Viability of complex self-interacting scalar field as dark matter.

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We study the viability of a complex scalar field $\chi$ with self-interacting potential $V = m_\chi^2/2|\chi|^2 + h|\chi|^4$ as dark matter. The scalar field is produced at reheating through the decay of the inflaton field and then, due to the self-interaction, a Bose-Einstein condensate of $\chi$ particles forms. The condensate represents dark matter in that model. We analyze the cosmological evolution of the model, stressing how, due to the presence of the self-interaction, the model naturally admits dark matter domination at late times, thus avoiding any fine tuning on the energy density of the scalar field at early times. Finally we give a lower bound for the size of dark matter halos at present time and we show that our model is compatible with dark matter halos greater than $0.1\, Kpc$ and with BBN and CMB bounds on the effective number of extra neutrinos $\Delta_{\chi}^{\nu}$. Therefore, the model is viable and for $h \simeq 10^{-4} - 10^{-12}$ one obtains a mass $m_\chi^2 \simeq m_\nu^2 \simeq 1 - 10^{-2}\, eV$ for dark matter particles from radiation-matter equality epoch to present time, but at temperatures $T_\chi \gg 10\, eV$, where $T_\chi$ is the photons temperature, thermal corrections to $m_\chi^2$ due to the self-coupling $h$ are dominant.

I. INTRODUCTION

Dark matter is one of the most fundamental ingredients of modern cosmology. Evidence for its existence comes from cosmological and astrophysical observations, e.g. cosmic microwave background (CMB) temperature anisotropy, large scale structures of the universe and measurements of galaxy rotation curves. There are many models that aim to explain the nature of dark matter, e.g. weakly interacting massive particles (WIMPS), axions and modified versions of general relativity. A valid alternative that has been widely studied, is to consider a scalar field as dark matter candidate and recently it has been studied the case in which a non-self-interacting scalar field forms a Bose-Einstein condensate. Scalar field dark matter (SFDM) models are competitive with $\Lambda CDM$ model to explain observational evidence of dark matter at cosmological level, up to linear perturbations.

Here we examine the possibility of describing dark matter through a complex self-interacting scalar field $\chi$ that forms a Bose-Einstein condensate at early times just after reheating. The scalar field has a renormalizable self-interacting potential $v(\chi, \bar{\chi}) = m_\chi^2/2|\chi|^2 + h|\chi|^4$ and, as we will discuss extensively, the presence of the self-interaction has many important consequences for the model. The first consequence it that it allows the formation of a $\chi$ particle condensate at early times. As was first studied in ref. [1], if the $\chi$ field is coupled with the inflaton field it is possible that at reheating the $\chi$ field is generated with a charge asymmetry $Q^\chi \equiv n^\chi - \bar{n}^\chi$, where $n^\chi$ and $\bar{n}^\chi$ are the number density of $\chi$ and $\bar{\chi}$ particles. If $Q^\chi$ is larger than some critical value, then a $\chi$-particle condensate forms just after reheating and the scalar field configuration will be that of a condensate in $\chi$-particle condensate at early times. Finally we give a lower bound for the size of dark matter halos at present time and we study the dependence of $\Delta_{\chi}^{\nu}$ on $L_H$, where $\Delta_{\chi}^{\nu}$ is the contribution of the scalar field $\chi$ to the effective number of extra neutrinos. We show that $\Delta_{\chi}^{\nu} \simeq 3.34 (L_H/Mpc)^{2/3}$ and does not depend on the coupling $h$, therefore it is possible to lower $\Delta_{\chi}^{\nu}$ below the big bang nucleosynthesis (BBN)
forms, assuming that we discuss the conditions under which the condensate cosmological evolution of the condensate. In section IV thermalized particles. In section III we describe the distribution of the 0\chi=b/\hbar\chi\hbar^2 \sim T^2_\chi,\ T_\chi is the photons temperature, namely just before radiation matter equality. This makes it possible to obtain the expected value of \rho_\chi=\rho_{DM}=0.232eV^4 at equality epoch without any fine tuning.

