Multiplicty and stability of closed geodesics on positively curved Finsler 4-spheres

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Abstract

In this paper, we prove that for every Finsler 4-dimensional sphere $(S^4, F)$ with reversibility $\lambda$ and flag curvature $K$ satisfying $\frac{\lambda}{9} \left(\frac{2}{1+\lambda}\right)^2 < K \leq 1$ with $\lambda < \frac{3}{2}$, either there exist at least four prime closed geodesics, or there exist exactly three prime non-hyperbolic closed geodesics and at least two of them are irrationally elliptic.

Key words: Positively curved, closed geodesic, irrationally elliptic, Finsler metric, spheres.

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1 Introduction and main result

Let $(M, F)$ be a Finsler manifold. A closed curve on $(M, F)$ is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve. As usual, a closed geodesic $c : S^1 = \mathbb{R}/\mathbb{Z} \to M$ is prime if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the $m$-th iteration $c^m$ of $c$ is defined by $c^m(t) = c(mt)$. The inverse curve $c^{-1}$ of $c$ is defined by $c^{-1}(t) = c(1-t)$ for $t \in \mathbb{R}$. Note that unlike Riemannian manifold, the inverse curve $c^{-1}$ of a closed geodesic $c$ on an irreversible Finsler manifold need not be a geodesic. We call two prime closed geodesics $c$ and $d$ distinct if there is no $\theta \in (0, 1)$ such that $c(t) = d(t + \theta)$ for all $t \in \mathbb{R}$. On a reversible Finsler (or Riemannian) manifold, two closed geodesics $c$ and $d$ are called geometrically distinct if $c(S^1) \neq d(S^1)$, i.e., they have different image sets in $M$. We shall omit the word distinct when we talk about more than one prime closed geodesic.

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For a closed geodesic $c$ on an $(n+1)$-dimensional manifold $M$, denote by $P_c$ the linearized Poincaré map of $c$, which is a symplectic matrix, i.e., $P_c \in \text{Sp}(2n)$. We define the elliptic height $e(P_c)$ of $P_c$ to be the total algebraic multiplicity of all eigenvalues of $P_c$ on the unit circle $U = \{ z \in \mathbb{C} | |z| = 1 \}$ in the complex plane $\mathbb{C}$. Since $P_c$ is symplectic, $e(P_c)$ is even and $0 \leq e(P_c) \leq 2n$. A closed geodesic $c$ is called elliptic if $e(P_c) = 2n$, i.e., all the eigenvalues of $P_c$ locate on $U$; irrationally elliptic if, in the homotopy component $\Omega^0(P_c)$ of $P_c$ (cf. Section 2 below for the definition), $P_c$ can be connected to the $\circ$-product of $n$ rotation matrices $R(\theta_i)$ with $\theta_i$ being irrational multiple of $\pi$ for $1 \leq i \leq n$; hyperbolic if $e(P_c) = 0$, i.e., all the eigenvalues of $P_c$ locate away from $U$; non-degenerate if $1$ is not an eigenvalue of $P_c$. A Finsler metric $F$ is called bumpy if all the closed geodesics on $(M, F)$ are non-degenerate.

There is a famous conjecture in Riemannian geometry which claims the existence of infinitely many closed geodesics on any compact Riemannian manifold. This conjecture has been proved for many cases, but not yet for compact rank one symmetric spaces except for $S^2$. The results of Franks in [Fra92] and Bangert in [Ban93] imply that this conjecture is true for any Riemannian 2-sphere (cf. [Hin93] and [Hin97]). However for a Finsler manifold, the above conjecture does not hold due to the Katok’s examples. It was quite surprising when Katok in [Kat73] found some irreversible Finsler metrics on spheres with only finitely many closed geodesics and all of them are non-degenerate and irrationally elliptic (cf. [Zil82]).

Based on Katok’s examples, Anosov in [Ano74] proposed the following conjecture (cf. [Lon06])

$$\mathcal{N}(S^n, F) \geq 2 \left\lfloor \frac{n+1}{2} \right\rfloor$$

for any Finsler metric $F$ on $S^n$, (1.1)

where, denote by $\mathcal{N}(M, F)$ the number of the distinct closed geodesics on $(M, F)$, and $[a] = \max\{k \in \mathbb{Z} | k \leq a\}$. In 2005, Bangert and Long in [BL10] proved this conjecture for any Finsler 2-dimensional sphere $(S^2, F)$. Since then, the index iteration theory of closed geodesics (cf. [Bot56] and [Lon02]) has been applied to study the closed geodesic problem on Finsler manifolds.

When $n \geq 3$, the above conjectures in the Riemannian or Finsler case is still widely open in full of generality. About the multiplicity and stability problem of closed geodesics, two classes of typical conditions, including the positively curved condition and the non-degenerate (or bumpy) condition, have been used widely.

In [Rad04], Rademacher has introduced the reversibility $\lambda = \lambda(M, F)$ of a compact Finsler manifold defined by

$$\lambda = \max\{F(-X) | X \in TM, F(X) = 1\} \geq 1.$$ 

Then Rademacher in [Rad07] has obtained some results about the multiplicity and stability of closed geodesics. For example, let $F$ be a Finsler metric on $S^n$ with reversibility $\lambda$ and flag curvature $K$ satisfying $\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1$, then there exist at least $n/2 - 1$ closed geodesics with length $< 2n\pi$. If $\frac{9\lambda^2}{4(1+\lambda)^2} < K \leq 1$ with $\lambda < 2$, then there exists a closed geodesic of elliptic-parabolic, i.e., its
linearized Poincaré map split into 2-dimensional rotations and a part whose eigenvalues are ±1. These results are some generalization of those in [BTZ82] and [BTZ83] in the Riemannian case.

Recently, Wang in [Wan12] proved the conjecture (1.1) for \((S^n, F)\) provided that \(F\) is bumpy and its flag curvature \(K\) satisfies \(\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1\). Also in [Wan12], Wang showed that for every bumpy Finsler metric \(F\) on \(S^n\) satisfying \(\frac{9\lambda^2}{4(1+\lambda)^2} < K \leq 1\), there exist two prime elliptic closed geodesics provided the number of closed geodesics on \((S^n, F)\) is finite. As some further generalization, Duan, Long and Wang in [DLW16] obtained the optimal lower bound of the number of distinct closed geodesics on a compact simply-connected Finsler manifold \((M, F)\) if \(F\) is bumpy and some much weak index conditions or positively curved conditions are satisfied.

The first author in [Dua15] and [Dua16] proved that for every Finsler \((S^n, F)\) for \(n \geq 3\) with reversibility \(\lambda\) and flag curvature \(K\) satisfying \(\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1\), either there exist infinitely many prime closed geodesics, or there exist exactly three prime closed geodesics and at least two of them are elliptic. In fact, the multiplicity and stability problem on high dimensional manifolds without the assumption of bumpy metrics is much difficult.

In this paper, we further consider the positively curved Finsler 4-dimensional sphere \((S^4, F)\) without the bumpy assumption, and obtain the following new progress about the multiplicity and stability of closed geodesics on \((S^4, F)\).

**Theorem 1.1.** For every Finsler metric \(F\) on a 4-dimensional sphere \(S^4\) with reversibility \(\lambda\) and flag curvature \(K\) satisfying \(\frac{25}{9} \left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1\) with \(\lambda < \frac{3}{2}\), either there exist at least four prime closed geodesics, or there exist exactly three prime non-hyperbolic closed geodesics and at least two of them are irrationally elliptic.

First, under the assumption of positively curved condition \(\frac{25}{9} \left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1\) with \(\lambda < \frac{3}{2}\), Theorem 1 and Theorem 4 in [Rad04] established the lower bound of the length of closed geodesics, which in turn gives the lower bound of \(i(c^m)\) and mean index \(\hat{i}(c)\) for any prime closed geodesic \(c\) on such \(S^4\) (cf. Lemma 3.1 below). Second, we shall make full use of the enhanced common index jump theorem established in [DLW16], which generalized the common index jump theorem in [LZ02], to obtain some crucial precise estimates of \(i(c^m)\) and \(\nu(c^m)\) (cf. Section 3.1 below). Note that Theorem 1.1 in [Dua15] showed the existence of three prime closed geodesics on \((S^4, F)\) with reversibility \(\lambda\) and flag curvature \(K\) satisfying \(\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1\). So, in order to prove our above Theorem 1.1, we assume the existence of exactly three prime closed geodesics on such \((S^4, F)\) (cf. the assumption (TCG) below). Finally, together with the Morse theory, under (TCG) we will carefully analyze the local and global information of these three prime closed geodesics and their iterates to complete the proof of Theorem 1.1 in Section 3.2.

In addition, under the assumption (TCG), in Theorem 4.2 in Section 4, we obtain more precise information about the third prime closed geodesic except that it is non-hyperbolic. These information will be greatly helpful to completely solve the conjecture (1.1) on the positively cured Finsler \((S^4, F)\) in the future.
Remark 1.2. (i) The irrationally ellipticity of closed geodesics is a kind of very important stability, in fact, it specially implies the non-degeneracy of closed geodesics. It is conjectured that all closed geodesics are irrationally elliptic if the number of prime closed geodesics on Finsler sphere $(S^n, F)$ with $n \geq 2$ is finite (cf. Conjecture of [DLW16]). In this direction, there are several well-known results. For example, Long and Wang in [LW08] proved that on every Finsler $S^2$ with only finitely many closed geodesics, there exist at least two irrationally elliptic one. Duan and Liu in [DL16] showed that if there exist exactly three prime closed geodesics on every Finsler $S^3$ satisfying $\frac{9}{4} \left(\frac{1}{\lambda} + \lambda\right)^2 < K \leq 1$ with $\lambda < 2$, then two of them are irrationally elliptic.

(ii) This paper continues to consider the case of $n = 4$ of $S^n$. Compared to previous works, and in particular to [Dua15] and [Dua16], the first substantial progress in this paper is to prove the existence of two prime irrationally elliptic closed geodesics on the above positively curved Finsler $S^4$ if there exist exactly three prime closed geodesics on it. Except for this, the second progress in this paper is to obtain five possible characterizations about the Morse index and the linearized Poincaré map of the third closed geodesic if there exist exactly three prime closed geodesics on such $S^4$ (cf. Theorem 4.2 below). These results is very difficult due to the high dimension of $n \geq 4$. In fact, in the case of $n = 4$, there possibly exist three $2 \times 2$ rotation matrices in the decomposition of $P_c$ (cf. (2.15) of Theorem 2.6 below), which brings high degeneracy of closed geodesics so that some methods and technical tools in [Dua15], [Dua16] and [DL16] are not enough to deal with these difficulties. In this paper, we make full use of the enhanced common index jump theorem established in [DLW16], together with some new ideas and estimates about indices of closed geodesics.

(iii) On one hand, note that the curvature pinching condition in Theorem 1.1 and Theorem 4.2 in this paper is a little restrictive than the corresponding assumption in [Dua15] and [Dua16], this is because that we need good index estimates in Lemma 3.1 and Lemma 3.2 to count the contributions of iterated closed geodesics to the non-zero Morse-type numbers, and the current pinching condition is a sufficient one for these estimates. On the other hand, we do not consider the case $n \geq 5$ because of more higher degeneracy. This maybe need some new ideas and different tools to deal with more difficulties.

In this paper, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively. We use only singular homology modules with $\mathbb{Q}$-coefficients. For an $S^1$-space $X$, we denote by $\overline{X}$ the quotient space $X/S^1$. We define the functions

$$E(a) = \min\{k \in \mathbb{Z} \mid k \geq a\}, \quad \varphi(a) = E(a) - \lfloor a \rfloor, \quad \{a\} = a - \lfloor a \rfloor.$$  \hfill (1.2)

Especially, $\varphi(a) = 0$ if $a \in \mathbb{Z}$, and $\varphi(a) = 1$ if $a \notin \mathbb{Z}$.
2 Morse theory and Morse indices of closed geodesics

2.1 Morse theory for closed geodesics

Let $M = (M, F)$ be a compact Finsler manifold, the space $\Lambda = \Lambda M$ of $H^1$-maps $\gamma : S^1 \to M$ has a natural structure of Riemannian Hilbert manifolds on which the group $S^1 = \mathbb{R}/\mathbb{Z}$ acts continuously by isometries. This action is defined by $(s \cdot \gamma)(t) = \gamma(t + s)$ for all $\gamma \in \Lambda$ and $s, t \in S^1$. For any $\gamma \in \Lambda$, the energy functional is defined by

$$E(\gamma) = \frac{1}{2} \int_{S^1} F(\gamma(t), \dot{\gamma}(t))^2 dt. \quad (2.1)$$

It is $C^{1,1}$ and invariant under the $S^1$-action. The critical points of $E$ of positive energies are precisely the closed geodesics $\gamma : S^1 \to M$. The index form of the functional $E$ is well defined along any closed geodesic $c$ on $M$, which we denote by $E''(c)$. As usual, we denote by $i(c)$ and $\nu(c)$ the Morse index and nullity of $E$ at $c$. In the following, we denote by

$$\Lambda^\kappa = \{d \in \Lambda \mid E(d) \leq \kappa\}, \quad \Lambda^\kappa^- = \{d \in \Lambda \mid E(d) < \kappa\}, \quad \forall \kappa \geq 0. \quad (2.2)$$

For a closed geodesic $c$ we set $\Lambda(c) = \{\gamma \in \Lambda \mid E(\gamma) < E(c)\}$.

Recall that respectively the mean index $\hat{i}(c)$ and the $S^1$-critical modules of $c^m$ are defined by

$$\hat{i}(c) = \lim_{m \to \infty} \frac{i(c^m)}{m}, \quad \overline{\mathcal{C}}_*(E, c^m) = H_*(\left(\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1; \mathbb{Q}\right). \quad (2.3)$$

We call a closed geodesic satisfying the isolation condition, if the following holds:

**Iso** For all $m \in \mathbb{N}$ the orbit $S^1 \cdot c^m$ is an isolated critical orbit of $E$.

Note that if the number of prime closed geodesics on a Finsler manifold is finite, then all the closed geodesics satisfy (Iso).

If $c$ has multiplicity $m$, then the subgroup $\mathbb{Z}_m = \{\frac{n}{m} \mid 0 \leq n < m\}$ of $S^1$ acts on $\overline{\mathcal{C}}_*(E, c)$.

As studied in p.59 of [Rad92], for all $m \in \mathbb{N}$, let $H_*(X, A)^{\mathbb{Z}_m} = \{[\xi] \in H_*(X, A) \mid T_\ast [\xi] = \pm [\xi]\}$, where $T$ is a generator of the $\mathbb{Z}_m$-action. On $S^1$-critical modules of $c^m$, the following lemma holds:

**Lemma 2.1.** (cf. Satz 6.11 of [Rad92] and [BL10]) Suppose $c$ is a prime closed geodesic on a Finsler manifold $M$ satisfying (Iso). Then there exist $U^-_{c^m}$ and $N_{c^m}$, the so-called local negative disk and the local characteristic manifold at $c^m$ respectively, such that $\nu(c^m) = \dim N_{c^m}$ and

$$\overline{\mathcal{C}}_q(E, c^m) \equiv H_q\left(\left(\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1\right)\right)$$

$$= \left(H_{i(c^m)}(U^-_{c^m} \cup \{c^m\}, U^-_{c^m}) \otimes H_{q-i(c^m)}(N_{c^m} \cup \{c^m\}, N_{c^m})\right)^{\mathbb{Z}_m},$$

(i) When $\nu(c^m) = 0$, there holds

$$\overline{\mathcal{C}}_q(E, c^m) = \begin{cases} \mathbb{Q}, & \text{if } i(c^m) - i(c) \in 2\mathbb{Z} \text{ and } q = i(c^m), \\ 0, & \text{otherwise}, \end{cases}$$

(Continued on next page...)
(ii) When $\nu(c^m) > 0$, there holds

$$
\overline{\c}(E, c^m) = H_{q - i(c^m)}(N_{c^m} \cup \{c^m\}, N_{c^m})\epsilon(c^m)\mathbb{Z}_m,
$$

where $\epsilon(c^m) = (-1)^{i(c^m) - i(c)}$.

