Mechanism Design for Fair Allocation

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Abstract

Mechanism design for a social utility being the sum of agents’ utilities (SoU) is a well-studied problem. There are, however, a number of problems of theoretical and practical interest where a designer may have a different objective than maximization of the SoU. One motivation for this is the desire for more equitable allocation of resources among agents. A second, more subtle, motivation is the fact that a fairer allocation indirectly implies less variation in taxes which can be desirable in a situation where (implicit) individual agent budgetary constraints make payment of large taxes unrealistic. In this paper we study a family of social utilities that provide fair allocation (with SoU being subsumed as an extreme case) and derive conditions under which Bayesian and Dominant strategy implementation is possible. Through a numerical example it is shown that the proposed method can result in significant gains both in allocation fairness and tax reduction.

1 Introduction

Mechanism design is a well-established framework for dealing with decentralized resource allocation problems in the presence of strategic agents. The corresponding literature is vast, especially in the domain of Dominant, Nash and Bayesian implementation, [7, 11, 8, 9]. In the area of Engineering—and in particular in the area of Networks—the majority of the works start with the assumption that the social objective is the sum of individual utilities (SoU) of all the system’s agents.

There are strong mathematical reasons for preferring SoU as the resource allocation objective, the main one being that for SoU, in conjunction with quasi-linear utilities, agents’ individual goals can be aligned directly with the overall social objective, so that when agents maximize their net utility they are simultaneously maximizing the overall social objective. The VCG mechanism [9] is the most prominent example of this. With social objective \( \sum_{i=1}^{N} v(\hat{x}(\theta_i); \theta_i) \) and individual utility of \( v(\hat{x}(\phi); \theta_i) - t_i(\phi) \), VCG taxes \( t_i(\phi) = -\sum_{j \neq i} v(\hat{x}(\phi); \theta_j) + f_i(\phi_{-i}) \) precisely ask the user to perform social objective maximization whilst maximizing self utility.

There are, however, a number of problems of theoretical and practical interest where a designer may have a different objective than maximization of the SoU. One obvious reason for such a preference is the desire of the social planner to introduce fairness in the allocation process. Consider for example, a network where optimizing SoU results in one (or a few) agents receiving almost all the available resources and everyone else receiving an appreciably lower portion. In instances of this kind, appealing to fairness or equality, a system designer may genuinely want allocations which are more equitable, even if this may come at the cost of reduced revenue.

A second—more subtle—reason for wanting a different social objective is the fact that the standard mechanism design framework does not provide any formal way of limiting the range of the taxes/subsidies required at equilibrium. This implies that when strong budget balance is imposed on the mechanism, the magnitude of the monetary transfers (taxes/subsidies) can vary greatly among agents (with those who benefit more from the allocation having to contribute more as well). This can be a significant practical problem since it does not take into account the budget constraints of individual agents. The work of [13] is attacking this problem by considering the dual version of the resource allocation problem and putting additional structure on the dual variables to ensure less variation between them. This would typically ensure less fluctuation in prices (and thus in the monetary transfers) since it is well-known that dual variables for resource allocation optimization problems act as prices in the corresponding markets. A modified social objective provides an alternative way for dealing with the issue of large tax variation. By making the social objective a more concave function of the utilities (compared to the SoU) a smaller variation of the taxes is expected.

As soon as one moves away from SoU objective, due to the social and individual objectives not aligning with each other, basic design techniques like VCG mechanism are not useful. This is one of the main reasons why there are significantly fewer results on fairness in the mechanism design literature. In [11], authors use the concept of “proportional fairness” (similar to the sum of log of utilities) to reduce disparity. The focus is on a tax-less mechanism where the contract proposes to throw away existing resources (“resource-burning”) in order to tax untruthful agents. This mechanism achieves at least \( \frac{1}{2} \) fraction of proportionally fair allocation. In [12], optimal auctions in the Bayesian setup are derived, such that instead of efficiency maximization or revenue maximization, a linearly combined efficiency metric is maximized which favors exchange at low prices (thereby ensuring fairer trade). “Envy-freeness” is another well-known criterion for equitable allocations (see [11]). This notion was originally proposed for exchange economies where an allocation is called envy-free if no agent is strictly better off by taking someone else’s allocation instead of their own.

Such a notion was argued in terms of the stability it provides, since each agent may be content with what they have comparing to the possible option of acquiring someone else’s allocation. This, however, is an ex-post notion and in general imposes quite stringent constraints on the design. For a large enough environment it may indeed be impossible to achieve envy-freeness in optimal allocation. It is interesting to note that the problem of mechanism design for risk-averse agents (see for e.g., [11, 5]) is in some respects the opposite problem:

\(^{1}\)Note that this exchange refers to both allocation of good and taxes.
of the one addressed here. In that case, the social planner’s objective is relatively more “aggressive” compared to the more risk averse individual objectives.

