Closed geodesics on positively curved spheres \( S^n \) with Finsler metric induced by \((\mathbb{R}P^n, F)\)

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Abstract

It’s well known that the \( n \)-sphere \( S^n \) is the universal double covering of the \( n \)-dimensional real projective space \( \mathbb{R}P^n \) and then any Finsler metric on \( \mathbb{R}P^n \) induces a Finsler metric of \( S^n \). In this paper, we prove that for every Finsler \((S^n, F)\) for \( n \geq 3 \) whose metric is induced by irreversible Finsler \((\mathbb{R}P^n, F)\) with reversibility \( \lambda \) and flag curvature \( K \) satisfying \((\frac{\lambda}{\lambda + 1})^2 < K \leq 1\), there exist at least \( n - 1 \) prime closed geodesics on \((S^n, F)\). Furthermore, if there exist finitely many distinct closed geodesics on \((S^n, F)\), then there exist at least \( 2\lfloor \frac{n}{2} \rfloor - 1 \) of them are non-hyperbolic.

Key words: Finsler spheres, Closed geodesics, Symmetric condition, Index iteration, Multiplicity.

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1 Introduction and main results

This paper is devoted to a study on closed geodesics on Finsler \( n \)-spheres endowed with some symmetric condition on the Finsler metric. For the definition of closed geodesics on a Finsler manifold, we refer readers to \( \text{[BCS]} \). As usual, on any Finsler \( n \)-sphere \( S^n = (S^n, F) \), a closed geodesic \( c : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow S^n \) is prime if it is not a multiple covering (i.e., iteration) of any other

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closed geodesics. Here the \( m \)-th iteration \( c^m \) of \( c \) is defined by \( c^m(t) = c(mt) \). The inverse curve \( c^{-1} \) of \( c \) is defined by \( c^{-1}(t) = c(1 - t) \) for \( t \in \mathbb{R} \). Note that on a non-symmetric Finsler manifold, the inverse curve of a closed geodesic is not a closed geodesic in general. We call two prime closed geodesics \( c \) and \( d \) distinct if there is no \( \theta \in (0, 1) \) such that \( c(t) = d(t + \theta) \) for all \( t \in \mathbb{R} \). We shall omit the word distinct when we talk about more than one prime closed geodesic. On a symmetric Finsler (or Riemannian) \( n \)-sphere, two closed geodesics \( c \) and \( d \) are called geometrically distinct if \( c(S^1) \neq d(S^1) \), i.e., their image sets in \( S^n \) are distinct.

For a closed geodesic \( c \) on \( (S^n, F) \), denote by \( P_c \) the linearized Poincaré map of \( c \). Then \( P_c \in \text{Sp}(2n - 2) \) is symplectic. For any \( M \in \text{Sp}(2k) \), we define the elliptic height \( e(M) \) of \( M \) to be the total algebraic multiplicity of all eigenvalues of \( M \) on the unit circle \( U = \{ z \in \mathbb{C} \mid |z| = 1 \} \) in the complex plane \( \mathbb{C} \). Since \( M \) is symplectic, \( e(M) \) is even and \( 0 \leq e(M) \leq 2k \). A closed geodesic is called elliptic if \( e(P_c) = 2(n - 1) \), i.e., all the eigenvalues of \( P_c \) locate on \( U \); hyperbolic if \( e(P_c) = 0 \), i.e., all the eigenvalues of \( P_c \) locate away from \( U \); non-degenerate if 1 is not an eigenvalue of \( P_c \). A Finsler sphere \( (S^n, F) \) is called bumpy if all the closed geodesics on it are non-degenerate. Following Rademacher in [Rad3], the reversibility \( \lambda = \lambda(M, F) \) of a compact Finsler manifold \((M, F)\) is defined to be

\[
\lambda := \max \{ F(-X) \mid X \in TM, F(X) = 1 \} \geq 1.
\]

We are aware of a number of results concerning closed geodesics on spheres. In [Fet1] of 1965, A. Fet proved that every bumpy Riemannian metric on a simply connected compact manifold carries at least two geometrically distinct closed geodesics. Motivated by the work [Kli1] of W. Klingenberg in 1969, W. Ballmann, G. Thorbergsson and W. Ziller studied in [BTZ1] and [BTZ2] of 1982-83 the existence and stability of closed geodesics on positively curved compact rank one symmetric spaces under pinching conditions. In [Hin1] of 1984, N. Hingston proved that a Riemannian metric on a sphere all of whose closed geodesics are hyperbolic carries infinitely many geometrically distinct closed geodesics. By the results of J. Franks in [Fran1] of 1992 and V. Bangert in [Ban1] of 1993, there are infinitely many geometrically distinct closed geodesics for any Riemannian metric on \( S^2 \) (cf. also [Hin2] for a different proof).

It was quite surprising when Katok [Kat] in 1973 found some non-reversible Finsler metrics on \textsc{cross} with only finitely many prime closed geodesics and all closed geodesics are non-degenerate and elliptic. The smallest number of closed geodesics on \( S^n \) that one obtains in these examples is \( 2^\left\lceil \frac{n+1}{2} \right\rceil \) (cf. [Zil]). Then Anosov in I.C.M. of 1974 conjectured that the lower bound of the number of closed geodesics on any Finsler sphere \( (S^n, F) \) should be \( 2^\left\lceil \frac{n+1}{2} \right\rceil \), i.e., the number of closed geodesics in Katok’s example. In [Rad4], Rademacher studied the existence and stability of closed geodesics
on positively curved Finsler manifolds. In particular, he proved that there are always \( \frac{n}{2} - 1 \) prime closed geodesics of length \( \leq 2n\pi \) on every Finsler \( n \)-sphere \( (S^n, F) \) satisfying \( \left( \frac{\lambda}{\lambda+1} \right)^2 < K \leq 1 \). In 2004, Bangert and Long \([BaL]\) (published in 2010) proved that on any Finsler 2-sphere \( (S^2, F) \), there exist at least two prime closed geodesics, which answers Anosov’s conjecture for \( S^2 \). Subsequently, such a multiplicity result for \( S^n \) with a bumpy Finsler metric was proved by Duan and Long \([DuL]\) and Rademacher \([Rad5]\) independently. Furthermore in a recent paper \([DLW1]\), Duan, Long and Wang proved the same conclusion for any compact simply-connected bumpy Finsler manifold. In \([LoD], [DuL2]\) of Long and Duan, they proved there exist at least two prime closed geodesics on any \( n \)-dimensional compact simply connected Finsler manifold for \( n = 3, 4 \). In \([Wan1]\), Wang proved Anosov’s conjecture for bumpy \( n \)-spheres satisfying \( \left( \frac{\lambda}{\lambda+1} \right)^2 < K \leq 1 \), furthermore, for \( 2n \)-spheres Wang \([Wan3]\) obtained the optimal lower bound estimation on the number of non-hyperbolic closed geodesics under the same condition. More recently, in \([DLW2]\), Duan, Long and Wang extend the main results of \([Wan1]\) and \([Wan3]\) to bumpy compact simply connected Finsler manifolds and get the optimal lower bound estimation on the number of closed geodesics under weaker curvature conditions.

