Nontrivial example of the composition of the brane product and coproduct on Gorenstein spaces

Shun Wakatsuki

Abstract
We give an example of a space with the nontrivial composition of the brane product and the brane coproduct, which we introduced in a previous article.

1 Introduction
Chas and Sullivan [CS99] introduced the loop product \( \mu : H_\ast(LM \times LM) \to H_{\ast-m}(LM) \) on the homology of the free loop space \( LM = \text{Map}(S^1, M) \) of a connected closed oriented manifold \( M \) of dimension \( m \). Constructing a 2-dimensional topological quantum field theory without counit, Cohen and Godin [CG04] generalized this product to other string operations, including the loop coproduct \( \delta : H_\ast(LM) \to H_{\ast-m}(LM \times LM) \). But Tamanoi [Tam10] showed that any string operation corresponding to a positive genus surface is trivial. In particular, the composition \( \mu \circ \delta \) is trivial. There are many attempts to find nontrivial and interesting operations.

Félix and Thomas [FT09] generalized the loop product and coproduct to the case \( M \) is a Gorenstein space. A Gorenstein space is a generalization of a manifold in the point of view of Poincaré duality. For example, connected closed oriented manifolds, classifying spaces of connected Lie groups, and their Borel constructions are Gorenstein spaces. Moreover, any 1-connected space \( M \) with \( \bigoplus_n \pi_n(M) \otimes \mathbb{Q} \) of finite dimension is a Gorenstein space. In spite of this huge generalization, string operations remain to tend to be trivial. For example, the loop product \( \mu \) is trivial over a field of characteristic zero for the classifying space of a connected Lie group [FT09 Theorem 14].

**Problem 1.1** ([FT09]). *Is there a Gorenstein space such that the composition \( \mu \circ \delta \) is nontrivial?*

This is the Gorenstein counterpart of the above result due to Tamanoi. But such an example is not found.

Sullivan and Voronov [CHV06 Part I, Chapter 5] generalized the loop product to the sphere space \( S^k M = \text{Map}(S^k, M) \) for \( k \geq 1 \). This product is called the brane product.
The brane coproduct, a generalization of the loop coproduct to the sphere spaces, is constructed by the author [Wak] in the case where the rational homotopy group $\bigoplus_n \pi_n(M) \otimes \mathbb{Q}$ is of finite dimension. In the construction, we assume the “finiteness” of the dimension of the $(k - 1)$-fold based loop space $\Omega^{k-1}M$ as a Gorenstein space. Moreover, the product and the coproduct were generalized to the mapping spaces from manifolds, by means of connected sums.

Here we briefly review the brane product and coproduct. See Section 2 for details. Let $\mathbb{K}$ be a field of characteristic zero, $S$ an oriented manifold of dimension $k$ with two disjoint base points, and $M$ a $k$-connected $m$-dimensional $\mathbb{K}$-Gorenstein space with $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ of finite dimension. Denote the “connected sum” and “wedge sum” of $S$ with itself along the two base points by $S \#$ and $S \vee$, respectively. Note that, by the definition of the connected sum, we have the canonical inclusion $S^{k-1} \hookrightarrow S \#$ and the quotient map $q: S \# \to (S \#)/S^{k-1} = S \vee$. Similarly we have $S^0 = \text{pt} \biguplus \text{pt} \hookrightarrow S$ and $p: S \to S/S^0 = S \vee$. Hence we have the following diagram

\[
\begin{array}{ccc}
P & \rightarrow & q \\
S & & S \vee \\
& \downarrow p & \\
S \# & & \\
\end{array}
\]

and its dual

\[
M^S \leftarrow M^S \vee \xrightarrow{\text{comp}} M^{S \#}, \tag{1.2}
\]

where the maps incl and comp are induced by $p$ and $q$.

Using this diagram, we can construct two operations, $S$-brane product $\mu_S$ and coproduct $\delta_S$:

\[
\begin{align*}
\mu_S &: H_*(M^S) \to H_{*-m}(M^{S \#}) \\
\delta_S &: H_*(M^{S \#}) \to H_{*-m}(M^S).
\end{align*}
\]

Note that, if $T$ and $U$ are oriented $k$-manifolds and we take $S = T \coprod U$ with one base point on $T$ and the other on $U$, then $\mu_S$ and $\delta_S$ have the form

\[
\begin{align*}
\mu_{T \coprod U} &: H_*(M^T \times M^U) \to H_{*-m}(M^{T \#U}) \\
\delta_{T \coprod U} &: H_*(M^{T \#U}) \to H_{*-m}(M^T \times M^U).
\end{align*}
\]

Moreover, if we take $T = U = S^1$, then $\mu_{S^1 \coprod S^1}$ and $\delta_{S^1 \coprod S^1}$ coincide with the usual loop product and coproduct, respectively. Hence the $S$-brane product and coproduct are generalizations of the loop product and coproduct.
In this article, we give examples that the composition \( \mu \circ \delta \) of the brane product and the brane coproduct is nontrivial.

