COINDUCTIVE PROPERTIES OF LIPSCHITZ FUNCTIONS ON STREAMS

JIHO KIM

Abstract. A simple hierarchical structure is imposed on the set of Lipschitz functions on streams (i.e. sequences over a fixed alphabet set) under the standard metric. We prove that sets of non-expanding and contractive functions are closed under a certain coiterative construction. The closure property is used to construct new final stream coalgebras over finite alphabets. For an example, we show that the 2-adic extension of the Collatz function and certain variants yield final bitstream coalgebras.

1. Introduction

In the realm of theoretical computer science, the categorical notion of coalgebras gives a mathematical foundation for computational dynamics. In the appropriate categories, the finality of coalgebras can be construed as denotational semantics of various models of computation such as automata [15], programming languages, recursive programming schemes [14], and other calculi. It also has connection to a diverse collection of other mathematical pursuits—the theory of non-wellfounded sets [1], modal logic [8, 13, 16, 17], fractals and self-similarity [12]—to name a few. Such connections raise interesting questions about the extent to which the theory of coalgebras may be useful in more “classical” mathematics.

According to a long tradition of children learning the 1, 2, 3’s, we learn at the very beginning—at least implicitly—that the sequence of natural numbers and many operations on them are defined via the principle of induction and iteration. Categorically speaking, these principles are shadows of the universal property of a certain initial algebra. Then we use the natural numbers to build other sequences (i.e. streams) of all sorts. Although the notion of streams is a basic one, it is ubiquitous in both mathematics and computer science and therefore worthy of extensive study. Analogously, the dual notions of coinduction and coiteration expressed as the universal property of certain final coalgebras lead to novel ways of expressing and understanding definitions of streams and operations on them.

This present paper focuses particularly on stream coalgebras and morphisms among them. Given the standard metric on the set of streams, we derive some coinductive closure properties on the set of non-expanding maps and the set of distance-preserving maps. Section 2 contains the definitions and constructions necessary for the paper. Of particular interest is the stratification of the set of Lipschitz functions on streams (each level of which we call $k$-causal functions). The stratification is achieved in several seemingly dissimilar ways which are proven to be
equivalent in Theorem 2.5. We go on to show that the family of isometric embeddings on streams can also be characterized by a similar set of criteria. Next, we define the notion of “woven functions” and explore some properties which will be essential in the main result.

Section 3 introduces the category theoretic notion of coalgebras and gives some clarifying examples beyond streams. Section 4 presents the coinductive closure property on sets of stream functions which can be stated roughly:

**Theorem.** A stream function coiteratively defined by a coalgebra woven from non-expanding (resp. distance-preserving) maps is non-expanding (resp. distance-preserving). The converse holds in the case of distance-preserving functions.

This theorem then is applied to the 2-adic extension of the Collatz function and its variants which figure largely in the $3x+1$ Problem. It gives an essentially new perspective on the 3x+1 conjugacy map as a coalgebra isomorphism between final stream coalgebras. The utility of the general approach explored in this paper may be limited in terms of resolving the 3x+1 Problem in particular. Nevertheless, it identifies additional structure within a large class of Collatz-like dynamical systems, which may shed light on these problems as a whole.

2. Streams

Let $A$ be some alphabet set (possibly infinite) and let $A^\omega$ be the set of sequences whose components come from $A$. Formally, these sequences are functions from the natural numbers $\omega = \{0, 1, 2, \ldots\}$ (which act as the indices) to the alphabet $A$. In this paper, we will refer to these sequences as $A$-streams and consider Lipschitz continuous functions on them.

2.1. $k$-causal functions.

**Definition 2.1 (Metric on $A^\omega$).** Given any $\sigma, \tau \in A^\omega$, define the distance between $\sigma$ and $\tau$ to be

$$d(\sigma, \tau) = \begin{cases} 0 & \text{if } \sigma = \tau \\ 2^{-i} & \text{if } \sigma \neq \tau \end{cases}$$

where $i$ is the least index such that $\sigma(i) \neq \tau(i)$.

The metric $d$ also satisfies the ultrametric inequality,

$$d(\sigma, \tau) \leq \max\{d(\sigma, \rho), d(\rho, \tau)\},$$

for any $\rho, \sigma, \tau \in A^\omega$. If $A$ has the discrete topology, the topology induced by this metric is the product topology on $A^\omega$.

**Definition 2.2 ($k$-causal function).** Let $k \in \mathbb{Z}$. A function $f: A^\omega \to B^\omega$ is $k$-causal if

$$d(f(\sigma), f(\tau)) \leq 2^{-k}d(\sigma, \tau).$$

In other words, $f$ is $k$-causal if and only if it is Lipschitz continuous with constant $2^{-k}$. Furthermore, let $\Gamma_k = \{f: A^\omega \to B^\omega \mid f \text{ is } k\text{-causal}\}$. Also let $\Gamma_{\text{bi}} \subset \Gamma_0$ be the subset which consists of distance-preserving functions. We will call maps in $\Gamma_{\text{bi}}$ bicausal.

**Example 2.3.** Consider the following examples:

(i) For all $\ell \leq k$, $k$-causal functions are $\ell$-causal. That is to say, $\Gamma_k \subseteq \Gamma_\ell$. 
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Lemma 2.4. We have the following simple observations.

(i) For any \( \sigma, \tau \in A^\omega \) and \( \ell \geq 0 \),

\[
p_\ell(\sigma) = p_\ell(\tau) \iff \sigma \equiv_\ell \tau \iff d(\sigma, \tau) \leq 2^{-\ell}
\]

(ii) For \( \ell \geq 0 \), \( p_\ell = \pi_\ell \circ p_{\ell + 1} \).

