THREE LENS SPACE SUMMANDS FROM THE POINCARÉ HOMOLOGY SPHERE

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Abstract. A regular fiber of the Seifert fibering of the Poincaré homology sphere admits a Dehn surgery to $L(2, 1)\#L(3, 2)\#L(5, 4)$. We prove that this is the only knot in the Poincaré homology sphere with a surgery to a connected sum of more than two lens spaces.

1. Introduction.

1.1. Background. A classical theorem, independently proved by Lickorish [12] and by Wallace [21], implies that for any pair of closed orientable 3-manifolds $Y$ and $Y'$, there exists a link $L \subset Y$ admitting a Dehn surgery to $Y'$. When can we characterize a knot in a given 3-manifold by the Dehn surgeries it admits?

Since Moser’s classification of surgeries on torus knots in $S^3$ almost 50 years ago [14], Dehn surgery characterization problems have sustained the interest of 3-manifold topologists for decades—for example, there is the Berge conjecture [11, Problem 1.78] and the cabling conjecture [5, Conjecture A], [11, Problem 1.79], among other problems. One of the most celebrated results in this direction to date is the Dehn surgery characterization of the unknot: the unknot is the only knot in $S^3$ that admits a non-trivial surgery to $S^3$ [6]. In this paper we give a Dehn surgery characterization of a knot in the Poincaré homology sphere $P$.

1.2. A notable surgery. Consider $P$ as the Seifert fibered space $M((2,−1),(3,1),(5,1))$ [19, Section 1.6], and note that any Seifert fibering of $P$ is isotopic to this one [2, Corollaire 4]. Let $F \subset P$ be a regular fiber of $P$, the isotopy class of which is unambiguous. Note that the two surgery diagrams in Figure 1 both present $L(2, 1)\#L(3, 2)\#L(5, 4)$. The designated surgery slope of 0 on $F$ specifies the slope represented by a regular fiber on the boundary of the exterior of $F$ in $P$, $P_F$. We are now ready to state the main result of the current work.

Theorem 1. Let $K \subset P$. Suppose $K$ admits a Dehn surgery to a connected sum of more than two lens spaces. Then $K = F$.

1.3. Changemakers reloaded. From the homological perspective, $P$ is one of the simplest non-trivial 3-manifolds, as $H_*(P) \cong H_*(S^3)$. From the cut and paste perspective, lens spaces—the 3-manifolds obtained by identifying two solid tori along their boundaries—are the simplest non-trivial 3-manifolds. From the perspective of Ozsváth–Szabó’s Heegaard Floer homology, $P$, lens spaces, and connected sums thereof are all as simple as possible, as they realize equality in the bound $\text{rk} \widehat{H}(Y) \geq |H_1(Y)|$. We call such a 3-manifold an $L$-space. We call a knot in any 3-manifold with a non-trivial surgery to an L-space an $L$-space knot.
Figure 1. The two surgery diagrams above are related by a slam-dunk in Kirby calculus. They both present $L(2,1)\#L(3,2)\#L(5,4)$.

Our proof of Theorem 1 relies on the same pair of complementary genus bounds as did Greene’s proof of the cabling conjecture for connected sums of lens spaces in [8]. In particular, we make use of the fact that $\mathcal{P}$ and $L(p,q)$ are L-spaces. Recall that for $K$ a knot in an integer homology 3-sphere there is a canonical identification of the set of slopes on $K$ with $\mathbb{Q}\cup\{1/0\}$. An appeal to the surgery exact triangle in Heegaard Floer homology gives us a favorable genus bound: for $K \subset P$ and $p/q > 0$, if $p/q$-surgery on $K$, denoted by $K(p/q)$, is an L-space, then $p/q \geq 2g(K) - 1$, where here $g(K)$ denotes the minimum genus of an orientable surface in $\mathcal{P}$ bounded by $K$. We first show that if $K(p/q)$ is a connected sum of more than two lens spaces, then $p/q > 2g(K) - 1$. With this strict inequality, we may use the work of Matignon–Sayari [13], building on work of Hoffman [9], to find an essential 2-sphere in $K(p/q)$ that meets the core of the surgery in two points. With this 2-sphere in mind, we obtain a Seifert fibering of $P$ from which we deduce Theorem 1. We show that $K(2g(K) - 1)$ is never a connected sum of more than two lens spaces by way of changemaker lattice embeddings.