**Theorem 1.3.** Let \( k \) be a positive even integer. Consider the case \( S = S^k \) (and hence \( S# = S^{k-1} \times S^1 \)). Let \( M \) be the Eilenberg-MacLane space \( K(\mathbb{Z}, 2n) \) with \( n > k/2 \). Then the composition \( \mu_{S^k} \circ \delta_{S^k} \) of the \( S^k \)-brane product

\[
\mu_{S^k} : H_*(\text{Map}(S^k, M)) \to H_{*+2n-1}(\text{Map}(S^{k-1} \times S^1, M))
\]

and the \( S^k \)-brane coproduct

\[
\delta_{S^k} : H_*\big(\text{Map}(S^{k-1} \times S^1, M)\big) \to H_{*-2n+k-1}(\text{Map}(S^k, M))
\]

is nontrivial.

This gives an answer to Problem 1.1 in the context of brane operations. Here it should be remarked that, the composition \( \mu_{S^k} \circ \delta_{S^k} \) corresponds to a cobordism without “genus”. In fact, if we take \( k = 1 \), the composition \( \mu_{S^1} \circ \delta_{S^1} \) is equal to the composition \( \delta \circ \mu \), not \( \mu \circ \delta \), of the loop product \( \mu \) and coproduct \( \delta \).

For a connected Lie group \( G \) and its closed connected subgroup \( H \), the homogeneous space \( M = G/H \) satisfies the assumption if the canonical map \( \pi_* (H) \otimes \mathbb{K} \to \pi_* (G) \otimes \mathbb{K} \) is not surjective.

By Theorem 1.3 and Theorem 1.4, we have the following corollary.

**Corollary 1.5.** Let \( k \) be a positive even integer, and \( M \) a \( k \)-connected (Gorenstein) space with \( \bigoplus_n \pi_n(M) \otimes \mathbb{K} \) of finite dimension. Assume that the minimal Sullivan model of \( M \) is pure and has at least one generator of odd degree. Then the \( S^k \)-brane coproduct is trivial for \( M \).
Sullivan model of $M$ is pure. Then the composition $\mu_S \circ \delta_S$ is nontrivial if and only if $M$ is a finite product $\prod K(\mathbb{Z}, 2n_i)$ of Eilenberg-MacLane spaces of even degrees.

Section 2 contains brief background material on brane operations. In Section 3 we construct rational models of the $S^k$-brane product and coproduct, which gives a method of computation. Next we review explicit constructions of the shriek maps in Section 4, which is necessary to accomplish the computation by the above models. Finally, in Section 5 we prove Theorem 1.3 and Theorem 1.4 using the above models.

### Contents

1. Introduction

2. Brane operations for the mapping space from manifolds

3. Models of the brane operations
   - 3.a Models of spaces
   - 3.b Models of operations

4. Explicit construction of shriek maps
   - 4.a Construction of $\Delta_i$
   - 4.b Construction of $c_i$

5. Proof of Theorem 1.3 and Theorem 1.4

### 2 Brane operations for the mapping space from manifolds

In this section, we review the constructions of the $S$-brane product and coproduct from [Wak]. Since the cochain models work well for fibrations, we define the duals of the $S$-brane product and coproduct at first, and then we define the $S$-brane product and coproduct as the duals of them.

Let $\mathbb{K}$ be a field of characteristic zero. This assumption enables us to make full use of rational homotopy theory. For the basic definitions and theorems on homological algebra and rational homotopy theory, we refer the reader to [FHT01].

**Definition 2.1** ([FHT88]). Let $m \in \mathbb{Z}$ be an integer.

1. An augmented dga (differential graded algebra) $(A, d)$ is called a ($\mathbb{K}$-) Gorenstein algebra of dimension $m$ if

   $$\dim \text{Ext}^l_A(\mathbb{K}, A) = \begin{cases} 
   1 & \text{(if } l = m) \\
   0 & \text{(otherwise)},
   \end{cases}$$
where the field \( \mathbb{K} \) and the dga \((A,d)\) are \((A,d)\)-modules via the augmentation map and the identity map, respectively.

