q-HEAT FLOW AND THE GRADIENT FLOW OF THE RENYI ENTROPY IN THE p-WASSERSTEIN SPACE

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ABSTRACT. Based on the idea of a recent paper by Ambrosio-Gigli-Savaré in Invent. Math. (2013), we show that flow of the q-Cheeger energy, called q-heat flow, solves the gradient flow problem of the Renyi entropy functional in the p-Wasserstein. For that, a further study of the q-heat flow is presented including a condition for its mass preservation. Under a convexity assumption on the upper gradient, which holds for all $q \geq 2$, one gets uniqueness of the gradient flow and the two flows can be identified. Smooth solution of the q-heat flow are solution the parabolic $q$-Laplace equation, i.e. $\partial_t f_t = \Delta_q f_t$.

The heat flow induced by a Dirichlet form is by now a well-understood concept. In [JKO98] Jordan, Kinderlehrer and Otto showed in the Euclidean setting that one can identify the heat flow with the gradient flow of the entropy functional in the 2-Wasserstein space. The main idea was to show that the solution of the gradient flow problem solves also the heat equation. Uniqueness of the solution implies that the two flows are identical. The identification of the heat flow and the gradient flow of the entropy functional on manifolds was later accomplished by Erbar [Erb10].

Otto [Ott96, Ott01] also gave a formal proof of how to use gradient flows in the $p$-Wasserstein spaces modeled on $\mathbb{R}^n$ in order to solve other equations like the porous media equation and the parabolic $q$-Laplace equation, i.e. the $q$-heat flow. Rigorous proofs were later given by Agueh [Agu02, Agu05]. Only recently Ohta and Takatsu [OT11a, OT11b] also showed that a similar construction works on manifolds if the functionals are $K$-convex.

All proofs until then required the contraction property which follows, at least in the Riemannian setting, from the curvature dimension condition introduced by Lott-Villani and Sturm [LV09, LV07, Stu06a, Stu06b]. Since this condition can be defined on any metric measure spaces it was believed that a similar identification holds also under such a condition. In [Gig09] Gigli gave a proof which did not require the contraction property. This proof let Ambrosio-Gigli-Savaré [AGS13] to define a new generalized gradient from which one gets a natural heat flow associated to a metric space. With the help of a calculus of the heat flow and its mass preservation they could show that the heat flow is a solution of the gradient flow problem of the entropy functional in the 2-Wasserstein space. Using a convexity of the square of the upper gradient of the entropy functional on gets uniqueness and hence the two flows are identical.

One of the main ingredient of the proof was the Kuwada lemma, i.e. if $\mu_t = f_t \mu$ is a solution of the heat flow and $|\dot{\mu}_t|$ is the metric derivative of $t \mapsto \mu_t$ in the
2-Wasserstein space $\mathcal{P}_2(M)$ then
\[ |\dot{\mu}_t|^2 \leq \int \frac{|
abla f_t|^2}{f_t} d\mu \]
where the right hand side is called the Fisher information of $f_t$. This was the “missing” ingredient, since it was long known that the derivative along the heat flow $t \mapsto f_t$ of the entropy functional is (minus) the Fisher information of $f_t$.

In [AGS11] Ambrosio-Gigli-Savaré showed the Kuwada lemma for $q \neq 2$, namely if $t \mapsto f_t$ is the $q$-heat flow such that the density is bounded from above and away from zero from below (implying the measure $\mu$ is finite), they showed
\[ |\dot{\mu}_t|^p \leq \int \frac{|
abla f_t|^q}{f_t^{p-1}} d\mu \]
where this time the metric derivative is taken in the $p$-Wasserstein space $\mathcal{P}_p(M)$, $t \mapsto \mu_t = f_t \mu$ is a solution of the $q$-heat flow and $p$ and $q$ are Hölder conjugates. A formal calculation reveals that the derivative of the following functional
\[ f \mapsto \frac{1}{(3-p)(2-p)} \int f^{3-p} - f d\mu, \]
called $(3-p)$-Rényi entropy, along the $q$-heat flow in the $p$-Wasserstein space is exactly minus the right hand side of the previous inequality, which can be called the $q$-Fisher information.

In this paper, we will follow [AGS13] and first develop a calculus of the $q$-heat flow to show mass preservation in the non-compact setting and that the formal calculation above holds in an abstract setting. In case $q > 2$ there is almost no restriction on the measure to get mass preservation besides a “not too bad” growth of the measure of a ball. The cases $q < 2$ are more restrictive. Using generalized exponential functions already known from information theory [OT11a, Section 3] one of the conditions can be stated as
\[ \int \exp_p(-Vp)d\mu < \infty \]
where $V(x) = C d(x, x_0)$ for some $C > 0$ and $\exp_p$ is the generalized exponential function which agrees with the usual exponential function and the condition with the condition stated in [AGS13]. In $\mathbb{R}^n$ this condition boils down to $q > \frac{2n}{n+1}$. However, the current proof requires the more restrictive condition
\[ \int V^p \exp_p(-Vp)d\mu < \infty. \]

In the second part under some assumptions on the functional, which hold assuming a curvature condition defined in a previous paper [Kel13], we show that the proof of [AGS13] can be adjusted to show that the $q$-heat flow solves the gradient flow problem of the Rényi entropy in $\mathcal{P}_p(M)$. For $q > 2$ we also get convexity of the $q$-the power of the upper gradient and hence uniqueness of the gradient flow. This implies that the $q$-heat flow and the gradient flow of the Rényi entropy can be identified. The current proof of the cases $q < 2$ requires the space to be compact and the measure be $n$-Ahlfors regular for some $n$ depending on $q$. However, this condition is satisfied on smooth manifolds if the the curvature condition $CD_p(0, N)$, defined in a previous paper [Kel13], holds for $N > n$. 
1. Preliminaries

In this part, we will introduce the main concepts used in this work. We will follow the notation used in [AGS13]. For a general introduction to the theory of optimal transport via 2-Wasserstein spaces see [Vil09], especially its Chapter 6 on Wasserstein spaces.

Let \((X,d)\) be a (complete) metric space and for simplicity we assume that \(X\) has no isolated points. As a convention we will always assume that \((M,d,\mu)\) is a locally compact metric space equipped with a locally finite Borel measure \(\mu\) and if not otherwise stated it is assumed to be geodesic (see below). Since we will also deal with spaces which are not locally compact (e.g. \((\mathcal{P}_p(M), w_p)\) with \(M\) non-compact), the sections below do not assume that \((X,d)\) is locally compact.

### Lipschitz constants and upper gradients.

Given a function \(f : X \to \mathbb{R} = [-\infty, \infty]\), the local Lipschitz constant \(|Df| : X \to [0, \infty]\) is given by

\[
|Df|(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y,x)}
\]

for \(x \in D(f) = \{x \in X \mid f(x) \in \mathbb{R}\}\), otherwise \(|Df|(x) = \infty\). The one sided versions \(|D^+f|\) and \(|D^-f|\), also called ascending slope (resp. descending slope)

\[
|D^+f|(x) := \limsup_{y \to x} \frac{[f(y) - f(x)]_+}{d(y,x)},
\]

\[
|D^-f|(x) := \limsup_{y \to x} \frac{[f(y) - f(x)]_-}{d(y,x)}
\]

for \(x \in D(f)\) and \(\infty\) otherwise, where \([r]_+ = \max\{0,r\}\) and \([r]_- = \max\{0,-r\}\). It is not difficult to see that \(|Df|\) is (locally) bounded iff \(f\) is (locally) Lipschitz.

The following lemma will be crucial to calculate the derivative of functionals along the gradient flow of the Cheeger energy.

**Lemma 1** ([AGS13, Lemma 2.5]). Let \(f, g : X \to \mathbb{R}\) be (locally) Lipschitz functions, \(\phi : \mathbb{R} \to \mathbb{R}\) a \(C^1\)-function with \(0 \leq \phi' \leq 1\) and \(\psi : [0, \infty) \to \mathbb{R}\) be a convex nondecreasing function. Setting

\[
\tilde{f} := f + \phi(g - f), \quad \tilde{g} = g - \phi(g - f)
\]

we have for every \(x \in X\)

\[
\psi(|D\tilde{f}|)(x) + \psi(|D\tilde{g}|)(x) \leq \psi(|Df|)(x) + \psi(|Dg|)(x).
\]

We say that \(g : X \to [0, \infty]\) is an upper gradient of \(f : X \to \mathbb{R}\) if for any absolutely continuous curve \(\gamma : [0, 1] \to D(f)\) the curve \(t \mapsto g(\gamma_s)|\dot{\gamma}_s|\) is measurable in \([0, 1]\) (with convention \(0 \cdot \infty = 0\)) and

\[
|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 g(\gamma_t) dt.
\]

It is not difficult to see that the local Lipschitz constant and the two slopes are upper gradients in case \(f\) is (locally) Lipschitz.
Relaxed slope and the Cheeger energy. In a metric space there is no natural gradient of $L^q$-functions which are not Lipschitz. Cheeger defined in [Cle99] a gradient via a relaxation procedure using slopes of Lipschitz function. In [AGS13, AGS11] Ambrosio-Gigli-Savaré used a more restrictive version of Cheeger’s original definition.

**Definition 2** ($q$-relaxed slope). A function $g \in L^q$ is a $q$-relaxed slope of $f \in L^2$ if there is a sequence of Lipschitz functions $f_n$ strongly converging to $f$ in $L^2$ such that $|Df_n|$ converges weakly (in $L^q$) to some $\tilde{g} \in L^q$ with $g \leq \tilde{g}$. We denote by $|\nabla f|_{*,q}$ the element of minimal $L^q$-norm among all $q$-relaxed slopes.

**Remark.** In order to apply the gradient flow theory of Hilbert spaces, we divert from the approach in [AGS11] and use approximations of $f$ in $L^2$ instead of $L^q$. Note that the proofs of [AGS11] also work in this setting if appropriate changes are made.

It was shown in [AGS11] that this definition, Cheeger’s original and two other definitions agree almost everywhere. However, if the space does not satisfy a local doubling condition and a local Poincaré inequality, then the $q$-relaxed slope might be different from the $q’$-relaxed slope if $q \neq q’$, see [DS13]. Nevertheless, we will drop the dependency on $q$ and just write $|\nabla f|_*$.

One can show that the relaxed slope is sublinear, i.e. $|\nabla(f+g)|_* \leq |\nabla f|_* + |\nabla g|_*$ almost everywhere, and satisfies a weak form of the chain rule, i.e. for any $C^1$-function $\phi : \mathbb{R} \to \mathbb{R}$, which is Lipschitz on the image of $f$, we have $|\nabla \phi(f)|_* \leq |\phi’(f)||\nabla f|_*$ with equality if $\phi$ is non-decreasing [AGS13, Proposition 4.8]. This can be easily proven for Lipschitz functions and their slopes, and follows by a cut-off argument also for functions and their relaxed slopes.

