VIOLATION OF SUM RULES FOR TWIST-3 PARTON DISTRIBUTIONS IN QCD

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Sum rules for twist-3 distributions are reexamined. Integral relations between twist-3 and twist-2 parton distributions suggest the possibility for a $\delta$-function at $x = 0$. We confirm and clarify this result by constructing $h_L$ and $h_3^L$ (quark-gluon interaction dependent part of $h_L$) explicitly from their moments for a one-loop dressed massive quark. The physics of these results is illustrated by calculating $h_L(x, Q^2)$ using light-front time-ordered pQCD to $O(\alpha_S)$ on a quark target.

1. Introduction

Ongoing experiments with polarized beams and/or targets conducted at RHIC, HERMES and COMPASS etc are providing us with important information on the spin distribution carried by quarks and gluons in the nucleon. They are also enabling us to extract information on the higher twist distributions which represent the effect of quark-gluon correlations. In particular, the twist-3 distributions $g_T(x, Q^2)$ and $h_L(x, Q^2)$ are unique in that they appear as a leading contribution in some spin asymmetries: For example, $g_T$ can be measured in the transversely polarized lepton-nucleon deep inelastic scattering and $h_L$ appears in the longitudinal-transverse spin asymmetry in the polarized nucleon-nucleon Drell-Yan process. The purpose of this paper is to reexamine the validity of the sum rules for these twist-3 distributions.

A complete list of twist-3 quark distributions is given by the light-cone correlation functions in a hadron with momentum $P$, spin $S$ and mass $M$:

$$
\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar{\psi}(0)\gamma_\mu\gamma_5\psi(\lambda n)|Q^2 |PS\rangle
= 2 \left[ g_1(x, Q^2)p_\mu(S \cdot n) + g_T(x, Q^2)S_\perp^\mu + M^2 g_3(x, Q^2)n^\mu(S \cdot n) \right],
$$

(1)
The light-like vectors \( p \) and \( n \) are introduced by the relation \( p^2 = n^2 = 0 \), \( n^+ = p^- = 0 \), \( P^\mu = p^\mu + M^2 n^\mu \) and \( S^\mu = (S \cdot n) p^\mu + (S \cdot p) n^\mu + S_1^\mu \). The variable \( x \in [-1,1] \) represents the parton’s light-cone momentum fraction. Antiquark distributions \( \bar{g}_{1,T,3}(x,Q^2) \) and \( \bar{h}_{1,L,3}(x,Q^2) \) are obtained by replacing \( \psi \) by its charge conjugation field \( C\bar{\psi}^T \) in (1)-(3) and are related to the quark distributions \( g_{1,T,3}(x,Q^2) = g_{1,T,3}(-x,Q^2) \) and \( h_{1,T,3}(x,Q^2) = -h_{1,T,3}(-x,Q^2) \). The sum rules in our interest are obtained by taking the first moment of the above relations. For example, from (1), one obtains

\[
(PS|\bar{\psi}(0)\gamma_\mu\gamma_5\psi(0)|Q^2|PS) = 2 \int_{-1}^{1} dx \left[ g_1(x,Q^2)p^\mu(S \cdot n) + g_T(x,Q^2)S_1^\mu + M^2 g_3(x,Q^2)n^\mu(S \cdot n) \right].
\]

From rotational invariance, it follows that the left hand side of (4) is proportional to the spin vector \( S^\mu \) and thus \( g_{1,T,3}(x,Q^2) \) must satisfy

\[
\int_{-1}^{1} dx g_1(x,Q^2) = \int_{-1}^{1} dx g_T(x,Q^2),
\]

(5)

\[
\int_{-1}^{1} dx g_1(x,Q^2) = 2 \int_{-1}^{1} dx g_3(x,Q^2).
\]

(6)

The same argument for (2) leads to the sum rule relations for \( h_{1,L,3}(x,Q^2) \):

\[
\int_{-1}^{1} dx h_1(x,Q^2) = \int_{-1}^{1} dx h_L(x,Q^2),
\]

(7)

\[
\int_{-1}^{1} dx h_1(x,Q^2) = 2 \int_{-1}^{1} dx h_3(x,Q^2).
\]

(8)

