SIX-LOOP DIVERGENCES
IN THE SUPERSYMMETRIC KÄHLER SIGMA MODEL

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Abstract

The two-dimensional supersymmetric σ-model on a Kähler manifold has a non-vanishing \( \beta \)-function at four loops, but the \( \beta \)-function at five loops can be made to vanish by a specific choice of renormalisation scheme. We investigate whether this phenomenon persists at six loops, and conclude that it does not; there is a non-vanishing six-loop \( \beta \)-function irrespective of renormalisation scheme ambiguities.
1. Introduction

Two-dimensional non-linear $\sigma$-models have been the object of intense study, in recent years largely because of their relationship with string theory. A string propagating on a manifold $M$ is described by a two-dimensional non-linear $\sigma$-model with $M$ as target manifold. Interest has naturally focussed on supersymmetric $\sigma$-models since the corresponding superstrings have desirable theoretical and phenomenological properties, such as finiteness and anomaly cancellation. Moreover, to obtain a realistic theory, the ten-dimensional space on which the superstring propagates must be compactified—in other words the string vacuum state must be a manifold of the form $M_4 \times K_6$, where $M_4$ is a maximally symmetric four-dimensional space and $K_6$ is a six-dimensional manifold representing internal compactified degrees of freedom. The requirement that the four-dimensional manifold retains $N = 1$ supersymmetry, which provides a possible resolution of the “naturalness” problem \cite{1}, then implies that $M_4$ is Minkowski space and that $K_6$ is a Ricci-flat Kähler manifold \cite{2}. We are thus led to consider supersymmetric $\sigma$-models with a Kähler manifold as target space; such theories in fact possess $N = 2$ supersymmetry \cite{3}.

We need to determine the conditions for a manifold $M_4 \times K_6$ to be a viable string vacuum state; in fact this requires the corresponding $\sigma$-model to be conformally invariant, which in turn implies that the renormalisation group $\beta$-functions for the Kähler metric should vanish \cite{4} (up to a diffeomorphism \cite{5}). It was initially believed that the $\beta$-function for a Ricci-flat supersymmetric Kähler $\sigma$-model automatically vanished to all orders \cite{6}. However, Grisaru, van de Ven and Zanon \cite{7} \cite{8} found a non-zero contribution to the $\beta$-function for the supersymmetric Kähler $\sigma$-model at the four-loop level, which did not vanish in the Ricci-flat case. In other words the natural metric on a Ricci-flat Kähler manifold, (i.e. the one which is Ricci-flat), does not satisfy the conformal invariance condition. Nevertheless Ricci-flat Kähler manifolds may still be of phenomenological interest, since a metric can be constructed on such manifolds (by adding non-local corrections to the standard metric) which does satisfy the conformal invariance condition \cite{9}. It was subsequently shown \cite{10}
that, remarkably, the five-loop divergence in the Kähler $\sigma$-model could be removed by a \textit{local} finite redefinition of the metric in terms of covariant quantities, equivalent to a change of renormalisation scheme. This result appears rather miraculous, and it is natural to ask whether it is an isolated occurrence; might it be that there exists a scheme in which there are no contributions to the $\beta$-function beyond four loops? With this motivation, we have carried out a partial computation of the six-loop contribution to the $\beta$-function for the Kähler $\sigma$-model. The terms we have calculated are sufficient to determine that the six-loop $\beta$-function cannot be cancelled by a local covariant field redefinition of the metric; there is no renormalisation scheme in which the $\beta$-function vanishes at six loops.

