From Dirac Notation to Probability Bracket Notation: 
Time Evolution and Path Integral under Wick Rotations

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Abstract

In this article, we continue to investigate the application of Probability Bracket Notation (PBN). We show that, under Special Wick Rotation (caused by imaginary-time rotation), the Schrodinger equation of a conservative system and its path integral in Dirac rotation are simultaneously shifted to the master equation and its Euclidean path integral of an induced micro diffusion in PBN. Moreover, by extending to General Wick Rotation and using the anti-Hermitian wave-number operator, we execute the path integral in Dirac notation side-by-side with the Euclidean path integral in PBN, and derive the Euclidean Lagrangian of induced diffusions and Smoluchowski equation.

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1. Quantum Mechanics and Path Integrals in Dirac Notation

Inspired by the great success of Dirac’s Vector Bracket Notation (VBN) for vectors in Hilbert spaces, we have proposed the Probability Bracket Notation (PBN) for probability modeling in probability spaces [1].

In this section, we will give a brief review of Quantum Mechanics (QM) and path integrals in Dirac notation (§1.4 of [2] and §4.3 of [3]). The expressions given here will be used to compare with and shifted to those given in PBN later. Readers who are familiar with these expressions may skip this section.

In Dirac’s VBN, the inner product of two Hilbert vectors is denoted by a bracket (we call it a v-bracket), which can be split into a v-bra and a v-ket:

\[ \langle \psi_A | \psi_B \rangle \Rightarrow \text{v-bra} : \langle \psi_A \rangle, \text{ v-ket} : | \psi_B \rangle \]  

(1.1)

where:

\[ \langle \psi_A | \psi_B \rangle = \langle \psi_B | \psi_A \rangle^*, \quad \langle \psi_A | = | \psi_A \rangle^\dagger \]  

(1.2)
Time-independent Observables and Bases: Suppose we have a bounded, time-independent standard system ([2], §2.1.1). We start with three complete orthogonal bases of the Hilbert space formed by three time-independent Hermitian observers: \( \hat{H} \) (Hamiltonian, with discrete eigenvalues), \( \hat{x} \) (position, with continuous eigenvalues) and \( \hat{p} \) (momentum, with continuous eigenvalues). We construct the powerful unit (identity) operators using their eigenvectors as follows:

\[
\text{Discrete } \varepsilon \text{-basis: } \hat{H} | \varepsilon_i \rangle = \varepsilon_i | \varepsilon_i \rangle, \quad \langle \varepsilon_i | \varepsilon_j \rangle = \delta_{ij}, \quad \hat{I} = \sum_i | \varepsilon_i \rangle \langle \varepsilon_i | \quad (1.3a)
\]

\[
\text{Continuous } x \text{-basis: } \hat{x} | x \rangle = x | x \rangle, \quad \langle x' | x \rangle = \delta(x-x'), \quad \hat{I} = \int dx | x \rangle \langle x | \quad (1.3b)
\]

\[
\text{Continuous } p \text{-basis: } \hat{p} | p \rangle = p | p \rangle, \quad \langle p' | p \rangle = \delta(p-p'), \quad \hat{I} = \int dp | p \rangle \langle p | \quad (1.3c)
\]

The stationary system state \( \Psi \)-ket now can be expanded by using these unit operators:

\[
\text{Discrete } \varepsilon \text{-basis: } | \Psi \rangle = \hat{I} | \Psi \rangle = \sum_i | \varepsilon_i \rangle \langle \varepsilon_i | \Psi \rangle \equiv \sum_i | \varepsilon_i \rangle c_i \quad (1.4a)
\]

\[
\text{Continuous } x \text{-basis: } | \Psi \rangle = \hat{I} | \Psi \rangle = \int dx | x \rangle \langle x | \Psi \rangle \equiv \int dx | x \rangle \psi(x) \quad (1.4b)
\]

\[
\text{Continuous } p \text{-basis: } | \Psi \rangle = \hat{I} | \Psi \rangle = \int dp | p \rangle \langle p | \Psi \rangle \equiv \int dp | p \rangle c(p) \quad (1.4c)
\]

Similarly, the stationary system state \( \Psi \)-bra can be expanded as follows:

\[
\langle \Psi | = \langle \Psi | \hat{I} = \sum_i \langle \Psi | \varepsilon_i \rangle \langle \varepsilon_i | = \sum_i \langle \varepsilon_i | c_i^* | \Psi \rangle \quad (1.5a)
\]

\[
\langle \Psi | = \langle \Psi | \hat{I} = \int dx \langle \Psi | x \rangle \langle x | \psi (x) \rangle = \int dx \langle x | \psi * (x) \rangle \quad (1.5b)
\]

\[
\langle \Psi | = \langle \Psi | \hat{I} = \int dp \langle \Psi | p \rangle \langle p | = \int dp \langle p | c^* (p) \rangle \quad (1.5c)
\]

The expectation value of the three observers now can be easily evaluated by inserting related unit operator in Eq. (1.3):

\[
\langle H \rangle \equiv \hat{H} \equiv \langle \Psi | \hat{H} | \Psi \rangle = \sum_i \langle \Psi | \hat{H} | \varepsilon_i \rangle \langle \varepsilon_i | = \sum_i \varepsilon_i | c_i \rangle^2 \quad (1.6a)
\]

\[
\langle x \rangle \equiv \hat{x} \equiv \langle \Psi | \hat{x} | \Psi \rangle = \int \langle \Psi | \hat{x} | x \rangle dx \langle x | \Psi \rangle = \int dx | x \rangle | \psi(x) \rangle^2 \quad (1.6b)
\]

\[
\langle p \rangle \equiv \hat{p} \equiv \langle \Psi | \hat{p} | \Psi \rangle = \int \langle \Psi | \hat{p} | p \rangle dp \langle p | \Psi \rangle = \int dp \langle p | c(p) \rangle^2 \quad (1.6c)
\]

Here \(| c_i \rangle^2\) is interpreted as the probability distribution function (PDF) of observable \( H \), \(| \psi(x) \rangle^2\) is the PDF of observable \( x \), and \(| c(p) \rangle^2\) is the PDF of observable \( p \).

Time evolution in Schrodinger Picture: The time evolution of state vectors is described by the Schrodinger equation. For a non-relativistic and spin-less particle in one-dimensional time-independent standard form, it reads:
\[
\frac{i\hbar}{\partial t} |\Psi(t)\rangle = \hat{H}(\hat{x}, \hat{p}) |\Psi(t)\rangle, \quad \hat{H}(\hat{x}, \hat{p}) = \hat{T} + \hat{V} = \frac{1}{2m} \hat{p}^2 + V(\hat{x}) = \hat{H}_t
\]  
(1.7a)

Eq. (1.7a) is independent of representations. We can expand it using bases of Eq. (1.3):

In \(\varepsilon\)-basis: \(i\hbar \frac{\partial}{\partial t} \langle \varepsilon_i | \Psi(t) \rangle = \langle \varepsilon_i | \hat{H} | \Psi(t) \rangle \Rightarrow i\hbar \frac{\partial}{\partial t} c_i(t) = \sum_k H_{ik} c_k(t)
\)  
(1.7b)

In \(x\)-basis: \(i\hbar \frac{\partial}{\partial t} \langle x | \Psi(t) \rangle = \langle x | \hat{H}(\hat{x}, \hat{p}) | \Psi(t) \rangle \Rightarrow i\hbar \frac{\partial}{\partial t} \psi(x,t) = \hat{H}(x, \frac{\hbar}{i} \frac{\partial}{\partial x}) \psi(x,t)
\)  
(1.7c)

In \(p\)-basis: \(i\hbar \frac{\partial}{\partial t} \langle p | \Psi(t) \rangle = \langle p | \hat{H}(\hat{x}, \hat{p}) | \Psi(t) \rangle \Rightarrow i\hbar \frac{\partial}{\partial t} c(p,t) = \hat{H}(i\hbar \frac{\partial}{\partial p}, p) c(p,t)
\)  
(1.7d)

Here we have used following matrix elements of the three operators:

\[
H_{ik} = \langle \varepsilon_i | \hat{H} | \varepsilon_k \rangle
\]  
(1.7e)

In \(x\)-basis: \(\langle x | \hat{\sigma} | x' \rangle = \frac{\hbar}{i} \frac{\partial}{\partial x} \delta(x-x')
\)  
(1.7f)

In \(p\)-basis: \(\langle p | \hat{\sigma} | p' \rangle = i\hbar \frac{\partial}{\partial p} \delta(p-p')
\)  
(1.7g)

Note that all three operators (\(\hat{H}, \hat{x}\) and \(\hat{p}\)) are Hermitian operators, and hence observables of the quantum system. But, as well known in QM, we cannot measure two observables at the same time if they do not commute. For example, we cannot measure \(\hat{x}\) and \(\hat{p}\) at the same time, because \([\hat{x}, \hat{p}] = i\hbar \neq 0\).

The Schrödinger equation (1.7a) has a symbolic normalized solution:

\[
|\Psi(t)\rangle = \hat{U}(t) |\Psi(0)\rangle, \quad \hat{U}(t) = \exp \left( -\frac{i}{\hbar} \int_0^t \hat{H} dt \right), \quad \hat{U}^\dagger(t) = \hat{U}^{-1}(t)
\]  
(1.8)

Normalization in \(\varepsilon\)-basis: \(1 = \langle \Psi(t) | \Psi(0) \rangle = \sum_i \langle \Psi(t) | \varepsilon_i \rangle \langle \varepsilon_i | \Psi(0) \rangle = \sum_i |c_i(t)|^2
\)  
(1.9a)

Normalization in \(x\)-basis: \(1 = \langle \Psi(t) | \Psi(t) \rangle = \int \langle \Psi(t) | x \rangle d\langle x | \Psi(t) \rangle = \int dx |\psi(x,t)|^2
\)  
(1.9b)

Normalization in \(p\)-basis: \(1 = \langle \Psi(t) | \Psi(t) \rangle = \int \langle \Psi(t) | p \rangle dp \langle p | \Psi(t) \rangle = \int dp |c(p,t)|^2
\)  
(1.9c)

This is called the Schrödinger picture. All three bases are equivalent and we can transform from one to another by unitary transformation. For example, a \(v\)-bracket can be transformed from \(x\)-basis to \(p\)-basis by using the following unitary transformation:

\[
\langle x | p \rangle = \psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}, \quad \langle p | x \rangle = \phi_x(p) = \psi^*_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}
\]  
(1.10a)

