On Finite Size Effects in $d = 2$ Quantum Gravity

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Abstract

A systematic investigation is given of finite size effects in $d = 2$ quantum gravity or equivalently the theory of dynamically triangulated random surfaces. For Ising models coupled to random surfaces, finite size effects are studied on the one hand by numerical generation of the partition function to arbitrary accuracy by a deterministic calculus, and on the other hand by an analytic theory based on the singularity analysis of the explicit parametric form of the free energy of the corresponding matrix model. Both these reveal that the form of the finite size corrections, not surprisingly, depend on the string susceptibility. For the general case where the parametric form of the matrix model free energy is not explicitly known, it is shown how to perform the singularity analysis. All these considerations also apply to other observables like susceptibility etc. In the case of the Ising model it is shown that the standard Fisher-scaling laws are reproduced.

A study of finite size effects in statistical systems is of importance for a variety of reasons. It is indispensable for a more reliable interpretation of numerical data. As shown by Cardy in the context of systems with conformal invariance, finite size effects also codify the spectrum of the theory. In most cases finite size effects are estimated or parametrised by extrapolating the results of simulations carried out for various sizes. It is the purpose of this article to show how such finite size effects for two dimensional quantum gravity systems can be studied systematically.

Let us begin with the case of so called "pure gravity" which is mapped
to the one-matrix model described by the partition function
\[
Z = e^{-F} = \int \mathcal{D}\phi \; \exp \left(-Tr\left[\frac{1}{2}\phi^2 + V(\phi)\right]\right) \tag{1}
\]
where \(\phi\) is a \(N \times N\) Hermitean matrix. When \(V(\phi) = g\phi^4\), the expression for \(F\) in the large-\(N\) limit is
\[
F = -\sum_n \frac{(-12g)^n(2n - 1)!}{n!(n + 2)!} \tag{2}
\]
interpreting the coefficient of \(g^n\) in this sum as the fixed area partition sum for an ensemble of random surfaces of fixed area \(n\), one finds
\[
Z_n = \frac{(-12g)^n(2n - 1)!}{n!(n + 2)!} \tag{3}
\]
This clearly holds for all \(n\). However, we are interested in the thermodynamic or the large-\(n\) limit. To this end we use
\[
\Gamma(z) \rightarrow z \rightarrow \infty \quad \frac{e^{-z}z^{z-1/2}}{\sqrt{2\pi}} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} + \ldots\right] \tag{4}
\]
for the Gamma-function \(\Gamma(z)\). This gives
\[
Z_n = \lim_{n \rightarrow \infty} \frac{(-12g)^n(2n - 1)!}{n!(n + 2)!} \left[1 - \frac{25}{8n} + \ldots\right] \tag{5}
\]
The exponential growth in \(n\) is non-universal and depends on \(V(\phi)\). The power law correction \(n^{-7/2}\) is universal and defines the string susceptibility \(\gamma\) to be \(\gamma = -b + 3\) when the power law correction is \(n^{-b}\). In this example the finite size corrections are of the form \([1 - \frac{25}{8n} + \ldots]\).

Next we consider the case of the Ising model coupled to random surfaces. This is described by the two matrix model defined by
\[
Z = e^{-F} = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \; \exp \left(-Tr\left[\frac{1}{2}(\phi_1^2 + \phi_2^2) - c\phi_1\phi_2 + (V(\phi_1) + V(\phi_2))\right]\right) \tag{6}
\]
This model is also exactly solvable. But the free energy is not known as an explicit function of \((c, g)\). However, a parametric form of the free energy is available in the form
\[
F = -\frac{1}{2} \log \frac{z}{g} - \frac{1}{g} \int_0^z \frac{dt}{t} g(t) + \frac{1}{2g^2} \int_0^z \frac{dt}{t} g(t)^2 + \frac{1}{2} \log(1 - c^2) + \frac{3}{4} \tag{7}
\]
where
\[ g(z) = \frac{z}{(1-3z)^2} - c^2z + 3c^2z^3 = g \]  

(8)