Define

$$
k_j(c^m) \equiv \dim H_j(N_{c^m} \cup \{c^m\}, N_{c^m}), \quad k_j^\pm(c^m) \equiv \dim H_j(N_{c^m} \cup \{c^m\}, N_{c^m})^{\pm\mathbb{Z}_m}.
$$

Then we have

**Lemma 2.2.** (cf. [Rad92], [LD09], [Wan12]) Let $c$ be a prime closed geodesic on a Finsler manifold $(M, F)$. Then

(i) For any $m \in \mathbb{N}$, there holds $k_j(c^m) = 0$ for $j \not\in [0, \nu(c^m)]$.

(ii) For any $m \in \mathbb{N}$, $k_0(c^m) + k_{\nu(c^m)}(c^m) \leq 1$ and if $k_0(c^m) + k_{\nu(c^m)}(c^m) = 1$ then there holds $k_j(c^m) = 0$ for $j \in (0, \nu(c^m))$.

(iii) For any $m \in \mathbb{N}$, there holds $k_0^+(c^m) = k_0(c^m)$ and $k_0^-(c^m) = 0$. In particular, if $c^m$ is non-degenerate, there holds $k_0^+(c^m) = k_0(c^m) = 1$, and $k_0^-(c^m) = k_j^\pm(c^m) = 0$ for all $j \neq 0$.

(iv) Suppose for some integer $m = np \geq 2$ with $n$ and $p \in \mathbb{N}$ the nullities satisfy $\nu(c^m) = \nu(c^n)$. Then there holds $k_j(c^m) = k_j(c^n)$ and $k_j^\pm(c^m) = k_j^\pm(c^n)$ for any integer $j$.

Let $(M, F)$ be a compact simply connected Finsler manifold with finitely many closed geodesics. It is well known that for every prime closed geodesic $c$ on $(M, F)$, there holds either $\hat{i}(c) > 0$ and then $i(c^m) \rightarrow +\infty$ as $m \rightarrow +\infty$, or $\hat{i}(c) = 0$ and then $i(c^m) = 0$ for all $m \in \mathbb{N}$. Denote those prime closed geodesics on $(M, F)$ with positive mean indices by $\{c_j\}_{1 \leq j \leq k}$. Rademacher in [Rad89] and [Rad92] established a celebrated mean index identity relating all the $c_j$s with the global homology of $M$ for compact simply connected Finsler manifolds (especially for $S^4$) as follows.

**Theorem 2.3.** (Satz 7.9 of [Rad92], cf. also [DuL10], [LD09] and [Wan12]) Assume that there exist finitely many prime closed geodesics on $(S^4, F)$ and denote prime closed geodesics with positive mean indices by $\{c_j\}_{1 \leq j \leq k}$ for some $k \in \mathbb{N}$. Then the following identity holds

$$
\sum_{j=1}^{k} \frac{\hat{\chi}(c_j)}{\hat{i}(c_j)} = \frac{2}{3},
$$

where

$$
\hat{\chi}(c_j) = \frac{1}{n(c_j)} \sum_{1 \leq m \leq n(c_j)} \chi(c_j^m) = \frac{1}{n(c_j)} \sum_{1 \leq m \leq n(c_j)} (-1)^{i(c_j^m)} k_l^\pm(c_j^m) \in \mathbb{Q},
$$

and the analytical period $n(c_j)$ of $c_j$ is defined by (cf. [LD09])

$$
n(c_j) = \min\{l \in \mathbb{N} | \nu(c_j^l) = \max_{m \geq 1} \nu(c_j^m), \quad i(c_j^{m+l}) - i(c_j^m) \in 2\mathbb{Z}, \quad \forall m \in \mathbb{N}\}.
$$
Set $\mathbb{T}^4 = \mathbb{T}^0 S^4 = \{\text{constant point curves in } S^4\} \cong S^4$. Let $(X, Y)$ be a space pair such that the Betti numbers $b_i = b_i(X, Y) = \dim H_i(X, Y; \mathbb{Q})$ are finite for all $i \in \mathbb{Z}$. As usual the Poincaré series of $(X, Y)$ is defined by the formal power series $P(X, Y) = \sum_{i=0}^{\infty} b_i t^i$. We need the following well known version of results on Betti numbers and the Morse inequality.

**Lemma 2.4.** (cf. Theorem 2.4 and Remark 2.5 of [Rad89] and [Hin84], Lemma 2.5 of [DuL10])

Let $(S^4, F)$ be a 4-dimensional Finsler sphere. Then, the Betti numbers are given by

\[
  b_j = \text{rank} H_j(\Lambda S^4 / S^1, \Lambda S^4 / S^1; \mathbb{Q}) = \begin{cases} 
  2, & \text{if } j \in K \equiv \{3k \mid 3 \leq k \in 2\mathbb{N} + 1\}, \\
  1, & \text{if } j \in \{2k + 3 \mid k \in \mathbb{N}_0\} \setminus K, \\
  0, & \text{otherwise} 
\end{cases} \tag{2.8}
\]

**Theorem 2.5.** (cf. Theorem I.4.3 of [Cha93])

Let $(M, F)$ be a Finsler manifold with finitely many prime closed geodesics, denoted by $\{c_j\}_{1 \leq j \leq k}$. Set

\[
  M_q = \sum_{1 \leq j \leq k, m \geq 1} \dim \overline{C}_q(E, c_j^m), \quad q \in \mathbb{Z}.
\]

Then for every integer $q \geq 0$ there holds

\[
  M_q - M_{q-1} + \cdots + (-1)^q M_0 \geq b_q - b_{q-1} + \cdots + (-1)^q b_0, \tag{2.9}
\]

\[
  M_q \geq b_q. \tag{2.10}
\]

### 2.2 Index iteration theory of closed geodesics

In [Lon99] of 1999, Y. Long established the basic normal form decomposition of symplectic matrices. Based on this result he further established the precise iteration formulae of indices of symplectic paths in [Lon00] of 2000. Note that this index iteration formulae works for Morse indices of iterated closed geodesics (cf. [Liu05] and Chap. 12 of [Lon02]). Since every closed geodesic on a sphere must be orientable. Then by Theorem 1.1 of [Liu05], the initial Morse index of a closed geodesic on a Finsler $S^4$ coincides with the index of a corresponding symplectic path.

As in [Lon00], denote by

\[
  N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad \text{for } \lambda = \pm 1, \ b \in \mathbb{R}, \tag{2.11}
\]

\[
  D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R} \setminus \{0, \pm 1\}, \tag{2.12}
\]

\[
  R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \text{for } \theta \in (0, \pi) \cup (\pi, 2\pi), \tag{2.13}
\]

\[
  N_2(e^{\theta\sqrt{-1}} B) = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}, \quad \text{for } \theta \in (0, \pi) \cup (\pi, 2\pi) \quad \text{and}
\]

\[
  B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \text{ with } b_j \in \mathbb{R}, \ \text{and } b_2 \neq b_3. \tag{2.14}
\]
Here $N_2(e^{\theta\sqrt{-1}}, B)$ is non-trivial if $(b_2 - b_3) \sin \theta < 0$, and trivial if $(b_2 - b_3) \sin \theta > 0$.

As in [Lon00], the $\circ$-sum (direct sum) of any two real matrices is defined by

$$
\begin{pmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{pmatrix}_{2i \times 2i} \circ 
\begin{pmatrix}
A_2 & B_2 \\
C_2 & D_2
\end{pmatrix}_{2j \times 2j} = 
\begin{pmatrix}
A_1 & 0 & B_1 & 0 \\
0 & A_2 & 0 & B_2 \\
C_1 & 0 & D_1 & 0 \\
0 & C_2 & 0 & D_2
\end{pmatrix}.
$$

For every $M \in \text{Sp}(2n)$, the homotopy set $\Omega(M)$ of $M$ in $\text{Sp}(2n)$ is defined by

$$
\Omega(M) = \{ N \in \text{Sp}(2n) \mid \sigma(N) \cap U = \sigma(M) \cap U \equiv \Gamma \text{ and } \nu_\omega(N) = \nu_\omega(M), \forall \omega \in \Gamma \},
$$

where $\sigma(M)$ denotes the spectrum of $M$, $\nu_\omega(M) \equiv \dim_C \ker_C (M - \omega I)$ for $\omega \in U$. The component $\Omega^0(M)$ of $P$ in $\text{Sp}(2n)$ is defined by the path connected component of $\Omega(M)$ containing $M$.

**Theorem 2.6.** (cf. Theorem 7.8 of [Lon99], Theorems 1.2 and 1.3 of [Lon00], cf. also Theorem 1.8.10, Lemma 2.3.5 and Theorem 8.3.1 of [Lon02]) For every $P \in \text{Sp}(2n - 2)$, there exists a continuous path $f \in \Omega^0(P)$ such that $f(0) = P$ and

$$
f(1) = N_1(1, 1)^{q_-} \circ I_{2p_0} \circ N_1(-1, 1)^{q_+} \circ ( - I_{2q_1}) \circ N_1(-1, 1)^{q_+} \\
\circ N_2(e^{\alpha_1\sqrt{-1}}, A_1) \circ \cdots \circ N_2(e^{\alpha_r\sqrt{-1}}, A_r) \circ N_2(e^{\beta_1\sqrt{-1}}, B_1) \circ \cdots \circ N_2(e^{\beta_0\sqrt{-1}}, B_{r_0}) \\
\circ R(\theta_1) \circ \cdots \circ R(\theta_{r'}) \circ R(\theta_{r'}^+) \circ \cdots \circ R(\theta_r) \circ H(2)^{\theta / 2},
$$

where $\frac{\theta}{2\pi} \in Q \cap (0, 1) \setminus \{ \frac{1}{2j} \}$ for $1 \leq j \leq r'$ and $\frac{\theta}{2\pi} \notin Q \cap (0, 1)$ for $r'+1 \leq j \leq r$; $N_2(e^{\alpha_j\sqrt{-1}}, A_j)$'s are non-trivial and $N_2(e^{\beta_j\sqrt{-1}}, B_j)$'s are trivial, and non-negative integers $p_-, p_+, q_-, q_0, q_+, r, r_0, h$ satisfy the equality

$$
p_- + p_0 + p_+ + q_- + q_0 + q_+ + r + 2r_0 + 2r_0 + h = n - 1.
$$

Let $\gamma \in \mathcal{P}_r(2n - 2) = \{ \gamma \in C([0, \tau], \text{Sp}(2n - 2)) \mid \gamma(0) = I \}$. We extend $\gamma(t)$ to $t \in [0, m\tau]$ for every $m \in \mathbb{N}$ by

$$
\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j \quad \forall j\tau \leq t \leq (j + 1)\tau \text{ and } j = 0, 1, \ldots, m - 1.
$$

Denote the basic normal form decomposition of $P \equiv \gamma(\tau)$ by (2.15). Then we have

$$
i(\gamma^m) = m(i(\gamma) + p_- + p_0 - r) + 2 \sum_{j=1}^{r} E \left( \frac{m\theta_j}{2\pi} \right) - r \\
- p_- - p_0 - \frac{1 + (-1)^m}{2} (q_- + q_0 + q_+) + 2 \sum_{j=1}^{r} \varphi \left( \frac{m\alpha_j}{2\pi} \right) - 2r_0,
$$

$$
\nu(\gamma^m) = \nu(\gamma) + \frac{1 + (-1)^m}{2} (q_- + 2q_0 + q_+) + 2\zeta(m, \gamma(\tau)),
$$

where we denote by

$$
\zeta(m, \gamma(\tau)) = r - \sum_{j=1}^{r} \varphi \left( \frac{m\theta_j}{2\pi} \right) + r_0 - \sum_{j=1}^{r} \varphi \left( \frac{m\alpha_j}{2\pi} \right) + r_0 - \sum_{j=1}^{r} \varphi \left( \frac{m\beta_j}{2\pi} \right).
$$
Let
\[ M \equiv \{ N_1(1,1); \ N_1(-1,a_2), a_2 = \pm 1; \ R(\theta), \ \theta \in [0, 2\pi); \ H(-2) \}. \] (2.21)

By Theorems 8.1.4-8.1.7 and 8.2.1-8.2.4 of [Lon02], we have

**Proposition 2.7.** Every path \( \gamma \in \mathcal{P}_r(2) \) with end matrix homotopic to some matrix in \( M \) has odd index \( i(\gamma) \). Every path \( \xi \in \mathcal{P}_r(2) \) with end matrix homotopic to \( N_1(1, -1) \) or \( H(2) \), and every path \( \eta \in \mathcal{P}_r(4) \) with end matrix homotopic to \( N_2(\omega, B) \) has even indices \( i(\xi) \) and \( i(\eta) \).

The common index jump theorem (cf. Theorem 4.3 of [LZ02]) for symplectic paths has become one of the main tools in studying the multiplicity and stability of periodic orbits in Hamiltonian and symplectic dynamics. Recently, the following enhanced common index jump theorem has been obtained by Duan, Long and Wang in [DLW16].

**Theorem 2.8.** (cf. Theorem 3.5 of [DLW16]) Let \( \gamma_k \in \mathcal{P}_{r_k}(2n) \) for \( k = 1, \ldots, q \) be a finite collection of symplectic paths. Let \( M_k = \gamma_k(\tau_k) \). We extend \( \gamma_k \) to \([0, +\infty)\) by (2.17) inductively. Suppose
\[ i(\gamma_k, 1) > 0, \quad \forall \ k = 1, \ldots, q. \] (2.22)

Then for any fixed integer \( \bar{m} \in \mathbb{N} \), there exist infinitely many \((q + 1)\)-tuples \( (N, m_1, \ldots, m_q) \in \mathbb{N}^{q+1} \) such that for all \( 1 \leq k \leq q \) and \( 1 \leq m \leq \bar{m} \), there holds
\[ \nu(\gamma_k, 2m_k - m) = \nu(\gamma_k, 2m_k + m) = \nu(\gamma_k, m), \] (2.23)
\[ i(\gamma_k, 2m_k + m) = 2N + i(\gamma_k, m), \] (2.24)
\[ i(\gamma_k, 2m_k - m) = 2N - i(\gamma_k, m) - 2(S_{M_k}^+(1) + Q_k(m)), \] (2.25)
\[ i(\gamma_k, 2m_k) = 2N + (S_{M_k}^+(1) + C(M_k) - 2\Delta_k), \] (2.26)

where \( S_{M_k}^\pm(\omega) \) is the splitting number of \( M_k \) at \( \omega \) (cf. Definition 9.1.4 of [Lon02]) and
\[ C(M_k) = \sum_{0<\theta<2\pi} S_{M_k}^-(e^{\sqrt{-1}\theta}), \Delta_k = \sum_{0<\{m_k\theta/\pi\}<\delta} S_{M_k}^-(e^{\sqrt{-1}\theta}), \] (2.27)
\[ Q_k(m) = \sum_{\substack{\theta \in (0, 2\pi), e^{\sqrt{-1}\theta} \in \sigma(M_k), \\{m_k\theta/\pi\} = 0}} S_{M_k}^-(e^{\sqrt{-1}\theta}). \]

More precisely, by (4.10), (4.40) and (4.41) in [LZ02], we have
\[ m_k = \left( \left\lfloor \frac{N}{M_i(\gamma_k, 1)} \right\rfloor + \chi_k \right) \bar{M}, \quad 1 \leq k \leq q, \] (2.28)
where \( \chi_k = 0 \) or \( 1 \) for \( 1 \leq k \leq q \) and \( \frac{M_i\theta}{\pi} \in \mathbb{Z} \) whenever \( e^{\sqrt{-1}\theta} \in \sigma(M_k) \) and \( \frac{\theta}{\pi} \in \mathbb{Q} \) for some \( 1 \leq k \leq q \). Furthermore, for any fixed \( M_0 \in \mathbb{N} \), we may further require \( M_0|N \), and for any \( \epsilon > 0 \), we can choose \( N \) and \( \{\chi_k\}_{1 \leq k \leq q} \) such that
\[ \left| \frac{N}{M_i(\gamma_k, 1)} - \chi_k \right| < \epsilon, \quad 1 \leq k \leq q. \] (2.29)
Then, for any \( m \in N \), there holds
\[
\nu(\gamma, m) - \frac{e(M)}{2} \leq i(\gamma, m + 1) - i(\gamma, m) - i(\gamma, 1) \leq \nu(\gamma, 1) - \nu(\gamma, m + 1) + \frac{e(M)}{2},
\]
where \( e(M) \) is the elliptic height defined in Section 1.