The sum rule (5) is known as Burkhard-Cottingham sum rule and (7) was first derived in Refs. \(^3\)\(^4\). Since the twist-4 distributions \( g_3, h_3 \) are unlikely to be measured experimentally, the sum rules involving those functions (6) and (8) are practically useless and will not be addressed in the subsequent discussions. As is clear from the above derivation, these sum rules are mere consequences of rotational invariance and there is no doubt in their validity in a mathematical sense. However, if one tries to confirm those sum rules by experiment, great care is required to perform the integral including \( x = 0 \). In DIS, \( x \) is identified as the Bjorken variable \( x_B = Q^2/2P \cdot q \) and \( x = 0 \) corresponds to \( P \cdot q \to \infty \).
which can never be achieved in a rigorous sense. Accordingly, if $h_L(x, Q^2)$
has a contribution proportional to $\delta(x)$ and $h_1(x, Q^2)$ does not, experimental
measurement would claim a violation of the sum rule.

In this paper we reexamine the sum rule involving the first moment of the
twist-3 distribution $h_L(x, Q^2)$. In particular we argue that $h_L(x, Q^2)$ has a
$\delta$-singularity at $x = 0$. Starting from the general QCD-based decomposition
of $h_L$, we show that it contains a function $h_L^n$ which has a $\delta(x)$-singularity.
In Sec.3, we construct $h_L$ for a massive quark from the moments of $h_3$
at the one-loop level and show that $h_3$ also has an $\delta(x)$-singularity, which together
with the singularity in $h_L^n$ gives rise to a $\delta(x)$ singularity in $h_L$ itself. In Sec.4,
we perform an explicit light-cone calculation of $h_L$ in the one-loop level to
confirm the result of the previous sections. Details can be found in Ref. 5.

2. $\delta(x)$-functions in $h_L(x, Q^2)$

The OPE analysis of the correlation function (2) allows us to decompose
$h_L(x, Q^2)$ into the contribution expressed in terms of twist-2 distributions
and the rest which we call $h_L^3(x, Q^2)$. Since the scale dependence of each
distribution is inessential in the following discussion, we shall omit it in this
section for simplicity. Introducing the notation for the moments on $[-1, 1]$,
$M_n[h_L] \equiv \int_{-1}^{1} dx x^n h_L(x)$, this decomposition is given in terms of the moment
relation 1:

\[ M_n[h_L] = \frac{2}{n + 2} M_n[h_1] + \frac{n}{n + 2} \frac{m_q}{M} M_{n-1}[g_1] + M_n[h_L^n], \quad (n \geq 1) \tag{1} \]

\[ M_0[h_L] = M_0[h_1], \tag{2} \]

with the conditions

\[ M_0[h_3] = M_1[h_3] = 0. \tag{3} \]

By inverting the moment relation, one finds

\[ h_L(x) = h_L^{WW}(x) + h_L^n(x) + h_L^3(x) \tag{4} \]

\[ = \begin{cases} 2x \int_x^1 dy \frac{h_1(y)}{y^2} + \frac{m_q}{M} \left[ \frac{g_1(x)}{x} - 2x \int_x^1 dy \frac{g_1(y)}{y^3} \right] + h_L^n(x) & (x > 0) \\ -2x \int_{-1}^x dy \frac{h_1(y)}{y^2} + \frac{m_q}{M} \left[ \frac{g_1(x)}{x} + 2x \int_{-1}^x dy \frac{g_1(y)}{y^3} \right] + h_L^3(x) & (x < 0) \end{cases} \tag{5} \]
where the first and second terms in Eq. (4) denote the corresponding terms in (5). In this notation the sum rule (2) and the condition (3) implies \(^\text{a}\)

\[
M_0[h^m_L] = 0. \tag{6}
\]

Naively integrating (5) over \(x\) for \(x > 0\), while dropping all surface terms \(^6\) one arrives at \(\int_0^1 dx h_L(x) = \int_0^1 dx h_1(x) + \int_0^1 dx h^2_L(x)\) and likewise for \(\int_{-1}^0 dx h_L(x)\). Together with (3), this yields \(\int_0^1 dx h_L = \int_0^1 dx h_1\). However, this procedure may be wrong due to the very singular behavior of the functions involved near \(x = 0\). Investigating this issue will be the main purpose of this paper.

