THE CENTERS OF IWAHORI-HECKE ALGEBRAS ARE FILTERED

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ABSTRACT. We show that the center of the Iwahori–Hecke algebra of the symmetric group \( S_n \) carries a natural filtered algebra structure, and that the structure constants of the associated graded algebra are independent of \( n \). A series of conjectures and open problems are also included.

1. INTRODUCTION

1.1. The main results. The class elements introduced by Geck-Rouquier \[GR\] form a basis for the center \( Z(\mathcal{H}_n) \) of the Iwahori-Hecke algebra \( \mathcal{H}_n \) of type \( A \) over the ring \( \mathbb{Z}[\xi] \), where the indeterminate \( \xi \) is related to the familiar one \( q \) by \( \xi = q - q^{-1} \). In this paper, we shall parameterize these class elements \( \Gamma_\lambda(n) \) by partitions \( \lambda \) satisfying \( |\lambda| + \ell(\lambda) \leq n \) which are the so-called modified cycle types, just as Macdonald \[Mac\] pp.131 does for the usual class sums of the symmetric group \( S_n \). Write the multiplication in \( Z(\mathcal{H}_n) \) as

\[
\Gamma_\lambda(n) \Gamma_\mu(n) = \sum_\nu k_{\lambda\mu}^\nu(n) \Gamma_\nu(n). \tag{1.1}
\]

The main result of this Note is the following theorem on these structure constants \( k_{\lambda\mu}^\nu(n) \).

**Theorem 1.1.**

(1) For any \( n \), \( k_{\lambda\mu}^\nu(n) \) is a polynomial in \( \xi \) with non-negative integral coefficients. Moreover, \( k_{\lambda\mu}^\nu(n) \) is an even (resp., odd) polynomial in \( \xi \) if and only if \( |\lambda| + |\mu| - |\nu| \) is even (resp., odd).

(2) We have \( k_{\lambda\mu}^\nu(n) = 0 \) unless \( |\nu| \leq |\lambda| + |\mu| \).

(3) If \( |\nu| = |\lambda| + |\mu| \), then \( k_{\lambda\mu}^\nu(n) \) is independent of \( n \).

It follows from (2) and (3) that the center \( Z(\mathcal{H}_n) \) is naturally a filtered algebra and the structure constants of the associated graded algebra are independent of \( n \). We in addition formulate several conjectures, including Conjecture 3.1 which simply states that \( k_{\lambda\mu}^\nu(n) \) are polynomials in \( n \), and further implications on the algebra generators of \( Z(\mathcal{H}_n) \).

1.2. Motivations and connections. Our motivation comes from the original work of Farahat-Higman \[FH\] on the structures of the centers of the integral symmetric group algebras \( \mathbb{Z}S_n \). Indeed, Theorem 1.1 specializes at \( \xi = 0 \) to some classical results of loc. cit.. In addition, Conjecture 3.1 in the specialization at \( \xi = 0 \) (which is a theorem in loc. cit.) when combined with the specialization of Theorem 1.1 allowed them to define a universal algebra \( K \) governing the structures of the centers \( Z(\mathbb{Z}S_n) \) for all \( n \) simultaneously. Farahat-Higman further developed

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this approach to establish a distinguished set of generators for $Z(\mathbb{Z}S_n)$, which is now identified as the first $n$ elementary symmetric polynomials in the Jucys-Murphy elements. This has applications to blocks of modular representations of $S_n$.

This Note arose from the hope that the results of Farahat–Higman might be generalized to the Iwahori-Hecke algebra setup and in particular it would provide a new conceptual proof of the Dipper-James conjecture. Recently, built on the earlier work of Mathas [14], the first author and Graham [FG] obtained a first positive step along the new line.