Motivated by the above results, we continue to study the multiplicity and non-hyperbolicity of closed geodesics on \( n \)-spheres satisfying \( \left( \frac{\lambda}{\lambda+1} \right)^2 < K \leq 1 \). Differently from the above papers, we don’t assume the bumpy condition, but we endow the Finsler metric some symmetric condition.

**Theorem 1.1.** On every Finsler \( (S^n, F) \) for \( n \geq 3 \) whose metric is induced by irreversible Finsler \( (\mathbb{R}P^n, F) \) with reversibility \( \lambda \) and flag curvature \( K \) satisfying \( \left( \frac{\lambda}{\lambda+1} \right)^2 < K \leq 1 \), there exist at least \( n - 1 \) prime closed geodesics on \( (S^n, F) \). Furthermore, if there exist finitely many distinct closed geodesics on \( (S^n, F) \), then there exist at least \( 2\left[\frac{n}{2}\right] - 1 \) of them are non-hyperbolic.

**Remark 1.3.** Our theorem is also motivated by Theorem 1.1 in \([LLZ]\) which was devoted to study the number of closed characteristics on symmetric compact convex hypersurfaces in \( \mathbb{R}^{2n} \) and the key point in the proof of Theorem 1.1 is the estimation \( (4.4) \) in Lemma 4.1.

In this paper, let \( \mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \) and \( \mathbb{C} \) denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively. We use only singular homology modules with \( \mathbb{Q} \)-coefficients. For an \( S^1 \)-space \( X \), we denote by \( \overline{X} \) the quotient space \( X/S^1 \). We define the function \( [a] = \max\{k \in \mathbb{Z} | k \leq a\} \).
2 Critical point theory for closed geodesics

Let $M = (M, F)$ be a compact Finsler manifold. Closed geodesics are critical points of the energy functional $E(\gamma) = \frac{1}{2} \int_M F(\gamma(t))^2 dt$ on the Hilbert manifold $\Lambda = \Lambda M$ of $H^1$-maps $\gamma : S^1 \to M$. An $S^1$ action is defined by $(s \cdot \gamma)(t) = \gamma(t+s)$ for all $\gamma \in \Lambda$ and $s, t \in S^1$. The index form of the functional $E$ is well defined along any closed geodesic $c$ on $M$, which we denote by $E''(c)$. As usual we denote by $i(c)$ and $\nu(c)$ the Morse index and nullity of $E$ at $c$. For $\kappa \in \mathbb{R}$ we denote by $\Lambda^\kappa = \{d \in \Lambda \mid E(d) \leq \kappa\}$. For a closed geodesic $c$, denote by $c^m$ the $m$-fold iteration of $c$ and $\Lambda(c^m) = \{\gamma \in \Lambda \mid E(\gamma) < E(c^m)\}$. Recall that respectively the mean index $\hat{i}(c)$ and the $S^1$-critical modules of $c^m$ are defined by

$$\hat{i}(c) = \lim_{m \to \infty} \frac{i(c^m)}{m}, \quad \mathcal{C}_x(E, c^m) = H_x((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1).$$

We call a closed geodesic satisfying the isolation condition, if the following holds:

**Iso** For all $m \in \mathbb{N}$ the orbit $S^1 \cdot c^m$ is an isolated critical orbit of $E$.

Note that if the number of prime closed geodesics on a Finsler manifold is finite, then all the closed geodesics satisfy (Iso).

If $c$ has multiplicity $m$, then the subgroup $\mathbb{Z}_m = \{\frac{n}{m} \mid 0 \leq n < m\}$ of $S^1$ acts on $\mathcal{C}_x(E, c)$. As studied in p.59 of [Rad1], for all $m \in \mathbb{N}$, let $H_*(X, A)^{\pm \mathbb{Z}_m} = \{[\xi] \in H_*(X, A) \mid T_*\xi = \pm[\xi]\}$, where $T$ is a generator of the $\mathbb{Z}_m$-action. On $S^1$-critical modules of $c^m$, the following lemma holds:

**Lemma 2.1.** (cf. Satz 6.11 of [Rad1] or Proposition 3.12 of [BaL]) Let $c$ be a prime closed geodesic on a Finsler manifold $(M, F)$ satisfying (Iso). Then there exist two sets $U^-_c$ and $N^-_c$, the so-called local negative disk and the local characteristic manifold at $c^m$ respectively, such that $\nu(c^m) = \dim N^-_c$ and

$$\mathcal{C}_q(E, c^m) \equiv H_q((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1)$$

$$= (H_{i(c^m)}(U^-_c \cup \{c^m\}, U^-_c) \otimes H_{q-i(c^m)}(N^-_c \cup \{c^m\}, N^-_c))^{\mathbb{Z}_m}$$

(i) When $\nu(c^m) = 0$, there holds

$$\mathcal{C}_q(E, c^m) = \begin{cases} Q, & \text{if } i(c^m) - i(c) \in 2\mathbb{Z}, \text{ and } q = i(c^m) \\ 0, & \text{otherwise}. \end{cases}$$

(ii) When $\nu(c^m) > 0$, there holds

$$\mathcal{C}_q(E, c^m) = H_{q-i(c^m)}(N^-_c \cup \{c^m\}, N^-_c)^{\beta(c^m)}_{\mathbb{Z}_m},$$

where $\beta(c^m) = 1$ (or $-1$), when $i(c^m) - i(c)$ is even (or odd).
Let
\[ k_j(c^m) \equiv \dim H_j(N_{c^m} \cup \{c^m\}, N_{c^m}), \quad k_j^{\pm 1}(c^m) \equiv \dim H_j(N_{c^m} \cup \{c^m\}, N_{c^m})^{\pm Z_m}. \]

Then we have

**Lemma 2.2.** (cf. [Rad1], [BaL]) Let \( c \) be a prime closed geodesic on a Finsler manifold \( (M, F) \) satisfying (Iso).

