HARDY AND POINCARÉ INEQUALITIES IN FRACTIONAL
ORLICZ-SOBOLEV SPACES

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Abstract. We provide sufficient conditions for boundary Hardy inequality to hold in bounded Lipschitz domains, complement of a point (the so called point Hardy inequality), domain above the graph of a Lipschitz function, complement of a bounded Lipschitz domain in fractional Orlicz-Sobolev setting. As a consequence we get sufficient conditions for regional fractional Orlicz Poincaré inequality in bounded Lipschitz domains. Necessary conditions for fractional Orlicz Hardy and regional fractional Orlicz Poincaré inequalities are also given for bounded Lipschitz domains. Various sufficient conditions on open sets are provided for fractional Orlicz Poincaré inequality and regional fractional Orlicz Poincaré inequality to hold.

1. Introduction

The aim of this article is to study two very well known inequalities, the Poincaré inequality and the Hardy inequality on fractional Orlicz-Sobolev setting. The classical Poincaré inequality [Eva10, Chapter 5.8.1] states that for any bounded domain \( D \subseteq \mathbb{R}^N \), \( q \geq 1 \) there exists \( c = c(q, N, D) > 0 \) such that for any \( f \in C^\infty_c(D) \),

\[
\|f\|_{L^q(D)} \leq c \|\nabla f\|_{L^q(D)}.
\]

(1.1)

For bounded domains it is a standard fact that the best constant is attained. Although generalized in many different direction as can be seen in [Beb03, J86, KZ08, LSY03], for the purpose of this paper we would like to refer Gossez [Gos74, Lemma 5.7], who generalized eq. (1.1) in Orlicz-Sobolev setting (defined below). The standard boundary Hardy inequality states that if \( D \) is a bounded Lipschitz domain in \( \mathbb{R}^N \), \( q \geq 1 \), then there exists a constant \( c = c(q) > 0 \) such that for \( f \in C^\infty_c(D) \),

\[
\|f/\delta_D\|_{L^q(D)} \leq c \|\nabla f\|_{L^q(D)},
\]

(1.2)

where \( \delta_D(x) := \inf\{|x-y| : y \in D^c\} \). We shall frequently denote \( \delta_D(x) \) by \( \delta_x \). This has been further developed in [BM97, BMS00, KMP07, MMP98] and several other works. For the generalization to Orlicz version of local Hardy inequality we refer to [Hei95, KaPPu09, MMOS14, MNOS11].

We shall primarily be concerned with the “Orlicz space” \( L^A(D) \) and the “fractional Orlicz-Sobolev space” \( W^{s,A}(D) \). We start by defining these spaces. A continuous, convex function \( A : [0, \infty) \rightarrow [0, \infty) \) such that \( \lim_{t \to 0} \frac{A(t)}{t} = 0 \) and \( \lim_{t \to \infty} \frac{A(t)}{t} = \infty \) is called an \( N \)-function or a Young function [AF03, Chapter 8.2].

Definition 1 ([AF03, Chapter 8.2]). We say that \( A \) satisfies the \( \Delta_2 \) condition globally or simply the \( \Delta_2 \) condition \( (A \in \Delta_2) \) if there exists constant \( p > 0 \) such that

\[
A(2t) \leq pA(t), \quad \forall \ t > 0.
\]

The set

\[
L^A(D) := \left\{ f : D \rightarrow \mathbb{R} \text{ measurable} \mid \exists \lambda > 0 \text{ such that } M_{L^A(D)}(f) \left( \frac{f}{\lambda} \right) < \infty \right\}
\]

is the Orlicz space with the Luxemburg norm. The fractional Orlicz-Sobolev space \( W^{s,A}(D) \) is defined as

\[
W^{s,A}(D) := \left\{ u \in L^A(D) : \nabla^s u \in L^A(D) \right\}
\]

with the norm

\[
\|u\|_{W^{s,A}(D)} := \|\nabla^s u\|_{L^A(D)}.
\]

These spaces are quite powerful tools in the study of partial differential equations and variational problems. They generalize the classical Sobolev, Orlicz, and fractional Sobolev spaces.
is called the Orlicz space and the set 
\[ W^{s,A}(D) := \left\{ f \in L^A(D) \mid \exists \lambda > 0 \text{ such that } M_{W^{s,A}(D)} \left( \frac{f}{\lambda} \right) < \infty \right\} \]
is called the fractional Orlicz-Sobolev space, where 
\[ M_{L^A(D)}(f) := \int_D A(|f(x)|)dx \quad \text{and} \quad M_{W^{s,A}(D)}(f) := \int_{D \times D} A \left( \frac{|f(x) - f(y)|}{|x-y|^s} \right) \frac{dx\,dy}{|x-y|^N}. \]

In the case \( A(t) = t^q \) for some \( q > 1 \), \( L^A(D) \) and \( W^{s,A}(D) \) are well known Lebesgue space \( L^q(D) \) and the fractional Sobolev space \( W^{s,q}(D) \) respectively [DNPV12, p. 524].

For \( s \in (0,1) \), we start by defining the following important quotients that will be used frequently throughout this article:
\[ H_{N,s,A}(D) := \inf_{f \in C_0^\infty(D)} \frac{M_{W^{s,A}(D)}(f)}{\int_D f(x)} \quad \text{and} \quad P_{N,s,A}^1(D) := \inf_{f \in C_0^\infty(D)} \frac{M_{W^{s,A}(D)}(f)}{\int_D f(x)} \]
and 
\[ P_{N,s,A}^2(D) := \inf_{f \in C_0^\infty(D)} \frac{M_{W^{s,A}(R^N)}(f)}{\int_R f(x)} \]

We shall say that
- fractional Orlicz Hardy inequality (denoted as \( FOHI(s,A) \)) holds if \( H_{N,s,A}(D) > 0 \);
- regional fractional Orlicz Poincaré inequality (\( RFOPI(s,A) \)) holds if \( P_{N,s,A}^1(D) > 0 \);
- fractional Orlicz Poincaré inequality (\( FOPI(s,A) \)) holds if \( P_{N,s,A}^2(D) > 0 \).

We start by recalling some literature on fractional Orlicz Hardy inequality for the case when \( A(t) = t^q \), \( q > 1 \). Kufner [KT78] proved \( FOHI(s,t^q) \) in one dimension when \( D = (0,\infty) \). For bounded Lipschitz domain \( D \), Dyda [Dyd04, Theorem 1.1 and Section 2] proved the following results:

**Theorem 1.1** (Dyda). Let \( \beta > 0 \) and \( q > 1 \). The Hardy inequality,
\[ \int_D \frac{|u(x)|^q}{\delta_D(x)^\beta} \, dx \leq c \int_{D \times D} \frac{|u(x) - u(y)|^q}{|x-y|^{N+\beta}} \, dx\,dy, \quad \text{for all } u \in C_c(D), \]
where \( c = c(D,\beta,N,q) < \infty \) is a constant, holds true in each of the following cases:
1. \( D \) is a bounded Lipschitz domain and \( \beta > 1 \);
2. \( D \) is a complement of a bounded Lipschitz domain, \( \beta \neq 1 \) and \( \beta \neq N \);
3. \( D \) is a domain above the graph of a Lipschitz function \( \mathbb{R}^{N-1} \to \mathbb{R} \) and \( \beta \neq 1 \);
4. \( D \) is a complement of a point and \( \beta \neq N \).

The question of best constant in fractional Hardy inequality, that is the exact value of \( FOHI(s,t^q) \), was first addressed in [BD11, Theorem 1] for upper half space. Heinig et al. [HKP97, Theorem 3.1] and Kufner et al. [KP03, Theorem 5.23] studied one dimensional \( FOHI(s,t^q) \) between two weighted \( L^q \) spaces. Reader may refer to [BC18, CS03, Dyd11, DF12, DV14, EHSV14, FL12, FS08, HKP97, HSV15, ILTV14, LS10] and references therein for more information related to fractional Hardy inequality.

Concerning \( RFOPI(s,A) \) and \( FOPI(s,A) \), we start with the trivial observation that \( P_{N,s,A}^2(D) \geq P_{N,s,A}^1(D) \), that is \( FOPI(s,A) \) holds whenever \( RFOPI(s,A) \) is true. For a bounded domain \( D \), \( RFOPI(s,t^q) \) is true if and only if \( 2s > 1 \) [Che18, Proposition 3.2] whereas \( FOPI(s,t^q) \) holds for all \( s \in (0,1) \) [CR20, Theorem 1.2]. It was first established in [Yer14, Lemma 1] that \( FOPI(s,t^q) \) is true for all values of \( s \), if the domain is an infinite strip i.e. \( D = (0,1) \times \mathbb{R}^{N-1} \), though the best constant was not established. The best constant for the above case is obtained in
\[ \liminf \] Let Theorem 1.2. \[ \text{CR20, Theorem 1.2} \] and \[ \text{CCRS21, Theorem 1.3} \]. Theorem 1.6 which can be regarded as the fractional Orlicz analogue of the results obtained in change of variable formula introduced in \[ \text{LS10, Lemma 2.4} \] is the key ingredient in the proof of examples of domains for which \( \text{RFOPI} \) results are proposition 5.1 and theorem 1.6. In proposition 5.1 we give complete characterization of domains in 1-dimension for which \( \text{RFOPI} \) results are true or false. In particular for \( N \)-functions \( t^q \), \( q > 1 \) and \( (1 + t) \log(1 + t) - t \) a complete answer, for all values of \( s \in (0, 1) \) can be provided for \( \text{FOHI}(s, A) \). At this point, we would like to mention that the line of argument in the proofs of theorems 1.2, 1.4 and 1.5 are adapted from \[ \text{Dyd04} \] where Dyda, in fact, predicted the possibility of his methods being generalized. Secondary objective of this article is to study \( \text{RFOPI}(s, A) \) and \( \text{FOPI}(s, A) \) for unbounded domains in \( \mathbb{R}^N \). In this direction our results are proposition 5.1 and theorem 1.6. In proposition 5.1 we give complete characterization of domains in 1-dimension for which \( \text{RFOPI}(s, A) \) holds, provided \( \lim \lambda s,A(\lambda) = 0 \) (see eq. (1.4)).

**Theorem 1.6** provides different sufficient criteria on domains for \( \text{FOPI}(s, A) \) and \( \text{RFOPI}(s, A) \) to hold. As an application of theorem 1.6, at the end of section 5 we provide several non-trivial examples of domains for which \( \text{RFOPI}(s, A) \) and \( \text{FOPI}(s, A) \) holds. An adapted version of a change of variable formula introduced in \[ \text{LS10, Lemma 2.4} \] is the key ingredient in the proof of theorem 1.6 which can be regarded as the fractional Orlicz analogue of the results obtained in \[ \text{CR20, Theorem 1.2} \] and \[ \text{CCRS21, Theorem 1.3} \].

Our main results are stated below.

