Strategic information transmission with sender’s approval

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Abstract
We consider sender–receiver games in which the sender has finitely many types and the receiver makes a decision in a compact set. The new feature is that, after the cheap talk phase, the receiver makes a proposal to the sender, which the latter can reject in favor of an outside option. We focus on situations in which the sender’s approval is absolutely crucial to the receiver, namely, on equilibria in which the sender does not exit at the approval stage. A nonrevealing equilibrium without exit may not exist. Our main results are that if the sender has only two types or if the receiver’s preferences over decisions do not depend on the type of the sender, there exists a (perfect Bayesian Nash) partitional equilibrium without exit, in which the sender transmits information by means of a pure strategy. The previous existence results do not extend: we construct a counter-example (with three types for the sender and type-dependent utility functions) in which there is no equilibrium without exit, even if the sender can randomize over messages. We establish additional existence results for (possibly mediated) equilibria without exit in the three type case.

This research started during the winter 2015–2016, while the first author was visiting Humboldt University, Berlin. We did not know of Shimizu (2013, 2017) at the time. These papers were pointed out to the first author by Daniel Krähmer on the occasion of a talk in Bonn in April 2018. Then we discovered Matthews (1989), among the references of Shimizu (2013, 2017).
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1 Introduction

We consider a general model of sender–receiver games. The specific feature of our games is that the sender has an outside option. After the cheap talk phase, the receiver proposes a decision to the sender; if the sender approves it, the decision is made; otherwise, the sender chooses his outside option, which can be interpreted as “exit”. Under complete information, the game reduces to an ultimatum game, in which one player makes a “take it or leave it” offer to the other. In our framework, this other player has private information and can send a costless message to the receiver before getting an offer.

We focus on situations in which the sender’s approval is absolutely crucial to the receiver. We thus assume that the receiver’s utility in case of exit is extremely low, as compared to what he can expect if the sender accepts his proposal. It is not difficult to find examples in which a decision-maker consults with an informed party before making a proposal that can be ultimately rejected and in which rejection has unvaluable, damaging consequences for the decision-maker. As examples of such unvaluable outcomes in case of proposal rejection, just think of workers’ unlimited strikes or killing of hostages by kidnappers.

Our solution concept is subgame perfect Nash equilibrium, in which the sender does not make uncredible threats at the approval stage, namely, accepts a proposal if and only if it gives him at least the utility of his outside option. 1 We simply refer to “equilibrium” and ask whether our sender–receiver game has an equilibrium in which exit does not occur.

To answer this question, we introduce an auxiliary, tractable, “limit game” \( \Gamma \), in which the receiver’s payoff, in case of exit, is \(-\infty\). We focus on the equilibria of \( \Gamma \) with finite payoff, namely, “no exit equilibria” (henceforth, NEE). The NEE of \( \Gamma \) are easily characterized by two sets of conditions: incentive compatibility and constrained optimization. Both sets of conditions are well-behaved but satisfying them jointly is demanding. Existence of a NEE in \( \Gamma \) is not obvious. For instance, as opposed to standard sender–receiver games, \( \Gamma \) may not have any nonrevealing NEE. In this case, without information transmission, the receiver cannot make any decision that would give at least his reservation utility to the sender, whatever his type. We identify various assumptions which guarantee that, in a situation like this, the sender can credibly reveal some information to the receiver, in such a way that exit will never happen.

We show that the study of the auxiliary game \( \Gamma \) can be motivated by two properties. First of all, if the game \( \Gamma \) has no NEE, then, at every equilibrium of every game \( \Gamma(\nu_0) \) in which the receiver obtains the finite payoff \( \nu_0 \) in case of exit, there is at least one type that rejects the receiver’s proposal. Conversely, if the limit game \( \Gamma \) does have a

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1 Except for the approval stage, our model behaves as a standard cheap talk game, in which Perfect Bayesian equilibrium is not restrictive.
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NEE, then, if the receiver gets a sufficiently low payoff \( v_0 \) in case of exit, he achieves his best equilibrium payoff in \( \Gamma(v_0) \) by maximizing his utility over the NEE of \( \Gamma \), without having to worry about the precise payoff \( v_0 \) he would get in case of exit.

We maintain the following assumptions on the game \( \Gamma \): the sender has finitely many types (which can be multidimensional, e.g., belong to \( \mathbb{R}^{n_1} \), for some \( n_1 \)), the receiver has a compact set of decisions (typically, a closed, bounded set in \( \mathbb{R}^{n_2} \), for some \( n_2 \)) and both players’ utility functions are continuous. We also make the *sine qua non* assumption that under complete information, i.e., when the receiver knows the sender’s type, there exists a decision that gives the sender at least his reservation utility.

We identify additional assumptions guaranteeing that the game \( \Gamma \) has a NEE in which the sender transmits information using a pure strategy. In this case, the sender’s strategy induces a partition of his set of types. Hence we refer to such a NEE as to a *partitional* NEE. Our main result (Theorem 8) is that the game \( \Gamma \) has a partitional NEE if the receiver’s utility function—when the sender participates—does not depend on the sender’s type. This assumption holds as soon as the receiver knows his own preferences over decisions, but is eager to make a choice that will ensure the—type-dependent—inform player’s participation.

We also show that the game \( \Gamma \) has a partitional NEE when the sender has two types only (Proposition 5). We generalize this rather straightforward result to the case of an arbitrary number of types by introducing the notion of “participation structure”, which consists of the maximal subsets of types (with respect to set inclusion) for which there is a decision inducing all of them to participate. For instance, if the sender has three types, 1, 2 and 3, there are 9 possible participation structures: the partitions of \{1, 2, 3\}, the set of all the pairs of types (namely, \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}) and the sets of two pairs of types (namely, \{\{1, 2\}, \{2, 3\}\}, etc.). We show that the game \( \Gamma \) has a partitional NEE if the participation structure is a partition of the type set (Proposition 6).

Finally, we establish that the game \( \Gamma \) has a partitional NEE if the decision set is a subset of the real line and for every type, the sender’s utility function is monotonic in the receiver’s decision (Proposition 7). The assumptions of Proposition 7 are satisfied in particular if the receiver’s decision is a lottery over two actions and utilities are expectations with respect to this lottery.

All the previous assumptions may look restrictive, but, without them, existence of a NEE in \( \Gamma \) cannot be guaranteed. We indeed propose an example, in which the sender has three types, the receiver has three actions, the utility functions are type-dependent and the participation structure is not a partition. In this example, there is *no NEE*, even if the sender can randomize over his messages and the receiver can propose a lottery over his actions. However, the problem disappears as soon as the information transmission stage is handled by a mediator, namely, a *mediated* equilibrium without exit does exist in this example.

We conclude with an analysis of equilibrium existence in \( \Gamma \) when the sender has three possible types. In this case, as observed above, there are two kinds of “participation structures” beyond the straightforward case of a partition. The first one—\{\{1, 2\}, \{2, 3\}, \{1, 3\}\}—arises in the example mentioned in the previous paragraph. Existence of a mediated equilibrium without exit can then be established. In the
other case—\{1, 2\}, \{1, 3\} or the like—another example shows that there may not be any partitional NEE. However we prove that there always exists a NEE (in which the sender possibly mixes over his messages) if the decision set is convex and the utility functions are affine (Proposition 9).

Here is a more detailed description of the paper. We discuss the related literature below. In Sect. 2, we make the sender–receiver game \( \Gamma \) and the solution concept fully precise. Propositions 3 and 4 (which are established in Sect. 6.1) allow us to argue that the game \( \Gamma \) is relevant to our study. In Sect. 3, we establish existence of a partitional NEE in \( \Gamma \) under the various assumptions above. Section 4 is devoted to examples. Sections 4.1 and 4.2 propose a family of kidnapping games. Section 4.1 illustrates partitional NEE. Section 4.2 proposes a game that has a NEE but no partitional NEE. Section 4.3 goes on with a game that does not have any NEE (i.e., any no exit Nash equilibrium) but has a no exit mediated equilibrium. The topic of Sect. 5 is the existence of possibly nonpartitional and mediated equilibria, in particular in the three type case. Section 6 is an appendix containing the proofs of Propositions 3, 4 (Sect. 6.1) and 9 (Sect. 6.2).

Related papers

Shimizu (2017) adds an approval stage to Crawford and Sobel’s (1982) sender–receiver game, in which types and decisions belong to a real interval. The setup is thus quite different from ours, in which types belong to a finite set and actions can be multidimensional. Shimizu (2017) shows that, specially in the case of uniform prior and quadratic utility functions, credible exit possibilities can make cheap talk informative even when the players’ conflict of interest is relatively large. To this aim, he assumes, as we do, that exit is damaging for the receiver. He insists on (but does not restrict to) “no exit equilibria”. Shimizu (2013) makes similar insights in the case where the receiver’s decision is binary.

Matthews (1989) studies a sender–receiver game motivated by a specific application, in which the sender is the U.S. President, the receiver is the Congress and the decision is about a practical matter, like the level of military expenditures. The President can veto the Congress’ proposal. Preferences are unimodal, as in Shimizu (2013, 2017), but the receiver’s utility does not depend on the sender’s type (as in the current paper, Theorem 8, Sect. 3.4). More importantly, in Matthews’ (1989) model, the sender’s rejection leads to status quo, rather than to exit, and does not necessarily yield a very low utility to the receiver. Matthews’ (1989) point is to show that thanks to incomplete information on the President’s type, veto can happen at equilibrium, i.e., without relying on uncredible threats.

Our model can be viewed as a principal-agent problem in which the principal—alias the receiver—cannot commit to a mechanism at the ex ante stage. This is an extreme case of Bester and Strausz’s (2001) principal-agent problem with limited commitment. In this context, it makes sense to allow the agent—alias the sender—to veto the principal’s decision. Under a mechanism design perspective, the principal looks for an equilibrium that gives him the best ex ante expected utility, which amounts to an equilibrium in which all types accept the principal’s proposal if the principal’s utility, when the agent chooses his outside option, is sufficiently low. This leads us to impose individual rationality conditions for the agent at the “posterior” stage, i.e., after
the principal has made a proposal. The relevance of posterior individual rationality and its impact on incentive compatibility have been stressed in a number of papers, e.g., Gresik (1991), Compte and Jehiel (2007, 2009) Forges (1990, 1999) and Matthews and Postlewaite (1989).

