Abstract

In this text, we provide a detailed exposition of the Algebraic Bethe ansatz for square ice (or six vertex model), which allows the construction of candidate eigenvectors for the transfer matrices of this model. We also prove some formula of V.E. Korepin for these vectors, which leads to an identification, up to a non-zero complex factor, with the vector obtained by coordinate Bethe ansatz.

1 Introduction

The model of ice was introduced as a model of statistical mechanics by L. Pauling [Pauling] in a three-dimensional setting, and consists in the set of possible arrangements of dihydrogen monoxide molecules on an infinite grid. He proposed a first very simple approximation of entropy for this model. This approximation was refined by further works [see for instance [Dimarzio Stillinger]], leading to an exact computation by E.H.Lieb [Lieb 1967], using mainly H. Bethe method [Bethe] for the diagonalisation of the Hamiltonian of the XXZ model, the analytic work of C.N. Yang and C.P. Yang [Yang Yang I] [Yang Yang II], and the diagonalisation of the XY model by E.H. Lieb, T. Shultz and D. Mattis [Lieb Shultz Mattis]. This work led to an extensive study of what are now called exactly solved models. Notably, the work of R.Baxter led to the discovery of a rich algebraic structure underlying these models, and an algebraisation of Bethe method, called now the algebraic Bethe ansatz, that he applied for the eight-vertex model [Baxter], an extension of the six-vertex model - itself a representation of square ice. However, the computation done by E.H.Lieb was not rigorous, in particular since it relied on some unproven hypothesis, and some arguments left partial in the texts on which it relied. Motivated by percolation theory, rigorous proofs of some parts of C.N. Yang and C.P. Yang statements were recently done by H. Duminil-Copin, M. Gagnebin, M. Harel, I. Manolescu and V. Tassion [Duminil-Copin et al.]. They also proposed an exposition of Bethe method, called coordinate Bethe ansatz [Duminil-Copin et al.]. We then provided a complete proof, which involves an exposition of E.H.Lieb’s arguments and some techniques developed later, as well as completion of several partial arguments, that square ice entropy is equal to $\frac{3}{2} \log_2(4/3)$. However, our aim was to extend the methods developed in this field to other models, called multidimensional subshifts of finite type, that lie outside of the scope of statistical physics, since they appear at the frontier between mathematics and computation theories. The proof that we presented makes use of the coordinate Bethe ansatz, which is highly specific to square ice, as it relies on some structures of $\mathbb{Z}^2$ that appear naturally when one represents these models as a discrete curves model. On the other hand, the algebraic Bethe ansatz can be formulated for the broad class of nearest-neighbour multidimensional subshifts of finite type, as soon as one can find a solution of the so-called Yang-Baxter equation for this model. We expect that this method, applied to particular subshifts of finite type, would lead to new exactly solved models. In the present text, we propose a complete exposition of the algebraic Bethe ansatz for square ice. Since the proof of square ice entropy relies on an expression of the candidate eigenvector provided by the coordinate Bethe ansatz, we prove a formula of V.E. Korepin which leads to an identification, up to a non-zero complex factor, to the vectors obtained by both ansatz. We also provide a proof of the commutation of the transfer matrices of the model with some Heisenberg Hamiltonian which relies on development of the algebraic Bethe ansatz. In the perspective of a generalisation to other models, this derivations would help us deriving all necessary properties of the vector only from the algebraic Bethe ansatz. Although these results are not completely new, since the literature on the subject is dispersed and does not provide complete proofs, we expect that this exposition will be useful for a broad audience.

The text is organized as follows: in Section 2 we present square ice and its representations [Section 2.1], define Lieb transfer matrices [Section 2.2], and state the coordinate Bethe ansatz [Section 2.3]. In Section 3 we present an overview of the remaining of the text.
2 Background

2.1 Representations of square ice

The most widely used representation of square ice is the six vertex model (whose name derives from that the elements of the alphabet represent vertices of a regular grid) and is presented in Section 2.1.1. In this text, we will use another representation, presented in Section 2.1.2, whose configurations consist of drifting discrete curves, representing possible particle trajectories.

2.1.1 The six vertex model

The six vertex model is the set of elements of $A_0^{Z^2}$, where

$$A_0 = \left\{ \begin{array}{c} \begin{array}{ccc} + & + & + \\ + & + & + \\ + & + & + \\ + & + & + \\ + & + & + \\ + & + & + \\ \end{array} \end{array} \right\},$$

such that for any two adjacent positions in $Z^2$, the arrows corresponding to the common edge of the symbols on the two positions have to be directed the same way: for instance, the pattern \begin{array}{ccc} + & + & + \\ + & + & + \end{array} is allowed, while \begin{array}{ccc} + & + & + \\ + & + & + \end{array} is not.

In an element of the model, the symbols draw a grid whose edges are oriented in such a way that all the vertices have two incoming arrows and two outgoing ones. This is called an Eulerian orientation of the square lattice. See an illustration on Figure 1.

![Figure 1: An example of pattern that appears in an element of the six vertex model.](image1)

2.1.2 Drifting discrete curves

From the six vertex model, we derive another representation of square ice through the application of the following invertible transformation on the elements of $A_0$:

$$A_0 = \left\{ \begin{array}{c} \begin{array}{ccc} + & + & + \\ + & + & + \\ + & + & + \\ + & + & + \\ + & + & + \\ + & + & + \\ \end{array} \end{array} \right\},$$

The set of image symbols is denoted $A_1$. The discrete drifting curves model is the subset, denoted $X^*$, of $A_1^{Z^2}$ that are obtained by the previous transformation of symbols. The pattern of the six vertex model on Figure 1 can be represented in the discrete drifting curves model as on Figure 2.

![Figure 2: Representation of pattern on Figure 1](image2)
2.2 Lieb path of transfer matrices

A pattern of $X^s$ is an element of $\mathcal{A}^s_{1}$ for some finite subset $U \subset \mathbb{Z}^2$ which appears in an element of $X^s$. A $(N,1)$-cylindric pattern of $X^s$ is a pattern in $\mathcal{A}^s_{1}(\ldots ; N ; \ldots )$ that can be wrapped on the cylinder $\mathbb{A}^s_{1}/n\mathbb{Z}^2$ without breaking the rules of $X^s$. Let us denote $\{0,1\}^N$ the set of length $N$ words on the alphabet $\{0,1\}$. For a word $\epsilon \in \{0,1\}^N$, we denote $|\epsilon|_1$ the number of integers $k \in \{1,\ldots, N\}$ such that $\epsilon_k = 1$.

Notation 1. Let $N \geq 1$ be an integer. Let us denote $\Omega_N$ the space $\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$, tensor product of $N$ copies of $\mathbb{C}^2$, whose canonical basis elements are denoted indirectly by $\epsilon = |\epsilon_1 \ldots \epsilon_N\rangle$ or the words $\epsilon_1 \ldots \epsilon_N$, for $(\epsilon_1,\ldots,\epsilon_N) \in \{0,1\}^N$, according to quantum mechanics notations, in order to distinguish them from the coordinate definition of vectors of $\Omega_N$. We also denote by $\Omega_N^{(n)}$ the vector space generated by the elements $\epsilon$ of the canonical basis such that $|\epsilon|_1 = n$.

Notation 2. For any $\epsilon$ in the canonical basis of $\Omega_N$ such that $|\epsilon|_1 = n$, we denote by $q_k[\epsilon]$ the $k$th position $j$ in $\{1,\ldots, N\}$ such that $\epsilon_j = 1$.

Notation 3. For all $N$ and $(N,1)$-cylindric pattern $w$, let $|w|$ denote the number of symbols

in this pattern. Such a pattern is said to link some $\epsilon \in \{0,1\}^N$ to $\eta \in \{0,1\}^N$ when there is a curve entering through the south (resp. outgoing through the north side) of $w$ at position $k$ if and only if $\epsilon_k = 1$ (resp. $\epsilon_k = 0$). This relation is denoted $\epsilon R[w] \eta$.

Definition 1. For all $t \geq 0$, $V_N(t) \in \mathcal{M}_{2^N}(\mathbb{C})$ denotes the matrix such that for all $\epsilon, \eta \in \{0,1\}^N$,

$$V_N(t)[\epsilon, \eta] = \sum_{\epsilon R[w] \eta} \ell^{|w|}$$

Let us notice that for all $N$, the map $t \mapsto V_N(t)$ is analytic, which we called Lieb path of transfer matrices in [Gangloff].

2.3 Coordinate Bethe ansatz

Since the entropy of $X^s$ is expressed according to the greatest eigenvalues on the subspaces $\Omega_N^{(n)}$ of $V_N(1)$, for all $N$, the strategy in order to compute the entropy is to search for an expression of these eigenvalues. This is the purpose of the coordinate Bethe ansatz, which produces candidate eigenvalues. Some auxiliary functions involved in the method depend on the position of $t$ according to 2. Since the method is independant provided the auxiliary functions, we present it only for the case $t < 2$, which corresponds to our interest. In the proof of the value of square ice entropy, these eigenvalues are identified with the maximal ones on a subdomain of the parameter $t$, and this information is transported to the parameter 1 through analyticity.

2.3.1 Auxiliary functions

Let us denote $\mu : (-1,1) \to (0,\pi)$ the inverse of the function $\cos : (0,\pi) \to (-1,1)$. For all $t \in [0,2]$, we will denote $\Delta_t = \frac{2\mu^t}{\pi}$, $\mu_t = \mu(-\Delta_t)$, and $I_t = (-\mu - \mu_t, \pi - \mu_t)$.

Notation 4. Let us denote $\Theta$ the unique analytic function $(t,x,y) \mapsto \Theta_t(x,y)$ from the set $\{(t,x,y) \in [0,2), x, y \in I_t\} \to \mathbb{R}$ such that $\Theta_t(0,0) = 0$ and for all $t,x,y$,

$$\exp(-i\Theta_t(x,y)) = \exp(i(x-y)) \frac{e^{-ix} + e^{-iy} - 2\Delta_t}{e^{-iy} + e^{ix} - 2\Delta_t}$$

By a unicity argument, for all $t,x,y$, $\Theta_t(x,y) = -\Theta_t(-x,-y)$.

Notation 5. Let us denote $\kappa$ the unique analytic map $(t,\alpha) \mapsto \kappa_t(\alpha)$ from $(0,2) \times \mathbb{R}$ to $\mathbb{R}$ such that $\kappa\sqrt{2}(0) = 0$ and for all $t,\alpha$,

$$e^{i\kappa_t(\alpha)} = \frac{e^{i\mu_t} - e^\alpha}{e^{i\mu_t + \alpha} - 1}.$$ 

Computation 1. For all $t \in (0,2)$, and $\alpha \in \mathbb{R}$, the derivative of $\kappa_t$ in $\alpha$ is given by:

$$\kappa_t'(\alpha) = \frac{\sin(\mu_t)}{\cosh(\alpha) - \cos(\mu_t)}.$$
Proof. \textbf{Computation of} \(\cos(\kappa_t(\alpha))\) \textbf{and} \(\sin(\kappa_t(\alpha))\):

\[
e^{\kappa_t(\alpha)} = \frac{(e^{-\mu t} - 1)(e^{\mu t} - e^\alpha)}{(e^{\mu t + \alpha} - 1)^2} = \frac{\cos(\mu t)e^\alpha - 1}{(\cos(\mu t)e^\alpha - 1)^2 + (\sin(\mu t)e^\alpha)^2}.
\]

Thus by taking the real part,

\[
\cos(\kappa_t(\alpha)) = \frac{2e^\alpha + (e^{2\alpha} - 1)\cos(\mu t)}{e^{2\alpha} - 2\cos(\mu t)e^\alpha + 1} = \frac{1 - \cos(\mu t)cosh(\alpha)}{cosh(\alpha) - cos(\mu t)}.
\]

where we factorized by \(2e^\alpha\) for the second equality. As a consequence:

\[
\cos(\kappa_t(\alpha)) = \frac{\sin^2(\mu t) + \cos^2(\mu t) - \cos(\mu t)\cosh(\alpha)}{\cosh(\alpha) - \cos(\mu t)} = \frac{\sin^2(\mu t)}{\cosh(\alpha) - \cos(\mu t)} - \cos(\mu t).
\]

A similar computation gives

\[
\sin(\kappa_t(\alpha)) = \frac{\sin(\mu t)\sinh(\alpha)}{\cosh(\alpha) - \cos(\mu t)}
\]

\textbf{Deriving the expression} \(\cos(\kappa_t(\alpha))\):

As a consequence, for all \(\alpha\):

\[
-k'\alpha(\alpha)\sin(\kappa_t(\alpha)) = -\frac{\sin^2(\mu t)\sin(\alpha)}{(\cosh(\alpha) - \cos(\mu t))^2} = -\frac{\sin(\kappa_t(\alpha))^2}{\sinh(\alpha)}.
\]

Thus, for all \(\alpha\) but in a discrete subset of \(\mathbb{R}\),

\[
k'(\alpha) = \frac{\sin(\mu t)}{\cosh(\alpha) - \cos(\mu t)}.
\]

This identity is thus verified on all \(\mathbb{R}\), by continuity. \(\square\)

From this, one deduces that \(\kappa_t\) is increasing. By another unicity argument, this function is antisymmetric: for all \(\alpha\), \(\kappa_t(-\alpha) = -\kappa_t(\alpha)\). It is also known that \(\kappa_t : \mathbb{R} \to I_t\) is an invertible map (see for instance \cite{Gangloff}, Proposition 7). Moreover:

\textbf{Computation 2. For all} \(t \in (0, 2)\) \textbf{and} \(\alpha \in I_t\),

\[
k'(\kappa^{-1}(\alpha)) = \frac{\cos(\alpha) + \cos(\mu t)}{\sin(\mu t)}
\]