In another direction, the results of Farahat-Higman have been partially generalized by the second author [W] to the centers of the group algebras of wreath products $G \wr S_n$ for an arbitrary finite group $G$ (e.g. a cyclic group $\mathbb{Z}_r$), and these centers are closely related to the cohomology ring structures of Hilbert schemes of points on the minimal resolutions. It will be interesting to develop the Farahat-Higman type results for the centers of the Iwahori-Hecke algebra of type $B$ or more generally of the cyclotomic Hecke algebras which are $q$-deformation of the group algebra $\mathbb{Z}(\mathbb{Z}_r \wr S_n)$.

1.3. This Note is organized as follows. In Section 2, we prove the three parts of our main Theorem 1.1 in Propositions 2.2, 2.4 and 2.6 respectively. In Section 3, we formulate several conjectures, which are Iwahori-Hecke algebra analogues of some results of Farahat-Higman. We conclude this Note with a list of open questions.

2. Proof of the main theorem

2.1. The preliminaries. Let $S_n$ be the symmetric group in $n$ letters generated by the simple transpositions $s_i = (i, i+1), i = 1, \ldots, n-1$.

Let $\xi$ be an indeterminate. The Iwahori-Hecke algebra $\mathcal{H}_n$ is the unital $\mathbb{Z}[\xi]$-algebra generated by $T_i$ for $i = 1, \ldots, n-1$, satisfying the relations
\begin{align}
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \\
T_i T_j &= T_j T_i, \quad |i-j| > 1 \\
T_i^2 &= 1 + \xi T_i.
\end{align}

The order relation (2.1) comes from the more familiar $(T_i - q)(T_i + q^{-1}) = 0$ via the identification $\xi = q - q^{-1}$. If $w = s_{i_1} \cdots s_{i_r} \in S_n$ is a reduced expression (where $r$ will be referred to as the length of $w$ in this case), then define $T_w := T_{i_1} \cdots T_{i_r}$.

It is well known that the Iwahori-Hecke algebra is a free $\mathbb{Z}[\xi]$-module with basis $\{T_w \mid w \in S_n\}$, and it is a deformation of the integral group algebra $\mathbb{Z}S_n$.

The Jucys-Murphy elements $L_i$ ($1 \leq i \leq n$) of $\mathcal{H}_n$ are defined to be $L_1 = 0$ and
\begin{align}
L_i &= \sum_{1 \leq k < i} T_{(k,i)}.
\end{align}

for $i \geq 2$.

Given $w \in S_n$ with cycle-type $\rho = (\rho_1, \ldots, \rho_t, 1, \ldots, 1)$ for $\rho_t > 1$, we define the modified cycle-type of $w$ to be $\tilde{\rho} = (\rho_1 - 1, \ldots, \rho_t - 1)$, following Macdonald [Mac pp.131]. Given a partition $\lambda$, let $C_\lambda(n)$ denote the conjugacy class of $S_n$ containing
all elements of modified type $\lambda$ if $|\lambda| + \ell(\lambda) \leq n$. Accordingly, let $c\lambda(n)$ denote the class sum of $C\lambda(n)$ if $|\lambda| + \ell(\lambda) \leq n$, and denote $c\lambda(n) = 0$ otherwise.

The center $Z(H_n)$ of the Iwahori-Hecke algebra is free over $\mathbb{Z}[\xi]$ of rank equal to the number of partitions of $n$; that is, it has a basis indexed by the conjugacy classes of $S_n$ (see [GR]). In this paper, we shall parameterize these Geck-Rouquier class elements $\Gamma\lambda(n)$ by the modified cycle types $\lambda$. The elements $\Gamma\lambda(n)$ for $|\lambda| + \ell(\lambda) \leq n$ are characterized by the following two properties among the central elements of $H_n$ [Fra]:

(i) The $\Gamma\lambda(n)$ specializes at $\xi = 0$ to the class sum $c\lambda(n)$;

(ii) The difference $\Gamma\lambda(n) - \sum_{w \in C\lambda(n)} T_w$ contains no minimal length elements of any conjugacy class.

In addition, we set $\Gamma\lambda(n) = 0$ if $|\lambda| + \ell(\lambda) > n$.

