Minimal sets of fibre-preserving maps in graph bundles

S. Kolyada · L’. Snoha · S. Trofimchuk

Abstract Topological structure of minimal sets is studied for a dynamical system \((E,F)\) given by a fibre-preserving, in general non-invertible, continuous selfmap \(F\) of a graph bundle \(E\). These systems include, as a very particular case, quasiperiodically forced circle homeomorphisms. Let \(M\) be a minimal set of \(F\) with full projection onto the base space \(B\) of the bundle. We show that \(M\) is nowhere dense or has nonempty interior depending on whether the set of so called endpoints of \(M\) is dense in \(M\) or is empty. If \(M\) is nowhere dense, we prove that either a typical fibre of \(M\) is a Cantor set, or there is a positive integer \(N\) such that a typical fibre of \(M\) has cardinality \(N\). If \(M\) has nonempty interior we prove that there is a positive integer \(m\) such that a typical fibre of \(M\), in fact even each fibre of \(M\) over a dense open set \(O \subseteq B\), is a disjoint union of \(m\) circles. Moreover, we show that each of the fibres of \(M\) over \(B \setminus O\) is a union of circles properly containing a disjoint union of \(m\) circles. Surprisingly, some of the circles in such “non-typical” fibres of \(M\) may intersect. We also give sufficient conditions for \(M\) to be a sub-bundle of \(E\).

Keywords Dynamical system · minimal set · graph bundle · skew product

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1 Introduction and statement of main results

Let \( f \) be a continuous selfmap of a compact metric space \( X \). The system \((X,f)\) is called minimal if there is no proper subset \( M \subseteq X \) which is nonempty, closed and \( f \)-invariant (i.e., \( f(M) \subseteq M \)). In such a case we also say that the map \( f \) itself is minimal. Clearly, a system \((X,f)\) is minimal if and only if the orbit \( \{x, f(x), f^2(x), \ldots \} \) of every point \( x \in X \) is dense in \( X \). Note that by an orbit we mean a forward orbit rather than a full orbit, even if \( f \) is a homeomorphism. The basic fact is that any compact dynamical system \((X,f)\) has minimal (closed) subsystems \((M,f|_M)\). Such closed sets \( M \) are called minimal sets of \( f \) or, more precisely, of \((X,f)\). The minimal sets, as ‘irreducible’ parts of a system, attract much attention and their topological structure is one of the central topics in topological dynamics.

The classification of compact metric spaces admitting minimal maps is a well-known open problem in topological dynamics \[2,10\]. For the state of the art of the problem see \[3,8,9,22\] and references therein.

It is folklore that if \( X \) is a compact zero-dimensional space, \( f : X \to X \) is continuous and \( M \subseteq X \) is a minimal set of \( f \) then \( M \) is either a finite set (a periodic orbit of \( f \)) or a Cantor set and this is in fact a characterization because also conversely, whenever \( M \subseteq X \) is a finite or a Cantor set then there is a continuous map \( f : X \to X \) such that \( M \) is a minimal set of \( f \). Among one-dimensional spaces, the characterization of minimal sets is known for graphs — minimal sets on graphs are finite sets, Cantor sets and unions of finitely many pairwise disjoint simple closed curves, see \[4\] or \[29\]. The full characterization of minimal sets on (local) dendrites has been found just recently \[3\].

In higher dimensions the topological structure of minimal sets is much more complicated and, besides some important examples (see e.g. \[9,13\]), only few results are known. One obvious fact is that if \( h \) is a homeomorphism of a connected space \( X \) then a minimal set of \( h \) either is nowhere dense or coincides with \( X \). It is interesting that the same conclusion is true for continuous endomorphisms of compact connected 2-manifolds \[25\] while it is an open problem whether this result holds also in dimensions \( n > 2 \). A related question is which manifolds admit minimal maps. Again, the answer is completely known only in dimension 2: among 2-manifolds, compact or not, with or without boundary, only finite unions of tori and finite unions of Klein bottles admit minimal maps \[8\]. In dimensions higher than 2 the tori and we know from \[13\] that also the odd-dimensional spheres admit minimal diffeomorphisms. Note that a non-compact manifold never admits a minimal map by \[16\]. This is because we define minimality as density of forward orbits. It does not exclude the possibility to have a homeomorphism of a non-compact manifold with all full orbits dense. In any case, 2-sphere without a finite set of points does not admit such a homeomorphism \[27\].

To find a full topological characterization of minimal sets on compact, connected 2-manifolds is a very difficult task. Very recently, a classification of minimal sets on 2-torus has been obtained for homeomorphisms \[22\].

The main contribution of the present paper is a partial description of minimal sets of fibre-preserving maps in graph bundles.
1.1 Fibre-preserving maps and their minimal sets

A dynamical system \((E,F)\) is called an extension of a base dynamical system \((B,f)\) if there is a continuous surjective map \(p : E \to B\), called a factor map or a projection, such that \(p \circ F = f \circ p\). We also say that the base \((B,f)\) is a factor of \((E,F)\). Note that for every \(b \in B\) we have \(F(p^{-1}(b)) \subseteq p^{-1}(f(b))\), i.e., \(F\) sends the fibre over \(b\) into the fibre over \(f(b)\). Therefore \(F\) is said to be fibre-preserving. Suppose that \((B,f)\) is minimal and \((E,F)\) is an extension of it. If we additionally assume that \(E\) is compact then always there is a minimal set \(M\) in the system \((E,F)\) and since \(M\) projects onto a minimal set of \((B,f)\), we necessarily have \(p(M) = B\).

A very special case of an extension is when \(E\) is a cartesian product, \(E = B \times Y\), and \(F(x,y) = (f(x),g(x,y))\). Then \(F\) is fibre-preserving, the fibres being the “vertical” copies of \(Y\), i.e. the sets \(\{b\} \times Y\) where \(b \in B\), and the factor map being the natural projection of \(E\) onto \(B\). The map \(F\) is also called a skew product map or sometimes a triangular map.

The study of fibre preserving maps and their minimal sets has a long tradition. Much attention has been paid to minimal sets of fibre-preserving maps on the torus, for instance in the case of quasi-periodically forced (qpf) circle homeomorphisms. These systems naturally appear in the study of the scalar linear quasi-periodic Schrödinger equations. In such a case the dynamics is given by the projective action of a quasi-periodic \(\text{SL}(2,\mathbb{R})\)-cocycle (the 2-torus is identified with \(S^1 \times \mathbb{P}^1(\mathbb{R})\) and the projective action of \(\text{SL}(2,\mathbb{R})\) is considered on \(\mathbb{P}^1(\mathbb{R})\)). The most interesting situation occurs when the mentioned Schrödinger equations are non-uniformly hyperbolic [17].

An old question by Herman [19, Section 4.14] concerns topological structure of the unique minimal set \(M\) in that case. Herman partially described the set \(M\). In particular, \(M\) is nowhere dense and the intersection \(M_{\theta}\) of \(M\) with a vertical fibre \(\{\theta\} \times \mathbb{P}^1(\mathbb{R})\) is, generically, a singleton. Herman’s question is whether also all the other fibres \(M_{\theta}\) are connected; for more details and related results see [5], [6], [7], [19] and references therein. Bjerklöv [2] shows that the question has an affirmative answer in some special cases. According to recent preprint by Hric and Jäger [20], in general the answer is negative.

In a more general setting of skew product circle flows (both continuous and discrete) over a minimal base (forcing) on a compact metric space \(Y\), a topological classification of minimal sets was recently given by Huang and Yi [21]. They showed that if \(M\) is a minimal set of such a system then either \(M\) is the whole space \(Y \times S^1\), or there is a positive integer \(N\) such that a typical fibre of \(M\) consists of \(N\) points, or a typical fibre of \(M\) is a Cantor set. Below in Theorem E, we amplify this result to general fibre-preserving (not necessarily invertible) maps in compact graph bundles over a minimal base.

Béguin, Crovisier, Jäger and Le Roux [5] have constructed transitive qpf circle homeomorphisms with complicated minimal sets. For example, the minimal set can be a Cantor set whose intersection with each vertical fibre (circle) is uncountable (the possibility that some of these intersections have isolated points in the topology of the fibre has not been excluded and is probable). Thus, minimal sets of fibre-preserving maps can be quite complicated. This is true even for triangular maps in the square. To illustrate this, recall that so called Floyd-Auslander minimal systems [17].
are homeomorphisms which are extensions of Cantor minimal homeomorphisms and their phase spaces are subsets of the unit square which are nonhomogeneous — some fibres are compact intervals while the others are singletons. Modifying the construction, one can obtain also a noninvertible nonhomogeneous system of this kind [33]. Note that, by the extension lemma from [23], all these systems can be embedded into systems given by triangular selfmaps of the square.

In the present paper we wish to shed more light on the problem of characterizing minimal sets of higher dimensional maps by studying minimal sets of continuous fibre-preserving (not necessarily invertible) maps in graph bundles. It does not seem easy to generalize the results to more general bundles.

1.2 Star-like interior points and end-points in graph bundles

To state our main results, we need some terminology. A fibre space is an object $(E, B, p)$ where $E$ and $B$ are topological spaces and $p : E \to B$ is a continuous surjection. Here $E$, $B$ and $p$ are called the total space, the base (space) and the projection (map) of the fibre space, respectively, and $p^{-1}(b)$ is called the fibre over the point $b \in B$. If $\Gamma$ is another topological space, the fibre space $(E, B, p)$ is called a fibre bundle with fibre $\Gamma$, and denoted by $(E, B, p, \Gamma)$, if the projection map $p : E \to B$ satisfies the following condition of local triviality: For every point $b \in B$ there is an open neighborhood $U$ of $b$ (which will be called a trivializing neighborhood) and a homeomorphism $h : p^{-1}(U) \to U \times \Gamma$ such that on $p^{-1}(U)$ it holds $p \circ h = p$. Here $p_1 : U \times \Gamma \to U$ is the canonical projection onto the first factor. We will always assume that both $E$ and $B$ are compact metric spaces and so we will speak on compact fibre bundles.

Given a fibre space $(E, B, p)$, consider dynamical systems $(E, F)$ and $(B, f)$ with $p \circ F = f \circ p$. Thus, $(E, F)$ is an extension of $(B, f)$ and $(B, f)$ is a factor of $(E, F)$, the projection map $p$ being the factor map. Then $F$ is fibre-preserving, it sends the fibre $p^{-1}(b)$ over $b \in B$ into the fibre $p^{-1}(f(b))$ over $f(b)$.

A graph is a (nonempty) compact metric space which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end-points. A graph need not be connected and a singleton is not a graph. A tree is a graph containing no circle (i.e. a simple closed curve). The number of arcs emanating from a point $x \in G$ is called the order of $x$ and is denoted by $\text{ord}(x,G)$. Points of order 1 are called end-points of $G$ and points of order at least 3 are called ramification points of $G$.

For $n \geq 1$ we will consider the notion of the $n$-star $S_n$, which can be described as the set of all complex numbers $z$ such that $z^n$ is in the real unit interval $[0,1]$, i.e., a central point (the origin) with $n$ copies of the interval $[0,1]$ attached to it. We will view the $n$-star as a tree with $n + 1$ vertices, one of them (the central point) having order $n$ and the other $n$ vertices (the end-points of $S_n$) having order 1. Any set homeomorphic to $S_n$ will also be called an $n$-star and also denoted by $S_n$. Note that both $S_1$ and $S_2$ are homeomorphic to a closed interval. By the open $n$-star $\Sigma_n$ we will mean $S_n$ without its $n$ end-points. In particular, $\Sigma_2$ is homeomorphic to an open interval (while $\Sigma_1$ to a half-closed interval).
Definition 1 Let $\Gamma$ be a graph and $Z \subseteq \Gamma$ be closed. A point $x \in Z$ is said to be a star-like interior point of $Z$ if there exists a $Z$-open neighborhood of $x$ (i.e., the intersection of $Z$ and a $\Gamma$-open neighborhood of $x$) which is homeomorphic to $\Sigma_k$ for some $k \geq 2$; we assume here that this homeomorphism sends the point $x$ to the central point of $\Sigma_k$ (then $k$ is uniquely determined). If $x \in Z$ is not a star-like interior point of $Z$ we say that it is an end-point of $Z$. Let $\text{Sint}(Z)$ and $\text{End}(Z)$ denote the set of all star-like interior points of $Z$ and the set of all end points of $Z$, respectively.

Figure 1 shows that a star-like interior point of $Z$ need not be an interior point of $Z$ in $\Gamma$ and an interior point of $Z$ need not be a star-like interior point of $Z$.

![Fig. 1 There is no connection between interior and star-like interior points.](image)

The set $\text{Sint}(Z)$ is open in $Z$ (but not necessarily in $\Gamma$) and so the set $\text{End}(Z)$ is closed in $Z$ (hence closed in $\Gamma$). If $Z$ is a subgraph of $\Gamma$, the set $\text{End}(Z)$ coincides with the usual set of end-points of the graph $Z$.

A graph bundle is a fibre bundle whose fibre $\Gamma$ is a graph. Given a graph bundle $(E, B, p, \Gamma)$, for $M \subseteq E$ and $b \in B$ we denote $M_b = M \cap p^{-1}(b)$; this set is said to be the fibre of $M$ over $b$. When speaking on the fibres of $M$ over points lying in a subset $U$ of $B$, we sometimes call them fibres of $M$ over the set $U$. If $M \subseteq E$ and $U \subseteq B$, we denote $M_U = M \cap p^{-1}(U)$. So, $M_U$ is the union of all fibres of $M$ over the set $U$.

Definition 2 Given a closed set $M$ in a compact graph bundle $(E, B, p, \Gamma)$ we define the set of star-like interior points of $M$ and the set of end-points of $M$ by, respectively,

$$\text{Sint}(M) = \bigcup_{b \in B} \text{Sint}(M_b)$$

and

$$\text{End}(M) = \bigcup_{b \in B} \text{End}(M_b).$$

Of course, $\text{End}(M) = M \setminus \text{Sint}(M)$. In general it is not true that $\text{Sint}(M)$ is open or $\text{End}(M)$ is closed in $E$ or $M$.

1.3 Main results

Throughout the paper, $(E, B, p, \Gamma)$ is a compact graph bundle, $(E, F)$ and $(B, f)$ are dynamical systems with $p \circ F = f \circ p$. We also assume that the base system $(B, f)$ is minimal or, equivalently, that $p(M) = B$ for each minimal set $M \subseteq E$ of $F$. Our first
main result is the following dichotomy for a minimal set $M$ formulated in terms of end-points of $M$.

**Theorem A.** Let $M$ be a minimal set (with full projection onto the base) of a fibre-preserving map in a compact graph bundle $(E, B, p, \Gamma)$. Then there are two mutually exclusive possibilities:

(A1) $\text{End}(M) = M$ (and this holds if and only if $M$ is nowhere dense in $E$);
(A2) $\text{End}(M) = \emptyset$ (and this holds if and only if $M$ has nonempty interior in $E$).

In particular, the fibre-preserving maps in tree bundles have only nowhere dense minimal sets.

The assumption that the base system $(B, f)$ is minimal is not restrictive. In fact, if $M$ is a minimal set of $(E, F)$ then its projection $p(M)$ is a minimal set of $(B, f)$ and so one can pass to the sub-bundle over $p(M)$ and to consider, instead of $(E, F)$, the system $(E^*, F|_{E^*})$ where $E^* = p^{-1}(p(M))$. As an application of this fact we get that though a minimal set of a triangular map in the square can contain a vertical interval (so that in general $\text{End}(M) \neq M$ in the case (A1)), the following corollary holds

$I$ denotes a real compact interval and $\text{pr}_1$ is the projection onto the first coordinate).

**Corollary B.** Let $F(x, y) = (f(x), g(x, y))$ be a continuous triangular map in the square $I^2$ and let $M$ be a minimal set of $F$. Then $M$ is nowhere dense in the space $\text{pr}_1(M) \times I$.

We know from the characterization of minimal sets on the interval that $\text{pr}_1(M)$ is either a finite set or a Cantor set. In the latter case the result in the corollary is nontrivial, it strengthens Theorem 1 from [14] (where the same result is obtained for a very particular and small subclass of the class of triangular selfmaps of the square) and answers in negative the question posed by J. Smítal whether a minimal set $M$ of a triangular map in the square can have nonempty interior in the space $\text{pr}_1(M) \times I$.

So, no direct-product $B \times I$ admits a minimal fibre-preserving map (with the fibre $I$). Cannot we remove the assumption that the maps are fibre-preserving? The answer is negative. In fact, if $S^1$ is a circle and $H$ is the Hilbert cube then the space $P = S^1 \times H$ admits a continuous minimal map (in the form of a skew product map with an irrational rotation in the base $S^1$ and homeomorphisms $H \to H$ as fibre maps, see [15]). However, $P$ can be written in the form $P = (S^1 \times H) \times I$. Thus we have a space of the form $B \times I$ admitting a minimal, of course not fibre-preserving map (with the fibre being $I$). Here dimension of $B$ is infinite. An interesting question is whether it is true that all minimal, not necessarily fibre-preserving, maps in interval bundles $B \times I$ have only nowhere dense minimal sets if we additionally assume that $B$ has finite dimension. Recall that, by the result from [25], this is true if $B$ is a one-dimensional manifold, possible with boundary, so that $B \times I$ is a 2-manifold with boundary.

In each of the cases (A1) and (A2) in Theorem A, there are severe restrictions for the topological structure of the minimal set $M$. In the case (A2), some of such restrictions are listed in Theorem C whose full version is given in Section 6. Here, in Introduction, we prefer to list just those of them which seem to be most important and whose statement is neither cumbersome nor involves the notion of strongly star-like
interior points which will be introduced in Section 4. To keep the shortened version of the theorem compatible with the full version, we do not renumber the items.

**Theorem C (shortened version).** Let $M$ be a minimal set (with full projection onto the base) of a fibre-preserving map in a compact graph bundle $(E, B, p, \Gamma)$. Assume that $M$ has nonempty interior. Then the following holds.

(C4) All the sets $M_b$, $b \in B$, are unions of circles. In fact there exist an open dense set $\mathcal{O} \subseteq B$ and a positive integer $m$ such that

- for each $z \in \mathcal{O}$, $M_z$ is a disjoint union of $m$ circles, and
- for each $z \in B \setminus \mathcal{O}$, $M_z$ is a union of circles which properly contains a disjoint union of $m$ circles.

In particular, if $M_z$ is a circle for some $z \in \mathcal{O}$, then $M_z$ is a circle for all $z$ in the open dense subset $\mathcal{O}$ of $B$.

(C6) The set $M_{\mathcal{O}}$ is dense in $M$.

(C8) If $z \in \mathcal{O}$ then the set $M_z$, which is a disjoint union of $m$ circles, is mapped by $F$ onto a disjoint union of $m$ circles in $M_{f(z)}$.

(C10) If $f$ is monotone then $\mathcal{O} = B$ (hence, $M$ is a sub-bundle of $E$).

(C11) If $E = B \times \Gamma$ and $B$ is locally connected then $\mathcal{O} = B$ (hence, $M$ is a sub-bundle of $E$ and if $B$ is also connected, then $M$ is a direct product).

Properties of the map $F|_M$ are partially described in Proposition 2. The next result shows that $\mathcal{O} \neq B$ is possible and that some circles in a fibre of $M$ over a point in $B \setminus \mathcal{O}$ can intersect.

**Theorem D.** There is a minimal selfmap $f$ of a Cantor set $B$, a connected graph $\Gamma$ and an extension $(B \times \Gamma, F)$ of $(B, f)$ with a minimal set $M$ such that, for some $b \in B$,

- $M_z$ is a circle for each $z \neq b$, and
- $M_b$ is a union of two circles. Depending on the choice of such a system, the union of any two different circles in any graph can appear as the set $M_b$.

Recall that a set in a Baire space is called residual if its complement is of 1st category, i.e. a countable union of nowhere dense sets. By saying that a typical (or generic) fibre of $M$ has some property we mean that there is a residual set in the base $B$ such that for each $b$ in this residual set, the fibre $M_b$ of $M$ has this property.

Notice that Theorem C, part (C4), describes a typical fibre of the minimal set $M$ in the case (A2). Also in the case (A1) we are able to describe a typical fibre of $M$.

**Theorem E.** Let $M$ be a minimal set (with full projection onto the base) of a fibre-preserving map in a compact graph bundle $(E, B, p, \Gamma)$. Assume that $M$ is nowhere dense. Then either

(E1) a typical fibre of $M$ is a Cantor set, or
(E2) there is a positive integer $N$ such that a typical fibre of $M$ has cardinality $N$.

The number $N$ in (E2) is given by the formula from Proposition 3 in Section 7. Even if $F$ is a homeomorphism, one cannot claim that all fibres of $M$ have the same cardinality, see examples in the next section.
In the special case when $E$ is a direct product $B \times \Gamma$, $\Gamma$ is the circle and $F : E \to E$ is a homeomorphism, Theorem E has been known from [21, Theorem 6.1].

Notice the following asymmetry: in the case (A2) we know from (C4) that a “non-typical” fibre of $M$ is a union of circles, while in the case (A1) the topological structure of a “non-typical” fibre is unknown even for the qpf circle homeomorphisms and the triangular maps in the square (as Floyd-Auslander systems show, some of these fibres can contain nondegenerate intervals).