Proof. From the first point of the proof of Computation\(\square\)

\[
\cos(\kappa_t(\alpha)) = \frac{\sin^2(\mu t)}{\cosh(\alpha) - \cos(\mu t)} - \cos(\mu t) = \sin(\mu t)k'\alpha(\alpha) - \cos(\mu t).
\]

As a consequence,

\[
k'(\kappa^{-1}(\alpha)) = \frac{\cos(\alpha) + \cos(\mu t)}{\sin(\mu t)}.
\]
2.3.2 Statement of the ansatz

**Notation 6.** For all \((p_1, ..., p_n) \in I^n_t\), let us denote \(\psi_{t,n,N}(p_1, ..., p_n)\) the vector in \(\Omega_N\) such that for all \(\epsilon \in \{0,1\}_N^t\),

\[
\psi_{t,n,N}(p_1, ..., p_n)[\epsilon] = \sum_{\sigma \in \Sigma_n} C_\sigma(t)[p_1, ..., p_n] \prod_{k=1}^n e^{ip_{\sigma(k)} q_k}[\epsilon],
\]

where (for \(\epsilon(\sigma)\) denoting the signature of \(\sigma\)):

\[
C_\sigma(t)[p_1, ..., p_n] = \epsilon(\sigma) \prod_{1 \leq k < l \leq n} e^{ip_{\sigma(k)}(e^{-ip_{\sigma(l)}} + e^{ip_{\sigma(l)}} - 2\Delta_t)}.
\]

**Notation 7.** For all \(t\) and \(z \neq 1\), we set

\[
L_t(z) = 1 + \frac{t^2 z}{1 - z}, \quad M_t(z) = 1 - \frac{t^2}{1 - z}.
\]

**Notation 8.** Let \((p_1, ..., p_n) \in I^n_t\) such that \(p_1 < ... < p_n\). If for all \(j, p_j \neq 0\), we denote

\[
\Lambda_{n,N}(t)[p_1, ..., p_n] = \prod_{k=1}^n L_t(e^{ip_k}) + \prod_{k=1}^n M_t(e^{ip_k}).
\]

If there exists some \(l\) such that \(p_l = 0\):

\[
\Lambda_{n,N}(t)[p_1, ..., p_n] = \left(2 + t^2(N - 1) + \sum_{k \neq l} \frac{\partial \Theta_t}{\partial x_l}(0, p_k)\right) \prod_{k=1}^n M_t(e^{ip_k}).
\]

**Theorem 1.** For all \(N\) and \(n \leq N/2\), and \((p_1, ..., p_n) \in I^n_t\) such that \(p_1 < ... < p_n\) and for all \(j\) the following equation is verified:

\[
(E_j)[t,n,N] : \quad Np_j = 2\pi j - (n + 1)\pi - \sum_{k=1}^n \Theta_t(p_j, p_k).
\]

Then we have:

\[
V_N(t).\psi_{t,n,N}(p_1, ..., p_n) = \Lambda_{n,N}(t)[p_1, ..., p_n] \psi_{t,n,N}(p_1, ..., p_n).
\]

3 Overview

In the following, we provide a proof of Theorem 1 using the algebraic Bethe ansatz. In Section 4 we expose an abstract version of it, which consists, given an analytic path of commuting transfer matrices that we call Yang-Baxter path and denote \(x \mapsto T^x_N\) (commuting means that for all \(x, y\), \(T^x_N\) and \(T^y_N\) commute) in constructing, under the existence of a sequence \((x_1, ..., x_n)\) which verifies a system of non-linear equations, a candidate eigenvector and eigenvalue for any matrix \(T^x\), using the commutation relations between the transfer matrices. These transfer matrices are constructed from the iteration of local matrices. We describe, in Section 5, the construction of a Yang-Baxter path proved a solution of the so-called Yang-Baxter equation on these local matrices. In Section 6, for all \(t\), we apply the algebraic Bethe ansatz to a trigonometric Yang-Baxter path, \(x \mapsto T^x_N(t)\), which takes the value \(V_N(t)\) for some parameter, and give a proof of Theorem 1. See an illustration on Figure 3. In the end, we derive, using the Yang-Baxter path of commuting matrices, the commutation of \(V_N(t)\) with some Heisenberg Hamiltonian [Section 7]. In the proof of the value of square ice, this property is used to identify the maximal eigenvalue of \(V_N(t)\).

4 Construction of Yang-Baxter paths

In this section, we explain how to construct more general transfer matrix [Section 4.2], and thus paths, deriving from local matrix functions [Section 4.1]. The interest of this construction of transfer matrices is that it allows a simple criterion for these matrices to commute, widely known as the Yang-Baxter equation [Section 4.3]. This criterion becomes simpler when the consider local matrix functions are strongly symmetric [Section 4.4]: it consists in selecting the local matrix functions in the same level surface of an operator on these functions. This simpler version helps to find some Yang-Baxter paths. We define the notion of Yang-Baxter path in Section 4.4.
In the following, for any finite sequence of matrices $M(i)$ having the same size, we denote by

$$\prod_{k=1}^{m} M^{(k)} = M^{(1)} \times \ldots \times M^{(m)}.$$  

### 4.1 Local matrix functions

**Definition 2.** A **local matrix function** of square ice is a function $R$ from $\{0, 1\}^2$ to $M_2(\mathbb{C})$ such that for all $u, v \in \{0, 1\}$, $(w, w') \in \{0, 1\}^2$, if there is no symbol of the discrete curves shift $X_s$ such that there is a curve crossing the west (resp. north, south, east) boundary if and only if $w'$ (resp. $v$, $u$, $w$) equals 1 (see an illustration on Figure 3), then 

$$R(u, v)[w, w'] = 0.$$  

An image of the local matrix function is called a **local matrix**. The **matricial representation** of this local matrix function is the element of $M_4(\mathbb{C})$ defined as:

$$R = \begin{pmatrix}
R(0,0) & R(0,1) \\
R(1,0) & R(1,1)
\end{pmatrix},$$

for some $a,b,c,d,e,f \in \mathbb{C}$.

**Fact 1.** A direct translation of Definition 2 is that when $u = v$, the matrix $R(u, v)$ is diagonal, and $R(0, 1)$ and $R(1, 0)$ have all coefficients equal to zero except for the respective coefficients $R(0, 1)[1, 0]$ and $R(1, 0)[0, 1]$. As a consequence, the local matrix function $R$ can be represented as

$$R = \begin{pmatrix}
a & 0 & 0 & 0 \\
0 & b & c & 0 \\
0 & d & e & 0 \\
0 & 0 & 0 & f
\end{pmatrix},$$

for some $a, b, c, d, e, f \in \mathbb{C}$.

**Remark 1.** Although this notion is defined as a function having matrix values, this notion is usually designated by the term of $R$-matrix, which refers more precisely to the representation of the local matrix function.

**Definition 3.** A **north-east local matrix** is a matrix $Q \in M_4(\mathbb{C})$ seen as the matrix of a linear operation from $\mathbb{C}^2 \otimes \mathbb{C}^2$ to itself, such that for all $t, u, v, w \in \{0, 1\}$, if there is no symbol of the discrete curves shift $X_s$
such that there is a curve crossing the west (resp. south, north, east) boundary if and only if \(u\) (resp. \(t, w, v\))
equals 1 (see an illustration on Figure 5), then
\[
Q[(t,u),(v,w)] = 0.
\]

The matrix \(Q\) is then equal to:
\[
\begin{pmatrix}
Q[(0,0),(0,0)] & Q[(0,0),(0,1)] & Q[(0,0),(1,0)] & Q[(0,0),(1,1)] \\
Q[(0,1),(0,0)] & Q[(0,1),(0,1)] & Q[(0,1),(1,0)] & Q[(0,1),(1,1)] \\
Q[(1,0),(0,0)] & Q[(1,0),(0,1)] & Q[(1,0),(1,0)] & Q[(1,0),(1,1)] \\
Q[(1,1),(0,0)] & Q[(1,1),(0,1)] & Q[(1,1),(1,0)] & Q[(1,1),(1,1)]
\end{pmatrix}.
\]

This means that the columns are indexed, from left to right (resp. from top to bottom) by the sequences 
\((0,0),(0,1),(1,0)\) and \((1,1)\).

![Figure 5: Definition of non-zero coefficients in a north east local matrix; the dashed segments designate possible curve crossing the border of the symbol.](image)

Property 1. From a local matrix function \(R\), it is possible to derive a north east local matrix \(Q\) as follows: for all \(t, u, v, w\),
\[
Q[(t,u),(v,w)] = R(t,w)[u,v].
\]

This association is bijective. As a consequence, if \(R\) is represented as
\[
R = \begin{pmatrix}
a & 0 & 0 & 0 \\
0 & b & c & 0 \\
0 & d & e & 0 \\
0 & 0 & 0 & f
\end{pmatrix},
\]
then \(Q\) is equal to :
\[
\begin{pmatrix}
R(0,0)[0,0] & R(0,1)[0,0] & R(0,0)[0,1] & R(0,1)[0,1] \\
R(0,0)[0,1] & R(0,1)[0,1] & R(0,0)[1,1] & R(0,1)[1,1] \\
R(1,0)[0,0] & R(1,1)[0,0] & R(1,0)[0,1] & R(1,1)[0,1] \\
R(1,0)[1,0] & R(1,1)[1,0] & R(1,0)[1,1] & R(1,1)[1,1]
\end{pmatrix} = \begin{pmatrix}
a & 0 & 0 & 0 \\
0 & c & b & 0 \\
0 & e & d & 0 \\
0 & 0 & 0 & f
\end{pmatrix},
\]

4.2 Derivation of a transfer matrix from a local matrix function

Let \(R\) be a local matrix function of square ice.

Definition 4. For all \(N\), the \(N\)th \textbf{monodromy matrix} relative to \(u, v \in \{0,1\}^N\) is the matrix \(M_N(u,v) \in \mathcal{M}_2(\mathbb{C})\) such that
\[
M_N(u,v) = \left( \prod_{k=1}^{N} R(u_k, v_k) \right).
\]

meaning that for all \((w, w') \in \{0,1\}^2\)
\[
M_N(u,v)[w, w'] = \sum_{w_2 \in \{0,1\}} \ldots \sum_{w_{N+1} \in \{0,1\}} \prod_{k=1}^{N} R(u_k, v_k)[w_k, w_{k+1}].
\]
where we denote \( w_1 = w \) and \( w_{N+1} = w' \). See an illustration on Figure 6. The matrix \( M_N(u,v) \) is thus represented by:

\[
M_N(u,v) = \begin{pmatrix}
M_N(u,v)[0,0] & M_N(u,v)[0,1] \\
M_N(u,v)[1,0] & M_N(u,v)[1,1]
\end{pmatrix}.
\]

This representation is usually called the Lax matrix.

![Figure 6: Illustration of the definition of the monodromy matrix relative to \( u,v \) defined from a local matrix. The coefficient \( w,w' \) of the matrix is the sum over all possible compositions of local matrices, from the leftmost square to the rightmost one.](image)

**Definition 5.** For all \( N \), the \( N \)th transfer matrix associated to the local matrix function \( R \) is the matrix \( T_N \in M_{2^N}(\mathbb{C}) \), thought as a matrix of an operator on \( \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 \), such that for all \( u,v \in \{0,1\}^*_N \),

\[
T_N[u,v] = \text{Tr}(M_N(u,v)).
\]

**Remark 2.** The fact that the entropy of \( X_N^* \) is equal to the entropy of \( \overline{X}_N \) is involved in this definition through the fact that the transfer matrix is defined as the trace of the monodromy matrix, which is crucial in the proof of the commutation criterion.

**Example 1.** For instance, the transfer matrix \( V_N(t) \) is obtained from the local matrix function represented as:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & t & 0 \\
0 & t & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

### 4.3 Yang-Baxter equation, commutation criterion for transfer matrices

For two matrices \( M,M' \in M_2(\mathbb{C}) \), we denote

\[
M \otimes M' = \begin{pmatrix}
M[0,0]M' & M[0,1]M' \\
M[1,0]M' & M[1,1]M'
\end{pmatrix}
\]

the Kronecker product of \( M \) and \( M' \). We also denote \( \overline{M} = \begin{pmatrix} M & 0 \\
0 & M \end{pmatrix} \). We use similar notations for matrices in \( M_n(\mathbb{C}) \) for all \( n \geq 1 \).

**Notation 9.** Let us consider \( R \) and \( R' \) two local matrix functions. For all \( t,v \in \{0,1\} \), we denote

\[
R' \circ R(t,v) = \sum_{u \in \{0,1\}} R(t,u) \otimes R'(u,v),
\]

which can be depicted as on Figure 7.

**Notation 10.** For two local matrix functions \( R,R' \), we denote \( M_N \otimes M'_N(t,v) \) the matrix in \( M_4(\mathbb{C}) \) such that:

\[
M_N \otimes M'_N(t,v) = \prod_{k=1}^{N} R' \circ R(t_k,v_k).
\]

where \( M_N \) and \( M'_N \) are the respective \( N \)th monodromy matrices of \( R \) and \( R' \).