2. Perturbative calculations for the Kähler sigma-model

Our calculational methods are based on the work of Refs. 8, 10. Firstly we describe the rudiments of Kähler geometry. A Kähler manifold is a complex manifold with a covariantly constant hermitian almost complex structure, i.e. there is a tensor $J^i_j$ satisfying

$$J^k_i J^j_k = -\delta^j_i,$$
$$J^k_i g_{kj} = -J^k_j g_{ki},$$
$$\nabla_i J^k_j = 0.$$ (2.1)

We can then choose a local complex co-ordinate system $\Phi^p$, $\bar{\Phi}^{\bar{p}}$, in which the metric takes the form

$$g_{p\bar{q}} = \frac{\partial}{\partial \Phi^p} \frac{\partial}{\partial \bar{\Phi}^{\bar{q}}} K(\Phi, \bar{\Phi}),$$
$$g_{pq} = g_{\bar{p}\bar{q}} = 0.$$ (2.2)
for some $K(\Phi, \bar{\Phi})$ which is referred to as the Kähler potential. Introducing the notation

$$K_p \equiv \frac{\partial K}{\partial \Phi^p}, \quad K_{\bar{p}} \equiv \frac{\partial K}{\partial \bar{\Phi}^{\bar{p}}},$$

(2.3)

the only non-vanishing Christoffel symbols are

$$\Gamma^p_{qr} = g^{pq}K_{\bar{q}r}, \quad \Gamma^p_{\bar{q}r} = g^{\bar{p}q}K_{p\bar{r}},$$

(2.4)

and the Riemann tensor is given by

$$R^{p\bar{p}qq} = K_{p\bar{p}qq} - g^{\bar{r}p}K_{r\bar{p}q}K_{r\bar{q}}.$$  

(2.5)

As we mentioned earlier, the $N = 1$ supersymmetric $\sigma$-model defined on a Kähler manifold in fact automatically possesses $N = 2$ supersymmetry\textsuperscript{[3]}. It can be expressed in terms of $N = 2$ chiral and anti-chiral superfields $\Phi^p(x, \theta, \bar{\theta})$ and $\bar{\Phi}^{\bar{p}}(x, \theta, \bar{\theta})$ as

$$S = \int d^2xd^2\theta d^2\bar{\theta}K(\Phi, \bar{\Phi}).$$

(2.6)

The chirality condition is

$$\bar{D}_\alpha \Phi^p = D_\alpha \bar{\Phi}^{\bar{p}} = 0,$$

(2.7)

where the superspace covariant derivatives $D, \bar{D}$ are defined by

$$D_\alpha = \frac{\partial}{\partial \theta_\alpha} + \frac{1}{2}i\bar{\theta}^\beta \partial_{\alpha\beta}, \quad \bar{D}_\beta = (D_\beta)^*, \quad \text{where} \quad \partial_{\alpha\beta} = \partial_\mu \sigma^{\mu}_{\alpha\beta}. $$

(2.8)

(For notation and conventions see “Superspace”\textsuperscript{[11]}. To perform perturbative calculations, we use the standard background field method, expanding around a background $\Phi_0, \bar{\Phi}_0$ using a linear quantum-background splitting

$$\Phi \rightarrow \Phi_0 + \Phi, \quad \bar{\Phi} \rightarrow \bar{\Phi}_0 + \bar{\Phi}. $$

(2.9)

The resulting expansion is then not manifestly covariant since the quantum fields $\Phi^p, \bar{\Phi}^{\bar{p}}$ are not vectors. This is in contrast to the normal co-ordinate method\textsuperscript{[12],[13]}
usually adopted for the quantisation of the non-linear $\sigma$-model; there the quantum field *is* a vector and consequently the coefficients of the expansion are functions of the Riemann tensor and its derivatives. This technique cannot be applied here due to the chirality constraints Eq. (2.7). In any case, any consequent loss of elegance is amply compensated by the many simplifications afforded by $N = 2$ perturbation theory. Moreover, the action Eq. (2.6) and its expansion are very compact and it is relatively easy to recover a covariant expression at the end of the calculation.

