Besov priors in density estimation: optimal posterior contraction rates and adaptation

Matteo Giordano
Department of Statistics, University of Oxford

Abstract

Besov priors are nonparametric priors that model spatially inhomogeneous functions. They are routinely used in inverse problems and imaging, where they exhibit attractive sparsity-promoting and edge-preserving features. A recent line of work has initiated the study of the asymptotic frequentist convergence properties of Besov priors. In the present paper, we consider the theoretical recovery performance of the associated posterior distributions in the density estimation model, under the assumption that the observations are generated by a spatially inhomogeneous true density belonging to a Besov space. We improve on existing results and show that carefully tuned Besov priors attain optimal posterior contraction rates. Furthermore, we show that a hierarchical procedure involving a hyper-prior on the regularity parameter leads to adaptation to any smoothness level.

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1 Introduction

Besov priors are a class of probability measures on functions spaces constructed via random wavelet expansions, with independently drawn random wavelet coefficients following weighted Laplace distributions. They were first systematically studied by Lassas et al. [33] and, over the last two decades, have enjoyed enormous popularity within the inverse problems [36, 13, 31, 32, 42, 9, 26, 27, 14, 2] and medical imaging [7, 41, 40, 49, 45] communities. Among the main advantages, Besov priors provide a ‘discretisation-invariant’ alternative [34, 33] to the popular total-variation prior of Rudin et al. [44], and give rise to sparse and edge-preserving reconstruction at the level of the maximum-a-posteriori estimator [31, 45, 2], observed to perform well in practice in the recovery of inhomogeneous objects with localised sharp irregularities such as images. See [41, 33, 13] and references therein. Alongside these qualities, Besov priors also possess a logarithmically concave structure, which facilitates posterior computation [9, 12] and theoretical analysis.

The aforementioned features stem from the $\ell^1$-type penalty induced by Besov priors on the wavelet coefficients. This furnishes a Bayesian model for functions in the Besov scale $B_{1,1}^s$, $s > 0$, in which the local variability is measured in an $L^1$-sense and the smoothness can be described by the $\ell^1$-decay of the wavelet coefficients (cf. Section 2.1 for definitions). These spaces are known to provide a mathematical characterisation of spatially inhomogeneous functions, namely ones that are flat or smooth in some parts of their domain while exhibiting high variation (or even jumps) in other areas, e.g., [15, Section 1]. Besov priors are thus naturally suited to a number of applications dealing with spatially inhomogeneous objects including, as mentioned above, imagining, but also geophysics, where parameters can exhibit jumps corresponding to changes in layered media, and signal processing, where signals can have extremely localised spikes. In contrast, the widely used Gaussian priors, which induce $\ell^2$-type penalties, model Sobolev-regular functions with less sharp variation, and are known to be unsuited to more structured recovery problems [3, 5, 24].

Despite the popularity in applications, a comprehensive study of the theoretical properties of Besov priors has been initiated only very recently. In particular, we are here interested in the asymptotic recovery performance of the associated posterior distributions in the large sample size setting, under the assumption that the available data is generated by a fixed true function (the ‘ground truth’). Such frequentist analysis of Bayesian nonparametric procedures, and the related theory of posterior contraction rates, have seen extremely vast developments since seminal work in the 2000s [16, 46, 18, 20, 47], leading to an extensive literature that encompasses a large number of prior distributions and statistical models; see the monograph [19]. For Besov priors, the first general results were derived in a recent paper by Agapiou et al. [3], based on a study of the concentration properties of product measures with heavier-than-Gaussian
tails (extending the results for Gaussian priors of [47]). They showed that, in the Gaussian white noise model, suitably ‘rescaled’ and ‘under-smoothing’ Besov prior contracts at optimal rates towards spatially inhomogeneous ground truths in the Besov scale. In the preprint [5], this result is extended to nonlinear inverse problems. A further related reference is [23], where optimal rates for Sobolev-regular ground truths are obtained in a reversible multi-dimensional diffusion model. We also refer to [10, 6, 42] for some earlier results, and to the preprint [43] for results on spike-and-slab and tree-type priors in nonparametric regression under a spatially varying Hölder smoothness assumption.

Notably, alongside the white noise setting, [3] also considered density estimation. In this case, however, the more involved concentration properties of Laplace priors compared to Gaussian priors leads to some intricate complexity bounds, ultimately resulting in their Theorem 6.7 in sub-optimal posterior contraction rates. More discussion can be found in Section 3.1.1.

The aim of the present paper is to contribute to this recent line of work. We focus on the density estimation setting, motivated by the investigation of [3]. Our first main result (Theorem 1) shows that, for spatially inhomogeneous true densities belonging to the Besov scale, carefully tuned Besov priors do attain optimal rates. We prove this via the general contraction rate theory for independent and identically distributed (i.i.d.) observations of Ghosal et al. [16], employing rescaled and under-smoothing Besov priors similar to those used in [3, Section 5] and in [5, 23], which we show in the proof to yield tighter complexity bounds. Theorem 1 thus extends the white noise result of [3] to the density estimation setting, reconciling the theory for Besov priors in the two statistical models.

An interesting aspect of the above result is the combination in the design of the prior of rescaling and under-smoothing, whose interplay allows to balance the bias associated to spatially inhomogeneous ground truths to the variance component; see the discussion after Theorem 1. This separates the existing theory for Besov priors to the one for Gaussian priors, where under traditional regularity assumptions, optimal rates are typically obtained with priors of matching smoothness, e.g., [47, 21, 19]. To shed additional light on this, we explore the role of rescaling and under-smoothing in Sections 3.1.2 and 3.1.3. In the former, we show that, for true densities in the Besov scale, the same optimal rates of Theorem 1 are attained (up to a logarithmic factor), by partially rescaled under-smoothing Besov priors. These are obtained by rescaling only the wavelets at resolutions larger than a prescribed threshold, providing an (arguably more natural) alternative prior construction that is not constrained to asymptotically shrink uniformly towards zero. In 3.1.3, we instead consider the recovery of spatially homogeneous ground truths, and show that in this case rescaling and under-smoothing are unnecessary, as (non-rescaled) Besov priors with matching regularity attain optimal posterior contraction rates.

For all the results described above, the specification of the prior requires knowledge of the smoothness of the unknown true density, which is often an unrealistic assumption. To overcome such limitation, we investigate adaptation to the unknown smoothness in Section 3.2. Following the well-established hierarchical Bayesian approach (e.g., [19, Chapter 10]), we consider a hierarchical, conditionally Besov prior, obtained by randomising the prior regularity parameter via a further (hyper-)prior. Theorem 4 shows that, for a carefully constructed hyper-prior (of the form considered in, e.g., [35, 17, 48]), optimal posterior contraction rates are obtained for true densities in the Besov scale, simultaneously for any smoothness level. To the best of our knowledge, this represent the first study of adaptation for Besov priors, and also the first instance in the literature of a prior achieving adaptive rates over the Besov scale. After the completion of this manuscript, we learned about independent ongoing work by Agapiou and Savva [4],
exploring adaptation for Besov priors in the white noise model.

We conclude the main body of the paper in Section 4, where a summary and further discussion of the presented results can be found, alongside an outline of potential directions for future research. In Section 4.2, we discuss implementation of posterior inference with Besov priors in the present density estimation setting. All the proofs of the main results are developed in Section 5. Finally, Appendix A contains some background material on Besov priors, as well as two auxiliary results used in the proofs.

2 Bayesian density estimation with Besov priors

2.1 Function spaces and other preliminaries

Throughout, the domain of interest is the \(d\)-dimensional unit cube \([0,1]^d\), \(d \in \mathbb{N}\). We denote the usual Lebesgue spaces on \([0,1]^d\) by \(L^p([0,1]^d)\), \(p \geq 1\), equipped with norm \(\| \cdot \|_p\), and denote by \(\langle \cdot , \cdot \rangle_2\) the inner product on \(L^2([0,1]^d)\). We write \(C([0,1]^d)\) for the space of continuous functions on \([0,1]^d\), equipped with the supremum-norm \(\| \cdot \|_\infty\).

Let \(\{\psi_r, l \in \mathbb{N}, r = 1,\ldots, 2^{ld}\}\) be an orthonormal tensor product wavelet basis of \(L^2([0,1]^d)\), constructed from \(S\)-regular, \(S \in \mathbb{N}\), compactly supported and boundary-corrected Daubechies wavelets in \(L^2([0,1])\); see, e.g., [21, Chapter 4.3] for details. In what follows, we tacitly assume \(S\) to be sufficiently large, in particular, greater than the smoothness parameter \(s > 0\) appearing below. For a resolution level \(L \in \mathbb{N}\), define the finite-dimensional approximation space

\[
V_L := \text{span}\{\psi_r, \ l = 1,\ldots, L, \ r = 1,\ldots, 2^{ld}\}
\]

and let \(P_L : L^2([0,1]^d) \to V_L\) be the associated projection operator. Note that \(V_L\) has dimension \(\dim(V_L) = O(2^{ld})\) as \(L \to \infty\). For a smoothness parameter \(s > 0\) and integrability indices \(p, q \in [1,\infty]\), define the Besov space \(B^s_{pq}([0,1]^d)\) via its wavelet characterisation (cf. [21], p.370f):

\[
B^s_{pq}([0,1]^d) := \left\{ w \in L^p([0,1]^d) : \| w \|_{B^s_{pq}}^p := \sum_{l=1}^{\infty} 2^{spq(s-\frac{d}{2}+\frac{d}{p})} \left( \sum_{r=1}^{2^{ld}} |\langle w, \psi_r \rangle_2|^q \right)^{\frac{p}{q}} < \infty \right\},
\]

where the above \(\ell_p\)- and \(\ell_q\)-sequence space norms are replaced by the \(\ell_\infty\)-norm if \(p = \infty\) or \(q = \infty\) respectively. Recall that the Besov scale contains the traditional Sobolev spaces \(H^s([0,1]^d)\) and Hölder spaces \(C^s([0,1]^d)\): for all \(s > 0\), \(B^s_{22}([0,1]^d) = H^s([0,1]^d)\) and \(C^s([0,1]^d) \subseteq B^{\infty}_{\infty\infty}([0,1]^d)\) (with equality holding when \(s \notin \mathbb{N}\)). As mentioned in the introduction, for \(p = q = 1\), the spaces \(B^s_{11}([0,1]^d)\) instead provide a mathematical model for spatially inhomogeneous functions (e.g., see [15, Section 1]).

When no confusion may arise, we suppress the dependence of the function spaces on the underlying domain, writing for example \(B^s_{pq}\) instead of \(B^s_{pq}([0,1]^d)\). We denote by \(\lesssim, \gtrsim, \preceq, \succeq\), respectively one- or two-sided inequalities holding up to multiplicative constants. We write \(N(\xi; \mathcal{F}, d)\), \(\xi > 0\), for the \(\xi\)-covering number of a set \(\mathcal{F}\) with respect to a semi-metric \(d\) on \(\mathcal{P}\), defined as the minimal number of balls of radius \(\xi\) in the metric \(d\) needed to cover \(\mathcal{P}\). Positive numerical constants in the proofs are denoted by \(c_1, c_2, \cdots > 0\).

2.2 Observation model, prior and posterior

Consider a sample \(X^{(n)} := (X_1,\ldots, X_n)\) of i.i.d. random variables with values in \([0,1]^d\), \(d \in \mathbb{N}\), from an unknown probability distribution \(P_0\) with density function \(p_0\).
relative to the $d$-dimensional Lebesgue measure $dx$. This paper is concerned with the problem of estimating $p_0$ from the random sample $X^{(n)}$. The main focus is on the case where $p_0$ is spatially inhomogeneous, e.g., flat in some parts of the domain and spiky in others. A natural mathematical model for such setting is then to assume a (Borel measurable) parameter space $p_0 \in \mathcal{P} \subseteq B^{s}_{1,1}([0,1]^d)$ for some $s > 0$; see Section 2.1 for definitions and details.

We adopt a Bayesian approach, regarding the data as conditionally independent given a density $p$, $X_1, \ldots, X_n \overset{iid}{\sim} p$, and assigning to $p$ a (possibly $n$-dependent) prior distribution $\Pi_n$ supported on the parameter space $\mathcal{P}$. This induces the posterior distribution $p|X^{(n)} \sim \Pi_n(\cdot|X^{(n)})$, which by Bayes’ formula (e.g., [19], p.7) equals

$$\Pi_n(A|X^{(n)}) = \frac{\int_A \prod_{i=1}^n p(X_i) d\Pi_n(p)}{\int_{\mathcal{P}} \prod_{i=1}^n p(X_i) d\Pi_n(p)}, \quad A \subseteq \mathcal{P} \text{ measurable.}$$

In the following, we study the (frequentist) consistency of $\Pi_n(\cdot|X^{(n)})$, assuming that the observations $X_1, \ldots, X_n$ are drawn from some fixed ‘true’ unknown density $p_0 \in \mathcal{P}$ to be estimated, and investigating the asymptotic concentration of the posterior distribution around $p_0$ when $n \to \infty$.

