Compact Composition Operators on the Bloch Space in Bounded Symmetric Domains

Zehua Zhou  Yan Liu
Department of Mathematics, Tianjin University, Tianjin 300072
E-mail: zehuazhou2003@yahoo.com.cn

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Abstract

Let $\Omega$ be a bounded symmetric domain except the two exceptional domains of $\mathbb{C}^N$ and $\phi$ a holomorphic self-map of $\Omega$. This paper gives a sufficient and necessary condition for the composition operator $C_\phi$ induced by $\phi$ to be compact on the Bloch space $\beta(\Omega)$.

Key words Bloch space, Bounded symmetric domains, Composition operator, Bergman metric.

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1 Introduction

Let $\mathcal{D}$ be a bounded homogeneous domain in $\mathbb{C}^N$. The class of all holomorphic functions with domain $\mathcal{D}$ will be denoted by $H(\mathcal{D})$. Let $\phi$ be a holomorphic self-map of $\mathcal{D}$. For $f \in H(\mathcal{D})$, we denote the composition $f \circ \phi$ by $C_\phi f$ and call $C_\phi$ the composition operator induced by $\phi$. 
Let $K(z, z)$ be the Bergman kernel function of $D$. The Bergman metric $H_z(u, u)$ in $D$ is defined by

$$H_z(u, u) = \frac{1}{2} \sum_{l,k=1}^{N} \frac{\partial^2 \log K(z, z)}{\partial z_l \partial z_k} u_l u_k,$$

where $z \in D$ and $u = (u_1, \ldots, u_N) \in \mathbb{C}^N$.

Following Timoney [1], we say that $f \in H(D)$ is in the Bloch space $\beta(D)$, if

$$\|f\|_{\beta(D)} = \sup_{z \in D} Q_f(z) < \infty,$$

(1)

where

$$Q_f(z) = \sup \left\{ \frac{\|\nabla f(z) u\|}{H_z^2(u, u)} : u \in \mathbb{C}^N - \{0\} \right\},$$

and $\nabla f(z) = \left( \frac{\partial f(z)}{\partial z_1}, \ldots, \frac{\partial f(z)}{\partial z_N} \right)$, $\nabla f(z) u = \sum_{l=1}^{N} \frac{\partial f(z)}{\partial z_l} u_l$.

Let $D$ be the unit disk in $\mathbb{C}$. Madigan and Matheson [2] proved that $C_{\phi}$ is always bounded on $\beta(D)$. They also gave the sufficient and necessary condition that $C_{\phi}$ is compact on $\beta(D)$.

Recently, Shi and Luo [3] proved that $C_{\phi}$ is always bounded on $\beta(D)$, where $D$ is a bounded homogeneous domain in $\mathbb{C}^N$. They also gave a sufficient condition for $C_{\phi}$ to be compact on $\beta(D)$ (i.e., Lemma 3). So this result leads us to ask whether the condition is also necessary. Zhou and Shi [4] give an affirmative answer to this question for classical bounded symmetric domains. In fact, the original purpose of the typescript for [4], is to answer the question in the bounded symmetric domains of $\mathbb{C}^N$, but in the final proof, the referee point out a fatal mistake which we aren’t able to correct that time, upon the suggestion of the referee, we cancel out the last part and published in the form of [4] which discussed only in the four types classical bounded symmetric domains. The following paper overcome difficulty and solve the problem, for the method used here we essentially follow [4], but some new techniques have been used.
symmetric domains. The other two types, called exceptional domains, consist of one domain each (a 16- and a 27-dimensional domain).

In what follows, $\Omega$ denotes a bounded symmetric domain except the two exceptional domains of $\mathbb{C}^N$, and $\phi$ a holomorphic self-map of $\Omega$. If $U = (u_{kl})_{m \times n}$ is a $m \times n$ complex matrix, write $u = (u_{11}, \ldots, u_{1n}, \ldots, u_{m1}, \ldots, u_{mn})$ as the corresponding vector of matrix $U$ and $\overline{u}'$ is the conjugate transpose of $u$. $C$ is a positive constant, not necessarily the same at each occurrence.

In this paper, we will give a sufficient and necessary condition that the composition operator $C \phi$ is compact on $\beta(\Omega)$.

Let $A = (a_{jk})_{m \times n}, B = (b_{lr})_{p \times q}$. The Kronecker product of $A$ and $B$, defined by $A \times B = C = (c_{jklr})$, is a $mp \times nq$ matrix, where $c_{jklr} = a_{jk}b_{lr}$.

It is well known [5] that the classical bounded symmetric domains $R_I, R_{II}, R_{III},$ and $R_{IV}$ can be expressed as follows:

$R_I(m,n) = \{ Z : Z$ is a $m \times n$ complex matrix, $I_m - ZZ' > 0 \}$, where $I_m$ is the $m \times m$ identity matrix ($m \leq n$).

