SYMMETRIC COHOMOLOGY OF GROUPS AND POINCARÉ DUALITY

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1. Introduction

Let $G$ be a group and $M$ be a $G$-module. In [9, 7] Zarelua and Staic proposed modified group cohomologies denoted respectively by $H^*_G(M, M)$ and $HS^n(G, M)$. According to [4], these groups together with the classical cohomology fit in the commutative diagram

$$
\begin{array}{ccc}
H^*_G(M, M) & \longrightarrow & HS^n(G, M) \\
\downarrow_{\gamma^n} & & \downarrow_{\alpha^n} \\
H^*(G, M) & \longrightarrow & H^n(G, M).
\end{array}
$$

All maps are isomorphisms in dimensions 0 and 1. By [4], the homomorphism $\gamma^n : H^*_G(M, M) \to HS^n(G, M)$ is a split monomorphism in general, and an isomorphism if $0 \leq n \leq 4$, or $M$ has no elements of order two. It follows that there is a natural decomposition

$$
HS^n(G, M) \cong H^n_G(M, M) \oplus H^n_0(G, M).
$$

Based on the results in [4] one can show that the map $\alpha^n$ vanishes on the mysterious summand $H^n_G(G, M)$ and hence all information in $HS^n$ relevant to the classical cohomology is already in Zarelua’s theory $H^n_G(G, M)$. By [8] and [4] the map $\beta^2$ (and hence $\gamma^2$) is a monomorphism. Moreover, if $p$ is an odd prime such that $G$ has no elements of order $q$, for $1 < q < p$, then $\beta^n$ is an isomorphism for all $k < p$ and $\beta^n$ is a monomorphism.

It follows from the very definition that if $G$ is a finite group of order $n$, then $H^*_G(M, M) = 0$ for all $k \geq n$. It was claimed [9, Theorem 3.2] that if $G$ is an oriented group (see Section 5) and $A$ is a trivial $G$-module, then for any $k > 0$ one has the Poincaré duality:

$$
H^1_A(G, A) \cong H^{-k-2}_A(G, A).
$$
However this cannot be true. In fact, if we take $G$ to be Klein’s Vier group $G = C_2 \times C_2$ and $k = 1$ then LHS equals to $H_1(G, A) = G \otimes A = (A/2A)^2$, while RHS $H^1(G, A) = \text{Hom}(G, A) = (2A)^2$ and these groups are not isomorphic in general. Here $2A = \{a \in A | 2a = 0\}$.

The aim of this work is to investigate Poincaré type duality in Zarelua’s cohomology. Unlike [9, Theorem 3.2] we do not restrict ourselves to oriented groups and trivial $G$-modules, but consider all finite groups and all modules over them. We construct groups $H^*_z(G, M)$ for which we have the Poincaré duality

$$H^*_z(G, M^{tw}) \cong H^{n-k-1}_z(G, M),$$

where $M^{tw}$ is a twisting of a $G$-module $M$ defined in Section 5. (For oriented groups, $M^{tw} = M$ for any $G$-module $M$.) The groups $H^*_z(G, M)$ come together with transformations from Tate cohomology

$$\kappa_k : \hat{H}^{-k}(G, M) \rightarrow H^*_z(G, M), \ k \geq 0,$$

which happen to be an isomorphism if $k = 0$ and hence we have

$$H^{n-k-1}_z(G, M) \cong \hat{H}^{0}(G, M^{tw}).$$

We will prove that the transformation $\kappa_1$ is always an epimorphism and $\text{Ker}(\kappa_1)$ is controlled by the elements of order two in $G$. As a consequence we obtain the following exact sequence

$$\bigoplus_{C \leq G} \hat{H}^{-1}(C, M^{tw}) \rightarrow \hat{H}^{-1}(G, M^{tw}) \rightarrow H^{n-2}_z(G, M) \rightarrow 0,$$

where the direct sum is taken over all subgroups of $G$ of order two.

For $k \geq 2$ the transformation $\kappa_k$ factors through Zarelua’s homology groups

$$\hat{H}^{-k}(G, M) = H_{k-1}(G, M) \xrightarrow{\beta_{k-1}} H^{k-1}_z(G, M) \rightarrow H^{k}_z(G, M)$$

and hence Zarelua’s claim is true iff $H_{k-1}^z(G, M) \rightarrow H^k_z(G, M)$ is an isomorphism. We will show that if $G$ is a group and $p \geq 5$ is a prime such that $G$ has no elements of order $q$ for all primes $q < p$, then Zarelua’s claim is true for all $0 < k \leq p - 3$.

2. Koszul complex and Poincaré duality for sets

For a group $G$ the exterior algebra $\Lambda^*(\mathbb{Z}[G])$ plays a prominent role in the definition of Zarelua’s exterior (co)homology of groups [9]. Hence we recollect some well-known facts about exterior algebras.

Recall that the exterior algebra

$$\Lambda^*(A) = \bigoplus_n \Lambda^n(A)$$

of an abelian group $A$ is the quotient algebra of the tensor algebra $T^*(A)$ with respect to the two-sided ideal generated by the elements of the form $a \otimes a \in T^2(A) = A \otimes A$.

