The reciprocals of tails of the alternating Riemann zeta function

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Abstract
In this paper, we give the integer parts of reciprocals of tails of the alternating Riemann zeta function at $s = 1, 2, 3, 4$ by using several new inequalities and elementary method.

Keywords Riemann zeta function, Alternating Riemann zeta function, inequality.

2010 Mathematics Subject Classification 11M06, 11B83, 11J70.

1 Introduction
In this paper, we study the reciprocals of tails of the alternating Riemann zeta function. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s},$$

where $s \in \mathbb{C}$. The above series converges absolutely for $\text{Re}(s) > 1$, and it can be analytically continued to the whole complex plane except for a simple pole $s = 1$ of residue 1. The alternating Riemann zeta function is defined by

$$\zeta^*(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s},$$

which converges for $\text{Re}(s) > 0$. As

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \zeta^*(s)$$

for $\text{Re}(s) > 1$, the above gives an explicit analytic continuation of the Riemann zeta function to the half plane $\text{Re}(s) > 0$ by the alternating Riemann zeta function.

There are various elegant properties of the Riemann zeta function. Recently, some authors started to study the tails of the Riemann zeta function and the tails of the alternating Riemann zeta function, which are defined respectively by

$$\zeta_n(s) = \sum_{k=n}^{\infty} \frac{1}{k^s},$$
where \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), and
\[
\zeta_n^*(s) = \sum_{k=n}^{\infty} \frac{(-1)^{k+1}}{k^s},
\]
where \( s \in \mathbb{C} \) with \( \text{Re}(s) > 0 \). For example, in 2016 Lin [5] studied the integer parts of reciprocals of tails of the Riemann zeta function at the integer point \( s \geq 2 \) and proved that
\[
\lfloor \zeta_n(2)^{-1} \rfloor = n - 1,
\]
and
\[
\lfloor \zeta_n(3)^{-1} \rfloor = 2n(n - 1),
\]
where \( n \) is any positive integer and \( \lfloor x \rfloor \) denotes the largest integer less than or equal to \( x \). Soon afterwards, Lin and Li [6] considered the computational formula for the case \( s = 4 \), and they obtained:
\[
\lfloor \zeta_n(4)^{-1} \rfloor = \begin{cases} 
24m^3 - 18m^2 + \left\lfloor \frac{3(5m-1)}{2} \right\rfloor & \text{if } n = 2m; \\
24m^3 - 54m^2 + \left\lfloor \frac{3(58m-17)}{4} \right\rfloor & \text{if } n = 2m - 1,
\end{cases}
\]
for any positive integer \( n \). Along this line, Xu [8] also proved two computation formulas for \( s = 4, 5 \), and Hwang and Song [1] obtained a complicated formula for the case \( s = 6 \), which depends on the residue of \( n \) modulo 48. In 2018, Kim and Song [4] studied the integer parts of the inverses of tails of the alternating Riemann zeta function for \( s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \), and they obtained that for any positive integer \( n \) and \( s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \),
\[
\lfloor \zeta_n^*(s)^{-1} \rfloor = \left\lfloor (-1)^{n+1} 2 \left( n - \frac{1}{2} \right)^s \right\rfloor.
\]
Later, Hwang and Song [2] proved that when \( s = \frac{1}{p} \) for any integer with \( p \geq 5 \) or \( s = \frac{2}{p} \) for any odd integer with \( p \geq 5 \), there exists an integer \( N > 0 \) such that the formula (1.1) still holds for every integer \( n \geq N \).