(iii) For \( \ell \geq 0 \), \( \sigma = p_\ell(\sigma) : t(\ell)(\sigma) \). In particular, \( \sigma = h(\sigma) : t(\ell)(\sigma) \). (Recall the convention that \( w : \sigma = c_w(\sigma) \) for any \( w \in A^\ell \) and \( \sigma \in A^\omega \).

(iv) For any \( \sigma \in A^\omega \) and \( n \geq 0 \), we have \( \sigma(n) = h(t(n)(\sigma)) \).

Theorem 2.5 \((k\text{-causal functions})\). Let \( k \in \mathbb{Z} \) and \( f : A^\omega \to B^\omega \). The following are equivalent.

(i) \( f \) is \( k \)-causal.

(ii) For all \( i, j \geq 0 \) such that \( k = i - j \),

\[
\sigma \equiv_j \tau \implies f(\sigma) \equiv_i f(\tau)
\]
(iii) For \( m, n \geq 0 \) with \( \min\{m, n\} = 0 \) and \( k = m - n \), \( f \) is the (unique) inverse limit of a chain of maps \( \{f_\ell\}_{\ell \geq 0} \) as follows:

\[
\begin{array}{cccccccc}
A^0 & A^1 & A^2 & \cdots & A^{n+\ell} & A^{n+(\ell+1)} & \cdots \\
\downarrow f_0 & \downarrow f_1 & \downarrow f_2 & \cdots & \downarrow f_\ell & \downarrow f_{\ell+1} & \\
B^0 & B^1 & B^2 & \cdots & B^{m+\ell} & B^{m+(\ell+1)} & \cdots \\
\end{array}
\]

where \( f_\ell \circ p_{n+\ell} = p_{m+\ell} \circ f \).

**Proof.** For (i) \( \Rightarrow \) (ii), suppose \( f \) is \( k \)-causal, and \( \sigma \equiv j \) \( \tau \) with \( k + j \geq 0 \). Then, \( d(\sigma, \tau) \leq 2^{-j} \).

Therefore,

\[
d(f(\sigma), f(\tau)) \leq 2^{-k}d(\sigma, \tau) \leq 2^{-(k+j)}.
\]

Letting \( i = k + j \), we have \( f(\sigma) \equiv_i f(\tau) \).

To show (ii) \( \Rightarrow \) (iii), for each \( \ell \geq 0 \), let the function \( f_\ell: A^{n+\ell} \to B^{m+\ell} \) be given by

\[
f_\ell(w) = p_{m+\ell}(f(w;\sigma))
\]

for \( w \in A^{n+\ell} \).

First we note that \( f_\ell \) is well-defined, i.e. it does not depend on the choice of \( \sigma \). Let \( w \in A^{n+\ell} \), then \( w;\sigma \equiv_{n+\ell} w;\tau \) for any \( \sigma, \tau \in A^* \). Since \( k = (m+\ell) - (n+\ell) \), we get \( f(w;\sigma) \equiv_{m+\ell} f(w;\tau) \) by (3).

Then by the definition of \( \equiv_{m+\ell} \), we have \( p_{m+\ell}(f(w;\sigma)) = p_{m+\ell}(f(w;\tau)) \), as required.

Next, we show that \( f_\ell \circ \pi_{n+\ell} = \pi_{m+\ell} \circ f_{\ell+1} \). For \( v \in A^{n+\ell+1} \), let \( v = wa \) where \( w \in A^{n+\ell} \) and \( a \in A \). Then,

\[
\pi_{m+\ell}(f_{\ell+1}(v)) = \pi_{m+\ell}(f_{\ell+1}(wa)) = \pi_{m+\ell}(p_{m+\ell+1}(f(wa;\sigma))) = p_{m+\ell}(f(w;\sigma)) = f_\ell(w) = f_\ell(\pi_{n+\ell}(v)) \quad [v = wa]
\]

We also need to show that \( f_\ell \circ q_{n+\ell} = p_{m+\ell} \circ f \).

\[
f_\ell(p_{n+\ell}(\sigma)) = p_{m+\ell}(f(p_{n+\ell}(\sigma); f^{(n+\ell)}(\sigma))) = p_{m+\ell}(f(\sigma)) = \sigma = p_\ell(\sigma); t^{(\ell)}(\sigma) \quad [\sigma = p_\ell(\sigma); t^{(\ell)}(\sigma)]
\]

Lastly, we need to verify the universal property of projective limits, namely that if there is a map \( e: Y_A \to Y_B \) and an associated sequence of maps, \( q_i^A: Y_A \to A^i \) and \( q_i^B: Y_B \to B^i \) so that the equations

\[
q_i = \pi_i \circ q_{i+1} \quad f_i \circ q_{n+i} = q_{m+i} \circ e \quad (4)
\]

hold for any \( i \), then there exists a unique pair of maps \( r^A: Y_A \to A^\omega \) and \( r^B: Y_B \to B^\omega \) so that the equations

\[
p_j \circ r = q_j \quad f \circ r = r \circ e \quad (5)
\]

hold for all \( j \). To show existence, given such \( e \) and \( q_i \) that satisfy (4), let \( r^A \) and \( r^B \) be given by

\[
r^A(x)(i) = q_i^A(x)(i) \quad (6)
\]

