The Continuous Classical Boundary Optimal Control of Triple Nonlinear Elliptic Partial Differential Equations with State Constraints

Jamil A. Ali Al-Hawasy*, Nabeel A. Thyab Al-Ajeeli
Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq

Received: 2/5/2020 Accepted: 16/6/2020

Abstract
Our aim in this work is to study the classical continuous boundary control vector problem for triple nonlinear partial differential equations of elliptic type involving a Neumann boundary control. At first, we prove that the triple nonlinear partial differential equations of elliptic type with a given classical continuous boundary control vector have a unique "state" solution vector, by using the Minty-Browder Theorem. In addition, we prove the existence of a classical continuous boundary optimal control vector ruled by the triple nonlinear partial differential equations of elliptic type with equality and inequality constraints. We study the existence of the unique solution for the triple adjoint equations related with the triple state equations. The Fréchet derivative is obtained. Finally we prove the theorems of both the necessary and sufficient conditions for optimality of the triple nonlinear partial differential equations of elliptic type through the Kuhn-Tucker-Lagrange's Multipliers theorem with equality and inequality constraints.

Keywords: optimal control vector, triple nonlinear elliptic equations, necessary and sufficient conditions for optimality

*Email: jhawassy17@uomustanriyah.edu.iq
1. Introduction

In many fields, the optimal control problems play a significant role in life. Different examples of the applications of such problems are presented in medicine [1], aircraft industry [2], electric power production [3], economic growth [4], and many other fields.

All these applications pushed many investigators to a higher level of interest in studying the optimal control problem for nonlinear ordinary differential equations [5], for different types of linear partial differential equations, including the hyperbolic, parabolic and elliptic [6-8], or for couple nonlinear partial differential equations of these three types [9-11]. While other authors [12, 13] studied these three types but included a Neumann boundary control. More recently, optimal control problems were studied for triple partial differential equations of these three types [14-16]. Also, the optimal control problem involving Neumann boundary control for triple partial differential equations of parabolic type was also recently investigated [17]. All these investigations motivated us to seek the optimal control problem, involving Neumann boundary control ruled by the triple nonlinear partial differential equations of elliptic type.

At first, our aim in this work is to prove that system of the triple nonlinear partial differential equations of elliptic type with a given classical continuous boundary control vector, which has a unique "state" solution vector, by using the Minty-Browder Theorem. Then, we prove the existence of a classical continuous boundary optimal control vector, ruled by the triple nonlinear partial differential equations of elliptic type with equality and inequality constraints.

We study the existence of the unique solution for the system of the triple adjoint equations related with the triple state equations. At the end, we prove the theorems of both the necessary and sufficient conditions for optimality of the triple nonlinear partial differential equations of elliptic type through the Kuhn-Tucker-Lagrange's Multipliers with equality and inequality constraints.

2. Problem Description

Let $\Psi$ be a bounded and open connected subset in $\mathbb{R}^2$ with Lipshitz boundary $\partial \Psi$. The optimal control problem is considered by the "state vector equation" which consists of the TNLEPDEs triple nonlinear elliptic partial differential equations with the Neumann boundary control.

$$
A_1 b_1 + b_1 - b_2 - b_3 + m_1(x, b_1) = \eta_1(x), \text{in } \Psi
$$

$$
A_2 b_2 + b_1 + b_2 + b_3 + m_2(x, b_2) = \eta_2(x), \text{in } \Psi
$$

$$
A_3 b_3 + b_1 - b_2 + b_3 + m_3(x, b_3) = \eta_3(x), \text{in } \Psi
$$

$$
\sum_{r=1}^{3} a_{1r} \frac{\partial \eta_1}{\partial \eta_1} = d_1, \text{on } \partial \Psi
$$

$$
\sum_{r=1}^{3} a_{2r} \frac{\partial \eta_2}{\partial \eta_2} = d_2, \text{on } \partial \Psi
$$

$$
\sum_{r=1}^{3} a_{3r} \frac{\partial \eta_3}{\partial \eta_3} = d_3, \text{on } \partial \Psi
$$

where

$$
A_r b_r = - \sum_{r=1}^{3} a_{r\sigma} \frac{\partial}{\partial \sigma} \left( a_{r\sigma}(x) \frac{\partial \eta_r}{\partial \eta_1} \right), r = 1, 2, 3 \quad a_{r\sigma} = a_{r\sigma}(x) \in C^\infty(\Psi), \quad \text{for } \sigma, r = 1, 2
$$

$$(d_1, d_2, d_3) = \left( d_1(x), d_2(x), d_3(x) \right) \in \left( L_2(\partial \Psi) \right)^3 \text{ is the Neumann boundary control vector.}
$$

The control constraints are

$$
\bar{d} \in \bar{E}, \Bar{E} \subset \left( L_2(\partial \Psi) \right)^3, \quad \text{where } \bar{d} = (d_1, d_2, d_3) \text{ and } \Bar{E} = E_1 \times E_2 \times E_3, \text{ with }
$$

$$
\bar{E} = \bar{E}_D = \left\{ \Bar{E} \in \left( L_2(\partial \Psi) \right)^3 | \Bar{E} = (E_1, E_2, E_3) \in \bar{D} \text{ a.e in } \partial \Psi \right\}, \text{ where } \bar{D} = D_1 \times D_2 \times D_3 , \text{ with } \bar{D} \subset \mathbb{R}^3 \text{ is a convex and compact set }.
$$

The cost function and the equality and inequality constraints are given by:

$$
\bar{v}_0(\bar{d}) = \int_\Psi \left[ v_{01}(x, b_1) + v_{02}(x, b_2) + v_{03}(x, b_3) \right] dx_1 dx_2
$$

$$
+ \int_{\partial \Psi} \left[ v_{04}(x, d_1) + v_{05}(x, d_2) + v_{06}(x, d_3) \right] dy_1
$$

(7)
where
\[ \tau_1(\tilde{d}) = \iint_{\Omega}[v_{11}(x, b_1) + v_{12}(x, b_2) + v_{13}(x, b_3)]\,dx_1\,dx_2 \]
\[ + \iint_{\Omega}[v_{14}(x, d_1) + v_{15}(x, d_2) + v_{16}(x, d_3)]\,dy = 0 \] (8)
\[ \tau_2(\tilde{d}) = \iint_{\Omega}[v_{22}(x, b_1) + v_{22}(x, b_2) + v_{23}(x, b_3)]\,dx_1\,dx_2 \]
\[ + \iint_{\Omega}[v_{24}(x, d_1) + v_{25}(x, d_2) + v_{26}(x, d_3)]\,dy \leq 0 \] (9)

The set of admissible control is
\[ \tilde{E}_A = \{ \tilde{d} \in \tilde{E} : \tau_1(\tilde{d}) = 0, \tau_2(\tilde{d}) \leq 0 \} \] (10)

The classical continuous boundary control vector problem is to minimize (7) subject to the state constraints (8) and (9), i.e. to find \( \tilde{d} \) such that
\[ \tilde{d} \in \tilde{E}_A \text{ and } \tau_0(\tilde{d}) = \min_{\tilde{d} \in \tilde{E}_A} \tau_0(\tilde{d}) \] .

