The dual Bonahon-Schläfli formula

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Abstract

Given a differentiable deformation of geometrically finite hyperbolic 3-manifolds \((M_t)\), the Bonahon-Schläfli formula \cite{Bon98a} expresses the derivative of the volume of the convex cores \((CM_t)\) in terms of the variation of the geometry of its boundary, as the classical Schläfli formula \cite{Sch58} does for the volume of hyperbolic polyhedra. Here we study the analogous problem for the dual volume, a notion that arises from the polarity relation between the hyperbolic space \(\mathbb{H}^3\) and the de Sitter space \(dS^3\). The corresponding dual Bonahon-Schläfli formula has been originally deduced from Bonahon’s work by Krasnov and Schlenker \cite{KS09}. Making use of the differential Schläfli formula \cite{SR99} and the properties of the dual volume, we give a (almost) self-contained proof of the dual Bonahon-Schläfli formula, without making use of the results in \cite{Bon98a}.

Introduction

The classical Schläfli formula expresses the derivative of the volume along a 1-parameter deformation of polyhedra in terms of the variation of its boundary geometry. It was originally proved by Schläfli \cite{Sch58} in the unit 3-sphere case, and later extended to polyhedra of any dimension sitting inside constant non-zero sectional curvature space forms of any dimension. Here we recall the statement in the 3-dimensional hyperbolic space \(\mathbb{H}^3\), which will be our case of interest:

**Theorem** (Schläfli formula). Let \((P_t)\) be a 1-parameter family of polyhedra in \(\mathbb{H}^3\) having the same combinatorics, obtained by taking a differentiable variation of the vertices of \(P = P_0\). Then the function \(t \mapsto \text{Vol}(P_t)\) is differentiable at \(t = 0\) and it verifies

\[
\frac{d}{dt} \text{Vol}(P_t) \bigg|_{t=0} = \frac{1}{2} \sum_{e \text{ edge of } \partial P} \ell(e) \delta \theta(e),
\]

where \(\ell(e)\) denotes the length of the edge \(e\) in \(P\) and \(\delta \theta(e)\) is the variation in \(t\) of the exterior dihedral angle along \(e\).

Bonahon \cite{Bon98a} proved an analogue of this result for variations of hyperbolic 3-manifolds. More precisely, consider a differentiable 1-parameter family of quasi-isometric geometrically finite hyperbolic 3-manifolds \((M_t)_t\); in any of such \(M_t\)’s there is a smallest convex subset \(CM_t\), called the convex core of \(M_t\), which plays the role of the polyhedron. It has a boundary \(\partial CM_t\) which is totally geodesic almost everywhere,
except for a closed subset $\lambda_t$ foliated by simple geodesics, where the surface $\partial CM_t$ is bent. The structure of $\partial CM_t$ is encoded in the datum of a hyperbolic metric $m_t$, obtained by gluing the metrics on the complementary regions of $\lambda_t$, and a measured lamination $\mu_t$, which describes the amount of bending of $\partial CM_t$ along $\lambda_t$. The geodesic lamination $\lambda_t$ is the analogue of the 1-skeleton in the boundary of the polyhedron, and the bending measure $\mu_t$ is the integral sum of the dihedral angles along the transverse arcs to $\lambda_t$.

The space of measured laminations $\mathcal{ML} (\partial CM) = \mathcal{ML} (\partial CM_t)$ is naturally endowed with a piecewise linear manifold structure, therefore the tangent directions at the point $\mu_0$ form in general a union of cones, each of which is sitting in the tangent space of some linear piece. Bonahon’s notion of Hölder cocycles (see [Bon97a], [Bon97b]) furnishes a natural way to describe these first order variations of measured laminations. In [Bon98b] the study of the dependence of $m_t$ and $\mu_t$ in terms of the hyperbolic structure $M_t$ is developed. In particular, the hyperbolic metric $m_t$ is shown to depend $C^1$ in the parameter, and the measure lamination always admits left and right derivatives in $t$, which is the best that can be expected in a piecewise linear setting. In light of these facts, Bonahon showed in [Bon98a] that, for a 1-parameter family of manifolds $(M_t)$ as above, the volume of the convex core $\text{Vol}(CM_t)$ always admits right (and left) derivative at $t = 0$, and verifies

$$\frac{d}{dt} \text{Vol}(CM_t) \bigg|_{t=0^+} = \frac{1}{2} \ell_{m_0}(\dot{\mu}_0^+) .$$

We will call this relation the Bonahon-Schläfli formula.

Another notion of volume can be introduced on the space of convex subsets sitting inside a convex co-compact hyperbolic manifold $M$ (for simplicity, here we require $CM_t$ to be not only of finite volume but also compact). Namely, we can define the dual volume of a compact convex subset $N$ of $M$ with smooth boundary by the following relation:

$$\text{Vol}^*(N) = -\text{Vol}(N) - \frac{1}{2} \int_{\partial N} H \, dA , \quad (1)$$

where $H$ denotes the trace of the shape operator of $\partial N$, defined by its exterior unitary normal vector field. This notion is related to the duality between the hyperbolic space $\mathbb{H}^3$ and the de Sitter space $\mathbb{dS}^3$, which allows to associate with a convex body $C$ in one geometry, a dual one $C^\Vert$ sitting in the other. By applying the definition (1) to a compact convex body $C \subset \mathbb{H}^3$, $\text{Vol}^*(C)$ is the de Sitter volume of $H \setminus C^\Vert$, where $H$ is a future-oriented half-space containing $C^\Vert$ (see Appendix for details).

In [KS09] the authors deduced a variation formula for the dual volume of the convex cores $(CM_t)$ from the Bonahon-Schläfli formula. More precisely, they showed that the derivative of $\text{Vol}^*(CM_t)$ exists and it verifies

$$\frac{d}{dt} \text{Vol}^*(CM_t) \bigg|_{t=0} = \frac{1}{2} d(L_{\mu_0})_{m_0} (\dot{m}_0) ,$$

where $L_{\mu_0}$ denotes the function on the Teichmüller space of $\partial CM$, which associates to a hyperbolic metric $m \in \mathcal{T}(\partial CM)$ the length of $\mu_0$ with respect to $m$. The remarkable property of this relation, which we call the dual Bonahon-Schläfli formula, is
that it does not involve the first variation of the bending measures $\mu_t$, but only the
derivative of the hyperbolic metric $m_t$. Therefore, contrary to the variation formula
of the standard volume, this relation does not require the notion of Hölder cocycle to
be stated. Motivated by this remark, in this paper we give an alternative proof of the
dual Bonahon-Schläfli formula, which does not involve the study the variation of the
measured lamination of $\partial CM_t$ and, apparently, the notion of Hölder cocycles.

Even if inspired by Bonahon’s work, our strategy of proof is quite different from
the one used in [Bon98a] and mainly relies on tools from differential geometry, as
the differential Schläfli formula [SR99] and the convexity properties of the equidistant
surfaces from the convex core. Without making use of the Hölder cocycles technology,
we will prove that the derivative of the dual volume of the convex core exists and it
verifies
\[
\frac{d}{dt} \text{Vol}^*(CM_t) \bigg|_{t=0} = \frac{1}{2} \frac{d}{dt} \ell_{m_t}(\mu_0) \bigg|_{t=0},
\]
where $\ell_{m_t}(\mu_0)$ is the length of the measured lamination $\mu_0$ realized inside the manifold
$M_t$, as $t$ varies in a neighborhood of 0. In order to deduce that the term $\frac{d}{dt} \ell_{m_t}(\mu_0) \bigg|_{t=0}$
coincides with $d(L_{\mu_0})_{m_0}(\dot{\mu}_0)$, and therefore the complete statement, we will need
Bonahon’s results about the $C^1$-dependence of the hyperbolic metric on the boundary
of the convex core with respect to the convex co-compact structure of $M$ (see [Bon98b]
Theorem 1). The original proof of this result heavily relies on the Hölder cocycles machinery, but we hope that a possible future develop of this work may furnish a more elementary proof of it, so that our presentation of the dual Bonahon-Schläfli formula
could be completely self-contained.

Finally, it is worth to mention that the dual Bonahon-Schläfli formula is the counter-
part “at the convex core” of another remarkable relation, which was proved in [Sch17]
and concerns the geometry “at infinity” of convex co-compact hyperbolic manifolds
and their renormalized volume $\text{RVol}$ (see [KS08]). More precisely, let $c_t \in \mathcal{T}(\partial_\omega M)$
be the conformal structure underlying the complex projective structure $\sigma_t$ at infinity
of $M_t$, and let $\sigma^0_t$ be the Fuchsian complex projective structure associated to $c_t$ by applying the Uniformization Theorem. The Schwarzian derivative of the identity map $(\partial_\omega M, \sigma^0_t) \to (\partial_\omega M, \sigma_t)$ is a holomorphic quadratic differential $-q_t$ on $(\partial_\omega M, c_t)$. Let $\mathcal{F}_t$ denote the horizontal measured foliation $\mathcal{F}_t$ of $q_t$. Then, the derivative at $t = 0$ of the renormalized volume $\text{RVol}$ of $M_t$ can be expressed as
\[
\frac{d}{dt} \text{RVol}(M_t) \bigg|_{t=0} = -\frac{1}{2} \frac{d}{dt}(\text{ext}\mathcal{F}_0)(\dot{c}_0),
\]
where $\text{ext}\mathcal{F}_0$ is the extremal length of $\mathcal{F}_0$, considered as a function over $\mathcal{T}(\partial_\omega M)$ (here the Teichmüller space is thought as space of Riemann surface structures over $\partial_\omega M$).

As described in [Sch17], this is one of several interesting results where the quantities
$m_t, \mu_t$ and $\text{Vol}^*$, at the boundary of the convex core $\partial CM_t$, behave between each other as $c_t, \mathcal{F}_t$ and $\text{RVol}$ do at the boundary at infinity $\partial_\omega M$ (see also [Sch13]).

We briefly outline the structure of the paper: in Section 1 we recall the notion
of convex co-compact hyperbolic 3-manifold, of equidistant surfaces from the convex
core $CM_t$, on which we will base large part of our analysis, and we describe a procedure
to locally approximate the boundary of the convex core $\partial CM$ by finitely bent pleated
surfaces. Section 2 is dedicated to the notion of dual volume and the description its properties. In the third Section we describe a formula for the derivative of the length of a measured lamination realized in a hyperbolic manifold $M$, which will be used to express the term $\frac{d}{dt} \ell_M(\mu_0)|_{t=0}$. Finally, in Section 4 we prove the dual Bonahon-Schläfli formula. In the Appendix we describe the notion of dual volume for compact convex bodies sitting inside $\mathbb{H}^3$. In this case we can characterize the dual volume in different ways, from which some of its properties are more manifest than in the case of convex domains sitting inside a convex co-compact hyperbolic manifold.

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1 Convex co-compact manifolds

Let $M$ be a complete hyperbolic 3-manifold, namely a 3-manifold endowed with a complete Riemannian metric having sectional curvature constantly equal to $-1$. A subset $C \subseteq M$ is convex if for any choice of distinct points and for every geodesic arc $\gamma$ in $M$ connecting them, $\gamma$ is fully contained in $C$. Then $M$ is said to be convex co-compact if $M$ has a non-empty compact convex subset $C$. It turns out that, if $M$ is a convex co-compact hyperbolic manifold, there exists a smallest compact convex subset with respect to the inclusion, called the convex core of $M$ and denoted by $CM$.

The boundary of the convex core is the union of a finite collection of connected surfaces, each of which is totally geodesic outside a subset having Hausdorff dimension 1. As described in [Can+06], the hyperbolic metrics on the flat parts "merge" together, defining a complete hyperbolic metric $m$ on $\partial CM$. The locus where $\partial CM$ is not flat is a geodesic lamination $\lambda$, namely a closed subset of $\partial CM$ which is union of disjoint simple $m$-geodesics, called the leaves of the lamination. The surface $\partial CM$ is bent along $\lambda$, and the amount of bending can be described by a measured lamination. More precisely, a measured lamination $\mu$ is a collection of regular positive measures, one for each arc transverse to a lamination $\lambda$, verifying two natural compatibility conditions: if $c$ is a transverse arc and $c'$ is a subarc of $c$, then the measure associated to $c'$ is the restriction to $c'$ of the measure of $c$; the measures are invariant under isotopies between transverse arcs. In particular, the bending measure of $\partial CM$ is a measured lamination that associates to each transverse arc $c$ an integral sum of the exterior dihedral angles along the leaves that $c$ meets. For a more detailed description we refer to [Can+06 Section II.1.11] (see also Section 3 for alternative definitions of these objects).

Definition 1.1. If $A$ is a subset of a metric space $(X, d)$, the $\varepsilon$-neighborhood of $A$ in $X$, which will be denoted by $N_\varepsilon A$, is the set of points of $X$ at distance $\leq \varepsilon$ from $A$. The $\varepsilon$-surface of $A$ in $X$, which will be denoted by $S_\varepsilon A$, is the set of points of $X$ at distance $\varepsilon$ from $A$.

Remark 1.2. If $C$ is a closed convex subset in $\mathbb{H}^3$, then the surfaces $S_\varepsilon C$ are strictly convex $C^1$ surfaces. Indeed, the distance function $d(C, \cdot): \mathbb{H}^3 \to \mathbb{R}_{\geq 0}$ is continuously differentiable on $\mathbb{H}^3 \setminus C$ (see [Can+06 Lemma II.1.3.6]) and its gradient is uniformly...
Lipschitz on \( N_\varepsilon C \setminus N_{\varepsilon'} C \)
for all \( \varepsilon > \varepsilon' > 0 \) (see [Can+06, Section II.2.11]). In particular, the equidistant surfaces from the convex core of a convex co-compact hyperbolic manifold \( M \) are \( C^{1,1} \)-surfaces.

Let \( \Sigma \) be a surface immersed in a Riemannian 3-manifold \( X \). The first fundamental form \( I \) of \( \Sigma \) is the symmetric \((2,0)\)-tensor obtained as pullback of the metric on \( X \). Given a choice of a normal vector field \( \nu \), the shape operator of \( \Sigma \) is the \( I \)-self-adjoint \((1,1)\)-tensor \( B \), defined by setting \( BU := -\mathcal{D}_U \nu \), where \( \mathcal{D} \) is the Levi-Civita connection of \( X \) and \( U \) is a tangent vector field to \( \Sigma \). The second fundamental form, denoted by \( \mathcal{II} \), is the symmetric \((2,0)\)-tensor \( \mathcal{II}(V,W) := I(BV,W) = I(V,BW) \), for any tangent vector fields \( V, W \) to \( \Sigma \). The mean curvature \( H \) is the trace of \( B \). The notions of second fundamental form, shape operator and mean curvature depend on the choice of a normal vector field on \( \Sigma \). Wherever we have to deal with surfaces which are boundaries of domains or with portions of \( \varepsilon \)-surfaces, we will always endow them with the exterior normal vector field pointing outwards the domain or the \( \varepsilon \)-neighborhood, respectively.

**Lines and planes** in \( \mathbb{H}^3 \) are 1 and 2-dimensional totally geodesic subspaces of \( \mathbb{H}^3 \), respectively. A **half-space** is the closure on one of the complementary regions of a plane inside \( \mathbb{H}^3 \). In the following we recall the geometric data of the equidistant surfaces from a plane and a line, respectively. For a proof of them, we refer for instance to [Can+06, Chapter II.2].

**Lemma 1.3.** Let \( P \) be a plane in \( \mathbb{H}^3 \), and fix \( \nu \) a unit normal vector field on \( P \). Then the map \( \eta_\varepsilon : P \to \mathbb{H}^3 \), defined by

\[
\eta_\varepsilon(p) := \exp_p(\varepsilon \nu(p)),
\]

parametrizes a connected component of the \( \varepsilon \)-surface from the hyperbolic plane \( P \) in \( \mathbb{H}^3 \), and in these coordinates we have

\[
I_\varepsilon = \cosh^2 \varepsilon \, g_P,
\]

\[
\mathcal{II}_\varepsilon = -\frac{\sinh 2 \varepsilon}{2} \, g_P = -\tanh \varepsilon \, I_\varepsilon,
\]

where we are choosing as unit normal vector field the one pointing outwards the \( \varepsilon \)-neighborhood of \( P \).

**Lemma 1.4.** Let \( \tilde{\gamma} : \mathbb{R} \to \mathbb{H}^3 \) be a unit speed complete geodesic, and denote by \( e_1(s), e_2(s) \) the vectors, tangent at \( \tilde{\gamma}(s) \), obtained as parallel translations of a fixed orthonormal basis \( e_1, e_2 \) of \( \tilde{\gamma}'(0) \perp T_{\tilde{\gamma}(0)} \mathbb{H}^3 \). Then the map \( \psi_\varepsilon : \mathbb{R} \times S^1 \to \mathbb{H}^3 \), defined by

\[
\psi_\varepsilon(s,e^\theta) := \exp_{\tilde{\gamma}(s)}(\varepsilon (\cos \theta \, e_1(s) + \sin \theta \, e_2(s))),
\]

parametrizes the \( \varepsilon \)-surface from the line \( \tilde{\gamma} \) and in these coordinates we have

\[
I_\varepsilon = \cosh^2 \varepsilon \, ds^2 + \sinh^2 \varepsilon \, d\theta^2,
\]

\[
\mathcal{II}_\varepsilon = -\cosh \varepsilon \sinh \varepsilon \, (ds^2 + d\theta^2),
\]

where we are choosing as unit normal vector field the one pointing outwards the \( \varepsilon \)-neighborhood of \( \tilde{\gamma} \).
We want to give a more precise description of the structure of the boundary of the convex core and, to do so, we need to remind the following notion:

**Definition 1.5** ([Bon96]). Let $S$ be a topological surface. A (abstract) pleated surface with topological type $S$ is a pair $(f, \rho)$, where $f: \tilde{S} \rightarrow \mathbb{H}^3$ is a continuous map from the universal cover $\tilde{S}$ of $S$ to $\mathbb{H}^3$ and $\rho: \pi_1(S) \rightarrow \text{Iso}^+(\mathbb{H}^3)$ is a homomorphism, verifying the following properties:

1. $\tilde{f}$ is $\rho$-equivariant;
2. the path metric on $\tilde{S}$, obtained by pullback of the metric on $\mathbb{H}^3$ under $\tilde{f}$, induces a hyperbolic metric $m$ on $S$;
3. there exists an $m$-geodesic lamination on $S$ such that $\tilde{f}$ sends every leaf of the preimage $\tilde{\lambda} \in \tilde{S}$ in a geodesic of $\mathbb{H}^3$ and such that $\tilde{f}$ is totally geodesic on each complementary region of $\tilde{\lambda}$ in $\tilde{S}$.

