Green functions for the TASEP with sublattice parallel update

S S Poghosyan\textsuperscript{1}, V B Priezzhev\textsuperscript{1} and G M Schütz\textsuperscript{2}

\textsuperscript{1} Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia
\textsuperscript{2} Institut für Festkörperforschung, Forschungszentrum Jülich, D-52428 Jülich, Germany
E-mail: spoghos@theor.jinr.ru, priezzvb@theor.jinr.ru and g.schuetz@fz-juelich.de

Received 25 February 2010
Accepted 31 March 2010
Published 29 April 2010

Online at stacks.iop.org/JSTAT/2010/P04022
doi:10.1088/1742-5468/2010/04/P04022

Abstract. We consider the totally asymmetric simple exclusion process (TASEP) in discrete time with the sublattice parallel dynamics describing particles moving to the right on the one-dimensional infinite chain with equal hopping probabilities. Using sequentially two mappings, we show that the model is equivalent to the TASEP with the backward-ordered sequential update in the case when particles start and finish their motion not simultaneously. The Green functions are obtained exactly in a determinant form for different initial and final conditions.

Keywords: driven diffusive systems (theory), exact results, stochastic particle dynamics (theory)

ArXiv ePrint: 1002.3456v4
1. Introduction

The totally asymmetric simple exclusion process (TASEP) is a stochastic interacting particle system which serves as a paradigmatic model for nonequilibrium behavior \[1\]–[3]. The dynamics of this lattice gas model is characterized by the updating law. In one dimension on the integer lattice \(\mathbb{Z}\) the most important cases of discrete-time updates are the backward-sequential, parallel and sublattice parallel updates \[4\]. For a finite number of particles these dynamics can be defined through a master equation of the form

\[
P(x, t + 1) = \sum_{x'} p_{x,x'} P(x', t)
\]

where \(x = \{x_i\}\) describes the positions of the particles and \(p_{x,x'}\) is the transition probability to go in one time step from a configuration \(x'\) to a configuration \(x\). This transition probability is different for the various update schemes. For the backward-sequential update, each particle may take one step to the right with probability \(v\) if the target site is vacant at the beginning of the time step or becomes vacant at the end of the time step (due to motion of the particle in front). For the parallel update, the motion to the right is allowed only if the target site is vacant at the beginning of the time step. By iterating (1) one obtains the solution of the master equation for any given initial configuration \(x^0\), i.e. the conditional probability to find a particle configuration \(x\) at time step \(t\), given that the process started from configuration \(x^0\). This stochastic many-body dynamics have a natural interpretation in field-theoretic terms \[3, 5\] where specific realizations of the process correspond to paths in the path integral representation of field-theoretic quantities. Therefore, in analogy to the corresponding terminology in field theory, we refer to this time-dependent conditional transition probability as the Green function.

For the first two cases, backward-sequential and parallel update, the Green functions of transition probabilities have been found by explicit solution of the master equations for the systems defined on an infinite lattice \[10, 18, 22\]. The Green function has a determinantal representation similar to the one first discovered for the continuous-time definition of the process \[17\] where particles jump independently after an exponentially distributed random time with fixed rate 1 \[1\]–[3]. This representation allows for a direct
derivation of the current distribution [8]–[10] and has inspired a considerable amount of subsequent detailed analysis of dynamical properties of the TASEP and related models, see, e.g., [11]–[14] and also of the ASEP where particles are allowed to jump in both directions [15, 16].

The third type of discrete-time update, sublattice parallel, was first considered in some detail in [19, 20] and has subsequently been studied for various applications both analytically and numerically [4, 21]. In this paper, we derive the Green function of the TASEP with sublattice parallel update which is defined as follows.

Consider the process on \( \mathbb{Z} \), i.e. the one-dimensional infinite chain. Each site labeled by an integer \( i \) is occupied by at most one particle which can hop only to the right in a discrete time. At the first moment of time, we look at all \((2i, 2i + 1)\) pairs. In each of them if the vertex \( 2i + 1 \) is free and the site \( 2i \) is occupied, the particle of the vertex \( 2i \) hops to the right with probability \( v \) and does not move with probability \( 1 - v \). If both sites in a pair are occupied or empty or if site \( 2i \) is empty and site \( 2i + 1 \) occupied, the pair remains unchanged at that time step. At the next time step we apply this rule of hopping to all pairs \((2i + 1, 2i + 2)\). Continuing, we apply the updating rule to \((2i, 2i + 1)\) pairs at each odd moment of time and to \((2i + 1, 2i + 2)\) pairs at each even moment.

