Higher rank graphs, $k$-subshifts and $k$-automata

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Given a $k$-graph $\Lambda$ we construct a Markov space $M_\Lambda$, and a collection of $k$ pairwise commuting cellular automata on $M_\Lambda$, providing for a factorization of Markov's shift. Iterating these maps we obtain an action of $\mathbb{N}^k$ on $M_\Lambda$ which is then used to form a semidirect product groupoid $M_\Lambda \rtimes \mathbb{N}^k$. This groupoid turns out to be identical to the path groupoid constructed by Kumjian and Pask, and hence its $C^*$-algebra is isomorphic to the higher rank graph $C^*$-algebra of $\Lambda$.

1. Introduction.

Given a row-finite $k$-graph $\Lambda$ (see [4]), Kumjian and Pask have constructed a path space $\Lambda^\infty$ and an action of $\mathbb{N}^k$ on $\Lambda^\infty$ such that the $C^*$-algebra of the path groupoid $\Lambda^\infty \rtimes \mathbb{N}^k$ is canonically isomorphic to the corresponding higher rank graph $C^*$-algebra $C^*(\Lambda)$.

Recall from [4] that a path in $\Lambda$ consists of a map $x : \Omega_k \to \Lambda$, where

$$\Omega_k = \{(n, m) \in \mathbb{N}^k \times \mathbb{N}^k, \ n \leq m\},$$

satisfying suitable conditions. In the paragraph after [4: Remarks 2.2], the authors observe that each path $x$ in $\Lambda^\infty$ is uniquely determined by a very small subset of its values, such as, for example, the values of the form

$$y(n) = x( (n,n,\ldots,n), (n+1,n+1,\ldots,n+1) ),$$

for every $n \in \mathbb{N}$. Noting that each $y(n)$ above is an element of $\Lambda$ of degree

$$d(y(n)) = (1,1,\ldots,1),$$

we consider the subset $\Sigma$ of $\Lambda$ formed by all elements possessing the above degree. Viewing $\Sigma$ as an alphabet, in the spirit of Symbolic Dynamics, one may easily see that each $y$ given above is in fact an element of a certain Markov subspace $M_\Lambda \subseteq \Sigma^\mathbb{N}$, the correspondence $x \to y$ in fact being a homeomorphism from $\Lambda^\infty$ to $M_\Lambda$.

We thus have two homeomorphic spaces, each carrying an action of a different monoid, namely of $\mathbb{N}^k$ in case of $\Lambda^\infty$, while $\mathbb{N}$ acts on $M_\Lambda$ by means of iterating Markov's shift. These actions are compatible in the sense that the above correspondence is covariant for the action of the submonoid

$$\{(n,n,\ldots,n) : n \in \mathbb{N}\} \subseteq \mathbb{N}^k,$$

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on $\Lambda^\infty$, but one may also use the above homeomorphism to extend the action of $\mathbb{N}$ on $M_\Lambda$ to an action of the much larger monoid $\mathbb{N}^k$. Alternatively, considering the action of the canonical basis vectors $e_i$ of $\mathbb{N}^k$ on $\Lambda^\infty$, we may define continuous maps

$$S_i : M_\Lambda \to M_\Lambda,$$

for $i = 1, \ldots, k$, which commute among themselves, giving a factorization of Markov’s shift $S$ on $M_\Lambda$, in the sense that

$$S_1S_2 \cdots S_k = S.$$

The well known Curtis-Hedlund-Lyndon Theorem in fact states that, when the alphabet is finite, every continuous map commuting with the shift on $\Sigma^\mathbb{N}$ is a cellular automaton, given by means of a sliding block code. Regardless of the size of our alphabet, we indeed show that the $S_i$ above are given by sliding block codes closely linked to the unique factorization property of $\Lambda$.

Motivated by this example, we introduce the notion of a weak $k$-automaton over a given alphabet $\Sigma$, as being a $k+1$-tuple

$$(Y; S_1, S_2, \ldots, S_k),$$

where $Y$ is a classical subshift (i.e., a closed subset of $\Sigma^\mathbb{N}$, invariant under the shift $S$), and the $S_i$ are pairwise commuting continuous maps from $Y$ to $Y$, providing for a factorization of the shift.

Since the path groupoid $\Lambda^\infty \rtimes \mathbb{N}^k$ may be built from nothing more than the information contained in the action of $\mathbb{N}^k$ on the path space, one sees that this groupoid is identical in all respects to the groupoid constructed from the associated weak $k$-automaton, and hence that the higher rank graph C*-algebra may be constructed solely based on the latter.

As an auxiliar gadget we also define a notion of a $k$-subshift, as being a closed subset of $\Sigma^{\mathbb{N}^k}$, invariant under the natural action of $\mathbb{N}^k$, and which is isomorphic to its image under the restriction to the diagonal. See (2.1) for the precise definition. The relevance of this notion resides in the fact that it has a more geometrical appeal, while being essentially the same thing as a weak $k$-automaton, as proved in (7.4).

The adjective “weak” above is nothing but a disclaimer highlighting the fact that the $S_i$ involved are not actually supposed to be cellular automata, although they share with the latter the important property of commuting with the shift. In (7.9) we then improve on (7.4) by precisely characterizing the $k$-subshifts giving rise to weak $k$-automata involving actual cellular automata.

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2. $P$-subshifts.

Let $\Sigma$ be a set, henceforth called the alphabet, viewed as a topological space with the discrete topology.
Given a monoid $P$, we will consider the set $\Sigma^P$ equipped with the product topology. Although we will not assume that $\Sigma$ is finite here, we observe that when $\Sigma$ is finite, then $\Sigma^P$ is a compact space by Tychonov's Theorem.

For each $p$ in $P$, we will moreover denote by $\theta_p : \Sigma^P \to \Sigma^P$, the map given by
\[
\theta_p(\xi)\big|_t = \xi(pt), \quad \forall \xi \in \Sigma^P, \quad \forall t \in P.
\]

It is then easy to prove that $\theta$ is a right action of $P$ on $\Sigma^P$, that is,
\[
\theta_p \theta_q = \theta_{qp}, \quad \forall p, q \in P,
\]
henceforth referred to as the Bernoulli action, or the full $P$-shift, on the alphabet $\Sigma$.

2.1. Definition. A $P$-subshift is any closed subset $X \subseteq \Sigma^P$ which is invariant under the Bernoulli action in the sense that $\theta_p(X) \subseteq X$, for every $p$ in $P$.

We will next describe an important source of $P$-subshifts given in terms of forbidden patterns.

2.2. Definition. By a pattern we shall mean a pair $(\pi, D_\pi)$, where $D_\pi$ is a finite subset of $P$, and $\pi : D_\pi \to \Sigma$ is any function. We shall frequently refer to $\pi$ as a pattern without mentioning $D_\pi$ explicitly. Given a pattern $\pi$ and an element $\xi$ in $\Sigma^P$, we will say that $\pi$ occurs in $\xi$, provided there exists some $p_0$ in $P$ such that $\pi(t) = \xi(pp_0t)$, for all $t$ in $D_\pi$.

2.3. Proposition. If the pattern $\pi$ occurs in $\theta_p(\xi)$, for some $\xi$ in $\Sigma^P$, then $\pi$ occurs in $\xi$.

