Abelian Quotients Arising from Extriangulated Categories via Morphism Categories

Zengqiang Lin

Received: 28 August 2020 / Accepted: 18 July 2021 / Published online: 10 August 2021
© The Author(s), under exclusive licence to Springer Nature B.V. 2021

Abstract

We investigate abelian quotients arising from extriangulated categories via morphism categories, which is a unified treatment for both exact categories and triangulated categories. Let $(\mathcal{C}, \mathcal{E}, \mathcal{s})$ be an extriangulated category with enough projectives $\mathcal{P}$ and $\mathcal{M}$ be a full subcategory of $\mathcal{C}$ containing $\mathcal{s}$. We show that a certain quotient category $\mathcal{s}$-def, the category of $\mathcal{s}$-deflations $f : M_1 \to M_2$ with $M_1, M_2 \in \mathcal{M}$, is abelian. Our main theorem has two applications. If $\mathcal{M} = \mathcal{C}$, we obtain that a certain ideal quotient category $\mathcal{s}$-tri/\mathcal{R}_2$ is equivalent to the category of finitely presented modules $\text{mod-}(\mathcal{C}/\mathcal{P})$, where $\mathcal{s}$-tri(\mathcal{C})$ is the category of all $\mathcal{s}$-triangles. If $\mathcal{M}$ is a rigid subcategory, we show that $\text{mod-}(\mathcal{M}/\mathcal{P})$ and $\text{mod-}(\mathcal{M}/\mathcal{P})^{\text{op}}$ provided that $\mathcal{M}$ is a cluster-tilting subcategory.

Keywords Extriangulated categories · Abelian categories · Morphism categories · Rigid subcategories

Mathematics Subject Classification (2010) 18E30

1 Introduction

In representation theory, there are a few quotient categories admitting natural abelian structures. For both triangulated categories and exact categories, cluster-tilting subcategories provide a way to construct abelian quotient categories. Let $\mathcal{C}$ be a triangulated category

Presented by: Anne Moreau

This work was partially supported by the national natural science foundation of Fujian Province (Grant No. 2020JQ01075) and the national natural science foundation of China (Grants No. 11871014 and No. 11871259)

Zengqiang Lin
lzq134@163.com

1 School of Mathematical sciences, Huaqiao University, Quanzhou, 362021, China
and $\mathcal{T}$ be a cluster-tilting subcategory of $\mathcal{C}$, then the quotient $\mathcal{C}/[\mathcal{T}]$ is abelian; for related works see [1, 8, 9]. The version of exact categories is given in [2]. Submodule categories provide another way to construct abelian quotient categories. Certain quotients of submodule categories are realized as categories of finitely presented modules over stable Auslander algebras [3, 16]. More generally, some quotients of categories of short exact sequences in exact categories are abelian; for related works see [4, 10]. For triangulated version, certain quotients of categories of triangles are abelian [14].

Recently, Nakaoka and Palu introduced the notion of extriangulated categories [15], which is a simultaneous generalization of exact categories and triangulated categories. They pointed out that the notion is a convenient setup for writing down proofs which apply to both exact categories and triangulated categories. For recent developments on extriangulated categories we refer to [6, 11–13, 17] etc.

In this paper, we focus our attention onto the abelian quotients arising from extriangulated categories via morphism categories, which is a unified treatment of abelian quotients for both exact categories and triangulated categories. Our approach to abelian quotients is based on identifying quotients of morphism categories as certain module categories.

Let $(\mathcal{C}, \mathcal{E}, \mathcal{s})$ be an extriangulated category and $\mathcal{M}$ be a full subcategory of $\mathcal{C}$. We denote by $\text{Mor} (\mathcal{M})$ the morphism category of $\mathcal{M}$ and by $\mathcal{s}$-$\text{def} (\mathcal{M})$ (resp. $\mathcal{s}$-$\text{inf} (\mathcal{M})$) the full subcategory of $\text{Mor} (\mathcal{M})$ consisting of $\mathcal{s}$-deflations (resp. $\mathcal{s}$-inflations). The full subcategory of $\mathcal{s}$-$\text{def} (\mathcal{M})$ consisting of split epimorphisms (resp. split monomorphisms) is denoted by $\mathcal{s}$-$\text{epi} (\mathcal{M})$ (resp. $\mathcal{s}$-$\text{mono} (\mathcal{M})$). We denote by $\mathcal{s}$-$\text{epi} (\mathcal{M})$ (resp. $\mathcal{s}$-$\text{mono} (\mathcal{M})$) the full subcategory of $\mathcal{s}$-$\text{def} (\mathcal{M})$ consisting of $(M \to M) \oplus (P \to M')$ (resp. $(M \to M) \oplus (M' \to I)$) with $P$ projective (resp. $I$ injective).

Our main theorem is the following (Theorem 3.2), which generalizes [10, Theorem 3.9].

**Theorem 1.1** Let $\mathcal{C}$ be an extriangulated category and $\mathcal{M}$ be a full subcategory of $\mathcal{C}$.

1. If $\mathcal{C}$ has enough projectives $\mathcal{P}$ and $\mathcal{M}$ contains $\mathcal{P}$, then $\mathcal{s}$-$\text{def} (\mathcal{M})/\mathcal{s}$-$\text{epi} (\mathcal{M}) \cong \text{mod} -(\mathcal{M}/[\mathcal{P}])$ and $\mathcal{s}$-$\text{def} (\mathcal{M})/\mathcal{s}$-$\text{epi} (\mathcal{M}) \cong \text{mod} -(\mathcal{M}/[\mathcal{P}])^{\text{op}}^{\text{op}}$.

2. If $\mathcal{C}$ has enough injectives $\mathcal{I}$ and $\mathcal{M}$ contains $\mathcal{I}$, then $\mathcal{s}$-$\text{inf} (\mathcal{M})/\mathcal{s}$-$\text{mono} (\mathcal{M}) \cong \text{mod} -(\mathcal{M}/[\mathcal{I}])^{\text{op}}^{\text{op}}$ and $\mathcal{s}$-$\text{inf} (\mathcal{M})/\mathcal{s}$-$\text{mono} (\mathcal{M}) \cong \text{mod} -(\mathcal{M}/[\mathcal{I}])$.

Theorem 1.1 has two interesting applications. We will investigate two special cases when $\mathcal{M} = \mathcal{C}$ and when $\mathcal{M}$ is rigid, that is, $\mathcal{E}(M, M') = 0$ for any $M, M' \in \mathcal{M}$.

For the first case, we denote by $\mathcal{s}$-$\text{tri} (\mathcal{C})$ the category of all $\mathcal{s}$-triangles, where the objects are the $\mathcal{s}$-triangles $X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{\delta} \cdots)$ and the morphisms from $X_\bullet$ to $Y_\bullet$ are the triples $\varphi_\bullet = (\varphi_1, \varphi_2, \varphi_3)$ such that the following diagram

$$
\begin{array}{cccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{\delta} \\
| & & | & & | & \\
\varphi_1 & \varphi_2 & \varphi_3 \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{\delta'}
\end{array}
$$

is a morphism of $\mathcal{s}$-triangles. Let $X_\bullet$ and $Y_\bullet$ be two $\mathcal{s}$-triangles, we denote by $\mathcal{R}_2 (X_\bullet, Y_\bullet)$ the class of morphisms $\varphi_\bullet : X_\bullet \to Y_\bullet$ such that $\varphi_3$ factors through $g_2$. It is easy...
abelian quotients via morphism categories

to see that $\mathcal{R}_2$ is an ideal of $s$-tri$(\mathcal{C})$, moreover, the following three quotient categories $s$-tri$(\mathcal{C})/\mathcal{R}_2$, $s$-def$(\mathcal{C})/(s$-epi$(\mathcal{C}))$ and $s$-inf$(\mathcal{C})/(s$-mono$(\mathcal{C}))$ are equivalent.

Given an $s$-triangle $\delta = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{\rho} )$, we define the contravariant defect $\delta^*$ and the covariant defect $\delta_*$ by the following exact sequence of functors

$$
\mathcal{C}(-, X_1) \xrightarrow{\mathcal{C}(-, f_1)} \mathcal{C}(-, X_2) \xrightarrow{\mathcal{C}(-, f_2)} \mathcal{C}(-, X_3) \rightarrow \delta^* \rightarrow 0,
$$

$$
\mathcal{C}(X_3, -) \xrightarrow{\mathcal{C}(f_2, -)} \mathcal{C}(X_2, -) \xrightarrow{\mathcal{C}(f_1, -)} \mathcal{C}(X_1, -) \rightarrow \delta_* \rightarrow 0.
$$

Our first application of Theorem 1.1 is the following (Theorem 4.1, Proposition 4.1 and Theorem 4.2), which generalizes [10, Theorem 4.1, Theorem 4.8, Theorem 5.1].

**Theorem 1.2** Let $\mathcal{C}$ be an extriangulated category.

1. The quotient $s$-tri$(\mathcal{C})/\mathcal{R}_2$ is abelian.
2. If $\mathcal{C}$ has enough projectives $\mathcal{P}$, then we have the following equivalence
   $$F : s$-tri$(\mathcal{C})/\mathcal{R}_2 \cong \text{mod-}(\mathcal{C}/[\mathcal{P}]), \delta \mapsto \delta^*.$$
3. If $\mathcal{C}$ has enough injectives $\mathcal{I}$, then we have the following equivalence
   $$G : s$-tri$(\mathcal{C})/\mathcal{R}_2 \cong (\text{mod-}(\mathcal{C}/[\mathcal{I}])^{op})^{op}, \delta \mapsto \delta_*.$$

We point out that the abelian quotient $s$-tri$(\mathcal{C})/\mathcal{R}_2$ admits nice properties. We describe the projectives, injectives and simple objects in $s$-tri$(\mathcal{C})/\mathcal{R}_2$; see Proposition 4.4 and Proposition 4.6. In particular, if $\mathcal{C}$ has enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$, then there is a duality between $\text{mod-}(\mathcal{C}/[\mathcal{P}])$ and $\text{mod-}(\mathcal{C}/[\mathcal{I}])^{op}$, which is used to derive Auslander-Reiten duality and defect formula for extriangulated categories; see Proposition 4.9.