**Lemma 1.** Let \( R \) and \( R' \) be two local matrix functions and \( T_N, T'_N \) their respective transfer matrices. Then, for all \( t,v \in \{0,1\}^*_N \):

\[
(T_N, T'_N)[t,v] = \text{Tr}(M_N \otimes M'_N(t,v)).
\]
Proof. From the expressions of the monodromy matrices, for all $t, v \in \{0,1\}^*_N$,

$$(T_N, T'_N)[t, v] = \sum_u \text{Tr}(M_N(t, u))\text{Tr}(M_N(u, v))$$

$$= \sum_u \sum_{w_1=w_{N+1}}^{w_1'=w_{N+1}} \sum_{w_2 \in \{0,1\}} \cdots \sum_{w_N \in \{0,1\}} \prod_{k=1}^{N} R(t_k, u_k)[w_k, w_{k+1}] \cdot R'(u_k, v_k)[w'_k, w'_{k+1}]$$

$$= \sum_{w_1=w_{N+1}}^{w_1'=w_{N+1}} \sum_{w_2 \in \{0,1\}} \cdots \sum_{w_N \in \{0,1\}} \left( \sum_{u} \prod_{k=1}^{N} R(t_k, u_k)[w_k, w_{k+1}] \cdot R'(u_k, v_k)[w'_k, w'_{k+1}] \right)$$

$$= \sum_{w_1=w_{N+1}}^{w_1'=w_{N+1}} \sum_{w_2 \in \{0,1\}} \cdots \sum_{w_N \in \{0,1\}} \left( \sum_{u} \prod_{k=1}^{N} R(t_k, u_k) \otimes R'(u_k, v_k) ([w_k, w_{k+1}], (w'_k, w'_{k+1})) \right)$$

$$= \text{Tr}(M_N \otimes M'_N(t, v)).$$

\[\square\]

![Graphical representation of the matrix $R' \circ R(t, v)$](image7)

![Graphical representation of the Yang-Baxter equation for the triple $(R, R', Q)$.](image8)

**Definition 6.** Consider $Q$ some north east local function. The triple $(R, R', Q)$ verifies the Yang-Baxter equation when for all $t, v \in \{0,1\}$, the matrices $R' \circ R(t, v)$ and $R \circ R'(v, t)$ are conjugated by $Q$:

$$R' \circ R(t, v) = Q \cdot R \circ R'(v, t).$$

This relation is illustrated on Figure 8.

**Remark 3.** The Yang-Baxter equation is also often called star-triangle equation, due to the shapes of diagrams that represent this equation under some simplifications (See for instance Figure 9).

**Lemma 2.** If there exists $Q$ an invertible north east local matrix function such that $(R, R', Q)$ verifies the Yang-Baxter equation, then for all $N$:

1. The $N$th monodromy matrices $M_N$ and $M'_N$ of $R$ and $R'$ verify:
   $$M_N \otimes M'_N = Q M'_N \otimes M_N Q^{-1}.$$
2. As a direct consequence the Nth transfer matrices associated to R and R' commute.

Proof. 1. Since \((R, R', Q)\) verifies Yang-Baxter equation, for all \(t, v\):

\[
M_N \otimes M'_N(t, v) = \prod_{k=1}^N Q.R \circ R'(v_k, t_k).Q^{-1}
\]

\[
= Q \left( \prod_{k=1}^N R \circ R'(v_k, t_k) \right) Q^{-1}
\]

This can be re-written directly

\[
M_N \otimes M'_N = QM'_N \otimes M_N Q^{-1}
\]

2. We have the following equalities, in virtue of Lemma 1:

\[
(T_N T'_N)[t, v] = Tr(M_N \otimes M'_N(t, v)),
\]

\[
(T'_N T_N)_{t, v} = Tr(M'_N \otimes M_N(t, v)).
\]

We deduce directly that

\[
T_N T'_N[t, v] = T'_N T_N[t, v].
\]

\[\square\]

4.4 Yang-Baxter paths

In the following, we will search for paths of local matrix function \(x \mapsto R^x\) such that for all \(x, y\), there exists some invertible matrix \(Q\) such that \((R^x, R^y, Q)\) verifies Yang-Baxter equation, with an additional constraint of rotational symmetry, which means that \(Q\) corresponds to some \(R^z\).

**Definition 7.** A Yang-Baxter path of local matrix functions is an analytic function \(x \mapsto R^x\) from \(\mathbb{R}\) to \(\mathcal{M}_2(\mathbb{C})\), such that there exists some analytic function \(\delta : \mathbb{R}^2 \mapsto \mathbb{R}\) for which for all \(x, y\), \((R^x, R^y, Q^{\delta(x,y)})\) verifies Yang-Baxter equation, where \(Q^z\) denotes the north east local matrix that corresponds to \(R^z\).

The Yang-Baxter path of \(N\)th transfer matrices corresponding to \(x \mapsto R^x\) is the analytic function \(x \mapsto T^x_N\), where for all \(x\), \(T^x_N\) is the transfer matrix constructed from \(R^x\).

For all \(u, v, w, w' \in \{0, 1\}\), we denote \(a[u, v, w, w'] \in \mathcal{A}\) the symbol in the alphabet of the six vertex model whose south (resp. north, west, east) arrow is oriented towards north (resp. north, east, west) when \(u = 0\) (resp. \(v = 0, w = 0, w' = 0\)), else towards south (resp. south, west, east). For instance, \(a[0, 1, 1, 0]\) is the symbol:

\[\square\]

**Definition 8.** To a path of local matrix function \(x \mapsto R^x\) we associate (in a bijective way) a **coefficient attribution function** \(\xi : \mathbb{R} \times \mathcal{A} \mapsto \mathbb{C}\) such that for all \(u, v, w, w' \in \{0, 1\}\):

\[
\xi(x, a[u, v, w, w']) = R^z(u, v)[w, w'].
\]

The number \(\xi(x, a)\) for \(x \in \mathbb{R}\) and \(a \in \mathcal{A}\) is the coefficient of a relative to \(x\).

For a given Yang-Baxter path \(x \mapsto R^x\), one can represent the Yang-Baxter equation for \((R^x, R^y, Q^z)\), where \(z \equiv \epsilon(x, y)\) in a geometrical way, as on Figure 9.

4.5 Strongly symmetric solutions of Yang-Baxter equation

In this section, we present a criterion based on a symmetry condition on \(R, R'\) that allows to ensure the existence of some \(Q\) (not necessarily invertible) such that \((R, R', Q)\) verifies Yang-Baxter equation. Implicitely, this criterion relies also on the symmetries verified by the model.
Definition 9. Let $R$ be a local matrix function represented as in Fact 1. This function is said to be strongly symmetric when $e = b$, $f = a$, $c = d$ and $ab \neq 0$. In this case, we denote
\[
\delta(R) = \frac{a^2 + b^2 - c^2}{2ab}.
\]

Let $R', R''$ be two other strongly symmetric local matrix functions obtained respectively by replacing $a, b, c$ by $a', b', c'$ and $a'', b'', c''$. We denote $(S)_{R, R', R''}$ the system of equations:
\[
(S)_{R, R', R''} : \begin{cases}
  a'c'a'' &= b'c'b'' + c'a'' \\
  a'b'c'' &= b'ac'' + cc'b'' \\
  c'ba'' &= c'^{ab''} + b'c'c''
\end{cases}
\]

Lemma 3. Let $R, R'$ and $R''$ three strongly symmetric local matrix functions and $Q$ the north east local matrix function corresponding to $R''$. Then $(R, R', Q)$ verifies Yang-Baxter equation if and only if $(S)_{R, R', R''}$ and $(S)_{R, R', R''}$ are verified.

Proof. Expression of the matrices involved:

Let us denote
\[
R = \begin{pmatrix}
  a & 0 & 0 & 0 \\
  0 & b & c & 0 \\
  0 & c & b & 0 \\
  0 & 0 & 0 & a
\end{pmatrix}, \quad R' = \begin{pmatrix}
  a' & 0 & 0 & 0 \\
  0 & b' & c' & 0 \\
  0 & c' & b' & 0 \\
  0 & 0 & 0 & a'
\end{pmatrix},
\]

Let us consider $Q$ a north east local matrix function represented as:
\[
Q = \begin{pmatrix}
  a'' & 0 & 0 & 0 \\
  0 & c'' & b'' & 0 \\
  0 & b'' & c'' & 0 \\
  0 & 0 & 0 & a''
\end{pmatrix}.
\]

One can represent $R' \circ R$ by a matrix in $\mathcal{M}_8(\mathbb{C})$: 
\[ R' \circ R = \begin{pmatrix}
R(0,0) \otimes R'(0,0) + R(0,1) \otimes R'(1,0) & R(0,0) \otimes R'(0,1) + R(0,1) \otimes R'(1,1)
R(1,0) \otimes R'(0,0) + R(1,1) \otimes R'(1,0) & R(1,0) \otimes R'(0,1) + R(1,1) \otimes R'(1,1)
\end{pmatrix}. \]

After direct computation:

\[
R' \circ R = \begin{pmatrix}
aa' & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & ab' & 0 & 0 & 0 & 0 & 0 \\
0 & cc' & ba' & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & bb' & 0 & 0 & 0 \\
0 & 0 & 0 & cb' & 0 & 0 & 0 \\
0 & 0 & 0 & ac' & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & aa'
\end{pmatrix}.
\]

We obtain \( R \circ R' \) by exchanging \( a \) (resp. \( b,c \)) with \( a' \) (resp. \( b', c' \)).

- **Reduction of Yang-Baxter equation to \((S)_{R,R';R'}\):**

The Yang-Baxter equation on \((R,R',Q)\) tells that \( Q \cdot R' \circ R = R \circ R' \cdot Q \).

When writing down the equations obtained writing Yang-Baxter equation block by block, we obtain a lot of trivial equations.

For instance, the north west block equation is written:

\[
\begin{pmatrix}
a'' & 0 & 0 & 0 \\
0 & c'' & b'' & 0 \\
0 & b'' & c'' & 0 \\
0 & 0 & 0 & a''
\end{pmatrix}
\begin{pmatrix}
aa' & 0 & 0 & 0 \\
0 & ab' & 0 & 0 \\
0 & cc' & ba' & 0 \\
0 & 0 & 0 & bb'
\end{pmatrix}
= \begin{pmatrix}
aa' & 0 & 0 & 0 \\
0 & a'b & 0 & 0 \\
0 & cc' & b'a & 0 \\
0 & 0 & 0 & bb'
\end{pmatrix}
\begin{pmatrix}
a'' & 0 & 0 & 0 \\
0 & c'b'' & 0 & 0 \\
0 & b'' & 0 & 0 \\
0 & 0 & 0 & a''
\end{pmatrix},
\]

which is equivalent to:

\[
\begin{pmatrix}
aa'a'' & 0 & 0 & 0 \\
0 & aa'c'' + cc'b'' & ba'b'' & 0 \\
0 & b''ab' + c''cc' & c''ba' & 0 \\
0 & 0 & 0 & bb'a''
\end{pmatrix}
= \begin{pmatrix}
aa'a'' & 0 & 0 & 0 \\
0 & a'bc'' & a'bb'' & 0 \\
0 & b''ab' + c''cc' & cc'b'' & b'ac'' \\
0 & 0 & 0 & bb'a''
\end{pmatrix},
\]

which reduces to the equation \( ba'c'' = ab'c'' + cc'b'' \).

After simplification of the other equations, the Yang-Baxter equation reduces to:

\[
\begin{align*}
(S)_{R,R';R'} : & \quad a'c'' = b'c'' + c'ae'' \\
& \quad a'bc'' = b'ac'' + cc'b'' \\
& \quad c'ba'' = c'ab'' + b'cc''
\end{align*}
\]

**Lemma 4.** Let \( R \) and \( R' \) be two strongly symmetric local matrix functions such that \( \delta(R) = \delta(R') \). There exists \( Q \) associated to a strongly symmetric local matrix function such that \((R,R',Q)\) verifies the Yang-Baxter equation.

**Proof.** Let \( Q \) be a north east local matrix function and \( R'' \) the local matrix function corresponding to \( Q \). The system \((S)_{R,R';R'}\) has a solution in \( a''', b''', c''' \) if and only if the determinant

\[
\begin{vmatrix}
-a'c & b'c & c'a \\
0 & cc' & (b'a - ba') \\
-c'b & c'a & b'c
\end{vmatrix} = -a'c(b'c^2 - c'a(b'-b'a')) - c'b(b'c(b'a - ba') - c^2ac)
\]

is zero, meaning

\[
a'b'c'c^3 - a'b'c'ea^2 + abcc'a^2 + abcc'b^2 - a'b'c'eb^2 - abcc'^2 = 0
\]

As a consequence, factoring by \( 2aa'b'bb'cc' \), this is equivalent to

\[
cc'\left(\delta(R') - \delta(R)\right) = 0,
\]

which is true by hypothesis. As a consequence, the system \((S)_{R,R';R'}\) has a solution. \(\square\)
5 Algebraic Bethe ansatz

Let us consider some analytic function $x \mapsto R^x$ from $\mathbb{C}$ to $\mathcal{M}_4(\mathbb{C})$ whose images are strongly symmetric local matrix functions and for all $x \in \mathbb{C}$, denote:

$$R^x = \begin{pmatrix} a(x) & 0 & 0 & 0 \\ 0 & b(x) & c(x) & 0 \\ 0 & c(x) & b(x) & 0 \\ 0 & 0 & 0 & a(x) \end{pmatrix}.$$ 

Let us denote $Q^x$ the north east local matrix function associated to $R^x$:

$$Q^x = \begin{pmatrix} a(x) & 0 & 0 & 0 \\ 0 & c(x) & b(x) & 0 \\ 0 & b(x) & c(x) & 0 \\ 0 & 0 & 0 & a(x) \end{pmatrix}.$$ 

We assume that:

1. there exists $\delta : \mathbb{C}^2 \mapsto \mathbb{C}$ such that for all $x, y$, $(R^x, R^y, Q^{x-y})$ verifies Yang-Baxter equation
2. $b(\delta(x,y)) = 0 \iff y \in \mathbb{C}$.
3. for all $x, y, z \in \mathbb{C}$, $\delta(\delta(x,y), \delta(x,z)) = \delta(y,z)$.