Let \( T = (T)^3 = (H^1(\Psi))^3 \), the notations (t,t) \text{--} (L_2(\Psi)) \text{, and } l \text{--} l \text{--} (L_2(\Psi)) \text{ and } l \text{--} l \text{--} (L_2(\partial\Psi)) \) refer to the inner product and the norm in \( L_2(\Psi) \text{ (L_2(\partial\Psi))} \text{ and } l \text{--} l \text{--} (H^1(\Psi)) \), the notations (t,t,t) \text{--} (L_2(\Psi))^3 \text{ and } l \text{--} l \text{--} (L_2(\Psi))^3 \text{ refer to the inner product and the norm in } (L_2(\Psi))^3 \text{, while the notations (t,t,t) \text{--} (L_2(\Psi))^3 = } \sum_{i=1}^{3} l_i l_i l_i (L_2(\Psi)) \text{ refer to the inner product and the norm in } T \text{, finally } T^{**} \text{ is referred to the dual of } T \text{.}

3. Weak formulation of the triple state equations
To find the weak formulation of problem (1–6), let
\[ \pi = T_1 \times T_2 \times T_3 = H^2(\Psi) \times H^2(\Psi) \times H^2(\Psi) \]
\[ = \{ (\tilde{t}, \tilde{v}) = (t_1, t_2, t_3) \in (H^2(\Psi))^3 \text{, with } t_1, t_2, t_3 \text{ satisfy (4)-(6), respectively on } \partial\Psi \} \].

By multiplying both sides of equations (1),(2) and (3) by \( t_1, t_2, t_3 \), respectively
integrating both sides of each one of the obtained equations with respect to \( \Omega \), and then using
the generalized Green's theorem, we get
\[ a_1(b_1, t_1) - (b_2 + b_3, t_1)_{L_2(\Psi)} + (m_1(x, b_1), t_1)_{L_2(\Psi)} = (g_1(x), t_1)_{L_2(\Psi)} + (d_1, t_1)_{L_2(\partial\Psi)}, \forall t_1 \in T_1 \] (11)
\[ a_2(b_2, t_2) + (b_1 + b_3, t_2)_{L_2(\Psi)} + (m_2(x, b_2), t_2)_{L_2(\Psi)} = (g_2(x), t_2)_{L_2(\Psi)} + (d_2, t_2)_{L_2(\partial\Psi)}, \forall t_2 \in T_2 \] (12)
\[ a_3(b_3, t_3) + (b_1 - b_2, t_3)_{L_2(\Psi)} + (m_3(x, b_3), t_3)_{L_2(\Psi)} = (g_3(x), t_3)_{L_2(\Psi)} + (d_3, t_3)_{L_2(\partial\Psi)}, \forall t_3 \in T_3 \] (13)

By adding equations (11), (12) and (13), we get
\[ a(\tilde{b}, \tilde{v}) + (m_1(x, b_1), t_1)_{L_2(\Psi)} + (m_2(x, b_2), t_2)_{L_2(\Psi)} + (m_3(x, b_3), t_3)_{L_2(\Psi)} = (g_1(x), t_1)_{L_2(\Psi)} + (d_1, t_1)_{L_2(\partial\Psi)} + (g_2(x), t_2)_{L_2(\Psi)} + (d_2, t_2)_{L_2(\partial\Psi)} \]
\[ + (g_3(x), t_3)_{L_2(\Psi)} + (d_3, t_3)_{L_2(\partial\Psi)}, \forall (t_1, t_2, t_3) \in T \] (14)

where
\[ a(\tilde{b}, \tilde{v}) = a_1(b_1, t_1) - (b_2 + b_3, t_1)_{L_2(\Psi)} + a_2(b_2, t_2) + (b_1 + b_3, t_2)_{L_2(\Psi)} \]
\[ + a_3(b_3, t_3) + (b_1 + b_2, t_3)_{L_2(\Psi)} \]
with
\[ a_r(b_r, t_r) = \int_{\Psi} \left( \sum_{i=1}^{2} a_{rij} \frac{\partial b_r}{\partial x_i} \frac{\partial t_r}{\partial x_i} + b_r t_r \right) \,dx \]
which satisfies
\[ a_r(b_r, t_r) \geq c_{t_r} \| b_r \|_{H^4(\Psi)}^2 \text{, where } c_{t_r} \geq 0, r = 1, 2, 3 \]
\[ |a_r(b_r, t_r)| \leq c_{2r} \| b_r \|_{H^4(\Psi)} \| T_r \|_{H^2(\Psi)}^2 \text{, where } c_{2r} \geq 0, r = 1, 2, 3 \].

The following assumptions are useful to prove the existence theorem of a unique solution of the weak form (14).

Assumption (I):
\[ a(\tilde{b}, \tilde{v}) \text{ is coercive, i.e. } \frac{a(\tilde{b}, \tilde{v})}{\| \tilde{b} \|_{(H^4(\Psi))^3}^2} \geq c \| \tilde{b} \|_{(H^4(\Psi))^3}^2 > 0, \forall \tilde{b} \in \tilde{T} \]
b) \( a(\tilde{b},\tilde{t}) \) is continuous, i.e.
\[ |a(\tilde{b},\tilde{t})| \leq \epsilon \| \tilde{b} \|_{(H^1(\Psi))^3} \| \tilde{t} \|_{(H^1(\Psi))^3} , \epsilon > 0 , \forall \tilde{b},\tilde{t} \in \mathbb{T} \]

c) \( m_1, m_2 \) and \( m_3 \) are of Carathéodory type on \( \Psi \times \mathbb{R} \) and the following sub linearity conditions with respect to \( b_1,b_2,b_3 \) are satisfied, respectively, i.e.
\[ |m_\sigma(\chi,b_\sigma)| \leq \phi_\sigma(\chi) + \bar{c}_\sigma|b_\sigma|, \forall (\chi,b_\sigma) \in \Psi \times \mathbb{R} \] with \( \phi_\sigma \in L_2(\Psi), \bar{c}_\sigma i > 0 \) for \( \sigma = 1,2,3 \)
d) \( m_\sigma(\chi,b_\sigma) \) are monotone with respect to \( b_\sigma \) for each \( \chi \in \Psi \), and \( m_\sigma(\chi,0) = 0 , \forall \chi \in \Psi \), \( \sigma = 1,2,3 \)
e) \( \eta_\sigma(\chi) \) are of the Carathéodory type on \( \Psi \) and satisfy \( |\eta_\sigma(\chi)| \leq \phi(\chi) , \forall \chi \in \Psi \), with \( \phi(\chi) \in L_2(\Psi) \), \( \sigma,j = 1,2,3 \)

**Theorem (1):** If assumption (I) is hold, and if one of the functions \( m_1,m_2 \) or \( m_3 \) in (14) is strictly monotone, then for each fixed classical continuous boundary optimal control vector \( \overline{\tilde{d}} \in \mathbb{F}_A \), the weak form of (14) has a unique "state" solution vector \( \overline{\tilde{b}} \in \mathbb{T} \).

**Proof:** It is clear that the existence of a unique solution of (14) is obtained after the usage of assumptions (I), then theorem (1) in reference [18] is applied.

4. **Existence of the Classical Continuous Boundary Optimal Control Vector**

In this section, the theorem of the existence of a classical continuous boundary optimal control vector under the suitable assumptions is proved. However, before proving it, it is necessary to deal with the following lemmas and assumptions.

**Lemma (I):** If the assumption (I) is hold, the functions \( m_1 , m_2 , m_3 \) are Lipschitz continuous with respect to \( b_1 , b_2 , b_3 \), respectively, and if \( \eta_1(\chi),\eta_2(\chi),\eta_3(\chi) \) are bounded, then the mapping \( \tilde{d} \rightarrow \overline{\tilde{b}}_{\tilde{d}} \) is Lipschitz continuous from \( \mathbb{E}^d \) into \( (L_2(\Psi))^3 \), i.e.
\[ \| \overline{\tilde{b}}_{\tilde{d}} \|_{(L_2(\Psi))^3} \leq L \| \tilde{d} \|_{(L_2(\partial\Psi))^3} \] with \( L > 0 \).