For describing the second application, we first give some notations. Let $\mathcal{C}'$ and $\mathcal{C}''$ be two full subcategories of $\mathcal{C}$. We denote by Cocone$(\mathcal{C}', \mathcal{C}'')$ (resp. Cone$(\mathcal{C}', \mathcal{C}'')$) the subcategory of objects $X$ admitting an $s$-triangle $X \xrightarrow{-} C' \xrightarrow{-} C'' \xrightarrow{-}$ (resp. $C' \xrightarrow{-} C'' \xrightarrow{-} X \xrightarrow{-}$) with $C' \in \mathcal{C}'$ and $C'' \in \mathcal{C}''$.

Let $\mathcal{M}$ be a rigid subcategory of $\mathcal{C}$. For convenience we let $\mathcal{M}_L = \text{Cocone}(\mathcal{M}, \mathcal{M})$ and $\mathcal{M}_R = \text{Cone}(\mathcal{M}, \mathcal{M})$. If $\mathcal{C}$ has enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$, then we let $\Omega \mathcal{M} = \text{Cocone}(\mathcal{P}, \mathcal{M})$ and $\Sigma \mathcal{M} = \text{Cone}(\mathcal{M}, \mathcal{I})$. It turns out that the quotient categories $s$-def$(\mathcal{M})/[s$-epi$(\mathcal{M})]$ and $s$-def$(\mathcal{M})/[sp$-epi$(\mathcal{M})]$ can be realized as subquotient categories of $\mathcal{C}$.

Our second application of Theorem 1.1 is the following (Theorem 5.4), which generalizes [2, Theorem 3.2, Theorem 3.4] and [7, Proposition 6.2].

**Theorem 1.3** Let $\mathcal{C}$ be an extriangulated category and $\mathcal{M}$ be a rigid subcategory of $\mathcal{C}$.

1. If $\mathcal{C}$ has enough projectives $\mathcal{P}$ and $\mathcal{M}$ contains $\mathcal{P}$, then $\mathcal{M}_L/[\mathcal{M}] \cong \text{mod-}(\mathcal{M}/[\mathcal{P}])$ and $\mathcal{M}_L/[\Omega \mathcal{M}] \cong (\text{mod-}(\mathcal{M}/[\mathcal{P}])^{op})^{op}$.
2. If $\mathcal{C}$ has enough injectives $\mathcal{I}$ and $\mathcal{M}$ contains $\mathcal{I}$, then $\mathcal{M}_R/[\mathcal{M}] \cong \text{mod-}(\mathcal{M}/[\mathcal{I}])$ and $\mathcal{M}_R/[\Sigma \mathcal{M}] \cong (\text{mod-}(\mathcal{M}/[\mathcal{I}])^{op})^{op}$.

In particular, if $\mathcal{M}$ is a cluster tilting subcategory of $\mathcal{C}$, then $\mathcal{M}_L = \mathcal{M}_R = \mathcal{C}$. Thus we have the following result (Corollary 5.7).
Corollary 1.4. Let \( \mathcal{C} \) be an extriangulated category with enough projectives \( \mathcal{P} \) and enough injectives \( \mathcal{I} \). If \( \mathcal{M} \) is a cluster tilting subcategory of \( \mathcal{C} \), then

1. \( \mathcal{C} / [\mathcal{M}] \cong \text{mod-}(\mathcal{M} / [\mathcal{P}]) \cong (\text{mod-}(\mathcal{M} / [\mathcal{I}])^{\text{op}})^{\text{op}}. \)

2. \( \mathcal{C} / [\Omega \mathcal{M}] \cong (\text{mod-}(\mathcal{M} / [\mathcal{P}])^{\text{op}})^{\text{op}}. \)

3. \( \mathcal{C} / [\Sigma \mathcal{M}] \cong \text{mod-}(\mathcal{M} / [\mathcal{I}]). \)

This paper is organized as follows. In Section 2 we make some preliminaries on morphism categories and extriangulated categories. In Section 3 we prove Theorem 1.1. In Section 4 we provide the first application. In Section 5 we provide the second application.

2 Definitions and Preliminaries

In this section, we first give some facts on morphism categories, then recall the definitions and basic properties on extriangulated categories from [11, 15] and [6].

2.1 Morphism Categories

Let \( \mathcal{C} \) be an additive category. The \textit{morphism category} of \( \mathcal{C} \) is the category \( \text{Mor}(\mathcal{C}) \) defined by the following data. The objects of \( \text{Mor}(\mathcal{C}) \) are all the morphisms \( f : X \to Y \) in \( \mathcal{C} \). The morphisms from \( f : X \to Y \) to \( f' : X' \to Y' \) are pairs \((a, b)\) where \( a : X \to X' \) and \( b : Y \to Y' \) such that \( bf = f'a \). The composition of morphisms is componentwise. For two objects \( f : X \to Y \) and \( f' : X' \to Y' \) in \( \text{Mor}(\mathcal{C}) \), we define \( \mathcal{R}(f, f') \) (resp. \( \mathcal{R}'(f, f') \)) to be the set of morphisms \((a, b)\) such that there is some morphism \( p : Y \to X' \) such that \( f'p = b \) (resp. \( pf = a \)). Then \( \mathcal{R} \) and \( \mathcal{R}' \) are ideals of \( \text{Mor}(\mathcal{C}) \). We denote by \( \text{s-epi}(\mathcal{C}) \) (resp. \( \text{s-mono}(\mathcal{C}) \)) the full subcategory of \( \text{Mor}(\mathcal{C}) \) consisting of split epimorphisms (resp. split monomorphisms).

Recall that a right \( \mathcal{C} \)-\textit{module} is a contravariantly additive functor \( F : \mathcal{C} \to Ab \), where \( Ab \) is the category of abelian groups. A \( \mathcal{C} \)-module \( F \) is called \textit{finitely presented} if there exists an exact sequence \( \mathcal{C}(\cdot, X) \to \mathcal{C}(\cdot, Y) \to F \to 0 \). We denote by \( \text{mod-}\mathcal{C} \) the category of finitely presented \( \mathcal{C} \)-modules, and by \( \text{proj-}\mathcal{C} \) (resp. \( \text{inj-}\mathcal{C} \)) the full subcategory of \( \text{mod-}\mathcal{C} \) consisting of projectives (resp. injectives).

Define a functor \( \alpha : \text{Mor}(\mathcal{C}) \to \text{mod-}\mathcal{C} \) by mapping \( f : X \to Y \) to \( F = \text{Coker}(\mathcal{C}(\cdot, f) : \mathcal{C}(\cdot, X) \to \mathcal{C}(\cdot, Y)) \). The functor \( \alpha \) induces the following equivalences, which are crucial in the proof of our main results.

Lemma 2.1 ([10, Lemma 3.1, Proposition 3.3]) Let \( \mathcal{C} \) be an additive category, then

1. \( \text{Mor}(\mathcal{C}) / \mathcal{R} \cong \text{Mor}(\mathcal{C}) / [\text{s-epi}(\mathcal{C})] \cong \text{mod-}\mathcal{C}. \)

2. \( \text{Mor}(\mathcal{C}) / \mathcal{R}' \cong \text{Mor}(\mathcal{C}) / [\text{s-mono}(\mathcal{C})] \cong (\text{mod-}\mathcal{C}^{\text{op}})^{\text{op}}. \)

2.2 Extriangulated Categories

Let \( \mathcal{C} \) be an additive category equipped with an additive bifunctor \( \mathbb{E} : \mathcal{C}^{\text{op}} \times \mathcal{C} \to Ab \). For any pair of objects \( A, C \in \mathcal{C} \), an object \( \delta \in \mathbb{E}(C, A) \) is called an \textit{\( \mathbb{E} \)-extension}. For any morphism \( a \in \mathcal{C}(A, A') \) and \( c \in \mathcal{C}(C', C) \), we denote the \( \mathbb{E} \)-extension \( \mathbb{E}(C, A)(\delta) \in \mathbb{E}(C, A') \) by \( as\delta \) and denote the \( \mathbb{E} \)-extension \( \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A) \) by \( c\delta \). Let \( \delta \in \mathbb{E}(C, A) \) and \( \delta' \in \mathbb{E}(C', A') \) be two \( \mathbb{E} \)-extensions. A morphism \( (a, c) : \delta \to \delta' \) of \( \mathbb{E} \)-extensions is a pair of morphisms \( a \in \mathcal{C}(A, A') \) and \( c \in \mathcal{C}(C, C') \) such that \( as\delta = c\delta' \).
Let $A, C \in C$ be any pair of objects. Two sequences of morphisms in $C$

$$A \xrightarrow{x} B \xrightarrow{y} C \quad \text{and} \quad A \xrightarrow{x'} B' \xrightarrow{y'} C'$$

are equivalent if there exists an isomorphism $b \in C(B, B')$ such that the following diagram is commutative.

$$\begin{array}{ccc}
A & \xrightarrow{x} & B & \xrightarrow{y} & C \\
\| & \| & \| & \| & \| \\
A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C
\end{array}$$

We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$.

**Definition 2.2** Let $\mathbb{E} : C^{\text{op}} \times C \to Ab$ be an additive bifunctor. A correspondence $s$ is called a realization of $\mathbb{E}$ if it associates an equivalence class $s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$ and associates a commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{x} & B & \xrightarrow{y} & C \\
\downarrow{a} & \downarrow{b} & \downarrow{c} & \downarrow{d} \\
A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C
\end{array}$$

to any morphism $(a, c) : \delta \to \delta'$ of $\mathbb{E}$-extensions, where $s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ and $s(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. In the above situation, we say the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ realizes $\delta$ and the triple $(a, b, c)$ realizes $(a, c)$.

**Definition 2.3** A realization $s$ of $\mathbb{E}$ is said to be additive if it satisfies the following two conditions.

1. Assume that $0 \in \mathbb{E}(C, A)$ is the zero element, then $s(0) = [A \xrightarrow{(0, 0)} A \oplus C \xrightarrow{(0, 1)} C]$.
2. Assume that $s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ and $s(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$, then $s(\delta \oplus \delta') = [A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C']$, where $\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$ is the element corresponding to $(\delta, 0, 0, \delta')$ under the isomorphism $\mathbb{E}(C \oplus C', A \oplus A') \cong \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A')$.

Let $s$ be an additive realization of $\mathbb{E}$. If $s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, then the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called an $s$-conflation, the morphism $x$ is called an $s$-inflation and $y$ is called an $s$-deflation. In this case, we say $A \xrightarrow{x} B \xrightarrow{y} C - \delta \rightarrow$ is an $s$-triangle.

