Imprints of interacting dark energy on cosmological perturbations

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We investigate the characteristic modifications in the evolving cosmological perturbations when dark energy interacts with dust-like matter, causing the latter’s background energy density fall off with time faster than usual. Focusing in particular to the late-time cosmic evolution, we show that such an interaction (of a specific form, arising naturally in a scalar-tensor formulation, or a wide range of modified gravity equivalents thereof), can have a rather significant effect on the perturbative spectrum, than on the background configuration which is not expected to get distorted much from ΛCDM. Specifically, the matter density contrast, which is by and large scale-invariant in the deep sub-horizon limit, not only gets dragged as the interaction affects the background Hubble expansion rate, but also receives a contribution from the perturbation in the (scalar field induced) dark energy, which oscillates about a non-zero mean value. As such, the standard parametrization ansatz for the matter density growth factor becomes inadequate. So we modify it suitably, and also find a numerical fit of the growth index in terms of the background parameters, in order to alleviate the problems that arise otherwise. Such a fit enables direct estimations of the background parameters, as well as the growth parameter and the reduced Hubble parameter, which we duly carry out using a redshift space distortion (RSD) subsample and its combination with the observational Hubble data.

On the whole, the parametric estimates show consistency with the general observational constraints on the background level cosmology, as well as the constraints on scalar-tensor gravity from astrophysical observations, apart from having significance in the domain of cosmological perturbations.

Keywords: Modified gravity theories; cosmological perturbations; parametric estimations

1. Introduction

One of the main compulsions in modern cosmology is to determine the extent to which the rate of formation of large-scale structures (LSS) is affected by the late-time cosmic acceleration, and the constraints imposed thereby on the dynamics of the dark energy (DE) which supposedly drives this acceleration. As the observations strongly favour the DE component to be a cosmological constant Λ, there is an
obvious dilution of any idea circumventing the dynamical evolution of the same. Nevertheless, such a dynamics may be reckoned, at least, within the proximity of mild (1σ level) distortions in the parametric estimations for the concordant ΛCDM model, with Λ and cold dark matter (CDM) as the dominant components of the universe.  

There lies a caveat though, which is even more suggestive in the sense that the effect of something ostensibly mild at the background cosmological level, and at low redshifts ($z \lesssim 1$), may not be so at a perturbative level and at moderate to high redshifts. It therefore necessitates one to test independently the mildness of the DE dynamics, sought in a given cosmological model, from perturbative studies extending to large enough redshifts. Admittedly, the analysis of the observational data on the growth of LSS (or that of the matter density perturbations) can place robust constraints on a given DE model, and possibly discriminate it among various other models.

Commonly known dynamical models of DE (quintessence, kessence, etc.), inspite of their merits, are plugged with some serious issues which make it imperative to search for wayouts in alternative scenarios. An area of considerable interest, in this respect, is that of the modified gravity (MG) theories, which go beyond the realm of General Relativity (GR). Such theories not only provide a geometric interpretation of the DE in the standard cosmological setup, but also alleviate the cosmic coincidence problem in the equivalent scalar-tensor formulations, by emulating effective scenarios in which the DE component can interact with matter fields (the CDM sources inclusive). In fact, many phenomenological studies involving the DE-matter (DEM) interactions draw proper physical explanation from scalar-tensor equivalent MG models, including those which lead to the coveted revelation of a unified cosmic dark sector.

Scalar-tensor formulations provide a natural perception of the DEM interactions, by virtue of the effective contact coupling(s) of the scalar field source(s) of the DE and matter field(s) in the conformally transformed frames. A given such interaction affects the evolution of the total background matter density $\rho^{(m)}(z)$, as well as the growth of the corresponding perturbation $\delta \rho^{(m)}(z)$, and hence the growth of the LSS. More specifically, the interaction (with a scalar field $\varphi$) makes $\rho^{(m)}(z)$ drift from its usual (dust-like) form, thereby leading to a drag force on $\delta \rho^{(m)}(z)$. Consequently, the evolution profile of the matter density contrast $\delta^{(m)}(z) := \delta \rho^{(m)}(z)/\rho^{(m)}(z)$ and the corresponding growth factor $f(z)$ differ from those in the non-interacting scenarios, such as quintessence. The field perturbation $\delta \varphi$, on the other hand, undergoes a damped oscillatory evolution in the sub-horizon regime, quite similar to that in the non-interacting scenarios. However, the oscillations are about a non-zero mean value, proportional to the strength of the

*For simplicity, in this paper, we consider matter content of the universe to be very nearly pressure-free. In particular, apart from the CDM, we consider only the visible matter in the form of a dust of baryons, the pressure due to which is negligibly small compared to the corresponding energy density and is safely ignorable even at the linear perturbative cosmological level.
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ϕ-coupling with matter. As such, δϕ contributes to the matter velocity divergence, affecting in turn the evolution of δ^{(m)}(z) by a further extent [19].

From the technical point of view, an intriguing outcome of a DEM interaction is that the matter perturbation growth factor f(z) can acquire a value > 1 at large z, contrary to the restriction set in the non-interacting scenarios. So there is a stringent need to look beyond the commonly known f(z) parametrizations which prohibit the breach of the f(z) = 1 barrier at any phase of evolution of the universe [68–70, 73–75]. In particular, it is worth looking for a suitable modification of the parametrization f(z) = \left[\Omega^{(m)}(z)\right]^{\gamma(z)} , where Ω^{(m)}(z) is the matter density parameter and γ(z) is the so-called growth index, which is not a constant even for ΛCDM [69, 70, 73–75]. This parametrization is well-motivated and widely used in the literature, since the assertion of the form of the function γ(z), and its value γ₀ at the present epoch (z = 0), from the Red-shift space distortion (RSD) observations, proves to be a convenient way of comparing cosmological models of various sort, as well as checking the viability of the same [69, 70, 73–83].

In attempting the modification of such a parametrization, in presence of a DEM interaction, our objective in this paper is to determine the observational constraints on the background level parameters, such as Ω^{(m)}₀ = Ω^{(m)}|_{z=0} and the DEM coupling strength n, in a scalar-tensor cosmological scenario. Nevertheless, we keep our attention to a scalar-tensor configuration of a specific sort, which is essentially meant for a typical case study in this paper, with a common motivation from the perspective of the equivalence with a wide range of MG formulations in the literature. Besides, we focus on studying the evolution of f(z) only at the sub-horizon scales, at which the late-time growth of the LSS is relevant observationally. However, our endeavor is to perform the growth analysis in a general way, without undermining the role of the field perturbation δϕ, as is quite often done in practice. Afterall, as mentioned above, the oscillations of δϕ are about a non-zero mean value in presence of a DEM interaction. So, depending on the strength of the latter, there can be an effect of some significance on f(z), and hence on the growth index γ(z), especially in the deep sub-horizon regime.

Now, the γ(z) stipulations in the literature are mostly power series expansions, about z = 0, Ω^{(m)} = 1, and so on [69, 70, 73–77]. Although effective, they have limited scope of applicability, for instance, only at very late times, or only in the deep matter-dominated era. Moreover, the power series coefficients increase the parameter space of the model, unless their dependence on the background level model parameters is explicitly worked out. It is nonetheless desirable to solve the evolution equation for f(z) directly by using the latter’s parametrization, in terms of the growth index γ(z). Doing this analytically is however a hard proposition. So, one may look for a numerical fit of γ(z) in terms of the background parameters, and check the consistency upto large redshifts. We attempt such a numerical fitting, in order to carry out subsequently the direct estimations of the parameters using the RSD observational data (or the GOLD sub-sample thereof [84]), and also the latter’s...
combination with the observational Hubble ($H(z)$) data. Apart from $n$ and $\Omega_{0}^{(m)}$, there are two parameters involved, viz. reduced Hubble constant $h$ and the RSD parameter $\sigma_{0}^{(8)} \equiv \sigma^{(8)}|_{z=0}$, where $\sigma^{(8)}(z)$ is the root-mean-square fluctuation of the mass distribution within a sphere of radius $8 \text{Mph}^{-1}$.

