Star-shaped distributions and their generalizations

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**Abstract**

Elliptically contoured distributions can be considered to be the distributions for which the contours of the density functions are proportional ellipsoids. We generalize elliptically contoured densities to “star-shaped distributions” with concentric star-shaped contours and show that many results in the former case continue to hold in the more general case. We develop a general theory in the framework of abstract group invariance so that the results can be applied to other cases as well, especially those involving random matrices.

**Key words**: elliptically contoured distribution, equivariance, global cross section, group action, Haar measure, invariance, isotropy subgroup, normalizer, orbital decomposition, star-shaped set.

1 Introduction

Elliptically contoured distribution is a convenient generalization of the multivariate normal distribution and now there exists substantial literature on elliptically contoured distribution, e.g., Chapter 2 of the third edition of [11] [8] [11] and [16]. Density $f(x)$ of an elliptically contoured distribution in $\mathbb{R}^p$ can be written as $f(x) = f_G(g(x))$ with $g(x) = (x^t\Sigma^{-1}x)^{1/2}$, where $\Sigma$ is a $p \times p$ positive definite matrix. Under this distribution, the “length” $g(x)$ and the “direction” $x/g(x)$ are independent. Moreover, by changing $f_G(\cdot)$, we can construct the elliptically contoured distribution with an arbitrary distribution of $g(x)$. Because the distribution of $x/g(x)$ is common to all the elliptically contoured distributions with the same $\Sigma$, distributional results concerning $x/g(x)$ derived under the assumption of normality continue to hold for all elliptically contoured distributions having the same $\Sigma$. This property is often referred to as “null robustness” and has been extensively discussed in the literature (e.g. [20]).
However, the class of elliptically contoured distributions seems to be too narrow. It does not include, e.g., a simple distribution with a density in $\mathbb{R}^2$ whose contours are concentric squares. Note that elliptically contoured distributions differ from the multivariate normal distribution only in the distribution of the one-dimensional length. Therefore, in the framework of elliptically contoured distributions, we can not consider non-normality which is exhibited in skewness or asymmetry of distributions.

As a matter of fact, some properties of elliptically contoured distributions, including the above-mentioned independence of length and direction and the null robustness, continue to hold beyond the class of elliptically contoured distributions if we define the “length” properly. We extend the class of elliptically contoured distributions to a class of distributions called star-shaped distributions, whose densities have arbitrary star-shaped sets as their contours.

In this paper, star-shaped distribution is developed in the general framework of group invariance, especially global cross sections and the associated orbital decompositions. Our primary concern is the star-shaped distribution, but the general theory can be applied to problems about distributions of random matrices, including the case where the group action is non-free. For group invariance in statistics in general, see a recent survey by [17]. Actually, the star-shaped distributions have also been considered under the name “$v$-spherical distributions” by [13] (see also [14]), but from a less algebraic point of view.

The material in the present paper is based on two earlier drafts of the authors, [28] [18]. We give a unified presentation of relevant and original results from these drafts in view of the current literature.

The organization of this paper is as follows. In Sections 2-3, we develop a general theory in the framework of group invariance. In Section 2 we study orbital decomposition and global cross sections. Based on the arguments there, we define decomposable distributions and investigate the associated distributional problems in Section 3. The results in those sections are applied to star-shaped distributions in Section 4. In Section 5, further applications to random matrices are presented. Some technical details are given in the Appendix.

2 Orbital decomposition and global cross sections

In this section we review some basic notions about group actions and investigate some properties of global cross sections. Our approach is based on global cross sections, but there is another approach—the one based on proper actions and quotient measures, for which the reader is referred to the significant papers [2] and Andersson, Brøns and Jensen (1983).

2.1 Orbital decomposition

Let a group $G$ act on a space $\mathcal{X}$ (typically the sample space) from the left $(g, x) \mapsto gx : G \times \mathcal{X} \to \mathcal{X}$. Let $Gx = \{gx : g \in G\}$ be the orbit containing $x \in \mathcal{X}$, and let $\mathcal{X}/G = \{Gx : x \in \mathcal{X}\}$ be the orbit space, i.e., the set of all orbits. When $\mathcal{X}$ consists of a single orbit $\mathcal{X} = Gx$, the action is said to be transitive.
Indicate by \( G_x = \{ g \in G : gx = x \} \) the \textit{isotropy subgroup} at \( x \in \mathcal{X} \). When \( G_x = \{ e \} \) for all \( x \in \mathcal{X} \), the action is said to be \textit{free}, where \( e \) denotes the identity element of \( G \). In general, the isotropy subgroups at two points on a common orbit are conjugate to each other:

\[
(1) \quad G_{gx} = gG_xg^{-1}, \quad g \in G, \quad x \in \mathcal{X}.
\]

The set of \textit{left cosets} \( gG_x = \{ gg' : g' \in G_x \} \), \( g \in G \), is called the \textit{left coset space} of \( G \) modulo \( G_x \), and is denoted by \( G/G_x = \{ gG_x : g \in G \} \). The group \( G \) acts on \( G/G_x \) by \( (g, hG_x) \mapsto (gh)G_x \), \( g, h \in G \). We define the \textit{canonical map} \( \pi : G \to G/G_x \) by \( \pi(g) = gG_x \), \( g \in G \).

We move on to the definitions concerning cross sections. A \textit{cross section} is defined to be a set \( Z \subset \mathcal{X} \) which intersects each orbit \( Gx \), \( x \in \mathcal{X} \), exactly once. Therefore, \( Z \) is in one-to-one correspondence with the orbit space. We denote this correspondence by \( \iota_Z : \mathcal{X}/G \to Z \), i.e., \( \iota_Z(Gx) = z \), where \( z \) is the unique point in \( Gx \cap Z \). A cross section \( Z \) is called a \textit{global cross section} if the isotropy subgroups are common at all points of \( Z : G_z = G_0 \), say, for all \( z \in Z \). Of course, there always exists a cross section, but this is not always the case with a global cross section.

Suppose there does exist a global cross section \( Z \) with the common isotropy subgroup \( G_0 \). It is well-known and easy to see that in this case, we have the following one-to-one correspondence, called the \textit{orbital decomposition}:

\[
(2) \quad \mathcal{X} \leftrightarrow \mathcal{Y} \times Z, \\
x \leftrightarrow (y, z), \quad x = gz, \quad y = \pi(g), \quad g \in G,
\]

where \( \mathcal{Y} = G/G_0 \) is the left coset space modulo \( G_0 \). It might help to regard \( y \) as the \textit{coordinate along the orbit} or the \textit{within-orbit coordinate}, and \( z \) as the \textit{orbit index}. In the orbital decomposition, we can think of \( y \) and \( z \) as functions \( y = y(x) \) and \( z = z(x) \) of \( x \). If \( x \leftrightarrow (y, z) \), then \( gx \leftrightarrow (gy, z) \), \( g \in G \). Therefore, \( y(x) \) is equivariant and \( z(x) \) is invariant:

\[
y(gx) = gy(x), \quad z(gx) = z(x), \quad g \in G, \quad x \in \mathcal{X}.
\]

Thus, the coordinate along the orbit \( y \in \mathcal{Y} \) is called the \textit{equivariant part}, and the orbit index \( z \in Z \) is called the \textit{invariant part}.

From now on we assume that a global cross section \( Z \) exists. We note that our results can be applied to the case of non-existence of a global cross section, by using the notion of orbit types. We discuss this point in Appendix B.

We end this subsection by giving two simple examples of the orbital decompositions.

First, consider the rotation group

\[
G = SO(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : 0 \leq \theta < 2\pi \right\}
\]

acting on \( \mathcal{X} = \mathbb{R}^2 - \{ 0 \} \) = \( \{ x = (x_1, x_2) \in \mathbb{R}^2 : x \neq 0 \} \) as

\[
\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} (\cos \theta)x_1 - (\sin \theta)x_2 \\ (\sin \theta)x_1 + (\cos \theta)x_2 \end{pmatrix}.
\]
In this case, $G_0$ is trivial, so $Y = G$. The orbit containing $x = (x_1, x_2)$ is the circle with center 0 and radius $\|x\| = \sqrt{x_1^2 + x_2^2}$. Therefore, any subset of $\mathbb{R}^2 - \{0\}$ intersecting each concentric circle with center 0 exactly once can serve as a cross section $Z$. We take the positive part of the $x_1$-axis as a standard cross section: $Z = \{(x_1, 0) : x_1 > 0\}$. Then the orbital decomposition of $x = (x_1, x_2)$ can be written as

$$\left(\frac{x_1}{\|x\|}, \frac{x_1}{\|x\|} - \frac{x_2}{\|x\|}\right)\left(\begin{array}{c} \|x\| \\ 0 \end{array}\right).$$

Thus we see that the equivariant part

$$\left(\frac{x_1}{\|x\|}, \frac{x_1}{\|x\|} - \frac{x_2}{\|x\|}\right)$$

can be labeled by the angle $\theta = \cos^{-1}(x_1/\|x\|) = \sin^{-1}(x_2/\|x\|)$ (the unique $\theta$ such that $\cos \theta = x_1/\|x\|$, $\sin \theta = x_2/\|x\|$), or the direction of $x$. The invariant part, on the other hand, can be indexed by the length $\|x\|$.

Now we move on to the next example. Let $G = \mathbb{R}^*_+$, the multiplicative group of positive real numbers. Then $G$ acts freely on $X = \mathbb{R}^2 - \{0\}$ by $(g, (x_1, x_2)) \mapsto (gx_1, gx_2)$. The orbit containing $x = (x_1, x_2)$ is the ray emanating from the origin in the direction of $x$. Hence, the cross sections are (boundaries of) “star-shaped” sets (see Section [4] for the precise definition). Here we take $Z = S^1 = \{x = (x_1, x_2) : \|x\| = 1\}$, the unit circle. For this cross section, $x = (x_1, x_2)$ can be factored as

$$\left(\frac{x_1}{\|x\|}, \frac{x_1}{\|x\|} - \frac{x_2}{\|x\|}\right),$$

so the equivariant part is the length and the invariant part is the direction.

### 2.2 Properties of global cross sections

For an arbitrary (not necessarily global) cross section $Z$, we can see that $gZ = \{gz : z \in Z\}$ is again a cross section for each $g \in G$. We call $gZ$ a cross section proportional to $Z$. Since $Z$ meets each orbit, we have

$$X = \bigcup_{g \in G} gZ.$$ (3)

We are interested in the case where (3) gives a partition of $X$, that is,

$$g_1Z \cap g_2Z \neq \emptyset \quad \Rightarrow \quad g_1Z = g_2Z$$

for $g_1, g_2 \in G$. The following proposition shows that a necessary and sufficient condition for (3) to give a partition of $X$ is that $Z$ be a global cross section.

**Proposition 2.1.** A cross section $Z$ is global if and only if $X = \bigcup_{g \in G} gZ$ gives a partition of $X$. 


Proof. Suppose that \( X = \bigcup_{g \in G} gZ \) gives a partition of \( X \). Let \( z_1 \) and \( z_2 \) be two arbitrary points of \( Z \). Let \( g \in G_{z_1} \). Then \( gz_1 = z_1 \in gZ \cap Z \neq \emptyset \), and hence \( gZ = Z \).

Thus there exists a \( z \in Z \) such that \( gZ = Z \). But since \( Z \) is a cross section, we have \( z_2 = z \) and hence \( gz_2 = z_2 \). This observation shows that \( g \in G_{z_1} \) implies \( g \in G_{z_2} \). By interchanging the roles of \( z_1 \) and \( z_2 \), we see that the converse is true as well and thus \( G_{z_1} = G_{z_2} \). Hence, \( Z \) is global.