**Proof:** Assume that \( \overline{\tilde{d}}, \overline{\tilde{d}}' \in \mathbb{E}^d \) are two given controls, then there corresponding "state" solution vectors (of the weak form (14)) are \( \overline{\tilde{b}}, \overline{\tilde{b}}' \). By subtracting the above three obtained weak forms from their corresponding ones in (14), putting \( \overline{\tilde{b}} = \overline{\tilde{b}}' - \overline{\tilde{b}} \) and \( \overline{\tilde{d}} = \overline{\tilde{d}}' - \overline{\tilde{d}} \), then adding the obtained three equations, we get
\[ a_1(\Delta b_1,\Delta b_1) + a_2(\Delta b_2,\Delta b_2) + a_3(\Delta b_3,\Delta b_3) + m_1(\chi,b_1 + \Delta b_1) - m_1(\chi,b_1) + \Delta b_1_{L_2(\Psi)} + (m_2(\chi,b_2 + \Delta b_2) - m_2(\chi,b_2) + \Delta b_2_{L_2(\Psi)}) + (m_3(\chi,b_3 + \Delta b_3) - m_3(\chi,b_3) + \Delta b_3_{L_2(\Psi)}) = (\Delta d_1,\Delta b_1_{L_2(\partial\Psi)}) + (\Delta d_2,\Delta b_2_{L_2(\partial\Psi)}) + (\Delta d_3,\Delta b_3_{L_2(\partial\Psi)}) \] (16)

By using assumption A-(a, d), taking the absolute value for both sides of (16), it becomes
\[ c\| \overline{\tilde{b}} \|_{(H^1(\Psi))^3}^2 \leq \theta_1 \| \Delta b_1 \|_{(H^1(\Psi))^3} + \theta_2 \| \Delta b_2 \|_{(H^1(\Psi))^3} + \theta_3 \| \Delta b_3 \|_{(H^1(\Psi))^3} \leq \| (\Delta d_1,\Delta b_1_{L_2(\partial\Psi)}) + (\Delta d_2,\Delta b_2_{L_2(\partial\Psi)}) + (\Delta d_3,\Delta b_3_{L_2(\partial\Psi)}) \| \] (17)

By using the Cauchy-Schwarz inequality and then the trace operator in the right side, on (17), we obtain
\[ c\| \overline{\tilde{b}} \|_{(H^1(\Psi))^3}^2 \leq 3c_1 \| \tilde{d} \|_{(L_2(\partial\Psi))^3} + \| \overline{\tilde{b}} \|_{(H^1(\Psi))^3} \Rightarrow \| \overline{\tilde{b}} \|_{(H^1(\Psi))^3} \leq L^2 \| \tilde{d} \|_{(L_2(\partial\Psi))^3} \] where \( L^2 = \frac{3c_1}{c} \) (18)

which gives
\[ \| \overline{\tilde{b}} \|_{(L_2(\Psi))^3} \leq L \| \tilde{d} \|_{(L_2(\partial\Psi))^3} \] (19)

**Assumption (II):**
Assume that \( \nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6 \) on \( \Psi \times \mathbb{R} \) and \( \nu_7,\nu_8,\nu_9,\nu_{10} \) on \( \Psi \times \mathbb{D} \) are of the Carathéodory type, then the following are satisfied for each \( P=0,1,2; \)
\[ |\nu_{P1}(\chi,b_1)| \leq Y_{P1}(\chi) + c_{P1} b_1^2 , |\nu_{P2}(\chi,b_2)| \leq Y_{P2}(\chi) + c_{P2} b_2^2 , |\nu_{P3}(\chi,b_3)| \leq Y_{P3}(\chi) + c_{P3} b_3^2 , |\nu_{P4}(\chi,d_1)| \leq Y_{P4}(\chi) + c_{P4} d_1^2 , |\nu_{P5}(\chi,d_2)| \leq Y_{P5}(\chi) + c_{P5} d_2^2 , |\nu_{P6}(\chi,d_3)| \leq Y_{P6}(\chi) + c_{P6} d_3^2 , \]
where \( Y_{P_1}, Y_{P_2}, Y_{P_3} \in L_1(\Psi), Y_{P_4}, Y_{P_5}, Y_{P_6} \in L_1(\partial \Psi) \) and \( c_{P_\sigma} \geq 0 \) for \( \sigma = 1,2,3,4,5,6. \)

**Lemma (2):** If assumption (II) is held, then the functional \( \mathcal{U}_P (d^-) \) is continuous on \((L_2(\partial \Psi))^3\) for each \( P=0,1,2 \). Proof: For any \( P = 0,1,2 \), we set

\[
\begin{align*}
\mathcal{P}_{P_1}(x, b) &= \mathcal{V}_{P_1}(x, b_1) + \mathcal{V}_{P_2}(x, b_2) + \mathcal{V}_{P_3}(x, b_3) \\
\mathcal{P}_{P_2}(x, d) &= \mathcal{V}_{P_4}(x, d_1) + \mathcal{V}_{P_5}(x, d_2) + \mathcal{V}_{P_6}(x, d_3)
\end{align*}
\]

To prove the continuity for any one of the above two integrals, the used technique will be similar. Thus, it is enough to prove one of them, which is in this case the second integral. Hence, let \( d = (d_1, d_2, d_3) \), with \( \mathcal{P}_{P_2}: \Psi \times \mathbb{R}^3 \rightarrow \mathbb{R} \), then from assumption (II), we have

\[
\| P_{P_2}(x, d) \| \leq Y_{P_7}(x) + c_{P_4}d_1^2 + c_{P_5}d_2^2 + c_{P_6}d_3^2
\]

Then, the \( \mathcal{F}_P P_{P_2}(x, d) \) is continuous on \((L_2(\partial \Psi))^3\) (by using Proposition (1) in reference [19]). Hence,

\[
\mathcal{F}_P P_{P_2}(x, d) = \int_{\Psi} P_{P_2}(x, d) \, dx_1 \, dx_2 + \int_{\Psi} P_{P_2}(x, d) \, dx_2
\]

**Theorem (2):** If the assumptions (I) and (II) are hold, \( \overline{E}_A \neq \emptyset \), \( m_1, m_2, m_3 \) are not dependent on \( d_1, d_2, d_3 \), respectively, and \( \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3 \) are bounded functions, so that

\[
\begin{align*}
|m_1(x, b_1)| &\leq \tilde{\sigma}_1(x) + \tilde{\sigma}_2|b_1|, \\
|m_2(x, b_2)| &\leq \tilde{\sigma}_2(x) + \tilde{\sigma}_2|b_2|, \\
|m_3(x, b_3)| &\leq \tilde{\sigma}_3(x) + \tilde{\sigma}_3|b_3|
\end{align*}
\]

where \( \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3 \) are not dependent on \( d_1, d_2, d_3 \), respectively. \( \mathcal{P}_{P_4}, \mathcal{P}_{P_5}, \mathcal{P}_{P_6} \) (\( P = 0,2 \)) are convex with respect to \( d_1, d_2, d_3 \), respectively, for fixed \( x \). Then there exists a continuous classical boundary optimal control vector.

**Proof:** The set \( E_{\tilde{\sigma}} \) and \( D_{\tilde{\sigma}} \) (\( \tilde{\sigma} = 1,2,3 \)) is convex and bounded, then \( E_{1 \times E_{2 \times E_{3}}} \) is convex and bounded. On the other hand, by using theorem (2) in reference [19], \( E_{\tilde{\sigma}} \) \( \forall \tilde{\sigma} = 1,2,3 \) is closed, since \( D_{\tilde{\sigma}} \) is closed, then \( E_{1 \times E_{2 \times E_{3}}} \) is closed, too. Therefore, we obtain that \( E_{1 \times E_{2 \times E_{3}}} \) is weakly compact.

