CENTRAL LIMIT THEOREMS FOR RANDOM WALKS ASSOCIATED WITH HYPERGEOMETRIC FUNCTIONS OF TYPE BC

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Abstract. Consider the non-compact Grassmann manifolds $G/K$ over the fields $\mathbb{R}, \mathbb{C}, \mathbb{H}$ with rank $q \geq 1$ and dimension parameter $p > q$. The associated spherical functions are Heckman–Opdam hypergeometric functions of type $BC$, where the double coset spaces $G//K$ are identified with the Weyl chambers $C^B_q \subset \mathbb{R}^q$ of type $B$. The associated double coset hypergroups on $C^B_q$ can be embedded into a continuous family of commutative hypergroups $(C^B_q, \ast_p)$ with $p \in [2q - 1, \infty[$ associated with these hypergeometric functions by Rösler (2010). Several limit theorems for random walks on these hypergroups were recently derived by Voit (2017). We here present further limit theorems when the time as well as $p$ tend to $\infty$. For integers $p$, this admits interpretations for group-invariant random walks on the Grassmannians $G/K$.

1. Introduction. In this paper we present several limit theorems for group invariant random walks on the non-compact Grassmann manifolds $G/K$ over $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$. We state these results via the associated double coset spaces $G//K$, which can be identified with the Weyl chambers $C^B_q \subset \mathbb{R}^q$ of type $B$. The associated spherical functions, regarded as functions on $C^B_q$, are hypergeometric functions of type $BC$, and it turns out that the limit theorems can be derived for a larger class of Markov chains on $C^B_q$ whose transition probabilities are related to these functions beyond the group cases.

Let us recapitulate the general setting. The Heckman–Opdam hypergeometric functions associated with root systems generalize the theory of spherical functions on Riemannian symmetric spaces; see $[H_{m}, HS, O]$ for the general theory, and $[R2, RKV, RV, SI, S2, Sch, NPP]$ for recent developments. We here are interested in the type $BC$ as well as in the $A$-case as a limit; see $[RKV, RV]$.

We recall that for the root system $A_{q-1}$, $q \geq 2$, the hypergeometric functions are connected with the groups $G := \mathrm{GL}(q, F)$ with maximal com-

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pact subgroups $K := U(q, \mathbb{F})$. Moreover, for the root system $BC_q$, $q \geq 1$, the hypergeometric functions are related with the non-compact Grassmann manifolds $G/K$ with $p > q$, where depending on $\mathbb{F}$, the group $G$ is one of the indefinite orthogonal, unitary or symplectic groups $SO_0(q, p)$, $SU(q, p)$ or $Sp(q, p)$ with $K = SO(q) \times SO(p)$, $SU(U(q) \times U(p))$ or $Sp(q) \times Sp(p)$, as maximal compact subgroup.

In all cases, the $K$-spherical functions on $G$ (i.e., the non-trivial, $K$-biinvariant, multiplicative continuous functions on $G$) are non-trivial, multiplicative continuous functions on the double coset space $G//K$ where depending on $F$, the group $G$ is one of the indefinite orthogonal, unitary or symplectic groups $SO_0(q, p)$, $SU(q, p)$ or $Sp(q, p)$ with $K = SO(q) \times SO(p)$, $SU(U(q) \times U(p))$ or $Sp(q) \times Sp(p)$, as maximal compact subgroup.

In all cases, the $K$-spherical functions on $G$ (i.e., the non-trivial, $K$-biinvariant, multiplicative continuous functions on $G$) are non-trivial, multiplicative continuous functions on the double coset space $G//K$ where $G//K$ carries commutative double coset hypergroup structure. The $KAK$-decomposition of $G$ shows that $G//K$ may be identified with the Weyl chambers $C^A_q := \{ x = (x_1, \ldots, x_q) \in \mathbb{R}^q : x_1 \geq \cdots \geq x_q \}$ of type $A$, $C^B_q := \{ x = (x_1, \ldots, x_q) \in \mathbb{R}^q : x_1 \geq \cdots \geq x_q \geq 0 \}$ of type $B$ respectively. This identification is based on a exponential mapping $x \mapsto a_x \in G$ from the Weyl chamber to a system of representatives $a_x$ of the double cosets in $G$

\begin{equation}
\tag{1.1}
a_x := e^x
\end{equation}

for $x \in C^A_q$ in the $A$-case, and

\begin{equation}
\tag{1.2}
a_x := \begin{pmatrix}
\cosh x & \sinh x & 0 \\
\sinh x & \cosh x & 0 \\
0 & 0 & I_{p-q}
\end{pmatrix}
\end{equation}

for $x \in C^B_q$ in the $BC$-case with the diagonal matrices $e^x := \text{diag}(e^{x_1}, \ldots, e^{x_q})$, $\cosh x := \text{diag}(\cosh x_1, \ldots, \cosh x_q)$, $\sinh x := \text{diag}(\sinh x_1, \ldots, \sinh x_q)$.

We identify $G//K$ with $C^A_q$ or $C^B_q$ respectively and fix $q$ and $p > q$.

For the spherical functions we follow [HS] and denote the Heckman–Opdam hypergeometric functions associated with the root systems

\begin{align*}
2 \cdot A_{q-1} & = \{ \pm 2(e_i - e_j) : 1 \leq i < j \leq q \} \subset \mathbb{R}^q, \\
2 \cdot BC_q & = \{ \pm 2e_i, \pm 4e_i, \pm 2e_i \pm 2e_j : 1 \leq i < j \leq q \} \subset \mathbb{R}^q
\end{align*}

by $F_A(\lambda, k; t)$ and $F_{BC}(\lambda, k; x)$ respectively, with spectral variable $\lambda \in \mathbb{C}^q$ and multiplicity parameter(s) $k$. Here, $e_1, \ldots, e_q$ are the unit vectors in $\mathbb{R}^q$.

The factor 2 in both root systems comes from the known connections of Heckman–Opdam theory with spherical functions on symmetric spaces in [HS] and references there. In the $A_{q-1}$-case, the spherical functions on $G//K \simeq C^A_q$ are then

\begin{align*}
\varphi^A_\lambda(a_x) & := \varphi^A_\lambda(x) \\
& := e^{i \cdot (x - \pi(x), \lambda)} \cdot F_A(i\pi(\lambda), d/2; \pi(x)) \quad (x \in \mathbb{R}^q, \lambda \in \mathbb{C}^q)
\end{align*}
with multiplicity $k = d/2$ where $d := \dim_{\mathbb{R}} \mathbb{F} \in \{1, 2, 4\}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and where
\[
\pi : \mathbb{R}^q \rightarrow \mathbb{R}^q_0 := \{ t \in \mathbb{R}^q : x_1 + \cdots + x_q = 0 \}
\]
is the orthogonal projection with respect to the standard scalar product as in [RKV] (6.7) and $a_t$ is identified with $x$. In the $BC$-case, the spherical functions on $G//K \simeq C^B_q$ are given by
\[
\varphi^p_\lambda(a_x) := \varphi^p_\lambda(x) := F_{BC}(i\lambda, k_p; x) \quad (x \in \mathbb{R}^q, \lambda \in \mathbb{C}^q)
\]
with multiplicity
\[
k_p = (d(p - q)/2, (d - 1)/2, d/2) \in \mathbb{R}^3
\]
corresponding to the roots $\pm 2e_i, \pm 4e_i$ and $2(\pm e_i \pm e_j)$ where $a_x$ is identified with $x$.

In the $BC$-case, the associated double coset convolutions $*_{p}$ of measures on $C^B_q$ are written down explicitly in [R2] for $p \geq 2q$; these convolutions and the associated product formulas for the associated hypergeometric functions $F_{BC}$ above can be extended to $p \in [2q - 1, \infty[$ by analytic continuation. The convolutions $*_{p}$ on the space $\mathcal{M}(C^B_q)$ of all bounded regular Borel measures on $C^B_q$ are associative, commutative, and probability-preserving, and they generate commutative hypergroups $(C^B_q, *_{p})$ in the sense of Dunkl, Jewett, and Spector with $0 \in C^B_q$ as identity by [R2]. For hypergroups we refer to [J, BH]. The non-trivial multiplicative continuous functions of these commutative hypergroups $(C^B_q, *_{p})$ are precisely the functions $\varphi^p_\lambda$ with $\lambda \in \mathbb{C}^q$ by [R2]. This means that for all $x, y \in C^B_q$ and $\lambda \in \mathbb{C}^q$,
\[
\varphi^p_\lambda(x) \varphi^p_\lambda(y) = \int_{C^B_q} \varphi^p_\lambda(t) d(\delta_x *_{p} \delta_y)(t)
\]
where the measures $\delta_x *_{p} \delta_y \in \mathcal{M}^1(C^B_q)$ with compact support are given by
\[
(\delta_x *_{p} \delta_y)(f) = \frac{1}{\kappa_p} \int_{B_q} \int_{U(q, \mathbb{F})} f \left( \text{arcosh}(\sigma_{\text{sing}}(\sinh x^{-1}w \sinh y + \cosh x^{-1}v \cosh y)) \right) dv dm_p(w)
\]
for $f \in C(C^B_q)$. Here, $dv$ means integration with respect to the normalized Haar measure on $U(q, \mathbb{F})$, $B_q$ is the matrix ball $\{ w \in M_q(\mathbb{F}) : w^*w \leq I_q \}$, and $dm_p(w)$ is the probability measure
\[
dm_p(w) := \frac{1}{\kappa_p} \Delta(I - w^*w)^{d(p/2 + 1/2 - q) - 1} dw \in \mathcal{M}^1(B_q)
\]
with the Lebesgue measure $dw$ on $B_q$, and $\kappa_p > 0$ a normalization constant. For $p = 2q - 1$ there exists a degenerate formula with $m_p \in \mathcal{M}^1(B_q)$ singular (see [R1] Section 3]).
For fixed $p \in [2q-1, \infty]$ and $d = 1, 2, 4$ we now study random walks on the hypergroups $(C_q^B, \ast_p)$ as follows: Fix a probability measure $\nu \in \mathcal{M}^1(C_q^B)$, and consider a time-homogeneous Markov process $(S_k^p)_{k \geq 0}$ on $C_q^B$ with start at the hypergroup identity $0 \in C_q^B$ and with the transition probability

$$P(S_{k+1}^p \in A \mid S_k^p = x) = (\delta_x \ast_p \nu)(A) \quad (x \in C_q^B, A \subset C_q^B \text{ a Borel set}).$$

Such Markov processes are called random walks on the hypergroup $(C_q^B, \ast_p)$ associated with $\nu$. Notice that we here use $p$ as a superscript, as it may vary below. The fixed parameters $q$, $d$ are suppressed.