2.2. The structure constants as positive integral polynomials. By inspection of the defining relations, $H_n$ as a $\mathbb{Z}$-algebra is $\mathbb{Z}_2$-graded by declaring that $\xi$ and $T_i$ ($1 \leq i \leq n - 1$) have $\mathbb{Z}_2$-degree (or parity) 1 and each integer has $\mathbb{Z}_2$-degree 0.

Lemma 2.1. Every $\Gamma\lambda(n)$ is homogeneous in the above $\mathbb{Z}_2$-grading with $\mathbb{Z}_2$-degree equal to $|\lambda|$ mod 2.

Proof. There is a constructive algorithm [Fra] pp.14] for producing the elements $\Gamma\lambda(n)$. This finite algorithm begins with the sum of $T_w$ with minimal length elements $w$ from the conjugacy class $C\lambda(n)$, then at each repeat of this algorithm, the only additions are of form (i) $T_w \rightarrow T_w + T_{s_iw} s_i$, (ii) $T_{s_iw} \rightarrow T_{s_iw} + T_w + \xi T_{s_iw}$, or (iii) $T_{w_1} \rightarrow T_{w_1} + T_{w_2} + \xi T_{w_1}$. The algorithm eventually ends up with the element $\Gamma\lambda(n)$. Each of the three type of additions clearly preserves the $\mathbb{Z}_2$-degree. As the minimal length elements have the same parity as $|\lambda|$, this proves the lemma.

Denote by $\mathbb{N}$ the set of non-negative integers.

Proposition 2.2. For any given $n$, $k_{\lambda\mu}^\nu(n)$ is a polynomial in $\xi$ with non-negative integral coefficients. Moreover, $k_{\lambda\mu}^\nu(n)$ is an even (respectively, odd) polynomial in $\xi$ if and only if $|\lambda| + |\mu| - |\nu|$ is even (respectively, odd).

Proof. As is seen in the proof of Lemma 2.1, the class elements are in the positive cone $Z(H_n) = Z(H_n) \cap \sum_{w \in S_n} \mathbb{N}[\xi] T_w$. Because of the positive coefficients in the order relation (2.1), the positive cone is closed under additions and products. Since the class elements contain minimal length elements from exactly one conjugacy class and contains those minimal length elements with coefficient 1 [Fra] (see Sect. 2.1 above), the coefficient of a class element in a central element $C$ is precisely the coefficient of the corresponding minimal length element in the $T_w$ expansion of $C$. This shows that $k_{\lambda\mu}^\nu(n) \in \mathbb{N}[\xi]$.

The more refined statement on when $k_{\lambda\mu}^\nu(n)$ is an even or odd polynomial in $\xi$ follows now from Lemma 2.1.

2.3. The filtered algebra structure on the center. Let $m_{\mu}(n)$ be the monomial symmetric polynomial in the (commutative) Jucys-Murphy elements $L_1, \ldots, L_n$, parameterized by a partition $\mu$. It is known that $m_{\mu}(n) \in Z(H_n)$. Some relations between the class elements and the monomial symmetric polynomials in Jucys–Murphy elements are summarized as follows (see [Mat, FrG, FrJ]).
Lemma 2.3. \(\lambda\) For any partition \(\lambda\), we can express \(m_\lambda(n)\) in terms of the class elements \(\Gamma_\mu(n)\) as
\[
m_\lambda(n) = \sum_{|\mu| \leq |\lambda|} b_{\lambda\mu}(n) \Gamma_\mu(n) \quad \text{for} b_{\lambda\mu}(n) \in \mathbb{Z}[\xi].
\]
(We set \(b_{\lambda\mu}(n) = 0\) if \(|\mu| + \ell(\mu) > n\), or equivalently if \(\Gamma_\mu(n) = 0\).)

(2) Proof. For any \(\lambda\), the coefficients \(b_{\lambda\mu}(n)\) with \(|\mu| = |\lambda|\) are independent of \(n\).

(3) Let \(\lambda\) be a partition with \(|\lambda| + \ell(\lambda) \leq n\). Then, each \(\Gamma_\lambda(n)\) is equal to \(m_\lambda(n)\) plus a \(\mathbb{Z}[\xi]\)-linear combination of \(m_\mu(n)\) with \(|\mu| < |\lambda|\).