From the assumption on \( \overline{E}_A \), there is a minimum sequence \( \{ \mathcal{O}_n \} = \{(d_{1n}, d_{2n}, d_{3n})\} \in \overline{E}_A \) for each \( n \), with \( \mathcal{V}_1(\mathcal{O}_n) = 0 \), \( \mathcal{V}_2(\mathcal{O}_n) \leq 0 \), so that

\[
\lim_{n \rightarrow \infty} \mathcal{V}_0(\mathcal{O}_n) = \inf_{\mathcal{O} \in \overline{E}_A} \mathcal{V}_0(\mathcal{O})
\]

But \( \overline{E}_A \) is weakly compact, then there is a subsequence of \( \{ \mathcal{O}_n \} \), which will be symbolized again by \( \{ \mathcal{O}_n \} \), that converges weakly to \( \mathcal{O} \) in \( \overline{E}_A \).

Then, corresponding to the \( \{ \mathcal{O}_n \} \), there is the sequence of the "state" solution vector \( \{ \mathcal{B}_n \} \) of the sequence of the weak form. Then, from the proof of Theorem (3), we have:

\[
\begin{align*}
&\mathcal{A}_1(b_{1n}, t_1) - (b_{2n} + b_{3n}, t_1)_{L_2(\Psi)} + a_2(b_1 + b_2, t_2) + (b_1 + b_2, t_2)_{L_2(\Psi)} + a_3(b_3, t_3) + (b_{1n} - b_{2n}, t_3)_{L_2(\Psi)} + (m_1(x, b_{1n}), t_1)_{L_2(\Psi)} + (m_2(x, b_{2n}), t_2)_{L_2(\Psi)} + (m_3(x, b_{3n}), t_3)_{L_2(\Psi)} \\
&+ (m_3(x, b_{3n}), t_3)_{L_2(\Psi)} + (m_3(x, b_{3n}), t_3)_{L_2(\Psi)}
\end{align*}
\]

With \( \| \mathcal{B}_n \|_{H^1(\Psi)}^3 \) for each \( n \) is bounded, then \( \{ \mathcal{B}_n \} \) has a subsequence, which will be symbolized again by \( \mathcal{B}_n \), such that \( \mathcal{B}_n \rightarrow \mathcal{B} \) weakly in \( \overline{V} \) (Alaoglu theorem [20]).

Now, we have to show that (20) converges to

\[
\begin{align*}
a_1(b_1, t_1) - (b_2 + b_3, t_1)_{L_2(\Psi)} + a_2(b_1 + b_3, t_2) + (b_1 + b_3, t_2)_{L_2(\Psi)} + a_3(b_3, t_3) + (b_1 - b_2, t_3)_{L_2(\Psi)} + (m_1(x, b_1), t_1)_{L_2(\Psi)} + (m_2(x, b_2), t_2)_{L_2(\Psi)} + (m_3(x, b_3), t_3)_{L_2(\Psi)} \\
+ (d_3, t_3)_{L_2(\Psi)}
\end{align*}
\]
First, let \((t_1, t_2, t_3) \in (C(\Psi))^3\), and, first for the left hand side, since \(b_{3n} \to b_3\) weakly in \(T_\sigma\), i.e. 
\[b_{2n} \to b_2\] weakly in \(L_2(\Psi)\), for each \(\sigma = 1, 2, 3\), then from the left hand side of (20) and (21) and by using the Cauchy-Schwarz inequality, one has 
\[
\left| a_1(b_{2n}, t_1) - (b_{2n} + b_{3n}, t_1) \right|_{L_2(\Psi)} + a_2(b_{2n}, t_2) + (b_{3n}, t_2) \right|_{L_2(\Psi)} - \leq (C_1\|b_{2n} - \|b_{3n}\|_{L_2(\Psi)} + \|b_{2n} - b_2\|_{L_2(\Psi)} + \|b_{3n} - b_3\|_{L_2(\Psi)})t_1\|_{L_2(\Psi)} + 
\]
\[
+ (c_2\|b_{2n} - b_2\|_{L_2(\Psi)} + \|b_{3n} - b_3\|_{L_2(\Psi)})t_2\|_{L_2(\Psi)} + 
\]
\[
+ (c_3\|b_{2n} - b_2\|_{H^1(\Psi)} + \|b_{3n} - b_3\|_{H^1(\Psi)})t_3\|_{L_2(\Psi)} \to 0 
\]
(22)

i. From assumption (II) and Proposition (1), the functions \(\int_\Psi m_1(x, b_{3n}) t_1 dx_1 dx_2\) and \(\int_\Psi m_2(x, b_{3n}) t_2 dx_1 dx_2\) and \(\int_\Psi m_3(x, b_{3n}) t_3 dx_1 dx_2\) are continuous with respect to \(b_{2n}, b_{3n}\) and \(b_{3n}\), respectively.

But \(\tilde{b}_n \to \tilde{b}\) weakly in \((L_2(\Psi))^3\), because \(\tilde{b}_n \to \tilde{b}\) weakly in \(\tilde{T}\), then by using the Rellich-Kondrachov theorem in [21], we get that \(\tilde{b}_n \to \tilde{b}\) strongly in \((L_2(\Psi))^3\), hence 
\[
\int_\Psi (m_1(x, b_{3n}), t_1)_{L_2(\Psi)} + \int_\Psi (m_2(x, b_{3n}), t_2)_{L_2(\Psi)} + \int_\Psi (m_3(x, b_{3n}), t_3)_{L_2(\Psi)} 
\]
\[
\to \int_\Psi (m_1(x, \tilde{b}), t_1)_{L_2(\Psi)} + \int_\Psi (m_2(x, \tilde{b}), t_2)_{L_2(\Psi)} + \int_\Psi (m_3(x, \tilde{b}), t_3)_{L_2(\Psi)} 
\]
(23a)

Second, since \(d_{1n} \to d_1, d_{2n} \to d_2\) and \(d_{3n} \to d_3\) weakly in \(L_2(\partial\Psi)\), then 
\[
(d_{1n} - d_1, t_1)_{L_2(\partial\Psi)} + (d_{2n} - d_2, t_2)_{L_2(\partial\Psi)} + (d_{3n} - d_3, t_3)_{L_2(\partial\Psi)} \to 0 
\]
(23b)

From (23a) and (23b), we obtain that (21) converges to (21).

Since \((C(\Psi))^3\) is dense in \(\tilde{V}\), then this convergence satisfies for any \((t_1, t_2, t_3) \in \tilde{T}\). This leads to 
\(\tilde{b}_n \to \tilde{b} = \tilde{b}_d\) is a solution of the weak form of the triple state.

From Lemma (2), the functional \(\mathcal{V}_P(\bar{d})\) is continuous on \((L_2(\partial\Psi))^3\). \(\forall P = 0, 1, 2\).

From the assumptions on \(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_6\), the integrals 
\[
\int_\Psi \mathcal{V}_P(x, b_{3n}) t_1 dx_1 dx_2 + \int_\Psi \mathcal{V}_P(x, b_{3n}) t_2 dx_1 dx_2 
\]
\[
\leq \int_\Psi \mathcal{V}_P(x, b_t) dx_1 dx_2 + \lim_{n \to \infty} \int_\Psi \mathcal{V}_P(x, d_{1n}) dy 
\]
\[
= \lim_{n \to \infty} \int_\Psi \mathcal{V}_P(x, b_{1n}) dx_1 dx_2 
\]
\[
+ \lim_{n \to \infty} \int_\Psi \mathcal{V}_P(x, d_{1n}) dy 
\]
\[
= \lim_{n \to \infty} (\int_\Psi \mathcal{V}_P(x, b_{1n}) dx_1 dx_2 + \int_\Psi \mathcal{V}_P(x, d_{1n}) dy) 
\]