The expansion of the Kähler potential is then

$$K(\Phi_0 + \Phi, \bar{\Phi}_0 + \bar{\Phi}) - K(\Phi_0, \bar{\Phi}_0) = K_p \Phi^p + K_{\bar{p}} \bar{\Phi}^{\bar{p}}$$

$$+ K_{pq} \Phi^p \Phi^q + \frac{1}{2} K_{pq} \Phi^p \Phi^q + \frac{1}{2} K_{pq} \Phi^{\bar{p}} \Phi^{\bar{q}} \tag{2.10}$$

where we omit the dependence of $K$ on $\Phi_0$ and $\bar{\Phi}_0$ on the right-hand side. The first quadratic term in Eq. (2.10) can be shown[^8] to give rise to an effective propagator

$$< \Phi^p(z) \bar{\Phi}^{\bar{p}}(z') > = - g^{pq} \partial^2 D^2 \delta(z - z') \bar{D}^2$$

(2.11)

where $z = (x, \theta)$. The remaining terms in the expansion then supply the vertices used to construct Feynman diagrams. After standard $D$-algebra manipulations, the diagrams can be written in momentum space form and hence evaluated. We use dimensional regularisation so that we work in $d$ dimensions and divergences appear as poles in $\epsilon = 2 - d$. We construct counterterm diagrams on a diagram-by-diagram basis, i.e. at each succeeding loop order, for each diagram we write down counterterm diagrams corresponding to the subdivergences of the original diagram. The remaining overall divergence is then cancelled loop by loop by adding corrections to the Kähler potential, writing

$$K \rightarrow K_B = K + \sum_{L=1}^{\infty} \sum_{n=1}^{L} \frac{1}{(4\pi)^L} \frac{K^{(n,L)}}{\epsilon^n} \tag{2.12}$$

The diagram-by-diagram subtraction method is of course equivalent to the standard method of constructing counterterms at each loop order from the lower-order
corrections in Eq. (2.12), but it obviates the need to consider also the wave-
function renormalisation of the quantum fields\textsuperscript{[14]}. Corresponding to Eq. (2.12),
the correction to the Kähler metric is given by

\[
g_{p\bar{q}}^{B} = g_{p\bar{q}} + \sum_{L=1}^{\infty} \sum_{n=1}^{L} \frac{K_{p\bar{q}}^{(n,L)}}{(4\pi)^{L} e^{n}}
\]  

(2.13)

and the \(\beta\)-function is then given by

\[
\beta_{p\bar{q}} = \sum_{L=1}^{\infty} LK_{p\bar{q}}^{(1,L)} = [\beta_{K}]_{p\bar{q}}
\]  

(2.14)

where the \(\beta\)-function for \(K\) is given by

\[
\beta_{K} = \sum_{L=1}^{\infty} LK^{(1,L)}.
\]  

(2.15)

3. Feynman diagram calculations up to five loops

It is straightforward to show\textsuperscript{[8]} from supergraph power counting that the divergent counterterms will not involve any superspace derivative \(D\) or \(\bar{D}\) acting on the background fields. Hence the \(D\)-algebra can be performed by integrating by parts the \(D, \bar{D}\) only on the internal quantum lines. We can then discard from the expansion Eq. (2.10) all terms containing only quantum \(\Phi\)'s or only \(\bar{\Phi}\)'s. The one-loop counterterm is given by

\[
K^{(1,1)} = -2\text{tr} \ln K_{p\bar{q}}
\]  

(3.1)

which leads to the well-known one-loop \(\beta\)-function

\[
\beta_{p\bar{q}}^{(1)} = 2R_{p\bar{q}}.
\]  

(3.2)