Consider $\tilde{C}$ the preimage of $CM$ inside $\mathbb{H}^3 \cong M$. The boundary $\partial \tilde{C}$ is parametrized by a pleated surface $\tilde{f}: \tilde{S} \rightarrow \mathbb{H}^3$ with bending locus $\tilde{\lambda}$, where $\tilde{S}$ is the universal cover of $\partial CM$, and with holonomy $\rho$ given by the composition of the homomorphism induced by the inclusion: $\partial CM \rightarrow M$ and the holonomy representation of $M$. In this situation, the pleated surface $\tilde{f}$ is locally convex, in the sense that the bending occurs always in the same direction, making $\tilde{f}$ locally bound a convex region (see also [Can+06, Section II.1.11]). In general $\tilde{f}$ is a covering of $\partial C$, which is non-trivial whenever $CM$ has compressible boundary.

It will be useful in our analysis to have a way to locally approximate $\partial CM$ by finitely bent surfaces. We briefly recall a procedure described in [Bon96, Section 7] which suits well for our purpose. We start by considering an arc $k$ in $S$ transverse to the bending lamination $\tilde{\lambda}$, having endpoints in two different flat pieces $P$ and $Q$ of $\tilde{S} \setminus \tilde{\lambda}$. We will assume $k$ to be short enough, so that we can find an open neighborhood $U$ of $k$ on which $\tilde{f}$ is a topological embedding, and all the leaves of $\tilde{\lambda}$ meeting $U$ intersect $k$. When this happens, we say that $\tilde{f}$ is a nice embedding near $k$. Let $\mathcal{P}_{PQ}$ be the set of those flat pieces in $\tilde{S} \setminus \tilde{\lambda}$ that separate $P$ from $Q$. For every finite subset $\mathcal{P}$ of $\mathcal{P}_{PQ}$, we label its elements by $P_0, \ldots, P_{n+1}$ following the order from $P = P_0$ to $Q = P_{n+1}$. Let $\Sigma_i$ be the closure of the region in $\tilde{S}$ which lies between $P_i$ and $P_{i+1}$, for $i = 0, \ldots, n$. If we orient the two leaves $\gamma, \gamma'$ lying in $\partial \Sigma_i$ accordingly, so that they can be deformed continuously from one to the other, then we call diagonals of $\Sigma_i$ the two unoriented lines in $\Sigma_i$ that connect two opposite endpoints of $\gamma$ and $\gamma'$.

We denote by $\tilde{\lambda}_P$ the geodesic lamination of $\tilde{S}$ obtained from $\tilde{\lambda}$ as follows: we maintain the geodesic lamination as it is outside $\bigcup \Sigma_i$ and, for every $i = 0, \ldots, n$, we erase all the leaves lying in the interior of the strip $\Sigma_i$ and we replace them by one of the two diagonals of $\Sigma_i$, say $d_i$. Now we define a pleated surface $f_P: S \rightarrow \mathbb{H}^3$, with bending locus $\tilde{\lambda}_P$, so that it coincides with $\tilde{f}$ outside the strips, and inside any $\Sigma_i$ it sends the chosen $d_i$ in the geodesic of $\mathbb{H}^3$ joining the endpoints of $f(\partial \Sigma_i)$ corresponding to the endpoints of $d_i$. Once we make a choice of a diagonal $d_i$ for any $i$, there is a unique way to extend $f_P$ on $S$ so that is becomes a pleated surface bent along $\tilde{\lambda}_P$. Moreover,
if the strips $\Sigma_i$ are thin enough and if the starting $\bar{f}$ is locally convex, then we can make a choice of the diagonals $d_0, \ldots, d_n$ so that the resulting $\bar{f}_n$ is still locally convex. Such $\bar{f}_n$ will not be equivariant anymore under the action of the holonomy of $\bar{f}$, but it will approximate the restriction of $\bar{f}$ on $U$.

Now, choose a sequence of increasing subsets $\mathcal{P}_n$ exhausting $\mathcal{P}_{PQ}$ and construct a corresponding sequence of convex pleated surfaces $f_n := \bar{f}_{P_n}$ as above. Every such $\bar{f}_n$ is finitely bent on the neighborhood $U$. Following the construction, we see that, given any $P'$ flat piece of $S$ intersecting $k$, there exists a large $N \in \mathbb{N}$ so that $f_n(P') = \bar{f}(P') \subset \partial C$ for every $n \geq N$. In particular, the functions $f_n$ are approximating $\bar{f}$ over the open set $U$. Moreover, following the proof of [Bon96, Lemma 22], we see that the bending measures $\mu_n(k)$ of $f_n$ on the arc $k$ are converging to $\mu(k)$, the bending measure of $k$ in $\partial C$.

Let now $r: \mathbb{H}^3 \to \tilde{C}$ denote the metric retraction of $\mathbb{H}^3$ over the convex set $\tilde{C}$ and let $d: \mathbb{H}^3 \to \mathbb{R}_{\geq 0}$ be the distance from $\tilde{C}$. We select an open neighborhood $V$ of $k$ so that $\nabla \subset U$ and, fixed $\rho > 0$, we define $W = W(V, \rho) := r^{-1}(V) \cap N_\rho \tilde{C}$. The surfaces $f_n(U)$ lie behind $\bar{f}(U) \subset \partial \tilde{C}$ if seen from $W$. Denote by $d_n: W \to \mathbb{R}_{\geq 0}$ the distance function from $f_n(U)$ on $W$. Since the surfaces $f_n(U)$ are convex, for every point $p \in W$ there exists a unique $q_n \in f_n(U)$ realizing $d_n(p) = d(p, q_n)$. Therefore, it makes sense to consider the metric retractions $r_n: W \to f_n(U)$, which will converge to $r$ over the compact sets of $W$ thanks to the convergence properties previously observed of the $f_n$’s. By the same argument as [Can+06, Lemma II.2.11.1], the distance functions $d_n$ are converging $C^{1,1}$-uniformly to $d$ on any compact set of $W$ (i.e. the gradients $\nabla d_n$ are uniformly Lipschitz and they converge to $\nabla d$). This shows that for every $\varepsilon < \rho$, the surface $d^{-1}(\varepsilon) \cap W = S_\varepsilon C \cap W$ is $C^{1,1}$-approximated by the sequence of surfaces $(d_n^{-1}(\varepsilon))_n \subset W$. Moreover, such surfaces $d_n^{-1}(\varepsilon) \subset W$ are the $\varepsilon$-equidistant surfaces from finitely bent convex pleated surfaces having bending measures on $k$ converging to $\mu(k)$.

**Definition 1.6.** Given $k$ an arc on which $\bar{f}$ is a nice embedding, we say that the sequence $f_n$ defined above is a standard approximation of $\partial C$ near $k$ and that the sequence of surfaces $S_{\varepsilon,n}$ is a standard approximation of $S_\varepsilon C$ over $k$.

## 2 The dual volume

**Definition 2.1.** Let $M$ be a convex co-compact hyperbolic manifold. If $N$ is a compact convex subset of $M$ with $C^{1,1}$-boundary, we define the dual volume of $N$ as

$$\text{Vol}'(N) := -\text{Vol}(N) - \frac{1}{2} \int_{\partial N} H \, dA.$$  

If $N = CM$, then we set $\text{Vol}'(CM) := -\text{Vol}(CM) + \frac{1}{2} \ell_m(\mu)$, where $m$ and $\mu$ are the hyperbolic metric and the bending measure of $\partial CM$, respectively.

The dual volume is involved in the notion of $W$-volume, defined in [KS08] and used by the authors to introduce the renormalized volume of a convex co-compact hyperbolic
manifold. In our convention of signs, if $N$ is a compact convex subset with \( C^{1,1} \)-boundary in a convex co-compact manifold $M$, we have

\[
W(N) := \text{Vol}(N) + \frac{1}{4} \int_{\partial N} H \, dA
\]

\[
\begin{align*}
&= \frac{1}{2} (\text{Vol}(N) - \text{Vol}^*(N)).
\end{align*}
\]

Krasnov and Schlenker [KS08] considered the interior normal vector field to define $II$ and $H$, which explains the difference in the signs between the relation above and the one in [KS08]. Moreover, their definition of dual volume $V^*$ also differ by a sign with respect to our $\text{Vol}^*$. Applying the definition of dual volume we just gave on compact convex bodies $C$ in $\mathbb{H}^3$, we obtain a function that is positive and monotonic increasing with respect to the inclusion, as follows from the arguments in the Appendix (see in particular the definition of $V^*_2$ in Subsection A.2). This justifies the choice we used here.

In addition, we mention that [BBB17, Lemma 3.3] the authors described a way to characterize the quantity $\int_{\partial N} H \, dA$ in terms of the metric at infinity $\rho_N$ associated to the equidistant foliation $(S_{\varepsilon} N)$. In this way the definition of dual volume (and of $W$-volume) can be given without any regularity assumption on $\partial N$. More precisely, they showed that

\[
\int_{\partial N} H \, dA = -\text{Area}(\rho_N) + 2\text{Area}(\partial N) + 2\pi \chi(\partial M).
\]

We remind that the mean curvature here is the trace of the shape operator $B$, which is defined using the exterior normal vector field to $\partial N$; this explains why the relation above differ by a factor $-2$ from the one in [BBB17]. In particular, the proof of [BBB17, Proposition 3.4] shows also:

**Proposition 2.2.** The dual volume is continuous on the space of compact convex subsets of $M$ with the Hausdorff topology.

In light of this fact, the following Proposition, besides its future usefulness, justifies the definition we gave of $\text{Vol}^*(CM)$.

**Proposition 2.3.** Let $M$ be a convex co-compact hyperbolic manifold, with convex core $CM$, bending lamination $\mu \in \mathcal{ML}(\partial CM)$ and hyperbolic metric $m$ on the boundary of $CM$. Then, for every $\varepsilon > 0$ we have

\[
\text{Vol}^*(N_\varepsilon CM) = \text{Vol}^*(CM) + \frac{\ell_m(\mu)}{4}(\cosh 2\varepsilon - 1) + \frac{\pi}{2} |\chi(\partial CM)| (\sinh 2\varepsilon - 2\varepsilon).
\]

As a consequence, we have

\[
\text{Vol}^*(N_\varepsilon CM) = \text{Vol}^*(CM) + O(|\chi(\partial CM)|, \ell_m(\mu); \varepsilon^2).
\]

**Proof.** First we study $\text{Vol}(N_\varepsilon CM) - \text{Vol}(CM)$. Let $\lambda$ be the support of $\mu$ and let $r': N_\varepsilon CM \to CM$ be the restriction of the metric retraction. We divide $N_\varepsilon CM \setminus CM$ in two regions, $(r')^{-1}(\partial CM \setminus \lambda)$ and $(r')^{-1}(\lambda)$.
If $F$ is the interior of a flat piece in $\partial CM$, then the portion of $N_\epsilon CM$ which retracts onto $F$ through $r'$ has volume equal to

$$
\int_0^\epsilon \int_F \cosh^2 t \, d\text{vol}_{\mathbb{H}^2} \, dt = \frac{\text{Area}(F)}{2} \left( \frac{\sinh 2\epsilon}{2} + \epsilon \right),
$$

where we are making use of the coordinates described in Lemma 1.3. Since the lamination $\lambda$ has Lebesgue measure 0 inside $\partial CM$, the sum of the areas of the flat pieces is

$$\text{Area}(\partial CM) = 2\pi |\chi(\partial CM)|. $$

Let $D$ be the closed convex subset in $\mathbb{H}^3$ obtained as the intersection of two half-spaces whose boundary planes meet with an exterior dihedral angle equal to $\theta_0$ and select $\gamma$ a geodesic arc lying inside the line along which $\partial D$ is bent. Then, the region in $N_\epsilon D$ which retracts over $\gamma$ has volume equal to

$$
\int_0^\epsilon \int_0^{\theta_0} \int_\gamma \cosh t \sinh t \, d\ell \, d\theta \, dt = \frac{\theta_0 \ell(\gamma)}{4} \left( \cosh \epsilon - 1 \right).
$$

(2)

An immediate consequence of this relation is that whenever $\partial CM$ is finitely bent, the volume of $(r'^{-1}(\lambda))$ coincides with $rac{\ell_m(\mu)}{4} \left( \cosh \epsilon - 1 \right)$, where $m$ is the hyperbolic metric of $\partial CM$. In the general case, we can select a suitable covering of $\partial CM$ by open sets on which we can apply the standard approximation argument of Definition 1.6. With this procedure, it is straightforward to see that the relation $\text{Vol}(r'^{-1}(\lambda)) = \frac{\ell_m(\mu)}{4} \left( \cosh \epsilon - 1 \right)$ still holds in the general case. Combining the relations we found, we obtain

$$\text{Vol}(N_\epsilon CM \setminus CM) = \pi |\chi(\partial CM)| \left( \frac{\sinh 2\epsilon}{2} + \epsilon \right) + \frac{\ell_m(\mu)}{4} \left( \cosh 2\epsilon - 1 \right).$$

Now we want to compute $\int_{S_\epsilon CM} H_\epsilon \, dA_\epsilon$. Using Lemmas 1.4 and 1.3 we immediately see that, in the finitely bent case the following holds:

$$\int_{S_\epsilon CM} H_\epsilon \, dA_\epsilon = -2\pi |\chi(\partial CM)| \sinh 2\epsilon - \ell_m(\mu) \cosh 2\epsilon.$$

The standard approximation procedure (see Definition 1.6) allows us again to prove this relation in the general case, with the only difference that the $C^{1,1}$-convergence is now crucial, because the expression of the mean curvature in chart involves the second derivatives in the coordinates system. Combining the relations we proved with the equality $\text{Vol}^\epsilon(\partial CM) = -\text{Vol}(CM) + \ell_m(\mu) / 2$, we deduce the relation in the statement.

As we will see in a moment, it will be convenient for us to differentiate the dual volume enclosed in a differentiable 1-parameter family of $C^{1,1}$-surfaces. In particular, we will make use of the following result, which is a corollary of the differential Schläfli formula proved in [SR99], [RS00]:

**Proposition 2.4.** Let $M$ be a convex co-compact hyperbolic manifold and let $(N_t)_t$ be a 1-parameter family of compact convex subsets in $M$ whose boundaries $\partial N_t$ are $C^{1,1}$.
and vary differentiably in \( t \). Then the variation of the dual volume of \( (N_t) \), at \( t = 0 \) can be expressed as

\[
\frac{d \text{Vol}^*(N_t)}{dt} \bigg|_{t=0} = -\frac{1}{4} \int_{\partial N_0} (\delta I_t H I - I) dA,
\]

where \( I, \mathbb{I}, H \) are the first and second fundamental forms and the mean curvature of the surface \( \partial N_0 \).

**Proof.** The same argument used in Proposition [A.17] to compute the variation of the function \( \text{Vol}^*(N) \) proves the statement, since the variation formula for the hyperbolic volume, recalled in [A.12], holds for \( \mathcal{C}^{1,1} \)-surfaces inside a generic hyperbolic manifold \( M \).

Contrary to the case of a convex body, it is not clear if the dual volume \( \text{Vol}^*(N) \) of a compact convex subset \( N \) in a convex co-compact manifold is or is not a non-negative quantity. However, as in the case of convex bodies (see Remark [A.7]), \( \text{Vol}^* \) shares, with the usual notion of volume, the property to be monotonic increasing with respect to the inclusion, as we see in the following:

**Proposition 2.5.** Let \( N, N' \) be two compact convex subsets inside a convex co-compact manifold \( M \). If \( N \subseteq N' \), then \( \text{Vol}^*(N) \leq \text{Vol}^*(N') \).

**Proof.** Thanks to Proposition [2.2] up to considering \( \varepsilon \)-neighborhoods and passing to the limit as \( \varepsilon \) goes to 0, we can assume that \( N \) and \( N' \) are compact convex subsets with \( \mathcal{C}^{1,1} \)-boundary. We will make use of the variation formula of Proposition [2.4]. Assume that \( \Sigma : I \times S \to M \) is a differentiable 1-parameter family of convex \( \mathcal{C}^{1,1} \)-surfaces \( \Sigma_t := \Sigma(t, \cdot) \), which parametrize the boundaries of an increasing family of compact convex subsets \( (N_t)_{t \in I} \) inside \( M \). Let \( V_t \) be the infinitesimal generator of the deformation at time \( t \), i.e., \( V_t \) is the vector field over \( S \) defined by \( V_t := \frac{d \Sigma_t}{dt} \). The tangential component of \( V_t \) does not contribute to the variation of the dual volume (compare with [RS00, Theorem 1]). Consequently, in order to compute the derivative of \( \text{Vol}^*(N_t) \), we can assume \( V_t \) to be along the exterior normal vector field \( \nu_t \) of \( \partial N_t \). Moreover, since the deformation \( (N_t) \) is increasing with respect to the inclusion, \( V_t \) is of the form \( f_t \nu_t \), for some \( f_t : S \to \mathbb{R} \), \( f_t \geq 0 \). Under this condition, the variation of the first fundamental form of \( \partial N_t \) is \( \delta I_t = -f_t \mathbb{I} \) (again, compare with [RS00, Theorem 1]). If \( k_{1,t}, k_{2,t} \) denote the principal curvatures of \( \partial N_t \), we obtain that

\[
(\delta I_t, H I_t - \mathbb{I}) = -f_t (\mathbb{I}, H I_t - \mathbb{I})
\]

\[
= -f_t ((k_{1,t} + k_{2,t})^2 - k_{1,t}^2 - k_{2,t}^2)
\]

\[
= -2f_t k_{1,t} k_{2,t} \leq 0,
\]

where, in the last step, we used the fact that the exterior curvature \( K_t^* = k_{1,t} k_{2,t} \) is non-negative since \( \partial N_t \) is convex. By Proposition [2.4], we deduce that \( \text{Vol}^* \) is non-decreasing along the deformation \( (N_t) \).