The paper is organized as follows: in section 2, we first recapitulate how the cosmological scenarios emerging from scalar-tensor theories (or their MG equivalents) naturally accommodate the DEM interactions, and then demonstrate the background cosmological solution, given in a parametric form, for a specific class of such theories (detailed in the Appendix). We go on to study the corresponding cosmological perturbations thereafter, in section 3, and obtain the evolution equations for $\delta^{(m)}(z)$, $f(z)$ and $\delta \varphi(z)$, in the well-known Newtonian gauge. Numerically solving those equations, for certain fiducial parametric settings, we examine the extent to which the effect of the DEM interaction on the evolution of $f(z)$ is accountable, as opposed to the latter’s (negligible) scale-dependence, at the sub-horizon scales. We follow this up, in section 4, with a proposed parametrization ansatz for $f(z)$ in presence of the DEM interaction, and subsequently obtain the requisite numerical fit of the growth index $\gamma(z)$ in terms of the background parameters. The statistical estimations of the requisite parameters ($n, \Omega_{0}^{(m)}, \sigma_{0}^{(8)}, h$) are carried out next, in section 5, using the Metropolis-Hastings algorithm for the Markov Chain Monte Carlo (MCMC) simulation with the two chosen datasets mentioned above. Finally, we summarize our findings and conclude in section 6.

**Conventions and Notations:** Throughout this paper, we use metric signature $(-, +, +, +)$ and natural units, with the speed of light $c = 1$. We denote and the gravitational coupling factor by $\kappa = \sqrt{8\pi G_{N}}$, where $G_{N}$ is the Newton’s constant, the metric determinant by $g$, and the values of parameters or functions at the present epoch by an affixed subscript ‘0’.

### 2. Interacting Dark Energy-Matter scenario

Let us recall that in GR the Einstein tensor $G_{\mu\nu}$ gets restricted by the (contracted) Bianchi identity $\nabla^{\mu} G_{\mu\nu} = 0$, which implies the consistency of the conservation relation for the energy-momentum tensor $T_{\mu\nu}$, i.e.

$$\nabla^{\mu} T_{\mu\nu} = 0,$$

with the Einstein’s equations $G_{\mu\nu} = \kappa^{2} T_{\mu\nu}$. However, a flexibility is there in the GR formulation itself. That is, for a multi-component system (such as the universe consisting of radiation, baryonic matter, CDM, DE etc.), the conservation of $T_{\mu\nu}$ does not necessarily imply the same for each of the component energy-momentum tensors $T_{\mu\nu}^{(i)}$. Therefore, re-expressing Eq. 2.1 in the form

$$\nabla^{\mu} T_{\mu\nu}^{(i)} = Q_{\nu}^{(i)} \quad \text{with} \quad \sum_{i} Q_{\nu}^{(i)} = 0,$$
one can always ponder on the scenarios in which some, or all, of these components may have mutual interactions, quantified in terms of the vectors $Q^{(i)}_{\mu}$. Now, from the cosmological perspective, it is reasonable to consider the evolution of the universe (particularly at late times, i.e. at moderate to low redshifts) to be driven by two interacting components, viz. a pressure-less matter component (comprised of the visible baryons and the CDM) and a DE component (presumably induced by some scalar field $\phi$). Eq. (2.2) then implies that the respective energy-momentum tensors, $T^{(m)}_{\mu\nu}$ and $T^{(\phi)}_{\mu\nu}$, satisfy the conservation relations

$$\nabla^\mu T^{(m)}_{\mu\nu} = Q_{\nu} \quad \text{and} \quad \nabla^\mu T^{(\phi)}_{\mu\nu} = -Q_{\nu},$$

(2.3)

with the vector $Q_{\nu}$ determining the extent of the interaction$^b$. Such an interaction is by no means ad-hoc — it appears naturally in scalar-tensor theories, which are characterized by explicit non-minimal gravitational couplings with effective scalar degree(s) of freedom in the Jordan frame. As is well-known, a suitable conformal transformation of the Jordan frame metric can lift the non-minimality, however, at the expense of leaving the effective matter Lagrangian dependent on the corresponding scalar field $\phi$, both implicitly and explicitly, in the Einstein frame. While making interpretations in the standard Friedmann-Robertson-Walker (FRW) cosmological setup, such a $\phi$-dependent matter Lagrangian can at once be identified as that which invokes an energy exchange between the ($\phi$-induced) DE and the matter sector, or in other words, a DEM interaction.

In principle, depending on the type of non-minimal coupling of the gravitational Lagrangian $R$ (or the Ricci curvature scalar) and a scalar field ($\phi$, say) in the Jordan frame, there can be a variety of functional forms of the interaction vector $Q_{\mu}$ appearing in the Einstein frame. For definiteness however, in this paper we shall consider a scalar-tensor formulation of a particular sort, marked by a $\phi^2 R$ term in the Jordan frame action, which then corresponds to the Brans-Dicke action (with a possible augmentation of a potential term for the scalar field). After a conformal transformation and a field re-definition $\phi = \kappa^{-1} e^{n\kappa \phi}$, one finds

$$Q_{\mu} = \kappa n \rho^{(m)} \partial_{\mu} \phi,$$

(2.4)

where $\rho^{(m)}$ is the matter density defined in the Einstein frame$^c$ and $n = (6 + w)^{-1/2}$, with $w$ denoting the Brans-Dicke parameter (see the Appendix for details). This sort of formulation is equivalent to that of a wide range of modified or alternative theories of gravity, albeit with a fixed value of the parameter $n$ in some cases. For instance, the $f(R)$ theories are well-known to have Brans-Dicke equivalent formulations only for $w = 0$, i.e. $n = 1/\sqrt{\pi}$ (fixed). We shall treat such fixed $n$ theories as exceptions though, or in other words, consider $n$ as a free parameter.

$^b$There could be interactions within the matter sector as well, i.e. among the baryons and various dark matter species. However, for simplicity, we exclude such a possibility in this work.

$^c$The physical entities are assumed to be definable and existent in the Einstein frame, and the cosmological matter under consideration is the pressure-less (baryonic + dark) matter, with the trace of the corresponding energy-momentum tensor equal to $-\rho^{(m)}$. 
throughout the rest of this paper. In fact, we shall estimate the value of \( n \) (or determine its upper bound) using observational data, in the cosmological perturbative analysis due to be carried out in the subsequent sections.

Refer now to the standard spatially flat FRW space-time geometry, described by the scale factor \( a(t) \) and the Hubble parameter \( H(t) = \dot{a}(t)/a(t) \), where the overdot \( \dot{\cdot} \) denotes differentiation with respect to the cosmic time \( t \). In accord with our presumption of the universe being comprised of a pressure-free matter component and a \( \phi \)-induced DE component, we have the Friedmann equation and the conservation relations (that follow from Eqs. (2.3) and (2.4)) given by

\[
H^2 = \frac{\kappa^2}{3} \left[ \rho^{(m)} + \rho^{(\phi)} \right],
\]

\[
\dot{\rho}^{(m)} + 3H \rho^{(m)} = -\kappa n \rho^{(m)} \dot{\phi},
\]

\[
\dot{\rho}^{(\phi)} + 3H \left[ \rho^{(\phi)} + p^{(\phi)} \right] = \kappa n \rho^{(m)} \dot{\phi},
\]

with the usual expressions for the energy density and pressure

\[
\rho^{(\phi)} = \frac{\dot{\phi}^2}{2} + U(\phi) \quad \text{and} \quad p^{(\phi)} = \frac{\dot{\phi}^2}{2} - U(\phi).
\]

corresponding to the field \( \phi \), with a potential \( U(\phi) \).

