Qubit Models of Black Hole Evaporation

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Recently, several simple quantum mechanical toy models of black hole evaporation have appeared in the literature attempting to illuminate the black hole information paradox. We present a general class of models that is large enough to describe both unitary and nonunitary evaporation, and study a few specific examples to clarify some potential confusions regarding recent results. We also generalize Mathur’s bound on small corrections to black hole dynamics. Conclusions are then drawn about the requirements for unitary evaporation of black holes in this class of models. We present a one-parameter family of models that continuously deforms nonunitary Hawking evaporation into a unitary process. The required deformation is large.

I. INTRODUCTION

Hawking’s calculation of black hole evaporation [1] leads to a direct conflict between general relativity and the unitary evolution of quantum mechanics [2]. This prompted the suggestion that the requirement of unitary evolution should be relaxed [3, 4], and pure states allowed to evolve into mixed states. Such nonunitary evolution, however, seems problematic [5].

Moreover, since black hole evaporation is a very slow process involving a large number of emitted particles, and since one only expects to start recovering information after about one half of the radiation has been emitted [8], one might imagine that unitarity is restored by the accumulation of small (say, nonperturbative) corrections over the course of the entire evolution process, cf. [9]. To evaluate this claim, Mathur [10] introduces a qubit model of black hole evaporation and then derives bounds on the entanglement entropy of the emitted radiation, which show that this scenario is not possible. Below we generalize his result to consider more general deformations of the pair creation dynamics. In particular, for small changes in the evaporation process the entanglement entropy continues to increase, therefore (barring remnants and variants thereof) the evolution is not unitary.

Thus, for unitarity to be restored, one needs to make large corrections to the semi-classical evaporation process described by Hawking. Recently, several specific unitary models have been introduced [11–13] as proposed alternatives to Hawking’s semiclassical evolution. As discussed in [14], models of this kind could be called “burning paper” models that involve large corrections to the Hawking evolution. While these models are unitary, they are typically written in a way that makes it difficult to compare to the semiclassical evolution and to see how Mathur’s bound operates. The difficulty arises because the bound is derived in terms of dynamics in an ever-enlarging Hilbert space, whereas unitary models are typically written as dynamics in a fixed-dimensional Hilbert space.

The primary goal of this paper is to clarify the meaning of Mathur’s bound in [10]. In particular, until one writes unitary models as (large) corrections to nonunitary, semiclassical evolution, the meaning of Mathur’s result remains obscure. Previous investigations of corrections in this context either directly introduce unitary models that are difficult to compare to Hawking evaporation [11–13]; or consider small corrections to Hawking evaporation that do not produce unitary evolution when made sufficiently large [13, 15], and thus are unconvincing illustrations of the result.

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1 See [6, 7], however, for criticisms of the arguments made in [5].
in [10]. A second goal is to characterize what kinds of corrections produce the desired unitary evolution. That the corrections need to be large is a necessary condition derived in [10], but sufficient conditions have not been discussed in the same way.

In order to address the above points, it is necessary to introduce a general model space that provides a uniform language to discuss both unitary and nonunitary black hole evaporation. This allows us, for example, to continuously deform the semiclassical Hawking evolution to unitary evolution. One can then explicitly see that the deformation is large in an appropriate sense, and therefore in agreement with a suitable generalization of Mathur’s argument. Let us emphasize that it is not our intention here to advocate for nonunitary evolution, only to demonstrate that unitarity demands there be a significant alteration of the traditional semiclassical evolution. To restrict ourselves to unitary evolution at this stage would be to beg the question.

In Section II, we introduce a very general framework for qubit models of black hole evaporation that is appropriate for both unitary and nonunitary evolution. Most of the discussion focuses on how to interpret the models, and their connection to ideas in quantum information theory. In Section III, we apply the formalism to a number of sample models; some of the models were chosen because they were discussed previously in the literature, and others because they illustrate some interesting issues. In Section IV, we briefly review Mathur’s argument against small corrections restoring unitary evolution, and generalize the main entanglement entropy bound to allow arbitrary deformations. Mathur’s original result [10] only explicitly considers one kind of perturbation. In Section V, using observations from Section III, we discuss what conditions ensure unitary evolution. In Section VI, we present a one-parameter family of models that continuously connects Hawking evaporation to a unitary model; one sees clearly that the deformation required is large. In Section VII, we conclude with some brief comments.

II. THE GENERAL MODEL

Before presenting our model, we make a few preliminary comments on nonunitary evolution in Section II.1. Then in Section II.2, using some results from quantum information theory, we explain how to describe nonunitary evolution that still has a good probabilistic interpretation. Along the way, we clarify some potential confusions regarding previous work. Finally, we present our general class of models in Section II.3 discussing the physical interpretation in Section II.4.

II.1. Nonunitarity

When we model the evolution of a closed quantum system, the state of the system is given by a ket \( |\psi(t)\rangle \) that satisfies

\[
|\psi(t)\rangle = U(t) |\psi(0)\rangle
\]  

(2.1)

for a unitary time evolution operator \( U(t) \). We may equivalently write the state of the system as a density matrix \( \rho(t) = |\psi(t)\rangle \langle \psi(t)| \) where \( \rho(t) \) satisfies

\[
\rho(t) = U \rho(0) U^\dagger.
\]  

(2.2)

Unitary evolution satisfies several nice conditions, namely:

1. Linearity

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*We do not claim that these are completely independent.*
2. Preservation of the norm: unit norm states evolve to unit norm states, ensuring that a probabilistic interpretation makes sense. For density matrices, the desired condition is that unit-trace, completely positive density matrices evolve to unit-trace, completely positive density matrices.

3. Invertibility: previous states can be found from the current state.

4. Purity: pure states evolve to pure states.

However, it has sometimes been suggested [2–4] that theories of quantum gravity (especially in the presence of black holes) will not be unitary. Let us note that the negation of unitary is ambiguous, since it is not clear which of the above conditions is relaxed. Let us consider three illustrative mappings.

1. This evolution does not conserve probability—the norm is not conserved; however, pure states still evolve to pure states, and the evolution is invertible:

$$|\psi\rangle \mapsto \frac{3}{4} |\psi\rangle.$$  \hspace{1cm} (2.3)

This kind of evolution is sometimes useful in modelling a system that decays into something that is outside the model. For instance, in modeling alpha decay. An equivalent way to describe this evolution is to say that the system has energies with an imaginary part. For a fundamental description that includes all degrees of freedom, however, the nonconservation of probability is nonsensical.

2. This evolution preserves the norm, is invertible, but evolves pure states to mixed states:

$$|\psi_1\rangle \mapsto \rho_1 = \frac{1}{2} |\psi_1\rangle \langle \psi_1| + \frac{1}{2} |\phi_1\rangle \langle \phi_1|$$

$$|\psi_2\rangle \mapsto \rho_2 = \frac{1}{2} |\psi_2\rangle \langle \psi_2| + \frac{1}{2} |\phi_2\rangle \langle \phi_2|$$ \hspace{1cm} (2.4)

While this model evolves pure states to mixed states, information is still preserved (in a weak sense) and probability is conserved.

3. This evolution conserves probability, evolves pure states to pure states, but is not invertible.

$$|\psi_1\rangle \mapsto |\psi_0\rangle \quad |\psi_2\rangle \mapsto |\psi_0\rangle$$ \hspace{1cm} (2.5)

In this model one cannot reconstruct the past from the current state, and therefore information is not preserved. In Section III.4, we introduce some models of this type.

In the black hole information paradox, one considers some initial configuration of matter $|\psi_m\rangle$ that collapses into a black hole, and then completely evaporates to radiation in a thermal mixed state $\rho \sim e^{-\beta H}$. Since the final state is both mixed and independent of the initial state, the evolution both fails to be invertible and pure in the above senses.

Suppose for the nonce that Hawking’s original argument is correct [2], and the fundamental theory of quantum gravity is not unitary. We can no longer write evolution as in (2.2). We restrict our considerations to evolution that satisfies conditions 1 and 2 but not necessarily condition 3 and 4. Then, assuming some very basic conditions that ensure a good probabilistic interpretation, we can write the most general possible evolution in the operator-sum representation (cf. [16])

$$\rho(t) = \sum_k E_k \rho(0) E_k^\dagger$$ \hspace{1cm} (2.6)

provided large ensembles of identical copies of the system, one can with some confidence distinguish distinct density matrices; however, this usage of the phrase “information preservation” is not canonical, and is problematic if one starts considering density matrices which are very close to each other. This would mean in an experiment repeated many times with identical initial conditions, one could reconstruct some information about the initial state from the final state. In a technical sense, however, quantum information is lost. We make the distinction here, since the semiclassical description implies that information is not preserved even in this weak sense.
for some set of operator $E_k$ that satisfy the completeness relation

$$\sum_k E_k(t)^\dagger E_k(t) = I. \tag{2.7}$$

This is one way to write the evolution of an open quantum system, and the transformation from $\rho(0)$ to $\rho(f)$ is called a “quantum operation” in the quantum information theory literature \[16\]. The operators $E_k$ determine the evolution of the density matrix. When there is only one $E_k$, the evolution is unitary.\(^4\)

II.2. Quantum Operations

As in \[10, 12–15\], we model the Hawking evaporation process as a discrete set of mappings on qubits. In the initial state, the system consists entirely of matter in a pure state. There have been some suggestions \[17\] in this context that the entanglement between the initial black hole-forming matter and the outside matter plays an important role; we do not address these issues at this time. The initial state is modeled as a set of $n$ “matter qubits”:

$$\rho_0 = |\psi_0\rangle \langle \psi_0| \in \text{span}\{ |\hat{q}_1 \hat{q}_2 \cdots \hat{q}_n\rangle\}, \tag{2.8}$$

where each $\hat{q}$ is a qubit, a quantum state labeled by 0 or 1. After a sequence of intermediate steps, the end state consists entirely of radiation (again, we are assuming no remnants), modeled as a (possibly mixed) density matrix acting on $n$ “radiation” qubits, $\rho_f$. Throughout the evolution, we keep the total dimension of the Hilbert space fixed. This is certainly true for the unitary evolution of closed systems, but here we put it in as a reasonable assumption. We are motivated in part by the black hole’s entropy. The black hole initially has $S \sim M^2$, which entirely radiates away on the time scale $\sim M^3$ with an emission every $\sim M$. This is consistent with a model having a fixed number of physical qubits.