### 3 Some index estimates and proof of Theorem 1.1

#### 3.1 Some index estimates for closed geodesics

Firstly we make the following assumption

**\textbf{(FCG)}** Suppose that there exist only finitely many prime closed geodesics \( \{c_k\}_{k=1}^q \) on \((S^4, F)\) with reversibility \( \lambda \) and flag curvature \( K \) satisfying \( \frac{25}{9} \left( \frac{\lambda}{1+\lambda} \right)^2 < K \leq 1 \) with \( \lambda < \frac{3}{2} \).

For any \( 1 \leq k \leq q \), we rewrite (2.15) as follows
\[
f_k(1) = N_1(1, 1)^{\text{pk}, -} \circ I_{2k, 0} \circ N_1(1, -1)^{\text{pk}, +} \\
\quad \circ N_1(-1, 1)^{\text{pk}, -} \circ (-I_{2q, 0}) \circ N_1(-1, -1)^{\text{pk}, +} \\
\quad \circ R(\theta_{k, 1}) \circ \cdots \circ R(\theta_{k, r_{k, 1}}) \circ R(\tilde{\theta}_{k, 1}) \circ \cdots \circ R(\tilde{\theta}_{k, r_{k, 2}}) \\
\quad \circ N_2(e^{\sqrt{-T_{\alpha_{k, 1}}} A_{k, 1}}) \circ \cdots \circ N_2(e^{\sqrt{-T_{\alpha_{k, 3}}} A_{k, r_{k, 3}}}) \\
\quad \circ N_2(e^{\sqrt{-T_{A_{k, 1}}} \tilde{A}_{k, 1}}) \circ \cdots \circ N_2(e^{\sqrt{-T_{A_{k, 4}}} \tilde{A}_{k, r_{k, 4}}}) \\
\quad \circ N_2(e^{\sqrt{-T_{B_{k, 1}}} \tilde{B}_{k, 1}}) \circ \cdots \circ N_2(e^{\sqrt{-T_{B_{k, 5}}} \tilde{B}_{k, r_{k, 5}}}) \\
\quad \circ N_2(e^{\sqrt{-T_{\delta_{k, 1}}} \tilde{B}_{k, 1}}) \circ \cdots \circ N_2(e^{\sqrt{-T_{\delta_{k, 6}}} \tilde{B}_{k, r_{k, 6}}}) \circ H(2)^{\text{oh}, k} \circ H(-2)^{\text{oh}, k}, \quad (3.1)
\]

where \( \frac{\delta_{k, j}}{2\pi} \in Q \cap (0, 1) \backslash \{\frac{1}{2}\} \) for \( 1 \leq j \leq r_{k, 1} \), \( \frac{\delta_{k, j}}{2\pi} \in (0, 1) \backslash Q \) for \( 1 \leq j \leq r_{k, 2} \), \( \frac{\alpha_{k, j}}{2\pi} \in Q \cap (0, 1) \backslash \{\frac{1}{2}\} \) for \( 1 \leq j \leq r_{k, 3} \), \( \frac{\beta_{k, j}}{2\pi} \in (0, 1) \backslash Q \) for \( 1 \leq j \leq r_{k, 6} \); \( N_2(e^{\sqrt{-T_{A_{k, j}}} A_{k, j}}) \)’s and \( N_2(e^{\sqrt{-T_{A_{k, j}}} \tilde{A}_{k, j}}) \)’s are nontrivial and \( N_2(e^{\sqrt{-T_{B_{k, j}}} B_{k, j}}) \)’s are trivial, and non-negative integers \( p_{k, -} \), \( p_{k, 0} \), \( p_{k, +} \), \( q_{k, -} \), \( q_{k, 0} \), \( q_{k, +} \), \( r_{k, 1} \), \( r_{k, 2} \), \( r_{k, 3} \), \( r_{k, 4} \), \( r_{k, 5} \), \( r_{k, 6} \), \( h_k = h_{k, +} + h_{k, -} \) satisfy the equality
\[
p_{k, -} + p_{k, 0} + p_{k, +} + q_{k, -} + q_{k, 0} + q_{k, +} + r_{k, 1} + r_{k, 2} + 2 \sum_{j=3}^{6} r_{k, j} + h_k = 3. \quad (3.2)
\]

**Lemma 3.1.** Under the assumption \((\text{FCG})\), for any prime closed geodesic \( c_k \), \( 1 \leq k \leq q \), there holds
\[
i(c_k^m) \geq 3 \left[ \frac{5m}{3} \right], \quad \forall \; m \in N
\]

\( (3.3) \)
and

\[ \hat{i}(c_k) > 5. \quad (3.4) \]

**Proof.** By the assumption (FCG), since the flag curvature \( K \) satisfies \( \frac{25}{9} \left( \frac{1}{\lambda + 1} \right)^2 < K \leq 1 \), we can choose \( \frac{25}{9} \left( \frac{1}{\lambda + 1} \right)^2 < \delta \leq K \leq 1 \). Then by Lemma 2 in [Rad07], it yields

\[ \hat{i}(c_k) \geq 3\sqrt{\frac{1 + \lambda}{\lambda}} > 5. \]

Note that it follows from Theorem 3 of [Rad04] that \( L(c_k^m) = mL(c_k) \geq m\pi \frac{1 + \lambda}{\lambda} > \frac{5m}{3}\pi/\sqrt{\delta} \) for \( m \geq 1 \) and \( 1 \leq k \leq q \). Then it follows from Lemma 3 of [Rad04] that \( i(c_k^m) \geq 3\left( \frac{5m}{\pi} \right) \).

Combining Lemma 3.1 with Theorem 2.9, it follows that

\[ i(c_k^{m+1}) - i(c_k^m) - \nu(c_k^m) \geq i(c_k) - \frac{e(P_{c_k})}{2} \geq 0, \quad \forall \ m \in \mathbb{N}, \ 1 \leq k \leq q. \quad (3.5) \]

Here the last inequality holds by the fact that \( e(P_{c_k}) \leq 6 \) and \( i(c_k) \geq 3 \).

It follows from (3.4), Theorem 4.3 in [LZ02] and Theorem 2.8 that for any fixed integer \( \bar{m} \in \mathbb{N} \), there exist infinitely many \((q + 1)\)-tuples \((N, m_1, \ldots, m_q) \in \mathbb{N}^{q+1}\) such that for any \( 1 \leq k \leq q \) and \( 1 \leq m \leq \bar{m} \), there holds

\[ i(c_k^{2m_1 - m}) + \nu(c_k^{2m_1 - m}) = 2N - i(c_k^m) - \left( 2S^+_{P_{c_k}}(1) + 2Q_k(m) - \nu(c_k^m) \right), \quad (3.6) \]

\[ i(c_k^{2m_1}) \geq 2N - \frac{e(P_{c_k})}{2}, \quad (3.7) \]

\[ i(c_k^{2m_1}) + \nu(c_k^{2m_1}) \leq 2N + \frac{e(P_{c_k})}{2}, \quad (3.8) \]

\[ i(c_k^{2m_1 + m}) = 2N + i(c_k^m), \quad (3.9) \]

where (3.7) and (3.8) follow from (4.32) and (4.33) in Theorem 4.3 in [LZ02] respectively.

Note that by List 9.1.12 of [Lon02], (2.27), (2.19) and \( \nu(c_k) = p_{k,-} + 2p_{k,0} + p_{k,+} \), we have

\[ S^+_{P_{c_k}}(1) = p_{k,-} + p_{k,0}, \quad (3.10) \]

\[ C(P_{c_k}) = q_{k,0} + q_{k,+} + r_{k,1} + r_{k,2} + 2r_{k,3} + 2r_{k,4}, \quad (3.11) \]

\[ Q_k(m) = \frac{1 + (-1)^m}{2} \left( q_{k,0} + q_{k,+} \right) + \left( r_{k,1} + r_{k,3} \right) - \sum_{j=1}^{r_{k,1}} \phi \left( \frac{m\theta_{k,j}}{2\pi} \right) - \sum_{j=1}^{r_{k,3}} \phi \left( \frac{m\alpha_{k,j}}{2\pi} \right), \quad (3.12) \]

\[ \nu(c_k^m) = p_{k,-} + 2p_{k,0} + p_{k,+} + \frac{1 + (-1)^m}{2} \left( q_{k,-} + 2q_{k,0} + q_{k,+} \right) + 2 \left( r_{k,1} + r_{k,3} + r_{k,5} \right) - 2 \left( \sum_{j=1}^{r_{k,1}} \phi \left( \frac{m\theta_{k,j}}{2\pi} \right) + \sum_{j=1}^{r_{k,3}} \phi \left( \frac{m\alpha_{k,j}}{2\pi} \right) + \sum_{j=1}^{r_{k,5}} \phi \left( \frac{m\beta_{k,j}}{2\pi} \right) \right). \quad (3.13) \]

By (3.10), (3.12) and (3.13), we obtain

\[ 2S^+_{P_{c_k}}(1) + 2Q_k(m) - \nu(c_k^m) = p_{k,-} - p_{k,+} - \frac{1 + (-1)^m}{2} \left( q_{k,-} - q_{k,+} \right) - 2r_{k,5} + 2 \sum_{j=1}^{r_{k,5}} \phi \left( \frac{m\beta_{k,j}}{2\pi} \right), \]
which, together with (3.6), gives

\[ i(c_k^{2m_k-m}) + \nu(c_k^{2m_k-m}) = 2N - i(c_k^m) - p_{k,-} + p_{k,+} + \frac{1 + (-1)^m}{2}(q_{k,-} - q_{k,+}) + 2r_{k,5} - 2 \sum_{j=1}^{r_{k,5}} \varphi \left( \frac{m \beta_k j}{2\pi} \right), \quad \forall \ 1 \leq m \leq \bar{m}. \]  

(3.14)

By (3.7)-(3.9), (3.14), (3.2), (3.3) and the fact \( e(P_{c_k}) \leq 6 \), there holds

\[ i(c_k^{2m_k-m}) + \nu(c_k^{2m_k-m}) \leq 2N + 3 - 3 \left[ \frac{5m}{3} \right], \quad \forall \ 1 \leq m \leq \bar{m}, \]  

(3.15)

\[ 2N - 3 \leq 2N - \frac{e(P_{c_k})}{2} \leq i(c_k^{2m_k}), \]  

(3.16)

\[ i(c_k^{2m_k}) + \nu(c_k^{2m_k}) \leq 2N + \frac{e(P_{c_k})}{2} \leq 2N + 3, \]  

(3.17)

\[ 2N + 3 \left[ \frac{5m}{3} \right] \leq i(c_k^{2m_k+m}), \quad \forall \ 1 \leq m \leq \bar{m}. \]  

(3.18)

Note that by (3.5), we have

\[ i(c_k^m) \leq i(c_k^{m+1}), \quad i(c_k^m) + \nu(c_k^m) \leq i(c_k^{m+1}) + \nu(c_k^{m+1}), \quad \forall m \in \mathbb{N}, \]

which, together with (3.15) and (3.18), implies

\[ i(c_k^m) + \nu(c_k^m) \leq i(c_k^{2m_k-m}) + \nu(c_k^{2m_k-m}) \leq 2N + 3 - 3 \left[ \frac{5m}{3} \right], \forall \ 1 \leq m \leq 2m_k - \bar{m}. \]  

(3.19)

\[ 2N + 3 \left[ \frac{5m}{3} \right] \leq i(c_k^{2m_k+m}) \leq i(c_k^m), \forall \ m \geq 2m_k + \bar{m}. \]  

(3.20)

In addition, by (2.26), (3.10), (3.11) and (3.13), the precise formulae of \( i(c_k^{2m_k}) + \nu(c_k^{2m_k}) \) can be computed as follows

\[ i(c_k^{2m_k}) + \nu(c_k^{2m_k}) = 2N + 2\Delta_k - (p_{k,-} + p_{k,0} + q_{k,0} + q_{k,+} + r_{k,1} + r_{k,2} + 2r_{k,3} + 2r_{k,4}) + p_{k,+} + 2p_{k,0} + p_{k,-} + q_{k,+} + q_{k,0} + q_{k,-} + 2q_{k,0} + q_{k,-} + 2r_{k,1} + 2r_{k,3} + 2r_{k,5} + r_{k,1} + 2r_{k,5} - 2r_{k,2} - 2r_{k,4} + 2\Delta_k, \quad k = 1, \ldots, q. \]  

(3.21)

where

\[ \Delta_k \equiv \sum_{0 \leq \{m_k \theta / \pi\} < \delta} S_{M_k}(e^{\sqrt{-1} \theta}) \leq r_{k,2} + r_{k,4}. \]  

(3.22)

by (2.27) and List 9.1.12 of [Lon02].

**Lemma 3.2.** Under the assumption (FCG), for \( k = 1, \ldots, q \) and \( 1 \leq m \leq \bar{m} \), we have

\[ i(c_k^{2m_k-1}) + \nu(c_k^{2m_k-1}) \leq 2N - 3, \]  

(3.23)

\[ i(c_k^{2m_k-2}) + \nu(c_k^{2m_k-2}) \leq 2N - 9. \]  

(3.24)
Proof. By (2.18), we have
\[ \hat{i}(c_k) = i(c_k) + p_{k,-} + p_{k,0} - r_{k,1} - r_{k,2} + \sum_{j=1}^{r_{k,1}} \frac{\theta_{k,j}}{\pi} + \sum_{j=1}^{r_{k,2}} \frac{\hat{\theta}_{k,j}}{\pi}. \]
Combining (3.25) with (3.4), there holds
\[ i(c_k) + p_{k,-} + p_{k,0} + r_{k,1} + r_{k,2} \geq 6. \] (3.26)
Then by (3.14), (3.26) and (3.2), we obtain
\[
i(c_k^{2m_k-1}) + \nu(c_k^{2m_k-1}) = 2N - i(c_k) - p_{k,-} + p_{k,+} \leq 2N - 6 + p_{k,0} + r_{k,1} + r_{k,2} + p_{k,\pm} \leq 2N - 3. \] (3.27)
By (3.14), (3.3) and (3.2), we get
\[
i(c_k^{2m_k-2}) + \nu(c_k^{2m_k-2}) = 2N - i(c_k^{2}) - p_{k,-} + p_{k,+) + q_{k,-} - q_{k,+) \leq 2N - 9 + 3 = 2N - 6. \] (3.28)
Now we assume \( i(c_k^{2m_k-2}) + \nu(c_k^{2m_k-2}) \geq 2N - 8, \forall k = 1, \ldots, q \), which, together with (3.28), gives
\[ i(c_k^{2m_k-2}) + \nu(c_k^{2m_k-2}) \in \{2N - 6, 2N - 7, 2N - 8\}. \]
We continue the proof by distinguishing three cases.

Case 1: \( i(c_k^{2m_k-2}) + \nu(c_k^{2m_k-2}) = 2N - 6 \).
In this case, by (3.3) and (3.28), we know that \( p_{k,+) + q_{k,-} = 3 \) and \( i(c_k^{2}) = 9 \). It follows from (2.18) and (3.2) that \( i(c_k^{2}) = 2i(c) \in 2\mathbb{N} \) since \( p_{k,+) + q_{k,-} = 3 \), thus Case 1 cannot happen.