Eq. (2.6) at once gives

\[
\rho^{(m)}(a) \propto a^{-3} e^{-\kappa n \phi(a)},
\]

which qualitatively shows how the matter density deviates from its usual dust-like form \((\sim a^{-3})\) because of the interaction. However, the determination of the complete solutions for \( \rho^{(m)}(a) \), \( \rho^{(\phi)}(a) \) and \( p^{(\phi)}(a) \) requires one to have the prior knowledge of the functional form of the potential \( U(\phi) \).

Commonly known forms of \( U(\phi) \) in the literature, for which exact analytic cosmological solutions have been worked out, are certain single or double exponentials.\(^{47-50}\) However, many such exponential forms lack proper physical motivation, and many such solutions involve quite a few free parameters, neither of which are desirable from theoretical perspectives. So, for a concrete case study in this paper, we shall limit ourselves to an exact solution of a specific sort, given by the set\(^{50}\)

\[
\varphi(a) = 2n \kappa^{-1} \ln a,
\]

\[
\rho^{(m)}(a) = \rho_0^{(m)} a^{-(3+2n^2)},
\]

\[
\rho^{(\phi)}(a) = 2n^2 \kappa^2 H^2(a) + \Lambda a^{-4n^2},
\]

\[
p^{(\phi)}(a) = 2n^2 \kappa^2 H^2(a) - \Lambda a^{-4n^2},
\]

where \( \Lambda \) is a constant of mass dimension \( = 4 \), \( \rho_0^{(m)} \) is the value of the matter density at the present epoch \((t = t_0, \text{ whence } a = 1)\), and

\[
H(a) = \frac{\kappa}{\sqrt{3 - 2n^2}} \left[ \frac{\rho_0^{(m)}}{a^{3+2n^2}} + \frac{\Lambda}{a^{4n^2}} \right]^{1/2}.
\]

This solution set is useful from the following points of view:
• It involves no more than two free parameters, since one parameter in the set \([n, \Lambda, \rho^{(m)}_0]\) always gets determined by the other two, by virtue of the constraint \(\Omega_0 = 1\), where \(\Omega_0\) is the total density parameter of the universe at the present epoch \(t_0\).

• It reduces to the standard \(\Lambda\)CDM solution just as the coupling parameter \(n \to 0\), or equivalently, \(w \to \infty\). Therefore, considering the solar system constraints on the scalar-tensor equivalent theories of gravity, which essentially imply a large lower bound on \(w\), we may infer that the dynamical evolution of the DE, due to a non-vanishing \(n\) here, has to be slow enough.

• It can be obtained under the assertion of a single exponential potential

\[
U(\varphi) = \Lambda e^{-2\kappa n \varphi},
\]

which amounts to just a mass term for the scalar field \(\phi\) in the Jordan frame. The mass parameter \(m = 2\kappa^2 \Lambda\), which one may verify easily (from the general formalism detailed in the Appendix). As a typical exemplary scenario, we may refer to that of the metric-scalar-torsion theory, in which such a mass term effectively originates from a norm-fixing condition on the axial mode vector \(A^\mu\) of torsion, or from a higher order term \((A^\mu A^\nu)^2\) augmented to the Lagrangian.\(^86\textsuperscript{–}88\)

From the above Eqs. (2.11) – (2.14), one derives the matter density parameter and the total equation of state (EoS) parameter of the system, respectively, as

\[
\Omega^{(m)}(a) := \frac{\rho^{(m)}}{\rho^{(\varphi)} + \rho^{(m)}} = \frac{(3 - 2n^2) \Omega^{(m)}_0}{(3 - 2n^2 - 3 \Omega^{(m)}_0) a^{3 - 2n^2} + 3 \Omega^{(m)}_0},
\]

\[
w(a) = \frac{\rho^{(\varphi)}}{\rho^{(\varphi)} + \rho^{(m)}} = \Omega^{(m)}(a) - 1 + \frac{4n^2}{3},
\]

where \(\Omega^{(m)}_0\) is the value of \(\Omega^{(m)}\) at the present epoch.

3. Evolution of cosmological perturbations

Let us study, in this section, the evolution of cosmological perturbations in the well-known Newtonian gauge in which the perturbed line element is given by

\[
\text{d}s^2 = e^{2N} \left[ -(1 - 2\Phi) \mathcal{H}^{-2} \text{d}N^2 + (1 + 2\Phi) \delta_{ij} \text{d}x^i \text{d}x^j \right],
\]

where \(N(t) = \ln a(t)\) is the number of e-foldings, \(\mathcal{H}(t) = a(t)H(t)\) is the conformal Hubble parameter and \(\Phi\) is the Bardeen potential\(^4\). For convenience, we shall use the units \(\kappa^2 = 8\pi G_N = 1\), and denote all the derivatives by subscripted commas (e.g. \(d\Phi/dN \equiv \Phi,_{N}\), \(d^2\Phi/dN^2 \equiv \Phi,_{NN}\), \(dU/d\varphi \equiv U,_{\varphi}\), etc.), from here on.

\(^4\)Due to the absence of anisotropic stress at late-times, we are free to consider both the temporal and spatial perturbations to be described by \(\Phi\).\(^89\textsuperscript{–}90\)
Given a fluid, which is assumed to be perfect even at a perturbative level, we have the perturbation in the corresponding energy-momentum tensor expressed as
\[ \delta T^\mu_\nu = [(1 + c_s^2) u^\mu u_\nu + c_s^2 \delta \xi^\mu] \delta \rho + (p + \rho) (u^\mu \delta u_\nu + u_\nu \delta u^\mu), \] (3.2)
where \( u^\alpha, \rho \) and \( p \) denote the background fluid velocity, energy density and pressure, with the respective perturbations \( \delta u^\alpha, \delta \rho \) and \( \delta p \), and \( c_s^2 := d\rho/d\rho \) defines the squared sound speed of the perturbations.

Resort now to the system of two interacting components, viz. the universe, with the DE induced by a scalar field \( \varphi \), and the matter sector considered as a pressure-less fluid, presumably, at all levels of perturbation, which implies that the corresponding sound speed \( c_s = 0 \). Recall the equation (2.6) which shows how the conservation of the background matter density \( \rho^{(m)} \) is affected by the coupling with \( \varphi \). Perturbing this equation, and following the standard procedure of dealing with the Fourier transforms of all the perturbations involved, we derive the conservation equation for the matter density perturbation \( \delta \rho^{(m)} \) as
\[ \delta^{(m)} = - \left[ \rho^{(m)} + 3\Phi,_{N} \right] - n (\delta \varphi),_{N}, \] (3.3)
where \( \delta^{(m)} = \delta \rho^{(m)}/\rho^{(m)} \) is the matter density contrast, and \( \delta \varphi \) and \( \theta^{(m)} \) respectively denote the scalar field perturbation and the matter velocity divergence
\[ \theta^{(m)} = - \frac{\theta^{(m)}}{2} \left[ 1 - 3w - 2n \varphi,_{N} \right] - \frac{1}{\lambda^2} \left( \Phi + n \delta \varphi \right), \] (3.4)
where \( w \) is the background EoS parameter given by Eq. (2.17), and \( \lambda \equiv \mathcal{H}/k \), with \( k \) being the comoving wavenumber.