Following \[12\], we use hats to distinguish the internal black hole qubits from the external radiation qubits. The hatted qubits represent all degrees of freedom that are inaccessible outside the black hole; unlike \[10\], we do not distinguish between degrees of freedom from the initial matter, from gravitational interactions, or any that arise during the evaporation process. We write basis elements for the final state as

$$\{|q_nq_{n-1}\cdots q_1\rangle\}, \tag{2.9}$$

where we have put the labels on the qubits in reverse order for reasons which should become clear. At the $i$th step, we have a density matrix acting on $n - i$ black hole qubits and $i$ radiation qubits, so that the total dimension of the Hilbert space is fixed. The evaporation concludes on the $n$th step when there is only radiation:

$$\rho_0 \rightarrow \rho_1 \rightarrow \cdots \rightarrow \rho_{n-1} \rightarrow \rho_n \tag{2.10}$$

In general the mapping from $\rho_0$ to $\rho_n = \rho_f$ should be a quantum operation, which is the composition of $n$ quantum operations (one for each emission). Therefore the total evolution from $\rho_0$ to $\rho_n$ (and each intermediate step) may be written in terms of some $E_k$'s, like in Equation (2.6). Quantum operations, however, may be written in an equivalent, alternative form that is more natural.

\(^4\) Note that we model evolution with discrete time evolution, and do not discuss continuous evolution, which might be governed by the Lindblad equation.
when discussing black hole evaporation. This form connects more directly with the discussion in [10].

We motivate this alternate form, by noting that any mixed density matrix may be “purified” by enlarging the Hilbert space. For example, a density matrix of the form

\[ \rho = p_1 |A\rangle \langle A| + p_2 |B\rangle \langle B| \]  

with orthonormal \{ |A\rangle , |B\rangle \} can be purified by introducing the orthonormal states \{ |\alpha\rangle , |\beta\rangle \}, and defining

\[ |\Psi\rangle = \sqrt{p_1} |A\rangle \otimes |\alpha\rangle + \sqrt{p_2} |B\rangle \otimes |\beta\rangle . \]  

Then, one sees that \( \rho \) is the reduced density matrix found by tracing out the new degrees of freedom. Let us emphasize purification is a formal mathematical operation, and not a dynamical process. In particular, for us, the new kets that we direct producted into the Hilbert space do not correspond to any physical degrees of freedom. Roughly speaking, then, we can imagine purifying each of the \( \rho_i \)s by enlarging the Hilbert space, and then the evolution in this enlarged Hilbert space would be unitary.

As it turns out, any quantum operation from say \( \rho_0 \) to \( \rho_n \) may be written in the following way:

\[ \rho_n = \text{tr}_{\text{aux}}[U (\rho_{\text{aux}} \otimes \rho_0) U^\dagger], \]  

for a unitary transformation \( U \) acting on some auxiliary degrees of freedom as well as the physical degrees of freedom. In particular, if \( \rho_0 \) acts on a \( d \)-dimensional Hilbert space, then we need introduce at most a \( d^2 \)-dimensional auxiliary Hilbert space to write the most general quantum operation in this form [16]. Thus for our \( n \)-qubit system, we need only introduce \( 2n \) auxiliary qubits to capture the most general evolution of density matrices.

In this language, we can think about the semiclassical Hawking evolution in the following way. We start with \( n \) black hole qubits initially in a pure state. We imagine that \( n \) might be

\[ 5 \text{ If one does treat the new degrees of freedom as physical in the final state, then one is considering a remnant scenario.} \]
roughly given by the entropy of the black hole. At the first time step, a pair of qubits is created at the horizon in an entangled state,

$$\frac{1}{\sqrt{2}}((\hat{0})|0\rangle + (\hat{1})|1\rangle).$$

The zero represents no particle and the one represents a particle. We refer the reader to [10, 15] for a thorough discussion on the origin of this description; see also [12]. We have now added two new qubits to the system, increasing the size of our Hilbert space. At each time step a new entangled pair is produced in the above state, and the Hilbert space keeps increasing in size. By the end of the evaporation process, on the $n$th step, we have added $2^n$ qubits to the initial $n$ qubits for a total $3n$ qubits. Since the black hole has completely evaporated and there are only the $n$ physical qubits of radiation, the remaining $2n$ hatted qubits should be interpreted as auxiliary degrees of freedom as in the above discussion. Since presumably the total number of physical degrees of freedom should remain fixed at $n$ qubits, at the $i$th step we have $2i$ auxiliary (hatted) qubits, but the semiclassical analysis does not make any clear identification of the auxiliary qubits at intermediate stages in the evaporation. It is clear, however, that by the end of the evaporation process all of the black hole (hatted) qubits must be auxiliary. Specifying the auxiliary subspace (in combination with giving the internal dynamics) corresponds to taking into account back reaction on the geometry.

The Hilbert space is illustrated in Figure 1, where the intermediate state is shown with some highlighted auxiliary degrees of freedom. In the figure, the highlighted region contains some of the initial matter qubits (circles) and some of the new infalling qubits (squares); this represents one possibility. One could also consider cases where for the first $n/2$ steps only the initial matter is auxiliary, for example. The parameterization of the auxiliary degrees of freedom at intermediate steps should be considered part of a model, so that one may trace out the auxiliary degrees of freedom to arrive at a fixed-dimensional Hilbert space description. Since we are mostly interested in the final state, where the auxiliary space is unambiguous, we do not always specify the auxiliary degrees of freedom. In the cases where the evolution is unitary, it should be clear what the auxiliary degrees of freedom are. Although nothing profoundly new has been said here, we hope this discussion may help clarify potential confusions regarding [10, 15], and other investigations [11–13].

II.3. The General Model

We are now ready to present the general class of models that we consider. We start with $n$ hatted qubits, and at each step we add a hatted qubit and an unhatted qubit. The number of qubits at each step is summarized in Table I.

We model the evolution in two steps: a creation step effected by operators $C_i$; and an internal evolution step effected by $\hat{U}_i$ acting on the hatted qubits and $U_i$ acting on the unhatted radiation qubits. Basis vectors at each step look like

$$\left\{ |\hat{q}_1\hat{q}_2\cdots\hat{q}_{n+i}q_{i-1}\cdots q_1\rangle \right\} \xrightarrow{C_i} \left\{ |\hat{q}_1\hat{q}_2\cdots\hat{q}_{n+i}\hat{q}_{n+i+1}q_{i+1}q_{i-1}\cdots q_1\rangle \right\}$$

$$\xrightarrow{\hat{U}_i \otimes U_i} \left\{ \hat{U} |\hat{q}_1\hat{q}_2\cdots\hat{q}_{n+i}\hat{q}_{n+i+1}q_{i+1}q_{i-1}\cdots q_1\rangle \right\}. \quad (2.15)$$

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6 If we want the initial state to model the matter just before it collapses into a black hole this may not be a good assumption, since in a suitably fine-grained description the number of degrees of freedom available to ordinary matter is parametrically smaller than the entropy of the black hole [18, 19]. We do not concern ourselves with this issue here, but the author is grateful to S. Giddings for pointing this out.

7 As a specific example, Reference [11] raises the issue of an ever-enlarging Hilbert space as potential issue in the analysis of [10].
TABLE I. Here we outline the discrete steps in our models. At the 0th or initial step there are $n$ black hole (BH) qubits and no radiation qubits. At each step in the evolution, the state is given by the ket $|\psi_i\rangle$ in an enlarging Hilbert space.

| step | no. BH qubits | no. rad. qubits | total no. qubits | no. aux. qubits | state |
|------|----------------|----------------|-----------------|----------------|-------|
| 0    | $n$            | 0              | $n$             | 0              | $|\psi_0\rangle$ |
| 1    | $n+1$          | 1              | $n+2$           | 2              | $|\psi_1\rangle$ |
| 2    | $n+2$          | 2              | $n+4$           | 4              | $|\psi_2\rangle$ |
| ...  | ...            | ...            | ...             | ...            | ...   |
| $i$  | $n+i$          | $i$            | $n+2i$          | $2i$           | $|\psi_i\rangle$ |
| ...  | ...            | ...            | ...             | ...            | ...   |
| $n$  | $2n$           | $n$            | $3n$            | $2n$           | $|\psi_n\rangle$ |

Of course, one can combine $C_i$ and $\hat{U}_i \otimes U_i$ into a single operator, but it is useful to break up the evolution in this way. Also, there is some physical motivation for thinking about the evolution in this way, since the pair creation time scale is roughly $\sim M$, the black hole mass, while there are some conjectures that the internal dynamics of the black hole should be as fast [20, 21]. (For a 3+1 dimensional Schwarzschild black hole, the scrambling time is speculated to be $\sim M \log M$, but the evolution time step is $\sim M$.)

For the majority of the discussion, we focus on the $C_i$ and are content to set $\hat{U}_i = U_i = I$. What properties should the $C_i$ satisfy? We want $C_i$ to preserve the norm and be linear, which means that we should require

$$(C_i)^\dagger C_i = I,$$  

where there is no sum on $i$. Note that this is not the same as $C_i(C_i)^\dagger$ since the $C_i$ have non-square matrix representations; the above requirement makes $C_i$ an isometric, but nonunitary mapping. We also assume that the $C_i$ act only on the hatted black hole qubits and not on the unhatted radiation qubits which are far away from the pair creation site.