It remains to show that, if \( N, N' \) are two convex subsets of \( M \) with \( \mathcal{C}^{1,1} \)-boundary and such that \( N \subseteq N' \), we can find a differentiable 1-parameter family, indexed by \( t \in [0, 1] \), of increasing convex subsets \( N_t \) with \( \mathcal{C}^{1,1} \)-boundary so that \( N_0 = N \) and \( N_1 = N' \). A way to produce such a path is described in the proof of [Sch13, Lemma 3.14].
3 The derivative of the length

From now on, $S$ will be a fixed closed surface of genus $g \geq 2$. We briefly recall the notions of [Bon88] that we will need. Given $m$ a hyperbolic metric on $S$, the universal cover $\tilde{S}$, endowed with the lifted metric $\tilde{m}$, is isometric to $\mathbb{H}^2$. As the topological boundary of the Poincaré disk sits at infinity of $\mathbb{H}^2$, also $\tilde{S}$ can be compactified by adding a topological circle $\partial_{\infty}S$ at infinity, and the resulting space does not depend on the chosen identification between them. The fundamental group naturally acts by isometries on $\tilde{S} \cong \mathbb{H}^2$, and since the isometries of $\mathbb{H}^2$ extend to $\partial \mathbb{H}$, the same does on $\partial_{\infty}S$. It turns out that the topological space $\partial_{\infty}S$, together with its action of $\pi_1(S)$, is independent of the hyperbolic metric $m$ we chose. In particular, all the spaces we are going to describe are intrinsically associated to the topological surface $S$, without prescribing any additional structure. Since a geodesic in $S$ is determined by its (distinct) endpoints in $\partial_{\infty}S$, the space $\mathcal{G}(S)$ of unoriented geodesics of $\tilde{S}$ can be naturally identified with

$$(\partial_{\infty}S \times \partial_{\infty}S \setminus \Delta)/\mathbb{Z}_2,$$

where $\Delta$ denotes the diagonal subspace of $(\partial_{\infty}S)^2$, and the action of $\mathbb{Z}_2$ exchanges the two coordinates in $(\partial_{\infty}S)^2$. Therefore, a geodesic lamination $\lambda$ of $S$ is identified with a closed, $\pi_1(S)$-invariant subset $\bar{\lambda}$ of disjoint geodesics in $\mathcal{G}(\tilde{S})$. In the same spirit, a measured lamination of $S$ is corresponds to a $\pi_1(S)$-invariant, locally finite Borel measure on $\mathcal{G}(S)$ with support contained in a geodesic lamination $\bar{\lambda}$ of $S$. We denote by $\mathcal{G}\mathcal{L}(S)$ and $\mathcal{M}\mathcal{L}(S)$ the spaces of geodesic laminations and measured laminations on $S$, respectively.

In the following, we recall the notion of length of measured laminations realized inside a fixed hyperbolic 3-manifold $M$ from [Bon97a, Section 7]. As in the case of $S$, we can define the space of unoriented geodesics of $M$, making use of the natural compactification of $\mathbb{H}^3$. The substantial difference is that the dynamical properties of the action of $\pi_1(M)$ depend in general on the hyperbolic metric we are considering on $M$. However, our interest will be to apply these notions to quasi-isometric deformations of hyperbolic manifolds. In this case, the holonomy representations turn out to be quasi-conformally conjugated in $\partial \mathbb{H}^3$, therefore the qualitative properties of the action of $\pi_1(M)$ on $\mathcal{G}(M)$ are preserved. Fix now a homotopy class of maps $[f_0]: S \to M$.

**Definition 3.1.** A geodesic lamination $\bar{\lambda}$ is realizable inside $M$ in the homotopy class $[f_0]$ if there exists a representative $f: S \to M$ of $[f_0]$ which sends each geodesic of $\bar{\lambda}$ homeomorphically in a geodesic of $M$. In such case, we say that $\bar{\lambda}$ is realized by $f$.

Let $\bar{\lambda}$ be a geodesic lamination on $S$ realized by a map $f$, and let $\rho: \pi_1(S) \to \pi_1(M)$ be the homomorphism induced by $[f_0]$ on the fundamental groups. Fixed a lift $\tilde{f}$ of $f$ to the universal covers, we can construct a function $r: \bar{\lambda} \to \mathcal{G}(M)$, associating to each leaf $g$ of $\bar{\lambda}$ the geodesic $\tilde{f}(g)$ sitting inside $M$. The map $r$ is $\rho$-equivariant and continuous with respect to the topologies of $\bar{\lambda}$ as subset of $\mathcal{G}(\tilde{S})$ and of $\mathcal{G}(M)$ (compare with [Bon97a, Section 7]). We claim that $r$ depends only on homotopy class $[f]$ and on the choice of a lift of any representative of $[f]$ realizing $\bar{\lambda}$. To see this, let $F_0 = f$ and $F_1 = f'$ be two such maps in $[f]$ homotopic through $(F_t)_{t \in I}$ (here $I$ denotes the interval $[0,1]$).
[0, 1]). Once we choose a lift \( \tilde{f} \) of \( f \), there exists a unique lift \( \tilde{F} \) of the homotopy so that \( \tilde{F}_0 = \tilde{f} \). This gives a preferred lift of \( f' \), namely \( \tilde{f}' := \tilde{F}_1 \). Because of the compactness of \( S \) and the existence of a homotopy \( F_t \) between them, the lifts \( \tilde{f} \) and \( \tilde{f}' \) must agree (up to parametrization) on any leaf \( g \) of \( \lambda \), since the geodesics \( \tilde{f}(g) \) and \( \tilde{f}'(g) \) are necessarily at bounded distance in \( \mathbb{H}^3 \) (see \cite{Thu79} Proposition 8.10.2). This is equivalent to saying that the definitions of \( r \) obtained using \( \tilde{f} \) and \( \tilde{f}' \) coincide. Moreover, different choices of lifts \( \tilde{f} \) produce maps \( r, r' \) which differ by post-composition by an element in \( \pi_1(M) \). The same argument as above shows that, if \( \lambda_1, \lambda_2 \) are two geodesic laminations realized by the maps \( f_1, f_2 \) respectively, which both contain the lamination \( \lambda \), then the two realizations \( f_1 \) and \( f_2 \) coincide on \( \lambda \).

Given \( \alpha \) a measured lamination on \( S \) with support contained in \( \lambda \), we denote by \( \bar{\alpha} := r_\ast \alpha \) the pushforward of \( \alpha \) under the map \( r \). \( \bar{\alpha} \) is a measure on \( \mathcal{G}(\bar{M}) \) with support \( r(\text{supp} \alpha) \), depending only on \( \alpha \in \mathcal{ML}(S) \), on the homotopy class \( [f] \) and on the choice of a lift of \( f \). Assume now that \( f(\lambda) \) lies inside some compact set \( K \) of \( M \) and let \( \mathcal{F}, \mathcal{F}' \) denote the geodesic foliations of the projective tangent bundles \( PTM, PT\bar{M} \), respectively. We can cover the preimage of \( K \) in \( PTM \) by finitely many \( \mathcal{F} \)-flow boxes \( \sigma_j : D_j \times I \to B_j \), where \( D_j \) is some topological space and \( \sigma_j \) is a homeomorphism sending each subset \( \{p\} \times I \subset D_j \times I \) in a subarc of a leaf in \( \mathcal{F} \), for any \( p \in D_j \). In addition, we fix a collection \( \{\xi_j\}_j \) of smooth functions with supports \( \text{supp} \xi_j \) contained in the interior of \( B_j \) for every \( j \), such that \( \sum_j \xi_j = 1 \) over the preimage of \( K \) in \( PTM \). If \( \sigma_j \) is a \( \mathcal{F} \)-flow box that meets \( f(\text{supp} \alpha) \), we can lift it to a \( \mathcal{F}' \)-flow box \( \bar{\sigma}_j : D_j \times I \to PT\bar{M} \) accordingly with the choice of the lift \( \tilde{f} \). The lift \( \bar{\sigma}_j \) induces an identification between the space \( \bar{D}_j \) with a subset in \( \bar{\mathcal{G}}(\bar{M}) \). Namely, a point \( p \in D_j \) corresponds to the complete leaf in \( \mathcal{F} \) extending the arc \( \bar{\sigma}_j(\{p\} \times I) \), which uniquely determines a geodesic in \( \bar{M} \). Through this identification, it makes sense to integrate the \( D_j \)-component of \( \bar{\sigma}_j \) with respect to the measure \( \bar{\alpha} \) previously defined on \( \bar{\mathcal{G}}(\bar{M}) \). If \( \sigma_j \) does not meet \( f(\text{supp} \alpha) \), then we choose an arbitrary lift \( \bar{\sigma}_j \). Finally, we select lifts \( \bar{\xi}_j \)'s of the \( \xi_j \)'s according with the choices of the lifts \( \bar{\sigma}_j \). The length of the realization of \( \alpha \) in \( M \) (in the homotopy class \( [f] \)) is

\[
\ell_M(\alpha) = \int_{\lambda} d\ell(\alpha) : = \sum_j \int_{D_j} \int_0^1 \bar{\xi}_j(\bar{\sigma}_j(p,s)) d\ell(s) d\bar{\alpha}(p),
\]

where \( d\ell \) denotes the length-measure along the leaves of \( \bar{\mathcal{F}} \).

Remark 3.2. By invariance of the length under reparametrization and by linearity of the integral, the choices of the functions \( \{\xi_j\}_j \) and the chosen \( \mathcal{F} \)-flow boxes \( \{\sigma_j\}_j \) are irrelevant; moreover, different lifts of \( f \) produce maps \( r \) which are conjugated by isometries in \( \pi_1(M) \). Therefore, the quantity \( \ell_M(\alpha) \) only depends on the measured lamination \( \alpha \), the hyperbolic metric on \( M \) and the homotopy class \( [f] : S \to M \). The notion makes sense as long as there exists a realizable geodesic lamination \( \lambda \) in the homotopy class \( [f] \) which contains \( \text{supp} \alpha \). Moreover, by what we observed before, this quantity does not depend on the specific representable lamination \( \lambda \) we chose, but it is determined only by \( \text{supp} \alpha \).

The aim of this section is to produce a variation formula for the length of the realization of a measured lamination inside a 1-parameter family of quasi-isometric con-
inside a fixed compact subset $K$. Therefore, any geodesic lamination on $S$ is realizable in the homotopy class $[f_0]$, and their realizations lie inside a fixed compact subset $K$ of $X$ (where $K$ contains $CM_t$ for every small $t$).

Let now $\lambda$ be any geodesic lamination containing $\text{supp} \alpha$ and assume that it is realized inside $M_t$ by a certain map $f_t : S \to M_t$, for any $t$. By what previously said, we are allowed to consider the length of the realization of $\alpha$ inside $M_t$ for every small value of $t$. Let $\{\sigma^0_j, \xi^0_j, \alpha^0_j, \tilde{\xi}^0_j\}$, be a collection of functions as in the definition of $\ell_{M_t}(\alpha)$. Then, in the same notations as above, we set
\[
\int_\lambda \int_D \tilde{\ell}_0 \, d\alpha := \sum_j \int_{D_j} \int_0^1 \tilde{\xi}^0_j(\tilde{\sigma}^0_j(p,s)) \frac{g_0 \left( \partial_s \tilde{\sigma}^0_j(p,s), \partial_s \tilde{\sigma}^0_j(p,s) \right)}{2g_0 \left( \partial_s \sigma^0_j(p,s), \partial_s \sigma^0_j(p,s) \right)} \, d\tilde{\sigma}_0(p),
\]
where $\frac{\partial_s \tilde{\sigma}_j}{\partial s}$. The result we want to prove is the following:

**Proposition 3.3.** Let $(g_t)$ be a differentiable 1-parameter family of convex co-compact hyperbolic metrics on a 3-manifold $X$, which are quasi-isometric to each other via the identity map of $X$. If $\alpha$ is a measured lamination on a surface $S$ and a homotopy class $[f : S \to X]$, then $\alpha$ is realizable in $M_t$ for $t$ sufficiently small, and the following holds
\[
\frac{d}{dt} \ell_{M_t}(\alpha) \big|_{t=0} = \int_\lambda \int_D \tilde{\ell}_0 \, d\alpha,
\]
where $\lambda$ is a geodesic lamination of $S$ containing $\text{supp} \alpha$.

We will prove the result using an approximation argument. First we need the following:

**Lemma 3.4.** When $\alpha \in \mathcal{ML}(S)$ is a rational lamination, then Proposition 3.3 holds.

**Proof.** Let $c$ be a free homotopy class of simple closed curves in $X$ and assume that $c$ admits a geodesic representative in $M_0$. Since we are considering a quasi-isometric deformation of convex co-compact manifolds, the homotopy class $c$ will admit a geodesic representative for all values of $t$. Moreover, we can find parametrizations $\gamma$ of the geodesic of $c$ in $M_t$, depending smoothly on $t$, because of the smooth dependence of the holonomy representation $\text{hol}_c(c)$. In other words, we can find a smooth map $\Sigma : (-\varepsilon, \varepsilon) \times I \to X$ such that $\Sigma(t,s) = \gamma(s)$ for every $t$ and $s \in I$. Let $\|\cdot\|_g$ denote the

\[13\]
norm with respect to the metric $g$. We have
\[
\frac{d}{dt} \| \partial_t \gamma \|_t \bigg|_{t=0} = \frac{\hat{g}_0(\partial_t \gamma^0, \partial_t \gamma^0) + 2\hat{g}_0(D_{\partial_t} \Sigma|_{t=0}, \partial_t \gamma^0)}{2\|\partial_t \gamma^0\|_0}
\]
\[
= \frac{\hat{g}_0(\partial_t \gamma^0, \partial_t \gamma^0)}{2\|\partial_t \gamma^0\|_0} + \hat{g}_0 \left( D_{\partial_t} \Sigma|_{t=0}, \frac{\partial_t \gamma^0}{\|\partial_t \gamma^0\|_0} \right)
\]
\[
= \frac{\hat{g}_0(\partial_t \gamma^0, \partial_t \gamma^0)}{2\|\partial_t \gamma^0\|_0} + \frac{d}{ds} \left[ \hat{g}_0 \left( \frac{\partial_t \Sigma|_{t=0}}{\|\partial_t \gamma^0\|_0}, \frac{\partial_t \gamma^0}{\|\partial_t \gamma^0\|_0} \right) \right],
\]
where in the last step we used the fact that $\gamma^0$ parametrizes a geodesic in $M_0$, and consequently the covariant derivative of $\frac{\partial_t \gamma^0}{\|\partial_t \gamma^0\|_0}$ vanishes. Once we integrate the last term in $t \in [0,1]$ we get 0, because the function of which we are taking the derivative coincides at the extremes (since the geodesics $\gamma'$ are closed). Hence we obtain
\[
\frac{d}{dt} \ell_M(c) \bigg|_{t=0} = \int_0^1 \frac{\hat{g}_0(\partial_t \gamma^0, \partial_t \gamma^0)}{2\|\partial_t \gamma^0\|_0} \, ds = \int_0^1 \frac{\hat{g}_0(\partial_t \gamma^0, \partial_t \gamma^0)}{2\|\partial_t \gamma^0\|_0} \, d\ell_0.
\]
Take now a rational lamination $\alpha \in M\mathcal{L}(S)$, i.e. the measure $\alpha$ is the weighted sum $\sum_i u_i \delta_{d_i}$, where the $d_i$ are homotopy classes of simple closed curves, the $u_i$ are positive weights, and $\delta_{d_i}$ is the transverse measure which counts the geometric intersection of an arc transverse to $d_i$ with $d_i$. Assume that $\alpha$ is realizable in $M_0$ or, equivalently, that the curves $c_i = f_0(d_i)$ admit a geodesic representative $\gamma^0_i$ in $M_0$. The same argument given above shows that the lamination $\alpha$ is realizable in $M_t$ for all $t$. Applying the definition of $\ell_M(\alpha)$, and denoting by $\gamma'_t : I \to M_t$ the geodesic representative of $c_i$, we see that
\[
\ell_M(\alpha) := \sum_i u_i \left( \int_0^1 \| \partial_t \gamma' (s) \|_t \, ds \right).
\]
Hence, taking the derivative in $t$ and using what observed above, we get
\[
\frac{d}{dt} \ell_M(\alpha) \bigg|_{t=0} = \sum_i u_i \left( \int_0^1 \frac{\hat{g}_0(\partial_t \gamma^0, \partial_t \gamma^0)}{2\|\partial_t \gamma^0\|_0} \, ds \right) = \int_\lambda d\ell_0 \, d\alpha,
\]
where $\lambda = \text{supp} \alpha = \bigcup_i d_i$.

We are now ready to deal with the proof of Proposition 3.3

\textbf{Proof of Proposition 3.3} Let $T$ be a train track in $S$ carrying $\alpha$ and consider a sequence of rational laminations $\alpha_n$ carried by $T$ and converging to $\alpha$ as measured laminations (see [Thu79] Proposition 8.10.7). Up to passing to a subsequence, we can assume that the laminations $\text{supp} \alpha_n$ converge in the Hausdorff topology to a lamination $\lambda$ carried by $T$. Since $\alpha_n$ is converging to $\alpha$, we must have $\lambda \supseteq \text{supp} \alpha$. We denote by $f_t : S \to X$ a realization of $\lambda$ in the homotopy class $[f]$ with respect to the metric $g_t$, and by $\tilde{f}_t : \tilde{S} \to \tilde{M}$ lifts of the $f_t$'s so that $t \to \tilde{f}_t$ is continuous with respect to the compact-open topology of $\mathcal{C}^0(\tilde{S}, \tilde{X})$.

Let now $K$ be a large compact in $X$ containing all the convex cores $CM_t$ for small values of $t$. Then, if $\mathcal{F}_t$ is the geodesic foliation of $PM_t$, we can choose $\mathcal{F}_t$-flow boxes
\{\sigma_j^i\}_j\) whose union of images contain the preimage of \(K\) in \(PTM_t\), and hence the realizations \(f_i(\lambda)\). We consequently construct maps \(\{\tilde{\sigma}_j^i\}_j\), \(\{\tilde{\xi}_j^i\}_j\) as in the definition of \(\ell_M(\cdot)\). We can ask these functions to vary smoothly in the parameter \(t\), since the hyperbolic metrics depends smoothly in \(n\). Now, we define

\[
\varphi_j^i(\cdot) := \int_0^1 \tilde{\xi}_j^i(\tilde{\sigma}_j^i(\cdot, s)) \, dt(s).
\]

In this notation, the length of the realization of \(\alpha_n\) in \(M_t\) can be expressed as

\[
\ell_M(\alpha_n) = \sum_j \int_{D_j} \varphi_j^i \, d\alpha_n.
\]

From this relation is clear that, as \(n\) goes to \(\infty\), \(\ell_M(\alpha_n)\) converges uniformly to \(\ell_M(\alpha)\) on a small interval \((-\varepsilon, \varepsilon)\) of the parameter \(t\). In the same way we see that \(\int dt \, d\alpha_n\) converges to \(\int dt \, d\alpha\) (here is even easier, because there is no dependence on \(t\)). Thanks to Lemma 3.4 the only thing left to conclude the proof is to show that

\[
\lim_{n \to \infty} \frac{d}{dt} \ell_M(\alpha_n) \bigg|_{t=0} = \frac{d}{dt} \ell_M(\alpha) \bigg|_{t=0}.
\]

Here we can argue as follows: the length of a homotopy class \(c\) of non-parabolic type can be expressed as the real part of its complex length \(\ell^c_\ast(c) \in \mathbb{C}/2\pi\mathbb{Z}\), which is holomorphic in the holonomy representation. The argument described above shows that the real lengths \(\ell^c_\ast(\alpha_n)\) are converging uniformly in a small neighborhood of \(\text{hol}_0\) (see also [Sul85, Theorem 2]). Since the real part of a holomorphic function determines (up to immaginary constant) the holomorphic function itself, we deduce that also the complex lengths \(\ell^c_\ast(\alpha_n)\) are converging uniformly, and hence \(C^\infty\)-uniformly. In particular this proves the convergence of the derivatives in \(t\).

\[
\boxed{\square}
\]

4 The variation formula

If \((M_t)_t\) is a smooth family of quasi-isometric convex co-compact manifolds, parametrized by \(t \in (-t_0, t_0)\), we can choose \(\phi_t : M_0 \to M_t\) so that:

i) \(\phi_t\) is a quasi-isometric diffeomorphism for any \(t\), and \(\phi_0 = id\);

ii) once we fix identifications between the universal covers of \(M_t\) with \(\mathbb{H}^3\) for every \(t\), then we can find lifts \(\tilde{\phi}_t : \mathbb{H}^3 \to \mathbb{H}^3\) of \(\phi_t\) so that \(\tilde{\phi}_0 = id_{\mathbb{H}^3}\), and the map \(t \to \tilde{\phi}_t\) is continuous with respect to the \(C^2\)-topology over the compact sets of \(\mathbb{H}^3\);

iii) the map \(\tilde{\phi}\) defined by \(\tilde{\phi}(t, \cdot) := \tilde{\phi}_t(\cdot)\), is smooth as a map from \((-t_0, t_0) \times \mathbb{H}^3\) to \(\mathbb{H}^3\).