We first address the potential singularity at \(x = 0\) in the integral expression for \(h^m_L(x)\) in (5). In order to regulate the region near \(x = 0\), we first multiply \(h^m_L(x)\) by \(x^\beta\), integrate from 0 to 1 and let \(\beta \to 0\). This yields

\[
\int_0^1 dx h^m_L(x) = \frac{m_q}{2M} \lim_{\beta \to 0} \beta \int_0^1 dx x^{\beta-1} g_1(y) = \frac{m_q}{2M} g_1(0+), \tag{7}
\]

while multiplying Eq. (5) by \(|x|^\beta\) and integration from \(-1\) to 0 yields

\[
\int_{-1}^0 dx h^m_L(x) = -\frac{m_q}{2M} \lim_{\beta \to 0} \beta \int_{-1}^0 dx |x|^{\beta-1} g_1(y) = -\frac{m_q}{2M} g_1(0-), \tag{8}
\]

where we have assumed that \(g_1(0\pm)\) is finite. Adding these results we have

\[
\int_{-1}^0 dx h^m_L(x) + \int_0^1 dx h^m_L(x) = \frac{m_q}{2M} (g_1(0+) - g_1(0-)). \tag{9}
\]

Eq. (6) and the fact that, in general, \(\lim_{x \to 0} g_1(x) - g_1(-x) \neq 0\), \(^b\) imply

\[
h^m_L(x) = h^m_L(x)_{\text{reg}} - \frac{m_q}{2M} (g_1(0+) - g_1(0-)) \delta(x), \tag{10}
\]

where \(h^m_L(x)_{\text{reg}}\) is defined by the integral in (5) at \(x > 0\) and \(x < 0\) and is regular at \(x = 0\). Eq. (10) indicates that \(h_L\) has a \(\delta(x)\) term unless \(h^2_L(x)\) has a \(\delta(x)\) term and it cancels the above singularity in \(h^m_L(x)\).

Eq. (10) demonstrates that the functions constituting \(h_L(x)\) are more singular near \(x = 0\) than previously assumed and great care needs to be taken when replacing integrals over nonzero values of \(x\) by integrals that involve the origin. In particular, if \(h_L(x)\) itself contains a \(\delta(x)\) term, then (2) implies

\[
\int_{0+}^1 (h_L(x) - h_1(x)) + \int_{-1}^0 (h_L(x) - h_1(x)) \neq 0, \tag{11}
\]

\(^a\)More precisely, the original OPE tells us \(M_0[h^1_L + h^m_L] = 0\). But as long as \(g_1(0\pm)\) is finite, which we will assume, this is equivalent to stronger relations (3) and (6).

\(^b\)For example, dressing a quark at \(O(\alpha_S)\) yields \(g_1(0+) \neq 0\) and \(g_1(0-) \equiv g_1(0+) = 0\).
and, since \( h_1(x) \) is singularity free at \( x = 0 \):

\[
\int_{-1}^{0-} dx h_1(x) + \int_{0+}^{1} dx h_1(x) = \int_{-1}^{1} dx h_1(x).
\]

(12)

Accordingly an attempt to verify the “\( h_L \)-sum rule” \(^3\) would obviously fail.

However, in order to see whether the \( \delta(x) \) identified in (10) eventually survives in \( h_L(x) \), we have to investigate the behavior of \( h_1^L(x) \) at \( x = 0 \). To this end we will explicitly construct \( h_L(x) \) for a massive quark to \( O(\alpha_S) \).

3. \( h_L(x, Q^2) \) from the moment relations

In this section we will construct \( h_L(x, Q^2) \) for a massive quark (mass \( m_q \)) to \( O(\alpha_S) \) from the one-loop calculation of \( M_n[h_3^L] \).

\[
h_L(x, Q^2) = h_L^{(0)}(x) + \frac{\alpha_S}{2\pi} C_F \ln \frac{Q^2}{m_q^2} h_L^{(1)}(x),
\]