$R_{II}(p) = \{ Z : Z$ is a $p \times p$ symmetric matrix $Z = Z', I_p - ZZ > 0 \}.$

$R_{III}(q) = \{ Z : Z$ is a $q \times q$ antisymmetric matrix $Z = -Z', I_q + ZZ > 0 \}.$

$R_{IV}(N) = \{ z : z = (z_1, \ldots, z_N), 1 + |zz'|^2 - 2z\overline{z}' > 0, |z\overline{z}'| < 1 \}.$

Their Bergman metrics are the following respectively [6]:

$$H^I_z(u,u) = (m + n)u(I_m - ZZ')^{-1} \times (I_n - \overline{Z}Z)^{-1}\overline{u},$$  \hspace{1cm} (2)$$

where $Z \in R_I(m,n)$ and $U$ is a $m \times n$ complex matrix, $u$ is the corresponding vector of $U$, $\overline{u}'$ is the conjugate transpose of $u$.

$$H^{II}_z(u,u) = (p + 1)u(I_p - Z\overline{Z})^{-1} \times (I_p - \overline{Z}Z)^{-1}\overline{u'},$$  \hspace{1cm} (3)
vector of $U$. 

$$H^H_z(u, u) = 2(q - 1)u(I_q + ZZ)^{-1} \times (I_q + ZZ)^{-1}u', \quad (4)$$

where $Z \in R_{III}(q)$ and $U$ is a $q \times q$ anti-symmetric complex matrix, $u$ is the corresponding vector of $U$.

$$H^{IV}_z(u, u) = \frac{2N}{(1 + |z z'|^2 - 2z \overline{z'})^2} \left(1 + |z z'|^2 - 2z \overline{z'}\right)I_N$$

$$-2 \left( \begin{array}{c} z \\ z' \end{array} \right)' \left( \begin{array}{cc} 1 - 2|z|^2 & z z' \\ z z' & -1 \end{array} \right) \left( \begin{array}{c} z \\ z' \end{array} \right) \right] u'. \quad (5)$$

where $z \in R_{IV}(N)$ and $u \in \mathbb{C}^N$.

Our main result is the following:

**Theorem** Let $\Omega \subset \mathbb{C}^N$ be a bounded symmetric domain except the two exceptional domains and $\phi$ a holomorphic self-map of $\Omega$. Then $C^\phi$ is compact on the Bloch space $\beta(\Omega)$ if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$H^\phi_{\phi(z)}(J\phi(z)u, J\phi(z)u) < \varepsilon,$$ 

for all $u \in \mathbb{C}^N - \{0\}$ whenever $\text{dist}(\phi(z), \partial \Omega) < \delta$, where $H_z(u, u)$ is the Bergman metric of $\Omega$.

**Remark** It is well known that the unit ball and unit polydiscs are bounded symmetric domains, so the above result holds in the unit balls and unit polydiscs. Furthermore, we can also obtain Theorem 2 in [2].

2 Some Lemmas

In order to prove the Theorem, we need the following Lemmas.

**Lemma 1** ([1], Theorem 2.12) Let $D \subset \mathbb{C}^N$ be a bounded homogeneous domain. Then
denotes the Bergman metric on $\mathcal{D}$, $J\phi(z) = \left( \frac{\partial \phi_l(z)}{\partial z_k} \right)_{1 \leq l, k \leq N}$ denotes the Jacobian matrix of $\phi$, and $J\phi(z)u$ denotes a vector, whose $l$th component is $(J\phi(z)u)_l = \sum_{k=1}^{N} \frac{\partial \phi_l(z)}{\partial z_k} u_k$, $l = 1, 2, \ldots, N$.

**Lemma 2** ([3], Lemma 3) Let $\mathcal{D}$ be a bounded homogeneous domain in $\mathbb{C}^N$. Then $C_\phi$ is compact on $\beta(\mathcal{D})$ if and only if for any bounded sequence $\{f_k\}$ in $\beta(\mathcal{D})$ which converges to zero uniformly on compact subsets of $\mathcal{D}$, we have $\|f_k \circ \phi\|_{\beta(\mathcal{D})} \to 0$, as $k \to \infty$.

**Lemma 3** ([3], Theorem 3) If $\phi : \mathcal{D} \to \mathcal{D}$ is a holomorphic self-map, where $\mathcal{D}$ is a bounded homogeneous domain in $\mathbb{C}^N$. Then $C_\phi$ is compact on $\beta(\mathcal{D})$ if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$\frac{H_{\phi(z)}(J\phi(z)u, J\phi(z)u)}{H_z(u,u)} < \varepsilon,$$

for all $u \in \mathbb{C}^N - \{0\}$ whenever $\text{dist}(\phi(z), \partial \mathcal{D}) < \delta$.