They are also the \(p\)-eigenvectors in \(x\)-basis and the \(x\)-eigenvectors in \(p\)-basis respectively. Using (1.10a), we can transform the state vector from \(x\)-basis to \(p\)-basis like:
\[
\psi(x, t) = \langle x | \psi(t) \rangle = \langle x | \left\{ \int p dp \langle p \rangle \right\} | \psi(t) \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp \, e^{i p x / \hbar} c(p, t) \quad (1.10b)
\]

**Heisenberg picture:** Using (1.8), the time-dependent expectation value of a time-independent observable \( \hat{O} \) can be expressed as:

\[
\langle O(t) \rangle \equiv \langle \Psi(t) | \hat{O} | \Psi(t) \rangle = \langle \Psi(0) | \hat{U}^{-1}(t) \hat{O} \hat{U}(t) | \Psi(0) \rangle \quad (1.11)
\]

If the Hamiltonian is time-independent as in Eq. (1.7a) and if the state is an eigenvector of the Hamiltonian, \( \hat{H}(\hat{x}, \hat{p}) | \Psi_E \rangle = E | \Psi_E \rangle \), we have from Eq. (1.8) and (1.11):

\[
\langle O(t) \rangle_E = \langle \Psi_E(0) | \exp \left( \frac{i}{\hbar} E t \right) \hat{O} \exp \left( \frac{-i}{\hbar} E t \right) | \Psi_E(0) \rangle = \langle O_S \rangle_E = \text{const.} \quad (1.12)
\]

Now we define a time-dependent operator and rewrite Eq. (1.11) to:

\[
\hat{O}_H(t) \equiv \hat{U}(t)^{-1} \hat{O} \hat{U}(t) \quad (1.13a)
\]

\[
\langle \hat{O}(t) \rangle = \langle \Psi | \hat{O}_H(t) | \Psi \rangle = \langle \Psi(t) | \hat{O} | \Psi(t) \rangle \quad (1.13b)
\]

This is called Heisenberg picture ([2], §1.8; [3], §11.12), where the state vector is time-independent, but the previously time-independent observable now becomes time-dependent. Applying Eq. (1.13) to the three operators in Eq. (1.7e-g), we have

\[
\langle H(t) \rangle = \langle \Psi | \hat{H}_H(t) | \Psi \rangle = \langle \Psi(t) | \hat{H} | \Psi(t) \rangle = \sum_i \alpha_i | \langle i | \Psi(0) \rangle |^2 = \sum_i \alpha_i | c(i, t) |^2 \quad (1.14a)
\]

\[
\langle x(t) \rangle = \langle \Psi | \hat{x}_H(t) | \Psi \rangle = \langle \Psi(t) | \hat{x} | \Psi(t) \rangle = \int dx x | \langle x | \Psi(t) \rangle |^2 = \int dx x | \psi(x, t) \rangle^2 \quad (1.14b)
\]

\[
\langle p(t) \rangle = \langle \Psi | \hat{p}_H(t) | \Psi \rangle = \langle \Psi(t) | \hat{p} | \Psi(t) \rangle = \int dp p | \langle p | \Psi(t) \rangle |^2 = \int dp p | c(p, t) |^2 \quad (1.14c)
\]

From Eq. (1.8), the operator in Eq. (1.13a) has the following time-dependence equation:

\[
\frac{i\hbar}{dt} \hat{O}_H(t) = \frac{i\hbar}{dt} [\hat{U}(t)^{\dagger} \hat{O}_S \hat{U}(t)] = [\hat{O}_H, \hat{H}] \quad (1.15)
\]

**Path integral:** In 1D-QM, the transition amplitude of a particle moving from point \((x_a, t_a)\) to point \((x_b, t_b)\) is defined as (see Eq. (2.1) of [4]):

\[
K(b, a) \equiv K(x_b, t_b | x_a, t_a) \equiv \langle x_b, t_b | x_a, t_a \rangle = \langle x_b | \hat{U}(t_b, t_a) | x_a \rangle \quad (1.16)
\]

Here \( \hat{U}(t, t_0) \) is the time evolution operator, given by Eq. (1.8). Starting from Eq. (1.16), one derives Feynman’s path integral (see: Eq. (2.54) and (2.60) of Ref. [2]):
\begin{align}
\langle x_b, t_b \mid x_a, t_a \rangle &= \int_{x(t_a)=x_a}^{x(t_b)=x_b} Dx \exp \left[ \frac{i}{\hbar} S_{cl}(x) \right] = \int_{x(t_a)=x_a}^{x(t_b)=x_b} Dx \exp \left[ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \mathcal{L}_{cl}(x, \dot{x}) \right] \tag{1.17}
\end{align}

Here $L_{cl}$ is the classical Lagrangian. For 1D particle, Feynman’s standard form (see (2.48) of [2]) of the Hamiltonian and its Lagrangian with time-dependent potential are:

\begin{align}
H(x, \dot{x}, t) &= T + V = \frac{m}{2} \dot{x}^2 + V(x, t), \quad H(x, p, t) = \frac{1}{2m} p^2 + V(x, t) \tag{1.18a} \\
L_{cl}(x, \dot{x}, t) &= T - V = \frac{m}{2} \dot{x}^2 - V(x, t) \tag{1.18b}
\end{align}

For a free particle, $V(x) = 0$, the Schrodinger equation and resulted transition amplitude from Feynman’s path integral are (see Eq. (2.125) of Ref. [2]):

\begin{align}
\frac{i\hbar}{\partial t} \psi(x) = \langle x \mid \hat{H} \mid \psi(t) \rangle &= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x), \quad \langle x \mid \psi(t) \rangle = \psi(x) \tag{1.19a} \\
\langle x_b, t_b \mid x_a, t_a \rangle &= \frac{m}{2\pi\hbar(t_b-t_a)} \exp \left[ \frac{i m (x_b-x_a)^2}{2\hbar(t_b-t_a)} \right] \tag{1.19b}
\end{align}

For a harmonic oscillator we have the following Schrodinger Equation in $x$-basis:

\begin{align}
\frac{i\hbar}{\partial t} \langle x \mid \psi(t) \rangle &= \langle x \mid \hat{H} \mid \psi(t) \rangle = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2} x^2 \right) \langle x \mid \psi(t) \rangle \tag{1.20}
\end{align}

Its transition amplitude is obtained from Feynman’s path integral (see Eq. (2.168) of [2]):

\begin{align}
\langle x_b, t_b \mid x_a, t_a \rangle &= \sqrt{\frac{m\omega}{2\pi\hbar \sin \omega(t_b-t_a)}} \\
&\cdot \exp \left\{ \frac{im\omega}{2\hbar \sin \omega(t_b-t_a)} \left[ (x_b^2 + x_a^2) \cos \omega(t_b-t_a) - 2x_a x_b \right] \right\} \tag{1.21}
\end{align}

In next section, we will briefly review $PBN$ proposed in Ref. [1], and apply it to describe expectation values and time evolution of continuous-time Homogeneous Markov chains (HMC) [4-6].

2. Probability Bracket Notation and Master Equations

In our previous article [1], we defined the symbols of $PBN$. The basic proposition is: the conditional probability of event $A$ given evidence $B$ in the sample space $\Omega$ is the
**probability bracket** or **P-bracket**, which, similar to Eq. (1.2) for the v-bracket in Dirac notation, can be split into a **P-bra** and a **P-ket**:

\[ P\text{-bracket: } P(A \mid B) \equiv (A \mid B) \quad \Rightarrow \quad P\text{-bra: } P(A) \equiv (A \mid , \quad P\text{-ket: } B) \]  

(2.1a)

The absolute and conditional probability are expressed in **PBN** as:

\[ P(A) = P(A \mid \Omega), \quad P(A \mid B) = \frac{P(A \cap B \mid \Omega)}{P(B \mid \Omega)} \]  

(2.1b)

In this article, we will always use \( P(A \mid B) \) as **P-bracket** and \( P(A) \) as **P-bra**. Also note:

\[ P(\Omega \mid B) = 1, \text{ if } \emptyset \subset B \subseteq \Omega \]  

(2.1c)

In the probability space \((\Omega, X, P)\) of a **continuous** random variable \(X\), we proposed [1] following properties that resembles the \(x\)-basis in Eq. (1.3b):

For \(\forall x \in \Omega:\quad X \mid x = x \mid x\), \quad (\Omega \mid x) = 1, \quad P: x \mapsto P(x) \equiv P(x \mid \Omega) \equiv (x \mid \Omega), \quad P(x \mid x') \equiv (x \mid x') = \delta(x-x'), \quad \int_{x \in \Omega} |x\rangle dx \: P(x) \equiv \int_{x \in \Omega} |x\rangle dx |x\rangle = I \]  

(2.2a)

\[ P(x \mid x') = \delta(x-x'), \quad \int_{x \in \Omega} |x\rangle dx \: P(x) = \int_{x \in \Omega} |x\rangle dx |x\rangle = I \]  

(2.2b)

The normalization of PDF of \(X, P(x)\), now can be expressed similar to Eq. (1.9b):

\[ P(\Omega \mid \Omega) = P(\Omega \mid I \mid \Omega) = \int \: P(\Omega \mid x) dx \: P(x \mid \Omega) = \int_{x \in \Omega} dx \: P(x \mid \Omega) = \int_{x \in \Omega} dx \: P(x) = 1 \]  

(2.3)

The expectation value of the observable \(X\) now can also be expanded similar to Eq. (1.6b):

\[ \langle X \rangle \equiv \bar{X} \equiv E(X) = P(\Omega \mid X \mid \Omega) = \int_{x \in \Omega} P(\Omega \mid X \mid x) dx \: P(x \mid \Omega) = \int_{x \in \Omega} dx \: x \: P(x) \]  

(2.4)

**Master Equation and Euclidean Hamiltonian**: The master equation for Homogeneous Markov Chains (HMC) with **continuous-time** in terms of probability vector can be written as [7-10]:

\[ \frac{\partial}{\partial t} \langle \psi(t) \rangle = \hat{G} \langle \psi(t) \rangle, \quad |\psi(t)\rangle = \hat{U}(t) |\psi(0)\rangle = e^{\hat{G}t} |\psi(0)\rangle \]  

(2.5)

The probability vector is equivalent to a **P-ket** in **PBN** and its time evolution reads:

\[ |\Psi(t)\rangle \Leftrightarrow |\Omega_t\rangle : \quad \frac{\partial}{\partial t} |\Omega_t\rangle = \hat{G} |\Omega_t\rangle, \quad |\Omega_t\rangle = \hat{U}(t) |\Omega_0\rangle = e^{\hat{G}t} |\Omega_0\rangle \]  