**Theorem 1.2.** Let \( D \subseteq \mathbb{R}^N \) be a bounded Lipschitz domain, \( A \in \Delta_2, s \in (0, 1) \) and \( \liminf_{\lambda \to 0^+} \alpha_{s,A}(\lambda) = 0 \), where \( \alpha_{s,A} : [0, \infty) \to \mathbb{R} \) is defined by

\[
\alpha_{s,A}(\lambda) := \sup_{t \in [0, \infty)} \frac{A(\lambda t)}{\lambda^\frac{N}{2} A(t)}.
\]

Then \( \text{FOHI}(s, A) \) and \( \text{RFOPI}(s, A) \) holds, that is \( H_{N,s,A}(D) > 0 \) and \( P_{N,s,A}^1(D) > 0 \).
Theorem 1.3. Let \( D \subseteq \mathbb{R}^N \) be a bounded Lipschitz domain, \( A \) be an \( N \)-function and

\[
(1.5) \quad \lim_{\varepsilon \to 0^+} \varepsilon \int_0^\varepsilon A(z) \, dz = \beta \in \mathbb{R}.
\]

Then

(1) if \( \beta = 0 \), then both \( \text{FOHI}(s, A) \) and \( \text{RFOPI}(s, A) \) do not hold,

(2) if \( \beta \in (0, \infty) \), then \( \text{FOHI}(s, A) \) does not hold.

Theorem 1.4. Let \( s \in (0, 1) \), \( A \in \Delta_2 \) and \( \alpha_{s,A} \) be as in eq. (1.4). Assume \( \liminf \lambda^{\frac{1-N}{s}} \alpha_{s,A}(\lambda) = 0 \) or \( \liminf \lambda^{\frac{1-N}{s}} \alpha_{s,A}(\lambda) = 0 \). Then \( \text{FOHI}(s, A) \) holds in \( \mathbb{R}^N \setminus \{0\} \), that is \( \lambda_{N,s,A}(\mathbb{R}^N \setminus \{0\}) > 0 \).

Theorem 1.5. Suppose \( D \subseteq \mathbb{R}^N \) be an open set, \( A \in \Delta_2 \), \( s \in (0, 1) \) and \( \alpha_{s,A} \) be as in eq. (1.4). Then \( \text{FOHI}(s, A) \) holds true, that is \( \lambda_{N,s,A}(D) > 0 \), in each of the following cases:

(1) \( D = \{ (x', x_N) \in \mathbb{R}^N \mid x' \in \mathbb{R}^{N-1}, x_N > \Phi(x') \} \), where \( \Phi : \mathbb{R}^{N-1} \to \mathbb{R} \) is a Lipschitz map and \( s, A \) are such that

\[
\liminf_{\lambda \to 0^+} \alpha_{s,A}(\lambda) = 0 \quad \text{or} \quad \liminf_{\lambda \to \infty} \alpha_{s,A}(\lambda) = 0;
\]

(2) \( D^c \) is closure of some bounded Lipschitz domain and \( s, A \) are such that

\[
\liminf_{\lambda \to 0^+} \lambda^{\frac{1-N}{s}} \alpha_{s,A}(\lambda) = 0 \quad \text{or} \quad \liminf_{\lambda \to \infty} \lambda^{\frac{1-N}{s}} \alpha_{s,A}(\lambda) = 0.
\]

Before giving the statement of our last result, that is theorem 1.6, we define some terminologies that will be required to formulate it in a precise manner.

Definition 2. We say that a set \( D \subseteq \mathbb{R}^N \) satisfies the finite ball condition if \( D \) does not contain arbitrarily large balls, that is if

\[
BC(D) := \sup \{ r : B(x, r) \subseteq D, \ x \in D \} < \infty.
\]

Definition 3. Let \( \{ D_\beta \}_\beta \) be a family of sets in \( \mathbb{R}^N \), where \( \beta \in \Lambda \) (some indexing set). We say that the \( \text{FOPI}(s, A) \) holds uniformly for \( \{ D_\beta \}_\beta \) if \( \inf_{\beta} D_\beta^{s}_{N,s,A}(D_\beta) > 0 \).

Let \( \omega \in \mathbb{S}^{N-1} \) and \( x \in \omega^\perp \), define \( L_D(x, \omega) := \{ t \mid x + tw \in D \} \subseteq \mathbb{R} \).

Definition 4. We say an open set \( D \subseteq \mathbb{R}^N \) is of type \( \text{LS}(s, A) \) if there exists \( \Sigma \subseteq \mathbb{S}^{N-1} \) with positive \( (N-1) \)-dimensional Hausdorff measure, such that uniform \( \text{FOPI}(s, A) \) holds for the family \( \{ L_D(x, \omega) \}_{\omega \in \Sigma, x \in \omega^\perp} \).

Theorem 1.6. Let \( D \subseteq \mathbb{R}^N \) be an open set, \( s \in (0, 1) \), \( A \) be an \( N \)-function and \( \alpha_{s,A} \) be as in eq. (1.4).

(1) Assume that \( A \in \Delta_2 \) and \( \lim_{\lambda \to 0^+} \alpha_{s,A}(\lambda) = 0 \). Let there exist \( \Sigma \subseteq \mathbb{S}^{N-1} \) with positive \( (N-1) \)-dimensional Hausdorff measure such that \( \sup_{\omega \in \Sigma, x \in \omega^\perp} BC(L_D(x, \omega)) < \infty \). Then the \( \text{RFOPI}(s, A) \) holds true in \( D \).

(2) Assume that there exist \( R, c_1 > 0 \) such that \( L^N(B(x, R) \cap D^c) > c_1 \) for any \( x \in D \). Then \( \text{FOPI}(s, A) \) holds in \( D \forall s \in (0, 1) \).

(3) \( \text{FOPI}(s, A) \) holds if \( D \) is an \( \text{LS}(s, A) \) domain.

This paper is arranged in the following way: In section 2, some necessary preliminaries are discussed. Proof of theorems 1.2 and 1.3 followed by some applications are given in section 3. Theorems 1.4 and 1.5 are proved in section 4. In section 5 we prove proposition 5.1 and theorem 1.6.
Throughout the paper the following conventions and notations will be followed, unless mentioned otherwise explicitly:

\(D\) will denote an open set in \(\mathbb{R}^N\), \(s \in (0, 1)\), \(A\) will denote an \(N\)-function, \(\mathcal{L}^k\) will denote the Lebesgue measure on \(\mathbb{R}^k\), \(S^{k-1}\) will denote the unit sphere in \(\mathbb{R}^k\), \(c\) will denote a generic constant which may change from line to line, \(X^c\) will stand for the complement of the set \(X\) in appropriate universal set (to be understood from the context), for any real valued function \(f\) or for any Lipschitz domain \(D\), \(\text{Lip}(f)\) or \(\text{Lip}(D)\) will denote the Lipschitz constant, \(\alpha_{s,A}\) will be as in eq. (1.4), \(p\) will be as in definition 1.

We start with some basic facts about \(N\)-functions. One may refer to [AF03, KR61] for detailed discussion on the topic.

**Lemma 2.1** ([AF03, Chapter 8.2]). \(A\) is an \(N\)-function if and only if there exists a non-decreasing, right continuous function \(a : [0, \infty) \to [0, \infty)\) satisfying \(a(0) = 0\), \(a(t) > 0\) for \(t > 0\), \(\lim_{t \to 0^+} a(t) = \infty\) such that

\[
A(x) = \int_0^x a(t)dt.
\]

The following two lemmas will be used frequently in the rest of the article.

**Lemma 2.2** ([AF03, Chapter 8.2]). Let \(A\) be an \(N\)-function, then \(A\) and \(t \mapsto \frac{A(t)}{t}\) both are strictly increasing function on \((0, \infty)\).

**Lemma 2.3.** Let \(A \in \Delta_2\). Then

\[
A(\lambda t) \leq \lambda^p A(t) \text{ for } t \in [0, \infty), \quad \forall \lambda \geq 1,
\]

where \(p\) is the constant in definition 1. The above inequality is equivalent to

\[
A(\lambda t) \geq \lambda^p A(t) \text{ for } t \in [0, \infty), \quad \forall 0 \leq \lambda \leq 1.
\]

**Proof.** Since \(A \in \Delta_2\), there exists \(p > 0\) such that \(A(2t) \leq pA(t), \quad \forall t > 0\). Now for \(a\) as in lemma 2.1, using the non decreasing property of \(a\),

\[
pA(t) \geq A(2t) = \int_0^{2t} a(\tau)d\tau > \int_t^{2t} a(\tau)d\tau > ta(t).
\]

This implies for any \(\lambda > 1\),

\[
\log \left( \frac{A(\lambda t)}{A(t)} \right) = \int_t^{\lambda t} \frac{a(\tau)}{A(\tau)}d\tau < \int_t^{p\tau} \frac{\tau}{\tau}d\tau = p\log \left( \frac{\lambda t}{t} \right) = \log(\lambda^p).
\]

The lemma follows. \(\square\)

**Lemma 2.4.** Let \(D \subseteq \mathbb{R}^N\) be a bounded domain, \(A \in \Delta_2\). Then for some constant \(c = c(D, A) > 0\),

\[
P_{N,s,A}^1(D) \geq cH_{N,s,A}(D).
\]

**Proof.** Let \(f \in C^\infty_c(D)\). First assume \(\text{diam}(D) \leq 1\). By the monotonicity of \(A\),

\[
\int_D A \left( \frac{|f(x)|}{\delta_x^s} \right) dx \geq \int_D A(|f(x)|) dx.
\]

In the case \(\text{diam}(D) > 1\), we exploit the \(\Delta_2\) condition of \(A\) and use lemma 2.3 to get

\[
\int_D A \left( \frac{|f(x)|}{\delta_x^s} \right) dx \geq \int_D A \left( \text{diam}(D)^{-s}|f(x)| \right) dx \geq \text{diam}(D)^{-sp} \int_D A(|f(x)|) dx.
\]

Hence the lemma follows. \(\square\)
**Proposition 2.1.** Let $s \in (0, 1)$ and $p > 1$ be as in lemma 2.3.

(1) If $D_1 \subseteq D_2 \subseteq \mathbb{R}^N$, then $P_{N,s,A}^2(D_2) \leq P_{N,s,A}^2(D_1)$.

