RIEMANNIAN AND SUB-RIEMANNIAN GEODESIC FLOWS

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Abstract. In the present paper we show that the geodesic flows of a sub-Riemannian metric and that of a Riemannian extension commute if and only if the extended metric is parallel with respect to a certain connection. This helps us to describe the geodesic flow of sub-Riemannian metrics on totally geodesic Riemannian submersions. As a consequence we can characterize sub-Riemannian geodesics as the horizontal lifts of projections of Riemannian geodesics.

1. Introduction

Since the introduction of sub-Riemannian geometry in 1986, see [18], one of the main topics of research has been finding important geometric invariants. A sub-Riemannian manifold is a connected manifold $M$ with a smoothly varying inner product $h$ defined only on a subbundle $H$ of the tangent bundle. Such spaces have a metric structure by considering the distance between two points as the infimum of the length of all curves tangent to $H$ connecting them. Understanding what the proper generalization of curvature should be for such spaces, has been a topic of great interest in recent years.

There are currently two main ways of attacking this problem. One approach considers symplectic invariants of the (normal) geodesic flow on sub-Riemannian spaces. For results in this direction, see e.g. [1] [2] [13] [19]. The second approach tries to understand curvature in terms of properties of the heat flow corresponding to a second order differential operator, known as sub-Laplacian, which is the sub-Riemannian analogue of the Laplace-Beltrami operator. Unlike the first mentioned approach, the second one requires a extension of $h$ to a Riemannian metric satisfying certain properties. Such an extension will not be unique in general, see [10] Section 4.5. However, once an appropriate choice has been made, one has at hand powerful results such as a parabolic Harnack-inequality and a Bonnet-Myers theorem, see e.g. [3] [4] [5] [11].

In the present paper we give an initial step to bring together both approaches, by studying the Riemannian and sub-Riemannian geodesic flows.

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Our main result states that these flows commute if and only if a Riemannian metric taming $h$ is parallel with respect to an appropriate connection. This requirement also appears as a hypothesis in the second approach to curvature described above, see Remark 2.3. Moreover, we prove that requiring that the projection of the Riemannian and sub-Riemannian geodesic flows to the base space of a submersion coincide is equivalent to requiring the fibers of the submersion to be totally geodesic. This generalizes a result found in [14], where it was shown that the trajectories of particles with a given gauge in a Yang-Mills field can be considered as projections of both sub-Riemannian and Riemannian geodesics on a principal bundle, given that the gauge group has a bi-invariant metric. See Example 2.9 for more details.

The structure of the paper is as follows. In Section 2 we introduce the main concepts that we will use and state our results. We postpone the proofs to the next section, for the sake of clarity. Section 2 concludes with two relevant examples. The technical tools and the proofs of our results are presented in Section 3.

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1.2. Notation and conventions. All manifolds are smooth and connected. For any vector bundle $E \to M$ over a manifold $M$, we will use $\Gamma(E)$ for the space of all smooth sections of $E$. For any vector field $X \in \Gamma(TM)$, we will write $L_X$ for the Lie derivative with respect to $X$ and $e^{tX}$ for its local flow on $M$. If $E$ is a subbundle of the tangent bundle $TM$, then $\text{Ann}(E)$ denotes the subbundle of $T^*M$ of all covectors that vanish on $E$.

2. Geodesic flows: Statement of the results

2.1. Sub-Riemannian manifolds. A sub-Riemannian manifold is a triple $(M, \mathcal{H}, h)$, where $M$ is a (connected) manifold, $\mathcal{H}$ is a subbundle of the tangent bundle $TM$ and $h$ is a metric tensor defined only on $\mathcal{H}$. Equivalently, it can be considered as a pair $(M, h^*)$, where $M$ is a manifold and $h^*$ is a bilinear positive semidefinite tensor of the cotangent bundle that vanishes on a subbundle of $T^*M$. We will call $h^*$ a sub-Riemannian cometric. The relation between $(\mathcal{H}, h)$ and $h^*$ can be described as follows. Let $\mathcal{H}$ be the image of the map $\sharp h^*$ given by

\begin{equation}
\sharp h^* : T^*M \to TM, \quad p \mapsto h^*(p, \cdot).
\end{equation}

and endow $\mathcal{H}$ with a metric tensor $h$ determined by equation

\begin{equation}
h(\sharp h^* p_1, \sharp h^* p_2) := h^*(p_1, p_2), \quad p_j \in T_x M, x \in M, j = 1, 2.
\end{equation}

Conversely, given the pair $(\mathcal{H}, h)$, the cometric $h^*$ is uniquely determined by (2.1) and (2.2). The kernel of $\sharp h^*$ will be the subbundle $\text{Ann}(\mathcal{H})$ of $T^*M$.