Now the $q$-Cheeger energy of the metric measure space $(M,d,\mu)$ is defined as

$$\text{Ch}_q(f) = \frac{1}{q} \int |\nabla f|^q d\mu$$

for all $f$ admitting a relaxed slopes, otherwise $\text{Ch}_q(f) = \infty$. Similarly, given a convex increasing function $L : [0,\infty) \to [0,\infty)$ with $L(0) = 0$, the $L$-Cheeger energy is defined as $\text{Ch}_L(f) = \int L(|\nabla f|)d\mu$. Then the $q$-Cheeger energy is nothing but the $L$-Cheeger energy for $L(r) = r^q/q$.

**Proposition 3.** Let $f, g \in D(\text{Ch}_q)$ and $\phi : \mathbb{R} \to \mathbb{R}$ be a nondecreasing contraction (with $\phi(0) = 0$ if $\mu(M) = \infty$) then $\mu$-almost everywhere in $M$

$$|\nabla(f + \phi(g - f))|^2 + |\nabla(g - \phi(g - f))|^2 \leq |\nabla(f)|^2 + |\nabla(g)|^2.$$

**Proof.** The proof follows along the lines of the proof of [AGS13, Proposition 4.8] using Lemma 1 (see [AGS13, Lemma 2.5]). \hfill $\square$

**Fisher information.** The Fisher information is the derivative of the entropy functional along the heat flow. The Kuwada lemma, a key tool of [AGS13] to identify the heat flow and the gradient flow of the entropy functional, shows that the square of the metric derivative in the 2-Wasserstein space along the heat flow is bounded from above by the Fisher information. In a different paper [AGS11] they showed that in the compact setting with density of the measure bounded from below and above, there is also a version of this along the $q$-heat flow in the $p$-Wasserstein space (see Lemma 19 for a precise version). For that reason we define the following $q$-Fisher information as follows.
Definition 4 (q-Fisher information). Let $q \in \left(\frac{1+\sqrt{5}}{2}, \infty\right)$. For a Borel function $f : M \to [0, \infty]$ we define the q-Fisher information $F_q(f)$ as

$$F_q(f) := r^{-q} \int |\nabla f^r|_q^q d\mu = qr^{-q} Ch_q(f^r)$$

where $q \neq \frac{1+\sqrt{5}}{2}$ and

$$r = 1 - \frac{p - 1}{q} = 1 - \frac{(p - 1)^2}{p}.$$

In case $q = \frac{1+\sqrt{5}}{2}$, note $q = p - 1$ and thus we define

$$F_q(f) = \int |\nabla \log f|_q^q d\mu = q Ch_q(\log f).$$

Remark. For $q \in \left(\frac{1+\sqrt{5}}{2}, \infty\right)$, we also have $r \in (0, 1)$, which will be our main interest for technical reasons. Nevertheless, all case $q \geq 2$ are covered. In the following, we will just write $r > 0$. Furthermore, notice that $N \geq 2$ and $1 - \frac{1}{N} = 3 - p$ implies $p = 2 + \frac{1}{N} \leq 2.5 < \frac{3+\sqrt{5}}{2}$. Thus only the cases $N \in (1, 2)$ remain to be covered. In the smooth setting $CD_p(K, N)$ with $N \in (1, 2)$ can only hold for 1-dimensional spaces.

Proposition 5. Let $r > 0$. Then for every Borel function $f : M \to [0, \infty]$ we have the equivalence

$$f \in D(F_q) \iff f \in L^{2r}(M, \mu) \quad \text{and} \quad \int_{\{f > 0\}} \frac{|\nabla f|^q}{f^{p-1}} d\mu < \infty$$

and in this case we have

$$F_q(f) = \int_{\{f > 0\}} \frac{|\nabla f|^q}{f^{p-1}} d\mu.$$

In addition, the functional is sequentially lower semicontinuous w.r.t. the strong convergence in $L^{2r}(M, \mu)$ and $L^2(M, \mu)$. If $p < 2$ then the functional is also convex.

Remark. Compare this to [AGS11, Remark 6.2] and [AGS13, Lemma 4.10]. And note that the statement $|\nabla f|_w \in L^1$ follows already from $f \in L^1$ and $\int \frac{|\nabla f|^q}{f^{p-1}} d\mu < \infty$ by applying the reverse Hölder inequality.

Proof. Similar to [AGS13, Lemma 4.10] first assume $f$ is bounded. Then note that $f \in D(F_q)$ requires $f^r \in L^2(M, \mu)$, i.e. $f \in L^{2r}(M, \mu)$ and by chain rule

$$|\nabla f^r|^q = r^q \frac{|\nabla f|^q}{f^{p-1}}.$$

Conversely, just use $\phi(r) = \sqrt{r + \epsilon - \sqrt{\epsilon}}$, apply the chain rule and let $\epsilon \to 0$. Convexity for $p < 2$ follows from [Bor97]. Since in that case $q \geq p$, we know $(x, y) \mapsto x^q/y^{p-1}$ is convex in $\mathbb{R}^2$. 

□
Absolutely continuous curves and geodesics. If \( I \subset \mathbb{R} \) is an open interval then we say that a curve \( \gamma : I \to X \) is in \( AC^p(I, X) \) (we drop the metric \( d \) for simplicity) for some \( p \in [1, \infty) \) if

\[
d(\gamma_s, \gamma_t) \leq \int_s^t g(r)dr \quad \forall s, t \in J : s < t
\]

for some \( g \in L^p(J) \). In case \( p = 1 \) we just say that \( \gamma \) is absolutely continuous. It can be shown [AGS08, Theorem 1.1.2] that in this case the metric derivative

\[
|\dot{\gamma}_t| := \limsup_{s \to t} \frac{d(\gamma_s, \gamma_t)}{|s-t|}
\]

with \( \lim \) for a.e. \( t \in I \) is a minimal representative of such a \( g \). We will say \( \gamma \) has constant (unit) speed if \( |\dot{\gamma}_t| \) is constant (resp. 1) almost everywhere in \( I \).

It is not difficult to see that \( AC^p(I, X) \subset C(\bar{I}, X) \) where \( C(\bar{I}, X) \) is equipped with the sup distance \( d^* \)

\[
d^*(\gamma, \gamma') := \sup_{t \in \bar{I}} d(\gamma_t, \gamma'_t).
\]

For each \( t \in \bar{I} \) we can define the evaluation map \( e_t : C(\bar{I}, X) \to X \) by

\[
e_t(\gamma) = \gamma_t.
\]

We will say that \((X, d)\) is a geodesic space if for each \( x_0, x_1 \in X \) where is a constant speed curve \( \gamma : [0, 1] \to X \) with \( \gamma_t = x_t \) and

\[
d(\gamma_s, \gamma_t) = |t-s|d(\gamma_0, \gamma_1).
\]

In this case, we say that \( \gamma \) is a constant speed geodesic. The space of all constant speed geodesics \( \gamma : [0, 1] \to X \) will be donated by \( \text{Geo}(X) \). Using the triangle inequality it is not difficult to show the following.

**Lemma 6.** Assume \( \gamma : [0, 1] \to X \) is a curve such that

\[
d(\gamma_s, \gamma_t) \leq |t-s|d(\gamma_0, \gamma_1)
\]

then \( \gamma \) is a geodesic from \( \gamma_0 \) to \( \gamma_1 \).

A weaker concept is a length space: In such spaces the distance between point \( x_0 \) and \( x_1 \in X \) is given by

\[
d(x_0, x_1) = \inf \int_0^1 |\dot{\gamma}_t|dt
\]

where the infimum is taken over all absolutely continuous curves connecting \( x_0 \) and \( x_1 \). In case \( X \) is complete and locally compact, the two concepts agree. Furthermore, Arzela-Ascoli also implies:

**Lemma 7.** If \((X, d)\) is locally compact then so is \((\text{Geo}(X), d^*)\) where \( d^* \) is the sup-distance on \( C(\bar{I}, X) \).
Geodesically convex functionals and gradient flows. A functional $E : X \to \mathbb{R} \cup \{+\infty\}$ is said to be $K$-geodesically convex for some $K \in \mathbb{R}$ if for each $x_0, x_1 \in D(E)$ there is a geodesic $\gamma \in \text{Geo}(X)$ connecting $x_0$ and $x_1$ such that

$$E(\gamma_t) \leq (1-t)E(\gamma_0) + tE(\gamma_1) - \frac{K}{2}(1-t)t^2(\gamma_0, \gamma_1).$$

In such a case it can be shown ([AGS08, Section 2.4]) that the descending slope is an upper gradient of $E$ and can be express as

$$|D^-E|(x) = \sup_{y \in X \setminus \{x\}} \left( \frac{E(x) - E(y)}{d(x, y)} + \frac{K}{2}d(x, y) \right)$$

In particular, it is lower semicontinuous if $E$ is. Furthermore, if $x : [0, \infty) \to D(E)$ is a locally absolutely continuous curve then

$$E(x_t) \geq E(x_s) - \int_s^t |\dot{x}_r||D^-E|(y_r)dr$$

for every $s, t \in [0, \infty)$ and $s < t$. Note by Young’s inequality we also have for any $p \in (1, \infty)$

$$E(x_t) \geq E(x_0) - \frac{1}{p} \int_0^t |\dot{x}_r|^p dt - \frac{1}{q} \int_0^t |D^-E|^q(x_r)dr.$$

Definition 8 (($E, p$)-dissipation inequality and metric gradient flows). Let $E : X \to \mathbb{R} \cup \{\infty\}$ be a functional on $X$ then we say that a locally absolutely continuous curve $t \mapsto q_t \in D(E)$ satisfies the $(E, p)$-dissipation inequality if for all $t \geq 0$

$$E(x_0) \geq E(x_t) + \frac{1}{p} \int_0^t |\dot{x}_r|^p dt + \frac{1}{q} \int_0^t |D^-E|^q(x_r)dr.$$

t $\mapsto x_t$ is a gradient flow of $E$ starting at $y_0 \in D(E)$ if

$$E(y_0) = E(x_0) + \frac{1}{p} \int_0^t |\dot{x}_r|^p dt + \frac{1}{q} \int_0^t |D^-E|^q(x_r)dr.$$

In the geodesically convex case we immediately see that if $t \mapsto x_t$ satisfies the $(E, p)$-dissipation inequality then it is a (generalized) gradient flow and

$$\frac{d}{dt}E(x_t) = -|\dot{x}_t|^p = -|D^-E|^q(x_t)$$

for almost all $t \in (0, 1)$.

