cap A_j)^{p_{j_i}} \supseteq \cdots \supseteq (K \cap A_j)^{p_{j_s}} \supseteq \emptyset \]

be the polar sequence of \( K \). From \( m_{j+1} - k_{j+1} - 1 = m_{j+1} - k_{j+1} - 1 \), it is followed that

\[ m_{j_0} - k_{j_0} - 1 > \cdots > m_j - k_j - 1 > \cdots > m_{j_s} - k_{j_s} - 1 > m_{j+1} - k_{j+1} - 1 = -1. \]

It is easily seen that \( m_{j_0} - k_{j_0} - 1 = n - k - 1 \). We set \( W_{-1} = \emptyset \) and

\[ W_{m_{j_0} - k_{j_0} - 1} = (K \cap A_j)^{p_{j_i}} \]
for $0 \leq i \leq s$. Regarding the equalities
\[ \dim(K \cap A_j)^{p_{ji}} = m_j - k_j + k_{j+1} \quad \text{and} \quad \dim A_{j+1} = m_{j+1}, \]
for each $c$,
\[ 1 \leq c < (m_j - k_j - 1) - (m_{j+1} - k_{j+1} - 1) = (m_j - k_j + k_{j+1}) - m_{j+1}, \]
assume that $A_{j+1} + Z_i^c$ is an $(m_{j+1} + c)$-dimensional subspace of $(K \cap A_j)^{p_{ji}}$ such that
\[ A_{j+1} + Z_i^{m_j - k_j + k_{j+1} - m_{j+1} - 1} \supseteq \cdots \supseteq A_{j+1} + Z_i^1 \supseteq \cdots \supseteq A_{j+1} + Z_i^1. \]
For each $c$, set
\[ W_{m_{j+1} - k_{j+1} - 1 + c} = A_{j+1} + Z_i^c. \]
We claim that the Schubert variety associated with $\{W_i\}_{i=1}^{n-k-1}$ is the Schubert variety claimed in the theorem. Due to Theorem 3.8, it suffices to show that for a given total polar $K^\perp$ of $K$ we have
\[ \dim(K^\perp \cap (A_{j+1} + Z_i^c)) \geq m_{j+1} - k_{j+1} - 1 + c. \]
From $\dim(K^\perp \cap (K \cap A_j)^{p_{ji}}) \geq m_j - k_j - 1$, it is followed that $\dim(K^\perp + (K \cap A_j)^{p_{ji}}) \leq n - k + k_{j+1}$. This gives $\dim(K^\perp + A_{j+1} + Z_i^c) \leq n - k + k_{j+1}$. Since $m_{j+1} - k_{j+1} = m_{j+1} - k_{j+1}$, we get $\dim(K^\perp + A_{j+1} + Z_i^c) \leq n - k + m_{j+1} - m_{j+1} + k_{j+1}$, and then $\dim(K^\perp \cap (A_{j+1} + Z_i^c)) \geq m_{j+1} - k_{j+1} - 1 + c$. \[ \square \]

**Definition 3.10.** For a given subspace $K$ of a Cayley-Klein space, the set of all total polars of $K$ is called the polar variety of $K$.

Schubert varieties are non-empty. Therefore, every subspace of a Cayley-Klein space has at least one total polar. The next proposition presents sufficient and necessary condition for the uniqueness of such a total polar.

**Proposition 3.11.** Consider a Cayley-Klein space of dimension $n$ with the absolute figure
\[ \hat{Q}_0 \supseteq \hat{A}_1 \supseteq \hat{Q}_1 \supseteq \hat{A}_2 \supseteq \cdots \supseteq \hat{A}_r \supseteq \hat{Q}_r \supseteq \hat{A}_{r+1} = \emptyset. \]
If $K$ is a subspace of $P^n(\mathbb{R})$ satisfying $K \cap A_l \neq \emptyset$ and $K \cap A_{l+1} = \emptyset$ for an integer $l$ with $0 \leq l \leq r$, $K$ has a unique total polar $(K \cap A_l)^{p_l}$ if and only if it is a regular subspace.

**Proof.** Assume that
\[ (K \cap A_{j_0})^{p_{j_0}} \supseteq \cdots \supseteq (K \cap A_{j_s})^{p_{j_s}} \supseteq \cdots \supseteq (K \cap A_j)^{p_j} \supseteq \emptyset \]
is the polar sequence of $K$ and $\dim K = k$. Let $m_j = \dim A_j$ and $k_j = \dim(K \cap A_j)$ for $0 \leq j \leq r$. Due to Theorem 3.8, it is followed that $K$ has exactly one total polar if and
only if \( \dim(K \cap A_{j_0})^{p_{j_0}} = n - k - 1 \). In this case, the total polar of \( K \) is \( (K \cap A_{j_0})^{p_{j_0}} \).

Since \( m_{j_0} - k_{j_0} = n - k \), it is followed that \( k_{j_0 + 1} = -1 \). This implies that \( j_0 = l \) or \( j_0 = l + 1 \). If \( j_0 = l \), then \( \dim(K + A_l) = k + m_l - k_l = n \). Otherwise, we have \( (K \cap A_l)^{p_l} = A_{l+1} = (K \cap A_{j_0})^{p_{j_0}} \). This gives \( m_l - k_l = m_{l+1} - k_{l+1} = n - k \), implying \( \dim(K + A_l) = n \). In both cases, the total polar of \( K \) equals \( (K \cap A_l)^{p_l} \).

**Proposition 3.12.** Let \( K \) and \( K' \) be subspaces of a Cayley-Klein space such that \( K \subseteq K' \). Then

1. Every total polar of \( K \) contains a total polar of \( K' \).
2. Every total polar of \( K' \) can be extended to a total polar of \( K \).