An absolutely continuous curve in $(M, \mathcal{H}, h)$ is called horizontal if $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$ for almost every $t$. The distance in a sub-Riemannian manifold is
Proposition 2.1. Let $\mathbf{h}$ be the Carnot-Carathéodory metric defined so that $\mathbf{h}(x, y)$ is the infimum of all integrals $\int_0^1 \mathbf{h}(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2} \, dt$ taken over all horizontal curves $\gamma$ satisfying $\gamma(0) = x$ and $\gamma(1) = y$. This distance can only be finite if any two points can be connected by a horizontal curve. A sufficient condition for this to hold, is that $\mathcal{H}$ is bracket-generating, i.e. that the sections of $\mathcal{H}$ and their iterated brackets span $T\mathbb{M}$.

Minimizers of the distance $\mathbf{h}$ are either normal geodesics or abnormal curves. Normal geodesics are projections of integral curves of the Hamiltonian vector field $\bar{\mathcal{H}}^\mathbf{h}$ of the Hamiltonian $H^\mathbf{h}(p) := \frac{1}{2} \mathbf{h}^*(p, p)$. Such curves are always locally length minimizers and smooth. For the definition of abnormal curves and more details on sub-Riemannian manifolds in general, we refer to [15].

For future computations, the following description of normal geodesics with respect to an arbitrary affine connection $\nabla$ will be convenient.

**Proposition 2.1.** Let $\nabla$ be any affine connection on $M$ with torsion $T^\nabla$. Then a curve $\lambda(t)$ in $T^*M$ with projection $\gamma$ is an integral curve of $\bar{\mathcal{H}}^\mathbf{h}$ if and only if

\[
\dot{\gamma}(t) = \mathbf{h}^* \lambda(t), \quad \nabla_\gamma \lambda(t) = -\lambda(t)T^\nabla(\dot{\gamma}, \dot{\gamma}) + (\nabla, \mathbf{h}^*)\langle \lambda(t), \lambda(t) \rangle.
\]

Let $\Pi^M : T^*M \to M$ be the canonical projection of the cotangent bundle. For every $x \in M$, following [18], we define the sub-Riemannian exponential $\exp^\mathbf{hr}(x, \dot{\gamma}) : U_x \subseteq T^*M \to M$ by the formula

\[
(2.3) \quad \exp^\mathbf{hr}(x, p) := \lambda(1), \quad \lambda(t) := (\Pi^M \circ e^{t\bar{\mathcal{H}}^\mathbf{h}})(p),
\]

where $U_x$ is the collection of all $p \in T^*_x M$ such that (2.3) is well defined.

2.2. Taming sub-Riemannian metrics. Let $(M, \mathcal{H}, \mathbf{h})$ be a a sub-Riemannian manifold and let $\mathbf{g}$ denote a Riemannian metric on $M$ such that $\mathbf{g} |_\mathcal{H} = \mathbf{h}$. Such a Riemannian metric $\mathbf{g}$ is said to tame $\mathbf{h}$. Let $\nabla^\mathbf{g}$ denote the Levi-Civita connection associated to $\mathbf{g}$ and define $\mathcal{V}$ as the orthogonal complement of $\mathcal{H}$ with respect to $\mathbf{g}$. We will use $\text{pr}_\mathcal{H}$ and $\text{pr}_\mathcal{V}$ for the respective orthogonal projections to $\mathcal{H}$ and $\mathcal{V}$. We introduce a connection $\nabla$, which will play a central role in our results, as follows

\[
(2.4) \quad \nabla^\mathbf{v} X := \text{pr}_\mathcal{H} \nabla^\mathbf{g} \text{pr}_\mathcal{H} X \text{pr}_\mathcal{H} Y + \text{pr}_\mathcal{V} \nabla^\mathbf{g} \text{pr}_\mathcal{V} X \text{pr}_\mathcal{V} Y + \text{pr}_\mathcal{H} [\text{pr}_\mathcal{V} X, \text{pr}_\mathcal{H} Y] + \text{pr}_\mathcal{V} [\text{pr}_\mathcal{H} X, \text{pr}_\mathcal{V} Y].
\]

Define a metric tensor $\mathbf{v}$ on $\mathcal{V}$ by $\mathbf{v} := \mathbf{g} |_\mathcal{V}$. This corresponds to a (degenerate) cometric $\mathbf{v}^*$ on the cotangent bundle through the relations (2.1) and (2.2). This cometric defines a Hamiltonian function $H^\mathbf{v}(p) := \frac{1}{2} \mathbf{v}^*(p, p)$ for any $p \in T^*M$. Write $H^\mathbf{g} = H^\mathbf{h} + H^\mathbf{v}$, which is the Hamiltonian of the Riemannian metric $\mathbf{g}$. The following result relates the connection $\nabla$ with the Hamiltonian functions defined by the cometrics $\mathbf{h}^*$ and $\mathbf{v}^*$. 
Lemma 2.2. Let \(\{\cdot, \cdot\}\) denote the Poisson bracket with respect to the canonical symplectic form on \(T^* M\). Then \(\{H^h, H^v\} = \{H^h, H^g\} = 0\) if and only if \(\nabla g = 0\).