Case 2: \( i(c_k^{2m_k-2}) + \nu(c_k^{2m_k-2}) = 2N - 7 \).
In this case, by (3.3) and (3.28), one of the following cases may happen.
(i) \( i(c_k^{2}) = 10 \) and \( p_{k,+) + q_{k,-} = 3 \).
(ii) \( i(c_k^{2}) = 9, p_{k,+) + q_{k,-} = 2 \) and \( p_{k,-} + q_{k,+) = 0 \).
For (i), by (2.18), we have \( i(c_k^{2}) = 2i(c_k) \) and \( \hat{i}(c_k) = i(c_k) \) since \( p_{k,+) + q_{k,-} = 3 \). However, there holds \( \hat{i}(c_k) > 5 \) by (3.4). So we have \( i(c_k^{2}) = 2i(c_k) = 2i(c_k) > 10 \), thus (i) of Case 2 cannot happen.
For (ii), by (3.2), there holds
\[ p_{k,0} + q_{k,0} + r_{k,1} + r_{k,2} + h_k = 1. \] (3.29)
It follows from (2.18) and (1.2) that
\[
i(c_k^{2}) = 2i(c_k) + p_{k,0} - q_{k,0} - 3(r_{k,1} + r_{k,2}) + 2 \sum_{j=1}^{r_{k,1}} E \left( \frac{\theta_{k,j}}{\pi} \right) + 2 \sum_{j=1}^{r_{k,2}} E \left( \frac{\hat{\theta}_{k,j}}{\pi} \right) \]
\[ > 2i(c_k) + p_{k,0} - q_{k,0} - 3(r_{k,1} + r_{k,2}) + 2 \sum_{j=1}^{r_{k,1}} \frac{\theta_{k,j}}{\pi} + 2 \sum_{j=1}^{r_{k,2}} \frac{\hat{\theta}_{k,j}}{\pi}. \] (3.30)
Combining (3.30) and (3.29), it yields
\[
\hat{i}(c_k) = i(c_k) + p_{k,0} - r_{k,1} - r_{k,2} + \sum_{j=1}^{r_{k,1}} \frac{\theta_{k,j}}{\pi} + \sum_{j=1}^{r_{k,2}} \frac{\tilde{\theta}_{k,j}}{\pi}
\]
which, together with (3.4), yields
\[
\hat{i}(c_k) < \frac{1}{2} \left( i(c_k^2) + p_{k,0} + q_{k,0} + r_{k,1} + r_{k,2} \right) \leq \frac{1}{2} \left( \hat{i}(c_k^2) + 1 \right),
\]
(3.31)
which, together with (3.4), yields \(i(c_k^2) > 9\), thus Case (ii) cannot happen.

**Case 3:** \(i(c_k^{2m_k-2}) + \nu(c_k^{2m_k-2}) = 2N - 8\).

In this case, by (3.3) and (3.28), one of the following cases may happen.

(i) \(i(c_k^2) = 11\) and \(p_{k,+} + q_{k,-} = 3\).

(ii) \(i(c_k^2) = 10\), \(p_{k,+} + q_{k,-} = 2\) and \(p_{k,-} + q_{k,+} = 0\).

(iii) \(i(c_k^2) = 9\), \(p_{k,+} + q_{k,-} = 2\) and \(p_{k,-} + q_{k,+} = 1\).

(iv) \(i(c_k^2) = 9\), \(p_{k,-} + q_{k,+} = 0\) and \(p_{k,+} + q_{k,-} = 1\).

For (i), similar to the arguments in Case 1, it can be shown that this case cannot happen.

For (ii), similar to (3.29) and the first equality in (3.30), we have
\[
p_{k,0} + q_{k,0} + r_{k,1} + r_{k,2} + h_k = 1,
\]
(3.32)
\[
i(c_k^2) = p_{k,0} + q_{k,0} + r_{k,1} + r_{k,2} \pmod{2},
\]
(3.33)
which yields
\[
i(c_k^2) = 1 + h_k \pmod{2}.
\]
(3.34)
Then we get \(h_k = 1\) by \(i(c_k^2) = 10\) and (3.32). By (2.18) and (3.2), we have \(i(c_k^2) = 2i(c_k)\) and \(\hat{i}(c_k) = i(c_k)\) since \(p_{k,+} + q_{k,-} = 2\) and \(h_k = 1\). However \(\hat{i}(c_k) > 5\) by (3.4), thus Case (ii) cannot happen.

For (iii), by (2.18) and (3.2), we have \(i(c_k^2) = 2i(c_k) + p_{k,-} - q_{k,+}\) and \(\hat{i}(c_k) = i(c_k) + p_{k,-}\) since \(p_{k,-} + q_{k,+} = 1\) and \(p_{k,+} + q_{k,-} = 2\). Then we obtain \(2\hat{i}(c_k) = i(c_k^2) + 1 = 10\). However \(\hat{i}(c_k) > 5\) by (3.4), thus Case (iii) cannot happen.

For (iv), similar to the proof of (3.33) in (ii) of Case 3, we have
\[
i(c_k^2) = p_{k,0} + q_{k,0} + r_{k,1} + r_{k,2} \pmod{2}.
\]
(3.35)
Therefore by (3.2), (3.35), \(i(c_k^2) = 9\) and \(p_{k,+} + q_{k,-} = 1\), we get
\[
p_{k,0} + q_{k,0} + r_{k,1} + r_{k,2} = 1.
\]
(3.36)
Then similar to the proof of (3.31) in (ii) of Case 2, we have
\[
\hat{i}(c_k) < \frac{1}{2} \left( i(c_k^2) + p_{k,0} + q_{k,0} + r_{k,1} + r_{k,2} \right) = \frac{1}{2} \left( \hat{i}(c_k^2) + 1 \right),
\]
(3.37)
which, together with (3.4), yields \(i(c_k^2) > 9\), thus Case (iv) cannot happen.
This completes the proof of Lemma 3.2.

Under the assumption (FCG), Theorem 1.1 of [Dua16] shows that there exist at least two elliptic closed geodesics $c_1$ and $c_2$ on $(S^4, F)$ whose flag curvature satisfies \( \left( \frac{\lambda_{-1}}{1+\lambda} \right)^2 < K \leq 1 \). The following Lemma gives some properties of these two closed geodesics which will be useful in the proof of Theorem 1.1.

**Lemma 3.3.** (cf. Lemma 3.1 and Lemma 3.3 of [Dua15] and Section 3 of [Dua16]) Under the assumption (FCG), there exist at least two prime elliptic closed geodesics $c_1$ and $c_2$ on $(S^4, F)$ whose flag curvature satisfies \( \left( \frac{\lambda_{-1}}{1+\lambda} \right)^2 < K \leq 1 \). Moreover, there exist infinitely many pairs of $(q+1)$-tuples $(N, m_1, m_2, \ldots, m_q) \in \mathbb{N}^{q+1}$ and $(N', m'_1, m'_2, \ldots, m'_q) \in \mathbb{N}^{q+1}$ such that

\[
i(c_1^{2m_1}) + \nu(c_1^{2m_1}) = 2N + 3, \quad \mathcal{C}_{2N+3}(E, c_1^{2m_1}) = Q, \quad (3.38)
\]
\[
i(c_2^{2m_2}) + \nu(c_2^{2m_2}) = 2N' + 3, \quad \mathcal{C}_{2N'+3}(E, c_2^{2m_2}) = Q, \quad (3.39)
\]
\[p_{k,-} = q_{k,+} = r_{k,3} = r_{k,6} = h_k = 0, \quad k = 1, 2, \quad (3.40)
\]
\[r_{1,2} = \Delta_1 \geq 1, \quad r_{2,2} = \Delta_2' \geq 1, \quad (3.41)
\]
\[\Delta_k + \Delta_k' = r_{k,2}, \quad k = 1, 2, \quad (3.42)
\]

where we can require $3|N$ or $3|N'$ as remarked in Theorem 2.8 and

\[
\Delta_k' = \sum_{0<\{m_k^i\theta/\pi\} < \delta} S_{-M_k} (e^{\sqrt{-1}\theta}), \quad k = 1, 2. \quad (3.43)
\]

In addition, for these two closed geodesics $c_1$ and $c_2$, there holds

\[
k^\varepsilon(c_k^{n(c_k)}) (c_k^{n(c_k)}) = 1, \quad k^\varepsilon(c_k^{n(c_k)}) (c_k^{n(c_k)}) = 0, \quad \forall 0 \leq j < \nu(c_k^{n(c_k)}), \quad k = 1, 2. \quad (3.44)
\]

### 3.2 Proof of Theorem 1.1

In this section, let $(S^4, F)$ be a Finsler sphere of dimension 4 with its reversibility $\lambda$ and flag curvature $K$ satisfying \( \frac{2\sqrt{3}}{9} \left( \frac{\lambda_{-1}}{1+\lambda} \right)^2 < K \leq 1 \) with $\lambda < \frac{3}{2}$. In order to prove Theorem 1.1, according to Lemma 3.3 and Theorem 1.1 of [Dua15], we make the following assumption.

**(TCG)** Suppose that there exist exactly two prime elliptic closed geodesics $c_1$ and $c_2$ possessing all properties listed in Lemma 3.3, and the third prime closed geodesic $c_3$ on such $(S^4, F)$.

In order to count the contribution of $c_k^m$ to the Morse-type number $M_q$, for the sake of convenience, we set

\[
M_q(k, m) = \dim \mathcal{C}_q(E, c_k^m), \quad \forall 1 \leq k \leq 3, \quad m \geq 1, \quad q \in \mathbb{N}_0. \quad (3.45)
\]

Next we fix $\bar{m} = 4$. Before proving Theorem 1.1, firstly we establish several crucial lemmas.
Lemma 3.4. For an integer $q$ satisfying $2N - 9 \leq q \leq 2N + 17$, there holds

\[
M_q = \begin{cases} 
\sum_{1 \leq k \leq 3} M_k(k, 2m_k - m), & \text{if } q = 2N - 9, \\
\sum_{1 \leq k \leq 3} M_k(k, 2m_k - 1), & \text{if } 2N - 8 \leq q \leq 2N - 4, \\
\sum_{1 \leq k \leq 3, 0 \leq m \leq 1} M_k(k, 2m_k - m), & \text{if } q = 2N - 3, \\
\sum_{1 \leq k \leq 3} M_k(k, 2m_k), & \text{if } 2N - 2 \leq q \leq 2N + 2, \\
\sum_{1 \leq k \leq 3} M_k(k, 2m_k + m), & \text{if } q = 2N + 3, \\
\sum_{1 \leq k \leq 3} M_k(k, 2m_k + 1), & \text{if } 2N + 4 \leq q \leq 2N + 8, \\
\sum_{1 \leq k \leq 3, 1 \leq m \leq 2} M_k(k, 2m_k + m), & \text{if } 2N + 9 \leq q \leq 2N + 14, \\
\sum_{1 \leq k \leq 3} M_k(k, 2m_k + m), & \text{if } 2N + 15 \leq q \leq 2N + 17. 
\end{cases}
\]  

(3.46)

Proof. According to Lemma 2.1, (2.4) and (i) of Lemma 2.2, we have

\[
M_q = \sum_{1 \leq k \leq 3, m \geq 1} M_k(k, m) = \sum_{1 \leq k \leq 3, m \geq 1} k^e_{q-i(c_k^m)}(c_k^m) \\
= \sum_{m \in \{m \in \mathbb{N} \mid i(c_k^m) + \nu(c_k^m) \leq 2N - 9, \text{if } m = 2m_k - 2\}} k^e_{q-i(c_k^m)}(c_k^m) = \sum_{m \in \{m \in \mathbb{N} \mid i(c_k^m) + \nu(c_k^m) \leq 2N - 15, \text{if } 1 \leq m \leq 2m_k - 4\}} M_k(k, m).\]

(3.47)

On one hand, by (3.19), (3.15), (3.24), (3.23) and (3.17), it yields

\[
i(c_k^m) + \nu(c_k^m) \leq \begin{cases} 
2N - 15, & \text{if } 1 \leq m \leq 2m_k - 4, \\
2N - 12, & \text{if } m = 2m_k - 3, \\
2N - 9, & \text{if } m = 2m_k - 2, \\
2N - 3, & \text{if } m = 2m_k - 1, \\
2N + 3, & \text{if } m = 2m_k.
\end{cases}
\]

(3.48)

On the other hand, by (3.16), (3.18) and (3.20), it yields

\[
i(c_k^m) \geq \begin{cases} 
2N - 3, & \text{if } m = 2m_k, \\
2N + 3, & \text{if } m = 2m_k + 1, \\
2N + 9, & \text{if } m = 2m_k + 2, \\
2N + 15, & \text{if } m = 2m_k + 3, \\
2N + 18, & \text{if } m \geq 2m_k + 4.
\end{cases}
\]

(3.49)

Combining (3.47)-(3.49), we get Lemma 3.4. 

Lemma 3.5. For some tuple $(k, m)$ with $k = 1, 2, 3$ and $m \in \mathbb{N}$, if there exist some integers $q_1, q_2 \in \mathbb{N}$ satisfying

\[
q_1 \leq i(c_k^m) \quad \text{and} \quad i(c_k^m) + \nu(c_k^m) \leq q_2,
\]

then there holds

\[
M_{q_1}(k, m) + M_{q_2}(k, m) \leq 1.
\]

(3.50)
Furthermore, if \( M_{q_1}(k,m) + M_{q_2}(k,m) = 1 \), then

\[
M_q(k,m) = 0, \quad \forall \ q \neq q_1, q_2. \tag{3.51}
\]

**Proof.** This follows directly from Lemma 2.1, (2.4), (i) and (ii) of Lemma 2.2. \(\blacksquare\)

**Lemma 3.6.** For some \( k \in \{1,2,3\} \), assume that either \( M_{2N-q+1}(k,2m_k-1) \geq 1 \) or \( M_{2N+q+1}(k,2m_k+1) \geq 1 \) for some even \( q \in \mathbb{N} \), then there exists a continuous path \( f_k \in C([0,1], \Omega^0(\mathcal{P}_c)) \) such that \( f_k(0) = P_{c_k} \) and \( f_k(1) \) belongs to one of the following five cases:

(i) \( I_4 \circ H(2) \),
(ii) \( N_1(1,1) \circ I_2 \circ N_1(1,-1) \),
(iii) \( I_4 \circ N_1(1,-1) \),
(iv) \( N_1(1,1) \circ I_4 \),
(v) \( I_6 \).

And in either case, the index iteration formula of \( c_k^m \) can be written as follows:

\[
i(c_k^m) = qm - p_{k,-} - p_{k,0}, \quad \forall \ m \geq 1. \tag{3.52}
\]

**Proof.** We only give the proof under the assumption \( M_{2N-q+1}(k,2m_k-1) \geq 1 \). The proof under the assumption \( M_{2N+q+1}(k,2m_k+1) \geq 1 \) is similar.

First, by Lemma 3.5 and the assumption \( M_{2N-q+1}(k,2m_k-1) \geq 1 \) in Lemma 3.6, we have

\[
i(c_k^{2m_k-1}) \leq 2N - q - 2, \quad i(c_k^{2m_k-1}) + \nu(c_k^{2m_k-1}) \geq 2N - q + 2, \tag{3.53}
\]

which, together with \( \nu(c_k^{2m_k-1}) = \nu(c_k) \) by (2.23), implies \( \nu(c_k) = p_{k,-} + 2p_{k,0} + p_{k,+} \in \{4,5,6\} \).

If \( \nu(c_k) = 4 \), by (3.53), we have \( i(c_k^{2m_k-1}) = 2N - q - 2 \) and \( i(c_k^{2m_k-1}) + \nu(c_k^{2m_k-1}) = 2N - q + 2 \). And we also have \( p_{k,-} + p_{k,0} + p_{k,+} = 2 \) or \( 3 \) by (3.2). Since \( i(c_k^{2m_k-1}) = 2N - q - 2 \in 2\mathbb{N} \), we get \( i(c_k) \in 2\mathbb{N} \) by (2.25), then by Proposition 2.7 and the symplectic additivity of symplectic paths (cf. Theorem 6.2.6 of [Lon02]), we must have \( p_{k,+} + h_{k,+} = 1 \). Therefore, if \( p_{k,-} + p_{k,0} + p_{k,+} = 2 \), we must have \( p_{k,0} = 2 \) since \( \nu(c_k) = 4 \) and \( h_{k,+} = 1 \), i.e., \( f_k(1) = I_4 \circ H(2) \). If \( p_{k,-} + p_{k,0} + p_{k,+} = 3 \), we must have \( p_{k,0} = 1, p_{k,+} = 1 \) and \( p_{k,-} = 1 \), i.e., \( f_k(1) = N_1(1,1) \circ I_2 \circ N_1(1,-1) \).