This Letter is organized as follows: in section II we describe the physics of a system composed of the Bose-Einstein \chi-particle condensate in equilibrium with \chi and \bar{\chi} thermalized particles. In section III we describe the cosmological evolution of the condensate. In section IV we discuss the conditions under which the condensate forms, assuming that \chi particles are produced at reheating via inflaton decay. In section V we derive the contribution of the \chi condensate and of the thermalized gas of \chi and \bar{\chi} particles to the effective number of extra neutrinos \Delta_\nu/f. In section VI we determine the lower bound L_H for dark matter halos at present times and in section VII we present a choice of the model parameters that gives a realistic model. Finally in section VIII we conclude.

II. BOSE-EINSTEIN CONDENSATE

Consider a scalar field with Lagrangian

$$L = \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - \frac{1}{2} m_\chi^2 |\chi|^2 - h |\chi|^4 \tag{1}$$

with \(h \ll 1\). We assume that the \chi particles are weakly self-interacting and that their mass varies adiabatically. One can then define the phase space distributions \(f_\chi\) and \(\bar{f}_\chi\) of the \chi particles and \bar{\chi} antiparticles so that the energy, number and charge density of the complex \chi field are respectively

$$\rho_\chi = (2\pi)^{-3} \int d^3p\ E_\chi(p) \ [f_\chi(p) + \bar{f}_\chi(p)] \tag{2}$$

$$n_\chi = (2\pi)^{-3} \int d^3p\ [f_\chi(p) + \bar{f}_\chi(p)]$$

$$Q_\chi = (2\pi)^{-3} \int d^3p\ [f_\chi(p) - \bar{f}_\chi(p)]$$

The phase space distributions for a \chi-particle condensate in equilibrium with thermalized \chi and \bar{\chi} particles with temperature \(T_\chi\) are

$$f_\chi(p) = f_\chi^{BE}(p) + (2\pi)^3 Q_\epsilon \ \delta^3(p) \tag{3}$$

$$f_\chi(p) = f_\chi^{BE}(p)$$

where \(f_\chi^{BE}(p) = 1/ [e^{\beta(E-\mu)} - 1]\) and \(f_\chi(p) = 1/ [e^{\beta(E+\mu)} - 1]\), \(\beta = 1/T_\chi\), \(\mu\) is the chemical potential and \(Q_\epsilon\) is the number density of the \chi particles of the condensate. Following ref.[8] one can calculate the thermal correction to the \chi mass as \([m_\chi^{th}]^2 \sim 4h \int \frac{d^3p}{(2\pi)^3} E_\chi(p) \approx (2hQ_\epsilon + T_\chi^2/3m_\chi^{th}),\) that gives

$$m_\chi^{th} \simeq \alpha T_\chi, \ \alpha \equiv \left[ h \left( \frac{2Q_\epsilon}{\chi} + \frac{1}{2} \right) \right]^{1/3} \tag{4}$$

Therefore the effective mass \(m_\chi\) of the \chi and \bar{\chi} particles will be

$$m_\chi \simeq m_\chi^{th} \quad for \quad T_\chi \leq \frac{m_\chi^{th}}{\alpha} \tag{5}$$

$$m_\chi \simeq m_\chi^{th}(Q_\epsilon, T_\chi, h) \quad for \quad T_\chi \gg \frac{m_\chi^{th}}{\alpha}$$

The number density of \chi particles is

$$n_\chi = Q_\epsilon + n_\chi^{th}, \quad n_\chi^{th} = (2\pi)^{-3} \int d^3p\ f_\chi^{BE}(p), \tag{6}$$

while the energy density of the \chi field is

$$\rho_\chi = m_\chi Q_\epsilon + \rho_\chi^{th} \tag{7}$$

$$\rho_\chi^{th} = (2\pi)^{-3} \int d^3p\ E_\chi(p) \left[ f_\chi^{BE}(p) - f_\chi^{BE}(p) \right]. \tag{8}$$

Also the charge density is

$$Q_\chi = Q_\epsilon + Q_\chi^{th} \tag{9}$$

$$Q_\chi^{th} = n_\chi^{th} - n_\chi = (2\pi)^{-3} \int d^3p\ \left[ f_\chi^{BE}(p) - f_\chi^{BE}(p) \right]. \tag{10}$$