**Proof.** Let

\[
\hat{Q}_0 \supseteq \hat{A}_1 \supseteq \hat{Q}_1 \supseteq \hat{A}_2 \supseteq \cdots \supseteq \hat{A}_r \supseteq \hat{Q}_r \supseteq \hat{A}_{r+1} = \emptyset.
\]

be the absolute figure of the space. Set \( m_j = \dim A_j \) and \( k_j = \dim(K \cap A_j) \) for \( 0 \leq j \leq r \). To prove the proposition, it suffices to consider the case \( \dim K' - \dim K = 1 \).

In this case, there exists a point \( X \in (K' \setminus K) \) and some integer \( l, 0 \leq l \leq r \), such that \( K \cap A_j = K' \cap A_j \) for \( l < j \leq r + 1 \) and \( (K \cap A_j) + X = K' \cap A_j \) for \( 0 \leq j \leq l \). This gives \( \dim(K \cap A_l)^{p_l} - \dim(K' \cap A_l)^{p_l} = 1 \) and \( (K \cap A_j)^{p_j} = (K' \cap A_j)^{p_j} \) for \( j \neq l \). Let

\[
(K \cap A_{j_0})^{p_{j_0}} \supseteq \cdots \supseteq (K \cap A_{j_r})^{p_{j_r}} = (K \cap A_l)^{p_l} \supseteq \cdots \supseteq (K \cap A_{j_s})^{p_{j_s}} \supseteq \emptyset
\]

be the polar sequence of \( K \). We have

1. For \( K' \), a total polar of \( K \), extend an arbitrary maximal independent subset of \( K' \cap (K' \cap A_l)^{p_l} \) to a maximal independent subset of \( K' \cap (K \cap A_l)^{p_l} \) and then to a maximal independent subset of \( K' \cap (K \cap A_{j_{i-1}})^{p_{j_{i-1}}} \). Continue in this way until getting a maximal independent subset \( B \) of \( K' \cap (K \cap A_{j_0})^{p_{j_0}} = K' \). Suppose that \( B' \) is obtained by deleting one of the elements of \( B \) added in the last step. Let \( (K')' \) be the subspace spanned by \( B' \). This gives \( \dim((K')' \cap (K' \cap A_{j_i})^{p_{j_i}}) \geq m_{j_i} - k_{j_i} - 1 \) for \( i < t \leq s \) and \( \dim((K')' \cap (K' \cap A_{j_i})^{p_{j_i}}) \geq m_{j_i} - k_{j_i} - 2 \) for \( 0 \leq t \leq i \).

2. Let \( (K')' \) be a total polar of \( K' \) and \( r, 0 \leq r \leq s \), be the maximum number that \( (K \cap A_{j_r})^{p_{j_r}} \nsubseteq (K')' \). Set \( K' = (K')' + X \) for a point \( X \in (K \cap A_{j_r})^{p_{j_r}} \setminus (K')' \).

It is easy to check that \( \dim(K' \cap (K \cap A_{j_r})^{p_{j_r}}) \geq m_{j_r} - k_{j_r} - 1 \) for \( 0 \leq t \leq s \).

\[
\square
\]

**Theorem 3.13.** In a Cayley-Klein space of dimension \( n \), let \( K \) be a subspace of \( P^n(\mathbb{R}) \). If \( K^\bot \) is a total polar of \( K \), then \( K \) is also a total polar of \( K^\bot \).

**Proof.** Let

\[
\hat{Q}_0 \supseteq \hat{A}_1 \supseteq \hat{Q}_1 \supseteq \hat{A}_2 \supseteq \cdots \supseteq \hat{A}_r \supseteq \hat{Q}_r \supseteq \hat{A}_{r+1} = \emptyset
\]

be the absolute figure of the space. We prove the theorem by induction on dimension \( K \). Since the subspaces \( P^n(\mathbb{R}) \) and \( \emptyset \) are the only total polars of each other, the theorem holds for \( K = \emptyset \). Now suppose that it is true for every \((k-1)\)-dimensional subspace of \( P^n(\mathbb{R}) \). Let \( K \) be a \( k \)-dimensional subspace satisfying \( K \cap A_l \neq \emptyset \) and
Let $K \cap A_{l+1} = \emptyset$ for some $l$, $0 \leq l \leq r$. Assume that $K^\perp$ is an arbitrary total polar of $K$. We show that $K$ is a total polar of $K^\perp$. If $K^\perp \not\supset A_i$, then $K \cap (K^\perp \cap A_i)^{p_i} \neq \emptyset$. Let $X$ be a point of $K \cap (K^\perp \cap A_i)^{p_i}$ in this case and a point of $K \cap A_i$ otherwise. Assume that $K - X$ is a complement of $X$ in $K$. Due to Proposition 3.12 there exists a total polar of $K - X$ that contains $K^\perp$. Let $K^\perp + Z$ be such a total polar for some point $Z \not\in K^\perp$. By the induction hypothesis, we conclude that $K - X$ is also a total polar of $K^\perp + Z$. Repeated application of Proposition 3.12 ensures the existence of a total polar $K'$ of $K^\perp$ that contains $K - X$. After this preliminary step, we return to prove the assertion. We know that $K^\perp \supseteq (K \cap A_i)^{p_i} \supseteq A_{i+1}$. Let $K^\perp \supseteq A_{j+1}$ and $K^\perp \not\supset A_j$ for some $j \leq l$. It is easily seen that $(K^\perp \cap A_j)^{p_j} \neq A_{j+1}$ and $(K^\perp \cap A_i)^{p_i} = A_{i+1}$ for $j < i$. Let

$$(K^\perp \cap A_{j+1})^{p_{j+1}} \supseteq \cdots \supseteq (K^\perp \cap A_j)^{p_j} = (K^\perp \cap A_j)^{p_j} \supseteq \emptyset$$

be the polar sequence of $K^\perp$ and $d_j = \dim(K^\perp \cap A_j)$ for $0 \leq j \leq r$. Since $K'$ is a total polar of $K^\perp$ and contains $K - X$, it is followed that $\dim(K' \cap (K^\perp \cap A_j)^{p_i}) \geq m_j - d_j - 1$, and consequently, $\dim((K - X) \cap (K^\perp \cap A_j)^{p_i}) \geq m_j - d_j - 2$ for $0 \leq i \leq s$. On the other hand $X \in K \cap (K^\perp \cap A_j)^{p_i}$ for $0 \leq i \leq s$, implying that $\dim(K \cap (K^\perp \cap A_j)^{p_i}) \geq m_j - d_j - 1$ for $0 \leq i \leq s$.