where \( j > \max\{i, m, n\} \). The index \( j \) must be greater than both \( m \) and \( n \) so that \( q_j \) is meaningful. Also, \( j \) must be greater than \( i \) so that \( q_j(x) \) is a word that has a
Proposition 2.7. Let \( a \) be well-defined map \( D_k \) trivial. We define \( w \) to be some \( w \in \Gamma_k \). The related notion of supercausal functions, however, is not exactly the same as 1-causal. Unlike supercausal functions as given in [1], if \( f: A^\omega \to B^\omega \) is 1-causal, then \( f(\sigma) \equiv_1 f(\tau) \) for any \( \sigma, \tau \in A^\omega \). More generally, when \( k \geq 1 \), the image of \( k \)-causal functions must have a common \( k \)-prefix. Consequently, we have a well-defined map \( D_k: \Gamma_k \to B_k \) given by \( D_k(f) = p_k(f(\sigma)) \).

**Proposition 2.7.** Let \( f: A^\omega \to B^\omega \) be \( k \)-causal for \( k > 0 \). Then \( f = c_w \circ \hat{f} \) for some \( w \in B_k \) and \( 0 \)-causal function \( \hat{f}: A^\omega \to B^\omega \).

**Proof.** Let \( w = D_k(f) \) and \( \hat{f} = t^{(k)} \circ f \).

2.2. **Woven functions.** The following definition presents a way to construct non-trivial \( k \)-causal maps from a set of \((k+1)\)-causal maps by, in a manner of speaking, “weaving them together.”
At this last inequality, we require the fact that $k \in \mathbb{N}$. Let $\mathcal{F} = \{f_a\}_{a \in A}$ be an $A$-indexed set of maps from $A^\omega$ to $B^\omega$. With a slight abuse of notation, we can think of $\mathcal{F}$ as a map $\mathcal{F}: A \times A^\omega \to B^\omega$ via $\mathcal{F}(a, \sigma) = f_a(\sigma)$. We define $T_\mathcal{F}: A^\omega \to B^\omega$ by

$$T_\mathcal{F}(\sigma) = \mathcal{F}(h(\sigma), t(\sigma)) = f_{h(\sigma)}(t(\sigma)).$$

$T_\mathcal{F}$ is said to be woven from $\mathcal{F}$.

Intuitively, for any input stream $\sigma$, the function $T_\mathcal{F}$ gives the image of $t(\sigma)$ under a function which $h(\sigma)$ picks out from $\mathcal{F}$.

**Example 2.9 (Woven function).** Consider the following examples.

(i) If $f_a = \text{id}$ for all $a \in A$, the function woven from $\{f_a\}_{a \in A}$ is the tail function $t$.

(ii) If $f_a = c_a$ for all $a \in A$, the function woven from $\{f_a\}_{a \in A}$ is the identity.

(iii) If $A$ is finite, we can rewrite (7) in Definition 2.8 as a definition by cases. For instance, suppose $A = \{0, 1, 2\}$. For $\mathcal{F} = \{f_0, f_1, f_2\}$, we have

$$T_\mathcal{F}(\sigma) = \begin{cases} f_0(t(\sigma)) & \text{if } h(\sigma) = 0 \\ f_1(t(\sigma)) & \text{if } h(\sigma) = 1 \\ f_2(t(\sigma)) & \text{if } h(\sigma) = 2 \end{cases}$$

**Lemma 2.10.** A function $T: A^\omega \to B^\omega$ is woven from a family of $(k+1)$-causal functions if it is $k$-causal. The converse holds if $k \leq 0$.

**Proof.** Let $T$ be $k$-causal. For each $a \in A$, let $f_a: A^\omega \to B^\omega$ be a function given by $f_a(\sigma) = T(a; \sigma) = T(c_a(\sigma))$. Since $c_a$ is $1$-causal, each $f_a$ is $(k+1)$-causal. Let $S$ be a function woven from $\{f_a\}_{a \in A}$. Then,

$$S(\sigma) = f_{h(\sigma)}(t(\sigma)) = T(h(\sigma); t(\sigma)) = T(\sigma).$$

Therefore, $T = S$ is woven from a family of $(k+1)$-causal functions.

Conversely, let $\mathcal{F} = \{f_a\}_{a \in A}$ be an $A$-indexed set of $(k+1)$-causal functions for some $k \leq 0$. Let $\sigma, \tau \in A^\omega$. Then, $\sigma = a; \sigma'$ and $\tau = b; \tau'$ for some $a, b \in A$ and $\sigma', \tau' \in A^\omega$. On one hand, suppose $a \neq b$. Then, $d(\sigma, \tau) = 1$, and

$$d(T_\mathcal{F}(a; \tau'), T_\mathcal{F}(b; \tau')) \leq 1 = d(\sigma, \tau) \leq 2^{-k}d(\sigma, \tau).$$

At this last inequality, we require the fact that $k \leq 0$ so that $2^{-k} \geq 1$. On the other hand, suppose $a = b$. Then,

$$d(T_\mathcal{F}(\sigma), T_\mathcal{F}(\tau)) = d(T_\mathcal{F}(a; \sigma'), T_\mathcal{F}(b; \tau')) = d(T_\mathcal{F}(a; \sigma'), T_\mathcal{F}(a; \tau')) = d(f_a(\sigma'), f_a(\tau')) \leq 2^{-(k+1)}d(\sigma', \tau') \quad [f_a \text{ is } (k+1)-\text{causal}]$$

$$= 2^{-k}d(a; \sigma', b; \tau') \quad [a = b]$$

$$= 2^{-k}d(\sigma, \tau)$$

The calculation shows that $T_\mathcal{F}$ is $k$-causal. \qed
2.3. 0-causal and bicausal functions. The case where \( k = 0 \) is particularly interesting because the set of 0-causal functions \( \Gamma_0 \) is closed under composition. Furthermore, \( \Gamma_0 \) contains a subfamily of functions \( \Gamma_k \) of bicausal functions which is also closed under composition. From Theorem 2.5 we immediately derive the following characterization of 0-causal functions.