Finally, Forges and Horst’s (2018) concept “talk and cooperate (perfect Bayesian) equilibrium” (TCE, Sect. 5.3) is motivated by the same questions as the present paper, but is defined in a different model: the sender also has to make a decision, which is relevant to his own payoff only. At a TCE, the receiver (who can be interpreted as a principal) proposes a joint decision, which the sender accepts whatever his type. Should player 1 reject player 2’s proposal, both players would choose an action, independently of each other. By contrast, in the present paper, the sender just chooses an outside option. Forges and Horst (2018) establish an existence result for another solution concept—“cooperate and talk (perfect Bayesian) equilibrium” (CTE)—by relying on tools introduced to study the equilibria of infinitely repeated games by Aumann et al. (1968), Sorin (1983), Simon et al. (1995) and Renault (2000). In the current paper, related tools only appear in the proof of Proposition 9, but, as we suggest in concluding remarks, could be useful to extend our results.

2 Model

2.1 Sender–receiver games

We start with a family of games \( \Gamma(v_0) \), \( v_0 \in \mathbb{R} \), between a sender (player 1) and a receiver (player 2). \( \Gamma(v_0) \) is described as follows:

- A type \( k \in K \) is chosen according to a prior probability \( p \in \Delta(K) \).
- The sender is informed of \( k \) and sends a message \( m \in M \) to the receiver.
- The receiver proposes a decision \( x \in X \) to the sender.
- If the sender accepts the proposal, the decision \( x \) is enforced, the sender gets \( U^k(x) \) and the receiver gets \( V^k(x) \).
- If the sender rejects the proposal, he chooses an outside option and gets \( u^k_0 \). The receiver gets \( v_0 \).

We assume that:

- The set of types \( K \) is finite \(^2\) and \( p^k > 0 \ \forall k \in K \).
- The set of messages \( M \) is finite, such that \( |M| \geq |K| \).
- The set of decisions \( X \) is compact. \(^3\)
- The utility functions \( U^k : X \to \mathbb{R} \) and \( V^k : X \to \mathbb{R} \) are continuous.

We further assume that:
- For every \( k \in K \), there exists \( x \in X \) such that \( U^k(x) \geq u^k_0 \).

\(^2\) We do not make any assumption beyond the fact that there are finitely many types; in particular, types can be “multidimensional,” with \( K \subset \mathbb{R}^{n_1} \), for some \( n_1 \geq 1 \).

\(^3\) As a typical example, \( X \subset \mathbb{R}^{n_2} \), for some \( n_2 \geq 1 \); for instance, player 2 has a finite set of actions \( A \) and \( X = \Delta(A) \) corresponds to the set of mixed strategies of player 2. In the latter case, observe that, according to the timing of \( \Gamma(v_0) \), the proposal to player 1 is \( x \in X \), as opposed to \( a \in A \) selected according to \( x \).
For every \( k \in K \), for every \( x \in X \), \( V^k(x) \geq v_0 \), namely,
\[
v_0 \leq \min_{k \in K} \min_{x \in X} V^k(x).
\]

We are interested in situations in which the sender’s approval is crucial to the receiver, namely, in which \( v_0 \) can be arbitrarily low. Let us denote as \( \Gamma(v_0) \) the “limit game,” in which \( v_0 = -\infty \). We will show that \( \Gamma(v_0) \) is a tractable tool, which is appropriate to study \( \Gamma(v_0) \) when \( v_0 \) is small enough.

Let us set, for every \( L \subseteq K \)
\[
X(L) = \{ x \in X : U^k(x) \geq u_0^k \text{ for every } k \in L \}. \tag{1}
\]

Given a subset of types \( L \), \( X(L) \) is the set of decisions that are acceptable by all types in \( L \). We write \( X(k) \) for \( X(\{k\}) \), so that \( X(L) = \bigcap_{k \in L} X(k) \).

### 2.2 Equilibria

Our solution concept, in \( \Gamma(v_0) \) and \( \Gamma \), is basically subgame perfect Nash equilibrium, but perfect Bayesian equilibrium would not be more demanding: as in standard sender–receiver games, finding beliefs rationalizing player 2’s choices is not an issue. What is crucial here is to avoid noncredible threats from player 1.

At a subgame perfect Nash equilibrium, player 1 of type \( k \) accepts (resp., rejects) player 2’s proposal \( x \) when \( U^k(x) > u_0^k \) (resp., \( U^k(x) < u_0^k \)). We further assume that player 1 accepts the proposal when he is indifferent, which is consistent with our interest in situations in which player 2 strictly prefers that player 1 participates. By proceeding backwards, \( \Gamma(v_0) \) amounts to a standard sender–receiver game, with the following utility functions (in which \( I \) denotes the indicator function):
\[
U^k_+(x) = U^k(x)I(U^k(x) \geq u_0^k) + u_0^kI(U^k(x) < u_0^k) = \max\{U^k(x), u_0^k\} \tag{2}
\]
for player 1 of type \( k \) and
\[
W^k(v_0, x) = V^k(x)I(U^k(x) \geq u_0^k) + v_0I(U^k(x) < u_0^k) \tag{3}
\]
for player 2, when player 1 is of type \( k \).\(^4\) In the latter sender–receiver game, the receiver’s utility function is not necessarily continuous, but it is upper-semi-continuous.

**Lemma 1** For every \( k \in K \) and \( v_0 \in \mathbb{R} \), the utility function \( W^k(v_0, \cdot) \) defined by (3) is upper-semi-continuous.

**Proof** Let \( x_n \in X, x_n \to x \). The only possibly delicate case is when \( U^k(x_n) < u_0^k \) for every \( n \) and \( U^k(x) = u_0^k \). Then \( W^k(v_0, x_n) = v_0 \leq V^k(x) = W^k(v_0, x) \), using our assumption. \( \square \)

\(^4\) This observation is made in Chen et al. (2008), in their account of Matthews (1989).
Subgame perfectness being taken into account by player 1’s behavior at the approval stage, we simply define a strategy for player 1 (in $\Gamma(v_0)$ and $\Gamma$) as a mapping $\sigma : K \rightarrow \Delta(M)$. We interpret $\sigma(k)(m)$ as the probability that player 1 sends message $m$ when his type is $k$, and denote it as $\sigma(m \mid k)$. We adopt the following notations:

For every $m \in M$, $P_{\sigma}(m) = \sum_k p_k^k \sigma(m \mid k)$. \hfill (4)

For every $k \in K$ and $m \in M$ s.t. $P_{\sigma}(m) > 0$, $p_m^k(\sigma) = \frac{p_k^k \sigma(m \mid k)}{P_{\sigma}(m)}$. \hfill (5)

$p_m^k(\sigma)$ is thus the posterior probability of type $k$ computed from $p$ and $\sigma$; let $p_m(\sigma) = (p_m^k(\sigma))_{k \in K}$ denote the corresponding posterior probability distribution over $K$. We have $\sum_m p_m(\sigma) p_m(\sigma) = p$. \hfill (6)

We say that $\sigma$ is nonrevealing if player 1 sends his message in a type-independent way, namely, if $\sigma(m \mid k) = \sigma(m \mid k')$ for every $m \in M$, $k, k' \in K$. In this case, $p_m(\sigma) = p$ for every $m$ s.t. $P_{\sigma}(m) > 0$.

For player 2, a strategy is a mapping $\tau : M \rightarrow X$.\hfill (5)

We say that $(\sigma, \tau)$ is a “no exit equilibrium” (henceforth, NEE) if

$$U^k(\tau(m)) \geq u_0^k \quad \forall k \in K, \forall m \in M \text{ s.t. } \sigma(m \mid k) > 0,$$ \hfill (6)

namely, if

$$U^k_+(\tau(m)) = U^k(\tau(m)) \quad \forall k \in K, \forall m \in M \text{ s.t. } \sigma(m \mid k) > 0.$$

Recalling (1) and denoting by supp $q$ the support of a probability distribution $q \in \Delta(K)$, condition (6) is equivalent to

$$\tau(m) \in X(\text{supp } p_m(\sigma)) \quad \forall m \in M \text{ s.t. } P_{\sigma}(m) > 0,$$ \hfill (7)

which obviously implies that

$$X(\text{supp } p_m(\sigma)) \neq \emptyset.$$ \hfill (8)

Let us set, for every $q \in \Delta(K)$,

$$Y(q) = \arg \max_{x \in X(\text{supp } q)} \sum_k q^k V^k(x).$$ \hfill (9)

In the limit game $\Gamma$, we focus on equilibria in which player 2’s expected utility is finite ($> -\infty$), namely, on NEE. The conditions for $(\sigma, \tau)$ to be a NEE are simpler in $\Gamma$ than in $\Gamma(v_0)$:

$$U^k(\tau(m)) \geq U^k(\tau(m')) \quad \forall k \in K, \forall m \in M \text{ s.t. } \sigma(m \mid k) > 0, \forall m' \in M$$ \hfill (10)

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\footnote{Pure strategies of player 2 (with respect to the set $X$) will suffice for our existence results. As indicated above, $X$ can be a set of lotteries.}
and \[ \tau(m) \in Y(p_m(\sigma)) \forall m \in M \text{ s.t. } P_\sigma(m) > 0. \] (11)

Condition (10) is player 1’s incentive compatibility condition. Condition (11) is player 2’s best reply condition at every posterior belief \( p_m(\sigma) \), i.e., given every message \( m \) of positive probability. We refer to the latter condition as to “constrained optimization”.