Note that, since the thermal corrections to \(m_\chi^{th}\) depend on \(Q_\epsilon, T_\chi\) and \(h\), then \(f_\chi^{BE}(p)\) and \(f_\chi^{BE}(p)\) will depend on \(Q_\epsilon\) and \(h\) via the effective mass \(m_\chi(Q_\epsilon, T_\chi, h)\). Therefore \(n_\chi^{th}, \rho_\chi^{th}\) and \(Q_\chi^{th}\) in general also depend on \(Q_\epsilon, T_\chi, h\) in any case, at temperatures \(T_\chi \gg m_\chi \geq \mu > 0\) one can neglect both \(m_\chi\) and \(\mu\) and recover the usual results 9.

$$n_\chi^{th} = n_\chi^{th} = \frac{\zeta(3)}{\pi^2} \ T_\chi^3, \ \rho_\chi^{th} = \frac{\pi^2}{15} \ T_\chi^4, \ Q_\chi^{th} = \frac{\mu T_\chi}{3} \ T_\chi^2 \tag{11}$$

For simplicity we also define the condensate contribution to the number, charge and energy density as

$$n_\epsilon = Q_\epsilon = Q_\chi, \ \rho_\epsilon^L = m_\chi Q_\epsilon \tag{12}$$
III. COSMOLOGICAL EVOLUTION

In this section we resume the main features of the cosmological evolution of the scalar field \( \chi \). The phase space distributions given in Eq.13 are solutions of the relativistic Boltzmann equations in FRW metric for \( T_\chi \sim 1/a(t) \), \( Q_\chi \sim T_\chi^3 \sim 1/(a(t))^3 \) and \( \mu = 0 \), at any temperature \( T_\chi \gg m^\chi \geq \mu > 0 \) and for any initial value of \( Q_\chi/T_\chi^3 \). Note also that \( Q_\chi/T_\chi^3 \) remains constant as long as \( T_\chi \gg m^\chi \), therefore from Eq.(11) it is evident that \( \alpha \) is constant in the same range of temperatures. In what follows we assume that the coupling \( h \) is small enough to give \( \alpha \ll 1 \). Therefore from Eq.(11) it follows that \( T_\chi \gg m^\chi_{th} \) at any time. We define the parameter

\[
k \equiv T_\chi/T_\gamma \tag{12}
\]

where \( T_\gamma \) is the photons temperature, and we note that \( k \) is also constant for \( T_\chi \gg m^\chi_{th} \). Moreover we define the following temperatures

\[
T_{1\chi} \equiv m^\chi_{th}/\alpha, \quad T_{2\chi} \equiv m^\chi_0
\tag{13}
\]

\[
T_{1\gamma} \equiv T_{1\chi}/k, \quad T_{2\chi} \equiv T_{2\chi}/k
\]

with \( T_{1\chi} > T_{2\chi} \) and \( T_{1\gamma} > T_{2\gamma} \) since \( \alpha \ll 1 \). Finally we define \( t_1 \) as the time when \( T_\chi = T_{1\chi} \) and \( T_\gamma = T_{1\gamma} \) and \( t_2 \) as the time when \( T_\chi = T_{2\chi} \) and \( T_\gamma = T_{2\gamma} \).

Armed with these definitions we describe the cosmological evolution of the scalar field \( \chi \). There are three important epochs in which the \( \chi \) field behave differently.

— At early times \( t \ll t_1 \), when \( T_\chi \gg T_{1\chi} \) the \( \chi \) mass is dominated by thermal corrections so \( m^\chi \simeq m^\chi_{th} = aT_\chi \ll T_\gamma \). That implies that the energy density of the condensate evolves as radiation since \( \rho^\chi \simeq m^\chi_{th} Q_\chi \sim T_\chi^4 \).

Of course, since \( T_\chi \gg m^\chi \), Eqs.13 are valid and \( \rho^\chi_{th} \sim T_\chi^4 \). In conclusion the whole \( \chi \) field evolves as radiation with \( \rho^\chi \sim T_\chi^4 \).