Given \(\varepsilon > 0\) and \(t \in (-t_0, t_0)\), we define

\[
\nu^*_\varepsilon(t) := \text{Vol}^*_{M_t}(N_\varepsilon CM_t), \quad u^*_\varepsilon(t) := \text{Vol}^*_{M_t}(\phi(N_\varepsilon CM_0)).
\]
The diffeomorphisms $\Phi_t$ are $\varepsilon^2$-close to $id_{\Sigma^3}$ on compact sets, and the regions $N_\varepsilon CM_0$ are strictly convex, with second fundamental form uniformly bounded away from 0 (depending on $\varepsilon$). Therefore, for small values of the parameter $t$ (possibly depending on $\varepsilon$), the region $\Phi_t(N_\varepsilon CM_0)$ will be convex in $M_t$. In particular this shows that the function $u_{\varepsilon}^t$ is well defined for small values of $t$. In fact, in Lemma 1.3 we will study more precisely how the convexity of these surfaces changes with respect to the parameter.

**Lemma 4.1.** The functions $u_{\varepsilon}^t$ are continuously differentiable for every $\varepsilon > 0$. Moreover, they verify

$$(u_{\varepsilon}^t)'(0) = -\frac{1}{4} \int_{S_0} \left( \delta I_{\varepsilon} - I_{\varepsilon} - \mathbb{I}_{\varepsilon} \right)_{\varepsilon} dA_{\varepsilon}.$$  

**Proof.** The smoothness of the functions $u_{\varepsilon}^t$ follows immediately from the smoothness requirements we made on the map $\Phi$. The first order variation at $t = 0$ is an immediate consequence of the differential Schläfli formula, stated in Proposition 2.4 and the fact that $\Phi_0 = id$. \hfill $\square$

**Proposition 4.2.** Assume $(M_t)$ is a 1-parameter family of convex co-compact manifolds as above. Then $u_{\varepsilon}^t$ admits derivative at $t = 0$ and the following holds:

$$\lim_{\varepsilon \to 0} (u_{\varepsilon}^t)'(0) = \frac{1}{2} \int_{\lambda} d\ell_0 d\mu_0,$$

where $\mu_0$ is the bending measure of $\partial CM_0$ and $\lambda$ is a geodesic lamination containing $\text{supp} \mu_0$.

**Proof.** As already observed, we can divide the surface $S_\varepsilon CM_0$ in two regions:

- the open set $S_\varepsilon^T := r^{-1}(\partial CM_0 \setminus \lambda) \cap S_\varepsilon CM_0$, namely the portion of $S_\varepsilon CM_0$ that projects onto the union of the interior of the flat pieces of $\partial CM_0$;
- the closed set $S_\varepsilon^C := r^{-1}(\lambda)$, namely the portion of $S_\varepsilon CM_0$ that projects to the bending lamination.

On the portion $S_\varepsilon^T$ we have an explicit description of all the geometric quantities, by Lemma 1.5. In particular, we can write the integral in terms of the hyperbolic metric on the flat parts, obtaining

$$\int_{S_\varepsilon^T} (\delta I_{\varepsilon} - I_{\varepsilon} - \mathbb{I}_{\varepsilon})_{\varepsilon} dA_{\varepsilon} = \sum_{F \subset \partial CM_0 \setminus \lambda} \int_F (\delta I_{\varepsilon} - \tanh \varepsilon I_{\varepsilon} \circ r) \cosh \varepsilon \d A_F$$

$$= -\sinh \varepsilon \cosh \varepsilon \int_{\partial CM_0 \setminus \lambda} (\delta I_{\varepsilon} - I_{\varepsilon})_{\varepsilon} \circ r \d A,$$

where the sum is taken over all the flat pieces $F$ in $\partial CM_0 \setminus \lambda$. The variation of the first fundamental form $\delta I_{\varepsilon}$ is the restriction of $\delta \Phi_\varepsilon$ to the tangent space of $S_\varepsilon CM_0$. In particular, since $S_\varepsilon CM_0$ lies in a compact set $K$ of $M_0$, the function $(\delta I_{\varepsilon} - I_{\varepsilon})_{\varepsilon}$ is uniformly bounded. In conclusion, we obtain

$$\lim_{\varepsilon \to 0} \int_{S_\varepsilon^T} (\delta I_{\varepsilon} - I_{\varepsilon} - \mathbb{I}_{\varepsilon})_{\varepsilon} dA_{\varepsilon} = -\lim_{\varepsilon \to 0} \sinh \varepsilon \cosh \varepsilon \int_{\partial CM_0 \setminus \lambda} (\delta I_{\varepsilon} - I_{\varepsilon})_{\varepsilon} \circ r \d A = 0.$$

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Observe that the gradient of the distance. Therefore, they define two orthogonal oriented foliations
expression
for any fixed \( \eta \). The preimage \( \pi^{-1}(S^b_\eta) \), which coincides with \( S^b_\eta \cap \tilde{\mathcal{C}} \), will be denoted by \( \tilde{S}^b_\eta \). Consider a short arc \( k \) in \( \tilde{S} \) with a neighborhood \( U \) on which \( \tilde{f} \) is a nice embedding and set \( W := \text{int}(\tilde{f}^{-1}(U)) \subseteq \mathbb{H}^3 \setminus \tilde{\mathcal{C}} \). Our actual goal is to compute

\[
\lim_{\epsilon \to 0} \int_{W \cap \tilde{S}^b_\eta} (\delta I_e, H_e I_e - \mathbb{1}_e) e dA_\epsilon. \tag{5}
\]

We will make use of a construction described in [Can+06, Section II.2.4]: there the authors illustrate an explicit way to extend the lamination \( \tilde{\mathcal{L}} \) to a partial foliation \( \mathcal{L} = \mathcal{L}_\eta \) of \( \partial \tilde{\mathcal{C}} \), defined in the \( \eta \)-neighborhood (with respect its hyperbolic path metric) of \( \tilde{\lambda} \), for any fixed \( \eta < \log 3/2 \). Up to taking a smaller neighborhood \( U \) of \( k \), we can assume \( \tilde{f}(U) \subset \bigcup \mathcal{L} \) and we can choose a continuous orientation of the foliation \( \mathcal{L} \cap \tilde{f}(U) \). Analogously to what is done in [Can+06, Section II.2.11], we define three orthonormal vector fields on \( W \) as follows:

1. the first vector field \( v \) is given by the gradient of the distance from \( \tilde{\mathcal{C}} \);
2. the second vector field \( E_1 \) is defined in terms of the oriented foliation \( \mathcal{L} \cap \tilde{f}(U) \).
   If \( p \) lies in \( W \), its projection \( r(p) \) belongs to an oriented leaf \( \tilde{f}(g) \) of \( \mathcal{L} \cap \tilde{f}(U) \).
   We denote by \( v \) the unitary vector of \( T_r^g(\mathbb{H}^3) \) tangent to \( \tilde{f}(g) \), and we define \( E_1(p) \) to be the parallel translation of \( v \) along the geodesic arc in \( \mathbb{H}^3 \) connecting \( r(p) \) to \( p \);
3. the last vector field \( E_2 \) is defined requiring that \( (E_1, E_2, v) \) is a positively oriented orthonormal frame of \( T^1 \mathbb{H}^3 \) in \( W \) (assume we have fixed an orientation of \( \mathbb{H}^3 \) since the beginning).

Observe that the \( E_i \)'s are tangent to the surfaces \( \tilde{S}_\epsilon \cap \tilde{\mathcal{C}} \cap W \), since they are orthogonal to the gradient of the distance. Therefore, they define two orthogonal oriented foliations on \( \tilde{S}_\epsilon \cap W \) for every \( \epsilon \). Moreover, if \( r(p) \in \tilde{\lambda} \), then \( E_1(p) \) is a principal direction for the equidistant surface \( \tilde{S}_\epsilon \cap \tilde{\mathcal{C}} \) passing through \( p \). In particular, we have that \( \mathbb{1}_e(E_1, E_1) \equiv -\tanh \epsilon \) (it is a direct consequence of the relations in Lemma 1.4). Expanding the expression \( (\delta I_e, H_e I_e - \mathbb{1}_e) e \) in terms of this orthonormal frame over \( W \cap \tilde{S}^b_\eta \) we have

\[
(\delta I_e, H_e I_e - \mathbb{1}_e) e = (\delta I_e)(E_1, E_1)\mathbb{1}_e(E_2, E_2) + (\delta I_e)(E_2, E_2)\mathbb{1}_e(E_1, E_1) + O(\tilde{g}_0|\epsilon|; \epsilon).
\]

Since the area of \( W \cap \tilde{S}^b_\eta \) goes to 0 as \( \epsilon \) goes to 0, and in particular is bounded, the integral of the term \( O(\tilde{g}_0|\epsilon|; \epsilon) \) in the expression (5) has limit 0. In the end, it remains to study

\[
\lim_{\epsilon \to 0} \int_{W \cap \tilde{S}^b_\eta} (\delta I_e)(E_1, E_1)\mathbb{1}_e(E_2, E_2) dA_\epsilon = \lim_{\epsilon \to 0} \int_{W \cap \tilde{S}^b_\eta} (\delta I_e)_{11}(\mathbb{1}_e)_{22} dA_\epsilon.
\]
We denote by $\mathcal{L}^1_\epsilon, \mathcal{L}^2_\epsilon$ the foliations on $\tilde{S}_\epsilon^1 \cap W$ tangent to $E_1, E_2$, and by $dt^1_\epsilon, dt^2_\epsilon$ their length elements, respectively. Then we can write

$$\int_{\tilde{W} \cap \tilde{S}^1_\epsilon} (\delta I)e_{11}(\mathcal{I}e)_{22} dA_\epsilon = \int_{\tilde{L}^1_\epsilon} \left( \int_{\tilde{L}^1_\epsilon} (\delta I)e_{11} dt^1_\epsilon (\mathcal{I}e)_{22} dt^2_\epsilon. \right. \tag{6}$$

Now it is time to see how this expression behaves in the finitely bent case. Assume that $\tilde{f}(U)$ meets a unique geodesic arc $\gamma$ in $\tilde{\lambda}$ with bending angle $\theta_0$. Then, in the coordinates described in Lemma 1.4, the vector fields $E_1$ and $E_2$ can be written as $E_1 = (\cosh \epsilon)^{-1} \partial_\epsilon, E_2 = (\sinh \epsilon)^{-1} \partial_\epsilon$. Therefore the following relations hold

$$(\delta I)e_{11} dt^1_\epsilon = \frac{g_0(\partial^\epsilon_1, \partial^\epsilon_2)}{\cosh^2 \epsilon} d(\cosh \epsilon) \quad (\mathcal{I}e)_{22} dt^2_\epsilon = \left( -\frac{\cosh \epsilon}{\sinh \epsilon} \right) d(\sinh \epsilon) \theta_0.$$ 

In particular, the limit as $\epsilon \to 0$ of the expression (6) becomes

$$\lim_{\epsilon \to 0} \int_{\tilde{L}^1_\epsilon} \left( \int_{\tilde{L}^1_\epsilon} (\delta I)e_{11} dt^1_\epsilon (\mathcal{I}e)_{22} dt^2_\epsilon = -\theta_0 \int_{\gamma} \hat{g}_0(\gamma', \gamma) d\ell = -2 \int_{\tilde{\lambda} \cap W} d\ell_0 d\mu_0. \right.$$ 

To prove this relation in the general case, we make use of the standard approximations of Definition 1.6. The bending measures along the arc $k$ of the finitely bent approximations $\tilde{f}_n$ weak*-converge to $\mu_0$ along $k$; the $\epsilon$-surfaces from the $\tilde{f}_n$’s converge $C^{1,1}$ uniformly to $W \cap S_\epsilon C$; the vector fields $E_{1,n}, E_{2,n}$ and $\nu_n$, defined by starting from the surface $\tilde{f}_n(U)$, converge uniformly to $E_1, E_2$ and $\nu$ over all the compact subsets of $W$. From these properties, the relation we proved in the finitely bent case extends to the general one.

Finally, a suitable choice of a covering and a partition of unity for a neighborhood of the bending lamination $\mu_0$, combined with Lemma 4.1 proves the statement.

We are interested in showing that, by letting $\epsilon$ varying linearly in $|t|$, the surfaces $\phi_t(S_\epsilon CM_0)$ and $\phi_t^{-1}(S_\epsilon CM_t)$ will remain convex. In what follows, the deformation parameter $t$ will vary inside the interval $(-t_0, t_0)$ and $\epsilon$ will be $\leq \epsilon_0$, for some fixed values of $t_0, \epsilon_0 > 0$. We want to prove the following:

**Lemma 4.3.** There exist constants $K, \tau > 0$, with $0 < \tau \leq t_0$, which depend only on the quasi-isometric deformation $(M_t)$, and on the fixed family of diffeomorphisms $(\phi_t)$, such that, for every $t \in (-\tau, \tau) \setminus \{0\}$ the regions $\phi_t(N_{Kt})(CM_0)$ and $\phi_t^{-1}(N_{Kt})(CM_t)$ are strictly convex in $M_t$ and $M_0$, respectively, with second fundamental forms uniformly bounded away from 0, depending on $t$. As a consequence, we have

$$\phi_t(N_{Kt})(CM_0) \subset CM_t \quad \text{and} \quad N_{Kt}(CM_t) \subset \phi_t(CM_0).$$

Let $\tilde{\phi}_t : \mathbb{H}^3 \to \mathbb{H}^3$ be a lift of $\phi_t$ verifying the properties mentioned at the beginning of the section. Denote by $\pi_t : \mathbb{H}^3 \to M_t$ the universal cover of $M_t$, and by $C_t \subset \mathbb{H}^3$ the preimage of the convex core $CM_t$ under $\pi_t$. The diffeomorphisms $\tilde{\phi}_t$ are converging $C^{2,2}$-uniformly to $id_{\mathbb{H}^3}$ over all compact sets of $\mathbb{H}^3$. Since $M_0$ is convex co-compact, for sufficiently small values of $t$ the subset $\phi_t(N_{t_0}CM_0) \subset M_t$ will remain convex, and
Figure 1: A schematic picture of the surface $S_\varepsilon \mathcal{H}_{\varepsilon,q}$

hence it will contain the convex core $CM_f$. Consequently, fixed $q_0$ a basepoint in $\mathbb{H}^3$, we can find a large $R > 0$ so that the ball $B_R = B(q_0, R)$ in $\mathbb{H}^3$ verifies

$$\pi \Phi(B_R) = \phi_0(B_R) \supseteq N_{t_0}CM_f$$

and $\Phi(B_R) \subseteq B_{R+1}$, whenever $t$ is small enough. Up to choosing a smaller $t_0$, we will assume that these properties hold for every $t \in (-t_0,t_0)$.

Let $r_t(q)$ be the nearest point retraction of $\mathbb{H}^3$ onto the convex subset $\bar{C}_t$. Given a point $q$ of $S_\varepsilon \bar{C}_t$, we denote by $\mathcal{H}_{t,q}$ the unique support half-space of $\bar{C}_t$ at $r_t(q)$ whose boundary $\partial \mathcal{H}_{t,q} = H_{t,q}$ is orthogonal to the geodesic segment connecting $r_t(q)$ to $q$ (see [Can+06] and Figure 1). By construction, we have the inclusion $N_{t_0} \mathcal{H}_{t,q} \subseteq N_{t_0} \bar{C}_t$, and the surfaces $S_\varepsilon \mathcal{H}_{t,q}$, $S_\varepsilon \bar{C}_t$ are tangent to each other at the point $q$.

Obviously Lemma 4.3 reduces to the study of the surfaces $\Phi(S_\varepsilon \bar{C}_0 \cap B_R)$ and $\Phi^{-1}(S_\varepsilon \bar{C}_t \cap B_R)$ in $\mathbb{H}^3$. Instead of dealing directly with these, which are only $C^{1,1}$-surfaces, we can work on equidistant surfaces from planes in $\mathbb{H}^3$. Indeed, given $q \in S_\varepsilon \bar{C}_t$, $S_\varepsilon \mathcal{H}_{t,q}$ lies outside $\text{int}(N_{t_0} \bar{C}_t)$, it approximates $S_\varepsilon \bar{C}_t$ at first order at $q$ and it is strictly convex, with second fundamental form described in Lemma 1.3. Therefore, for the convexity of $\Phi(S_\varepsilon \bar{C}_0 \cap B_R)$ it is enough to show that, for every $q \in S_\varepsilon \bar{C}_0 \cap B_R$ and $t \in (-t_0,t_0)$, the surface $\Phi(S_\varepsilon \mathcal{H}_{t,q})$ remains convex at $\Phi(q)$, with second fundamental form uniformly bounded away from 0. Analogously, the convexity of the surfaces $\Phi^{-1}(S_\varepsilon \mathcal{H}_{t,q})$ at $\Phi^{-1}(q)$, as $q$ varies in $S_\varepsilon \bar{C}_t \cap B_R$, would imply the convexity of $\Phi^{-1}(S_\varepsilon CM_f)$.

Let us introduce some notation we will make use in the following. Given $U$ an open set of $\mathbb{H}^3$, we denote by $S(U, \varepsilon_0)$ the collection of those surfaces $\Sigma$ which are embedded in $U$ and they are obtained as $\Sigma = S_\varepsilon \mathcal{H} \cap U$, for some $\mathcal{H}$ half-space of $\mathbb{H}^3$ intersecting $U$ and for some $0 < \varepsilon < \varepsilon_0$.