Proof. By hypothesis there exists $p_0$ in $P$ such that
\[
\pi(t) = \theta_p(\xi)\big|_{p_0t} = \xi(pp_0t), \quad \forall t \in D_\pi,
\]
whence the conclusion. \qed

Given a pattern $\pi$, it is easy to see that the set of all $\xi$ in $\Sigma^P$ such that $\pi$ occurs in $\xi$ is open in $\Sigma^P$.

2.4. Proposition. Given a collection $\Pi$ of patterns, let $X_\Pi$ be the set of all elements $\xi$ in $\Sigma^P$ such that no pattern in $\Pi$ occurs in $\xi$. Then $X_\Pi$ is a $P$-subshift.

Proof. $X_\Pi$ is closed by the observation made just before the statement, and it is invariant under the Bernoulli action by the contrapositive of (2.3). \qed

The following is a well known result in the theory of classical subshifts. It is usually stated for finite alphabets, but it works just as well for infinite ones.

2.5. Proposition. If $X$ is any $P$-subshift then there exists a collection $\Pi$ of patterns such that $X = X_\Pi$. 

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Proof. Let $\Pi$ be the collection of all patterns which do not occur in any $\xi$ in $X$. It is then obvious that $X \subseteq X_\Pi$, and we next claim that $X$ is dense in $X_\Pi$. To see this, choose any $\eta$ in $X_\Pi$, and let $V$ be any open subset of $\Sigma^P$ containing $\eta$. Since $\Sigma^P$ has the product topology, there exists a finite set $D \subseteq P$ such that

$$\eta \in W := \{\zeta \in \Sigma^P : \zeta(t) = \eta(t), \text{ for all } t \in D\} \subseteq V.$$ 

Setting $\pi = \eta|_D$, we have that $\pi$ is a pattern, obviously occurring in $\eta$, whence $\pi$ is certainly not in $\Pi$. By definition of $\Pi$, it follows that $\pi$ occurs in some $\xi$ in $X$, so there exists $p_0$ in $P$ such that

$$\pi(t) = \xi(p_0 t) = \theta_{p_0}(\xi)|_t,$$

for every $t$ in $D$. This says that $\theta_{p_0}(\xi) \in W \cap X$, proving the desired density, namely that $X_\Pi \subseteq X$. Since $X$ is closed by hypothesis, the proof is concluded. □

If $Q$ is a submonoid of $P$, always assumed to share the neutral element, we may consider the restriction mapping $\rho_Q$ from $\Sigma^P$ to $\Sigma^Q$, namely

$$\rho_Q(\xi) = \xi|_Q, \quad \forall \xi \in \Sigma^P. \quad (2.6)$$

Clearly $\rho_Q$ is a continuous map.

2.7. Proposition. Let $X \subseteq \Sigma^P$ be a $P$-subshift, and let $Q$ be a submonoid of $P$. Then

(i) $\rho_Q(X)$ is invariant under the Bernoulli action of $Q$,

(ii) if $\Sigma$ is finite, then $\rho_Q(X)$ is a $Q$-subshift.

Proof. Letting $\theta'$ denote the full $Q$-shift, observe that, for all $q$ in $Q$, and all $\xi$ in $X$, one has that

$$\theta'_q(\xi|_Q) = \theta_q(\xi)|_Q,$$

from where it easily follows that $\rho_Q(X)$ is invariant under the Bernoulli action of $Q$. Assuming that $\Sigma$ is finite, we have that $\Sigma^P$ is compact, whence so is $X$. Observing that $\rho_Q$ is continuous, we see that $\rho_Q(X)$ is compact, hence closed in $\Sigma^Q$. This concludes the proof. □

We will soon discuss an important class of examples in which $\rho_Q(X)$ is closed, even though the alphabet might be infinite. It will then follow from (2.7.i) that $\rho_Q(X)$ is a $Q$-subshift. Incidentally, we do not have any example in which $\rho_Q(X)$ fails to be closed.

3. Cellular Automata.

As before we let $\Sigma$ be any set, which we view as a discrete topological space. From now on we shall be concerned with metric aspects, most notably with the notion of uniform continuity, so we shall equip $\Sigma$ with the metric defined by

$$d(a, b) = \begin{cases} 
0, & \text{if } a = b, \\
1, & \text{otherwise},
\end{cases}$$

for all $a$ and $b$ in $\Sigma$. The topology on $\Sigma$ induced by this metric is clearly the discrete topology. For this reason $d$ is sometimes called the discrete metric. However, while many
other metrics on $\Sigma$ also induce the discrete topology, we observe that $d$ induces the discrete topology in a uniform way, meaning that there exists $r > 0$, namely $r = 1/2$ such that for every $a$ in $\Sigma$, the ball centered at $a$ with radius $r$ coincides with the singleton $\{a\}$. The fact that $r$ does not depend on $a$ is what makes $d$ a uniformly discrete metric.

In this section we shall be concerned with the monoid $\mathbb{N}$, formed by all natural numbers, including zero, and hence we will be working with $\mathbb{N}$-subshifts, also known simply as subshifts.

The most popular metric considered on the product space $\Sigma^\mathbb{N}$ (as usual also denoted by $d$, by abuse of language) is as follows: given $x$ and $y$ in $\Sigma^\mathbb{N}$, one puts $d(x, y) = 2^{-k}$, where $k$ is the largest integer such that $x_i = y_i$, for all $i \leq k$. If $x = y$, then obviously no such $k$ exists, in which case we set $d(x, y) = 0$. It is well known that $d$ defines a metric on $\Sigma^\mathbb{N}$, which is compatible with the product topology.

The role of uniform continuity is evidenced by our next result.

3.1. Lemma.

(i) Each projection $p_k : \Sigma^\mathbb{N} \to \Sigma$ is uniformly continuous.

(ii) For every nonempty $X \subseteq \Sigma^\mathbb{N}$, and for every uniformly continuous map $\varphi : X \to \Sigma$, one has that $\varphi$ depends only on finitely many coordinates, meaning that there exists some $k \in \mathbb{N}$, and a map $\psi : \Sigma^{k+1} \to \Sigma$ such that

$$\varphi(x) = \psi(x_0, x_1, \ldots, x_k), \quad \forall x \in X.$$  

Proof. Regarding the projection $p_k$, and given $\varepsilon > 0$, choose $\delta = 2^{-k}$. Then, for every $x$ and $y$ in $\Sigma^\mathbb{N}$, with $d(x, y) \leq \delta$, we necessarily have that $x_i = y_i$, for all $i \leq k$, hence

$$d(p_k(x), p_k(y)) = d(x_k, y_k) = 0 < \varepsilon,$$

proving (i). With respect to (ii), let $\delta > 0$ be such that

$$d(x, y) < \delta \Rightarrow d(\varphi(x), \varphi(y)) < 1/2,$$

for every $x, y \in \Sigma^\mathbb{N}$, and choose an integer $k$ such that $2^{-k} < \delta$. Given $x$ and $y$ in $\Sigma^\mathbb{N}$ such that

$$(x_0, x_1, \ldots, x_k) = (y_0, y_1, \ldots, y_k),$$

we then have that $d(x, y) \leq 2^{-k} < \delta$, so $d(\varphi(x), \varphi(y)) < 1/2$, which clearly implies that $\varphi(x) = \varphi(y)$, since $\Sigma$ has the 0-1 metric. This proves that $\varphi(x)$ depends only on $(x_0, x_1, \ldots, x_k)$. \qed

Observe that when $\Sigma$ is finite and $X$ is closed in $\Sigma^\mathbb{N}$, then $X$ is compact by Tychonov’s Theorem, so every continuous function on $X$ is necessarily uniformly continuous. Consequently the conclusion of (3.1.ii) holds for every continuous function $\varphi$.

