Exact algorithms for dominating induced matchings

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Abstract

Say that an edge of a graph $G$ dominates itself and every other edge adjacent to it. An edge dominating set of a graph $G = (V, E)$ is a subset of edges $E' \subseteq E$ which dominates all edges of $G$. In particular, if every edge of $G$ is dominated by exactly one edge of $E'$ then $E'$ is a dominating induced matching. It is known that not every graph admits a dominating induced matching, while the problem to decide if it does admit is NP-complete. In this paper we consider the problem of finding a minimum weighted dominating induced matching, if any, of a graph with weighted edges. We describe two exact algorithms for general graphs. The algorithms are efficient in the cases where $G$ admits a known vertex dominating set of small size, or when $G$ contains a polynomial number of maximal independent sets.

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1 Introduction

By \(G(V, E)\) we denote a \textit{simple undirected graph} with vertex set \(V\) and edge set \(E\), \(n = |V|\) and \(m = |E|\). We consider \(G\) as a \textit{weighted} graph, that is, one in which there is a non-negative real weight assigned to each edge of \(G\). If \(v \in V\) and \(W \subseteq V\), then denote by \(N(v)\), the set of vertices adjacent (neighbors) to \(v\), denote by \(G[W]\) the subgraph of \(G\) induced by \(W\), and write \(N_W(v) = N(v) \cap W\). Say that \(D \subseteq V\) is a \textit{(vertex) dominating set of} \(G\) if \(D \cup N(D) = V\), where \(N(D) = \bigcup_{v \in D} N(v)\).

Given an edge \(e \in E\), say that \(e\) \textit{dominates} itself and every edge sharing a vertex with \(e\). Subset \(E' \subseteq E\) is an \textit{induced matching} of \(G\) if each edge of \(G\) is dominated by at most one edge in \(E'\). A \textit{dominating induced matching (DIM)} of \(G\) is a subset of edges which is both dominating and an induced matching. Not every graph admits a DIM, and the problem of determining whether a graph admits it is also known in the literature as \textit{efficient edge domination problem}. The weighted version of DIM problem is to find a DIM such that the sum of weights of its edges is minimum among all DIM’s, if any.

The (unweighted version) of the dominating induced matching problem is known to be NP-complete \([5]\), even for planar bipartite graphs \([7]\) or regular graphs \([2]\). There are polynomial time algorithms for some classes, as chordal graphs \([7]\), generalized series-parallel graphs \([7]\) (both for the weighted problem), claw-free graphs \([3]\), graphs with bounded clique-width \([3]\), convex graphs \([6]\), bipartite permutation graphs \([8]\) (see also \([1]\)).

In this paper, we describe two exact (exponential time) algorithms for the weighted problem. The first runs in linear time for a given vertex dominating set of fixed size of the graph. The second runs in polynomial time if the graph admits a polynomial number of maximal independent sets.

We will use an alternative definition, taken from \([4]\), of the problem of finding a dominating induced matching. It asks to determine if the vertex set of a graph \(G\) admits a partition into two subsets \(W\) and \(B\) such that \(W\) is an independent set and \(B\) induces an 1-regular graph. The vertices of \(W\) are called \textit{white} and those of \(B\) are \textit{black}.

We consider only graphs without isolated vertices, because isolated vertices must be white in any black-white partition. So, we can delete them and solve the problem in the residual graph.

Assigning one of the two possible colors to vertices of \(G\) is called a coloring of \(G\). A coloring is \textit{partial} if only part of the vertices of \(G\) have been assigned.

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colors, otherwise it is total. A partial coloring is valid if no two white vertices are adjacent and no black vertex has more than one black neighbor. A black vertex is single if it has no black neighbors, otherwise, it is paired. A total coloring is valid if no two white vertices are adjacent and every black vertex is paired. Clearly, $G$ admits a DIM if and only if it admits a total valid coloring. In fact, a total valid coloring defines exactly one DIM, given by the set $B$.

