HAHN-BANACH OPERATORS

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Abstract. We consider real spaces only.

Definition. An operator $T : X \to Y$ between Banach spaces $X$ and $Y$ is called a Hahn-Banach operator if for every isometric embedding of the space $X$ into a Banach space $Z$ there exists a norm-preserving extension $\tilde{T}$ of $T$ to $Z$.

A geometric property of Hahn-Banach operators of finite rank acting between finite-dimensional normed spaces is found. This property is used to characterize pairs of finite-dimensional normed spaces $(X, Y)$ such that there exists a Hahn-Banach operator $T : X \to Y$ of rank $k$. The latter result is a generalization of a recent result due to B. L. Chalmers and B. Shekhtman.

Everywhere in this paper we consider only real linear spaces. Our starting point is the classical Hahn-Banach theorem ([H], [B1]). The form of the Hahn-Banach theorem we are interested in can be stated in the following way.

Hahn-Banach Theorem. Let $X$ and $Y$ be Banach spaces, $T : X \to Y$ be a bounded linear operator of rank $1$ and $Z$ be a Banach space containing $X$ as a subspace. Then there exists a bounded linear operator $\tilde{T} : Z \to Y$ satisfying

(a) $||\tilde{T}|| = ||T||$;

(b) $\tilde{T}x = Tx$ for every $x \in X$.

Definition 1. An operator $\tilde{T} : Z \to Y$ satisfying (a) and (b) for a bounded linear operator $T : X \to Y$ is called a norm-preserving extension of $T$ to $Z$.

The Hahn-Banach theorem is one of the basic principles of linear analysis. It is quite natural that there exists a vast literature on generalizations of the Hahn-Banach theorem for operators of higher rank. See the papers G. P. Akilov [A], J. M. Borwein [Bor], B. L. Chalmers and B. Shekhtman [CS], G. Elliott and I. Halperin [EH], D. B. Goodner [Go], A. D. Ioffe [I], S. Kakutani [Kak], J. L. Kelley [Kel], J. Lindenstrauss [L1], [L2], L. Nachbin [N1] and M. I. Ostrovskii [O], representing different directions of such generalizations, and references therein. There exist two interesting surveys devoted to the Hahn-Banach theorem and its generalizations, see G. Buskes [Bus] and L. Nachbin [N2].

We shall use the following natural definition.

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Definition 2. An operator \( T : X \to Y \) between Banach spaces \( X \) and \( Y \) is called a Hahn-Banach operator if for every isometric embedding of the space \( X \) into a Banach space \( Z \) there exists a norm-preserving extension \( \tilde{T} \) of \( T \) to \( Z \).

The existence of non-Hahn-Banach operators was mentioned in the remarks to Chapter IV of Banach’s book, see [B2, p. 234]. S. Banach and S. Mazur [BM] proved that the identity operator on \( l_1 \) is a non-Hahn-Banach operator (in fact, this operator does not have even continuous extensions for some isometric embeddings). It has been known for a long time that there exist non-Hahn-Banach operators of rank 2 (see F. Bohnenblust [Boh], an important relevant result was proved relatively recently by H. König & N. Tomczak-Jaegermann [KT]). A problem of characterization of Hahn-Banach operators arises in a natural way.

Factorizational characterizations of Hahn-Banach operators are well known. In particular, using by now a standard technique (that goes back to G. P. Akilov [A], D. B. Goodner [Go, pp. 92–93] and R. Phillips [P, p. 538]) it is easy to show that an operator \( T : X \to Y \) is a Hahn-Banach operator if and only if for some set \( \Gamma \) there exist operators \( T_1 : X \to \ell_\infty(\Gamma) \) and \( T_2 : \ell_\infty(\Gamma) \to Y \) such that \( T_2T_1 = T \) and \( ||T_2||||T_1|| = ||T|| \). (See [J] for the undefined terminology from the theory of Banach spaces.)

One of the main purposes of the present paper is to find a geometric property of Hahn-Banach operators of finite rank acting between finite-dimensional normed spaces (see Theorem 1). This property does not imply that the operator is a Hahn-Banach operator (see the remark after Theorem 1), but it can be used to answer the following question: given \( k \in \mathbb{N} \), for which pairs of finite-dimensional spaces \((X, Y)\) does there exist a Hahn-Banach operator \( T : X \to Y \) of rank \( k \)? (See Theorem 2.) This result is a generalization of a recent result due to B. L. Chalmers and B. Shekhtman [CS].