If \( \nu(c_k) = 5 \), we have \( p_{k,-} + p_{k,0} + p_{k,+} = 3 \). Then we must have \( p_{k,0} = 2 \) and \( p_{k,-} + p_{k,+} = 1 \). So either \( i(c_k^{2m_k-1}) = 2N - q - 2 \) when \( p_{k,+} = 1 \), or \( i(c_k^{2m_k-1}) = 2N - q - 3 \) when \( p_{k,-} = 1 \) by Proposition 2.7, the symplectic additivity and (3.53), i.e., \( f_k(1) = I_4 \circ N_1(1,1) \) or \( N_1(1,1) \circ I_4 \).

If \( \nu(c_k) = 6 \), we must have \( p_{k,0} = 3 \), and then we have \( i(c_k^{2m_k-1}) = 2N - q - 3 \) by Proposition 2.7, the symplectic additivity and (3.53), i.e., \( f_k(1) = I_6 \).

Note that by (2.25), (3.10) and (3.12), we get

\[
i(c_k^{2m_k-1}) = 2N - i(c_k) - 2p_{k,-} - 2p_{k,0},
\]
and by the above arguments in either case, we have \(i(c_k) = q - p_{k,-} - p_{k,0}\). Then by (2.18), we have

\[
i(c_k^m) = qm - p_{k,-} - p_{k,0}.
\]

This completes the proof of Lemma 3.6. \(\qed\)

**Proof of Theorem 1.1.**

At first, we consider the contribution of \(c_k^m\) with \(k = 1, 2, 3\) and \(m \in \mathbb{N}\) to the Morse-type numbers in Claim 1 and Claim 2. And then we use Claim 3 and Claim 4 to complete the proof of Theorem 1.1.

**Claim 1:** For \(2N - 2 \leq q \leq 2N + 2\), there holds: (i) \(M_q(1, m) = 0\) for any \(m \in \mathbb{N}\), and (ii) \(M_q(k, m) = 0\) for \(k = 2, 3\) and \(m \neq 2m_k\). In addition, \(M_q(2, 2m_2) = 0\) for \(q = 2N - 2, 2N, 2N + 2\) and \(M_{2N-1}(2, 2m_2) + M_{2N+1}(2, 2m_2) \leq 1\).

**Proof.** By Lemma 3.4, we know that \(M_q(k, m) = 0\) for \(2N - 2 \leq q \leq 2N + 2\), \(m \neq 2m_k\) and \(k = 1, 2, 3\). Also note that by (3.38), (3.44), Lemma 2.1 and (2.4), we have

\[
M_q(1, 2m_1) = 0 \quad \text{for} \quad 2N - 2 \leq q \leq 2N + 2.
\]

On one hand, there holds

\[
\nu(c_2^{2m_2}) = \nu(c_2^{2m_2}')
\]

by the choices of \(m_2\) and \(m_2'\) in (2.28) of Theorem 2.8. On the other hand, it yields

\[
i(c_2^{2m_2}) = i(c_2^{2m_2}') \quad \text{(mod 2)}
\]

by (2.18) of Theorem 2.6. So, \(i(c_2^{2m_2}) + \nu(c_2^{2m_2})\) is odd since \(i(c_2^{2m_2}') + \nu(c_2^{2m_2})\) is odd by (3.39) of Lemma 3.3, which implies that \(M_q(2, 2m_2) = 0\) for \(q = 2N - 2, 2N, 2N + 2\) and \(M_{2N-1}(2, 2m_2) + M_{2N+1}(2, 2m_2) \leq 1\) by (3.44), Lemma 2.1 and (2.4). Hence, Claim 1 holds.

**Claim 2:** \(M_{2N-1}(2, 2m_2) = M_{2N+1}(2, 2m_2) = 0\).

**Proof.** Otherwise, by Claim 1 we have

\[
M_{2N-1}(2, 2m_2) + M_{2N+1}(2, 2m_2) = 1.
\]

Then by (3.16) and Lemma 3.5, it yields

\[
M_{2N-3}(2, 2m_2) = 0.
\]

By (2.8) and Theorem 2.5, we have \(M_{2N-1} \geq b_{2N-1} = 1\) and \(M_{2N+1} \geq b_{2N+1} = 1\), then we get \(M_{2N-1}(3, 2m_3) + M_{2N+1}(3, 2m_3) \geq 1\) by Lemma 3.4, (i) of Claim 1 and (3.54). Thus by (3.16), (3.17) and Lemma 3.5, it yields

\[
M_{2N-3}(3, 2m_3) = M_{2N+3}(3, 2m_3) = 0.
\]
So, by Lemma 3.4, Claim 1, (3.54) and (3.56), we obtain

\[ \sum_{q=2N-2}^{2N} (-1)^q M_q = \sum_{q=2N-2}^{2N} (-1)^q \sum_{1 \leq k \leq 3} M_q(k, 2m_k) = \sum_{q=2N-3}^{2N+3} (-1)^q M_q(3, 2m_3) - 1. \]  \hspace{1cm} (3.57)

And by (3.16), (3.17), Lemma 2.1 and (2.4), we have

\[ \sum_{q=2N-3}^{2N+3} (-1)^q M_q(3, 2m_3) = \sum_{0 \leq l \leq 6} (-1)^l c_{c_3}^{2m_3} k_l c_{c_3}^{2m_3} (c_3^{2m_3}). \]  \hspace{1cm} (3.58)

On the other hand, by (2.8) and Theorem 2.5, we have

\[ \sum_{q=2N-2}^{2N+2} (-1)^q b_q = -2. \]  \hspace{1cm} (3.59)

Combining (3.57)-(3.59), we get

\[ \chi(c_3^{2m_3}) = \sum_{0 \leq l \leq 6} (-1)^l c_{c_3}^{2m_3} k_l c_{c_3}^{2m_3} (c_3^{2m_3}) \geq -1. \]  \hspace{1cm} (3.60)

Note that since \( n(c_3)|2m_3 \) and \( \nu(c_3^{2m_3}) = \nu(c_3^{n(c_3)}) \) by (2.7) and (2.28), there holds

\[ \chi(c_3^{n(c_3)}) = \chi(c_3^{2m_3}) \geq -1. \]  \hspace{1cm} (3.61)

By (3.38), (3.44), Lemma 2.1 and (2.4), it yields \( M_{2N-3}(1, 2m_1) = 0 \). Together with (3.55) and (3.56), we have

\[ \sum_{1 \leq k \leq 3} M_{2N-3}(k, 2m_k) = 0. \]  \hspace{1cm} (3.62)

Then combining Lemma 3.4 and (3.62), we obtain that

\[ M_{2N-3} = \sum_{1 \leq k \leq 3} M_{2N-3}(k, 2m_k - 1). \]  \hspace{1cm} (3.63)

One one hand, by (3.23) and Lemma 3.5 it yields

\[ M_{2N-3}(k, 2m_k - 1) \leq 1, \quad \forall \ k = 1, 2, 3, \]  \hspace{1cm} (3.64)

then it follows from (3.63) and (3.64) that \( M_{2N-3} \leq 3 \). On the other hand, we have \( M_{2N-3} \geq b_{2N-3} = 2 \) by (2.8) and Theorem 2.5. So it yields \( M_{2N-3} \in \{2, 3\} \).

We continue the proof by distinguishing two cases.

**Case 1:** \( M_{2N-3} = 3 \).

In this case, it follows from (3.63) and (3.64) that \( M_{2N-3}(k, 2m_k - 1) = 1 \) for \( k = 1, 2, 3 \). Then according to (3.23) and Lemma 3.5, there holds

\[ M_{2N-5}(k, 2m_k - 1) = 0, \quad \forall \ k = 1, 2, 3. \]  \hspace{1cm} (3.65)
Combining (3.65) and Lemma 3.4, we get $M_{2N-5} = 0$. But by (2.8) and Theorem 2.5, we have $M_{2N-5} \geq b_{2N-5} = 1$. This is a contradiction.

**Case 2: $M_{2N-3} = 2$.**

In this case, it follows from (3.63) and (3.64) that one of the following cases may happen:

(i) $M_{2N-3}(3,2m_3 - 1) = 0$ and $M_{2N-3}(1,2m_1 - 1) = M_{2N-3}(2,2m_1 - 1) = 1$.

(ii) $M_{2N-3}(3,2m_3 - 1) = 1$ and $M_{2N-3}(1,2m_1 - 1) + M_{2N-3}(2,2m_2 - 1) = 1$.

For (i), by (3.23) and Lemma 3.5, there holds

$$M_q(k,2m_k - 1) = 0, \quad \forall q \neq 2N - 3, \quad k = 1, 2.$$  \hspace{1cm} (3.66)

So, according to Lemma 3.4 and (3.66), we have

$$M_{2N-5} = \sum_{1 \leq k \leq 3} M_{2N-5}(k,2m_k - 1) = M_{2N-5}(3,2m_3 - 1), \quad \text{and} \quad (3.67)$$

$$M_{2N-7} = \sum_{1 \leq k \leq 3} M_{2N-7}(k,2m_k - 1) = M_{2N-7}(3,2m_3 - 1). \quad \text{and} \quad (3.68)$$

By (2.8) and Theorem 2.5, we have $M_{2N-5} \geq b_{2N-5} = 1$ and $M_{2N-7} \geq b_{2N-7} = 1$, then it follows from (3.67) and (3.68) that

$$M_{2N-5}(3,2m_3 - 1) \geq 1, \quad M_{2N-7}(3,2m_3 - 1) \geq 1. \quad \text{and} \quad (3.69)$$

So the assumption with $q = 6$ in Lemma 3.6 is satisfied, and then by (3.52) and (2.7), we have

$$i(c_3^m) = 6m - p_{3,-} - p_{3,0}, \quad \text{and} \quad (3.70)$$

$$n(c_3) = 1. \quad \text{and} \quad (3.71)$$

Notice that $\nu(c_3^{2m_3-1}) = \nu(c_3^{2m_3-2}) = \nu(c_3)$ by (2.19) for each of five cases in Lemma 3.6. Together with Lemma 2.1, (2.4), (iv) of Lemma 2.2, (3.70), (3.71) and (i) of Case 2, it yields

$$M_{2N-9}(3,2m_3 - 2) = k^i(c_3^{2m_3-2}) + i(c_3^{2m_3-2})(c_3^{2m_3-2})$$

$$= k^i(c_3^{2m_3-1}) + i(c_3^{2m_3-2})(c_3^{2m_3-1})$$

$$= M_{2N-9}(i(c_3^{2m_3-2}))(i(c_3^{2m_3-1}),3,2m_3 - 1)$$

$$= M_{2N-3}(3,2m_3 - 1) = 0. \quad \text{and} \quad (3.72)$$

Through comparing (3.9) and (3.70), we get $2N = 12m_3$. Then by (3.70) and (3.2) it yields

$$i(c_3^{2m_3-1}) = 12m_3 - 6 - p_{3,-} - p_{3,0} \geq 2N - 9, \quad \text{and} \quad (3.73)$$

Then by (3.69) and Lemma 3.5, there holds

$$M_{2N-9}(3,2m_3 - 1) = 0. \quad \text{and} \quad (3.74)$$
Finally, combining (3.66), (3.72), (3.74) and Lemma 3.4, we obtain

\[ M_{2N-9} = \sum_{1 \leq k \leq 3} \sum_{1 \leq m \leq 2} M_{2N-9}(k, 2m_k - m) = M_{2N-9}(1, 2m_1 - 2) + M_{2N-9}(2, 2m_2 - 2). \]  

(3.75)

By (3.24) and (ii) of Lemma 2.2, we get \( M_{2N-9}(k, 2m_k - 2) \leq 1 \) for \( k = 1, 2 \). Then it follows from (3.75) that \( M_{2N-9} \leq 2 \). However by (2.8) and Theorem 2.5, we have \( M_{2N-9} \geq b_{2N-9} = 2 \). Thus we get \( M_{2N-9} = 2 \).

Now, in this case, we have

\[ M_{2N-3} = b_{2N-3}, \quad M_{2N-9} = b_{2N-9}. \]  

(3.76)

Combining (2.8), Theorem 2.5 and (3.76), we obtain

\[ \sum_{q=2N-8}^{2N-4} (-1)^q M_q = \sum_{q=2N-8}^{2N-4} (-1)^q b_q = -2. \]  

(3.77)

Note that for \( 2N-8 \leq q \leq 2N-4 \), it follows from (3.23) and (i) of this case that \( M_q(k, 2m_k - 1) = 0 \) for \( k = 1, 2 \). So we get \( M_q = M_q(3, 2m_3 - 1) \) by Lemma 3.4, and then it yields

\[ \sum_{q=2N-8}^{2N-4} (-1)^q M_q = \sum_{q=2N-8}^{2N-4} (-1)^q M_q(3, 2m_3 - 1). \]  

(3.78)

Note that \( 2N - 9 \leq i(c_3^{2m_3-1}) \leq i(c_3^{2m_3-1}) + \nu(c_3^{2m_3-1}) \leq 2N - 3 \) by (3.73) and (3.23), according to (3.74), (i) of this case, Lemma 2.1 and (2.4), we obtain

\[ \sum_{q=2N-8}^{2N-4} (-1)^q M_q(3, 2m_3 - 1) = \sum_{q=2N-9}^{2N-3} (-1)^q M_q(3, 2m_3 - 1) \]

\[ = \sum_{0 \leq l \leq 6} (-1)^i(c_3^{2m_3-1}) + l_k^2(c_3^{2m_3-1}) (c_3^{2m_3-1}). \]  

(3.79)

Combining (3.77)-(3.79), we get

\[ \chi(c_3^{2m_3-1}) = \sum_{0 \leq l \leq 6} (-1)^i(c_3^{2m_3-1}) + l_k^2(c_3^{2m_3-1}) (c_3^{2m_3-1}) = -2. \]  

(3.80)

However, since \( n(c_3) = 1 \) by (3.71), it follows from (iv) of Lemma 2.2 and (3.80) that \( \chi(c_3) = \chi(c_3^{2m_3-1}) = -2 \), which contradicts to (3.61), thus Case (i) cannot happen.

For (ii), without loss of generality, we assume that \( M_{2N-3}(1, 2m_1 - 1) = 0 \). Then by (3.23) and Lemma 3.5, \( M_q(k, 2m_k - 1) = 0 \) for \( q \neq 2N-3 \) and \( k = 2, 3 \). So, according to Lemma 3.4, we have

\[ M_{2N-5} = M_{2N-5}(1, 2m_1 - 1), \quad M_{2N-7} = M_{2N-7}(1, 2m_1 - 1). \]  

(3.81)

By (2.8) and Theorem 2.5, we have \( M_{2N-5} \geq b_{2N-5} = 1 \) and \( M_{2N-7} \geq b_{2N-7} = 1 \), then it follows from (3.81) that \( M_{2N-5}(1, 2m_1 - 1) \geq 1 \) and \( M_{2N-7}(1, 2m_1 - 1) \geq 1 \). Thus by Lemma 3.6, there
holds \( f_1(0) = P_{c_1} \) and \( f_1(1) \) belongs to one of five cases in Lemma 3.6, which contradicts (3.40) in Lemma 3.3.

This completes the proof of Claim 2.

Claim 3: \( c_1 \) and \( c_2 \) are irrationally elliptic.

Proof. By (3.41) and (3.42), there holds \( \Delta_2 = 0 \). Then, together with the fact that \( r_{2,3} = r_{2,4} = 0 \) from (3.40), it follows from (3.41) and (3.21) that

\[
2N + 1 \geq i(c_2^{2m_2}) + \nu(c_2^{2m_2}) \\
= 2N + (p_{2,0} + p_{2,+} + q_{2,-} + q_{2,0} + r_{2,1} + 2r_{2,5} - r_{2,2}) \\
\geq 2N - 3, \tag{3.83}
\]

where (3.82) holds by the fact that \( p_{2,0} + p_{2,+} + q_{2,-} + q_{2,0} + r_{2,1} + 2r_{2,5} \leq 2 \) from (3.2) and (3.41), and \( r_{2,2} \geq 1 \) from (3.41), and the equality in (3.83) holds if and only if \( r_{2,2} = 3 \).