From Eqs. (3.3) and (3.4), one obtains the second order differential equation
\[ \delta^{(m)} = - \frac{3}{2} \left[ 3\Phi,_{N} + \delta^{(m)} + n (\delta \varphi),_{N} \left( 1 - 3w - 2n \varphi,_{N} \right) \right] \]
\[ - 3\Phi,_{N} + n (\delta \varphi),_{N} + \frac{1}{\lambda^2} \left( \Phi + n \delta \varphi \right), \] (3.5)
which shows that the dependence of \( \delta^{(m)} \) on the coupling parameter \( n \) is compounded with its scale \( (\lambda) \) dependence, because of that of the Bardeen potential \( \Phi \).

However, more strikingly, both these dependencies have an added influence of \( \delta \varphi \) which depends on \( n \) and \( \lambda \) as well. This can be inferred from the equation
\[ (\delta \varphi),_{N} + \left( 2 + \frac{\mathcal{H},_{N}}{\mathcal{H}} \right) (\delta \varphi),_{N} + \left[ \frac{1}{\lambda^2} + \frac{(U,_{\varphi} \varphi)^2}{\mathcal{H}^2} \right] \delta \varphi \]
\[ = 3n \left( \delta^{(m)} - 2\Phi \right) \Omega^{(m)} + \frac{2\Phi U,_{\varphi}}{\mathcal{H}^2} + 4\Phi,_{N} \varphi,_{N}, \] (3.6)
\[ \delta \]To be specific, Eq. (3.3) follows from the conservation of \( \delta T^0_0 \) component of Eq. (2.3), whereas the Euler equation (3.4) is obtained from that of the \( \delta T^i_i \) component, in addition to the use of the form the perturbed metric (3.1) and the background matter conservation equation (2.6). Of course, the final forms of the expressions are derived by expanding each perturbed quantity, say \( h(\vec{x}, N) \), in the Fourier space, viz. \( h(\vec{x}, N) = \int d^3 k h(\vec{k}, N) e^{i \vec{k} \cdot \vec{x}}. \)
which one obtains in a similar way (by perturbing the field conservation equation (2.7), and taking note that $c_s^2 = 1$ for the field perturbations).

On the other hand, the scale-dependence of $\Phi$ is evident from the time-time and time-space components of the perturbed Einstein’s equations, viz.

$$
\Phi_{,N} + \Phi = -\frac{1}{2} \left[ \varphi_{,N} \delta \varphi + 3 \lambda^2 \Omega^{(m)} \theta^{(m)} \right],
$$

(3.7)

$$
\left( 1 + \frac{\lambda^2 U}{H^2} \right) \Phi = \frac{\lambda^2}{2} \left[ 3 \Omega^{(m)} \theta^{(m)} + \frac{U \varphi}{H^2} \delta \varphi + \left\{ \left( \delta \varphi \right)_{,N} + 3 \delta \varphi \right\} \varphi_{,N} + 9 \lambda^2 \Omega^{(m)} \theta^{(m)} \right].
$$

(3.8)

In fact, together with Eq. (3.5), these equations make it clear that $\Phi$ is scale-dependent, and so is $\delta^{(m)}$, even when there is no scalar field $\varphi$. It also follows that in such a case the scale-dependence of $\delta^{(m)}$ can be ignored only in the deep sub-horizon regime ($k \gg H$ or $\lambda \ll 1$), whence $\Phi$ becomes negligible. This is of course well-known, and can be inferred in presence of the field $\varphi$ as well. However, the scenario is rather intriguing when $\varphi$ couples to matter non-minimally, and thereby leads to the DEM interaction. One may then ask the natural question as to whether such a coupling can be of any significance at all, in a study of cosmological perturbations limited to the sub-horizon scales only. That is to say, whether the dependence of $\delta^{(m)}$ on the coupling strength (or the parameter $n$) can lead to a stronger effect on its evolution profile, than that due to the scale-dependence, which is nonetheless suppressed in the deep sub-horizon regime.

We endeavor to address this here, by first using Eqs. (3.7) and (3.8) to eliminate $\Phi$ and its derivatives from Eqs. (3.5) and (3.6), and then solving the resulting equations numerically, for certain chosen values of $n$ and $\lambda$, at least an order of magnitude below unity. As the background cosmological solution, we consider the one given by the set (2.10) - (2.14) and (2.16), (2.17), corresponding to the chosen potential (2.15). Resorting then to the fiducial parametric settings

$$
\Omega_0^{(m)} = 0.3, \quad h \equiv H_0 /[100 \text{Km s}^{-1}\text{Mpc}^{-1}] = 0.67,
$$

and the adiabatic initial conditions, viz.

$$
\delta^{(m)} \simeq \delta_{N}^{(m)} \simeq e^N \simeq 0.0009 \quad \text{and} \quad \delta \varphi \simeq \delta \varphi_{,N} \simeq 0
$$

at the recombination epoch $N \simeq -7$ (i.e. at redshift $z \equiv e^{-N} - 1 \simeq 1100$), we obtain the evolution profiles of $\delta^{(m)}$ and $\delta \varphi$ for those choices of $n$ and $\lambda$.

Figs. 1(a) and (b) depict the evolution of $\delta^{(m)}$ in the range $N \in [-7, 0]$, respectively, for $n = 0.1$ (fixed) and two values of $\lambda$ ($= 0.01, 0.1$), and for $\lambda = 0.1$ (fixed) and two values of $n$ ($= 0.01, 0.1$). The evolution profile is exponential, which gets

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The sound speed of field perturbations refers to the coefficient of the second order spatial derivative of $\varphi$. In the Fourier space, the coefficient of $\lambda^{-2} \delta \varphi$ determines that sound speed, which identically turns out to be unity for the perturbed configuration we are dealing here.
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Fig. 1. Evolution profiles of the matter density contrast $\delta^{(m)}(N)$ in the range $N \in [-7, 0]$, for (a) fixed coupling strength $n = 0.1$ and different perturbation scales $\lambda = 0.01, 0.1$, and (b) fixed perturbation scale $\lambda = 0.1$ and different coupling strengths $n = 0.01, 0.1$. The insets show the respective (log scale) evolution profiles of the field perturbation $\delta \varphi(N)$, in the same range.

(a) $\delta^{(m)}$ and $\delta \varphi$ for a fixed $n$ and different $\lambda$  
(b) $\delta^{(m)}$ and $\delta \varphi$ for a fixed $\lambda$ and different $n$

steeper with the increase of $\lambda$ from 0.01 to 0.1, or with the increase of $n$ by the same amount, in the respective cases. However, the margin of such an enhancement in the steepness is quite less in the former, than in the latter, as noticed particularly from the deviation at late times (i.e. close to the present epoch $N = 0$). This at least shows qualitatively that a very nearly scale-independent evolution of $\delta^{(m)}$ at small scales does not necessarily imply that the effect on it due to a non-minimal $\varphi$-matter coupling is ignorable.