Definition 2. Let $\{e_0, e_1, \ldots, e_n\}$ be an orthonormal basis for $-\mathbb{Z}^{n+1}$. A vector $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_n) \in -\mathbb{Z}^{n+1}$ is a changemaker if $0 \leq \sigma_0 \leq \ldots \leq \sigma_n$ and for all $i \in \{1, \ldots, n\}$, $\sigma_i \leq \sum_{j=1}^{i-1} \sigma_j + 1$. A lattice $L$ is a changemaker lattice if $L$ embeds in $-\mathbb{Z}^{rkL+1}$ as the orthogonal complement to a changemaker $\sigma \in -\mathbb{Z}^{rkL+1}$.

For $X$ a compact 4-manifold, denote by $Q_X$ the free $\mathbb{Z}$-module $H_2(X)/\text{Tors}$ equipped with the integer-valued symmetric bilinear form given by the intersection pairing of surfaces in $X$. Along the way to Theorem 1, we prove the following.

Theorem 3. Let $K \subset \mathcal{P}$ be an L-space knot. If $K(2g(K) - 1)$ bounds a sharp (cf. Definition 6) 4-manifold $X$ with $\text{rk } H_2(X) = n$ and $H_1(X)$ torsion-free, then $Q_X$ embeds in $-\mathbb{Z}^{n+1}$ as the orthogonal complement to a changemaker $\sigma$ with $\langle \sigma, \sigma \rangle = -(2g(K) - 1)$. 
In particular, we make use of the following variant of Donaldson’s theorem originally due to Frøyshov [4] as Greene suggests in [7, Section 1.9].

**Theorem 4.** Let \( Z \) be a smooth, oriented, compact 4-manifold with \( \text{rk} \ Z = n \) and \( \partial Z = P \), oriented as above. If \( Z \) is negative definite, then \( Q Z \cong -\mathbb{Z}^n \) or \( Q Z \cong -E_8 \oplus -\mathbb{Z}^{n-8} \), where \(-E_8\) is the integer lattice whose pairing is given by the matrix

\[
\begin{bmatrix}
-2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

1.4. A generalized cabling conjecture for connected sums of lens spaces. Contrast Theorem 1 with the abundance of knot surgeries from \( S^3 \) to connected sums of lens spaces. The 3-sphere admits infinitely many distinct Seifert fiberings over \( S^2 \) with two exceptional fibers; each torus knot appears as a regular fiber of some Seifert fibering of \( S^3 \). Each torus knot admits a surgery to a connected sum of lens spaces—as do appropriate cables thereof. Greene proved that torus knots and their cables account for all knots in \( S^3 \) with surgeries to connected sums of lens spaces. A lens space also admits infinitely many Seifert fiberings over \( S^2 \) with two exceptional fibers, and thus contains infinitely many knots with surgeries to a connected sum of lens spaces. Moreover, Baker showed that for any pair of integers \( r \) and \( s \), there are infinitely many lens spaces containing hyperbolic knots admitting surgeries to \( L(r,1)\#L(s,1) \) [11, Corollary 2], subsuming all previously known examples of hyperbolic knots in lens spaces with surgeries to connected sums of lens spaces. The uniqueness of the surgery in Theorem 1 leads us to pose the following conjecture.

**Conjecture 5.** Let \( K \) be a knot in a 3-manifold \( Y \) admitting a Seifert fibering over \( S^2 \) with \( n \geq 3 \) exceptional fibers. If Dehn surgery on \( K \) yields a connected sum of more than \( n - 1 \) lens spaces, then there is a Seifert fibering of \( Y \) where \( K \) is a regular fiber.

In private correspondence with Baker, we learned of a conjecture more general than Conjecture 5. It posits that if a one-cusped hyperbolic 3-manifold admits two exceptional Dehn fillings with one of the fillings non-prime, then the non-prime filling has only two summands.

1.5. **Organization.** In Section 2 we collect the necessary input from Heegaard Floer homology and lattice theory to prove that \( K(2g(K) - 1) \) is never a connected sum of more than two lens spaces for \( K \subset P \). In Section 3 we invoke the main theorem of Matignon–Sayari [13, Theorem 1.1] to show that a knot satisfying the hypotheses of Theorem 1 is a regular fiber of \( P \).
1.6. **Conventions.** All manifolds are assumed to be smooth and oriented. All homology groups are taken with integer coefficients. Denote by $U$ the unknot in $S^3$. We orient the lens space $L(p, q)$ as $U(-p/q)$.

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2. Input from Floer homology.