**Lemma 4** ([4], Lemma 4) Let $\mathcal{D}$ be a bounded homogeneous domain of $\mathbb{C}^N$, and let $T(z,z)$ denote its metric matrix. If $T(0,0) = \lambda I_N$, where $\lambda$ is a constant depending only on $\mathcal{D}$, then a holomorphic function $f$ on $\mathcal{D}$ is in $\beta(\mathcal{D})$ if and only if

$$\sup_{z \in \mathcal{D}} \left\{ \nabla f(z)T^{-1}(z,z)\overline{\nabla f(z)} \right\} < \infty. \quad (7)$$

If (7) holds, then there exists a constant $C$ depending only on $\mathcal{D}$ such that

$$\|f\|_{\beta(\mathcal{D})} \leq C \sup_{z \in \mathcal{D}} \left\{ \nabla f(z)T^{-1}(z,z)\overline{\nabla f(z)} \right\}.$$

**Lemma 5** ([1], Proposition 4.5) Let $\mathcal{D}$ be a bounded homogeneous domain in $\mathbb{C}^N$. If $f$ is a bounded holomorphic function in $\mathcal{D}$, then $f \in \beta(\mathcal{D})$ and there exists a constant $C$ depending only on $\mathcal{D}$, such that
It is well known that every $m \times n$ ($m \leq n$) matrix $A$ may be written as $A = U(\sum_{k=1}^{m} \lambda_k E_{kk})V$, where $U$ and $V$ are $m \times m$ and $n \times n$ unitary matrices respectively, and $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$, $E_{kk}$ is a $m \times n$ matrix, the element of $k$th row and $k$th column is 1, and other elements are 0. Hence for every $P \in R_I(m,n)$ ($m \leq n$), there exist $m \times m$ unitary matrix $U$ and $n \times n$ unitary matrix $V$, such that $P = U(\sum_{k=1}^{m} \lambda_k E_{kk})V$; $(1 \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0)$.

**Lemma 6** ([4], Lemma 8) Let $P = U \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_m & 0 & \cdots & 0 \end{pmatrix} V \in R_I$, and write

$$Q = U \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \sqrt{1 - \lambda_1^2} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \sqrt{1 - \lambda_2^2} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sqrt{1 - \lambda_m^2} \end{pmatrix} V,$$

$$R = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \sqrt{1 - \lambda_1^2} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \sqrt{1 - \lambda_2^2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sqrt{1 - \lambda_m^2} & 0 & \cdots & 0 \\ 0_{(n-m) \times m} & 0 & \cdots & 0 & I_{n-m} \end{pmatrix},$$

where $U$ and $V$ are $m \times m$ and $n \times n$ unitary matrices respectively, and $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$.

For $Z \in R_I$, denote $\Phi^{(I)}_P(Z) = Q(P - Z)(I_n - V^T Z)^{-1}R^{-1}$, then

1. $\Phi^{(I)}_P \in \text{Aut}(R_I)$;
2. $\Phi^{(I)}_P$ is $I$-annihilating;
\( IV \) \( d\Phi_P^{(I)}(Z)|_{Z=P} = -QdZR, d\Phi_P^{(I)}(Z)|_{Z=0} = -Q^{-1}dZR^{-1}; \)

\( V \) \( \Phi_P^{(I)}(Z) = Q^{-1}(I_m - Z\overline{T}^{-1})(P - Z)R, \) for \( Z \in R_I; \)

\( VI \) \( (I_m - Z\overline{T}^{-1})Q \left( I_m - \Phi_P^{(I)}(Z)\Phi_P^{(I)}(Z) \right) \overline{Q}(I_m - P\overline{T}') = I_m - ZZ', \) for \( Z \in R_I. \)

### 3 An Important Proposition

**Proposition** Let \( R_A(A = I, II, III, IV) \) be a classical bounded symmetric domain. If \( a_j \in R_A, \ d(a_j, \partial R_A) \to 0 \) as \( j \to \infty, \) and \( w_j \in \mathbb{C}^N - \{0\}, \) then exists a sequence of functions \( \{f_j\} \) satisfying the following three conditions:

(i) \( \{f_j\} \) is a bounded sequence in \( \beta(R_A); \)

(ii) \( \{f_j\} \) tends to zero uniformly on any compact subsets of \( R_A; \)

(iii) \( \left| \frac{\nabla f_j(a_j)w_j}{H_{a_j}^I(w_j, w_j)} \right| \geq C. \)

**Proof** Note that the construction of the test functions in [4], and replace \( \phi(Z_j) \) by \( a_j \) at a time, we can construct a sequence of functions \( \{f_j\} \) satisfying the above three conditions. For example, for the reader’s convenience, we give the proof for the domain \( R_I. \) But the proof completely follows from [4], if necessary, the proof can be also omitted.