**Corollary 2.11** (0-causal functions, [6]). Let \( f : A^\omega \to B^\omega \) be a stream function. The following are equivalent.

(i) \( f \) is 0-causal (i.e. non-expanding).

(ii) For all \( n \geq 0 \),

\[
\sigma \equiv_n \tau \implies f(\sigma) \equiv_n f(\tau) \tag{8}
\]

(iii) \( f \) is the (unique) inverse limit of a chain of maps as follows:

\[
\begin{array}{cccccccc}
A^0 & \to & A^1 & \to & \cdots & \to & A^j & \to & \cdots \\
\pi_0 & \pi_1 & \pi_2 & \cdots & \pi_j & \cdots \\
A^0 & \downarrow & A^1 & \downarrow & \cdots & \downarrow & A^j & \downarrow & \cdots \\
A^0 & \to & A^1 & \to & \cdots & \to & A^j & \to & \cdots \\
\end{array}
\tag{9}
\]

where \( f_j \circ p_j = p_j \circ f \).

A similar result holds for bicausal functions with minimal changes. Moreover, if the domain and codomain are streams over the same finite alphabet, we can strengthen the result slightly.

**Corollary 2.12** (Bicausal functions with arbitrary alphabet). Let \( f : A^\omega \to B^\omega \) be a stream function. The following are equivalent.

(i) \( f \) is bicausal (i.e. distance-preserving).

(ii) For all \( n \geq 0 \),

\[
\sigma \equiv_n \tau \iff f(\sigma) \equiv_n f(\tau) \tag{10}
\]

(iii) \( f \) is the (unique) inverse limit of a chain of injective maps \( \{f_j\}_{j \geq 0} \) where \( f_j \circ p_j = p_j \circ f \), as arranged in (9).

**Proof.** In light of Corollary 2.11 we only need to extend the proof for the extra conclusions.

For (i)\(\Rightarrow\)(ii), assume \( f(\sigma) \equiv_n f(\tau) \). Then, \( d(f(\sigma), f(\tau)) \leq 2^{-n} \), but since \( f \) is distance-preserving \( d(\sigma, \tau) = d(f(\sigma), f(\tau)) \). Therefore \( \sigma \equiv_n \tau \).

For (ii)\(\Rightarrow\)(iii), fix some \( j \geq 0 \) and suppose \( f_j(w) = f_j(v) \) for some \( w, v \in A^j \). Then, \( p_j(f(w:\sigma')) = p_j(f(v:\sigma')) \), or equivalently, \( f(w:\sigma) \equiv_j f(v:\sigma') \). Finally, we have \( w:\sigma \equiv_j v:\sigma' \), i.e. \( w = j \), by (10). This shows that \( f_j \) is injective for any \( j \).

For (iii)\(\Rightarrow\)(i), we already know that \( f \) must be 0-causal, therefore

\[
d(f(\sigma), f(\tau)) \leq d(\sigma, \tau).
\]

Suppose that \( d(\sigma, \tau) = 2^{-j} \) for some \( j \in \mathbb{N} \) and \( \sigma, \tau \in A^\omega \). Because \( d(\sigma, \tau) = 2^{-j} \), we have \( p_{j+1}(\sigma) \neq p_{j+1}(\tau) \) by the definition of the metric \( d \). Since \( f_{j+1} \) is injective, \( f_{j+1}(p_{j+1}(\sigma)) \neq f_{j+1}(p_{j+1}(\tau)) \), and because of the universal property of \( f \) (in particular, \( f \circ p_{i+1} = p_{i+1} \circ f \)), we have \( p_{j+1}(f(\sigma)) \neq p_{j+1}(f(\tau)) \). That is to say, \( f(\sigma) \) and \( f(\tau) \) first differ at an index \( i < j + 1 \), so

\[
d(f(\sigma), f(\tau)) = 2^{-i} \geq 2^{-j} = d(\sigma, \tau).
\]

This shows that the distance must be preserved by \( f \), i.e. \( f \) is bicausal. \( \square \)
Corollary 2.13 (Bicausal functions with finite alphabet, [6]). Let \( f: A^\omega \to A^\omega \) be a stream function where \( A \) is finite. The following are equivalent:

(i) \( f \) is bicausal.
(ii) For all \( n \geq 0, \sigma \equiv_n \tau \) if and only if \( f(\sigma) \equiv_n f(\tau) \).
(iii) For any \( k \geq 0 \), \( f \) is the (unique) inverse limit of a chain of bijective (or equivalently, surjective) maps \( \{f_j\}_{j \geq 0} \) where \( f_j \circ p_j = p_j \circ f \), as arranged in [9].
(iv) \( f \) is a 0-causal bijection.

Proof. If \( A \) is finite, \( A^j \) is finite for any finite \( j \). Consequently, \( f_j: A^j \to A^j \) is injective if and only if it is bijective. In light of Corollary 2.12, this shows the equivalence of (i), (ii), and (iii).