We can write the $C_i$ in the following form

$$C_i = |\varphi_1\rangle \otimes \hat{P}_1 + |\varphi_2\rangle \otimes \hat{P}_2 + |\varphi_3\rangle \otimes \hat{P}_3 + |\varphi_4\rangle \otimes \hat{P}_4,$$  

(2.17)

where $|\varphi_j\rangle$ are an orthonormal basis for the created pair qubits, and the $\hat{P}^i$s are linear operators which act on the hatted qubits (with implicit $i$ dependence). Following [12, 13], we use the basis

$$|\varphi_1^i\rangle = \frac{1}{\sqrt{2}} (|\hat{0}_{n+i+1}\rangle |0_{i+1}\rangle + |\hat{1}_{n+i+1}\rangle |1_{i+1}\rangle)$$

$$|\varphi_2^i\rangle = \frac{1}{\sqrt{2}} (|\hat{0}_{n+i+1}\rangle |0_{i+1}\rangle - |\hat{1}_{n+i+1}\rangle |1_{i+1}\rangle)$$

$$|\varphi_3^i\rangle = |\hat{0}_{n+i+1}\rangle |1_{i+1}\rangle$$

$$|\varphi_4^i\rangle = |\hat{1}_{n+i+1}\rangle |0_{i+1}\rangle,$$

(2.18)

for the newly created pair. The constraint in Equation (2.16) implies the following condition on the $\hat{P}$s:

$$(C_i)^\dagger C_i = \hat{P}_1^i \hat{P}_1 + \hat{P}_2^i \hat{P}_2 + \hat{P}_3^i \hat{P}_3 + \hat{P}_4^i \hat{P}_4 = \hat{I}.$$  

(2.19)

Note that this defines the $\hat{P}$s as a set of generalized measurement operators acting on the black hole Hilbert space.

A fully specified model, then, entails
1. A set of $\hat{P}$s at each step $i$ that satisfy the completeness relation (2.19).
2. The unitary operators $\hat{U}_i$ and $U_i$ for each $i$.
3. A clear delineation of the auxiliary subspace at each step $i$.

The last item is frequently omitted in our discussion; it should be clear for unitary models, and it does not make significant differences for the nonunitary models. If one wants to acquire the fixed-dimensional Hilbert space description, however, then one must trace out the auxiliary degrees of freedom at each step. This gives a very general model space that makes it easy to compare and contrast different models of evolution.

II. Physical Motivations

At this point, the class of models introduced above may seem fairly abstract with little contact with the original black hole problem. Let us review the physical motivations for this type of model as laid out in [10] and in [12].

We consider an initial configuration of spherically symmetric matter that forms a black hole, which we expect should be well described by the Schwarzschild solution. The model is based on the semiclassical evolution of fields in the background of such a Schwarzschild black hole. In order to give a Hilbert space description of evolution, it is necessary to specify a spacelike slicing of the geometry so that we can specify the quantum state of fields on each slice. It is important to the arguments advanced in [10] that there exists a “nice slicing” of the black hole geometry [22]. This slicing avoids the geometry’s strong curvature, has sub-Planck scale extrinsic and intrinsic curvature, and yet cuts through the initial matter, horizon, and outgoing Hawking radiation in a smooth way [10, 22]. Thus, all quantum gravity effects seem to be under control.

Our model should be considered an effective description of the dynamics on the slicing. As is well known, in the presence of curved backgrounds the quantum field theory notion of particle becomes observer dependent. If we expand our fields on the slice into modes inside the horizon and outside the horizon, then one finds that pairs of particles are created inside and outside of the horizon. More explicitly there is a Bogoliubov transformation such that the in vacuum evolves to a state of the form $\exp(\gamma a_{\text{inside}}^\dagger a_{\text{outside}}^\dagger)$ acting on the out vacuum. We reduce our problem to an essentially two-dimensional one by expanding the modes in spherical harmonics. From a two-dimensional perspective, each harmonic corresponds to a different field. Then following [1, 23], as emphasized in this context in [12], we can use a set of modes that are localized wavepackets so that we can talk about locality. This is implicit in the discussion of [10]. Moreover, we can truncate the Fock space to occupation numbers zero or one. We then are effectively left with a discussion of qubits, with $|0\rangle$ representing no excitation and $|1\rangle$ representing an excitation.

In this description, a pair of particles are created roughly every $M$ in Planck units, with the outgoing particles traveling freely outward on the slices and the ingoing particles traveling freely inward toward the initial matter that is very far away on each slice. The pair of particles are created entangled, as should be clear from the above exponential. This all suggests an effective, discrete time evolution with [10, 12–15]

\[
\hat{P}_1 = \hat{I} \quad \hat{P}_{2,3,4} = 0 \quad \hat{U} = U = I.
\]

Because the particles are well-separated on the slice, we do not expect interparticle interactions to be significant which is represented by the choice $\hat{U} = U = I$. We call this point in model space, the Hawking model. Location of the particles on the slice can then be read in the following way from the states. Consider, for illustrative purposes, $n = 3$ with a state of the form

\[
|\hat{q}_1\hat{q}_2\hat{q}_3\rangle_{\text{initial}} |\hat{1}_4\hat{0}_5\rangle_{\text{infalling}} |0_21_1\rangle_{\text{outgoing}}.
\]
The first three hatted qubits represent the initial infalling matter. Note that in the Hawking model this matter plays no role in the evolution. In general, we imagine that we find some qubit description of the initial matter, the details of which are irrelevant to our concerns here. The next two hatted qubits represent the infalling Hawking radiation as it travels inward on the slices. We see that on the first time step, a particle was emitted but not on the second time step. The two unhatted bits represent the outgoing radiation, so the above implies an outgoing particle was emitted on the first time step and then no particle on the second time step. We have written the qubits in the above order so that reading from left to right loosely corresponds to traveling outward on the slice. This allows us to talk about a coarse form of locality \[10, 12\]. Note that in the Hawking model the above state would be superposed with several other direct product states.

As discussed in the Introduction and for this context in \[10, 12, 14, 15\], the above semiclassical description of Hawking evaporation is incomplete. In particular, one expects quantum gravity effects, backreaction, and interactions to play a role. Because of the nature of the nice slicing and the low curvature at the horizon, however, one generally expects all of these corrections to be small. That is to say, on the pair creation time scale, one expects from naive estimates that the dynamics be \(\epsilon\) away from the above. This expectation is in considerable tension with the expectation that the dynamics are unitary. For instance, we might introduce a set of \(\hat{P}\)s that act on the last emitted ingoing particle. This would suggest that the horizon is not effectively the vacuum and still “remembers” the previous emission. One might also allow some mild nonlocal interactions inside the black hole via some nearest neighbor \(\hat{U}\)s. Or, motivated by holography and fast scrambling \[20, 21, 24\], one might consider general \(\hat{U}\)s, in which case one gives up all notions of locality on the slice inside the black hole. The distinction between the initial matter and the infalling particles is also lost. Allowing general internal dynamics does not affect the argument of \[10\] and its generalization in Section \[IV\] which only relies on the pair creation, \(\hat{P}\)s, being close to the Hawking model. We will always use \(\hat{U} = I\), since there is no physical motivation to consider strong interactions among the outgoing radiation. (Such corrections would also be irrelevant to Mathur’s bound on the entanglement entropy.) This should become clear after we examine some examples.

### III. EXAMPLES

In this section, we highlight some special points in model space that may be of interest and/or were discussed in the recent literature. One of the main results of this paper is the one-parameter family of models presented in Section \[VI\] that continuously deforms the Hawking pair production in Section \[III.1\] into the unitary evolution in Section \[III.3\] however, it is also useful to write the various models in a common form, so that the similarities and differences are manifest. This is especially true when one wants to compare unitary models to nonunitary models. We start with the canonical Hawking evaporation model. This should be thought of as the baseline model to which all other models should be compared.

#### III.1. The Hawking Model

The standard Hawking evaporation corresponds to creating a new pair in the state \(|\varphi_1\rangle\) irrespective of the state of the system, as discussed at length in Section \[II.4\]. Thus, it can be written as

\[
C_i^H = |\varphi_1\rangle \otimes \hat{I} \quad \hat{U} = U = I, \quad (3.1)
\]
and so we can write the $\hat{P}$s as

$$\hat{P}_1 = \hat{I} \quad \hat{P}_{2,3,4} = 0. \quad (3.2)$$

In [10], Mathur showed that if the created pair is at most $\epsilon$ away from $|\varphi_1\rangle$, then the entanglement entropy of the radiation continues to grow with each step and thus the final state is mixed and unitarity is lost. In this language, the bound shows that if the $\hat{P}$s are small deformations from the above, then the final state will be mixed and the evolution will not be unitary. In the sequel, we demonstrate this quite explicitly.