Let us denote $\mathcal{D}$ the set $(x,y) \in \mathbb{C}^2$ such that $b(\delta(x,y)) \neq 0$. We also assume that for all $(x, y) \in \mathcal{D}$,

$$\frac{c(\delta(x,y))}{b(\delta(x,y))} = -\frac{c(\delta(y,x))}{b(\delta(y,x))}.$$

We describe in this section an ansatz (meaning a candidate eigenvector) for the eigenequation of the transfer matrix associated to any $R^x$. In this version of the Algebraic Bethe ansatz, we extracted the conditions on the parameters of the path of local matrix functions allowing the ansatz, in order to understand how the ansatz works in general, and in the perspective of extending it to other models. In Section 5.1, we derive some commutation relations on the components of the monodromy matrices that come from Yang-Baxter equation. In Section 5.2, we state the Algebraic Bethe ansatz for the path $x \mapsto R^x$ using these commutation relations. In the end, we apply this method to a particular trigonometric Yang-Baxter path [Section ??].

Let us recall that for all $x, y, z = \delta(x,y)$, Yang-Baxter equation is reduced to

$$\begin{cases} 
  a(x)c(y)a(z) = b(x)c(y)b(z) + c(x)a(y)c(z) \\
  a(x)b(y)c(z) = b(x)a(y)c(z) + c(x)c(y)b(z) \\
  c(x)b(y)a(z) = c(x)a(y)b(z) + b(x)c(y)c(z) 
\end{cases}$$

5.1 Commutation of Lax matrices components

Notation 11. We denote $M_N^x$ the $N$th monodromy matrix associated to $R^x$, represented as:

$$M_N^x = \begin{pmatrix} M_N^x(0,0) & M_N^x(0,1) \\ M_N^x(1,0) & M_N^x(1,1) \end{pmatrix} = \begin{pmatrix} A_N(x) & C_N(x) \\ B_N(x) & D_N(x) \end{pmatrix}.$$ 

Notation 12. Let us also denote for all $(x,y) \in \mathcal{D}$:

$$\lambda(x,y) = \frac{a(\delta(x,y))}{b(\delta(x,y))} \quad \text{and} \quad \mu(x,y) = -\frac{c(\delta(x,y))}{b(\delta(x,y))}.$$ 

Fact 2. The function $\mu : (x,y) \mapsto \mu(x,y)$ is antisymmetric: for all $x, y$, $\mu(x,y) = -\mu(y,x)$, by hypothesis on the antisymmetry of $(c/b) \circ \delta$.

Lemma 5. For all $x, y$ such that $b(\delta(x,y)) \neq 0$:

$$\begin{cases} 
  A_N(x)B_N(y) = \lambda(x,y)B_N(y)A_N(x) + \mu(x,y)B_N(x)A_N(y) \\
  D_N(x)B_N(y) = \lambda(x,y)B_N(y)D_N(x) + \mu(x,y)B_N(x)D_N(y) \\
  B_N(x)A_N(y) = \lambda(x,y)A_N(y)B_N(x) + \mu(x,y)A_N(x)B_N(y) 
\end{cases}$$

Moreover, for all $x, y$ such that $a(\delta(x,y)) \neq 0$:

$$\begin{cases} 
  B_N(x)B_N(y) = B_N(y)B_N(x) \\
  A_N(x)A_N(x) = A_N(y)A_N(x). 
\end{cases}$$
Lemma 6. For all \(x, y\),
\[
M_N^x \otimes M_N^y . Q^{\delta(x,y)} = Q^{\delta(x,y)} . M_N^x \otimes M_N^y.
\]

For all \(x, y\) the matrix \(M_N^x \otimes M_N^y\) is:
\[
\begin{pmatrix}
A_N(y)A_N(x) & A_N(y)C_N(x) & C_N(y)A_N(x) & C_N(y)C_N(x) \\
A_N(y)B_N(x) & A_N(y)D_N(x) & C_N(y)B_N(x) & C_N(y)D_N(x) \\
B_N(y)A_N(x) & B_N(y)C_N(x) & D_N(y)A_N(x) & D_N(y)C_N(x) \\
B_N(y)B_N(x) & B_N(y)D_N(x) & D_N(y)B_N(x) & D_N(y)D_N(x)
\end{pmatrix}
\]

1. Equation on the monodromy matrices:

We know by Lemma for all \(x, y\),
\[
M_N^x \otimes M_N^y . Q^{\delta(x,y)} = Q^{\delta(x,y)} . M_N^x \otimes M_N^y.
\]

For all \(x, y\) the matrix \(M_N^x \otimes M_N^y\) is:
\[
\begin{pmatrix}
(0,0) & (0,1) & (1,0) & (1,1)
\end{pmatrix}
\]

2. On coefficients ((1, 0), (0, 0)):

In particular, from the equality on the coefficient ((1, 0), (0, 0)) of this last equality:
\[
a(\delta(x, y)).B_N(y)A_N(x) = b(\delta(x, y)).A_N(x)B_N(y) + c(\delta(x, y)).B_N(x)A_N(y)
\]

By dividing by \(b(\delta(x, y))\), we get the first equation stated in the lemma.

3. Coefficients ((1, 1), (0, 1)) we obtain:
\[
c(\delta(x, y)).B_N(y)D_N(x) + b(\delta(x, y)).D_N(y)B_N(x) = a(\delta(x, y)).B_N(x)D_N(y).
\]

After exchanging \(x\) and \(y\):
\[
b(\delta(y, x)).D_N(x)B_N(y) = a(\delta(y, x)).B_N(y)D_N(x) - c(\delta(y, x)).B_N(x)D_N(y).
\]

We obtain the second equation stated in the lemma by dividing by \(b(\delta(x, y))\).

4. The third equality is obtained from coefficients ((0, 1), (0, 0)). The second system is obtained from coefficients ((1, 1), (0, 0)) and ((0, 0), (0, 0)).

As a consequence, we have the following statement:

**Lemma 6.** For all \(N\), \(1 \leq n \leq N\), \(x \in \mathbb{C}\) and a finite sequence \((x_1, ..., x_n) \in \mathbb{C}^n\) of distinct numbers and distinct from \(x\):
\[
A_N(x). \left( \prod_{k=1}^{n} B_N(x_k) \right) = \left( \prod_{k=1}^{n} \lambda(x, x_k)B_N(x_k) \right) . A_N(x)
\]
\[
+ B_N(x) \left( \sum_{j=1}^{n} \mu(x, x_j) . \left( \prod_{k \neq j} \lambda(x_j, x_k)B_N(x_k) \right) \right) . A_N(x_j)
\]

A similar equality is verified in which \(A\) is replaced by \(D\) and in each copy of \(\lambda\) and \(\mu\) the two arguments are exchanged. Another similar equality is obtained by exchanging only \(B\) and \(A\).

**Proof.**

- **Auxiliary equality derived from Yang-Baxter equation:**

For all \(x, y, z\) all distinct:
\[
\mu(x, z)\lambda(z, y) = \lambda(x, y)\mu(x, z) + \mu(x, y)\mu(y, z).
\]

This equation derives from the third equation of the system \((S)_{R, R', R''}\), where \((R, R', R'')\) is equal to \((R^{(z,x)}, R^{(z,y)}, R^{(\delta(z,x), \delta(z,y))}) = (R^{(z,x)}, R^{(z,y)}, R^{(x,y)})\):
\[
c(\delta(z, x))b(\delta(z, y))a(\delta(x, y)) = c(\delta(z, x))a(\delta(z, y))b(\delta(x, y)) + c(\delta(z, y))c(\delta(x, y))b(\delta(z, x))
\]

After dividing by \(b(\delta(z, y))b(\delta(z, x))b(\delta(z, x), \delta(z, y))\):
\[
\frac{a(\delta(x,y)) c(\delta(z,x))}{b(\delta(x,y)) b(\delta(z,x))} = \frac{c(\delta(z,x)) a(\delta(z,y))}{b(\delta(z,x)) b(\delta(z,y))} + \frac{c(\delta(z,y)) c(\delta(x,y))}{b(\delta(z,y)) b(\delta(x,y))}
\]

This can be rewritten:

\[
\lambda(x,y)\mu(z,x) = \mu(z,x)\lambda(z,y) - \mu(x,y)\mu(z,y).
\]

We then use the antisymmetry of \(\mu\) and then multiply by \(-1\) to get:

\[
\lambda(x,y)\mu(z,x) = \mu(x,z)\lambda(x,y) + \mu(x,y)\mu(y,z).
\]

- **Proof of the statement by recursion:**
  The statement of the lemma for \(n = 1\) derives directly from Lemma [5]. Let us assume that it is verified for some \(n \geq 1\), and consider a real number \(x\) and a sequence \((x_1, ..., x_{n+1})\).

- **Application of recursion hypothesis on \(1, x, (x_1)\):**
  We first apply the equality for \(x, (x_1)\):

  \[
  A_N(x). \left( \prod_{k=1}^{n+1} B_N(x_k) \right) = \lambda(x, x_1) B_N(x_1). A_N(x). \left( \prod_{k=2}^{n+1} B_N(x_k) \right) + B_N(x) \mu(x, x_1). A_N(x_1). \left( \prod_{k=2}^{n+1} B_N(x_k) \right)
  \]

- **Application of recursion hypothesis on \(n\):**
  We apply then the equality on \(n, x, (x_2, ..., x_{n+1})\) and \(n, x_1, (x_2, ..., x_{n+1})\) and obtain:

  \[
  A_N(x). \left( \prod_{k=1}^{n+1} B_N(x_k) \right) = \lambda(x, x_1) B_N(x_1). \left( \prod_{k=2}^{n+1} \lambda(x, x_k) B_N(x_k) \right). A_N(x)
  \]

- **Application of the auxiliary equation:**
  We group the second and last terms of the second member of this equality, and use the auxiliary equality replacing \(x, y, z\) by \(x, x_1, x_j\):

  \[
  \lambda(x, x_1) \mu(x, x_j) + \mu(x, x_1) \mu(x_1, x_j) = \mu(x, x_j) \lambda(x_j, x_1)
  \]

  Thus we have:

  \[
  A_N(x). \left( \prod_{k=1}^{n+1} B_N(x_k) \right) = \lambda(x, x_1) B_N(x_1). \left( \prod_{k=2}^{n+1} \lambda(x, x_k) B_N(x_k) \right). A_N(x)
  \]

  \[
  + B_N(x_1). B_N(x). \left( \sum_{j=2}^{n+1} \mu(x, x_j) \lambda(x, x_1) \left( \prod_{k=2}^{n+1} \lambda(x, x_k) B_N(x_k) \right) A_N(x_j) \right)
  \]

  \[
  + B_N(x) \mu(x, x_1). \left( \prod_{k=2}^{n+1} \lambda(x, x_k) B_N(x_k) \right). A_N(x_1)
  \]

  We then group the two last terms in order to obtain the formula.
5.2 Statement of the ansatz

The algebraic Bethe ansatz consists in proposing eigenvectors in the space $\Omega_N$ for a transfer matrix $T_N(x)$, $x \in \mathbb{C}$, using the commutation relations on the components of the monodromy matrices proved in Section 5.1.

**Notation 13.** In the following, for all $N \geq 1$, we denote $\nu_N = (0...0)$ in the canonical basis of $\Omega_N$. This vector is often called vacuum state (since it is empty from 1 symbols).

Let us fix some $x \in \mathbb{C}$, some integer $n \leq N$ and a sequence $x = (x_1,...,x_n) \in \mathbb{C}^n$ such that none of the numbers $b(\delta(x_j,x_k)), b(\delta(x,x_j)), j,k \leq n$ is zero.

**Notation 14.** Let us denote $\psi_{n,N}(x)$ the vector:

$$\psi_{n,N}(x) = (B_N(x_1)...B_N(x_n)) \nu_N.$$

**Theorem 2** (Algebraic Bethe ansatz). Let us assume that for all $j$:

$$a(x_N) \prod_{k=1}^{n} \frac{a(\delta(x_j,x_k))}{b(\delta(x_j,x_k))} = b(x_N) \prod_{k=1}^{n} \frac{a(\delta(x_N,x_k))}{b(\delta(x_N,x_k))}.$$

Then $T_N(x) \psi_{n,N}(x) = \Lambda_{x,n,N}(x_1,...,x_n) \psi_{n,N}(x)$, where

$$\Lambda_{x,n,N}(x_1,...,x_n) = a(x_N) \prod_{k=1}^{n} \frac{a(\delta(x_N,x_k))}{b(\delta(x_N,x_k))} + b(x_N) \prod_{k=1}^{n} \frac{a(\delta(x_N,x_k))}{b(\delta(x_N,x_k))}.$$

**Remark 4.** The vector $\psi_{n,N}(x)$ is a candidate eigenvector, since we don’t know if this vector is zero or not.

**Proof.** We have, by definition of the transfer matrix, that:

$$T_N(x) \cdot \psi_{n,N}(x) = (A_N(x) + D_N(x)) \cdot \psi_{n,N}(x).$$

By Lemma 1 and since for all $z$, $A_N(z) \cdot \nu_N = a(z)^N \cdot \nu_N$ and $D_N(z) \cdot \nu_N = b(z)^N \cdot \nu_N$:

$$A_N(x) \cdot \psi_{n,N}(x) = a(x_N) \prod_{k=1}^{n} \lambda(x_N,x_k) \psi_{n,N}(x) + \left( \sum_{j} a(x_j)^N \mu(x_j,x_j) \prod_{k \neq j} \lambda(x_j,x_k) \right) \cdot B_N(x) \cdot \prod_{k \neq j} B_N(x_k) \cdot \nu_N.$$

Similarly we have:

$$D_N(x) \cdot \psi_{n,N}(x) = b(x_N) \prod_{k=1}^{n} \lambda(x_N,x_k) \psi_{n,N}(x) + \left( \sum_{j} b(x_j)^N \mu(x_j,x_j) \prod_{k \neq j} \lambda(x_N,x_k) \right) \cdot B_N(x) \cdot \prod_{k \neq j} B_N(x_k) \cdot \nu_N.$$

Since $\mu(x_j,x_j) = -\mu(x_j,x)$ (by antisymmetry of $(c/b) \circ \delta$), and from the definition of $\lambda$, we obtain that

$$T_N(x) \cdot \psi_{n,N}(x) = \Lambda_{x,n,N}(x_1,...,x_n) \cdot \psi_{n,N}(x).$$

\[
\square
\]

5.3 Expression of the candidate eigenvector

This part is the development of an idea that one can find in [Korepin et al.] (Appendix VII.2). The formula (A.2.4) that according to the authors, helped to derive the result of the coordinate Bethe ansatz from the algebraic one, was proved in this text for $N = 2$ (Formula 5.9).