Proof. Part (1) is a consequence of \cite[Theorems 2.7 and 2.26]{Mat}.

By \cite[Lemma 5.2]{FrG}, the coefficient of \(T_w\) (for an increasing \(w \in S_n\) of the right length) in a so-called quasi-symmetric polynomial in Jucys–Murphy elements is independent of \(n\). Monomial symmetric polynomials are just sums of the corresponding quasi-symmetric polynomials, independently of \(n\). This proves (2).

Part (3) can be read off from the proof of \cite[Theorem 4.1]{FrJ}. \(\square\)

Proposition 2.4. We have \(k^\nu_\lambda(n) = 0\) unless \(|\nu| \leq |\lambda| + |\mu|\).

Proof. By Lemma 2.3 (3), the product \(\Gamma_\lambda(n) \Gamma_\mu(n)\) is equal to \(m_\lambda(n) m_\mu(n)\) plus a linear combination of products \(m_\lambda(n) m_{\mu'}(n)\) satisfying \(|\lambda'| + |\mu'| < |\lambda| + |\mu|\). A product of monomial symmetric polynomials \(m_\alpha(n) m_{\beta}(n)\) is a sum of monomial symmetric polynomials \(m_\gamma(n)\) satisfying \(|\alpha| + |\beta| = |\gamma|\). Consequently, \(\Gamma_\lambda(n) \Gamma_\mu(n)\) is a sum of \(m_\gamma(n)\) with partitions \(\gamma\) of size at most \(|\lambda| + |\mu|\). The proposition follows now by Lemma 2.3 (1).

Assign degree \(|\lambda|\) to the basis element \(\Gamma_\lambda(n)\) and let \(Z(\mathcal{H}_n)_m\) be the \(\mathbb{Z}[\xi]\)-span of \(\Gamma_\lambda(n)\) of degree at most \(m\), for \(m \geq 0\). Then, Proposition 2.4 provides a filtered algebra structure on the center \(Z(\mathcal{H}_n) = \bigcup_m Z(\mathcal{H}_n)_m\).

Remark 2.5. Denote by \(A_n\) the (commutative) \(\mathbb{Z}[\xi]\)-subalgebra of \(\mathcal{H}_n\) generated by the Jucys–Murphy elements \(L_1, \ldots, L_n\). The algebra \(A_n\) is filtered by the subspaces \(A_n^{(m)}\) \((m \geq 0)\) spanned by all products \(L_{i_1} \cdots L_{i_m}\) of \(m\) Jucys–Murphy elements. The filtrations on \(A_n\) and \(Z(\mathcal{H}_n)\) are compatible with each other by the inclusion \(Z(\mathcal{H}_n) \subset A_n\). However the algebra \(\mathcal{H}_n\) does not seem to admit a natural filtration which is compatible with the one on \(A_n\) by inclusion \(A_n \subset \mathcal{H}_n\). This is very different from the symmetric group algebra \(\mathbb{Z}S_n\), which admits such a filtration by assigning degree 1 to every transposition \((i, j)\).

2.4. The graded algebra \(\text{gr}Z(\mathcal{H}_n)\).

Proposition 2.6. If \(|\nu| = |\lambda| + |\mu|\), then \(k^\nu_\lambda(n)\) is independent of \(n\).

(In this case, we shall write \(k^\nu_\lambda(n)\) as \(k^\nu_{\mu\lambda}(n)\).)

Proof. By the definition of \(k^\nu_\lambda(n)\), we can assume without loss of generality that \(|\lambda| + \ell(\lambda) \leq n\) and \(|\mu| + \ell(\mu) \leq n\).