(The Ricci tensor \(R_{p\bar{q}}\) is given by \(R_{p\bar{q}} = -g^{rs}R_{p\bar{s}r\bar{q}}\). The two and three loop
simple pole counterterms $K^{(1,2)}$ and $K^{(1,3)}$ are zero in minimal subtraction, leading to vanishing $\beta$-function at two and three loops; however Grisaru, van de Ven and Zanon\textsuperscript{[8]} showed that $K^{(1,4)}$ is non-zero and is given by
\begin{equation}
K^{(1,4)} = \frac{\zeta(3)}{3} (R_p^q R^R_{s} R^v_{q} R^v_{s} R^w_{u} R^r_{v} R^r_{u} R^p_{q} + R_p^q R^R_{s} R^r_{v} R^u_{u} R^w_{q} R^p_{r})
\end{equation}
which implies a non-vanishing contribution to the $\beta$-function for the supersymmetric Kähler $\sigma$-model at four loops. Subsequently, Grisaru, Kazakov and Zanon\textsuperscript{[10]} computed the simple pole contribution at five loops, i.e. $K^{(1,5)}$. They found a non-vanishing contribution within minimal subtraction, given by
\begin{equation}
K^{(1,5)} = \frac{3\zeta(4)}{10} (R_p^q R^R_{s} R^r_{u} R^u_{v} R^w_{w} R^w_{x} R^w_{q} R^w_{q} - R_p^q R^R_{s} R^r_{u} R^w_{u} R^w_{x} R^w_{q} R^w_{q})
+ R_p^q R^R_{s} R^r_{v} R^w_{w} R^w_{x} R^w_{x} + R_p^q R^R_{s} R^r_{v} R^w_{u} R^w_{x} R^w_{x} R^w_{p}
+ \nabla_w R_p^q R^R_{s} R^r_{v} R^u_{u} R^v_{v} R^v_{q} R^v_{q} + \nabla^w R_p^q R^R_{s} R^r_{v} R^u_{u} R^v_{v} R^v_{q} R^v_{q}
+ 2 \nabla_w R_p^q R^R_{s} R^r_{v} R^u_{u} R^v_{v} R^v_{q} R^v_{q}).
\end{equation}

They then observed that the resulting contribution to the $\beta$-function could in fact be removed by a local field redefinition of the metric, equivalent to a change in renormalisation scheme. The effect of a change $\delta g_{\bar{p}q}$ in the metric $g_{\bar{p}q}$ on the $\beta$-function is given by
\begin{equation}
\delta \beta_{\bar{p}q} = \beta \frac{\partial}{\partial g} \delta g_{\bar{p}q} - \delta g \frac{\partial}{\partial g} \beta_{\bar{p}q}.
\end{equation}
Using Eqs. (2.14), (2.15), it is easy to see that if $\delta g_{\bar{p}q}$ is generated according to Eq. (2.3) by a shift $\delta K$ in the Kähler potential, the corresponding effect on $\beta^K$ in Eq. (2.15) is given by
\begin{equation}
\delta \beta^K = \beta \frac{\partial}{\partial g} \delta K - \delta g \frac{\partial}{\partial g} \beta^K.
\end{equation}
Using
\begin{equation}
R_{uvw} \frac{\partial}{\partial g^{uv}} R_p^q R^R_{s} R^r_{r} = \nabla^q \nabla^R_p R^r_{s} - R_u^q R^R_{r} R^u_{r}
\end{equation}
together with Eqs. (3.1), and (3.2), Grisaru, Kazakov and Zanon\textsuperscript{[10]} showed that
taking

$$\delta K = \frac{3}{4} \frac{\zeta(4)}{\zeta(3)} K^{(1,4)}$$

(3.8)

induced a change in the five-loop $\beta$-function given by

$$\delta \beta^{K(5)} = -5 K^{(1,5)}$$

(3.9)

(with $K^{(1,5)}$ as in Eq. (3.4)), and hence the five-loop contribution to the $\beta$-function is removed by the field redefinition Eq. (3.8); in other words there is a renormalisation scheme in which the $\beta$-function is zero at five loops.