For (iv)\(\Rightarrow\)(iii), assume \( f \) is bijective. It is surjective, in particular, for any word \( w \in A^j \) and \( \sigma \in A^\omega \), there is some \( \tau \in A^\omega \) so that \( f(\tau) = w:\sigma \). Then

\[
f_j(p_j(\tau)) = p_j(f(\tau)) = p_j(w:\sigma) = w
\]

That is to say, \( f_j \) is surjective. (In fact, this shows that surjective 0-causal functions are inverse limits of surjections in general.) Since \( A \) is finite, \( f_j \) is also bijective.

For (iii)\(\Rightarrow\)(iv), we do not require that \( A \) be finite. Notice that \( \{f_j^{-1}\}_{j \geq 0} \) and \( \{f_j^{-1} \circ f_j = \text{id}_{A^j}\}_{j \geq 0} \) both have inverse limits. Let \( g: A^\omega \to A^\omega \) be the inverse limit of the former. The latter inverse limit is \( \text{id}: A^\omega \to A^\omega \), but by uniqueness, \( \text{id} = g \circ f \). Similarly, \( \text{id} = f \circ g \), which shows \( f \) is bijective. \(\square\)

Example 2.14 (Functions on \( \mathbb{Z}_2 \)). In order to discuss concrete examples, we will often fix \( A = \mathbb{Z}_2 \), the two-element set \( \{0, 1\} \). The set of bitstreams \( 2^\omega \) underlies the ring \( \mathbb{Z}_2 \) of 2-adic integers. The following examples show how 2-adic arithmetic operations correspond to the notion of \( k \)-causal functions on the underlying streams.

(i) The mappings \( x \mapsto 2x \) and \( x \mapsto 2x + 1 \), respectively, correspond to \( c_0 \) and \( c_1 \) on \( 2^\omega \), and consequently, they are both 1-causal.
(ii) The function \( t: \mathbb{Z}_2 \to \mathbb{Z}_2 \) given by

\[
t(x) = \begin{cases} 
\frac{x}{2} & \text{if } x \equiv 0 \pmod{2} \\
\frac{x-1}{2} & \text{if } x \equiv 1 \pmod{2}
\end{cases}
\]  

(11)

corresponds exactly to the tail map on streams \( [2] \) and is \((-1)\)-causal. In the case of 2-adic integers (i.e. bitstreams), we call \( x \) even or odd, depending on whether \( h(x) = 0 \) or \( h(x) = 1 \). The two cases in (11) differentiate between the two possibilities.

(iii) For any \( k \in \mathbb{Z}_2 \), the mapping \( x \mapsto x + k \) is bicausal. For any \( k \in \mathbb{Z}_2 \), the mapping \( x \mapsto k \cdot x \) is causal, but only bicausal if \( k \) is a unit (via Corollary 2.13[3]). Since the composition of bicausals is bicausal, the mapping \( x \mapsto ax + b \) is bicausal for any \( b \in \mathbb{Z}_2 \) and \( a \in 2\mathbb{Z}_2 + 1 \).

3. Coalgebras and Coinduction

In a very general sense, coinduction is a notion which is categorically dual to induction. Here we will introduce the bare minimum of ideas in the theory of coalgebras, but more thorough introductions are available elsewhere [19]. Though we start with the most general (category theoretic) definition, we will quickly focus on a particular instance of stream coalgebras.
Definition 3.1. Given an endofunctor $F$ on a category $C$, an $F$-coalgebra is a $C$-morphism $X \xrightarrow{f} FX$. An $F$-coalgebra morphism from $X \xrightarrow{f} FX$ to $Y \xrightarrow{g} FY$ is a $C$-morphism $X \xrightarrow{m} Y$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & FX \\
\downarrow{m} & & \downarrow{Fm} \\
Y & \xrightarrow{g} & FY
\end{array}
$$

commutes (i.e. $g \circ m = Fm \circ f$). The class of all $F$-coalgebras and $F$-coalgebra morphisms forms a category $\text{Coalg}(F)$. Terminal objects in $\text{Coalg}(F)$, if they exist at all, are called final $F$-coalgebras. If $C$ is the category Set of sets—as assumed in the rest of this paper—we often refer to the coalgebra as the structure map and its domain as the carrier set.

Example 3.2 (Trivial). One of the most trivial examples is the coalgebra where the endofunctor is the identity functor. Coalgebras in this context are endofunctions where the domain and codomain coincide. A coalgebra morphism from $X \xrightarrow{f} X$ to $Y \xrightarrow{g} Y$ is a function $X \xrightarrow{m} Y$ so that $g \circ m = m \circ f$. Any singleton set with the identity function is a final coalgebra.

Example 3.3 (Mealy automaton). For a more substantial example, consider the endofunctor $M_{A,B}$ given by $M_{A,B}X = (B \times X)^A$ for a pair of sets $A$ and $B$. While this example will only be incidental to the main results of this paper, it is interesting to note that 0-causal functions appear naturally in other coalgebraic situations. Coalgebras in this context have the form $X \xrightarrow{\alpha} (B \times X)^A$ and are called Mealy automata with input and output taken from sets $A$ and $B$, respectively. The coalgebra structure map $\alpha$ corresponds to a (deterministic) transition function $A \times X \xrightarrow{\hat{\alpha}} B \times X$ on the state space $X$. Here $\hat{\alpha}$ gives an output symbol from $B$ and the next state in $X$ given an input symbol from $A$ and the current state in $X$.