Conversely, suppose that \( Z \) is global, and let \( G_0 \) be the common isotropy subgroup. Suppose \( g_1 Z \cap g_2 Z \neq \emptyset \) for \( g_1, g_2 \in G \). Then, there exist \( z_1, z_2 \in Z \) such that \( g_1 z_1 = g_2 z_2 \).

Since \( Z \) is a cross section, we have \( z_1 = z_2 \) and thus \( g_1 z_1 = g_2 z_1 \). Therefore, \( g_1^{-1} g_2 \in G_0 \) and \( g_1 z = g_2 z \) for all \( z \in Z \). Thus we obtain \( g_1 Z = g_2 Z \).

For a global cross section \( Z \), we call \( \{gZ : g \in G\} \) the family of proportional global cross sections.

In the preceding discussions, a global cross section \( Z \) was given first and the equivariant function \( y \) was induced by the orbital decomposition with respect to \( Z \). Conversely, we can construct a global cross section from a given equivariant function in the following way. The proof of the following proposition is not difficult and is omitted.

**Proposition 2.2.** Let a group \( G \) act on a space \( Y \) as well as on \( X \), and let \( \tilde{y} : X \to Y \) be an equivariant function. Suppose that the action of \( G \) on \( Y \) is transitive and that \( \tilde{y} \) satisfies the following condition:

\[
(4) \quad \tilde{y}(x) = \tilde{y}(gx) \iff x = gx
\]

for \( g \in G \) and \( x \in X \). Then, the inverse image \( \tilde{y}^{-1}(\{y_0\}) \subset X \) of each \( y_0 \in Y \) is a global cross section. Moreover, global cross sections \( \tilde{y}^{-1}(\{y\}), \ y \in Y \), are all proportional to one another.

**Remark 2.1.**

1. When \( Y \) is the coset space \( Y = G/G_0 \) modulo a subgroup \( G_0 \), the common isotropy subgroup of the global cross section \( \tilde{y}^{-1}(\{y_0\}) \) with \( y_0 = G_0 \) coincides with \( G_0 \). Furthermore, in this case \( \tilde{y}(x) \) is the equivariant part of \( x \in X \) with respect to this global cross section. Note that since the action of \( G \) on \( Y \) is assumed to be transitive, we may assume without loss of generality that \( Y \) is a coset space.

2. If the action of \( G \) on \( Y \) is free, then \( \tilde{y}(x) = \tilde{y}(gx) = g\tilde{y}(x) \) implies \( g = e \), so condition (4) is satisfied for any equivariant function \( \tilde{y} \). In particular, when \( Y = G \), condition (4) is automatically satisfied and the action of \( G \) on \( X \) is free, as long as an equivariant function \( \tilde{y} : X \to G \) exists.

When \( Y \) is a coset space \( G/G_0 \), we will call the global cross section \( \tilde{y}^{-1}(\{G_0\}) \) the unit global cross section.

We now consider the variety of global cross sections. From a given global cross section \( Z \), we can construct a general cross section \( Z' \) by moving the points of \( Z \) within their orbits. For example, in the case of star-shaped distributions discussed in Section 4, we consider transforming an ellipse \( Z \) centered at the origin to the unit circle \( Z' \) by the
transformation \( x \mapsto x/\|x\| \). In the case of non-free actions, for \( Z' \) to be global, i.e., for the isotropy subgroups to be the same on the whole of \( Z' \), movements of the points within the orbits have to be made subject to some restriction. Let
\[
\mathcal{N} = \{g \in \mathcal{G} : gG_0g^{-1} = G_0\}
\]
denote the normalizer of the common isotropy subgroup \( G_0 \) of a global cross section \( Z \). Note that \( G_0 \) is a normal subgroup of \( \mathcal{N} \) so that we can think of the factor group \( \mathcal{M} = \mathcal{N}/G_0 \) (Appendix A). We can characterize a general global cross section in terms of the normalizer \( \mathcal{N} \).

**Theorem 2.1.** Let \( Z \) be a global cross section with the common isotropy subgroup \( G_0 \). Then \( Z' \subset X \) is a global cross section if and only if it can be written as
\[
Z' = \{g_0n_zz : z \in Z\}
\]
for some \( g_0 \in \mathcal{G} \) and \( n_z \in \mathcal{N}, \ z \in \mathcal{Z} \).

The proof is given in Appendix A (Corollary A.3). As can be seen there, it is easy to show that \( Z' = \{g_0n_zz : z \in Z\} \) is a global cross section. The point is the proof of the converse. Characterization of a general global cross section, including the proof of the converse and the question of the uniqueness of \( n_z \) in representation (5), is fully discussed in Appendix A. Note that \( g_0 \) in (5) is not essential, since \( Z' = \{g_0n_zz : z \in Z\} \) and \( g_0^{-1}Z' = \{n_zy : z \in Z\} \) are proportional and thus induce the same family of proportional global cross sections.

**Remark 2.2.** When the action is free, we have \( \mathcal{N} = \mathcal{G} \) so that (5) becomes
\[
Z' = \{gz : z \in Z\}
\]
for some \( g_z \in \mathcal{G}, \ z \in \mathcal{Z} \).

We finish this subsection by explicitly writing down how the equivariant part transforms by the construction of a general global cross section in (5). Let \( x \leftrightarrow (y, z) \) be the orbital decomposition with respect to the global cross section \( Z \) with the common isotropy subgroup \( G_0 \), and let \( x \leftrightarrow (y', z') \) be the orbital decomposition with respect to the \( Z' \) in (5). This \( Z' \) has the isotropy subgroup \( G_0' = g_0G_0g_0^{-1} \). Now the equivariant part based on \( Z' \) is given as follows:

**Proposition 2.3.** Let \( Z \) be a global cross section with the common isotropy subgroup \( G_0 \). Moreover, let \( Z' \) be as in (5), and \( x \leftrightarrow (y', z') \) the orbital decomposition with respect to \( Z' \). Write \( x \in X \) as \( x = gz = g'z' \) with \( z \in \mathcal{Z}, \ z' \in \mathcal{Z}' \) and \( g, g' \in \mathcal{G} \). Then we have \( y' = yn_z^{-1}g_0^{-1} \), where \( y = gG_0 \) and \( y' = g'G_0' \).

**Proof.** We can write \( x = gz \) as \( x = gn_z^{-1}g_0^{-1}z' \) in terms of \( z' = g_0n_zz \). This implies \( y' = (gn_z^{-1}g_0^{-1})(gG_0g_0^{-1}) = gG_0n_z^{-1}g_0 = yn_z^{-1}g_0^{-1} \).

**Remark 2.3.** When the action is free and a general cross section is given by (6), the equivariant part transforms as \( g' = gg_z^{-1} \), where \( g \) and \( g' \) are the equivariant parts with respect to \( Z \) and \( Z' \), respectively.
3 Decomposable distributions

In this section we define a class of distributions called decomposable distributions and study some distributions induced by them. The general discussion here is applied to particular cases in the next two sections. Especially, an extension of elliptically contoured distributions called star-shaped distributions is discussed in Section 4.

3.1 Assumptions and the definition

In order to make distributional arguments, we need to make topological and measure-theoretic assumptions. In this paper, measurability of topological spaces refers to Borel measurability.

Assumption 3.1.

1. \( X \) is a locally compact Hausdorff space.
2. \( G \) is a second countable, locally compact Hausdorff topological group acting continuously on \( X \).
3. \( G_0 \) is compact.
4. Global cross section \( Z \) is locally compact, and the bijection \( x \leftrightarrow (y, z) \) with respect to \( Z \) is bimeasurable, where the topology on \( Z \) is the relative topology of \( Z \) as a subset of \( X \).

We agree that a quotient space receives the quotient topology when regarded as a topological space. This applies to the coset space \( Y = G/G_0 \) as well as to the orbit space \( X/G \). Because of 2 of Assumption 3.1 there exists a left Haar measure \( \mu_G \) on \( G \), which is unique up to a multiplicative constant.

We consider densities with respect to a dominating measure \( \lambda \) on \( X \) which is relatively invariant with multiplier \( \chi \):

\[
\lambda(dgx) = \chi(g)\lambda(dx), \ g \in G.
\]

Note that the relative invariance of \( \lambda \) only determines its behavior within each orbit so that for any nonnegative \( f_Z(z(x)) \),

\[
(7) \quad \tilde{\lambda}(dx) = f_Z(z(x))\lambda(dx)
\]

is again a relatively invariant measure with the same multiplier \( \chi \) as \( \lambda \).

We are now in a position to define the decomposable distributions.

Definition 3.1. A distribution on \( X \) is said to be decomposable with respect to a global cross section \( Z \) if it is of the form

\[
f(x)\lambda(dx) = f_Y(y(x))f_Z(z(x))\lambda(dx).
\]

In particular, it is said to be cross-sectionally contoured if \( f_Z(z) \) is constant. In contrast, it is said to be orbitally contoured if \( f_Y(y) \) is constant.
Obviously, a distribution $f(x)\lambda(dx)$ is cross-sectionally contoured with respect to $\mathcal{Z}$ if and only if $f(x)$ is constant on each proportional global cross section $g\mathcal{Z}$, $g \in \mathcal{G}$. Similarly, $f(x)\lambda(dx)$ is orbitally contoured if and only if $f(x)$ is constant on each orbit $\mathcal{G}x$, $x \in \mathcal{X}$. Before examining the distributions of the invariant and equivariant parts, we observe the following two points.

First, a decomposable distribution $f_Y(y(x))f_Z(z(x))\lambda(dx) = f_Y(y(x))\tilde{\lambda}(dx)$ can always be thought of as a cross-sectionally contoured distribution in view of (7).

Next, we can take various global cross sections, in addition to “standard” ones like the unit sphere. This enables us to consider the cross-sectionally contoured distributions associated with a variety of global cross sections. In contrast, once an action is given, there is no room for choosing the orbits; the orbits are determined by the action in question, and usually those orbits are familiar subsets of $\mathcal{X}$. Hence, we can not produce the orbitally contoured distributions based on the orbits which are unfamiliar subsets of $\mathcal{X}$.

For these reasons, we will be concerned with the cross-sectionally contoured distributions from now on.

### 3.2 Distributions of invariant and equivariant parts

First, we confirm the independence of invariant and equivariant parts. This corresponds to the independence of “direction” and “length” in elliptically contoured distributions. Thanks to the assumption that $\mathcal{G}_0$ is compact, we have the induced measure $\mu_Y = \pi(\mu_G) = \mu_G\pi^{-1}$ on $\mathcal{Y}$ (Proposition 2.3.5 and Corollary 7.4.4 of [31]). Also, by the same assumption we can define $\tilde{\chi}(y)$, $y \in \mathcal{Y}$, by $\tilde{\chi}(y) = \chi(g)$ with $g \in \pi^{-1}(\{y\})$, where $\chi$ is the multiplier of $\lambda$. With some abuse of notation, we will write $\chi(y)$ for $\tilde{\chi}(y)$.

Now, $\lambda(dx)$ is factored as

$$\lambda(dx) = \chi(y)\mu_Y(dy)\nu_Z(dz)$$

(Theorem 7.5.1 of [31], Theorem 10.1.2 of [12]). By changing the weights of the orbit as (7) if necessary, from now on we assume that $\nu_Z(dz)$ is (standardized to be) a probability measure on $\mathcal{Z}$. The following theorem is an immediate consequence of the factorization of $\lambda(dx)$ in (8).

**Theorem 3.1.** Suppose that $x$ is distributed according to a cross-sectionally contoured distribution $f_Y(y(x))\lambda(dx)$. Then we have:

1. $y = y(x)$ and $z = z(x)$ are independently distributed.
2. The distribution of $z$ does not depend on $f_Y$.
3. The distribution of $y$ is $f_Y(y)\chi(y)\mu_Y(dy)$.