2. A path-connected topological space \( M \) is called a \((\mathbb{K})\)-Gorenstein space of dimension \( m \) if the singular cochain algebra \( C^*(M) \) of \( M \) is a Gorenstein algebra of dimension \( m \).

Here, \( \text{Ext}_A(L,N) \) is defined using a semifree resolution of \((L,d)\) over \((A,d)\), for a dga \((A,d)\) and \((A,d)\)-modules \((L,d)\) and \((N,d)\). \( \text{Tor}_A(L,N) \) is defined similarly. See [FHT01, Section 1] for details of semifree resolutions.

An important example of a Gorenstein space is given by the following proposition.

**Proposition 2.2** ([FHT88, Proposition 3.4]). A 1-connected topological space \( M \) is a \((\mathbb{K})\)-Gorenstein space if \( \bigoplus_n \pi_n(M) \otimes \mathbb{K} \) is finite dimensional. Similarly, a Sullivan algebra \((\bigwedge V,d)\) is a Gorenstein algebra if \( V \) is finite dimensional.

Note that this proposition is proved only for \( \mathbb{Q}\)-Gorenstein spaces in [FHT88], but the proof can be applied for any field \( \mathbb{K} \) of characteristic zero and Sullivan algebras over \( \mathbb{K} \).

We use the following theorem to construct the brane operations.

**Theorem 2.3** ([FT09, Theorem 12] for \( k = 1 \), [Wak, Corollary 3.2] for \( k \geq 2 \)). Let \( M \) be a \((k-1)\)-connected (and 1-connected) space with \( \bigoplus_n \pi_n(M) \otimes \mathbb{K} \) of finite dimension, for \( k \geq 1 \). Then we have an isomorphism

\[
\text{Ext}^*_C(C^*(M), C^*(S^{k-1}M)) \cong H^{*-\bar{m}}(M),
\]

where \( \bar{m} \) is the dimension of \( \Omega^{k-1}M \) as a Gorenstein space.

Now we can define the \( S \)-brane coproduct as follows. Let \( S \) be an oriented manifold with two distinct base points, \( M \) a \( k \)-connected \( m \)-dimensional \( \mathbb{K} \)-Gorenstein space with \( \bigoplus_n \pi_n(M) \otimes \mathbb{K} \) of finite dimension. Consider the diagram, extending (1.2),

\[
\begin{array}{ccc}
M^S & \xleftarrow{\text{comp}} & M^S \\
\text{incl} & & \text{incl} \\
S^{k-1}M & \xrightarrow{c} & M.
\end{array}
\]
where the map EM is the Eilenberg-Moore map, which is an isomorphism since S^{k-1}M is 1-connected (see [FHT01, Theorem 7.5] for details). By this, we define the dual of the S-brane coproduct as the composition

\[ \delta^\lor: H^\ast(M^S) \xrightarrow{\text{incl}^\lor} H^{*+\bar{m}}(M^S^\lor) \xrightarrow{\text{comp}^\lor} H^{*+\bar{m}}(M^S^\lor). \]

Similarly we can define the S-brane product using the generator \( \Delta \in \text{Ext}_{C^\ast(M^2)}^{m}(C^\ast(M),C^\ast(M^2)) \) and the diagram

\[ M^S \xleftarrow{\text{incl}} M^S^\lor \xrightarrow{\text{comp}} M^S^\lor \]

The diagram above is as follows:

\[ \begin{array}{ccc}
M^S & \xleftarrow{\text{incl}} & M^S^\lor \\
\downarrow & & \downarrow \\
M \times M & \xrightarrow{\Delta} & M.
\end{array} \]

Note that, for the brane product and the loop coproduct, we can replace the assumption \( \bigoplus_n \pi_n(M) \otimes \mathbb{K} \) is of finite dimension with the assumption \( \pi_\ast(M) \otimes \mathbb{K} \) is of finite type by using [PT09, Theorem 12] instead of Theorem 2.3.

### 3 Models of the brane operations

In this section, we consider the case \( S = S^k \) and give rational models of the \( S^k \)-brane operations, for an integer \( k \geq 1 \). In Section 5, we will prove Theorem 1.3 and Theorem 1.4 using these models.