We want to mention that some similar questions on Fibonacci numbers have already been considered. For example, in 2008 Ohtsuka and Nakamura [7] studied the properties of infinite sums of reciprocal Fibonacci numbers, and they proved that
\[
\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} 
F_{n-2} & \text{if } n \geq 2 \text{ is even}; \\
F_{n-2} - 1 & \text{if } n \geq 1 \text{ is odd},
\end{cases}
\]
and
\[
\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} 
F_{n-1}F_n - 1 & \text{if } n \geq 2 \text{ is even}; \\
F_{n-1}F_n & \text{if } n \geq 1 \text{ is odd}.
\end{cases}
\]
Here, the Fibonacci sequence \( \{F_k\} \) is defined by the recursive formula \( F_{k+1} = F_k + F_{k-1} \) with \( k \geq 1 \) and the initial values \( F_0 = 0 \) and \( F_1 = 1 \). And then in 2013, Kuhapatanakul [3] considered the infinite sums of reciprocal generalized Fibonacci numbers defined by
the recursive formula $V_{k+1} = aV_k + bV_{k-1}$ and the initial values $V_0 = c$ and $V_1 = 1$. And it was shown in [3] that

$$\left( \sum_{k=n}^{\infty} \frac{(-1)^k}{V_k} \right)^{-1} = (-1)^n(V_n + V_{n-1}) - 1,$$

for any positive integer $n$, integers $a, b$ with $1 \leq b \leq a$ and $c = 0$. A direct corollary of the above formula is the following result for the Fibonacci numbers:

$$\left( \sum_{k=n}^{\infty} \frac{(-1)^k}{F_k} \right)^{-1} = (-1)^nF_{n+1} - 1,$$

where $n$ is any positive integer.

Inspired by their results, we try to find the closed formulas of the integer parts of reciprocals of tails of the alternating Riemann zeta function at integer points. By using elementary method and several new inequalities, we obtain four interesting computational formulas for $\lfloor \zeta_n^*(s)^{-1} \rfloor$ with $s = 1, 2, 3, 4$. That is, we shall prove the following four theorems.

**Theorem 1.1.** For a positive integer $n$, we have

$$\lfloor \zeta_n^*(1)^{-1} \rfloor = \begin{cases} -2n & \text{if } n \geq 2 \text{ is even;} \\ 2n - 1 & \text{if } n \geq 1 \text{ is odd.} \end{cases}$$

**Theorem 1.2.** For a positive integer $n$, we have

$$\lfloor \zeta_n^*(2)^{-1} \rfloor = \begin{cases} -(2n^2 - 2n + 1) - 1 & \text{if } n \geq 2 \text{ is even;} \\ 2n^2 - 2n + 1 & \text{if } n \geq 1 \text{ is odd.} \end{cases}$$

**Theorem 1.3.** For a positive integer $n$, we have

$$\lfloor \zeta_n^*(3)^{-1} \rfloor = \begin{cases} -(2n^3 - 3n^2 + \frac{9}{2}n - \frac{5}{2}) - \frac{3}{2} & \text{if } n \geq 22 \text{ is even;} \\ 2n^3 - 3n^2 + \frac{9}{2}n - \frac{5}{2} & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

**Theorem 1.4.** For a positive integer $n$, we have

$$\lfloor \zeta_n^*(4)^{-1} \rfloor = \begin{cases} -(2n^4 - 4n^3 + 8n^2 - 6n - 8) - 1 & \text{if } n \geq 10 \text{ is even;} \\ 2n^4 - 4n^3 + 8n^2 - 6n - 8 & \text{if } n \geq 11 \text{ is odd.} \end{cases}$$

The structure of this paper is as follows. In Section 2, we construct several inequalities which are necessary to the proofs of our theorems. In Section 3, we firstly prove Theorem 3.1 which is a unified idea for all integers $s \geq 1$, and then the proofs of Theorems 1.1 - 1.4 are given respectively.
2 Several Inequalities

For a fixed integer $s$ with $1 \leq s \leq 4$, we want to prove there exist functions $f_s(k), g_s(k)$ with $\lim_{k \to \infty} f_s(k) = \lim_{k \to \infty} g_s(k) = \infty$ and integers $k_{s,\text{even}}, k_{s,\text{odd}} \geq 1$ such that

$$\frac{1}{f_s(k) + 1} - \frac{1}{f_s(k+1) + 1} < -\frac{1}{(2k)^s} + \frac{1}{(2k+1)^s} < \frac{1}{f_s(k)} - \frac{1}{f_s(k+1)}$$

holds for any integer $k \geq k_{s,\text{even}}$, and

$$\frac{1}{g_s(k) + 1} - \frac{1}{g_s(k+1) + 1} < -\frac{1}{(2k-1)^{s'}} + \frac{1}{(2k)^{s'}} < \frac{1}{g_s(k)} - \frac{1}{g_s(k+1)}$$

holds for any integer $k \geq k_{s,\text{odd}}$.