(13)

where the scale \( Q^2 \) is introduced as an ultraviolet cutoff and the \( C_F = 4/3 \) is the color factor. \( h_L^{WW,3m(0,1)} \) are defined similarly. \( g_1^{(0)}(x) = h_1^{(0)}(x) = \delta(1-x) \) gives \( h_L^{(0)}(x) = \delta(1-x) \), as it should. One loop calculations for \( g_1(x) \) and \( h_1(x) \) for a quark yield the well known splitting functions\(^7,8\):

\[
g_1^{(1)}(x) = \frac{1 + x^2}{[1-x]_+} + \frac{3}{2} \delta(1-x),
\]

(14)

\[
h_1^{(1)}(x) = \frac{2x}{[1-x]_+} + \frac{3}{2} \delta(1-x).
\]

(15)

Inserting these equations into the defining equation in (5), one obtains

\[
h_L^{WW(1)}(x) = 3x + 4x \ln \frac{1-x}{x},
\]

(16)

\[
h_L^{m(1)}(x) = \frac{2}{(1-x)_+} - 4x \ln \frac{1-x}{x} - 3 + 3x - \frac{1}{2} \delta(x).
\]

(17)

In the first line of (17), the term \( (3x - \frac{3}{2} \delta(1-x)) \) comes from the self-energy correction, i.e. from expanding \( M = m_q \left[ 1 + \frac{\alpha_S}{2\pi} C_F \frac{3}{2} \ln \frac{Q^2}{m_q^2} \right] \) in Eq. (4), and

\[
-\frac{1}{2} \delta(x) = -\frac{1}{2} g_1(0+) \delta(x) \text{ in } h_L^{m(1)}(x) \text{ accounts for the second term on the r.h.s. of (10).}
\]

We also note that \( h_L^{WW(1)} \) does not have any singularity at \( x = 0 \) and satisfies \( \int_0^1 dx h_L^{WW(1)}(x) = \int_0^1 dx h_1^{(1)}(x) \) as it should.

\( h_L^{(1)}(x) \) can be constructed if we know the purely twist-3 part \( h_3^{(1)}(x) \) at the one-loop level. One-loop renormalization of \( h_L \) was completed in \(^9\) and the mixing matrix for the local operators contributing to the moments of \( h_L(x, Q^2) \)
was presented. We obtain for the moment of the quark distribution
\[ \int_{-1}^{1} dx x^n h^{(1)}_L(x) = \frac{3}{n+1} - \frac{6}{n+2} + \frac{1}{2} \]
for \( n \geq 2 \). From this result, together with (3), we can construct \( h^{(1)}_L(x) \) as
\[ h^{(1)}_L(x) = 3 - 6x + \frac{1}{2} \delta(1-x) - \frac{1}{2} \delta(x). \]
We emphasize that the \(-\frac{1}{2} \delta(x)\) in (19) is necessary to reproduce the \( n=0 \) moment of \( h^{(1)}_L(x) \). From (16), (17) and (19), one obtains
\[ h_L(x,Q^2) = \delta(1-x) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{m_q^2} C_F \left[ \frac{2}{[1-x]^+} + \frac{1}{2} \delta(1-x) - \delta(x) \right]. \]
We remark that the above calculation indicates that the \( \delta(x) \) terms appear not only in \( h^n_L \) but also in \( h^3_L \). Furthermore they do not cancel but add up to give rise to \( -\delta(x) \) in \( h_L(x,Q^2) \) itself. In the next section we will confirm Eq. (20) through a direct calculation of \( h_L(x,Q^2) \) for a quark.

4. Light-cone calculation of \( h_L(x,Q^2) \)

In order to illustrate the physical origin of the \( \delta(x) \) terms in \( h_L(x) \) and to develop a more convenient procedure for calculating such terms, we now evaluate \( h_L(x) \) using time-ordered light-front (LF) perturbation theory. The method has been outlined in Ref. 10 and we will restrict ourselves here to the essential steps only. There are two equivalent ways to perform time-ordered LF perturbation theory: one can either work with the LF Hamiltonian for QCD and perform old-fashioned perturbation theory10, or one can start from Feynman perturbation theory and integrate over the LF-energy \( k^- \) first. In the following, we will use the latter approach for the one-loop calculation of \( h_L(x) \).

