The Quasi Curvature-Dimension Condition
with Applications to Sub-Riemannian Manifolds

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Abstract

We obtain the best known quantitative estimates for the $L^p$-Poincaré and log-Sobolev inequalities on domains in various sub-Riemannian manifolds, including ideal Carnot groups and in particular ideal generalized H-type Carnot groups and the Heisenberg groups, corank 1 Carnot groups, the Grushin plane, and various H-type foliations, Sasakian and 3-Sasakian manifolds. Moreover, this constitutes the first time that a quantitative estimate independent of the dimension is established on these spaces. For instance, the Li-Yau/ Zhong-Yang spectral-gap estimate holds on all Heisenberg groups of arbitrary dimension up to a factor of 4.

We achieve this by introducing a quasi-convex relaxation of the Lott-Sturm-Villani $CD(K, N)$ condition we call the “quasi curvature-dimension condition” $QCD(Q, K, N)$. Our motivation stems from a recent interpolation inequality along Wasserstein geodesics in the ideal sub-Riemannian setting due to Barilari and Rizzi. We show that on an ideal sub-Riemannian manifold of dimension $n$, the measure contraction property $MCP(K, N)$ implies $QCD(Q, K, N)$ with $Q = 2^{N-n} \geq 1$, thereby verifying the latter property on the aforementioned ideal spaces; a result of Balogh-Kristály-Sipos is used instead to handle nonideal corank 1 Carnot groups. By extending the localization paradigm to completely general interpolation inequalities, we reduce the study of various analytic and geometric inequalities on $QCD$ spaces to the one-dimensional case. Consequently, we deduce that while (strictly) sub-Riemannian manifolds do not satisfy any type of $CD$ condition, many of them satisfy numerous functional inequalities with exactly the same quantitative dependence (up to a factor of $Q$) as their $CD$ counterparts. © 2020 Wiley Periodicals LLC.

1 Introduction

The curvature-dimension condition $CD(K, N)$ was first introduced in the 1980s by Bakry and Émery [8,10] in the context of diffusion generators, having in mind primarily the setting of weighted Riemannian manifolds, namely smooth Riemannian manifolds endowed with a smooth density with respect to the Riemannian volume. The $CD(K, N)$ condition serves as a generalization of the classical condition

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in the nonweighted Riemannian setting of having Ricci curvature bounded below by $K \in \mathbb{R}$ and dimension bounded above by $N \in [1, \infty]$ (see, e.g., [67, 72] for further possible extensions). Numerous consequences of this condition have been obtained over the past decades, extending results from the classical nonweighted setting and at times establishing new ones directly in the weighted one. These include diameter bounds, volume comparison theorems, heat kernel and spectral estimates, Harnack inequalities, topological implications, Brunn-Minkowski-type inequalities, and isoperimetric, functional, and concentration inequalities—see, e.g., [11, 55, 87] and the references therein.

Being a differential and Hilbertian condition, it was for many years unclear how to extend the Bakry-Émery definition beyond the smooth Riemannian setting. A satisfactory definition was finally found based on the theory of optimal transport [2, 3, 39, 66, 76, 84, 86, 87]. Given two probability measures $\mu_0, \mu_1$ on a common geodesic space $(X, d)$ and a prescribed cost of transporting a single mass from point $x$ to $y$, the Monge-Kantorovich idea is to optimally couple $\mu_0$ and $\mu_1$ by minimizing the total transportation cost, and as a by-product obtain a Wasserstein geodesic $[0, 1] \ni t \mapsto \mu_t$ connecting $\mu_0$ and $\mu_1$ in the space of probability measures $\mathcal{P}(X)$. This gives rise to the notion of displacement convexity of a given functional on $\mathcal{P}(X)$ along Wasserstein geodesics, introduced and studied by McCann [64]. Following the works of Cordero-Erausquin–McCann–Schmuckenschläger [34], Otto-Villani [73], and von Renesse-Sturm [88], it was realized that the $\text{CD}(K, \infty)$ condition in the smooth setting may be equivalently formulated synthetically as a certain convexity property of an entropy functional along $W_2$ Wasserstein geodesics (associated to $L^2$-optimal-transport, when the transport cost is given by the squared-distance function).

This idea culminated in the seminal works of Lott, Sturm, and Villani [61, 82, 83], where a synthetic definition of $\text{CD}(K, N)$ was proposed on a general (complete, separable) metric space $(X, d)$ endowed with a (locally finite Borel) reference measure $m$ (“metric-measure space”); it was moreover shown that the latter definition coincides with the Bakry-Émery one in the smooth Riemannian setting (and in particular in the classical nonweighted one), that it is stable under measured Gromov-Hausdorff convergence, and that it implies various geometric and analytic inequalities relating metric and measure, in complete analogy with the smooth setting. It was subsequently also shown [71, 75] that Finsler manifolds and Alexandrov spaces satisfy the curvature-dimension condition. Thus emerged an overwhelmingly convincing notion of Ricci curvature lower bound $K$ and dimension upper bound $N$ on metric-measure spaces, leading to a rich and fruitful theory exploring the geometry of such spaces by means of optimal transport.

However, one interesting setting in which the $\text{CD}$ theory is not applicable (at least, not directly) is the sub-Riemannian one. It was first shown by Juillet [50] that the $d$-dimensional Heisenberg group $\mathbb{H}^d$, which is the simplest example of a nontrivial sub-Riemannian manifold, equipped with the Carnot-Carathéodory metric and left-invariant Lebesgue measure, does not satisfy the $\text{CD}(K, N)$ condition
for any $K, N \in \mathbb{R}$. In [51], Juillet extended this observation to completely general (strictly) sub-Riemannian manifolds (in which the rank of the distribution is nowhere maximal) equipped with an arbitrary smooth positive measure. On the other hand, Juillet showed in [50] that the Heisenberg group $\mathbb{H}^d$ (of topological dimension $n = 2d + 1$) does satisfy the property $\text{MCP}(0, N)$ for $N = n + 2$. The latter is a particular case of the measure contraction property $\text{MCP}(K, N)$, introduced independently by Ohta [70] and Sturm [83] as a weaker variant of the $\text{CD}(K, N)$ condition. More general Carnot groups were subsequently shown to satisfy $\text{MCP}(0, N)$ for appropriate $N$ by Rifford, Barilari, and Rizzi [15, 79, 80]. Additional examples of sub-Riemannian spaces verifying $\text{MCP}(\cdot, K, N)$ have been found in [7, 16, 17, 21, 56], such as generalized H-type groups, the Grushin plane, and various H-type foliations, and Sasakian and 3-Sasakian structures.

In the past year, the study of MCP spaces has seen some increased activity, starting from the work of Cavalletti and Santarcangelo [33], who obtained sharp isoperimetric inequalities, and continuing with the work of Han-Milman [46] and Han [45], who obtained sharp Poincaré and $L^p$-Poincaré inequalities, respectively, for $\text{MCP}(K, N)$ spaces whose diameter is upper-bounded by $D \in (0, \infty)$. While these results are sharp for the class of MCP spaces, as witnessed by equipping $(\mathbb{R}, | \cdot |)$ with an appropriate measure $m$, it remained unclear whether they provide good quantitative estimates for the above specific examples from the sub-Riemannian setting, which certainly have more structure than general MCP spaces. Moreover, the recent Jacobian interpolation inequalities à la Cordero-Erausquin–McCann–Schmuckenschläger [34], obtained by Balogh, Kristály, and Sipos [13, 14] for the Heisenberg group and more general corank 1 Carnot groups and by Barilari and Rizzi [16] in the ideal sub-Riemannian setting (see below), strongly suggest that more information can be extracted in these cases than by merely employing the MCP property.

In this work, we introduce a new property we call quasi curvature-dimension $\text{QCD}(Q, K, N)$ ($Q \geq 1$), which constitutes a “quasi-convex” relaxation of the $\text{CD}(K, N)$ condition (the latter is recovered when the “slack” parameter $Q$ is set to 1), and serves as a bridge between the CD and MCP conditions. We draw our nomenclature from the theory of quasi-Banach spaces—recall that a 1-homogeneous functional $\| \cdot \|$ on a linear space $E$ is called a quasi-norm if there exists a constant $Q \geq 1$ so that

$$\|(1 - t)x_0 + tx_1\| \leq Q((1 - t)\|x_0\| + t\|x_1\|) \quad \forall x_0, x_1 \in E, \forall t \in (0, 1).$$

Roughly speaking, our main results in this work are as follows:

- In the above sub-Riemannian examples, and more generally, whenever an appropriate $(X, d, m)$ satisfies a “Jacobian” interpolation inequality and $\text{MCP}(K, N)$ holds, then $\text{QCD}(Q, K, N)$ holds as well with $Q = 2^{n-n}$, where $n$ denotes the topological dimension (in fact, modulo the results of [13, 14, 16], this will be essentially trivial). This extends the well-known fact [83, cor. 5.5] that when $N = n$ (so that $Q = 2^{n-n} = 1$),
the MCP($K, n$) condition on unweighted Riemannian manifolds is equivalent to the QCD(1, $K, n$) = CD($K, n$) condition (i.e., to a lower bound $K$ on the Ricci curvature).

- Any property of CD($K, N$) spaces that is amenable to localization and in dimension one is stable under perturbations, and also holds (up to constants depending only on $Q$) for QCD($Q, K, N$) spaces (which are in addition essentially nonbranching and also satisfy the MCP($K', n'$) condition for some $K' \in \mathbb{R}$ and $n' \in (1, \infty)$—see below for more details). For example, this applies to $L^p$-Poincaré inequalities, Sobolev and log-Sobolev inequalities, as well as isoperimetric inequalities.

Consequently, we deduce that while (strictly) sub-Riemannian manifolds do not satisfy any type of CD condition, in the ideal and corank 1 Carnot group settings, they satisfy (up to constants) most geometric and analytic properties as their CD counterparts. Moreover, the latter constants do not directly depend on the topological dimension $n$ nor the geodesic dimension $N$, but rather on their difference via the formula $Q = 2^{N-n}$, and so for any family of spaces for which $N - n$ is bounded above, they are dimension-independent (!); for example, $Q = 2^{N-n} = 4$ for all $d$-dimensional Heisenberg groups $\mathbb{H}^d$, regardless of $d$.

As a taste of the type of results one can obtain using these observations, we state the following consequence of our main theorem, Theorem 2.9. We refer to the next sections for precise definitions, and at this point only introduce the notation $\text{geo}(/)$ for the geodesic hull of a set, namely the union of all geodesics starting at $x \in \Omega$ and ending at $y \in \Omega$. Note that $\text{geo}(\Omega)$ need not be geodesically convex, and that $\text{geo}(B_r(x)) \subset B_{2r}(x)$ by the triangle inequality (where $B_r(x)$ denotes a geodesic ball around $x$ of radius $r > 0$)—the latter is the prototypical example the reader should bear in mind below.

**THEOREM 1.1.** Let $X$ be an ideal generalized H-type group of dimension $n$ and corank $k$, equipped with its Carnot-Carathéodory metric $d$ and canonical left-invariant volume measure $m$. Then for all closed subsets $\Omega \subset X$ with $\text{diam}(\Omega) \leq D < \infty$ and for any (locally) $d$-Lipschitz function $f : (X, d) \rightarrow \mathbb{R}$:

- The following Poincaré inequality holds:

\[
\int_\Omega f \, m = 0 \Rightarrow \frac{\pi^2}{4^k \cdot D^2} \int_\Omega f^2 \, m \leq \int_{\text{geo}(\Omega)} |\nabla_X f|^2 \, m.
\]

- More generally, the following $L^p$-Poincaré inequality holds for any $p \in (1, \infty)$:

\[
\int_\Omega |f|^{p-2} f \, m = 0 \Rightarrow \frac{p - 1}{4^k} \left( \frac{2\pi}{p \cdot \sin(\pi / p) \cdot D} \right)^p \int_\Omega |f|^p \, m \leq \int_{\text{geo}(\Omega)} |\nabla_X f|^p \, m.
\]
The following log-Sobolev inequality holds (for some universal numeric $C > 1$):

$$
\int_{\Omega} (f^2 - 1) m = 0 \Rightarrow \frac{\pi^2}{4k^2 CD^2} \int_{\Omega} f^2 \log(f^2) m \leq \int_{\text{geo}(\Omega)} |\nabla_X f|^2 m.
$$

In particular, this applies to all Heisenberg groups $\mathbb{H}^d$ with $k = 1$ (independently of $d$).

Analogous results hold for ideal Carnot groups, the (ideal) Grushin plane, (ideal) Sasakian and 3-Sasakian manifolds (under appropriate curvature lower bounds), (ideal) H-type foliations with completely parallel torsion and nonnegative horizontal sectional curvature, and general (possibly nonideal) Carnot groups of corank 1—see Section 2. To put these results into context, note that the Poincaré inequality (1.1) on the Heisenberg group $\mathbb{H}^d$ coincides up to a factor of 4 with the celebrated Li-Yau/Zhong-Yang sharp spectral-gap estimate [58,91], which applies to geodesically convex subsets of $CD(0,N)$ spaces [12,31,54]. Instead of assuming that $\Omega$ is geodesically convex, we use an arbitrary set $\Omega$ but take its geodesic hull $\text{geo}(\Omega)$ on the energy side of the inequality—this variant, originating in our previous work with B. Han [46], is crucial in the sub-Riemannian setting, where nontrivial geodesically convex sets are known to be scarce; for instance, even for the simplest case of the Heisenberg group $\mathbb{H}^1$, it was shown in [68] that the smallest geodesically convex set containing three distinct points that do not lie on a common geodesic is $\mathbb{H}^1$ itself, implying in particular that there are no nontrivial geodesically convex balls in $\mathbb{H}^1$. Similarly, up to the factor of $4^k$, our estimates for the $L^p$-Poincaré inequality (spectral gap of the $p$-Laplacian) and for the log-Sobolev inequality are known to be best possible on geodesically convex subsets of $CD(0,N)$ spaces.

To the best of our knowledge, Theorem 1.1 entails the best known quantitative estimates for the $L^p$-Poincaré and log-Sobolev inequalities in the above-mentioned sub-Riemannian setting, and moreover, constitutes the first time that a dimension-independent quantitative estimate (not depending on $n$) has been established on the above spaces.

While the validity of a local Poincaré inequality in the sub-Riemannian setting is well-known, starting from the work of D. Jerison on vector fields satisfying the Hörmander’s condition [49] (see also [35] and the references therein), we are almost not aware of any explicit constants in any of these inequalities. This includes the subelliptic curvature-dimension approach developed by Baudoin-Garofalo [20], which was used by Baudoin-Bonnefont-Garofalo in [19, theorem 4.2] to obtain a local Poincaré inequality on various sub-Riemannian manifolds satisfying a nonnegative generalized Ricci curvature bound—namely, for $\Omega = B_r(x)$ and with $B_{2r}(x)$ on the energy side of the Poincaré inequality (in place of $\text{geo}(B_r(x)))$, these authors obtained a Poincaré constant of the form $C/r^2$ for all
$r > 0$ and some nonexplicit constant $C > 0$ depending on various additional curvature parameters and the underlying dimension. Note that by [44], it is always possible to tighten (i.e., replace $B_{2r}(x)$ by $B_r(x)$ on the energy side) a local $L^p$-Poincaré inequality on any length space (see also [49]), but this would result in a further loss of explicit constants and dependence on the underlying dimension (via the doubling constant). The results of [19] were extended to a possibly negative generalized Ricci bound by Kim [53, theorem 1.1]. We remark that when the generalized Ricci curvature is strictly positive in the sense of [20], the global situation is simpler as the underlying measure is necessarily finite, and a global Poincaré as well as (a variant of) a log-Sobolev inequality, with explicit constants, were obtained by Baudoin-Bonnefont in [18]; however, it is not clear how to localize these estimates to geodesic balls. For gradient estimates on the heat-kernel on the Heisenberg group and its associated (global) Poincaré and log-Sobolev inequalities, see [9, 24, 36, 47, 57].

The only prior explicit estimates we are aware of for Poincaré and $L^p$-Poincaré inequalities on geodesic balls in the sub-Riemannian setting were just recently obtained in [46] and [45], respectively, but these only employed the MCP information, and thus are inevitably worse than the estimates of Theorem 1.1 by a factor exponential in the dimension $n$.

We refer the reader to the next section for the definition of the QCD($Q, K, N$) condition and statement of our main results. The rest of this work is organized as follows. In Section 3 we recall some preliminaries from sub-Riemannian geometry and the theory of optimal transport. In Section 4 we prove a localization theorem for general interpolation coefficients. In Section 5 we study one-dimensional QCD densities. In Section 6 we prove our main result on the equivalence (up to a factor of $Q$) between the best constants in various functional inequalities on QCD spaces and their CD counterparts. In Section 7 we provide some concluding remarks.

2 Statement of the Results

2.1 Curvature via interpolation

The starting point of this work is the following interpolation inequality along $W_2$ geodesics. It will be more convenient to state it using a dynamical plan $\nu$, namely a probability measure on $\text{Geo}(X, d)$, the space of constant speed geodesics $\gamma$ parametrized on the unit interval $[0, 1]$. We know that any $W_2$ geodesic $(\mu_t)_{t \in [0, 1]}$ can be lifted to an optimal dynamical plan $\nu$ so that $(\nu_t)_{t \in [0, 1]} = \mu_t$ for all $t \in [0, 1]$, where $\nu_t(\gamma) = \gamma_t$ denotes the evaluation map.

We will say that a metric-measure space $(X, d, m)$ is Monge if for any two probability measures with finite second moments $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with $\mu_0 \ll m$ and $\text{supp}(\mu_1) \subseteq \text{supp}(m)$, there exists a unique $W_2$ geodesic $\mu_t \in \mathcal{P}(X)$ connecting $\mu_0, \mu_1$; it is given by a map (there exists $S : X \to \text{Geo}(X, d)$ so that $\nu = S_\nu \mu_0$ is the associated optimal dynamical plan), and $\mu_t = (\nu_t)_{t \in [0, 1]}$. We refer to Section 5 for missing definitions and assertions, and only
presently remark that in this work, a geodesic is always meant to mean minimizing geodesic, and that $\mathcal{P}_c(X)$ denotes the space of (Borel) probability measures on $X$ with bounded support.

