Singular Charge Density at the Center of the Pion?

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We relate the three-dimensional infinite momentum frame spatial charge density of the pion to its electromagnetic form factor \( F_\pi(Q^2) \). Diverse treatments of the measured form factor data including phenomenological fits, non-relativistic quark models, the application of perturbative QCD, QCD sum rules, holographic QCD and the Nambu Jona-Lasinio (NJL) model all lead to the result that the charge density at the center of the pion has a logarithmic divergence. Relativistic constituent quark models do not display this singularity. Future measurements planned for larger values of \( Q^2 \) may determine whether or not a singularity actually occurs.

Understanding the pion is a necessary step to learning how QCD describes the interaction and existence of elementary particles. As a nearly massless excitation of the QCD vacuum with pseudoscalar quantum numbers, the pion plays a central role in particle and nuclear physics as a harbinger of spontaneous symmetry breaking and as the carrier of the longest range force between nucleons.

The importance of the pion has been recognized by a huge level of both experimental and theoretical activity aimed at measuring its properties and understanding its structure. New measurements of the pion electromagnetic form factor, \( F_\pi(Q^2) \), have been performed \(^1\)\(^2\) and are planned \(^3\). Here we present the first phenomenological analysis of existing data to determine the charge density of the pion in a model independent manner.

A proper determination of a charge density requires the measurement of a density operator. We shall show that measurements of the pion form factor directly involve the three-dimensional charge density of partons, in the infinite momentum frame, \( \hat{\rho}_\infty(x^-, b) \). In this frame \(^4\), the electromagnetic charge density \( J^\mu \) becomes \( J^+ \) and

\[
\hat{\rho}_\infty(x^-, b) = J^+(x^-, b) = \sum_q e_q \bar{q}(x^-, b) \gamma^+ q(x^-, b) = \sum_q e_q \sqrt{2} q^+_\gamma(x^-, b) q(x^-, b),
\]

where \( q^+_\gamma(x^+) = \gamma^0 \gamma^+ / \sqrt{2} q(x^+) \), the independent part of the quark-field operator \( q(x^+) \). We set the time variable, \( x^+ = (t + z)/\sqrt{2}, \) to zero, and do not display it in any function.

We are concerned with the relationship between charge density and the electromagnetic form factor \( F_\pi(Q^2) \), which is determined from the current density via the relation:

\[
F_\pi(Q^2) = \frac{\langle p^+, p' | J^+(0) | p^+, p \rangle}{2p^+},
\]

where states are normalized as \( \langle p'^+, p'| p^+, p \rangle = 2p^+ (2\pi)^3 \delta(p'^+ - p^+) \delta(2\gamma)(p' - p) \). We take the momentum transfer \( q_\alpha = p_\alpha' - p_\alpha \) to be space-like, with \( Q^2 \equiv -q^2 > 0 \), and use the Drell-Yan (DY) frame with \( (q^+ = 0, Q^2 = q^2) \). The matrix element appearing in Eq. \(2\) involves the combination of creation and destruction operators: \( b\gamma b - d\gamma d \) for each flavor of quark, so that the valence charge density is probed. Note also that the form factor \( F_1 \) is independent of renormalization scale because the vector current \( q\gamma q \) is conserved \(^5\).

The spatial structure of a hadron can be examined if one uses \(^6\)\(^7\) states that are transversely localized. The state with transverse center of mass \( R \) set to \( 0 \) is formed by taking a linear superposition of states of transverse momentum:

\[
|p^+, R = 0, \lambda \rangle \equiv \mathcal{N} \int \frac{d^2p}{(2\pi)^2} |p^+, p, \lambda \rangle,
\]

where \( |p^+, p, \lambda \rangle \) are light-cone helicity eigenstates \(^8\) and \( \mathcal{N} \) is a normalization factor satisfying \( |\mathcal{N}|^2 \int \frac{d^2p}{(2\pi)^2} = 1 \). Wave packet representations can be used to avoid states normalized to \( \delta \) functions \(^8\)\(^9\), but this leads to the same results as using Eq. \(3\). Considering that \( 2p^+p^- - p^2 = m^2 > 0 \), one finds that \( p^+ \) must approach infinity. This ultra-large value of \( p^+ \) (infinite momentum frame) maintains the interpretation of a pion moving with well-defined longitudinal momentum\(^9\). It is in just such a frame that the interpretation of a hadron as a set of a large number
of partons is valid. Setting the transverse center of momentum of a state of total very large momentum $p^+$ to zero as in Eq. (3), allows the transverse distance $b$ relative to $R$ to be defined.