Remark. The theory developed in [AGS08] covers mainly the case $p = 2$ and only mentioned the required adjustments. For a comprehensive treatment of the case $p \neq 2$ and even more general situations see [RMS08].

Wasserstein spaces. In this section, we will give a short introduction to the Wasserstein space $P_p(M)$; for an overview of its general properties see [Vil09, Chapter 6].

Fix some $x_0 \in M$ and let $P(M)$ be the set of probability measures on $M$. Denote by $P_p(M)$ the following set

$$\left\{ \mu \in P(M) \mid \int d^p(x, x_0)d\mu(x) \right\}.$$
It can be shown that the following object \( w_p(\cdot, \cdot) \) defines a complete metric on \( \mathcal{P}_p(M) \).

\[
\|x,y\|_p = \inf_{\pi \in \Pi(\mu_0, \mu_1)} \frac{1}{p} \int d^p(x, y) \, d\pi(x, y)
\]

where \( \Pi(\mu_0, \mu_1) \) is the set of all \( \pi \in \mathcal{P}(M \times M) \) with \( (p_1)_* \pi = \mu_0 \) and \( (p_2)_* \pi = \mu_1 \) with \( p_i \) be the projections to the \( i \)-th coordinate. We will say that \( (\mathcal{P}_p(M), w_p) \) is the \( p \)-Wasserstein space (modeled on \( (M, d) \)).

Furthermore, it is well-known that \( \mathcal{P}_p(M) \) is a geodesic/length space if \( M \) is. However, it is compact if and only if \( M \) is. In that case it agrees with the space of probability measures and the topology induced by \( w_p \) agrees with the weak topology on \( \mathcal{P}(M) \). Nevertheless, we have the following nice property:

**Lemma 9** ([Kel11, Theorem 6]). Let \( (M, d) \) be a proper metric space, then every bounded set in \( \mathcal{P}_p(M) \) is precompact w.r.t. to the weak topology induced by \( \mathcal{P}_p(M) \subset \mathcal{P}(M) \).

**Proof.** Let \( x_0 \) be some fixed measure in \( M \). By [Vil09, Lemma 4.3] we know that the \( \|x, y\|_p \) is lower semicontinuous w.r.t. to the weak convergence of measures. Thus we only need to prove tightness of every \( \|x, y\|_p \)-ball \( B^p_R(\delta_{x_0}) \subset \mathcal{P}_p(M) \), i.e. for every \( \varepsilon > 0 \) there is a compact set \( K_\varepsilon \subset M \) such that every \( \mu \in B^p_R(\delta_{x_0}) \)

\[
\mu(M \setminus K_\varepsilon) \leq \varepsilon.
\]

If we set \( K_\varepsilon = B^p_R(x_0) \) then

\[
\mu(M \setminus K_\varepsilon) \leq \varepsilon^p \int_{d(x, x_0) \geq \frac{1}{\varepsilon}} d^p(x, x_0) d\mu(x)
\]

\[
\leq \varepsilon \mu(pB^p_R(\delta_{x_0}, \mu) \leq \varepsilon pR^p
\]

which implies tightness since any ball in \( M \) is compact.

We say that a function \( E : \mathcal{P}_p(M) \rightarrow \mathbb{R} \cup \{\infty\} \) is weakly lower semi-continuous, if it is lower semicontinuous w.r.t. the weak topology on \( \mathcal{P}_p(M) \subset \mathcal{P}(M) \). In particular, the weak closure of bounded subset of sublevels of \( E \) are contained in that sublevel.

**Theorem 10.** Let \( (M, d) \) be a proper geodesic metric space and \( E \) be a functional on \( \mathcal{P}_p(M) \) such that \( E \) and \( |D^-E| \) are weakly lower semi-continuous. Then for all \( \mu_0 \in D(E) \) there exists a gradient flow \( t \mapsto \mu_t \) of \( E \) starting at \( \mu_0 \).

**Proof.** Just note by the previous lemma the assumptions [AGS08, Assumption 2.4a,c] hold and thus [AGS08, Corollary 2.4.12] can be applied.

**Remark.** The requirement \(|D^-E|\) to be weakly lower semi-continuous is rather restrictive in the non-compact case. Note, however, below we only need lower semi-continuity, which follows from \( K \)-convexity. Existence will follow from existence of the \( q \)-heat equation.

## 2. The functional

Given a function \( U \in \mathcal{D}^N \) for \( N \in [1, \infty) \) we write \( U'(\infty) = \lim_{r \rightarrow \infty} \frac{U(r)}{r} \). Let \( \mu \in \mathcal{P}(M) \) be some reference measure, we define the functional \( U_\mu : \mathcal{P}(M) \rightarrow \mathbb{R} \cup \{\infty\} \) by

\[
U_\mu(\nu) = \int U(\rho) d\mu + U'(\infty) \nu_s(M)
\]
where $\nu = \rho \mu + \mu_s$ the the Lebesgue decomposition of $\nu$ w.r.t. $\mu$.

In the following we usually fix a metric measure space $(M, d, \mu)$ and drop the subscript $\mu$ from the functional $\mathcal{U}_\mu$. In addition, we use $\mathcal{U}_m, \mathcal{U}_\alpha$ etc. to denote the functional generated by $\mathcal{U}_m, \mathcal{U}_\alpha$, etc.

Now let
\[
\mathcal{U}_{p}(x) = \frac{1}{(3-p)(2-p)}(x^{3-p} - x)
\]
and let $\mathcal{U}_p$ be the associated functional.

**Remark.** The linear term in $\mathcal{U}_p$ is just for cosmetic reasons, it does not have any influence. Take $U = c \cdot x$ with $c > 0$ and let $\mathcal{U}$ be the associated functional, then $U'(\infty) = c$ for $p \in (2, 3)$ and thus
\[
\mathcal{U}(\nu) = c \int f d\mu + c \cdot \nu^s(M) = c
\]
where $\nu = \rho \mu + \mu_s$ is the Lebesgue decomposition w.r.t. $\mu$. Therefore, we have $\mathcal{U}_p(\nu) = \mathcal{U}_p(\nu) - \frac{1}{(3-p)(2-p)}$ with $\mathcal{U}_p(x) = \frac{1}{(3-p)(2-p)}x^{3-p}$. For $p \in (1, 2)$ we have $\mathcal{U}_p(\nu) < \infty$ iff $\nu^s = 0$ and hence the linear term is constant as well.

Following the strategy in [AGS13, Section 7.2 and 8] we will show that under a curvature conditions the $q$-heat flow can be identified with the gradient flow of the function $\mathcal{U}_p$ in the $p$-Wasserstein space: More precisely, if $p \in (1, 2)$ then $3-p \in (1, 2)$ and the functional is displacement convex if the strong version of $CD_{p}(K, \infty)$ holds for some $K \geq 0$ (see [Kel13] for definition of $CD_{p}(K, N)$). If $p \in (2, 3)$ we have $3-p \in (0, 1)$ so that $\mathcal{U}_p$ is displacement convex if $CD_{p}(0, N)$ holds with $1 - \frac{1}{N} = 3-p$.

**Remark.** Note, that in contrast to the case $p = 2$, the strong version of $CD_{p}(K, \infty)$ does not imply $K$-convexity of functionals in $\mathcal{D}_{\infty}$ for $K < 0$ and $p < 2$. We get $K'$-convexity in those cases if the space is bounded (see [Kel13]). Also Ohta and Takatsu could show that on a weighted Riemannian manifold of non-negative Ricci curvature the functional $\mathcal{U}_p$ with $p \in (2, 3)$ is $K$-convex in $P_2(M)$ if $CD(K, N)$ holds (see [OT11a, Theorem 4.1]). This is, however, not enough for $p \neq 2$.

Recall from the introduction that $r > 0$ will be an abbreviation for $q \in \left(\frac{1+r}{2}, \infty\right)$, or equivalently $p \in (1, \frac{3+r}{2})$.

**Lemma 11.** Assume $r > 0$ then
\[
\nu \mapsto \mathcal{U}_p(\nu)
\]
is lower semicontinuous in $P_p(M)$.

**Proof.** Just note that $\mathcal{U}_p$ is convex and for $r > 0$ we have $p \in (1, \frac{3+r}{2}) \subset (1, 3)$ and thus $3-p > 0$. \qed

**Remark.** The functional $\mathcal{U}_p$ appeared in a similar form already in [Gig12, Proof of Lemma 3.13] and Otto’s preprint [Ott96] and also Augeh’s thesis [Agu02, Agu05]. Gigli used the functional and the gradient flow of the $q$-Cheeger energy to show that all gradients of $q$-Sobolev functions can be weakly represented by a plan. In the Euclidean case, Otto and Augeh showed that the parabolic $q$-Laplace equation, which is the $q$-heat flow for smooth solutions, can be solved using the gradient flow of $\mathcal{U}_p$ in the $p$-Wasserstein case. This should also be compared to [OT11a, OT11b],
where the (parabolic) porous media equation is solved via a gradient flow of a similar functional in the 2-Wasserstein space for Riemannian manifolds of non-negative Ricci curvature. Note, however, no identification is done. Furthermore, our approach shows that the abstract solution of the $q$-heat flow solves the gradient flow problem in the $p$-Wasserstein space.

3. **Gradient flow of the Cheeger energy in $L^2$**

We assume now that $\text{Ch}_q$ is the $q$-Cheeger energy on $(M, d, \mu)$ where $(M, d)$ is a proper metric space and $\mu$ is a $\sigma$-finite measure. From [AGS13, Proposition 4.1] we know that the domain of $\text{Ch}_q$ is dense in $L^2(M, \mu)$.

Since $L^2(M, \mu)$ is Hilbert and $\text{Ch}_q$ is convex and lower semicontinuous, we can apply the classical theory of gradient flows developed in [Bre73] (see also [AGS08]). For that recall that the subdifferential $\partial^- \text{Ch}_q$ at $f \in D(\text{Ch}_q)$ is defined as (possibly empty) set of functions $\ell \in L^2(M, \mu)$ such that for all $g \in L^2(M, \mu)$

$$
\int (\ell(g - f)) d\mu \leq \text{Ch}_q(g) - \text{Ch}_q(f).
$$

If $f \notin D(\text{Ch}_q)$ then $\partial \text{Ch}_q(f) = \emptyset$. The domain $D(\partial \text{Ch}_q)$ of $\partial^- \text{Ch}_q$ will be all $f \in L^2(M, \mu)$ such that $\partial^- \text{Ch}_q \neq \emptyset$, which is dense in $L^2(M, \mu)$ (see [Bre73, Proposition 2.11]).

