COMPACT MANIFOLDS
WITH A LITTLE NEGATIVE CURVATURE

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1. Bochner's Theorem states that a compact oriented Riemannian manifold \((M, g)\) with positive Ricci curvature has \(H^1(M; \mathbb{R}) = 0\). Myers' Theorem implies the stronger result that \(\pi_1(M)\) is finite under the same hypothesis. Both theorems fail if the Ricci curvature is positive except on a set of arbitrarily small diameter, since every compact manifold admits such a metric of volume one. Nevertheless, we can extend these theorems and the Bochner Theorem for \(\ell^2\)-forms, yielding topological obstructions to manifolds admitting metrics with a little negative curvature.

2. Results for \(H^1(M; \mathbb{R})\). The Laplacian on \(p\)-forms has the Weitzenböck decomposition \(\Delta^p = \nabla^* \nabla + R^p\); here \(\nabla\) is the Levi-Civita connection and \(R^p \in \text{End}(\Lambda^p T^*M)\) with \(R^1 = \text{Ricci}\). We write \(R^p(x) \geq R_0\) for \(x \in M\) if the lowest eigenvalue of \(R^p(x)\) is at least \(R_0\). In what follows, we normalize all metrics to have volume one.

**Theorem 1.** Pick \(R_0 > 0\) and \(K < 0\). There exists \(\varepsilon = \varepsilon(R_0, K, \dim M) > 0\) such that if \(\text{Ric}(x) \geq R_0\) except on a set \(A\), with diameter \(\text{diam}(A) \leq \varepsilon\), where \(\text{Ric}(x) \geq K\), then \(H^1(M; \mathbb{R}) = 0\).

In other words, if the metric has a deep well of negative Ricci curvature, we may still conclude \(H^1(M; \mathbb{R}) = 0\) provided the well is narrow enough. Notice that there is no restriction on the topology of \(A\).

Theorem 1 is a consequence of the following weaker version about metrics with a shallow well of negative Ricci curvature.

**Theorem 1'.** Pick \(R_0 > 0\). There exists \(\varepsilon' = \varepsilon'(R_0, \dim M) > 0\) and \(\delta = \delta(R_0, \dim M) < 0\) such that if \(\text{Ric}(x) \geq R_0\) except on a set \(A\), with \(\text{diam}(A) \leq \varepsilon'\), where \(\text{Ric}(x) \geq \delta\), then \(H^1(M; \mathbb{R}) = 0\).

We sketch a proof of Theorem 1'. By semigroup domination for the heat flow on one forms, it is enough to show that \(\Delta^0 + \text{Ric}' > 0\), where \(\text{Ric}'(x)\) is the lowest eigenvalue of Ricci at \(x\). By an elementary argument, we have

**Lemma 2.** Let \(V: M \to \mathbb{R}\) be continuous. If (i) \(\int_M V \, d\text{vol}(g) > 0\) and

(ii) \(\lambda_1 \geq -V_{\min} + \frac{\|V - V_{\min}\|^2}{\int_M V}\),

then \(\Delta^0 + V > 0\).

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Here $\lambda_1$ is the first nonzero eigenvalue of $\Delta^0$, $V_{\min}$ is the minimum of $V$, $V_{av} = \text{vol}(M)^{-1} \int V$ and $\| \cdot \|$ is the $L^2$-norm. We set $V = \min \{ R_0, \text{Ric}' \}$. Then for $\varepsilon'$ and $\delta$ sufficiently small, (i) holds and the right side of (ii) is arbitrarily close to zero. However, by Myers' Theorem, the diameter of $M - A$ and hence of $M$ is bounded above. By Gromov [5] or Li and Yau [8], this keeps $\lambda_1$ bounded away from zero as $\varepsilon', \delta$ go to zero. Thus $\Delta^0 + V > 0$ and hence $\Delta^0 + \text{Ric}' > 0$.

To derive Theorem 1, we strengthen Lemma 2. If $\Delta^0 f = \lambda_1 f$, then we apparently need $\lambda_1 \geq -V_{\min}$ to show $\langle (\Delta^0 + V) f, f \rangle > 0$. However, we can do much better provided $f$ is not concentrated near $V_{\min}$. In fact, by estimates of Li [7] and Croke [2] we can estimate how concentrated any function in the span of the first $m$ eigenfunctions of $\Delta^0$ may be near $V_{\min}$. Roughly speaking, this allows $V_{\min} = \text{Ric}'_{\min}$ to be arbitrarily negative and to replace $\lambda_1$ by $\lambda_m$ in Lemma 2(ii). Now we can mimic the proof of Theorem 1' using the estimates in Li and Yau [9] for $\lambda_m$. The method of proof yields explicit upper bounds for $\varepsilon, \varepsilon'$, and $|\delta|$ in terms of the geometric data.

A different method of coupling geometric information with semigroup domination may be found in [1].

3. Results for $\pi_1(M)$. Here the results for deep and shallow wells differ.

**Theorem 3.** Assume $M$ admits a metric $g$ with $\text{Ric}(x) \geq R_0 > 0$ except on a set $A$, with $\text{diam}(A) \leq \varepsilon$, where $\text{Ric}(x) \geq K$, for $\varepsilon$ as in Theorem 1. If $\pi_1(M)$ contains a solvable subgroup of finite index, then $\pi_1(M)$ is finite. In particular, if $\pi_1(M)$ has polynomial growth, then $\pi_1(M)$ is finite [4].