Next we relate the charge density

$$\rho_\infty(x^-,b) = \left(\frac{\langle p^+, R = 0, \lambda | \hat{\rho}_\infty(x^-,b) | p^+, R = 0, \lambda \rangle}{\langle p^+, R = 0, \lambda | p^+, R = 0, \lambda \rangle}\right),$$  \hspace{1cm} (4)$$
to $F_\pi(Q^2)$. In the DY frame no momentum is transferred in the plus-direction, so that information regarding the $x^-$ dependence of the distribution is not accessible. Therefore we integrate over $x^-$, using the relationship

$$q_\perp(x^-,b)q_\perp(x^-,b) = e^{i\vec{p} \cdot \vec{x} - i\vec{p} \cdot \vec{b}} Q_\perp(0)q_\perp(0)e^{i\vec{p} \cdot \vec{b}} e^{-i\vec{p}^+ x^-},$$  \hspace{1cm} (5)$$
to find

$$\rho(b) \equiv \int dx^- \rho_\infty(x^-,b) = \langle p^+, R = 0, \lambda | \hat{\rho}_\infty(0,b) | p^+, R = 0, \lambda \rangle / (2p^+).$$  \hspace{1cm} (6)$$
Furthermore, the use of Eqs. (4-7) leads to the simplification of the right-hand-side of the above equation:

$$\rho(b) = \int \frac{d^2q}{(2\pi)^2} F_\pi(Q^2) = q^2 e^{-i\vec{q} \cdot \vec{b}},$$  \hspace{1cm} (7)$$
where $\rho(b)$ is termed the transverse charge density, giving the charge density at a transverse position $b$, irrespective of the value of the longitudinal position or momentum. This relation between an integral of the three-dimensional infinite momentum frame density and the electromagnetic form factor is our principal new formula. Previous results involved the integral over the longitudinal momentum fraction $x$ of the impact parameter parton distribution function (pdf) $q(x,b)$, which gives the charge density for a quark at position $b$ for a momentum fraction (of the plus-component) $x$. The equality of the respective integrals over $x^-$ or $x$ of the quantities $\rho_\infty(x^-,b)$ and $q(x,b)$ is an example of Parseval’s theorem. The central charge density of the pion is determined by $\rho(b = 0)$, because the longitudinal dimension is Lorentz contracted to essentially zero in the infinite momentum frame.

Recent pion data provide an accurate measurement of the pion form factor up to a value of $Q^2 = 2.45$ GeV$^2$. Their analysis includes an assessment of the influence of the necessary model dependence caused by extracting the form factor from the measured cross sections on the experimental error bars. The existing data for the pion form factor show that it is well represented by the monopole form

$$F_\pi(Q^2) = 1/(1 + R^2Q^2/6),$$  \hspace{1cm} (8)$$
with $R^2 = 0.431$ fm$^2$. A better representation of the data may be a monopole plus dipole which involves the square of the term of Eq. (5), but any form involving the monopole term leads to a singular central charge density. This is because the use Eq. (5) in Eq. (7) gives the result:

$$\rho(b) = \frac{3K_0\left(\frac{\sqrt{R^2}}{\pi}\right)}{\pi R^2},$$  \hspace{1cm} (9)$$
where $K_0$ is modified Bessel function of rank zero. For small values of $b$ this function diverges as $\sim \log(b)$. This divergence is very surprising because the charge density we are considering measures a valence quark operator between eigenstates of the full Hamiltonian. The divergences of quark distribution functions that occur at small values of Bjorken $x$ do not occur here. Any model, such as vector meson dominance or holographic QCD which yields a monopole form factor has a central density with a logarithmic divergence.