We shall mainly present two different types of CLTs for $(S_k^p)_{k \geq 0}$.

For the first type, in Section 5 we start with some probability measure $\nu$ having classical second moments. For each constant $c \in [0, 1]$ we consider the compression mapping $D_c(x) := cx$ on $C_q^B$ as well as the compressed probability measures $\nu_c := D_c(\nu)$ and the associated random walks $(S_k^{(p,c)})_{k \geq 0}$. We prove in Section 4 that $S_n^{(p,n^{-1/2})}$ converges in distribution as $n \to \infty$ to some “Gaussian” measure $\gamma_{t_0} \in \mathcal{M}^1(C_q^B)$ which depends on $p$ where the time $t_0 \geq 0$ can be computed via $\nu$. Triangular CLTs of this type are well-known in probability theory; see e.g. [BH] for some examples. Moreover, for integers $p \geq 2q$, this result is known for biinvariant random walks on non-compact Grassmannians; see [G1] [G2] [Te1] [Te2] [Ri].

For the second CLTs, in Section 4 we study the random walks $(S_k^p)_{k \geq 0}$ for a given $\nu \in \mathcal{M}^1(C_q^B)$ where the time $k$ as well as the dimension $p$ tend to infinity in a coupled way. It turns out that under suitable moment conditions on $\nu$ and for any sequence $(p_n)_n \subset [2q, \infty]$ with $p_n \to \infty$, there are normalizing vectors $m(n) \in \mathbb{R}^q$ such that $(S_n^{(p_n)} - m(n))/\sqrt{n}$ tends in distribution to some $q$-dimensional normal distribution $N(0, \Sigma^2)$ where the $m(n)$ and the covariance matrix $\Sigma^2$ are explicitly known and depend on $\nu$. For $q = 1$, such CLTs were given in [Gr1] [V1] by completely different methods. Both proofs for $q = 1$ however are based on the fact that for $p \to \infty$, the hypergroup structures $(C_1^B = [0, \infty[ \ast_p)$ tend to some commutative semigroup on $C_1^B = [0, \infty]$ which is isomorphic to $([0, \infty[, +)$. This observation finally shows that for large $p$, $(S_n^{(p_n)})_n$ behaves like a sum of iid random variables, which then leads to the CLT. For $q \geq 2$, the situation is much more involved, as here for $p \to \infty$ the hypergroups $(C_q^B, \ast_p)$ converge to the double coset structures $G//K$ in the case $A_{q-1}$, where the parameter $d = 1, 2, 4$ remains unchanged; see [RK, RV]. As a consequence, we need stronger conditions either on the moments of $\nu$ or on the rate of $p \to \infty$ than in [Gr1]; see Theorems 4.1, 4.4 below. The CLTs in [Gr1] [V1] and here for non-compact Grassmannians are related to other CLTs for radial random walks on Euclidean spaces of large dimensions in [Gr2]. Moreover, our CLTs for $p \to \infty$ are related to a CLT in the case $A_{q-1}$ in [V2] which uses the concept of
moment functions on commutative hypergroups from [BH, Z1]. In fact, we need these moment functions for the hypergroups \((C_B^q, \ast_p)\) and for their limits associated with the case \(A_{q-1}\). These moment functions are essential to describe the drift \(m(n)\) and covariance matrix \(\Sigma^2\) above. We collect several results on these functions now.

2. Modified moments. In general, examples of moment functions on a commutative hypergroup can be obtained as partial derivatives of the multiplicative functions of the hypergroup with respect to the spectral variables at the identity character; see [BH]. To obtain explicit formulas for moment functions for our particular examples on Weyl chambers, we start with explicit integral representations of the multiplicative functions in [RV] which follow from the Harish–Chandra integral representation of spherical functions.

We start with some notations from matrix analysis; we usually follow [HJ]. For a Hermitian matrix \(A = (a_{ij})_{i,j=1}^q\) over \(\mathbb{F}\) we denote by \(\Delta(A)\) the determinant of \(A\), and by \(\Delta_r(A) = \det((a_{ij})_{1 \leq i,j \leq r})\) the \(r\)th principal minor of \(A\) for \(r = 1, \ldots, q\). For \(\mathbb{F} = \mathbb{H}\), these determinants are taken in the sense of Dieudonné, i.e. \(\det(A) = (\det C(A))^{1/2}\), when \(A\) is considered as a complex matrix. For each positive Hermitian \(q \times q\)-matrix \(A\) and for \(\lambda \in \mathbb{C}^q\) we consider the power function

\[
\Delta_\lambda(A) := \Delta_1(A)^{\lambda_1} \cdots \Delta_{q-1}(A)^{\lambda_{q-1}} \cdot \Delta_q(A)^{\lambda_q}.
\]

We shall also need the singular values \(\sigma_1(a) \geq \cdots \geq \sigma_q(a)\) of a \(q \times q\)-matrix \(a\), which are ordered by size and which are the ordered eigenvalues of \(a^*a\). Finally, for \(x \in C_B^q\), \(u \in U_q(\mathbb{F})\), and \(w \in B_q\), we define

\[
g(x, u, w) := u^*(\cosh x + \sinh x \cdot w)(\cosh x + \sinh x \cdot w)^*u.
\]

We recapitulate the following facts; see [RV] Lemmas 4.10 and 4.8:

**Lemma 2.1.**

1. Consider the probability measures \(m_p\) from (1.6). Then for each \(n \in \mathbb{N}\) there exists a constant \(C := C(q, n, \mathbb{F})\) such that for all \(p \geq 2q\),

\[
\int_{B_q} \frac{\sigma_1(w)^{2n}}{\Delta(I - w^*w)^{2n}} \, dm_p(w) \leq \frac{C}{p^n}.
\]

2. Let \(x \in C_B^q\), \(w \in B_q\), \(u \in U(q, \mathbb{F})\) and \(r = 1, \ldots, q\). Then

\[
\frac{\Delta_r(g(x, u, w))}{\Delta_r(g(x, u, 0))} \in [(1 - \tilde{x}\sigma_1(w))^{2r}, (1 + \tilde{x}\sigma_1(w))^{2r}] \quad \text{with} \quad \tilde{x} := \min(x_1, 1).
\]

We now recall from [V2] the moment functions in the \(A\)- and \(BC\)-cases.
Definition 2.2. The spherical functions of type A in (1.3) satisfy
\begin{equation}
\varphi_A^A(x) = \int_{U(q, \mathbb{R}^n)} \Delta(i\lambda - i\rho^A)/2(u^{-1}e^{2x}u) \, du \quad (x \in C_q^A)
\end{equation}
with the half-sum of positive roots
\begin{equation}
\rho^A := (\rho_1^A, \ldots, \rho_q^A) \in C_q^A \quad \text{with} \quad \rho_l^A := \frac{d}{2}(q + 1 - 2l) \quad (l = 1, \ldots, q)
\end{equation}
(see [RV, Section 3]). Equations (2.4) in particular implies that \( \varphi_{-i\rho^A} \equiv 1 \), and that for \( \lambda \in \mathbb{R}^n \) and \( x \in C_q^A \), we have \( |\varphi_{\lambda - i\rho^A}(x)| \leq 1 \).

By [V2], for multiindices \( l = (l_1, \ldots, l_q) \in \mathbb{N}_0^q \) we define the moment functions
\begin{equation}
m_l^A(x) := \frac{\partial^{|l|}}{\partial \lambda^{|l|}} \varphi_{-i\rho^A - i\lambda}(x) \bigg|_{\lambda=0} = \frac{\partial^{|l|}}{(\partial \lambda_1)^{l_1} \ldots (\partial \lambda_q)^{l_q}} \varphi_{-i\rho^A - i\lambda}(x) \bigg|_{\lambda=0}
\end{equation}
of order \( |l| := l_1 + \cdots + l_q \) for \( t \in C_q^A \). Notice that the last equality in (2.6) follows from (2.4) by interchanging integration and derivatives. We denote the \( j \)th unit vector by \( e_j \in \mathbb{Z}_+^q \) and the moment functions of order 1 and 2 by \( m_{e_j} \) and \( m_{e_j + e_k} \) \((j, k = 1, \ldots, q)\). The \( q \) moment functions of first order lead to the vector-valued moment function
\begin{equation}
m_1^A(x) := (m_{e_1}^A(x), \ldots, m_{e_q}^A(x))
\end{equation}
of first order. Moreover, the moment functions of second order can be grouped into
\begin{equation}
m_2^A(x) := \begin{pmatrix}
m_{2e_1}^A(x) & \cdots & m_{e_1 + e_q}^A(x) \\
\vdots & \ddots & \vdots \\
m_{e_q + e_1}^A(x) & \cdots & m_{2e_q}^A(x)
\end{pmatrix} \quad \text{for} \ x \in C_q^A.
\end{equation}
We now form the \( q \times q \)-matrices \( \Sigma^A(x) := m_2^A(x) - m_1^A(x)^t \cdot m_1^A(x) \).

These moment functions have the following basic properties (see [V2, Section 2]):

Lemma 2.3.

1. There is a constant \( C = C(q) \) such that \( \|m_1^A(x) - x\| \leq C \) for all \( x \in C_q^A \).
2. For each \( t \in C_q^A \), \( \Sigma^A(x) \) is positive semidefinite.
3. For \( x = c \cdot (1, \ldots, 1) \in C_q^A \) with \( c \in \mathbb{R} \), \( \Sigma^A(x) = 0 \). For all other \( x \in C_q^A \), \( \Sigma^A(x) \) has rank \( q - 1 \).
(4) All second moment functions $m_{e_i+e_j}^A(x)$ are growing at most quadratically, and $m_{2e_1}^A(x)$ and $m_{2e_q}^A(x)$ are in fact growing quadratically.