Rutten [18] showed that the set $\Gamma_0$ of all 0-causal functions from $A^\omega$ to $B^\omega$ forms the carrier set of a final coalgebra $\Gamma_0 \xrightarrow{\gamma} (B \times \Gamma_0)^A$ for the Mealy automaton endofunctor $M_{A,B}$. The structure map $\gamma$ is given by

$$
\gamma(f)(a) = (D_1(f \circ c_a), t \circ f \circ c_a)
$$

for $f \in \Gamma_0$, $a \in A$, and $\sigma \in A^\omega$. The finality in $\text{Coalg}(M_{A,B})$ amounts to the idea that given any Mealy automaton, each state $x_0$ can be assigned a 0-causal function which encodes the output of the automaton for any stream of inputs that might be presented to the automaton starting at state $x_0$. This illustrates a theme in the theory of coalgebras that morphisms into final coalgebras often encapsulate infinite dynamics of any given coalgebra.

Example 3.4 (Power set functor). For a more pathological example, consider the functor $P$ given $PX = P(X)$ (the power set of $X$). $P$-coalgebras are exactly graphs, encoded as functions of the type $X \to PX$. Cantor’s theorem tells us that $PX$ has a strictly greater cardinality than $X$. However, Lambek’s Lemma asserts that final coalgebras must be isomorphisms (i.e. bijections in Set). This proves that $\text{Coalg}(P)$ has no final coalgebra.
For the rest of the paper, we will focus our attention exclusively on stream coalgebras (over Set). In the case where the endofunctor $S_B$ is given by $S_B X = B \times X$ for set $B$, we call these coalgebras $B$-stream coalgebras. Regardless of what $B$ might be, final stream coalgebras do exist, and the standard example is

$$B^\omega \xrightarrow{(h,t)} B \times B^\omega$$

where $h : B^\omega \to B$ and $t : B^\omega \to B^\omega$ are the head and tail maps defined earlier. Finality for $\text{Coalg}(S_B)$ can be stated in the following way. Given any $B$-stream coalgebra, $X \xrightarrow{(g,s)} B \times X$, there exists a unique map $\Phi = \Phi^{(g,s)} : X \to B^\omega$ so that the diagram

$$\begin{array}{ccc}
X & \xrightarrow{(g,s)} & B \times X \\
\downarrow{\Phi} & & \downarrow{\text{id}_B \times \Phi} \\
B^\omega & \xrightarrow{(h,t)} & B \times B^\omega
\end{array}$$

commutes, i.e. $g = h \circ \Phi$ and $t \circ \Phi = \Phi \circ s$. We say the unique map $\Phi$ above is coinductively induced (or equivalently, coiteratively defined) by $(g,s)$. We can actually define $\Phi$ from $(g,s)$ explicitly:

$$\Phi(x)(n) = h(t^n(x)) = g(s^n(x))$$

As a stream, $\Phi(x)$ records the image of the $s$-orbit of $x$ under $g$.

4. Results

For the main theorem, we consider the properties of a function $\varphi$ coinductively induced by a coalgebra $A^\omega \xrightarrow{(H,T)} B \times A^\omega$ whose carrier set is the set of streams and whose structure map incorporates a function $T$ woven from a set of 0-causal functions. The resulting map $\varphi$ is therefore a stream function, and we explore its relationship to the set of stream function from which it was derived.

**Theorem 4.1.** Let $A^\omega \xrightarrow{(H,T)} B \times A^\omega$ be a $B$-stream coalgebra so that

$$h(\sigma) = h(\tau) \implies H(\sigma) = H(\tau)$$

for all $\sigma, \tau \in A^\omega$. Let $\varphi : A^\omega \to B^\omega$ be the (unique) coalgebra morphism induced by $(H,T)$.

(i) The coalgebra morphism $\varphi$ is causal if $T$ is woven from a family of 0-causal functions.

(ii) If the converse of (i) also holds, then the coalgebra morphism $\varphi$ is bicausal if and only if $T$ is woven from a family of bicausal functions.

**Remark 4.2.** Because $\varphi$ is coinductively induced by $(H,T)$, we have two identities: $h \circ \varphi = H$ and $t \circ \varphi = \varphi \circ T$ (or more generally, $t^n \circ \varphi = \varphi \circ T^n$ for all $n \geq 0$). The proof uses the second characterization of 0-causal and bicausal functions from Corollaries 2.11 and 2.12, respectively. For each $n \geq 0$, 0-causal and bicausal functions must preserve $n$-prefix-equivalence. The proofs proceed by induction on $n \geq 0$. 
Lemma 2.10. Recall from the proof of the lemma that $T$ consider the converse, and suppose $T_i.e. Therefore $\sigma_a \equiv_{n+1} \tau$. First of all, since $T$ is woven from 0-causal functions, it is $(-1)$-causal, and therefore $T^{(n)}$ is $(-n)$-causal. By Theorem 2.5, $T^{(n)}(\sigma) \equiv_1 T^{(n)}(\tau)$, or equivalently, $h(T^{(n)}(\sigma)) = h(T^{(n)}(\tau))$. The premise [14] of the theorem therefore gives us:

$$H(T^{(n)}(\sigma)) = H(T^{(n)}(\tau)).$$

(16)

Secondly, $\sigma \equiv_{n+1} \tau$ implies that $\sigma \equiv_n \tau$. By the induction hypothesis, $\varphi(\sigma) \equiv_n \varphi(\tau)$. Therefore, to check that $\varphi(\sigma) \equiv_{n+1} \varphi(\tau)$, it is only necessary to verify that $\varphi(\sigma)$ and $\varphi(\tau)$ agree at index $n$:

\[
\varphi(\sigma)(n) = h(t^{(n)}(\varphi(\sigma))) = h(\varphi(T^{(n)}(\sigma))) = H(T^{(n)}(\sigma)) = H(T^{(n)}(\tau)) = h(T^{(n)}(\varphi(\tau))) = \varphi(\tau)(n)
\]

[Lemma 2.10]

Therefore $\varphi(\sigma) \equiv_{n+1} \varphi(\tau)$. This completes the proof of (i).