Notice that, under (11), \( U_k^*(\tau(m)) \geq u_{0k}^* \) for every \( k \in K, m \in M \) such that \( \sigma(m \mid k) > 0 \), so that (10) holds as well with \( U_k^*(\tau(m')) \) in the right hand sight of the inequalities:

\[ U_k^*(\tau(m)) \geq U_k^*(\tau(m')) \forall k \in K, \forall m \in M \text{ s.t. } \sigma(m \mid k) > 0, \forall m' \in M. \] (12)

**Proposition 2** For every \( v_0 \in \mathbb{R} \), the game \( \Gamma(v_0) \) has a nonrevealing equilibrium (possibly with exit). The game \( \Gamma \) has a nonrevealing NEE if and only if \( X(K) \neq \emptyset \). Hence \( \Gamma \) may not have any nonrevealing NEE.

**Proof** The following strategies define a nonrevealing equilibrium in \( \Gamma(v_0) \): player 1 sends the same message \( m \in M \) whatever his type and then accepts \( x \) if and only if \( U_k^*(x) \geq u_{0k}^* \); whatever the message, player 2 chooses \( x^* \in X \) to maximize \( \sum_k p^k W_k(v_0, x) \), which is well-defined thanks to Lemma 1.

In \( \Gamma \), if \( X(K) \neq \emptyset \), a nonrevealing NEE can be achieved as above, provided that player 2 chooses \( x^* \in X \) to maximize \( \sum_k p^k V_k(x) \) subject to \( x \in X(K) \). If \( X(K) = \emptyset \) and player 1’s message is type-independent, condition (7) cannot be satisfied. \( \square \)

The next two propositions (which are established in Sect. 6.1) give us some foundations to study the NEE of the limit game \( \Gamma \) by making precise relationships between the latter and the NEE of \( \Gamma(v_0) \).

**Proposition 3** Let \((\sigma, \tau)\) be a NEE in \( \Gamma(v_0) \), for some \( v_0 \in \mathbb{R} \). Then \((\sigma, \tau)\) is a NEE in \( \Gamma(z_0) \) for every \( z_0 \in \mathbb{R} \) such that \( z_0 \leq v_0 \) and is also a NEE in \( \Gamma \), with the same interim expected utility as in \( \Gamma(v_0) \) for both players.

In other words, if \( \Gamma \) has no NEE (which indeed may happen, see Sect. 4.3), then, whatever \( v_0 \in \mathbb{R} \), \( \Gamma(v_0) \) has no NEE, that is, all equilibria of \( \Gamma(v_0) \) must involve non-participation of at least one type.

**Proposition 4** Let \((\sigma, \tau)\) be a NEE in \( \Gamma \). Then there exists \( v_0 \in \mathbb{R} \) such that, for every \( z_0 \leq v_0 \), \((\sigma, \tau)\) is a NEE of \( \Gamma(z_0) \), with the same interim expected utility as in \( \Gamma \) for both players.

The previous properties are useful under a mechanism design perspective. Assume player 2 is a “principal” who cannot commit to a mechanism \( \mu : K \rightarrow X \) but receives a message from the agent (player 1) and then, makes a decision in \( X \) subject to the agent’s participation constraints. With this interpretation, which turns out to be an extreme case of Bester and Strausz’s (2001) model, an optimal mechanism amounts to an equilibrium of \( \Gamma(v_0) \) which maximizes player 2’s ex ante expected utility. At a given \( v_0 \), the equilibrium of \( \Gamma(v_0) \) that is best for player 2 may involve the exit of some types of player 1. By contrast, player 2’s best equilibrium payoff in the limit game
Γ, when it exists, is achieved at an equilibrium \((σ^*, τ^*)\) without exit. Let us denote player 2’s corresponding payoff as \(v^*_{NE}\). By Proposition 4, if \(v_0\) is small enough, \((σ^*, τ^*)\) is an equilibrium without exit in \(Γ(v_0)\), giving the same expected utility \(v^*_{NE}\) to player 2. By Proposition 3, \(v^*_{NE}\) is the best equilibrium payoff player 2 can achieve at an equilibrium without exit in \(Γ(v_0)\). Moreover, as we show in detail in Sect. 6.1, if \(v_0\) is sufficiently low, player 2 cannot expect an expected utility higher than \(v^*_{NE}\) at an equilibrium of \(Γ(v_0)\) with exit of some types.\(^6\) Summing up, if the limit game \(Γ\) has a NEE, an optimal mechanism, when the principal’s utility \(v_0\) in case of exit is sufficiently low, can be found by maximizing player 2’s utility over the equilibria of \(Γ\), without worrying about the precise level \(v_0\).

### 3 Existence of partitional NEE

Henceforth, we focus on the game \(Γ\). In this section, we identify various sufficient conditions for the existence of a partitional NEE in \(Γ\), in which player 1 uses a pure strategy, namely a mapping \(σ : K → M\), to send his message to player 2. The strategy \(σ\) then induces the partition \(\{K_m, m ∈ σ(K)\}\) of \(K\), with \(K_m = σ^{-1}(m) = \{k ∈ K : σ(k) = m\}\). In this case, (4) and (5) become respectively:

\[
\text{For every } m ∈ M, \quad P_σ(m) = \sum_{k ∈ K_m} p^k. \tag{13}
\]

\[
\text{For every } k ∈ K \text{ and } m ∈ M \text{ s.t. } P_σ(m) > 0, \quad p^k_m(σ) = \frac{p^k I(k ∈ K_m)}{P_σ(m)}. \tag{14}
\]

As in Sect. 2, for player 2, we focus on strategies of the form \(τ : M → X\). At a partitional equilibrium of \(Γ\), player 2’s equilibrium condition is (11), with \(p_m(σ)\) defined by (14).

Player 1’s equilibrium conditions reduce to incentive compatibility conditions expressing that given player 2’s strategy \(τ\), player 1 of type \(k\) prefers the message \(σ(k)\) to any other message \(m\), namely,

\[
U^k(τ(σ(k))) \geq U^k(τ(m)) \quad \text{for every } k ∈ K \text{ and } m ∈ M. \tag{15}
\]

#### 3.1 Two types

If only two types are possible, we show that either there is a decision giving both types at least their reservation utility or full revelation of information is credible and allows to avoid exit.

**Proposition 5** Let us assume that \(|K| = 2\). Then \(Γ\) has a partitional NEE.

\(^6\) The observation that the principal’s ex ante expected utility is maximized when all types of the agent participate is also made in Bester and Strausz (2001), footnote 8.
Proof If $X(1) \cap X(2) \neq \emptyset$, let

$$x^* \in \arg\max_{x \in X(1) \cap X(2)} \left[ p_1 V^1(x) + p_2 V^2(x) \right]$$

and let $m^*$ be an arbitrary element of $M$. Then $\sigma(1) = \sigma(2) = m^*$ and $\tau(m) = x^*$ for every $m \in M$ defines a nonrevealing NEE of $\Gamma$.

Otherwise, if $X(1) \cap X(2) = \emptyset$, let

$$x_k \in \arg\max_{x \in X(k)} V^k(x), \ k = 1, 2$$

and let $m_1 \neq m_2$ be two distinct elements of $M$. Then $\sigma(k) = m_k$, $\tau(m_k) = x_k$, $k = 1, 2$, defines a fully revealing NEE of $\Gamma$. Indeed, constrained optimization (11) holds by construction; to see that incentive compatibility (15) also holds, observe that $x_k \notin X(\ell)$ for $\ell \neq k$, since $X(1) \cap X(2) = \emptyset$. In other words, $U^\ell(x_k) < u_0^k \leq U^\ell(x_\ell)$ for $\ell \neq k$. \qed

3.2 Straightforward partitional NEE

In this section, we propose an easy generalization of Proposition 5 when the sender has an arbitrary number of types. Recalling the definition of $X(L)$ (see (1)), let us set

$$T = \{ \emptyset \neq L \subseteq K : X(L) \neq \emptyset \}$$

and define $T^*$ as the set of maximal elements of $T$ for set inclusion, namely,

$$T^* = \{ L \in T : [L' \in T \text{ and } L \subseteq L'] \Rightarrow L' = L \}.$$  \hspace{1cm} (16)

We refer to $T^*$ as to the “participation structure” of the game $\Gamma$.

Proposition 6 If the participation structure of $\Gamma$ is a partition of $K$, $\Gamma$ has a partitional NEE.

Proof Let $T^* = \{ K_r \}$. Consider the strategy of player 1 consisting of revealing the cell $K_r$ containing his type. Let $x^*_r \in X(K_r)$ be an optimal decision of player 2 when he learns that player 1’s type belongs $K_r$, namely,

$$x^*_r \in \arg\max_{x \in X(K_r)} \left( \sum_{k \in K_r} p_k \sum_{j \in K_r} p_j V^k(x) \right).$$

Constrained optimization (11) holds by construction. Incentive compatibility (15) is also immediate, because if $k \in K_r$, $x^*_r \in X(K_r)$ while for $j \neq r$, $x^*_j \notin X(K_r)$. \qed
3.3 Unidimensional decisions and monotonic utility functions for the sender

The next result holds in particular when player 2's decision can be interpreted as a probability distribution over two possible actions (i.e., $X = \Delta(A), |A| = 2$) and the utility $U^k(x)$ of player 1 of type $k$ is expected utility with respect to $x$.

**Proposition 7** Let us assume that the decision set $X$ is a compact subset of the real line and every utility function $U^k$, $k \in K$, is monotonic over $X$. Then $\Gamma$ has a partitional NEE.

**Proof** Write $a = \min X$ and $b = \max X$, we have $X \subseteq [a, b]$. Define the following partition of $K$:

$K_- = \{ k \in K : U^k$ is weakly decreasing and not constant $\}$

$K_+ = \{ k \in K : U^k$ is weakly increasing or constant $\}$

For $k \in K_-$, define $x_k^0 = \max\{ x \in X, U^k(x) \geq u_k^0 \} \in X$ and for $k \in K_+$ define $x_k^0 = \min\{ x \in X, U^k(x) \geq u_k^0 \} \in X$. For all $x$ in $X$ we have: if $k \in K_-$, $U^k(x) \geq u_k^0 \iff x \leq x_k^0$ and if $k \in K_+$, $U^k(x) \geq u_k^0 \iff x \geq x_k^0$. We define next $x_-$ and $x_+$ in $X$ by:

$x_- = \min_{k \in K_-} x_k^0$ if $K_- \neq \emptyset$; $x_- = b$ if $K_- = \emptyset$.

