The Volume of the Past Light-Cone and the Paneitz Operator

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ABSTRACT

We study a conjecture involving the invariant volume of the past light-cone from an arbitrary observation point back to a fixed initial value surface. The conjecture is that a 4th order differential operator which occurs in the theory of conformal anomalies gives $8\pi$ when acted upon the invariant volume of the past light-cone. We show that an extended version of the conjecture is valid for an arbitrary homogeneous and isotropic geometry. First order perturbation theory about flat spacetime reveals a violation of the conjecture which, however, vanishes for any vacuum solution of the Einstein equation. These results may be significant for constructing quantum gravitational observables, for quantifying the back-reaction on spacetime expansion and for alternate gravity models which feature a timelike vector field.

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1 Introduction

Suppose $S$ is a Cauchy surface for the usual fields of physics and let $M$ stand for a globally hyperbolic spacetime manifold comprising $S$ and its future. We will often think of $S$ as the locus of points $x^\mu = (0, \vec{x})$, with $M$ as the set of all $x^\mu = (t, \vec{x})$ with $t \geq 0$. Of course points are just labels, geometry derives from the metric field, $g_{\mu\nu}(t, \vec{x})$, which we shall take to be spacelike.

A quantity of great geometrical significance is the invariant volume of the past light-cone of an arbitrary point $x^\mu \in M$. It can be expressed as an integral involving some other geometrical quantities which each require a little explanation,

$$V[g](x) = \int_M d^4x' \sqrt{-g(x')} \Theta(-\sigma[g](x, x')) \Theta(F[g](x, x')) .$$  \hspace{1cm} (1)

(Our notation is that functional dependence upon fields appears in square brackets, whereas dependence upon coordinates and other parameters is parenthesized.) Of course $g(x')$ is the determinant of $g_{\mu\nu}(x')$. The quantity $\sigma[g](x, x')$ was introduced by DeWitt and Brehme [1]. It is one half the square of the geodesic length from $x^\mu$ to $x'^\mu$ and can be expressed in terms of the geodesic $\chi^\mu[g](\tau, x, x')$ which runs between $x^\mu$ (at $\tau = 0$) to $x'^\mu$ (at $\tau = 1$),

$$\sigma[g](x, x') = \frac{1}{2} \int_0^1 d\tau g_{\mu\nu}(\chi) \dot{\chi}^\mu \dot{\chi}^\nu .$$  \hspace{1cm} (2)

If more than one geodesic connects $x^\mu$ and $x'^\mu$ then $\sigma[g](x, x')$ is defined to be the value for which the right hand side of (2) is smallest; if no geodesic connects the two points then $\sigma[g](x, x')$ is $\frac{1}{2}$ times the minimum distance between them. Because our metric is spacelike we see that $\sigma[g](x; x')$ is positive when $x^\mu$ and $x'^\mu$ are spacelike separated, and negative when they are timelike separated. The condition $F[g](x, x') > 0$ in expression (1) restricts the integration to points $x'^\mu$ in the past of $x^\mu$. Owing to the factor of $\Theta(-\sigma)$ we need only define $F[g](x, x')$ for the case where $x^\mu$ and $x'^\mu$ are timelike related: it is $+1$ when extending the geodesic to $\tau \geq 1$ eventually hits the Cauchy surface $S$; otherwise it is $-1$.

The invariant volume of the past light-cone is interesting for a number of reasons. First, if we consider $S$ to be the initial value surface (defined invariantly in some way) on which a quantum gravitational state is specified, then $V[g](x)$ at some invariantly defined point $x^\mu$ ought to be an observable.
because a local observer should be able to look back into his past. It is notoriously difficult to identify physically meaningful observables in quantum gravity \[2, 3\]. A second potential application is quantifying the back-reaction to spacetime expansion. Suitable observables already exist for the important case of scalar-driven inflation \[4\] but these do not apply for pure quantum gravity and $V[g](x)$ may have a role to play in invariantly fixing the observation point \[5\]. A final application concerns alternate gravity models which involve a timelike vector field \[6, 7\]. Because $V[g](x)$ necessarily grows as one evolves, its gradient is timelike, and can serve to define a timelike vector field based upon the metric, without the complications associated with introducing new dynamical degrees of freedom. It has been suggested that such a term might arise from quantum corrections to the effective field equations \[8\].

The purpose of this paper is to study a conjecture concerning $V[g](x)$ and a certain 4th order differential operator. To motivate the conjecture, consider the flat space limit $g_{\mu\nu}(t, \vec{x}) \to \eta_{\mu\nu}$,

$$\sigma[\eta](x, x') = \frac{1}{2} (x - x')^2 , \quad F[\eta](x, x') = \text{sgn}(t - t') , \quad V[\eta](x) = \frac{\pi}{3} t^4 . \quad (3)$$

Acting the square of the d’Alembertian ($\partial^2 \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$) on $V[\eta](x)$ gives a simple constant,

$$\partial^4 V[\eta](x) = 8\pi . \quad (4)$$

The conjecture is that a known differential operator $D_P$ allows us to extend relation (4) to an arbitrary, globally hyperbolic metric (and for a general Cauchy surface $S$),

$$D_P V[g](x) = 8\pi . \quad (5)$$

Of course there is no guarantee that any local differential operator has this property. However, we will show that an extended version of (5) pertains for an arbitrary homogeneous and isotropic cosmology. We will also show that the conjecture fails for a general first order perturbation about flat spacetime, although only by terms which vanish with the vacuum Einstein equations. This suggests that some modified version of the conjecture might still be valid.

The Paneitz operator $D_P$ of our conjecture (5) is known from the theory of conformal anomalies \[9, 10\]. For a general metric $g_{\mu\nu}(t, \vec{x})$ it takes the form,

$$D_P \equiv \Box^2 + 2 D_\mu \left[ R^{\mu\nu} - \frac{1}{3} g^{\mu\nu} R \right] D_\nu , \quad (6)$$
where $R_{\mu\nu}$ is the Ricci tensor, $R$ is the Ricci scalar, $D_\mu$ is the covariant derivative operator and $\Box$ is the covariant d’Alembertian,

$$\Box \equiv g^{\mu\nu}D_\mu D_\nu \longrightarrow \frac{1}{\sqrt{-g}} \partial_\mu \left[ \sqrt{-g} g^{\mu\nu} \partial_\nu \right] \quad \text{acting on a scalar.}$$

(7)

The Paneitz operator occurs in the nonlocal effective actions which represent conformal anomalies [9, 10] owing to its special behavior under a conformal transformation,

$$g_{\mu\nu}(x) = \Omega^2(x) \overline{g}_{\mu\nu}(x) \quad \Rightarrow \quad \Omega^{-4} \times \overline{D}_P .$$

(8)

(Here $\overline{g}_{\mu\nu}$ is the conformally rescaled metric and $\overline{D}_P$ is the Paneitz operator constructed from it.) The fact that all matter theories engender conformal anomalies means that logarithms of $D_P$ are ubiquitous in the quantum effective action, and inverses of $D_P$ must appear in the quantum-corrected, effective field equations. So our conjecture represents one way that the invariant volume of the past light-cone can arise in the effective field equations of gravity without introducing new physics.