As usual, we denote by $S$ the shift on $\Sigma^\mathbb{N}$, defined by

$$S(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots), \quad \forall x = (x_0, x_1, x_2, \ldots) \in \Sigma^\mathbb{N},$$

so that a closed subspace $X \subseteq \Sigma^\mathbb{N}$ is an $\mathbb{N}$-subshift if and only if $X$ is invariant under $S$ in the sense that $S(X) \subseteq X$.  

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3.2. Definition. Let $X \subseteq \Sigma^\mathbb{N}$ be a subshift.

(i) The language of $X$, denoted $L(X)$, is the set of all finite words occurring as a contiguous block of characters in some $x$ in $X$.

(ii) Given $k \in \mathbb{N}$, the subset formed by all words in $L(X)$ of length $k + 1$ will be denoted by $L_k(X)$.

(iii) By a (sliding block) code for $X$ we mean any function $\psi : L_k(X) \to \Sigma$. The integer $k$ is called the anticipation of $\psi$.

(iv) Given a code $\psi$, we define

$$T_\psi : (x_0, x_1, \ldots) \in X \mapsto (y_0, y_1, \ldots) \in \Sigma^\mathbb{N},$$

where $y_n = \psi(x_n, x_{n+1}, \ldots, x_{n+k})$, and $k$ is the anticipation of $\psi$. One says that $T_\psi$ is the cellular automaton associated to the code $\psi$.

Another reason for our interest in uniform continuity is in order:

3.3. Proposition. Let $X \subseteq \Sigma^\mathbb{N}$ be a subshift and let $\psi : L_k(X) \to \Sigma$ be a code for $X$. The cellular automaton $T_\psi$ is a uniformly continuous map.

Proof. Given $\varepsilon > 0$, choose $p$ such that $2^{-p} < \varepsilon$. For $x$ and $x'$ in $X$, let

$$y = T_\psi(x), \quad \text{and} \quad y' = T_\psi(x').$$

If $d(x, x') \leq 2^{-(p+k)}$, we have that $x_i = x'_i$, for all $i \leq p + k$, from where one deduces that $y_i = y'_i$, for all $i \leq p$, whence

$$d(y, y') \leq 2^{-p} < \varepsilon.$$

This proves that $T_\psi$ is uniformly continuous. \hfill \Box

If $\psi$ is defined on $\Sigma^2$ by $\psi(a, b) = b$, then $T_\psi$ is clearly the shift itself. From the above we then deduce the elementary fact that the shift is uniformly continuous.

Given a subshift $X$ and a code $\psi$ for $X$, it easy to prove that

$$T_\psi(S(x)) = S(T_\psi(x)), \quad \forall x \in X. \quad (3.4)$$

The next result, known as the Curtis-Hedlund-Lyndon Theorem, says that the only uniformly continuous maps commuting with the shift are the cellular automata. The corresponding result for the bilateral shift is discussed in [5:13.9].

3.5. Proposition. Let $X$ be a subshift and let $T : X \to X$ be a continuous mapping which commutes with $S$. Then $T$ is uniformly continuous if and only if $T$ is a cellular automaton.
Proof. The “if” part having already been dealt with in (3.3), we move on to the “only if” part. Consider the function \( \varphi : X \to \Sigma \) given by \( \varphi = p_0 \circ T \), where \( p_0 \) is the projection on the leftmost coordinate. By (3.1.i) we have that \( \varphi \) is uniformly continuous, and hence

\[
p_0(T(x)) = \psi(x_0, x_1, \ldots, x_k), \quad \forall x \in X,
\]

for some \( k \) and some \( \psi : \Sigma^{k+1} \to \Sigma \), by (3.1.ii). For every \( x \) in \( X \) it then follows that

\[
p_n(T(x)) = p_0(S^n(T(x))) = p_0(T(S^n(x))) =
\]

\[
p_0(T(x_n, x_{n+1}, \ldots)) = \psi(x_n, x_{n+1}, \ldots, x_{n+k}).
\]

Noticing that \((x_n, x_{n+1}, \ldots, x_{n+k})\) necessarily lies in \( L_k(X) \), and upon restricting \( \psi \) to \( L_k(X) \), we then have that \( T \) is the cellular automaton associated to \( \psi \). \( \square \)

4. Higher rank graphs.

Given any integer \( k \geq 1 \), let \( \Lambda \) be a \( k \)-graph (see [4]). That is, \( \Lambda \) is a (small) category equipped with a degree functor \( d: \Lambda \to \mathbb{N}^k \), satisfying the unique factorization property, namely, if \( d(\lambda) = m + n \), then there are unique \( \alpha, \beta \in \Lambda \) with \( \lambda = \alpha \beta \) and \( d(\alpha) = m \) and \( d(\beta) = n \).

Recall that \( \Omega_k \) is the \( k \)-graph consisting of all pairs \((m, n)\) \in \( \mathbb{N}^k \times \mathbb{N}^k \) such that \( m \leq n \), equipped with the degree map defined by \( d(m, n) = n - m \) and with allowed products \((n, r)(m, n) = (m, r)\). By definition, a path in \( \Lambda \) is a functor from \( \Omega_k \) to \( \Lambda \), compatible with the degree maps. The set of all paths in \( \Lambda \) is denoted by \( \Lambda^\infty \).

Recall from [4: Definitions 2.1] that, for each \( p \) in \( \mathbb{N}^k \), one defines \( \sigma^p : \Lambda^\infty \to \Lambda^\infty \), by

\[
\sigma^p(x)(m, n) = x(p + m, p + n), \quad \forall x \in \Lambda^\infty, \quad \forall (m, n) \in \Omega_k.
\]

In what follows we would like to relate the path space \( \Lambda^\infty \) to an \( \mathbb{N}^k \)-subshift. In order to do so we begin by introducing the notation \( \mathbb{1} := (1, 1, \ldots, 1) \), which we use in defining our alphabet \( \Sigma \), by

\[
\Sigma = \{ \lambda \in \Lambda : d(\lambda) = \mathbb{1} \}.
\]

Given any path \( x \in \Lambda^\infty \), we may consider the element \( \xi_x \) of \( \Sigma^{\mathbb{N}^k} \), defined by

\[
\xi_x(n) = x(n, n + \mathbb{1}), \quad \forall n \in \mathbb{N}^k,
\]
For every \( p \) in \( \mathbb{N}^k \), and every \( x \) in \( \Lambda^\infty \), one easily checks that

\[
\theta_p(\xi_x) = \xi_{\sigma^p(x)},
\]

from where one deduces that the subset of \( \Sigma^{\mathbb{N}^k} \), given by

\[
X_\Lambda = \{ \xi_x : x \in \Lambda^\infty \}
\]
is invariant under the Bernoulli action of \( \mathbb{N}^k \).

**4.2. Proposition.** The set \( X_\Lambda \) introduced above is an \( \mathbb{N}^k \)-subshift and the correspondence

\[
\Xi : x \in \Lambda^\infty \mapsto \xi_x \in X_\Lambda
\]
is a homeomorphism. In addition, \( \Xi \) is covariant relative to the action \( \sigma \) on \( \Lambda^\infty \) and the Bernoulli action on \( X_\Lambda \).