$x_+ = \max_{k \in K_+} x_k^0$ if $K_+ \neq \emptyset$; $x_+ = a$ if $K_+ = \emptyset$.

(1) If $x_+ \leq x_-$(in particular, if $K_- \text{ or } K_+ = \emptyset$), let

$x^* \in \arg \max_{x \in [x_-, x_-] \cap X} \sum_{k \in K} p^k V^k(x)$

and let $m^*$ be an arbitrary element of $M$. Then $\sigma(k) = m^*$ for every $k \in K$ and $\tau(m) = x^*$ for every $m \in M$ defines a nonrevealing NEE of $\Gamma$.

(2) If $x_+ > x_-$, then both $K_-$ and $K_+$ are non empty. Let $m^*_- \neq m^*_+$ be two distinct elements of $M$. Take $\sigma(k) = m^*_-$ if $k \in K_-$, $\sigma(k) = m^*_+$ if $k \in K_+$, namely, $\sigma$ induces the partition $\{K_-, K_+\}$. Player 2's corresponding posterior probability distribution on $K$ can be computed as in (14):

$p^k_{m^*_-} = \frac{p^k I(k \in K_-)}{\sum_{j \in K_-} p^j}$ and $p^k_{m^*_+} = \frac{p^k I(k \in K_+)}{\sum_{j \in K_+} p^j}$.

Let then

$x^*_- \in \arg \max_{x \leq x_-, x \in X} \sum_{k \in K_-} p^k_{m^*_-} V^k(x)$ and $x^*_+ \in \arg \max_{x \geq x_+, x \in X} \sum_{k \in K_+} p^k_{m^*_+} V^k(x)$. 
Constrained optimization (11) holds by construction. There remains to check incentive compatibility (15). Observe that $x_\pm^* < x_\pm^+$; for $k \in K_-$, $U^k$ is decreasing, hence $U^k(x_\pm^*) \geq U^k(x_\pm^+)$. Similarly for $k \in K_+$, $U^k$ is increasing so that $U^k(x_\pm^*) \leq U^k(x_\pm^+)$. \hfill \Box

3.4 Type-independent utility function for the receiver

In this section, we assume that, when the sender accepts the receiver’s proposal, the utility function of the receiver does not depend on the sender’s type, namely, that $V^k(x) = V(x)$ for every $k$ and $x$. This assumption is sometimes referred to as “private values” or “known-own payoff.” Matthews (1989) formulates it in the context of a game of information transmission with sender’s approval.

Theorem 8 Let us assume that player 2’s utility function does not depend on player 1’s type, namely, that there exists a continuous function $V : X \rightarrow \mathbb{R}$ such that $V^k = V$ for every $k \in K$. Then $\Gamma$ has a partitional NEE.

The proof consists of an algorithm, which constructs a partitional NEE that is as revealing as possible, given the incentive compatibility constraints to be fulfilled. More precisely, we start with player 2’s constrained best reply to full revelation by player 1. Imagine that type $k$ would envy type $\ell$ if one tried to implement these fully revealing strategies, while type $\ell$ would not envy any type. By merging type $\ell$ and type $k$, one reduces the incentives problem. A key property is that, if player 2’s utility function is independent of player 1’s type, then player 2’s optimal decision $x_\ell$ when facing type $\ell$ remains optimal when facing type $\ell$ or type $k$. Before making use of this property, we first show, by relying on the same kind of argument, that the envy relation cannot have any cycle.

**Proof** Let us fix, for every $k \in K$,

$$x_k \in \arg \max_{x \in X(k)} V(x). \quad (17)$$

The existence of such $x_k$’s is guaranteed by our assumptions. If the previous optimization problem has several solutions, we take $x_k$ to maximize $U^k(x)$.

For every pair of types $j, k \in K$, we say that “type $k$ envies type $j$”—and write $kRj$—if $U^k(x_j) > U^k(x_k)$. An immediate property is that

$$\text{for every } j, k \in K, kRj \Rightarrow V(x_k) > V(x_j). \quad (18)$$

To show this, observe that, by definition, $x_k \in X(k)$, i.e., $U^k(x_k) \geq u_0^k$. Hence, if $kRj$, we must have $U^k(x_j) > u_0^k$, which implies $x_j \in X(k)$ (so that $x_j \in X(j) \cap X(k)$) and $V(x_k) \geq V(x_j)$. But $V(x_k) = V(x_j)$ cannot arise, because $U^k(x_j) > U^k(x_k)$ and, in case of multiple solutions to $\max_{x \in X(k)} V(x)$, we choose $x_k$ to maximize $U^k(x)$.

The previous property implies that the envy relation $R$ has no cycle.
We will gradually construct a subset \( L \subseteq K \) of leader types which do not envy any other type in \( L \) and a subset \( F = K \setminus L \) of follower types which envy a type in \( L \).\(^7\)

We start with \( L = F = \emptyset \). Let us denote as \( \alpha_1 < \cdots < \alpha_n \) the distinct values among \( V(x_k), k \in K \). Necessarily, \( n \leq |K| \). Define then

\[
K_j = \{ k \in K : V(x_k) = \alpha_j \} \quad j = 1, \ldots, n. \tag{19}
\]

**Step 1** Consider every type \( k \in K_1 \): \( V(x_k) = \alpha_1 \) is strictly below any other \( \alpha_j \). By (18), type \( k \) cannot envy any other type. We modify \( L \) into \( L = K_1 \), while \( F \) does not change (\( F = \emptyset \)).

**Step 2** Consider every type \( k \in K_2 \). If \( k \) does not envy any type, put \( k \) in \( L \). Otherwise, again by (18), \( k \) can only envy a type in \( L \) (as defined at the end of step 1, namely, \( K_1 \)), put \( k \) in \( F \).

\[ \ldots \]

**Step \( j \)** Let \( L \) and \( F \) be the sets of leaders and followers constructed so far. \( L \cup F = K_1 \cup \cdots \cup K_{j-1} \) so that by (18) and (19), types in \( L \cup F \) cannot envy types in \( K_j \). Consider every such type \( k \in K_j \). If \( k \) envies a type in \( L \), put \( k \) in \( F \). Otherwise, put \( k \) in \( L \). \( L \) and \( F \) are thus updated at the end of step \( j \).

\[ \ldots \]

**Step \( n \)** Proceed as for step \( j \). Deduce the final sets of leaders and followers.

For instance, if \(|K| = 3\) and \( R \) is fully described by \( 3R2 \) and \( 2R1 \), the previous construction results in \( K_1 = \{1\}, K_2 = \{2\}, K_3 = \{3\}, L = \{1, 3\} \).

Using the \( x_\ell \)'s defined by (17) and the set \( L \), we construct a NEE \((\sigma, \tau)\) of \( \Gamma \). For simplicity, we rename the messages in \( M \) so that \( L \subseteq M \). Player 1’s strategy is such that \( \sigma(K) = L \). More precisely, \( \sigma : K \rightarrow L \) is defined by

\[
\sigma(k) = \begin{cases} 
    k & \text{if } k \in L \\
    \arg \max_{j \in L, k \in R_j} U^k(x_j) & \text{if } k \in K \setminus L.
\end{cases}
\]

In other words, leader types announce themselves, while non leader types report the leader type they most envy. Player 2’s strategy is defined by \( \tau : L \rightarrow X : \tau(\ell) = x_\ell \), with \( x_\ell \) defined by (17).

Incentive compatibility (15) follows from the fact that player 2’s strategy \( \tau \) restricts his decisions to the subset \( \{x_\ell, \ell \in L\} \). Hence types in \( L \), who cannot envy any other type in \( L \), are truthful. Types in \( K \setminus L \) behave as well as they can given the player 2’s restricted decision set.

If player 1 follows \( \sigma \), then, given message \( \ell \in L \), player 2 deduces that player 1’s type \( k \in \sigma^{-1}(\ell) \). The set \( \sigma^{-1}(\ell) \) contains \( \ell, x_\ell \in X(\ell) \) by (17) and all other types in \( \sigma^{-1}(\ell) \) envy \( \ell \), so that \( x_\ell \in \bigcap_{k \in \sigma^{-1}(\ell)} X(k) \). Since \( x_\ell \) is a maximizer of \( V(x) \) over \( X(\ell) \), it is also a maximizer of \( V(x) \) over \( \bigcap_{k \in \sigma^{-1}(\ell)} X(k) \). \( \square \)

\(^7\) This part of the proof can be derived from a property going back to von Neumann and Morgenstern (1944), namely, that the kernel of an acyclic relation is nonempty (see also Richardson 1953).
Remarks

– A main feature of the proof of Theorem 8 is that, in the partitional NEE that is constructed, the receiver makes a decision in a subset of \( \{x_k, k \in K\} \) where \( x_k \) is the optimal decision he would make if he were sure to face type \( k \). The receiver’s private values guarantee that if type \( k \) envies type \( \ell \), then \( x_\ell \), the receiver’s optimal choice when he faces type \( \ell \) for sure (i.e., under the constraint \( x \in X(\ell) \)), is still optimal when he faces type \( k \) or type \( \ell \) (i.e., under the constraint \( x \in X(\ell) \cap X(k) \)). This property may no longer hold when player 2’s utility is type-dependent.

– Theorem 8 does not depend on the underlying utility representation: the result holds if the receiver’s von Neumann–Morgenstern preferences over \( X \) given type \( k \) are equivalent for every \( k \in K \).