2.1. A negative definite 4-manifold with boundary $\mathcal{P}$. Recall that to a rational homology sphere $Y$ equipped with a spin$^c$ structure $t$, Ozsváth–Szabó associated a numerical invariant $d(Y, t) \in \mathbb{Q}$, called a correction term, satisfying $d(-Y, t) = -d(Y, t)$. They also proved that if $Y$ is the boundary of a negative definite 4-manifold $X$, then

$$c_1(s)^2 + b_2(X) \leq 4d(Y, t)$$

for every $s \in \text{Spin}^c(Y)$ extending $t \in \text{Spin}^c(Y)$.

**Definition 6.** A negative definite 4-manifold $X$ is **sharp** if, for every $t \in \text{Spin}^c(Y)$, there exists some extension $s \in \text{Spin}^c(X)$ attaining equality in the bound (1).

Following Greene’s construction in [8, Section 2], let us consider a knot $K$ in an integer homology sphere $Y$ and suppose that $K(p)$ bounds a negative definite 4-manifold $X$ with $H_1(X)$ torsion-free, where $p$ is some positive integer. Denote the trace cobordism of $p$-surgery on $K$ by $W_p(K)$, and let $W = -W_p(K)$. The homology class of the closed surface obtained by capping off a Seifert surface for $K$ with the core of the 2-handle attachment, $\Sigma$, generates $H_2(W)$ and satisfies $\langle [\Sigma], [\Sigma] \rangle = -p$. Form the oriented 4-manifold

$$Z := W \cup_{K(p)} X,$$

and note that $Z$ is negative definite with $\text{rk} Q_Z = \text{rk} Q_W + \text{rk} Q_X$ and $\partial Z = Y$. We identify $\text{Spin}^c(K(p)) \cong \mathbb{Z}/p\mathbb{Z}$ as follows. Every $t \in \text{Spin}^c(K(p))$ extends to some $s \in \text{Spin}^c(W_p(K))$, and the residue of $\langle c_1(s), [\Sigma] \rangle + p \text{ mod } 2p$ is an even integer that is independent of the choice of extension $s$. The assignment $t \mapsto i$ gives the desired identification. With this notation in place we state the following lemma. Its proof is no different than that of [8, Lemma 2.3], which treats the case when $Y = S^3$.

**Lemma 7.** Suppose that $K(p)$ bounds a smooth, negative definite 4-manifold $X$ with $\text{rk} H_2(X) = n$ and $H_1(X)$ torsion-free, and form $Z = W \cup X$ as in (2). Then every $i \in \text{Spin}^c(K(p))$ extends to some $s \in \text{Spin}^c(Z)$, and

$$c_1(s)^2 + (n + 1) \leq 4d(K(p), i) - 4d(U(p), i).$$

Furthermore, if $X$ is sharp, then for every $i$ there exists some extension $s$ that attains equality in (3). $\square$
We now assume that $Y$ is an L-space. Let $K \subset Y$ be an L-space knot, and consider the Alexander polynomial of $K$,

$$\Delta_K(T) = \sum_{j=0}^{g} a_j \cdot T^j, \quad g := \deg(\Delta_K),$$

and define the torsion coefficient

$$t_i(K) = \sum_{j \geq 1} j \cdot a_{|i|+j}.$$

Tange noted that the torsion coefficients of an L-space knot in any L-space integer homology sphere $Y$ with irreducible exterior have many of the same properties as they do for L-space knots in $S^3$. Namely, for $i \geq 0$, they form a non-increasing sequence of non-negative integers [17], with $t_i(K) = 0$ if and only if $i \geq g = g(K)$ [16]. Tange also states the following result explicitly. Its proof follows from that of [15, Corollary 7.5], replacing $S^3$ by an arbitrary integer homology sphere L-space.

**Theorem 8.** Let $K$ be an L-space knot in an integer homology sphere L-space $Y$, and let $p$ be a positive integer. Then the torsion coefficients and correction terms satisfy

$$d(Y) - 2t_i(K) = d(K(p), i) - d(U(p), i),$$

for all $|i| \leq p/2$.

Greene arrived at the notion of a changemaker lattice from a careful analysis of how Lemma 9 says elements of $\text{Char}(Q_Z)$ of self pairing $- (n + 1)$ must pair against $\tau$ when $Y = S^3$, in which case Donaldson’s theorem implies $Q_Z \cong -Z^{n+1}$. Changemaker lattices also arise for us by considering Theorem 4 in the case that a small positive integer surgery on an L-space knot in $\mathcal{P}$ bounds a sharp 4-manifold.
Lemma 10. Let \( Y = \mathcal{P} \), and let \( Z \) be as in (2). If \( p \leq 2g(K) - 1 \), then \( Q_Z \cong -\mathbb{Z}^{n+1} \).