Remark. Let \( \alpha \in \mathbb{R}, \alpha \neq 0 \). It is clear that \( \alpha T \) is a Hahn-Banach operator if and only if \( T \) is a Hahn-Banach operator. Hence studying Hahn-Banach operators it is enough to consider Hahn-Banach operators of norm 1.

We need the following notation. By \( S(X) \) and \( B(X) \) we denote the unit sphere and the unit ball of a Banach space \( X \) respectively. Let \( X \) be a finite-dimensional Banach space. An intersection of \( B(X) \) with a supporting hyperplane of \( B(X) \) will be called a support set of \( B(X) \). By the dimension of a set in a finite-dimensional space we mean the dimension of its affine hull. (See [S] for the undefined terminology from the theory of convex bodies.) We define \( f(X) \) to be the maximal dimension of a support set of \( B(X) \). For \( x \in S(X) \) we define \( d(x) \) to be the dimension of the set \( \{ x^* \in S(X^*) : x^*(x) = 1 \} \). It is clear that \( d(x) = 0 \) if and only if \( x \) is a smooth point; in the general case \( d(x) \) indicates the number of linearly independent directions of non-smoothness of the norm at \( x \).

Theorem 1. Let \( X \) and \( Y \) be finite-dimensional Banach spaces and \( T : X \to Y \) be a Hahn-Banach operator of rank \( k \). Assume that \( ||T|| = 1 \) and let \( x_0 \in S(X) \) be such that \( ||Tx_0|| = 1 \). Then \( Tx_0 \) belongs to a support set of \( B(Y) \) of dimension \( \geq k - 1 - d(x_0) \).

Proof. Let \( C(S(X^*)) \) denote the space of all continuous functions on \( S(X^*) \) with the sup norm. We identify \( X \) with a subspace of \( C(S(X^*)) \) in the following way:
every vector is identified with its restriction (as a function on $X^*$) to $S(X^*)$. We introduce the following notation: $C = C(S(X^*))$ and $B_C = B(C(S(X^*)))$.

Since $T$ is a Hahn-Banach operator, there exists $\tilde{T} : C \to Y$ such that $\tilde{T}|_X = T$ and $\|\tilde{T}\| = 1$. We shall use $\tilde{T}$ to find a “large” support set of $B(Y)$.

Since $\|Tx_0\| = 1$, there exists $h \in S(Y^*)$ such that $h(Tx_0) = 1$. Let $F = \{x^* \in S(X^*) : x^*(x_0) = 1\}$. Observe that

$$T^*h \in F.$$  \hspace{1cm} (1)

Choose a basis $\{y_1, \ldots, y_m\}$ in $Y$ such that $y_1 = Tx_0$ and $y_2, \ldots, y_m \in \ker h$.

The operator $\tilde{T}$ can be represented in the form

$$\tilde{T} = \sum_{i=1}^{m} \mu_i \otimes y_i, \quad \mu_i \in C^*.$$ 

By the F.Riesz representation theorem (see e.g. [DS], p. 265) we may identify $\mu_i$ with (signed) measures on $S(X^*)$.

Our first purpose is to show that $\mu_1$ is supported on $F \cup (-F)$. We have $\tilde{T}(B_C) \subset B(Y) \subset \{y : |h(y)| \leq 1\}$, $h(y_1) = 1$ and $y_2, y_3, \ldots, y_m \in \ker h$. Therefore for every $z \in B_C$ we have

$$\mu_1(z) = h(\sum_{i=1}^{m} \mu_i(z)y_i) = h(\tilde{T}(z))$$

and $|\mu_1(z)| = |h(\tilde{T}(z))| \leq 1$. Hence

$$||\mu_1|| \leq 1$$  \hspace{1cm} (2)

Also, since $\tilde{T}x_0 = y_1$, we have

$$\mu_1(x_0) = 1.$$  \hspace{1cm} (3)

Conditions (2), (3) and $||x_0|| = 1$ imply that $\mu_1$ is supported on $F \cup (-F)$. (By this we mean that the restriction of $\mu_1$ to $S(X^*)\setminus(F \cup (-F))$ is a zero measure.)

We decompose $\mu_i = \nu_i + \omega_i$, where $\nu_i$ is the restriction of $\mu_i$ to $F \cup (-F)$. Since $\mu_1$ is supported on $F \cup (-F)$, then $\omega_1 = 0$.