In LF gauge, \( A^+ = 0 \), parton distributions can be expressed in terms of LF momentum densities (\( k^+ \)-densities). Therefore, one finds for a parton distribution, characterized by the Dirac matrix \( \Gamma \) at \( O(\alpha_S) \) and for \( 0 < x < 1 \)
\[ f_{\Gamma}(x) \bar{u}(p) \Gamma u(p) = -ig^2 \bar{u}(p) \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{1}{k-m_q+i\epsilon} \frac{1}{k-m_q+i\epsilon} \gamma^\nu u(p) \times \delta \left( x - \frac{k^+}{p^+} \right) D_{\mu\nu}(p-k), \]
where \( D_{\mu\nu}(q) = \frac{1}{q^2+i\epsilon} g_{\mu\nu} - \frac{g_{\mu\nu}}{q^2+i\epsilon} - \frac{q_{\mu} q_{\nu}}{q^2} \) is the gauge field propagator in LF gauge, and \( n^\mu \) is a light-like vector such that \( n A = A^+ \sim (A^0 + A^3) / \sqrt{2} \) for
any four vector $A^\mu$. The $k^-$ integrals in expressions like Eq. (21) are performed using Cauchy’s theorem, yielding for $0 < k^+ < p^+$

$$\int \frac{dk^-}{2\pi} \frac{1}{(k^2 - m_q^2 + i\epsilon)^2} \frac{1}{(p-k)^2 + i\epsilon}$$

$$= \frac{1}{(2k^+)^2} \frac{1}{2(p^+-k^+)^2} \left( p^- - \frac{m_q^2 + k^2}{2k^+} - \frac{(p^- - k^-)^2}{2(p^+ - k^+)^2} \right)^2 \left( \frac{k_\perp \to \infty}{2p^+ - k^+} \right)^2$$

where we used $k^+ = xp^+$. In order to integrate all terms in Eq. (21) over $k^-$, Cauchy’s theorem is used to replace any factors of $k^-$ in the numerator of Eq. (21) containing $k^-$ by their on-shell value at the pole of the gluon propagator

$$k^- \to \tilde{k}^- \equiv p^- - \frac{(p_\perp - k^-)^2}{2(p^+ - k^+)}$$

In the following we will focus on the UV divergent contributions to the parton distribution only. This helps to keep the necessary algebra at a reasonable level. We find for $0 < x < 1$ to $O(\alpha_S)$

$$h_L(x, Q^2) = \frac{\alpha_S}{2\pi} \frac{C_F}{m_q^2} \ln \frac{Q^2}{m_q^2} \frac{2}{1 - x}$$

where the usual $+$-prescription for $\frac{1}{|1-x|_+}$ applies at $x = 1$, i.e. $\frac{1}{|1-x|_+} = \frac{1}{1-x}$ for $x < 1$ and $\int_0^1 dx \frac{1}{|1-x|_+} = 0$. Furthermore, $h_L(x) = 0$ for $x < 0$, since anti-quarks do not occur in the $O(\alpha_S)$ dressing of a quark. In addition to Eq. (24), there is also an explicit $\delta(x-1)$ contribution at $x = 1$. These are familiar from twist-2 distributions, where they reflect the fact that the probability to find the quark as a bare quark is less than one due to the dressing with gluons. For higher-twist distributions, the wave function renormalization contributes $\frac{\alpha_S}{2\pi} \frac{C_F}{m_q^2} \ln \frac{Q^2}{m_q^2} \frac{2}{1 - x}$. The same wave function renormalization also contributes at twist-3. However, for all higher twist distributions there is an additional source for $\delta(x-1)$ terms which has, in parton language, more the appearance of a vertex correction, but which arises in fact from the gauge-piece of self-energies connected to the vertex by an ‘instantaneous fermion propagator’ $\gamma_{2\mu}$. For $g_T(x, Q^2)$ these have been calculated in Ref. 10 where they give an additional contribution $-\frac{\alpha_S}{2\pi} \frac{C_F}{m_q^2} \ln \frac{Q^2}{m_q^2} \frac{2}{1 - x}$, i.e. the total contribution at $x = 1$ for $g_T(x, Q^2)$ was found to be $\frac{\alpha_S}{2\pi} \frac{C_F}{m_q^2} \ln \frac{Q^2}{m_q^2} \frac{1}{2} \delta(x-1)$. We found the same $\delta(x-1)$ terms also for $h_L(x, Q^2)$. Combining the $\delta(x-1)$ piece with Eq.