By the same manner, and for each \(P = 0, 2\), we get the following two convergences:
\[
\int_\Psi \mathcal{V}_P(x, b_{2n}) dx_1 dx_2 + \int_\Psi \mathcal{V}_P(x, d_{2n}) dy 
\]
\[
\leq \lim_{n \to \infty} (\int_\Psi \mathcal{V}_P(x, b_{2n}) dx_1 dx_2 + \int_\Psi \mathcal{V}_P(x, d_{2n}) dy) 
\]
and 
\[
\int_\Psi \mathcal{V}_P(x, b_{3n}) dx_1 dx_2 + \int_\Psi \mathcal{V}_P(x, d_{3n}) dy 
\]
\[
\leq \lim_{n \to \infty} (\int_\Psi \mathcal{V}_P(x, b_{3n}) dx_1 dx_2 + \int_\Psi \mathcal{V}_P(x, d_{3n}) dy) 
\]
From the above inequalities, one gets that \(\mathcal{V}_P(\bar{d})\), \(\forall P = 0, 2\), is weakly lower semicontinuous with respect to \((\tilde{b}, \bar{d})\). Thus 
\[
\mathcal{V}_P(\tilde{d}) \leq \lim_{n \to \infty} \mathcal{V}_P(\tilde{d}_n) \leq 0, 
\]
\[ \tau_0(\tilde{d}) = \lim_{n \to \infty} \tau_0(\tilde{d}_n) = \lim_{n \to \infty} \tau_0(\tilde{d}_n) = \inf_{w \in \hat{W}} \tau_0(w) \]

\( \tilde{d} \) is a continuous classical boundary optimal control vector.

5. The Necessary and Sufficient Conditions for Optimality of the Continuous Classical Boundary Optimal Control Vector

The following assumptions are useful in this section to derive the Fréchet derivative of the Hamiltonian.

Assumption (III)

a) \( m_{1b_1}, m_{2b_2}, m_{3b_3} \) are of the Carathéodory type on \( \Psi \times \mathbb{R} \) and satisfy

\[ m_{1b_1}(\tilde{x}, b_1) \leq \tilde{c}_1, \quad m_{2b_2}(\tilde{x}, b_2) \leq \tilde{c}_2, \quad m_{3b_3}(\tilde{x}, b_3) \leq \tilde{c}_3, \quad \text{with} \quad \tilde{c}_1, \tilde{c}_2, \tilde{c}_3 \geq 0 \]

b) \( b_1, b_2, b_3 \) are of the Carathéodory type on \( \Psi \) and satisfy

\[ |b_1(\tilde{x})| \leq \tilde{c}_4, \quad |b_2(\tilde{x})| \leq \tilde{c}_5, \quad |b_3(\tilde{x})| \leq \tilde{c}_6, \quad \text{with} \quad \tilde{c}_4, \tilde{c}_5, \tilde{c}_6 \geq 0 \]

c) \( v_{p1b_1}, v_{p2b_2}, v_{p3b_3}, v_{p4d_1}, v_{p5d_2}, v_{p6d_3} (\forall P = 0,1,2) \) are of the Carathéodory type on \( \Psi \times \mathbb{R} \) and satisfy

\[ v_{p1b_1} \leq Y_{p_1} + c_{p_1}|b_1|, \quad v_{p2b_2} \leq Y_{p_2} + c_{p_2}|b_2|, \quad v_{p3b_3} \leq Y_{p_3} + c_{p_3}|b_3|, \quad v_{p4d_1} \leq Y_{p_4} + c_{p_4}|d_1|, \quad v_{p5d_2} \leq Y_{p_5} + c_{p_5}|d_2|, \quad v_{p6d_3} \leq Y_{p_6} + c_{p_6}|d_3| \]

where \( c_{p_0} \geq 0, Y_{p_1}, Y_{p_2}, Y_{p_3} \in L_2(\Psi) \) and \( Y_{p_4}, Y_{p_5}, Y_{p_6} \in L_2(\partial\Psi) \), for \( \sigma = 1,2,3,4,5,6 \) and \( P = 0,1,2 \).

Theorem (3): If the assumptions (I), (II), and (III) are hold, the Hamiltonian is given as:

\[ H_1(\tilde{x}, b_1, b_2, b_3, z_1, z_2, z_3, d_1, d_2, d_3) = z_1(z_1(\tilde{x}) - m_1(\tilde{x}, b_1)) + v_{p1}(\tilde{x}, b_1) + v_{p2}(\tilde{x}, b_2) + v_{p3}(\tilde{x}, b_3) + v_{p4}(\tilde{x}, d_1) + v_{p5}(\tilde{x}, d_2) + v_{p6}(\tilde{x}, d_3) \]

The triple adjoint equations of the triple state equations (1-6) are:

\[ A_1 \frac{\partial z_1}{\partial \tilde{y}} = 0, \quad \text{in} \quad \tilde{y} \]
\[ A_2 \frac{\partial z_2}{\partial \tilde{y}} = 0, \quad \text{in} \quad \tilde{y} \]
\[ A_3 \frac{\partial z_3}{\partial \tilde{y}} = 0, \quad \text{in} \quad \tilde{y} \]

Then the Fréchet derivative of \( \tau_0 \) is

\[ \bar{\nu}_0(\tilde{d}) = \int_{\tilde{W}} H'_{\tilde{d}} \cdot \Delta \tilde{d} d\tilde{y}, \quad \text{where} \]

\[ H'_{\tilde{d}} = \left( \begin{array}{c}
H_{1\tilde{d}}(x, z_1, z_2, z_3, b_1, b_2, b_3, d_1, d_2, d_3) \\
H_{2\tilde{d}}(x, z_1, z_2, z_3, b_1, b_2, b_3, d_1, d_2, d_3) \\
H_{3\tilde{d}}(x, z_1, z_2, z_3, b_1, b_2, b_3, d_1, d_2, d_3)
\end{array} \right) = \left( \begin{array}{c}
\frac{\partial z_1}{\partial \tilde{y}} + v_{p4d_1} \\
\frac{\partial z_2}{\partial \tilde{y}} + v_{p5d_2} \\
\frac{\partial z_3}{\partial \tilde{y}} + v_{p6d_3}
\end{array} \right) \]

\( \tilde{z} = \tilde{z}_{\tilde{d}} \) is the triple adjoint equation of the triple state equation \( \tilde{y}_{\tilde{d}} \).

Proof: Formulating the triple adjoint equations (24-29) by their weak forms, then adding them, and then setting \( \tilde{q} = \Delta \tilde{d} \) in the resulting equation, yield

\[ \begin{align*}
(\tilde{z}_1, \Delta b_1) + (\tilde{z}_2, \Delta b_2) + (\tilde{z}_3, \Delta b_3) & = a_1(\tilde{z}_1, \Delta b_1) + a_2(\tilde{z}_2, \Delta b_2) + a_3(\tilde{z}_3, \Delta b_3) \\
- (\tilde{z}_1, - \Delta b_1) + (\tilde{z}_2, - \Delta b_2) & = (\tilde{z}_1, - \Delta b_1) + (\tilde{z}_2, - \Delta b_2)
\end{align*} \]