2. The equivalence of the TASEP with sublattice parallel and the backward-sequential updates

As noted above, the conditional probability to find \( N \) particles at positions \( x_1 < x_2 < \cdots < x_N \) at discrete time \( t \) if these are in positions \( x_1^0 < x_2^0 < \cdots < x_N^0 \) at the initial moment of time is called the Green function of the process. The discrete spacetime dynamics can be described by a set of trajectories on a triangular lattice which is obtained from the square lattice by adding a diagonal bond between the upper left and the lower right corners of each elementary square. Being occupied by a trajectory, diagonal bonds have a statistical weight \( v \) and vertical ones can have weights 1 or \( 1 - v \). It is convenient to draw trajectories of particles on a chessboard (figure 1), where black circles show the initial positions of particles. We notice that diagonal bonds of trajectories can be located only on white squares. If we select a sublattice which contains upper left and lower right sites of white squares of the chessboard denoted by white circles in figure 1, we can see that particles effectively move on the sublattice of white vertices. There are some exceptions at the start and at the end of trajectories. Then, we have to consider four different cases to find a generalized determinant formula of the Green function.

Consider first the case when spacetime trajectories of \( N \) particles start and end on the sublattice with white vertices, figure 2(a) (the case of arbitrary initial conditions will be considered in the next section). If we choose initial points on the white vertices of the first row, coordinates of particles \( \{x_i^0\} \) at initial times \( \{T_i^0 = 0\} \) \((i = 1, 2, \ldots, N)\) are even.

The first transformation we use is a rotation of the set of trajectories by \( \pi/4 \) clockwise around the initial point \( \{x = 0, T = 0\} \). Considering the vertical axis as the new time coordinate, we obtain a new set of trajectories (figure 2(b)). This set represents a new discrete-time process on the square lattice with the unit time and space intervals

\[ \text{doi:10.1088/1742-5468/2010/04/P04022} \]
corresponding to vertical and horizontal distances between neighboring sites. Starting points change their spacetime coordinates as

\[ T_i^0 = x_i^0 / 2, \quad x_i^0 = x_i^0 / 2. \]  

(2)

Now vertical bonds have weights \( v \) and diagonal ones have weights 1 or \( 1 - v \). We want to map them to the spacetime paths of particles of the TASEP with backward-sequential update. To this end, we shift the coordinates in each row with respect to the above previous row by \( i \to i + 1 \). Due to the second transformation, vertical and diagonal bonds are interchanged as is shown in figure 3. The transformation of coordinates (in new units) can be written as

\[ (x'', T'') = \left( x, \frac{x + T}{2} \right). \]  

(3)
From figure 3 we see that worldlines on the transformed lattice represent trajectories of particles of the TASEP with backward-sequential update with initial spacetime coordinates \( \{x_i^0, T_i^0\} \) and final coordinates \( \{x_i'', T_i''\} \), \( i = 1, 2, \ldots, N \). The transition probability from spacetime coordinates \( \{x_i'', T_i''\} \) to \( \{x_i^0, T_i^0\} \) is given by the generalized determinant formula [22]

\[
P(\mathbf{x}^0, T^0 | \mathbf{x}^0, T^0) = \det M^{(N)},
\]

where the matrix elements of the \( N \times N \) matrix \( M^{(N)} \) are

\[
M_{ij}^{(N)} = F_{i-j} \left( x_i'' - x_j'', T_i'' - T_j'' \right),
\]

with the function \( F_m(x, T) \) introduced in [22]

\[
F_m(x, T) = \frac{1}{2\pi i} \int_{|z|=1-\epsilon} dz \left( 1 - v + \frac{v}{z} \right)^T (1-z)^{-m} z^{x-1}.
\]

Substituting transformation (3) to the determinant formula (4) we obtain the Green function of the TASEP with sublattice parallel update:

\[
P(\mathbf{x}, T | \mathbf{x}^0, T^0) = \det M^{(N)}
\]

with matrix elements

\[
M_{ij}^{(N)} = F_{i-j} \left( x_i - x_j, \frac{T + x_i}{2} - \frac{x_j}{2} \right).
\]

3. Other cases of starting and ending points

Consider the case when the \( i \)th particle starts its motion from an odd site and ends on an even one (figure 4(a)).