Let \(\exp^r(x, \cdot) : V_x \subseteq T_x M \to M\) denote the Riemannian exponential map with respect to \(g\) from the point \(x\). Let \(\sharp\) be the identification of \(T^* M\) with \(TM\) using \(g\). It is clear that \(\exp^r(x, t\sharp p) = (\Pi^M \circ e^{t\tilde{H}^h} \circ e^{t\tilde{H}^v})(p)\), since projections of the solutions to the Hamiltonian system with Hamiltonian \(H^g(p) = \frac{1}{2} g(\sharp p, \sharp p)\) are exactly the Riemannian geodesics. It follows from Lemma 2.2 that if \(\nabla g = 0\), then for \(p \in T^* M\), we have
\[
\exp^r(x, t\sharp p) = (\Pi^M \circ e^{t\tilde{H}^h} \circ e^{t\tilde{H}^v})(p) = (\Pi^M \circ e^{t\tilde{H}^v} \circ e^{t\tilde{H}^h})(p),
\]
for any value of \(t\) such that the above terms are well defined. Furthermore, \(e^{s\tilde{H}^h} \circ e^{t\tilde{H}^v}(p) = e^{t\tilde{H}^v} \circ e^{s\tilde{H}^h}(p)\) for any \(p \in T^* M\), and \(t, s \in \mathbb{R}\) such that both sides are well defined.

Remark 2.3. The connection \(\nabla\) was first introduced in \([5, 10, 11]\) as a tool to obtain generalized curvature-dimension inequalities for sub-Riemannian manifolds. Such inequalities connect a Riemannian metric \(g\) taming \(h\) with the second order operator \(\Delta^h\) given by
\[
\Delta^h f = \text{div} h^* df, \quad f \in C^\infty(M),
\]
where the divergence is with respect to the volume form of \(M\) defined by \(g\). It turns out that the possibility of choosing a Riemannian extension such that \(\nabla g = 0\) is essential for obtaining results such as a parabolic Harnack inequality for the heat flow of \(\Delta^h\). However, note that in this setting we also need the requirement that the trace of the map
\[
X \mapsto \text{pr}_{\mathcal{H}}[\text{pr}_V Y, \text{pr}_V[\text{pr}_{\mathcal{H}} Y, \text{pr}_{\mathcal{H}} X]],
\]
vanishes for any vector field \(Y\). Although the map (2.5) is written with vector fields \(X\) and \(Y\), it is in fact tensorial in both arguments. This latter requirement is needed to ensure that \(\Delta^h\) commutes with the Laplace-Beltrami operator of \(g\). See \([11\text{, Appendix A}]\) for details. Note that (2.5) vanishes identically if \(V\) is integrable, which is the case considered in Section 2.3.

Remark 2.4. All the results in this paper are still valid if we only require that \(v\) is a nondegenerate metric tensor on \(\mathcal{V}\). The same is also true if we consider pseudo sub-Riemannian metrics on \(\mathcal{H}\), assuming only that \(h\) is nondegenerate.

2.3. Totally geodesic Riemannian foliations. Let \((M, \mathcal{H}, h)\) be a sub-Riemannian manifold and let \(g\) be a Riemannian metric taming \(h\). Assume that the orthogonal complement \(\mathcal{V}\) of \(\mathcal{H}\) with respect to \(g\) is integrable, i.e. we assume that for any pair of vector fields \(Z, W \in \Gamma(\mathcal{V})\) we have \([Z, W]_x \in \mathcal{V}_x, x \in M\). By the Frobenius theorem, we know that there exists a foliation \(\mathcal{F}\) of \(M\) with leaves tangent to \(\mathcal{V}\).
Define $\nabla_v g$ as in (2.4). It is simple to verify that for any $v, w \in T_x M$, $x \in M$,
\[
(2.6) \quad (\nabla_v g)(w, w) = (\nabla_{pr_H v} g)(pr_H w, pr_H w) + (\nabla_{pr_V v} g)(pr_H w, pr_H w)
\]
It follows that $\nabla g = 0$ if and only if for any $X \in \Gamma(H)$ and $Z \in \Gamma(V)$, we have
\[
(2.7) \quad (\nabla X g)(Z, Z) = (L_X g)(Z, Z) = 0,
\]
and
\[
(2.8) \quad (\nabla Z g)(X, X) = (L_Z g)(X, X) = 0.
\]
To explain the geometric meaning of (2.7), let $II$ be the second fundamental form of the leaves of the foliation $F$. This is a symmetric, bilinear vector valued tensor defined by $II(Z, W) := pr_H \nabla^g pr_V Z pr_V W$. If $II = 0$, then $F$ is called totally geodesic. This means that the leaves of $F$ are totally geodesic submanifolds of $M$, i.e. if $F_x$ is the leaf of $F$ containing $x \in M$ and $v \in T_x F_x = \mathcal{V}_x \subseteq T_x M$, then the curve $\gamma(t) = \exp(x, tv)$ is contained in $F_x$. Using the definition of the Levi-Civita connection, it follows that $(L_X g)(Z, Z) = -2 g(X, II(Z, Z))$, hence, (2.7) is equivalent to $F$ being a totally geodesic foliation.