### 3.3 Distributions generated via two global cross sections

Next, we investigate distributions generated by considering two global cross sections at a time. The relation between two global cross sections was given in Theorem [27]. Let $\mathcal{Z}$ and $\mathcal{Z}'$ be two global cross sections. By choosing an appropriate global cross section
from the family of proportional global cross sections, we assume without essential loss of
generality that the common isotropy subgroups for \( \mathcal{Z} \) and \( \mathcal{Z}' \) are the same: \( \mathcal{G}_z = \mathcal{G}_{z'} = \mathcal{G}_0, \ z \in \mathcal{Z}, \ z' \in \mathcal{Z}' \).

The invariant and equivariant parts \( z = z(x), \ y = y(x) \) with respect to \( \mathcal{Z} \) are given
via the orbital decomposition (2) as before. In a similar manner, define the invariant and
equivariant parts \( z' = z'(x), \ y' = y'(x) \) with respect to \( \mathcal{Z}' \). Note that by our assumption
the coset spaces are the same, \( \mathcal{Y} = \mathcal{G}/\mathcal{G}_0, \) in both cases.

Denote by \( g : \mathcal{Y} \to \mathcal{G} \) an arbitrary selection \( g(y) \in y \subset \mathcal{G} \). From now on, we will write
\( g(x) \) for \( g(y(x)) : x = g(x)z(x), \ x \in \mathcal{X} \). Define \( g'(x) \) in the same way: \( x = g'(x)z'(x), \ x \in \mathcal{X} \).
Here we define the map \( w : \mathcal{X} \to \mathcal{X} \) by
\[
w = w(x) = g(x)z'(x) = g(x)g'(x)^{-1}x = g(x)g'(z(x))^{-1}z(x) , \quad x \in \mathcal{X}.
\]
Note that since \( \mathcal{G}_{z'} = \mathcal{G}_0, \ z' \in \mathcal{Z}' \), \( w \) does not depend on the choice of the selection \( g(y) \).
We call \( w \) the within-orbit bijection, because \( x \) and \( w(x) \) are on the same orbit and we are transforming \( x \) to \( w(x) \) in each orbit separately. The within-orbit bijection is a basic tool for deriving a new cross-sectionally contoured distribution from a given cross-sectionally contoured distribution.

**Theorem 3.2.** Suppose that \( x \) is distributed according to a cross-sectionally contoured
distribution \( f_\mathcal{Y}(y(x))\lambda(dx) \). Then the distribution of \( w = w(x) \) is
\[
f_\mathcal{Y}(y'(w))\chi(g(w)^{-1}g'(w))\Delta^\mathcal{G}(g(w)^{-1}g'(w))\lambda(dw),
\]
where \( \Delta^\mathcal{G} \) is the right-hand modulus of \( \mathcal{G} : \mu_\mathcal{G}(d(gg_1)) = \Delta^\mathcal{G}(g_1)\mu_\mathcal{G}(dg), \ g_1 \in \mathcal{G} \).

**Proof.** We regard \( w = w(x) \) as a function of \( y = y(x) \) and \( z = z(x) : w = w(x) = w(y, z) \). Noting that the integration over \( \mathcal{Y} \) can be carried out by the integration over \( \mathcal{G} \),
we have for an arbitrary measurable subset \( B \subset \mathcal{X} \) that
\[
P(w \in B) = \int_\mathcal{X} I_B(w(x))f_\mathcal{Y}(y(x))\lambda(dx)
= \int_\mathcal{X} \int_\mathcal{Z} I_B(w(y, z))f_\mathcal{Y}(y)\chi(y)\mu_\mathcal{Y}(dy)\nu_\mathcal{Z}(dz)
= \int_\mathcal{X} \int_\mathcal{Z} I_B(gg'(z)^{-1}z)\hat{f}_\mathcal{Y}(g)\chi(g)\mu_\mathcal{G}(dg)\nu_\mathcal{Z}(dz).
\]
where \( \hat{f}_\mathcal{Y} = f_\mathcal{Y} \circ \pi \) and \( I_B \) is the indicator function of \( B \). Let \( \mathcal{G}_B(z) = \{g \in \mathcal{G} : gz \in B\} \), \( z \in \mathcal{Z} \), and \( \mathcal{Z}_B = \{z(x) \in \mathcal{Z} : x \in B\} \). Then \( I_B(x) = I_{\mathcal{G}_B(z)}(g) \cdot I_{\mathcal{Z}_B}(z) \) for \( x = gz \),
and we can write (9) as
\[
\int_\mathcal{Z} I_{\mathcal{Z}_B}(z) \int_g I_{\mathcal{G}_B(z)}(gg'(z)^{-1})\hat{f}_\mathcal{Y}(g)\chi(g)\mu_\mathcal{G}(dg)\nu_\mathcal{Z}(dz)
= \int_\mathcal{Z} I_{\mathcal{Z}_B}(z)\chi(g'(z))\Delta^\mathcal{G}(g'(z)) \int_g I_{\mathcal{G}_B(z)}(g)\hat{f}_\mathcal{Y}(gg'(z))\chi(g)\mu_\mathcal{G}(dg)\nu_\mathcal{Z}(dz)
= \int_\mathcal{X} I_B(x)\hat{f}_\mathcal{Y}(g'(x))\chi(g(x)^{-1}g'(x))\Delta^\mathcal{G}(g(x)^{-1}g'(x))\lambda(dx)
= \int_B f_\mathcal{Y}(y'(w))\chi(g(w)^{-1}g'(w))\Delta^\mathcal{G}(g(w)^{-1}g'(w))\lambda(dw).
\]
For notational simplicity, we will write
\[ \Delta(g) = \chi(g)\Delta\hat{\gamma}(g), \quad g \in \mathcal{G}, \]
which is a continuous homomorphism from \( \mathcal{G} \) to \( \mathbb{R}_+^* \). Because of 3 of Assumption 3.1, we have \( \Delta(g) = 1 \) for all \( g \in \mathcal{G}_0 \), so \( \Delta(g(w))^{-1}g'(w) \) does not depend on the choice of the selections \( g(w) \) and \( g'(w) \).

Let \( \mathcal{E} \) be the set of all measurable equivariant functions \( \tilde{y} : \mathcal{X} \to \mathcal{G}/\mathcal{G}_0 \) satisfying (11) of Proposition 2.2 and let \( \tilde{y} : \mathcal{X} \to \mathcal{G} \) be an arbitrary selection of \( \tilde{y} : \tilde{y}(x) \in \tilde{y}(x), \ x \in \mathcal{X} \). By Theorem 3.2 when \( f_{y}(y(x)) \) is a density function with respect to \( \lambda(dx) \), we can define a non-parametric family of distributions dominated by \( \lambda \):
\[
(10) \quad \left\{ f(x; h, \tilde{y}) = \frac{1}{\chi(h)}f_{y}(h^{-1}\tilde{y}(x))\Delta(g(x)^{-1}\tilde{y}(x)) : h \in \mathcal{G}, \ \tilde{y} \in \mathcal{E} \right\}.
\]

Note that \( \tilde{y}(x) \) is the equivariant part of \( x \) with respect to the unit global cross section \( \tilde{Y}^{-1}(\{\mathcal{G}_0\}) \) (Remark 2.1). We can see that distributions in (10) have cross-sectionally contoured densities \( \chi^{-1}f_{y}(g^{-1}\tilde{y}(x)) \) with respect to global cross section \( \tilde{Y}^{-1}(\{\mathcal{G}_0\}) \) and dominating measure \( \lambda(dx) = \Delta(g(x)^{-1}\tilde{y}(x))\lambda(dx) \). This \( \lambda \) is relatively invariant with the same multiplier \( \chi \) as \( \lambda \), and \( \lambda \) and \( \lambda \) are absolutely continuous with respect to each other because \( 0 < \Delta(g(x)^{-1}\tilde{y}(x)) < \infty \).

Now we turn to the distribution of \( z' = z'(x) \). Note that we may instead obtain the distribution of \( z'(w) \) because \( z'(x) = z'(w(x)) \), for \( x \) and \( w(x) \) are on the same orbit. Corresponding to the orbital decomposition with respect to \( \mathcal{Z}' \), \( \lambda(dx) \) is factored as
\[ \lambda(dx) = \chi(y')\mu_Y(dy')\nu_{Z'}(dz'). \]

Here we use the same \( \mu_Y \) as in (8). Recall that in (8) we have chosen \( \nu_{Z'}(dz) \) to be a probability measure on \( \mathcal{Z} \). Therefore, \( \nu_{Z'} \) is not necessarily a probability measure on \( \mathcal{Z}' \). In terms of \( \nu_{Z'} \), the distribution of \( z' \) is written as follows.

**Theorem 3.3.** Suppose that \( x \) is distributed according to a cross-sectionally contoured distribution \( f_{y}(y(x))\lambda(dx) \). Then the distribution of \( z' = z'(x) \) is
\[
(11) \quad \frac{1}{\Delta(g(z'))}\nu_{Z'}(dz').
\]
In addition, \( z' = z'(x) \) is independently distributed of \( y = y(x) \).

**Proof.** We have \( \Delta(g(w))^{-1}g'(w)) = \Delta(g(z'(w)))^{-1} \). Writing \( y' = y'(w) \) and \( z' = z'(w) \), we have by Theorem 3.2 that the distribution of \( w \) is
\[
(12) \quad f_{y}(y'(w))\Delta(g(z'(w)))^{-1}\lambda(dw) = f_{y}(y')\Delta(g(z'))^{-1}\chi(y')\mu_Y(dy')\nu_{Z'}(dz')
\]
Accordingly, the distribution of \( z' = z'(w) \) is \( \Delta(g(z'))^{-1}\nu_{Z'}(dz') \).
Since $x$ and $w = g(x)g'(x)^{-1}x$ are on the same orbit, we have $z'(x) = z'(w)$ so that the distribution of $z'(x)$ is the same as that of $z'(w)$. Moreover, we can see from \cite{12} that $y'(w) = g(x)g'(x)^{-1}y'(x) = y(x)$ and $z'(w) = z'(x)$ are independent. \hfill $\blacksquare$

From \cite{11} we can construct various distributions on $Z'$ by appropriately choosing the global cross sections $Z$. Here we can ask the following question: Given a density $f(z')$ on $Z'$, can we find a global cross section $Z$ such that the distribution of $z'(x)$ when $x$ is distributed as a cross-sectionally contoured distribution $f_Y(y(x))\lambda(dx)$ with respect to $Z$ coincides with $f(z')$? Recall that the distribution of $z'(x)$ depends only on $Z$ and not on $f_Y$. The following corollary gives the answer.

**Corollary 3.1.** Let $f(z')\nu_{Z'}(dz')$ be a distribution on $Z'$ such that $f(z')$ is almost everywhere positive on $Z'$ with respect to $\nu_{Z'}$. Suppose there exists a coset $g\mathcal{N}', g \in \mathcal{G}$, with respect to the normalizer $\mathcal{N}'$ of $\mathcal{G}'_0 = \mathcal{G}_{z'}$, $z' \in Z'$, such that

\begin{equation}
\Delta^{-1}(\{f(z')\}) \cap g\mathcal{N}' \neq \emptyset
\end{equation}

for each $z' \in Z'$ with positive $f(z')$. Then there exists a global cross section $Z$ such that the distribution of $z' = z'(x)$ coincides with $f(z')\nu_{Z'}(dz')$ for $x$ having an arbitrary cross-sectionally contoured distribution with respect to $Z$.