(5) There exists a constant $C = C(p)$ such that for all $x \in C^A_q$ and $\lambda \in \mathbb{R}^q$,

$$|\varphi_{-i\rho}^A - \lambda(x) - e^{i(\lambda, m_1^A(x))}| \leq C||\lambda||^2.$$ 

Let $\nu \in M^1(C^A_q)$. For $k \in \mathbb{N}$ we say that $\nu$ admits $k$th moments of type $A$ if for all $l \in \mathbb{N}_0^q$ with $|l| \leq k$ we have $m_l^A \in L^1(C^A_q, \nu)$. We then call $m_l^A(\nu) := \int_{C^A_q} m_l^A(x) \, d\nu(x)$ the $l$th multivariate moment of $\nu$. The vector

$$m_1^A(\nu) := \int_{C^A_q} m_1^A(x) \, d\nu(x) \in C^A_q \subset \mathbb{R}^q$$

is called the dispersion of $\nu$. We form the modified symmetric $q \times q$-covariance matrix

$$\Sigma^A(\nu) := \int_G m_2^A(x) \, d\nu(x) - m_1^A(\nu)^t \cdot m_1^A(\nu).$$

We are interested in the $A$-case only as a limit of the $BC$-case for $p \to \infty$. For this we need an additional transformation

$$T : C^B_q \to C^B_q, \quad x = (x_1, \ldots, x_q) \mapsto \ln \cosh x := (\ln \cosh x_1, \ldots, \ln \cosh x_q)$$

(cf. [5,13]). We define the modified moment functions $\hat{m}_l(x) := m_l^A(T(x))$ which admit modified integral representations similar to (2.6). Moreover, for $\nu \in M^1(C^B_q)$ we consider the image measure $T(\nu) \in M^1(C^B_q) \subset M^1(C^A_q)$. As $|x - \ln \cosh x| \leq \ln 2$ for all $x \in [0, \infty[ \quad \text{by an elementary calculation, we see that for all multiindices } l, \quad \text{the } l\text{th moment of type } A \text{ of } \nu \text{ exists if and only if the } l\text{th moment of type } A \text{ of } T(\nu) \text{ exists. We put } \hat{m}_l(\nu) := m_l^A(T(\nu)) \quad \text{and } \hat{\Sigma}(\nu) := \Sigma^A(T(\nu)).$

We next turn to the $BC$-case.

DEFINITION 2.4. For all $p > 2q - 1$, $x \in C^B_q$, and $\lambda \in \mathbb{C}^q$, the $\varphi_{\lambda}^p$ in (1.4) satisfy

$$\varphi_{\lambda}^p(x) = \int_{B_q U(\lambda, \mathbb{R})} \Delta_{(i\lambda - \rho)/2}(g(x, u, w)) \, du \, dm_p(w)$$

with $\Delta_\lambda$ from (2.1), the half-sum of positive roots

$$\rho = \rho^{BC}(p) := \sum_{i=1}^q \left(\frac{d}{2}(p + q + 2 - 2i) - 1\right) e_i,$$

g as above, and with $m_\rho(w) \in M^1(B_q)$ from (1.6) (see [13]). As in [4] we
define the moment functions for \( l = (l_1, \ldots, l_q) \in \mathbb{N}_0^q \) by

\[
m^{p}_l(x) := \left. \frac{\partial^{||l||}}{\partial \lambda^{l_1} \cdots \partial \lambda^{l_q}} \varphi^p_{-i\rho - i\lambda}(x) \right|_{\lambda = 0} = \frac{1}{2^{||l||}} \int_{B_q(U(q,F))} \left( \ln \Delta_1(g(x,u,w)) \right)^{l_1} \cdot \left( \ln \Delta_2(g(x,u,w)) \right)^{l_2} \cdots \left( \ln \Delta_{q-1}(g(x,u,w)) \right)^{l_q} \, du \, dm_p(w)
\]

for \( x \in C^B_q \). We also form the first moment function \( m^{p}_1 \), the matrix-valued second moment function \( m^{p}_2 \), and \( \Sigma^p(x) := m^{p}_2(x) - m^{p}_1(x)^t \cdot m^{p}_1(x) \) as above.

We have the following basic properties (see [V2, Section 3]):

**Lemma 2.5.**

1. There is a constant \( C = C(p,q) \) such that for all \( x \in C^B_q \),
   \[
   \|m^{p}_1(x) - x\| \leq C.
   \]
2. For each \( x \in C^B_q \), \( \Sigma^p(x) \) is positive-semidefinite.
3. \( \Sigma^p(0) = 0 \), and for \( x \in C^B_q \setminus \{0\} \), \( \Sigma^p(x) \) has full rank \( q \).
4. All second moment functions \( m^{p}_{e_j+e_t}(x) \) are growing at most quadratically, and \( m^{p}_{2e_1} \) is growing quadratically.
5. There exists a constant \( C = C(p,q) \) such that for all \( x \in C^B_q \) and \( \lambda \in \mathbb{R}^q \),
   \[
   |\varphi^p_{-i\rho - i\lambda}(x) - e^{i(\lambda m^{p}_1(x))}| \leq C\|\lambda\|^2_2.
   \]

Similarly to the \( A \)-case, we also define multivariate \( l \)th moments, dispersions, and covariance matrices of type \( BC(p) \) for measures \( \nu \in \mathcal{M}^1(C^B_q) \).

We next derive estimates for \( |\tilde{m}_l(\nu) - m^{p}_l(\nu)| \) for \( l \in \mathbb{N}_0^q \) and large \( p \) under the assumption that these moments exist. For this we first show that for a given \( \nu \in \mathcal{M}^1(C^B_q) \) the existence of moments of some maximal order \( \lambda \) is independent of taking classical moments, moments of type \( A \), or moments of type \( BC \). For our purpose it will be sufficient to study the case with \( |l| \) even.

Let \( k \in \mathbb{N}_0 \) and \( \nu \in \mathcal{M}^1(C^B_q) \). We say that \( \nu \) admits finite \( A \)-type moments of order at most \( 2k \) if

\[
\tilde{m}_{2k\cdot e_1}, \ldots, \tilde{m}_{2k\cdot e_q} \in L^1(C^B_q, \nu).
\]

Indeed, by the definition of moment functions in (2.6) and Hölder’s inequality, in this case all moments of order at most \( 2k \) are \( \nu \)-integrable. Similarly, if

\[
m^{p}_{2k\cdot e_1}, \ldots, m^{p}_{2k\cdot e_q} \in L^1(C^B_q, \nu)
\]

then we say that \( \nu \) admits finite \( BC(p) \)-type moments of order at most \( 2k \).
**Proposition 2.6.** For \( k \in \mathbb{N} \) and \( \nu \in \mathcal{M}^1(C_q^B) \) the following are equivalent:

1. \( \nu \) has all classical moments of order at most \( 2k \), i.e. \( \int_{C_q^B} x_1^{l_1} \cdots x_q^{l_q} \, d\nu(x) < \infty \) for all \( l = (l_1, \ldots, l_q) \in \mathbb{N}_0^q \) with \( |l| \leq 2k \).
2. \( \nu \) admits all moments of type \( A \) of order at most \( 2k \).
3. \( T(\nu) \) admits all moments of type \( A \) of order at most \( 2k \).
4. For each \( p \geq 2q - 1 \), \( \nu \) admits all moments of type \( BC(p) \) of order at most \( 2k \).

**Proof.** For (1)\( \Rightarrow \) (2) we prove that \( m_{2k-c_1}^A, \ldots, m_{2k-c_q}^A \in L^1(C_q^B, \nu) \). From (2.6) we have

\[
m_{2k-c_j}^A(\nu) = \frac{1}{2^k} \int_{C_q^B} \int_{U(q, \mathbb{F})} (\ln \Delta_{j+1}(u^* e^{2x} u) - \ln \Delta_j(u^* e^{2x} u))^{2k} \, du \, d\nu(x).
\]

We now deduce from [V2, Lemma 4.2] that \( jx_q \leq \ln \Delta_j(u^* e^{2x} u) \leq jx_1 \) for \( u \in U(q, \mathbb{F}), x \in C_q^B \), and \( j = 1, \ldots, q \). Therefore, by elementary inequalities,

\[
m_{2k-c_j}^A(\nu) \leq \frac{1}{2^{2k}} \int_{C_q^B} |(j(x_1 - x_q) + x_q)^{2k} \, d\nu(x) < \infty.
\]

To prove (2)\( \Rightarrow \) (1) it suffices to show that \( \int_{C_q^B} x_1^{2k} \, d\nu(x) < \infty \). It can be easily seen that for every \( u \in U(q, \mathbb{F}) \) there exist coefficients \( c_i(u) \geq 0 \) for \( i = 1, \ldots, q \) with \( \sum_{i=1}^q c_i(u) = 1 \) and \( \Delta_1(u^* e^{2x} u) = \sum_{i=1}^q c_i(u) e^{2x_i} \geq c_1(u) e^{2x_1} \). We now use the inequalities \( 2^{2k}(a^{2k} + b^{2k}) \geq (a + b)^{2k} \) for \( a = \ln(c_1(u) e^{2x_1}) \) and \( b = -\ln c_1(u) \). Hence

\[
\int_{U(q, \mathbb{F})} \int_{C_q^B} (\ln \Delta_1(u^* e^{2x} u))^{2k} \, du \, d\nu(x)
\geq \int_{U(q, \mathbb{F})} \int_{C_q^B} (\ln(c_1(u) e^{2x_1}))^{2k} \, du \, d\nu(x)
\geq -\int_{U(q, \mathbb{F})} |\ln c_1(u)|^{2k} \, du + \int_{C_q^B} x_1^{2k} \, d\nu(x).
\]

As by [V2, Lemma 5.1 and Proposition 4.4] for small \( \varepsilon > 0 \) and some \( C > 0 \),

\[
\int_{U(q, \mathbb{F})} |\ln c_1(u)|^{2k} \, du \leq C \int_{U(q, \mathbb{F})} ((1 + c_1(u)^{-\varepsilon})^{2k} \, du
\leq \sum_{l=0}^{2k} C \binom{2k}{l} \int_{U(q, \mathbb{F})} c_1(u)^{-l\varepsilon} \, du < \infty,
\]

we get \( \int_{C_q^B} x_1^{2k} \, d\nu(x) < \infty \) as claimed.
The equivalence of (2) and (3) follows from
\[ \frac{1}{4} u^* e^{2x} u \leq u^*(\cosh x)^2 u \leq \frac{1}{2} u^* e^{2x} u, \]
which yields \[ |\ln \Delta_j(u^*(\cosh x)^2 u) - \ln \Delta_j(u^*e^{2x} u)| \leq j \cdot \ln 4. \]