(i) There hold \( 0 \leq k_j^{\pm 1}(c^m) \leq k_j(c^m), \forall m \in \mathbb{N}, \ j \in \mathbb{Z}, \ k_j(c^m) = 0 \) whenever \( j \notin [0, \nu(c^m)] \) and \( k_{\nu(c^m)}(c^m) \leq 1 \). If \( k_{\nu(c^m)}(c^m) = 1 \), then \( k_j(c^m) = 0 \) when \( j \in [0, \nu(c^m)]. \)

(ii) Suppose for some integer \( m = np \geq 2 \) with \( n \) and \( p \in \mathbb{N} \) the nullities satisfy \( \nu(c^m) = \nu(c^n) \).

Then there hold \( k_j(c^m) = k_j(c^n) \) and \( k_j^{\pm 1}(c^m) = k_j^{\pm 1}(c^n) \) for any integer \( j \).

Next we recall the Fadell-Rabinowitz index in a relative version due to [Rad2]. Let \( X \) be an \( S^1 \)-space, \( A \subseteq X \) a closed \( S^1 \)-invariant subset. Note that the cup product defines a homomorphism

\[ H^*_{S^1}(X) \otimes H^*_{S^1}(X, A) \to H^*_{S^1}(X, A) : (\zeta, z) \to \zeta \cup z, \]

where \( H^*_{S^1} \) is the \( S^1 \)-equivariant cohomology with rational coefficients in the sense of A. Borel (cf. Chapter IV of [Bor1]). We fix a characteristic class \( \eta \in H^2(CP^\infty) \). Let \( f^* : H^*(CP^\infty) \to H^*_{S^1}(X) \) be the homomorphism induced by a classifying map \( f : X_{S^1} \to CP^\infty \). Now for \( \gamma \in H^*(CP^\infty) \) and \( z \in H^*_{S^1}(X, A), \) let \( \gamma \cdot z = f^*(\gamma) \cup z. \) Then the order \( ord_{\eta}(z) \) with respect to \( \eta \) is defined by

\[ ord_{\eta}(z) = \inf \{ k \in \mathbb{N} \cup \{ \infty \} \mid \eta^k \cdot z = 0 \}. \]

By Proposition 3.1 of [Rad2], there is an element \( z \in H^*_{S^1+1}(\Lambda, \Lambda^0) \) of infinite order, i.e., \( ord_{\eta}(z) = \infty \). For \( \kappa \geq 0 \), we denote by \( j_\kappa : (\Lambda^\kappa, \Lambda^0) \to (\Lambda \Lambda^0) \) the natural inclusion and define the function \( d_z : \mathbb{R}^{\geq 0} \to \mathbb{N} \cup \{ \infty \} : \)

\[ d_z(\kappa) = ord_{\eta}(j_\kappa^*(z)). \]

Denote by \( d_z(\kappa-) = \lim_{\kappa \searrow 0} d_z(\kappa - \epsilon) \), where \( t \searrow a \) means \( t > a \) and \( t \to a \).

Then we have the following property due to Section 5 of [Rad2]

**Lemma 2.3.** The function \( d_z \) is non-decreasing and \( \lim_{\lambda \searrow \kappa} d_z(\lambda) = d_z(\kappa) \). Each discontinuous point of \( d_z \) is a critical value of the energy functional \( E \). In particular, if \( d_z(\kappa) - d_z(\kappa-) \geq 2 \), then there are infinitely many prime closed geodesics \( c \) with energy \( \kappa \).

For each \( i \geq 1 \), we define

\[ \kappa_i = \inf \{ \delta \in \mathbb{R} \mid d_z(\delta) \geq i \}. \]
Then we have the following.

**Lemma 2.4.** (cf. Lemma 2.3 of [Wan2]) Suppose there are only finitely many prime closed geodesics on \((S^n, F)\). Then each \(\kappa_i\) is a critical value of \(E\). If \(\kappa_i = \kappa_j\) for some \(i < j\), then there are infinitely many prime closed geodesics on \((S^n, F)\).

**Lemma 2.5.** (cf. Lemma 2.4 of [Wan2]) Suppose there are only finitely many prime closed geodesics on \((S^n, F)\). Then for every \(i \in \mathbb{N}\), there exists a closed geodesic \(c\) on \((S^n, F)\) such that

\[
E(c) = \kappa_i, \quad \mathcal{C}_{2i + \dim(z) - 2}(E, c) \neq 0,
\]

(2.1)

where \(\dim(z) = n + 1\).

**Definition 2.6.** A prime closed geodesic \(c\) is \((m, i)\)-variationally visible: if there exist some \(m, i \in \mathbb{N}\) such that (2.1) holds for \(c^m\) and \(\kappa_i\). We call \(c\) infinitely variationally visible: if there exist infinitely many \(m, i \in \mathbb{N}\) such that \(c\) is \((m, i)\)-variationally visible. We denote by \(V_\infty(S^n, F)\) the set of infinitely variationally visible closed geodesics.