Let $(\mathcal{D}, g)$ denote a sub-Riemannian structure on a smooth $n$-dimensional connected manifold $M$, and let $d$ denote the associated Carnot-Carathéodory sub-Riemannian metric. Assume that $(M, \mathcal{D}, g)$ is ideal, namely that it admits no nontrivial abnormal geodesics and that $(M, d)$ is complete. Let $\mu$ denote a measure with smooth positive density with respect to some (any) volume measure on $M$. It follows from the work of McCann [65] and Cordero-Erausquin–McCann–Schmuckenschläger [34] in the complete Riemannian setting and of Figalli and Rifford [40] in the ideal sub-Riemannian one that $(M, d, \mu)$ is a Monge space. The following interpolation inequality was first established in the Riemannian setting by Cordero-Erausquin–McCann–Schmuckenschläger [34], and very recently extended to the ideal sub-Riemannian setting by Barilari and Rizzi [16]:

**THEOREM 2.1** (Interpolation inequality for ideal (sub-)Riemannian manifolds [16, 34]). Let $\mu_0, \mu_1 \in \mathcal{P}_c(M)$ with $\mu_0, \mu_1 \ll \mu$, where $(M, \mathcal{D}, \mu)$ denotes an ideal sub-Riemannian manifold as above, and let $\nu$ be the associated optimal dynamical plan. Denoting by $\rho_t := \frac{d\nu_t}{d\mu}$ the corresponding densities along the $W_2$ geodesic from $\mu_0$ to $\mu_1$, one has for any $t \in (0, 1)$:

\[
\rho_t \frac{1}{\nu}(\gamma_t) \geq \beta_t^{1/2}(\gamma_1, \gamma_0) \rho_0 \frac{1}{\nu}(\gamma_0) + \beta_t^{1/2}(\gamma_0, \gamma_1) \rho_1 \frac{1}{\nu}(\gamma_1)
\]

for $\nu$-a.e. $\gamma \in \text{Geo}(M, d)$. Here $\beta_t(x, y)$ denotes the measure distortion coefficient from $x \in M$ to $y \in M$, defined as

\[
\beta_t(x, y) := \limsup_{r \to 0+} \frac{\mathcal{m}(Z_t(\{x\}, B_r(y)))}{\mathcal{m}(B_r(y))},
\]

where $Z_t(A, B)$ denotes the set of all $t$-midpoints between points $a \in A$ and $b \in B$ (if $A, B$ are Borel measurable, $Z_t(A, B)$ is analytic and hence $\mathcal{m}$-measurable).

On an $N$-dimensional Riemannian manifold whose Ricci curvature is bounded below by $K \in \mathbb{R}$, classical comparison theorems verify that

\[
\beta_t^{1/N}(x, y) \geq \tau_{K,N}^{(t)}(d(x, y))
\]

(with equality on model spaces of constant sectional curvature $\frac{K}{N-1}$), where

\[
\tau_{K,N}^{(t)}(\theta) := t^{\frac{1}{N}} \left( \sigma_{K,N-1}(\theta) \right)^{1 - \frac{1}{N}},
\]

\[
\sigma_{K,N-1}(\theta) := \begin{cases} +\infty & \text{if } K\theta^2 \geq \pi^2(N-1), \\ \frac{s_{K/(N-1)}(\theta)}{s_{K/(N-1)}(\theta)} & \text{otherwise}, \end{cases}
\]

where
and

\[ s_\kappa(\theta) := \begin{cases} 
\frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} \theta), & \text{if } \kappa > 0, \\
\theta, & \text{if } \kappa = 0, \\
\frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} \theta), & \text{if } \kappa < 0.
\end{cases} \]

The definitions of \( \text{CD}(K, N) \) given by Sturm \footnote{82, 83} and Lott–Villani \footnote{60, 61} may then be described in analogy to the above (sub-)Riemannian interpolation inequality. While their definitions are more involved (and slightly differ) on general metric-measure spaces \( (X, d, m) \) and for general \( N \in [1, \infty) \), when \( N \in (1, \infty) \) and on Monge spaces, the condition simplifies to requiring that for all \( \mu_0, \mu_1 \in \mathcal{P}_c(X) \) with \( \mu_0 \ll m \) and for all \( t \in (0, 1) \):

\[ \rho_t^{-\frac{1}{N}}(\gamma_t) \geq \tau_{K, N}^{(1-t)}(d(\gamma_0, \gamma_1)) \rho_0^{-\frac{1}{N}}(\gamma_0) + \tau_{K, N}^{(t)}(d(\gamma_0, \gamma_1)) \rho_1^{-\frac{1}{N}}(\gamma_1) \]

for \( \nu \)-a.e. \( \gamma \in \text{Geo}(X, d) \).

Similarly, the (weaker) \( \text{MCP}(K, N) \) condition on Monge spaces is defined by requiring that for all \( \mu_0, \mu_1 \in \mathcal{P}_c(X) \) with \( \mu_0 \ll m \) and \( \text{supp}(\mu_1) \subset \text{supp} m \), for all \( t \in (0, 1) \):

\[ \rho_t^{-\frac{1}{N}}(\gamma_t) \geq \tau_{K, N}^{(1-t)}(d(\gamma_0, \gamma_1)) \rho_0^{-\frac{1}{N}}(\gamma_0) \]

for \( \nu \)-a.e. \( \gamma \in \text{Geo}(X, d) \).

Equivalently, it is enough to check this for \( \mu_0 = \frac{1}{m(B)} m_{\mathcal{L}B} \) with bounded \( B \) (0 < \( m(B) < \infty \)) and for \( \mu_1 = \delta_o \) with \( o \in \text{supp}(m) \). In particular, it follows (since \( \int_{\text{supp} \mu_t} \rho_t = 1 \)) that

\[ m(Z_{1-t}(\{o\}, B)) = m(Z_t(B, \{o\})) \geq m(\text{supp} \mu_t) \geq \tau_{K, N}^{(1-t)}(\Theta_{o, B})^N m(B), \]

where:

\[ \Theta_{o, B} := \begin{cases} 
\inf_{x \in B} d(o, x), & K \geq 0, \\
\sup_{x \in B} d(o, x), & K < 0,
\end{cases} \]

and we immediately conclude from \footnote{2.2} that on \( \text{MCP}(K, N) \) spaces

\begin{equation}
(2.3) \quad \beta_t(x, y) \geq \tau_{K, N}^{(t)}(d(x, y))^N \quad \forall t \in (0, 1).
\end{equation}

It is presently not known whether the ideal assumption in Theorem \footnote{2.1} can be removed (say, replaced with being complete and Monge). However, there is one nonideal (yet still Monge) sub-Riemannian setting in which an analogous result has been established. The following was very recently shown by Balogh–Kristály–Sipos \footnote{14}:

**Theorem 2.2** (Interpolation inequality for corank 1 Carnot groups \footnote{14}). Let \( M \) denote an \( n \)-dimensional corank 1 Carnot group, endowed with its Carnot-Carathéodory sub-Riemannian metric \( d \) and left-invariant measure \( m \). Then for
any $\mu_0, \mu_1 \in \mathcal{P}_e(M)$ with $\mu_0, \mu_1 \ll m$, \((2.1)\) holds. Furthermore,
\begin{equation}
(2.4) \quad \beta_{1-t}(\gamma_1, \gamma_0) \geq (1 - t)^{n+2} \quad \text{and} \quad \beta_t(\gamma_0, \gamma_1) \geq t^{n+2}
\end{equation}
for $\nu$-a.e. $\gamma \in \text{Geo}(M, d)$.

In fact, \((2.4)\) was previously shown by Rizzi \cite{80}, thereby deducing that corank 1 Carnot groups satisfy MCP$(0, n + 2)$. Note that a corank 1 Carnot group $(M, d, m)$ as above is indeed a Monge space, even though it may not be ideal—see Section 3.3 below.

2.2 The quasi curvature-dimension condition

We are now ready to introduce the following definition and establish the subsequent proposition; we continue using the standard notation from the previous subsection.

**Definition 2.3** (Quasi curvature-dimension QCD$(Q, K, N)$). A Monge space $(X, d, m)$ is said to satisfy the QCD$(Q, K, N)$ condition, if for all $\mu_0, \mu_1 \in P_e(X)$ with $\mu_0, \mu_1 \ll m$ and for all $t \in (0, 1)$:
\begin{equation}
(2.5) \quad \underset{\nu}{\rho_t^{-\frac{1}{Q}}} (\gamma_t) \geq \frac{1}{Q^{1/N}} \left( \frac{1 - t}{Q} (d(\gamma_0, \gamma_1)) \rho_0^{-\frac{1}{K}} (\gamma_0) + \frac{t}{Q} (d(\gamma_0, \gamma_1)) \rho_1^{-\frac{1}{N}} (\gamma_1) \right)
\end{equation}
for $\nu$-a.e. $\gamma \in \text{Geo}(X, d)$.

**Proposition 2.4.** Let $(M, D, g)$ denote an $n$-dimensional ideal sub-Riemannian manifold, let $d$ denote the associated Carnot-Carathéodory sub-Riemannian metric, and let $m$ be a measure with smooth positive density on $M$. If $(M, d, m)$ satisfies MCP$(K, N)$, then it also satisfies QCD$(Q, K, N)$ with $Q = 2^{N-n}$.

**Proof.** By the preceding comments, we know that the MCP$(K, N)$ condition implies \((2.3)\). Note that necessarily $N \geq n$, since otherwise this would mean that $\beta_t(x, y) \gg t^n$ as $t \to 0$, which is easily seen to be impossible (see, e.g., \cite[theorem 5]{16}). Plugging this into the interpolation theorem \((2.1)\) and applying Jensen’s inequality
\[ a, b \geq 0, \quad \alpha \in (0, 1] \quad \Rightarrow \quad (a + b)^\alpha \geq 2^{\alpha - 1} (a^\alpha + b^\alpha) \]
with $\alpha = \frac{n}{N} \in (0, 1]$, we deduce that with the same notation used there, for all $t \in (0, 1)$, for $\nu$-a.e. $\gamma \in \text{Geo}(X, d)$:
\[ \rho_t^{-\frac{1}{Q}} (\gamma_t) \geq \frac{1}{2^{\frac{1}{n} - \frac{n}{N}}} \left( \frac{1 - t}{Q} (d(\gamma_0, \gamma_1)) \rho_0^{-\frac{1}{K}} (\gamma_0) + \frac{t}{Q} (d(\gamma_0, \gamma_1)) \rho_1^{-\frac{1}{N}} (\gamma_1) \right). \]

It was shown in \cite{15, 17, 21, 56, 79, 80} that general ideal Carnot groups, ideal generalized H-type groups, and the Heisenberg group in particular, the (ideal) Grushin plane, (ideal) Sasakian and 3-Sasakian manifolds (under appropriate curvature lower bounds), and (ideal) H-type foliations with completely parallel torsion
and nonnegative horizontal sectional curvature, when endowed with their canonical sub-Riemannian metric and volume measure, all satisfy $\text{MCP}(0, N)$ for appropriate $N \in (1, \infty)$ (see these references and also \cite{7} for additional nonideal classes). It follows by Proposition \ref{prop:2.4} that in addition, they also satisfy $\text{QCD}(Q, 0, N)$ for appropriate $Q > 1$. We will only record the following particular instance that follows by combining Proposition \ref{prop:2.4} with \cite{15, theorem 3} (cf. \cite{16, sec. 7.2}).

**Corollary 2.5.** Any ideal generalized H-type group $X$ of dimension $n$ and corank $k$, equipped with its Carnot-Carathéodory sub-Riemannian metric $d$ and left-invariant measure $m$, satisfies $\text{MCP}(0, n + 2k)$ and hence $\text{QCD}(4^k, 0, n + 2k)$. In particular, this applies to all Heisenberg groups $\mathbb{H}^d$ with $n = 2d + 1$ and $k = 1$.

In the nonideal setting, by invoking Theorem \ref{thm:2.2} instead of Theorem \ref{thm:2.1} above, a completely identical argument for corank 1 Carnot groups yields the following:

**Corollary 2.6.** A corank 1 Carnot group $X$ of dimension $n$, equipped with its Carnot-Carathéodory sub-Riemannian metric $d$ and left-invariant measure $m$, satisfies $\text{MCP}(0, n + 2)$ and hence $\text{QCD}(4, 0, n + 2)$.

See Section 7.3 for a discussion of the optimality of the constant $Q = 2^N \to n$ in Proposition \ref{prop:2.4} (and in particular the constant $4^k$ in Corollary \ref{cor:2.5}) as well as the constant $Q = 4$ in Corollary \ref{cor:2.6}.

### 2.3 One-dimensional QCD spaces

Up until now we have not really done anything of substance, besides applying Jensen’s inequality and introducing the QCD definition, so we must now justify its usefulness. The latter stems from the following one-dimensional observation. We denote by $\mathcal{L}^1$ the Lebesgue measure on $\mathbb{R}$.

**Proposition 2.7.** Let $h$ be a density on $\mathbb{R}$ that is continuous on its support. Then $(\mathbb{R}, | \cdot |, h\mathcal{L}^1)$ is a QCD($Q, K, N$) space iff there exists a density $f$ on $\mathbb{R}$, continuous on its support, with

$$h \leq f \leq Qh,$$

so that $(\mathbb{R}, | \cdot |, f\mathcal{L}^1)$ is a CD($K, N$) space.

This is proved in Corollary \ref{cor:5.4} and Proposition \ref{prop:5.7} by taking $f$ to be the “CD($K, N$) upper envelope” of $h$. It is not too hard to realize that \ref{eq:2.6} is a genuinely one-dimensional property, and that an analogous necessary condition need not hold in higher-dimensional settings without some dimension-dependence in the estimate; indeed, by Carathéodory’s theorem, the convex hull in $\mathbb{R}^n$ can be realized by $n + 1$ points but no less in general, and so any penalty incurred for “quasi-concavity” between two points will be amplified as the dimension increases. Consequently, we need an apparatus for reducing the study of QCD spaces to the one-dimensional case.
2.4 General localization theorem

We achieve this apparatus by extending the localization method—a paradigm that reduces the task of establishing various analytic and geometric inequalities on an $n$-dimensional space to the one-dimensional setting—to spaces satisfying general interpolation inequalities that include the QCD case.

In the Euclidean setting, the localization method has its roots in the work of Payne and Weinberger [74] on the spectral gap for convex domains in Euclidean space, and has been further developed by Gromov and V. Milman [43] and Kannan, Lovász, and Simonovits [22]. In a groundbreaking work [54], B. Klartag reinterpreted the localization paradigm as a measure disintegration adapted to $L^1$-optimal-transport, and extended it to weighted Riemannian manifolds satisfying $CD(K, N)$. In a subsequent breakthrough, Cavalletti and Mondino [30] (cf. [32]) have succeeded in extending this technique to Monge spaces satisfying $CD(K, N)$ with $N < \infty$. The localization method is also available on Monge spaces satisfying $\text{MCP}(K, N)$ with $N < \infty$ [26, 32], starting from the work of Bianchini and Cavalletti in the nonbranching setting [23].

In Theorem 4.1, we extend the localization method to Monge spaces for completely general interpolation coefficients. With our usual notation, it applies assuming that the Monge space is $\text{MCP}(K', N')$ for some $K' \in \mathbb{R}$ and $N' \in (1, \infty)$, and that for a fixed $N \in (1, \infty)$ and coefficients $(0, 1) \times \mathbb{R}_+ \ni (t, \theta) \mapsto \sigma_i(t)(\theta) \in [0, +\infty]$, $i = 0, 1$, which are continuous in each variable, the following interpolation property holds for all $t \in (0, 1)$:

$$\rho_t^{-\frac{1}{N'}}(\gamma_t) \geq \sigma_0(1-t)(d(\gamma_0, \gamma_1))\rho_0^{-\frac{1}{N}}(\gamma_0) + \sigma_1(t)(d(\gamma_0, \gamma_1))\rho_1^{-\frac{1}{N}}(\gamma_1)$$

for $\nu$-a.e. $\gamma \in \text{Geo}(X, d)$.

The proof is based on the proof of the localization theorem for $CD(K, N)$ spaces by Cavalletti and Mondino [30] theorem 5.1, with one crucial difference—in [30], the fact that the $CD(K, N)$ condition on a one-dimensional geodesic enjoys the local-to-global property was extensively used, and so it was enough to establish it locally on geodesics participating in the localization. In contrast, the above condition employing general functions $\sigma_0, \sigma_1$ will typically not satisfy the local-to-global property even on a one-dimensional space (this is the case for $QCD(Q, K, N)$ when $Q > 1$ and even $\text{MCP}(K, N)$), and so we are required to directly obtain the global property on the geodesics.

2.5 Functional inequalities on QCD spaces

Combining all of the above ingredients, we are able to conclude that any property that is amenable to localization and stable under perturbations as in (2.6) will be shared by $QCD(Q, K, N)$ spaces together with their $CD(K, N)$ counterparts, up to constants depending solely on $Q$. Fortunately, this includes a multitude of fundamental analytic and geometric properties; we will only demonstrate this for the $L^p$-Poincaré and log-Sobolev inequalities. We remark that the Poincaré inequality...
is sometimes also referred to as the “Poincaré-Wirtinger” or “Poincaré-Neumann” inequality in the literature.

Given a metric-measure space \((X, d, m)\), let \(|\nabla_X f| : X \to \mathbb{R}\) denote the local Lipschitz constant of \(f\), defined as

\[
|\nabla_X f|(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y, x)}
\]

(and 0 if \(x\) is an isolated point). Throughout this work, by “locally Lipschitz function” we mean a locally \(d\)-Lipschitz function. Assume that \(\text{supp}(m)\) is geodesically convex (any two points in \(\text{supp}(m)\) can be connected by a geodesic in \(\text{supp}(m)\)).

Given a subset \(\text{supp}(m)\), recall that \(\text{geo}(\cdot)\) denotes its geodesic hull.

We denote by \(\lambda_p[(X, d, m), \Omega]\) the best constant \(\lambda_p\) so that for any (locally) Lipschitz function \(f : (X, d) \to \mathbb{R}\), the following \(L^p\)-Poincaré inequality holds:

\[
\int_{\Omega} |f|^p - f m = 0 \quad \Rightarrow \quad \lambda_p \int_{\Omega} |f|^p m \leq \int_{\text{geo}(\Omega)} |\nabla_X f|^p m.
\]

We denote by \(\lambda_{LS}[(X, d, m), \Omega]\) the best constant \(\lambda_{LS}\) so that for any (locally) Lipschitz function \(f : (X, d) \to \mathbb{R}\), the following log-Sobolev inequality holds:

\[
\int_{\Omega} (f^2 - 1) m = 0 \quad \Rightarrow \quad \frac{\lambda_{LS}}{2} \int_{\Omega} f^2 \log(f^2) m \leq \int_{\text{geo}(\Omega)} |\nabla_X f|^2 m.
\]

The idea to use \(\text{geo}(\Omega)\) instead of \(\Omega\) on the energy side of the functional inequalities above originated in our previous work with B. Han [46], and enables us to get a meaningful inequality without imposing various extra conditions on \(\Omega\). Indeed, if we were to replace \(\text{geo}(\Omega)\) by \(\Omega\), the best constants above would clearly be 0 for (say) disconnected \(\Omega\), or even if \(\Omega\) just contains arbitrarily small necks.

One way to resolve this is to require that \(\Omega\) be geodesically convex, but as already mentioned in the Introduction, this is too strong of an imposition on many spaces, especially in the sub-Riemannian setting, where geodesically convex subsets are known to be scarce.