There are several (co)chain complexes arising from this construction. All of them depend on some choices. Firstly, we choose an element $a \in A$. Since $a \wedge a = 0$, we have a cochain complex $(\Lambda^*(A), \delta)$:

$$0 \rightarrow \mathbb{Z} \xrightarrow{\delta^0} \Lambda^2(A) \rightarrow \cdots \rightarrow \Lambda^{n-1}(A) \xrightarrow{\delta^{n-1}} \Lambda^n(A) \rightarrow 0$$

where $\delta^k : \Lambda^k(A) \rightarrow \Lambda^{k+1}(A)$ is given by

$$\delta^k(a_1 \wedge \cdots \wedge a_k) = (-1)^k a \wedge a_1 \wedge \cdots \wedge a_k.$$

Secondly, we can choose a homomorphism of abelian groups $f : A \rightarrow \mathbb{Z}$. Then we have the Koszul complex [2, § 9, p. 147] $(\Lambda^*(A), \partial)$, where $\partial_{m-1} : \Lambda^m(A) \rightarrow \Lambda^{m-1}(A)$, $m \geq 0$, is given by

$$\partial_{m-1}(a_1 \wedge \cdots \wedge a_m) = \sum_{i=1}^{m} (-1)^{i+1} f(a_1) a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_m.$$

Here, as usual, the hat $\hat{\cdot}$ denotes a missing value.
Next, we will specify the above general constructions.

Let us fix a set $S$ and take $A = \mathbb{Z}[S]$ to be a free abelian group generated by $S$. We take $f : \mathbb{Z}[S] \to \mathbb{Z}$ the homomorphism for which $f(s) = 1$ for all $s \in S$. In this case the Koszul complex looks as follows:

$$\cdots \to \Lambda^3(\mathbb{Z}[S]) \xrightarrow{\partial_2} \Lambda^2(\mathbb{Z}[S]) \xrightarrow{\partial_1} \mathbb{Z}[S] \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

and for any $s_1, \cdots, s_m \in S$ we have

$$\partial(s_1 \wedge \cdots \wedge s_m) = \sum_{i=1}^{m} (-1)^{i+1} s_1 \wedge \cdots \wedge \hat{s}_i \wedge \cdots \wedge s_m.$$  

We call it the Koszul complex of $S$ and it will be denoted by $\text{Kos}_s(S)$. The following fact is well-known, see for example [2, §9, Proposition 1, p.147].

**Lemma 2.1.** Let $S$ be a nonempty set. Then for any element $c \in S$ one has

$$\delta_c^{k-1} \delta_{k-1} - \delta_0 \delta^*_c = (-1)^{k-1} \text{Id}_{\Lambda^k(\mathbb{Z}[S])}.$$  

In particular $H_i(\text{Kos}_s(S)) = 0$.

Assume now that $S$ is finite and denote by $n$ the cardinality of $S$. It is well-known that $\Lambda^k(\mathbb{Z}[S])$ is a free abelian group of rank $\binom{n+k}{k}$. In particular, $\Lambda^k(\mathbb{Z}[S]) = 0$ for all $k \geq n + 1$ and $\Lambda^n(\mathbb{Z}[S])$ is an infinite cyclic group and hence has two generators. We call the set $S$ with chosen generator $\omega$ of $\Lambda^n(\mathbb{Z}[S])$ an oriented set.

Clearly any order $S = \{t_1 < t_2 < \cdots < t_k\}$ on $S$ gives an orientation $\omega = t_1 \wedge \cdots \wedge t_k$. Moreover, if for a $k$-element subset $I = \{1 \leq i_1 < \cdots < i_k \leq n\}$ we set

$$t_I = t_{i_1} \wedge \cdots \wedge t_{i_k} \in \Lambda^k(\mathbb{Z}[S]),$$

then the collection of $t_I$, $|I| = k$ form a basis of $\Lambda^k(\mathbb{Z}[S])$ as an abelian group. Two orders on $S$ define the same orientation if the corresponding permutation is even.

The Koszul complex in our situation is simply the following exact sequence

$$(2.0.2) \quad 0 \to \Lambda^k(\mathbb{Z}[S]) \xrightarrow{\partial_k} \Lambda^{k-1}(\mathbb{Z}[S]) \to \cdots \to \Lambda^2(\mathbb{Z}[S]) \xrightarrow{\partial_1} \mathbb{Z}[S] \xrightarrow{\partial_0} \mathbb{Z} \to 0.$$

Denote by $N$ the sum $\sum_{s \in S} s \in \mathbb{Z}[S]$. We can look again to the cochain complex $(\Lambda^*(\mathbb{Z}[S]), \delta_N)$. Recall that the homomorphism $\delta^*_N : \Lambda^*(\mathbb{Z}[S]) \to \Lambda^{k+1}(\mathbb{Z}[S])$ is given by

$$\delta^*_N(s_1 \wedge \cdots \wedge s_k) = (-1)^k N \wedge s_1 \wedge \cdots \wedge s_k.$$  

In this way we obtain the cochain complex:

$$(2.0.3) \quad 0 \to \mathbb{Z} \xrightarrow{\delta^*_N} \mathbb{Z}[S] \xrightarrow{\delta_k^*} \Lambda^2(\mathbb{Z}[S]) \to \cdots \to \Lambda^{n-1}(\mathbb{Z}[S]) \xrightarrow{\partial_{n-1}} \Lambda^n(\mathbb{Z}[S]) \to 0.$$  

Our next goal is to prove that the cochain complexes (2.0.2) and (2.0.3) are isomorphic. The isomorphism is given by the Hodge star operator, which depends on the orientation on $S$. Recall that the Hodge star operator is an important operation on differential forms on a Riemannian manifold, see for example [3]. In the context of group cohomology it already appeared in [9].

To describe the Hodge star operator in our circumstances, we choose an order $S = \{t_1 < t_2 < \cdots < t_n\}$ compatible with a given orientation. Thus $\omega = t_1 \wedge \cdots \wedge t_n$ and then we define the Hodge star operator $\alpha_k : \Lambda^k(\mathbb{Z}[G]) \to \Lambda^{n-k}(\mathbb{Z}[G])$ by the conditions

$$(2.0.4) \quad t_I \wedge \alpha_k(t_I) = \omega.$$  

(Classically, the image of $\nu \in \Lambda^k(\mathbb{Z}[G])$ under $\alpha_k$ is denoted by $\ast \nu$.) It is well-known that this map depends only on the orientation on $S$.