Proof. Since \( Q_Z \) is negative definite, it follows by Theorem 4 that \( Q_Z \cong -\mathbb{Z}^{n+1} \) or \( Q_Z \cong -E_8 \oplus -\mathbb{Z}^{n-7} \). Suppose \( Q_Z \cong -E_8 \oplus -\mathbb{Z}^{n-7} \), and let \( \tau = (s, \sigma) \) where \( s \in E_8 \), \( \sigma \in -\mathbb{Z}^{n-7} \), and by change of basis we have \( 0 \leq \sigma_0 \leq \sigma_1 \leq \ldots \leq \sigma_{n-8} \), and let \( c = (0, \ldots, 0, 1, \ldots, 1) \).

Then

\[
0 \geq \langle \epsilon, \tau \rangle \geq \langle \tau, \tau \rangle = -p.
\]

Since \( 0 = c^2 + n + 1 - 8 \leq -8t_i(K) \) for \( i = \frac{(c, \tau) + p}{2} \) by Lemma 9, it follows that \( t_i(K) = 0 \), so \( i \geq g(K) \), \( (c, \tau) + p \geq 2g(K) \), and therefore \( p > 2g(K) - 1 \).

**Proof of Theorem 3.** By Lemma 10, we have an embedding \( Q_W \oplus Q_X \hookrightarrow Q_Z = -\mathbb{Z}^{n+1} \), where the generator of \( Q_W \) is sent to some \( \tau \in -\mathbb{Z}^{n+1} \) satisfying \( 0 \leq \tau_0 \leq \cdots \leq \tau_n \) and \( \langle \tau, \tau \rangle = -(2g(K) - 1) \). By Lemma 9, we have that \( c^2 + (n + 1) - 8 \leq -8t_i(K) \) for all \( |i| \leq (2g(K) - 1)/2 \) and \( c \in \text{Char}(-Z^{n+1}) \) such that \( (c, \tau) + 2g(K) - 1 \equiv 2i \mod (2g(K) - 1) \), and, by the sharpness of \( X \), for every \( |i| \leq (2g(K) - 1)/2 \) there exists \( c \) attaining equality. Note that for any \( c \in \{\pm 1\}^{n+1} \), we have \( c^2 + (n + 1) - 8 = -8 \) and \( -8t_i(K) \leq -8 \) for all \( |i| \leq g(K) - 1 \). Let \( f(K) = \min\{i \geq 0 \mid t_i(K) = 1\} \). Then we have the equality

\[
(6) \quad \{\langle \epsilon, \tau \rangle \mid \epsilon \in \{\pm 1\}^{n+1}\} = \{j \in [(2g(K) - 1) - 2f(K), (2g(K) - 1) + 2f(K)] \mid j \equiv 1 \mod 2\},
\]

which we rewrite as

\[
(7) \quad \{|\langle \epsilon, \tau \rangle| \mid \epsilon \in \{0, 1\}^{n+1}\} = \{0, 1, \ldots, |\tau|\},
\]

from which see that \( \tau \) is a changemaker [8 Lemma 3.2].

To see that \( Q_X = (\tau)\perp \) on the nose, observe that \( \text{rk} Q_X = \text{rk} (\tau)\perp \) and \( \text{disc}(Q_X) = |H_1(K(2g(K) - 1))| = 2g(K) - 1 = |\tau| = \text{disc}((\tau)\perp) \) using [7] Lemma 3.10] at the last step, so the two lattices coincide.

**2.2. Linear lattices.** Let \( p > q > 0 \) be integers. There is a unique continued fraction expansion

\[
p/q = [x_1, x_2, \ldots, x_n] = x_1 - \frac{1}{x_2 - \frac{1}{\ddots - \frac{1}{x_n}}}
\]

with each \( x_i \geq 2 \) an integer. The lens space \( L(p, q) \) is the oriented boundary of the negative definite 4-manifold \( X(p, q) \) given by attaching 4-dimensional 2-handles to a linear chain of \( n \) unknots in the boundary of \( B^4 \), where the framing of the \( i^{th} \) handle attachment is \(-x_i\) (see Figure 2). We note that \( X(p, q) \) is sharp, and that \( Q_{X(p, q)} \) is indecomposable.