4. The six-loop calculation

In this section we present details of a six-loop calculation performed with the aim of investigating whether the six-loop $\beta$-function could also be removed by field redefinitions. In fact we can show by carrying out only a small fraction of the full six-loop calculation that the six-loop $\beta$-function cannot be eradicated. It is sufficient to focus attention on diagrams with the topology shown in Fig. 1. The reason for selecting these particular diagrams is that they are the only ones with three or fewer vertices which contribute to the $\beta$-function. All other diagrams with three or fewer vertices can easily be reduced using $D$-algebra to standard Feynman diagrams containing tadpoles, which do not contribute to the $\beta$-function in minimal subtraction. Hence these diagrams will turn out to determine all terms in $\beta^{K(6)}$ with three or fewer Riemann tensors. It is straightforward to show using $D$-algebra that any superspace diagram with the topologies shown in Fig. 1 must contain at least two $\Phi$ quantum lines and at least two $\bar{\Phi}$ quantum lines at each vertex, otherwise it can be reduced to a diagram with tadpoles. Hence the only superspace diagrams with the topologies of Fig. 1 which contribute to the $\beta$-function are
those shown in Fig. 2. Using $D$-algebra, each of the superspace diagrams Figs. 2(a)-(e) can be reduced to the momentum integral Fig. 3(a) and each of Figs. 2(f)-(l) can be reduced to Fig. 3(b). The evaluation of the momentum integrals is tedious but straightforward. As mentioned earlier, we subtract from each six-loop diagram the lower order diagrams with counterterm insertions corresponding to divergent subdiagrams of the six-loop diagram. We regulate infra-red divergences by replacing potentially infra-red divergent propagators $\frac{1}{k^2}$ by $^{[10][15]}$

$$\frac{1}{k^2} + \frac{2}{\epsilon} \delta(k).$$ (4.1)

This avoids the necessity for massive propagators, thereby simplifying the calculation enormously. Denoting by $G_a$, $G_b$ the momentum integrals corresponding to Figs. 3(a), (b), (together with their subtraction diagrams), we find

$$G_a = -\frac{4}{5} \frac{1}{(4\pi\epsilon)^6} \left( \frac{496}{3} + 40\zeta(3)\epsilon^3 - 15\zeta(4)\epsilon^4 - 7\zeta(5)\epsilon^5 \right)$$ (4.2a)

$$G_b = -\frac{2}{15} \frac{1}{(4\pi\epsilon)^6} (8 - 4\zeta(3)\epsilon^3 + 3\zeta(4)\epsilon^4 + 3\zeta(5)\epsilon^5).$$ (4.2b)

The resulting contributions to $K^{(1,6)}$ arising from the diagrams in Fig. 2 will be non-covariant, consisting of products of derivatives of $K$ contracted together. However it can be proved using $N = 2$ supersymmetry $^{[8]}$ that the final complete result for $K^{(1,6)}$ should be covariant. This implies that contributions from graphs with more than three vertices will appear in such a way as to “covariantise” the contributions of Fig. 2. Since there are no graphs with only two vertices contributing to $K^{(1,6)}$, it is clear from Eq. (2.5) that every term in $K^{(1,6)}$ must contain at least three Riemann tensors. Hence every vertex in the diagrams of Fig. 2 must correspond to the linear term on the right-hand side of Eq. (2.5) (or derivatives of it); graphs with additional vertices must supply the quadratic terms on the right-hand side of Eq. (2.5) so as to reconstitute the Riemann tensor. It follows that we can uniquely reconstruct the terms cubic in the Riemann tensor in the final covariant
result by substituting in the contribution from Fig. 2:\(^{[10]}\):

\[
K_{pq¯p} \rightarrow R_{p¯pq¯q} \\
K_{pqr¯p} \rightarrow \nabla_r R_{p¯pq¯q} \\
K_{pqr¯p} \rightarrow \nabla_r \nabla_s R_{p¯pq¯q} \\
K_{pqr¯s} \rightarrow \nabla_r \nabla_s R_{p¯pq¯q}.
\]  

Combining symmetry factors and \(D\)-algebra factors for the graphs in Fig. 2 with the results for the momentum integrals in Eq. (4.2), and then reconstituting the covariant expression \textit{via} the substitutions Eq. (4.3), we obtain

\[
K^{(1,6)} = \frac{-7}{30} \zeta(5)(\nabla_u \nabla_v R_{pqrs} p^r q^s \nabla^u \nabla^v R_{u^p v^q} + 2\nabla^u \nabla_v R_{pqrs} p^r q^s \nabla^u \nabla^v R_{u^p v^q} + 2\nabla^u \nabla_v R_{pqrs} R_{pqrs} p^r q^s R_{u^p v^q} + 2\nabla^u \nabla_v R_{pqrs} R_{pqrs} p^r q^s R_{u^p v^q}) \\
+ \frac{1}{5} \zeta(5)(\nabla_u R_{pqrs} p^r q^s \nabla^u R_{u^p v^q} + 2\nabla^u \nabla_v R_{pqrs} p^r q^s \nabla^u R_{u^p v^q} + 2\nabla^u \nabla_v R_{pqrs} R_{pqrs} p^r q^s R_{u^p v^q} + 2\nabla^u \nabla_v R_{pqrs} R_{pqrs} p^r q^s R_{u^p v^q}) \ldots
\]  