In this paper, we ask if and how we can design mechanisms that implement social objectives that are especially designed for fairness and go beyond the standard paradigm of SoU. We seek a methodology that is flexible enough to create space for the designer when envy-freeness may not be feasible. In particular we concentrate on a specific form of the social objective given by the additive function \(\sum_{i=1}^{N} g_i(x; \theta_i)\) where \(g_i(x; \theta_i)\) is the utility of the \(i\)-th user with allocation \(x\) and type \(\theta_i\), and \(g_i(\cdot) = z - \varepsilon f(z)\) is a family of concave functions parameterized by \(\varepsilon > 0\), with \(f(\cdot)\) an arbitrary convex function. In this setup, the SoU is a special case with \(\varepsilon = 0\), while as the parameter \(\varepsilon\) increases, more fairness is built into the allocation. Within this framework we ask for which values of the parameter \(\varepsilon\) and under what conditions for the convex function \(f(\cdot)\) is Dominant strategy implementation possible and when is Bayesian NE implementation possible. We show that indeed mechanism design is possible provided \(\varepsilon\) is not too large, by providing an upper bound on the range of \(\varepsilon\). This is done by formulating the incentive compatibility constraints as a set of linear inequalities on the design variables and checking whether this system (together with strong budget balance and/or individual rationality) is feasible. Our proving techniques follow closely the work of [2][3]. Not surprisingly, the results are derived under certain assumptions on the prior beliefs, \(p_i(\theta_{-i}|\theta_i)\), of agents, which are trivially satisfied for the case where \(p_i(\theta_{-i}|\theta_i) = p_i(\theta_{-i})\).

We finally demonstrate (through a numerical example) that the range of \(\varepsilon\) is sufficient to provide quite significant gains in fairness - as measured by the decrease in Gini Coefficient of the utilities. In addition, and this relates to the second reason we mentioned above regarding our motivation for this work, the results show a significant decrease in the variance of required taxes to achieve incentive compatibility.

This remaining of this paper is organized as follows. Section 2 defines the Centralized problem which has been modified for fairness. The next two sections prove the existence of mechanisms that implement the aforementioned social objectives for type sets of size two in Dominant strategy (Section 3) and for general type sets in Bayesian Nash equilibria (Section 4). The numerical example is presented in Section 5. Finally, Section 3 discusses future work and immediate extensions of the results in here.

### 2 Centralized Problem

For a system of agents \(N = \{1, \ldots, N\}\), efficient allocation is calculated via the following optimization problem:

\[
\hat{x}(\theta) = \arg \max_{x \in X} \sum_{i \in N} g(x; \theta_i),
\]

where \(X \subset \mathbb{R}_+^N\) is the constraint set, \(\theta = (\theta_i)_{i \in N} \in \Theta \triangleq \times_{i \in N} \Theta_i\) is the profile of agents and \(\Theta_i\) is the discrete type set for agent \(i\) with \(|\Theta_i| = L_i\). The utility function \(v(x; \theta_i)\) measures agent’s \(i\) satisfaction at allocation \(x\) with private type being \(\theta_i\). A concave transformation \(f(\cdot)\) is applied for making the allocation fairer compared to the SoU setup. It is further assumed that the functions \(g_i: \mathbb{R} \to \mathbb{R} \) and \(v(\cdot; \theta_i): \mathbb{R}_+^N \to \mathbb{R}\) are such that the optimization has a unique solution (e.g., if \(v(\cdot; \theta_i)\) is concave and \(g_i(\cdot)\) is concave and increasing and \(X\) is a convex set). Specifically the form, \(g_i(\cdot) = z - \varepsilon f(z)\), for \(\varepsilon \geq 0\) is considered here, where \(f(\cdot)\) is assumed to be a bounded, convex function. Note that at \(\varepsilon = 0\) optimization becomes the SoU problem, while as \(\varepsilon\) increases from 0, the function \(g_i\) has a stronger concave component.

Define, \(\forall \ i \in N, J_i := \{\psi_i, \xi_i\} \in \Theta_i^2 | \psi_i \neq \xi_i\). With this we define the difference, \(\forall \ i \in N, \ (\theta_i, \phi_i) \in J_i, \ \theta_{-i} \in \Theta_{-i}, \ v > 0, \)

\[
K_i(\theta_i, \phi_i, \theta_{-i}, \varepsilon) := \sum_{j \in N} g_j(v(\hat{x}(\theta_i, \theta_{-i}); \theta_j)) - \sum_{j \in N} g_j(v(\hat{x}(\phi_i, \theta_{-i}); \theta_j)),
\]

(2)

Optimality conditions from [1] give that the above difference is always non-negative. However, due to the finite type spaces, the difference above is expected to be strictly positive. We make appropriate assumptions in this regard later.

### 3 Dominant Strategy Implementation

The Mechanism Design problem in this section is to find a message space \(M = x \times_{i \in N} \mathbb{M}_i\) and allocation, tax functions \((\hat{x}, t): M \to X \times \mathbb{R}^N\) such that the induced game for agents in \(N\) with action space \(M\) and quasi-linear utilities

\[
\hat{u}_i(m; \theta_i) = v(\hat{x}(m); \theta_i) - t_i(m) \quad \forall m \in M_i, \forall i \in N
\]

(3)

has a dominant strategy equilibrium\(^{\dagger}\) \(m^*\) for which \(\hat{x}(m^*) = \hat{x}(\theta)\), where \(\theta = (\theta_i)_{i \in N}\) is the true type profile. This is known as Dominant Strategy Incentive Compatibilility (DSIC).