Just as Gauss’s law has a differential and an integral form, so too our conjecture (5) can be expressed in terms of an integral. The retarded Green’s function $G[g](x, x')$ of $D_P$ obeys,

$$\sqrt{-\overline{g}} \overline{D}_P \, G[g](x; x') = \delta^4(x-x') \quad \text{and} \quad \Theta(-\mathcal{F}[g](x, x')) \mathcal{G}[g](x, x') = 0 .$$

(9)

Because the characteristics of the highest derivative term in $D_P$ are set by the metric, in the same way as for typical second order operators, $\mathcal{G}[g](x, x')$ must vanish for any point $x'^\mu$ outside the past light-cone of $x^\mu$. Hence we should get a finite result from integrating $\mathcal{G}[g](x, x')$ over $\mathcal{M}$ back to the initial value surface $\mathcal{S}$. We define this integral as the functional $\mathcal{P}[g](x)$,

$$\mathcal{P}[g](x) \equiv \int_{\mathcal{M}} d^4x' \sqrt{-g(x')} \mathcal{G}[g](x, x') ,$$

(10)

One can regard $\mathcal{P}[g](x)$ to be $D_P^{-1}$ acting on 1, so the integral expression of our conjecture (5) is,

$$\mathcal{V}[g](x) = 8\pi \mathcal{P}[g](x) .$$

(11)

In section 2 we demonstrate that an extended version of (11) pertains for an arbitrary homogeneous and isotropic geometry. In section 3 we consider the conjecture for first order perturbations about flat spacetime. Although
the conjecture is violated in general, it is valid for any first order perturbation which obeys the vacuum Einstein equations. We discuss the implications of this work in section 4. An appendix summarizes some useful but tedious integral identities.

2 FRW Geometries

The purpose of this section is to verify the conjecture $\Box$ for an arbitrary homogeneous and isotropic geometry,

$$g_{\mu\nu}(t)dx^\mu dx^\nu = a^2(\eta) \left[-d\eta^2 + \frac{dr^2}{1 - kr^2} + r^2d\Omega \right] \equiv a^2\tilde{g}_{\mu\nu}dx^\mu dx^\nu. \quad (12)$$

Here $\eta$ is the conformal time, $a(\eta)$ is the scale factor and $k$ is the spatial curvature. The case of $k > 0$ corresponds to positive spatial curvature; $k = 0$ is spatial flatness; and $k < 0$ is negative spatial curvature. For $k > 0$ it should be noted that $r$ has the finite range $0 \leq r \leq 1/\sqrt{k}$, and that any value of $r$ within this range corresponds to two distinct points on the manifold. We first work out $V[g](x)$, then construct $P[g](x)$ and demonstrate that $V[g](x) = 8\pi P[g](x)$.

Because the geometry is homogeneous we can choose the spatial origin to coincide with the point from which we are computing the invariant volume of the past light-cone. Recall that the invariant volume of the past light-cone from $x^\mu = (\eta, \vec{0})$ is the integral of $d^4x'\sqrt{-g(x')}$ over all points $x'^\mu = (\eta', \vec{x}')$ which are in the past of $x^\mu$ and timelike related to it. This obviously requires $\eta' < \eta$. To enforce the timelike relation we first compute the coordinate radius $r(\eta, \eta')$ which is traveled by a light ray emitted at $x'^\mu$ and received at $x^\mu$,

$$\int_0^r dx = \int_{\eta'}^\eta ds \quad \Rightarrow \quad r(\eta, \eta') = \frac{1}{\sqrt{k}} \left| \sin(\sqrt{k}\Delta\eta) \right|. \quad (13)$$

Here $\Delta\eta \equiv \eta - \eta'$, and we should call attention to the fact that the formula for $r(\eta, \eta')$ remains valid no matter what is the sign $k$. However, one should note that for $\sqrt{k}\Delta\eta > \pi$ the light-cone has wrapped all the way around the spatial manifold.

The points $x'^\mu = (\eta', \vec{x}')$ which are lightlike related to $x^\mu = (t, \vec{0})$ can be written as,

$$\vec{x}' = r(\eta, \eta') \times \hat{r}(\theta', \phi'), \quad (14)$$
where the radial unit vector is the same as in flat space,

\[
\hat{r}(\theta', \phi') \equiv \left( \sin(\theta') \cos(\phi'), \sin(\theta') \sin(\phi'), \cos(\theta') \right) .
\]  \hspace{1cm} (15)

Suppose the initial value surface is at \( \eta' = \eta_I \) and that, for the positive curvature case, the observation time \( \eta \) is not so late that the light-cone has wrapped all the way around the spatial manifold. It follows that the invariant volume of the past light-cone (in the background geometry) is,

\[
V[\text{g}](t, \vec{0}) = \int_{\eta_I}^\eta d\eta' \int d^3 \vec{x}' \sqrt{-g(\eta', \vec{x}') \Theta\left( r(\eta, \eta') - r' \right)} ,
\]  \hspace{1cm} (16)

\[
= 4\pi \int_{\eta_I}^\eta d\eta' a^4(\eta') \times \int_0^{r(\eta, \eta')} r'^2 dr' \sqrt{1 - kr'^2} ,
\]  \hspace{1cm} (17)

\[
= \frac{\pi}{k^{\frac{3}{2}}} \int_{\eta_I}^\eta d\eta' a^4(\eta') \left[ 2\sqrt{k} \Delta \eta - \sin(2\sqrt{k} \Delta \eta) \right] .
\]  \hspace{1cm} (18)

For the case of positive curvature and \( \sqrt{k}(\eta - \eta_I) > \pi \) the result is more complicated,

\[
k > 0 \quad \text{and} \quad \sqrt{k}(\eta - \eta_I) > \pi \quad \Rightarrow \quad V = \frac{\pi}{k^{\frac{3}{2}}} \left\{ \int_{\eta_I}^{\eta - \pi/\sqrt{k}} d\eta' a^4(\eta') \left[ 2\sqrt{k} \Delta \eta - \sin(2\sqrt{k} \Delta \eta) \right] + 2\pi \int_{\eta_I}^{\eta - \pi/\sqrt{k}} d\eta' a^4(\eta') \right\}.
\]  \hspace{1cm} (19)

This falsifies the original conjecture, but a very simple extension of it can be made for which (18) remains correct at all times. The extension is just to redefine the “the volume of the past light-cone” to mean the sum of the volumes of the past light-cone from the observation point and from any focal points at which past-directed, null geodesics from the observation point converge. Finally, note that expressions (18) and (19) are valid as well for \( \vec{x} \neq \vec{0} \) owing to the homogeneity of the geometry.