(2) Let $D \subseteq \mathbb{R}^N$ be an open set and $u \in W^{s,A}(D)$. Assume that $A \in \Delta_2$. For $t > 0$, define $v_t \in W^{s,A}(D)$ by $v_t(x) = u(tx)$. Then

$$
\iint_{D \times D} A \left( \frac{v_t(x) - v_t(y)}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N} \leq \begin{cases} 
\int_{tD \times tD} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N}, & t < 1, \\
\int_{tD \times tD} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N}, & t \geq 1,
\end{cases}
$$

and also

$$
\iint_{D \times D} A \left( \frac{v_t(x) - v_t(y)}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N} \geq \begin{cases} 
\int_{tD \times tD} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N}, & t \geq 1, \\
\int_{tD \times tD} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N}, & t < 1.
\end{cases}
$$

Furthermore,

$$
P_{N,s,A}^1(D) \leq P_{N,s,A}^1(tD) \leq \frac{P_{N,s,A}^1(D)}{t^s} \text{ if } t < 1,
$$

$$
P_{N,s,A}^1(D) \geq P_{N,s,A}^1(tD) \geq \frac{P_{N,s,A}^1(D)}{t^s} \text{ if } t \geq 1.
$$

(3) Let $D \subseteq \mathbb{R}^N$ be an open set and $t > 0$. Assume that $A \in \Delta_2$. Then

$$
P_{N,s,A}^2(D) \leq P_{N,s,A}^2(tD) \leq \frac{P_{N,s,A}^2(D)}{t^s} \text{ if } t < 1,
$$

$$
P_{N,s,A}^2(D) \geq P_{N,s,A}^2(tD) \geq \frac{P_{N,s,A}^2(D)}{t^s} \text{ if } t \geq 1.
$$

(4) Let $D \subseteq \mathbb{R}^N$ be such that $BC(D) = \infty$, then $P_{N,s,A}^1(D) = 0 = P_{N,s,A}^2(D)$.

**Proof.**

(1) Directly follows from the definition.

(2) For the first inequality, take the change of variable $X = tx, Y = ty$ and in case of $t < 1$, use lemma 2.2 to infer $A((t^s f(x,y)) \leq A(f(x,y))$. In case of $t \geq 1$ use lemma 2.3. For the second inequality replace $t$ by $\frac{1}{t}$.

(3) Similar as case (2).

(4) $BC(D) = \infty$ implies there exist a positive sequence $\{r_n\}_n$ and a sequence $\{x_n\}_n$ such that $r_n \to \infty$ and $B(x_n, r_n) \subseteq D$. Note that $P_{N,s,A}^2(D) \leq P_{N,s,A}^2(D)$. Then by (1) and (3) we infer that

$$
P_{N,s,A}^2(D) \leq P_{N,s,A}^2(B(x_n, r_n)) = P_{N,s,A}^2(B(0, r_n)) \leq \frac{P_{N,s,A}^2(B(0,1))}{r_n} \to 0.
$$

This finishes the proof. 

**Lemma 2.5.** Let $D \subseteq \mathbb{R}$ be an open set with $BC(D) < \infty$ and $D = \bigcup_{k=1}^{\infty} I_k$, where $I_k$’s are disjoint intervals. Then for any $k \in \mathbb{N}$, $P_{1,s,A}^1(I_k) \geq \min \{BC(D)^{-sp}, 1\} P_{1,s,A}^1((0,1))$, where $p > 1$ is as in lemma 2.3.

**Proof.** Using (2) of proposition 2.1, we have $P_{1,s,A}^1(I_k) \geq \text{diam}(I_k)^{-\beta} P_{1,s,A}^1((0,1))$, where

$$
\beta = \begin{cases} 
s, & \text{if } \text{diam}(I_k) < 1, \\
sp, & \text{if } \text{diam}(I_k) \geq 1.
\end{cases}
$$

If $\text{diam}(I_k) < 1$, $P_{1,s,A}^1(I_k) \geq P_{1,s,A}^1((0,1))$. On the other hand if $\text{diam}(I_k) \geq 1$,

$$
P_{1,s,A}^1(I_k) \geq \text{diam}(I_k)^{-sp} P_{1,s,A}^1((0,1)) \geq BC(D)^{-sp} P_{1,s,A}^1((0,1)).
$$

Combining the two cases we finally get the desired result. 

\[\square\]
3. \textit{FOHI}(s, A) and \textit{RFOPI}(s, A) on Bounded Lipschitz Domains

In this section we shall prove theorems 1.2 and 1.3. Our proof of theorem 1.2 is motivated by [Dyd04]. We start this section by proving some technical lemmas which will be used in the proof of theorem 1.2. Let \( D \subseteq \mathbb{R}^N \) be a non-empty open set. Throughout the section we shall assume, unless stated otherwise, \( \Omega \subseteq D \), \( A \in \Delta_2 \) and \( p \) is as in definition 1.

We now fix a function \( f \in C_0^\infty(D) \) and define

\[
G = G(f; \Omega; l_1, l_2) := \left\{ x \in \Omega \mid A \left( \left| f(x) \right| \frac{1}{\delta_x^s} \right) > \frac{2^{p+1}}{l_2 \delta_x^s} \int_{B(x, l_1 \delta_x) \cap \Omega} A \left( \frac{\left| f(x) - f(y) \right|}{\delta_y^s} \right) \, dy \right\};
\]

where \( l_1 > 1, l_2 \) are positive numbers, independent of \( f \), whose values are given later.

**Lemma 3.1.** Let \( f, l_1, l_2 \) be as above. Then

\[
\int_{\Omega \setminus G} A \left( \left| f(x) - f(y) \right| \frac{1}{|x-y|^s} \right) \frac{dy}{|x-y|^N} \geq \frac{1}{l_1 \delta_x^s} \int_{B(x, l_1 \delta_x) \cap \Omega} A \left( \frac{\left| f(x) - f(y) \right|}{\delta_y^s} \right) \frac{dy}{(l_1 \delta_x)^N} \geq \frac{l_2}{l_1 \delta_x^s} \int_{B(x, l_1 \delta_x) \cap \Omega} A \left( \frac{\left| f(x) - f(y) \right|}{\delta_y^s} \right) \frac{dy}{\delta_x^s}.
\]

Integrating over \( \Omega \setminus G \), we get the desired result. \( \square \)

In the view of the above lemma, we can says that the set \( \Omega \setminus G \) is a good set for fractional Orlicz Hardy inequality in the sense that the required inequality holds in it.

For any \( x \in G \), let us define the set \( E^*(x) = \{ y \in E \mid \frac{|f(y)|}{2} \leq |f(y)| \leq \frac{3}{2} |f(x)| \} \).

**Lemma 3.2.** Let \( x \in G \) and \( E \subseteq B(x, l_1 \delta_x) \cap \Omega \). If \( \mathcal{L}^N(E) \geq l_2 \delta_x^N \), then

\[
\frac{\mathcal{L}^N(E)}{2} \leq \mathcal{L}^N(E) - \frac{l_2 \delta_x^N}{2} \leq \mathcal{L}^N(E^*(x)).
\]

**Proof.** Since \( x \in G \), \( E \subseteq B(x, l_1 \delta_x) \cap \Omega \), by eq. (3.1) we have

\[
A \left( \frac{|f(x)|}{\delta_x^s} \right) \geq \frac{2^{p+1}}{l_2 \delta_x^s} \int_{B(x, l_1 \delta_x) \cap \Omega} A \left( \frac{|f(x) - f(y)|}{\delta_y^s} \right) \, dy \geq \frac{2^{p+1}}{l_2 \delta_x^s} \int_{E \setminus E^*(x)} A \left( \frac{|f(x) - f(y)|}{\delta_y^s} \right) \, dy.
\]

Note that, \( y \in E \setminus E^*(x) \) implies \( |f(x) - f(y)| \geq |f(x)| - |f(y)| \geq |f(x)|/2 \). Thus by using lemmas 2.2 and 2.3 we get

\[
A \left( \frac{|f(x)|}{\delta_x^s} \right) \geq \frac{2}{l_2 \delta_x^s} N A \left( \frac{|f(x)|}{\delta_x^s} \right) \left\{ \mathcal{L}^N(E) - \mathcal{L}^N(E^*(x)) \right\}.
\]

Using the hypothesis on measure of \( E \), we get \( \frac{\mathcal{L}^N(E)}{2} \leq \mathcal{L}^N(E) - \frac{l_2 \delta_x^N}{2} \leq \mathcal{L}^N(E^*(x)). \) \( \square \)

**Lemma 3.3.** Let \( E_1 \subseteq \Omega \) and \( E_2 \subseteq B(x, l_1 \delta_x) \cap \Omega \) be such that \( \mathcal{L}^N(E_2) \geq l_2 \delta_x^N \) for all \( x \in E_1 \). Then

\[
\int_{E_1 \cap G} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx \leq \frac{2^{p+1} \mathcal{L}^N(E_1)}{\mathcal{L}^N(E_2)} \int_{E_2} A \left( \frac{\sup \{ \delta_x^s \mid x \in E_2 \} \left| f(y) \right|}{\inf \{ \delta_y^s \mid x \in E_1 \} \delta_y^s} \right) \, dy.
\]


**Proof.** Assume that $E_1 \cap G$ is non empty. First, we fix $\eta > 1$ and pick $x_0 \in E_1 \cap G$ such that
$$\sup_{x \in E_1 \cap G} |f(x)| \leq \eta |f(x_0)|.$$ Also $|f(x_0)| \leq 2|f(y)|$ for any $y \in E_2(x_0)$. Hence, for any $y \in E_2(x_0)$, using lemmas 2.2 and 2.3,
$$\int_{E_1 \cap G} A \left( \frac{|f(x)|}{\delta_x^s} \right) dx \leq \mathcal{L}^N(E_1 \cap G) A \left( \frac{\sup_{x \in E_1 \cap G} |f(x)|}{\inf \{\delta_x^s \mid x \in E_1 \cap G\}} \right) \leq \mathcal{L}^N(E_1 \cap G) A \left( \frac{2\eta |f(y)|}{\inf \{\delta_x^s \mid x \in E_1 \cap G\}} \right) \leq 2^p \eta^p \mathcal{L}^N(E_1) A \left( \frac{\sup \{\delta_y^s \mid y \in E_2(x_0)\} |f(y)|}{\inf \{\delta_x^s \mid x \in E_1\} \delta_y^s} \right).$$
Integrating over $y \in E_2(x_0)$, using lemma 3.2 and the fact that $E_2(x_0) \subseteq E_2$, we obtain
$$\int_{E_1 \cap G} A \left( \frac{|f(x)|}{\delta_x^s} \right) dx \leq \frac{2^{p+1}\eta^p \mathcal{L}^N(E_1)}{\mathcal{L}^N(E_2)} \int_{E_2} A \left( \frac{\sup \{\delta_y^s \mid y \in E_2\} |f(y)|}{\inf \{\delta_x^s \mid x \in E_1\} \delta_y^s} \right) dy.$$
Letting $\eta \to 1$ the proof follows. \qed

**Lemma 3.4.** Let $\Omega \subseteq D \subseteq \mathbb{R}^N$ be two open sets, $f \in C^\infty_c(D)$, $l_1, l_2 > 0$, $G = G(f, \Omega; l_1, l_2)$, $0 < \gamma < 1$ and $m \in \mathbb{N}$. Assume $\Omega = \cup_{j=0}^\infty A_j$, $\mathcal{L}^N(A_i \cap A_j) = 0 \forall j \neq i$ and $\exists n_0 \in \mathbb{N}$ such that $f \equiv 0$ on $A_j$ for $j \geq n_0$.