For a coloring $C$ of the vertices of $G$, denote by $C^{-1}(\text{white})$ and $C^{-1}(\text{black})$, the subsets of vertices colored white and black. A coloring $C'$ is an extension of a $C$ if $C^{-1}(\text{black}) \subseteq C'^{-1}(\text{black})$ and $C^{-1}(\text{white}) \subseteq C'^{-1}(\text{white})$.

## 2 An Algorithm Based on Vertex Domination

Next, we will propose an exact algorithm for solving the weighted dominating induced matching problem, for general graphs, based on vertex dominations.

Let $C$ be a partial valid coloring of $G = (V, E)$. Such as in [4], this coloring can be further propagated according to the following rules:

(i) each neighbor of a white vertex must be black

(ii) Except for its pair, the neighbors of a paired black vertex must be white

(iii) each vertex with two black neighbors must be white

(iv) if a single black vertex has exactly one uncolored neighbor then this neighbor must be black

We can propagate the coloring iteratively, until it becomes no more possible to color new vertices, simply by the application of the above rules. Then we check for validity. A valid coloring so obtained is then called stable.

Let $C$ be a partial valid coloring of $G$, and $C'$ be a stable coloring obtained from $C$, by the applications of rules (i)-(iv), above. Denote by $D$ and $D'$, respectively the subsets of vertices of $G$ which are colored in $C$ and $C'$. Clearly, $D' \supseteq D$. For our purposes, assume that the initial set $D$ of colored vertices is a vertex dominating set of $G$.

### Lemma 2.1

Let $C'$ be a stable coloring. Then

(i) If there are no single (black) vertices then $C'$ is a total coloring,

(ii) Any uncolored vertex has exactly one black neighbor, and such a neighbor must be single.

**Proof.** Recall that $D$ is the initial colored vertices and is a dominating set of the graph $G$. $C'$ is not a total coloring if only if there is some uncolored vertex $v$. Clearly, $v \notin D$ and $N(v) \cap D \neq \emptyset$. Let $w$ be some neighbor of
v in D. If the color of w is white then v must be colored black by rule (i) which is a contradiction. Hence w is a black vertex. Again, if w is a paired black vertex or v has another black neighbor w' ≠ w, v must get color white by rules (ii) and (iii) and this is a contradiction. Consequently, v has exactly one black neighbor and which is single black vertex. Therefore, if there are no single black vertices then there are not uncolored vertices and C' is a total coloring.

Let D' be the colored vertices of the stable coloring C', let S = \{s_1, \ldots, s_p\} be the set of single vertices, and U the set of still uncolored vertices of G, that is, U = V \ D'. The above lemma implies that U admits a partition into (disjoint) parts:

\[ U = (N(s_1) \cap U) \cup \ldots \cup (N(s_p) \cap U) \]

**Theorem 2.2** Let C be a coloring of the vertices of G, C' a stable extension of it, and \( D = \text{C}^{-1}(\text{black}) \cup \text{C}^{-1}(\text{white}) \) a dominating set of G. Then (i) \( S \subseteq \text{C}^{-1}(\text{black}) \); and (ii) if C extends to a valid total coloring \( C'' \) then \( C'' \) is an extension of \( C' \).

**Proof.** Suppose that \( S \not\subseteq \text{C}^{-1}(\text{black}) \) which means that exists a vertex \( s_i \in S \) and \( s_i \not\in \text{C}^{-1}(\text{black}) \). By definition of S, \( s_i \) is a single black vertex. If \( s_i \in D \) then \( s_i \in \text{C}^{-1}(\text{black}) \), contradiction. Therefore \( s_i \not\in D \).

Since \( D \) is a dominating set, then \( \exists v \in D \) such that \( s_i \in N(v) \). If \( v \) is black then \( s_i \) is not a single black vertex, again a contradiction. Hence \( v \) must be white.

- If \( s_i \) has no uncolored neighbors then C is not extensible to a total valid coloring because \( s_i \) can not become a paired vertex, contrary to the hypothesis.
- Otherwise, let \( y \) be an uncolored neighbor of \( s_i \). Clearly, \( y \not\in D \). Since \( y \) is uncolored then it has exactly one neighbor in \( D' \). That is, \( s_i \) is the unique neighbor of \( y \) in \( D' \). Since \( D \subseteq D' \) and \( s_i \in D' \setminus D \), it follows that \( D \) is not a dominating set, contradiction.