To prove (3) \(\Rightarrow\) (4) we see from the preceding estimate and [V2] Lemma 6.3 that
\[ |\ln \Delta_j g(x, u, w) - \ln \Delta_j(u^*(\cosh x)^2 u)| \leq (j + 1) \cdot \ln 4 + 2j \cdot \max(|\ln(1 - \sigma_1(w))|, \ln(\sigma_1(w) + 1)) =: H_j(w). \]

It can be easily seen that \[ \int_{B_q} \ln(1 + \sigma_1(w))^{2k} dm_p(w) \text{ is finite.} \] Moreover, as \[ 1 \geq \sigma_1(w) \geq \cdots \geq \sigma_q(w) \geq 0 \text{ for } w \in B_q, \] we obtain
\[ \frac{1}{1 - \sigma_1(w)} \leq \frac{2}{1 - \sigma_1(w)^2} \leq 2 \prod_{r=1}^{q} \frac{1}{1 - \sigma_r(w)^2} \leq \frac{2}{\Delta(I - w^*w)}. \]

Now, from Lemma 2.1 and (2.14) together with the elementary inequality
\[ |\ln(1 + z)| \leq \frac{|z|}{1 - |z|} \text{ for } |z| < 1 \]
we get
\[ \int_{B_q} |\ln(1 - \sigma_1(w))|^{2k} dm_p(w) \leq 2^{2k} \int_{B_q} \sigma_1(w)^{2k} \cdot \Delta(I - w^*w)^{-2k} dm_p(w) < \infty. \]

Hence, \[ \int_{B_q} |H_j(q)|^{2k} dm_p(w) < \infty \text{ for } j = 1, \ldots, q. \] Therefore, using the elementary inequality \[ 3^{2k}(a^{2k} + b^{2k} + c^{2k}) \geq (a + b + c)^{2k} \] we have
\[ m_{2k-\varepsilon_j}^p(\nu) \leq \left( \frac{3}{2} \right)^{2k} \int_{B_q \times U(x, \bar{v}) \times C_q^B} \left( |\ln \Delta_{j+1} g(x, u, w) - \ln \Delta_{j+1}(u^*(\cosh x) u)|^{2k} + |\ln \Delta_{j+1}(u^*(\cosh x) u) - \ln \Delta_j(u^*(\cosh x) u)|^{2k} + |\ln \Delta_j g(x, u, w) - \ln \Delta_j(u^*(\cosh x) u)|^{2k} \right) dm_p(w) du d\nu(x). \]

If we use (2.13), (2.16) and the assumption, we see that the right hand side above is finite, which shows that \[ m_{2k-\varepsilon_j}^A(\nu) < \infty. \]
Finally, (4) ⇒ (3) follows analogously from
\[
m_{2k-ε}^A(ν) \leq \left( \frac{3}{2} \right)^{2k} \int_{B_q \times U(q,F) \times C_q^B} \left[ \left| \ln \Delta_{j+1}(u^*(cosh x)u) - \ln \Delta_{j+1}g(x, u, w) \right|^{2k} \right. \\
+ \left| \ln \Delta_{j+1}g(x, u, w) - \ln \Delta_{j}g(x, u, w) \right|^{2k} \\
+ \left. \left| \ln \Delta_{j}(u^*(cosh x)u) - \ln \Delta_{j}g(x, u, w) \right|^{2k} \right] \, dm_p(w) \, du \, dν(x). \]
\]

**Proposition 2.7.** Let \( l = (l_1, \ldots, l_q) \in \mathbb{N}_0^q \) with \( |l| \geq 3 \) and \( ν \in \mathcal{M}(C_q^B) \). Assume that \( ν \) admits finite moments of order \( 4(|l| - 2) \). Then there exists a constant \( C := C(|l|, q, ν) \) with \( |\tilde{m}_l(ν) - m_l^P(ν)| \leq C/\sqrt{p} \).

**Proof.** Consider the \( |l| \) factors of the integrand in (2.11) and (2.6). For \( i = 1, \ldots, |l| \) these factors have the form:
\[
f_i(x, u, w) := \ln \Delta_r(g(x, u, w)) - \ln \Delta_{r-1}(g(x, u, w)),
\]
\[
\tilde{f}_i(x, u, w) := \ln \Delta_r(g(x, u, 0)) - \ln \Delta_{r-1}(g(x, u, 0))
\]
with \( Δ_0 := 1 \) where \( r \in \{1, \ldots, q\} \) is the smallest integer with \( i \leq l_1 + \cdots + l_r \).

Then, by Lemma 2.1(2) and (2.15) for \( i = 1, \ldots, |l|, x \in C_q^B, u \in U(q, F), w \in B_q \), we have
\[
|f_i(x, u, w) - \tilde{f}_i(x, u, w)| \leq 2 \max_{r=1, \ldots, q} \left| \ln \Delta_r(g(x, u, w)) - \ln \Delta_r(g(x, u, 0)) \right|
\]
\[
\leq 4q \bar{x} \sigma_1(w) \frac{\bar{x}}{1 - \bar{x} \sigma_1(w)} \leq 4q \bar{x} \frac{\sigma_1(w)}{1 - \sigma_1(w)}
\]
where \( \bar{x} = \min(1, x) \). Thus, by (2.14),
\[
|f_i(x, u, w) - \tilde{f}_i(x, u, w)| \leq 8q \bar{x} \frac{\sigma_1(w)}{Δ(I - w^*w)}.
\]

Now, notice that \( |\tilde{m}_l(ν) - m_l^P(ν)| \) is equal to
\[
\left| \frac{1}{2^{|l|}} \int_{B_q \times U(q,F) \times C_q^B} \left( \prod_{i=1}^{|l|} f_i(x, u, w) - \prod_{i=1}^{|l|} \tilde{f}_i(x, u, w) \right) \, du \, dm_p(w) \, dν(t) \right|
\]
and by a telescopic sum argument,
\[
(2.17) \quad |\tilde{m}_l(ν) - m_l^P(ν)|
\]
\[
\leq \frac{1}{2^{|l|}} \sum_{i=1}^{|l|} \int_{B_q \times U(q,F) \times C_q^B} \left| (f_i(x, u, w) - \tilde{f}_i(x, u, w)) \right|
\]
\[
\times \prod_{j=i+1}^{|l|} f_j(x, u, w) \prod_{k=1}^{i-1} \tilde{f}_k(x, u, w) \, du \, dm_p(w) \, dν(x).
\]
We estimate the summands of the right hand side of (2.17) in two ways: For \( i = 1 \), the Cauchy–Schwarz inequality and Lemma 2.1 show that

\[
(2.18)\quad \int \limits_{B_q \times U(q, End_q)} |(f_1(x, u, w) - \tilde{f}_1(x, u, w)) \prod_{j=2} \left| f_j(x, u, w) \right| du \, dm_\nu(x) \leq \left( \int \limits_{B_q \times U(q, End_q)} \sum_{i=1} \left| f_i(x, u, w) - \tilde{f}_i(x, u, w) \right|^2 du \, dm_\nu(x) \right)^{1/2} \times \left( \int \limits_{B_q \times U_0(q, End_q)} \prod_{j=2} \left| f_j(x, u, w) \right|^2 du \, dm_\nu(x) \right)^{1/2} \leq M_1 \cdot 8q \left( \int \limits_{B_q} \frac{\sigma_1(w)^2}{\Delta(I - w^* w)^2} \, dm_\nu(w) \right)^{1/2} \leq M_1 \cdot \frac{C}{\sqrt{p}}
\]

with \( M_1 := M_1(\nu, |l|, q) = 8q \max_{r \in \mathbb{N}_0^q, |r| \leq 2(|l| - 1)} \max(\tilde{m}_r(\nu), m^p_r(\nu)) \), which is finite by our assumption and Proposition 2.6. Similarly, the same upper bound holds for \( i = |l| \) in (2.17). Now, let \( i = 2, \ldots, q - 1 \). Here Hölder’s inequality yields

\[
(2.19)\quad \int \limits_{B_q \times U_0(q, End_q)} \left| (f_i(x, u, w) - \tilde{f}_i(x, u, w)) \prod_{j=i+1} \left| f_j(x, u, w) \right| du \, dm_\nu(x) \right| \leq \left( \int \limits_{B_q \times U_0(q, End_q)} \left| (f_i(x, u, w) - \tilde{f}_i(x, u, w)) \right|^2 du \, dm_\nu(x) \right)^{1/2} \times \left( \int \limits_{B_q \times U_0(q, End_q)} \prod_{j=i+1} \left| f_j(x, u, w) \right|^4 du \, dm_\nu(x) \right)^{1/4} \times \left( \int \limits_{B_q \times U_0(q, End_q)} \prod_{k=1} \left| \tilde{f}_k(x, u, w) \right|^4 du \, dm_\nu(x) \right)^{1/4} \leq M_2 \cdot \frac{C}{\sqrt{p}}
\]

with \( M_2 := M_2(\nu, |l|, q) = 8q \max_{r \in \mathbb{N}_0^q, |r| \leq 4(|l| - 2)} \max(\tilde{m}_r(\nu), m^p_r(\nu)) \), which again is finite by our assumption and Proposition 2.6. Thus, (2.18) and (2.19) yield the claim. ■

3. Spherical Fourier transform. We here collect some known facts about the spherical Fourier transform of types A and BC. We start with the multiplicative functions and the dual space as in \( \text{R2} \) and \( \text{NPP} \) for \( p \geq 2q - 1 \). The set of continuous multiplicative functions,
\[ \chi(C_q^B, *_p) := \left\{ f : C_q^B \to \mathbb{C} : f \text{ continuous, } \int_{C_q^B} f d(\delta_x *_p \delta_y) = f(x)f(y) \right\}, \]

is given by \( \{ \varphi^p_{\lambda} : \lambda \in \mathbb{C}^q \} \). Moreover, the set \( \chi_b(C_q^B, *_p) \) of bounded functions in \( \chi(C_q^B, *_p) \) is equal to \( \{ \varphi^p_{\lambda} : \exists \lambda \in \text{co}(W_q \cdot \rho) \} \) where \text{co} denotes the convex hull, and \( W_q^B \) the Weyl group of type \( B_q \) acting on \( \mathbb{C}^q \). The dual space

\[ (C_q^B, *_p)^\wedge := \{ f \in \chi_b(C_q^B, *_p) : f(x^-) = \overline{f(x)} \} \]

is \( \{ \varphi^p_{\lambda} : \lambda \in C_q^B \) or \( \lambda \in i \cdot \text{co}(W_q^B \cdot \rho) \). Finally, the support of the Plancherel measure is the set \( \{ \varphi^p_{\lambda} : \lambda \in C_q^B \} \).