Let $A \xrightarrow{x} B \xrightarrow{y} C - \delta \rightarrow$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' - \delta' \rightarrow$ be any pair of $s$-triangles. Let $(a, c) : \delta \to \delta'$ be a morphism of $\mathbb{E}$-extensions. If a triple $(a, b, c)$ realizes $(a, c)$, then we say $(a, b, c)$ is a morphism of $s$-triangles.

**Definition 2.4** ([15, Definition 2.12]) A triple $(\mathcal{C}, \mathbb{E}, s)$ is an extriangulated category if the following conditions are satisfied.

1. $\mathbb{E} : C^{\text{op}} \times C \to Ab$ is an additive bifunctor.
(ET2) $s$ is an additive realization of $E$.

(ET3) Each commutative diagram

\[
A \xrightarrow{\ x \ } B \xrightarrow{\ y \ } C \xrightarrow{\ \delta \ } \\
\downarrow{\ a \ } \downarrow{\ b \ } \downarrow{\ \ } \\
A' \xrightarrow{\ x' \ } B' \xrightarrow{\ y' \ } C' \xrightarrow{\ \delta' \ }
\]

whose rows are $s$-triangles can be completed to a morphism of $s$-triangles.

(ET3)$^{op}$ Each commutative diagram

\[
A \xrightarrow{\ x \ } B \xrightarrow{\ y \ } C \xrightarrow{\ \delta \ } \\
\downarrow{\ b \ } \downarrow{\ c \ } \downarrow{\ \ } \\
A' \xrightarrow{\ x' \ } B' \xrightarrow{\ y' \ } C' \xrightarrow{\ \delta' \ }
\]

whose rows are $s$-triangles can be completed to a morphism of $s$-triangles.

(ET4) Let $A \xrightarrow{\ f \ } B \xrightarrow{\ f' \ } D \xrightarrow{\ \delta \ }$ and $B \xrightarrow{\ g \ } C \xrightarrow{\ g' \ } F \xrightarrow{\ \delta' \ }$ be $s$-triangles. There exists a commutative diagram

\[
A \xrightarrow{\ f \ } B \xrightarrow{\ f' \ } D \xrightarrow{\ \delta \ } \\
\downarrow{\ h \ } \downarrow{\ \ } \downarrow{\ } \\
A \xrightarrow{\ g \ } C \xrightarrow{\ g' \ } E \xrightarrow{\ \delta'' \ } \\
\downarrow{\ \ } \downarrow{\ } \downarrow{\ } \\
F \xrightarrow{\ e \ } F
\]

such that the second row and the third column are $s$-triangles, moreover, $\delta = d^*\delta''$ and $f_*\delta'' = e^*\delta'$.

(ET4)$^{op}$ Let $D \xrightarrow{\ f' \ } A \xrightarrow{\ f \ } B \xrightarrow{\ \delta \ }$ and $F \xrightarrow{\ g' \ } B \xrightarrow{\ g \ } C \xrightarrow{\ \delta' \ }$ be $s$-triangles. There exists a commutative diagram

\[
D \xrightarrow{\ d \ } E \xrightarrow{\ e \ } F \xrightarrow{\ g^*\delta \ } \\
\downarrow{\ h' \ } \downarrow{\ \ } \downarrow{\ } \\
D \xrightarrow{\ f' \ } A \xrightarrow{\ f \ } B \xrightarrow{\ \delta \ } \\
\downarrow{\ h \ } \downarrow{\ g \ } \downarrow{\ \ } \\
C \xrightarrow{\ \ } C \\
\downarrow{\ } \downarrow{\ } \downarrow{\ } \\
\downarrow{\ \ } \downarrow{\ } \\
\end{align}

such that the first row and the second column are $s$-triangles, moreover, $\delta' = e_*\delta''$ and $d_*\delta = g^*\delta''$. 

$\mathcopyright$ Springer
Definition 2.5 Let \((C, E, s)\) be an extriangulated category.

1. An object \(P \in C\) is called \textit{projective} if for any \(s\)-deflation \(y : B \to C\) and any morphism \(c : P \to C\), there exists a morphism \(b : P \to B\) such that \(yb = c\). The full subcategory of projectives is denoted by \(\mathcal{P}\).

2. We say that \(C\) has enough projectives if for any object \(C \in C\) there exists an \(s\)-triangle \(\xymatrix{A \ar[r]^x & P \ar[r]^y & C \ar[r]^\delta & \text{ with } P \in \mathcal{P}.\)

Example 2.6 ([15, Example 3.26])

1. Let \(C\) be an exact category, then \(C\) is an extriangulated category with \(E^1(-, -) = \text{Ext}_C^1(-, -)\). In particular, if \(C\) is an exact category with enough projectives, then \(C\) is an extriangulated category with enough projectives.

2. Let \(C\) be a triangulated category with shift functor \([1]\), then \(C\) is an extriangulated category with \(E^1(-, -) = C(-, [1])\). Moreover, \(C\) has enough projectives. In this case, \(\mathcal{P}\) consists of zero objects.

The following lemmas will be used frequently later.

Lemma 2.7 ([15, Corollary 3.5]) Let \(C\) be an extriangulated category. Assume that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \\
\downarrow{a} & & \downarrow{b} \\
A' & \xrightarrow{x'} & B' \\
\end{array}
\]

is a morphism of \(s\)-triangles. Then the following statements are equivalent.

1. \(a\) factors through \(x\).
2. \(a \cdot \delta = c \cdot \delta' = 0\).
3. \(c\) factors through \(y'\).

Lemma 2.8 ([11, Proposition 1.20]) Let \(C\) be an extriangulated category. Assume that \(\xymatrix{A \ar[r]^x & B \ar[r]^y & C \ar[r]^\delta & \text{ is an } s\text{-triangle, } f : A \to D \text{ is a morphism and } D \ar[r]^d & E \ar[r]^e & C \ar[r]^\delta & \text{ is an } s\text{-triangle, then there is a morphism } g : B \to E \text{ which gives a morphism of } s\text{-triangles}
\]

and moreover, \(\xymatrix{A \ar[r]^{(f)} & D \ar[r]^{(d, g)} & B \ar[r]^{(d, -g)} & E \ar[r]^{e \cdot \delta} & \text{ is an } s\text{-triangle.}\)
Lemma 2.9 ([15, Corollary 3.12]) Let $\mathcal{C}$ be an extriangulated category. Then for any $s$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} 0$, the following two sequences are exact.

$$
\begin{align*}
&\mathcal{C}(-, A) \to \mathcal{C}(-, B) \to \mathcal{C}(-, C) \to \mathcal{E}(-, A) \to \mathcal{E}(-, B) \to \mathcal{E}(-, C), \\
&\mathcal{C}(C, -) \to \mathcal{C}(B, -) \to \mathcal{C}(A, -) \to \mathcal{E}(C, -) \to \mathcal{E}(B, -) \to \mathcal{E}(A, -).
\end{align*}
$$

Let $\mathcal{C}$ be an extriangulated category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$. Let $X$ be any object in $\mathcal{C}$. It admits an $s$-triangle

$$
X \xrightarrow{1^0} \Sigma X \xrightarrow{\delta_X} \cdots \xrightarrow{(\text{resp. } \Omega X \xrightarrow{P_0} X \xrightarrow{\delta_X} )}
$$

with $1^0 \in \mathcal{I}$ (resp. $P_0 \in \mathcal{P}$). We can get $s$-triangles

$$
\Sigma^i X \xrightarrow{1^i} \Sigma^{i+1} X \xrightarrow{\delta^{i+1}_X} \cdots \xrightarrow{(\text{resp. } \Omega^{i+1} X \xrightarrow{P_i} \Omega^i X \xrightarrow{\delta^{i+1}_X} )}
$$

with $1^i \in \mathcal{I}$ (resp. $P_i \in \mathcal{P}$) for $i > 0$ recursively.

For convenience, we denote $\mathcal{E}(X, \Sigma^i Y) \cong \mathcal{E}(\Omega^i X, Y)$ by $\mathcal{E}^i(X, Y)$, where the equivalence follows from [11, Lemma 5.1].

The following result extends the exact sequences appeared in Lemma 2.9.

**Lemma 2.10** ([11, Proposition 5.2]) Let $\mathcal{C}$ be an extriangulated category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$. Then for any $s$-triangle

$$
A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} 0,
$$

the following two sequences are exact.

$$
\begin{align*}
&\mathcal{C}(-, A) \to \mathcal{C}(-, B) \to \mathcal{C}(-, C) \to \mathcal{E}(-, A) \to \mathcal{E}(-, B) \to \mathcal{E}(-, C) \to \mathcal{E}^2(-, A) \\
&\to \mathcal{E}^2(-, B) \to \mathcal{E}^2(-, C) \to \cdots \to \mathcal{E}^i(-, A) \to \mathcal{E}^i(-, B) \to \mathcal{E}^i(-, C) \to \cdots,
\end{align*}
$$

\[\begin{align*}
&\mathcal{C}(C, -) \to \mathcal{C}(B, -) \to \mathcal{C}(A, -) \to \mathcal{E}(C, -) \to \mathcal{E}(B, -) \to \mathcal{E}(A, -) \to \mathcal{E}^2(C, -) \\
&\to \mathcal{E}^2(B, -) \to \mathcal{E}^2(A, -) \to \cdots \to \mathcal{E}^i(C, -) \to \mathcal{E}^i(B, -) \to \mathcal{E}^i(A, -) \to \cdots.
\end{align*}\]

**Lemma 2.11** Let $\mathcal{C}$ be an extriangulated category with enough projectives $\mathcal{P}$ and $f : X \to Y$ be a morphism in $\mathcal{C}$.

1. If $\pi : P \to Y$ is an $s$-deflation with $P \in \mathcal{P}$, then $(f, -\pi) : X \oplus P \to Y$ is an $s$-deflation and $(f, -\pi) \cong f$ in $\text{Mor}(\mathcal{C}/[\mathcal{P}])$.
2. If $h : X \to Z$ is a morphism in $\mathcal{C}$ and $g : Y \to Z$ is an $s$-deflation such that $gf = h$ in $\text{Mor}(\mathcal{C}/[\mathcal{P}])$, then there exists an object $P \in \mathcal{P}$ and two morphisms $u : X \to P$ and $v : P \to Y$ such that $g(f - vu) = h$.