We construct a sequence of the functions according to the following four parts.

Part A: To construct the sequence of \( \{f_j\}, \) we first assume that

\[ a_j = r_j E_{11}, j = 1, 2, \ldots, \]

where \( E_{kl} \) is a \( m \times n \) matrix, the element of \( k \)th row and \( l \)th column is 1, and other elements are 0. It is clear that \( 0 < r_j < 1 \) and \( r_j \to 1 \) as \( j \to \infty. \)

Denote \( w_j = (w_{11}, \ldots, w_{1n}, w_{21}, \ldots, w_{2n}, \ldots, w_{m1}, \ldots, w_{mn}). \) Using the formula (2), we have

\[ H_{a_j}^I(w_j, w_j) = H_{r_j E_{11}}^I(w_j, w_j) \]
\[ Z.H.ZHOU \]

\[ \begin{align*}
= (m + n)w^j
\end{align*} \]

\[ \begin{pmatrix}
(1 - r_j^2)^{-1} & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
(1 - r_j^2)^{-1} & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

\[ = (m + n) \left[ \frac{|w_{11}^j|^2}{(1 - r_j^2)^2} + \frac{1}{1 - r_j^2} \left( \sum_{l=2}^{n} |w_{ll}^j|^2 + \sum_{k=2}^{m} |w_{k1}^j|^2 \right) + \sum_{2 \leq k \leq m, 2 \leq l \leq n} |w_{kl}^j|^2 \right]. \]

Denote

\[ A_j^I = \frac{|w_{11}^j|^2}{(1 - r_j^2)^2}, \]

\[ B_j^I = \frac{1}{1 - r_j^2} \left( \sum_{l=2}^{n} |w_{ll}^j|^2 + \sum_{k=2}^{m} |w_{k1}^j|^2 \right), \]

\[ C_j^I = \sum_{2 \leq k \leq m, 2 \leq l \leq n} |w_{kl}^j|^2, \]

then

\[ H_{\omega}^{I} (w^j, w^j) = (m + n)(A_j^I + B_j^I + C_j^I). \] (8)

We construct the functions according to three different cases:

Case 1 If for some \( j, \)

\[ \max(B_j^I, C_j^I) \leq A_j^I, \] (9)

then set

\[ f_j(Z) = \log \left( 1 - e^{-a(1-r_j)}z_{11} \right) - \log (1 - z_{11}), \] (10)

where \( Z = (z_{kl}), 1 \leq k \leq m, 1 \leq l \leq n \) and \( a \) is any positive number.

Case 2 If for some \( j, \)

\[ \max(A_j^I, C_j^I) \leq B_j^I, \] (11)

then set

\[ f_j(Z) = \left( \sum_{l=2}^{n} e^{-i\theta_{ll}^j}z_{1l} + \sum_{k=2}^{m} e^{-i\theta_{k1}^j}z_{1k} \right) \left( \frac{1}{\sqrt{1 - e^{-a(1-r_j)}z_{11}}} - \frac{1}{\sqrt{1 - z_{11}}} \right), \] (12)
Case 3 If for some \( j \),
\[
\max(A_j^l, B_j^l) \leq C_j^l,
\] (13)
then set
\[
f_j(Z) = \left( \sum_{2 \leq k \leq m, 2 \leq l \leq n} e^{-i\theta_{kl}^j} z_{kl} \right) \sqrt{1 - z_{11}} \left( \frac{1}{\sqrt{1 - e^{-a(1-r_j)} z_{11}}} - \frac{1}{\sqrt{1 - z_{11}}} \right),
\]
(14)
where \( a \) is any positive number, and \( \theta_{kl}^j = \arg w_{kl}^j, 2 \leq k \leq m, 2 \leq l \leq n \). If \( w_{kl}^j = 0 \) for some \( k \) or \( l \), replace the corresponding term \( e^{-i\theta_{kl}^j} z_{kl} \) by 0.

Let \( E \) be a compact subset of \( R_J \), then there exists a \( \rho \in (0, 1) \) such that \( |z_{11}| \leq \rho \), for any \( Z = (z_{kl}) \in E \). It is easy to show that the sequence of functions defined by (10), (12) and (14) respectively, converges to zero uniformly on \( E \) as \( j \to \infty \), so the sequence satisfies the conditions (ii).

Now we prove the above sequence satisfies the condition (i).