The paper is organized as follows. In Section 2 we present several illustrating examples of minimal sets of fibre-preserving maps in graph bundles and we also prove Theorem D. Section 3 contains some dynamical and topological preliminaries. Then, in Section 4 we introduce the key notion of our paper, namely that of a strongly star-like interior point of a subset of a graph bundle, and we study the structure of open neighborhoods of those compact subsets of a fibre which entirely consist of strongly star-like interior points of a given subset of the bundle. The proofs of Theorems A, C and E are given in Sections 5, 6 and 7, respectively.

2 Some examples and proof of Theorem D

Theorems C and E give necessary conditions for subsets of graph bundles to be minimal for a fibre-preserving map. Observe the following.

Suppose that the base $B$ is a singleton and so $E$ is just $\Gamma$. Then Theorems C and E imply that minimal sets on graphs are finite sets, Cantor sets or disjoint unions of (finitely many) circles. This is already a characterization of minimal sets on graphs, as shown in [4] or [29]. If $B$ is finite (and so the minimal base system is just a periodic orbit) we get that each fibre of $M$ either is a Cantor set or consists of the same finite number of points or the same finite number of disjoint circles. Again, one can easily show that this is a characterization of minimal sets (with full projection) of fibre-preserving maps in graph bundles with finite base.

However, we do not know how far we are from a topological characterization of minimal sets (with full projection) of fibre-preserving maps in graph bundles with infinite base. Indeed, if typical fibres of some compact set $M \subseteq E$ are as described in Theorems C and E (and $M$ has no isolated point, which would be an obvious obstacle for $M$ to be minimal) then it is not easy to check when there exists a fibre-preserving map $F$ in $E$ such that $M$ is a minimal set of $F$.

2.1 Examples of nowhere dense minimal sets

Only nowhere dense minimal sets can appear if $\Gamma$ is a tree. Say, a triangular map in the square can have a minimal set which is the direct product of a Cantor set with itself. More interesting are the following examples of nowhere dense minimal sets which are not totally disconnected.

Example 1 (Floyd-Auslander minimal sets). By the extension lemma from [23] one can extend any Floyd-Auslander minimal system $(M,H)$ (see [17]) to a triangular map defined on the product of the Cantor set (the projection of $M$) and a compact
interval. Though in this example \( H \) is a homeomorphism on \( M \), it is not true in general that if \( f \) is a homeomorphism then \( F|_M \) is monotone — to see it, replace \((M,H)\) in this construction by a noninvertible modification of it from \([33]\). Other examples can be obtained in a similar way, by replacing a Floyd-Auslander minimal system by some other cantoroids (for the definition of a cantoroid see \([3]\)). □

**Example 2** (Boundary of the Möbius band as a minimal set). Imagine, in \( \mathbb{R}^1 \), a circle \( S^1 \) in a horizontal plane and a vertical straight line segment \( I \) whose center is a point of \( S^1 \) and the length of \( I \) is smaller than the radius of \( S^1 \). By moving \( I \) periodically along \( S^1 \) in such a way that the center of \( I \) is always in \( S^1 \) and during one period, when the center of \( I \) comes back to its initial position, we turn \( I \) upside down to obtain the Möbius band \( E \). Here \( E \) is an interval bundle, \( S^1 \) being the base space and the positions of \( I \) being the fibres over points of \( S^1 \). The described movement, when considering time from \(-\infty\) to \(+\infty\), gives a flow on \( E \) and each time-\( t \) map of this flow is a fibre-preserving map on \( E \).

We can move \( I \) in such a way that for the time-\( t \) map \( F \) of the mentioned flow, the restriction \( f = F|_{S^1} \) is an irrational rotation, by some angle \( \alpha \), of \( S^1 \). Hence \( S^1 \) is a minimal set of \( F \). Then the boundary \( \partial E \) of \( E \) is also a minimal set of \( F \), since the restriction of \( F \) to \( \partial E \) is conjugate to \( \alpha/2 \) rotation of the circle.

Notice that the simple closed curve \( \partial E \) is a sub-bundle of \( E \) (the fibre having cardinality 2) but it is not a direct product of the base space \( S^1 \) with a two-point set. □

**Example 3** (Sturmian minimal sets). Consider a Sturmian minimal system \((\mathcal{S},\sigma)\), see e.g. \([34]\) pp. 200–203, satisfying the following properties: it is a minimal subshift of \( \{0,1\}^\mathbb{Z} \) and it is an almost one-to-one extension of a system \((\hat{S}^1,\text{rot}_{\alpha})\), where \( \hat{S}^1 \) is the circle and \( \text{rot}_{\alpha} \) is an irrational rotation. More precisely, if \( \Sigma: (\mathcal{S},\sigma) \rightarrow (\hat{S}^1;\text{rot}_{\alpha}) \) is the corresponding factor map, then there is a countable dense set \( D \subset \hat{S}^1 \) such that for all \( z \in \hat{S}^1 \setminus D \) the fibre \( \Sigma^{-1}(z) \) consists of just one point of \( \mathcal{S} \) and for all \( z \in D \) the fibre \( \Sigma^{-1}(z) \) consists of two points of \( \mathcal{S} \). We may think of \( \mathcal{S} \) as being a minimal set of a fibre-preserving map in \( \hat{S}^1 \times [0,1] \), whose base map is \( \text{rot}_{\alpha} \). Let us explain this.

The point inverses of \( \Sigma \) are the fibres of the mentioned almost 1-to-1 extension and the homeomorphism \( \sigma \) sends fibres to fibres. Topologically, \( \mathcal{S} \) is a Cantor set (since the Sturmian system is an uncountable minimal subshift) and so we may assume that \( \mathcal{S} \subseteq [0,1] \). Consider the map \( H : \mathcal{S} \rightarrow \hat{S}^1 \times [0,1] \) sending \( s \in \mathcal{S} \) to \((s(s),s) \in \hat{S}^1 \times [0,1] \). Then \( H \) is continuous and injective, so it is an embedding of the set \( \mathcal{S} \) into the cylinder \( \hat{S}^1 \times [0,1] \). Moreover, vertical fibres of the Cantor set \( H(\mathcal{S}) \subseteq \hat{S}^1 \times [0,1] \) correspond to point inverses of \( \Sigma \) which means that \( H \) induces fibre-preserving dynamics on \( H(\mathcal{S}) \) which is topologically conjugate to \( \sigma \).

Again, by the extension lemma from \([23]\), one can extend this dynamics on \( H(\mathcal{S}) \) to a fibre-preserving map \( F : \hat{S}^1 \times [0,1] \rightarrow \hat{S}^1 \times [0,1] \). Then \( H(\mathcal{S}) \) is a minimal set of \( F \) having singleton fibres with the exception of countably many fibres, each of them consisting of two points. □
2.2 Examples of minimal sets with nonempty interior

This case can occur only if the graph $\Gamma$ contains a circle. As an example, consider an irrational rotation of the torus ($M$ is the whole torus). To produce some more general “direct product” examples with $B$ being a general compact metric space admitting a minimal map, one can use Proposition 1 and Corollary 2 below.

To prove Proposition 1 let us start by recalling a theorem due to H. Weyl (see e.g. [26, Chapter I, Theorem 4.1]) saying that if $(a_n)_{n=1}^\infty$ is a sequence of distinct integers then for almost all (with respect to the Lebesgue measure) real numbers $x$ the sequence $(a_n x)_{n=1}^\infty$ is uniformly distributed modulo 1. As an obvious consequence of this theorem we get that for any sequence of positive integers $n_1 < n_2 < \ldots$ there is an angle $\alpha$ such that the rotation $g$ of $S^1$ by the angle $\alpha$ is minimal with respect to the sequence $(n_k)_{k=1}^\infty$. This means that for every $s \in S^1$ the set $\{g^{n_k}(s) : k = 1, 2, \ldots\}$ is dense in $S^1$. Of course, any such rotation $g$ is necessarily irrational.

The following simple proposition dealing with direct product maps (rather than with skew product minimal systems as for instance in [15]) is, though not most general possible, sufficient for our purposes. We present here a short proof, based on the Weyl’s theorem mentioned above.

**Proposition 1** Let $(B, f)$ be a minimal dynamical system, $B$ being a metric space. Then there exists an irrational rotation $g$ of the circle $S^1$ such that the direct product system $(B \times S^1, f \times g)$ is minimal.

**Proof** Fix $x_0 \in B$ and positive integers $n_1 < n_2 < \ldots$ such that $f^{n_i}(x_0) \to x_0$ when $k \to \infty$. By the Weyl’s theorem, there is an irrational rotation $g$ of $S^1$ such that for every $s \in S^1$ the set $\{g^{n_i}(s) : k = 1, 2, \ldots\}$ is dense in $S^1$. We claim that $F = f \times g$ is minimal. It is sufficient to prove that the $\omega$-limit set $\omega_F(x, s) = B \times S^1$ for every $(x, s) \in B \times S^1$.

From the choice of $x_0$ and $g$ it follows that for every $y \in S^1$, $\omega_F(x_0, y) \supseteq \{x_0\} \times S^1$. Since the $f$-orbit of $x_0$ is dense in $B$ and $F(\omega_F(x_0, y)) \subseteq \omega_F(x_0, y)$ and $g$ is surjective, the closed set $\omega_F(x_0, y)$ contains the union of a dense family of fibres. We have thus proved that $\omega_F(x_0, y) = B \times S^1$ for every $y \in S^1$.

Now fix any point $(x, s) \in B \times S^1$. Since $\omega_F(x) = B$ and $S^1$ is compact, the set $\omega_F(x, s)$ contains at least one point $(x_0, y) \in \{x_0\} \times S^1$. Then $\omega_F(x, s) \supseteq \omega_F(x_0, y) = B \times S^1$.

**Corollary 1** Let $E = B \times \Gamma$ be a graph bundle such that $B$ is a compact metric space admitting a minimal map and $\Gamma$ be a graph containing a circle $C$. Then there exists a fibre-preserving mapping $F : E \to E$ such that $B \times C$ is a minimal set of $F$.

**Proof** Using Proposition 1 extend a minimal map $f : B \to B$ to a minimal map $f \times g : B \times C \to B \times C$. Then use the fact that there is a retraction $r : \Gamma \to C$ and put $F = f \times (g \circ r)$.

However, for a general (i.e., not direct product) graph bundle $(E, B, p, \Gamma)$, where $B$ is a compact metric space admitting a minimal map and $\Gamma$ contains a circle, the existence of fibre-preserving maps having minimal sets with nonempty interior is not
clear at all. For instance, already the construction of such a minimal homeomorphism on the Klein bottle is not easy, see \([11]\) or \([32]\). We do not know whether in any graph bundle which is not a tree bundle and whose base admits a minimal map there exists a fibre-preserving map having a minimal set with nonempty interior.

Recall that \((X, f)\) is a totally minimal system if \((X, f^n)\) is minimal for \(n = 1, 2, \ldots\).

**Corollary 2** Let \((B, f)\) be a totally minimal dynamical system, \(B\) being a metric space. Let \(\Gamma\) be a graph which contains \(m\) disjoint circles. Denote the union of these circles by \(S\). Then there exists a continuous map \(h : \Gamma \to \Gamma\) such that \(B \times S\) is a minimal set in the direct product system \((B \times \Gamma, f \times h)\).

**Proof** Let \(g\) be the irrational rotation by angle \(\alpha\), which can be assigned to the minimal system \((B, f^m)\) by Proposition 1. Fix a circle \(C\) in \(S\). Let \(\tilde{g}\) be the map \(S \to S\) whose restriction to \(C\) is conjugate to \(g\) and which is identity on \(S \setminus C\). Then compose \(\tilde{g}\) with a homeomorphism on \(S\), which cyclically permutes the \(m\) circles in \(S\).

Finally, extend the selfmap of \(S\) obtained in such a way to a continuous selfmap \(h\) of \(\Gamma\) (this is possible, see e.g. \([4]\)). By Proposition 1 the set \(B \times C\) is minimal for \((f \times h)^m = f^m \times h^m\) since \(h^m|_C\) is conjugate to \(g\). Then \(B \times S\) is minimal for \(f \times h\). \(\square\)

**Example 4** (Torus attached to the boundary of the Möbius band as a minimal set).

We construct a space \(E\) similarly as the Möbius band in Example 2 with only one difference — now, instead of moving the straight line segment \(I\) along the circle \(S^1\), we move the graph \(\Gamma\) which is the segment \(I\) with two identical circles attached to \(I\) at the endpoints of \(I\) in such a way that the intersections of the circles with the straight line segment joining the centers of the circles are the endpoints of \(I\). We assume that the diameter of \(\Gamma\) is smaller than that of \(S^1\). So, \(E\) is a Möbius band whose boundary simple closed curve is replaced by a topological torus \(T^2\).

As in Example 2 we consider the time-1 map \(F\) of the flow induced by the mentioned “movement” and put \(f = F|_I\), an irrational rotation of \(S^1\) by some angle \(\alpha\). The map \(F\) is fibre-preserving and we are going to extend it to a fibre-preserving continuous map \(G : E \to E\) for which the torus \(T^2\) will be a minimal set.

Let \(\phi : \Gamma \to \Gamma\) be any continuous map such that the points of \(\Gamma\) which are symmetrical with respect to the center of \(I\) are mapped to symmetrical points (hence the center of \(I\) is a fixed point) and the restriction of \(\phi\) to each of the two circles in \(\Gamma\) is an irrational rotation. The symmetry condition requires that both circles rotate by the same angle \(\beta\) and with the same “orientation”. Further, let \(\Phi : E \to E\) be a continuous map which maps each of the fibres of \(E\) into itself in such a way that the restriction of \(\Phi\) to each of the fibres is an isometric copy of \(\phi\) (the fibres of \(E\) are isometric to \(\Gamma\)). Simply, in one of the fibres we choose an orientation of the circles (the same orientation), hence also the “orientation” of the \(\beta\)-rotations on them. The continuity of \(\Phi\) then determines the “orientation” of the rotations on the circles in all other fibres. (Since we have the same orientation of the circles in \(\Gamma\), one can see that this is a correct construction, we really get a well defined map \(\Phi\).)

Put \(G = \Phi \circ F\). Then \(G\) is a fibre-preserving map on the graph-bundle \(E\) and the restriction of \(G\) to the torus \(T^2\) is a double rotation — irrational \(\alpha/2\)-rotation in one direction and \(\beta\)-rotation in the other direction. Now we restrict ourselves to \(\beta\) for which \(G\) is a minimal map on \(T^2\). Notice that, in contrast to Corollary 2, the obtained minimal set \(T^2\) is not a direct product of the base space \(S^1\) with a union of circles. \(\square\)
2.3 Proof of Theorem D

Given a set $A \subseteq \mathbb{R}^k$ and a vector $v \in \mathbb{R}^k$, by $A + v$ we mean the set $\{a + v : a \in A\}$.

**Theorem D.** There are a minimal selfmap $f$ of a Cantor set $B$, a connected graph $\Gamma$ and an extension $(B \times \Gamma, F)$ of $(B, f)$ with a minimal set $M$ such that, for some $b \in B$,

- $M_b$ is a circle for each $z \neq b$, and
- $M_b$ is a union of two circles. Depending on the choice of such a system, the union of any two different circles in any graph can appear as the set $M_b$.

**Proof.** Case I: $M_b$ is a union of two disjoint circles.

Let $(C, f)$, with $C$ being a subset of the real line, be a Cantor minimal system such that $C$ has exactly two pre-images and all the other points have only one pre-image each. Such systems appear for instance in symbolic and interval dynamics. It will be convenient to give an explicit construction of such a system in order to introduce the notation which will be used throughout the whole proof. Start with the dyadic adding machine on the Cantor ternary set. Recall that it is often viewed as a restriction of the adding machine to the invariant Cantor set, usually a restriction of the map shown for instance in [33, Fig. 1]; notice that then the adding machine is increasing at each point except at the rightmost one where it is decreasing. Choose a point $a$ in this Cantor set which does not belong to the countable set consisting of the endpoints of the contiguous intervals (including the leftmost and the rightmost points of the Cantor set). Hence the points $a_{-j} := f^{-j}(a)$, $j = 1, 2, \ldots$ do not belong to this countable set, too. Now blow up the backward orbit of $a$, i.e., for $j = 1, 2, \ldots$, replace the point $a_{-j}$ by a compact interval with length $L_{-j}$ with convergent sum $\sum_{j=1}^{\infty} L_{-j}$ and remove the interior of this interval. This means that the points $a_{-j}$, $j = 1, 2, \ldots$ are “doubled”, i.e. replaced by pairs of points $a_{-j}^- < a_{-j}^+$. What we get is again a Cantor set. Consider the dynamics on it which is inherited from the adding machine, except for the “new” points $a_{-j}^-a_{-j}^+$, $j = 1, 2, \ldots$ where we still need to define the dynamics. To this end, send both $a_{-j}^-$ and $a_{-j}^+$ to $a$ and, since the adding machine is increasing at each $a_{-j}$ and we want a continuous dynamics, for $j = 2, 3, \ldots$ send $a_{-j}^-$ to $a_{-j+1}^-$ and $a_{-j}^+$ to $a_{-j+1}^+$. The map defined in such a way is continuous and the system is minimal.

Recall that, up to a homeomorphism, there is only one Cantor set and it is homogeneous. Therefore, no matter which of the Cantor minimal systems $(C, f)$ (such that one point has two pre-images and all the other points have only one pre-image) we choose, we may think of $C$ as a Cantor set on the real line, with the point having two pre-images being for instance the rightmost point of $C$. For the same reason we can also assume that the two-preimages, denote them $c_l < c_r$, are the endpoints of a contiguous interval (this is important for geometry of our construction below).

Applying Proposition [1] we extend $(C, f)$ to a minimal system $(C \times S_1, f \times g)$ where $g$ is an irrational rotation of the circle $S_1 = \{y, z) \in \mathbb{R}^2 : y^2 + z^2 = 1\}$. Denote by $a_1$ and $b_1$ the $g$-images of the points $(0, 1)$ and $(0, -1)$, respectively. Let $J_1$ be one of the half-circles determined by $a_1, b_1$.

The set $C$ is the union of $C_L = \{x \in C : x \leq c_l\}$ and $C_R = \{x \in C : x \geq c_r\}$. Put $C_R = C_R - (c_r - c_l)$. Then $C_L \cup C_R$ is a Cantor set with $C_L \cap C_R = \{c_l\}$. Further put $s_2 = s_1 + (0, 3), a_2 = a_1 + (0, 3), b_2 = b_1 + (0, 3)$ and $J_2 = J_1 + (0, 3)$. Finally,
denote \( M = (C_L \times S_1) \cup (C_R \times S_2) \). The dynamical system \((C \times S_1, f \times g)\) induces in a natural way a (minimal) dynamical system \((M, F)\) which is topologically conjugate to \((C \times S_1, f \times g)\) and is obtained from \((C \times S_1, f \times g)\) by just replacing \((C_R \times S_1)\) by its translate \((C_R \times S_2)\), ‘without changing dynamics’. In the new system \((M, F)\) the map \( F \) preserves ‘vertical’ fibres; the fibre over \( c_l \) consists of two circles, each of the other fibres is just a circle. Denote by \( \varphi \) the base map of \( F \). It is clear that \((M, F)\) can be considered as a minimal extension of the dynamical system \((C_l \cup C_R, \varphi)\) obtained from \((C, f)\) by identifying points \( c_l \) and \( c_r \). Let \( \Gamma = S_1 \cup J \cup S_2 \) where \( I \subseteq \mathbb{R}^2 \) is the ‘vertical’ interval with end-points \((0, 1)\) and \((0, 2)\). Put \( E = (C_L \cup C_R) \times \Gamma \). Then \( \Gamma \) is a connected graph and \( E \) is a graph bundle with fibre \( \Gamma \).

We claim that the map \( F \) can be extended to a continuous fibre-preserving map \( G : E \to E \). We are going to define \( G \). Of course, \( G|_M = F \). Further, for \( x \in C_L \setminus \{c_l\} \) and \( y, z \in S_2 \) put \( G(x, y, z) = F(x, y, z - 3) \) and for \( x \in C_R \setminus \{c_l\} \) and \( y, z \in S_1 \) put \( G(x, y, z) = F(x, y, z + 3) \). So, \( G \) is already defined on \((C_L \cup C_R) \times (S_1 \cup S_2)\). It remains to define \( G \) on \((C_L \cup C_R) \times (I \setminus \{(0, 1), (0, 2)\})\). So, fix \( x \in C_L \cup C_R \). Then \( G(x) \) is the “geometrical” circle such that each fibre of \( G \) has to be a graph (i.e., a simple closed curve) such that \( \varphi(x) \) is topologically conjugate to \( \varphi(x) \times \{a_i\} \) and \( G(x, 0, z) = \varphi(x) \times \{b_i\} \). For \( 1 < z < 2 \) let \( G(x, 0, z) \) be the point of \( \varphi(x) \times J_i \) such that the length of the sub-arc of \( \varphi(x) \times J_i \) with end-points \( \varphi(x) \times \{a_i\} \) and \( G(x, 0, z) \) equals \( z - 1 \).

Then \( G \) maps \( E \) continuously onto its unique minimal set \( M \). Here \( M_{c_l} \) is the union of two circles and \( M_b \) for \( b \neq c_l \) is a circle. So, \( M \) is not a sub-bundle of \( E \).

**Case II:** \( M_b \) consists of two arbitrarily intersecting circles whose union is a graph.

Before giving such a construction we wish to mention that if \( E \) were not required to be a graph bundle, it would be sufficient to consider a skew product minimal map on the pinched torus from \([2]\). In that example, one fibre is “figure eight” (two circles intersecting in one point), all the other fibres are circles (simple closed curves).