where the ellipsis represents terms with more than three Riemann tensors. We now need to consider the effects of field redefinitions. A five-loop field redefinition

\[
\delta g_{pq}^{(5)} = \partial_p \partial_q \delta K^{(5)}
\]  

produces a change in \(\beta K^{(6)}\) given according to Eqs. (3.1), (3.2) and (3.6) by

\[
\delta \beta K^{(6)} = \mathcal{O} \delta K^{(5)}
\]  

where

\[
\mathcal{O} = \nabla_u \nabla^u + R_{u^v \bar{v}} \cdot \frac{\partial}{\partial g_{u^v \bar{v}}}
\]  

9
Using the identity
\[
\nabla_u \nabla^u R_p{}^q{}^r{}^s = R_u{}^q R^u{}^r{}^s - \nabla^q \nabla_p R^r
\]
\[
+ R_p{}^v{}^u R_v{}^r{}^s + R_r{}^v{}^u R_p{}^u{}^s - R_v{}^s{}^u R_p{}^u{}^v
\]  
(4.8)

(which follows from the Bianchi identity), together with Eq. (3.7), we have
\[
\mathcal{O} R_p{}^q{}^r{}^s = R_p{}^v{}^q R^v{}^r{}^s + R_r{}^v{}^u R_p{}^u{}^s - R_v{}^s{}^u R_p{}^u{}^v
\]  
(4.9a)
\[
\mathcal{O} \nabla^w R_p{}^q{}^r{}^s = \nabla^w (R_p{}^v{}^q R^v{}^r{}^s + R_r{}^v{}^u R_p{}^u{}^s - R_v{}^s{}^u R_p{}^u{}^v)
\]
\[
+ R_p{}^v{}^u \nabla^u R^r{}^s - R_v{}^q{}^u \nabla^u R^v{}^s + R_r{}^v{}^u \nabla^u R_p{}^u{}^v
\]
\[
- R_v{}^s{}^u \nabla^u R^v{}^r
\]  
(4.9b)
\[
\mathcal{O} \nabla^w \nabla^w R_p{}^q{}^r{}^s = \nabla_x \nabla^w (R_p{}^v{}^q R^v{}^r{}^s + R_r{}^v{}^u R_p{}^u{}^s - R_v{}^s{}^u R_p{}^u{}^v)
\]
\[
+ \nabla_x (R_p{}^v{}^u \nabla^u R^r{}^s - R_v{}^q{}^u \nabla^u R_p{}^u{}^v + R_r{}^v{}^u \nabla^u R_p{}^u{}^s
\]
\[
- R_v{}^s{}^u \nabla^u R^v{}^r)
\]
\[
+ (R_v{}^w{}^u \nabla^u R^q{}^r{}^s - R_p{}^v{}^x \nabla^u R_v{}^q{}^r{}^s + R_v{}^q{}^x \nabla^u \nabla^w R_p{}^v{}^r{}^s
\]
\[
- R_r{}^v{}^x \nabla^u \nabla^w R_p{}^q{}^r{}^s + R_v{}^s{}^x \nabla^u \nabla^w R_p{}^q{}^r
\]  
(4.9c)