For the functions defined by (10), it is easy to see
\[
\nabla f_j(Z) = \left( \frac{\partial f_j}{\partial z_{11}}(Z), \ldots, \frac{\partial f_j}{\partial z_{1n}}(Z), \ldots, \frac{\partial f_j}{\partial z_{m1}}(Z), \ldots, \frac{\partial f_j}{\partial z_{mn}}(Z) \right) = \left( \frac{\partial f_j}{\partial z_{11}}(Z), \ldots, 0, \ldots, 0, \ldots, 0 \right).
\]

From formula (2), it is easy to know that the metric matrix of \( R_I(m, n) \) is
\[
T(Z, Z) = (m + n)(I_m - \overline{Z} Z)^{-1} \times (I_n - \overline{Z} Z)^{-1},
\]
so \( T(0, 0) = (m + n)I_{mn} \), and
\[
T^{-1}(Z, Z) = (m + n)^{-1}(I_m - \overline{Z} Z)^{-1} \times (I_n - \overline{Z} Z).
\]

Thus
\[
\nabla f_j(Z) T^{-1}(Z, Z) \overline{f_j(Z)} = (m + n)^{-1} \left| \frac{\partial f_j}{\partial z_{11}}(Z) \right|^2 \left( 1 - \sum_{l=1}^n |z_{1l}|^2 \right) \left( 1 - \sum_{k=1}^n |z_{k1}|^2 \right) = (m + n)^{-1} \left( 1 - \sum_{l=1}^n |z_{1l}|^2 \right) \left( 1 - \sum_{k=1}^n |z_{k1}|^2 \right).
\]
Now Lemma 4 gives

$$\|f_j\|_{\beta(R)} \leq 16C(m + n)^{-1}.$$  

This proves that the functions (10) satisfy condition (i).

For the functions defined by (12), If $Z \in R_I(m, n)$, we have

$$I_m - ZZ' = \left( \delta_{st} - \sum_{k=1}^{n} z_{sk} \bar{z}_{tk} \right)_{1 \leq s, t \leq m} > 0.$$  

$$I_n - ZZ' = \left( \delta_{st} - \sum_{k=1}^{m} z_{ks} \bar{z}_{kt} \right)_{1 \leq s, t \leq n} > 0.$$  

Hence

$$\delta_{11} - \sum_{l=1}^{n} z_{1l} \bar{z}_{1l} = 1 - \sum_{l=1}^{n} |z_{1l}|^2 > 0. \quad (15)$$  

$$\delta_{11} - \sum_{k=1}^{m} z_{k1} \bar{z}_{k1} = 1 - \sum_{k=1}^{m} |z_{k1}|^2 > 0. \quad (16)$$  

Now (15) and (16) imply

$$|f_j(Z)| = \left| \left( \sum_{l=2}^{n} e^{-i\theta_l} z_{1l} + \sum_{k=2}^{m} e^{-i\theta_k} z_{k1} \right) \left( \frac{1}{\sqrt{1 - e^{-a(1-r_j)z_{11}}}} - \frac{1}{\sqrt{1 - |z_{11}|}} \right) \right| \leq \left( \sum_{l=2}^{n} |z_{1l}|^2 + \sum_{k=2}^{m} |z_{k1}|^2 \right)^{\frac{1}{2}} \left( \frac{1}{\sqrt{1 - e^{-a(1-r_j)z_{11}}}} + \frac{1}{\sqrt{1 - |z_{11}|}} \right) \lesssim (\sqrt{n} + \sqrt{m}) \sqrt{1 - |z_{11}|^2} \left( \frac{1}{\sqrt{1 - |z_{11}|}} + \frac{1}{\sqrt{1 - |z_{11}|}} \right) \leq 4(\sqrt{m} + \sqrt{n}). \quad (17)$$  

From Lemma 5, (17) means that $\{f_j\}$ satisfy the condition (i). Similarly, we may prove the functions defined by (14) satisfy the condition (i).

At last, we prove that the sequence of functions defined by (10) satisfies the conditions (iii). In fact, by (8) and (9),

$$H_{\alpha_j}(w^j, \omega^j) = (m + n)(A_j^I + B_j^I + C_j^I) \leq 3(m + n)A_j^I.$$  

where

$$\Phi_{I} \subset \Psi_{I} (\text{iii}).$$

and

$$\lim_{j \to \infty} \left[ 1 - \frac{(1 - r_j)e^{-a(1-r_j)}}{1 - e^{-a(1-r_j)r_j}} \right] = \frac{a}{a+1} \neq 0.$$ 

This proves the sequence of functions defined by (10) satisfies the conditions (iii). Similarly, we can prove the sequence of functions defined by (12) or (14) satisfies the conditions (iii).

Part B: We assume that

$$a^j = r_j^{(1)} E_{11} + r_j^{(2)} E_{22},$$

where $1 > r_j^{(1)} \geq r_j^{(2)} \geq 0$. By $a^j \to \partial R_I$, we may assume that $r_j^{(1)} \to 1, r_j^{(2)} \to \lambda_0(\leq 1)$.

If $\lambda_0 = 1$, using the same methods as in Part A, we can construct a sequence of functions $\{f_j(Z)\}$ satisfying the three conditions (i), (ii) and (iii).