Given a family \(\mathcal{X}\) of metric-measure spaces \((X, d, m)\) so that \(\text{supp}(m)\) is geodesically convex and \(D \in (0, \infty)\), we denote by \(\Xi_{\mathcal{X}, D}\) the collection of all \((\mathcal{X}, \Omega)\) where \(\mathcal{X} = (X, d, m) \in \mathcal{X}\) and \(\Omega\) is a closed subset of \(\text{supp}(m)\) with \(\text{diam}(\Omega) \leq D\). For any of our constants \(\lambda_* \in \{\lambda_p, \lambda_{LS}\}\), we set

\[
\lambda_*[\mathcal{X}, D] := \inf\{\lambda_*[\mathcal{X}, \Omega], (\mathcal{X}, \Omega) \in \Xi_{\mathcal{X}, D}\}, \quad \widetilde{\lambda}_*[\mathcal{X}, D] := \inf\{\lambda_*[\mathcal{X}, \text{supp}(m, X)], (\mathcal{X}, \text{supp}(m, X)) \in \Xi_{\mathcal{X}, D}\}.
\]

Clearly \(\lambda_*[\mathcal{X}, D] \leq \widetilde{\lambda}_*[\mathcal{X}, D]\). Note that the \(\widetilde{\lambda}_*\) definition corresponds to simply integrating over \(X\) (or equivalently \(\text{supp}(m)\)) in both sides of the above inequalities; thus \(\widetilde{\lambda}_*[\mathcal{X}, D]\) is the best constant in these standard versions for all members of \(\mathcal{X}\) so that \(\text{diam}(\text{supp}(m)) \leq D\), whereas the \(\lambda_*[\mathcal{X}, D]\) variant gives us the
added flexibility of considering arbitrary closed subsets of supp(m) of diameter at most D. In the one-dimensional setting, we additionally abbreviate for a density h on \( \mathbb{R} \) and a closed interval \( I \subset \mathbb{R} \):

\[
\lambda_\ast[h, I] := \lambda_\ast[\mathbb{R}, |\cdot|, hL^1], I).
\]

**Definition 2.8 (QCD\textsubscript{reg}(Q, K, N), CD\textsubscript{reg}(K, N), and CD\textsubscript{1}(K, N)).** Given \( K \in \mathbb{R}, N \in (1, \infty) \) and \( Q \geq 1 \), we denote by QCD\textsubscript{reg}(Q, K, N) the family of all Monge spaces \((X, d, m)\) satisfying QCD\( (Q, K, N) \) and MCP\( (K', N') \) for some \( K' \in \mathbb{R} \) and \( N' \in (1, \infty) \); note that QCD\textsubscript{reg}(1, K, N) coincides with the family CD\textsubscript{reg}(K, N) of Monge spaces satisfying CD\( (K, N) \) (and hence MCP\( (K, N) \)). We also denote by CD\textsubscript{1}(K, N) the family of one-dimensional spaces \((\mathbb{R}, |\cdot|, hL^1)\) satisfying CD\( (K, N) \). Note that

\[
\text{CD}_1(K, N) \subset \text{CD}_{\text{reg}}(K, N) \subset \text{QCD}_{\text{reg}}(Q, K, N).
\]

It is known that supp\( (m) \) is geodesically convex on MCP\( (K, N) \) spaces, and hence for all of the above spaces. In the one-dimensional setting, it is not too hard to show that \( \tilde{\lambda}_\ast[\text{CD}_1(K, N), D] = \lambda_\ast[\text{CD}_1(K, N), D] \) (see Corollary 6.2). We can now state the following:

**Theorem 2.9.** For all \( K \in \mathbb{R}, N \in (1, \infty), Q \geq 1 \), and \( D \in (0, \infty) \):

\[
\tilde{\lambda}_p[\text{CD}_1(K, N), D] \geq \lambda_p[\text{QCD}_{\text{reg}}(Q, K, N), D] \geq \frac{1}{Q} \tilde{\lambda}_p[\text{CD}_1(K, N), D]
\]

for all \( p \in (1, \infty) \).

\[
\tilde{\lambda}_{L,S}[\text{CD}_1(K, N), D] \geq \lambda_{L,S}[\text{QCD}_{\text{reg}}(Q, K, N), D] \geq \frac{1}{Q} \tilde{\lambda}_{L,S}[\text{CD}_1(K, N), D].
\]

The case \( Q = 1 \) with the \( \lambda_\ast \) middle term above replaced by (the a priori larger) \( \tilde{\lambda}_\ast \) is not new, and was obtained by Cavalletti–Mondino \[31\] as an immediate corollary of their localization theorem for CD\textsubscript{reg}(K, N) spaces; the possibility to extend this from \( \tilde{\lambda}_\ast \) to \( \lambda_\ast \) as above was anticipated in our previous work \[46\], and is in itself new. The case \( Q > 1 \) is the main novelty of Theorem 2.9 and constitutes the main result of this work.

The constants \( \tilde{\lambda}_p[\text{CD}_1(K, N), D] \) have been well studied in the literature and completely determined:

\[
(2.7) \quad \tilde{\lambda}_p[\text{CD}_1(K, N), D] = \lambda_p[c_{\kappa}^{N-1}(t), [-D/2, D/2]],
\]

where

\[
c_{\kappa}(t) := \begin{cases} \cos(\sqrt{\kappa}t) [-\frac{\pi}{2}, \frac{\pi}{2}] (\sqrt{\kappa}t) & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ \cosh(\sqrt{-\kappa}t) & \text{if } \kappa < 0. \end{cases}
\]

This follows from the results of Bakry-Qian \[12\] when \( p = 2 \) (see also \[6,25\]), and Matei \[63\], Valtorta \[85\], Esposito–Nitsch–Trombetti \[38\] and Naber–Valtorta \[69\].
for general \( p \in (1, \infty) \) (see also [25, chap. 6] and [90]); in fact, these authors directly showed in the weighted Riemannian setting that \( \lambda_p[\mathcal{CD}_{\text{reg}}(K, N), D] = \lambda_p[\mathcal{CD}_1(K, N), D] \) prior to Klartag’s extension of the localization method to the Riemannian setting. In particular (see [12, 85]):

\[
\lambda_p[\mathcal{CD}_1(0, N), D] = \lambda_p[1, [\pm 2, D/2]] = (p - 1) \left( \frac{2\pi}{p \sin(\pi/p)D} \right)^p ;
\]

(2.8) \( K > 0, D \geq \pi(1 - 1/K) \Rightarrow \lambda_2[\mathcal{CD}_1(K, N), D] = \frac{N}{N - 1}K. \)

Note that even in the simplest case of \( p = 2 \) and \( K \geq 0 \), Theorem 2.9 constitutes a sharp and stable extension of the celebrated Li–Yau / Zhong–Yang (\( K = 0 \)) and Lichnerowicz (\( K > 0 \)) estimates [37, 58, 59, 89, 91] to the QCD setting—indeed, setting \( Q = 1 \) and applying Theorem 2.9 to a geodesically convex (so \( \text{geo}(\Omega) = \Omega \)) of diameter at most \( D \), the latter sharp spectral-gap estimates are immediately recovered from (2.8) and (2.9), respectively. The same holds if we set \( Q > 1 \) and let \( Q \to 1 \).

To the best of our knowledge, the model densities on which the sharp constants \( \lambda_{\text{LS}}[\mathcal{CD}_1(K, N), D] \) are attained have not been completely determined, although the natural conjecture is that the answer is the same as for \( \lambda_p \) in (2.7). Up to numeric constants \( C, C' > 1 \), this conjecture has been verified for \( N = \infty \) by E. Calderon [25, chap. 7], who showed that

\[
\lambda_{\text{LS}}[\mathcal{CD}_1(\infty, D) \geq \frac{1}{C} \lambda_{\text{LS}}[\exp(-Kt^2/2), [-D/2, D/2]]
\]

\[
\geq \frac{1}{C'} \left\{ \begin{array}{ll}
( -K \right) \frac{3}{2} D e^{K \frac{D^2}{4}}, & K < -\frac{1}{D^2} \\
\max(K, \frac{1}{D^2}), & \text{otherwise}
\end{array} \right.
\]

Note that \( \lambda_{\text{LS}}[\mathcal{CD}_1(K, N), D] = \frac{N}{N - 1}K \) when \( K > 0 \) and \( D \geq \pi(1 - 1/K) \) by the Bakry–Émery estimate [10]. The case most interesting for us, \( K = 0 \), is well-known to experts, and in particular

\[
\lambda_{\text{LS}}[\mathcal{CD}_1(0, N), D] = \frac{1}{C} \lambda_{\text{LS}}[1, [-D/2, D/2]] = \frac{1}{C} \frac{\pi^2}{D^2}.
\]

In conjunction with Corollary 2.5, Theorem 2.9 thus immediately yields Theorem 1.1 from the Introduction, which is the main application we have chosen to highlight in this work. Analogously, by invoking Theorem 2.9 in conjunction with Proposition 2.4 (and recalling the subsequent comments), or alternatively in conjunction with Corollary 2.6, the \( L^p \)-Poincaré and log-Sobolev inequalities of Theorem 1.1 equally hold on the sub-Riemannian manifolds listed below (with their canonical sub-Riemannian metric and volume measure). Each of these spaces satisfies MCP(0, \( N \)), QCD(\( Q, 0, N \)) and the inequalities of Theorem 1.1 with the values of \( N, Q, \) and \( k \) given by the following table:
Specializing Theorem 1.1 to geodesic balls $\Omega = B_r(x)$, recall that geo($\Omega$) appearing on the energy side of the inequalities satisfies geo($\Omega$) $\subset B_{2r}(x)$, and so we obtain $L^p$-Poincaré and log-Sobolev inequalities on geodesic balls in nontight form. As mentioned in the Introduction, it is always possible to tighten (i.e., replace $B_{2r}(x)$ by $B_r(x)$ on the energy side) a local $L^p$-Poincaré inequality on any length space, but this would result in a loss of explicit constants and dependence on the underlying dimension (via the doubling constant).

3 Preliminaries

3.1 Sub-Riemannian structures

We refer to [1, 15, 16, 40] and the references therein for more precise information and missing definitions pertaining to sub-Riemannian structures, as these will not be directly required in this work. Below we briefly describe some rudimentary notions.

A sub-Riemannian structure on a smooth, connected $n$-dimensional manifold $M$ ($n \geq 3$) is defined by a set of $m$ global smooth vector fields $X_1, \ldots, X_m$, called a generating frame. The distribution $\mathcal{D}$ at the point $x \in M$ is defined as

$$\mathcal{D}_x = \text{span}\{X_1(x), \ldots, X_m(x)\} \subset T_x M.$$ 

The generating frame induces a natural inner product $g_x$ on $\mathcal{D}_x$. It is always assumed that the distribution satisfies Hörmander’s bracket-generating condition (each tangent space $T_x \mathcal{D}$ is spanned by the vector fields $\{X_i\}$ and their iterated Lie brackets evaluated at $x$). Being slightly imprecise, an absolutely continuous map $\xi : [0, 1] \to M$ is called a horizontal curve if $\xi(t) \in \mathcal{D}_x(\xi(t))$ for almost every $t \in [0, 1]$. Its length is defined by

$$\ell(\xi) := \int_0^1 \sqrt{g(\dot{\xi}(t), \dot{\xi}(t))} dt.$$

*Assuming appropriate curvature lower bounds detailed in [16, 17, 56].

†Assuming completely parallel torsion and nonnegative horizontal sectional curvature [21, §3.7].
The Carnot-Carathéodory sub-Riemannian metric $d$ is then defined as
\[ d(x, y) := \inf \{ \ell(\xi) \mid \xi(0) = x, \xi(1) = y, \xi \text{ is horizontal} \}. \]

By the Chow-Rashevskii theorem, the bracket-generating condition implies that $d : M \times M \to \mathbb{R}$ is finite and continuous. We will always assume that $(M, d)$ is complete, in which case the infimum above is always attained; a constant velocity horizontal curve realizing this infimum and parametrized on $[0, 1]$ is called a geodesic. If in addition the sub-Riemannian structure $(\mathcal{D}, g)$ admits no abnormal geodesics between distinct points, it is called ideal; roughly speaking, this means that the differential of the endpoint map $\xi \mapsto \xi(1)$ on horizontal paths $\xi$ with fixed initial point $\xi(0)$ is nonsingular for any geodesic $\gamma$ of positive length. It is known that complete fat sub-Riemannian structures are ideal, and that the ideal assumption is generic when the distribution $\mathcal{D}$ has constant rank at least 3.

In various places, we have emphasized how our results apply to generalized $H$-type groups. These are certain step 2 Carnot groups, which include the Kaplan H-type groups and the Heisenberg groups, as well as all corank 1 Carnot groups. A Carnot group of rank $r \geq 1$ and step $s \geq 1$ is a connected, simply connected nilpotent Lie group $G$, whose associated Lie algebra $\mathfrak{g}$ admits a stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ such that $\mathfrak{g}_1, \ldots, \mathfrak{g}_s$ are linear subspaces of $\mathfrak{g}$ satisfying $\mathfrak{g}_s \neq \{0\}$, $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{j+i}$ for all $i = 1, \ldots, s-1$, $[\mathfrak{g}_1, \mathfrak{g}_s] = \{0\}$, and the degree $1$ stratum $\mathfrak{g}_1$ has dimension $r$. A left-invariant sub-Riemannian structure is obtained by equipping $\mathfrak{g}_1$ with an inner product. Note that a corank 1 Carnot group is necessarily of step 2.

The Heisenberg group $\mathbb{H}^d$ is an ideal Carnot group of corank 1. Its elements are $(z_1, \ldots, z_d, t) \in \mathbb{C}^d \times \mathbb{R} \simeq \mathbb{R}^{2d+1}$, with the group structure given by
\[(z_1, \ldots, z_d, t) \cdot (z'_1, \ldots, z'_d, t') = \left( z_1 + z'_1, \ldots, z_d + z'_d, t + t' + \frac{1}{2} \sum_{j=1}^d \text{Im}(\overline{z}_j z'_j) \right).\]

Its bi-invariant Haar measure is just the Lebesgue measure $\mathcal{L}^{2d+1}$. Its sub-Riemannian structure is given by the global set of left-invariant generating fields:
\[ X_j = \partial_{x_j} - \frac{y_j}{2} \partial_t, \quad Y_j = \partial_{y_j} + \frac{x_j}{2} \partial_t, \]
where $z_j = x_j + iy_j$. They satisfy the bracket relations $[X_j, Y_l] = \delta_{jl} Z$ and $[X_j, Z] = [Y_l, Z] = 0$, where $Z = \partial_t$.

### 3.2 Optimal transport
Let $(X, d)$ be a complete separable metric space endowed with a locally finite Borel measure $\mu$—such triplets $(X, d, \mu)$ are called metric-measure spaces. We refer to [3, 4, 42, 85, 87] for background on metric-measure spaces in general, and the theory of optimal transport on such spaces in particular.
We denote by $\text{Geo}(X, d)$ the set of all closed directed constant-speed geodesics parametrized on the interval $[0, 1]$. We regard $\text{Geo}(X, d)$ as a subset of all Lipschitz maps $\text{Lip}([0, 1], X)$ endowed with the uniform topology. Recall that $(X, d)$ is called geodesic if for any $x, y \in X$ there exists $\gamma \in \text{Geo}(X, d)$ with $\gamma_0 = x$ and $\gamma_1 = y$. Given a subset $A$ of a geodesic space $(X, d)$, we denote by $\text{geo}(A)$ the geodesic hull of $A$, namely,

$$\text{geo}(A) := \bigcup_{\gamma \in \text{Geo}(X, d), \gamma_0, \gamma_1 \in A} \gamma;$$

note that $(\text{geo}(A), d)$ need not be a geodesic space itself.

The space of all Borel probability measures on $(X, d)$ is denoted by $\mathcal{P}(X)$. It is naturally equipped with its weak topology, in duality with bounded continuous functions $C_b(X)$ over $X$. The subspace of those measures having bounded support is denoted by $\mathcal{P}_e(X)$, and those with finite second moment is denoted by $\mathcal{P}_2(X)$. The weak topology on $\mathcal{P}_2(X)$ is metrized by the $L^2$-Wasserstein distance $W_2$, defined as follows for any $\mu_0, \mu_1 \in \mathcal{P}(X)$:

$$W_2^2(\mu_0, \mu_1) := \inf_{\pi} \int_{X \times X} d^2(x, y) \pi(dx, dy),$$

where the infimum is taken over all $\pi \in \mathcal{P}(X \times X)$ having $\mu_0$ and $\mu_1$ as the first and the second marginals, respectively; such candidates $\pi$ are called transference plans. It is known that the infimum in (3.1) is always attained for any $\mu_0, \mu_1 \in \mathcal{P}(X)$; when this minimum is finite, the collection of transference plans realizing it, called optimal transference plans between $\mu_0$ and $\mu_1$, is denoted by $\text{Opt}(\mu_0, \mu_1)$.

When $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, then necessarily $W_2(\mu_0, \mu_1) < \infty$. In this case, it is known that a transference plan $\pi$ is optimal if and only if it is supported on a $d^2$-cyclically monotone set. A set $\Lambda \subseteq X \times X$ is said to be $c$-cyclically monotone if for any finite set of points $\{(x_i, y_i)\}_{i=1}^N \subseteq \Lambda$ it holds that

$$\sum_{i=1}^N c(x_i, y_i) \leq \sum_{i=1}^N c(x_i, y_{i+1}),$$

with the convention that $y_{N+1} = y_1$.

As $(X, d)$ is a complete and separable metric space then so is $(\mathcal{P}_2(X), W_2)$. Under these assumptions, it is known that $(X, d)$ is geodesic if and only if $(\mathcal{P}_2(X), W_2)$ is geodesic. Let $e_t$ denote the evaluation map

$$e_t : \text{Geo}(X, d) \ni \gamma \mapsto \gamma_t \in X.$$ 

A measure $\nu \in \mathcal{P}(\text{Geo}(X, d))$ is called an optimal dynamical plan if $(e_0, e_1)_\# \nu$ is an optimal transference plan; it easily follows in that case that $[0, 1] \ni t \mapsto (e_t)_\# \nu$ is a geodesic in $(\mathcal{P}_2(X), W_2)$. It is known that any geodesic $(\mu_t)_{t \in [0, 1]}$ in $(\mathcal{P}_2(X), W_2)$ can be lifted to an optimal dynamical plan $\nu$ so that $(e_t)_\# \nu = \mu_t$ for all $t \in [0, 1]$ (cf. [3] theorem 2.10). We denote by $\text{OptGeo}(\mu_0, \mu_1)$ the space of all optimal dynamical plans $\nu$ so that $(e_t)_\# \nu = \mu_i$, $i = 0, 1$. By the preceding remarks, it
follows that for any closed \( \Omega \subset X \) so that \((\Omega, d)\) is geodesic, \(\text{OptGeo}(\mu_0, \mu_1)\) is nonempty for all \(\mu_0, \mu_1 \in \mathcal{P}_2(\Omega)\).

### 3.3 Monge space

**Definition 3.1 (Monge space).** A metric-measure space \((X, d, m)\) will be called a Monge space if for any \(\mu_0, \mu_1 \in \mathcal{P}_2(X)\) with \(\mu_0 \ll m\) and \(\text{supp}(\mu_1) \subset \text{supp}(m)\), the following holds:

- There exists a unique optimal dynamical plan \(\nu \in \text{OptGeo}(\mu_0, \mu_1)\), and hence a unique optimal transference plan \(\pi \in \text{Opt}(\mu_0, \mu_1)\).
- \(\nu\) is induced by a map, namely, there exists \(S : X \to \text{Geo}(X, d)\) such that \(\nu = S_t \mu_0\).
- Denoting \(\mu_t = (e_t)_\# \nu\), we have \(\mu_t \ll m\) for all \(t \in [0, 1]\).