One can easily prove that the weak form (30), with fixed continuous classical boundary optimal control vector \( \tilde{d} \in \tilde{E} \), has a unique “state” solution vector \( \tilde{z} = \tilde{z}_{\tilde{d}} \), by applying the same manner employed in the proof of theorem (3).
Now, by setting once the solution $b_1$ in the weak forms of the state equations (11) and once again the solution $b_2 + \Delta b_2$, then subtracting the 1st obtained weak form from the other one, we obtain
$$a_1 (\Delta b_1, t_1) - (\Delta b_2 + \Delta b_3, t_1)_{L^2(\psi)} + (m_1(b_1 + \Delta b_1) - m_1(b_1), t_1)_{L^2(\psi)} = (\Delta d_1, t_1)_{L^2(\psi)} \forall t_1 \in T_1$$
(31)
The same above substituting and subtracting are repeated but from a side with the solutions $b_2$ and $b_2 + \Delta b_2$ in the weak form of equation (12) and from the other side with the solutions $b_3$ and $b_3 + \Delta b_3$ in the weak form of the state equation (13), respectively, to obtain
$$a_2 (\Delta b_2, t_2) + (\Delta b_3 + \Delta b_2, t_2)_{L^2(\psi)} + (m_2(b_2 + \Delta b_2) - m_2(b_2), t_2)_{L^2(\psi)} = (\Delta d_2, t_2)_{L^2(\psi)} \forall t_2 \in T_2$$
(32)
$$a_3 (\Delta b_3, t_3) + (\Delta b_1 - \Delta b_3, t_3)_{L^2(\psi)} + (m_3(b_3 + \Delta b_3) - m_3(b_3), t_3)_{L^2(\psi)} = (\Delta d_3, t_3)_{L^2(\psi)} \forall t_3 \in T_3$$
(33)
Adding (31), (32) and (33), then substituting $\tilde{t} = (\zeta_1, \zeta_2, \zeta_3)$ in the resulting equation, yield
$$a_1 (\Delta b_1, \zeta_1) - (\Delta b_2 + \Delta b_3, \zeta_1)_{L^2(\psi)} + a_2 (\Delta b_2, \zeta_2) + (\Delta b_1 + \Delta b_3, \zeta_2)_{L^2(\psi)} + a_3 (\Delta b_3, \zeta_3) + (\Delta b_1 - \Delta b_2, \zeta_3)_{L^2(\psi)} + (m_1(b_1 + \Delta b_1), \zeta_1) - m_1(b_1), \zeta_1)_{L^2(\psi)} + ((m_2(b_1, b_2 + \Delta b_2, \zeta_2)) - m_2(b_2, \zeta_2)_{L^2(\psi)} + ((m_3(b_3 + \Delta b_3, \zeta_3)) - m_3(b_3, \zeta_3)_{L^2(\psi)} = (\Delta d_1, \zeta_1)_{L^2(\psi)} + (\Delta d_2, \zeta_2)_{L^2(\psi)} + (\Delta d_3, \zeta_3)_{L^2(\psi)}$$
(34)
From the assumptions on $m_1, m_2, m_3$ and by using Proposition (2) in reference [19], the Fréchet derivative of $m_1, m_2, m_3$ exists. Hence, from Lemma (1) and the Minkowski inequality, (34) becomes
$$a_1 (\Delta b_1, \zeta_1) - (\Delta b_2 + \Delta b_3, \zeta_1)_{L^2(\psi)} + a_2 (\Delta b_2, \zeta_2) + (\Delta b_1 + \Delta b_3, \zeta_2)_{L^2(\psi)} + a_3 (\Delta b_3, \zeta_3) + (\Delta b_1 - \Delta b_2, \zeta_3)_{L^2(\psi)} + (m_1(b_1 + \Delta b_1), \zeta_1) - m_1(b_1), \zeta_1)_{L^2(\psi)} + ((m_2(b_1, b_2 + \Delta b_2, \zeta_2)) - m_2(b_2, \zeta_2)_{L^2(\psi)} + ((m_3(b_3 + \Delta b_3, \zeta_3)) - m_3(b_3, \zeta_3)_{L^2(\psi)} = (\Delta d_1, \zeta_1)_{L^2(\psi)} + (\Delta d_2, \zeta_2)_{L^2(\psi)} + (\Delta d_3, \zeta_3)_{L^2(\psi)}$$
(35)
where $\tilde{\varepsilon}_1(\Delta d), \tilde{\varepsilon}_2(\Delta d)$ and $\tilde{\varepsilon}_3(\Delta d) \rightarrow 0$ and $\|\Delta d\|_{L^2(\psi)} \rightarrow 0$ as $\Delta d \rightarrow 0$. Subtracting (30) from (35), to get
$$(\nu_0 b_1, (b_1), \Delta b_1)_{L^2(\psi)} + (\nu_0 b_2, (b_2), \Delta b_2)_{L^2(\psi)} + (\nu_0 b_3, (b_3), \Delta b_3)_{L^2(\psi)} + \tilde{\varepsilon}_1(\Delta d)\|\Delta d\|_{L^2(\psi)}^3 + \tilde{\varepsilon}_2(\Delta d)\|\Delta d\|_{L^2(\psi)}^3 + \tilde{\varepsilon}_3(\Delta d)\|\Delta d\|_{L^2(\psi)}^3 = (\Delta d_1, \zeta_1)_{L^2(\psi)} + (\Delta d_2, \zeta_2)_{L^2(\psi)} + (\Delta d_3, \zeta_3)_{L^2(\psi)}$$
(36)
Now, from the assumptions on $\nu_{01}, \nu_{02}, \nu_{03}, \nu_{04}, \nu_{05}, \nu_{06}$, Proposition (2) in reference [19], and then using the result of Lemma (1), we have
$$\tau_0 (d + \Delta d) - \tau_0 (d) = \int_{\Omega} (\nu_{01} b_1, (\chi, b_1) \Delta d_1 + \nu_{02} b_2, (\chi, b_2) \Delta d_2 + \nu_{03} b_3, (\chi, b_3) \Delta d_3) d\chi d\gamma$$
(37)
where $\varepsilon_4(\Delta d) \rightarrow 0$ as $\Delta d \rightarrow 0$. By substituting (36) in the above equality, we get
$$\tau_0 (d + \Delta d) - \tau_0 (d) = \int_{\Omega} (\nu_{04} d_1, (\chi, \nu_{04} d_1) \Delta d_1 d\gamma + f_{\nu_{04}} (\nu_{05} d_2, (\chi, \nu_{05} d_2) \Delta d_2 + \nu_{06} d_3, (\chi, \nu_{06} d_3) \Delta d_3) d\gamma + \varepsilon_5(\Delta d)\|\Delta d\|_{L^2(\psi)}^3$$
(38)
Note: In the proof of the above theorem, we have found the Fréchet derivative for the functional $\tau_0$, so the same technique is used to find the Fréchet derivative for $\tau_1$ and $\tau_2$.  

3027
Theorem (4):
(a) If assumptions (I), (II), and (III) are held, then $\bar{E}$ is convex, and if $\bar{d} \in \bar{E}_A$ is a continuous classical boundary optimal control vector, then $\forall \, P = 0,1,2$ and there exist multipliers $\xi_P \in \mathbb{R}$, with $\xi_0 \geq 0$, $\xi_1 \geq 0$, $\sum_{p=1}^{P} |\xi_P| = 1$, so that the following Kuhn-Tucker-Lagrange's Multipliers conditions are held:

$$\int_{\partial \Psi} \overline{H}_d \cdot \Delta d \, d\gamma \geq 0 \, , \quad \text{with} \quad \Delta d = \bar{w} - \bar{d} \, , \quad \forall \, \bar{c} \in \bar{E}$$

where $v_{adj} = \sum_{p=0}^{P} \xi_P \gamma_{adj} \& \gamma_j = \sum_{p=0}^{P} \xi_P \gamma_{adj}$, (for $j = 0,1,2, \sigma = 4,5,6$) in (Theorem (5)),

$$\xi_2 \overline{U}_2(d) = 0 \, , \quad \text{(Transversality conditions)}$$

(b) (Minimum Principle in point wise weak form): The inequality (39a) is equivalent to

$$H_d \tilde{d} = \min_{\bar{c} \in \bar{E}} H_d \tilde{c} \quad \text{a.e. in } \partial \Psi$$