It follows from the definition that for any subset $I = \{i_1 < \cdots < i_k\}$ one has

$$\alpha_k(t_I) = (-1)^{\phi(t_I)} t_I,$$
where \( I^c \) is the complement of \( I \) in \( \{1, \cdots, n\} \) and \( \psi(I) = (i_1 - 1) + \cdots + (i_k - k) \).

The following fact is probably well-known and we are not claiming any originality. In any case, it owes very much to [29 Lemma 3.6].

**Proposition 2.2.** The collection of Hodge star operators \( \alpha_k \) are compatible with differentials \( \partial_k \) and \( \delta^k_N \) in the sense that one has an isomorphism of (co)chain complexes of abelian groups:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \Lambda^n(\mathbb{Z}[S]) & \overset{\partial_k}{\longrightarrow} & \Lambda^{n-1}(\mathbb{Z}[S]) & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z}[S] & \overset{\partial_0}{\longrightarrow} & \mathbb{Z} & \longrightarrow & 0 \\
\alpha_0 & \downarrow & a_{n-1} & \downarrow & a_n & \downarrow & \cdots & \downarrow & a_1 & \downarrow & a_0 & \downarrow & 0
\end{array}
\]

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\alpha_0 & \downarrow & a_{n-1} & \downarrow & a_n & \downarrow & \cdots & \downarrow & a_1 & \downarrow & a_0 & \downarrow & 0
\end{array}
\]

**Proof.** Take a subset \( I = \{i_1 < \cdots < i_k\} \), \( k < n \). We need to show that

\[
\partial_{n-k-1} \alpha_k(t) = \alpha_{k+1} \delta^k_N(t).
\]

Since \( \alpha_k \) depends only on the orientation of \( S \), we can change the ordering on \( S \), if it is necessary, to assume \( I = \{1 < \cdots < k\} \). In this case we obtain

\[
\alpha_{k+1} \delta^k_N(t) = \alpha_{k+1} \left( (-1)^k \sum_{i=1}^{n-k} t_{k+i} \wedge \cdots \wedge t_k \right)
\]

\[
= \sum_{k=1}^{n-k} \alpha_{k+1} (t_1 \wedge \cdots \wedge t_k \wedge t_{k+i})
\]

\[
= \sum_{i=1}^{n-k} (-1)^{i-1} t_{k+1} \wedge \cdots \wedge \hat{t}_{k+i} \wedge \cdots \wedge t_n.
\]

On the other hand, we also have

\[
\partial_{n-k-1} \alpha_k(t) = \partial_{n-k-1} (t_1 \wedge \cdots \wedge t_n)
\]

\[
= \sum_{i=1}^{n-k} (-1)^{i-1} t_{k+1} \wedge \cdots \wedge \hat{t}_{k+i} \wedge \cdots \wedge t_n
\]

and we are done. \( \square \)

### 3. Classical and Zarelua’s (co)homology of groups

In this section we recall the definition of the classical and Zarelua’s (co)homology of groups.

Let \( G \) be a group. We consider \( \mathbb{Z} \) as a \( G \)-module, with trivial action of \( G \). Denote \( P_k = \mathbb{Z}[G^{k+1}] \), which is considered as a \( G \)-module via the action

\[
g(g_0, \cdots, g_k) = (gg_0, \cdots, g_k)
\]

and consider the standard projective resolution of \( \mathbb{Z} \) by \( G \)-modules [11]

\[
(3.0.1) \quad \cdots \rightarrow P_k \overset{\partial}{\rightarrow} P_{k-1} \rightarrow \cdots \rightarrow P_0 \overset{\epsilon}{\rightarrow} \mathbb{Z},
\]

where the boundary map is given by

\[
\partial(g_0, \cdots, g_k) = \sum_{i=0}^{k} (-1)^i (g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_k).
\]

and the mapping \( \epsilon \) sends each generator \( g \) to \( 1 \in \mathbb{Z} \). For any \( G \)-module \( M \) we apply the functor \( \text{Hom}_G(-, M) \) (resp. \( (\_)^G : M \) ) to the standard projective resolution to obtain the cochain complex \( K^*(G, M) \) (resp. chain complex \( K_*(G, M) \)) and then one defines the cohomology and homology of a group \( G \) with coefficients in \( M \) by

\[
H^*(G, M) = H^*(K^*(G, M)) \quad \text{and} \quad H_*(G, M) = H_*(K_*(G, M)).
\]
Zarelua uses the Koszul complex \( \text{Kos}^*(G) \) of Section 2 to define his (co)homology groups as follows. Firstly, the abelian groups \( \Lambda^k (\mathbb{Z}[G]) \), \( k \geq 0 \), have the natural \( G \)-module structure given by \( g \cdot x_1 \wedge \cdots \wedge x_k = gx_1 \wedge \cdots \wedge gx_k \).

Let us denote by \( \text{Kos}^*_G(G) \) the chain complex

\[
\cdots \to \Lambda^3 (\mathbb{Z}[G]) \overset{\partial}{\to} \Lambda^2 (\mathbb{Z}[G]) \overset{\partial}{\to} \mathbb{Z}[G].
\]

By Lemma 2.1 one can look at \( \text{Kos}^*_G(G) \) as a (non-projective) resolution of \( \mathbb{Z} \) in the category of \( G \)-modules and then one puts

\[
K^*_G(M, M) = \text{Hom}_G(\text{Kos}^*_G(G), M), \quad K^*_G(M, M) = \text{Kos}^*_G(G) \otimes_G M.
\]

Zarelua’s (co)homology groups are defined by

\[
H^*_G(M, M) := H^*(K^*_G(M, M)) \quad \text{and} \quad \hat{H}^*_G(M, M) := H_*(K^*_G(M, M)).
\]

We have a morphism of projective resolutions, see [4, Lemma 3.3],

\[
\cdots \to \mathbb{Z}[G^3] \to \mathbb{Z}[G^2] \to \mathbb{Z}[G] \to 0,
\]

and hence one obtains natural transformations

\[
\beta^* : H^*_G(M, M) \to H^*(G, M) \quad \text{and} \quad \beta_* : H_*(G, M) \to H^*_G(M, M).
\]

We refer to [4] for extensive information about \( \beta^* \). It was not stated in [4] but all results obtained in [4] have obvious analogues for \( \beta_* \). In particular, \( \beta_0, \beta_1 \) are isomorphisms, while \( \beta_2 \) fits in the exact sequence

\[
\bigoplus_{C \in G} H_2(C_2, M) \to H_2(G, M) \overset{\beta_2}{\to} \hat{H}^2_2(G, M) \to 0,
\]

where \( C_2 \) runs through all subgroups of order two, etc.