where \(\mathcal{O}\) is defined in Eq. (4.7). One point to notice is that all terms involving the Ricci tensor have cancelled on the right-hand sides of Eq. (4.9). This is a useful property since \(K^{(1,6)}\) does not contain the Ricci tensor. (In fact the Ricci tensor never appears in the simple pole counterterms when using minimal subtraction, since it corresponds to tadpole diagrams.) In order to correspond to a change in renormalisation prescription, \(\delta K^{(5)}\) must be a local quantity constructed from covariant quantities, namely the Riemann tensor and its covariant derivatives. It is convenient to represent both \(\delta K^{(5)}\) and the resultant \(\delta \beta K^{(6)}\) diagrammatically, with each Riemann tensor (or its covariant derivative) denoted by a vertex with a number of legs corresponding to the number of indices, and contractions represented by lines joining the vertices. Of course the leading covariant term obtained by the substitution Eq. (4.3) in the counterterm for a given diagram is then represented according to this prescription by the original diagram. When evaluating
the contribution to $\delta \beta^K(6)$ from any term in $\delta K(5)$, constructed from a string of Riemann tensors and their derivatives contracted together, we see from Eqs. (4.6), (4.7) that there are two possibilities:

1.) The two derivatives in $\nabla_u \nabla^u$ act on different Riemann tensors (or derivatives of Riemann tensors). The diagram for the term thus obtained is constructed from the original diagram for the term in $\delta K(5)$ by adding an extra line between two vertices.

2.) Both derivatives in $\nabla_u \nabla^u$ act on the same Riemann tensor (or derivative of a Riemann tensor), in which case the contribution to $\delta \beta^K(6)$ is easily obtained using Eq. (4.9). All terms obtained in this way are represented by diagrams constructed from the diagram for the term in $\delta K(5)$ by “opening out” one of its vertices. By this we mean replacing a vertex with $m$ legs with two vertices joined by two lines, such that the total number of free legs is still $m$ and with each new vertex having at least two free legs.

These two possibilities are depicted in Fig. 4. For the moment we are only concerned with the topology of the diagrams and we ignore the orientation of the propagators. Let us now consider how we might construct a $\delta K(5)$ which would produce a $\delta \beta^K(6)$ with the possibility of cancelling the original $\beta^K(6)$ derived from Eq. (4.4). A moment’s thought shows that the only possible covariant terms in $\delta K(5)$ which could produce terms of the topology depicted in Fig. 1 according to the two rules given above, and which could thus have a chance of cancelling the terms in $\beta^K(6)$ given explicitly in Eq. (4.4), have the topology shown in Fig. 5. However, Fig. 5(a) also produces contributions to $\delta \beta^K(6)$ with the topology of Fig. 6(a), and Fig. 5(b) also produces contributions with the topology Fig. 6(b). The $D$-algebra for diagrams of the type shown in Fig. 6 always leads to momentum space integrals which reduce to tadpoles; hence the original $\beta^K(6)$ contains no terms of the topology Fig. 6(a), (b). Moreover there is no other possible term in $\delta K(5)$ which produces the topologies of Fig. 6. Hence if we aim to cancel $\beta^K(6)$ we must omit terms with the topology of Figs. 5(a), (b) from $\delta K(5)$. This means that the
terms of topology Fig. 1 in $\beta^{K(6)}$ must be cancelled solely by a $\delta K^{(5)}$ generated by terms of the topology Fig. 5(c). However, any term in $\delta K^{(5)}$ represented by Fig. 5(c) is of the form

$$a \nabla_w R^q_p \nabla^w R^p_r R^u_s R^v_q + b \nabla_w R^q_p \nabla^w R^p_r R^s_t R^u_v R^t_q + c \nabla_w R^q_p \nabla^w R^r_s R^u_v R^u_q$$