It follows from the work of McCann [65] and Cordero-Erausquin–McCann–Schmuckenschläger [34] that (smooth, connected) complete Riemannian manifolds \((M, g)\) equipped with their induced geodesic distance \(d\) and volume measure \(\text{Vol}_g\) are Monge spaces (strictly speaking, this was shown for \(\mu_0, \mu_1 \in \mathcal{P}_c(X)\), but the extension from \(\mathcal{P}_c(X)\) to \(\mathcal{P}_2(X)\) is nowadays standard—see, e.g., [40, sec. 3.4]). It was shown by Figalli and Rifford [40, secs. 3, 4] that very general (smooth, connected) complete sub-Riemannian manifolds \((M, D, g)\) equipped with their volume measure are also Monge spaces; for instance, this holds for all ideal sub-Riemannian structures [40, theorem 5.9] (see also [16, theorem 39]).

Another sub-Riemannian setting where the results of [40] ensure the first two properties above is for general (possibly nonideal) step 2 Carnot groups (cf. [15, sec. 1.3] or [14, sec. 2.5]). Ensuring the third property requires additional justification. This has been established in the literature under additional assumptions, like being corank 1 [14, prop. 2.4] or generalized H-type [15, cor. 4]—the idea is to combine the known MCP information for these spaces with the fact that step 2 Carnot groups do not admit branching minimizing geodesics, and invoke (3.2) below. As pointed out to us by the referee, the same argument actually applies to general step 2 Carnot groups without any further assumptions, since these spaces satisfy MCP\((0, N)\) for some \(N > 1\) by the results of Badreddine-Rifford [7, theorem 4].

Clearly, the Monge property continues to hold when the volume measure is replaced by any measure \(m\) having smooth positive density with respect to the former.

### 3.4 Essentially nonbranching spaces

**Definition 3.2 (Essentially nonbranching).** A subset \(G \subset \text{Geo}(X, d)\) of geodesics is called nonbranching if for any \(\gamma^1, \gamma^2 \in G\) the following holds:

\[
\exists t \in (0, 1) \quad \gamma^1_s = \gamma^2_s \quad \forall s \in [0, t] \quad \implies \quad \gamma^1_s = \gamma^2_s \quad \forall s \in [0, 1].
\]

\((X, d)\) is called nonbranching if \(\text{Geo}(X, d)\) is nonbranching. \((X, d, m)\) is called essentially nonbranching if for any \(\mu_0, \mu_1 \ll m\) in \(\mathcal{P}_2(X)\), any \(\nu \in \text{OptGeo}(\mu_0, \mu_1)\) is concentrated on a Borel nonbranching subset \(G \subset \text{Geo}(X, d)\).
Recall that a measure $\nu$ on a measurable space $(\Omega, \mathcal{F})$ is said to be concentrated on $A \subset \Omega$ if $\exists B \subset A$ with $B \in \mathcal{F}$ so that $\nu(\Omega \setminus B) = 0$.

The above definition was introduced in [78] by Rajala and Sturm, who showed that RCD$(K, \infty)$ spaces are essentially nonbranching. The restriction to essentially nonbranching spaces is natural and facilitates avoiding pathological cases: as an example of possible pathological behaviour we mention the failure of the local-to-global property of CD$(K, N)$ within this class of spaces; in particular, a heavily branching metric-measure space verifying a local version of CD$(0, 4)$ that does not verify CD$(K, N)$ for any fixed $K \in \mathbb{R}$ and $N \in [1, \infty]$ was constructed by Rajala in [77], while the local-to-global property of CD$(K, N)$ has been recently verified in [28] for essentially nonbranching metric-measure spaces (with finite $m$).

It is easy to realize that a Monge space is necessarily essentially nonbranching (e.g., [28, cor. 6.15]). Conversely, it was shown by Cavalletti and Mondino in [29] that an essentially nonbranching space satisfying the measure contraction property MCP$(K, N)$ (for some $K \in \mathbb{R}$ and $N \in (1, \infty)$, defined next) is a Monge space:

\begin{equation}
\text{essentially nonbranching} + \text{MCP}(K, N) \Rightarrow \text{Monge} \\
\Rightarrow \text{essentially nonbranching}.
\end{equation}

### 3.5 MCP$(K, N)$

The measure contraction property MCP$(K, N)$, $N \in (1, \infty)$, introduced by Ohta [70] and Sturm [83], is a certain weak variant of the curvature-dimension condition CD$(K, N)$. On general metric-measure spaces the two definitions slightly differ, but on essentially nonbranching (and hence Monge) spaces they coincide. Recall the definition of the functions $\sigma_{K,N}^{(t)}$ and $\tau_{K,N}^{(t)}$ from Section 2.

**Definition 3.3 (Measure contraction property MCP$(K, N)$ on Monge spaces).** A Monge space $(X, d, m)$ is said to satisfy MCP$(K, N)$ if for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, $\mu_0 \ll m$, and $\text{supp}(\mu_1) \subset \text{supp}(m)$, writing $\mu_t = (e_t)_\# \nu = \rho_t m$ where $\nu$ is the unique element of OptGeo$(\mu_0, \mu_1)$, we have for all $t \in [0, 1)$:

\begin{equation}
\rho_t^{-1/N}(\gamma_t) \geq \tau_{K,N}^{(t)}(d(\gamma_0, \gamma_1))\rho_0^{-1/N}(\gamma_0) \quad \text{for } \nu\text{-a.e. } \gamma \in \text{Geo}(X, d).
\end{equation}

In fact, as follows from, e.g., [28, prop. 9.1], it is enough to test the above for

\begin{equation}
\mu_0 = \frac{1}{m(B)} m_{\ll B} \text{ with } 0 < m(B) < \infty, \quad \mu_1 = \delta_o \text{ with } o \in \text{supp}(m).
\end{equation}

Since some of our results are formulated on essentially nonbranching spaces, we also mention for completeness the a priori weaker (but by (3.2), equivalent) definition on the latter spaces (see [28, prop. 9.1]): for any $\mu_0, \mu_1$ as in (3.4), one should require the existence of $\nu \in \text{OptGeo}(\mu_0, \mu_1)$ so that $\mu_t := (e_t)_\# \nu \ll m$ for all $t \in [0, 1)$, and so that writing $\mu_t = \rho_t m$, (3.3) holds for each $t \in [0, 1)$. 

It was shown in [70, 83] that the following (sharp) Bonnet-Myers diameter bound holds:

\[
\text{diam}(\text{supp } m) \leq D_{K, N} := \begin{cases} \frac{\pi}{\sqrt{K/(N-1)}} & \text{if } K > 0, \\ +\infty & \text{otherwise;} \end{cases}
\]

we remark that while this is obvious from our present definition and the fact that \(\tau_{K, N}(\theta) = +\infty\) if \(\theta \geq D_{K, N}\), the above bound was shown in [70] under an a priori weaker (but ultimately equivalent) definition of MCP\((K, N)\) where the set \(B\) above is assumed to be a subset of \(B(o, D_{K, N})\) and in addition \((\text{supp } m, d)\) is a priori assumed to be a length space.

### 3.6 CD\((K, N)\)

The curvature-dimension condition CD\((K, N)\) has been defined on a general metric-measure space independently in several seminal works by Sturm and Lott-Villani: the case \(N = 1\) and \(K \geq R\) was defined in [61, 82], the case \(N > 1\) in [83] for \(K \in R\), and in [61] for \(K = 0\) (and subsequently for \(K \in R\) in [60]).

In this work, we will only require the definition for Monge spaces with \(N \in (1, \infty)\).

**Definition 3.4 (CD\((K, N)\) for Monge spaces).** A Monge space \((X, d, m)\) is said to satisfy CD\((K, N)\) if for any \(\mu_0, \mu_1 \in \mathcal{P}_2(X)\) with \(\mu_0 \ll m\), writing \(\mu_t = (e_t)_*\mu = \rho_t m\) where \(\nu\) is the unique element of OptGeo\((\mu_0, \mu_1)\), we have for all \(t \in [0, 1]\):

\[
\rho_t^{-1/N}(\gamma_t) \geq e_{K,N}(d(\gamma_0, \gamma_1))\rho_0^{-1/N}(\gamma_0) + e_{K,N}(d(\gamma_0, \gamma_1))\rho_1^{-1/N}(\gamma_1)
\]

for \(\nu\)-a.e. \(\gamma \in \text{Geo}(X, d)\).

When \(N \in (1, \infty)\), if \((X, d, m)\) satisfies either CD\((K, N)\) or MCP\((K, N)\), then (supp\((m), d)\) is proper (every closed bounded set is compact) and geodesic; in addition, by approximating \(\delta_o\) by \(\mu_t = m(B(o, \varepsilon))^{-1}m_{B(o, \varepsilon)}\), it is also known that the CD\((K, N)\) condition implies the MCP\((K, N)\) one (e.g., [28, sec. 6]).

**Remark 3.5.** Note that the definitions of MCP\((K, N)\) and CD\((K, N)\) given in this section employ \(\mu_0, \mu_1 \in \mathcal{P}_2(X)\), whereas the ones given in Section 2 employed \(\mu_0, \mu_1 \in \mathcal{P}_c(X)\). On Monge spaces so that (supp\((m), d)\) is proper (the proof of properness is valid for either variant), these two variants are completely equivalent—see, e.g., the proof of [28, prop. 9.1].

### 3.7 MCP\((K, N)\) densities

**Definition 3.6 (MCP\((K, N)\) density).** A nonnegative \(h \in L^1_{\text{loc}}(R, \mathcal{L}^1)\) is called an MCP\((K, N)\) density if

\[
h(tx_1 + (1-t)x_0) \geq c_{K,N-1}^{(1-t)}(|x_1 - x_0|)^{N-1}h(x_0)
\]

for all \(x_0, x_1 \in \text{supp } h\) and \(t \in [0, 1]\).
We use supp\(h\) throughout this work to denote \(\text{supp}(h \mathcal{L}^1)\), where recall, \(\mathcal{L}^1\) denotes the Lebesgue measure on \(\mathbb{R}\). The following is well-known (see, e.g., \cite{46} lemma 4.1):

**Lemma 3.7.** The one-dimensional metric-measure space \((\mathbb{R}, |\cdot|, h \mathcal{L}^1)\) satisfies MCP\((K, N)\) if and only if (up to modification on a null set) \(h\) is a MCP\((K, N)\) density.

**Lemma 3.8.** Let \(h\) be an MCP\((K, N)\) density. Then \(\text{supp}\ h \subset \mathbb{R}\) is a closed interval, \(h\) is locally bounded above on \(\text{supp}\ h\), it is positive and locally Lipschitz on its interior \(\text{int}\ \text{supp}\ h\), and we may modify \(h\) at the endpoints \(\text{supp}\ h \setminus \text{int}\ \text{supp}\ h\) so that \(h\) is continuous on \(\text{supp}\ h\).

**Proof.** By definition of MCP\((K, N)\) density, \(\text{supp}\ h\) is clearly convex, and is thus a closed interval. As follows from \cite{28} lemmas A.8 and A.9 (which were stated for CD\((K, N)\) densities of finite mass, but the proof only uses the defining property of MCP\((K, N)\) densities, and the local properties only require locally finite mass), \(h\) is locally bounded above on \(\text{supp}\ h\), and is positive and locally Lipschitz on \(\text{int}\ \text{supp}\ h\). Lastly, since an MCP\((K, N)\) density is clearly lower semi-continuous, we may modify the values of \(h\) at the endpoints if necessary to ensure that \(h\) is continuous on the entire \(\text{supp}\ h\). \(\square\)

### 3.8 Localization on MCP spaces

Recall that given a measure space \((X, \mathcal{X}, m)\), a set \(A \subset X\) is called \(m\)-measurable if \(A\) belongs to the completion of the \(\sigma\)-algebra \(\mathcal{X}\), generated by adding to it all subsets of null \(m\)-sets; similarly, a function \(f : (X, \mathcal{X}, m) \to \mathbb{R}\) is called \(m\)-measurable if all of its sublevel sets are \(m\)-measurable. We denote by \(\mathcal{M}(X, \mathcal{X})\) the collection of measures on \((X, \mathcal{X})\). We denote by \(\mathcal{H}^1\) the one-dimensional Hausdorff measure on the underlying metric space.

**Definition 3.9 (Disintegration on sets).** Let \((X, \mathcal{X}, m)\) denote a measure space. Given any family \(\{X_q\}_{q \in \mathcal{Q}}\) of subsets of \(X\), a disintegration of \(m\) on \(\{X_q\}_{q \in \mathcal{Q}}\) is a measure-space structure \((\mathcal{Q}, \mathcal{D}, q)\) and a map

\[ q \in \mathcal{Q} \mapsto m_q \in \mathcal{M}(X, \mathcal{X}) \]

so that:

- For \(q\)-a.e. \(q \in \mathcal{Q}\), \(m_q\) is concentrated on \(X_q\).
- For all \(B \in \mathcal{D}\), the map \(q \mapsto m_q(B)\) is \(q\)-measurable.
- For all \(B \in \mathcal{D}\), \(m(B) = \int_{\mathcal{Q}} m_q(B) \, dq\); this is abbreviated by \(m = \int_{\mathcal{Q}} m_q(q)\).

**Theorem 3.10 (Localization on MCP\((K, N)\) spaces).** Let \((X, d, m)\) be an essentially nonbranching metric-measure space satisfying the MCP\((K, N)\) condition for some \(K \in \mathbb{R}\) and \(N \in (1, \infty)\). Let \(g : X \to \mathbb{R}\) be \(m\)-integrable with \(\int_X g \, dm = 0\) and \(\int_X |g(x)| \, d(\mathcal{X}, x_0)\, m(dx) < \infty\) for some (equivalently, all) \(x_0 \in X\). Then there exists an \(m\)-measurable subset \(T \subset X\) and a family \(\{X_q\}_{q \in \mathcal{Q}} \subset X\) such that:
(1) There exists a disintegration of $\mathfrak{m}_T \tau$ on $\{X_q\}_{q \in Q}$:

$$\mathfrak{m}_T = \int_Q m_q(qdq), \quad q(Q) = 1.$$ 

(2) For $q$-a.e. $q \in Q$, $X_q$ is a closed geodesic in $(X, d)$.

(3) For $q$-a.e. $q \in Q$, $m_q$ is a Radon measure supported on $X_q$ with $m_q \ll \mathcal{H}^1_{X_q}$.

(4) For $q$-a.e. $q \in Q$, the metric-measure space $(X_q, d, m_q)$ verifies MCP($K, N$).

(5) For $q$-a.e. $q \in Q$, $\int g m_q = 0$, and $g \equiv 0$ $m$-a.e. on $X \setminus T$.

The localization paradigm on MCP($K, N$) spaces has its roots in the work of Bianchini and Cavalletti in the nonbranching setting (cf. [23, theorem 9.5]), and was extended to essentially nonbranching MCP($K, N$) spaces with $N < \infty$ and finite $m$ in [28, theorem 7.10 and remark 9.2] (building upon [26]) and for general $m$ in [32, theorem 3.5]. The idea to use $L^1$-transport between the positive and negative parts $g_+ := \max(g, 0)$ and $g_- := (-g)_+$ of the balanced function $g$ to ensure that it remains balanced along the localization is due to Klartag [54] (see [30] for an adaptation to the metric-measure space setting).

**Proof of Theorem 3.10.** Simply combine [32, theorem 3.5] with the proof of [30, theorem 5.1]. Up to modification on a $m$-null set, the set $T$ is the transport set of the $1$-Lipschitz Kantorovich potential $u$ associated to the $L^1$-optimal-transport between $g_+ m$ and $g_- m$, which consists of geodesics $\{X_q\}$ on which the function $u$ is affine with slope $1$; for more details, see the proof of Theorem 4.1 below.

\section{4 A General Localization Theorem}

Our first observation in this work is the following:

**Theorem 4.1 (General localization theorem).** Let $(X, d, m)$ be an essentially nonbranching metric-measure space satisfying the MCP($K', N'$) condition for some $K' \in \mathbb{R}$ and $N' \in (1, \infty)$; in particular, the space is Monge by (3.2).

Let $N \in (1, \infty)$, and let $(0, 1) \times \mathbb{R}_+ \ni (t, \theta) \to \sigma_i^{(t)}(\theta) \in [0, +\infty]$, $i = 0, 1$, be continuous in each variable. Assume that

- For all $\mu_0, \mu_1 \in \mathcal{P}_c(X)$ with $\mu_0, \mu_1 \ll m$, writing $\mu_t = (c_t)_*\nu = \rho_t m$ where $\nu$ is the unique element of $\text{OptGeo}(\mu_0, \mu_1)$, we have for all $t \in (0, 1)$:

$$\rho_t^{-\frac{1}{K}}(\gamma_t) \geq (1 - t)^{\frac{1}{K}} \sigma_0^{(1-t)}(d(\gamma_0, \gamma_1))^{\frac{N-1}{K}} \rho_0^{-\frac{1}{K}}(\gamma_0)$$

$$+ t^{\frac{1}{K}} \sigma_1^{(t)}(d(\gamma_0, \gamma_1))^{\frac{N-1}{K}} \rho_1^{-\frac{1}{K}}(\gamma_1),$$

for $\nu$ a.e. $\gamma \in \text{Geo}(X, d)$.
Let $g : X \to \mathbb{R}$ be $m$-integrable with $\int_X g \, m = 0$ and $\int_X |g(x)| \, d(x, x_0) \, m(dx) < \infty$ for some (equivalently, all) $x_0 \in X$. Then all the conclusions of Theorem 3.10 hold, and in addition:

(6) For $q$-a.e. $q \in \mathcal{Q}$, $m_q = h_q \mathcal{H}^{1 \downarrow} X_q$ with continuous density $h_q : X_q \to \mathbb{R}^+$ satisfying

$$h_q^{1\frac{1}{1-t}}(x_t) \geq \sigma_0(1-t)^{-d}(d(x_0, x_1)) h_q^{1\frac{1}{1-t}}(x_0) + \sigma_1^0(t)^d(d(x_0, x_1)) h_q^{1\frac{1}{1-t}}(x_1)$$

$$\forall x_0, x_1 \in X_q, \forall t \in (0, 1),$$

where $x_t$ denotes the unique point on $X_q$ so that $d(x_t, x_0) = t d(x_0, x_1)$ and $d(x_t, x_1) = (1-t)d(x_0, x_1)$.

Remark 4.2. To handle infinite values of $\sigma_i$, we use the convention that $\infty \cdot 0 = 0$.

Remark 4.3. The assumption that the space is MCP($K', N'$) may be relaxed, and is only included to guarantee some a priori good properties of the space, like being Monge, being proper, and having absolutely continuous conditional measures $m_q \ll \mathcal{H}^{1 \downarrow} X_q$ with continuous densities in the disintegration of $m \cdot \tau$. For reasonable choices of $\sigma_0, \sigma_1$, this would in any case be guaranteed, but we avoid this extraneous generality, especially since we would like to apply the localization theorem in the QCD setting to functions for which $\sigma_0^{(1)}, \sigma_1^{(1)} < 1$.