The connected sum of lens spaces \( \#_{i=1}^m L(p_i, q_i) \) bounds a canonical sharp 4-manifold \( X := \#_{i=1}^m X(p_i, q_i) \), a Kirby diagram for which is given by the disjoint union of the surgery diagrams for \( X(p_i, q_i) \), \( 1 \leq i \leq m \). The lattice \( Q_X \cong \oplus_{i=1}^m Q_{X(p_i, q_i)} \) then contains \( m \) indecomposable summands. Improving on our initial appeal to the surgery exact triangle in Heegaard Floer homology, the basic result that a changemaker lattice has at most two indecomposable summands [7 Lemma 5.1] allows us to conclude the following.
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\[ -x_1 -x_2 -x_3 -x_{n-1} -x_n \]

Figure 2. A Kirby diagram for \( X(p, q) \).

**Proposition 11.** If \( K(p/q) \) is a connected sum of more than two lens spaces, then \( p/q > 2g(K) - 1 \).

**Proof.** If \( K(p/q) \) is a connected sum of more than two lens spaces, it is an L-space bounding a sharp 4-manifold whose intersection form has more than two indecomposable summands, so if \( p/q \) is positive, then \( p/q > 2g(K) - 1 \) by combining an appeal to the surgery exact triangle, Theorem 3, and \[7, Lemma 5.1\]. If \( p/q < 0 \), we may build a negative definite 4-manifold with boundary \(-\mathcal{P}\) —a contradiction since a familiar application of Donaldson’s theorem would then imply that \(-E_8\) embeds in a diagonal lattice, but it does not. By combining an appeal to the surgery exact triangle and the observation that \( |H_1(K(p/q))| = |p| \), we see that \( |p/q| > 1 \), since if \( g(K) = 0 \), then \( K \) bounds a disk and thus \( K(p/q) = \mathcal{P}\# - L(p, q) \). Let \( x_i \geq 2 \), \( 1 \leq i \leq n \), be integers satisfying \([x_1, \ldots, x_n] = |p/q| \). Let \( L = K_1 \cup \cdots \cup K_n \) be the framed link in \( \mathcal{P} \) where \( K_1 = K \), \( K_i \) is a meridian of \( K_{i-1} \) for \( 2 \leq i \leq n \), and the framing of \( K_i \) is \(-x_i \), \( 1 \leq i \leq n \). Denote by \( W_{p/q}(L) \) the trace cobordism of surgery on the framed link \( L \), which has negative definite intersection form and \( \partial W_{p/q}(L) = -\mathcal{P} \coprod K(p/q) \). Let \( X \) be the canonical sharp 4-manifold with boundary a connected sum of lens spaces oriented as \(-K(p/q) \) described above. Then \( Z := W_{p/q}(L) \cup_{K(p/q)} X \) is negative definite with boundary \(-\mathcal{P}\). We conclude that \( p/q > 2g(K) - 1 \). \( \square \)

3. AN ESSENTIAL ANNULUS IN \( \mathcal{P}_K \).

With Proposition 11 in hand, if \( K(p/q) \) is a connected sum of more than two lens spaces, then we may use the following theorem of Matignon–Sayari to complete the proof of Theorem 1.

**Theorem 12.** \[13, Theorem 1.1\] Let \( M \) be an irreducible 3-manifold with boundary a torus \( T \). Let \( \lambda \) be a slope on \( T \) that bounds an orientable surface in \( M \) and \( g \) the genus of this surface. If there exists a reducing slope \( r \), then either \( \Delta(r, \lambda) \leq 2g - 1 \) or else the minimum geometric intersection number of an essential 2-sphere in \( M(r) \) with the core of \( r \)-Dehn filling is 2. \( \square \)

Suppose that \( K \subset \mathcal{P} \) admits a surgery to a connected sum of more than two lens spaces. For us, \( \lambda \) is the 0-framing of \( K \), and the slope of this surgery \( p/q \) is strictly greater than \( 2g - 1 \) by Proposition 11. In particular, \( \Delta(0, p/q) = p > 2g - 1 \), so \( K(p/q) \) contains an essential 2-sphere which meets the core of \( p/q \)-Dehn surgery in exactly 2 points by Theorem 12. We use this information to complete the proof of Theorem 1 with the following two lemmas.
Lemma 13. Let \( \frac{p}{q} > 2g - 1 \) with \( q > 1 \), and suppose that \( K(p/q) \) is reducible for \( K \subset P \). Then \( K \) is an exceptional fiber of \( P \) and the \((p,q)\)-cable of \( K \) is a regular fiber.

