Quantum key distribution protocol based on contextuality monogamy

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The security of quantum key distribution (QKD) protocols hinges upon features of physical systems that are uniquely quantum in nature. We explore the role of quantumness as qualified by quantum contextuality, in a QKD scheme. A new QKD protocol based on the Klyachko-Can-Binicioğlu-Shumovsky (KCBS) contextuality scenario using a three-level quantum system is presented. We explicitly show the unconditional security of the protocol by a generalized contextuality monogamy relationship based on the no-disturbance principle. This protocol provides a new framework for QKD which has conceptual and practical advantages over other protocols.

I. INTRODUCTION

The existence of pre-defined values for quantum observables that are independent of any measurement settings, has been a matter of debate ever since quantum theory came into existence. While Einstein made a case for looking for hidden variable theories that would give such values [1], the work of John Bell proved that such local hidden variable theories cannot be compatible with quantum mechanics [2]. This points towards a fundamental departure of the behaviour of quantum correlations from the ones that can be accommodated within classical descriptions. While the departure from classical behaviour indicated by Bell's inequalities requires composite quantum systems and the assumption of locality, the contradiction between assignment of pre-defined measurement-independent values to observables and quantum mechanics, goes deeper and was brought out more vividly by the discovery of quantum contextuality [3]. In a non-contextual classical description, a joint probability distribution exists for the results of any joint measurements on the system, and the results of a measurement of a variable do not depend on other compatible variables being measured. Quantum mechanics precludes such a description of physical reality; on the contrary in the quantum description, there exists a context among the measurement outcomes, which forbids us from arriving at joint probability distributions of more than two observables. Given a situation where an observable \( A \) commutes with two other observables \( B \) and \( C \) which do not commute with each other; a measurement of \( A \) along with \( B \) and a measurement of \( A \) along with \( C \), may lead to different measurement outcomes for \( A \). Thus, to be able to make quantum mechanical predictions about the outcome of a measurement, the context of the measurement needs to be specified.

The first proof that the quantum world is contextual, was given by Kochen and Specker and involved 117 different vectors in a 3-dimensional Hilbert space [3]. Subsequently, the number of observables required for such a ‘no-coloring’ proof was brought down to 31 by Conway and Kochen [4], while Peres provided a compact proof based on cubic symmetry using 33 observables [5]. In higher dimensions the number of observables can be further reduced and more compact proofs are possible [6, 7].

Klyachko et. al. found a minimal set of 5 observables for a qutrit for which the predicted value for quantum correlation exceeds the bound (the KCBS inequality) imposed by non-contextual deterministic models [8]. The violation observed is state dependent and one can find states that do not allow for stronger than classical correlations for the same set of observables. A state independent violation of a non-contextuality inequality implies that stronger correlations than classical are possible for all states for the same set of observables [9]. In a 3-dimensional Hilbert space the minimum number of observables required to achieve such a violation is 13 [9, 10] and can be brought down to 9 if one excludes the maximally mixed state [11]. Recently graph theory has also been used to describe contextuality scenarios, where vertices describe unit vectors and edges describe the orthogonality relationships between them [12, 13].

While at the level of individual measurements quantum mechanics is contextual, the probability distribution for an observable \( A \) does not depend upon the context and is not disturbed by other compatible observables being measured. This is called the ‘no-disturbance’ principle and leads to interesting monogamy relations for contextuality inequalities [14] similar to those obeyed by Bell-type inequalities [15]. These monogamy relations are a powerful expression of quantum constraints on correlations without involving a tensor product structure, and we shall exploit them in our work.

Non-trivial quantum features of the world play an important role in quantum information processing [16] and in particular in making the QKD protocols [17] fundamentally secure as opposed to their classical counterparts [18–20]. QKD protocols can be categorized into two distinct classes, namely the ‘prepare and measure schemes’, and the ‘entanglement assisted schemes’. In the prepare and measure schemes whose prime example is the BB84 [21] protocol, one party prepares a quantum state and transmits it to the other party who performs

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suitable measurements to generate a key. On the other hand, the entanglement assisted protocols utilize entanglement between two parties and a prime example of such a protocol is the Ekert protocol \[22\]. One distinct advantage in the entanglement assisted QKD protocols is the ability to check security based on classical constraints on correlations between interested parties via Bell’s inequalities. It has also been shown that any two non-orthogonal states suffice for constructing a QKD protocol \[23\]. The idea has been extended to qutrits \[24\] to allow four mutually unbiased bases for QKD. Quantum cryptography protocols have been proven to be robust against eavesdropping and noise \[18–27\].