In the next proposition all total polars of a hyperplane of a Cayley-Klein space are determined.

**Proposition 3.14.** Let $H$ be a hyperplane of a Cayley-Klein space with the absolute figure

$$Q_0 \supseteq A_1 \supseteq \hat{Q}_1 \supseteq A_2 \supseteq \cdots \supseteq A_r \supseteq \hat{Q}_r \supseteq \hat{A}_{r+1} = \emptyset.$$

Suppose $H$ contains $A_i$ and does not contain $A_{i-1}$ for some $i$, $0 < i \leq r+1$. A point $X$ is a total polar of $H$ if and only if $X \in (H \cap A_i)^{p_i-1}$.

**Proof.** We first show that $(H \cap A_j)^{p_j} = A_{j+1}$ for $j \neq i - 1$. Set $m_j = \dim A_j$ for $0 \leq j \leq r + 1$. If $j < i - 1$, then

$$\dim(H \cap A_j)^{p_j} = m_j - \dim(H \cap A_j) + \dim(H \cap A_{j+1}).$$

Since $H$ does not contain neither $A_j$ nor $A_{j+1}$, we get

$$\dim(H \cap A_j)^{p_j} = m_j - (m_j - 1) + m_j + 1 - 1 = m_{j+1}.$$

$(H \cap A_j)^{p_j}$ contains $A_{j+1}$ and so, equals it. If $i - 1 < j$, then $H$ contains $A_j$ and it is obvious that $(H \cap A_j)^{p_j} = A_{j+1}$. For $j = i - 1$, it is concluded that $(H \cap A_j)^{p_j} \neq A_{j+1}$ due to Lemma 3.5. In view of Theorem 3.8 and above considerations, the validity of the proposition can be easily verified.

In the remaining part of this section, we compare our definition for total polars with the other definitions introduced in section two. It is easily seen that in each Cayley-Klein plane, the set of all total polars of a point or a line of the plane equals to the set of polars corresponded to it in $[4]$. We show that the set of all total polars of a
subspace in a Cayley-Klein space includes all polars associated to it in \([5]\). Consider a Cayley-Klein space of dimension \(n\) with the absolute figure

\[
Q_0 \supseteq \hat{A}_1 \supseteq \hat{Q}_1 \supseteq \hat{A}_2 \supseteq \cdots \supseteq \hat{A}_r \supseteq \hat{Q}_r \supseteq \hat{A}_{r+1} = \emptyset.
\]

Let \(K\) be a \(k\)-dimensional subspace of \(P^n(\mathbb{R})\) satisfying \(K \cap A_l \neq \emptyset\) and \(K \cap A_{l+1} = \emptyset\) for some integer \(l\), \(0 \leq l \leq r\). If \(K^\perp\) is a polar of \(K\) according to \([3]\), then \((K \cap A_j)^{P_l} = (K^\perp \cap A_j) + A_{j+1}\) for \(0 \leq j \leq r\). Since \(K \cap A_j = \emptyset\) for \(l < j \leq r\), it is followed that \(K^\perp \supseteq (K \cap A_j)^{P_l}\). This implies that \(\dim(K^\perp \cap (K \cap A_j)^{P_l}) = m_l - k_l - 1\). Now, assume that \(\dim(K^\perp \cap (K \cap A_j)^{P_l}) \geq m_j - k_j - 1\) for an integer \(j\), \(0 < j \leq l\). We have

\[
\begin{align*}
\dim(K^\perp \cap (K \cap A_{j-1})^{P_{l-1}}) &= \dim K^\perp + \dim((K \cap A_{j-1})^{P_{l-1}}) - \dim((K^\perp \cap A_{j-1})^{P_{l-1}}), \\
&= \dim K^\perp + \dim((K \cap A_{j-1})^{P_{l-1}}) - \dim((K^\perp \cap A_{j-1}) + A_j), \\
&= \dim((K \cap A_{j-1})^{P_{l-1}}) - \dim A_j + \dim(K^\perp \cap A_j) \\
&\geq m_{j-1} - k_{j-1} + k_j - m_j + m_j - k_j - 1 = m_{j-1} - k_{j-1} - 1.
\end{align*}
\]

Regarding Theorem \([3, 5]\) it is concluded that \(K^\perp\) is a total polar of \(K\).

### 4. Tangent Cones and Spheres

In this section, we define tangent subspaces to the absolute figure of a Cayley-Klein space. We will determine all tangent lines through a given point not in the second vertex. It is proved that the set of all points on these lines is a quadric called a tangent cone. Finally, we obtain the equations of all spheres using tangent cones.

**Definition 4.1.** Consider a Cayley-Klein space of dimension \(n\). A subspace \(K\) of \(P^n(\mathbb{R})\) is tangent to the absolute figure of the space if there exists a total polar \(K^\perp\) of \(K\) with \(K \cap K^\perp \neq \emptyset\). Briefly, we say that \(K\) is a tangent subspace.

**Proposition 4.2.** Consider a Cayley-Klein space with the absolute figure

\[
Q_0 \supseteq \hat{A}_1 \supseteq \hat{Q}_1 \supseteq \hat{A}_2 \supseteq \cdots \supseteq \hat{A}_r \supseteq \hat{Q}_r \supseteq \hat{A}_{r+1} = \emptyset,
\]

such that \(A_1\) is a hyperplane. Let \(X\) be a point not in \(A_1\). A line \(L\) through \(X\) is tangent to the absolute figure if and only if it passes through a point of \(Q_1\).