In general Dominant strategy implementation is very restrictive (note that the well studied VCG mechanisms are no longer applicable since this is not the maximization of SoU). Following [1], the special case of \(L_i = 2 \ \forall \ i \in N\) is considered in this section. In particular, \(\Theta_i = \{\theta_i^1, \theta_i^2\}\ \forall \ i \in N\).

The proposed mechanism is a direct mechanism, thus \(M_i = \Theta_i, \ \forall i \in N\). Agents report their types (possibly untruthfully) and the allocation they receive on the basis of this is the optimal allocation \(\hat{x}(\phi)\) for the quoted type profile \(\phi\), where \(\hat{x}(\cdot)\) is defined in (1).

Assuming that for not participating in the mechanism, an agent receives \(0\) utility value (including \(0\) tax), the voluntary participation condition for Dominant strategy implementation is

\[
\forall (\theta_i, \theta_{-i}) \in \Theta_i \times \Theta_{-i}, \ v(\hat{x}(\theta_i, \theta_{-i}); \theta_i) - t_i(\theta_i, \theta_{-i}) \geq 0.
\]

(4)

This is the ex-post version of the individual rationality (IR).

Next we state the assumption on \([1]\) under which the results are derived.

**Condition (A_0)** Assume that \(\exists \ \varepsilon_{max} > 0\) such that for all \(0 \leq \varepsilon < \varepsilon_{max}, \ \forall i \in N, (\theta_i, \phi_i) \in J_i, \ \theta_{-i} \in \Theta_{-i}, \)

\[
K_i(\theta_i, \phi_i, \theta_{-i}, \varepsilon) + K_i(\phi_i, \theta_i, \theta_{-i}, \varepsilon) > 0.
\]

(5)

**Condition (A_D)** and the ones below in Corollary 3.2, 4.3, 4.4 and Condition (A_0) can be checked only at \(\varepsilon = 0\). By continuity of the optimization on parameter \(\varepsilon\), this will imply that these conditions continue to hold for all \(0 \leq \varepsilon < \varepsilon_{max}\) for some \(\varepsilon_{max} > 0\).

The first contribution of this paper is summarized in the following Theorem.

\(^{\dagger}\)For any \(\theta\), message \(m^* \in M\) is a dominant strategy equilibrium if it satisfies \(\hat{u}_i(m^*, m_{-i}; \theta_i) \geq \hat{u}_i(m_i, m_{-i}; \theta_i) \ \forall m_i \in M_i, \forall m_{-i} \in M_{-i}, \ \forall i \in N\).
\textbf{Theorem 3.1.} If Condition (A_D) is satisfied then for all \(0 \leq \varepsilon < \varepsilon_{\text{max}}\) there exist taxes \((t_i(\theta))_{\theta \in \Theta, i \in N}\) that satisfy DSIC (which implies implementation in Dominant strategies) and IR.

\textit{Proof.} Note that the taxes are a finite collection of variables, since \(\Theta, N\) are both finite sets. For the DSIC constraints to be satisfied, the following constraints must hold \(\forall i \in N, \forall \theta_{-i} \in \Theta_{-i},\)

\[v(\hat{x}(\theta_i^H, \theta_{-i}); \theta_i^H) - t_i(\theta_i^H, \theta_{-i}) \geq v(\hat{x}(\theta_i^L, \theta_{-i}); \theta_i^H) - t_i(\theta_i^L, \theta_{-i}),\quad (6a)\]

\[v(\hat{x}(\theta_i^L, \theta_{-i}); \theta_i^L) - t_i(\theta_i^L, \theta_{-i}) \geq v(\hat{x}(\theta_i^H, \theta_{-i}); \theta_i^L) - t_i(\theta_i^H, \theta_{-i}).\quad (6b)\]

This gives truth-telling as a dominant strategy for agent \(i\) regardless of types of others. For IR, the following constraints must be satisfied \(\forall i \in N, \forall \theta_{-i} \in \Theta_{-i},\)

\[t_i(\theta_i^H, \theta_{-i}) \leq v(\hat{x}(\theta_i^H, \theta_{-i}); \theta_i^H),\quad (7a)\]

\[t_i(\theta_i^L, \theta_{-i}) \leq v(\hat{x}(\theta_i^L, \theta_{-i}); \theta_i^L).\quad (7b)\]

From the above sets of constraints, it is clear that one can design \((t_i(\theta_i^H, \theta_{-i}), t_i(\theta_i^L, \theta_{-i}))\) separately for each \(i \in N, \theta_{-i} \in \Theta_{-i}.\) So for any \(i, \theta_{-i},\) the constraints can be rewritten in the form

\[
\begin{bmatrix}
1 & -1 \\
-1 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
t_1 \\
t_2
\end{bmatrix}
\leq
\begin{bmatrix}
A_{HH} - A_{LH} \\
A_{HH} \\
A_{LH}
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{bmatrix}
\]

(8)

where \((t_1, t_2) = (t_i(\theta_i^H, \theta_{-i}), t_i(\theta_i^L, \theta_{-i}))\) and

\[A_{HH} = v(\hat{x}(\theta_i^H, \theta_{-i}); \theta_i^H),\quad A_{LL} = v(\hat{x}(\theta_i^L, \theta_{-i}); \theta_i^L),\quad A_{HL} = v(\hat{x}(\theta_i^L, \theta_{-i}); \theta_i^H).\quad (9)\]