For the bicausal case, suppose $T$ is woven from bicausal functions. The basic step of the induction is the same as above. For the induction step, assume for some $n \geq 0$,

$$\sigma \equiv_n \tau \iff \varphi(\sigma) \equiv_n \varphi(\tau)$$

(17)

We have already showed above that this implies $\sigma \equiv_{n+1} \tau \implies \varphi(\sigma) \equiv_{n+1} \varphi(\tau)$, so consider the converse, and suppose $\varphi(\sigma) \equiv_{n+1} \varphi(\tau)$. In particular, $\varphi(\sigma) \equiv_1 \varphi(\tau)$, i.e. $h(\varphi(\sigma)) = h(\varphi(\tau))$. Since $h \circ \varphi = H$, we have $H(\sigma) = H(\tau)$. By the premise of the theorem—the converse of [14], to be precise—we can conclude that $h(\sigma) = h(\tau)$.

Let $a = h(\sigma) = h(\tau)$, and proceed:

\[
\begin{align*}
\varphi(\sigma) \equiv_{n+1} \varphi(\tau) & \implies t(\varphi(\sigma)) \equiv_n t(\varphi(\tau)) \quad \text{[}t \text{ is } (-1)\text{-causal]} \\
& \iff \varphi(T(\sigma)) \equiv_n \varphi(T(\tau)) \quad \text{[}t \circ \varphi = \varphi \circ T]\n\end{align*}
\]

(16)

\[
\begin{align*}
& \iff T(\sigma) \equiv_n T(\tau) \quad \text{[induction hypothesis [17]]} \\
& \iff f_h(\sigma)(t(\sigma)) \equiv_n f_h(\tau)(t(\tau)) \quad \text{[}T \text{ is woven from } \{f_\alpha\}\] \\
& \iff t(\sigma) \equiv_n t(\tau) \quad \text{[}f_\alpha \text{ is bicausal]} \\
& \iff a : t(\sigma) \equiv_{n+1} a : t(\tau) \quad \text{[}c_a \text{ is } 1\text{-causal]} \\
& \iff \sigma \equiv_{n+1} \tau \quad \text{[}a = h(\sigma) = h(\tau)\]
\]

For the other direction of (ii), suppose $\varphi$ is bicausal. Then, for $\sigma, \tau \in A^{\omega}$,

\[
\begin{align*}
d(T(\sigma), T(\tau)) &= d(\varphi(T(\sigma)), \varphi(T(\tau))) \quad \text{[}\varphi \text{ is bicausal]} \\
&= d(t(\varphi(\sigma)), t(\varphi(\tau))) \quad \text{[}t \circ \varphi = \varphi \circ T]\n\end{align*}
\]

(16)

\[
\leq 2d(\varphi(\sigma), \varphi(\tau)) \quad \text{[}t \text{ is } (-1)\text{-causal]} \\
= 2d(\sigma, \tau) \quad \text{[}\varphi \text{ is bicausal]} \\
\]

This shows that $T$ is $(-1)$-causal and therefore woven from 0-causal functions, by Lemma 2.10. Recall from the proof of the lemma that $T$ is woven from $\{f_\alpha\}_{\alpha \in A}$.
where each \( f_a \) is given by \( f_a(\sigma) = T(a;\sigma) \). We must show that \( f_a \) is bicausal for any \( a \in A \), but first recall that since \( \varphi \) is 0-causal, the map \( \varphi \circ c_a \) is 1-causal. In particular, the head of \( \varphi(c_a(\sigma)) \) does not depend on \( \sigma \). Therefore we have

\[
  \varphi(c_a(\sigma)) \equiv_j \varphi(c_a(\tau)) \iff \varphi(c_a(\sigma)) \equiv_{j+1} \varphi(c_a(\tau)) \tag{18}
\]

for all \( \sigma, \tau \in A^\omega \), \( j \in \mathbb{N} \). Then we can proceed:

\[
  f_a(\sigma) \equiv_n f_a(\tau) \iff T(a;\sigma) \equiv_n T(a;\tau) \quad \text{[def. of } f_a]\]
\[
  \iff \varphi(T(a;\sigma)) \equiv_n \varphi(T(a;\tau)) \quad \text{[\( \varphi \) bicausal]}\]
\[
  \iff t(\varphi(a;\sigma)) \equiv_n t(\varphi(a;\tau)) \quad \text{[\( t \circ \varphi = \varphi \circ T \)]}\]
\[
  \iff \varphi(a;\sigma) \equiv_{n+1} \varphi(a;\tau) \quad \text{[18]}\]
\[
  \iff a;\sigma \equiv_{n+1} a;\tau \quad \text{[\( \varphi \) bicausal]}\]
\[
  \iff \sigma \equiv_n \tau \quad \text{[\( t \) is \((-1)\)-causal]}
\]

This calculation shows that \( f_a \) is distance-preserving. \( \square \)

By taking \( A = B \) and \( H = h \), we can immediately get the following corollary.