The union \( P \cup Q \) of disjoint sets will sometimes be denoted by \( P \cup Q \). We will also keep the notations from Case I. Starting with the minimal system \((C \times S_1, f \times g)\) we are going to produce a fibre-preserving selfmap \( G^* \) of a direct product graph bundle \( E^* \subseteq \mathbb{R}^2 \) with the following properties:

1. \( E^* = (C_L \cup C_R) \times \Gamma^* \)
2. \( \Gamma^* = S_1 \cup S_1' \) where \( S_1 \) is the “geometrical” circle \( y^2 + z^2 = 1 \) and \( S_1' \) is a “topological” circle (i.e., a simple closed curve) such that
   - \( \emptyset \neq S_1 \cap S_1' \neq S_1 \) has finitely many connected components (just because we want \( E^* \) to be a graph bundle, i.e. \( \Gamma^* \) has to be a graph),
   - \( S_1' \) is a subset of the closed disc bounded by the circle \( S_1 \) and each radius of \( S_1 \) contains exactly one point of \( S_1' \),
3. \( M^* = (C_L \times S_1) \cup (C_R \times S_1') \) is a minimal set for \( G^* \).

Note that each fibre of \( M^* \) consists of one circle, except of \( M_{c_l}^* \) which consists of two intersecting circles \( \{c_l\} \times \{S_1\} \) and \( \{c_l\} \times \{S_1'\} \). Though the only restrictions for the choice of \( S_1' \) are those in (2), let us explicitly mention three simplest cases:

(121) \( S_1 \) and \( S_1' \) intersect just in one point (hence \( M_{c_l}^* \) is homeomorphic to the “figure eight”), or
(122) \( S_1 \) and \( S_1' \) intersect in an arc (\( M_{c_l}^* \) is homeomorphic to the “figure \( \Theta \)”), or
(123) \( S_1 \) and \( S_1' \) intersect in two points.
Then \( \sigma: C \times S_1 \to (C_L \times S_1) \cup (C_R \times S_1) \) is a homeomorphism and so the map

\[
\sigma(x, y, z) := \begin{cases} (x, y, z), & \text{if } (x, y, z) \in C_L \times S_1, \\ (x, \alpha(y, z)), & \text{if } (x, y, z) \in C_R \times S_1. \end{cases}
\]

Then \( \sigma: C \times S_1 \to (C_L \times S_1) \cup (C_R \times S_1) \) is a homeomorphism and so the map

\[
F_1^* := \sigma \circ F_1 \circ \sigma^{-1},
\]

being topologically conjugate to \( F_1 \), is a continuous minimal selfmap of \( (C_L \times S_1) \cup (C_R \times S_1) \). Since \( f(c_l) = f(c_r) = \max C \), the definition of \( F_1^* \) gives that

\[
F_1^*(\{c_l\} \times S_1) = \{\max C\} \times S_1^* = F_1^*(\{c_r\} \times S_1^*), \quad \text{(2.1)}
\]

\[
F_1^*(c_l, y, z) = \sigma(F_1(c_l, y, z)) = \sigma(F_1(c_l, y, z)) = F_1^*(c_r, \alpha(y, z)) \quad \text{for } (y, z) \in S_1. \quad \text{(2.2)}
\]

Then (2.1) and (2.2) imply that

\[
F_1^*(c_l, y, z) = F_1^*(c_r, y, z) \in \{\max C\} \times S_1^*, \quad \text{for } (y, z) \in S_1 \cap S_1^*. \quad \text{(2.3)}
\]

Now the idea is to identify the pairs of points \((c_l, y, z), (c_r, y, z)\) where \((y, z) \in S_1 \cap S_1^*\) with the same \( F_1^* \)-images and to produce in such a way a new map \( F_2^* \) on a new space \( M^* \). Since we wish to keep under control the geometry of our example, we proceed geometrically. In view of (2.3), the mentioned pairs of points become identified if we translate \( C_R \times S_1^* \) by the vector \((-c_r - c_l)\), \(0, 0\). Therefore denote \( M^* := (C_L \times S_1) \cup (C_R \times S_1^*) \) and let \( T: (C_L \times S_1) \cup (C_R \times S_1^*) \to M^* \) be defined by

\[
T(x, y, z) := \begin{cases} (x, y, z), & \text{if } (x, y, z) \in C_L \times S_1, \\ (x - (c_r + c_l), y, z), & \text{if } (x, y, z) \in C_R \times S_1^*. \end{cases}
\]

As already indicated, due to (2.3) there is a unique continuous map \( F_2^*: M^* \to M^* \) such that the following diagram commutes:

\[
\begin{array}{ccc}
(C_L \times S_1) \cup (C_R \times S_1^*) & \xrightarrow{F_1^*} & (C_L \times S_1) \cup (C_R \times S_1^*) \\
\downarrow T & & \downarrow T \\
M^* & \xrightarrow{F_2^*} & M^*
\end{array}
\]

A straightforward analysis of the map \( F_2^* \) shows that a point in \( \{c_l\} \times S_1 \subseteq M^* \) and a point in \( \{c_l\} \times S_1^* \subseteq M^* \) lying on the same radius of the circle \( \{c_l\} \times S_1 \) have always the same \( F_2^* \)-image:

\[
F_2^*(c_l, y, z) = F_2^*(c_l, \alpha(y, z)) \quad \text{whenever } (y, z) \in S_1. \quad \text{(2.4)}
\]

To finish the study of the properties of \( F_2^* \), notice that \( F_2^* \) is fibre-preserving and, being a factor of the minimal map \( F_1^* \), is also minimal.
Now define $E^*$ and $\Gamma^*$ as in 1. and 2. at the beginning of the proof of Case II. To finish our construction, it is sufficient to extend $F^*_{2} : M^* \to M^*$ to a continuous fibre-preserving map $G^* : E^* \to E^*$. Here is one such extension:

$$G^*(x,y,z) := \begin{cases} 
F^*_{2}(x,y,z), & \text{if } (x,y,z) \in M^*, \\
F^*_{2}(x,\alpha^{-1}(y,z)), & \text{if } (x,y,z) \in (C_L \setminus \{c\}) \times S^1, \\
F^*_{2}(x,\alpha(y,z)), & \text{if } (x,y,z) \in (C_R \setminus \{c\}) \times S^1.
\end{cases} \tag{2.5}$$

The definition is correct. In fact, the first and the second case are compatible, because if $(x,y,z) \in M^*$ and simultaneously $x \in C_L \setminus \{c\}$ and $(y,z) \in S^1$, then $(y,z) \in S^1 \cap S^1$ and so $(\alpha^{-1})(x,y,z) = (x,y,z)$. Analogously, the first and the third case are compatible. Hence, $G^* : E^* \to E^*$ is a well defined extension of $F^*_{2}$. It is obviously fibre-preserving. To show that it is continuous, it is sufficient to show that the restrictions of $G^*$ to the closed sets $C_L \times (S_1 \cup S^1_1)$ and $C_R \times (S_1 \cup S^1_1)$ are continuous. Since the arguments for both cases are analogous, we prove only the continuity of $G^*$ on the former set. It is the union of two closed sets $C_L \times S_1$ and $C_L \times S^1$ and so the continuity of $G^*|C_L \times (S_1 \cup S^1_1)$ follows from the following two facts:

- On the set $C_L \times S_1$, since it is a subset of $M^*$, the map $G^*$ is continuous because it coincides there, by (2.5), with the continuous map $F^*_{2}$.
- On the set $C_L \times S^1_1$ the map $G^*$ is also continuous, because it coincides there with the continuous map $F^*_{2} \circ (\text{id}_{C_L} \times \alpha^{-1})$ where $\text{id}_{C_L}$ is the identity on $C_L$. To see this, first notice that for $x \in C_L \setminus \{c\}$ the coincidence works by (2.5). Further, if $(y^*,z^*) \in S^1_1$ then $(c_1,y^*,z^*) \in M^*$ and so, using (2.5) and (2.4) we get $G^*(c_1,y^*,z^*) = F^*_2(c_1,y^*,z^*) = F^*_{2}(c_1,\alpha^{-1}(y^*,z^*))$, as required.

The construction is now completed. In the case (121) it gives the space $E^*$ made of two "tubes" $(C_L \cup C_R^*) \times S_1$ and $(C_L \cup C_R^*) \times S^1_1$, the second tube lying "inside" the first one and so they touch "internally". If one wishes that they touch "externally", i.e. that $M^*_{\alpha}$ is a geometric, not only topological "figure eight", it is sufficient to use an appropriate conjugacy. Similarly, in (122) the tubes can "touch externally" along an arc. Also in (123) we can get that $S^1_1$ is not anymore a subset of the closed disc bounded by the circle $S_1$, but $S_1$ and $S^1_1$ are two geometric circles having two points in common. 

\[ \square \]

### 3 Dynamical and topological preliminaries

For convenience of the reader, we collect below several dynamical and topological facts which will be used throughout the rest of the paper. The reader should at least pay attention to the concepts of a redundant open set and the homeo-part of a minimal system since they are instrumental in the paper.

#### 3.1 Some basic facts on minimality

In this subsection we always assume that $X$ is a compact metric space and $f : X \to X$ is a continuous map. The facts here, if not obvious, are mostly results from our
In the introduction we gave two equivalent definitions of minimality (in terms of invariant subsets and in terms of density of forward orbits). For a compact metric space $X$ and a continuous map $f : X \to X$ also the following are equivalent:

1. $(X, f)$ is minimal,
2. $f(X) = X$ and every backward orbit of every point in $X$ is dense (by a backward orbit of $x_0 \in X$ we mean any set $\{x_0, x_1, \ldots, x_n, \ldots\}$ with $f(x_{i+1}) = x_i$ for $i \geq 0$),
3. the only closed subsets $A$ of $X$ with $f(A) \supseteq A$ are $\emptyset$ and $X$,
4. for every non-empty open set $U \subseteq X$, there is $N \in \mathbb{N}$ such that $\bigcup_{n=0}^{N} f^{-n}(U) = X$.

We will also need some necessary conditions for minimality. If $(X, f)$ is minimal then

a. for every non-empty open set $U \subseteq X$, there is $N \in \mathbb{N}$ such that $\bigcup_{n=0}^{N} f^n(U) = X$,

b. $f$ is feebly open, i.e. it sends non-empty open sets to sets with non-empty interior,

c. $f$ is almost one-to-one, which means that the set $\{x \in X : \text{card } f^{-1}(x) = 1\}$ is a $G_\delta$-dense set in $X$,

d. if $A \subseteq X$ is nowhere dense (dense, of 1st category, of 2nd category, residual) then both $f(A)$ and $f^{-1}(A)$ are nowhere dense (dense, of 1st category, of 2nd category, residual), respectively.

A set $G \subseteq X$ is said to be a redundant open set for a map $f : X \to X$ if $G$ is nonempty, open and $f(G) \subseteq f(X \setminus G)$ (i.e., its removal from the domain of $f$ does not change the image of $f$).

**Lemma 1** ([24]) Let $X$ be a compact metric space and $f : X \to X$ continuous. Suppose that there is a redundant open set for $f$. Then the system $(X, f)$ is not minimal.

### 3.2 Homeo-part of a minimal system

**Definition 3** Let $f$ be a continuous selfmap of a compact metric space $X$. Let $H \subseteq X$ be the set of all points $x_0 \in X$ whose full orbit $\{x \in X : \exists i, j \geq 0 \text{ with } f^i(x) = f^j(x_0)\}$ is of the form $\{x_0, x_1, \ldots, x_{n-1}, x_0, x_1, x_2, \ldots\}$ where $f(x_n) = x_{n+1}$ for every integer $n$. Then the system $(H, f|_H)$ is said to be the homeo-part of the system $(X, f)$. We also shortly say that $H$ is the homeo-part of $f$.

One can show that $H$ is always a $G_\delta$ set (possibly empty). For minimal maps this is easier to prove and we can say even more.

**Lemma 2** Let $X$ be a compact metric space and $f : X \to X$ be a minimal map. Then the homeo-part $H$ of $f$ is a dense $G_\delta$ set.

**Proof** Set $D = \{x \in X : \text{card } f^{-1}(x) > 1\}$. By [24] Theorem 2.8, the homeo-part of a minimal map is residual and $D$ is an $F_\sigma$-set of first category. It is straightforward to check that $H = X \setminus \bigcup_{n=0}^{+\infty} f^n(D)$. By (d) from Subsection 3.1 we get that $H$ is $G_\delta$. □
Lemma 3 Let $X$ be a compact metric space and $f : X \to X$ be a continuous minimal map. Let a set $D = \{ \ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots \}$ be such that $f(x_k) = x_{k+1}$ for every integer $n$ (i.e., $D$ is a union of the forward orbit of $x_0$ and one of the backward orbits of $x_0$). Suppose that there is a point in $D$ that has more than one $f$-preimage in $X$ (or, equivalently, an $f$-preimage in $X \setminus D$). Then $(f|_D)^{-1}$ is not continuous.

Proof Suppose, on the contrary, that $g := (f|_D)^{-1} : D \to D$ is continuous. Without loss of generality we may assume that the mentioned point with two preimages is $x_0$. Denote by $z$ a point in $X \setminus D$ with $f(z) = x_0$. Choose two disjoint open neighborhoods $U_{-1}$ and $U_z$ of the points $x_{-1}$ and $z$, respectively. Denote $V_{-1} := U_{-1} \cap D$. Due to the continuity of $g$ at the point $x_0$, we can find a neighborhood $U_0$ of $x_0$ such that for $V_0 := U_0 \cap D$ we have $g(V_0) \subseteq V_{-1}$. Now use the continuity of $f$ at the point $z$ to get an open neighborhood $U_z^* \subseteq U_z$ of $z$ with $f|U_z^* \subseteq U_0$. Since $f$ is minimal, there is $k > 0$ with $x_k \in U_z^*$, whence $x_{k+1} = f(x_k) \in V_0$. Then $g(x_{k+1}) = x_k \in U_z^*$ which contradicts the facts that $g(V_0) \subseteq V_{-1}$ and $U_z^*$ is disjoint with $V_{-1}$. □

The next description of properties of the homeo-part of a minimal map follows partially from Theorem 2.8 and its proof in [24]. Note that the notion of a full orbit of a point (for a not necessarily invertible map) was introduced in Definition 3.

Lemma 4 Let $X$ be a compact metric space and $f : X \to X$ be a continuous minimal map. Let $H \subseteq X$ be the homeo-part of $f$. Then:

1. $f(H) = H = f^{-1}(H)$ (equivalently, $H$ is a union of full orbits of the map $f$),
2. every point of the set $H$ has just one $f$-pre-image (and this pre-image lies in $H$),
3. both $f|H$ and $(f|H)^{-1}$ are minimal homeomorphisms $H \to H$,
4. $H$ is a $G_\delta$ dense subset of $X$,
5. $H$ is a maximal subset of $X$ with the properties (1) and (2),
6. $H$ is a maximal subset of $X$ with the property (3).

Proof The equivalence in (1) is obvious. The properties (1) and (2) follow from the definition of the homeo-part, see Definition 3. For the property (3) see Theorem 2.8 in [24] and its proof. Lemma 2 gives (4). The property (5) is obvious since if we add something to $H$, we have to add another full orbit (because we want (1)). This full orbit contains, due to the definition of the homeo-part, a point with two preimages. Then the enlarged set will not satisfy (2). Similarly, Lemma 3 shows that if we add something to $H$ then the enlarged set will not satisfy (3) and so we get (6). □

Lemma 5 Let $X$ be a compact metric space and $f : X \to X$ be minimal. Let $H$ be the homeo-part of $f$ and $P$ be a residual set in $X$. Then there is a set $R \subseteq X$ such that

1. $R \subseteq P \cap H$ and $R$ is residual in $X$,
2. $f(R) = R = f^{-1}(R)$,
3. both $f|_R$ and $(f|_R)^{-1}$ are minimal homeomorphisms $R \to R$.

In particular, the inclusion $R \subseteq H$ and (2) give that $R$ is a union of some of the full (i.e., forward and backward) orbits of the homeomorphism $f|_R$.

Proof Put $R = H \cap \bigcap_{n \in \mathbb{Z}} f^n(P)$. The minimal map $f$ preserves residuality in both forward and backward direction. Therefore the set $R$, being the intersection of countably many residual sets, is residual in $X$. The rest is obvious. □
3.3 Locally closed sets and a generalization of Baire category theorem

A subset $S$ of a topological space $X$ is \textit{locally closed} if every $x \in S$ has a neighborhood $U$ such that the intersection $S \cap U$ is closed in the subspace $U$ of $X$. The following conditions are equivalent, see e.g. [12, p. 112]:

1. The set $S$ is locally closed.
2. The difference $\overline{S} \setminus S$ is closed (i.e. $S$ is open in $\overline{S}$).
3. $S$ is a difference of two closed sets (intersection of a closed set with an open set).

\textbf{Lemma 6} Let $X$ be a topological space and $S \subseteq X$ a locally closed set. If $S$ is not nowhere dense then $S$ has nonempty interior.

\textit{Proof} By the assumption (if $\text{Int}_X$ denotes the interior in $X$) we have $V := \text{Int}_X(S) \neq \emptyset$. The set $V$ is open in $X$ and hence, being a subset of $\overline{S}$, obviously also open in $\overline{S}$. Now the fact that $S$ is dense in $\overline{S}$ gives that $S \cap V \neq \emptyset$. Further, since $S$ is locally closed, $\overline{S}$ is open in $\overline{S}$. Therefore there is a set $U$ open in $X$ such that $S = U \cap \overline{S}$. Since $S \subseteq U$ and $S \cap V \neq \emptyset$, we have $U \cap V \neq \emptyset$. This set is open in $X$ and since $U \cap V \subseteq \overline{S}$ we get that $\text{Int}_X(S) \neq \emptyset$. \hfill \Box

Recall that a \textit{Baire space} is a topological space having the property that whenever a countable union of closed sets has nonempty interior then one of them has nonempty interior (i.e. so called Baire category theorem works). The following lemma gives a generalization of Baire category theorem: it shows that closed sets can be replaced by locally closed ones.

\textbf{Lemma 7} Let $X$ be a Baire topological space and $\{S_\lambda : \lambda \in \Lambda\}$ a countable family of subsets of $X$. Assume that

(i) $\bigcup_{\lambda \in \Lambda} S_\lambda$ has nonempty interior in $X$ and
(ii) for every $\lambda \in \Lambda$, $S_\lambda$ is locally closed.

Then there is $\lambda_0 \in \Lambda$ such that $S_{\lambda_0}$ has nonempty interior in $X$.

\textit{Proof} By applying Baire category theorem to the closed sets $\overline{S}_\lambda$, $\lambda \in \Lambda$, we get that there is $\lambda_0 \in \Lambda$ such that $\overline{S}_{\lambda_0}$ has nonempty interior. So, $S_{\lambda_0}$ is not nowhere dense and since it is locally closed, it has nonempty interior in $X$ by Lemma[6] \hfill \Box

4 Strongly star-like interior points

We introduce the notion of a \textit{strongly} star-like interior point which is more restrictive than that of a star-like interior point of $M$ and, though not appearing in the statement of Theorem A, will play a key role in the proof of it.

First of all recall that, when speaking on a graph bundle, we always assume that it is a (compact) \textit{metric space}, as it was already said in Introduction. To avoid cumbersome formulations, we will often make no distinction between homeomorphic spaces. If $(E, B, p, \Gamma)$ is a graph bundle and $Q \subseteq E$ and $Z \subseteq \Gamma$, then we say that $Q$ is \textit{canonically homeomorphic} to $U \times Z$, if $p(Q) = U$ and there is a homeomorphism
\(h : Q \to U \times Z\) such that on \(Q\) we have \(pr_1 \circ h = p\) (here \(h\) is said to be a \textit{canonical homeomorphism}). Notice that, in this terminology, in the above definition of the fibre bundle it is required that \(p^{-1}(U)\) be canonically homeomorphic to \(U \times \Gamma\).

Recall that if \((E, B, p, \Gamma)\) is a graph bundle and \(M \subseteq E\) and \(b \in B\), then the the fibre of \(M\) over \(b\) is \(M_b = M \cap p^{-1}(b)\). Further, by \(\Gamma_b\) we will denote the set \(p^{-1}(b)\), the fibre over \(b\) (now we slightly abuse the already adopted notation \(M_b\), since \(\Gamma\) is not a subset of \(E\)). Note that \(\Gamma_b = E_b \subseteq E\) is a graph homeomorphic to \(\Gamma\) and if \(E = B \times \Gamma\) then \(\Gamma_b = \{b\} \times \Gamma\). Also subsets of \(\Gamma_b\) will be sometimes denoted by, say, \(I_b\), \(T_b\), etc.

We believe that this will not cause any misunderstanding because always when using notation like \(X_b\) it will be clear what kind of a set it is. Recall also that if \(M \subseteq E\) and \(U \subseteq B\), we denote \(M_U = M \cap p^{-1}(U)\).

By an arc we mean a homeomorphic image of a compact real interval. Sometimes we call it a closed arc, since in an obvious way we also use the notions of an open or a half-closed arc. For \(N \geq n \geq 2\) let \(\Sigma_n \subseteq \Sigma_N\) be two open stars with the same central point. Suppose that \(\Sigma_n\) is the union of some of the half-closed branches of \(\Sigma_N\) (i.e., \(\Sigma_n\) is obtained from \(\Sigma_N\) by removing \(N - n \geq 0\) open branches of \(\Sigma_N\)). Then we will say that \(\Sigma_n\) is a \textit{full sub-star of} \(\Sigma_N\). Here ‘full’ does not mean that \(n = N\); it just refers to the fact that \(\Sigma_n\) consists of ‘whole’ branches of \(\Sigma_N\) (rather than of just subsets of them) and so it can be \(n < N\). Note also that we consider only the case when \(N \geq n \geq 2\) (though, formally, such a definition would make sense for \(N \geq n \geq 1\)).