(2.6)

We see that the master equation in **PBN** is also representation-independent, just like Schrodinger Eq. (1.7a) in Dirac notation, and operator \(\hat{G}\) is the **Euclidean Hamiltonian**, corresponding to the Hamiltonian in Eq. (1.7a).
If our probability space is from a *Fock space*, we have the following discrete basis from the *occupation numbers* [1]:

\[ \hat{n}_i | \bar{n} \rangle = n_i | \bar{n} \rangle, \quad P(\bar{n} | \bar{n}') = \delta_{\bar{n}, \bar{n}'} = \prod_n \delta_{n_i, n_i'} \sum_{\bar{n}} P(\bar{n}) \equiv \sum_{\bar{n}} | \bar{n} \rangle (\bar{n}) = I \tag{2.7} \]

Using the basis in Eq. (2.7), the state $P$-ket, state $P$-bra and expectation of $\hat{F}(\bar{n})$ become:

\[ | \Omega_\pi \rangle = \sum_{\bar{n}} | \bar{n} \rangle P(\bar{n} | \Omega_\pi) = \sum_{\bar{n}} m(\bar{n}, t) | \bar{n} \rangle, \quad P(\Omega) = \sum_{\bar{n}} P(\bar{n}) \]

\[ \therefore \langle \hat{F}(\bar{n}) \rangle = P(\Omega | F(\bar{n}) | \Omega_\pi) = \sum_{\bar{n}, \bar{n}'} P(\bar{n} | F(\bar{n}) m(\bar{n}, t) | \bar{n}) = \sum_{\bar{n}} F(\bar{n}) m(\bar{n}, t) \tag{2.8a} \]

They resample Eq. (1.4a), (1.5a) and (1.6a) in Dirac VBN for discrete basis. For one-dimensional *Fock space*, we can act on Eq. (2.5) from left by a $P$-bra $P(n) :$

\[ \frac{\partial}{\partial t} P(n | \Omega_\pi) = P(n | \hat{G} | \Omega_\pi) = \sum_m P(n | \hat{G} | m) P(m | \Omega_\pi) = \sum_m G_{mn} P(m | \Omega_\pi) \tag{2.9} \]

Or:

\[ \frac{\partial}{\partial t} P_n(t) = \sum_m G_{mn} P_m(t), \quad \text{where } P_n(t) \equiv P(n | \Omega_\pi) \tag{2.10} \]

Eq. (2.9-10) resembles Eq. (1.7b), the Schrodinger Equation in the $\varepsilon$-representation.

The master equation of (2.5) in continuous x-basis of (2.2) looks similar to Eq. (1.7c), the Schrodinger equation in $x$-basis:

\[ \frac{\partial}{\partial t} P(x | \Omega_\pi) = P(x | \hat{G} | I | \Omega_\pi) = \int P(x | \hat{G} | x') d x' P(x' | \Omega_\pi) \tag{2.11a} \]

Or:

\[ \frac{\partial}{\partial t} P(x, t) = \int G(x, x') d x' P(x', t) \tag{2.11b} \]

If a stochastic process $X(t) \equiv X_\pi$ is a HMC, we can always set $X(0) = 0$ [4-6]. We also have the formal solutions of time-dependent $x$-basis (see §5.2 of [1]):

\[ P(X_\pi = x | X, t) = P(x, t | \hat{U}(t), \quad | X_\pi = x) = | x, t) = \hat{U}^{-1}(t) | x \rangle \]

\[ \therefore P(X_\pi = x | \Omega_\pi) = P(x, t | \Omega_\pi) = P(x | \hat{U}(t) | \Omega_\pi) = P(x | \Omega_\pi) = P(x, t) \tag{2.12a} \]

Then the evolution of the absolute probability distribution function (PDF) can be derived from the transition probability:

\[ P(x, t) = P(x | \Omega_\pi) = P(X_\pi = x | \Omega_\pi) = \int d x_0 P(X_\pi = x \cap X_0 = x_0 | \Omega_\pi) \]

\[ = \int d x_0 P(X_\pi = x | X_0 = x_0) P(x_0 | \Omega_\pi) = P(X_\pi = x | X_0 = 0) = P(x, t | 0, 0) \tag{2.13a} \]
Here we have used the definition of the conditional probability (2.1b) and the initial distribution corresponding to $X(0) = x_0 = 0$:

$$P(x_0 \mid \Omega_0) = P(x = x_0 = 0, t = 0 \mid \Omega_0) = \delta(x_0 = 0) = \delta(0) \quad (2.13b)$$

**Langevin Equation and Smoluchowski equation:** In our later discussion, we will try to find the relation between Schrödinger equation (1.7a) and the master equation (2.5) for a subset of HMC: the diffusion process. The equation of motion for a general diffusion is described in the form of Langevin equation (see [7], §5.1-§5.4):

$$\frac{dy}{dt} = A(y, t) + B(y, t)\xi(t) \quad (2.14)$$

Here $A(y, t)$ and $B(y, t)$ are often referred to as the drift and diffusion terms, and $\xi(t)$ is chosen to be the Langevin stochastic process (white noise) with the following statistical properties (related to the diffusion coefficient $D$):

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t_1)\xi(t_2) \rangle = 2D\delta(t_1 - t_2), \quad (2.15a)$$

The Fokker-Planck equation [9] then can be written in x-basis as:

$$\frac{\partial}{\partial t} P(x, t) = \hat{G} P(x, t) = -\frac{\partial}{\partial x} \left[ A(x, t) + DB(x, t)\frac{\partial B(x, t)}{\partial x} \right] P(x, t) + D \frac{\partial^2}{\partial x^2} \left[ B^2(x, t) P(x, t) \right] \quad (2.16)$$

For a Brownian motion in an external potential $V(x, t)$, the Newton-Langevin equation becomes (see Eq. (5.17) of [7]):

$$m \frac{d^2x}{dt^2} = -m\gamma \frac{dx}{dt} - \frac{\partial V}{\partial x} + mg(t) \quad (2.17a)$$

The Fokker-Planck equation now becomes famous Klein-Kramers equation (see Eq. (5.20) of [7]):

$$\frac{\partial}{\partial t} P = -v \frac{\partial P}{\partial x} + V' \frac{\partial P}{\partial v} + \left[ \gamma \frac{\partial}{\partial v} v + D \frac{\partial^2}{\partial v^2} \right] P, \text{ where: } v = \dot{x}, \quad V' = \hat{\partial}_x V \quad (2.17b)$$

For over-damped particles, Eq. (2.17b) leads to Smoluchowski equation or ordinary Fokker-Planck equation (see Eq. (18.288) of [2] and Eq. (5.22) of [7]):
\[ \frac{\partial}{\partial t} P(x,t) = \hat{G} P(x,t) = \left[ D \frac{\partial^2}{\partial x^2} + \frac{1}{m \gamma} \frac{\partial}{\partial x} V'(x) \right] P(x,t) \] (2.18a)

where: \( V' = \partial_x V \), \( D = k_B T / (m \gamma) \) (2.18b)

**Heisenberg Picture**: up to now, all master equations are displayed in Schrodinger picture. Similar to Eq. (1.13) and (1.15), we can also introduce the Heisenberg picture in PBN (more see §5.2 of [1]):

\[ | \Omega_x \rangle = \hat{U}(t) | \Omega_0 \rangle = e^{\hat{H}t}, \quad \Rightarrow \quad \hat{\Omega}_x(t) = \hat{U}^{-1}(t) \hat{\Omega}_x(t) \] (2.19)

\[ \frac{d}{dt} \hat{\Omega}_x(t) = \frac{d}{dt} [\hat{U}(t)^{-1} \hat{\Omega}_x(t)] = [\hat{\Omega}_x, \hat{G}] \] (2.20)

### 3. Special Wick Rotation, Time Evolution and Induced Diffusions

It is well known [8] that the imaginary time Schrodinger equation becomes a Diffusion-like equation. Specifically, the free particle Schrodinger equation becomes Einstein-Brown-like equation. In this section, we will show that, under **Special Wick Rotation** (SWR) caused by imaginary time rotation, many expressions of §1 in Dirac VBN for Hilbert space are naturally shifted to expressions of §2 in PBN for the probability space.

We will discuss only the following diffusion-like process, which in \( x \)-basis reads:

\[ \frac{\partial}{\partial t} (x | \Omega_x) = \hat{G}(x, -\partial_x) | \Omega_x) = \{ D \partial_x^2 - W(x, -\partial_x, t) \} (x | \Omega_x) \]

Or: \[ \frac{\partial}{\partial t} P(x,t) = \hat{G} P(x,t), \quad \hat{G} = \hat{T}_D + \hat{V}_e = D \partial_x^2 - \hat{W} \] (3.1a)

where Euclidean potential: \( \hat{V}_e = -W(x, -\partial_x, t) \) (3.1b)

Here we introduced *Euclidean potential* \( \hat{V}_e \) to make the *Euclidean Hamiltonian* \( \hat{G} \) more resembling the standard Hamiltonian \( \hat{H} \) in Eq. (1.7a) or (1.18a).

To simplify our discussion, we will use following operators and constants:

\[ \hat{k} \equiv \frac{\hat{p}}{\hbar} = \frac{1}{i} \frac{\partial}{\partial x}, \quad \mu_x \equiv \frac{m}{\hbar}, \quad u(x) \equiv \frac{V(x)}{m}, \quad \hat{k} \equiv -i \hat{\kappa} \] (3.2a)

The Hermitian wave-number operator (\( \hat{k} \)) is well known in QM and the imaginary operator \( \hat{\kappa} \) (more see Eq. (3.18, 4.11) below) is closely related to operator (\( \hat{\kappa} \)):

\[ \hat{\kappa} \equiv \frac{\hat{c}}{i}, \quad \text{in } x\text{-basis: } P(x | \hat{\kappa} | x') = -\frac{\partial}{\partial x} \delta(x - x') \] (3.2b)
With (3.2a), Eq. (1.7a) now reads:

\[
 i \frac{\partial}{\partial t} \psi(x,t) = \{ \hat{T}_h + V_h \} \psi(x,t) = \left\{ \frac{1}{2\mu_h} \hat{k}^2 + \mu_h u(x) \right\} \psi(x,t) \equiv \frac{1}{\hbar} \hat{H}(x,\hat{k}) \psi(x,t)
\]

Or:

\[
 \frac{\partial}{i \partial t} \psi(x,t) = -\{ \hat{T}_h + V_h \} \psi(x,t) = \left\{ \frac{1}{2\mu_h} \frac{\partial^2}{\partial x^2} - \mu_h u(x) \right\} \psi(x,t),
\]

where:

\[
 \hat{T}_h \equiv \frac{1}{\hbar} \hat{T} = \frac{1}{2\mu_h} \hat{k}^2 = -\frac{1}{2\mu_h} \frac{\partial^2}{\partial x^2}, \quad V_h \equiv \frac{V}{\hbar} = \frac{mV}{\hbar m} = \mu_h u(x)
\]

We hide Planck constant (\(h\)) in the effective mass (\(\mu_h\)), so when we shift from microscopic world to macroscopic world, we can easily get rid of Planck constant.