Using the Farkas Lemma, the above system is feasible in \(t\) iff

\[
\begin{bmatrix}
1 & -1 \\
-1 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{bmatrix}
= 0 \implies
\begin{bmatrix}
A_{HH} - A_{LH} \\
A_{HH} \\
A_{LH}
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{bmatrix}
\geq 0.
\]

(10)

The equality constraints on \(\lambda\) give that

\[\lambda_1 + \lambda_2 = \lambda_2 + \lambda_4 = \lambda_1 \implies \lambda_3 = \lambda_4 = 0 \implies \lambda_3 = \lambda_4 = 0.\quad (11)\]

So \(\lambda \in \mathbb{R}_{+}^4\) can be parametrized as \(\lambda = (\xi, \xi, 0, 0)^T\) for \(\xi \in \mathbb{R}_{+}.\)

Thus for feasibility, using Farkas Lemma, the condition that must be satisfied is

\[\{A_{HH} - A_{LH}, A_{LL} - A_{HL}, A_{HH}, A_{LL}\} \cdot (\xi, \xi, 0, 0) \geq 0 \implies \xi (A_{HH} + A_{LL} - A_{HL} - A_{HH}) \geq 0 \implies A_{HH} + A_{LL} - A_{HL} - A_{HH} \geq 0.\quad (12)\]

From the definition in (2), we have

\[v(\hat{x}(\theta_i, \theta_{-i}); \theta_i) - \varepsilon f(v(\hat{x}(\theta_i, \theta_{-i}); \theta_i)) + \sum_{j \neq i} v(\hat{x}(\theta_j, \theta_{-i}); \theta_j) - \varepsilon f(v(\hat{x}(\theta_j, \theta_{-i}); \theta_j)) = v(\hat{x}(\theta_i, \theta_{-i}); \theta_i) - \varepsilon f(v(\hat{x}(\theta_i, \theta_{-i}); \theta_i)) + K_i(\theta_i, \phi_i, \theta_{-i}, \varepsilon).\quad (13)\]

Using the above twice, first with \((\theta_i, \phi_i) = (\theta_i^H, \theta_{-i}), \theta_{-i} = \theta_{-i}\) and then with \((\theta_i, \phi_i) = (\theta_i^L, \theta_{-i}), \theta_{-i} = \theta_{-i},\) and adding the two results in

\[A_{HH} - \varepsilon f(A_{HH}) + A_{LL} - \varepsilon f(A_{LL}) = K_i(\theta_i^H, \theta_{-i}, \varepsilon) + K_i(\theta_i^L, \theta_{-i}, \varepsilon) + A_{HL} - \varepsilon f(A_{HL}) + A_{LH} - \varepsilon f(A_{LH}).\quad (14)\]

This can be rewritten as

\[\Leftrightarrow A_{HH} + A_{LL} - A_{HL} - A_{LH} = K_i(\theta_i^H, \theta_{-i}, \varepsilon) + K_i(\theta_i^L, \theta_{-i}, \varepsilon) + \varepsilon (f(A_{HH}) + f(A_{LL}) - f(A_{HL}) - f(A_{LH})).\quad (15)\]

Thus it is sufficient to prove that the RHS above is non-negative. Owing to Condition (A_D), \(\exists \varepsilon_{\text{max}} > 0\) such that the sum of the first two terms in RHS is strictly positive for all \(0 \leq \varepsilon < \varepsilon_{\text{max}}\) and clearly the second term can be made arbitrarily small in magnitude (by choosing a small \(\varepsilon_{\text{max}}\)). Hence the condition in (12) is satisfied\(\square\) for all \(0 \leq \varepsilon < \varepsilon_{\text{max}}\).

Stated below is a corollary to the above result, which uses a stricter condition.

\textbf{Corollary 3.2.} If \(\exists \varepsilon_{\text{max}} > 0\) such that for all \(0 \leq \varepsilon < \varepsilon_{\text{max}},\)

\[
\min_{\theta_i \in \Theta_i} \min_{\varepsilon \in \mathbb{R}_{+}} \min_{\theta_{-i} \in \Theta_{-i}} K_i(\theta_i, \phi_i, \theta_{-i}, \varepsilon) > 0.\quad (16)
\]

then for all \(0 \leq \varepsilon < \varepsilon_{\text{max}}\) there exist taxes that satisfy DSIC and IR.

\textit{Proof.} As (16) implies (5), the Corollary follows from Theorem 3.1\(\square\)

4 Bayesian Implementation

In this section the type sets are of arbitrary size. Dominant strategy implementation is too restrictive for the general scenario, hence the next best reasonable solution concept - Bayesian implementation, is considered.

In a Bayesian set up, agents have a prior distribution on the type profile. For agent \(i,\) prior is \(p_i \in \Delta(\Theta)\). For basic regularity assume that the prior gives non-zero probability on all points of \(\Theta\) (this is only a technical condition and the ensuing results can be proved without it as well). These priors are assumed to be common knowledge between agents and designer - hence there is no need to introduce second order beliefs over the priors and so on.