**Example.** Based on the exact sequence in [4, Corollary 4.4] and [6, Exercise 5, p. 66] we see that \( H^*_G(D_n, \mathbb{F}_2) = 0 \) for all dihedral groups \( D_n \).

4. **Tate Cohomology and \( \hat{H}^*_G(M, M) \) Groups**

In what follows we restrict ourselves to the case when \( G \) is a finite group of order \( n \). Recall that in this case we also have the Tate cohomology groups \( \hat{H}^*(G, M) \), which combine the aforementioned homology and cohomology of groups. They are defined by

\[
\hat{H}^k(G, M) = \begin{cases} 
H^k(G, M) & \text{if } k > 0 \\
\text{Coker } N & \text{if } k = 0 \\
\text{Ker } N & \text{if } k = -1 \\
H_{-k-1}(G, M) & \text{if } k < -1
\end{cases}
\]

where \( N \) is the norm homomorphism

\[
N : H_0(G, M) \to H^0(G, M)
\]

induced by the map \( M \to M \) given by \( m \mapsto \sum_{g \in G} gm = Nm \).

We also need the following equivalent description of Tate cohomology \([1]\). For any abelian group \( A \) denote by \( A^\vee \) the "dual" abelian group \( \text{Hom}(A, \mathbb{Z}) \). It is well-known that the natural map \( A \to (A^\vee)^{\vee} \) is an isomorphism if \( A \) is a finitely generated free abelian group. Observe that if \( A \) is a \( G \)-module, then \( A^\vee \) is also a \( G \)-module with the action

\[
(g \cdot \psi)(a) = \psi(g^{-1}a)
\]

where \( a \in A \) and \( \psi \in A^\vee \).
Since the standard resolution $P_* \to \mathbb{Z} \to 0$ splits as a chain complex of abelian groups, after dualizing we still obtain an exact sequence

$$0 \to \mathbb{Z} \to P_*^\vee.$$ 

We can eliminate $\mathbb{Z}$ to get an exact sequence

$$\cdots \to P_1 \to P_0 \to P_0^\vee \to P_1^\vee \to \cdots$$

and after reindexing $P_k := P_{-k-1}$ for $k < 0$, one obtains [11, p.103]

$$\hat{H}^k(G, M) = H^k(\text{Hom}_G(P_*, M)), \ k \in \mathbb{Z}.$$ 

Equivalently, for any $k \leq 0$ for the Tate cohomology groups $\hat{H}^k(G, M)$ we have

$$(4.0.1) \quad \hat{H}^k(G, M) \cong H_k(\text{Hom}_G(\mathbb{Z}, M) \leftarrow \text{Hom}_G(P_0^\vee, M) \leftarrow \text{Hom}_G(P_1^\vee, M) \leftarrow \cdots).$$

Now we introduce groups $H^*_s(G, M)$ which are defined as follows. We consider the exact sequence (4.0.3) for $S = G$:

$$0 \to \mathbb{Z} \xrightarrow{\delta_N} \mathbb{Z}[G] \xrightarrow{\delta_N} \Lambda^2(\mathbb{Z}[G]) \to \cdots \to \Lambda^{n-1}(\mathbb{Z}[G]) \xrightarrow{\delta_N^{n-1}} \Lambda^n(\mathbb{Z}[G]) \to 0$$

which is denoted by $C^*(G)$. Thus $C^k = \Lambda^k\mathbb{Z}[G], 0 \leq k \leq n$.

**Lemma 4.1.** The boundary maps $\delta_N^i$ are $G$-homomorphisms, $0 \leq i \leq n - 1$.

**Proof.** It is required to check

$$N(gx_1 \wedge \cdots \wedge gx_i) = g(N(x_1 \wedge \cdots \wedge x_i)).$$

Since $\sum_{h \in G} h = \sum_{h \in G} gh$, we have

$$N(gx_1 \wedge \cdots \wedge gx_i) = \sum_{h \in G} h \wedge gx_1 \wedge \cdots \wedge gx_i$$

$$= \sum_{h \in G} gh \wedge gx_1 \wedge \cdots \wedge gx_i$$

$$= gN(x_1 \wedge \cdots \wedge x_i).$$

\[ \square \]

Since $C^*(G)$ is a cochain complex of $G$-modules we can apply the functor $\text{Hom}_G(-, M)$ to obtain a new chain complex

$$\text{Hom}_G(\mathbb{Z}, M) \leftarrow \text{Hom}_G(\mathbb{Z}[G], M) \leftarrow \text{Hom}_G(\Lambda^2\mathbb{Z}[G], M) \leftarrow \cdots \leftarrow \text{Hom}_G(\Lambda^n\mathbb{Z}[G], M) \leftarrow 0,$$

which is denoted by $K^*_s(G, M)$. Now we set

$$H^*_s(G, M) := H_*(K^*_s(G, M)).$$

We are now going to construct natural transformations from the Tate cohomology and Zarelua’s homology into $H^*_s(G, M)$. The construction is based on the following well-known lemma.