**Proof**

1. The first assertion follows from [15, Corollary 3.16] or the dual of Lemma 2.8. The second assertion is clear.
2. Since $gf = h$, there is an object $P \in \mathcal{P}$ and two morphisms $u : X \to P$ and $w : P \to Z$ such that $gf - h = wu$. Since $g : Y \to Z$ is an $s$-deflation, there exists a morphism $v : P \to Y$ such that $w = gv$. Therefore, $g(f - vu) = h$. 

\[\square\]

\[\square\] Springer
3 Proof of Theorem 1.1

Throughout this paper, we assume that \((C, \mathcal{E}, s)\) is an extriangulated category.

Let \(\mathcal{M}\) be a full subcategory of \(C\). We denote by \(s\)-def(\(\mathcal{M}\)) (resp. \(s\)-inf(\(\mathcal{M}\))) the full subcategory of \(\text{Mor}(\mathcal{M})\) consisting of \(s\)-deflations (resp. \(s\)-inflations). Recall that the full subcategory of \(s\)-def(\(\mathcal{M}\)) consisting of split epimorphisms (resp. split monomorphisms) is denoted by \(s\text{-epi}(\mathcal{M})\) (resp. \(s\text{-mono}(\mathcal{M})\)). We denote by \(s\text{-epi}(\mathcal{M})\) (resp. \(s\text{-mono}(\mathcal{M})\)) the full subcategory of \(s\)-def(\(\mathcal{M}\)) consisting of \((M \xrightarrow{1} M) \oplus (P \rightarrow M')\) (resp. \((M \xrightarrow{1} M) \oplus (M' \rightarrow I)\)) with \(P \in \mathcal{P}\) (resp. \(I \in \mathcal{I}\)).

Lemma 3.1 Let \(C\) be an extriangulated category with enough projectives \(\mathcal{P}\) and \(\mathcal{M}\) be a full subcategory of \(C\) containing \(\mathcal{P}\). Assume that the following diagram is a morphism of \(s\)-triangles with \(M_i, M'_i \in \mathcal{M}\). Then

(1) The following statements are equivalent.

(a) The morphism \(b\) factors through \(f'\) in \(\mathcal{M}/[\mathcal{P}]\).
(b) The morphism \(b\) factors through \(f'\).
(c) The morphism \((a, b)\) factors through some object in \(s\text{-epi}(\mathcal{M})\).

(2) The following statements are equivalent.

(a) The morphism \(a\) factors through \(f\) in \(\mathcal{M}/[\mathcal{P}]\).
(b) The morphism \((a, b)\) factors through some object in \(s\text{-epi}(\mathcal{M})\).

Proof (1) Since (b)\(\Leftrightarrow\)(c) follows from Lemma 2.1 and (b)\(\Rightarrow\)(a) is clear, we only prove (a)\(\Rightarrow\)(b). Suppose that there is a morphism \(p : M_2 \rightarrow M'_1\) such that \(f'p = b\). By Lemma 2.11, there exists an object \(P \in \mathcal{P}\) and two morphisms \(u : M_2 \rightarrow P\) and \(v : P \rightarrow M'_1\) such that \(f'(p - vu) = b\). Thus \(b\) factors through \(f'\).

(2) (a)\(\Rightarrow\)(b). Suppose that there is a morphism \(p : M_2 \rightarrow M'_1\) such that \(pf = a\). Since \(C\) has enough projectives, there is an \(s\)-deflation \(a_1 : P \rightarrow M'_1\) with \(P \in \mathcal{P}\). It is easy to see that \(a - pf\) factors through \(a_1\). We assume that \(a - pf = a_1a_2\) where \(a_2 : M_1 \rightarrow P\). Since \((b - f'p)f = f'a - f'pf = f'a_1a_2\), we have the following commutative diagram.
In other words, \((a, b)\) factors through \((M_2 \oplus P \xrightarrow{(1 \ 0 \ f_{a_1})} M_2 \oplus M'_2) \in \text{sp-epi}(\mathcal{M})\).

(b) \(\Rightarrow\) (a). Assume that the morphism \((a, b)\) factors through \((M \oplus P \xrightarrow{(1 \ 0 \ \pi)} M \oplus M') \in \text{sp-epi}(\mathcal{M})\). Suppose that the following diagram

\[
\begin{array}{ccc}
M_1 & \xrightarrow{f} & M_2 \\
\downarrow^{a_1} & & \downarrow^{b_1} \\
M \oplus P & \xrightarrow{(1 \ 0 \ \pi)} & M \oplus M' \\
\downarrow^{a_2} & & \downarrow^{b_2} \\
M'_1 & \xrightarrow{f'} & M'_2 \\
\end{array}
\]

is commutative. Let \(p = a_2 b_1 : M_2 \to M'_1\), then \(p f = a_2 b_1 f = a_2 a_1\), thus \(a = a_2 a_1 = p f\).

**Theorem 3.2** Let \(\mathcal{C}\) be an extriangulated category and \(\mathcal{M}\) be a full subcategory of \(\mathcal{C}\).

1. If \(\mathcal{C}\) has enough projectives \(\mathcal{P}\) and \(\mathcal{M}\) contains \(\mathcal{P}\), then \(\text{s-def}(\mathcal{M})/\text{[s-epi}(\mathcal{M})]) \cong \text{mod-}(\mathcal{M}/[\mathcal{P}])\) and \(\text{s-def}(\mathcal{M})/\text{[sp-epi}(\mathcal{M})]) \cong (\text{mod-}(\mathcal{M}/[\mathcal{P}]))^{\text{op}}\).

2. If \(\mathcal{C}\) has enough injectives \(\mathcal{I}\) and \(\mathcal{M}\) contains \(\mathcal{I}\), then \(\text{s-inf}(\mathcal{M})/\text{[s-mono}(\mathcal{M})]) \cong (\text{mod-}(\mathcal{M}/[\mathcal{I}])^{\text{op}}\) and \(\text{s-inf}(\mathcal{M})/\text{[si-mono}(\mathcal{M})]) \cong \text{mod-}(\mathcal{M}/[\mathcal{I}])\).

**Proof** Since (2) is dual to (1), we only prove (1).

Define a functor

\[
F : \text{s-def}(\mathcal{M}) \to \text{Mor}(\mathcal{M}/[\mathcal{P}]), \quad (M_1 \xrightarrow{f} M_2) \mapsto (M_1 \xrightarrow{f} M_2).
\]

For any object \(f : M_1 \to M_2\) in \(\text{Mor}(\mathcal{M}/[\mathcal{P}])\), by Lemma 2.11 there is an object \(P \in \mathcal{P}\) and an \(\text{s-deflation} (f, -\pi) : M_1 \oplus P \to M_2\) such that \((f, -\pi) \cong f\). Therefore, \(F(f, -\pi) \cong f\) and \(F\) is dense.

Assume that \(f : M_1 \to M_2\) and \(f' : M'_1 \to M'_2\) are objects in \(\text{s-def}(\mathcal{M})\) and \((a, b)\) is a morphism in \(\text{Mor}(\mathcal{M}/[\mathcal{P}])\) from \(f\) to \(f'\). Then \(b f = f' a\). By Lemma 2.11, there exists an object \(Q \in \mathcal{P}\) and two morphisms \(u : M_1 \to Q\) and \(v : Q \to M'_1\) such that \(f'(a - vu) = b f\). Thus, \(F(a - vu, b) = (a, b)\) and the functor \(F\) is full.

The functor \(F\) induces a full and dense functor \(\tilde{F} : \text{s-def}(\mathcal{M}) \to \text{Mor}(\mathcal{M}/[\mathcal{P}])/\mathcal{R}\). By Lemma 3.1(1), we have \(\text{s-def}(\mathcal{M})/\text{[s-epi}(\mathcal{M})]) \cong \text{Mor}(\mathcal{M}/[\mathcal{P}])/\mathcal{R}\). It follows that \(\text{s-def}(\mathcal{M})/\text{[s-epi}(\mathcal{M})]) \cong \text{mod-}(\mathcal{M}/[\mathcal{P}])\) by Lemma 2.1(1).
The functor $F$ induces a full and dense functor $\overline{F} : s\text{-}\text{def}(\mathcal{M}) \to \text{Mor}(\mathcal{M}/[P])/\mathcal{R}'$. By Lemma 3.1(2), we have $s\text{-}\text{def}(\mathcal{M})/[s\text{-}\text{epi}(\mathcal{M})] \cong \text{Mor}(\mathcal{M}/[P])/\mathcal{R}'$. It follows that $s\text{-}\text{def}(\mathcal{M})/[s\text{-}\text{epi}(\mathcal{M})] \cong \text{(mod-}(\mathcal{M}/[P])^{\text{op}})^{\text{op}}$ by Lemma 2.1(2).

4 Application to Category of $s$-Triangles

In this section, we will investigate the first application of Theorem 3.2 in the case when $\mathcal{M} = \mathcal{C}$.

We denote by $s\text{-}\text{tri}(\mathcal{C})$ the category of $s$-triangles in $\mathcal{C}$, where the objects are the $s$-triangles $X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{\delta} )$ and the morphisms from $X_\bullet$ to $Y_\bullet$ are the triples $\varphi_\bullet = (\varphi_1, \varphi_2, \varphi_3)$ such that the following diagram is commutative

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & X_2 \xrightarrow{f_2} X_3 \xrightarrow{\delta} \\
\varphi_1 & & \varphi_2 \\
Y_1 & \xrightarrow{g_1} & Y_2 \xrightarrow{g_2} Y_3 \xrightarrow{\delta'}
\end{array}
\]

and $\varphi_1 \delta = \varphi_3^* \delta'$. Let $X_\bullet$ and $Y_\bullet$ be two $s$-triangles, we denote by $\mathcal{R}_2(X_\bullet, Y_\bullet)$ (resp. $\mathcal{R}_1'(X_\bullet, Y_\bullet)$) the class of morphisms $\varphi_\bullet : X_\bullet \to Y_\bullet$ such that $\varphi_3$ factors through $g_2$ (resp. $\varphi_1$ factors through $f_1$). It is easy to see that $\mathcal{R}_2$ and $\mathcal{R}_1'$ are ideals of $s\text{-}\text{tri}(\mathcal{C})$.