To construct the Paneitz operator on the background geometry we first extract the conformal factor \( \Omega = a(\eta) \) and exploit the simple scaling rule (8),

\[
\bar{D}_P = \frac{1}{a^4} \tilde{D}_P .
\]  \hspace{1cm} (20)

Recall that \( \tilde{D}_P \) is the Paneitz operator constructed in the conformally related metric,

\[
d\bar{s}^2 \equiv \bar{g}_{\mu\nu}dx^\mu dx^\nu = -d\eta^2 + \frac{dr^2}{1 - kr^2} + r^2d\Omega .
\]  \hspace{1cm} (21)
Now consider the action of the scalar d’Alembertian on a function which depends only on the conformal time \( \eta \),

\[
\square f(\eta) = \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_{\nu} f \right) = -\frac{d^2 f}{d\eta^2} .
\]  

To get the curvature part of the Paneitz operator we recall the simple form of the Ricci tensor and its trace in the conformally rescaled geometry \( \tilde{g} \),

\[
\tilde{R}^{00} = 0 , \quad \tilde{R}^{0j} = 0 , \quad \tilde{R}^{ij} = 2k\tilde{g}^{ij} , \quad \tilde{R} = 6k .
\]

Now consider the action of the curvature terms on the same function \( f(\eta) \),

\[
2\tilde{D}_{\mu} \left[ \tilde{R}^{\mu\nu} - \frac{1}{3} \tilde{g}^{\mu\nu} \tilde{R} \right] \tilde{D}_{\nu} f(\eta) = \frac{2}{\sqrt{-g}} \partial_{\mu} \left[ \sqrt{-\tilde{g}} \left( \tilde{R}^{\mu\nu} - \frac{1}{3} \tilde{g}^{\mu\nu} \tilde{R} \right) \partial_{\nu} f \right] = 4k \frac{d^2 f}{d\eta^2} .
\]

Combining relations \( (20), (22) \) and \( (24) \) gives,

\[
\square_P f(\eta) = \frac{1}{a^4} \left( \frac{d}{d\eta} \right)^2 \left[ \left( \frac{d}{d\eta} \right)^2 + 4k \right] f(\eta) .
\]

Now recall from \( (9-10) \) that constructing \( P[g](x) \) amounts to solving the differential equation,

\[
D_P P[g](x) = 1 ,
\]

subject to retarded boundary conditions. From \( (25) \) we see that this requires inverting the product of two second order, differential operators. The associated retarded Green’s functions are,

\[
\left( \frac{d}{d\eta} \right)^2 G_1(\eta, \eta') = \delta(\eta-\eta') \implies G_1 = \theta(\eta-\eta') \times (\eta-\eta') ,
\]

\[
\left[ \left( \frac{d}{d\eta} \right)^2 + 4k \right] G_2(\eta, \eta') = \delta(\eta-\eta') \implies G_2 = \theta(\eta-\eta') \times \frac{\sin[2\sqrt{k}(\eta-\eta')]}{2\sqrt{k}} .
\]

It follows that the unique solution for \( P[g](x) \) is,

\[
\begin{align*}
P[g](x) &= \int_{\eta_l}^{\eta_u} d\eta' G_2(\eta, \eta') \int_{\eta_l}^{\eta_u} d\eta'' G_1(\eta', \eta'') a^4(\eta'') , \\
&= \int_{\eta_l}^{\eta_u} d\eta'' a^4(\eta'') \int_{\eta_l}^{\eta_u} d\eta' G_1(\eta', \eta'') G_2(\eta, \eta') , \\
&= \frac{1}{8k^2} \int_{\eta_l}^{\eta_u} d\eta' a^4(\eta') \left\{ 2\sqrt{k}(\eta-\eta') - \sin[2\sqrt{k}(\eta-\eta')] \right\} .
\end{align*}
\]
Multiplying \((31)\) by \(8\pi\) gives precisely \((18)\). Note that expression \((31)\) is correct for \(\mathcal{P}[\mathcal{F}](x)\) for all \(k\) and \(\eta\), whereas expression \((18)\) must be replaced by \((19)\) to give \(\mathcal{V}[\mathcal{F}](x)\) for the case of positive curvature and times so late that the light-cone has wrapped all the way around the spatial manifold. Hence the conjecture will not remain valid unless we modify the volume of the past light-cone to multiply count points which have been multiply covered.

3 Perturbations about Flat Spacetime

The purpose of this section is to compare \(\mathcal{V}[g](x)\) with \(8\pi\mathcal{P}[g](x)\) by using first order perturbation theory around flat space. That means we write the metric as,

\[
g_{\mu\nu}(t, \vec{x}) = \eta_{\mu\nu} + h_{\mu\nu}(t, \vec{x}) .
\]

(32)

The field \(h_{\mu\nu}(t, \vec{x})\) is known as the graviton field. By convention its indices are raised and lowered with the Lorentz metric,

\[
h^\mu_\nu \equiv \eta^{\mu\rho} h_{\rho\nu} , \quad h^{\mu\nu} \equiv \eta^{\mu\rho} \eta^{\nu\sigma} h_{\rho\sigma} \quad \text{and} \quad h \equiv \eta^{\mu\nu} h_{\mu\nu} .
\]

(33)

In the first subsection we work out \(\mathcal{V}[\eta + h](x)\) at first order in \(h_{\mu\nu}\); the corresponding first order variation in \(8\pi\mathcal{P}[\eta + h](x)\) is derived in subsection 3.2. In the final subsection we reduce the difference of the two expressions to an invariant form.

3.1 First order perturbation of \(\mathcal{V}[\eta + h](x)\)

One computes the first order correction to \(\mathcal{V}[g](x)\) from expression \((1)\) by expanding the measure factor and the theta function which enforces that \(x^\mu\) and \(x'^\mu\) are timelike separated,

\[
\sqrt{-g(x')} = 1 + \frac{1}{2} h(x') + O(h^2) ,
\]

(34)

\[
\Theta(-\sigma[g](x'; x')) = \Theta\left(-\frac{1}{2}(x'-x)^2\right) - \delta\left(\frac{1}{2}(x'-x)^2\right) \delta\sigma(x; x') + O(h^2) .
\]

(35)

Note that there is no first order correction to the functional \(\mathcal{F}[g](x; x')\) whose sign determines whether \(x'^\mu\) is in the past \((\mathcal{F} = +1)\) or future \((\mathcal{F} = -1)\) of \(x^\mu\). Indeed, it is not changed to any order in perturbation theory,

\[
\mathcal{F}[g](x, x') = \text{sgn}(t-t') .
\]

(36)
From expression (2) we see that the variation of \( \sigma[g](x, x') \) about any metric consists of the metric variation, plus the endpoint variation and a term proportional to the geodesic equation,

\[
\delta \sigma[g](x, x') = \frac{1}{2} \int_0^1 d\tau \delta g_{\mu\nu}(\chi) \dot{\chi}^\mu \dot{\chi}^\nu + g_{\mu\nu}(\chi) \dddot{\chi}^\mu \delta \chi^\nu \bigg|_0^1 - \int_0^1 d\tau g_{\mu\nu}(\chi) \left[ \dddot{\chi}^\mu + \Gamma^\mu_{\rho\sigma}(\chi) \dot{\chi}^\rho \dot{\chi}^\sigma \right] \delta \chi^\nu .
\] (37)