The insets of Figs. (a) and (b) show the corresponding evolution profiles of $\delta \varphi$, characterized by initial oscillations, with a gradual smoothening out, due to damping, as $N$ increases. The initial oscillations however become less rapid as $\lambda$ is increased from 0.01 to 0.1 for $n = 0.1$, whereas their pattern remains the same as $n$ is increased from 0.01 to 0.1 for $\lambda = 0.1$. There is an overall enhancement of $\delta \varphi$ though, at all epochs (regardless of the oscillations), in both the cases. In fact, the enhancement is much larger in the former case than in the latter, which is unlike what happens for $\delta^{(m)}$. Nevertheless, this is of not much significance, as $\delta \varphi$ is always subdued (compared to $\delta^{(m)}$) at small scales, due to its unit sound speed ($c_s^2 = 1$), which makes the size of its sound horizon of the order of the Hubble radius $H^{-1}$ at the present epoch ($N = 0$). However, as shown above, the $\varphi$-coupling with matter can lead to a fairly considerable effect of $\delta \varphi$ on the evolution of $\delta^{(m)}$, surmounting the latter’s scale-dependence by quite an extent, in the deep sub-horizon regime.
4. Analysis of the matter perturbation growth

Let us now examine closely, and more systematically, the evolution of the matter density contrast \( \delta(m) \), by referring back to the equation (3.5) derived in the previous section. From the subsequent equations therein, viz. (3.6) - (3.8), it is clear that the field perturbation \( \delta \Phi \) and the Bardeen potential \( \Phi \) are both proportional to \( \hat{\lambda}^2 \), and therefore suppressed immensely at the deep sub-horizon scales (\( \hat{\lambda} \ll 1 \)):

\[
\delta \Phi \simeq 3n \hat{\lambda}^2 \Omega(m) \delta(m), \tag{4.1}
\]

\[
\Phi \simeq \frac{3\hat{\lambda}^2}{2} \Omega(m) \delta(m). \tag{4.2}
\]

So, from the observational perspective, since the late-time growth of the LSS is well within the horizon, it suffices to limit one’s attention to the evolution of \( \delta(m) \) only. However, the crucial point to note here is that \( \Phi \) and \( \delta \Phi \), inspite of their small scale suppression, may not have a negligible effect on the \( \delta(m) \) evolution. This is evident from the rightmost term of Eq. (3.5), given by \( (\Phi + n \delta \Phi) \), modulo the factor \( \hat{\lambda}^{-2} \) which gets neutralized by the \( \hat{\lambda}^2 \) appearing on the right hand sides of Eqs. (4.1) and (4.2). These equations further imply a practically scale-independent evolution of \( \delta(m) \), satisfying the following approximated form of Eq. (3.5), for \( \hat{\lambda} \ll 1 \):

\[
\delta(m)_{\frac{N}{N}} + \left[ 2 \left( 1 - 2n^2 \right) - \frac{3\Omega(m)}{2} \right] \delta(m)_{\frac{N}{N}} = \frac{3 \left( 1 + 2n^2 \right)}{2} \Omega(m) \delta(m), \tag{4.3}
\]

which can be converted to the first-order differential equation

\[
f(N) + f^2 + \left[ 2 \left( 1 - 2n^2 \right) - \frac{3\Omega(m)}{2} \right] f = \frac{3 \left( 1 + 2n^2 \right)}{2} \Omega(m), \tag{4.4}
\]

where \( f(N) := \left[ \ln \delta(m)(N) \right]_{\frac{N}{N}} \) is known as the growth factor.\(^{91}^{95}\)

4.1. Growth factor parametrization

Setting now, certain fiducial values of the matter density parameter at the present epoch, \( \Omega_0^{(m)} \), and the coupling parameter \( n \), we solve the equations (4.3) and (4.4) numerically, to determine the variation of \( \delta(m) \) and \( f \) with the redshift \( z \equiv e^{-N} - 1 \).

Figs. 2(a) and (b) show the corresponding plots in the redshift range \( z \in [0, 5] \), respectively, for \( \Omega_0^{(m)} = 0.3 \) (fixed) and \( n = 0, 0.1 \), and for \( n = 0.1 \) (fixed) and \( \Omega_0^{(m)} = 0.25, 0.3 \).

As we see, with decreasing \( z \), both \( \delta(m)(z) \) and its growth rate get boosted for a non-zero \( n \) and for an increased \( \Omega_0^{(m)} \) in the respective cases. However, the same does not hold for \( f(z) \) and its decay rate correspondingly. While \( f(z) \) gets enhanced in both the cases, its decay rate gets boosted in the former and diminished in the latter. Note further that \( f(z) \) exceeds unity at high redshifts, when \( n \) is non-vanishing. This is really of importance, in the sense that it demands one to go beyond the standard prescription followed in the matter perturbation growth analysis, in presence the DEM interaction. To be more specific, the well-known (and widely used) growth
factor parametrization ansatz $f(N) = \left[\Omega^{(m)}(N)\right]^{\gamma(N)}$, with $\gamma(N)$ dubbed as the growth index, would only suit the non-interacting scenarios which forbid to any breach of the $f = 1$ barrier. One is therefore required to look for a modified or alternative parametrization in presence of the DEM interaction (i.e. for $n \neq 0$). The following is what we propose, by noting that the pre-factor $(1 + 2n^2)$ on the right hand side of Eq. (4.4) plays the key role in allowing $f$ to exceed unity at large $z$ (or high $|N|$), whence $\Omega^{(m)} \to 1$:

$$f(N) = (1 + 2n^2) \left[\Omega^{(m)}(N)\right]^{\gamma(N)}. \quad (4.5)$$

So the task now boils down to asserting the functional form of the growth index $\gamma(N)$, in an appropriate way, which we discuss below.

### 4.2. Growth index fitting

In the literature, there has been a plethora of functional forms of $\gamma(N)$ considered from various standpoints. However, usually they are either suitable for the non-interacting DE models only, or valid only at some particular cosmological regimes. For instance, the linear order Taylor expansion of $\gamma(N)$ about $z = 0$ remains valid only up to very low redshifts, whereas that about $[1 - \Omega^{(m)}]$ holds good only in the deep matter dominated era. Hence, taking the direct route, we substitute Eq. (4.5) in Eq. (4.4) and obtain the following functional fit of the solution of the
resulting equation numerically, in terms of parameters \( n \) and \( \Omega_0^{(m)} \):

\[
\gamma(N) = \left[ 0.3994 \left( \Omega_0^{(m)} \right)^{-0.08} - 0.9844 n^2 \right] (1 + e^N)^{0.3345}.
\] (4.6)

This has the obvious advantage of not requiring one to determine first the observational constraints on the Taylor coefficients of \( \gamma(N) \), while carrying out the statistical estimations of \( n \) and \( \Omega_0^{(m)} \), as well as the other parameters involved in the analysis, such as the RSD parameter \( \sigma_0^{(8)} \) and the reduced Hubble constant \( h \).

Note also that in the ΛCDM limit \( (n \to 0) \), and for \( \Omega_0^{(m)} = 0.3 \), the above fit \( 4.6 \) gives \( \gamma_0 = \left. \gamma \right|_{N=0} = 0.553 \), which is very close to the theoretically predicted value \( 6/11 \simeq 0.545 \) (for ΛCDM). In fact, the fit is fairly accurate overall, as one may see by examining the percentage error

\[
E_f(z) = \frac{100 \left[ f_F(z) - f(z) \right]}{f(z)},
\] (4.7)

where \( f_F(z) \) denotes the fitted growth factor obtained by plugging Eq. (4.6) in Eq. (4.5) and casting the result as a function of \( z \), and \( f(z) \) is the exact growth factor obtained by numerically solving Eq. (4.5) in the preceding subsection.

Figs. 3(a) and (b) show the variation of \( E_f(z) \) in the redshift range \( z \in [0, 100] \), respectively, for \( \Omega_0^{(m)} = 0.3 \) (fixed) and \( n = 0, 0.1 \), and for \( n = 0.1 \) (fixed) and \( \Omega_0^{(m)} = 0.25, 0.3 \).
\( \Omega_0^{(m)} = 0.25, 0.3 \). Except for \( z \lesssim 1.5 \), the error indeed remains quite low (\( \lesssim 0.5\% \)), and does not change too much with the alteration of the value of \( n \) or of \( \Omega_0^{(m)} \) in the respective cases. In particular, even for \( n \) as high as 0.1, the maximum error is only about 1.65\%, which implies that it is quite reasonable to carry out the parametric estimation relying on this fit.