**Proof.** Having already observed that \( X_\Lambda \) is invariant under the Bernoulli action, we will next prove that \( X_\Lambda \) is closed. We then pick any \( \xi \) in the closure of \( X_\Lambda \), so there exists a sequence \( \{ x_i \}_i \) in \( \Lambda^\infty \) such that \( \{ x_{i,i} \}_i \) converges to \( \xi \).

For each \( j \) in \( \mathbb{N} \), write \( j \mathbb{1} \) for the element \( (j, j, \ldots, j) \) of \( \mathbb{N}^k \), and put \( \lambda_j = \xi(j \mathbb{1}) \). We then claim that \( s(\lambda_j) = r(\lambda_{j+1}) \), for every \( j \), where \( s \) and \( r \) refer to the source and range maps relative to the category \( \Lambda \). To see this, let \( i_0 \) be large enough, so that

\[
\xi_{x_i}(j \mathbb{1}) = \xi(j \mathbb{1}), \quad \text{and} \quad \xi_{x_i}(j \mathbb{1}+1) = \xi(j \mathbb{1}+1),
\]
for every \( i \geq i_0 \). Choosing any \( i \geq i_0 \), we then have that

\[
s(\lambda_j) = s(\xi(j \mathbb{1})) = s(\xi_{x_i}(j \mathbb{1})) = s(x_i(j \mathbb{1}, j \mathbb{1}+1)) = \cdots
\]
Recall that $x_i$ is a path, hence a functor from $\Omega_k$ to $\Lambda$. Since the morphisms $(j \mathbb{1}, j \mathbb{1}+1)$ and $(j+1 \mathbb{1}, j \mathbb{1}+1)$ may be composed in $\Omega_k$, we have that $x_i(j \mathbb{1}, j \mathbb{1}+1)$ and $x_i(j+1 \mathbb{1}, j \mathbb{1}+1)$ may be composed in $\Lambda$, so the source of the former must coincide with the range of the latter, whence the above equals

$$\cdots = r(x_i(j \mathbb{1}+1, j \mathbb{1}+2)) = r(\xi_{x_i}(j \mathbb{1}+1)) = r(\xi(j \mathbb{1}+1)) = r(\lambda_j),$$

thus proving our claim. By the paragraph after [4: Remarks 2.2], we conclude that there exists a path $x$ in $\Lambda^\infty$ such that $x(j \mathbb{1}, j \mathbb{1}+1) = \lambda_j$, for all $j$, and we next claim that $\xi_x = \xi$. Given any $n$ in $\mathbb{N}^k$, choose some $j$ in $\mathbb{N}$ such that $n \leq j \mathbb{1}$, and observe that

$$\lambda_0 \lambda_1 \ldots \lambda_j = x(0 \mathbb{1}, \mathbb{1}) x(1, 2 \mathbb{1}) \cdots x(j \mathbb{1}, j \mathbb{1}+1)$$

$$= x(0 \mathbb{1}, j \mathbb{1}+1)$$

$$= x(0 \mathbb{1}, n) x(n, n+1) x(n+1, j \mathbb{1}+1)$$

$$= x(0 \mathbb{1}, n) \xi_x(n) x(n+1, j \mathbb{1}+1).$$

We next choose $i_0$ large enough, so that $\xi_{x_i}$ coincides with $\xi$ on $\{0 \mathbb{1}, \ldots, j \mathbb{1}, n\}$, for every $i \geq i_0$. Therefore

$$\lambda_0 \lambda_1 \ldots \lambda_j = \xi(0 \mathbb{1}) \xi(1) \cdots \xi(j \mathbb{1})$$

$$= \xi_{x_i}(0 \mathbb{1}) \xi_{x_i}(1) \cdots \xi_{x_i}(j \mathbb{1})$$

$$= x_i(0 \mathbb{1}, i) x_i(1, 2 \mathbb{1}) \cdots x_i(j \mathbb{1}, j \mathbb{1}+1)$$

$$= x_i(0 \mathbb{1}, j \mathbb{1}+1)$$

$$= x_i(0 \mathbb{1}, n) x_i(n, n+1) x_i(n+1, j \mathbb{1}+1).$$

Contrasting our last two calculations, and invoking the unique factorization property, we deduce that

$$\xi_x(n) = x_i(n, n+1) = \xi_{x_i}(n) = \xi(n).$$

This concludes the proof of the claim according to which $\xi_x = \xi$, whence $\xi$ lies in $X_\Lambda$, and so we see that $X_\Lambda$ is closed.

The last sentence of the statement has already been verified in (4.1), so we are finally left with the task of proving our correspondence $\Xi$ to be a homeomorphism. In order to do this we first observe that $\Xi$ is injective by the paragraph after [4: Remarks 2.2]. We will next prove that $\Xi$ is an open mapping. For this, recall from [4: Definitions 2.4] that the topology of $\Lambda^\infty$ is generated by the cylinders,

$$Z(\lambda) = \{x \in \Lambda^\infty : x(0, d(\lambda)) = \lambda\},$$

as $\lambda$ range in $\Lambda$. Given any $y$ in $\Lambda^\infty$, and given any open set $U \subseteq \Lambda^\infty$ containing $y$, one may then find some $\lambda$ such that

$$y \in Z(\lambda) \subseteq U.$$
Choosing $j$ such that $j \mathbb{1} \geq d(\lambda)$, and setting $\mu = y(0, j \mathbb{1})$, we then have that
\[
\mu = y(0, j \mathbb{1}) = y(0, d(\lambda)) \ y(d(\lambda), j \mathbb{1}) = \lambda y(d(\lambda), j \mathbb{1}),
\]
from where we see that
\[
y \in Z(\mu) \subseteq Z(\lambda),
\]
and we conclude that the cylinders of the form $Z(\mu)$, with $d(\mu)$ a multiple of $\mathbb{1}$, also form a basis for the topology of $\Lambda^\infty$. In order to prove that $\Xi$ is an open map, it is therefore enough to verify that $\Xi(Z(\mu))$ is open for every such $\mu$.

Given $\mu$ in $\Lambda$ as above, i.e. with $d(\mu) = j \mathbb{1}$, use the factorization property to write
\[
\mu = \lambda_0 \lambda_1 \ldots \lambda_{j-1},
\]
with $d(\lambda_i) = \mathbb{1}$, and observe that
\[
x \in Z(\mu) \iff x(0, j \mathbb{1}) = \lambda_0 \lambda_1 \ldots \lambda_{j-1}
\]
\[
\iff x(i \mathbb{1}, i \mathbb{1} + \mathbb{1}) = \lambda_i, \quad \forall i = 0, \ldots, j-1
\]
\[
\iff \xi_x(i \mathbb{1}) = \lambda_i, \quad \forall i = 0, \ldots, j-1.
\]
Setting
\[
V = \{ \xi \in \Sigma^{N^k} : \xi(i \mathbb{1}) = \lambda_i, \ \forall i = 0, \ldots, j-1 \},
\]
which is clearly open in $\Sigma^{N^k}$, we then deduce that $x \in Z(\mu)$ if and only if $\xi_x \in V$. It then follows that $\Xi(Z(\mu)) = X_\Lambda \cap V$, proving that $\Xi$ is an open mapping, as claimed.

Finally, leaving for the reader the easy task of verifying that $\Xi$ is continuous, the proof is concluded. \(\square\)

Having twice resorted to the paragraph after [4: Remarks 2.2], we have not yet exhausted its consequences from our point of view. A further, and major consequence is the content of our next result.