**Theorem 4.1** Let $\mathcal{C}$ be an extriangulated category.

1. If $\mathcal{C}$ has enough projectives $\mathcal{P}$, then $s\text{-}\text{tri}(\mathcal{C})/\mathcal{R}_2 \cong \text{mod-}(\mathcal{C}/[\mathcal{P}])$.
2. If $\mathcal{C}$ has enough injectives $\mathcal{I}$, then $s\text{-}\text{tri}(\mathcal{C})/\mathcal{R}_2 \cong \text{mod-}(\mathcal{C}/[\mathcal{I}])^{\text{op}}$.

**Proof** (1) We have $s\text{-}\text{tri}(\mathcal{C})/\mathcal{R}_2 \cong s\text{-}\text{def}(\mathcal{C})/[s\text{-}\text{epi}(\mathcal{C})]$ by Lemma 3.1. Thus the desired result follows from Theorem 3.2(1).

(2) We note that $\mathcal{R}_2 = \mathcal{R}_1'$ by Lemma 2.7. Thus $s\text{-}\text{tri}(\mathcal{C})/\mathcal{R}_2 = s\text{-}\text{tri}(\mathcal{C})/\mathcal{R}_1' \cong s\text{-}\text{inf}(\mathcal{C})/[s\text{-}\text{mono}(\mathcal{C})] \cong \text{(mod-}(\mathcal{C}/[\mathcal{I}])^{\text{op}})^{\text{op}}$, where the last equivalence follows from Theorem 3.2(2).

**Lemma 4.2** Let $\mathcal{C}$ be an extriangulated category. Assume that the following

\[
\begin{array}{ccc}
X_\bullet & X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{\delta} \\
\varphi_\bullet & \varphi_1 & \varphi_2 \\
Y_\bullet & Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} Y_3 \xrightarrow{\delta'}
\end{array}
\]

is a morphism of $s$-triangles. Then

1. The following statements are equivalent.
   a. $\varphi_\bullet = 0$ in $s\text{-}\text{tri}(\mathcal{C})/\mathcal{R}_2$.
   b. $\varphi_1$ factors through $f_1$.
   c. $\varphi_3$ factors through $g_2$. 

\[\text{Springer}\]
The following are morphisms of $s$-triangles.

Moreover, $\varphi_* = i_* \pi_*$ in $s$-tri$(\mathcal{C})/\mathcal{R}_2$.

(3) The following statements are equivalent.

(a) $\varphi_*$ is a monomorphism in $s$-tri$(\mathcal{C})/\mathcal{R}_2$.

(b) $(f_1, \psi_1) : X_1 \to X_2 \oplus Y_1$ is a section.

Proof

(1) It follows from Lemma 2.7.

(2) Note that $\pi_* : X_* \to I(\varphi_*)$ and $i_* : I(\varphi_*) \to Y_*$ are morphisms of $s$-triangles since $\varphi_1 \pi_* = \varphi_* \delta$. Lemma 2.8 and its dual imply that $K(\varphi_*)$ and $C(\varphi_*)$ are $s$-triangles, thus $k_* : K(\varphi_*) \to X_*$ and $c_* : Y_* \to C(\varphi_*)$ are morphisms of $s$-triangles. It follows that $\varphi_* = i_* \pi_*$ by (1).

(3) (a) $\Rightarrow$ (b). There is a morphism $p_1 = (0, 1) : X_2 \oplus Y_1 \to Y_1$ such that $\varphi_* = p_1 \left( \frac{f_1}{\psi_1} \right)$, so $\varphi_* k_* = i_* \pi_* k_* = 0$ by (2) and (1). Since $\varphi_* k_*$ is a monomorphism, $k_* = 0$. It follows from (1) that $(\frac{f_1}{\psi_1})$ is a section.

(b) $\Rightarrow$ (a). Assume that there exists a morphism $(f_1', \psi_1') : X_2 \oplus Y_1 \to X_1$ such that $(f_1', \psi_1') \left( \frac{f_1}{\psi_1} \right) = 1$. Suppose that $\psi_* : Z_* \to X_*$ is a morphism of $s$-triangles such that $\varphi_* \psi_* = 0$.

By (1) there is a morphism $p_1 : Z_2 \to Y_1$ such that $\varphi_* \psi_1 = p_1 h_1$. Thus there exists a morphism $q_1 = (f_1', \psi_1') \left( \frac{f_1}{\psi_1} \right) : Z_2 \to X_1$ such that $q_1 h_1 = (f_1' \psi_2 + \psi_1' p_1) h_1 = (f_1' f_1 + \psi_1' \psi_1) \psi_1 = \psi_1$. We infer that $\psi_* = 0$ by (1).
If $\mathcal{C}$ has enough projectives $\mathcal{P}$, then $s\text{-tri}(\mathcal{C})/\mathcal{R}_2 \cong \text{mod-}(\mathcal{C}/[\mathcal{P}])$ is abelian by Theorem 4.1. The following result implies that $s\text{-tri}(\mathcal{C})/\mathcal{R}_2$ is always abelian for general case.

**Proposition 4.3** Let $\mathcal{C}$ be an extriangulated category. Then $s\text{-tri}(\mathcal{C})/\mathcal{R}_2$ is an abelian category.

**Proof** The proof is an adaption of [10, Theorem 4.8]. Assume that $\varphi_\bullet : X_\bullet \to Y_\bullet$ is a morphism of $s$-triangles. According to Lemma 4.2, it is routine to check that $k : K(\varphi_\bullet) \to X_\bullet$ is a kernel of $\varphi_\bullet$, $\epsilon : Y_\bullet \to C(\varphi_\bullet)$ is a cokernel of $\varphi_\bullet$ and

$$\text{Coker}(\text{Ker}(\varphi_\bullet)) \cong I(\varphi_\bullet) \cong \text{Ker}(\text{Coker}(\varphi_\bullet)).$$

**Proposition 4.4** Let $\mathcal{C}$ be an extriangulated category. Then an $s$-triangle

$$P_X = (\Omega X \xrightarrow{f_1} P \xrightarrow{f_2} X \xrightarrow{\rho} \to)$$

with $P$ projective is a projective object in $s\text{-tri}(\mathcal{C})/\mathcal{R}_2$. Moreover, if $\mathcal{C}$ has enough projectives, then each projective object in $s\text{-tri}(\mathcal{C})/\mathcal{R}_2$ is of the form $P_X$.

**Proof** Assume that $f_1 : X_1 \to X_2$ is an epimorphism and $\varphi_\bullet : P_X \to Z_\bullet$ is a morphism in $s\text{-tri}(\mathcal{C})/\mathcal{R}_2$. By the dual of Lemma 4.2 (3), the morphism $(\varphi_3, h_2) : Y_3 \oplus Z_2 \to Z_3$ is a retraction. There exists a morphism $\left(\begin{array}{c} \varphi_3^i \\ h_2' \end{array}\right) : Z_3 \to Y_3 \oplus Z_2$ such that $\varphi_3 \varphi_3^i + h_2 h_2' = 1$.

Suppose that $\varphi_3^i = \varphi_3 \varphi_3^i$. Then there exists a morphism $\phi_\bullet : P_X \to Y_\bullet$ of $s$-triangles since $P$ is projective. Note that $\varphi_3 - \varphi_3 \varphi_3 = h_2(h_2' \varphi_3)$, we obtain $\psi_\bullet = \varphi_\bullet \phi_\bullet$ by Lemma 4.2 (1). Therefore, $P_X$ is projective.

Assume that $\mathcal{C}$ has enough projectives. For each $s$-triangle

$$X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{\delta} \to),$$

there exists an $s$-triangle

$$P_{X_\bullet} = (\Omega X_3 \xrightarrow{g_1} P \xrightarrow{g_2} X_3 \xrightarrow{\rho} \to)$$

with $P$ projective. We have a morphism $\varphi_\bullet : P_{X_\bullet} \to X_\bullet$ of $s$-triangles with $\varphi_3 = 1$. Consequently, $\varphi_\bullet : P_{X_\bullet} \to X_\bullet$ is an epimorphism and $P_{X_\bullet}$ is projective. Furthermore, suppose that $X_\bullet$ is a projective object in $s\text{-tri}(\mathcal{C})/\mathcal{R}_2$, then the epimorphism $\varphi_\bullet : P_{X_\bullet} \to X_\bullet$ is split. Thus each projective object of $s\text{-tri}(\mathcal{C})/\mathcal{R}_2$ is of the form $P_X$ for some object $X$ in $\mathcal{C}$. \qed

**Definition 4.5** ([6, 17]) An $s$-triangle $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{\delta} \to$ is called Auslander-Reiten $s$-triangle if the following holds:

1. $\delta \in \mathbb{E}(\mathcal{C}, A)$ is non-split.
2. If $g : X_1 \to Y$ is not a section, then $g$ factors through $f_1$.
3. If $h : Z \to X_3$ is not a retraction, then $h$ factors through $f_2$.

**Proposition 4.6** Let $\mathcal{C}$ be a Krull-Smidt extriangulated category. Assume that $X_\bullet : X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{\delta} \to$ is a non-split $s$-triangle such that $X_1$ and $X_2$ are indecomposable. Then $X_\bullet$ is a simple object in $s\text{-tri}(\mathcal{C})/\mathcal{R}_2$ if and only if $X_\bullet$ is an Auslander-Reiten $s$-triangle in $\mathcal{C}$.
Proof For the “only if” part, assume that \( \varphi_1 : X_1 \to Y_1 \) is not a section. We have the following morphism

\[
\begin{array}{c}
X_* & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{\delta} & Y_* \\
\downarrow{\varphi_*} & & \downarrow{\varphi_1} & & \downarrow{\varphi_2} & & \downarrow{\varphi_*} \\
Y_* & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{\delta'} & 
\end{array}
\]

of \( s \)-triangles by Lemma 2.8. Since \( f_1 \) and \( \varphi_1 \) are not sections and \( X_1 \) is indecomposable, the morphism \( (f_1, \varphi_1) \) is not a section. Thus \( \varphi_* \) is not a monomorphism by Lemma 4.2 (3). We infer that \( \varphi_* = 0 \) since \( X_* \) is a simple object. It follow from Lemma 4.2 (1) that \( \varphi_1 \) factors through \( f_1 \). Dually, we can prove that if \( \varphi_3 : Z_3 \to X_3 \) is not a retraction, then \( \varphi_3 \) factors through \( f_2 \). Thus \( X_* \) is an Auslander-Reiten \( s \)-triangle.