Since $T$ is of rank $k$, there exists a subspace $L \subset X$ of dimension $k$ such that $T|_L$ is an isomorphism. Let

$$M = \{x \in L : \forall x^* \in F, \ x^*(x) = 0\}.$$ 

Then $\dim M \geq k - d(x_0) - 1$.

Let $x \in B(M)$. The definitions of $M$ and $\nu_i$ imply that

$$\nu_i(x) = 0, \quad i \in \{1, \ldots, m\}.$$  \hspace{1cm} (4)
(Recall that we identify vectors in \( M \) with the corresponding functions in \( C \).)

Now we construct a “mixture” of \( x \) and \( x_0 \).

It is clear that for each \( \delta > 0 \) there exists a function \( g_\delta \in B_C \) such that

\[
g_\delta|_{F \cup (-F)} = x_0|_{F \cup (-F)}
\]

and the restrictions of \( g_\delta \) and \( x \) to the complement of the \( \delta \)–neighbourhood of \( F \cup (-F) \) coincide.

We have

\[
\lim_{\delta \downarrow 0} \tilde{T} g_\delta = \lim_{\delta \downarrow 0} \sum_{i=1}^{m} \mu_i(g_\delta) y_i = \lim_{\delta \downarrow 0} \sum_{i=1}^{m} (\nu_i(g_\delta) + \omega_i(g_\delta)) y_i.
\]

We have \( \nu_i(g_\delta) = \nu_i(x_0) \) for every \( \delta > 0 \) and \( i \in \{1, \ldots, m\} \).

It is clear that \( \omega_i(F \cup (-F)) = 0 \). By the definition of \( g_\delta \) it follows that \( \lim_{\delta \downarrow 0} g_\delta(x^*) = x(x^*) \) for \( x^* \in S(X^*) \setminus (F \cup (-F)) \) and that the functions \( g_\delta \) are uniformly bounded. By the Lebesgue dominated convergence theorem we get

\[
\lim_{\delta \downarrow 0} \omega_i(g_\delta) = \omega_i(x).
\]

Therefore

\[
\lim_{\delta \downarrow 0} \tilde{T} g_\delta = \sum_{i=1}^{m} (\nu_i(x_0) + \omega_i(x)) y_i.
\]

Equation (4) implies that \( \nu_i(x_0) = \nu_i(x_0 + x) \). Using this and the fact that \( \omega_1 = 0 \), we get

\[
\lim_{\delta \downarrow 0} \tilde{T} g_\delta = \sum_{i=1}^{m} \mu_i(x_0 + x) y_i - \sum_{i=2}^{m} \omega_i(x_0) y_i = T(x_0 + x) - \sum_{i=2}^{m} \omega_i(x_0) y_i.
\]

Since \( g_\delta \in B_C \), \( ||\tilde{T}|| = 1 \) and \( B(Y) \) is closed, then

\[
T(x_0 + x) - \sum_{i=2}^{m} \omega_i(x_0) y_i \in B(Y)
\]

for every \( x \in B(M) \).

By (1) we have \( h(Tx) = 0 \) for every \( x \in M \). Recall, also, that \( y_2, \ldots, y_m \in \ker h \).

Therefore

\[
h \left( T(x_0 + x) - \sum_{i=2}^{m} \omega_i(x_0) y_i \right) = hTx_0 = 1
\]

for every \( x \in M \). Since \( T|_M \) is an isomorphism and the vector \( \sum_{i=2}^{m} \omega_i(x_0) y_i \) does not depend on \( x \), the intersection of \( B(Y) \) with the supporting hyperplane \( \{ y : h(y) = 1 \} \) has dimension \( \geq \dim M \geq k - d(x_0) - 1 \). □
Theorem 2. Let the space $C$ be Banach operator it is enough to show that it has a norm-preserving extension to spaces such that there exists a Hahn-Banach operator $T$ restriction on the dimension (spaces of dimension $\geq X, Y$ their results and to characterize pairs $(L, S)$ of finite-dimensional normed linear spaces such as $L$ being an embedding of the considered spaces into $S$. The approach is an embedding of the considered spaces into $S$. The extension is isomorphisms that are Hahn-Banach operators. One of the steps in their approach is the identity mapping of $X = l_1^k$ onto the space $Y$ whose unit ball is the intersection of $(1 + \varepsilon)B(l_1^2)$, $(\varepsilon > 0)$ and $B(l_1^\infty)$. It is easy to see that

1. the norm of this operator is 1;
2. the only points where the operator attains its norm are $\pm e_1, \pm e_2, \ldots, \pm e_n$, where $\{e_1, \ldots, e_n\}$ is the unit vector basis;
3. the points $\pm e_1, \pm e_2, \ldots, \pm e_n$ are contained in $(n - 1)$-dimensional support sets of $B(Y)$.