Intuition regarding a possible singularity in the central charge density may be improved by considering other examples. Suppose that the non-relativistic (NR) limit in which the quark masses are heavy is applicable. In this case, the pion would be a pure $q\bar{q}$ object and the charge density is the Fourier transform of the form factor. Given the form factor of Eq. (5) the three-dimensional density is uniquely given by

$$\rho_{NR}(r) = \frac{3}{2\pi r R^2} e^{-\frac{r^2}{R^2}},$$  \hspace{1cm} (10)$$
where $r$ is the distance relative to the pion center of mass. If one takes $r = \sqrt{b^2 + z^2}$ as demanded by the rotational invariance of the non-relativistic wave function, then one finds $\int_{-\infty}^{\infty} dz \rho_{NR}(r)$ is equal to $\rho(b)$ of Eq. (7). This is expected because in the NR limit the charge density is the same in all frames, including the infinite momentum
frame. The meaning of the 1/r behavior of the density can be understood by considering that for a q̅q pion, the wave function is the square root of the density so that the short distance wave function ψNR ~ 1/√r. Using the Schrödinger equation, one finds that the potential must contain terms proportional to 1/r^2. There is no evidence that the strong interaction potential behaves in this manner. Thus a non-relativistic viewpoint tells us that the central singularity derived from the form factor falling as 1/Q^2 requires unsupported assumptions regarding the nature of the short-distance interactions between quarks. On the other hand, the lowest-energy solution of the Dirac equation for hydrogenic atoms has a singular radial behavior, ψ_D(r) ~ 1/r^{1−γ}e^{−r} (γ ≃ 1 − Z^2α^2), near the origin at r = 0. Consider ρ_D(b) ≡ \int d^2 \psi_D(r)^2 and define η ≡ 2−2γ, which ranges between 0 and 2. We find for small values of b (in units of twice the appropriate Bohr radius) that ρ_D(b) is well behaved for 0 < η < 1, behaves (for all b) as K_0(b) for η = 1, and behaves as 1/b^η for 1 < η < 2. Thus there are physical examples with a singular central density.

The divergence of the central transverse charge density encountered here may be the consequence of using a simple parametrization, so we shall consider the predictions of a variety of different approaches. We begin with perturbative QCD (pQCD) which provides a prediction for asymptotically large values of Q^2 that

\[ \lim_{Q^2 \to \infty} F_\pi(Q^2) = 16πα_s(Q^2)f_\pi^2/Q^2, \]

with the pion decay constant f_π = 93 MeV, and in leading order:

\[ α_s(Q^2) = \frac{4π}{(11 - \frac{2}{3}n_f) \ln Q^2/Q_0^2}, \]

with n_f the number of quarks of mass smaller than Q and Λ is a parameter fixed by data. One might think that the \( Q^2 \) term in the denominator would lead to a non-singular behavior of \( ρ(b) \) for small values of b. This is not the case. To see this, consider the integral:

\[ \int_{Q_0}^{Q_{max}} \frac{dQ}{Q \log Q/Λ} J_0(Qb), \]

for the case \( Q_{max} > Q_0 > Λ > 0 \). In the limit that \( Q_{max} \) approaches infinity, this is the contribution of the integral of Eq. 17 arising from values of \( Q > Q_0 \), assuming that the value of \( Q_0 \) is large enough for Eq. 12 to be valid. Take \( Q_{max} = 1/b = \epsilon/b, \) where \( \epsilon \) is a small positive number such that \( J_0(\epsilon) = 1 \) to any specified degree of numerical precision. Then

\[ \int_{Q_0}^{\epsilon/b} \frac{dQ}{Q \log Q/Λ} J_0(Qb) = \log \log \left( \frac{\epsilon}{b} \right) - \log \log \left( \frac{Q_0}{Λ} \right), \]

\[ Q_{max}b \ll 1. \]

In the limit that \( b \) approaches zero Eq. 14 becomes

\[ \lim_{b \to 0} \int_{Q_0}^{∞} \frac{dQ}{Q \log Q/Λ} J_0(Qb) = \log \log(1/b) + \cdots, \]

We see that the pQCD form factor corresponds to a singularity at short distance. The same feature would arise in any model form factor such as those based on sum rules e.g. 18 that joins smoothly to the pQCD result at very large values of \( Q^2 \).