For the “if” part, assume that \( \varphi_* : X_* \to Y_* \) is a morphism of \( s \)-triangles. If \( \varphi_1 \) is a section, then \( (f_1, \varphi_1) \) is a section, thus \( \varphi_* \) is a monomorphism. If \( \varphi_1 \) is not a section, then \( \varphi_1 \) factors through \( f_1 \) since \( X_* \) is an Auslander-Reiten \( s \)-triangle, thus \( \varphi_* = 0 \). Therefore, each morphism \( \varphi_* : X_* \to Y_* \) is either a monomorphism or a zero morphism. It means that \( X_* \) is a simple object in \( s \)-tri(C)/\( R_2 \).

From now on to the end of this section we assume that \( C \) is an extriangulated category with enough projectives \( \mathcal{P} \) and enough injectives \( \mathcal{I} \).

Given an \( s \)-triangle \( \delta = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{\rho} \to) \), we define the contravariant defect \( \delta^* \) and the covariant defect \( \delta_* \) by the following exact sequence of functors

\[
\begin{align*}
C(\cdot, X_1) & \xrightarrow{C(-, f_1)} C(\cdot, X_2) \xrightarrow{C(-, f_2)} C(\cdot, X_3) \to \delta^* \to 0, \\
C(X_3, \cdot) & \xrightarrow{C(f_2, \cdot)} C(X_2, \cdot) \xrightarrow{C(f_1, \cdot)} C(X_1, \cdot) \to \delta_* \to 0.
\end{align*}
\]

Example 4.7 (1) Let \( \delta = P_X = (\Omega X \xrightarrow{f_1} P \xrightarrow{f_2} X \xrightarrow{\rho} \to) \) with \( P \in \mathcal{P} \). Then \( \delta^* = (C(\cdot, [P]))(\cdot, X) \) and \( \delta_* = \text{End}(X, \cdot) \).

(2) Let \( \delta = I_X = (X \xrightarrow{f_1} I \xrightarrow{f_2} \Sigma X \xrightarrow{\rho} \to) \) with \( I \in \mathcal{I} \). Then \( \delta^* = \text{End}(\cdot, X) \) and \( \delta_* = (C(\cdot, [I]))(X, \cdot) \).

The following result gives an explanation of [6, Theorem 4.1].

**Theorem 4.8** Let \( C \) be an extriangulated category with enough projectives \( \mathcal{P} \) and enough injectives \( \mathcal{I} \).

(1) We have the following equivalences

\[
\text{s-tri}(C)/R_2 \cong \text{mod-}(C/[\mathcal{P}]) \cong (\text{mod-}(C/[\mathcal{I}])^{\text{op}})^{\text{op}}.
\]

Moreover, the equivalence \( F : \text{s-tri}(C)/R_2 \cong \text{mod-}(C/[\mathcal{P}]) \) is given by \( \delta \mapsto \delta^* \) and

the equivalence \( G : \text{s-tri}(C)/R_2 \cong (\text{mod-}(C/[\mathcal{I}])^{\text{op}})^{\text{op}} \) is given by \( \delta \mapsto \delta_* \).

(2) The abelian category \( \text{mod-}(C/[\mathcal{P}]) \) has enough projectives and enough injectives. Moreover, each projective object is of the form \( (C/[\mathcal{P}])(\cdot, X) \), and each injective object is of the form \( \text{End}(\cdot, X) \).

\( \square \) Springer
(3) The abelian category \( \text{mod-}(\mathcal{C}/[\mathcal{I}])^\text{op} \) has enough projectives and enough injectives. Moreover, each projective object is of the form \((\mathcal{C}/[\mathcal{I}]) (X, -)\), and each injective object is of the form \( \mathbb{E}(X, -) \).

Proof (1) The first assertion follows from Theorem 4.1. Assume that

\[
\delta = ( X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{-} )
\]

is an \( s \)-triangle. Recall that \( F(\delta) = \text{Coker}(\mathcal{C}/[\mathcal{P}]) (-, f_2) \) and \( \delta^* = \text{Coker}(\mathcal{C}(-, f_2)) \).
Since \( \delta^*(\mathcal{P}) = 0 \), we view \( \delta^* \) as a finitely presented \( \mathcal{C}/[\mathcal{P}] \)-module. Thus \( F(\delta) = \delta^* \).
Similarly, we have \( G(\delta) = \delta_* \).

(2) and (3) follows from (1), Proposition 4.4 and Example 4.7.

Corollary 4.9
Let \( \mathcal{C} \) be an extriangulated category with enough projectives \( \mathcal{P} \) and enough injectives \( \mathcal{I} \). Then there is a duality

\[
\Phi : \text{mod-}(\mathcal{C}/[\mathcal{P}]) \rightarrow \text{mod-}(\mathcal{C}/[\mathcal{I}])^\text{op}, \quad \delta^* \mapsto \delta_*.
\]

Moreover, by restrictions, we obtain the following two dualities

\[
\Phi : \text{proj-}(\mathcal{C}/[\mathcal{P}]) \rightarrow \text{inj-}(\mathcal{C}/[\mathcal{I}])^\text{op}, \quad (\mathcal{C}/[\mathcal{P}])(-, X) \mapsto \mathbb{E}(X, -).
\]

\[
\Phi : \text{inj-}(\mathcal{C}/[\mathcal{P}]) \rightarrow \text{proj-}(\mathcal{C}/[\mathcal{I}])^\text{op}, \quad \mathbb{E}(-, X) \mapsto (\mathcal{C}/[\mathcal{I}])(X, -).
\]

Proof It is a direct consequence of Theorem 4.8.

Corollary 4.10 ([6, Proposition 4.9])
Let \( \mathcal{C} \) be an extriangulated category with enough projectives \( \mathcal{P} \) and enough injectives \( \mathcal{I} \).

(1) There is an isomorphism between \( (\mathcal{C}/[\mathcal{P}]) (Y, X) \) and the group of natural transformations from \( \mathbb{E}(X, -) \) to \( \mathbb{E}(Y, -) \).

(2) There is an isomorphism between \( (\mathcal{C}/[\mathcal{I}]) (X, Y) \) and the group of natural transformations from \( \mathbb{E}(-, X) \) to \( \mathbb{E}(-, Y) \).

Now we have the following Auslander-Reiten duality and defect formula for extriangulated categories.

Proposition 4.11
Let \( \mathcal{C} \) be an Ext-finite extriangulated category with enough projectives \( \mathcal{P} \) and enough injectives \( \mathcal{I} \). Assume that either \( \mathcal{C}/[\mathcal{P}] \) or \( \mathcal{C}/[\mathcal{I}] \) is a dualizing \( k \)-variety. Then there is an equivalence \( \tau : \mathcal{C}/[\mathcal{P}] \cong \mathcal{C}/[\mathcal{I}] \) satisfying the following properties:

(1) \( D\mathbb{E}(-, X) \cong (\mathcal{C}/[\mathcal{P}]) (\tau^{-1} X, -) \), \( D\mathbb{E}(X, -) \cong (\mathcal{C}/[\mathcal{I}]) (-, \tau X) \).

(2) \( D\delta_* = \delta^* \tau^{-1} \), \( D\delta^* = \delta_* \tau \) for each \( s \)-triangle \( \delta \).

Proof Without loss of generality, we assume that \( \mathcal{C}/[\mathcal{I}] \) is a dualizing \( k \)-variety. The composition of \( \Phi : \text{mod-}(\mathcal{C}/[\mathcal{P}]) \rightarrow \text{mod-}(\mathcal{C}/[\mathcal{I}])^\text{op} \) and \( D : \text{mod-}(\mathcal{C}/[\mathcal{I}])^\text{op} \rightarrow \text{mod-}(\mathcal{C}/[\mathcal{I}]) \) defines an equivalence

\[
\Theta : \text{mod-}(\mathcal{C}/[\mathcal{P}]) \xrightarrow{\Phi} \text{mod-}(\mathcal{C}/[\mathcal{I}])^\text{op} \xrightarrow{D} \text{mod-}(\mathcal{C}/[\mathcal{I}]).
\]
It follows that \( \Theta((\mathcal{C}/[\mathcal{P}])(-, X)) = D\mathbb{E}(X, -) \cong (\mathcal{C}/[\mathcal{I}])(-, Y) \) for some \( Y \in \mathcal{C} \). Therefore, there is an equivalence \( \tau : \mathcal{C}/[\mathcal{P}] \cong \mathcal{C}/[\mathcal{I}] \) mapping \( X \) to \( Y \). The equivalence \( \tau \) induces an equivalence \( \tau_*^{-1} : \text{mod}-(\mathcal{C}/[\mathcal{P}]) \cong \text{mod}-(\mathcal{C}/[\mathcal{I}]), F \mapsto F\tau^{-1} \), such that \( D\Phi = \tau_*^{-1} \). Assume that \( \delta \) is an \( s \)-triangle, then \( D\Phi(\delta^*) = D\delta_* \). On the other hand, \( \tau_*^{-1}(\delta^*) = \delta^*\tau^{-1} \). Hence, we have \( D\delta_* = \delta^*\tau^{-1} \). It follows that \( D\delta^* = \delta_*\tau \). If \( \delta = I_X \), then by Example 4.7 we have \( \delta^* = \mathbb{E}(\mathcal{C}/[\mathcal{I}]) \) and \( \delta_* = (\mathcal{C}/[\mathcal{I}])I_X \). Therefore, \( D\mathbb{E}(X, -) \cong (\mathcal{C}/[\mathcal{I}])(X, -) \) since \( D\delta^* = \delta_*\tau \).

5 Application to Rigid Subcategories

In this section, we will investigate the second application of Theorem 3.2 in the case when \( \mathcal{M} \) is a rigid subcategory of \( \mathcal{C} \), that is, \( \mathbb{E}(M, M') = 0 \) for any objects \( M, M' \in \mathcal{M} \).