As the weight of the first bond of the \( i \)th trajectory is 1, we can set the beginning of motion at the nearest white (sublattice) site and then rotate the sublattice as in the previous case (figure 4(b)). Applying the shift transformation, we obtain for the initial
Figure 4. Spacetime trajectories of particles in the case when the \(i\)th worldline starts from the non-white vertex and ends on a white one (a) and rotated version of trajectories by 45° clockwise (b).

coordinates of the \(i\)th particle:
\[
(x''_i, T''_i) = (x_i, \left\lceil \frac{x}{2} \right\rceil) \tag{9}
\]
where \([x]\) is the ceiling function. Substituting expressions of all starting and end points into the determinant formula (4), we derive the Green function for this case.

The third case is when the \(i\)th trajectory starts from the white (even) vertex and ends on a non-white one. We see that, if we add an additional vertical bond with weight 1 to the last node of that trajectory, the total weight of the whole path will not be changed. Then we can set the endpoint of the \(i\)th particle at time \(T + 1\). Repeating two transformations, we obtain for coordinates of the end point:
\[
(x''_i, T''_i) = \left( x_i, \frac{T + x_i}{2} \right) \tag{10}
\]

The last case when the \(i\)th trajectory starts and ends on non-white vertices is an obvious combination of the second and third case.

Generalizing all four cases of boundary conditions, we derive the following transformations for the initial and final coordinates for all types of trajectories:
\[
T''_i = \left\lceil \frac{T}{2} \right\rceil, \quad x''_i = x_i, \quad T''_i = \left\lceil \frac{T + x_i}{2} \right\rceil, \quad x''_i = x_i. \tag{11}
\]

Substituting new coordinates (11) into the determinant formula (4) we obtain the Green function of the TASEP with sublattice parallel update:
\[
P(x_1, x_2, \ldots, x_N|x_0^0, x_0^0, \ldots, x_N^0; T) = \det M^{(N)}, \tag{12}
\]
with matrix elements
\[
M^{(N)}_{ij} = F_{i-j} \left( x_i - x_j, \left\lceil \frac{T + x_i}{2} \right\rceil - \left\lceil \frac{x_j}{2} \right\rceil \right). \tag{13}
\]
4. Discussion

Having explicit determinant expressions for the Green function, we can compare their relative advantages and disadvantages for the three basic updates, the backward-sequential, parallel and sublattice parallel ones. Criteria for the comparison follow from practical use of the Green function in probabilistic calculations. To find a probability distribution for a selected particle or a correlation function for several particles in the TASEP, detailed information contained in the function \( P(x_1, x_2, \ldots, x_N | x_1^0, x_2^0, \ldots, x_N^0; T) \) should be reduced by summation over a part of the final coordinates \( \{x_i^0\} \) for fixed initial coordinates \( \{x_0^i\} \) (see, e.g., [10]). Then, the first criterion for the comparison is the simplicity of the summation procedure in different cases. The second criterion is the simplicity of the matrix \( M^{(N)}_{ij} \) itself, because asymptotic calculations for large \( N \) and \( T \) need an elaborate analysis of resulting determinant expressions (see, e.g., [13, 14]). The third criterion is the presence or lack of particle–hole symmetry, which is essential for the derivation of single-particle probability distributions in some particular cases [10].

(A) The backward-sequential update. The form of the matrix elements \( M^{(N)}_{ij} \) in this case is especially simple:

\[
M^{(N)}_{ij} = F_{i-j}(x_i - x_j^0, T) \tag{14}
\]

where function \( F_m(x, T) \) is given by equation (6). The Green function \( P(x_1, x_2, \ldots, x_N | x_1^0, x_2^0, \ldots, x_N^0; T) \) is uniform in variables \( \{x_i\} \), so the summation procedure is straightforward [10]. A shortcoming of this update is a lack of the particle–hole symmetry. Indeed, due to possible transitions for one time step \( x_i \rightarrow x_i + 1, x_{i+1} = x_i + 1 \rightarrow x_{i+1} + 1, \ldots, x_{i+k} = x_{i+k-1} + 1 \rightarrow x_{i+k} + 1, \) a hole can move in the opposite direction by jumps of length \( k > 1 \).

(B) The parallel update. The form of the matrix \( M^{(N)}_{ij} \) in this case is more complicated [18]:

\[
M^{(N)}_{ij} = \tilde{F}_{i-j}(x_i - x_j^0, T) \tag{15}
\]

where

\[
\tilde{F}_{\pm m}(N, T) = \sum_{n=0}^{m} \sum_{k=-n}^{\infty} (\pm 1)^n \frac{m(m+k+n-1)!}{(k+n)!n!(m-n)!} \left( \pm \frac{v}{1-v} \right)^n F_0(N \pm k, T). \tag{16}
\]

The Green function obeys the particle–hole symmetry, but a drawback is in the determinant formula

\[
P(x_1, x_2, \ldots, x_N | x_1^0, x_2^0, \ldots, x_N^0; T) = (1-v)^n \det M^{(N)} \tag{17}
\]

which depends on the number of pairs \( n \) of neighboring particles in the final configuration. Therefore, the sum over \( \{x_i\} \) splits into groups by the number of clusters of connected particles.