On the other hand, \( C' \) and \( C'' \) are extensions of \( C \). Then the vertices of D have the same color in these colorings. Any colored vertex \( v \not\in D \) of \( C' \) was obtained by some propagation rule base on previous colored vertices. The rules are correct and deterministic. Hence, \( v \) must have the same color in \( C'' \) and \( C'' \) is an extension of \( C' \).

Clearly, given a partial valid coloring C, we can compute efficiently a stable extension \( C' \) of it. In addition, if D is a dominating set then we can try
to obtain a total valid coloring from the stable coloring $C'$ by appropriately choosing exactly one vertex from each subset $N_U(s_i)$, to be black, that is, to be the pair of the so far single vertex $s_i$.

**Lemma 2.3** Let $U$ and $S$, respectively be the sets of uncolored and single vertices, relative to some stable coloring $C'$ of graph $G$. If $C'$ extends to a total valid coloring then, for each $s_i \in S$, $G[N_U(s_i)]$ is a union of a star and an independent set, any of them possibly empty. Moreover, the pair of $s_i$ must be a maximum degree vertex in $G[N_U(s_i)]$.

**Proof.** Suppose by contrary that $G[N_U(s_i)]$ is not a union of a star and an independent set. Then $G[N_U(s_i)]$ contains either two non-adjacent edges, or a $K_3$.

- Let $\{(u_1, u_2), (v_1, v_2)\}$ be two disjoint edges in $G[N_U(s_i)]$. Since no white vertices can be adjacent, let $u'$ be the black vertex from $\{u_1, u_2\}$ and $v$ the black vertex from $\{v_1, v_2\}$. Then $\{u, s_i, v\}$ is a black $P_3$ or $K_3$ and therefore can not be extended to a valid coloring.
- Let $\{(u_1, u_2, u_3)\}$ be a $K_3$ in $G[N_U(s_i)]$. Therefore $\{s_i, u_1, u_2, u_3\}$ is a $K_4$ and therefore $G$ has no valid coloring.

Consequently, $G[N_U(s_i)]$ must be a union of a star and an independent set. Now, suppose by contrary that the pair of $s_i$ is a vertex $v \in N_U(s_i)$ and $v$ has not maximum degree in $G[N_U(s_i)]$. Clearly, the rest of vertices in $N_U(s_i)$ are white vertices. In particular, a maximum degree vertex $u$ in $G[N_U(s_i)]$ is white. But, there is some neighbor $z \neq v$ of $u$ in $N_U(s_i)$ and $z$ is not adjacent to $v$. Hence, $z$ and $u$ are white adjacent vertices, which is a contradiction. \(\square\)

We can repeatedly execute the procedure below described for choosing the vertices to be paired to the single vertices $s_i$ of the partial colorings. The procedure is repeated until all parts of the partition $U = N_U(s_1) \cup \ldots \cup N_U(s_p)$ have selected their paired black vertices or the coloring becomes invalid.

Let $s_i \in S$ be a single vertex. **Case 1:** $N_U(s_i) = \emptyset$: then stop, it will not lead to a valid one. **Case 2:** There is exactly one maximum degree vertex in $G[N_U(s_i)]$: then clearly, the only alternative is to choose this vertex. **Case 3:** There is no edge $vw$, where $v \in N_U(s_i)$ and $w \in N_U(s_j)$, for any $j \neq i$: then the choice of the neighbor of $s_i$ to become black is independent on the choices of the others parts of the partition. Choose the vertex $w$ of maximum degree in $G[N_U(s_i)]$ that minimizes the weight of the edge $ws_i$. **Case 4:** There is an edge $vw$, where $v \in N_U(s_i)$ and $w \in N_U(s_j)$, for some $j \neq i$: then $v$ may become white if and only if $w$ may become black. Each of these two
choices may lead to valid or invalid total colorings. So, we proceed with both alternatives, as if in parallel.