5. Parametric estimation using the growth data

Let us now proceed to carry out the statistical parametric estimation using the Metropolis-Hastings algorithm for MCMC. Our interest is in the RSD observations which measure the quantity

\[
[f \sigma^{(8)}](z) = f(z) \sigma_0^{(8)} \frac{\delta^{(m)}(z)}{\delta_0^{(m)}},
\]

where \( \sigma_0^{(8)} \equiv \sigma_0^{(8)}|_{z=0} \) and \( \delta^{(m)} \equiv \delta^{(m)}|_{z=0} \). Therefore, using the parametrization (4.5) of the growth factor, and its relation to the matter density contrast, we express Eq. (5.1) in the following form (as a function of \( N \)):

\[
[f \sigma^{(8)}](N) = (1 + 2n^2) \left[ \Omega_0^{(m)}(N) \right]^{\gamma(N)} \sigma_0^{(8)} \times \exp \left[ (1 + 2n^2) \int_0^N dN \left[ \Omega_0^{(m)}(N) \right]^{\gamma(N)} \right],
\]

where \( \gamma(N) \) is given by its numerical fit (4.6) in terms of \( n \) and \( \Omega_0^{(m)} \).

Now, the statistical analysis is essentially the standard minimization of

\[
\chi^2 := \sum_{i,j} A_{obs}(z_i) - A_{th}(z_i) \cdot C^{-1}_{ij} \cdot A_{obs}(z_i) - A_{th}(z_i),
\]

where \( A_{obs}(z_i) \) denotes the observed \( \sigma^{(8)} \) at a particular redshift \( z_i \), at which the theoretical value \( A_{th} \) of \( \sigma^{(8)} \) can be obtained from Eq. (5.2), and \( C_{ij} \) is the covariance matrix for three WiggleZ data points\( \text{GOLD} \) in the refined RSD subsample, viz. the Growth GOLD dataset\( \text{GOLD} \) we are using in this paper. There are three parameters, viz. \( n \), \( \Omega_0^{(m)} \) and \( \sigma_0^{(8)} \), that are to be constrained in this analysis. An additional parameter, viz. the reduced Hubble constant \( h \), appears (and needs to be constrained as well) in the analysis with the GOLD dataset and the observational Hubble \( (H(z)) \) dataset\( \text{GOLD} \) combined.

Setting the prior ranges

\[
n \in [-0.35, 0.35], \quad \Omega_0^{(m)} \in [0.1, 0.5],
\]

\[
\sigma_0^{(8)} \in [0.6, 0.95], \quad h \in [0.55, 0.85],
\]

we carry out the analysis, with the GOLD and the GOLD\( +H(z) \) datasets, and determine the allowed parametric domains up to 3\( \sigma \). Figs.\( 4(a) \) and \( b \) show the same, for the respective cases, whereas the Table\( 1 \) shows the corresponding parametric estimates, up to 1\( \sigma \) confidence limits, as well as the minimized values of \( \chi^2 \).
Fig. 4. Upto 3σ contour levels for (a) the GOLD dataset, and (b) its combination with the observational Hubble dataset. The dashed lines on the histograms indicate the 1σ levels, whereas the solid lines on each plot denote the median best-fits of the respective parameters.
Table 1. Estimates of the parameters $\Omega_0^{(m)}$, $\sigma_0^{(8)}$, $h$ and $n$, upto 1\(\sigma\) limits, alongwith the minimized $\chi^2$, for the GOLD and GOLD$+H(z)$ datasets.

| Observational Datasets | Parametric estimates (best fit and 1\(\sigma\) limits) | $\chi^2_{\text{min}}$ |
|------------------------|--------------------------------------------------------|----------------------|
|                        | $\Omega_0^{(m)}$ | $\sigma_0^{(8)}$ | $h$   | $|n|$   |
| GOLD                   | $0.269^{+0.042}_{-0.038}$ | $0.766^{+0.041}_{-0.040}$ | –     | $< 0.074$ | 13.152 |
| GOLD$+H(z)$            | $0.296^{+0.035}_{-0.032}$ | $0.748^{+0.036}_{-0.034}$ | $0.691^{+0.022}_{-0.021}$ | $< 0.065$ | 30.662 |

The parameters are more tightly constrained by the GOLD$+H(z)$ dataset, than by the GOLD dataset, which is of course expected with the increased number of uncorrelated data points in the former. However, note that even for the GOLD$+H(z)$ dataset, the marginalized $\Omega_0^{(m)}$ shown in Table 1 is not much deviated from that for $\Lambda$CDM, such as $0.3111 \pm 0.0056$ obtained from the Planck 2018 TT,TE,EE+lowE+Lensing+BAO combined analysis.\cite{12} It is also worth pointing out here that the best fit value of the DEM coupling parameter $n$ turns out to be quite insignificant (about three orders of magnitude below unity). This is the reason why in the Table 1 we have only quoted the 1\(\sigma\) limit of $n$, which is quite small as well. All these imply only a very mild effect of such a coupling on the $\Lambda$CDM background evolution, attained in the limit $n \rightarrow 0$ of the solution (2.14).

On the whole, therefore, we infer that the background dynamics of the effective DE component, described by Eq. (2.14), is constrained to be very weak by the Growth GOLD subsample, and even more so, by the combination of the latter with the $H(z)$ data. This corroborates to not only the general indications of the cosmological observations, but also the solar system constraints on the scalar-tensor theories. More specifically, the smallness of $n$, which is the root cause of such a weak DE dynamics, tallies with the very large upper bound on the effective Brans-Dicke parameter $\omega$ which is linear in $n^{-2}$ (see Eq. (A.3) in the Appendix).\cite{32,103,106}

Nevertheless, the consequences of the DEM interaction in the cosmological perturbative spectrum do not seem to be that mild, from the amount of deviation of the derived estimate of the growth index at the present epoch, $\gamma_0 \equiv \gamma|_{z=0}$, from the corresponding predicted value $6/11 \simeq 0.545$ for $\Lambda$CDM.\cite{15} Figs. 5(a) and (b) show such a deviation in each of the plotted evolution profiles of the best fit $\gamma(z)$ and the corresponding 1\(\sigma\) margins derived using the parametric estimations with the GOLD dataset and with the GOLD$+H(z)$ dataset, respectively. Specifically, the estimates are $\gamma_0 = 0.559 \pm 0.007$ for GOLD and $\gamma_0 = 0.556 \pm 0.005$ for GOLD$+H(z)$, which imply that the predicted $\Lambda$CDM value 0.545 is not within the 1\(\sigma\) error limits in either case, even though the coupling parameter $n$ is estimated to be very small.

One may note that these estimates of $\gamma_0$ are consistent with those obtained for many dynamical DE models in the literature, the non-interacting ones as well.
as those in which the DE interacts with matter (albeit, mostly phenomenologically). However, it may cause an inherent deception if we infer a very significant change over ΛCDM just from what shown in Figs. 5(a) and (b). In particular, unlike the ΛCDM value of \( \gamma_0 \), the corresponding \( f(z) \) and \( f(z)\sigma^8(z) \) values at the present epoch do not fall beyond the 1σ confidence limit of our estimates for both the datasets, as shown in Figs. 6(a),(b) and in Figs. 7(a),(b), respectively. The contradictory aspect of \( \gamma_0 \) is actually due to the fact that it is not an independent parameter.
in our formulation, but is a redefined one, the error on which is primarily dependent on the estimates of the parameters $n$ and $\Omega^{(m)}_0$. Hence, the error-propagation method leads to a very small $1\sigma$ interval of $\gamma_0$, of the order of $10^{-3}$. On the other hand, the $f(z)$ and $f(z)\sigma^{(8)}(z)$ plots in Figs. 6(a),(b) and 7(a),(b) are obtained by using our growth factor ansatz (4.5) for the range $z \in [0, 2.5]$, upto $1\sigma$, for either dataset. The $1\sigma$ uncertainties in the respective values at the present epoch, viz. $f_0$ and $f^{(8)}_0 |_0$, turn out to be of the order of $10^{-2}$, within which the $f_0$ and $f^{(8)}_0 |_0$ values corresponding to $\Lambda$CDM get easily accommodated. This in fact confirms that the ansatz (4.5) is consistent with the $\Lambda$CDM model as well.