To understand the meaning of (2.8), let us first start with the definition of a Riemannian submersion. Let $\pi : (M, g) \to (B, \tilde{g})$ be a submersion between two Riemannian manifolds with $\mathcal{V} = \ker \pi$, and $H = \mathcal{V}^\perp$. The submersion $\pi$ is called Riemannian if
\[
g(v, w) = \tilde{g}(\pi_* v, \pi_* w), \quad \text{for any } v, w \in H.
\]
Let $F$ be a foliation on a Riemannian manifold $(M, g)$ satisfying (2.8). Then for every $x \in M$, there exists a neighborhood $U$ of $x$ such that the quotient $B = U/F|_U$ is a well defined manifold that can be given a metric $\tilde{g}$ such that the quotient map $\pi : (U, g|_U) \to (B, \tilde{g})$ is a Riemannian submersion. Hence, we call such foliations Riemannian, since they can locally be obtained from a Riemannian submersion, see e.g. [17] for more details.

Using Lemma 2.2, we have the following result

**Theorem 2.5.** Let $(M, g)$ be a Riemannian manifold and let $F$ be a totally geodesic Riemannian foliation of $M$ corresponding to an integrable subbundle $\mathcal{V}$. Define a sub-Riemannian manifold $(M, H, h)$ where $H$ is the orthogonal complement of $\mathcal{V}$ and $h = g|_H$. Then, for any $x \in M$ and $p \in T^* M$, if $P_t$ denotes parallel transport of vectors along $\exp^r(x, t^* p)$ with respect to the Levi-Civita connection, we have
\[
\exp^{sr}(x, tp) = \exp^r \left( \exp^r(x, t^* p), -t \right) P_t \right| P_t \right| p,
\]
for any $t$ such that both sides are well defined.

**Remark 2.6.** It follows from the proof of this theorem, found in Section 3.5, that we could have also written
\[
\exp^{sr}(x, tp) = \exp^r \left( \exp^r(x, -t \right) P_t \right| p, t P_t \right| p,
\]
where \( \tilde{P}_t \) denotes the parallel transport along \( \exp^r(x, -\text{pr}_V t\sharp p) \) with respect to the Levi-Civita connection.

2.4. Submersions and sub-Riemannian geometry. Let \((B, \tilde{g})\) be a Riemannian manifold and let \( \pi : M \to B \) be a submersion into \( B \) with vertical bundle \( \mathcal{V} := \ker \pi_\ast \). Since the vector fields with values in \( \mathcal{V} \) are exactly the vector fields on \( M \) that are \( \pi \)-related to the zero section of \( TB \), \( \mathcal{V} \) is an integrable subbundle. The leaves of the corresponding foliation are given by submanifolds \( M_b := \pi^{-1}(b) \), \( b \in B \). Moreover, \( M_b \) is an embedded submanifold since any \( b \in B \) is a regular value of \( \pi \).

An Ehresmann connection \( H \) on \( \pi \) is a subbundle satisfying \( TM = H \oplus V \).

Each Ehresmann connection \( H \) on \( \pi \) gives us a sub-Riemannian structure \((H, h)\) by lifting \( \tilde{g} \), i.e.

\[
h(v, w) = \tilde{g}(\pi_\ast v, \pi_\ast w), \quad v, w \in H_x, x \in M.
\]

Let \( g \) be any Riemannian metric on \( M \) taming \( h \). We are interested in a “good” way of choosing the Riemannian metric \( g \), in the sense that we want to consider when \( g \) satisfies the property

\[
\text{For any } x \in M, \text{ there is a neighborhood } U_x \text{ of } 0 \in T_x^* M \text{ such that for any } p \in U_x, \text{ the curves } \gamma(t) = \exp^{sr}(x, tp) \text{ and } \\
\eta(t) = \exp^r(x, t\sharp p), 0 \leq t \leq 1, \text{ have the same projection in } M.
\]

Related to this condition, we have the following result for submersions.

**Theorem 2.7.** The condition \((A)\) holds if and only if

1. \( \mathcal{V} \) is the orthogonal complement of \( H \).
2. The leaves of the foliation of \( \mathcal{V} \) are totally geodesic.

Note that the largest neighborhood \( U_x \) in condition \((A)\) is exactly the neighborhood of elements \( p \in T_x^* M \) such that both \( \exp^{sr}(x, p) \) and \( \exp^r(x, t\sharp p) \) are well defined. If both \( d\mathbb{H} \) and the distance induced by the Riemannian metric \( g \) are complete metrics, the latter being a sufficient condition for the first, then we may choose \( U_x = T_x^* M \), see [13, Section 7], thus obtaining a global version of Theorem 2.7.

**Remark 2.8.** An equivalent formulation of Theorem 2.7 is that any curve \( \exp^{sr}(x, tp) \) is the horizontal lift of the projection of the curve \( \exp^r(x, t\sharp p) \) if and only if (a) and (b) hold. For the definition of horizontal lifts of curves, see Section 3.1. Another equivalent formulation is that \((A)\) holds if and only if the foliation \( \mathcal{F} = \{M_b : b \in B\} \) is a totally geodesic Riemannian foliation.