Therefore $T$ satisfies the condition of Theorem 1 with $k = n$. On the other hand, the operator $T$ is not a Hahn-Banach operator if $n \geq 3$ and $\varepsilon$ is small enough. In fact, $||T^{-1}|| = 1 + \varepsilon$. Therefore, if $T$ were a Hahn-Banach operator it would imply that for every Banach space $Z$ containing $l_1^k$ as a subspace there exists a projection onto $l_1^k$ with the norm $\leq 1 + \varepsilon$. It remains to apply the well-known result of B. Grünbaum (see [Gr] or [J, p. 81]).

B. L. Chalmers and B. Shekhtman [CS] characterized 2-dimensional spaces having isomorphisms that are Hahn-Banach operators. One of the steps in their approach is an embedding of the considered spaces into $L_1$. This is why they got the restriction on the dimension (spaces of dimension $\geq 3$ may be non-isometric to any subspace of $L_1$, see J. Lindenstrauss [L2, p. 494]). Our next purpose is to extend their results and to characterize pairs $(X, Y)$ of finite-dimensional normed linear spaces such that there exists a Hahn-Banach operator $T : X \to Y$ of rank $k$.

Theorem 2. Let $X$ and $Y$ be finite-dimensional normed linear spaces and let $k$ be a positive integer satisfying $k \leq \min\{\dim X, \dim Y\}$. There exists a Hahn-Banach operator $T : X \to Y$ of rank $k$ if and only if $f(X^*) + f(Y) \geq k - 1$.

Proof. The necessity has been already proved (see the corollary).

Sufficiency. Suppose that

$$k \leq \min\{\dim X, \dim Y, f(X^*) + f(Y) + 1\}.$$

It is well known (see e.g. [KS]) that in order to show that $T : X \to Y$ is a Hahn-Banach operator it is enough to show that it has a norm-preserving extension to the space $C = C(S(X^*))$ (The space $X$ is embedded into $C$ in the same way as in Theorem 1). Therefore, if an operator $Q : C \to Y$ is such that the restriction of $Q$ to $X$ has rank $k$ and $||Q|| = ||Q|_X|| = 1$, then $T = Q|_X$ is a Hahn-Banach operator of rank $k$.

Our purpose is to construct such $Q$. Let $n = f(Y)$ and $m = f(X^*)$. Let $y_0, y_1, \ldots, y_n \in Y$ be linearly independent and such that

$$\{y : y = \theta y_0 + \sum_{i=1}^{n} a_i y_i, \text{ where } \theta = \pm 1, |a_i| \leq 1\} \subset S(Y).$$
Let $x_0^*, x_1^*, \ldots, x_m^* \in X^*$ be linearly independent and such that
\[\{ x^*: x^* = \theta x_0^* + \sum_{i=1}^{m} b_i x_i^*, \text{ where } \theta = \pm 1, \ |b_i| \leq 1 \} \subset S(X^*).\]

Let $x_0 \in S(X)$ be such that $x_0^*(x_0) = 1$. Let
\[x_0^*, x_1^*, \ldots, x_m^*, x_{m+1}^*, \ldots, x_r^*,\]
where $r = \dim X - 1$ be a basis in $X^*$ satisfying the condition $x_{m+1}^*(x_0) = \cdots = x_r^*(x_0) = 0$. (Observe that the condition $x_1^*(x_0) = \cdots = x_m^*(x_0) = 0$ follows from our choice of the vectors.) Let $x_0, x_1, \ldots, x_r$ be its biorthogonal vectors.

Let $y_0, y_1, \ldots, y_s$, where $s = \dim Y - 1$, be a basis in $Y$.
We suppose that $k > m + 1$. (It will be clear from our argument which changes should be made if it is not the case.)