On the other hand, by Claim 2, we have \( i(c_2^{2m_2}) + \nu(c_2^{2m_2}) \notin \{2N - 1, 2N + 1\} \). Note that \( i(c_2^{2m_2}) + \nu(c_2^{2m_2}) = i(c_2^{2m_2}) + \nu(c_2^{2m_2}) = 1 \) (mod 2) by (3.39). Thus, by (3.82), we obtain \( i(c_2^{2m_2}) + \nu(c_2^{2m_2}) \leq 2N - 3 \), which together with (3.83) implies \( r_{2,2} = 3 \), i.e., \( c_2 \) is irrationally elliptic. By the symmetric properties of \( c_1 \) and \( c_2 \) in Lemma 3.3 (or, more precisely, replacing \( N \) with \( N' \) in the above arguments), we conclude that \( c_1 \) is also irrationally elliptic.

This completes the proof of Claim 3.

Claim 4: \( c_3 \) is non-hyperbolic.

Proof. Assume that \( c_3 \) is hyperbolic, which, together with the assumption (TCG) and Claim 3, implies the Finsler metric \( F \) on \( S^4 \) is bumpy. Then by Theorem 1.1 in [DLW16], we know that there exist at least four distinct non-hyperbolic prime closed geodesics, which contradicts the assumption (TCG). Thus by \( c_3 \) is non-hyperbolic.

Therefore, by the assumption (TCG), Claims 3 and 4 complete the proof of Theorem 1.1.

4 Some further information about the third closed geodesic

In this section, under the assumption (TCG), we further study the third closed geodesic \( c_3 \) and obtain some much precise information about it (cf. Theorem 4.2 below). At first, we establish a similar result as Lemma 3.6.

Lemma 4.1. For some \( k \in \{1,2,3\} \), assume that either \( M_{2N-q+1}(k, 2m_k - 2) \geq 1 \) or \( M_{2N+q+1}(k, 2m_k + 2) \geq 1 \) for some even \( q \in \mathbb{N} \), then there exists a continuous path \( f_k \in C([0,1], \Omega^\partial(P_{c_k})) \) such that \( f_k(0) = P_{c_k} \) and \( f_k(1) \) belongs to one of the following cases:

(i) \( I_{2p_{k,0}} \circ (-I_{2q_{k,0}}) \circ H(2) \) with \( p_{k,0} + q_{k,0} = 2 \),

(ii) \( N_1(1,1)^{op}_{k,-} \circ N_1(-1,1)^{op}_{k,+} \circ I_{2p_{k,0}} \circ (-I_{2q_{k,0}}) \circ N_1(1,1)^{op}_{k,+} \circ N_1(-1,1)^{op}_{k,-} \) with \( p_{k,+} = 1, \) \( q_{k,+} = 1 \) and \( p_{k,-} + q_{k,-} = 1 \),
(iii) $I_{2p_{k,0}} \diamond (-I_{2q_{k,0}}) \diamond N_1(1,-1)^{\circ p_{k,+}} \diamond N_1(-1,1)^{\circ q_{k,-}}$ with $p_{k,0} + q_{k,0} = 2$ and $p_{k,+} + q_{k,-} = 1$.

(iv) $N_1(1,1)^{\circ p_{k,-}} \diamond N_1(-1,-1)^{\circ q_{k,+}} \diamond I_{2p_{k,0}} \diamond (-I_{2q_{k,0}})$ with $p_{k,0} + q_{k,0} = 2$ and $p_{k,-} + q_{k,+} = 1$.

(v) $I_{2p_{k,0}} \diamond (-I_{2q_{k,0}})$ with $p_{k,0} + q_{k,0} = 3$.

And in either case, the index iteration formula of $c_k^m$ can be written as follows:

$$i(c_k^m) = \frac{q}{2} m - (p_{k,-} + p_{k,0}) - \frac{1}{2}(-1)^m(q_{k,0} + q_{k,+}).$$

**Proof.** We only give the proof under the assumption $M_{2N+q\pm 1}(k,2m_k + 2) \geq 1$. The proof under the assumption $M_{2N-q\pm 1}(k,2m_k - 2) \geq 1$ is similar.

First, by Lemma 3.5 and the assumption $M_{2N+q\pm 1}(k,2m_k + 2) \geq 1$ in Lemma 4.1, we have

$$i(c_k^{2m_k+2}) \leq 2N + q - 2, \quad i(c_k^{2m_k+2}) + \nu(c_k^{2m_k+2}) \geq 2N + q + 2,$$

which, together with $\nu(c_k^{2m_k+2}) = \nu(c_k^2)$ by (2.23), implies $\nu(c_k^2) = p_{k,-} + 2p_{k,0} + p_{k,+} + q_{k,-} + 2q_{k,0} + q_{k,+} \in \{4,5,6\}$.

If $\nu(c_k^2) = 4$, by (4.2), we have $i(c_k^{2m_k+2}) = 2N + q - 2$ and $i(c_k^{2m_k+2}) + \nu(c_k^{2m_k+2}) = 2N + q + 2$.

And by (3.2) we also have

$$p_{k,-} + p_{k,0} + p_{k,+} + q_{k,-} + q_{k,0} + q_{k,+} \in \{2,3\}.$$  (4.3)

Since $i(c_k^{2m_k+2}) = 2N + q - 2 \in 2\mathbb{N}$, we get $i(c_k^2) \in 2\mathbb{N}$ by (3.9). Note that by (2.18) and (3.2), we have

$$i(c_k^2) = p_{k,-} + p_{k,0} + q_{k,0} + q_{k,+} + r_{k,1} + r_{k,2} \pmod{2},$$

Then, by (4.4)

$$i(c_k^2) = 1 + p_{k,+} + q_{k,-} + h_k \pmod{2}.$$  (4.4)

So, by (4.3) and (3.2), we get $p_{k,+} + q_{k,-} + h_k = 1$ since $\nu(c_k^2) = 4$. Therefore, if $p_{k,-} + p_{k,0} + p_{k,+} + q_{k,-} + q_{k,0} + q_{k,+} = 2$, we must have $p_{k,0} + q_{k,0} = 2$ and $h_k = 1$, if $p_{k,-} + p_{k,0} + p_{k,+} + q_{k,-} + q_{k,0} + q_{k,+} = 3$, we must have $p_{k,0} + q_{k,0} = 1$, $p_{k,+} + q_{k,-} = 1$ and $p_{k,-} + q_{k,+} = 1$.

If $\nu(c_k^2) = 5$, we have $p_{k,-} + p_{k,0} + p_{k,+} + q_{k,-} + q_{k,0} + q_{k,+} = 3$. Then we must have $p_{k,0} + q_{k,0} = 2$ and $p_{k,-} + q_{k,+} + p_{k,+} + q_{k,-} = 1$. And by (3.9) and (4.4) we have $i(c_k^{2m_k+2}) = 2N + q - 2$ when $p_{k,+} + q_{k,-} = 1$, $i(c_k^{2m_k+2}) = 2N + q - 3$ when $p_{k,-} + q_{k,+} = 1$.

If $\nu(c_k) = 6$, we must have $p_{k,0} + q_{k,0} = 3$, and then we have $i(c_k^{2m_k+2}) = 2N + q - 3$ by (4.4) and (3.9).

Note that by (3.9), we get $i(c_k^{2m_k+2}) = 2N + i(c_k^2)$, and in either case, we have

$$i(c_k^2) = q - p_{k,-} - p_{k,0} - q_{k,+} - q_{k,0}.$$  (4.5)

Then by (2.18) and according to the precise cases in Lemma 4.1, we have

$$i(c_k^2) = 2i(c_k) + p_{k,-} + p_{k,0} - q_{k,0} - q_{k,+}.$$  (4.6)
Combining (2.18), (4.5) and (4.6), it yields

\[ i(c_k^m) = \frac{q}{2}m - (p_{k,-} + p_{k,0}) - \frac{1 + (-1)^m}{2}(q_{k,0} + q_{k,+}). \]

This completes the proof of Lemma 4.1.

**Theorem 4.2.** For every Finsler metric \( F \) on \( S^4 \) with reversibility \( \lambda \) and flag curvature \( K \) satisfying \( \frac{25}{9}(\frac{1}{\lambda + 1})^2 < K \leq 1 \) with \( \lambda < \frac{3}{2} \), suppose that there exist precisely three prime closed geodesics \( c_1, c_2 \) and \( c_3 \), then both \( c_1 \) and \( c_2 \) are irrationally elliptic with \( i(c_1) = 3 \) and \( i(c_2) = 9 \), and \( c_3 \) is non-hyperbolic and must belong to one of the following precise classes:

(i) \( i(c_3) = 3 \) and \( P_{c_3} \approx N_1(1,1) \circ I_4 \),

(ii) \( i(c_3) = 3 \) and \( P_{c_3} \approx I_6 \),

(iii) \( i(c_3) = 4 \) and \( P_{c_3} \approx N_1(1,1) \circ I_2 \circ N_1(1,-1) \),

(iv) \( i(c_3) = 4 \) and \( P_{c_3} \approx I_4 \circ N_1(1,-1) \),

(v) \( i(c_3) = 4 \) and \( P_{c_3} \approx I_4 \circ H(2) \),

where and below, "\( P_{c_k} \approx A \)" means that there exists a continuous path \( f_k \in C([0,1], \Omega^0(P_{c_k})) \) such that \( f_k(0) = P_{c_k} \) and \( f_k(1) = A \) in Theorem 2.6.

**Proof.** Under the assumption (TCG), it follows from Theorem 1.1 that both \( c_1 \) and \( c_2 \) are irrationally elliptic. Then there holds \( P_{c_k} \approx R(\tilde{\theta}_{k,1}) \circ R(\tilde{\theta}_{k,2}) \circ R(\tilde{\theta}_{k,3}) \) for some \( \tilde{\theta}_{k,1}, \tilde{\theta}_{k,2}, \tilde{\theta}_{k,3} \in (0,1) \setminus Q \) for \( k = 1, 2 \), respectively.

Then by (2.18), we have

\[ i(c_k^m) = m(i(c_k) - 3) + 2 \sum_{j=1}^{3} E \left( \frac{m\tilde{\theta}_{k,j}}{2\pi} \right) - 3, \quad \nu(c_k^m) = 0, \quad k = 1, 2, \quad (4.7) \]

\[ \hat{i}(c_k) = i(c_k) - 3 + \sum_{j=1}^{3} \frac{\tilde{\theta}_{k,j}}{\pi}, \quad k = 1, 2, \quad (4.8) \]

and then by (2.7), we get

\[ n(c_k) = 1, \quad k = 1, 2. \quad (4.9) \]

By (2.6), (iii) of Lemma 2.2, (4.7) and (4.9), for the average Euler numbers of \( c_1 \) and \( c_2 \) we have

\[ \hat{\chi}(c_1) = -1, \quad \hat{\chi}(c_2) = -1. \quad (4.10) \]

Noticing that \( \nu(c_k^m) = 0, \forall \ m \in \mathbb{N} \) and \( k = 1, 2 \), so by Lemma 2.1, (2.4), and (iii) of Lemma 2.2, we have

\[ M_q(k,m) = \begin{cases} 1, & \text{if } q = i(c_k^m), \\ 0, & \text{if } q \neq i(c_k^m), \end{cases} \quad \text{for } k = 1, 2, \ m \in \mathbb{N}. \quad (4.11) \]

By Claim 3 of the proof of Theorem 1.1, (4.7) and (4.11), we have

\[ i(c_2^{2m_2}) + \nu(c_2^{2m_2}) = i(c_2^{2m_2}) = 2N - 3, \quad M_{2N-3}(2,2m_2) = 1. \quad (4.12) \]
By (3.38) and (4.7), we have
\[ i(c_1^{2m_1}) + \nu(c_1^{2m_1}) = i(c_1^{2m_1}) = 2N + 3, \quad M_{2N+3}(1, 2m_1) = 1. \] (4.13)

It follows from (4.11)-(4.13) and Lemma 3.4 that
\[ M_q = \sum_{1 \leq k \leq 3} M_q(k, 2m_k) = M_q(3, 2m_3) \quad \text{for } 2N - 2 \leq q \leq 2N + 2 \] (4.14)
and
\[ M_{2N+3}(2, 2m_2) = 0. \] (4.15)

By (2.8), Theorem 2.5 and (4.14), we have \( M_{2N-1} = M_{2N-1}(3, 2m_3) \geq b_{2N-1} = 1 \) and \( M_{2N+1} = M_{2N+1}(3, 2m_3) \geq b_{2N+1} = 1 \), then by (3.16), (3.17) and Lemma 3.5, there holds
\[ M_{2N-3}(3, 2m_3) = M_{2N+3}(3, 2m_3) = 0. \] (4.16)

By (4.13), (4.15), (4.16) and Lemma 3.4, we have
\[ M_{2N+3} = \sum_{1 \leq k \leq 3} M_{2N+3}(k, 2m_k + 1) + 1. \] (4.17)

By (3.18) and Lemma 3.5, \( M_{2N+3}(k, 2m_k + 1) \leq 1 \) for \( k = 1, 2, 3 \). Then it follows from (4.17) that \( M_{2N+3} \leq 4 \). On the other hand, by (2.8) and Theorem 2.5, we have \( M_{2N+3} \geq b_{2N+3} = 2 \). Thus \( M_{2N+3} \in \{2, 3, 4\} \).

We continue the proof by distinguishing three cases.

**Case 1:** \( M_{2N+3} = 4 \).

In this case, by (4.17), it yields
\[ M_{2N+3}(k, 2m_k + 1) = 1, \quad \forall \, k = 1, 2, 3. \] (4.18)

Then by (3.18) and Lemma 3.5, we have
\[ M_{2N+5}(k, 2m_k + 1) = 0, \quad \forall \, k = 1, 2, 3. \] (4.19)

It follows from (4.19) and Lemma 3.4 that \( M_{2N+5} = 0 \). However by (2.8) and Theorem 2.5, we have \( M_{2N+5} \geq b_{2N+5} = 1 \), which is a contradiction.

**Case 2:** \( M_{2N+3} = 3 \).

In this case, by (4.17), one of the following cases may happen.

(i) \( M_{2N+3}(3, 2m_3 + 1) = 0 \) and \( M_{2N+3}(1, 2m_1 + 1) = M_{2N+3}(2, 2m_2 + 1) = 1 \).

(ii) \( M_{2N+3}(3, 2m_3 + 1) = 1 \) and \( M_{2N+3}(1, 2m_1 + 1) = M_{2N+3}(2, 2m_2 + 1) = 1 \).
For (i), by (4.11), we have $i(c_1^{2m_1+1}) = 2N + 3$ and $i(c_2^{2m_2+1}) = 2N + 3$, then by (3.9), it yields $i(c_1) = i(c_2) = 3$. So, by (3.4) and (4.8), we get

$$5 < \hat{i}(c_1) < 6, \quad 5 < \hat{i}(c_2) < 6.$$  \hfill (4.20)

By (3.18), Lemma 3.5 and (i) of Case 2, there holds

$$M_q(k,2m_k + 1) = 0 \quad \text{for } q \neq 2N + 3 \text{ and } k = 1, 2.$$  \hfill (4.21)

So, according to Lemma 3.4 and (4.21), we have

$$M_{2N+5} = \sum_{1 \leq k \leq 3} M_{2N+5}(k,2m_k + 1) = M_{2N+5}(3,2m_3 + 1),$$  \hfill (4.22)

$$M_{2N+7} = \sum_{1 \leq k \leq 3} M_{2N+7}(k,2m_k + 1) = M_{2N+7}(3,2m_3 + 1).$$  \hfill (4.23)

By (2.8) and Theorem 2.5, we have $M_{2N+5} \geq b_{2N+5} = 1$ and $M_{2N+7} \geq b_{2N+7} = 1$, then it follows from (4.22) and (4.23) that

$$M_{2N+5}(3,2m_3 + 1) \geq 1, \quad M_{2N+7}(3,2m_3 + 1) \geq 1.$$  \hfill (4.24)

Thus the assumption with $q = 6$ in Lemma 3.6 is satisfied, and then by (3.52) we have

$$i(c_3^{2m}) = 6m - p_3, - p_3, 0.$$  \hfill (4.25)

Then we have $n(c_3) = 1$ in either case of $P_{c_3}$ by (2.7). By (4.25), we have

$$\hat{i}(c_3) = 6.$$  \hfill (4.26)

By (2.5), we have the following identity

$$\sum_{k=1}^{3} \frac{\hat{\chi}(c_k)}{i(c_k)} = B(4,1) = -\frac{2}{3}.$$  \hfill (4.27)

Combining (4.10), (4.20), (4.26) and (4.27), we obtain $-2 < \hat{\chi}(c_3) = -4 + \frac{6}{i(c_1)} + \frac{6}{i(c_2)} < -\frac{8}{5}$, which contradicts to $\hat{\chi}(c_3) \in \mathbb{Z}$, where the latter is due to $n(c_3) = 1$ and the definition of $\hat{\chi}(c_3)$.