Naito [Nai13] constructed a rational model of the duals of the loop product and coproduct in terms of Sullivan models using the torsion functor description of [KMN15]. The author [Wak] constructed a rational model of the duals of the brane product and coproduct as a generalization of it. Here we give a rational model of the \( S^k \)-brane operations by a similar method.

#### 3.a Models of spaces

Let \( M \) be a \( k \)-connected space with \( \bigoplus_n \pi_n(M) \otimes \mathbb{K} \) of finite dimension. Take a Sullivan model \((\wedge V, d)\) of \( M \) with \( V^{\leq k} = 0 \) and \( \text{dim} V < \infty \). For simplicity, we sometimes denote \((\wedge V, d)\) by \( M \). Denote \((S^k)_\# = S^{k-1} \times S^1 \) by \( T^{(k)} \) and \((S^k)_\lor = (S^{k-1} \times S^1)/S^{k-1} \) by \( U^{(k)} \). For an integer \( l \in \mathbb{Z} \), let \( s^l V \) be a graded module defined by \((s^l V)^n = V^{n+l}\) and \( s^l v \) denotes the element in \( s^l V \) corresponding to an element \( v \in V \). Here we recall models of mapping spaces from the interval, sphere, and disk.

(3.1) Consider \( s \) as an derivation on the algebra \( \wedge V^\otimes 2 \otimes \wedge V \) with \( s \circ s = 0 \). Define a derivation \( d \) on the algebra by

\[ d(sv) = 1 \otimes v - v \otimes 1 - \sum_{i=1}^{\infty} \frac{(sd)^i}{i!} (v \otimes 1), \]
inductively. Denote the dga \((\wedge V^\otimes 2 \otimes \wedge s V, d)\) by \(\mathcal{M}(I)\). This is a Sullivan model of the path space \(M^I (\simeq \bar{M})\). Moreover, define a map \(\bar{\varepsilon}: \mathcal{M}(I) \to \mathcal{M}\) by \(\bar{\varepsilon}(v \otimes 1) = \bar{\varepsilon}(1 \otimes v) = v\) and \(\bar{\varepsilon}(sv) = 0\) for \(v \in V\). Then it is a relative Sullivan model (resolution) of the product map \(\wedge V^\otimes 2 \wedge s V\). See [PHT10, Section 15 (c)] or [Wak16, Appendix A] for details.

(3.2) Assume \(k \geq 2\). Define derivations \(s^{(k-1)}\) and \(d\) on the graded algebra \(\wedge V \otimes \wedge s^{k-1} V\) by

\[
\begin{align*}
  s^{(k-1)}(v) &= s^{k-1}v, \quad s^{(k-1)}(s^{k-1}v) = 0, \\
  d(v) &= dv, \quad d(s^{k-1}v) = (-1)^{k-1}s^{(k-1)}dv.
\end{align*}
\]

Denote the dga \(\wedge V \otimes \wedge s^{k-1} V\) by \(\mathcal{M}(S^{k-1})\). This is a Sullivan model of the space \(M^{S^{k-1}}\). See [Wak, Section 5] for details.

(3.3) Assume \(k \geq 2\). Define derivations \(s^{(k)}\) and \(d\) on the graded algebra \(\wedge V \otimes \wedge s^{k-1} V \otimes \wedge s^k V\) by

\[
\begin{align*}
  s^{(k)}(v) &= s^k v, \quad s^{(k)}(s^{k-1}v) = s^{(k)}(s^k v) = 0, \\
  d(v) &= dv, \quad d(s^{k-1}v) = d(s^k v), \quad d(s^k v) = s^{k-1}v + (-1)^k s^{(k)}dv.
\end{align*}
\]

Denote the dga \(\wedge V \otimes \wedge s^{k-1} V \otimes \wedge s^k V\) by \(\mathcal{M}(D^k)\). This is a Sullivan model of the space \(M^{D^k} (\simeq \bar{M})\). Moreover, define a map \(\bar{\varepsilon}: \mathcal{M}(D^k) \to \mathcal{M}\) by \(\bar{\varepsilon}(v) = v\), \(\bar{\varepsilon}(s^{k-1}v) = \bar{\varepsilon}[k]v = 0\) for \(v \in V\). Then it is a relative Sullivan model (resolution) of the map \(\varepsilon: \mathcal{M}(S^{k-1}) \to \mathcal{M}\), where \(\varepsilon(v) = v\) and \(\varepsilon(s^{k-1}v) = 0\). In particular, \(\bar{\varepsilon}\) is a quasi-isomorphism. See [Wak, Section 5] for details.