Chiral quark models (see the review 19) present other examples of transverse charge densities that are singular at the center. In those models, the pion form factor takes the monopole form of Eq. 3 so that the central density diverges as \( \log b \) at the origin. Nevertheless all physical observables, including \( f_π \) and structure functions, are computed to be finite. We consider two such models. The first is the spectral quark model SQM 21,22 In this model \( F_\pi(Q^2) \) takes the form of Eq. 3 with \( R^2/6 = m_π^2 \). The impact parameter dependent parton distribution function is given by 21

\[ q(x,b) = \frac{m_π^2}{2π(1−x)^2} \left[ −bm_πK_1 \left( \frac{bm_π}{1−x} \right) (1−x) + K_0 \left( \frac{bm_π}{1−x} \right) \right]. \]

For small values of b this diverges as \( \log b \) for all values of x. Nonetheless, the SQM produces reasonable structure functions and quark distribution functions 20.

Another example is the NJL model, as regulated by two Pauli-Villars subtractions. The form factor is given by 19

\[ F_\pi(Q^2) = \int_0^1 dx F_\pi(Q^2, x) \]
with \( F_x(Q^2, x) = \frac{1}{f} \frac{d^2M^2}{d^2y} \ln(M^2 + \Lambda^2 + x(1-x)Q^2)_{\text{reg}} \), where \( M \) is the quark mass, \( \Lambda \) is a parameter related to regularization and \( f \) is the pion decay constant. The subscript \( \text{reg} \) denotes the regularization procedure \[ Q_{\text{reg}}(\Lambda^2) = O(0) - O(\Lambda^2) + \frac{dQ}{d\Lambda^2}. \] The phenomenologically determined values are \( M = 280 \text{ MeV}, f = 93.3 \text{ MeV}, \Lambda = 870 \text{ MeV} \). The impact parameter dependent pdf is the two-dimensional Fourier transform of \( F_x(Q^2, x) \):

\[
q(x, b) = -\frac{3M^2}{(2\pi)^3f^2} \int_0^\infty dQ Q J_0(Qb) \left[ \ln \frac{M^2 + x(1-x)Q^2}{M^2 + \Lambda^2 + x(1-x)Q^2} + \frac{\Lambda^2}{M^2 + \Lambda^2 + x(1-x)Q^2} \right].
\]

(18)

This gives a well-behaved expression for \( b \to 0 \) for all non-zero values of \( x(1-x) \). Indeed:

\[
q(x, 0) = \frac{3M^2}{2(2\pi)^3f^2} \left( \Lambda^2 + M^2 \log \frac{M^2}{M^2 + \Lambda^2} \right) \frac{1}{x(1-x)}.
\]

(19)

Thus a logarithmic divergence appears upon integrating on \( x \).

Gaussian models with generalized parton distributions \( H(x, 0, Q^2) = (1 \times H(x, 0, Q^2) = F(Q^2)) \) dominated by behavior near \( x = 1 \) present a set of examples that also yield a form factor with a \( 1/Q^2 \) asymptotic behavior, and have a impact parameter distribution that is well behaved at each value of \( x \) for all \( b \). The key asymptotic features are captured in the simple formula \[ H(x, 0, Q^2)_{\text{reg}} = \frac{1}{1-x}e^{-\alpha(1-x)^2} \text{ so that } q(x, b)_{\text{reg}} = \frac{1}{2\pi(1-x)}e^{-b^2/(4\alpha(1-x)^2)}. \] This form shows that \( q(x, b) \) is well behaved for all values of \( b \) and for each value of \( x \), but the integral over \( x \) contains a logarithmic divergence.