For a further clarification, let us recall that the fitting function (4.6) involves a linear dependence of the growth index $\gamma$ on $n^2$. However, since the estimated $n$ is as large as $O(10^{-2})$ upto the $1\sigma$ limit, we may safely argue that the dependence of $\gamma$ on $n$ alone would not be sufficient enough to account for the fall-out of the $\Lambda$CDM value beyond the estimated $1\sigma$ level of $\gamma$. It is actually the latter’s dependence on $\Omega^{(m)}_0$ which plays a major role here. If suppose this dependence is ignored, i.e. the same $\Omega^{(m)}_0$ value (say, 0.29) is assumed for all values of $n$, then for $n = 0$ ($\Lambda$CDM) the fitting gives $\gamma_0 = 0.5561$, whereas for $n = 0.01$ (say) one gets $\gamma_0 = 0.5559$. This clearly shows how crucial is the $\Omega^{(m)}_0$-dependence, in comparison to the $n$-dependence of $\gamma$.

Let us also point out that one should not get deceived by the maximum percentage error ($\sim 1.65\%$) in our fitting, as shown by the plots in Figs. 3(a) and (b). Such a deviation is due to the fiducial value of $n$ ($= 0.1$) we have assigned while obtaining such plots. The actual estimates of $n$ shown in Table 1 is far less than that value, even upto the $1\sigma$ limit, for both the datasets we have considered. If we use these estimates, then the percentage error in the fit would have a much lesser maximum value than $\sim 1.65\%$, and would be negligible at the present epoch ($z = 0$).
On the whole, therefore, we may infer that the exclusion of the ΛCDM growth index value $\gamma_0 \simeq 0.545$ from the estimated $1\sigma$ domain of $\gamma$ is not because of the error in the fitting but due mainly to the uncertainty in the parametric estimation of $\Omega_0^{(m)}$, with a supporting role played by the uncertainty in the parameter $n$ as well as its covariance with $\Omega_0^{(m)}$.

6. Conclusion
We have thus made a comprehensive study of the evolution of the cosmological perturbations in a dark energy-matter (DEM) interacting scenario that emerges from a class of scalar-tensor theories, motivated from the perspective of a wide range of modified gravity theories. While giving a general account on its formulation, we have emphasized on the plausible forms of the potential term for the scalar field, that would lead to exact solutions of the cosmological equations in the standard setup. Resorting to one such form of the potential, which is simply a mass term in the (original) Jordan frame, or equivalently an exponential term in the Einstein frame, we have discussed the merits of the cosmological solution one obtains thereby. One advantage is that the DEM coupling strength is measured by just the one parameter $n$, which is inversely related to square root of the Brans-Dicke parameter $\omega$. The dynamics of the DE, induced by the scalar field $\varphi$ which the solution describes, is however expected to be quite mild, in view of the strong observational favour on ΛCDM (that is attained by switching off $n$). Nevertheless, in spite of the mildness of the DE dynamics, the DEM interaction may leave some imprint on the perturbative spectrum, while the background cosmology is governed by the abovementioned solution. This is what we have contemplated on in this paper.

Choosing to carry out the perturbative analysis in the well-known Newtonian gauge, we have derived the evolution equations for the matter density contrast $\delta^{(m)}(z)$, the growth factor $f(z)$ and the scalar field perturbation $\delta \varphi$. These equations are highly coupled though, and an effective decoupling could be achieved only at the deep sub-horizon scales, on which we have kept our attention. In fact, remember that even after making a specific gauge choice (to eliminate the unreal perturbations) one is left to choose the size of the horizon, as the perturbations behave differently depending on that size. Now, the deep sub-horizon regime is of relevance in the sense that $\delta^{(m)}(z)$ is nearly scale-independent therein, so that one can make the most of out of the RSD observations for the LSS formed well inside the radius of the horizon. Moreover, in this regime, the perturbation in the $\varphi$-induced DE, i.e. $\delta \varphi(z)$, is highly suppressed compared to the total matter density perturbation $\delta \rho^{(m)}(z)$, and hence the onus is completely on $\delta^{(m)}(z)$. Nevertheless, the DEM interaction changes the scenario a bit, as $\delta \varphi(z)$ makes an additional, albeit small, contribution to $\delta^{(m)}(z)$, and hence to the growth factor $f(z)$. To be more precise, the perturbation $d \varphi(z)$ is oscillatory at the sub-horizon scales. However, the oscillations are about a mean value is proportional to the DEM coupling parameter $n$. So, for $n \neq 0$ a contribution would come from the time-averaged $\delta \varphi(z)$. 
Now, the point to be reckoned here is that $n$ is expected to be quite small, as otherwise the background DE dynamics would not be considered as mild. Therefore, to what extent would the effect of the overall $n$-dependence of $\delta^{(m)}(z)$ be of significance (if at all), compared to the latter’s scale-dependence, which is anyway neglected in the deep sub-horizon regime? We have addressed this issue by simultaneously solving the evolution equations for $\delta^{(m)}(z)$ and $\delta \varphi(z)$ numerically, for some fiducial settings of $n$ and $\hat{\lambda}$, the perturbation scale. As it turned out, the deviation in the $\delta^{(m)}$ evolution profile is quite less for a single order-of-magnitude change in $\hat{\lambda}$ with $n$ kept fixed, than for the exact converse. Although $n$ and $\hat{\lambda}$ are characteristically different, this gives at least a qualitative idea that the coupling $(n)$-dependence is quite overwhelming than the scale $(\hat{\lambda})$-dependence of $\delta^{(m)}$. In other words, the scale-independence of $\delta^{(m)}$ in the deep sub-horizon regime does not necessarily mean a negligible effect of the DE interaction.

While the deep sub-horizon limit allows a scale-independent formulation of $\delta^{(m)}(z)$, one finds the desired decoupling of the evolution equations. The equation for the growth factor $f(z)$ can in principle be solved by using a suitable parametrization ansatz. As $\delta^{(m)}(z)$ essentially depends on the matter density parameter $\Omega^m(z)$, and so does $f(z)$, it is quite justified to consider the parametrization $f(z) = [\Omega^m(z)]^{\gamma(z)}$, where $\gamma(z)$ is referred to as the growth index. Such a parametrization is well-known and widely used in the literature, as it provides a convenient way of discriminating various DE models and checking their viability. Nevertheless, it has the drawback of not allowing $f(z)$ to exceed unity, and as such incompatible with the configuration we have in presence of the DE interaction. So we have modified this parametrization to $f(z) = (1 + 2n^2)[\Omega^m(z)]^{\gamma(z)}$, and looked to carry on with the formulation in terms of the growth index $\gamma(z)$. However, contrary to the common practice, we have refrained from resorting to any Taylor expansion of $\gamma(z)$, as it usually remains valid only for small range of redshifts or does not take into account the explicit dependence on the coupling parameter $n$. Instead, we have taken the direct approach, i.e. to substitute the parametrized form of $f(z)$ in its evolution equation and solve the latter to obtain a numerical fitting function for $\gamma(z)$ in terms of the parameters $\Omega^m_0 = \Omega^m(z)\big|_{z=0}$ and $n$. The fitting function is found to be quite accurate even at redshifts $z \simeq 100$ or more. Hence, both the issues with the Taylor expanded forms are alleviated.