Next we construct models of mapping spaces which appear in the definition of brane operations, using the above models.

(3.4) Since \(M^{T^{(k)}} = (M^{S^{k-1}})^{S^1}\), we have a Sullivan model \(\mathcal{M}(T^{(k)}) = (\wedge V \otimes \wedge s^{k-1} V \otimes \wedge s V \otimes \wedge s s^{k-1} V, d)\) of \(M^{T^{(k)}}\) iterating the construction in (3.2).

(3.5) Since \(U^{(k)}\) is homotopy equivalent to \(S^k \vee S^1\), the mapping space \(M^{U^{(k)}}\) is homotopy equivalent to \(M^{S^k} \times_M M^{S^1}\), and hence we have a Sullivan model \(\mathcal{M}(U^{(k)}) = (\wedge V \otimes \wedge s V, d) \otimes (\wedge V \otimes \wedge s V, d)\).

### 3.b Models of operations

Here we give a model of the \(S^k\)-brane product and coproduct in a similar way to [Nai13] and [Wak].

First we give a model of the \(S^k\)-brane coproduct. Recall that the dual \(\delta^V_{S^k}\) of the \(S^k\)-brane coproduct is the composition

\[
\delta^V_{S^k}: H^*(M^{S^k}) \xrightarrow{\text{incl}^*} H^*[m] \xrightarrow{\text{comp}_*} H^{*+[m]}(M^{U^{(k)}}) \xrightarrow{\text{comp}_*} H^{*+[m]}(M^{T^{(k)}}).
\]
First the map $\text{incl}^*: H^*(M^{S^k}) \to H^*+(M^{U(k)})$ is induced by the canonical inclusion $\mathcal{M}(S^k) \to \mathcal{M}(U(k))$, which we also denote by $\text{incl}^*$. Next the map $\text{comp}_1: H^*+(M^{U(k)}) \to H^*+(M^{T(k)})$ is computed as follows. Let

$$\gamma \in \text{Hom}_{\mathcal{M}(S^{k-1})}(\mathcal{M}(D^k), \mathcal{M}(S^{k-1}))$$

be a representative of the nontrivial element (see Theorem 2.3)

$$c! \in \text{Ext}_{\mathcal{M}(S^{k-1})}^m(C^*(M), C^*(S^{k-1} M))$$

$$\cong H^m(\text{Hom}_{\mathcal{M}(S^{k-1})}(\mathcal{M}(D^k), \mathcal{M}(S^{k-1}))).$$

Then the map $\text{Tor}_1(c!, \text{id}): \text{Tor}_{\mathcal{M}(S^{k-1})}^*(C^*(M), C^*(M^{S^2})) \to \text{Tor}_{\mathcal{M}(S^{k-1})}^{*+m}(C^*(S^{k-1} M), C^*(M^{S^2}))$ is induced by the cochain map

$$\gamma \otimes \text{id}: \mathcal{M}(D^k) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^k) \to \mathcal{M}(S^{k-1}) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^k),$$

since $\mathcal{M}(D^k)$ is a resolution of $\mathcal{M}$ over $\mathcal{M}(S^{k-1})$. The map $\text{comp}_1$ is computed by this combined with the quasi-isomorphism

$$\tilde{\epsilon} \otimes \text{id}: \mathcal{M}(D^k) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^k) \cong \mathcal{M} \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^k). \quad (3.6)$$

Hence the dual of the $S^k$-brane coproduct is induced by the composition

$$\mathcal{M}(S^k) \xrightarrow{\text{incl}} \mathcal{M}(U(k)) \cong \mathcal{M} \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^k)$$

$$\xleftarrow{\tilde{\epsilon} \otimes \text{id}} \mathcal{M}(D^k) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^k)$$

$$\xrightarrow{\gamma \otimes \text{id}} \mathcal{M}(S^{k-1}) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^k) \cong \mathcal{M}(T^k). \quad (3.7)$$