The numerical generation of \( Z_n \) is done through the following sequence of steps:i) eliminate the logarithms in \( F \) by considering instead \( \frac{\partial F}{\partial g} \), ii) eliminate inverse powers of \( 1/(1-3z) \) by repeated use of eqn(). These steps result in a polynomial of degree 8 for \( \frac{\partial F}{\partial g} \) with coefficients depending on \((c, g)\). Again repeated use of eqn() written as a quintic in \( z \) results in \( \frac{\partial F}{\partial g} \) being expressed as another quintic. Now the numerical technique to solve this quintic consists of writing

\[ z = \sum a_1(n)g^n; \quad z^2 = \sum a_2(n)g^n; \quad \ldots; \quad z^5 = \sum a_5(n)g^n \]

(9)

Then the quintic(eqn()) is equivalent to the recursion relation

\[ a_1(n) = \xi_1a_1(n-1) + \xi_2a_2(n) + \xi_3a_2(n-1) + \xi_4a_3(n) + \xi_6a_5(n) \]

(10)

where \( \xi_i \) are coefficients depending on \( c \). It should be noted that all \( a - i(n) \) vanish for negative \( n \) and in fact as \( z \simeq g \) for small \( g \) is the branch we are interested in, it follows that \( a_i(n) = 0 \) for \( n < i \) and that \( a_i(i) = a_1(1)^i \). It is also easy to see that \( a_1(1) = (1 - c^2)^{-1} \).

A naive method of solving these recursion relations would make use of the identities

\[ a_2(n) = \sum_{m=1}^{n-1} a_1(m) \cdot a_1(n-m) \]
\[ a_3(n) = \sum_{m=1}^{n-1} a_1(m) \cdot a_2(n-m) \]
\[ \ldots = \ldots \]
\[ a_5(n) = \sum_{m=1}^{n-1} a_1(m) \cdot a_4(n-m) \]

(11)

The computational requirements then grow as \( \simeq n^6 \) if all \( a_i(m); m \leq n \) are to be evaluated. This is prohibitive. On the other hand the structure of the eqns(11) shows that computation of \( a_2(n) \) only requires the knowledge of \( a_1(m) \) for \( m \leq n - 1 \), computation of \( a_3(n) \) requires only the knowledge
of $a_2(m)$ for $m \leq n - 1$ and hence of $a_1(m)$ for $m \leq n - 2$ etc. Thus the eqns (10) and (11) can be solved iteratively simultaneously and the computational requirement only grows as $\approx n^2$ which is entirely manageable.

**The Cosmological Constant Problem**

The main difficulty with the abovementioned method is that $a_i(n)$ grow very rapidly with $n$ reflecting the (non-universal) exponential growth of $Z_n$. For example, $Z_{10} \approx 10^{100}$. In fact by the time one gets to around 100, various coefficients exceed the typical machine limits. Of course, special purpose arithmetic can be used, but as we shall see soon, a more elegant option is available. The idea is to go up to some $n_{max}$ such that the coefficients are approaching the machine limits and get an estimate for the "cosmological constant" $\bar{\mu}$ where $Z_n \approx e^{\bar{\mu} n}$. Then one scales all $a_i(n)$ according to

$$\tilde{a}_i(n) = e^{-\bar{\mu} n} a_i(n)$$

for all $i$. This results in the scaling

$$\tilde{\xi}_1 = e^{-\bar{\mu}} \xi_1; \quad \tilde{\xi}_3 = e^{-\bar{\mu}} \xi_3$$

with all other $\xi$'s remaining the same. In the statistical mechanical language this is equivalent to tuning the system to "criticality". I will not discuss the results here. They can be seen in [1]. That reference also carries the details of how a similar analysis can be carried out for the scaling behaviour of magnetic susceptibility also.

**Singularity analysis of finite size effects**

This is based on the observation that in a sense $n$ and $\log g$ are "conjugate" to each other and that the large-$n$ behaviour of $Z_n$ controls the convergence of the perturbative expansion in $g$. In particular this is related to the behaviour of $F$ near its singular points i.e points where some derivative of $F$ blows up.

Let us illustrate this first with the example of pure gravity. Here the explicit parametric form of $F$ is

$$F = -\frac{1}{2} \ln z + \frac{1}{24} (z - 1)(9 - z)$$

with

$$g(z) = \frac{1 - z}{12 z^2} = g$$
The singular point $g_c$ corresponds to $z_0$ where $g'(z_0) = 0$. Hence
\[ z_0 = 2 \quad g_c = -1/48 \] (16)

Expanding $g(z)$ around $z_0$ one finds
\[ g - g_c \simeq a(z - z_0)^2 + b(z - z_0)^3 + ... \] (17)