In LF gauge, different components of the fermion field acquire different wave function renormalization. However, since all twist-3 parton distributions involve one LC-good and one LC-bad component, one finds the same wave function renormalization for all three twist-3 distributions.
(24) we thus find for $0 < x < 1$

$$h_L(x, Q^2) = \delta(x - 1) + \frac{\alpha_s}{2\pi} C_F \ln \frac{Q^2}{m_q^2} \left[ \frac{2}{1 - x} + \frac{1}{2} \delta(x - 1) \right].$$

Comparing this result with the well known result for $h_1$

$$h_1(x, Q^2) = \delta(x - 1) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{m_q^2} \left[ \frac{2x}{1 - x} + \frac{3}{2} \delta(x - 1) \right];$$

one realizes that

$$\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{1} dx \left[ h_L(x, Q^2) - h_1(x, Q^2) \right] = \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{m_q^2} C_F \neq 0,$$

i.e. if one excludes the possibly problematic region $x = 0$, then the $h_L$-sum rule is violated already for a quark dressed with gluons at order $O(\alpha_s)$.

In the above calculation, we carefully avoided the point $x = 0$. For most values of $k^+$, the denominator in (21) contains three powers of $k^-$ when $k^- \rightarrow \infty$. However, when $k^+ = 0$, $k^2 - m_q^2$ becomes independent of $k^-$ and the denominator in (21) contains only one power of $k^-$. Therefore, for those terms in the numerator which are linear in $k^-$, the $k^-$-integral diverges linearly. Although this happens only for a point of measure zero (namely at $k^+ = 0$), a linear divergence is indicative of a singularity of $h_L(x, Q^2)$ at that point. To investigate the $k^+ \approx 0$ singularity in these terms further, we consider

$$f(k^+, k_\perp) \equiv \int dk^- \frac{k^-}{(k^2 - m_q^2 + i\varepsilon)^2 (p - k)^2 + i\varepsilon} = \int dk^- \frac{k^-}{(k^2 - m_q^2 + i\varepsilon)^2 [(p - k)^2 + i\varepsilon]} = f_{can}(k^+, k_\perp) + f_{sin}(k^+, k_\perp),$$

where the ‘canonical’ piece $f_{can}$ is obtained by substituting for $k^-$ its on energy-shell value $k^- = p^- - \frac{(p - k)^2}{2(p^+ - k^+)}$ [the value at the pole at $(p - k)^2 = 0$, Eq. (23)]. For $k^+ = xp^+ \neq 0$, it is only this canonical piece which contributes. To see this, we note that $k^- - k^- = -\frac{(p - k)^2}{2(p^+ - k^+)}$, and therefore

$$f_{sin}(k^+, k_\perp) = \int dk^- \frac{k^-}{(k^2 - m_q^2 + i\varepsilon)^2 (p - k)^2 + i\varepsilon} = \frac{1}{2(p^+ - k^+)} \int dk^- \frac{1}{(k^2 - m_q^2 + i\varepsilon)^2}.$$

Obviously $12 \int dk^- \frac{1}{(2k^+ k^- - m_q^2 + i\varepsilon)^2} = 0$ for $k^+ \neq 0$ because then one can always avoid enclosing the pole at $k^- = \frac{m_q^2 + k^2 - i\varepsilon}{2k^+}$ by closing the contour in

\(^d\)Note that the divergence at $k^+ = p^+$ is only logarithmic.
the appropriate half-plane of the complex $k^- - \text{plane}$. However, on the other hand $\int \frac{d^2k_L}{(k_\perp^2 - k_\perp^2 + m_q^2 + i\epsilon)^2} = \frac{i\pi}{k_\perp^2 + m_q^2}$ and therefore

$$f_{\sin}(k^+, k_\perp) = \frac{1}{2p^+} \frac{i\pi \delta(k^+)}{k_\perp^2 + m_q^2}.$$  

(30)

Upon collecting all terms $\propto k^-$ in the numerator of (21), and applying (30) to those terms we find for the terms in $h_L(x, Q^2)$ that are singular at $x = 0$

$$h_{L,\sin}(x, Q^2) = -\frac{\alpha_S}{2\pi} \ln \frac{Q^2}{m_q^2} C_F \delta(x).$$  