After applying any the above Cases 2, 3 or 4, perform the propagation rules again and validate the coloring so far obtained. Proceed so until eventually the coloring becomes invalid, or a valid solution is obtained. At the end, choose the minimum weight solution obtained throughout the process.

As for the complexity, it is clear that it depends on the cardinality of the dominating set $D$ and on the number of parallel iterations, considered sequentially. Next, we describe bounds for these parameters.

**Lemma 2.4** There are at most $2^q$ parallel computations where $q \leq p = |S| \leq |D|$, and $q \leq \frac{n}{3}$.

**Proof.** By Theorem 1, it follows that $p \leq |D|$. On the other hand, we can apply the above Cases 1-4, in such an ordering that we keep applying Cases 1 and 2, with propagation until all remaining single vertices $s_i$ satisfy $|N(s_i) \cap U| \geq 2$. Let $S' \subset S$ denote the set of remaining single vertices, and $q = |S'|$. Consequently, $q \leq \frac{n}{3}$.

Next, examine the parallel computations. They are generated by Case 4. Let $vw$ be an edge of $G$, where $v \in N(s_i) \cap U$ and $w \in N(s_j) \cap U$, $i \neq j$. In one of the instances, $v$ is black, meaning that $s_i$ becomes paired, while in the other one $w$ is black, implying that $s_j$ becomes paired. This means that the size of the set $S'$ of single vertices always decreases by at least one unit in all computations. Hence there are at most $2^q$ parallel computations. \qed

Considering that the remaining operations involved in each parallel thread of the algorithm can be performed in linear time, it is not hard to conclude that there is an $O(2^q m)$ time algorithm to obtain a minimum DIM, if any, extensible from a partial valid coloring $C$ of a weighted graph $G = (V, E)$ such that $D = C^{-1}($black$) \cup C^{-1}($white$)$ is a dominating set of $G$.

The complexity of the algorithm depends on the size of the dominating set $D$ employed. We remark that if $G = (V, E)$ has no isolated vertices then we can easily find in linear time a dominating set with at most half the vertices. Just determine a maximal independent set $I$. Clearly, $I$ and $V \setminus I$ are both dominating sets of $G$ and one of them has at most $\frac{n}{2}$ vertices.

Finally, in order to obtain the minimum weighted DIM of the graph $G$, we have to apply the described algorithm for all possible bi-colorings of $D$. There are exactly $2^{|D|}$ such colorings. Therefore
Theorem 2.5 There is an algorithm of complexity $O(\min\{2^{2|D|}, 2^{\frac{5n}{6}}\} \cdot m) \approx O^*(\min\{4^{|D|}, 1.7818^n\})$ to compute a minimum weighted DIM of a weighted graph, if existing.

Proof. The complexity is $O(2^{|D|} \cdot 2^q \cdot m) = O(2^{|D|} \cdot \min\{2^{|D|}, 2^{\frac{n}{3}}\} \cdot m) = O(\min\{2^{|D|}, 2^{\frac{n}{2}} + \frac{n}{3}\} \cdot m) = O(\min\{2^{|D|}, 2^{\frac{5n}{6}}\} \cdot m)$. □

Corollary 2.6 The above algorithm solves the minimum weighted DIM problem in $O(m)$ time given a dominating set of fixed size.

3 An algorithm based on maximal independent sets

In this section, we describe an exact algorithm for finding a minimal weighted DIM of a graph, based on enumerating maximal independent sets. We consider a weighted graph $G = (V, E)$.

Any maximal independent set $I \subseteq V$ induces a partial bi-coloring in $G$ as follows:

- color as black all vertices of $V \setminus I$
- color as white the vertices of $I$ except those having exactly one single neighbor.

Observation 1 If all vertices of $G$ have degree $\neq 1$ then the above partial coloring is total.

The algorithm is then based on the following lemma.

Lemma 3.1 Let $G$ be a graph, $I$ a maximal independent set of it and $C$ the partial bi-coloring induced by $I$. Then $C$ is extensible to a DIM if and only if $C$ is a valid coloring and each single vertex, if existing, has at least one uncolored neighbor in $C$.