The proof below is based on the proof of the localization theorem for essentially nonbranching CD($K, N$) spaces by Cavalletti and Mondino [30, theorem 5.1]. However, as already mentioned in Section 2, there is one crucial difference—in [30], the authors extensively used the fact that the CD($K, N$) condition on a one-dimensional metric-measure space enjoys the local-to-global property, and so it is enough to establish it locally on the geodesic $X_q$. Consequently, the authors only required the local CD_{loc}($K, N$) condition to deduce their localization theorem. In contrast, the above condition employing general functions $\sigma_0, \sigma_1$ will typically not satisfy the local-to-global property even on a one-dimensional space (for example, this is the case for MCP($K, N$) when $\sigma_1 = 0$ or for QCD($Q, K, N$) when $Q > 1$), and so we are required to directly obtain the global property on $X_q$. This requires modifying the argument in several places and taking care of some additional technical points.

**Proof of Theorem 4.1** We may assume that $\text{supp}(m) = X$ without loss of generality; otherwise we restrict from $(X, d, m)$ to $(\text{supp}(m), d, m)$ without altering any of the above properties of the space (see, e.g., [28, sec. 6]). The MCP($K', N'$) assumption implies that $(X, d)$ is proper and geodesic. Recall from Theorem 3.10 the disintegration

$$m_{\cdot \tau} = \int_{\mathcal{Q}} m_q q(dq),$$

where for $q$-a.e. $q \in \mathcal{Q}$, $m_q$ is a Radon measure (in particular, finite on compact sets) supported on the closed geodesic $X_q$, $m_q \ll \mathcal{H}^{1 \downarrow} X_q$, and $(X_q, d, m_q)$ satisfies...
MCP($K', N'$). Of course, we may identify $(X_q, d, m_q)$ with $(I_\alpha, |.|.\bar{\nu} L^1)$ for an
appropriate closed interval $I_\alpha \subset \mathbb{R}$ via a unit-speed parametrization of the geodesic
$X_q$. Consequently, Lemmas 3.7 and 3.8 imply that for $q$-a.e. $q \in Q$, we may write
$$m_q = h_q \mathcal{H}_{\perp}^{1} x_q,$$
with $h_q$ being an MCP($K', N'$) density that is continuous on $X_q$ and positive on its relative interior relint $X_q$.

It remains to establish assertion (6) of Theorem 4.1. To this end, let us recall
from the work of Cavalletti and Mondino how the geodesics $X_q$ are constructed and how the disintegration (4.2) is obtained (see [30, sec. 3], [28, sec. 7], and [27] for the case that $m$ is finite, and [32, sec. 3] for an adaptation to the case when $m$ is only assumed locally finite, and hence $\sigma$-finite by properness). Let $\nu$ denote the Kantorovich potential associated to the $L^1$-optimal-transport (corresponding to the cost $c(x, y) = d(x, y)$) between $g_+ m$ and $g_- m$. Let $\Gamma := \{(x, y) \in X \times X, \nu(x) - \nu(y) = d(x, y)\}$ and $\Gamma^{-1} := \{(x, y) \in X \times X, (y, x) \in \Gamma\}$. The transport relation $R$ and the transport set $\mathcal{T}$ are defined as:
$$R := \Gamma \cup \Gamma^{-1}, \quad \mathcal{T} := P_1(R \setminus \{x = y\}),$$
where $P_i$ is the projection onto the $i$th component. Note that $R$ is closed, and it is
easy to show that $\mathcal{T}$ is $\sigma$-compact. The nonbranched transport set $\mathcal{T}^b$ is defined
as $\mathcal{T} \setminus (A_+ \cup A_-)$, where $A_\pm$ denote the sets of forward and backward branching
points, respectively (see [30]). The nonbranched transport relation is defined as
$R^b := R \cap (\mathcal{T}^b \times \mathcal{T}^b)$. One can show that $A_\pm$ are $\sigma$-compact and hence $\mathcal{T}^b$
and $R^b$ are Borel. A crucial observation is that on Monge spaces of full support, $m(\mathcal{T} \setminus \mathcal{T}^b) = m(A_+ \cup A_-) = 0$.

It turns out that $R^b$ is an equivalence relation over $\mathcal{T}^b$, and that for all $x \in \mathcal{T}^b$,
$(R(x), d)$ (where $R(x) := \{y; (x, y) \in R\}$) is isometric to a closed interval in
$(\mathbb{R}, |.|)$. Denote by $Q$ the set of equivalence classes induced by $R^b$ over $\mathcal{T}^b$, and
let $\Omega : \mathcal{T}^b \to Q$ denote the quotient map. A disintegration theorem guarantees the
existence of the disintegration (4.2) of $m_{\mathcal{T}^b} = m_{\mathcal{T}^b}$ strongly consistent with the
partition of $\mathcal{T}^b$ given by the equivalence classes $\{R^b(q)\}_{q \in Q}$ of $R^b$. The geodesics
$\{X_q\}$ are obtained as the closure of each equivalence class in $\mathcal{T}^b$, and hence have
disjoint relative interiors $\{\text{relint } X_q\}$. Note that the function $\nu$ is affine on each $X_q$
with slope 1.

As explained in [30, sec. 3] and [32, sec. 3], up to modifying $\mathcal{T}^b$ and $Q$ on $m$-null
and $\nu$-null sets, respectively, the set $Q$ can in fact be realized as a Borel subset of
$\mathcal{T}^b$ so that (equipping $Q$ with the trace $\sigma$-algebra) the quotient map $\Omega : \mathcal{T}^b \to Q$
is Borel measurable and so that $q$ is a Borel probability measure on $Q$. By inner
regularity of Borel probability measures, it follows that, up to modification on a
$q$-null set, $Q$ is $\sigma$-compact; we write $Q = \bigcup_{k \geq 1} Q^k$ with $Q^k$ compact in $(X, d)$.

The ray map $r : Q \times \mathbb{R} \supset \text{Dom}(r) \to \mathcal{T}^b$ is defined by
$$\text{graph}(r) := \{(q, t, x) \in Q \times [0, \infty) \times \mathcal{T}^b, (q, x) \in \Gamma, d(q, x) = t\}$$
By definition \( \text{Dom}(r) := r^{-1}(T^b) \). It is known that \( r \) is a Borel map. After these preparations, we can finally commence the proof of assertion (6).

Given \( k \) and real parameters \( a_0 < a_1 \) and \( \varepsilon_0, \varepsilon_1 > 0 \), denote

\[
Q_{a_0, a_1, \varepsilon_0, \varepsilon_1}^k := \left\{ q \in Q^k, \left[ \min(a_0 - \varepsilon_0, a_1 - \varepsilon_1), \max(a_0 + \varepsilon_0, a_1 + \varepsilon_1) \right] \right\}
\]

is in the interior of \( u(X_q) \)

Note that:

\[
Q_{a_0, a_1, \varepsilon_0, \varepsilon_1}^k = Q^k \cap \bigcup_{n \geq 1} \left( P_1^{-1}(u^{-1}(\min(a_0 - \varepsilon_0, a_1 - \varepsilon_1) - 1/n)) \cap P_1^{-1}(u^{-1}(\max(a_0 + \varepsilon_0, a_1 + \varepsilon_1) + 1/n)) \right).
\]

Since \( Q^k \) is compact, since \( u \) is Lipschitz and \( r \) is Borel, since the projection of a Borel set is analytic and hence universally measurable [81, sec. 4.3], and since \( q \) is a Borel measure, it follows that \( Q_{a_0, a_1, \varepsilon_0, \varepsilon_1}^k \) is q-measurable.

Let \( a_0 < a_1 \) and \( \varepsilon_0, \varepsilon_1 > 0 \) be such that \( q(Q_{a_0, a_1, \varepsilon_0, \varepsilon_1}^k) > 0 \). Consider the measures:

\[
\mu_i := \frac{1}{q(Q_{a_0, a_1, \varepsilon_0, \varepsilon_1}^k)} \int_{Q_{a_0, a_1, \varepsilon_0, \varepsilon_1}^k} \frac{1}{2\varepsilon_i} \cdot H^1_{\text{relint} X_q} \left[ \left \{ u - a_i \leq \varepsilon_i \right \} q \right](d\sigma).
\]

We postpone showing that \( \mu_i \) are well-defined Borel measures on \((X, d)\) (namely, the q-measurability of \( q \mapsto H^1_{\text{relint} X_q}(B) \) given a Borel set \( B \subset X \)) to Lemma 4.5. Since

\[
[a_i - \varepsilon_i, a_i + \varepsilon_i] \subset u(X_q) \quad \text{for all } q \in Q_{a_0, a_1, \varepsilon_0, \varepsilon_1}^k.
\]

we see that \( \mu_i \) are probability measures. Since \( Q^k \) is compact and \( u \) is affine with slope 1 on each \( X_q \), it follows that \( \mu_i \) are compactly supported. Moreover, we claim that \( \mu_i \ll m \) with Radon-Nykodim derivative \( \rho_i \) given by

\[
\rho_i(x) := \frac{1}{q(Q_{a_0, a_1, \varepsilon_0, \varepsilon_1}^k)} \frac{1}{2\varepsilon_i} \cdot H^1_{\text{relint} X_q} \left[ \left \{ u(x) - a_i \leq \varepsilon_i \right \} q \right](x)
\]

for \( x \in \text{relint} X_q, q \in Q_{a_0, a_1, \varepsilon_0, \varepsilon_1}^k, i = 0, 1, \)

and \( \rho_i(x) = 0 \) otherwise. Indeed, this is a good definition for m-a.e. \( x \), since the relative interiors of \( X_q \) are disjoint (after perhaps removing a q-null set of \( q \)'s), and \( m \ll H^1 \) (and hence does not charge \( X_q \) \setminus \text{relint} X_q \)) for q-a.e. \( q \). Establishing the m-measurability of \( \rho_i \) is postponed to Lemma 4.5. It follows by (4.2) that necessarily \( \mu_i = \rho_i m \). Consider the map \( T : X \to X \) that given \( x \in \text{relint} X_q \) with \( q \in Q_{a_0, a_1, \varepsilon_0, \varepsilon_1}^k \) produces the unique \( T(x) \in X_q \) so that

\[
\frac{u(T(x)) - a_1}{\varepsilon_1} = \frac{u(x) - a_0}{\varepsilon_0}.
\]
Let $G \subseteq X$ be the set on which $T$ is well-defined as described above. By the above arguments, $G$ has full $\mu_0$-measure (and hence may be assumed Borel), and on $G$ we have

$$T(x) = r\left(\Omega(x), u(\Omega(x)) - \frac{\varepsilon_1}{\varepsilon_0} a_0 - a_1\right),$$

so that $T$ is Borel measurable (as $r$, $\Omega$, and $u$ are), and we have $T \mu_0 = \mu_1$. Denote by $\pi \in \mathcal{P}(X \times X)$ the transference plan between $\mu_0$ and $\mu_1$ given by $(\text{Id} \times T) \mu_0$. We now use the following crucial observation due to Cavalletti [26, lemma 4.4] (cf. [30, lemma 4.1]), which connects the $L^1$-optimal-transport induced by $u$ with $L^2$-optimal-transport, and lies at the heart of the proof.

**Lemma 4.4.** If $\Delta \subseteq X \times X$ is a set so that

$$(x_0, y_0), (x_1, y_1) \in \Delta \implies (u(y_1) - u(y_0))(u(x_1) - u(x_0)) \geq 0,$$

then $\Delta$ is $d^2$-cyclically monotone.

Denoting by $\gamma_T(x)$ the geodesic from $x$ to $T(x)$ in $X_q$ (for $x \in G$), it follows that $\nu := (\gamma_T)_\# \mu_0$ is the (unique) optimal dynamical plan between $\mu_0$ and $\mu_1$. Setting $\mu_t = (e_t)_\# \nu$, we clearly have for all $t \in [0, 1]$ that

$$\mu_t := \frac{1}{q(Q^k_{a_0, a_1, \varepsilon_0, \varepsilon_1})} \int_{\delta Q^k_{a_0, a_1, \varepsilon_0, \varepsilon_1}} \frac{1}{2\varepsilon_t} h_q^{1/2} \chi_{q}^{1/2} |u - a_1| \leq \varepsilon_t \chi q(dq),$$

where $a_t := (1-t)a_0 + ta_1$ and $\varepsilon_t := (1-t)\varepsilon_0 + t\varepsilon_1$. Writing $\mu_t = \rho_t \mu$, we deduce from (4.2) as before the following representation for the densities:

$$\rho_t(x) = \frac{1}{q(Q^k_{a_0, a_1, \varepsilon_0, \varepsilon_1})} \frac{1}{2\varepsilon_t} h_q^{1/2} |u - a_1| \leq \varepsilon_t \chi q(x),$$

for $x \in \text{relint } X_q$ and $q$-a.e. $q \in Q^k_{a_0, a_1, \varepsilon_0, \varepsilon_1}$.

For notational convenience, given a closed geodesic $X_q$, we identify it with the closure $L_q$ of the interval $(\inf u(X_q), \sup u(X_q)) \subseteq \mathbb{R}$ (by mapping $x \in X_q$ to the unique $s \in L_q$ so that $u(x) = s$). Applying our assumption (4.1), it follows that given $t \in (0, 1)$, for $q$-a.e. $q \in Q^k_{a_0, a_1, \varepsilon_0, \varepsilon_1}$, and for $H^{1}$-a.e. $s_0 \in [a_0 - \varepsilon_0, a_0 + \varepsilon_0]$, we have

$$\varepsilon_t^{1/2} h_q^{1/2}(s_t) \geq (1-t)^{1/2} \sigma_0^{(1-t)}(s_1 - s_0) \frac{\varepsilon_t^{1/2} h_q^{1/2}(s_0)}{\varepsilon_0^{1/2} h_q^{1/2}(s_0)}$$

$$+ t^{1/2} \sigma_1^{(t)}(s_1 - s_0) \frac{\varepsilon_t^{1/2} h_q^{1/2}(s_1)}{\varepsilon_1^{1/2} h_q^{1/2}(s_1)},$$

where $s_t = (1-t)s_0 + ts_1$ and $s_1$ is given by

$$\frac{s_0 - a_0}{\varepsilon_0} = \frac{s_1 - a_1}{\varepsilon_1}.$$

Since $\sigma_0^{(1-t)}$ and $\sigma_1^{(t)}$ are assumed continuous, and since $h_q$ is continuous and positive on relint $L_q$ for $q$-a.e. $q$, the above holds for all $s_0 \in [a_0 - \varepsilon_0, a_0 + \varepsilon_0]$ (and,
in particular, for \( s_0 = a_0 \), for \( q \)-a.e. \( q \in Q^{k}_{a_0, a_1, \varepsilon_0, \varepsilon_1} \). Namely, given \( t \in (0, 1) \), for any \( k, a_0 < a_1 \), and \( \varepsilon_0, \varepsilon_1 > 0 \), we have

\[
(4.6) \quad \varepsilon_t \frac{1}{h_q^{(1-i)}}(a_t) \geq (1-t) \frac{1}{h_q^{(1-i)}}(a_1 - a_0) \frac{1}{x^q} (a_0) + t \frac{1}{h_q^{(1-i)}}(a_1 - a_0) \frac{1}{x^q} (a_1),
\]

for \( q \)-a.e. \( q \in Q^{k}_{a_0, a_1, \varepsilon_0, \varepsilon_1} \). Enumerating over \( k \) and all rational values of \( a_0 < a_1 \) and \( \varepsilon_0, \varepsilon_1 > 0 \), and using the continuity of \( h_q^{(1-i)} \) and \( h_q^{(i)} \) and also of \( h_q \) on \( \text{relint} \, L_q \), it follows that given \( t \in (0, 1) \), there exists a single \( q \)-null set \( N_t \), so that for all \( q \in Q \setminus N_t \), (4.6) holds for all \( a_0 < a_1 \) in \( \text{relint} \, L_q \) and \( \varepsilon_0, \varepsilon_1 > 0 \) small enough. Optimizing on the choice of \( \varepsilon_t > 0 \), we set

\[
\varepsilon_0 := \frac{\delta}{1-t} \cdot \frac{\sigma^{(1-i)}_0(a_1 - a_0)h_q^{(1-i)}(a_0)}{\sigma^{(1-i)}_0(a_1 - a_0)h_q^{(1-i)}(a_0) + \sigma^{(i)}_1(a_1 - a_0)h_q^{(1-i)}(a_1)},
\]

\[
\varepsilon_1 := \frac{\delta}{t} \cdot \frac{\sigma^{(i)}_1(a_1 - a_0)h_q^{(1-i)}(a_1)}{\sigma^{(1-i)}_0(a_1 - a_0)h_q^{(1-i)}(a_0) + \sigma^{(i)}_1(a_1 - a_0)h_q^{(1-i)}(a_1)},
\]

for some small enough \( \delta > 0 \), and thus deduce from (4.6) that given \( t \in (0, 1) \), for all \( q \in Q \setminus N_t \):

\[
(4.7) \quad h_q^{(1-i)}(a_t) \geq \sigma^{(1-i)}_0(a_1 - a_0)h_q^{(1-i)}(a_0) + \sigma^{(i)}_1(a_1 - a_0)h_q^{(1-i)}(a_1) \quad \forall a_0, a_1 \in \text{relint} \, L_q.
\]

In fact, since \( h_q \) was modified to be continuous on the entire \( L_q \), the above holds for all \( a_0, a_1 \in L_q \) if we interpret \( \infty \cdot 0 \) as 0 (recall that \( \sigma_i \) are allowed to be infinite). It remains to apply this to all rational \( t \in (0, 1) \), and by invoking the continuity of \( (0, 1) \ni t \mapsto \sigma_i(t) \) and of \( h_q \), we deduce that for \( q \)-a.e. \( q \in Q \), (4.7) holds for all \( a_0, a_1 \in L_q \) and \( t \in (0, 1) \). This concludes the proof. \( \square \)

It remains to address a couple of measurability issues that arose during the proof above; we continue using the same notation as there (see also an alternative argument in Remark 4.6 below).

**Lemma 4.5.**

1. For any Borel set \( B \subset X \), \( \exists q \mapsto \mathcal{H}^1_{L_x q}(B) \) is \( q \)-measurable.
2. The map \( \text{Dom}(t) \ni (q, t) \mapsto h_q((q, t)) \) is \( q \otimes L^1 \)-measurable.
3. The densities \( \rho_i \) defined in (4.4) are \( \mathcal{L}^1 \)-measurable.

**Proof.**

1. Note that for \( q \)-a.e. \( q \in Q \), we have

\[
\mathcal{H}^1_{L_x q}(B) = \mathcal{H}^1_{\text{relint} q}(B) = \int_{\text{Dom}(r(q, \cdot))} 1_B(r(q, t)) \mathcal{L}^1(dt).
\]
Since $\text{Dom}(r) \ni (q,t) \mapsto 1_B(r(q,t))$ is a Borel function (as $r$ and $B$ are Borel), and since $\text{Dom}(r)$ is Borel, the first assertion follows.

(2) It will be convenient to extend the definition of $h_q(r(q,t))$ to the entire $\mathbb{Q} \times \mathbb{R}$ by setting $H(q,t) := h_q(r(q,t))1_{\text{Dom}(r)}(q,t)$. Given a compact interval $I \subset \mathbb{R}$, note that for $q$-a.e. $q$:

$$\int_I H(q,t) \mathcal{L}^1(dx) = m_q(B_I), \quad B_I := \{ x \in \mathcal{T}^b, u(\Omega(x)) - u(x) \in I \}.$$  

Since $\Omega, u, \mathcal{T}^b, \text{Dom}(r)$ are Borel, it follows that $B_I$ is Borel as well, so by the measurability property of the disintegration (4.2) we deduce that

$$Q \ni q \mapsto \int_I H(q,t) \mathcal{L}^1(dx)$$

is $q$-measurable.