The **induced microscopic diffusion** is a subset of Eq. (3.1a):

\[
 \frac{\partial}{\partial t} P(x,t) = \hat{G}(x,\hat{k}) P(x,t), \quad \hat{G} = \frac{1}{2\mu_h} \frac{\partial^2}{\partial x^2} - \mu_h u(x) = \frac{\kappa^2}{2\mu_h} - \mu_h u(x)
\]

The **Special Wick Rotation** (SWR), caused by imaginary time rotation, is defined by:

\[
 \text{SWR}: \quad \text{i} t \rightarrow t, \quad |\psi(t)\rangle \rightarrow |\Omega_t\rangle, \quad \langle x_b, t_b | x_a, t_a \rangle \rightarrow P(x_b, t_b | x_a, t_a)
\]

Under SWR, Schrödinger Eq. (3.3a) is indeed shifted to induced micro diffusion (3.4):

\[
 \frac{\partial}{\partial t} | \psi(t) \rangle = -\frac{1}{\hbar} \hat{H}(\hat{x}, \hat{k}) | \psi(t) \rangle \rightarrow \frac{\partial}{\partial t} | \Omega_t \rangle = \hat{G}(\hat{x}, \hat{k}) | \Omega_t \rangle
\]

\[
 \therefore -\frac{1}{\hbar} \hat{H}(\hat{x}, \hat{k}) = -\frac{\kappa^2}{2\mu_h} - \mu_h u(x) \rightarrow \hat{G}(\hat{x}, \hat{k}) = \frac{1}{2\mu_h} \frac{\partial^2}{\partial x^2} - \mu_h u(x) = \frac{\kappa^2}{2\mu_h} - \mu_h u(x)
\]

\[
 \therefore \hat{G}(\hat{x}, \hat{k}) = -\frac{1}{\hbar} \hat{H}(\hat{x}, i\hat{k}) = \frac{\kappa^2}{2\mu_h} - W(x), \quad W(x) = \mu_h u(x) = -V_c(x)
\]

Moreover, under SWR (3.5), the path integral of transition amplitude in Eq. (1.16) now is shifted to the path integral of transition probability of an induced micro diffusion as follows:

\[
 \langle x_b, t_b | x_a, t_a \rangle = \langle x_b | \hat{U}(t_b, t_a) | x_a \rangle = \langle x_b | \exp \left\{ \int i dt \frac{-1}{\hbar} \hat{H}(\hat{x}, \hat{k}) \right\} | x_a \rangle
\]

\[
 \rightarrow P(x_b, t_b | x_a, t_a) = P(x_b | \hat{U}(t_b, t_a) | x_a) = P(x_b | \exp \left\{ \int dt \hat{G}(\hat{x}, \hat{k}) \right\} | x_a)
\]

This is an expected shift, because they share following properties:
1. Both time evolution operators $\hat{U}(t, t_0)$ for the Master equation (2.6) and that for the Schrodinger equation (1.8) satisfy the composition law (see §1.7 of [2]):

$$\hat{U}(t_n, t_0) = \hat{U}(t_n, t_{n-1}) \hat{U}(t_{n-1}, t_{n-2}) \ldots \hat{U}(t_1, t_0) \quad (3.8)$$

2. For the transition amplitude $K(x_a, t_b | x_a, t_a) = \langle x_a | \hat{U}(t_b, t_a) | x_a \rangle$ in QM, we can insert between each pairs of the $\hat{U}$ 's a unit operator from $v$-bra and $v$-ket of Eq. (1.3b):

$$\hat{U}(t_i, t_{i-1}) \hat{U}(t_{i-1}, t_{i-2}) = \hat{U}(t_i, t_{i-1}) I \hat{U}(t_{i-1}, t_{i-2}) , \quad I = \int_{-\infty}^{\infty} dx \mid x \mid \langle x \mid$$

Similarly, for the transition probability $P(x_b, t_b | x_a, t_a) = \langle x_a | \hat{U}(t_b, t_a) | x_a \rangle$, we can also insert between each pairs of the $\hat{U}$ 's a unit operator from $P$-bra and $P$-ket of Eq. (2.2b):

$$\hat{U}(t_i, t_{i-1}) \hat{U}(t_{i-1}, t_{i-2}) = \hat{U}(t_i, t_{i-1}) I \hat{U}(t_{i-1}, t_{i-2}) , \quad I = \int_{-\infty}^{\infty} dx \mid x \mid P(x)$$

(3.9)

(3.10)

Therefore, under SWR, the Schrodinger equation and its path integral of a conservative system are simultaneously shifted to the master equation and its path integral of an induced micro diffusion. Now we show two simple but very useful examples of path integrals under SWR (3.5-3.7).

**Induced Micro Einstein-Brown motion**: The simplest example is the induced micro Einstein-Brown motion, when $u(x) = 0$ in Eq. (3.3-3.4). Applying (3.5) to Eq. (1.19b) for free particle, we get the transition probability:

$$P(x_b, t_b | x_a, t_a) = \frac{1}{\sqrt{4\pi D_h(t_b - t_a)}} \exp \left[ -\frac{(x_b - x_a)^2}{4D_h(t_b - t_a)} \right]$$

(3.11)

Here we have introduced the induced micro diffusion coefficient,

$$D_h \equiv 1/(2\mu_h) = \hbar/(2m)$$

(3.12a)

It is related to the macroscopic diffusion coefficient $D$, derived from Newton-Langevin equation with potential $V(x) = 0$, a viscous force $-m\gamma \dot{x}$ ($m\gamma = 6\pi \mu a$) and a fluctuation force $\xi(t)$ (see Eq. (2.17), (2.18b) or §1.1.2 of [7]):

$$D = \frac{k_B T}{m\gamma} \equiv \frac{1}{2\mu_p}$$

(3.12b)

Comparing Eq. (3.12a) and (3.12b), we derive the microscopic friction coefficient:

$$D_h = D \quad \rightarrow \quad \frac{\hbar}{2m} = \frac{k_B T}{m\gamma} \quad \rightarrow \quad \gamma = \frac{2k_B T}{\hbar}$$

(3.13)
In the vacuum of outer space, the micro friction coefficient of induced micro diffusion can be calculated from Eq. (3.13) with $T \approx 2.7^\circ K$.

Applying Eq. (2.13a) to Eq. (3.11), we have the absolute PDF initially at $x = 0$:

$$P(x, t) = P(x | \Omega_r) = P(x, t | 0, 0) = \sqrt{\frac{1}{4\pi D_h t}} \exp \left[ -\frac{x^2}{4D_h t} \right]$$  \hspace{1cm} (3.14)

**Induced Micro diffusion of Harmonic Oscillator**: Our second example is the induced micro diffusion of a Harmonic Oscillator. Applying (3.5) to Eq. (1.21), we have the transition probability for harmonic MC:

$$P(x_b, t_b | x_a, t_a) = \sqrt{\frac{\mu_b \omega}{2\pi i \sin[\omega(t_b - t_a)/i]}} \cdot \exp \left\{ \frac{i\mu_b \omega}{2\sin[\omega(t_b - t_a)/i]} \left[ (x_b^2 + x_a^2) \cos[\omega(t_b - t_a)/i] - 2x_a x_b \right] \right\}$$ \hspace{1cm} (3.15)

Using $\sin(y/i) = (\sinh y)/i$ and $\cos(y/i) = (\cosh y)$, we have:

$$P(x_b, t_b | x_a, t_a) = \sqrt{\frac{D_h \omega}{4\pi \sinh[\omega(t_b - t_a)]}} \cdot \exp \left\{ \frac{-D_h \omega}{4 \sinh[\omega(t_b - t_a)]} \left[ (x_b^2 + x_a^2) \cosh[\omega(t_b - t_a)] - 2x_a x_b \right] \right\}$$ \hspace{1cm} (3.16)

Here $D_h$ is defined in Eq. (3.12a). Following Ref [2], the transition amplitude of Harmonic oscillator, Eq. (2.168) or our Eq. (1.19) is shifted by an analytic continuation of the time difference $t_b - t_a$ to imaginary values $-i(t_b - t_a)$. The resulted equation in [2], Eq. (2.405), is identical to our Eq. (3.16). It is used to evaluate the quantum statistical partition of harmonic oscillators in contact with a reservoir of temperature $T$ (see Eq. (2.399) of [2]). The partition function is given by the following integral of transition probability (3.16) from “time” $t = 0$ to $t = h\beta \equiv h/(k_aT)$:

$$Z_\omega = \int_{-\infty}^{\infty} dx P(x, h\beta | x, 0) = \frac{1}{2\sinh[\beta \hbar \omega / 2]}, \quad \beta \equiv \frac{1}{k_aT}$$ \hspace{1cm} (3.17)

**The anti-Hermitian wave-number operator $\hat{k}$**: The operator $\hat{k}$ introduced in Eq. (3.2b) is anti-Hermitian, because:

$$\hat{k} = i\hat{\gamma}, \quad [\hat{x}, \hat{k}] = i \rightarrow [\hat{x}, \hat{k}] = 1, \quad \therefore \hat{k}^{\dagger} = \hat{k} \rightarrow \hat{k}^{\dagger} = -\hat{k}$$ \hspace{1cm} (3.18)

Using this anti-Hermitian operator, the transformation (3.6b) can be expressed as:
\[-\frac{1}{\hbar} \hat{H}(x, \hat{k}) = -\frac{1}{2\mu_h} \hat{k}^2 - \mu_h u(x) \quad \overset{\hat{k}\rightarrow i\hat{\epsilon}}{\rightarrow} \quad \hat{G}(x, \hat{\epsilon}) = \frac{1}{2\mu_h} \hat{\epsilon}^2 - \mu_h u(x) \quad (3.19)\]

The Schrödinger equation of harmonic oscillator, Eq. (1.20), now is shifted under SWR to an induced micro diffusion as following:

\[ \frac{\partial}{i\partial t} \psi(x,t) = -\frac{1}{\hbar} \hat{H}(x, \hat{k}) \psi(x,t) = \left\{ -\frac{1}{2\mu_h} \hat{k}^2 - \mu_h \frac{\omega^2}{2} \right\} \psi(x,t) \quad (3.20) \]

\[ \rightarrow \frac{\partial}{\partial t} P(x,t) = \hat{G}(x, \hat{\epsilon}) P(x,t) = \left\{ \frac{1}{2\mu_h} \hat{\epsilon}^2 - \mu_h \frac{\omega^2}{2} \right\} P(x,t) \quad (3.21) \]

Thus, under Special Wick Rotation, the Schrödinger equation of harmonic oscillator (1.20) or (3.20) and its path integral (1.21) are simultaneously shifted to the master equation (3.21) and its path integral solution (3.16) of an induced micro diffusion.