Let \( \mathcal{C}' \) and \( \mathcal{C}'' \) be two full subcategories of \( \mathcal{C} \). We denote by \( \text{Cocone}(\mathcal{C}', \mathcal{C}'') \) the full subcategory of \( \mathcal{C} \) of objects \( X \) admitting an \( s \)-triangle \( \xymatrix{ X \ar[r] & C' \ar[r] & C'' \ar[r] & - \to } \) with \( C' \in \mathcal{C}' \) and \( C'' \in \mathcal{C}'' \). We denote by \( \text{Cone}(\mathcal{C}', \mathcal{C}'') \) the full subcategory of objects \( X \) admitting an \( s \)-triangle \( \xymatrix{ C' \ar[r] & C'' \ar[r] & X \ar[r] & - \to } \) with \( C' \in \mathcal{C}' \) and \( C'' \in \mathcal{C}'' \).

For convenience we let \( \mathcal{M}_L = \text{Cocone}(\mathcal{M}, \mathcal{M}) \) and \( \mathcal{M}_R = \text{Cone}(\mathcal{M}, \mathcal{M}) \). If \( \mathcal{C} \) has enough projectives \( \mathcal{P} \), then we let \( \Omega \mathcal{M} = \text{Cocone}(\mathcal{P}, \mathcal{M}) \). If \( \mathcal{C} \) has enough injectives \( \mathcal{I} \), then we let \( \Sigma \mathcal{M} = \text{Cone}(\mathcal{M}, \mathcal{I}) \).

Lemma 5.1 Let \( \mathcal{C} \) be an extriangulated category and \( \mathcal{M} \) be a rigid subcategory of \( \mathcal{C} \). If \( \xymatrix{ X \ar[r]^k & M_1 \ar[r]^f & M_2 \ar[r]^-{\delta} & - \to } \) is an \( s \)-triangle with \( M_i \in \mathcal{M} \), then \( k \) is a left \( \mathcal{M} \)-approximation of \( X \).

Proof For any \( M \in \mathcal{M} \), by Lemma 2.9 we have the following exact sequence
\[
\mathbb{C}(M_2, M) \to \mathbb{C}(M_1, M) \to \mathbb{C}(X, M) \to \mathbb{E}(M_2, M) = 0.
\]

Hence, \( k \) is a left \( \mathcal{M} \)-approximation of \( X \). \( \Box \)

Lemma 5.2 Let \( \mathcal{C} \) be an extriangulated category with enough projectives \( \mathcal{P} \) and \( \mathcal{M} \) be a rigid subcategory of \( \mathcal{C} \) containing \( \mathcal{P} \). Assume that the following diagram

\[
\xymatrix{ X \ar[r]^k & M_1 \ar[r]^f & M_2 \ar[r]^-{\delta} & - \to \\
| | | | \\
X' \ar[r]^{k'} & M'_1 \ar[r]^{f'} & M'_2 \ar[r]^-{\delta'} & - \to }
\]

is a morphism of \( s \)-triangles with \( M_i, M'_i \in \mathcal{M} \). Then

1. The following statements are equivalent.
   (a) The morphism \( b \) factors through \( f' \).
   (b) The morphism \( (a, b) \) factors through some object in \( s\text{-epi}(\mathcal{M}) \).
   (c) The morphism \( g \) factors through some object in \( \mathcal{M} \).

2. The following statements are equivalent.
   (a) The morphism \( a \) factors through \( f \) in \( \mathcal{M}/[\mathcal{P}] \).
(b) The morphism \((a, b)\) factors through some object in \(\text{sp-epi}(\mathcal{M})\).

(c) The morphism \(g\) factors through some object in \(\Omega\mathcal{M}\).

Proof (1) We note that \((a) \Leftrightarrow (b)\) follows from Lemma 3.1(1).

(a)\(\Rightarrow\)(c). Assume that \(b\) factors through \(f'\). It follows that \(g\) factors through \(k\) by Lemma 2.7, which implies that \(g\) factors through \(M_1 \in \mathcal{M}\).

(c)\(\Rightarrow\)(a). Suppose that \(g\) has a factorization \(X \xrightarrow{g_1} M \xrightarrow{g_2} X'\) with \(M \in \mathcal{M}\). By Lemma 5.1, we have \(g_1\) factors through \(k\). Thus \(g\) factors through \(k\). It follows that \(b\) factors through \(f'\) by Lemma 2.7.

(2) We note that \((a) \Leftrightarrow (b)\) follows from Lemma 3.1(2).

(a)\(\Rightarrow\)(c). Suppose that there is a morphism \(p : M_2 \to M_1'\) such that \(p f = a\). Since \(C\) has enough projectives, there is an \(s\)-triangle \(\Omega M_1' \xrightarrow{a_1'} \Omega M_1 \xrightarrow{a_1} M_1' \xrightarrow{k'} \) with \(P \in \mathcal{P}\). It is easy to see that \(p f - a\) factors through \(a_1\). Assume that \(p f - a = a_1 a_2\) where \(a_2 : M_1 \to P\). By (ET4)\(\text{op}\), we have the following diagram

\[
\begin{array}{ccc}
\Omega M_1 & \xrightarrow{d'} & Y \\
\downarrow & & \downarrow \text{id} \\
\Omega M_1' & \xrightarrow{a_1'} & P \\
\downarrow c & & \downarrow a_1 \\
M_2' & \xrightarrow{c'} & M_2
\end{array}
\]

where the first row and the second column are \(s\)-triangles. It follows that \(Y \in \Omega\mathcal{M}\). Since the upper-right square of the above diagram obtained by (ET4)\(\text{op}\) is a weak pullback and \((k', a_1) (\sigma_{a_2 k}) = ak + a_1 a_2 k = pf k = 0\), there exists a morphism \(h : X \to Y\) such that \(dh = g\) and \(ch = a_2 k\). Therefore, \(g\) factors through \(Y \in \Omega\mathcal{M}\).

(c)\(\Rightarrow\)(a). Suppose that \(g\) has a factorization \(X \xrightarrow{g_1} \Omega M \xrightarrow{g_2} X'\) with \(\Omega M \in \Omega\mathcal{M}\). Then by Lemma 5.1 and (ET3) we complete the following morphism of \(s\)-triangles

\[
\begin{array}{ccc}
X & \xrightarrow{k} & M_1 \\
\downarrow g_1 & \downarrow & \downarrow a_1 \\
\Omega M & \xrightarrow{i} & P \\
\downarrow g_2 & \downarrow a_2 & \downarrow b_2 \\
X' & \xrightarrow{k'} & M_1' \\
\downarrow & \downarrow & \downarrow \\
\Omega M & \xrightarrow{\pi} & M_2 \\
\downarrow & \downarrow & \downarrow \\
\Omega M & \xrightarrow{\varphi} & M_2'
\end{array}
\]

where \(P \in \mathcal{P}\). Since \((a - a_2 a_1)k = k'(g - g_2 g_1) = 0\), there exists a morphism \(p : M_2 \to M_2'\) such that \(a - a_2 a_1 = pf\). Therefore, \(a = pf\). \(\square\)
Lemma 5.3  Let $\mathcal{C}$ be an extriangulated category with enough projectives $\mathcal{P}$ and $\mathcal{M}$ be a rigid subcategory of $\mathcal{C}$ containing $\mathcal{P}$. Then

1. $\text{s-def}(\mathcal{M})/[\text{s-epi}(\mathcal{M})] \cong \mathcal{M}_L/[\mathcal{M}]$.
2. $\text{s-def}(\mathcal{M})/[\text{sp-epi}(\mathcal{M})] \cong \mathcal{M}_L/[\Omega \mathcal{M}]$.

Proof (1) For any object $f : M_1 \to M_2$ in $\text{s-def}(\mathcal{M})$, there exists an $s$-triangle $X \xrightarrow{k} M_1 \xrightarrow{f} M_2 \xrightarrow{\delta} \Rightarrow$ where $X$ is unique under isomorphism. By (ET3)$^{\text{op}}$, the following commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{k} & M_1 \\
\downarrow g & & \downarrow a \\
X' & \xrightarrow{k'} & M'_1
\end{array}
\begin{array}{ccc}
 & & \\
\downarrow b & & \\
 & & \\
 & & \\
M'_2 & \xrightarrow{f'} & M_2
\end{array}
$$

whose rows are $s$-triangles can be completed to a morphism of $s$-triangles. The morphism $g : X \to X'$ is not unique in general. Assume that $g' : X \to X'$ is another morphism such that $(g', a, b)$ is a morphism of $s$-triangles. Then $(g - g', 0, 0)$ is also a morphism of $s$-triangles. Lemma 2.7 implies that $g - g'$ factors through $k$, that is, $g - g'$ factors through $M_1 \in \mathcal{M}$. It follows that $g = g'$ in $\mathcal{M}_L/[\mathcal{M}]$.

Hence the assignment $(M_1 \xrightarrow{f} M_2) \mapsto (a, b)$ defines a well-defined functor $F : \text{s-def}(\mathcal{M}) \to \mathcal{M}_L/\mathcal{M}$. Note that $F$ is dense. The functor $F$ is full by Lemma 5.1 and (ET3). Lemma 5.2(1) implies that $F$ induces an equivalence $\text{s-def}(\mathcal{M})/\mathcal{R} \cong \mathcal{M}_L/\mathcal{M}$.

(2) For any $X \in \mathcal{M}_L$, we fix an $s$-triangle $X \xrightarrow{k} M_1 \xrightarrow{f} M_2 \xrightarrow{\delta} \Rightarrow$ with $M_1, M_2 \in \mathcal{M}$. Assume that $g : X \to X'$ is a morphism in $\mathcal{M}_L$. Then by Lemma 5.1 and (ET3) there exist two morphisms $a : M_1 \to M'_1$ and $b : M_2 \to M'_2$ such that the following diagram is a morphism of $s$-triangles

$$
\begin{array}{ccc}
X & \xrightarrow{k} & M_1 \\
\downarrow g & & \downarrow a \\
X' & \xrightarrow{k'} & M'_1
\end{array}
\begin{array}{ccc}
 & & \\
\downarrow b & & \\
 & & \\
 & & \\
M'_2 & \xrightarrow{f'} & M_2
\end{array}
$$

where $M'_1, M'_2 \in \mathcal{M}$. Suppose that the morphisms $a' : M_1 \to M'_1$ and $b' : M_2 \to M'_2$ satisfy that $(g, a', b')$ is also a morphism of $s$-triangles. Then $(0, a - a', b - b')$ is a morphism of $s$-triangles. It follows that $(a - a', b - b')$ factors through some object in $\text{sp-epi}(\mathcal{M})$ by Lemma 5.2(2). We have $(a, b) = (a', b')$ in $\text{s-def}(\mathcal{M})/[\text{sp-epi}(\mathcal{M})]$. Therefore, the assignment $X \mapsto (M_1 \xrightarrow{f} M_2), g \mapsto (a, b)$ defines a well-defined functor $G : \mathcal{M}_L \to \text{s-def}(\mathcal{M})/[\text{sp-epi}(\mathcal{M})].$

It is easy to see that $G$ is full and dense. Lemma 5.2(2) implies that $G$ induces an equivalence $\mathcal{M}_L/[\Omega \mathcal{M}] \cong \text{s-def}(\mathcal{M})/[\text{sp-epi}(\mathcal{M})]$. \qed

Theorem 5.4  Let $\mathcal{C}$ be an extriangulated category and $\mathcal{M}$ be a rigid subcategory of $\mathcal{C}$.