**Proof.** The line \(L\) intersects \(A_1\) at a point \(Y\). Suppose that \(Y \in A_k \setminus A_{k+1}\) for some integer \(k \geq 1\). We show that \(L\) is a tangent line if and only if \(Y \in Q_1\). If \(k = 1\), \(L\) has a unique total polar \(Y^{P_1}\). So \(L\) is tangent to the absolute figure if and only if \(Y^{P_1}\) contains \(Y\). That is \(Y \in Q_1\). If \(k > 1\), then every hyperplane of \(A_1\) containing \(A_2\) is a total polar of \(L\) by Theorem \([3, 5]\). Such a total polar intersects \(L\) at \(Y\).

**Proposition 4.3.** Consider a Cayley-Klein space with the absolute figure

\[
Q_0 \supseteq \hat{A}_1 \supseteq \hat{Q}_1 \supseteq \hat{A}_2 \supseteq \cdots \supseteq \hat{A}_r \supseteq \hat{Q}_r \supseteq \hat{A}_{r+1} = \emptyset,
\]
such that $A_1$ is not a hyperplane. Let $X$ be a point not in $A_1$ and $L$ a line through it. Then

(1) For $X \notin Q_0$, $L$ is a tangent line if and only if it passes through a point of $X^p \cap Q_0$.

(2) For $X \in Q_0$, $L$ is a tangent line if and only if $L \subseteq X^p$.

**Proof.**

(1) Since $X \notin Q_0$, $X \notin X^p$. So $L$ intersects $X^p$ at a point such as $Y$. Similar to the proof of the previous proposition we can show that $L$ is a tangent line if and only if $Y \in X^p \cap Q_0$.

(2) Assume that $L \notin X^p$. We show that $L$ is not a tangent line. Since $A_1 \subseteq X^p$, $L \cap A_0 \neq \emptyset$ and $L \cap A_1 = \emptyset$. Thus $L$ is regular and has a unique total polar $X^p \cap Y^p$ for some point $Y \neq X$ of $L$. If $L$ is a tangent line, it intersects $X^p \cap Y^p$ at $X$. This gives $X \in Y^p$ and we get $Y \in X^p$ which is a contradiction. Now, suppose that $L \subseteq X^p$. We consider two cases $L \cap A_1 = \emptyset$ and $L \cap A_1 \neq \emptyset$. If $L \cap A_1 = \emptyset$, $L$ has a unique total polar $X^p \cap Y^p$ for some point $Y \neq X$ of $L$. Since $X \in X^p \cap Y^p$, $L$ intersects its total polar and is tangent to the absolute figure of the space. If $L \cap A_1 \neq \emptyset$, $L$ intersects $A_1$ at a point $Y$. Regarding Theorem 3.8 it is concluded that any hyperplane of $X^p$ containing $A_1$ is a total polar of $L$. Such a total polar intersects $L$ at $Y$.

\[ \square \]

**Remark 2.** In a Cayley-Klein space of dimension $n$ with the absolute figure

$$Q_0 \supseteq \hat{A}_1 \supseteq \hat{Q}_1 \supseteq \hat{A}_2 \supseteq \cdots \supseteq \hat{A}_r \supseteq \hat{Q}_r \supseteq \hat{A}_{r+1} = \emptyset,$$

set $n_i = \dim A_i - \dim A_{i+1}$ for $0 \leq i \leq r$. Let $E_i$ be a diagonal matrix of order $n_i$ where the diagonal entries equal 1 or $-1$. Assume that the number of $-1$ elements is $q_i$. There exists a projective coordinate system for $P^n(\mathbb{R})$ in which the quadrics of the absolute figure and their vertices have the following equations:

$$Q_0 : X_0^2 E_0 X_0 = x_0^2 + \cdots + x_{n_0-1}^2 - x_{n_0}^2 - \cdots - x_{n_0-1}^2 = 0;$$

$$X_0 = \begin{pmatrix} x_0 & \cdots & x_{n_0-1} \end{pmatrix}^T.$$

$$A_1 : x_0 = \cdots = x_{n_0-1} = 0.$$

$$Q_1 : X_1^T E_1 X_1 = x_{n_0}^2 + \cdots + x_{n_0+n_1-1}^2 - x_{n_0+n_1-1}^2 - \cdots - x_{n_0+n_1-1}^2 = 0;$$

$$E_0 X_0 = O_0, \quad X_1 = \begin{pmatrix} x_{n_0} & \cdots & x_{n_0+n_1-1} \end{pmatrix}^T.$$

$$A_2 : x_0 = \cdots = x_{n_0-1} = \cdots = x_{n_0+n_1-1} = 0.$$

$$Q_r : X_r^T E_r X_r = x_{n_0+\cdots+n_{r-1}}^2 + \cdots + x_{n_0+\cdots+n_r-1}^2 - x_{n_0+\cdots+n_r-1}^2 - \cdots - x_{n_0+\cdots+n_r-1}^2 = 0,$$

$$E_0 X_0 = O_0, \quad E_1 X_1 = O_1, \ldots, \quad E_{r-1} X_{r-1} = O_{r-1};$$

$$X_r = \begin{pmatrix} x_{n_0+\cdots+n_{r-1}} & \cdots & x_n \end{pmatrix}^T.$$

This coordinate system is called the normal projective coordinate system.
Theorem 4.4. In a Cayley-Klein space of dimension $n$ with the absolute figure $Q_0 \supset A_1 \supset \hat{Q}_1 \supset \hat{A}_2 \supset \cdots \supset \hat{A}_r \supset \hat{Q}_r \supset \hat{A}_{r+1} = \emptyset,$

let $Z(z)$ be a point not in $A_1$. The set of all points on tangent lines through $Z$ is a quadric.