The recovery performance of the posterior distribution is known to depend on the choice of the prior, which represents a key modelling step in the Bayesian approach. To reflect the assumed spatial inhomogeneity of $p_0$, we employ Besov priors. These were introduced by Lassas et al. [33] as random wavelet expansions with random coefficients following the Laplace distribution, and were recently shown by Agapiou et al. [3] to yield optimal estimation of spatially inhomogeneous functions in the white noise model. In accordance with the latter references, we define Besov priors for densities starting from a random function

$$W_n(x) = \sum_{l=1}^{\infty} \sum_{r=1}^{2^d} \sigma_{n,lr} W_{lr}(x), \quad x \in [0,1]^d, \quad (1)$$

where $\{\psi_{lr}, \ l \in \mathbb{N}, \ r = 1, \ldots, 2^d\}$ is an orthonormal tensor product wavelet basis of $L^2([0,1]^d)$ as described in Section 2.1, $\sigma_{n,lr} > 0$ are deterministic (possibly $n$-dependent) scaling factors satisfying

$$\sigma_{n,lr} = O(2^{-l(t+d/2)}), \quad \text{as } l \to \infty, \quad \text{some } t > 0, \quad (2)$$

and $W_{lr}$ are i.i.d. random coefficients following the Laplace distribution, whose density equals

$$\lambda(z) = \frac{1}{2} e^{-|z|}, \quad z \in \mathbb{R}. \quad (3)$$

The decay of the scaling factors determines the regularity properties of realisations of the random function $W_n$: in particular, a simple computation shows that condition (2) implies that $W_n \in C([0,1]^d) \cap B^s_{1,1}([0,1]^d)$ almost surely for all $0 < t' < t$ and $r \in [1, \infty]$ (cf. Lemma 5.2 and Proposition 6.1 in [3], or also Lemma 7.1 in [5]). Under this condition, we may regard $W_n$ as a Borel measurable random element in the separable Banach space $C([0,1]^d)$, whose law we denote by $\Pi_{W_n}$. Following the terminology of [3, 5], we call $\Pi_{W_n}$ a \textit{t-regular Besov prior} on $C([0,1]^d)$.

Given $W_n$ as in (1), we then construct a prior on the set of densities functions on $[0,1]^d$ by taking the law $\Pi_n$ of the random function

$$\phi_{W_n}(x) := \frac{\phi(W_n(x))}{\int_{[0,1]^d} \phi(W_n(x')) dx'}, \quad x \in [0,1]^d,$$
where $\phi : \mathbb{R} \rightarrow (0, \infty)$ is a positive, strictly increasing and smooth link function. A common choice is the exponential link function $\phi(z) = e^z$, $z \in \mathbb{R}$, but for some of the results to follow it will prove useful to employ link functions affording better control over the tails. In a slight abuse of terminology, we call $\Pi_n$ a $t$-regular Besov prior on densities. Posterior computation with Besov prior in the density estimation model is discussed in Section 4.2.

3 Main results

In this section we present our main results concerning the asymptotic behaviour of posterior distributions $\Pi_n(\cdot|X^{(n)})$ arising from the Besov priors on densities $\Pi_n$ introduced in Section 2.2. We quantify the speed at which $\Pi_n(\cdot|X^{(n)})$ concentrates around the true density $p_0$ generating the data according to the usual notion of posterior contraction rates, that is sequences $\xi_n \rightarrow 0$ such that, for large enough $M > 0$,

$$\Pi_n \left( p : d(p, p_0) > M \xi_n \big| X^{(n)} \right) \rightarrow 0$$

in $P_0$-probability as $n \rightarrow \infty$. Above, $d$ is a distance between probability densities. In the present paper, we will mostly deal with the total variation distance $d_{TV}(p, p') = \frac{1}{2} \| p - p' \|_1$, which, due to its characterisation as (half) $L^1$-norm, is naturally aligned with the $L^1$-structure underlying the Besov spaces $B^{s}_{1,1}([0,1]^d)$.

3.1 Posterior contraction rates for Besov priors with fixed regularity

3.1.1 Under-smoothing rescaled Besov priors

We first consider the case where the scaling factors $\sigma_{n,lr}$ in (1) are chosen as deterministic sequences. Specifically, for fixed $s > d$, we take

$$\sigma_{n,lr} = 2^{-l(s-d/2)} n^{-d/(2s+d)},$$

resulting in a rescaled $(s-d)$-regular Besov prior $\Pi_{W_n}$ arising as the law of

$$W_n = \frac{1}{n^{2s+d}} \sum_{l=1}^{\infty} \sum_{r=1}^{2^{ld}} 2^{-l(s-d/2)} W_{lr} \psi_{lr}, \quad W_{lr} \overset{iid}{\sim} \text{Laplace}. \quad (4)$$

We then construct a prior $\Pi_n$ on densities as in (3). We here allow for any log-Lipschitz link function (e.g., the commonly used exponential link function, but also any of the more restrictive ones appearing in the results below). The next theorem shows that the posterior distribution resulting from $\Pi_n$ contracts around the true density $p_0 \in B^{s}_{1,1}([0,1]^d)$ in total variation distance at optimal rate.

**Theorem 1.** For fixed $s > d$, let the prior $\Pi_n$ be constructed as in (3) for $W_n$ as in (4) and $\phi : \mathbb{R} \rightarrow (0, \infty)$ a strictly increasing, injective and smooth function with uniformly Lipschitz logarithm. Let $X_1, \ldots, X_n$ be a random sample from a probability density $p_0 \in B^{s}_{1,1}([0,1]^d)$ satisfying $p_0(x) > 0$ for all $x \in [0,1]^d$. Then, for $M > 0$ large enough,

$$\Pi_n \left( p : d_{TV}(p, p_0) > M n^{-s/d} \big| X^{(n)} \right) \rightarrow 0$$

in $P_0$-probability as $n \rightarrow \infty$. 

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The obtained posterior contraction rate $n^{-s/(2s+d)}$ is known to be the minimax optimal rate for estimating $p_0 \in B^s_{2,1}(0,1)^d$ from the random sample $X^{(n)}$ in $L^p$-loss for any $p < 2s + d$; see [25, Theorem 10.3] (whose proof techniques naturally extend to the multi-dimensional case $d \geq 2$). Due to the identity $d_{TV}(p, p') = \frac{1}{2} \|p - p'\|_1$, the rate is optimal for the total variation distance. Furthermore, an inspection of the proof shows that the claim of Theorem 1 remains valid also if the total variation distance is replaced by either the Hellinger distance (cf. (15)) or the $L^2$-distance; see the discussion preceding Lemma 5 below for details.

Theorem 1 improves upon the density estimation results in [3, Section 6], which only considers the case of spatially homogeneous densities $p_0 \in B_{2,\infty}^s([0,1])$ and where only polynomially suboptimal posterior contraction rates are obtained. Here we show that, even for spatially inhomogeneous $p_0 \in B^s_{2,1}(0,1)^d$, suitably tuned Besov priors achieve optimal rates. The proof follows a similar strategy to the results in [3], based on the general posterior contraction rate theory of Ghosal et al. [16], but crucially uses a different tuning of the prior which combines an under-smoothing effect with rescaling by the factor $n^{-d/(2s+d)}$. In particular, the rescaling implies that the prior $\Pi_n$ asymptotically concentrates over sieve sets of densities that are uniformly bounded and bounded away from zero (cf. Lemma 5, for which, compared to [3], tighter complexity bounds with respect to the relevant distances (respectively, the total variation or Hellinger distances) can be obtained.

Expanding on the tuning of the prior in (4), draws from $\Pi_n$ almost surely lie in $B^s_{2,1}([0,1]^d)$ only for $0 < t < s - d$; see Section A.1. Since in Theorem 1 the true density $p_0$ is assumed to be in $B^s_{1,1}([0,1]^d)$, the prior is seen to be under-smoothing. This contrasts with a number of results in the literature on Gaussian priors, where, under Sobolev- or Hölder-type regularity assumptions on the ground truth, optimal contraction rates are obtained with priors matching the true smoothness. See Section 3.1 in van der Vaart and van Zanten [47] for results in density estimation, and [21, Chapter 7.3] or [19, Chapter 11] for a general overview on the theory. In the case of spatially inhomogeneous $p_0 \in B^s_{2,1}(0,1)^d$ and Besov priors, the necessity of under-smoothing and rescaling can be explained in terms of the bias-variance tradeoff: as shown in the calculations of [3], it turns out that for Besov priors with matching regularity the bias term appearing in the proof is too large due to the misalignment between the $L^1$-structure underlying the spaces $B^s_{2,1}(0,1)^d$ and the Kullback-Leibler divergence and variation in the prior mass condition (9) below. Under-smoothing allows to reduce the bias (in particular, sidestepping the necessity of approximating $p_0$), while the corresponding variance increase is balanced via the rescaling by the decaying factor $n^{-d/(2d+d)}$. This joint effect was already observed in [3, Section 5.2], where in the white noise model optimal contraction rates towards spatially inhomogeneous signals are obtained using under-smoothing rescaled Besov priors with a similar tuning to (4). See the recent work by Agapiou and Wang [5] for an extension to nonlinear inverse problems. In fact, under traditional regularity assumptions on the ground truth, under-smoothing rescaled Gaussian priors have been successfully employed in the context of nonlinear inverse problems, e.g., [37, 1, 22, 39, 28], and reversible diffusion models [23].

### 3.1.2 Partially-rescaled Besov priors

As observed in the previous discussion, the rescaling factor $n^{-d/(2s+d)}$ in (4) balances the increased variance resulting from under-smoothing. Furthermore, it also asymptotically implies a bound on the norm of prior draws, which yields in the proof tight complexity bounds for the associated sieve sets. On the other hand, from a Bayesian perspective, the rescaling is arguably not satisfactory as it is causes the prior to shrink towards zero:
in particular, \( \| W_n \|_\infty \to 0 \) almost surely as \( n \to \infty \). A more refined investigation of the prior geometry (cf. the proof of Theorem 2 in Section 5.2 below) however reveals that the aforementioned variance increase is - in some sense - only effective at resolution levels \( l \) such that \( 2^{id} \) is larger than the usual optimal dimension \( \eta d/(2s+d) \) for estimating an \( s \)-regular function. This motivates the construction of the following partially rescaled \((s-d)\)-regular Besov prior:

\[
W_n = \sum_{l=1}^{L_n} \sum_{r=1}^{2^d} 2^{-l(s-\frac{d}{2})} W_{lr} \psi_{lr} + \frac{1}{n \log n} \sum_{l=L_n+1}^{\infty} \sum_{r=1}^{2^d} 2^{-l(s-\frac{d}{2})} W_{lr} \psi_{lr}, \quad W_{lr} \overset{iid}{\sim} \text{Laplace},
\]

where \( L_n \in \mathbb{N} \) is chosen so that \( 2^{L_n} \simeq n^{1/(2s+d)} \). A prior \( \Pi_n \) on densities is then constructed as previously following (3). Here, to deal with the weaker control over the norm of \( \phi_{W_n} \) resulting from the partial rescaling, we employ uniformly Lipschitz link functions that are bounded away from zero. This allows in the proof to construct sieve sets with sufficiently small complexity. The next theorem shows that the resulting posterior distribution attains (up to a log-factor) the same optimal rate of Theorem 1.

**Theorem 2.** For fixed \( B > 0 \) and \( s > d \), let the prior \( \Pi_n \) be constructed as in (3) for \( W_n \) as in (5) and \( \phi : \mathbb{R} \to (B, \infty) \) a strictly increasing, bijective, smooth and uniformly Lipschitz function. Let \( X_1, \ldots, X_n \) be a random sample from a probability density \( p_0 = f_0/\int_{[0,1]^d} f_0(x) \, dx \) for some \( f_0 \in B_{11}([0,1]^d) \) satisfying \( f_0(x) > B \) for all \( x \in [0,1]^d \). Then, for \( M > 0 \) large enough,

\[
\Pi_n \left( p : d_{TV}(p, p_0) > Mn^{-\frac{d}{2s+d}} \sqrt{\log n} \| \phi \|_2 \right) \to 0
\]

in \( P_0 \)-probability as \( n \to \infty \).

Instances of link functions satisfying the assumptions of Theorem 2 are certain regular link functions appearing in the Bayesian inverse problems literature \([1, 38, 22, 28]\). As a concrete example, take

\[
\phi(z) = B + \frac{1-B}{g * \eta(0)} g * \eta(z), \quad \eta(z) = e^z 1_{\{z < 0\}} + (1 + z) 1_{\{z \geq 0\}},
\]

where \( g : \mathbb{R} \to [0, \infty) \) is a smooth and compactly supported function with \( \int_R g(z) \, dz = 1 \) (cf. Example 8 in [38]).

The conclusion of Theorem 2 is restricted to densities that are bounded away from zero, with ‘core’ \( f_0 \) point-wise greater than the constant \( B \) lower bounding the range of the link function \( \phi \). The latter is a relatively mild assumption as any arbitrary small but fixed \( B > 0 \) is allowed, and choosing smaller values of \( B \) only affects the constant premultiplying the obtained contraction rate. In fact, a direct adaptation of the proof of Theorem 2 implies that taking an \( n \)-dependent lower bound \( B = B_n \simeq 1/\log n \) leads to posterior contraction towards any density \( p_0 \in B_{11}([0,1]^d) \) bounded away from zero at a rate that deteriorates only by an additional log-factor \((\log n)^c\) for some \( c > 0 \).