**Lemma 4.2.** Let $M$ be a $G$-module and $L_*$ a chain complex of finitely generated $G$-modules. Assume each $L_k$ is free as an abelian group. Then the map

$$\iota : L_* \otimes G M \to \text{Hom}_G(L_*^\vee, M)$$

is a natural chain map, which is an isomorphism if each $L_k$ is a projective $G$-module, where $\iota$ is the composite of the following maps

$$L_* \otimes G M = H_0(G, L_* \otimes M) \xrightarrow{N} H^0(G, L_* \otimes M) \xrightarrow{\iota} H^0(G, \text{Hom}(L_*^\vee, M)) = \text{Hom}_G(L_*^\vee, M)$$

and $\iota : L_* \otimes M \to \text{Hom}(L_*^\vee, M)$ is given by

$$\iota(x \otimes m)(\psi) = \psi(x)m.$$ 

Here $x \in L_*, m \in M$ and $\psi \in L_*^\vee = \text{Hom}(L_*, \mathbb{Z}).$
Proof. See [1, p.103] for an explicitly written proof of this fact, which implicitly is there. \qed

By Lemma 2.1 and Proposition 2.2 we see that the exact sequence $C^*(G)$ splits as a chain complex of abelian groups. Hence we can dualize it to get an exact sequence

$$0 \leftarrow \mathbb{Z} \leftarrow C_1^\vee \leftarrow \cdots \leftarrow C_n^\vee \leftarrow 0.$$ 

In fact, this is an exact sequence of $G$-modules and hence it can be considered as a (nonprojective) resolution of $\mathbb{Z}$. By a well-known property of projective resolutions, the identity map $\mathbb{Z} \rightarrow \mathbb{Z}$ has a lifting as a morphism of chain complexes

$$0 \leftarrow \mathbb{Z} \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$$

By dualizing, we obtain the diagram

$$0 \rightarrow \mathbb{Z} \rightarrow C_1 \rightarrow C_2 \rightarrow \cdots$$

By applying the functor $\text{Hom}_G(\cdot, M)$ and taking the homology functor one obtains natural homomorphisms

$$\zeta_k : \hat{H}^k(G, M) \rightarrow H_k(G, M), \quad k \geq 0.$$ 

The next aim is to prove that the maps $\zeta_k$ factor through Zarelua’s homology groups. To this end, we use Lemma 4.2 first when $L_* = C_*^\vee$ and then when

$$L = P_\text{aug}^* = 0 \leftarrow \mathbb{Z} \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$$

to obtain the commutative diagram of chain complexes

$$P_\text{aug}^* \otimes_G M \rightarrow \text{Hom}_G(P_\text{aug}^\vee, M)$$

$$C_*^\vee \otimes_G M \rightarrow \text{Hom}_G(C_*^\vee, M)$$

Since $P_n$ are projective $G$-modules, we see that the top map after taking homology induces an isomorphism in dimensions $\geq 2$. So we obtain the diagram

$$H_k(G, M) \xrightarrow{\sim} \hat{H}^{k-1}(G, M)$$

$$\beta_k \downarrow \quad \zeta_k \downarrow$$

$$H_k^\lambda(G, M) \xrightarrow{\lambda_k} H_{k+1}(G, M)$$

Hence $\zeta_{k+1} = \zeta_k \beta_k, \quad k \geq 1$.

Lemma 4.3. i) For any group $G$ and any $G$-module $M$ the homomorphism

$$\zeta_0 : \hat{H}^0(G, M) \rightarrow H_0^\lambda(G, M)$$

is an isomorphism.

ii) Let $k \geq 2$ and let $G$ be a group such that the $G$-modules $N^i(\mathbb{Z}[G])$ are projective $G$-modules for all $2 \leq i \leq k$. Then for any $G$-module $M$ the homomorphism $\zeta_i$ is an isomorphism for all $0 \leq i < k$.

Proof. i) By definition we have an exact sequence

$$M \xrightarrow{\eta} H^0(G, M) \rightarrow \hat{H}^0(G, M) \rightarrow 0.$$
Next, we have
\[ K^n_0(G, M) = \text{Hom}_G(\mathbb{Z}, M) = H^0(G, M) \]
and
\[ K^n_1(G, M) = \text{Hom}_G(\mathbb{Z}[G], M) = M. \]
Since the boundary map \( K^n_i(G, M) \to K^n_{i-1}(G, M) \) is induced by the norm map, we obtain
\[ H^n(G, M) = \hat{H}^0(G, M). \]

ii) It follows from the basic properties of projective resolutions that the augmented chain complexes \( P_* \to \mathbb{Z} \) and \( C_\omega^i \) are homotopy equivalent up to dimension \( k \) and hence the result. □

**Corollary 4.4.** If the group has no elements of order two, then
\[ \alpha_1 : \hat{H}^{-1}(G, M) \to H^1_{\omega}(G, M) \]
is an isomorphism. More generally, if \( p \) is an odd prime, such that for all prime \( q < p \) the group \( G \) has no elements of order \( q \), then \( \alpha_k \) is an isomorphism for all \( k < p - 1 \).

**Proof.** In this case all \( G \)-modules \( \Lambda^i(\mathbb{Z}[G]), 2 \leq i \leq p - 1 \), are free, see the proof [4, Theorem 4.2]. □

5. **A Poincaré Duality**

In this section we will still assume that \( G \) is a finite group of order \( n \). We would like to apply the results of Section 2 to group cohomology. Observe that in general the Hodge star operator \( \alpha_k \) is not compatible with the action of \( G \). To avoid this difficulty we will twist the action.

For an element \( g \in G \) denote by \( \ell_g : G \to G \) the map given by \( \ell_g(h) = gh, h \in G \). Since \( \ell_g \) is a bijection, we can consider the sign of this permutation, which is denoted by \( \epsilon(g) \). In this way we obtain a group homomorphism
\[ \epsilon : G \to \{\pm 1\} \]

Following [9], \( G \) is called **oriented** if \( \epsilon \) is the trivial homomorphism. It is clear that any group of odd order or any perfect group is oriented, because there are no nontrivial homomorphisms into \( \{\pm 1\} \).