The mechanism used here is a direct mechanism with allocation function \(\hat{x}(\cdot)\) (same as before). Given the allocation and tax functions \((\hat{x}, t) : \Theta \rightarrow \mathbb{R}^N,\) the utility function in the Bayesian set up for strategies \((\sigma_j : \Theta_j \rightarrow \Theta_j)_{j \in N}\) is given by, \(\forall i \in N,\)

\[
\hat{u}_i(\sigma | \theta_i) = \mathbb{E}_{p_i(\cdot | \theta_i)}[v(\hat{x}(\sigma(\theta)); \theta_i) - t_i(\sigma(\theta))].\quad (17)
\]

where \(\theta_i\) is the true type of agent \(i\). The Bayesian implementation condition - also known as Bayesian Strategy Incentive Compatibility (BSIC) - for the direct mechanism is that the truthful strategy

\(^{\text{1a}}\)Overall the behaviour of \(K_i(\theta, \phi, \theta_{-i}, \varepsilon)\) w.r.t. \(\varepsilon\) will dictate the value of \(\varepsilon_{\text{max}}\). This in turn will effect the usefulness of this method, since a designer might want to ensure certain minimum gains in fairness for which he/she might want to choose \(\varepsilon\) as large as possible.
\(\sigma^*_i(\theta_i) = \theta_i, \ \forall \ \theta_i \in \Theta_i, \ \forall \ i \in N\) must be a Bayesian Nash equilibrium (BNE)4 for the induced Bayesian game.

In addition, it is required the tax function to have the Strong Budget Balance (SBB) property, i.e.,

\[
\sum_{i \in N} t_i(\psi) = 0, \ \forall \ \psi \in \Theta. \quad (18)
\]

We restrict attention to optimization (1) and priors \(\{p_i(\cdot)\}_{i \in N}\) that satisfy the following conditions.

**Condition (A0)** Assume that \(\exists \ \varepsilon_{\text{max}} > 0\) such that \(H(\varepsilon) > 0\) for all \(0 \leq \varepsilon < \varepsilon_{\text{max}}\), where

\[
H(\varepsilon) := \min_{i \in N} \min_{(\theta_i, \phi_i) \in \mathcal{T}_i} \mathbb{E}_{p_i(\cdot | \theta_i)} \left[ h_i(\theta_i, \phi_i, \theta_{-i}, \varepsilon) \right] \quad (19a)
\]

\[
h_i(\theta_i, \phi_i, \theta_{-i}, \varepsilon) := \sum_{\psi \in \Theta} \left( v(\hat{x}(\theta_i, \theta_{-i}); \theta_i) - v(\hat{x}(\phi_i, \theta_{-i}); \theta_i) \right). \quad (19b)
\]

This is the most general form of the assumption needed about the optimization here; after proving Theorem 4.1, corollaries are stated with stricter assumptions than above.

**Condition (B)** Assume that for any non-zero vector \(R := (R(\psi))_{\psi \in \Theta}\) with \(R(\psi) \in \mathbb{R}\), there does not exist any \(\lambda := (\lambda_k(\theta, \psi_k))_{k \in N, (\theta_k, \psi_k) \in \mathcal{X}_k}\) with \(\lambda_k(\theta, \psi_k) \in \mathbb{R}_+\) such that \(\forall \ i \in N, \ \forall \ \psi \in \Theta, \ \psi \neq \psi_i\)

\[
p_i(\psi_{-i} | \psi_i) \sum_{\phi_i \in \Theta_i} \lambda_i(\phi_i, \psi_i) - \sum_{\theta_i \in \Theta_i} p_i(\psi_{-i} | \theta_i) \lambda_i(\theta_i, \psi_i) = R(\psi). \quad (20)
\]

This condition was first introduced in [4] and subsumes the case of conditionally independent priors, i.e., \(p_i(\psi_{-i} | \psi_i) = p_j(\psi_{-j})\) (refer to [4] for an example of priors which are not conditionally independent but still satisfy the condition above).

The second contribution of this paper is summarized in the following Theorem.

**Theorem 4.1.** If Conditions (A0) and (B) are satisfied then for all \(0 \leq \varepsilon < \varepsilon_{\text{max}}\), there exist taxes \(\{t_i(\phi)\}_{\phi \in \Theta, i \in N}\) that satisfy BSIC (which implies implementation in BNE) and SBB.

**Proof.** The utility for any agent \(i\) when other agents are truth-telling is

\[
\hat{u}_i(\phi_i | \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p_i(\theta_{-i} | \theta_i) \left[ v(\hat{x}(\phi_i, \theta_{-i}); \theta_i) - t_i(\theta_i, \theta_{-i}) \right] \quad (21)
\]

where agent \(i\)’s true and quoted types are \(\theta_i, \phi_i \in \Theta_i\), respectively. Here we consider taxes in the d’AgV form

\[
t_i(\psi) = z_i(\psi) - \frac{1}{N-1} \sum_{j \in N \setminus i} z_j(\psi) \quad \forall \ i \in N, \ \psi \in \Theta. \quad (22)
\]

Taxes in this form always satisfy SBB and any tax function which satisfies SBB can be written in this form. Therefore WLOG, the design variables from here onwards will be \(\{z_j(\psi)\}_{j \in N, \psi \in \Theta}\).