By [Bre73] the gradient flow of $\text{Ch}_q$ gives for all $f_0 \in L^2(M, \mu)$ a locally Lipschitz map $t \mapsto f_t = H_t(f_t)$ (we drop $q$ if no confusion arises) from $(0, \infty)$ to $L^2(M, \mu)$ and $f_t \to f_0$ in $L^2(M, \mu)$ as $t \to 0$ and the derivative satisfies

$$
\frac{d}{dt} f_t \in -\partial^- \text{Ch}_q(f_t) \quad \text{for a.e. } t \in (0, \infty).
$$

**Definition 12** ($q$-Laplacian). Let $f \in D(\partial \text{Ch}_q)$ then $\Delta_q f$ is defined as the element $\ell \in -\partial^- \text{Ch}_q(f)$ of minimal $L^2$-norm.

By [Bre73, Theorem 3.2] we have the regularization effect that $\frac{d}{dt} f_t$ exists everywhere in $(0, \infty)$ and is the element $\ell \in -\partial^- \text{Ch}_q(f_t)$ with minimal $L^2$-norm, i.e. $\frac{d}{dt} f_t = \Delta_q f_t$.

**Remark.** We can also define the $L$-Laplacian using the same theory where $L$ is a convex increasing function with $L(0) = 0$. Since such flows might be interesting in combination with Orlicz-Wasserstein spaces, we will analyze these flows in the future.

**Proposition 13** (Properties of the Laplacian). If $f \in D(\Delta_q)$ and $g \in D(\text{Ch}_q)$ then

$$
- \int g \Delta_q f d\mu \leq \int |\nabla g|_s |\nabla f|_s^{-1} d\mu.
$$

Equality holds if $g = \phi(f)$ for some Lipschitz function $\phi : J \to \mathbb{R}$ with $J$ a closed interval containing the image of $f$ (and $\phi(0) = 0$ if $\mu(M) = \infty$). In that case one also has

$$
- \int \phi(f) \Delta_q f d\mu = \int \phi'(f) |\nabla f|_s^2 d\mu.
$$

If, in addition, $g \in D(\Delta_q)$ and $\phi$ is nondecreasing and Lipschitz on $\mathbb{R}$ with $\phi(0) = 0$ then

$$
\int (\Delta_q g - \Delta_q f) \phi(g - f) d\mu \leq 0.
$$
Proof. The first two parts were already proven in [AGS11, Proposition 6.5] for \( C^1 \)-functions \( \phi \). However, using the proof of [AGS13, Proposition 4.15], adapted to \( p \neq 2 \), it can be proven in the same way. For convenience we include the full proof:

Since \( -\Delta q f \in \partial^- \text{Ch}_q(f) \) we have for all \( \epsilon > 0 \)

\[
\text{Ch}_q(f) - \epsilon g \Delta_q f \leq \text{Ch}_q(f + \epsilon g).
\]

Furthermore, \(|\nabla f|^q_\ast + \epsilon|\nabla g|^q_\ast\) is a relaxed slope of \( f + \epsilon g \), we get

\[
- \int \epsilon g \Delta f \leq \frac{1}{q} \int (|\nabla f|^q_\ast + \epsilon|\nabla g|^q_\ast)^q - |\nabla f|^q_\ast d\mu
\]

\[
= \epsilon \int |\nabla g|_\ast |\nabla f|^q_\ast - 1 d\mu + o(\epsilon).
\]

Dividing by \( \epsilon \) and letting \( \epsilon \to 0 \) we obtain the result.

In case \( g = \phi(f) \) we apply chain rule and get \(|\nabla (f + \epsilon \phi(f))|^q_\ast = (1 + \epsilon \phi'(f))|\nabla f|^q_\ast\), and thus

\[
\text{Ch}_q(f + \epsilon \phi(f)) - \text{Ch}_q(f) = \frac{1}{q} \int |\nabla f|^q_\ast ((1 + \epsilon \phi(f))^q - 1) d\mu
\]

\[
= \epsilon \int \phi'(f)|\nabla f|^q_\ast d\mu + o(\epsilon).
\]

For the third part, just set \( h = \phi(g - f) \), then \( h \in \text{D}(\text{Ch}_q) \) and for \( \epsilon > 0 \)

\[
- \epsilon \int (\Delta_q f - \Delta_q g) h d\mu = - \epsilon \int \Delta_q f \cdot h d\mu - \epsilon \int \Delta_q g \cdot (-h) d\mu
\]

\[
\leq \text{Ch}_q(f + \epsilon h) - \text{Ch}_q(f) + \text{Ch}_q(g - \epsilon h) - \text{Ch}_q(g).
\]

Taking \( \epsilon \) sufficiently small such that \( \epsilon \phi \) is a contraction, we can apply Proposition 3 and conclude. \( \square \)

Actually with the help of Proposition 3 we can also prove:

**Proposition 14.** If \( f, g \in \text{D}(\Delta_L) \) and \( \phi \) is nondecreasing and Lipschitz on \( \mathbb{R} \) with \( \phi(0) = 0 \) then

\[
\int (\Delta_L g - \Delta_L f) \phi(g - f) d\mu \leq 0.
\]

**Proof.** As above set \( h = \phi(g - f) \), then \( h \in \text{D}(\text{Ch}_q) \) and for \( \epsilon > 0 \)

\[
- \epsilon \int (\Delta_L f - \Delta_L g) h d\mu = - \epsilon \int \Delta_L f \cdot h d\mu - \epsilon \int \Delta_L g \cdot (-h) d\mu
\]

\[
\leq \text{Ch}_L(f + \epsilon h) - \text{Ch}_L(f) + \text{Ch}_L(g - \epsilon h) - \text{Ch}_L(g).
\]

Then conclude by taking \( \epsilon \) sufficiently small and applying Proposition 3. \( \square \)

Using these results we can generalize [AGS13, Theorem 4.16] to the case \( p \neq 2 \) (and also [AGS11, Proposition 6.6] where \( 0 < c \leq f_0 \leq C < \infty \) is required).

**Theorem 15** (Comparison principle and contraction). Let \( f_t = H_t(f_0) \) and \( g_t = H_t(g_0) \) be the gradient flows of \( \text{Ch}_q \) starting from \( f_0, g_0 \in L^2(M, \mu) \) respectively. Then the following holds:

1. *(Comparison principle)* Assume \( f_0 \leq C \) (resp. \( f_0 \geq c \)). Then \( f_t \leq C \) (resp. \( f_t \geq c \)) for every \( t \geq 0 \). Similarly, if \( f_0 \leq g_0 + C \) for some constant \( C \in \mathbb{R} \), then \( f_t \leq g_t + C \).
(2) (Contraction) If \( e : \mathbb{R} \to [-1, \infty) \) is a convex lower semicontinuous function and \( E(f) = \int e(f) \, d\mu \) is the associated convex and lower semicontinuous functional in \( L^2(M, \mu) \) then

\[
E(f_t) \leq E(f_0) \quad \text{for every } t \geq 0,
\]

and

\[
E(f_t - g_t) \leq E(f_0 - g_0) \quad \text{for every } t \geq 0.
\]

In particular, \( H_t : L^2(M, \mu) \to L^2(M, \mu) \) is a contraction on \( L^2(M, \mu) \cap L^r(M, \mu) \) w.r.t. the \( L^r(M, \mu) \)-norm for all \( r \geq 1 \), i.e. for all \( f_0, g_0 \in L^2(M, \mu) \cap L^r(M, \mu) \) then

\[
\|H_t(f_0) - H(g_0)\|_r \leq \|f_0 - g_0\|_r.
\]

(3) If \( e : \mathbb{R} \to [0, \infty) \) is locally Lipschitz in \( \mathbb{R} \) and \( E(f_0) < \infty \) then

\[
E(f_t) + \int_0^t \int e''(f_s) |\nabla f_s|^2 \, d\mu \, ds = E(f_0) \quad \forall t \geq 0.
\]

(4) When \( \mu(M) < \infty \) we have

\[
\int f_t \, d\mu = \int f_0 \, d\mu \quad \text{for every } t \geq 0.
\]

Remark. (1) The first two assertions also hold for the gradient flow of the \( L \)-Cheeger energy, we will leave the details to the reader.

Proof. The proof follows along the lines of [AGS13, Theorem 4.16]. We will only show the result assuming \( e' \) is bounded and globally Lipschitz. By the same approximation as in [AGS13, Theorem 4.16] the result follows.

Note first that the first statement follows by choosing \( e(r) = \max\{r - C, 0\} \) (resp. \( e(r) = \max\{c - r, 0\} \)).

So let \( e' \) be bounded and Lipschitz on \( \mathbb{R} \) then for \( x, y \in \mathbb{R} \) we have

\[
|e'(x)| \leq |e'(0)| + \text{Lip}(e')|x|,
\]

\[
|e(y) - e(x) - e'(x)(y - x)| \leq \frac{1}{2} \text{Lip}(e')|y - x|^2
\]

\[
|e(y) - e(x)| \leq (|e'(0)| + \text{Lip}(e')) (|x| + |y - x|)|y - x|,
\]

where we assume \( e'(0) = e(0) = 0 \) if \( \mu(M) = \infty \). Furthermore, we will assume w.l.o.g. \( E(f_0 - g_0) < \infty \) (which forces \( e(0) = 0 \) if \( \mu(M) = \infty \)).

By convexity of \( \text{Ch}_q \) the maps \( t \mapsto f_t \) and \( t \mapsto g_t \) are locally Lipschitz continuous in \((0, \infty)\) with values in \( L^2(M, \mu) \) (see [AGS08, Theorem 2.4.15] and [Bre73, Theorem 3.2]). Thus, the map \( t \mapsto e(f_t - g_t) \) is locally Lipschitz in \((0, \infty)\) with values in \( L^1(M, \mu) \), in particular, wherever \( t \mapsto f_t \) and \( t \mapsto g_t \) are commonly differentiable, we have

\[
\frac{d}{dt} e(f_t - g_t) = e'(f_t - g_t) \frac{d}{dt} (f_t - g_t)
\]

\[
= e'(f_t - g_t)(\Delta_q f_t - \Delta_q g_t) \leq 0.
\]

Hence the function is \( t \mapsto E(f_t - g_t) \) is locally Lipschitz in \((0, \infty)\). Integrating we see that the second assertion holds.

For the third statement, set \( g_0 = g_t = 0 \). Absolute continuity of \( t \mapsto E(f_t) \) and the previous theorem yields for \( \phi = e' \)

\[
\frac{d}{dt} \int e(f_t) \, d\mu = \int e'(f_t) \Delta_q f_t \, d\mu = -\int e''(f_t) |\nabla f_t|^2 \, d\mu.
\]
In case $\mu(M) < \infty$ we can choose $e(r) = r$ and thus
\[
\frac{d}{dt} \int f_t d\mu = -\int 0 \cdot |\nabla f_t|^q d\mu
\]
and hence $\int f_t d\mu = \int f_0 d\mu$. □

In order to prove mass preservation for $\mu(M) = \infty$ we adjust [AGS13, Section 4.4]. First we recall some facts about the $p$-logarithm (see also [OT11a, Section 3]) which will make the notation below easier.