The metric perturbation is just \( \delta g_{\mu\nu} = h_{\mu\nu} \) and the other two terms vanish because the endpoints are fixed and \( \chi^\mu \) is a geodesic. The zeroth order geodesic is, \( \chi^\mu(\tau) = x^\mu + (x' - x)^\mu \tau \), so the first order correction to (1) is,

\[
\delta V(x) = \frac{1}{2} \int_\mathcal{M} d^4x' \Theta(t-t') \Theta \left( -\frac{1}{2} (x' - x)^2 \right) h(x')
- \frac{1}{2} \int_\mathcal{M} d^4x' \Theta(t-t') \delta \left( \frac{1}{2} (x' - x)^2 \right) \int_0^1 d\tau h_{\mu\nu} \left( x + (x'-x)\tau \right) (x' - x)^\mu (x' - x)^\nu ,
\] (38)

\[
= \frac{1}{2} \int_0^t dt' \int d^3x' \Theta(t-t' - \|\vec{x} - \vec{x}'\|) h(t', \vec{x}') - \frac{1}{2} \int_0^1 d\tau \int d^3x' \Theta(t - \|\vec{x} - \vec{x}'\|) \frac{\delta \left( x' - x \right)^\mu (x' - x)^\nu \delta \chi^\nu .}.
\] (39)

Note that the temporal differences in (39) contain no factors of \( \tau \),

\[
(x' - x)^0 \equiv -\|\vec{x}' - \vec{x}\| \equiv -\Delta x .
\] (40)

So expanding out the double contraction in (39) gives,

\[
h_{\mu\nu} \left( t - \Delta x \tau, \vec{x} + \Delta x \vec{\tau} \right) (x' - x)^\mu (x' - x)^\nu = \Delta x^2 \left\{ h_{00} - 2h_{0i} \vec{\tau}^i + h_{ij} \vec{\tau}^i \vec{\tau}^j \right\} .
\] (41)

Here and subsequently the radial unit vector is,

\[
\vec{\tau} \equiv \frac{\vec{x}' - \vec{x}}{\Delta x} .
\] (42)

The final form is obtained by changing variables in the second term of (39) from \( \tau \) to the retarded time,

\[
\tau \equiv \frac{t-t'}{\Delta x} \equiv \frac{\Delta t}{\Delta x} \quad \leftrightarrow \quad t' \equiv t - \Delta x \tau .
\] (43)
This allows us to perform the radial integration,
\[
\int_0^1 d\tau \int d^3 x' \Theta(t - \Delta x) \Delta x f(t - r\tau, \bar{x} + \Delta x\hat{r})
\]
\[= \int d\Omega \int_0^t dr r^3 \int_0^1 d\tau f(t - r\tau, \bar{x} + \Delta x\hat{r}), \tag{44}\]
\[= \int d\Omega \int_0^t dr r^2 \int_{t-r}^t dt' f(t', \bar{x} + \Delta t\hat{r}), \tag{45}\]
\[= \int_0^t dt' \int d\Omega f(t', \bar{x} + \Delta t\hat{r}) \int_{\Delta t}^t dr r^2, \tag{46}\]
\[= \frac{1}{3} \int_0^t dt' (t^3 - \Delta t^3) \int d\Omega f(t', \bar{x} + \Delta t\hat{r}). \tag{47}\]

Hence our final form for the first order perturbation of \(V[g](x)\) is,
\[
\delta V(x) = \frac{1}{2} \int_0^t dt' \int d^3 x' \Theta(\Delta t - \Delta x) h(t', \bar{x}') - \frac{1}{6} \int_0^t dt' (t^3 - \Delta t^3) \int d\Omega
\]
\[\times \left\{ h_{00}(t', \bar{x} + \Delta t\hat{r}) - 2 h_{0i}(t', \bar{x} + \Delta t\hat{r})\hat{r}^i + h_{ij}(t', \bar{x} + \Delta t\hat{r})\hat{r}^i\hat{r}^j \right\}. \tag{48}\]

### 3.2 First order perturbation of \(8\pi P[\eta + h](x)\)

Recall that \(P[g](x)\) can be expressed as the inverse of the Paneitz operator acting on unity,
\[
P[g](x) \equiv \int_M d^4 x' \sqrt{-g(x')} G[g](x, x') = \frac{1}{D_P}[1](x), \tag{49}\]

If we write,
\[
D_P = D_P + \delta D_P + O(h^2), \tag{50}\]

then the functional inverse becomes,
\[
\frac{1}{D_P} = \frac{1}{D_P} - \frac{1}{D_P} \times \delta D_P \times \frac{1}{D_P} + O(h^2). \tag{51}\]

The first order correction we are seeking is accordingly,
\[
\delta P(x) = - \int_M d^4 x' G[\eta](x, x') \times \delta D' \times P[\eta](x'), \tag{52}\]
\[= - \int_M d^4 x' \frac{1}{8\pi} \Theta(t-t')\Theta(-(x-x')^2) \times \delta D' \times \frac{1}{24} t'^4. \tag{53}\]
It remains to work out the first order variation of the Paneitz operator (54). Because the Ricci tensor vanishes for flat space the background value of the Paneitz operator is just the square of the flat space d'Alembertian, 

$$\overline{D_P} = (\partial^2)^2.$$  

(54)

Expanding the scalar d'Alembertian in powers of the graviton field gives,

$$\Box \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left[ \sqrt{-g} g^{\mu\nu} \partial_\nu \right] = \partial^2 + \frac{1}{2} h^\mu_\mu \partial_\mu - \partial_\mu h^{\mu\nu} \partial_\nu + O(h^2).$$  

(55)

Therefore the expansion of $$\Box^2$$ is,

$$\Box^2 = \partial^4 + \partial^2 \left[ \frac{1}{2} h^\mu_\mu \partial_\mu - \partial_\mu h^{\mu\nu} \partial_\nu \right] + \left[ \frac{1}{2} h^\mu_\mu \partial_\mu - \partial_\mu h^{\mu\nu} \partial_\nu \right] \partial^2 + O(h^2).$$  

(56)

The Riemann tensor is first order in the graviton field,

$$R^\rho_\sigma\mu\nu = -\frac{1}{2} \left( h^\rho_{\mu\sigma\nu} - h_{\mu\sigma\nu\rho} + h_{\sigma\nu\rho\mu} - h_{\nu\rho\mu\sigma} \right) + O(h^2).$$  

(57)

Hence the expansions of the Ricci tensor and the Ricci scalar are,

$$R_{\mu\nu} = \frac{1}{2} \left( h^\rho_{\mu\nu,\rho} + h^\rho_{\nu,\mu\rho} - h_{\mu\nu} - h_{\mu\nu,\rho} \right) + O(h^2),$$  

$$R = h^\rho_{\rho,\sigma} - h^\rho_\rho + O(h^2).$$  

(58)  

(59)

Because the curvature terms are already first order in the graviton field we do not need to worry about the distinction between covariant differentiation and ordinary differentiation in computing the expansions of the two curvature terms in the Paneitz operator,

$$2D_\mu R_{\mu\nu} D_\nu = \partial_\mu \left( h^\rho_{\mu\nu,\rho} + h^\rho_{\nu,\mu\rho} - h_{\mu\nu} - h_{\mu\nu,\rho} \right) \partial_\nu + O(h^2),$$  

$$-\frac{2}{3} \partial_\mu \left( h^\rho_{\rho,\sigma} - h^\rho_\rho \right) \partial_\mu + O(h^2).$$  

(60)  

(61)