(31)

Together with Eq. (25), this gives our final result for $h_L$, up to $O(\alpha_S)$, valid also for $x = 0$

$$h_L(x, Q^2) = \delta(x - 1) + \frac{\alpha_S}{2\pi} C_F \ln \frac{Q^2}{m_q^2} \left[-\delta(x) + \frac{2}{[1-x]^+} + \frac{1}{2} \delta(x - 1)\right].$$  

(32)

As expected, $h_L$ from Eq. (32) does now satisfy the $h_L$-sum rule, provided of course the origin is included in the integration.

This result is important for several reasons. First of all it confirms our result for $h_L(x, Q^2)$ as determined from the moment relations. Secondly, it provides us with a method for calculating these $\delta(x)$ terms and thus enabling us to address the issue of validity of the naive sum rules more systematically. And finally, it shows that there is a close relationship between these $\delta(x)$ terms and the infamous zero-modes in LF field theory

Ref. 11, where canonical Hamiltonian light-cone perturbation theory is used to calculate $h_L(x)$ and, for $x \neq 0$ the result obtained in Ref. 11 agrees with ours which provides an independent check of the formalism and the algebra. However, the canonical light-cone perturbation theory used in Ref. 11 is not adequate for studying the point $x = 0$. From the smooth behaviour of $h_L(x)$ near $x = 0$ the authors of Ref. 11 conclude that the sum rule for the parton distribution $h_L(x)$ is violated to $O(\alpha_S)$. Our explicit calculation for $h_L(x)$ not only proves that the sum rule for $h_L(x)$ is not violated to this order if the point $x = 0$ is properly included, but also shows that it is incorrect to draw conclusions from smooth behaviour near $x = 0$ about the behaviour at $x = 0$.

5. Summary

We have investigated the twist-3 distribution $h_L(x)$, and found that the sum-rule for its lowest moment is violated if the point $x = 0$ is not properly included.

For a massive quark, to $O(\alpha_S)$ we found

$$h_L(x, Q^2) = \delta(x - 1) + \frac{\alpha_S}{2\pi} C_F \ln \frac{Q^2}{m_q^2} \left[-\delta(x) + \frac{2}{[1-x]^+} + \frac{1}{2} \delta(x - 1)\right].$$  

(1)
At \( O(\alpha_S) \), \( h_L(x, Q^2) \) does not satisfy its sum rule if one excludes the origin from the region of integration (which normally happens in experimental attempts to verify a sum rule). Of course, QCD is a strongly interacting theory and parton distribution functions in QCD are nonperturbative observables. Nevertheless, if one can show that a sum rule fails already in perturbation theory, then this is usually a very strong indication that the sum rule also fails nonperturbatively (while the converse is often not the case!).

From the QCD equations of motion, we were able to show nonperturbatively that the difference between \( h_L(x, Q^2) \) and \( h_3^L(x, Q^2) \) contains a \( \delta(x) \) term

\[
[h_L(x, Q^2) - h_3^L(x, Q^2)]_{\text{singular}} = \frac{m_q}{2M} (g_1(0^+, Q^2) - g_1(0^-, Q^2)) \delta(x). \tag{2}
\]

Since \( g_1(0^+, Q^2) - g_1(0^-, Q^2) \equiv \lim_{x \to 0} g_1(x, Q^2) - \bar{g}_1(x, Q^2) \) seems to be nonzero (it may even diverge), one can thus conclude that either \( h_L(x, Q^2) \) or \( h_3^L(x, Q^2) \) or both do contain such a singular term.

We checked the validity of this relation to \( O(\alpha_S) \) and found that, to this order, both \( h_3^L \) and \( h_L \) contain a term \( \propto \delta(x) \). We also verified that even though the sum rule for \( h_L(x, Q^2) \) is violated if \( x = 0 \) is not included, it is still satisfied to \( O(\alpha_S) \) if the contribution from \( x = 0 \) (the \( \delta(x) \) term) is included.

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\*In the next-to-leading order QCD for a quark, \( \lim_{x \to 0} g_1(x) - \bar{g}_1(x) \) is logarithmically divergent \(^{14}\).
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