**Definition 3.1.** Let \( \nu \in \mathcal{M}^1(C_q^B) \). The BC-type spherical (or hypergroup) Fourier transform is given by

\[ F_{BC}^p(\nu)(\lambda) := \int_{C_q^B} \varphi^p_{\lambda}(x) d\nu(x) \quad \text{for } \lambda \in \{ \lambda \in \mathbb{C}^q : \exists \lambda \in \text{co}(W_q^B \cdot \rho) \}. \]

We recapitulate some estimates from [V2].

**Lemma 3.2.** For all \( x \in C_q^B \), \( \lambda \in \mathbb{R}^q \), and \( l \in \mathbb{N}_0^q \),

\[ \left| \frac{\partial^{|l|}}{\partial \lambda^{|l|}} \varphi^p_{\lambda - i\rho}(x) \right| \leq m^p_l(x). \]

**Lemma 3.3.** Let \( k \in \mathbb{N}_0 \) and assume that \( \nu \in \mathcal{M}^1(C_q^B) \) admits finite \( k \)th modified moments. Then, for all \( \lambda \in \mathbb{C}^q \) with \( \exists \lambda \in \text{co}(W_q^B \cdot \rho) \), \( F_{BC}^p(\nu)(\cdot) \) is \( k \) times continuously differentiable, and for all \( l \in \mathbb{N}_0^n \) with \( |l| \leq k \),

\[ (3.1) \quad \frac{\partial^{|l|}}{\partial \lambda^{|l|}} F_{BC}^p(\nu)(\lambda) = \int_{C_q^B} \frac{\partial^{|l|}}{\partial \lambda^{|l|}} \varphi^p_{\lambda}(x) d\nu(x). \]

In particular,

\[ \frac{\partial^{|l|}}{\partial \lambda^{|l|}} F_{BC}(\nu)(-i\rho) = \int_{C_q^B} m^p_l(x) d\nu(x). \]

**Remark 3.4.** There are corresponding results to the Lemmas 3.2 and 3.3 for the A-case with the corresponding moment functions \( m^A_l \) for \( l \in \mathbb{N}_0^n \) and the Fourier transform \( F_A \) and \( \nu \in \mathcal{M}^1(C_q^A) \); see [V2] Lemmas 7.1, 7.2.

**4. Limit theorems for growing \( p \).** We now derive several limit theorems when the time and \( p \) tend to infinity. The statements are similar, but the assumptions on the moments and the relation between the time and \( p \) are different. We first present a CLT with a condition on \( (p_n)_{n \geq 1} \):

**Theorem 4.1.** Let \( (p_n)_{n \geq 1} \subset ]2q-1, \infty[ \) be increasing with \( \lim_{n \to \infty} n/p_n = 0 \). Let \( \nu \in \mathcal{M}^1(C_q^B) \) with \( \nu \neq \delta_0 \) and with second moments. Consider the
associated random walks \((S_n^p)_{n \geq 0}\) on \(C_q^B\). Then
\[
S_n^p - n \cdot \tilde{m}_1(\nu) / \sqrt{n}
\]
converges in distribution to \(\mathcal{N}(0, \tilde{\Sigma}(\nu))\).

**Proof.** We know from [RV, Theorem 4.2(2)] that there exists a constant \(C > 0\) such that for all \(p > 2q - 1\), \(x \in C_q^B\), \(\lambda \in \mathbb{R}^q\),
\[
|\varphi_{\lambda - i\rho}^p(x) - \varphi_{\lambda - i\rho}^A(\ln \cosh x)| \leq C \cdot \|\lambda\|_1 \cdot \tilde{x}
\]
where \(\|\lambda\|_1 := |\lambda_1| + \cdots + |\lambda_q|\) and \(\tilde{x} := \min(x_1, 1) \geq 0\). Hence, denoting the half-sums of positive roots of type \(BC\) associated with \(S_n^p\) as described in (2.10) by \(\rho(n) := \rho_{BC}(p_n)\) for all \(\nu \in \mathcal{M}^1(C_q^B)\), we get
\[
\left| \int_{C_q^B} \varphi_{\lambda - i\rho}^p(n)(x) d\nu(x) - \int_{C_q^B} \varphi_{\lambda - i\rho}^A(\ln \cosh x) d\nu(x) \right| \leq C \cdot \|\lambda\|_1 \cdot \sqrt{p_n},
\]

(4.1)

Let \(\nu^{(n,p)} \in \mathcal{M}^1(C_q^B)\) be the law of \(S_n^p\). Then \(T(S_n^p)\) has distribution \(T(\nu^{(n,p_n)})\) whose \(A\)-type spherical Fourier transform satisfies
\[
\mathcal{F}_A(T(\nu^{(n,p_n)}))(\lambda - i\rho^A) = \int_{C_q^A} \varphi_{\lambda - i\rho^A}^A(x) dT(\nu^{(n,p_n)})(x)
\]
\[
= \int_{C_q^B} \varphi_{\lambda - i\rho}^A(\ln \cosh x) d\nu^{(n,p_n)}(x)
\]
for \(\lambda \in \mathbb{R}^q\). Furthermore, by plugging \(\nu^{(n,p_n)}\) into (4.1) we get
\[
\mathcal{F}_A(T(\nu^{(n,p_n)}))(\lambda - i\rho^A)
\]
\[
= \int_{C_q^B} \varphi_{\lambda - i\rho}^p(n)(x) d\nu^{(n,p_n)}(x) + O\left(\frac{\|\lambda\|_1}{p_n^{1/2}}\right)
\]
\[
= \mathcal{F}_{BC}^p(\nu^{(n,p_n)})(\lambda - i\rho(n)) + O\left(\frac{\|\lambda\|_1}{p_n^{1/2}}\right)
\]
\[
= \left(\mathcal{F}_{BC}^p(\nu)(\lambda - i\rho(n))\right)^n + O\left(\frac{\|\lambda\|_1}{p_n^{1/2}}\right)
\]
\[
= \left(\int_{C_q^B} \varphi_{\lambda - i\rho^A}(\ln \cosh x) d\nu(x)\right)^n + O\left(\frac{\|\lambda\|_1}{p_n^{1/2}}\right)
\]
\[
= \left(\mathcal{F}_A(T(\nu))(\lambda - i\rho^A) + O\left(\frac{\|\lambda\|_1}{p_n^{1/2}}\right)\right)^n + O\left(\frac{\|\lambda\|_1}{p_n^{1/2}}\right).
\]

(4.2)
Using the the initial moment assumption and Lemma 2.6 we see that the first and second modified moments \( \tilde{m}_1 \) and \( \tilde{m}_2 \) exist. Moreover, all entries of the modified covariance matrix \( \tilde{\Sigma}(\nu) = \tilde{m}_2(\nu) - \tilde{m}_1(\nu)^2 \cdot \tilde{m}_1(\nu) \) are finite. By Lemma 3.3 the Taylor expansion of \( \mathcal{F}_A(T(\nu))(\lambda - i\rho^A) \) as \( |\lambda| \to 0 \) is given by

\[
(4.3) \quad \mathcal{F}_A(T(\nu))(\lambda - i\rho^A) = 1 - i\langle \lambda, \tilde{m}_1(\nu) \rangle - \lambda \tilde{m}_2(\nu) \lambda^t + o(|\lambda|^2).
\]

Using the initial assumption \( O(1/\sqrt{np_n}) = o(1/n) \) we obtain

\[
E(\varphi^A_{\lambda/\sqrt{n} - i\rho^A}(T(S^n_{np_n}))|\langle \lambda, \tilde{m}_1(\nu) \rangle|<1) \left( \frac{\lambda}{\sqrt{n}} - i\rho^A \right) e^{i\langle \lambda, \tilde{m}_1(\nu) \rangle} + O\left( \frac{\|\lambda\|_1}{\sqrt{np_n}} \right) = e^{i\langle \lambda, \tilde{m}_1(\nu) \rangle/\sqrt{n}}
\]

Thus,\( n \to \infty \)

\[
\lim_{n \to \infty} E(\varphi^A_{\lambda/\sqrt{n} - i\rho^A}(T(S^n_{np_n})) \cdot \exp(i\langle \lambda, \tilde{m}_1(\nu) \rangle /\sqrt{n})) = \exp(-\lambda \tilde{\Sigma}(\nu) \lambda^t/2).
\]

On the other hand, from Lemma 2.3(5) we have

\[
\lim_{n \to \infty} E(\varphi^A_{\lambda/\sqrt{n} - i\rho^A}(T(S^n_{np_n})) - \exp(-i\langle \lambda, \tilde{m}_1(\nu) \rangle /\sqrt{n})) = 0.
\]

Formulas (4.4) and (4.5) and the fact that \( |e^{i\langle \lambda, \sqrt{n}\tilde{m}_1(\nu) \rangle}| \leq 1 \) together imply that for all \( \lambda \in \mathbb{R}^q \),

\[
\lim_{n \to \infty} \exp(-i\langle \lambda, (\tilde{m}_1(S^n_{np_n}) - n \cdot \tilde{m}_1(\nu)) \rangle /\sqrt{n}) = \exp(-\lambda \tilde{\Sigma}(\nu) \lambda^t/2).
\]

Lévy’s continuity theorem for the classical \( q \)-dimensional Fourier transform implies that \( (\tilde{m}_1(S^n_{np_n}) - n \cdot \tilde{m}_1(\nu))/\sqrt{n} \) tends to the normal distribution \( \mathcal{N}(0, \tilde{\Sigma}(\nu)) \).