Proof. $\Rightarrow$ It is easy to see that if $C$ is not a valid coloring, then it is not extensible to a DIM. Besides, if $C$ has a single vertex $v$ with no uncolored neighbors then all neighbors of $v$ are white in $C$ and in any extension of $C$. Also, $C$ is not extensible to a DIM because $v$ can not ever get its pair.

$\Leftarrow$ Let $C$ be a valid coloring where each single black vertex has at least one uncolored neighbor. Then for each single black vertex $v$, choose any uncolored neighbor $w$ to be its pair ($w$ has exactly one single neighbor) and the remaining of uncolored vertices get color white. In this total coloring, the black vertices induce an 1-regular subgraph and the white vertex set is an independent set because it is part of $I$. Hence, the total coloring is valid and hence a DIM. □
The algorithm can then be formulated as follows. Generate the maximal independent sets $I$ of $G$. For each $I$, find its induced coloring $C$. If $C$ is invalid or some single vertex has no uncolored neighbor then do nothing. Otherwise, for each single vertex $v$ in $C$, if any, choose the minimum weight $vw$, among the uncolored neighbors of $v$; then color $w$ as black and the remaining neighbors of $v$ as white. The set of black vertices then forms a DIM of $G$. At the end select the minimum weight among all DIMs obtained in the process, if any.

Clearly, this algorithm determines the minimum weight DIM of a weighted graph $G = (V, E)$ because given any DIM $E' \subseteq E$ of $G$, the vertex set formed by those vertices not incident to any of the edges of $E'$ is an independent set and as such, is clearly a subset of some maximal independent set of $G$. So, any DIM $E'$ is considered in the algorithm.

All the operations performed by the algorithm relative to a fixed maximal independent set can be performed in linear time $O(m)$. If $G$ has $\mu$ maximal independent sets, we can generate them all in time $O(nm\mu)$ time \[^9\]. Therefore the complexity of the entire algorithm is $O(nm^2\mu)$. On the other hand, $\mu \leq O(3^\frac{3}{2}n^2)$, leading to a worst case of $O(3^\frac{3}{2}nm^2) \approx O^*(1.44225^n)$ time. In particular, if $G$ is a bipartite graph then $\mu \leq 2^\frac{3}{2}n^2$ and the complexity reduces to $O^*(1.41421^n)$. In any case, if $G$ has a polynomial number of maximal independent sets then the algorithm terminates within polynomial time.

Finally, we observe the following additional relation between maximal independent sets and DIM's.

**Lemma 3.2** Let $G(V, E)$ be a graph with no isolated edges, $E' \subseteq E$ a DIM of $G$, and $I \subseteq V$ the independent set formed by those vertices not incident to any of the edges of $E'$. Then $I$ is contained in exactly one maximal independent set of $G$.

**Proof.** If $I$ is a maximal independent set there is nothing to prove. Otherwise, suppose the lemma is false and let $I_1, I_2$ be two distinct maximal independent sets properly containing $I$. Let $V_1 = I_1 \setminus I$, and $V_2 = I_2 \setminus I$. Choose any $v_2 \in V_2$. Clearly, $\{v_2\} \cup I$ is an independent set, and we know that $I_1 = V_1 \cup I$ is a maximal one. Consequently, there must be some vertex $v_1 \in V_1$ adjacent to $v_2$. However, both $v_1$ and $v_2$ are vertices incident to edges of the DIM $E'$. Consequently, $v_1v_2 \in E'$. In this case, $v_1v_2$ must form an isolated edge of $G$, a contradiction. Therefore the lemma holds. \[ \square \]

Based on the above lemma and that fact that every isolated edge must be part of any DIM, it is simple to extend the exact algorithm proposed in this section, so as to count the number of distinct DIM's (unweighted or minimum...
weighted) of $G$, in the same complexity as deciding whether $G$ admits a DIM. Observe that $G$ may contain an exponential number of DIM’s.

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