— At temperatures \( T_{1\chi} \gg T_\chi \gg T_{2\chi} \) one has \( m^\chi_{th} \gg m^\chi_0 \), therefore the mass of the \( \chi \) particles is simply \( m^\chi_0 \). Moreover one still have \( T_\chi \gg m^\chi_0 \), therefore Eqs.13 are still valid. Then in this case \( \rho^\chi_{th} \sim T_\chi^4 \) still evolves as radiation but \( \rho^\chi \simeq m^\chi_0 Q_\chi \sim T_\chi^3 \) evolves as matter. Therefore at the temperature \( T_{1\chi} \) the condensate passes from a radiation-like to a matter-like evolution.

— At temperatures \( T_{2\chi} < T_\chi < T_{2\gamma} \) below \( m^\chi_0 \), also the thermalized \( \chi \) and \( \bar{\chi} \) particles begins to evolve as matter with \( \rho^\chi_{th} \sim a^{-3} \). Therefore the temperature \( T_{2\chi} \) characterizes the transition of \( \rho^\chi_{th} \) from radiation-like to matter-like fluid. In this range of temperatures the total energy density of the scalar field evolves as matter, i.e. \( \rho^\chi \sim a^{-3} \). In particular, if at \( T_\chi \simeq T_{2\chi} \) one has \( \rho^\chi \gg \rho^\chi_{th} \) one can take \( \rho^\chi \sim \rho^\chi_{th}(t_2)/(a(t_2)/a(t)^3) \) for any time \( t > t_2 \).

To summarize, the thermalized gas of \( \chi \) particles and \( \bar{\chi} \) antiparticles becomes non-relativistic at temperatures below \( m^\chi_0 \) as usual, but the condensate still evolves as matter at temperatures \( T_{1\chi} \gg T_\chi \gg T_{2\chi} \) well above \( m^\chi_0 \).

This last feature is typical of this model and it is due to the fact that thermal corrections to the mass are important only at very high temperatures, i.e. above \( T_{1\chi} \). Of course the thermal corrections to \( m^\chi_0 \) are due to the presence of the \( h |\chi|^4 \) self-coupling. If self-interactions are turned off, there are no thermal corrections to \( m^\chi_0 \), therefore the condensate always evolves as matter and this implies a severe fine tuning on its energy density at early times. Moreover the self-interaction is important for a second reason. Since the condensate is formed right after reheating, one should explain why it starts to dominate just at radiation-matter equality. This question is easily answered in that context. In fact, because of the self-interaction, the condensate evolves as a relativistic fluid at high temperatures and it cannot dominate over radiation before \( t_1 \), i.e. at temperatures \( T_\chi > T_{1\chi} \) (or \( T_\gamma > T_{1\chi} \)). Therefore one can choose the coupling constant \( h \) and \( m^\chi_0 \) properly, in order to ensure a dark matter domination at temperatures \( T_\gamma \sim 0.698 \, eV \). This helps to explain the cosmological coincidence problem without any fine tuning on \( \rho^\chi \) at early times.

IV. CONDENSATE FORMATION

In the model that we are presenting we suppose that the scalar field \( \chi \) is produced at reheating via the inflaton decay. The \( \chi \) and \( \bar{\chi} \) particles are produced with a charge asymmetry \( Q^\chi > 0 \) via an Affleck-Dine mechanism [10], and then, due to self-interactions, they forms a \( \chi \)-particle condensate. The conditions under which the condensate is formed are studied in ref.[8]. Since the charge and energy densities are conserved, the quantity \( R \equiv Q^\chi/\rho^\chi_{3/4} \) remains constant as long as \( T_\chi \gg m^\chi \). In ref.[8] it is found that the condensate forms if the \( \chi \) field is produced at reheating with

\[
R \geq 0.2 h^{1/2} \tag{14}
\]

It is also found that, if \( R \geq 1/2 \) one has \( Q^\chi \gg n^\chi_{th} \), i.e., the majority of the \( \chi \) particles are in the condensate. After the condensate formation, the phase space distributions of the \( \chi \) and \( \bar{\chi} \) particles are given by Eq.13 and Eq.(14) reads

\[
R \equiv \frac{Q^\chi}{\rho^\chi_{3/4}} = \frac{Q_\chi T_\chi^3 + \mu(T_\chi)/3T_\chi}{(\alpha Q_\chi T_\chi^3 + \pi^2/15)^{3/4}} > 0.2 h^{1/2} \tag{15}
\]