For $q$-a.e. $q$, supp $H(q,\cdot)$ coincides with the closure of $\text{Dom}(r)(q,\cdot)$. Since in addition $t \mapsto H(q,t)$ is continuous on its support for $q$-a.e. $q$, by applying (4.8) to $I_\varepsilon = [t - \varepsilon, t + \varepsilon]$ for a fixed $t \in \mathbb{R}$ and $\varepsilon > 0$, dividing by $\mathcal{L}^1(I_\varepsilon \cap \text{Dom}(r)(q,\cdot))$, and taking the limit as $\varepsilon \to 0$ (assuming the denominator is positive for all $\varepsilon > 0$), it follows that for all $t \in \mathbb{R}$, the function

$$Q \ni q \mapsto H(q,t)$$

is $q$-measurable (we have used the standard fact that the pointwise limit of measurable functions is measurable, e.g., [81, prop. 3.1.27]).

Now given $s > 0$, note that that continuity of $t \mapsto H(q,t)$ on its support for $q$-a.e. $q$ implies that up to a $q \otimes \mathcal{L}^1$ null set:

$$\{(q,t) \in \mathbb{Q} \times \mathbb{R}, H(q,t) \geq s\} =$$

$$\bigcup_{n \in \mathbb{N}, n > 1/s} \left\{ q \in \mathbb{Q}, H(q,t) \geq s - \frac{1}{n} \right\} \times \left\{ t \in \mathbb{R}, |t - \tau| \leq \frac{1}{n} \right\}$$

(compare with the proof of [81, theorem 3.1.30]). Since each of the product sets on the right-hand side is $q \otimes \mathcal{L}^1$-measurable, it follows that so is the left-hand side, concluding the proof of the second assertion.

(3) Note that the disintegration formula (4.2) and the fact that $m_q \ll \mathcal{H}^1_{\mathcal{L}^1_X q}$ for $q$-a.e. $q \in \mathbb{Q}$ together imply that if $D \subset \text{Dom}(r)$ is such that $q \otimes \mathcal{L}^1(D) = 0$, then

$$m(r(D)) = \int m_q(r(D))q(dq) = \int_D h_q(r(q,t))q \otimes \mathcal{L}^1(dq dt) = 0.$$  

In particular, we see that $\bigcup_{q \in \mathbb{Q}} (X_q \setminus \text{relint } X_q)$, and $\Omega^{-1}(Q_0)$ for any $q$-null set $Q_0$ are ($m$-measurable) $m$-null sets. It follows that for $m$-a.e. $x$ we have

$$p_t(x) = \frac{1}{q(Q_{\alpha_0, \alpha_1, \varepsilon_0, \varepsilon_1})} \frac{1}{2\varepsilon_1} \int \left[ u(x) - |a_i| \leq \varepsilon_1 \right] \frac{1}{h_\Omega(x)(x)} \frac{1}{\Omega^{-1}(\varepsilon_1^0, \alpha_0, \varepsilon_0, \varepsilon_1)}(x),$$
where \( B^k_{a_0, a_1, e_0, e_1} \subset Q \) is a Borel set that coincides with \( Q^k_{a_0, a_1, e_0, e_1} \) up to a \( q \)-null set. Since \( \Omega : T^b \to Q \) is Borel and \( u \) is Lipschitz, this reduces the task of establishing that \( \rho_i \) is \( m \)-measurable to showing that \( h_{\Omega(x)}(x) \) is \( m \)-measurable. Given \( s > 0 \), the second assertion of the lemma ensures that \( \{ (q, t) \in Q \times \mathbb{R}, h_{q}(r(q, t)) \geq s \} \) is \( q \otimes L^1 \) measurable, and hence may be written as \( D_0 \triangle D \) where \( D_0 \) is a \( q \otimes L^1 \)-null set, and \( D \) is a Borel subset of \( Q \times \mathbb{R} \). Since \( m(r(D_0)) = 0 \), it follows that up to an \( m \)-null set:

\[
\{ x \in T^b, h_{\Omega(x)}(x) \geq s \} = r\{ (q, t) \in Q \times \mathbb{R}, h_{q}(r(q, t)) \geq s \} = r(D).
\]

Since \( D \) and \( r \) are Borel, \( r(D) \) is analytic (see [28, theorem 6.18]) and hence \( m \)-measurable [81] sec. 4.3, thereby concluding the proof of the third assertion. □

**Remark 4.6.** Using essential uniqueness of disintegration, it is possible to avoid establishing the last two assertions of Lemma 4.5 directly, and argue in the proof of Theorem 4.1 as follows. First, note that \( \mu_i \ll m \), since if \( m(B) = 0 \), then by the disintegration (4.2) if follows that \( m_q(B) = 0 \) for \( q \)-a.e. \( q \), and since \( m_q \) and \( H^1_{L_X} \) are mutually absolutely continuous for \( q \)-a.e. \( q \), it follows that \( \mu_i(B) = 0 \) directly from the definition (4.3). Using the disintegration (4.2) again, we write

\[
(4.9) \quad \mu_i = \frac{d\mu_i}{dm} m = \int_Q \frac{d\mu_i}{dm} m_q(q(dq)) = \int_Q \frac{d\mu_i}{dm} h_{q} H^1_{L_X} q(dq).
\]

Since \( X_q \) have disjoint relative interiors and \( H^1 \) does not charge their endpoints, and since \( \mu_i \) is a Borel probability measure on our Polish space, it follows by [22, theorem A.7] (cf. [28, theorem 6.18]) that the disintegration must be essentially unique, meaning that for any other disintegration

\[
\mu_i = \int_Q \tilde{m}_q q(dq),
\]

with \( \tilde{m}_q \) concentrated on \( \text{relint } X_q \) for \( q \)-a.e. \( q \), we must have \( \tilde{m}_q = \frac{d\mu_i}{dm} h_{q} H^1_{L_X} q(dq) \) for \( q \)-a.e. \( q \). Comparing (4.9) with the definition of \( \mu_i \) from (4.3), it immediately follows that \( \frac{d\mu_i}{dm} = \rho_i h_{1} H^1_{L_X} \)-a.e. for \( q \)-a.e. \( q \), which by the disintegration (4.2) means that \( \frac{d\mu_i}{dm} = \rho_i m \)-a.e. and in particular establishes the \( m \)-measurability of \( \rho_i \).

### 4.1 Characterization of one-dimensional case

Before concluding this section, it is worth noting that, at least in the one-dimensional setting, Theorem 4.1 admits the following (standard) converse.

**Lemma 4.7.** Let \( N, \sigma_0 \), and \( \sigma_1 \) be as in Theorem 4.1 and let \( h : \mathbb{R} \to \mathbb{R}_+ \) be continuous on its support. Then the one-dimensional metric-measure space \((\mathbb{R}, |\cdot|, m = h L^1)\) satisfies (4.1) if and only if \( h \) satisfies

\[
(4.10) \quad h \frac{1}{\sigma_0^{(1-\tau)}} ((1-t)x_0 + tx_1) \geq \sigma_0^{(1-\tau)} (|x_1 - x_0| h \frac{1}{\sigma_0^{(1-\tau)}} (x_0) + \sigma_1^{(\tau)} (|x_1 - x_0| h \frac{1}{\sigma_0^{(1-\tau)}} (x_1)).
\]
for all $x_0, x_1 \in \text{supp} h$ and $t \in (0, 1)$.

**Proof.** The “only if” direction follows immediately from the proof of Theorem 4.1 (after localization to dimension $1$, the MCP($K'$, $N'$) assumption was only used there to guarantee that the density $h$ is continuous on its support). The “if” direction is standard, but for completeness, we sketch the proof. Let $\rho_0, \rho_1 : \text{supp} h \to \mathbb{R}_+$ be two probability densities w.r.t. $Q$; $K$, $N$ are in $\mathcal{P}_c(\mathbb{R})$. The $W_2$ optimal transport between $\mu_0$ and $\mu_1$ is obtained by a monotone map $T_1 : \text{supp} h \to \text{supp} h$, and by the change-of-variables formula, we have

$$J_1(x_0) := T_1'(x_0) = \frac{\rho_0(x_0) h(x_0)}{\rho_1(x_1) h(x_1)} \text{ for } \mu_0 \text{-a.e. } x_0,$$

where we denote $x_1 := T_1(x_0)$. The $W_2$ geodesic $\mu_t := \rho_t m$ is obtained by pushing forward $\mu_0$ via $T_t(x) = (1 - t)x + t T_1(x)$, and so by the change-of-variables formula, we have for each $t \in [0, 1]$ that for $\mu_0$-a.e. $x_0$:

$$J_t(x_0) := (1 - t) + t J_1(x_0) = \frac{\rho_0(x_0) h(x_0)}{\rho_t(x_t) h(x_t)},$$

where $x_t := T_t(x_0) = (1 - t) x_0 + t x_1$. Abbreviating $C_{x_0}^{-1} := \rho_0(x_0) h(x_0)$, it follows that for $\mu_0$-a.e. $x_0$, by (4.10) and H"older’s inequality:

$$(C_{x_0} \rho_t(x_t))^{-\frac{1}{N}} = J_t^{-\frac{1}{N}}(x_0) h^\frac{1}{N}(x_t)$$

$$\geq (1 - t) J_0(x_0) + t J_1(x_0))^{-\frac{1}{N}}$$

$$\cdot \left( \sigma_0^{(1 - t)}(|x_1 - x_0| h^{\frac{1}{N-1}}(x_0) + \sigma_1^{(t)}(|x_1 - x_0| h^{\frac{1}{N-1}}(x_1)) \right)^{\frac{N-1}{N}}$$

$$\geq (1 - t) J_0(x_0))^{-\frac{1}{N}} \sigma_0^{(1 - t)}(|x_1 - x_0| h^{\frac{1}{N-1}}(x_0) \right)^{\frac{N-1}{N}}$$

$$+ (t J_1(x_0))^{-\frac{1}{N}} \sigma_1^{(t)}(|x_1 - x_0| h^{\frac{1}{N-1}}(x_1))^{\frac{N-1}{N}}$$

$$= (1 - t) \frac{1}{N} \sigma_0^{(1 - t)}(|x_1 - x_0| h^{\frac{1}{N-1}}(x_0) \right)^{\frac{N-1}{N}}$$

$$+ t \frac{1}{N} \sigma_1^{(t)}(|x_1 - x_0| h^{\frac{1}{N-1}}(x_1))^{\frac{N-1}{N}},$$

establishing 4.1.

□

**Remark 4.8.** By employing Lebesgue’s differentiation theorem and allowing to modify $h$ on a null set, one may show (e.g., as in [25, lemma 3.3.10]) that Lemma 4.7 remains valid for general $h \in L^1_{\text{loc}}(\mathbb{R})$ without requiring continuity. We refrain from this generality here, as it will not be needed.

5 One-Dimensional QCD Densities

**Definition 5.1** (One-dimensional QCD density). Let $K \in \mathbb{R}$, $N \in (1, \infty)$, and $Q \geq 1$. We say that a function $h : \mathbb{R} \to \mathbb{R}_+$ that is continuous on its support is a QCD($Q$, $K$, $N$) density if:
(5.1) \[ h^\frac{1}{Q-1}(tx_1 + (1-t)x_0) \geq \]
\[ \frac{1}{Q^\frac{1}{Q-1}} \left( \sigma_{K,N-1}^{(1-t)}(|x_1 - x_0|)h^\frac{1}{Q-1}(x_0) + \sigma_{K,N-1}^{(t)}(|x_1 - x_0|)h^\frac{1}{Q-1}(x_1) \right) , \]
for all \( x_0, x_1 \in \text{supp} h \) and \( t \in (0, 1) \).

**Remark 5.2.** Clearly, the support of a QCD density \( h \) is always an interval and \( h \) is strictly positive in its interior. Note that a function \( h \) satisfying (5.1) with \( Q > 1 \) may in general be discontinuous at every point of its support, and hence we in addition require continuity above.

**Remark 5.3.** When \( Q = 1 \), \( h \) as above is said to be a CD\((K, N)\) density. In this case, there is no need to a priori assume that \( h \) is continuous on its support; any \( h : \mathbb{R} \to \mathbb{R}_+ \) satisfying (5.1) with \( Q = 1 \) is automatically lower semicontinuous on its support and continuous in its interior (see, e.g., [28, app. A]), and so up to modifying the value of \( h \) at the endpoints, such an \( h \) is already continuous.

Applying Lemma 4.7 with \( Q = 1 \), \( K \geq 1 \), \( N \in (1, \infty) \), we immediately obtain the following:

**Corollary 5.4.** Given \( K \in \mathbb{R}_+, N \in (1, \infty) \), and a function \( h : \mathbb{R} \to \mathbb{R}_+ \) that is continuous on its support, the one-dimensional metric-measure space \((\mathbb{R}, \| \cdot \|, hL^1)\) satisfies QCD\((Q, K, N)\) if and only if \( h \) is a QCD\((Q, K, N)\) density.

Note that when \( K > 0 \), \( \mathbb{R}_+ \ni \theta \mapsto \sigma_{K,N-1}^{(t)}(\theta) \) is not continuous for \( t = 0, 1 \), as it jumps from 0, 1 (respectively) to \( +\infty \) at \( \theta = D_{K,N} \). However, the values \( t = 0, 1 \) were (deliberately) excluded from consideration in all of the statements of the previous section, and so Lemma 4.7 applies.

For later use, we introduce the following one-dimensional members of the family QCD\(_{\text{reg}}\)(\(Q, K, N\)) defined in Section 2.

**Definition 5.5 (QCD\(_1\)(\(Q, K, N\))).** We denote by QCD\(_1\)(\(Q, K, N\)) the one-dimensional metric-measure spaces \((\mathbb{R}, \| \cdot \|, hL^1)\) that satisfy QCD\((Q, K, N)\) and MCP\((K', N')\) for some \( K' \in \mathbb{R} \) and \( N' \in (1, \infty) \).

As usual, note that when \( Q = 1 \), QCD\(_1\)(1, \(K, N\)) coincides with CD\(_1\)(\(K, N\)), defined in Section 2. We can now remove the continuity assumption in Corollary 5.4 (without invoking Remark 4.8), and obtain it as part of the conclusion:

**Corollary 5.6.** Given \( K \in \mathbb{R}_+, N \in (1, \infty) \), \( Q \geq 1 \), and \( h \in L^1_{\text{loc}}(\mathbb{R}) \), \((\mathbb{R}, \| \cdot \|, hL^1) \in \text{QCD}\(_1\)(Q, K, N)\) if and only if (up to modification on a null set) \( h \) is both a QCD\((Q, K, N)\) and MCP\((K', N')\) density, for some \( K' \in \mathbb{R} \) and \( N' \in (1, \infty) \).
PROOF. The “if” direction follows from the “if” directions of Corollary 5.4 and Lemma 3.7. The “only if” direction follows by first using the MCP\((K', N')\) property of the space to invoke Lemmas 3.7 and 3.8 and conclude that up to modification on a null set, \(h\) is continuous on its support, and then applying the “only if” direction of Corollary 5.4.

5.1 One-dimensional QCD and CD densities are equivalent

By Theorem 4.1 and Corollary 5.4 we can already reduce the study of any property of QCD spaces that is amenable to localization to the one-dimensional case. To treat the one-dimensional case, our second main observation in this work is as follows:

PROPOSITION 5.7 (One-dimensional QCD and CD densities are equivalent). \(h\) is a QCD\((Q, K, N)\) density iff there exists a CD\((K, N)\) density \(f\) so that

\[ h \leq f \leq Qh. \]

Contrary to the results of the previous section, Proposition 5.7 is rather particular to the functions \(D_{K,1}\). The reason is that \(D_{K,1}\) satisfies the following second-order ODE:

\[ \sigma''(t) + \theta^2 \frac{K}{N-1} \sigma(t) = 0 \quad \text{on } t \in [0, 1], \sigma(0) = 0, \sigma(1) = 1. \]

Consequently, we will construct \(f\) above as a “CD\((K, N)\) upper envelope” of \(h\). For the proof, we will require the following:

DEFINITION 5.8 (CD\((K, N)\) model density). A function \(f_m : \mathbb{R} \rightarrow \mathbb{R}_+\) that is smooth on its support and satisfies

\[ (f_m^{\frac{1}{N-1}})''(t) + \frac{K}{N-1} f_m^{\frac{1}{N-1}}(t) = 0 \quad \text{on } \text{supp} \ f_m \]

is called a CD\((K, N)\) model density.

By using (5.2), one immediately verifies that a CD\((K, N)\) model density is a CD\((K, N)\) density that satisfies (5.1) with equality (and \(Q = 1\)). Note that the maximal interval \(I_{f_m}\) on which a solution to (5.3) exists and coincides with \(f_m\) on \(\text{supp} \ f_m\) is of diameter \(D_{K,N}\), and hence \(\text{diam} \ (\text{supp} \ f_m) \leq D_{K,N}\). We will say that \(f_m\) is of maximal support if \(\text{supp} \ f_m = I_{f_m}\); note that in that case, \(f_m\) is continuous on the entire \(\mathbb{R}\). For more on the well-known differential characterization of CD\((K, N)\) densities as satisfying (5.3) with \(\leq 0\) instead of \(= 0\) (in the sense of distributions) we refer to [28, app. A].

Note that contrary to CD\((K, N)\) densities, QCD\((Q, K, N)\) densities do not and cannot satisfy any differential characterization whenever \(Q > 1\). To see this, take any CD\((K, N)\) density \(f\) supported on an interval \(I\) of positive length, and multiply it by any continuous function \(p\) that oscillates on \(I\) between the values of \(1\) and \(Q\); the resulting density \(h = fp\) is a QCD\((Q, K, N)\) density by (the trivial direction of) Proposition 5.7. Obviously, by making \(p\) oscillate as violently as one
desires, no differential characterization of \(QCD(Q, K, N)\) densities is possible, and furthermore, it is possible to arrange so that \(h\) does not satisfy any \(CD(K', N')\) condition for any \(K' \in \mathbb{R}\) and \(N' \in (1, \infty)\). A concrete example of a density that satisfies \(QCD(2, 0, 2)\) but not \(CD(K', N')\) for any \(K' \in \mathbb{R}\) and \(N' \in (1, \infty)\) is given; e.g., by \(h(x) = 1 + |x|\) on the interval \([-1, 1]\); the latter is easily seen after noting that the distributional second derivative of \(h\) on \([-1, 1]\) is the delta measure \(2\delta_0\).