**Notation 15.** For all $n$ integer, $x = (x_1,...,x_n)$ a sequence of complex numbers, and $j \leq n$, we denote $x^j$ the sequence obtained by suppressing the $j$th element:

$$x^j = (x_1,...,x_{j-1},x_{j+1},...,x_n).$$

For a sequence $\epsilon$ in $\{0,1\}^N$, we denote

$$\epsilon_\ast = (\epsilon_1,...,\epsilon_{N-1}).$$
Lemma 7. Let $N \geq 1$ be an integer, $1 \leq n \leq N$ and $(x_1, \ldots, x_n)$ a sequence of complex numbers with $n \leq N$. We have the following equality:

$$
\prod_{k=1}^{n+1} B_{N+1}(x_k) = \left( \prod_{k=1}^{n} B_N(x_k) \right) \otimes \left( \prod_{k=1}^{n} A_1(x_k) \right) + \sum_{j=1}^{n} \prod_{k \neq j} \lambda(x_k, x_j) \left( \prod_{k \neq j} B_N(x_k) D_N(x_j) \right) \otimes \left( B_1(x_j) \prod_{k \neq j} A_1(x_k) \right)
$$

Remark 5. Let us notice that the proof, although simply relying on commutation relations, is dependent upon the fact that for all $x, y$

$$
B_{N+1}(x) = B_N(x) \otimes A_1(x) + D_N(x) \otimes B_1(x).
$$

Proof. This statement is true for all $N$ and $(x_1, \ldots, x_n)$ when $n = 1$: this comes from the fact that for all $x$

$$
B_{N+1}(x) = B_N(x) \otimes A_1(x) + D_N(x) \otimes B_1(x)
$$

Let us assume the statement for some $n$, and prove it for $n + 1$. Let us consider some sequence $(x_1, \ldots, x_{n+1})$. Using the hypothesis for the sequence $(x_2, \ldots, x_{n+1})$, the expression

$$
B_{N+1}(x_1) = B_N(x_1) \otimes A_1(x_1) + D_N(x_1) \otimes B_1(x_1),
$$

and the fact that for all $x, y$, $B_1(x)B_1(y) = 0$, we get the following equality:

$$
\prod_{k=1}^{n+1} B_{N+1}(x_k) = \left( \prod_{k=1}^{n+1} B_N(x_k) \right) \otimes \left( \prod_{k=1}^{n+1} A_1(x_k) \right) + \sum_{j=2}^{n+1} \prod_{k \neq j} \lambda(x_k, x_j) \left( \prod_{k \neq j} B_N(x_k) D_N(x_j) \right) \otimes \left( A_1(x_1) B_1(x_j) \prod_{k \neq j} A_1(x_k) \right) + \left( D_N(x_1) \prod_{k=2}^{n+1} B_N(x_k) \right) \otimes \left( B_1(x_1) \prod_{k=2}^{n+1} A_1(x_k) \right)
$$

We know that:

$$
D_N(x_1) \prod_{k=2}^{n+1} B_N(x_k) = \prod_{k=2}^{n+1} \lambda(x_k, x_1) B_N(x_k) D_N(x_1)
$$

and

$$
A_1(x_1) B_1(x_j) = \lambda(x_1, x_j) B_1(x_j) A_1(x_1) + \mu(x_1, x_j) B_1(x_1) A_1(x_j).
$$

As a consequence of Lemma 5.

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The statement follows by antisymmetry of \( \mu \).

**Lemma 8.** For all \( N, 0 \leq n \leq N, \) and \( x = (x_1, ..., x_n) \in \mathbb{C}^n \):

\[
\psi_{n,N}(x) = \left( \prod_{k=1}^{n} c(x_k) a(x_k)^N \right) \sum_{\epsilon \in \Omega^{(n)}_N} \left( \sum_{\sigma \in \Sigma_n} \prod_{k=1}^{n} \left( \frac{b(x_k)}{a(x_k)} \right)^{q_{\sigma(\epsilon)}} \prod_{\sigma(k) < \sigma(j)} \lambda(x_k, x_j) \right) \epsilon
\]

**Proof.**

1. **A recursion equation deriving from recursion on monodromy matrices components:**

   Let us denote \( \epsilon \) the empty sequence, and set the convention \( \psi_{0,0}(\epsilon) \) is neutral for the tensor product (for all vector \( v, \psi_{0,0}(\epsilon) \otimes v = v \)). Since for all \( N \) and \( 1 \leq n \leq N \) and \( x = (x_1, ..., x_n) \),

   \[
   \psi_{n,N}(x) = B_N(x_1)...B_N(x_n) \nu_N,
   \]

   by application of Lemma 7 we have:

   \[
   \psi_{n,N+1}(x) = \left( \prod_{k=1}^{n} a(x_k) \right) \cdot \psi_{n,N}(x) \otimes |0\rangle
   + \sum_{j=1}^{n} b(x_j)^N c(x_j) \left( \prod_{k \neq j} a(x_k) \right) \left( \prod_{k \neq j} \lambda(x_k, x_j) \right) \psi_{n-1,N}(x'_j) \otimes |1\rangle,
   \]

   where we applied the equalities \( A_1(x_k), |0\rangle = a(x_k), |0\rangle, D_N(x_k) \nu_N = b(x_k)^N \nu_N \) and \( B_1(x_j), |0\rangle = b(x_j). |0\rangle \).

2. **Application to a recursion on coordinates according to some \( \epsilon \):**

   It is straightforward to see that \( \psi_{n,N}(x) \) lies in the subspace \( \Omega^{(n)}_N \), since for any \( x \in \mathbb{C} \), and \( \epsilon \in \Omega_N \), we have

   \[
   |B_N(x) \cdot \epsilon|_1 \leq |\epsilon|_1 + 1.
   \]

   As a consequence, in order to prove the lemma, it is sufficient to compute the coordinate of \( \psi_{n,N}(x) \) according to any \( \epsilon \in \Omega^{(n)}_N \). As a consequence of the first point, if \( \epsilon_{N+1} = 0 \), then:
$$\psi_{n,N+1}(x|\epsilon) = \left( \prod_{k=1}^{n} a(x_k) \right) \cdot \psi_{n,N}(x|\epsilon_*) \otimes |0\rangle$$

else, if $\epsilon_{N+1} = 1$:

$$\psi_{n,N+1}(x|\epsilon) = \sum_{j=1}^{N} b(x_j)^N c(x_j) \left( \prod_{k \neq j} a(x_k) \right) \left( \prod_{k \neq j} \lambda(x_k, x_j) \right) \psi_{n-1,N}(x_j|\epsilon_*) \otimes |1\rangle,$$

3. Expression of the coefficient of $\psi_{n,N}(x)$ relative to $\epsilon$:

As a direct consequence of the last point, this coefficient is:

$$\sum_{S} \prod_{k \in k_h=1} \left( b(x_{k_h}[S])^k c(x_{k_h}[S]) \prod_{j \in S_{k_h+1}} a(x_j) \prod_{j \in S_{k+1}} \lambda(x_j, x_{k_h}[S]) \prod_{l \in S_k} a(x_j) \right),$$

where the sum is over the sequences $S = (S_k)_{k \in \{1, N\}}$ of subsets of $\{1, ..., n\}$ such that $S_N = \emptyset$ and for all $k \leq N - 1$, $S_{k+1} \subset S_k$ and $|S_k \setminus S_{k+1}| = \epsilon_k$. In this formula, when $\epsilon_k = 1$, we denote $l_k[S]$ the element of $S_k \setminus S_{k+1}$.

4. Change of the variable $S$ into a permutation of $\Sigma_n$:

Moreover, a sequence $S = (S_k)_{k \in \{1, N\}}$ of subsets of $\{1, ..., n\}$ verifies the previous hypotheses if and only if there exists a permutation $\sigma \in \Sigma_n$ such that $S_{k+1} = S_k$ whenever $\epsilon_k = 0$ and for all $l \in \{1, ..., n\}$, $S_q[l] = S_q[l+1] \cup \{\sigma(l)\}$. For this permutation, for all $l \in \{1, ..., n\}$, $S_q[l] = \{\sigma(1), ..., \sigma(l)\}$.

As a consequence, the coefficient is equal to

$$\sum_{\sigma \in \Sigma_n} \prod_{l=1}^{n} \left( b(x_{\sigma(l)})^q[l] c(x_{\sigma(l)}) \prod_{j < l} a(x_{\sigma(j)}) \prod_{j < l} \lambda(x_{\sigma(j)}, x_{\sigma(l)}) \prod_{k \epsilon_k = 0} \left( \prod_{j \in S_k} a(x_j) \right) \right),$$

This can be rewritten:

$$\sum_{\sigma \in \Sigma_n} \prod_{l=1}^{n} \left( b(x_{\sigma(l)})^q[l] c(x_{\sigma(l)}) \prod_{j < l} \lambda(x_{\sigma(j)}, x_{\sigma(l)}) \prod_{k=1}^{N} \left( \prod_{j \in S_{k+1}} a(x_j) \right) \right),$$

where $S_{N+1} = \emptyset$. We rewrite again:

$$\sum_{\sigma \in \Sigma_n} \prod_{l=1}^{n} \left( b(x_{\sigma(l)})^q[l] c(x_{\sigma(l)}) \prod_{j < l} \lambda(x_{\sigma(j)}, x_{\sigma(l)}) \prod_{l=1}^{n} (a(x_{\sigma(l)}))^{N-q[l]} \right),$$

which yields the statement, by the change of variable $\sigma \mapsto \sigma^{-1}$.

6 Application to trigonometric Yang-Baxter paths

In this section, we choose particular Yang-Baxter paths for the Lieb path $t \mapsto V_N(t)$ defined in Section 2.2 corresponding to widely known trigonometric solutions of Yang-Baxter equation [Section 6.1]. We then prove Theorem 1 applying the algebraic Bethe ansatz to these Yang-Baxter paths [Section 6.2].

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6.1 Trigonometric Yang-Baxter path of commuting matrices

For all $\gamma \in (0, \pi)$ and $x \in \mathbb{C}$, consider the local matrix function $R_x^\gamma$ defined by:

$$R_x^\gamma = \frac{1}{\sin(\gamma/2)} \begin{pmatrix} \sin(\gamma - x) & 0 & 0 & 0 \\ 0 & \sin(x) & \sin(\gamma) & 0 \\ 0 & \sin(\gamma) & \sin(x) & 0 \\ 0 & 0 & 0 & \sin(\gamma - x) \end{pmatrix}$$

Let us denote $T_{\gamma,N}^x$ the corresponding transfer matrix and $M_{\gamma,N}^x$ the monodromy matrix. We denote $B_{\gamma,N}(x)$ the north east components of the corresponding monodromy matrices. Let us also denote $\delta(x,y)$, the candidate eigenvector provided by the algebraic Bethe ansatz for the path $x \mapsto T_{\gamma,N}^x$ at $\gamma/2$, provided a solution $(x_1, ..., x_n)$ of the system of Bethe equations, and $A_{\gamma,n,N}(x_1, ..., x_n)$ the corresponding candidate eigenvalue. Let us also denote $\delta_0 : (x, y) \mapsto y - x$.

Lemma 9. Let us fix some $\gamma \in \mathbb{R}\setminus\mathbb{Z}$. For all $x, y$, $T_{\gamma,N}^x$ and $T_{\gamma,N}^y$ commute.

Remark 6. These local matrix functions are obtained by a parameterization of

$$\{(a, b, c) \in \mathbb{C}^3 : a^2 + b^2 - c^2 - 2\Delta ab = 0\},$$

where $\Delta = -\cos(\gamma)$.

Proof. • Verification of some Yang-Baxter equation: For all $x, y$, the triple of matrices $(R_x^\gamma, R_y^\gamma, Q_{\gamma,N}^{\delta(x,y)})$ verifies the Yang-Baxter equation. Indeed, the system $(S)_{R_x^\gamma, R_y^\gamma, Q_{\gamma,N}^{\delta(x,y)}}$ is written:

$$\begin{cases}
\sin(\gamma - x) \sin(\gamma) \sin(\gamma - (y - x)) = \sin(x) \sin(y - x) \sin(\gamma) + \sin(\gamma)^2 \sin(\gamma - y) \\
\sin(\gamma - x) \sin(y) \sin(\gamma) = \sin(y) \sin(\gamma - y) \sin(\gamma) + \sin(\gamma)^2 \sin(y - x) \\
\sin(\gamma) \sin(y) \sin(\gamma - x) = \sin(\gamma) \sin(\gamma - y) \sin(y - x) + \sin(x) \sin(\gamma)^2 \sin(y - x)
\end{cases}$$

By factoring by $\sin(\gamma)$, and developing, one can check that this system is verified.

• Invertibility of the north east local matrix: When $y$ is not in the set

$$(\{x + \gamma + \pi\mathbb{Z}\} \cup \{\pi - \gamma + 2\pi\mathbb{Z}\},)$$

$Q_{\gamma,N}^{\delta - x}$ is invertible. As a consequence, the matrices $T_{\gamma,N}^x$ and $T_{\gamma,N}^y$ commute. In the other cases, we obtain this commutation by continuity.