Next, we choose a generator \( \omega \) of \( \Lambda^n(\mathbb{Z}[G]) \). We can take \( \omega = 1 \wedge x_2 \wedge \cdots \wedge x_n \), where \( < \) is a chosen order on \( G \):
\[ G = \{1 = x_1 < x_2 < \cdots < x_n\}. \]

Then for any \( g \in G \) we have
\[ g\omega = \epsilon(g)\omega. \]
Comparing the definitions, we see that \( G \) is oriented if the action of \( G \) on \( \Lambda^n(\mathbb{Z}[G]) \) is trivial, in other words if for any \( 1 \leq i \leq n \) one has
\[ 1 \wedge x_2 \wedge \cdots \wedge x_n = x_i \wedge x_2 \wedge \cdots \wedge x_i x_n. \]

Now we introduce the twisting operation on \( G \)-modules.

If \( A \) is a \( G \)-module, then \( A^{\text{tw}} \) is the \( G \)-module which is \( A \) as an abelian group, while the new action is given by
\[ g \ast a = \epsilon(g)(ga). \]

For oriented groups \( A^{\text{tw}} = A \). In general, we have the following isomorphism of \( G \)-modules
\[ A^{\text{tw}} \cong A \otimes \mathbb{Z}^{\text{tw}} \]
and the map \( \alpha_0 \) from Section 2 induces the isomorphism of \( G \)-modules
\[ \mathbb{Z}^{\text{tw}} \overset{\alpha_0}{\longrightarrow} \Lambda^n(\mathbb{Z}[G]), \quad 1 \mapsto \omega. \]

Our next goal is to extend the isomorphism (5.0.2) to other exterior powers. We claim that for any \( x \in G \) one has
\[ \alpha_k(gx) = \epsilon(g)\alpha_k(x). \]
Proposition 5.2. For any finite group $G$ and any $G$-module $M$ one has a Poincaré type duality:

$$\sigma : \Lambda^2(\mathbb{Z}[G]) \cong \Lambda^{n-k}(\mathbb{Z}[G])$$

and hence we have an isomorphism of cochain complexes of $G$-modules:

$$0 \to \Lambda^n(\mathbb{Z}[G]) \to \Lambda^{n-1}(\mathbb{Z}[G]) \to \cdots \to \mathbb{Z}[G] \to \Lambda^1(\mathbb{Z}[G]) \to \Lambda^0(\mathbb{Z}[G]) \to 0$$

After applying the functor $\text{Hom}_G(-, M)$ we see that the cochain complex $K^*_G(M, M)$ is isomorphic to the complex

$$0 \leftarrow \text{Hom}_G(\mathbb{Z}^\text{tw}, M) \leftarrow \text{Hom}_G(\mathbb{Z}[G]^\text{tw}, M) \leftarrow \cdots \leftarrow \text{Hom}_G(\Lambda^n(\mathbb{Z}[G], M^\text{tw}).$$

Since the assignment $A \mapsto A^\text{tw}$ is an autoequivalence of the category of $G$-modules, we have $\text{Hom}_G(\mathbb{Z}^\text{tw}, Y) = \text{Hom}_G(X, Y^\text{tw})$. Hence $K^*_G(M, M)$ is isomorphic to truncated $K^*_G(M, M)$, which is obtained by deleting the last group $\text{Hom}_G(\Lambda^n(\mathbb{Z}[G], M^\text{tw}).$ Thus we obtain the following result.

Proposition 5.2. For any finite group $G$ and any $G$-module $M$ one has a Poincaré type duality:

$$\Delta_k : H^{n-k-1}_G(M, M) \to H^k(G, M^\text{tw}),$$

for all $0 \leq k \leq n-2$.

In particular, by Lemma 5.3 and Corollary 5.4 we have the following facts.

Corollary 5.3. Let $G$ be a finite group of order $n$ and $M$ be a $G$-module.

i) We have an isomorphism

$$\Delta_0 : H^{n-1}_G(M, M) \cong \hat{H}^0(G, M^\text{tw}).$$

ii) If $p$ is a prime such that the group has no elements of order $q$, for any prime $q$ such that $q < p$, then one has an isomorphism

$$\Delta_i : H^{n-i-1}_G(M, M) \cong \hat{H}^i(G, M^\text{tw}), 0 \leq i < p-1.$$
Observe that for \( o(x) = 2 \), we have
\[
f_x(1 + x) = 1 \wedge x + x \wedge 1 = 0.
\]
Thus, for such \( x \) the map \( f_x \) factors through
\[
\hat{f}_x : \mathbb{Z}[G] \mathbb{Z}[G](x + 1) \rightarrow \Lambda^2(\mathbb{Z}[G]).
\]
Now we set
\[
W_1 = \bigoplus_{o(x) = 2} \mathbb{Z}[G] \mathbb{Z}[G](x + 1)
\]
and
\[
W_2 = \bigoplus_{\xi \in \mathcal{C}_2(G)} \mathbb{Z}[G],
\]
where \( x \) (resp. \( \xi \)) is running through the set of elements of order two (resp. \( \mathcal{C}_2(G) \)).

It would be convenient to write a general element of \( W_1 \) (resp. \( W_2 \)) as a sum \( \sum_{o(x) = 2} f_x(u) \) (resp. \( \sum_{\xi} i_\xi(v) \)). Here \( u \in \mathbb{Z}[G] / \mathbb{Z}[G](x + 1), v \in \mathbb{Z}[G] \) and \( f_x : \mathbb{Z}[G] / \mathbb{Z}[G](x + 1) \rightarrow W_1 \) and \( i_\xi : \mathbb{Z}[G] \rightarrow W_2 \) are standard inclusions into the direct sum.