**Proof.** For any $z' \in Z'$ with $f(z') > 0$, we can choose $g(z') \in \mathcal{G}$ such that $g(z')^{-1} \in \Delta^{-1}(\{f(z')\}) \cap g\mathcal{N}'$. Take a cross section $Z = \{g(z')^{-1}z' : z' \in Z'\}$. Then $Z$ is global with the common isotropy subgroup $g\mathcal{G}'_0g^{-1}$. Writing $z' \in Z'$ as $z' = g(z') \cdot g(z')^{-1}z'$, we see that $g(z')$ can serve as a selection of the equivariant part of $z'$ with respect to $Z$. Theorem \ref{thm:3.3} implies that the density of $z' = z'(x)$ is $\Delta(g(z')^{-1}) = f(z')$. \hfill $\blacksquare$

**Remark 3.1.** When the action is free, we have $\mathcal{N}' = \mathcal{G}$ and condition \cite{13} is satisfied as long as $\Delta$ is not identically equal to 1 since $\Delta$ is a continuous homomorphism from $\mathcal{G}$ to $\mathbb{R}_{+}^*$. If in addition $\mathcal{G}$ is unimodular $\Delta^0 \equiv 1$ (e.g., abelian or compact), then condition $\Delta = \chi \neq 1$ is equivalent to $\lambda$ not being an invariant measure.

### 4 Star-shaped distributions

In this section, we define star-shaped distributions in $\mathbb{R}^p$ and investigate their properties. Most results presented here are easy consequences of the general arguments in the preceding section, but also included here are results which can be obtained only after regarding the orbits and cross sections as submanifolds of $\mathbb{R}^p$.

Let $\mathcal{G} = \mathbb{R}_+^*$ and define its action on $\mathcal{X} = \mathbb{R}^p - \{0\}$ by

\begin{equation}
(g, (x_1, \ldots, x_p)) \mapsto (gx_1, \ldots, gx_p).
\end{equation}

Under this action, the Lebesgue measure $dx$ is relatively invariant with multiplier $\chi(g) = g^p$. We take $\lambda(dx) = dx$ as the dominating measure. Note that the origin has Lebesgue measure zero and that omitting it in the sample space $\mathcal{X} = \mathbb{R}^p - \{0\}$ does not affect the
discussion about the distributions in \( X \). By so doing, we have made the sample space have just one orbit type (Appendix B) and made our action (14) free.

Since the action is free, we know from Remark 2.1 that choosing a unit cross section \( Z \) is equivalent to choosing an equivariant function from \( X \) to \( \mathbb{R}^*_+ \). Now, let \( g : X \to \mathbb{R}^*_+ \) be an equivariant function. We call distributions with the densities of the form

\begin{equation}
(15) \quad f(x) = f_g(g(x))
\end{equation}

star-shaped distributions. Obviously, this reduces to the elliptically contoured distributions when \( g(x) = (x^t \Sigma^{-1} x)^{1/2} \) with \( \Sigma \in PD(p) \) (the set of \( p \times p \) positive definite matrices).

The orbits under (14) are rays emanating from the origin, so the unit cross section \( Z = \{ x : g(x) = 1 \} \) associated with \( g \) is a set which meets each ray exactly once. Hence,

\begin{equation}
(16) \quad \bigcup_{0 \leq c \leq 1} cZ
\end{equation}

contains every line segment connecting the origin and a point on \( Z \). Namely, (16) is a star-shaped set with respect to the origin. This is why we call the distributions with densities of the form (15) star-shaped. (For the term “star-shaped,” see also Definition 3.1 of [27].)

Throughout this section, we assume \( x \) is distributed according to \( f_g(g(x))dx \).

A version of the Haar measures on \( G = \mathbb{R}^*_+ \) is given by \( g^{-1} dg \). By Theorem 3.1 \( g = g(x) \) and \( z = x/g(x) \) are independent and the joint distribution of \( g \) and \( z \) can be written as

\[
\frac{1}{c_0} f_g(g) g^{p-1} dg \times \nu_Z(dz),
\]

where \( c_0 = \int_0^\infty f_g(g) g^{p-1} dg \), and \( \nu_Z \) is a probability measure on \( Z \). Note that we have taken \( \mu_G(dg) = c_0^{-1} g^{-1} dg \).

For action (14), the most standard cross section is the unit sphere \( Z' = \mathbb{S}^{p-1} = \{ x \in X : g'(x) = 1 \} \), where \( g'(x) = \| x \| = (x^t x)^{1/2} \) is the usual Euclidean length of \( x \in \mathbb{R}^p - \{ 0 \} \). Now, \( dx \) obviously factors as

\[
dx = g^{p-1} dg' dz' = \frac{1}{c_0} g^{p-1} dg' \times c_0 dz',
\]

where \( dz' \) is the volume element of \( \mathbb{S}^{p-1} \). Since \( G = \mathbb{R}^*_+ \) is abelian, we have \( \Delta(g) = \chi(g) = g^p \). Thus, by Theorem 3.3 the distribution of the direction vector \( z' = x/\| x \| \) is obtained as

\begin{equation}
(17) \quad c_0 g(z')^{-p} dz',
\end{equation}

from which another expression of \( c_0 \) can be given: \( c_0 = 1 / \int_{\mathbb{S}^{p-1}} g(z')^{-p} dz' \).

When (15) is an elliptically contoured density, (17) becomes \( c_0 (z^t \Sigma^{-1} z')^{-p/2} dz' \). Normalizing constant \( c_0 \), being independent of the choice of \( f_g(\cdot) \), can be obtained by considering the particular case of normality \( f_g(g) = (2\pi)^{-p/2} (\det \Sigma)^{-1/2} \exp(-g^2/2) \) as

\[
c_0 = (2\pi)^{-\frac{p}{2}} (\det \Sigma)^{-\frac{1}{2}} \int_0^\infty e^{-\frac{g^2}{2}} g^{p-1} dg = \omega_p^{-1} (\det \Sigma)^{-\frac{1}{2}},
\]

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where \( \omega_p = 2\pi^{p/2}/\Gamma(p/2) = \int_{S^{p-1}} dz' \) is the total volume of \( S^{p-1} \). This distribution

\[
(18) \quad \frac{1}{\omega_p (\det \Sigma)^{\frac{1}{2}}} (z^T \Sigma^{-1} z')^{-\frac{1}{2}} dz'
\]

is derived in Section 3.6 of [30]. Our (17) is a generalization of (18) to the case of an arbitrary (i.e., not necessarily elliptically contoured) star-shaped density.

**Remark 4.1.** Distribution (18) has been studied in several parts of the literature. [30] also notes that (18) can be thought of as a special case of the angular Gaussian distribution, and discusses some of its properties. Several arguments about statistical inferences based on this model are given in [29]. See also Sections 9.4.4, 10.3.5 and 10.7.1 of [26].

The special case \( p = 2 \) is treated in [22] and Section 3.5.6 of [26].

The distribution (18) of \( z' = x/\|x\| \), as well as that of \( z^T \Sigma^{-1} z' = x^T \Sigma^{-1} x/x^T x \), plays an important role in null robust testing problems. See, e.g., [19] and [23].

We now investigate star-shaped distributions more closely by viewing the orbits and cross sections as submanifolds of \( \mathbb{R}^p \). We make the additional assumption that \( g(x) \) is piecewise of class \( C^1 \).

Fix \( z_0 \in \mathcal{Z} \) and call \( M_C(z_0) = \mathcal{Z} \) the cross section manifold and \( M_O(z_0) = \{ u_1 z_0 : u_1 > 0 \} \) the orbit manifold through \( z_0 \). The tangent vector of \( M_O(z_0) \) at \( z_0 = v_1 = z_0 \) is \( v_1 = z_0 \). Choose local coordinates \( u_2, \ldots, u_p \) of \( M_C(z_0) = \mathcal{Z} \) such that \( v_j = \frac{\partial}{\partial u_j} z(0, \ldots, 0, u_j, 0, \ldots, 0)|_{u_j=0} \), \( j = 2, \ldots, p \), are orthonormal vectors. Then \( du_2 \cdots du_p \) is the volume element of \( M_C(z_0) \) at \( z_0 \). Writing \( x = u_1 z(u_2, \ldots, u_p) \), we see that

\[
(19) \quad dx = |\det(v_1, \ldots, v_p)| \times du_1 \times du_2 \cdots du_p,
\]

where \( (v_1, \ldots, v_p) \) denotes the matrix consisting of columns \( v_1, \ldots, v_p \).

Let \( n_{z_0} \) be the unit normal vector of \( \mathcal{Z} \) at \( z_0 \) pointing outward of the star-shaped set \( \bigcup_{0 \leq x \leq 1} c\mathcal{Z} \). Write \( v_1 = z_0 \) as a linear combination of the orthonormal vectors \( n_{z_0}, v_2, \ldots, v_p \) as \( z_0 = a_1 n_{z_0} + a_2 v_2 + \cdots + a_p v_p \). Then \( |\det(v_1, \ldots, v_p)| = a_1 = \langle z_0, n_{z_0} \rangle = z_0^T n_{z_0} \), and (19) is written as \( dx = du_1 \times du_2 \cdots du_p \times \langle z_0, n_{z_0} \rangle \). For the rest of this section \( \langle \cdot, \cdot \rangle \) denotes the standard inner product of \( \mathbb{R}^p \). Rewrite this further as

\[
(20) \quad dx = \|z_0\|du_1 \times du_2 \cdots du_p \times \left\langle \frac{z_0}{\|z_0\|}, n_{z_0} \right\rangle.
\]

Note that the first term \( \|z_0\|du_1 = \sqrt{\langle v_1, v_1 \rangle} du_1 \) in (20) is the volume element of \( M_O(z_0) \) around \( z_0 \). The second term \( du_2 \cdots du_p \) is the volume element of \( M_C(z_0) \) as mentioned above. Concerning the third term, let \( \theta \) denote the angle between \( z_0 \) and \( T_{z_0}(M_C) \), where \( T_{z_0}(M_C) \) stands for the tangent space of \( M_C(z_0) \) at \( z_0 \). Then \( \pi/2 - \theta \) is the angle between \( z_0 \) and \( n_{z_0} \), and the third term in (20) can be written as \( \langle z_0/\|z_0\|, n_{z_0} \rangle = \sin \theta \). Therefore, (20) means that \( dx \) can be factored into the volume elements of \( M_O(z_0) \) and \( M_C(z_0) \) and the sine of the angle between \( T_{z_0}(M_O) \) and \( T_{z_0}(M_C) \).

We note in passing that the unit normal vector \( n_{z_0} \) coincides with the normalized gradient of \( g(x) \), i.e., \( n_{z_0} = \nabla g(z_0)/\|\nabla g(z_0)\| \). We also note the following fact. Let \( H_{z_0} = z_0 + T_{z_0}(M_C) \) be the tangent hyperplane of \( \mathcal{Z} \) at \( z_0 \). Then

\( \langle z_0, n_{z_0} \rangle = \) Euclidean distance from the origin to \( H_{z_0} \).
which is the support function at \(z_0\) (Section 8.1 of [15]).

Now consider the translation by \(g \in \mathcal{G} = \mathbb{R}^+_1\) from \(x = z_0\) to \(x = g z_0\). Since this translation is just the scale change, its effect is straightforward. The volume element of the orbit manifold \(M_0(z_0)\) is multiplied by \(g\), and the volume element of the cross section manifold \(M_C(z_0)\) is multiplied by \(g^{p-1}\), with \(p - 1\) being the dimensionality of \(M_C(z_0)\). Furthermore, the angle between these two manifolds remains unchanged under the translation. Therefore, around \(x = g z_0\) the volume element \(dx\) is

\[
dx = \|z_0\|dg \times g^{p-1}dz \times \left\langle \frac{z_0}{\|z_0\|}, n_{z_0} \right\rangle = \frac{1}{c_0}g^{p-1}dg \times c_0 \langle z_0, n_{z_0} \rangle dz,
\]

where \(dg = g du_1\) is the volume element of \(\mathcal{G} = \mathbb{R}^+_1\) around \(g \in \mathcal{G}\), and \(dz\) is the volume element of \(\mathcal{Z} = M_C(z_0)\). Therefore, the distribution \(\nu_\mathcal{Z}\) of \(z\) can be expressed as \(\nu_\mathcal{Z}(dz) = c_0 \langle z_0, n_{z_0} \rangle dz\).