Our focus in this work is to explore the utility of quantum contextuality for QKD. While contextuality has already been exploited for QKD \[28\], we propose a new QKD protocol which is based on the KCBS scenario and the related monogamy relationships \[8, 14\]. Our protocol falls in the class of ‘prepare and measure schemes’ but still allows a security check based on conditions on correlations shared between the the two parties Alice and Bob. In fact in our protocol it is the monogamy relation of the KCBS inequality which is responsible for unconditional security.

We first devise a QKD protocol between Alice and Bob utilizing the KCBS scenario of contextuality as a resource with post-processing of outcomes allowed on Alice’s site. Considering Eve as an eavesdropper and using the novel graph theoretic approach \[12, 14\] we then derive an appropriate monogamy relation between Alice-Bob and Alice-Eve correlations for the optimal settings of Eve. From this monogamy relationship, we then explicitly calculate the bounds on correlation to be shared among Alice and Bob demonstrating the security of the protocol. Although our protocol is not device independent, it adds a new angle to the QKD protocol research.

The material in the paper is arranged as follows: In Section II we provide a brief review of the KCBS inequality. In Section III A we describe our protocol, in Section III B we derive the monogamy relations for the required measurement settings and in Section III C we discuss the security of the protocol. Section IV offers some concluding remarks.

\[ \Pi_0 \Pi_1 \Pi_2 \Pi_3 \Pi_4 \]

**FIG. 1.** The KCBS orthogonality graph. Each vertex corresponds to a projector and the edge linking two projectors indicates their orthogonality relationship.

The KCBS inequality is used as a test of contextuality in systems with Hilbert space dimension three and more. In this section we review two equivalent formulations of the inequality, one of which will be directly used in our QKD protocol to be described later.

Consider a set of five observables which are projectors in a 3-dimensional Hilbert space. The projectors are related via an orthogonality graph as given in Figure 1. The vertices in the graph correspond to the projectors, and two projectors are orthogonal to each other if they are connected by an edge. A set of projectors which are mutually orthogonal also commute pairwise and can therefore be measured jointly. Such a set of co-measurable observables is called a context. Therefore, in the KCBS scenario, every edge between two projectors denotes a measurement context and each projector appears in two different contexts. However, a non-contextual model will not differentiate between different contexts of a measurement and will deterministically assign values to the vertices irrespective of the context.

A deterministic non-contextual model must assign a value 0 or 1 to the \(i^{th}\) vertex and therefore the probability that the vertex is assigned a value 1, denoted by \(P_i\), takes values 0 or 1 (and the corresponding probability for a vertex to have value 0 is \(1 - P_i\)). In such a non-contextual assignment the maximum number of vertices that can be assigned the probability \(P_i = 1\) (constrained by the orthogonality relations), is 2 irrespective of the state. Therefore,

\[ \tilde{K}(A, B) = \frac{1}{5} \sum_{i=0}^{4} P_i \leq \frac{2}{5}. \] (1)

This is the KCBS inequality \[8, 12\], which is a state-dependent test of contextuality utilizing these projectors, and is satisfied by all non-contextual deterministic models. In a quantum mechanical description, given a quan-
One can then reformulate Eqn. (1) in terms of antici-
related to the projectors considered above as
quantum theory is for a pure state and turns out to be
equality. The maximum value that can be achieved in
quantum theory can exhibit violation of the above in-
non-contextual and deterministic models. However,
yields anti-correlated outcomes. Eqn. (4) is obeyed by
the probability that a joint measurement of
observables which take values
in the interval \([0, 1]\). The bounds so imposed by
non-contextuality, quantum theory and the exclusivity
principle can be identified with graph theoretic invariants
of the exclusivity graph of the five projectors, which in
this case is also a pentagon \([12]\).

