Bodies of zero total cross section and bodies invisible in one direction

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Abstract

We consider a body in a parallel flow of non-interacting particles. The interaction of particles with the body is perfectly elastic. We introduce the notions of a body of zero resistance and an invisible body and prove that all such bodies do exist.

Introduction

Suppose that there is parallel flow of non-interacting particles falling on $B \subset \mathbb{R}^3$ which is a bounded connected set with piecewise smooth boundary. Initially, the velocity of a particle equals $v_0 \in S^2$; then it makes several reflections from $B$, and finally moves freely with the velocity $v_B^+(x, v_0)$, where $x \in \{v_0\}^\perp$ indicates the initial position of the particle. One can imagine that the flow is highly rarefied or consists of light rays. The force of pressure of the flow on the body, or resistance of the body in the direction $v_0$, is proportional to

$$R_{v_0}(B) := \int_{\{v_0\}^\perp} (v_0 - v_B^+(x, v_0)) \, dx,$$

where the ratio equals the density of the flow/medium and $dx$ means the Lebesgue measure in $\{v_0\}^\perp$. The problem of minimal resistance is concerned with minimizing the resistance in a prescribed class of bodies. There is a large literature on this problem, starting from the famous Newton’s aerodynamic problem \cite{1}.

We say that the body is invisible in the direction $v_0$ if the trajectory of each particle outside a prescribed bounded set coincides with a straight line. We prove that there exist bodies of zero resistance and bodies invisible in one direction.

Consider the class of bodies $B$ that are contained in the cylinder $\Omega \times [0, h]$ and contain a cross section $\Omega \times \{c\}$, $c \in [0, h]$. For the sake of brevity, we shall call them bodies inscribed in the cylinder. Multiple reflections are allowed. If $\Omega$ is the unit circle then the infimum of resistance equals zero, $\inf_B |R_{v_0}(B)| = 0$ (see \cite{2}). The infimum is not attained, that is, zero resistance bodies do not exist. This follows from the following simple proposition.

Lemma. Let $\Omega$ be a convex set with nonempty interior and let $B$ be a body inscribed in the cylinder $\Omega \times [0, h]$. Then $R_{v_0}(B) \neq 0$.

Proof. Using that the particle trajectory does not intersect the section $\Omega \times \{c\}$ and $\Omega$ is convex, one concludes that the particle initially moves in the cylinder above this section, then intersects the lateral surface of the cylinder and moves freely afterwards. This implies that $v_B^+(x, v_0) \neq v_0$, hence $R_{v_0}(B) \neq 0$.

1 Zero resistance bodies and invisible bodies

Theorem. There exist (a) a body that has zero resistance in the direction $v_0$; (b) a body invisible in the direction $v_0$.

Proof. (a) Consider two identical coplanar equilateral triangles $ABC$ and $A'B'C'$, with $C$ being the midpoint of the segment $A'B'$, and $C'$, the midpoint of $AB$. The vertical line $CC'$ is parallel to $v_0$. Let $A''$ ($B''$) be the point of intersection of segments $AC$ and $A'C'$ ($BC$ and $B'C'$, respectively); see Fig. 1. The body $B$ generated by rotation of the triangle $AA'A''$ (or $BB'B''$) around the axis $CC'$ is shown on Fig. 2a. It has zero resistance in the direction $v_0$.

Let the particle first hit the segment $A'A''$ at a

Figure 1: The basic construction.

\begin{itemize}
\item The particle first hits the segment $A'A''$ at a
point E. (If the particle first hits $B'B'', \text{ the argument is the same.})$ After the reflection, the direction of motion forms the angle $\pi/3$ with the vertical. Next, the particle hits the segment $B'B$ at the point F such that $|A'E| = |B'F|$, and after the second reflection moves vertically downward. That is, the final velocity equals $v_0$.

(b) A body invisible in the direction $v_0$ can be obtained by doubling a zero resistance body; see Fig. 3.

Denote by $m = m(\mathcal{B}, v_0)$ the maximal number of reflections of an individual particle from the body.

**Theorem** (a) If the body $\mathcal{B}$ has zero resistance in the direction $v_0$ then $m(\mathcal{B}, v_0) \geq 2$. (b) If $\mathcal{B}$ is invisible in the direction $v_0$ then $m(\mathcal{B}, v_0) \geq 4$. These inequalities are sharp: there exist zero resistance bodies and invisible bodies with exactly 4 reflections.

**Proof.** (a) If $m = 1$ then the final velocity of each particle does not coincide with the initial one, $v_0^+(x, v_0) \neq v_0$, therefore $R_{v_0}(\mathcal{B}) \neq 0$. That is, a zero resistance body requires at least two reflections.

(b) Note that a thin parallel beam of particles changes the orientation under each reflection. To be more precise, let $x(t) = x + v_0t$, $v(t) = v_0$ be the initial motion of a particle, and let $x(t) = x^{(i)}(x) + v^{(i)}(x)t$, $v(t) = v^{(i)}(x)$ be its motion between the $i$th and $(i + 1)$th reflections, $i = 0, 1, \ldots, m$. Let the body be invisible in the direction $v_0$; then one has $v^{(0)} = v^{(m)} = v_0$, $x^{(0)} = x$, and $x^{(m)} - x \perp v_0$.

At each reflection and for any fixed $x$, the orientation of the triple $(\frac{\partial x^{(i)}}{\partial x_1}, \frac{\partial x^{(i)}}{\partial x_2}, v^{(i)})$ changes. The initial and final orientations, $(\frac{\partial x^{(0)}}{\partial x_1}, \frac{\partial x^{(0)}}{\partial x_2}, v^{(0)})$ and $(\frac{\partial x^{(m)}}{\partial x_1}, \frac{\partial x^{(m)}}{\partial x_2}, v^{(m)})$, coincide, therefore $m$ is even.

One easily sees that with $m = 2$, the initial and final parts of a trajectory cannot belong to a straight line; this proves that $m = 4$.

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