Hence, by Lemma 2.3(1), the definition of \( T \), and by \( \lim_{x \to \infty} (x - \ln \cosh x) = \ln 2 \) we conclude that \( (S^n_{np_n} - n\tilde{m}_1(\nu))/\sqrt{n} \to \mathcal{N}(0, \tilde{\Sigma}(\nu)) \) as claimed. ■

For the weak law of large numbers (LLN) we only need first moments:

**Theorem 4.2.** Let \( (p_n)_{n \geq 1} \subset [2q - 1, \infty] \) be increasing with \( \lim_{n \to \infty} n/p_n = 0 \). Let \( \nu \in \mathcal{M}^1(C^B_q) \) be with \( \nu \neq \delta_0 \) and first moments. Consider the associated random walks \( (\tilde{S}^p_n)_{n \geq 0} \) on \( C^B_q \) for \( p > 2q - 1 \) and let \( \varepsilon > 1/2. \)
Then
\[ \frac{1}{n^\varepsilon}(\tilde{S}_{n}^{p} - n \cdot \tilde{m}_1(\nu)) \to 0 \quad \text{in probability.} \]

In particular, \( \tilde{S}_{n}^{p}/n \to \tilde{m}_1(\nu) \) in probability.

**Proof.** The proof is very similar to that of Theorem 4.1. In fact, (4.2), (4.3), \( \varepsilon > 1/2 \) and \( O(1/\sqrt{np_n}) = o(1/n) \) show that
\[
E(\varphi_{\lambda/n^\varepsilon - i\rho_A}(T(\tilde{S}_{n}^{p})))e^{i(\lambda,n^{1-\varepsilon}\tilde{m}_1(\nu))} = \mathcal{F}_A(T(\nu^{(n,p_n)}))(\lambda/n^\varepsilon - i\rho_A) \cdot e^{i(\lambda,n^{1-\varepsilon}\tilde{m}_1(\nu))}
\]
\[
= \left[ \left( \mathcal{F}_A(T(\nu)) \left( \frac{\lambda}{n^\varepsilon} - i\rho_A \right) + O\left( \frac{\|\lambda\|_1}{\sqrt{np_n}} \right) \right)^n + O\left( \frac{\|\lambda\|_1}{\sqrt{np_n}} \right) \right] \cdot e^{i(\lambda,\tilde{m}_1(\nu)/n^\varepsilon)n}
\]
\[
= \left( 1 + o\left( \frac{\|\lambda\|^2}{n} \right) \right)^n.
\]
Thus,
(4.6) \[ \lim_{n \to \infty} E(\varphi_{\lambda/n^\varepsilon - i\rho_A}(T(\tilde{S}_{n}^{p})))e^{i(\lambda,n^{1-\varepsilon}\tilde{m}_1(\nu))} = 1 \]

for all \( \lambda \in \mathbb{R}^q \). On the other hand, from Lemma 2.3(5) we have
(4.7) \[ \lim_{n \to \infty} E(\varphi_{\lambda/n^\varepsilon - i\rho_A}(T(\tilde{S}_{n}^{p}))) - \exp(-i\langle \lambda, \tilde{m}_1(\tilde{S}_{n}^{p}) \rangle/n^\varepsilon) = 0. \]

Formulas (4.6), (4.7), and \( |e^{i(\lambda,\sqrt{\tilde{m}_1(\nu)})}| \leq 1 \) imply that for all \( \lambda \in \mathbb{R}^q \),
(4.8) \[ \lim_{n \to \infty} E(\exp(-i\langle \lambda, (\tilde{m}_1(\tilde{S}_{n}^{p}) - n \cdot \tilde{m}_1(\nu)) \rangle/n^\varepsilon)) = 1. \]

Hence, by Lévy’s continuity theorem, \( (\tilde{m}_1(\tilde{S}_{n}^{p}) - n \cdot \tilde{m}_1(\nu))/n^\varepsilon \to 0 \) in distribution and in probability. The proof can be now completed as that of Theorem 4.1. \( \blacksquare \)

**Remark 4.3.** For rank \( q = 1 \) the CLT 4.1 was derived in \([\text{Gr1}]\) with different techniques under weaker assumptions, namely without the restriction \( n/p_n \to 0 \). The proof in \([\text{Gr1}]\) relies on the convergence of the moment functions
(4.8) \[ (m_1^p(x))^2 - m_2^p(x) \to 0 \]
on \([0, \infty[ \) for \( p \to \infty \). However, for \( q \geq 2 \) this convergence is no longer available.

We next try to get rid of the restriction \( n/p_n \to 0 \) by using fourth moments.

**Theorem 4.4.** Let \( (p_n)_{n \geq 1} \subset [2q - 1, \infty[ \) be increasing with \( \lim_{n \to \infty} p_n = \infty \). Let \( \nu \in \mathcal{M}^1(C_q^B) \) with \( \nu \neq \delta_0 \) and with fourth moments. Then for the
associated random walks \((S_n^p)_{n \geq 0}\) on \(C_q^B\), \((S_n^p - n \cdot m_1^p(\nu))/\sqrt{n}\) tends in distribution to \(\mathcal{N}(0, \tilde{\Sigma}(\nu))\).

**Proof.** Taylor’s theorem and Proposition 2.7 show that for \(p \geq 2q - 1\),

\[
\left| E(\varphi_{\lambda/\sqrt{n} - i\rho(n)}(S_n^p)) - \left(1 - i\langle \lambda, m_1^p(\nu) \rangle - \frac{\lambda m_2^p(\nu) \lambda^t}{2n}\right)\right| \\
\leq \sum_{l \in \mathbb{N},|l|=3} m_l^p(\nu) \frac{\lambda_1 \cdots \lambda_q}{l_1! \cdots l_q!} \leq \frac{1}{n^{3/2}} \sum_{l \in \mathbb{N},|l|=3} (\bar{m}_l(\nu) + C/\sqrt{p}) \frac{\lambda_1 \cdots \lambda_q}{l_1! \cdots l_q!} \\
\leq K_1 \frac{\|\lambda\|_\infty^3}{n^{3/2}}
\]

for some constant \(K_1 > 0\) independent of \(p\). Analogously, for all \(p \geq 2q - 1\),

\[
\left| e^{i\langle \lambda, \sqrt{nm_1^p}(\nu) \rangle} - \left(1 + \frac{i\langle \lambda, m_1^p(\nu) \rangle}{\sqrt{n}} - \frac{\langle \lambda, m_1^p(\nu) \rangle^2}{2n}\right)\right| \leq K_2 \frac{\|\lambda\|_\infty^3}{n^{3/2}}
\]

for some \(K_2 > 0\). Using (4.9), (4.10) we follow the proof of Theorem 4.1. We however use the \(BC\)-type Fourier transform and moments instead of type \(A\), and approximate \(A\)-type moments by \(BC\)-type moments via Proposition 2.7.

We have

\[
E(\varphi_{\lambda/\sqrt{n} - i\rho(n)}(S_n^p))e^{i\langle \lambda, \sqrt{nm_1^p}(\nu) \rangle} \\
= F_{BC}(\nu^{(n,p_n)})(\lambda/\sqrt{n} - i\rho(n)) \cdot e^{i\langle \lambda, \sqrt{nm_1^p}(\nu) \rangle} \\
= \left[\left(1 - \frac{i\langle \lambda, m_1^p(\nu) \rangle}{\sqrt{n}} - \frac{\lambda m_2^p(\nu) \lambda^t}{2n} + o\left(\frac{1}{n}\right)\right) \\
\times \left(1 + \frac{i\langle \lambda, m_1^p(\nu) \rangle}{\sqrt{n}} - \frac{\langle \lambda, m_1^p(\nu) \rangle^2}{2n} + o\left(\frac{1}{n}\right)\right)\right]^n \\
= \left(1 - \frac{\lambda \Sigma_{\nu}(\nu) \lambda^t}{2n} + o\left(\frac{1}{n}\right)\right)^n.
\]

By Lemma 2.7 \(|\lambda \Sigma_{\nu}(\nu) \lambda^t - \lambda \tilde{\Sigma}(\nu) \lambda^t| = O(|\lambda|^2/\sqrt{p_n})\) as \(p_n \to \infty\). Therefore,

\[
\lim_{n \to \infty} E(\varphi_{\lambda/\sqrt{n} - i\rho(n)}(S_n^p))e^{i\langle \lambda, \sqrt{nm_1^p}(\nu) \rangle} \\
= \lim_{n \to \infty} \left(1 - \frac{\lambda \tilde{\Sigma}(\nu) \lambda^t}{2n} + \frac{\lambda (\Sigma_{\nu}(\nu) - \tilde{\Sigma}(\nu)) \lambda^t}{2n} + o\left(\frac{1}{n}\right)\right)^n = \exp(-\lambda \tilde{\Sigma}(\nu) \lambda^t/2)
\]

On the other hand, from Lemma 2.5(5) we have

\[
\lim_{n \to \infty} E(\varphi_{\lambda/\sqrt{n} - i\rho(n)}(S_n^p)) - \exp(-i\langle \lambda, m_1^p(S_n^p) \rangle/\sqrt{n}) = 0.
\]

The rest of the proof is now analogous to that of Theorem 4.1.

We next consider a weak LLN whenever second moments exist:
THEOREM 4.5. Let \((p_n)_{n \geq 1} \subset \{2q - 1, \infty\}\) be increasing with \(\lim_{n \to \infty} p_n = \infty\). Let \(\nu \in \mathcal{M}^1(C_q^B)\) with \(\nu \neq \delta_0\) and with second moments. Consider the associated random walks \((\tilde{S}_n^p)_{n \geq 0}\) on \(C_q^B\) for \(p > 2q - 1\). Let \(\varepsilon > 1/2\). Then

\[
\frac{1}{n^\varepsilon}(\tilde{S}_n^p - n \cdot m_1^p(\nu)) \to 0 \quad \text{in probability.}
\]

Proof. As in the proof of the preceding theorem, for \(p > 2q - 1\) we have

\[
(4.12) \quad \left| E(\varphi_{\lambda/n^\varepsilon - i\rho(n)}(\tilde{S}_n^p)) - \left(1 - \frac{i\langle \lambda, m_1^p(\nu) \rangle}{n^\varepsilon}\right) \right| \leq K_1 \frac{||\lambda||_3^3}{n^{2\varepsilon}}
\]

for some \(K_1 > 0\) independent of \(p\). Moreover, in the same way,

\[
(4.13) \quad \left| e^{i\langle \lambda, n^\varepsilon \cdot m_1^p(\nu) \rangle} - \left(1 + \frac{i\langle \lambda, m_1^p(\nu) \rangle}{n^\varepsilon}\right) \right| \leq K_2 \frac{||\lambda||_3^3}{n^{2\varepsilon}}.
\]

Using (4.12) and (4.13) we now follow the proof of Theorem 4.4. For \(\lambda \in \mathbb{R}^q\) we have

\[
E(\varphi_{\lambda/n^\varepsilon - i\rho(n)}(\tilde{S}_n^p))e^{i\langle \lambda, n^\varepsilon \cdot m_1^p(\nu) \rangle}
\]

\[
= \mathcal{F}_{BC}(\nu^{(n,p_n)})(\lambda/n^\varepsilon - i\rho(n)) \cdot e^{i\langle \lambda, n^\varepsilon \cdot m_1^p(\nu) \rangle}
\]

\[
= \left[\left(1 - \frac{i\langle \lambda, m_1^p(\nu) \rangle}{n^\varepsilon}\right) + o\left(\frac{1}{n}\right)\right] \left(1 + \frac{i\langle \lambda, m_1^p(\nu) \rangle}{n^\varepsilon} + o\left(\frac{1}{n}\right)\right)^n
\]

\[
= \left(1 + o\left(\frac{1}{n}\right)\right)^n.
\]

Therefore, for \(\lambda \in \mathbb{R}^q\), \(\lim_{n \to \infty} E(\varphi_{\lambda/n^\varepsilon - i\rho(n)}(\tilde{S}_n^p))e^{i\langle \lambda, n^\varepsilon \cdot m_1^p(\nu) \rangle} = 1\).