### 3 Index iteration theory for closed geodesics on \(S^n\)

Let \(c\) be a closed geodesic on a Finsler n-sphere \(S^n = (S^n, F)\). Denote the linearized Poincaré map of \(c\) by \(P_c \in \text{Sp}(2n - 2)\). Then \(P_c\) is a symplectic matrix. Note that the index iteration formulae in [Lon2] of 2000 (cf. Chap. 8 of [Lon3]) work for Morse indices of iterated closed geodesics (cf. [LiL], Chap. 12 of [Lon3]). Since every closed geodesic on a sphere must be orientable. Then by Theorem 1.1 of [Liu] of C. Liu, the initial Morse index of a closed geodesic \(c\) on a \(n\)-dimensional Finsler sphere coincides with the index of a corresponding symplectic path introduced by C. Conley, E. Zehnder, and Y. Long in 1984-1990 (cf. [Lon3]). Hence in this section, we recall briefly the index theory for symplectic paths. All the details can be found in [Lon3].

As usual, the symplectic group \(\text{Sp}(2n)\) is defined by

\[
\text{Sp}(2n) = \{ M \in \text{GL}(2n, \mathbb{R}) \mid M^T JM = J \},
\]

whose topology is induced from that of \(\mathbb{R}^{4n^2}\), where \(J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\) and \(I_n\) is the identity matrix in \(\mathbb{R}^n\). For \(\tau > 0\) we are interested in paths in \(\text{Sp}(2n)\):

\[
\mathcal{P}_\tau(2n) = \{ \gamma \in C([0, \tau], \text{Sp}(2n)) \mid \gamma(0) = I_{2n} \},
\]

which is equipped with the topology induced from that of \(\text{Sp}(2n)\). The following real function was introduced in [Lon1]:

\[
D_\omega(M) = (-1)^{n-1} \omega^n \det(M - \omega I_{2n}), \quad \forall \omega \in \mathbb{U}, M \in \text{Sp}(2n).
\]
Thus for any $\omega \in U$ the following codimension 1 hypersurface in $\text{Sp}(2n)$ is defined in [Lon1]:

$$\text{Sp}(2n)_\omega^0 = \{ M \in \text{Sp}(2n) \mid D_\omega(M) = 0 \}.$$ 

For any $M \in \text{Sp}(2n)_\omega^0$, we define a co-orientation of $\text{Sp}(2n)_\omega^0$ at $M$ by the positive direction $\frac{d}{dt} M e^{tJ}$ of the path $Me^{tJ}$ with $0 \leq t \leq 1$ and $\epsilon > 0$ being sufficiently small. Let

$$\text{Sp}(2n)^*_\omega = \text{Sp}(2n) \setminus \text{Sp}(2n)_\omega^0,$$

$$\mathcal{P}_{\tau,\omega}^*(2n) = \{ \gamma \in \mathcal{P}_\tau(2n) \mid \gamma(\tau) \in \text{Sp}(2n)^*_\omega \},$$

$$\mathcal{P}_{\tau,\omega}^0(2n) = \mathcal{P}_\tau(2n) \setminus \mathcal{P}_{\tau,\omega}^*(2n).$$

For any two continuous arcs $\xi$ and $\eta : [0, \tau] \to \text{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, it is defined as usual:

$$\eta * \xi(t) = \begin{cases} 
\xi(2t), & \text{if } 0 \leq t \leq \tau/2, \\
\eta(2t - \tau), & \text{if } \tau/2 \leq t \leq \tau.
\end{cases}$$

Given any two $2m_k \times 2m_k$ matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\
C_k & D_k \end{pmatrix}$ with $k = 1, 2$, as in [Lon3], the $\diamond$-product of $M_1$ and $M_2$ is defined by the following $2(2m_1 + m_2) \times 2(2m_1 + m_2)$ matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\
0 & A_2 & 0 & B_2 \\
C_1 & 0 & D_1 & 0 \\
0 & C_2 & 0 & D_2 \end{pmatrix}.$$ 

Denote by $M^{\otimes k}$ the $k$-fold $\diamond$-product $M \diamond \cdots \diamond M$. Note that the $\diamond$-product of any two symplectic matrices is symplectic. For any two paths $\gamma_j \in \mathcal{P}_\tau(2n_j)$ with $j = 0$ and 1, let $\gamma_0 \diamond \gamma_1(t) = \gamma_0(t) \diamond \gamma_1(t)$ for all $t \in [0, \tau]$.

A special path $\xi_n$ is defined by

$$\xi_n(t) = \left( \begin{array}{cc} 2 - \frac{t}{\tau} & 0 \\
0 & (2 - \frac{t}{\tau})^{-1} \end{array} \right)^{\otimes n}$$

for $0 \leq t \leq \tau$.

**Definition 3.1.** (cf. [Lon1], [Lon3]) For any $\omega \in U$ and $M \in \text{Sp}(2n)$, define

$$\nu_\omega(M) = \dim \ker C(M - \omega I_{2n}).$$

For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, define

$$\nu_\omega(\gamma) = \nu_\omega(\gamma(\tau)).$$
If $\gamma \in \mathcal{P}_{\tau,\omega}^*(2n)$, define
\[ i_{\omega}(\gamma) = [\text{Sp}(2n)^0_{\omega} : \gamma * \xi_n], \tag{3.1} \]
where the right hand side of (3.1) is the usual homotopy intersection number, and the orientation of $\gamma * \xi_n$ is its positive time direction under homotopy with fixed end points.

If $\gamma \in \mathcal{P}_{\tau,\omega}^0 (2n)$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of $\gamma$ in $\mathcal{P}_{\tau}(2n)$, and define
\[ i_{\omega}(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf \{ i_{\omega}(\beta) | \beta \in U \cap \mathcal{P}_{\tau,\omega}^*(2n) \}. \]
Then
\[ (i_{\omega}(\gamma), \nu_{\omega}(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\}, \]
is called the index function of $\gamma$ at $\omega$. For any symplectic path $\gamma \in \mathcal{P}_{\tau}(2n)$ and $m \in \mathbb{N}$, we define its $m$-th iteration $\gamma^m : [0, m\tau] \to \text{Sp}(2n)$ by
\[ \gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j, \quad \text{for } j\tau \leq t \leq (j+1)\tau, \ j = 0, 1, \ldots, m - 1. \]
We still denote the extended path on $[0, +\infty)$ by $\gamma$.