### 3.1.3 Posterior contraction rates for spatially homogeneous densities

We conclude the investigation on Besov priors with fixed regularity considering the case of spatially homogeneous densities \( p_0 \in B_{11}([0,1]^d) \), which is the setting studied in Section 6 of [3]. Under this assumption, the Kullback-Leibler divergence and variation are well-aligned to the regularity structure of the ground truth, so that, as opposed to the inhomogeneous case considered previously (cf. the discussion after Theorem 1),
non-rescaled Besov priors with matching regularity attain the necessary balance of bias and variance. This paves the way to obtaining optimal posterior contraction rates, as we illustrate in the next theorem considering the following (truncated) $s$-regular Besov prior

$$W_n = \sum_{l=1}^{L_n} \sum_{r=1}^{2^d} 2^{-l(s+\frac{d}{2})} W_{lr} \psi_{lr}, \quad W_{lr} \overset{iid}{\sim} \text{Laplace},$$

where $L_n \in \mathbb{N}$ is such that $2^{L_n} \simeq n^{1/(2s+d)}$.

**Theorem 3.** For fixed $s > d$, let the prior $\Pi_n$ be constructed as in (3) for $W_n$ as in (6) and $\phi : \mathbb{R} \to (0, \infty)$ a strictly increasing, bijective and smooth function with uniformly Lipschitz logarithm. Let $X_1, \ldots, X_n$ be a random sample from a probability density $p_0 \in B_{s,\infty}^d([0,1]^d)$ satisfying $p_0(x) > 0$ for all $x \in [0,1]^d$. Then, for $M > 0$ large enough,

$$\Pi_n \left( p : d_{\text{TV}}(p, p_0) > Mn^{-\frac{s+d}{s}} \right) \to 0$$

in $P_0$-probability as $n \to \infty$.

Truncating the prior at level $L_n$ is here natural since, as remarked before Theorem 2, $2^{L_n} \simeq n^{d/(2s+d)}$ is the optimal dimension for estimating a (spatially homogeneous) $s$-regular function. In the proof, the truncation reduces the complexity of the sieve sets associated to $W_n$, allowing the derivation of suitable bounds on their metric entropy. For ground truths with Sobolev regularity, similar truncated Besov priors were recently shown to yield optimal contraction rates in a reversible diffusion model [23].

### 3.2 Adaptive posterior contraction rates

The optimal contraction rates obtained in Theorems 1 - 3 (up to a log-factor in Theorem 2) depend in an essential way on the appropriate specification of the prior regularity relative to the smoothness of the ground truth: for $p_0 \in B_{s,1}^d([0,1]^d)$, Theorems 1 and 2 employ $(s-d)$-regular Besov priors, while for $p_0 \in B_{s,\infty}^d([0,1]^d)$, Theorem 3 assumes prior regularity $s$. In each case, choosing different smoothness parameters (and/or different rescalings in Theorems 1 and 2) would yield through our proofs sub-optimal rates, in accordance to the results of Agapiou et al. [3], where rates for various prior regularities are obtained, and to related literature on Gaussian priors, e.g., [47, 11, 30]. The results presented in the previous sections are thus non-adaptive, in that the construction of the prior requires knowledge of the smoothness of the true density. As this is often an unrealistic assumption, it is of interest to construct a Bayesian procedure based on Besov priors that, not requiring knowledge of the regularity of $p_0$, automatically adapts to it attaining optimal contraction rates.

An established method to achieve adaptation in Bayesian procedures is by randomising the prior regularity parameter, assigning to it a further (hyper-)prior; see, e.g., [19, Chapter 10]. Here, we pursue this approach. In particular, we employ a hierarchical, conditionally Besov prior $\Pi_{W_n}$ arising as the law of

$$W_n = \frac{1}{n^{s+d/2}} \sum_{l=1}^{\infty} \sum_{r=1}^{2^d} 2^{-l(s-\frac{d}{2})} W_{lr} \psi_{lr}, \quad W_{lr} \overset{iid}{\sim} \text{Laplace}, \quad S \sim \Sigma_n,$$

where $\Sigma_n$ is an absolutely continuous ($n$-dependent) distribution on $(0, \infty)$ with density $\sigma_n$. We take a specific choice for $\Sigma_n$, assuming it to be supported on the increasing
interval \((d, \log n)\), with density
\[
\sigma_n(s) = \frac{e^{-n^{d/(2s+d)}}}{\zeta_n}, \quad s \in (d, \log n],
\]
where \(\zeta_n \simeq \log n\) is the normalising constant. Conditionally given \(S\), \(W_n|S\) thus corresponds to the rescaled under-smoothing Besov prior considered in Section 3.1.1. For any \(S > d\), \(W_n \in C([0,1]^d)\) with conditional probability given \(S\) equal to one (cf. Section A.1), implying that the hierarchical prior \(\Pi_{W_n}\) is supported on \(C([0,1]^d)\).

Given \(W_n\) as in (7), a hierarchical Besov prior on densities \(\Pi_n\) is constructed as in (3), via a link function \(\phi\) that, as in Section 3.1.2, we require to be uniformly Lipschitz and bounded away from zero. The next theorem shows that the associated posterior distribution attains optimal posterior contract rates towards densities \(p_0\) of any smoothness \(s_0 \in (d, \infty)\).

**Theorem 4.** For fixed \(B > 0\), let the prior \(\Pi_n\) be constructed as in (3) for \(W_n\) as in (7) and \(\phi : \mathbb{R} \to (B, \infty)\) a strictly increasing, bijective, smooth and uniformly Lipschitz function. Let \(X_1, \ldots, X_n\) be a random sample from a probability density \(p_0 = f_0/\int_{[0,1]^d} f_0(x)dx\) for some \(f_0 \in B_{11}^s([0,1]^d)\), any \(s_0 > d\), satisfying \(f_0(x) > B\) for all \(x \in [0,1]^d\). Then, for \(M > 0\) large enough,

\[
\Pi_n \left( p : d_{TV}(p, p_0) > M n^{-\frac{s_0}{2s_0+d}} \left| X^{(n)} \right| \right) \to 0
\]
in \(P_0\)-probability as \(n \to \infty\).

Similar to Theorem 2, the stronger requirements on the link function \(\phi\) play a crucial role in the proof to handle the weaker control, resulting from the hierarchical construction, over the norm of \(W_n\) in (7) compared to the rescaled Besov priors of fixed regularity of Theorem 1. We remark that the above lower bound \(B\) can be chosen arbitrarily small, and in fact that taking \(B = B_n \simeq 1/\log n\) would allow to extend Theorem 4 to any density \(p_0 \in B_{11}^s([0,1]^d)\) bounded away from zero, with the rate \(n^{-s_0/(2s_0+d)}\) replaced by \(n^{-s_0/(2s_0+d)}(\log n)^c\) for some \(c > 0\).

Regarding the specific choice of the hyper-prior, note that \(\sigma_n(s)\) is proportional to \(e^{-n^{1/2s_0}}\), where \(\epsilon_{s,n} := n^{-s/(2s+d)}\) is the optimal rate for estimating \(p_0 \in B_{11}^s([0,1]^d)\) obtained in Theorem 1 using a rescaled \((s-d)\)-regular Besov prior. This choice is motivated by previous findings in the literature that showed that, under some generality, hyper-priors of this kind can lead to adaptation in various statistical models, including in density estimation [35, 17] and drift estimation for diffusion processes [48]. In accordance to the latter references, Theorem 4 shows that this is indeed the case in the setting of Besov priors and spatially inhomogeneous densities.

### 4 Summary and discussion

#### 4.1 Outlook

In this paper we have studied the recovery performance of Besov priors in density estimation. Our main results show that, for spatially inhomogeneous densities \(p_0 \in B_{11}^s([0,1]^d)\), suitably calibrated, rescaled and under-smoothing Besov priors attain optimal posterior contraction rates, and that randomising the smoothness parameter in the prior construction leads to adaptation. While we focused on density estimation, we expect our techniques to be applicable to other nonparametric statistical models such as classification and nonparametric regression. In particular, the hierarchical prior...
construction in Section 3.2 should yield adaptive rates in the latter models under Besov regularity assumptions on the ground truth.

The role of under-smoothing and rescaling has been discussed after Theorem 1, and subsequently explored in Sections 3.1.2 and 3.1.3. It is an interesting open question, not limited to the density estimation setting, whether non-rescaled Besov priors with matching regularity can achieve optimal rates for spatially inhomogeneous ground truths, as the observed sub-optimality [3] might be an artefact of the existing proofs. For Gaussian priors, the techniques developed by Castillo [11] allow to precisely lower bound the associated posterior contraction rates, based essentially on a converse of the decentering inequality (29). Such result is however currently unavailable for Besov priors (cf. [3, Remark 2.14]), due to their more complicated information geometry.

Regarding Theorem 4, it is conceivable that other hyper-priors on the smoothness could lead to adaptation, including potentially hyper-priors not depending on the sample size $n$, such as the ones employed in Knapik et al. [29]. Our proof strategy, based on the general contraction rate theory of [16], heavily relies on the specific choice (8), and extensions to other hyper-priors appear to require substantial modifications or different mathematical techniques. These issues represent interesting directions for future research.

4.2 Posterior computation

The general Besov prior $\Pi_{W_n}$ introduced in Section 2.2 naturally lends itself to discretisation by truncating the series in (1) at some fixed resolution level $L \in \mathbb{N}$. Identifying a function $u = \sum_{l=1}^{L} \sum_{r=1}^{2^d} u_{lr} \psi_{lr}$ with the vector $w = (w_{lr}, \ l = 1, \ldots, L, \ r = 1, \ldots, 2^d) \in \mathbb{R}^{\dim(V_L)}$, this yields a prior distribution $\Pi_{W_n}^L$ on $w$ given by a product of Laplace distributions, with density

$$
\pi_{W_n}^L(w) = \prod_{l=1}^{L} \prod_{r=1}^{2^d} \frac{1}{2\sigma_{n,lr}} e^{-\frac{|w_{lr}|}{\sigma_{n,lr}}} = e^{-\frac{\sum_{l=1}^{L} \sum_{r=1}^{2^d} |w_{lr}|}{\sum_{l=1}^{L} \sum_{r=1}^{2^d} \sigma_{n,lr}}}.
$$

Given observations $X^{(n)} = (X_1, \ldots, X_n)$, and a fixed link function $\phi : \mathbb{R} \to [0, \infty)$, the corresponding posterior distribution $\Pi_{W_n}^L(\cdot|X^{(n)})$ of $w|X^{(n)}$ has density

$$
\pi_{W_n}^L(w|X^{(n)}) \propto \prod_{i=1}^{n} \phi^{\sum_{l=1}^{L} \sum_{r=1}^{2^d} u_{lr} \psi_{lr}(X_i)} \pi_{W_n}^L(w) \int_{[0,1]^d} \phi^{\sum_{l=1}^{L} \sum_{r=1}^{2^d} u_{lr} \psi_{lr}(x')} dx'.
$$

For concrete choices of $\sigma_{lr}$, the right hand side can be computed from the data, numerically approximating the integrals in the denominator. While potentially computationally intensive for large dimensions $d$, this provides a starting point for implementing a Markov chain Monte Carlo (MCMC) algorithm (e.g., of Metropolis-Hastings type) to sample from $\Pi_{W_n}^L(\cdot|X^{(n)})$.

In order to sample from the posterior distribution resulting from the hierarchical prior of Section 3.2, MCMC methods for Besov priors of fixed regularity (such as the one outlined above) can be employed within a Gibbs-type sampling scheme that exploits the conditionally Besov structure of the prior. The algorithm would then alternate, for a given regularity $S$, an MCMC step targeting the marginal posterior distribution of $w|(X^{(n)}, S)$, followed by, given the actual sample of $w$, a second MCMC run targeting the marginal posterior distribution of $S|(X^{(n)}, w)$. 

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5 Proofs

The proofs of Theorems 1 - 4 are based on the general theory for posterior contraction rates in the i.i.d. sampling model of Ghosal et al. [16], asserting that if for some sequence $\xi_n \to 0$ such that $n\xi_n^2 \to \infty$, some constant $C > 0$, and all $n \in \mathbb{N}$ large enough,

$$\Pi_n \left( p : -E_{p_0} \left( \frac{P}{P_0}(X) \right) \leq \xi_n^2, \quad E_{p_0} \left( \frac{P}{P_0}(X) \right) \geq e^{-CN_n^2} \right) \geq \epsilon$$

(9)

and there exists sets of densities $\mathcal{P}_n$ such that

$$\Pi_n(\mathcal{P}_n) \leq e^{-(C+4)n\xi_n^2},$$

(10)

and

$$\log N(\xi_n; \mathcal{P}_n, d_{TV}) \lesssim n\xi_n^2,$$

(11)

then, for sufficiently large $M > 0$, $\Pi_n (p : d_{TV}(p, p_0) > M \xi_n \mid X^{(n)}) \to 0$ in $P_0$-probability as $n \to \infty$.