### III.2. A Burning Paper Model

Here, we present a unitary “burning paper” model that is equivalent to the one given in [13].

$$\hat{P}_1 = \hat{P}_2 = \left[ \frac{1}{\sqrt{2}} |\hat{00}\rangle \langle \hat{00}| + \frac{1}{2} |\hat{10}\rangle \langle \hat{10}| - \frac{1}{2} |\hat{01}\rangle \langle \hat{01}| \right]_{n+i-1,n+i}$$

$$\hat{P}_3 = \left[ |\hat{10}\rangle \langle \hat{11}| + \frac{1}{\sqrt{2}} |\hat{00}\rangle \langle \hat{10}| + \frac{1}{\sqrt{2}} |\hat{00}\rangle \langle \hat{01}| \right]_{n+i-1,n+i}$$

$$\hat{P}_4 = 0,$$ \quad (3.3)

where the subscript on the brackets indicates that the qubits referred to are the $(n+i-1)$th and the $(n+i)$th qubits. It should be clear that this model is quite far from the Hawking model. This models the creation step as

$$C_i = |\hat{0}_{n+i+1}\rangle |1_i\rangle \otimes \left[ |\hat{10}\rangle \langle \hat{11}| + \frac{1}{\sqrt{2}} |\hat{00}\rangle \langle \hat{10}| + \frac{1}{\sqrt{2}} |\hat{00}\rangle \langle \hat{01}| \right]_{n+i-1,n+i}$$

$$\quad + |\hat{0}_{n+i+1}\rangle |0_i\rangle \otimes \left[ |\hat{00}\rangle \langle \hat{00}| + \frac{1}{\sqrt{2}} |\hat{10}\rangle \langle \hat{10}| - \frac{1}{\sqrt{2}} |\hat{10}\rangle \langle \hat{01}| \right]_{n+i-1,n+i}.$$ \quad (3.4)

The important property of the evolution to note is that the $(n+i+1)$th black hole qubit is always $\hat{0}$, as is the $(n+i)$th qubit. Thus these two qubits are “zeroed,” and effectively deactivated. We use the word zeroed in this sense, even if the qubit under discussion is deactivated to a different value. (It could even be something like $i \mod 2$. In this situation, the information is sometimes said to be “bleached” out of the state.) It is clear, then, that these two qubits should be thought of as the auxiliary qubits at intermediate steps. We’ll discuss this a bit more in Section [V]

We also introduce some interesting internal dynamics. First, we need to move the auxiliary qubits out of the way, so that they don’t affect the next radiation step. So we first cyclically shift all of the qubits 2 positions to the right, thus shoving the two $\hat{0}$s to the two leftmost positions, 1 and 2. Then, we introduce some dynamics for the physical degrees of freedom. We cyclically shift only the nonauxillary qubits to the right by one unit. This defines $\hat{U}$ so that the model agrees with the burning paper model studied in [13]. If we chop off the zeroed qubits, we recover the model exactly.

The first model in [12] is in the same class of models. It too zeroes two qubits, one of which is the newly created black hole qubit. The main difference being that instead of the radiation being

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Throughout the discussion, the reader may assume that the identity acts on any subspaces which are not explicitly shown.
determined by the two rightmost hatted qubits, it is instead determined by the leftmost qubit. This model may be written as

\[
\hat{P}_1 = \hat{P}_2 = \frac{1}{\sqrt{2}} |0_{i+1}\rangle \langle 0_{i+1}| \otimes \hat{u} \hat{u}^\prime
\]

\[
\hat{P}_3 = |0_{i+1}\rangle \langle 1_{i+1}| \otimes \hat{u},
\]

where \(\hat{u}\) and \(\hat{u}'\) are unitary operators acting on the remaining hatted qubits. While it is not stated in [12], we should require that \(\hat{u}\) and \(\hat{u}'\) do not mix the first \(i+1\) or the last \(i\) auxiliary hatted qubits with the remaining physical qubits. The simplest case is to take \(\hat{u} = \hat{u}' = I\). In this model, the entanglement entropy of the radiation is always zero, which contrasts with our expectations from [8].

### III.3. “Nonlocal” Unitary Evolution

In [12], Giddings presents three unitary models of evolution. We focus on the second model that he presents. This second model can be written in our notation as

\[
\hat{P}_1 = |\hat{0}_{2i+1}\rangle \langle \hat{0}_{2i+2}| \hat{0}_{2i+1} \hat{0}_{2i+2}
\]

\[
\hat{P}_2 = |\hat{0}\rangle \langle \hat{1}| \hat{1}
\]

\[
\hat{P}_3 = |\hat{0}\rangle \langle \hat{0}| \hat{0}
\]

\[
\hat{P}_4 = |\hat{0}\rangle \langle \hat{1}| \hat{1}
\]

\[\hat{U} = I, \quad \hat{\theta} = \frac{\pi}{2}\]

where we have suppressed the \((2i + 1)\) and \((2i + 2)\) subscripts in all but the first \(\hat{P}\). One sees that as in the models presented in Section III.2, two hatted qubits are zeroed at each step. In this case, they are the \((2i + 1)\)th and \((2i + 2)\)th qubits. Thus as the evolution progresses the hatted qubits are gradually put into a fiducial form. By the \(i\)th step, the first \(2i\) qubits are zeroed, and should be thought of as auxiliary. Note that the above evolution rule breaks down on the penultimate step, when there is only one nonauxillary qubit left. By then, we expect the black hole to be on the Planck scale, and so we can just emit the last qubit freely. This model corresponds to \(\theta = \frac{\pi}{2}\) in the model presented in Section VI.

This model is nonlocal when one considers the model in the original nice slicing of the black hole. In this context, the \((2i + 1)\)th and \((2i + 2)\)th qubit are very far from the pair creation site at the horizon. Note that this property is shared by the model in Equation (3.5), and to a lesser extent the model in Equation (3.3). The difference being how far away the zeroed qubits are. One can either interpret these unitary models as nonlocal interactions transmitting information far down the nice slice to the horizon [12], or in terms of fuzzball microstates altering the state at the horizon, or as burning paper; these information theoretic models are too crude to distinguish. This point is discussed further in the Conclusion.

### III.4. A Pure, but Not Invertible Model

There are several of models of this kind. The simplest to consider is

\[
\hat{P}_{1,2,4} = 0 \quad \hat{P}_3 = I \quad \hat{\theta} = I
\]

In this model, regardless of the state of the system, the new pair is created in the state \(|\varphi_3\rangle\). The state \(|\varphi_3\rangle\) is not entangled, and thus we can think of this model as zeroing the new black hole
qubit and the new radiation qubit. Instead of putting the internal qubits into a fiducial form, we put the radiation into a fiducial form. We hope that this convincingly demonstrates that purity of the final state does not ensure unitarity.

Another interesting example of (almost) pure but not invertible evolution is the model in Equation (3.5), with
\[ \hat{u} = \hat{u}' = \hat{S}_{i+2,n+1}, \]
where \( \hat{S}_{i+2,n+1} \) is the operator that cyclically shifts the \( (i+2) \)th through \( (n+i) \)th qubits to the left. For example, consider the evolution:
\[
\begin{align*}
|0\hat{q}_1\hat{q}_2\cdots\hat{q}_{n-2}\rangle \\
\text{C}_0 \rightarrow |\hat{0}\hat{0}\hat{q}_1\hat{q}_2\cdots\hat{q}_{n-2}\rangle |0\rangle \\
\text{C}_1 \rightarrow |\hat{0}\hat{0}\hat{0}\hat{q}_1\hat{q}_2\cdots\hat{q}_{n-2}\rangle |00\rangle \\
\text{C}_2 \rightarrow |\hat{0}\hat{0}\hat{0}\hat{0}\hat{q}_1\hat{q}_2\cdots\hat{q}_{n-2}\rangle |000\rangle \\
\vdots
\end{align*}
\]
In the above example, the radiation is never entangled with the black hole degrees of freedom, but its state is only determined by the first and last qubit of the initial state. Thus, the evolution is not unitary. This is a potential problem with the model even as it is defined in [12]. What happened? In essence, the model keeps “reading” and zeroing the same qubit, which has the net effect of zeroing the radiation qubits as well as the new black hole qubits. This illustrates that unitary evolution can be lost when auxiliary qubits mix with nonauxillary qubits.

This model is not quite pure for arbitrary initial states, since
\[
\begin{align*}
|\hat{1}\hat{q}_1\hat{q}_2\cdots\hat{q}_{n-2}\rangle \\
\text{C}_0 \rightarrow |\hat{0}\hat{1}\hat{q}_1\hat{q}_2\cdots\hat{q}_{n-2}\rangle |1\rangle \\
\text{C}_1 \rightarrow |\hat{0}\hat{0}\hat{0}\hat{q}_1\hat{q}_2\cdots\hat{q}_{n-2}\rangle |11\rangle \\
\text{C}_2 \rightarrow |\hat{0}\hat{0}\hat{0}\hat{0}\hat{q}_1\hat{q}_2\cdots\hat{q}_{n-2}\rangle |011\rangle \\
\vdots
\end{align*}
\]
which gives a pure final state, but if one considers a nontrivial superposition of the above initial state and the one in (3.9) then the final state is mixed. Note that the radiation only carries two qubits of information about the initial state, and so the entropy of the final state is very small although nonvanishing on some initial states. It is in this sense that we call the evolution almost pure.

### III.5. An Impure Model

A generic model that one constructs leads to impure evolution. When thinking about the different forms that \( \hat{P} \)s can take, a particularly natural variation on the Section III.3 model to consider might be
\[
\begin{align*}
\hat{P}_1 &= |\hat{0}\hat{0}\rangle \langle \hat{0}\hat{0}| \\
\hat{P}_2 &= |\hat{1}\hat{1}\rangle \langle \hat{1}\hat{1}| \\
\hat{P}_3 &= |\hat{0}\hat{1}\rangle \langle \hat{0}\hat{1}| \\
\hat{P}_4 &= |\hat{1}\hat{0}\rangle \langle \hat{1}\hat{0}| \\
\hat{P}_5 &= |\hat{1}\hat{0}\rangle \langle \hat{1}\hat{0}| \\
\end{align*}
\]
where as in (3.6) the above operators act on the \((2i + 1)\)th and \((2i + 2)\)th qubits. It should be clear that this model leads to mixed states. Note that it is also a large deformation from the Hawking model. It is easy to see that this model is not invertible either, by considering \(|\hat{0}\hat{0}\rangle\) and \(|\hat{1}\hat{1}\rangle\) as initial states.

