Generalized Sampling on Graphs With Subspace and Smoothness Priors
Yuichi Tanaka and Yonina C. Eldar

Abstract—We consider a framework for generalized sampling of graph signals that parallels sampling in shift-invariant (SI) subspaces. This framework allows for arbitrary input signals, which are not constrained to be bandlimited. Furthermore, the sampling and reconstruction filters can be different. We present design methods of the correction filter that compensates for these differences and can be obtained in closed form in the graph frequency domain. This paper considers two priors on graph signals: The first is a subspace prior, where the signal is assumed to lie in a periodic graph spectrum (PGS) subspace. The PGS subspace is proposed as a counterpart of the SI subspace used in standard sampling theory. The second is a smoothness prior that imposes a smoothness requirement on the graph signal. We suggest recovery methods both for the case when the recovery filter can be optimized and in the setting in which a predefined filter must be used. Sampling is performed in the graph frequency domain, which is a counterpart of “sampling by modulation” in SI subspaces. We compare our approach with existing sampling methods in graph signal processing. The effectiveness of the proposed generalized sampling is validated numerically through several experiments.

I. INTRODUCTION

Sampling theory for graph signals has been recently studied with the goal of building parallels of sampling results in standard signal processing [1]–[10]. Since the pioneering Shannon–Nyquist sampling theorem [11], [12], sampling theories that encompass more general signal spaces beyond that of bandlimited signals in shift-invariant (SI) spaces have been studied widely with many promising applications [13]–[19]. More relaxed priors have also been considered such as smoothness priors. These theories allow for sampling and recovery of signals in arbitrary subspaces using almost arbitrary sampling and recovery kernels. These results are particularly useful in the SI setting in which sampling and recovery reduce to simple filtering operations.

Graph signal processing (GSP) [20], [21] is a relatively new field of signal processing that studies discrete signals defined on a graph. Recent work on GSP ranges from theory to practical applications including wavelet/filter bank design [22]–[25], learning graphs from observed data [26]–[28], restoration of graph signals [29], [30], image/point cloud processing [31], and deep learning on graphs [32].

One of the topics of interest in GSP is graph sampling theory [1]–[10] which is aimed at recovering a graph signal from its sampled version. Most studies on sampling graph signals considered recovery of discrete graph signals from their sampled version [1]–[7], [10]. Current approaches generally rely on vertex subsampling. In contrast to time domain uniform sampling, graphs are naturally discrete and samples on the vertices are distributed nonuniformly. This fact implies that the maximum bandwidth, which is typically measured by the number of nonzero coefficients in the graph Fourier spectrum [1], [8], allowing for perfect recovery will differ depending on the sampling set. Therefore, sampling set selection for graph signals, often referred to as sensor position selection in machine learning and sensor network communities, is required in the context of GSP [1], [8], [10], [33], [34]. While many deterministic and random sampling set selection methods have been studied, they still focus primarily on vertex subsampling.

Here, our goal is to build a generalized graph sampling framework that allows for (perfect) recovery of arbitrary graph signals beyond bandlimited signals, and parallels SI sampling for time domain signals. In SI sampling, the input subspace has a particular SI structure. Sampling is modeled by uniformly sampling the output of the signal convolved with an arbitrary sampling filter. Under a mild condition on the sampling filter, recovery is obtained by a correction filter having an explicit closed-form frequency response. Herein, we demonstrate how one can extend these ideas to graphs by defining an appropriate input space of graph signals and sampling in the graph frequency domain [35].

We consider two priors characterizing the graph signals:

1) Subspace Prior: The signal lies in a known subspace characterized by a given generator; and
2) Smoothness Prior: The signal is smooth on a given graph.

Both priors parallel those considered in SI sampling [13], [19].

For the subspace prior, we define an input signal subspace which we define as a periodic graph spectrum (PGS) subspace, and is a counterpart of the SI subspace. This subspace maintains the repeated graph frequency spectra similar to that in SI signals. In particular, the spectral domain characteristics of such graph signals mimic that of SI time domain signals. In the smoothness prior, we assume that the quadratic form of the graph signal is small for a given smoothness function. In this setting, perfect recovery is no longer possible. Nonetheless, following the work in general Hilbert space sampling, we show how to design graph filters that allows us to best approximate the input signal under several different criteria [19], [36]–[38].

Generalized sampling for standard and graph sampling paradigms allows for the use of arbitrary sampling and reconstruction filters that are not necessarily ideal low-pass filters. It also allows for the use of fixed recovery filters that may have implementation advantages. In all settings, and under
Our framework relies on graph sampling performed in the graph frequency domain \[35\] as a counterpart of “sampling by modulation” in the SI setting \[19, 39\]. This sampling method maintains the shape of the graph spectrum. Whereas in SI sampling, sampling in the time and frequency domains coincide, in the graph setting, vertex domain sampling and frequency domain sampling are in general different. Sampling by modulation enables a generalized graph sampling framework that is analogous to SI sampling—exhibiting a symmetric structure where the sampling and reconstruction steps contain similar building blocks as those in SI sampling. Our approach reduces to the standard SI results in the case of a graph representing the conventional time axis whose graph Fourier basis is the discrete Fourier transform (DFT).

In the context of subspace sampling with a PGS prior, our results allow for perfect recovery of graph signals beyond bandlimited ones for almost all signal and sampling spaces. In particular, we require these subspaces to satisfy a direct-sum (DS) condition, as in generalized sampling. When the DS condition does not hold, we design a correction filter that best approximates the input under both least-squares (LS) and minimax (MX) criteria. We then introduce LS and MX strategies for recovery under a smoothness prior. In all cases, the graph filters have explicit graph frequency responses that parallel those in the SI setting.

Due to the generality of our results, they allow in particular for recovery of non-bandlimited graph signals. This is in contrast to most studies on graph sampling theory \[1–3, 5\] which focus on recovery of bandlimited signals or recovery of noisy bandlimited signals \[1, 3, 5, 25\]. We validate the reconstruction error of our generalized sampling framework for non-bandlimited graph signals through numerical experiments. In the special case in which the underlying graph is bipartite, we show that perfect recovery of a non-bandlimited graph signal is possible with vertex domain sampling and reconstruction.

An earlier work focused on generalized sampling of graph signals has been reported in \[4\]. This approach is based on the framework of generalized Hilbert space sampling \[13, 19\] and demonstrates the possibility of perfect recovery of graph signals that are not necessarily bandlimited. However, sampling operator inversion is in general required for the reconstruction process. Similar matrix inversions can be found in many graph sampling studies \[1–7, 10\]. Such inversion can be computationally demanding especially for large graphs. In addition, most previous work considered vertex domain subsampling, which also leads to different building blocks in the sampling and reconstruction steps. Our framework, in contrast, leads to simple closed form recovery methods based on graph filters in both the sampling and recovery steps. We expand on the similarities and differences between our work and previous approaches in Section \[VII\].

Our preliminary work \[40\] studies generalized graph sampling with a subspace prior with a DS condition. This paper significantly expands its results by introducing an integrated framework, studying different criteria, and considering the smoothness prior.

The remainder of this paper is organized as follows. Section \[II\] reviews generalized sampling in Hilbert spaces and in the SI setting. The notations and basics of GSP are introduced in Section \[III\]. A framework for generalized graph sampling is presented in Section \[IV\]. Section \[V\] proposes signal recovery methods assuming a PGS subspace prior. We relax this prior in Section \[VI\] to a smoothness prior. Section \[VII\] describes the relationships between our work and existing methods. Numerical experiments are presented in Section \[VIII\]. Finally, Section \[IX\] concludes the paper.

II. GENERALIZED SAMPLING IN HILBERT SPACE

This section introduces prior results on generalized sampling in Hilbert spaces \[14, 15, 19\] and corresponding results in the SI setting, which are fundamental for our generalized graph sampling approach. Detailed derivations of the results can be found in \[19\] and the references therein. Table \[I\] summarizes the main results of this section in the SI setting.