4 Examples (including a counter-example)

4.1 Partitional NEE

Let the informed player have three possible types, i.e., \( K = \{1, 2, 3\} \) and let the uninformed player’s decision set be

\[
X = \{(x_a, x_b) : x_a \geq 0, x_b \geq 0, x_a + x_b \leq 100\}.
\]

Let the utility function and reservation utility of the informed player be

\[
U^1(x) = x_a - x_b \quad u^1_0 = 30, \\
U^2(x) = x_b - x_a \quad u^2_0 = 40, \\
U^3(x) = x_a + 2x_b \quad u^3_0 = 20.
\]

Let the uninformed player’s utility function be type-independent:

\[
V^k(x) = V(x) = -(x_a + x_b), \quad k = 1, 2, 3.
\]

There are two goods, \( a \) and \( b \), \( X \) accounts for the decision-maker’s resource constraints. Type 1 likes good \( a \), dislikes good \( b \); type 2 has symmetric preferences; type 3 likes both goods, and likes good \( b \) more than good \( a \).

As a possible interpretation, player 1 is a kidnapper who can either have political motivations (type 1), or just look for a monetary ransom (type 2) or be opportunistic (type 3). Good \( a \) stands for political prisoners who can be released while good \( b \) stands for money. The rationale for the preferences above is that idealist type 1 feels insulted when receiving money, while type 2’s political views are opposite to those of type 1. In any case, if player 1 does not accept player 2’s offer, the hostage is killed, leading to an invaluable loss for player 2.

Recalling (1) and using “\( Co \)” for convex hull, we have here

\[
X(\{1\}) = X([1, 3]) = Co \{(30, 0), (100, 0), (65, 35)\},
\]
Let us modify the uninformed player’s utility function in the previous example, to make it depend on the informed player’s type:

\[
\begin{align*}
V^1(x) &= \frac{x_a}{3}, \\
V^2(x) &= \frac{x_b}{3}, \\
V^3(x) &= -(x_a + x_b).
\end{align*}
\]

A possible interpretation is that the decision-maker sympathizes with the kidnapper when the latter has sharp preferences: should he face type 1, he would share his political views and be happy to support them; should he face type 2, he would understand his need for money; but in front of an opportunistic type, he is only affected by the amount he has to pay.
Let as above $x_k^*$ denote the uninformed player’s optimal decision (in $X$) when he faces type $k$; we have now

$$x_1^* = (100, 0), x_2^* = (0, 100), x_3^* = (0, 10).$$

(21)

Let us take $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. There is no nonrevealing NEE, since $X(\{1, 2, 3\}) = \emptyset$. There is no completely revealing NEE: given (21), type 3 would pretend to be type 2. More generally, there is no partitional NEE. Given the above description of the sets $X(L)$, two possible partitions must still be considered: $\{\{1\}, \{2, 3\}\}$ and $\{\{1, 3\}, \{2\}\}$. \{\{1\}, \{2, 3\}\}: if the uninformed player believes he faces type 1 (posterior $(1, 0, 0)$), his optimal choice is $x_1^* = (100, 0)$; if he believes he faces type 2 or type 3 (posterior $(0, \frac{1}{2}, \frac{1}{2})$), his optimal choice is

$$x_{23}^* = \arg \min_{x \in X(\{2, 3\})} \left[ x_a + \frac{2}{3}x_b \right] = (0, 40).$$

This cannot be incentive compatible for type 3:

$$100 = U^3(x_1^*) > U^3(x_{23}^*) = 80.$$

$\{\{1, 3\}, \{2\}\}$: if the uninformed player believes he faces type 2 (posterior $(0, 1, 0)$), his optimal choice is $x_2^* = (0, 100)$; if he believes he faces type 1 or type 3 (posterior $(\frac{1}{2}, 0, \frac{1}{2})$), his optimal choice is

$$x_{13}^* = \arg \min_{x \in X(\{1, 3\})} \left[ \frac{2}{3}x_a + x_b \right] = (100, 0).$$

Again, this cannot be incentive compatible for type 3:

$$200 = U^3(x_2^*) > U^3(x_{13}^*) = 100.$$

This illustrates that Theorem 8 does not extend to the case where player 2’s utility function depends on player 1’s type.

Let us show that if player 1 uses a mixed strategy to transmit information, a partially revealing NEE exists in this example: type 1 reports that his type belongs to $\{1, 3\}$, type 2 reports that his type belongs to $\{2, 3\}$, type 3 reports that his type belongs to $\{1, 3\}$ (resp., $\{2, 3\}$) with probability $\frac{1}{3}$ (resp., $\frac{2}{3}$). If the informed player follows this reporting strategy, the uninformed player’s posterior upon receiving $\{1, 3\}$ is $(\frac{3}{4}, 0, \frac{1}{4})$ while upon receiving $\{2, 3\}$, it is $(0, \frac{3}{4}, \frac{1}{4})$. Given $\{1, 3\}$, the uninformed player’s problem reduces to $\min_{x \in X(\{1, 3\})} x_b$. Every $x = (x_a, 0)$ with $x_a \in [30, 100]$ is optimal. Let us take $x_{13}^* = (80, 0)$. Given $\{2, 3\}$, the uninformed player’s optimal choice is

$$x_{23}^* = \arg \min_{x \in X(\{2, 3\})} [2x_a + x_b] = (0, 40).$$
There remains to check incentive compatibility. Type 1 prefers \( x_{13}^* = (80, 0) \) to \( x_{23}^* = (0, 40) \), and vice-versa for type 2. Type 3 must be indifferent between sending \( \{1, 3\} \) or \( \{2, 3\} \), because he must randomize between these two outcomes. Indeed we have \( U^3(x_{13}^*) = U^3(x_{23}^*) = 80 \). Proposition 9 in Sect. 5 states that the previous construction can be generalized.

4.3 No NEE at all

In the following example, none of the existence results of Sect. 3 can be applied. We will show that there is no NEE, even if player 1 can transmit information with the help of a mixed strategy. The game is described by:

- \(|K| = 3\), a prior \( p = (p^1, p^2, p^3) \), \( X = \Delta(A) \), where \( A = \{a, b, c\} \), and \( u^k_0 = 0 \), \( k = 1, 2, 3 \).
- The following payoff matrices describe \((U^k(\alpha), V^k(\alpha))\) for every \( \alpha \in A \):

|   | a  | b  | c  |
|---|----|----|----|
| 1 | 10 | -2 | 0  |
| 2 | 21 | 10 | -2 |
| 3 | -2 | 0  | 1  |

The utility functions over \( X = \Delta(A) \) are obtained as expected utilities with respect to \( x = (x_a, x_b, x_c) \).

If \( p^k = 1 \), namely, if player 2 knows that he faces type \( k \) (for some \( k = 1, 2, 3 \)), he maximizes his utility and guarantees player 1’s approval by choosing \( a \) if \( k = 1 \), \( b \) if \( k = 2 \), \( c \) if \( k = 3 \).

Let us turn to the case \( p^k > 0 \) for every \( k \). A nonrevealing NEE requires that there is a decision \( x \in \Delta(A) \) such that \( U^k(x) \geq 0 \) for \( k = 1, 2, 3 \). However, \( \sum_k U^k(\alpha) < 0 \) for every \( \alpha \in A \), so that \( \sum_k U^k(x) < 0 \) for every \( x \in \Delta(A) \). Hence there is no nonrevealing NEE. A fully revealing NEE does not exist either. Indeed, in this case, player 2 would choose \( a \) if \( k = 1 \), \( b \) if \( k = 2 \), \( c \) if \( k = 3 \) and player 1’s incentive compatibility would be violated.

Looking for a partially revealing NEE, we first check that there is a unique, nonrevealing NEE, as soon as only two types are possible. Take, e.g., \( p^3 = 0 \). Then

\[
X(\{1, 2\}) = \{x \in X : -2x_b + x_c \geq 0 \text{ and } x_a - 2x_c \geq 0\}
= Co \left\{ (1, 0, 0), \left( \frac{2}{3}, 0, \frac{1}{3} \right), \left( \frac{4}{7}, \frac{1}{7}, \frac{2}{7} \right) \right\}
\]

where, as above, “Co” denotes the convex hull. Player 2’s optimization problem is:

\[
\max p^1(x_a + x_c) + p^2(x_a + 2x_b) \text{ s.t. } x \in X(\{1, 2\}).
\]

\[8\] The reasoning below holds, and is much simpler, if player 2’s proposal has to be in \( A \).
For every $p$ such that $p^1 > 0$ and $p^2 > 0$, the unique solution is achieved at $x = (1, 0, 0)$, namely, action $a$ with probability 1. Similarly, action $b$ is the only solution if $p^2 > 0$ and $p^3 > 0$, action $c$ is the only solution if $p^1 > 0$ and $p^3 > 0$.

Let us come back to $p$ such that $p^k > 0$ for every $k$. By sending his message according to a mixed strategy $\sigma$, player 1 “splits” the prior belief $p$ into posteriors $p_m(\sigma)$ such that $\sum_m P_\sigma (m) p_m(\sigma) = p$ (see (5)); $p$ cannot be split into $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$, because there is no fully revealing NEE. And none of the posteriors can be interior (namely, such that $p^k_m(\sigma) > 0, k = 1, 2, 3$), because there is no nonrevealing NEE at such posteriors. Hence, at least one of the posteriors $p_m$ must be on an edge, say $p_m = (p^1_m, p^2_m, 0)$ with $p^1_m > 0$ and $p^2_m > 0$; by the reasoning above, player 2’s decision when he receives message $m$ must be $\tau(m) = a$. There should be another posterior $p_m'$ such that $p^3_m > 0$ and thus $\tau(m') = b$ or $c$. To achieve the posteriors $p_m$ and $p_m'$, message $m$ must be sent with positive probability by types 1 and 2, while message $m'$ must be sent with positive probability by at least type 3. If $\tau(m') = c$, type 1 strictly prefers $m'$ to $m$. If $\tau(m') = b$, it must be that $p^2_m > 0$, so that type 2 must send $m'$ with positive probability; but type 2 strictly prefers $m$ to $m'$. Hence there is no incentive compatible splitting and thus no NEE at all, even if player 1 can use a mixed strategy.