Therefore, any realistic choice of the model parameters should fulfill Eq.(15) for any \( T_\chi > T_{1\chi} \). We stress that the presence of the self-interaction is fundamental in this model for the \( \chi \)-particle condensate formation.
V. EFFECTIVE NUMBER OF EXTRA NEUTRINOS

In the range \( T_\chi > T_{2\chi} \approx m_\chi^0 \), the energy density of thermalized \( \chi \) particles evolves as radiation and therefore the contribution of \( \rho_{\chi}^0 \) to the effective number of extra neutrino \( \Delta_\nu^{eff} \) is

\[
\Delta_\nu^{th} = \frac{16}{7} \left( \frac{T_\chi}{T_\nu} \right)^4.
\]

The equation for the radial function \( \sigma \) is

\[
\Delta \sigma + (1 - 4\Phi) \omega^2 \sigma - (1 - 2\Phi) \left( \frac{m_\chi^2}{2} \sigma + h\sigma^3 \right) = 0
\]

In the limit \( \Lambda \equiv h^2 \left( \frac{4\pi G m_\chi^2}{\rho} \right) \gg 1 \) and neglecting baryon contribution, one has the approximate solution for \( r/L_H \leq \pi \)

\[
\sigma(r) = \sigma_0 \sqrt{\sin(r/L_H)/(r/L_H)} - \cos(r/L_H)
\]

\[
L_H \equiv h^{1/2} \frac{M_p}{m_0}
\]

where \( M_p \) is the reduced Planck mass. The solution (23) is valid with the requirement that \( \Lambda^{-1} \ll \sigma_0/M_p \ll \Lambda^{-1/2} \). The mass of the \( \chi \) particles can be expressed as a function of \( L_H \) and of the coupling constant \( h \) as

\[
m_\chi = h^{1/4} \sqrt{\frac{M_p}{L_H}} = 3 h^{1/4} \sqrt{\frac{Kpc}{L_H}} \text{eV}
\]

The approximate solution (23) is valid for \( r/L_H \leq \pi \) where one has \( \rho^\lambda(r)/\rho^\lambda(0) = \sigma^2/\sigma_0^2 \sim \sin(r/L_H)/(r/L_H) \) [12]. We can say nothing about \( \rho^\lambda(r) \) for \( r/L_H \gg \pi \) and we cannot exclude that the exact solution can have a value of \( \rho^\lambda(r)/\rho^\lambda(0) \sim 1 \) at some \( r \gg L_H \). Therefore the size of the halo could be many orders of magnitude greater than \( L_H \). That means that at this level of analysis, \( L_H \) gives a lower limit for the dark matter halo sizes. Therefore any value of \( L_H \leq 100 \text{Kpc} \) is acceptable, since there is no evidence of halos of size much less than \( 100 \text{Kpc} \). In the next section we will show that \( \Delta_\nu^{eff} \) is very sensitive to the choice of \( L_H \) and that any value of \( L_H \leq 0.1 \text{Kpc} \) gives a \( \Delta_\nu^{eff} \) compatible with BBN bounds.

VI. DARK MATTER HALOS

Following the analysis given in [12] we can study the formation of dark matter halos in our model and compare the results with observations. We assume matter domination and we take \( T_\chi < m_\chi^0 \), since we want the \( \chi \) field to represent dark matter. In that limit we can describe the \( \chi \) field as a classical complex field. We consider a spherically-symmetric metric

\[
ds^2 = e^{2u}dt^2 - e^{2v}dr^2 - r^2 d\Omega^2
\]

It is shown in ref. [13] that all stable field configurations have the form

\[
\chi = \frac{\sigma(r)}{\sqrt{2}} e^{i\omega t}.
\]

Moreover one can take the Newtonian limit for the gravitational field taking \( u \approx -v \approx \Phi \). Under these assumptions, the evolution of the gravitational potential is given [12] by

\[
\Delta \Phi = 4\pi G \left( \rho_{\chi}^{eff} + \rho_b \right)
\]

where \( \rho_b \) is the baryon energy density and

\[
\rho_{\chi}^{eff} = 2\omega^2 \sigma^2 - \frac{1}{2} m_\chi^2 \sigma^2 - \frac{1}{2} h\sigma^4
\]