A. Framework

Let \(x\) be a vector in a Hilbert space \(\mathcal{H}\) and \(c[n]\) be its \(n\)th sample given by \(c[n] = \langle s_n, x \rangle\), where \(\{s_n\}\) is a Riesz basis and \(\langle \cdot, \cdot \rangle\) is an inner product. Denoting by \(S\) the set transformation corresponding to \(\{s_n\}\) we can write the samples as \(c = S^* x\), where \(\cdot^*\) represents the adjoint. The subspace generated by \(\{s_n\}\) is denoted by \(S\).

In the SI setting, \(s_n = s(t - nT)\) for a real function \(s(t)\) and a given period \(T\). The samples in this setting can be expressed as

\[
c[n] = \langle s(t - nT), x(t) \rangle = x(t) * s(-t)_{t=nT},
\]

(1)

where \(*\) represents convolution. The continuous-time Fourier transform (CTFT) of the samples \(c[n]\), denoted \(C(\omega)\), can be written as

\[
C(\omega) = R_{SX}(\omega),
\]

(2)

where

\[
R_{SX}(\omega) := \frac{1}{T} \sum_{k=-\infty}^{\infty} S^* \left( \frac{\omega - 2\pi k}{T} \right) X \left( \frac{\omega - 2\pi k}{T} \right)
\]

(3)

is the sampled cross correlation. Thus, we can view sampling in the Fourier domain as multiplying the input spectrum by the filter’s frequency response and subsequently aliasing the result with uniform intervals that depend on the sampling period. In bandlimited sampling, \(s(-t) = \text{sinc}(t/T)\), where \(\text{sinc}(t) = \sin(\pi t)/(\pi t)\). However, \(s(t)\) can be arbitrary in the generalized sampling framework.

The recovery of the sampled signal \(c\) is represented as

\[
\hat{x} = WHc = WH(S^* x),
\]

(4)

where \(W\) is a set transformation corresponding to a basis \(\{w_n\}\) for the reconstruction space, which spans a closed subspace \(\mathcal{V}\) of \(\mathcal{H}\). The transform \(H\) is called the correction transformation and operates on the samples \(c\) prior to recovery.
TABLE I
Correction and Reconstruction Filters for Shift-Invariant and Graph Spectral Filters where CF and RF are abbreviations of correction filter and reconstruction filter, respectively. DS, LS, and MX refer to direct-sum, least squares, and minimax solutions, respectively. Spectra $R_{XY}(\omega)$ and $R_{X}(\lambda_1)$ are defined in (3) and (54), respectively.

| Filter | Shift-invariant subspace | Periodic graph spectrum subspace |
|--------|--------------------------|--------------------------------|
|        | Unconstrained            | Unconstrained                  |
|        | Predefined ($W(\omega)$ is fixed) | Predefined ($W(\lambda_1)$ is fixed) |
| Subspace | CF | RF | DS, MX | CF | RF | DS, MX |
| Prior | $1/R_{SA}(\omega)$ | $A(\omega)$ | $R_{WA}(\omega)/R_{SA}(\omega)R_{WW}(\omega)$ | $1/R_{SA}(\lambda_1)$ | $A(\lambda_1)$ | $R_{WA}(\lambda_1)/R_{SA}(\lambda_1)R_{WW}(\lambda_1)$ |
| Smoothness | $1/R_{SW}(\omega)$ | $S(\omega)/|V(\omega)|^2$ | $1/R_{SW}(\omega)$ | $S(\lambda_1)/|V(\lambda_1)|^2$ | $1/R_{SW}(\lambda_1)$ | $S(\lambda_1)/|V(\lambda_1)|^2$ |

In the SI setting, the frequency response of the correction filter is a discrete-time Fourier transform (DTFT) of the sequence $d[n]$, and is $2\pi/T$ periodic.

1) Unconstrained Case: We first consider the case in which the recovery is unconstrained, so that $\mathcal{W}$ can be any transformation. In this case, we may recover a signal in $\mathcal{A}$ by choosing $\mathcal{W} = A$ in (4). If $S^*A$ is invertible, then perfect recovery of any $x \in \mathcal{A}$ is possible by choosing $H = (S^*A)^{-1}$. Invertibility can be ensured by the DS condition: $\mathcal{A}$ and $\mathcal{S}^\perp$ intersect only at the origin and span $\mathcal{H}$ jointly. This requirement is formally written as

$$\mathcal{H} = \mathcal{A} \oplus \mathcal{S}^\perp.$$  

Under the DS condition, a unique recovery is obtained by an oblique projection operator onto $\mathcal{A}$ along $\mathcal{S}^\perp$ given by

$$\tilde{x} = A(S^*A)^{-1} S^*x = x.$$  

In the SI setting, the frequency response of the correction filter is

$$H(\omega) = \frac{1}{R_{SA}(\omega)}.$$  

If $\mathcal{A}$ and $\mathcal{S}^\perp$ intersect, then there is more than one signal in $\mathcal{A}$ that matches the sampled signal $c$. We may then consider several selection criteria to obtain an appropriate signal out of (infinitely) many candidates. Widely accepted strategies are the LS and MX approaches.

The LS recovery is the minimum energy solution obtained as

$$\tilde{x} = \arg \min_{x \in \mathcal{A}} \|x\|^2,$$

and is given by

$$\tilde{x} = A(S^*A)^\dagger S^*x.$$  

Here, $H(\omega) = 0$ for $\omega$ with $R_{SA}(\omega) = 0$. The MX criterion minimizes the worst-case error from the original signal:

$$\tilde{x} = \arg \min_{\tilde{x}} \max_{x \in \mathcal{A}, S^*x = c} \|\tilde{x} - x\|^2.$$  

The solution with a subspace prior is the same as that in (12).
2) Predefined Case: When the reconstruction transformation $W$ is predefined, perfect recovery is not possible in general. However, we can still design a correction transformation $H$ such that the solution is close to $x$ in some sense.

With the DS condition in (8), a minimal-error recovery can be obtained:

$$
\min_H \| \tilde{x} - x \|^2 = \min_H \| WHS^*x - x \|^2 \quad \text{for } x \in \mathcal{A}. \quad (14)
$$

The optimal $H$ is given by

$$
H = (W^*W)^{-1}W^*A(S^*A)^{-1}. \quad (15)
$$

The recovered signal is $\tilde{x} = W(W^*W)^{-1}W^*A(S^*A)^{-1}S^*x$, which is the orthogonal projection of the unconstrained solution onto $\mathcal{W}$. In the SI setting, the cost function becomes

$$
H(\omega) = \frac{R_{WA}(\omega)}{R_{SA}(\omega)R_{WW}(\omega)} \cdot (16)
$$

When the DS condition does not hold, the LS and MX strategies can be considered as in the unconstrained case. The LS methodology becomes

$$
\hat{x} = \arg\min_{x \in \mathcal{W}} \| S^*x - c \|^2. \quad (17)
$$

The solution is $H = (S^*W)^\dagger$, which results in the following reconstruction:

$$
\hat{x} = W(S^*W)^\dagger S^*x. \quad (18)
$$

This solution is the same as that in (12) by replacing $A$ with $W$.

In the MX criterion, we consider the error between $\tilde{x}$ and the best approximation of $x$ in $\mathcal{W}$:

$$
\tilde{x} = \arg\max_{x \in \mathcal{W}, \tilde{x} = x} \max_{x \in \mathcal{A}, S^*x = c} \| \tilde{x} - W(W^*W)^{-1}W^*x \|^2 \quad (19)
$$

where the right-most term is the orthogonal projection of $x$ onto $\mathcal{W}$. The MX solution is given by

$$
\hat{x} = W(W^*W)^{-1}W^*A(S^*A)^\dagger S^*x, \quad (20)
$$

with $H = (W^*W)^{-1}W^*A(S^*A)^\dagger$. The corresponding SI solution is the same as that in (16) with $H(\omega) = 0$ when the denominator is zero.