Adding the first order contributions from expressions (56) and (60-61) gives $$\delta D_P,$$

$$\delta D_P = \partial^2 \left[ \frac{1}{2} h^\mu_\mu \partial_\mu - \partial_\mu h^{\mu\nu} \partial_\nu \right] + \left[ \frac{1}{2} h^\mu_\mu \partial_\mu - \partial_\mu h^{\mu\nu} \partial_\nu \right] \partial^2$$

$$+ \partial_\mu \left( h^\rho_{\mu\nu,\rho} + h^\rho_{\nu,\mu\rho} - h_{\mu\nu} - h_{\mu\nu,\rho} \right) \partial_\nu - \frac{2}{3} \partial_\mu \left( h^\rho_{\rho,\sigma} - h^\rho_\rho \right) \partial_\mu.$$  

(62)

We have assigned each of the ten operators of (62) an arbitrary number and listed them in Table 1. We shall employ this notation, $$(\delta D)_I$$ for $$I$$ from 1 to 10, in the reductions of the subsequent subsection.
$\frac{1}{2} \partial^2 h_{\mu\nu} \partial_\mu$  

$-\partial^2 \partial_\mu h_{\mu\nu} \partial_\nu$  

$+\frac{1}{2} h_{\mu\nu} \partial_\mu \partial^2$  

$-\partial_\mu h_{\mu\nu} \partial_\nu \partial^2$  

$+\partial_\mu h^{\rho\nu,\mu} \partial_\nu$  

$-\partial_\mu h^{\rho\mu,\nu} \partial_\nu$  

$+\frac{1}{2} \partial^2 h_{\mu\nu} \partial_\mu$  

$-\partial_\mu h_{\mu\nu} \partial_\nu \partial^2$  

$-\frac{2}{3} \partial_\mu h^{\rho\sigma,\mu} \partial_\nu \partial_\rho$  

$+\frac{2}{3} \partial_\mu h^{\rho\mu,\nu} \partial_\nu \partial^2$  

Table 1: First order perturbations of the Paneitz operator.

### 3.3 The deficit term

Recall that expression (48) for $\delta \mathcal{V}(x)$ gives the first order perturbation of the left hand side of our conjecture (11). Combining equations (53) and (62) from the previous subsection gives an expression for the first order perturbation of the right hand side,

$$8\pi \delta P(x) = -\frac{1}{24} \int_0^t dt' \int d^3x' \Theta(\Delta t - \Delta x) \sum_{I=1}^{10} (\delta D'_P)_I t'^I,$$  

where $\Delta t \equiv t - t'$, $\Delta x \equiv ||\vec{x} - \vec{x}'||$, and the operators $(\delta D_P)_I$ are listed in Table 1. Although (48) and (63) are correct and complete, it is not obvious whether or not they agree. To compare them we will reduce (63) to the same form as (48). This can be accomplished by the following steps:

1. Act any derivatives from $(\delta D'_P)_I$ which stand to the right of the $h_{\mu\nu}(x')$ on the factor of $t'^I$; then

2. Integrate by parts to remove all the derivatives from the graviton fields.

Step 2 produces volume terms which are integrated throughout the light-cone and surface terms restricted to its boundary. If (48) is correct then the sum of all the volume terms must agree with the first integral of (48), and the sum of all the surface terms must agree with the second integral of (48).

It turns out that only $(D_P)_3$ produces a volume term, and this volume term agrees with the first integral in (48). Tables 2-5 summarize our results for the surface terms. To illustrate the reduction procedure consider $(D_P)_1 =$

| I | $(\delta D_P)_I$ | I | $(\delta D_P)_I$ |
|---|-----------------|---|-----------------|
| 1 | $+\frac{1}{2} \partial^2 h_{\mu\nu} \partial_\mu$ | 6 | $+\partial_\mu h^{\rho\nu,\mu} \partial_\nu$ |
| 2 | $-\partial^2 \partial_\mu h_{\mu\nu} \partial_\nu$ | 7 | $-\partial_\mu h^{\rho\mu,\nu} \partial_\nu$ |
| 3 | $+\frac{1}{2} h_{\mu\nu} \partial_\mu \partial^2$ | 8 | $-\partial_\mu h^{\rho\mu,\nu} \partial_\nu$ |
| 4 | $-\partial_\mu h_{\mu\nu} \partial_\nu \partial^2$ | 9 | $-\frac{2}{3} \partial_\mu h^{\rho\sigma,\mu} \partial_\nu \partial_\rho$ |
| 5 | $+\partial_\mu h^{\rho\nu,\mu} \partial_\nu$ | 10 | $+\frac{2}{3} \partial_\mu h^{\rho\mu,\nu} \partial_\nu \partial^2$ |
| #  | Coef. of $h_{00}$                  | Coef. of $\hat{r}^3 h_{00,i}$ | Coef. of $\hat{r}^3 \hat{r}^3 h_{00,ij}$ |
|----|------------------------------------|---------------------------------|------------------------------------------|
| 1  | $\frac{1}{6} t^3 - \frac{1}{2} t^2 \Delta t$ | $\frac{1}{6} t^3 \Delta t$    | 0                                        |
| 2  | $-\frac{1}{3} t^3$                  | $-\frac{1}{3} t^3 \Delta t$    | 0                                        |
| 3  | $\frac{1}{2} t' \Delta t^2$        | 0                               | 0                                        |
| 4  | $-t' \Delta t^2$                   | 0                               | 0                                        |
| 5  | $\frac{1}{3} t^3 - 2t^2 \Delta t + t' \Delta t^2$ | $\frac{2}{3} t^3 \Delta t - t^2 \Delta t^2$ | $\frac{1}{6} t^3 \Delta t^2$              |
| 6  | $\frac{1}{3} t^3 - 2t^2 \Delta t + t' \Delta t^2$ | $\frac{1}{3} t^3 \Delta t - \frac{1}{2} t^2 \Delta t^2$ | 0                                        |
| 7  | $-\frac{1}{3} t^3 + 2t^2 \Delta t - t' \Delta t^2$ | $-\frac{1}{3} t^3 \Delta t + \frac{1}{2} t^2 \Delta t^2$ | 0                                        |
| 8  | $-\frac{1}{3} t^3 + 2t^2 \Delta t - t' \Delta t^2$ | $-\frac{1}{3} t^3 \Delta t + t^2 \Delta t^2$ | 0                                        |
| 9  | $-\frac{2}{9} t^3 + \frac{1}{3} t^2 \Delta t - \frac{2}{3} t' \Delta t^2$ | $-\frac{4}{9} t^3 \Delta t + \frac{2}{3} t^2 \Delta t^2$ | $-\frac{1}{9} t^3 \Delta t^2$              |
| 10 | $+\frac{2}{9} t^3 - \frac{1}{3} t^2 \Delta t + \frac{2}{3} t' \Delta t^2$ | $\frac{2}{9} t^3 \Delta t - \frac{2}{3} t^2 \Delta t^2$ | 0                                        |
| Sum| $-\frac{1}{6} t^3 - \frac{1}{2} t^2 \Delta t - \frac{1}{2} t' \Delta t^2$ | $-\frac{1}{18} t^3 \Delta t$ | $\frac{1}{18} t^3 \Delta t^2$             |

Table 2: Reductions involving $h_{00}$. Each coefficient appears in the form $\int_0^t dt' \int d\Omega \times \text{Coeff.} \times f(t', \bar{x} + \Delta t \hat{r})$. 