Proof: (a) from Theorem (3), $\nabla_{\bar{d}}(P \bar{d}) \forall \bar{P} = 0,1,2$ and at any $\bar{d} \in \bar{E}$ has a continuous Fréchet derivative. Since the continuous classical boundary optimal control vector $\bar{d} \in \bar{E}_A$ is optimal, then by using the Kuhn-Tucker-Lagrange's Multipliers theorem $\forall \bar{P} = 0,1,2$, there exist multipliers $\xi_P \in \mathbb{R}$ with $\xi_0 \geq 0$, $\xi_1 \geq 0$, $\sum_{p=1}^{P} |\xi_P| = 1$, such that

$$\left(\sum_{p=0}^{P} \xi_P \overline{U}_p(d)\right)_{L_2(\partial \Psi)} \geq \left(\bar{c} - \bar{d}\right) \geq 0 \, , \quad \forall \, \bar{c} \in \bar{E} \, , \quad \text{Hence (41a)}$$

Then, from Theorem (3), (41a) with the setting $\Delta d_1 = e_1 - d_1, \Delta d_2 = e_2 - d_2, \Delta d_3 = e_3 - d_3$, we can rewrite $\forall \, \bar{c} \in \bar{E}$ as

$$\int_{\partial \Psi} \left(\Delta d_1 + (\xi_2 + v_{adj}) \Delta d_2 + (\xi_3 + v_{adj}) \Delta d_3\right) d\gamma \geq 0$$

where $\gamma_j = \sum_{p=0}^{P} \xi_P \gamma_{adj} = \sum_{p=0}^{P} \xi_P \gamma_{adj}$, for $j = 1,2,3, \sigma = 4,5,6$

$\Rightarrow \int_{\partial \Psi} H_d \overline{H}_d \cdot \Delta d \, d\gamma \geq 0 \, , \quad \forall \, \bar{c} \in \bar{E} \, , \quad \Delta d = \bar{c} - \bar{d}$

(b) Let $\{\bar{d}_n\}$ be a sequence, dense in $\bar{E}_D$, and $S \subset \partial \Psi$ be a measurable set, such that

$$\bar{E}(\chi) = \left\{ \bar{d}_n(\chi), \text{for } \chi \text{ belong in } S \right\}$$

$$\bar{E}(\bar{c}(\chi)), \text{for } \chi \text{ not belong in } S$$

Hence (41a) becomes

$$\int_{\bar{E}} \overline{H}_d \overline{H}_d \cdot (\bar{d}_n - \bar{d}) \, d\gamma \geq 0 \, , \quad \text{for any } S \subset \partial \Psi \, .$$

Then, by using Theorem (2) in reference [19]. we obtain

$$H_d \Delta d \cdot (\bar{d}_n - \bar{d}) \geq 0 \, , \quad \text{a.e on } \partial \Psi.$$}

The above inequality satisfies on $\partial \Psi$, except in a subset $\partial \Psi_n$ with $\tau(\partial \Psi_n) = 0$, for each $n$, where $\tau$ is a Lebesgue measure, then this equality holds on $\partial \Psi$ except in $\bigcup_n \partial \Psi_n$ with $\tau(\bigcup_n \partial \Psi_n) = 0$. But $\{d_n\}$ is a dense in $\bar{E}$, then there exists $d \in \bar{E}$, such that

$$H_d \overline{H}_d \cdot d = \min_{\bar{c} \in \bar{E}} H_d \overline{H}_d \overline{c} \, , \quad \text{a.e. on } \partial \Psi \, .$$

6. The Sufficient Conditions for Optimality of the Continuous Classical Boundary Optimal Control Vector

Theorem (7): In addition to assumptions (I), (II), and (III), if $m_1, m_2, m_3, v_{11}, v_{12}, v_{13}$ are affine with respect to $\bar{b}, v_{14}, v_{15}, v_{16}$ are affine with respect to $\bar{d}$, $\xi_1, \xi_2, \xi_3$ are bounded functions for $\chi$. Also, if $\gamma_{adj}(P = 0, \sigma = 1,2,3,4,5,6)$ are convex with respect to $b_1, b_2, b_3, d_1, d_2, d_3$, respectively, for each $\chi$, then the necessary and sufficient conditions for optimality in the previous theorem (6), with $\xi_0 > 0$, are sufficient.

Proof: Assume that $\bar{d} \in \bar{E}_A$, $\bar{d}$ satisfies the conditions (39a) and (39b).

Let

$$\gamma(d) = \sum_{p=0}^{P} \xi_P \gamma(d) \Rightarrow \overline{\gamma(d)} \Delta d = \sum_{p=0}^{P} \xi_P \int_{\partial \Psi} \left(\xi_1 + v_{adj}\right) \Delta d_1 + (\xi_2 + v_{adj}) \Delta d_2$$

$$+ (\xi_3 + v_{adj}) \Delta d_3) d\gamma$$

$$= \int_{\partial \Psi} H_d(\chi, \xi_1, \xi_2, \xi_3, d_1, d_2, d_3) \cdot \Delta d \, d\gamma \geq 0$$

and

$$m_1(\chi, b_1) = m_{11}(\chi)b_1 + m_{12}(\chi), \quad m_2(\chi, b_2) = m_{21}(\chi)b_2 + m_{22}(\chi).$$
ions of elliptic type

Let \( \bar{d} = (d_1, d_2, d_3) \), \( \bar{\bar{d}} = (\bar{d}_1, \bar{d}_2, \bar{d}_3) \) be given two continuous classical boundary optimal control vectors, then the corresponding "state" solution vector are \( \bar{b} = (b_1, b_2, b_3) \), \( \bar{b} = (\bar{b}_1, \bar{b}_2, \bar{b}_3) \). By substituting \( (\bar{b}, \bar{\bar{d}}) \) in (1)-(6) and multiplying the resulting equations by \( \theta \in [0, 1] \) once, we again the substitution of the pair \( (\bar{b}, \bar{\bar{d}}) \) in (1)-(6). By multiplying the result by \( \bar{\theta} = (1 - \theta) \), and finally summing each pair from the corresponding equations together, we get:

\[
A_1(\theta b_1 + \bar{\theta} \bar{b}_1) + (\theta b_2 + \bar{\theta} \bar{b}_2) - (\theta b_3 + \bar{\theta} \bar{b}_3) + m_{12}(\chi) (\theta b_1 + \bar{\theta} \bar{b}_1)
\]

\[
\sum_{i=0}^{2} a_{i1} \frac{\partial}{\partial n_1} (\theta b_1 + \bar{\theta} \bar{b}_1) = \theta d_1 + \bar{\theta} \bar{d}_1
\]

\[
A_2(\theta b_2 + \bar{\theta} \bar{b}_2) + (\theta b_2 + \bar{\theta} \bar{b}_2) + (\theta b_3 + \bar{\theta} \bar{b}_3) + m_{22}(\chi) (\theta b_2 + \bar{\theta} \bar{b}_2)
\]

\[
\sum_{j=0}^{2} a_{2j} \frac{\partial}{\partial n_2} (\theta b_2 + \bar{\theta} \bar{b}_2) = \theta d_2 + \bar{\theta} \bar{d}_2
\]

and

\[
A_3(\theta b_3 + \bar{\theta} \bar{b}_3) + (\theta b_3 + \bar{\theta} \bar{b}_3) + (\theta b_3 + \bar{\theta} \bar{b}_3) + m_{32}(\chi) (\theta b_3 + \bar{\theta} \bar{b}_3)
\]

\[
\sum_{i=0}^{2} a_{3i} \frac{\partial}{\partial n_3} (\theta b_3 + \bar{\theta} \bar{b}_3) = \theta d_3 + \bar{\theta} \bar{d}_3
\]

Now, if we have the continuous classical boundary optimal control vector \( \bar{\bar{d}} = (\bar{d}_1, \bar{d}_2, \bar{d}_3) \) with \( \bar{d}_1 = \theta u_1 + \bar{\theta} \bar{u}_1 \), \( \bar{d}_2 = \theta u_2 + \bar{\theta} \bar{u}_2 \) and \( \bar{d}_3 = \theta u_3 + \bar{\theta} \bar{u}_3 \). Then, from (42a,b), (43a,b), and (44a,b), one gets that the "state" solution vector \( \bar{b}_1 = b_1 d_1, \bar{b}_2 = b_2 d_2, \bar{b}_3 = b_3 d_3 \) with \( \bar{b}_1 = \theta b_1 u_1 + \theta b_1 \bar{b}_1 \), \( \bar{b}_2 = \theta b_2 u_2 + \bar{\theta} \bar{b}_2 \), \( \bar{b}_3 = \theta b_3 u_3 + \theta b_3 \bar{b}_3 \) are their corresponding solution, i.e. they satisfy (1)-(6), respectively. So, the operators \( d_1 \mapsto b_1 d_1, d_2 \mapsto b_2 d_2, \) and \( d_3 \mapsto b_3 d_3 \) are convex-linear with respect to \( (b_1, d_1), (b_2, d_2) \) and \( (b_3, d_3) \), respectively.