C. Smoothness Prior

The smoothness prior is a less restrictive assumption than the subspace prior because the actual signal subspace $\mathcal{A}$ is not necessarily known. Instead, we assume the signal is smooth, which is formulated as $\| Vx \| \leq \rho$ for some invertible operator $V$:

In the SI setting $V = V(\omega)$ is nonzero for all $\omega$. Smoothness is often measured by low energy in high frequency components:

$$
\int_{-\infty}^{\infty} |V(\omega)X(\omega)|^2 d\omega \leq \rho^2. \quad (21)
$$

In general, with a smoothness prior, there are infinitely many solutions. Two approaches to select a solution are the LS and MX methods, which can be applied in both the unconstrained and constrained settings.

1) Unconstrained Case: Suppose that $V^*V$ is a bounded operator. In the LS method, the objective function is formulated by choosing the smoothest signal among all the possibilities:

$$
\hat{x} = \arg\min_{x \in \{x | S^*x = c, \| Vx \| \leq \rho\}} \| Vx \|^2. \quad (22)
$$

The solution to (22) is given by

$$
\hat{x} = \tilde{W}(S^*\tilde{W})^{-1}S^*x \quad (23)
$$

where $\tilde{W} = (V^*V)^{-1}S$. In the SI setting, the correction filter in (23) reduces to

$$
H(\omega) = \frac{1}{R_{SW}(\omega)} \cdot (24)
$$

with

$$
\tilde{W}(\omega) = \frac{S(\omega)}{|V^2(\omega)|}. \quad (25)
$$

The MX solution is formulated as

$$
\hat{x} = \arg\min_{\tilde{x} \in \{x | S^*x = c, \| Vx \| \leq \rho\}} \| \tilde{x} - x \|^2, \quad (26)
$$

and its solution coincides with (23).

2) Predefined Case: When the recovery space is predefined, the constraint on the feasible set is slightly different from that in (22). First, we require that the recovery is in $\mathcal{W}$. Second, this implies that we may not be able to have $S^*x = c$ but only $S^*x = P c$ where $P = S^*W(S^*W)^\dagger$ is the orthogonal projection onto the range space of $S^*W$. This leads to the modified LS objective:

$$
\hat{x} = \arg\min_{x \in \{x | S^*x = P c, \| Vx \| \leq \rho\}} \| Vx \|^2. \quad (27)
$$

The solution can be shown to be given by

$$
\hat{x} = \tilde{W}(S^*\tilde{W})^\dagger S^*x \quad (28)
$$

where $\tilde{W} = W(W^*V^*VW)^{-1}W^*S$. In the SI setting, (27) reduces to the use of $H(\omega) = 1/R_{SW}(\omega)$ prior to reconstruction with $W(\omega)$ [19] Section 7.2.1. Therefore, the constrained recovery under the LS objective is the same in the subspace and smoothness priors and the smoothness constraint is not included in the solution.

The MX criterion with a smoothness prior can be formulated as

$$
\hat{x} = \arg\min_{\tilde{x} \in \{x | S^*x = P c, \| Vx \| \leq \rho\}} \max_{x \in \mathcal{W}, \tilde{x} = x} \| \tilde{x} - W(W^*W)^{-1}W^*x \|^2. \quad (29)
$$

This solution is given by

$$
\hat{x} = W(W^*W)^{-1}W\tilde{W}(S^*\tilde{W})^{-1}S^*x. \quad (30)
$$

This is the orthogonal projection onto $\mathcal{W}$ of the unconstrained solution in (23). The correction transformation is $H = (W^*W)^{-1}W\tilde{W}(S^*\tilde{W})^{-1}$. In the SI setting, it reduces to

$$
H(\omega) = \frac{R_{SW}(\omega)}{R_{SW}(\omega)R_{WW}(\omega)}. \quad (31)$$
III. Graph Signal Processing and Sampling of Graph Signals

A. Spectral Graph Theory and Basics of GSP

A graph \( G \) is represented as \( G = (V, E) \), where \( V \) and \( E \) denote sets of vertices and edges, respectively. The number of vertices is given as \( N = |V| \) unless otherwise specified. We define an adjacency matrix \( A \) with elements \( a_{mn} \) that represents the weight of the edge between the \( m \)th and \( n \)th vertices; \( a_{mn} = 0 \) for unconnected vertices. The degree matrix \( D \) is a diagonal matrix, with the \( m \)th diagonal element \( D_{mm} = \sum_{n} a_{mn} \).

GSP uses different variation operators \([20], [21]\) depending on the application and assumed signal and/or network models. Here, for concreteness, we use the graph Laplacian \( L := D - A \) or its symmetrically normalized version \( L := D^{-1/2}LD^{-1/2} \). The extension to other variation operators (e.g., adjacency matrix) is possible with a slight modification for properly ordering its eigenvalues as long as the graph is undirected without self-loops. Because \( L \) always possesses an eigendecomposition \( U \Lambda U^T \) with its eigenvalues \( \lambda_i \) and \( U \) is a unitary matrix containing the eigenvectors \( u_i \), and \( A = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) \) consists of the eigenvalues \( \lambda_i \). We refer to \( \lambda_i \) as the graph frequency.

A graph signal \( x : V \rightarrow \mathbb{C} \) is a signal that assigns a value to each vertex. It can be written as a vector \( x \) in which the \( n \)th element \( x[n] \) represents the signal value at the \( n \)th vertex. The graph Fourier transform (GFT) is defined as

\[
\hat{x}[i] = \langle u_i, x \rangle = \sum_{n=0}^{N-1} u_i^*[n]x[n]. \tag{32}
\]

Graph filtering can be defined in two categories. One is vertex domain filtering, which is defined as a linear combination of the neighborhood samples

\[
x_{\text{out}}[n] := \sum_{i \in N_n} g_{n,i} \hat{x}[i], \tag{33}
\]

where \( g_{n,i} \) is the weight, i.e., filter coefficient, for the \( i \)th sample, and \( N_n \) represents neighborhood vertex indices around the \( n \)th vertex. This can also be represented as \( x_{\text{out}} = Gx \) where \( G = \Psi \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{N-1})U^* \).

The other is graph frequency domain filtering, which can be defined as a generalized convolution \([41]\):

\[
x_{\text{out}}[n] := \sum_{i=0}^{N-1} \hat{x}[i]G(\lambda_i)u_i[n] \tag{34}
\]

where \( G(\lambda_i) \) is the filter response in the graph frequency domain given by \( G(\lambda_i) \in \mathbb{R} \). This filtering is equivalently written as

\[
x_{\text{out}} = UG(\Lambda)U^*x, \tag{35}
\]

where \( G(\Lambda) := \text{diag}(G(\lambda_0), G(\lambda_1), \ldots) \). If \( G(\lambda_i) \) is a \( P \)th order polynomial, \([34]\) coincides with vertex domain filtering \([33]\) with a \( P \)-hop local neighborhood \([20]\). Hereinafter, \([33] \) and \([34] \) are together denoted as \( x_{\text{out}} = \hat{G}x \) for convenience.

B. Sampling of Graph Signals

Two notions of sampling over graphs have been proposed: 1) Sampling in the vertex domain \([1], [2]\) and 2) in the graph frequency domain \([35]\).

1) Sampling in the Vertex Domain: For sampling in the vertex domain, samples on a predetermined vertex set \( T \) are selected. This corresponds to nonuniform subsampling in the time domain. In contrast to the SI setting, vertex domain sampling is performed nonuniformly because vertex indices do not reflect the structure of the signal. Many approaches have been proposed to select the “best” sampling set from a given graph under different criteria \([1], [5], [6], [10]\).