2.5. Examples.

**Example 2.9** (Principal bundles). Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \), and let \( \langle \cdot, \cdot \rangle_\mathfrak{g} \) be a bi-invariant inner product on \( \mathfrak{g} \). Let \( \pi : M \to B \) be a right principal \( G \)-bundle over a Riemannian manifold \((B, \tilde{g})\). On \( M \), for each \( A \in \mathfrak{g} \), we have the canonical vector field \( \xi_A \) associated to the group
action, defined by
\[ \xi_A|_x = \left. \frac{d}{dt}(x \cdot \exp^G(tA)) \right|_{t=0}, \quad x \in M, \]
where \( \exp^G \) is the group exponential of \( G \). If \( \mathcal{V} = \ker \pi_* \), any element in \( \mathcal{V}_x \) can uniquely be represented as \( \xi_A|_x \) for some element \( A \in \mathfrak{g} \).

Let \( \mathcal{H} \) be an Ehresmann connection on \( \pi \) satisfying \( \mathcal{H}_x \cdot a = \mathcal{H}_{x-a} \) for any \( a \in G \). Then we have a corresponding connection form \( \omega \), which is a \( \mathfrak{g} \)-valued one-from uniquely determined by the properties \( \ker \omega = \mathcal{H} \) and \( \omega(\xi_A) = A \). Conversely, if \( \omega \) is any \( \mathfrak{g} \)-valued one-from satisfying \( \omega(\xi_A) = A \) and \( \omega(v \cdot a) = \text{Ad}(a^{-1})\omega(v) \), \( \ker \omega \) will always be an Ehresmann connection on \( \pi \) invariant under the group action.

Given a connection from \( \omega \), we define a Riemannian metric \( g \) on \( M \) by
\[
g(v, w) = \tilde{g}(\pi_*v, \pi_*w) + \langle \omega(v), \omega(w) \rangle_g.
\]
Let \( \nabla^g \) and \( \nabla^{\tilde{g}} \) denote the Levi-Civita connections for \( M \) and \( B \), respectively. It can then be verified that for any vector field \( \tilde{X} \) and \( \tilde{Y} \) on \( B \) and any \( A, A_1, A_2 \in \mathfrak{g} \),
\[
\begin{align*}
\nabla^{\tilde{g}}_{hX} hY &= h\nabla_X^{\tilde{Y}} + \frac{1}{2} \mathcal{R}(hX, hY), \\
\nabla^{\tilde{g}}_{\xi_A} \xi_{A_1} &= \frac{1}{2} g(\xi_{[A_1, A_2]}), \\
\nabla^{\tilde{g}}_{hX} \xi_A &= -\nabla^{\tilde{g}}_{\xi_A} hX = -\frac{1}{2} \tilde{g}(\xi_A, \mathcal{R}(hX, \cdot)),
\end{align*}
\tag{2.9}
\]
and these relations uniquely determine \( \nabla^g \). From (2.9) it follows that \( \{ M_b : b \in B \} \) is a totally geodesic Riemannian foliation.

We also have the following two observations.
(i) For any vector \( v \in TM \), \( (\nabla^g_\omega)(v) = 0 \), so for any geodesic \( \gamma \) in \( M \), \( \omega(\dot{\gamma}) = \omega(\text{pr}_Y \dot{\gamma}) \) is a constant.
(ii) Since each \( M_b \) is a totally geodesic submanifold of \( M \) and its metric comes from a bi-invariant metric, we have that for any \( v \in \mathcal{V}_x \),
\[
\exp^r(x, v) = x \cdot \exp^G(\omega(v)).
\]
We use (i) and (ii) to write the result of Theorem 2.5 as
\[
\exp^r(x, tp) = \exp^r(x, t\dot{p}) \cdot \exp^r(-t \text{pr}_Y P(t\dot{p})) = \exp^r(x, t\dot{p}) \cdot \exp^G(-t \omega(P(t\dot{p}))) = \exp^r(x, t\dot{p}) \cdot \exp^G(-t \omega(\dot{p})).
\tag{2.10}
\]
The latter relation was first observed in [15] Theorem 11.8. Projections of the curves in (2.10) are the trajectories of particles in \( B \) with gauge in \( \mathfrak{g}^* \) and with Yang-Mills field given by \( -\omega(\mathcal{R}(\cdot, \cdot)) \). For more information, see also [14] or [15] Chapter 12. See also [9] Section 2 for a generalization of this idea to general submersions.

Example 2.10 (Octonionic Hopf fibration). Let us consider the case of a Riemannian submersion \( \pi : S^m \to B \) with connected totally geodesic fibers, where \( S^m \) is the unit sphere with its usual round metric. According to [8] Theorem 3.5, such a Riemannian submersion is necessarily a Hopf fibration.
whenever $1 \leq \dim B \leq m - 1$. To be more specific, let $\mathbb{H}$ denote the division algebra of quaternions, and let $\mathbb{C}P^n$ and $\mathbb{H}P^n$ denote the complex and quaternionic projective $n$-spaces, respectively. Let $S^m(r)$ denote the $m$-dimensional sphere of radius $r$. Then any such submersion $\pi : S^m \to B$ is contained in the list

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n, \quad S^3 \to S^{4n+3} \to \mathbb{H}P^n, \quad n \geq 2,$$

$$S^1 \to S^3 \to S^2(\frac{1}{2}), \quad S^3 \to S^7 \to S^4(\frac{1}{2}), \quad S^7 \to S^{15} \to S^8(\frac{1}{2}).$$

With the exception of $S^7 \to S^{15} \to S^8(\frac{1}{2})$, all of these submersions can be given structures of principal $U(1)$- or $SU(2)$-bundles. The formulas of Example 2.3 have been successfully applied to study the normal sub-Riemannian geodesics for these fibrations listed above, see [8].