We can also play with operator algebra. In the Heisenberg picture (1.13) of QM, it is well known that (see §11.12 of [3]):

\[ \frac{d}{dt} \hat{x}(t) = \frac{d}{dt} \left[ U^{-1}(t) \hat{x} U(t) \right] = \frac{1}{i\hbar} \left[ \hat{x}(t), \hat{H} \right] = \frac{1}{2m\hbar} \left[ \hat{x}(t), \hat{\rho} \right] = \frac{\hat{p}(t)}{m} = \frac{\hat{k}(t)}{\mu_h} \quad (3.22) \]

We have similar relation in probability space: using Heisenberg picture (2.19-20), the evolution equation (2.5) and the commutation relation in Eq. (3.18), we have:

\[ \frac{d}{dt} \hat{x}(t) = \frac{d}{dt} \left[ U^{-1}(t) \hat{x} U(t) \right] = \left[ \hat{x}(t), \hat{G} \right] = \frac{1}{2\mu_h} \left[ \hat{x}(t), \hat{\epsilon}^2 \right] = \frac{\hat{k}(t)}{\mu_h} \quad (3.23) \]

It is interesting to investigate induced Micro Einstein-Brown motion with constant drift \( v \):\n
\[ P(x,t) = P(x \mid \Omega_t) = \frac{1}{\sqrt{4\pi D_h t}} \exp \left[ -\frac{(x-vt)^2}{4D_h t} \right], \quad D_h = \frac{1}{2\mu_h} \quad (3.24) \]

Using Eq. (3.23-24), (3.1b) and (3.12a), we get the real expectation value of “velocity”:

\[ \langle \hat{x} \rangle = \left\langle \frac{\hat{k}}{\mu_h} \right\rangle = P(\Omega \mid \frac{\hat{k}}{\mu_h} \mid \Omega_t) = \int dx P(\Omega \mid x) \left[ -2D_h \frac{\partial}{\partial x} \delta(x-x') \right] dx' P(x' \mid \Omega_t) \]

\[ = \int dx \frac{-(x-vt)}{t\sqrt{4\pi D_t}} \exp \left[ -\frac{(x-vt)^2}{4D_t} \right] = \int dx \frac{v}{\sqrt{4\pi D_h t}} \exp \left[ -\frac{(x-vt)^2}{4D_h t} \right] = v \quad (3.25) \]

This suggests that we should use the anti-Hermitian operator \( \hat{\epsilon} \) instead of the Hermitian operator \( \hat{k} \) in Eq. (3.2), which would have given an imaginary expectation value:
\[ \langle \dot{x} \rangle = \langle \hat{\kappa} / \mu_h \rangle = P(\Omega \mid i\hat{\kappa} / \mu_h \mid \Omega_t) = i \nu \] (3.26)

**Implication of Special Wick Rotation**: Our study shows that, SWR (3.5) is not just an imaginary time rotation. It also transforms the Schrodinger equation (1.7a) and its path integral (1.16) in Dirac VBN to the master equation (3.4) and it path integral (3.7) of an induced micro diffusion in PBN. According to Eq. (3.8-10), this implies that the \( x \)-basis (1.3b) in Dirac notation is mapped to the \( x \)-basis (2.2b) in PBN under SWR:

\[ \langle x' \mid x \rangle = \delta(x-x'), \quad \hat{I} = \int dx \mid x \rangle \langle x \mid \xrightarrow{\text{SWR}} P(x' \mid x) = \delta(x-x'), \quad \hat{I} = \int dx \mid x \rangle P(x) \] (3.27)

4. General Wick Rotation, Path Integral and Imaginary Wave Number

In this section, we consider the following shift of time-evolution equations:

\[ \frac{\partial}{i \partial t} \psi(x,t) = -\frac{1}{\hbar} H(x,k,t) \psi(x,t) = \left[ \frac{1}{2\mu_h} \frac{\partial^2}{\partial x^2} - V_h(x,t) \right] \psi(x,t) \] (4.1a)

\[ \rightarrow \frac{\partial}{\partial t} P(x,t) = \hat{G} P(x,t) = (\hat{T}_D + \hat{V}_e) P(x,t) = \left[ \hat{T}_D - W(x,\hat{k},t) \right] P(x,t) \]

where: \( \hat{\kappa} = -\frac{\partial}{\partial x}, \quad \hat{T}_D(\hat{\kappa}) = \frac{1}{2\mu_D} \frac{\partial^2}{\partial x^2} = \frac{1}{2\mu_D} \hat{\kappa}^2, \quad \mu_D \equiv \frac{1}{2D} = \text{const.} \) (4.1b)

Note again that we used Euclidean potential \( \hat{V}_e = -W \) as we did in Eq. (3.1b).

This shift is accomplished by following **General Wick Rotation** (GWR):

- **SWR**: \( it \rightarrow t, \mid \psi(t) \rangle \rightarrow \mid \Omega_t \rangle, \langle x_{t_b} \mid x_{t_a} \rangle \rightarrow P(x_{t_b} \mid x_{t_a}, t_a) \) (4.2a)
- **Wave Number Shift**: \( \hat{k} \rightarrow i\hat{\kappa}, \hat{k} \rightarrow i\kappa, \mid \hat{k} \rangle \rightarrow \mid i\kappa \rangle, \langle \hat{k} \mid \rightarrow P(i\kappa) \) (4.2b)
- **Diffusion Shift**: \( \mu_h \rightarrow \mu_D, \quad V_h(x) = \frac{1}{\hbar} V(x) \rightarrow W(x,\hat{k},t) = -V_e(x,\hat{k},t) \) (4.2c)

There is no unified form for Diffusion shift (4.2c). We only discuss following diffusions:

- **Induced Micro Diffusion**: \( \mu_h = \mu_D, \quad V_h(x) = \mu_h u(x) \rightarrow W(x) = \mu_h u(x) \) (4.3a)
- **Induced Macro Diffusion**: \( \mu_h \rightarrow \mu_D, \quad V_h(x) = \mu_h u(x) \rightarrow W(x) = \mu_D u(x) \) (4.3b)
- **Strong Damping**: \( \mu_h \rightarrow \mu_D, \quad V_h(x) \rightarrow W(x,\hat{k}) = \frac{-1}{m\gamma} \frac{\partial}{\partial x} V'(x) = \frac{1}{m\gamma} \hat{k} V'(x) \) (4.3c)

**Path Integral for QM and MC, Side by Side**: To see this more clearly, let us go several steps in path integral for MC of continuous states, following the path integral process in

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QM (§2.1, [2]). We assume the Hamiltonian takes Feynman’s standard form in Eq. (1.18). Its Schrödinger equation in Dirac VBN reads:

\[
\frac{\partial}{\partial t} \psi(t) = -\frac{i}{\hbar} \hat{H}(\hat{x}, \hat{p}, t) \psi(t) = \left\{ -\hat{T}_h(\hat{k}) - V(x, t) \right\} \psi(t)
\]

(4.4a)

Then, under GWR, it is shifted to the following master equation in PBN:

\[
\rightarrow \frac{\partial}{\partial t} |\Omega_\tau\rangle = \hat{G}(\hat{x}, \hat{k}, t) |\Omega_\tau\rangle = \left\{ \hat{T}_D(\hat{k}) - W(x, \hat{k}, t) \right\} |\Omega_\tau\rangle = \left\{ \hat{T}_D + \hat{V}_c \right\} |\Omega_\tau\rangle
\]

(4.4b)

To begin with, we divide the time interval \([t_a, t_b]\) into \(N+1\) small piece:

\[
\Delta t = t_{n+1} - t_n = (t_b - t_a)/(N + 1) > 0
\]

(4.5)

Applying Eq. (3.8) to (1.16) with (4.5), we have for QM:

\[
\langle x_b, t_b | x_a, t_a \rangle = \langle x_a | \hat{U}(t_b, t_a) | x_a \rangle = \langle x_a | \hat{U}(t_b, t_N) \cdots \hat{U}(t_n, t_{n-1}) \cdots \hat{U}(t_1, t_a) | x_a \rangle
\]

(4.6a)

Applying Eq. (3.8) to (4.1) with (4.5), we have for MC:

\[
P(x_b, t_b | x_a, t_a) = P(x_b | \hat{U}(t_b, t_a) | x_a) = P(x_b | \hat{U}(t_b, t_N) \cdots \hat{U}(t_n, t_{n-1}) \cdots \hat{U}(t_1, t_a) | x_a)
\]

(4.6b)

Now we apply the solution of \(\hat{U}(t_n, t_{n-1})\) in Eq. (1.8) to (4.6a) for QM:

\[
\langle x_b, t_b | x_a, t_a \rangle = \prod_{n=1}^{N} \int_{-\infty}^{\infty} dx_n \langle x_a | \hat{U}(t_n, t_{n-1}) | x_{n-1} \rangle = \prod_{n=1}^{N} \int_{-\infty}^{\infty} dx_n \langle x_n | e^{-\frac{i}{\hbar} \hat{H}_\Delta} | x_{n-1} \rangle
\]

(4.7a)

Similarly, we apply the solution of \(\hat{U}(t_n, t_{n-1})\) in Eq. (2.5) to (4.6b) for MC:

\[
P(x_b, t_b | x_a, t_a) = \prod_{n=1}^{N} \int_{-\infty}^{\infty} dx_n P(x_n | \hat{U}(t_n, t_{n-1}) | x_{n-1}) = \prod_{n=1}^{N} \int_{-\infty}^{\infty} dx_n P(x_n | e^{\hat{G}_\Delta} | x_{n-1})
\]

(4.7b)