For all $t$, we have:

$$R_{\mu t}^{\mu t/2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which means that the Yang-Baxter path of transfer matrices corresponding to $x \mapsto R_{\mu t}^x$ intersects the Lieb path at $x = \mu t/2$. For all $t, x$, we denote

$$R_{\mu t}^x \equiv \begin{pmatrix} a_t(x) & 0 & 0 & 0 \\ 0 & b_t(x) & c_t(x) & 0 \\ 0 & c_t(x) & b_t(x) & 0 \\ 0 & 0 & 0 & a_t(x) \end{pmatrix}.$$

We consider the Yang-Baxter path associated to the Lieb path $t \mapsto V_N(t)$ defined as

$$x \mapsto T_{\gamma,N}^x(t) = T_{\mu t,N}^x.$$

6.2 Proof of Theorem 1

In this section, we provide a proof of Theorem 1 which is a consequence of successive applications of Theorem 1, Theorem 2, and Theorem 3.
Proof. 1. Expressing terms in the second member with exponentials:
By definition of \( a_t \) and \( b_t \), for all \( j \):
\[
2i \sin(\mu t/2)a_t(x_j) = e^{i(\mu t - (\frac{\alpha_j}{2} + \frac{\alpha}{2}))} - e^{-i(\mu t - (\frac{\alpha_j}{2} + \frac{\alpha}{2}))}
\]
\[
= e^{\mu t} - e^{-\mu t} 
\]
\[
2i \sin(\mu t/2)b_t(x_j) = e^{i(\frac{\alpha_j}{2} + \frac{\alpha}{2})} - e^{-i(\frac{\alpha_j}{2} + \frac{\alpha}{2})} 
\]
\[
= e^{\mu t} - e^{-\mu t}.
\]

As a consequence, for all \( j, k \) such that \( j \neq k \):
\[
(2i)^2 \sin^2(\mu t/2)a_t(x_j)a_t(x_k) = e^{-i\mu t} - \frac{\alpha_j}{2} - \frac{\alpha}{2} + e^{i\mu t} + \frac{\alpha_j}{2} + \frac{\alpha}{2} - e^{-i\mu t} - \frac{\alpha_j}{2} - \frac{\alpha}{2} - e^{i\mu t} - \frac{\alpha_j}{2} - \frac{\alpha}{2} - e^{-i\mu t} + \frac{\alpha_j}{2} + \frac{\alpha}{2}
\]
\[
(2i)^2 \sin^2(\mu t/2)b_t(x_j)b_t(x_k) = e^{i\mu t} - \frac{\alpha_j}{2} - \frac{\alpha}{2} + e^{-i\mu t} + \frac{\alpha_j}{2} + \frac{\alpha}{2} - e^{i\mu t} - \frac{\alpha_j}{2} - \frac{\alpha}{2} - e^{-i\mu t} + \frac{\alpha_j}{2} + \frac{\alpha}{2}
\]
\[
(2i)^2 \sin^2(\mu t/2)a_t(x_j)b_t(x_k) = e^{i\mu t} - \frac{\alpha_j}{2} - \frac{\alpha}{2} + e^{-i\mu t} + \frac{\alpha_j}{2} + \frac{\alpha}{2} - e^{i\mu t} - \frac{\alpha_j}{2} - \frac{\alpha}{2} - e^{-i\mu t} + \frac{\alpha_j}{2} + \frac{\alpha}{2}
\]

2. Expression of the term in \( \Delta_t \):
Thus, since by definition of \( \mu t \) we have \(-2\Delta_t = e^{i\mu t} + e^{-i\mu t} \):
\[
-2\Delta_t a_t(x_j)b_t(x_k)(2i)^2 \sin^2(\mu t/2) = \frac{1}{2i\sin(\mu t/2)} \left( e^{2i\mu t} - \frac{\alpha_j}{2} - \frac{\alpha}{2} + e^{-2i\mu t} + \frac{\alpha_j}{2} + \frac{\alpha}{2} - e^{2i\mu t} + \frac{\alpha_j}{2} + \frac{\alpha}{2} - e^{-2i\mu t} - \frac{\alpha_j}{2} - \frac{\alpha}{2} \right)
\]

3. Expression of the second member:
The second member in the equality of the statement is thus
\[
\frac{1}{2i\sin(\mu t/2)} \left( e^{2i\mu t} - \frac{\alpha_j}{2} - \frac{\alpha}{2} + e^{-2i\mu t} + \frac{\alpha_j}{2} + \frac{\alpha}{2} - e^{2i\mu t} + \frac{\alpha_j}{2} + \frac{\alpha}{2} - e^{-2i\mu t} - \frac{\alpha_j}{2} - \frac{\alpha}{2} \right)
\]

4. Expression of the first member:
On the other hand:
\[
2i \sin(\mu t/2)a_t(x_k-x_j) = e^{i(\mu t - i(\frac{\alpha_k}{2} + \frac{\alpha}{2}))} - e^{-i(\mu t - i(\frac{\alpha_k}{2} + \frac{\alpha}{2}))}
\]
\[
= e^{\mu t} - e^{-\mu t} 
\]
\[
\]
Thus, using \(-2\Delta_t = e^{i\mu t} - e^{-i\mu t} \):
\[
-2\Delta_t \cdot 2i \sin(\mu t/2)a_t(x_k-x_j) = e^{2i\mu t} - \frac{\alpha_k}{2} - \frac{\alpha}{2} + e^{-2i\mu t} + \frac{\alpha_k}{2} + \frac{\alpha}{2} - e^{2i\mu t} + \frac{\alpha_k}{2} + \frac{\alpha}{2} - e^{-2i\mu t} - \frac{\alpha_k}{2} - \frac{\alpha}{2}
\]

This yields the statement of the lemma.

\[\square\]

Fact 3. For all \( j \in \{1, \ldots, n\} \), we have, by direct computation, that:
\[
a_t(x_j) = \frac{e^{\mu t} - e^{-\mu t} + \frac{\alpha_j}{2}}{e^{\mu t} - e^{-\mu t} + \frac{\alpha_j}{2}} = \frac{e^{\mu t} - e^{-\mu t} - \mu t}{e^{\mu t} - e^{-\mu t} - \mu t} = e^{-i\kappa_t(\alpha_j)} = e^{-i\eta_j} = z_j.
\]
Notation 17. Let us also denote $x_0 = \frac{4\pi}{T}$. We have directly that:

$$z_0 \equiv \frac{a_t(x_0)}{b_t(x_0)} = 1.$$ 

Theorem 3. Let $(p_1, ..., p_n) \in I^n$ such that $p_1 < ... < p_n$ and for all $j$, the equation $(E_j)[t, n, N]$ is verified. Then $(x_1, ..., x_n)$ verifies the equations

$$(E_j')[t, n, N]: \quad a_t(x_j)^N \prod_{k \neq j} a_t(x_k - x_j) b_t(x_k - x_j) = b_t(x_j)^N \prod_{k \neq j} a_t(x_j - x_k) b_t(x_j - x_k).$$

Proof. 1. Rewriting the equations $(E_j')[t, n, N]$:

By antisymmetry of $(a_t/b_t) \circ \delta_0$, and thus $b_t \circ \delta_0$, for all $j, k$ such that $j \neq k$:

$$\frac{a_t(x_k - x_j)b_t(x_j - x_k)}{a_t(x_j - x_k)b_t(x_k - x_j)} = \frac{a_t(x_k - x_j)}{a_t(x_j - x_k)}.$$

By application of Lemma 10 and Fact 3

$$\frac{a_t(x_k - x_j)b_t(x_j - x_k)}{a_t(x_j - x_k)b_t(x_k - x_j)} = -\frac{b_t(x_j)b_t(x_k) - 2\Delta_t a_t(x_j)b_t(x_k) + a_t(x_k)a_t(x_j)}{b_t(x_k)b_t(x_j) - 2\Delta_t a_t(x_k)b_t(x_j) + a_t(x_j)a_t(x_k)} = -\frac{1 - 2\Delta_t z_j + z_j z_k}{1 - 2\Delta_t z_k + z_j z_k}.$$

As a consequence, the equation $(E_j')[t, n, N]$ is equivalent to:

$$z_j^N = \prod_{k \neq j} \left( -\frac{1 - 2\Delta_t z_j + z_j z_k}{1 - 2\Delta_t z_k + z_j z_k} \right).$$

$$z_j^N = (-1)^{n+1} \prod_{k \neq j} \frac{z_k}{z_j} \left( \frac{1/z_j - 2\Delta_t + z_k}{1/z_k - 2\Delta_t + z_j} \right).$$

2. Rewriting Bethe equations:

On the other hand, the equation $(E_j)[t, n, N]$ implies, taking the exponential and then using the definition of $\Theta_t$:

$$e^{-iNp_j} = (-1)^{n+1} \cdot e^{-i\sum_{k=1}^n \Theta_t(-p_j, -p_k)}.$$

$$e^{-iNp_j} = (-1)^{n+1} \cdot \prod_{k \neq j} e^{-i(p_j - p_k)} \frac{e^{-ip_j} + e^{ip_k} - 2\Delta_t}{e^{-ip_k} + e^{ip_j} - 2\Delta_t}. $$

$$z_j^N = (-1)^{n+1} \cdot \prod_{k \neq j} \frac{z_j}{z_k} \left( \frac{1/z_j - 2\Delta_t + z_k}{1/z_k - 2\Delta_t + z_j} \right).$$

Theorem 4. Let $(p_1, ..., p_n) \in I^n$ such that $p_1 < ... < p_n$ and for all $j$, the equation $(E_j)[t, n, N]$ is verified. Then there exists some $\rho \in \mathbb{C}^*$ such that:

$$\psi_{t, n, N}(x_1, ..., x_n) = \rho \cdot \psi_{t, n, N}(p_1, ..., p_n).$$
Proof. By an application of Lemma \[3\] for all \( n \) and \( \epsilon \in \Omega^{(n)}_N \):

\[
\psi_{\mu, n, N}(x)[\epsilon] = \left( \prod_{k=1}^{n} c_t(x_k) a_t(x_k)^N \right) \left( \sum_{\sigma \in \Sigma_n} \prod_{k=1}^{n} \left( \frac{b_t(x_k)}{a_t(x_k)} \right)^{q_{\sigma(k)}[\epsilon]} \prod_{k<j} a_t(x_j - x_k) \right).
\]

By the change of variable \( \sigma \mapsto \sigma^{-1} \), and that for all \( k \), \( b_t(x_{\sigma(k)})/a_t(x_{\sigma(k)}) = 1/z_{\sigma(k)} \):

\[
\psi_{\mu, n, N}(x)[\epsilon] = \left( \prod_{k=1}^{n} c_t(x_k) a_t(x_k)^N \right) \left( \sum_{\sigma \in \Sigma_n} \prod_{k=1}^{n} \left( \frac{b_t(x_{\sigma(k)})}{a_t(x_{\sigma(k)})} \right)^{q_{\sigma(k)}[\epsilon]} \prod_{k<j} a_t(x_j - x_k) \right)
= \left( \prod_{k=1}^{n} c_t(x_k) a_t(x_k)^N \right) \left( \sum_{\sigma \in \Sigma_n} \prod_{k=1}^{n} e^{q_{\sigma(k)}[\epsilon]} \prod_{k<j} a_t(x_j - x_k) \right)
\]

Furthermore, by an application of Lemma \[10\]

\[
\frac{a_t(x_{\sigma(j)} - x_{\sigma(k)})}{b_t(x_{\sigma(j)} - x_{\sigma(k)})} = -\frac{b_t(x_{\sigma(j)}) b_t(x_{\sigma(k)})}{2\Delta_t \left(e^{(\alpha_{\sigma(j)} - \alpha_{\sigma(k)})/2} - e^{(\alpha_{\sigma(j)} - \alpha_{\sigma(k)})/2}\right)}
= \frac{1}{2\Delta_t} \left(\frac{1}{z_{\sigma(k)}} + \frac{1}{z_{\sigma(j)}} - 2\Delta_t\right).
\]

Moreover,

\[
\prod_{k<j} \left( -\frac{b_t(x_{\sigma(j)}) b_t(x_{\sigma(k)})}{2\Delta_t \left(e^{(\alpha_{\sigma(j)} - \alpha_{\sigma(k)})/2} - e^{(\alpha_{\sigma(j)} - \alpha_{\sigma(k)})/2}\right)} \right)
\]

is equal to

\[
\frac{\prod_{k=1}^{n} b_t(x_k)^{N-1}}{(2\Delta_t)^{(n-1)n/2}(-1)^{\left|\{(k,j): k<j, \sigma(k) > \sigma(j)\}\}\right|} \cdot \sqrt{\prod_{k \neq j} \left| e^{(\alpha_{k} - \alpha_{j})/2} - e^{(\alpha_{j} - \alpha_{k})/2}\right|^{N-1}}
\]

which is equal to

\[
\frac{\prod_{k=1}^{n} b_t(x_k)^{N-1}}{(2\Delta_t)^{(n-1)n/2} e(\sigma) \cdot \sqrt{\prod_{k \neq j} \left| e^{(\alpha_{k} - \alpha_{j})/2} - e^{(\alpha_{j} - \alpha_{k})/2}\right|^{N-1}}}
\]

We deduce, since for all \( t \), \( z_{\sigma(t)} = e^{-\rho t} \) that

\[
\psi_{\mu, n, N}(x)[\epsilon] = \rho \cdot \psi_{\sigma, n, N}(p_1, ..., p_n),
\]

where

\[
\rho = \frac{\prod_{k=1}^{n} c_t(x_k) a_t(x_k)^N b_t(x_k)^{N-1}}{(2\Delta_t)^{(n-1)n/2} e(\sigma) \cdot \sqrt{\prod_{k \neq j} \left| e^{(\alpha_{k} - \alpha_{j})/2} - e^{(\alpha_{j} - \alpha_{k})/2}\right|^{N-1}}}
\]