By considering the Poincaré disk model, we can identify $\mathbb{H}^3$ with the open unit ball $\Delta$ of $\mathbb{R}^3$, and functions $f : \mathbb{H}^3 \to \mathbb{H}^3$ as maps from $\Delta \subset \mathbb{R}^3$ to itself. If $U$ is an open set
of \( \mathbb{R}^n \), \( K \subset U \) is compact and \( f : U \to \mathbb{R}^m \) is a smooth map, we define

\[
\|f\|_{\mathcal{C}^0(K)} := \max_{p \in K} \|f(p)\|_0,
\]

\[
\|f\|_{\mathcal{C}^k(K)} := \|f\|_{\mathcal{C}^0(K)} + \sum_{h=1}^{k} \left\| D^h f \right\|_{\mathcal{C}^0(K)}
\]

for \( k \geq 1 \), where \( \|\cdot\|_0 \) is the Euclidean (operator) norm and \( D \) is the flat connection.

**Remark 4.4.** In the notations we introduced above, for every \( \varepsilon \leq \varepsilon_0 \) and for every \( q \in S_q \hat{C}_1 \), the surface \( S_q \mathcal{H}_{t,q} \cap B_R \) belong to the family \( S(B_R, \varepsilon_0) \).

**Lemma 4.5.** Let \( B \) be an open ball in \( \mathbb{H}^3 \), \( F : (-t_0, t_0) \times \mathbb{H}^3 \to \mathbb{H}^3 \) a smooth family of diffeomorphisms \( F_t = F(t, \cdot) \), with \( F_0 = id_{\mathbb{H}^3} \) and \( \|F\|_{\mathcal{C}^4((-t_0, t_0) \times \mathbb{H}^3)} < \infty \), and \( \varepsilon_0 \) a positive number. Given \( \Sigma \in S(B, \varepsilon_0) \), we label \( I_1^\Sigma \) and \( I_2^\Sigma \) the first and second fundamental forms of \( F_t(\Sigma) \), respectively, as \( t \in (-t_0, t_0) \). Then we can find \( t'_0 \in (0, t_0] \) and \( D > 0 \), depending only on the ball \( \bar{B} \) and on \( \|F\|_{\mathcal{C}^4((-t_0, t_0) \times \mathbb{H}^3)} \), such that, for every surface \( \Sigma = S_q \mathcal{H}(\infty) \cap B \) in \( S(B, \varepsilon_0) \), we have

\[
I_2^\Sigma + \tanh(I_1^\Sigma) \leq D|t| I_1^\Sigma, \tag{7}
\]

where we are considering the unit normal vector field on \( F_t(\Sigma) \) pointing outwards \( F_t(N_q \mathcal{H}(\infty) \cap B) \).

Assuming for the moment this fact, we can prove Lemma 4.3.

**Proof of Lemma 4.3.** First we study the surfaces \( \phi_\varepsilon(S_q \mathcal{H}_{0,q} \cap B_R) \). Following the argument described above, we need to measure the convexity of the surfaces \( \phi_\varepsilon(S_q \mathcal{H}_{0,q} \cap B_R) \).

We apply Lemma 4.5 to \( F_t : = \hat{\phi}_\varepsilon \) and \( B : = B_R \), obtaining two positive constants \( t'_0 \leq t_0 \) and \( D \), which depend only on \( \|\phi_\varepsilon\|_{\mathcal{C}^4((-t_0, t_0) \times \mathbb{H}^3)} \), so that the relation (7) holds for every \( \Sigma \in S(B_R, \varepsilon_0) \). Now we choose \( K_1, \tau_1 > 0 \), which will depend only on \( D \) and \( t'_0 \), so that \( K_1 \tau_1 \leq \varepsilon_0 \), \( \tau_1 \leq t'_0 \) and

\[
-\tanh(K_1|t|) + |t| \leq 0 \quad \text{for every } t \in (-\tau_1, \tau_1).
\]

In the following, we are going to prove that \( \phi_\varepsilon(S_q \mathcal{H}_{0,q} \cap B_R) \) is convex for every \( t \in (-\tau_1, \tau_1) \), with second fundamental forms uniformly bounded away from zero. Let \( t \) be in \( (-\tau_1, \tau_1) \) and consider \( \varepsilon = K_1|t| \). By the choices we made, if \( q \) is a point in \( S_{K_1|t|} \hat{C}_1 \cap B_R \), then the surface \( S_q \mathcal{H}_{0,q} \cap B_R \) belongs to \( S(B_R, \varepsilon_0) \). In particular, the first and second fundamental forms \( I_1, I_2 \) of \( \phi_\varepsilon(S_q \mathcal{H}_{0,q} \cap B_R) \) verify the relation (7) with \( \varepsilon = K_1|t| \), which can be rewritten as

\[
I_2 + \frac{\tanh(K_1|t|)}{2} I_1 \leq \left( \frac{-\tanh(K_1|t|)}{2} + D|t| \right) I_1.
\]

Because of the choices we made, the right hand side is negative semi-definite. Therefore we have

\[
I_2 \leq -\frac{\tanh(K_1|t|)}{2} I_1.
\]
In particular, the surface $\tilde{\phi}_t(S_{K_1} \mathcal{H}(0_{q} \cap B_R))$ is strictly convex at the point $\tilde{\phi}_t(q)$. Since the choice of $q \in S_{K_1} \mathcal{H}(0_{q} \cap B_R)$ was arbitrary, the argument previously mentioned proves the convexity of $\phi_t(S_{K_1} \mathcal{H}(CM_0))$ for every $t \in (-\tau_1, \tau_1)$.

Now we have to deal with the case of $\varphi_t^{-1}(S_tCM_t)$. Fixed $t \in (-t_0, t_0)$, we define

$$M_s^{(t)} := M_{t+s},$$

$$\psi_s^{(t)} := \phi_{t+s} \circ \varphi_t^{-1}: M_0 = M_t \rightarrow M_s = M_{t+s}$$

for every $s \in (-s_0, s_0)$, with $s_0 = s_0(t) = t_0 - |t|$. Then we apply Lemma 4.5 to the 1-parameter family of diffeomorphisms $(\psi_s^{(t)})_s$, where $\psi_s^{(t)} := \phi_{t+s} \circ \tilde{\phi}_t^{-1}$. By construction, the constants $s_0'$ and $D'$ only depend on $\mathcal{B}_{R+1}$ and $\| \psi(t)\|_{\mathcal{H}^4((-s_0,s_0) \times \mathcal{B}_{R+1})}$.

Since we can find a uniform upper bound for $\| \psi(t)\|_{\mathcal{H}^4((-s_0,s_0) \times \mathcal{B}_{R+1})}$, we can assume that $s_0'$ and $D'$ are independent of $t \in (-\tau_1, \tau_1)$. Therefore, applying the argument of the previous case to the 1-parameter deformation $(M_s^{(t)})$ and the diffeomorphisms $(\psi_s^{(t)})_s$, we can select $\tau \leq s_0'$ and $K$, both independent of $t$, so that the surfaces $\psi_s^{(t)}(S_{K_1} \mathcal{H}(CM_0))$ are convex for every $s \in (-\tau, \tau)$. Moreover, it is not restrictive to ask that $\tau \leq \tau_1$ and $K \geq K_1$ (this ensures that $K$ and $\tau$ work also for $\phi_t(S_{K_1} \mathcal{H}(CM_0))$). Therefore, if $t \in (-\tau, \tau)$, then $s = -t \in (-\tau, \tau)$ and the surface

$$\psi_s^{(t)}(S_{K_1} \mathcal{H}(CM_0)) \bigg|_{s=-t} = \varphi_t^{-1}(S_{K_1} \mathcal{H}(CM_t))$$

is convex, as desired.

The second part follows because of the minimality of the convex core in the family of convex subsets.

Proof of Lemma 4.5. Let $\alpha$ be a curve lying on some surface $\Sigma = S_p \mathcal{H} \cap B \in \mathcal{S}(B, e_0)$. We denote by $\alpha_t$ the curve $F_t \circ \alpha$, by $v_t$ the unit normal vector field of $F_t(\Sigma)$ pointing outwards $F_t(N_p \mathcal{H} \cap B)$, by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ the norm and the scalar product in the hyperbolic metric of $\mathbb{H}^3$.

Assume momentarily that we could find two universal constants $C_1, C_2 > 0$ (depending only on the ball $\overline{B} \subset \mathbb{R}^3$) and a $\tilde{t}_0 > 0$ (depending only on $\overline{B}$ and on the family $(F_t)_t$), such that

$$\left| \| \alpha_t' \|^2 - \| \alpha_t' \|^2 \right| \leq C_1 \| \alpha_t' \|^2 \| F_t - id \|_{\mathcal{H}^2(\overline{B})},$$

$$\left| \langle D_{\alpha_t'}v_t, \alpha_t' \rangle - \langle D_{\alpha_t'}v_0, \alpha_t' \rangle \right| = \left| \langle D_{\alpha_t'}v_t, \alpha_t' \rangle - \tanh \epsilon \| \alpha_t' \|^2 \right| \leq C_2 \| \alpha_t' \|^2 \| F_t - id \|_{\mathcal{H}^2(\overline{B})}$$

for all $t \in (-\tilde{t}_0, \tilde{t}_0)$ (in the last line we used the fact that $S_p \mathcal{H}$ has second fundamental form as in Lemma 1.3). With such estimates, we deduce that

$$\langle L^2 + \tanh \epsilon \tilde{L}^2 \rangle (\alpha_t', \alpha_t') = - \langle D_{\alpha_t'}v_t, \alpha_t' \rangle + \tanh \epsilon \| \alpha_t' \|^2$$

$$\leq - \tanh \epsilon \| \alpha_t' \|^2 + C_2 \| \alpha_t' \|^2 \| F_t - id \|_{\mathcal{H}^2(\overline{B})} + \tanh \epsilon \| \alpha_t' \|^2$$

$$+ C_1 \tanh \epsilon \| \alpha_t' \|^2 \| F_t - id \|_{\mathcal{H}^1(\overline{B})}$$

$$\leq (C_1 + C_2) \tilde{L}^2 (\alpha_t', \alpha_t') \| F_t - id \|_{\mathcal{H}^2(\overline{B})}$$
and therefore that $\| F_t^2 + \tanh \epsilon F_t^2 \|_{\| \cdot \|_{\mathbb{R}^2(\mathcal{B})}} \leq (C_1 + C_2) \| F_t - \text{id} \|_{\varphi^2(\mathcal{B})}$ for every $t \in (-\bar{t}_0, \bar{t}_0)$. Since the map $F$ is regular in $t$, where $F_t = F(t, \cdot)$, we can find two constants $\bar{t}_0$ and $D$, depending only on $\| F \|_{\varphi^4((-\bar{t}_0, \bar{t}_0) \times \mathcal{B})}$ and $\mathcal{B}$, for which the statement holds (the $\mathcal{C}^4$-norm arises because, in order to obtain a bound of the type $O(t)$ from Taylor’s Theorem, we need the $\mathcal{C}^2$-regularity in $t$ of the terms $F_t, DF_t, D^2F_t$).

The only thing left is to prove the two relations above. Let $g_0$ denote the Euclidean metric of $\mathbb{R}^3$ and $g$ the hyperbolic metric on $\Delta \cong \mathbb{H}^3$. Identifying $\mathbb{H}^3$ with an open set of $\mathbb{R}^3$, it make sense to compute a tensor $T_p$ at $p$ on vectors (or forms) lying in the tangent (or cotangent) space at a different point $q$, via the identifications $T_p \mathbb{H}^3 \cong T_p \mathbb{R}^3 \cong T_q \mathbb{R}^3 \cong T_q \mathbb{H}^3$. Therefore we can write:

$$\| \alpha' \|_0 - \| \alpha'' \|_0 \leq \| (g \circ F_1)(D\alpha F_1, D\alpha F_1) - g(\alpha', \alpha') \|_0$$

$$\leq \| (g \circ F_1)(D\alpha F_1, D\alpha F_1 - \alpha'') \|_0 + \| (g \circ F_1)(D\alpha F_1 - \alpha', \alpha') \|_0$$

$$+ \| (g \circ F_1)(\alpha', \alpha') - g(\alpha', \alpha') \|_0$$

$$\leq (\| g \circ F_0 \|_0 \| DF_1 \|_0 \| DF_1 - \text{id} \|_0 + \| g \circ F_1 \|_0 \| DF_1 - \text{id} \|_0$$

$$+ \| g \circ F_1 - g(0) \|_0 \| \alpha' \|_0^2,$$

where $\| \cdot \|_0$ is the operator norm with respect to the Euclidean metric in $\mathbb{R}^3$. The terms $\| DF_1 - \text{id} \|_0$ and $\| g \circ F_1 - g(0) \|_0$ can be bounded by some universal constant times $\| F_t - \text{id} \|_{\varphi^1(\mathcal{B})}$. The terms $\| g \circ F_1 \|_0$. $\| DF_1 \|_0$ are controlled, since $F_1$ is $\mathcal{C}^1$-close to $\text{id}$. Since $\mathcal{B}$ is compact and the $F_t$’s are diffeomorphisms $\mathcal{C}^1$-close to $\text{id}$, the norms $\| \cdot \|_0$, $\| DF_t \|_0$ and $\| \cdot \|$ are uniformly equivalent between each other on $\mathcal{B}$. Combining these facts together we obtain the first inequality.

For the second relation, the way to proceed is analogous, but there are some additional ingredients we need to use. The vector field $v_0$ is the restriction to $\Sigma$ of the gradient of the signed distance from the plane $\partial \mathcal{H}$ (oriented in the suitable way), independently on $\epsilon$. Given a fixed half-space, we can two other vector fields $V_1, V_2$ so that $V_1, V_2$ span the tangent space to every surface $S_\epsilon \mathcal{H}$ as $\epsilon \leq \bar{t}_0$. The vector fields $V_1, V_2$ and $\nabla d$ have first derivatives which are uniformly bounded, as we vary $\mathcal{H}$, since the half-spaces $\mathcal{H}$ must meet $\mathcal{B}$. The vector field $v_t$ can be obtained as

$$\frac{(F_t)_* (V_1) \times (F_t)_* (V_2)}{(F_t)_* (V_1) \times (F_t)_* (V_2)},$$

where $\times$ denotes the vector product. Therefore, once we know that $\| F_t - \text{id} \|_{\varphi^2(\mathcal{B})} \leq 1$, we can say that the first derivatives of $v_t$ are close to the ones of $v_0 = V_1 \times V_2 / \| V_1 \times V_2 \|$, again uniformly in the half-space $\mathcal{H}$ meeting $\mathcal{B}$. Using the expression $\langle Z, D_x Y \rangle = g_{ij} Z^i \partial_k Y^j + Y^k \Gamma_{jk}^{(h)_k}$ and proceeding similarly to what we did before, we can prove the second inequality, which leads to the complete proof of the statement.

**Proposition 4.6.** Assume $(M_t)$ is a 1-parameter family of convex co-compact manifolds as above. Then there exists the derivative of $\text{Vol}^*_M(CM_t)$ at $t = 0$ and it verifies

$$\frac{d}{dt} \left. \text{Vol}^*_M(CM_t) \right|_{t=0} = \lim_{\epsilon \to 0} (u^*_\epsilon)'(0).$$
Proof. The left-hand side is nothing but the limit of the incremental ratio of the function \( v_0^* \) at \( t = 0 \). Let \( K, \tau \) be the constants furnished by Lemma 4.3. We split our incremental ratio as

\[
\frac{v_0^*(t) - v_0^*(0)}{t} = \frac{u_{K|t}^*(t) - u_{K|t}^*(0)}{t} + \frac{v_{K|t}^*(0) - v_0^*(0)}{t} - \frac{u_{K|t}^*(t) - v_0^*(t)}{t},
\]

where we used the fact that \( u_{\varepsilon}^*(0) = v_{\varepsilon}^*(0) \) for all \( \varepsilon > 0 \). The functions \( u_{\varepsilon}^* \) are continuously differentiable, as proved in Lemma 4.1. Using a first order expansion of \( u_{K|t}^* \) at \( t = 0 \), we see that the limit, as \( t \) goes to 0, of first term is \( \lim_{t \to 0} (u_{K|t}^*)' \). Therefore it is enough to show that the other terms are converging to 0 as \( t \) goes to 0.

By Proposition 2.3 applied to the 3-manifold \( M_t \), for every \( \varepsilon > 0 \) we have

\[
v_{\varepsilon}^*(s) - v_0^*(s) = \text{Vol}_{M_t}^*(N_{\varepsilon}CM_t) - \text{Vol}_{M_t}^*(CM_t) = O((\ell_{m_t}(\mu_t), \chi(\partial CM_t); \varepsilon^2)). \tag{8}
\]

In particular, for \( s = 0 \) and \( \varepsilon = K|t| \), this relation proves that the second term goes to 0.

Let \( L > 1 \) be a constant so that all the diffeomorphisms \( \phi_t \) are \( L \)-Lipschitz on a large compact set in \( M_0 \) containing the convex core \( CM_0 \). It is immediate to see that the following properties hold:

\[
\phi_t(N_{\varepsilon}CM_0) \subseteq N_{L\varepsilon}(CM_0) \quad \text{for every } \varepsilon > 0,
\]

\[
N_{\varepsilon'}N_{\varepsilon}CM_0 \subseteq N_{\varepsilon'\varepsilon}CM_t \quad \text{for every } \varepsilon', \varepsilon > 0.
\]

Applying Lemma 4.3 to the 3-manifold \( M_t \) and the properties above, we obtain the following chain of inclusions:

\[
CM_t \subseteq \phi_t(N_{K|t}CM_0) \subseteq N_{LK|t}(CM_0) \subseteq N_{LK|t}(N_{K|t}CM_t) \subseteq N_{(L+1)K|t}CM_t.
\]

for all \( t \in (-\tau, \tau) \). All the submanifolds involved are compact convex subsets of \( M_t \); hence we are allowed to consider their dual volumes. Using the monotonicity of \( \text{Vol}_{M_t}^* \), proved in Proposition 2.5, we get

\[
v_0^*(t) \leq u_{K|t}^*(t) \leq v_{(L+1)K|t}^*(t) \quad \text{for all } t \in (-\tau, \tau).
\]

Applying this to estimate the third term, we obtain

\[
0 \leq \frac{u_{K|t}^*(t) - v_0^*(t)}{t} \leq \frac{v_{(L+1)K|t}^*(t) - v_0^*(t)}{t}, \tag{9}
\]

Since the constants \( K \) and \( L \) only depend on the family \( (\phi_t)_t \), if we apply the equation \( 8 \) with \( s = t \) and \( \varepsilon = (L+1)K|t| \), we get

\[
v_{(L+1)K|t}^*(t) - v_0^*(t) = O((\phi_t)_t, \ell_{m_t}(\mu_t), \chi(\partial CM_t); t^2).
\]

Consequently, the right side in the inequality \( 9 \) goes to 0 as \( t \) goes to 0, and so does the third term, which concludes the proof. \qed

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Given \( \mu \in \mathcal{ML}(S) \), we define the length function of \( \mu \) as the map \( L_\mu : \mathcal{T}(S) \to \mathbb{R}_{\geq 0} \) from the Teichmüller space of \( S \) to \( \mathbb{R}_{\geq 0} \) which associates to the hyperbolic metric \( m \in \mathcal{T}(S) \) the length of \( \mu \) with respect to the metric \( m \). The functions \( L_\mu \) are real-analytic, since they are restrictions of holomorphic functions over the set of quasi-Fuchsian groups (see [Ker85] Corollary 2.2).