(1) If $\forall j \in \mathbb{N}$,
$$\int_{G \cap A_j} A \left( \frac{|f(x)|}{\delta_x^s} \right) dx \leq \gamma \int_{A_{j+m}} A \left( \frac{|f(x)|}{\delta_x^s} \right) dx,$$
then there exists a constant $c > 0$ such that
$$\int_{\Omega} A \left( \frac{|f(x)|}{\delta_x^s} \right) dx \leq c \int_{\Omega} \int_{\Omega} A \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right) dx dy.$$

(2) If $\forall j \geq m$,
$$\int_{G \cap A_j} A \left( \frac{|f(x)|}{\delta_x^s} \right) dx \leq \gamma \int_{A_{j-m}} A \left( \frac{|f(x)|}{\delta_x^s} \right) dx,$$
then there exists a constant $c > 0$ such that
$$\int_{\Omega} A \left( \frac{|f(x)|}{\delta_x^s} \right) dx \leq c \int_{\Omega} \int_{\Omega} A \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right) dx dy + c \int_{A_0 \cup \cdots \cup A_{m-1}} A \left( \frac{|f(x)|}{\delta_x^s} \right) dx.$$

**Proof.** (1)
Here we used the fact that $f \equiv 0$ for large $j$ forcing the iterative process to terminate. So we have

$$\int_{\Omega \cap G} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx \leq \sum_{k=1}^{\infty} \gamma^k \int_{\Omega \cap G} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx.$$ 

Hence by using above estimate and lemma 3.1 we get

$$\int_{\Omega} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx = \int_{\Omega \cap G} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx + \int_{\partial D} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx$$

$$\leq \sum_{k=0}^{\infty} \gamma^k \int_{\Omega \cap G} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx \leq c \int_{\Omega} A \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right) \, dx \, dy.$$ 

This completes the proof of (1).

(2) Let for each $j \geq 0$, $k_j$ be the largest nonnegative integer such that $j - k_jm \geq 0$, so that $0 \leq j - k_jm \leq m - 1$. Then $k_j$-many repeated applications of the inequality in the hypothesis gives

$$\int_{A_j \cap G} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx \leq \gamma \int_{A_{j-m}} A \left( \frac{|f(y)|}{\delta_y^s} \right) \, dy$$

$$= \gamma \left[ \int_{A_{j-m} \backslash G} A \left( \frac{|f(y)|}{\delta_y^s} \right) \, dy + \int_{A_{j-m} \cap G} A \left( \frac{|f(y)|}{\delta_y^s} \right) \, dy \right]$$

$$\leq \cdots \leq \gamma^{k_j} \int_{A_{j-k_jm} \cap G} A \left( \frac{|f(y)|}{\delta_y^s} \right) \, dy + \sum_{k=1}^{k_j} \gamma^k \int_{A_{j-km} \cap G} A \left( \frac{|f(y)|}{\delta_y^s} \right) \, dy$$

Summing over $j$,

$$\int_{\Omega \cap G} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx \leq \gamma \left[ \int_{A_{j} \cup \cdots \cup A_{m-1}} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx + \int_{\partial D} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx \right].$$ 

Proceeding as in part (1), the lemma follows.

Before proving theorem 1.2, we need a geometric decomposition of a bounded Lipschitz domain $D$, given in [Dyd04, p. 581]. We outline the construction for the sake of completeness.

Let us denote $x = (x_1, x_2, \cdots, x_{N-1}, x_N) = (\tilde{x}, x_N) \in \mathbb{R}^N$ with $\tilde{x} \in \mathbb{R}^{N-1}$, $x_N \in \mathbb{R}$. $D \subseteq \mathbb{R}^N$ shall be assumed to be a bounded Lipschitz domain throughout the rest of the section. For any $z \in \partial D$ there are linear isometry $L_z : \mathbb{R}^N \rightarrow \mathbb{R}^N$ and Lipschitz function $\phi_z : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ such that $\text{Lip}(\phi_z) \leq \text{Lip}(D)$ and

$$L_z(D) \cap B(L_z(z), r_0) = \{ x \in \mathbb{R}^N \mid x_N > \phi_z(\tilde{x}) \} \cap B(L_z(z), r_0),$$

for some positive $r_0$ which depends only on $D$(existence of $r_0$ is guaranteed because $\partial D$ is compact).

Without loss of generality, we can assume $L_z$ to be the identity map. Because otherwise we can work with the Lipschitz domain $L_z(D)$ and the point $L_z(z)$, then pull back the construction to the original domain $D$ and the point $z$ via $L_z^{-1}$. For $x \in \mathbb{R}^N$ we set

$$V_z(x) := |x_N - \phi_z(\tilde{x})|.$$
For \( E \subseteq \mathbb{R}^{N-1} \) and \( r > 0 \) define
\[
Q_z(E, r) := \{x \in D \mid \hat{x} \in E, 0 < V_z(x) \leq r\}.
\]
Set\[
K_r := \left\{ x \in \mathbb{R}^{N-1} \mid |x_\ell - z_\ell| \leq \frac{r}{2}, \ \ell = 1, 2, \ldots, N - 1 \right\}.
\]
That is \( K_r \) is an \( N - 1 \) dimensional square of side-length \( r \). \( Q_z(K_r, r) \) is referred to as a Lipschitz box and is denoted by \( Q_z(r) \). We now choose \( \rho > 0 \) small enough such that \( Q_z(\rho) \subseteq D \cap B(\hat{z}, r_0/2) \). Let \( x \in Q_z(\rho) \). Then using the fact that \( \phi_z(w) = w_N \) for any \( w \in \partial D \cap B(\hat{z}, r_0) \),
\[
V_z(x) = |x_N - \phi_z(\hat{x})| \leq |x_N - w_N| + |\phi_z(\tilde{w}) - \phi_z(\hat{x})| \leq |x_N - w_N| + \text{Lip}(D)|\tilde{w} - \hat{x}| \leq (1 + \text{Lip}(D))|w - x|.
\]
Since \( w \) is arbitrary, we get
\[
\frac{V_z(x)}{(1 + \text{Lip}(D))} \leq \delta_x \leq V_z(x), \quad \forall \ x \in Q_z(\rho).
\]
For \( j \in \mathbb{N} \cup \{0\} \) consider the dyadic decomposition of \( K_\rho \) (by dissecting sides first and then decomposing the cube into smaller cubes and proceeding by induction) into the union of \( (N - 1) \)-dimensional cubes \( K_i^j \) for \( 1 \leq i \leq 2^{k(N-1)} \). Now for each \( j \), \( K_i^j \)'s have disjoint interiors and have sides-length \( \frac{\rho}{2^j} \). Set \( Q_i^j := Q_z(K_i^j, \rho) \) and define
\[
A_k = Q_z(K_\rho, \rho/2^k) \setminus Q_z(K_\rho, \rho/2^{k+1}), \quad \text{for } k \in \mathbb{N} \cup \{0\}.
\]
Then \( A_k \)'s are mutually disjoint sets with \( \bigcup_{k=0}^\infty A_k = Q_z(\rho) \). Thus the Lipschitz box \( Q_z(\rho) \) is the union of sets \( A_j \cap Q_i^j \), \( i, j \in \mathbb{N} \) whose pairwise intersections are measure zero sets. Moreover using eqs. (3.2) and (3.3), for \( k \geq j \), we have
\[
\frac{\rho}{2^{k+1}(1 + \text{Lip}(D))} \leq \delta_x \leq \frac{\rho}{2^j}, \quad \text{for } x \in A_k \cap Q_i^j
\]
and
\[
\mathcal{L}^N(A_k \cap Q_i^j) = \left( \frac{\rho}{2^j} - \frac{\rho}{2^{k+1}} \right) \left( \frac{\rho}{2^j} \right)^{N-1} = \frac{\rho^N}{2^{k+1+jN-j}}.
\]
Let \( x \in A_k \cap Q_i^j \) and \( y \in A_j \cap Q_i^j \) where \( k \geq j \). Then \( \frac{\rho}{2^{k+1}} \leq |x_N - \phi_z(\hat{x})| \leq \frac{\rho}{2^j} \) and \( \frac{\rho}{2^j} + \frac{\rho}{2^{k+1}} \leq |y_N - \phi_z(\hat{y})| \leq \frac{\rho}{2^j} \). Moreover both \( x \) and \( y \) lie 'above' the same dyadic cube \( Q_i^j \) which has side length \( \frac{\rho}{2^j} \). So for \( 1 \leq \ell \leq N - 1, |x_\ell - y_\ell| \leq \frac{\rho}{2^j} \). Therefore \( |\hat{x} - \hat{y}| \leq \sqrt{N - 1} \frac{\rho}{2^j} \). Again, using these
\[
|x_N - y_N| = |x_N - \phi_z(\hat{x}) + \phi_z(\hat{x}) - \phi_z(\hat{y}) + \phi_z(\hat{y}) - y_N| \leq \frac{\rho}{2^j} + \text{Lip}(D)|\hat{x} - \hat{y}| + \frac{\rho}{2^j} \leq \frac{\rho}{2^{j-1}} + \text{Lip}(D)\sqrt{N - 1} \frac{\rho}{2^j}.
\]
So we finally get
\[
|x - y|^2 \leq \left( \sqrt{N - 1} \frac{\rho}{2^j} \right)^2 + \left( \frac{\rho}{2^{j-1}} + \text{Lip}(D)\sqrt{N - 1} \frac{\rho}{2^j} \right)^2 = \left( \frac{\rho}{2^j} \right)^2 \left( N - 1 + \left(2 + \text{Lip}(D)\sqrt{N - 1}\right)^2 \right).
\]
This gives us for \( k \geq j \),
\[
|x - y| \leq \frac{\rho}{2^j} \sqrt{N - 1 + \left(\text{Lip}(D)\sqrt{N - 1} + 2\right)^2} \quad \text{for } x \in A_k \cap Q_i^j \text{ and } y \in A_j \cap Q_i^j.
\]
For the next result we consider $Q = Q_z(\rho)$ defined above.