We list some examples of star-shaped distributions.

(a) Elliptically contoured distribution: When \(g(x) = (x'\Sigma^{-1}x)^{1/2}\), we have \(\langle z, n_z \rangle = \langle z, \Sigma^{-1}z/\|\Sigma^{-1}z\| \rangle = (z'\Sigma^{-2}z)^{-1/2}\) for \(z \in \mathcal{Z} = \{x : g(x) = 1\}\). So in this case, \(\nu_\mathcal{Z}(dz)\) has density \(\omega_p^{-1}(\det \Sigma)^{-1/2}(z'\Sigma^{-2}z)^{-1/2}\) with respect to the volume element of the ellipsoid \(\{z \in \mathbb{R}^p : z'\Sigma^{-1}z = 1\}\).

(b) "Hypercube distribution": Take \(g(x) = \max(|x_1|, \ldots, |x_p|), \ x = (x_1, \ldots, x_p)\). Then the unit cross section \(\mathcal{Z}\) is the surface of the hypercube \(C_p\) in \(\mathbb{R}^p\) and we have \(\langle z, n_z \rangle = 1\) on the relative interiors of the facets of \(C_p\). Hence \(\nu_\mathcal{Z}(dz)\) is the uniform distribution on \(\partial C_p\). Constant value of the density is \(c_0 = 1/\text{Vol}_{p-1}(\partial C_p) = 1/(2p \times 2^{p-1}) = 1/(2^p)\).

(c) "Crosspolytope distribution," also known as \(\ell_1\)-norm symmetric distribution ([9] [8] [10]): Let \(g(x) = |x_1| + \ldots + |x_p|, \ x = (x_1, \ldots, x_p)\). The associated unit cross section \(\mathcal{Z}\) is the surface of the crosspolytope \(C_p^\Delta\), which is polar to \(C_p\) (Chapter 0 of [32]). Since \(\langle z, n_z \rangle\), the distances of the facets of \(C_p^\Delta\) from the origin, are constant \((= 1/\sqrt{p})\) by symmetry, \(\nu_\mathcal{Z}(dz)\) has constant density \(c_0/\sqrt{p} = 1/\text{Vol}_{p-1}(\partial C_p^\Delta) = 1/(2^p \times \sqrt{p}/(p-1)!\} = (p-1)!/(2^p \sqrt{p})\).

(d) Take \(\mathcal{Z}\) to be the surface of a \(p\)-dimensional polytope \(P\) \((0 \in \text{int}(P))\) whose facets are not equidistant from the origin. Then we obtain a non-uniform distribution of \(z\) on \(\partial P\).

We now summarize our results in this section in the following theorem.

**Theorem 4.1.** Suppose the distribution of \(x \in \mathbb{R}^p - \{0\}\) has a star-shaped density \(f_G(g(x))\) with respect to \(dx\). Then \(g = g(x)\) and \(z = x/g(x)\) are independent and the joint distribution of \(g\) and \(z\) is written as

\[
\frac{1}{c_0}f_G(g)g^{p-1}dg \times \nu_{\mathcal{Z}}(dz),
\]

(21)
where \( c_0 = \int_0^\infty f_G(g) g^{p-1} dg \) and \( \nu_z \) is a probability measure on \( \mathcal{Z} = \{ x : g(x) = 1 \} \).

Let \( dz' \) denote the volume element of the unit sphere \( S^{p-1} \subset \mathbb{R}^p \). Then \( c_0 \) can also be written as \( c_0 = 1/ \int_{S^{p-1}} g(z')^{-p} dz' \), and the distribution of the direction \( z' = x/\|x\| \) is given by \( c_0 g(z')^{-pdz'} \).

Under the additional assumption that \( g(x) \) is piecewise of class \( C^1 \), we can write the \( \nu_z (dz) \) in (21) as \( \nu_z (dz) = c_0 (z, \nu_z) dz, \) where \( \nu_z \) is the outward unit normal vector of \( \mathcal{Z} \) and \( dz \) on the right-hand side is the volume element of \( \mathcal{Z} \).

In addition, Corollary 3.1 together with Remark 3.1 yields the following result for the case of star-shaped distributions.

**Corollary 4.1.** Suppose we are given an arbitrary distribution on \( S^{p-1} \) which has almost everywhere positive density \( f(z') \) with respect to the volume element \( dz' \) on \( S^{p-1} \). Then we can realize this distribution \( f(z')dz' \) as the distribution of the direction \( z' = x/\|x\| \) of \( x \in \mathbb{R}^p - \{0\} \) which is distributed according to a star-shaped distribution.

**Proof.** Since \( g(z')^{-1} \in \Delta^{-1}(\{f(z')\}) = \{f(z')^{1/p}\} \), we may take \( \mathcal{Z} = \{f(z')^{1/p}z' : z' \in S^{p-1}\} \) and \( g(x) = \|x\|g(x/\|x\|) = \|x\|f(x/\|x\|)^{-1/p} \).

5 Applications to random matrices

In this section we consider cross-sectionally contoured distributions of random matrices. For illustrative purposes, we consider a generalization of matrix beta distribution by taking actions of the triangular group and the general linear group. These groups are not commutative. Furthermore, the action of the general linear group is not free. Therefore, the results of Sections 2 and 3 can be fully illustrated by this example. Other examples of decomposable distributions of random matrices are given in [28] and [18]. See also [8] for a generalization of elliptically contoured distribution to random matrices.

Let \( W_1 = (w_{1,ij}) \) and \( W_2 = (w_{2,ij}) \) be two \( p \times p \) positive definite matrices. The sample space \( \mathcal{X} \) is \( \{W = (W_1, W_2) : W_1, W_2 \in PD(p)\} \) (Section 5.1) or essentially this set but with some exceptional null subset removed (Section 5.2).

As a dominating measure on \( \mathcal{X} \), we consider

\[
\lambda(dW) = (\det W_1)^{a-p/2} (\det W_2)^{b-p/2} dW_1 dW_2,
\]

where \( a, b > (p-1)/2 \) and \( dW_1 = \prod_{1 \leq i < j \leq p} dw_{1,ij}, \) \( dW_2 = \prod_{1 \leq i < j \leq p} dw_{2,ij}. \)

5.1 Action of the triangular group

First we consider the action of the lower triangular group. Let \( LT(p) \) denote the group consisting of \( p \times p \) lower triangular matrices with positive diagonal elements. Then \( \mathcal{G} = LT(p) \) acts on

\[
\mathcal{X} = \{(W_1, W_2) : W_1, W_2 \in PD(p)\}
\]

by

\[
(T, (W_1, W_2)) \mapsto (TW_1T^t, TW_2T^t), \quad T \in LT(p).
\]
This action is free and any cross section under this action is global.

It is interesting to note that there are two common cross sections used in the literature. Let \( TT^t = W_1 + W_2 \) be the Cholesky decomposition of \( W_1 + W_2 \). Then \( T \) itself is an equivariant function and \( U = T^{-1}W_1(T^{-1})^t \) is the associated invariant function. If \( W_1 \) and \( W_2 \) are independent Wishart matrices, then \( U \) has the matrix beta distribution. On the other hand, let \( TT^t = W_2 \) be the Cholesky decomposition of \( W_2 \). Then the invariant \( F = T^{-1}W_1(T^{-1})^t \) has the matrix F distribution ([7], Chapter 5 of [12]).

Here we prefer to consider the Cholesky decomposition of \( W_1 + W_2 \) and use the following beta-type cross section:

\[
Z' = \{(U, I_p - U) : 0 < U < I_p\} \subset PD(p) \times PD(p),
\]

where \( I_p \) denotes the \( p \times p \) identity matrix and \( A < B \) means \( B - A \in PD(p) \) for \( p \times p \) symmetric \( A \) and \( B \). The orbital decomposition of \( W = (W_1, W_2) \) with respect to \( Z' \) is written as

\[
(W_1, W_2) = (TUT^t, T(I_p - U)T^t), \quad T = T(W), \ U = U(W).
\]

Next we move on to a general cross section. By using Remark 2.2 in the opposite direction, we obtain a general cross section \( Z' \) :

\[
Z = \{z_U : 0 < U < I_p\}
\]

with

\[
z_U = (S(U)US(U)^t, S(U)(I_p - U)S(U)^t),
\]

where \( S(U) = (s_{ij}(U)) \) is a function from \( \{U : 0 < U < I_p\} \) to \( LT(p) \). Then the associated equivariant function is

\[
g(W) = T(W)S(U(W))^{-1}
\]

by Remark 2.3 and the invariant part is

\[
z(W) = z_U(W) = (S(U(W))U(W)S(U(W))^t, S(U(W))(I_p - U(W))S(U(W))^t).
\]

Using a density of the form

\[
f(W) = f_G(g(W))
\]

with respect to \( \lambda(dW) \) in (22), we obtain a cross-sectionally contoured distribution with respect to \( Z \). Now the application of Theorems 3.1 and 3.3 gives the following results about the distributions of \( G = g(W), \ Z = z(W) \) and \( U = U(W) \). Note that \( U(W) \) is in one-to-one correspondence with the invariant part \( z'(W) = (U(W), I_p - U(W)) \) with respect to \( Z' \).

**Theorem 5.1.** Suppose that the distribution of \( W = (W_1, W_2) \) is given as

\[
f_G(g(W))(\det W_1)^{\frac{a+1}{2}}(\det W_2)^{\frac{b+1}{2}}dW_1dW_2
\]
with some $f_G: LT(p) \rightarrow \mathbb{R}$. Then $G = (g_{ij}) = g(W)$ and $Z = z(W)$ are independent, and their joint distribution is given by

$$
\frac{1}{c_0} f_G(G) \prod_{i=1}^{p} g_{ii}^{2(a+b)-i} dG \times \nu_Z(dZ),
$$

where $c_0 = \int_{LT(p)} f_G(G) \prod_{i=1}^{p} g_{ii}^{2(a+b)-i} dG$ and $\nu_Z$ is a probability measure on $Z$. Furthermore, $U = (u_{ij}) = U(W)$ is independent of $G = g(W)$, and its distribution is given by

$$
2^p c_0 \prod_{i=1}^{p} s_{ii}(U)^{2(a+b)+p-2i+1} (\det U)^{a-\frac{p+1}{2}} (\det(I_p - U))^{b-\frac{p+1}{2}} dU,
$$

where $dU = \prod_{1 \leq i \leq j \leq p} du_{ij}$.

**Proof.** Remember the following well-known facts: (a) The multiplier of relatively invariant measure $\lambda(dW)$ in (22) is $\chi(T) = (\det T)^{2(a+b)}$ (31 (9.1.4)), so $\chi(T) = \prod_{i=1}^{p} t_{ii}^{2(a+b)}$ for $T = (t_{ij}) \in LT(p)$; (b) For $LT(p)$, the left Haar measure is a multiple of $\prod_{i=1}^{p} t_{ii}^{-i} dT$ (31 (7.7.2)) and the right-hand modulus is $\Delta(T)^{LT(p)} = \prod_{i=1}^{p} t_{ii}^{-p+2i+1}$ (31 (7.7.6)); (c) With respect to the standard cross section $\mathcal{Z}$, we have the factorization $dW_1 dW_2 = 2^p \prod_{i=1}^{p} t_{ii}^{2p+2-i} dTdU$, $T = (t_{ij})$, for (23) (12) (10.3.5). With the help of these facts, the theorem follows immediately from Theorems 3.1 and 3.3.