Proof. Consider the normal projective coordinate system for the space. We first assume that $A_1$ is a hyperplane. For a given point $Y(y) \neq Z$, the line $Y + Z$ is a tangent line if and only if it passes through $Q_1$. It means that $(Y + tZ)(y + tz)$ is a point of $Q_1$ for a scalar $t$. For some integer $n_0 \geq 1$, $Q_1$ has the equation $x_0 = 0$, $X_1^T E_1 X_1 = 0$ where $X_1 = (x_1 \ldots x_{n_1})^T$. Consequently $(Y_1 - y_0 Z_1)^T E_1 (Y_1 - y_0 Z_1) = 0$ in which $Z_1 = (z_1 \ldots z_{n_1})^T$ and $Y_1 = (y_1 \ldots y_{n_1})^T$. This gives

$$y_0^2 Z_1^T E_1 Z_1 + Y_1^T E_1 Y_1 - 2y_0 Y_1^T E_1 Z_1 = 0.$$ 

It is a homogeneous equation of degree 2 describing a quadric in $P^n(R)$. Now, suppose that $A_1$ is not a hyperplane. Let $Z \notin Q_0$ and $Y(y)$ be a point with $Y \neq Z$. The line $Y + Z$ is a tangent line if and only if it passes through $Z^p \cap Q_0$. It means that $(Y + tZ)(y + tz) \in Z^p \cap Q_0$ for a scalar $t$. We have

$$Q_0 : X_0^T E_0 X_0 = 0, \quad X_0 = (x_0 \ldots x_{n_0-1})^T$$

for some integer $n_0$. Since $Y + tZ \in Z^p$, $(Y_0 + tZ_0)^T E_0 Z_0 = 0$ where $Y_0 = (y_0 \ldots y_{n_0-1})^T$ and $Z_0 = (z_0 \ldots z_{n_0-1})^T$. It gives $t = -Y_0^T E_0 Z_0 / Z_0^T E_0 Z_0$. From $Y + tZ \in Q_0$, it is followed that $(Y_0 + tZ_0)^T E_0 (Y_0 + tZ_0) = 0$. Therefore,

$$(Z_0^T E_0 Z_0)(Y_0^T E_0 Y_0) - (Y_0^T E_0 Z_0)^2 = 0$$

that is a homogeneous equation of degree 2. If $Z \in Q_0$, the desired point set is equal to the hyperplane $Z^p$ which is a quadric in $P^n(R)$. \hfill \square

Definition 4.5. Suppose that the assumptions of Theorem 4.4 hold. The set of all points on tangent lines through $Z$ is called the tangent cone with vertex $Z$.

Definition 4.6. Consider a Cayley-Klein space with the absolute figure $Q_0 \supset A_1 \supset \hat{Q}_1 \supset \hat{A}_2 \supset \cdots \supset \hat{A}_r \supset \hat{Q}_r \supset \hat{A}_{r+1} = \emptyset.$

Let $Z$ be a point in $P^n(R) \setminus Q_0$. Suppose that $T_Z$ is the tangent cone with vertex $Z$. Every quadric in the pencil spanned by $Q_0$ and $T_Z$ is called a sphere with center $Z$.

Under the assumptions of Definition 4.6, a quadric $\lambda Q_0 + \mu T_Z$ is a sphere with center $Z$ for some real numbers $\lambda$ and $\mu$. Suppose that $A_1$ is not a hyperplane. In the normal projective coordinate system, $Q_0$ and $T_Z$ have the following equations

$$Q_0 : X_0^T E_0 X_0 = 0, \quad T_Z : (Z_0^T E_0 Z_0)(X_0^T E_0 X_0) - (X_0^T E_0 Z_0)^2 = 0$$
where $X_0 = (x_0 \ldots x_{n-1})^T$ and $Z_0 = (z_0 \ldots z_{n-1})^T$. Therefore, the equation of $\lambda Q_0 + \mu T_Z$ is as follows

$$\lambda(X_0^T E_0 X_0) + \mu((Z_0^T E_0 Z_0)(X_0^T E_0 X_0) - (X_0^T E_0 Z_0)^2) = 0.$$ 

This gives

$$1 + \frac{\lambda}{\mu(Z_0^T E_0 Z_0)} = \frac{(X_0^T E_0 Z_0)^2}{(Z_0^T E_0 Z_0)(X_0^T E_0 X_0)}.$$ 

Suppose that $1 + \frac{\lambda}{\mu(Z_0^T E_0 Z_0)}$ equals $\alpha^2$ or $(\alpha i)^2$ for some real number $\alpha > 0$. Regarding the distance formulas for points of the space, the radius of the sphere $\lambda Q_0 + \mu T_Z$ is defined to be the complex number $\arccos(\alpha)$ or $\arccos(\alpha i)$. If $A_1$ is a hyperplane, we have

$$Q_0 : x_0^2 = 0, \quad T_Z : x_0^2(Z_1^T E_1 Z_1) + X_1^T E_1 X_1 - 2x_0(X_1^T E_1 Z_1) = 0.$$ 

Therefore, the sphere $\lambda Q_0 + \mu T_Z$ has the following equation

$$\frac{\lambda}{\mu}x_0^2 + (Z_1^T E_1 Z_1)x_0^2 + X_1^T E_1 X_1 - 2x_0(X_1^T E_1 Z_1) = 0.$$ 

If $-\frac{\lambda}{\mu}$ equals $\alpha^2$ or $(\alpha i)^2$ for some real number $\alpha > 0$, the radius of the sphere is defined to be $\alpha$ or $\alpha i$. In this way, we are able to obtain all spheres of a Cayley-Klein space without using the metric of the space.

### 5. Reflections

Every reflection in a Cayley-Klein space is defined by means of two subspaces which are total polar to each other. In this section, we give the formal definition of this notion and prove that every reflection of a Cayley-Klein space is a motion of the space. Also we show that in a Cayley-Klein space of dimension $n$ each motion is a product of at most $n + 1$ reflections in point-hyperplane pairs.

**Definition 5.1.** Let $K$ and $K'$ be two disjoint subspaces of $P^n(\mathbb{R})$ satisfying $\dim K + \dim K' = n - 1$. For a point $X$ not in $K \cup K'$, there exists a unique line passing through $X$ and intersecting $K$ and $K'$. The intersections of the line with $K$ and $K'$ are called the projection of $X$ onto $K$ in the direction of $K'$ and the projection of $X$ onto $K'$ in the direction of $K$ respectively.