**Definition 4** Suppose that \(M\) is a closed subset of a \textit{product} graph bundle \(E = B \times \Gamma\). Then we define \(\text{Sint}_s(M)\), the set of strongly star-like interior points of \(M\), as follows. A point \(x = (x_1, x_2) \in M\) is said to be a \textit{strongly star-like interior point} of \(M\), if

- \(x\) has order \(N \geq 2\) in the graph \(\Gamma_{x_2} = \{(x_1) \times \Gamma\}\),
- there exists an \(E\)-open neighborhood \(O \times \Sigma_N\) of \(x\) such that \(x_2\) is the central point of \(\Sigma_N\) and the corresponding \(M\)-open neighborhood \(\mathcal{G} = M \cap (O \times \Sigma_N)\) of \(x\) has the following structure:

\[
\mathcal{G}_{x_1} = \{x_1\} \times \Sigma_k
\]

where \(k \geq 2\) and \(\Sigma_k\) is a full sub-star of \(\Sigma_N\), and for every \(z \in p(\mathcal{G}) \subseteq O\) we have \(\mathcal{G}_z = \{z\} \times \Sigma_{k(z), z} \subseteq \{z\} \times \Sigma_k\), where \(k(z) \in \{2, \ldots, k\}\) and \(\Sigma_{k(z), z}\) is a full sub-star of \(\Sigma_k\). (Notice that \(\Sigma_{k(x_1), x_1} = \Sigma_k\).

We will say that \(\mathcal{G}\) is a \textit{canonical} \(\text{Sint}_s(M)\)-neighborhood of \(x\) (note that, among others, \(\mathcal{G} \subseteq \text{Sint}_s(M)\)).

Above, \(\text{Sint}_s(M)\) was defined for a closed subset \(M\) of \(E = B \times \Gamma\). Since each graph bundle is locally trivial and the above definition has a local character, the concept of a strongly star-like interior point has an obvious extension to the case when the graph bundle \(E\) is not a direct product space. For a closed set \(M\) in an arbitrary graph bundle we set \(\text{End}_s(M) = M \setminus \text{Sint}_s(M)\).

**Example 5** Let \(E = B \times \Gamma\) where \(B = [0, 1]\) and \(\Gamma = ([−1, 1] \times \{0\}) \cup \{(0) \times [0, 1]\}\). Put \(A = [0, 1] \times [-1, 1] \times \{0\}\) and

\[
M^1 = A \cup \{(x, 0, x) : x \in [0, 1]\}, \quad M^2 = M^1 \cup \{(0, 0, z) : z \in [0, 1]\}, \quad M^3 = A \cup \{(0, 0, z) : z \in [0, 1]\}, \quad M^4 = A \cup \{(x, 0, 1-x) : x \in [0, 1]\}.
\]

Then \((0, 0, 0) \notin \text{Sint}_s(M^i)\) for \(i = 1, 2\) and \((0, 0, 0) \in \text{Sint}_s(M^i)\) for \(i = 3, 4\). \(\square\)
In the definition we write $\Sigma_{k(z)}$ rather than $\Sigma_{k(c)}$ because it may happen that $\Sigma_{k(z_1)}$ and $\Sigma_{k(z_2)}$, considered as subgraphs of $\Gamma$, are different even when $k(z_1) = k(z_2)$. The following instructive example illustrates this fact.

**Example 6** Let $E = B \times S_4$ where $B = [0, 1]$. Let $(C_n)_{n=1}^\infty$ be a sequence of pairwise disjoint Cantor sets in $(0, 1]$ converging, in the Hausdorff metric, to the singleton $\{0\}$. Denote three of the four closed branches of $S_4$ by $J_1, J_2, J_3$ and the central point of $S_4$ by $c$. Let $M$ be the set with

$$M_k = \begin{cases} \{x\} \times (J_1 \cup J_2 \cup J_3) & \text{if } x = 0 \\ \{x\} \times (J_1 \cup J_2) & \text{if } x \in C_n \text{ for } n \equiv 1 \mod 3, \\ \{x\} \times (J_2 \cup J_3) & \text{if } x \in C_n \text{ for } n \equiv 0 \mod 3, \\ \{x\} \times (J_3 \cup J_1) & \text{if } x \in C_n \text{ for } n \equiv 2 \mod 3, \\ \emptyset & \text{otherwise,} \end{cases}$$

see Fig. 2. Then $M$ is compact and $\{0\} \times \{c\} \in \text{Sint}_t(M)$. In fact all the points of $M$ except of the end-points of the stars $M_x, x \in p(M)$, belong to $\text{Sint}_t(M)$. □

![Fig. 2](image)

Notice that $\text{Sint}_t(M)$ is open in $M$ (but not necessarily in $E$) and $\text{End}_t(M)$ is closed in $M$ (hence closed in $E$). By comparing Definitions 2 and 4 observe that

$$\text{Sint}_t(M) \subseteq \text{Sint}(M) = \bigcup_{b \in B} \text{Sint}(M_b), \quad \text{End}_t(M) \supseteq \text{End}(M) = \bigcup_{b \in B} \text{End}(M_b). \quad (4.1)$$

In general neither of these two inclusions is an equality. For $M \subseteq E$ and $b \in B$ we will further use the notation

$$M_b^{M_t} = M_b \cap \text{Sint}_t(M) = (\text{Sint}_t(M))_b.$$

**Example 7** Let $E = B \times \Gamma$ with $B = [-1, 1]$ and $\Gamma = [0, 3]$. Let $C$ be a Cantor set with $\min C = 0$, $\max C = 1$ and let $M = ([[-1, 0] \times [0, 1]) \cup (C \times [1, 2]) \cup \{0\} \times [2, 3])$. Then $\text{Sint}_t(M) = ([[-1, 0] \times (0, 1)) \cup (C \times (1, 2)) \cup \{0\} \times (2, 3))$. So, $M_0^{M_t} = \{0\} \times ((0, 1) \cup (1, 2) \cup (2, 3))$ while $\text{Sint}(M_0) = \{0\} \times (0, 3)$. □
Lemma 8 Let $(E,B,p,G)$ be a compact graph bundle and $M \subseteq E$ a compact set. Then

$$\text{End}_sM = \text{End}M.$$  

Proof Without loss of generality we may assume that $E = B \times G$. One inclusion is trivial by \cite{4,1}. To prove the other one, suppose that there is a point $x \in \text{End}_s(M) \setminus \text{End}M$. Then, if the second coordinate of $x$ has order $m$ in $G$, we have $m \geq 2$ (otherwise $x$ would be in $\text{End}(M)$) and some $E$-open neighborhood $O \times \Sigma_m$ of $x$ is disjoint with $\text{End}(M)$. Hence, if $z \in O$ then the set $(\{z\} \times \Sigma_m) \cap M$ is empty or is of the form $(\{z\} \times \Sigma_{k(z)} z)$ where $k(z) \in \{2,\ldots,m\}$ and $\Sigma_{k(z)} z$ is a full sub-star of $\Sigma_m$ (otherwise it would necessarily contain a point from $\text{End}(M_x)$). It follows that $x \in \text{Sint}_s(M)$, a contradiction.

Lemma 9 Let $(E,B,p,G)$ be a compact graph bundle and $M \subseteq E$ a compact set. If $\text{End}_s(M) = M$ then $M$ is nowhere dense in $E$.

Proof If $M$ is somewhere dense in $E$ then, being closed, has nonempty interior in $E$. It is clear that this interior contains a point which belongs to $\text{Sint}_s(M)$.

Lemma 10 Let $(E,B,p,G)$ be a compact graph bundle and $M \subseteq E$ a compact set with $p(M) = B$. If $\text{End}(M) = \emptyset$ then $M$ has nonempty interior in $E$.

Proof We may assume that $E = B \times G$. Let $K_1,K_2,\ldots,K_k$ be the list of circles in $G$. For $i = 1,2,\ldots,k$, let $B^{(i)}_j$ be the set of points $b \in B$ such that $M_b$ contains $b \times K_i$. The set $M$ is closed and so all the sets $B^{(i)}_j$ are closed. Since $p(M) = B$ and $\text{End}(M) = \emptyset$, we have $B = \bigcup_{i=1}^k B^{(i)}_j$ and since the metric space $B$ is compact (hence second category), there is $j \in \{1,2,\ldots,k\}$ such that the (closed) set $B^{(i)}_j$ has nonempty interior. Since $G$ is a graph, it follows that $M$ has nonempty interior in $E$.

Trivial examples show that the converse statements to the previous two lemmas are not true.

Lemma 11 Let $E = B \times G$ be a compact graph bundle, $M \subseteq E$ a compact set and $a \in B$. Suppose that $\Delta = \{a\} \times \Delta^G$ is a compact subset of $M_0^G$. If $W$ is a sufficiently small open neighborhood of $a$ and $U$ is a sufficiently small open neighborhood of $\Delta^G$ then the $E$-open neighborhood $W \times U$ of $\Delta$ has the following properties:

- The corresponding $M$-open neighborhood $\mathcal{D} = M \cap (W \times U)$ of $\Delta$ is a subset of $\text{Sint}_s(M)$.
- If we write $\mathcal{D}_z = \{z\} \times \mathcal{D}^G$, then $\mathcal{D}_z \subseteq \mathcal{D}^G$ and $\mathcal{D}^G_\mathcal{D}_z \subseteq \overline{\mathcal{D}^G_\mathcal{D}_z}$ whenever $z \in p(\mathcal{D})$.
- The set $p(\mathcal{D})$ is closed in $W$, hence it is a Baire space.

Proof Since $\Delta$ is compact, it can be covered by a finite family of $M$-open sets $\mathcal{D}^j = M \cap (O_j \times \Sigma_{N(j)})$, $j = 1,\ldots,r$, where $\mathcal{D}^j$ are some canonical $\text{Sint}_s(M)$-neighborhoods of points in $\Delta$. Put $W = \bigcap_{j=1}^r O_j$ and $U = \bigcup_{j=1}^r \Sigma_{N(j)}$. We prove that $\mathcal{D} = M \cap (W \times U)$ satisfies all the requirements. First, it is obvious that $\mathcal{D}$ is an $M$-open neighborhood of $\Delta$ and $\mathcal{D} \subseteq \text{Sint}_s(M)$. Further notice that if we denote, for $j = 1,\ldots,r$, $M^j = M \cap (W \times \Sigma_{N(j)})$
then \( a \in p(M'), (M')_a = \{ a \} \times \Sigma^j \) where \( \Sigma^j \) is a full sub-star of \( \Sigma_{N(j)} \) and, for \( z \in p(M'), (M')_z = \{ z \} \times \Sigma_{j(z)} \) where \( \Sigma_{j(z)} \) is a full sub-star of \( \Sigma^j = \Sigma_{j(a),a} \). Thus

\[
\mathcal{G} = M \cap (W \times U) = \bigcup_{j=1}^r M^j = \bigcup_{j=1 \in p(M')} (\{ z \} \times \Sigma_{j(z)}) .
\]

Fix \( z \in p(\mathcal{G}) = \bigcup_{j=1}^r p(M^j) \). Since

\[
\mathcal{G}_a^j = \bigcup_{j=1}^r \Sigma_{j(z)} \quad \text{in particular} \quad \mathcal{G}_a^j = \bigcup_{j=1}^r \Sigma^j,
\]

we get \( \mathcal{G}_a^j \subseteq \mathcal{G}_a^r \). Hence \( \mathcal{G}_a^j \subseteq \mathcal{G}_a^r \) and so, to prove that \( \mathcal{G}_a^j \subseteq \mathcal{G}_a^r \), it is sufficient to show that the assumption that some point \( q \in \mathcal{G}_a^j \setminus \mathcal{G}_a^r \) belongs to \( \mathcal{G}_a^r \), leads to a contradiction. To this end consider such a point \( q \). Since \( q \in \mathcal{G}_a^j \), there is \( j \in \{ 1, \ldots, r \} \) such that \( q \in \Sigma^j \) and so \( q \in U \). On the other hand, \( q \in \mathcal{G}_a^r \) and so \( (z,q) \in \mathcal{G} \subseteq M = M \). Also, \( (z,q) \in W \times U \) because \( z \in p(\mathcal{G}) \subseteq W \) and \( q \in U \). Thus, \( (z,q) \in M \cap (W \times U) = \mathcal{G} \) which implies that \( q \in \mathcal{G}_a^r \), a contradiction.

Now we prove that \( p(\mathcal{G}) \) is closed in \( W \). It can be seen from the definition that if \( \mathcal{G} \) is a canonical \( \text{Sint},(M) \)-neighborhood of a point \( x \in M \subseteq B \times \Gamma \) and for each \( z \in p(\mathcal{G}) \) we put \( \mathcal{G}_z = \{ z \} \times \mathcal{G}_a^r \), then the family \( \{ \mathcal{G}_z : z \in p(\mathcal{G}) \} \) is finite. Since \( \mathcal{G} \) was defined using only finitely many such canonical \( \text{Sint},(M) \)-neighborhoods, we get that also the family \( \{ \mathcal{G}_z : z \in p(\mathcal{G}) \} \) is finite. Therefore, if \( p(\mathcal{G}) \ni z \rightarrow \rightarrow z \in W \), we may (passing to a subsequence if necessary) assume that all sets \( \mathcal{G}_z^r \) are the same. But then, since \( M \) is closed, obviously \( \mathcal{G}_z \) is nonempty and so \( z \in p(\mathcal{G}) \).

So, the set \( p(\mathcal{G}) \) is closed (hence is of type \( G_d \)) in the metric space \( W \). Since \( W \) is open in \( B \), this implies that \( p(\mathcal{G}) \) is \( G_d \) in the compact space \( B \). Thus \( p(\mathcal{G}) \) is a topologically complete (i.e. completely metrizable) space, hence a Baire space (see, e.g., [31, Theorems 12.1 and 9.1]).

In the situation from Lemma[11], let \( \Delta \subseteq M^j \) be connected. Then it is a graph and obviously there exist \( m, n \geq 0 \) such that every sufficiently small connected open \( I_a \)-neighborhood \( V \) of \( \Delta \) has the following properties:

- \( V \) is connected and (see Lemma[1]) \( M \cap V \subseteq M^j \),
- \( V \setminus \Delta \) consists of pairwise disjoint open arcs \( \{ a \} \times I_1, \ldots, \{ a \} \times I_m, \{ a \} \times J_1, \ldots, \{ a \} \times J_n \) where the arcs \( \{ a \} \times I_1 \) are subsets of \( M^j \) and the arcs \( \{ a \} \times J_1 \) are disjoint from \( M_a \). Each of these arcs is attached to \( \Delta \) at an end-point of \( \Delta \) or at a ramification point of \( I_a \) (an end-point of \( \Delta \) can simultaneously be a ramification point of \( I_a \)).

We extend the notion of a ramification point as follows. If \( G \) is a (not necessarily closed and not necessarily connected) subset of a graph \( \Gamma \) and \( g \in G \), we say that \( g \) is a ramification point of \( G \) if there is a \( G \)-open neighborhood of \( g \) which has the form of an open \( r \)-star with \( r \geq 3 \) and with central point \( g \).

By an open graph we mean a graph without its end-points if it has any. So, since a graph is a union of finitely many connected graphs, an open graph is a union of finitely connected graphs.
many connected open graphs, whose closures are pairwise disjoint. Notice that, by
this definition, a graph having no end-points (in particular, a circle) is also an open
graph and that a circle with one point removed is not an open graph. If an open graph
$G$ is a subset of a graph $\Gamma$ then $G$ need not be an open set in $\Gamma$. Each ramification
point of $G$ is a ramification point of $\Gamma$ but the converse is not true in general. If $\Gamma$ is
a graph and $G \subseteq \Gamma$ is an open graph, by the end-points of $G$ we mean the end-points
of the (closed) graph $\overline{G}$. It follows from the definition of strongly star-like interior
points that the set $M^\Delta_a$ is open in the topology of $M_a$ (though not necessarily open
in the topology of $\Gamma_a$). Its connected components are not necessarily open graphs.
For instance, $M^\Delta_a$ can be a circle with one point removed. In any case, $M^\Delta_a$ is a subset of
$\Gamma_a$ and so the notion of a ramification point can be applied to it.

In the following two technical lemmas we keep the notation from Lemma 11.

**Lemma 12** Let $E = B \times \Gamma$ be a compact graph bundle, $M \subseteq E$ a compact set and
$a \in B$. Suppose that $\Delta = \{a\} \times \Delta^\Gamma \subseteq M^\Delta_a$ is an arc or a circle and does not contain
any ramification point of $M^\Delta_a$. Then for any sufficiently small open neighborhood $W$ of
$a$ and any sufficiently small connected open neighborhood $U$ of $\Delta^\Gamma$ as in Lemma 11 it
holds that $\mathscr{D} = W^* \times U^*$, i.e. $\mathscr{D}$ has the structure of a direct product. Here $a \in W^* \subseteq
W$ is some not necessarily $B$-open set. If $\Delta$ is a circle then $\{a\} \times U^*$ coincides with $\Delta$
and if $\Delta$ is an arc then $\{a\} \times U^*$ is an open arc containing $\Delta$ (and still containing no
ramification point of $M^\Delta_a$).

**Proof** Every point from $\Delta$ has an $M_a$-neighborhood in the form of an open arc and
so, since $\Delta \subseteq \text{Sint}_a(M)$, $\Delta$ can be covered by a finite family of canonical $\text{Sint}_a(M)$-
neighborhoods of points from $\Delta$ which have the form (see the proof of Lemma 11)
\[ q^j = M \cap (O_j \times \Sigma_{N(j)}) = V^j \times \Sigma^j_a. \]

Here $V^j$ is a (not necessarily $B$-open) set containing $a$ and $\Sigma^j_a$ is an open arc in $\Gamma$
such that $\{a\} \times \Sigma^j_a \subseteq \text{Sint}_a(M)$ contains no ramification point of $M^\Delta_a$.

If two open arcs $\Sigma^j_a$ and $\Sigma^j_b$ intersect and $z \in \bigcap O_j$ then $z \in V^j$ if and only if $z \in V^b$.
This together with the fact that $\Delta$ is connected gives that if $z \in \bigcap O_j$ then $z$ belongs to
all of the sets $V^j$ whenever it belongs to one of them. Now let $U$ be any sufficiently
small connected open neighborhood of $\Delta^\Gamma$ so that $(\{a\} \times U) \cap M_a \subseteq \{a\} \times \bigcup \Sigma^j_a$.

Further, let $W \subseteq \bigcap O_j$ be any open neighborhood of $a$. Then the claim holds with
$U^* = U \cup \bigcup \Sigma^j_a$ and $W^* = \{z \in W : z \in V^j$ for some (hence for all) $j\}$. \qed

**Example 8** Let us return to Example 6. Denoting by $\Delta$ an arc in $M_0$ containing the
ramification point $c$, we see that without assuming that $\Delta$ contains no ramification
point of $M^\Delta_a$, in Lemma 11 one cannot ensure the existence of $\mathscr{D}$ in the form of a
direct product. Further, if $\Delta$ does not contain the ramification point $c$ and is a sub-arc
of, say, $J_1$ we see that one cannot claim that $W^*$ exists in the class of $B$-open sets. \qed

**Example 9** Let us return to Example 7 and put $\Delta = \{0\} \times \{1/2, 3/2, 5/2\}$. Then
$\Delta \subseteq M^\Delta_a$ and it does not contain any ramification point of $M^\Delta_0$ (even any ramification
point of $\Gamma_0$). However, $\Delta$ is disconnected and there is no $M$-open neighborhood of $\Delta$
of the product form $W^* \times U^*$. \qed
Lemma 13 Let $E = B \times \Gamma$ be a compact graph bundle, $M \subseteq E$ a compact set and $a \in B$. Suppose that $\Delta = \{a\} \times \Delta^\Gamma \subseteq M^\Delta_a$ is a graph possibly degenerate to a singleton (and possibly containing ramification points of $M^\Delta_a$, which may or may not be ramification points of $\Delta$). Then for any sufficiently small open neighborhood $W$ of $a$ and any sufficiently small connected open neighborhood $U$ of $\Delta^\Gamma$ as in Lemma 11, the following holds.