Next, we insert the unit operator (3.9) into each \(v\)-bracket in (4.7a) for QM:

\[
\langle x_a | e^{-\frac{i}{\hbar} \hat{H}(\hat{p}, \hat{x}_a)_\Delta} | x_{n-1} \rangle = \int_{-\infty}^{\infty} dx \langle x_a | e^{-\frac{i}{\hbar} \varepsilon(\hat{p}, \hat{x}_a)_\Delta} | x \rangle \langle x | e^{-\frac{i}{\hbar} \hat{H}(\hat{p})_\Delta} | x_{n-1} \rangle
\]

(4.8a)

\[
= \int_{-\infty}^{\infty} dx e^{-\frac{i}{\hbar} \varepsilon(\hat{x}, \hat{p})_\Delta} \delta(x - x_n) \langle x | e^{-\frac{i}{\hbar} \hat{T}(\hat{p})_\Delta} | x_{n-1} \rangle = e^{\frac{i}{\hbar} \varepsilon(\hat{x}_a, \hat{p})_\Delta} \langle x_n | e^{-\hat{T}_h(\hat{k})_\Delta} | x_{n-1} \rangle
\]

Similarly, we insert the unit operator (3.10) into each \(P\)-bracket in (4.7b) for MC:
\[ P(x_n \mid e^{\hat{G}(\hat{\kappa}, \hat{\omega}, \Delta t)} \mid x_{n-1}) = \int_{-\infty}^{\infty} dx P(x_n \mid e^{-W(i)\Delta t} \mid x) P(x \mid e^{T_0(i)\Delta t} \mid x_{n-1}) \]
\[ = \int_{-\infty}^{\infty} dx e^{-W(x, \Delta t)} \delta(x-x_n) P(x \mid e^{T_0(i)\Delta t} \mid x_{n-1}) = e^{W(x, \Delta t)} P(x_n \mid e^{T_0(i)\Delta t} \mid x_{n-1}) \]

(4.8b)

To evaluate the v-bracket in (4.8a), we need the unit operator from momentum operator \( \hat{p} \) and its eigenvectors as given in Eq. (1.3c), (1.7e) and (1.10a):

\[ \hat{p} \mid p \rangle = p \mid p \rangle, \quad \langle p \mid p' \rangle = \delta(p-p'), \quad \int dp \mid p \rangle \langle p \mid = I \]
\[ \langle x \mid \hat{p} \mid x' \rangle = \frac{\hbar}{i} \frac{\partial}{\partial x} \delta(x-x'), \quad \langle x \mid p \rangle = \frac{1}{\sqrt{2\pi}} \exp[\i\pi x/h] \]

(4.9)

From Eq. (4.9), we have the properties of wave-number operator \( \hat{k} \equiv \hat{p} / \hbar \):

\[ \hat{k} \mid k \rangle = k \mid k \rangle, \quad \langle k \mid k' \rangle = \delta(k-k'), \quad \int dk \mid k \rangle \langle k \mid = I \]
\[ \langle x \mid \hat{k} \mid x' \rangle = \frac{\i}{\partial_x} \delta(x-x'), \quad \langle x \mid k \rangle = \frac{1}{\sqrt{2\pi}} \exp[ikx], \quad \langle k \mid x \rangle = \frac{1}{\sqrt{2\pi}} \exp[-ikx] \]

(4.10a)

To evaluate the P-bracket in (4.8b), we need related properties of the anti-Hermitian operator \( \hat{\kappa} \) in probability space, which is given by the wave number shift (4.2b). The properties of operator \( \hat{k} \) in Eq. (4.10) now are shifted to the properties of \( \hat{\kappa} \) as follows:

(4.10a) \( \rightarrow i\hat{\kappa} \mid \iota \kappa \rangle = \iota \kappa \mid \iota \kappa \rangle, \quad P(i\kappa \mid i\kappa') = \delta(i\kappa-i\kappa'), \quad \int i\iota \kappa \mid i\iota \kappa \rangle P(i\kappa) = I \)

(4.11a)

(4.10b) \( \rightarrow (x \mid i\hat{\kappa} \mid x') = \frac{1}{i} \frac{\partial}{\partial x} \delta(x-x'), \quad P(x \mid i\kappa) = \frac{1}{\sqrt{2\pi}} \exp[-ixx], \quad P(i\kappa \mid x) = \frac{1}{\sqrt{2\pi}} \exp[+ixx] \)

(4.11b)

Renaming \( \mid i\kappa \rangle \rightarrow \mid \kappa \rangle \), \( P(i\kappa \rightarrow P(\kappa \mid \kappa) \) and using the property of delta function, we get:

\[ \hat{\kappa} \mid \kappa \rangle = \kappa \mid \kappa \rangle, \quad P(\kappa \mid \kappa') = -i \delta(\kappa-\kappa'), \quad \int i\iota \kappa \mid \iota \kappa \rangle P(\kappa) = I \]
\[ P(x \mid \hat{\kappa} \mid x') = -\frac{\partial}{\partial x} \delta(x-x'), \quad P(x \mid \kappa) = \frac{1}{\sqrt{2\pi}} \exp[-ixx], \quad P(\kappa \mid x) = \frac{1}{\sqrt{2\pi}} \exp[+ixx] \]

(4.12a)

(4.12b)

We can verify the properties related to operator \( \hat{\kappa} \) in (4.12) as follows:

\[ P(x \mid \hat{\kappa} \mid \kappa) = \int P(x \mid \hat{\kappa} \mid x') dx' P(x' \mid \kappa) = \int -\frac{\partial}{\partial x} \delta(x-x') dx' \frac{1}{\sqrt{2\pi}} \exp[-ixx] = \kappa P(x \mid \kappa) \]

(4.13)

\[ P(\kappa \mid \kappa') = \int P(\kappa \mid x) dx P(x \mid \kappa') = \frac{1}{2\pi} \int dx e^{-x(x-\kappa')} = \frac{1}{2\pi} \int_{\text{imaginary}} dx e^{\i(x-k')} = \delta(k'-k) = \delta(\i\kappa'-\i\kappa) = -i \delta(\kappa'-\kappa) \]

(4.14a)
Our wave-number operator $\hat{k}$ is an \textit{imaginary} observable, because its eigenvalues are all imaginary numbers. It is also an \textit{auxiliary} variable, because it is used to complete path integral (4.8b), which has a $\mathcal{P}$-bracket involving $\hat{T}_D$ with differential operator $\hat{k}$. It plays a similar auxiliary role as the momentum operator $\hat{p}$ used to complete path integral (4.8a), which has a $\mathcal{V}$-bracket involving $\hat{T}$ with differential operator $\hat{p}$.

Now let us go back to Eq. (4.8a), using the operator $\hat{k}$ and $\langle x | k \rangle$ in Eq. (4.10) to evaluate the $\mathcal{V}$-bracket involving $\hat{T}$:

\[
\langle x_n | e^{-iT_n(k)\Delta t} | x_{n-1} \rangle = \int dk_n \langle x_n | k_n \rangle \langle k_n | e^{-iT_n(k_n)\Delta t/h} | x_{n-1} \rangle = \int \frac{dk_n}{2\pi} e^{ik_n x_n} e^{-iT_n(k_n)\Delta t/h} \langle k_n | x_{n-1} \rangle = \int \frac{dk_n}{2\pi} e^{ik_n (x_n - x_{n-1})} e^{-iT_n(k_n)\Delta t} \quad (4.15a)
\]

Similarly, we can evaluate the $\mathcal{P}$-bracket involving $\hat{T}_D$ in Eq. (4.8b) by using the expression of unit operator, $\hat{k}$ and $P(x|\kappa)$ in Eq. (4.12):

\[
P(x_n | e^{\mathcal{P}(k)\Delta t} | x_{n-1}) = \int \frac{id\kappa_n}{2\pi} P(x_n | e^{\mathcal{P}(k)\Delta t} | \kappa_n) P(\kappa_n | x_{n-1}) = \int \frac{id\kappa_n}{2\pi} e^{-\kappa_n x_n} e^{\mathcal{P}(\kappa_n)\Delta t} P(\kappa_n | x_{n-1}) = \int \frac{id\kappa_n}{2\pi} e^{-\kappa_n (x_n - x_{n-1}) + \mathcal{P}(\kappa_n)\Delta t} \quad (4.15b)
\]

Inserting (4.15a) back to (4.8a), we obtain the Feynman’s path integral formula (see Eq. (2.14) of [2]) for transition amplitude:

\[
\langle x_b, t_b | x_a, t_a \rangle \approx \prod_{n=1}^N \int_{-\infty}^\infty dx_n \prod_{n=1}^{N+1} \int_{-\infty}^\infty \frac{dp_n}{2\pi\hbar} \exp \left\{ \frac{i}{h} \left[ p_n (x_n - x_{n-1}) - \Delta t H(p_n, x_n, t_n) \right] \right\} \quad (4.16a)
\]

Inserting (4.15b) back to (4.8b), we obtain the following Euclidean path integral for transition probability:

\[
P(x_b, t_b | x_a, t_a) \approx \prod_{n=1}^N \int_{-\infty}^\infty dx_n \prod_{n=1}^{N+1} \int_{-\infty}^\infty \frac{id\kappa_n}{2\pi} \exp \left\{ -\kappa_n (x_n - x_{n-1}) + \Delta t G(\kappa_n, x_n, t_n) \right\} \quad (4.16b)
\]

Note that we can get (4.16b) from (4.16a) by using transformation (3.19) and (4.14)
\[ i\Delta t \rightarrow \Delta t, \quad k_n \equiv \frac{p_n}{\hbar} \rightarrow ik_n, \quad -\frac{\hat{H}}{\hbar} \rightarrow \hat{G} = \hat{T}_D - \hat{W} = \hat{T}_D + \hat{V}_e \quad (4.16c) \]

**Time Evolution Equation from Path Integral:** we can “recover” the Schrödinger equation and master equation from (4.16a) and (4.16b). First, from Eq. (4.16a), we have following approximation for transition amplitude (see §2.1.3 of [2]):

\[ \langle x_b, t_b \mid x_a, t_a \rangle \approx \int_{-\infty}^{\infty} dx_N \langle x_b, t_b \mid x_N, t_N \rangle \langle x_N, t_N \mid x_a, t_a \rangle, \quad t_N = t_b - \Delta t \quad (4.17a) \]

where

\[ \langle x_b, t_b \mid x_N, t_N \rangle \approx \int_{-\infty}^{\infty} \frac{dp_b}{2\pi i} \exp \left\{ \frac{i}{\hbar} [p_b(x_b - x_N) - \Delta t H(p_b, x_b, t_b)] \right\} \]

\[ = \exp \left\{ -\frac{i}{\hbar} \Delta t H(-ih\partial_{x_b}, x_b, t_b) \right\} \int_{-\infty}^{\infty} \frac{dp_b}{2\pi i} \exp \left\{ \frac{i}{\hbar} [p_b(x_b - x_N)] \right\} \]

\[ = \exp \left\{ -\frac{i}{\hbar} \Delta t H(-ih\partial_{x_b}, x_b, t_b) \right\} \delta(x_b - x_N) \quad (4.18a) \]