The correlation can be further analyzed if one considers
observables which take values \(X_i \in \{-1, +1\}\) and are
related to the projectors considered above as
\[X_i = 2\Pi_i - I.\] (3)

One can then reformulate Eqn. (1) in terms of anti-
correlation between two measurements as \([29]\]
\[K(A, B) = \frac{1}{5} \sum_{i=0}^{4} P(X_i \neq X_{i+1}) \leq \frac{3}{5}.\] (4)

Where \(i + 1\) is sum modulo 5 and \(P(X_i \neq X_{i+1})\) denotes
the probability that a joint measurement of \(X_i\) and \(X_{i+1}\)
yields anti-correlated outcomes. Eqn. (4) is obeyed by
all non-contextual and deterministic models. However,
quantum theory can exhibit violation of the above in-
nequality. The maximum value that can be achieved in
quantum theory is for a pure state and turns out to be
\[\frac{1}{5} \sum_{i=0}^{4} P_{QM}(X_i \neq X_{i+1}) = \frac{4\sqrt{5} - 5}{5} > \frac{3}{5}.\] (5)

It should be noted that the maximum algebraic value of
the expression on the left hand side of the KCBS in-
nequality as formulated in Eqn. (4) is one. We shall use
this formulation of the KCBS inequality directly in our
protocol in the next section as it allows evaluation of
(anti-)correlation between two joint measurements.

III. THE QKD PROTOCOL, CONTEXTUALITY
MONOGAMY AND SECURITY

A. The protocol

In a typical key-distribution situation, two separated
parties Alice and Bob want to share a secret key securely.

They both have access to the KCBS scenario of five pro-
jectors. Alice randomly selects a vertex \(i\) and prepares
the corresponding pure state \(\Pi_i\) and transmits the state
to Bob. Bob on his part, also randomly selects a vertex \(j\)
and performs a measurement \(\{\Pi_j, I - \Pi_j\}\) on the state.
We denote \(i\) and \(j\) as the settings of Alice and Bob re-
spectively. The outcome of Bob’s measurement depends
on whether he ended up measuring in the context of Al-
ice’s state or not. The outcome \(\Pi_j\) is assigned the value 1
and the outcome \(I - \Pi_j\) is assigned the value 0. After the
measurement, Bob publicly announces his measurement
setting, namely the vertex \(j\). Three distinct cases arise:

C1: \(i, j\) are equal \((i = j)\): By definition Bob is assured
to get the outcome 1. Alice notes down a 0 with
herself and publicly announces that the transmis-
sion was successful. Both of them thus share an
anti-correlated bit.

C2: \(i, j\) are in context but not equal: Bob’s projector is
in the context of Alice’s state. Since the state Alice
is sending is orthogonal to Bob’s chosen projector,
he is assured to get the outcome 0. Alice then notes
down 1 with herself and publicly announces that
the transmission was successful and Bob uses his
outcome as part of the key. This way they both
share an anti-correlated bit. It should be noted
that Alice does not note down her part of the key
until Bob has announced his choice of setting.

C3: \(i, j\) are not in context: Bob’s projector does not
lie in the context of Alice’s state. Alice publicly
announces that the transmission was unsuccessful
and they try again. However they keep this data,
as it may turn out to be useful to detect Eve.

Using the protocol, Alice and Bob can securely share a
random binary key. Their success depends on chances
that Bob’s measurements are made in the context of Al-
ice’s state. Whenever Bob measures in the correct con-
text which happens three-fifths of the time, Alice is able
to ensure that they have an anti-correlated key bit. When
Bob measures in the same context but not the same pro-
jector as Alice, she notes down a 1 with herself and thus
they share a 1-0 anti-correlation. On the other hand,
when Bob measures the same projector as Alice’s state,
she notes down a 0 with herself and again they share a
0-1 anti-correlation. At no stage Alice needs to reveal her
part of the key in public or to Bob. The QKD scenario is depicted
in Figure 2.

In the ideal scenario without any eavesdropper, Alice
and Bob will always get an anti-correlated pair of out-
comes and therefore will violate the KCBS inequality to
its algebraic maximum value which is one. It should be
noted that they are able to achieve the algebraic bound
because when Bob ends up measuring the same projec-
tor as Alice, she notes down 0 on her side which is not
the quantum outcome of her state. Thus this in no way
is a demonstration that quantum theory reaches the al-
gebraic bound of KCBS inequality which in fact it does
not. However, in the presence of an eavesdropper the violation of the KCBS inequality can be used as a test for security as will be shown later. The presence of Eve is bound to decrease the Alice Bob anti-correlation and that can be checked by sacrificing part of the key.