- $\left(\{a\} \times (U \setminus \Delta^\Gamma)\right) \cap M \subseteq M^\Delta_a$ is empty or consists of pairwise disjoint open arcs $\{a\} \times I_1^\Gamma, \ldots, \{a\} \times I_m^\Gamma$ ($m \geq 0$ being finite and independent on $U$, since $U$ is small enough; $m = 0$ means that the described set is empty).
- For each $i = 1, \ldots, m$ the open arc $\{a\} \times I_i^\Gamma$ is attached to $\Delta$ at a point $p_i = (a, p_i^\Gamma)$ which is an end-point of $\Delta$ or a ramification point of $M^\Delta_a$ (an end-point of $\Delta$ can simultaneously be a ramification point of $M^\Delta_a$ and it can be $p_i = p_j$ even if $i \neq j$), and at each of the end-points of $\Delta$ there is at least one such open arc attached to it. Here for every $i$, the closure of $\{a\} \times I_i^\Gamma$ is an arc and any two of the sets $\Delta, \{a\} \times I_i^\Gamma$, $i = 1, \ldots, m$ are either disjoint or intersect only at one of the ‘attaching’ points $p_i$.
- $\mathcal{D}_a = \Delta$ or $\mathcal{D}_a = \Delta \cup \bigcup_{i=1}^m (\{a\} \times I_i^\Gamma)$, depending on whether $m = 0$ or $m \geq 1$. So, $\mathcal{D}_a$ is an open graph.
- The structure of the corresponding $M$-open neighborhood $\mathcal{D} = M \cap (W \times U) \subseteq \text{Sint}_z(M)$ of $\Delta$ is such that for any $z \in p(\mathcal{D})$, $\mathcal{D}^z_a \subseteq \mathcal{D}^z$ is a union of finitely many open graphs whose closures are pairwise disjoint, $\mathcal{D}^z_a \subseteq \mathcal{D}^z_\Delta$ and $\text{End}(\mathcal{D}^z_a) \subseteq \text{End}(\mathcal{D}^z_\Delta)$.
- For any $z \in p(\mathcal{D})$, each of the connected components of $\mathcal{D}_z$ is the union of a (nonempty, closed) possibly degenerate subgraph of $\{z\} \times \Delta^\Gamma$ and some (possibly zero) of the open arcs $\{z\} \times I_i^\Gamma$ with the ‘attaching’ points $(z, p_i^\Gamma)$ belonging to $\mathcal{D}_z$. If this subgraph is nondegenerate and does have one or more end-points, then at each of these end-points there is at least one of these open arcs attached to it. If the subgraph is a singleton (which may happen even if $\Delta$ is nondegenerate) then at least two of these open arcs are attached to it.

In particular, if $\Delta$ is a tree, possibly degenerate to a singleton, then:

- For each $z \in p(\mathcal{D})$, the set $\mathcal{D}_z$ contains (a nonempty closed subgraph of $\{z\} \times \Delta^\Gamma$, possibly disconnected, possibly degenerate to a finite set, and) at least two of the open arcs $\{z\} \times I_i^\Gamma$, with the ‘attaching’ points $(z, p_i^\Gamma)$ belonging to $\mathcal{D}_z$.

Of course, if $\Delta$ is a singleton, then the last statement of the lemma does not say anything more than the definition of a strongly star-like interior point of $M$.

Proof The arguments are completely analogous to those used in the proof of Lemma 11. In fact, the first three parts are just consequences of our definitions of $M^\Delta_a$, ramification points, endpoints and open graphs. The rest follows from Lemmas 11 and the remarks above Lemma 11 (Note that a key role is played by the fact that $\mathcal{D} \subseteq \text{Sint}_z(M)$. For instance, if the intersection of $\mathcal{D}_z$ with $\{z\} \times \Delta^\Gamma$ is a singleton, then at least two open arcs have to be attached to this singleton, otherwise $\mathcal{D}_z$ could not be a subset of $\text{Sint}_z(M)$.)
5 Proof of Theorem A

We will use the notation $F_z = F|_{I_z}$. So, $F_z$ is a map from $I_z$ into $\Gamma_f(z)$.

We start with the following result partially describing $F$ on its minimal sets in case (A2) of our Theorem A. Its use simplifies arguments in the proof of Theorem A.

**Proposition 2** Let the assumptions of Theorem A be satisfied. Let $I_a$ be a closed arc and $T_b$ be a tree such that $I_a \subseteq M^S_a$, $T_b \subseteq M_b$ and $F(I_a) \subseteq T_b$. If the interior of $I_a$ does not contain any ramification point of $M^S_a$ then $F|_{I_a}$ is monotone (hence $F(I_a)$ is an arc or a point).

The statement in the parentheses is obvious since a monotone image of an arc cannot be a nondegenerate tree. Both cases (i.e., $F(I_a)$ is an arc or a point) occur in the example of a noninvertible fibre-preserving minimal map on the torus in [24] (the base is a ‘horizontal’ circle, the fibres are ‘vertical’ circles). Since in this example there is a vertical arc mapped by $F$ into a point while the vertical circle containing this arc is mapped onto a circle, the example also shows that the proposition would not be true if $T_b$ were allowed to contain a circle.

**Proof** It is sufficient to prove a weaker version of the proposition which is obtained by adding the assumption that neither the end-points of $I_a$ are ramification points of $M^S_a$. For if one or both end points of $I_a$ are ramification points of $M^S_a$ then, by applying such a weaker proposition to all sub-arcs $J_a$ of $I_a$ which do not contain end-points of $I_a$, we get the monotonicity of $F$ on the whole interior of $I_a$. Since the $F$-image of this interior is a point or a (not necessarily closed) arc and $T_b$ does not contain a circle, $F$ is obviously monotone on $I_a$.

So, let $I_a$ contain no ramification point of $M^S_a$ and suppose, on the contrary, that $F|_{I_a}$ is not monotone. Then there exists $q \in T_b$ such that $(F|_{I_q})^{-1}(q) \subseteq I_q$ is not connected. Take two points $u, v$ in two different connected components of $(F|_{I_q})^{-1}(q)$ and consider the (unique) arc $J_a \subset I_a$ with the end-points $u, v$. From the choice of $u, v$ it follows that there is a point $w \in J_a$ with $F(w) \neq q$. This point $w$ partitions $J_a$ into two nondegenerate closed sub-arcs $J^1_a$ and $J^2_a$. The set $F(J_a) = F_a(J_a) \subseteq T_b$ is a nontrivial continuum (hence a tree) and each of the sets $F(J^1_a)$ and $F(J^2_a)$ contains the (unique) arc in $T_b$ having the end-points $F(w)$ and $q$. It follows that the arc $F_a$ contains two disjoint closed nondegenerate sub-arcs $T^1_a, T^2_a$ such that $F(T^1_a)$ and $F(T^2_a)$ are closed arcs with $F(T^1_a) \subseteq \text{Int} F(T^2_a)$ (where by $\text{Int} F(T^2_a)$ we mean the arc $F(T^2_a)$ without its end-points).

Now, since we will work only with some neighborhood of $a$, we may assume that $E$ has the structure of a product space, i.e. $E = B \times I$. So $I_a$ has the form $\{a\} \times I$ and similarly $T^1_a = \{a\} \times T^1$ and $T^2_a = \{a\} \times T^2$. By Lemma [12] there is an $M$-open neighborhood $\mathcal{O}$ of $I_a$ which has the product form $\mathcal{O} = W^* \times U^*$ for some (not necessarily $B$-open) set $W^* \ni a$ and some open arc $U^*$ containing $I$.

Since $F_a(\{a\} \times T^1) \subseteq \text{Int} F_a(\{a\} \times T^2)$ and since (by replacing $T^1$ by a smaller arc if necessary) we may assume that the arc $F_a(\{a\} \times T^1)$ does not contain any ramification point of $\Gamma_b$, we have $F_a(\{x\} \times T^1) \subseteq \text{Int} F_a(\{x\} \times T^2)$ also for all $x$ sufficiently close to $a$. By replacing $W^*$ by its intersection with a small open neighborhood of $a$
if necessary, we may assume that this is the case for all \( x \in W^* \). Then
\[
F|_{\mu(W^* \times \text{Int} T^1)} \subseteq F|_{\mu(W^* \times T^2)} \subseteq F|_{\mu(M \setminus (W^* \times \text{Int} T^1))}.
\]
Hence the nonempty \( M \)-open set \( W^* \times \text{Int} T^1 \) is redundant for \( F|_{\mu} \) which contradicts the minimality of \( F|_{\mu} \).

When \( M \subseteq E \) and \( \beta \in \text{End}(M) \), i.e. \( \beta \in \text{End}(M_b) \) where \( b = p(\beta) \), then still it can happen that there is an open arc \( J \subseteq M_b \) such that \( \beta \in J \) (e.g., let \( \Gamma_b \) be a 3-star \( S_3 \) with central point \( \beta \), \( M_b \) be the union of a 2-star \( S_2 \) with the same central point \( \beta \) and a sequence of points lying in \( S_3 \setminus S_2 \) and converging to \( \beta \)). However, the following lemma holds.

**Lemma 14** Let the assumptions of Theorem A be satisfied. Suppose that there exists a point \( \beta \in \text{End}(M) \setminus \text{F}(\text{End}_e(M)) \) such that no open arc containing \( \beta \) exists in \( M_b \), \( b = p(\beta) \).

**Proof** Choose any \( \beta' \in \text{End}(M) \setminus \text{F}(\text{End}_e(M)) \) and denote \( p(\beta') = b \). Suppose that \( \beta' \) is contained in an open arc \( J \subseteq M_b \). Then, since \( \beta' \in \text{Sint}(M_b) \), the point \( \beta' \) is necessarily a ramification point of \( \Gamma_b \) and in one of the small open branches emanating from \( \beta' \) there are both a sequence of points in \( M_b \) converging to \( \beta' \) and a sequence of points in \( \Gamma_b \setminus M_b \) converging to \( \beta' \). Then this branch obviously contains also a sequence of points \( \beta_n \rightarrow \beta' \) such that, for every \( n \), \( \beta_n \in \text{End}(M_b) \) and no open arc in \( M_b \) contains \( \beta_n \). Now it is sufficient to put \( \beta = \beta_n \) for a sufficiently large \( n \), because \( F(\text{End}_e(M)) \) is a closed set which does not contain \( \beta' \).

We are finally ready to prove our Theorem A.

**Theorem A.** Let \( M \) be a minimal set (with full projection onto the base) of a fibre-preserving map in a compact graph bundle \((E,B,p,\Gamma)\). Then there are two mutually exclusive possibilities:

1. (A1) \( \text{End}(M) = M \) (and this holds if and only if \( M \) is nowhere dense in \( E \));
2. (A2) \( \text{End}(M) = \emptyset \) (and this holds if and only if \( M \) has nonempty interior in \( E \)).

In particular, the fibre-preserving maps in tree bundles have only nowhere dense minimal sets.

**Proof** Also the last claim is obvious, since if \( \Gamma \) is a tree then \( \text{End}(M) \neq \emptyset \) and we are therefore in the case (A1). Thus, taking into account Lemmas 8, 9 and 10 it remains to prove the dichotomy: either \( \text{End}(M) = M \) or \( \text{End}(M) = \emptyset \). To this end suppose that \( \text{End}(M) \neq \emptyset \). To prove that then \( \text{End}(M) = M \), it suffices to show that every point in \( \text{End}(M) \) has an \( F \)-pre-image in \( \text{End}_e(M) \). Indeed, suppose for a moment that \( F(\text{End}_e(M)) \supseteq \text{End}(M) \). Then \( F(\text{End}_e(M)) \supseteq \text{End}(M) = \text{End}_e(M) \) by Lemma 8. It follows that the nonempty closed set \( \text{End}_e(M) \) is not a proper subset of \( M \), otherwise \((M,F|_M)\) is not minimal, see the equivalence (1) \( \Leftrightarrow \) (3) in Subsection 3.1. So, \( \text{End}_e(M) = M \) whence by Lemma 8 we get \( \text{End}(M) = M \).

Thus, to finish the proof, we suppose that there is a point \( \beta \in \text{End}(M) \setminus \text{F}(\text{End}_e(M)) \) and we want to get a contradiction. If we denote \( p(\beta) = b \), by Lemma 14 we can assume that

\[
\text{there is no open arc in } M_b \text{ containing } \beta.
\]  

(5.1)
Since $F(M) = M$ and $\beta \notin F(\text{End}_b(M))$, there is a point $\alpha \in \text{Sint}_t(M)$ with $F(\alpha) = \beta$. Denote $p(\alpha) = a$. From now on we will work only with neighborhoods of $\Gamma_a$ and $\Gamma_b$ and so, due to the local triviality of the graph bundle, we may assume that $E = B \times \Gamma$. Let $\text{ord}(\beta, \Gamma_b) = r \geq 1$, i.e. $\beta = (b, \beta^T)$ where $\beta^T$ is the central point of an open $r$-star in $\Gamma$. Since the set $F(\text{End}_b(M))$ is closed in $E$ and does not contain $\beta$, for some $B$-open neighborhood $O$ of $b$ and some open $r$-star $\Sigma_b$ with the central point $\beta^T$ the open $E$-neighborhood $\Theta^* = O \times \Sigma_b$ and hence also the $M$-open neighborhood $\Theta = \Theta^* \cap M$ of $\beta$ are disjoint from $F(\text{End}_b(M))$. In view of (5.1),

the connected component of $M_b \cap \Theta$ containing $\beta$ is either the singleton $\beta$ or a (half-closed or closed) arc whose one end-point is $\beta$. \hspace{0.5cm} (5.2)

Recall that $F_z = F|_{\Gamma_z}$. Consider the map $F_a : \Gamma_a \to \Gamma_b$ and choose that connected component $\Delta$ of the set $F_{\Gamma_a}^{-1}(\beta) \cap M$ which contains the point $\alpha$. Since $\beta \notin F(\text{End}_b(M))$, we have $\Delta \subseteq \text{Sint}_t(M)$. The set $\Delta$ is closed, so it is the singleton $\alpha$ or a (nondegenerate closed) connected subgraph of $\Gamma_a$ containing $\alpha$. Let $\overline{\Delta}$ be the counterpart of $\Delta$ in $\Gamma$, i.e., $\Delta = \{a\} \times \overline{\Delta}$.

Let $W$ be a $B$-open neighborhood of $a$ and $U$ be a connected $\Gamma$-open neighborhood of $\overline{\Delta}$, both as small as Lemma 13 requires. In what follows, $\mathcal{D} = M \cap (W \times U) \subseteq \text{Sint}_t(M)$, $H_i$ and $p_i = (a, p_i^T)$ will have the meaning from this lemma. We will also consider the half-closed arcs $A_i^T = \{p_i^T\} \cup H_i$, $i = 1, \ldots, m$. Since $F(\Delta)$ is just the singleton $\beta$, we may also assume that $W$ and $U$ are small enough to give $F(\mathcal{D}) \subseteq \Theta$, hence none of the sets $F(\mathcal{D}_z), z \in W$, contains a circle. \hspace{0.5cm} (5.3)

Claim. There is a $d \in \mathcal{D}$ such that $\mathcal{D}_d^T$ contains no circle (and each component of $\mathcal{D}_d^T$ is nondegenerate since $\mathcal{D} \subseteq \text{Sint}_t(M)$ and $\mathcal{D}$ is $M$-open). Moreover, $m \geq 2$ and $\mathcal{D}_d$ contains at least two different half-closed arcs from the list $\{d\} \times A_i^T$, $i = 1, \ldots, m$.

Prof of Claim. Let $C_1^T, \ldots, C_q^T, q \geq 0$, be the list of all (not necessarily pairwise disjoint) circles in $\Delta^T$. If $z \in p(\mathcal{D})$ then, by Lemma 13, $\mathcal{D}_z^T \subseteq \Delta^T \cup \bigcup_{i=1}^m H_i^T$ and $\mathcal{D}_z^T$ contains only those circles which are contained in $\Delta^T$. So, if $D_z^T$ contains a circle, it is necessarily a circle from the list $C_1^T, \ldots, C_q^T$. Denote $K_i = \{z \in p(\mathcal{D}) : \mathcal{D}_z^T \supseteq C_i^T\}, \ i = 1, \ldots, q.$

To prove the claim suppose, on the contrary, that for every $z \in p(\mathcal{D})$, $\mathcal{D}_z^T$ contains a circle. Then $q \geq 1$ and

$$p(\mathcal{D}) = \bigcup_{i=1}^q K_i.$$ 

Each of the sets $K_i, i = 1, \ldots, q$, is obviously closed in the set $p(\mathcal{D})$ which is, by Lemma 11, a Baire space. Hence there is $s \in \{1, \ldots, q\}$ with

$$\text{Int}_{p(\mathcal{D})} K_s \neq \emptyset. \hspace{0.5cm} (5.4)$$

Now fix an arbitrary $j \in \{1, \ldots, q\}$ and an open arc $L_j^T$ in $C_j^T$ such that the closure of $L_j^T$ contains only points of order 2 in $\Gamma$ (in particular, $L_j^T$ has positive distance
from the set \( \{ p_i : i = 1, \ldots, m \} \). Observe that then for every \( z \in K_j \) the map \( F_z \) is, by Proposition 2 (see also (5.3)), monotone on \( \{ z \} \times L_j^f \) and so \( F_z(\{ z \} \times L_j^f) \) is an open, closed or half-closed arc, possibly degenerate to a point. Since \( F_z(\mathcal{D}_f) \) is by (5.3) a tree (which is a uniquely arcwise connected space), we have that \( F_z(\{ z \} \times (C_j^f \setminus L_j^f)) \supseteq F_z(\{ z \} \times L_j^f) \). Hence

\[
F(S \times L_j^f) \subseteq F(M \setminus (S \times L_j^f)) \quad \text{for any set } S \subseteq K_j, \ j \in \{1, \ldots, q\}. \quad (5.5)
\]

Note also that here \( S \times L_j^f \subseteq M \).

Then by (5.3), for \( j = s \) and \( S = \text{Int}_{p(\mathcal{D})}K_s \), we obtain \( F(\text{Int}_{p(\mathcal{D})}K_s \times L_j^f) \subseteq F(M \setminus (\text{Int}_{p(\mathcal{D})}K_s \times L_j^f)) \). Therefore, since the set \( \emptyset \neq \text{Int}_{p(\mathcal{D})}K_s \times L_j^f \subseteq M \) is obviously open in the topology of \( M \), the set \( \text{Int}_{p(\mathcal{D})}K_s \times L_j^f \) is a redundant open set for \( F|M \), which contradicts the minimality of \( F|M \). We have thus proved that there exists \( d \in p(\mathcal{D}) \) such that \( \mathcal{D}_d \) contains no circle.

Applying now the last assertion of Lemma 13, we find that \( \mathcal{D}_d \) contains at least two different half-closed arcs from the list \( \{ d \} \times A_i^f, i = 1, \ldots, m \). Thus \( m \geq 2 \) which finishes the proof of the claim.

Next, we will replace \( W \) by a smaller open neighborhood of \( a \) and \( U \) by a smaller connected open neighborhood of \( \Delta^f \) so that \( \mathcal{D} \) have an additional nice property. We are going to show how to do that. Note also that the Claim will still work.

Recall that, by the Claim, \( m \geq 2 \). The attaching points \( p_i = (a, p_i^f), i = 1, 2, \ldots, m \) belong to \( \Delta \) and so are mapped to the point \( \beta \). On the other hand, \( \Delta \) is disjoint with the open arcs \( \{ a \} \times I_i^f \). Therefore each of the sets \( F(\{ a \} \times I_i^f) \) is a nondegenerate connected set in \( M_0 \) containing \( \beta \). Taking into account (5.2), we see that each of these sets is in fact a closed or half-closed arc containing \( \beta \) as one of its end-points (so that the connected component of \( M_0 \cap \mathcal{D} \) containing \( \beta \) is not a singleton, see (5.2)), and \( F(\{ a \} \times A_i^f) \subseteq F(\{ a \} \times A_i^f) \) or \( F(\{ a \} \times A_i^f) \subseteq F(\{ a \} \times A_i^f) \) whenever \( i, j \in \{1, \ldots, m\} \). By replacing the half-closed arcs \( A_i^f \) by shorter ones (i.e., by replacing \( U \) by a smaller connected open neighborhood of \( \Delta^f \)) if necessary, we may assume that each of the half-closed arcs \( \{ a \} \times A_i^f \) is monotonically (see (5.3) and Proposition 2) mapped by \( F \) onto the same half closed arc \( H \) with the end-point \( \beta \in F(\{ a \} \times A_i^f) \) and another end-point \( \beta^* \notin F(\{ a \} \times A_i^f) \).

Now fix \( k \in \{1, \ldots, m\} \) and choose a small open arc \( J_k = \{ a \} \times J_k^f \) such that the closure of \( J_k \) lies in the interior of \( \{ a \} \times A_i^f \) and the closure of \( F(J_k) \) lies in the interior of \( H \). Then the closure of \( F(\{ a \} \times J_k^f) \) lies in the interior of \( F(\{ a \} \times A_i^f) \) for every \( i = 1, 2, \ldots, m \). By continuity, and replacing \( W \) by a smaller neighborhood of \( a \) if necessary, we may assume that

\[
F(\{ z \} \times J_k^f) \subseteq F(\{ z \} \times A_i^f) \quad \text{for every } z \in W \text{ and } i = 1, 2, \ldots, m. \quad (5.6)
\]

Note that this holds (i.e., such a \( J_k^f \) exists) for any \( k \in \{1, \ldots, m\} \).