**Definition 3.2.** (cf. [Lon1], [Lon3]) For any $\gamma \in \mathcal{P}_{\tau}(2n)$, we define
\[ (i(\gamma, m), \nu(\gamma, m)) = (i_1(\gamma^m), \nu_1(\gamma^m)), \quad \forall m \in \mathbb{N}. \]
The mean index $\hat{i}(\gamma, m)$ per $m\tau$ for $m \in \mathbb{N}$ is defined by
\[ \hat{i}(\gamma, m) = \lim_{k \to +\infty} \frac{i(\gamma, mk)}{k}. \]
For any $M \in \text{Sp}(2n)$ and $\omega \in \mathcal{U}$, the splitting numbers $S_{M}^\pm(\omega)$ of $M$ at $\omega$ are defined by
\[ S_{M}^\pm(\omega) = \lim_{\epsilon \to 0^+} \frac{i_{\omega}\exp(\pm\sqrt{-1}\epsilon)}{i_{\omega}}(\gamma) - i_{\omega}(\gamma), \]
for any path $\gamma \in \mathcal{P}_{\tau}(2n)$ satisfying $\gamma(\tau) = M$.

For a given path $\gamma \in \mathcal{P}_{\tau}(2n)$ we consider to deform it to a new path $\eta$ in $\mathcal{P}_{\tau}(2n)$ so that
\[ i_1(\gamma^m) = i_1(\eta^m), \quad \nu_1(\gamma^m) = \nu_1(\eta^m), \quad \forall m \in \mathbb{N}, \tag{3.2} \]
and that $(i_1(\eta^m), \nu_1(\eta^m))$ is easy enough to compute. This leads to finding homotopies $\delta : [0,1] \times [0, \tau] \to \text{Sp}(2n)$ starting from $\gamma$ in $\mathcal{P}_{\tau}(2n)$ and keeping the end points of the homotopy always stay in a certain suitably chosen maximal subset of $\text{Sp}(2n)$ so that (3.2) always holds. In fact, this set
was first discovered in [Lon1] as the path connected component $\Omega^0(M)$ containing $M = \gamma(\tau)$ of the set

$$\Omega(M) = \{ N \in \text{Sp}(2n) \mid \sigma(N) \cap U = \sigma(M) \cap U \text{ and } \nu_{\lambda}(N) = \nu_{\lambda}(M) \forall \lambda \in \sigma(M) \cap U \}.$$ 

Here $\Omega^0(M)$ is called the homotopy component of $M$ in $\text{Sp}(2n)$.

In [Lon1]-[Lon3], the following symplectic matrices were introduced as basic normal forms:

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda = \pm 2,$$  \hspace{1cm} (3.3)

$$N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad \lambda = \pm 1, b = \pm 1, 0,$$  \hspace{1cm} (3.4)

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi),$$  \hspace{1cm} (3.5)

$$N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi),$$  \hspace{1cm} (3.6)

where $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in \mathbb{R}$ and $b_2 \neq b_3$.

Splitting numbers possess the following properties:

**Lemma 3.3.** (cf. [Lon1] and Lemma 9.1.5 of [Lon3]) Splitting numbers $S_{M}^\pm(\omega)$ are well defined, i.e., they are independent of the choice of the path $\gamma \in \mathcal{P}_r(2n)$ satisfying $\gamma(\tau) = M$. For $\omega \in U$ and $M \in \text{Sp}(2n)$, splitting numbers $S_{N}^\pm(\omega)$ are constant for all $N \in \Omega^0(M)$.

**Lemma 3.4.** (cf. [Lon1], Lemma 9.1.5 and List 9.1.12 of [Lon3]) For $M \in \text{Sp}(2n)$ and $\omega \in U$, there hold

$$S_{M}^\pm(\omega) = 0, \quad \text{if } \omega \notin \sigma(M).$$

$$S_{N_1(1,\omega)}^\pm(1) = \begin{cases} 1, & \text{if } a \geq 0, \\
0, & \text{if } a < 0. \end{cases}$$

For any $M_i \in \text{Sp}(2n_i)$ with $i = 0$ and 1, there holds

$$S_{M_0 \circ M_1}^\pm(\omega) = S_{M_0}^\pm(\omega) + S_{M_1}^\pm(\omega), \quad \forall \omega \in U.$$ 

We have the following

**Theorem 3.5.** (cf. [Lon1] and Theorem 1.8.10 of [Lon3]) For any $M \in \text{Sp}(2n)$, there is a path $f : [0, 1] \to \Omega^0(M)$ such that $f(0) = M$ and

$$f(1) = M_1 \diamond \cdots \diamond M_k,$$
where each $M_i$ is a basic normal form listed in (3.3)-(3.6) for $1 \leq i \leq k$.

We have the following properties in the index iteration theory.

**Theorem 3.6.** (cf. Theorem 2.2 of [LoZ]) Let $\gamma \in \mathcal{P}_f(2n)$, then for any $m \in \mathbb{N}$, there holds

$$\nu(\gamma, m) - \frac{e(M)}{2} \leq i(\gamma, m + 1) - i(\gamma, m) - i(\gamma, 1) \leq \nu(\gamma, 1) - \nu(\gamma, m + 1) + \frac{e(M)}{2}$$

where $e(M)$ is the elliptic height defined in §1.

The following is the common index jump theorem of Y. Long and C. Zhu.

**Theorem 3.7.** (cf. Theorems 4.1-4.3 of [LoZ]) Let $\gamma_k \in \mathcal{P}_{\tau_k}(2n)$ for $k = 1, \ldots, p$ be a finite collection of symplectic paths. Let $M_k = \gamma_k(\tau_k)$. Suppose $\hat{i}(\gamma_k, 1) > 0$, for all $k = 1, \ldots, p$. Then there exist infinitely many $(N, m_1, \ldots, m_p) \in \mathbb{N}^{p+1}$ such that

$$\nu(\gamma_k, 2m_k - 1) = \nu(\gamma_k, 1),$$
$$\nu(\gamma_k, 2m_k + 1) = \nu(\gamma_k, 1),$$
$$i(\gamma_k, 2m_k - 1) + \nu(\gamma_k, 2m_k - 1) = 2N - \left( \hat{i}(\gamma_k, 1) + 2S^{+}_{M_k}(1) - \nu(\gamma_k, 1) \right),$$
$$i(\gamma_k, 2m_k + 1) = 2N + \hat{i}(\gamma_k, 1),$$
$$i(\gamma_k, 2m_k) \geq 2N - \frac{e(M_k)}{2} \geq 2N - n,$$
$$i(\gamma_k, 2m_k) + \nu(\gamma_k, 2m_k) \leq 2N + \frac{e(M_k)}{2} \leq 2N + n,$$

for every $k = 1, \ldots, p$.