A mediated equilibrium, in which information transmission is monitored by a mediator, can nevertheless be achieved in the previous example. Consider the following three lotteries over $A$: $\delta^1 = (\frac{1}{2}, 0, \frac{1}{2}), \delta^2 = (\frac{1}{2}, \frac{1}{2}, 0), \delta^3 = (0, \frac{1}{2}, \frac{1}{2})$. Assume that, instead of selecting a message by himself, player 1 can just choose among these three lotteries. If player 1 expects player 2 to pick the action selected by the lottery, player 1 prefers $\delta^k$ over the other two lotteries when his type is $k$. Similarly, player 2 is happy to choose the action recommended by the lottery if he believes that player 1 reveals his type truthfully to the mediator. This procedure will be generalized in the next section.

5 Mixed and mediated information transmission

In Sect. 3, we have identified conditions guaranteeing the existence of a partitional no exit (Nash) equilibrium in $\Gamma$, in which the informed player sends information by means of a pure strategy. In the examples of Sects. 4.2 and 4.3, there is no partitional NEE, but there is an equilibrium without exit if the sender uses a mixed strategy or relies on a mediator to choose his message. In this section, we propose existence results for such mixed and mediated equilibria without exit in $\Gamma$, when the informed player has three possible types. Before that, we make precise the notion of no exit mediated equilibrium in $\Gamma$. We conclude the section by suggesting possible connections with repeated games with incomplete information.
5.1 Definition of no exit mediated equilibrium

Recalling the functions $U_k^+ (\text{see (2)})$ and the sets $Y(q)$ (see (9)) introduced in Sect. 2, a no exit mediated equilibrium is a mapping $\gamma : K \rightarrow \Delta(X)$, such that $S = \bigcup_{k \in K} \text{supp } \gamma^k$ is finite, and

\[
\forall k, j \in K, \sum_{x \in S} \gamma^k(x)U^k(x) \geq \sum_{x \in S} \gamma^j(x)U^k(x) \tag{22}
\]

\[
\forall x \in S, x \in Y(p_x) \tag{23}
\]

where, in (23), $p_x \in \Delta(K)$ denotes the posterior over $K$ given $x \in S$.

The interpretation is that player 1 announces $k$ to a mediator who then selects $x \in X$ according to $\gamma^k$. At equilibrium, player 1 truthfully reveals his type to the mediator, player 2 proposes the decision $x$ that is recommended by the mediator and player 1 accepts. Player 1’s equilibrium condition (22) reflects the fact that this player can lie about his type and/or reject player 2’s proposal. This is a “veto-incentive compatibility condition” (see, e.g., Forges 1999), which implies posterior individual rationality, by taking $k = j$ in (22):

For every $k \in K$ and $x \in \text{Supp}(\gamma^k)$, $U^k(x) \geq u_0^k$. \tag{24}

A mediated equilibrium is an instance of the more general notion of communication equilibrium (see, e.g., Forges 1986).

Condition (22) is significantly weaker than player 1’s Nash equilibrium condition (i.e., (10) or (12)) since in a mediated equilibrium, player 1 does not control the randomization $\gamma^k$ performed by the mediator but just chooses between $|K|$ probability distributions $\gamma^k, k \in K$. One can easily check that, as expected, a (Nash) equilibrium $(\sigma, \tau)$, as defined in Sect. 2.2, induces a mediated equilibrium, by defining $\gamma : K \rightarrow X$ as follows: $\forall k \in K, \gamma^k$ chooses $m$ according to $\sigma^k$ and finally $x = \tau(m)$.10

5.2 The case of three types

In this section, we assume that $|K| = 3$. In this case, the participation structure $T^*$ of $\Gamma$ (see (1) and (16)) can be of the following typical forms:

1. $T^*$ is a partition of $K$.
2. $T^* = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.
3. $T^* = \{\{1, 2\}, \{1, 3\}\}$.

In case 1, by Proposition 6, $\Gamma$ has a partitional NEE. Case 2 means that player 2 is able to obtain the approval of every pair of types but cannot ensure the participation of the three types simultaneously. In this case, as illustrated in Sect. 4.3, $\Gamma$ may have no

10 As noted in Sect. 2.2, for Nash equilibrium, player 1’s incentive compatibility condition (12), with $U^k_+ (x)$ in the right hand sight, is equivalent to (10), which only uses $U^k (x)$. By contrast, for mediated equilibrium, (22) implies the condition obtained by replacing $U^k_+ (x)$ by $U^k (x)$ in the right hand sight of (22) but the reverse is not true.
NEE. We will show below that $\Gamma$ always has a mediated equilibrium without exit. In case 3, player 2 can only guarantee the approval of two pairs of types. We will establish that under further assumptions, $\Gamma$ always has a NEE (which may not be partitional, as in Sect. 4.2).

5.2.1 $T^* = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$

When $K = \{1, 2, 3\}$ and $T^* = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, we construct a mediated equilibrium in $\Gamma$, as defined by (22) and (23). Recall that $p^k > 0$ for every $k$. For every pair $(j, k)$ of types, let $x^*_jk \in X((j, k))$ be an optimal decision for player 2 when he learns that player 1 is of type $j$ or $k$, namely,

$$x^*_jk \in \arg \max \left[ \frac{p^j}{p^j + p^k} V^j(x) + \frac{p^k}{p^j + p^k} V^k(x) \right].$$ (25)

Consider the following mediator: for every $k = 1, 2, 3$, if player 1 reports type $k$, he selects $x^*_ik$ or $x^*_jk$, $i, j \neq k$, with equal probability $\frac{1}{2}$ and recommends it to player 2. If player 1 reports his type $k$ truthfully, player 2 learns, with equal probability, that player 1’s type is in $\{i, k\}$ or in $\{j, k\}$ for $i, j \neq k, i \neq j$. Condition (25) guarantees that player 2 follows the mediator’s recommendation.

For player 1, let us consider $k = 1$. By reporting his type truthfully, player 1 obtains

$$\frac{1}{2} U^1(x^*_{12}) + \frac{1}{2} U^1(x^*_{13}).$$

If he lies by, say, pretending to be of type 2, he obtains

$$U^1_2 = \frac{1}{2} \max \left\{ U^1(x^*_{12}), u_0^1 \right\} + \frac{1}{2} \max \left\{ U^1(x^*_{23}), u_0^1 \right\}.$$ By construction, $x^*_1 \in X((1, 2))$ and $x^*_3 \in X((1, 3))$. Hence $U^1(x^*_{12}) \geq u_0^1$ and $U^1(x^*_{13}) \geq u_0^1$. But $x^*_3 \notin X(1)$ because $\{2, 3\}$ is maximal. Hence, $U^1(x^*_{23}) < u_0^1$.

$$U^1_2 = \frac{1}{2} U^1(x^*_{12}) + \frac{1}{2} u_0^1 \leq \frac{1}{2} U^1(x^*_{12}) + \frac{1}{2} U^1(x^*_{13}).$$

The other incentive compatibility conditions of player 1 can be checked in a symmetric way. ☐

5.2.2 $T^* = \{\{1, 2\}, \{1, 3\}\}$

Proposition 9 Let us assume that $K = \{1, 2, 3\}$, the participation structure $T^* = \{\{1, 2\}, \{1, 3\}\}$, the decision set $X$ is compact and convex and the utility functions $U^k$ and $V^k$, $k \in K$, are affine. Then $\Gamma$ has a partially revealing NEE, in which player 1 sends his message using a mixed, possibly not pure, strategy.
**Proof** See Sect. 6.2. We establish that there must exist an equilibrium in which type 2 reports that his type belongs to \{1, 2\}, type 3 reports that his type belongs to \{1, 3\} and type 1 reports that his type belongs to \{1, 2\} (resp., \{1, 3\}) with some probability \(\delta \in (0, 1)\) (resp., \(1 - \delta\)). Incentive compatibility requires that type 1 be indifferent between reporting \{1, 2\} or \{1, 3\}. A similar equilibrium is shown to be the only possible one in the example of Sect. 4.2.

5.3 Concluding remarks

Let us sum up our findings on (mixed) NEE in the case where player 2 proposes a lottery over finitely many actions, namely, \(X = \Delta(A)\) and \(U^k, V^k\) are affine for every \(k \in K\). We could easily prove that a NEE exists when player 1 has 2 types (Proposition 5). For 3 types, the existence of a NEE is immediate when the participation structure \(T^*\) is a partition (Proposition 6), but no NEE may exist when \(T^* = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}\) (Sect. 4.3). However, a NEE does exist if \(T^* = \{\{1, 2\}, \{1, 3\}\}\) (Proposition 9). The proof of the latter result is rather involved, but has some similarities with Sorin’s (1983) proof to establish the existence of an equilibrium in two-person infinitely repeated games with a single informed player having 2 possible types.

Indeed, the NEE conditions in \(\Gamma\), namely, (10) and (11), are not only related to standard equilibrium conditions in sender–receiver games, as pointed out in Sect. 2, but also remind of the equilibrium conditions formulated by Aumann et al. (1968) (see Aumann and Maschler 1995) for “joint plan” equilibria in infinitely repeated games with incomplete information on one side. These conditions appear as conditions (i), (ii) and (iii) in Proposition 1 in Sorin (1983). More precisely, condition (10) is the exact analog of condition (ii) in Sorin (1983) and condition (11) is similar to condition (i) in Sorin (1983), mutatis mutandis in the definition of the set \(Y(p)\) of acceptable decisions by player 2.11

One might thus be tempted to use here the tools that have first been used by Sorin (1983) in the two type case and then developed by Simon et al. (1995)) and Renault (2000) for an arbitrary number of types. The difficulty is that, in the present paper, player 1’s effective utility functions \(U^k, k \in K\), are not affine. Furthermore, the mapping \((p \mapsto \max\{\sum_k p^k V^k(x), x \in X(\text{supp}\ p)\})\) is not continuous and has value \(-\infty\) in the interior of \(\Delta(K)\) as soon as no nonrevealing NEE exists.

Regarding the more general solution concept of mediated equilibrium, existence is guaranteed in the framework above when \(T^* = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}\) but for an arbitrary number of states, the existence of a mediated equilibrium is still an open question, and we could not even solve the case of four types when \(T^* = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}\).