5.1 Proof of Theorem 1

Set $\xi_n := n^{-s/(2s+d)}$, and write $W_n$ in (4) as $W_n = (n\xi_n^2)^{-1}W$, where $W$ is the non-rescaled $(s - d)$-regular Besov random element

$$W := \sum_{i=1}^{\infty} \sum_{r=1}^{2^d} 2^{-l(s-d)}W_{lr} \psi_{lr}, \quad W_{lr} \overset{iid}{\sim} \text{Laplace}.$$

(12)

Let $\Pi_W$ denote the law of $W$. We verify conditions (9) - (11) employing tools for Besov priors largely due to Agapiou et al. [3]. Starting with (9), write $p_0 = f_0/\int_{[0,1]^d} f_0(x')dx'$ for some strictly positive $f_0 \in B_{1,1}^s$. Since $\phi : \mathbb{R} \to (0, \infty)$ is strictly increasing, bijective and smooth, it posses a strictly increasing, bijective and smooth inverse $\phi^{-1} : (0, \infty) \to \mathbb{R}$, so that $f_0 = \phi \circ w_0 = \phi^{-1} \circ f_0 \in B_{1,1}^s$. This follows from Theorem 10 in [8], upon writing $w_0(x) = \phi^{-1}((f_0(x) - \phi(0)) + \phi(0))$ and noting that $f_0 = \phi(0) \in B_{1,1}^s$ and that $\phi^{-1}(\cdot + \phi(0)) : (-\phi(0), \infty) \to \mathbb{R}$ is smooth and vanishes at zero. Thus,

$$p_0(x) = \frac{\phi(w_0(x))}{\int_{[0,1]^d} \phi(w_0(x'))dx'} = \phi_{w_0}(x), \quad x \in [0,1]^d.$$

By construction (cf. (3)), each density $p$ in the support of $\Pi_n$ takes the form $p = \phi_w$ for some $w \in C([0,1]^d)$. For all such densities, since $\phi$ is uniformly log-Lipschitz, a standard computation (e.g., Problem 2.4 in [19]) shows that for some $c_1 > 0$,

$$\max \left\{ -E_{p_0} \left( \frac{P}{P_0}(X) \right), \quad E_{p_0} \left( \frac{P}{P_0}(X) \right) \right\} \leq \|w - w_0\|_\infty e^{-c_1 \|w-w_0\|_\infty^2}.$$

Therefore, the prior probability in (9) is lower bounded by

$$\Pi_{W_n}(w : \|w - w_0\|_\infty \leq c_2 \xi_n) = \Pi_W (w : \|w - n\xi_n^2w_0\|_\infty \leq c_2n\xi_n^3)$$

for some $c_2 > 0$. By construction, $W$ in (12) has associated ‘decentering’ space $\mathcal{Z} = B_{1,1}^s$ with norm $\|\cdot\|_\mathcal{Z} = \|\cdot\|_{B_{1,1}^s}$ (cf. Section A.1). By the centering inequality (29), it follows that the latter probability is greater than

$$e^{-n\xi_n^2w_0 \|w\|_\infty} \Pi_W (w : \|w\|_\infty \leq c_2n\xi_n^3) = e^{-\|w_0\|_{B_{1,1}^s} n\xi_n^2} \Pi_W (w : \|w\|_\infty \leq c_2n\xi_n^3).$$
Finally, by Proposition 6.3 in [3] (which straightforwardly extends to the multi-dimensional case \(d \geq 2\), noting \(n^{-d/(2s+d)} \to 0\) as \(n \to \infty\) if \(s > d\),
\[
\Pi_W \left( w : \|w\|_{\infty} \leq c_2 n^{-\frac{d}{2}} \right) \geq e^{-c_3 (c_2 n^{-\frac{d}{2}})^{-d/(s-d)}} = e^{-c_4 n^{d/(2s+d)}} = e^{-c_4 n^{-d}},
\]
for some \(c_3, c_4 > 0\). Combining the last two displays yields (9) with \(C = \|w_0\|_{B_1^s} + c_4\).

Turning to conditions (10) and (11), define the sieves \(\mathcal{P}_n := \{\phi_w, w \in W_n\}\), where
\[
W_n := \left\{ w = w^{(1)} + w^{(2)} : \|w^{(1)}\|_2 \leq R \xi_n, \|w^{(2)}\|_{B_{11}^s} \leq R \right\} \cap \{ w : \|w\|_{\infty} \leq R \}.
\]
Then
\[
\Pi_n(\mathcal{P}_n^c) \leq \Pi_{W_n}(\mathcal{W}_n^c)
\]
\[
\leq 1 - \Pi_{W_n}(w = w^{(1)} + w^{(2)} : \|w^{(1)}\|_2 \leq R \xi_n, \|w^{(2)}\|_{B_{11}^s} \leq R)
\]
\[
+ 1 - \Pi_{W_n}(w : \|w\|_{\infty} \leq R),
\]
which, by Lemma 5, fixing \(K > C + 4\) and choosing sufficiently large \(R > 0\), is smaller than
\[
2 e^{-K n \xi_n^2} \leq e^{-(C+4) n \xi_n^2}.
\]
Finally, since \(d_{TV}(p, p') = \frac{1}{2} \|p - p'\|_1\), \(\log N(\xi_n; \mathcal{P}_n, d_{TV}) = \log N(2 \xi_n; \mathcal{P}_n, \|\cdot\|_1)\). Also, by Lemma 10, we have for all \(w, w' \in W_n\), for constants \(c_5, c_6 > 0\),
\[
\|\phi_w - \phi_{w'}\|_1 \leq c_5 e^{c_6 \|w - w'\|_{\infty}} \|w - w'\|_1 \lesssim \|w - w'\|_1,
\]
since \(\|w\|_{\infty}, \|w'\|_{\infty} \leq R\). Thus, for some \(c_7 > 0\),
\[
\log N(2 \xi_n; \mathcal{P}_n, \|\cdot\|_1) \leq \log N(c_7 \xi_n; W_n, \|\cdot\|_1)
\]
and since \(W_n \subset \{ w = w^{(1)} + w^{(2)} : \|w^{(1)}\|_2 \leq R \xi_n, \|w^{(2)}\|_{B_{11}^s} \leq R \}\), using Theorem 4.3.36 in [21] and the fact that that \(\|w^{(1)}\|_1 \leq \|w^{(1)}\|_2\), the latter metric entropy is bounded by a multiple of
\[
\log N(\xi_n; \{ w : \|w\|_{B_{11}^s} \leq R \}, \|\cdot\|_1) \lesssim \xi_n^{-\frac{d}{2}} = n^{\frac{d}{2d+2}} = n \xi_n^2,
\]
concluding the verification of (11) and the proof of Theorem 1.

An inspection of the above proof reveals that Theorem 1 remains valid also if the total variation distance is replaced by either the Hellinger distance
\[
d_H(p, p') := \sqrt{\int_{[0,1]^d} \left( \sqrt{p(x)} - \sqrt{p'(x)} \right)^2 dx}
\]
(15)
or the \(L^2\)-distance. Indeed, for the Hellinger distance, the complexity bound (11) can be verified with \(d_{TV}\) replaced by \(d_H\) noting that for \(W_n\) as in (14), the sieves \(\mathcal{P}_n = \{\phi_w, w \in W_n\}\) contain densities that are uniformly bounded and bounded away from zero, whence the equivalence \(d_H(\phi_w, \phi_{w'}) \sim \|\phi_w - \phi_{w'}\|_2\) for all \(w, w' \in W_n\) (following, e.g., from Lemma B.1 in [19]). Similar computations as in the proof of Lemma 10 further show that \(\|\phi_w - \phi_{w'}\|_2 \lesssim \|w - w'\|_2\) for all \(w, w' \in W_n\), so that
\[
\log N(\xi_n; \mathcal{P}_n, d_H) \lesssim \log N(\xi_n; \{ w : \|w\|_{B_{11}^s} \leq R \}, \|\cdot\|_2) \lesssim \xi_n^{-\frac{d}{2}} = n \xi_n^2.
\]
Via Theorem 2.1 in [16], this yields 
\[ \Pi_n \left( p : d_H(p, p_0) > M n^{-\frac{d}{2+2d}} \left| X(n) \right| \right) \to 0 \]
in \( P_0 \)-probability as \( n \to \infty \). For the \( L^2 \)-distance, note that via Theorem 8.20 in [19], the last display can be strengthen to 
\[ \Pi_n \left( \phi_w \in \mathcal{P}_n : d_H(\phi_w, p_0) > M n^{-\frac{d}{2+2d}} \left| X(n) \right| \right) \to 0 \]
in \( P_0 \)-probability as \( n \to \infty \), implying posterior contraction in \( L^2 \)-distance since \( \| \phi_w - p_0 \|_2 \lesssim d_H(p, p_0) \) for all \( w \in \mathcal{W}_n \).

**Lemma 5.** For \( s > d \), let \( \Pi_{W_n} \) be the rescaled Besov prior arising as the law of \( W_n \) in (4). Then, for all \( K > 0 \) there exist sufficiently large \( R > 0 \) such that, for \( n \in \mathbb{N} \) large enough,

1. 
\[ \Pi_{W_n} \left( w = w^{(1)} + w^{(2)} : \| w^{(1)} \|_2 \leq R n^{-\frac{d}{2+2d}}, \| w^{(2)} \|_{B_{11}^s} \leq R \right) \geq 1 - e^{-K n^{d/(2+2d)}} \]

2. 
\[ \Pi_{W_n} (w : \| w \|_\infty \leq R) \geq 1 - e^{-K n^{d/(2+2d)}} \]

**Proof.** To prove point 1., letting \( \xi_n \), \( W \) and \( \Pi_W \) be defined as at the beginning of the proof of Theorem 1, the probability of interest equals 
\[ \Pi_W \left( w = w^{(1)} + w^{(2)} : \| w^{(1)} \|_2 \leq R n^{-\frac{d}{2+2d}}, \| w^{(2)} \|_{B_{11}^s} \leq R \right) \] (16)

By construction, the spaces associated to \( W \) in (12) are respectively \( Z = B_{11}^s \), with norm \( \| \cdot \| = \| \cdot \|_{B_{11}^s} \), and \( Q = H^{s-d/2} \); \( \| \cdot \|_Q = \| \cdot \|_{H^{s-d/2}} \); see Section A.1. By the two-level concentration inequality (30) it follows that for some \( c_1 > 0 \), for all \( \overline{R} > 0 \),
\[ \Pi_W \left( w = w^{(1)} + w^{(2)} + w^{(3)} : \| w^{(1)} \|_2 \leq n_1^{-\frac{d}{2+2d}}, \| w^{(2)} \|_{H^{s-d/2}} \leq \sqrt{R n_3^{-\frac{d}{2}}}, \right. \]
\[ \left. \| w^{(3)} \|_{B_{11}^s} \leq R n_3^{-\frac{d}{2}} \right) \geq 1 - \frac{1}{\Pi_W (w : \| w \|_2 \leq n_1^{-\frac{d}{2}})} e^{-c_1 R n_3^{-\frac{d}{2}}} \]

Since \( \| w \|_2 \leq \| w \|_\infty \), using (13), we have that \( \Pi_W \left( w : \| w \|_2 \leq n_1^{-\frac{d}{2}} \right) \geq e^{-c_2 n_1^{\frac{d}{2}}} \) for some \( c_2 > 0 \) as \( n \to \infty \), so that the right hand side of the last display is lower bounded by 
\[ 1 - e^{-c_1 R n_3^{-\frac{d}{2}}} \geq 1 - e^{-K n_3^{\frac{d}{2}}} \]
(17)

upon choosing sufficiently large \( \overline{R} > 0 \). Now for \( \overline{w}^{(2)} \) as in the second to last display, let \( P_{L_n} \overline{w}^{(2)} \) be its wavelet approximation at resolution \( L_n \in \mathbb{N} \) satisfying \( 2L_n \approx n^{1/(2r+d)} \); see Section 2.1 for definitions. Then, by the properties of wavelet projections, 
\[ \| \overline{w}^{(2)} - P_{L_n} \overline{w}^{(2)} \|_2 \leq 2^{-L_n \left( s-\frac{d}{2} \right)} \| \overline{w}^{(2)} \|_{H^{s-d/2}} \lesssim n \frac{d/2}{2r+d} n^{d/2} = n_3^{\frac{d}{2}} \]
and moreover, by the wavelet characterisation of \( \| \cdot \|_{B_{11}^s} \) (cf. Section 2.1) and Hölder’s inequality,
\[ \| P_{L_n} \overline{w}^{(2)} \|_{B_{11}^s} \leq \sum_{l=1}^{L_n} 2^{l \left( s-\frac{d}{2} \right)} \sum_{r=1}^{2^d} |\langle \overline{w}^{(2)}, \psi_{l,r} \rangle | \]
\[ \leq \sqrt{\text{dim}(V_{L_n})} \left( \sum_{l=1}^{L_n} 2^{2l \left( s-\frac{d}{2} \right)} \sum_{r=1}^{2^d} |\langle \overline{w}^{(2)}, \psi_{l,r} \rangle |^2 \right)^{1/2} \]
\[ \lesssim \sqrt{2^{L_n d}} \| \overline{w}^{(2)} \|_{H^{s-d/2}} \lesssim n \frac{d/2}{2r+d} n^{d/2} = n_3^{\frac{d}{2}} \]
Taking \( w^{(1)} := \overline{w}^{(1)} + (\overline{w}^{(2)} - P_{L_n} \overline{w}^{(2)}) \) and \( w^{(2)} := P_{L_n} \overline{w}^{(2)} + \overline{w}^{(3)} \) thus implies \( \| w^{(1)} \|_2 \lesssim n \xi_n^2 \) and \( \| w^{(2)} \|_{B_{11}^d} \lesssim n \xi_n^2 \), so that taking \( R > 0 \) large enough the probability of interest (16) is lower bounded by the right hand side of (17), concluding the proof of the first claim.