Define the \( G \)-module homomorphisms
\[
f_1 : W_1 \rightarrow \Lambda^2(\mathbb{Z}[G]) \quad \text{and} \quad f_2 : W_2 \rightarrow \Lambda^2(\mathbb{Z}[G])
\]
by
\[
f_1 \left( \sum_{o(x) = 2} f_x(u) \right) = \sum_x f_x(u) \quad \text{and} \quad f_2 \left( \sum_{\xi} i_\xi(v) \right) = \sum_{\xi} i_\xi(\xi).
\]

The maps \( f_1 \) and \( f_2 \) define the \( G \)-module homomorphism
\[
f : W_1 \oplus W_2 \rightarrow \Lambda^2(\mathbb{Z}[G]),
\]
which is \( f_1 \) on \( W_1 \) and \( f_2 \) on \( W_2 \).

**Lemma 6.1.** The map \( f : W_1 \oplus W_2 \rightarrow \Lambda^2(\mathbb{Z}[G]) \) is an isomorphism of \( G \)-modules.

**Proof.** By construction, \( f \) is a \( G \)-module homomorphism. Hence we need to show that \( f \) is an isomorphism of free abelian groups. To this end, also decompose the RHS as a direct sum of two summands \( U \oplus V \), where \( U \) (resp. \( V \)) is spanned as an abelian group by \( a \wedge b, a, b \in G \) such that \( o(a^{-1}b) \neq 2 \) (resp. \( o(a^{-1}b) = 2 \)). We have \( f(W_1) \subset U \) and \( f(W_2) \subset V \).

Take \( \xi \in \mathcal{C}_2(G) \). Denote by \( \tau_\xi \) the inclusion of \( \mathbb{Z}[G] \) in \( W_2 \) corresponding to the summand indexed by \( \xi \). The collection \( \tau_\xi(g) \) form a basis of \( W_2 \), where \( g \in G \). By definition \( f \) sends this element to \( g \wedge g\sigma(\xi) \), which up to sign is a free generator of \( V \) (because \( o(g^{-1}g\sigma(\xi)) \neq 2 \)). Conversely, take any basis element \( a \wedge b \) of \( V \), where \( a, b \in G \) and \( a < b \) in a chosen order of \( G \). We have \( a \wedge b = a(1 \wedge c) \), where \( c = a^{-1}b \). By definition of \( V \) we have \( o(c) \neq 2 \). Denote by \( \xi \) the class of \( c \) in \( \mathcal{C}_2(G) \). Then \( c = \sigma(\xi) \) or \( c^{-1} = \sigma(\xi) \). In the first case \( f_\sigma(\xi)(a) = a \wedge b \) and in the second case \( f_\sigma(\xi)(b) = a \wedge b \). This shows that the matrix corresponding to \( f \) in the chosen bases is a diagonal matrix with \( \pm 1 \) on the diagonal. Hence \( f \) induces an isomorphism \( W_2 \rightarrow V \).

Let \( x \in G \) be an element of order two. We set \( C_2x = \{1, x\} \), the cyclic subgroup generated by \( x \). First, we choose a \section \( \tau_x : G / C_2(x) \rightarrow G \) of the canonical map \( G \rightarrow G / C_2(x) \). Then the collection \( \tau_x(\eta) \) form a basis of the abelian group \( \mathbb{Z}[G] / \mathbb{Z}[G](x + 1) \), where \( \eta \) runs through the set \( G / C_2(x) \). Hence, the elements \( \tau_x(\eta) \) form a basis of \( W_1 \), where \( o(x) = 2 \). By definition the map \( f \) sends a basis element \( \tau_x(\eta) \) to \( \tau(\eta) \wedge \tau(\eta)x \), which is a basis element (up to sign) of \( U \). Conversely, take any basis element \( a \wedge b \) of \( U \), where \( a, b \in G \) and \( a < b \) in a chosen order of \( G \). We have \( a \wedge b = a(1 \wedge c) \), where \( c = a^{-1}b \) and \( o(c) = 2 \). Denote by \( \tilde{a} \) the class of \( a \) in \( G / C_2(c) \). Then \( \tau(\tilde{a}) \) is either \( a \) or \( ac \) and we have \( f_\sigma(\tau(\tilde{a})) = a \wedge ac = a \wedge b \) or \( f_\sigma(\tau(\tilde{a})) = ac \wedge abc = ac \wedge a = a \wedge b \) and we see that \( f \) is a bijection on basis elements. Thus \( f \) induces an isomorphism \( W_1 \rightarrow U \) and the Lemma follows. \( \square \)
Lemma 6.2. Define the $G$-module homomorphisms $\theta_1 : \mathbb{Z}[G] \to W_1$ and $\theta_2 : \mathbb{Z}[G] \to W_2$ by

$$\theta_1(g) = \sum_{\sigma(x) = 2} j_x(\bar{g}),$$
$$\theta_2(g) = \sum_{\xi} i_\xi(g - g\sigma(\xi)^{-1}).$$

Here $\bar{g}$ is the image of $g \in G$ in $\mathbb{Z}[G]/\mathbb{Z}[G](x + 1)$. Then one has the following commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}[G] & \xrightarrow{\delta_N^1} & \Lambda^2(\mathbb{Z}[G]) \\
\downarrow f & & \downarrow f \\
\mathbb{Z}[G] & \xrightarrow{\theta} & W_1 \oplus W_2
\end{array}$$

where the components of $\theta$ are $\theta_1$ and $\theta_2$.