The dependence of the geometry of the convex core \( CM \) on the hyperbolic structure of \( M \) is a subtle problem. In [KS95] the authors established the continuity of the hyperbolic metric and the bending measure of \( \partial CM \) with respect to the structure of \( M \). A much more sophisticated analysis, involving the notion of Hölder cocycles, allowed Bonahon to describe more precisely the regularity of these maps, as done in [Bon98b]. In the following, we recall a parametrization result from [Bon96], which was an essential tool in the study of [Bon98b].

Fixed a maximal lamination \( \lambda \) on a surface \( S \), we say that a representation \( \rho \) of \( \pi_1(S) \) in \( \text{Iso}^+(\mathbb{H}^3) \) realizes \( \lambda \) if there exists a pleated surface \( \tilde{f} \) with holonomy \( \rho \) and pleating locus contained in \( \lambda \). Let \( \mathcal{R}(\lambda) \) be the set of conjugacy classes of homomorphisms realizing \( \lambda \), which is open in the character variety of \( \pi_1(S) \) and in bijection with the space of pleated surfaces with bending locus \( \lambda \), up to a natural equivalence relation. [Bon96] Theorem 31] describes a biholomorphic parametrization of \( \mathcal{R}(\lambda) \) in terms of the hyperbolic metric and the bending cocycle of the pleated surface realizing \( \rho \in \mathcal{R}(\lambda) \). In particular, we denote by \( \psi_\lambda : \mathcal{R}(\lambda) \to \mathcal{T}(S) \) the map associating to \( \rho \) the hyperbolic metric of the pleated surface with holonomy \( \rho \).

Now, let \( M \) be a hyperbolic convex co-compact manifold. Denote by \( \mathcal{OD}(M) \) the space of quasi-isometric deformations of \( M \), and by \( \mathcal{R}(\partial CM) \) the representation variety of \( \pi_1(\partial CM) \) in \( \text{Iso}^+(\mathbb{H}^3) \). We have a natural map \( R : \mathcal{OD}(M) \to \mathcal{R}(\partial CM) \) which associates to a convex co-compact hyperbolic structure \( M' \) on \( M \) the conjugacy class of the holonomy \( [\rho'] \) of \( \partial CM' \). If \( \lambda \) is a maximal lamination of \( \partial CM' \) extending the support of the bending measure of \( \partial CM' \), then \( \psi_\lambda \) is defined on a open neighborhood of \( [\rho'] \), therefore we are allowed to consider the map \( \psi_\lambda \circ R \). The result of [Bon98b] we need is the following:

**Theorem 4.7** ([Bon98b] Theorem 1]). Let \( M \) be a hyperbolic convex co-compact manifold and denote by \( \mathcal{OD}(M) \) the space of quasi-isometric deformations of \( M \). Then the map \( Q : \mathcal{OD}(M) \to \mathcal{T}(\partial CM) \) associating to the structure \( M' \) the hyperbolic metric on \( \partial CM' \), is continuously differentiable. Moreover, given any maximal lamination extending the support of the bending measure of \( \partial CM' \), the differential of \( Q \) at \( M' \) coincides with the differential of the map \( \psi_\lambda \circ R \) at \( M' \).

We are finally ready to prove the variation formula for the dual volume of the convex core of a convex co-compact hyperbolic manifold:

**Theorem 8.** Let \( (M_t)_t \) be a smooth 1-parameter family of quasi-isometric hyperbolic convex co-compact manifolds. Denote by \( \mu_0 \in \mathcal{ML}(\partial CM_0) \) the bending measure of the convex core of \( M_0 \) and let \( t \mapsto m_t \in \mathcal{T}(\partial CM) = \mathcal{T}(CM_0) \) be the family of hyperbolic metrics \( m_t \) associated to the boundary of the convex core \( CM_t \) at the time \( t \). Then the derivative of the dual volume of \( CM_t \) at \( t = 0 \) exists and it verifies

\[
\frac{d}{dt} \text{Vol}^*_t(CM_t) \bigg|_{t=0} = \frac{1}{2} \langle d(L_{\mu_0}) \rangle_{m_0}(\dot{m}_0).
\]
Proof. By Proposition 4.6, the derivative of $\text{Vol}^*_M(CM_t)$ at $t = 0$ exists and it coincides with $\lim_{\varepsilon \to 0} (u^*_\varepsilon)'(0)$. By Proposition 4.2, we have the equality

$$\left. \frac{d}{dt} \text{Vol}^*_M(CM_t) \right|_{t=0} = \frac{1}{2} \int_{\lambda_0} \int d\ell_0 d\mu_0,$$

where $\lambda_0 = \text{supp} \mu_0$. By Theorem 4.7, given a maximal lamination $\lambda$ containing $\lambda_0 = \text{supp} \mu_0$, the variation of the hyperbolic metric $\tilde{m}_t$ of the pleated surface in $M_t$ realizing $\lambda$ coincides with the variation of the hyperbolic metric $m_t$ on the boundary of the convex core $CM_t$. By definition, the quantity $\int d\ell_0 d\mu_0$ is $\frac{1}{2} \left. \frac{d}{dt} L_{\mu_0}(\tilde{m}_t) \right|_{t=0}$. Therefore, we obtain that

$$\left. \frac{d}{dt} L_{\mu_0}(m_t) \right|_{t=0} = \frac{1}{2} \int_{\lambda} d\ell_0 d\mu_0,$$

which proves the statement. \qed
A Appendix

Let $\mathbb{R}^{n,1}$ denote the $n$-dimensional Minkowski space, namely the vector space $\mathbb{R}^{n,1}$ endowed with the Lorentzian scalar product $\langle \cdot, \cdot \rangle$ defined as follows

$$\langle x, x \rangle = x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2$$

for any $x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n,1}$. Let $H_1$ and $H_{-1}$ be the subsets of $\mathbb{R}^{n,1}$ given by the elements $x$ verifying $\langle x, x \rangle = 1$ and $\langle x, x \rangle = -1$, respectively. The subspace $H_{-1} \cap \{x_{n+1} > 0\}$ defines a connected $n$-manifold embedded in $\mathbb{R}^{n,1}$ and diffeomorphic to $\mathbb{R}$, which can be naturally endowed with a Riemannian metric, defined simply as the restriction of the scalar product $\langle \cdot, \cdot \rangle$ on the tangent space $T_x H_{-1} = \ker \langle x, \cdot \rangle$, at any point $x \in H_{-1} \cap \{x_{n+1} > 0\}$. We denote by $\mathbb{H}^n$ the resulting Riemannian manifold. This will be our standard model for the $n$-dimensional hyperbolic space.

In the same way, the subset $H_1$ is diffeomorphic to $S^{n-1} \times \mathbb{R}$ and it admits a structure of Lorentzian $n$-manifold, i.e. for every point $x^* \in H_1$, the restriction of the scalar product $\langle \cdot, \cdot \rangle$ to the tangent space $T_{x^*} H_1 = \ker \langle x^*, \cdot \rangle$ defines a scalar product of signature $(n-1,1)$. We will call the resulting Lorentzian manifold the $n$-dimensional de Sitter space and it will be denoted by $\text{dS}^n$.

In the group $O(n,1)$ of isometries of $\mathbb{R}^{n,1}$, we find a 2-index subgroup $O(n,1)^+$, given by those elements that leave $H_{-1} \cap \{x_{n+1} > 0\}$ invariant. These transformations act on the hyperbolic space by isometries, and their induced action on the set of orthonormal frames with basepoint in $\mathbb{H}^n$ is faithful and transitive. In other words, $O(n,1)^+$ is identified with the group of isometries of the hyperbolic space, and denoted by $\text{Iso}(\mathbb{H}^n)$. For the de Sitter space, the isometry group $\text{Iso}(\text{dS}^n)$ is the entire $O(n,1)$ and $O(n,1)^+$ is the subgroup of those isometries that preserve a time-orientation on $\text{dS}^n$.

The hyperbolic space possesses totally geodesic subspaces of any codimension $k$, and they are all obtained as the intersection of $H_{-1} \cap \{x_{n+1} > 0\}$ with codimension $k$ vector subspaces of $\mathbb{R}^{n,1}$ containing a time-like direction. Similarly, the codimension $k$ totally geodesic subspaces of $\text{dS}^n$ are intersections of $H_1$ with codimension $k$ vector subspaces containing a space-like direction. In both cases, a codimension 1 subspace is also called hyperplane (if $n = 3$ we will simply call it plane). While in the hyperbolic space all the totally geodesic subspaces with the same dimension are (ambient) isometric, in the de Sitter space two subspaces are (ambient) isometric if and only if their metrics have the same signature. We say that a subspace of $\text{dS}^n$ is space-like if its induced metric is positive definite. A vector tangent vector $v \neq 0$ to $\text{dS}^n$ is space-like if $\langle v, v \rangle > 0$, time-like if $\langle v, v \rangle < 0$ and light-like if $\langle v, v \rangle = 0$. A half-space in $\text{dS}^n$ or in $\mathbb{H}^n$ is the closure of one of the two components of the complementary of a hyperplane.

A.1 Dual convexes and dual hypersurfaces

In this subsection we describe a duality between closed convex subsets with non-empty interior and strictly convex hypersurfaces sitting inside the geometric spaces $\mathbb{H}^n$ and $\text{dS}^n$, which can be interpreted as an "oriented" instance of the polarity of $\mathbb{R}P^n$ with respect to the quadric $\{x_1^2 + \cdots + x_n^2 = x_{n+1}^2\}$.
Given a point \( x \in \mathbb{H}^n \) and a unitary vector \( v \in T^1_x \mathbb{H}^n \), we can associate to \((x,v)\) a corresponding point \((x',v')\) in the tangent bundle of \(dS^n\). To see this, the vector \( v \in T^1_x \mathbb{H}^n \subset \mathbb{R}^{n,1} \) verifies \( \langle v, v \rangle = 1 \) so, as an element of \( \mathbb{R}^{n,1} \), it belongs to \(dS^n\). On the other hand, since \( v \) is tangent to \( \mathbb{H}^n \) at \( x \), we must have \( \langle x, v \rangle = 0 \). In other words, \( x \), as element of \( \mathbb{R}^{n,1} \), belongs to \( \ker(v, \cdot) \), which is nothing but \( T_x dS^n \). Therefore, the couple \((x',v') := (v,x)\) defines a point in the time-like future-directed tangent bundle \( T^+_1 dS^n \) (since \( \langle x, x \rangle = -1 \)). In the same way, if \((x,v)\) is a point in \( T^-_1 dS^n \), the couple \((x,v) := (v',x')\) defines an element in \( T^1 \mathbb{H}^n \). This correspondence \( T^1 \mathbb{H}^n \rightarrow T^+_1 dS^n \) is clearly one-to-one, and it can be interpreted also as a duality between oriented hyperplanes with basepoint of \( \mathbb{H}^n \) and future-oriented space-like hyperplanes with basepoint of \(dS^n\). A couple \((x,v)\) in \( T^1 \mathbb{H}^n \) corresponds to a hyperplane (namely \( \ker(v, \cdot) \cap \mathbb{H}^n \)) with basepoint \( x \) in \( \mathbb{H}^n \), together with the choice of a normal direction \( v \) at \( x \). In the same way, a point \((x',v')\) in \( T^-_1 dS^n \) is equivalent to the datum of a space-like hyperplane (namely \( \ker(v', \cdot) \)) with basepoint \( x' \in dS^n \) endowed with its future-directed normal vector \( v' \).

As already described in [HR93], this correspondence induces a duality between closed convex subsets with non-empty interior in the two geometries. In the following we rapidly recall the definitions and the results in [HR93] Section 2] that we will use:

**Definition A.1.** A convex body in \( \mathbb{H}^n \) or \( dS^n \) is a subset \( C \) with non-empty interior, which is an intersection of closed half-spaces.

Observe that any convex body is closed. We define a dual operation between subsets of \( \mathbb{H}^n \) and \( dS^n \) as follows: given \( C \subseteq \mathbb{H}^n \), we set

\[
C^\wedge := \{v' \in dS^n \subseteq \mathbb{R}^{n,1} \mid \forall w \in C \quad \langle v', w \rangle \leq 0\}.
\]

In the same way, if \( C' \subseteq dS^n \), then

\[
(C')^\wedge := \{v \in \mathbb{H}^n \subseteq \mathbb{R}^{n,1} \mid \forall w' \in C' \quad \langle v, w' \rangle \leq 0\}.
\]

**Lemma A.2 ([HR93] Section 2]).** Let \( C, D \) be two subsets of \( \mathbb{H}^n \) (of \( dS^n \)). Then

i) \( C^\wedge \) is a convex body;

ii) if \( C \subseteq D \), then \( C^\wedge \supseteq D^\wedge \);

iii) if \( C \) is a convex body, then \( C^{\wedge\wedge} = C \).

Assuming that \( C \) is a convex body with regular boundary, we want to investigate in detail the relation between the hypersurfaces \( \partial C \) and \( \partial(C^\wedge) \). First we need to introduce some notations. Let \( D \) and \( D^* \) denote the Levi-Civita connections of \( \mathbb{H}^n \) and \( dS^n \), respectively. In what follows, \( \Sigma \) will be an orientable smooth manifold of dimension \( n - 1 \) and \( \iota: \Sigma \rightarrow \mathbb{H}^n \) an immersion of \( \Sigma \) in \( \mathbb{H}^n \). Fixed a unitary normal vector field \( v: \Sigma \rightarrow T^1 \mathbb{H}^n \) on \( \Sigma \), we denote by \( I \) and \( II \) its first and second fundamental forms, by \( B \) its shape operator and by \( H \) its mean curvature (see Section I for the definitions). Moreover, we introduce the third fundamental form as the symmetric \((2,0)\)-tensor \( III(v,\cdot) = I(B\cdot, v) \). If \( \nabla \) is the Levi-Civita connection of \( \Sigma \), then we have

\[
\nabla_U V = D_U V - II(U, V) v
\]
for any tangent vector fields $U, V$ of $\Sigma$. We assume in addition that $\mathcal{I}$ is negative definite at every point, which in particular implies $\Sigma$ to be strictly convex. Now we define the dual couple $(t^*, v^*)$ of $(t, v)$ as follows: the maps $t^*: \Sigma \rightarrow dS^n$ and $v^*: \Sigma \rightarrow T_{t^*}^{-1}dS^n$ associate to any $p \in \Sigma$ the elements $t^*(p)$ and $v^*(p)$, respectively, verifying $(p, v_p) = (t^*(p), v^*(p))$, where $(p, v_p)$ denotes the image of $(p, v_p)$ under the duality map $T^1\mathbb{H}^n \rightarrow T^*_{t^*}dS^n$. In other words, making use of the correspondence, the Gauss map $v$ becomes the immersion of a dual hypersurface $t^*$ in $dS^n$, and the immersion $t$ becomes the Gauss map of the immersed hypersurface $t^*$. As we will see in the proof of Proposition [A.4], the fact that $t^*$ is an immersion is implied by the fact that $\mathcal{I}$ is negative definite. Frequently, instead of referring to the dual hypersurface as the map $t^*$, we will simply denote it, with abuse, by $\Sigma^*$. The pull-back metric $I^*$ on $\Sigma^*$ turns out to be positive definite. As in the Riemannian case, $I^*$ is called the first fundamental form of $\Sigma^*$. In general, when $\Sigma^*$ is an immersed hypersurface in $dS^n$, we say that $\Sigma^*$ is space-like if its first fundamental form is positive definite. In order to define the second fundamental form $\mathcal{I}^*$, we require the following relation to hold:

$$\nabla^*_U V^* = D^*_U V^* - \mathcal{I}^* (U^*, V^*) V^*$$

for any tangent vector fields $U^*, V^*$ of $\Sigma^*$, where $\nabla^*$ is the Levi-Civita connection of $\Sigma^*$. The shape operator $B^*$ is again the $I^*$-self-adjoint operator associated to $\mathcal{I}^*$. We remark that, since the normal vector field $v^*$ on $\Sigma^*$ has $\langle v^*, v^* \rangle = -1$, the shape operator of $\Sigma^*$ verifies $B^*U^* = -D^*_U v^*$, while the one of $\Sigma$ is $BU = -D_U v$. The third fundamental form is defined, as the one of $\Sigma$, by asking $\mathcal{I}^*(-, \cdot) = I^*(B^*, B^*)$.

With exactly the same procedure, we see that, given a space-like hypersurface $\Sigma^*$ immersed via $t^*$ in $dS^n$, together with a future-directed normal vector field so that $\mathcal{I}^*$ is positive definite, we can construct a dual hypersurface in $\mathbb{H}^3$ with $\mathcal{I} < 0$. We clearly have $t^{**} = t$.

When a hypersurface arises as the boundary of a domain, we will always choose the normal vector field to the the one pointing outwards. In this terms, we can restate the analysis done by Hodgson and Rivin as follows:

**Lemma A.3 ([HR93], Section 2)].** Let $C$ be a convex body in $\mathbb{H}^n$ with smooth boundary and with $\mathcal{I} > 0$. Then the boundary of $C^\wedge$ in $dS^n$ is parametrized by the dual hypersurface associated to $\partial C$. In other words, we have

$$(\partial C^\wedge)^* = \partial (C^\wedge).$$

In the same way, if $C'$ is a convex body in $dS^n$ with smooth, space-like convex boundary and with $\mathcal{I}^* > 0$, then $(\partial C'^\wedge)^* = \partial (C'^\wedge)$.

The following statement describes explicitly the relations between the fundamental forms of $(t, v)$ and $(t^*, v^*)$:

**Proposition A.4 ([Sch06], Proposition 1.6]).** If $\Sigma$ is an immersed, strictly convex hypersurface in $\mathbb{H}^n$, then its dual $\Sigma^*$ is an immersed, space-like, strictly convex hypersurface in $dS^n$, and viceversa. Moreover, under the duality correspondence between $\Sigma$ and $\Sigma^*$, we have that:

- $I = \mathcal{I}^*$;
• $\mathcal{I} = -\mathcal{J}$;

• $\mathfrak{I} = \mathfrak{J}$.