### 5.2 Action of the general linear group

Consider the action of the general linear group $G = GL(p)$ consisting of all $p \times p$ nonsingular matrices. In this case, for there to exist a global cross section, we restrict the sample space as

$$
\mathcal{X} = \{(W_1, W_2) \in PD(p) \times PD(p) : \text{the p roots of } \det(W_1 - l(W_1 + W_2)) = 0 \text{ are all distinct}\}.
$$

If there are multiple roots in (24), there are more than one orbit type (Appendix B). As in the case of $LT(p)$, the action of $GL(p)$ is $(B, (W_1, W_2)) \mapsto (BW_1B^t, BW_2B^t)$, $B \in GL(p)$.

As a standard global cross section, we can take

$$
\mathcal{Z}' = \{(L, I_p - L) : L = \text{diag}(l_1, \ldots, l_p), \ 1 > l_1 > \cdots > l_p > 0\}.
$$

For this $\mathcal{Z}'$, the common isotropy subgroup is

$$
G_0' = \{\text{diag}(\epsilon_1, \ldots, \epsilon_p) : \epsilon_1 = \pm 1, \ldots, \epsilon_p = \pm 1\},
$$

and the normalizer of $G_0'$ is given as

$$
N' = \{P \in GL(p) : P \text{ has exactly one nonzero element in each row and in each column}\},
$$

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which is the group generated by permutation matrices and nonsingular diagonal matrices. The orbital decomposition of $W = (W_1, W_2)$ with respect to $Z'$ can be written as

$$
(W_1, W_2) = (B_L B^t, B(I_p - L) B^t) = (B(W) L(W) B(W)^t, B(W)(I_p - L(W)) B(W)^t).
$$

In this representation, $L(W) = \text{diag}(l_1(W), \ldots, l_p(W))$ is uniquely determined by $W$, but $B(W)$ is unique only up to the sign of each column of $B(W)$. We can use an arbitrary selection $B(W)$, e.g., the selection $B(W)$ such that $B(W) \in GL(p)/2^p$, where $GL(p)/2^p$ denotes the set of $p \times p$ nonsingular matrices whose first nonzero element in each column is positive. Note that it seems more convenient here to work with a selection $B(W)$ rather than the cosets of $G_0'$, although they are equivalent.

Now we turn to a general global cross section. Using Theorem 2.1 in the opposite direction, we find that a general global cross section $Z$ is of the form

$$Z = \{z_L : L = \text{diag}(l_1, \ldots, l_p), 1 > l_1 > \cdots > l_p > 0\},$$

where

$$z_L = (B_0 P(L) L P(L)^t B_0, B_0 P(L)(I_p - L) P(L)^t B_0^t)$$

with $B_0 \in GL(p)$ and $P(L) \in N'$. Without loss of generality, we take $B_0 = I_p$ in (27) so that the isotropy subgroup for $Z$ is also the $G_0'$ in (25). By Proposition 2.3 a selection of the equivariant part with respect to $Z$ is given by

$$B(W) P(L(W))^{-1},$$

but here we take the selection $g(W)$ which is given by changing the sign of each column of (28) if necessary so that $g(W) \in GL(p)/2^p$. The invariant part, on the other hand, is

$$z(W) = z_{L(W)} = (P(L(W)) L(W) P(L(W))^t, P(L(W))(I_p - L(W)) P(L(W))^t).$$

Consider a density of the form

$$f(W) = t(g(W)),$$

where $t : GL(p) \to \mathbb{R}$ satisfies $t(B B_1) = t(B)$, $B \in GL(p)$, $B_1 \in G_0'$. Then $t(g(W)) \lambda(dW)$ is a cross-sectionally contoured distribution with respect to $Z$. Applying Theorems 3.1 and 3.3 we obtain the following results about the distributions of $G = g(W)$, $Z = z(W)$ and $(l_1, \ldots, l_p) = (l_1(W), \ldots, l_p(W))$. Notice that $(l_1(W), \ldots, l_p(W))$ is in one-to-one correspondence with $z'(W) = (L(W), I_p - L(W)).$

**Theorem 5.2.** Suppose that the distribution of $W = (W_1, W_2)$ is

$$t(g(W)) (\det W_1)^{a - \frac{p+1}{2}} (\det W_2)^{b - \frac{p+1}{2}} dW_1 dW_2,$$

where $t : GL(p) \to \mathbb{R}$ is a real-valued function such that $t(B)$, $B \in GL(p)$, does not depend on the sign of each column of $B$. Then $G = g(W)$ and $Z = z(W)$ are independent, and their joint distribution is given by

$$\frac{1}{c_0} t(G) (\det G)^{2(a+b)-p} dG \times \nu_Z(dZ),$$
where \( c_0 = \int_{GL(p)/2^p} t(G)(\det G)^{2(a+b)-p} dG = 2^{-p} \int_{GL(p)} t(G)(\det G)^{2(a+b)-p} dG \) and \( \nu_Z \) is a probability measure on \( Z \). Furthermore, \((l_1, \ldots, l_p) = (l_1(W), \ldots, l_p(W))\) is independent of \( G = g(W)\), and its distribution is given by

\[
2^p c_0 (\det P(\text{diag}(l_1, \ldots, l_p)))^{2(a+b)} \prod_{i=1}^{p} \frac{a - \frac{i-1}{2}}{b - \frac{i-1}{2}} \prod_{i<j} (l_i - l_j) dl_1 \cdots dl_p.
\]

**Proof.** This theorem is a direct consequence of Theorems 3.1 and 3.3. We only have to recall the following easy or well-known facts: (a) The (left) Haar measure \( \mu_{GL(p)} \) on \( GL(p) \) is a multiple of \( (\det B)^{-p} dB \); (b) \( GL(p) \) is unimodular, so \( \Delta(B) = \chi(B) = (\det B)^{2(a+b)}, B \in GL(p) \); (c) In terms of the standard global cross section \( Z' \), we have the factorization \( dW_1 dW_2 = 2^p(\det B)^{p+2} dB \prod_{i<j}(l_i - l_j) dl_1 \cdots dl_p \) for [26] Theorem 13.2.1.

### Appendix

**A Variety of global cross sections**

In this Appendix, we discuss the construction of general global cross sections from a given global cross section and characterize the class of all global cross sections in terms of the normalizer of the common isotropy subgroup. The proof of Theorem 2.1 is provided in this Appendix in particular, but a thorough investigation into the variety of global cross sections, including the uniqueness of \( n_z \) in [3], is also conducted here. This material was partly discussed in [19] but here we give a complete characterization.

#### A.1 Action of a factor group on each \( \mathcal{G} \)-orbit

We begin by confining our discussion to the action of \( \mathcal{G} \) on each \( \mathcal{G} \)-orbit \( \tilde{X} = \mathcal{G} x_0 \) with \( x_0 \in X \). For an arbitrary point \( x \in \tilde{X} \), let

\[
N_x := \{ g \in \mathcal{G} : g \mathcal{G}_x g^{-1} = \mathcal{G}_x \}
\]

be the normalizer of \( \mathcal{G}_x = \{ g \in \mathcal{G} : gx = x \} \) in \( \mathcal{G} \). Then, since \( \mathcal{G}_{gx} = g \mathcal{G}_x g^{-1} \) for \( g \in \mathcal{G} \) and \( x \in \tilde{X} \), we have that the normalizers satisfy [21] p.33

\[
N_{gx} = gN_x g^{-1}, \quad g \in \mathcal{G}, \quad x \in \tilde{X}.
\]

Now, consider the factor group

\[
\mathcal{M}_x := N_x / \mathcal{G}_x = \{ n\mathcal{G}_x : n \in N_x \}
\]

for each \( x \in \tilde{X} \). Then we have the following proposition:

**Proposition A.1.** All factor groups \( \mathcal{M}_x, x \in \tilde{X} \), are isomorphic to one another.
Proof. For given $M_x$ and $M_{x'}$, $x, x' \in \tilde{X}$, take an element $g \in G$ such that $x' = gx$. Then we have $G_{x'} = gG_xg^{-1}$, so Lemma 1.51 of [21] implies that the mapping
\[ \tau_{x',x} : M_x \rightarrow M_{x'} \]
defined as $m = nG_x \mapsto (gng^{-1})(gG_xg^{-1}) = gmg^{-1}$, $n \in N_x$, serves as an isomorphism. $\blacksquare$

Fix $z \in \tilde{X}$ as a reference point, and write
\[ G_0 = G_z, \quad N = N_z \quad \text{and} \quad M = M_z. \]

Now we define an action of $M$ on $\tilde{X}$ as follows:

Let $\tau_x$, $x \in \tilde{X}$, be an arbitrary selection of $\tau_{x,z}$:
\[ \tau_x(m) \in \tau_{x,z}(m) \subset N_x, \quad m \in M. \]

Using this $\tau_x$, we define $xm$, $x \in \tilde{X} = Gz$, $m \in M$, as
\[ xm := \tau_x(m)x. \]

If we write $x = gz$, $g \in G$ and $m = nG_0$, $n \in N$, we can express $xm$ as
\[ (29) \quad xm = \tau_x(m)x = (gng^{-1})(gz) = gnz. \]

We can confirm that this is well-defined: For $m = nG_0 = ng_0G_0$, $g_0 \in G_0$, and $x = gz = gg_0z$, $g_0' \in G_0$, we have $xm = (gg_0')(ng_0)z = gg'_0nz = gng_0'z = gnz$ since $g'_0n = ng''_0$ for some $g''_0 \in G_0$.

Moreover, $xm$, $x \in \tilde{X}$, $m \in M$, has the following property.

Lemma A.1. For any $x \in \tilde{X}$ and $m_1, m_2 \in M$, we have
\[ (xm_1)m_2 = x(m_1m_2). \]

Proof. For $x = gz \in \tilde{X}$, $g \in G$, and $m_i = n_iG_0 \in M$, $n_i \in N$, $i = 1, 2$, we have
\[ (xm_1)m_2 = (gn_1z)m_2 = (gn_1)n_2z = g(n_1n_2)z = x(m_1m_2) \]
since $m_1m_2 = (n_1n_2)G_0$. $\blacksquare$

So we have a right action of $M$ on $\tilde{X}$:

Proposition A.2. The mapping
\[ (x, m) \mapsto xm, \quad x \in \tilde{X}, \quad m \in M, \]
is a right action of $M$ on $\tilde{X}$. Moreover, this action is free.

Proof. It is easy to see that $e_M$ (the identity element of $M$) satisfies $xe_M = x$ for all $x \in \tilde{X}$. This fact together with lemma A.1 implies that the mapping $(x, m) \mapsto xm$ is a right action.

Next we show that this action is free. Suppose $xm = x$ for $m = nG_0$, $n \in N$ and $x = gz$, $g \in G$. Then, $gnz = gz$ or $nz = z$. So $n \in G_0$, and $m = nG_0 = G_0 = e_M$. $\blacksquare$

As the following proposition shows, the $M$-orbits $xM$, $x \in \tilde{X}$, can be characterized in terms of the isotropy subgroups $G_x$, $x \in \tilde{X}$, under the action of $G$ on $\tilde{X}$.
Proposition A.3. For \( x, x' \in \tilde{X} \), we have that
\[
xM = x'M \quad \text{if and only if} \quad G_x = G_{x'}.
\]

Proof. Suppose \( xM = x'M \). Then, writing \( x' = xm, \ x = gz, \ g \in G, \ m = nG, \ n \in N \), we can calculate \( G_{x'} = G_{xm} = G_{gz}^{-1}g^{-1} = G_{x}g^{-1} = G_{g} = G_{x} \).

Conversely, suppose \( G_{x} = G_{x'} \). Then, writing \( x' = g'x, \ g' \in G, \ x = gz, \ g \in G \), we have \( G_{x'} = G_{g'x} = g'G_{x}g'^{-1} = G_{x} \), and thus \( g' \in N_x = gNz^{-1} \). Accordingly, we can write \( g' = gng^{-1}, \ n \in N, \) and hence \( x' = g'x = (gng^{-1})(gz) = gnz = xm \) for \( m = nG \in M \). Therefore, \( xM = x'M \).