**Lemma 3.5.** Let $\lim_{\lambda \to 0^+} \alpha_{s,A}(\lambda) = 0$, where $\alpha_{s,A}$ is as in eq. (1.4). Then there exists a constant $c = c(D, N, A, s) > 0$ such that for all $f \in C_c^\infty(D)$ we have

$$\int_{Q_z(\rho)} A\left(\frac{|f(x)|}{\delta_x^s}\right) dx \leq c \int_{Q_z(\rho) \times Q_z(\rho)} A\left(\frac{|f(x) - f(y)|}{|x - y|^s}\right) dx dy.$$

**Proof.** From the hypothesis on $s, A$ it follows that there exists $m \in \mathbb{N}$ such that

$$2^{2+p}(1 + \text{Lip}(D)) \alpha_{s,A}(2^{s-m}s(1 + \text{Lip}(D))^s) < \frac{1}{2}.$$

Set

$$l_1 = 2(1 + \text{Lip}(D))\sqrt{N - 1 + (\text{Lip}(D))\sqrt{N - 1 + 2}^2} \quad \text{and} \quad l_2 = \frac{1}{2^{m+1}}.$$

Let $f \in C_c^\infty(D)$ and $G = G(f, Q_z(\rho); l_1, l_2)$. Set $E_1 := A_j \cap Q_i^j$ and $E_2 := A_{j+m} \cap Q_j^i$. For these choices of $l_1$ and $l_2$, from eqs. (3.4) and (3.5), we get $\mathcal{L}^N(E_2) = \frac{\rho^N}{2^{m+1}}$ and $\delta_x \leq \frac{\rho}{2^{m+1}}$ for $x \in E_2$. Also utilizing eqs. (3.4) and (3.6), $E_2 \subseteq B(x, l_1 \delta_x)$. Thus the sets $E_1$ and $E_2$ satisfy the hypotheses of lemma 3.3. Thus we obtain, using lemma 3.3, eq. (3.4) and the choice of $m$,

$$\int_{(A_j \cap Q_i^j) \cap G} A\left(\frac{|f(x)|}{\delta_x^s}\right) dx \leq 2^{1+p+m} \int_{A_j \cap Q_i^j} A\left(2^{s-m}s(1 + \text{Lip}(D))^s\frac{|f(y)|}{\delta_y^s}\right) dy$$

$$= 2^{2+p}(1 + \text{Lip}(D)) \int_{A_{j+m} \cap Q_j^i} A\left(2^{s-m}s(1 + \text{Lip}(D))^s\frac{|f(y)|}{\delta_y^s}\right) dy$$

$$\leq 2^{2+p}(1 + \text{Lip}(D)) \alpha_{s,A}(2^{s-m}s(1 + \text{Lip}(D))^s) \int_{A_{j+m} \cap Q_j^i} A\left(\frac{|f(y)|}{\delta_y^s}\right) dy \leq \frac{1}{2} \int_{A_{j+m} \cap Q_j^i} A\left(\frac{|f(y)|}{\delta_y^s}\right) dy.$$
Proof of Theorem 1.2. Consider the following two sets

\[ D_1 = \{ x \in D \mid \delta_x \geq \tilde{\rho} \}, \quad D_2 = \{ x \in D \mid \delta_x < \tilde{\rho} \}, \]

where \( \tilde{\rho} > 0 \) is sufficiently small. Then by Lemma 3.5 and compactness of \( \partial D \), we have

\[ \int_{D_2} A \left( \frac{\|f(x)\|}{\delta_x^s} \right) dx \leq c \int_{D_1 \cap G} \int_{D_2} A \left( \frac{\|f(x) - f(y)\|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N}, \]

since \( D_2 \) may be covered by sets of the form \( Q_{\delta_x}(\rho) \) such that every \( x \in D_2 \) belongs to at-most \( \sigma = \sigma(D,N) \in \mathbb{N} \) sets of type \( Q_{\delta_x}(\rho) \). This is possible for sufficiently small \( \tilde{\rho} < \rho \), and such a \( \tilde{\rho} \) may be chosen to depend only on \( \text{Lip}(D), N, r_0 \). We now take \( G = G(f,D;l_1,l_2) \), where \( l_1 = \frac{\text{diam}(D)}{\rho} \) and \( l_2 = \frac{L^N(D_2)}{\text{diam}(D)^N} \). Set \( E_1 = D_1 \) and \( E_2 = D_2 \). Then for any \( x \in E_1 \),

\[ l_1 \delta_x = \text{diam}(D) \frac{\delta_x}{\rho} \geq \text{diam}(D), \]

which implies \( E_2 \subseteq B(x,l_1\delta_x) \) and

\[ L^N(E_2) = \text{diam}(D)^N l_2 \geq \delta_x'^N l_2. \]

We can now apply Lemma 3.3 to get

\[ \int_{D_1 \cap G} A \left( \frac{\|f(x)\|}{\delta_x^s} \right) dx \leq c(E_1,E_2) \int_{D_2} \int_{D_1 \cap G} A \left( \frac{\|f(x) - f(y)\|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N}, \]

and also from Lemma 3.1 we get

\[ \int_{D_1 \setminus G} A \left( \frac{\|f(x)\|}{\delta_x^s} \right) dx \leq c \int_{D_1 \setminus G} \int_{D_1 \setminus G} A \left( \frac{\|f(x) - f(y)\|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N}. \]

Combining the estimates eq. (3.8), eq. (3.9) and eq. (3.10) we conclude \( H_{N,s,A}(D) > 0 \). If \( A \in \Delta_2 \), then using Lemma 2.4, \( P_{N,s,A}(D) > 0 \). This completes the proof.

Proof of Theorem 1.3. For any \( \varepsilon \in (0,1) \) denote \( D_\varepsilon := \{ x \in D \mid \text{dist}(x,\partial D) < \varepsilon \} \). Let \( f_\varepsilon \in C^\infty_c(D) \) be such that \( f_\varepsilon \equiv 1 \) on \( D \setminus D_\varepsilon \), \( 0 \leq f_\varepsilon \leq 1 \) on \( D \) and \( \nabla f_\varepsilon < \frac{\rho}{2} \) on \( D \) for some constant \( c_1 > 0 \). Now

\[
\begin{align*}
&\iint_{D \times D} A \left( \frac{|f_\varepsilon(x) - f_\varepsilon(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N} = \iint_{D_\varepsilon \times D_\varepsilon} A \left( \frac{|f_\varepsilon(x) - f_\varepsilon(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N} \\
&\quad + 2 \iint_{D_\varepsilon \times D_\varepsilon} A \left( \frac{|f_\varepsilon(x) - f_\varepsilon(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N} \leq 3 \iint_{D \times D} A \left( \frac{|f_\varepsilon(x) - f_\varepsilon(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N} \\
&\quad \leq 3 \int_{D_\varepsilon} \left\{ \iint_{y \in D_\varepsilon,|x-y|<\varepsilon} A \left( \frac{|f_\varepsilon(x) - f_\varepsilon(y)|}{|x - y|^s} \right) \frac{dy}{|x - y|^N} \right\} dx \\
&\quad + \int_{y \in D_\varepsilon,|x-y|>\varepsilon} A \left( \frac{|f_\varepsilon(x) - f_\varepsilon(y)|}{|x - y|^s} \right) \frac{dy}{|x - y|^N} dx = 3 \int_{D_\varepsilon} (I_1(x) + I_2(x)) dx.
\end{align*}
\]
Using $|\nabla f_\varepsilon| < \frac{c_1}{\varepsilon}$ on $D$, we get

$$I_1(x) = \int_{y \in D, |x-y| < \varepsilon} A\left(\frac{|f_\varepsilon(x) - f_\varepsilon(y)|}{|x-y|^s}\right) \frac{dy}{|x-y|^N} \leq \int_{|x-y| < \varepsilon} A\left(\frac{c_1|x-y|^{1-s}}{\varepsilon}\right) \frac{dy}{|x-y|^N}$$

$$= \int_{|y| < \varepsilon} A\left(\frac{c_1|y|^{1-s}}{\varepsilon}\right) \frac{dy}{|y|^N} \leq c \int_{0}^{\varepsilon} A\left(\frac{c_1r^{1-s}}{\varepsilon}\right) \frac{dr}{r} = c \int_{0}^{\varepsilon} \frac{A(z)}{z} dz.$$

The fact that $0 \leq f_\varepsilon(x) \leq 1$ on $D$ gives

$$I_2(x) = \int_{y \in D, |x-y| > \varepsilon} A\left(\frac{|f_\varepsilon(x) - f_\varepsilon(y)|}{|x-y|^s}\right) \frac{dy}{|x-y|^N} \leq \int_{|x-y| > \varepsilon} A(2|x-y|^{-s}) \frac{dy}{|x-y|^N}$$

$$= c \int_{\varepsilon}^{\infty} \frac{A(2r^{-s})}{r} dr \leq c \int_{0}^{2\varepsilon} \frac{A(z)}{z} dz.$$

The hypothesis on $D$ that it has bounded Lipschitz boundary implies that $\mathcal{L}^N(D_\varepsilon)$ is bounded above as well as bounded below by a constant multiple of $\varepsilon$. Let $\frac{1}{c_2} := \max\{c_1, 2\}$. Then we have, as $\varepsilon \to 0$,

$$\int_{D \times D} A\left(\frac{|f_\varepsilon(x) - f_\varepsilon(y)|}{|x-y|^s}\right) \frac{dx dy}{|x-y|^N} \leq c \mathcal{L}^N(D_\varepsilon) \int_{0}^{(c_2\varepsilon)^{-s}} \frac{A(z)}{z} dz \leq c \int_{0}^{(c_2\varepsilon)} \frac{A(z)}{z} dz \to \beta.$$ 

Now $f_\varepsilon \in C^c_c(D)$, $f_\varepsilon \to 1$ pointwise a.e. in $D$ and $A$ is continuous. Therefore by Fatou’s lemma,

$$\lim_{\varepsilon \to 0} \int_{D} A\left(\frac{|f_\varepsilon(x)|}{\delta^s_x}\right) \frac{dx}{\delta^s_x} \geq \int_{D} A\left(\frac{1}{\delta^s_x}\right) \frac{dx}{\delta^s_x} > 0$$

and

$$\lim_{\varepsilon \to 0} \int_{D} A(|f_\varepsilon(x)|) \frac{dx}{\delta^s_x} \geq A(1) \mathcal{L}^N(D) > 0.$$

Now to prove (1) assume $\beta = 0$. Then LHS of eq. (3.11), which is also the numerator in the definition of both $H_{N,s,A}$ and $P_{N,s,A}$, converges to 0. We now use eqs. (3.12) and (3.13) to conclude $H_{N,s,A}(D) = 0$ and $P_{N,s,A}(D) = 0$ respectively. This proves (1).