**Theorem 5.** Let \((p_1, ..., p_n) \in P^n_\sigma\) such that \( p_1 < ... < p_n \) and for all \( j \), the equation \((E_j)[t, n, N]\) is verified. Then we have the equality:

\[\Lambda_{n, N(t)}[p_1, ..., p_n] = \Lambda_{\mu, n, N}(x_1, ..., x_n).\]

**Proof.**

- When for all \( j \), \( p_j \neq 0 \):

1. Recall of the definition of \( \Lambda_{\mu, n, N}(x_1, ..., x_n) \):

\[
\Lambda_{\mu, n, N}(x_1, ..., x_n) = a_t(x_0)^N \prod_{k=1}^{n} \frac{a_t(x_k - x_0)}{b_t(x_k - x_0)} + b_t(x_0)^N \prod_{k=1}^{n} \frac{a_t(x_0 - x_k)}{b_t(x_0 - x_k)}.
\]
2. Factors in the first product expressed with $L_t$: for all $j \neq 0$,\
\[
\frac{a_t(x_j - x_0)}{b_t(x_j - x_0)} = L_t(z_j).
\]
Indeed, for all $j \neq 0$:
\[
\frac{a_t(x_j - x_0)}{b_t(x_j - x_0)} = \frac{a_t(i\frac{\alpha_j}{2})}{b_t(i\frac{\alpha_j}{2})} = \frac{e^{i\mu t + \frac{\alpha_j}{2}^2} - e^{-i\mu t - \frac{\alpha_j}{2}^2}}{e^{\frac{\alpha_j}{2}^2} - e^{-\frac{\alpha_j}{2}^2}}.
\]
On the other hand, since $z_i = a_t(x_i)/b_t(x_i)$:
\[
L_t(z_i) = 1 + \frac{t^2 z_i}{1 - z_i} = 1 + \frac{t^2 a_t(x_i)/b_t(x_i)}{1 - a_t(x_i)/b_t(x_i)} = \frac{t^2 a_t(x_i) + b_t(x_i) - a_t(x_i)}{b_t(x_i) - a_t(x_i)}.
\]
Since $\Delta_t = (2 - t^2)/2$, $t^2 = 2 - 2\Delta_t$:
\[
L_t(z_i) = \frac{b_t(x_i) + a_t(x_i) - 2\Delta_t a_t(x_i)}{b_t(x_i) - a_t(x_i)} - 2a_t(x_i)\Delta_t = (e^{i\mu t} + e^{-i\mu t})(e^{i\frac{\alpha_j}{2} + \frac{\alpha_j}{2}} - e^{-i\frac{\alpha_j}{2} - \frac{\alpha_j}{2}})
\]
\[
L_t(z_i) = \frac{e^{i\frac{\alpha_j}{2} + \frac{\alpha_j}{2}} - e^{-i\frac{\alpha_j}{2} - \frac{\alpha_j}{2}} + (e^{i\frac{\alpha_j}{2} + \frac{\alpha_j}{2}} - e^{-i\frac{\alpha_j}{2} - \frac{\alpha_j}{2}})}{e^{i\frac{\alpha_j}{2} - \frac{\alpha_j}{2}} - e^{-i\frac{\alpha_j}{2} + \frac{\alpha_j}{2}} - (e^{i\frac{\alpha_j}{2} + \frac{\alpha_j}{2}} - e^{-i\frac{\alpha_j}{2} - \frac{\alpha_j}{2}})} + \frac{e^{i\frac{\alpha_j}{2} - \frac{\alpha_j}{2}} - e^{-i\frac{\alpha_j}{2} + \frac{\alpha_j}{2}} + e^{i\frac{\alpha_j}{2} + \frac{\alpha_j}{2}} - e^{-i\frac{\alpha_j}{2} + \frac{\alpha_j}{2}} - e^{i\frac{\alpha_j}{2} - \frac{\alpha_j}{2}} - e^{-i\frac{\alpha_j}{2} - \frac{\alpha_j}{2}}}{e^{i\frac{\alpha_j}{2} - \frac{\alpha_j}{2}} - e^{-i\frac{\alpha_j}{2} + \frac{\alpha_j}{2}} - (e^{i\frac{\alpha_j}{2} + \frac{\alpha_j}{2}} - e^{-i\frac{\alpha_j}{2} - \frac{\alpha_j}{2}})}
\]
It is simplified into:
\[
L_t(z_i) = \frac{e^{i\mu t + \frac{\alpha_j}{2}} - e^{-i\mu t - \frac{\alpha_j}{2}} + (e^{i\mu t + \frac{\alpha_j}{2}} - e^{-i\mu t - \frac{\alpha_j}{2}})}{e^{\frac{\alpha_j}{2} - \frac{\alpha_j}{2}} - e^{-\frac{\alpha_j}{2} + \frac{\alpha_j}{2}} - (e^{\frac{\alpha_j}{2} + \frac{\alpha_j}{2}} - e^{-\frac{\alpha_j}{2} + \frac{\alpha_j}{2}})} = \frac{(e^{i\mu t - e^{-\frac{\alpha_j}{2}}} - e^{-i\mu t - e^{-\frac{\alpha_j}{2}}})}{e^{\frac{\alpha_j}{2} - \frac{\alpha_j}{2}} - e^{-\frac{\alpha_j}{2} + \frac{\alpha_j}{2}}} e^{i\mu t - \frac{\alpha_j}{2}} - e^{-i\mu t - \frac{\alpha_j}{2}}
\]
\[
= \frac{e^{i\mu t + \frac{\alpha_j}{2}} - e^{-i\mu t - \frac{\alpha_j}{2}}}{e^{\frac{\alpha_j}{2} - \frac{\alpha_j}{2}} - e^{-\frac{\alpha_j}{2} + \frac{\alpha_j}{2}}},
\]
3. Factors in the second product expressed with $M_t$:
Using similar arguments, we get to:
\[
\frac{a_t(x_0 - x_j)}{b_t(x_0 - x_j)} = M_t(z_i).
\]
4. Equality to $\Lambda_{n,N}(t)[p_1, ..., p_n]$:
As a consequence, since $a_t(x_0) = b_t(x_0) = 1$:
The candidate eigenvalue is then equal to
\[
\prod_{k=1}^{n} L_t(z_k) + \prod_{k=1}^{n} M_t(z_k) = \Lambda_{n,N}(t)[p_1, ..., p_n].
\]
\[\boxed{\text{• When there exists (a unique) } l \text{ such that } p_l = 0:\n1. Notations:}\]
Since \( p_l = 0 \), this means, since \( \kappa_l \) is increasing and antisymmetric, that \( \alpha_l = 0 \), and thus \( x_l = x_0 \). In order to prove the theorem, the strategy is to introduce a perturbation to \( x_0 \). Let us denote, for all \( \varepsilon > 0 \), \( x_0^{(\varepsilon)} = \frac{\mu}{2} + i \frac{\varepsilon}{2} \). Let us also denote

\[
\Lambda^{(\varepsilon)} = \frac{a_t(x_l - x_0^{(\varepsilon)})}{b_t(x_l - x_0^{(\varepsilon)})} \Lambda_0^{(\varepsilon)} + \frac{a_t(x_0^{(\varepsilon)} - x_l)}{b_t(x_0^{(\varepsilon)} - x_l)} \Lambda_1^{(\varepsilon)}
\]

where

\[
\Lambda_0^{(\varepsilon)} = \prod_{k=1}^{n} \frac{a_t(x_k - x_0^{(\varepsilon)})}{b_t(x_k - x_0^{(\varepsilon)})}
\]

and

\[
\Lambda_1^{(\varepsilon)} = \prod_{k=1}^{n} \frac{a_t(x_0^{(\varepsilon)} - x_k)}{b_t(x_0^{(\varepsilon)} - x_k)}.
\]

2. Perturbed eigenvalue \( \Lambda^{(\varepsilon)} \) is in the spectrum of \( T_{\mu_l,N}(x_0^{(\varepsilon)}) \):

By Theorem 2 for all \( \varepsilon > 0 \) sufficiently small,

\[
T_{\mu_l,N}(x_0^{(\varepsilon)}) \cdot \psi_{\mu_l,n,N}(x) = \Lambda^{(\varepsilon)} \cdot \psi_{\mu_l,n,N}(x)
\]

It is natural to expect that \( \Lambda^{(\varepsilon)} \) admits a limit when \( \varepsilon \) tends to zero. However, this computation is slightly more complex than in the first case (when all the \( p_j \) are non zero).

3. Developing the expression of \( \Lambda^{(\varepsilon)} \):

We have the following:

\[
\frac{a_t(-\varepsilon/2)}{b_t(-\varepsilon/2)} = \frac{e^{i\mu t - \varepsilon/2} - e^{-i\mu t + \varepsilon/2}}{e^{\varepsilon/2} - e^{-\varepsilon/2}} = -e^{it} + \frac{e^{i\mu t} - e^{-i\mu t}}{e^\varepsilon - 1}
\]

Similarly:

\[
\frac{a_t(\varepsilon/2)}{b_t(\varepsilon/2)} = -e^{it} - \frac{e^{i\mu t} - e^{-i\mu t}}{e^\varepsilon - 1}.
\]

As a consequence:

\[
\Lambda^{(\varepsilon)} = -e^{it} \left( \Lambda_0^{(\varepsilon)} + \Lambda_1^{(\varepsilon)} \right) + e^{it} \frac{e^{i\mu t} - e^{-i\mu t}}{e^\varepsilon - 1} \left( \Lambda_0^{(\varepsilon)} - \Lambda_1^{(\varepsilon)} \right).
\]

The only term whose limit is unknown is the third one, on which we will focus.

4. An expression of \( \Lambda_0^{(\varepsilon)}/\Lambda_1^{(\varepsilon)} \) using the function \( \Theta_l \):

Let us denote \( \chi \) the function

\[
\chi : \alpha \mapsto (n + 1)\pi + N\kappa_l(\alpha) + \sum_{k \neq l}^{n} \Theta_l(k_l(\alpha), l_p).
\]

By definition of \( \Theta_l \), antisymmetry of \( \kappa_l \) and Lemma 10.
5. First term of Taylor expansion:
We deduce that \( \frac{\Lambda_0^{(c)}}{\Lambda_1^{(c)}} = e^{-\chi_0^{(c)}} \).
When \( \epsilon = 0 \), by Bethe equations, \( \chi(\epsilon) = 0 \).

Since \( \epsilon \to 0 \), \( \Lambda_0^{(c)} \) converges towards \( \prod_{k \neq \ell} M(z_k) \), which is equal by Bethe equations to \( \prod_{k \neq \ell} L(z_k) \) - the limit of \( \Lambda_1^{(c)} \cdot \Lambda^{(c)} \) tends towards
\[
\Lambda_{n,N}(t)[p_1, ..., p_n] = -(e^{i\mu t} + e^{-i\mu t}) \cdot \prod_{k \neq \ell} M(z_k) - i(e^{i\mu t} - e^{-i\mu t}) \cdot \chi'(0) \cdot \prod_{k \neq \ell} M(z_k)
\]

Since \( \kappa'_0(0) = \frac{\sin(\mu)}{1 - \cos(\mu)} \), and using trigonometric equalities the equalities
\[
\Delta_t = -\cos(\mu_t) = (2 - t^2)/2,
\]
we have the following:
\[
\Lambda_{n,N}(t)[p_1, ..., p_n] = \left( -2 \cos(\mu_t) + 2 \cdot \frac{\sin^2(\mu_t)}{1 - \cos(\mu_t)} \cdot \left( N + \sum_{k \neq \ell} \frac{\partial \Theta_t}{\partial x}(0, p_k) \right) \right) \cdot \prod_{k \neq \ell} M(z_k)
\]
\[
\quad = \left( -2 \cos(\mu_t) + 2(1 + \cos(\mu_t)) \cdot \left( N + \sum_{k \neq \ell} \frac{\partial \Theta_t}{\partial x}(0, p_k) \right) \right) \prod_{k \neq \ell} M(z_k)
\]
\[
\quad = \left( 2 - t^2 \right) + t^2 \cdot \left( N + \sum_{k \neq \ell} \frac{\partial \Theta_t}{\partial x}(0, p_k) \right) \prod_{k \neq \ell} M(z_k)
\]
\[
\quad = \left( 2 + t^2(N - 1) + t^2 \sum_{k \neq \ell} \frac{\partial \Theta_t}{\partial x}(0, p_k) \right) \prod_{k \neq \ell} M(z_k)
\]
7 Commutation of the transfer matrix with some hamiltonian

The aim of this section is to prove that for all \( t \), the matrix \( V_N(t) \) commutes with a Heisenberg hamiltonian \( H_N(t) \) [Section 7.2], which is defined in Section 7.1. The proof relies on the paths of commuting matrices \( x \mapsto T^{x}_{\mu_x N} \). We also provide an expression of the eigenvalue of \( H_N(t) \) associated to the candidate eigenvector obtained by the algebraic Bethe ansatz.

7.1 Definitions

Let us recall that \( \Omega_N = C^2 \otimes \cdots \otimes C^2 \). In this section, for the purpose of notation, we identify \( \{1, ..., N\} \) with \( \mathbb{Z}/N\mathbb{Z} \).