Similarly, the dual of the $S^k$-brane product is induced by the composition

$$\mathcal{M}(T^k) \xrightarrow{\text{comp}^*} \mathcal{M}(U^k) \cong \mathcal{M} \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(S^k)$$

$$\xleftarrow{\tilde{\epsilon} \otimes \text{id}} \mathcal{M}(I) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(S^k)$$

$$\xrightarrow{\eta \otimes \text{id}} \mathcal{M} \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(S^k)$$

$$\cong \mathcal{M}(I) \otimes_{\mathcal{M}} \mathcal{M}(S^k) \xrightarrow{\tilde{\epsilon} \otimes \text{id}} \mathcal{M} \otimes_{\mathcal{M}} \mathcal{M}(S^k) \cong \mathcal{M}(S^k). \quad (3.8)$$

Here $\eta \in \text{Hom}_{\mathcal{M}(S^{k-1})}(\mathcal{M}(I), \mathcal{M})$ is a representative of the nontrivial element $\Delta_1 \in \text{Ext}^m_{\mathcal{M}(S^{k-1})}(C^*(M), C^*(M^2))$ and $\text{comp}^*: \mathcal{M}(T^k) \to \mathcal{M}(U^k)$ is the canonical quotient map.

### 4 Explicit construction of shriek maps

Models of $S^k$-brane operations are constructed in Section 3 using the representatives of the shriek maps $\gamma$ and $\eta$. They are constructed by Theorem 2.3 which only states the existence of the shriek maps. In this section, we recall methods to construct shriek maps explicitly from [Nai13], [Wak16] and [Wak].

Recall the definition of a pure Sullivan algebra. Here we denote $V^{even} = \bigoplus_n V^{2n}$ and $V^{odd} = \bigoplus_n V^{2n+1}$.
Definition 4.1 (c.f. [FHT01, Section 32]). A Sullivan algebra \((\wedge V, d)\) with \(\dim V < \infty\) is called pure if \(d(V^{\text{even}}) = 0\) and \(d(V^{\text{odd}}) \subset \wedge V^{\text{even}}\).

In the rest of this section, let \((\wedge V, d)\) be a pure minimal Sullivan algebra, \(\{x_1, \ldots, x_p\}\) a basis of \(V^{\text{even}}\), and \(\{y_1, \ldots, y_q\}\) a basis of \(V^{\text{odd}}\).

4.a Construction of \(\Delta_t\)

Here we recall the description of \(\Delta_t\) in [Wak16], which is a generalization of that of Naito [Nai13]. Note that, although the description holds if the Sullivan model \((\wedge V, d)\) is semi-pure (see [Wak16, Definition 1.5] for the definition), we only refer and use it in the case \((\wedge V, d)\) is pure.

Proposition 4.2 ([Wak16, Theorem 5.6 (2)]). Take \((\wedge V \otimes \wedge V \otimes \wedge V, \wedge V \otimes \wedge V \otimes \wedge V) = M(I)\) as in (3.1). If a cocycle \(\eta \in \text{Hom} \wedge V \otimes \wedge V \otimes \wedge V \otimes \wedge V \otimes \wedge V\) satisfies
\[
\eta(s x_1 \cdots s x_p) = (1 \otimes y_1 - y_1 \otimes 1) \cdots (1 \otimes y_q - y_q \otimes 1),
\]
then we have
\[
[\eta] \neq 0 \in H^* \text{Hom} \wedge V \otimes \wedge V \otimes \wedge V \otimes \wedge V \otimes \wedge V) \cong \text{Ext} \wedge V \otimes \wedge V \otimes \wedge V \otimes \wedge V \otimes \wedge V.
\]

This proposition gives a construction of the map \(\Delta_t\).

4.b Construction of \(c_t\)

Next we recall the description of \(c_t\) in [Wak]. The following proposition gives it completely when \(k\) is even.

Proposition 4.3 ([Wak Proposition 6.2]). Assume that \(k\) is even. Define an element
\[
\gamma \in \text{Hom} \mathcal{M}(S^{k-1}) \mathcal{M}(D^k), \mathcal{M}(S^{k-1}))
\]
by \(\gamma(s^k y_1 \cdots s^k y_q) = s^{k-1} x_1 \cdots s^{k-1} x_p\) and \(\gamma(s^k y_j \cdots s^k y_j) = 0\) for \(l < q\). Then \(\gamma\) defines a non-trivial element in \(\text{Ext} \mathcal{M}(S^{k-1}) \mathcal{M}(S^{k-1}))\).

Note that, although the proposition is proved only when \(k = 2\) in [Wak], the same proof also applies when \(k > 2\) as long as \(k\) is even.