So the isotropy subgroups \( G_x, \ x \in \tilde{X} \), are constant on each \( M \)-orbit and different on different \( M \)-orbits. Therefore, the \( M \)-orbits can be labeled by the isotropy subgroups \( G_x, \ x \in \tilde{X} \), which do not depend on the choice of the reference point \( z \).

Now we have two groups \( G \) and \( M \) acting on \( \tilde{X} \). These two actions commute with each other:

Proposition A.4. The actions of \( G \) and \( M \) on \( \tilde{X} \) commute:
\[
(30) \quad g(xm) = (gx)m, \ g \in G, \ m \in M, \ x \in \tilde{X}.
\]

Proof. Writing \( x = gz, \ g' \in G, \) and \( m = nG, \ n \in N \), we can deduce \( g(xm) = g(g'zn) = (g'z)n = (g'z)m = (gx)m \).

Thus, we can say that \( \tilde{X} \) is a \( G-M \) bispace. We will write \( g(xm) \) and \( (gx)m \) as \( gxm \) without ambiguity.

As we saw in Proposition A.3, the \( M \)-orbits \( xM, \ x \in \tilde{X} \), can be labeled without referring to the reference point \( z \), but the \( M = M_z \) itself does depend on \( z \). By using the commutativity of the actions of \( G \) and \( M \) in Proposition A.4 and considering proportional translations by the action of \( G \), we can identify the elements of \( M \) in terms of relative positions of two points of \( \tilde{X} \) and thereby get rid of \( z \) as follows:

Since the action of \( M \) on \( \tilde{X} \) is free by Proposition A.2, we know that \( M \) can be identified with an \( M \)-orbit:

\[
\begin{align*}
M & \leftrightarrow xM, \quad x \in \tilde{X}. \\
\end{align*}
\]

So we can see \( M \) as \( zM \) in particular, and hence as \( \{z\} \times zM \) by \( m \leftrightarrow (z, zm) \):

\[
\begin{align*}
M & \leftrightarrow \{z\} \times zM, \quad m \leftrightarrow (z, zm), \ m \in M. \\
\end{align*}
\]

Indicate by \( \Pi \) the set of ordered pairs of points of \( \tilde{X} \) on the same \( M \)-orbits:

\[
\Pi = \{(x_1, x_2) \in \tilde{X} \times \tilde{X} : x_1M = x_2M\},
\]

and define an equivalence relation \( \sim_\Pi \) among the elements of \( \Pi \) as follows:

\[
(x_1, x_2) \sim_\Pi (x_1', x_2') \iff g(x_1, x_2) = (x_1', x_2') \quad \text{for some} \quad g \in G,
\]

where \( g(x_1, x_2) \) is the proportional translate of \( (x_1, x_2) \) by \( g \):

\[
g(x_1, x_2) := (gx_1, gx_2),
\]
i.e., the diagonal action. We denote the equivalence class under \( \sim \) by \([ \cdot ]_\Pi\). Then, we can think of \( \Pi/ \sim \) as a group with the following product:

\[
([x_1, x_2])_\Pi \cdot ([x_3, x_4])_\Pi = ([x_1, gx_4])_\Pi,
\]

where \( g \) is an arbitrary element of \( G \) satisfying \( x_2 = gx_3 \). That is, \( ([x_1, x_2])_\Pi \cdot ([x_2, gx_4])_\Pi = ([x_1, gx_4])_\Pi \). We can check that operation (31) is well-defined.

**Proposition A.5.** The factor group \( \mathcal{M} \) is isomorphic to the group \( \Pi/ \sim \).

**Proof.** Consider the following two mappings:

\[
(32) \quad \mathcal{M} \ni m \mapsto ([z, zm])_\Pi \in \Pi/ \sim,
\]

\[
(33) \quad \Pi/ \sim \ni ([x_1, x_2])_\Pi \mapsto \text{the unique } m \in \mathcal{M} \text{ such that } x_2 = x_1 m.
\]

It can be verified that (32) and (33) are the inverse mappings of each other. Moreover, we can show that (32) is a homomorphism in the following way: For \( m_1, m_2 \in \mathcal{M} \), we have \( ([z, z(m_1 m_2)])_\Pi = ([z, (n_1 n_2)z])_\Pi = ([z, n_1 (zm_2)])_\Pi = ([z, zm_1])_\Pi \cdot ([z, zm_2])_\Pi \) with \( n_1 \in m_1 \) and \( n_2 \in m_2 \).

Note that thanks to Proposition A.3, the group \( \Pi/ \sim \) does not depend on the reference point \( z \). Proposition A.5 implies that an element of \( \mathcal{M} \) can be specified by an ordered pair \((x_1, x_2)\) of points of \( \tilde{X} \) having the same isotropy subgroup \( G_{x_1} = G_{x_2} \) if we identify all proportional translates \( g(x_1, x_2), \ g \in G \).

**A.2 Global cross sections on the whole sample space**

Let us get back to the action of \( G \) on the whole of \( X \). Throughout this subsection, we assume that there exists a global cross section. We agree that a global cross section always refers to the one under the action of \( G \) on \( X \).

For an arbitrary global cross section \( Z' \), we write

\[
\mathcal{G}_{0, Z'} = \mathcal{G}_{Z'}, \quad \mathcal{N}_{Z'} = \mathcal{N}_{Z'} \quad \text{and} \quad \mathcal{M}_{Z'} = \mathcal{M}_{Z'}
\]

with \( Z' \in Z' \).

First we note that the difference between two proportional global cross sections \( Z' \) and \( gZ' \) is not essential since they induce the same family of proportional global cross sections. So we introduce the equivalence relation \( \sim_{\text{gcs}} \) among the global cross sections by proportionality:

\[
Z_1 \sim_{\text{gcs}} Z_2 \iff Z_1 = gZ_2 \quad \text{for some } g \in G.
\]

The equivalence class under \( \sim_{\text{gcs}} \) is indicated by \([\cdot]_{\text{gcs}}\).

Fix \( Z \) as a reference global cross section, and put

\[
\mathcal{G}_0 = \mathcal{G}_{0, Z}, \quad \mathcal{N} = \mathcal{N}_Z \quad \text{and} \quad \mathcal{M} = \mathcal{M}_Z.
\]

For this \( Z \), let \( \iota = \iota_Z \), i.e., the natural one-to-one correspondence between \( X/\mathcal{G} \) and \( Z \). Based on the reference global cross section \( Z \), we can generate a global cross section from another global cross section in the following way.
Let $Z'$ be an arbitrary global cross section, and $M$ a mapping from $\mathcal{X}/\mathcal{G}$ to $\mathcal{M}$. Consider the following subset of $\mathcal{X}$:

\[(34)\] 

\[Z'M := \{ z'm_z : z' \in Z' \}, \]

where $m_x := M(Gx)$ for $x \in \mathcal{X}$. Note that $m_{gx} = m_x$, $g \in \mathcal{G}$, $x \in \mathcal{X}$.

In (34), $z'm_z$ is defined as in (29) under the action of $M$ on $\tilde{\mathcal{X}} = \mathcal{G}z'$ with $z = \iota(Gz') \in \mathcal{G}z' \cap Z$ as the reference point of $\mathcal{G}z'$. If we write $z' = g_z z$, $g_z \in \mathcal{G}$, and use an arbitrary $n_z \in m_z = m_{z'} \in M$ for each $z' \in Z'$, we obtain a more direct expression of definition (34):

\[Z'M = \{ g_z n_z z : z \in Z \}. \]

Now we have that $Z'M$ is also a global cross section:

**Theorem A.1.** Let $Z'$ be an arbitrary global cross section, and let $M$ be an arbitrary mapping from $\mathcal{X}/\mathcal{G}$ to $\mathcal{M}$. Then, $Z'M$ is a global cross section.

**Proof.** It is clear that $Z'M = \{ z'm_z : z' \in Z' \}$ is a cross section, since $z'$ and $z'm_z \in z'M \subset \mathcal{G}z'$ are on the same $\mathcal{G}$-orbit $\mathcal{G}z'$. Further, the cross section $Z'M$ is global because for $z' = g_z z$ and $n_z \in m_z$, we have $\mathcal{G}_{g_x n_z} = g_z n_z \mathcal{G} 0 n_z^{-1} g_z^{-1} = g_z \mathcal{G} z' = \mathcal{G}_{g_z z} = \mathcal{G}_{0, z'}$, common for all $z \in Z$.

Theorem A.1 implies, in particular, that for arbitrary $g \in \mathcal{G}$ and $M : \mathcal{X}/\mathcal{G} \to \mathcal{M}$, the subset $(g Z)M \subset \mathcal{X}$ is a global cross section. We will show below that the converse is true as well. That is, an arbitrary global cross section $Z'$ must be of this form:

\[Z' = (g Z)M, \quad g \in \mathcal{G}, \quad M : \mathcal{X}/\mathcal{G} \to \mathcal{M}. \]

Moreover, we want to study the uniqueness of $M$ in such an expression. To make the arguments succinct, we will introduce an action on the set of equivalence classes of global cross sections as follows.

Using $Z'M$ in (34), we define $[Z']_{\text{gcs}}M$ as

\[[Z']_{\text{gcs}}M := [Z'M]_{\text{gcs}} \]

for a global cross section $Z'$ and a mapping $M : \mathcal{X}/\mathcal{G} \to \mathcal{M}$. Note that $Z'M$ is a global cross section because of Theorem A.1. Thanks to the next lemma, this $[Z']_{\text{gcs}}M$ is well-defined.

**Lemma A.2.** Let $Z'$ be an arbitrary global cross section. Then, we have

\[(gZ')M = g(Z'M) \]

for any $g \in \mathcal{G}$ and $M : \mathcal{X}/\mathcal{G} \to \mathcal{M}$. 

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Proof. This is a direct consequence of Proposition [A.4]. Taking $x' \in Z'$ and $m = m_{gz'} = m_{z'}$ in (30), we obtain $(gz')m_{gz'} = g(z'm_{z'})$. The result follows from (31).

By this lemma, we can write $(gZ')M$ and $g(Z'M)$ as $gZ'M$.

Now, the set of mappings $M : \mathcal{X}/G \to \mathcal{M}$ forms a group with the product defined pointwise. The identity element is the constant map onto $e_M$, and the inverse elements are pointwise inverses in $\mathcal{M}$.

Viewing the set of mappings $M : \mathcal{X}/G \to \mathcal{M}$ in this way, we can regard

\[(35) \quad ([Z']_{gcs}, M) \mapsto [Z']_{gcs}M\]

as a right action of the group of mappings $M : \mathcal{X}/G \to \mathcal{M}$ on the set of equivalence classes of global cross sections $[Z']_{gcs}$. This can be seen from the following lemma.

Lemma A.3. For an arbitrary global cross section $Z'$ and mappings $M_1, M_2 : \mathcal{X}/G \to \mathcal{M}$, we have

\[(36) \quad ([Z']_{gcs}M_1)M_2 = [Z']_{gcs}(M_1M_2).\]

Proof. This follows essentially from Lemma [A.1]. Denoting $m_{i,x} := M_i(\mathcal{G}x)$, $x \in \mathcal{X}$, for $i = 1, 2$, we can write $Z'M_1 = \{z'm_{1,z'} : z' \in Z'\}$ and hence $(Z'M_1)M_2 = \{(z'm_{1,z'}m_{2,z} : z' \in Z'\}$. But since $m_{2,z}m_{1,z'} = m_{2,z'}$, we obtain $(Z'M_1)M_2 = \{(z'm_{1,z'}m_{2,z} : z' \in Z'\} = \{z'(m_{1,z'}m_{2,z}) : z' \in Z'\}$ as $g(Z'M_1)M_2$ by Lemma [A.1].