Let us define \( I_T \in \{0, 1\}^{K \times N} \) as a submatrix of the identity matrix \( I_N \), whose rows are determined by the sampling set \( T \) that identifies the vertices that remain after sampling, i.e., row indices in \( I_N \). Sampling in the vertex domain is defined as follows:

**Definition 1** (Sampling of graph signals in the vertex domain \([1], [2]\)). Let \( x \in \mathbb{C}^N \) be the original graph signal, and let \( I_T \) be the submatrix of the identity matrix \( I_N \) extracting \( K = |T| \) rows corresponding to the sampling set \( T \). The sampled graph signal \( c \in \mathbb{C}^K \) is given by

\[
c = I_T Gx. \tag{36}
\]

Aggregation sampling \([3]\) is a variant of vertex sampling that defines sampling as filtered observations gathered at a single vertex \( i \).

**Definition 2** (Aggregation sampling of graph signals \([3]\)). Let \( x \in \mathbb{C}^N \) and the sampling matrix \( I_T \) be the same as in Definition 1. The sampled graph signal with aggregation sampling is given by \([36]\) with

\[
G = \Psi \text{diag}(u_0^*(\lambda_i), u_1^*(\lambda_i), \ldots)U^*, \tag{37}
\]

where \( [\Psi]_{\ell k} = \lambda_{\ell k}^i \).

The definitions above based on vertex domain operations result in nonuniform sampling in general. When the signal is bandlimited (in a graph frequency sense), perfect recovery is guaranteed if \( T \) is a uniqueness set \([1], [8]\). However, the reconstruction step typically requires a matrix inversion. Instead, we use frequency domain sampling in this study to build the parallel of generalized SI sampling to the graph setting.

2) Sampling in the Graph Frequency Domain: To define sampling over a graph, we extend sampling in SI subspaces expressed by \([2]\) to the graph frequency domain \([35]\). In particular, the graph Fourier transformed input \( \hat{x} \) is first multiplied by a graph frequency filter \( S(\Lambda) \); the product is subsequently aliased with period \( K \). This results in the following definition:

**Definition 3** (Sampling of graph signals in the graph frequency domain). Let \( \hat{x} \in \mathbb{C}^N \) be the original signal in the graph frequency domain, i.e., \( \hat{x} = U^*x \), and let \( S(\Lambda) \) be an arbitrary sampling filter in the graph frequency domain. For any sampling ratio \( M \in \mathbb{Z} \), the sampled graph signal in
the graph frequency domain is given by $\hat{\mathbf{c}} \in \mathbb{C}^K$, where $K = N/M$, and
\begin{equation}
\hat{\mathbf{c}}(\lambda_i) = \sum_{k=0}^{M-1} S(\lambda_{i+k}M) \tilde{x}(\lambda_{i+k}M).
\end{equation}

In matrix form, the sampled graph signal can be represented as $\hat{\mathbf{c}} = \mathbf{D}_{\text{samp}} S(\mathbf{A}) \tilde{\mathbf{x}}$ where $\mathbf{D}_{\text{samp}} = \begin{bmatrix} \mathbf{I}_K & \mathbf{I}_K & \cdots \end{bmatrix}$.

Hereafter, we denote the sampling matrix $\mathbf{S}^*$ as follows.
\begin{equation}
\mathbf{S}^* = \mathbf{D}_{\text{samp}} S(\mathbf{A}) \mathbf{U}^*.
\end{equation}

Suppose that $\mathbf{U}^*$ is the DFT matrix: For example, the DFT matrix diagonalizes the graph Laplacian $\mathbf{L}$ of the circular graph $\mathbb{C}[i] = \mathbb{C}(2\pi i/N)$ ($i = 0, \ldots, N-1$) yields the same output as in (38).

IV. SAMPLING AND RECOVERY FRAMEWORK
A. Framework

Our framework for generalized graph sampling is shown in Fig. 2. It parallels that of SI sampling illustrated in Fig. 1 [13], [15] where sampling, filtering, and reconstruction are all performed in the graph frequency domain. As in standard sampling theory, three filters are critical in the recovery problem: sampling, correction, and reconstruction filters.

To sample $\mathbf{x}$, we transform the input to the GFT domain resulting in $\tilde{\mathbf{x}} = \mathbf{U}^* \mathbf{x}$. The output is subsequently filtered by the sampling filter $S(\mathbf{A})$. The filtered signal is downsampled to yield the sampled signal $\hat{\mathbf{c}} = \mathbf{S}^* \tilde{\mathbf{x}} = \mathbf{D}_{\text{samp}} S(\mathbf{A}) \tilde{\mathbf{x}}$. In the reconstruction step, $\hat{\mathbf{c}}$ is filtered by the correction filter $\mathbf{H} = H(\mathbf{A})$. Finally, $\hat{\mathbf{d}} = \mathbf{H}(\mathbf{A}) \hat{\mathbf{c}}$ is upsampled to the original dimension by $\mathbf{D}_{\text{samp}}$, and the reconstruction filter $W(\mathbf{A})$ is applied to the upsampled signal. After performing an inverse GFT, we obtain the recovered signal $\hat{\mathbf{x}}$. This can be written as $\hat{\mathbf{x}} = \mathbf{W} \mathbf{H} \hat{\mathbf{c}} := \mathbf{U}^* \mathbf{W}(\mathbf{A}) \mathbf{D}_{\text{samp}}^\dagger \mathbf{H} \hat{\mathbf{c}}$.

The primary objective in this framework is to consider the design method of the correction and reconstruction filters, $\mathbf{H}$ and $\mathbf{W}$, respectively that recovers the original signal as accurately as possible with a given prior and constraint. We will follow the same strategies as that of generalized sampling in Hilbert spaces introduced in Section II-B, DS, LS, and MX. The solutions with subspace and smoothness priors are presented in Sections II-B and II-C, respectively.

Suppose that $\mathbf{U}^*$ is the DFT matrix for a graph $\mathcal{G}$. Then, the following relationship holds:
\begin{equation}
\mathbf{U}_{\text{reduced}} \mathbf{D}_{\text{samp}} \mathbf{U}^* = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} = \mathbf{I}_{\mathcal{V}_1}.
\end{equation}

This relationship indicates that vertex and graph frequency domain sampling coincide when no sampling filter is performed.

Before describing the filter design methods in the following sections, we introduce a special case where vertex and spectral domain sampling coincide.

B. Sampling on Bipartite Graphs

Suppose that a graph is bipartite having two equal-size vertex sets. Formally, let $\mathcal{B} = (\mathcal{V}_1, \mathcal{V}_2, \mathcal{E})$ be a bipartite graph that contains two disjoint vertex sets $\mathcal{V}_1$ and $\mathcal{V}_2$ where $|\mathcal{V}_1| = |\mathcal{V}_2| = N/2$, i.e., edges only exist between $\mathcal{V}_1$ and $\mathcal{V}_2$. Without loss of generality, we assume that its first $N/2$ vertices correspond to $\mathcal{V}_1$ and its last ones to $\mathcal{V}_2$. We also assume that the GFT matrix is the eigenvector matrix of the symmetric normalized graph Laplacian.

Fig. 3 illustrates the vertex domain representation of our generalized sampling framework of Fig. 2 where the sampled signal is transformed back into the vertex domain. Suppose that the reduced-size graph $\mathcal{G}_{\text{reduced}}$ of size $N/2$ is obtained by reconnecting edges within $\mathcal{V}_1$ with Kron reduction. Specifically, it is obtained by 1) selecting $\mathcal{V}_1$ as kept vertices and 2) reconnecting edges by Kron reduction. The symmetric normalized graph Laplacian of $\mathcal{G}_{\text{reduced}}$ can be obtained from $\mathbf{L}$ of $\mathcal{B}$ by
\begin{equation}
\mathbf{L}_{\text{reduced}} = \mathbf{L}_{\mathcal{V}_1 \mathcal{V}_1} - \mathbf{L}_{\mathcal{V}_1 \mathcal{V}_2} \mathbf{L}_{\mathcal{V}_2 \mathcal{V}_2} \mathbf{L}_{\mathcal{V}_2 \mathcal{V}_1},
\end{equation}
where $\mathbf{L}_{\mathcal{X} \mathcal{Y}}$ is a submatrix of $\mathbf{L}$ whose extracted rows and columns from $\mathbf{L}$ are specified by $\mathcal{X}$ and $\mathcal{Y}$, respectively.