Not all models that describe the existing form factor data have a singular central charge density. Relativistic light-front constituent quark models \[ \text{23, 22, 27} \] are able to describe the pion phenomenology and the current form factor data. These models produce a non-singular transverse charge density as we shall illustrate. These models can be most simply derived \[ \text{20} \] by using the impulse approximation (evaluating the triangle diagram). One starts by evaluating the integral over the minus component of the loop momentum \( k^\mu \), and then cutting off the remaining integral over \( x = k^+/(p^+) \), \( k_\perp \) using a phenomenological wave function that depends on the combination \( (k^2 + m^2)/x(1-x) \), with \( m \) as the assumed constituent quark mass. We illustrate these models by computing the form factor using the model of \[ \text{27} \]. The wave function chosen to be a power-law form, and the model is able to describe all of the existing form factor data in both the time-like and space like regions, \( f_x \), and the transition form factor \( f_{x\gamma} \) in which a virtual photon transforms a real pion into a real photon. The model form factor of \[ \text{27} \] and the monopole fit of Eq. \[ 8 \] are shown along with the measured data in Fig. 1. Both models provide a good fit to the data, but present very different predictions for larger values of \( Q^2 \) where measurements remain to be done. The corresponding versions of \( \rho(b) \) and \( b_0(b) \) are shown in Fig. 2. The singularity contained in Eq. \[ 9 \] appears as a rapidly rising function as \( b \) approaches zero, while the relativistic constituent quark model provides a \( \rho(b) \) that is smooth for small values of \( b \).

We summarize. The high \( Q^2 \) behavior of the form factor determines the short distance behavior of the transverse charge density \( \rho(b) \). If the form factor really behaves as the monopole form of Eq. \[ 5 \], then \( \rho(b) \) maintains a logarithmic singularity at the origin. A variety of models predict this behavior, \[ \text{13, 14, 15, 19, 20, 23, 24} \] as well as any non-relativistic constituent quark model that predicts a monopole behavior of \( F_x(Q^2) \) and solutions of the Dirac equation.

If the form factor falls asymptotically as perturbative QCD predicts, the \( \rho(b) \) behaves singularly as \( \ln \ln b \) for small values of \( b \). It seems reasonable that \( \rho(b) \), a property of the valence quark density, should have no singularity. The relativistic constituent quark model produces transverse charge densities that are free of singularities, while providing a generally good phenomenology of the pion \[ \text{27} \].

It is therefore absolutely and manifestly clear that obtaining data at higher values of \( Q^2 \) is essential to providing further understanding. Such data could provide support for or rule out either constituent quark models or current pQCD evaluations of \( F_x \). If an assumption that the central density is non-singular is correct, the form factor will fall as described by constituent quark models. On the other hand, if asymptotic pQCD is valid, the central charge density would be singular— a remarkable fact of nature.

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FIG. 1: (Color online) $Q^2 F_\pi (Q^2)$. Pion form factor data as plotted in [2]. The data labeled Jlab are from [2]. The data Brauel et al. [27] and that of Ackermann et al. [28] have using the method of [2]. The Amendola data et al. are from [30]. The data point labeled PionCT is from [31]. The (red) dashed curve uses the monopole fit Eq. (8) and the (black) solid line the constituent quark model of [27].

[4] Our notation is that $x^\pm \equiv (x^0 \pm x^3)/\sqrt{2}$, $p^\pm \equiv (p^0 \pm p^3)/\sqrt{2}$, and $p_\mu x^\mu = p^- x^- + p^+ x^+ - \mathbf{p} \cdot \mathbf{b}$. The coordinates perpendicular to the 0 and 3 directions are denoted as $\mathbf{b}$ and $\mathbf{p}$.

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FIG. 2: (Color online) $\rho(b)$ (upper panel) and $b\rho(b)$ (lower panel) corresponding to the two models shown in Fig. 1. The (red) dashed curve uses the monopole fit and the (blue) solid-line the relativistic constituent quark model of\[27\].

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