The key as generated by the above protocol although completely anticorrelated is not completely random, there are more ones in the key than zeros. Therefore, the actual length of the effective key is smaller than the number of successful transmissions. In order to calculate the actual key rate we compute the Shannon information of the transmitted string. Given the fact that $P_0 = \frac{1}{2}$ and $P_1 = \frac{2}{3}$ for the string generated out of successful transmission, the Shannon information turns out be

$$S = -P_0 \log_2 P_0 - P_1 \log_2 P_1 = 0.9183$$

The probability of success (i.e., when Bob chooses his measurement in the context of Alice’s state) is $\frac{2}{3}$ as stated earlier. Thus the average key rate per transmission can be obtained as $S/3 = 0.306$. We tabulate the average key rate of a few QKD protocols in the absence of an eavesdropper in Table I.

| QKD protocol | Success probability (per transmission) | Av. key rate in bits (per transmission) |
|--------------|---------------------------------------|----------------------------------------|
| BB84 (2 basis) | $\frac{1}{2}$ | 0.50 |
| BB84 (3 basis) | $\frac{1}{3}$ | 0.50 |
| Ekert(EPR pairs) | $\frac{1}{2}$ | 0.50 |
| 3-State [24] | $\frac{1}{4}$ | 0.50 |
| KCBS | $\frac{3}{5}$ | 0.55 |

TABLE I. The key rate for various QKD protocols in the absence of an eavesdropper. As can be seen the KCBS protocol offers a little higher key rate compared to the other protocols.

Alice randomly chooses a measurement setting $i$ and implements the measurement $\{\Pi_i, I-\Pi_i\}$ on her part of the entangled state. In the situation when she gets a positive answer and her states collapses to $\Pi_i$ Bob’s state collapses to $\Pi_i$ too. This then becomes equivalent to the situation where Alice prepares the state $\Pi_i$ and sends it to Bob. The probability of this occurrence is $1/3$. Bob too randomly chooses a measurement setting $j$ and implements the corresponding measurement. The rest of the protocol proceeds exactly as in the case of prepare and measure scenario.

Although there are a number of possible choices for the projectors $\Pi_i$, we detail below a particular choice of vectors $|v_i\rangle$ (un-normalized) corresponding to the projectors $\Pi_i$, on which the above assertions can be easily verified.

$$|v_0\rangle = \left(1, 0, \sqrt{\cos \frac{\pi}{5}}\right)$$

$$|v_1\rangle = \left(\cos \frac{4\pi}{5}, -\sin \frac{4\pi}{5}, \sqrt{\cos \frac{\pi}{5}}\right)$$

$$|v_2\rangle = \left(\cos \frac{2\pi}{5}, \sin \frac{2\pi}{5}, \sqrt{\cos \frac{\pi}{5}}\right)$$

$$|v_3\rangle = \left(\cos \frac{2\pi}{5}, -\sin \frac{2\pi}{5}, \sqrt{\cos \frac{\pi}{5}}\right)$$

$$|v_4\rangle = \left(\cos \frac{4\pi}{5}, \sin \frac{4\pi}{5}, \sqrt{\cos \frac{\pi}{5}}\right)$$

With

$$\Pi_i = \frac{|v_i\rangle\langle v_i|}{\langle v_i|v_i\rangle}, \quad i = 0, 1, 2, 3, 4.$$  

Thus our ‘prepare and measure’ protocol can be translated into an ‘entanglement assisted’ protocol. We have provided this mapping for the sake of completeness and in our further discussions we will continue to consider the prepare and measure scheme.

B. Contextuality monogamy

In quantum mechanics, given observables $A, B, C$, such that $A$ can be jointly measured both with $B$ and $C$ (i.e. it is compatible with both) the marginal probability distribution $P(A)$ for $A$ as calculated from both the joint probability distributions $P(A, B)$ and $P(A, C)$ is the same:

$$\sum_b P(A = a, B = b) = \sum_c P(A = a, C = c) = P(A = a).$$

This is called the ‘no-disturbance’ principle and it reduces to the ‘no-signaling’ principle when the measurements $B$ and $C$ are performed on spatially separate systems.

The ‘no-disturbance’ principle can be used to construct contextuality monogamy relationships of a set of observables if they can be partitioned into disjoint subsets each of which can reveal contextuality by themselves but cannot be simultaneously used as tests of contextuality.
Consider the situation where Alice and Bob are different parties who make preparations and measurements as detailed in Section III A. We consider the possibility of a third party Eve who tries to eavesdrop on the conversation between them. As will be detailed in Section III C Eve will have to violate the KCBS inequality with Alice to gain substantial information about the key.