For (ii), without loss of generality, we assume that $M_{2N+5}(1,2m_1 + 1) = 0$ and $M_{2N+3}(2,2m_1 + 1) = 1$. Thus, similarly, by (3.18) and Lemma 3.5 and Lemma 3.4, we have

$$M_{2N+5} = M_{2N+5}(1,2m_1 + 1), \quad M_{2N+7} = M_{2N+7}(1,2m_1 + 1).$$  \hfill (4.28)

By (2.8) and Theorem 2.5, we have $M_{2N+5} \geq b_{2N+5} = 1$ and $M_{2N+7} \geq b_{2N+7} = 1$. Then it follows from (4.28) that $M_{2N+5}(1,2m_1 + 1) \geq 1$ and $M_{2N+7}(1,2m_1 + 1) \geq 1$. Thus by Lemma 3.6, there are five cases for $P_{c_1}$, which contradicts the fact that $c_1$ is irrationally elliptic.

**Case 3:** $M_{2N+3} = 2.$
In this case, by (4.17), one of the following cases may happen:

(i) \( M_{2N+3}(3,2m_3+1) = 1 \) and \( M_{2N+3}(1,2m_1+1) = M_{2N+3}(2,2m_2+1) = 0 \).

(ii) \( M_{2N+3}(3,2m_3+1) = 0 \) and \( M_{2N+3}(1,2m_1+1) + M_{2N+3}(2,2m_2+1) = 1 \).

For (i), by (3.18) and Lemma 3.5, there holds

\[
M_q(3,2m_3+1) = 0 \quad \text{for} \ q \neq 2N+3, \tag{4.29}
\]

and then by Lemma 3.4, for \( q = 2N + 5, 2N + 7 \), we have

\[
M_q = \sum_{1 \leq k \leq 3} M_q(k,2m_k+1) = M_q(1,2m_1+1) + M_q(2,2m_2+1). \tag{4.30}
\]

On the other hand, by (2.8) and Theorem 2.5, we have \( M_{2N+5} \geq b_{2N+5} = 1 \) and \( M_{2N+7} \geq b_{2N+7} = 1 \), which, together with (4.11), implies

\[
M_q(1,2m_1+1) = \begin{cases} 1, & \text{if} \ q = 2N + 5, \\ 0, & \text{if} \ q \neq 2N + 5, \end{cases} \quad M_q(2,2m_2+1) = \begin{cases} 1, & \text{if} \ q = 2N + 7, \\ 0, & \text{if} \ q \neq 2N + 7, \end{cases} \tag{4.31}
\]

or

\[
M_q(1,2m_1+1) = \begin{cases} 1, & \text{if} \ q = 2N + 7, \\ 0, & \text{if} \ q \neq 2N + 7, \end{cases} \quad M_q(2,2m_2+1) = \begin{cases} 1, & \text{if} \ q = 2N + 5, \\ 0, & \text{if} \ q \neq 2N + 5. \end{cases} \tag{4.32}
\]

So it follows from Lemma 3.4, (4.29) and (4.31)-(4.32) that

\[
M_{2N+9} = \sum_{1 \leq k \leq 3} M_{2N+9}(k,2m_k+m) = \sum_{1 \leq k \leq 3} M_{2N+9}(k,2m_k+2). \tag{4.33}
\]

Without loss of generality, we assume that (4.31) holds. So we get \( i(c_1^{2m_1+1}) = 2N + 5 \) and \( i(c_2^{2m_2+1}) = 2N + 7 \) by (4.11). Then by (3.9), it yields \( i(c_1) = 5 \) and \( i(c_2) = 7 \).

Since \( i(c_2) = 7, \) by (4.7), we have the index iteration formula of \( c_2 \) as follows

\[
i(c_2^{3m}) = 4m - 3 + 2 \sum_{j=1}^{3} E \left( \frac{m \tilde{\theta}_{2,j}}{2\pi} \right). \tag{4.34}
\]

By (4.34), it yields \( i(c_2^3) \geq 11, \) then by (3.9), we get \( i(c_2^{2m_2+2}) \geq 2N + 11. \) Thus by (4.11), we have

\[
M_{2N+9}(2,2m_2 + 2) = 0. \tag{4.35}
\]

By (3.18) and Lemma 3.5, we have

\[
M_{2N+9}(k,2m_k+2) \leq 1, \quad \forall \ k = 1, 2, 3, \tag{4.36}
\]

which, together with (4.33) and (4.35), implies \( M_{2N+9} \leq 2. \) However by (2.8) and Theorem 2.5, we have \( M_{2N+9} \geq b_{2N+9} = 2, \) which implies \( M_{2N+9} = 2. \) So we have

\[
M_{2N+9}(1,2m_1+2) = M_{2N+9}(3,2m_3+2) = 1. \tag{4.37}
\]
It follows from (4.11) and (4.37) that
\[ M_q(1, 2m_1 + 2) = M_q(3, 2m_3 + 2) = 0 \quad \text{for} \quad q = 2N + 11, 2N + 13. \] (4.38)

Combining (4.29), (4.31), (4.38) and Lemma 3.4, we obtain
\[ M_q = \sum_{1 \leq k \leq 3, \quad 1 \leq m \leq 2} M_q(k, 2m_k + m) = M_q(2, 2m_2 + 2) \quad \text{for} \quad q = 2N + 11, 2N + 13. \] (4.39)

Then by (4.11), we obtain that \( M_{2N+11} = 0 \) or \( M_{2N+13} = 0 \), which contradicts to \( M_{2N+11} \geq b_{2N+11} = 1 \) and \( M_{2N+13} \geq b_{2N+13} = 1 \) by (2.8) and Theorem 2.5.

For (ii), without loss of generality, we assume that
\[ M_{2N+3}(2, 2m_2 + 1) = 0, \quad M_{2N+3}(1, 2m_1 + 1) = 1. \] (4.40)

By (4.11), we have \( i(c_1^{2m_1+1}) = 2N + 3 \) and
\[ M_q(1, 2m_1 + 1) = 0, \quad \forall \ q \neq 2N + 3. \] (4.41)

Then by (3.9), it yields \( i(c_1) = 3 \). Then by (4.7), we obtain
\[ i(c_1^m) = 2 \sum_{j=1}^{3} E \left( \frac{m \tilde{\theta}_{1,j}}{2\pi} \right) - 3. \] (4.42)

By (4.42), it yields \( i(c_1^2) \leq 9 \). Then by (3.3), we get \( i(c_1^2) = 9 \). And then \( i(c_1^{2m_1+2}) = 2N + 9 \) by (3.9). Thus by (4.11), we have
\[ M_q(1, 2m_1 + 2) = \begin{cases} 1, & \text{if} \quad q = 2N + 9, \\ 0, & \text{if} \quad q \neq 2N + 9. \end{cases} \] (4.43)

**Claim 1**: \( i(c_1^{2m_2+1}) \leq 2N + 9 \), or equivalently, \( i(c_1^2) \leq 9 \) by (3.9).

If \( i(c_1^{2m_2+1}) > 2N + 9 \), by (3.5) we get \( i(c_2^{2m_2+2}) \geq i(c_1^{2m_2+1}) > 2N + 9 \). Then by (4.11) we know
\[ M_q(2, 2m_2 + 1) = M_q(2, 2m_2 + 2) = 0 \quad \text{for} \quad q = 2N + 5, 2N + 7, 2N + 9. \] (4.44)

Combining Lemma 3.4, (4.41) and (4.44), we have
\[ M_{2N+5} = \sum_{1 \leq k \leq 3} M_{2N+5}(k, 2m_k + 1) = M_{2N+5}(3, 2m_3 + 1), \] (4.45)
\[ M_{2N+7} = \sum_{1 \leq k \leq 3} M_{2N+7}(k, 2m_k + 1) = M_{2N+7}(3, 2m_3 + 1). \] (4.46)

On the other hand, by (2.8) and Theorem 2.5, we have \( M_{2N+5} \geq b_{2N+5} = 1 \) and \( M_{2N+7} \geq b_{2N+7} = 1 \). Then by (4.45) and (4.46), we know that
\[ M_{2N+5}(3, 2m_3 + 1) \geq 1, \quad M_{2N+7}(3, 2m_3 + 1) \geq 1. \] (4.47)
Thus the assumption with $q = 6$ in Lemma 3.6 is satisfied, and then we have the index iteration formula of $c_3$ as follows

$$i(c_3^n) = 6m - p_{3,} - p_{3,0}. \quad (4.48)$$

Then by the fact $\nu(c_3) = p_{3,} + 2p_{4,0} + p_{3,+}$ and (3.2), we get

$$i(c_3) + \nu(c_3) = 6 + p_{3,0} + p_{3,} \leq 9. \quad (4.49)$$

Then by (3.9) and (2.23), it yields $i(c_3^{2m_3+1}) + \nu(c_3^{2m_3+1}) \leq 2N + 9$. So, according to Lemma 3.5 and (4.47), there holds

$$M_{2N+9}(3, 2m_3 + 1) = 0. \quad (4.50)$$

Similar to (3.72), we obtain

$$M_{2N+9}(3, 2m_3 + 2) = M_{2N+3}(3, 2m_3 + 1) = 0. \quad (4.51)$$

Combining (4.41), (4.43), (4.44), (4.50), (4.51) and Lemma 3.4, we obtain that

$$M_{2N+9} = \sum_{\substack{1 \leq k \leq 3, \\ 1 \leq m \leq 2}} M_{2N+9}(k, 2m_k + m) = M_{2N+9}(1, 2m_1 + 2) = 1, \quad (4.52)$$

which gives a contradiction $1 = M_{2N+9} \geq b_{2N+9} = 2$ by (2.8) and Theorem 2.5. This finished the proof of Claim 1.

Note that $i(c_2) \neq 3$ by (4.40), (4.11) and (3.9), it yields $i(c_2) \in \{5, 7, 9\}$ by Claim 1 since $i(c_2)$ is odd. Next we have three subcases according to the value of $i(c_2)$.

**Subcase 3.1: $i(c_2) = 5$.**

In this subcase, by (4.11) and (3.9), we have

$$M_q(2, 2m_2 + 1) = \begin{cases} 1, & \text{if } q = 2N + 5, \\ 0, & \text{if } q \neq 2N + 5. \end{cases} \quad (4.53)$$

By Lemma 3.4, (4.41) and (4.53), we obtain that

$$M_{2N+7} = \sum_{1 \leq k \leq 3} M_{2N+7}(k, 2m_k + 1) = M_{2N+7}(3, 2m_3 + 1). \quad (4.54)$$

Together with $M_{2N+7} \geq b_{2N+7} = 1$ by (2.8) and Theorem 2.5, we get

$$M_{2N+7}(3, 2m_3 + 1) \geq 1. \quad (4.55)$$

**Claim 2:** $M_q(3, 2m_3 + 1) = 0$ for $q \geq 2N + 11$.

If $M_{q_0}(3, 2m_3 + 1) \geq 1$ for some $q_0 \geq 2N + 11$, then by (4.55) and Lemma 3.5, we know that $i(c_3^{2m_3+1}) \leq 2N + 6$ and $i(c_3^{2m_3+1}) + \nu(c_3^{2m_3+1}) \geq q_0 + 1 \geq 2N + 12$, which, together with the fact
\( \nu(c_{3m+1}^3) \leq 6 \), implies \( i(c_{3m+1}^3) = 2N + 6 \) and \( \nu(c_{3m+1}^3) = 6 \) and \( q_0 \) only can be 2N + 11. Now by (3.9) and (2.23), we have \( i(c_3) = 6 \) and \( \nu(c_3) = 6 \), which implies that \( P_{c_3} \approx I_{10} \) by (3.1), (3.2) and the fact \( \nu(c_3) = p_{3,-} + 2p_{3,0} + p_{3,+} \). Then \( i(c_3) \) must be odd by Proposition 2.7 and the symplectic additivity of symplectic paths. This contradicts to \( i(c_3) = 6 \) and completes the proof of Claim 2.

In summary, by Lemma 3.4, (4.41), (4.53), Claim 2 and (4.43), there holds

\[
M_q = \sum_{1 \leq k \leq 3} M_q(k, 2m_k + 1) = M_q(3, 2m_3 + 1) \quad \text{for } 2N + 4 \leq q \leq 2N + 8, \ q \neq 2N + 5, \quad \text{(4.56)}
\]

\[
M_{2N+5} = \sum_{1 \leq k \leq 3} M_{2N+5}(k, 2m_k + 1) = 1 + M_{2N+5}(3, 2m_3 + 1), \quad \text{(4.57)}
\]

\[
M_{2N+9} = \sum_{1 \leq k \leq 3, \ 1 \leq m \leq 2} M_{2N+9}(k, 2m_k + m) = 1 + M_{2N+9}(3, 2m_3 + 1) + \sum_{2 \leq k \leq 3} M_{2N+9}(k, 2m_k + 2). \quad \text{(4.58)}
\]

\[
M_{2N+10} = \sum_{1 \leq k \leq 3, \ 1 \leq m \leq 2} M_{2N+10}(k, 2m_k + m) = M_{2N+10}(3, 2m_3 + 1) + \sum_{2 \leq k \leq 3} M_{2N+10}(k, 2m_k + 2). \quad \text{(4.59)}
\]

\[
M_q = \sum_{1 \leq k \leq 3, \ 1 \leq m \leq 2} M_q(k, 2m_k + m) = M_q(2, 2m_2 + 2) + M_q(3, 2m_3 + 2), \quad 2N + 11 \leq q \leq 2N + 14. \quad \text{(4.60)}
\]

Note that by (4.7) and \( i(c_2) = 5 \), we have the index iteration formula of \( c_2 \) as follows

\[
i(c_2^n) = 2m - 3 + 2 \sum_{3}^{3} \left( \frac{m \bmod 2}{2\pi} \right). \quad \text{(4.61)}
\]

It follows from (4.61) that \( i(c_2^2) \leq 13 \). Then by (3.3) and the fact that \( i(c_2^2) \) is odd, we get \( i(c_2^2) \in \{9, 11, 13\} \).

We continue the proof by distinguishing three values of \( i(c_2^2) \).