On the other hand from the Lemma 2.5(5) for all \(\lambda \in \mathbb{R}^q\) we have

\[
(4.14) \quad \lim_{n \to \infty} E(\varphi_{\lambda/n^\varepsilon - i\rho(n)}(\tilde{S}_n^p) - \exp(-i\langle \lambda, m_1^p(\tilde{S}_n^p)/n^\varepsilon \rangle)) = 0.
\]

Hence, by Lévy’s continuity theorem, \((\tilde{m}_1(\tilde{S}_n^p) - n \cdot m_1^p(\nu))/n^\varepsilon \to 0\) in distribution. As in the proof of Theorem 4.1, this implies the claim. \(\blacksquare\)

5. A CLT with inner normalization. We here present a CLT for fixed \(p\) in the following setting: Fix some non-trivial \(\nu \in \mathcal{M}^1(C_q^B)\) with some moment condition; for \(d \in [0, 1]\), consider the component-wise compression map \(D_d : x \mapsto d \cdot x\) on \(C_q^B\) and the compressed measure \(\nu_d := D_d(\nu) \in \mathcal{M}^1(C_q^B)\). For given \(\nu\) and \(d\) we study the random walks \((S_n^{(p,d)})_{n \geq 0}\) associated with \(\nu_d\) and in particular the limit \((S_n^{(p,n^{-1/2})})_{n \geq 1}\). This case can be seen as a CLT with inner standardization in contrast to the case \((S_n^p)_{n \geq 0}\) in Section 3 with CLTs with outer standardization. Limits for \((S_n^{(p,n^{-1/2})})_{n \geq 1}\) for \(q = 1\) were studied in [Z1]. In the group cases, our CLT is related to the CLTs in [G1, G2, T1, T2, Ri].
Definition 5.1. Let $p \geq 2q - 1$ and $t \geq 0$. A probability measure $\gamma_t = \gamma_t(p) \in \mathcal{M}^1(C^B_q)$ is called $BC(p)$-Gaussian with time parameter $t$ and shape parameter $p$ if

$$F_{BC}^p(\gamma_t)(\lambda) = \exp\left(-\frac{t(\lambda_1^2 + \cdots + \lambda_q^2 + \|\rho\|_2^2)}{2}\right)$$
for $\lambda \in C^B_q \cup i \cdot \co(W^B_q \cdot \rho) \subset C^q$.

The injectivity of the hypergroup Fourier transform (see [J]) ensures that the measures $\gamma_t$ are determined uniquely, and that they form a convolution semigroup $(\gamma_t)_{t \geq 0}$, i.e. for all $s, t \geq 0$,

$$\gamma_s * \gamma_t = \gamma_{s+t}$$
and $\gamma_0 = \delta_0$. The existence of the measures $\gamma_t$ for $t > 0$ is not obvious at the beginning, but it will be a consequence of the proof of the following CLT.

Theorem 5.2. Let $\nu \in \mathcal{M}^1(C^B_q)$ with $\nu \neq \delta_0$ and with finite second moments. Let

$$t_0 := \frac{2}{qd} \int_{C^B_q} \|x\|_2^2 \, d\nu(x).$$

Then $(S_{n}^{(p,n^{-1/2})})_{n \geq 1}$ tends in distribution to $\gamma_{t_0/(p+1)}$ as $n \to \infty$.

For the proof we need some information on $\varphi^p_\lambda$:

Lemma 5.3. Fix $p \in [2q - 1, \infty[$.

1. For $i, j = 1, \ldots, q$ with $i \neq j$ and $\lambda \in C^q$, we have $\frac{\partial}{\partial x_i} \varphi^p_\lambda(0) = 0$ and $\frac{\partial^2}{\partial x_i \partial x_j} \varphi^p_\lambda(0) = 0$.

2. For $i = 1, \ldots, q$, and $\lambda \in C^B_q \cup i \cdot \co(W^B_q \cdot \rho)$,

$$\frac{\partial^2}{\partial x_i^2} \varphi^p_\lambda(0) = -\frac{2(\lambda_1^2 + \cdots + \lambda_q^2 + \|\rho\|_2^2)}{(p+1)qd} < 0.$$

Proof. The functions $\varphi^p_\lambda(x)$ are invariant under the action of the Weyl group of type $BC$ with respect to $x$. Hence, $\varphi^p_\lambda(x_1, \ldots, x_q)$ is even in each $x_i$, which yields (1). Moreover, as $\varphi^p_\lambda(x_1, \ldots, x_q)$ is invariant under permutations, $\frac{\partial^2}{\partial x_i^2} \varphi^p_\lambda(0)$ is independent of $i$. To complete the proof of (2), we recall from [HS, (1.2.6)] that for all $\lambda \in C^q$, $F_{BC}(\lambda, k_p, \cdot)$ is the unique solution to the eigenvalue problem

$$Lf = -(\lambda_1^2 + \cdots + \lambda_q^2 + \|\rho\|_2^2) f$$
on $\text{int}(C^B_q) = \{x \in C^B_q : x_1 > \cdots > x_q > 0\}$ with $f(0) = 1$ with the operator
(5.2) \[ L := \sum_{1 \leq i \leq q} \left[ \frac{\partial^2}{\partial x_i^2} + (k_1 \coth x_i + 2k_2 \coth 2x_i) \frac{\partial}{\partial x_i} \right] + k_3 \sum_{1 \leq i < j \leq q} \left[ \coth(x_i + x_j) \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) + \coth(x_i - x_j) \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \right]. \]

Now, using part (1), \( \varphi^p_\lambda(x) = F_{BC}(i\lambda, k_p, x) \), and the Taylor expansion of \( \coth \) at 0, we get

\[-(\lambda_1^2 + \ldots + \lambda_q^2 + \|\rho\|^2)\varphi^p_\lambda(0) = \lim_{\|x\| \to 0} L\varphi^p_\lambda(x) \]
\[= (q + qk_1 + 2qk_2 + q(q - 1)k_3) \frac{\partial^2}{\partial x_1^2} \varphi^p_\lambda(x) \bigg|_{x=0} = \frac{(p + 1)qd}{2} \cdot \frac{\partial^2}{\partial x_1^2} \varphi^p_\lambda(x) \bigg|_{x=0} \]

for all \( \lambda \in \mathbb{C}^q \). Finally, as \( \co(W_B^B \cdot \rho) \) is contained in \( \{ x \in \mathbb{R}^q : \|x\|_2 \leq \|\rho\|^2 \} \), the final statement of (2) is also clear.

**Proof of Theorem 5.2** Let \( \lambda \in \mathbb{R}^q \cup i \cdot \co(W_B^B \cdot \rho) \). Then \( \varphi^p_\lambda \) is \( \mathbb{R} \)-valued. Lemma 5.3, \( \varphi^p_\lambda(0) = 1 \), and \( |\varphi^p_\lambda(x)| \leq 1 \) for \( x \in C_B^q \) ensure that there exists \( c > 0 \) with \( \varphi^p_\lambda(x) - 1 \geq -c(x_1^2 + \cdots + x_q^2) \) for \( x \in C_B^q \). Consequently, by Taylor’s expansion, \( n \left| \varphi^p_\lambda \left( \frac{x}{\sqrt{n}} \right) - 1 + \frac{\lambda_1^2 + \ldots + \lambda_q^2 + \|\rho\|^2}{(p + 1)qd} \cdot \frac{\|x\|^2}{n} \right| \leq C \|x\|^2 \) for some constant \( C > 0 \), where \( \|x\|^2 \) is integrable with respect to \( \nu \) by our assumption. Thus, by dominated convergence,

\[ \lim_{n \to \infty} n \int_{C_B^q} \left( \varphi^p_\lambda \left( \frac{x}{\sqrt{n}} \right) - 1 + \frac{\lambda_1^2 + \ldots + \lambda_q^2 + \|\rho\|^2}{(p + 1)qd} \cdot \frac{\|x\|^2}{n} \right) d\nu(x) = 0. \]