4Strategy \(\sigma^* = \left( \sigma^*_i : \Theta_i \rightarrow \Theta_i \right)_{i \in N}\) is a BNE if \(\forall i \in N, \ \forall \ \theta_i \in \Theta_i, \ \forall \ \sigma_i' : \Theta_i \rightarrow \Theta_i; \ \hat{u}_i(\sigma_i^*, \sigma_{-i}^* | \theta_i) \geq \hat{u}_i(\sigma_i', \sigma_{-i}^* | \theta_i).\)
Proof.

\[
\sum_{\phi_j \in \Theta_j} \lambda_j(\theta_j, \psi_j) - \sum_{\phi_j \notin \Theta_j} \lambda_j(\psi_j, \phi_j) - \sum_{\phi_j \notin \Theta_j} p_j(\psi_j \mid \phi_j) \lambda_j(\theta_j, \psi_j) \tag{29a}
\]
\[
= \sum_{\psi_j \in \Theta_j} \left[ p_j(\psi_j \mid \psi_j) \sum_{\phi_j \in \Theta_j} \lambda_j(\theta_j, \psi_j) - \sum_{\phi_j \notin \Theta_j} p_j(\psi_j \mid \phi_j) \lambda_j(\theta_j, \psi_j) \right] \tag{29b}
\]
\[
= \sum_{\psi_j \notin \Theta_j} p_j(\psi_j \mid \psi_k) \lambda_k(\theta_k, \psi_k) - \sum_{\phi_k \notin \psi_k} p_k(\psi_k \mid \phi_k) \lambda_k(\theta_k, \phi_k), \quad k \neq j \tag{29c}
\]
\[
= 0. \tag{29d}
\]

Here 29c) follows by application of 27 and other equations are just by rearranging summation terms.

With the application of above Lemma, one can rewrite LHS of (27) to get that \( \forall j, \psi \),
\[
p_j(\psi_j \mid \psi_j) \sum_{\phi_j \in \Theta_j} \lambda_j(\phi_j, \psi_j) - \sum_{\phi_j \notin \Theta_j} p_j(\psi_j \mid \phi_j) \lambda_j(\theta_j, \psi_j) = R(\psi) \tag{30}
\]

Condition (B) on priors states that there exist no \( \lambda \in \Lambda \) such that above holds for a non-zero \( R \). Hence \( A^T \lambda = 0 \) implies that \( R \equiv 0 \), therefore (by 27) \( \forall j \in \mathbb{N}, \forall \psi \in \Theta \),
\[
p_j(\psi_j \mid \psi_j) \sum_{\phi_j \in \Theta_j} \lambda_j(\phi_j, \psi_j) = \sum_{\phi_j \notin \Theta_j} p_j(\psi_j \mid \theta_j) \lambda_j(\theta_j, \psi_j) \tag{31}
\]

Next we show that for all \( \lambda \in \Lambda \) that satisfy (31) we have \( b^T \lambda \geq 0 \).\n
\[
\sum_{i \in \mathbb{N}} \sum_{(\theta_i, \phi_i) \in \mathcal{F}_i} b(i, \theta_i, \phi_i) \lambda_i(\theta_i, \phi_i) \geq 0 \tag{32a}
\]
\[
\Leftrightarrow \sum_{i \in \mathbb{N}} \sum_{(\theta_i, \phi_i) \in \mathcal{F}_i} \lambda_i(\theta_i, \phi_i) \sum_{\theta_{-i} \notin \Theta_{-i}} p_i(\theta_{-i} \mid \theta_i) \cdot \tag{32b}
\]
\[
\left[ v(\hat{x}(\theta_i, \theta_{-i})); \theta_i \right] - v(\hat{x}(\phi_i, \theta_{-i}); \theta_i) \right] \geq 0. \tag{32c}
\]

Proving this will finish the proof by Farkas Lemma.

For any \( i, \theta_i, \phi_i, \theta_{-i} \), denote \( x = \hat{x}(\theta_i, \theta_{-i}), \hat{x} = \hat{x}(\phi_i, \theta_{-i}) \).

Rearranging terms from (2), gives
\[
v(x; \theta_i) - v(\hat{x}; \theta_i) = \sum_{j \in \mathbb{N}} \left( v(x; \theta_j) - v(\hat{x}; \theta_j) \right) + \varepsilon \sum_{j \in \mathbb{N}} \left( f(v(x; \theta_j)) - f(v(\hat{x}; \theta_j)) \right) + K_i(\theta_i, \phi_i, \theta_{-i}, \varepsilon). \tag{33}
\]

Denote the RHS expression in (33) as \( \eta1 = \eta1 + \eta2 \), where
\[
\eta1 = \sum_{j \in \mathbb{N}} \left( v(x; \theta_j) - v(\hat{x}; \theta_j) \right), \tag{34a}
\]
\[
\eta2 = \varepsilon \sum_{j \in \mathbb{N}} \left( f(v(x; \theta_j)) - f(v(\hat{x}; \theta_j)) \right) + K_i(\theta_i, \phi_i, \theta_{-i}, \varepsilon). \tag{34b}
\]

Now continuing from LHS of (32b),
\[
\text{LHS of (32b)} = \sum_{i \in \mathbb{N}} \lambda_i(i, \phi_i) \sum_{\theta_{-i} \in \Theta_{-i}} p_i(\theta_{-i} \mid \theta_i) (\eta1 + \eta2). \tag{35}
\]