(C) The sublattice parallel update. The Green function for this update is free of shortcomings of two previous cases. It is uniform in variables \( \{x_i\} \), obeys the particle–hole symmetry and has a relatively simple analytical form (13). A fee for this advantage is a rather complicated time dependence in (13) which involves both initial and final coordinates and the ceiling function \( \lceil x \rceil \). Thus, we may conclude that a proper choice of the discrete-time Green function strongly depends on peculiarities of the corresponding probabilistic problem.

\[ \text{doi:10.1088/1742-5468/2010/04/P04022} \]
Acknowledgments

This work was supported by the RFBR grant nos. 07-02-91561-a and 09-01-00271-a and DFG grant no. 436 RUS 113/909/0-1(R).

References

[1] Spohn H, 1991 Large Scale Dynamics of Interacting Particles (Berlin: Springer)
[2] Liggett T M, 1999 Stochastic Interacting Systems: Contact, Voter and Exclusion Processes (Berlin: Springer)
[3] Schütz G M, Exactly solvable models for many-body systems far from equilibrium, 2001 Phase Transitions and Critical Phenomena vol 19, ed C Domb and J Lebowitz (London: Academic) pp 1–251
[4] Rajewsky N, Santen L, Schadschneider A and Schreckenberg M, The asymmetric exclusion process: comparison of update procedures, 1998 J. Stat. Phys. 92 151
[5] Mattis D C and Glasser M L, 1998 Rev. Mod. Phys. 70 979
[6] Kandel D, Domany E and Nienhuis B, 1990 J. Phys. A: Math. Gen. 23 L755
[7] Honecker A and Peschel I, 1997 J. Stat. Phys. 88 319
[8] Johansson K, Shape fluctuations and random matrices, 2000 Commun. Math. Phys. 209 437
[9] Nagao T and Sasamoto T, Asymmetric simple exclusion process and modified random matrix ensembles, 2004 Nucl. Phys. B 699 487
[10] Rajek A and Schütz G M, Current distribution and random matrix ensembles for an integrable asymmetric fragmentation process, 2005 J. Stat. Phys. 118 511
[11] Sasamoto T, Spatial correlations of the 1D KPZ surface on a flat substrate, 2005 J. Phys. A: Math. Gen. 38 L549
[12] Sasamoto T, Fluctuations of the one-dimensional asymmetric exclusion process using random matrix techniques, 2007 J. Stat. Mech. P07007
[13] Borodin A, Ferrari P L, Prähofer M and Sasamoto T, Fluctuation properties of the TASEP with periodic initial configuration, 2007 J. Stat. Phys. 129 1055
[14] Borodin A and Ferrari P L, Large time asymptotics of growth models on space-like paths I: PushASEP, 2008 Electr. J. Prob. 13 1380
[15] Tracy C A and Widom H, Integral formulas for the asymmetric simple exclusion process, 2008 Commun. Math. Phys. 279 815
[16] Tracy C A and Widom H, A Fredholm determinant representation in ASEP, 2008 J. Stat. Phys. 132 291
[17] Schütz G M, Exact solution of the master equation for the asymmetric exclusion process, 1997 J. Stat. Phys. 88 427
[18] Povolotsky A M and Priezzhev V B, Determinant solution for the totally asymmetric exclusion process with parallel update, 2006 J. Stat. Mech. P07002
[19] Schütz G M, Generalized Bethe Ansatz solution of a one-dimensional asymmetric exclusion process on a ring with blockage, 1993 J. Stat. Phys. 71 471
[20] G M Schütz, Time-dependent correlation functions in a one-dimensional asymmetric exclusion process, 1993 Phys. Rev. E 47 4265
[21] Jalarpour F H, Ghafari F E and Masharian S R, Exact shock profile for the ASEP with sublattice-parallel update, 2005 J. Phys. A: Math. Gen. 38 4579
[22] Brankov J, Priezzhev V B and Shelest R V, Generalized determinant solution of the discrete-time totally asymmetric exclusion process and zero-range process, 2004 Phys. Rev. E 69 066136

doi:10.1088/1742-5468/2010/04/P04022