III.6. Mathur–Plumberg Shift–Anti-Shift Models

In [13], Mathur and Plumberg present several models. We can write their “Model A,” in the following way. Let \(\hat{T}_j\) be the operator that cyclically shifts only the newly created (not the first \(n\)) hatted qubits to the right by \(j\). Then, the model is of the form

\[
\hat{P}_1 = \lambda_1 \hat{T}_1 \quad \hat{P}_2 = \lambda_2 \hat{T}_{-1} \quad \hat{P}_{3,4} = 0, \tag{3.12}
\]

where the completeness relation requires

\[
|\lambda_1|^2 + |\lambda_2|^2 = 1. \tag{3.13}
\]

More generally, we can replace \(\hat{T}_1\) and \(\hat{T}_{-1}\) by other unitary transformations that act on the hatted qubits. The smallness of the corrections to Hawking is loosely determined by \(\lambda_2\); see [13] for more details and numerical results.

The “Model B” of [13] may be written as

\[
\hat{P}_1 = \lambda_1 \hat{T}_1 \otimes |\hat{1}_{i+1}\rangle \langle \hat{1}_{i+1}| + \hat{I} \otimes |\hat{0}_{i+1}\rangle \langle \hat{0}_{i+1}| \quad \hat{P}_2 = \lambda_2 \hat{T}_{-1} \otimes |\hat{1}_{i+1}\rangle \langle \hat{1}_{i+1}| \quad \hat{P}_{3,4} = 0, \tag{3.14}
\]

where one can confirm that the completeness relation imposes the same constraint (3.13).

Note that neither of the above models zeroes any qubits, and so one does not expect information about the initial matter to be transmitted out in the radiation.

III.7. Mathur “Ising” Model

In [15], Mathur presents a model that can be mapped onto the one-dimensional Ising model and thus solved analytically. The model can be written in the form

\[
\hat{P}_1 = \lambda_1 [\hat{\text{0}}] \langle \text{0} | + |\hat{1}\rangle \langle n_{+i}| \quad \hat{P}_2 = \lambda_2 [\hat{\text{0}}] \langle \text{0} | - |\hat{1}\rangle \langle n_{+i}| \quad \hat{P}_{3,4} = 0, \tag{3.15}
\]

where \(\lambda_1\) and \(\lambda_2\) are related to the parameters \(a\) and \(b\) in [15] via

\[
\lambda_1 = \frac{e^a + e^b}{2}, \quad \lambda_2 = \frac{e^a - e^b}{2}. \tag{3.16}
\]

The above rule is not valid for the first step, for which we use the Hawking model (\(\lambda_1 = 1\) and \(\lambda_2 = 0\)). The parameters \(\lambda_1\) and \(\lambda_2\) are subject to the same constraint as before

\[
|\lambda_1|^2 + |\lambda_2|^2 = 1. \tag{3.16}
\]

In fact, this model is qualitatively the same as the model in Section III.6: both set \(\hat{P}_1\) and \(\hat{P}_2\) to unitary transformations, and \(\hat{P}_3 = \hat{P}_4 = 0\). The expression for the final state entropy can be written in the form [15]

\[
S = n \log 2 - (n - 1) \left( ae^{2a} + be^{2b} \right). \tag{3.17}
\]
Let us note that for no value of $a$ and $b$ that solves the completeness relation does this vanish. Thus, no matter how large the corrections are in this model, unitarity is lost; however, one can set $\lambda_1 = \lambda_2$ in which case the entanglement entropy of the radiation is $\log 2$ for all time. In this case, the pair creation (after the first step) is given by

$$C_i = |\hat{0}_{n+i+1}\rangle |0_i\rangle \otimes |\hat{0}_{n+i}\rangle + |\hat{1}_{n+i+1}\rangle |1_i\rangle \otimes |\hat{1}_{n+i}\rangle.$$  \hspace{1cm} (3.18)

One sees that after the first step, no further entanglement between the hatted and unhatted qubits is generated. One might think there would be more entanglement generated when the above acts on qubits which are in a superposition of 1 and 0, but since the above only acts on previously created pairs we do not have to worry about that issue. One can imagine that the model breaks down when the black hole becomes very small and the last qubit is emitted freely, so this model is effectively pure in this case. Let us note that keeping the entanglement entropy of the radiation constant is in stark contrast with expectations we have from Page [8]; there is no characteristic rise and fall of the entanglement entropy of the radiation. Furthermore, the final state consisting entirely of radiation is completely independent of the initial matter that formed the black hole; the evolution is (almost) pure but far from invertible like the model discussed in Section III.4. This model, while interesting to consider, never leads to unitary evolution, even when one considers arbitrarily large deformations from the Hawking point in model space.

**IV. REVIEW AND GENERALIZATION OF MATHUR’S ARGUMENT**

In [10], Mathur argues that small corrections to the pair creation process are insufficient to restore unitarity. More specifically, he demonstrates that small corrections don’t accumulate. Let us define $S_i$ as the entanglement entropy of the radiation with the rest of the system at step $i$. If the black hole evaporation process is unitary, then in the limit of large $n$ we expect $S_i$ to rise linearly with $i$ until about the halfway point, $i = n/2$, and then rapidly turn over and fall linearly to zero on the final step [9]. On the other hand, if one uses the Hawking model of evolution $C_H^i$, then one sees that $S_i$ increases by $\log 2$ at each time step:

$$S_i^{\text{Hawking}} = i \log 2;$$  \hspace{1cm} (4.1)

there is no turnover. This is also the maximum entropy that is possible for the $i$ radiation qubits, which indicates that the radiation carries no information about the initial state.

The central insight in Mathur’s argument [10] is that the marginal increase in entanglement entropy,

$$\Delta S_i = S_{i+1} - S_i,$$  \hspace{1cm} (4.2)

varies smoothly with small deformations away from the Hawking model, and thus a large deformation is needed to make $\Delta S$ negative if one starts from the Hawking model’s $\log 2$. In [10], only a $\hat{P}_2$-type deformation to the Hawking model was explicitly considered; however, we demonstrate below that the the argument generalizes to all deformations in the model space.

In particular, we claim in the class of models discussed in this paper, if

$$\|C_i - C_i^H\| < \varepsilon < 1,$$  \hspace{1cm} (4.3)

then

$$\Delta S_i \geq \log 2 - k\varepsilon,$$  \hspace{1cm} (4.4)
where \( k_\varepsilon \) is parametrically small, positive, and vanishes as \( \varepsilon \) goes to zero. In fact, we demonstrate below that \( k_\varepsilon \) behaves as no worse than \( k_\varepsilon \sim -9\varepsilon \log \varepsilon \) as \( \varepsilon \) approaches zero. Above, \( \| \cdot \| \) is the operator norm. For an operator \( O \), \( \|O\| \) is the square root of the largest eigenvalue of \( O^\dagger O \)\(^9\).

The proof follows that in [10] with some minor modifications. To begin, we use strong subadditivity of (von Neumann) entanglement entropy. Throughout, without loss of generality, we set \( \hat{U} = U = I \) since they do not affect the entanglement entropies. Let us consider the \((i + 1)\)th state

\[
|\psi_{i+1}⟩ = C_i |ψ_i⟩
\]

Let \( R_i \) denote the first \( i \) emitted radiation qubits, \( r \) denote the \((i + 1)\)th emitted radiation qubit, \( B_i \) denote the first \( n + i \) black hole qubits, and \( b \) denote the \((n + i + 1)\)th black hole qubit. Then, strong subadditivity implies

\[
S(R_i \cup r) + S(r \cup b) ≥ S(R_i) + S(b).
\]

By definition, \( S(R_i \cup r) \) is simply \( S_{i+1} \), whereas since the \( C_i \) acts trivially on emitted radiation \( S(R_i) = S_i \). Thus, we can write the above as

\[
\Delta S_i ≥ S(b) - S(r \cup b).
\]

Note that for the Hawking model \( S(b) = \log 2 \) and \( S(r \cup b) = 0 \), and the bound is saturated.

Now, we need to use the condition on \( C_i \) to place bounds on \( S(b) \) and \( S(r \cup b) \). The operator norm is compatible with the Hilbert space norm, which means

\[
\| (C_i - C_i^H) \| \lambda \| ≤ \| C_i - C_i^H \| \| \lambda \| \|
\]

for all \( |\lambda⟩ \). Applying this to \( |ψ_i⟩ \) gives the condition

\[
\| |ψ_{i+1}⟩ - C_i^H |ψ_i⟩ \| < \varepsilon.
\]

We may write the two kets in the form

\[
|ψ_{i+1}⟩ = \alpha_1 |ϕ_1⟩ |Λ_1⟩ + \alpha_2 |ϕ_2⟩ |Λ_2⟩ + \alpha_3 |ϕ_3⟩ |Λ_3⟩ + \alpha_4 |ϕ_4⟩ |Λ_4⟩,
\]

\[
C_i^H |ψ_i⟩ = |ϕ_1⟩ |Λ_0⟩,
\]

where the \(|Λ_i⟩\) are normalized, but not necessarily orthogonal kets in the \( n + 2i \) qubit space \( R_i \cup B_i \). Of course normalization demands that

\[
|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 = 1.
\]

One can show that the condition (4.9) implies that

\[
\text{Re} (\alpha_1 ⟨Λ_0 |Λ_1⟩) > 1 - \frac{\varepsilon^2}{2},
\]

and since \( |⟨Λ_0 |Λ_1⟩| ≤ 1 \), we see

\[
1 - \frac{\varepsilon^2}{2} < |\alpha_1| ≤ 1.
\]

This, in turn, implies that

\[
1 - \delta^2 < |\alpha_1|^2 < 1, \quad |\alpha_2|, |\alpha_3|, |\alpha_4| < \delta = \varepsilon \sqrt{1 - \frac{\varepsilon^2}{4}},
\]

\(^9\) In fact, the result is unchanged by using any other norm that is compatible with the Hilbert space norm.
where we have defined $\delta$ for convenience. Note that $\delta$ is what should be directly compared with $\varepsilon$ in [10].