Now we prove (2). Applying L’Hospital rule to eq. (1.5), we get $\lim_{\varepsilon \to 0} A(\varepsilon^{-s}) = \beta$. So there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $A(n^s) \geq \frac{\beta n}{n+1}$. Using eq. (3.12) we have

$$\lim_{n \to 0} \int_{D} A\left(\frac{|f_{\frac{1}{n}}(x)|}{\frac{1}{n}}\right) \frac{dx}{\frac{1}{n}} \geq \int_{D} A\left(\frac{1}{\frac{1}{n}}\right) \frac{dx}{\frac{1}{n}} \geq \sum_{n=n_0}^{\infty} \int_{D} A\left(\frac{1}{\frac{1}{n}}\right) \frac{dx}{\frac{1}{n}} \geq \sum_{n=n_0}^{\infty} A(n^s) \frac{dx}{\frac{1}{n}} = \sum_{n=n_0}^{\infty} \frac{\beta n}{2} \mathcal{L}^N\left(D \cap D_{\frac{1}{n+1}}\right) = \sum_{n=n_0}^{\infty} \frac{\beta n}{2} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \infty.$$ 

The proof follows after observing that $\beta > 0$ in eq. (3.11). □

The conclusions we can draw as an application of theorems 1.2 and 1.3, for any bounded Lipschitz domain $D$ and for any $q > 1$ are shown yellow in table 1.
Table 1. Conclusions we can draw from theorems 1.2 and 1.3

| \( A(t) \) | \( H_{N,s,A}(D) > 0 \) | \( H_{N,s,A}(D) = 0 \) | \( P_{N,s,A}^{1}(D) > 0 \) | \( P_{N,s,A}^{1}(D) = 0 \) |
|----------|------------------|------------------|------------------|------------------|
| \( t^q \) | \( s \in (\frac{1}{q}, 1) \) | \( s \in (0, \frac{1}{q}) \) | \( s \in (\frac{1}{q}, 1) \) | \( s \in (0, \frac{1}{q}) \) |
| \( t^q(1 + |\log t|) \) | \( s \in (\frac{1}{q}, 1) \) | \( s \in (0, \frac{1}{q}) \) | \( s \in (\frac{1}{q}, 1) \) | \( s \in (0, \frac{1}{q}) \) |
| \( \frac{t^q}{\log(e+t)} \) | \( s \in (\frac{1}{q}, 1) \) | \( s \in (0, \frac{1}{q}) \) | \( s \in (\frac{1}{q}, 1) \) | \( s \in (0, \frac{1}{q}) \) |
| \( (1 + t) \log(1 + t) - t \) | NA | \( s \in (0, 1) \) | NA | \( s \in (0, 1) \) |

4. Proof of Theorems 1.4 and 1.5

In this section we prove theorems 1.4 and 1.5. The notations and conventions followed in this section will be the same as that in section 3. We need lemmas 4.1 and 4.2 to prove theorem 1.4 and to prove theorem 1.5 we further need lemma 4.3.

Lemma 4.1. Suppose \( \liminf \lambda^{-\frac{1}{N}} \alpha_{s,A}(\lambda) = 0 \). Then there exists a constant \( c = c(s, A, N) > 0 \) such that for any \( r > 0 \) and any \( f \in C_c^\infty(\mathbb{R}^N) \),

\[
\int_{B(0,r)^c} A \left( \frac{|f(x)|}{|x|^s} \right) dx \leq c \int_{B(0,r)^c \times B(0,r)^c} A \left( \frac{|f(x) - f(y)|}{|x - y|^q} \right) \frac{dydx}{|x - y|^N}.
\]

Proof. We can assume \( f \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \), because for the sake of proving the above inequality the value of \( f \) inside \( B(0, r) \) does not matter. Let \( D := \mathbb{R}^N \setminus \{0\} \) so that \( \delta_x = |x|, \Omega := B(0, r)^c, l_1 = 2^{m+1}, l_2 = \mathcal{L}^N(B(0, 1))2^{mN}(2^N - 1), G := G(f, \Omega; l_1, l_2) \) and \( A_k = B(0, 2^{k+1}r) \setminus B(0, 2^kr) \) for \( k \in \mathbb{N} \cup \{0\} \). For a fixed \( j \in \mathbb{N} \cup \{0\} \), set \( E_1 = A_j \) and \( E_2 = A_{j+m} \). Then for \( x \in E_1, 2^{j}r \leq \delta_x \leq 2^{j+1}r \), which gives \( \text{dist}(x, A_{j+m}) \leq 2^{j+1}r - 1 \leq 2^{j+1}r \). Thus \( \mathcal{L}^N(A_k) = \mathcal{L}^N(B(0, 1))2^{k+N}(2^N - 1) \). So \( E_1, E_2 \) satisfy the hypotheses of lemma 3.3. We can then conclude

\[
\int_{A_j \cap G} A \left( \frac{|f(x)|}{\delta_y^s} \right) dx \leq 2^{p+1-mN} \int_{A_{j+m}} \frac{\mathcal{L}^N(A_j)}{\mathcal{L}^N(A_{j+m})} \int_{A_{j+m}} \left( \frac{\sup\{\delta_y^s \ | \ x \in A_{j+m} \} \cdot |f(y)|}{\inf\{\delta_y^s \ | \ x \in A_j \}} \right) dy
\]

\[
= 2^{p+1-mN} \int_{A_{j+m}} A \left( \frac{(2^{j+m+1}r)^s |f(y)|}{(2^j)^s \delta_y^s} \right) dy = 2^{p+1-mN} \int_{A_{j+m}} A \left( \frac{(2^{m+1}r)^s |f(y)|}{\delta_y^s} \right) dy
\]

\[
= 2^{p+1+N} \int_{A_{j+m}} A \left( \frac{(2^{m+1}r)^s |f(y)|}{2^{(m+1)N} A \left( \frac{|f(y)|}{\delta_y^s} \right)} \right) A \left( \frac{|f(y)|}{\delta_y^s} \right) dy
\]

\[
= 2^{p+1+N}2^{(m+1)(1-N)} \alpha_{s,A}(2^{(m+1)s}) \int_{A_{j+m}} A \left( \frac{|f(y)|}{\delta_y^s} \right) dy.
\]

Now we use the hypothesis and choose large enough \( m = m(s, A, N) \in \mathbb{N} \), so that

\[
\int_{A_j \cap G} A \left( \frac{|f(x)|}{\delta_y^s} \right) dx \leq \frac{1}{2} \int_{A_{j+m}} A \left( \frac{|f(y)|}{\delta_y^s} \right) dy.
\]

Now we can apply lemma 3.4 to complete the proof. □
Lemma 4.2. Suppose \( \liminf_{\lambda \to 0+} \lambda^{\frac{1-N}{s}} \alpha_{s,A}(\lambda) = 0 \). Then there exists a constant \( c = c(s,A,N) > 0 \) and \( m = m(s,A,N) \in \mathbb{N} \) such that for any \( r > 0 \) and any \( f \in C^\infty_c(\mathbb{R}^N) \),

\[
\int_{B(0,2^m r)^c} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx \\
\leq c \left[ \int_{B(0,2^m r)^c \times B(0,2^m r)} A \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right) \, dy \, dx + \int_{B(0,2^m r) \setminus B(0,r)} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx \right].
\]

Proof. From the hypothesis on \( s \) and \( A \), there exists \( m \in \mathbb{N} \) such that

\[
2^{p+1+N} 2(1-m)(1-N) \alpha_{s,A} \left( 2(1-m)s \right) < \frac{1}{2}.
\]

We fix this \( m \). Let \( D = C^\infty_c(\mathbb{R}^N \setminus \{0\}) \), \( \Omega = B(0,r)^c \), \( l_1 = 2 \), \( l_2 = \mathcal{L}^N(B(0,1))2^{-mN}(2^N - 1) \), \( G = G(f,\Omega;l_1,l_2) \) and \( A_k = B(0,2^{k+1} r) \setminus B(0,2^k r) \), then \( \mathcal{L}^N(A_k) = \mathcal{L}^N(B(0,1))2^{kN}r^N(2^N - 1) \). As in the previous lemma, we assume \( f \in C^\infty_c(D) \). For \( j \geq m \), set \( E_j = A_j \) and \( E_2 = A_j - m \). So for \( x \in A_j \), dist\((x,A_j - m) \leq 2^{j+1}r < l_1 \delta_x \). Also \( \frac{\alpha_{s,A}}{\delta_x} \geq \mathcal{L}^N(B(0,1))2^{-mN}(2^N - 1) = l_2 \). Hence, for any \( j \geq m \), we can apply lemma 3.3 to get

\[
\int_{A_j \cap G} A \left( \frac{|f(x)|}{|x|^s} \right) \, dx \\
\leq 2^{p+1+mN} \int_{A_j - m} A \left( \frac{2(1-m)s}{|y|^s} |f(y)| \right) \, dy \\
= 2^{p+1+N} \int_{2^{(1-m)N}} A \left( 2(1-m)s |f(y)| \right) \, dy \\
\leq 2^{p+1+N} 2(1-m)(1-N) \alpha_{s,A} \left( 2(1-m)s \right) \int_{A_j - m} A \left( |f(y)| \right) \, dy < \frac{1}{2} \int_{A_j - m} A \left( |f(y)| \right) \, dy.
\]

Finally we apply lemma 3.4 to complete the proof. \( \square \)

Proof of Theorem 1.4. Set \( D = \mathbb{R}^N \setminus \{0\} \), take \( f \in C^\infty_c(D) \). Then \( f \equiv 0 \) near \( 0 \). First let us assume \( \liminf_{\lambda \to 0} \lambda^{\frac{1-N}{s}} \alpha_{s,A}(\lambda) = 0 \). Choose \( r > 0 \) small enough, such that, supp\((f) \subseteq B(0,r)^c \). Then lemma 4.1 implies \( H_{N,s,A}(D) > 0 \).

If, however, \( \liminf_{\lambda \to 0} \lambda^{\frac{1-N}{s}} \alpha_{s,A}(\lambda) = 0 \), we choose \( r > 0 \) so small that supp\((f) \subseteq B(0,2^m r)^c \). Proceed exactly same as in the previous case and apply lemma 4.2. The restriction on supp\((f) \) ensures that the last term in the inequality is zero. Hence \( H_{N,s,A}(\mathbb{R}^N \setminus \{0\}) > 0 \). \( \square \)

We shall the notations as in the geometric decomposition of \( D \), done after lemma 3.3, for lemma 4.3.