The allowed parametric spaces are finally determined using the Metropolis-Hastings algorithm for MCMC, for two chosen datasets, viz. the RSD GOLD subsample and that combined with the observational $H(z)$ data. The combined dataset constrains the parameters more tightly, however, even for this dataset the marginalized $\Omega^m_0$ shows only a mild deviation from that for $\Lambda$CDM, as quoted in the Planck 2018 results for example. The estimated $n$, up to $1\sigma$, is very small (0.074 and 0.065 for the GOLD and Gold+$H(z)$ respectively). So the DEM coupling, or in fact the DE dynamics, is very weak. Notwithstanding the general agreement with the cosmological observations, this corroborates to the astrophysical constraints on the scalar-tensor theories. However, the overall constraints on the parameters (includ-
Imprints of interacting dark energy on...

In the presence of interacting dark energy (DE) and matter, the reduced Hubble constant $h$, and the growth index $\gamma_0$ do not indicate that mild an effect of the DE interaction on the growth of the matter density contrast $\delta_m(z)$. This is noticed from the amount of deviation of the growth index at the present epoch $\gamma_0$, estimated using such constraints, from the corresponding predicted value for $\Lambda$CDM. Nevertheless, such a deviation does not make anything clear on the individual contributions to $\delta_m(z)$, i.e. the one due to the drag force on the latter, as the coupling makes the background matter density drift from its usual dust-like form, and the other due to the fluctuations of DE perturbation $\delta\phi(z)$ about a non-zero mean value proportional to the coupling parameter $n$. A more in-depth study is required to assert this. Such a study is presently being done in a subsequent work, which we hope to report soon.

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Appendix A. DEM interaction in a class of Scalar-Tensor theories

The Jordan frame action for scalar-tensor theories is generally expressed, in presence of matter fields $\{\psi\}$, as

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ f(\phi) R(g_{\mu\nu}) - w(\phi) g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - V(\phi) \right]$$

$$+ \int d^4x \sqrt{-\tilde{g}} \mathcal{L}^{(m)}(g_{\mu\nu}, \psi), \quad (A.1)$$

where $g$ is the determinant of the metric $g_{\mu\nu}$ of the Jordan frame, $f(\phi)$ and $w(\phi)$ are two differentiable coupling functions of the scalar field $\phi$, with $V(\phi)$ being the corresponding (self-interacting) potential, and $\mathcal{L}^{(m)}(g_{\mu\nu}, \psi)$ is the matter Lagrangian density. Note that, a non-trivial $f(\phi)$ implies a non-minimal coupling of $\phi$ and the Ricci curvature scalar $R$, to which the matter fields $\{\psi\}$ are minimally coupled.

Moreover, one can always redefine the field $\phi$ in a way that $w$ is a constant.

Consider now a class of such theories, characterized by a quadratic form of the non-minimal coupling function, i.e. $f(\phi) = \phi^2$. This has equivalent formulations in a wide range of modified gravity theories, and also of the Brans-Dicke (BD) theory in absence of the potential $V(\phi)$. In fact, it is easy to see that for $f(\phi) = \phi^2$, a field re-definition $\phi^2 = \tilde{\phi}$ renders the free Jordan frame action to the BD action expressed in terms of $\tilde{\phi}$. So the constant $w$ is nothing but the BD parameter.

Under the stipulation

$$\phi_0 \equiv \phi(t_0) = \kappa^{-1}, \quad \text{where} \quad \kappa^2 = 8\pi G_N,$$  \quad (A.2)

one has the effective (running) gravitational coupling factor equal to the Newton’s constant $G_N$ at the present epoch $(t = t_0)$. On the other hand, a conformal trans-
formation $g_{\mu \nu} \rightarrow \hat{g}_{\mu \nu} = \kappa^{2} \phi^{2} g_{\mu \nu}$, followed by another field re-definition
\[
\phi := \kappa^{-1} e^{n \kappa \varphi}, \quad \text{with} \quad n = (6 + w)^{-1/2},
\]
(A.3)
reduces the Jordan frame action to the Einstein frame action in the form
\[
\hat{S} = \int d^{4}x \sqrt{-\hat{g}} \left[ \frac{\hat{R}(\hat{g}_{\mu \nu})}{2\kappa^{2}} - \frac{1}{2} \hat{g}^{\mu \nu} \partial_{\mu} \varphi \partial_{\nu} \varphi - U(\varphi) + \hat{L}^{(m)}(\hat{g}_{\mu \nu}, \varphi, \psi) \right],
\]
(A.4)
where $\hat{g}$ is the determinant of the corresponding (transformed) metric $\hat{g}_{\mu \nu}$, and
\[
U(\varphi) := \frac{e^{-4\kappa n \varphi}}{2} V(\phi(\varphi)),
\]
(A.5)
\[
\hat{L}^{(m)}(\hat{g}_{\mu \nu}, \varphi, \psi) := e^{-4\kappa n \varphi} \mathcal{L}^{(m)}(\hat{g}_{\mu \nu}, \phi(\varphi), \psi),
\]
(A.6)
are, respectively, the scalar field potential and the effective matter Lagrangian density in the Einstein frame.

Therefore, with the usual definitions of the energy-momentum tensors corresponding to matter and the scalar field $\varphi$, viz.
\[
\hat{T}_{\mu \nu}^{(m)} := \frac{2}{\sqrt{-\hat{g}}} \frac{\delta}{\delta \hat{g}^{\mu \nu}} \left[ \sqrt{-\hat{g}} \hat{L}^{(m)} \right],
\]
(A.7)
\[
\hat{T}_{\mu \nu}^{(\varphi)} := \frac{2}{\sqrt{-\hat{g}}} \frac{\delta}{\delta \hat{g}^{\mu \nu}} \left[ \sqrt{-\hat{g}} \left\{ \frac{1}{2} \hat{g}^{\mu \nu} \partial_{\mu} \varphi \partial_{\nu} \varphi + U(\varphi) \right\} \right]
\]
\[
= \partial_{\mu} \varphi \partial_{\nu} \varphi - \frac{1}{2} \hat{g}_{\mu \nu} \left[ \hat{g}^{\alpha \beta} \partial_{\alpha} \varphi \partial_{\beta} \varphi + 2U(\varphi) \right],
\]
(A.8)
the variation of the action (A.4) leads to the relation
\[
\hat{\nabla}^{\mu} \hat{T}_{\mu \nu}^{(m)} = - \hat{\nabla}^{\mu} \hat{T}_{\mu \nu}^{(\varphi)} = \mathcal{Q}_{\nu},
\]
(A.9)
where
\[
\mathcal{Q}_{\nu} = - \kappa n \hat{T}_{\mu \nu}^{(m)} \partial_{\nu} \varphi,
\]
(A.10)
and $\hat{\nabla}^{\mu} = \hat{g}^{\mu \nu} \hat{\nabla}_{\nu}$ and $\hat{T}^{(m)} = \hat{g}^{\mu \nu} \hat{T}_{\mu \nu}^{(m)}$ are, respectively, the covariant derivative and the trace of the matter energy-momentum tensor in the Einstein frame.

Eq. (A.9) shows that $\hat{T}^{(m)}_{\mu \nu}$ and $\hat{T}^{(\varphi)}_{\mu \nu}$ are not conserved, i.e. there exists an interaction between matter and the field $\varphi$, the extent of which is determined by the vector $\mathcal{Q}_{\nu}$. This is nothing but the consequence of the $\varphi$-dependence of the matter Lagrangian density $\hat{L}^{(m)}$, defined by Eq. (A.6), in the Einstein frame. Hence, $\hat{L}^{(m)}$ is strictly speaking, an interaction Lagrangian density. In the standard cosmological framework, such an interaction can inevitably be looked upon as the DE-matter (DEM) interaction, under the legitimate assumption that the field $\varphi$ sources the DE component of the universe.
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