Let us use the above to place an upper bound on $S(r \cup b)$. The corresponding reduced density matrix is given by

\[
\rho_{rb} = \begin{pmatrix}
|\alpha_1|^2 & \alpha_1\alpha_2^* \langle A_1|\Lambda_1 \rangle & \alpha_1\alpha_3^* \langle A_1|\Lambda_2 \rangle & \alpha_1\alpha_4^* \langle A_1|\Lambda_3 \rangle \\
\alpha_1\alpha_2 \langle A_2|\Lambda_1 \rangle & |\alpha_2|^2 & \alpha_2\alpha_3^* \langle A_2|\Lambda_2 \rangle & \alpha_2\alpha_4^* \langle A_2|\Lambda_3 \rangle \\
\alpha_1\alpha_3 \langle A_3|\Lambda_1 \rangle & \alpha_2\alpha_3 \langle A_3|\Lambda_2 \rangle & |\alpha_3|^2 & \alpha_3\alpha_4^* \langle A_3|\Lambda_3 \rangle \\
\alpha_1\alpha_4 \langle A_4|\Lambda_1 \rangle & \alpha_2\alpha_4 \langle A_4|\Lambda_2 \rangle & \alpha_3\alpha_4 \langle A_4|\Lambda_3 \rangle & |\alpha_4|^2
\end{pmatrix}. \quad (4.15)
\]

We can now use Audenaert’s optimal generalization [25] of Fannes’ inequality, which places a bound on the difference in entropy of two $d$-dimensional density matrices $\rho$ and $\sigma$ [10]. Let $T$ be the trace distance between $\rho$ and $\sigma$, then [25]

\[
|S(\rho) - S(\sigma)| \leq T \log(d-1) - T \log T - (1 - T) \log(1 - T),
\]

where the trace distance is defined as

\[
T = \frac{1}{2} \text{tr} \left[ \sqrt{(\rho - \sigma)\dagger(\rho - \sigma)} \right], \quad (4.17)
\]

or one-half the sum of the absolute value of the eigenvalues of $\rho - \sigma$. Note that the above definition of the trace distance differs by a factor of 2 from some references. With the above normalization $0 \leq T \leq 1$ for all unit-trace density matrices $\rho$ and $\sigma$. In this case we consider the $\sigma$ to be the density matrix with $|\alpha_1| = 1$, and $\rho$ to be $\rho_{rb}$. Next, we may use Gersgorin’s circle theorem to bound the eigenvalues, $\lambda$, and therefore the trace distance, $T$. Gershgorin’s theorem tells us all the eigenvalues must lie in the union of discs in the complex plane centered on the diagonal entries with radii given by the sum of the absolute value of off-diagonal entries for each row. For instance the first row gives a disc

\[
D_1 : |\lambda - (|\alpha_1|^2 - 1)| \leq |\alpha_1||\alpha_2||\langle A_2|\Lambda_1 \rangle| + |\alpha_1||\alpha_3||\langle A_3|\Lambda_1 \rangle| + |\alpha_1||\alpha_4||\langle A_4|\Lambda_1 \rangle|. \quad (4.18)
\]

Applying our inequalities on the components and the hermiticity of $\rho - \sigma$ (and therefore reality of its spectrum), we can find an interval that must include any eigenvalues in the above disc:

\[
I_1 = (-3\delta - \delta^2, 3\delta). \quad (4.19)
\]

One finds the remaining three rows can be encompassed by the interval

\[
I_2 = (-\delta - 2\delta^2, \delta + 3\delta^2), \quad (4.20)
\]

and so all eigenvalues must satisfy

\[
|\lambda| < 3\delta + \delta^2. \quad (4.21)
\]

Thus, we may conclude that the trace distance must satisfy

\[
T < 2(3\delta + \delta^2), \quad (4.22)
\]

and therefore

\[
S(r \cup b) \leq -x_1 \log(\frac{1}{2}x_1) - (1 - x_1) \log(1 - x_1) \quad x_1 = \min(\frac{3}{4}, 2(3\delta + \delta^2)). \quad (4.23)
\]

\[10\] One could instead use Fannes’ inequality for a weaker bound with stronger restrictions on $\varepsilon$. 

The 3/4 comes by finding critical point of the right-hand side of (4.16) as a function of $T$.

The bound on $S(b)$ can be derived in analogous fashion. The state $|\psi_{i+1}\rangle$ may be written out as

$$|\psi_{i+1}\rangle = |0\rangle |\chi_0\rangle + |1\rangle |\chi_1\rangle$$

$$|\chi_0\rangle = \frac{\alpha_1}{\sqrt{2}} |0\rangle |\Lambda_1\rangle + \frac{\alpha_2}{\sqrt{2}} |0\rangle |\Lambda_2\rangle + \alpha_3 |1\rangle |\Lambda_3\rangle$$

$$|\chi_1\rangle = \frac{\alpha_1}{\sqrt{2}} |1\rangle |\Lambda_1\rangle - \frac{\alpha_2}{\sqrt{2}} |1\rangle |\Lambda_2\rangle + \alpha_4 |0\rangle |\Lambda_4\rangle.$$  \hfill (4.24)

The reduced density matrix can be written as

$$\rho_b = \begin{pmatrix} \langle \chi_0 |\chi_0\rangle & \langle \chi_1 |\chi_0\rangle \\ \langle \chi_0 |\chi_1\rangle & \langle \chi_1 |\chi_1\rangle \end{pmatrix}.$$ \hfill (4.25)

As before we can place bounds on the above components

$$\left| \langle \chi_0 |\chi_0\rangle - \frac{1}{2} \right| < \delta + \frac{\delta^2}{2}$$

$$\left| \langle \chi_1 |\chi_1\rangle - \frac{1}{2} \right| < \delta + \frac{\delta^2}{2}$$

$$| \langle \chi_1 |\chi_0\rangle | < \sqrt{2}(\delta + \delta^2).$$ \hfill (4.26)

One can once again use the Fannes–Audenaert inequality along with Gershgorin’s theorem to bound $S(b)$, where $\sigma$ is the $\alpha_1 = 1$ density matrix, $I/2$. One finds the trace distance satisfies

$$T < (1 + \sqrt{2})\delta + \left(\frac{1}{2} + \sqrt{2}\right)\delta^2,$$ \hfill (4.27)

and this gives

$$S(b) \geq \log 2 + x_2 \log x_2 + (1 - x_2) \log(1 - x_2) \quad x_2 = \min\left(\frac{1}{2}, (1 + \sqrt{2})\delta + \left(\frac{1}{2} + \sqrt{2}\right)\delta^2\right).$$ \hfill (4.28)

Finally, this allows us to write

$$k_\varepsilon = -x_1 \log\left(\frac{1}{2}x_1\right) - x_2 \log x_2 - (1 - x_1) \log(1 - x_1) - (1 - x_2) \log(1 - x_2),$$ \hfill (4.29)

where recall $x_1$ is defined in Equation (4.23), $x_2$ in Equation (4.28), and $\delta$ in Equation (4.14). For asymptotically small $\varepsilon$, $k_\varepsilon \sim -(7 + \sqrt{2})\varepsilon \log \varepsilon$; and in fact for small but finite $\varepsilon$, $k_\varepsilon < -9\varepsilon \log \varepsilon$.

Numerically, one finds that $k_\varepsilon$ first surpasses $\log 2$, thus allowing the entanglement entropy to decrease for $\varepsilon \approx .02$. Furthermore, one finds $k_\varepsilon$ reaches $2 \log 2$, thus allowing the maximal marginal decrease of entanglement entropy for $\varepsilon \approx .05$. For larger $\varepsilon$, the inequality with $k_\varepsilon$ given in (4.29) places no restriction on the marginal change in entanglement.

Since we are not especially interested in making the tightest possible bound or even the above numerical values, we may as well write the bound in slightly less unwieldy form,

$$\Delta S_i \geq \log 2 + 9\varepsilon \log \varepsilon \quad \varepsilon \ll 1.$$ \hfill (4.30)

This establishes the claim. Let us note that the above bound’s asymptotic behavior is weaker than the inequality derived in \cite{10} as a consequence of using more general arguments to include arbitrary perturbations. On the other hand, the result presented here is stronger in the sense that \cite{10} finds only a leading order result valid to order $O(\varepsilon^2)$ whereas the bound (4.4) with $k_\varepsilon$ given in (4.29) is valid for finite $\varepsilon \in (0, 1)$. The above bounds could possibly be strengthened with more work,\footnote{An equivalently strong bound here would be $k_\varepsilon = 2\delta$.} however, that is irrelevant to the basic claim that small corrections to the low energy pair creation process cannot restore unitarity.

\footnote{For example, one might make progress by direct computation of $S(r \cup b)$ and $S(b)$ as was performed in \cite{10}, but this would involve solving an eigenvalue problem for a four-dimensional density matrix with arbitrary coefficients.}
V. REQUIREMENTS FOR UNITARITY

While it is interesting to think about the different kinds of evolution that one could have, perhaps the most interesting question to ask is what kinds of models are unitary or, equivalently what sorts of corrections to the Hawking evolution can restore unitarity. Above we see that small corrections to the evolution cannot restore unitarity; this gives a necessary condition that the corrections are large. It would be nice to also have some sufficient conditions, since it is clear that not every large correction one could consider leads to unitary evolution.