**Lemma 16.** The following inequality holds for $p \in (2, 3), x \geq 0$ and $V \geq 0$
\[
x \ln_p x \geq x - \exp_p(-V^p) - (p - 2)V^p \exp_p(-V^p) + (p - 3)V^p x
\]
where
\[
\exp_p(t) = \{1 + (2 - p)t\}^{\frac{1}{p-2}}
\]
and
\[
\ln_p(s) = \frac{s^{p-2} - 1}{2 - p}
\]
which are inverse of each other for $t \in (-\infty, \frac{1}{1-p^2}]$. Note also that $\exp_p$ is monotone on its domain and for sufficiently small $h$
\[
\exp_p(h) \cdot \exp_p(-h) \leq 2.
\]

**Proof.** Note first that $x \ln_p x$ is convex and thus
\[
x \ln_p x \geq x_0 \ln_p x_0 + (\ln_p x_0 + x_0^{2-p})(x - x_0)
\]
\[
= x \ln_p x_0 + x_0^{2-p}(x - x_0).
\]
Now choosing $x_0 = \exp_p(-V^p) \geq 0$ then
\[
x_0^{2-p}(x - x_0) = \{1 + (p - 2)V^p\}^{\frac{2-p}{p-2}} (x - \exp_p(-V^p))
\]
\[
= x - \exp_p(-V^p) - (p - 2)V^p \exp_p(-V^p) + (p - 2)V^p x
\]
Since $x \ln_p x_0 = -V^p x$ we see that
\[
x \ln_p x \geq x - \exp_p(-V^p) - (p - 2)V^p \exp_p(-V^p) + (p - 3)V^p x.
\]

□

**Lemma 17** (Momentum-entropy estimate). Assume $p \in (1, 3)$. Let $\mu$ be a finite measure and $V : X \to [0, \infty)$ be a Lipschitz function with $V \geq \epsilon > 0$ such that
\[
I_p := \begin{cases} 0 & \text{if } p \in (1, 2) \\ \frac{p-2}{3-p} \int V^p \exp_p(-V^p) d\mu & \text{if } p \in (2, 3) \end{cases}
\]
is finite and if $p \in (2, 3)$ assume in addition
\[
\int \exp_p(-V^p) d\mu \leq 1
\]
Let $f_0 \in L^2(X, \mu)$ be non-negative with
\[
\int V^p f_0 d\mu < \infty
\]
and for some $z > 0$
\[ z \int \exp_p(-V^p) \, d\mu \leq \int f_0 \, d\mu \]
if $p \in (2,3)$ and otherwise choose $z \leq 1$. Then $t \mapsto \int V^p f_t \, d\mu$ is locally absolutely continuous in $[0,\infty)$ and for every $t \geq 0$
\[ \int V^p f_t \, d\mu \leq S_t \]
and
\[ \int_0^t \int_{\{f_s > 0\}} |\nabla f_s|^q \, f_s^{q-1} \, d\mu ds \leq \frac{4}{3-p} S_t \]
where
\[ S_t = e^{C_p \operatorname{Lip}(V) \epsilon t} \left( I_p + \int \frac{1}{(2-p)} (f_0^{3-p} - f_0) + (pV^p + z^{-1}l_p) f_0 \, d\mu \right) \]
with $C_p = (p \cdot (3-p)^{-1})^q / q$ and $l_p = \max\{\frac{1}{2-p}, 1\}$.

**Proof.** Define the following
\[ M^q(t) := \int V^p f_t \, d\mu, \quad E(t) := \frac{1}{(3-p)(2-p)} \int f_t^{3-p} - f_t \, d\mu, \]
\[ F^p(t) := \int_{\{f_t > 0\}} \frac{v|f_s|^q}{f_t^{q-1}} \, d\mu. \]
Applying Theorem 15 (see remark below that theorem) to $(f_t + \epsilon) = H_t(f_t + \epsilon)$ and letting $\epsilon \to 0$ we see that $F \in L^p(0,T)$ for every $T > 0$ and
\[ \frac{d}{dt} E(t) = -F^p(t) \ \text{a.e. in} \ (0,T). \]
Furthermore, by the Lemma above, conservation of mass and the assumption $\int \exp_p(-V^p) \, d\mu \leq 1$, we have for $p \in (2,3)$
\[ (3-p)E(t) = \int f_t \ln_p f_t \, d\mu \]
\[ \geq \int f_t - \exp_p(-V^p) \, d\mu - (p-2) \int V^p \exp_p(-V^p) \, d\mu \]
\[ + (p-3)M^q(t) \]
\[ \geq \int f_0 - \exp_p(-V^p) \, d\mu - I_p - (p-1)M^q(t) \]
\[ \geq (1 - z^{-1}) \int f_0 \, d\mu - I_p + (p-3)M^q(t) \]
\[ \geq -z^{-1}l_p \int f_0 \, d\mu - I_p + (p-3)M^q(t) \]
For $p \in (1,2)$ note that $\frac{1}{(2-p)} x^{3-p} \geq 0$ and hence
\[ (3-p)E(t) = \frac{1}{(2-p)} \int f_t^{3-p} - f_t \, d\mu \]
\[ \geq -\frac{1}{(2-p)} \int f_0 \, d\mu \]
\[ \geq -z^{-1}l_p \int f_0 \, d\mu - I_p + (p-3)M^q(t). \]
In order to estimate the derivative of $M(t)$ we introduce a truncated weight $V_k(x) = \min\{V(x), k\}$ and the corresponding functional $M_k^q(t)$ as above. We know that the function $t \mapsto M_k^q(t)$ is locally Lipschitz continuous and thus for a.e. $t > 0$

$$\left| \frac{d}{dt} M_k^q(t) \right| = \left| \int V_k^p \Delta_q f_t d\mu \right| \leq p \int V_k^{p-1} \|\nabla V_k\| \|\nabla f_t\|^{q-1} d\mu$$

$$\leq p L \int \left( V_k^{p-1} f_t^{\frac{q}{q-1}} \right) \left( \frac{\|\nabla f_t\|^{q-1}}{f_t} \right) d\mu$$

$$\leq p LF(t) M_k(t)$$

using $\text{Lip} V_k \leq L$ and Hölder inequality (note $(p - 1)q = p$).

Since by mass preservation $M_k(t) \geq \hat{c} := \epsilon \int f_0 d\mu$, we can apply Gronwall’s inequality and get

$$M_k^q(t) \leq M_k^q(0) \exp \left( \int_0^t \frac{p LF(s)}{M_k^{q-1}(s)} ds \right) \leq M^q(0) \exp \left( \int_0^t \frac{p LF(s)}{\epsilon q^{-1}} ds \right)$$

for $t \in [0, N]$. Thus $M_k^q(t)$ is uniformly bounded and by monotone convergence, we obtain the same differential inequality for $M^q(t)$, i.e. for $t \in [0, \infty)$

$$\left| \frac{d}{dt} M^q(t) \right| = p LF(t) M(t).$$

Now combining this with the result above we get

$$\frac{d}{dt} ((3 - p) E + p M^q) + (3 - p) F^p \leq p^2 LFM \leq (3 - p) F^p + C_p L^q M^q$$

where

$$C_p = (p \cdot (3 - p)^{-1})^q / q.$$ 

Combining this with the inequality above, we get by the Gronwall inequality

$$-z^{-1} I_p \int f_0 d\mu - I_p + (2p - 3) M^q(t) \leq (3 - p) E(t) + p M^q(t)$$

$$\leq e^{C_p L^q t} \left( (3 - p) E(0) + p M^q(0) \right)$$

$$\leq e^{C_p L^q t} \left( (3 - p) E(0) + p M^q(0) \right)$$

$$+ z^{-1} I_p \int f_0 d\mu + I_p.$$ 

Furthermore, we have

$$(3 - p) \int_0^t F^p(s) ds \leq (3 - p)(E(0) - E(t)) \leq (3 - p) E(0) + I_q + z^{-1} I_p \int f_0 d\mu + (3 - p) M^q(t).$$

Having established this, similar to [AGS13, Theorem 4.20] we can show that the gradient flow of the $q$-Cheeger energy is mass preserving even if the measure $\mu$ is just $\sigma$-finite. The proof relies on an approximation procedure developed in [AGS13, Section 4.3]. We will freely use the concepts and results during the proof. The reader may consult [AGS13, Section 4.3] for further reference.
Theorem 18. Assume $p \in (1, \infty)$. If $\mu$ is a $\sigma$-finite measure such that for some Lipschitz function $V : X \to [\epsilon, \infty]$ for some $\epsilon > 0$ such that for $p \in (2, 3)$

$$\int \exp_p(-V) d\mu \leq 1$$

and

$$\int V^p \exp_p(-V) d\mu < \infty$$

and for $p \in (1, 2)$ there is an increasing function $\Phi : \mathbb{R} \to [0, \infty]$ such that

$$\int \Phi(-V) d\mu \leq 1.$$

Then the gradient flow $H_t$ of the $q$-Cheeger energy is mass preserving, i.e. for $f_t = H_t(f_0)$ with $\int f_0 d\mu < \infty$

$$\int f_t d\mu = \int f_0 d\mu.$$

Moreover, if $f_0 \in L^2(M, \mu)$ is nonnegative and

$$\int V^p f_0 d\mu, \int f_0 d\mu < \infty$$

then the bound of the previous Lemma hold.

Proof. We will use the construction of [AGS13, Theorem 4.20], see in particular, [AGS13, Proposition 4.17]. By homogeneity of the $H_t$, i.e. $H_t \lambda f = \lambda^q H_t f$, we can assume $\int f_0 d\mu \leq 1$ if $\int f_0 d\mu < \infty$. In case $\int f_0 d\mu = \infty$, we can find a sequence $f_n \leq f_0$ such that $n \leq \int f_0 d\mu < \infty$. Since mass preservation holds for those functions, we can use the comparison principle to show that $\int f_t d\mu \geq n$ for all $n$ and hence it also holds in the case $\int f_0 d\mu = \infty$. So w.l.o.g. $\int f_0 d\mu \leq 1$.