To prove point 2., the probability of interest equals \( \Pi_W (w : \| w \|_\infty \leq R_n \xi_n^2) \), to which we directly apply the concentration inequality (28) to deduce the lower bound, holding for \( R > 0 \) large enough,

\[
1 - c_3 e^{-c_4 R_n \xi_n^2} \geq 1 - e^{-K n \xi_n^2}.
\]

\[\square\]

5.2 Proof of Theorem 2

We verify conditions (9) - (11) with \( \xi_n := n^{-s/(2s+d)} \sqrt{\log n} \). Since \( \phi : \mathbb{R} \rightarrow (B, \infty) \) is strictly increasing, bijective and smooth, it posses a strictly increasing, bijective and smooth inverse \( \phi^{-1} : (B, \infty) \rightarrow \mathbb{R} \). Hence, for \( p_0 \propto f_0 \) with \( f_0 \in B_{11}^d \) satisfying \( f_0(x) > B \) for all \( x \in [0,1]^d \), we have \( p_0 = \phi w_0 \) with \( w_0 = \phi^{-1} \circ f_0 \in B_{11}^d \) (following again by Theorem 10 in [8] applied to the smooth function \( \phi^{-1}(\cdot + \phi(0)) \) vanishing at zero). Since \( \phi \) is uniformly Lipschitz, by point 1. in Lemma 11, for each \( w \in C([0,1]^d) \) such that \( \| w - w_0 \|_\infty \leq \xi_n \), since then \( \| w \|_\infty \leq \| w_0 \|_\infty + 1 \), we have

\[
\max \left\{ - \mathbb{E}_{p_0} \left( \log \frac{\phi w}{p_0}(X) \right), \mathbb{E}_{p_0} \left( \log \frac{\phi w}{p_0}(X) \right)^2 \right\} \leq \frac{1}{B^2} \mathbb{E}_{w_0} \left( \| w - w_0 \|_\infty \right)^2 \leq \frac{\| p_0 \|_\infty}{B/\phi(\| w_0 \|_\infty + 1)} \| w - w_0 \|_2 \lesssim \| w - w_0 \|_\infty^2.
\]

Thus, for some \( c_1 > 0 \),

\[
\Pi_n \left( \rho : - \mathbb{E}_{p_0} \left( \log \frac{p}{p_0}(X) \right) \leq \xi_n^2, \mathbb{E}_{p_0} \left( \log \frac{p}{p_0}(X) \right)^2 \leq \xi_n^2 \right) \geq \Pi_{W_n}(w : \| w - w_0 \|_\infty \leq c_1 \xi_n).
\]

By construction, the decentering space \( Z_n \) associated to \( W_n \) in (5) satisfies \( Z_n = B_{11}^s \), with norm

\[
\| w \|_{Z_n} := \sum_{l=1}^{L_n} \sum_{r=1}^{2^d} 2^l(s-\frac{d}{2}) \| \langle w, \psi_{l',r} \rangle \| + n \xi_n^2 \sum_{l=1}^{L_n} \sum_{r=1}^{2^d} 2^l(s-\frac{d}{2}) \| \langle w, \psi_{l',r} \rangle \| \leq n \xi_n^2 \| w \|_{B_{11}^d}.
\]

Thus by the decentering inequality (29), the right hand side of the second to last display is lower bounded by \( e^{-\| w_0 \|_{B_{11}^d}} n \xi_n^2 \Pi_{W_n}(w : \| w \|_\infty \leq c_1 \xi_n) \). Decompose \( W_n \) in (5) as \( W_n = W_n^{(1)} + (n \xi_n^2)^{-1} W_n^{(2)} \), where

\[
\begin{align*}
W_n^{(1)} := & \sum_{l=1}^{L_n} \sum_{r=1}^{2^d} 2^{-l(s-\frac{d}{2})} W_{l',r} \psi_{l',r}; \quad W_n^{(2)} := \sum_{l=L_n+1}^{L_n} \sum_{r=1}^{2^d} 2^{-l(s-\frac{d}{2})} W_{l',r} \psi_{l',r}, \quad W_{l',r} \overset{iid}{\sim} \text{Laplace},
\end{align*}
\]

and note that by construction \( W_n^{(1)} \) and \( W_n^{(2)} \) are independent. Then,

\[
\Pi_{W_n}(w : \| w \|_\infty \leq c_1 \xi_n) \geq \Pr \left( \| W_n^{(1)} \|_\infty \leq \frac{c_1}{2} \xi_n, \| (n \xi_n^2)^{-1} W_n^{(2)} \|_\infty \leq \frac{c_1}{2} \xi_n \right) \geq \Pr \left( \| W_n^{(1)} \|_\infty \leq \frac{c_1}{2} \xi_n \right) \Pr \left( \| W_n^{(2)} \|_\infty \leq \frac{c_1}{2} n \xi_n^3 \right).
\]
By the continuous embedding $B^1_{d/2} \subset C([0,1]^d)$, holding since $s > d$ (e.g., [21], p.370), the first probability is lower bounded, for some $c_2 > 0$, by

$$\Pr \left( \|W_n^{(1)}\|_{B^1_{d/2}} \leq c_2 \xi_n \right) = \Pr \left( \sum_{l=1}^{L_n} \sum_{r=1}^{d'} |W_{lr}| \leq c_2 \xi_n \right) \geq \Pr \left( \dim(V_{L_n}) \max_{1 \leq i \leq L_n} \max_{r=1, \ldots, d'} |W_{lr}| \leq c_2 \xi_n \right) = \prod_{l=1}^{L_n} \prod_{r=1}^{d'} \Pr \left( |W_{lr}| \leq c_2 \frac{\xi_n}{\dim(V_{L_n})} \right).$$

Since for all $z \in (0,1)$, $\Pr(|W_{lr}| \leq z) = 1 - e^{-z} \geq z/2$, recalling $\dim(V_{L_n}) = O(2^{L_n d}) = O(n^{d/(2s+d)})$, the last line is greater than

$$\left( c_3 n^{-\frac{4+d}{2+d}} \sqrt{\log n} \right)^{c_4 n^{d/(2s+d)}} \geq e^{c_4 n^{d/(2s+d)} \log(n^{-c_5})} = e^{-c_6 n^{d/(2s+d)} \log n} = e^{-c_6 n^2},$$

for $c_3, \ldots, c_6 > 0$. On the other hand, as in (13), noting that the proof of Proposition 6.3 in [3] also applies with the first $L_n$ terms in the series removed,

$$\Pr \left( \|W_n^{(2)}\|_\infty \leq \frac{C_1}{2} n^{3/\xi_n^2} \right) \geq \Pr \left( \|W_n^{(2)}\|_\infty \leq \frac{C_1}{2} n^{d/(2s+d)} \right) \geq e^{-c_7 n^{d/(2s+d)}} \geq e^{-c_7 n^2},$$

as $n \to \infty$ for some $c_7 > 0$. Combining the last two displays yields condition (9) with $C = \|w_0\|_{B^1_{d/2}} + c_6 + c_7$. Next, define the sieves $P_n := \{\phi_w, w \in W_n\}$, where

$$W_n = \left\{ w = w^{(1)} + w^{(2)} : w^{(1)} \in W_n^{(1)}, w^{(2)} \in W_n^{(2)} \right\}$$

with

$$W_n^{(1)} = \left\{ w^{(1)} \in V_{L_n} : \|w^{(1)}\|_2 \leq Rn^{\xi_n^2} \right\}$$

and

$$W_n^{(2)} = \left\{ w^{(2)} = w^{(2,1)} + w^{(2,2)} : \|w^{(2,1)}\|_2 \leq Rn^{-\frac{d}{2s+d}}, \|w^{(2,2)}\|_{B^1_{d/2}} \leq R \right\} \cap \left\{ w^{(2)} : \|w^{(2)}\|_\infty \leq R \right\}.$$

Condition (10) then follows since

$$\Pi_n(P_n^c) \leq \Pi_{W_n^c}(W_n^c) \leq \Pr \left( W_n^{(1)} \not\in W_n^{(1)} \right) + \Pr \left( (n^{\xi_n^2})^{-1} W_n^{(2)} \not\in W_n^{(2)} \right) \leq 3 - \Pr \left( W_n^{(1)} \in V_{L_n} : \|w^{(1)}\|_2 \leq Rn^{\xi_n^2} \right) - \Pr \left( W_n^{(2)} = W_n^{(2,1)} + W_n^{(2,2)} : \|w^{(2,1)}\|_2 \leq Rn^{-\frac{d}{2s+d}} \log n, \|w^{(2)}\|_{B^1_{d/2}} \leq Rn^{\xi_n^2} \right) - \Pr \left( \|W_n^{(2)}\|_\infty \leq Rn^{\xi_n^2} \right)$$

which, by Lemma 6, for $n \in \mathbb{N}$ large enough, fixing $K > C + 4$ and choosing sufficiently large $R > 0$, is smaller than

$$3e^{-Kn^2} \leq e^{-(C+4)n^2}. $$

Finally, let $\{\gamma_j, j = 1, \ldots, J_n\}$, with $J_n = N(\xi_n ; W_n^{(1)}, \| \cdot \|_2)$, be a minimal $\xi_n$-covering of $W_n^{(1)}$ in $L^2$-distance, and let $\{\chi_k, k = 1, \ldots, K_n\}$, with $K_n = N(\xi_n ; W_n^{(2)}, \| \cdot \|_1)$, be
a minimal $\xi_n$-covering of $W_n^{(1)}$ in $L^1$-distance. Then for each $w = w^{(1)} + w^{(2)} \in W_n$ there exist $j \in \{1, \ldots, J_n\}$ and $k \in \{1, \ldots, K_n\}$ such that

$$\|w^{(1)} - \gamma_j\|_2 \leq \xi_n; \quad \|w^{(2)} - \chi_k\|_1 \leq \xi_n,$$

which by point 2. in Lemma 11 implies that

$$d_{TV}(\phi_w, \phi_{\gamma_j + \chi_k}) \leq \|w - (\gamma_j + \chi_k)\|_1 \leq \|w^{(1)} - \gamma_j\|_2 + \|w^{(2)} - \chi_k\|_1 \leq \xi_n.$$

It follows that for some $c_8 > 0$ the set $\{\gamma_j + \chi_k, \ j = 1, \ldots, J_n, \ k = 1, \ldots, K_n\}$ is a $c_8\xi_n$-covering of $W_n$ in total variation distance, whence

$$\log N(\xi_n; W_n, d_{TV}) \leq \log(J_nK_n) = \log N(\xi_n; W_n^{(1)}, \|\cdot\|_2) + \log N(\xi_n; W_n^{(2)}, \|\cdot\|_1). \quad (19)$$

Recalling $W_n \subset V_n$, with $\dim(V_n) = O(n^{-d/8})$, the first metric entropy equals

$$\log N(\xi_n; \{y \in \mathbb{R}^{\dim(V_n)} \mid |y|_2 \leq Rn\xi_n^2\}, |\cdot|_2) \leq \dim(V_n) \log \left(\frac{3Rn\xi_n^2}{\xi_n}\right) \leq n^{\frac{d}{2d+e}} \log(n^{c_9}) \approx n\xi_n^2,$$

having used the usual metric entropy bound for balls in Euclidean spaces (e.g., [21, Theorem 4.3.34]). Arguing as in the conclusion of the proof of Theorem 1, the second metric entropy in (19), is bounded by a multiple of

$$\log N(\xi_n; \{w : \|w\|_{B_t} \leq R\}, \|\cdot\|_1) \lesssim \xi_n^{\frac{d}{2d+e}} = n^{\frac{d}{2d+e}} \log(n) - \frac{d}{2d+e} \leq n\xi_n^2.$$

The last two displays and (19) conclude the verification of (11) and the proof of Theorem 2.

**Lemma 6.** For $s > d$, let $W_n^{(1)}, W_n^{(2)}$ be the random functions in (18). Then, for all $n \in \mathbb{N}$ large enough, all $K > 0$, there exist sufficiently large $R > 0$ such that

1. $\Pr \left( W_n^{(1)} \in V_n, \|W_n^{(1)}\|_\infty \leq Rn^{\frac{d}{2d+e}} \log n \right) \geq 1 - e^{-Kn^{s/(2s+d)} \log n};$

2. $\Pr \left( W_n^{(2)} = W_n^{(2,1)} + W_n^{(2,2)} : \|W_n^{(2,1)}\|_2 \leq Rn^{\frac{d}{2d+e}} \sqrt{\log n}, \|W_n^{(2)}\|_{B_t} \leq Rn^{\frac{d}{2d+e}} \log n \right) \geq 1 - e^{-Kn^{d/(2s+d)} \log n};$

3. $\Pr \left( \|W_n^{(2)}\|_\infty \leq Rn^{\frac{d}{2d+e}} \log n \right) \geq 1 - e^{-Kn^{d/(2s+d)} \log n}.$

**Proof.** For point 1., setting $\xi_n := n^{-s/(2s+d)} \sqrt{\log n}$, recalling $W_n^{(1)} \in V_n$ by construction, the probability of interest equals

$$\Pr \left( \|W_n^{(1)}\|_\infty \leq Rn\xi_n^2 \right) \geq 1 - e^{-Kn\xi_n^2}$$

having used the sup-norm concentration inequality (28) and chosen $R > 0$ large enough. Points 2. and 3. follow from exactly the same argument as in the proof of Lemma 5.
upon noting that $W_n^{(2)}$ is also a $(s - d)$-regular Besov random element with associated spaces

$$Q_n := \left\{ w = \sum_{l=L_n+1}^{\infty} \sum_{r=1}^{2^l} w_{lr} \psi_{lr} : \|w\|_{Q_n} = \|w\|_{H^{s-d/2}} < \infty \right\},$$

and

$$Z_n := \left\{ w = \sum_{l=L_n+1}^{\infty} \sum_{r=1}^{2^l} w_{lr} \psi_{lr} : \|w\|_{Z_n} = \|w\|_{B_{11}^s} < \infty \right\},$$

and that for each $w \in Q_n$,

$$\|w\|^2_{L^2} = \sum_{l=L_n+1}^{\infty} \sum_{r=1}^{2^l} 2^{-2(t(s-d)/2)2(t(s-d)/2)}|w_{lr}|^2 \leq 2^{-2L_n(s-d)}\|w\|^2_{H^{s-d/2}} \lesssim n^{-\frac{2(s-d)}{s+d}}\|w\|^2_{H^{s-d/2}}.$$ 

The details are omitted for brevity.