If $\lambda_0 < 1$, by Lemma 6, there exist $\Phi_{r_j^{(1)} E_{11} + r_j^{(2)} E_{22}}^{I} \in R_I$ and $\Phi_{r_j^{(1)} E_{11},v}^{I} \in R_I$, such that $\Phi_{r_j^{(1)} E_{11} + r_j^{(2)} E_{22}}^{I}(r_j^{(1)} E_{11} + r_j^{(2)} E_{22}) = 0$ and $\Phi_{r_j^{(1)} E_{11}}^{I}(r_j^{(1)} E_{11}) = 0 (j = 1, 2, \cdots)$. If we denote $\Psi^{(j)}(Z) = \left( \Phi_{r_j^{(1)} E_{11}}^{I} \right)^{-1} \circ \Phi_{r_j^{(1)} E_{11} + r_j^{(2)} E_{22}}^{I}$, then $\Psi^{(j)} \in R_I$ and $\Psi^{(j)}(a^j) = \Psi^{(j)}(r_j^{(1)} E_{11} + r_j^{(2)} E_{22}) = r_j^{(1)} E_{11} = r_j E_{11}$, where $r_j = r_j^{(1)}$.

Set $g_j = f_j \circ \Psi^{(j)}$, where $\{f_j\}$ are the functions obtained in Part A. Since $\Psi^{(j)}(Z) \in Aut(R_I)$, it is clear that

$$H_{a^j}(w^j, w^j) = H_{\Psi^{(j)}(a^j)}^{I}(J\Psi^{(j)}(a^j)w^j, J\Psi^{(j)}(a^j)w^j) = H_{r_j E_{11}}^{I}(v^j, v^j),$$

(18)

where $w^j = J\phi(Z^j)u^j, v^j = J\Psi^{(j)}(\phi(Z^j))w^j$. It follows from (18) that

$$\frac{\left| \nabla(g_j)(a^j)w^j \right|}{H_{a^j}^{\frac{1}{a}}(w^j, w^j)} = \frac{\left| \nabla(f_j)(r_j E_{11}) J\Psi^{(j)}(\phi(Z^j))w^j \right|}{H_{r_j E_{11}}^{\frac{1}{r_j}}(J\Psi^{(j)}(\phi(Z^j))w^j, J\Psi^{(j)}(\phi(Z^j))w^j)} = \frac{\left| \nabla(f_j)(r_j E_{11})v^j \right|}{H_{r_j E_{11}}^{\frac{1}{r_j}}(v^j, v^j)}.$$

Now the discussion in Part A shows that
that is, \( \{g_j\} \) satisfies condition (iii).

We prove that \( \{g_j\} \) is a bounded sequence in \( \beta(R_I) \); In fact, since \( \Psi^{(j)}(Z) \in Aut(R_I) \),

\[
Q_{g_j}(Z) = Q_{f_j \circ \Psi^{(j)}}(Z) = Q_{f_j} \left( \Psi^{(j)}(Z) \right),
\]
so \( \|g_j\|_{\beta(R_I)} = \|f_j\|_{\beta(R_I)} \) is bounded.

Now we prove \( \{g_j\} \) tends to zero uniformly on compact subset \( E \) of \( R_I \).

If we write \( \Psi^{(j)}(Z) = \left( \psi_{k_1}^{(j)}(Z) \right)_{1 \leq l \leq m, 1 \leq k \leq n} \), by the definition of \( \Psi^{(j)} \) and Lemma 6, a direct calculation shows that

\[
\psi_{11}^{(j)}(Z) = z_{11} + r_j^{(2)} \frac{z_{12} z_{21}}{1 - r_j^{(2)} z_{22}}.
\]  
(19)

It is easy to show that \( \psi_{11}^{(j)}(Z) \) converges uniformly to \( \psi_{11}(Z) = z_{11} + \lambda_0 \frac{z_{12} z_{21}}{1 - \lambda_0 z_{22}} \) on \( R_I(m,n) \).

Since \( \lambda_0 < 1 \), \( \lambda_0 E_{11} + \lambda_0 E_{22} \in R_I \), similarly, there exists \( \Psi(Z) \in Aut(R_I) \), such that \( \Psi(\lambda_0 E_{11} + \lambda_0 E_{22}) = \lambda_0 E_{11} \), and the first component of \( \Psi(Z) \) is \( \psi_{11}(Z) \). It is clear that \( \psi_{11}(Z) \) is holomorphic on \( R_I \). Let \( M_1 = \sup_{Z \in E} |\psi_{11}(Z)| = |\psi_{11}(Z_0)|, (Z_0 \in E) \). From \( \Psi(Z) \in Aut(R_I) \), we know \( M_1 = |\psi_{11}(Z_0)| < 1 \), so we may choose \( M_0 > 0 \) with \( M_1 < M_0 < 1 \). So for \( j \) large enough, \( |\psi_{11}^{(j)}(Z_0)| < M_0 \), from this it follows that

\[
1 - |\psi_{11}^{(j)}(Z)| > 1 - M_0 > 0,
\]
by the definition of \( f_j(Z) \), it is easy to know \( g_j(Z) = f_j \circ \Psi^{(j)}(Z) \) tends to zero uniformly on \( E \). So \( g_j(Z) \) satisfies the three conditions (i), (ii) and (iii).