By Lemma 2.3 (3), \(\Gamma_\lambda(n) \Gamma_\mu(n) = m_\lambda(n) m_\mu(n) + X\), where \(X\) is a linear combination of products of monomials whose combined partition size is less than \(|\lambda| + |\mu|\). The monomials appearing in \(X\) correspond to partitions of size less than \(|\nu| = |\lambda| + |\mu|\), and thus will not contribute to \(k^\nu_\lambda(n)\) by Lemma 2.3 (1). The product \(m_\lambda(n) m_\mu(n)\) is a sum of monomials \(m_\alpha(n)\) satisfying \(|\alpha| = |\lambda| + |\mu|\) with coefficients independent of \(n\); the contribution of each such \(m_\alpha(n)\) to \(k^\nu_{\mu\lambda}(n)\)
is independent of \( n \) by Lemma 2.3 (2). Summing all these contributions produces \( k_{\lambda \mu}^n(n) \) which is independent of \( n \).

Proposition 2.6 is equivalent to the statement that all the structure constants of the graded algebra \( \text{gr} \mathcal{Z}(\mathcal{H}_n) \) associated to the filtered algebra \( \mathcal{Z}(\mathcal{H}_n) \) are independent of \( n \).

2.5. Examples. We provide some explicit calculations of the structure constants \( k_{\lambda \mu}^n(n) \) for the multiplication between \( \Gamma_{\lambda}(n) \), with \( n = 3, 4, 5 \). For the sake of notational simplicity, we will write \( \Gamma_{\lambda}(n) \) as \( \Gamma_{\lambda} \) with \( n \) dropped in the following examples. We also drop parentheses in the subscripts of class elements, denoting \( \Gamma_{(\lambda_1, \ldots, \lambda_k)} \) by \( \Gamma_{\lambda_1, \ldots, \lambda_k} \). The square brackets denote the top-degree parts of each product. The compatibility of these examples with Theorem 1.1 is manifest.

(1) Let \( n = 3 \). In \( \mathcal{Z}(\mathcal{H}_3) \), we have
\[
\Gamma_1 \Gamma_1 = [(\xi^2 + 3)\Gamma_2] + 2\xi \Gamma_1 + 3\Gamma_0.
\]

(2) Let \( n = 4 \). In \( \mathcal{Z}(\mathcal{H}_4) \), we have
\[
\begin{align*}
\Gamma_1 \Gamma_1 &= [(\xi^2 + 3)\Gamma_2 + (\xi^2 + 2)\Gamma_1,1] + 3\xi \Gamma_1 + 6\Gamma_0, \\
\Gamma_1 \Gamma_2 &= [(\xi^4 + 4\xi^2 + 4)\Gamma_3] + (2\xi^3 + 6\xi)\Gamma_2 + (2\xi^3 + 4\xi)\Gamma_{1,1} \\
&+ (3\xi^2 + 4)\Gamma_1 + 4\xi \Gamma_0, \\
\Gamma_1 \Gamma_{1,1} &= [(\xi^2 + 2)\Gamma_3] + 2\xi \Gamma_2 + 4\xi \Gamma_{1,1} + \Gamma_1.
\end{align*}
\]