The dependence of \( \Delta_\nu^{eff} \) and \( \Delta_\nu^{th} \) on \( L_H \), we parameterize \( L_H \) and \( m_\chi^0 \) as

\[
L_H \simeq \frac{0.1}{n^2} \text{Kpc}, \quad m_\chi^0 \simeq 10 n h^{1/4}
\]
where $n$ will be fixed later. First we constrain the temperature of the thermalized $\chi$ particles with the BBN bounds on $\Delta_\nu^{eff}$. Since we expect that $m_0^\chi \ll T_B^{BBN} \simeq k T_B^{BBN}$, where $T_B^{BBN} \simeq 0.1 - 10 \, MeV$ is the photons temperature at BBN, therefore $\rho_{th}^\chi$ will evolve as radiation at BBN and then it will contribute to the effective number of extra neutrinos. Imposing the condition $\Delta_\nu^{th} \leq 1$ [11], from Eq. (14) one obtains

$$\frac{T_\chi}{T_\gamma} \equiv k \leq 0.8 \tag{26}$$

Note that as long as $T_\chi > m_0^\chi$ one has $T_\chi \sim 1/a$ and $k$ maintains constant. Moreover from Eq. (26) one has that $\rho_{th}^\chi/\rho_{red} \leq k^4/g_r \leq 0.3 \cdot k^4$, where $g_r \geq 3.36$ is the relativistic degree of freedom and $\rho_{red}$ is the energy density of relativistic particles. Now we should impose that at radiation-matter equality the condensate evolves as matter, and this implies that

$$T_\chi^{eq} = k T_\gamma^{eq} \leq T_1 \chi \tag{27}$$

where $T_\gamma^{eq} \simeq 0.69 \, eV$ is the temperature of photons at radiation-matter equality. Moreover, as consistence condition for the model, one has to require that

$$T_1 \chi \gg T_2 \chi \tag{28}$$

We will check the validity of (27) and (28) later on.

We want to stress two important facts. First the condition for the condensate to evolve as radiation, i.e. $\rho_{th}^\chi \sim 1/a^4$ is $T_\chi \gg T_1 \chi \equiv m_0^\chi/a \gg m_0^\chi$, therefore the condensate can evolve as matter at temperatures well above $m_0^\chi$. Second the thermalized $\chi$ particles do not have the same temperature of radiation but $T_\chi/T_\gamma = k \leq 0.8$. This implies that at radiation-matter equality one has $T_\chi^{eq} = k T_\gamma^{eq} \leq 0.56 \, eV$. Since we want the condensate to represent dark matter, we impose that at radiation matter equality $\rho_{th}^{\chi^{eq}} = m_0^\chi Q_c^{eq} = \rho_{DM}^{eq} \simeq 0.323 \, eV^4$. Since $Q_c/T_\chi^3$ is constant for $T_\chi \gg T_2 \chi \equiv m_0^\chi$ and using Eq. (25) one has

$$\frac{Q_c}{T_\chi^3} = \frac{Q_c^{eq}}{T_\chi^{eq}^3} \simeq \frac{9.5 \times 10^{-2}}{n k^3 h^{1/4}} \tag{29}$$

Assuming that $Q_c/T_\chi^3 \geq 1$ (since $h \ll 1$) and using Eq. (29) again, one also has

$$\alpha \simeq \frac{0.57 h^{1/4}}{n^{1/3} k} \tag{30}$$

and from Eq. (17) one has

$$\Delta_\nu^{\chi} \simeq \frac{0.72}{n^{4/3}} \tag{31}$$

Note that $\Delta_\nu^{\chi}$ depends only on $n$ and it is independent of the other parameters of the model. This implies that $\Delta_\nu^{\chi}$ is determined only from the choice of $L_H$. Therefore one can choose $L_H$ in such a way that it gives a value of $\Delta_\nu^{\chi}$ in the BBN bound. Imposing $\Delta_\nu^{\chi} \leq 1$ [11] one obtains the $n \geq 0.78$. We should check the conditions given in Eqs. (27) and (28). By use of the expressions

$$T_1 \chi \simeq 17.6 \, k n^{4/3} \, eV \tag{32}$$

$$T_2 \chi \simeq 10 \, n \, h^{1/4} \, eV$$

one can check that Eq. (27) implies that $n \geq 0.088$ and Eq. (28) implies that $h \leq 10^4 n^{4/3}$.