### 5.3 Proof of Theorem 3

We verify conditions (9) - (11) with $\xi_n := c_1n^{-s/(2s+d)}$, for sufficiently large $c_1 > 0$ to be chosen below. Arguing as in the proof of Theorem 1,

$$\Pi_n \left( p : -E_{p_0} \left( \log \frac{p}{p_0} (X) \right) \leq \xi_n^2, \ E_{p_0} \left( \log \frac{p}{p_0} (X) \right)^2 \leq \xi_n^2 \right) \geq \Pi_{W_n}(w : \|w - w_0\|_{\infty} \leq c_2 \xi_n)$$

for some $c_2 > 0$ and $w_0 = \phi^{-1} \circ f_0 \in B_{11}^{s}$. The decentering space $Z_n$ associated to $W_n$ in (6) equals the approximation space $V_{q_n}$, with norm $\| \cdot \|_{Z_n} = \| \cdot \|_{B_{11}^{s}}$. Since $w_0 \in B_{\infty}^{s}$, its wavelet projection $P_{L_n}w_0 \in V_{L_n}$ satisfies

$$\|w_0 - P_{L_n}w_0\|_{\infty} \leq 2^{-L_n s}\|w_0\|_{B_{\infty}^{s}} \simeq \|w_0\|_{B_{\infty}^{s}} n^{-\frac{s}{s+d}}$$

as well as

$$\|P_{L_n}w_0\|_{B_{11}^{s+d}} = \sum_{l=1}^{L_n} 2^{l(s-d)} \sum_{r=1}^{2^l} \sum_{l=1}^{L_n} 2^{l(s-d)} \|w_0\|_{B_{\infty}^{s+d}} 2^{L_n d} \simeq \|w_0\|_{B_{\infty}^{s+d}} n^{s} \xi_n^2.$$

By the triangle and the decentering inequality (29), taking $c_1 > 0$ large enough in the definition of $\xi_n$, for some $c_3, c_4 > 0$,

$$\Pi_{W_n}(w : \|w - w_0\|_{\infty} \leq c_2 \xi_n) \geq \Pi_{W_n}(w : \|w - P_{L_n}w_0\|_{\infty} \leq c_3 \xi_n) \geq e^{-c_4\|w_0\|_{B_{\infty}^{s}} n^{s} \xi_n^2} \Pi_{W_n}(w : \|w\|_{\infty} \leq c_3 \xi_n).$$

By (13) (with $s$ replaced by $s + d$),

$$\Pi_{W_n}(w : \|w\|_{\infty} \leq c_3 \xi_n) \geq e^{-c_5(c_3 \xi_n)^{-d/s}} = e^{-c_6 n \xi_n^2}, \quad (20)$$

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for $c_5, c_6 > 0$ as $n \to \infty$, whence condition (9) follows for $C = c_4 \|w_0\|_{B_{s,d}^l} + c_6$. Next, take the sieves $\mathcal{P}_n := \{\phi_w, \ w \in \mathcal{W}_n\}$, where

$$\mathcal{W}_n := \left\{ w = w^{(1)} + w^{(2)} : \|w^{(1)}\|_\infty \leq R\xi_n, \ \|w^{(2)}\|_{B_{s,d}^l} \leq n\xi_n^2 \right\}.$$ 

Using Lemma 7, choosing $R > 0$ large enough,

$$\Pi_n(\mathcal{P}_n^c) \leq 1 - \Pi_{\mathcal{W}_n}(W = w^{(1)} + w^{(2)} : \|w^{(1)}\|_\infty \leq R\xi_n, \ \|w^{(2)}\|_{B_{s,d}^l} \leq n\xi_n^2)$$

$$\leq e^{-(C+4)n\xi_n^2}$$

verifying condition (10). Finally, since the link function $\phi$ is uniformly log-Lipschitz, by Problem 2.4 in [19], for all $w, w' \in \mathcal{W}_n$,

$$d_{TV}(\phi_w, \phi_{w'}) \leq 2d_{H}(\phi_w, \phi_{w'}) \leq \|w - w'\|_\infty e^{-c_7\|w - w'\|_\infty},$$

for some $c_7 > 0$, implying

$$\log N(\xi_n; \mathcal{P}_n, d_{TV}) \leq \log N(c_8\xi_n; \mathcal{W}_n, \|\cdot\|_\infty),$$

for $c_8 > 0$. Using Theorem 4.3.36 in [21], the latter metric entropy is bounded by a multiple of

$$\log N\left(\xi_n; \{w : \|w\|_{B_{s,d}^l} \leq n\xi_n^2 \}, \|\cdot\|_\infty \right) \lesssim (n\xi_n)^{\frac{2d}{d+2}} \simeq n\xi_n^2,$$

concluding the verification of (11) and the proof of Theorem 3. \hfill \Box

**Lemma 7.** For $s > d$, let $\Pi_{\mathcal{W}_n}$ be the non-rescaled truncated Besov prior arising as the law of $W_n$ in (6). Then, for all $n \in \mathbb{N}$ large enough, all $K > 0$, there exist sufficiently large $R > 0$ such that

$$\Pi_{\mathcal{W}_n}\left(w = w^{(1)} + w^{(2)} : \|w^{(1)}\|_\infty \leq R\xi_n^{-\frac{d}{s+d}}, \ \|w^{(2)}\|_{B_{s,d}^l} \leq Rn\xi_n^2\right)$$

$$\geq 1 - e^{-Kn^d/(2s+d)}.$$ 

**Proof.** The claim follows similarly to the proof of point 1. in Lemma 5 and point 2. in Lemma 6, using the two-level concentration inequality (30), the centred small ball estimate (20) and noting that the spaces associated to $W_n$ satisfy $Z_n = Q_n = V_{L_n}$ with norms $\|w\| = \|w\|_{B_{s,d}^l}$ and $\|w\| = \|w\|_{H_{s+d/2}}$, and further that for each $w \in V_{L_n}$ satisfying $\|w\|_{H_{s+d/2}} \lesssim n^{d/(4s+2d)},$

$$\|w\|_{B_{s,d}^l} = \sum_{l=1}^{L_n} 2^l(s+\frac{d}{2}) \sum_{r=1}^{2^d} |w_{lr}|$$

$$\leq \sqrt{\dim(V_{L_n})} \sum_{l=1}^{L_n} 2^{2l(s+\frac{d}{2})} \sum_{r=1}^{2^d} |w_{lr}|^2 \simeq n^{\frac{d}{2s+d}} \|w\|_{H_{s+d/2}} = n^{\frac{d}{2s+d}}.$$ \hfill \Box
5.4 Proof of Theorem 4

We verify conditions (9) - (11) with \( \xi_n := c_1 n^{-s_0/(2s_0 + d)} \) for sufficiently large \( c_1 > 0 \) to be chosen below. Arguing as in the proof of Theorem 2,

\[ \Pi_n \left( p : -E_{p_0} \left( \log \frac{p}{p_0}(X) \right) \leq \xi_n^2, \ E_{p_0} \left( \log \frac{p}{p_0}(X) \right)^2 \leq \xi_n^2 \right) \geq \Pi_{W_n}(w : \|w - w_0\|_{\infty} \leq c_2 \xi_n) \]

for some \( c_2 > 0 \) and \( w_0 = \phi^{-1} \circ f_0 \in B^s_{11} \). Condition (9) then follows from Lemma 8 for a large enough constant \( C > 0 \), upon taking sufficiently large \( c_1 \) in the definition of \( \xi_n \).

Next, define the sieves \( P_n = \{ \phi_w, \ w \in W_n \} \), where

\[ W_n := \left\{ w = w^{(1)} + w^{(2)} : \|w^{(1)}\|_1 \leq R n^{-\frac{d}{2s_0 + d}}, \ |w^{(2)}|_{B^{s_0 + d}_{11}} \leq R n^{\frac{d}{2s_0 + d}} \right\}, \]

with \( s^* := s_0/(1 + M/\log n) \) and \( R, M > 0 \). By Lemma 9, taking sufficiently large \( M \) and \( R \), for all \( n \) large enough,

\[ \Pi_n(P_n) \leq \Pi_{W_n}(W_n^c) \leq e^{-(C+4)n^2}, \]

which verifies Condition (10). Finally, recalling that \( \phi \) is assumed to be uniformly Lipschitz and bounded away from zero, by point 2. in Lemma 11 we have \( d_{TV}(\phi_w, \phi_{w'}) \lesssim \|w - w'\|_1 \), implying that \( \log N(\xi_n; P_n, d_{TV}) \leq \log N(c_3 \xi_n; W_n, \| \cdot \|_1) \) for some \( c_3 > 0 \). By construction of \( W_n \), using (23) below and Theorem 4.3.36 in [21], the latter metric entropy is bounded by a multiple of

\[ \log N(\xi_n; \left\{ w : \|w\|_{B^{s_0 + d}_{11}} \leq R n^{-\frac{d}{2s_0 + d}} \right\}, \| \cdot \|_1) \lesssim \frac{R n^{\frac{d}{2s_0 + d}}}{\xi_n} \lesssim \frac{n^{\frac{d}{2s_0 + d}}}{n^{\frac{d}{2s_0 + d}}} = n^{\frac{d}{2s_0 + d}} \lesssim n\xi_n^2. \]

This concludes the verification of Condition (11), and via an application of Theorem 2.1 in [16], the proof of Theorem 4.

\[ \square \]

Lemma 8. Let \( \Pi_{W_n} \) be the hierarchical rescaled Besov prior arising as the law of \( W_n \) in (7). Let \( w_0 \in B^s_{11}([0,1]^d) \), any \( s > d \). Then, for sufficiently large \( D_1, D_2 > 0 \),

\[ \Pi_{W_n}(w : \|w - w_0\|_{\infty} \leq D_1 n^{-\frac{s_0}{2s_0 + d}}) \geq e^{-D_2 n^{d/(2s_0 + d)}}. \]

Proof. For each fixed \( s > d \), let \( \varepsilon_{s,n} := n^{-s/(2s + d)} \) and let \( \Pi_{W_{s,n}} \) be the rescaled \((s - d)\)-regular Besov prior arising as the law of

\[ W_{s,n} := \frac{W_s}{n^{\frac{s}{2s + d}}}, \quad W_s := \sum_{l=1}^{2^d} \sum_{r=1}^{2^d} 2^{-l-s - \frac{d}{2}} W_{l,l} \psi_{l,r}, \quad W_{l,l} \text{ iid Laplace.} \]

Denote by \( \Pi_{W_s} \) the law of \( W_s \). The probability of interest equals

\[ \int_d \log n \Pi_{W_{s,n}}(w : \|w - w_0\|_{\infty} \leq D_1 \varepsilon_{s,n}) \sigma_n(s) ds \geq \int_{s_0}^{s_0 + \frac{d}{2s_0 + d}} \Pi_{W_{s,n}}(w : \|w - w_0\|_{\infty} \leq D_1 \varepsilon_{s,n}) \sigma_n(s) ds. \]

(22)
For $\Lambda_n \in \mathbb{N}$ in to be chosen below, let $P_{\Lambda_n}w_0$ be the wavelet projection of $w_0 \in B_{11}^{s_0}$ onto the approximation space $V_{\Lambda_n}$. Then, by standard wavelet properties,

$$
\|w_0 - P_{\Lambda_n}w_0\|_\infty \leq \sum_{l > \Lambda_n} \left( \sum_{r=1}^{2^d} |\langle w_0, \psi_{lr} \rangle_2|^2 \right)^{\frac{1}{2}} 
\lesssim \sum_{l > \Lambda_n} 2^{\frac{ld}{2}} \sup_{r=1, \ldots, 2^d} |\langle w_0, \psi_{lr} \rangle_2| \leq 2^{-\Lambda_n(s_0-d)}\|w_0\|_{B_{11}^{s_0}}.
$$

Thus, taking $\Lambda_n \in \mathbb{N}$ such that $2^{\Lambda_n} \approx n^{s_0/(2s_0+d)(s_0-d)}$ (note that this is higher than the usual order $n^{1/(2s_0+d)}$), we have

$$
\|w_0 - P_{\Lambda_n}w_0\|_\infty \lesssim 2^{-\Lambda_n(s_0-d)}\|w_0\|_{B_{11}^{s_0}} \lesssim n^{-\frac{s_0}{2s_0+d}} = \varepsilon_{s_0,n}.
$$

On the other hand, for all $s \in [s_0, s_0 + 1 / \log n]$,

$$
\|P_{\Lambda_n}w_0\|_{B_{11}^{s_0}} = \sum_{l \leq \Lambda_n} 2^{l(s - s_0)} 2^{l(s_0 - \frac{d}{2})} \sum_{r=1}^{2^d} |\langle w_0, \psi_{lr} \rangle_2| 
\leq 2^{\Lambda_n(s - s_0)}\|w_0\|_{B_{11}^{s_0}} \lesssim n^{2s_0+2s_0-2d(\log n)^{-1}} = e^{2s_0+2s_0-d-2d}(\log n)^{-1} \lesssim 1.
$$