**Notation 18.** Let us denote \( a \) and \( a^* \) the matrices in \( M_2(\mathbb{C}) \) equal to

\[
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

For all \( j \in \mathbb{Z}/N\mathbb{Z} \), we denote \( a_j \) (creation operator at position \( j \)) and \( a_j^* \) (anihilation operator at position \( j \)) the matrices in \( M_{2N}(\mathbb{C}) \) equal to

\[
a_j \equiv id \otimes \cdots \otimes a \otimes \cdots \otimes id, \quad a_j^* \equiv id \otimes \cdots \otimes a^* \otimes \cdots \otimes id.
\]

where \( id \) denotes the identity, and \( a \) acts on the \( j \)-th copy of \( \mathbb{C}^2 \).

In other words, the image of a vector \( |\epsilon_1 \ldots \epsilon_N\rangle \) in the basis of \( \Omega_N \) by \( a_j \) (resp. \( a_j^* \)) is as follows:

- if \( \epsilon_j = 0 \) (resp. \( \epsilon_j = 1 \)), then the image vector is \( 0 \);
- if \( \epsilon_j = 1 \) (resp. \( \epsilon_j = 0 \)), then the image vector is \( |\eta_1 \ldots \eta_N\rangle \) such that \( \eta_j = 0 \) (resp. \( \eta_j = 1 \)) and for all \( k \neq j \), \( \eta_k = \epsilon_k \).

**Remark 7.** The term creation (resp. anihilation) refer to the fact that for two elements \( \epsilon, \eta \) of the basis of \( \Omega_N \), \( a_j[\epsilon, \eta] \neq 0 \) (resp. \( a_j^*[\epsilon, \eta] \neq 0 \)) implies that \( |\eta_1| = |\eta| + 1 \) (resp. \( |\eta_1| = |\eta| - 1 \)). If we think of \( 1 \) symbols as particles, this operator acts by creating (resp. annihilating) a particle.

**Definition 10.** Let us denote \( H_N \) the matrix in \( M_{2N}(\mathbb{C}) \) defined as:

\[
H_N(t) = \sum_{j \in \mathbb{Z}/N\mathbb{Z}} (a_j^* a_{j+1} + a_j a_{j+1}^*) + D_t,
\]

where \( D_t \) is the diagonal matrix such that for all \( \epsilon \) in the canonical basis of \( \Omega_N \),

\[
D_t[\epsilon, \epsilon] = \Delta_t \cdot |\{ i \in \mathbb{Z}/N\mathbb{Z} : \epsilon_i = \epsilon_{i+1} \}|.
\]

**Remark 8.** Let us notice that the hamiltonian defined in Definition 10 differs by a multiple of the identity matrix from the hamiltonian defined in [Duminil-Copin et al] (equation (2.9)). Hence, the commutation property which is proved in the present text is equivalent to the commutation of \( V_N(t) \) with the Hamiltonian defined in [Duminil-Copin et al].

7.2 Proof of the commutation

In this section, we fix an integer \( N \geq 2 \) and a positive number \( t \).

**Notation 19.** In the following of this section, we denote \( R : x \mapsto R^x_{\mu} \) and \( T : x \mapsto T^{x}_{\mu_x N} \).

**Proposition 1.** We have the equality

\[
H_N(t) = t \sin(\mu t/2)T^t(0)T(0)^{-1}.
\]

Moreover, \( H_N(t) \) and \( V_N(t) \) commute:

\[
H_N(t) \cdot V_N(t) = V_N(t) \cdot H_N(t).
\]
Idea of the proof: Another proof of this proposition can be found in [Duminil-Copin et al.] (Lemma 5.1). The proof that we propose here uses the commutation of the transfer matrices of the Yang-Baxter path \(x \mapsto T_{\mu, N}^{x}\). From this commutation property, we obtain the commutation of the transfer matrix \(V_{N}(t)\) with \(T'(0)\) and \(T(0)\). Since \(H_{N}(t)\) is expressed with \(T'(0)\) and \(T(0)\), \(H_{N}(t)\) commutes with \(V_{N}(t)\). This expression of \(H_{N}(t)\) derives from the analysis of the action of the matrices \(T(0)\) and \(T'(0)\).

Proof. 1. The matrix \(T(0)\):

This transfer matrix is constructed from the local matrix function whose representation is:

\[
R^0_{\mu} = \frac{\sin(\mu t)}{\sin(\mu t/2)} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = t \cdot P,
\]

where we denote

\[
P \equiv \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Equivalently, this matrix has entry \(T(0)[\epsilon, \eta] = t^{N}\) when there is a \((N, 1)\)-cylindric pattern of \(X^*\) that connects \(\epsilon\) to \(\eta\) which does not contain the symbols \(\square\) and \(\square\); in other words, \(\eta\) is obtained from \(\epsilon\) by shifting all its symbols 1 by one position to the left. Otherwise, \(T(0)[\epsilon, \eta] = 0\).

2. The matrix \(T'(0)\):

The derivative of \(R : x \mapsto R_{\mu, t}^{x}\) is the function that to \(x\) associates:

\[
R'(x) = \frac{1}{\sin(\mu t/2)} \begin{pmatrix}
-\cos(\mu t - x) & 0 & 0 & 0 \\
0 & \cos(x) & 0 & 0 \\
0 & 0 & \cos(x) & 0 \\
0 & 0 & 0 & -\cos(\mu t - x)
\end{pmatrix}.
\]

In particular it has the following value in 0:

\[
R'(0) = \frac{1}{\sin(\mu t/2)} \begin{pmatrix}
\Delta_t & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \Delta_t
\end{pmatrix} = \begin{pmatrix}
R'(0)(0, 0) & R'(0)(0, 1) \\
R'(0)(1, 0) & R'(0)(1, 1)
\end{pmatrix}.
\]

Let us recall that for all \(\epsilon, \eta\), by definition of the transfer matrix from the monodromy matrix:

\[
T(x)[\epsilon, \eta] = \prod_{k=0}^{N-1} R_{\mu, t}^{x}(\epsilon_k, \eta_k)[0, 0] + \prod_{k=0}^{N-1} R_{\mu, t}^{x}(\epsilon_k, \eta_k)[1, 1].
\]

As a consequence,

\[
T'(0)[\epsilon, \eta] = \sum_{j=0}^{N-1} \left( \prod_{k=0}^{j-1} R_{\mu, t}^{0}(\epsilon_k, \eta_k) \right) R'(0)(\epsilon_j, \eta_j) \left( \prod_{k=j+1}^{N-1} R_{\mu, t}^{0}(\epsilon_k, \eta_k) \right)[0, 0]
\]

\[
+ \left( \prod_{k=0}^{j-1} R_{\mu, t}^{0}(\epsilon_k, \eta_k) \right) R'(0)(\epsilon_j, \eta_j) \left( \prod_{k=j+1}^{N-1} R_{\mu, t}^{0}(\epsilon_k, \eta_k) \right)[1, 1].
\]

With a similar interpretation as for \(T(0)\), the \(j\)th term of this sum can be different from 0 only if \(\eta\) is obtained from \(\epsilon\) by shifting any of its 1 symbols by one position to the left, except potentially for the \(j - 1\)th and \(j\)th positions. Considering these positions, the following cases are possible:

(a) if \(\epsilon_{j-1} = 1\) and \(\epsilon_j = 0\): \(\eta\) has to be obtained from \(\epsilon\) by shifting all the curves by one position to the left, except the one on position \(j\), which is shifted to position \(j+1\). In this case, the \(j\) term in the sum is \(t^{N-1} \frac{1}{\sin(\mu t/2)}\).
In this proof, we denote

\[ \psi \] is obtained from \( \epsilon \) by shifting any of its \( 1 \) symbols by one position to the left, and the \( j \)th term is equal to \( t^{N-1} \frac{\Delta_i}{\sin(\mu_i/2)} \).

(c) if \( \epsilon_{j-1} = 0 \) and \( \epsilon_j = 1 \): \( \psi \) is obtained from \( \epsilon \) by shifting any of its \( 1 \) symbols by one position to the left, except the one on position \( j \), which is fixed, and the \( j \)th term of the sum is equal to \( t^{N-1} \frac{1}{\sin(\mu_i/2)} \).

(d) if \( \epsilon_{j-1} = 1 \) and \( \epsilon_j = 1 \): \( \psi \) is obtained from \( \epsilon \) by shifting any of its \( 1 \) symbols by one position to the left, and the \( j \)th term is equal to \( t^{N-1} \frac{\Delta_i}{\sin(\mu_i/2)} \).

3. Expression of \( H_N(t) \) with \( T(0)^{-1} \) and \( T'(0) \):
From the definition of \( T(0) \), this matrix is invertible. Moreover, as a consequence of the last point, we have:

\[ H_N(t) = t \sin(\mu_t/2) \cdot T(0)^{-1} \cdot T'(0). \]

4. Commutation of \( H_N(t) \) with \( V_N(t) \):
Since for all \( x \), \( T(x) \) commutes with \( V_N(t) \) (by construction of the Yang-Baxter path):

\[ V_N(t) \cdot T(x) - T(0) = T(x) - T(0) \cdot V_N(t). \]

As a consequence, taking the limit \( x \to 0 \), \( V_N(t) \) commutes with \( T'(0) \), thus with \( T(0)^{-1} \cdot T'(0) \), and thus with \( H_N(t) \).

\[ \Box \]

Proposition 2. Let \( n \leq N \) and \( (p_1, \ldots, p_n) \in \mathbb{I}_n^N \) such that \( p_1 < \ldots < p_n \) and for all \( j \), the equation \((E_j)[t,n,N]\) is verified. Then:

\[ H_N(t) \cdot \psi_{t,n,N}(p_1, \ldots, p_n) = t \sin(\mu_t/2) \cdot \left( N \frac{\cos(\mu_t)}{\sin(\mu_t)} + 2 \sum_{j=1}^{n} (\cos(\mu_j) + \cos(p_j)) \right) \cdot \psi_{t,n,N}(p_1, \ldots, p_n). \]

Remark 9. In particular, when \( t = \sqrt{2} \), since \( \mu_t = \pi/2 \) we have:

\[ H_N(\sqrt{2}) \cdot \psi_{\sqrt{2},n,N}(p_1, \ldots, p_n) = \left( 2 \sum_{k=1}^{n} \cos(p_k) \right) \cdot \psi_{\sqrt{2},n,N}(p_1, \ldots, p_n), \]

used in the proof of the value of square ice entropy \([Gangloff]\).

Proof. In this proof, we denote \( \psi \equiv \psi_{t,n,N}(p_1, \ldots, p_n) \)

1. Eigenvalue of \( \psi \) for \( T(0) \):
By applying Theorem 2 to the trigonometric local matrix function \( R_{\mu_t}^n \) and \( x = (x_1, \ldots, x_n) \) such that for all \( j \), \( x_j = \frac{\mu_j}{\pi} + i \frac{\alpha_j}{\pi} \), with \( \alpha_j = \kappa_t^{-1}(p_j) \), for all \( x \),

\[ T(ix/2) \cdot \psi = \left( a_t(ix/2)^N \prod_{k=1}^{n} \frac{a_t(x_k - ix/2)}{b_t(x_k - ix/2)} + b_t(ix/2)^N \prod_{k=1}^{n} \frac{a_t(ix/2 - x_k)}{b_t(ix/2 - x_k)} \right) \cdot \psi. \]

As a consequence, we have:

\[ T(0) \cdot \psi_{t,n,N}(p_1, \ldots, p_n) = \sin(\mu_t)^N \cdot \left( \prod_{k=1}^{n} \frac{a_t(x_k)}{b_t(x_k)} \right) \cdot \psi_{t,n,N}(p_1, \ldots, p_n), \]

\[ = \sin(\mu_t)^N \cdot \left( \prod_{k=1}^{n} e^{-ip_k} \right) \cdot \psi_{t,n,N}(p_1, \ldots, p_n). \]

2. Eigenvalue of \( \psi \) for \( T'(0) \):
For all \( k \) and \( x \neq 0 \) sufficiently close to 0,

\[ \frac{a_t(x_k - ix/2)}{b_t(x_k - ix/2)} = \frac{e^{ix/2} - \frac{\alpha_k}{\pi} + \frac{\mu_t}{2} - \frac{\pi}{2} + \frac{\pi}{2} - \frac{\alpha_k}{\pi}}{e^{-ix/2} - \frac{\alpha_k}{\pi} + \frac{\mu_t}{2} - \frac{\pi}{2} - \frac{\pi}{2} - \frac{\alpha_k}{\pi}} = e^{i\kappa_t(x-x_k)}. \]
Deriving the equality of the first equality in the first point relatively to \( x \) and evaluating in 0 (let us notice that since \( N \geq 2 \), the derivative of the second sum in 0 is equal to zero), we obtain, using the antisymmetry of \( \kappa_t \) and thus the symmetry of \( \kappa_t' \):

\[
\frac{i}{2} T'(0) \cdot \psi = \left( \frac{i}{2} N \sin(\mu t) N^{-1} \cos(\mu t) \prod_{k=1}^{n} e^{-ip_k} + i \sum_{j=1}^{n} \kappa'_t(\alpha_j) \prod_{k=1}^{n} e^{-ip_k} \right) \cdot \psi.
\]

\[
T'(0) \cdot \psi = \left( N \sin(\mu t) N^{-1} \cos(\mu t) + 2 \sum_{j=1}^{n} \kappa'_t(\alpha_j) \prod_{k=1}^{n} e^{-ip_k} \right) \psi.
\]

**Eigenvalue of \( \psi \) for \( H_N(t) \):**

Since for all \( j \), \( \kappa'_t(\kappa_t^{-1}(p_j)) = (\cos(p_j) + \cos(\mu t))/\sin(\mu t) \), and by an application of Proposition 1:

\[
H_N(t) \psi = t \sin(\mu t/2) \cdot \left( \frac{N \cos(\mu t)}{\sin(\mu t)} + 2 \frac{1}{\sin(\mu t)} \sum_{j=1}^{n} (\cos(\mu t) + \cos(p_j)) \right) \cdot \psi.
\]

\[\square\]

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