**Lemma 2.1.** Let $f_1(k) = -4k$ and $g_1(k) = 4k - 3$. For any positive integer $k$, we have

$$\frac{1}{f_1(k) + 1} - \frac{1}{f_1(k+1) + 1} < -\frac{1}{2k} + \frac{1}{2k+1} < \frac{1}{f_1(k)} - \frac{1}{f_1(k+1)} \tag{2.1}$$

and

$$\frac{1}{g_1(k) + 1} - \frac{1}{g_1(k+1) + 1} < -\frac{1}{2k} - \frac{1}{2k} < \frac{1}{g_1(k)} - \frac{1}{g_1(k+1)}. \tag{2.2}$$

**Proof.** As for any positive integer $k$ it holds

$$-\frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{4k} - \frac{1}{4k+4} = -\frac{1}{4k(k+1)(2k+1)} < 0,$$

the right-hand side of (2.1) is proved. For the left-hand side of (2.1), we use

$$-\frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{4k-1} - \frac{1}{4k+3} = \frac{3}{2k(2k+1)(4k-1)(4k+3)} > 0$$

which holds for any positive integer $k$.

Similarly, let’s prove the inequality (2.2). The right-hand side follows from

$$\frac{1}{2k-1} - \frac{1}{2k} - \frac{1}{4k-3} + \frac{1}{4k+1} = -\frac{3}{2k(2k-1)(4k-3)(4k+1)} < 0,$$

and the left-hand side follows from

$$\frac{1}{2k-1} - \frac{1}{2k} - \frac{1}{4k-2} + \frac{1}{4k+2} = \frac{1}{2k(2k-1)(2k+1)} > 0.$$

This completes the proof of this lemma. \qed

**Lemma 2.2.** Let $f_2(k) = -2(2k - \frac{1}{2})^2 - \frac{3}{2}$ and $g_2(k) = 2(2k - \frac{3}{2})^2 + \frac{1}{2}$. For any positive integer $k$, we have

$$\frac{1}{f_2(k) + 1} - \frac{1}{f_2(k+1) + 1} < -\frac{1}{(2k)^2} + \frac{1}{(2k+1)^2} < \frac{1}{f_2(k)} - \frac{1}{f_2(k+1)} \tag{2.3}$$

and

$$\frac{1}{g_2(k) + 1} - \frac{1}{g_2(k+1) + 1} < -\frac{1}{(2k-1)^2} - \frac{1}{(2k)^2} < \frac{1}{g_2(k)} - \frac{1}{g_2(k+1)}. \tag{2.4}$$
Proof. Since
\[
-\frac{1}{(2k)^2} + \frac{1}{(2k+1)^2} = -\frac{4k+1}{16k^4 + 16k^3 + 4k^2}
\]
and
\[
\frac{1}{f_2(k)} - \frac{1}{f_2(k+1)} = -\frac{16k+4}{64k^4 + 64k^3 + 16k^2 + 12},
\]
the right-hand side of (2.3) is equivalent to
\[(16k + 4) \left(16k^4 + 16k^3 + 4k^2\right) < (4k + 1) \left(64k^4 + 64k^3 + 16k^2 + 12\right).
\]
The above inequality is just \(12 > 0\), which holds for any positive integer \(k\). Using
\[
\frac{1}{f_2(k) + 1} - \frac{1}{f_2(k+1) + 1} = -\frac{16k+4}{64k^4 + 64k^3 - 8k + 5},
\]
we find the left-hand side of (2.3) is equivalent to
\[(16k + 4) \left(16k^4 + 16k^3 + 4k^2\right) > (4k + 1) \left(64k^4 + 64k^3 - 8k + 5\right).
\]
Through proper simplification, the above inequality is just
\[16k^2 + 8k - 5 > 0,
\]
which holds for any positive integer \(k\). So this completes the proof of the inequality (2.3).