Inverting this one gets
\[ z - z_0 \simeq c(g - g_c)^{1/2} + d(g - g_c) + .. \] (18)

Precise values of $a,b,c,d$ can be found in [1]. It should be noted that $F'(z_0) = 0$. Expanding $F$ around $z_0$ one has an expression of the type
\[ F(z) - F(z_0) = A(z - z_0)^2 + B(z - z_0)^3 + ... \] (19)

Combining the last eqns one gets for the singular part of the free energy $F$ near $g_c$ the expansion
\[ F(g)_{sing} - F(g_c) = \frac{12283\sqrt{3}}{5}(g - g_c)^{5/2} + \frac{1769472\sqrt{3}}{7}(g - g_c)^{7/2} + .. \] (20)

It should be remarked that on the basis of eqns(18,19) one would have expected this expansion to start with $(g - g_c)^{3/2}$. It turns out that the coefficient of this term vanishes. At this stage it appears accidental, but later when we present our improved singularity analysis, we will see that such cancellations (even more miraculous ones happen in the two-matrix model) have a natural explanation. Now we can use the expansion
\[ (g - g_c)^{\alpha} = \sum_n g^n (-g_c)^{\alpha-n} \frac{\Gamma(\alpha + 1)}{\Gamma(n + 1)\Gamma(\alpha - n + 1)} \] (21)

and the asymptotic expansion of Gamma functions(eqn()) to get the asymptotic expansion for $G_n$ defined by
\[ (g - g_c)^{\alpha} = \sum_n G_n \] (22)

\[ G_{nn \rightarrow \infty} = (-1)^{n+1}(-g_c)^{\alpha-n} \Gamma(\alpha + 1) n^{-(1+\alpha)} \frac{\sin \pi \alpha}{\pi} e^{\alpha(1+\alpha)/2n} \] (23)
Combining eqns(19,21,22) yields eqn(5).

In the case of the two-matrix model, the singularity analysis is a bit more involved. In the case of the $\phi^4$-potential, $g'(z_0) = 0$ yields $z_0 = -1/3$ in the low temperature phase and a known function $z_0(c)$ in the high temperature phase. From the parametric representation it is easy to see that $F'(z_0) = 0$ always. Boulatov and Kazakov [2] had claimed that $F''(z_0)$ is also zero. But we find that this is so only at the Ising critical point $c = c_{cr}$. Now we present the singularity analysis for the cases $c \neq c_{cr}$ and $c = c_{cr}$ separately.

**Non-critical Case**

Here the situation is identical to the pure gravity case except that all the coefficients in the expansion depend on $c$. Again, the coefficient of $(g - g_c)^{3/2}$ in $F$, which is now a function of $c$, vanishes for all $c$! Apart from reproducing the correct string susceptibility, the form of the finite size corrections one obtains are:

\[
\begin{align*}
\text{High Temp Phase (c = .36)} & : 1 - \frac{8.76}{n} + ... \\
\text{Low Temp Phase (c = .20)} & : 1 - \frac{72.69}{n} + ...
\end{align*}
\] (24)

**Critical Case**

In this case we have $F''(z_0) = 0, g''(z_0) = 0$ in addition to the vanishing of the corresponding first derivatives. Thus eqns(17-19) are now replaced by

\[
\begin{align*}
g - g_c & \simeq a(z - z_0)^3 + b(z - z_0)^4 + ... \quad (25) \\
z - z_0 & \simeq c(g - g_c)^{1/3} + d(g - g_c)^{2/3} + h(g - g_c) + .. \quad (26) \\
F(z) - F(z_0) & = A(z - z_0)^3 + B(z - z_0)^4 + ... \quad (27)
\end{align*}
\]

Again the $(g - g_c)^{4/3}, (g - g_c)^{5/3}$ terms in the singular part of the free energy drop out and the leading singular part $\simeq (g - g_c)^{7/3}$ characteristic of the string exponent $\gamma = -1/3$. The form of the finite size corrections are

\[
\begin{align*}
V(\phi) & = \phi^4 : 1 + \frac{0.4287}{n^{1/3}} - \frac{3.01}{n} - \frac{1.298}{n^{4/3}} + ...
\end{align*}
\]

These eqns represent the most important results of our analysis, namely, that the form of the finite size corrections is system-dependent and in this particular example depend on the string susceptibility $\gamma$. It should be noted that
in eqn(28) there are no terms of the type $n^{-2/3}$. This will become clearer
with our improved singularity analysis. It should also be noted from eqn(28)
that the coefficients in the finite size factor are non-universal. This means
it is possible to choose an appropriate potential that will minimise the finite
size corrections. This situation is well known in the study of lattice gauge
theories("improved action"-principle).