We first show the case $p \in (2, 3)$. For that use the following approximation: $\mu^0 := \exp_p(-V) \mu$ and $\mu^k := \exp_p(-V^p) \mu^0 = \mu_k$ for $V_k := \min(V, k)$. Then $\mu^k$ is an increasing family of finite measures and

$$\lim \mu^k(B) = \mu(B) \quad \forall B \in B(M).$$

Since $V$ is Lipschitz we see that the density of $\mu$ w.r.t. $\mu^0$ is bounded from below and above on any bounded set.

For each $\mu^k$ let $f^k_t = H^k_t(f_0)$ be the gradient flow starting at $f_0$. Then since $\int \exp_p(-V) d\mu^k \leq 1$ we can apply the previous lemma with $z_k = \int f^k_t d\mu^k$ for all $t \geq 0$ and obtain

$$\int V^p f^k_t d\mu^k \leq e^{2 \text{Lip}(V)^2 t} \left( I_p + \int \frac{1}{(2-p)} (f_0^{3-p} - f_0) + (pV^p + z^{-1} I_p) f_0 d\mu^k \right).$$

Since $f^k_t \to f_t$ strongly in $L^2(X, \mu^0)$ (see [AGS13, Proposition 4.17]) we can assume up to changing to a subsequence $f^k_t \to f_t$ $\mu$-almost everywhere, and thus Fatou’s lemma and monotonicity of $\mu^k$ implies

$$\int V^p f_t d\mu \leq \liminf_{k \to \infty} \int V^p f^k_t d\mu^k$$

and the bound of the previous lemma holds since $z_k \nrightarrow z = \int f_0 d\mu = 1$. 

Now consider \( A_h = \{ x \in M \mid V(x) \leq h \} \). Since we assume \( \int \exp_p(-V^p) d\mu \leq 1 \) we can choose \( h \) such that \( \exp_p(h) \exp_p(-h) \leq 2 \) and get by monotonicity

\[
\mu(A_h) \leq \int 2 \exp_p(h^p) \exp_p(-V^p) d\mu \leq 2 \exp_p(h^p) < \infty
\]

and thus by (4.42) of \cite{AGS13, Proposition 4.17}

\[
\int_{A_h} f_t d\mu = \lim_{k \to \infty} \int_{A_h} f_t^k d\mu^k.
\]

From the bound on the \( p \)-th moment we obtain for every \( t > 0 \) a constant \( C > 0 \) such that

\[
h \int_{X \setminus A_h} f_t^k d\mu^k \leq C
\]

for every \( h > 0 \) and hence

\[
\int f_t d\mu \geq \int_{A_h} f_t d\mu = \lim_{k \to \infty} \int_{A_h} f_t^k d\mu^k \geq z - \limsup_{k \to \infty} \int_{X \setminus A_h} f_t^k d\mu^k \geq z - C/h^p.
\]

Since \( h \) is arbitrary and the integral of \( f_t \) does not exceed \( z \) we see that \( \int f_t d\mu = z \).

The second inequality of the previous lemma follows by lower semicontinuity of the Cheeger energy (see 5).

Mass preservation for signed initial data \( f_0 \) follows by the same arguments as in \cite{AGS13, Theorem 4.20}.

In order to treat the case \( p \in (1, 2) \) let \( \Phi \) be increasing such that \( \int \Phi(-V) d\mu \leq 1 \) and construct a monotone approximation \( \mu^k = \Phi(-V_k)\mu^0 \) and proceed as above.

\[\square\]

Remark. Let \( p \in (2, 3) \) if \( p \to 2 \) then the condition

\[
\int \exp_p(-V^p) d\mu \leq 1
\]

converges to

\[
\int \exp(-V^2) d\mu \leq 1
\]

which is precisely the condition used in \cite{AGS13, (4.2)}. Note, however, it is stronger:

Assuming \( p \in (2, 3) \) and \( (p - 2)V^p \geq 1 \) we have

\[
\exp_p(-V^p) = \{ 1 + (2 - p)(-V^p) \}^{\frac{1}{2-p}} \\
\leq \{ 2(p-2)V^p \}^{\frac{1}{2-p}} \\
= CV^{\frac{1}{2-p}} \geq C \exp(-V^2)
\]

if \( V \) is sufficiently large. In the Euclidean setting with \( V(x) \approx \|x\| \) we get

\[
\int_{\mathbb{R}^n \setminus B_1(0)} \exp_p(-V^p) d\lambda \approx \int_{\mathbb{R} \setminus B_1(0)} \|x\|^{-\frac{p}{p-2}} d\lambda \\
\approx \int_1^\infty r^{-\frac{n}{p-2} - 1} dr
\]
which is finite if \( p < \frac{2n}{(n-1)} \), i.e. \( q > \frac{2n}{n+1} \). However, note that we currently need the more restrictive condition
\[
\int V^p \exp_p(-V^p)d\mu \approx \int_1^\infty r^{p-\frac{p}{p-2}} r^{n-1} d\mu
\]
which is finite iff
\[ p - \frac{p}{p-2} + n < 1, \]
i.e. \( p < \frac{1}{2} (3-n + \sqrt{n^2 + 2n + 9}) \approx 2 + \frac{2}{n} - \frac{2}{n^2} + \mathcal{O}(\frac{1}{n^3}) \) as \( n \to \infty \).

4. Gradient flow in the \( p \)-Wasserstein space

Throughout this section we will assuming that the gradient flow of the \( q \)-Cheeger energy is mass preserving, i.e. the conditions of Theorem 18 hold. Furthermore, we assume that all slopes of Lipschitz functions are equal almost everywhere, i.e.
\[
|Df| = |D^+ f| \quad \mu\text{-almost everywhere.}
\]
This condition holds if the space satisfies a local doubling and Poincaré condition, in particular if \( CD_p(K, N) \) holds with \( N < \infty \).

Our motivation for the functional \( \mathcal{U}_p \) and the identification is the Kuwada lemma. It appeared the first time in [Kuw10] for \( p = 2 \) and was extended by Ambrosio-Gigli-Savaré to \( p \neq 2 \) for finite measures and \( 0 < c \leq f_0 \leq C < \infty \).

**Lemma 19** (Kuwada lemma). Let \( f_0 \in L^q(M, \mu) \) be non-negative and \((f_t)_{t \in [0, \infty)}\) be the gradient flow of the \( q \)-Cheeger energy starting from \( f_0 \). Assume \( \int f_0 d\mu = 1 \). Then the curve \( t \mapsto d\mu_t = f_t d\mu \) is absolutely continuous in \( P_p(M) \) and
\[
|\dot{\mu}_t|^p \leq \int |\nabla f_t|^q \frac{q}{p-1} d\mu \quad \text{for almost every } t \in (0, 1).
\]

**Proof.** The proof follows from [AGS11, Lemma 7.2] using Theorem 15 above the requirement \( 0 < c \leq f_0 \leq C < \infty \) can be easily dropped. \( \square \)

**Remark.** Formally this lemma can be extended to cover \( \partial_t f_t = \Delta \phi(f_t) \), which includes the porous media equation, \( \phi(r) = c_m \cdot r^m \). The theorems below hold with minor adjustments as well. However, since a general existence theory of such equations on abstract metric spaces is not available, an identification is difficult using our approach. This is exactly why Ohta-Takatsu [OT11a, OT11b] can only use the gradient flows in \( P_2 \) to get a solution, but they do not identify the two flows.

**Proposition 20.** Let \( M \) be a proper metric measure space. In case \( p \in (2, 3) \) assume, in addition, that \( M \) is compact and \( n \)-Ahlfors regular for \( 3 - p > 1 - \frac{1}{n} \), i.e. \( n(p - 2) < 1 \). If \( r > 0 \) and \( \mathcal{U}_p(\mu_0) < \infty \), then \( |D^- \mathcal{U}_p| (\mu_0) < \infty \) implies \( \mu_0 \) is absolutely continuous w.r.t. \( \mu \) and if there is a sequence of absolutely continuous measure \( \mu_n \) such that \( \mathcal{w}_p(\mu_0, \mu_n) \to 0 \) and
\[
|D^- \mathcal{U}_p| (\mu_0) = \lim_{n \to 0} \frac{\mathcal{U}_p(\mu_0) - \mathcal{U}_p(\mu_n)}{\mathcal{w}_p(\mu_0, \mu_n)}.
\]

**Remark.** (1) The proof is extracted from [OT11a, Proof of Claim 7.7 and Remark 7.8]. It is stated in the smooth setting but also works in the Ahlfors regular case. The proof depends on the Ahlfors regularity to show that \( \mathcal{U}_p(\mu_\infty) < \mathcal{U}_p(\mu_0) \), but it might be interesting to know if Ahlfors regularity is really needed.
(2) The only time where this proposition is needed is during the proof of Theorem 22 which is based on [AGS13, Theorem 7.5]. In order to use the coupling technique and convexity absolute continuity of \( \mu_n \) is essential.

**Proof.** Let \( m = 3 - p \). In case \( m > 1 \) the measures \( \mu_0 \) and \( \mu_n \) must be absolutely continuous. So we are left to show the cases \( 0 < m < 1 \).

First assume \( \mu_0 \) has non-trivial singular part, i.e. \( \mu_0 = f_0 \mu + \mu^s \) where \( \mu^s \) and \( \mu \) are mutually singular. Define for each \( r > 0 \) a measure \( \hat{\mu}_r \) as follows

\[
d\hat{\mu}_r(x) = \rho_r(x) d\mu(x) := \left\{ f_0(x) + \int \frac{\chi_{B_r(y)}(x)}{\mu(B_r(y))} d\mu^s(y) \right\} d\mu(x).
\]

Then we have

\[
\int \rho_r(x)^m d\mu = \int \left[ \int \left\{ \frac{f_0(x)}{\mu^s(M)} + \frac{\chi_{B_r(y)}(x)}{\mu(B_r(y))} \right\} d\mu^s(y) \right]^m d\mu(x)
\]

\[
\geq \mu^s(M)^{m-1} \int \left[ \int \left\{ \frac{f_0(x)}{\mu^s(M)} + \frac{\chi_{B_r(y)}(x)}{\mu(B_r(y))} \right\} d\mu^s(y) \right] d\mu(x)
\]

\[
\geq \mu^s(M)^{m-1} \left[ \int f_0 \mu + \int_{B_r(y)} \frac{1}{\mu(B_r(y))} d\mu \right] d\mu^s(y)
\]

\[
+ \mu^s(M)^{m-1} \int \mu(B_r(y))^{1-m} d\mu^s(y).
\]

Ahlfors regularity implies that for some \( C, c > 0 \)

\[
c \cdot r^n \leq \mu(B_r(y)) \leq C \cdot r^n
\]

and thus

\[
\mu^s(M)^{m-1} \int \mu(B_r(y))^{1-m} d\mu^s(y) \geq c^{1-m} \mu^s(M)^m \cdot r^{n(1-m)}.
\]