Proof. Since the morphisms involved are $G$ homomorphisms, we only need to control the image of $1 \in \mathbb{Z}[G]$. We have

$$\delta_N^1(1) = -N \wedge 1 = \sum_{x \neq 1} 1 \wedge x = \sum_{\sigma(x) = 2} 1 \wedge x + \sum_{x \neq 1} 1 \wedge x.$$ 

The second summand can be rewritten as follows. If $x^2 \neq 1$, then $x$ and $x^{-1}$ are two representatives of a class in $\mathbb{C}_2(G)$. Hence

$$\delta_N^1(1) = \sum_{\sigma(x) = 2} 1 \wedge x + \sum_{\xi} \left( 1 \wedge \sigma(\xi) + 1 \wedge \sigma(\xi)^{-1} \right).$$

On the other hand, $f \circ \theta = f_1 \theta_1 + f_2 \theta_2$. It follows that

$$f(\theta(1)) = f_1 \left( \sum_{\sigma(x) = 2} j_x(\bar{1}) \right) + f_2 \left( \sum_{\xi} f_{\sigma(\xi)}(1 - \sigma(\xi)^{-1}) \right),$$

$$= \sum_{\sigma(x) = 2} f_1(1) + \sum_{\xi} f_{\sigma(\xi)}(1 - \sigma(\xi)^{-1})$$

$$= \sum_{\sigma(x) = 2} 1 \wedge x + \sum_{\xi} \left( 1 \wedge \sigma(\xi) - \sigma(\xi)^{-1} \wedge \sigma(\xi)^{-1} \sigma(\xi) \right)$$

$$= \sum_{\sigma(x) = 2} 1 \wedge x + \sum_{\xi} \left( 1 \wedge \sigma(\xi) + 1 \wedge \sigma(\xi)^{-1} \right)$$

and the result is proved. \qed

According to Lemma 4.3, we know that $\chi_1 : \check{H}^1(G, M) \to H^1_c(G, M)$ is an isomorphism if $G$ has no elements of order two. Now we will consider arbitrary groups. To state the corresponding fact, we fix some notation. If $o(x) = q$ we denote by $C_q(x)$ the cyclic subgroup generated by $x$.

Proposition 6.3. Let $G$ be a finite group of order $n$ and $M$ be a $G$-module. Then one has the following exact sequence

$$\bigoplus_{o(x) = 2} \check{H}^1(C_2(x), M) \to \check{H}^1(G, M) \xrightarrow{\chi_1} H^1_c(G, M) \to 0.$$ 

Proof. It follows from the above lemmata that the complex $K^*_c(G, M)$ in dimensions 0, 1, 2 looks as follows:

$$\begin{array}{cccc}
\check{H}^0(G, M) & \xrightarrow{\cdot} & M & \xleftarrow{\cdot} & W^1(G, M) \oplus W^2(G, M)
\end{array},$$

where

$$W^1(G, M) = \text{Hom}_G(W_1, M).$$
After these preliminaries, we will now prove Proposition 6.3. By definition we have the norm map
\[ W^2(G, M) = \text{Hom}_G(W_2, M) \]
is the set of functions \( \ell_1 \) defined on the set \( \xi \in C_2(G) \) with values in \( M \). The 0-th boundary map \( d : M \to H^0(G, M) \) is the norm map \( m \mapsto \sum_{x \in G} x m \), while the next boundary map \( \delta \) is given by
\[ \delta(\ell_1, \ell_2) = \sum_{\alpha(x) = 2} \ell_1(x) + \sum_{\xi} (1 - \sigma(\xi)^{-1}) \ell_2(\xi). \]

After these preliminaries, we will now prove Proposition 6.3. By definition we have
\[ H^{-1}(G, M) = \frac{\text{Ker}(N : M \to H^0(G, M))}{V} \quad \text{and} \quad H^1(G, M) = \frac{\text{Ker}(N : M \to H^0(G, M))}{U}, \]
where \( V \) is generated by elements of the form \( gm - m, g \in G, m \in M \), while \( U = U_1 + U_2 \). Here \( U_1 \) is generated by \( n \in M \), where \( xn + n = 0 \) for an element \( x \in G \) of order two and \( U_2 \) is generated by elements of the form \( gm - m \), where \( g = \sigma(\xi) \) for an element \( \xi \in C_2(G) \) and \( m \in M \). Our claim is that \( V \subseteq U \). In other words we have to show \( gm - m \in U \) for all \( g \in G \) and \( m \in M \). By definition, this holds automatically if \( g = \sigma(\xi)^{-1} \) for an element \( \xi \in C_2(G) \). Since \( gm - m = -(g^{-1}n - n) \) for \( n = gm \), we see that \( gm - m \in U \) for all \( g \) with \( o(g) \neq 2 \). Assume now \( o(g) = 2 \). Then for \( n = gm - m \) we have \( (g + 1)n = 0 \) and hence \( n \in U_1 \subseteq U \). So we proved that \( V \subseteq U \). It implies that the map \( \kappa_1 \) is surjective. It is also clear that \( \text{Ker}(\kappa_1) = U/V \). Since \( U = U_1 + U_2 \) and \( U_2 
subseteq U \), we see that \( \text{Ker}(\kappa_1) = U_1/U_1 \cap V \). On the other hand the image of \( \bigoplus_{\alpha(x) = 2} H^{-1}(C_2(\lambda), M) \) in \( H^{-1}(G, M) \) is \( U_1/V_1 \), where \( V_1 \subseteq M \) is a subgroup generated by elements of the form \( xn - n \), where \( o(x) = 2, n \in M \). Obviously \( V_1 \subseteq U_1 \cap V \) and Proposition 6.3 follows.

**Corollary 6.4.** Let \( G \) be a finite group of order \( n \) and \( M \) be a \( G \)-module. Then one has the following exact sequence
\[ \bigoplus_{C_2 \subseteq G} H^{-1}(C_2, M^m) \to H^{-1}(G, M^m) \to H^0_n(G, M) \to 0, \]
where the sum is taken over all subgroups of order two.

This follows directly from the Proposition 6.3 and the Poincaré duality 6.2.

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