It follows that for all $s \in [s_0, s_0 + 1 / \log n]$, by the triangle inequality (choosing $D_1 > 0$ above large enough), for some $c_1 > 0$,

$$
\Pi_{W_{s,n}}(w : \|w - w_{0}\|_\infty \leq D_1\varepsilon_{s_0,n}) \geq \Pi_{W_{s,n}}(w : \|w - P_{\Lambda_n}w_0\|_\infty \leq c_1\varepsilon_{s_0,n}),
$$

and since $W_{s,n}$ in (21) is a fixed Besov random element with associated decentering space equal to $Z_{n} = B_{11}^{s_0}$ and norm $\| \cdot \|_{Z_{n}} = \varepsilon_{s_0,n}^2 \cdot \| \cdot \|_{B_{11}^{s_0}}$ (cf. Section A.1), by the decentering inequality (29) the latter probability is lower bounded by

$$
e^{-\|P_{\Lambda_n}w_0\|_{B_{11}^{s_0}} n\varepsilon_{s_0,n}^2} \Pi_{W_{s,n}}(w : \|w\|_\infty \leq c_1n\varepsilon_{s_0,n}^2) \geq e^{-c_2n\varepsilon_{s_0,n}^2} \Pi_{W_{s,n}}(w : \|w\|_\infty \leq c_1n\varepsilon_{s_0,n}^2) .
$$

By the centred small ball inequality (13) (noting that $W_s$ coincides with $W$ there),

$$
\Pi_{W_s}(w : \|w\|_\infty \leq c_1n\varepsilon_{s_0,n}^2) \geq e^{-c_3(c_1n\varepsilon_{s_0,n}^2)^{-d/(s-d)}} \geq e^{-c_4n\varepsilon_{s_0,n}^2},
$$

for $c_2, c_3, c_4 > 0$, since

$$
(n\varepsilon_{s_0,n}^2)^{-\frac{d}{s-d}} = \left( \frac{4s_0+2d+2d_0+2d^2-2s_0-2d_0-s_0-2d}{(2^d+2d_0+2d^2)} \right)^{-\frac{d}{s-d}} = \left( \frac{2s_0+2d_0-s_0-2d}{(2^d+2d_0+2d^2)} \right)^{-\frac{d}{s-d}} = \left( \frac{2s_0+2d_0-s_0-2d}{(2^d+2d_0+2d^2)} \right)^{-\frac{d}{s-d}} \leq n^{2s_0+n^2}.
$$

as for all $s \geq s_0 > d$

$$(s-d)(2s+d) - (2s_0 - s_0 - d^2) = 2s(s-s_0) - d(s-s_0) = (s-s_0)(2s-d) \geq 0.
$$

Combining the previous bounds, we find that for all $s \in [s_0, s_0 + 1 / \log n]$, for sufficiently large $D_1 > 0$,

$$
\Pi_{W_{s,n}}(w : \|w - w_0\|_\infty \leq D_1\varepsilon_{s_0,n}) \geq e^{-c_5n\varepsilon_{s_0,n}^2}.
$$
The lower bound (22) then implies that the probability of interest is greater than
\[
\int_{s_0}^{s_0 + \frac{1}{M} \log n} e^{-c_5 n \varepsilon_{s_0,n}^2} \sigma_n(s) ds = e^{-c_5 n \varepsilon_{s_0,n}^2} \int_{s_0}^{s_0 + \frac{1}{M} \log n} e^{-n \varepsilon_{s_0,n}^2} \zeta_n ds.
\]

Using that \( \sigma_n(s) \) is increasing in \( s \) and that the normalisation constant satisfies \( \zeta_n \simeq \log n \), the integral on the right hand side is bounded below by
\[
\frac{1}{\zeta_n} e^{-n \varepsilon_{s_0,n}^2} \geq (\log n)^{-2} e^{-n \varepsilon_{s_0,n}^2} \geq e^{-c_6 n \varepsilon_{s_0,n}^2},
\]
for some \( c_6 > 0 \). The claim then follows taking \( D_2 = c_5 + c_6 > 0 \). \( \square \)

**Lemma 9.** Let \( \Pi_{W_n} \) be the hierarchical rescaled Besov prior arising as the law of \( W_n \) in (7). For fixed \( s_0 > d \), and \( M, R > 0 \), let \( s^* = s_0/(1 + M/\log n) \) and define the set
\[
\mathcal{W}_n = \left\{ w = w^{(1)} + w^{(2)} : \|w^{(1)}\|_1 \leq R n^{-\frac{s}{2(s+d)}} \|w^{(2)}\|_{B_{s,n}^{1+s+d}} \leq R n^{-\frac{s}{2(s+d)}} \right\}.
\]

Then, for all \( K > 0 \), there exist sufficiently large \( M, R \), such that for \( n \in \mathbb{N} \) large enough,
\[
\Pi_{W_n}(\mathcal{W}_n^c) \leq e^{-K n^{d/(2s+n)} d}.
\]

**Proof.** For any \( s > d \), let \( \varepsilon_{s,n}, \Pi_{W_{s,n}}, \Pi_{W_s}, W_s, W_n \) and \( s^* \) be defined as at the beginning of the proof of Lemma 8. For \( s^* = s_0/(1 + M/\log n) \), some algebra shows
\[
\frac{n \varepsilon_{s^*,n}^2}{n \varepsilon_{s_0,n}^2} = n^{dM/d + \log n} \frac{d}{2s + d} \frac{2dM}{2s + d} = n^{dM/d + \log n + 2dM/(2s + d)} = e^{dM(2s + d)/(2s + d)} > 1,
\]
which implies, bounding the exponential above and below, that
\[
\sqrt{c_1 n \varepsilon_{s_0,n}^2} \leq n \varepsilon_{s^*,n}^2 \leq \sqrt{c_1 n \varepsilon_{s_0,n}^2}; \quad (c_1) \frac{1}{2} \varepsilon_{s_0,n} \leq \varepsilon_{s^*,n} \leq \sqrt{c_1 \varepsilon_{s_0,n}}, \quad c_1 := e^{2dM/(2s + d)} > 1,
\]
the lower bounds holding, for any fixed \( M > 0 \), for all \( n \in \mathbb{N} \) large enough. The probability of interest is equal to
\[
\int_d^{s^*} \Pi_{W_{s,n}}(\mathcal{W}_n^c) \sigma_n(s) ds + \int_{s^*}^{\log n} \Pi_{W_{s,n}}(\mathcal{W}_n^c) \sigma_n(s) ds
\]
\[
\leq e^{-(K+1)n \varepsilon_{s_0,n}^2} + \int_{s^*}^{\log n} \Pi_{W_{s,n}}(\mathcal{W}_n^c) \sigma_n(s) ds,
\]
(24)

having chosen \( M \) large enough and used that \( \sigma_n(s) \) is increasing in \( s \), that \( \zeta_n \simeq \log n \), and (23) to bound the first integral by
\[
\int_d^{s^*} \sigma_n(s) ds \leq s^* e^{-n \varepsilon_{s^*,n}^2} \zeta_n \leq e^{-c_2 n \varepsilon_{s^*,n}^2} \leq e^{-c_2 \sqrt{c_1 n} \varepsilon_{s_0,n}^2} \leq e^{-(K+1)n \varepsilon_{s_0,n}^2}.
\]

We proceed bounding the second integral in (24). To do so, note
\[
\Pi_{W_{s,n}}(\mathcal{W}_n)
\]
\[
= \Pi_{W_s}(w = w^{(1)} + w^{(2)} : \|w^{(1)}\|_1 \leq R n \varepsilon_{s^*,n}^2 \varepsilon_{s,n}^2; \|w^{(2)}\|_{B_{s,n}^{1+s+d}} \leq R n^2 \varepsilon_{s^*,n}^2 \varepsilon_{s,n}^2),
\]

22
The spaces associated to $W_s$ are respectively $Z = B_{11}^s$, with norm $\| \cdot \|_Z = \| \cdot \|_{B_{11}^s}$, and $Q = H^{s-d/2}$, with $\| \cdot \|_Q = \| \cdot \|_{H^{s-d/2}}$ (cf. Section A.1). Letting
\[
\overline{W}_n = \left\{ \overline{\{w = \overline{w}^{(1)} + \overline{w}^{(2)} + \overline{w}^{(3)} : \| \overline{w}^{(1)} \|_1 \leq n \varepsilon_{s^*, n}^2 \varepsilon_{s, n}^2, \| \overline{w}^{(2)} \|_{H^{s-d/2}} \leq \sqrt{Rn \varepsilon_{s, n}^2}, \| \overline{w}^{(3)} \|_{B_{11}^{s^*}} \leq \sqrt{Rn \varepsilon_{s, n}^2} \right\},
\]
the two-level concentration inequality (30) implies, using again (23), for $c_3, c_4 > 0$,
\[
\Pi_{W_s}(\overline{W}_n) \geq 1 - \frac{1}{\Pi_{W_s}(w : \| w \|_1 \leq n \varepsilon_{s^*, n}^2 \varepsilon_{s, n}^2)} e^{-c_3 Rn \varepsilon_{s, n}^2},
\]
\[
\geq 1 - \frac{1}{\Pi_{W_s}(w : \| w \|_1 \leq n \varepsilon_{s^*, n}^2 \varepsilon_{s, n}^2)} e^{-c_4 Rn \varepsilon_{s, n}^2}.
\]
As $\| w \|_1 \leq \| w \|_\infty$, noting that $s \geq s^* = s_0 \log n / (M + \log n) > d$ for all $n$ large enough since $s_0 > d$, by the centred small ball inequality (13),
\[
\Pi_{W_s}(w : \| w \|_1 \leq n \varepsilon_{s^*, n}^2 \varepsilon_{s, n}^2) \geq e^{-c_5 (n \varepsilon_{s^*, n}^2 \varepsilon_{s, n}^2)^{-d/(s-d)}} \geq e^{-c_6 n \varepsilon_{s^*, n}^2} \geq e^{-c_7 n \varepsilon_{s, n}^2},
\]
having used (23) and the fact that
\[
(n \varepsilon_{s^*, n}^2 \varepsilon_{s, n}^2)^{-d/(s-d)} \leq \left( \frac{n^{-2s^* - d} \varepsilon_{s^*, n}^2 + 2s^* \varepsilon_{s, n}^2}{(2s^* + d)(s^* - d)} \right)^{-d/(s-d)} \leq n \varepsilon_{s, n}^2,
\]
since the last exponent is smaller than one. For sufficiently large $R > 0$, it follows that
\[
\Pi_{W_s}(\overline{W}_n) \geq 1 - e^{-c_8 (R - c_7) n \varepsilon_{s, n}^2} \geq 1 - e^{-(K+1) n \varepsilon_{s, n}^2}. \tag{25}
\]
Next, approximate $\overline{w}^{(2)}$ in the definition of $\overline{W}_n$ by its wavelet projection $P_{L_n} \overline{w}^{(2)}$ at resolution $L_n \in \mathbb{N}$ with $2L_n \approx n \varepsilon_{s, n}^2$. Then,
\[
\| \overline{w}^{(2)} - P_{L_n} \overline{w}^{(2)} \|_1 \leq 2 - L_n (s - \frac{d}{2}) \| \overline{w}^{(2)} \|_{H^{s-d/2}} \leq n \delta \varepsilon_{s^*, n} = n \delta \varepsilon_{s, n}^2 \varepsilon_{s^*, n} = n \delta \varepsilon_{s, n}^2 \varepsilon_{s^*, n}.
\]
Also, as shown in the conclusion of the proof of Lemma 5,
\[
\| P_{L_n} \overline{w}^{(2)} \|_{B_{11}^{s^*}} \leq \sqrt{2L_n d} \| \overline{w}^{(2)} \|_{H^{s-d/2}} \leq n \frac{d}{2+2s^*+d} \varepsilon_{s^*, n} = n \frac{d}{2+(2s^*+d) \varepsilon_{s, n}^2} \varepsilon_{s^*, n} \leq n \varepsilon_{s, n}^2 \varepsilon_{s^*, n},
\]
since the exponent is smaller than $d/(2s^* + d)$ when $s \geq s^*$. For $\overline{w}^{(1)}$, $\overline{w}^{(3)}$ in the definition of $\overline{W}_n$, setting $\overline{w}^{(1)} := \overline{w}^{(1)} + (\overline{w}^{(2)} - P_{L_n} \overline{w}^{(2)})$ and $\overline{w}^{(2)} := \overline{w}^{(3)} + P_{L_n} \overline{w}^{(2)}$, then shows that for all $s \in [s^*, \log n]$ and all $n$ and $R$ large enough,
\[
\overline{W}_n \subseteq \overline{W}_n := \{ \overline{w} = \overline{w}^{(1)} + \overline{w}^{(2)} : \| \overline{w}^{(1)} \|_1 \leq \sqrt{Rn \varepsilon_{s, n}^2 \varepsilon_{s^*, n}}, \| \overline{w}^{(2)} \|_{B_{11}^{s^*}} \leq \sqrt{Rn \varepsilon_{s, n}^2 \varepsilon_{s^*, n}} \}.
\]
In view of (25),
\[
\Pi_{W_s}(\overline{W}_n) \geq 1 - e^{-(K+1) n \varepsilon_{s, n}^2}. \tag{26}
\]
We conclude showing that, choosing sufficiently large $R > 0$,
\[
\overline{W}_n \subseteq \{ w = w^{(1)} + w^{(2)} : \| w^{(1)} \|_1 \leq Rn \varepsilon_{s, n}^2 \varepsilon_{s^*, n}, \| w^{(2)} \|_{B_{11}^{s^*+d}} \leq Rn \varepsilon_{s, n}^2 \varepsilon_{s^*, n} \} \tag{27}
\]
for all \( s \in [s^*, \log n] \) and all \( n \in \mathbb{N} \) large enough. First consider the case \( s \in [s^* + d, \log n] \). Then
\[
\| \tilde{w}^{(2)} \|_{B_{1t}^{s} + d} \leq \| \tilde{w}^{(2)} \|_{B_{1t}^{s}} \leq \tilde{R} n e_{s^*, n}^{2} \leq \tilde{R} n^{2} e_{s, n}^{2}
\]
since \( n e_{s, n}^{2} \to \infty \). The inclusion (27) thus follows with \( \tilde{w}^{(1)} = \tilde{w}^{(1)} \), \( \tilde{w}^{(2)} = \tilde{w}^{(2)} \), and \( R = \tilde{R} \). Next consider the range \( s \in [s^*, s^* + d] \). Approximate \( \tilde{w}^{(2)} \) in the definition of \( \tilde{W}_n \), by its wavelet projection \( P_{\Lambda_n} \tilde{w}^{(2)} \) with \( \Lambda_n \in \mathbb{N} \) satisfying \( 2^{\Lambda_n} \simeq n^{(2s^* + d + d^*)/s^*} \). Then,
\[
\| P_{\Lambda_n} \tilde{w}^{(2)} \|_{B_{1t}^{s} + d} \leq 2^{\Lambda_n(s^* + d - s)} \| \tilde{w}^{(2)} \|_{B_{1t}^{s}} \leq n^{d/s^*} n e_{s^*, n}^{2} = n^{2} e_{s, n}^{2}
\]
and
\[
\| \tilde{w}^{(2)} - P_{\Lambda_n} \tilde{w}^{(2)} \|_{1} \lesssim 2^{-\Lambda_n s} \| \tilde{w}^{(2)} \|_{B_{1t}^{s}}
\]
\[
\lesssim n^{- \frac{d}{s^*} - \frac{d^*}{s^*} - \frac{d^*}{s^*}} = n^{- \frac{2s^* + d + d^*}{s^*} - \frac{d^*}{s^*}}.
\]
The inclusion (27) thus follows showing that the right hand side is smaller than
\[
ne_{s, n}^{2} \lesssim n^{d/s^*} n^{- s^*} = n^{d/s^*} n^{- \frac{2s^* + d + d^*}{s^*}} = n^{\frac{2s^* + d + d^*}{s^*} + \frac{2d^*}{s^*}}.
\]
Indeed, the difference between the numerators of the exponents equals
\[
\Delta(s) = -2d s^* + ds^* + d^2 - 2s^* s^* - s[-2(s^*)^2 - 3ds^* - d^2] - d(s^*)^2 - 2d^2 s^* - d^3
\]
\[
= -2(s^* + d)s^* + s[2(s^*)^2 + 3ds^* + d^2] - d(s^*)^2 - 2d^2 s^* + d^3
\]
which, as a function of \( s \), is a downward-pointing parabola with maximum attained at
\[
s_v := \frac{2(s^*)^2 + 3ds^* + d^2}{4(s^* + d)} < s^*
\]
since, recalling \( s^* = s_0 \log n/(M + \log n) > d \) for all \( n \) large enough as \( s_0 > d \),
\[
2(s^*)^2 + 3ds^* + d^2 - 4(s^* + d)s^* = -2(s^*)^2 - ds^* + d^2 \leq -2(s^*)^2 < 0.
\]
Hence, since \( \Delta(s) \) is decreasing for \( s > s_v \), for all \( s \in [s^*, s^* + d] \),
\[
\Delta(s) \leq \Delta(s^*)
\]
\[
= -2(s^* + d)(s^*)^2 + s^*[2(s^*)^2 + 3ds^* + d^2] - d(s^*)^2 - 2d^2 s^* + d^3
\]
\[
= -d(d - 1)s^* \leq 0.
\]
This shows as required that \( \| \tilde{w}^{(2)} - P_{\Lambda_n} \tilde{w}^{(2)} \|_{1} \lesssim ne^{(K+1)n^2}e_{s^*, n} \), so that taking \( w^{(1)} := \tilde{w}^{(1)} + P_{\Lambda_n} \tilde{w}^{(2)} \) and \( w^{(2)} := P_{\Lambda_n} \tilde{w}^{(2)} \), the desired inclusion (27) follows for large enough \( R > 0 \). By (26), we then conclude
\[
\Pi_{W_{s,n}}(W_n) \geq \Pi_{W_s}(\tilde{W}_n) \geq 1 - e^{-(K+1)n^2e_{s^*, n}}.
\]
Combined with (24) this yield
\[
\Pi_{W_n}(W_n) \leq e^{-(K+1)n^2e_{s^*, n}} + \int_{s^*}^{\log n} e^{-(K+1)n^2e_{s^*, n}}\sigma_n(s)ds
\]
\[
\leq 2e^{-(K+1)n^2e_{s^*, n}} \leq e^{-Ke_{s^*, n}}.
\]
A Additional material