Therefore:

\[ \langle x_b, t_b \mid x_a, t_a \rangle \approx \exp \left\{ -\frac{i}{\hbar} \Delta t H(-ih\partial_{x_b}, x_b, t_b) \right\} \langle x_b, t_b - \Delta t \mid x_a, t_a \rangle \quad (4.19a) \]

\[ \approx \frac{1}{\Delta t} \left[ \langle x_b, t_b \mid x_a, t_a \rangle - \langle x_b, t_b - \Delta t \mid x_a, t_a \rangle \right] \]

\[ \approx \frac{1}{\Delta t} \left\{ \exp \left[ -\frac{i}{\hbar} \Delta t H(-ih\partial_{x_b}, x_b, t_b) \right] - 1 \right\} \langle x_b, t_b \mid x_a, t_a \rangle \quad (4.20a) \]

In the limit \( \Delta t \rightarrow 0 \), Eq. (4.20a) leads to the Schrödinger equation (1.7a) for transition amplitude and the momentum operator in the \( x \)-basis:

\[ i\hbar \partial_t \langle x_b, t_b \mid x_a, t_a \rangle = H \left( -ih\partial_{x_b}, x_b, t_b \right) \langle x_b, t_b \mid x_a, t_a \rangle, \quad \hat{p} = -ih\partial_{x_b} \]

Or:

\[ -i\partial_t \langle x_b, t_b \mid x_a, t_a \rangle = (-1/\hbar)H \left( \hat{h}k, x_b, t_b \right) \langle x_b, t_b \mid x_a, t_a \rangle, \quad \hat{k} = -i\partial_{x_b} \quad (4.21a) \]

On the other hand, from Eq. (4.16b), we have similar expression for transition \( P \)-bracket:

\[ P(x_b, t_b \mid x_a, t_a) \approx \int_{-\infty}^{\infty} dx_N P(x_b, t_b \mid x_N, t_N) P(x_N, t_N \mid x_a, t_a), \quad t_N = t_b - \Delta t \quad (4.17b) \]

where

\[ P(x_b, t_b \mid x_N, t_N) \approx \int_{-\infty}^{\infty} \frac{id\kappa_b}{2\pi} \exp \left\{ -\kappa_b(x_b - x_N) + \Delta t G(\kappa_b, x_b, t_b) \right\} \]

\[ = \exp \left\{ \Delta t G(-\partial_{x_b}, x_b, t_b) \right\} \int_{-\infty}^{\infty} \frac{dk_b}{2\pi} \exp \left\{ \left[ -ik_b(x_b - x_N) \right] \right\} \]

\[ = \exp \left\{ \Delta t G(-\partial_{x_b}, x_b, t_b) \right\} \delta(x_b - x_N) \quad (4.18b) \]

Therefore:

\[ P(x_b, t_b \mid x_a, t_a) \approx \exp \left\{ \Delta t G(-\partial_{x_b}, x_b, t_b) \right\} P(x_b, t_b - \Delta t \mid x_a, t_a) \quad (4.19b) \]
In the limit $\Delta t \to 0$, Eq. (3.26b) leads to the master equation (4.1a) for transition probability and the imaginary wave-number operator in the $x$-basis:

$$\partial_x P(x_b, t_b | x_a, t_a) = G(\kappa, x_b, t_b) P(x_b, t_b | x_a, t_a), \quad \kappa = -\partial_{x_b}$$

(4.21b)

Note again that we can get (4.21b) from (4.21a) by using transformation (4.16c).

**Implication of General Wick Rotation:** The properties of the anti-Hermitian operator $\hat{\kappa}$ are given by the shift (4.2b) of GWR. Its properties play a crucial role in performing the Euclidean path integral. This implies that, under GWR (4.2), not only time (related to the energy), but also the wave-number (related to momentum) becomes imaginary:

$$it \to t, \quad k \to ik$$

(4.22)

### 5. Euclidean Path Integral and Euclidean Lagrangian in PBN

To continue the path integrals in Eq. (4.16a-b), let us consider following shift from Feynman’s standard form (4.4a-b) to *induced macro diffusion* under GWR:

$$\frac{1}{i} \partial_{i\bar{t}} |\psi_i\rangle = \frac{-1}{\hbar} \hat{H} |\psi_i\rangle, \quad \frac{-1}{\hbar} \hat{H} = -\hat{T}_h - V(x, t) = \frac{-1}{2\mu_h} \hat{k}^2 - \mu_h u(x, t)$$

(5.1a)

$$\rightarrow \frac{\partial}{\partial t} |\Omega_i\rangle = \hat{G} |\Omega_i\rangle, \quad \hat{G} = \frac{1}{2\mu_D} \hat{k}^2 - \mu_D u(x) = \hat{T}_D(\hat{k}) - W(x) = \hat{T}_D + V_c$$

(5.1b)

Now, in Eq. (4.16a), we can integrate with momentum as follows (see §2.1.7, [2]):

$$\int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} \exp \left\{ \frac{i}{\hbar} \left[ p_n (x_n - x_{n-1}) - \Delta t \frac{p_n^2}{2m} - \Delta t V(x_n, t_n) \right] \right\}$$

$$= \int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} \exp \left\{ -\frac{i}{\hbar} \frac{\Delta t}{2m} \left[ p_n - \frac{m}{\Delta t} \left( \frac{x_n - x_{n-1}}{\Delta t} \right) \right]^2 + \frac{m}{2} \Delta t \left( \frac{x_n - x_{n-1}}{\Delta t} \right)^2 - \Delta t V(x_n, t_n) \right\}$$

$$= \sqrt{\frac{m}{2\pi\hbar i\Delta t}} \exp \left\{ \frac{m}{2} \Delta t \left( \frac{x_n - x_{n-1}}{\Delta t} \right)^2 - \Delta t V(x_n, t_n) \right\}$$

(5.2a)

Then Eq. (4.16a) becomes:
\[
\langle x_b, t_b \mid x_a, t_a \rangle \approx \frac{1}{\sqrt{2\pi\hbar\Delta t}} \lim_{m \to \infty} \prod_{n=1}^{N} \int_{-\infty}^{\infty} \frac{dx_n}{\sqrt{2\pi\hbar\Delta t}} \exp\left\{ i \frac{\hbar}{\Delta t} A_N \right\},
\]

where:

\[
A_N = \Delta t \sum_{n=1}^{N+1} \left[ \frac{m}{2} \left( \frac{x_n - x_{n-1}}{\Delta t} \right)^2 - V(x_n, t_n) \right]
\]

\[
\rightarrow A[x] = \int_{t_a}^{t_b} dtL_c(x, \dot{x}), \quad L_c(x, \dot{x}) = \frac{m}{2} \dot{x}^2 - V(x, t)
\]

(5.3a)

Symbolically, it can be written as:

\[
\langle x_b, t_b \mid x_a, t_a \rangle = \int_{x(t_a) = x_a}^{x(t_b) = x_b} Dx \exp\left\{ \frac{i}{\hbar} S_f(x) - \frac{1}{2} \int_{t_a}^{t_b} L_c(x, \dot{x}) \right\}
\]

\[
= \int Dx \exp\left\{ \int_{t_a}^{t_b} dt L_h(x, \dot{x}) \right\}, \quad L_h(x, \dot{x}) = \frac{\mu_B}{2} \dot{x}^2 - V_h(x)
\]

(5.4a)

Similarly, in Eq. (4.16b), we can integrate with imaginary wave number as follows:

\[
\int_{-\infty}^{\infty} \frac{dk_n}{2\pi} \exp\left\{ -\frac{\kappa_n^2}{\Delta t} - \Delta t W(x_n, t_n) \right\} = \int_{-\infty}^{\infty} \frac{dk_n}{2\pi} \exp\left\{ -\Delta t \left[ k_n^2 - \frac{\Delta t}{\mu_D} \left( \frac{x_n - x_{n-1}}{i\Delta t} \right)^2 \right] - \Delta t W(x_n, t_n) \right\}
\]

\[
= \sqrt{\frac{\mu_D}{2\pi\Delta t}} \exp\left\{ -\Delta t \left[ \left( \frac{x_n - x_{n-1}}{\Delta t} \right)^2 \right] - \Delta t W(x_n, t_n) \right\}
\]

(5.2b)

**Euclidean Lagrangian:** Now Eq. (4.16b) becomes a Euclidean path integral:

\[
P(x_b, t_b \mid x_a, t_a) \approx \sqrt{\frac{\mu_D}{2\pi\Delta t}} \prod_{n=1}^{N} \int_{-\infty}^{\infty} \frac{dx_n}{\sqrt{2\pi\hbar\Delta t}} \exp\left\{ -A_N \right\},
\]

\[
A_N = \Delta t \sum_{n=1}^{N+1} \left[ \frac{\mu_D}{2} \left( \frac{x_n - x_{n-1}}{\Delta t} \right)^2 + W(x_n, t_n) \right]
\]

\[
\rightarrow A[x] = \int_{t_a}^{t_b} dtL_c(x, \dot{x}), \quad L_c(x, \dot{x}) = \frac{\mu_D}{2} \dot{x}^2 + W(x, t) = \frac{\mu_D}{2} \dot{x}^2 - V_c(x, t)
\]

(5.3b)

Symbolically, it can be written as following Euclidean path integral:

\[
P(x_b, t_b \mid x_a, t_a) = \int_{x(t_a) = x_a}^{x(t_b) = x_b} Dx \exp\left\{ -\int_{t_a}^{t_b} dt \left[ \frac{\mu_D}{2} \dot{x}^2 - V_c(x, t) \right] \right\}
\]

(5.4b)
Compare Eq. (4.27b, 5.4b) with (4.27a, 5.4a), we see that we shift from amplitude to probability by making the following maps induced by GWR:

\[ \begin{align*}
  \text{it} & \rightarrow t, \quad \hat{x} \rightarrow i\hat{x}, \quad \mu_h \rightarrow \mu_D, \quad V_h \rightarrow W = -V_e, \quad \langle x_b, t_b \mid x_a, t_a \rangle \rightarrow P(x_b, t_b \mid x_a, t_a) \\
\end{align*} \]

(5.5a)

Moreover, compare Eq. (4.21a, 5.4a), with (4.21b, 5.4b), we have the following cycle:

\[ \begin{align*}
  \hat{x} = \frac{p}{m} \quad \rightarrow \quad \hat{p} = \frac{\hat{k}}{\mu_h} \quad \rightarrow \quad i\hat{k} \quad \rightarrow \quad i\hat{k} \rightarrow i\hat{x} \\
\end{align*} \]

(5.5b)

At the same time, the Lagrangian in (5.4a) for Schrödinger Equation (5.1a) is shifted by GWR to the Euclidian Lagrangian in (5.4b) for induced macro diffusion (5.1b):

\[ L_h = \frac{\mu_h}{2} \hat{x}^2 - V_h(x) \quad \rightarrow \quad L_e = \frac{\mu_D}{2} \hat{x}^2 + W(x) = \frac{\mu_D}{2} \hat{x}^2 - V_e(x) \]

(5.5c)

Here we see why Euclidean potential \( V_e = -W \) was introduced in Eq. (3.1b) and (5.1b).