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\[ \frac{1}{2} \partial^2 h^\mu \partial_\mu. \] Step 1 gives,

\[ -\frac{1}{24} \int_0^t dt' \int d^3x' \Theta(\Delta t - \Delta x) \left[ -\frac{1}{2} \partial^2 \dot{h}(t', \vec{x'}) \partial_0' \right] t'^4 \]
\[ = \frac{1}{12} \int_0^t dt' \int d^3x' \Theta(\Delta t - \Delta x) \partial^2 \left[ \dot{h}(t', \vec{x'}) t'^3 \right]. \quad (64) \]

The next step is to partially integrate the \( \partial^2 \). It would be silly to act this on the \( \dot{h}(t', \vec{x'}) t'^3 \) because we must throw all derivatives off the graviton field in order to reach the same form as (48). So we instead partially integrate it immediately. Note also that the only surface terms lie on the boundary of the light-cone:

- Surface terms at spatial infinity are zero from the \( \Theta(\Delta t - \Delta x) \);
- Surface terms at \( t' = 0 \) vanish on account of the factor of \( t'^3 \); and
- Surface terms at \( t' = t \) vanish because the theta function becomes \( \Theta(0 - \Delta x) \), which restricts \( \vec{x}' \) to a region of zero volume around \( \vec{x} \).

The only contribution comes from when the \( \partial^2 \) acts on the theta function,

\[ \partial^2 \Theta(\Delta t - \Delta x) = -\frac{2}{\Delta x} \delta(\Delta t - \Delta x). \quad (65) \]

Substituting (65) in (64) gives,

\[ -\frac{1}{24} \int_0^t dt' \int d^3x' \Theta(\Delta t - \Delta x) \left[ -\frac{1}{2} \partial^2 \dot{h}(t', \vec{x'}) \partial_0' \right] t'^4 \]
\[ = -\frac{1}{6} \int_0^t dt' t'^3 \int \Omega \int_0^\infty dr r \delta(\Delta t - r) \dot{h}(t', \vec{x} + r \hat{r}), \quad (66) \]
\[ = -\frac{1}{6} \int_0^t dt' t'^3 \Delta t \int \Omega \dot{h}(t', \vec{x} + \Delta t \hat{r}). \quad (67) \]

Note that the time derivative in \( \dot{h}(t', \vec{x} + \Delta t \hat{r}) \) in expression (67) is only with respect to the first argument; it does not include the \( t' \) dependence of \( \Delta t = t - t' \) in the spatial argument. The full derivative with respect to \( t' \) is,

\[ \frac{\partial}{\partial t'} h(t', \vec{x} + \Delta t \hat{r}) = \dot{h}(t', \vec{x} + \Delta t \hat{r}) - \hat{r} \cdot \hat{\nabla} \dot{h}(t', \vec{x} + \Delta t \hat{r}). \quad (68) \]
Table 3: Reductions involving $h_{0i}$. Each coefficient appears in the form
\[ \int_{t_0}^{t_0'} dt' \int d\Omega \times \text{Coef.} \times f(t', \mathbf{x} + \Delta t \mathbf{r}). \]

The final result is,
\[ -\frac{1}{24} \int_{t_0}^{t_0'} dt' \int d^3x' \Theta(\Delta t - \Delta x) \left[ -\frac{1}{2} \delta^2 h(t', \mathbf{x}') \partial_0' \right] t'^4 = \int_{t_0}^{t_0'} dt' \left[ -\frac{1}{6} t'^3 + \frac{1}{2} t'^2 \Delta t \right] \]
\[ \times \int d\Omega h(t', \mathbf{x} + \Delta t \mathbf{r}) - \frac{1}{6} \int_{t_0}^{t_0'} dt' t'^3 \Delta t \int d\Omega \mathbf{\hat{r}} \cdot \nabla h(t', \mathbf{x} + \Delta t \mathbf{r}) . \] (69)

Upon substituting the $3 + 1$ decomposition $h = -h_{00} + h_{ii}$ we have the first row of entries for Tables 2 and 4.

Although Tables 2-5 reduce $8\pi \delta^2 \mathcal{P}(x)$ to a sum of surface terms roughly like those of $\delta V(x)$, we have still not reached an irreducible form from which a definitive comparison can be made. The key to attaining such a form is to expand the graviton fields in powers of $\Delta t$ and then perform the angular integrations. The details of this procedure are explained in the Appendix but the results for the three surface terms of expression (48) for $\delta V(x)$ are simple enough to quote,
\[ -\frac{1}{6} \int_{t_0}^{t_0'} dt' (t'^3 - \Delta t^3) \int d\Omega h_{00}(t', \mathbf{x} + \Delta t \mathbf{r}) = \int_{t_0}^{t_0'} dt' \left[ -\frac{1}{6} t'^3 - \frac{1}{2} t'^2 \Delta t - \frac{1}{2} t' \Delta t^2 \right] \]

| #  | Coef. of $\mathbf{\hat{r}}^i h_{0i}$ | Coef. of $h_{0i,i}$ | Coef. of $\mathbf{\hat{r}}^j h_{0j,ij}$ |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 0 | $\frac{1}{3} t'^3 \Delta t$ | 0 |
| 3 | 0 | 0 | 0 |
| 4 | $t' \Delta t^2$ | 0 | 0 |
| 5 | $-t' \Delta t^2$ | $-\frac{2}{3} t'^3 \Delta t + \frac{3}{2} t'^2 \Delta t^2$ | $-\frac{1}{3} t'^3 \Delta t^2$ |
| 6 | 0 | $-\frac{1}{3} t'^3 \Delta t + \frac{1}{2} t'^2 \Delta t^2$ | 0 |
| 7 | 0 | 0 | 0 |
| 8 | $t' \Delta t^2$ | $\frac{1}{3} t'^3 \Delta t - t'^2 \Delta t^2$ | 0 |
| 9 | 0 | $\frac{4}{3} t'^3 \Delta t - \frac{2}{3} t'^2 \Delta t^2$ | $\frac{2}{3} t'^3 \Delta t^2$ |
| 10 | 0 | 0 | 0 |
| Sum | $t' \Delta t^2$ | $\frac{1}{3} t'^3 \Delta t + \frac{1}{3} t'^2 \Delta t^2$ | $-\frac{1}{3} t'^3 \Delta t^2$ |
Hence our final result takes the form,

\begin{align}
\times 4\pi \sum_{n=0}^{\infty} \frac{\Delta t^{2n} \nabla^{2n}}{(2n+1)!} h_{00}(t', \vec{x}), \\
\frac{1}{3} \int_{t}^{t'} dt' (t'^3 - \Delta t^3) \int d\Omega \tilde{\omega} h_{0i}(t', \vec{x} + \Delta t \vec{\tilde{r}}) = \int_{t}^{t'} dt' \left[ \frac{1}{3} t'^3 + \frac{1}{2} t'^2 \Delta t + t' \Delta t^2 \right] \\
\times 4\pi \sum_{n=0}^{\infty} \frac{\Delta t^{2n+1} \nabla^{2n}}{(2n+1)! (2n+3)} h_{0i,i}(t', \vec{x}), \\
- \frac{1}{6} \int_{t}^{t'} dt' (t'^3 - \Delta t^3) \int d\Omega \tilde{\omega} \tilde{r} h_{ij}(t', \vec{x} + \Delta t \vec{\tilde{r}}) = \int_{t}^{t'} dt' \left[ -\frac{1}{6} t'^3 - \frac{1}{2} t'^2 \Delta t - \frac{1}{2} t' \Delta t^2 \right] \\
\times 4\pi \sum_{n=0}^{\infty} \frac{\Delta t^{2n} \nabla^{2n-2}}{(2n+1)! (2n+3)} \left[ h_{ii,jj}(t', \vec{x}) + 2n h_{ij,ij}(t', \vec{x}) \right].
\end{align}