and generates a contribution to $\delta \beta^{K(6)}$ given by

$$a(\nabla^x \nabla_w R^q_p \nabla^x \nabla^w R^p_r R^u_s R^v_q + \nabla^x \nabla^w R^q_p \nabla^x \nabla^w R^p_r R^q_s R^u_v + 2\nabla^x \nabla^w R^q_p \nabla^y \nabla^w R^p_r \nabla^y R^q_s R^u_v) + b(\nabla^x \nabla_w R^q_p \nabla^x \nabla^w R^p_r R^q_s R^u_v + \nabla^x \nabla^w R^q_p \nabla^x \nabla^w R^p_r \nabla^x R^q_s R^u_v + \nabla^x \nabla^w R^q_p \nabla^x \nabla^w R^p_r \nabla^x R^q_s R^u_v + 2\nabla^x \nabla^w R^q_p \nabla^x \nabla^w R^p_r \nabla^x R^q_s R^u_v) + c(\nabla^x \nabla_w R^q_p \nabla^x \nabla^w R^p_r \nabla^x R^q_s R^u_v R^u_q + \nabla^x \nabla^w R^q_p \nabla^x \nabla^w R^p_r \nabla^x R^q_s R^u_v R^u_q + 2\nabla^x \nabla^w R^q_p \nabla^x \nabla^w R^p_r \nabla^x R^q_s R^u_v R^u_q).$$

(4.11)

It is immediately apparent that there is no choice of $a$, $b$ and $c$ which could cancel the terms in $\beta^{K(6)}$ given by Eq. (4.4). Hence $\beta^{K(6)}$ cannot be removed by any covariant field redefinition.
5. Conclusions

We have shown that there is a well-defined set of terms in $\beta^{K(6)}$ (namely those containing only three Riemann tensors) which are non-zero and moreover cannot be removed by field redefinitions: so there is no renormalisation scheme in which the $\beta$-function for the Kähler potential in the Kähler $\sigma$-model vanishes at six loops. It follows that the $\beta$-function for the metric, calculated according to Eq. (2.13), is also non-zero irrespective of renormalisation scheme at six loops. Now in general, the metric $\beta$-function is ambiguous up to diffeomorphisms (or in other words coordinate changes on the two-dimensional worldsheet) given by

$$\beta_{ij} \to \beta_{ij} + \nabla(v^j)$$

for some vector $v$ (where $i, j$ are real indices). One might conceivably entertain the hope that some combination of field redefinition and diffeomorphism might result in a vanishing $\beta$-function. Aside from the implausibility of this scenario, in any case the important quantity to consider from the point of view of string theory is not the metric $\beta$-function itself but rather $B_{ij}$ defined by

$$B_{ij} = \beta_{ij} + \nabla(iS_j)$$

where $S$ is a well-defined, calculable vector quantity. It is the vanishing of $B_{ij}$ which is the condition for conformal invariance, and $B_{ij}$ is invariant under diffeomorphisms, since under Eq. (5.1), we also have

$$S_i \to S_i - v_i.$$  

Now for the supersymmetric Kähler $\sigma$-model, it is known that $S$ is zero to all orders when calculated in the usual complex coordinates in which the metric $\beta$-function is given by Eq. (2.14). Hence, from Eq. (5.2), we see that $B_{ij}$ is non-vanishing in every renormalisation scheme at six loops, which is a coordinate-independent
(or diffeomorphism-invariant) statement. If it had turned out that there was a scheme in which \( B_{ij} \) vanished at six loops, then the non-vanishing \( B_{ij} \) in any other scheme would in some sense have been an artefact—the six-loop divergence would have been generated by the lower-order divergences. As it is, we conclude that in fact there is a new and independent contribution to the \( \beta \)-function at six loops. The fact that this did not occur at five loops remains mysterious.

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Figure Captions

Fig. 1 Topology of 3-point diagrams contributing to the $\beta$-function.

Fig. 2 Superspace diagrams contributing to the $\beta$-function (arrows pointing towards $\Phi$ lines at vertices).

Fig. 3 Six-loop momentum integrals (pairs of similar arrows representing contracted momenta in the numerator).

Fig. 4 Examples of operations generating a contribution to $\delta \beta^{K(6)}$ from a term in $\delta K^{(5)}$.

Fig. 5 Diagrams representing possible covariant terms in $\delta K^{(5)}$ which would produce in $\delta \beta^{K(6)}$ 3-Riemann terms like those in $\beta^{K(6)}$.

Fig. 6 Diagrams representing additional contributions to $\delta \beta^{K(6)}$, not already in $\beta^{K(6)}$, produced by terms in $\delta K^{(5)}$ shown in Figs 5(a), (b).
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