Furthermore, notice

\[
\int_{B_r(y)} f_0^m d\mu \leq \left( \int_{B_r(y)} f_0 d\mu \right)^m \left( \int_{B_r(y)} d\mu \right)^{1-m}
\]

\[
\leq \left( \int_{B_r(y)} f_0 d\mu \right)^m C^{1-m} r^{n(1-m)}.
\]

Since \( \lim_{r \to 0} \sup_{y \in M} \int_{B_r(y)} f_0 d\mu = 0 \) we see that for sufficiently small \( r > 0 \) (note \( (m-1) < 1 \))

\[
\mathcal{U}_m(\hat{\mu}_r) \leq \mathcal{U}_m(\mu_0) - \tilde{C} \mu^s(M)^m \cdot r^{n(1-m)}.
\]

Furthermore, by our assumption

\[
n(1-m) = n(p-2) < 1.
\]

To estimate \( w_\rho(\mu_0, \hat{\mu}_r) \) note that the density of \( \hat{\mu}_r \) is defined as follows

\[
\rho^*_r(x) := \int \frac{\chi_{B_r(y)}(x)}{\mu(B_r(y))} d\mu^s(y).
\]
Now choose the following coupling $\pi$ between $\mu^s$ and $\rho^r_\mu$

$$d\pi(x, y) = \int \frac{\chi_{B_r(z)}(x)}{\mu(B_r(z))} d\mu(x) d(Id \times Id)_\ast \mu^s(z, y)$$

Then

$$w_p^p(\mu_0, \mu_r) \leq \frac{1}{p} \int d^p(x, y) d\pi(x, y)$$

$$\leq \frac{1}{p} \int \int d^p(x, y) \frac{\chi_{B_r(z)}(x)}{\mu(B_r(z))} d\mu(x) d(Id \times Id)_\ast \mu^s(z, y)$$

$$\leq \frac{1}{p} \int r^p d\mu^s$$

and thus

$$w_p^p(\mu_0, \mu_r) \leq r \left( \frac{\mu^s(M)}{p} \right)^{\frac{1}{p}}.$$

Combining these we get

$$|D^- U_n| (\mu_0) \geq \limsup_{r \to 0} \frac{U_N(\mu_0) - U_N(\mu_r)}{w_p(\mu_0, \mu_r)}$$

$$\geq \limsup_{r \to 0} \frac{\tilde{C} \mu^s(M)^m \cdot r^{n(1-m)}}{r \left( \frac{\mu^s(M)}{p} \right)^{\frac{1}{p}}} = \infty$$

since $n(1 - m) < 1$. Which implies that $\mu_0$ must be absolutely continuous.

For the second part, a similar argument works. Given $\mu_n$ we can construct $\tilde{\mu}_n^r$ similar to $\tilde{\mu}_r$. The estimates for $U_n$ hold without any change. For the rest just note

$$\frac{w_p(\mu_0, \tilde{\mu}_n^r)}{w_p(\mu_0, \mu_n)} \leq \frac{1}{w_p(\mu_0, \mu_n)} \left\{ w_p(\mu_0, \mu_n) + w_p(\mu_n, \tilde{\mu}_n^r) \right\}$$

$$\leq 1 + \frac{1}{w_p(\mu_0, \mu_n)} \left( \frac{\mu^s(M)}{p} \right)^{\frac{1}{p}}.$$

Thus

$$\frac{U_n(\mu_0) - U_n(\tilde{\mu}_n^r)}{w_p(\mu_0, \tilde{\mu}_n^r)} \geq \frac{U_n(\mu_0) - U_n(\mu_n) + \tilde{C} \mu^s(M) r^{n(1-m)}}{w_p(\mu_0, \mu_n)} \left( 1 + \frac{r \mu^s(M)^{\frac{1}{p}}}{p \mu^s(M)^{\frac{1}{p}}} \right)^{-1}.$$
Theorem 21. Assume $r > 0$ and let $\mu_0 \in D(U_p)$ with $|D^{-1}U_p|(\mu_0) < \infty$. Then $\mu_0 = \rho \mu$, $\rho \in D(Ch_q)$ and
\[
|D^{-1}U_p|^{r}(\mu_0) \leq r^{-q} \int |\nabla \rho|_{2}^{r} d\mu.
\]

Proof. We will follow the strategy of [AGS13, Theorem 7.4]. First assume $\rho \in L^2(M, \mu)$ and let $(\rho_t)_{t \in (0, \infty)}$ be the gradient flow of the $q$-Cheeger energy starting from $\rho$. Let $\mu_t = \rho_t \mu$, then according to the definition of the $q$-Fisher information we have by Lemma 15 and 19
\[
U_p(\mu_0) - U_p(\mu_t) \geq \frac{1}{q} \int_{0}^{t} F_q(\rho_s) ds + \frac{1}{p} \int_{0}^{t} |\frac{\partial}{\partial s} \rho_s|^{p} d\mu
\geq \frac{1}{q} \left( \frac{1}{t} \int_{0}^{t} \sqrt{F_q(\rho_s)} ds \right)^{q} + \frac{1}{p} \left( \frac{1}{t} \int_{0}^{t} |\frac{\partial}{\partial s} \rho_s| ds \right)^{p}
\geq \frac{1}{t} \left( \int_{0}^{t} \sqrt{F_q(\rho_s)} ds \right) w_p(\mu_0, \mu_t).
\]
Thus dividing by $w_p(\mu_0, \mu_t)$ and letting $t \to 0^+$ we get the result, since lower-semicontinuity of $F_q$ implies
\[
\sqrt{F_q(\rho_0)} \leq \liminf_{t \to 0^+} \frac{1}{t} \int_{0}^{t} \sqrt{F_q(\rho_s)} ds.
\]

In case just $U_p(\mu_0) < \infty$ holds we prove the result by approximation: Let $\rho^n = \min\{\rho, n\}$ and $(\rho^n_t)$ be the corresponding gradient flow of the $q$-Cheeger energy. Using the comparison principle we see that $\rho_t = \lim_{n \to \infty} \rho^n_t$ almost everywhere. Thus using the fact that $z_n = \int \rho^n d\mu = \int \rho^n_t d\mu$ we deduce that $\mu^n_t = \frac{1}{z_n} \rho^n_t \mu$ converges to $\mu_t = \rho t \mu$ in $\mathcal{P}_p(M)$. Now using the lower semicontinuity properties of $U_p$ we deduce
\[
U_p(\mu_0) - U_p(\mu_t) \geq \frac{1}{t} \left( \int_{0}^{t} \sqrt{F_q ds} \right) w_p(\mu_0, \mu_t)
\]
and conclude as above. \qed

Theorem 22. Assume $\mu$ is finite and, in addition if $p > 2$, assume also that $(M, d, \mu)$ is as in Proposition 20. Let $\mu_0 = \rho \mu \in D(U_p)$ and assume $\rho$ is a bounded Lipschitz continuous map with $\rho \geq \epsilon$. Then
\[
|D^{-1}U_p|^{q}(\mu_0) \leq \int \frac{|D\rho|^q}{\rho^{q-1}} d\mu = r^{-q} \int |D\rho|^q d\mu,
\]
where $|D\rho|(x) = \max\{|D^+\rho|(x), |D^-\rho|(x)\}$.

Remark. For $p < 2$, we have $2 - p > 0$ and the idea of [AGS13, Theorem 7.5] can be followed in a similar way using the approximation function $\Phi$ (see Theorem 18) so that a similar version to that theorem follows. For $p > 2$, we have $2 - p < 0$, so that an appropriate version requires further work. Note, however, that Proposition 20 requires $M$ to be compact and hence $\mu$ to be finite.

Proof. Recall that
\[
U_p(r) = \frac{1}{(3-p)(2-p)}(x^{3-p} - x)
\]
\[
\hat{U}_p(r) = \frac{1}{(3-p)(2-p)} x^{3-p}.
\]
Define
\[ L(x,y) := \begin{cases} \frac{1}{2} \frac{\rho^2 - p^2(x) - \frac{1}{2} \rho^2(y)}{d(x,y)} & \text{if } x \neq y \\ \frac{1}{2} \frac{\rho^2 - p^2}{\rho^2} & \text{if } x = y. \end{cases} \]

Note that \( L \) is measurable and for fixed \( x \in M \) the map \( y \mapsto L(x,y) \) is upper semicontinuous. Furthermore, since \( \rho \) is Lipschitz and \( \epsilon \leq \rho \leq M \), \( L \) is bounded.

Now take a sequence of absolutely continuous measures \( \mu_n \) with \( w_p(\mu_0, \mu_n) \to 0 \) and
\[ |D^r U_p|\left( \mu_0 \right) = \lim_{n \to 0} \frac{U_p(\mu_0) - U_p(\mu_n)}{w_p(\mu_0, \mu_n)}. \]

Let \( \rho_n \) be the density of \( \mu_n \) w.r.t. \( \mu \) and \( \pi_n \) be some \( c_\rho \)-optimal transport plan of \( (\mu_0, \mu_n) \). Because \( r \mapsto U_p(r) \) is convex we have
\[
U_p(\mu_0) - U_p(\mu_n) = \int (U_p(\rho) - U_p(\rho_n)) d\mu \leq \int U'_p(\rho)(\rho - \rho_n) d\mu \\
= \int U'_p(\rho) d\mu_0 - \int U'_p(\rho) d\mu_n = \int \left( \tilde{U}'_p(\rho(x)) - \tilde{U}'_p(\rho(y)) \right) d\pi_n(x, y) \\
\leq \int L(x, y) d\pi_n(x, y) \leq w_p(\mu_0, \mu_n) \left( \int L^q(x, y) d\pi_n(x, y) \right)^{1/q} \\
= w_p(\mu_0, \mu_n) \left( \int \left( \int L^q(x, y) d\pi_{n,x}(y) \right) d\mu_0(x) \right)^{1/q}
\]
where \( \pi_{n,x} \) is the disintegration of \( \pi_n \) w.r.t. the first marginal \( \mu_0 \) and \( \tilde{U}_p(x) = \frac{1}{(3-p)(2-p)} x^{2-p} \). Since \( \int (\int d^p(x, y) d\pi_{n,x}(y)) d\mu_0(x) \to 0 \) we can assume w.l.o.g. that for \( \mu_0 \)-a.e. \( x \in M \)
\[
\lim_{n \to \infty} \int d^p(x, y) d\pi_{n,x}(y) = 0
\]
and in particular
\[
\int_{M \setminus B_r(x)} L^q(x, y) d\pi_{n,x}(y) \to 0
\]
for all \( r > 0 \). Furthermore, notice
\[
\lim_{n \to \infty} \sup_{B_r(x)} L^q(x, y) d\pi_{n,x}(y) \leq \lim_{n \to \infty} \sup_{B_r(x)} L^q(x, y) d\pi_{n,x}(y) + \lim_{n \to \infty} \sup_{M \setminus B_r(x)} L^q(x, y) d\pi_{n,x}(y) \\
\leq \lim_{n \to \infty} \sup_{B_r(x)} L^q(x, y) d\pi_{n,x}(y) \leq \sup_{y \in B_r(x)} L^q(x, y).
\]
By upper semicontinuity of \( L(x, \cdot) \) we immediately get \( \lim \sup_n \int L^q(x, y) d\pi_{n,x}(y) \leq L^q(x, x) \) for \( \mu_0 \)-almost every \( x \in M \). Since \( L \) is bounded, we can use Fatou’s lemma.
and conclude
\[
|D^- \mathcal{U}_p|(\mu_0) = \lim_{n \to 0} \frac{U_p(\mu_n) - U_p(\mu_0)}{w_p(\mu_0, \mu_n)}
\leq \lim_{n \to \infty} \sup_{n \in \mathbb{N}} \left( \int |\nabla \rho_n|^{2q} d\mu_n(x) \right)^{1/q}
\leq \left( \int |\nabla \rho|^{2q} d\mu(x) \right)^{1/q}
= \left( \int \frac{|D\rho|^q}{\rho^{p-1} q d\mu} \right)^{1/q} = \left( \int \frac{|D\rho|^q}{\rho^{p-1} d\mu} \right)^{1/q}.
\]

\[
\square
\]