First, we assume the simplest case where no sampling filter is performed. The relationship between sampling in the vertex domain and the vertex domain representation of sampling in the graph frequency domain is given in the following theorem, taken from [25]:

**Theorem 1.** Suppose that the GFT matrix $\mathbf{U}_{\mathcal{B}}$ is the eigenvector matrix of the symmetric normalized graph Laplacian of $\mathcal{B}$ and $\mathbf{U}_{\text{reduced}}$ is the eigenvector matrix of $\mathbf{L}_{\text{reduced}}$ in (40). Then, the following relationship holds:
\begin{equation}
\mathbf{U}_{\text{reduced}} \mathbf{D}_{\text{samp}} \mathbf{U}^* = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} = \mathbf{I}_{\mathcal{V}_1}.
\end{equation}
Corollary 1. Suppose that the same setup as in Theorem 7 is used, whereas a sampling filter \( G \) is performed before subsampling like (36). If the sampling filter \( G \) is diagonalizable by \( U_B \), \( U_{\text{reduced}}S^* \), i.e., the vertex domain representation of graph frequency domain sampling, is identical to \( I_{V_1}G \).

Proof. From the assumption, \( G = U_B S(\Lambda) U_B^\top \). (42)

Subsequently, the sampling matrix \( S^* \) in (39) is rewritten as

\[
S^* = D_{\text{samp}} S(\Lambda) U_B^\top = D_{\text{samp}} U_B^\top U_B S(\Lambda) U_B^\top = D_{\text{samp}} U_B^\top G.
\]

(43)

Using (41),

\[
U_{\text{reduced}}S^* = U_{\text{reduced}} D_{\text{samp}} U_B^\top G = I_{V_1}G,
\]

completing the proof. \( \square \)

Similarly, in the reconstruction phase, \( WHU_{\text{reduced}}^\top \) can be represented with vertex domain upsampling:

\[
WHU_{\text{reduced}}^\top = U_B W(\Lambda) D_{\text{samp}}^\top H(\Lambda) U_{\text{reduced}}^\top = U_B W(\Lambda) \text{diag}(H(\Lambda), H(\Lambda)) D_{\text{samp}}^\top U_{\text{reduced}}^\top = U_B W'(\Lambda) U_B^\top U_{\text{reduced}}^\top = U_B W'(\Lambda) U_B^\top [I \ 0]^\top = W'I_{V_1}.
\]

(45)

where \( W'(\lambda_i) := W(\lambda_i) H(\lambda_i \text{mod } N/2) \). This is illustrated in Fig. 4

Interestingly, from (44) and (45), the sampling-then-reconstruction step is represented as

\[
\hat{x} = W'I_{V_1} \Lambda_{V_1} Gx.
\]

(46)

This results in perfect recovery of a non-bandlimited graph signal with vertex domain sampling by utilizing our framework.

V. SIGNAL RECOVERY WITH SUBSPACE PRIOR

In this section, we assume that the signal lies in a known PGS subspace that depends on the given graph. Subsequently, we present two solutions of the correction filter. One is the unconstrained solution that guarantees perfect recovery of the graph signal with an arbitrary choice of the sampling filter. The other is the predefined solution where a given filter must be used for reconstruction.

A. PGS Subspace

We first consider a graph signal subspace that parallels the generation process in SI subspaces shown in (6) and (7). As discussed in the previous section, vertex domain sampling is in general a nonuniform sampling operator in contrast to the uniform SI sampling of (6). Hence, we utilize graph frequency domain sampling that mimics “sampling by modulation” in (2).

In (7), the \( 2\pi/T \)-periodic spectrum of the expansion coefficients \( D(e^{j\omega T}) \) is multiplied by the (non-periodic) generator \( A(\omega) \) to obtain the signal spectrum \( X(\omega) \). We reflect this characteristic into the signal subspace in the graph setting.

The spectrum of the graph considered herein is finite and discrete. Suppose that we have a length \( K \) spectrum \( \hat{d}(\lambda_i) \) \( (i = 0, \ldots, K - 1, K \leq N) \) as the expansion coefficients. While its original length is finite, we assume \( \hat{d} \) is periodic beyond \( i \geq K \) like (7), i.e.,

\[
\hat{d}(\lambda_i) = \hat{d}(\lambda_i \text{mod } K).
\]

(47)

With this assumption, we can naturally define the signal subspace for graph signals as a counterpart of the SI subspace, as follows:

Definition 4. A PGS subspace of a given graph \( G \) is a space of graph signals that can be expressed as a GFT spectrum filtered by a given generator:

\[
\Lambda_{\text{PGS}} = \left\{ x[n] | x[n] = \sum_{i=0}^{N-1} \hat{d}(\lambda_i \text{mod } K) A(\lambda_i) u_i[n] \right\},
\]

(48)

where \( A(\lambda_i) \) is the graph frequency domain response of the generator and \( \hat{d}(\lambda_i) \) is an expansion coefficient.

This signal can be represented in matrix form as

\[
x := A\hat{d} = UA(\Lambda)D_{\text{samp}}^\top \hat{d}
\]

(49)

where \( \hat{d} := [\hat{d}(\lambda_0), \ldots, \hat{d}(\lambda_{K-1})]^\top \).

In fact, bandlimited graph signals are a special case of signals in a PGS subspace. Suppose that \( A(\lambda_i) \) is a bandlimiting low-pass filter \( G_{\text{BL}, K}(\lambda_i) \), i.e.,

\[
G_{\text{BL}, K}(\lambda_i) = \begin{cases} 1 & i \in [0, K - 1], \\ 0 & \text{otherwise}. \end{cases}
\]

(50)

The graph signal \( x \) generated by (49) completely maintains \( \hat{d} \): It is \( K \)-bandlimited under the GFT basis \( U \). However, the graph signal generated by (48) with an arbitrary \( A(\lambda_i) \) is in general not necessarily bandlimited; thus, our generalized sampling introduced in the following sections allows for reconstruction of non-bandlimited graph signals.
Suppose that $T$ in (5) is a positive integer, i.e., the spectra $D(e^{j\omega T})$ is repeated $T$ times within $\omega \in [0, 2\pi]$, and $A(\omega)$ in (5) has the support $\omega \in [0, 2\pi]$. In this case, a sequence $X[i] = D(e^{j\omega T})A(\omega)|_{\omega=2\pi i/N}$ $(i = 0, \ldots, N - 1)$ corresponds to the DFT spectrum of length $N$. Therefore, this $X[i]$ can be regarded as a graph signal spectrum in a PGS subspace when $U^*$ is the DFT matrix, e.g., the graph $G$ is the circular graph.

B. Unconstrained Case

Our solutions for generalized graph sampling can be defined following the general Hilbert space results of Section 11. Owing to the definition of the PGS subspace and sampling in the graph frequency domain, the sampling, correction, and reconstruction filters can all be implemented in the graph frequency domain.

1) Solutions: For the unconstrained solution, we can use a reconstruction filter $W(\lambda_i) = A(\lambda_i)$ in (48). Suppose that the DS condition (8) is satisfied for the signal and sampling subspaces. Following the expression in (9), the signal recovery is given as

$$
\hat{x} = A(S^*A)^{-1}S^*x
= A(S^*A)^{-1}S^*Ad
= A\hat{d} = x,
$$

where the correction filter is

$$
H = (S^*A)^{-1}.
$$

Its graph frequency response is

$$
H(\lambda_i) = \frac{1}{R_{SA}(\lambda_i)},
$$

where

$$
R_{SA}(\lambda_i) := \sum_{\ell} S(\lambda_i+K\ell)A(\lambda_i+K\ell).
$$

The inverse of $R_{SA}(\lambda_i)$ is well defined with the DS condition. Note the similarity with (10).