**Definition 5.2.** Let $K$ and $K'$ be two nonempty disjoint subspaces of $P^n(\mathbb{R})$ satisfying $\dim K + \dim K' = n - 1$. The involution in the pair $(K, K')$ is a collineation of $P^n(\mathbb{R})$ that fixes each point of $K \cup K'$ and maps any other point $X$ to the harmonic conjugate of $X$ with respect to the two projections of $X$ onto $K$ and $K'$. If $K$ and $K'$ are total polar with respect to the absolute figure of a Cayley-Klein space, this involution is called the reflection in the pair $(K, K')$. 

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Assume that \( \{X_0, \ldots, X_k\} \) is an independent set of points of \( K \) and \( \{H_0, \ldots, H_k\} \) is an independent set of hyperplanes containing \( K' \). Suppose that in a projective coordinate system, \((x^0_0 \ldots x^0_n)^T \) and \((b^0_0 \ldots b^0_J)^T \) are the coordinate matrices of \( X_i \) and \( H_j \) respectively for \( 0 \leq i, j \leq k \). By setting

\[
K = \begin{pmatrix}
x^0_0 & x^1_0 & \cdots & x^k_0 \\
\vdots & \vdots & \ddots & \vdots \\
x^0_n & x^1_n & \cdots & x^k_n
\end{pmatrix}
\quad \text{and} \quad
K' = \begin{pmatrix}
b^0_0 & b^1_0 & \cdots & b^n_0 \\
\vdots & \vdots & \ddots & \vdots \\
b^0_k & b^1_k & \cdots & b^n_k
\end{pmatrix},
\]

the matrix of the involution in the pair \((K, K')\) is equal to \( l_{n+1} - 2K(K'K)^{-1}K' \) [10].

Consider a Cayley-Klein space of dimension \( n \) with the absolute figure

\[
\hat{Q}_0 \supset \hat{A}_1 \supset \hat{Q}_1 \supset \hat{A}_2 \supset \cdots \supset \hat{A}_r \supset \hat{Q}_r \supset \hat{A}_{r+1} = \emptyset.
\]

The matrix associated to a motion of the space in the normal projective coordinate system, introduced in Remark [2] is a block lower triangular matrix of order \( n + 1 \) such as

\[
\begin{pmatrix}
U_0 & 0 \\
& \ddots \\
& & U_r
\end{pmatrix}
\]

where \( U_i \) is a matrix of dimension \( n_i \) satisfying \( U_i^T E_i U_i = E_i \) for \( 0 \leq i \leq r \).

**Theorem 5.3.** Every reflection in a Cayley-Klein space is a motion of the space.

**Proof.** Consider a Cayley-Klein space of dimension \( n \) with the absolute figure

\[
\hat{Q}_0 \supset \hat{A}_1 \supset \hat{Q}_1 \supset \hat{A}_2 \supset \cdots \supset \hat{A}_r \supset \hat{Q}_r \supset \hat{A}_{r+1} = \emptyset.
\]

Let \( K \) be a \( k \)-dimensional subspace of the space and \( K' \) a total polar of \( K \) satisfying \( K \cap K' = \emptyset \). We show that the reflection in the pair \((K, K')\) is a motion of the space. We have \( K \cap A_l \neq \emptyset \) and \( K \cap A_{l+1} = \emptyset \) for some integer \( 0 \leq l \leq r \). Assume that \( K \subseteq A_d \) and \( K \nsubseteq A_{d+1} \) for some \( 0 \leq d \leq r \). Set \( m_j = \dim(A_j) \) and \( k_j = \dim(K \cap A_j) \) for \( 0 \leq j \leq r \). Let \( B = \{X_i \mid 0 \leq i \leq k \} \) be an independent subset of \( K \) satisfying \( B \cap A_j = \{X_0, \ldots, X_k\} \) for \( d \leq j \leq l \) and \( K' = \bigcap_{i=0}^k H_i \), where \( H_i \) is a total polar of \( X_i \) for \( 0 \leq i \leq k \). First, we show that for each \( j, d \leq j \leq l \) and each \( i, k_{j+1} < i \leq k_j \), the hyperplane \( H_i \) does not contain \( A_j \). We will prove this by induction on \( j \). Assume that the number of hyperplanes \( H_i \) with \( 0 \leq i \leq k_l \) not containing \( A_l \) equals \( k_l + 1 - t \) for some integer \( t \geq 0 \). These hyperplanes intersect \( A_l \) in \( k_l + 1 - t \) independent hyperplanes \( \{X_i^h \mid H_i \nsubseteq A_l, \ 0 \leq i \leq k_l \} \) of \( A_l \). So

\[
\dim(K' \cap A_l) = \dim\left(\bigcap_{i=0}^{k_l} H_i \cap A_l\right) = m_l - (k_l + 1 - t).
\]
Since $K \cap K^\perp = \emptyset$, it is followed that $(K \cap A_t) \cap (K^\perp \cap A_t) = \emptyset$. This gives

$$k_l + m_l - (k_l + 1 - t) - \dim((K \cap A_t) + (K^\perp \cap A_t)) = -1,$$

implying $t = 0$. Now suppose that for each $j, s < j \leq l$, we have $H_i \not\supset A_j$ for $k_{j+1} \leq i \leq k_j$. To complete the proof it suffices to show that $H_i \not\supset A_s$ for $k_{s+1} < i \leq k_s$. Notice that for each $j, s < j \leq l$, the set

$$\{H_i \cap A_j | k_{j+1} < i \leq k_j\} = \{X_i^p | k_{j+1} < i \leq k_j\}$$

is an independent set of hyperplanes of $A_j$ which are containing $A_{j+1}$. This implies that the set $\{H_i \cap A_s | H_i \not\supset A_s, 0 \leq i \leq k_s\}$, which is equal to

$$\{X_i^p | H_i \not\supset A_s, k_{s+1} < i \leq k_s\} \cup \{H_i \cap A_s | 0 \leq i \leq k_{s+1}\},$$

is an independent set of hyperplanes of $A_s$. From $(K \cap A_s) \cap (K^\perp \cap A_s) = \emptyset$, it follows that $(K \cap A_s) \cap (\bigcap_{i=0}^k H_i \cap A_s) = \emptyset$. If $t$ is the number of hyperplanes $H_i$ not containing $A_s$ for $k_{s+1} < i \leq k_s$, then

$$k_s + m_s - (k_s + 1 - t) - \dim((K \cap A_s) + (K^\perp \cap A_s)) = -1,$$

implying $t = 0$.