Applying the same reduction to the terms of Tables 2-5 and carrying out some judicious partial integrations with respect to $t'$, allows us to reach a definitive expression for the difference of $8\pi \delta \mathcal{P}(x)$ and $\delta \mathcal{V}(x)$,

\begin{align}
8\pi \delta \mathcal{P}(x) - \delta \mathcal{V}(x) = \int_{t}^{t'} dt' t'^3 \Delta t^4 \times 4\pi \sum_{n=0}^{\infty} \frac{\Delta t^{2n} \nabla^{2n}}{(2n+1)! (2n+3)(2n+5)} \\
\times \left\{ \frac{1}{18} \nabla^4 h_{00}(t', \vec{x}) - \frac{1}{9} \nabla^2 h_{0i,i}(t', \vec{x}) - \frac{1}{36} \nabla^2 \tilde{h}_{ii}(t', \vec{x}) \\
+ \frac{1}{36} \nabla^4 h_{ii}(t', \vec{x}) - \frac{1}{36} \nabla^2 h_{ij,ij}(t', \vec{x}) + \frac{1}{12} \tilde{h}_{ij,ij}(t', \vec{x}) \right\}.
\end{align}

The various graviton fields in (73) can be assembled into components of the linearized curvature tensor,

\begin{align}
\frac{1}{18} \nabla^4 h_{00} - \frac{1}{9} \nabla^2 h_{0i,i} - \frac{1}{36} \nabla^2 \tilde{h}_{ii} + \frac{1}{36} \nabla^4 h_{ii} - \frac{1}{36} \nabla^2 h_{ij,ij} + \frac{1}{12} \tilde{h}_{ij,ij} \\
= -\frac{1}{9} \nabla^2 \left[ h_{ii,ii} - \frac{1}{2} h_{00,0i} - \frac{1}{2} h_{0i,0i} \right] \\
- \frac{1}{36} \nabla^2 \left[ h_{ij,ij} - h_{ii,ij} \right] + \frac{1}{12} \partial_0^2 \left[ h_{ij,ij} - h_{ii,ij} \right],
\end{align}

\begin{align}
= \frac{1}{18} \nabla^2 \delta R - \frac{1}{12} \partial_0^2 \delta R_{ij,ij}.
\end{align}

Hence our final result takes the form,

\begin{align}
8\pi \delta \mathcal{P}(x) - \delta \mathcal{V}(x) = \int_{t}^{t'} dt' t'^3 \Delta t^4 \times 4\pi \sum_{n=0}^{\infty} \frac{\Delta t^{2n} \nabla^{2n}}{(2n+1)! (2n+3)(2n+5)} \\
\times \left\{ \frac{1}{18} \nabla^2 \delta R(t', \vec{x}) - \frac{1}{12} \left[ \partial_0^2 + \nabla^2 \right] \delta R_{ij,ij}(t', \vec{x}) \right\}.
\end{align}
Table 4: Reductions involving $h_{ii}$. Each coefficient appears in the form $\int_0^t dt' \int d\Omega \times \text{Coef.} \times f(t', \vec{x} + \Delta t \hat{r})$.

| # | Coef. of $h_{ii}$                               | Coef. of $\hat{r}^j h_{ii,j}$ |
|---|-----------------------------------------------|---------------------------------|
| 1 | $-\frac{1}{6}t'^3 + \frac{1}{2}t'^2 \Delta t$ | $-\frac{1}{6}t'^3 \Delta t$    |
| 2 | 0                                            | 0                               |
| 3 | $-\frac{1}{2}t' \Delta t^2$                  | 0                               |
| 4 | 0                                            | 0                               |
| 5 | 0                                            | 0                               |
| 6 | 0                                            | 0                               |
| 7 | $\frac{1}{3}t'^3 - 2t'^2 \Delta t + t' \Delta t^2$ | $\frac{1}{3}t'^3 \Delta t - \frac{1}{2}t'^2 \Delta t^2$ |
| 8 | 0                                            | 0                               |
| 9 | 0                                            | 0                               |
| 10 | $-\frac{2}{9}t'^3 + \frac{1}{3}t'^2 \Delta t - \frac{2}{3}t' \Delta t^2$ | $-\frac{2}{9}t'^3 \Delta t + \frac{2}{3}t'^2 \Delta t^2$ |
| Sum | $-\frac{1}{18}t'^3 - \frac{1}{6}t'^2 \Delta t - \frac{1}{6}t' \Delta t^2$ | $-\frac{1}{18}t'^3 \Delta t + \frac{1}{6}t'^2 \Delta t^2$ |
Table 5: Reductions involving $h_{ij,j}$. Each coefficient appears in the form \( \int_0^t dt' \int d\Omega \times \text{Coef.} \times f(t', \vec{x} + \Delta t \vec{r}) \).

| #  | Coef. of $\tilde{\tau}^i h_{ij,j}$ | Coef. of $h_{ij,ij}$ |
|----|-----------------------------------|----------------------|
| 1  | 0                                 | 0                    |
| 2  | 0                                 | 0                    |
| 3  | 0                                 | 0                    |
| 4  | 0                                 | 0                    |
| 5  | $-\frac{1}{2} t'^2 \Delta t^2$   | $\frac{1}{6} t'^3 \Delta t^2$ |
| 6  | 0                                 | 0                    |
| 7  | 0                                 | 0                    |
| 8  | 0                                 | 0                    |
| 9  | 0                                 | $-\frac{1}{9} t'^3 \Delta t^2$ |
| 10 | 0                                 | 0                    |
| Sum| $-\frac{1}{2} t'^2 \Delta t^2$   | $\frac{1}{18} t'^3 \Delta t^2$ |
4 Discussion

The invariant volume of the past light-cone is an interesting quantity because it provides a partial solution to the tough problem of constructing observables for quantum gravity [2, 3], because it can play a role in characterizing the quantum field theoretic back-reaction on spacetime expansion [8, 5], and because its gradient can provide an alternative to the timelike vector field involved in certain alternate gravity models [6, 7] without introducing new dynamical degrees of freedom. It is well known that nonlocal functionals of the metric arise from quantum corrections to the effective field equations and a number of authors have considered nonlocal gravity models [8, 11, 12].