We denote the Alice-Bob KCBS test by $K(A, B)$ with projectors $\{\Pi_i\}$ and Alice-Eve KCBS test by $\tilde{K}(A, E)$ with projectors $\{\Pi^E_i\}$. We have assumed different projectors in the two KCBS tests for clarity in derivation of a monogamy relationship, but essentially the measurements to be performed by Eve would have to be the same as that of Bob to mimic Alice and Bob’s KCBS scenario as will be detailed in Section III C where we take up the security analysis of our protocol. In this joint scenario the $\Pi^E_i$ projector is connected by an edge to $\Pi_{i+1}$, $\Pi^E_{i+1}$, $\Pi_{i-1}$, $\Pi^E_{i-1}$ and $\Pi^E_i$, where $i + 1$ and $i - 1$ are taken modulo 5 and the presence of an edge denotes commutativity between the two connected vertices. These relationships follow from the fact that the projectors used by Eve will follow the same commutativity relationships as the original KCBS scenario. By introducing herself in the channel, Eve has created an extended scenario which will have to obey contextuality monogamy due to the no-disturbance principle. The no-disturbance principle guarantees that the marginal probabilities as calculated from the joint probability distribution do not depend on the choice of the joint probability distribution used.

We follow the graph theoretical approach developed to derive generalized monogamy relationships based only on no-disturbance principle in reference [14]. A joint commutation graph representing a set of $n$ KCBS-type in-
the protocol based on monogamy of the KCBS inequality. The analysis is inspired by the security proof for QKD protocols based on the monogamy of violations of Bell’s inequality [20].

Alice and Bob perform the protocol a large number of times and share the probability distribution $P(a, b|i, j)$, which denotes the probability of Alice and Bob obtaining outcomes $a, b \in \{0, 1\}$ when their settings are $i, j \in \{0, 1, 2, 3, 4\}$ respectively. In the ideal case they obtain $a \neq b$ when $j = i + 1$, where addition is taken modulo 5. However in the presence of Eve, the secrecy of correlation between Alice and Bob has to be ensured even if Eve is distributing the correlation between them. On the other hand, Eve would like to obtain information about the correlation between Alice and Bob and the associated key. Eve can attempt to accomplish this in several ways which might include intercepting the information from Alice and re-sending to Bob after gaining suitable knowledge about the key. It could also be that she is correlated to Alice’s preparation system or to Bob’s measurement devices. In other words Eve has access to a tripartite probability distribution $P(a, b, e|i, j, k)$, where Alice, Bob and Eve obtain outcomes $a, b$ and $e$ when their settings are $i, j$ and $k$ respectively. It is required that the marginals to this probability distribution correspond to the observed correlation between Alice and Bob as will be shown below. In general it is not easy to characterize the strategy of an eavesdropper without placing some constraints on her.

For the following security analysis we place fairly minimal restrictions on the eavesdropper. It is required of her to obey the no-disturbance principle and as a consequence her correlation with Alice will be limited by monogamy (14). Such a constraint is well motivated because it is a fundamental law of nature and will have to be obeyed at all times.

We assume that the correlation observed by Alice and Bob, $P(a, b|i, j)$ as defined above, is a consequence of marginalizing over an extended tripartite probability distribution $P(a, b, e|i, j, k)$, distributed by an eavesdropper Eve:

$$P(a, b|i, j) = \sum_c P(a, b, c|i, j, k) = \sum_c P(c|e)P(a, b|i, j, k, e).$$

(15)

Where the second equality is a consequence of the no-disturbance principle: Eve’s output is independent of the settings used by Alice and Bob. We can also analyze the correlation between Alice and Eve in a similar manner:

$$P(a, e|i, k) = \sum_b P(a, b, e|i, j, k) = \sum_b P(b|j)P(a, e|i, j, k).$$

(16)

Where the second equality also follows from the no-disturbance principle and implies that Eve can decide on her output based on the settings disclosed by Bob. Bob’s outcome, however, cannot be used as it is never disclosed in the protocol. The natural question that arises now is how strong does the correlation between Alice and Bob need to be such that the protocol is deemed secure. As will be seen the question can be answered by monogamy of contextuality.

The QKD scenario now is as follows: Alice and Bob utilize the preparations and measurements as detailed in Section III A, while an eavesdropper Eve limited only by the no-disturbance principle is assumed to distribute the correlation between them. Whenever Eve distributes the correlation between herself and Alice she uses the same measurement settings as Bob to guess the bit of Alice. This way Eve can hope to gain some information about the key. However, contextuality monogamy limits the amount to which Eve can be correlated to Alice without disturbing the correlation between Alice and Bob significantly as shown in Section III B.