Another avenue for further research would be to investigate existence of NEE in the game \(\Gamma\) when the decision set \(X\) is a real interval and utility functions satisfy Crawford and Sobel’s (1982) assumptions. In this context, Shimizu (2017) proposes a characterization of partitional NEE in \(\Gamma(v_0)\), for \(v_0\) sufficiently small, with a focus on the uniform quadratic case.

11 Condition (iii) in Sorin (1983) is an individual rationality condition for player 1 which has no exact counterpart in the above formulation of our equilibrium conditions. Indeed, once the final approval stage is captured in the utility functions \(U^k(\cdot), k \in K\), player 1 has no action explicitly influencing the payoff.
6 Appendix

6.1 Proof of Propositions 3 and 4

For the sake of completeness, we first explicitly recall the conditions to be satisfied by an equilibrium of $\Gamma(v_0)$ whether they involve exit of some types on equilibrium path or not.

Let us fix a pair of strategies $\sigma : K \rightarrow \Delta(M)$ and $\tau : M \rightarrow X$. Player 1’s equilibrium conditions can be written as

$$U_k^x(\tau(m)) \geq U_k^x(\tau(m')) \quad \forall k \in K, \forall m \in M \text{ s.t. } \sigma(m \mid k) > 0, \forall m' \in M. \quad (26)$$

Player 2’s equilibrium conditions can be written as

$$\sum_k p_k^m(\sigma) W_k^x(v_0, \tau(m)) \geq \sum_k p_k^m(\sigma) W_k^x(\tau(m), x) \quad \forall m \in M \text{ s.t. } P_\sigma(m) > 0, \forall x \in X.$$

We deduce that the necessary and sufficient conditions for $(\sigma, \tau)$ to be a NEE in $\Gamma(v_0)$ are (10) for player 1 and, recalling (4),

$$\forall m \in M \text{ s.t. } P_\sigma(m) > 0, \tau(m) \in \left[ \arg \max_{x \in X} \sum_k p_k^m(\sigma) W_k^x(v_0, x) \right] \cap X(\text{supp } p_m(\sigma)) \quad (27)$$

for player 2. These conditions imply constrained optimization, namely (11).

Proof of Proposition 3 If $(\sigma, \tau)$ satisfies (27) in $\Gamma(v_0)$, the same holds in $\Gamma(z_0)$ since for every $x \in X$,

$$\sum_k p_k^m(\sigma) W_k^x(z_0, x) \leq \sum_k p_k^m(\sigma) W_k^x(v_0, x)$$

and for every $x \in X(\text{supp } p_m(\sigma))$,

$$\sum_k p_k^m(\sigma) W_k^x(z_0, x) = \sum_k p_k^m(\sigma) W_k^x(v_0, x) = \sum_k p_k^m(\sigma) V_k^x(x).$$

Furthermore, (11) must hold and player 1’s equilibrium conditions (10) are the same in $\Gamma(v_0)$ and $\Gamma(z_0)$ or $\Gamma$ as long as player 2’s strategy remains unchanged. □

Proof of Proposition 4 Let $(\sigma, \tau)$ be a NEE in $\Gamma$. By definition, constrained optimization (11) holds, so that in particular $\tau(m) \in X(\text{supp } p_m(\sigma))$ for every $m$ such that $P_\sigma(m) > 0$. Let us keep player 1’s strategy, $\sigma$, fixed. Player 2’s strategy $\tau$ remains a

12 As a refinement of subgame perfect equilibrium, Matthews (1989) strengthens (26) to (10) in the case of equilibria which typically involve exit on path.
best reply to \( \sigma \) in \( \Gamma(v_0) \), with \( v_0 \leq \min_{k \in K} \min_{x \in X} V^k(x) \), provided that \( v_0 \) is such that optimality of no exit holds, namely, recalling (27),

\[
\forall m \in M \text{ s.t. } P_\sigma(m) > 0, \sum_k p_m^k(\sigma)V^k(\tau(m)) \geq \max_{x \in X} \sum_k p_m^k(\sigma)W^k(v_0, x).
\]

These can be viewed as finitely many inequalities over \( v_0 \), which have a solution in \( \mathbb{R} \), since the right hand sights are well-defined, for every \( v_0 \leq \min_{k \in K} \min_{x \in X} V^k(x) \), by Lemma 1.13. Hence there exists \( v_0 \) such that (\( \sigma, \tau \)) is a NEE of \( \Gamma(v_0) \) and from Proposition 3, in \( \Gamma(z_0) \), for every \( z_0 \leq v_0 \).

**Application to a mechanism design problem**

In Sect. 3, we have established that, under various reasonable assumptions, \( \Gamma \) has a partitional NEE \((\sigma, \tau)\), in which both \( \sigma \) and \( \tau \) are pure. In this case, we can easily compute the highest ex ante expected utility that player 2, interpreted here as the principal, can obtain at a partitional NEE of \( \Gamma \). \(^{14}\) We will show below that there exists \( v_0 \in \mathbb{R} \) such that, for every \( z_0 \leq v_0 \), the highest ex ante expected utility player 2 can obtain at an arbitrary partitional equilibrium of \( \Gamma(z_0) \) (which can involve exit or not) is the same as in \( \Gamma \). More precisely,

**Corollary of Propositions 3 and 4** There exists \( v_0 \in \mathbb{R} \) such that, for every \( z_0 \leq v_0 \), the best partitional NEE for player 2 in \( \Gamma \) remains the best partitional equilibrium for player 2 in \( \Gamma(z_0) \). \( \square \)

**Proof** Let \( v_{NE}^* \) be the highest ex ante expected utility player 2 can obtain at a partitional NEE of \( \Gamma \). This number is well-defined if \( \Gamma \) has a partitional NEE. Let \((\sigma^*, \tau^*)\) achieve the expected utility \( v_{NE}^* \) for player 2. Using Proposition 4, there exists \( v_0 \) sufficiently small such that for every \( z_0 \leq v_0 \), \((\sigma^*, \tau^*)\) is a NEE of \( \Gamma(z_0) \) with the same expected utility \( v_{NE}^* \) for player 2. By Proposition 3, for every such \( z_0 \), there does not exist any equilibrium without exit giving a higher expected utility to player 2 (because such an equilibrium would still be a NEE of \( \Gamma \), with the same expected utilities).

Let us consider the partitional equilibria \((\sigma, \tau)\) of \( \Gamma(v_0) \) in which exit possibly occurs, i.e., in which the set

\[
K_E = \left\{ k \in K : U^k(\tau \circ \sigma(k)) < u_0^k \right\} \neq \emptyset,
\]

\(^{13}\) The right hand sight of the inequalities can be rewritten as

\[\max_{L \subset \text{supp}(p_m(\sigma))} \max_{x \in X(L)} \left\{ \sum_{k \in L} p_m^k(\sigma)V^k(x) + v_0 \sum_{k \in \text{supp}(p_m(\sigma)) \setminus L} p_m^k(\sigma) \right\} \]

\(^{14}\) The number of pure equilibrium payoffs is finite, in the same way as the number of partitions of \( K \). Hence as soon as there is a pure equilibrium in \( \Gamma \), there is an equilibrium achieving the highest expected payoff for the receiver.
i.e., \( p_E = \text{def} \sum_{k \in K_E} p^k > 0 \). The highest expected utility player 2 can achieve at such an equilibrium is

\[
p_E v_0 + (1 - p_E) \overline{v}
\]

where

\[
\overline{v} = \max_{k \in K} \max_{x \in X} V^k(x).
\]

If \( v_0 \) is such that, for every \( p_E \) that can arise given the prior \( p \),

\[
p_E v_0 + (1 - p_E) \overline{v} \leq v_{NE}^*,
\]

namely, \( v_0 \leq \frac{1}{p_E} \left[ v_{NE}^* - (1 - p_E) \overline{v} \right] \) (28)

then \((\sigma^*, \tau^*)\) will guarantee the highest possible equilibrium utility to player 2, in every game \( \Gamma(z_0) \) with \( z_0 \leq v_0 \).

Let \( k \) be the type with the smallest prior probability, namely, such that \( p^k = \min \{ p^1, \ldots, p^K \} \). The inequality (28) will hold at every \( p_E \) that can arise given the prior \( p \) as soon as it holds at \( p_E = p^k \): we just have to require

\[
v_0 \leq \frac{1}{p^k} \left[ v_{NE}^* - (1 - p^k) \overline{v} \right].
\]

The previous result is quite intuitive: an upper bound on the receiver’s expected utility at an equilibrium of \( \Gamma(v_0) \) with exit is obtained when the receiver’s proposal is rejected by only the least likely type, while the best possible utility is achieved at all the other types. If \( v_0 \) is sufficiently low, the best equilibrium utility for the receiver in \( \Gamma(v_0) \) will be not be achieved at an equilibrium with exit, but rather at an equilibrium without exit, which is in turn is necessarily a NEE of \( \Gamma \).

\[\square\]

6.2 Proof of Proposition 9

For simplicity, in this section, we assume that \( u_k^0 = 0 \) for each \( k \). This is w.l.o.g. since we can translate the payoffs of each type of the sender. We start with preliminaries.

6.2.1 Mappings and multi-valued mappings

Recalling definitions (1) and (9), let for each \( p \) in \( \Delta(K) \):

\[
f(p) = \sup \{ \sum_{k \in K} p^k V^k(x), x \in X(\text{supp } p) \} \in \mathbb{R} \cup \{-\infty\} \quad \text{and}
\]

\[
\Phi(p) = \{(U^k(x))_{k \in K}, x \in Y(p)\} \subset \mathbb{R}^K.
\]

The sets \( Y(p) \subset X(\text{supp } p) \) and \( \Phi(p) \) are convex compact subsets of \( \mathbb{R} \) and \( \mathbb{R}^K \), respectively. If \( X(\text{supp } p) \neq \emptyset \), then \( f(p) \in \mathbb{R}, Y(p) \neq \emptyset \) and \( \Phi(p) \neq \emptyset \). For each
Otherwise, assume that \( \limsup_{n \to \infty} \) and \( X \). Strategic information transmission with sender’s approval

**Lemma 11**

Consider a converging sequence \( p_n \to p \) with \( \text{supp } p_n = \supp p \) for each \( n \), then \( f(p_n) \to f(p) \).