### 4 Proof of the main theorem

In this section, we give the proof of Theorems 1.1 by using the techniques similar to those of Theorem 1.1 in [LLZ]. We will assume the following

(F) There are only finitely many prime closed geodesics $\{c_j\}_{1 \leq j \leq p}$ on a Finsler $n$-sphere $(S^n, F)$ which is the universal double covering of $(\mathbb{R}P^n, F)$.

First note that if the flag curvature $K$ of $(S^n, F)$ satisfies $\left( \frac{\lambda}{\lambda + 1} \right)^2 < K \leq 1$, then every nonconstant closed geodesic must satisfy

$$i(c) \geq n - 1, \quad \hat{i}(c) > n - 1,$$

where (4.1) follows from Theorem 3 and Lemma 3 of [Rad3], (4.2) follows from Lemma 2 of [Rad4].

Now it follows from Theorem 3.6 that

$$i(c^{m+1}) - i(c^m) - \nu(c^m) \geq i(c) - \frac{e(P_c)}{2} \geq 0, \quad \forall m \in \mathbb{N}. \quad (4.3)$$
Here the last inequality holds by (4.1) and the fact that $e(P_c) \leq 2(n - 1)$.

We also note that a prime closed geodesic $c$ on $S^n$ is a contractible closed geodesic $c$ on $\mathbb{RP}^n$ or 2-iterates of a non-contractible closed geodesic $d$ on $\mathbb{RP}^n$, i.e., $c = d^2$, and when $c$ is contractible on $\mathbb{RP}^n$, then $-c$ is also a prime closed geodesic on $S^n$ which is different from $c$ and plays the same role in the variational setting of the energy functional $E$ on $\Lambda S^n$ as $c$. Specially, the $m$-th iterates $c^m$ and $(-c)^m$ have precisely the same Morse indices, nullities and critical modules.

In the following we call a prime closed geodesic $c$ on $S^n$ symmetric if it is 2-iterates of a non-contractible closed geodesic on $\mathbb{RP}^n$, non-symmetric if it is a contractible closed geodesic on $\mathbb{RP}^n$.

Lemma 4.1. For a prime closed geodesic $c$ on $S^n$ which is symmetric, we have

$$i(c) + 2SP_c(1) - \nu(c) \geq n - 1.$$  \hspace{1cm} (4.4)

Proof. Let $c = d^2$, where $d$ is a closed geodesic on $(\mathbb{RP}^n, F)$. Since $\left(\frac{\pi}{\lambda + 1}\right)^2 < K \leq 1$, by Theorem 1 of [Rad3] we have

$$L(c) \geq \pi(1 + \frac{1}{\lambda}).$$  \hspace{1cm} (4.5)

Let $\tilde{c}$ be a closed geodesic on $(S^n, \tilde{F})$ with constant sectional curvature $\left(\frac{\pi}{L(c)}\right)^2$ and $L(\tilde{c}) = L(c)$, i.e., $\tilde{c}$ is a great circle, let $\tilde{d}$ be its corresponding prime closed geodesic on $(\mathbb{RP}^n, \tilde{F})$, then $\tilde{c} = \tilde{d}^2$.

Note that the sectional curvature of $(\mathbb{RP}^n, \tilde{F})$ is $\left(\frac{\pi}{L(c)}\right)^2$, which is less than the flag curvature $K$ of $(\mathbb{RP}^n, F)$ by (4.5), similar to Lemma 3 of [Rad3] and using a comparison principle, we obtain

$$I_d(-1) \geq I_{\tilde{d}}(-1),$$  \hspace{1cm} (4.6)

where $I_d(-1)$ and $I_{\tilde{d}}(-1)$ is defined as $I(\omega)$ in Chapter 12 of [Lon3] for $\omega = -1$ and the closed geodesics $d$ and $\tilde{d}$ respectively. Note that $\tilde{d}$ is a minimal closed geodesic on $(\mathbb{RP}^n, \tilde{F})$ and $\tilde{c}$ is a great circle on $(S^n, \tilde{F})$, then $i(\tilde{d}) = 0$ and $i(\tilde{c}) = n - 1$ (cf. Page 186 of [Kli2]). Thus by Theorem 12.1.1(i) of [Lon3] we have $I_{\tilde{d}}(-1) = i(\tilde{c}) - i(\tilde{d}) = n - 1$, which together with (4.6) implies

$$I_d(-1) \geq n - 1.$$  \hspace{1cm} (4.7)

Noticing that Theorem 12.1.1 and Corollary 12.1.4 of [Lon3], by a similar proof of Lemma 15.6.3 of [Lon3] and the crucial estimation (4.7), we obtain (4.3).

Since $\hat{i}(c_j) > n - 1$ for $1 \leq j \leq p$ by (4.2), we can use the common index jump theorem of Theorem 3.7 to obtain infinitely many $(N, m_1, \ldots, m_p) \in \mathbb{N}^{p+1}$ such that the following hold for
In fact, we have

\( \nu(c_j^{2m_j-1}) = \nu(c_j), \)  

\( i(c_j^{2m_j}) \geq 2N - \frac{e(P_{c_j})}{2} \geq 2N - (n - 1), \)  

\( i(c_j^{2m_j}) + \nu(c_j^{2m_j}) \leq 2N + \frac{e(P_{c_j})}{2} \leq 2N + (n - 1), \)  

\( i(c_j^{2m_j-1}) + \nu(c_j^{2m_j-1}) = 2N - (i(c_j) + 2S^+_{P_{c_j}}(1) - \nu(c_j)), \)  

\( i(c_j^{2m_j+1}) = 2N + i(c_j), \)

where \( S^+_{P_{c_j}}(1) \) denotes the splitting number of \( c_j \) at 1.