Now we can finish the proof. By the Claim, there exists \( d \in p(\mathcal{D}) \) such that \( \mathcal{D}_d \) does not contain any circle and contains at least two different half-closed arcs, say \( \{ d \} \times A_i^f \) and \( \{ d \} \times A_j^f \). Both these properties are shared by all the points \( z \in p(\mathcal{D}) \) sufficiently close to the point \( d \). Indeed, \( M \) is closed and \( \Gamma \) contains only finitely many circles and so, if \( z \in p(\mathcal{D}) \) is close to \( d \), neither the set \( \mathcal{D}_d \) can contain a circle. But
then, using the same argument as for the point $d$ (see the very end of the proof of the Claim), the set $\mathcal{D}_t$ also contains at least two of the half-closed arcs $\{z\} \times A_i^t$. It follows that for any $z \in \mathcal{P}(\mathcal{D})$ close to $d$ there is at least one $i \neq 1$ such that $\{z\} \times A_i^t \subseteq M$ and so, regardless of whether $\{z\} \times J_i^t \subseteq \{z\} \times A_i^t$ is a subset of $M$ or is disjoint from $M$, the condition (5.6) applied to $k = 1$ gives $F(M_c \setminus (\{z\} \times J_i^t)) \supseteq F(M_c \cap (\{z\} \times J_i^t))$. Hence, for sufficiently small neighborhood $W_1 \subseteq W$ of $d$ we have $F(M \setminus (W_1 \times J_i^t)) \supseteq F(M \cap (W_1 \times J_i^t))$ and so the nonempty $M$-open set $M \cap (W_1 \times J_i^t)$ is redundant for $F|_M$, a contradiction with minimality of $F|_M$. □

6 Proof of Theorem C

If $M \subseteq E$ is a closed set with $\text{End}(M) = \emptyset$, we have $\text{End}(M_b) = \emptyset$ for every $b \in \mathcal{B}$ and so every set $M_b$ is a (possibly disconnected) graph without end-points (this in particular means that for every $b \in \mathcal{B}$ the set $M_b$ contains at least one circle). We will be interested in whether such a graph $M_b$ has a ramification point or not. Of course, $M_b$ does not have any ramification point if and only if it is a union of disjoint circles. Denote

$\mathcal{R}_B(M) := \{ b \in \mathcal{B} : M_b \text{ has a ramification point} \}$,

$\mathcal{R}_E(M) := \{ \gamma \in E : \gamma \text{ is a ramification point of } M_{p(\gamma)} \}$.

Lemma 15 Let $E = B \times \Gamma$ be a compact graph bundle and $M \subseteq E$ a closed set with $\text{End}(M) = \emptyset$.

(a) If $U$ is an open ball in $\mathcal{B}$ with $U \subseteq \mathcal{R}_B(M)$ then there are an open ball $V \subseteq U$ and a ramification point $q$ of $\Gamma$ such that $V \times \{q\} \subseteq \mathcal{R}_E(M)$.

(b) Let $q$ be a ramification point of $\Gamma$ of order $N$ and $V$ be an open ball in $\mathcal{B}$ with $V \times \{q\} \subseteq \mathcal{R}_E(M)$. Let an open star $\Sigma_N \subseteq \Gamma$ with central point $q$ be a $\Gamma$-open neighborhood of $q$ (i.e., $\Sigma_N$ contains no ramification point of $\Gamma$ different from $q$). Then there are a full sub-star $\Sigma_k$ of $\Sigma_N$ with $k \geq 3$ and an open ball $W \subseteq V$ in $\mathcal{B}$ such that $(W \times \Sigma_N) \cap M = W \times \Sigma_k$ (hence $W \times \Sigma_k$ is an $M$-open set).

Proof (a) For each $u \in U$ there is $q_u$ in $\Gamma$ such that $(u,q_u) \in M$ is a ramification point of $M_u$. Since there are only finitely many ramification points in $\Gamma$ and the set $M$ is closed, we get that the same $q$ works for all $u$ in a subset of $U$ with nonempty interior.

(b) For all $v \in V$, $(v,q)$ is a ramification point of $M_v$. The neighborhood $\Sigma_N$ of $q$ is a disjoint union of the point $q$ and $N$ open arcs emanating from $q$. If $v \in V$ and $I$ is one of these open arcs then $\{v\} \times I$ is either a subset of $M_v$ or disjoint with $M_v$ (because the graph $M_v$ possibly disconnected, has no end-points). Let $I_i, i = 1, \ldots, r$ be the list of those of the $N$ open arcs for which $\{v\} \times I_i \subseteq M_v$ for at least one $v \in V$. We say that $v \in V$ has signature $\lambda = \{i_1, \ldots, i_k\}$ if $M_v$ contains from this list just the open arcs $\{v\} \times I_{i_1}, \ldots, \{v\} \times I_{i_k}$. So, the signature $\lambda$ is a subset (with cardinality at least three) of $\{1, \ldots, r\}$. Let $A$ be the family of signatures of all points $v \in V$. Then $A$ is finite and if $S_\lambda$ is the set of all points $v \in V$ with signature $\lambda$, then $V = \bigcup_{\lambda \in A} S_\lambda$. Until the end of the proof we will work in (the topology of) the Baire space $V$. Denote by $\overline{S}_\lambda$ the closure (in $V$) of $S_\lambda$. We claim that $\overline{S}_\lambda \setminus S_\lambda$ is closed in $\overline{S}_\lambda$, i.e., $S_\lambda$ is locally closed (in $V$). The reason is as follows. If $x \in \overline{S}_\lambda \setminus S_\lambda$ has signature $\mu$ then, since $M$ is closed,
\(\mu \supseteq \lambda\). If \(x \in S_{1k} \setminus S_{k} \), then \(\mu \supseteq \lambda\). This property of having the signature strictly larger than \(\lambda\) is obviously inherited by the limit of a sequence of points from \(S_{1k} \setminus S_{k}\). It follows that \(S_{1k} \setminus S_{k}\) is closed. So, applying Lemma 7 to the Baire space \(V\) we get that there is an open ball \(W\) in \(V\) (hence \(W\) is an open ball in \(B\)) such that all points \(w \in W\) have the same signature \(\{i_1, \ldots, i_k\}\) (of cardinality \(k \geq 3\)). It follows the existence of a full sub-star \(\Sigma_k\) of \(\Sigma_N\) with the required properties. 

\(\square\)

**Lemma 16** Let \(M\) be a minimal set (with full projection onto the base) of a fibre-preserving map \(F\) in a direct product graph bundle \(E = B \times \Gamma\). Assume that \(\text{End}(M) = \emptyset\). Suppose that an open ball \(V\) in \(B\) and a ramification point \(q\) are such that \(V \times \{q\} \subseteq \mathcal{R}_E(M)\). Then there are an open ball \(V^* \subseteq V\) and a ramification point \(\tilde{q}\) of \(\Gamma\) such that \(F(V^* \times \{q\}) = f(V^*) \times \{\tilde{q}\} \subseteq \mathcal{R}_E(M)\). The same is true for closed balls instead of open ones.

**Proof** Choose \(\Sigma_N, \Sigma_k\) and \(W\) by Lemma 15(b). It is obviously sufficient to show that \(F(W \times \{q\}) \subseteq \mathcal{R}_E(M)\). Indeed, then (since \(\Gamma\) has only finitely many ramification points and \(F\) is continuous) for any sufficiently small open ball \(V^* \subseteq B\) such that \(V^* \subseteq W\), the second projection of the set \(F(V^* \times \{q\}) \subseteq \mathcal{R}_E(M)\) will be just a singleton \(\tilde{q}\) (a ramification point of \(\Gamma\)).

So, fix any \(a \in W\) (from now on we will write \(W_a\) instead of \(W\), to indicate that it contains \(a\)) and put \(\alpha = (a, q)\), \(\beta = F(\alpha) = (b, p)\) (of course, \(b = f(a)\) and \(\beta \in M = \text{Sint}_t(M)\)). We are going to prove that \(\beta \in \mathcal{R}_E(M)\).

Suppose, on the contrary, that \(\beta \notin \mathcal{R}_E(M)\). Then \(\beta \in \text{Sint}_t(M) \setminus \mathcal{R}_E(M)\) and so one can apply Lemma 12 to a small arc in \(M_0\) containing \(\beta\), to obtain that there is an \(M\)-open neighborhood of \(\beta\) in the form \(W_b \times \Sigma_2\) where \(W_b\) contains \(b\) but it need not be a \(B\)-neighborhood of \(b\) and \(p\) is the central point of \(\Sigma_2\). Recall that \(W_b\) is a \(B\)-open neighborhood of \(a\) and \(W_b \times \Sigma_2\) is an \(M\)-open neighborhood of \(a \in \text{Sint}_t(M)\).

Since \(F\) is continuous, we may assume that \(W_b\) and \(\Sigma_2\) are small enough so that \(F(W_b \times \Sigma_2) \subseteq W_b \times \Sigma_2\). We are going to show that there exists a redundant open set for \(F|_M\), which will contradict the minimality of \(F|_M\). To this end consider two cases.

First assume that there exists \(x \in W_b\) such that at least three different (half-closed) branches of \(\{x\} \times \Sigma_2\) are mapped by \(F\) onto nondegenerate sets, i.e., onto (not necessarily closed) arcs containing the point \(F(x, q)\). Then there is a point in \(\{f(x)\} \times \Sigma_2\) different from \(F(x, q)\) which is \(F\)-covered twice, by points \(P, Q\) belonging to different branches of \(\{x\} \times \Sigma_2\). Hence, some open arc \(\{x\} \times J\) in the branch containing \(P\) is such that the closure of its image lies in the interior (in topology of \(M_{t(J)}\)) of the \(F\)-image of the branch containing \(Q\). Since such a property carries over to all fibres close to the fibre over \(x\), the existence of a redundant open set for \(F|_M\) easily follows.

So, for every \(x \in W_b\) there are at most two \(k\) branches of \(\{x\} \times \Sigma_2\) which are mapped by \(F\) to nondegenerate sets. If we denote by \(J_1, \ldots, J_k\) the branches of \(\Sigma_2\) and by \(W^j\) the set of all \(x \in W_b\) with \(F(\{x\} \times J_j) = F(x, q)\), then \(W^j\) is closed in \(W_b\) and, since \(k \geq 3\), we have \(W_b = \bigcup_{j=1}^k W^j\). Since \(W_b \times \Sigma_2 = \bigcup_{j=1}^k (W^j \times \Sigma_2)\) and the sets \(W^j \times \Sigma_2\) are closed in \(W_b \times \Sigma_2\), there is \(i_0\) such that \(W^{i_0} \times \Sigma_2\) has nonempty interior in \(W_b \times \Sigma_2\). It follows that \(W^{i_0}\) has nonempty interior in \(W_b\). Thus there is a set \(\emptyset \neq \Omega \subseteq W^{i_0}\) open in \(W_b\). So, if \(A\) is an open arc lying in \(J_{i_0}\), the set \(\Omega \times A\) is open in \(W_b \times \Sigma_2\), hence open in \(M\). Since it is redundant for \(F|_M\), the proof is finished.  

\(\square\)
Theorem C (full version). Let $M$ be a minimal (with full projection onto the base) of a fibre-preserving map in a compact graph bundle $(E, B, p, \Gamma)$. Assume that $M$ has nonempty interior. Then the following holds.

(C1) $M = \text{Sint}_B(M)$.

(C2) If $B$ is infinite then $M$ exhibits the following kind of ‘perfectness’:
- If $U$ is a trivializing neighborhood, $h : p^{-1}(U) \to U \times \Gamma$ a canonical homeomorphism and $M_U = h(M_U)$, then for every $(z, p) \in M_U$ there is a sequence of points $U \ni z_n \to z$, $z_n \neq z$, such that $(z_n, p) \in M_U$ for all $n$.

(C3) $\mathcal{B}_B(M)$ is a closed nowhere dense subset of $B$.

(C4) All the sets $M_b \cap B$, are unions of circles. In fact there exist an open dense set $\mathcal{O} \subseteq B$ and a positive integer $m$ such that
- for each $z \in \mathcal{O}$, $M_z$ is a disjoint union of $m$ circles, and
- for each $z \in B \setminus \mathcal{O}$, $M_z$ is a union of circles which properly contains a disjoint union of $m$ circles.

In particular, if $M_1$ is a circle for some $z \in B$, then $M_z$ is a circle for all $z$ in the open dense subset $\mathcal{O}$ of $B$.

(C5) For each $z \in \mathcal{O}$ there exists a trivializing neighborhood $z \in U \subseteq \mathcal{O}$ such that if $h : p^{-1}(U) \to U \times \Gamma$ is a canonical homeomorphism then $M_U = h(M_U)$ has the structure of a direct product. It means that $M_U = U \times \bigcup_{i=1}^m C_i$ where $C_1, \ldots, C_m$ are pairwise disjoint circles in $\Gamma$. Consequently,
- if $\mathcal{O} = B$, then $M$ is a sub-bundle of $E$ whose fibre is a disjoint union of $m$ circles, and
- if $\mathcal{O} = B$, $E = B \times \Gamma$ and $B$ is connected, then $M$ is a direct product of $B$ and a disjoint union of $m$ circles.

(C6) The set $M_{\mathcal{O}}$ is dense in $M$.

(C7) Call a circle $\mathcal{X} \subseteq M_b$, $b \in B$, a generating circle if there are circles $\mathcal{X}_n \subseteq M_b$, $b_n \in \mathcal{O}$, $n = 1, 2, \ldots$, such that $\mathcal{X}_n \to \mathcal{X}$ with respect to the Hausdorff metric in $E$. Then the set $M$ is the union of all generating circles. If $b \in \mathcal{O}$ then $M_b$ is a disjoint union of $m$ circles and each of them is in fact generating. If $b \in B \setminus \mathcal{O}$, the set $M_b$ may contain a circle that is not generating but it always contains at least $m + 1$ generating circles, at least $m$ of them being pairwise disjoint.

(C8) If $z \in \mathcal{O}$ then the set $M_z$, which is a disjoint union of $m$ circles, is mapped by $F$ onto a disjoint union of $m$ circles in $M_{F(z)}$.

(C9) If $z \in B \setminus \mathcal{O}$ then a generating circle in $M_z$ is mapped by $F$ onto a generating circle in $M_{F(z)}$. A non-generating circle in $M_z$ need not be mapped onto a circle.

(C10) If $f$ is monotone then $\mathcal{O} = B$ (hence, $M$ is a sub-bundle of $E$).

(C11) If $E = B \times \Gamma$ and $B$ is locally connected then $\mathcal{O} = B$ (hence, $M$ is a sub-bundle of $E$ if and if $B$ is also connected, then $M$ is a direct product).

Concerning (C8), let us remark that if $z \in \mathcal{O}$ and $S$ is a circle in $M_z$ then the map $F|_S : S \to M_{f(z)}$ need not be injective even if $f$ is a homeomorphism (see the non-invertible skew-product torus map in [24]) and the map $F|_{M_z} : M_z \to M_{f(z)}$ need not be surjective (see Theorem D).

In (C9), two different/disjoint generating circles in $M_z$ can be mapped onto the same generating circle in $M_{f(z)}$ (again, see Theorem D).

Proof (C1) Since $\text{End}(M) = \emptyset$, this follows from Lemma $8$. 
(C2) Since the argument is local (concerns only that part of the minimal set which projects onto $U$), we may simply assume that $E = B \times \Gamma$, $M_U = U \times \Gamma$ and to work with $M_U$ rather than with $\tilde{M}_U$.

We have $(z, p) \in \text{Sint}_t(M)$. Consider an $M$-open neighborhood $\mathcal{G}$ of $(z, p)$, mentioned in the definition of a strongly star-like interior point. One of the properties of $\mathcal{G}$ is that if $b \in p(\mathcal{G})$ then $\mathcal{G}_b$ contains the point $(b, p)$. Thus, it is sufficient to prove that $z \in p(\mathcal{G})$ is a limit point of $p(\mathcal{G})$. Suppose, on the contrary, that $z$ is an isolated point of $p(\mathcal{G})$. Then $\mathcal{G} \cap M_t$ is an $M$-open neighborhood of $(z, p)$. Since $F|_M$ is minimal, $(z, p)$ returns to $\mathcal{G} \cap M_t$, whence we obviously get that $z$ is a periodic point of $f$. However, $f$ is minimal and so $B$ is just the periodic orbit of $z$ under $f$, a contradiction with the infiniteness of $B$.

(C3) We claim that the set $\mathcal{E}_E(M)$ is closed (in $E$, hence also in $M$). To show this, let $\mathcal{E}_E(M) \ni \gamma_n \to \gamma \in E$. Since we work only with a neighborhood of the fibre containing $\gamma$, we may assume that $E = B \times \Gamma$. Denote $p(\gamma_n) = b_n$ and $p(\gamma) = b$. Since $M$ is closed, $\gamma \in M$. However, $M = \text{Sint}_t(M)$ and so, by the definition of a star-like interior point, for large $n$ the point $\gamma_n$ has an $M_{b_n}$-open neighborhood whose second projection is a subset of the second projection of an $M_b$-open neighborhood of the point $\gamma$. Since $\gamma_n \in \mathcal{E}_E(M)$, this obviously implies that also $\gamma \in \mathcal{E}_E(M)$. We have thus proved that $\mathcal{E}_E(M)$ is closed, hence compact. Then also its projection $\mathcal{E}_B(M) = p(\mathcal{E}_E(M))$ is compact.

To prove that the (closed) set $\mathcal{E}_B(M)$ is nowhere dense, suppose, on the contrary, that some closed ball $C$ is a subset of $\mathcal{E}_B(M)$ (closed balls here and in the rest of the proof of (C3)) and are always closed balls in the topology of $B$.

Combining Lemma (15)a and Lemma (16) we get that there are a closed ball $C_1 \subseteq C$ and ramification points $q_1, q_2 \in \Gamma$ such that

$$C_1 \times \{q_1\} \subseteq \mathcal{E}_E(M) \quad \text{and} \quad F(C_1 \times \{q_1\}) = f(C_1) \times \{q_2\} \subseteq \mathcal{E}_E(M).$$

The set $f(C_1) \subseteq \mathcal{E}_B(M)$ has nonempty interior in $B$ because $C_1$ has nonempty interior in $B$ and $f : B \to B$, being a minimal map, is feebly open. Then, by Lemma (16)b, there is a closed ball $C_2$ and a ramification point $q_3$ of $\Gamma$ such that

$$C_2 \subseteq f(C_1) \quad \text{and} \quad F(C_2 \times \{q_2\}) = f(C_2) \times \{q_3\} \subseteq \mathcal{E}_E(M).$$

Again, as above, $f(C_2)$ has nonempty interior in $B$ and so we can apply Lemma (16) to find $C_3$ and $q_4$. Continuing in this way, we obtain a sequence of closed balls $(C_n)_{n=1}^{\infty}$ in $B$ and a sequence $(q_n)_{n=1}^{\infty}$ of ramification points of $\Gamma$ such that

$$C_n \times \{q_n\} \subseteq \mathcal{E}_E(M) \quad \text{and} \quad F(C_n \times \{q_n\}) \supseteq C_{n+1} \times \{q_{n+1}\} \quad \text{for every } n.$$

Now choose a point $\gamma$ in the nonempty compact set

$$(C_1 \times \{q_1\}) \cap F^{-1}(C_2 \times \{q_2\}) \cap F^{-2}(C_3 \times \{q_3\}) \cap \ldots .$$

Then all the points $\gamma, F(\gamma), F^2(\gamma), \ldots$ belong to $\mathcal{E}_E(M)$. By minimality of $F|_M : M \to M$, the set $\mathcal{E}_E(M) \subseteq M$ containing the $F$-orbit of $\gamma$ is dense in $M$. Since $\mathcal{E}_E(M)$ is also closed in $M$ (see the beginning of the proof of (C3)), we get that $\mathcal{E}_E(M) = M$. However, this contradicts the fact that $\mathcal{E}_E(M)$ is nowhere dense in $M$. Indeed,
if \((z,g) \in \mathcal{R}_E(M)\) then it is a ramification point of \(M\) and a small connected \(\Gamma\)-neighborhood of \((z,g)\) (which has the form of a star, a full sub-star of which is a subset of \(M\)) contains no other ramification points of \(\Gamma\) while containing points from \(M\) different from \((z,g)\). It obviously follows that in every \(M\)-open neighborhood of \((z,g)\) there is an \(M\)-ball disjoint with \(\mathcal{R}_E(M)\).

(C4) Let \(H\) be the homeo-part of the minimal system \((B,f)\). Both the \(f\)-image and the \(f\)-pre-image of a nowhere dense set are nowhere dense (see Subsection 3.1). Therefore, since \(\mathcal{R}_B(M)\) is nowhere dense in \(B\) by (C3), the set

\[
H^* = H \setminus \bigcup_{n=-\infty}^{\infty} f^n(\mathcal{R}_B(M))
\]

is residual, \(f(H^*) = H^*\), every point of \(H^*\) has just one \(f\)-pre-image, and both \(f|_{H^*}\) and \((f|_{H^*})^{-1}\) are minimal homeomorphisms. For any \(w \in H^*\), the set \(M_w\) is a graph without end-points which, by definition of \(H^*\), has no ramification point and so \(M_w\) is a circle or a disjoint union of several circles for all \(w \in H^*\).

Suppose that, for some \(a \in B\), the set \(M_a\) is not a union of circles. In our argument only \(E_1\) for a small neighborhood \(U\) of \(a\) will play a role, therefore we may assume that \(E = B \times \Gamma\). So, \(M_a = \{a\} \times M_a^\Gamma\) for some subgraph \(M_a^\Gamma\) of \(\Gamma\). Choose \(z_0 \in M_a^\Gamma\) such that \((a,z_0) \in M_a\) does not belong to any circle contained in \(M_a\) (it may belong to a circle in \(\Gamma\)). Then for all \(b \in B\) sufficiently close to \(a\), in the set \(M_b\) there is no circle containing the point \((b,z_0)\), since otherwise (due to closedness of \(M\)) and the fact that there are only finitely many circles in \(\Gamma\) also \(M_b\) would contain a circle containing \((a,z_0)\). Fix a point \(y^* \in H^*\). Then its forward orbit under \(f\) is a subset of \(H^*\) and so, if we choose a point \(z \in \Gamma\) with \((y^*,z) \in M_{y^*}\), for each \(n = 0,1,2,\ldots\) the point \(F^n(y^*,z)\) belongs to one of the circles forming the set \(M_{F^n(y^*)}\). It follows that the trajectory of \((y^*,z)\) under \(F\) does not approach the point \((a,z_0)\), which contradicts the minimality of \(F|_M\). Thus we have proved that all the sets \(M_b, b \in B\), are unions of circles.