Similarly, for the inequality (2.4), we have
\[
\frac{1}{(2k - 1)^2} - \frac{1}{(2k)^2} = \frac{4k - 1}{16k^4 - 16k^3 + 4k^2}.
\]
Since
\[
\frac{1}{g_2(k)} - \frac{1}{g_2(k+1)} = \frac{16k - 4}{64k^4 - 64k^3 + 8k + 5},
\]
through proper simplification, the right-hand side of (2.4) is just
\[16k^2 - 8k - 5 > 0,
\]
which holds for any positive integer \(k\). For the left-hand side, we have
\[
\frac{1}{g_2(k) + 1} - \frac{1}{g_2(k+1) + 1} = \frac{16k - 4}{64k^4 - 64k^3 + 16k^2 + 12}.
\]
Then the left-hand side of (2.4) is just \(12 > 0\), which holds for any positive integer \(k\). So this completes the proof of the inequality (2.4).

Lemma 2.3. Let \(f_3(k) = -16k^3 + 12k^2 - 9k + 1\). For any positive integer \(k \geq 11\), we have
\[
\frac{1}{f_3(k) + 1} - \frac{1}{f_3(k+1) + 1} < -\frac{1}{(2k)^3} + \frac{1}{(2k+1)^3} < \frac{1}{f_3(k)} - \frac{1}{f_3(k+1)}. \quad (2.5)
\]
Proof. We have
\[
-\frac{1}{(2k)^3} + \frac{1}{(2k+1)^3} = -\frac{12k^2 + 6k + 1}{64k^6 + 96k^5 + 48k^4 + 8k^3}.
\]
Since
\[
\frac{1}{f_3(k)} - \frac{1}{f_3(k+1)} = -\frac{48k^2 + 24k + 13}{256k^6 + 384k^5 + 240k^4 + 104k^3 + 117k^2 + 75k - 12},
\]
the right-hand side of (2.5) is equivalent to
\[
(12k^2 + 6k + 1) (256k^6 + 384k^5 + 240k^4 + 104k^3 + 117k^2 + 75k - 12) >
(48k^2 + 24k + 13) (64k^6 + 96k^5 + 48k^4 + 8k^3).
\]
The above inequality is just
\[
288k^5 + 1452k^4 + 1602k^3 + 423k^2 + 3k - 12 > 0,
\]
which holds for all integers \( k \geq 1 \). Since
\[
\frac{1}{f_3(k+1)} - \frac{1}{f_3(k+2)} = -\frac{48k^2 + 24k + 13}{256k^6 + 384k^5 + 240k^4 + 72k^3 + 93k^2 + 33k - 22},
\]
the left-hand side of (2.5) is equivalent to
\[
(12k^2 + 6k + 1) (256k^6 + 384k^5 + 240k^4 + 72k^3 + 93k^2 + 33k - 22) <
(48k^2 + 24k + 13) (64k^6 + 96k^5 + 48k^4 + 8k^3).
\]
The above inequality is just
\[
96k^5 - 972k^4 - 922k^3 - 27k^2 + 99k + 22 > 0,
\]
which holds for all integers \( k \geq 11 \). So this completes the proof.

Lemma 2.4. Let \( g_3(k) = 16k^3 - 36k^2 + 33k - 12 \). For any positive integer \( k \geq 4 \), we have
\[
\frac{1}{g_3(k+1)} - \frac{1}{g_3(k+2)} < \frac{1}{(2k-1)^3} - \frac{1}{(2k)^3} < \frac{1}{g_3(k)} - \frac{1}{g_3(k+1)}, \tag{2.6}
\]
Proof. We have
\[
\frac{1}{(2k-1)^3} - \frac{1}{(2k)^3} = \frac{12k^2 - 6k + 1}{64k^6 - 96k^5 + 48k^4 - 8k^3}.
\]
Since
\[
\frac{1}{g_3(k)} - \frac{1}{g_3(k+1)} = \frac{48k^2 - 24k + 13}{256k^6 - 384k^5 + 240k^4 - 104k^3 + 117k^2 - 75k - 12},
\]
the right-hand side of (2.6) is equivalent to
\[
(12k^2 - 6k + 1) (256k^6 - 384k^5 + 240k^4 - 104k^3 + 117k^2 - 75k - 12) <
\[(48k^2 - 24k + 13) \cdot (64k^6 - 96k^5 + 48k^4 - 8k^3)\].