We can show that action (35) is transitive as follows: Let $Z'$ be an arbitrary global cross section. Then, since $Z'$ and $\mathcal{Z}$ are cross sections, we can write $Z' = \{g_z : z \in Z\}$ for some $g_z \in G$. Fix an arbitrary $z_0 \in Z$ and then we have $\mathcal{G}g_z = \mathcal{G}g_{z_0} = g_z\mathcal{G}g_{z_0}^{-1} = g_z\mathcal{G}g_z^{-1}$ or $g_z^{-1}g_z \in \mathcal{N}$ for all $z \in Z$. Putting $g = g_{z_0}$, we can represent $g_z = gn_z$ for certain $n_z \in \mathcal{N}$, and thus obtain

\[(37) \quad Z' = \{ gn_z : z \in Z\}. \]

Now define $M : \mathcal{X}/G \to \mathcal{M}$ as $M(\mathcal{G}z) = n_z\mathcal{G}0 \in \mathcal{M}$, $z \in Z$. With this $M$, we can express (30) as $Z' = gZM$ and then arrive at

\[(38) \quad [Z']_{gcs}M = [Z]_{gcs}M. \]

This proves that action (35) is transitive.

On the other hand, (35) is not a free action. However, we have a certain kind of uniqueness of $M$ in (37). To state this fact about uniqueness, we introduce an equivalence relation $\sim_m$ among mappings $M : \mathcal{X}/G \to \mathcal{M}$ in the following way:

\[(39) \quad M_1 \sim_m M_2 \Leftrightarrow M_1(\cdot) = \tilde{m}M_2(\cdot) \quad \text{for some } \tilde{m} \in \mathcal{M}. \]

We denote the equivalence class under $\sim_m$ by $[\cdot]_m$. Now we verify the uniqueness of $[M]_m$ for $M$ in (37):

Suppose $M' : \mathcal{X}/G \to \mathcal{M}$ also satisfies $[Z']_{gcs}M = [Z]_{gcs}M'$. Then we have $[Z]_{gcs}M = [ZM']_{gcs}$ and thus

\[(40) \quad \forall z \in Z : n_zz = gn'_z z \]
for some $g \in G$, where $n_z \in M(Gz)$ and $n'_z \in M'(Gz)$. By considering the isotropy subgroup at $n_z = gn'_z$, we can easily see that the above $g$ is in $N$. It follows from $\ref{3}$ that $n_z G_0 = gn'_z G_0$ or

$$M(Gz) = \bar{m}M'(Gz) \quad \text{for all } z \in Z$$

with $\bar{m} := gG_0 \in M$, and this means $[M]_m = [M']_m$.

Thus we have proved the following theorem.

**Theorem A.2.** Let $\mathcal{Z}'$ be an arbitrary global cross section. Then, there exists a mapping $M : \mathcal{X}/G \to M$ such that

$$[\mathcal{Z}']_{gcs} = [\mathcal{Z}]_{gcs} M.$$

Here, $[M]_m$ is uniquely determined.

Statements in Theorem $\ref{A.2}$ can be translated into relations between two arbitrary global cross sections $\mathcal{Z}_1$ and $\mathcal{Z}_2$. To do this, we need to introduce some notation.

For arbitrary mappings $M : \mathcal{X}/G \to M$ and global cross sections $\mathcal{Z}'$, we write

$$M Z':= M'M$$

with $M' : \mathcal{X}/G \to M$ such that $[\mathcal{Z}']_{gcs} = [\mathcal{Z}]_{gcs} M'$. Note that because of the uniqueness part of Theorem $\ref{A.2}$, $[MZ']_m$ does not depend on the choice of such $M'$.

**Corollary A.1.** Let $\mathcal{Z}_1$, $\mathcal{Z}_2$ be arbitrary global cross sections. Then, there exists a mapping $M_{12} : \mathcal{X}/G \to M$ such that

$$[\mathcal{Z}_2]_{gcs} = [\mathcal{Z}_1]_{gcs} M_{12}.$$

Here, $M_{12}$ is unique in the sense that for two such $M_{12}$ and $M'_{12}$, we have

$$[(M_{12})^{Z_1}]_m = [(M'_{12})^{Z_1}]_m.$$

**Proof.** As was shown above, action $\ref{35}$ is transitive. This implies the existence of $M_{12}$.

Now we show the uniqueness of $M_{12}$. Suppose there exists another $M'_{12} : \mathcal{X}/G \to M$ such that $[\mathcal{Z}_2]_{gcs} = [\mathcal{Z}_1]_{gcs} M'_{12}$. Then we have $[\mathcal{Z}_1]_{gcs} M_{12} = [\mathcal{Z}_1]_{gcs} M'_{12}$. But by virtue of Lemma $\ref{A.3}$, this can be written as $[\mathcal{Z}]_{gcs}(M_1 M_{12}) = [\mathcal{Z}]_{gcs}(M_1 M'_{12})$ with $M_1$ satisfying $[\mathcal{Z}_1]_{gcs} = [\mathcal{Z}]_{gcs} M_1$. Now, the uniqueness part of Theorem $\ref{A.2}$ yields $[M_1 M_{12}]_m = [M_1 M'_{12}]_m$. $lacksquare$

**Corollary A.2.** Suppose two global cross sections $\mathcal{Z}_1$, $\mathcal{Z}_2$ are related by

$$[\mathcal{Z}_2]_{gcs} = [\mathcal{Z}_1]_{gcs} M_{12}, \quad [\mathcal{Z}_1]_{gcs} = [\mathcal{Z}_2]_{gcs} M_{21}$$

for $M_{12}, M_{21} : \mathcal{X}/G \to M$. Then, these $M_{12}$ and $M_{21}$ are the inverse elements of each other in the sense that

$$[(M_{12})^{Z_1}]_m = [(M_{21}^{-1})^{Z_1}]_m, \quad [(M_{21})^{Z_2}]_m = [(M_{12}^{-1})^{Z_2}]_m.$$
Proof. Since \([Z_2]_{gcs} = [Z_1]_{gcs}M_{12} = [Z_1]_{gcs}M_{21}^{-1}\), we have directly from Corollary A.1 that \([(M_{12})Z_1]\)_m = \([(M_{21}^{-1})Z_2]\)_m. The relation \([(M_{21})Z_2]\)_m = \([(M_{12}^{-1})Z_1]\)_m is shown in a similar manner.

Now we have a characterization of global cross sections. Theorem A.2 implies that an arbitrary global cross section \(Z'\) must be written as \(Z' = gZM\) for some \(g \in G\) and \(M : \mathcal{X}/G \to \mathcal{M}\). Together with the remark after Theorem A.1 this leads to the following characterization:

**Corollary A.3.** A subset \(Z' \subset \mathcal{X}\) is a global cross section if and only if it is of the form \(Z' = gZM, \ g \in G, \ M : \mathcal{X}/G \to \mathcal{M}\).

Therefore, \(Z'\) is a global cross section if and only if it can be written as \(Z' = \{gn_z : z \in Z\}\) for some \(g \in G\) and \(n_z \in N, \ z \in Z\).

### B Orbit types

In this paper we have discussed properties of a global cross section, assuming one exists. However, a global cross section does not always exist. In discussing star-shaped distributions in Section [4] we omitted the origin from the sample space to guarantee the existence of a global cross section. Similarly, in Section [5.2] we assumed the distinctness of the roots of the characteristic equation. In these examples, the excluded sets are of measure zero and can be ignored. However, there are some cases where “singular sets” have positive measure and cannot be ignored. An example of this case is given by the orthogonal projection of a random matrix \(U\) onto the cone of nonnegative definite matrices [24, 25].

A global cross section exists if and only if all the isotropy subgroups \(G_x, \ x \in \mathcal{X}\), are conjugate to one another. One can confirm this easily by recalling property (1).

Now define the equivalence relation \(\sim_{\mathcal{X}}\) in \(\mathcal{X}\) by the conjugacy of the isotropy subgroups:

\[ x \sim_{\mathcal{X}} x' \iff G_x = gG_{x'}g^{-1} \text{ for some } g \in G. \]

Then, even when a global cross section does not exist for the action of \(G\) on the whole of \(\mathcal{X}\), there does exist a global cross section if we restrict our attention to the action of \(G\) on each equivalence class under \(\sim_{\mathcal{X}}\). These equivalence classes are called the orbit types. See Section 1.8 of [21] or Section 1.4 of [5]. We assume that the number of orbit types is at most countable. It is known that if \(G\) is compact, the number of orbit types is actually finite (see Section 4.1 of [5]).

Let \(\{\mathcal{X}_i : i \geq 1\}\) be the partition of \(\mathcal{X}\) into orbit types. By restricting the action \((G, \mathcal{X})\) of \(G\) on \(\mathcal{X}\), we obtain the action \((G, \mathcal{X}_i)\) of \(G\) on each \(\mathcal{X}_i, \ i \geq 1\). For each \(i \geq 1\), let \(Z_i\) be a global cross section for \((G, \mathcal{X}_i)\), and denote by \(G_i\) the common isotropy subgroup at the points of \(Z_i\). Then for each \(i \geq 1\), we have the orbital decomposition of \(\mathcal{X}_i:\)

\[ \mathcal{X}_i \leftrightarrow Y_i \times Z_i, \]

\[ x_i \leftrightarrow (y_i, z_i), \quad x_i = g_i z_i, \quad y_i = g_i G_i \in Y_i = G/G_i. \]
Write $\mathcal{Y} = \bigcup_i \mathcal{Y}_i$ and $\mathcal{Z} = \bigcup_i \mathcal{Z}_i$, and define the functions $y : \mathcal{X} \to \mathcal{Y}$ and $z : \mathcal{X} \to \mathcal{Z}$ by

$$y(x) = y_i(x), \quad z(x) = z_i(x) \quad \text{if} \ x \in \mathcal{X}_i, \quad i \geq 1.$$ 

Note that $\mathcal{Z}$ is a cross section for $(\mathcal{G}, \mathcal{X})$.

Concerning topological questions, we assume 1, 3 and 4 of Assumption 3.1 with $\mathcal{X}$, $\mathcal{G}_0$, $\mathcal{Z}$ and $x \leftrightarrow (y, z)$ replaced by $\mathcal{X}_i$, $\mathcal{G}_i$, $\mathcal{Z}_i$ and $x_i \leftrightarrow (y_i, z_i)$, respectively. On $\mathcal{X}_i$ we consider a dominating measure $\lambda_i$ which is relatively invariant with multiplier $\chi_i$. We note that $\mathcal{G}$ is metrizable by 2 of Assumption 3.1 (4, A5.16 Theorem). We regard the elements of $\mathcal{Y}$ as subsets of $\mathcal{G}$. By endowing $\mathcal{Y}$ with the Hausdorff distance, we make $\mathcal{Y}$ a metric space. (For details, see Appendix A.3 of our technical report.)

Let $\lambda(dx) = \sum_i I_{\mathcal{X}_i}(x)\lambda_i(dx)$, where $I_{\mathcal{X}_i}$ is the indicator function of $\mathcal{X}_i$. Note that $\lambda$ is not necessarily a relatively invariant measure. Now suppose that $x$ is distributed according to $f_{\mathcal{Y}}(y(x))\lambda(dx)$ for some $f_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{R}$. Here we assume $\int_{\mathcal{X}_i} f_{\mathcal{Y}}(y(x))\lambda(dx) > 0$ for each $i \geq 1$. Under these conditions, it is easy to show that for each $i \geq 1$,

$$P(y(x) \in A, z(x) \in B \mid x \in \mathcal{X}_i) = P(y(x) \in A \mid x \in \mathcal{X}_i)P(z(x) \in B \mid x \in \mathcal{X}_i)$$

for each measurable $A \subset \mathcal{Y}_i$ and $B \subset \mathcal{Z}_i$. Therefore, $y(x)$ and $z(x)$ are conditionally independent given $x \in \mathcal{X}_i$.

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