In order for the evolution to be pure, we need the final state (including the auxiliary qubits) to be a direct product of the form

\[ |\psi_n\rangle = |\hat{\phi}\rangle \otimes |\chi\rangle, \]

(5.1)

where \(|\hat{\phi}\rangle\) is a state in the \(2n\)-qubit auxiliary space and \(|\chi\rangle\) is the state of the physical radiation qubits. Our first observation is that \(|\hat{\phi}\rangle\) should be independent of the initial state. Suppose that this were not true:

\[ |\phi_0^{(1)}\rangle \mapsto |\hat{\phi}^{(1)}\rangle \otimes |\chi^{(1)}\rangle \]

(5.2)

\[ |\phi_0^{(2)}\rangle \mapsto |\hat{\phi}^{(2)}\rangle \otimes |\chi^{(2)}\rangle, \]

which seems fine until one considers an initial state which is a superposition of the above two states; the final state is then mixed when one traces out the hatted qubits. (This argument assumes that the \(|\chi\rangle\)'s are linearly independent so that the evolution is invertible.) This is basically a variant of the no-cloning theorem.

The next observation is that we need the final radiation state \(|\chi\rangle\) to be a unitary transformation of the initial state \(|\psi_0\rangle\). Let \(F\) be the total map from initial state to the final state, then \(F\) is a linear, isometric (norm-preserving) mapping from \(n\) hatted qubits to \(2n\) hatted plus \(n\) unhatted qubits. From the above, unitarity demands that

\[ F_{\text{unitary}} : |\psi\rangle \mapsto |\hat{\phi}\rangle \otimes |\chi\rangle \quad |\chi\rangle = U |\psi\rangle, \]

(5.3)

where \(U\) in the above is a unitary transformation from the initial \(n\) hatted qubits to the final \(n\) unhatted qubits, and \(|\hat{\phi}\rangle\) is fixed. The total map \(F\) is just the product of all the \(C_i\)'s and \(\hat{U}_i \otimes U_i\)'s. All of the hatted qubits have to be zeroed or bleached, and the information stored in the initial matter transferred to the radiation.

Let us think about how we can zero or bleach the hatted qubits. We have \(n\) steps to project \(2n\) qubits to a unique state with the \(\hat{P}\)'s; the unitary \(\hat{U}\)'s clearly cannot zero qubits. At each step, we can zero at most two qubits. If \(C\) bleaches some subspace to a state \(|\hat{\alpha}\rangle\), then it may be written as

\[ C = |\hat{\alpha}\rangle \otimes O, \]

(5.4)

for an unspecified operator \(O\). If \(|\hat{\alpha}\rangle\) is a \(p\)-qubit subspace, then \(O\) maps \(n+i\) qubits to \(n+i+2-p\) qubits and must satisfy

\[ O^\dagger O = I; \]

(5.5)

this is only possible if \(n+i+2-p \geq n+i\), immediately implying \(p \leq 2\).

Our key observation is that the desire to zero the hatted qubits is in tension with the completeness relation (2.19). Since we only have four \(\hat{P}\)'s, at any given step the best we can do is project out a four-dimensional subspace, or two qubits. It is this tension that connects the need to zero qubits with the requirement to have large corrections to the Hawking model. This might help elucidate
the results in [10]. Note that this is very much in agreement with the picture presented in Figure 1, wherein at each stage there are two new auxillary qubits. For the state to be pure, these auxillary qubits must be zeroed.

Let us note that there are two different ways to zero two qubits at each step, although the distinction is not actually that sharp when one considers the full $C_i$s. In the burning paper model of Section III.2 we use only three $\hat{P}$s, which zero one qubit. The three $\hat{P}$s were chosen to ensure that the newly created $\hat{q}$ is also zeroed. The second way is illustrated in Section III.3 in which all four $\hat{P}$s are used to zero two old $\hat{q}$s. One can consider various unitary transformations, however, these are the only two qualitative kinds of models that lead to a pure radiation final state. Remember that which $\hat{P}$s get used tell us which pair state is created at the horizon. It is impossible to preferentially use only $\hat{P}_1$ and simultaneously have unitary evolution.

As we saw in Section III.4, it is possible for the evolution to be pure, but not invertible. In the model in Equation (3.8), qubits that were zeroed in previous steps mixed with nonzeroed qubits. When we zero the qubits, we are then thinking of them as auxillary degrees of freedom that should be erased in the operator-sum description (2.6). Thus, it does not make physical sense to allow mixing with the auxillary degrees of freedom if one wants unitary evolution.

The requirements outlined above for purity and invertibility should ensure unitary evolution.

VI. A ONE-PARAMETER INTERPOLATING MODEL

We can interpolate between the Hawking model in Section III.1 and the unitary model in Section III.3 via

$$\begin{align*}
\hat{P}_1 &= \cos \theta \hat{1} + (1 - \cos \theta) |\hat{0}_{2i+1}\hat{1}_{2i+2} \rangle \langle \hat{1}_{2i+1}\hat{0}_{2i+2}| \\
\hat{P}_2 &= \sin \theta |\hat{0}\hat{0} \rangle \langle \hat{1}\hat{1}| \\
\hat{P}_3 &= \sin \theta |\hat{0}\hat{0} \rangle \langle \hat{0}\hat{1}| \\
\hat{P}_4 &= \sin \theta |\hat{0}\hat{0} \rangle \langle \hat{1}\hat{0}|
\end{align*}$$

with $\theta = 0$ giving Hawking’s evolution and $\theta = \frac{\pi}{2}$ giving the unitary evolution in Equation (3.6). (Once again we suppressed subscripts on the qubits after the first line.) We may write

$$\begin{align*}
(C - C^H)^\dagger(C - C^H) = 2\hat{I} - \hat{P}_1 - \hat{P}_1^\dagger = 2(1 - \cos \theta)(\hat{I} - |\hat{0}\rangle \langle \hat{0}|),
\end{align*}$$

so that one finds

$$\|C - C^H\| = 2|\sin \frac{\theta}{2}|.$$  \hspace{1cm} (6.3)

We clearly see that this is in accord with Mathur’s argument and its generalization in Section IV.

One of the main results of this paper is the above model, which continuously connects the Hawking model to a unitary model, clearly illustrating that they are far apart in model space. Previous efforts to illuminate Mathur’s bound [13, 15], considered different types of small corrections and showed that they did not significantly affect the entropy of the final state; however, they did not consider corrections that when made large would give a unitary “burning paper” type model. The above model fills this gap. In Figure 2, we plot the second Rényi entanglement entropies of the radiation as a function of $\theta$ for three different initial states. Recall that the second Rényi entropy is defined as

$$S_2(\rho_{\text{red}}) = -\log \text{tr}(\rho_{\text{red}}^2),$$  \hspace{1cm} (6.4)
and is a positive-definite measure of entanglement that vanishes if and only if $\rho_{\text{red}}$ is pure. For computational purposes, however, it is a bit more convenient than the traditional von Neumann entropy. It should also be noted that $S_2$ gives a lower bound for the von Neumann entropy. Moreover, when one examines the von Neumann entropy it behaves qualitatively similarly.

FIG. 2. Here we present the second Rényi entropies of the unhatted radiation qubits as a function of $\theta$ for the model in Equation (6.1) with three different initial states. Intermediate steps are dashed curves and the final step is solid; the steps are shown in the color order (red, yellow, green, blue, purple, black). The point $\theta = 0$ corresponds to the canonical Hawking evolution, while $\theta = \frac{\pi}{2}$ corresponds to the unitary model in Equation (3.6).

In Figure 2, one sees that at $\theta = 0$ (Hawking model) the entropy rises by one at each step, except for the final step where it drops down by one. This is an artifact of the way we end the evolution, since the model breaks down on the penultimate step. We chose to just emit the last qubit freely, which makes sense on physical grounds since the black hole should be quite small by that point; one can easily imagine large corrections in the final stage(s) of black hole evaporation. At $\theta = \frac{\pi}{2}$, we have the unitary model, and the entanglement entropy of the radiation has the expected rise and fall. For no $\theta$ close to $\theta = 0$, however, does the entanglement entropy of the radiation fail to increase (excluding the final step). Of course, a 5 qubit initial state is not realistic for the macroscopically large black holes we are thinking about; however, it suffices to demonstrate the qualitative behaviour which should not change as $n$ is increased. The author was limited by the computational power required, which grows quite rapidly with $n$.

VII. CONCLUSION

We have presented a very general framework that provides a natural, unifying language to compare information-theoretic models of black hole evaporation. The framework involves describing the dynamics of pure state vectors $|\psi\rangle$ in an ever enlarging Hilbert space. The only constraint on the models is the completeness relation (2.19). In Section II.2, we explain how to interpret this model in terms of potentially mixed evolution in a fixed dimensional Hilbert space: one must trace out some auxiliary degrees of freedom to arrive at a dynamical equation like in Equation (2.6). Part of a full model, then, involves specifying the auxiliary degrees of freedom. While at intermediate steps this can be ambiguous, by the end of the evolution we are left only with radiation and thus any nonradiation degrees of freedom are by default auxilary. This excludes remnants or other scenarios where one can identify physical degrees of freedom that the Hawking radiation is entangled with at the end of the evaporation.

In Section III, we show how to write a number of interesting models in our unifying notation. Many of them had been introduced and studied previously in the literature. These models illustrate some of the key obstructions and requirements to have unitary evolution.
In Section V, we discuss the requirements for unitarity, and summarize a set of sufficient conditions. A key backdrop to our discussion, and indeed the whole paper, is a recent theorem [10] showing that small corrections to the Hawking model cannot give unitary evolution. A corollary is that large corrections are necessary to have unitary evolution. As we show, this is, however, not a sufficient condition; unsurprisingly, there are many large corrections that fail to give unitary evolution. One interesting observation is how the unitary requirement that internal qubits be zeroed becomes connected to corrections to the pair creation via the completeness relation (2.19). This may help elucidate the results in [10].

Finally, in Section VI, we give a one-parameter family of models that continuously interpolates between the Hawking model and a unitary model. In terms of this parameter, one can clearly see that the unitary model is far from the Hawking model, thus illustrating the theorem in [10].