The above inequality is just
\[288k^5 - 1452k^4 + 1602k^3 - 423k^2 + 3k + 12 > 0,\]
which holds for all integers \(k \geq 4\). Since
\[
\frac{1}{g_3(k) + 1} - \frac{1}{g_3(k + 1) + 1} = \frac{48k^2 - 24k + 13}{256k^6 - 384k^5 + 240k^4 - 72k^3 + 93k^2 - 33k - 22},
\]
the left-hand side of (2.6) is equivalent to
\[
(12k^2 - 6k + 1) \cdot (256k^6 - 384k^5 + 240k^4 - 72k^3 + 93k^2 - 33k - 22) >
(48k^2 - 24k + 13) \cdot (64k^6 - 96k^5 + 48k^4 - 8k^3).
\]
The above inequality is just
\[96k^5 + 972k^4 - 922k^3 + 27k^2 + 99k - 22 > 0,\]
which holds for all integers \(k \geq 1\). This proves the inequality (2.6). □

**Lemma 2.5.** Let \(f_4(k) = -32k^4 + 32k^3 - 32k^2 + 12k + 7\). For any positive integer \(k \geq 5\), we have
\[
\frac{1}{f_4(k) + 1} - \frac{1}{f_4(k + 1) + 1} < -\frac{1}{(2k)^4} + \frac{1}{(2k + 1)^4} < \frac{1}{f_4(k)} - \frac{1}{f_4(k + 1)}. \tag{2.7}
\]

**Proof.** We have
\[
-\frac{1}{(2k)^4} + \frac{1}{(2k + 1)^4} = -\frac{32k^3 + 24k^2 + 8k + 1}{256k^8 + 512k^7 + 384k^6 + 128k^5 + 16k^4}.
\]
Since
\[
\frac{1}{f_4(k)} - \frac{1}{f_4(k + 1)} = -\frac{128k^3 + 96k^2 + 96k + 20}{1024k^8 + 2048k^7 + 2048k^6 + 1280k^5 + 448k^4 + 64k^3 - 1488k^2 - 744k - 91},
\]
the right-hand side of (2.7) is equivalent to
\[2048k^7 + 3584k^6 - 45312k^5 - 58880k^4 - 32608k^3 - 9624k^2 - 1472k - 91 > 0,\]
which holds for all integers \(k \geq 5\). Since
\[
\frac{1}{f_4(k) + 1} - \frac{1}{f_4(k + 1) + 1} = -\frac{128k^3 + 96k^2 + 96k + 20}{1024k^8 + 2048k^7 + 2048k^6 + 1280k^5 + 384k^4 - 1648k^2 - 816k - 96},
\]
the left-hand side of (2.7) is equivalent to
\[52480k^5 + 65600k^4 + 35840k^3 + 10480k^2 + 1584k + 96 > 0,\]
which holds for all integers \(k \geq 1\). This completes the proof. □
Lemma 2.6. Let \( g_4(k) = 32k^4 - 96k^3 + 128k^2 - 84k + 12 \). For any positive integer \( k \geq 6 \), we have

\[
\frac{1}{g_4(k) + 1} \frac{1}{g_4(k + 1) + 1} < \frac{1}{(2k - 1)^4} \frac{1}{(2k)^4} < \frac{1}{g_4(k)} \frac{1}{g_4(k + 1)},
\]

(2.8)

Proof. We have

\[
\frac{1}{(2k - 1)^4} - \frac{1}{(2k)^4} = \frac{32k^3 - 24k^2 + 8k - 1}{256k^8 - 512k^7 + 384k^6 - 128k^5 + 16k^4}.
\]