Identifying the monoid $\mathbb{N}$ as a submonoid of $\mathbb{N}^k$ via the correspondence $i \leftrightarrow i \mathbb{1}$, we will shortly refer to the restriction map $\rho_\mathbb{N}$, introduced in (2.6).

**4.3. Proposition.** Let $A = \{A_{\lambda \mu}\}_{\lambda, \mu \in \Sigma}$ be the matrix given by

\[
A_{\lambda \mu} = \begin{cases} 
1, & \text{if } s(\lambda) = r(\mu), \\
0, & \text{otherwise},
\end{cases}
\]

and let $X_A \subseteq \Sigma^\mathbb{N}$ be the Markov space for $A$. Then $\rho_\mathbb{N}(X_\Lambda) = X_A$, and $\rho_\mathbb{N}$ is a homeomorphism from $X_\Lambda$ onto $X_A$.

*Proof.* We first observe that, for every $x$ in $\Lambda^\infty$, one has

\[
\rho_\mathbb{N}(\xi_x) = (x(i \mathbb{1}, i \mathbb{1} + \mathbb{1}))_{i \in \mathbb{N}},
\]
which is evidently in $X_A$, so we see that $\rho_\Sigma(X_A) \subseteq X_A$.

Given any $\lambda = (\lambda_i)_{i \in \mathbb{N}} \in X_A$, the paragraph after [4: Remarks 2.2] produces a path $x$ in $\Lambda^\infty$ such that $x(i1, i1+1) = \lambda_i$, for all $i \in \mathbb{N}$, so that $\rho_\Sigma(\xi_x) = \lambda$, which in turn proves that $\rho_\Sigma(X_A) = X_A$.

The uniqueness of the path $x$ obtained above implies that $\rho_\Sigma$ is one-to-one, so it remains to prove that $\rho_\Sigma$ is a homeomorphism. Continuity not being an issue, we focus on proving continuity of the inverse map. In order to do so let us review the above construction of $x$ from any given $\lambda$ in $X_A$: given any $(m, n) \in \Omega_k$, one chooses $j$ such that $j1 \geq n$, and uses the unique factorization property to write

$$\lambda_0 \lambda_1 \ldots \lambda_{j-1} = \mu x(m, n) \nu,$$

with $d(\mu) = m$, $d(x(m, n)) = n - m$, and $d(\nu) = j1 - n$. The resulting map $x : \Omega_k \rightarrow \Lambda$ is then a path satisfying $\rho_\Sigma(\xi_x) = \lambda$, whence $\xi_x = \rho_\Sigma^{-1}(\lambda)$, and we must then prove that $\xi_x$ varies continuously with $\lambda$. Since $\xi_x$ lives in the product space $\Sigma^{\mathbb{N}^k}$, all we need to do is show that $\xi_x(n)$ is continuous as a function of $\lambda$, for every $n \in \mathbb{N}^k$. Recalling that $\xi_x(n) = x(n, n+1)$, notice that the above recipe to construct $x(n, n+1)$ depends only on $(\lambda_0, \lambda_1, \ldots, \lambda_{j-1})$, where $j$ is any integer such that $j1 \geq n+1$. The function producing $\xi_x(n)$ from $\lambda$ therefore factors as the composition

$$X_A \rightarrow \Sigma^j \rightarrow \Sigma,$$

where the leftmost arrow is the projection on the first $j$ coordinates while the rightmost one corresponds to the above recipe producing $x(n, n+1)$ from $(\lambda_0, \lambda_1, \ldots, \lambda_j)$. Since the projection is continuous and $\Sigma^j$ carries the discrete topology, continuity follows. \qed

5. Higher rank subshifts.

Motivated by the example of the $\mathbb{N}^k$-subshift discussed in the previous section, we make the following:

5.1. Definition. Let $k$ be a positive integer. By a subshift of rank $k$, or a $k$-subshift, we shall mean a $\mathbb{N}^k$-subshift $X$ on a given alphabet $\Sigma$ such that

(i) the restriction map $\rho_\Sigma$ is injective on $X$,
(ii) $\rho_\Sigma(X)$ is closed in $\Sigma^{\mathbb{N}^k}$, and
(iii) $\rho_\Sigma$ is a homeomorphism from $X$ to $\rho_\Sigma(X)$.
Since Markov spaces are automatically closed, we conclude from (4.3) that the $\mathbb{N}^k$-subshift $X_\Lambda$ built from a $k$-graph $\Lambda$ is an example of a $k$-subshift.

Observe also that, in case the alphabet $\Sigma$ is finite, then any $\mathbb{N}^k$-subshift is compact, hence (5.1.ii) is automatically true, while (5.1.iii) follows from (5.1.i). In other words, when the alphabet is finite, (5.1.ii-iii) could be omitted from the above definition without any consequences.

Let us now fix a subshift $X$ of rank $k$ on the alphabet $\Sigma$.

Setting $Y = \rho_\mathbb{N}(X)$, we have by (2.7.i) that $Y$ is invariant under the Bernoulli action of $\mathbb{N}$, that is, invariant under the usual shift map

$$S : \Sigma^\mathbb{N} \to \Sigma^\mathbb{N},$$

and since $Y$ is also closed by assumption, we have that $Y$ is a $\mathbb{N}$-subshift, that is, a classical subshift.

*A priori*, it does not make sense to ask whether or not $\rho_\mathbb{N}$ is covariant, since the monoids acting on the Bernoulli spaces $\Sigma^{\mathbb{N}^k}$ and $\Sigma^\mathbb{N}$ are not the same. But if we consider only the smaller monoid, namely $\mathbb{N}$, then covariance clearly holds, and in particular

$$\rho_{\mathbb{N}} \theta = S \rho_\mathbb{N},$$

as the reader may easily verify.

Furthermore, since $\rho_\mathbb{N}$ is a homeomorphism from $X$ to $Y$, the Bernoulli action of $\mathbb{N}^k$ on $X$ may be transferred to $Y$, so we get an action $\tau$ of $\mathbb{N}^k$ on $Y$ such that the diagram

\[
\begin{array}{ccc}
X & \overset{\rho_\mathbb{N}}{\longrightarrow} & Y \\
\downarrow{\theta_\mathbb{N}} & & \downarrow{\tau} \\
X & \overset{\rho_\mathbb{N}}{\longrightarrow} & Y \\
\end{array}
\]

Diagram (5.3)

commutes for every $n$ in $\mathbb{N}^k$.

5.4. Theorem. Let $X$ be a $k$-subshift on the alphabet $\Sigma$, and put $Y = \rho_\mathbb{N}(X)$. Then:

(i) $Y$ is a classical subshift.

(ii) For any integer $i$ with $1 \leq i \leq k$, let $e_i$ be the canonical basis vector of $\mathbb{N}^k$, and put $S_i = \tau_{e_i}$. Then the $S_i$ are pairwise commuting, continuous maps from $Y$ to $Y$, and

$$S_1 S_2 \cdots S_k = S,$$

where $S$ is the restriction of the shift map to $Y$ (here denoted simply by $S$, by abuse of language).
Proof. The first point is an obvious consequence of the definitions and of (2.7.i). It was included here only for future reference. Regarding (ii) we have

\[ S_1 S_2 \cdots S_k = \tau(e_1) \cdots \tau(e_k) = \tau(e_1 + \cdots + e_k) = \tau(1) = \rho_{\tau} \theta \rho_{\tau}^{-1} \overset{(5.2)}{=} S. \]

Given that the \( S_i \) commute among themselves, it follows that \( S_i \) also commutes with \( S \), so this is turns out to be strongly related to (3.4), and hence also to cellular automata by (3.5).