Now let \(m\) be the maximum number of (disjoint) circles in \(M_w\) for \(w \in H^*\). Then \(m \geq 1\). Fix a point \(w \in H^*\) such that \(M_w\) consists of \(m\) circles. Since \(w\) has just one \(f\)-pre-image (and this pre-image belongs to \(H^*\)) and \(F : M \to M\) is surjective, also \(M_{f^{-1}(w)}\) consists of \(m\) disjoint circles (less than \(m\) circles cannot be continuously mapped onto \(m\) disjoint circles). By induction, \(M_{f^{-1}(w)}\) consists of \(m\) disjoint circles for every \(j = 0,1,2,\ldots\). Since \(B\) is a compact metric space and \(f : B \to B\) is minimal, the backward orbit \(\{f^{-j}(w) : j = 0,1,2,\ldots\}\) is dense in \(B\) (see Subsection 3.1). Since \(M\) is closed, the fact that \(M_w\) consists of \(m\) disjoint circles for every \(w\) in a dense subset of \(H^*\) implies (in view of the fact that there are only finitely many possibilities for a choice of \(m\) disjoint circles in \(\Gamma\)) that \(M_w\) consists of \(m\) disjoint circles for all \(w \in H^*\) and \(M_w\) contains \(m\) disjoint circles (and perhaps some other circles) for all \(w \in B \setminus H^*\). So, if we put

\[
\mathcal{O} = \{z \in B : M_z\text{ is a disjoint union of }m\text{ circles}\},
\]

then \(B \setminus \mathcal{O} = \{z \in B : M_z\text{ is a union of circles properly containing }m\text{ disjoint circles}\}\).

Since \(\mathcal{O} \supseteq H^*\), \(\mathcal{O}\) is dense in \(B\). To prove that \(\mathcal{O}\) is open we are going to show that \(B \setminus \mathcal{O}\) is closed. So, let \(B \setminus \mathcal{O} \ni x_m \to x \in B\). Since we may assume that all the
points $x_n$ are in a trivializing neighborhood of $x$, we may also assume that $E = B \times \Gamma$. Further, by passing to a subsequence if necessary, we may assume that for some disjoint circles $C_1, \ldots, C_m$ in $\Gamma$ we have $M_n \supseteq \{ x_n \} \times \bigcup_{i=1}^m C_i$ for every $n$. Taking into account that the points $x_n$ belong to $B \setminus \mathcal{O}$ and again passing to a subsequence if necessary, we may assume that there is a circle $S$ in $\Gamma$ different from all $C_i$, $i = 1, \ldots, m$, such that $M_n \supseteq \{ x_n \} \times S$ for every $n$. Then, since $M$ is closed, $M \supseteq \{ x \} \times (\mathcal{S} \cup \bigcup_{i=1}^m C_i)$ which implies that $x \in B \setminus \mathcal{O}$.

(C5) Let $z \in V \subseteq \mathcal{O}$ be a trivializing neighborhood. We may simply assume that $p^{-1}(V) = V \times \Gamma$. Then $M_z = \{ z \} \times \bigcup_{i=1}^m C_i$ for some pairwise disjoint circles $C_i$ in $\Gamma$. If a circle $C \subseteq \Gamma$ is different from these $m$ circles, then $M_z$ does not contain $\{ v \} \times C$ whenever $v \in V$ is sufficiently close to $z$ (otherwise the closed set $M_z$ would contain $\{ z \} \times C$). Thus, it is sufficient to choose a sufficiently small neighborhood $z \in U \subseteq V$.

From what we have just proved it follows that if $\mathcal{O} = B$ then $M$ is a bundle with fibre equal to a disjoint union of $m$ circles. Now additionally assume that $E = B \times \Gamma$ and $B$ is connected. For every $x \in B$ the set $M_x$ is a disjoint union of $m$ circles (where $m$ does not depend on $x \in B$). There are only finitely many $m$-tuples of circles in $\Gamma$ and so, using the closedness of $M$ and connectedness of $B$, we get that $M$ is the product of $B$ and some $m$-tuple of disjoint circles in $\Gamma$.

(C6) Since $f$ is minimal, the $f$-pre-image of a residual set is residual and so there is a point $x \in \mathcal{O}$ whose forward orbit is a subset of $\mathcal{O}$. Choose a point in $M_x$. Since its forward orbit is dense in $M$ and is a subset of $M_x$, the result follows.

(C7) If $b \in \mathcal{O}$ then $M_b$ is a disjoint union of $m$ circles and each of them is generating by definition (even if the point $b$ is isolated in $B$). Then (C4), (C6) show that every $M_b$ is the union of generating circles (even if $b \in B \setminus \mathcal{O}$).

If $b \in B \setminus \mathcal{O}$, it is possible that each circle in $M_b$ is generating as in Theorem D. However, it may contain also a non-generating circle. To see this, consider the case (12\text{3}) in the proof of Theorem D. There, in one fibre of a minimal set, we can have two “geometric” circles having two points in common. This gives 6 circles altogether but only two of them, namely (in the notation from the proof of Theorem D) $\{ c_i \} \times S_1$ and $\{ c_i \} \times S_1'$, are generating ones. However, at least $m$ of the circles in $M_b$, $b \in B \setminus \mathcal{O}$ are disjoint generating circles. Indeed, consider a trivializing neighborhood $W$ of $b$ and think of $E_w$ as being the product $W \times \Gamma$. Then just choose a sequence of points $b_n \in \mathcal{O}$, $b_n \to b$ such that every $M_{b_n} = \{ b_n \} \times A$ for the union $A$ of some fixed $m$ disjoint circles in $\Gamma$ (this is possible since $\Gamma$ contains only finitely many combinations of disjoint $m$ circles). Then $M_{b} \supseteq \{ b \} \times A$ and so $M_{b}$ contains at least $m$ disjoint generating circles. Since $b \notin \mathcal{O}$, $M_{b}$ cannot be just the union of these $m$ circles and since we already know that $M$ is a union of generating circles, $M_{b}$ has to contain another generating circle.

(C8) Fix $z \in \mathcal{O}$. First we prove that if $S$ is a circle in $M_z$ then $F(S)$ is a circle in $M_{f(z)}$. We will work only with small neighborhoods of $z$ and $f(z)$, therefore we may assume that $E = B \times \Gamma$. By (C5), we may fix a neighborhood $z \in U \subseteq \mathcal{O}$ such that

$$M_U = U \times \bigcup_{i=1}^m C_i$$

where $C_1, \ldots, C_m$ are pairwise disjoint circles in $\Gamma$.  \hfill (6.1)
Set $S = \{ z \} \times C$ where $C$ is one of the circles $C_i$. We need to prove that $F(S) \subseteq M_{f(z)}$ is a circle.

Let us start by considering the case when $f(z) \in \emptyset$. Then $M_{f(z)}$ is a disjoint union of circles and so $F(S)$ is necessarily a connected subset of one of them, call it $T$. To prove that $F(S) = T$ suppose, on the contrary, that $F(S)$ is a proper subset of the circle $T$. We are going to prove that then there exists a redundant open set for $F|_M$ (which will contradict the minimality of $F|_M$). If $F(S)$ is an arc in $T$, there are two non-overlapping arcs in $S$ such that each of them is mapped onto $F(S)$. Hence there are also two disjoint arcs $\{ z \} \times J_1$ and $\{ z \} \times J_2$ in $S$ such that $F(\{ z \} \times J_1)$ is in the interior of $F(\{ z \} \times J_2)$. Then (6.1) and the fact that the mentioned property of the point $z$ carries over to all the points sufficiently close to $z$, easily imply the existence of a redundant open set for $F|_M$, as desired. It remains to check the case when $F(S)$ is only a singleton in $T$. Then the existence of a redundant open set for $F|_M$ is obvious if also for all $v$ in a neighborhood of $z$ we have that $F(\{ v \} \times C)$ is a singleton. If such a neighborhood of $z$ does not exist, then arbitrarily close to $z$ there are points $v \in \emptyset$ for which $F(\{ v \} \times C)$ is not a singleton. By choosing such a point $v$ close enough to $z$ we can guarantee that $F(\{ v \} \times C)$ is a proper subset of a circle, i.e. an arc. To find a redundant open set for $F|_M$, one can simply repeat the argument which was used above in the case when $F(S)$ was an arc. We have thus proved that $F(S) \subseteq M_{f(z)}$ is a circle if $f(z) \in \emptyset$. It is a generating circle by definition, since $f(z) \in \emptyset$.

Now consider the case when $f(z) \in B \setminus \emptyset$. In $U \setminus \{ z \}$ there is a sequence $z_n \to z$ such that $f(z_n) \in \emptyset$ (otherwise some neighborhood of $z$ would be mapped into $B \setminus \emptyset$ which would contradict the fact that a minimal map sends open sets to sets with nonempty interior). Put $S_n = \{ z_n \} \times C$ and $F(S_n) = \{ f(z_n) \} \times K_n, n = 1, 2, \ldots$. Then, by what we have proved above (note that both $z_n$ and $f(z_n)$ are in $\emptyset$), we know that $K_n \subseteq \Gamma$ is a circle for every $n$. However, there are only finitely many circles in $\Gamma$ and so, by passing to a subsequence if necessary, we may assume that $K_n = K$ does not depend on $n$. Then obviously also $F(S) = \{ f(z) \} \times K$ and so $F(S)$ is a circle, in fact a generating circle (because $f(z_n) \in \emptyset$).

To finish the proof of (C8), it remains to show that different, hence disjoint, circles in $M_z$ are mapped onto disjoint circles in $M_{f(z)}$.

Again, we start by considering a particular case when $f(z) \in \emptyset$. By replacing $U$ in (6.1) by a smaller neighborhood of $z$ if necessary, we may assume, due to (C5), that $M_{f(U)} = f(U) \times \bigcup_{i=1}^{\infty} Q_i$, where $Q_1, \ldots, Q_m$ are pairwise disjoint circles in $\Gamma$. Let $S = \{ z \} \times C, S' = \{ z \} \times C'$ be disjoint circles in $M_z$ (here $C, C'$ are in $\{ C_1, \ldots, C_m \}$, see (6.1)). To prove that also the circles $F(S)$ and $F(S')$ are disjoint, suppose on the contrary that $F(S) = F(S') = \{ f(z) \} \times Q$ for some $Q \in \{ Q_1, \ldots, Q_m \}$. The circle $\{ f(z) \} \times Q$ has positive distance from the rest of $M_{f(z)}$. Therefore, in view of (6.1), for all $v$ sufficiently close to $z$ it holds that both $\{ v \} \times C$ and $\{ v \} \times C'$ are mapped by $F$ onto the same circle $\{ f(v) \} \times Q$. The existence of a redundant open set for $F|_M$ easily follows; a contradiction.

Finally, consider the case when $f(z) \in B \setminus \emptyset$. Again, let $S = \{ z \} \times C, S' = \{ z \} \times C'$ be disjoint circles in $M_z$. Choose a sequence $U \setminus \{ z \} \ni z_n \to z$ such that $f(z_n) \in \emptyset$. Consider the circles $S_n = \{ z_n \} \times C$ and $S'_n = \{ z_n \} \times C'$. For each $n$, both $z_n$ and $f(z_n)$ are in $\emptyset$ and therefore, as we already know, $F(S_n) = \{ f(z_n) \} \times P_n$ and $F(S'_n) = \{ f(z_n) \} \times P'_n$ are disjoint circles. By passing to a subsequence if necessary, we may
We claim that the family of 

\[ \text{assume that } P_n = P \text{ and } P'_n = P' \text{ do not depend on } n. \text{ Then obviously } F(S) = \{ f(z) \times P \text{ and } F(S') = \{ f(z) \times P' \text{ which means that } F(S) \text{ and } F(S') \text{ are disjoint circles.} } \]

\( (C9) \) Let \( S \subseteq M_z \) be a generating circle. So, there are circles \( S_n \subseteq M_{z_n} \), \( z_n \in \mathcal{O} \) (hence \( z_n \neq z \)), \( n = 1, 2, \ldots \), such that \( S_n \to S \) with respect to the Hausdorff metric.

By \( (C8), F(S_n) \) is a generating circle for every \( n \). Since \( F(S_n) \to F(S) \) in the Hausdorff metric, \( F(S) \) is a generating circle. Now see the proof of Theorem D, the case (12). The set \( M^*_n \) consists of two circles, one “inside” the other. Together there are six circles there, two generating and four non-degenerating. Straightforward analysis shows that images of two non-degenerating circles are just arcs, not circles.

\( (C10) \) Let \( f \) be monotone. Suppose that \( B \setminus \mathcal{O} \neq \emptyset \). To show that this leads to a contradiction, consider two cases.

If for every \( z \in B \setminus \mathcal{O} \) the set \( f^{-1}(z) \) intersects \( B \setminus \mathcal{O} \), then there is a backward orbit of \( f \) lying in \( B \setminus \mathcal{O} \). However, \( B \setminus \mathcal{O} \) is nowhere dense while every backward orbit of a minimal map is dense, a contradiction.

If there exists \( z_0 \in B \setminus \mathcal{O} \) such that the connected set \( f^{-1}(z_0) \) is a subset of \( \mathcal{O} \), we get a contradiction as follows. Fix a point \( a \in f^{-1}(z_0) \). Since now we are going to find a special neighborhood of \( a \) by considering just small neighborhoods of \( a \) and \( z_0 \), we may assume for a moment that \( E = B \times \Gamma \). By \( (C5), there is a small neighborhood \( U_a \) of \( a \) such that \( U_a \subseteq \mathcal{O} \) and \( M \mathcal{O}_a = U_a \times \bigcup_{i=1}^{m} C_i \) where \( C_0, \ldots, C_m \) are pairwise disjoint circles in \( \Gamma \). By \( (C8), F\{a \times C_i\} = \{ f(a) \} \times K_i, i = 1, \ldots, m, \) for some pairwise disjoint circles \( K_1, \ldots, K_m \) in \( \Gamma \). Since there are only finitely many circles in \( \Gamma \), there is \( \varepsilon_0 > 0 \) such that any two different (not necessarily disjoint) circles in \( \Gamma \) have Hausdorff distance at least \( \varepsilon_0 \). Therefore, if \( f \in \{1, \ldots, m\} \) and if \( u \in U_a \) is sufficiently close to \( a \) then the set \( F\{u \times C_i\}, \) which is a circle by \( (C8), \) equals \( \{ f(u) \} \times K_i \). By replacing \( U_a \) by a smaller neighborhood if necessary, we may assume that the last claim works for all \( u \in U_a \). Finally, consider the relative neighborhood of \( a \) in \( f^{-1}(z_0) \) of the form \( V_a = U_a \cap f^{-1}(z_0) \). Denote also \( S^n = \{ z_0 \} \times K_i \). Then we have that

\[
\text{for every } v \in V_a, \quad F(M_v) = \bigcup_{i=1}^{m} S_i^n \subseteq M_{z_0}.
\]

Without our above temporary assumption that \( E = B \times \Gamma \), of course still a small relative neighborhood \( V_a \) of \( a \) exists such that \( (6.2) \) works for some pairwise disjoint circles \( S^n_1, \ldots, S^n_m \) in \( M_{z_0} \). Remember that, given \( a \in f^{-1}(z_0), \) the family of these circles does not depend on the choice of \( v \in V_a \).

Let \( V_{a(1)}, \ldots, V_{a(r)} \) be a finite cover of the compact space \( f^{-1}(z_0) \) (in the relative topology), chosen from the open cover \( \{ V_a : a \in f^{-1}(z_0) \} \). Then, since \( F(M) = M \), we have by \( (6.2), \)

\[
M_{z_0} = \bigcup_{a \in f^{-1}(z_0)} F(M_a) = \bigcup_{j=1}^{r} F(M_{V_{a(j)}}) = \bigcup_{j=1}^{r} \bigcup_{i=1}^{m} S_i^{a(j)}.
\]

We claim that the family of \( m \) disjoint circles \( \{ S_1^{a(j)}, \ldots, S_m^{a(j)} \} \) does not depend on \( j \). To see it, fix \( j, k \in \{1, \ldots, r\}, j \neq k. \) In particular case when \( V_{a(j)} \cap V_{a(k)} \neq \emptyset \) it suffices
to choose \( x \in V_{a(j)} \cap V_{a(k)} \) and to use that, by (6.2), it holds \( \bigcup_{i=1}^m S^i_{a(j)} = F(M_a) = \bigcup_{i=1}^m S^i_{a(k)} \). In general case realize that in the family \( V_{a(1)}, \ldots, V_{a(n)} \) there is a finite chain of sets starting with \( V_{a(j)} \) and ending with \( V_{a(k)} \) such that any two consecutive elements of the chain intersect (if such a chain did not exist, the connected set \( f^{-1}(z_0) \) would be a union of two disjoint nonempty sets open in the topology of \( f^{-1}(z_0) \)). Hence also in the general case we have \( \bigcup_{i=1}^m S^i_{a(j)} = \bigcup_{i=1}^m S^i_{a(k)} \). Then (6.3) implies that \( M_{\partial} \) is a union of just \( m \) disjoint circles. Hence \( z_0 \in \partial \), a contradiction.

(C11) We claim that to prove \( \partial = B \) we may without loss of generality assume that \( B \) is also connected. In fact, suppose for a moment that we have proved \( \partial = B \) under the additional assumption of connectedness of \( B \). Then we can finish the proof as follows. The space \( B \), being compact and locally connected, has finitely many components \( B_1, \ldots, B_r \) and these are locally connected. The map \( f_j \), being minimal, cyclically permutes them and \( f_j \) is minimal on each of them. Then, for \( i = 1, \ldots, r \), the set \( M_{B_i} \) is a minimal set of \( F_{B_i} \times \Gamma \). Hence, using our temporary assumption that \( \partial \) is the whole base space provided the base space is locally connected and connected, we get that for every \( x \in B_i \) the set \( M_x \) is a disjoint union of \( m_i \) circles (where \( m_i \) does not depend on \( x \in B_i \)). There are only finitely many \( m_i \)-tuples of circles in \( \Gamma \) and so, using the closedness of \( M_{B_i} \) and connectedness of \( B_i \), we get that \( M_{B_i} \) is the product of \( B_i \) and some \( m_i \)-tuple of disjoint circles in \( \Gamma \). Further, by (C4) the positive integer \( m \) does not depend on \( i \), i.e., there is \( m \) with \( m_i = m \) for all \( i = 1, \ldots, r \). Thus, \( \partial = B \) (still, if \( r > 1 \), the \( m_i \)-tuple of circles may depend on \( i \)).

So, assume that the locally connected space \( B \) is also connected. We are going to prove that then \( \partial = B \).

Consider the open set \( \partial \subseteq B \) defined in (C4). Since \( B \) is locally connected, so is \( \partial \). Recall that a space \( X \) is locally connected if and only if for every open set \( U \) of \( X \), each component of \( U \) is open. It follows that the set \( \partial \) can be represented as a disjoint union \( \partial = \bigcup \mathcal{W}_j \) of a countable family of its components \( \mathcal{W}_j \), each \( \mathcal{W}_j \) being \( B \)-open, locally connected and connected. Let \( m \) be the positive integer from (C4).

Due to the connectedness of \( \mathcal{W}_j \subseteq \partial \), by (C5) we obtain the direct product structure of each \( M_{\mathcal{W}_j} \), i.e., there exist pairwise disjoint circles \( \Gamma \) such that

\[
M_{\mathcal{W}_j} = \bigcup_{i=1}^m C_i \times \bigcup_{i=1}^m C_i.
\] (6.4)

The circles \( C_i, \ldots, C_m \) in general depend on \( j \), but \( m \) does not. Let \( L \) be the (finite) set of all circles \( C_i \) (for all \( j \) and all \( i = 1, \ldots, m \)).

Since \( M \) is a closed set, for the closure of \( M_{\mathcal{W}_j} \) we have \( \overline{M_{\mathcal{W}_j}} = \overline{\mathcal{W}_j} \times \bigcup_{i=1}^m C_i \subseteq M \). The set \( \overline{\mathcal{W}_j} \) is connected. We call each of \( m \) connected components \( \overline{\mathcal{W}_j} \times C_i \) of the closure \( \overline{M_{\mathcal{W}_j}} \), a prime cylinder (more precisely, \( \mathcal{W}_j \times C_i \) is a prime cylinder corresponding to the circle \( C_i \)). Each prime cylinder has nonempty \( E \)-interior. Notice also that each prime cylinder is a union of generating circles and is of course a connected subset of \( M \). For a fixed circle \( C \) in \( \Gamma \), consider the set of indices \( I(C) := \{ j : \mathcal{W}_j \times C \subseteq M \} \).
Let $[C_\alpha]$ be the components of the set $\bigcup_{j \in I(C)} W_j \times C$, so

$$\bigcup_{j \in I(C)} W_j \times C = \bigcup_{\alpha} [C_\alpha].$$  \tag{6.5}

We will say that each $[C_\alpha]$ is a maximal cylinder corresponding to the circle $C$ (note that it is a subset of $M$). Observe that $[C_\alpha]$ has the form

$$[C_\alpha] = \bigcup_{\gamma \in \gamma} P_\gamma$$

where $P_\gamma$, $\gamma \in \gamma$, are some prime cylinders corresponding to $C$. \tag{6.6}

By definition, $[C_\alpha] \cap [C_\beta] = \emptyset$ for $\alpha \neq \beta$. We will also need the following claim.