One of the key points emphasized in [10, 14] is that the nice slicing of the Schwarzschild solution implies at best small corrections to the Hawking model in Equation (3.1), and therefore a loss of unitarity evolution. To restore unitarity, large corrections are required of the form discussed here; however, one must show why these corrections arise in the black hole and not in all of our earth-based experiments and observations. It seems quite difficult to do this, since in the nice slice construction no geometric quantity is large. The only quantity that seems to be large is the number of degrees of freedom or number of particles required to form the black hole, but this is not a basic geometric quantity.

Let us further note that for our discussion there is a factorization of the internal dynamics and the pair creation dynamics. The internal black hole dynamics can be as nonlocal, or scramble as rapidly as one wants, but whether the evaporation process is unitary or not is (modulo a few caveats mentioned in Sections III and V) entirely determined by the pair creation process. Thus, the discussion in [20, 21, 24, 26] is not directly relevant to our concerns here, although it is important for better understanding black holes. The pair creation process is localized near the horizon, where for a large black hole the geometry suggests one can trust the semiclassical approximation even if one might doubt its validity deep within the black hole. On the pair creation time scale, however, it is precisely this physics that needs an order unity correction [10].

One other issue that one may wish to raise in our discussion is the issue of conservation laws [11, 27]. While we hope we have sufficiently addressed the issue of an expanding Hilbert space as raised in [11], one may still be concerned that we haven’t discussed conservation of energy (or angular momentum, electric charge, etc.). These issues are rebutted in [14]. Let us note, however, by not discussing the original spacetime physics and the resulting at most small corrections, the fundamental issue has been totally elided. While the results presented in [11, 27] are interesting unto themselves, they do not provide a plausible physical mechanism to modify the pair creation process to get the dynamics they suggest.

If, as suggested by string theory, or from other considerations, we think black hole evaporation is a unitary process; then, the pair creation process must not strictly adhere to the causal structure on the Schwarzschild nice slicing. There are two obvious frameworks (ignoring the possibility of remnants) to discuss this deviation: fuzzballs or nonlocality.

The fuzzball proposal (see [28–32] for reviews) suggests that the black hole metric is only an effective geometry that approximates $e^{\mathrm{Si}}$ microstates. The microstates differ from each other on the horizon scale, thus large corrections to the Hawking evolution are anticipated and information is transmitted from local excitations. How the fuzzball proposal relates to these qubit models is discussed in [10, 13, 15]. The main point being that since the geometry at the would-be horizon depends on the internal state, their is a physical mechanism to get large corrections to the pair creation process. Since the fuzzball’s interior geometry (and the whole causal structure) is quite different from the original black hole solution in which the nice slices were constructed, one should probably not interpret them with the original notion of locality for the internal qubits.
discussed in Section II.4. Moreover, let us note that by adding nontrivial dynamics of the internal fuzzball structure via a $\hat{U}$ to the model in (3.6), one can effectively change which qubits get emitted. This is the point referred to at the end of Section III.3.

There is one explicit family of (non-extremal) fuzzball microstates [33] for which one can understand the bulk Hawking emission process. As first suggested in [34], the geometry’s ergoregion instability [35] can be interpreted as a Bose enhanced version of the Hawking instability for the corresponding black hole. This explanation was justified by comparing gravitational emission to the dual CFT emission process for both ergoregion emission from the fuzzballs and Hawking radiation from the corresponding black hole [34, 36–39]. In [34], a toy model was presented for the ergoregion emission, based on the CFT description. The toy model consists of a set of two-level atoms that can spontaneously emit or absorb photons. In the geometric description, the de-exciting atoms correspond to accumulating particles in the ergoregion that decrease the geometry’s mass and angular momentum. Loosely, in our language, the toy model is in the class discussed in Section III.2. To properly capture the Bose enhancement, however, is a bit trickier.

In [12], several unitary models of evolution (some of which were discussed here) were presented, motivated by proposed nonlocal physics on the Schwarzschild background. As mentioned in [40], it is unclear what sets the scale of the proposed nonlocality, so as to ensure it operates in the black hole background but not in everyday low-energy experiments. While the sorts of models discussed here remain too crude to distinguish between nonlocal physics or fuzzball microstates, their utility lies in their generality, which serves to sharpen our information theoretic understanding of black hole evaporation. One obvious task that remains is to translate Mathur’s bound and its generalization in Section IV into a sharper, quantitative statement about the breakdown of the semiclassical limit of quantum gravity. Since the entire discussion has been in a Hamiltonian framework, it would be especially nice to have analogous bounds on the path integral.

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[1] S. W. Hawking, “Particle Creation by Black Holes,” Commun. Math. Phys. 43 (1975) 199–220.
[2] S. W. Hawking, “Breakdown of Predictability in Gravitational Collapse,” Phys. Rev. D14 (1976) 2460–2473.
[3] S. W. Hawking, “The Unpredictability of Quantum Gravity,” Commun. Math. Phys. 87 (1982) 395.
[4] D. N. Page, “Is BLACK HOLE EVAPORATION PREDICTABLE?” Phys. Rev. Lett. 44 (1980) 301.
[5] T. Banks, L. Susskind, and M. E. Peskin, “Difficulties for the Evolution of Pure States Into Mixed States,” Nucl. Phys. B244 (1984) 125.
[6] W. G. Unruh and R. M. Wald, “On evolution laws taking pure states to mixed states in quantum field theory,” Phys. Rev. D52 (1995) 2176–2182, arXiv:hep-th/9503024 [hep-th].
[7] W. Unruh, “Decoherence without Dissipation,” arXiv:1205.6750 [quant-ph].
[8] D. N. Page, “Average entropy of a subsystem,” Phys. Rev. Lett. 71 (1993) 1291–1294, arXiv:gr-qc/9305007.
[9] D. N. Page, “Information in black hole radiation,” Phys. Rev. Lett. 71 (1993) 3743–3746, arXiv:hep-th/9306083.
[10] S. D. Mathur, “The information paradox: A pedagogical introduction,” Class. Quant. Grav. 26 (2009) 224001, arXiv:0909.1038 [hep-th].
[11] B. Czech, K. Larjo, and M. Rozali, “Black Holes as Rubik’s Cubes,” arXiv:1106.5229 [hep-th].
[12] S. B. Giddings, “Models for unitary black hole disintegration,” arXiv:1108.2015 [hep-th].
[13] S. D. Mathur and C. J. Plumberg, “Correlations in Hawking radiation and the infall problem,” arXiv:1101.4899 [hep-th].
[14] S. D. Mathur, “What the information paradox is not,” arXiv:1108.0302 [hep-th].
[15] S. D. Mathur, “The information paradox and the infall problem,” Class. Quant. Grav. 28 (2011) 125010.
[16] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information. Cambridge University Press, 2000.
[17] S. D. Mathur and C. J. Plumberg, “Correlations in Hawking radiation and the infall problem,” arXiv:1101.4899 [hep-th].
[18] S. D. Mathur, “What the information paradox is not,” arXiv:1108.0302 [hep-th].
[19] S. D. Mathur, “The information paradox and the infall problem,” Class. Quant. Grav. 28 (2011) 125010.
[20] P. Hayden and J. Preskill, “Black holes as mirrors: quantum information in random subsystems,” JHEP 09 (2007) 120.
[21] Y. Sekino and L. Susskind, “Fast Scramblers,” JHEP 10 (2008) 065.
[22] D. A. Lowe, J. Polchinski, L. Susskind, L. Thorlacius, and J. Uglum, “Black hole complementarity versus locality,” Phys. Rev. D52 (1995) 6997–7010.
[23] S. B. Giddings and W. M. Nelson, “Quantum emission from two-dimensional black holes,” Phys. Rev. D46 (1992) 2486–2496.
[24] L. Susskind, “Addendum to Fast Scramblers,” arXiv:1101.6048 [hep-th].
[25] K. M. R. Audenaert, “A sharp continuity estimate for the von neumann entropy,” Journal of Physics A: Mathematical and Theoretical 40 no. 28, (2007) 8127.
[26] N. Lashkari, D. Stanford, M. Hastings, T. Osborne, and P. Hayden, “Towards the Fast Scrambling Conjecture,” arXiv:0808.2096 [hep-th].
[27] S. L. Braunstein and M. K. Patra, “Black hole evaporation rates without spacetime,” Phys. Rev. Lett. 107 (2011) 071302.
[28] I. Bena and N. P. Warner, “Black holes, black rings and their microstates,” Lect. Notes Phys. 755 (2008) 1–92.
[29] V. Balasubramanian, J. de Boer, S. El-Showk, and I. Messamah, “Black Holes as Effective Geometries,” Class. Quant. Grav. 25 (2008) 214004.
[30] S. D. Mathur, “The quantum structure of black holes,” Class. Quant. Grav. 23 (2006) R115.
[31] V. Cardoso, O. J. C. Dias, and R. C. Myers, “On the gravitational stability of D1-D5-P black holes,” Phys. Rev. D76 (2007) 105015.
[32] B. D. Chowdhury and S. D. Mathur, “Radiation from the non-extremal fuzzball,” Class. Quant. Grav. 25 (2008) 135005.
[33] B. D. Chowdhury and S. D. Mathur, “Pair creation in non-extremal fuzzball geometries,” Class. Quant. Grav. 25 (2008) 225021.
[34] B. D. Chowdhury and S. D. Mathur, “Non-extremal fuzzballs and ergoregion emission,” Class. Quant. Grav. 26 (2009) 035006.
[35] S. G. Avery, B. D. Chowdhury, and S. D. Mathur, “Emission from the D1D5 CFT,” JHEP 10 (2009) 065.
[36] S. G. Avery and B. D. Chowdhury, “Emission from the D1D5 CFT: Higher Twists,” JHEP 01 (2010) 087.
[37] B. D. Chowdhury and S. D. Mathur, “Non-locality vs. complementarity: a conservative approach to the information problem,” Class. Quant. Grav. 28 (2011) 025002.