We have studied the relation between the invariant volume of the past light-cone $V_g(x)$ and the Paneitz operator $D_P$, a 4th order differential operator which occurs in the theory of conformal anomalies. Based on their flat space limits we conjectured that acting $D_P$ on $V_g(x)$ might give $8\pi$ for a general metric. We checked this conjecture in its integral form by comparing $V_g(x)$ with $8\pi$ times $P_g(x)$, the integral of the retarded Green’s function of the Paneitz operator. If the same operator whose logarithm occurs in the ubiquitous conformal anomalies [9, 10] could be shown to give the invariant volume of the past light-cone then alternate gravity models which involve the latter would become considerably more plausible.

Section 2 considered the case of an arbitrary homogeneous and isotropic geometry, which has great significance for cosmology. We explicitly constructed the invariant volume of the past light-cone $V_\eta(x)$ and $8\pi$ times the integral of the Paneitz Greens function $P_\eta(x)$. Some trivial calculus manipulations suffice to show that the two expressions agree exactly for the case of zero or negative spatial curvature. For positive spatial curvature the two expressions agree when the observation point occurs less than one Hubble time later than the initial value surface. After one Hubble time $V$ does not agree with $8\pi P$ unless one modifies $V$ to be the sum of the volumes of the past light-cone from the observation point and from every focal point at which past-directed, null geodesics from the observation point converge.

In section 3 we compared $V[\eta + h](x)$ with $8\pi P[\eta + h](x)$ at first order in perturbation theory about flat spacetime. An explicit expression (48) was derived for $\delta V(x)$, and another expression (63) was obtained for $8\pi \delta P(x)$. It was not so easy to compare the two relations but we eventually obtained a definitive result (76) for their difference. Although expression (76) is not zero, it does vanish for an arbitrary linearized solution of the vacuum Einstein
equations because they imply,
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \quad \implies \quad \delta R = 0 \quad \text{and} \quad \partial^2 \delta R_{\rho\sigma\mu\nu} = 0 \, . \quad (77) \]

We do not yet know what the vanishing of (76) with the linearized Einstein equations means. That \( 8\pi \delta P(x) - \delta \mathcal{V}(x) \) must involve the linearized curvature tensor follows because \( \mathcal{V}[\eta + h](x) \) and \( 8\pi P[\eta + h](x) \) agree for \( h_{\mu\nu} = 0 \), and both transform as scalars under any diffeomorphism which preserves the initial value surface \( \mathcal{S} \). However, not all components of the linearized curvature tensor vanish with the linearized Einstein equations — for example, \( \delta R_{ijij} \) does not, nor does \( \delta R_{0i0i} \). Yet only vanishing combinations appeared in the difference (76). This seems unlikely to have been an accident, but we do not understand its significance.

One might wonder if \( D_P \) can be changed by some local operator to make the difference (76) go away. The answer is no. If there were such an operator then acting \( \partial^4 \) on (76) would give this operator acting on \( t^4/24 \). However, direct computation shows that acting \( \partial^4 \) on a nonlocal expression of the form (76) fails to localize it,
\[
\partial^4 \int_0^t dt' t'^3 \Delta t^4 \times 4\pi \sum_{n=0}^{\infty} \frac{\Delta t^{2n} \nabla^{2n}}{(2n+1)!(2n+3)(2n+5)} f(t', \vec{x}) = \int_0^t dt' t'^3 \times 4\pi \sum_{n=0}^{\infty} \frac{\Delta t^{2n} \nabla^{2n}}{(2n+1)!(2n+3)(2n+5)} f(t', \vec{x}) \, . \quad (78)
\]

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5 Appendix

The purpose of this appendix is to derive some relations which apply to the angular integral of functions over the surface of the flat space light-cone. One can represent such a function as $f(\vec{x} + \Delta t \hat{r})$, and the relations all derive from expanding in powers of $\Delta t$,

$$f(\vec{x} + \Delta t \hat{r}) = \sum_{n=0}^{\infty} \frac{\Delta t^n}{n!} (\hat{r} \cdot \nabla)^n f(\vec{x}).$$

(79)

This brings all factors of the unit vector $\hat{r}$ outside the function, whereupon we can evaluate the angular integrations using the relation,

$$\int d\Omega \hat{r}_i \hat{r}_j \cdots \hat{r}_k f(\vec{x} + \Delta t \hat{r}) = 4\pi \left\{ \begin{array}{ll} 0 & n \text{ odd} \\ \frac{1}{n+1} \delta(i_1 i_2 \cdots \delta i_{n-1} i_n) & n \text{ even} \end{array} \right..$$

(80)

The reductions of section 3.3 necessitate consideration of $f(\vec{x} + \Delta t \hat{r})$ by itself, or multiplied with up to three unit vectors,

$$\int d\Omega \hat{r}_i f(\vec{x} + \Delta t \hat{r}) = 4\pi \sum_{n=0}^{\infty} \frac{\Delta t^{2n}}{(2n+1)!} \nabla^{2n} f(\vec{x}),$$

(81)

$$\int d\Omega \hat{r}_i \hat{r}_j f(\vec{x} + \Delta t \hat{r}) = 4\pi \sum_{n=0}^{\infty} \frac{\Delta t^{2n+1} \nabla^{2n}}{(2n+1)!(2n+3)} \partial_i \partial_j f(\vec{x}),$$

(82)

$$\int d\Omega \hat{r}_i \hat{r}_j \hat{r}_k f(\vec{x} + \Delta t \hat{r}) = 4\pi \sum_{n=0}^{\infty} \frac{\Delta t^{2n+2} \nabla^{2n+2} \delta^{ij} \delta^{kl} \delta^{mn}}{(2n+1)!(2n+3)(2n+5)} f(\vec{x}),$$

(83)

$$\int d\Omega \hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_l f(\vec{x} + \Delta t \hat{r}) = 4\pi \sum_{n=0}^{\infty} \frac{\Delta t^{2n+1} [3\delta^{ijkl} \nabla^{2n} + 2n \partial_i \partial_j \partial_k \partial_l \nabla^{2n-2}]}{(2n+1)!(2n+3)(2n+5)} f(\vec{x}).$$

(84)

By combining and comparing these expressions one can derive the following identities which were used in preparing Tables 2-5,

$$\int d\Omega \left[ \nabla^2 - (\hat{r} \cdot \nabla)^2 \right] f(\vec{x} + \Delta t \hat{r}) = \frac{2}{\Delta t} \int d\Omega \hat{r} \cdot \nabla f(\vec{x} + \Delta t \hat{r}),$$

(85)
\[
\int d\Omega \left[ \nabla^2 - (\vec{r} \cdot \nabla)^2 \right] \vec{r}^i f(\vec{x} + \Delta t \vec{r}) = \frac{2}{\Delta t} \int d\Omega \left[ \partial_t - 3\vec{r}^i \right] f(\vec{x} + \Delta t \vec{r}), \quad (86)
\]

\[
\int d\Omega \left[ \partial_t - \vec{r}^i \vec{r} \cdot \nabla \right] f_i(\vec{x} + \Delta t \vec{r}) = \frac{2}{\Delta t} \int d\Omega \vec{r}^i f_i(\vec{x} + \Delta t \vec{r}). \quad (87)
\]