**Proof** Suppose \( p_n \to p \). Then for \( n \) large enough, \( \text{supp } p_n \supset \supp p \) so \( X(\text{supp } p_n) \subset X(\text{supp } p) \). It follows that \( \limsup p_n f(p_n) \leq f(p) \). (whether \( f(p) = -\infty \) or not)

If \( p = \lambda p_1 + (1 - \lambda) p_2 \) with \( \lambda \in (0, 1) \), then \( \text{supp } p = \text{supp } p_1 \cup \text{supp } p_2 \) and \( X(\text{supp } p) = X(\text{supp } p_1) \cap X(\text{supp } p_2) \).

If \( f(p) = -\infty \) then \( \lambda f(p_1) + (1 - \lambda) f(p_2) \geq f(p) \). Consider \( x \) in \( X(\text{supp } p) \), we have \( f(p_1) \geq \sum_{k \in K} p^k \nu^k(x) \) and \( f(p_2) \geq \sum_{k \in K} p^{2k} \nu^k(x) \), so \( \lambda f(p_1) + (1 - \lambda) f(p_2) \geq \sum_{k \in K} p^k \nu^k(x) \), and taking the supremum for \( x \) in \( X(\text{supp } p) \) we get \( \lambda f(p_1) + (1 - \lambda) f(p_2) \geq f(p) \). Hence \( f \) is convex.

**Lemma 12**

Let \( F : [0, 1] \to \mathbb{R} \) be a correspondence with non-empty convex values and compact graph. If \( F(0) \subset \{ x \in \mathbb{R}, x < 0 \} \) and \( F(1) \subset \{ x \in \mathbb{R}, x > 0 \} \), there exists \( t \in (0, 1) \) such that \( 0 \in F(t) \).
Proof The sets $C_+ = \{ t \in [0, 1], F(t) \cap \mathbb{R}_+ \neq \emptyset \}$ and $C_- = \{ t \in [0, 1], F(t) \cap \mathbb{R}_- \neq \emptyset \}$ are closed because $F$ is u.s.c. Since $F$ has non empty values, $C_+$ and $C_-$ are non empty, and $C_+ \cup C_- = [0, 1]$. By connexity of $[0, 1]$, one can find $t$ in both sets, that is such that $F(t)$ intersects both $\mathbb{R}_+$ and $\mathbb{R}_-$. Since $F(t)$ is convex, it contains 0.

6.2.2 Existence of an equilibrium

For $k = 2, 3$, define $\delta_k$ as the Dirac measure on the state $k$, and $p_{-k}$ as the conditional probability on $K$ knowing the state is not $k$:

$$\delta_2 = (0, 1, 0), \quad \delta_3 = (0, 0, 1), \quad p_{-2} = \left( \frac{p_1}{p_1 + p_3}, 0, \frac{p_3}{p_1 + p_3} \right)$$

and

$$p_{-3} = \left( \frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}, 0 \right).$$

Choose $u_2$ in $\Phi(\delta_2)$, $u_3$ in $\Phi(\delta_3)$, $u_{1,2}$ in $\Phi(p_{-3})$ and $u_{1,3}$ in $\Phi(p_{-2})$. These are vectors in $\mathbb{R}^3$, and to simply notations we write:

$$u_2 = \begin{pmatrix} a \\ - \\ + \end{pmatrix}, u_3 = \begin{pmatrix} b \\ - \\ + \end{pmatrix}, u_{1,2} = \begin{pmatrix} c \geq 0 \\ + \\ - \end{pmatrix}, u_{1,3} = \begin{pmatrix} d \geq 0 \\ - \\ + \end{pmatrix}.$$

with $a = u_1^{1,2}$, $b = u_1^{1,3}$, $c = u_1^{1,2}$ and $d = u_1^{1,3}$, Here $+$ means $\geq 0$, and $-$ means $< 0$. We have $u_2^{2} \geq 0$, $u_3^{2} \geq 0$, $c \geq 0$, $u_1^{2,2} \geq 0$, $d \geq 0$ and $u_1^{2,3} \geq 0$ since for each $p$ and $u \in \Phi(p)$, we have $u^k \geq 0$ for each $k$ in $\text{supp } p$. The subset $\{2,3\}$ is not in $\mathcal{T}$, this gives $u_3^2 < 0, u_3^3 < 0, u_{1,2}^3 < 0$ and $u_{1,3}^2 < 0$.

Suppose $a \leq d$. Then a simple equilibrium exists. Player 1 uses the partition $\{(2), \{1,3\}\}$ to communicate: he sends the message $m = 2$ if the state is 2, and the message $m = \{1,3\}$ if the state is 1 or 3. Player 2 proposes $x_2$ in $Y(\delta_2)$ such that $u_2 = (U^k(x_2))_{k \in K}$ after receiving $m = 2$, and proposes $x_{1,3}$ in $Y(p_{-2})$ such that $u_{1,3} = (U^k(x_{1,3}))_{k \in K}$ after receiving $m = \{1,3\}$. By definition of $Y(\delta_2)$ and $Y(p_{-2})$, player 2 is in best reply. And no type of player 1 has an incentive to deviate, so we have an equilibrium where player 1 plays pure. If we suppose $b \leq c$, we have a similar equilibrium where player 1 uses the partition $\{(3), \{1,2\}\}$.

From now on, we assume that $a > d \geq 0$ and $b > c \geq 0$. Then $a \geq 0$. Consider any sequence $(p_n)$ converging to the Dirac measure on state 2 such that $\text{supp } p_n = \{1,2\}$ for each $n$. By Lemma 11 part (b), we must have $\lim \sup_n f(p_n) \geq f(p)$, and since $f$ is u.s.c., $\lim \sup_n f(p_n) = f(p)$. This being true for any such sequence, $f(p_n) \xrightarrow{n \to \infty} f(p)$. That is, the restriction of $f$ to the set $\{p, \text{supp } p \subset \{1,2\}\}$ is continuous at $\delta_2$. And by Lemma 11 part (a), the restriction of $\Phi$ to the segment $[p_{-3}, \delta_2]$ has a closed graph. Similarly we have $b \geq 0$, and we can prove that the restriction of $f$ to the set $\{p, \text{supp } p \subset \{1,3\}\}$ is continuous at $\delta_3$, and the restriction of $\Phi$ to the segment $[p_{-2}, \delta_3]$ has a closed graph.

The initial probability $p$ is on the segment $[\delta_2, p_{-2}]$, and also on the segment $[\delta_3, p_{-3}]$. For $t \in [0, 1], define q_t = t\delta_2 + (1-t)p_{-3}$ and $q'_t$ in $[p_{-2}, \delta_3]$ such that

\begin{quote}
\textcircled{3} Springer
\end{quote}
\( p \) belongs to the segment \([q_t, q'_t]\). \( q'_t \) is uniquely defined for each \( t \), \( q'_0 = \delta_3 \) and \( q'_1 = p - 2 \). We are going to construct an equilibrium with posteriors \( q_t \) and \( q'_t \) for some appropriate \( t \). We need player 1 of type 1 to be indifferent between splitting to \( q_t \) and \( q'_t \).

Define the correspondence \( F : [0, 1] \rightarrow \mathbb{R} \), with for each \( t \) in \([0, 1]\):

\[
F(t) = \{u^1 - v^1, u \in \Phi(q_t), v \in \Phi(q'_t)\}.
\]

\( F \) clearly has non empty convex compact values. We have seen that the restrictions of \( \Phi \) to the segments \([p - 3, \delta_2]\) and \([p - 2, \delta_3]\) have closed graphs, moreover \( q_t \) and \( q'_t \) are continuous in \( t \), hence \( F \) has a closed graph. \( F(0) = \{u^1 - v^1, u \in \Phi(p - 3), v \in \Phi(\delta_3)\} \). If \( F(0) \cap \mathbb{R}_+ \neq \emptyset \), there exists a pure equilibrium where player 1 uses the partition \([3], \{1, 2\}\), so we assume that \( F(0) \) is a subset of \([x \in \mathbb{R}, x < 0]\). Similarly, we assume that \( F(1) = \{u^1 - v^1, u \in \Phi(\delta_2), v \in \Phi(p - 2)\} \) is a subset of \([x \in \mathbb{R}, x > 0]\) (otherwise there exists an equilibrium where player 1 uses the partition \([2], \{1, 3\}\)). Then by Lemma 12 we can find \( t^* \) in \([0, 1]\) such that \( 0 \in F(t^*) \).

We can now conclude the proof. We can find \( x \) in \( Y(q_{t^*}) \), \( y \) in \( Y(q'_{t^*}) \), \( u = (U^k(x))_k \in \Phi(q_{t^*}) \) and \( u' = (U^k(y))_k \in \Phi(q'_{t^*}) \) such that for some \( e \geq 0 \):

\[
u = \begin{pmatrix} e \\ + \\ - \\ + \end{pmatrix}, u' = \begin{pmatrix} e \\ + \\ - \\ + \end{pmatrix}
\]

We have an equilibrium as follows. Player 1 sends a message so as to induce the posteriors \( q_{t^*} \) and \( q'_{t^*} \) (type 2 sends the message 2, type 3 sends the message 3, and type 1 randomizes between the messages 2 and 3 so that the posteriors are \( q_{t^*} \) after \( m = 2 \) and \( q'_{t^*} \) after \( m = 3 \)). Player 2 then proposes \( x \) at \( q_{t^*} \), and \( y \) at \( q'_{t^*} \). Player 2 is in best reply by construction. Type 1 of player 1 is indifferent. If type 2 of player
1 deviates and sends $q'_r^t$, player 2 will propose $y$ and type 2 will reject it, having the reserve payoff of 0, which is not better than the payoff without deviating. Similarly, player 1 of type 3 has no profitable deviation, and we have an equilibrium. □

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