Since

\[
\frac{1}{g_4(k)} - \frac{1}{g_4(k + 1)} = \frac{128k^3 - 96k^2 + 96k - 20}{1024k^8 - 2048k^7 + 2048k^6 - 1280k^5 + 384k^4 - 16k^2 + 32}.
\]

the denominator of the above formula is greater than zero if \( k \geq 2 \). Then for any integer \( k \geq 2 \), the right-hand side of (2.8) is equivalent to

\[
52480k^5 - 65600k^4 + 35840k^3 - 10480k^2 + 1584k - 96 > 0,
\]

which holds for all integers \( k \geq 2 \). Since

\[
\frac{1}{g_4(k) + 1} - \frac{1}{g_4(k + 1) + 1} = \frac{128k^3 - 96k^2 + 96k - 20}{1024k^8 - 2048k^7 + 2048k^6 - 1280k^5 + 384k^4 - 16k^2 + 32},
\]

the denominator of the above formula is greater than zero if \( k \geq 2 \). Then for any integer \( k \geq 2 \), the left-hand side of (2.8) is equivalent to

\[
2048k^7 - 3584k^6 - 45312k^5 + 58880k^4 - 32608k^3 + 9624k^2 - 1472k + 91 > 0,
\]

which holds for all integers \( k \geq 6 \). This completes the proof.

\[\square\]

3 Proofs of the Theorems

For proving Theorems 1.1-1.4, we give a unified idea of their proofs.

Theorem 3.1. Assume that for any positive integer \( s \geq 1 \), there exist functions \( f_s(x), g_s(x) \in \mathbb{Q}[x] \), such that

(i) \( \lim_{k \to \infty} f_s(k) = \lim_{k \to \infty} g_s(k) = \infty \);

(ii) there exists a positive integer \( k_{s,even} \), such that

\[
\frac{1}{f_s(k) + 1} - \frac{1}{f_s(k + 1) + 1} < \frac{1}{(2k)^s} + \frac{1}{(2k + 1)^s} < \frac{1}{f_s(k)} - \frac{1}{f_s(k + 1)} \quad (3.1)
\]

holds for any integer \( k \geq k_{s,even} \);
(iii) there exists a positive integer $k_{s,\text{odd}}$, such that

$$\frac{1}{g_s(k) + 1} - \frac{1}{g_s(k + 1) + 1} < \frac{1}{(2k - 1)^s} - \frac{1}{(2k)^s} < \frac{1}{g_s(k)} - \frac{1}{g_s(k + 1)}$$

(3.2)

holds for any integer $k \geq k_{s,\text{even}}$.

Then

$$\frac{1}{f_s \left(\frac{n}{2}\right) + 1} < \zeta_n^*(s) < \frac{1}{f_s \left(\frac{n}{2}\right)}$$

holds for any positive even number $n$ with $n \geq 2k_{s,\text{even}}$, and

$$\frac{1}{g_s \left(\frac{n+1}{2}\right) + 1} < \zeta_n^*(s) < \frac{1}{g_s \left(\frac{n+1}{2}\right)}$$

holds for any positive odd number $n$ with $n \geq 2k_{s,\text{odd}} - 1$.

**Proof.** Let $n$ be a positive even integer. We have

$$\zeta_n^*(s) = \sum_{k=\frac{n}{2}}^{\infty} \left( -\frac{1}{(2k)^s} + \frac{1}{(2k+1)^s} \right).$$

Using assumption (ii), we get

$$\sum_{k=\frac{n}{2}}^{\infty} \left( \frac{1}{f_s(k) + 1} - \frac{1}{f_s(k + 1) + 1} \right) < \sum_{k=\frac{n}{2}}^{\infty} \left( -\frac{1}{(2k)^s} + \frac{1}{(2k+1)^s} \right)$$

$$< \sum_{k=\frac{n}{2}}^{\infty} \left( \frac{1}{f_s(k)} - \frac{1}{f_s(k + 1)} \right),$$

which holds for any positive even number $n$ with $n \geq 2k_{s,\text{even}}$. From the formulas above and assumption (i), we obtain

$$\frac{1}{f_s \left(\frac{n}{2}\right) + 1} < \zeta_n^*(s) < \frac{1}{f_s \left(\frac{n}{2}\right)}$$

which holds for any positive even number $n$ with $n \geq 2k_{s,\text{even}}$.