6. Cellular automaton factorization of Markov subshifts associated to \( k \)-graphs.

We have already mentioned that the space \( X_\Lambda \) built from a \( k \)-graph \( \Lambda \) is a subshift of rank \( k \). Moreover, by (4.3), we have that \( \rho_{\tau}(X_\Lambda) \) is a Markov space. We may then use (5.4) to produce a factorization of Markov’s shift and we will now show that each \( S_i \) occurring in (5.4) is in fact a cellular automaton with anticipation 1. This could be obtained by (3.5), but we may in fact produce the block codes directly.

We first observe that the language of a Markov subshift, such as \( \rho_{\tau}(X_\Lambda) \), is governed by its matrix, and in particular

\[ \mathcal{L}_1(\rho(X_\Lambda)) = \{ (\lambda_0, \lambda_1) \in \Sigma^2 : s(\lambda_0) = r(\lambda_1) \}. \]

We therefore define, for each \( i = 1, \ldots, k \),

\[ \varphi_i : \mathcal{L}_1(\rho(X_\Lambda)) \to \Sigma \]

as follows: given \((\lambda, \mu)\) in \( \mathcal{L}_1(\rho(X_\Lambda)) \), we have that \( \lambda \mu \in \Lambda \), and clearly \( d(\lambda \mu) = 1 + 1 \).

Writing

\[ 1 + 1 = e_i + 1 + (1 - e_i), \]

the unique factorization property allows us to write \( \lambda \mu = \alpha \beta \gamma \), where \( \alpha \), \( \beta \) and \( \gamma \) lie in \( \Lambda \), \( d(\alpha) = e_i \), \( d(\beta) = 1 \), and \( d(\gamma) = 1 - e_i \). We then set

\[ \varphi_i(\lambda, \mu) = \beta. \]

We will now show that each \( S_i \) coincides with the cellular automaton relative to the sliding block code \( \varphi_i \). In order to do this, observe that each \( S_i \) is officially defined as

\[ S_i = \tau_{e_i} = \rho_{\tau} \theta_{e_i} \rho_{\tau}^{-1}. \]

Given any \( y \) in \( \rho_{\tau}(X_\Lambda) \), we may write \( y = \rho_{\tau}(\xi) \), for some \( \xi \) in \( X_\Lambda \), and we may further write \( \xi = \xi_x \), for some \( x \in \Lambda^\infty \). In other words, \( y = \rho_{\tau}(\xi_x) \). We then have for every \( j \) in \( \mathbb{N} \), that

\[ S_i(y)_{j} = \rho_{\tau} \theta_{e_i} \rho_{\tau}^{-1}(y)_{j} = \theta_{e_i} \rho_{\tau}^{-1}(y)_{j} = \rho_{\tau}^{-1}(y)_{j + e_i} = \xi_x(j 1 + e_i) = x(j 1 + e_i, j 1 + e_i 1). \]

(6.1)
On the other hand, notice that

\[
y(j) y(j+1) = \xi_x(j) \xi_x(j+1) = x(j+1, j+1) x(j+1, j+2) = x(j, j+2) = x(j, j+e_i) x(j+e_i, j+e_i+1) x(j+e_i+1, j+1).
\]

By the unique factorization property we then have that

\[
\varphi_i(y(j), y(j+1)) = x(j+e_i, j+e_i+1) (6.1) S_i(y)_{|j},
\]

thus proving that indeed \(S_i\) is the cellular automaton associated to the block code \(\varphi_i\), as claimed.

7. Constructing \(k\)-subshifts from factorizations of the shift.

Motivated by (5.4) we introduce the following concept:

7.1. Definition. A weak \(k\)-automaton over a given alphabet \(\Sigma\) is a \(k+1\)-tuple

\[
(Y; S_1, S_2, \ldots, S_k),
\]

where \(Y\) is a closed subset of \(\Sigma^\mathbb{N}\), invariant under the shift \(S\) (i.e., \(Y\) is a classical subshift), and the \(S_i\) are pairwise commuting continuous maps from \(Y\) to \(Y\) such that

\[
S_1 S_2 \cdots S_k = S.
\]

As seen in (5.4), every \(k\)-subshift leads to a weak \(k\)-automaton, and it is our plan to show that \(k\)-subshifts are essentially the same thing as weak \(k\)-automata. As a first step let us show how to construct a \(k\)-subshift given a weak \(k\)-automata \((Y; S_1, S_2, \ldots, S_k)\), which we consider fixed for the time being.

For each \(n = (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k\), put

\[
\alpha_n = S_1^{n_1} S_2^{n_2} \cdots S_k^{n_k},
\]

so that \(\alpha\) is an action of \(\mathbb{N}^k\) on \(Y\). We next let \(\Psi : Y \rightarrow \Sigma^\mathbb{N}^k\) be defined by

\[
\Psi(y)_{|n} = \alpha_n(y)_{|0}, \quad \forall y \in Y, \quad \forall n \in \mathbb{N}^k.
\]

7.2. Proposition. Setting \(X = \Psi(Y)\), one has that \(X\) is a subshift of rank \(k\). In addition we have that \(\rho_{\pi}(X) = Y\) and, regarding the canonical action \(\tau\) introduced in Diagram (5.3), one has that \(\tau_{\pi_i} = S_i\), for every \(i = 1, \ldots, k\).

Proof. Given \(y \in Y\), and given \(p\) and \(t\) in \(\mathbb{N}^k\), observe that

\[
\theta_p(\Psi(y))_{|t} = \Psi(y)_{|t+p} = \alpha_t(\alpha_p(y))_{|0}, \quad \forall y \in Y, \quad \forall n \in \mathbb{N}^k.
\]

so we deduce that \(\theta_p(\Psi(y)) = \Psi(\alpha_p(y))\), from where we see that the range of \(\Psi\), also known as \(X\), is invariant under the Bernoulli action of \(\mathbb{N}^k\).
In order to prove that \( \rho(C_6)(X) = Y \), it suffices to verify that

\[
\rho(C_6)(\Psi(y)) = y, \quad \forall y \in Y, \quad (7.2.2)
\]

but this follows from the following computation, where \( j \in \mathbb{N} \):

\[
\rho(C_6)(\Psi(y))|_j = \Psi(y)|_{j|} = \alpha_{j|}(y)|_0 = S_j^i S_{j}^{j} \cdots S_{k}^{j}(y)|_0 = S(y)|_0 = y(j).
\]

In order to prove that \( X \) is closed, suppose that \( \{y_i\}_i \) is a sequence in \( Y \) such that \( \{\Psi(y_i)\}_i \) converges to some \( x \) in \( \Sigma\mathbb{N}^k \). Then

\[
y := \rho(C_6)(x) = \lim_i \rho(C_6)(\Psi(y_i)) \quad (7.2.2) \lim_i y_i,
\]

so \( y \in Y \), and we claim that \( x = \Psi(y) \). In fact, for every \( n \in \mathbb{N}^k \), we have that

\[
x(n) = \lim_i \Psi(y_i)|_n = \lim_i \alpha_n(y_i)|_0 = \alpha_n(y)|_0 = \Psi(y)|_n,
\]

proving the claim.