The fibration $S^7 \to S^{15} \to S^8(\frac{1}{2})$ is called the octonionic Hopf fibration and requires some more explanation. Here we follow [7, Section 6]. Let $\mathcal{O}$ denote the algebra of octonions with $\mathcal{O}P^1$ being the octonionic projective line. Consider the subsets of $\mathcal{O} \times \mathcal{O}$ given by

$$L_m = \{(u, mu) : u \in \mathcal{O}\} \quad \text{for } m \in \mathcal{O}, \quad L_\infty = \{(0, u) : u \in \mathcal{O}\}.$$

Given any point $(x, y) \in S^{15} \subset \mathcal{O} \times \mathcal{O} = \mathbb{R}^{16}$, we map it to the unique $m \in \mathcal{O}P^1$ such that $(x, y) \in L_m$. The octonionic Hopf fibration has no principal bundle structure, in fact, there are no non-vanishing vertical vector fields, see [16, Theorem A]. However our results, Theorem 2.5 and Theorem 2.7, do hold in this case.

3. Proofs

3.1. Connections, horizontal and vertical lifts. Let $\pi : M \to B$ be a submersion with Ehresmann connection $\mathcal{H}$ as defined in Section 2.4. Then for a given vector $\hat{v} \in T_xB$, the unique element $v \in \mathcal{H}_x$ such that $\pi_*v = \hat{v} \in T_{\pi(x)}B$ is called the horizontal lift of $\hat{v}$. We will write this element as $v = h_x\hat{v}$. Furthermore, if $\hat{X}$ is a vector field on $B$, then $h\hat{X}$ is the vector field on $M$ with values in $\mathcal{H}$ given by the formula $h\hat{X}|_x := h_x\hat{X}|_{\pi(x)}$.

For a given absolutely continuous curve $\hat{\gamma} : [0,T] \to B$, the horizontal lift $\gamma$ of $\hat{\gamma}$ to $x \in M_{\hat{\gamma}(0)}$ is the solution of the initial value problem

$$\dot{\gamma}(t) = h_{\gamma(t)}\hat{\gamma}, \quad \gamma(0) = x.$$  

This problem clearly has a unique solution, but the horizontal lift $\gamma(t)$ may in general only exist for sufficiently small values of $t$.

Let $\mathcal{V} = \ker \pi_*$ be the vertical bundle and let $\text{pr}_\mathcal{H}$ and $\text{pr}_\mathcal{V}$ be the respective projections to $\mathcal{H}$ and $\mathcal{V}$ with respect to the decomposition $TM = \mathcal{H} \oplus \mathcal{V}$. Then the curvature $\mathcal{R}$ of $\mathcal{H}$ is the vector valued two-form, given by equation

$$(3.1) \quad \mathcal{R}(v, w) = \text{pr}_\mathcal{V}[\text{pr}_\mathcal{H} X, \text{pr}_\mathcal{H} Y]|_x, \quad v, w \in T_xM,$$

where $X$ and $Y$ are any vector fields satisfying $X|_x = v$ and $Y|_x = w$. It is simple to verify that formula (3.1) is independent of the choice of vector fields $X$ and $Y$. 
We can use the same terminology in a more general setting and define the curvature $R$ of $\mathcal{H}$ by (3.1) whenever we have some decomposition of the tangent bundle into a direct sum $TM = \mathcal{H} \oplus \mathcal{V}$. However, in this case, $\mathcal{H}$ will also have a cocurvature $\overline{R}$ analogously given by

$$\overline{R}(v, w) = \text{pr}_H[\text{pr}_V X, \text{pr}_V Y]|_x, \quad X|_x = v, Y|_x = w.$$ 

Clearly $\overline{R} = 0$ if and only if $\mathcal{V}$ is integrable. For more information, see [12, Chapter III].

In what follows, we will also need vertical lifts, that exist whenever we have a vector bundle $\Pi : E \to M$ over a manifold. For any $e_1, e_2 \in E_x$, we define the vertical lift of $e_2$ at $e_1$ as

$$\text{vl}_{e_1} e_2 = \frac{d}{dt}(e_1 + te_2)|_{t=0} \in T_{e_1}E.$$ 

Note that any element of $\ker \Pi^*_\ast$ can be written as a vertical lift. Similarly, we can lift a section $\alpha \in \Gamma(E)$ of $E$ to a vector field $\text{vl} \alpha$ on $E$ by $\text{vl} \alpha|_p = \text{vl}_e \alpha|_{\Pi(e)}$. For us, the particular case of $\Pi^\ast M : T^\ast M \to M$ will be important and the fact that $\ker \Pi^\ast M$ are spanned by vertical lifts of one-forms.