**Proof of Proposition 5.7.** The “if” direction is trivial by using that \(f\) is a \(CD(K, N)\) density and passing from \(f\) to \(h\) using \(h f \geq h\). For the “only if” direction, let \(h\) be a \(QCD(Q, K, N)\) density. Its support is a closed interval, and we may assume it is nonempty (and thus of positive length), otherwise there is nothing to prove. Define

\[
\bar{f} := \inf \{f_m, f_m\text{ is a }CD(K, N)\text{ model density with }\text{supp }f_m = \text{supph and }f_m \geq h\},
\]

where the infimum is interpreted pointwise. Note that by definition of \(CD(K, N)\) density, the pointwise infimum of a set of \(CD(K, N)\) densities having common support \(I \subset \mathbb{R}\) is itself a \(CD(K, N)\) density (whose support is in general a subset of \(I\)); note that the infimum will automatically be continuous on \(I\) since it is upper semicontinuous (being an infimum of continuous functions) and lower semicontinuous (satisfying \(5.1\) with \(Q = 1\)). Hence, assuming the infimum above is over a nonempty set, then \(\bar{f}\) is a \(CD(K, N)\) density satisfying \(\bar{f} \geq h\), and in particular \(\text{supp }\bar{f} = \text{supph}\).

In addition, define:

\[
f(x) := \sup \left\{ \left( \sigma_{K, N}^{(1, t)}([x_1 - x_0]) h \frac{1}{x_1 - x_0}(x_0) \right) + \sigma_{K, N}^{(t)}([x_1 - x_0]) h \frac{1}{x_1 - x_0}(x_1) \right\}^{N-1},
\]

if \(x \in \text{supph}\) and \(f(x) = 0\) otherwise. Note that by definition of \(QCD\) density, \(f \leq Qh\).

We will show that \(\bar{f} = f\) on \(\text{int supph}\), and so setting \(f = \bar{f}\) will conclude that \(f\) is a \(CD(K, N)\) density on \(\text{supph}\) with \(h \leq f \leq Qh\) on \(\text{int supph}\) (and hence on \(\text{supph}\) by continuity of \(h\)), as desired. To this end, we require the following:

**Lemma 5.9.** For all \(x \in \text{int supph}\), there exists a \(CD(K, N)\) model density \(f_m\) so that \(f_m(x) = f(x)\) and \(f_m \geq h\).

Once this lemma is established, it first follows that the infimum in the definition of \(\bar{f}\) is indeed over a nonempty set (by choosing any \(x \in \text{int supph}\), applying the lemma, and restricting \(f_m\) to \(\text{supph}\)). Moreover, the lemma immediately implies that \(\bar{f} \leq f\) on \(\text{int supph}\). On the other hand, we also have \(\bar{f} \geq f\) on \(\text{supph}\),
since if \( f_m \) is a CD\((K, N)\) model density with \( f_m \geq h \), then for any \( t \in [0, 1] \) and \( x, x_0, x_1 \in \text{supp} \, h \) so that \( x = (1 - t)x_0 + tx_1 \), we have

\[
f_m^{\frac{1}{\pi^{-1}}} (x) = \sigma_{K, N-1}^t ([x_1 - x_0]) f_m^{\frac{1}{\pi^{-1}}} (x_0) + \sigma_{K, N-1}^t ([x_1 - x_0]) f_m^{\frac{1}{\pi^{-1}}} (x_1)
\]

\[
\geq \sigma_{K, N-1}^t ([x_1 - x_0]) h^{\frac{1}{\pi^{-1}}} (x_0) + \sigma_{K, N-1}^t ([x_1 - x_0]) h^{\frac{1}{\pi^{-1}}} (x_1),
\]

and so taking supremum over \( t, x_0, x_1 \) as above, it follows that \( f_m (x) \geq \underline{f} (x) \), and taking infimum over \( f_m \) as above, we indeed verify that \( \overline{f} \geq f \). This implies that \( \overline{f} = f \) on int supp \( h \), and so all that remains is to establish the lemma.

Given \( x \in \text{int} \, \text{supp} \, h \), assume in the contrapositive that there is no CD\((K, N)\) model density \( f_m \) so that \( f_m(x) = \underline{f}(x) \) and \( f_m \geq h \). Hence, for any CD\((K, N)\) model density \( f_m \) of maximal support (and therefore continuous on \( \text{supp} \, h \)) so that \( f_m(x) = \underline{f}(x) \), either there exists \( x_1 > x \) so that \( 0 < f_m(x_1) < h(x_1) \) or there exists \( x_0 < x \) so that \( 0 < f_m(x_0) < h(x_0) \), but it is impossible that both possibilities occur simultaneously, since otherwise, as \( x_0, x_1 \in \text{supp} \, f_m \cap \text{supp} \, h \), we would have (for \( t \in (0, 1) \)) so that \( x = (1 - t)x_0 + tx_1 \):

\[
f_m^{\frac{1}{\pi^{-1}}} (x) = \sigma_{K, N-1}^t ([x_1 - x_0]) f_m^{\frac{1}{\pi^{-1}}} (x_0) + \sigma_{K, N-1}^t ([x_1 - x_0]) f_m^{\frac{1}{\pi^{-1}}} (x_1)
\]

\[
< \sigma_{K, N-1}^t ([x_1 - x_0]) h^{\frac{1}{\pi^{-1}}} (x_0) + \sigma_{K, N-1}^t ([x_1 - x_0]) h^{\frac{1}{\pi^{-1}}} (x_1)
\]

\[
\leq f^{\frac{1}{\pi^{-1}}} (x),
\]

a contradiction. Let us denote the first possibility above by \( R \) and the second by \( L \).

By the second-order ODE description \((5.3)\), the set of CD\((K, N)\) model densities \( f_m \) of maximal support with a given value of \( f_m(x) \) is parametrized by its slope \( s = f_m'(x) \in \mathbb{R} \) and varies continuously in \( s \). Consequently, \( L \) and \( R \) are complementing open conditions with respect to \( s \in \mathbb{R} \), and so by connectedness of \( \mathbb{R} \), either \( L \) or \( R \) must hold for all \( f_m \) with \( f_m(x) = \underline{f}(x) \) simultaneously. But this is impossible: fixing \( x_0 < x < x_1 \) so that \( x_0, x_1 \in \text{int} \, \text{supp} \, h \) (i.e., \( h(x_0), h(x_1) > 0 \)), it is immediate to show (see \([67]\) lemma 3.1)) that \( f_m(x_0) \to 0 \) when \( s \to +\infty \) and that \( f_m(x_1) \to 0 \) when \( s \to -\infty \), and so both possibilities \( L \) and \( R \) can occur, a contradiction. Note that this argument is also valid when \( K > 0 \), even though the (maximal) support of \( f_m \) may not contain \( \text{supp} \, h \).

This concludes the proof of the lemma, and hence of the proposition. \( \square \)

## 6 Functional Inequalities on QCD Spaces

### 6.1 Equivalent formulation, monotonicity, and stability

We begin this section by rewriting the \( L^p \)-Poincaré and log-Sobolev inequalities we consider in this work in an equivalent form. Note that since \( \Omega \) is always assumed bounded, (\( \text{supp}(m), d \)) is proper by the underlying MCP\((K', N')\) assumption, \( m \) is locally finite, and the test function \( f \) is locally Lipschitz, then all integrals involved in these inequalities are necessarily finite. We formulate the inequalities a
bit more generally, using a bounded $\Lambda \supset \Omega$ instead of $\text{geo}(\Omega)$ on the energy side of the inequalities.

- The $L^p$-Poincaré constant $\lambda_p[(X, d, m), \Omega, \Lambda]$ is defined as the best constant $\lambda_p$ so that for any (locally) Lipschitz function $f : (X, d) \to \mathbb{R}$:

$$
\int_\Omega |f|^{p-2} f m = 0 \Rightarrow \lambda_p \int_\Omega |f|^p m \leq \int_\Lambda |\nabla_X f|^p m.
$$

Note that it coincides with the best constant $\lambda_p$ so that for any (locally) Lipschitz function $f : (X, d) \to \mathbb{R}$:

$$
\lambda_p \min_{c \in \mathbb{R}} \int_\Omega |f - c|^p m \leq \int_\Lambda |\nabla_X f|^p m.
$$

Indeed, this is immediate after noting that the unique minimizing $c$ above (since $p \in (1, \infty)$) satisfies $\int_\Omega |f - c|^{p-2} (f - c)m = 0$, and of course $|\nabla_X f| = |\nabla_X (f - c)|$.

- The log-Sobolev constant $\lambda_{LS}[(X, d, m), \Omega, \Lambda]$ is defined as the best constant $\lambda_{LS}$ so that for any (locally) Lipschitz function $f : (X, d) \to \mathbb{R}$:

$$
\int_\Omega (f^2 - 1)m = 0 \Rightarrow \lambda_{LS} \frac{1}{2} \int_\Omega f^2 \log(f^2) m \leq \int_\Lambda |\nabla_X f|^2 m.
$$

It coincides (when $m(\Omega) > 0$) with the best constant $\lambda_{LS}$ so that for any (locally) Lipschitz function $f : (X, d) \to \mathbb{R}$:

$$
\lambda_{LS} \int_\Omega \left( \Phi(f^2) - \Phi \left( \frac{1}{m(\Omega)} \int_\Omega f^2 m \right) \right) m \leq \int_\Lambda |\nabla_X f|^2 m,
$$

where $\Phi(x) := x \log(x)$. Indeed, this is immediate to check by applying (6.2) to $f/\sqrt{c}$ with $c = \int_\Omega f^2 m / m(\Omega)$ whenever $c > 0$ on one hand, and noting that $\Phi(1) = 0$ on the other. Furthermore, the convexity of $\Phi : \mathbb{R}_+ \to \mathbb{R}$ ensures (see Holley-Stroock [48] or the proof of [55, prop. 5.5]) that for all nonnegative $g$ for which the integrals below are finite:

$$
\int_\Omega \left( \Phi(g) - \Phi \left( \frac{1}{m(\Omega)} \int_\Omega g m \right) \right) m = \inf_{t \in \mathbb{R}_+} \int_\Omega \left( \Phi(g) - \Phi(t) - \Phi'(t)(g - t) \right) m,
$$

and that the integrand on the right-hand side is nonnegative for each $t$.

We conclude that we can express each of our functional inequalities (6.1) and (6.2) in the form

$$
\lambda_*[(X, d, m), \Omega, \Lambda] \inf_{\alpha \in A} \int_\Omega F_{\alpha}(f) m
\leq \int_\Lambda G(|\nabla_X f|) m \quad \forall \text{ locally Lipschitz } f,
$$

for an appropriate $G$ and family $\{F_{\alpha}\}_{\alpha \in A}$ of nonnegative functionals (depending on $\lambda_* \in \{\lambda_p, \lambda_{LS}\}$), with identical best constants in either formulation. Two immediate crucial consequences are the following:
Lemma 6.1. The best constant \( \lambda_*[(X, d, m), \Omega, \Lambda] \) in (6.3) satisfies:

1. **Monotonicity**: if \( \Omega_2 \subset \Omega_1 \) and \( \Lambda_2 \supset \Lambda_1 \), then
   \[ \lambda_*[(X, d, m), \Omega_2, \Lambda_2] \geq \lambda_*[(X, d, m), \Omega_1, \Lambda_1] \].
2. **Stability**: if \( m_2 \leq c_1 m_1 \) on \( \Omega \) and \( m_1 \leq c_2 m_2 \) on \( \Lambda \), then
   \[ \lambda_*[(X, d, m_2), \Omega, \Lambda] \geq \frac{1}{c_1 c_2} \lambda_*[(X, d, m_1), \Omega, \Lambda] \].

6.2 One-dimensional case

As an immediate corollary, we obtain the following:

**Corollary 6.2.** For any family \( \mathcal{X} \) of one-dimensional metric-measure spaces \( (\mathbb{R}, |\cdot|, m) \) for which \( \text{supp}(m) \) is an interval and which is closed under restrictions to intervals, and for any \( D \in (0, \infty) \), we have \( \lambda_*[\mathcal{X}, D] = \lambda_*[\mathcal{X}, D] \). In particular, this applies to \( \mathcal{X} = \text{CD}^1(K, N) \) and \( \mathcal{X} = \text{QCD}^1(Q, K, N) \).

**Proof.** The inequality \( \lambda_*[\mathcal{X}, D] \leq \tilde{\lambda}_*[\mathcal{X}, D] \) always holds, so we just need to show the converse. Given \( (\mathbb{R}, |\cdot|, m) \in \mathcal{X} \) and a closed \( \Omega \subset \text{supp}(m) \) of diameter at most \( D \), the monotonicity assertion ofLemma 6.1 implies

\[
\lambda_*[(\mathbb{R}, |\cdot|, m), \Omega, \text{geo}(\Omega)] \\
\geq \lambda_*[(\mathbb{R}, |\cdot|, m_{\text{geo}(\Omega)}), \text{geo}(\Omega)] \\
= \lambda_*[(\mathbb{R}, |\cdot|, m_{\text{geo}(\Omega)}), \text{geo}(\Omega)] \geq \tilde{\lambda}_*[\mathcal{X}, D],
\]

since \( (\mathbb{R}, |\cdot|, m_{\text{geo}(\Omega)}) \in \mathcal{X} \) and \( \text{supp}(m_{\text{geo}(\Omega)}) = \text{geo}(\Omega) \) is an interval of diameter at most \( D \). Taking infimum over all \( (\mathbb{R}, |\cdot|, m) \) and \( \Omega \) as above concludes the proof. \( \square \)

Since \( \text{geo}(\Omega) \) is not necessarily geodesically convex in dimension greater than 1, we do not know how to extend the identification between \( \lambda_* \) and \( \tilde{\lambda}_* \) asserted in Corollary 6.2 to general families of metric-measure spaces. However, for families which admit localization to one-dimensional geodesics like \( \text{CD}^\text{reg}(K, N) \) or more generally \( \text{QCD}^\text{reg}(Q, K, N) \), we can in fact extend it as described in Theorem 6.4 below.

Together with Proposition 5.7, we can already conclude the one-dimensional case of Theorem 2.9:

**Theorem 6.3.** For all \( K \in \mathbb{R}, N \in (1, \infty), D \in (0, \infty), Q \geq 1, \) and \( \lambda_* \in \{\lambda_p, \lambda_{LS}\} \):

\[ \tilde{\lambda}_*[\text{CD}^1(K, N), D] \geq \lambda_*[\text{QCD}^1(Q, K, N), D] \geq \frac{1}{Q} \tilde{\lambda}_*[\text{CD}^1(K, N), D]. \]

**Proof.** The first inequality is trivial since \( \text{CD}^1(K, N) \subset \text{QCD}^1(Q, K, N) \). Taking into account Corollary 6.2, it remains to establish

\[ \tilde{\lambda}_*[\text{QCD}^1(Q, K, N), D] \geq \frac{1}{Q} \tilde{\lambda}_*[\text{CD}^1(K, N), D]. \]
Let \((\mathbb{R}, |\cdot|, hL^1) \in \text{QCD}_1(Q, K, N)\) with \(I = \text{supp} \ h\) having diameter at most \(D\). By Corollary 5.6, up to modifications on a null set, \(h\) is a \(\text{QCD}(Q, K, N)\) density.

By Proposition 5.7, there exists a \(\text{CD}(K, N)\) density \(f\) so that \(h \leq f \leq Qh\). Consequently, the stability assertion of Lemma 6.1 implies that

\[
\lambda_*[(\mathbb{R}, |\cdot|, hL^1), I, I] \geq \frac{1}{Q} \lambda_*[(\mathbb{R}, |\cdot|, fL^1), I, I] \geq \frac{1}{Q} \lambda_*[\text{CD}_1(K, N), D].
\]

Taking infimum over all \((\mathbb{R}, |\cdot|, hL^1)\) as above concludes the proof. \(\square\)

### 6.3 Localization

It remains to establish:

**Theorem 6.4.** For all \(K \in \mathbb{R}, N \in (1, \infty), D \in (0, \infty), Q \geq 1, \) and \(\lambda_* \in \{\lambda_p, \lambda_{LS}\}:

\[
\lambda_*[\text{QCD}_{\text{reg}}(Q, K, N), D] = \lambda_*[\text{QCD}_{\text{reg}}(Q, K, N), D] = \lambda_*[\text{QCD}_1(Q, K, N), D] = \lambda_*[\text{QCD}_1(Q, K, N), D] = \lambda_*[\text{QCD}_1(Q, K, N), D].
\]

In conjunction with Theorem 6.3, this will establish our main Theorem 2.9.

**Proof of Theorem 6.4.** Since \(\text{QCD}_1(Q, K, N) \subset \text{QCD}_{\text{reg}}(Q, K, N)\) and \(\lambda_* \geq \lambda_*\) always, we trivially have

\[
\lambda_*[\text{QCD}_1(Q, K, N), D] \geq \lambda_*[\text{QCD}_{\text{reg}}(Q, K, N), D] \geq \lambda_*[\text{QCD}_1(Q, K, N), D].
\]

so it remains to establish that

\[
\lambda_*[\text{QCD}_{\text{reg}}(Q, K, N), D] \geq \lambda_*[\text{QCD}_1(Q, K, N), D]
\]

to close the chain of inequalities and conclude that they are in fact all equalities. Denote by \(Z_* : \mathbb{R} \rightarrow \mathbb{R}, *, \in \{p, LS\}\), the function \(Z_*(t) := |t|^{p-2}t\) and \(Z_{LS}(t) := t^2 - 1\). Given \((X, d, m) \in \text{QCD}_{\text{reg}}(Q, K, N)\), a closed \(\Omega \subset \text{supp}(m)\) with \(\text{diam}(\Omega) \leq D\), and a (locally) Lipschitz function \(f\) on \((X, d)\) with \(\int_{\Omega} Z_*(f)m = 0\), set \(g = Z_*(f)1_\Omega\). As \(\Omega\) is bounded, \((\text{supp}(m), d)\) is proper by MCP(K', N'), and \(m\) is locally finite, the integrability assumption \(\int_X |g(x)||d(x, x_0)m(dx) < \infty\) is clearly satisfied, and we may apply the generalized localization theorem 4.1 with the QCD interpolation weights

\[
\sigma_i^{(q)}(\theta) = Q^{-\frac{1}{|\cdot|}} \sigma_i^{(q)}(K, N - 1)
\]

(recalling in addition Corollary 5.4).

It follows that an \(m\)-measurable subset \(T \subset X\) and a family \(\{X_q\}_{q \in Q} \subset X\) exist so that the following disintegration of \(m_{\Lambda_T}\) on \(\{X_q\}_{q \in Q}\) holds:

\[
m_{\Lambda_T} = \int_Q m_q q(dq), \quad q(Q) = 1,
\]

and for \(q\)-a.e. \(q \in Q\):
(1) \( X_q \) is a closed geodesic in \((X,d)\).

(2) \( \mu_q \) is a Radon measure supported on \( X_q \) with \( \mu_q \ll \mathcal{H}^1_{L_q} \).

(3) \( \int_{X_q \cap \Omega} Z_*(f) \mu_q = \int g \mu_q = 0 \).

(4) \( (X_q, d, \mu_q) \) verifies MCP\((K', N')\).

(5) \( (X_q, d, \mu_q) \) verifies QCD\((\tilde{Q}, K, N)\).

In addition, \( g \equiv 0 \) \( \text{m-a.e. on } X \setminus \mathcal{T} \), implying that \( Z_*(f) \equiv 0 \) \( \text{m-a.e. on } \Omega \setminus \mathcal{T} \).