Lemma 4.3. For a fixed \( z \in \partial D \), define \( Q_{k} := \cup_{\lambda \to \infty} A_{\lambda} \), where \( A_{\lambda} \) are as in eq. (3.3). Let \( \liminf_{\lambda \to \infty} \alpha_{s,A}(\lambda) = 0 \). Then there exists \( m = m(Lip(D),s,A) \) and \( c = c(Lip(D),s,A) \) such that for any \( f \in C^\infty_c(D) \),

\[
\int_{Q_m} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx \\
\leq c \left[ \int_{Q_m \times Q_m} A \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right) \, dy \, dx + \int_{Q_{\rho} \setminus Q_m} A \left( \frac{|f(x)|}{\delta_x^s} \right) \, dx \right].
\]

Proof. From the hypothesis on \( s \) and \( A \), there exists \( m \in \mathbb{N} \) such that

\[
2^{2+p}(1 + \text{Lip}(D))\alpha_{s,A}(2^{s+m}s(1 + \text{Lip}(D))^s) < \frac{1}{2}.
\]

We fix this \( m \). Let \( l_1 = 2(1 + \text{Lip}(D))\sqrt{N - 1 + (\text{Lip}(D))\sqrt{N-1} + 2^2} \), \( l_2 = 2^{m-1} \) and \( G = G(f,Q_z(\rho); l_1,l_2) \). For \( j \geq m \), set \( E_1 = A_j \cap Q_j^i \) and \( E_2 = A_j \cap \mathbb{R} \setminus Q_j^i \). Equations (3.4) and (3.5) imply that we can apply lemma 3.3 on \( E \). Thus, we get, utilizing the choice of \( m \),

\[
\int_{(A_j \cap Q_j^i) \cap G} A\left(\frac{|f(x)|}{\delta_x^s}\right) dx \leq 2^{1+p-m} \int_{A_j \cap \mathbb{R} \setminus Q_j^i} A\left(2^{s+m}(1 + \text{Lip}(D))^s\frac{|f(y)|}{\delta_y^s}\right) dy
\]

\[
= 2^{2+p}(1 + \text{Lip}(D)) \int_{A_j \cap \mathbb{R} \setminus Q_j^i} A\left(2^{s+m}(1 + \text{Lip}(D))^s\frac{|f(y)|}{\delta_y^s}\right) A\left(\frac{|f(y)|}{\delta_y^s}\right) dy
\]

\[
\leq 2^{2+p}(1 + \text{Lip}(D))\alpha_{s,A}(2^{s+m}(1 + \text{Lip}(D))^s) \int_{A_j \cap \mathbb{R} \setminus Q_j^i} A\left(\frac{|f(y)|}{\delta_y^s}\right) dy < \frac{1}{2} \int_{A_j \cap \mathbb{R} \setminus Q_j^i} A\left(\frac{|f(y)|}{\delta_y^s}\right) dy.
\]

Summing over \( 1 \leq i \leq 2^{j(N-1)} \), we get

\[
\int_{A_j \cap G} A\left(\frac{|f(x)|}{\delta_x^s}\right) dx \leq \frac{1}{2} \int_{A_j \cap \mathbb{R} \setminus Q_j^i} A\left(\frac{|f(y)|}{\delta_y^s}\right) dy.
\]

Applying lemma 3.4 the lemma follows. \( \square \)

**Proof of Theorem 1.5.** (1) Without loss of generality, we can assume \( D = \mathbb{R}^{N-1} \times \{0\} \). Let \( z \in D \). First assume the case \( \liminf_{\lambda \to 0} \alpha_{s,A}(\lambda) = 0 \). Take \( f \in C_c^\infty(D) \). Note that for the choice of \( D \) in this case, we are free to choose \( \rho \) as large as we want and we still have \( Q_z(\rho) \subset D \). Choose \( \rho > 0 \) so that \( \text{supp}(f) \subset Q_z(\rho) \). Then applying lemma 3.5 we get

\[
\int_D A\left(\frac{|f(x)|}{\delta_x^s}\right) dx = \int_{Q_z(\rho)} A\left(\frac{|f(x)|}{\delta_x^s}\right) dx \leq c \int_{Q_z(\rho) \times Q_z(\rho)} A\left(\frac{|f(x) - f(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N}
\]

\[
\leq c \int_{D \times D} A\left(\frac{|f(x) - f(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N}.
\]

Now assume \( \liminf_{\lambda \to \infty} \alpha_{s,A}(\lambda) = 0 \). Lemma 4.3 implies

\[
\int_{0 < x_N < \frac{\rho_N}{m\rho}} A\left(\frac{|f(x)|}{\delta_x^s}\right) dx \leq c \int_{0 < x_N, y_N < \frac{\rho_N}{m\rho}} A\left(\frac{|f(x) - f(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N} + \int_{x_N > \frac{\rho_N}{m\rho}} A\left(\frac{|f(x)|}{\delta_x^s}\right) dx,
\]

where \( m, \rho \) is the integer considered in lemma 4.3. Note that \( m \) does not depend on \( \rho \). So we choose \( \rho \) big enough, so that \( \text{supp}(f) \subset \{x \in \mathbb{R}^N \mid x_N < \frac{\rho_N}{m\rho}\} \) forcing the remainder term in the above equation to vanish.

(2) Let \( f \in C_c^\infty(D) \). We choose \( R > 0 \) large enough so that \( D^c \subset B(0,R) \). Now \( D \cap B(0,R) \) is a bounded Lipschitz domain. A careful observation of the proof of theorem 1.2 reveals that, under the assumption \( \liminf_{\lambda \to 0} \alpha_{s,A}(\lambda) = 0 \), we can use the same technique to get a constant \( c = c(A,s,N,D) \) such that

\[
(4.1) \quad \int_{D \cap B(0,R)} A\left(\frac{|f(x)|}{\delta_x^s}\right) dx \leq c \int_{D \cap B(0,R) \times D \cap B(0,R)} A\left(\frac{|f(x) - f(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N},
\]
even if $f \notin C_c^\infty(D \cap B(0, R))$.

First we assume $\liminf_{\lambda \to \infty} \lambda^{-N} \alpha_{s,A}(\lambda) = \liminf_{\lambda \to 0^+} \lambda^{-N} \alpha_{s,A}(\lambda) = 0.$

We apply lemma 4.1 to get, for any $r = R$,

$$\int_{B(0,R)^c} A \left( \frac{|f(x)|}{\delta_x^s} \right) dx \leq c \int_{B(0,R)^c \times B(0,R)^c} A \left( \frac{|f(x) - f(y)|}{|x-y|^s} \right) dydx.$$ 

Adding eqs. (4.1) and (4.2), the claim follows.

Now we assume $\liminf_{\lambda \to 0^+} \lambda^{-N} \alpha_{s,A}(\lambda) = 0$. Note that in this case $\liminf_{\lambda \to 0^+} \alpha_{s,A}(\lambda) = 0$ is implied. Consider the extension of $f$ by 0 to whole of $\mathbb{R}^N$. Applying lemma 4.2 with $r = \frac{R}{2^m}$ and then proceeding similarly as above, the claim follows.

Finally, assume $\liminf_{\lambda \to \infty} \alpha_{s,A}(\lambda) = 0$. Denote $\forall \eta > 0, D_\eta := \{ x \in D \mid \text{dist}(x, \partial D) < \eta \}$. An application of lemma 4.3, as lemma 3.5 was used in the proof of theorem 1.2, gives

$$\int_{D_\delta} A \left( \frac{|f(x)|}{\delta_x^s} \right) dx \leq c \left[ \int_{D_\delta \times D_\delta} A \left( \frac{|f(x) - f(y)|}{|x-y|^s} \right) dydx + \int_{D_\delta \backslash D_\delta} A \left( \frac{|f(x)|}{\delta_x^s} \right) dx \right],$$

for some fixed $\varepsilon > 0$ and a constant $c = c(s, N, A, D) > 0$ and or any $f \in C_c^\infty(D)$. Now set $\Omega := D \backslash D_\delta$ and let $R > 0$ be such that $\Omega^c \subseteq B(0, R)$. Before proceeding further note that if, for some constant $c = c(s, N, A, D) > 0$ and for all $f \in C_c^\infty(D)$, we can show

$$\int_\Omega A \left( \frac{|f(x)|}{\delta_x^s} \right) dx \leq c \int_{\Omega \times \Omega} A \left( \frac{|f(x) - f(y)|}{|x-y|^s} \right) dydx,$$

then the proof will be complete. Set $A_j := B(0, 2^j R) \cap B(0, 2^{j-1} R)$ and $A_0 := B(0, R) \cap \Omega$. Then for $k \in \mathbb{N}$, $\mathcal{L}_c^N(A_k) = \mathcal{L}_c^N(B(0, 1) R^{2k-1} N - 1)$ and for any $x \in A_k$, $\varepsilon > 0$, $A_k \subseteq B(0, R)$. Whereas $\mathcal{L}_c^N(A_0) = \mathcal{L}_c^N(B(0, R)) - \mathcal{L}_c^N(\Omega^c)$, for $x \in A_0 \cap A_1, \varepsilon \leq \delta_x \leq R$ and for $x \in A_1$, $\varepsilon \leq \delta_x \leq 2R$. Now let $x \in A_j, y \in A_{j+m}$. Then for any $j \geq 0$, $\text{dist}(x, y) \leq \delta_c x \leq 2^{j+m} R = 2^m 2^{j+m} e^{-1}$. So it is clear that there exists $l_1, l_2 > 0$ such that the hypotheses of lemma 3.3 are satisfied. So, we get

$$\int_{A_j \cap G} A \left( \frac{|f(x)|}{\delta_x^s} \right) dx \leq 2^{p+1} \mathcal{L}_c^N(A_j) \int_{A_{j+m}} A \left( \frac{\sup \delta_x^s \{ f(x) \mid x \in A_{j+m} \}}{\inf \delta_y^s \{ f(x) \mid x \in A_j \}} \right) dy$$

$$\leq c(R, \mathcal{L}_c^N(\Omega^c), N) 2^{-mN} \int_{A_{j+m}} A \left( \frac{\sup \delta_x^s \{ f(x) \mid x \in A_{j+m} \}}{\inf \delta_y^s \{ f(x) \mid x \in A_j \}} \right) dy$$

$$\leq c 2^{-mN} \int_{A_{j+m}} A \left( \frac{c(R) 2^{mN} \delta_x^s \{ f(x) \}}{\delta_y^s \{ f(x) \}} \right) dy.$$
which implies
Lemma 5.1.
Let $x \in G$ another change of variable to polar coordinates. We then get
\[ \int_{A_{j+m}} A \left( \frac{|f(y)|}{\delta_y^s} \right) dy \leq \int_{A_{j+m}} A \left( \frac{|f(y)|}{\delta_y^s} \right) dy. \]
The proof follows as an application of lemma 3.4.
\[ \square \]

5. **RFOPI**(s, A) and **FOPI**(s, A) on unbounded domains

In this section we study **RFOPI**(s, A) and **FOPI**(s, A) for general domains in $\mathbb{R}^N$.

**Proposition 5.1.** Let $D \subseteq \mathbb{R}$ be an open set and $\lim_{\lambda \to 0} \alpha_{s,A}(\lambda) = 0$. Then $P_{N,s,A}(D) > 0$ if and only if $BC(D) < \infty$.

**Proof.** Suppose $BC(D) < \infty$. There exists a countable number of disjoint open intervals, say $I_k$, such that $D = \bigcup_{k=1}^{\infty} I_k$. Let $f \in C_{c}^{\infty}(D) \setminus \{0\}$. Using theorem 1.2 followed by lemma 2.5, we get
\[
\begin{align*}
\iiint_{D \times D} A \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right) \, dx \, dy \geq & \sum_{k=1}^{\infty} \int_{I_k} \int_{I_k} A \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right) \, dx \, dy \\
& \geq \sum_{k=1}^{\infty} P_{1,s,A}(I_k) \int_{I_k} A(|f(x)|) \, dx \geq \min\{BC(D)^{-sp}, 1\} P_{1,s,A}((0, 1)) \sum_{k=1}^{\infty} \int_{I_k} A(|f(x)|) \, dx \\
& = \min\{BC(D)^{-sp}, 1\} P_{1,s,A}((0, 1)) \int_{D} A(|f(x)|) \, dx,
\end{align*}
\]
which implies $P_{1,s,A}(D) \geq cP_{1,s,A}((0, 1)) > 0$. The other part follows from proposition 2.1. \[ \square \]

The next lemma is an important change of variable formula for the fractional Orlicz seminorm.