Part C: Assume that

\[
\phi(Z^j) = \sum_{k=1}^{m} r_j^{(k)} E_{kk}, (1 > r_j^{(1)} \geq r_j^{(2)} \geq \ldots \geq r_j^{(m)} \geq 0).
\]

By the same discussion as in Part B, we can construct a sequence of functions \( f_j(z) \) which
Part D: In the general situation, \( \phi(Z^j) \in R_I(m,n) \), so there exist \( m \times m \) unitary matrix \( P_j \) and \( n \times n \) unitary matrix \( Q_j \), such that

\[
P_j \left( \phi(Z^j) \right) Q_j = \sum_{k=1}^{m} r^{(k)}_j E_{kk}.
\]

We may assume that \( P_j \to P \) and \( Q_j \to Q \), as \( j \to \infty \). Let \( P_j = (p_{kl}^j) \), \( P = (p_{kl}) \), \( P_j \to P \) means that \( p_{kl}^j \to p_{kl} \) as \( j \to \infty \) for any \( 1 \leq k \leq m, 1 \leq l \leq n \). Let \( \psi^j(Z) = P_j Z Q_j \), \( \Psi(Z) = P Z Q, Z \in R_I(m,n) \). It is easy to show that \( P \) is a \( m \times m \) unitary matrix, \( Q \) is a \( n \times n \) unitary matrix and \( \psi^j(Z) \) converges uniformly to \( \psi(Z) \) on \( R_I \).

Let \( g_j(Z) = f_j \left( \psi^j(Z) \right) \), where \( \{f_j\} \) are the functions obtained in Part C. From the same discussion as that of Part B, we know \( g_j(Z) \) satisfies conditions (i) and (iii). For the compact subset \( E \subset R_I \), it is easy to know \( \psi(E) \) is also a compact subset of \( R_I \), so we can choose an open subset \( D_1 \) of \( R_I \) such that \( \psi(E) \subset D_1 \subset \overline{D_1} \subset R_I \). Since \( \psi^j(Z) \) converges uniformly to \( \psi(Z) \) on \( R_I \), so as \( j \to \infty \), \( \psi^j(E) \subset D_1 \). Since \( f_j(Z) \) tends to zero uniformly on \( D_1 \), we know \( g_j(Z) = f_j \left( \psi^j(Z) \right) \) tends to zero uniformly on \( E \subset R_I \), i.e., \( g_j \) satisfies condition (iii).

The last claim follows from the discussion in the above. This completes the proof.

4 Proof of Theorem

It is well known that a bounded symmetric domain except the two exceptional domains can be expressed as a topological product of the first four types of irreducible domains \( R_A(A = I, II, III, IV) \). So we may assume that \( \Omega = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_n \), where \( \Omega_k \subset \mathbb{C}^{N_k}, (k = 1, 2, \ldots, n) \) are the classical bounded symmetric domains and \( N_1 + N_2 + \ldots + N_n = N \). It is obvious \( \Omega \) is homogeneous, so by Lemma 3, we only need to prove the condition (6) is necessary.
three conditions:

We may construct a sequence of functions \( \{ \phi \} \) satisfying the following

\[
K_\Omega(z, z) = K_{\Omega_1}(z_1, z_1) \times K_{\Omega_2}(z_2, z_2) \times \cdots \times K_{\Omega_n}(z_n, z_n),
\]

\[
T_\Omega(z, z) = \begin{bmatrix}
T_{\Omega_1}(z_1, z_1) & 0 & \cdots & 0 \\
0 & T_{\Omega_2}(z_2, z_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & T_{\Omega_n}(z_n, z_n)
\end{bmatrix}
\]

\[
H^\Omega_z(u, u) = H^\Omega_{z_1}(u_1, u_1) + H^\Omega_{z_2}(u_2, u_2) + \cdots + H^\Omega_{z_n}(u_n, u_n),
\] (20)

where \( u = (u_1, \ldots, u_n), u_k \in \mathbb{C}^k (k = 1, 2, \ldots, n) \), and \( H^\Omega_z(u, u) = H_z(u, u) \).