A.1 General properties of Besov priors

In this section we record, for ease of exposition, a number of properties of Besov priors employed throughout the paper, largely based on the results of Agapiou et al. [3]. For $t > 0$, consider a (possibly $n$-independent) $t$-regular Besov prior $\Pi_{W_n}$ on $C([0,1]^d)$, arising as the law of

$$W_n = \sum_{l=1}^{\infty} \sum_{r=1}^{d^l} \sigma_{n,lr} W_{lr} \psi_{lr}, \quad W_{lr} \sim \text{Laplace},$$

with $\sigma_{n,lr} > 0$ satisfying (2). An analogous argument as in Lemma 5.2 and Proposition 6.1 in [3] (see also Lemma 7.1 in [5]) shows that $W_n \in C([0,1]^d) \cap B_{r,t}^d$ almost surely for all $0 < t' < t$ and $r \in [1,\infty]$. On the contrary, $\Pr(W_n \in B_{r,t}^d) = 0$. As $\Pi_{W_n}$ is supported on $C([0,1]^d)$, its log-concavity (cf. Lemma 3.4 in [2]) implies (via a Fernique-like theorem [13, Section 2] and the exponential Markov inequality) the following supremum concentration inequality: for some constants $c_1, c_2 > 0$,

$$\Pr (\|W_n\|_\infty > R) \leq c_1 e^{-c_2 R}, \quad \text{all } R > 0. \quad (28)$$

The finer information geometry properties of $\Pi_{W_n}$ are characterised by two associated function spaces,

$$Q_n := \left\{ w = \sum_{l=1}^{\infty} \sum_{r=1}^{d^l} w_{lr} \psi_{lr} : \|w\|_{Q_n}^2 := \sum_{l=1}^{\infty} \sum_{r=1}^{d^l} \sigma_{n,lr}^{-2} |w_{lr}|^2 < \infty \right\},$$

and

$$Z_n := \left\{ w = \sum_{l=1}^{\infty} \sum_{r=1}^{d^l} w_{lr} \psi_{lr} : \|w\|_{Z_n} := \sum_{l=1}^{\infty} \sum_{r=1}^{d^l} \sigma_{n,lr}^{-1} |w_{lr}| < \infty \right\}.$$

Note that $Z_n \subset Q_n$ with continuous embedding. The weighted $\ell^2$-space $Q_n$ contains the admissible shifts $w$ for which the law of the random function $W_n + w$ is absolutely continuous with respect to $\Pi_{W_n}$ (cf. [3, Proposition 2.7]). On the other hand, the weighted $\ell^1$-norm $\|w\|_{Z_n}$ quantifies the loss in prior probability of non-centred balls compared to centred ones: by Proposition 2.11 in [3], for all $w \in Z_n$, all $\xi > 0$,

$$\Pr (\|W_n - w\|_\infty \leq \xi) \geq e^{-\|w\|_{Z_n}} \Pr (\|W_n\|_\infty \leq \xi). \quad (29)$$

In the proofs we often refer to $Z_n$ as the ‘decentering’ space. Via the two-level concentration inequality in Proposition 2.15 in [3], the bulk of the prior probability mass is seen to be contained in an enlargement of the sum of sufficiently large balls in $Q_n$ and $Z_n$: for some constant $c_3 > 0$, for all Borel measurable $A \subset C([0,1]^d)$ and all $R > 0$,

$$\Pr \left( W_n = W_n^{(1)} + W_n^{(2)} + W_n^{(3)} : W_n^{(1)} \in A, \|W_n^{(2)}\|_{Q_n} \leq \sqrt{R}, \|W_n^{(3)}\|_{Z_n} \leq R \right) \geq 1 - \frac{1}{\Pr(W_n \in A)} e^{-R/c_3}. \quad (30)$$
A.2 Auxiliary results

In this section we collect two auxiliary results used in the proofs of the main theorems.

**Lemma 10.** Let \( \phi : \mathbb{R} \to (0, \infty) \) be a strictly increasing and continuous function with uniformly Lipschitz logarithm with Lipschitz constant \( \mathcal{L} > 0 \). For \( w, w' \in C([0, 1]^d) \), let \( \phi_w, \phi_{w'} \) be the associated probability density functions defined according to (3). Then,

\[
\| \phi_w - \phi_{w'} \|_1 \leq \frac{2\mathcal{L}e^{\|w-w'\|_\infty}}{\phi(\|w'\|_\infty)} \| w - w' \|_1.
\]

**Proof.** Some algebra yields

\[
\| \phi_w - \phi_{w'} \|_1 = \left\| \phi \circ w \|_1 - \| \phi \circ w' \|_1 \right\|
\]

\[
\leq \frac{2}{\| \phi \circ w' \|_1} \| \phi \circ w - \phi \circ w' \|_1 \leq \frac{2}{\phi(\|w'\|_\infty)} \| \phi \circ w - \phi \circ w' \|_1.
\]

The latter norm equals

\[
\int_{[0,1]^d} |\phi(w(x)) - \phi(w'(x))| dx
\]

\[
= \int_{[0,1]^d} \left| e^{\log \phi(w(x))} - 1 \right| dx
\]

\[
\leq \int_{[0,1]^d} |\log \phi(w(x)) - \log \phi(w'(x))| e^{\log \phi(w(x)) - \log \phi(w'(x))} dx
\]

having used that for all \( z \in \mathbb{R} \), \(|e^z - 1| \leq |z| e^{|z|}\). Recalling that \( \log \phi \) is uniformly Lipschitz with Lipschitz constant \( \mathcal{L} > 0 \), the claim follows upper bounding the integral in the last line by

\[
\mathcal{L} \int_{[0,1]^d} |w(x) - w'(x)| e^{\mathcal{L}|w(x) - w'(x)|} dx \leq \mathcal{L} e^{\mathcal{L}|w-w'|_\infty} \| w - w' \|_1.
\]

\(\square\)

**Lemma 11.** For fixed \( B > 0 \), let \( \phi : \mathbb{R} \to (B, \infty) \) be a strictly increasing and uniformly Lipschitz function with Lipschitz constant \( \mathcal{L} > 0 \). For \( w, w' \in C([0, 1]^d) \), let \( \phi_w, \phi_{w'} \) be the associated probability density functions defined according to (3). Then,

1. 

\[
\max \left\{ -E_{\phi_w} \left( \log \frac{\phi_w}{\phi_{w'}}(X) \right), E_{\phi_{w'}} \left( \log \frac{\phi_w}{\phi_{w'}}(X) \right) \right\} \leq \frac{\mathcal{L}^2}{B^2} \| \phi_{w'} \|_\infty \| w - w' \|_2^2.
\]

2. 

\[
\| \phi_w - \phi_{w'} \|_1 \leq \frac{2\mathcal{L}}{B} \| w - w' \|_1.
\]

**Proof.** For point 1., by Lemma B.2 in [19],

\[
\max \left\{ -E_{\phi_w} \left( \log \frac{\phi_w}{\phi_{w'}}(X) \right), E_{\phi_{w'}} \left( \log \frac{\phi_w}{\phi_{w'}}(X) \right) \right\} \leq \| \phi_{w'} \|_\infty \| \phi_w \|_\infty d_2(H, \phi_w, \phi_{w'}),
\]
where $d_H$ is the Hellinger distance (cf. (15)). Using that

$$d_H(\phi w', \phi w) = \left\| \sqrt{\phi w'} - \sqrt{\phi w} \right\|_2 \leq \frac{2}{\left\| \sqrt{\phi w'} \right\|_2} \left\| \sqrt{\phi w'} - \sqrt{\phi w} \right\|_2,$$

and that

$$\left\| \sqrt{\phi w'} - \sqrt{\phi w} \right\|_2 = \frac{1}{\sqrt{\phi w' + \sqrt{\phi w}}} \left\| \phi w - \phi w' \right\|_2 \leq \frac{1}{\sqrt{\phi(\|w\|_\infty)} + \sqrt{\phi(\|w\|_\infty)}} \left\| \phi w - \phi w' \right\|_2,$$

the claim follows since $\phi(z) > B$ for all $z \in \mathbb{R}$ and $\left\| \phi w - \phi w' \right\|_2 \leq L \| w - w' \|_2$. For point 2., arguing as in the proof of Lemma 10,

$$\left\| \phi w - \phi w' \right\|_1 \leq \frac{2}{\phi(\|w\|_\infty)} \left\| \phi w - \phi w' \right\|_1,$$

whence the claim follows since $\phi$ is bounded below by $B$ and uniformly Lipschitz. \qed

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