(3) Let \( n = 5 \). In \( \mathcal{Z}(\mathcal{H}_5) \), we have
\[
\begin{align*}
\Gamma_1 \Gamma_1 &= [(\xi^2 + 3)\Gamma_2 + (\xi^2 + 2)\Gamma_1,1] + 4\xi \Gamma_1 + 10\Gamma_0, \\
\Gamma_1 \Gamma_2 &= [(\xi^4 + 4\xi^2 + 4)\Gamma_3 + (\xi^4 + 2\xi^2 + 1)\Gamma_{2,1}] \\
&+ (3\xi^3 + 8\xi)\Gamma_2 + (3\xi^3 + 4\xi)\Gamma_{1,1} + (6\xi^2 + 6)\Gamma_1 + 10\xi \Gamma_0, \\
\Gamma_1 \Gamma_{1,1} &= [(\xi^2 + 2)\Gamma_3] + (2\xi^2 + 3)\Gamma_{2,1} + 2\xi \Gamma_2 + 4\xi \Gamma_{1,1} + 3\Gamma_1, \\
\Gamma_1 \Gamma_3 &= [(\xi^6 + 6\xi^4 + 10\xi^2 + 5)\Gamma_4] \\
&+ (2\xi^5 + 10\xi^3 + 13\xi)\Gamma_3 + (2\xi^5 + 8\xi^3 + 7\xi)\Gamma_{2,1} \\
&+ (3\xi^4 + 10\xi^2 + 6)\Gamma_2 + (3\xi^4 + 8\xi^2 + 4)\Gamma_{1,1} + (4\xi^3 + 6\xi)\Gamma_1 + 5\xi \Gamma_0, \\
\Gamma_2 \Gamma_2 &= [(\xi^6 + 7\xi^5 + 16\xi^4 + 15\xi^2 + 5)\Gamma_4] \\
&+ (2\xi^7 + 14\xi^5 + 29\xi^3 + 19\xi)\Gamma_3 + (2\xi^7 + 13\xi^5 + 22\xi^3 + 11\xi)\Gamma_{2,1} \\
&+ (3\xi^6 + 20\xi^4 + 32\xi^2 + 7)\Gamma_2 + (3\xi^6 + 19\xi^4 + 26\xi^2 + 8)\Gamma_{1,1} \\
&+ (4\xi^5 + 25\xi^3 + 27\xi)\Gamma_1 + (5\xi^4 + 30\xi^2 + 20)\Gamma_0, \\
\Gamma_2 \Gamma_{1,1} &= [(\xi^6 + 6\xi^4 + 10\xi^2 + 5)\Gamma_4] \\
&+ (2\xi^5 + 10\xi^3 + 11\xi)\Gamma_3 + (2\xi^5 + 9\xi^3 + 9\xi)\Gamma_{2,1} \\
&+ (3\xi^4 + 11\xi^2 + 6)\Gamma_2 + (3\xi^4 + 9\xi^2 + 4)\Gamma_{1,1} + (4\xi^3 + 7\xi)\Gamma_1 + 5\xi \Gamma_0.
\end{align*}
\]
2.6. A universal graded algebra. Introduce a graded $\mathbb{Z}[\xi]$-algebra $\mathcal{G}$ with a basis given by the symbols $\Gamma_\lambda$, where $\lambda$ runs over all partitions, and with multiplication given by
\[
\Gamma_\lambda \Gamma_\mu = \sum_{|\nu| = |\lambda|+|\mu|} k_{\lambda \mu}^\nu \Gamma_\nu.
\]
By Propositions 2.2 and 2.6 the structure constants $k_{\lambda \mu}^\nu$ are independent of $n$ and actually lie in $\mathbb{N}[\xi^2]$. Furthermore, we have surjective homomorphisms $\mathcal{G} \to \text{gr}\mathcal{Z}[\mathcal{H}_n]$ for all $n$, which send each $\Gamma_\lambda$ to $\Gamma_\lambda(n)$. The following proposition is immediate.

**Proposition 2.7.** The $\mathbb{Z}[\xi]$-algebra $\mathcal{G}$ is commutative and associative.

Below for the one-row partition $(m)$, we shall write $\Gamma_{(m)}$ simply as $\Gamma_m$.

**Theorem 2.8.** The $\mathbb{Q}(\xi)$-algebra $\mathbb{Q}(\xi) \otimes_{\mathbb{Z}[\xi]} \mathcal{G}$ is a polynomial algebra with generators $\Gamma_m$, $m = 1, 2, \ldots$.