On the basis of eqn(28) it had been conjectured in [1] that if the string
susceptibility is of the form $\gamma = p/q$ with p,q relatively prime, the form of the
finite size corrections will be $1 + \frac{a}{n^{1/q}} + \ldots$. We will take up this issue now.

**Improved Singularity Analysis**

The singularity analysis presented above relied on the availability of the
explicit form of the parametrised free energy. In most cases such explicit
forms are not available and the above-presented method will fail in those
cases. There are many models of interest like multi-critical matrix models,
models with $\gamma > 0$ [3] whose finite size analysis one may wish to perform.

In all such cases where the method of orthogonal polynomials [4] can be
applied, the free energy is expressible as

$$F = \int_0^1 d\xi \ (1 - \xi) \ln f(\xi) \quad (29)$$

where the function $f(\xi)$ is defined through the method of orthogonal polynomials

$$\int dx \ w(x) P_i(x) P_j(x) = h_i \delta_{ij} \quad \text{OneMatrixModels}$$

$$\int dx \ dy \ w(x, y) P_i(x) P_j(y) = h_i \delta_{ij} \quad \text{TwoMatrixModels} \quad (30)$$

$$f_i = h_i / h_{i-1} \rightarrow_{N \rightarrow \infty} f(\xi) \quad (31)$$

In what follows we shall only consider one matrix models but consider arbitrary potentials. The analog of the defining eqns for $g(z)$ is now

$$g_\xi = w(f(\xi)) \quad (32)$$

In all these eqns the explicit form of $w(x)$ is dictated by the potential. The multcritical points are determined by the vanishing of various derivatives of
$w(x)$. The upshot of all this is that multcritical models are characterised by

$$g_\xi - g_c = A(f(\xi) - B)^p + C(f(\xi) - D)^{p+1} + \ldots \quad (33)$$
The meaning of this equation is that all parameters but $g$ have been fixed at their critical values and $g$ is near $g_c$. Further, if $w(x)$ is chosen to be the smallest polynomial satisfying multicriticality, there is only one term in eqn(38).

The parametric representation given earlier in terms of the variable $z$ can be recovered on identifying $f(1)$ with $z$ whence eqn(37) reads as $g = w(z) = g(z)$. Inverting eqn(38)

$$f(\xi) = B' + A'(g\xi - g_c)^{1/p} + ..$$

On using this with eqn(29) one finds

$$F_{sing} \simeq \sum_{m=1}^{m=p-1} (g - g_c)^{2+m/p}$$

A naive expectation for the exponent would be $1 + m/p$, but the $(1 - \xi)$ factor in the integrand suppresses this dominant exponent. This is basically what was behind the surprising cancellations we alluded to earlier.

Thus we see that in the fractional powers of $1/n$ characterising finite size corrections, two terms will be missing, one corresponding to the first term in eqn(35) which controls the leading behaviour and one corresponding to the term $m = p$ which is regular. Now we quote results for various cases:

| $n$ | $\gamma$ | Finite Size Correction |
|-----|----------|------------------------|
| 2   | $-1/2$   | $1 + \frac{a_1}{n} + \frac{a_2}{n^2} + ...$ |
| 3   | $-1/3$   | $1 + \frac{a_1}{n^{1/3}} + \frac{a_2}{n^{2/3}} + ...$ |
| 4   | $-1/4$   | $1 + \frac{a_1}{n^{1/4}} + \frac{a_2}{n^{2/4}} + ...$ |

Likewise, the results for models with positive $\gamma$ are given below:
Indeed there are many interesting issues that need to be properly understood in this context. For example, in the continuum version of Distler and Kawai [5], and of David [6], the liouville field is integrated over all its possible values. In such a treatment no "finite size" corrections will ever appear. The fixed area formulae hold for all areas. It would be interesting to formulate the problem of scaling violations in such approaches. Other issues of both practical and theoretical interest are the finite size analysis for other observables like the Hausdorff dimension, loop length distributions, resistivity of random networks etc. Another outstanding issue is to understand what happens in the $c = 1$ case where $\gamma = 0$ and there are logarithmic modifications to the fixed area partition sum.

**References**

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