1. If $\mathcal{C}$ has enough projectives $\mathcal{P}$ and $\mathcal{M}$ contains $\mathcal{P}$, then $\mathcal{M}_L/[\mathcal{M}] \cong \text{mod}(\mathcal{M}/[\mathcal{P}])$

and $\mathcal{M}_L/[\Omega \mathcal{M}] \cong (\text{mod}(\mathcal{M}/[\mathcal{P}])^{\text{op}})^{\text{op}}$.
(2) If $\mathcal{C}$ has enough injectives $\mathcal{I}$ and $\mathcal{M}$ contains $\mathcal{I}$, then $\mathcal{M}_R / [\Sigma \mathcal{M}] \cong \text{mod-}(\mathcal{M}/[\mathcal{I}])$ and $\mathcal{M}_R / [\mathcal{M}] \cong (\text{mod-}(\mathcal{M}/[\mathcal{I}])^{\text{op}})^{\text{op}}$.

Proof We only prove (1). It follows from Theorem 4.1 and Lemma 5.3.

Definition 5.5 ([11, Definition 5.3]) Let $\mathcal{C}$ be an extriangulated category with enough projectives and enough injectives. A full subcategory $\mathcal{M}$ is called $n$-cluster tilting for some integer $n \geq 2$, if it satisfies the following conditions.

1. $\mathcal{M}$ is functorially finite in $\mathcal{C}$.
2. $X \in \mathcal{M}$ if and only if $\mathcal{E}^i(X, \mathcal{M}) = 0$ for $i \in \{1, 2, \ldots, n-1\}$.
3. $X \in \mathcal{M}$ if and only if $\mathcal{E}^i(\mathcal{M}, X) = 0$ for $i \in \{1, 2, \ldots, n-1\}$.

In particular, a 2-cluster tilting subcategory of $\mathcal{C}$ is simply called cluster tilting subcategory.

Assume that $\mathcal{M}$ is an $n$-cluster tilting subcategory. Define

\[
\mathcal{M}^{\perp_{n-2}} = \{ X \in \mathcal{C} | \mathcal{E}^i(X, \mathcal{M}) = 0, i \in \{1, 2, \ldots, n-2\} \}
\]

and

\[
\mathcal{M}^{\perp_{n-2}} = \{ X \in \mathcal{C} | \mathcal{E}^i(\mathcal{M}, X) = 0, i \in \{1, 2, \ldots, n-2\} \}.
\]

Proposition 5.6 Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of an extriangulated category $\mathcal{C}$, then $\mathcal{M}_L = \mathcal{M}^{\perp_{n-2}}$ and $\mathcal{M}_R = \mathcal{M}^{\perp_{n-2}}$.

Proof We only prove the first equation. Let $X \in \mathcal{M}^{\perp_{n-2}}$ and $X \xrightarrow{x} I \xrightarrow{y} C - \delta \Rightarrow$ be an $s$-triangle with $I \in \mathcal{I}$. Assume that $f : X \to M$ is a left $\mathcal{M}$-approximation of $X$. By Lemma 2.8, we have the following morphism of $s$-triangles

\[
\begin{array}{c}
X \xrightarrow{x} I \xrightarrow{y} C - \delta \\
\downarrow f \quad \downarrow g \\
M \xrightarrow{x'} C' \xrightarrow{y'} C - \delta \Rightarrow
\end{array}
\]

such that $X \xrightarrow{(f, g)} M \oplus I \xrightarrow{(x', -g)} C' \xrightarrow{y' + \delta} \Rightarrow$ is an $s$-triangle. We claim that $C' \in \mathcal{M}$, thus $X \in \mathcal{M}_L$. In fact, for each $M' \in \mathcal{M}$, by Lemma 2.10 we have the following exact sequence

\[
\mathcal{E}(C', M') \to \mathcal{E}^i(M \oplus I, M') \to \mathcal{E}(X, M') \to \mathcal{E}^i(C', M') \to \mathcal{E}(M \oplus I, M') \to \mathcal{E}(X, M') \to \cdots \to \mathcal{E}^i(M \oplus I, M') \to \mathcal{E}^i(X, M') \to \mathcal{E}^i+1(C', M') \to \mathcal{E}^i+1(M \oplus I, M') \to \cdots.
\]

Since $\mathcal{E}^i(M \oplus I, M') = \mathcal{E}^i+1(M \oplus I, M') = 0$, we get $\mathcal{E}^i(X, M') = \mathcal{E}^i+1(C', M')$ for $i \in \{1, 2, \ldots, n-2\}$. Thus, as $X \in \mathcal{M}^{\perp_{n-2}}$, we have $\mathcal{E}^i(X, M') = 0$ if $2 \leq i \leq n-1$. Noting that $f : X \to M$ is a left $\mathcal{M}$-approximation, we have $\mathcal{E}(C', M') = 0$. Therefore $C' \in \mathcal{M}$.

Conversely, suppose that $X \in \mathcal{M}_L$. There exists an $s$-triangle

\[
X \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 - \delta \Rightarrow
\]

with $M_1, M_2 \in \mathcal{M}$. For any $M \in \mathcal{M}$, we have an exact sequence

\[
0 = \mathcal{E}^i(M_1, M) \to \mathcal{E}^i(X, M) \to \mathcal{E}^i+1(M_2, M) = 0
\]

for $i \in \{1, 2, \ldots, n-2\}$. Therefore, $X \in \mathcal{M}^{\perp_{n-2}}$.

 Springer
Corollary 5.7 Let \( \mathcal{C} \) be an extriangulated category with enough projectives \( \mathcal{P} \) and enough injectives \( \mathcal{I} \). If \( \mathcal{M} \) is a cluster tilting subcategory of \( \mathcal{C} \), then

1. \( \mathcal{C}/[\mathcal{M}] \cong \text{mod-}(\mathcal{M}/[\mathcal{P}]) \cong (\text{mod-}(\mathcal{M}/[\mathcal{I}])^{\text{op}})^{\text{op}} \).
2. \( \mathcal{C}/[\mathcal{OM}] \cong (\text{mod-}(\mathcal{M}/[\mathcal{P}])^{\text{op}})^{\text{op}} \).
3. \( \mathcal{C}/[\Sigma \mathcal{M}] \cong \text{mod-}(\mathcal{M}/[\mathcal{I}]) \).

Remark 5.8 Let \( \mathcal{M} \) be a cluster tilting subcategory of an extriangulated category \( \mathcal{C} \). Then the fact that the quotient category \( \mathcal{C}/[\mathcal{M}] \) is abelian follows from the theory of cotorsion pairs on extriangulated categories or that of one-sided triangulated categories; see [11, Theorem 3.2] and [5, Theorem 3.3].

Acknowledgements The author would like to thank the anonymous referee for helpful comments and suggestions to improve this paper.

Data availability This paper has no associated data.

References

1. Buan, A.B., Marsh, R., Reiten, I.: Cluster-tilted algebras. Trans. Amer. Math. Soc. 359(1), 323–332 (2007)
2. Demonet, L., Liu, Y.: Quotients of exact categories by cluster tilting subcategories as module categories. J. Pure Appl. Algebra 217(12), 2282–2297 (2013)
3. Eiriksson, Ö.: From submodule categories to the stable Auslander algebra. J. Algebra 486, 98–118 (2017)
4. Gentle, R.: A study of the sequence category. University of British Columbia (1982)
5. He, J., Zhou, P.: Abelian quotients of extriangulated categories, Proc. Indian Acad. Sci. Math. Sci. 129(4), Art. 61, 11 (2019)
6. Iyama, O., Nakaoka, H., Palu, Y.: Auslander-Reiten theory in extriangulated categories, arXiv:1805.03776v1 (2018)
7. Iyama, O., Yoshino, Y.: Mutation in triangulated categories and rigid Cohen-Macaulay modules. Invent. Math. 172(1), 117–168 (2008)
8. Keller, B., Reiten, I.: Cluster-tilted algebras are Gorenstein and stably Calabi-Yau. Adv. Math. 211(1), 123–151 (2007)
9. Koenig, S., Zhu, B.: From triangulated categories to abelian categories: cluster tilting in a general framework. Math. Z. 258(1), 133–160 (2008)
10. Lin, Z.: Abelian quotients of category of short exact sequences. J. Algebra 551, 61–92 (2020)
11. Liu, Y., Nakaoka, H.: Hearts of twin cotorsion pairs on extriangulated categories. J. Algebra 528, 96–149 (2019)
12. Liu, Y., Zhou, P.: Abelian categories arising from cluster tilting subcategories. Appl. Categ. Struct. 28, 575–594 (2020)
13. Liu, Y., Zhou, P.: Abelian categories arising from cluster-tilting subcategories II, Quotient Functors. Proc. Roy. Soc. Edinburgh Sect. A 150(6), 2721–2756 (2020)
14. Neeman, A.: Triangulated Categories Annals of Mathematics Studies, vol. 148. Princeton University Press, Princeton (2001)
15. Nakaoka, H., Palu, Y.: Extriangulated categories, Hovey twin cotorsion pairs and model structures. Cah. Topol. Géom. Diffr. Catég. 60(2), 117–193 (2019)
16. Ringel, C.M., Zhang, P.: From submodule categories to preprojective algebras. Math. Z. 278(1-2), 55–73 (2014)
17. Zhou, P., Zhu, B.: Triangulated quotient categories revisited. J. Algebra 502, 196–232 (2018)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.