Now, consider the normal projective coordinate system for the space. Set $p_j = \sum_{i=0}^j n_i$ for $0 \leq j \leq r$ where $n_i = m_i - m_{i+1}$. Suppose that for each $i$, $0 \leq i \leq b$, the coordinate matrix of $X_1$ is $0 \ldots 0 x^i_{p_j} \ldots x_{p_{j}} x^i_{p_{j-1}} \ldots x_{p_{j-1}}$ where $X_i \in A_j \setminus A_{j+1}$ for some $d \leq j \leq l$. If $X_j^i = (x^i_{p_j} \ldots x^{i}_{p_{j-1}})^T$, the coordinate matrix of $H_i$ takes the form $(b_0^i \ldots b_{p_j}^i \alpha^i(E_jX_j)^T 0 \ldots 0)$ for some real number $\alpha$. Since $H_i \not\supset A_j$, it is followed that $\alpha \neq 0$. We can assume that $\alpha = 1$. Assume that for each $j, d \leq j \leq l$, $X_j$ is an $n_j$ by $(k_j - k_{j-1})$ matrix $X_j^{k_{j-1}+1} \ldots X_j^{k_j}$. Let $K$ and $K^\perp$ be

$$\begin{pmatrix}
\mathbf{0}_{p_{l-1} \times (k_{l-1} - k_l)} & \mathbf{0}_{p_{l-2} \times (k_{l-2} - k_{l-1})} & \mathbf{0}_{p_{l-3} \times (k_{l-3} - k_{l-2})} & \cdots & \mathbf{0}_{p_{d-1} \times (k_d - k_{d+1})} \\
X_l & X_{l-1} & \cdots & \mathbf{0}_{d-1} & \mathbf{X}_d
\end{pmatrix}$$
and

\[
\begin{pmatrix}
* & * & ... & (E_i X_i)^T & 0_{(k_i-k_{i+1}) \times (n+1-p_i)} \\
* & * & ... & (E_{i-1} X_{i-1})^T & 0_{(k_{i-1}-k_i) \times (n+1-p_{i-1})} \\
* & * & ... & (E_{i-2} X_{i-2})^T & 0_{(k_{i-2}-k_{i-1}) \times (n+1-p_{i-2})} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
* & (E_d X_d)^T & \cdots & \cdots & 0_{(k_d-k_{d+1}) \times (n+1-p_d)}
\end{pmatrix}
\]

respectively in which \((i+1)\)-th column of \(K\) is the coordinate matrix of \(X_i\) and \((i+1)\)-th row of \(K^\perp\) is the coordinate matrix of \(H_i\). The matrix of the reflection in the pair \((K, K^\perp)\) equals \(I_{n+1} - 2K(K^\perp K)^{-1}K^\perp\). The product \(K^\perp K\) and its inverse are the following block upper triangular matrices

\[
K^\perp K = \begin{pmatrix}
X_i^T E_i X_i & * \\
0 & X_d^T E_d X_d
\end{pmatrix}
\]

and

\[
(K^\perp K)^{-1} = \begin{pmatrix}
(X_i^T E_i X_i)^{-1} & * \\
0 & (X_d^T E_d X_d)^{-1}
\end{pmatrix}.
\]

It can be seen that the matrices \(K(K^\perp K)^{-1}\) and \(K(K^\perp K)^{-1}K^\perp\) are equal to

\[
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

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respectively. Therefore the matrix \( I_{n+1} - 2K(K^\perp K)^{-1}K^\perp \) equals
\[
\begin{pmatrix}
I_{p_d-1} & \cdots & \cdots & 0_{n_d \times (n+1-p_d-1)} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
I_{n_d-2X_d(X_d^T E_d X_d)^{-1}(E_d X_d)^T} & \cdots & \cdots & 0_{n_d \times (n+1-p_d)} \\
\end{pmatrix}
\]
which is a block lower triangular matrix. It is easy to see that
\[
(l_{n_d} - 2X_d(X_d^T E_d X_d)^{-1}X_d^T X_d) E_d l_{n_d} = E_d
\]
for \( d \leq j \leq l \).

**Theorem 5.4.** Every motion in a Cayley-Klein space of dimension \( n \) is a composition of at most \( n+1 \) reflections in point-hyperplane pairs.

**Proof.** To prove the theorem we show that the matrix associated to a given motion is a product of at most \( n+1 \) matrices of the form
\[
I_{n+1} - 2X(X^\perp X)^{-1}X^\perp,
\]
in which \( X \) and \( X^\perp \) are the coordinate matrices of a point and a total polar of it respectively. In each Cayley-Klein space of dimension \( n \), with the exception of \( n = 0 \), such a matrix represents a reflection in a point-hyperplane pair. For \( n = 0 \), the only motion of the space is the identity mapping which is a product of zero reflection. This motion is equal to \( I_1 - 2X(X^\perp X)^{-1}X^\perp \) for \( X = (1) \) and \( X^\perp = (1) \). Suppose that the assertion is true in each Cayley-Klein space of dimension \( n - 1 \). Now, consider a Cayley-Klein space of dimension \( n \) with the absolute figure
\[
\bigcirc \supseteq \hat{A}_1 \supseteq \bigcirc \supseteq \hat{A}_2 \supseteq \cdots \supseteq \hat{A}_r \supseteq \bigcirc \supseteq \hat{A}_{r+1} = \emptyset.
\]
Let \( g \) be a motion of the space and \( U \) the matrix associated to it in the normal projective coordinate system. Thus \( U \) is a block lower triangular matrix such as
\[
\begin{pmatrix}
U_0 & 0 \\
\vdots & \ddots \\
0 & \cdots & U_r
\end{pmatrix}
\]
where \( U_i \) is a matrix of dimension \( n_i \) satisfying \( U_i^T E_i U_i = E_i \) for \( 0 \leq i \leq r \). Let \( X \) be the point with the coordinate matrix \( X = (1 \ 0 \ \ldots \ 0)^T \). If \( g(X) = X \), the first column of \( U \) takes the form \( (\alpha \ 0 \ \ldots \ 0)^T \) for some \( \alpha \neq 0 \). Since \( U_0^T E_0 U_0 = E_0 \), it
is followed that