**Subcase 3.1.1: \( i(c_2^2) = 9 \).**

In this subcase, by (3.9), we have \( i(c_2^{2m_2+2}) = 2N + 9 \), then by (4.11), we have

\[
M_q(2, 2m_2 + 2) = \begin{cases} 1, & \text{if } q = 2N + 9, \\ 0, & \text{if } q \neq 2N + 9. \end{cases} \quad \text{(4.62)}
\]

It follows from (4.60) and (4.62) that

\[
M_{2N+11} = M_{2N+11}(3, 2m_3 + 2), \quad M_{2N+13} = M_{2N+13}(3, 2m_3 + 2), \quad \text{(4.63)}
\]

which, together with \( M_{2N+11} \geq b_{2N+11} = 1 \) and \( M_{2N+13} \geq b_{2N+13} = 1 \) by (2.8) and Theorem 2.5, implies

\[
M_{2N+11}(3, 2m_3 + 2) \geq 1, \quad M_{2N+13}(3, 2m_3 + 2) \geq 1. \quad \text{(4.64)}
\]

Thus the assumption with \( q = 12 \) in Lemma 4.1 is satisfied, and then we have

\[
i(c_3^m) = 6m - (p_{3,-} + p_{3,0}) - \frac{1 + \bmod 2}{2}(q_{3,0} + q_{3,+}), \quad \text{(4.65)}
\]

\[
\nu(c_3^m) = p_{3,-} + 2p_{3,0} + p_{3,+} + \frac{1 + \bmod 2}{2}(q_{3,-} + 2q_{3,0} + q_{3,+}). \quad \text{(4.66)}
\]
Then we can know that
\[ \hat{i}(c_3) = 6 \] (4.67)
and
\[ n(c_3) \in \{1, 2\}. \] (4.68)

By (3.2), we obtain
\[ i(c_3^2) + \nu(c_3^2) = 12 + p_{3,0} + p_{3,+} + q_{3,-} + q_{3,0} \leq 15, \]
which implies
\[ i(c_{3m+2}^2) + \nu(c_{3m+2}^2) \leq 2N + 15 \] (4.69)
by (3.9) and (2.23). Then by Lemma 3.5 and (4.64), there holds
\[ M_{2N+15}(3, 2m_3 + 2) = 0, \quad M_{2N+17}(3, 2m_3 + 2) = 0. \] (4.70)

It follows from Lemma 3.4, (4.41), (4.43), (4.53), Claim 2, (4.62) and (4.70) that
\[ M_{2N+15}(k, 2m_k + 3) \leq 1, \quad \forall \ k = 1, 2, 3. \] (4.71)

By (3.18) and Lemma 3.5, we have
\[ M_{2N+15}(k, 2m_k + 3) \leq 1, \quad \forall \ k = 1, 2, 3. \] (4.72)

Then by (4.71) we get \( M_{2N+15} \leq 3 \). We claim that \( M_{2N+15} \neq 3 \). In fact, if \( M_{2N+15} = 3 \), we have
\[ M_{2N+15}(k, 2m_k + 3) = 1, \quad k = 1, 2, 3. \]
Then by (3.18) and Lemma 3.5, there holds \( M_{2N+17}(k, 2m_k + 3) = 0, \quad k = 1, 2, 3, \) which, together with (4.72), implies \( M_{2N+17} = 0 \). This gives a contradiction
\[ 0 = M_{2N+17} \geq b_{2N+17} = 1 \] by (2.8) and Theorem 2.5. Hence \( M_{2N+15} \leq 2 \). However, again by (2.8) and Theorem 2.5, we have \( M_{2N+15} \geq b_{2N+15} = 2 \). So we get \( M_{2N+15} = 2 \).

In summary, in Case 3, we have
\[ M_{2N+3} = b_{2N+3}, \quad M_{2N+15} = b_{2N+15}. \] (4.73)

Combining (4.74), (2.8) and Theorem 2.5, we obtain that
\[ \sum_{q=2N+4}^{2N+14} (-1)^q M_q = \sum_{q=2N+4}^{2N+14} (-1)^q b_q = -6. \] (4.75)

By (4.56)-(4.60), (ii) of Case 3, Claim 2, (4.62) and (4.70), we have
\[ \sum_{q=2N+4}^{2N+14} (-1)^q M_q = -2 + \sum_{q=2N+4}^{2N+10} (-1)^q M_q(3, 2m_3 + 1) + \sum_{2N+14}^{2N+15} (-1)^q \sum_{2k \leq 3} M_q(k, 2m_k + 2) \\
= -3 + \sum_{q=2N+3}^{2N+15} (-1)^q M_q(3, 2m_3 + 1) + \sum_{2N+9}^{2N+15} (-1)^q M_q(3, 2m_3 + 2). \] (4.76)
Note that $2N + 3 \leq i(c_3^{2m+1}) \leq i(c_3^{2m+1}) + \nu(c_3^{2m+1}) \leq 2N + 15$ and $2N + 9 \leq i(c_3^{2m+2}) \leq i(c_3^{2m+2}) + \nu(c_3^{2m+2}) \leq 2N + 15$ by (3.18), (3.5) and (4.69), then according to Lemma 2.1 and (2.4), we obtain

$$
\sum_{q=2N+3}^{2N+15} (-1)^q M_q(3, 2m_3 + 1) + \sum_{2N+9}^{2N+15} (-1)^q M_q(3, 2m_3 + 2)
\leq \sum_{0 \leq l \leq 6} (-1)^i(c_3^{2m+1}) + \sum_{0 \leq l \leq 6} (-1)^i(c_3^{2m+2}) \leq 3.
\tag{4.77}
$$

Combining (4.75), (4.76) and (4.77), by (2.6), we get

$$
\chi(c_3^{2m+1}) + \chi(c_3^{2m+2}) = \sum_{0 \leq l \leq 6} (-1)^i(c_3^{2m+1}) + \sum_{0 \leq l \leq 6} (-1)^i(c_3^{2m+2}) = -3.
\tag{4.78}
$$

Note that by (4.68) and (iv) of Lemma 2.2, there holds

$$
k_j^{c_3^{2m+1}}(c_3^{2m+1}) = k_j^{c_3^{2m+2}}(c_3^{2m+2}) = k_j^{c_3^{2m+1}}(c_3^{2m+2}), \quad \forall \ 0 \leq j \leq 6.
\tag{4.79}
$$

Then by (2.6) it yields

$$
\chi(c_3) = \chi(c_3^{2m+1}), \quad \chi(c_3^2) = \chi(c_3^{2m+2}),
\tag{4.80}
$$

which, together with (4.78) and (4.68), implies

$$
\chi(c_3) \neq \chi(c_3^2)
\tag{4.81}
$$

since $\chi(c_3^m) \in \mathbb{Z}$, and

$$
n(c_3) = 2.
\tag{4.82}
$$

It follows from (2.6), (4.78), (4.80) and (4.82) that

$$
\frac{\dot{\chi}(c_3)}{\dot{i}(c_3)} = \frac{1}{2} \left( \chi(c_3) + \chi(c_3^2) \right) = \frac{1}{2} \left( \chi(c_3^{2m+1}) + \chi(c_3^{2m+2}) \right) = -\frac{3}{2}.
\tag{4.83}
$$

Note that $i(c_k) > 5$, $k = 1, 2$ by (3.4), so it follows from (4.10), (4.67) and (4.83) that

$$
\frac{\dot{\chi}(c_1)}{i(c_1)} = -\frac{1}{i(c_1)} > -\frac{1}{5}, \quad \frac{\dot{\chi}(c_2)}{i(c_2)} = -\frac{1}{i(c_2)} > -\frac{1}{5}, \quad \frac{\dot{\chi}(c_3)}{i(c_3)} = -\frac{1}{4}.
\tag{4.84}
$$

which, together with (2.5), yields

$$
-\frac{2}{3} = \frac{\dot{\chi}(c_1)}{i(c_1)} + \frac{\dot{\chi}(c_2)}{i(c_2)} + \frac{\dot{\chi}(c_3)}{i(c_3)} > -\frac{13}{20}.
\tag{4.85}
$$

This is a contradiction.

**Subcase 3.1.2:** $i(c_3^2) = 11$. 

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In this subcase, by (3.9), it yields \(i(e_2^{2m_2+2}) = 2N + 11\) and then by (4.11), we have
\[
M_q(2, 2m_2 + 2) = \begin{cases} 
1, & \text{if } q = 2N + 11, \\
0, & \text{if } q \neq 2N + 11,
\end{cases}
\]  
which, together with (4.60), implies
\[
M_{2N+13} = M_{2N+13}(3, 2m_3 + 2). \tag{4.87}
\]
However by (2.8) and Theorem 2.5, we have \(M_{2N+13} \geq b_{2N+13} = 1\), so we have
\[
M_{2N+13}(3, 2m_3 + 2) \geq 1. \tag{4.88}
\]
Thus by (3.18) and Lemma 3.5, there holds
\[
M_{2N+9}(3, 2m_3 + 2) = 0. \tag{4.89}
\]
Combining (4.58), (4.86) and (4.89), we obtain that
\[
M_{2N+9} = 1 + M_{2N+9}(3, 2m_3 + 1). \tag{4.90}
\]
Since \(M_{2N+9} \geq b_{2N+9} = 2\) by (2.8) and Theorem 2.5, by (4.90), we know that
\[
M_{2N+9}(3, 2m_3 + 1) \geq 1. \tag{4.91}
\]
Noticing that we also have \(M_{2N+7}(3, 2m_3 + 1) \geq 1\) by (4.55), then by Lemma 3.6, we obtain that
\[
i(c_3^{m}) = 8m - p_{3,-} - p_{3,0}. \tag{4.92}
\]
and
\[
n(c_3) = 1. \tag{4.93}
\]
Similar to (3.72), we have
\[
M_{2N+5}(3, 2m_3 + 1) = k^{i(c_3^{2m_3+1})}_{2N+5-i(c_3^{2m_3+1})}(c_3^{2m_3+1}) \]
\[
= k^{i(c_3^{2m_3+2})}_{2N+5-i(c_3^{2m_3+1})}(c_3^{2m_3+2}) \]
\[
= M_{2N+5-i(c_3^{2m_3+1})+i(c_3^{2m_3+2})}(3, 2m_3 + 2) \]
\[
= M_{2N+13}(3, 2m_3 + 2). \tag{4.94}
\]
On the other hand, by (4.92) and (3.2), it yields \(i(c_3) \geq 5\), then by (3.9), we have \(i(e_3^{2m_3+1}) \geq 2N + 5\). So, by (4.91) and Lemma 3.5, there holds
\[
M_{2N+5}(3, 2m_3 + 1) = 0, \tag{4.95}
\]
which, together with (4.94), contradicts to (4.88).
Subcase 3.1.3: $i(c_2^2) = 13$.

In this subcase, by (3.9), it yields $i(c_2^{2m_2+2}) = 2N + 13$ and then by (4.11), we have

$$M_q(2, 2m_2 + 2) = \begin{cases} 
1, & \text{if } q = 2N + 13, \\
0, & \text{if } q \neq 2N + 13,
\end{cases}$$

(4.96)

which, together with (4.60), implies

$$M_{2N+11} = M_{2N+11}(3, 2m_3 + 2).$$

(4.97)

However by (2.8) and Theorem 2.5, we have $M_{2N+11} \geq b_{2N+11} = 1$, so we have

$$M_{2N+11}(3, 2m_3 + 2) \geq 1.$$  

(4.98)

Thus by (3.18) and Lemma 3.5, there holds

$$M_{2N+9}(3, 2m_3 + 2) = 0.$$  

(4.99)

Similar to Subcase 3.1.2, we have

$$i(c_3^m) = 8m - p_{3,-} - p_{3,0},$$  

(4.100)

and we obtain a contradiction

$$0 = M_{2N+3}(3, 2m_3 + 1) = M_{2N+11}(3, 2m_3 + 2) \geq 1.$$  

(4.101)

Subcase 3.2: $i(c_2) = 7$.

In this subcase, by (3.9), it yields $i(c_2^{2m_2+1}) = 2N + 7$ and then by (4.11), we have

$$M_q(2, 2m_2 + 1) = \begin{cases} 
1, & \text{if } q = 2N + 7, \\
0, & \text{if } q \neq 2N + 7,
\end{cases}$$  

(4.102)

which, together with (4.41) and Lemma 3.4, implies

$$M_{2N+5} = \sum_{1 \leq k \leq 3} M_{2N+5}(k, 2m_k + 1) = M_{2N+5}(3, 2m_3 + 1).$$  

(4.103)

However by (2.8) and Theorem 2.5, we have $M_{2N+5} \geq b_{2N+5} = 1$, so we have

$$M_{2N+5}(3, 2m_3 + 1) \geq 1.$$  

(4.104)

Claim 3: $M_q(3, 2m_3 + 1) = 0, \forall q \geq 2N + 9.$

If $M_{q_0}(3, 2m_3 + 1) \geq 1$ for some $q_0 \geq 2N + 9$, then by (4.104) and Lemma 3.5, we know that $i(c_3^{2m_3+1}) \leq 2N + 4$ and $i(c_3^{2m_3+1}) + \nu(c_3^{2m_3+1}) \geq q_0 + 1 \geq 2N + 10$, which, together with the fact $\nu(c_3^{2m_3+1}) \leq 6$, implies $i(c_3^{2m_3+1}) = 2N + 4$ and $\nu(c_3^{2m_3+1}) = 6$ and $q_0$ only can be $2N + 9$. Now by (3.9) and (2.23), we have $i(c_3) = 4$ and $\nu(c_3) = 6$, which implies that $P_{c_3} \approx I_6$ by (3.1), (3.2) and
the fact $\nu(c_3) = p_{3,-} + 2p_{3,0} + p_{3,+}$. So $i(c_3)$ must be odd by Proposition 2.7 and the symplectic additivity of symplectic paths. This contradicts to $i(c_3) = 4$ and completes the proof of Claim 3.

Since $i(c_2) = 7$ in this subcase, similar to the proof of (4.35), we have

$$M_{2N+9}(2,2m_2 + 2) = 0. \quad (4.105)$$

Then by Lemma 3.4, (4.41), (4.102), Claim 3, (4.43) and (4.105), we obtain

$$M_{2N+9} = \sum_{1 \leq k \leq 3, 1 \leq m \leq 2} M_{2N+9}(k,2m_k + m) = 1 + M_{2N+9}(3,2m_3 + 2), \quad (4.106)$$

which, together with $M_{2N+9} \geq b_{2N+9} = 2$ by (2.8) and Theorem 2.5, implies that

$$M_{2N+9}(3,2m_3 + 2) \geq 1. \quad (4.107)$$

Then by (3.18) and Lemma 3.5 we get

$$M_q(3,2m_3 + 2) = 0, \quad \forall q \neq 2N + 9. \quad (4.108)$$

By Lemma 3.4, (4.41), (4.102), Claim 3, (4.43) and (4.108), we obtain

$$M_q = \sum_{1 \leq k \leq 3, 1 \leq m \leq 2} M_q(k,2m_k + m) = M_q(2,2m_2 + 2), \quad \text{for } q = 2N + 11, 2N + 13. \quad (4.109)$$

So there holds $M_{2N+11} = 0$ or $M_{2N+13} = 0$ by (4.11), which contradicts to $M_{2N+11} \geq b_{2N+11} = 1$ and $M_{2N+13} \geq b_{2N+13} = 1$ by (2.8) and Theorem 2.5.

**Subcase 3.3: $i(c_2) = 9$.**

In this subcase, by (3.9), it yields $i(c_{2m_2+1}) = 2N + 9$ and then by (4.11), we have

$$M_q(2,2m_2 + 1) = \begin{cases} 1, & \text{if } q = 2N + 9, \\ 0, & \text{if } q \neq 2N + 9, \end{cases} \quad (4.110)$$

which, together with (4.41) and Lemma 3.4, implies

$$M_q = \sum_{1 \leq k \leq 3} M_q(k,2m_k + 1) = M_q(3,2m_3 + 1), \quad \text{for } q = 2N + 5, 2N + 7. \quad (4.111)$$

However by (2.8) and Theorem 2.5, we have $M_{2N+5} \geq b_{2N+5} = 1$ and $M_{2N+7} \geq b_{2N+7} = 1$, so we have $M_{2N+5}(3,2m_3 + 1) \geq 1$ and $M_{2N+7}(3,2m_3 + 1) \geq 1$. Thus the assumption with $q = 6$ of Lemma 3.6 is satisfied, and then by Lemma 3.6 we conclude that $c_3$ must belong to one of the classes in Theorem 4.2 and the index of $c_3$ is the following

$$i(c_3) = 6 - p_{3,-} - p_{3,0}. \quad (4.112)$$

This completes the proof of Theorem 4.2.

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