**Proposition 23.** If $|D^- \mathcal{U}_p|$ is sequentially lower semicontinuous w.r.t. $\mathcal{P}_p(M)$ then
\[
|D^- \mathcal{U}_p|^q(\mu_0) = r^{-q} \int |\nabla \rho|^q d\mu \quad \forall \mu_0 = \rho \mu \in D(\mathcal{U}_p).
\]

**Remark.** In [AGS13, Theorem 7.6] Ambrosio-Gigli-Savaré proved also that the converse holds for the entropy functional. We are not able to prove the converse in case $2r > 1$, i.e. $p > 2$.

**Proof.** By the above results we only need to show that $|D^- \mathcal{U}_p|(\mu_0) \leq r^{-q} \int |\nabla \rho|^q d\mu$.

First assume $\rho$ is bounded and find a sequence of measures $\mu_n \in \mathcal{P}_p(M)$ with Lipschitz densities $\rho_n$ bounded from below by $\frac{1}{n}$ converging in $L^2(M)$ to $\rho$ (by compactness $\rho_n^* \to \rho^*$ in $L^2$) such that
\[
\lim_{n \to \infty} \frac{1}{n} \int |\nabla \rho_n|^q d\mu = \text{Ch}_q(\rho^*).
\]

Since $|\nabla \rho_n^*| = |D \rho_n^*$ almost everywhere, we see that
\[
|D^- \mathcal{U}_p|(\mu_0) \leq \liminf_{n \to \infty} |D^- \mathcal{U}_p|(\mu_n)
\leq \liminf \int r^{-q} \int |\nabla \rho_n|^q d\mu = r^{-q} \int |\nabla \rho|^q d\mu.
\]

In case $\rho$ is unbounded we can truncate $\rho$ without increasing the $q$-Cheeger energy use the lower semicontinuity again to conclude the result. \square

**Corollary 24.** Assume one of the following holds:
- $p \in (1, 2)$ and the strong $CD_p(K, \infty)$ condition holds for some $K \geq 0$
- $p \in (2, \frac{3+\sqrt{5}}{2})$, the $CD_p(0, N)$ condition holds such that $p = \frac{2N+1}{N}$ and $M$ is $n$-Ahlfors regular for some $n < N$.

Then $|D^- \mathcal{U}_p|$ is lower semicontinuous and an upper gradient of $\mathcal{U}_p$.

**Proof.** In case $p \in (1, 2)$ note that $3 - p \in (0, 1)$ and thus $U_p \in \mathcal{D}_\infty$. In case $p \in (2, \frac{3+\sqrt{5}}{2})$ we have $3 - p \in (0, 1)$ and thus $U_p \in \mathcal{D}_N$ for $3 - p = 1 - \frac{1}{n}$. In both cases displacement convexity, i.e. $K$-convexity with $K = 0$, follows. Which implies that $|D^- \mathcal{U}_p|$ is lower semicontinuous and an upper gradient of $\mathcal{U}_p$. \square
that it is possible to show that \(|D^- \mu_p|q\) is lower semicontinuous and an upper gradient of \(\mu_p\) if one of the curvature condition holds.

**Theorem 25** (Uniqueness of the gradient flow of \(\mu_p\)). Let \(r > 0\) and assume that \(|D^- \mu_p|q\) is lower semicontinuous and convex w.r.t. linear interpolation. Then for every \(\mu_0 \in \mathcal{P}_p(M)\) there exists at most one gradient flow of \(\mu_p\) starting from \(\mu_0\).

**Remark.** By Lemma 5 and [AGS13, Theorem 7.8] convexity of \(|D^- \mu_p|q\) holds if \(p \leq 2\).

**Proof.** Assume that \((\mu^1_t)\) and \((\mu^2_t)\) are two distinct gradient flows starting from \(\mu_0\). Then we have for \(i = 1, 2\) and all \(T \geq 0\)

\[
\mathcal{U}_p(\mu_0) = \mathcal{U}_p(\mu_T^i) + \frac{1}{p} \int_0^T |\mu_t^i|^q dt + \frac{1}{q} \int_0^T |D^- \mathcal{U}_p|^q(\mu_t^i) dt.
\]

Note that the curve \(t \mapsto \mu_t = (\mu^1_t + \mu^2_t)/2\) is absolutely continuous in \(\mathcal{P}_p(M)\) and

\[
|\dot{\mu}_t|^p \leq \frac{(|\dot{\mu}_t^1|^p + |\dot{\mu}_t^2|^p)}{2}.
\]

Using the strict convexity of \(\mathcal{U}_p\) and the convexity of \(|D^- \mathcal{U}_p|^q\) we conclude

\[
\mathcal{U}_p(\mu_0) > \mathcal{U}_p(\mu_T) + \frac{1}{p} \int_0^T |\dot{\mu}_t|^q dt + \frac{1}{q} \int_0^T |D^- \mathcal{U}_p|^q(\mu_t) dt \\
\geq \mathcal{U}_p(\mu_T) + \int_0^T |\mu_t||D^- \mathcal{U}_p|(\mu_t) dt
\]

But this is a contradiction to

\[
\mathcal{U}_p(\mu_t) \geq \mathcal{U}_p(\mu_s) - \int_s^t |\mu_t||D^- \mathcal{U}_p|(\mu_t) dt
\]

for \(s, t \in [0, \infty)\) (note \(|D^- \mathcal{U}_p|\) is an upper gradient).

Finally we can identify the two flows. The theorem and its proof is similar to [AGS13, Theorem 8.5].

**Theorem 26** (Identification of the gradient flows). Let \(r > 0\) and assume that \(\mathcal{U}_p\) is \(K\)-convex in \(\mathcal{P}_p(M)\). Then for all \(f_0 \in L^2(M, \mu)\) such that \(\mu_0 = f_0 \mu \in \mathcal{P}_p(M)\) the following is equivalent:

1. If \(f_t\) is the gradient flow of \(\mu_t\) in \(L^2(M, \mu)\) starting from \(f_0\), then \(\mu_t = f_t \mu\) is the gradient flow of \(\mu_p\) in \(\mathcal{P}_p(M)\) starting from \(\mu_0\), the map \(t \mapsto \mathcal{U}_p(\mu_t)\) is absolutely continuous in \((0, \infty)\) and

\[
-\frac{d}{dt}\mathcal{U}_p(\mu_t) = |\dot{\mu}_t|^p = |D^- \mathcal{U}_p|^q \quad \text{for a.e.} t \in (0, \infty).
\]

2. Conversely, if we assume in addition that \(|D^- \mathcal{U}_p|^q\) is convex w.r.t. linear interpolation, then whenever \(\mu_t\) is the gradient flow of \(\mathcal{U}_p\) in \(\mathcal{P}_p(M)\) starting from \(\mu_0\), then \(\mu_t\) is absolutely continuous and its density \(f_t\) w.r.t. \(\mu\) is the gradient flow of \(\mu_0\) in \(L^2(M, \mu)\) starting from \(f_0\). The same holds if the gradient flow of \(\mathcal{U}_p\) in \(\mathcal{P}_p(M)\) starting \(\mu_0\) is unique.
Proof. By $K$-convexity of $\mathcal{U}_p$ we know that $|D^-\mathcal{U}_p|$ is an upper gradient and

$$|D^-\mathcal{U}_p|^q(\rho\mu) = F_q(\rho)$$

thus by the Kuwada lemma we know that if $f_t$ is the gradient flow of the $q$-Cheeger energy then

$$|\dot{\mu}_t|^p \leq \int \frac{|
abla f_t|^q}{f_t^{q-1}} d\mu = F_q(f_t)$$

and

$$t \mapsto \mathcal{U}_p(\mu_t)$$

is absolutely continuous with

$$\frac{d}{dt} \mathcal{U}_p(\mu_t) = -\int \frac{|
abla f_t|^q}{f_t^{q-1}} d\mu.$$ 

Hence

$$\int \frac{|
abla f_t|^q}{f_t^{q-1}} d\mu \geq \frac{1}{p} |\dot{\mu}_t|^p + \frac{1}{q} |D^-\mathcal{U}_p|^p$$

so that $\mu_t$ satisfies the $\mathcal{U}_p$-dissipation inequality, i.e.

$$\mathcal{U}_p(\mu_0) - \mathcal{U}_p(\mu_t) \geq \int_0^t \int \frac{|
abla f_s|^q}{f_s^{q-1}} d\mu ds$$

and $\mu_t$ is the gradient flow of $\mathcal{U}_p$ in $\mathcal{P}_p(M)$ starting at $\mu_0$. Absolute continuity of $t \mapsto \mathcal{U}_p(\mu_t)$ in $(0, \infty)$ implies

$$\frac{d}{dt} \mathcal{U}_p(\mu_t) = -|\dot{\mu}_t||D^-\mathcal{U}_p|$$

For the second part, assume that $t \mapsto \tilde{f}_t$ is the gradient flow of the $q$-Cheeger energy starting at $\tilde{f}_0$. By the previous part we know that $\tilde{\mu}_t = \tilde{f}_t \mu$ is also a gradient flow of $\mathcal{U}_p$. Uniqueness (Theorem 25 above) implies that $\mu_t = \tilde{\mu}_t$ for all $t \geq 0$. □

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