Similarly, let $n$ be a positive odd number. We have

$$\zeta_n^*(s) = \sum_{k=\frac{n+1}{2}}^{\infty} \left( \frac{1}{(2k-1)^s} - \frac{1}{(2k)^s} \right).$$

Using assumption (iii), we obtain

$$\sum_{k=\frac{n+1}{2}}^{\infty} \left( \frac{1}{g_s(k) + 1} - \frac{1}{g_s(k + 1) + 1} \right) < \sum_{k=\frac{n+1}{2}}^{\infty} \left( \frac{1}{(2k-1)^s} - \frac{1}{(2k)^s} \right)$$

$$< \sum_{k=\frac{n+1}{2}}^{\infty} \left( \frac{1}{g_s(k)} - \frac{1}{g_s(k + 1)} \right),$$

which holds for any positive odd number $n$ with $n \geq 2k_{s,\text{odd}} - 1$. From the formulas above and assumption (i), we find

$$\frac{1}{g_s \left(\frac{n+1}{2}\right) + 1} < \zeta_n^*(s) < \frac{1}{g_s \left(\frac{n+1}{2}\right)}$$

which holds for any positive even number $n$ with $n \geq 2k_{s,\text{odd}} - 1$. □
3.1 Proof of Theorem 1.1

For \( s = 1 \), we take \( k_{1,\text{even}} = k_{1,\text{odd}} = 1 \), \( f_1(k) = -4k \) and \( g_1(k) = 4k - 3 \). Then \( f_1(k) \) and \( g_1(k) \) satisfy the three conditions of Theorem 3.1 by Lemma 2.1. Hence, using Theorem 3.1, we obtain that

\[-\frac{1}{2n-1} < \zeta^*_n(1) < -\frac{1}{2n}\]

holds for any positive even number \( n \), and

\[\frac{1}{2n} < \zeta^*_n(1) < \frac{1}{2n-1}\]

holds for any positive odd number \( n \). This completes the proof of Theorem 1.1. \( \square \)

3.2 Proof of Theorem 1.2

For \( s = 2 \), we take \( k_{2,\text{even}} = k_{2,\text{odd}} = 1 \), \( f_2(k) = -2(2k - \frac{1}{2})^2 - \frac{3}{2} \) and \( g_2(k) = 2(2k - \frac{3}{2})^2 + \frac{1}{2} \). Then using Lemma 2.2 and Theorem 3.1, we obtain

\[\frac{1}{f_2\left(\frac{n}{2}\right)} + 1 < \zeta^*_n(2) < \frac{1}{f_2\left(\frac{n}{2}\right)}\]

or equivalently

\[-\frac{1}{2n^2 - 2n + 1} < \zeta^*_n(2) < -\frac{1}{2n^2 - 2n + 2}\]

holds for any positive even number \( n \), and

\[\frac{1}{g_2\left(\frac{n+1}{2}\right)} + 1 < \zeta^*_n(2) < \frac{1}{g_2\left(\frac{n+1}{2}\right)}\]

or equivalently

\[\frac{1}{2n^2 - 2n + 2} < \zeta^*_n(2) < \frac{1}{2n^2 - 2n + 1}\]

holds for any positive odd number \( n \). So this proves Theorem 1.2. \( \square \)

3.3 Proof of Theorem 1.3

For \( s = 3 \), we take \( k_{3,\text{even}} = 11 \), \( k_{3,\text{odd}} = 4 \), \( f_3(k) = -16k^3 + 12k^2 - 9k + 1 \) and \( g_3(k) = 16k^3 - 36k^2 + 33k - 12 \). Then from Lemma 2.3, Lemma 2.4 and Theorem 3.1, we get

\[\frac{1}{f_3\left(\frac{n}{3}\right)} + 1 < \zeta^*_n(3) < \frac{1}{f_3\left(\frac{n}{3}\right)}\]