Proof. We denote by $\alpha : U \to \Sigma \subset \mathbb{H}^n$ a local parametrization, where $U$ is an open set in $\mathbb{R}^{n-1}$. Let $(E_k)_k$ be the orthonormal frame on $\mathbb{R}^{n,1}$ corresponding to a fixed orthonormal basis $\mathcal{B}$ of $\mathbb{R}^{n,1}$ (the $E_k$’s are the sections of $T\mathbb{R}^{n,1}$ associated with $\mathcal{B}$ under the identification $T_p\mathbb{R}^{n,1} \cong \mathbb{R}^{n,1}$). For convenience, we introduce the following notation: if $f$ is a map from $U$ to $\mathbb{R}^{n,1}$, we denote by $X_f$ the element of $\Gamma(\alpha^*T\mathbb{R}^{n,1})$ defined as $X_f = \sum_k f^k E_k \circ \alpha$, where $\Gamma(\alpha^*T\mathbb{R}^{n,1})$ is the space of sections of the pullback bundle of $T\mathbb{R}^{n,1}$ over $\alpha$.

Let $v \in \Gamma(\alpha^*T\mathbb{R}^{n,1})$ be the exterior unitary normal vector field of $\Sigma$. Then, by definition of the duality $T^1\mathbb{H}^n \leftrightarrow T_{-1}dS^n$, we can construct a parametrization $\alpha^* : U \to dS^n$ of $\Sigma^*$ by requiring that $X_{\alpha^*} = v$. In other words, the $k$-th component of $\alpha^*$ with respect to $\mathcal{B}$ coincides with the $k$-th component of $v$ with respect to the frame $(E_k)_k$, for all $k$. Analogously we have $v^*_\alpha = X_{\alpha^*}$, where $v^*$ is the future-directed normal vector field to $\Sigma^*$. Since $(E_k)_k$ is an orthonormal frame of parallel vector fields with respect to the Levi-Civita connection $D$ of $\mathbb{R}^{n,1}$, for every coordinate vector field $\partial_i$ of $\alpha$ we have

$$D_{\partial_i} v = D_{\partial_i} X_{\alpha^*} = \partial_i^* v$$

and, in the dual hypersurface

$$D_{\partial_i} v^* = D_{\partial_i} X_{\alpha} = \partial_i^*,$$

where $\partial_i^*$ is the $i$-th coordinate vector field of $\Sigma^*$ associated to the parametrization $\alpha^*$. Observe also that the normal direction to $\mathbb{H}^n$ at the point $\alpha(p)$ is given by $X_{\alpha}(p)$. This implies that, if $D$ is the Levi-Civita connection of $\mathbb{H}^n$, we have

$$D_{\partial_i} v = D_{\partial_i} v - \frac{\langle D_{\partial_i} v, X_{\alpha} \rangle}{\langle X_{\alpha}, X_{\alpha} \rangle} X_{\alpha}$$

$$= D_{\partial_i} v + \langle D_{\partial_i} v, X_{\alpha} \rangle X_{\alpha}$$

$$= D_{\partial_i} v + \langle \partial_i^* v, X_{\alpha} \rangle X_{\alpha}$$

$$= D_{\partial_i} v. \quad (X_{\alpha} = v^* \perp \Sigma^*)$$

This equality, combined with the relation (10), shows that the shape operator $B$ of $\Sigma$ verifies

$$B \partial_i = -D_{\partial_i} v = -D_{\partial_i} v = -\partial_i^*. \quad (\text{relation (10)})$$

In the same way we see that $D_{\partial_i} v^* = D_{\partial_i} v^*$, with $\mathcal{D}^*$ the connection on $dS^n$, so the shape operator $B^*$ of $\Sigma^*$ verifies

$$B^* \partial_i^* = +D_{\partial_i} v^* = +D_{\partial_i} v^* = +\partial_i. \quad (\text{relation (10)})$$

The tangent spaces $T_{\Sigma} \Sigma$ and $T_{\Sigma^*} \Sigma^*$, as linear subspaces of $\mathbb{R}^{n,1}$, are both orthogonal to the 2-plane generated by $X_{\alpha}$ and $X_{\alpha^*}$, so they must coincide. Therefore, the shape operators $B$ and $B^*$ are both endomorphisms of $T_{\Sigma} \Sigma = T_{\Sigma^*} \Sigma^*$ and, by the relations we
just proved, they verify $B^{-1} = -B^*$. All the relations in the statement can be deduced from this equality, in the following we prove $I = -I^*$, the others are analogous:

$$I(\partial_i, \partial_j) = \langle B \partial_i, \partial_j \rangle = -\langle \partial_i^*, B^* \partial_j^* \rangle = -I^*(\partial_i^*, \partial_j^*).$$

\[\square\]

**Remark A.5.** If we considered the past-directed time-like tangent bundle $T_{-1}dS^n$ instead of the future-directed one, we would have $B^{-1} = B^*$ and, consequently, the relations of the statement would still be verified except for the one of the second fundamental forms, which would be replaced by $I = I^*$.

### A.2 The dual volume of convex bodies

From now on, we will focus on the case $n = 3$, i.e., the 3-dimensional hyperbolic and de Sitter spaces. Thanks to the correspondence between convex bodies in the hyperbolic and de Sitter geometries, it is possible to define a notion of dual volume for convex bodies in $H^3$. In what follows, we will describe different and complementary ways to introduce this quantity.

Let $S$ be a space-like plane in $dS^3$. We denote by $t_S: dS^3 \to \mathbb{R}$ the signed future-directed time-like distance from the plane $S$. Given such a $S$ in $dS^3$, we can find global coordinates $(S^2 \times \mathbb{R}, h_S)$ on $dS^3$ so that the submanifold $S^2 \times \{0\}$, sitting inside $S^2 \times \mathbb{R}$, corresponds to the space-like plane $S$, and the $\mathbb{R}$-component of the coordinate system is given by the function $t_S$ defined above. Then the Lorentzian metric of $dS^3$ can be written as

$$h_S = -dt_S^2 + \cosh^2 t_S g_{S^2},$$

where $g_{S^2}$ denotes the standard Riemannian metric on the 2-sphere of radius 1. Once we fix an orientation on $dS^3$, we can define $\omega_S$ to be the 2-form given by

$$\left(\int_0^{t_S} \cosh^2 \rho \, d\rho\right) d\text{vol}_{S^2},$$

where we are choosing $d\text{vol}_{S^2}$ so that

$$d\omega = \cosh^2 t_S dt_S \wedge d\text{vol}_{S^2} = d\text{vol}_{dS^3}.$$

**Definition A.6.** Let $C$ be a compact convex body in $H^3$ with $C^{1,1}$-boundary. Then

- given a fixed point $p$ in the interior of $C$, we define
  $$V_1^*(C) = \text{Vol}_{dS^3}(p^\land \backslash C^\land),$$

  where $p^\land$ denotes (with abuse) the convex in $dS^3$ dual to $\{p\} \subset H^3$;

- given a fixed space-like plane $S$ in $dS^3$, we define
  $$V_2^*(C) = \int_{\partial(C^\land)} \omega_S,$$

  where $(\partial C)^* = \partial(C^\land) \subset dS^3$ is future-oriented;
• choosing as normal vector field to $\partial C$ the one pointing outwards $C$, we define

$$V_3^* (C) = - \operatorname{Vol}_{H^3}(C) - \frac{1}{2} \int_{\partial C} H \, dA.$$ 

**Remark A.7.** Given $p$ be a point in $H^3$, the set $p^\wedge$ coincides with the upper (i.e., future-directed) half-space bounded by the polar space-like plane of $p$. If $C$ is a compact convex body and $p$ lies in the interior of $C$, then there exists a radius $r > 0$ such that the ball $B_r$ of radius $r$ centered at $p$ is contained in $C$. By Lemma A.2 we deduce that $p^\wedge \supset B_r \supset C^\wedge$. This implies in particular that $C^\wedge$ lies in the interior of $p^\wedge$. The subset $p^\wedge \setminus C^\wedge$ is the region of $dS^3$ bounded from above by $\partial (C^\wedge)$ and from below by the polar plane to $p$. Since $C$ is compact, we can find a $R$-ball ball $B_R$ at $p$ containing $C$. Again by Lemma A.2 we have

$$p^\wedge \setminus B_r \subseteq p^\wedge \setminus C^\wedge \subseteq p^\wedge \setminus B_R.$$ 

It is immediate to check that $p^\wedge \setminus B_R$ is compact, therefore the same holds for $p^\wedge \setminus C^\wedge$. This proves that $0 < V_1^* (C) < \infty$. In fact, the same kind of argument shows that $V_1^*$ is monotonic increasing with respect to the inclusion. Contrary to the standard hyperbolic volume, $V_1^*$ is not additive, as one can easily see by considering, for instance, two simplices glued along a face to build a convex polytope (see relation (13) below).

We will see in Remark A.11 a proof of the independence of $V_1^*$ and $V_2^*$ on the chosen point $p$ and plane $S$, respectively. The request of $C^{1,1}$-regularity of the boundary is technical and it will appear later when we will consider variation formulas. Observe that all the results in the previous subsection hold also in the $C^{1,1}$-case, up to replacing any equality with an equality almost everywhere whenever order 2 derivatives are involved (e.g., $H$, $B$, $I$, and $II$).

The remainder of this subsection will be dedicated to the proof of the equivalence of these quantities. More precisely, we will see in Proposition A.17 that, for every compact convex body in $H^3$ with $C^{1,1}$-boundary and with $II < 0$, we have

$$V_1^* (C) = V_2^* (C) = V_3^* (C)$$

Therefore, combining this with Proposition A.19 we will be allowed to define:

**Definition A.8.** If $C$ is a compact convex body in $H^3$, we define its dual volume to be $\operatorname{Vol}^* (C) = V_1^* (C)$. If $C$ has $C^{1,1}$-boundary, we equivalently set $\operatorname{Vol}^* (C) = V_i^* (C)$, $i = 1, 2, 3$.

Before going into the details, we want to make some remarks about the convenience of these different descriptions. The definition $V_1^*$ is useful because it does not require the convex body to have $C^{1,1}$-boundary. The expression $V_2^*$ will be convenient to show the independence of $V_1^*$ on the chosen point $p$. Lastly, the third definition gives an explicit link between the notions of dual and standard volumes in terms of the geometry of the boundary of the domain. In addition, $V_3^*$ can be trivially extended to the case of convex subsets with regular boundary sitting inside a general 3-dimensional hyperbolic manifold, as we did in Definition 2.1.
Lemma A.9. For any choice of space-like planes $S, S'$ we have
$$\int_S \omega_{S'} = 0.$$ 

Proof. Let $F : dS^3 \to dS^3$ be the antipodal map, i.e., $F(v) = -v$ for all $v \in dS^3$. Since the subspaces of $dS^3$ are intersections of vector subspaces of $R^3$ with $dS^3$, every subspace of $dS^3$ is invariant under $F$. The degree of $F$ as a diffeomorphism of $dS^3$ is equal to $(-1)^4 = 1$, while the degree of the restriction of $F$ on a plane in $dS^3$ is equal to $(-1)^3 = -1$. Moreover, we observe that, if $t_S$ is the signed distance from $S'$, then we have $t_S \circ F = -t_S$. Then
$$F^* \omega_{S'} = \left( \int_{t_S \circ F} \cosh^2 \rho \, d\rho \right) F^* \text{dvol}_{S^2} = \left( \int_{-t_S} \cosh^2 \rho \, d\rho \right) (-1) \text{dvol}_{S^2} = \omega_{S'}.$$ 

Now, using this and the fact that $F$ has degree $-1$ on $S$, we get
$$\int_S \omega_{S'} = - \int_S F^* \omega_{S'} = - \int_S \omega_{S'},$$
and so $\int_S \omega_{S'} = 0$, as desired. \qed

Corollary A.10. For every compact convex body $C$ in $H^3$ with $C^1$ boundary we have
$$V^*_1(C) = V^*_2(C).$$

Proof. The proof goes as follows:
$$V^*_1(C) := \int_{\partial p \setminus C^0} \text{dvol}_{dS^3} = \int_{\partial p \setminus C^0} \omega_S = \int_{\partial (p \setminus C^0)} \omega_S =: V^*_2(C).$$
The first equality holds by definition of the 2-form $\omega_S$; the second one is simply an application of the Stokes' Theorem, where the sign $+$ and $-$ stand for future and past-oriented, respectively; in the third one we are using the fact that $\partial (p \setminus C)$ is a plane, therefore $\int_{\partial (p \setminus C)} \omega_S$ vanishes by Lemma A.9. \qed

Remark A.11. The chain of equalities in the previous proof shows at the same time that $V^*_1(C)$ does not depend on the choice of $p$ (since it is equal to $\int_{\partial (C^0)} \omega_S$), and $V^*_2(C)$ does not depend on the choice of $S$ (since it is equal to $\int_{p \setminus C^0} \text{dvol}_{dS^3}$). In fact, the proof of Corollary A.10 can be immediately generalized to $H^n$, $dS^n$ for any $n \geq 3$. On the contrary, the equality between $V^*_1$ and the other two definitions is specific of the 3-dimensional case (see [SS03] for higher dimensional analogues). In order to prove that $V^*_1$ coincides with $V^*_1 = V^*_2$, we will use an analytic approach based on the following result:
Theorem A.12 ([SR99]). Let $\Sigma$ be an immersed $C^{1,1}$-surface in a hyperbolic manifold $M$ and let $V$ be a section of the restriction of $TM$ over $\Sigma$. The vector field $V$ define a deformation of $\Sigma$ inside $M$. Denote by $I$, $II$, $H$ the first and second fundamental forms and the mean curvature of $\Sigma$, respectively, where we choose as normal vector field the one pointing outwards the volume bounded by $S$. If $\delta T$ denotes the first order variation of the object $T$ under the deformation, then

$$\delta \text{Vol}_M = -\frac{1}{2} \int_{\Sigma} \left( \delta H + \frac{1}{2} (\delta I, II) \right) dA.$$ 

Analogously, if $\Sigma^*$ is an immersed, space-like $C^{1,1}$-surface in a Lorentzian manifold $M^*$ with constant sectional curvature equal to $+1$, and $V^*$ is a section of the restriction of $TM^*$ to $\Sigma^*$, then

$$\delta \text{Vol}_{M^*} = \frac{1}{2} \int_{\Sigma^*} \left( \delta H^* + \frac{1}{2} (\delta I^*, II^*)^* \right) dA^*,$$

where $I^*$, $II^*$, $H^*$ are the first and second fundamental forms and the mean curvature of $\Sigma^*$, respectively, and $\delta T^*$ denotes the first order variation of the object $T^*$ under the deformation.

Remark A.13. There is a confusion in the sign of [SR99] for the Lorentzian case. By following the proof of [RS00, Theorem 3] given in the Riemannian setting, when we switch to the Lorentzian context we have a difference of sign on the right hand side because $BX = \nabla_X n$ instead of $-\nabla_X n$. However, there is a different sign on the left hand side too, because the divergence theorem, applied to a domain with space-like boundary and with exterior normal direction, has a different sign with respect to the Riemannian one.

Remark A.14. The request of $C^{1,1}$-regularity of the boundary is needed here in order to have a notion of mean curvature. This quantity will be a function in $L^\infty(\Sigma)$, therefore defined almost everywhere. Nevertheless, the relations above still hold and make sense, since the integrals of $H$ and its variation are well defined quantities.

The relations in Theorem A.12 are called the differential Schlafli formulas in $\mathbb{H}^3$ and $dS^3$, respectively. This name comes from a much classical result about the variation of the volume of a family of polyhedra, called the classical Schlafli formula:

Theorem A.15 ([Sch58], [Mil94], [Vin+13]). Consider $P$ a convex polyhedron inside $\mathbb{H}^3$. By taking a smooth variation of the vertices and considering their convex hull, we obtain a $1$-parameter family $(P_t)_t$ of polyhedra in $\mathbb{H}^3$, with $P_0 = P$. Assume that the polyhedra $P_t$ share the same combinatorial structure for $t$ sufficiently close to $0$. Then the function $t \mapsto \text{Vol}(P_t)$ admits derivative $\delta \text{Vol}(P)$ at $t = 0$, and it verifies

$$\delta \text{Vol}(P) = \frac{1}{2} \sum_{e \text{ edge of } P} \ell(e) \delta \theta(e),$$

where the sum is taken over the $1$-dimensional edges $e$ of $P$, $\ell(e)$ denotes the length of $e$ in $P$ and $\delta \theta(e)$ is the variation of the exterior dihedral angle along $e$ in the family $(P_t)_t$, (since the combinatorics of $P_t$ does not change, any edge $e$ of $P$ has a corresponding edge $e_t$ in $P_t$).
In [RS00, Theorem 2] it is shown how to deduce the classical Schläfli formula in $\mathbb{H}^3$ from the differential one stated in Theorem A.12 above. We briefly recall the procedure. Consider the convex body $N_\varepsilon P_t$ given by the set of points at distance $\leq \varepsilon$ from $P_t$ (which has $C^{1,1}$ boundary), then apply the differential Schläfli formula to $t \mapsto N_\varepsilon P_t$ and take the limit as $\varepsilon$ goes to 0. In particular, in [RS00] it is proved that:

$$\lim_{\varepsilon \to 0} \int_{(P_t)^\varepsilon} H_{t,x} \, dA_{t,x} = - \sum_{e \text{ edge of } P_t} \ell(e_t) \, \theta(e_t). \quad (12)$$

Therefore the integral of the mean curvature can be considered as the analogous, in the $C^{1,1}$ case, of the weighted length of the codimension 1 bending locus of $\partial P$, where the weights are given by the exterior dihedral angles along the edges.

Assuming temporarily the equivalence of the definitions $V_\ast^i$, $i = 1, 2, 3$, proved in Proposition A.17, we can easily deduce a dual Schläfli formula (see [San76] and [Suá00]) for the $V_\ast^i$ in the case of a polyhedron. As above, consider a 1-parameter family of polyhedra $(P_t)$. We can approximate any polyhedron $P_t$ with its $\varepsilon$-neighborhoods $N_\varepsilon P_t$. Using the relation (12) and the fact that $V_\ast^3(N_\varepsilon P_t) = V_\ast^1(N_\varepsilon P_t)$, we deduce that

$$\Vol^\ast(P_t) = -\Vol_{\mathbb{H}^3}(P_t) + \frac{1}{2} \sum_{e \text{ edge of } P_t} \ell(e_t) \, \theta(e_t). \quad (13)$$

Now, differentiating this relation in $t$ and applying the classical Schläfli formula, we obtain

$$\delta \Vol^\ast(P) = -\frac{1}{2} \sum_{e \text{ edge of } P} \ell(e) \, \delta \theta(e) + \frac{1}{2} \sum_{e \text{ edge of } P} \left( \delta \ell(e) \, \theta(e) + \ell(e) \, \delta \theta(e) \right)$$

$$= \frac{1}{2} \sum_{e \text{ edge of } P} \delta \ell(e) \, \theta(e).$$

Therefore, the variation of the dual volume of $(P_t)_t$ is in fact the "dual" of the variation of the hyperbolic volume of $(P_t)_t$, in the sense that, instead of involving the variation of the angles along the edges $e$ and the length of $e$ at the time $t = 0$, we have the variation of the length of $e$ and the angle along $e$ at the time $t = 0$. A remarkable fact is that the dual Schläfli formula does not need the combinatorial structure of $P_t$ to be preserved along the deformation, but only the set of vertices. Indeed, if an edge $e_t$ of $P_t$ collapses into a face at $t = 0$, its dihedral angle $\theta(e)$ in $P$ is 0, and therefore $e$ does not contribute to $\delta \Vol^\ast(P)$. Heuristically this suggests that, in order to generalize the result to the convex core of hyperbolic convex co-compact manifolds, the way the bending locus of the convex core will vary should not play a central role in the variation of the dual volume. This is not the case for the hyperbolic volume of the convex core, where more care has to be taken already to determine the correct space in which the variation of the bending measure $b_0$ is defined (compare with [Bon98a]).