**Proof.** Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, the product is of the form
\[
\Gamma_{\lambda_1} \Gamma_{\lambda_2} \cdots = \sum_{\mu} d_{\lambda \mu}(\xi) \Gamma_\mu
\]
for $d_{\lambda \mu}(\xi) \in \mathbb{N}[\xi]$. As $\xi$ goes to 0, $\Gamma_\mu$ goes to the class sum $c_\mu$, and $d_{\lambda \mu}(\xi)$ specifies to the structure constant $d_{\lambda \mu}$ as defined in [Mac] pp.132 which we recall:
\[
c_{\lambda_1} c_{\lambda_2} \cdots = \sum_{\mu} d_{\lambda \mu} c_\mu.
\]
It is known therein that the (integral) matrix $[d_{\lambda \mu}]$ for $|\lambda| = |\mu| = k$ with any $k$ is triangular with respect to the dominance ordering of partitions and all its diagonal entries are nonzero, thus the matrix $[d_{\lambda \mu}]$ is invertible over $\mathbb{Q}$. This forces the matrix $[d_{\lambda \mu}(\xi)]$ invertible over the field $\mathbb{Q}(\xi)$. Thus each $\Gamma_\mu$ is generated by $\Gamma_1, \Gamma_2, \ldots$ over $\mathbb{Q}(\xi)$. By definition, the elements $\Gamma_\mu$ for all partitions $\mu$ are linearly independent. Thus the theorem follows by comparing the graded dimensions of the algebra $\mathbb{Q}(\xi) \otimes_{\mathbb{Z}[\xi]} \mathcal{G}$ and the polynomial algebra in $\Gamma_m$, $m = 1, 2, \ldots$. 

It is not clear though if the matrix $[d_{\lambda \mu}(\xi)]$ remains triangular. A similar type of phenomenon with a negative answer appears in the example of $M_3$ in Mathas [Mat] p.310.

3. Conjectures and discussions

3.1. Several conjectures. We expect the following conjecture to hold.

**Conjecture 3.1.** Given partitions $\lambda, \mu$ and $\nu$, there exists a polynomial $f_{\lambda \mu}^\nu$ in one variable with coefficients in $\mathbb{Q}[\xi]$, such that $f_{\lambda \mu}^\nu(n) = k_{\lambda \mu}^\nu(n)$ for all $n$.

Recall $b_{\lambda \mu}(n)$ from Lemma 2.3 [FH]. Similarly, we conjecture that there exists polynomials $g_{\lambda \mu}(x)$ with coefficients in $\mathbb{Q}[\xi]$, such that $g_{\lambda \mu}(n) = b_{\lambda \mu}(n)$ for all $n$.

The specialization at $\xi = 0$ of Conjecture 3.1 is a result in [FH]. Below we shall assume that Conjecture 3.1 holds.

Set $\mathcal{B}$ to be the ring of polynomials in $\mathbb{Q}[x]$ which take integer values at integers. We can define a $\mathcal{B}[\xi]$-algebra $\mathcal{K}$ with a basis given by the symbols $\Gamma_\lambda$, where $\lambda$ runs over all partitions, and the multiplication given by
\[
\Gamma_\lambda \Gamma_\mu = \sum_{\nu} f_{\lambda \mu}^\nu \Gamma_\nu.
\]
Since $f_{\lambda\mu}^\nu = 0$ unless $|\nu| \leq |\lambda| + |\mu|$, the algebra $\mathcal{K}$ is an algebra filtered by $\mathcal{K}_m$ ($m \geq 0$) which is the $\mathbb{B}[\xi]$-span of $\Gamma_\lambda$ with $|\lambda| \leq m$. We have a natural surjective homomorphism of filtered algebras

$$p_n : \mathcal{K} \longrightarrow \mathcal{Z}(\mathcal{H}_n)$$

given by

$$p_n \left( \sum f_\lambda \Gamma_\lambda \right) = \sum f_\lambda(n) \Gamma_\lambda(n).$$

The algebra $\mathcal{G}$ introduced earlier becomes the associated graded algebra for the filtered algebra $\mathcal{K}$ up to a base ring change, i.e., gr$\mathcal{K} = \mathbb{B}[\xi] \otimes_{\mathbb{Z}[\xi]} \mathcal{G}$. For $r \geq 1$, set

$$E_r := \sum_{|\lambda| = r} \Gamma_\lambda.$$

**Conjecture 3.2.** The $\mathbb{B}[\xi]$-algebra $\mathcal{K}$ is generated by $E_r$, $r = 1, 2, \cdots$.

Conjecture 3.2 in the specialization with $\xi = 0$ is a main theorem of Farahat-Higman [FH].