By (4.1), (4.3), and (4.8) and (4.11), for \( m \geq 2 \) we obtain

\[
i(c_j^{2m_j-m}) + \nu(c_j^{2m_j-m}) \leq i(c_j^{2m_j-m+1}) \\
\leq i(c_j^{2m_j-1}) = 2N - (i(c_j) + 2S^+_{P_{c_j}}(1)) \\
\leq 2N - n + 1.
\]

By (4.1), (4.3), and (4.12), for all \( m \geq 1 \) we obtain

\[
i(c_j^{2m_j+m}) \geq i(c_j^{2m_j+1}) = 2N + i(c_j) \geq 2N + n - 1.
\]

By Lemma 2.5, for every \( i \in \mathbb{N} \), there exist some \( \rho(i) \in \{1, \ldots, p\} \) and \( \lambda(i) \in \mathbb{N} \) such that

\[
E^{\lambda(i)}_{\rho(i)} = \kappa_i, \quad \overline{C}_{2i+\dim(z)-2}(E, c^{\lambda(i)}_{\rho(i)}) \neq 0,
\]

and by §2, we have \( \dim(z) = n + 1 \). These \( c^{\lambda(i)}_{\rho(i)} \) possess the following properties.

Claim 1. The closed geodesics \( c^{\lambda(i)}_{\rho(i)} \) are pairwise distinct. More precisely, for \( i \neq j \) only one of the following two possibilities happens:

(i) \( \rho(i) \neq \rho(j) \);

(ii) \( \rho(i) = \rho(j) \) and \( \lambda(i) \neq \lambda(j) \).

In fact, Claim 1 follows directly from the definitions of the maps \( \rho \) and \( \lambda \) and the Lemma 2.4. Here the key point is that \( c^{\lambda(i)}_{\rho(i)} = -c^{\lambda(j)}_{\rho(j)} \) can not hold for any \( i \neq j \), and thus the claimed (i) and (ii) are the only possibilities.

Claim 2. We have the following

\[
\lambda(i) \in \{2m_{\rho(i)} - 1, 2m_{\rho(i)}\}, \quad \text{if} \quad 2i + \dim(z) - 2 \in (2N - n + 1, 2N + n - 1),
\]

In fact, we have

\[
\overline{C}_q(E, c^{\lambda(i)}_{\rho(i)}) = 0, \quad \text{if} \quad q \in (2N - n + 1, 2N + n - 1)
\]
for $1 \leq i \leq p$ and $\lambda(i) \notin \{2m_{\rho(i)} - 1, 2m_{\rho(i)}\}$ by (4.13), (4.14) and Lemma 2.2. Thus in order to satisfy (4.15), we must have $\lambda(i) \in \{2m_{\rho(i)} - 1, 2m_{\rho(i)}\}$.

Since we have infinitely many $N$ satisfying (4.8)-(4.12) and the number of prime closed geodesics is finite, by Definition 2.6 and (4.15) we obtain

$$c_{\rho(i)} \in \mathcal{V}_\infty(S^n, F), \quad N - n + 2 \leq i \leq N - 1,$$

for large $N$.

**Claim 3.** If $c_{\rho(i)}$ is symmetric for some $N - n + 2 \leq i \leq N - 1$, then $\lambda(i) = 2m_{\rho(i)}$.

In fact, by (4.11) and Lemma 4.1 for symmetric closed geodesics, we have

$$i(c_{\rho(i)}(2m_{\rho(i)} - 1)) + \nu(2m_{\rho(i)} - 1) \leq 2N - n + 1.$$

Thus Claim 3 holds by (4.15), (4.16) and Lemma 2.2.

**Claim 4.** If $\lambda(i) = 2m_{\rho(i)} - 1$ for some $N - n + 2 \leq i \leq N - 1$, then $c_{\rho(i)}$ is non-symmetric and non-hyperbolic.

The first statement follows directly from Claim 3. We prove the latter.

In fact, suppose $c_{\rho(i)}$ for some $N - n + 2 \leq i \leq N - 1$ is hyperbolic. Then $\nu(c_{\rho(i)}) = S^+_{P_{c\rho(i)}} (1) = 0$, which together with (4.11) yields

$$i(c_{\rho(i)}(2m_{\rho(i)} - 1)) + \nu(2m_{\rho(i)} - 1) \leq 2N - n + 1 < 2i + n - 1.$$

Hence by (4.11) we obtain

$$i(c_{\rho(i)}(2m_{\rho(i)} - 1)) + \nu(2m_{\rho(i)} - 1) \leq 2N - n + 1 < 2i + n - 1.$$

This contradicts (4.15) by Lemma 2.2. Thus Claim 4 holds.

**Claim 5.** When $n$ is even and $\lambda(i) = 2m_{\rho(i)}$ for some $N - n + 2 \leq i \leq N - 1$, $c_{\rho(i)}$ is non-hyperbolic.

Suppose $c_{\rho(i)}$ is hyperbolic. Then we have $\nu(c_{\rho(i)}) = 0$ and $e(P_{\rho(i)}) = 0$. By (4.9), (4.10) and Lemma 2.2, correspondingly we have

$$\overline{C}_q(E, c_{\rho(i)}(2m_{\rho(i)} - 1)) = 0, \quad \text{if } q \neq 2N. \quad (4.17)$$

This contradicts to (4.15), because $n$ is even. Hence Claim 5 holds.

**Claim 6.** When $n$ is odd, let

$$\mathcal{I} = \{i \in \{N - n + 2, \ldots, N - 1\} \mid \lambda(i) = 2m_{\rho(i)} \text{ holds for } c_{\rho(i)}^\lambda\}.$$
Then there exists at most one $i \in I$ such that $c_{\rho(i)}$ is hyperbolic. Here we do not require specially $c_{\rho(i)}$ is symmetric or not.

In fact, suppose $c_{\rho(i)}$ is hyperbolic for some $i \in I$, and then by (4.17) and (4.15) we must have $2i + n - 1 = 2N$. Assume $c_{\rho(j)}$ is also hyperbolic for some $j \in I \setminus \{i\}$. Then we obtain $2i + n - 1 \neq 2j + n - 1 = 2N$. Thus (4.15) and (4.17) with $i$ replaced by $j$ imply $c_{\rho(j)}$ can not be hyperbolic. This completes the proof of Claim 6.