5 Proof of Theorem 1.3 and Theorem 1.4

In this section, we give a proof of Theorem 1.3 and Theorem 1.4 using the models constructed above.

Proof of Theorem 1.3 We compute the \(S^k\)-brane coproduct using (3.7). Since \(M = K(\mathbb{Z}, 2n)\), we take the Sullivan model \((\wedge V, d) = (\wedge x, 0)\) where \(x\) is the generator of degree \(2n\). Note that, in this case, the differentials in \(\mathcal{M}(S^k)\) and
\( \mathcal{M}(T^{(k)}) \) are zero, and hence they are identified with the cohomology groups \( H^*(M^{S^k}) \) and \( H^*(M^{T^{(k)}}) \).

By Proposition [3.3], we have a representative \( \gamma \) of the shriek map \( c_1 \) defined by \( \gamma(1) = s^{k-1}x \) and \( \gamma((s^k)x^l) = 0 \) for \( l \geq 1 \).

Since any Sullivan algebra satisfies the lifting property for a surjective quasi-isomorphism, there is a section \( \varphi \) of \( \bar{\varepsilon} \otimes \text{id} \) in (3.5), which is also a quasi-isomorphism. It is given explicitly by \( \varphi(1 \otimes x) = 1 \otimes x \), \( \varphi(1 \otimes s^kx) = 1 \otimes ss^{k-1}x \), and \( \varphi(1 \otimes sx) = 1 \otimes sx \).

Using these maps, we compute the composition (3.7). Since all maps in the composition are \( \wedge V \)-linear, it is enough to compute the image for the elements \( (s^kx)^n \) for \( n \geq 0 \). Applying \( \text{incl}^* \) and the section \( \varphi \) to the element, we have that it is mapped to \( 1 \otimes (ss^{k-1}x)^n \in \mathcal{M}(D^k) \otimes \mathcal{M}(S^{k-1}) \mathcal{M}(T^{(k)}) \). Then the map \( \gamma \otimes \text{id} \) send it to \( s^{k-1}x \otimes (ss^{k-1}x)^n \in \mathcal{M}(S^{k-1}) \otimes \mathcal{M}(S^{k-1}) \mathcal{M}(T^{(k)}) \).

Hence the \( S^k \)-brane coproduct \( \delta^v_{S^k} \) is the map determined by \( \delta^v_{S^k}(\alpha) = s^{k-1}x\iota(\alpha) \), where \( \iota: \mathcal{M}(S^k) \to \mathcal{M}(T^{(k)}) \) is the algebra map defined by \( \iota(x) = x \) and \( \iota(s^kx) = ss^{k-1}x \).

Similarly, we can compute the \( S^k \)-brane coproduct. Define a \( \wedge V \otimes 2 \)-linear map \( \eta: \mathcal{M}(I) \to \wedge V \otimes 2 \) by \( \eta(1) = 0 \) and \( \eta(sx) = 1 \). By Proposition [3.3], \( \eta \) is a representative of the shriek map \( \Delta \). We have a section \( \psi \) of \( \bar{\varepsilon} \otimes \text{id} \) in (3.8), which is defined by \( \psi(x \otimes 1) = 1 \otimes (x_1 \otimes 1) \), \( \psi(1 \otimes sx) = 1 \otimes (sx \otimes 1) - sx \otimes 1 \), and \( \psi(1 \otimes s^2x) = 1 \otimes (1 \otimes s^2x) \). Here we denote the element \( x \otimes 1 \in \mathcal{M}(I) \) by \( x_1 \).

As a result, the \( S^k \)-brane product \( \mu_{S^k}^v \) is the map determined by \( \mu_{S^k}^v(\beta) = 0 \), \( \mu_{S^k}^v(sx \cdot \beta) = -\rho(\beta) \), and \( \mu_{S^k}^v(s^{k-1}x \cdot \beta) = \mu_{S^k}^v(sx \cdot s^{k-1}x \cdot \beta) = 0 \), for \( \beta \in \wedge x \otimes \wedge ss^{k-1}x \). Here \( \rho: \wedge x \otimes \wedge ss^{k-1}x \to \mathcal{M}(S^k) \) is the algebra map defined by \( \rho(x) = x \) and \( \rho(ss^{k-1}x) = s^kx \).

Composing these two, we have \( \delta_{S^k} \circ \mu_{S^k} = 0 \). In fact, \( \delta_{S^k} \circ \mu_{S^k}(sx) = -s^{k-1}x \neq 0 \in \mathcal{M}(T^{(k)}) \cong H^*(M^{T^{(k)}}) \). This proves the theorem.