Let us take $n \simeq 0.8$ and $k \simeq 0.3$ in what follows. With such a value of $n$ one obtains $L_H \simeq 0.17 \, Kpc$ that is well below the typical size for dark matter halos and therefore it is compatible with astrophysical observations. From Eq. (29) one has $Q_c/T_\chi^3 \sim 10 h^{-1/4} \geq 1$ for any $h < 1$, therefore Eq. (30) is correct. Moreover one has $R \sim h^{-1/4} \geq 0.2 h^{1/2}$, so Eq. (15) is fulfilled and the values of $k \simeq 0.3$ and $n \simeq 0.8$ are compatible with the condensate formation at early times. We also obtain $\alpha \simeq 2 h^{1/4}$, $m_0^\chi \simeq 8 h^{1/4} \, eV$ and $\Delta_\nu^{\chi} \simeq 0.97$. Since $n \gg 0.088$ the condition (27) is fulfilled and Eq. (27) implies that $h \ll 10^{-2}$ for $n \simeq 0.8$ and $k \simeq 0.3$. Though it is not necessary, one can ask that at equality time $\rho_{th}^\chi$ still evolves as radiation, i.e. $T_\chi^{eq} \gg m_0^\chi$, obtaining $h \ll 10^{-7}$. The values of $T_\chi$ and $T_7$ at $t_1$, $t_2$ and at matter-radiation equality are resumed in table I. In table II we show the values of $\alpha$, $m_0^\chi$ and $T_\chi$ for different values of $h$. We stress that $m_0^\chi \sim 1 - 10^{-2} \, eV$ for $h \sim 10^{-4} - 10^{-12}$, though in the case of a scalar field with no self-interaction, one needs an extremely light mass $m_0^\chi \sim 10^{-22} \, eV$ to avoid the formation of dark matter halos of an excessively small size.

As we have already stressed, both the values of $\Delta_\nu^{\chi}$ and $L_H$ only depends on $n$ as

$$\Delta_\nu^{\chi} \simeq 3.34 \left( \frac{L_H}{Mpc} \right)^{2/3}, \quad L_H \simeq 0.1 \frac{n^2}{Mpc} \tag{33}$$

and from Eq. (33) it is evident how it is possible to lower the value of $\Delta_\nu^{\chi}$ diminishing $L_H$. This means that any value of $n \geq 0.8$ will give a $\Delta_\nu^{eff}$ in the BBN bounds and a value of $L_H \leq 0.16 \, Kpc$ well below the typical size of dark matter halos.

We note that in [12] the authors take $L_H$ of the order of the core of dark matter halos, i.e. $L_H \simeq 10 \, Kpc$. They also take the coupling in the interval $h \sim 1 - 10^{-4}$, obtaining a mass $m_0^\chi \sim 1 \, eV$ and they show that such values of the mass and coupling give a number of effective extra neutrinos $\Delta_\nu^{eff}$ that exceeds the BBN bound. This result is in agreement with our analysis but we have shown that it is possible to take smaller $L_H \leq 0.1 \, Kpc$ to
lower the value of $\Delta^{\text{eff}}$ below BBN bounds. Of course this is possible since, as we have discussed in section [VII] $L_H$ is a lower bound for the typical size of dark matter halos and therefore any $L_H \leq 100 \, \text{Kpc}$ is in agreement with astrophysical observations.

We stress that we have used the approximate (and incomplete) solution given in Eq.(23) to have a lower limit for the dark matter halo sizes. Of course this analysis is incomplete in many respects, since it does not take into account the formation of dark matter halos as evolving from linear perturbations nor how the cosmological evolution of the universe influences this process. Moreover the presence of a residual $\rho_0^{\chi}$ as well as a cosmological constant, baryons and radiation, were not considered. Therefore we think that an analysis of dark matter halos formation that takes into account all of these considerations will be very useful to further constrain the model.