So far we have thus proven that \( X \) is a \( \mathbb{N}^k \)-subshift. In order to show that it is a subshift of rank \( k \), we must verify (5.1.i–iii). By (7.2.2) we see that \( \Psi \) is one-to-one, and since it is onto \( X \), by definition, it follows that \( \Psi \) is bijective. Employing (7.2.2) once more we deduce that \( \rho(C_6)|_X = \Psi^{-1} \), from where (5.1.i) follows.

Noticing that both \( \rho \) and \( \Psi \) are clearly continuous, we obtain (5.1.iii), while (5.1.ii) follows from the facts that \( \rho(C_6)(X) = Y \), and that \( Y \) is closed by assumption.

The last part of the statement may now be proved as follows: for \( y \in Y \), and \( j \in \mathbb{N} \), one has

\[
\tau_{e_i}(y)|_j = \rho(C_6)\theta_{e_i} \rho^{-1}(y)|_j = \theta_{e_i} \rho^{-1}(y)|_{j|} = \theta_{e_i} \Psi(y)|_{j|} = \Psi(S_i(y))|_{j|} = \alpha_{j|}(S_i(y))|_0 = S_j^i(S_i(y))|_0 = S_i(y)|_j;
\]

so \( \tau_{e_i} = S_i \).

\(\square\)

**7.3. Proposition.** Suppose that \( X \) and \( X' \) are \( k \)-subshifts over the same alphabet \( \Sigma \) such that \( \rho_C(X) = \rho_C(X') \), and such that the canonical actions \( \tau \) and \( \tau' \) coincide. Then \( X = X' \).

**Proof.** Given \( x \in X \), and \( n \in \mathbb{N}^k \), we have seen in (5.3) that \( \tau_n \rho_C(x) = \rho_C \theta_n(x) \). Therefore

\[
x(n) = \theta_n(x)|_{0|} = \rho_C \theta_n(x)|_0 = \tau_n \rho_C(x)|_0.
\]

This says that \( x \) may be reconstructed from \( \rho_C(x) \) together with the canonical action \( \tau \), from where the result follows. \(\square\)

Summarizing our last two results we have:
7.4. Corollary. Given \( k \geq 1 \) and an alphabet \( \Sigma \), the correspondence
\[
X \mapsto (\rho_{\mathcal{N}}(X); \tau_{e_1}, \tau_{e_2}, \ldots, \tau_{e_k})
\]
establishes a one-to-one correspondence from the collection of all \( k \)-subshifts \( X \subseteq \Sigma^\mathbb{N}^k \) onto the collection of all weak \( k \)-automata on the alphabet \( \Sigma \).

The adjective “weak” employed in Definition (7.1) is meant to highlight the fact that the \( S_i \) mentioned there are not actually cellular automata, although they share with the latter the important property of commuting with the shift, a property we saw in (3.5) to characterize true cellular automata in the uniformly continuous case.

Nevertheless it is interesting to determine necessary and sufficient conditions on a given \( k \)-subshift for the maps \( S_i \) in the weak \( k \)-automata associated to it by (7.4) to be actual cellular automata. In order to do this we must first consider a metric on \( \Sigma^\mathbb{N}^k \) as follows.

7.5. Definition. We shall say that two elements \( x \) and \( y \) of \( \Sigma^\mathbb{N}^k \) agree on a given subset \( A \subseteq \mathbb{N}^k \), whenever \( x(n) = y(n) \), for all \( n \) in \( A \). If \( p \) is the largest integer such that \( x \) and \( y \) agree on the subset
\[
B_p := \{ n = (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k : n_i \leq p, \text{ for all } i \},
\]
we put
\[
d(x, y) = 2^{-p},
\]
with the understanding that if \( x = y \), then \( p = \infty \), in which case \( d(x, y) = 0 \).

As in the case of section (3), one proves that \( d \) is a metric on \( \Sigma^\mathbb{N}^k \), compatible with the product topology, and we may then speak of uniformly continuous functions defined, or taking values, in \( \Sigma^\mathbb{N}^k \).

The above choice of the \( B_p \) is not so crucial except for the fact that the \( B_p \) form an increasing sequence of finite subsets of \( \mathbb{N}^k \), whose union coincides with \( \mathbb{N}^k \). Any other choice of finite subsets with these properties may also be used to define a metric on \( \Sigma^\mathbb{N}^k \), which in turn induce the same uniform structure [3: Chapter 6] on \( \Sigma^\mathbb{N}^k \). The common underlying uniform structure is in fact what really matters here, and it says that two points \( x \) and \( y \) are close if and only if they agree on a large finite subset of \( \mathbb{N}^k \).

We shall however not make any explicit use of uniform structures in this work, beyond the elementary observation that
\[
d(x, y) \leq 2^{-p} \iff x \text{ and } y \text{ agree on } B_p,
\]
for every \( x \) and \( y \) in \( \Sigma^\mathbb{N}^k \).

7.7. Proposition. The restriction map
\[
\rho_{\mathbb{N}} : \Sigma^\mathbb{N}^k \to \Sigma^\mathbb{N},
\]
defined in (2.6) is uniformly continuous.
Proof. Given $x$ and $y$ in $\Sigma^{\mathbb{N}}$, let $p$ be the largest integer such that $x$ and $y$ agree on $B_p$. Therefore $\rho_N(x)$ and $\rho_N(y)$ obviously agree on $\{0, 1, \ldots, p\}$, so

$$d(\rho_N(x), \rho_N(y)) \leq 2^{-p} = d(x, y).$$

This proves that $\rho_N$ is in fact contractive, hence uniformly continuous. \qed

If $X \subseteq \Sigma^{\mathbb{N}}$ is a $k$-subshift, then $\rho_N$ is a homeomorphism on $X$ by definition, so $\rho_N^{-1}$ is continuous on $\rho_N(X)$, although perhaps not uniformly.

7.8. Definition.
(a) By a uniform $k$-subshift we shall mean a $k$-subshift $X \subseteq \Sigma^{\mathbb{N}}$ such that the inverse of $\rho_N$ is uniformly continuous on $\rho_N(X)$.
(b) By a $k$-automaton we shall mean a weak $k$-automaton $(Y; S_1, S_2, \ldots, S_k)$ such that each $S_i$ is actually a cellular automaton.

In what follows we will show that the two concepts just defined are related to each other by the same process involved in (7.4).

7.9. Proposition. Let $X \subseteq \Sigma^{\mathbb{N}}$ be a $k$-subshift. Then $X$ is a uniform $k$-subshift if and only if its associated weak $k$-automaton $(\rho_N(X); \tau_{e_1}, \tau_{e_2}, \ldots, \tau_{e_k})$ is an actual $k$-automaton.

Proof. Recall from Diagram (5.3) that

$$\tau_n = \rho_N \theta_n \rho_N^{-1},$$

for every $n$ in $\mathbb{N}^k$. Assuming that $X$ is a uniform $k$-subshift, we have that $\rho_N^{-1}$ is uniformly continuous. Leaving for the reader the easy task of proving that $\theta_n$ is also uniformly continuous, we deduce that $\tau_n$ is likewise uniformly continuous. Since we have already seen that $\tau_n$ commutes with the shift, we deduce from (3.5) that $\tau_n$ is a cellular automaton, and in particular so are the $\tau_{e_i}$. This completes the proof of the “only if” part of the statement.