### 3.2. Connections and the symplectic form

Let $\vartheta$ be the Liouville one-form on the cotangent bundle $T^\ast M$ given by $\vartheta|_p(v) = p(\Pi^\ast M v)$, and let $\varsigma = -d\vartheta$ be the canonical symplectic form. Consider any affine connection $\nabla$ on $M$ with torsion tensor $T^\nabla$. There is a unique Ehresmann connection $\mathcal{E}^\nabla$ on $\Pi^\ast M$ such that a smooth curve $\lambda(t)$ in $T^\ast M$ is tangent to $\mathcal{E}^\nabla$ if and only if $\lambda(t)$ is parallel along $\gamma(t) = \Pi^\ast M(\lambda(t))$, see [12, Chapter III]. Then we can write $T(T^\ast M) = \mathcal{E}^\nabla \oplus (\ker \Pi^\ast M)$, where $\mathcal{E}^\nabla$ is spanned by horizontal lifts of vector fields on $M$, while $\ker \Pi^\ast M$ is spanned by vertical lifts of forms on $M$. As a consequence, we can completely describe $\varsigma$ by its values on such elements.

Let $X$ and $Y$ be vector fields on $M$ with horizontal lifts $hX$ and $hY$ and let $\alpha$ and $\beta$ be forms on $M$ with vertical lifts $\text{vl} \alpha$ and $\text{vl} \beta$. Then it is simple to verify from the definition of $\varsigma$ that

$$\varsigma(hX|_p, hY|_p) = -p(T^\nabla(X, Y)), \quad \varsigma(hX|_p, \text{vl} \alpha|_p) = \alpha(X|_{\Pi^\ast M(p)}), \quad \varsigma(\text{vl} \alpha|_p, \text{vl} \beta|_p) = 0. \tag{3.2}$$

### 3.3. Proof of Proposition 2.1

For any (possibly degenerate) cometric $s^* \in \Gamma(\text{Sym}^2 TM)$, define the Hamiltonian $H^{s^*}(p) := \frac{1}{2} s^*(p, p)$. Define the vector $\sharp s^* p$ by $s^*(\alpha, p) = \alpha(\sharp s^* p)$. Let $\nabla$ be any connection on $M$ and write $\tilde{H}^{s^*} = A + B$ where $A$ and $B$ have values in $\mathcal{E}^\nabla$ and $\ker \Pi^\ast M$, respectively.
Then, for any vector field \( X \) and one-form \( \alpha \) on \( M \), we use (3.2) to get
\[
\begin{align*}
\tilde{H}^s(vl \alpha)|_p &= s^*(\alpha, p) = \alpha(h^s p) = \varsigma(h^s, vl \alpha) = \alpha((\Pi^* M) X)|_p, \\
\tilde{H}^s(hX)|_p &= \frac{1}{2}(\nabla_X s^*)(p, p) = \varsigma(h^s, hX)|_p \\
&= -p(T\nabla(\Pi^* M X)) - \varsigma(hX, B).
\end{align*}
\]
It follows that
\[
(3.3) \quad \tilde{H}^s = h_p s^* p - vl_p \left( pT\nabla(z^s p, \cdot) + \frac{1}{2}(\nabla, s^*)(p, p) \right).
\]
In order to obtain the result, put \( s^* = h^s \) and use that for any curve \( \lambda(t) \) in \( T^* M \) with projection \( \gamma(t) \), we have
\[
\dot{\lambda}(t) = h_{\lambda(t)} \dot{\gamma}(t) + vl_{\lambda(t)} \nabla \dot{\gamma}(t).
\]