The condition for a secure key distribution between Alice and Bob in terms of Alice-Bob mutual information $I(A : B)$ and Alice-Eve mutual information $I(A : E)$ is [30]:

$$I(A : B) > I(A : E).$$

(17)

For individual attacks and binary outputs of Alice it essentially means that the probability $P_B$ that Bob guesses the bit of Alice should be greater than the probability, $P_E$ for Eve to correctly guess the bit of Alice. Thus the above condition simplifies to [31]

$$P_B > P_E.$$  

(18)

Bob can correctly guess the bit of Alice with probability $P_B = K(A, B)$. For $K(A, B) = 1$ Bob has perfect knowledge about the bit of Alice while for $K(A, B) = 0$ he has no knowledge. For any other values of $K(A, B)$ they may have to perform a security check.

We assume that Eve has a procedure that enables her to distribute correlation according to Eqns. (15-16). The procedure takes an input $k$ among the five possible inputs according to the KCBS scenario and outputs $e$. She uses this outcome to determine the bit of Alice when Alice’s setting was $i$. The probability that Eve correctly guesses the bit of Alice is denoted by $P_{ik}$. Since there are 5 possible settings for Alice and Eve each, the average probability for Eve to be successful $P_E$ is

$$P_E = \frac{1}{15} \sum_{i=0}^4 (P_{ii} + P_{i+1} + P_{i-1})$$

$$\leq \max\{P_{ii}, P_{i+1}, P_{i-1}|\forall i\}.$$  

(19)

The terms in the above equation denote the success probability of Eve when she uses the same setting as Alice and when she measures in the context of Alice, respectively. For all other cases she is unsuccessful. Without loss of generality we can assume that $P_{0i}$ is the greatest term appearing in Eqn (19). This corresponds to the success
probability of Eve when her setting is 1 and Alice’s setting is 0. However, Alice’s setting is not known to Eve as it is never disclosed in the protocol. Therefore the best strategy that Eve can employ is to always choose her setting to be 1 irrespective of Alice’s settings and try to violate the KCBS inequality with her. The probabilities that appear in the KCBS inequality would then be,

\[ P(a \neq e|i = 0, k = 1) = P_{01} = P_{01}, \]
\[ P(a \neq e|i = 1, k = 1) = P_{11} = 1 - P_{01}, \]
\[ P(a \neq e|i = 2, k = 1) = P_{21} \leq P_{01}, \]
\[ P(a \neq e|i = 3, k = 1) = P_{31} \leq P_{01}, \]
\[ P(a \neq e|i = 4, k = 1) = P_{41} \leq P_{01}. \] (20)

The probability for Eve to get a particular outcome is independent of Alice’s choice of settings. Her best strategy to eavesdrop can at most yield all the preceding probabilities to be equal (except the second term which will show a correlation instead of the required anti-correlation) which will maximize \( K(A, E) \). Evaluating the KCBS violation for Alice and Eve, we get,

\[ K(A, E) = \frac{3}{5}P_{01} + \frac{1}{5} > \frac{3}{5}P_E + \frac{1}{5}. \] (21)

Using the monogamy relationship given by Eqn. (14), we get,

\[ \frac{3}{5}P_E + \frac{1}{5} \leq \frac{6}{5} - P_B. \] (22)

For the protocol to work Eqn. (18) must hold and the above condition implies that it happens only if

\[ K(A, B) > \frac{5}{8}. \] (23)

Therefore the protocol is unconditionally secure if Alice and Bob share KCBS correlation greater than \( \frac{5}{8} \). It is worth mentioning that \( \frac{5}{8} \) is smaller than the maximum violation of the KCBS inequality in quantum theory.

As shown in reference [14] the monogamy relation (14) is a minimal condition and no stronger conditions exist. This implies that any QKD protocol whose security is based on the violation of the KCBS inequality cannot offer security if the condition given in Eqn. (23) is not satisfied. This quantifies the minimum correlation required for unconditional security. We conjecture that no key distribution scheme based on the violation of the KCBS inequality can perform better than our protocol since we utilize post-processing on Alice’s side to extend the maximum violation of the KCBS inequality up to its algebraic maximum.