Now assume the condition (6) fails, then there exists a sequence \( \{ z^j \} \) in \( \Omega \) with

\( \phi(z^j) \to \partial \Omega, \) as \( j \to \infty, u^j \in \mathbb{C}^N - \{0\} \), and an \( \varepsilon_0 \), such that

\[
\frac{H^\Omega_{\phi(z^j)}(J\phi(z^j)u^j, J\phi(z^j)u^j)}{H^\Omega_{z^j}(u^j, u^j)} \geq \varepsilon_0,
\] (21)

for all \( j = 1, 2, \ldots \). Write \( J\phi(z^j)u^j = w^j \), by (20), we have

\[
H^\Omega_{\phi(z^j)}(w^j, w^j) = H^\Omega_{\phi_1(z^j)}(w^j_1, w^j_1) + H^\Omega_{\phi_2(z^j)}(w^j_2, w^j_2) + \cdots + H^\Omega_{\phi_n(z^j)}(w^j_n, w^j_n),
\] (22)

where \( w^j = (w^j_1, w^j_2, \ldots, w^j_n), w_k^j \in \mathbb{C}^N, (k = 1, 2, \ldots, n) \), and \( \phi = (\phi_1, \phi_2, \ldots, \phi_n) \). It is obvious that for some \( k \), without loss of generality, we may assume for \( k = 1 \), there exists a subsequence \( \{ j_s \} \) which we still denote by \( \{ j \} \), such that

\[
H^\Omega_{\phi(z^j)}(w^j_l, w^j_l) \leq H^\Omega_{\phi_1(z^j)}(w^j_1, w^j_1), \quad (l = 2, \ldots, n).
\]

So by (22)

\[
H^\Omega_{\phi(z^j)}(w^j_l, w^j_l) \leq nH^\Omega_{\phi_1(z^j)}(w^j_1, w^j_1).
\] (23)

Since \( \Omega_1 \) is a classical bounded symmetric domain, denote \( a^j = \phi_1(z^j) \), by the important Proposition we may construct a sequence of functions \( \{ f_j \} \) satisfying the following
(b) \( \{f_j\} \) tends to zero uniformly on any compact subsets of \( \Omega_1 \);

(c) \[
\frac{\left| \nabla f_j(a_j)w^j_1 \right|}{H_\phi^2(w^1_j, u^j_1)} \geq C.
\]

We first prove that \( \{f_j\} \) as a function sequence on \( \Omega \) is a bounded sequence in \( \beta(\Omega) \).

In fact,

\[
\frac{|\nabla (f_j)(z)u|}{H_\phi^2(u,u)} = \frac{|\nabla (f_j)(z_1)u_1|}{\left( H_{\phi_1}^1(u_1,u_1) + \ldots + H_{\phi_n}^1(u_n,u_n) \right)^{\frac{1}{2}}}
\leq \frac{|\nabla (f_j)(z_1)u_1|}{\left( H_{\phi_1}^1(u_1,u_1) \right)^{\frac{1}{2}}} \leq \|f_j\|_{\beta(\Omega)},
\]

so \( \{f_j\} \) is a bounded sequence in \( \beta(\Omega) \).

For the compact subset \( E \subset \Omega \), let \( E_1 = \{z_1 : (z_1,z_2,\ldots,z_n) \in E \subset \Omega\} \). It is easy to know \( E_1 \) is a compact subset of \( \Omega_1 \). We also have

\[
\sup_{z \in E} |f_j(z)| = \sup_{z_1 \in E_1} |f_j(z_1)| \to 0,
\]
as \( j \to \infty \) i.e., \( \{f_j\} \) tends to zero uniformly on compact subsets of \( \Omega \).

Now we prove \( \|C_\phi f_j\|_{\beta(\Omega)} \not\to 0 \). In fact, by (22) and (23), we have

\[
\|C_\phi f_j\|_{\beta(\Omega)} = \|f_j \circ \phi\|_{\beta(\Omega)} \geq Q_{f_j \circ \phi}(z^j) \geq \|f_j \circ \phi\|_{\beta(\Omega)} \geq Q_{f_j \circ \phi}(z^j).
\]

\[
\geq \frac{|\nabla (f_j \circ \phi)(z^j)w^j|}{\left( H_{\phi(z)}^1(u^1_j,u^1_j) + \ldots + H_{\phi(z)}^1(u^1_n,u^1_n) \right)^{\frac{1}{2}}} \geq \frac{|\nabla (f_j \circ \phi)(z^j)w^j|}{\left( H_{\phi(z)}^1(u^1_j,u^1_j) \right)^{\frac{1}{2}}} \geq \frac{\|H_{\phi(z)}^1(u^1_j,u^1_j)\|}{\|\phi(z)\|_{\beta(\Omega)}} \geq 0.
\]

by (c), we know \( \|C_\phi f_j\|_{\beta(\Omega)} \not\to 0 \) as \( j \to \infty \). This contradicts the compactness of \( C_\phi \) by Lemma 2. Now the proof of the Theorem is complete.

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