The solution for the LS and MX strategies when $A$ and $S$ intersect can be derived from (12) as

$$
\hat{x} = A(S^*A)^{\dagger}S^*x.
$$

The correction filter in this case is $H = (S^*A)^{\dagger}$ and it has the same graph frequency response as (56) but $H(\lambda_i) = 0$ for $\lambda_i$ with $R_{SA}(\lambda_i) = 0$.

2) Special Cases: Suppose that both the generator and sampling filters are $A(\Delta) = S(\Delta) = G_{BL,K}(\Delta)$ in (50). Subsequently $H(\lambda_i) = 1$ and no correction filter is required. This is equivalent to the perfect recovery condition for bandlimited graph signals using graph frequency domain sampling (55).

Another interesting case is the bipartite graph introduced in Section IV-B. For example, suppose that $S(\lambda_i)$ in (42) is $G_{BL,N/2}(\lambda_i)$ and the generator is $A(\lambda_i) = G_{IR}(\lambda_i)$ with

$$
G_{IR}(\lambda_i) = \begin{cases} 
1 & \lambda_0 \leq \lambda_i \leq 2/\lambda_{max}, \\
-\frac{2\lambda_i}{\lambda_{max}} & \lambda_i > 2/\lambda_{max}, 
\end{cases}
$$

where the correction filter again becomes $H(\lambda_i) = 1$; therefore, $W(\lambda_i) = A(\lambda_i) = G_{IR}(\lambda_i)$. This implies that the non-bandlimited graph signal can be perfectly reconstructed from bandlimited measurements by performing the same filtering as in the generation process without an explicit correction filter.

C. Predefined Case

Suppose that the reconstruction filter $W(\lambda_i)$ is predefined. The reconstructed signal $\hat{x}$ will in general be different from $x$ in this case. As in the unconstrained setting introduced in the previous subsection, the correction transforms in our framework are given by graph spectral filters.

If $A$ and $S$ satisfy the DS condition in (8), the solution in (55) is reduced to

$$
H = (W^*W)^{-1}W^*A(S^*A)^{-1}.
$$

The corresponding graph filter is

$$
H(\lambda_i) = \frac{\tilde{R}_{WA}(\lambda_i)}{R_{SA}(\lambda_i)\tilde{R}_{WW}(\lambda_i)}.
$$

If $W(\lambda_i) = A(\lambda_i)$, the response above is identical to that of the unconstrained case shown in (56).

Without the DS condition, we can apply the LS and MX strategies. The LS solution is

$$
\hat{x} = W(W^*W)^{-1}W^*A(S^*A)^{\dagger}S^*x,
$$

where the correction filter $H = (W^*W)^{\dagger}$ has spectral response

$$
H(\lambda_i) = \begin{cases} 
\frac{1}{R_{SW}(\lambda_i)} & \tilde{R}_{SW}(\lambda_i) \neq 0, \\
0 & \text{otherwise}.
\end{cases}
$$

The MX solution becomes

$$
\hat{x} = (W^*W)^{-1}W^*A(S^*A)^{\dagger}S^*x,
$$

with

$$
H = (W^*W)^{-1}W^*A(S^*A)^{\dagger}.
$$

The spectral response of the filter now is the same as in (56) but $H(\lambda_i) = 0$ if the denominator is zero.

The graph correction filters are summarized in Table I. The table demonstrates nicely the similarities with the SI sampling.

VI. SIGNAL RECOVERY WITH SMOOTHNESS PRIOR

The subspace prior introduced in the previous section enables the input graph signal to be recovered perfectly; however, it requires the full knowledge of the given graph and generator. In this section, we consider a less restrictive assumption. We still assume that the GFT basis $U$ is given; however, the generator function $A(\lambda_i)$ is unknown.

We assume that the graph signal is smooth on the given graph where smoothness is measured by the signal energy in the high graph-frequency components as in the SI setting (21). Although several possible operators exist for measuring signal smoothness on a graph [20], we consider a simple quadratic form of $x$:

$$
||Vx||_2^2 = x^*V^2x = \sum_{i=0}^{N-1} V^2(\lambda_i)||\hat{x}(\lambda_i)||^2
$$
where \( V := UV(\Lambda)U^* \) is an arbitrary graph filter with spectral response \( V(\lambda_i) \). If we assume \( V = L^{1/2} \), \( \|Vx\|_2^2 = x^*Lx \), which is a Laplacian quadratic form. Hereinafter, we assume \( V(\lambda_i) \neq 0 \) for all \( i \) for simplicity.

For the unconstrained case, the LS recovery is given from (23) by

\[
\hat{x} = \tilde{W}(S^*\tilde{W})^{-1}S^*x, \tag{64}
\]

where \( \tilde{W} = (V^*V)^{-1}S = UV(\Lambda)U^* \) and \( S^*V^{-2}S = D_{\text{samp}}S^2(\Lambda)V^{-2}(\Lambda)D_{\text{samp}}^T \) is invertible if \( \tilde{R}_{SS}(\lambda_i) \neq 0 \) for all \( i \). This results in \( H(\lambda_i) = \frac{1}{\tilde{R}_{SW}(\lambda_i)} \) where the spectral response is

\[
H(\lambda_i) = \frac{\tilde{R}_{WW}(\lambda_i)}{\tilde{R}_{SW}(\lambda_i)\tilde{R}_{WW}(\lambda_i)}. \tag{65}
\]

The MX solution coincides with (65) as in the SI solution.

We next consider the predefined case. For the LS criterion, the solution in Hilbert space (27) reduces to the constrained LS solution with subspace prior (60). This does not depend on \( V(\lambda_i) \), i.e., the smoothness prior does not affect the solution.

The MX solution can be obtained from (30):

\[
\hat{x} = W(W^*W)^{-1}W^*\tilde{W}(S^*\tilde{W})^{-1}S^*x. \tag{66}
\]

This results in

\[
H = (W^*W)^{-1}W^*\tilde{W}(S^*\tilde{W})^{-1}, \tag{67}
\]

with spectral response

\[
H(\lambda_i) = \frac{\tilde{R}_{WW}(\lambda_i)}{\tilde{R}_{SW}(\lambda_i)\tilde{R}_{WW}(\lambda_i)}. \tag{68}
\]

The smoothness prior \( V(\lambda_i) \) is incorporated appropriately in the correction filter, in contrast to the LS solution.

These correction filters are summarized in Table I.

VIII. SIGNAL RECOVERY EXPERIMENTS

In this section, we validate the proposed generalized sampling methods through signal recovery experiments.

A. Recovery Experiments for Bandlimited and Non-bandlimited Samplings

The graph used is a random sensor graph with \( N = 64 \). We downsampled the input signal by a factor of two such that \( K = 32 \). We used the following functions:

- Generator function,

\[
A(\lambda_i) = 1 - \lambda_i/(\lambda_{\text{max}} + \epsilon) \tag{71}
\]

- Sampling functions,

\[
S(\lambda_i) = \begin{cases} G_{\text{BL},K}(\lambda_i) & \text{for bandlimited sampling} \\ G_{\text{IR}}(\lambda_i) & \text{for non-bandlimited sampling} \end{cases} \tag{72}
\]

- Reconstruction function (used only for the predefined solutions),

\[
W(\lambda_i) = \cos\left(\frac{\pi}{2}, \frac{i}{\lambda_{\text{max}} + \epsilon}\right) \tag{73}
\]

- Smoothness function (used only for the smoothness prior),

\[
V(\lambda_i) = \lambda_i/\lambda_{\text{max}} + 1 \tag{74}
\]

We set \( \epsilon = 0.1 \). All the functions are visualized in Fig. 5. It is noteworthy that \( A(\lambda_i) \) is not bandlimited; therefore, the original signal retains its full band. Each element in the expansion coefficients \( d \) is a random variable drawn from
Table II

| Prior | Solution/Strategy | BL sampling | Non-BL sampling |
|-------|-------------------|-------------|-----------------|
|       | No noise | Noisy | No noise | Noisy |
| Subspace | Unconstrained | -297.86  | -12.56  | -298.80  | -12.81  |
|        | Predefined: DS and MX | -20.16  | -11.91  | -20.24  | -12.11  |
|        | Predefined: LS* | -18.83  | -11.36  | -20.15  | -12.17  |
| Smoothness | Unconstrained | -10.44  | -8.55  | -24.26  | -12.39  |
|        | Predefined: MX | -14.98  | -11.34  | -20.21  | -12.06  |
| BL sampling and reconstruction | -10.44  | -8.55  | -8.42  | -6.67  |

* Same as the predefined solution for smoothness prior with LS strategy.