IV. CONCLUSIONS

The cryptography protocol we presented is a direct application of the simplest known test of contextuality namely the KCBS inequality and the related monogamy relation. For the protocol to work, Alice and Bob try to achieve the maximum possible anti-correlation amongst themselves. They achieve the algebraic maximum of the KCBS inequality by allowing post-processing on Alice’s site. We then showed that any eavesdropper will have to share a monogamous relationship with Alice and Bob severely limiting her eavesdropping. For this purpose we derived a monogamy relationship for the settings of Eve which allow her to gain optimal information. We found that the optimal information gained by Eve cannot even allow her to maximally violate the KCBS inequality as allowed by quantum theory. Such an unconditional security provides a significant advantage to our protocol since it does not utilize the costly resource of entanglement. Furthermore, being a prepare and measure scheme of QKD it also allows for a check of security via the violation of the KCBS inequality much like the protocols based on the violation of Bell’s inequalities. Finally, we note that our protocol is a consequence of contextuality monogamy relationship, which are expected to play an interesting role in quantum information processing.

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[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[2] J. S. Bell, Physics 1, 195 (1964).
[3] S. Kochen and E. P. Specker, Journal of Mathematics and Mechanics 17, 59 (1967).
[4] A. Peres, Quantum Theory: Concepts and Methods, Fundamental Theories of Physics (Springer, 1995).
[5] A. Peres, Journal of Physics A Mathematical General 24 (1991).
[6] N. D. Mermin, Rev. Mod. Phys. 65, 803 (1993).
[7] A. Cabello, Phys. Rev. Lett. 101, 210401 (2008).
[8] A. A. Klyachko, M. A. Can, S. Binicioğlu, and A. S. Shumovsky, Phys. Rev. Lett. 101, 020403 (2008).
[9] A. Cabello, M. Kleinmann, and C. Budroni, Phys. Rev. Lett. 114, 250402 (2015).
[10] S. Yu and C. H. Oh, Phys. Rev. Lett. 108, 030402 (2012).
[11] P. Kurzyński and D. Kaszlikowski, Phys. Rev. A 86, 042125 (2012).
[12] A. Cabello, S. Severini, and A. Winter, Phys. Rev. Lett. 112, 040401 (2014).
[13] A. Acín, T. Fritz, A. Leverrier, and A. B. Sainz, Communications in Mathematical Physics 334, 533 (2015).
[14] R. Ramanathan, A. Soeda, P. Kurzyński, and D. Kaszlikowski, Phys. Rev. Lett. 109, 050404 (2012).
[15] M. Pawłowski and i. c. v. Brukner, Phys. Rev. Lett. 102, 030403 (2009).
[16] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information: 10th Anniversary Edition, 10th ed. (Cambridge University Press, New York, NY, USA, 2011).
[17] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. 74, 145 (2002).
[18] V. Scarani, H. Bechmann-Pasquinucci, N. J. Cerf, M. Dušek, N. Lütkenhaus, and M. Peev, Rev. Mod. Phys. 81, 1301 (2009).
[19] A. Acín, N. Gisin, and L. Masanes, Phys. Rev. Lett. 97, 120405 (2006).
[20] M. Pawłowski, Phys. Rev. A 82, 032313 (2010).
[21] C. H. Bennett and G. Brassard, in Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing (IEEE Press, New York, 1984) pp. 175–179.
[22] A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[23] C. H. Bennett, Phys. Rev. Lett. 68, 3121 (1992).
[24] H. Bechmann-Pasquinucci and A. Peres, Phys. Rev. Lett. 85, 3313 (2000).
[25] P. W. Shor and J. Preskill, Phys. Rev. Lett. 85, 441 (2000).
[26] C. Branciard, N. Gisin, B. Kraus, and V. Scarani, Phys. Rev. A 72, 032301 (2005).
[27] C.-H. F. Fung, K. Tamaki, and H.-K. Lo, Phys. Rev. A 73, 012337 (2006).
[28] A. Cabello, V. D’Ambrosio, E. Nagali, and F. Sciarrino, Phys. Rev. A 84, 030302 (2011).
[29] Y.-C. Liang, R. W. Spekkens, and H. M. Wiseman, Physics Reports 506, 1 (2011).
[30] I. Csiszar and J. Korner, IEEE Transactions on Information Theory 24, 339 (1978).
[31] M. Pawłowski, Phys. Rev. A 85, 046302 (2012).