**Corollary 4.3.** Let \( T : A^\omega \to A^\omega \) be a function. Let \( \varphi : A^\omega \to A^\omega \) be the (unique) coalgebra morphism induced by the coalgebra \( A^\omega \xrightarrow{\langle h, T \rangle} A \times A^\omega \).

(i) The coalgebra morphism \( \varphi \) is causal if \( T \) is woven from a family of 0-causal functions.

(ii) The coalgebra morphism \( \varphi \) is bicausal if and only if \( T \) is woven from a family of bicausal functions.

In essence, this corollary asserts that the sets \( \Gamma_0 \) of 0-causal functions and \( \Gamma_{bi} \) of bicausal functions on a set of streams are both closed under a particular coinductive construction. If \( T : A^\omega \to A^\omega \) is woven from a subset of \( \Gamma_0 \) (resp. \( \Gamma_{bi} \)), then the coalgebra morphism \( \varphi \) coinductively induced by \( \langle h, T \rangle \) lies in \( \Gamma_0 \) (resp. \( \Gamma_{bi} \)).

Theorems 4.1 and Corollary 4.3 generalize a result for \( A = B = 2 \) and “solenoidal bijections” in a paper by Bernstein and Lagarias [5] in several ways. First, it is useful to know that the ring structure of \( \mathbb{Z}_2 \) is not strictly necessary. Second, the cardinality of the underlying set is not required to be two, prime, or even finite. Lastly, if one direction of the implication is abandoned, instead of requiring bicausal functions, the premise of the theorem can be weakened to 0-causal functions.

**Connection to the 3x + 1 Problem.** In the 2-adic context, if we specify that \( f_0(x) = x \) and \( f_1(x) = 3x + 2 \), the resulting woven function is

\[
  T(x) = \begin{cases} 
  \frac{x}{2} & \text{if } h(x) = 0 \\
  \frac{3x-1}{2} + 2 & \text{if } h(x) = 1 
  \end{cases} \tag{19}
\]

Noting that \( \frac{3x-1}{2} + 2 = \frac{3x+1}{2} \), we can see that (19) is the definition of the 2-adic extension of Collatz function, which on the integers is given by:

\[
  C(n) = \begin{cases} 
  \frac{n}{2} & \text{if } n \text{ even} \\
  \frac{3n+1}{2} & \text{if } n \text{ odd} 
  \end{cases}
\]

The famed 3x + 1 Problem is to determine whether or not for all \( n > 0 \), there exists a \( k \) where \( C^{(k)}(n) = 1 \). All computational evidence point to an affirmative answer;
as of June 2008 [13], the conjecture has been verified up to \( n = 18 \cdot 2^{58} \) by machine, but the problem is still unsolved in general [10, 11, 2].

Noting that \( f_0 \) and \( f_1 \) are both bicausal, we conclude that the coalgebra morphism \( Q: \mathbb{Z}_2 \to 2^\omega \) coinductively induced by \( \langle h, T \rangle \) is bicausal. Since \( A = 2 \) is finite, \( Q \) is a bijection. Moreover, bijective coalgebra morphisms are also coalgebra isomorphisms [18]. In other words, we have the following result.

**Theorem 4.4.** Let \( T \) be the 2-adic extension of the \( 3x + 1 \) function given in (19). In the category of 2-stream coalgebras, \( \langle h, T \rangle \) is terminal. More specifically, for any 2-stream coalgebra \( X \xrightarrow{f} 2 \times X \), there exists a unique coalgebra morphism \( X \xrightarrow{\psi} \mathbb{Z}_2 \) so that \( h \circ \psi = h \) and \( T \circ \psi = \psi \circ f \).

**Remark 4.5.** The unique coalgebra isomorphism \( Q \) here is the \( 3x + 1 \) conjugacy map denoted in Lagarias [9] as \( Q_\infty \). It produces the parity vector for 2-adic integers, i.e. \( Q(x)(n) \) is the parity of the \( n \)th iterate of \( x \in \mathbb{Z}_2 \) under \( T \). It should be clear that this is a special case of (13). In addition, Bernstein [4] exhibited a formula for the inverse of \( Q \) (using the notation from this paper):

\[
Q^{-1}(\sigma) = -\frac{1}{3} \sum_{\sigma(\ell)=1} \frac{1}{3^{\ell}} 2^\ell.
\]

This coinductive observation about \( T \) also applies to any function woven from bicausals on streams with a finite alphabet. For instance, for \( m, n \in \mathbb{Z} \), let \( T_{m,n}: \mathbb{Z}_2 \to \mathbb{Z}_2 \) be given by

\[
T_{m,n}(x) = \begin{cases} 
Q^{(m)}(\frac{x}{2}) & \text{if } h(x) = 0 \\
Q^{(n)}(\frac{x-1}{2}) & \text{if } h(x) = 1 
\end{cases}
\]  

(20)

woven from \( f_0 = Q^{(m)} \) and \( f_1 = Q^{(n)} \). By the same argument as for \( T \), we can conclude that \( \langle h, T_{m,n} \rangle \) is a final stream coalgebra. Given the plethora of other examples, co-algebraic finality in and of itself cannot yield a solution to the \( 3x + 1 \) problem. Nonetheless, it is worth noting that many of the interesting cousins of the \( 3x + 1 \) Problem are based on the dynamics of woven functions (often woven from bicausals).

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Department of Mathematics, Indiana University, Rawles Hall, 831 East 3rd St, Bloomington, IN 47405
E-mail address: jihokim@indiana.edu