Based on the preparations above, now we can give the proof of Theorems 1.1.

Note that by Claim 1 and Claim 2, we have

$$0 \leq \#\{\rho^{-1}(l)\} \leq 2, \quad \forall 1 \leq l \leq p.$$ 

We introduce first three sets of integers in the domain \{N − n + 2, . . . , N − 1\} of the map $\rho$.

$$\Theta_1 \equiv \{h \in \{1, \ldots, p\} \mid \#\{\rho^{-1}(h)\} = 2\},$$

$$\Theta_2 \equiv \{h \in \{1, \ldots, p\} \mid \#\{\rho^{-1}(h)\} = 1 \text{ and } \lambda(\rho^{-1}(h)) = 2m_h - 1\},$$

$$\Theta_3 \equiv \{h \in \{1, \ldots, p\} \mid \#\{\rho^{-1}(h)\} = 1 \text{ and } \lambda(\rho^{-1}(h)) = 2m_h\}.$$ 

Then clearly the sets $\Theta_1$, $\Theta_2$ and $\Theta_3$ are pairwise disjoint. Thus we have

$$\{N − n + 2, . . . , N − 1\} = \rho^{-1}(\Theta_1) \cup \rho^{-1}(\Theta_2) \cup \rho^{-1}(\Theta_3).$$

Because $\rho|_{\rho^{-1}(\Theta_1)} : \rho^{-1}(\Theta_1) \to \Theta_1$ is a two to one map, and $\rho|_{\rho^{-1}(\Theta_i)} : \rho^{-1}(\Theta_i) \to \Theta_i$ are bijections for $i = 2, 3$, we obtain

$$2\#\Theta_1 + \#\Theta_2 + \#\Theta_3 = n - 2. \quad (4.18)$$

We carry our the proof by counting non-hyperbolic closed geodesics corresponding to integers in the three sets $\Theta_1$, $\Theta_2$, and $\Theta_3$.

**Case 1. Study on $\Theta_1$.**

In this case, for each $h \in \Theta_1$, there exist $i \neq j \in \{N − n + 2, . . . , N − 1\}$ such that $\rho(i) = \rho(j) = h$, $\lambda(i) = 2m_{\rho(i)} - 1$, and $\lambda(j) = 2m_{\rho(i)}$. Thus $c_h$ is non-symmetric and non-hyperbolic by Claim 4. Thus $-c_h$ is also non-hyperbolic and geometrically distinct from $c_h$. Thus we obtain two geometrically distinct non-hyperbolic closed geodesics for each $h \in \Theta_1$. Then by Claim 2 we have $2\#\Theta_1$ non-hyperbolic closed geodesics in this case.

**Case 2. Study on $\Theta_2$.**

In this case $\rho|_{\rho^{-1}(\Theta_2)} : \rho^{-1}(\Theta_2) \to \Theta_2$ is a bijection. By Claim 4, the orbit $c_h$ is non-symmetric and non-hyperbolic for each $h \in \Theta_2$. Thus we obtain two non-hyperbolic closed geodesics for each $h \in \Theta_2$ as in Case 1. Hence by Claim 2 we have $2\#\Theta_2$ non-hyperbolic closed geodesics in this case.
Case 3. Study on $\Theta_3$ when $n$ is even.

In this case $\rho|_{\rho^{-1}(\Theta_3)} : \rho^{-1}(\Theta_3) \to \Theta_3$ is a bijection. For each $h \in \Theta_3$, $c_h$ is non-hyperbolic by Claim 5. Hence we have at least $\#\Theta_3$ non-hyperbolic closed geodesics in this case.

Case 4. Study on $\Theta_3$ when $n$ is odd.

In this case $\rho|_{\rho^{-1}(\Theta_3)} : \rho^{-1}(\Theta_3) \to \Theta_3$ is a bijection. By Claim 6, $c_{\rho(i)}^\lambda$ is non-hyperbolic for all $\rho(i) = h \in \Theta_3$ but at most one. Hence by Claim 2 we have at least $\#\Theta_3 - 1$ non-hyperbolic closed geodesics in this case.

By Cases 1-4, by Claim 2 the number of closed geodesics is at least

$$2\#\Theta_1 + 2\#\Theta_2 + \#\Theta_3 \geq 2\#\Theta_1 + \#\Theta_2 + \#\Theta_3 = n - 2, \tag{4.13}$$

where we have used (4.13). Note that by Lemma 4.1 and Lemma 4.3 of [Wan4], there is an elliptic(also non-hyperbolic) closed geodesic $c_{j_0}$ such that $i(c_{j_0}^{2mj_0}) + \nu(c_{j_0}^{2mj_0}) = 2N + (n - 1)$ and $C_{2N+(n-1)}(E, c_{j_0}^{2mj_0}) \neq 0$. Then by Claim 3 and Case 1, we get that $c_{j_0}$ is different from the closed geodesics in $\Theta_1$ and $\Theta_3$. Then we get one more closed geodesic $c_{j_0}$ which belongs to $\Theta_2$ or do not belong to any of $\Theta_i$ for $i = 1, 2, 3$. Thus the number of closed geodesics is at least $n - 1$.

Now when $n$ is even, by Cases 1, 2, and 3, the number of non-hyperbolic closed geodesics is at least

$$2\#\Theta_1 + 2\#\Theta_2 + \#\Theta_3 \geq 2\#\Theta_1 + \#\Theta_2 + \#\Theta_3 = n - 2, \tag{4.14}$$

which together with the above considerations yields that there exist $n - 1$ non-hyperbolic closed geodesics. It proves the conclusion of Theorem 1.1 when $n$ is even.

Now when $n$ is odd, by Cases 1, 2, and 4, the number of non-hyperbolic closed geodesics is at least

$$2\#\Theta_1 + 2\#\Theta_2 + \#\Theta_3 - 1 \geq 2\#\Theta_1 + \#\Theta_2 + \#\Theta_3 - 1 = n - 3, \tag{4.15}$$

which together with the above considerations yields that there exist $n - 2$ non-hyperbolic closed geodesics. It proves the conclusion of Theorem 1.1 when $n$ is odd.

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