Note (cf. e.g. [FrG]) that

$$E_r(n) := \sum_{|\lambda| = r} \Gamma_\lambda(n) \in \mathcal{Z}(\mathcal{H}_n)$$

can be interpreted as the $r$-th elementary symmetric function in the $n$ Jucys-Murphy elements. If Conjecture 3.2 holds, then the surjectivity of the homomorphism $p_n : \mathcal{K} \rightarrow \mathcal{Z}(\mathcal{H}_n)$ implies that the center $\mathcal{Z}(\mathcal{H}_n)$ is generated by $E_r(n)$, $1 \leq r \leq n - 1$. That would provide a new and conceptual proof of the Dipper-James conjecture [DJ, FrG] along with additional results of independent interest.

### 3.2. Open questions and discussions.

A fundamental difficulty in pursuing the approach of [FH] for the Iwahori-Hecke algebras is present in the following.

**Question 3.3.** Find an explicit expression for the elements $\Gamma_\lambda(n)$ for all $\lambda$.

The more challenging Conjecture 3.2 is likely to follow from an affirmative answer to Question 3.4 below, as a similar calculation in the case of symmetric groups plays a key role in the original approach of [FH].

**Question 3.4.** Calculate the structure constants $k_{\lambda\mu}^\nu$ with $|\nu| = |\lambda| + m$.

Recall the positivity and integrality from Theorem 1.1 (1).

**Question 3.5.** Find a combinatorial or geometric interpretation of the positivity and integrality of the structure constants $k_{\lambda\mu}^\nu(n)$ as polynomials in $\xi$.

With the connections between results of Farahat-Higman and cohomology rings of Hilbert schemes of $n$ points on the affine plane in mind (cf. [W]), we post the following.

**Question 3.6.** Are there any connections between the center $\mathcal{Z}(\mathcal{H}_n)$ and the equivariant $K$-group of Hilbert schemes of $n$ points on the affine plane?

Let $W$ be an arbitrary finite Coxeter group. The group algebra $\mathbb{Z}W$ is naturally filtered by assigning degree 1 to each reflection (not just simple reflection) and degree $r$ to any element $w \in W$ with a reduced expression in terms of reflections of
minimal length $r$. This induces a filtered algebra structure on the center $\mathcal{Z}(ZW)$ as elements of a conjugacy class have the same degree. In the cases of types $A$ and $B$, this definition agrees with the notion of degree for general wreath products introduced in [W]. (In the case of symmetric groups, the degree of $c_{\lambda}(n)$ coincides with $|\lambda|$.) The Geck-Rouquier basis has been defined for centers of the integral Iwahori-Hecke algebras $\mathcal{H}_W$ associated to any such $W$ [GR], and its characterization (as in Sect. 2.1) holds in this generality [Fra]. Note that the generalization of Theorem 1.1(1) to all such $\mathcal{H}_W$ holds with the same proof. We ask for a generalization of Theorem 1.1(2) as follows.

Question 3.7. Let $W$ be an arbitrary finite Coxeter group. If we apply the notion of degree above to the Geck-Rouquier elements in the center $\mathcal{Z}(\mathcal{H}_W)$, does it provide an algebra filtration on $\mathcal{Z}(\mathcal{H}_W)$?

We expect that the answer to Question 3.7 at least for Iwahori-Hecke algebras of type $B$, is positive. More generally, we ask the following.

Question 3.8. Establish and characterize an appropriate basis of class elements for the centers of the integral cyclotomic Hecke algebras associated to the complex reflection groups $\mathbb{Z}_r \wr S_n$. Furthermore, generalize the results of this Note and [W] to the cyclotomic Hecke algebra setup. Are there any connections between these centers and equivariant $K$-groups of Hilbert schemes of points on the minimal resolution $\mathbb{C}^2/\mathbb{Z}_r$?

It will be already nontrivial and of considerable interest to answer the question for the Iwahori-Hecke algebras of type $B$ corresponding to $r = 2$.

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