We define an operator $Q_1 : C \to Y$ as follows. Let $\mu_0, \mu_{m+1}, \ldots, \mu_k$ be norm-preserving extensions of $x_0^*, x_{m+1}^*, \ldots, x_k^*$ to $C$. Let
\[Q_1(f) = \mu_0(f)y_0 + \sum_{i=1}^{k-m-1} \frac{\mu_{m+i}(f)}{\|\mu_{m+i}\|} y_i.\]

It is clear that for $x \in X$
\[Q_1(x) = x_0^*(x)y_0 + \sum_{i=1}^{k-m-1} \frac{1}{\|\mu_{m+i}\|} x_{m+i}^*(x)y_i.\] (5)

We have supposed that $k - m - 1 \leq n$. This and the choice of $y_0, \ldots, y_n$ implies that $\|Q_1\| \leq 1$.

Our next step is to show that there exist signed measures $\nu_0, \nu_1, \ldots, \nu_m$ of norm 1 on $S(X^*)$ satisfying the conditions
\[\nu_i(x_j) = \delta_{i,j}, \ i = 0, \ldots, m, \ j = 0, \ldots, r\] (6)
and
\[\forall f \in B_C \ \forall j \in \{1, \ldots, m\} \ |\nu_j(f)| \leq 1 - |\nu_0(f)|.\] (7)

Let us verify that the following measures satisfy these conditions. We define $\nu_j, \ j = 1, \ldots, m$ as atomic measures with atoms at $x_0^* + \sum_{i=1}^{m} \theta_i x_i^*$, $\theta_i = \pm 1$ satisfying $\nu_j(x_0^* + \sum_{i=1}^{m} \theta_i x_i^*) = 2^{-m} \delta_j$ and $\nu_0$ as an atomic measure satisfying $\nu_0(x_0^* + \sum_{i=1}^{m} \theta_i x_i^*) = 2^{-m}$.

Condition (6) follows from the fact that the sequences $\{x_0, \ldots, x_r\}$ and $\{x_0^*, \ldots, x_r^*\}$ are biorthogonal.

Let us verify condition (7). Denote by $\mathbb{1}$ the function that is identically 1 on $S(X^*)$. Let $f \in B_C, \ j \in \{1, \ldots, m\}$. Then $\nu_j(-f) = \nu_j(\mathbb{1} - f) \leq (\text{since the function } \mathbb{1} - f \text{ is nonnegative}) \leq \nu_0(\mathbb{1} - f) = 1 - \nu_0(f)$. (Here we explicitly use the fact that the spaces are real.)
This proves (7) in the case when $\nu_j(f)$ is negative and $\nu_0(f)$ is positive. It remains to observe that

A. Since we may consider $-f$ instead of $f$ it is enough to prove (7) for functions with positive $\nu_0(f)$.

B. For each function $f \in B_C$ and $j \in \{1, \ldots, m\}$ there exists a function $f_j \in B_C$ such that $\nu_0(f_j) = \nu_0(f)$ and $\nu_j(f_j) = -\nu_j(f)$.

We introduce $Q_2 : C \rightarrow Y$ by the equality

$$Q_2(f) = \nu_0(f)y_0 + \sum_{i=1}^{m} \alpha_i \nu_i(f)y_{k-m-1+i}.$$  

Condition (7) implies that there exists $\{\alpha_i\}_{i=1}^{m}$ such that $\alpha_i \neq 0$ for every $i$ and $||Q_2|| \leq 1$.

Condition (6) implies that for $x \in X$ we have

$$Q_2(x) = x_0^*(x)y_0 + \sum_{i=1}^{m} \alpha_i x_i^*(x)y_{k-m-1+i}. \tag{8}$$

Now, let $Q = \frac{1}{2} (Q_1 + Q_2)$. Our estimates for $||Q_1||$ and $||Q_2||$ immediately imply $||Q|| \leq 1$. On the other hand, equations (5) and (8) imply that $Q(x_0) = y_0$. Hence $||Q|| = ||Q|_X|| = 1$.

Also, from (5) and (8) we get for every $x \in X$:

$$Q(x) = x_0^*(x)y_0 + \sum_{i=1}^{k-m-1} \frac{1}{2||\mu_{m+i}||} x_{m+i}^*(x)y_i + \sum_{i=1}^{m} \frac{1}{2} \alpha_i x_i^*(x)y_{k-m-1+i}.$$  

Since the sequences $\{y_0, \ldots, y_s\}$ and $\{x_0^*, \ldots, x_i^*\}$ are linearly independent, it follows that $Q|_X$ is of rank $k$. □

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