or equivalently

\[-\frac{1}{2n^3 - 3n^2 + \frac{9}{2}n - 2} < \zeta^*_n(3) < -\frac{1}{2n^3 - 3n^2 + \frac{9}{2}n - 1}\]

holds for any positive even number \( n \geq 22 \). So we have the inequality

\[-\left(2n^3 - 3n^2 + \frac{9}{2}n - 1\right) < \zeta^*_n(3)^{-1} < -\left(2n^3 - 3n^2 + \frac{9}{2}n - 2\right).\]
Since \(- (2n^3 - 3n^2 + \frac{9}{2}n - 1)\) and \(- (2n^3 - 3n^2 + \frac{9}{2}n - 2)\) are two consecutive integers, it follows that for any positive even number \(n \geq 22\),
\[
\lfloor \zeta_n^*(3)^{-1} \rfloor = - \left( 2n^3 - 3n^2 + \frac{9}{2}n - 1 \right).
\]
Similarly, for any positive odd number \(n \geq 7\), we get
\[
\frac{1}{f_3 \left( \frac{n+1}{2} \right) + 1} < \zeta_n^*(3) < \frac{1}{f_3 \left( \frac{n+1}{2} \right)},
\]
or equivalently
\[
\frac{1}{2n^3 - 3n^2 + \frac{9}{2}n - \frac{5}{2}} < \zeta_n^*(3) < \frac{1}{2n^3 - 3n^2 + \frac{9}{2}n - \frac{5}{2}}.
\]
Then we have the inequality
\[
2n^3 - 3n^2 + \frac{9}{2}n - \frac{5}{2} < \zeta_n^*(3)^{-1} < 2n^3 - 3n^2 + \frac{9}{2}n - \frac{3}{2}.
\]
Since \(2n^3 - 3n^2 + \frac{9}{2}n - \frac{3}{2}\) and \(2n^3 - 3n^2 + \frac{9}{2}n - \frac{5}{2}\) are two consecutive positive integers, it follows that for any positive odd number \(n \geq 7\),
\[
\lfloor \zeta_n^*(3)^{-1} \rfloor = 2n^3 - 3n^2 + \frac{9}{2}n - \frac{5}{2}.
\]
This completes the proof of Theorem 1.3.

3.4 Proof of Theorem 1.4

For \(s = 4\), we take \(k_{4,\text{even}} = 5\), \(k_{4,\text{odd}} = 6\), \(f_4(k) = -32k^4 + 32k^3 - 32k^2 + 12k + 7\) and \(g_4(k) = 32k^4 - 96k^3 + 128k^2 - 84k + 12\). Hence by Lemma 2.5, Lemma 2.6 and Theorem 3.1, we have
\[
\frac{1}{f_4 \left( \frac{n}{4} \right) + 1} < \zeta_n^*(4) < \frac{1}{f_4 \left( \frac{n}{4} \right)},
\]
or equivalently
\[
\frac{1}{2n^4 - 4n^3 + 8n^2 - 6n - 8} < \zeta_n^*(4) < \frac{1}{2n^4 - 4n^3 + 8n^2 - 6n - 7}
\]
holds for any positive even number \(n \geq 10\). Therefore, we find
\[
\lfloor \zeta_n^*(4)^{-1} \rfloor = - (2n^4 - 4n^3 + 8n^2 - 6n - 7)
\]
holds for any positive even number \(n \geq 10\). Similarly, for any positive odd number \(n \geq 11\), we have
\[
\frac{1}{g_4 \left( \frac{n+1}{2} \right) + 1} < \zeta_n^*(4) < \frac{1}{g_4 \left( \frac{n+1}{2} \right)},
\]
or equivalently
\[
\frac{1}{2n^4 - 4n^3 + 8n^2 - 6n - 7} < \zeta_n^*(4) < \frac{1}{2n^4 - 4n^3 + 8n^2 - 6n - 8}
\]
Then it follows that for any positive odd number \(n \geq 11\),
\[
\lfloor \zeta_n^*(4)^{-1} \rfloor = 2n^4 - 4n^3 + 8n^2 - 6n - 8.
\]
This completes the proof of Theorem 1.4.
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