\[ U = \begin{pmatrix} \alpha & 0 & \ldots & 0 & 0 \\ 0 & V_0 & \ldots & 0 & 0 \\ 0 & 0 & U_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & * & \ldots & \ldots & U_r \end{pmatrix} \]

in which \( \alpha^2 = 1 \) and \( V_0^T E_0' V_0 = E_0' \) for \( E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). It can be assumed that \( \alpha = 1 \).

Consider a Cayley-Klein space of dimension \( n - 1 \) with the absolute figure

\[ \hat{Q}_0 \supseteq \hat{A}_1 \supseteq \hat{Q}_1 \supseteq \hat{A}_2 \supseteq \ldots \supseteq \hat{A}_r \supseteq \hat{Q}_r \supseteq \hat{A}_{r+1} = \emptyset. \]

where \(( x_1 \ldots x_{n_0-1} ) E_0' ( x_1 \ldots x_{n_0-1} )^T = 0\) is the equation of \( Q_0' \). By the induction hypothesis, for the matrix

\[ U' = \begin{pmatrix} V_0 & 0 \\ U_1 & \end{pmatrix} \]

there exist matrices \( S'_1, \ldots, S'_m \) for \( m \leq n \) such that \( U' = S'_1 \cdots S'_m \) in which

\[ S'_i = I_n - 2X'_i (X'_i)^\perp (X'_i)^\perp \]

for some point \( X'_i \) and a total polar \( (X'_i)^\perp \) of it with the coordinate matrices \( X'_i \) and \( (X'_i)^\perp \) respectively. It is easily seen that the hyperplane \( X^\perp_i \) with the coordinate matrix \( X^\perp_i = (0 (X'_i)^\perp) \) is a total polar of the point \( X_i \) with the coordinate matrix \( \begin{pmatrix} 0 \\ X'_i \end{pmatrix} \) in the \( n \)-dimensional Cayley-Klein space. By setting

\[ S_i = I_{n+1} - 2X_i (X^\perp_i X_i)^{-1} X^\perp_i, \]

we get

\[ S_i = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & S'_i \\ 0 & & & 1 \end{pmatrix} \]

for \( 1 \leq i \leq m \), implying \( U = S_1 \cdots S_m \).

If \( g(X) \neq X \), then \( g(X) - X \notin Q_0 \) or \( g(X) + X \notin Q_0 \) in which \( g(X) \pm X \) is the point with the coordinate matrix \( UX \pm X \). In the former case, the unique total polar \( (g(X) - X)^p \) of \( g(X) - X \) contains \( g(X) + X \) and in the latter case we get \( g(X) - X \in (g(X) + X)^p \). Since \( g(X) \) is the harmonic conjugate of \( X \) with respect to \( g(X) - X \) and \( g(X) + X \), it is followed that \( g(X) \) is mapped to \( X \) under the reflection
Therefore, in each Cayley-Klein space the set of all reflections in point-hyperplane pairs is a generator of the group of motions of the space. We show that the reflections in point-hyperplane pairs that both point and hyperplane are not regular are necessary for generating motions. Consider a Cayley-Klein space of dimension three with the absolute figure

\[ \hat{Q}_0 \supseteq \hat{A}_1 \supseteq \hat{Q}_1 \supseteq \hat{A}_2 \supseteq \hat{Q}_2 \supseteq \hat{A}_3 \supseteq \hat{Q}_3 \supseteq \emptyset, \]

in which each \( Q_i \), \( 0 \leq i \leq 3 \), is a hyperplane of \( A_i \). All points of the space not in \( A_1 \) and all planes not containing \( A_3 \) are the only regular points and planes respectively. The total polar of each regular point is the plane \( A_1 \) and the total polar of each regular plane is the point \( A_3 \). In the normal projective coordinate system, the coordinate matrix of a regular point \( X \) and a regular plane \( H \) takes the form \( ( 1 \ x \ y \ z )^T \) and \( ( a \ b \ c \ 1 ) \) respectively. Also the coordinate matrices of the plane \( A_1 \) and the point \( A_3 \) are \( ( 1 \ 0 \ 0 \ 0 ) \) and \( ( 0 \ 0 \ 0 \ 1 )^T \) respectively. So the matrices associated to the reflections in the pairs \( (X, A_1) \) and \( (A_3, H) \) are

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
-2x & 1 & 0 & 0 \\
-2y & 0 & 1 & 0 \\
-2z & 0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-2a & -2b & -2c & -1
\end{pmatrix}.
\]

We show that the set of all such matrices can not span the group of motions of this Cayley-Klein space. For each matrix of the set the product of the second and the third diagonal entries is positive. Since each matrix is lower triangular, this positivity holds for every product of them. Let \( X' \) be a point of \( A_1 \setminus A_2 \) and \( H' \) a total polar of it different from \( A_1 \). The coordinate matrices of \( X' \) and \( H' \) are of the forms \( ( 0 \ 1 \ y' \ z' )^T \) and \( ( a' \ 1 \ 0 \ 0 ) \) respectively. The matrix associated to the reflection in the pair \( (X', H') \) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-2a' & -1 & 0 & 0 \\
-2a'y' & -2y' & 1 & 0 \\
-2a'z' & -2z' & 0 & 1
\end{pmatrix}.
\]

This matrix can not be a product of the above matrices. The authors report there are no competing interests to declare.

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