**Lemma 5.1.** Let $0 < s < 1$ and $D \subseteq \mathbb{R}^N$ be a measurable set. Then for any $f \in C_{c}^{\infty}(D)$,
\[
\begin{align*}
2 \iiint_{D \times D} A \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right) \, dx \, dy \, dz \, N &= \int_{\omega \in S^{N-1}} \int_{x \in \omega^{\perp}} \int_{t \in L_D(x, \omega)} \int_{t \in L_D(x, \omega)} A \left( \frac{|f(x + \ell \omega) - f(x + t \omega)|}{|\ell - t|^s} \right) \, dt \, d\delta(x) \, d\sigma(\omega),
\end{align*}
\]
where $L_D(x, \omega) := \{ t \in \mathbb{R}^N : x + t \omega \in D \}$.

**Proof.** For a fixed $f \in C_{c}^{\infty}(D)$ and $x \in D$, consider the change of variable $y = x + z$ followed by another change of variable to polar coordinates. We then get
\[
\begin{align*}
2 \iiint_{D \times D} A \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right) \, dx \, dy \, dz \, N &= 2 \int_{x \in D} \int_{z \in D-x} A \left( \frac{|f(x) - f(x + z)|}{|z|^s} \right) \, dx \, dz \, N \\
&= 2 \int_{x \in D} \int_{\omega \in S^{N-1}} \int_{r > 0} A \left( \frac{|f(x) - f(x + r \omega)|}{r^s} \right) \, dr \, d\sigma(\omega) \, dx \\
&= \int_{\omega \in S^{N-1}} \int_{x \in D} \int_{h \in L_D(x, \omega)} A \left( \frac{|f(x) - f(x + h \omega)|}{|h|^s} \right) \, dh \, d\delta(x) \, d\sigma(\omega).
\end{align*}
\]
Now, for a fixed $\omega \in S^{N-1}$ we can write $x = (x - (x \cdot \omega) \omega) + (x \cdot \omega) \omega$. Using this we can split the integral over $x \in D$ by considering the change of variable $\ell = x \cdot \omega$ and $z = x - (x \cdot \omega) \omega$, where $\ell \in L_D(z, \omega)$ and $z \in \omega^{\perp}$. So we get
Proof of Theorem 1.6. Let $f \in C^\infty_c(D)$. (1) Applying lemma 5.1 we obtain

\[
2 \int_{D \times D} A \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N} = \int_{\mathbb{S}^{N-1}} \int_{x \in \mathbb{S}^{N-1}} \int_{\mathbb{T} \in L_D(x,\omega)} \int_{\mathbb{T} \in L_D(z,\omega)} A \left( \frac{|f(z + \ell \omega) - f(z + (\ell + h)\omega)|}{h^s} \right) dhdld\sigma(z) d\sigma(\omega)
\]

\[
= \int_{\mathbb{S}^{N-1}} \int_{x \in \mathbb{S}^{N-1}} \int_{\mathbb{T} \in L_D(x,\omega)} \int_{\mathbb{T} \in L_D(z,\omega)} A \left( \frac{|f(z + \ell \omega) - f(z + t \omega)|}{|\ell - t|^s} \right) dhdld\sigma(z) d\sigma(\omega)
\]

for any $\omega \in \mathbb{S}^{N-1}$, $x \in \mathbb{S}^{N-1}$, $t \in L_D(x,\omega)$, $h \in L_D(z,\omega)$.

We conclude the proof of this lemma with the observation that $L_D(z, \omega) = L_D(z + \ell \omega, \omega) + \ell$. This finishes the proof of the theorem.

\[\square\]

\textbf{Proof of Theorem 1.6.} Let $f \in C^\infty_c(D)$. (1) Applying lemma 5.1 we obtain

\[
2 \int_{D \times D} A \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N} = \int_{\mathbb{S}^{N-1}} \int_{x \in \mathbb{S}^{N-1}} \int_{\mathbb{T} \in L_D(x,\omega)} \int_{\mathbb{T} \in L_D(z,\omega)} A \left( \frac{|f(x + \ell \omega) - f(x + t \omega)|}{|\ell - t|^s} \right) dhdld\sigma(x) d\sigma(\omega)
\]

\[
\geq \sum_{k=1}^\infty \int_{\mathbb{S}^{N-1}} \int_{x \in \mathbb{S}^{N-1}} \int_{\mathbb{T} \in L_D(x,\omega)} \int_{\mathbb{T} \in L_D(z,\omega)} A \left( \frac{|f(x + \ell \omega) - f(x + t \omega)|}{|\ell - t|^s} \right) dhdld\sigma(z) d\sigma(\omega)
\]

where $L_D(x, \omega) = \cup_{k=1}^\infty I_k$, for mutually disjoint family of intervals $\{I_k\}_k$. From lemma 2.5 we have, for any $k$,

\[
P_{1,s,A}(I_k) \geq \min\{BC(L_D(x, \omega))^{-sp}, 1\} P_{1,s,A}((0, 1)).
\]

We can now use the hypothesis of the lemma to get a constant $c = c(D) > 0$ such that $c < \min\{BC(L_D(x, \omega))^{-sp}, 1\}$. Again as an application of theorem 1.2 $P_{1,s,A}((0, 1)) > 0$. Using these we get

\[
\int_{D \times D} A \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N} \geq c \int_{\mathbb{S}^{N-1}} \int_{x \in \mathbb{S}^{N-1}} A(|f(x + t \omega)|) dtd\sigma(x) d\sigma(\omega)
\]

\[
\geq c \int_{\mathbb{S}^{N-1}} \int_{x \in \mathbb{S}^{N-1}} A(|f(x)|) dxd\sigma(\omega) = c \int_{D} A(|f(x)|) dx.
\]

which implies that $P_{N,s,A}(D) > 0$. This completes the first part of the proof.

(2) We use the positivity, increasing property of $A$, the fact that $f \equiv 0$ on $D^c$ and suitable change of variable in the following calculation.

\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} A \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right) \frac{dydx}{|x - y|^N} = \frac{1}{R^N} \int_{\mathbb{R}^N \times \mathbb{R}^N} A \left( \frac{R^s|f(x/R)| - f(y)}{|x - y|^s} \right) \frac{dydx}{|x - y|^N}
\]

\[
\geq \frac{1}{R^N} \int_{x \in \mathbb{R}^N} \int_{y \in B(x,R) \cap D^c} A \left( \frac{R^s|f(x/R)|}{|x - y|^s} \right) \frac{dydx}{|x - y|^N} \geq \frac{\mathcal{L}^N(B(x,R) \cap D^c)}{R^N} \int_{\mathbb{R}^N} A \left( \frac{|f(x/R)|}{R} \right) dx
\]
\[= \frac{c_1}{R^N} \int_{\mathbb{R}^N} A(\|f(x)\|) \, dx = \frac{c_1}{R^N} \int_D A(\|f(x)\|) \, dx.\]

This shows \(P^2_{N,s,A}(D) > 0\) and the proof of the second part follows.

(3) This is a straightforward use of lemma 5.1 as in the first part of the proof. \(\Box\)

As an application of theorem 1.6 we discuss here some examples of some unbounded domains for which Poincaré inequality holds true, provided \(\lim_{\lambda \to 0^+} \alpha_{s,A}(\lambda) = 0\) for the first condition (see also [CCRS21, CR20]). Although verification of the claims in these examples is straightforward, we work it out in the second example.

**Example 5.1** (Domain between graphs of two functions). Let \(f_i : \mathbb{R}^{N-1} \to [m, M] \quad (i = 1, 2)\) be two bounded continuous function such that \(f_1 < f_2\). We then define \(D\) as

\[D = \{ (\bar{x}, x_N) \in \mathbb{R}^N \mid f_1(\bar{x}) < x_N < f_2(\bar{x}) \}.\]

It is easy to verify that all the hypotheses of theorem 1.6 are true.

**Example 5.2** (Countable union of parallel strips). Let \(D := I \times \mathbb{R} \subseteq \mathbb{R}^2\), where \(I := \bigcup_{k=1}^\infty I_k\) and \(I_k\)'s are disjoint intervals with \(\text{dist}(I_i, I_j) \geq \gamma\) for a constant \(\gamma > 0\), where \(I_i, I_j\) are any two distinct intervals. Also assume that \(\sup(\mathcal{L}^1(I_k)) = M < \infty\). Then it is easy to check that the second condition of theorem 1.6 holds. This gives \(P^2_{1,s,A}(I) > 0\), for any \(s \in (0, 1)\). Now choose \(\Sigma = \{ \omega_\theta := (\cos \theta, \sin \theta) \mid \theta \in (0, \pi/4) \}\). Then for \(x \in \omega_\theta^k\), the set \(L_D(x, \omega_\theta)\) can be expressed as the countable union of disjoint intervals, i.e., \(L_D(x, \omega_\theta) = \bigcup_{k=1}^\infty J_k(x, \omega_\theta)\). Observe that \(\mathcal{L}^1(J_k(x, \omega_\theta)) = \frac{c^1(I_k)}{\cos \theta}\) for any \(k\), which is independent of \(x\). Thus, we have for some constant \(c = c(x, \omega_\theta)\),

\[(5.1) \quad L_D(x, \omega_\theta) = c + (\sec \theta)I.\]

This gives \(\sup_{\omega_\theta \in \Sigma, x \in \omega_\theta^k} BC(L_D(x, \omega_\theta)) \leq M\sqrt{2}\) and which shows that the first condition holds. Consequently RFOP\(I(s, A)\) holds in \(D\) when \(\lim_{\lambda \to 0^+} \alpha_{s,A}(\lambda) = 0\). For third condition, Using (3) of proposition 2.1 and eq. (5.1) we obtain, since \(\theta \in (0, \pi/4)\),

\[P^2_{1,s,A}(L_D(x, \omega_\theta)) \geq \frac{P^2_{1,s,A}(I)}{(\cos \theta)^s} \geq 2^{2s} P^2_{1,s,A}(I).\]

So, uniform FOP\(I(s, A)\) holds for the collection \(\{L_D(x, \omega_\theta)\}_{\omega_\theta \in \Sigma, x \in \omega_\theta^k}\). Hence the domain \(D\) is of type LS\((s, A)\) for any \(s \in (0, 1)\). Consequently FOP\(I(s, A)\) holds in \(D\) for any \(s \in (0, 1)\).

**Example 5.3** (Concentric annulus). The following domain satisfies condition 2 for all \(s \in (0, 1)\) of theorem 1.6:

\[D = \bigcup_{k=1}^\infty B(0, 2k) \setminus B(0, 2k - 1).\]

**Example 5.4** (Domain with holes around points of \(\mathbb{Z} \times \mathbb{Z}\)). The following domain satisfies the condition 2 of theorem 1.6:

\[D = \mathbb{R}^2 \setminus \left( \bigcup_{x \in \mathbb{Z}^2} B(x, 1/10) \right).\]
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