Writing $S_i$ for $\tau_{e_i}$, assume now that each $S_i$ is a cellular automaton, hence a uniformly continuous map. Setting $Y = \rho_N(X)$, recall from the proof of (7.2) that the inverse of $\rho_N$ on $Y$ is the map $\Psi$ given by

$$\Psi(y)|_n = \alpha_n(y)|_0, \quad \forall y \in Y, \quad \forall n \in \mathbb{N}^k,$$

where

$$\alpha_n = S_1^{n_1} S_2^{n_2} \cdots S_k^{n_k},$$

for each $n = (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$.

Since each $S_i$, is uniformly continuous, the same is true for $\alpha_n$, so it follows from (3.1) that $\Psi(y)|_n$ depends only on finitely many coordinates of $y$.

Given $\varepsilon > 0$, choose $p \in \mathbb{N}$ such that $2^{-p} < \varepsilon$. Based on the conclusion of the above paragraph, let $q$ be a positive integer such that for every $n \in B_p$, one has that $\Psi(y)|_n$ depends only on

$$(y(0), y(1), \ldots y(q)).$$
Setting $\delta = 2^{-q}$, assume that $y, y' \in Y$ are such that $d(y, y') \leq \delta$. Then $y$ and $y'$ agree on \{0, 1, \ldots, q\}, whence
\[ \Psi(y)|_n = \Psi(y')|_n, \quad \forall n \in B_p, \]
and we then conclude that
\[ d(\Psi(y), \Psi(y')) \leq 2^{-p} < \varepsilon. \]
This shows that $\Psi$, and hence also $\rho_{\xi}^{-1}$ is uniformly continuous, which in turn says that $X$ is a uniform $k$-subshift. \qed

Given a $k$-graph $\Lambda$, recall from the paragraph immediately after (5.1), that the associated $\mathbb{N}^k$-subshift $X_\Lambda$ built from $\Lambda$ according to (4.2) is an example of a $k$-subshift. Moreover, as seen in Section (6), the weak $k$-automaton
\[ (\rho_{\xi}(X_\Lambda); S_1, \ldots, S_k) \]
associated to $X_\Lambda$ via (7.4) is such that the $S_i$ are cellular automata with anticipation 1, and hence we in fact have a $k$-automaton, according to Definition (7.8.b). It then follows from (7.9) that $X_\Lambda$ is a uniform $k$-subshift.

It would therefore be interesting to determine conditions on a given $k$-automaton $(Y; S_1, \ldots, S_k)$, which would imply that it comes from a $k$-graph in the sense that the associated action of $\mathbb{N}^k$ on $Y$ is conjugate to one coming from the $k$-automaton arising from a $k$-graph. By (4.3), requiring $Y$ to be a finite type subshift will likely be among these conditions.

8. Renault-Deaconu groupoids.

A submonoid $P$ of a discrete group $G$ satisfying $P^{-1}P \subseteq PP^{-1}$ is called an Ore monoid. Given a right action $\alpha$ of $P$ on a locally compact, Hausdorff, topological space $X$ by means of local homeomorphisms, as in [2: Section 2] (see also [6]), one may build the corresponding Renault-Deaconu groupoid
\[ X \rtimes_\alpha P = \{(x, g, y) \in X \times G \times X : \exists n, m \in P, \alpha_n(x) = \alpha_m(y), g = nm^{-1}\}, \]
also called the transformation, or semidirect product groupoid. By [2: 3.2], one has that $X \rtimes_\alpha P$ is an étale groupoid with the topology generated by the sets of the form
\[ U(n, m, A, B) = \{(x, nm^{-1}, y) : x \in A, y \in B, \alpha_n(x) = \alpha_m(y)\}, \]
where $A, B \subseteq X$ are open subsets, and $n, m \in P$.

If $X$ is a given subshift of rank $k$, we of course have a natural action of the Ore monoid $\mathbb{N}^k$ on $X$, but this might not put us in the situation of the paragraph above since $k$-subshifts are not necessarily locally compact, and neither is the action of $\mathbb{N}^k$ by local homeomorphisms. In fact, even when $k = 1$ and the alphabet is finite (in which case at least $X$ is compact), namely in the case of a classical subshift, the shift map itself may
fail to be a local homeomorphisms; even worse, it may fail to be an open mapping. By [1: 2.5], only subshifts of finite type are open.

The $k$-subshift $X_{\Lambda}$ arising from a row-finite $k$-graph $\Lambda$ fortunately does not suffer from such tribulations: the path space $\Lambda^{\infty}$ is locally compact by [4: Lemma 2.6], and each $\sigma^p$ is a local homeomorphisms by [4: Remarks 2.5]. In view of the commutative diagram

\[
\begin{array}{cccc}
\Lambda^{\infty} & \xrightarrow{\Xi} & X_{\Lambda} & \xrightarrow{\rho_N} & M_{\Lambda} \\
\downarrow{\sigma^p} & & \downarrow{\theta_p} & & \downarrow{\tau_p} \\
\Lambda^{\infty} & \xrightarrow{\Xi} & X_{\Lambda} & \xrightarrow{\rho_N} & M_{\Lambda}
\end{array}
\]

where we write $M_{\Lambda}$ for $\rho_N(X_{\Lambda})$, in which all vertical arrows are homeomorphisms, we then have that $X_{\Lambda}$ and $M_{\Lambda}$ are locally-compact spaces, and both $\theta_p$ and $\tau_p$ are local homeomorphisms. We may therefore form the Renault-Deaconu groupoids relative to the actions $\sigma$, $\theta$ and $\tau$, obtaining the following three evidently isomorphic groupoids:

\[\Lambda^{\infty} \rtimes_{\sigma} \mathbb{N}, \quad X_{\Lambda} \rtimes_{\theta} \mathbb{N}^k, \quad \text{and} \quad M_{\Lambda} \rtimes_{\tau} \mathbb{N}^k.\]

The first one above has already explicitly appeared in [4: Definition 2.7], where it was called the path groupoid of $\Lambda$, while playing a prominent role given that its groupoid $C^*$-algebra is isomorphic [4: Corollary 3.5] to the higher rank graph $C^*$-algebra $C^*(\Lambda)$.

Evidently we now see that $C^*(\Lambda)$ may also be modeled by the weak $k$-automaton $M_{\Lambda}$. So let us formally state this as one of our main conclusions.

**8.1. Theorem.** Given a row-finite $k$-graph $\Lambda$, let $\Sigma$ be the alphabet consisting of all morphisms $\lambda$ with $d(\lambda) = (1, 1, \ldots, 1)$, and let $A = \{A_{\lambda\mu}\}_{\lambda, \mu \in \Sigma}$ be the 0-1 matrix such that $A_{\lambda\mu} = 1$, if and only if $s(\lambda) = r(\mu)$. Then there are $k$ pairwise commuting cellular automata $S_1, S_2, \ldots, S_k$, whose product coincide with Markov’s shift on the Markov space $X_{\Lambda}$. Each $S_i$ is moreover a local homeomorphism and, denoting by $\tau$ the action of $\mathbb{N}^k$ on $X_{\Lambda}$ obtained by iterating the $S_i$, one has that the semidirect product groupoid $X_{\Lambda} \rtimes_{\tau} \mathbb{N}^k$ is a model for the higher rank graph $C^*$-algebra in the sense that $C^*(X_{\Lambda} \rtimes_{\tau} \mathbb{N}^k)$ and $C^*(\Lambda)$ are isomorphic.

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