Since \( \text{supp}(g \mu) \subset \Omega \), we know that \( \text{diam}(\text{supp}(g \mu)) \leq D \). Let \( q \in \mathcal{Q} \) be such that all of the above properties hold, and denote:

\[
L_q := \text{geo}_{X_q}(X_q \cap \text{supp}(g \mu)),
\]

where the geodesic (convex) hull is taken in the metric space \((X_q,d)\), which is isometric to a closed subinterval of \((\mathbb{R}, |\cdot|)\). It follows that \( \text{diam}(L_q) \leq D \), and we have

\[
(6.5) \quad X_q \cap \text{supp}(g \mu) \subset L_q \subset X_q \cap \text{geo}(\text{supp}(g \mu)).
\]

Since \( \mu_{\mathcal{T}}(\{g \neq 0\} \setminus \text{supp}(g \mu)) = 0 \), the above disintegration and Fubini’s theorem imply that for \( q \)-a.e. \( q \in \mathcal{Q}, g \equiv 0 \) \( \mu_q \text{-a.e. on } X \setminus \text{supp}(g \mu) \), and in particular on \( X_q \setminus L_q \). It follows by property (3) that for \( q \)-a.e. \( q \in \mathcal{Q} \):

\[
(6) \quad Z_*(f) \equiv 0 \text{ \( \mu_q \text{-a.e. on } X_q \cap \Omega \setminus (L_q \cap \Omega) \) and } \int_{L_q \cap \Omega} Z_*(f) \mu_q = 0.
\]

We therefore add this requirement from \( q \) to our previous requirements, as they all hold for \( q \)-a.e. \( q \in \mathcal{Q} \).

Since the QCD\((\tilde{Q}, K, N)\) and MCP\((K', N')\) conditions are closed under restrictions onto geodesically convex subsets, it follows that \( (L_q, d, \mu_q \ll L_q) \) verifies both conditions; however, since \( \Omega \) was not assumed to be geodesically convex, note that \( (L_q \cap \Omega, d, \mu_q \ll (L_q \cap \Omega)) \) may not satisfy QCD\((\tilde{Q}, K, N)\) nor MCP\((K', N')\). Nevertheless, by the monotonicity property established in Lemma \(6.1\):

\[
\lambda_*[(L_q, d, \mu_q \ll L_q), L_q \cap \Omega, L_q] \geq \lambda_*[(L_q, d, \mu_q \ll L_q), L_q, L_q]
\]

\[
\geq \lambda_*[\text{QCD}_1(\tilde{Q}, K, N), D],
\]

where the last inequality is due to the fact that \((L_q, d, \mu_q \ll L_q)\) is (isometric to) a one-dimensional metric-measure space satisfying QCD\((\tilde{Q}, K, N)\) and MCP\((K', N')\) and \( \text{diam}(L_q) \leq D \).

Since \( \int_{L_q \cap \Omega} Z_*(f) \mu_q = 0 \) by property (6), we may revert back from the infimum formulation \(6.3\) of our functional inequality to the standard one in \(6.1\) or \(6.2\) for \( \lambda_* \in \{\lambda_p, \lambda_{L_S}\} \), respectively. We conclude that

\[
\lambda_p[\text{QCD}_1(\tilde{Q}, K, N), D] \int_{L_q \cap \Omega} |f|^p \mu_q \leq \int_{L_q} |\nabla L_q | f|^p \mu_q,
\]

in the first case, and

\[
\frac{\lambda_{L_S}[\text{QCD}_1(\tilde{Q}, K, N), D]}{2} \int_{L_q \cap \Omega} f^2 \log(f^2) \mu_q \leq \int_{L_q} |\nabla L_q | f|^2 \mu_q,
\]

in the second case.
in the second. Recall that \( Z_*(f) = 0 \) m\(_q\)-a.e. on \( X_q \cap \Omega \setminus (L_q \cap \Omega) \) by property (6), and hence the integrand on the left-hand sides above vanishes m\(_q\)-a.e. on \( X_q \cap \Omega \setminus (L_q \cap \Omega) \) (also in the \( LS \) case, since \( Z_{LS}(f) = 0 \) iff \( f^2 = 1 \) iff \( \log(f^2) = 0 \)). It follows that we may enlarge the domain of integration on the left-hand sides to \( X_q \cap \Omega \); on the right-hand sides we may enlarge the domain of integration to \( X_q \cap \text{geo}(\text{supp}(g_m)) \) thanks to the nonnegativity of the integrand and (6.5).

Using \( |\nabla_{Le} f| \leq |\nabla_X f| \) and integrating the resulting inequalities with respect to \( q \), we deduce from the disintegration formula that:

\[
\bar{\lambda}_p[\text{QCD}_1(Q, K, N), D] \int_{\mathcal{T} \cap \Omega} |f|^p m \leq \int_{\mathcal{T} \cap \text{geo}(\text{supp}(g_m))} |\nabla_X f|^p m,
\]

and

\[
\frac{\bar{\lambda}_{LS}[\text{QCD}_1(Q, K, N), D]}{2} \int_{\mathcal{T} \cap \Omega} f^2 \log(f^2) m \leq \int_{\mathcal{T} \cap \text{geo}(\text{supp}(g_m))} |\nabla_X f|^2 m,
\]

respectively. Recalling that \( Z_*(f) = 0 \) m-a.e. on \( \Omega \setminus \mathcal{T} \), we may enlarge as before the domain of integration on the left-hand sides to \( \Omega \); on the right-hand sides we may enlarge it to \( \text{geo}(\Omega) \supset \text{geo}(\text{supp}(g_m)) \). This establishes (6.4), thereby concluding the proof.

\section{Concluding Remarks}

\subsection{Curvature geodesic-topological dimension condition}

Before concluding, we mention an alternative path for deriving the exact same results we obtain in this work, which is more tailored to the ideal sub-Riemannian setting.

\begin{definition}[Curvature geodesic-topological dimension condition \( \text{CGTD}(K, N, n) \)] A Monge space \((X, d, m)\) is said to satisfy the \( \text{CGTD}(K, N, n) \) condition, \( K \in \mathbb{R}, n \in [1, \infty), n \leq N \in (1, \infty) \), if for all \( \mu_0, \mu_1 \in P_{\mathcal{C}}(X) \) with \( \mu_0, \mu_1 \ll m \) and for all \( f \in (0, 1) \):

\[
\rho_{t^{-1}}^{1/n}(\gamma_1) \geq \tau_{K,N}(d(\gamma_0, \gamma_1)) \frac{n}{K} \rho_{t^{-1}}^{1/n}(\gamma_0) + \tau_{K,N}(d(\gamma_0, \gamma_1)) \frac{n}{K} \rho_{t^{-1}}^{1/n}(\gamma_1)
\]

for \( \nu\)-a.e. \( \gamma \in \text{Geo}(X, d) \).
\end{definition}

Note that the \( \text{CGTD}(K, N, n) \) condition implies both the \( \text{MCP}(K, N) \) condition (by dropping the rightmost term above) and the \( \text{QCD}(2^{N-n}, K, N) \) condition (by applying Jensen’s inequality as in the proof of Proposition 2.4). Repeating the argument in Section 2, Theorem 2.1 implies that the \( \text{MCP}(K, N) \) condition on an ideal \( n \)-dimensional sub-Riemannian manifold automatically self-improves to \( \text{CGTD}(K, N, n) \), and so all the ideal \( n \)-dimensional sub-Riemannian manifolds mentioned in Section 2.2 satisfy \( \text{CGTD}(0, N, n) \) for some appropriate \( N > n \).

We may now apply the general localization Theorem 4.1 to deduce that the \( \text{CGTD}(K, N, n) \) condition localizes to one-dimensional geodesics; consequently, it
is enough to study the properties of one-dimensional CGTD($K, N, n$) densities $h$, which by Lemma 4.7 are characterized by

$$h^{1-\frac{1}{N}}(tx_1 + (1-t)x_0) \geq \sigma_{K, N}^{(1-\frac{1}{N})}\left(|x_1 - x_0|\right)^{\frac{N-1}{N}}h^{\frac{1}{N}}(x_0)$$

$$+ \sigma_{K, N}^{(1-\frac{1}{N})}\left(|x_1 - x_0|\right)^{\frac{N-1}{N}}h^{\frac{1}{N}}(x_1),$$

for all $x_0, x_1 \in \text{supp} h$ and $t \in (0, 1)$. Note that the case $n = 1$ is understood in the limiting sense, namely as taking the maximum between the two terms on the right and thus recovering the MCP($K, N$) density characterization. We see again, now on the level of one-dimensional densities, that a CGTD($K, N, n$) density is simultaneously both an MCP($K, N$) density (by dropping the rightmost term above) and a QCD($2^{N-n}, K, N$) density (by Jensen’s inequality).

Repeating the argument of Section 6, we immediately deduce that

$$\lambda_*(\text{CGTD}(K, N, n), D) = \lambda_*(\text{CGTD}_1(K, N, n), D) = \overline{\lambda}_*(\text{CGTD}_1(K, N, n), D)$$

(\text{using the obvious analogues of our usual definitions and notation}).

While this approach has the clear advantage of providing us with more information on the resulting one-dimensional densities after localization, we do not know how to use this additional information beyond what the QCD($2^{N-n}, K, N$) condition tells us. In other words, we only know to deduce the existence of an equivalent CD($K, N$) one-dimensional density $f$ so that $h \leq f \leq 2^{N-n}h$, yielding

$$\overline{\lambda}_*(\text{CGTD}_1(K, N, n), D) \geq \frac{1}{2^{N-n}}\overline{\lambda}_*(\text{CD}_1(K, N), D),$$

and thus arriving at the same conclusion as before. For this reason, we have chosen to present our results using the more general QCD condition, in the hope that it would also be applicable in more general settings beyond the sub-Riemannian one, when the CGTD condition is inapplicable.

### 7.2 Additional properties and variants

Continuing in the same vein, one can engage in a more comprehensive study of the QCD or CGTD conditions: determining what would be a good definition without a priori assuming that the space is Monge or essentially nonbranching, studying the stability of the resulting definition under measured Gromov-Hausdorff convergence and tensorization, rewriting it in terms of the $N$-Renyi entropy, extending the definition to include $N = \infty$, etc. (\text{in analogy to the Lott-Sturm-Villani program for the CD case}). One can also introduce the RQCD and RCQTD conditions, in analogy to the RCD condition, by adding the assumption that the space is infinitesimally Hilbertian [5][41], as it is known that sub-Riemannian Carnot groups are indeed infinitesimally Hilbertian [62]. Another interesting direction suggested by the referee is to investigate the relation between QCD($Q, K, N$) or CGTD($K, N, n$) and the Baudoin-Garofalo CD($\rho_1, \rho_2, \kappa, m$) condition [20]. We refrain from pursuing these directions here.
7.3 Optimality of $Q = 2^{N-n}$ and Brunn-Minkowski inequalities

It was shown in the various references mentioned in Section 2.2 that the corresponding sub-Riemannian manifolds satisfy MCP(0, $N$) with $N$ being best possible (i.e., minimal). It is also clear from applying the (optimal) Jensen inequality (as in the proof of Proposition 2.4) that the constant $Q = 2^{N-n}$ is the best possible when transitioning from the CGTD($K, N, n$) condition to the QCD($Q, K, N$) one. However, one may wonder whether the overall optimality of the constant $2^{N-n}$ is lost when transitioning from the optimal MCP(0, $N$) condition to the QCD($2^{N-n}, 0, N$) one. We mention here that this is not the case and that the value $Q = 2^{N-n}$ is indeed optimal (i.e., minimal) in the QCD($Q, 0, N$) condition, at least whenever the parameter $N$ from the MCP(0, $N$) condition coincides with the minimal geodesic dimension $N$ (see [16, theorem 5] for the precise definition of the latter)—by [14–17, 56, 80], this is the case for generalized H-type and corank 1 Carnot groups, the Grushin plane, and Sasakian and 3-Sasakian manifolds (under appropriate curvature lower bounds).

To see the aforementioned optimality, observe that a standard application of the localization argument from the previous section would verify that the QCD($Q, 0, N$) condition implies the following “quasi Brunn-Minkowski inequality”:

$$(7.1) \quad m(Z_t(A, B))^{\frac{1}{n}} \geq \frac{1}{Q^{\frac{1}{n}}} \left( (1-t)m(A)^{\frac{1}{n}} + t m(B)^{\frac{1}{n}} \right),$$

for all Borel sets $A, B \subset X$ of finite positive measure. On the other hand, it was shown by Juillet in [51] that on any strictly sub-Riemannian manifold $M$ equipped with its sub-Riemannian Carnot-Carathéodory metric $d$ and any positive smooth measure $m$, and for any $\varepsilon, \delta > 0$, there exist Borel sets $A, B \subset M$ of finite positive measure and $t \in (0, 1)$, so that

$$\frac{m(B)}{m(A)} \in [1-\varepsilon, 1+\varepsilon], \quad m(Z_t(A, B)) \leq \frac{1}{2^{N-n}} (1+\varepsilon)m(A).$$

Juxtaposing this with (7.1), it follows that necessarily $Q \geq 2^{N-n}$, and hence the value $Q = 2^{N-n}$ is optimal in both our QCD($Q, 0, N$) condition and in Juillet’s construction whenever $N' = N$ (note that we always have $N \geq N$ as a consequence of [16, theorem 5]). As a side note, we mention that Juillet’s construction guarantees that diam$(A \cup B) < R$ for any given $R > 0$, which shows that $(M, d, m)$ as above does not satisfy CD($K', N'$) for any $K', N' \in \mathbb{R}$.

Note that (7.1) with $Q = 2^{N-n}$ in the ideal sub-Riemannian setting and with $Q = 4$ ($N = n+2$) for corank 1 Carnot groups, follows immediately by Jensen’s inequality from the Brunn-Minkowski inequalities of Barilari-Rizzi [16, theorem 9] and Balogh-Kristály-Sipos [14, theorem 4.2], respectively:

$$(7.2) \quad m(Z_t(A, B))^{\frac{1}{N}} \geq (1-t)^{\frac{N}{n}} m(A)^{\frac{1}{n}} + t^{\frac{N}{n}} m(B)^{\frac{1}{n}}.$$
Juillet’s construction therefore demonstrates the optimality of (7.2) not only when one of the sets degenerates to a point (as e.g., in [16]), but also for sets of equal measures.

### 7.4 Equivalent characterization of the QCD condition

Finally, we conclude this work by mentioning an essentially equivalent characterization of the QCD condition that highlights again the connection to the CD definition. The simplest case to examine is when $K = 0$.

**Definition 7.2 (QCD$^m(Q, 0, N)$).** A Monge space $(X, d, m)$ is said to satisfy the QCD$^m(Q, 0, N)$ condition, $Q \geq 1$, $N \in (1, \infty)$, if for all $\mu_0, \mu_1 \in P_c(X)$ with $\mu_0, \mu_1 \ll m$, there exists a family of Borel measures $(m_t)_{t \in [0, 1]}$ with $\mu \leq m_t \leq Qm$ on supp $\mu_t$ so that the $W_2$ geodesic $(\mu_t)$ satisfies the CD$(0, N)$ interpolation inequality with respect to $(m_t)$—namely, denoting $\vec{\rho}_t := \frac{d\mu_t}{d\mu}$, we have for all $t \in (0, 1)$

\[
(7.3) \quad \vec{\rho}_t^{-\frac{1}{N}}(\gamma_t) \geq (1 - t)\vec{\rho}_0^{-\frac{1}{N}}(\gamma_0) + t\vec{\rho}_1^{-\frac{1}{N}}(\gamma_1) \quad \text{for } \nu\text{-a.e. } \gamma \in \text{Geo}(X, d).
\]

We claim that this definition is equivalent to the original QCD$(Q, 0, N)$ definition on Monge spaces satisfying MCP$(K', N')$ for some $K' \in \mathbb{R}$ and $N' \in (1, \infty)$. Indeed, since $\rho_t := \frac{d\mu_t}{d\mu}$ satisfies $\vec{\rho}_t \leq \rho_t \leq Q\vec{\rho}_t$, if QCD$(Q, 0, N)$ holds then clearly QCD$(Q, 0, N)$ holds as well by passing from (7.3) to (2.5). In the other direction, the MCP$(K', N')$ condition guarantees that given $(\mu_t)$ as above, we may choose versions of the densities $\rho_t := \frac{d\mu_t}{d\mu}$ so that $(0, 1) \ni t \mapsto \rho_t(\gamma_t)$ is continuous and upper semicontinuous at the endpoints for $\nu$-a.e. $\gamma \in \text{Geo}(X, d)$ (see [28, cor. 9.5 and remark 9.9]). If the space in addition satisfies QCD$(Q, 0, N)$, then by considering all rational $t \in (0, 1)$ and employing the latter continuity, it follows that there is a subset $G$ of geodesics $\gamma$ having full $\nu$-measure, so that $1/\rho_t(\gamma_t)$ satisfies (2.5) for all $t \in (0, 1)$ and is therefore almost a QCD$(Q, 0, N + 1)$ density on $[0, 1]$ (this is where the assumption $K = 0$ comes in handy)—it satisfies all requirements but is only lower semicontinuous at the endpoints $t \in \{0, 1\}$. Nevertheless, inspecting the proof of Proposition [57], it follows that there exists a continuous CD$(0, N + 1)$ density $f_\gamma : [0, 1] \to \mathbb{R}_+$ so that $1/\rho_t(\gamma_t) \leq f_\gamma(t) \leq Q/\rho_t(\gamma_t)$ for all $t \in (0, 1)$, and also $1/\rho_t(\gamma_t) \leq f_\gamma(t)$ for $t \in \{0, 1\}$ by lower semicontinuity. Since the space is Monge, one knows that there is a subset $H$ of geodesics of full $\nu$-measure for which $H \ni \gamma \mapsto \gamma_t$ is injective for all $t \in [0, 1]$ (see, e.g., [28, cor. 6.15]). Consequently, denoting $\xi_t = \xi_1 = 1$ and $\xi_t(\gamma_t) := f_\gamma(t)\rho_t(\gamma_t) \in [1, Q]$ for $\gamma \in G \cap H$ and $\xi_t = 0$ elsewhere for $t \in (0, 1)$, it follows that $\xi_t$ is well-defined, and standard arguments imply that $\xi_t$ is measurable. We can now define $m_t = \xi_t m$, and it readily follows that $m \leq m_t \leq Qm$ on supp $\mu_t$. Since $\vec{\rho}_t = \rho_t/\xi_t$ so that $1/\vec{\rho}_t(\gamma_t) = f_\gamma(t)$ for $\gamma \in G \cap H$ and $t \in (0, 1)$, and $1/\vec{\rho}_t(\gamma_t) = 1/\rho_t(\gamma_t)$ for
$t \in \{0,1\}$, it follows that for all $t \in (0,1)$, for $\nu$-a.e. $\gamma \in \text{Geo}(X,d)$:
\[
\rho_{t}^{\frac{1}{n}}(\gamma_{t}) = f_{\gamma}^{\frac{1}{n}}(t) \geq (1-t)f_{\gamma}^{\frac{1}{n}}(0) + tf_{\gamma}^{\frac{1}{n}}(1)
\]
\[
\geq (1-t)\rho_{0}^{\frac{1}{n}}(\gamma_{0}) + t\rho_{1}^{\frac{1}{n}}(\gamma_{1}),
\]
and so we confirm that (7.3) is satisfied, i.e., that the space verifies $\text{QCD}_{m}(Q,0,N)$.

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