In particular, we note that the conclusions of this section are based on the relation given in Eq.(23) between $m_\chi^3$ and $L_H$. An analysis of dark matter halos formation different from that described in section [VII] can considerably change Eq.(23) and therefore it can give less stringent constraints on the parameters of our model.

| $t$ | $t_1$ | $t_2$ | $t_{eq}$ |
|-----|-------|-------|---------|
| $T_\chi$ | 0.39 eV | $8 \, h^{1/4} \, eV$ | 0.21 eV |
| $T_\gamma$ | 13 eV | $26.7 \, h^{1/4} / eV$ | 0.698 eV |

TABLE I: Values of $T_\chi$ and $T_\gamma$ at three different times: $t_1$ when $T_\chi = T_{1\chi}$, $t_2$ when $T_\chi = T_{2\chi}$ and $t_{eq}$ at radiation-matter equality.

| $h$ | $10^{-4}$ | $10^{-8}$ | $10^{-12}$ |
|-----|------------|------------|------------|
| $\alpha$ | 0.2 | 0.02 | 0.002 |
| $m_\chi^3$ | 0.8 eV | 0.08 eV | 0.008 eV |
| $T_{2\chi}$ | 2.67 eV | 0.267 eV | 0.002 eV |

TABLE II: Values of $\alpha$, $m_\chi^3$ and $T_{2\chi}$ for different values of the coupling $h$.

VIII. CONCLUSIONS

We have shown how it is possible to use a complex scalar field with self-interactions $h |\chi|^4$ in order to obtain a realistic model for dark matter. In this model, dark matter is described as a condensate of $\chi$ particles that forms just after reheating and dominates at late times.

As pointed out, the presence of the self-interaction is very important in the model. First, it is essential to explain the condensate formation. In fact the $\chi$ field is produced at reheating with a charge asymmetry, and under the conditions discussed in section [III] self-interactions drive the formation of a $\chi$-particle condensate.

Second, due to self-interactions, the $\chi$ mass has thermal corrections that are important at high temperatures $T_\chi \gg T_{1\chi} \gg m_\chi^0$, where $m_\chi^0 \simeq m_\chi^3 \sim T_\chi$. This implies that as long as $T_\chi \gg T_{1\chi}$ one has $\rho_\chi^3 \sim T_{1\chi}$ and the $\chi$ field behaves as radiation. This explains why dark matter dominates at late times. In fact, one can choose the parameters of the model properly in order to ensure that the $\chi$ condensate begins to evolve as matter with $\rho_\chi^3 \sim T_{1\chi}$ just before radiation matter equality and that at equality one has the right value $\rho_\chi^3 = \rho_{DM} \simeq 0.323 \, eV^4$. We have also given a lower bound $L_H$ for the size of dark matter halos and we have studied the dependence of the contribution of the $\chi$ field to the effective number of extra neutrinos $\Delta^{\text{eff}}$ on $L_H$. We have shown that, according to Eq.(23), it is possible to diminish $\Delta^{\text{eff}}$ just lowering the value of $L_H$ and that any $L_H \leq 100 \, \text{Kpc}$ gives a $\Delta^{\text{eff}}$ within BBN bounds. Since $L_H$ is a lower bound for dark matter halo sizes, any value $L_H \leq 100 \, \text{Kpc}$ is acceptable.

In section [VIII] we have constructed a realistic model, choosing $L_H \simeq 0.1 \, \text{Kpc}$, $T_{1\chi}/T_{\gamma} \simeq 0.3$, and the coupling in the interval $h \simeq 10^{-4} - 10^{-12}$. With such values of the parameters the condensate begins to evolve as matter with $\rho_\chi^3 \sim T_{1\chi}$ at $T_{\gamma} \simeq 13 \, eV$ and it gives the right value $\rho_\chi^3 = \rho_{DM} \simeq 0.323 \, eV^4$ at equality epoch $T_{eq} \simeq 0.698 \, eV$.

Therefore, we conclude that, at the present level of analysis, our model is in agreement with cosmological and astrophysical observations. Of course a more profound analysis of the evolution and growth of cosmological perturbations in that model is still missing, but we think that such a study would be very useful to further constrain the model with cosmological data.

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