Moreover, to get the Euclidian Lagrangian of induced micro diffusion in Eq. (3.4), we can either apply Eq. (4.3a) to (5.5c), or apply time rotation directly to Eq. (5.4):

\[ \begin{align*}
  \text{it} & \rightarrow t, \quad \hat{x} \rightarrow i\hat{x} : \exp \left[ \int_{t_a}^{t_b} \! \! i dt \left[ \frac{\mu_h}{2} \hat{x}^2 - V_h(x) \right] \right] \rightarrow \exp \left[ -\int_{t_a}^{t_b} \! \! dt \left[ \frac{\mu_h}{2} \hat{x}^2 + V_h(x) \right] \right] \\
\end{align*} \]

(5.5d)

This implies that our integration procedure in PBN is self-consistent.

**Macro Einstein-Brown motion**: The simplest example of GWR is the Einstein-Brown motion from induced macro diffusion (4.3b) with \( V = W = 0 \). Applying (5.5a) to Eq. (1.19b) for free particle, we get:

\[ P(x_b, t_b \mid x_a, t_a) = \frac{1}{\sqrt{4\pi D(t_b-t_a)}} \exp \left[ -\frac{(x_b-x_a)^2}{4D(t_b-t_a)} \right] \]

(5.6)

It is identical to Eq. (3.11), but the diffusion coefficient \( D \) here should be identified from Newton-Langevin equation, with potential \( V(x) = 0 \) in Eq. (2.17a) and (2.18b):

\[ \begin{align*}
  m \frac{d^2x}{dt^2} &= -m\gamma \frac{dx}{dt} + \xi(t), \quad \langle x\xi \rangle = 0 \\
  \langle mx^2/2 \rangle &= k_B T/2 \equiv \beta/2 \quad \Rightarrow \quad D = k_B T / m\gamma = 1 / (m\gamma\beta) \\
\end{align*} \]

(5.7a)

(5.7b)

Applying Eq. (2.13a) to Eq. (5.6), we obtain the absolute PDF initially at \( x = 0 \):
\[ P(x,t) = P(x | \Omega_t) = P(x,t | 0,0) = \sqrt{\frac{1}{4\pi D t}} \exp \left[ -\frac{x^2}{4Dt} \right] \] (5.7d)

This is identical to the Eq. (1.5) in Ref. [7]. Similar to Eq. (3.24-25), we can calculate the expectation value of “velocity” in a macro Einstein-Brown motion with constant drift:

\[ P(x,t) = P(x | \Omega_t) = \frac{1}{\sqrt{4\pi D t}} \exp \left[ -\frac{(x-\nu t)^2}{4Dt} \right] \] (5.8a)

Using Eq. (3.23), (4.1b), (4.12) and (5.8a), we get a real value as expected:

\[ \langle \dot{x} \rangle = P(\Omega | (\dot{k} / \mu_D) | \Omega_t) = \int dx P(\Omega | x) \left[ -2D \partial_x \delta(x-x') \right] dx' P(x'| \Omega_t) = \nu \] (5.8b)

**Strong Damping Harmonic Oscillator**: Our next example of GWR is the strong damping diffusion in an external potential, as in Eq. (4.3c). The transformation (5.5a) now reads:

\[ it \to t, \quad \mu_n = \frac{m}{\hbar} \to \mu_D = \frac{1}{2D}, \quad V_n(x) \to W = \hat{k} \left[ \frac{1}{m\gamma} \right] \] (5.9)

The master equation is the *Smoluchowski equation* (2.18), not an induced macro diffusion of (5.1b). So we shall re-integrating Eq. (4.16b) by using Eq. (5.9) as follows:

\[ \int_{-\infty}^{\infty} id\kappa_n \exp \left\{ -\kappa_n(x_n - x_{n-1}) + \Delta t \frac{\kappa_n^2}{2\mu_D} - \Delta t \frac{\kappa_n}{m\gamma} V'(x_n) \right\} \]

\[ = \int_{-\infty}^{\infty} \frac{dk_n}{2\pi} \exp \left\{ ik_n(x_n - x_{n-1}) - \Delta t \frac{k_n^2}{2\mu_D} + \Delta t \frac{ik_n}{m\gamma} V'(x_n) \right\} \] (5.10a)

\[ = \int_{-\infty}^{\infty} \frac{dk_n}{2\pi} \exp \left\{ -\Delta t \frac{k_n}{2\mu_D} \left[ k_n - \frac{i \mu_D}{m\gamma} \frac{x_n - x_{n-1}}{\Delta t} - \frac{i \mu_D V'(x_n)}{m\gamma} \right]^2 \right\} \]

\[ \times \exp \left\{ -\frac{\mu_D}{2\Delta t} \left( \frac{x_n - x_{n-1}}{\Delta t} + \frac{1}{m\gamma} V'(x_n) \right)^2 \right\} \] (5.10b)

\[ = \sqrt{\frac{\mu_D}{2\pi\Delta t}} \exp \left\{ -\frac{\mu_D}{2\Delta t} \left[ \frac{x_n - x_{n-1}}{\Delta t} + \frac{1}{m\gamma} V'(x_n) \right]^2 \right\} \] (5.10c)

Compared with Eq. (5.2b) and (5.3b), Eq. (5.10c) gives us the following Euclidean Lagrangian in the limit of \( \Delta t \to 0 \):

\[ L_e(x, \dot{x}) = \frac{\mu_D}{2} \left[ \dot{x} + \frac{1}{m\gamma} V'(x) \right]^2 = \frac{1}{4D} \left[ \dot{x} + \frac{1}{m\gamma} V'(x) \right]^2 \] (5.11)
This is almost identical to the Euclidean Lagrangian of strong damping in Ref. [2] (see §18.9.3 of [2]), except that $\dot{x}$ in (5.11) is replaced by a retarded coordinate $\dot{x}^R$. But the approach in ref [2] is different from ours: it starts with the square of absolute value of the standard Feynman path integral for QM (see §18.8 of [2]):

$$P(x_b, t_b | x_a, t_a) = \left| \langle x_b, t_b | x_a, t_a \rangle \right|^2 = \left| \int \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[ \frac{m}{2} \dot{x}^2 - V(x) \right] \right\} \right|^2 \quad (5.12)$$

With some assumptions (see §18.8-9 of [2]) and calculations, the Euclidean Lagrangian (5.11) is reached, and with an auxiliary momentum integration, it leads to the path integral solution of Smoluchowski equation (2.18). On the other hand, we started with Smoluchowski equation (2.18), and then applied it into our path integral formulas (4.16b) to get the Euclidean Lagrangian (5.11).

For harmonic oscillator ($V(x) = \frac{m\omega^2 x^2}{2}$), Smoluchowski Eq. (2.18) becomes:

$$\partial_t P(x_b, t_b | x_a, t_a) = \left\{ D \frac{\partial^2}{\partial x_b^2} + \frac{\omega^2}{\gamma} \left[ 1 + x_b \frac{\partial}{\partial x_b} \right] \right\} P(x_b, t_b | x_a, t_a) \quad (5.14a)$$

$$= \left\{ D \frac{\partial^2}{\partial x_b^2} + \eta \left( \frac{\partial}{\partial x_b} x_b \right) \right\} P(x_b, t_b | x_a, t_a), \quad \eta \equiv \frac{\omega^2}{\gamma} \quad (5.14b)$$

The path integral solution of Eq. (5.14) is well-known (see §18.9.3 of [2], Eq. (4.23) of [7] or Eq. (3.120) of [10]):

$$P(x_b, t_b | x_a, t_a) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{[x_b - \bar{x}(t_b-t_a)]^2}{2\sigma^2(t_b-t_a)} \right\} \quad (5.15)$$

Here $\bar{x}(t)$, $\sigma^2(t)$ are the averages defined as:

$$\bar{x}(t) = \langle x(t) \rangle = x_a e^{-\eta t}, \quad \sigma^2(t) = \langle [x(t) - \bar{x}(t)]^2 \rangle = (D/\eta)(1 - e^{-\eta t}) \quad (5.16)$$

Because this is a HMC, using Eq. (2.13a), we have the following absolute PDF:

$$P(x, t) = P(x, t | 0, 0) = \frac{\sqrt{\eta}}{\sqrt{2\pi D(1-e^{-2\eta t})}} \exp \left\{ -\frac{\eta x^2}{2D(1-e^{-2\eta t})} \right\} \quad (5.17)$$

Note: If $V(x) = 0$ in Eq. (2.18), we can get the Euclidean Lagrangian for Einstein-Brown motion either from Eq. (5.11) with $V(x) = 0$, or from Eq. (5.5c) with $W(x) = 0$:
\[ L_\epsilon(x, \dot{x}) = \frac{\mu_D}{2} \dot{x}^2 = \frac{1}{4D} \dot{x}^2 \]  

(5.18a)

Under the limit \( \omega = 0 \rightarrow \eta = 0 \), Eq. (5.14b) produces the same PDF of Einstein-Brown motion, identical to Eq. (5.7d), as an example of induced macro diffusion.

Summary

We demonstrated that, under Special Wick Rotation, the Schrodinger equation and its path integral of a conservative system in Dirac notation were simultaneously shifted to the master equation and its Euclidean path integral of an induced micro diffusion in PBN. Next, we extended to the General Wick Rotation and introduced the anti-Hermitian wave number operator \( \hat{\kappa} \). Then, in parallel with the path integral in Dirac notation, we performed step-by-step the Euclidean path integral in PBN and derived the Euclidean Lagrangian of induced diffusions and Smoluchowski equation. Our study reveals that many expressions in PBN are closely tied to those in Dirac Notation by Wick rotations.

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