**Claim (Properties of maximal cylinders).**

(a) Two maximal cylinders $\mathcal{M}_1, \mathcal{M}_2$ corresponding to the same circle $C$ either are disjoint or coincide.

(b) If $b \in \mathcal{O}$ and $\mathcal{M}_1 \cap \mathcal{M}_2 \cap M_b \neq \emptyset$, then $\mathcal{M}_1 = \mathcal{M}_2$.

(c) If $\mathcal{M}$ and $\mathcal{M}_\lambda$, $\lambda \in \Lambda$, are maximal cylinders with $M \subseteq \bigcup_{\lambda \in \Lambda} M_\lambda$, then $\mathcal{M} = \mathcal{M}_\lambda_0$ for some $\lambda_0 \in \Lambda$.

(d) For any $k \geq 1$, the $F^k$-image of a prime cylinder $P = W_j \times C \subseteq M$ is a subset of a maximal cylinder.

(e) The family of all maximal cylinders is finite and its union equals $M$.

(f) Any maximal cylinder is mapped by $F$ into a maximal cylinder.

**Proof of Claim**

(a) Each of the sets $\mathcal{M}_1, \mathcal{M}_2$ is a component of the set $\bigcup_{j \in I(C)} W_j \times C$. Two non-disjoint components coincide.

(b) By (C4), $M_b$ is a disjoint union of circles. One of them, call it $C$, is such that $\mathcal{M}_1 \cap \mathcal{M}_2$ intersects $\{b\} \times C$ and, by definition of maximal cylinders, $\mathcal{M}_b = \mathcal{M}_2 = \{b\} \times C$. Now apply (a).

(c) By definition of a maximal cylinder, there exists $b \in \mathcal{O}$ and $C \subseteq L$ such that $\mathcal{M} \supseteq \{b\} \times C$. There is $\lambda_0 \in \Lambda$ such that $\mathcal{M}_\lambda_0$ intersects $\{b\} \times C$. By (b), $\mathcal{M} = \mathcal{M}_\lambda_0$.

(d) The set $P$ is a union of generating circles and, by (C8) and (C9), a generating circle is mapped onto a (generating) circle. It follows that if $S$ is a circle in $\Gamma$ then, due to continuity of $F^k$ and the fact that $\Gamma$ contains only finitely many circles, the set of those $z \in \overline{W_j}$ for which $F^k(\{z\} \times C) = \{f^k(z)\} \times S$, is open in $\overline{W_j}$. However, the set $\overline{W_j}$ is connected. Therefore there exists one circle $S$ such that

$$F^k(P) = f^k(\overline{W_j}) \times S \subseteq M.$$  \tag{6.7}

Since $f : B \to B$ is minimal, it is feebly open. Hence $f^k$ is feebly open. Therefore the set $U_i := \text{Int} f^k(\overline{W_j})$ of all $B$-interior points of $f^k(\overline{W_j})$ is dense in $f^k(\overline{W_j})$, hence also dense in $\overline{f^k(W_j)} = f^k(\overline{W_j})$. So, $U_j = f^k(\overline{W_j})$. On the other hand, $U_i$ is open and $\mathcal{O} = \bigcup_j W_j$ is dense and open, hence

$$f^k(\overline{W_j}) = \overline{U_i} = \overline{\bigcup U_i} = (\bigcup_j W_j) \cap U_i = \bigcup_j (W_j \cap U_i).$$  \tag{6.8}

Further, by (6.7), $U_i \times S \subseteq M$. So, if $\mathcal{W}_i \cap U_i \neq \emptyset$ for some $i$, then $\mathcal{W}_i \times S \subseteq M$, i.e. $l \in I(S)$. It follows, taking into account (6.7) and (6.8), that

$$F^k(P) = f^k(\overline{W_j}) \times S = \bigcup_{i \in I(S)} (\mathcal{W}_i \cap U_i) \times S \subseteq \bigcup_{i \in I(S)} \mathcal{W}_i \times S = \bigcup_{\alpha} [C_\alpha]$$
where \([S_a]\) are the components of the set \(\bigcup_{i \in (S)} F_i \times S\) (see (6.5)). Since \(F^k(P)\) is connected, it is a subset of one \(S_a\), which finishes the proof that \(F^k\)-image of a prime cylinder is a subset of a maximal cylinder.

(c) Now, since the prime cylinder \(P\) has nonempty interior in \(M\) and \(F\vert_M\) is minimal, we have that \(M = \bigcup_{i=1}^N F^k(P)\) for some \(N\) (this is a property of minimal systems, see Subsection 3.1). This together with (d) give that \(M\) is covered by \(N\) (not necessarily distinct) maximal cylinders. Then, using (c), we get that the family of all maximal cylinders is finite (has at most \(N\) elements) and its union equals \(M\).

(f) Let \(\mathcal{M}_1, \ldots, \mathcal{M}_r\) be the list of all (pairwise distinct) maximal cylinders (at the moment we do not know whether they are pairwise disjoint). For \(i = 1, \ldots, r\) put \(\mathcal{M}_i = B_i \times S_i\), where \(B_i \subseteq B\) is closed and connected set (containing at least one of the sets \(F_j\)) and \(S_i\) is a circle in \(\Gamma\) (in fact \(S_i \in L\), see (6.4) and the notation \(L\) after it). We prove that, for instance, \(F(\mathcal{M}_1)\) is a subset of a maximal cylinder. By \(6.6\),

\[\mathcal{M}_1 = \bigcup_{\gamma \in \Psi_1} \mathcal{P}_\gamma\]

where \(\mathcal{P}_\gamma, \gamma \in \Psi\) is the family of prime cylinders contained in \(\mathcal{M}_1\) (of course, all these prime cylinders \(P_\gamma\) correspond to the circle \(S_\gamma\)). We know, by (d), that for each \(\gamma \in \Psi\) there is a maximal cylinder, call it \(\mathcal{M}_\gamma\), with \(F(P_\gamma) \subseteq \mathcal{N}_{\gamma}\). In the particular case when all these maximal cylinders are the same, i.e. when there is \(\gamma_0 \in \Psi\) such that \(\mathcal{N}_{\gamma} = \mathcal{N}_{\gamma_0}\), for all \(\gamma \in \Psi\), we get the desired relation:

\[F(\mathcal{M}_1) = F(\bigcup_{\gamma \in \Psi_1} \mathcal{P}_\gamma) = \bigcup_{\gamma \in \Psi_1} F(\mathcal{P}_\gamma) \subseteq \mathcal{N}_{\gamma_0}.
\]

To finish the proof, we are going to show that the assumption that not all maximal cylinders \(\mathcal{N}_{\gamma}\) are the same, leads to a contradiction.

So, let \(d \geq 2\) and \(\mathcal{M}^1, \ldots, \mathcal{M}^d\) be the list of all pairwise distinct maximal cylinders in the family \(\mathcal{N}_{\gamma}, \gamma \in \Psi\). Then there is a decomposition \(\Psi = \Psi^1 \cup \cdots \cup \Psi^d\) such that \(F(P_\gamma) \subseteq \mathcal{N}^j\) for all \(\gamma \in \Psi^j\). Denote \(\Pi_j := \bigcup_{\gamma \in \Psi^j} \mathcal{P}_\gamma, j = 1, 2, \ldots, d\). Of course,

\[F(\Pi_j) = F(\bigcup_{\gamma \in \Psi^j} \mathcal{P}_\gamma) \subseteq \mathcal{N}^j.
\]

We claim that the sets \(\Pi_j\) are pairwise disjoint. To show this, suppose on the contrary that \(\Pi_i \cap \Pi_k \neq \emptyset\) for some \(i \neq k\). Then, in view of the fact that all prime cylinders \(P_\gamma\) correspond to the circle \(S_\gamma\), there is \(b_0 \in B\) such that \(\{b_0\} \times S_1 \subseteq \Pi_i \cap \Pi_k\). Obviously, \(\{b_0\} \times S_1\) is a generating circle and, by (C8) and (C9), its \(F\)-image is some circle \(\{f(b_0)\} \times S^e\). Then \(\mathcal{N}^e = B^e \times S^e\) and \(\mathcal{N}^k = B^k \times S^e\) for some closed sets \(B^e, B^k\) containing \(f(b_0)\). Here \(\mathcal{N}^e, \mathcal{N}^k\) are different, but not disjoint, maximal cylinders corresponding to the same circle \(S^e\). This contradicts already proved part (a) of the Claim. So, we have proved that the sets \(\Pi_j\) are pairwise disjoint. Then

\[\mathcal{M}_1 = \bigcup_{\gamma \in \Psi_1} \mathcal{P}_\gamma = \bigcup_{j=1}^d \bigcup_{\gamma \in \Psi^j} \mathcal{P}_\gamma = \bigcup_{j=1}^d \Pi_j
\]

is the decomposition of the connected set \(\mathcal{M}_1\) into finitely many closed nonempty sets, a contradiction. This finishes the proof of Claim.

Now we are ready to finish the proof of (C11).

Similarly as in the proof of the part (f) of Claim, let \(\mathcal{M}_1, \ldots, \mathcal{M}_r\) be the list of all (pairwise distinct) maximal cylinders, where \(\mathcal{M}_i = B_i \times S_i\). By the part (b) of Claim,
two different maximal cylinders may intersect only in fibres above the set $B \setminus \mathcal{O}$. Therefore
\[
\mathcal{M}_1 \setminus (\mathcal{M}_2 \cup \cdots \cup \mathcal{M}_r)
\text{ has nonempty interior in } M.
\]
(6.9)
Since the map $F|_M$ is minimal, there exists a positive integer $j$ with
\[
F^j(\mathcal{M}_1) \cap (\mathcal{M}_1 \setminus (\mathcal{M}_2 \cup \cdots \cup \mathcal{M}_r)) \neq \emptyset.
\]
However, every maximal cylinder is mapped by $F$ into a maximal cylinder, therefore we necessarily have $F^j(\mathcal{M}_1) \subseteq \mathcal{M}_1$. It follows that $F^j|_{\mathcal{M}_1} : \mathcal{M}_1 \to \mathcal{M}_1$ is minimal. (Indeed, if in a minimal system $(M, F)$ there is a closed and connected set $Y \neq \emptyset$ with $F^j(Y) \subseteq Y$ for some $j \geq 1$, then $Y$ is a minimal set of $F^j$. This is probably well known and explicitly can be found in [28].)

However, $f$ is minimal on the connected space $B$ (see the discussion at the beginning of the proof of (C11)), hence it is totally minimal (this is well known, see e.g. [28]). Since the minimal map $f^j$ is the base map of $F^j$, the fact that $F^j|_{\mathcal{M}_1}$ is minimal implies that $B_1 = B$. In the same way we get $B_i = B$ for all $i = 1, \ldots, r$. But then $M$ is a finite union (see Claim (e)) of maximal cylinders in the form $\mathcal{M}_i = B \times S_i$, $i = 1, \ldots, r$. Since the maximal cylinders $\mathcal{M}_i$ of this particular form are assumed to be pairwise distinct and $\mathcal{O} \neq \emptyset$, by Claim (b) they are pairwise disjoint (i.e., the circles $S_i$ are pairwise disjoint). Thus $\mathcal{O} = B$. The sets $\mathcal{M}_i$ are the components of the minimal set $M$ and so they are cyclically permuted by $F$. \hfill \Box

7 Proof of Theorem E

Lemma 17 Let $M$ be a nowhere dense closed subset of a compact graph bundle $(E, B, p, \Gamma)$. Then a typical fibre of $M$ is totally disconnected.

Proof Suppose, on the contrary, that $A = \{ b \in B : M_b \text{ is not totally disconnected}\}$ is of 2nd category in $B$. Of course, $M_b$ is not totally disconnected if and only if it contains an arc and since $\Gamma$ is a graph, this is if and only if $M_b$ contains a ball in $I_b$. Therefore, since $A$ is of 2nd category, there is $n_0 \in \mathbb{N}$ such that
\[
A_{n_0} = \{ b \in B : M_b \text{ contains a ball in } I_b \text{ with radius } \geq 1/n_0 \}
\]
is of 2nd category. Since $B$ is covered by finitely many trivializing neighborhoods, there is a trivializing neighborhood $U$ such that $A_{n_0} \cap U$ is of 2nd category. To get a desired contradiction, it is sufficient to show that $M \cap p^{-1}(U)$ is somewhere dense. Of course, we may without loss of generality assume that $p^{-1}(U) = U \times \Gamma$. To prove that $M \cap (U \times \Gamma)$ is somewhere dense, fix a countable dense set $S \subseteq \Gamma$. For $b \in U$ and $s \in S$, a ball in $I_b = \{ b \} \times \Gamma$ whose radius is $\geq 1/n_0$ and whose center has distance from $\{ b \} \times \{ s \}$ less than $1/(2n_0)$ is in the sequel said to be a big ball centered close to level $s$. Let
\[
A_{n_0}^{U, s} = \{ b \in U : M_b \text{ contains a big ball centered close to level } s \}.
\]
It is obvious that $A_{n_0} \cap U = \bigcup_{s \in S} A_{n_0}^{U, s}$ and so there is $s_0 \in S$ such that $A_{n_0}^{U, s_0}$ is of 2nd category, hence dense in some nonempty open set $G_{s_0} \subseteq U$. On the other hand,
any ball in $\Gamma$ whose radius is $\geq 1/n_0$ and whose center has distance from $z_0$ less than $1/(2n_0)$, contains the ball $G_2$ with center $z_0$ and radius $1/(2n_0)$. Then $M$, being closed, contains the open set $G_1 \times G_2$. This contradicts nowhere density of $M$. □

**Proposition 3** In Theorem A, suppose that $\text{card} M_ε < \infty$ for some $z$ in the homeo-part $H$ of $f$. Then a typical fibre of the minimal set $M$ has cardinality $N := \min\{\text{card} M_ε : x \in H\} < \infty$.

**Proof** By the assumption, $N \leq \text{card} M_ε$ is a positive integer and there is $z_0 \in H$ with $\text{card} M_{z_0} = N$. Denote

$$B^{(\leq N)} := \{x \in B : \text{card} M_ε \leq N\}.$$  

Then $z_0 \in B^{(\leq N)}$ and we claim that $B^{(\leq N)} = \bigcap_{n=1}^{\infty} G_{1/n}^{(N)}$ where $G_{1/n}^{(N)}$ is the set of those points $x \in B$ for which $M_ε$ can be covered by a disjoint union of $N$ open sets in the fibre $p^{-1}(x)$, each of these sets having diameter $< 1/n$. The inclusion $B^{(\leq N)} \subseteq \bigcap_{n=1}^{\infty} G_{1/n}^{(N)}$ is trivial. To prove the converse inclusion, realize that simultaneous assumptions $x \in \bigcap_{n=1}^{\infty} G_{1/n}^{(N)}$ and $\text{card} M_ε \geq N + 1$ obviously give a contradiction.

Since $M$ is compact, $G_{1/n}^{(N)}$ is open. So, $B^{(\leq N)}$ is a $G_δ$ set in $B$. Moreover, we claim that it is dense in $B$. To see this, realize that the set $H$ is $f$-invariant and for $x \in H$ we have $\text{card} M_ε \geq \text{card} M_{f(x)} \geq N$. Hence, since for $z_0 \in H$ we have $\text{card} M_{z_0} = N$, we get that $\text{card} M_{f^k(z_0)} = N$ for all $k = 0, 1, \ldots$. So the set $B^{(\leq N)}$ contains the whole (forward) orbit of $z_0$, which is dense by minimality of $f$. We have proved that $B^{(\leq N)}$ is a $G_δ$ dense set in $B$.

For each $x$ in the $G_δ$ dense set $H \cap B^{(\leq N)}$ it obviously holds $\text{card} M_ε = N$. □

**Theorem E.** Let $M$ be a minimal set (with full projection onto the base) of a fibre-preserving map in a compact graph bundle $(E, B, p, \Gamma)$. Assume that $M$ is nowhere dense. Then either

- (E1) a typical fibre of $M$ is a Cantor set, or
- (E2) there is a positive integer $N$ such that a typical fibre of $M$ has cardinality $N$.

**Proof** Below, we use some ideas from the proof of [21, Theorem 6.1].

By Lemma [17] a typical fibre of $M$ is totally disconnected. If (E1) does not hold, $B^{\text{isol}} := \{b \in B : M_b \text{ has an isolated point}\}$ is a 2nd category set. Since $B$ can be covered by a finite number of trivializing neighborhoods, one of them, call it $W$, is such that $B^{\text{isol}} \cap W$ is of 2nd category in $W$ (even, of 2nd category in $B$).

We fix a homeomorphism $h : p^{-1}(W) \to W \times \Gamma$ such that on $p^{-1}(W)$ it holds $p_1 \circ h = p$.

Let $\mathcal{T}$ be a countable family of subtrees of the fibre $\Gamma$ such that the interiors, in the topology of $\Gamma$, of them are connected (i.e., the interior of such a tree is obtained from the tree by possible removing of some or all of the endpoints of the tree; no
point which is not an endpoint is removed) and these interiors form a base of the topology on $T$. Consider the countable set
\[
\mathcal{D} := \{ (T_i^+, T_i^-) : T_i^+, T_i^- \in \mathcal{D} \text{ and } T_i^+ \subseteq \text{Int } T_i^- \}. 
\]
Note that the homeomorphism $h$ induces corresponding families of trees in each fibre $p^{-1}(b)$, $b \in W$. For each pair $(T_i^+, T_i^-) \in \mathcal{D}$, put
\[
W(T_i^+, T_i^-) := \{ b \in W : M_b \cap T_{2,b} = M_b \cap T_{1,b} \text{ is a singleton} \} \tag{7.1}
\]
where $T_{i,b} := h^{-1}(\{b \times T_i^+ \})$ is the tree in $p^{-1}(b)$ corresponding to $T_i^+$, $i = 1, 2$. Of course,
\[
B^\text{isol} \cap W = \bigcup_{(T_i^+, T_i^-) \in \mathcal{D}} W(T_i^+, T_i^-).
\]
Since $B^\text{isol} \cap W$ is a 2nd category set, there is a pair $(\tilde{T}_1^+, \tilde{T}_2^-) \in \mathcal{D}$ such that $W(\tilde{T}_1^+, \tilde{T}_2^-)$ is dense in an open subset $U \subseteq W$.

Let $\mathcal{X}(E)$ be the (compact) space of all compact subsets of $E$ endowed with the Hausdorff distance generated by the original metric in $E$. Since $M$ is compact, the map $\Theta : B \rightarrow \mathcal{X}(E)$ defined by $\Theta(b) = M_b$, $b \in B$ is upper semicontinuous. Hence, see e.g. [14, Theorem 1.4.13], the set $C(\Theta)$ of continuity points of $\Theta$ is residual in $B$.

By Lemma 5 there is an invariant residual set $R$ in $B$ such that $R \subseteq C(\Theta) \cap H$ where $H$ is the homeo-part of $f$.

Denote $V := \text{Int } \tilde{T}_1^\circ$. We claim that for any $b \in U \cap R \subseteq W(\tilde{T}_1^+, \tilde{T}_2^-) \cap R$ it holds that $M_b^\circ \cap V$ is a singleton. In fact, each such point $b$ is a limit of points from $W(\tilde{T}_1^+, \tilde{T}_2^-)$ and so $M_b^\circ \cap T_1^\circ$ contains a point. Suppose that $M_b^\circ \cap V$ contains more than one point. Then, since $b$ is a point of continuity of $\Theta$, also for those points $c \in W(\tilde{T}_1^+, \tilde{T}_2^-) \cap U$ which are sufficiently close to $b$, we get that $M_c^\circ \cap V$ contains at least two points, which contradicts (7.1).

The set $\mathcal{O} := M \cap h^{-1}(U \times V)$ is a nonempty open subset of $M$. Hence, by the well known property of compact minimal systems, there is a positive integer $n_0$ such that every point from $M$ visits $\mathcal{O}$ not later than after $n_0$ iterations.

Now fix $y \in R$ and $e \in M_y$. By what was said above, $F^{n(e)}(y) \in \mathcal{O}$ for some $n(e) \leq n_0$. Hence $F^{n(e)}(G(e)) \subseteq \mathcal{O}$ for some neighborhood $G(e)$ of $e$ in $M_y$. It follows that
\[
F^{n(e)}(G(e)) \subseteq h^{-1}(\{ F^{n(e)}(y) \} \times V) \cap M \subseteq \mathcal{O}. \tag{7.2}
\]

By definition of $\mathcal{O}$ and the fact that $y$ has been chosen in the invariant set $R$, we get that $F^{n(e)}(y) \in U \cap R$. Therefore, by (7.2), $F^{n(e)}(G(e))$ is a singleton. Then also $F^{n_0}(G(e))$ is a singleton. Since $M_0$ is compact, there are finitely many points $e_1, \ldots, e_k \in M_0$ such that $G(e_1) \cup \cdots \cup G(e_k) = M_0$. It follows that $F^{n_0}(M_0)$ is a finite set (it is a subset of $M_{f_0^{n_0}}$ with cardinality $\leq k$). Since $y \in H$ and $F(M) = M$, also $f^{n_0}(y) \in H$ and $M_{f_0^{n_0}}(y) = M_{f_0^{n_0}}(y)$ is finite. So, one can apply Proposition D to get (E2).

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