Let now come back to the proof of the equivalence of the definitions of $V_\ast^i$. 


Lemma A.16. Let \((\Sigma_t)\) be a smooth deformation of immersed surfaces in \(\mathbb{H}^3\), with \(I I < 0\), and denote by \(\Sigma^*_t\) the dual surface of \(\Sigma_t\). Then the variation of the volume in \(dS^3\) bounded by the surfaces \(\Sigma^*_t\) can be expressed as

\[
\delta \text{Vol}_{dS^3} = -\frac{1}{4} \int_{\Sigma} (\delta I, HI - I I) \, dA,
\]

where \(\Sigma = \Sigma_0\).

**Proof.** By Theorem A.12 we have that

\[
\delta \text{Vol}_{dS^3} = \frac{1}{2} \int_{\Sigma^*} \left( \delta H^* + \frac{1}{2} (\delta I^*, II^*)^* \right) \, dA^*.
\]

To prove the statement, we will apply Proposition A.4 in order to translate this expression on \(\Sigma^*\) in terms of one on \(\Sigma\). By Proposition A.4 we have that

\[
H^* = -\frac{H}{\det(B)}
\]

and \(dA^* = \det(B) \, dA\) (observe that, since \(\Sigma\) is strictly convex, \(\det(B)\) is different from 0 everywhere). Therefore, we can compute the variation of the mean curvature as follows

\[
\delta H^* = -\delta \left( \frac{H}{\det(B)} \right) = -\frac{\text{tr}(\delta B)}{\det(B)} + \frac{\text{tr}(B) \, \text{tr}(B^{-1} \delta B)}{\det(B)}
\]

\[
= \det(B)^{-1} \left( \text{tr}(B^{-1} \delta B) \, \text{tr}(B) - \text{tr}(\delta B) \right)
\]

\[
= \text{tr}(B^{-1} \delta B B^{-1}),
\]

where in the last step we used the identity

\[
\text{tr}(M^{-1}N) = \det(M)^{-1} \left( \text{tr}(M) \, \text{tr}(N) - \text{tr}(MN) \right) \quad \forall M, N \in \text{GL}(2, \mathbb{R}). \tag{14}
\]

Using the relation \(\text{tr}(MN) = \text{tr}(NM)\) and the fact that \(B\) is \(I\)-selfadjoint, we see that

\[
(\delta I^*, II^*)^* = -2 \, \text{tr}(B^{-1} \delta B B^{-1}) - \text{tr}(B^{-1} I^{-1} \delta I).
\]

On the other hand, we have

\[
(\delta I, HI - I I) = \text{tr}(I^{-1} \delta I I^{-1} (HI - I I)) = \text{tr}(B) \, \text{tr}(I^{-1} \delta I) - \text{tr}(I^{-1} \delta IB)
\]

\[
= \det(B) \, \text{tr}(B^{-1} I^{-1} \delta I),
\]

where in the last step we used again the relation (14) with \(M = B\) and \(N = I^{-1} \delta I\). Now, putting these equalities together, we see that

\[
\frac{1}{2} \left( \delta H^* + \frac{1}{2} (\delta I^*, II^*)^* \right) \, dA^* = \frac{1}{2} \left( -\frac{1}{2} \, \text{tr}(B^{-1} I^{-1} \delta I) \right) \, dA^*
\]

\[
= -\frac{1}{4} \, \text{tr}(B^{-1} I^{-1} \delta I) \, \det(B) \, dA
\]

\[
= -\frac{1}{4} (\delta I, HI - I I) \, dA,
\]

as desired. \ \square
**Proposition A.17.** The three definitions of the dual volume given above coincide on all compact convex bodies in $\mathbb{H}^3$ with $\mathcal{C}^{1,1}$-boundary and $\mathbb{I} < 0$.

**Proof.** In order to prove the remaining equality, we first show that $V_1^*$ and $V_3^*$ have the same variation formula. Let $C_t$ be a differentiable family of compact convex bodies in $\mathbb{H}^3$ with $\mathcal{C}^{1,1}$-boundary and $\mathbb{I} < 0$. If $p$ lies in the interior of $C_0$, then it will be an internal point of $C_t$ for small values of the parameter. In particular $p$ can be used to define $V_1^*(C_t) = \text{Vol}_{3\mathbb{H}}(p^\circ \setminus (C_t)^\circ)$ whenever $t$ is sufficiently close to 0. Since $p$ is fixed, the only component of the boundary that is varying is $\partial(C_t^*)$. Applying Lemma A.16 we get

$$\frac{d}{dt} V_1^*(C_t) \bigg|_{t=0} = -\frac{1}{4} \int_{\partial C_0} (\delta I, H - \mathbb{I}) \, dA.$$

On the other side, by Theorem A.12 the variation of $V_3^*$ is

$$\frac{d}{dt} V_3^*(C_t) \bigg|_{t=0} = +\frac{1}{2} \int_{\partial C_0} \left( \delta H + \frac{1}{2}(\delta I, \mathbb{I}) \right) \, dA - \frac{1}{2} \frac{d}{dt} \int_{\partial C_t} H \, dA \bigg|_{t=0}.$$

In local coordinates $(x^1, x^2)$ the volume form can be written as $\sqrt{\det((g_t)_{ij})} \, dx^1 \wedge dx^2$, where $\det((g_t)_{ij})$ denotes the determinant of the $2 \times 2$-matrix representing the tensor $g_t$ with respect to the basis $\partial_1, \partial_2$. The differential of the function $\det$ at a point $A \in \text{GL}(2, \mathbb{R})$ verifies

$$d(\det)_A (H) = \det(A) \text{tr}(A^{-1}H).$$

Using this fact combined with the relation $(\delta I, I) = \text{tr}(I^{-1}\delta I)$, we see that the variation of $H_t \, dA_t$ is given by $\delta H + \frac{1}{2}(\delta I, I)$. Therefore we get

$$\frac{d}{dt} V_3^*(C_t) \bigg|_{t=0} = +\frac{1}{4} \int_{\partial C_0} (\delta I, \mathbb{I} - H) \, dA,$$

which gives the equality between the derivatives in $t$ at $t = 0$ of $V_1^*(C_t)$ and $V_3^*(C_t)$.

Assuming that any convex body with $\mathcal{C}^{1,1}$-boundary and $\mathbb{I} < 0$ can be differentiably deformed, through convex bodies with $\mathcal{C}^{1,1}$-boundary and $\mathbb{I} < 0$, in a small geodesic ball, it would be enough to show that $V_1^*$ and $V_3^*$ coincides on any geodesic ball in $\mathbb{H}^3$.

A way to prove the first claim is to reduce the problem to the Euclidean setting, and then perform the deformation using convex combinations. To do so, we work in the projective model, instead of the hyperboloid one. We recall that the projective model of the hyperbolic space can be described, in a suitable affine chart, as the interior of a Euclidean open ball $B$ centered in the origin. In this description, the half-spaces are nothing but intersections of $B$ with Euclidean half-spaces. It immediately follows that a compact convex body $C$ in $\mathbb{H}^3$ corresponds to a Euclidean compact convex body lying inside $B$. Up to acting by isometries of $\mathbb{H}^3$, we can always assume that the interior part of the convex body $C$ contains the point $0 \in B$. Therefore, it is enough to prove that we can find a differentiable deformation of convex $\mathcal{C}^{1,1}$-surfaces $(C_t)$, such that $\mathbb{I}_t < 0$, $\Sigma_0 = \partial C$ and $\Sigma_1 = \partial D_r$, where $D_r$ is a small closed ball of radius $r$ centered at 0 and contained in the interior of $C$. To do so, we consider $t \mapsto N^E_t((1-t)\cdot C)$ (where $s \cdot C = \{sx \mid x \in C\} \subset \mathbb{R}^3$, as $t$ varies in $I = [0,1]$), with $N^E$ that stands for Euclidean neighborhood. Since the boundary of $(1-t)\cdot C$ has negative definite fundamental form
for all $t \neq 1$ and it is $\mathcal{C}^{1,1}$, the boundary of $N^E_{tr}((1-t) \cdot C)$ has $\mathcal{I}_t < 0$ too, and it has $\mathcal{C}^{1,1}$-boundary (in the Euclidean setting, the convexity of $C$ is not enough to have the strict convexity of $N^E_C$). At time $t = 0$ we get $N_0C = C$ and at $t = 1$ $N^E_r(0) = D_r$. It is not difficult to see that the boundaries $S^E_{tr}((1-t) \cdot C)$ are varying differentiably in $t$, and therefore that this deformation works for our purposes.

It remains to prove that any geodesic ball $B_\varepsilon = B_\varepsilon(p)$ of radius $\varepsilon$ in $\mathbb{H}^3$ verifies $V_1^*(B_\varepsilon) = V_3^*(B_\varepsilon)$. Choosing the gradient of the distance as normal vector field on $\partial B_\varepsilon$, we can easily see that the following relations hold

$$I_\varepsilon = \sinh^2 \varepsilon g_{S^2},$$
$$\mathbb{I}_\varepsilon = - \coth \varepsilon I_\varepsilon,$$
$$\text{Vol}(B_\varepsilon) = \int_0^\varepsilon \sinh^2 t \, dt \, \text{Vol}(S^2).$$

Then we have

$$V_3^*(B_\varepsilon) = - \int_0^\varepsilon \sinh^2 t \, dt \, \text{Vol}(S^2) - \frac{1}{2} (-2 \coth \varepsilon) \sinh^2 \varepsilon \, \text{Vol}(S^2)$$
$$= \frac{1}{2} \left( \frac{\sinh 2\varepsilon}{2} + \varepsilon \right) \text{Vol}(S^2).$$

The dual convex of $B_\varepsilon$ is the future of the time-like $\varepsilon$-equidistant surface $S_\varepsilon$ from $p^\wedge$ (we are considering signed distances), and the dual surface of $\partial B_\varepsilon$ is $S_\varepsilon$. Using the equality $d\text{vol}_{dS^3} = \cosh^2 t \, d\text{vol}_{S^2}$, we obtain

$$V_1^*(B_\varepsilon) = \text{Vol}_{dS^3}^* (p^\wedge \setminus B_\varepsilon^\wedge)$$
$$= \int_0^\varepsilon \cosh^2 t \, dt \, \text{Vol}(S^2)$$
$$= \frac{1}{2} \left( \frac{\sinh 2\varepsilon}{2} + \varepsilon \right) \text{Vol}(S^2),$$

which proves $V_1^*(B_\varepsilon) = V_3^*(B_\varepsilon)$. \qed

Let $\mathcal{C}B_\varepsilon$ denote the family of compact convex bodies of $\mathbb{H}^3$ endowed with the Hausdorff distance $d_{3\varepsilon}$, which is defined as follows

$$d_{3\varepsilon}(C, D) := \inf\{ \varepsilon > 0 \mid N_\varepsilon C \supseteq D \text{ and } C \subseteq N_\varepsilon D \}$$

where we are using the notation introduced in 1.1.

Remark A.18. By Remark 1.2, any compact convex body $C$ can be approximated in $\mathcal{C}B_\varepsilon$ by a sequence of compact convex bodies with $\mathcal{C}^{1,1}$-boundary and $\mathcal{I} < 0$, e. g. we can take $N_{1/n}C$, with $n \in \mathbb{N}$.

Proposition A.19. The function $V_1^* : \mathcal{C}B_\varepsilon \to \mathbb{R}_{>0}$ is continuous.
By definition of the Hausdorff distance, it is enough to prove that, for any compact convex body $C$ we have

$$\lim_{\epsilon \to 0} V^*_1(N_{\epsilon} C) = V^*_1(C).$$

Since $(N_{\epsilon} C)^{\wedge}$ is the $\epsilon$-neighborhood of $C^{\wedge}$ with respect to the time-like distance from $C^{\wedge}$, the fact follows from the continuity of $\text{Vol}_{\mathbb{S}^3}$ with respect to the Hausdorff distance. Alternatively, the same argument of [BBB17, Proposition 3.4] applies, where now the corresponding metric at infinity is defined on the full Riemann sphere $\mathbb{C}P^1$.

Proposition A.19 implies that the dual volume of a compact convex body can be approximated by the dual volume of strictly convex bodies with $C^1$-boundary which converge to $C$ with respect to the Hausdorff distance. For the existence of such a sequence, we can consider $C_n := N_{1/n} C$, for $n \in \mathbb{N} \setminus \{0\}$, as observed in Remark 1.2. This shows the consistence of the different definitions $V^*_i$ we gave.

References

[BBB17] M. Bridgeman, J. Brock, and K. Bromberg. “Schwarzian derivatives, projective structures, and the Weil-Petersson gradient flow for renormalized volume”. In: ArXiv e-prints (Apr. 2017). arXiv:1704.06021 [math.DG]

[Bon88] F. Bonahon. “The geometry of Teichmüller space via geodesic currents”. In: Invent. Math. 92.1 (1988), pp. 139–162.

[Bon96] F. Bonahon. “Shearing Hyperbolic Surfaces, Bending Pleated Surfaces And Thurston’s Symplectic Form”. In: Ann. Fac. Sci. Toulouse 6 (1996), pp. 233–297.

[Bon97a] F. Bonahon. “Geodesic laminations with transverse Hölder distributions”. In: Ann. Sci. École Norm. Sup. 30.2 (1997), pp. 205–240.

[Bon97b] F. Bonahon. “Transverse Hölder distributions for geodesic laminations”. In: Topology 36.1 (1997), pp. 103–122.

[Bon98a] F. Bonahon. “A Schlafli-type formula for convex cores of hyperbolic 3-manifolds”. In: J. Differential Geom. 50.1 (1998), pp. 25–58.

[Bon98b] F. Bonahon. “Variations of the boundary geometry of 3-dimensional hyperbolic convex cores”. In: J. Differential Geom. 50.1 (1998), pp. 1–24.

[Can+06] R. D. Canary et al. Fundamentals of Hyperbolic Manifolds: Selected Expositions. London Mathematical Society Lecture Note Series. Cambridge University Press, 2006. ISBN: 9780521615587.

[HR93] C. D. Hodgson and I. Rivin. “A characterization of compact convex polyhedra in hyperbolic 3-space”. In: Invent. Math. 111.1 (Dec. 1993), pp. 77–111. issn: 1432-1297.

[Ker85] S. P. Kerckhoff. “Earthquakes are analytic”. In: Comment. Math. Helv. 60 (1985), pp. 17–30.
[KS08] K. Krasnov and J.-M. Schlenker. “On the Renormalized Volume of Hyperbolic 3-Manifolds”. In: Commun. Math. Phys 279.3 (May 2008), pp. 637–668. ISSN: 1432-0916.

[KS09] K. Krasnov and J.-M. Schlenker. “A symplectic map between hyperbolic and complex Teichmüller theory”. In: Duke Math. J. 150.2 (Nov. 2009), pp. 331–356.

[KS95] L. Keen and C. Series. “Continuity of convex hull boundaries”. In: Pacific J. Math. 168.1 (1995), pp. 183–206.

[Mil94] J. W. Milnor. “The Schlafli differential equality”. In: Collected papers Vol. 1: Geometry (1994).

[RS00] I. Rivin and J.-M. Schlenker. “On the Schlafli differential formula”. In: ArXiv Mathematics e-prints (Jan. 2000). eprint: math/0001176.

[San76] L. A. Santaló. Integral Geometry and Geometric Probability. Vol. 1. Encyclopedia of Mathematics and its applications. Addison-Wesley Publishing Company, 1976.

[Sch06] J.-M. Schlenker. “Hyperbolic manifolds with convex boundary”. In: Invent. Math. 163.1 (Jan. 2006), pp. 109–169. ISSN: 0020-9910.

[Sch13] J.-M. Schlenker. “The renormalized volume and the volume of the convex core of quasifuchsian manifolds”. In: Math. Res. Lett. 20.4 (2013), pp. 773–786.

[Sch17] J.-M. Schlenker. “Notes on the Schwarzian tensor and measured foliations at infinity of quasifuchsian manifolds”. In: ArXiv e-prints (Aug. 2017). arXiv:1708.01852 [math.GT]

[Sch58] L. Schläfli. “On the multiple integral \( \int dx_1 \cdots dz \), whose limits are \( p_1 = a_1 x + b_1 y + \cdots + h_1 z > 0 \), \( p_2 > 0 \), \( \cdots \), \( p_n > 0 \) and \( x^2 + y^2 + \cdots + z^2 < 1 \)”. In: Quart. J. Pure Appl. Math 168 (1858), pp. 269–301.

[SR99] J.-M. Schlenker and I. Rivin. “The Schlafli formula in Einstein manifolds with boundary”. In: Electron. Res. Announc. Amer. Math. Soc. 5 (Mar. 1999), pp. 18–23.

[SS03] J.-M. Schlenker and R. Souam. “Higher Schlafli Formulas and Applications”. In: Compos. Math. 135.1 (Jan. 2003), pp. 1–24. ISSN: 0010-437X.

[Suá00] E. Suárez-Peiró. “A Schlafli differential formula for simplices in semi-Riemannian hyperquadrics, Gauss-Bonnet formulas for simplices in the de Sitter sphere and the dual volume of a hyperbolic simplex”. In: Pacific Journal of Mathematics 194.1 (2000), pp. 229–255.

[Sul85] D. P. Sullivan. “Quasiconformal homeomorphisms and dynamics II: Structural stability implies hyperbolicity for Kleinian groups”. In: Acta Math. 155 (1985), pp. 243–260.

[Thu79] W. P. Thurston. The geometry and topology of three-manifolds, lecture notes. Princeton University, 1976-79.
[Vin+13] E. B. Vinberg et al. *Geometry II: Spaces of Constant Curvature*. Encyclopaedia of Mathematical Sciences. Springer Berlin Heidelberg, 2013. ISBN: 9783662029015.