The non-bandlimited sampling results are summarized in Table II together with the visualization of the reconstructed signals in Fig. 7. The results tend to be similar to those of the bandlimited case; however, the unconstrained solution for the smoothness prior now presents a significant gain compared to the bandlimited reconstruction, and it also outperforms the predefined reconstruction, as expected.

B. Recovery Experiment on Bipartite Graphs with Vertex Domain Sampling

The second experiment demonstrates recovery of non-bandlimited graph signals from only vertex domain sampling, as described in Section IV-B and V-B. To the best of our knowledge, this example is the first attempt to recover non-bandlimited graph signals only from vertex domain operations.

In the signal recovery of this experiment, we set $S(\Lambda) = G_{BL,N/2}(\Lambda)$ and $A(\lambda_i) = W(\lambda_i) = G_{IR}(\lambda_i)$ as described in Section IV-B. Because the subsampling itself is performed in the vertex domain, all operations can be represented as the vertex domain operators if $G$ and $W'$ in (40) can be represented as vertex domain filters. Although the filters we used cannot be represented as vertex domain filters, we can always perform polynomial approximations such as the Chebyshev polynomial approximation (CPA) [22], [45]. As described in Section III-A the $P$th order CPA of an arbitrary graph spectral filter corresponds to a vertex domain filter with $P$-hop localization.

In this experiment, the CPA is applied to the sampling and reconstruction filters $S(\Lambda)$ and $W'(\Lambda)$ in the sampling-then-reconstruction operation as they output a non-polynomial spectral response. Consequently, we can approximately recover the full-band graph signal $x$.

The original signal $x$ is obtained as follows:

$$x = W'V_d,$$

where each element in $d$ is a random variable drawn from the normal distribution $\mathcal{N}(1, 1)$. Figure 8 shows the average MSEs of the reconstructed signals after 100 independent runs according to the polynomial order. For comparison, we also plot the MSE of the bandlimited reconstruction where we use $S(\Lambda) = G_{BL,N/2}(\Lambda)$ as the reconstruction filter, in which $\tilde{z}$ denotes the polynomial approximated filter. As shown in Fig. 8, the reconstruction error decreases monotonically as $P$ becomes larger. The reconstructed signals are also shown in

2Note that the generation process in (75) uses the non-polynomial $W'$. 

$N(1, 1)$. Examples of $x$ generated by $A(\lambda_i)$ in (71) are shown in Figs. 6(a) and 7(a).

For comparison, we perform bandlimited signal recovery: $S(\lambda_i) = W(\lambda_i) = G_{BL,K}(\lambda_i)$ with no correction filter $H(\lambda_i) = 1$. We perform two samplings in the experiment to highlight the difference between the proposed sampling and the bandlimited sampling as in (72).

We performed 1000 independent runs and calculated the average MSEs. Furthermore, we also repeated the experiments with zero-mean Gaussian noise with variance $\sigma^2 = 0.1$ added to $x$.

Table II summarizes the average MSEs for bandlimited sampling.

**Noiseless Signals**: The unconstrained solution for the subspace prior perfectly recovers the original signal with machine precision. The predefined solutions for both the subspace and smoothness priors contain some reconstruction errors; however, they are much smaller than those in the bandlimited reconstruction. The unconstrained solution with a smoothness prior (64) yields the same results as those in bandlimited sampling and reconstruction when using $G_{BL,K}(\lambda_i)$ as the sampling filter.

**Noisy Signals**: All methods contain increased errors for noisy cases, as expected. The unconstrained solution for the subspace prior demonstrated a significantly worse performance than that of the noiseless case, as it did not assume any smoothness of the reconstructed graph signals. The predefined filters, both with the subspace and smoothness priors, demonstrated performances that were close to those of unconstrained solutions because their reconstruction filters yielded a smooth signal. The MX solution with the smoothness prior in (68) exhibited an extremely close MSE to that of the other predefined solutions. The bandlimited reconstruction still exhibits a significant gap from generalized sampling.
Fig. 6. Signal recovery experiments with bandlimited sampling.

(h) Spectra: SS and SM refer to the subspace and smoothness priors. UNC and PD also refer to the unconstrained and predefined solutions.

Fig. 7. Signal recovery experiments with non-bandlimited sampling.
The right and left vertex sets correspond to $V_1$ (retained) and $V_2$ (discarded), respectively. The expansion coefficients $d$ are drawn from $N(0.25 \times 10^{-2}, 1)$ for clear visualization. Chebyshev polynomial approximation of order 16 is used both for $W(A)$ and $S(A)$.

Fig. 9. The bandlimited reconstruction yields large errors while the proposed reconstruction exhibits very similar signal values to the original ones.

Future studies on this type of generalized sampling, especially for the non-bipartite case, is an interesting topic for research.

IX. CONCLUSIONS

In this study, we proposed a framework for generalized sampling of graph signals. We assumed that the graph signals lie in the PGS subspace that is an extension of the SI subspace in standard signal processing to the graph setting. Sampling is defined in the graph frequency domain. We considered two priors of the graph signals, the subspace and smoothness priors, which are parallel to those studied for signals in the SI subspace. All filters used in our framework can be represented as graph spectral filters. Numerical experiments revealed that our proposed sampling can recover a class of sampled signals that is broader than that obtained with the existing graph sampling theories. Finally, we also presented perfect recovery of non-bandlimited graph signals on bipartite graphs without involving the GFT domain.

REFERENCES

[1] A. Anis, A. Gadde, and A. Ortega, “Efficient sampling set selection for bandlimited graph signals using graph spectral proxies,” IEEE Trans. Signal Process., vol. 64, no. 14, pp. 3775–3789, Jul. 2016.

[2] S. Chen, R. Varma, A. Sandryhaila, and J. Kovacevic, “Discrete signal processing on graphs: Sampling theory,” IEEE Trans. Signal Process., vol. 63, no. 24, pp. 6510–6523, Dec. 2015.

[3] A. G. Marques, S. Segarra, G. Leus, and A. Ribeiro, “Sampling of graph signals with successive local aggregations,” IEEE Trans. Signal Process., vol. 64, no. 7, pp. 1832–1843, 2016.

[4] S. P. Chepuri, Y. C. Eldar, and G. Leus, “Graph sampling with and without input priors,” in Proc. IEEE Int. Conf. Acoust., Speech and Signal Process. (ICASSP), 2018, pp. 4564–4568.

[5] G. Puy, N. Tremblay, R. Gribonval, and P. Vandergheynst, “Random sampling of bandlimited signals on graphs,” Applied and Computational Harmonic Analysis, vol. 44, no. 2, pp. 446–475, Mar. 2018.

[6] M. Tsitsvero, S. Barbarossa, and P. Di Lorenzo, “Signals on graphs: Uncertainty principle and sampling,” IEEE Trans. Signal Process., vol. 64, no. 18, pp. 4845–4860, Sep. 2016.

[7] D. Vallesia, G. Pracastoro, and E. Magli, “Sampling of graph signals via randomized local aggregations,” IEEE Trans. Signal Inf. Process. Netw., pp. 1–1, 2018, in press.

[8] I. Pesenson, “Sampling in Paley–Wiener spaces on combinatorial graphs,” Transactions of the American Mathematical Society, vol. 360, no. 10, pp. 5603–5627, 2008.

[9] I. Pesenson and M. Pesenson, “Sampling, filtering and sparse approximations on combinatorial graphs,” Journal of Fourier Analysis and Applications, vol. 16, no. 6, pp. 921–942, 2010.