### 3.4. Proof of Lemma 2.2

We begin by noting that the torsion \( T\nabla \) of \( \nabla \) is given by \( T\nabla = -R - \overline{R} \), where \( R \) and \( \overline{R} \) denote the curvature and the cocurvature of \( \mathcal{H} \) respectively. From equation (3.3), we get that the Hamiltonian vector fields are given by
\[
\begin{align*}
\tilde{H}^h|_p &= h_p z^h p + vl_p \left( pR(\cdot^h p, \cdot) + (\nabla, h^s)(p, p) \right) \\
\tilde{H}^v|_p &= h_p z^v p + vl_p \left( p\overline{R}(\cdot^v p, \cdot) + (\nabla, v^s)(p, p) \right).
\end{align*}
\]
It now follows that from equation (3.2) that
\[
\begin{align*}
\{ \tilde{H}^h|_p, \tilde{H}^v|_p \} &= \varsigma(\tilde{H}^h|_p, \tilde{H}^v|_p) \\
&= -pT\nabla(\cdot^h p, \cdot^v p) + p\overline{R}(\cdot^v p, h^s p) + (\nabla, h^s p, v^s)(p, p) \\
&= -pR(\cdot^h p, \cdot^v p) - (\nabla, h^s p, v^s)(p, p) \\
&= -(\nabla, h^s p, v^s)(p, p) - (\nabla, v^s p, h^s)(p, p)
\end{align*}
\]
Since \( \mathcal{H} \) and \( \mathcal{V} \) are orthogonal with respect to \( g \), we obtain that \( z^h p = pr_{\mathcal{H}} z p = pr^*_{\mathcal{H}} p \), and similar relations hold for \( z^v p \). Then
\[
(\nabla^{\mathcal{H}}_{\cdot^h} v^s)(\alpha, \alpha) = (pr_{\mathcal{H}} z\alpha) g(\alpha, pr_{\mathcal{V}} z\alpha) - 2(\nabla_{pr_{\mathcal{H}} z\alpha}) g(pr_{\mathcal{V}} z\alpha) = -2(\nabla_{pr_{\mathcal{H}} z\alpha}) g(pr_{\mathcal{V}} z\alpha),
\]
and similarly \((\nabla^{\mathcal{V}}_{\cdot^v} h^s)(\alpha, \alpha) = -(\nabla_{pr_{\mathcal{V}} z\alpha}) g(pr_{\mathcal{H}} z\alpha, pr_{\mathcal{V}} z\alpha)\). It follows that \( \{ \tilde{H}^h, \tilde{H}^v \} = 0 \) if and only if the map
\[
(3.5) \quad v \mapsto (\nabla_{pr_{\mathcal{V}} v} g)(pr_{\mathcal{H}} v, pr_{\mathcal{V}} v) - (\nabla_{pr_{\mathcal{V}} v} g)(pr_{\mathcal{V}} v, pr_{\mathcal{V}} v), \quad v \in TM,
\]
vanishes. Notice that the first term in the above map is bilinear in \( pr_{\mathcal{H}} v \) and linear in \( pr_{\mathcal{V}} v \) and vice versa for the second term. The map (3.5) is hence zero if and only if both
\[
(\nabla_{pr_{\mathcal{H}} v} g)(pr_{\mathcal{V}} w, pr_{\mathcal{V}} w) = 0 \quad \text{and} \quad (\nabla_{pr_{\mathcal{V}} v} g)(pr_{\mathcal{H}} w, pr_{\mathcal{H}} w) = 0
\]
holds for any \( v, w \in T M \). The result now follow from the identity (2.4).

3.5. Proof of Theorem 2.5. Write \( H^h = H^h + H^v \). From Lemma 2.2 we know that
\[
e^{-tH^v} = e^{tH^h} \circ e^{tH^v}(p),
\]
whenever both sides are defined. Now, since \( H^h \) is the Hamiltonian of a Riemannian metric, we have that \( \alpha(t) = e^{tH^h}(p) \) satisfies \( \Pi^M(\alpha(t)) = \exp_r(x, t\sharp p) \). Furthermore, if \( P_t \) is the parallel transport of vectors with respect to the Levi-Civita connection, then \( \sharp \alpha(t) = P_t \sharp p \).

Next, from equation (3.4), we know that \( \gamma(t) = e^{tH^v}(p) \) is a solution to
\[
\Pi^M(\gamma(t)) = \gamma(t), \quad \dot{\gamma}(t) = \nabla^*_{\gamma(t)} \lambda(t), \quad \nabla^*_\gamma \lambda(t) = 0, \quad \lambda(0) = p.
\]
However, since \( \nabla^*_\gamma \) preserves \( \text{Ann}(H) \) and \( \text{Ann}(V) \), we might as well consider equation (3.6) with \( \lambda(t) \) replaced by \( \lambda^V(t) = \pi^V_* \lambda(t) \), which satisfies \( \nabla^V_\gamma(0) = \pi^V_* \sharp p \). Finally, since \( \lambda(t) \) is a curve in \( \text{Ann}(H) \), \( \gamma(t) \) is tangent to \( V \) and \( F \) is a totally geodesic foliation, we have \( \nabla^*_\gamma \lambda(t) = \nabla^V_\gamma \lambda(t) = 0 \). This completes the proof.

3.6. Proof of Theorem 2.7. By Proposition 2.1 if \( \gamma(t) = \exp^{sr}(x, tp) \) and \( \eta(t) = \exp^v(x, t\sharp p) \), then \( \dot{\gamma}(0) = \sharp^h p \), while \( \dot{\eta}(0) = \sharp p \).

First assume that (A) holds. Let us pick an element \( p \in U_x \), such that \( \sharp^h p = 0 \). Then \( \gamma(t) = \exp^{sr}(x, tp) \) is a constant curve, so \( \eta(t) = \exp(x, t\sharp p) \) must be a contained in \( M_b \) with \( b = \pi(x) \). Hence, we must have \( \pi_\gamma \dot{\eta}(0) = \pi_* \sharp p = 0 \), so it follows that \( \sharp \) maps \( \text{Ann}(H) \) into \( V \), and so \( V \) is orthogonal to \( \mathcal{H} \). Furthermore, since any geodesic in \( M \) which starts in and is tangent to \( M_b \) remains in \( M_b \), we know that \( M_b \) is a totally geodesic submanifold.

The converse statement follows from Theorem 2.5 and the fact that (a) and (b) imply that the foliation \( \{ M_b : b \in B \} \) is a totally geodesic Riemannian foliation.

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