Proof. We invoke Theorem 12 to identify an essential 2-sphere \( \hat{A} \subset K(p/q) \) which intersects the core of the surgery solid torus in two points. Let \( A \) denote the essential annulus \( \hat{A} \cap P_K \). Then \( A \) is separating in \( P_K \), and \( \partial A \) separates \( \partial P_K \) into two annuli \( A_1, A_2 \) with \( \partial A_i = \partial A \) for \( i = 1, 2 \). Observe that each boundary component of \( A, A_1, \) and \( A_2 \) is a \((p,q)\)-curve on \( \partial P_K \).

Since \( P \) is atoroidal, each \( T_i \) bounds a solid torus \( V_i \in P \). We will show that in fact each \( V_i \) is contained in \( P_K \), giving a decomposition of \( P_K \) as \( V_1 \cup A \cap V_2 \). Then \( P_K \) admits a Seifert fibering with two exceptional fibers, so \( K \) is an exceptional fiber of \( P \).

To see that \( V_1 \) is contained in \( P_K \), suppose to the contrary that \( \nu(K) \subset V_1 \). Then \( A_2 \subset V_1 \). Consider the Seifert fibering of \( V_1 \) over \( \nu(K) \) to see that \( A \) is isotopic to \( A_2 \) in \( V_1 \setminus \nu(K) \), and therefore \( A \) is not essential in \( P_K \).

By the same argument, we see that \( V_2 \) is also contained in \( P_K \), hence \( P_K = V_1 \cup A \cup V_2 \) as desired. Furthermore, since \( A \) is not boundary parallel in \( P_K \), the Seifert fibering of \( P_K \) induced by those on \( V_1 \) and \( V_2 \) has two exceptional fibers (see Figure 4). Equip \( \nu(K) \) with the Seifert fibering induced by \((p,q)\)-curves on \( \partial \nu(K) \) to see that \( K \) is an exceptional fiber of \( P = P_K \cup A \cup V_2 \), and that the \((p,q)\)-cable of \( K \) is a regular fiber. \( \square \)

Lemma 14. Let \( p \geq 2g \) be an integral reducing slope for \( K \subset P \). Suppose that \( K(p) \) is a connected sum of more than two lens spaces. Then \( K \) is a regular fiber of \( P \).

Proof. We again invoke Theorem 12 to identify an essential properly embedded annulus \( A \) in \( P_K \) with boundary slope \( p \). Since \( p \) is integral, there is an annulus \( A' \subset \nu(K) \) with \( \partial A' = \partial A \) and \( K \subset A' \). Let \( T = A \cup A' \). Since \( P \) is atoroidal, it follows that \( T \) bounds a solid torus.
Denote the core of this solid torus by $E$, and observe that $K$ is the $(p, q)$-cable of $E$ for some $p, q \in \mathbb{Z}$ with $|q| \geq 2$ such that $E(p/q)$ is a connected sum of at least two lens spaces. That $K$ is a regular fiber of $P$ then follows from Lemma 13. \[\square\]

**Proof of Theorem 1.** Let $K \subset P$, and suppose that $K(p/q)$ is a connected sum of more than two lens spaces. By Proposition 11, $p/q > 2g(K) - 1$. By Lemma 13, we conclude that $q = 1$, since if $p/q > 2g(K) - 1$ is non-integral and $K(p/q)$ is reducible, then $K(p/q)$ is a connected sum of two lens spaces. By Lemma 14, we conclude that $K = F$. We may furthermore identify that $p/q = |H_1(L(2, 1)\#L(3, 2)\#L(5, 4))| = 30$. \[\square\]

**Remark 15.** If we can identify a reducing sphere that meets the core of surgery on $K$ in two points, for $K$ satisfying the hypotheses of Conjecture 5, the then topological arguments in this section can be adapted to prove the conjecture with help from the classification of embedded tori in Seifert fibered spaces [10, Theorem VI.34]. For a Seifert fibered space over $S^2$ that is not an integer homology sphere L-space, i.e. not $S^3$ or $\pm P[3]$, parts of our argument break down. For example, if $K$ is non-trivial in homology then there need not be a slope on $\partial \nu(K)$ bounding an orientable surface in the exterior of $K$. Even if $K$ is nullhomologous, a positive L-space slope on $K$ need not be bounded below by $2g(K) - 1$—in fact, every knot $K$ that is doubly primitive in a Brieskorn sphere that is not an L-space admits an integral surgery slope $p$ to a lens space, and it follows from [18, Theorem 1] that $|p| \leq 2g(K) - 1$.

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