[10] A. Sakiyama, Y. Tanaka, T. Tanaka, and A. Ortega, “Eigendecomposition-free sampling set selection for graph signals,” IEEE Trans. Signal Process., vol. 67, no. 10, pp. 2679–2692, May 2019.

[11] C. E. Shannon, “Communication in the presence of noise,” Proc. Inst. Radio Eng., vol. 37, no. 1, pp. 10–21, 1949.

[12] A. J. Jerri, “The Shannon sampling theorem—its various extensions and applications: A tutorial review,” Proc. IEEE, vol. 65, no. 11, pp. 1565–1596, 1977.

[13] Y. C. Eldar and T. Michaeli, “Beyond bandlimited sampling,” IEEE Signal Process. Mag., vol. 26, no. 3, pp. 48–68, May 2009.

[14] Y. C. Eldar, “Sampling with arbitrary sampling and reconstruction spaces and oblique dual frame vectors,” J. Fourier Analysis and Applications, vol. 9, no. 1, pp. 77–96, 2003.

[15] Y. C. Eldar and T. G. Dvorkind, “A minimum squared-error framework for generalized sampling,” IEEE Trans. Signal Process., vol. 54, no. 6, pp. 2155–2167, Jun. 2006.

[16] M. Unser, “Sampling—50 years after Shannon,” Proc. IEEE, vol. 88, no. 4, pp. 569–587, 2000.

[17] M. Unser and A. Aldroubi, “A general sampling theory for nonideal acquisition devices,” IEEE Trans. Signal Process., vol. 42, no. 11, pp. 2915–2925, Nov. 1994.

[18] Y. C. Eldar, A. Ben-Tal, and A. Nemirovski, “Linear minimax regret estimation of deterministic parameters with bounded data uncertainties,” IEEE Trans. Signal Process., vol. 52, no. 8, pp. 2177–2188, Aug. 2004.

[19] Y. C. Eldar, Sampling theory: Beyond bandlimited systems. Cambridge University Press, 2015.

[20] D. I. Shuman, S. K. Narang, P. Frossard, A. Ortega, and P. Vandergheynst, “The emerging field of signal processing on graphs: Extending high-dimensional data analysis to networks and other irregular domains,” IEEE Signal Process. Mag., vol. 30, no. 3, pp. 83–98, Oct. 2013.

[21] A. Ortega, P. Frossard, J. Kovacevic, J. M. F. Moura, and P. Vandergheynst, “Graph signal processing: Overview, challenges, and applications,” Proc. IEEE, vol. 106, no. 5, pp. 808–828, May 2018.

[22] D. K. Hammond, P. Vandergheynst, and R. Gribonval, “Wavelets on graphs via spectral graph theory,” Applied and Computational Harmonic Analysis, vol. 30, no. 2, pp. 129–150, Mar. 2011. [Online]. Available: [http://wiki.epfl.ch/swg]

[23] S. K. Narang and A. Ortega, “Compact support biorthogonal wavelet filterbanks for arbitrary undirected graphs,” IEEE Trans. Signal Process., vol. 61, no. 19, pp. 4673–4685, Oct. 2013. [Online]. Available: [http://biron.usc.edu/wiki/index.php/Graph_Filterbanks]

[24] Y. Tanaka and A. Sakiyama, “M-channel oversampled graph filter banks,” IEEE Trans. Signal Process., vol. 62, no. 14, pp. 3578–3590, Jul. 2014.

[25] A. Sakiyama, K. Watanabe, Y. Tanaka, and A. Ortega, “Two-channel critically-sampled graph filter banks with spectral domain sampling,” IEEE Trans. Signal Process., vol. 67, no. 6, pp. 1447–1460, Mar. 2019.

[26] X. Dong, D. Thanou, P. Frossard, and P. Vandergheynst, “Learning Laplacian matrix in smooth graph signal representations,” IEEE Trans. Signal Process., vol. 64, no. 23, pp. 6160–6173, 2016.

[27] V. Kalofolias, “How to learn a graph from smooth signals,” in Proc. AISTATS'16, 2016.
[28] H. E. Egilmez, E. Pavez, and A. Ortega, “Graph learning from data under laplacian and structural constraints,” *IEEE J. Sel. Topics Signal Process.*, vol. 11, no. 6, pp. 825–841, Sep. 2017.

[29] M. Onuki, S. Ono, M. Yamagishi, and Y. Tanaka, “Graph signal denoising via trilateral filter on graph spectral domain,” *IEEE Trans. Signal Inf. Process. Netw.*, vol. 2, no. 2, pp. 137–148, Jun. 2016.

[30] S. Ono, I. Yamada, and I. Kumazawa, “Total generalized variation for graph signals,” in *Proc. IEEE Int. Conf. Acoust. Speech, Signal Process.*, 2015, pp. 5456–5460.

[31] G. Cheung, E. Magli, Y. Tanaka, and M. Ng, “Graph spectral image processing,” *Proc. IEEE*, vol. 106, no. 5, pp. 907–930, May 2018.

[32] M. M. Bronstein, J. Bruna, Y. LeCun, A. Szlam, and P. Vandergheynst, “Geometric deep learning: Going beyond euclidean data,” *IEEE Signal Process. Mag.*, vol. 34, no. 4, pp. 18–42, Jul. 2017.

[33] A. Krause and C. Guestrin, “Near-optimal observation selection using submodular functions,” in *Proc. AAAI*, vol. 7, 2007, pp. 1650–1654.

[34] A. Krause, A. Singh, and C. Guestrin, “Near-optimal sensor placements in gaussian processes: Theory, efficient algorithms and empirical studies,” *Journal of Machine Learning Research*, vol. 9, pp. 235–284, 2008.

[35] Y. Tanaka, “Spectral domain sampling of graph signals,” *IEEE Trans. Signal Process.*, vol. 66, no. 14, pp. 3752–3767, Jul. 2018.

[36] Y. C. Eldar and T. Werther, “General framework for consistent sampling in Hilbert spaces,” *Int. J. Wavelets, Multiresolution, Inform. Process.*, vol. 3, no. 3, pp. 347–359, Sep. 2005.

[37] A. Hirabayashi and M. Unser, “Consistent sampling and signal recovery,” *IEEE Trans. Signal Process.*, vol. 55, no. 8, pp. 4104–4115, Aug. 2007.

[38] T. G. Dvorkind and Y. C. Eldar, “Robust and consistent sampling,” *IEEE Signal Process. Lett.*, vol. 16, no. 9, pp. 739–742, Sep. 2009.

[39] M. Vetterli, J. Kovacevic, and V. K. Goyal, *Foundations of Signal Processing*. Cambridge University Press, 2014.

[40] Y. Tanaka and Y. C. Eldar, “Generalized sampling on graphs with a subspace prior,” in *Proc. International Conference on Sampling Theory and Applications (SampTA)*, 2019.

[41] D. I. Shuman, B. Ricaud, and P. Vandergheynst, “Vertex-frequency analysis on graphs,” *Applied and Computational Harmonic Analysis*, vol. 40, no. 2, pp. 260–291, Mar. 2016.

[42] G. Strang, “The discrete cosine transform,” *SIAM Rev.*, vol. 41, no. 1, pp. 135–147, 1999.

[43] F. Dorfler and F. Bullo, “Kron reduction of graphs with applications to electrical networks,” *IEEE Trans. Circuits Syst. I*, vol. 60, no. 1, pp. 150–163, Jan. 2013.

[44] D. I. Shuman, M. J. Faraji, and P. Vandergheynst, “A multiscale pyramid transform for graph signals,” *IEEE Trans. Signal Process.*, vol. 64, no. 8, pp. 2119–2134, Apr. 2016.

[45] D. I. Shuman, P. Vandergheynst, and P. Frossard, “Chebyshev polynomial approximation for distributed signal processing,” in *Proc. Int. Conf. Distrib. Comput. Sensor Syst.*, 2011, pp. 1–8.