As opposed to Myers' theorem, the proof uses $H^1 = 0$ to show $\pi_1$ is finite. In the tower of coverings $M \to M_k \to M_{k-1} \to \cdots \to M_0 \to M$ associated to the solvable subgroup, we argue inductively that $H^1(M_j; \mathbf{R}) = 0$ implies $M_{j+1}$ is a finite cover of $M_j$, noting that $\Delta^0 + \text{Ric}'$ is still positive for finite covers of $M$.

If a manifold with infinite $\pi_1$ admits a shallow well metric, the metric must be very distorted, in the sense that either the injectivity radius is very small at each point, or a generator of $\pi_1$ has very long geodesic length. To be more precise, we fix a point $x_0$ of $M$.

**Theorem 3'.** Suppose $\pi_1(M, x_0)$ is infinite. For a set of generators $G = \{ \gamma_1, \ldots, \gamma_l \}$ for $\pi_1(M, x_0)$ and for positive numbers $l, \rho$ and $R_0$, there exist $\delta = \delta(R_0, \text{dim} M, G, l, \rho, \pi_1(M, x_0)) < 0$ and $\varepsilon = \varepsilon(R_0, \text{dim} M) > 0$ such that if $g$ is a metric satisfying

(i) some point of $M$ has injectivity radius larger than $\rho$,

(ii) the shortest geodesic in $\gamma_i$ has length less than $l$ for each $i$,

(iii) $\text{Ric}(g) \geq R_0$ except on a set of diameter less than $\varepsilon$,

then $\text{Ric}(g) < \delta$ somewhere on $M$.

Here we bound the growth function $\gamma(r)$ of $\pi_1$ by $C_1 \cdot \exp(C_2 \sqrt{-\delta r})$ for positive constants $C_1, C_2$ as in [4, 11]. For fixed $C_3 > 0$ and $N \in \mathbf{Z^+}$, this is bounded in turn by $C_3 \cdot r$ for $r = 1, 2, \ldots, N$ by taking $\delta$ close to zero.
For $N$ sufficiently large, this implies $\pi_1(M)$ contains a nilpotent subgroup of finite index [4] and Theorem 3 applies.

4. Results for $p$-forms. $H^p(M, \mathbb{R}) = 0$ if $R^p$ is positive. More generally, if we define $R^p$ analogously to $\text{Ric}'$, then $H^p(M, \mathbb{R}) = 0$ whenever $\nu^p = \lim_{t \to -\infty} t^{-1} \ln E[\exp(- \int_0^t R^p(x_s) \, ds)] < 0$. Here $E$ is expectation with respect to the Wiener measure for Brownian motion $x_\cdot$ on $M$. For the universal cover $\tilde{M}$, $\nu^p(M) = \nu^p(\tilde{M})$ with the pullback metric, so $\nu^p < 0$ implies the vanishing of the space of $L^2$ harmonic $p$-forms on $\tilde{M}$. By the weak Hodge Theorem, $\text{Im}[H^p(\tilde{M} ; \mathbb{R}) \to H^p(M;\mathbb{R})] = 0$, where $H^p$ denotes cohomology with compact supports. This implies that no nonzero class in $H^p(\tilde{M} ; \mathbb{R})$ has a representative differential form with compact support. For $p = 1$, we showed in [3] that in fact $H_1^1(\tilde{M};\mathbb{Z}) = 0$, so in particular a compact 3-manifold with infinite $\pi_1$ and admitting a metric as in Theorem 3 must be a $K(\pi, 1)$.

For higher dimensional manifolds, we fix generators of $\pi_1(M)$ with associated growth function $\gamma(r)$ and a function $f(r)$ with $\limsup_{r \to \infty} f(r)\gamma(kr) = 0$ for all $k \in \mathbb{Z}^+$. $f(r)$ is then independent of the choice of generators.

**Theorem 4.** Assume $R^p > 0$ or more generally that $\nu^p < 0$ on $M$. Let $r$ denote the distance from a fixed point in $\tilde{M}$. If $\pi_1(M)$ is infinite, no nonzero class in $H^p(\tilde{M};\mathbb{R})$ has a representative form which decays faster than $f(r)$.

By Micallef-Moore [10], a simply connected manifold with curvature operator positive on complex totally isotropic two-planes is homeomorphic to a sphere ($\dim M \geq 4$). It is known that this curvature condition implies $R^2 > 0$ if $\dim M$ is even, and it may be that it implies $R^p > 0$ for $p \neq 1, n - 1$. Thus Theorem 4 gives topological information on nonsimply connected manifolds with this type of curvature operator, at least for $p = 2$ and $\dim M$ even.

To prove Theorem 4, we use a notion of bounded homology $H_p^\infty$ and $l_1$-cohomology $H^p$ complementary to Gromov’s bounded cohomology [6]. As in [3, Theorem 5A], the integral of a compactly supported closed $p$-form over a bounded chain is unchanged under the heat flow and decays to zero as $t \to \infty$, so $\text{Im}[H^p_\infty \to H^p] = 0$. Using a Poincaré duality map in this theory and the fact that $\nu^p = \nu^{n-p}$, we conclude that every class $\alpha \in H^p(\tilde{M})$ is the boundary of an infinite chain $\sigma = \sum n_i \sigma_i$ with bounded coefficients. Let $\theta$ be a closed differential form which decays faster than $f(r)$. By estimates in [11], the boundary of suitable partial sums of $\sigma$ has volume growth bounded by $\gamma(kr)$ for some $k$, so the integral of $\theta$ over the boundary of these partial sums tends to zero at infinity. Thus $\int_{\partial} \theta = 0$ so $\theta$ is cohomologous to zero.

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