Rewriting this relation as

\[ \int_{C_B^q} \varphi^p_\lambda \left( \frac{x}{\sqrt{n}} \right) d\nu(x) = 1 - \frac{1}{n} \cdot \frac{\lambda_1^2 + \ldots + \lambda_q^2 + \|\rho\|^2}{(p + 1)qd} \cdot \int_{C_B^q} \|x\|^2 d\nu(x) + o \left( \frac{1}{n} \right) \]

we obtain

\[ F_{BC}(\mathbb{P}_{S^{(p,n-1/2)}})(\lambda) = \int_{C_B^q} \varphi^p_\lambda \left( \frac{x}{\sqrt{n}} \right) d\nu^{(n)}(x) = \left[ \int_{C_B^q} \varphi^p_\lambda \left( \frac{x}{\sqrt{n}} \right) d\nu(x) \right]^n \]
\[= \left( 1 - \frac{1}{n} \cdot \frac{\lambda_1^2 + \ldots + \lambda_q^2 + \|\rho\|^2}{(p + 1)qd} \cdot \int_{C_B^q} \|x\|^2 d\nu(x) + o \left( \frac{1}{n} \right) \right)^n \]

and
\[ \lim_{n \to \infty} \mathcal{F}_{BC}^p(\mathbb{P}_{S_n(p,n^{-1/2})})(\lambda) = \exp \left( \frac{-\lambda_1^2 + \cdots + \lambda_q^2 + \|\rho\|^2_2}{(p+1)d} \cdot \int_C \|x\|^2_2 \, d\nu(x) \right) \]

\[ = \exp \left( \frac{-t_0(\lambda_1^2 + \cdots + \lambda_q^2 + \|\rho\|^2_2)}{2(p+1)} \right) \]

for \( \lambda \in \mathbb{R}^q \cup i \cdot \text{co}(W_q^B \cdot \rho) \). Hence, by Lévy’s continuity theorem on commutative hypergroups [BH] Theorem 4.2.4(iv) there exists \( \nu \in \mathcal{M}_b^+(C_q^B) \) with

\[ \mathcal{F}_{BC}^p(\nu)(\lambda) = \exp \left( \frac{-t_0(\lambda_1^2 + \cdots + \lambda_q^2 + \|\rho\|^2_2)}{2(p+1)} \right) \]

for all \( \lambda \in \mathbb{R}^q \), and \( (\mathbb{P}_{S_n(p,n^{-1/2})})_{n \geq 1} \) converges to \( \nu \) weakly. As \( \mathcal{F}_{BC}^p(\nu)(-i\rho) = 1 \), the limit measure \( \nu \) is indeed a probability measure. This implies that \( (\mathbb{P}_{S_n(p,n^{-1/2})})_{n \geq 1} \) converges weakly to \( \nu = \gamma_{t_0/(p+1)} \) as desired. \( \blacksquare \)

**Remark 5.4.** The above proof shows that the measures \( \gamma_t \) in Definition 5.1 exist.

### 6. A law of large numbers for inner normalizations and growing parameters.

We now present a further limit theorem for \( (S_n(p,n^{-1/2}))_{n \geq 1} \) as \( p,n \to \infty \) where the limit is a point measure, i.e., a weak law of large numbers:

**Theorem 6.1.** Let \( \nu \in \mathcal{M}_b^1(C_q^B) \) with \( \nu \neq \delta_0 \) and finite second moments. Let \( t_0 := \frac{2}{q} \int_{C_q^B} \|x\|^2_2 \, d\nu(x) \) be as in Theorem 5.2 and \( (p_n)_{n \geq 1} \subset [2q-1, \infty[ \) be increasing with \( \lim_{n \to \infty} n/p_n = 0 \). Then \( (S_n(p_n,n^{-1/2}))_{n \geq 1} \) tends in probability as \( n \to \infty \) to

\[ \ln(e^{t_0/4} + \sqrt{e^{t_0/2} - 1}) \cdot (1, \ldots, 1). \]

For the proof we first rewrite the Taylor expansion for \( \varphi^A(x) \) at \( x = 0 \) from [GT] for \( d = 1 \) in our notation. For \( d = 2,4 \) this expansion also holds, and we have:

**Lemma 6.2.** As \( \|x\|_2 \to 0 \), \( \varphi^A(x) = 1 + \frac{i}{q} (\lambda_1 + \cdots + \lambda_q) \sum_{k=1}^q x_k + R_\lambda(x) \)

with \( R_\lambda(x) = \sum_{\alpha} f_\alpha(\lambda) P_\alpha(x) \) where the \( P_\alpha(x) \) are symmetric polynomials in \( x_1, \ldots, x_q \) which are homogeneous of order \( \geq 2 \).

**Lemma 6.3.** For \( p \geq 2q-1 \), the half-sum \( \rho := \rho^{BC}(p) \) satisfies \( \rho^A - \rho \in \text{co}(W_q^B \cdot \rho) \) with the Weyl group \( W_q^B \) of type \( B_q \).

**Proof.** Let \( \rho = (\rho_1, \ldots, \rho_q) \) and put \( \hat{\rho} := (\rho_q, \rho_{q-1}, \ldots, \rho_1) \). Then \( -\rho, -\hat{\rho} \) are in \( W_q^B \cdot \rho \). Moreover, by (2.5) and (2.10),

\[ \rho^A - \rho = \left( 1 - \frac{d}{2}(p+1) \right) (1, \ldots, 1) = \frac{1}{2} (-\rho - \hat{\rho}). \]

This proves the result. \( \blacksquare \)
Proposition 6.4. Let $\nu$, $t_0$ and $(p_n)_{n \geq 1}$ be as in Theorem 6.1. Consider the half-sum of positive roots $\rho(n) := \rho^{BC}(p_n)$ associated with the $p_n$ as in (2.10). Then, for all $\lambda \in \mathbb{C}^q$ with $\Im \lambda = \rho^A$,
\begin{align}
\varphi_{\lambda-i\rho(p)}^n \left( \frac{x}{\sqrt{n}} \right) d\nu(x) = 1 + i \frac{t_0}{4n} \sum_{k=1}^{q} (\lambda_k - i\rho_k^A) + o(1/n) \quad \text{as } n \to \infty.
\end{align}

Proof. Lemma 6.2 and the Taylor expansion $\ln \cosh x = x^2/2 + O(x^4)$ yield
\begin{align}
\varphi_{\lambda}^A \left( \ln \cosh \frac{x}{\sqrt{n}} \right) = 1 + i \sum_{k=1}^{q} \lambda_k \frac{\|x\|^2}{2ndq} + R_{\lambda} \left( \frac{\|x\|^2}{n} \right)
\end{align}
as $n \to \infty$ for $\lambda \in \mathbb{C}^q$. Moreover, [RV, Theorem 4.2(2)] states that $\varphi_{\lambda}^A \left( \ln \cosh \frac{x}{\sqrt{n}} \right)$ for all $\lambda \in \mathbb{C}^q$ with $\Im \lambda - \rho(n) \in \text{co}(W_q^B \cdot \rho(n))$. However, an analysis of the proof in [RV] shows that (6.3) is in fact valid only for $\lambda \in \{ \lambda \in \mathbb{C}^q : \Im \lambda - \rho(n) \in \text{co}(W_q^B \cdot \rho(n)) \}$ and $\Im \lambda - \rho^A \in \text{co}(W_q^A \cdot \rho^A)$.

If we use (6.2), (6.3), then as a consequence of Lemma 6.3 and $p_n/n \to \infty$ we see that
\begin{align}
\left| \varphi_{\lambda-i\rho(p)}^n \left( \frac{x}{\sqrt{n}} \right) - 1 - i \sum_{k=1}^{q} (\lambda_k - i\rho_k^A) \frac{\|x\|^2}{2ndq} \right| = o \left( \frac{\|x\|^2}{n} \right)
\end{align}
for all $\lambda \in \mathbb{C}^q$ with $\Im \lambda = \rho^A$. Integration with respect to $d\nu(x)$ and the moment condition on $\nu$ now yield the result. 

Proof of Theorem 6.1. Let $\nu^{(n,p_n)}$ be the $n$-fold $*_{p_n}$ convolution power of $\nu$. Proposition 6.4 shows that for all $\lambda \in \mathbb{C}^q$ with $\Im \lambda = \rho^A$,
\begin{align*}
\lim_{n \to \infty} \int_{C_q^q} \varphi_{\lambda-i\rho(p)}^n \left( \frac{x}{\sqrt{n}} \right) d\nu^{(n,p_n)}(x) &= \lim_{n \to \infty} \left( \int_{C_q^q} \varphi_{\lambda-i\rho(p)}^n \left( \frac{x}{\sqrt{n}} \right) d\nu(x) \right)^n \\
&= \lim_{n \to \infty} \left( 1 + i \frac{t_0}{4n} \sum_{k=1}^{q} (\lambda_k - i\rho_k^A) + o(1/n) \right)^n = e^{i(t_0/4) \sum_{k=1}^{q} (\lambda_k - i\rho_k^A)}.
\end{align*}
Thus, the distributions $\mathbb{P}_{T(S_{n}^{(p_n,n^{-1/2})})}$ of $T(S_{n}^{(p_n,n^{-1/2})})$ satisfy
\begin{align*}
\lim_{n \to \infty} \mathcal{F}^A(\mathbb{P}_{T(S_{n}^{(p_n,n^{-1/2})})}(\lambda - i\rho^A)) &= \lim_{n \to \infty} \int_{C_q^q} \varphi_{\lambda-i\rho^A}^n \left( \ln \cosh \frac{x}{\sqrt{n}} \right) d\nu^{(n,p_n)}(x) = e^{i(t_0/4) \sum_{k=1}^{q} (\lambda_k - i\rho_k^A)}
\end{align*}
for all $\lambda \in \mathbb{C}^q$ with $\Im \lambda = \rho^A$. The substitution $\lambda \mapsto \lambda + i\rho^A$ yields
\[
\lim_{n \to \infty} \mathcal{F}^A(\mathbb{P}_{T(S_n^{(p_n,n-1/2)})})(\lambda) = e^{(it_0/4)\sum_{k=1}^q \lambda_k}
\]
for all $\lambda \in \mathbb{R}^q$. On the other hand, by (1.3),
\[
\mathcal{F}^A(\delta_{(t_0/4)(1,...,1)})(\lambda) = \varphi^A_{\lambda}\left(\frac{t_0}{4}(1,\ldots,1)\right) = e^{(it_0/4)\sum_{k=1}^q \lambda_k}
\]
for $\lambda \in \mathbb{C}^q$. Since (6.5) holds on $\mathbb{R}^q$, i.e., on the support of the Plancherel measure, the continuity Theorem 4.2.11 in [BH] shows that $\mathbb{P}_{T(S_n^{(p_n,n-1/2)})}$ tends vaguely to $\delta_{(t_0/4)(1,...,1)}$. As $\mathbb{P}_{T(S_n^{(p_n,n-1/2)})}$ and $\delta_{(t_0/4)(1,...,1)}$ are probability measures, we even have weak convergence. Since $T^{-1}$ is continuous, the continuous mapping theorem shows that $\mathbb{P}_{S_n^{(p_n,n-1/2)}}$ tends weakly to $T^{-1}(\delta_{(t_0/4)(e_1,...,e_q)}) = \delta_{\ln(e^{t_0/4}+\sqrt{e^{t_0/2}-1})(1,...,1)}$.

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