Next we prove Theorem [1.4].

**Proof of Theorem [1.4].** Let \( \langle V, d \rangle \) be the minimal Sullivan model of \( M, \{x_1, \ldots, x_p\} \) a basis of \( V^\text{even} \), and \( \{y_1, \ldots, y_q\} \) a basis of \( V^\text{odd} \). Consider the part

\[
\mathcal{M} \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \xrightarrow{\bar{\varepsilon} \otimes \text{id}} \mathcal{M}(D^k) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \\
\xrightarrow{\gamma \otimes \text{id}} \mathcal{M}(S^{k-1}) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)})
\]

in (3.7). Define a section \( \varphi \) of \( \bar{\varepsilon} \otimes \text{id} \) by \( \varphi(1 \otimes v) = 1 \otimes v \), \( \varphi(1 \otimes sv) = 1 \otimes sv \), for \( v \in V \), \( \varphi(1 \otimes ss^{k-1}x_i) = 1 \otimes ss^{k-1}x_i \), and \( \varphi(1 \otimes ss^{k-1}y_j) = 1 \otimes ss^{k-1}y_j + (-1)^k s \sigma(dy_j \otimes 1) \). Here, in the last term \( s \sigma(dy_j \otimes 1) \), \( \sigma \) is the derivation which sends \( v \otimes 1 \) to \( ss^kv \otimes 1 \), for \( v \in V \), and the other generators to 0. The map \( s \) is also the derivation which sends \( v \) to \( sv \), \( ss^{k-1}v \) to \( ss^{k-1}v \), and others to 0. Then we have \( \text{Im} \varphi \subset \mathcal{N} \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \), where \( \mathcal{N} = \wedge V \otimes \wedge ss^{k-1}V \wedge ss^k \{x_1, \ldots, x_p\} \subset \mathcal{M}(D^k) \). Let \( \gamma \) be the representative of \( c_1 \) given by Proposition [3.3]. Since \( V \) has at least one generator of odd degree, \( \gamma \)
is zero on $\mathcal{N}$. This implies that the composition $(\gamma \otimes 1) \circ \varphi$ is zero, and hence the brane coproduct $\delta_{S^k}$ is zero.

Acknowledgment

I would like to express my gratitude to Katsuhiko Kuribayashi and Takahito Naito for productive discussions and valuable suggestions. Furthermore, I would like to thank my supervisor Nariya Kawazumi for the enormous support and comments. This work was supported by JSPS KAKENHI Grant Number 16J06349 and the Program for Leading Graduate School, MEXT, Japan.

References

[CG04] Ralph L. Cohen and Véronique Godin. A polarized view of string topology. In Topology, geometry and quantum field theory, volume 308 of London Math. Soc. Lecture Note Ser., pages 127–154. Cambridge Univ. Press, Cambridge, 2004.

[CHV06] Ralph L. Cohen, Kathryn Hess, and Alexander A. Voronov. String topology and cyclic homology. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2006. Lectures from the Summer School held in Almería, September 16–20, 2003.

[CS99] Moira Chas and Dennis Sullivan. String topology, 1999, arXiv:math/9911159.

[FHT88] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. Gorenstein spaces. Adv. in Math., 71(1):92–112, 1988.

[FHT01] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. Rational homotopy theory, volume 205 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2001.

[FT09] Yves Félix and Jean-Claude Thomas. String topology on Gorenstein spaces. Math. Ann., 345(2):417–452, 2009.

[KMN15] Katsuhiko Kuribayashi, Luc Menichi, and Takahito Naito. Derived string topology and the Eilenberg-Moore spectral sequence. Israel J. Math., 209(2):745–802, 2015.

[Nai13] Takahito Naito. String operations on rational Gorenstein spaces, 2013, arXiv:1301.1785.

[Tam10] Hirotaka Tamanoi. Loop coproducts in string topology and triviality of higher genus TQFT operations. J. Pure Appl. Algebra, 214(5):605–615, 2010.
[Wak] Shun Wakatsuki. Coproducts in brane topology. To appear in Algebr. Geom. Topol., also available at arXiv:1802.04973.

[Wak16] Shun Wakatsuki. Description and triviality of the loop products and coproducts for rational Gorenstein spaces, 2016, arXiv:1612.03563.