For any fixed \( i \), consider the summation with only \( \eta1 \) first
\[
\sum_{(\theta_i, \phi_i) \in \mathcal{F}_i} \lambda_i(\theta_i, \phi_i) \sum_{\theta_{-i} \notin \Theta_{-i}} p_i(\theta_{-i} \mid \theta_i) \cdot \tag{36}
\]

By (31), the inside summation in the first term is equal to \( p_i(\theta_{-i} \mid \phi_i) \sum_{\phi_i} \lambda_i(\phi_i, \psi_i) \). Incorporating this and changing variables of summation appropriately gives the overall summation from (36) equal to 0. Now consider the term in RHS of (35) with \( \eta2 \)
\[
\sum_{i \in \mathbb{N}} \lambda_i(\theta_i, \phi_i) \sum_{\theta_{-i} \notin \Theta_{-i}} p_i(\theta_{-i} \mid \theta_i) \cdot \eta2. \tag{37}
\]

Rearranging terms in \( \eta2 \), we can write
\[
\eta2 = \sum_{j \in \mathbb{N}} \left( v(\hat{x}(\theta_i, \theta_{-i}); \theta_j) - v(\hat{x}(\phi_i, \theta_{-i}); \theta_j) \right). \tag{38}
\]

Therefore by Condition (A.B), \( \exists \varepsilon_{\text{max}} > 0 \) such that for all \( 0 \leq \varepsilon < \varepsilon_{\text{max}} \), the inside summation in (37) is non-negative \( \forall i \in \mathbb{N}, (\theta_i, \phi_i) \in \mathcal{F}_i \). This finishes the proof by Farkas Lemma, since the expression in (32a) is now shown to be non-negative for all positive \( 0 \leq \varepsilon < \varepsilon_{\text{max}} \).

Stated below are corollaries which successively use stricter conditions.

**Corollary 4.3.** If \( \exists \varepsilon_{\text{max}} > 0 \) such that for all \( 0 \leq \varepsilon < \varepsilon_{\text{max}} \),
\[
\min_{i \in \mathbb{N}} \min_{(\theta_i, \phi_i) \in \mathcal{F}_i} \min_{\theta_{-i} \in \Theta_{-i}} h_i(\theta_i, \phi_i, \theta_{-i}, \varepsilon) > 0 \tag{39}
\]
and Condition (B) is satisfies then for all \( 0 \leq \varepsilon < \varepsilon_{\text{max}} \) there exist taxes which satisfy BSC and SBB.

**Corollary 4.4.** If \( \exists \varepsilon_{\text{max}} > 0 \) such that for all \( 0 \leq \varepsilon < \varepsilon_{\text{max}} \), and \( \forall i \in \mathbb{N}, (\theta_i, \phi_i) \in \mathcal{F}_i, \theta_{-i} \in \Theta_{-i}, \)
\[
K(\varepsilon) + \varepsilon g_i(\theta_i, \phi_i, \theta_{-i}, \varepsilon) > 0. \tag{40a}
\]
where \( K(\varepsilon) := \min_{i \in \mathbb{N}} \min_{(\theta_i, \phi_i) \in \mathcal{F}_i} \min_{\theta_{-i} \in \Theta_{-i}} K_i(\theta_i, \phi_i, \theta_{-i}, \varepsilon), \tag{40b} \]
\[
g_i(\theta_i, \phi_i, \theta_{-i}, \varepsilon) := \sum_{j \in \mathbb{N}} f(v(\hat{x}(\theta_i, \theta_{-i}); \theta_j)) - f(v(\hat{x}(\phi_i, \theta_{-i}); \theta_j)) \tag{40c}
\]
and Condition (B) is satisfied then for all \( 0 \leq \varepsilon < \varepsilon_{\text{max}} \) there exist taxes which satisfy BSC and SBB.

Since (40a) \( \Rightarrow \) (39) \( \Rightarrow \) Condition (A.B), hence these Corollaries follow from Theorem 4.1.
5 Numerical Results / Examples

This section contains numerical examples which have been evaluated to ascertain the scope of application of the existence results provided in the previous sections. In particular we are interested in evaluating the gain in overall fairness attained with the proposed method.

Consider $\Theta_i = S \triangleq \{\theta^H, \theta^L\} \forall i \in \mathbb{N}$, where $\theta^H > \theta^L$ and utilities are quadratic with private consumption i.e. $\forall i \in \mathbb{N}$, $\forall \theta_i \in S, \forall x \in \mathbb{R}_+^N$, 

$$v(x; \theta_i) := 2\theta_i x_i - \theta_i x_i^2.$$  \hspace{1cm} (41)

The constraint set is $X = \{x \in \mathbb{R}_+^N \mid \sum_i x_i = 1\}$ and $g_\varepsilon(z) = z - \varepsilon z^2$. Thus the Centralized optimization problem is

$$\hat{x}(\theta) = \arg\max_{\varepsilon \in X} \sum_{i \in \mathbb{N}} (2\theta_i x_i - \theta_i x_i^2) - \varepsilon (2\theta_i x_i - \theta_i x_i^2)^2.$$  \hspace{1cm} (42)

We consider the well known Gini coefficient (GC) as a measure of disparity in allocation. At $\varepsilon = 0$ the objective is exactly the SoU. As $\varepsilon$ starts increasing from 0 onwards, the optimization problem is transformed such that higher utilities will be weighed less than lower ones - thereby giving closer to equal distribution of allocation.