Now, from this result and since \( v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16} \) are affine with respect to \( b_1, b_2, b_3, d_1, d_2, d_3 \), respectively, on \( \Psi \), we get that \( \forall \chi \in \Psi \), \( \bar{v}(\bar{d}) \) is convex-linear in \( (\bar{b}, \bar{\bar{d}}) \). Also, since \( (\forall \chi \in \Psi \) \( v_{p1}, v_{p2}, v_{p3}, v_{p4}, v_{p5}, v_{p6} \) are convex with respect to \( b_1, b_2, b_3, d_1, d_2 \) and \( d_3 \), respectively, i.e. \( \bar{v}(\bar{d}) \) is convex with respect to \( \bar{b} \) and \( \bar{\bar{d}} \), then \( \bar{v}(\bar{d}) \) is convex in \( \bar{b} \) and \( \bar{\bar{d}} \) in the convex set \( \bar{E} \) and has a continuous Fréchet derivative that satisfies

\[
\bar{\bar{v}}(\bar{d}) \cdot \Delta \bar{d} \geq 0 \Rightarrow \bar{v}(\bar{d}) \text{ has a minimum at } \bar{d}, \text{ i.e. } \bar{v}(\bar{d}) \leq \bar{v}(\bar{\bar{d}}) \text{, } \forall \bar{\bar{d}} \in \bar{E}, \text{ then we have}
\]

\[
\sum_{i=0}^{2} \xi_i \bar{\bar{v}}(\bar{d}) \leq \sum_{i=0}^{2} \xi_i \bar{v}(\bar{\bar{d}})
\]

Now, let \( \bar{E} \) be also admissible and satisfies the Transversality condition, then (45) becomes \( \bar{v}(\bar{d}) \leq \bar{v}(\bar{\bar{d}}) \), \( \forall \bar{\bar{d}} \in \bar{E} \), i.e. \( \bar{d} \) is a classical continuous boundary control vector problem.

Conclusions

The existence and uniqueness theorem for the "state" solution vector of the triple nonlinear partial differential equations of elliptic type is proved successfully, when the classical continuous boundary control vector is given. The proof of the existence of the classical continuous boundary control vector, ruling by the considered triple nonlinear partial differential equations of elliptic type, is demonstrated with the equality and inequality constraints. The studying of the existence solution of the triple adjoint equations related with the triple nonlinear partial differential equations of elliptic type is demonstrated with the equality and inequality constraints. Finally, the theorems of both the necessary and sufficient conditions for optimality of the triple nonlinear partial differential equations of elliptic type, through the Kuhn-Tucker-Lagrangian’s Multipliers with equality and inequality constraints, is demonstrated.
References
1. Grigorenko, N. L., Grigorieva, É. V., Roi, P. K., and Khailov, E. N. 2019. Optimal control problems for a mathematical model of the treatment of psoriasis. *Computational mathematics and modeling*, 30(4): 352-363.

2. Kahina, L., Spiteri, P., Demim, F., Mohamed, A., Nemra, A., and Messine, F. 2018 Application optimal control for a problem aircraft flight. *Journal of engineering science and technology review*, 11(1): 156-164.

3. Aderinto, Y. O., Afolabi, A. O., and Issa, I. T. 2017. On optimal planning of electric power generation systems. *Journal of mathematics*, 50(1): 89-95.

4. Kryazhimskii, A. V., and Taras’ev, A. M. 2016. *Optimal control for proportional economic growth*. Pleiades publishing. Ltd, 293(1): S101-S119.

5. Afshar, M., Merrikh-Bayat, F., and Razvan, M. R. 2016. Stepwise solution for optimal Control problems. *Journal of science and engineering*, 13(2): 024-037.

6. Mabonzo.V. D., Ampini, D. 2019. Existence of optimal control for a nonlinear partial Differential equation of hyperbolic-type. *European Journal of Pure And Applied Mathematics*, 12(4):1595-1601.

7. Kadhem, G.M. 2015. The continuous classical optimal control problem of partial differential equations. M.Sc. Thesis, Department of mathematics, College of Science, Mustansiriya University, Iraq.

8. Al-Rawdanee, E.H.M. 2015. The continuous classical optimal control problem of a non-Linear partial differential equations of elliptic type. M.Sc. Thesis, Department of mathematics, College of Science, Mustansiriya University, Iraq.

9. Al-Hawasy, J. 2019. The continuous classical boundary optimal control of couple nonlinear hyperbolic boundary value problem with equality and inequality constraints. *Baghdad science journal*, 16(4): 1064-1074.

10. Kadhem, G.M. 2015. The continuous classical optimal control problem of partial Differential equations. M.Sc. Thesis, Department of mathematics, College of Science, Mustansiriya University, Iraq.

11. Al-Hawasy, J. and Al-Qaisi, S. 2019. The solvability of the continuous classical Boundary optimal control of couple nonlinear elliptic partial differential equations with State constraints. *Al-Mustansiriya journal of science*, 30(1):143-151.

12. Al-Hawasy, J. and Naeif, A. 2017. The continuous classical boundary optimal control of A couple nonlinear parabolic partial differential equations. 1st Scientific international conference, Special Issus: 123-136.

13. Al-Hawasy, J. and Ali, Lamya. 2020. Constraints Optimal Control Governing by Triple Nonlinear Hyperbolic Boundary Value Problem. *Journal of Applied Mathematics, Hindawi, 2020*: 1-14.

14. Al-Hawasy, J. and Jaber, M. 2020 The continuous classical optimal control governing by triple parabolic boundary value problem. *Ibn Al-Haitham for Pure and Applied Science*, 33(1): 129-142.

15. Al-Hawasy, J. and Jasim, D. 2020. The continuous classical optimal control problems of a triple elliptic partial differential equations. *Ibn Al-Haitham for pure and applied science*, 33(1):143-151.

16. Al-Hawasy, J. 2019. Solvability for continuous classical optimal control associated with triple hyperbolic boundary value problem. Accepted in Ijpam.

17. Al-Hawasy, J. and Jaber, M. 2019. The continuous classical boundary optimal control Vector governing by triple linear partial differential equations of parabolic type. Accepted in *Ibn Al-Haitham for pure and applied science*.

18. Borzabadi, A. H., Kamyad, A. V., and Farahi, M. H. G. 2 Optimal Control of the Heat Equation in an Inhomogeneous Body. *J. Appl. Math. and Computing*, 15(1-2): 127-146

19. Chryssoverghi, I. and Bacopoulos, A. 1993. Approximation of Relaxed Nonlinear Parabolic Optimal Control Problems. *Journal of Optimization Theory and Approximations*, 77(1).

20. Bacopoulos, A. and Chryssoverghi, I. 2003. Numerical Solutions of Partial Differential Equations by Finite Elements Methods. *Symeon Publishing Co, Athens.*

21. Brezis, H. 2011. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, New York-USA