For the numerical analysis we consider Bayesian implementation (although Dominant implementation is also possible in this two-type set up) with $\theta^H = 1, \theta^L = 0.75$ and two cases $N = 10, 90$ and vary $\varepsilon$. Figures 1-2 depict the GC of the utility at optimal allocation, $\{v(\hat{x}(\theta); \theta_i)\}_{i \in \mathbb{N}}$ at $N = 10$ and $N = 90$, respectively. This is done for various type profiles, which due to symmetry can be defined by the number, $m \in \{0, 1, \ldots, N\}$, of agents with type $\theta^H$. Also included in the plots is the mean GC where $m$ is chosen with Binomial($N, 0.5$) distribution for $N = 10$ and Binomial($N, 0.1$) for $N = 90$.

In general there are two sources of upper bound on $\varepsilon$. One is through the $\varepsilon_{\max}$ defined in Condition (A_B) and other is through the well-defined-ness of the optimization problem (12). For $\varepsilon < \frac{1}{2\theta^H} = 0.5$, optimization (12) is a convex optimization problem and hence has a unique optimizer. One can verify however that the optimization continues to have a unique optimizer even beyond 0.5 and for all values of $\varepsilon$ within the range of interest for us. For $N = 10$, Condition (A_B) gives $\varepsilon_{\max} \approx 1.504$ and for $N = 90$ we get $\varepsilon_{\max} > 5$. With this we limit our plots to $\varepsilon \leq 1.5$ for $N = 10$ and $\varepsilon \leq 5$ for $N = 90$.

In the figures we see that the mean GC can be reduced by 42 and 80 percentage points for $N = 10$ and $N = 90$, respectively.

Finally, Figures 3-4 depict the standard deviation of the tax vector $t(\theta)$ for various type profiles as well as their mean. Here in each case the tax is chosen such that it minimizes the average variance within the feasible space of taxes, as dictated by the BSIC constraints. As $\varepsilon$ increases, for $N = 10$ the standard deviation in taxes paid can be driven from 0.134 to 0.0001 over the range of permissible $\varepsilon$. As mentioned in the Introduction, this means that tax fluctuation between various agents can be made lower, which can prevent a situation where taxes required to be paid from agents are not within their means. For $N = 90$, standard deviation reduces by more than one order of magnitude, here it goes from 0.046 to 0.0035. $^6$

$^6$Defined as the ratio of mean of the difference between every possible pair of data points with mean size. The lower the value of GC, the more equitable the allocation. GC of 0 is perfectly fair and GC of 1 is absolutely unfair - everyone except an individual receiving 0. GC is independent of scale, hence can also be used in comparing fairness across different settings.

![Figure 1: Gini Coefficient of $\{v(\hat{x}(\theta); \theta_i)\}_{i \in \mathbb{N}}$ vs $\varepsilon$, $N = 10$.](image1)

![Figure 2: Gini Coefficient of $\{v(\hat{x}(\theta); \theta_i)\}_{i \in \mathbb{N}}$ vs $\varepsilon$, $N = 90$.](image2)

![Figure 3: Standard Deviation of $t(\theta)$ vs $\varepsilon$, $N = 10$.](image3)

![Figure 4: Standard Deviation of $t(\theta)$ vs $\varepsilon$, $N = 90$.](image4)
Discussion and Conclusions

This paper introduces a concept of fairness in resource allocation problems pertaining to SoU maximization. The fairness aspect is adjustable (through the selection of the parameter $\varepsilon$ and function $f(\cdot)$) in a family of functions, thus giving a wide variety of criteria that a designer may choose at their own discretion. The main result in this paper is the proof of existence of mechanisms that implement the fairer allocation in Dominant and Bayesian equilibria (in respective cases). Numerical results indicate that through the proposed techniques there are significant gains in fairness of allocation, within the permissible limits of the design method.

Although the form considered here is $g_\varepsilon(z) = z - \varepsilon f(z)$, it is easy to see that the results can be extended even with $f$ depending “mildly” on $\varepsilon$. Ideally one would like to consider the class of $g_\varepsilon(z) = z^{1-\varepsilon}$, so as to reconcile with known fair social utilities such as the geometric mean and min utility. This however may not be a practical necessity since as indicated by the results in Section 5, even the form $g_\varepsilon(z) = z - \varepsilon z^2$ provides a significant reduction in GC, as well as the standard deviation of taxes.

In this work, truth-telling is an efficient BNE. One may wonder if there are any other BNE, and if so whether they are efficient. The justification for not studying extraneous BNE is via a focusing argument: the designer announces the contract and the equilibrium that he/she wants agents to play. So long as the announced strategy is indeed an equilibrium, agents will play it since they are anticipating others to be focussed towards it as well, through the common announcement. We are currently researching a modification of our mechanism that guarantees truth-telling as the only BNE by adding one continuous message per agent, other than his/her type. This would be especially needed in situations where selection of equilibria is too complex to predict.

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